Exact Normal Modes for a System with Three Luttinger Liquids Interacting in an Elongated Trap

B. Altschul

Department of Physics
Indiana University
Bloomington, IN 47405 USA

Abstract

We consider a system of spinless fermions in a nearly one-dimensional cylindrical trap, with the Fermi level such that only the lowest-lying states with angular momenta $\ell = +\hbar$, 0, or $-\hbar$ about the axis of the trap are occupied. We treat the particles in these states as comprising three separate Luttinger liquids, with the possibility for forward and backward scattering between them. We determine the normal modes in the system in the presence of this scattering, finding two sets of linear harmonic fluid modes and one set of modes that are self-coupled through a sine-Gordon interaction.
The problem of spinless fermionic particles trapped in a cylindrical potential with a very high aspect ratio possesses many interesting aspects. If the fermions are constrained to move almost exclusively along the direction of the traps’ axis, then the system becomes effectively one-dimensional in character. This will affect the qualitative nature of the system’s behavior, since fermions in one dimension have many curious properties.

In particular, a one-dimensional system of low-temperature fermions exhibits bosonization properties [1, 2, 3], as the coherent fermion-hole excitations behave like bosons. This phenomenon is described by the Luttinger model. The main physical applications of this model occur in the context of the quantum Hall effect [4, 5]. However, the model is clearly also relevant to a trapped fermion system such as we have described. The field of fermion trapping is advancing very rapidly [6, 7, 8, 9], and it may be possible in the foreseeable future to observe Luttinger-liquid-type behavior in trapped fermionic gasses.

We consider a cylindrical trap with hard walls, of length \( L \) and axial radius \( b \ll L \). The fermion modes in this trap are described by three quantum numbers—\( k, \ell, \) and \( n_\rho \)—corresponding to the cylindrical coordinates \((z, \theta, \rho)\). \( k \) is the momentum along the axis of the trap (setting \( \hbar = 1 \)), while \( \ell \) is the angular momentum about this axis, and \( n_\rho \) is the number of radial nodes in the wavefunction. The wavefunction is periodic in the \( z \)-direction with period \( L \) and vanishes at \( \rho = b \). In the absence of other interactions, the energy of a trapped particle is

\[
E_{k,\ell,n_\rho} = \frac{1}{2m} \left[ k^2 + \left( \frac{\ell_{\rho,n_\rho}}{b} \right)^2 \right],
\]

where \( m \) is the fermion mass and \( \ell_{\rho,n_\rho} \) is the \( n_\rho \)-th root of the Bessel function \( J_\ell \). For fixed \( k \), the three lowest-lying levels have \( n_\rho = 0 \) and \( \ell = +1, 0, \) and \(-1\). The energy gap for the two \( |\ell| = 1 \) states is \( \Delta = \frac{(j_{1,1})^2 - (j_{0,1})^2}{2mb^2} \approx \frac{44}{mb^2} \).

Previously, we have considered the effects of virtual particles in radially and angularly excited states of the trap on a Luttinger liquid lying entirely in the ground state [10]. In this paper, we consider a different regime, in which the Fermi energy \( \epsilon_F \) is larger than \( \Delta \), so that the lowest-lying \( \ell = \pm 1 \) states are filled with large numbers of particles, while all the higher excited states of the trap remain empty. We shall treat this system as consisting of three interacting Luttinger liquids and shall study the normal modes of excitation. For particular choices of couplings, there can exist extremely simple exact solutions of this model.

In accordance with the Luttinger model [11, 12], we linearize the fermion spectrum and append an infinite negative energy Dirac sea to the physical system. If we fix the number of particles with each value of \( \ell \), then we can take the noninteracting Hamiltonian (ignoring any additive constant) to be

\[
H_0 = \sum_{\ell = -1}^{+1} \left\{ \frac{2\pi v_F^\ell}{L} \sum_{q > 0} \left[ \rho_+^\ell(q)\rho_+^\ell(-q) + \rho_-^\ell(-q)\rho_-^\ell(q) \right] \right\},
\]

