On a low energy bound in a class of chiral field theories with solitons

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Abstract

A low energy bound in a class of chiral solitonic field theories related the infrared physics of the SU(N) Yang-Mills theory is established.

1. The model. Consider $N-1$ smooth fields $n_a = n_a(x)$ in spacetime taking their values in the Lie algebra of $SU(N)$. The fields are chosen to be commutative $[n_a, n_b] = 0$ and orthonormal $(n_a, n_b) = \delta_{ab}$ with respect the Cartan-Killing form in the Lie algebra. For any two Lie algebra elements $\xi$ and $\eta$, the Cartan-Killing form is defined as $(\xi, \eta) = \text{tr}(\hat{\xi}\hat{\eta})$ where the operator $\hat{\xi}$ acts on the Lie algebra as a Lie derivative $\hat{\xi}\eta = [\xi, \eta]$. There can only be $N-1$ mutually commuting and linearly independent elements in the Lie algebra of $SU(N)$ because the rank of $SU(N)$ is $r = N-1$ (the dimension of the Cartan subalgebra). If $h_a$ form an orthonormal basis in the Cartan subalgebra in a matrix representation of $SU(N)$, then $n_a(x) = U^\dagger(x)h_aU(x)$, (1)

where $U(x) \in SU(N)$. In Eq. (1) $U(x)$ is defined modulo the left multiplication by elements from the Cartan subgroup generated by $h_a$ (the maximal Abelian subgroup $T = U(1)^{N-1}$). So, in fact, $U(x) \in SU(N)/T$ since any group element can be represented as a product of an element of $T$ and an element of the quotient $SU(N)/T$. Under the condition that $n_a$ approach fixed constant values at the spatial infinity, $n_a(x) \to h_a$, i.e., $U(x)$ approaches the group unity, the fields $n_a$ define a map of a spatial three-sphere $S^3$ into the manifold $SU(N)/T$ for every moment of time. The third homotopy group of this map is nontrivial $\pi_3(SU(N)/T) \sim \mathbb{Z}$, $G = SU(N)$. When $N = 2$, the only field $n_1$ can be regarded as a unit 3-vector. It is a Hopf map: $S^3 \to S^2 \sim SU(2)/U(1)$. The corresponding topological number is the Hopf invariant which can also be interpreted as a linking number of two curves in $S^3$ being preimages of two distinct points of $S^2$. The two-forms $F^a = F^a_{jk}dx^j \wedge dx^k$, $j, k = 1, 2, 3$, where

$$F^a_{jk} = iN\sum_b(n_a, [\partial_j n_b, \partial_k n_b]),$$

(2)

are closed, that is, $F^a_{jk} = \partial_j C^a_k - \partial_k C^a_j$. This is proved at the end of next section. The forms $F^a$ may not be exact. This follows from the fact that the cohomology ring $H^*(G/T)$ is rationally generated by $H^2(G/T)$. The topological number of the map $S^3 \to G/T$ should be constructed out of the 2-forms $F^a = iN(n_a, \sum_b[dn_b, dn_b])$ on $G/T$. Introducing the field
$B^a_i = \frac{1}{2} \epsilon_{ijk} F^a_{jk}$ with $\epsilon_{ijk}$ being the Levi-Chebita tensor, the topological number of the above map can be written as

$$Q = (16\pi^2 N)^{-1} \int dx \sum_i a C^a_i B^a_i .$$  

(3)

For SU(2), $Q$ is a Hopf invariant. Since $SU(2)/U(1) \subset G/T$, the normalization coefficient in (3) can be chosen so that $Q$ is an integer when $n_a$ realize a Hopf map.

