BISTABLE WAVES OF A RECURSIVE SYSTEM ARISING FROM SEASONAL AGE-STRUCTURED POPULATION MODELS

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(Communicated by Xiaoqiang Zhao)

Abstract. This paper is devoted to the existence, uniqueness and stability of bistable traveling waves for a recursive system, which is defined by the iterations of the Poincaré map of a yearly periodic age-structured population model derived in the companion paper [8]. The existence of the wave is established by appealing to a monotone dynamical system theory, and the uniqueness and stability are obtained by employing a squeezing method.

1. Introduction. In this paper, we are concerned with the recursive system

\[ u_{n+1} = Q[u_n], \quad n \geq 0, \]  

(1)

where \( Q : BC(\mathbb{R}, \mathbb{R}) \to BC(\mathbb{R}, \mathbb{R}) \) is the Poincaré map of the following time periodic and nonlocal reaction-diffusion model that is derived in the companion work [8] in order to study the invasion dynamics of yearly generated age-structured species with distinct breeding and maturation seasons:

\[ \frac{\partial u}{\partial t} = D_M(t) \frac{\partial^2 u}{\partial x^2} - d_M(t) u + R(t, u(t - \tau(t), \cdot)), \quad t > 0, x \in \mathbb{R}. \]  

(2)

In (2), \( D_M \) and \( d_M \) are, respectively, the yearly periodic diffusion and death rates of the mature population \( u(t, x) \), which consists of all individuals with age being equal or greater than the positive yearly periodic maturation age \( \tau(t) \). The nonlocal term \( R \), meaning the recruitment of mature population, has the following expression:

\[ R(t, \phi) = [1 - \tau'(t)] b(t - \tau(t), \phi) \ast k_I(t, t - \tau(t), \cdot). \]  

(3)

Here \( b(t, u) \) is the birth rate at time \( t \), and \( k_I(t, s, x) \) is the Green function of

\[ \partial_t \rho = D_I(t) \partial_{xx} \rho - d_I(t) \rho, \]  

(4)

where \( D_I \) and \( d_I \) are the diffusion and death rates of immature population that consists of all individuals with age being less than \( \tau(t) \).

In [8], the following biological scenario was introduced for the study of seasonal influence on the species invasion with yearly generation and distinct breeding and maturation seasons.

\[ 2010 \text{ Mathematics Subject Classification. Primary: 39A30; Secondary: 92D25, 45P05, 45M10.} \]

\[ \text{Key words and phrases. Bistable wave, exponentially asymptotical stability, recursive system.} \]

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(A1) (Seasonality) $D_M > 0, D_I \geq 0, d_M > 0, d_I > 0, \tau > 0, b \geq 0$ are all $C^1$ functions and $T$-periodic in time.

(A2) (Distinct breeding and maturation seasons) Assume that $0 < \alpha \leq \beta < t_\alpha \leq t_\beta < T,$ (5)

where $t_\alpha$ and $t_\beta$ satisfy

$$t_\alpha - \tau(t_\alpha) = \alpha, \quad t_\beta - \tau(t_\beta) = \beta.$$ (6)

Further, we assume that $b(t, u) = p(t)h(u)$ is Lipschitz continuous, where $h \in C^1$ and $p(t) = p(t + T) \geq 0$ with $\int_0^T p(t)dt = 1$ and $p(t) \equiv 0$ for $t \in [0, \alpha] \cup [\beta, T].$

(A3) (Ordering in maturation) $\tau'(t) < 1, t \in \mathbb{R}.$

With (A1)-(A3), we are able to derive an explicit relation between the initial value $\varphi$ and time-$T$ solution map $u(T, \cdot; \varphi).$ It reads

$$Q[\varphi] = k_M(T, 0, \cdot) * \varphi + \int_{t_\alpha}^{t_\beta} k_M(T, s, \cdot) * R(s, k_M(s - \tau(s), 0, \cdot) * \varphi)ds,$$ (7)

where $R$ is defined as in (3) and $k_M(t, s, x)$ is the Green function of

$$\partial_t \rho = D_M(t)\partial_{xx} \rho - d_M(t)\rho.$$ (8)

We refer to [8] for the biological interpretations of (A1)-(A3), the derivation of (7), as well as the invasion speed of (1) when $Q$ has a monostable structure.

The purpose of this paper is to study the propagation dynamics of (1) when $Q$ has a bistable structure, which will be stated in the next section.

For the study of bistable waves, we recall some related works that motivated our research. Lui [6, 7] studied the bistable waves of a monotone nonlinear integral operator arising the model in population genetics. Chen [1] introduced a squeezing method to study the existence, uniqueness and the global exponential stability of bistable waves for a class of nonlocal evolution equations admitting the comparison principle. This method is then refined to study time delayed reaction-diffusion equations by Smith and Zhao [9] and spatiotemporally nonlocal reaction-diffusion equations by Wang, Li and Ruan [10]. For the extended systems of bistable maps with the spatial coupling given by convolution with distribution functions, Coutinho and Fernandez [2] proved the existence of bistable waves as well as the uniqueness of wave speed. Fang and Zhao [4] established a monotone dynamical system theory on the existence of bistable waves. Zhang and Zhao [12] applied the aforementioned theory to get the existence of bistable waves for a competitive recursion system, and then they utilized the dynamical system convergence result [15, Theorem 2.2.4] and Chen’s squeezing method to obtain the uniqueness and Lyapunov, exponential stability of waves. These ideas are shown to be extendable to the model system with quiescent stages or seasonal successions [13, 14]. For the traveling waves of monotone iterative systems with monostable structure, we refer to [11, 5, 3].

We plan to apply the dynamical system theory in [4] to establish the existence of bistable waves. As for the uniqueness and stability, we intend to integrate and refine the ideas and methods in [1, 9, 10, 12] so that they are adapted to our recursive system. The multiple convolutions contained in the expression of iterative map $Q$ make the proofs more involved, especially the constructions of upper and lower solutions for the exponential stability.

The rest of this paper is organized as follows. In section 2, we state the bistability structure of $Q$ and provide an example that generates such a structure. In section 3,
we establish the existence of bistable waves by appealing to the monotone dynamical system theory in [4]. In section 4, we first prove a strong comparison principle and the strict monotonicity of wave profiles, which are then used to construct super and sub solutions for the Lyapunov stability and uniqueness of waves up to a phase shift. In section 5, we show the exponentially asymptotical stability of the wave.

2. Bistability structure. Restricting $Q$ on $\mathbb{R}$ yields a map $\overline{Q}: \mathbb{R} \to \mathbb{R}$ that is defined by

$$\overline{Q}[z] = z\overline{k}_M(T,0) + \int_{t_\alpha}^{t_*} \overline{k}_M(T,s)R(s,z\overline{k}_M(s - \tau(s),0))ds$$  \hspace{1cm} (9)$$

where $R$, defined as in (3), reduces to

$$R(s,z) = [1 - \tau'(s)]\overline{k}_I(s, z - \tau(s))b(s - \tau(s), z)$$  \hspace{1cm} (10)$$

with $\overline{k}_I(t,s) := \int_{t}^{t_*} k_I(t,s,y)dy$ and $\overline{k}_M(t,s) := \int_{t}^{t_*} k_M(t,s,y)dy$, where $k_I$ and $k_M$ are defined to be the Green functions of (4) and (8), respectively.

We impose the following structure for $\overline{Q}$:

(A4) (Bistability) $\overline{Q}$ admits exactly three fixed points $u^* > u_+ > 0$, and $\overline{Q}[z]$ is nondecreasing in $z \in [0, u^*]$. Moreover, fixed points 0 and $u^*$ are stable and $u_+$ is unstable in the sense that

$$\overline{Q}'[0] < 1, \quad \overline{Q}'[u^*] < 1, \quad \overline{Q}'[u_+] > 1.$$  \hspace{1cm} (11)$$

By (A4), we can infer that the limit $\lim_{n \to \infty} \overline{Q}^{[n]}[z]$ equals to 0 if $z \in (0, u_+)$ and $u^*$ if $z \in (u_+, u^*)$. $\overline{Q}[z]$ is nondecreasing in $z \in [0, u^*]$ if and only if $h(z)$ is nondecreasing in $z \in [0, u^*/\overline{k}_M(\alpha, 0)]$.

From the expression of $\overline{Q}$ it is not immediate to see under what specific conditions the bistability structure is achieved. In the following, we provide an example that is motivated by the biological scenario that the breeding season is short and there is an Allee effect in birth. That is, $\beta - \alpha$ is small and there are no newborns when the species density is small. More precisely,

(A4') Let $\theta > 0$. Assume that

$$h(u) = \begin{cases} 0, & \theta \geq u \geq 0, \\ h_1(u - \theta), & u \geq \theta, \end{cases}$$  \hspace{1cm} (12)$$

where $h_1: [0, +\infty) \to [0, +\infty)$ satisfies

$$h_1(0) = 0, \quad h_1'(u) > 0, \quad h_1''(u) < 0, \quad h_1(+\infty) \in (0, +\infty).$$  \hspace{1cm} (13)$$

We obtain the following condition under which the bistability structure (A4) is realized.

