Thermal Bosonic and Fermionic Quantum Fields in Static Background Gauge Potentials

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Abstract

We study at finite temperature the energy-momentum tensor $T_{\mu\nu}(x)$ of (i) a scalar field in arbitrary dimension, and (ii) a spinor field in 1+1 dimensions, interacting with a static background electromagnetic field. $T_{\mu\nu}$ separates into an UV divergent part $T_{\mu\nu}^{\text{sea}}$ representing the virtual sea, and an UV finite part $T_{\mu\nu}^{\text{plasma}}$ describing the thermal plasma of the matter field. $T_{\mu\nu}^{\text{sea}}$ remains uniform in the presence of a uniform

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electric field $\vec{E}$, while $T_{\mu\nu}^{\text{plasma}}$ becomes a periodic function with period $\Delta x = 2\pi T/E$ in the direction parallel to $\vec{E}$. A related periodicity is found for a uniform static magnetic field if one spatial direction perpendicular to the magnetic field is compactified to a circle.
1 Introduction

A finite-temperature ($T > 0$) or thermal quantum field can be visualized as a sea of virtual particles occupying space together with a thermal gas of real field excitations. The virtual particle sea is independent of the temperature $T$. It can, however, be deformed by coupling the field to static background structures of various kinds. This is generally known as the static vacuum Casimir effect. Somewhat less widely known is that, for boundaries and other static backgrounds, the thermal gas is ”mechanically” distorted along with the sea.

Abelian (and non-Abelian) gauge theories present Casimir problems with particular features arising from the underlying gauge invariance. Indeed, the restrictions on the class of allowed gauge transformations imposed by $T > 0$ are found to have remarkable consequences for the spatial energy distribution of a charged thermal matter field coupled to a static background electromagnetic field ($\vec{E}, \vec{B}$). The local distortion by ($\vec{E}, \vec{B}$) of the virtual sea and thermal plasma (as we now refer to the thermal gas consisting of both particles and antiparticles) can be revealed by a local analysis in terms of the thermal stress energy momentum tensor.

The problem of charged quantum fields coupled to a uniform electromagnetic background field is an old one, going back to famous papers of Euler and Heisenberg [1] and Schwinger [2]. There has been done a great deal of subsequent research, much of it reviewed in refs. [3]-[6]. However, global aspects of the problem have received most of the attention, while local aspects seem to have been neglected.

In the present paper we try to gain general insight into the local response of gauged thermal matter fields to a background electromagnetic field. Using the Matsubara formalism (see e.g. refs. [7, 8, 9]), we work on a hypercylinder of circumference $\beta = 1/T$ in the euclidean time direction, choosing space to
be flat and infinite. Our results reveal some surprising features: We find for both scalar and spinor fields that a uniform background electric field $\vec{E}$ causes the thermal plasma to become non-uniform – in fact periodic in its spatial distribution along the direction of the $\vec{E}$, with period $\Delta x = 2\pi T/|E|$. The sea nevertheless remains spatially uniform.

The organization of this paper is as follows: We begin by considering thermal scalar fields, for which the discussion can easily be carried out in arbitrary dimension. By coupling the scalar field to an arbitrary static gauge potential $A_\mu(x)$ we show that the characteristic effects arising from the minimal coupling are common to all dimensions. Special attention is drawn to periodicity features related to gauge invariance and to the topology $S^1 \times R^d$ of space-time. We then compute for the specific gauge potential $A_\mu = (Ex_1 + \text{const}, \vec{0})$ the thermal stress tensor $T^\beta_{\mu\nu}(x)$. The periodicity of the thermal plasma along $x_1$ with period $\Delta x_1 = 2\pi T/|E|$ is thereby made explicit.

In section 3 we turn to the Schwinger model at $T > 0$ with the linear background potential referred to above. The discussion is now more involved, but the calculations reveal features similar to those encountered in the scalar case. We conclude in section 4 by commenting on a claim in the literature concerning the factorization of the thermal heat kernel, and drawing an analogy between our findings and the Quantum Hall Effect.

2 Thermal scalar field

Scalar electrodynamics is useful as a theoretical laboratory for studying gauge theory phenomena in arbitrary space-time dimension. We first consider the general problem of a thermal scalar field $\hat{\phi}$ coupled to an arbitrary static i) background potential $V(\vec{x})$ and ii) gauge potential $A_\mu(\vec{x})$. We then specialize to the potential $A_\mu = (Ex_1 + \text{const}, \vec{0})$ for a uniform background electric field $\vec{E} = (E, 0, ..., 0)$ and compute explicitly the thermal stress tensor of $\hat{\phi}$. The
case of a uniform magnetic field is also discussed briefly.

2.1 Scalar field in a static background Schrödinger potential

To set the stage we briefly review the case of a scalar quantum field interacting with a static background potential $V(\vec{x})$ in $d$-dimensional free space $R^d$. We wish to study the thermodynamical properties and vacuum Casimir energy of this system. To this end it will be convenient to work in the imaginary time or Matsubara formalism. Euclidean space-time is then a hyper-cylinder $S^1 \times R^d$. Correspondingly we impose periodic boundary conditions in euclidean time on the scalar field $\phi(x_0, \vec{x})$:

$$\phi(x_0, \vec{x}) = \phi(x_0 + \beta, \vec{x})$$ (2.1)

where $\beta = 1/T$ and $T$ is the temperature. The spectral operator for the theory in question is $[-\partial_0^2 - \Delta + V(\vec{x})]$ with $\Delta$ the Laplacian in $d$ dimensions. The vacuum and thermodynamical properties of the system can be computed from the bilocal heat kernel

$$h^{(\beta)}(t; x, y) = \sum_k e^{-t \lambda_k^2} \phi_k(x) \phi_k^*(y),$$

where $\lambda_k^2$ and $\phi_k(x_0, \vec{x})$ are the eigenvalues and respective eigenfunctions of the spectral operator,

$$[-\partial_0^2 - \Delta + V(\vec{x})] \phi_k(x) = \lambda_k^2 \phi_k(x).$$

With $\phi(x)$ subject to the boundary condition (2.1) we have

$$\phi_k(x) \rightarrow \phi_{mn}(x) = \frac{1}{\sqrt{\beta}} e^{i \frac{2\pi m}{\beta} x_0} \phi_n(\vec{x}),$$

$$\lambda_k^2 \rightarrow \lambda_{mn}^2 = \left(\frac{2\pi m}{\beta}\right)^2 + \omega_n^2,$$ (2.2)

4Throughout this section we use the euclidean notation $x = \{x_\mu\} = (x_0, \vec{x})$. 

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where the spatial modes $\varphi_n(\vec{x})$ and associated spectrum $\{\omega^2_n\}$ are determined by the spatial mode equation

$$[-\Delta + V(\vec{x})] \varphi_n(\vec{x}) = \omega^2_n \varphi_n(\vec{x}) .$$

The Euclidean thermal Green function then has the spectral representation

$$\langle \hat{\phi}(x) \hat{\phi}(y) \rangle = \sum_{m,n} \frac{\phi_{mn}(x) \phi_{mn}(y)^*}{\left( (\frac{2\pi m}{\beta})^2 + \omega_n^2 \right)}$$

$$= \int_0^\infty dt \frac{1}{\beta} \sum_m e^{-t(\frac{2\pi m}{\beta})^2} e^{i\frac{2\pi m}{\beta}(x_0-y_0)} \sum_n e^{-t\omega_n^2} \varphi_n(\vec{x}) \varphi_n^*(\vec{y}).$$

We may perform the Matsubara sum by using the Jacobi identity

$$\sum_{m=-\infty}^{\infty} e^{-b(m-a)^2} = \sqrt{\frac{\pi}{b}} \sum_{l=-\infty}^{\infty} e^{-\frac{\pi l^2}{b}} e^{-i2\pi al}$$

with the result

$$\langle \hat{\phi}(x) \hat{\phi}(y) \rangle = \int_0^\infty dt h(t; x, y) \big|_{T=0}$$

$$\times \sum_{l=-\infty}^{\infty} e^{-\frac{\pi l^2}{4t}} e^{i\frac{l^2}{4}(x_0-y_0)}$$

$$\langle \hat{\phi}(x) \hat{\phi}(y) \rangle_{\text{sea}} + \langle \hat{\phi}(x) \hat{\phi}(y) \rangle_{\text{gas}} .$$

We denote operators by a "hat". 

