RIEMANNIAN FOLIATIONS ON QUATERNION CR-SUBMANIFOLDS OF AN ALMOST QUATERNION KÄHLER PRODUCT MANIFOLD

GABRIEL EDUARD VİLCU

Dedicated to Professor Stere IAnuș on the occasion of his 70th birthday.

Abstract. The purpose of this paper is to study the canonical foliations of a quaternion CR-submanifold of an almost quaternion Kähler product manifold.

AMS Mathematics Subject Classification. 53C15.

Keywords. quaternion CR-submanifold, quaternion Kähler manifold, Riemannian foliation.

1. Introduction

The notion of CR-submanifold, first introduced in Kähler geometry by Bejancu (see [3]), was extended in the quaternion settings by Barros, Chen and Urbano in [2]. They consider CR-quaternion submanifolds of quaternion Kählerian manifolds as generalizations of both quaternion and totally real submanifolds. Some foliations on this kind of submanifolds have been studied in [6] and [10].

The natural product of two Kählerian manifolds is also a Kählerian manifold ([15]) and the geometry of CR-submanifolds of Kählerian product manifolds is an interesting subject which was studied in [1] and [13]. On the other hand, the product of two quaternion Kähler manifolds does not become a quaternion Kähler manifold, but it is an almost quaternion Kähler product manifold (see [9]).

The study of quaternion CR-submanifolds of an almost quaternion Kähler product manifold was initiated in [8] by Kang and Lee. If \( M \) is a such kind of submanifold then two distributions, denoted by \( D \) and \( D^\perp \), are defined on \( M \). Moreover, if \( D^\perp \) is invariant with respect to the canonical almost product structure \( F \) on \( M \), it follows that \( D^\perp \) is always integrable and so we have a foliation on \( M \), called canonical totally real foliation on \( M \); on the other hand, if \( M \) is a \( D \)-geodesic CR-submanifold and \( D \) is \( F \)-invariant then \( D \) is also integrable and so we have another foliation on \( M \), called quaternion foliation. The object of this note is to study these foliations.

2. Preliminaries

Let \( \overline{M} \) be a smooth manifold endowed with a tensor \( F \) of type \((1,1)\) such that \( F^2 = Id \). Then \( (\overline{M}, F) \) is said to be an almost product manifold with almost product structure \( F \). Moreover if \( \overline{g} \) is a Riemannian metric on \( M \) such that:

\[
\overline{g}(FX, FY) = \overline{g}(X, Y), \quad \forall X, Y \in \Gamma(T\overline{M}),
\]

then we say that \( (\overline{M}, F, \overline{g}) \) is an almost product Riemannian manifold.

**Definition 2.1.** ([9]) Let \( (\overline{M}, F, \overline{g}) \) be an almost product Riemannian manifold of dimension \( 4n \) and assume that there is a rank 3-subbundle \( \sigma \) of \( \text{End}(T\overline{M}) \) satisfying
the following conditions:

i. A local basis \( \{J_1, J_2, J_3\} \) exists of sections of \( \sigma \) such that:
\[
J_1^2 = J_2^2 = J_3^2 = -Id, \quad J_1 J_2 = -J_2 J_1 = J_3
\]

and
\[
\bar{g}(J_\alpha X, J_\alpha Y) = \bar{g}(X, Y), \quad \alpha \in \{1, 2, 3\}
\]

for all local vector fields \( X, Y \) on \( \overline{M} \).

ii. The Levi-Civita connection \( \nabla \) of \( \bar{g} \) satisfies:
\[
(\nabla_X J_1)Y = k[\omega_3(X)J_2 Y - \omega_2(X)J_3 Y + \omega_3(FX)J_2(FY) - \omega_2(FX)J_3(FY)]
\]
\[
(\nabla_X J_2)Y = k[-\omega_3(X)J_1 Y + \omega_1(X)J_3 Y - \omega_3(FX)J_1(FY) + \omega_1(FX)J_3(FY)]
\]
\[
(\nabla_X J_3)Y = k[\omega_2(X)J_1 Y - \omega_1(X)J_2 Y + \omega_2(FX)J_1(FY) - \omega_1(FX)J_2(FY)]
\]

for some non-zero constant \( k \) and all vector field \( X, Y \) on \( \overline{M} \), \( \omega_1, \omega_2, \omega_3 \) being local 1-forms over the open for which \( \{J_1, J_2, J_3\} \) is a local basis of \( \sigma \).

