A canonical barycenter via Wasserstein regularization *

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Abstract

We introduce a weak notion of barycenter of a probability measure $\mu$ on a metric measure space $(X,d,m)$, with the metric $d$ and reference measure $m$. Under the assumption that optimal transport plans are given by mappings, we prove that our barycenter $B(\mu)$ is well defined; it is a probability measure on $X$ supported on the set of the usual metric barycenter points of the given measure $\mu$. The definition uses the canonical embedding of the metric space $X$ into its Wasserstein space $P(X)$, pushing a given measure $\mu$ forward to a measure on $P(X)$. We then regularize the measure by the Wasserstein distance to the reference measure $m$, and obtain a uniquely defined measure on $X$ supported on the barycentric points of $\mu$. We investigate various properties of $B(\mu)$.

1 Introduction

In this paper, we propose a canonical weak notion of barycenter of a probability measure on a compact metric measure space $(X,d,m)$, where we assume that $m$ is a probability measure.

For a separable compact metric space $X$, we let $P(X)$ denote the space of Borel probability measures equipped with the weak-* topology. Given a measure $\mu \in P(X)$, a barycenter (or Fréchet mean) on $\mu$ is a minimizer of the average

\[ \int_X d(x,y)^2 \, d\mu(x) \]
squared distance to points in the support of \( \mu \); that is, an element of
\[
b(\mu) = \arg\min \left( y \mapsto \int_X d^2(x, y)d\mu(x) \right).
\] (1.1)

These metric barycenters are natural generalizations of centers of mass of distributions on Euclidean space. As the function \( y \mapsto \int_X d^2(x, y)d\mu(x) \) is in general highly non-convex, except under strong additional criteria (for example, that \( X \) a simply connected, non-positively curved space), the minimizer may not be unique; that is, \( b(\mu) \) may not be a singleton. Choosing a unique canonical minimizer in any reasonable sense is clearly impossible; for example, if \( \mu \) is a uniform measure on the equator of the round sphere, the north and south pole are both barycenters and neither is any more natural than the other. To develop a canonical notion of barycenter, we utilize the geometry of \( P(X) \) as follows.

Via the natural isometry \( x \mapsto \delta_x \), one may isometrically embed the set \( X \) into \( P(X) \) equipped with Wasserstein metric
\[
W_2(\mu, \nu) := \inf \int_{X \times X} d^2(x, y)d\gamma(x, y) \tag{1.2}
\]
where the infimum is over the set of all probability measures \( \gamma \) on \( X \times X \) with marginals \( \mu \) and \( \nu \). The minimization (1.2) is the well known Monge-Kantorovich problem, reviewed for example, in books of Villani [16, 17], and more recently Santambrogio [12]; our particular interest here lies in the fact that the Wasserstein distance extends the underlying geometry on \( X \) to \( P(X) \). It is also relevant here (and well known) that \( P(X) \) with this metric inherits compactness from \( X \), and that the Wasserstein metric topology coincides with the weak-* topology.

One can then consider instead the barycenter of the image measure \( (x \mapsto \delta_x) \# \mu \) on the metric space \( (P(X), W_2) \), which amounts to finding a minimizer of
\[
\nu \mapsto \int_X W_2^2(\nu, \delta_x)d\mu(x) = \int_X \int_X d^2(y, x)d\nu(y)d\mu(x) \tag{1.3}
\]
\[
= \int_X \int_X d^2(y, x)d\mu(x)d\nu(y).
\]

The general study of barycenters in the Wasserstein space was initiated by Agueh-Carlier [1] when the underlying space \( X = \mathbb{R}^n \) is Euclidean, and continued by the present authors to the setting where \( X \) is a smooth Riemannian manifold [8]. These represent a natural, non-linear way to interpolate between several (or infinitely many) probability measures, and have received a great deal of attention in recent years, due in part to important applications in image processing and statistics; see, for example, the work of Rabin et al. [11] and Bigot and Klein [2] among others. Clearly, if \( x \in BC(\mu) \) is a barycenter of \( \mu \), then \( \delta_x \) is a barycenter of \( (x \mapsto \delta_x) \# \mu \), and so this minimization can be considered a relaxation of the barycenter problem on \( X \). Although the later problem is on the infinite dimensional space \( P(X) \) rather than \( X \), it has the
advantage of being a linear minimization. On the other hand, it certainly does not resolve the non-uniqueness issue; the minimizers of (1.3) are exactly those measures which are supported on $b(\mu)$, while only the Dirac masses supported on the same set correspond to classical barycenters.