(2)
where the $\rho^\ell_\pm$ operators are

$$
\rho^\ell_+(q) = \sum_k a^\dagger_{k+q} a^\ell_k,
$$

$$
\rho^\ell_-(q) = \sum_k b^\dagger_{k+q} b^\ell_k.
$$

(3)

The $a^\ell$ ($b^\ell$) operators correspond to right- (left-)moving (spinless) fermions of angular momentum $\ell$ about the axis of the trap, and $v_F^\ell$ is the Fermi velocity for the Luttinger liquid of angular momentum $\ell$. (For quantities, such as the Fermi velocity, which depend only upon $|\ell|$, we shall never write a negative superscript; only “0” or “1” will be used.)

For $\ell = 0$, we have $v_F^0 = \sqrt{2k_F/m}$, while $v_F^1 = \sqrt{2(k_F - \Delta)/m}$. The $\rho^\ell_\pm$ obey the Bose commutation relations $[\rho^\ell_+(q), \rho^\ell_+(q')] = [\rho^\ell_-(q), \rho^\ell_-(q')] = \delta_{qq'}\delta_{\ell\ell'} q^2/(2\pi)$ and $[\rho^\ell_+(q), \rho^\ell_-(q')] = 0$, so they create and annihilate phonons with energies $|q|v_F^\ell$.

We may also introduce fermion-fermion scattering terms which are bilinear in the $\rho^\ell_\pm$ operators. At low temperatures, nearly all the scattered particles lie near the Fermi surface, with momenta $k_F^\ell = mv_F^\ell$. We shall consider both forward scattering interactions of the types $(k_F^\ell; k_F^\ell) \rightarrow (k_F^\ell; k_F^\ell)$ or $(k_F^\ell; -k_F^\ell) \rightarrow (k_F^\ell; -k_F^\ell)$ and backward scattering $(k_F^\ell; -k_F^\ell) \rightarrow (k_F^\ell; -k_F^\ell)$. The backward scattering is forbidden by momentum conservation unless $\ell = \pm \ell'$, and it is truly distinct from the forward scattering only if $\ell \neq \ell'$. [“Inelastic” scattering processes of the form $(k_F^\ell; -k_F^\ell) \rightarrow (k_F^\ell; -k_F^\ell')$ with $|\ell| \neq |\ell'|$ are also allowed by angular and linear momentum conservation, but they lead to substantially different and more complicated effects, in part because they change the numbers of particles in the three Luttinger liquids. We shall ignore these interactions in the body of this paper, although a brief discussion of them is located in the Appendix.]

We write the forward scattering terms as generalizations of the interactions in the Tomonaga-Luttinger model [13]:

$$
H_{fs} = \frac{1}{2L} \sum_q \left[ \begin{array}{c}
\rho^0_+(q) \\
\rho^0_-(q) \\
\rho^1_+(q) \\
\rho^1_-(q) \\
\rho^-_+(q) \\
\rho^-_-(q)
\end{array} \right]^T \left[ \begin{array}{cccccc}
g_{00} & g_{00} & g_{01} & g_{01} & g_{01} & g_{01} \\
g_{00} & g_{00} & g_{01} & g_{01} & g_{01} & g_{01} \\
g_{01} & g_{01} & g_{11} & g_{11} & g_{11} & g_{11} \\
g_{01} & g_{01} & g_{11} & g_{11} & g_{11} & g_{11} \\
g_{01} & g_{01} & g_{11} & g_{11} & g_{11} & g_{11} \\
g_{01} & g_{01} & g_{11} & g_{11} & g_{11} & g_{11}
\end{array} \right] \left[ \begin{array}{c}
\rho^0_+(q) \\
\rho^0_-(q) \\
\rho^1_+(q) \\
\rho^1_-(q) \\
\rho^-_+(q) \\
\rho^-_-(q)
\end{array} \right].
$$

(5)

