Is Goldbach Conjecture true?

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Abstract

We answer the question positively. In fact, we believe to have proved that every even integer \(2N \geq 3 \times 10^6\) is the sum of two odd distinct primes. Numerical calculations extend this result for \(2N\) in the range \(8 - 3 \times 10^6\). So, a fortiori, it is shown that every even integer \(2N > 2\) is the sum of two primes (Goldbach conjecture). Of course, we would be grateful for comments and objections.

1 Introduction

Goldbach conjecture (1742) states that:

Statement 1.1 (Goldbach form). Every integer \(N > 5\) is the sum of three primes.

An equivalent formulation due to Euler, called strong form, has replaced in literature the Goldbach form:

Statement 1.2 (strong form). Every even integer \(2N > 2\) is the sum of two primes.

Numerical calculations have verified it up to \(4 \times 10^{18}\) [5]; for a remarkable theoretical result see the reference [2]. The strong Golbach conjecture is also called binary or even. It implies the following weaker form (also called odd or ternary):

Statement 1.3 (weak form). Every odd integer \(2N + 1 > 5\) is the sum of three primes.
This formulation has been proved in the asymptotic case [7], while a general proof [4] is to the author knowledge still under consideration by the mathematical community. Here we consider a further formulation, implying the strong form, that we call very strong:

**Statement 1.4** (very strong form). *Every even integer* $2N > 6$ *is the sum of two odd distinct primes.*

Reference [5] holds also for this very strong form (Oliveira e Silva, personal communication). Explicit experimental evidence of Statement 1.4 up to $5 \times 10^8$ can be found in [6]: in fact, indicating with $r(2N)$ the number of Goldbach partitions of $2N$ (i.e. the number of unordered pairs of primes having sum equal to $2N$) it results $r(2N) > 1$ in the range $4 - 5 \times 10^8$, excluding $r(4) = r(6) = r(8) = r(12) = 1$.

We will show:

**Statement 1.5** (our result). *Every integer* $2N \geq 3 \times 10^6$ *is the sum of two odd distinct primes.*

Finally, Goldbach conjecture in its very strong form is shown combining our result with that found in [6] or [5].

## 2 Preliminary remarks in order to prove the conjecture

Without explicit definitions all the numbers considered in what follows must be taken as strictly positive integers.

**Definition 2.1.** Primes of type $Q$ related to $2N$ (symbol: $Q_j(2N)$): are primes that divide $2N$ ($2 = Q_1(2N) < Q_2(2N) < ... < Q_t(2N) \leq N$).

**Definition 2.2.** Primes of type $P$ related to $2N$ (symbol: $P_j(2N)$): are primes less than $2N - 2$ and non-divisors of $2N$ ($3 \leq P_1(2N) < P_2(2N) < ... < P_h(2N) < 2N - 2$). Their set is indicated as $P_{2N}$. The cardinality of the set $P_{2N}$ is $\text{card}P_{2N} = h$.

**Note 2.1.** Primes of type $P$ and $Q$ relative to $2N$ are a partition of the set of the primes less than $2N - 2$. 
Definition 2.3. Composites of type P related to $2N$ (symbol: $X_j(2N)$): are composites less than $2N - 2$, factorized into prime factors only of type P related to $2N$ ($2N - 2 > X_1(2N) > X_2(2N) > ... > X_s(2N) = P_1^2(2N)$). Their set is indicated as $X_{2N}$ and $\text{card}X_{2N} = s$.

Definition 2.4. Integers of type P related to $2N$ (symbol: $a_n(2N)$): are primes or composites of type P related to $2N$ ($2N - 2 > a_1(2N) > a_2(2N) > ... > a_w(2N) = P_1(2N)$). Their set is indicated as $A_{2N} = X_{2N} \cup P_{2N}$ and $\text{card}A_{2N} = s + h$.

Note 2.2. In absence of ambiguity we will indicate, for example, $P_j$ instead of $P_j(2N)$.