On the other hand, one can try to define a canonical, distinguished minimizer of (1.3). For that purpose, throughout the paper we assume that $(X, d, m)$ satisfies a regularity condition (see Definition 2.1); that is, the minimizers of (1.2) in the definition of $W_2(m, \nu)$ are realized as mappings pushing $m$ forward to $\nu$. Examples of regular spaces $X$ include any smooth Riemannian manifold (with $m$ absolutely continuous with respect to local coordinates) due to the Brenier-McCann theorem [3, 10], and more general (singular) metric measure spaces including the $CD(K, N)$ space in the sense of Lott-Villani-Sturm [9, 14, 15]; see the work of Cavaletti and Huesmann [4], Gigli [6], and Gigli, Rajala, and Sturm [7]. In this case, we show that one can pick the unique minimizer which is the most spread out, or the closest to the reference measure $m$. This is the approach adopted in this paper.

For $\epsilon > 0$, consider the function on $P(X)$ defined by

$$F_\epsilon(\nu) := \int_X \int_X d^2(y, x)d\mu(x)d\nu(y) + \epsilon W_2^2(m, \nu).$$ (1.4)

It is not hard to establish using regularity of $X$, that $F_\epsilon$ has a unique minimizer $\mu_\epsilon$. In our main theorem (Theorem 2.2) we show that as $\epsilon \to 0$, this minimizer converges weakly-* to a unique measure $\nu$ which minimizes $W_2^2(m, \cdot)$ among all minimizers of (1.3). We call this distinguished minimizer the Wasserstein regularized barycenter of $\mu$, and denote it by $B(\mu)$. This Wasserstein regularized barycenter is a weak notion of the metric barycenter; moreover, we verify that $B(\mu)$ is the unique minimizer of $\nu \mapsto W_2(m, \nu)$ among all $\nu \in P(X)$ with $\text{supp} \nu \subset b(\mu)$.

The mapping $\mu \mapsto B(\mu)$ induces a subtle dynamical structure on $P(X)$, and we go on to establish some notable properties of it: the mapping reduces the variance (Corollary 3.3), has fixed points (including Dirac masses, but also others, see Remark 2.3 and Example 3.5), has periodic orbits of period 2 (Examples 3.4 and 3.5) but not greater (Corollary 3.6) and is monotone in convex order (Corollary 3.7).

In the next section, we prove our main theorem regarding $B(\mu)$, while Section 3 is reserved for the development of various properties of the mapping $B : P(X) \to P(X)$. 

2 Wasserstein regularized Riemannian barycenter $B(\mu)$: existence, uniqueness and characterization.

In this section, we prove that the Wasserstein regularized barycenter $B(\mu)$ of $\mu \in P(X)$ is well defined; that is, the weak-* limit $B(\mu)$ of the $\mu_\epsilon$ as $\epsilon \to 0$ exists.
Moreover, we characterize $B(\mu)$ as the probability measure supported on the set $b(\mu)$ of barycentric points of $\mu$ which is closest to the reference measure $m$ in the Wasserstein distance.

**Definition 2.1 (Regularity of $X$).** We say that the metric measure space $(X, d, m)$ is regular if for every $\nu \in \mathcal{P}(M)$, any minimizer $\gamma$ in the definition (1.2) of $W_2(m, \nu)$ is concentrated on the graph of a function; that is, $\gamma = (Id, T)_# m$ for a map $T : X \to X$.

As mentioned in the introduction, many metric measure spaces are regular, including any compact smooth Riemannian manifold where the reference measure $m$ is absolutely continuous with respect to volume, and other more singular spaces such as $CD(K, N)$ spaces. The case where $m$ is (normalized) Riemannian volume is our motivating example.