Theorem 2.1. Let (A1)-(A3) and (A4') hold. Assume that

$$h_1'(0) > \frac{1 - \overline{k}_M(T,0)}{\overline{k}_M(T,0)} \left\{ \int_{t_\alpha}^{t_*} \frac{[1 - \tau'(s)]\overline{k}_I(s, s - \tau(s))p(s - \tau(s))}{\overline{k}_M(s, s - \tau(s))}ds \right\}^{-1}.$$  \hspace{1cm} (14)$$

Then there exist $\theta^* > 0$ and $\delta^* > 0$ such that (A4) holds when $\theta \in (0, \theta^*)$ and $\beta \in (\alpha, \alpha + \delta^*)$.

Before the proof, we remark that (14) is the necessary and sufficient condition such that 0 is not globally stable for all $\theta > 0$. 
In view of (13) in (A4′), we first consider the limiting case where \( \beta \to \alpha \). Define

\[
\mathcal{R}(t) := [1 - \tau'(t)]p(t - \tau(t)).
\]

By (A2) and (A3), we change variable to obtain

\[
\int_{t_\alpha}^{t_\beta} \mathcal{R}(s) ds = \int_{t_\alpha}^{\beta} p(s) ds = 1.
\]

Letting \( \beta \to \alpha \), we get \( \mathcal{R}(s) \to \delta(s - t_\alpha) \) for \( s \in [0, T] \) with \( \delta(\cdot) \) being the Dirac function. And hence, as \( \beta \to \alpha \), the map \( Q \) reduces to \( \hat{Q} \) defined as follows.

\[
\hat{Q}[z] = z\mathcal{K}_M(T, 0) + \mathcal{K}_M(T, t_\alpha)\mathcal{K}_I(t_\alpha, \alpha)h(z\mathcal{K}_M(\alpha, 0)).
\]

Meanwhile, the inequality (14) has the following limit as \( \beta \to \alpha \),

\[
h'_1(0) > \frac{1 - \mathcal{K}_M(T, 0)}{\mathcal{K}_M(T, 0)} \left\{ \frac{\mathcal{K}_I(t_\alpha, \alpha)}{\mathcal{K}_M(t_\alpha, \alpha)} \right\}^{-1}.
\]

Define an auxiliary function

\[
F(z) := \frac{1}{z} \left\{ \hat{Q}[z] - z \right\} = [\mathcal{K}_M(T, 0) - 1] + \frac{\mathcal{K}_M(T, 0)\mathcal{K}_I(t_\alpha, \alpha)}{\mathcal{K}_M(t_\alpha, \alpha)} \frac{h(z\mathcal{K}_M(\alpha, 0))}{z\mathcal{K}_M(\alpha, 0)}.
\]

By (A4′), we know that

\[
\frac{h(u)}{u} = \begin{cases} 
0, & \theta \geq u > 0, \\
0, & u = +\infty,
\end{cases}
\]

which implies that

\[
F(z) = \mathcal{K}_M(T, 0) - 1, \quad \forall z \in (0, \theta/\mathcal{K}_M(\alpha, 0)) \text{ and } F(\infty) = \mathcal{K}_M(T, 0) - 1.
\]

**Claim.** \( \hat{Q} \) has exactly two positive fixed points if \( \theta \) is sufficiently small. It is equivalent to prove that \( F \) has exactly two positive zeros provided that \( \theta \) is sufficiently small.

Indeed, we firstly study the shape of \( F \) on the interval \( [\theta/\mathcal{K}_M(\alpha, 0), \infty) \). By direct computations, we obtain

\[
\text{Sign}\{F'(z)\} = \text{Sign}\left\{ z\mathcal{K}_M(\alpha, 0)h'_1(z\mathcal{K}_M(\alpha, 0) - \theta) - h_1(z\mathcal{K}_M(\alpha, 0) - \theta) \right\}.
\]

Define

\[
F_1(z) := z\mathcal{K}_M(\alpha, 0)h'_1(z\mathcal{K}_M(\alpha, 0) - \theta) - h_1(z\mathcal{K}_M(\alpha, 0) - \theta).
\]

In view of (13) in (A4′), we have

\[
F'_1(z) = z\mathcal{K}_M^2(\alpha, 0)h''_1(z\mathcal{K}_M(\alpha, 0) - \theta) < 0.
\]

Combining \( F'_{1}(< 0) > 0 \) and \( F_{1}(\infty) < 0 \), we may conclude that there exists \( z_0 \in (\theta/\mathcal{K}_M(\alpha, 0), \infty) \) such that \( F'(z_0) = 0 \) and

\[
F'(z) > 0, \quad \forall z \in (\theta/\mathcal{K}_M(\alpha, 0), z_0) \quad \text{and} \quad F'(z) < 0, \quad \forall z \in (z_0, \infty).
\]

In conclusion, \( F(z) = \mathcal{K}_M(T, 0) - 1 < 0 \) for \( z \in (0, \theta/\mathcal{K}_M(\alpha, 0)) \), and \( F(z) \) strictly increases to its maximum value first and then strictly decreases to \( \mathcal{K}_M(T, 0) - 1 \), which is negative.

It follows from (13) that \( F \) converges uniformly on any closed subset of \( (0, \infty) \) as \( \theta \to 0 \). The limit reads

\[
F_2(z) := [\mathcal{K}_M(T, 0) - 1] + \frac{\mathcal{K}_M(T, 0)\mathcal{K}_I(t_\alpha, \alpha)}{\mathcal{K}_M(t_\alpha, \alpha)} \frac{h(z\mathcal{K}_M(\alpha, 0))}{z\mathcal{K}_M(\alpha, 0)}.
\]
By (17) and (13), we know that \( \lim_{z \to 0} F_2(z) > 0 \), \( F_2(\infty) < 0 \) and that \( F_2 \) is strictly decreasing. Therefore, \( F(\zeta_0) > 0 \) if \( \theta \) is small enough. Together with (18), the continuity implies that \( F \) has exactly two positive zeros. The claim is proved.

Note that \( \overline{Q} \) converges to \( Q \) uniformly on \([0, \infty)\) as \( \beta \to \alpha \). It follows that there exist \( \theta^* > 0 \) and \( \delta^* > 0 \) such that \( \overline{Q} \) has exactly two positive fixed points \( u^* > u_* \).

Note that \( \overline{Q}[0] = \overline{K}_M(T, 0) < 1 \), then the monotonicity of \( \overline{Q} \) implies that (11) is valid. The proof of the theorem is completed.

\[ \square \]

3. Existence of traveling waves. In this section we shall apply the dynamical system theory in [4] to establish the existence of waves for (1). For this purpose, we need some notations.

Let \( C := BC(\mathbb{R}, \mathbb{R}) \) be all bounded and continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \) equipped with the compact open topology, which can be induced by the norm

\[ \| \phi \| = \sum_{k=1}^{\infty} 2^{-k} \max_{|x| \leq k} |\phi(x)|, \quad (19) \]

For \( \phi, \psi \in C \), we write \( \phi \geq \psi \) if \( \phi(x) - \psi(x) \geq 0 \) for \( x \in \mathbb{R} \). Denote \([a, b]_C := \{ \phi \in C : b \geq \phi \geq a \} \) and \( C_* := [0, 1]_C \).

In order to apply [4, Theorem 3.1], it suffices to check that the \( Q \) satisfies the following six assumptions.

(H1) (Translation invariance) \( T_y Q[\phi] = QT_y[\phi], \forall \phi \in C_{u^*}, y \in \mathbb{R} \), where \( T_y \) is defined by \( T_y[\phi](x) = \phi(x - y) \).

(H2) (Continuity) \( Q : C_{u^*} \to C_{u^*} \) is continuous with respect to compact open topology.

(H3) (Monotonicity) \( Q \) is order preserving in the sense that \( Q[\phi] \geq Q[\psi] \) whenever \( \phi \geq \psi \) in \( C_{u^*} \).

(H4) (Compactness) \( Q : C_{u^*} \to C_{u^*} \) is compact with respect to compact open topology.

(H5) (Bistability) \( \overline{Q} : \mathbb{R} \to \mathbb{R} \) have three fixed points \( 0 < u_* < u^* \), among which \( u_* \) is unstable, and \( 0, u^* \) are strongly stable from above and below, respectively, in the sense that there exists \( \delta > 0 \) such that

\[ \overline{Q}[\eta] < \eta, \quad \overline{Q}[u^* - \eta] > u^* - \eta, \quad \forall \eta \in (0, \delta). \]

(H6) (Counter-propagation) \( c_+(u_*, u^*) + c_-(0, u_*) > 0 \), where \( c_+(u_*, u^*) \) and \( c_-(0, u_*) \) are the leftward and rightward spreading speed of the map \( Q \) on \([u_*, u^*]_C \) and \([0, u_*]_C \), respectively.

