Polynomial Sequences of Binomial Type and Path Integrals

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Abstract

Polynomial sequences $p_n(x)$ of binomial type are a principal tool in the umbral calculus of enumerative combinatorics. We express $p_n(x)$ as a path integral in the “phase space” $\mathbb{N} \times [-\pi, \pi]$. The Hamiltonian is $h(\phi) = \sum_{n=0}^{\infty} p'_n(0)/n! e^{i\phi}$ and it produces a Schrödinger type equation for $p_n(x)$. This establishes a bridge between enumerative combinatorics and quantum field theory. It also provides an algorithm for parallel quantum computations.

Contents

1 Introduction on Polynomial Sequences of Binomial Type ........................................... 2
2 Preliminaries on Path Integrals .................................................................................. 4
3 Polynomial Sequences from Path Integrals ................................................................. 6
4 Some Applications ....................................................................................................... 10

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Polynomial Sequences and Path Integrals

Under the inspiring guidance of Feynman, a shorthand way of expressing—and of thinking about—these quantities have been developed.
Lewis H. Ryder [22, Chap. 5].

1 Introduction on Polynomial Sequences of Binomial Type

The umbral calculus [18, 19, 21] in enumerative combinatorics uses polynomial sequences of binomial type as a principal ingredient. A polynomial sequence $p_n(x)$ is said to be of binomial type [18, p. 102] if $p_0(x) = 1$ and if it satisfies the binomial identity,

$$p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$$

for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$.

Example 1.1 Among many examples [18, § 5] of polynomial sequences of binomial type we mention only very few:

1. The power monomials $p_n(x) = x^n$.
2. The rising factorial sequence $p_n(x) = x(x+1) \cdots (x+n-1)$.
3. The falling factorial sequence $p_n(x) = x(x-1) \cdots (x-n+1)$.
4. The Abel polynomials $A_n(x) = x(x-an)^{n-1}$.
5. The Laguerre polynomials

$$L_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{k-1} (-x)^k.$$ (2)

Using delta functionals one can show that [18, p. 106] that for any polynomial sequence of binomial type $p_n(x)$:

$$p_n(0) = \delta_{n0} = \begin{cases} 1, & n = 0; \\ 0, & n > 0. \end{cases}$$ (3)

For a polynomial sequence $p_n(x)$ we could consider its coefficients $a_{n,k}$ in its decomposition over power monomials $x^k$:

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k.$$
The coefficients $a_{n,k}$ are an example of connecting constants [18, § 6] between sequences $p_n(x)$ and $x^n$. The definition (1) of polynomial sequences of binomial type can be expressed in the term of coefficients $a_{n,k}$ as follows [18, Prop. 4.3]:

$$
\left( i + j \right) a_{n,i+j} = \sum_{k=0}^{n} \binom{n}{k} a_{k,i}a_{n-k,j}.
$$

(4)

Therefore these coefficients are highly interdependent. Particularly from (3) it follows that $a_{n,0} = \delta_{n,0}$. Moreover one can observe from (1) that the knowledge of the sequence $c_n = a_{n,1}$, $n = 0, 1, \ldots$ allows to reconstruct all $a_{n,k}$ by a recursion. Sequence $c_n = a_{n,1}$ is known as cumulants associated with polynomials $p_n(x)$ and the related probability measure, see e.g. [15] for a survey. One may state the following

**Problem 1.2** Give a direct expression for a polynomial sequence $p_n(x)$ of binomial type from the sequence of its cumulants $c_n = a_{n,1} = p_n'(0)$, $n = 0, 1, \ldots$.

There are several well known solutions to that problem. The umbral calculus suggests the following procedure (see [12] for an interesting exposition). Let us define the formal power series:

$$
f(t) = \sum_{n=1}^{\infty} c_n \frac{t^n}{n!},
$$

(5)

then the function $e^{xf(t)}$ is a generating function for polynomials $p_n(x)$:

$$
\sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!} = e^{xf(t)}.
$$

(6)