It is well known that regularity ensures that the function $\nu \mapsto \epsilon W_2^2(\nu, m)$ is strictly convex with respect to linear interpolation between measures in $\mathcal{P}(X)$ (see, e.g. [12, Proposition 7.19]). Since the functional $\nu \mapsto \int_X \int_X d^2(y, x)d\mu(x)d\nu(y)$ is linear, we see that $F_\epsilon$ in (1.4) is strictly convex in $\nu$ (with respect to linear interpolation); therefore its minimizer, $\mu_\epsilon$ is unique. Then, from the weak-* compactness of probability measures as $\epsilon \to 0$, a limit point of the $\mu_\epsilon$ exists. The following result establishes uniqueness of this limit point (that is, the limit exists) and a characterization of it.

**Theorem 2.2 (Characterization of $B(\mu)$).** As $\epsilon \to 0$, $\mu_\epsilon$ converges weak-* to a unique limit $B(\mu)$ and we have

$$\text{supp}(m) \cap b(\mu) \subseteq \text{supp} B(\mu) \subseteq b(\mu),$$

where $b(\mu)$ is the set of barycentric points defined in (1.1). Moreover, $B(\mu)$ is the unique minimizer of the functional $\nu \mapsto W_2(\nu, m)$ among $\nu \in \mathcal{P}(X)$ with $\text{supp} \nu \subset b(\mu)$, i.e.

$$\{B(\mu)\} = \arg\min_{\text{supp} \nu \subset b(\mu)} W_2(\nu, m).$$

**Proof.** For notational simplicity let us denote

$$d_0 = \min \left( y \mapsto \int_X d^2(x, y)d\mu(x) \right).$$

**Step 1:** A standard continuity-compactness argument yields the existence of a minimizer $\bar{\mu}$ in (2.1). Now note that the set of probability measures supported on $b(\mu)$ is convex (with respect to linear interpolation) and so as the functional in (2.1) is strictly convex by [12] Proposition 7.19, the minimizer $\bar{\mu}$ is unique.

**Step 2:** Now, let $\mu_0$ be any limit point of the minimizers $\mu_\epsilon$ of $F_\epsilon$ as $\epsilon \to 0$. We will show that $\text{supp} \mu_0 \subset b(\mu)$. Suppose $z \in \text{supp} \mu_0$, but $z \notin b(\mu)$. As the set $b(\mu)$ is closed, there exists $r > 0$ such that for the metric ball $B_r(z)$ of radius
Fix a point $\bar{\mu}$ and so $\mu_0(B_r(z)) > 0$. Since $B_{2r}(z)$ is disjoint from $b(\mu)$, we have $\eta > 0$ such that
\[
\int_X d^2(x, y) d\mu(x) > d_0 + \eta \quad \text{for all } y \in B_r(z),
\]
and so
\[
\int_X \int_X d^2(x, y) d\mu(x) d\mu_0(y) \geq d_0(1 - \mu_0(B_r(z))) + (d_0 + \eta)\mu_0(B_r(z)) = d_0 + \eta\mu_0(B_r(z)).
\]

Fix a point $\bar{x} \in b(\mu)$ and define $\tilde{\mu} = \delta_{\bar{x}}$. Then,
\[
F_\epsilon(\mu_\epsilon) \leq F_\epsilon(\tilde{\mu}) = d_0 + \epsilon W_2^2(\mu, \tilde{\mu}) \\
\leq \int_X \int_X d^2(x, y) d\mu(x) d\mu_0(y) - \eta\mu_0(B_r(z)) + \epsilon W_2^2(\mu, \mu_0) + \epsilon W^2(\mu_\epsilon, \mu_0) \\
\leq F_\epsilon(\mu_0) - \frac{\eta}{2}\mu_0(B_r(z)) \quad \text{for sufficiently small } \epsilon.
\]

As $F_\epsilon(\nu)$ is continuous in both $\nu$ (with respect to the weak-* topology) and $\epsilon$, and $\mu_0$ is a limit point of the $\mu_\epsilon$ converges, this is a contradiction for small $\epsilon$, establishing supp $\mu_0 \subset b(\mu)$.