Lemma 3.1. Assume that (A1)-(A4) hold. Then \( Q : C_{u^*} \to C_{u^*} \) satisfies (H1)-(H6).

Proof. Since \( Q \) is spatially homogeneous, (H1) holds. The monotonicity (H3) follows from the monotonicity of \( \overline{Q} \). The bistability (H5) is satisfied due to (A4). In the following we prove the continuity (H2), the compactness (H4) and the counter-propagation (H6).

For (H2) and (H4) we employ the same ideas as in [8, Lemma 4.1] with some modifications. We first verify the continuity, that is, \( \overline{Q}[\phi_n] \to \overline{Q}[\phi] \) as \( \phi_n \to \phi \) in \( C_{u^*} \). Indeed, let \( L \) be the Lipschitz constant of \( h(\phi) \) and define

\[ J(y) = k_M(T, 0, y) + L \int_{t_n}^{t_s} \overline{P}(s) k_M(T, s, \cdot) * k_M(s - \tau(s), 0, \cdot) * k_I(s, s - \tau(s), \cdot) ds. \]
Clearly, \( J \in L^1(\mathbb{R}, \mathbb{R}) \). It then follows that
\[
|Q[\phi_n](x) - Q[\phi](x)| \leq J * |\phi_n - \phi|(x),
\]
and hence,
\[
|Q[\phi_n] - Q[\phi]| = \sum_{k=1}^{\infty} 2^{-k} \max_{|x| \leq k} |Q[\phi_n](x) - Q[\phi](x)|
\]
\[
\leq \sum_{k=1}^{\infty} 2^{-k} \max_{|x| \leq k} J * |\phi_n - \phi|(x)
\]
\[
= \sum_{k=1}^{\infty} 2^{-k} \max_{|x| \leq k} \left\{ \left( \int_{|y| \geq C} + \int_{|y| \leq C} \right) J(y)|\phi_n - \phi|(x-y)dy \right\}
\]
\[
\leq 2u^* \int_{|y| \geq C} J(y)dy + 2C \int_{\mathbb{R}} J(y)dy \|\phi_n - \phi\|, \quad C > 0.
\]
Since \( C > 0 \) is arbitrary and \( J \in L^1(\mathbb{R}, \mathbb{R}) \), we obtain the continuity.

Next we verify the compactness. Indeed, by the Ascoli-Arzela theorem it suffices to show that \( Q[\phi](x) \) is equi-continuous for \( \phi \in C_{u^*} \) and \( x \) in any compact set. For any \( \phi \in C_{u^*} \) and \( x_1, x_2 \in \mathbb{R} \), by inequality (20), we have
\[
|Q[\phi](x_1) - Q[\phi](x_2)| \leq u^* \int_{\mathbb{R}} |J(x_1 - x_2 + y) - J(y)|dy \to 0, \quad \text{as } x_1 - x_2 \to 0. \tag{21}
\]
The compactness is obtained.

Finally we verify the counter-propagation (H6). It suffices to show that \( c_+^* (u_+, u^*) > 0 \) and \( c_+^*(0, u_+) > 0 \). By [5, Theorem 3.10(ii)], we see that \( c_+^* (u_+, u^*) \) is bounded below by
\[
\inf_{\mu > 0} \frac{1}{\mu} \ln \left\{ \int_{\mathbb{R}} e^{\mu y} \mathcal{K}(y)dy \right\}, \tag{22}
\]
where \( \mathcal{K} \) is defined by
\[
\mathcal{K} = k_M(T, 0, \cdot) + \int_{\alpha}^{\alpha} \partial_x R(s, u_+)k_M(T, s, \cdot) \ast k_M(s, \tau(s), \cdot) \ast k_M(s - \tau(s), 0, \cdot)ds.
\]
Further, by [8, Theorem 4.3], we obtain \( c_+^* (u_+, u^*) > 0 \). Similarly, we can obtain \( c_+^*(0, u_+) > 0 \).

Now, by [4, Theorem 3.1] and the above lemma we have the existence of bistable waves.

**Theorem 3.2.** Assume that (A1)-(A4) hold. Then there exists \( c \in \mathbb{R} \) such that \( \{Q^n\}_{n \geq 1} \) admits a nondecreasing traveling wave \( U(\cdot) \in \mathcal{C} \) in the sense that
\[
U(x - cn) = Q^n[U](x), \quad U(-\infty) = 0, \quad U(+\infty) = u^*. \tag{23}
\]

4. Lyapunov stability and uniqueness. We first present two lemmas that will be of useful in what follows. The first one is the strong comparison principle.

**Lemma 4.1.** Assume that the sequences \( \{\varphi_n\}_{n \geq 0}, \{\psi_n\}_{n \geq 0} \subset \mathcal{C} \) with \( \varphi_0 \geq \psi_0 \neq \psi_0 \) satisfy \( \varphi_{n+1} \geq Q[\varphi_n] \) and \( \psi_{n+1} \leq Q[\psi_n] \) for all \( n \geq 0 \). Then \( \varphi_n(x) > \psi_n(x) \) for \( n \geq 1 \) and \( x \in \mathbb{R} \).
Lemma 4.2. Let (H3) and the expression of \( Q \) defined in (7), we immediately obtain that \( \varphi_1 \geq \psi_1 \) provided that \( \varphi_0 \geq \psi_0 \). Inductively, \( \varphi_n \geq \psi_n \) for \( n \geq 1 \). Further, by the monotonicity of \( h \) we have

\[
\varphi_1(x) - \psi_1(x) \geq \int_{\mathbb{R}} k_M(T, 0, x - y)(\varphi_0(y) - \psi_0(y))dy.
\]

Now we prove \( \varphi_1 > \psi_1 \) for all \( x \in \mathbb{R} \). If not, then there exists \( x_0 \) such that \( \varphi(x_0) = \psi(x_0) \), and hence,

\[
\int_{\mathbb{R}} k_M(T, 0, x_0 - y)(\varphi_0(y) - \psi_0(y))dy = 0.
\]

Since \( \varphi_0 - \psi_0 \geq 0 \) is continuous, there must be \( \varphi_0 \equiv \psi_0 \), which is a contradiction. Thus \( \varphi_1 > \psi_1 \) for all \( x \in \mathbb{R} \). Inductively, we have \( \varphi_n(x) > \psi_n(x) \) for all \( x \in \mathbb{R} \) and \( n \geq 1 \).

The second lemma is about the strict monotonicity of wave profiles, which is one of the keys to prove the uniqueness and stability.

**Lemma 4.2.** Let \((U, c)\) be a monotone traveling wave. Then

\[
U \in C^1, \quad U' > 0, \quad \lim_{|x| \to \infty} U'(x) = 0.
\]

**Proof.** From the definition we see that \( U \) satisfies

\[
U(x - c) = [k_M(T, 0, \cdot) * U](x) + \int_{t_a}^{t_b} R(s) [K(s, \cdot) * h (k_M(s - \tau(s), 0, \cdot) * U)](x)ds.
\]  

(24)

Note that \( k_M(t, s, \cdot) \in C^\infty \) and \( U \) is continuous. It follows that \( k_M(T, 0, \cdot) * U \in C^\infty \). For second part of the righthand side of (24), by the Lebesgue dominated convergence theorem we infer that it is differentiable. Thus, \( U \in C^1 \).

Now, we prove \( U' > 0 \) on \( \mathbb{R} \). Note that \( U \) is nondecreasing. It then follows that \( U' \geq 0 \). By the monotonicity of \( h \), we have \( h' \geq 0 \) almost everywhere. Consequently, we obtain

\[
U'(x) \geq \int_{\mathbb{R}} k_M(T, 0, x + c - y)U'(y)dy.
\]

Now we argue by the way of contradiction. Assume that there exists \( x_0 \) such that \( U'(x_0) = 0 \). Then

\[
0 = U'(x_0) \geq \int_{\mathbb{R}} k_M(T, 0, x_0 + c - y)U'(y)dy \geq 0,
\]

which implies that \( U' = 0 \) almost everywhere. Note that \( U \) is continuous. We then obtain \( U' \equiv 0 \), and hence, \( U \) is a constant, which is a contradiction. Therefore, \( U' > 0 \).