Then \( \overline{M}, F, \sigma, \bar{g} \) is said to be an almost quaternion Kähler product manifold.

**Remark 2.2.** The natural product manifold of two quaternion Kähler manifolds is an almost quaternion Kähler product manifold. In this case we have \( k = \frac{1}{2} \) (see [9]).

**Definition 2.3.** ([2]) A submanifold \( M \) of an almost quaternion Kähler product manifold \( \overline{M}, F, \sigma, \bar{g} \) is called a quaternion CR-submanifold if there exist two orthogonal complementary distributions \( D \) and \( D^\perp \) on \( M \) such that:

i. \( D \) is invariant under quaternion structure, that is:
\[
J_\alpha(D_x) \subseteq D_x, \forall x \in M, \forall \alpha \in \{1, 2, 3\}; \tag{2.4}
\]

ii. \( D^\perp \) is totally real, that is:
\[
J_\alpha(D^\perp_x) \subseteq T_x M^\perp, \forall x \in M, \forall \alpha \in \{1, 2, 3\}. \tag{2.5}
\]

A submanifold \( M \) of an almost quaternion Kähler product manifold \( \overline{M}, F, \sigma, \bar{g} \) is a quaternion submanifold (respectively, a totally real submanifold) if \( \dim D^\perp = 0 \) (respectively, \( \dim D = 0 \)).

A submanifold \( M \) of an almost quaternion Kähler product manifold \( \overline{M}, F, \sigma, \bar{g} \) is called \( F \)-invariant if \( F(T_x M) \subseteq T_x M, \forall x \in M \).

The distribution \( D \) (respectively \( D^\perp \)) is said to be \( F \)-invariant if \( F(D) \subseteq D \) (respectively \( F(D^\perp) \subseteq D^\perp \)).

**Definition 2.4.** ([2]) Let \( M \) be a quaternion CR-submanifold of an almost quaternion Kähler product manifold \( \overline{M}, F, \sigma, \bar{g} \). Then \( M \) is called a QR-product if \( M \) is locally the Riemannian product of a quaternion submanifold and a totally real submanifold of \( \overline{M} \).

**Remark 2.5.** For a submanifold \( M \) of an almost quaternion Kähler product manifold \( \overline{M}, F, \sigma, \bar{g} \), we denote by \( g \) the metric tensor induced on \( M \). If \( \nabla \) is the covariant differentiation induced on \( M \), the Gauss and Weingarten formulas are given by:
\[
\nabla_X Y = \nabla_X Y + B(X, Y), \forall X, Y \in \Gamma(TM) \tag{2.6}
\]

and
\[
\nabla_X N = -A_N X + \nabla_X^\perp N, \forall X \in \Gamma(TM), \forall N \in \Gamma(TM^\perp) \tag{2.7}
\]
where \( B \) is the second fundamental form of \( M \), \( \nabla^\perp \) is the connection on the normal bundle and \( A_N \) is the shape operator of \( M \) with respect to \( N \). The shape operator \( A_N \) is related to \( B \) by:

\[
g(A_N X, Y) = g(B(X, Y), N),
\]

for all \( X, Y \in \Gamma(TM) \) and \( N \in \Gamma(TM^\perp) \).

**Definition 2.6.** ([2]) Let \( M \) be a quaternion CR-submanifold of an almost quaternion Kähler product manifold \((\ol{M}, \sigma, \ol{g})\). We say that:

i. \( M \) is \( D \)-geodesic if \( B(X, Y) = 0, \forall X, Y \in \Gamma(D) \).
ii. \( M \) is \( D^\perp \)-geodesic if \( B(X, Y) = 0, \forall X, Y \in \Gamma(D^\perp) \).
iii. \( M \) is mixed geodesic if \( B(X, Y) = 0, \forall X \in \Gamma(D), Y \in \Gamma(D^\perp) \).

We recall now the following results which we shall need in the sequel.

**Theorem 2.7.** ([8]) If \( M \) is a quaternion CR-submanifold of an almost quaternion Kähler product manifold \((\ol{M}, F, \sigma, \ol{g})\) such that the totally real distribution \( D^\perp \) is \( F \)-invariant, then \( D^\perp \) is integrable.