The main goal of this paper is to present a different solution to the Problem 1.2. It is probably not the best from the computation point of view unless computations are made in parallel by a quantum computer, see Remark 4.1. We will express $p_n(x)$ as a path integral (23) in the “phase space” $\mathbb{N} \times [-\pi, \pi]$. The Fourier transform $h(\phi) = \sum_{n=0}^{\infty} c_n e^{i n \phi}$ of the sequence $c_n/n! = p_n'(0)/n!$ plays the rôle of a Hamiltonian. Thus we establish a connection between enumerative combinatorics and quantum field theory. This connection could be useful in both areas. It is interesting to note that quantum mechanical commutation relations $PQ - QP = I$ was applied to umbral calculus by J. Cigler [2]. Existence of a “yet-to-be-discovered stochastic process”

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1 I am grateful to the first anonymous referee for its indication.
connected with Laguerre polynomials was conjectured by S.M. Roman and G.-C. Rota \[18\]. Note also that the approach to the umbral calculus developed in \[16, 18, 19, 21\] can be a prototype for a general construction of coherent states (wavelets) in Banach spaces \[10\].

2 Preliminaries on Path Integrals

Let a physical system has the configuration space \(Q\). This means that we can label results of our measurements performed on the system at any time \(t_0 \in \mathbb{R}\) by points of \(Q\). For a quantum system dynamics the principal quantity is the propagator \([5, \S 2.2], [22, \S 5.1]\) \(K(q_2, t_2; q_1, t_1)\)—a complex valued function defined on \(Q \times \mathbb{R} \times Q \times \mathbb{R}\). It is a probability amplitude for a transition \(q_1 \to q_2\) from a state \(q_1\) at time \(t_1\) to \(q_2\) at time \(t_2\). The density of probability of this transition is postulated to be

\[
P(q_2, t_2; q_1, t_1) = |K(q_2, t_2; q_1, t_1)|^2.
\]

The fundamental assumption about the quantum world is the absence of trajectories for a system’s evolution through the configurational space \(Q\): the system at any time \(t_i\) could be found at any point \(q_i\). Then the transition amplitude \(q_1 \to q_2\) is a result of all possible transitions amplitudes \(q_1 \to q_i \to q_2\) integrated over \(Q\).

Mathematically this can expressed in the term of propagator \(K\) as follows. Let us fix any \(t_i, t_1 < t_i < t_2\). There exists a measure \(dq\) on \(Q\) such that

\[
K(q_2, t_2; q_1, t_1) = \int_Q K(q_2, t_2; q_i, t_i)K(q_i, t_i; q_1, t_1) \, dq_i.
\] (7)

This identity could be thought as a recurrence formula for the propagator \(K\).

Another assumption on propagator is its skew symmetry

\[
K(q_2, t_2; q_1, t_1) = \overline{K(q_1, t_1; q_2, t_2)},
\] (8)

it ensures that all transitions \(q_1 \to q_2\) are done by unitary operators.

It is useful to consider \(K(q_2, t_2; q_1, t_1)\) as a kernel for the associated integral operator \(K : L_2(Q, dq) \to L_2(Q, dq)\) defined as

\[
[K \phi](q, t) = \int_Q K(q, t; q_i, t_i)\phi(q_i, t_i) \, dq_i,
\] (9)

with \(t_i\) is fixed. Operator \(K\) is defined on functions \(\phi(q, t) \in L_2(Q, dq)\) such that the right-hand side of (9) does not depend on \(t_i\). Then (7) is read as
Polynomial Sequences and Path Integrals

Another consequence of (7): functions \( \phi(g_2, t_2) \) which belongs to image of operator \( K \) are fixed points for \( K \):

\[
\phi(q, t) = [K \phi](q, t) = \int_Q K(q, t; q_i, t_i) \phi(q_i, t_i) \, dq_i.
\]

These are wave functions and describe actual states of the system. It follows from (7) that all functions \( \kappa(q, t) = K(q, t; q_0, t_0) \) for arbitrary fixed \( q_0 \in Q \), \( t_0 \in \mathbb{R} \) are wave functions too. The function \( |\phi(q, t)|^2 \) gives the probability distribution over \( Q \) to observe the system in a point \( q \) at time \( t \) if it was in \( q_0 \) at time \( t_0 \).