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Now suppose that we have calculated from \( \langle \hat{\phi}(x)\hat{\phi}(y) \rangle_{sea} \) the \( T = 0 \) vacuum stress tensor

\[
T_{\mu\nu}^{sea} = \langle \hat{T}_{\mu\nu} \rangle = \sum_n T_n^{\mu\nu}
\]

(2.6)
as a (still to be renormalized) spatial mode sum. Repeating the calculation at finite temperature we obtain from eq. (2.4)

\[
T^{(\beta)\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle_{\beta} = \sum_n T_n^{\mu\nu} \frac{1 + e^{-\beta \omega_n}}{1 - e^{-\beta \omega_n}} = T_{\mu\nu}^{sea} + \sum_n T_n^{\mu\nu} \frac{2}{e^{\beta \omega_n} - 1} = T_{\mu\nu}^{sea} + T_{\mu\nu}^{gas}.
\]

(2.7)

At finite temperature the mode sum is thus modified by the familiar Bose-Einstein distribution, in agreement with one’s expectations. This completes our brief review of a usual Casimir problem at \( T > 0 \).

2.2 Scalar field in a static background gauge potential

Let us now consider a (massive or massless) scalar quantum field \( \phi(x) \) coupled to a static background Abelian gauge potential \( A_\mu(\vec{x}) \). We shall again be interested in studying the thermodynamical properties and Casimir energy of this system.

On a space-time hyper-cylinder \( S^1 \times \mathbb{R}^d \), both \( \phi(x_0,\vec{x}) \) and \( A_\mu(x_0,\vec{x}) \) are required to satisfy periodic boundary conditions in euclidean time:

\[
\phi(x_0,\vec{x}) = \phi(x_0 + \beta,\vec{x}) \ , \ A_\mu(x_0,\vec{x}) = A_\mu(x_0 + \beta,\vec{x})
\]

(2.8)

The relevant spectral operator in this case is the gauged Laplacian in \( d + 1 \) dimensions \(-D_\mu^2\), where \( D_\mu = \partial_\mu - i A_\mu \) couples the quantum scalar field to a static background gauge potential \( A_\mu(\vec{x}) \). (We have absorbed the electric charge into \( A_\mu \).) The thermodynamical and vacuum properties of the system
can again be computed from the bilocal heat kernel

\[ h^{(\beta)}(t; x, y) = \sum_k e^{-t\lambda_k^2} e^{-tM^2} \phi_k(x) \phi^*_k(y) \]

where \( \lambda_k^2 \) are the eigenvalues of \(-D^2\), \( M \) is the mass and \( \phi(x) \) is subject to the boundary condition (2.8). Periodicity in \( x_0 \) implies

\[ \phi_k(x) \to \phi_{mn} = \frac{1}{\sqrt{2\pi}} e^{i\frac{2\pi m}{\beta} x_0} \varphi_{mn}(\vec{x}) , \]

where \( \varphi_{mn}(\vec{x}) \) now satisfies the associated eigenvalue problem

\[ \left[ -(\vec{\nabla} + i\vec{A}(\vec{x}))^2 + V_m(\vec{x}) \right] \varphi_{mn}(\vec{x}) = \lambda_{mn}^2 \varphi_{mn} \quad (2.9) \]

with

\[ V_m(\vec{x}) = [A_0(\vec{x}) - \frac{2\pi m}{\beta}]^2 . \quad (2.10) \]

Notice that the \( m \)-dependence of the Schrödinger-like background potential \( V_m(\vec{x}) \) leads to a coupling of spatial position \( \vec{x} \) with the Matsubara frequencies. Eqs. (2.8) thus represent a different equation for each Matsubara frequency \( \frac{2\pi m}{\beta} \), with \( n \) labelling the complete set of normalizable solutions \( \{\varphi_{mn}\} \) of this Schrödinger problem for a given potential \( V_m(\vec{x}) \). The situation is thus very different from the scalar case discussed previously. It is characteristic of gauge theories and has very important consequences as we shall see.

On the cylinder \( S^1 \times R^d \) we can always gauge a static \( A_0(\vec{x}) \) to the interval \([0, \frac{2\pi}{\beta}]\), but not in general to zero, if we respect the periodicity property (2.8) of \( A_\mu \) and \( \phi \). The only exception is when \( A_0 = N\frac{2\pi}{\beta} \), in which case we may gauge \( A_0 \) to zero by performing the (allowed) gauge transformation \( A_0 \to A_0 + \partial_0 \lambda \) with \( \lambda = (\frac{2\pi N}{\beta}) x_0 \).

Another way of stating this is to observe that in the exceptional case \( A_0 = N\frac{2\pi}{\beta} \) the gauge transformation can be absorbed into the Matsubara index \( m \) via the transformation \( m \to m + N \). For this reason \( A_0(\vec{x}) \) is always gauge equivalent to a configuration taking values in the range \([0, \frac{2\pi}{\beta}]\). In
the zero-temperature limit, on the other hand, we may always gauge $A_0(\vec{x})$ to zero. Indeed, the discrete Matsubara frequencies $k_0 = \frac{2\pi m}{\beta}$ become a continuous variable $k_0$ in the range $-\infty < k_0 < \infty$, and $A_0(\vec{x})$ may be absorbed into a shift in $k_0$ under the integral $\int dk_0$.

**Green function**

Following the steps of the previous section, we have this time for the thermal Green function

$$< \hat{\phi}(x)\hat{\phi}(y) >_\beta = \sum_{m,n} \frac{\phi_{mn}(x)\phi^*_{mn}(y)}{[\lambda^2_{mn} + M^2]}$$

$$= \int_0^\infty dt \ e^{-tM^2} \frac{1}{\beta} \sum_m e^{i\frac{2\pi m}{\beta}(x_0-y_0)} h_m(t; \vec{x}, \vec{y}) \quad (2.11)$$

where

$$h_m(t; \vec{x}, \vec{y}) = \sum_n e^{-t\lambda^2_{mn}} \varphi_{mn}(\vec{x})\varphi^*_{mn}(\vec{y}) \quad (2.12)$$

is the spatial bilocal heat kernel for the $m$'th spatial background potential $V_m(\vec{x})$ in (2.10). Notice that because of the coupling of Matsubara frequencies with the spatial modes the Green function can no longer be straightforwardly separated into “sea” and “gas” contributions as in the potential problem discussed previously. In special cases this separation can nevertheless be made explicit, as we shall see.

**Energy-momentum tensor**

The symmetric canonical energy-momentum tensor for the complex scalar field $\hat{\phi}(x)$ coupled to a background $A_\mu$ is formally given by (in Minkowski space-time)

$$\hat{T}_{\mu\nu} = \frac{1}{2}[(D_\mu \hat{\phi})^\dagger(D_\nu \hat{\phi}) + (D_\nu \hat{\phi})^\dagger(D_\mu \hat{\phi})] - \eta_{\mu\nu}\hat{\mathcal{L}}$$

where

$$\hat{\mathcal{L}} = \frac{1}{2}[(D_\mu \hat{\phi})^\dagger(D^\mu \hat{\phi}) - M^2 \hat{\phi}\hat{\phi}^\dagger].$$

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Using the equation of motion \( (D_{\mu}D^{\mu} + M^2)\hat{\phi} = 0 \) we have for the divergence of \( \hat{T}_{\mu\nu} \),

\[
\partial^\mu \hat{T}_{\mu\nu} = F_{\mu\nu} \frac{i}{2} [\hat{\phi}^\dagger (D^\mu \hat{\phi}) - (D^\mu \hat{\phi})^\dagger \hat{\phi}] 
\]

and for the trace,

\[
\hat{T}^\mu_\mu = -\frac{1}{2}(d-1)(D_{\mu}\hat{\phi})^\dagger (D^{\mu} \hat{\phi}) + \frac{1}{2}(d+1)M^2 \hat{\phi}^\dagger \hat{\phi}. \tag{2.13}
\]

From (2.13) we see that \( \hat{T}_{\mu\nu} \) is traceless only in \( d = 1 \) spatial dimension for \( M = 0 \). We furthermore see that \( \hat{T}_{\mu0} \) is conserved if \( F_{i0} = 0 \), that is, if the electric field vanishes. This still allows for a static magnetic field, which makes sense, since a static magnetic field cannot do work on charges.

From (2.13) the thermal stress energy tensor \( T_{\mu\nu}^{(\beta)} = \langle \hat{T}_{\mu\nu} \rangle_\beta \) is now easily written down. We have for the separate components

\[
\langle D_0\hat{\phi}(x)[D_0\hat{\phi}(x)]^\dagger \rangle_\beta = \int_0^\infty dt \frac{e^{-tM^2}}{\beta^2} \frac{1}{\beta} \sum_m [m - a(\bar{x})] h_m(t; \bar{x}, \bar{y}) \tag{2.14}
\]

\[
\langle D_i\hat{\phi}(x)[D_j\hat{\phi}(x)]^\dagger \rangle_\beta = \int_0^\infty dt \frac{e^{-tM^2}}{\beta^2} \sum_m \lim_{\bar{x} \to \bar{y}} \{ (D_i^x)^\dagger D_j^y h_m(t; \bar{x}, \bar{y}) \} \tag{2.15}
\]

\[
\langle D_0\hat{\phi}(x)[D_j\hat{\phi}(x)]^\dagger \rangle_\beta = \int_0^\infty dt \frac{e^{-tM^2}}{\beta^2} \frac{1}{\beta} \sum_m \lim_{\bar{x} \to \bar{y}} \{ (D_j^y)^\dagger h_m(t; \bar{x}, \bar{y}) \} \tag{2.16}
\]

\[
\langle |\hat{\phi}(x)|^2 \rangle_\beta = \int_0^\infty dt \frac{e^{-tM^2}}{\beta} \sum_m h_m(t; \bar{x}, \bar{x}) \tag{2.17}
\]

where

\[
a(\bar{x}) = \frac{\beta}{2\pi} A_0(\bar{x})
\]

is the rescaled temporal component \( A_0 \).