All the interaction strengths $g_{|\ell||\ell'|}$ are presumed to have the same underlying origin; however, their values depend upon the radial density profiles of the two interacting particles, so they depend upon $|\ell|$ and $|\ell'|$. The $g_{|\ell||\ell'|}$ may most generally be functions of the momentum transfer $q$, but their large-$q$ behavior is irrelevant to the low-energy physics, and we shall take the interactions to be momentum-independent. The symmetries of the matrix in (5) are physically very reasonable. The $\ell = 0$ states interact symmetrically with the $\ell = \pm 1$ states, and the forward scattering interactions among $|\ell| = 1$ states are independent of the sign of the $\ell$. 2
We shall introduce the remaining Hamiltonian for backward scattering later, after we look a bit further at the forward-scattering effects. We may simplify the interactions by introducing symmetric and antisymmetric combinations of the $|\ell|=1$ operators: $\rho_\pm^S = \frac{1}{\sqrt{2}} \left( \rho_{\pm}^1 + \rho_{\mp}^1 \right)$ and $\rho_\pm^A = \frac{1}{\sqrt{2}} \left( \rho_{\pm}^1 - \rho_{\mp}^1 \right)$. This change of basis mixes states which are degenerate under $H_0$, and it simplifies $H_{fs}$ to

$$H_{fs} = \frac{1}{2L} \sum_q \begin{bmatrix} \rho_+^S(q) \\ \rho_-^S(q) \end{bmatrix}^T \begin{bmatrix} g_{00} & g_{00} & \sqrt{2}g_{01} \\ g_{00} & g_{00} & \sqrt{2}g_{01} \\ \sqrt{2}g_{01} & \sqrt{2}g_{01} & 2g_{11} \end{bmatrix} \begin{bmatrix} \rho_+^S(-q) \\ \rho_-^S(-q) \end{bmatrix}. \quad (6)$$

The antisymmetric modes are completely decoupled from the others.

The separation of the antisymmetric $|\ell|=1$ modes is extremely fortuitous, because the same separation also occurs when backward scattering is included. The backscattering Hamiltonian is

$$H_{bs} = \frac{1}{L} \sum_{k,p,q,\ell,\ell'} \sum_{q} g_{bs} \rho_{k}^{\ell,+} a_{p}^{\ell,+} a_{p+2kF+q}^{\ell'} b_{p-2kF-q}^{\ell'}.$$ \quad (7)

We have neglected the backscattering for the $\ell=0$ modes, because it may be rewritten as an effective forward scattering (i.e. as a contribution to $g_{00}$). $H_{bs}$ adds additional terms to the linear coupling matrix and introduces a sine-Gordon coupling $\Pi$ as well:

$$H_{fs+bs} = \frac{1}{2L} \sum_q \begin{bmatrix} \rho_+^A(q) \\ \rho_-^A(q) \end{bmatrix}^T \begin{bmatrix} g_{00} & g_{00} & \sqrt{2}g_{01} \\ g_{00} & g_{00} & \sqrt{2}g_{01} \\ \sqrt{2}g_{01} & \sqrt{2}g_{01} & 2g_{11} - g_{bs} \end{bmatrix} \begin{bmatrix} \rho_+^A(-q) \\ \rho_-^A(-q) \end{bmatrix} - \frac{g_{bs}}{L} \sum_q \rho_+^A(q)\rho_-^A(-q) + \frac{2g_{bs}}{(2\pi\alpha^1)^2} \int dx \cos \left( \sqrt{8} \phi^A \right). \quad (8)$$

$\phi^A$ is the boson field associated with the antisymmetric modes,

$$\phi^A(x) = -\frac{i\pi}{L} \sum_{p\neq 0} e^{-\alpha^1 |p|/2 - ipx} \left[ \rho_+^A(p) + \rho_-^A(p) \right]. \quad (9)$$

The parameter $\alpha^1$ is a short-distance cutoff, and $v_F \alpha^1$ is the bandwidth of the $\ell = \pm 1$ Luttinger liquids. (In $[10]$, we found that $\alpha^\ell = 1/k_F^\ell$ for a system of trapped fermions such as we are considering.)

