SCALAR CURVATURE RIGIDITY WITH A VOLUME CONSTRAINT

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Abstract. Motivated by Brendle-Marques-Neves’ counterexample to the Min-Oo’s conjecture, we prove a volume constrained scalar curvature rigidity theorem which applies to the hemisphere.

1. Introduction

Recently, Brendle, Marques and Neves \cite{BrendleMarquesNeves} have solved the long-standing Min-Oo’s conjecture \cite{MinOo} by constructing a counterexample.

**Theorem 1.1** (Brendle, Marques and Neves \cite{BrendleMarquesNeves}). Suppose \( n \geq 3 \). Let \( \bar{g} \) be the standard metric on the hemisphere \( S_+^n \). There exists a smooth metric \( g \) on \( S_+^n \), which can be made to be arbitrarily close to \( \bar{g} \) in the \( C^\infty \)-topology, satisfying

- the scalar curvature of \( g \) is at least that of \( \bar{g} \) at each point in \( S_+^n \)
- \( g \) and \( \bar{g} \) agree in a neighborhood of \( \partial S_+^n \),

but \( g \) is not isometric to \( \bar{g} \).

In this paper, we observe that if the metric \( g \) in Theorem 1.1 is assumed to satisfy an additional volume constraint, then it must be isometric to \( \bar{g} \). Precisely, we have

**Theorem 1.2.** Let \( \bar{g} \) be the standard metric on \( S_+^n \). Let \( g \) be another metric on \( S_+^n \) with the properties

- \( R(g) \geq R(\bar{g}) \) in \( S_+^n \)
- \( H(g) \geq H(\bar{g}) \) on \( \partial S_+^n \)
- \( g \) and \( \bar{g} \) induce the same metric on \( \partial S_+^n \)

where \( R(g) \), \( R(\bar{g}) \) are the scalar curvature of \( g \), \( \bar{g} \), and \( H(g) \), \( H(\bar{g}) \) are the mean curvature of \( \Sigma \) in \( (\Omega, g) \), \( (\Omega, \bar{g}) \). Suppose in addition

\[ V(g) \geq V(\bar{g}) , \]

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where $V(g), V(\bar{g})$ are the volume of $g, \bar{g}$. If $||g - \bar{g}||_{C^2(\Omega)}$ is sufficiently small, then there is a diffeomorphism $\varphi : \Omega \to \Omega$ with $\varphi|_\Sigma = \text{id}$, the identify map on $\Sigma$, such that $\varphi^*(g) = \bar{g}$.

Theorem 1.2 is indeed a special case of a more general result:

**Theorem 1.3.** Let $(\Omega, \bar{g})$ be an $n$-dimensional compact Riemannian manifold, of constant sectional curvature 1, with smooth boundary $\Sigma$. Suppose $\Pi + H\bar{\gamma} \geq 0$ (i.e. $\Pi + H\bar{\gamma}$ is positive semi-definite), where $\bar{\gamma}$ is the induced metric on $\Sigma$ and $\Pi, H$ are the second fundamental form, the mean curvature of $\Sigma$ in $(\Omega, \bar{g})$. Suppose the first nonzero Neumann eigenvalue $\mu$ of $(\Omega, \bar{g})$ satisfies $\mu > n - \frac{2}{n+1}$.

Consider a nearby metric $g$ on $\Omega$ with the properties

- $R(g) \geq n(n - 1)$ where $R(g)$ is the scalar curvature of $g$
- $H(g) \geq \bar{H}$ where $H(g)$ is the mean curvature of $\Sigma$ in $(\Omega, g)$
- $g$ and $\bar{g}$ induce the same metric on $\Sigma$
- $V(g) \geq V(\bar{g})$ where $V(g), V(\bar{g})$ are the volumes of $g, \bar{g}$.

If $||g - \bar{g}||_{C^2(\Omega)}$ is sufficiently small, then there is a diffeomorphism $\varphi$ on $\Omega$ with $\varphi|_\Sigma = \text{id}$, such that $\varphi^*(g) = \bar{g}$.

As a by-product of the method used to derive Theorem 1.3, we obtain a volume estimate for metrics close to the Euclidean metric in terms of the scalar curvature.

**Theorem 1.4.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\Sigma$. Suppose $\Pi + H\bar{\gamma} > 0$ (i.e. $\Pi + H\bar{\gamma}$ is positive definite), where $\Pi, H$ are the second fundamental form, the mean curvature of $\Sigma$ in $\mathbb{R}^n$ and $\bar{\gamma}$ is the metric on $\Sigma$ induced from the Euclidean metric $\bar{g}$. Let $g$ be another metric on $\Omega$ satisfying

- $H(g) \geq \bar{H}$, where $H(g)$ is the mean curvature of $\Sigma$ in $(\Omega, g)$
- $g$ and $\bar{g}$ induce the same metric on $\Sigma$.

Given any point $a \in \mathbb{R}^n$, there exists a constant $\Lambda > \frac{\max_{q \in \Omega}||g - a||^2}{4(n-1)}$, depending only on $\Omega$ and $a$, such that if $||g - \bar{g}||_{C^3(\Omega)}$ is sufficiently small, then

$$ (1.1) \quad V(g) - V(\bar{g}) \geq \int_{\Omega} R(g) \Phi \, d\text{vol}_{\bar{g}} $$

where $\Phi(x) = -\frac{1}{4(n-1)}|x - a|^2 + \Lambda > 0$ on $\bar{\Omega}$.

Theorem 1.4 may be compared to a previous theorem of Bartnik [2], which estimates the total mass [1] of an asymptotically flat metric that is a perturbation of the Euclidean metric.
Theorem 1.5 (Bartnik [2]). Let \( g \) be an asymptotically flat metric on \( \mathbb{R}^3 \). If \( g \) is sufficiently close to the Euclidean metric \( \bar{g} \) (in certain weighted Sobolev space), then
\[
16\pi m(g) \geq \int_{\mathbb{R}^3} R(g) \, d\text{vol}_g
\]
where \( m(g) \) is the total mass of \( g \).

Our proofs of Theorems 1.2 - 1.4 follow a recent perturbation analysis of Brendle and Marques in [5], where they established a scalar curvature rigidity theorem for “small” geodesic balls in \( S^n \).

Theorem 1.6 (Brendle and Marques [5]). Let \( \Omega \subset S^n \) be a geodesic ball of radius \( \delta \). Suppose
\[
\cos \delta \geq \frac{2}{\sqrt{n+3}}
\]
Let \( \bar{g} \) be the standard metric on \( S^n \). Let \( g \) be another metric on \( \Omega \) with the properties
\begin{itemize}
  \item \( R(g) \geq n(n-1) \) at each point in \( \Omega \)
  \item \( H(g) \geq \bar{H} \) at each point on \( \partial \Omega \)
  \item \( g \) and \( \bar{g} \) induce the same metric on \( \partial \Omega \)
\end{itemize}
where \( R(g) \) is the scalar curvature of \( g \), and \( H(g) \), \( \bar{H} \) are the mean curvature of \( \partial \Omega \) in \( (\Omega, g) \), \( (\Omega, \bar{g}) \). If \( g - \bar{g} \) is sufficiently small in the \( C^2 \)-norm, then \( \varphi^*(g) = \bar{g} \) for some diffeomorphism \( \varphi : \Omega \to \Omega \) such that \( \varphi|_{\partial \Omega} = \text{id} \).

In Theorem 1.6, the condition (1.3) is equivalently to
\[
H \geq 4 \tan \delta
\]
because the mean curvature \( \bar{H} \) of \( \partial B(\delta) \) is \( (n-1) \frac{\cos \delta}{\sin \delta} \). As another application of the formulas in Section 2, we obtain a generalization of Theorem 1.6 to convex domains in \( S^n \).

Theorem 1.7. Let \( \Omega \subset S^n \) be a smooth domain contained in a geodesic ball \( B \) of radius less than \( \frac{\pi}{2} \). Let \( \bar{g} \) be the standard metric on \( S^n \). Let \( \bar{H} \) be the second fundamental form, the mean curvature of \( \partial \Omega \) in \( (\Omega, \bar{g}) \). Suppose \( \Omega \) is convex, i.e. \( \bar{H} \geq 0 \). At \( \partial \Omega \), suppose
\[
\bar{H} \geq 4 \tan r
\]
where \( r \) is the \( \bar{g} \)-distance to the center of \( B \). Then the conclusion of Theorem 1.6 holds on \( \Omega \).
Theorem 1.7 is an immediate consequence of Theorem 5.1 in Section 5. In a simpler setting, where the background metric $\bar{g}$ is a flat metric, we have

**Theorem 1.8.** Let $\Omega$ be a compact manifold with smooth boundary $\Sigma$. Suppose there is a flat metric $\bar{g}$ on $\Omega$ such that $\bar{H} + \bar{\gamma} \geq 0$ (i.e. $\bar{\Pi} + \bar{H} \bar{\gamma}$ is positive semi-definite), where $\bar{\Pi}$, $\bar{H}$ are the second fundamental form, the mean curvature of $\Sigma$, and $\bar{\gamma}$ is the induced metric on $\Sigma$. Given another metric $g$ on $\Omega$ such that

- $R(g) \geq 0$ on $\Omega$
- $H(g) \geq \bar{H}$ at $\Sigma$
- $g$ and $\bar{g}$ induce the same metric on $\Sigma$,

if $||g - \bar{g}||_{C^2(\Omega)}$ is sufficiently small, then $\varphi^*(g) = \bar{g}$ for some diffeomorphism $\varphi : \Omega \rightarrow \Omega$ with $\varphi|_{\Sigma} = \text{id}$.

Similar calculation at the infinitesimal level provides examples of compact 3-manifolds of nonnegative scalar curvature whose boundary surface does not have positive Gaussian curvature but still has positive Brown-York mass [7, 8]. We include this in the end of the paper to compare with known results in [17].

**Theorem 1.9.** Let $\Sigma \subset \mathbb{R}^n$ be a connected, closed hypersurface satisfying $\bar{\Pi} + \bar{H} \bar{\gamma} \geq 0$, where $\bar{\Pi}$, $\bar{H}$ are the second fundamental form, the mean curvature of $\Sigma$, and $\bar{\gamma}$ is the induced metric on $\Sigma$. Let $\Omega$ be the domain enclosed by $\Sigma$ in $\mathbb{R}^n$. Let $h$ be any nontrivial $(0,2)$ symmetric tensor on $\Omega$ satisfying

$$(1.6) \quad \text{div}_g h = 0, \quad \text{tr}_g h = 0, \quad h |_{T\Sigma} = 0.$$ 

Let $\{g(t)\}_{|t| < \epsilon}$ be a 1-parameter family of metrics on $\Omega$ satisfying

$$(1.7) \quad g(0) = \bar{g}, \quad g'(0) = h, \quad R(g(t)) \geq 0, \quad g(t)|_{T\Sigma} = \bar{g}|_{T\Sigma}.$$ 

Then

$$(1.8) \quad \int_{\Sigma} \bar{H} d\sigma_{\bar{g}} > \int_{\Sigma} H(g(t)) d\sigma_g$$

for small $t \neq 0$, where $H(g(t))$ is the mean curvature of $\Sigma$ in $(\Omega, g(t))$.