**Step 3:** We now show that any limit point $\mu_0$ of the $\mu_\epsilon$ must coincide with $\tilde{\mu}$; this will show that the limit $B(\mu)$ is well defined and equal to $\tilde{\mu}$.

To see this, suppose by contradiction that there is a limit point $\mu_0 \neq \tilde{\mu}$ of the $\mu_\epsilon$. Then by steps 1 and 2, there is a $\delta > 0$ with
\[
W_2^2(\tilde{\mu}, \mu_0) \leq W_2^2(\mu_0, \mu) - \delta.
\]

Choose $\epsilon > 0$ sufficiently small so that $W_2(\mu_0, \mu_\epsilon) \leq \frac{\delta}{4D}$, where $D$ is the diameter of $X$. We have that
\[
W_2^2(\mu_0, \mu) - W_2^2(\mu_0, \mu_\epsilon) = [W_2(\mu_0, \mu) + W_2(\mu, \mu_\epsilon)][W_2(\mu_0, \mu) - W_2(\mu, \mu_\epsilon)] \\
\leq 2DW_2(\mu_0, \mu_\epsilon) \leq \frac{\delta}{2},
\]
and so
\[
F_\epsilon(\mu_\epsilon) = \int_X \int_X d^2(x, y) d\mu(x) d\mu_\epsilon(y) + \epsilon W_2^2(\mu_\epsilon, \mu) \\
\geq \int_X \int_X d^2(x, y) d\mu(x) d\tilde{\mu}(y) + \epsilon W_2^2(\mu_0, \mu) - \frac{\delta}{2} \\
\geq \int_X \int_X d^2(x, y) d\mu(x) d\tilde{\mu}(y) + \epsilon W_2^2(\tilde{\mu}, \mu) + \delta - \frac{\delta}{2} > F_\epsilon(\tilde{\mu}) + \delta/2,
\]
which contradicts that $\mu_\epsilon$ is a minimizer of $F_\epsilon$, and therefore establishes that $\mu_\epsilon$ converges to $B(\mu) = \tilde{\mu}$ as $\epsilon \to 0$. 

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Step 4. Finally, we verify that \( b(\mu) \cap \text{supp}(m) \subset \text{supp} \bar{\mu} \). Suppose not; then there exists a barycenter point \( \bar{x} \in b(\mu) \) and \( r > 0 \) such that the metric ball \( B_r(\bar{x}) \) satisfies \( m(B_r(\bar{x})) > 0 \) and \( \bar{\mu}(B_{2r}(\bar{x})) = 0 \). Now, let \( \gamma \in \Gamma(m, \bar{\mu}) \) be an optimal transport plan from \( m \) to \( \bar{\mu} \). Define \( \bar{\mu} \) as

\[
\bar{\mu} = \bar{\mu} - \pi^2 B_r(\bar{x}) \times X + m(B_r(\bar{x})) \delta_{\bar{x}}
\]

where \( \pi^2 \) is the projection \( X \times X \ni (x, y) \mapsto y \in X \) and \( \gamma_{B_r(\bar{x}) \times X} \) is the restriction of \( \gamma \) to the set \( B_r(\bar{x}) \times X \). Then, as \( \bar{\mu} \) is supported on \( b(\mu) \), so is \( \bar{\mu} \). Define \( \bar{\gamma} \), a transport plan from \( m \) to \( \bar{\mu} \), as

\[
\bar{\gamma} = \gamma - \gamma_{B_r(\bar{x}) \times X} + m_{B_r(\bar{x})} \otimes \delta_{\bar{x}}.
\]

Note this plan \( \bar{\gamma} \) modifies \( \gamma \) by transporting the mass on \( B_r(\bar{x}) \) to the Dirac at \( \bar{x} \). From the assumption \( \bar{\mu}(B_{2r}(\bar{x})) = 0 \), we see that

\[
\int_{B_r(\bar{x}) \times X} d^2(x, y) d\gamma(x, y) > 4 m(B_r(\bar{x})) r^2.
\]

Therefore, from the obvious inequality

\[
\int d^2(x, y) dm_{B_r(x)} \otimes \delta_{\bar{x}}(x, y) \leq r^2 m(B_r(\bar{x})),
\]

we get

\[
W_2^2(m, \bar{\mu}) \leq W_2^2(m, \bar{\mu}) - 3 m(B_r(\bar{x})) r^2,
\]

contradicting the characterization of \( \bar{\mu} \) as the minimizer of \( \nu \mapsto W_2^2(m, \nu) \) among probability measures supported on \( b(\mu) \).