Concerning the values of $\text{card}P_{2N} = h$ we give:

Theorem 2.1. If $2N > 6$, then $h \geq 2$.

Proof. The strongest formulation of Bertrand’s postulate [3, p. 373], states that: for every $N > 3$ there exists an odd prime $P_r$ satisfying $N < P_r < 2N - 2$. It is remarkable that, for Definition 2.2, $P_r$ is a prime of type P. Besides, $2N - P_r = a_r$ is an integer of type P; otherwise a prime of type $Q$, let it be $Q_v$, divides $a_r$ and so $Q_v | P_r$, i.e. $Q_v = P_r$, in contradiction with Note 2.1. Since $a_r < N < P_r$, there is at least a prime of type P different from $P_r$ and so $h \geq 2$.

Theorem 2.2. If $2N \geq 3 \times 10^6$, then $\text{card}X_{2N} > \text{card}P_{2N}$.

Proof. We consider the following two relations concerning the functions $\pi(x)$ (number of primes $\leq x$) and $\phi(x)$ (totient Euler’s function):

(i) $\pi(x) < 1.25506 \frac{x}{\ln x}$ for $x > 1$ [1, p. 233, theorem 8.8.1]

(ii) $\phi(x) > \frac{x}{e^{\gamma} \ln x + \ln \ln x}$ for $x \geq 3$ [1, p. 234, theorem 8.8.7]; $\gamma = 0.577...$ is the Euler-Mascheroni constant.

The function $\phi(x)$ counts the number of the positive integers less than $x$ and prime to $x$. In this way $\text{card}A_{2N} = \phi(2N) - 2$, because $\phi(2N)$ considers also $2N - 1$ and $1$; but these numbers are not integers of type P. We have

(iii) $\text{card}P_{2N} = h < \pi(2N)$

(iv) $\text{card}X_{2N} = s = \text{card}A_{2N} - \text{card}P_{2N} = \phi(2N) - 2 - h$. 

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Therefore, from (i)-(iv) it results
\[ s - h = \phi(2N) - 2 - 2h > f(2N) \]
where
\[ f(2N) = \frac{2N}{1.781 ln ln 2N + \frac{3}{ln ln 2N}} - 2 - 2.510 \frac{2N}{ln 2N}. \]
f(2N) is a divergent sequence; by numerical computations it is increasing for \( 2N > 10^6 \) and for \( 2N \geq 3 \times 10^6 \) its values are greater than \( 10^3 \). In this way, a fortiori, from (1) it follows the proof.

We introduce now an essential concept for our purposes:

**Definition 2.5.** G-system related to \( 2N \): it is the system
\[
\begin{align*}
\Gamma_1 &: 2N - P_1 = a_{n_1} = a_1 \\
\Gamma_2 &: 2N - P_2 = a_{n_2} \\
&\vdots \\
\Gamma_j &: 2N - P_j = a_{n_j} \\
&\vdots \\
\Gamma_{h-1} &: 2N - P_{h-1} = a_{n_{h-1}} \\
\Gamma_h &: 2N - P_h = a_{n_h}
\end{align*}
\]

Fixed in (2) \( 2N > 6 \), we remark that:

**Note 2.3.** Existence of the G-system is guaranteed by Theorem 2.1.

**Note 2.4.** \( (a_{n_j})_{j=1,2,\ldots,h} \) is the subsequence of the sequence \( (a_n)_{n=1,2,\ldots,h+s} \) that does not contain the terms generated by \( 2N - X_{s-r}, r = 0, 1, \ldots, s - 1 \); \( a_1 = a_{n_1} \) because \( 2N - P_1 \) is the greatest integer of type \( P \), while, for example, if \( X_s = P_1^2 < P_2 \), then \( a_2 > a_{n_2} \).