R. Feynman developing ideas of A. Einstein, Smoluhovski and P.A.M. Dirac proposed an expression for the propagator via the "integral over all possible paths". In a different form it can be written as follows [22, (5.13)]

\[
K(q_2, t_2; q_1, t_1) = \int \frac{Dq \, Dp}{\hbar} \exp \left( \frac{i}{\hbar} \int_{t_1}^{t_2} dt \left( p \dot{q} - H(p, q) \right) \right). \tag{11}
\]

Here \( H(p, q) \) is the classical Hamiltonian [1, Chap. 9] of the system—a function defined over the phase space. In the simplest case the phase space is the Cartesian product of the configuration space \( Q \) and the space of momenta dual to it. Points in phase space have a double set of coordinates \( (q, p) \), where \( q \) is points of the configuration space \( Q \) and \( p \) is a coordinate in the space of momenta. The inner integral in (11) is taken over a path in the phase space. The outer integral is taken over "all possible paths with respect to a measure \( Dq \, Dp \) on paths in the phase space". We construct an example of path integral (21)–(22) similar to (11) in the next section.

From reading of physics textbooks [1, 22] one can obtain an impression that the measure \( Dq \, Dp \) could be well defined from an unspecified complicated mathematical construction. But this is not true up to now: there is no a general mathematically rigorous definition of the measure in the path

\[\text{For a model of relativistic quantisation generated by two orthoprojections see [1].}\]

\[\text{The editor preface to the Russian translation of [3] tells: “There is no a rigorous definition of the path integral in the book of Feynman and Hibbs... However it is not very important for a physicist in most cases; he only need an assurance that a rigorous proof could be obtained”. The recent textbook [22] which were already reprinted 9 times in two editions refereed for a rigorous path integral construction to the old paper [3], however it is known that this approach is not complete, see e.g. the review of this article MR $\#17,1261c$ as well as [3] and [17, Notes X.11].}\]
space. However there are different approaches of regularisation of path integrals, see \[4, 23\] for recent accounts.

It is known that path integrals give fundamental solutions of evolution-
ary equations \[3\]. Particularly wave functions \(10\) are solutions to the Schrödinger equation \[22, § 5.2\], \[5, § 4.1\]

\[
i\hbar \frac{\partial \phi(q, t)}{\partial t} = \hat{H}(q, p)\phi(q, t),\]

(12)

where \(\hat{H}(q, p)\) is an operator which is the quantum Hamiltonian. It was mentioned that functions \(\kappa(q, t) = K(q, t; q_0, t_0)\) are wave functions and thus are solutions to \(12\). In fact it follows from \(10\) that they are fundamental solutions, i.e. any other solution is a linear combination (superposition in physical language) of those. Therefore the propagator \(K(q_2, t_2; q_1, t_1)\) contains the complete information on dynamics of a quantum system defined by the equation \(12\).

3 Polynomial Sequences from Path Integrals

Let the configuration space \(\mathbb{Q}\) be a semigroup with an operation denoted by + and let a propagator \(K(q_2, t_2; q_1, t_1)\) be homogeneous in time and space, i.e.

\[
K(q_2 + q, t_2 + t; q_1 + q, t_1 + t) = K(q_2, t_2; q_1, t_1),
\]

(13)

for all \(q, q_1, q_2 \in \mathbb{Q}, t, t_1, t_2 \in \mathbb{R}\). Such a propagator corresponds to a free time-shift invariant system. Then \(K(q_2, t_2; q_1, t_1)\) is completely defined by values of the function \(S(q, t)\) such that \(S(q_2 - q_1, t_2 - t_1) = K(q_2, t_2; q_1, t_1)\). From \(7\) we can deduce a “recurrence” identity for the function \(S(q, t)\) \[3\]:

\[
S(q, t_1 + t_2) = \int_{\mathbb{Q}} S(q, t_1)S(q - q_i, t_2) \, dq_i.
\]

(14)

An equation of the form \(14\) was taken in \[11\] as the definition of tokens between two cancellative semigroups \(\mathbb{Q}\) and \(\mathbb{R}^+\). It allows to express a time shift by the integral of all space shifts over \(\mathbb{Q}\). Tokens are closest relatives to sectional coefficients considered by Henle \[7\]. A polynomial sequence \(p_n(x)\) of binomial type is another example of token \[11\]: the sequence \(q_n(x) = p_n(x)/n!\) is a token between cancellative semigroup \(\mathbb{N}\) and \(\mathbb{R}\), i.e. from \(11\) follows a realization of \(14\) in the form

\[
q_n(x + y) = \sum_{k=0}^{\infty} q_k(y)q_{n-k}(x),
\]

(15)
i.e. in this case the configuration space $\mathbb{Q}$ is the discrete set of nonnegative numbers and the integration reduces to summation.