An explicit realization of the Hopf map by the fields $n_a$ is as follows. Consider the Cartan-Weyl basis in the Lie algebra. Let $\alpha$ be a positive root. For every positive root $\alpha$, there are two basis elements $e_{\alpha}$ and $e_{-\alpha} = \bar{e}_{\alpha}$ such that for any element $h$ from the Cartan subalgebra

$$[h, e_{\alpha}] = (h, \alpha) e_{\alpha} ,$$

(4)

$$[e_{\alpha}, e_{-\alpha}] = \alpha , \quad [e_{\alpha}, e_{\beta}] = N_{\alpha, \beta} e_{\alpha+\beta} ,$$

(5)

where $N_{\alpha, \beta} \neq 0$ if $\alpha + \beta$ is a root. Note that the elements $e_{\alpha}$ and $e_{-\alpha}$ form a basis of an SU(2) subalgebra (associated with the root $\alpha$). Let $U_\omega (x) \in SU(2)/U(1) \subset SU(N)/T$ where the subgroup SU(2) is associated with a simple root $\omega$. One can always choose $h_1 = N^{1/2} \omega$. The norm of any root of SU(N) is $1/N$ with respect to the Cartan-Killing form (see next section). Then $n_1(x) = N^{1/2}U_\omega U_\omega / U_\omega$ is a Hopf map. The other fields realize a trivial map, $n_a = h_a, a > 1$. Indeed, $U_\omega (x) = \exp [iu_\omega (x)]$ where $u_\omega (x) = \varphi_\omega (x) e_\omega + \bar{\varphi}_\omega (x) e_{-\omega}$. For $a > 1$, it follows from (4) that

$$n_a = U_\omega U_\omega h_a = h_a$$

because $(\omega, h_a) \sim (h_1, h_1) = 0$. Now, if we introduce an orthonormal basis in the SU(2) subgroup, $\tau_1 = i(e_\omega - e_{-\omega})/\sqrt{2}$, $\tau_2 = (e_\omega + e_{-\omega})/\sqrt{2}$ and $\tau_3 = \sqrt{N} \omega$, then

$$[\tau_j, \tau_k] = iN^{-1/2} \epsilon_{jkn}\tau_n .$$

Let $n$ be an isotopic unit three-vector whose components are $(\tau_j, n_1)$. It defines the Hopf map by construction. From (2) we infer that $F^a_{jk} = \delta^{a1}\sqrt{N} n \cdot (\partial_j n \times \partial_k n)$. Hence our $B^a_j$ and $C^a_j$ contain an extra factor $\sqrt{N}$ when the fields $n_a$ realize a Hopf map associated with an SU(2) subgroup of SU(N). This explains the normalization factor $N^{-1}$ in (3). Since all the root have the same norm in SU(N), the normalization coefficient in (3) for any SU(2) subgroup has to be the same. The root system is invariant under the Weyl symmetry, and so should be $Q$. The sum over $a$ in (3) provides this invariance.

The dynamics of the fields $n_a$ is determined by the Lagrangian density

$$\mathcal{L} = m^2 \sum_{\mu, \nu} \left( \partial_\mu n_a, \partial_\mu n_a \right) - \frac{g}{4} \sum_{\mu, \nu} F^a_{\mu \nu} F^a_{\mu \nu} ,$$

(6)

$\mu, \nu = 0, 1, 2, 3$; and $\partial_0$ stands for the time derivative. In the case of SU(2), this Lagrangian density describes the Faddeev model [2] for knot solitons. The knot solitons have been extensively studied numerically [3]. The model [3] has been introduced in [4] and may also have solitonic solutions. The Lagrangian [3] is believed to describe (in a certain approximation) the infrared physics of the SU(N) Yang-Mills theory [4, 5, 6]. Recent analytical [7] and lattice [8, 9] studies of this correspondence in the SU(2) case look promising.

