THE INTEGRAL (ORBIFOLD) CHOW RING OF TORIC DELIGNE-MUMFORD STACKS

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Abstract. In this paper we study the integral Chow ring of toric Deligne-Mumford stacks. We prove that the integral Chow ring of a semi-projective toric Deligne-Mumford stack is isomorphic to the Stanley-Reisner ring of the associated stacky fan. The integral orbifold Chow ring is also computed. Our results are illustrated with several examples.

1. Introduction

Chow groups with integer coefficients of algebraic stacks were defined by Edidin and Graham [EG] and Kresch [Kr], using Totaro’s idea [To] of integral Chow ring of classifying spaces. In [EG], the authors constructed an intersection theory of stack quotient $[X/G]$ of a quasi-projective variety $X$ by an algebraic group $G$. In the case of Deligne-Mumford stacks, the authors proved that the equivariant Chow ring $A^*_G(X)$ with integer coefficients is isomorphic to the integral Chow ring of the quotient stack $[X/G]$.

Toric Deligne-Mumford stacks were introduced by Borisov, Chen and Smith [BCS] via a generalization of the quotient construction [Cox] of simplicial toric varieties. A construction of toric stacks using logarithmic geometry can be found in [Iwa1]. The purpose of this paper is to compute the (orbifold) Chow ring with integer coefficients of a toric Deligne-Mumford stack. A toric Deligne-Mumford stack is defined in terms of a stacky fan $\Sigma = (N, \Sigma, \beta)$, where $N$ is a finitely generated abelian group, $\Sigma \subset N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ is a simplicial fan and $\beta : \mathbb{Z}^n \to N$ is a map determined by the elements $\{b_1, \ldots, b_n\}$ in $N$. By assumption, $\beta$ has finite cokernel and $\{\vec{b}_1, \ldots, \vec{b}_n\}$ generate the simplicial fan $\Sigma$, where $\vec{b}_i$ is the image of $b_i$ under the natural map $N \to N_{\mathbb{Q}}$. The toric Deligne-Mumford stack $X(\Sigma)$ associated to $\Sigma$ is defined to be the quotient stack $[Z/G]$, where $Z$ is the open subvariety $\mathbb{C}^n \setminus \bigcup(J_\Sigma)$, $J_\Sigma$ is the irrelevant ideal of the fan, $G$ is the product of an algebraic torus and a finite abelian group. The $G$-action on $Z$ is given via a group homomorphism $\alpha : G \to (\mathbb{C}^*)^n$, where $\alpha$ is obtained by taking $\text{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ functor to the Gale dual $\beta^\vee : \mathbb{Z}^n \to N^\vee$ of $\beta$ and $G = \text{Hom}_\mathbb{Z}(N^\vee, \mathbb{C}^*)$.

Each ray $\rho_i$ in the fan $\Sigma$ gives a line bundle $L_i$ over $X(\Sigma)$ which is defined by the quotient $Z \times \mathbb{C}/G$ and the action of $G$ on $\mathbb{C}$ is through the $i$-th component of the map $\alpha$. The Picard group of $X(\Sigma)$ is seen to be isomorphic to $N^\vee$.

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Every stacky fan $\Sigma$ has an underlying reduced stacky fan $\Sigma_{\text{red}} = (\overline{N}, \Sigma, \beta)$, where $\overline{N} = N/\text{torsion}$, $\beta: \mathbb{Z}^n \to \overline{N}$ is the natural projection given by the vectors $\{\overline{b}_1, \ldots, \overline{b}_n\} \subseteq \overline{N}$. The toric Deligne-Mumford stack $\mathcal{X}(\Sigma_{\text{red}})$ is a toric orbifold. By construction $\mathcal{X}(\Sigma_{\text{red}}) = [Z/G]$, where $G = \text{Hom}_\mathbb{Z}(\overline{N}', \mathbb{C}^*)$ and $\overline{N}'$ is the Gale dual $\overline{\beta}' : \mathbb{Z}^n \to \overline{N}'$ of the map $\overline{\beta}$. The stack $\mathcal{X}(\Sigma_{\text{red}})$ can be obtained by the rigidification construction (see e.g. [ACV]). Each ray $\rho_i$ in the fan $\Sigma$ also gives a line bundle $L_i$ over the toric orbifold which corresponds to the divisor $D_i$. This line bundle $L_i$ also has a quotient construction $\mathbb{Z} \times \mathbb{C}/\overline{G}$ where $\overline{G}$ acts on $\mathbb{C}$ via the $i$-th component of the map $\overline{\alpha}: \overline{G} \to \mathbb{C}^*$, obtained by taking $\text{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ to the map $\overline{\beta}'$.