Remark 2.3 (Instability). The mapping \( \mu \mapsto B(\mu) \) is highly unstable. To see this, consider uniform measure \( \mu \) on the round sphere \( X \) with \( m = \text{vol} \). Symmetry considerations easily imply that \( B(\mu) = \mu \). Now, set \( \mu^\epsilon := \epsilon p_n + (1 - \epsilon) \mu \), where \( p_n \) is the north pole. Then for any \( \epsilon > 0 \), it is easy to see that \( B(\mu^\epsilon) = \delta_{p_n} \). As \( \mu^\epsilon \) is close to \( \mu \), but \( B(\mu^\epsilon) = p_n \) is far from \( B(\mu) = \mu \) in any reasonable topology, we conclude that \( B \) is not stable.

3 Further properties of \( B(\mu) \).

We now investigate some properties of \( B(\mu) \). First, as an immediate corollary of the characterization of \( B(\mu) \) in Theorem 2.2 we have the following result:

Corollary 3.1 (Support determines \( B(\mu) \)). If \( \text{supp} B(\mu) = \text{supp} B(\nu) \), then \( B(\mu) = B(\nu) \).

Next, note that we can interpret (2.1) as characterizing \( B(\mu) \) as the projection of the reference measure \( m \) to the set of measures supported on \( b(\mu) \), with respect to Wasserstein distance. In the Riemannian setting, the following result makes this characterization more explicit.
Corollary 3.2. Assume $X$ is Riemannian and smooth. For almost all $x$ (with respect to $\text{vol}$), there is a unique $y \in \arg\min_{y \in b(\mu)} d^2(x, y)$. Denoting the unique minimizer $y = T(x)$, and assuming $m$ is absolutely continuous with respect to $\mu$, this $T$ is the optimal transport mapping from $m$ to $B(\mu)$; in particular,

$$T#m = B(\mu).$$

Proof. Set $f(x) = \min_{y \in b(\mu)} d^2(x, y)$. By a now standard argument of McCann [10], $f$ is Lipschitz and hence differentiable almost everywhere (with respect to $\text{vol}$) by Rademacher’s theorem. Another argument in [10] implies that $x \mapsto d^2(x, y)$ is differentiable whenever $f$ is, and, as $f(x) - d^2(x, y) \leq 0$ for all $y$, with equality for $y \in \arg\min_{y \in b(\mu)} d^2(x, y)$, we have, for all $x$ at which $f$ is differentiable and all $y \in \arg\min_{y \in b(\mu)} d^2(x, y)$

$$\nabla f(x) = \nabla_x (d^2(x, y)).$$

Equivalently, $y = \exp_x (2D f(x)) := T(x)$; that is, $y$ is uniquely determined by $x$. This holds wherever $f$ is differentiable, and therefore $m$-a.e.

Now, for any other $\nu$ supported on $b(\mu)$ letting $T_\nu$ be the optimal map from $m$ to $\nu$, we have $d(x, T(x)) \leq d^2(x, T_\nu(x))$ for almost all $x$ and so

$$W_2(m, T#m) \leq \int_X d^2(x, T(x))d\text{vol}(x) \leq \int_X d^2(x, T_\nu(x))d\text{vol}(x) = W_2(m, \nu)$$

which establishes minimality of $T#m$ and therefore that $T#m = B(\mu)$, by Theorem 2.2 and that $T$ is the optimal map between $m$ and $B(\mu).$ □

Next, we note that, although $B(\mu)$ may not be supported on a single point, it is at least no more spread out than $\mu$, in the sense that it has lower variance, $\text{var}(\mu) := \min_{y \in X} \int_X d^2(x, y)d\mu(x)$.