Much of the remaining work we shall do on this model involves Bogoliubov transforming the operators in order to simplify the interactions, replacing the operators $\rho_\pm(q)$ with new operators $\tilde{\rho}_\pm(q)$ that obey the same commutation relations $[16]$. Our transformations
are formally unitary only if the interactions decrease sufficiently quickly as functions of $|q|$. However, we have previously stated that the large-momentum behavior of the couplings is irrelevant to the low-energy physics, and for momenta above $1/\alpha^4$, the model is unphysical in any case. So we shall continue to treat the couplings as constants.

Performing an appropriate Bogoliubov transformation on the antisymmetric modes transforms the corresponding portion of the Hamiltonian into

$$H' = \frac{2\pi u_A}{L} \sum_{q>0} \left[ \tilde{\rho}_+^A(q)\tilde{\rho}_+^A(-q) + \tilde{\rho}_-^A(-q)\tilde{\rho}_-^A(q) \right] + \frac{2g_{bs}}{(2\pi\alpha^1)^2} \int dx \cos \left[ \left( \frac{64}{2\pi v_F^1 + g_{bs}} \right)^{1/4} \tilde{\phi}^A \right],$$

where the $\tilde{\rho}^A$ and $\tilde{\phi}^A$ are the transformed operators and the corresponding field, and $u_A = \sqrt{(v_F^1)^2 - (g_{bs}/2\pi)^2}$. For repulsive interactions ($g_{bs} > 0$), the cosine term is renormalized to zero at long wavelengths. For $g_{bs} < 0$, the flow is not toward $g_{bs} = 0$, but for a particular choice of couplings, the Hamiltonian is exactly solvable [17].

The self-coupled sine-Gordon field $\tilde{\phi}^A$ describes one set of fundamental modes of the system. We now turn our attention to the remaining modes. With another Bogoliubov transformation, the remainder of the Hamiltonian becomes

$$H'' = \frac{2\pi}{L} \sum_{q>0} \left[ \begin{array}{c} \tilde{\rho}_+^0(q) \\ \tilde{\rho}_0^0(q) \\ \tilde{\rho}_-^S(q) \\ \tilde{\rho}_-^S(q) \end{array} \right]^T \left[ \begin{array}{cccc} u_0 & 0 & \bar{g} & \bar{g} \\ 0 & u_0 & \bar{g} & \bar{g} \\ \bar{g} & \bar{g} & u_S & 0 \\ \bar{g} & \bar{g} & 0 & u_S \end{array} \right] \left[ \begin{array}{c} \tilde{\rho}_+^0(-q) \\ \tilde{\rho}_0^0(-q) \\ \tilde{\rho}_-^S(-q) \\ \tilde{\rho}_-^S(-q) \end{array} \right].$$

(11)

The transformed coupling is

$$\bar{g} = \left[ 4 \left( \frac{2\pi v_F^0}{2\pi v_F^1 + 2g_{00}} \right) \left( \frac{2\pi v_F^1 + g_{bs}}{2\pi v_F^1 + 4g_{11} - g_{bs}} \right) \right]^{1/4} \frac{g_{01}}{2\pi},$$

(12)

and the wave speeds at $\bar{g} = 0$ are

$$u_0 = \sqrt{(v_F^0)^2 + v_F^0 g_{00}/\pi}$$

(13)

$$u_S = \sqrt{(v_F^0 + g_{bs}/2\pi)(v_F^1 + 2g_{11}/\pi - g_{bs}/2\pi)}$$

(14)

We shall denote the matrix appearing in (11) by $M$.

If the interactions $g_{00}$, $g_{11}$, and $g_{bs}$ conspire to produce a degeneracy, with $u_0 = u_S$, then $M$ may be diagonalized easily. We again take symmetric and antisymmetric combinations $\tilde{\rho}_\pm^B = \frac{1}{\sqrt{2}} (\tilde{\rho}_\pm^0 + \tilde{\rho}_\pm^S)$ and $\tilde{\rho}_\pm^C = \frac{1}{\sqrt{2}} (\tilde{\rho}_\pm^0 - \tilde{\rho}_\pm^S)$, which transforms $H''$ into the form