Since $N$ is a finitely generated abelian group, we write it as the invariant factor form

$$N = \mathbb{Z}^d \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r},$$

where $m_1 | m_2 | \cdots | m_r$. Let $\Sigma$ be a stacky fan. Suppose that the map $\beta$ generates the torsion part of $N$, then $r \leq n - d$. We prove that the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is a nontrivial $\mu = \mu_{m_1} \times \cdots \times \mu_{m_r}$-gerbe over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$ obtained as the stack of roots of line bundles $M_i$. In the Picard group of the toric orbifold, there exist $n - d$ line bundles $M_1, \ldots, M_{n-d}$ such that the $\mathcal{X}(\Sigma)$ can be constructed as $\mu_{m_i}$-gerbes over the line bundle $M_i$ for $i = 1, \ldots, n - d$. These $n - d$ line bundles form the canonical generators of the Picard group.

Iwanari [Iwa1], [Iwa2] proved that the integral Chow ring of the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$ is isomorphic to the Stanley-Reisner ring $SR(\Sigma_{\text{red}})$ of the reduced stacky fan $\Sigma_{\text{red}}$, where

$$SR(\Sigma_{\text{red}}) := \frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{(I_\Sigma + C(\Sigma_{\text{red}}))},$$

the ideal $I_\Sigma$ is generated by

$$(1.2) \quad \{x_{i_1} \cdots x_{i_k} : \rho_{i_1} + \cdots + \rho_{i_k} \notin \Sigma\},$$

and $C(\Sigma_{\text{red}})$ is the ideal generated by linear relations:

$$(1.3) \quad \left(\sum_{i=1}^n \theta(b_i)x_i\right)_{\theta \in N^*}.$$

In the toric orbifold case we have that

$$\frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{(C(\Sigma_{\text{red}}))} \cong \overline{N'},$$

where $\overline{N'}$ is the Picard group of the toric orbifold. Let $t_1, \ldots, t_{n-d}$ be the canonical generators of the Picard group of $\mathcal{X}(\Sigma_{\text{red}})$. Then $t_i$'s generate $x_i$'s and $x_i$'s generate $t_i$'s, let $\mathbf{x} = (x_i), \mathbf{t} = (t_i)$ be column vectors, then there exist integral matrices $A = (a_{i,j})$ and $C = (c_{i,j})$ such that $\mathbf{x} = A\mathbf{t}$, and $\mathbf{t} = C\mathbf{x}$. Let $\tilde{\mathbf{x}} = (\tilde{x}_i)$
be a column vector and
\[ M = \begin{bmatrix} m_1 & \cdots & \cdot \cdot \cdot & m_{n-d} \end{bmatrix} \]
a diagonal matrix such that
\[ \tilde{x} := AMt = AMCx = Ex, \]
where \( E = AMC \) is a \( n \times n \) integral matrix. So every \( \tilde{x}_i \) is a integral linear combination of \( x_i \)'s. Let \( I_\Sigma \) be an ideal in \( \mathbb{Z}[x_i : \rho_i \in \Sigma(1)] \) generated by
\[ \{ \tilde{x}_{i_1} \cdots \tilde{x}_{i_k} : \rho_{i_1} + \cdots + \rho_{i_k} \notin \Sigma \}. \]
We define the Stanley-Reisner ring \( SR(\Sigma) \) of the stacky fan \( \Sigma \) as follows:
\[ SR(\Sigma) := \frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{(I_\Sigma + C(\Sigma))}, \]
where \( C(\Sigma) \) is the same as the ideal \( C(\Sigma_{\text{red}}) \) in (1.3).