This paper is organized as follows. In Section 2, we derive a basic formula concerning a perturbed metric (Theorem 2.1), which corresponds to [5, Theorem 10] of Brendle and Marques. In Section 3, we prove Theorem 1.3, which implies Theorem 1.2. In Section 4, we give a proof of Theorem 1.4. In Section 5, we consider other applications of the formulas in Section 2 and prove Theorem 1.7 - 1.9.
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2. Basic formulas for a perturbed metric

Let $\Omega$ be an $n$-dimensional, smooth, compact manifold with boundary $\Sigma$. Let $\bar{g}$ be a fixed smooth Riemannian metric on $\Omega$. Given a tensor $\eta$, let $\|\eta\|$ denote the length of $\eta$ measured with respect to $\bar{g}$. Denote the covariant derivative with respect to $\bar{g}$ by $\nabla$. Indices of tensors are raised by $\bar{g}$. Let $\bar{R}_{ijkl}$ denote the curvature tensor of $\bar{g}$ such that if $\bar{g}$ has constant sectional curvature $\kappa$, then $\bar{R}_{ijkl} = \kappa (g_{ij} g_{kl} - g_{il} g_{kj})$. Consider a nearby Riemannian metric $g = \bar{g} + h$ where $h$ is a symmetric $(0,2)$ tensor with $|h|$ very small, say $|h| \leq \frac{1}{2}$.

The following pointwise estimates of the scalar curvature of $g$ and the mean curvature of $\Sigma$ were derived by Brendle and Marques in [5].

**Proposition 2.1** (Brendle and Marques [5]). The scalar curvatures $R(g)$, $R(\bar{g})$ of the metrics $g$, $\bar{g}$ satisfy

$$
\left| R(g) - R(\bar{g}) + \langle \text{Ric}(\bar{g}), h \rangle - \langle \text{Ric}(\bar{g}), h^2 \rangle \right|
+ \frac{1}{4} |\nabla h|^2 - \frac{1}{2} |\bar{g}^{ij} g^{kl} \nabla_i h_{kp} \nabla_l h_{jq} + \frac{1}{4} |\nabla (\text{tr}_g h)|^2
+ \nabla_i [g^{ik} g^{jl} (\nabla_k h_{jl} - \nabla_l h_{jk})] |$$

$$\leq C (|h| |\nabla h|^2 + |h|^3)$$

where $\text{Ric}(\bar{g})$ is the Ricci curvature of $\bar{g}$, $h^2$ is the $\bar{g}$-square of $h$, i.e. $(h^2)_{ik} = \bar{g}^{ij} h_{ij} h_{kl}$, $\langle \cdot, \cdot \rangle$ is taken with respect to $\bar{g}$, and $C$ is a positive constant depending only on $n$.

**Remark 2.1.** If the background metric $\bar{g}$ is Ricci flat, i.e $\bar{R}_{ik} = 0$, then there will be no $|h|^3$ term in the above estimate. That is because

$$R(g) = g^{ik} \bar{R}_{ik} - g^{ik} g^{lj} (\nabla_i h_{jl} - \nabla_l h_{ij}) + g^{ik} g^{jl} g^{pq} (\Gamma^q_{il} \Gamma^p_{jk} - \Gamma^q_{jl} \Gamma^p_{ik}),$$

where each term on the right, except $g^{ik} \bar{R}_{ik}$, involves derivatives of $h$.

**Proposition 2.2** (Brendle and Marques [5]). Assume that $g$ and $\bar{g}$ induce the same metric on $\Sigma$, i.e. $h|_{T \Sigma} = 0$ where $T \Sigma$ is the tangent bundle of $\Sigma$. Then the mean curvatures $H(g)$, $H(\bar{g})$ of $\Sigma$ in $(\Omega, g)$,
(Ω, ̃g), each with respect to the outward normals, satisfy

\[
2 [H(g) - H(\bar{g})] - \left( h(\nabla, \nabla) - \frac{1}{4} h(\nabla, \nabla)^2 + \sum_{\alpha=1}^{n-1} h(e_\alpha, \nabla)^2 \right) H(\bar{g}) \\
+ \left( 1 - \frac{1}{2} h(\nabla, \nabla) \right) \sum_{\alpha=1}^{n-1} \left[ 2 \nabla_{e_\alpha} h(e_\alpha, \nabla) - \nabla_{e_\alpha} h(e_\alpha, e_\alpha) \right]
\]

\[
\leq C (|h|^2 |\nabla h| + |h|^3)
\]

where \( \{e_\alpha \mid 1 \leq \alpha \leq n - 1\} \) is a local orthonormal frame on \( \Sigma \), \( \nabla \) is the \( \bar{g} \)-unit outward normal vector to \( \Sigma \), and \( C \) is a positive constant depending only on \( n \).

To derive the main formula (2.23) in this section, we let

\[
(2.1) \quad DR_\bar{g}(h) = -\Delta_\bar{g}(\text{tr}_g h) + \text{div}_\bar{g}\text{div}_\bar{g} h - \langle \text{Ric}(\bar{g}), h \rangle
\]

be the linearization of the scalar curvature at \( \bar{g} \) along \( h \). Here “\( \Delta_\bar{g} \), \( \text{div}_\bar{g} \)” denote the Laplacian, the divergence with respect to \( \bar{g} \).

**Lemma 2.1.** With the same notations in Proposition 2.1, assume in addition \( \text{div}_g h = 0 \), then

\[
R(g) - R(\bar{g}) = DR_\bar{g}(h) - \frac{1}{2} DR_\bar{g}(h^2) + \langle h, \nabla^2 \text{tr}_g h \rangle - \frac{1}{4} \left( |
abla h|^2 + |\nabla (\text{tr}_g h)|^2 \right)
\]

\[
+ \frac{1}{2} h^{ij} h^{kl} R_{ijkl} + E(h) + \nabla_i (E^i(h))
\]

where \( E(h) \) is a function and \( E_1(h) \) is a vector field on \( \Omega \) satisfying

\[
|E(h)| \leq C (|h| |
abla h|^2 + |h|^3), \quad |E_1(h)| \leq C |h|^2 |\nabla h|
\]

for a positive constant \( C \) depending only on \( n \).

**Proof.** First note that

\[
(2.2) \quad - \nabla_i \left[ \bar{g}^{ik} \bar{g}^{jl} (\nabla_k h_{jl} - \nabla_l h_{jk}) \right] - \langle \text{Ric}(\bar{g}), h \rangle = DR_\bar{g}(h).
\]

Suppose \( \bar{g}^{ik} = \bar{g}^{ik} + \tau^{ik} \). Then \( \tau^{ik} = -h^{ik} + E_2^{ik}(h) \) where \( h^{ik} = \bar{g}^{ij} h_{jl} \bar{g}^{ik} \) and \( |E_2(h)| \leq C |h|^2 \). Hence,

\[
\bar{g}^{ik} \bar{g}^{jl} - \bar{g}^{ik} \bar{g}^{jl} = -h^{ik} h^{jl} - \bar{g}^{ik} \bar{g}^{jl} + E_3^{ikjl}(h)
\]

where \( |E_3(h)| \leq C |h|^2 \). Therefore,

\[
(2.3) \quad - \nabla_i \left[ (\bar{g}^{ik} \bar{g}^{jl} - \bar{g}^{ik} \bar{g}^{jl}) (\nabla_k h_{jl} - \nabla_l h_{jk}) \right]
\]

\[
= \nabla_i \left[ (\bar{g}^{ik} h^{jl} + \bar{g}^{jl} h^{ik} - E_3^{ikjl}(h)) (\nabla_k h_{jl} - \nabla_l h_{jk}) \right]
\]

\[
= \frac{1}{2} \Delta_\bar{g} |h|^2 + \langle h, \nabla^2 \text{tr}_g(h) \rangle \bar{g} - \text{div}_\bar{g}\text{div}_\bar{g} h^2 - \nabla_i \left( E_3^{ikjl}(\nabla_k h_{jl} - \nabla_l h_{jk}) \right).
\]
Applying the Ricci identity, one has

\begin{equation}
\frac{1}{2} \bar{g}^{ij} \bar{g}^{kl} \bar{g}^{pq} \nabla_i h_{kp} \nabla_l h_{jq} = \frac{1}{2} \text{div}_\bar{g} \text{div}_\bar{g} (h^2) - \frac{1}{2} \langle \text{Ric}(\bar{g}), h^2 \rangle + \frac{1}{2} h^{ij} h^{kl} R_{ikjl}.
\end{equation}

The lemma follows from Proposition 2.1, (2.2), (2.3) and (2.4).

Next, let \( DH_\bar{g}(h) \) denote the linearization of the mean curvature at \( \bar{g} \) along \( h \). Proposition 2.2 implies

\begin{equation}
DH_\bar{g}(h) = \frac{1}{2} \left[ h(\bar{\nu}, \bar{\nu}) H(\bar{g}) - \sum_{\alpha=1}^{n-1} \left( 2 \nabla_{e_\alpha} h(e_\alpha, \bar{\nu}) - \nabla_{\bar{\nu}} h(e_\alpha, e_\alpha) \right) \right].
\end{equation}

For later use, we note the following equivalent expression of \( DH_\bar{g}(h) \) (see [13, (34)] for instance)

\begin{equation}
DH_\bar{g}(h) = \frac{1}{2} \left\{ [d(\text{tr}_\bar{g} h) - \text{div}_\bar{g} h](\bar{\nu}) - \text{div}_\Sigma X \right\},
\end{equation}

where \( X \) is the vector field on \( \Sigma \) dual to the 1-form \( h(\bar{\nu}, \cdot) |_{T\Sigma} \).

Let \( DR_\bar{g}^*(\cdot) \) denote the formal \( L^2 \bar{g} \)-adjoint of \( DR_\bar{g}(\cdot) \), i.e.

\begin{equation}
DR_\bar{g}^*(\lambda) = -(\Delta_\bar{g} \lambda) \bar{g} + \nabla^2_\bar{g} \lambda - \lambda \text{Ric}(\bar{g})
\end{equation}

where \( \lambda \) is a function and \( \nabla^2_\bar{g} \lambda \) denotes the Hessian of \( \lambda \) with respect to \( \bar{g} \). The content of the following lemma had been used in [13].