Due to the Lorentz symmetry of the Lagrangian density, a Lorentz transformation of a static solution is a time dependent solution of the Euler-Lagrange equations for (3). Solutions that describe interacting solitons are not static (even modulo Lorentz transformations). In this paper a low energy bound for static solitons with a topological number $Q$ is established:

$$E \geq c_N |Q|^{3/4} ,$$

(7)

$$c_N = 8\pi^2 3^{3/8} \left( \frac{2N^3}{N^2 - N - 1} \right)^{1/4} \sqrt{m^2 g} ,$$

(8)
where \( E \) is the energy functional

\[
E = m^2 \int dx \sum_{j,a} (\partial_j n_a, \partial_j n_a) + \frac{q}{2} \int dx \sum_{j,a} B^a_j B^a_j \tag{9}
\]

\[
\equiv \int dx (E_2(x) + E_4(x)) \equiv E_2 + E_4. \tag{10}
\]

For the Faddeev-Hopf knot solitons the low energy bound was found in [10] and improved in [11, 12] (meaning a larger constant \( c_2 \)). Beyond conventional perturbation theory, the Yang-Mills quantum dynamics can be studied by the large \( N \) expansion method with the purpose to establish a relation (duality) to a string theory on some manifold. Therefore it is of interest to investigate the \( N \) dependence of the low energy bound for solitons in the model (6).

2. Notations and necessary facts. We would need the following algebraic inequalities. Let \( a_i, b_i \geq 0, a = \sum_i a_i, b = \sum_i b_i \) and \( \gamma \geq 1 \). Then

\[
a_1^\gamma + a_2^\gamma + \cdots + a_r^\gamma \leq a^{\gamma}; \tag{11}
\]

\[
\sqrt{a} \leq \sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_r} \leq \sqrt{r} \sqrt{a}; \tag{12}
\]

\[
\sum_i a_i^p b_i^q \leq a^p b^q, \quad p + q = 1. \tag{13}
\]

Define \( p_i = a_i/a \leq 1 \). Then \( \sum_i p_i = 1 \). The inequality (11) follows from an obvious inequality \( p_i^\gamma \leq p_i \) if one takes the sum over \( i \). The second inequality is proved by squaring it and applying the basic algebraic inequality \( \sqrt{a_i} \sqrt{a_j} \leq \frac{1}{2}(a_i + a_j) \). The third inequality is an algebraic Hölder inequality (see, e.g., [13]).

An arrow is used to denote vectors in space, e.g., \( \vec{\partial} \phi = (\partial_1 \phi, \partial_2 \phi, \partial_3 \phi) \) for the gradient. The scalar product for two vector fields is

\[
\langle \vec{u}, \vec{v} \rangle = \int dx \vec{u} \cdot \vec{v}. \tag{14}
\]

The \( L_p \) norm of a vector field reads

\[
\| \vec{u} \|_p = \left( \int dx (\vec{u} \cdot \vec{u})^{p/2} \right)^{1/p}. \tag{15}
\]

The following functional inequalities are used in the sequel

\[
|\langle \vec{u}, \vec{v} \rangle| \leq \| \vec{u} \|_p \| \vec{v} \|_q, \quad p^{-1} + q^{-1} = 1, \tag{16}
\]

\[
\| \vec{u} \|_{6/5} \leq \| \vec{u} \|_{1}^{2/3} \| \vec{u} \|_2^{1/3}, \tag{17}
\]

\[
\| \vec{u} \|_{6} \leq \lambda_1 \| \text{curl} \vec{u} \|_2, \quad \lambda_1 = (48)^{1/6}(3\pi)^{-2/3}. \tag{18}
\]

The first two inequalities are Hölder type inequalities [13]. The third one follows from Rosen’s result for scalar functions [14] (cf. also [11])

\[
\| \phi \|_6 \leq \lambda_1 \| \vec{\partial} \phi \|_2, \tag{19}
\]

where the \( L_p \) norm for scalar functions is defined by (13) for one-dimensional vectors. Let \( \phi = (\vec{u} \cdot \vec{u})^{1/2} \). We have

\[
\vec{\partial} \phi \cdot \vec{\partial} \phi = \phi^{-2} \sum_j (\partial_j \vec{u} \cdot \vec{u})^2 \leq \sum_j \partial_j \vec{u} \cdot \partial_j \vec{u}. \]
Making use of this inequality, we infer (18) from (19):

\[ \|u\|_6 = \|\phi\|_6 \leq \lambda_1 \|\partial_\phi\|_2 \leq \lambda_1 \left[ \int dx \sum_{i,j} (\partial_j u_i)^2 \right]^{1/2} = \lambda_1 \|\nabla u\|_2 . \] (20)

The last equality in (20) is true if \( \text{div} u = 0 \) and \( u \) decreases sufficiently fast at spatial infinity, which we require in (18). The coefficient \( \lambda_1 \) is the least possible coefficient in inequality (18) [14].