Let \( A^*(\mathcal{X}(\Sigma), \mathbb{Z}) \) be the integral Chow ring of the toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) \). Then we have:

**Theorem 1.1.** Let \( \mathcal{X}(\Sigma) \) be a toric Deligne-Mumford stack with semi-projective coarse moduli space. Suppose that in the map \( \beta \) of the stacky fan \( \Sigma \), the vectors \( b_1, \cdots, b_n \) generate the torsion part of \( N \). Then there is an isomorphism of rings:
\[ A^*(\mathcal{X}(\Sigma), \mathbb{Z}) \cong SR(\Sigma). \]
where \( y \) is a formal variable and the ideal \( \mathcal{I}_\Sigma \) is the ideal generated by the elements in (1.3) replacing \( \tilde{x}_i \) by \( \tilde{y}^{b_i} \), i.e. by the elements

\[
\{ \tilde{y}^{b_1} \cdots \tilde{y}^{b_k} : \rho_{i_1} + \cdots + \rho_{i_k} \notin \Sigma \}.
\]

We define a ring

\[
\mathbb{Z}[\Sigma] = S_{\Sigma}[y^{v_1}, \ldots, y^{v_k}],
\]

which is a ring over the Stanley-Reisner ring. The product is defined as follows:

For any two \( v_1, v_2 \in \text{Box}(\Sigma) \), let \( v_1 + v_2 = \sum_{\rho_i \subseteq \sigma(v_1, v_2)} a_i b_j \) and let \( I \) be the set of rays \( \rho_i \) such that \( a_i > 1 \), \( J \) the set of rays \( \rho_j \) such that \( \rho_j \) belongs to \( \sigma(\overline{v}_1), \sigma(\overline{v}_2) \), but not \( \sigma(\overline{v}_3) \). Then

\[
y^{v_1} \cdot y^{v_2} := \begin{cases} 
\tilde{y}^{b_1} \prod_{i \in I} \tilde{y}^{b_i} \cdot \prod_{i \in J} \tilde{y}^{b_i} & \text{if there is a cone } \sigma \in \Sigma \text{ such that } \overline{v}_1 \in \sigma, \overline{v}_2 \in \sigma, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( \text{Cir}(\Sigma) \) be the ideal generated by the elements in (1.3) replacing \( x_i \) by \( y^{b_i} \), i.e. by the linear relations:

\[
\left( \sum_{i=1}^n \theta(b_i)y^{b_i} \right) \quad \theta \in \mathbb{N}^*.
\]

Let \( A^*_{\text{orb}}(\mathcal{X}(\Sigma), \mathbb{Z}) \) be the integral orbifold Chow ring of the toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) \). Then we have:

**Theorem 1.2.** Let \( \mathcal{X}(\Sigma) \) be a toric Deligne-Mumford stack associated to the stacky fan \( \Sigma \) such that the coarse moduli space is semi-projective and the map \( \beta \) generates the torsion part of \( N \) in the stacky fan \( \Sigma \). Then we have an isomorphism of graded rings:

\[
A^*_{\text{orb}}(\mathcal{X}(\Sigma), \mathbb{Z}) \cong \frac{\mathbb{Z}[\Sigma]}{\text{Cir}(\Sigma)}.
\]

This is the first nontrivial examples of the formula for integral orbifold Chow rings. Using Theorem 1.1, the proof of Theorem 1.2 is similar to [BCS], except that we work with integer coefficients.