**Lemma 2.2.** Let \( p \) be any smooth \( (0, 2) \) symmetric tensor on \( \Omega \), then

\begin{equation}
\int_\Omega DR_\bar{g}(p) \lambda \ d\text{vol}_\bar{g} = \int_\Omega \langle DR_\bar{g}^*(\lambda), p \rangle \ d\text{vol}_\bar{g} - \int_\Sigma 2DH_\bar{g}(p) \lambda \ d\sigma_\bar{g}
+ \int_\Sigma \lambda_{\bar{\nu}}(\text{tr}_\bar{g}(p) - p(\bar{\nu}, \bar{\nu})) \ d\sigma_\bar{g}
\end{equation}

where \( \lambda_{\bar{\nu}} = \partial_{\bar{\nu}} \lambda \) denotes the directional derivative of \( \lambda \) along \( \bar{\nu} \).
Proof. Let \( Y \) be the vector field on \( \Sigma \) dual to the 1-form \( p(\nabla, \cdot)|_{T\Sigma} \). Integrating by parts, one has

\[
(2.9) \quad \int_{\Omega} DR_{\tilde{g}}(p) \lambda \, d\text{vol}_{\tilde{g}} = \int_{\Omega} \langle DR_{\tilde{g}}^*(\lambda), p \rangle \, d\text{vol}_{\tilde{g}}
\]

\[
= \int_{\Sigma} \lambda \partial_{\lambda} \langle tr_{\tilde{g}}p \rangle + \langle tr_{\tilde{g}}p \partial_{\lambda} \rangle + \lambda \partial_{\lambda} \langle \nabla_{\Sigma} \lambda \rangle - \lambda \partial_{\lambda} \langle \nabla \tilde{g} \rangle \, d\sigma_{\tilde{g}}
\]

\[
= \int_{\Sigma} \lambda \partial_{\lambda} \langle tr_{\tilde{g}}p \rangle + \langle \nabla_{\Sigma} \langle tr_{\tilde{g}}p \rangle \rangle - \langle Y, \nabla_{\Sigma} \lambda \rangle \, d\sigma_{\tilde{g}} + \int_{\Sigma} \lambda \partial_{\lambda} \langle \nabla_{\Sigma} \lambda \rangle \, d\sigma_{\tilde{g}} + \int_{\Sigma} \lambda \partial_{\lambda} \langle \nabla \tilde{g} \rangle \, d\sigma_{\tilde{g}}
\]

where \( \nabla_{\Sigma}(\cdot) \) denotes the gradient on \( \Sigma \) with respect to the induced metric. From this and (2.6) the Lemma follows. \( \square \)

Using Lemma 2.2, we can estimate \( \int_{\Omega} [R(g) - R(\tilde{g})] \lambda \, d\text{vol}_{\tilde{g}} \).

**Proposition 2.3.** Suppose \( g \) and \( \tilde{g} \) induce the same metric on \( \Sigma \) and \( h \) satisfies \( \text{div}_{\tilde{g}} h = 0 \). Given any \( C^2 \) function \( \lambda \) on \( \Omega \), one has

\[
\int_{\Omega} [R(g) - R(\tilde{g})] \lambda \, d\text{vol}_{\tilde{g}} = \int_{\Omega} \langle h, DR_{\tilde{g}}^*(\lambda) \rangle \, d\text{vol}_{\tilde{g}} - \frac{1}{2} \int_{\Omega} \langle h^2, DR_{\tilde{g}}^*(\lambda) \rangle \, d\text{vol}_{\tilde{g}}
\]

\[
+ \int_{\Omega} \left[ \langle tr_{\tilde{g}} h \rangle \langle h, \nabla_{\tilde{g}}^2 \lambda \rangle + \frac{1}{2} h_{ij} h_{kl} R_{ijkl} \lambda - \frac{1}{4} (|\nabla h|^2 + |\nabla (tr_{\tilde{g}} h)|^2) \lambda \right] \, d\text{vol}_{\tilde{g}}
\]

\[
+ \int_{\Sigma} \left[ -\left( h_{nn} \right)^2 - \frac{1}{2} |X|^2 \right] \lambda_{nn} \, d\sigma_{\tilde{g}} - \int_{\Sigma} h_{nn} \langle X, \nabla_{\Sigma} \lambda \rangle \, d\sigma_{\tilde{g}}
\]

\[
+ \int_{\Sigma} \left[ -\frac{1}{2} h_{nn}^2 \|\tilde{g}\| - \frac{1}{2} \Pi(X, X) - \frac{3}{2} |X|^2 \bar{H} \right] \lambda \, d\sigma_{\tilde{g}} - \int_{\Sigma} (2 - 2tr_{\tilde{g}} h) D\bar{H}_{\tilde{g}}(h) \lambda \, d\sigma_{\tilde{g}}
\]

\[
+ \int_{\Omega} E(h) \lambda \, d\text{vol}_{\tilde{g}} - \int_{\Omega} E_1^i(h) \nabla_i \lambda \, d\text{vol}_{\tilde{g}} + \int_{\Sigma} F_1(h) \lambda \, d\sigma_{\tilde{g}}
\]

where \( \Pi \) is the second fundamental form of \( \Sigma \) in \( (\Omega, \tilde{g}) \) with respect to \( \tilde{g} \), \( X \) is the vector field on \( \Sigma \) that is dual to the 1-form \( h(e_n, \cdot)|_{T\Sigma} \), \( E(h) \) and \( E_1(h) \) are as in Lemma 2.1, and \( F_1(h) \) is a function on \( \Sigma \) satisfying

\[
|F_1(h)| \leq c |h|^2 |\nabla h|
\]

for a positive constant \( c \) depending only on \( n \).
Proof. By (2.8) with \( p = h \), using the fact that \( h|_{\mathcal{T}(\Sigma)} = 0 \), we have

\[
\int_{\Omega} DR_{\bar{g}}(h) \lambda \, d\text{vol}_{\bar{g}} = \int_{\Omega} \langle DR_{\bar{g}}(\lambda), h \rangle \, d\text{vol}_{\bar{g}} - \int_{\Sigma} 2DH_{\bar{g}}(h) \lambda \, d\sigma_{\bar{g}}. \tag{2.10}
\]

By the second line in (2.9) with \( p = h^2 \), and integrating by parts, we also have

\[
\int_{\Omega} -\frac{\lambda}{2} DR_{\bar{g}}(h^2) + \lambda \langle h, \nabla^2 \text{tr} \bar{g} h \rangle \, d\text{vol}_{\bar{g}}
= \int_{\Omega} -\frac{1}{2} \langle DR_{\bar{g}}^*(\lambda), h^2 \rangle + \text{tr} \bar{g} h \langle h, \nabla^2 \lambda \rangle \, d\text{vol}_{\bar{g}} + \mathcal{B} \tag{2.11}
\]

where

\[
\mathcal{B} = \int_{\Sigma} \frac{1}{2} \left[ \lambda \partial_{\Sigma}(\langle h^2 \rangle) - |h|^2 \partial_{\Sigma} \lambda - \lambda(\text{div}_{\bar{g}} h^2)(\nabla) + (h^2)(\nabla \lambda) \right] \, d\sigma_{\bar{g}}
+ \int_{\Sigma} \left[ \lambda h(\nabla \text{tr} \bar{g} h) - \text{tr} \bar{g} h \nabla \lambda \right] \, d\sigma_{\bar{g}}.
\tag{2.12}
\]

To compute \( \mathcal{B} \), let \( \{e_\alpha \mid 1 \leq \alpha \leq n-1\} \) be an orthonormal frame on \( \Sigma \) and let \( e_n = \nabla \). Denote \( \nabla \) also by “\( ; \)”, thus \( h_{ij;\kappa} = \nabla_k h_{ij} \). The assumptions \( h|_{\mathcal{T}\Sigma} = 0 \) and \( \text{div}_{\bar{g}} h = 0 \) imply the following facts on \( \Sigma \):

\[
|h|^2 = (h_{nn})^2 + 2|X|^2, \quad (h^2)_{nn} = (h_{nn})^2 + |X|^2, \quad (h^2)_{n\alpha} = h_{nn} h_{n\alpha}, \tag{2.13}
\]

\[
(h^2)(\nabla \lambda) = [(h_{nn})^2 + |X|^2] \lambda_{n} + h_{nn} \langle X, \nabla \lambda \rangle, \tag{2.14}
\]

\[
h_{\beta\gamma;\alpha} = h_{\beta n} \nabla_{\alpha} + h_{n\gamma} \nabla_{\beta}, \tag{2.15}
\]

\[
h_{nn;\alpha} = (\text{tr} \bar{g} h)_{;\alpha} - \sum_{\beta=1}^{n-1} h_{\beta\beta;\alpha} = (\text{tr} \bar{g} h)_{;\alpha} - 2\Pi(X, e_\alpha), \tag{2.16}
\]

\[
0 = (\text{div} h)_\alpha = h_{\alpha n; n} + \sum_{\beta=1}^{n-1} h_{\alpha\beta;\beta} = h_{\alpha n; n} + h_{n\alpha} H(\bar{g}) + \Pi(X, e_\alpha), \tag{2.17}
\]

\[
0 = (\text{div}_{\bar{g}} h)_{n} = h_{nn; n} + \sum_{\alpha=1}^{n-1} h_{n\alpha;\alpha} = h_{nn; n} + \text{div}_\Sigma X + h_{nn} H(\bar{g}), \tag{2.18}
\]

\[
2DH_{\bar{g}}(h) = (\text{tr} \bar{g} h)_{;n} - \text{div}_\Sigma X, \tag{2.19}
\]
where \((2.19)\) follows from \((2.6)\). By \((2.16)-(2.18)\), we have

\begin{equation}
(2.20) \quad \partial_\nu(h^2) - \langle \text{div}_g h^2, \nabla \rangle = 3h_{mn}h_{n;\alpha} + h_{\alpha\beta}h_{mn;\beta} - h_{\alpha\beta}h_{mn;\beta}
\end{equation}

\begin{equation}
= - \Pi(X, X) - 3H(\bar{g})|X|^2 - H(\bar{g})(h_{mn})^2 - h_{mn}\text{div}_\Sigma X - \langle X, \nabla\Sigma \text{tr}_g h \rangle.
\end{equation}

By \((2.12), (2.13), (2.14), (2.20)\) and integration by parts, we have

\begin{equation}
(2.21) \quad B = \int_\Sigma \left[ -(h_{mn})^2 - \frac{1}{2}|X|^2 \right] \lambda \, d\nu - \int_\Sigma h_{mn} \langle X, \nabla\Sigma \lambda \rangle
+ \int_\Sigma \left[ -\frac{1}{2} \Pi(X, X) - \frac{3}{2} H(\bar{g})|X|^2 - \frac{1}{2} H(\bar{g})(h_{mn})^2 + 2h_{mn} \text{D}H_\bar{g}(h) \right] \lambda \, d\sigma_{\bar{g}}.
\end{equation}

Note that

\begin{equation}
(2.22) \quad \int_\Omega (\nabla_i E_i^\alpha(h)) \lambda \, d\text{vol}_{\bar{g}} = -\int_\Omega E_i^\alpha(h)\nabla_i \lambda \, d\text{vol}_{\bar{g}} + \int_\Sigma \lambda F_1(h) \, d\sigma_{\bar{g}}
\end{equation}

where \(|F_1(h) = \langle E_1(h), \nabla \rangle| \leq C|h|^2|\nabla h|\). Proposition \((2.3)\) now follows from Lemma \((2.1, 2.10, 2.11, 2.21,\) and \((2.22)\). \(\square\)

The formula \((2.23)\) next is a general form of [5, Theorem 10], which Brendle and Marques derived for geodesic balls in \(\mathbb{S}^n\).