The above expressions giving the energy-momentum tensor in terms of the heat kernel need of course to be properly UV-regularized. We see that these results differ essentially from the nongauge results given earlier, since in the
present case $T_{\mu\nu}$ is expressed as a Matsubara sum over nonidentical spatial problems, the latter depending on Matsubara label $m$. Thus an explicit separation into a sea and thermal gas (or more properly, thermal plasma) contributions is in general not possible. We now consider specific background potentials for which the problem is completely soluble, and the separation into gas and thermal plasma contributions can be displayed.

2.2.1 Constant gauge potential

To begin with we consider a constant background gauge potential

$$A_0 = \frac{2\pi}{\beta} a, \quad \vec{A} = 0.$$  \hspace{1cm} (2.18)

As already pointed out, on the cylinder $S^1 \times R^d$ this gauge potential cannot be gauged to zero; however, it is gauge equivalent to a potential with $a$ in the range $0 \leq a \leq 1$. On the other hand, the spatial component $\vec{A}$ of a constant $A_\mu$ can always be gauged to zero on this cylinder, so that we may choose $\vec{A} = 0$.

Following our general notation we have in this case (for infinite volume the index $n$ becomes the continuous momentum label $\vec{k}$)

$$\phi_{mk}(\vec{x}) = \frac{1}{(2\pi)^{d/2}} e^{i\vec{k} \cdot \vec{x}},$$  

$$\lambda_{mk}^2 = \left(\frac{2\pi}{\beta}\right)^2 (m-a)^2 + \vec{k}^2,$$  \hspace{1cm} (2.19)

and

$$V_m(\vec{x}) = \left(\frac{2\pi}{\beta}\right)^2 (m-a)^2,$$

$$h_m(t; \vec{x}, \vec{y}) = e^{-(\frac{2\pi}{\beta})^2 (m-a)^2} h_0(t; \vec{x} - \vec{y}),$$  \hspace{1cm} (2.20)

where

$$h_0(t; \vec{x} - \vec{y}) = \frac{1}{(2\pi)^d} \int \! d^d k \, e^{-i\vec{k} \cdot \vec{x}} e^{i\vec{k} \cdot (\vec{y} - \vec{y})} = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|\vec{x} - \vec{y}|^2}{4t}}.$$
is the infinite volume, zero temperature heat kernel of the free scalar field.

The factorization of the heat kernel $h_m$ into an $m$-dependent and and $m$-independent factor now enables one to perform Matsubara sums explicitly by making use of the identity

$$
\sum_{-\infty}^{\infty} (m-a)^2 e^{-b(m-a)^2} = \frac{1}{2b} \sum_{-\infty}^{\infty} e^{-b(m-a)^2} - 2\left(\frac{\pi}{b}\right)^{\frac{d}{2}} \sum_{n=1}^{\infty} n^2 e^{-\frac{n^2\beta^2}{b}} \cos(2\pi an) \tag{2.21}
$$

obtained from (2.3) by differentiation with respect to $b$. One finds for the Minkowskian energy density after some simple algebra

$$
T_{00}^{(\beta)} = T_{00}^{sea} + T_{00}^{plasma} \tag{2.22}
$$

where the temperature-independent part representing the sea is given by the UV-divergent integral

$$
T_{00}^{sea} = \frac{1}{2} \int_0^{\infty} dt \ e^{-tM^2(4\pi t)^{-\left(\frac{d+1}{2}\right)}} \left[\frac{d-1}{2t} + M^2\right], \tag{2.23}
$$

and the temperature dependent part representing the thermal plasma carries all the dependence on the gauge potential and is finite:

$$
T_{00}^{plasma} = \int_0^{\infty} dt \ e^{-tM^2(4\pi t)^{-\left(\frac{d+1}{2}\right)}} \sum_{n=1}^{\infty} \cos(2\pi an) \left\{\frac{n^2\beta^2}{4t^2} + \frac{d-1}{2t} + M^2\right\} e^{-\frac{n^2\beta^2}{4t}}. \tag{2.24}
$$

The integral in (2.24) can be evaluated in terms of the modified Bessel function $K_\nu(z)$, but we shall not do so. It is important to observe that the thermal part vanishes exponentially as $T \to 0$, and that it exhibits the expected periodicity property in the parameter $a$, in line with our earlier observation that $a$ can always be chosen to lie in the interval $[0, 1]$.

The remaining diagonal stress tensor elements $T_{ii}(\beta)$ with $i = 1, 2, \ldots, d$ are given by $T_{00}(\beta)$ with the signs of the $(d-1)/2t$ and $M^2$ terms in the

\[\text{Note that the transition from Euclidean to Minkowskian space-time involves a minus sign: } \langle -|D_0\phi|^2 \rangle_\beta \rightarrow \langle |D_0\phi|^2 \rangle_\beta. \]

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curly brackets of eqs. (2.23), (2.24) reversed. All nondiagonal elements of $T^{(\beta)}_{\mu\nu}$ vanish. This tensor is obviously conserved.

Finally observe that for $d = 1$ and $M = 0$ we have $T^{\text{sea}}_{\mu\nu} = 0$, and there is no need to perform an UV renormalization. Furthermore $T^{\text{plasma}}_{00} = T^{\text{plasma}}_{11}$ in this case, in accordance with the trace condition (2.9).

### 2.2.2 Constant electric field

Next we consider the linear background potential

$$A_0(x_1) = Ex_1 + 2\pi a/\beta, \quad \vec{A} = 0$$

(2.25)

corresponding to a constant background electric field $\vec{E} = (-E, 0, ..., 0)$ in the $x_1$ direction. Here the constant term $2\pi a/\beta$ in $A_0$ has the obvious physical significance of determining the location $x_1 = -2\pi a/\beta E$ of the zero of this linear potential. Global aspects of what now follows have been investigated previously for scalar fields as well as for spinor fields (see e.g. refs. [3]-[6]). We focus here on local aspects of this problem. Note that the limit $E \to 0$ is distinctly nontrivial. For that reason we have chosen to analyse separately the $E \neq 0$ problem here and the $E = 0$ problem in subsection 2.2.1 above.

Continuing with our general notation (where now $n \to (n, \vec{k}_\perp)$) we have the spatial modes

$$\varphi_{mn\vec{k}_\perp}(\vec{x}) = \varphi_n(x_m)(2\pi)^{-\frac{1}{2}(d-1)}e^{i\vec{k}_\perp \cdot \vec{x}_\perp}$$

(2.26)

with $\vec{k}_\perp = (k_2, ..., k_d)$ and $\vec{x}_\perp = (x_2, ..., x_d)$ representing momentum and position perpendicular to $x_1$. Inserting $\varphi_{mn\vec{k}_\perp}$ into the spatial mode equation (2.9) one obtains

$$\left[ -\frac{d^2}{dx_m^2} + E^2 x_m^2 \right] \varphi_n(x_m) = \epsilon_n \varphi_n(x_m),$$

(2.27)

where

$$x_m \equiv x_1 + \frac{2\pi}{\beta E}(a - m).$$

(2.28)
This is just the harmonic oscillator eigenvalue problem in Schrödinger theory with orthonormal eigenfunctions

\[ \varphi_n(x_m) = 2^{-n/2} \frac{1}{\sqrt{n!}} \left( \frac{E}{\pi} \right)^{\frac{1}{4}} e^{-\frac{1}{2} Ex_m^2} H_n(\sqrt{E}x_m), \]

\[ \epsilon_n = 2E(n + \frac{1}{2}), \quad n = 0, 1, 2, ... \] (2.29)

Here \( H_n(z) \) are Hermite polynomials satisfying \( y'' - 2zy' + 2ny = 0 \). In eq. (2.3) the \( m \)-dependent backgrounds \( V_m(x_1) = E^2 x_m^2 \) are identical harmonic oscillator potentials centered at equidistant positions \( x_1 = (m - a)2\pi/\beta E \). As we shall see this periodic arrangement of identical potentials leads to a periodic structure along \( x_1 \) (with period \( \Delta x_1 = 2\pi/\beta E \)) in \( T_{\mu\nu}(x) \) and in other local quantum functions. Notice that the \( m \) dependence of \( \varphi_{mnk_\perp}(\vec{x}) \) resides entirely in the argument \( x_m \) of the harmonic oscillator wave function \( \varphi_n(x_m) \) – i.e. entirely in the position \( x_1 = (m - a)2\pi/\beta E \) of the zero of \( V_m(x_1) \). One consequence of this fact is that the spectrum of \( -D^2 \) given by

\[ \lambda_{mnk_\perp}^2 = 2E(n + \frac{1}{2}) + \vec{k}_\perp^2 \] (2.30)

does not depend on the Matsubara label \( m \). This is in sharp contrast with the nongauge scalar theory of section 2.1 with its spectrum (2.2), and with the constant \( A_\mu \) problem above with spectrum (2.19).