Lastly, we prove \( \lim_{|x| \to \infty} U'(x) = 0 \). Differentiating both sides of (24), we obtain

\[
U'(x) = [k'_M(T, 0, \cdot) * U](x + c)
\]

\[
+ \int_{t_a}^{t_b} R(s) [K(s, \cdot) * h'(k_M(s - \tau(s), 0, \cdot) * U)k'_M(s - \tau(s), 0, \cdot) * U)](x + c)ds.
\]

Since \( 0 \leq u \leq u^* \), we have \( 0 \leq k_M(s - \tau(s), 0, \cdot) * U \leq u^* \) for all \( s \in [t_a, t_b] \). And hence, there exists \( r_m > 0 \) satisfying \( h'(z) \leq r_m \) a.e. for \( z \in [0, u^*] \) such that

\[
U'(x) \leq [k'_M(T, 0, \cdot) * U](x + c) + \int_{t_a}^{t_b} R(s) r_m [K(s, \cdot) * |k'_M(s - \tau(s), 0, \cdot) * U|](x + c)ds.
\]
To finish the proof of \( \lim_{|x| \to \infty} U'(x) = 0 \), we only need to prove \( \lim_{|x| \to \infty} |k'_M(t, 0, \cdot) \ast U|(x) = 0 \) uniformly for \( t \in [\alpha, \beta] \cup \{T\} \). Indeed, for all \( t \in (0, T] \),

\[
k_M(t, 0, x) = \frac{e^{-\int_0^t dM(s) \, dc}}{\sqrt{4\pi e^{-\int_0^t D_1(s) \, dc}}} \exp \left\{ -\frac{x^2}{4 \int_0^t d_1(s) \, dc} \right\}.
\]

It follows that \( |k'_M(t, 0, \cdot)| \in L^1 \) is an odd function. So, for any \( \varepsilon > 0 \), there exists \( R_1 = R_1(\varepsilon) > 0 \) such that

\[
\int_{|y| \geq R_1} |k'_M(t, 0, y)| u^* \, dy \leq \varepsilon/2, \quad \forall t \in [\alpha, \beta] \cup \{T\}.
\]

By the fact that \( \lim_{x \to \pm \infty} U(x) = u^*, 0 \) and the Cauchy theorem, there exists \( R_2 = R_2(R_1, \varepsilon) > 0 \) such that

\[
|U(x - y) - U(x + y)| \leq \varepsilon/(2R_1 A), \quad \forall |x| \geq R_2, \ y \in [-R_1, 0],
\]

where \( A := \max\{k'_M(t, 0, z) : z \in [-R_1, 0], t \in [\alpha, \beta] \cup \{T\}\} \). And hence, by the property of odd function and changing variable, we obtain that

\[
||k'_M(t, 0, \cdot) \ast U|(x)| = \left| \left( \int_{-R_1}^{+R_1} + \int_{|y| \geq R_1} \right) k'_M(t, 0, y) U(x - y) \, dy \right|
\]

\[
\leq \left| \int_{-R_1}^{+R_1} k'_M(t, 0, y) U(x - y) \, dy \right| + \varepsilon/2
\]

\[
= \left| \int_{-R_1}^{0} k'_M(t, 0, y) [U(x - y) - U(x + y)] \, dy \right| + \varepsilon/2
\]

\[
\leq \left| \int_{-R_1}^{0} k'_M(t, 0, y) [U(x - y) - U(x + y)] \, dy + \varepsilon/2 \right|
\]

\[
\leq \varepsilon, \quad \forall t \in [\alpha, \beta] \cup \{T\},
\]

provided that \( |x| \geq R_2 \) with \( R_2 \) only depending on \( \varepsilon \). This completes the proof. \( \square \)

The following definition of upper and lower solutions will be used.

**Definition 4.3.** A function sequence \( W^+_n \in \mathcal{C}, n \geq 0 \), is an upper solution of system (1) if \( W^+_n(x) \) satisfies

\[
W^+_{n+1}(x) \geq Q^n(W^+_n)(x), \quad n \geq 0.
\]

A function sequence \( W^-_n \in \mathcal{C}, n \geq 0 \), is an lower solution of (7) if \( W^-_n(x) \) satisfies

\[
W^-_{n+1}(x) \leq Q^n(W^+_n)(x), \quad n \geq 0.
\]

Motivated by [1, 12], we have the following results on upper and lower solutions for the iterative system (1).

**Lemma 4.4.** Let the traveling wave solution \( (U, c) \) be obtained in Theorem 3.2. There exist positive number \( \sigma \) and \( \rho_0, \eta_0 \in (0, 1) \) such that for any \( x_0 \in \mathbb{R}, \rho \in (0, \rho_0) \) and \( \eta \in (0, \eta_0) \),

\[
W^+_n(x) = U(x - cn + x_0 \pm \eta(1 - e^{-\sigma n})) \pm \eta \rho e^{-\sigma n}, \quad \forall x \in \mathbb{R}, \ n \geq 0.
\]

are upper and lower solutions of system (1), respectively.
Proof. Without loss of generality, we assume that the translation $x_0 = 0$. For the sake of convenience, we define $R$ as in (15) and

$$K(s, y) := [k_M(T, s) * k_I(s, s - \tau(s), \cdot)](y)$$

and define

$$z_n := -cn + \eta(1 - e^{-\sigma n}), n \geq 0,$$

and define

$$D^+_{n}(x) := W^+_{n+1}(x) - Q[W^+_n](x).$$

Since $U(x) = Q^n[U](x + cn)$, using Newton-Leibniz formula, we may compute to have

$$\begin{align*}
D^+_{n}(x) &= W^+_{n+1}(x) - Q[W^+_n](x) \\
&= U(x + z_{n+1}) + \eta e^{-\sigma n} - U(x - c + z_n) - k_M(T, 0)\eta e^{-\sigma n} \\
&- \int_{t_n}^{t_{n+1}} R(s)K(s, \cdot) * h(k_M(s - \tau(s), 0, \cdot) * (U(\cdot + z_n) + \eta e^{-\sigma n}))ds \\
&+ \int_{t_n}^{t_{n+1}} R(s)K(s, \cdot) * h(k_M(s - \tau(s), 0, \cdot) * (U(\cdot + z_n))))ds \\
&= U(x + z_{n+1}) - U(x - c + z_n) + \eta e^{-\sigma n} \left\{ e^{-\sigma} - k_M(T, 0) \\
&- \int_{t_n}^{t_{n+1}} R(s)K(s - \tau(s), 0)K(s, \cdot) * H(\cdot)ds \right\},
\end{align*}$$

where $H$ is defined by

$$H(\cdot) = \int_{0}^{1} h'(k_M(s - \tau(s), 0, \cdot) * (U(\cdot + z_n) + \zeta \eta e^{-\sigma n})) ds.$$ 

Define

$$\bar{K}_0(s) := k_M(T, s)k_I(s, s - \tau(s))k_M(s - \tau(s), 0).$$

By assumption (A4), i.e. $\bar{Q}'(u^*) < 1$ and $\bar{Q}'(0) < 1$, we know that there exist sufficiently small $\kappa > 0$ and $\vartheta \in (0, 1)$ such that

$$\bar{Q}'(u^*) + \kappa \int_{t_n}^{t_{n+1}} R(s)\bar{K}_0(s)ds < \vartheta \quad \text{and} \quad \bar{Q}'(0) + \kappa \int_{t_n}^{t_{n+1}} R(s)\bar{K}_0(s)ds < \vartheta.$$ 

It follows from the properties of birth rate $\bar{h} \in \mathcal{C}^1$ that, there exists $\delta > 0$ such that

$$|h'(u\bar{k}_M(s - \tau(s), 0)) - h'(0)| \leq \kappa, \quad \forall |u - 0| \leq \delta, s \in [t_\alpha, t_\beta],$$

and

$$|h'(u\bar{k}_M(s - \tau(s), 0)) - h'(u\bar{k}_M(s - \tau(s), 0))| \leq \kappa, \quad \forall |u - u^*| \leq \delta, s \in [t_\alpha, t_\beta].$$

Define

$$L(s, \eta, x) := \bar{K}^{-1}(s - \tau(s), 0)k_M(s - \tau(s), 0, \cdot) * (U(\cdot) + \zeta \eta e^{-\sigma n}))(x),$$

then we have the following claim.