If polynomial sequence $q_n(x)$ represents a particular propagator for a configuration space $\mathbb{Q} = \mathbb{N}$ then one may wish to obtain a path integral representation in the form (14) for it. We do it in a way parallel to the path integral deduction in [22, § 5.1].

Let $x \in \mathbb{R}$ be fixed and $N \in \mathbb{N}$ be an arbitrary positive integer. Then repetitive use of (15) allows to write

$$q_n(x) = \sum_{k_1+k_2+\cdots+k_N = n} q_{k_1} \left( \frac{x}{N} \right) q_{k_2} \left( \frac{x}{N} \right) \cdots q_{k_N} \left( \frac{x}{N} \right).$$

The first two terms in the Taylor expansion of $q_{k_j}, j = 1, \ldots, N$ are:

$$q_{k_j} \left( \frac{x}{N} \right) = \delta_{k_j0} + q'_{k_j}(0) \frac{x}{N} + o \left( \frac{x}{N} \right),$$

where the Kronecker delta as the first term come from (3). We use the Pontrjagin duality between $\mathbb{Z}$ and the unit disk $\mathbb{T} = [-\pi, \pi]$ to construct an integral resolution of the Kronecker delta:

$$\delta_{k0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ipk} \, dp.$$  

It is also convenient to introduce the Fourier transform $h(p)$ for the numerical sequence $q'_{k}(0)$ from the same Pontrjagin duality:

$$h(p) = \sum_{k=0}^{\infty} q'_{k}(0) e^{ipk} \iff q'_{k}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ipk} h(p) \, dp.$$  

Then we can transform the Taylor expansion (17) as follows:

$$q_{k} \left( \frac{x}{N} \right) = \delta_{k0} + q'_{k}(0) \frac{x}{N} + o \left( \frac{x}{N} \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ipk} \, dp + \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ipk} h(p) \, dp \frac{x}{N} + o \left( \frac{x}{N} \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ipk} \left( 1 + h(p) \frac{x}{N} \right) \, dp + o \left( \frac{x}{N} \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left( -ipk + h(p) \frac{x}{N} \right) \, dp + o \left( \frac{x}{N} \right),$$

(20)
where the last transformation is based on the estimation

$$\exp \left( h(p) \frac{x}{N} \right) = \left( 1 + h(p) \frac{x}{N} \right) + o \left( \frac{x}{N} \right).$$

Substituting $N$ copies of \((20)\) to \((16)\) we obtain

$$q_n(x) = \sum_{k_1 + \cdots + k_N = n} \prod_{i=1}^{N} q_k \left( \frac{x}{N} \right)$$

$$= \sum_{k_1 + \cdots + k_N = n} \left( \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp \left( -i p_1 k_1 + h(p_1) \frac{x}{N} \right) \right)$$

$$\times \cdots \times \int_{-\pi}^{\pi} \exp \left( -i p_N k_N + h(p_N) \frac{x}{N} \right) dp_N + o \left( \frac{x}{N} \right)$$

$$= \sum_{k_1 + \cdots + k_N = n} \left( \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp \left( \sum_{l=1}^{n} \left( -i p_l k_l + h(p_l) \frac{x}{N} \right) \right) \right)$$

$$\times \left. \int_{-\pi}^{\pi} \exp \left( -i p_N k_N + h(p_N) \frac{x}{N} \right) dp_N \right) + o \left( \frac{x}{N} \right)$$

$$= \sum_{k_1 + \cdots + k_N = n} \left( \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp \left( \sum_{l=1}^{n} \left( -i p_l k_l + h(p_l) \frac{x}{N} \right) \right) \right)$$

$$\times \left. \int_{-\pi}^{\pi} \exp \left( -i p_N k_N + h(p_N) \frac{x}{N} \right) dp_N \right) + o \left( \frac{x}{N} \right)$$

$$= \sum_{k_1 + \cdots + k_N = n} \left( \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp \left( \sum_{l=1}^{n} \left( -i p_l k_l + h(p_l) \frac{x}{N} \right) \right) \right)$$