Corollary 3.3 (Variance reduction). For $\mu \in P(X)$,

$$\text{var}(B(\mu)) \leq \text{var}(\mu).$$

Moreover, the equality holds if and only if $\text{supp}\mu \subset \text{supp}B(\mu)$.

Proof. Observe that $\text{var}(\mu) = \int_X \int_X d^2(x, y)d\mu(x)dB(\mu)(y)$ since from Theorem 2.2 $B(\mu) = b(\mu)$ where $b(\mu)$ is the set of barycenter points of $\mu$. Now note that

$$\text{var}(B(\mu)) = \min_{x \in X} \int_X d^2(x, y)dB(\mu)(y)$$

$$= \min_{\nu \in P(X)} \int_X \int_X d^2(x, y)d\text{vol}(x)dB(\mu)(y)$$

$$\leq \int_X \int_X d^2(x, y)d\mu(x)dB(\mu)(y)$$

$$= \text{var}(\mu).$$

The equality holds if and only if $\mu$ is a minimizer of $\nu \mapsto \int_X \int_X d^2(x, y)dB(\mu)(x)d\nu(y)$.

This is equivalent to $\text{supp}\mu \subset \text{supp}B(\mu)$. □
The equality case above is illustrated in the following simple examples:

**Example 3.4.** Let $X$ be the $n$-dimensional Riemannian round sphere with $m = \text{vol}$, and let $\mu = \frac{1}{2} \delta_{p_s} + \frac{1}{2} \delta_{p_n}$, where $p_s$ and $p_n$ are the south and north poles, respectively. Then, $B(\mu)$ is the uniform probability measure on the equator, and $B(B(\mu)) = \mu$.

**Example 3.5.** Let $X = S_1$ be the circle, $m$ be the normalized arc-length and $\mu = \sum_{i=1}^{N} \frac{1}{N} \delta_{x_i}$, where $\{x_1, ..., x_N\}$ are evenly spaced points on $X$. The set $b(\mu)$ of minimal points of the function $y \mapsto \sum_{i=1}^{N} \frac{1}{N} d^2(y, x_i)$ depends on the parity of $N$:

1. If $N$ is odd, the function is minimized at each $x_i$, and so by rotational symmetry the regularized barycenter is $B(\mu) = \sum_{i=1}^{N} \frac{1}{N} \delta_{x_i} = \mu$; that is, $\mu$ is a fixed point of $B$.

2. If $N$ is even, the minimizing points are exactly those points $y_i, i = 1, 2, ..., N$ which are halfway in between two neighbouring $x_i$s, and so $B(\mu) = \sum_{i=1}^{N} \frac{1}{N} \delta_{y_i}$.

An identical argument then yields $B(B(\mu)) = \mu$.

As we see from the past two examples, it is possible that the operation $\mu \mapsto B(\mu)$ in $P(X)$ may have a periodic orbit. We next prove that no orbit can be periodic with period greater than two.

**Corollary 3.6 (Period is at most 2.).** Suppose that $B^N(\mu) = \mu$ for some positive integer $N$. Then $B^2(\mu) = \mu$.

**Proof.** The general inequality $\text{var}(B(\mu)) \leq \text{var}(\mu)$ combined with periodicity easily implies that $\text{var}(B^k(\mu)) = \text{var}(\mu)$ for all positive integers $k$. In particular, $\text{var}(\mu) = \text{var}(B(\mu))$, and Corollary 3.3 implies that $\text{supp}(\mu) \subseteq \text{supp}(B^2(\mu)) \subseteq \text{supp}(B^4(\mu)) \subseteq ... \subseteq \text{supp}(B^{2N}(\mu)) = \text{supp}(\mu)$. As the first and last terms in this string of inclusions coincide, we must have equality throughout; in particular, $\text{supp}(\mu) = \text{supp}(B^2(\mu))$. Therefore, as $\mu = B^N(\mu) = B(B^{N-1}(\mu))$, we have