**Theorem 2.8.** ([8]) If \( M \) is a quaternion CR-submanifold of an almost quaternion Kähler product manifold \((\ol{M}, F, \sigma, \ol{g})\) such that the quaternion distribution \( D \) is \( F \)-invariant, then \( D \) is integrable if and only if \( M \) is \( D \)-geodesic.

### 3. Totally real foliation on a quaternion CR-submanifold

Let \( M \) be a quaternion CR-submanifold of an almost quaternion Kähler product manifold \((\ol{M}, F, \sigma, \ol{g})\). Then we have the orthogonal decomposition:

\[
TM = D \oplus D^\perp.
\]

We have also the following orthogonal decomposition:

\[
TM^\perp = \mu \oplus \mu^\perp,
\]

where \( \mu \) is the subbundle of the normal bundle \( TM^\perp \) which is the orthogonal complement of:

\[
\mu^\perp = J_1 D^\perp \oplus J_2 D^\perp \oplus J_3 D^\perp.
\]

If the totally real distribution \( D^\perp \) is \( F \)-invariant, by Theorem 2.7, we can consider the foliation \( \mathfrak{F}^\perp \) on \( M \), with structural distribution \( D^\perp \) and transversal distribution \( D \), called the canonical totally real foliation on \( M \).

We can illustrate now some of the techniques in this paper on the following theorem (see also [3], [5], [6], [8]).

**Theorem 3.1.** If \( M \) is a quaternion CR-submanifold of an almost quaternion Kähler product manifold \((\ol{M}, F, \sigma, \ol{g})\) such that the totally real distribution \( D^\perp \) is \( F \)-invariant, then the next assertions are equivalent:

i. The canonical totally real foliation \( \mathfrak{F}^\perp \) is totally geodesic;
ii. \( B(X, Y) \in \Gamma(\mu), \forall X \in \Gamma(D), Y \in \Gamma(D^\perp) \);
iii. \( A_N X \in \Gamma(D^\perp), \forall X \in \Gamma(D^\perp), N \in \Gamma(\mu^\perp) \);
iv. \( A_N Y \in \Gamma(D), \forall Y \in \Gamma(D), N \in \Gamma(\mu^\perp) \).
Proof. i. ⇔ ii. For $X, Z \in \Gamma(D^\perp)$ and $Y \in \Gamma(D)$ we have:

$$
\mathfrak{g}(J_\alpha(\nabla_X Z), Y) = -\mathfrak{g}(\nabla_X Z - B(X, Z), J_\alpha Y)
$$

$$
= \mathfrak{g}(\nabla_X J_\alpha Z + \nabla_X J_\alpha Z, Y)
$$

$$
= k\mathfrak{g}(\omega_\beta(X)J_\gamma Z - \omega_\gamma(X)J_\beta Z, Y)
$$

$$
+ k\mathfrak{g}(\omega_\beta(F X)J_\gamma(F Z) - \omega_\gamma(F X)J_\beta(F Z), Y)
$$

$$
+ \mathfrak{g}(-A_{J_\alpha}Z + \nabla_X J_\alpha Z, Y)
$$

$$
= \mathfrak{g}(\nabla_X J_\alpha Z, Y)
$$

where $(\alpha, \beta, \gamma)$ is an even permutation of $(1,2,3)$, and taking into account (2.7) we obtain:

$$
\mathfrak{g}(J_\alpha(\nabla_X Z), Y) = -\mathfrak{g}(B(X, Y), J_\alpha Z).
$$

(3.1)

The equivalence is now clear from (3.1).

ii. ⇔ iii. This equivalence follows from (2.8).

iii. ⇔ iv. This equivalence is true because $A_N$ is a self-adjoint operator.

Proposition 3.2. If $M$ is a quaternion CR-submanifold of an almost quaternion Kähler product manifold $(\overline{M}, F, \sigma, \overline{g})$ such that the totally real distribution $D^\perp$ is $F$-invariant and $M$ is mixed geodesic, then the canonical totally real foliation $\mathfrak{F}^\perp$ is totally geodesic.

Proof. The assertion follows from Theorem 3.1.

A submanifold $M$ of a Riemannian manifold $(\overline{M}, \overline{g})$ is said to be a ruled submanifold if it admits a foliation whose leaves are totally geodesic immersed in $(\overline{M}, \overline{g})$.