Now we can observe that the expression $\sum_{l=1}^{n} \left( -i p_l k_l + h(p_l) \frac{x}{N} \right)$ looks like an integral sum for the function $-i p(t) k'(t) + h(p(t))$ on the interval $t \in [0, x]$. At this point we say together with physicists \([22, \S \, 5.1]\) magic words and \((21)\) “could be symbolically written for $N \to \infty$” as

$$q_n(x) = \int \mathcal{D}k \mathcal{D}p \exp \left( \int_{0}^{x} \left( -i p(t) k'(t) + h(p(t)) \right) dt \right),$$

where the first integration is taken over all possible paths $k(t) : [0, x] \to \frac{1}{x} \mathbb{N}$, such that $k(0) = 0$ and $k(x) = \frac{n}{x}$, and the path $p(t) : [0, x] \to [-\pi, \pi]$ is unrestricted. It is enough to consider only paths $k(t)$ with monotonic grow—other paths make the zero contributions. Indeed the identity $q_l(x) \equiv 0$ for $l < 0$ (made by an agreement) implies that contribution of all paths with $k'(t) < 0$ at some point $t$ vanish. Here (and in the inner integral of \((22)\)) $k'(t)$ means the derivative of the path $k(t)$ in the distributional sense, i.e. it
is the Dirac delta function times \( \frac{1}{x} \) in the points where \( k(t) \) jumps from one integer (mod \( \frac{1}{x} \)) value \( k(t - 0) = \frac{m}{x} \) to another \( k(t + 0) = \frac{m + j}{x} \).

In fact there is no any magic in the above path integral. Due to the property \( q_n(x) \equiv 0 \) for \( n < 0 \) paths \( k(t) \) take values on the finite set \( k_0, k_1, \ldots, k_n \), otherwise they make the zero contribution to the path integral as was mentioned above. Therefore the set of paths \( k(t) \) is countable. Under such a condition the situation is equivalent to a finite Markov process with a well-defined measure on the path space according to Kolmogorov theorem, see for example [3, § 1] or [25, Th. 2.1]. Moreover at the beginning of the next section we show how to calculate the path integral (22) as an ordinary integral.

Now we can give a solution to Problem 1.2 using the relation \( p_n(x) = n! q_n(x) \) between \( p_n(x) \) and \( q_n(x) \):

\[
p_n(x) = n! \int \mathcal{D}k \mathcal{D}p \exp \left( \int_0^x (-ipk' + h(p)) \, dt \right), \quad h(p) = \sum_{k=0}^{\infty} p'_k(0) \frac{e^{ipk}}{k!}. \tag{23}
\]

Note that a comparison of our Hamiltonian \( h(p) \) with the formal power series \( f(t) \) (2) made out of cumulants shows that \( h(p) = f(e^{ip}) \).

**Example 3.1** Here is a list of Hamiltonians producing the polynomial sequences from Example 1.1.

1. The power monomials \( p_n(x) = x^n \) are produced by \( h(p) = e^{ip} \).

2. The rising factorial sequence \( p_n(x) = x(x+1) \cdots (x+n-1) \) is produced by the Heaviside function \( h(p) = \chi(p) = \sum_{k=1}^{\infty} \frac{e^{ipk}}{k} \).

3. The falling factorial sequence \( p_n(x) = x(x-1) \cdots (x-n+1) \) is produced by the shifted Heaviside function

\[
h(p) = \chi(p - \pi) = \sum_{k=1}^{\infty} \frac{(-1)^k e^{ipk}}{k} = \sum_{k=1}^{\infty} \frac{e^{i(p-\pi)k}}{k}.
\]

4. The Abel polynomials \( A_n(x) = x(x-an)^{n-1} \) are produced by \( h(p) = \sum_{k=1}^{\infty} \frac{(ak)^{k-1}}{k!} e^{ipk} \).

5. The Laguerre polynomials (2) are generated by the delta function \( h(p) = \delta(p) = \sum_{k=1}^{\infty} e^{ipk} \).
4 Some Applications

Let us start from the question of calculational applicability of the path integral formula (23). We transform this path integral into an ordinary integration. Because we could not make any seriously restricting assumptions about the function \( h(p) \) the following arguments have an algebraical-combinatorial nature rather than analytic. They are similar to heuristic manipulations with formal power series which are used in a deduction of formula (6).

For a convenience we scale by \( \frac{1}{x} \) the parameter \( t \) in the inner integral of (22):

\[
\int_0^x (-ip(t)k'(t) + h(p(t))) \, dt = \int_0^1 (-ip_1(t_1)k'_1(t_1) + xh(p_1(t_1))) \, dt_1,
\]

where on the right-hand side new paths are \( k_1(t_1) : [0, 1] \to \mathbb{N} \) with the endpoint \( k_1(0) = 0 \) and \( k_1(x) = n \), and the path \( p_1(t_1) : [0, 1] \to [-\pi, \pi] \) is unrestricted. We continue renaming variables of integration \( t_1, k_1, \) and \( p_1 \) by \( t, k \) and \( p \) respectively.