The spatial heat kernel (2.12) constructed from the spatial modes (2.26) is

\[ h_m(t; \vec{x}, \vec{y}) = k_E(t; x_m, y_m) h_0(t; |\vec{x}_\perp - \vec{y}_\perp|) \] (2.31)

where

\[ k_E(t; x_m, y_m) = \sum_{n=0}^{\infty} e^{-t\lambda_n^2} \varphi_n(x_m)\varphi_n^*(y_m) \]

\[ = \left[ \frac{E}{2\pi \sinh(2Et)} \right]^{\frac{1}{2}} e^{-\frac{1}{2} E(x_1 - y_1)^2 \coth(2Et)} \]

\[ \times e^{-E x_m y_m \tanh(Et)} \] (2.32)

and \( h_0 \) is the free-space heat kernel for \( d - 1 \) dimensions. The mode sum \( \sum_n \) here has been performed with the help of the identity (see e. g. ref. [10], p. 14).
\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{z}{2} \right)^n H_n(x) H_n(y) = (1-z^2)^{-1/2} \exp \left\{ \frac{z}{1-z^2} [2xy(1-z) - z(x-y)^2] \right\}.
\]

Alternatively one can use known formulae for the propagator in the harmonic oscillator problem (see section 3). Note that in the limit \( E \to 0 \) the heat kernel \([2.32]\) smoothly becomes the one-dimensional free-space heat kernel as it should

\[
k_E(t; x_m, y_m) \to \frac{1}{\sqrt{4\pi}} e^{-(x_1-y_1)^2/4t}, \quad E \to 0.
\]

The limit \( E \to 0 \) is nonetheless far from being uniform: As \( E \to 0 \) the potentials \( V_m(x_1) = E^2 x^2_m \to (2\pi m/\beta)^2 \) change into \( m \)-dependent constants (like mass terms) independent of \( x_1 \). The background returns to the constant potential \( A_\mu = (2\pi a/\beta, \vec{0}) \) of the preceeding subsection with its spectrum \( \lambda^2_{mk} = (m-a)^2(2\pi/\beta)^2 + k^2 \). Remarkably, the dependence on \( m \) so conspicuously absent from the \( E > 0 \) spectrum \( \lambda^2_{mnk} \) reenters the \( E = 0 \) spectrum.

The Green function \([2.11]\) and local quantities derived from it possess the sea + plasma structure one expects to find. Let us display this for the Green function \([2.11]\) in the limit \( y \to x \);

\[
< |\hat{\phi}(x)|^2 > \beta = \int_0^\infty dt e^{-t M^2} (4\pi t)^{-d/2} \left[ \frac{2Et}{\sinh(2Et)} \right]^{1/2} \times \frac{1}{\beta} \sum_m e^{-EX_m^2 \tanh(Et)}
\]

where, using the identity \([2.3]\), the Matsubara sum can be evaluated with the result

\[
\frac{1}{\beta} \sum_{m=-\infty}^{\infty} e^{-EX_m^2 \tanh Et} = \left[ \frac{E}{4\pi \tanh Et} \right]^{1/2} \sum_{n=-\infty}^{\infty} e^{-n^2 \beta^2 E/4 \tanh Et} \times e^{-in(2\pi a + x_1 \beta E)}.
\]

Thus we find

\[
< |\hat{\phi}(x)|^2 >_{sea} = < |\hat{\phi}(x)|^2 >_{T=0} = \int_0^\infty dt e^{-t M^2} (4\pi t)^{-d/2} \frac{E}{4\pi \sinh Et},
\]

15
\[ <|\hat{\phi}(x)|^2 >_{\beta}^{\text{plasma}} = \int_0^\infty dt e^{-tM^2}(4\pi t)^{-\frac{1}{2}} \frac{E}{4\pi \sinh Et} \times \sum_{n=1}^{\infty} 2 \cos n(2\pi a + x_1 \beta E) e^{-n^2 \beta^2 E/4 \tanh Et}. \]  

(2.36)

One easily verifies that \(<|\hat{\phi}(x)|^2 >_{\text{sea}}\) coincides with the corresponding \(T=0\) quantity \(<|\hat{\phi}(x)|^2 >_{T=0}\) as it should. This function of course needs UV renormalization. All dependence on temperature is in \(<|\hat{\phi}(x)|^2 >_{\beta}^{\text{plasma}}\). The latter function is finite and it vanishes exponentially as \(T \to 0\). Moreover it is periodic in \(x_1\) with period \(\Delta x_1 = 2\pi/\beta E\), reflecting the equidistant arrangement of potentials \(V_m(x_1) = E^2 x_m^2\).

We now proceed to the straightforward calculation of \(T^{(\beta)}_{\mu\nu}\). Using eqs. (2.14)-(2.17) one easily verifies

\[ T^{(\beta)}_{00}(x) = T^{\text{sea}}_{00} + T^{\text{plasma}}_{00}(x_1) \]  

(2.37)

where

\[ T^{\text{sea}}_{00} = \frac{1}{2} \int_0^\infty dt e^{-tM^2}(4\pi t)^{-\frac{1}{2}(d-1)} \frac{E}{4\pi \sinh Et} \left[ \frac{d-1}{2t} + M^2 \right], \]  

(2.38)

and

\[ T^{\text{plasma}}_{00}(x_1) = \int_0^\infty dt e^{-tM^2}(4\pi t)^{-\frac{1}{2}(d-1)} \frac{E}{4\pi \sinh Et} \times \sum_{n=1}^{\infty} e^{-n^2 \beta^2 E/4 \tanh Et} \cos n(2\pi a + x_1 \beta E) \times \left[ \frac{d-1}{2t} + M^2 + n^2 \left( \frac{E \beta}{2 \sinh Et} \right)^2 \right]. \]  

(2.39)

The energy densities of the virtual sea and thermal plasma have been obtained by performing the Matsubara sums in eq. (2.37) with the help of the identities (2.3), (2.21). For the reader’s convenience we give the form in which the latter identity is used here:

\[ \frac{E^2}{\beta} \sum_{m=-\infty}^{\infty} x_m^2 e^{-x_m^2 E \tanh Et} = \left[ \frac{E}{4\pi \tanh Et} \right]^{\frac{1}{2}} \frac{E}{2 \tanh Et} \times \left\{ \sum_{n=-\infty}^{\infty} e^{-n^2 \beta^2 E/4 \tanh Et} e^{-in(2\pi a + x_1 \beta E)} \right\} \]  

(2.40)
Of course, \( T^{\text{sea}}_{00} \) needs UV renormalization. In the limit \( E \to 0 \), \( T^{\text{sea}}_{00} \) and \( T^{\text{plasma}}_{00} \) above smoothly become the \( E = 0 \) functions \((2.23), (2.24)\).

The other diagonal components of \( T_{\mu\nu}^{(\beta)} \) can be similarly obtained. For brevity we do not write them down. Again \( T_{11}^{(\beta)} \) is given by eqs. \((2.37)\) - \((2.39)\) with the signs of the \((d-1)/2t\) and \(M^2\) terms reversed. Thus \( T^{\text{sea}}_{00} = -T^{\text{sea}}_{11} \) for any spatial dimension \( d \geq 1 \). All nondiagonal components of \( T_{\mu\nu}^{(\beta)} \) vanish. Interestingly \( T^{\text{sea}}_{\mu\nu} \) still vanishes for a massless scalar field in \( d = 1 \) spatial dimensions. Indeed the tracelessness of \( T_{\mu\nu} \) implies \( T^{\text{sea}}_{00} = T^{\text{sea}}_{11} \) and this, combined with \( T^{\text{sea}}_{00} = -T^{\text{sea}}_{11} \), leads to \( T^{\text{sea}}_{\mu\nu} = 0 \). However, for \( d > 1 \) this tensor does not vanish.