$$H'' = \frac{2\pi}{L} \sum_{q>0} \left[ \begin{array}{c} \tilde{\rho}_+^B(q) \\ \tilde{\rho}_0^B(q) \\ \tilde{\rho}_-^C(q) \\ \tilde{\rho}_-^C(q) \end{array} \right]^T \left[ \begin{array}{cccc} u + \bar{g} & \bar{g} & 0 & 0 \\ \bar{g} & u + \bar{g} & 0 & 0 \\ 0 & 0 & u - \bar{g} & -\bar{g} \\ 0 & 0 & -\bar{g} & u - \bar{g} \end{array} \right] \left[ \begin{array}{c} \tilde{\rho}_+^B(-q) \\ \tilde{\rho}_0^B(-q) \\ \tilde{\rho}_-^C(-q) \\ \tilde{\rho}_-^C(-q) \end{array} \right].$$

(15)
One further Bogliubov transformation gives the decoupled normal modes, with frequencies
\[ \omega = |q| \sqrt{u^2 + 2\bar{g}}. \]

In the presence of a \( u_0 = u_S \) degeneracy, we find exact energy shifts containing terms linear in \( \bar{g} \). In the absence of this special degeneracy, we expect the lowest-order correction to the frequency to be \( \mathcal{O}(\bar{g}^2) \). We may find this leading term by treating \( \bar{g} \) perturbatively.

Through a completely straightforward calculation, we find shifted frequencies
\[ \omega_0 \approx |q|u_0 + |q|\bar{g} \frac{2u_S}{u_0^2 - u_S^2} \]
\[ \omega_S \approx |q|u_S - |q|\bar{g} \frac{2u_0}{u_0^2 - u_S^2} \]
for the \( \bar{\rho}^0 \) and \( \bar{\rho}^S \) modes, respectively.

If we wish to write \( H'' \) (for general \( u_0, u_S \), and \( \bar{g} \)) as a sum of noninteracting harmonic oscillator Hamiltonians, we must find a \( U(2, 2) \) matrix that diagonalizes \( M \). That is, we need to find a matrix \( V \) such that \( V^\dagger MV \) is diagonal, and \( V^\dagger I_{2,2} V = I_{2,2} \), where \( I_{2,2} = \text{diag}[1 -1 1 -1] \). [Our subsidiary condition on \( V \)—which makes it a member of \( U(2, 2) \), by definition—ensures that the transformation preserves the correct commutation relations. The Bogliubov transformation that led to the form (11) for \( H'' \) corresponds to an element of the block diagonal \( U(1,1) \times U(1,1) \) subgroup of \( U(2,2) \).]

The \( U(2,2) \) diagonalization problem for \( M \) is equivalent to the ordinary diagonalization of
\[ M' = \begin{bmatrix} u_0 & 0 & \bar{g} & i\bar{g} \\ 0 & -u_0 & i\bar{g} & -\bar{g} \\ \bar{g} & i\bar{g} & u_S & 0 \\ i\bar{g} & -\bar{g} & 0 & -u_S \end{bmatrix}, \]
which is related to \( M \) by a phase rotation of the \( \bar{\rho}^- \) terms. The positive eigenvalues of \( M' \) are exactly the normal mode frequencies, and the eigenvectors of \( M' \) give (upon an appropriate undoing of the phase rotation) the weights of the various \( \bar{\rho} \) in the normal mode creation and annihilation operators.

The positive eigenvalues of \( M' \) give normal mode frequencies of
\[ \omega = \frac{|q|}{\sqrt{2}} \sqrt{u_0^2 + u_S^2 + \sqrt{(u_0^2 - u_S^2)^2 + 16\bar{g}^2u_0u_S}}. \]
When \( u_0 = u_S \), we recover our earlier result, and for \( u_0 \neq u_S \), the exact expression reduces to the perturbative result up to \( \mathcal{O}(\bar{g}^2) \). The corresponding eigenvectors of \( M' \) are fairly complicated, but they may be found by elementary means.