Let \( \omega_a, a = 1, 2, \ldots, r, \) be simple roots of SU(N). They have the same norm \( (\omega_a, \omega_a) \equiv \gamma^2 \). The angle between \( \omega_a \) and \( \omega_{a+1} \) is \( 2\pi/3 \), and otherwise the roots are perpendicular. Any positive root can be written in the form \( \alpha = \omega_a + \omega_{a+1} + \cdots + \omega_{a+q} \) for \( a + q \leq r \). From this it is easy to deduce that all roots have the same norm with respect to the Cartan-Killing form, \( (\alpha, \alpha) = \gamma^2 \).

To find the actual norm \( \gamma \), one should compute, say, the matrix \( \omega_1 \) in the Cartan-Weyl basis and take the trace of its square. From (4) it follows that \( \omega_1 \) is block diagonal. The block associated with the Cartan subalgebra is zero because \( \omega_a \) commute amongst each other. The nontrivial blocks come from the subspaces spanned by \( e_\alpha \) and \( e_{-\alpha} \) where the positive root \( \alpha \) is either equal to \( \omega_1 \) or contains \( \omega_2 \) or \( \omega_1 + \omega_2 \). There are \( r - 1 \) roots containing \( \omega_2 \) and \( r - 1 \) roots containing \( \omega_1 + \omega_2 \). Then \( \gamma^2 = \text{tr} (\omega_1^2) = \gamma^4 N \) as is deduced from (4). Hence

\[ (\omega_a, \omega_a) = N^{-1} . \] (21)

As a consequence of (21), the following identity holds for any Lie algebra element \( v \)

\[ v = \sum_a n_a (n_a, v) + N \sum_a [n_a, [n_a, v]] \equiv v_\parallel + v_\perp . \] (22)

The proof is based on the following observation. Relation (22) is covariant under the adjoint action of SU(N). So, according to (4), \( n_a \) can be replaced by \( h_a \) after a corresponding adjoint rotation of \( v \). Decomposing \( v \) in the Cartan-Weyl basis, one can see that the first term in (22) is the Cartan component of \( v \). The double commutator in the second term can be computed by means of (4) and gives rise to the factor \( \sum_a (\alpha, h_a)^2 = (\alpha, \alpha) = N^{-1} \) for every basis element \( e_\alpha \). Thus the second term in (22) is nothing but a projector onto the subspace orthogonal to the Cartan subalgebra spanned by \( n_a \).

By differentiating (4) one finds

\[ \partial_\mu n_a + i[A_\mu, n_a] = 0 , \] (23)

\[ i\partial_\mu U^\dagger U \equiv A_\mu - \sum_a n_a C_a^\mu , \quad (n_a, A_\mu) = 0 . \] (24)

Equation (23) can be interpreted as: The fields \( n_a \) are transported parallel with respect to the connection \( A_\mu \). Taking a commutator of (23) with \( n_a \), summing over \( a \) and making use of the identity (23), the connection \( A_\mu \) can be explicitly written via \( n_a \),

\[ A_\mu = iN \sum_a \partial_\mu n_a, n_a \] . (25)

The connection (23) has been introduced by Cho to study monopoles in the Yang-Mills theory for SU(2) and SU(3) [17]. By multiplying (23) by \( n_b \) using the Cartan-Killing form, one deduces that the derivatives of \( n_a \) are orthogonal to the fields themselves

\[ (\partial_\mu n_a, n_b) = 0 . \] (26)
Now we show that the tensor (2) is an Abelian gauge field tensor (cf. [1, 3]), that is, the two-forms $F^a$ are closed, $dF^a = 0$. Consider the following algebraic transformations

$$
\partial_\mu A_\nu - \partial_\nu A_\mu = 2iN\sum_a[\partial_\mu n_a, \partial_\nu n_a] = 2iN\sum_a[[A_\mu, n_a], [A_\nu, n_a]] = 2iN\sum_a[[A_\nu, [n_a, A_\mu]], [n_a, [A_\mu, A_\nu]]] = -i[A_\mu, A_\nu] - i\sum_a n_a (n_a, [A_\mu, A_\nu]).
$$