As expected \( T^{\text{sea}}_{00} \) is independent of position: the uniform electric field leaves the virtual sea spatially uniform. Physically this seems reasonable. Virtual particles do not have the prolonged existence needed to participate in e.g. thermal equilibrium: the sea remains temperature-independent.

Things are quite different for the thermal plasma. The particles of the thermal plasma do have prolonged existence, and they do participate in thermal equilibrium. Also they are nonuniformly redistributed by the background electric field \( E \) as eq. \((2.36)\) shows. The plasma becomes spatially nonuniform – in fact periodic as already described with period \( \Delta x_1 = 2\pi T/E \). As temperature \( T \to 0 \) with \( E \) fixed, the plasma ceases to exist and spatial uniformity is restored. Alternatively, holding \( T \) fixed and letting \( E \to 0 \), the period \( \Delta x_1 = 2\pi/\beta E \) becomes infinite, thereby restoring spatial uniformity to the plasma. In fact, the total energy contained in the plasma changes discontinuously as we let \( E \) tend to zero. Indeed, if we integrate \( T^{\text{plasma}}_{00}(x) \) over an integral number of periods the result is zero. Hence, the total energy of the thermal plasma vanishes in the presence of the uniform electric field. However, in the limit \( E \to 0 \) the nonzero energy density \((2.24)\) characterizes the uniform plasma throughout space.
2.2.3 Constant magnetic field

To investigate the effect of a uniform background magnetic field on the virtual sea and thermal gas of a scalar field it is of particular interest to consider the case \( d = 3 \) spatial dimensions. We choose for the static background potential \( A_\mu = (0, 0, Bx_1, 0) \) leading to the magnetic field \( \vec{B} = (0, 0, B) \). Following our general notation we then have for the spatial modes (now \( n \rightarrow (n, k_2, k_3) \))

\[
\phi_{mnk_2k_3}(\vec{x}) = \frac{1}{2\pi} e^{i(k_2x_2 + k_3x_3)} \varphi_n(x_{k_2}),
\]

where \( k_2, k_3 \) are continuous momentum labels in finite space and

\[
x_{k_2} = x_1 + \frac{k_2}{B}.
\]

The modes \( \varphi_n(x_{k_2}) \) are the harmonic oscillator wavefunctions (2.29) satisfying

\[
[-\partial^2 + B^2 x_{k_2}^2] \varphi_n(x_{k_2}) = 2B(n + \frac{1}{2}) \varphi_n(x_{k_2}).
\]

The eigenvalues of \(-D^2\) are now

\[
\lambda^2_{mnk_2k_3} = (\frac{m 2\pi}{\beta})^2 + 2B(n + \frac{1}{2}) + k_3^2,
\]

and are independent of \( k_2 \).

It is already apparent that the uniform magnetic field does not introduce spatial non-uniformity into either the virtual sea or the thermal gas. Indeed, all mode sums involve an integration in \( k_2 \) over the infinite interval \([-\infty, \infty]\). Since \( k_2 \) is a continuous variable we can perform the shift \( k_2 \rightarrow k_2 - Bx_1 \) in the integration variable, thereby absorbing the \( x_1 \)-dependence into the integration. Thus \( T_{\mu\nu} \) and other local quantum functions will not depend on \( x_1 \).

If, however, we compactify the \( x_2 \) direction perpendicular to the magnetic field to a circle of perimeter \( L \), we are led to a problem paralleling the one with constant electric field discussed in the previous subsection. Compact \( x_2 \) corresponds to discrete momenta \( k_2 = p(\frac{2\pi}{L}) \), with \( p \) running over all
integers (as in the case of the Matsubara index). The harmonic oscillator mode equation above becomes
\[
-\partial_1^2 + B^2 x_p^2 \varphi_n(x_p) = 2B(n + \frac{1}{2})\varphi_n(x_p),
\]
where \( x_p = x_1 + p \frac{2\pi}{BL} \). Again we have an infinite set of harmonic-oscillator potentials equally spaced at intervals \( \Delta x_1 = \frac{2\pi}{BL} \) along the \( x_1 \)-axis. Much as in section 2.2.2, local quantities such as \( T_{\mu\nu} \) are periodic in \( x_1 \) with period \( \Delta x_1 \). Clearly one is observing here something akin to the Quantum Hall Effect.

One could of course extend this discussion to include backgrounds with both constant electric and magnetic fields. We shall not pursue these interesting matters here, but rather turn to the case of massless fermions in 1+1 dimensional space-time.

### 3 The finite temperature Schwinger model

Electrodynamics of massless fermions in 1+1 dimensions (the Schwinger Model (SM)) is described by the Lagrangian density
\[
\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \bar{\psi}(x) (i\partial - eA) \psi(x).
\]

Our discussion will follow the general lines of the previous section. Aside from the usual complications of spin, we shall have to face in the infinite volume limit the existence of an infinite number of normalizable zero modes.

#### 3.1 The case of no electric field

We choose again (2.18) for our gauge field configuration. We emphasize once more that at finite temperature the \( A_0 \) field cannot be gauged to zero. Hence \( A_0 = \text{const} \) implies observable effects. The corresponding eigenvalue

\[7\text{The same applies to } A_1 \text{ if we were to compactify space to a circle } S^1.\]
equation for the Dirac field may be written in the form

\[
\left( i\partial_0 + \frac{2\pi}{\beta} a + \gamma^5 \partial_1 \right) \psi(x) = \lambda \gamma_0 \psi(x) \tag{3.2}
\]

with the solution

\[
\psi_{m,k}(x) = \frac{1}{\sqrt{\beta}} e^{i(2m+1)\frac{\pi}{\beta} x^0} \begin{pmatrix} \varphi_k(x_1) \\ \bar{\varphi}_k(x_1) \end{pmatrix} \tag{3.3}
\]

where

\[
\begin{pmatrix} \varphi_k(x_1) \\ \bar{\varphi}_k(x_1) \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} A \\ \bar{A} \end{pmatrix} e^{ikx_1}
\]

and the eigenvalues

\[
\lambda_m(k) = \pm \sqrt{a_m^2 + k^2} \tag{3.4}
\]

where

\[
a_m = -(2m+1)\frac{\pi}{\beta} + \frac{2\pi}{\beta} a
\]

and

\[
\bar{A} = \pm \sqrt{\frac{a_m - ik}{a_m + ik}} A,
\]

respectively. We normalize the eigenfunctions by choosing \(|A| = \frac{1}{\sqrt{2}}\). Note that zero modes are absent, unless \(a = (M + \frac{1}{2})\).

Three local quantities are of primary interest to us: The heat kernel and corresponding zeta-function densities, as well as the external field Green function for the construction of the energy-momentum tensor.

**Local heat kernel and zeta-function**

From its definition we have for the matrix valued heat kernel

\[
h^{(\beta)}(t; x, y)_{\alpha,\beta} = \langle x | e^{-t (iD / \beta)^2} \rangle_{\alpha,\beta} | y \rangle
\]

\[
= \sum_m \int \frac{dk}{2\pi} e^{-t \lambda_m^2(k)} \psi_{m,k}(x) \psi_{m,k}^*(y) \beta. \tag{3.5}
\]

---

8 Our euclidean conventions are \(\gamma_0 = -\sigma_1, \gamma_1 = \sigma_2, \gamma^5 = \sigma_3\).
with the corresponding expression for the local zeta function, defined by

$$\zeta(s; x, y) = \frac{1}{\Gamma(s)} \int dt \, t^{s-1} h^{(\beta)}(t; x, y). \quad (3.6)$$

Since the eigenvalues come in pairs of positive and negative numbers, the "chiral" parts of the $2 \times 2$ matrix cancel, leaving us with a diagonal spinor structure:

$$h^{(\beta)}(t; x, y) = \frac{1}{2\beta} \sum_m \int \frac{dk}{2\pi} e^{-\alpha_0^2(k)} e^{i(2m+1)\phi(x^0-y^0)} e^{ik(x_1-y_1)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.7)$$

We shall be interested only in the diagonal part, $x = y$ of the heat kernel. Taking the trace we have

$$\text{tr} h^{(\beta)}(t; x, x) = \frac{1}{\beta} \sum_m \int \frac{dk}{2\pi} e^{-\alpha_0^2(k)}. \quad (3.8)$$

Making use of the Jacobi identity (2.3) we may cast $\text{tr} h^{(\beta)}(t; x, x)$ into the form

$$\text{tr} h^{(\beta)}(t; x, x) = \text{tr} h(t; x, x) \left[ 1 + 2 \sum_{m=1}^{\infty} (-1)^m \cos(2\pi ma) e^{-m^2 \beta^2} \right]. \quad (3.9)$$

where $h(t; x, x)$ is the zero-temperature diagonal heat kernel

$$\text{tr} h(t; x, x) = \frac{1}{\sqrt{4\pi t}} \int \frac{dk}{2\pi} e^{-k^2}. \quad (3.10)$$

Indeed, for $\beta \to \infty$,

$$\sum_m \frac{1}{\beta} e^{-\alpha_0^2} \to \int \frac{dk_0}{2\pi} e^{-k_0^2} = \frac{1}{\sqrt{4\pi t}},$$

so that we are lead smoothly from (3.9) to (3.10) in the limit $\beta \to \infty$.