This gives a complete description of the modes of the three interacting Luttinger liquids. Two set of modes—which are linear combinations of \( \rho^0 \) and \( \rho^S \)—describe linear harmonic fluids, with the two different wave speeds given by (19). The remaining modes are coupled through a sine-Gordon interaction \cite{15, 18}. If we fix the number of particles
having each value of $\ell$, then, under the approximations of the Luttinger model, the $\rho^\ell$ operators generate a complete set of states, and the modes we have found are the only low-energy excitations for the system. Because the interactions $H_{fs}$ and $H_{bs}$ are fairly general and physically quite reasonable, the spectrum we have derived should be highly relevant to the study of trapped fermions in situations where Luttinger liquid behavior is observable.

Acknowledgments

The author is grateful to K. Huang for many helpful discussions. This work is supported in part by funds provided by the U. S. Department of Energy (D.O.E.) under cooperative research agreement DE-FG02-91ER40661.

Appendix: Hamiltonian for “inelastic” scattering

In the body of this paper, we have considered only those scattering processes which preserve the individuals particles’ $\ell$ values. However, there exists another type of scattering interaction, which does not have this property. Interactions of the type $(k^\ell_F; -k^-\ell_F) \rightarrow (k^\ell_F; -k^-\ell^\prime_F)$ with $|\ell| \neq |\ell'|$ are also allowed by linear and angular momentum conservation. In this appendix, we shall examine the effects of such terms. We shall find that they lead to interactions that are similar to sine-Gordon interactions but more complicated.

We begin with an interaction Hamiltonian that describes the scattering process in question:

$$H_{in} = \frac{1}{L} \sum_{k,p,q} \sum_{\ell=\pm 1} g_{ln} a^\ell_k b^{-\ell}_{-k+q} a^0_p b^0_{-p+q} + \text{h.c.},$$

where “h.c.” denotes the Hermitian conjugate. We may simplify this by Fourier transforming the $a$ and $b$ operators and expanding the resultant fermion operators in terms of the boson field. For example, $a^+_k$ may be written as

$$a^+_k = \frac{1}{\sqrt{2\pi \alpha^1 L}} U_{+1}^1 \int_0^L dx e^{-ikx} \psi^+_1(x)$$

$$= \frac{1}{\sqrt{2\pi \alpha^1 L}} U_{+1}^1 \int_0^L dx e^{-ikx} e^{ik\frac{1}{2}x} e^{-i\phi_{+1}(x)+i\theta_{+1}(x)}.$$

The operator $U_{+1}^1$ carries the fermion number of $a^+_k$ but does not affect the bosonic state of the system, while $\phi_{+1}$ is the boson field corresponding to the $\ell = +1$ fermion states, and $\theta_{+1}$ is the integral of the momentum conjugate to $\phi_{+1}$. In general, the operators $\phi^\ell$ and $\theta^\ell$ have the forms

$$-i\phi^\ell = -\frac{\pi}{L} \sum_{p \neq 0} \frac{1}{p} e^{-\alpha |p|/2} e^{-ipx} [\rho^+_p(p) + \rho^-_p(p)] + i\pi \frac{N^\ell + N^-}{L} x$$

($22$)
\[ i\theta^\ell = -\frac{\pi}{L} \sum_{\ell \neq 0} \frac{1}{p} e^{-\alpha |p|/2} e^{-ipx} [\rho^\ell_+(p) - \rho^\ell_-(p)] + i\pi \frac{N^\ell_+ - N^\ell_-}{L} x, \tag{23} \]

where \( N^\ell_+ \) and \( N^\ell_- \) are the numbers of excess right- and left-moving fermions of angular momentum \( \ell \), respectively.

If we express all the operators in \( H_{\text{in}} \) in this (bosonized) fashion, we get

\[
H_{\text{in}} = \frac{g_{\text{in}}}{(2\pi)^2 \alpha^0 \alpha^1 L^3} \sum_{k, p, q} \sum_{\ell = \pm 1} \int_0^L dx_1 dx_2 dx_3 dx_4 e^{ik(x_2 - x_1)} e^{-ik^\ell_0(x_2 - x_1)} e^{iq(x_4 - x_2)} e^{ip(x_3 - x_4)}
\]

\[
e^{-ik^\ell_0(x_3 - x_4)} e^{-i\phi^\ell(x_1) + i\theta^\ell(x_1)} e^{i\phi^{-\ell}(x_2) + i\theta^{-\ell}(x_2)} e^{i\theta^0(x_3) - i\theta^0(x_3)} e^{-i\theta^0(x_4) - i\theta^0(x_4)} + \text{h.c.} \tag{24} \]