$$\text{supp}(B(B(\mu))) = \text{supp}(B^2(\mu)) = \text{supp}(\mu) = \text{supp}(B(B^{N-1}(\mu)))$$

and so Corollary 3.3 implies that $B(\mu) = B^{N-1}(\mu)$. Applying $B$ to both sides we arrive at

$$\mu = B^N(\mu) = B(B^{N-1}(\mu)) = B(B(\mu)) = B^2(\mu).$$

This completes the proof. $\square$

Finally, we record a version of Jensen’s inequality for the Wasserstein regularized barycenter, which might alternatively be interpreted as expressing a convex order between $B(\mu)$ and $\mu$ (see Remark 3.8 below). Recall that a function $\phi : X \to \mathbb{R}$ is said to be geodesically convex if for each geodesic segment $\sigma : [0,1] \to X$, the function $\phi(\sigma(t))$ is convex.
Corollary 3.7 (Monotonicity in convex order). Assume that $X$ is a Riemannian manifold and $\mu$ is absolutely continuous with respect to vol. For any geodesically convex function $\phi$ on $X$, we have

$$\int_X \phi(x)d(B(\mu))(x) \leq \int_X \phi(x)d\mu(x).$$

Proof. First, we recall that the classical Jensen’s inequality extends to Riemannian manifolds (see, for instance, [5, Proposition 2]), asserting that, for any $y \in b(\mu)$, and geodesically convex function $\phi$,

$$\phi(y) \leq \int_X \phi(x)d\mu(x). \tag{3.1}$$

Now, as $\text{supp}(B(\mu)) \subseteq b(\mu)$, integrating (3.1) yields the desired result. We conclude the paper with a brief remark, offering some perspective on the preceding corollary.

Remark 3.8 (Martingales and convex order on Riemannian manifolds). Recall that a coupling $\pi$ between two probability measures $\mu$ and $\nu$ on $\mathbb{R}^n$ is a (discrete) martingale for $(\mu, \nu)$ if for $\mu$ almost every $x$,

$$x = \int_{\mathbb{R}^n} yd\pi_x(y)$$

where $\pi_x$ represents the disintegration of $d\pi(x, y) = d\pi_x(y)d\mu(x)$ with respect to $\mu$. Strassen’s coupling theorem [3] asserts that there exists a martingale coupling of $\mu$ and $\nu$ if and only if $\nu$ dominates $\mu$ in convex order; that is $\int_{\mathbb{R}^n} \phi(x)d\mu(x) \leq \int_{\mathbb{R}^n} \phi(x)d\nu(x)$ for all convex $\phi : \mathbb{R}^n \to \mathbb{R}$.

On a metric space $X$, it is natural to define a martingale coupling of $\mu, \nu \in P(X)$ to be a coupling $d\pi(x, y) = d\pi_x(y)d\mu(x)$ such for $\mu$-a.e. $x$, we have

$$x \in b(\pi_x).$$

By analog with the Euclidean case, we will say that $\mu$ dominates $\nu$ in (geodesically) convex order if we have $\int_X \phi(x)d\mu(x) \leq \int_X \phi(x)d\nu(x)$ for all geodesically convex $\phi$.

On a smooth Riemannian manifold, it is not hard to see (using (3.1)) that, if there exists a martingale coupling of $\nu$ and $\mu$ then $\mu$ dominates $\nu$ in convex order; that is, one implication of Strassen’s theorem extends to manifolds. The converse fails in general; on the sphere, for example, the only geodesically convex functions are constants, so any $\mu$ dominates any $\nu$ in convex order.

The preceding proposition asserts that on any manifold, $\mu$ dominates $B(\mu)$ in convex order. In this case, it is worth noting that product measure is a martingale between them. So the collection of pairs $\{(B(\mu), \mu)\}_{\mu \in P(X)}$ is a collection of marginals for which the conclusion of Strassen’s theorem extends to manifolds. We also note that when $B^2(\mu) = \mu$, product measure is a martingale with respect to either order; that is, it is a martingale for $(B(\mu), \mu)$ and for $(\mu, B(\mu)) = (B(B(\mu)), B(\mu))$. 

9
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