Claim 1. Let $\eta_0 \leq \delta$, there exists $\zeta > 0$ such that if $x \leq -\zeta$, then for all $\zeta \in [0, 1]$ and $s \in [t_\alpha, t_\beta]$ we have

$$|L(s, \zeta, x) - 0| \leq \delta,$$

and, if $x \geq \zeta$, then for all $\zeta \in [0, 1]$ and $s \in [t_\alpha, t_\beta]$ we have

$$|L(s, \zeta, x) - u^*| \leq \delta.$$
Indeed, considering the fact that 
\[ \int_{\mathbb{R}} K_M(s - \tau(s), 0) = \int_{\mathbb{R}} k_M(s - \tau(s), 0, y) dy, \]
we know
\[ \int_{\mathbb{R}} K_M^{-1}(s - \tau(s), 0) k_M(s - \tau(s), 0, y) dy = 1. \]

Since \( \eta_0 < \delta \) and \( \eta \leq \eta_0 < 1, \rho \leq \rho_0 < 1, \) we know \( \zeta \eta e^{-\alpha n} \leq \delta \) for all \( n \geq 0. \) And hence, by the Lebesgue dominated convergence theorem, we obtain
\[ \lim_{x \to +\infty} L(s, \eta, x) = U(+\infty) + \zeta \eta e^{-\alpha n} \leq u^* + \delta, \quad \forall n \geq 0, \]
and
\[ \lim_{x \to -\infty} L(s, \eta, x) = U(-\infty) + \zeta \eta e^{-\alpha n} \leq \delta, \quad \forall n \geq 0, \]
uniformly for \( \zeta \in [0, 1] \) and \( s \in [t_\alpha, t_\beta]. \) Note that \( U(+\infty) = u^*, U(-\infty) = 0, \) it follows that the claim is valid.

By above claim, we further have
\[ |h'(i(s, \eta, y)K_M(s - \tau(s), 0)) - h'(0)| \leq \kappa, \quad \forall y \leq -\zeta, \zeta \in [0, 1], s \in [t_\alpha, t_\beta], \]
and
\[ |h'(u*K_M(s - \tau(s), 0)) - h'(i(s, \eta, y)K_M(s - \tau(s), 0))| \leq \kappa, \forall y \geq \zeta, \zeta \in [0, 1], s \in [t_\alpha, t_\beta]. \]

Note that \( K(s, \cdot) \in L^1. \) It follows that there exist a sufficiently large \( \xi > 0 \) such that
\[ \left( \int_{-\infty}^{-\xi} + \int_{\xi}^{+\infty} \right) K(s, y) dy < \eta, \quad \text{uniformly for } s \in [t_\alpha, t_\beta]. \]

Claim 2. \( D^+_n(x) \geq 0. \) We proceed with three cases.

Case 1. \( x > \zeta + \xi - z_n. \) It is clear that \( x + z_n - \zeta > \xi. \) By the monotonicity and boundedness of \( U, \) we have
\[
D^+_n(x) \geq \eta e^{-\alpha n} \left\{ e^{-\sigma} - K_M(T, 0) - \int_{t_\alpha}^{t_\beta} R(s)K_M(s - \tau(s), 0)K(s, \cdot) * H(\cdot) ds \right\}
\]
\[
\geq \eta e^{-\alpha n} \left\{ e^{-\sigma} - \left[ K_M(T, 0) + \int_{t_\alpha}^{t_\beta} R(s)K_M(s - \tau(s), 0) \left( \int_{-\infty}^{-\zeta} + \int_{-\zeta}^{\xi} \right) \right.ight.
\]
\[
\left. + \int_{\zeta}^{+\infty} K(s, x + z_n - y) \int_{0}^{1} h'(i(s, \eta, y)K_M(s - \tau(s), 0)) dy dy ds \right] \right\}
\]
\[
\geq \eta e^{-\alpha n} \left\{ e^{-\sigma} - \left[ K_M(T, 0) + \int_{t_\alpha}^{t_\beta} R(s)K_M(s - \tau(s), 0) \left( \int_{-\infty}^{-\zeta} + \int_{-\zeta}^{\xi} \right) \right.ight.
\]
\[
\left. + \int_{\zeta}^{+\infty} K(s, x + z_n - y)(\kappa + h'(u*K_M(s - \tau(s), 0))) dy dy ds \right] \right\}
\]
\[
\geq \eta e^{-\alpha n} \left\{ e^{-\sigma} - \left[ Q(u^*) + \kappa \int_{t_\alpha}^{t_\beta} R(s)R_0(s) ds + B\eta \right] \right\}
\]
\[
\geq \eta e^{-\alpha n} (e^{-\sigma} - \vartheta - B\eta) \geq 0.
\]
provided that \( \sigma \in (0, -\ln \nu) \) and \( \eta \) is small enough.

**Case 2.** \( x \leq -\zeta - \xi - z_n \): That is \( x + z_n + \zeta \leq -\xi \), By the monotonicity and boundedness of \( U \), then we have

\[
D_n^+(x) \geq \eta e^{-\sigma n} \left\{ e^{-\sigma} - \bar{K}_M(T, 0) - \int_{t_n}^{t_\beta} R(s) \bar{K}_M(s - \tau(s), 0) K(s, \cdot) H(\cdot) ds \right\}
\]

\[
\geq \eta e^{-\sigma n} \left\{ e^{-\sigma} - \left[ \bar{K}_M(T, 0) + \int_{t_n}^{t_\beta} R(s) \bar{K}_M(s - \tau(s), 0) \left( \int_{-\infty}^{-\zeta} + \int_{-\zeta}^{\zeta} + \int_{\zeta}^{+\infty} \right) K(s, x + z_n - y) \left( \kappa + h'(0)dy + B\eta \right) ds \right] \right\}
\]

\[
\geq \eta e^{-\sigma n} \left\{ e^{-\sigma} - \left[ \bar{K}_M(T, 0) + \int_{t_n}^{t_\beta} R(s) \bar{K}_M(s - \tau(s), 0) \left( \int_{-\infty}^{-\zeta} + \int_{-\zeta}^{+\infty} \right) \right] K(s, x + z_n - y) \right\}
\]

\[
\geq \eta e^{-\sigma n} \left\{ e^{-\sigma} - \frac{Q'(0) + \kappa}{\int_{t_n}^{t_\beta} R(s) K_0(s) ds + B\eta} \right\}
\]

\[
\geq \eta e^{-\sigma n} (e^{-\sigma} - \vartheta - B\eta) \geq 0
\]

provided that \( \sigma \in (0, -\ln \nu) \) and \( \eta \) is small enough.

**Case 3.** \( x \in [-\zeta - \xi - z_n, \zeta + \xi - z_n] \). By the previous lemma, we know \( U \) is strictly increasing in compact set \( x \in [-\zeta - \xi - |c|, \zeta + \xi + |c| + 1] \), there exists \( \mu > 0 \) such that

\[
U(x) - U(y) \geq \mu(x - y), \quad x, y \in [-\zeta - \xi - |c|, \zeta + \xi + |c| + 1].
\]

Considering that \( -|c| < z_n + 1 - z_n = |c| + \eta e^{-\sigma n} (1 - e^{-\sigma}) < -c + 1 \), we know \( x + z_n + 1 - z_n \in [-\zeta - \xi, \zeta + \xi + 1] \) and \( x \in [-\zeta - \xi - |c|, \zeta + \xi + |c| + 1] \). Thus, we have

\[
U(x + z_{n+1}) - U(x + z_n) \geq \mu(z_{n+1} - z_n), \quad \forall x \in [-\zeta - \xi - z_n, \zeta + \xi - z_n].
\]

It follows that

\[
D_n^-(x) \geq \mu(z_{n+1} - z_n + c) + \eta e^{-\sigma n} \left\{ e^{-\sigma} - \bar{K}_M(T, 0) - B \int_{t_n}^{t_\beta} R(s) K_0(s) ds \right\}
\]

\[
= \eta e^{-\sigma n} \left\{ \mu(1 - e^{-\sigma}) + \rho e^{-\sigma} - \rho \bar{K}_M(T, 0) + B \int_{t_n}^{t_\beta} R(s) K_0(s) ds \right\}
\]

\[
\geq 0,
\]

provided that \( \rho \) is chosen sufficiently small.

Combining cases 1-3, we see that there exist \( \sigma > 0 \) and sufficiently small \( \eta_0, \rho_0 \in (0, 1) \) such that \( D_n^+(x) \geq 0, n \geq 0, x \in \mathbb{R} \). Hence \( W_n^+(x) \) is an upper solution of iterative system (1). By the similar arguments, we can prove that \( W_n^-(x) \) is a lower solution of (1).

With the help of upper and lower solutions in above lemma, we can observe the following Lyapunov stability theorem.