**Theorem 2.1.** Suppose \(g\) and \(\bar{g}\) induce the same metric on \(\Sigma\) and \(h\) satisfies \(\text{div}_g h = 0\). Given any \(C^2\) function \(\lambda\) on \(\Omega\), one has

\begin{equation}
(2.23) \quad \int_\Omega \left[ R(g) - R(\bar{g}) \right] \lambda \, d\text{vol}_{\bar{g}} + \int_\Sigma (2 - \text{tr}_g h) [H(g) - H(\bar{g})] \lambda \, d\sigma_{\bar{g}}
= \int_\Omega \left[ \langle h, \text{D}R^\alpha(g) \rangle \text{dvol}_{\bar{g}} - \frac{1}{2} \int_\Omega \langle h^2, \text{D}R^\alpha(g) \rangle \text{dvol}_{\bar{g}} 
+ \int_\Omega \left[ \text{tr}_g h \langle h, \nabla^2 \lambda \rangle + \frac{1}{2} h^{ij}h^{kl} \bar{R}_{ikjl} \lambda - \frac{1}{4} (|\nabla h|^2 + |\nabla(\text{tr}_g h)|^2) \lambda \right] \text{dvol}_{\bar{g}}
+ \int_\Sigma \left[ -\frac{1}{4} (h_{mn})^2 H(\bar{g}) - \frac{1}{2} \Pi(X, X) + H(\bar{g})|X|^2 \right] \lambda \, d\sigma_{\bar{g}}
+ \int_\Sigma \lambda n \left[ -(h_{mn})^2 - \frac{1}{2} |X|^2 \right] \, d\sigma_{\bar{g}} + \int_\Sigma (-1) h_{mn} \langle X, \nabla\Sigma \lambda \rangle \, d\sigma_{\bar{g}}
+ \int_\Omega E(h)\lambda \, d\text{vol}_{\bar{g}} + \int_\Omega Z(h)\nabla \lambda \, d\text{vol}_{\bar{g}} + \int_\Sigma F(h)\lambda \, d\sigma_{\bar{g}}
\end{equation}
where $E(h)$ is a function and $Z(h)$ is a vector field on $\Omega$ satisfying
\[ |E(h)| \leq C(|h||\nabla h|^2 + |h|^3), \quad |Z(h)| \leq C|h|^2|\nabla h|, \]
and $F(h)$ is some function on $\Sigma$ satisfying
\[ |F(h)| \leq C(|h|^2|\nabla h| + |h|^3). \]

Proof. Proposition 2.2 implies
\[ 2[H(g) - H(\bar{g})] = 2DH_{\bar{g}}(h) + J(h) + F_2(h) \]
where
\[ J(h) = \left[ \frac{1}{4}(h_{nn})^2 + |X|^2 \right] H(\bar{g}) - h_{nn}DH_{\bar{g}}(h) \]
and $F_2(h)$ is some function on $\Sigma$ satisfying $|F_2(h)| \leq C(|h|^2|\nabla h| + |h|^3)$.

Therefore
\[ (2 - h_{nn})[H(g) - H(\bar{g})] = (2 - 2h_{nn})D\bar{g}(h) + \left[ \frac{1}{4}(h_{nn})^2 + |X|^2 \right] H(\bar{g}) \]
\[ + F_2(h) - \frac{1}{2}h_{nn}[J(h) + F_2(h)]. \]

(2.23) now follows readily from Proposition 2.3 and (2.25). \hfill \Box

The term $D\mathcal{R}^*(\lambda)$ in (2.23) may suggest that one consider a background metric $\bar{g}$ which admits a nontrivial function $\lambda$ such that $D\mathcal{R}^*(\lambda) = 0$ (such metrics are known as static metrics [10]). For instance, if $\bar{g}$ is the standard metric on $S^n$ and $\lambda = \cos r$, where $r$ is the $\bar{g}$-distance to a point, then (2.23) reduces to the formula in [11 Theorem 10].

Besides static metrics, one can also consider those metrics $\bar{g}$ with the property that there exists a function $\lambda$ such that
\[ D\mathcal{R}^*(\lambda) = \bar{g}. \]

These metrics were studied by the authors in [13] and [14]. In this case, the terms
\[ \int_\Omega \langle h, D\mathcal{R}^*(\lambda) \rangle \, d\Omega_{\bar{g}} - \frac{1}{2} \int_\Omega \langle h^2, D\mathcal{R}^*(\lambda) \rangle \, d\Omega_{\bar{g}} \]
in (2.23) become
\[ \int_\Omega \text{tr}_g h \, d\Omega_{\bar{g}} - \frac{1}{2} \int_\Omega |h|^2 \, d\Omega_{\bar{g}}. \]

To compensate these terms, one can include the difference between the volumes of $g$ and $\bar{g}$ into (2.23).
Corollary 2.1. Suppose \( \bar{g} \) is a metric on \( \Omega \) with the property that there exists a function \( \lambda \) satisfying \( DR^{\ast}_{\bar{g}}(\lambda) = \bar{g} \). Let \( g = \bar{g} + h \) be a nearby metric such that \( g \) and \( \bar{g} \) induce the same metric on \( \Sigma \) and \( h \) satisfies \( \text{div}_\bar{g} h = 0 \). Let \( V(g) \), \( V(\bar{g}) \) denote the volume of \((\Omega, g)\), \((\Omega, \bar{g})\). Then
\begin{equation}
2(V(g) - V(\bar{g})) + \int_{\Sigma} [R(g) - R(\bar{g})] \lambda \, d\sigma_{\bar{g}} + \int_{\Sigma} (2 - \text{tr}\bar{g} h) [H(g) - H(\bar{g})] \lambda \, d\sigma_{\bar{g}}
\end{equation}
(2.27)
\begin{align*}
&= \int_{\Omega} \left[ -\frac{1}{4} \frac{1}{n-1} \right] (\text{tr}\bar{g} h)^2 \, d\nu_{\bar{g}} + \int_{\Omega} \left[ -\frac{1}{4} \left( |\nabla h|^2 + |\nabla_{\bar{g}} (\text{tr}\bar{g} h)|^2 \right) \lambda \right] \, d\sigma_{\bar{g}} \\
&\quad + \int_{\Omega} \left[ \frac{1}{1-n} R(\bar{g}) (\text{tr}\bar{g} h)^2 + \langle h, \text{Ric}(\bar{g}) \rangle (\text{tr}\bar{g} h) + \frac{1}{2} h_{ij} h_{kl} R_{ijkl} \right] \lambda \, d\sigma_{\bar{g}} \\
&\quad + \int_{\Omega} \left[ \frac{1}{4} (h_{nn})^2 H(\bar{g}) - \frac{1}{2} \langle \Pi(X, X) + H(\bar{g}) | X |^2 \rangle \right] \lambda \, d\sigma_{\bar{g}} \\
&\quad + \int_{\Omega} \lambda_{,n} \left[ -(h_{nn})^2 - \frac{1}{2} |X|^2 \right] \, d\sigma_{\bar{g}} + \int_{\Omega} (-1) h_{nn} \langle X, \nabla \lambda \rangle \, d\sigma_{\bar{g}} \\
&\quad + \int_{\Omega} G(h) \, d\nu_{\bar{g}} + \int_{\Omega} E(h) \lambda \, d\nu_{\bar{g}} + \int_{\Omega} Z(h) \nabla \lambda \, d\nu_{\bar{g}} + \int_{\Sigma} F(h) \lambda \, d\sigma_{\bar{g}}
\end{align*}
where \( G(h) \) and \( E(h) \) are functions on \( \Omega \) satisfying
\begin{equation}
|G(h)| \leq C|h|^3, \quad |E(h)| \leq C(|h||\nabla h|^2 + |h|^3),
\end{equation}
\( Z(h) \) is a vector field on \( \Omega \) satisfying
\begin{equation}
|Z(h)| \leq C|h|^2|\nabla h|,
\end{equation}
and \( F(h) \) is a function on \( \Sigma \) satisfying
\begin{equation}
|F(h)| \leq C(|h|^2|\nabla h| + |h|^3).
\end{equation}

Proof. The difference between the volumes of \( \bar{g} \) and \( g = \bar{g} + h \) is
\begin{equation}
V(g) - V(\bar{g}) = \int_{\Omega} \frac{1}{2} (\text{tr}\bar{g} h) + \left[ \frac{1}{8} (\text{tr}\bar{g} h)^2 - \frac{1}{4} |h|^2 \right] + G(h) \, d\nu_{\bar{g}},
\end{equation}
where \( G(h) \) is a function satisfying \(|G(h)| \leq C|h|^3\) for a constant \( C \) depending only on \( n \). Suppose \( DR^{\ast}_{\bar{g}}(\lambda) = \bar{g} \), i.e.
\begin{equation}
-(\Delta \lambda) \bar{g} + \nabla^{\ast}_{\bar{g}} \lambda - \lambda \text{Ric}(\bar{g}) = \bar{g}.
\end{equation}
Taking trace, one has \( \Delta \lambda = \frac{1}{1-n} [R(\bar{g}) \lambda + n] \). Thus,
\begin{equation}
\nabla^{\ast}_{\bar{g}} \lambda = \frac{1}{1-n} [R(\bar{g}) \lambda + 1] \bar{g} + \lambda \text{Ric}(\bar{g}).
\end{equation}
(2.27) follows from (2.23), (2.28) and (2.29). \( \square \)
3. VOLUME CONSTRAINED RIGIDITY

We prove Theorem 3.1 in this section. First, we recall its statement:

**Theorem 3.1.** Let $(\Omega, \bar{g})$ be an $n$-dimensional compact Riemannian manifold, of constant sectional curvature $1$, with smooth boundary $\Sigma$. Suppose $\bar{\Pi} + \bar{H} \bar{\gamma} \geq 0$ (i.e. $\bar{\Pi} + \bar{H} \bar{\gamma}$ is positive semi-definite), where $\bar{\gamma}$ is the induced metric on $\Sigma$ and $\bar{\Pi}$, $\bar{H}$ are the second fundamental form, the mean curvature of $\Sigma$ in $(\Omega, \bar{g})$. Suppose the first nonzero Neumann eigenvalue $\mu$ of $(\Omega, g)$ satisfies $\mu > n - \frac{2}{n+1}$.

Consider a nearby metric $g$ on $\Omega$ with the properties

- $R(g) \geq n(n-1)$ where $R(g)$ is the scalar curvature of $g$
- $H(g) \geq H$ where $H(g)$ is the mean curvature of $\Sigma$ in $(\Omega, g)$
- $g$ and $\bar{g}$ induce the same metric on $\Sigma$
- $V(g) \geq V(\bar{g})$ where $V(g)$, $V(\bar{g})$ are the volumes of $g$, $\bar{g}$.

If $||g - \bar{g}||_{C^2(\Omega)}$ is sufficiently small, then there is a diffeomorphism $\varphi$ on $\Omega$ with $\varphi|_\Sigma = \text{id}$, which is the identity map on $\Sigma$, such that $\varphi^*(g) = \bar{g}$.

**Proof.** Fix a real number $p > n$. By [3, Proposition 11], if $||g - \bar{g}||_{W^{2,p}(\Omega)}$ is sufficiently small, there exists a $W^{3,p}$ diffeomorphism $\varphi$ on $\Omega$ with $\varphi|_\Sigma = \text{id}$ such that $h = \varphi^*(g) - g$ is divergence free with respect to $\bar{g}$, and $||h||_{W^{2,p}(\Omega)} \leq N||g - \bar{g}||_{W^{2,p}(\Omega)}$ for some positive constant $N$ depending only on $(\Omega, \bar{g})$. Replacing $g$ by $\varphi^*(g)$, we may assume $g = \bar{g} + h$ with $\text{div}_\bar{g} h = 0$. We want to prove that if $||h||_{C^1(\Omega)}$ is sufficiently small and $g$ satisfies the conditions in the theorem, then $h$ must be zero.