The heat kernel (3.9) is interesting in two respects: i) It exhibits the remarkable periodicity property in $A_0$ already encountered in section 2. As commented in that section, this periodicity property has its origin in the fact that $A_0(\vec{x})$ is always gauge equivalent to a configuration taking values in $[0, \frac{2\pi}{\beta}]$. ii) This exact result provides a counterexample to a claim [12] made in the literature concerning the factorization of the temperature dependence in the thermal heat kernel of a gauged spinor field.
Energy density

The energy density is given by

\[ T_{00}(x) = \text{tr} \left[ i \gamma_0 D_0(x) G(x, y; A) \right]_{x=y} \quad (3.11) \]

where \( G(x, y; A)_{\alpha\beta} \) is the external field Green function having spectral representation

\[ G(x, y; A)_{\alpha\beta} = \sum_m \int \frac{dk}{2\pi} \psi_m(k)_{\alpha} \psi_m^*(y)_{\beta} \frac{1}{\lambda_m(k)} \quad (3.12) \]

with the sum extending over the positive and negative eigenvalues \((\lambda, -\lambda)\), which results in the off-diagonal structure

\[ G(x, y; A) = \frac{1}{\beta} \sum_m \int \frac{dk}{2\pi} e^{i(2m+1)\frac{\pi}{2}(x_0-y_0)} e^{ik(x_1-y_1)} \frac{-1}{|\lambda_m(k)|} \begin{pmatrix} 0 & \frac{a_m+ik}{\sqrt{a_m-ik}} \\ \frac{a_m-ik}{\sqrt{a_m+ik}} & 0 \end{pmatrix}. \quad (3.13) \]

Performing the differentiation in \((3.11)\) and combining terms, we obtain

\[ T_{00}(x) = \frac{2}{\beta} \sum_m \int \frac{dk}{2\pi} \frac{a_m^2}{a_m^2+k^2}. \quad (3.14) \]

Notice that this sum diverges and hence requires regularization. For the special case of a vanishing gauge-potential \((a = 0)\) the result \((3.14)\) has a familiar interpretation. Making use of the identity

\[ \sum_{m=-\infty}^{\infty} \frac{1}{(2m+1)^2 + x^2} = \frac{\pi}{2x} \tanh \frac{\pi x}{2} \]

we may write \((3.14)\) for \(A_\mu = 0\) in the form

\[ T_{00}(x) = \frac{2}{\beta} \sum_m \int \frac{dk}{2\pi} - \int \frac{dk}{2\pi} \omega(k) + 2 \int \frac{dk}{2\pi} \omega(k) \frac{1}{e^{\beta \omega(k)} + 1}. \quad (3.15) \]

where we have set \(|k| = \omega(k)\). The last term evidently corresponds to the usual Fermi-distribution for a gas of massless fermions, while the second term can be interpreted as the negative energy contribution of the Dirac sea. As for the first term, one can identify it with \(\delta^2(0)\), which is independent of \(\beta\);
hence this term is eliminated by temperature-independent renormalization, as should be the case.

**Effective action and thermal energy**

The effective (euclidian) action is given by

\[
S_{\text{eff}} = -\frac{1}{2} \ln \det (i\partial + eA)^2 = \frac{1}{2} \zeta'(0) \quad (3.16)
\]

where \( \zeta(s) \) is the zeta-function defined for the spectrum \((3.4)\),

\[
\zeta(s) = \sum_{-\infty}^{\infty} \int \frac{dk}{2\pi} \frac{1}{(\lambda_m^2(k))^s} \quad (3.17)
\]

and where the "prime" on \( \zeta(s) \) indicates differentiation with respect to the argument. With the identification \( S_{\text{eff}} = -\ln Z \), with \( Z \) the external field partition function, the average thermodynamic energy is thus given by

\[
U = \frac{1}{2} \left( \frac{\partial}{\partial \beta} \zeta'(s) \right)_{s=0}. \quad (3.18)
\]

In order to allow for a comparison with previous results, we first perform the differentiations and then take the limit \( s \to 0 \) in the resulting expression, which then takes the form

\[
\left( \frac{\partial}{\partial \beta} \zeta'(s) \right)_{s=0} = -2 \sum_m \int \frac{dk}{2\pi} \frac{1}{\lambda_m^2(k)} \lambda_m^2(k). \quad (3.18)
\]

It is now important to observe that our parametrization of the temporal part of the gauge field in terms of \( A_0 = \frac{2\pi}{\beta}a \) correctly exhibits the \( \beta \)-dependence of the eigenvalue spectrum. Indeed, as we already remarked above, \( A_0 \) may always be mapped into the interval \([0, \frac{2\pi}{\beta}]\) by an allowed gauge transformation, and correspondingly the parameter \( a \) is taken as lying in the interval \([0,1]\). With this observation we have

\[
\frac{\partial}{\partial \beta} \lambda_m^2(k) = -\frac{2}{\beta} a_m^2,
\]

so that finally

\[
U = \frac{1}{2} \frac{\partial}{\partial \beta} \zeta'(0) = \frac{2}{\beta} \sum_m \int \frac{dk}{2\pi} \frac{a_m^2}{(a_m^2 + k^2)},
\]

in agreement with \((3.14)\). These considerations show that it is appropriate to define the divergent sum \((3.14)\) in terms of the zeta-function via \((3.17)\).
3.2 A constant Electric Field

We now turn to the case of a constant electric field $\mathcal{E} = -E/e$, with the choice \([2.25]\) for the gauge potential. The eigenvalue equation (3.2) is now replaced by

$$[i\partial_0 + E x_1 + \frac{2\pi}{\beta} a + \gamma^5 \partial_1] \psi(x) = \lambda \gamma_0 \psi(x)$$ \hspace{1cm} (3.19)

Making again the ansatz

$$\psi(x) = \frac{1}{\sqrt{\beta}} e^{i(2m+1)\pi x_0} \begin{pmatrix} \varphi(x_1) \\ \bar{\varphi}(x_1) \end{pmatrix}$$ \hspace{1cm} (3.20)

and defining

$$x_m = x_1 - (2m+1) \frac{\pi}{E\beta} + \frac{2\pi a}{E\beta}$$ \hspace{1cm} (3.21)

we arrive at the coupled set of equations

$$\begin{pmatrix} E y + \frac{d}{dy} \end{pmatrix} \varphi = -\lambda \bar{\varphi}$$
$$\begin{pmatrix} E y - \frac{d}{dy} \end{pmatrix} \bar{\varphi} = -\lambda \varphi$$ \hspace{1cm} (3.22)

where we have set $y = x_m$. Define the operators

$$a = \frac{1}{\sqrt{2}} \left( \sqrt{|E|} y + \frac{1}{\sqrt{|E|}} \frac{d}{dy} \right)$$ \hspace{1cm} (3.23)
$$a^\dagger = \frac{1}{\sqrt{2}} \left( \sqrt{|E|} y - \frac{1}{\sqrt{|E|}} \frac{d}{dy} \right).$$ \hspace{1cm} (3.24)

These operators evidently satisfy the commutation relations of destruction and creation operators, respectively:

$$[a, a^\dagger] = 1$$

Substituting one equation into the other in (3.22) we have, depending on the sign of $E$,

$$\begin{align*}
E \text{ positive:} & \quad \begin{pmatrix} 2|E|aa^\dagger \varphi = \lambda^2 \varphi \\ 2|E|a^\dagger a \varphi = \lambda^2 \varphi \end{pmatrix} \hspace{1cm} (3.25)
\end{align*}$$

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and

\[
E_{\text{negative}} : \begin{cases}
2|E|a^\dagger a \varphi = \lambda^2 \varphi \\
2|E|aa^\dagger \varphi = \lambda^2 \varphi
\end{cases}
\] (3.26)