As was mentioned before \( k(t) \) is an increasing step functions uniquely defined by a collection of \( n \) points \( t_j \in [0, 1] \) (some of them could coincide) where it jumps by 1. Thus \( k'(t) = \sum_1^n \delta(t - t_j) \) and consequently:

\[
\int_0^1 -ip(t)k'(t) \, dt = -i \sum_{j=1}^n p(t_j).
\]

Thereafter we could transform (23) as follows:

\[
p_n(x) = n! \int \mathcal{D}k \mathcal{D}p \exp \left( \int_0^1 (-ip(t)k'(t) + xh(p(t))) \, dt \right)
\]

\[
= n! \int \mathcal{D}k \mathcal{D}p \exp \left( -i \sum_{j=1}^n p(t_j) + \int_0^1 xh(p(t)) \, dt \right), \quad (24)
\]

Now we use the rotational symmetry of the unit circle \([-\pi, \pi]\) and that paths \( p(t) \) are completely unrestricted, therefore the whole set \( P \) of such paths is also rotational invariant. Thus the integration over that set \( P \) acts just like an averaging over the unit circle \([-\pi, \pi]\). On the other hand a random path \( k(t) \) is defined by a random collection of \( n \) points \( t_j \in [0, 1] \) consequently the

\footnote{I am grateful to the second anonymous referee who suggested this question.}
integration of $p(t_j)$ for any $j$ over all paths $k(t)$ will produce $n$ independent equal uniformly distributed random variables on the unit circle. Thus we could finally express our path integral (24) as a simple integration:

$$p_n(x) = n! \int DkDp \exp \left( -i \sum_{j=1}^{n} p(t_j) + \int_0^1 xh(p(t)) \, dt \right)$$

$$= n! \int_{-\pi}^{\pi} \exp \left( -i \sum_{j=1}^{n} p + xh(p) \right) \, dp$$

$$= n! \int_{-\pi}^{\pi} e^{-inp} e^{xh(p)} \, dp. \quad \text{(25)}$$

If we recall the connection $h(p) = f(e^{ip})$ between our Hamiltonian $h(p)$ and the generating function of cumulants $f(t)$ (5) then the last formula (23) obtains the very simple meaning: it uses the Fourier transform on $[-\pi, \pi]$ to extract $n$-th term out of the generating function (6) for the polynomial sequence $p_n(x)$.

**Remark 4.1** As we see that our formula (23) can offer in conventional calculations the same (or slightly longer) procedure as the well-known umbral approach (1). The situation could change if hypothetical quantum computers [14] will be able to calculate quantum propagators directly in parallel processes. In this case the path integral formula (23) should have advantages over the generating function (6) where calculations must be done in a row.

Finally we consider a realisation of the Schrödinger equation (12). Homogeneous propagator $S(q, t)$ (14) are wave functions (10) itself. Thus they should satisfy to the Schrödinger like equation (12). For polynomial sequences $q_n(x)$ this equation takes the form

$$\frac{\partial}{\partial x} q_n(x) = \hat{H} q_n(x), \quad \text{(26)}$$

where the operator $\hat{H}$ is of the pseudodifferential type [24, 25]

$$\hat{H} a_n = \mathcal{F}^{p\rightarrow n} h(p) \mathcal{F}^{n\rightarrow p} a_n$$

$$= \int_{-\pi}^{\pi} e^{-inp} h(p) \sum_{k=0}^{\infty} a_n e^{inp} \, dp,$$

$$= \sum_{k=0}^{\infty} h_k a_{n-k}. \quad \text{(27)}$$
Polynomial Sequences and Path Integrals

where $\mathcal{F}^{p \to n}$ and $\mathcal{F}^{n \to p}$ the Fourier transform in the indicated variables. The function $h(p)$ above is the Hamiltonian from the path integral \(^{(22)}\) and

$$h_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ipn} h(p) \, dp.$$  

It turns to be just a convolution on $\mathbb{N}$ because Hamiltonian $h(p)$ depends only from $p$ and is independent from $k$. The equation \(^{(27)}\) express the property \(^{[11]}\) of tokens to intertwine shift invariant operators between two cancellative semigroups. In the present case the operators are the derivative with respect $x$ (i.e. the convolution with $\delta'(x)$ on $\mathbb{R}$) and the convolution with $h_n$ on $\mathbb{N}$.