Since $\bar{g}$ has constant sectional curvature $1$, we choose $\lambda = -\frac{1}{n-1}$ such that $DR^*_\bar{g}(\lambda) = \bar{g}$. Corollary 2.1 then shows

$$
(3.1) \\
- 2(V(g) - V(\bar{g})) - \frac{1}{n-1} \int_\Omega [R(g) - R(\bar{g})] \ dvol_{\bar{g}} \\
- \frac{1}{n-1} \int_\Sigma (2 - \text{tr}_\bar{g} h) [H(g) - H(\bar{g})] \ d\sigma_{\bar{g}} \\
\geq \frac{1}{4(n-1)} \int_\Omega [-(n+1)(\text{tr}_\bar{g} h)^2 + 2|h|^2 + |\nabla h|^2 + |\nabla (\text{tr}_\bar{g} h)|^2] \ dvol_{\bar{g}} \\
+ \frac{1}{4(n-1)} \int_\Sigma [(h_{mn})^2 H(g) + 2\bar{\Pi}(X, X) + H(g)|X|^2] \ d\sigma_{\bar{g}} \\
- C||h||_{C^2(\Omega)} \left[ \int_\Omega (|h|^2 + |\nabla h|^2) \ dvol_{\bar{g}} + \int_\Sigma |h|^2 d\sigma_{\bar{g}} \right]
$$

for a constant $C$ depending only on $(\Omega, \bar{g})$. 
Using the variational property of $\mu$, we have

\begin{equation}
(3.2) \int_{\Omega} |\nabla (\nabla g)|^2 \, d\nu_{\bar{g}} \geq \mu \left[ \left( \int_{\Omega} (\nabla g)^2 \, d\nu_{\bar{g}} \right) - \frac{1}{\nu(\bar{g})} \left( \int_{\Omega} \nabla g \, d\nu_{\bar{g}} \right)^2 \right].
\end{equation}

By (2.28), $\int_{\Omega} \nabla g \, d\nu_{\bar{g}}$ is related to $(V(g) - V(\bar{g}))$ by

\begin{equation}
(3.3) \int_{\Omega} \nabla g \, d\nu_{\bar{g}} = 2(V(g) - V(\bar{g})) - \int_{\Omega} \left\{ \frac{1}{4} (\nabla g)^2 - \frac{1}{2} |h|^2 \right\} + 2G(h) \, d\nu_{\bar{g}},
\end{equation}

where $G(h) \leq C|h|^3$.

Given any constant $0 < \epsilon < 1$, using (3.2) and the fact $|h|^2 \geq \frac{1}{n} (\nabla g)^2$ and $|\nabla h|^2 \geq \frac{1}{n} |\nabla (\nabla g)|^2$, we have

\begin{equation}
(3.4) \int_{\Omega} \left[ -(n+1)(\nabla g)^2 + 2|h|^2 + |\nabla h|^2 + |\nabla g(\nabla g)|^2 \right] \, d\nu_{\bar{g}}
\geq \int_{\Omega} \left( \epsilon |h|^2 + \epsilon |\nabla h|^2 + \left[ -(n+1) + \frac{2 - \epsilon}{n} \right] (\nabla g)^2 + \left[ \frac{(1-\epsilon)}{n} + 1 \right] |\nabla (\nabla g)|^2 \right) \, d\nu_{\bar{g}}
\geq \int_{\Omega} \left( \epsilon |h|^2 + \epsilon |\nabla h|^2 + \left[ -(n+1) + \frac{2 - \epsilon}{n} + \frac{(1-\epsilon)}{n} \mu + \mu \right] (\nabla g)^2 \right) \, d\nu_{\bar{g}}
\geq \mu \left[ \frac{(1-\epsilon)}{n} + 1 \right] \frac{1}{\nu(\bar{g})} \left( \int_{\Omega} \nabla g \, d\nu_{\bar{g}} \right)^2.
\end{equation}

Since $\mu > n - \frac{2}{n+1}$, we can chose $\epsilon$ (depending only on $\mu$ and $n$) such that

\begin{equation}
(3.5) \left[ -(n+1) + \frac{2 - \epsilon}{n} + \frac{(1-\epsilon)}{n} \mu + \mu \right] \geq 0.
\end{equation}

Then it follows from (3.3), (3.4) and (3.5) that

\begin{equation}
(3.6) \int_{\Omega} \left[ -(n+1)(\nabla g)^2 + 2|h|^2 + |\nabla h|^2 + |\nabla (\nabla g)|^2 \right] \, d\nu_{\bar{g}}
\geq \epsilon \int_{\Omega} \left( |h|^2 + |\nabla h|^2 \right) \, d\nu_{\bar{g}} - C_1 (V(g) - V(\bar{g}))^2 - C_1 \int_{\Omega} |h|^4 \, d\sigma_{\bar{g}}
\end{equation}

where $C_1$ is a positive constant depending only on $(\Omega, \bar{g})$.

At the boundary $\Sigma$, the assumption $\Pi + H(\bar{g}) \geq 0$ implies $H(\bar{g}) \geq 0$, therefore

\begin{equation}
(3.7) \int_{\Sigma} [(h_{nn})^2 H(\bar{g}) + 2(\Pi(X, X) + H(\bar{g})|X|^2)] \, d\sigma_{\bar{g}} \geq 0
\end{equation}
for any \( h \). By (3.1), (3.6) and (3.7), we have
\[
-8(n-1)(V(g) - V(\bar{g})) - 4 \int_{\Omega} [R(g) - R(\bar{g})] \, d\text{vol}_{\bar{g}}
-4 \int_{\Sigma} (2 - \text{tr}_{\bar{g}} h) [H(g) - H(\bar{g})] \, d\sigma_{\bar{g}}
\geq 4 \epsilon \int_{\Omega} (|h|^2 + |\nabla h|^2) \, d\text{vol}_{\bar{g}}
\]
(3.8)
for some positive constant \( C \) depending only on \( (\Omega, \bar{g}) \).

Finally, we note that
\[
(V(g) - V(\bar{g}))^2 \leq C \left( \int_{\Omega} |h| \, d\text{vol}_{\bar{g}} \right) (V(g) - V(\bar{g}))
\]
by (3.3) and the assumption \( V(g) \geq V(\bar{g}) \). Also, by the trace theorem,
\[
||h||_{L^2(\Sigma)} \leq C ||h||_{W^{1,2}(\Omega)}
\]
(3.10)
for some constant \( C \) only depending on \( \Omega \). Therefore, by (3.8), (3.9), (3.10) and the assumptions \( V(g) \geq V(\bar{g}) \), \( R(g) \geq R(\bar{g}) \) and \( H(g) \geq H(\bar{g}) \), we conclude that if \( ||h||_{C^1(\Omega)} \) is sufficiently small, then
\[
0 \geq \frac{\epsilon}{2} \int_{\Omega} (|h|^2 + |\nabla h|^2) \, d\text{vol}_{\bar{g}}
\]
which implies \( h \) must be identically zero. This completes the proof. \( \square \)

**Remark 3.1.** In Theorem 3.1 if \( \Sigma \) is indeed empty, i.e \( (\Omega, \bar{g}) \) is a closed space form, its first nonzero Neumann eigenvalue satisfies \( \mu \geq n \) as \( (\Omega, \bar{g}) \) is covered by \( S^n \). In this case, Theorem 3.1 says that \( V(g) \geq V(\bar{g}) \) implies \( g \) is isometric to \( \bar{g} \) for a nearby metrics \( g \) with \( R(g) \geq R(\bar{g}) \). This could be compared to a more profound theorem known in 3-dimension:
\"If \((M, g)\) is closed 3-manifold with \( R(g) \geq 6, \text{Ric}(g) \geq g \) and \( V(g) \geq V(S^3) \), then \((M, g)\) is isometric to \( S^3 \).\" (See [4, Corollary 5.4] and earlier reference of [3, 11])

When \( \Sigma \neq \emptyset \), the boundary assumption \( \mathbb{I} + \bar{H} \bar{\gamma} \geq 0 \) in Theorem 3.1 can be relaxed in certain circumstances. A detailed examination of the above proof shows, if
\[
\mathbb{I}(v, v) + \bar{H} \bar{\gamma} \geq -\beta \bar{\gamma}
\]
(3.12)
for some positive constant $\beta$, where $\beta$ is sufficiently small comparing to the constant $\epsilon$ in (3.5) and the constant $C$ in (3.10), then the conclusion of Theorem 3.1 still holds on such an $(\Omega, \bar{g})$. In particular, this shows Corollary 3.1.

Let $(M, \bar{g})$ be an $n$-dimensional Riemannian manifold of constant sectional curvature $1$. Suppose $\Omega \subset M$ is a bounded domain with smooth boundary $\Sigma$, satisfying the assumptions in Theorem 3.1, i.e $\mu > n - \frac{2}{n+1}$ and $\bar{H} \geq 0$ on $\Sigma$. Let $\tilde{\Omega} \subset \bar{M}$ be another bounded domain with smooth boundary $\tilde{\Sigma}$. If $\tilde{\Sigma}$ is sufficiently close to $\Sigma$ in the $C^2$ norm, then the conclusion of Theorem 3.1 holds on $\tilde{\Omega}$.

It is known that the first nonzero Neumann eigenvalue of $S^n_+$ is $n$ (see [9, Theorem 3]). Therefore, Theorem 1.2 follows from Theorem 3.1.

Moreover, by Corollary 3.1, Theorem 3.1 holds on a geodesic ball in $S^n$ whose radius is slightly larger than $\frac{\pi}{2}$.

By the next lemma, we know Theorem 3.1 also holds on any geodesic ball in $S^n$ that is strictly contained in $S^n_+$.

**Lemma 3.1.** Let $B(\delta) \subset S^n$ be a geodesic ball of radius $\delta$. Let $\mu(\delta)$ be the first nonzero Neumann eigenvalue of $B(\delta)$.

(i) $\mu(\delta)$ is a strictly decreasing function of $\delta$ on $(0, \frac{\pi}{2}]$.