**Note 2.5.** The \( h \) equations are not necessarily distinct; in fact, if \( a_{n_j} = P_k \), then \( \Gamma_j \) is equivalent to \( \Gamma_k \) (and \( 2N = P_j + P_k \)).

**Note 2.6.** \( P_j \nmid a_{n_j}, \forall j \); otherwise \( P_j \mid 2N \) in contradiction with Definition 2.2.

**Theorem 2.3.** Each term of \( (a_{n_j})_{j=1,2,\ldots,h} \) is composite \( \iff \) \( 2N \) is not the sum of two odd distinct primes.

**Proof.** Immediate.

**Note 2.7.** Theorem 2.3 holds also for \( N \) prime; in fact, from Note 2.6, \( P_j \neq a_{n_j}, \forall j \) and so the equation \( 2N - P_j = P_j \) does not belong to the G-system (in fact, in this case, \( P_j \) would not be a prime of type \( P \)).
3 Proof of the conjecture

**Theorem 3.1.** Every even integer \( 2N \geq 3 \times 10^6 \) is the sum of two odd distinct primes.

**Proof.** Let us suppose that \( 2N \geq 3 \times 10^6 \) is not the sum of two odd distinct primes. From Theorem 2.3 it follows that each term of \((a_{nj})_{j=1,2,...,h}\) is composite; so, in particular, the first relation at the top of system (2), being \( a_{n_1} = a_1 = X_1 \), is \( 2N - X_1 = P_1 \).

Let us suppose \( 2N - X_2 > P_2 \). Thus:

\[
\begin{align*}
2N - X_1 &= P_1 \\
2N - \alpha &= P_2 \\
2N - X_2 &> P_2 \\
&\vdots
\end{align*}
\]

Since \( X_2 < \alpha < X_1 \), \( \alpha \) is a prime (of type P) and this is impossible because \( 2N \) does not verify the conjecture. So (3) becomes

\[
\begin{align*}
2N - X_1 &= P_1 \\
2N - X_2 &\leq P_2 \\
&\vdots
\end{align*}
\]

Proceeding in analogous way the system (2) may be written as

\[
\begin{align*}
2N - X_1 &= P_1 \\
2N - X_2 &\leq P_2 \\
&\vdots \\
2N - X_j &\leq P_j \\
&\vdots \\
2N - X_{h-1} &\leq P_{h-1} \\
2N - X_h &\leq P_h
\end{align*}
\]

(5)

Starting now from the bottom of the system (2) we have

\[
\begin{align*}
&\vdots \\
2N - P_{h-1} &= a_{n_{h-1}} \geq X_{s-1} \\
2N - P_h &= a_{n_h} \geq X_s
\end{align*}
\]

(6)
It occurs because $a_{nh}$ is a composite of type P and $X_s$ is the smallest composite of the same type (and with similar consideration $a_{nh-1} \geq X_{s-1}$). So system (2) may be rewritten as

\[
\begin{align*}
2N - X_1 & \geq P_{h-s+1} \\
\vdots & \\
2N - X_{s-j} & \geq P_{h-j} \\
\vdots & \\
2N - X_{s-1} & \geq P_{h-1} \\
2N - X_s & \geq P_h
\end{align*}
\]

Considering $s - j = 1$ we obtain the relation at the top of the system (7). Comparing this relation with that at the top of system (5), we obtain $P_1 \geq P_{h-s+1}$. Since $P_1$ is the smallest prime of type P, we have $P_1 = P_{h-s+1}$ and, therefore, $1 = h - s + 1$. Thus $\text{card}X_{2N} = \text{card}P_{2N}$ (see Definitions 2.2 and 2.3) and this, by Theorem 2.2, is impossible. In this way it follows the proof. \hfill \square

At this point we obtain the aforementioned result:

**Theorem 3.2.** Every even integer $2N > 6$ is the sum of two odd distinct primes.

**Proof.** It follows immediately from Theorem 3.1 and [6] or [5]. \hfill \square

**References**

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