In (27) we have used (25); next, $\partial_\mu n_a$ has been transformed via (23); Eq. (28) follows from the Jacobi identity; to derive (29), the first term in (28) has been transformed by means of the algebraic identity (22), while the second one via the Jacobi identity; finally, by applying the algebraic identity (22) to (29), Eq. (30) has been deduced. Introducing the Yang-Mills field strength tensor

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu],
$$

it follows from (30) that

$$
F_{\mu\nu} = i\sum_a n_a (n_a, [A_\mu, A_\nu]) = \sum_a n_a F_{\mu\nu}^a.
$$

The last equality in (32) is deduced by multiplying (30) and the middle of (27) by $n_a$ using the Cartan-Killing form. Now observe that the field strength (curvature) of the pure gauge connection (24) is zero. Making use of the decomposition (24) of a pure gauge connection we obtain

$$
0 = F_{\mu\nu} - \sum_a n_a \left( \partial_\mu C^a_\nu - \partial_\nu C^a_\mu \right),
$$

where the identity (23) has been used again for algebraic transformations. Thus, $F_{\mu\nu}^a = \partial_\mu C^a_\nu - \partial_\nu C^a_\mu$. Note that (24) allows one to determine $C^a_\mu$ via the group element $U$ explicitly. In (3) the vector potential $C_i^a$ can always be chosen to satisfy the Coulomb gauge, $\partial_i C_i^a = 0$, thanks to the gauge freedom $C^a_\mu \rightarrow C^a_\mu + \partial_\mu \xi^a$.

3. A key algebraic inequality. In this section the following inequality is proved

$$
\mathcal{E}_4 \leq \kappa_N \mathcal{E}_2^2, \quad \kappa_N = \frac{gN}{4m^4} \left( 1 - \frac{1}{N^2 - N} \right).
$$

It is used in the next section to establish the low energy bound. Consider the $(N^2 - 1) \times (N^2 - 1)$ matrix

$$
G = \sum_{i,b} \partial_i n_b \otimes \partial_i n_b.
$$

It can be regarded as a linear operator on the Lie algebra, i.e., $G\eta = \sum_{i,b} \partial_i n_b (\partial_i n_b, \eta)$ for any Lie algebra element $\eta$. It has $N - 1$ zero eigenvalues because $Gn_a = 0$. The matrix $G$ satisfies

$$
\text{tr} G^2 \geq \frac{1}{N^2 - N} \left( \text{tr} G \right)^2.
$$

The proof is simple. Let $g_k$, $k = 1, 2, ..., n = N^2 - N$, be nonzero eigenvalues of $G$. They are real since $G = G^T$ with respect to the Cartan-Killing form. Consider a function of one real variable $\xi$, $f(\xi) = \sum_k (g_k - \xi)^2$. Computing the sum explicitly, one finds that $f(\xi) = \text{tr} G^2 - 2\xi \text{tr} G + \xi^2 n$. 


The function attains its absolute minimum at \( \xi = \xi_0 = \text{tr} \ G/n \). Since \( f(\xi) \geq 0 \) for all \( \xi \)'s, the inequality (33) follows from \( f(\xi_0) \geq 0 \).