(ii) For any $0 < \delta < \frac{\pi}{2}$,

$$
\mu(\delta) > n + \frac{(\sin \delta)^{n-2} \cos \delta}{\int_0^\delta (\sin t)^{n-1} dt} > \frac{n}{(\sin \delta)^2}.
$$

**Proof.** By [9, Theorem 2, p.44], $\mu(\delta)$ is characterized by the fact that

(3.13) \{ $(\sin t)^{n-1} J' \} + [\mu(\delta) - (n-1) (\sin t)^{-2}] (\sin t)^{n-1} J = 0$

has a solution $J = J(t)$ on $[0, \delta]$ satisfying

(3.14) \begin{align*}
J(0) &= 0, \quad J'(\delta) = 0, \
J'(t) &\neq 0, \quad \forall \ t \in [0, \delta).
\end{align*}

Given $0 < \delta_1 < \delta_2 < \frac{\pi}{2}$, let $J_i = J_i(t)$ be a solution to (3.13) with $\mu(\delta)$ replaced by $\mu(\delta_i)$, satisfying (3.14) on $[0, \delta_i]$, $i = 1, 2$. Replacing $J_i$ by $-J_i$ if necessary, we may assume that $J_i' > 0$ on $[0, \delta_i)$, hence $J_i > 0$ on $(0, \delta_i]$. Define

$$
\beta_i(t) = \left[ \frac{\mu(\delta_i) - n - 1}{(\sin t)^2} \right] (\sin t)^{n-1}.
$$

By (3.13), $f_i$ satisfies

$$
f_i' = -\beta_i - \frac{1}{(\sin t)^{n-1}} f_i^2.
$$
Therefore, on \((0, \delta_1]\),

\[
(3.15) \quad (f_1 - f_2)' = \frac{1}{(\sin t)^{n-1}}(f_2^2 - f_1^2) + [\mu(\delta_2) - \mu(\delta_1)](\sin t)^{n-1}.
\]

Note that \(f_1(t), f_2(t)\) can be extended continuously to 0 such that \(f_1(0) = f_2(0)\). Moreover, \(f_1 > 0, f_2 > 0\) on \((0, \delta_1)\), \(f_2(\delta_1) > 0 = f_1(\delta_1)\). Let \(0 \leq t_0 < \delta_1\) be such that \(f_1 = f_2\) at \(t_0\) and \(f_2 > f_1\) for \(t_0 < t \leq \delta_1\). On \((t_0, \delta_1]\), one would have \((f_1 - f_2)' > 0\) if \(\mu(\delta_2) \geq \mu(\delta_1)\), which is a contradiction to \(f_2 > f_1\). Therefore, \(\mu(\delta_2) < \mu(\delta_1)\). This proves (i).

To prove (ii), we further claim that \(t_0 = 0\), i.e. \(f_2 > f_1\) on \((0, \delta_1]\). If not, there would be a nonpositive local minimum of \((f_2 - f_1)\) at some \(\tilde{t}_0 \in (0, t_0]\). At \(\tilde{t}_0\), (3.15) implies

\[
(3.16) \quad 0 = (f_1 - f_2)' \leq [\mu(\delta_2) - \mu(\delta_1)](\sin \tilde{t}_0)^{n-1} < 0
\]

because \(0 < f_2(\tilde{t}_0) \leq f_1(\tilde{t}_0)\) and \(\mu(\delta_2) < \mu(\delta_1)\). Hence \(f_2 > f_1\) on \((0, \delta_1]\).

Integrating (3.15) on \([0, \delta_1]\), we have

\[
(3.17) \quad - f_2(\delta_1) = \int_0^{\delta_1} (f_1 - f_2)' dt > [\mu(\delta_2) - \mu(\delta_1)] \int_0^{\delta_1} (\sin t)^{n-1} dt.
\]

Therefore

\[
(3.18) \quad \mu(\delta_1) > \mu(\delta_2) + \frac{f_2(\delta_1)}{\int_0^{\delta_1} (\sin t)^{n-1} dt}.
\]

Now let \(\delta_1 = \delta \in (0, \frac{\pi}{2})\) and \(\delta_2 = \pi/2\). Applying the fact that \(\mu(\frac{\pi}{2}) = n\), \(J_2 = \sin t\), and

\[
f_2 = (\sin t)^{n-2} \cos t,
\]

we have

\[
(3.19) \quad \mu(\delta) > n + \frac{(\sin \delta)^{n-2} \cos \delta}{\int_0^{\delta} (\sin t)^{n-1} dt}
\]

\[
> n + \frac{(\sin \delta)^{n-2} \cos^2 \delta}{\int_0^{\delta} \cos t(\sin t)^{n-1} dt}
\]

\[
= \frac{n}{\sin^2 \delta}.
\]

Therefore, (ii) is proved.

\[\square\]

4. A Volume estimate on domains in \(\mathbb{R}^n\)

On \(\mathbb{R}^n\), the standard Euclidean metric \(\bar{g}\) satisfies \(DR_\bar{g}(\lambda) = \bar{g}\) with

\[
\lambda(x) = -\frac{1}{2(n-1)}|x - a|^2 + L
\]
where $| \cdot |$ denotes the Euclidean length, $a \in \mathbb{R}^n$ is any fixed point and $L$ is an arbitrary constant. In this section, we use this fact and Corollary 2.1 to prove Theorem 1.4 in the introduction. First we need some lemmas.

**Lemma 4.1.** On a compact Riemannian manifold $(\Omega, \bar{g})$ with smooth boundary $\Sigma$, there exists a positive constant $C$ depending only on $(\Omega, \bar{g})$ such that, for any Lipschitz function $\phi$ on $\Sigma$, there is an extension of $\phi$ to a Lipschitz function $\tilde{\phi}$ on $\Omega$ such that

$$
(4.2) \quad \int_{\Omega} \left( |\tilde{\phi}|^2 + |\nabla \phi|^2 \right) d\text{vol}_{\bar{g}} \leq C \int_{\Sigma} \left( \phi^2 + |\nabla^\Sigma \phi|^2 \right) d\sigma_{\bar{g}}
$$

where $\nabla$, $\nabla^\Sigma$ denote the gradient on $\Omega$, $\Sigma$ respectively.

**Proof.** Let $d(\cdot, \Sigma)$ be the distance to $\Sigma$. Let $\delta > 0$ be a small constant such that the tubular neighborhood $U_{2\delta} = \{ x \in \Omega \mid d(x, \Sigma) < 2\delta \}$ can be parametrized by $F : \Sigma \times [0, 2\delta) \to U_{2\delta}$, with $F(y, t) = \text{exp}_y(t\nu(y))$ where $\text{exp}_y(\cdot)$ is the exponential map at $y \in \Sigma$ and $\nu(y)$ is the inward unit normal at $y$. In $U_{2\delta}$, the metric $\bar{g}$ takes the form $dt^2 + \sigma^t$, where $\{\sigma^t\}_{0 \leq t < 2\delta}$ is a family of metrics on $\Sigma$. By choosing $\delta$ sufficiently small, one can assume $\sigma^t$ is equivalent to $\sigma^0$ in the sense that $\frac{1}{2} \leq \sigma^t(v, v) \leq 2$ for any tangent vector $v$ with $\sigma^0(v, v) = 1$, $\forall \ 0 \leq t < 2\delta$.

Let $\rho = \rho(t)$ be a fixed smooth cut-off function on $[0, \infty)$ such that $0 \leq \rho \leq 1$, $\rho(t) = 1$ for $0 \leq t \leq \delta$ and $\rho(t) = 0$ for $t \geq \frac{3}{2}\delta$. On $U_{2\delta}$, consider the function $\tilde{\phi}(y, t) = \phi(y)\rho(t)$. Since $\tilde{\phi}$ is identically zero outside $U_{\frac{3}{2}\delta} = \{ x \in \Omega \mid d(x, \Sigma) < \frac{3}{2}\delta \}$, $\tilde{\phi}$ can be viewed as an extension of $\phi$ on $\Omega$. For such an $\tilde{\phi}$, one has

$$
(4.3) \quad \int_{\Omega} |\tilde{\phi}|^2 d\text{vol}_{\bar{g}} \leq \int_{0}^{2\delta} \left( \int_{\Sigma} |\phi|^2 d\sigma^t \right) dt \leq C\delta \int_{\Sigma} |\phi|^2 d\sigma_{\bar{g}}
$$

and

$$
(4.4) \quad \int_{\Omega} |\nabla \tilde{\phi}|^2 d\text{vol}_{\bar{g}} \leq 2 \int_{U_{2\delta}} \left( |\nabla \rho|^2 \phi^2 + |\nabla \phi|^2 \rho^2 \right) d\text{vol}_{\bar{g}}
$$

$$
\leq C\delta \int_{\Sigma} |\phi|^2 d\sigma_{\bar{g}} + 2 \int_{0}^{2\delta} \left( \int_{\Sigma} |\nabla_t \phi|^2 d\sigma^t \right) dt
\leq C \left[ \int_{\Sigma} |\phi|^2 d\sigma_{\bar{g}} + \int_{\Sigma} |\nabla^\Sigma \phi|^2 d\sigma_{\bar{g}} \right]
$$

where $\nabla^\Sigma_t$ denotes the gradient on $(\Sigma, \sigma^t)$ and $C$ is a positive constant depending only on $(\Omega, \bar{g})$. (4.2) now follows from (4.3) and (4.4).

$\Box$
Lemma 4.2. On a compact Riemannian manifold \((\Omega, \bar{g})\) with smooth boundary \(\Sigma\), there exists a positive constant \(C\) depending only on \((\Omega, \bar{g})\) such that, for any smooth \((0, 2)\) symmetric tensor \(h\) on \(\Omega\), one has

\[
\int_{\Omega} |h|^3 d\sigma_{\bar{g}} \leq C \left( \int_{\Sigma} |h|^3 d\sigma_{\bar{g}} + ||h||_{C^2(\Omega)} \int_{\Sigma} |h|^2 d\sigma_{\bar{g}} + \int_{\Omega} ||h||^2 |\nabla h|^2 d\sigma_{\bar{g}} \right)
\]

Proof. On \(\Omega\), let \(\phi = |h|^\frac{3}{2}\). By lemma 4.1, there exists a Lipschitz function \(\tilde{\phi}\) on \(\Omega\) such that \(\tilde{\phi}|_\Sigma = \phi|_\Sigma\) and

\[
\int_{\Omega} (|\tilde{\phi}|^2 + |\nabla \tilde{\phi}|^2) d\sigma_{\bar{g}} \leq C \int_{\Sigma} \left( \phi^2 + |\nabla^\Sigma \phi|^2 \right) d\sigma_{\bar{g}}.
\]

Let \(\lambda_1 > 0\) be the first Dirichlet eigenvalue of \((\Omega, \bar{g})\), then

\[
\int_{\Omega} \phi^2 d\sigma_{\bar{g}} \leq 2 \int_{\Omega} \left[ \tilde{\phi}^2 + (\phi - \tilde{\phi})^2 \right] d\sigma_{\bar{g}}
\]

\[
\leq 2 \int_{\Omega} \tilde{\phi}^2 d\sigma_{\bar{g}} + 2\lambda_1^{-1} \int_{\Omega} |\nabla (\phi - \tilde{\phi})|^2 d\sigma_{\bar{g}}
\]

\[
\leq C \left[ \int_{\Sigma} \left( \phi^2 + |\nabla^\Sigma \phi|^2 \right) d\sigma_{\bar{g}} + \int_{\Omega} |\nabla \phi|^2 d\sigma_{\bar{g}} \right]
\]

where

\[
\int_{\Omega} |\nabla \phi|^2 d\sigma_{\bar{g}} = \int_{\Omega} |\nabla |h|^\frac{3}{2}|^2 d\sigma_{\bar{g}} \leq \frac{9}{4} \int_{\Omega} |h||\nabla h|^2 d\sigma_{\bar{g}}.
\]