Now, \(2|E|a^\dagger a\) is just the Hamiltonian of the harmonic oscillator with the zero-point energy omitted. Correspondingly \(\varphi\) and \(\bar{\varphi}\) are given by the harmonic oscillator eigenfunctions. Defining the ground state \(|0\rangle\) by \(a|0\rangle = 0\) we conclude that the eigenstates and corresponding eigenvalues are given by

\[
E_{\text{positive}} : |\Psi(\pm)\rangle = \begin{pmatrix}
|n\rangle \\
\mp |n-1\rangle
\end{pmatrix}, \quad \lambda_n = \pm \sqrt{2n|E|}
\]

and

\[
E_{\text{negative}} : |\Psi(\pm)\rangle = \begin{pmatrix}
|n-1\rangle \\
\mp |n\rangle
\end{pmatrix}, \quad \lambda_n = \pm \sqrt{2n|E|}
\]

where \(|n\rangle\) are the eigenstates of the harmonic oscillator and \(\lambda_n^2\) are the corresponding energy-eigenvalues without the “zero-point energy”. Denoting by \(\varphi_n(x_1)\) the eigenfunctions of the harmonic oscillator, normalized with respect to the interval \([-\infty, \infty]\) and setting \(y = x_m\), we have for \(n \geq 1\) the orthonormalized eigenfunctions of the Dirac operator (3.19),

\[
\psi_{m,n}^{(\pm)}(x) = \frac{1}{\sqrt{2\beta}} e^{i(2m+1)\frac{\pi}{2}x_0} \begin{pmatrix}
\varphi_n(x_m) \\
\mp \varphi_{n-1}(x_m)
\end{pmatrix}, \quad n \geq 1,
\] (3.27)

for positive \(E\), and

\[
\psi_{m,n}^{(\pm)}(x) = \frac{1}{\sqrt{2\beta}} e^{i(2m+1)\frac{\pi}{2}x_0} \begin{pmatrix}
\varphi_{n-1}(x_m) \\
\mp \varphi_n(x_m)
\end{pmatrix}, \quad n \geq 1,
\] (3.28)

for negative \(E\), each corresponding to the eigenvalues

\[
\lambda_n = \pm \sqrt{2n|E|},
\] (3.29)

respectively. Since the spectrum corresponds to the absence of the ”zero-point energy” of the harmonic oscillator, we have an infinite set of orthonor-
malized zero modes labelled by $m$ and chirality, of the form

$$\phi_m^{(+)}(x) = \frac{1}{\sqrt{\beta}} e^{i(2m+1)\pi x_0} \left( \begin{array}{c} \varphi_0(x_m) \\ 0 \end{array} \right)$$

(3.30)

for positive $E$, and

$$\phi_m^{(-)}(x) = \frac{1}{\sqrt{\beta}} e^{i(2m+1)\pi x_0} \left( \begin{array}{c} 0 \\ \varphi_0(x_m) \end{array} \right)$$

(3.31)

for negative $E$, each corresponding to the eigenvalue $\lambda_0 = 0$. This is in line with the Atiyah-Singer Index theorem in the infinite volume limit (see [6]). Notice that in the case of the zero-modes, the superscript denotes ”chirality”.

The wave functions (3.30) and (3.31) correspond to the ground state wave functions of the harmonic oscillator, localized at the positions $x_1 = \frac{(2m+1)\pi}{\beta} - \frac{2\pi a}{E\beta}$ with $m \in \mathbb{Z}$. This provides a physical interpretation of the degeneracy of the spectrum. In order to gain a further insight into the problem, we examine next the effective Lagrangian giving rise to this degeneracy, as defined in terms of the ”local” $\zeta$-function.

In order to simplify the discussion, we shall restrict ourselves in the following to the case where $E$ is positive.

**Effective Lagrangian density**

We begin by considering the local heat kernel. For the case in question it takes the form (we now take $E > 0$; we include the zero modes.)

$$h_{\alpha\beta}^{(\beta)}(t; x, y) = \sum_{m=-\infty}^{\infty} \sum_{|n|=1}^{\infty} e^{-2nEt} \psi_n^{(\sigma)}(x) \psi_n^{(\sigma)}(y) + \phi_m^{(+)}(x) \phi_m^{(+)}(y)$$

or explicitly

$$h_{\alpha\beta}^{(\beta)}(t; x, y) = \sum_{m=-\infty}^{\infty} \frac{1}{\beta} e^{i(2m+1)\pi (x_0 - y_0)}$$

(3.32)

$$\times \left( \begin{array}{cc} \sum_{n=0}^{\infty} e^{-2nEt}\varphi_n(x_m)\varphi_n^*(y_m) & 0 \\ 0 & \sum_{n=1}^{\infty} e^{-2nEt}\varphi_{n-1}(x_m)\varphi_{n-1}^*(y_m) \end{array} \right)$$

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The diagonal matrix structure is again a consequence of the existence of a pair of eigenfunctions $\psi_n^{(\pm)}$ corresponding to the eigenvalues $\pm \sqrt{2nE}$, if $n \neq 0$. We now observe that (note that the sum starts with $n = 0$)

$$
\sum_{n=0}^{\infty} e^{-2nEt} \varphi_n(x_m) \varphi_n^*(y_m) = e^{Et} < x_m | e^{-tH_{ho}} | y_m >
$$

where the matrix element on the r.h.s. is the propagation kernel of the harmonic oscillator known to be given by

$$
<x|e^{-tH_{ho}}|y> = \sqrt{\frac{E}{\pi}} e^{-Et} \sqrt{1 - e^{-4Et}} \exp \left\{ -\frac{E(x^2 + y^2)(1 + e^{-4Et}) - 4xye^{-2Et}}{1 - e^{-4Et}} \right\}.
$$

Going to the limit of coincident points $x = y$, and taking the trace in matrix space, we arrive at

$$
\text{tr} h^{(\beta)}(t; x, x) = 1 + 2 \sum_{m=-\infty}^{\infty} (-1)^m \cos \left[ m(\beta x + 2\pi a) e^{-\frac{m^2E^2}{4\tanh Et}} \right].
$$

Making use of the identity (2.34) we may thus write the heat kernel (3.33) in the form

$$
\text{tr} h^{(\beta)}(t; x, x) = \frac{E}{2\pi} \left( \frac{1}{\tanh Et} \right) \left\{ 1 + 2 \sum_{m=1}^{\infty} (-1)^m \cos \left[ m(\beta x + 2\pi a) e^{-\frac{m^2E^2}{4\tanh Et}} \right] \right\}.
$$

In order to compute the effective Lagrangian density we first need to subtract the zero-mode contribution:

$$
\text{tr} h^{(\beta)}(t; x, x) = \left[ \text{tr} h^{(\beta)}(t; x, x) - \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \varphi_0(x_m) \varphi_0^*(x_m) \right]
$$

where $\varphi_0(x)$ is the zero-energy harmonic oscillator wave function:

$$
\varphi_0(x) = \left( \frac{E}{\pi} \right)^{\frac{1}{4}} e^{-\frac{E}{2}x^2}.
$$

Using (2.3) we have

$$
\text{tr} h^{(\beta)}(t; x, x) = \frac{E}{2\pi} \left[ f_0(t) + 2 \sum_{m=1}^{\infty} (-1)^m \cos \left[ m(\beta x + 2\pi a) \right] f_m(t) \right],
$$

(3.36)

\footnote{the Hamiltonian in our case is of the form $H = p^2 + E^2y^2$, and thus corresponds to making the identifications $m = \frac{1}{2}$, $\omega = 2E$ in the conventional hamiltonian.}
where
\[ f_m(t) = \frac{1}{\tanh Et} e^{-\frac{m^2 \beta^2 E^2}{4}} - e^{-\frac{m^2 \beta^2 E^2}{4}}. \]

where the “prime” indicates the exclusion of zero-modes. From here we obtain for the effective Lagrangian\(^\text{10}\)

\[ \mathcal{L}_{\text{eff}}(x_1) = \frac{1}{2} \left[ \frac{d}{ds} \zeta^{(\beta)}(s; x, x) \right]_{s=0} + \frac{1}{2} \zeta(0; x, x) \ln \mu^2, \]

where
\[ \zeta^{(\beta)}(s; x, x) = \frac{1}{\Gamma(s)} \int_0^\infty dt s^{-1} h^{(\beta)}(t; x, x). \]

and \( \mu \) is an arbitrary scale parameter reflecting the usual ambiguity associated with a change in scale of the dimensionful eigenvalues \( \lambda_n \). We see that \( \mathcal{L}_{\text{eff}}(x_1) \) is again a periodic function of \( x_1 \) with period \( \Delta T = \frac{2\pi}{E\beta} \). The degeneracy of the \( \lambda_n \) spectrum with respect to the Matsubara index labelling the zero modes, reflects this fact. It is clear that correspondingly the energy density will exhibit the same periodicity. Notice that we have again obtained a clean separation of the \( \beta \)-independent (sea) and \( \beta \)-dependent (plasma) contributions.