**Theorem 4.5 (Liapunov Stability).** Let \( u_n \) satisfy the system (1). Then the traveling wave solution \((U, c)\) is Lyapunov stable in the sense that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \|u_n(\cdot) - U(\cdot - cn)\|_{C^0} \leq \varepsilon \) provided that \( \|u_0(\cdot) - U(\cdot)\|_{C^0} \leq \delta \).
Proof. Let \( W^+_n(x) \) be upper and lower solutions defined in Lemma 4.4 with \( x_0 = 0 \). By Lemma 4.2, we can define \( U'_m := \max_{x \in \mathbb{R}} U'(x) > 0 \). For any \( \varepsilon > 0 \), set
\[
\delta := \min \{ \frac{\varepsilon}{2}, \frac{\rho e}{2U'_m}, \rho \eta_0 \},
\]
where \( \rho, \eta_0 \) are defined in (26). By assumption \( \| u_0(\cdot) - U(\cdot) \|_{C^0} \leq \delta \) we have
\[
U(x) - \delta \leq u_0(x) \leq U(x) + \delta.
\]
Using the definition of \( \delta \) and Lemma 4.4, we obtain for any \( n \geq 0 \),
\[
U(x - cn - \delta \rho^{-1}(1 - e^{-\sigma})) - \delta e^{-\sigma} \leq u_n(x) \leq U(x - cn + \delta \rho^{-1}(1 - e^{-\sigma})) + \delta e^{-\sigma}.
\]
By the differential intermediate value theorem and the definition of \( U'_m, \delta \), we obtain
\[
\| u_n(x) - U(x - cn) \|_{C^0} \leq U'_m \delta \rho^{-1} + \delta \leq \varepsilon.
\]
This complete the proof of the proposition. \( \square \)

Now, we are ready to prove the uniqueness of bistable traveling waves.

**Theorem 4.6 (Uniqueness).** Assume (A1)-(A4) hold. If there are two bistable waves \( (U, c) \) and \( (\overline{U}, \overline{c}) \), then \( \overline{c} = c \) and \( \overline{U}(\cdot) = U(\cdot + x_0) \) for some \( x_0 \in \mathbb{R} \).

Proof. We divided the proof into two steps.

**Step 1.** Since \( \overline{U}(x) \) and \( U(x) \) are traveling waves with the same limit as \( |x| \to \infty \), there exist \( x_1 \in \mathbb{R} \) and \( \gamma > 1 \) such that, for \( \eta_0 \in (0, 1) \) defined in Lemma 4.4, we have
\[
U(\cdot + x_1) - \eta_0 \rho < \overline{U}(\cdot) < U(\cdot + x_1 + \gamma) + \eta_0 \rho \quad \text{on} \quad \mathbb{R}.
\]
By a translation, we can assume \( x_1 = 0 \). Comparing \( \overline{U}(x - \overline{c}n) \) with \( W^+_n \) of (26) (with \( x_0 = 0 \) for \( W^-_n \) and \( x_0 = \gamma \) for \( W^+_n \)) in lemma 4.4, we obtain, for all \( x \in \mathbb{R} \) and \( n \geq 0 \),
\[
U(x - cn - \eta_0(1 - e^{-\sigma})) - \eta_0 \rho e^{-\sigma} < \overline{U}(x - \overline{c}n)
\]
\[
< U(x - cn + \gamma + \eta_0(1 - e^{-\sigma})) + \eta_0 \rho e^{-\sigma}.
\]
Keeping \( \xi = x - \overline{c}n \) fixed, sending \( n \to \infty \), and using (23), we then obtain from the first inequality that \( c \geq \overline{c} \) and from the second inequality that \( c \leq \overline{c} \), so that \( c = \overline{c} \).

In addition, let \( n \to \infty \), we have
\[
U(\xi - \eta_0) \leq \overline{U}(\xi) \leq U(\xi + \gamma + \eta_0) \quad \forall \xi \in \mathbb{R}.
\] (27)

**Step 2.** Define
\[
\xi^* := \inf \{ \xi : \overline{U}(\cdot) \leq U(\cdot + \xi) \}, \quad \xi_* := \sup \{ \xi : \overline{U}(\cdot) \geq U(\cdot + \xi) \}.
\]
It follows from (27) that both \( \xi^* \) and \( \xi_* \) are well-defined. To finish the proof, it suffices to show that \( \xi^* = \xi_* \). To do this, we use a contradiction argument. Hence, assume \( \xi_* \neq \xi^* \), without losing generality, we assume that \( \xi_* < \xi^* \) (since \( \xi_* > \xi^* \) can be proved similarly) and \( \overline{U}(\cdot) \neq U(\cdot + \xi^*) \).

Fix \( \rho \in (0, \rho_0] \), where \( \rho_0 \) is defined as in Lemma 4.4. Since \( \lim_{|x| \to \infty} U'(x) = 0 \), there exists \( M = M(U, \rho) > 0 \) such that
\[
\eta_0 \rho^{-1} U'(x) \leq 1, \quad |x| \geq M.
\] (28)

Note that the definition of \( \xi^* \) implies \( \overline{U}(\cdot) \leq U(\cdot + \xi^*) \). Then by the strong comparison principle established in Lemma 4.1 we obtain \( \overline{U}(\cdot) < U(\cdot + \xi^*) \) on \( \mathbb{R} \). Consequently, by continuity of \( \overline{U} \) and \( U \), there exists a small constant \( \overline{\gamma} \in (0, \eta_0] \) with \( \eta_0 \) defined in lemma 4.4 such that
\[
\overline{U}(x) < U(x + \xi^* - \eta_0 \overline{\gamma}), \quad \forall x \in [-M - 1 - \xi^*, M + 1 - \xi^*].
\] (29)
Then, for any $\epsilon > 0$ system (5) holds, there exist $\gamma > 0$ such that
$$U(x + \xi^* - \eta_0 \gamma) - \bar{U}(x) > U(x + \xi^* - \eta_0 \gamma) - U(x + \xi^*)$$
$$= -\eta_0 \gamma U'(x + \xi^* - \theta \eta_0 \gamma)$$
$$\geq -\gamma \rho.$$

Hence, in conjunction with (29), we know $U(x + \xi^* - \eta_0 \gamma) + \gamma > \bar{U}(x)$ for all $x \in \mathbb{R}$. Then using Lemma 4.4 again, we have
$$U(x - cn + \xi^* - \eta_0 \gamma (1 - e^{-\sigma n})) + \gamma_0 e^{-\sigma n} > \bar{U}(x - cn), \quad \forall x \in \mathbb{R}, \quad n \geq 0.$$ Setting $\xi = x - cn$ fixed and sending $n \to \infty$, we obtain
$$U(\xi + \xi^* - \eta_0 \gamma) \geq \bar{U}(\xi), \quad \forall \xi \in \mathbb{R}.$$ But this contradict to the definition of $\xi^*$. Hence, $\xi^* = \xi^*$, which complete the proof of the theorem. \hfill \Box

5. Exponentially asymptotical stability. In this section, we will show that the monotonic traveling wave $(U, c)$ is globally exponentially asymptotically stable. We will modify the squeezing idea and technics in [1] to adapt our iterative system. For this purpose, we first prove some technical lemmas.

**Lemma 5.1.** Assume that (A1)-(A4) hold. Let $\{u_n\}_{n \geq 0}$ be a sequence satisfying system (1). If $u_0 \in [0, u^*]_c$ satisfies
$$\limsup_{x \to -\infty} u_0(x) < u_* < \liminf_{x \to +\infty} u_0(x). \quad (31)$$
Then, for any $\epsilon > 0$, there exist $M = M(\epsilon) > 0$ and an integer $N = N(\epsilon) > 0$ such that
$$u_N(x) \geq u^* - \epsilon, \quad \forall x \geq M \quad \text{and} \quad u_N(x) \leq \epsilon, \quad \forall x \leq -M. \quad (32)$$

**Proof.** Since (5.1) holds, there exist $M_1 > 0$ and $r \in (0, \min\{u_*, u^* - u_*\})$ such that
$$u^* \geq u_0(x) \geq u_* + r, \quad \forall x \geq M_1 \quad \text{and} \quad u_* - r \geq u_0(x) \geq 0, \quad \forall x \leq -M_1.$$ By Lemma 3.1, bistability assumption (A5) holds. It follows that, for any $\epsilon > 0$, there exists an integer $N = N(\epsilon) > 0$ such that
$$Q^N[u_* + r] \geq u^* - \epsilon/2 \quad \text{and} \quad Q^N[u_* - r] \leq \epsilon/2. \quad (33)$$ Assumption (A2) implies that $Q^N$ is continuous with respect to the compact open topology. And hence, using assumption (A1), we can obtain
$$\lim_{x \to -\infty} u_N(x) = \lim_{x \to -\infty} Q^N[u_0](x) = Q^N[\lim_{x \to -\infty} u_0(\cdot + x)](0).$$ Note that $\lim_{x \to -\infty} u_N(\cdot + x) \geq u_* + r$ and $\lim_{x \to -\infty} u_N(\cdot + x) \leq u_* - r$ locally uniformly. Then it follows from assumption (A3) that the following inequality holds locally uniformly,
$$\lim_{x \to -\infty} u_N(x) \geq Q^N[u_* + r] \quad \text{and} \quad \lim_{x \to -\infty} u_N(x) \leq Q^N[u_* - r].$$
As such, there exists $M = M(\epsilon) > 0$ such that
$$u_N(x) \geq Q^N[u_* + r] - \epsilon/2, \quad \forall x \geq M \quad \text{and} \quad u_N(x) \leq Q^N[u_* - r] + \epsilon/2, \quad \forall x \leq -M.$$ Then, applying (33) to the above inequality, we obtain (32). This completes the proof. \hfill \Box
Lemma 5.2. Assume that \( \varphi_1 \geq Q[\varphi_0] \) and \( \psi_1 \leq Q[\psi_0] \). If \( \varphi_0 \geq \psi_0 \), then there exists a strictly decreasing function \( \lambda = \lambda(m) \) in \( m \), such that for \( m > 0 \),

\[
\min_{x \in [-m, m]} \{ \varphi_1(x) - \psi_1(x) \} \geq \lambda(m) \int_0^1 [\varphi_0(y) - \psi_0(y)]dy.
\]

Proof. By monotonicity of birth rate \( h \), we compute to obtain

\[
\varphi_1(x) - \psi_1(x) \geq Q[\varphi_0](x) - Q[\psi_0](x)
\]

\[
\geq \int_{\mathbb{R}} k_M(T, 0, x-y)(\varphi_0(y) - \psi_0(y))dy
\]

\[
\geq \int_0^1 k_M(T, 0, x-y)(\varphi_0(y) - \psi_0(y))dy.
\]

Since \( k_M(T, 0, \cdot) \) is symmetrically strictly decreasing, we then obtain assertion of the lemma by setting \( \lambda(m) := k_M(T, 0, m+1) \).