Consider a local Cartan-Weyl basis which is obtained by an adjoint transformation of the basis (3), (5) with the group element \( U(x) \) defined in (1). Denoting \( n_{\alpha} = U^\dagger e_{\alpha} U \) and \( n_{-\alpha} = U^\dagger e_{-\alpha} U = \bar{n}_{\alpha} \) we have

\[
[n_{\alpha}, n_{\beta}] = (h_{\alpha}, \beta)n_{\beta}, \quad [n_{\alpha}, n_{-\alpha}] = U^\dagger \alpha U \equiv \alpha_U, \tag{36}
\]

where \((\alpha_U, n_{\alpha}) = (\alpha, h_{\alpha})\) and \((n_{\alpha}, n_{\beta}) = 0, (n_{\alpha}, n_{-\alpha}) = 1, (n_{\alpha}, \alpha) = (n_{-\alpha}, n_{-\alpha}) = 0\).

To establish a relation between \( E_4 \), \( E_2 \) and \( G \), we decompose the connection (25) in the local Cartan-Weyl basis

\[
A_i = \sum_{\alpha > 0} (A_i^\alpha n_{\alpha} + \text{c.c.}) \equiv H_i + \bar{H}_i. \tag{37}
\]

Then we obtain

\[
E_2 = m^2 \text{tr } G = -m^2 \sum_{i,b} ([A_i, n_b], [A_i, n_b]) = \frac{m^2}{N} \sum_{i} (A_i, A_i) = \frac{2m^2}{N} \sum_{i} (\bar{H}_i, H_i). \tag{38}
\]

The second equality follows from (23); the third one is a consequence of (22) and (24). Making use of (32) we also get

\[
E_4 = -\frac{g}{4} \sum_{a,i,j} (n_a, [A_i, A_j])^2 \tag{39}
\]

\[
= -\frac{g}{4} \sum_{\alpha,\beta > 0} \sum_{i,j} (A_i^\alpha \bar{A}_j^\beta - \text{c.c.}) (\alpha, \beta) (A_i^\beta \bar{A}_j^\beta - \text{c.c.})
\leq -\frac{g}{4N} \sum_{i,j} [(H_i, H_j) + \text{c.c.}]^2. \tag{40}
\]

Note that the local Cartan component of \([A_i, A_j]\) can only come from the second commutation relation in (36). Hence the sum over \( a \) in (39) yields the factor \( \sum_{a}(\alpha, h_a)(h_{a}, \beta) = (\alpha, \beta) = N^{-1}\cos \theta_{\alpha\beta} \leq N^{-1} \) for any two positive roots \( \alpha \) and \( \beta \) with the angle \( \theta_{\alpha\beta} \) between them. In a similar fashion we derive

\[
\text{tr } G^2 = \sum_{a,b} \sum_{i,j} ([A_i, n_a], [A_j, n_b])^2 = \sum_{a,b} \sum_{i,j} (n_a, [n_b, A_i], A_j)]^2 \tag{41}
\]

\[
= \sum_{\alpha,\beta > 0} \sum_{i,j} (A_i^\alpha \bar{A}_j^\beta + \text{c.c.}) (\alpha, \beta)^2 (A_i^\beta \bar{A}_j^\beta + \text{c.c.})
\leq \frac{1}{N^2} \sum_{i,j} [(H_i, H_j) + \text{c.c.}]^2. \tag{42}
\]

Combining (12) and (40), we infer

\[
4Ng^{-1}E_4 + N^2 \text{tr } G^2 \leq 4 \sum_{i,j} (H_i, \bar{H}_j)(H_j, \bar{H}_i) \leq 4 \left( \sum_{i}(H_i, \bar{H}_i) \right)^2 = 4 (\text{tr } G)^2, \tag{43}
\]

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where the Schwartz inequality
\[
\left| (H_i, H_j) \right|^2 \leq (H_i, H_i)(H_j, H_j)
\]
has been used. The inequality (34) immediately follows from (43), (38) and (35).