**Effective action**

Restricting the dimension of our system to \( k \) “potential wells” (\( k \) zero modes) we have the following relation between \( E\beta \) and the length \( L \) of our system:

\[ L = k \frac{2\pi}{E\beta}. \quad (3.37) \]

Integrating \( \mathcal{L}_{\text{eff}} \) over a space-time volume \( \beta L \) we thus have for the effective action

\[ S_{\text{eff}} = \frac{1}{2} \left[ \zeta'(0) + \zeta(0) \ln \mu^2 \right], \quad (3.38) \]

\(^{10}\)In the \( \zeta \)-function regularization the ambiguity in the calculation of the effective action is well known to be determined by \( \zeta(0) \): \( \ln \det A = -\zeta'(0) + \zeta(0) \ln \mu^2 \), where \( \mu \) is an arbitrary scale parameter.
where
\[ \zeta(s) = \int_0^\beta dx_0 \int_{-\frac{L}{2}}^{\frac{L}{2}} dx_1 \zeta^{(\beta)}(s; x, x). \] (3.39)
and the "prime" now means differentiation with respect to \( s \). From (3.36) we have (the cosine term does not contribute to the integral)
\[ \zeta(s) = \frac{k}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} f_0(t) \]
\[ = \frac{2k}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^\infty dt \ t^{s-1} e^{-2nEt} \]
\[ = \frac{2k}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{1}{(2nE)^s} = \frac{2k}{(2E)^s} \zeta_R(s) \] (3.40)
where \( \zeta_R(s) \) is the Riemann \( \zeta \)-function
\[ \zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \]
Differentiating with respect to \( s \), setting \( s = 0 \) and using \( \zeta_R(0) = -\frac{1}{2}, \zeta'(0) = -\frac{1}{2} \ln 2\pi, \) as well as (3.37), we obtain for the effective action
\[ S_{eff} = -\ln Z = \frac{E\beta L}{4\pi} \ln \left( \frac{E}{\pi \mu^2} \right) \] (3.41)
in agreement with the result obtained for the corresponding functional determinant on the torus in the presence of a finite number of zero modes [6].
From (3.41) we have for the average thermal energy
\[ U = -\frac{\partial}{\partial \beta} \ln Z = \frac{EL}{4\pi} \ln \left( \frac{E}{\pi \mu^2} \right). \] (3.42)
Since the energy is temperature independent there is no way of normalizing it relative to the \( T = 0 \) case. Notice that this temperature independence of the total thermal energy is a consequence of the spacial periodicity of \( L_{eff} \), which in turn is the result of gauge invariance and periodicity in the time direction.

4 Conclusion

We have investigated the effect that uniform background electric and (in less detail) magnetic fields have on the distribution of thermal matter fields
in unbounded space. By calculating the thermal stress tensor $T^{(\beta)}_{\mu\nu}(x)$ and effective Lagrangian density we found that in the presence of a constant electric field the thermal plasma distribution of both scalar and fermion fields becomes periodic along the direction of $\vec{E}$ with period $\Delta x = \frac{2\pi T}{E}$, while the virtual sea remains uniform. On the other hand, a background uniform magnetic field does not lead to a spatial non-uniformity in either plasma or sea unless we compactify one of the spatial directions perpendicular to the magnetic field (say $x_2$) to a circle. In this case periodicity of the energy density distribution is again obtained along a direction $x_1$ perpendicular to both $x_2$ and to the magnetic field. In both cases (electric and magnetic field) this periodicity can be traced to the gauge invariance of the theory and the possibility of mapping $A_0(\vec{x})$ and $A_2(\vec{x})$ into the intervals $[0, \frac{2\pi}{\beta}]$ and $[0, \frac{2\pi}{L}]$, respectively, by a bona fide gauge transformation. This periodic structure was shown to reflect an infinite degeneracy of the eigenvalue spectrum of the spectral operator.

It is interesting to note that the case of constant magnetic field in 3 spatial dimensions with one of the spatial dimensions in the plane orthogonal to the magnetic field compactified, i.e., the quantum Hall problem on a cylinder, is formally equivalent to the finite temperature case with constant electric field in 1+1 dimensions. This stems from the fact that at finite temperature we are working in 2 dimensional Euclidean space with temporal direction compactified. In the latter case the degeneracy of the Landau levels in the quantum Hall Effect corresponds to the number of zero modes of the Dirac operator as implied by the Atiyah-Singer Index theorem.

Our exact results for a constant background electromagnetic field have a natural extension to arbitrary static background fields $\vec{E}(\vec{x})$ and $\vec{B}(\vec{x})$. One relevant equation to consult is eq. (2.9). There we see that the space components of the vector potential are decoupled from the Matsubara index $m$. For an arbitrary static magnetic field $\vec{B} = \nabla \times \vec{A}$ with $A_0 = 0$ (recall
that at finite temperature $A_0 = 0$ cannot be generally achieved by a gauge transformation) one simply makes the replacements $\lambda_{mn}^2 \rightarrow (\frac{2\pi m}{\beta})^2 + \omega_n^2$ and $\varphi_{mn} \rightarrow \varphi_n$ in eq. (2.25). The spatial mode equation then becomes

$$\left[-(\nabla - i\vec{A})^2\right]\varphi_n(\vec{x}) = \omega_n^2\varphi_n(\vec{x}).$$

For arbitrary $\vec{B}(\vec{x})$ one thus has a situation much like the non-gauge theory of section 2.1, leading to some non-uniform distribution of both the sea and plasma components comparable to what physical boundaries would cause.

The situation is quite different when $A_0$ is non-zero. Then the static background field affects the sea and plasma quite differently. While plasma periodicity along the direction of $\vec{E}$ is strictly true only for constant $\vec{E}$, one would expect for any electric field which is only weakly dependent on $\vec{x}$ a roughly periodic response from the plasma, and a nearly uniform distribution of the sea-component. Mathematically we have an infinite set of eigenvalue equations with a different Schrödinger like background potential $V_m(\vec{x}) = [A_0(\vec{x}) - 2\pi m/\beta]^2$ for each Matsubara frequency. Note that $m \rightarrow m + N$ corresponds to performing an allowed gauge transformation with gauge function $\lambda = x_0(2\pi N/\beta)$, and that $V_{m+N}(\vec{x})$ and $V_m(\vec{x})$ are connected by this gauge transformation. This situation differs fundamentally from the non-gauge case, where the potential $V(\vec{x})$ does not bear the label $m$. Since Green functions, the energy momentum tensor, etc., are given in terms of equally weighted sums over all the individual problems labelled by $m$, they are explicitly gauge invariant.

In general the diagonal heat kernel of a scalar or fermion quantum field at finite temperature $T > 0$ in a static background is expected to factorize in the following way:

$$h^{(\beta)}(t; x, x) = h(t; x, x)_{T=0} [1 + f(t; x; T)]$$

where $h(t; x, x)_{T=0}$ is the temperature-zero heat kernel for the same background, and $f(t, x, T)$ is some function of the temperature $T$, the diffusion or
“proper” time \( t \) and the spatial position \( \vec{x} \). This function \( f \) vanishes exponentially as either \( T \to 0 \) or \( t \to 0 \). The factorization above is motivated by the expectation that \( h^{(\beta)}(t; x, x) \) separates quite generally for a static background into an UV divergent sea part, and an UV finite gas part

\[
h^{(\beta)}(t; x, x) = h(t; x, x)_{\text{sea}} + h(t; \vec{x})_{\text{gas}}
\]

where

\[
h(t; x, x)_{\text{sea}} = h(t; x, x)_{T=0}.
\]

Defining \( f(t; x; T) \) by

\[
h(t; \vec{x})_{\text{gas}} = f(t; x; T)h(t; x, x)_{T=0}
\]

we arrive at the factorization above. Known properties of \( h(t; \vec{x})_{\text{gas}} \) lead to the stated properties of \( f \).

It is a matter of some interest to study the function \( f(t, \vec{x}; T) \). Let us list the explicit examples computed in this paper.

Nongauge scalar theory:

\[
1 + f = \sum_{n=-\infty}^{\infty} e^{-n^2 \beta^2 / 4t}.
\]

Gauged scalar theory with \( A_0 = Ex_1 + 2\pi a/\beta \):

\[
1 + f = \sum_{n=-\infty}^{\infty} e^{-n^2 \beta^2 E/4 \tanh Et} e^{-in\beta A_0}.
\]

Schwinger model with \( A_0 = Ex_1 + 2\pi a/\beta \):

\[
1 + f = \sum_{n=-\infty}^{\infty} (-)^n e^{-n^2 \beta^2 E/4 \tanh Et} e^{-in\beta A_0}.
\]

From the gauge theory examples we see that \( f(t, \vec{x}; T) \) in general depends on \( A_0 \) when a gauge potential background is involved.

In ref. [12] it is argued that \( f \) has the simple form

\[
1 + f = \sum_{n=-\infty}^{\infty} (-)^n e^{-n^2 \beta^2 / 4t}
\]

for a gauged spinor theory in a general (even time-dependent) background gauge potential \( A_\mu \). This claim is incorrect: the result for the Schwinger model provides a simple, explicit counterexample.
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