Definition 3.3. A quaternion CR-submanifold of an almost quaternion Kähler product manifold which is a ruled submanifold with respect to the foliation $\mathfrak{F}^\perp$ is called totally real ruled quaternion CR-submanifold.

Theorem 3.4. Let $M$ be a quaternion CR-submanifold of an almost quaternion Kähler product manifold $(\overline{M}, F, \sigma, \overline{g})$ such that $D^\perp$ is $F$-invariant. The next assertions are equivalent:

i. $M$ is a totally real ruled quaternion CR-submanifold.

ii. $M$ is $D^\perp$-geodesic and:

$$
B(X, Y) \in \Gamma(\mu), \ \forall X \in \Gamma(D), \ Y \in \Gamma(D^\perp).
$$

iii. The subbundle $\mu^\perp$ is $D^\perp$-parallel, i.e:

$$
\nabla_X J_\alpha Z \in \Gamma(\mu^\perp), \ \forall X, Z \in D^\perp, \ \alpha \in \{1,2,3\}
$$

and the second fundamental form satisfies:

$$
B(X, Y) \in \Gamma(\mu), \ \forall X \in \Gamma(D^\perp), \ Y \in \Gamma(TM).
$$

iv. The shape operator satisfies:

$$
A_{J_\alpha}Z = 0, \ \forall X, Z \in D^\perp, \ \alpha \in \{1,2,3\}
$$

and

$$
A_N X \in \Gamma(D), \ \forall X \in \Gamma(D^\perp), \ N \in \Gamma(\mu).
$$
Proof. i. $\iff$ ii. For any $X, Z \in \Gamma(D^\perp)$ we have:
\[
\nabla_X Z = \nabla_X Z + B(X, Z) = \nabla^{D^\perp} X + h^\perp(X, Z) + B(X, Z)
\]
and thus we conclude that the leaves of $D^\perp$ are totally geodesic immersed in $\overline{M}$ if and only if $h^\perp = 0$ and $M$ is $D^\perp$-geodesic. The equivalence is now clear from Theorem 3.1.

i. $\iff$ iii. For any $X, Z \in \Gamma(D^\perp)$, and $U \in \Gamma(D)$ we have:
\[
g(\nabla_X Z, U) = g((-\nabla X J_\alpha)Z + \nabla_X J_\alpha Z, J_\alpha U) = k g(\omega_\beta(X)J_\gamma Z - \omega_\gamma(X)J_\beta Z, J_\alpha U) + k g(\omega_\beta(FX)J_\gamma(FZ) - \omega_\gamma(FX)J_\beta(FZ), J_\alpha U) + g(-A_{J_\alpha Z}X + \nabla^\perp_X J_\alpha Z, J_\alpha U) = -g(A_{J_\alpha Z}X, J_\alpha U)
\]
where $(\alpha, \beta, \gamma)$ is an even permutation of $(1,2,3)$, and from (2.8) we obtain:
\[
g(\nabla_X Z, U) = g(-A_{J_\alpha Z}X + \nabla^\perp_X J_\alpha Z, J_\alpha U). (3.2)
\]

On the other hand, for any $X, Z, W \in \Gamma(D^\perp)$ we have:
\[
g(\nabla_X Z, J_\alpha W) = g(B(X, J_\alpha U), J_\alpha Z). (3.3)
\]

If $X, Z \in \Gamma(D^\perp)$ and $N \in \Gamma(\mu)$, then we have:
\[
g(\nabla_X Z, J_\alpha N) = g(-\nabla X J_\alpha Z + \nabla_X J_\alpha Z, J_\alpha N) = k g(\omega_\beta(X)J_\gamma Z + \omega_\gamma(X)J_\beta Z, J_\gamma N) + k g(\omega_\beta(FX)J_\gamma(FZ) + \omega_\gamma(FX)J_\beta(FZ), J_\gamma N) + g(-A_{J_\alpha Z}X + \nabla^\perp_X J_\alpha Z, J_\alpha N)
\]
and thus we obtain:
\[
g(\nabla_X Z, J_\alpha N) = g(\nabla^\perp_X J_\alpha Z, J_\alpha N). (3.4)
\]

Finally, $M$ is a totally real ruled quaternion CR-submanifold if and only if $\nabla X Z \in \Gamma(D^\perp)$, $\forall X, Z \in \Gamma(D^\perp)$ and by using (3.2), (3.3) and (3.4) we deduce the equivalence.

ii. $\iff$ iv. This is clear from (2.8). 