4. The low energy bound. Let \( \lambda = (16\pi^2 N)^{-1} \). Then Eq. (3) can be written as \( Q = \lambda_0 \sum_a \langle \tilde{C}^a, \tilde{B}^a \rangle \). Making use of (10) one gets (cf. the case \( N = 2 \) in (10))
\[
|Q| \leq \lambda_0 \sum_a \| \tilde{C}^a \|_p \| \tilde{B}^a \|_{p'}
\]
(44)
\[
\leq \lambda_0 \lambda_1 \sum_a \| \text{curl} \tilde{C}^a \|_2 \| \tilde{B}^a \|_{6/5}
\]
(45)
\[
= \lambda_0 \lambda_1 \sum_a \| \tilde{B}^a \|_2 \| \tilde{B}^a \|_{6/5}
\]
(46)
\[
\leq \lambda_0 \lambda_1 \sum_a \| \tilde{B}^a \|_2 \| \tilde{B}^a \|^{1/3} \| \tilde{B}^a \|^{2/3}_1 .
\]
To get (45), Eq. (18) has been used, which dictated the choice \( p = 6 \) in (44), and also \( \partial_i C_i^a = 0 \); then the substitution \( \tilde{B}^a = \text{curl} \tilde{C}^a \) has been made; the last inequality (46) is a consequence of (17). The energy can be written as
\[
E = m^2 \sum_a \| \partial_a n_a \|_2^2 + \frac{g}{2} \sum_a \| \tilde{B}^a \|_2^2 = \sum_a (E_{2a} + E_{4a}) = E_2 + E_4 .
\]
(47)
Hence, continuing (46) we get
\[
|Q| \leq \lambda_0 \lambda_1 (2/g)^{2/3} \sum_a \left( E_{4a}^{4/3} \right)^{1/2} \left( \| \tilde{B}^a \|_1^{4/3} \right)^{1/2}
\]
(48)
\[
\leq \lambda_0 \lambda_1 (2/g)^{2/3} \left[ \left( \sum_a E_{4a} \right)^{4/3} \right]^{1/2} \left[ \left( \sum_b \| \tilde{B}^b \|_1 \right)^{4/3} \right]^{1/2}
\]
(49)
\[
= \lambda_0 \lambda_1 (2/g)^{2/3} \frac{1}{E_4^{2/3}} \left[ \sum_b \int dx \sqrt{\tilde{B}_b \cdot \tilde{B}_b} \right]^{2/3}
\]
\[
\leq \lambda_0 \lambda_1 g^{-1} (2E_4)^{2/3} (N - 1)^{1/3} \left[ \int dx \sqrt{2E_4} \right]^{2/3}
\]
(50)
\[
\leq \lambda_0 \lambda_1 g^{-1} (2E_4 E_2)^{2/3} [2(N - 1)\kappa N]^{1/3}
\]
(51)
\[
\leq \lambda_0 \lambda_1 g^{-1} 2^{-2/3} [2(N - 1)\kappa N]^{1/3} E_4^{4/3}
\]
(52)
\[
= c_N^{-4/3} E_4^{4/3} ,
\]
where the constant \( c_N \) is given in (8). To get (49), the Hölder inequality (13) for \( p = q = 1/2 \) and then (11) for \( \gamma = 4/3 \) have been applied; (54) follows from (12); the algebraic inequality (34) has been used to deduce (51); the final result (52), which is equivalent to (7), comes from the basic inequality \( E_1 E_2 \leq E_2^2 / 4 \).

If the Lagrangian (3) defines an effective theory of the SU(N) gauge fields in some approximation, then the coefficients \( m \) and \( g \) should depend on \( N \), the Yang-Mills coupling constant and a mass scale that determines the energy range in which the approximation is valid.
The result (7) is trivially generalized to the case when the mass scales and coupling constants are different for each mode \( n_a \), that is, \( m^2 \) and \( g \) are replaced by \( m^2_a \) and \( g_a \), respectively, and inserted into the corresponding sums over \( a \) in (3) (cf. [4]). In this case, \( m^2 = \max_a \{ m^2_a \} \) and \( g = \max_a \{ g_a \} \) in (8). Indeed, all the inequalities in sections 3 and 4 still hold as a consequence of \( m^2_a \leq m^2 \) and \( g_a \leq g \). Equality (38) becomes an inequality \( m^2 \text{tr} G \leq E^2 \), which, however, does not affect our derivation of (34) from (43).

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