To handle the boundary term \(\int_{\Sigma} |\nabla^\Sigma \phi|^2 d\sigma_{\bar{g}}\), given any constant \(\epsilon > 0\), one considers

\[
\int_{\Sigma} |\nabla^\Sigma (|h|^2 + \epsilon)^\frac{3}{2}|^2 d\sigma_{\bar{g}} = -\int_{\Sigma} (|h|^2 + \epsilon)^\frac{3}{2} \Delta_{\Sigma} (|h|^2 + \epsilon)^\frac{3}{4} d\sigma_{\bar{g}}
\]

where \(\Delta_{\Sigma}\) denotes the Laplacian on \(\Sigma\). Let \(\{e_\alpha \mid \alpha = 1, \ldots, n-1\}\) be a local orthonormal frame on \(\Sigma\) and \(e_n\) be the outward unit normal to \(\Sigma\). Let \(\bar{H}\) be the mean curvature of \(\Sigma\) with respect to \(e_n\). Denote covariant differentiation \(\Omega\) by “; “. Let \(i, j\) run through \(\{1, \ldots, n\}\). One has

\[
\Delta_{\Sigma} |h|^2 = \sum_{\alpha} (|h|^2)_{,\alpha\alpha} - \bar{H} (|h|^2)_{,n}
\]

\[
= \sum_{\alpha, i,j} 2(h_{ij}h_{ij,\alpha\alpha} + h_{ij,\alpha}^2) - \bar{H} \sum_{i,j} 2h_{ij}h_{ij,n}
\]

\[
\geq - C||h||_{C^2(\bar{g})} |h|.
\]
Therefore,

\[
\Delta_\Sigma (|h|^2 + \epsilon)^{\frac{3}{4}} = \frac{3}{4} (|h|^2 + \epsilon)^{-\frac{1}{4}} \Delta_\Sigma |h|^2 - \frac{3}{16} (|h|^2 + \epsilon)^{-\frac{3}{4}} |\nabla_\Sigma |h|^2|^2 \\
\geq - C||h||_{C^2(\Omega)} (|h|^2 + \epsilon)^{-\frac{1}{4}} |h| - \frac{3}{16} (|h|^2 + \epsilon)^{-\frac{3}{4}} |\nabla_\Sigma |h|^2|^2.
\]

(4.10)

It follows from (4.8) and (4.10) that

\[
\int_\Sigma |\nabla_\Sigma (|h|^2 + \epsilon)^{\frac{3}{4}}|^2 d\sigma_\bar{g} \leq C||h||_{C^2(\Omega)} \int_\Sigma (|h|^2 + \epsilon)^{\frac{3}{4}} |h| d\sigma_\bar{g} \\
+ \frac{1}{3} \int_\Sigma |\nabla_\Sigma (|h|^2 + \epsilon)^{\frac{3}{4}}|^2 d\sigma_\bar{g}.
\]

(4.11)

Letting \( \epsilon \to 0 \), one has

\[
\int_\Sigma |\nabla_\Sigma |h|^2|^2 d\sigma_\bar{g} \leq C||h||_{C^2(\Omega)} \int_\Sigma |h|^2 d\sigma_\bar{g}.
\]

(4.12) now follows from (4.6), (4.7) and (4.12).

We recall the statement of Theorem 1.4 and give its proof.

**Theorem 4.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary \( \Sigma \). Suppose \( \Pi + \bar{H}\bar{\gamma} > 0 \) (i.e. \( \Pi + \bar{H}\bar{\gamma} \) is positive definite), where \( \Pi \), \( \bar{H} \) are the second fundamental form, the mean curvature of \( \Sigma \) in \( \mathbb{R}^n \) and \( \bar{\gamma} \) is the metric on \( \Sigma \) induced from the Euclidean metric \( \bar{g} \). Let \( g \) be another metric on \( \bar{\Omega} \) satisfying

- \( g \) and \( \bar{g} \) induce the same metric on \( \Sigma \).
- \( H(g) \geq \bar{H} \), where \( H(g) \) is the mean curvature of \( \Sigma \) in \( (\Omega, g) \).

Given any point \( a \in \mathbb{R}^n \), there exists a constant \( \Lambda > \frac{\max_{q \in \bar{\Omega}} |q - a|^2}{4(n-1)} \), which depends only on \( \Omega \) and \( a \), such that if \( ||g - \bar{g}||_{C^3(\Omega)} \) is sufficiently small, then

\[
V(g) - V(\bar{g}) \geq \int_\Omega R(g) \Phi \, d\vol_\bar{g}
\]

where \( \Phi = -\frac{1}{4(n-1)}|x - a|^2 + \Lambda > 0 \) on \( \bar{\Omega} \).

**Proof.** Fix a number \( p > n \). By the proof of [5, Proposition 11], one knows if \( ||g - \bar{g}||_{W^{4,p}(\Omega)} \) is sufficiently small, then there exists a \( W^{4,p} \) diffeomorphism \( \varphi : \bar{\Omega} \to \Omega \) such that \( \varphi|_{\Sigma} = \text{id} \), \( h = \varphi^*(g) - \bar{g} \) is divergence free with respect to \( \bar{g} \), and \( ||h||_{W^{3,p}(\Omega)} \leq N||g - \bar{g}||_{W^{3,p}(\Omega)} \) for a positive constant \( N \) depending only on \( (\Omega, \bar{g}) \). In what follows, we will work with \( \varphi^*(g) \). For convenience, we still denote \( \varphi^*(g) \) by \( g \).
Given $a \in \mathbb{R}^n$, consider $\lambda(x) = -\frac{1}{2(n-1)}|x-a|^2 + L$ where $L$ is a constant to be determined. First, we require $L > \frac{1}{2(n-1)} \max_{q \in \Omega} |q-a|^2$ so that $\lambda > 0$ on $\bar{\Omega}$. Since $\lambda$ satisfies $DR_g^*(\lambda) = \bar{g}$, Corollary 2.1 shows

\begin{equation}
(4.14)
\end{equation}

\begin{equation}
-2(V(g) - V(\bar{g})) + \int_{\Omega} R(g) \lambda \, d\sigma_g + \int_{\Sigma} (2 - \text{tr}_{\bar{g}} h) \left[ H(g) - \bar{H} \right] \lambda \, d\sigma_g
\leq - \frac{1}{4} \int_{\Omega} |\nabla h|^2 \lambda \, d\sigma_g + \int_{\Sigma} \left[ -\frac{1}{4}(h_{nn})^2 \bar{H} - \frac{1}{2}(\Pi(X, X) + \bar{H}|X|^2) \right] \lambda \, d\sigma_g
\end{equation}

\begin{equation}
+ \int_{\Sigma} \lambda_n \left[ -(h_{nn})^2 - \frac{1}{2}|X|^2 \right] \, d\sigma_g + \int_{\Sigma} (-1) h_{nn} \langle X, \bar{\nabla}^\Sigma \lambda \rangle \, d\sigma_g
\end{equation}

\begin{equation}
+ \int_{\Omega} G(h) \, d\sigma_g + \int_{\Omega} E(h) \lambda \, d\sigma_g + \int_{\Omega} Z^i(h) \bar{\nabla}^i \lambda \, d\sigma_g + \int_{\Omega} F(h) \lambda \, d\sigma_g
\end{equation}

where $|G(h)| \leq C|h|^3$, $|E(h)| \leq C(|h||\nabla h|^2 + |h|^3)$, $|Z(h)| \leq C|h|^2 |\nabla h|$, $|F(h)| \leq C(|h|^2 |\nabla h| + |h|^3)$ for some constant $C$ depending only on $\Omega$.

At $\Sigma$, $\lambda_n$ and $\bar{\nabla}^\Sigma \lambda$ are determined solely by $\Omega$ and $a$ (in particular they are independent on $L$). Apply the assumption $\Pi + \bar{H} > 0$ (which implies $\bar{H} > 0$) and the fact $|h|^2 = (h_{nn})^2 + 2|X|^2$, we have

\begin{equation}
(4.15)
\end{equation}

\begin{equation}
\left[ -\frac{1}{4}(h_{nn})^2 \bar{H} - \frac{1}{2}(\Pi(X, X) + \bar{H}|X|^2) \right] \lambda
\end{equation}

\begin{equation}
+ \lambda_n \left[ -(h_{nn})^2 - \frac{1}{2}|X|^2 \right] + (-1) h_{nn} \langle X, \bar{\nabla}^\Sigma \lambda \rangle
\end{equation}

\begin{equation}
\leq - LC_1|h|^2 + C_2|h|^2
\end{equation}

where $C_1$, $C_2$ are positive constants depending only on $\Omega$ and $a$. We fix $L$ such that

\begin{equation}
(4.16)
LC_1 - C_2 > 0
\end{equation}

and let $m = \frac{1}{4} \min_{\Omega} \lambda$ (note that $\lambda$ is fixed now). (4.14)-(4.16) imply

\begin{equation}
(4.17)
\end{equation}

\begin{equation}
-2(V(g) - V(\bar{g})) + \int_{\Omega} R(g) \lambda \, d\sigma_g + \int_{\Sigma} (2 - \text{tr}_{\bar{g}} h) \left[ H(g) - \bar{H} \right] \lambda \, d\sigma_g
\leq - m \int_{\Omega} |\nabla h|^2 \, d\sigma_g - (LC_1 - C_2) \int_{\Sigma} |h|^2 \, d\sigma_g
\end{equation}

\begin{equation}
+ C_3 \left( \int_{\Omega} (|h||\nabla h|^2 + |h|^3) \, d\sigma_g + \int_{\Sigma} (|h|^2 |\nabla h| + |h|^2) \, d\sigma_g \right)
\end{equation}
where $C_3$ depends only on $\Omega$, $a$ and $L$. Apply Lemma 4.2 to the term $\int_{\Omega} |h|^3 \ d\text{vol}_g$ on the right side of (4.17), we have

$$-2(V(g) - V(\bar{g})) + \int_{\Omega} R(g) \lambda \ d\text{vol}_g + \int_{\Sigma} (2 - \text{tr}_g h) [H(g) - \bar{H}] \lambda \ d\sigma_{\bar{g}}$$

$$\leq -m \int_{\Omega} |\nabla h|^2 \ d\text{vol}_{\bar{g}} - (LC_1 - C_2) \int_{\Sigma} |h|^2 \ d\sigma_{\bar{g}}$$

$$+ C||h||_{C^2(\Omega)} \left( \int_{\Omega} |\nabla h|^2 \ d\text{vol}_{\bar{g}} + \int_{\Sigma} |h|^2 \ d\sigma_{\bar{g}} \right).$$

where $C$ is independent on $h$. From this, we conclude that if $||h||_{C^2(\Omega)}$ is sufficiently small, then (4.13) holds with $\Phi = \frac{1}{2} \lambda$. This completes the proof. \[\square\]

**Remark 4.1.** When $\Omega \subset \mathbb{R}^n$ is a ball of radius $R$, one can take $a$ to be the center of $\Omega$. In this case, by computing $\bar{H}, \bar{\Pi}$ and $\lambda_n$ explicitly in (4.16), the constant $L$ can be chosen to be any constant satisfying

$$L > \left[ \frac{1}{2(n-1)} + \frac{4}{(n-1)^2} \right] R^2.$$ 

**Remark 4.2.** By the results in [12,17] based on the positive mass theorem [16,18], a metric $g$ on $\Omega$ satisfying the boundary conditions in Theorem 4.1 must be isometric to the Euclidean metric if $R(g) \geq 0$. Therefore, a nontrivial metric $g$ in Theorem 4.1 necessarily has negative scalar curvature somewhere. For such a $g$, Theorem 4.1 shows if the weighted integral $\int_{\Omega} R(g) \Phi \ d\text{vol}_g$ is nonnegative, then $V(g) \geq V(\bar{g})$.