Corollary 3.5. Let $M$ be a quaternion CR-submanifold of an almost quaternion Kähler product manifold $(\overline{M}, F, \sigma, \overline{g})$ such that $D^\perp$ is $F$-invariant. If $M$ is totally geodesic, then $M$ is a totally real ruled quaternion CR-submanifold.

Proof. The assertion is clear from Theorem 3.4. 

4. QR-products and foliations with bundle-like metric

From Theorem 2.8 we deduce that any $D$-geodesic CR-submanifold of an almost quaternion Kähler product manifold such that $D$ is $F$-invariant, admits a $\sigma$-invariant totally geodesic foliation, which we denote by $\mathcal{F}$, called quaternion foliation.
Proposition 4.1. If $M$ is a totally geodesic quaternion CR-submanifold of an almost quaternion Kähler product manifold $(M, F, \sigma, \overline{\mathcal{F}})$ such that $D$ and $D^\perp$ are $F$-invariant, then $M$ is a ruled submanifold with respect to both foliations $\mathcal{F}$ and $\overline{\mathcal{F}}$.

Proof. The assertion follows from Theorem 2.8 and Corollary 3.5.

Theorem 4.2. Let $M$ be a quaternion CR-submanifold of an almost quaternion Kähler product manifold $(M, F, \sigma, \overline{\mathcal{F}})$ such that $D$ and $D^\perp$ are $F$-invariant. Then $M$ is a QR-product if and only if the next two conditions are satisfied:

(i) The induced metric is bundle-like for the foliation $\mathcal{F}$.

(ii) The second fundamental form $B(X, Y) \in \Gamma(\mu)$, for all $X, Y \in \Gamma(D^\perp)$, where $\mu = \Gamma(D)$.

Proof. The proof is immediate from Theorems 2.8 and 3.1.

Let $(M, g)$ be a Riemannian manifold and $\mathcal{F}$ a foliation of $M$. The metric $g$ is said to be bundle-like for the foliation $\mathcal{F}$ if the induced metric on the transversal distribution $D^\perp$ is parallel with respect to the intrinsic connection on $D^\perp$. This is true if and only if the Levi-Civita connection $\nabla$ of $(M, g)$ satisfies (4.1):

$$g(\nabla Q_X Q X, Q^\perp Z) + g(\nabla Q_Z Q X, Q^\perp Y) = 0, \forall X, Y, Z \in \Gamma(TM),$$

where $Q^\perp$ is the projection morphism of $TM$ on $D^\perp$.

If for a given foliation $\mathcal{F}$ there exists a Riemannian metric $g$ on $M$ which is bundle-like for $\mathcal{F}$, then we say that $\mathcal{F}$ is a Riemannian foliation on $(M, g)$.

Theorem 4.3. Let $M$ be a quaternion CR-submanifold of an almost quaternion Kähler product manifold $(M, F, \sigma, \overline{\mathcal{F}})$ such that $D^\perp$ is $F$-invariant. The next assertions are equivalent:

(i) The induced metric $g$ on $M$ is bundle-like for the totally real foliation $\mathcal{F}^\perp$.

(ii) The second fundamental form $B$ of $M$ satisfies:

$$B(U, J_\alpha V) + B(V, J_\alpha U) \in \Gamma(\mu) + J_\beta(D^\perp) \oplus J_\gamma(D^\perp), \forall U, V \in \Gamma(D)$$

for $\alpha = 1, 2$ or $3$, where $(\alpha, \beta, \gamma)$ is an even permutation of $(1, 2, 3)$.