Let $\widehat{M}$ be the operator on sequences $\widehat{M}: \{a_n\} \mapsto \{a_n/n!\}$. Then $q_n(x) = \widehat{M}p_n(x)$ and because $\widehat{M}$ and $\frac{\partial}{\partial x}$ commute we obtain for a polynomial sequence $p_n(x)$ of binomial type the equation

$$\frac{\partial}{\partial x} p_n(x) = \widehat{M}^{-1} \hat{H} \widehat{M} p_n(x) \iff \frac{\partial}{\partial x} p_n(x) = n! \sum_{k=0}^{\infty} h_k \frac{p_{n-k}(x)}{(n-k)!} \quad (28)$$

Of course the above equation follows from the differentiation of identity \(^{[1]}\) with respect to $y$ at the point $y = 0$. While this formula and the next example are rather elementary we emphasise their relations to the quantum mechanical framework.

**Example 4.2** The polynomial sequences from Examples \(^{[1]}\) and \(^{3.1}\) satisfy to the following realizations of equation \(^{(28)}\):

1. The power monomials $p_n(x) = x^n$ satisfy to

$$p'_n(x) = n! \sum_{k=0}^{\infty} \delta_k! \frac{p_{n-k}(x)}{(n-k)!} \cdot \frac{n!}{(n-1)!}p_{n-1}(x) = n! p_{n-1}(x).$$

2. The rising factorial sequence $p_n(x) = x(x+1) \cdots (x+n-1)$ satisfies to

$$p'_n(x) = n! \sum_{k=0}^{\infty} \frac{1}{k} \frac{p_{n-k}(x)}{(n-k)!}.$$  

3. The falling factorial sequence $p_n(x) = x(x-1) \cdots (x-n+1)$ satisfies to

$$p'_n(x) = n! \sum_{k=0}^{\infty} \frac{(-1)^k}{k} \frac{p_{n-k}(x)}{(n-k)!}.$$
4. The Abel polynomials $A_n(x) = x(x - an)^{n-1}$ satisfy to

$$p'_n(x) = n! \sum_{k=0}^{\infty} \frac{(ak)^{k-1} p_{n-k}(x)}{k! (n-k)!} = \sum_{k=0}^{\infty} \binom{n}{k} (ak)^{k-1} p_{n-k}(x).$$

5. The Laguerre polynomials (2) satisfy to

$$p'_n(x) = n! \sum_{k=0}^{\infty} \frac{p_{n-k}(x)}{(n-k)!}.$$

**Remark 4.3** In our consideration we used primary the definition of token (14) and the Pontrjagin duality in (18)–(19). Thus formula (22) has the same form for many other tokens [8], which we shall not consider here however. It follows from general properties of path integrals that for any admissible function $h(p)$ formula (22) gives a token $q_n(x)$ and (23) produce a polynomial sequence $p_n(x)$ of binomial type.

We obtain further generalisation if drop the homogeneity assumption (13). Let combinatorial quantities $p_{n,k}(x, y)$ where $n, k \in \mathbb{N}$ and $x, y \in \mathbb{R}$ be polynomials in the variable $x$ of the degree $n$ and in the variable $y$ of the degree $k$. We assume that they satisfy the identities

$$p_{n,k}(x, y) = \sum_{m=0}^{\infty} p_{n,m}(x, y) p_{m,k}(y, z),$$

for any fixed $y$ such that $x \leq y \leq z$. Obviously this is an inhomogeneous version of the identities (1) and (14). The above quantum mechanical derivation of path integral (22) can be carried (under some necessary assumptions) also in this case. The main difference in the resulting formulae is that the Hamiltonian $h(p, k)$ will depend from both variables $p \in [-\pi, \pi]$ and $k \in \mathbb{N}$. Consequently the Schrödinger equation (28) will not be any more a trivial convolution over $\mathbb{N}$. We left consideration of this case for some further papers.

*In the conclusion:* the main result of this paper is yet another illustration to the old observation that mathematics is indivisible.

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Polynomial Sequences and Path Integrals

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