5. Other related results

In this section, we collect some other by-products of the formulas derived in Section 2. First, we discuss a scalar curvature rigidity result for general domains in $\mathbb{S}^n$.

**Theorem 5.1.** Let $\Omega \subset \mathbb{S}^n$ be a smooth domain contained in a geodesic ball $B$ of radius less than $\frac{\pi}{2}$. Let $\bar{g}$ be the standard metric on $\mathbb{S}^n$. Let $\bar{\Pi}, \bar{H}$ be the second fundamental form, the mean curvature of $\Sigma = \partial \Omega$ in $(\Omega, \bar{g})$ with respect to the outward unit normal $\bar{\nu}$. Suppose $\bar{\Pi} \geq -c\bar{g}$, where $c \geq 0$ is a function on $\Sigma$ and $\bar{g}$ is the induced metric on $\Sigma$. Let $q$ be the center of $B$. Suppose at $\Sigma \setminus \{q\}$,

$$\bar{H} - c \geq \left[ \frac{5 \cos \theta + \sqrt{\cos^2 \theta + 8}}{2} \right] \tan r$$

where $r$ is the $\bar{g}$-distance to $q$ and $\theta$ is the angle between $\bar{\nu}$ and $\nabla r$. Then the conclusion of Theorem 4.1 holds on $\Omega$. 


Proof. As before, replacing $g$ by $\varphi^*(g)$ for some diffeomorphism $\varphi$, we may assume $\div g h = 0$ where $h = g - \bar{g}$. On $\Omega$, let $\lambda = \cos r > 0$, where $r$ is the $\bar{g}$-distance to $q$. At $\Sigma \setminus \{q\}$, we have

$$\lambda_n = -\sin r \cos \theta, \quad |\nabla^\Sigma \lambda| = \sin r \sin \theta.$$  

Apply Theorem 2.1, using the fact $D\mathcal{R}_{\bar{g}}^*(\lambda) = 0$ and the assumptions on $R(g)$ and $H(g)$, we have

$$\int_{\Omega} \left[ \frac{1}{4}(|\nabla h|^2 + |\nabla (\tr g h)|^2) + \frac{1}{2}(|h|^2 + (\tr g h)^2) \right] \cos r \ d\vol_{\bar{g}}$$

$$\leq \int_{\Sigma} \left[ -\frac{1}{4}(h_{nn})^2 \bar{H} - \frac{1}{2}(\bar{g}(X, X) + \bar{H}|X|^2) \right] \cos r \ d\sigma_{\bar{g}}$$

$$+ \int_{\Sigma \setminus \{q\}} \left[ (h_{nn})^2 + \frac{1}{2}|X|^2 \right] (\sin r \cos \theta) \ d\sigma_{\bar{g}} + \int_{\Sigma \setminus \{q\}} |h_{nn}| |X| (\sin r \sin \theta) \ d\sigma_{\bar{g}}$$

$$+ C||h||_{C^1(\bar{\Omega})} \left\{ \int_{\Omega} (|h|^2 + |\nabla h|^2) \ d\vol_{\bar{g}} + \int_{\Sigma} |h|^2 \ d\sigma_{\bar{g}} \right\}$$

$$\leq - \int_{\Sigma \setminus \{q\}} \left[ \left( \frac{1}{4}(\bar{H} - c) \cos r - \sin r \cos \theta \right) (h_{nn})^2 + \frac{1}{2} \left( (\bar{H} - c) \cos r - \sin r \cos \theta \right) |X|^2 \right.$$

$$- |h_{nn}| |X| (\sin r \sin \theta) \ d\sigma_{\bar{g}}$$

$$+ C||h||_{C^1(\bar{\Omega})} \left\{ \int_{\Omega} (|h|^2 + |\nabla h|^2) \ d\vol_{\bar{g}} + \int_{\Sigma} |h|^2 \ d\sigma_{\bar{g}} \right\}$$

for some positive constant $C$ independent on $h$.

Note that the assumption (5.1) implies

$$\frac{1}{4}(\bar{H} - c) \cos r - (\sin r \cos \theta) \geq 0$$

and

$$\bar{H} - c \cos r - (\sin r \cos \theta) \geq 0.$$  

By (5.1), (5.4) and (5.5), we have

$$0 \leq \left( \frac{1}{4}(\bar{H} - c) \cos r - \sin r \cos \theta \right) (h_{nn})^2 - |h_{nn}| |X| (\sin r \sin \theta)$$

$$+ \frac{1}{2} \left( (\bar{H} - c) \cos r - \sin r \cos \theta \right) |X|^2$$

for any $h_{nn}$ and $X$. The result now follows from (5.3) and (5.6). □

Remark 5.1. It is clear from the proof of Theorem 5.1 that the center $q$ of $B$ does not need to be inside $\Omega$. 

Theorem 5.1 directly implies Theorem 1.7 in the introduction.

Proof of Theorem 1.7. Choose $c = 0$ in Theorem 5.1. Since
\[
4 \geq \frac{5 \cos \theta + \sqrt{\cos^2 \theta + 8}}{2}
\]
for any $\theta$, the result follows from Theorem 5.1. \(\square\)

Next, we consider a corresponding scalar curvature rigidity result when the background metric $\bar{g}$ is a flat metric.

Theorem 5.2. Let $\Omega$ be a compact manifold with smooth boundary $\Sigma$. Suppose $\bar{g}$ is a smooth Riemannian metric on $\Omega$ such that $\bar{g}$ has zero sectional curvature and $\bar{\Pi} + \bar{H} \bar{\gamma} \geq 0$ on $\Sigma$, where $\bar{\Pi}$, $\bar{H}$ are the second fundamental form, the mean curvature of $\Sigma$, and $\bar{\gamma}$ is the induced metric on $\Sigma$. Suppose $g$ is another metric on $\Omega$ satisfying
- $R(g) \geq 0$ where $R(g)$ is the scalar curvature of $g$
- $g$ and $\bar{g}$ induce the same metric on $\Sigma$
- $H(g) \geq H$ where $H(g)$ is the mean curvature of $\Sigma$ in $(\Omega, g)$.

If $\|g - \bar{g}\|_{C^2(\bar{\Omega})}$ is sufficiently small, then there is a diffeomorphism $\varphi$ on $\Omega$ with $\varphi|_\Sigma = \text{id}$ such that $\varphi^*(g) = \bar{g}$.

Proof. As before, we may assume $\text{div}_{\bar{g}} h = 0$ where $h = g - \bar{g}$. Choose $\lambda = 1$ in (2.23), one has
\[
\int_{\Omega} \left[ \frac{1}{4}(|\nabla h|^2 + |\nabla(\text{tr}_g h)|^2) \right] \, d\text{vol}_{\bar{g}} + \int_{\Sigma} \left[ \frac{1}{4}(h_{nn})^2 H(\bar{g}) + \frac{1}{2} \bar{\Pi}(X, X) + H(\bar{g})|X|^2 \right] \, d\sigma_{\bar{g}} 
\leq \int_{\Omega} E(h) \, d\text{vol}_{\bar{g}} + \int_{\Sigma} F(h) \, d\sigma_{\bar{g}}
\]
where $|F(h)| \leq C(|h|^2|\nabla h| + |h|^3)$ and $|E(h)| \leq C|h||\nabla h|^2$ by Remark 2.1. The result follows from (5.7). \(\square\)

To finish, we mention that the positive Gaussian curvature condition of the boundary surface in [17] is not a necessary condition for the positivity of its Brown-York mass.

Theorem 5.3. Let $\Sigma \subset \mathbb{R}^n$ be a connected, closed hypersurface satisfying $\bar{\Pi} + \bar{H} \bar{\gamma} \geq 0$, where $\bar{\Pi}$, $\bar{H}$ are the second fundamental form, the mean curvature of $\Sigma$, and $\bar{\gamma}$ is the induced metric on $\Sigma$. Let $\Omega$ be the domain enclosed by $\Sigma$ in $\mathbb{R}^n$. Let $h$ be any nontrivial $(0, 2)$ symmetric tensor on $\Omega$ satisfying
\[
\text{div}_{\bar{g}} h = 0, \quad \text{tr}_g h = 0, \quad h|_{T\Sigma} = 0.
\]
Let \( \{g(t)\}_{|t|<\epsilon} \) be a 1-parameter family of metrics on \( \Omega \) satisfying
\[
g(0) = \bar{g}, \quad g'(0) = h, \quad R(g(t)) \geq 0, \quad g(t)|_{\Sigma} = \bar{g}|_{\Sigma}.\tag{5.9}
\]
Then
\[
\int_{\Sigma} H d\sigma_{\bar{g}} > \int_{\Sigma} H(g(t)) d\sigma_{\bar{g}} \tag{5.10}
\]
for small \( t \neq 0 \), where \( H(g(t)) \) is the mean curvature of \( \Sigma \) in \((\Omega, g(t))\).

Proof. By Lemma 2.2, one knows
\[
\frac{d}{dt} \left( \int_{\Omega} [R(g(t)) - R(\bar{g})] \ d\text{vol}_{\bar{g}} - 2 \int_{\Sigma} [\bar{H} - H(g(t))] \ d\sigma_{\bar{g}} \right) \bigg|_{t=0} = 0.
\]
Direct calculation using Lemma (2.2) (2.17) and (5.8) shows
\[
\frac{d^2}{dt^2} \left( \int_{\Omega} [R(g(t)) - R(\bar{g})] \ d\text{vol}_{\bar{g}} - 2 \int_{\Sigma} [\bar{H} - H(g(t))] \ d\sigma_{\bar{g}} \right) \bigg|_{t=0} = -\frac{1}{2} \int_{\Omega} |\nabla h|^2 \ d\text{vol}_{\bar{g}} - \int_{\Sigma} [\frac{1}{2}(\bar{\Pi}(X,X) + H(\bar{g})|X|^2) \ d\sigma_{\bar{g}}
\]
which is negative by the assumption on \( \bar{\Pi} + \bar{H}\bar{\gamma} \). Thus, for small \( t \),
\[
2 \int_{\Sigma} [\bar{H} - H(g(t))] \ d\sigma_{\bar{g}} > \int_{\Omega} [R(g(t)) - R(\bar{g})] \ d\text{vol}_{\bar{g}} \geq 0. \tag{5.11}
\]

Given an \( h \) satisfying (1.6), a family of deformation \( \{g(t)\} \) satisfying (1.7) is given by \( g(t) = u(t)\bar{g} \rightarrow (\bar{g} + th) \) for small \( t \), where \( u(t) > 0 \) is a conformal factor such that \( R(g(t)) = 0 \) (see [13, Lemma 4]).

An example of a non-convex surface \( \Sigma \subset \mathbb{R}^3 \), which is topologically a 2-sphere and satisfies the condition \( \bar{\Pi} + \bar{H}\bar{\gamma} \geq 0 \), is given by a capsule-shaped surface with its middle slightly pinched.

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