Lemma 5.3. Assume that (A1)-(A4) hold. Let \( \{u_n\}_{n \geq 0} \) be a sequence defined by system (1). If there exists \( N > 0 \) such that, for some \( x_0 \in \mathbb{R} \), \( \eta \in (0, \eta_0/2] \), \( \rho \in (0, \rho_0] \) and \( \xi > 0 \) there holds

\[
U(x + x_0) - \eta \rho \leq u_N(x) \leq U(x + x_0 + \xi) + \eta \rho, \quad \forall x \in \mathbb{R}. \tag{34}
\]

Then there exists a small positive \( \varepsilon^* \) such that for any \( n > 1 \), there holds

\[
U(x - cn + \hat{x}_n) - \delta_n \leq u_{N+n}(x) \leq U(x - cn + \hat{x}_n + \hat{\xi}_n) + \hat{\delta}_n, \quad \forall x \in \mathbb{R}. \tag{35}
\]

where \( \hat{x}_n \), \( \xi_n \) and \( \delta_n \) satisfying

\[
\hat{x}_n := x_0 + \varepsilon^* \min\{\xi, 1\}/\rho - \eta,
\]

\[
\hat{\delta}_n := e^{-\sigma(n-1)}[\eta \rho + \varepsilon^* \min\{\xi, 1\}],
\]

\[
\hat{\xi}_n := \xi + \eta(2 - e^{-\sigma n}) - \varepsilon^*/\rho.
\]

Proof. First of all, comparing \( W_0^* \) in Lemma 4.4 with (34) and denoting \( \delta_0 = \eta \rho \), we see that for all \( x \in \mathbb{R} \) and \( n \geq 0 \),

\[
U(x + x_0 - cn - \eta(1 - e^{-\sigma n})) - \delta_0 e^{-\sigma n} \leq u_{N+n}(x) \leq U(x + x_0 + \xi - cn + \eta(1 - e^{-\sigma n})) + \delta_0 e^{-\sigma n}.
\]

By Lemma 4.2, we have \( \lim_{|x| \to \infty} U'(x) = 0 \). Then we can fix a positive number \( M_1 \) such that \( U'(x) \leq \rho/2 \) for all \( |x| \geq M_1 \). Set

\[
\bar{\xi} := \min\{\xi, 1\}, \quad \varepsilon_1 := \frac{1}{2} \min\{U'(x) : -|x_0| \leq x \leq |x_0| + 2\}.
\]

Then the mean value theorem implies that

\[
\int_0^1 U(y + x_0 + \bar{\xi}) - U(y + x_0)dy \geq 2\varepsilon_1 \bar{\xi},
\]

and hence, at least one of the following is true

(i) \( \int_0^1 u_N(y) - U(y + x_0)dy \geq \varepsilon_1 \bar{\xi} \);

(ii) \( \int_0^1 U(y + x_0 + \bar{\xi}) - u_N(y)dy \geq \varepsilon_1 \bar{\xi} \).

In what follows, we consider only the case (i). The case (ii) is similar and thus omitted. Considering the first part of (34) and using Lemma 5.2 we obtain, for \( \lambda = \lambda(M_1 + 2 + |c| + |x_0|) \) and every \( x \in [-M_1 - 2 - |x_0| - |c|, M_1 + 2 + |x_0| + |c|] \),

\[
u_{N+1}(x) - [U(x + x_0 - \eta(1 - e^{-\sigma})) - \delta_0 e^{-\sigma}] \geq \lambda \int_0^1 u_N(y) - [U(y + x_0) - \delta_0]dy \geq \lambda \varepsilon_1 \bar{\xi}.
\]
Define 
\[ \varepsilon^* = \min \left\{ \frac{\eta_0 \rho}{2}, \min_{x \in [-M_1-2|c|-2,M_1+2|c|+2]} \frac{\rho \lambda \xi_1}{2U''(x)} \right\} \].

Then for all \( x \in [-M_1-1-|c|-|x_0|, M_1+1+|c|+|x_0]| \), we have
\[ U(x + x_0 - c - \eta(1 - e^{-\sigma}) + \frac{2}{\rho} \varepsilon^* \xi) - U(x + x_0 - c - \eta(1 - e^{-\sigma})) \leq U'(-\theta) \frac{2}{\rho} \varepsilon^* \xi \leq \lambda \varepsilon_1 \xi. \]

Therefore, for \( x \in [-M_1-1-|c|-|x_0|, M_1+1+|c|+|x_0]| \), we have
\[ u_{N+1}(x) \geq U(x + x_0 - c - \eta(1 - e^{-\sigma}) + \frac{2}{\rho} \varepsilon^* \xi) - \delta_0 e^{-\sigma}. \]

When \( |x| \geq M_1 + 1 + |c| + |x_0| \), by the definition of \( M_1 \) and the mean value theorem, we have
\[ U(x + x_0 - c - \eta(1 - e^{-\sigma})) \geq U(x + x_0 - c - \eta(1 - e^{-\sigma}) + \frac{2}{\rho} \varepsilon^* \xi) - \varepsilon^* \xi. \]

In conclusion, the above analysis implies
\[ u_{N+1}(x) \geq U(x + x_0 - c - \eta(1 - e^{-\sigma}) + \frac{2}{\rho} \varepsilon^* \xi) - [\delta_0 e^{-\sigma} + \varepsilon^* \xi], \quad \forall x \in \mathbb{R}. \] (36)

Define \( q = \delta_0 e^{-\sigma} + \varepsilon^* \xi \), the definition of \( \varepsilon^* \) and \( \delta_0 \) implies \( q/\rho \leq \eta_0 \). Therefore, using Lemma 4.4 to (36) again, we have, for \( n > 1 \),
\[ u_{N+n}(x) \geq U(x + x_0 - cn - \eta(1 - e^{-\sigma}) + \frac{2}{\rho} \varepsilon^* \xi - (q/\rho)(1 - e^{-\sigma(n-1)})) - qe^{-(n-1)} \]
\[ \geq U(x - cn + x_0 + \varepsilon^* \xi/\rho - \eta) - e^{-\sigma(n-1)}[\delta_0 + \varepsilon^* \xi]. \]

Thus, we can define \( \hat{x}_n := x_0 + \varepsilon^* \xi/\rho - \eta \) and \( \hat{\delta}_n = e^{-\sigma(n-1)}(\delta_0 + \varepsilon^* \xi) \).

On the other hand, for \( n \geq 1 \),
\[ u_{N+n}(x) \leq U(x + x_0 + \xi - cn + \eta(1 - e^{-\sigma n})) + \delta_0 e^{-\sigma n} \]
\[ \leq U(x + x_0 + \xi - cn + \eta(1 - e^{-\sigma n})) + \hat{\delta}_n. \]

And hence, \( \hat{\xi}_n = x_0 + \xi + \eta(1 - e^{-\sigma n}) - \hat{x}_n = \xi + \eta(2 - e^{-\sigma n}) - \varepsilon^*/\rho \) is defined. This complete the proof of the lemma. \(\square\)

Now, we are in a position to prove the main result in this section.