Proof. From (4.1) we deduce that $g$ is bundle-like for totally real foliation $\mathcal{F}^\perp$ if and only if:

$$g(\nabla_U X, V) + g(\nabla_V X, U) = 0, \forall X \in \Gamma(D^\perp), U, V \in \Gamma(D).$$

On the other hand, for any $X \in \Gamma(D^\perp)$, $U, V \in \Gamma(D)$ we have:

$$g(\nabla_U X, V) + g(\nabla_V X, U) = g(\nabla_U X - B(U, X), V) + g(\nabla_V X - B(V, X), U)$$

$$= g(\nabla_U X, V) + g(\nabla_V X, U)$$

$$= g(-\nabla_U J_\alpha X + \nabla_U J_\alpha X, J_\alpha V)$$

$$+ g(-\nabla_V J_\alpha X + \nabla_V J_\alpha X, J_\alpha U)$$

$$= k g(\omega_\beta(U) J_\gamma X - \omega_\gamma(U) J_\beta X, J_\alpha V)$$

$$+ k g(\omega_\beta(U) J_\gamma X - \omega_\gamma(U) J_\beta X, J_\alpha U)$$

$$+ k g(\omega_\beta(U) J_\gamma X - \omega_\gamma(U) J_\beta X, J_\alpha U)$$

$$+ g(\nabla_U J_\alpha X, J_\alpha V) + g(\nabla_U J_\alpha X, J_\alpha U)$$

$$= -g(A_{J_\alpha X} U, J_\alpha V) - g(A_{J_\alpha X} V, J_\alpha U)$$
where \((\alpha, \beta, \gamma)\) is an even permutation of \((1,2,3)\), and taking into account \((2.8)\) we derive:
\[
g(\nabla U X, V) + g(\nabla V X, U) = -\overline{g}(B(U, J\alpha V) + B(V, J\alpha U), J\alpha X),
\]
for any \(X \in \Gamma(D^\bot), \ U, V \in \Gamma(D)\).

The proof is now complete from (4.2) and (4.3).

\[\square\]

Acknowledgement

The author expresses his gratitude to the referee for carefully reading the manuscript and giving useful comments. This work was partially supported by a PN2-IDEI grant, no. 525/2009.

References

[1] Atçeken M, CR-submanifolds of Kaehlerian product manifolds, Balkan J. Geom. Appl. 12(2) (2007) 8–20
[2] Barros M, Chen B Y and Urbano F, Quaternion CR-submanifolds of quaternion manifolds, Kodai Math. J. 4 (1981) 399–417
[3] Bejancu A, CR submanifolds of a Kaehler manifold, Proc. Am. Math. Soc. 69 (1978), 135–142
[4] Bejancu A and Farran H R, Foliations and geometric structures (Dordrecht : Springer-Verlag) (2006)
[5] Ianuș S and Pastore A M, Some foliations and harmonic morphisms, Rev. Roum. Math. Pures Appl. 50(5-6) (2005) 671–676
[6] Ianuș S, Ionescu A M and Vilcu G E, Foliations on quaternion CR-submanifolds, Houston J. Math., 34(3) (2008) 739–751.
[7] Ishihara S, Quaternion Kählerian manifolds, J. Diff. Geometry 9 (1974) 483–500
[8] Kang T H and Lee Y H, Quaternionic CR-submanifolds of an almost quaternionic Kähler product manifold, Indian J. Pure Appl. Math. 30(9) (1999) 871–884
[9] Kang T H and Nam H C, Submanifolds of an almost quaternionic Kaehler product manifold, Bull. Korean Math. Soc. 34(4) (1997) 635–665
[10] Papantoniou B and Hasan Shahid M, Quaternion CR-submanifolds of a quaternion Kaehler manifold, Int. J. Math. Math. Sci. 27(1) (2001) 27–37
[11] Reinhart B L, Foliated manifolds with bundle-like metrics, Ann. Math. 69(2) (1959) 119–132
[12] Rovenskii V, Foliations, submanifolds, and mixed curvature, J. Math. Sci., New York 99(6) (2000) 1699–1787
[13] Hasan Shahid M, CR-submanifolds of Kaehlerian product manifolds, Indian J. Pure Appl. Math. 23(12) (1992) 873–879
[14] Tondeur P, Geometry of foliations (Basel: Birkhäuser) (1997)
[15] Yano K and Kon M, Submanifolds of Kaehlerian product manifolds, Atti Accad. Naz. Lincei, Mem. Cl. Sci. Fis. Mat. Nat., VIII. Ser., Sez. I, 15 (1979) 267–292

1Department of Mathematics and Computer Science, Petroleum-Gas University of Ploiești, Bulevardul București, Nr. 39, Ploiești 100680, Romania
2Research Center in Geometry, Topology and Algebra, Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei, Nr.14, Sector 1, Bucharest 70109, Romania

E-mail: gvilcu@mail.upg-ploiesti.ro