The sums over \( k, p, \) and \( q \) give \( \delta \)-functions according to \( \sum_k e^{ikx} = L\delta(x) \), and these simplify the expression substantially. We find that

\[
H_{\text{in}} = \frac{g_{\text{in}}}{(2\pi)^2 \alpha^0 \alpha^1} \sum_{\ell = \pm 1} \int_0^L dx \, e^{-i\phi^\ell(x) + i\theta^\ell(x)} e^{i\phi^{-\ell}(x) + i\theta^{-\ell}(x)} e^{-2i\theta^0(x)} + \text{h.c.} \tag{25} \]

Since \( \phi^\ell \) and \( \theta^\ell \) are Hermitian, this is equivalent to

\[
H_{\text{in}} = \frac{2g_{\text{in}}}{(2\pi)^2 \alpha^0 \alpha^1} \sum_{\ell = \pm 1} \int_0^L dx \cos \left[ \phi^\ell(x) - \phi^{-\ell}(x) - \theta^{-\ell}(x) + 2\theta^0(x) \right] \cdot \cos \left[ \theta^+ - \theta^{-1}(x) - 2\theta^0(x) \right]. \tag{26} \]

This a new nonlinear interaction. It is similar to a sine-Gordon interaction, but it contains substantial additional complexity. The effects of this interaction would presumably need to be determined perturbatively.

References

[1] H. J. Schulz, cond-mat/9503150

[2] J. Sólyom, Adv. Phys. 28, 209 (1979).

[3] V. J. Emery, in Highly Conducting One-Dimensional Solids, edited by J. T. Devreese, R. P. Evrard, V. E. van Doren (Plenum, New York, 1979), p.258.

[4] X. G. Wen, Phys. Rev. B 41 12838 (1990).

[5] X. G. Wen, Phys. Rev. B 43 11025 (1991).

[6] B. DeMarco, D. S. Kim, Science 285, 1703.
[7] F. Schreck, L. Khaykovich, K. L. Corwin, G. Ferrari, T. Bourdel, J. Cubizolles, C. Salomon, Phys. Rev. Lett. 87, 080403 (2001).

[8] S. R. Granade, M. Gehm, K. M. O’Hara, J. E. Thomas, Phys. Rev. Lett. 88, 120405 (2002).

[9] Z. Hadzibabic, C. A. Stan, K. Dieckmann, S. Gupta, M. W. Zwierlein, A. Görlitz, W. Ketterle, Phys Rev. Lett. 88, 160401 (2002).

[10] B. Altschul, cond-mat/0309629.

[11] J. M. Luttinger, J. Math. Phys. 4, 1154 (1963).

[12] D. C. Mattis, E. H. Lieb, J. Math. Phys 6, 304 (1965).

[13] S. Tomonaga, Prog. Theor. Phys. 5, 544 (1950).

[14] S.-T. Chui, P. A. Lee, Phys Rev. Lett. 35, 315 (1975).

[15] S. Coleman, Phys. Rev. D. 11, 2088 (1975).

[16] N. N. Bogoliubov, J. Phys. USSR 11, 23 (1947).

[17] A. Luther, V. J. Emery, Phys. Rev. Lett. 33, 589 (1974).

[18] R. Rajaraman, Solitons and Instantons (Elsevier, Amsterdam, 1982), p. 16–23, 291–298.

[19] D. C. Mattis, J. Math. Phys. 15, 609 (1974).

[20] A. Luther, I. Peschel, Phys. Rev. B 9, 2911 (1974).

[21] R. Heidenreich, B. Schoer, R. Seiler, D. Uhlenbrock, Phys. Lett. A 54, 119 (1975).

[22] F. D. M. Haldane, J. Phys. C 12, 4791 (1979).

[23] F. D. M. Haldane, J. Phys. C 14, 2585 (1981).