**Theorem 5.4 (Exponentially Asymptotically Stability).** Assume that (A1)-(A4) hold. Let \( (U, c) \) be the traveling wave of \( Q \) obtained in theorem 3.2. Also assume \( \{u_n\}_{n \geq 0} \) be a sequence satisfying system (1). If \( u_0 \in [0, u^*] \) be such that
\[ \limsup_{x \to -\infty} u_0(x) < u_* < \liminf_{x \to +\infty} u_0(x). \]

Then there exist a constant \( a > 0 \) independent of \( u_0 \) and two constants \( A, \xi \) dependent on \( u_0 \) such that
\[ \|u_n(\cdot) - U(\cdot - cn + \xi)\|_{C^0} \leq Ae^{-\alpha n}, \quad \forall n \geq 0. \] (37)

**Proof.** We shall prove this theorem in several steps.

**Step 1.** By Lemma 5.1, choosing \( \varepsilon = \eta \rho \) with \( \eta \in (0, \eta_0/2] \) and \( \rho \in (0, \rho_0] \) defined in Lemma 4.4, we know there exist \( x_0 \in \mathbb{R}, \xi > 0 \) and an integer \( N > 0 \) such that
\[ U(x + x_0) - \eta \rho \leq u_N(x) \leq U(x + x_0 + \xi) + \eta \rho, \quad \forall x \in \mathbb{R}. \] (38)
Step 2. We define
\[\delta^* := \min\{\eta_0\rho/2, \varepsilon^*/4\}, \quad \kappa^* := (\varepsilon^* - 2\delta^*)/\rho \geq \varepsilon^*/(2\rho).\]

Clearly, the definition of \(\varepsilon^*\) implies \(\kappa^* \leq 1\). Also, we fix the integer \(n^* > 1\) such that
\[e^{-\sigma(n^*-1)}(1 + \varepsilon^*/\delta^*) \leq 1 - \kappa^*.\]

We take \(\eta \rho = \delta^*\) and \(\eta = \delta^*/\rho\) in (38). Further, we can define \(\xi \geq 1\), otherwise, we directly go to Step 3 by choosing \(\xi = 1\).

Applying Lemma 5.3 to (38), we can obtain
\[U(x - cn^* + \hat{x}_{n^*}) - \delta_{n^*} \leq U(x - cn^* + \hat{x}_{n^*} + \hat{\xi}_{n^*}) + \hat{\delta}_{n^*} (39)\]
where
\[
\hat{x}_{n^*} = x_0 + \varepsilon^*/\rho - \delta^*/\rho,
\]
\[
\hat{\delta}_{n^*} = e^{-\sigma(n^*-1)}[\delta^* + \varepsilon^*] \leq (1 - \kappa^*)\delta^* \leq \delta^*,
\]
\[
\hat{\xi}_{n^*} = \xi + \delta^*(2 - e^{-\sigma n^*})/\rho - \varepsilon^*/\rho \leq \xi - \varepsilon^*/\rho + 2\delta^*/\rho \leq \xi - \kappa^*.
\]

Using Lemma 5.3 again to (39) with \(\hat{\delta}_{n^*}\) replaced by \(\delta^*\), there holds (39) with \(n^*, \hat{\delta}_{n^*}\) and \(\hat{\xi}_{n^*}\) replaced by \(jn^*, \delta^*\) and \(\xi - j\kappa^*\), respectively, for \(j \geq 1\). Note that there exists integer \(j_0 > 0\) such that \(0 < \xi - j_0\kappa^* < 1\). So, without loss of generality, we assume \(\xi = 1\), and thus go to the next step.

Step 3. Now, we use the mathematical induction method to show that for every positive integer \(j\), (39) holds for some \(\hat{x}_j = x_j \in \mathbb{R}\) and
\[n^* = N^j := jn^*, \quad \hat{\delta}_{n^*} = \delta^j := (1 - \kappa^*)^j\delta^*, \quad \hat{\xi}_{n^*} = \xi^j := (1 - \kappa^*)^j\]
Clearly, by Step 2, the assertion is true for \(j = 1\). Now, assume the assertion is true for some \(j = k \geq 1\). We want to show that the assertion holds for \(j = k + 1\). Indeed, applying Lemma 5.3 to (39) with \(n^* = N^k\), \(\hat{\delta}_{n^*} = \delta^k\) and \(\hat{\xi}_{n^*} = \xi^k\) and \(\hat{x}_{n^*} = \hat{x}^k\), it can be derived that (39) holds with \((n^*, \hat{x}_{n^*}, \hat{\delta}_{n^*}, \hat{\xi}_{n^*})\) replaced by \((N^{k+1}, \hat{x}, \hat{\delta}, \hat{\xi})\)
where \((\hat{x}, \hat{\delta}, \hat{\xi})\) satisfies
\[
\hat{x} = x^k + \varepsilon^*\xi^k/\rho - \delta^k/\rho,
\]
\[
\hat{\delta} = e^{-\sigma(n^*-1)}[\delta^k + \varepsilon^*] \leq (1 - \kappa^*)\delta^k = (1 - \kappa^*)^{k+1}\delta^* = \delta^{k+1},
\]
and
\[
\hat{\xi} = \xi^k + \delta^k(2 - e^{-\sigma n^*})/\rho - \varepsilon^*/\rho \leq \xi^k - \varepsilon^*/\rho - 2/\rho = \xi^k - \xi^k + \kappa^*(1 - \kappa^*)\delta^*
\]
\[
\leq (1 - \kappa^*)^{k+1} = \xi^{k+1}.
\]
That is, the assertion holds for \(j = k + 1\). This completes the induction.

Step 4. The previous step tells us that (39) holds for \((n^*, \hat{x}_{n^*}, \hat{\delta}_{n^*}, \hat{\xi}_{n^*})\) replaced by \((N^j, \hat{x}^j, \hat{\delta}^j, \hat{\xi}^j)\) for all \(j \geq 1\). In addition, from Lemma 5.3, we know (39) holds for all \(n \geq N^j\), \((\hat{x}_n, \hat{\delta}_n, \hat{\xi}_n) = (x^j, \delta^j, \xi^j)\). That is,
\[U(x - cn + x^j) - \delta^j \leq U(x - cn + x^j + \xi^j) + \delta^j \quad \forall n \geq N + jn^*.
\]
And hence, setting \(x(n) = x^j, \delta(n) = \delta^j\) and \(\xi(n) = \xi^j\) for all \(n \in [N + N^j, N + N^{j+1})\) for all \(j \geq 1\), we have
\[U(x - cn + x(n)) - \delta(n) \leq U(x - cn + x(n) + \xi(n)) + \delta(n). (40)\]
Moreover, define
\[ a := -\frac{\ln(1 - \kappa^*)}{\kappa^*} > 0. \]
we have
\[ \delta(n) = \delta^j = (1 - \kappa^*)^j \leq e^{\alpha N} e^{-an}, \]
\[ \xi(n) = \xi^j = (1 - \kappa^*)^j \leq e^{\alpha N} e^{-an}, \]
and
\[ x(n) = x^j = n^{-1} + \varepsilon^* i^{-1} / \rho - \delta^j / \rho. \]
Since \( x^{j+1} - x^j = \varepsilon^* \xi^j / \rho - \delta^j / \rho = \frac{[(\varepsilon^* - \delta^j) / \rho]}{(1 - \kappa^*)^j} \), the sequence \( x^j \) convergent as \( j \to \infty \). So, \( x(n) \) convergent to \( x(\infty) \) as \( n \to \infty \). What is more, for \( n \in [N + N^j, N + N^j+1) \),
\[ |x(\infty) - x(n)| = \frac{\varepsilon^* - \delta^j}{\rho} \sum_{i = j}^{\infty} (1 - \kappa^*)^i = \frac{\varepsilon^* - \delta^*}{\rho \kappa^*} (1 - \kappa^*)^j \leq \frac{\varepsilon^* - \delta^*}{\rho \kappa^*} e^{\alpha N} e^{-an}. \]
As a conclusion, it follows from (40) that
\[ \|u_{N+n}(\cdot) - U(\cdot - cn + x(\infty))\|_{C^0} \leq U'_n[|x(\infty) - x(n)| + \xi(n)] + \delta(n) \leq A_1 e^{-an}, \quad \forall n > 1. \]
where \( U'_n = \max\{U'(x) : x \in \mathbb{R}\} \) and \( A_1 = e^{\alpha N} (\delta^* + 1 + \frac{\varepsilon^* - \delta^*}{\rho \kappa^*}). \)
Further, by the boundedness of \( u_n \) and \( U \), there exists \( A_2 > 0 \) such that
\[ \|u_n(\cdot) - U(\cdot - cn + x(\infty) + cN)\|_{C^0} \leq A_2 e^{-an}, \quad \forall 0 \leq n \leq N. \]
Consequently, (37) is valid provided that \( A = \max\{A_1, A_2\} \) and \( \xi = x(\infty) + cN. \)
This completes the proof of the theorem. \( \square \)

Acknowledgments. This work received fundings from the National Natural Science Foundation of China (11771108, 11771109) and the Fundamental Research Funds for the Central Universities of China. We are very grateful to the anonymous referee for careful reading and valuable comments which led to an improvement of our original manuscript.

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Received December 2017; revised January 2018.

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