This paper is organized as follows. In Section 2 we recall the construction of Gale duality for finitely generated abelian groups in [BCS] and use it to study toric Deligne-Mumford stacks. We give a construction of toric Deligne-Mumford stacks in this section. In Section 3 we discuss line bundles over toric Deligne-Mumford stacks and determine the Picard group of toric Deligne-Mumford stacks. In Section 4 we study the integral Chow ring of toric Deligne-Mumford stacks. We prove that the integral Chow ring of a toric Deligne-Mumford stack is isomorphic to the Stanley-Reisner ring of its stacky fan. We study the integral orbifold Chow ring in Section 5 and in Section 6 we compute some examples.
Conventions. In this paper we work entirely algebraically over the field of complex numbers. Chow rings and orbifold Chow rings are taken with integer coefficients. By an orbifold we mean a smooth Deligne-Mumford stack with trivial generic stabilizer.

We use $N^*$ to denote the dual $\text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ and $\mathbb{C}^*$ the multiplication group $\mathbb{C} - \{0\}$. We denote by $N \to \overline{N}$ the natural map modulo torsion. Since $N$ is a finitely generated abelian group, we write

$$N = \mathbb{Z}^d \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r},$$

where $m_1 | m_2 | \cdots | m_r$. This is called the invariant factor decomposition.

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2. Trivial and Nontrivial Gerbes

In this section we first recall the construction of Gale duality for finitely generated abelian groups. We use the properties of Gale duality to classify toric Deligne-Mumford stacks as trivial and nontrivial gerbes over the toric orbifolds.

2.1. Gale duality and toric Deligne-Mumford stacks. We recall the construction of Gale duality according to [BCS]. Let $N$ be a finitely generated abelian group with rank $d$. Let

$$\beta : \mathbb{Z}^n \to N$$

be a map determined by $n$ integral vectors $\{b_1, \cdots, b_n\}$ in $N$. Taking $\mathbb{Z}^n$ and $N$ as $\mathbb{Z}$-modules, from the homological algebra, there exist projective resolutions $\hat{E}$ and $\hat{F}$ of $\mathbb{Z}^n$ and $N$ satisfying the following diagram

$$\begin{array}{ccc}
\hat{E} & \longrightarrow & \mathbb{Z}^n \\
\downarrow & & \downarrow \beta \\
\hat{F} & \longrightarrow & N.
\end{array}$$

Let $\text{Cone}(\beta)$ be the mapping cone of the map between $\hat{E}$ and $\hat{F}$. Then we have an exact sequence of the mapping cone:

$$0 \longrightarrow \hat{F} \longrightarrow \text{Cone}(\beta) \longrightarrow \hat{E}[1] \longrightarrow 0,$$

where $\hat{E}[1]$ is the shifting of $\hat{E}$ by 1. Since $\hat{E}$ is projective as $\mathbb{Z}$-modules, so we have the exact sequence obtained by applying $\text{Hom}_\mathbb{Z}(-, \mathbb{Z})$:

$$0 \longrightarrow \hat{E}[1]^* \longrightarrow \text{Cone}(\beta)^* \longrightarrow \hat{F}^* \longrightarrow 0.$$

Taking cohomology of the above sequence we get the exact sequence

$$(2.1) \quad N^* \xrightarrow{\beta^*} (\mathbb{Z}^n)^* \xrightarrow{\partial^*} H^1(\text{Cone}(\beta)^*) \longrightarrow \text{Ext}_\mathbb{Z}^1(N, \mathbb{Z}) \longrightarrow 0.$$
**Definition 2.1.** Let $N^\vee = H^1(\text{Cone}(\beta)^*)$. The map

$$\beta^\vee : (\mathbb{Z}^n)^* \rightarrow N^\vee$$

is called the Gale dual of the map $\beta$.

According to [BCS], both $N^\vee$ and $\beta^\vee$ are well defined up to natural isomorphism.

This construction can be made more clear. Since $N$ has rank $d$ and $\mathbb{Z}^n$ is a free $\mathbb{Z}$-module, the projection resolutions can be chosen as:

$$0 \rightarrow \mathbb{Z}^n \rightarrow 0 = \hat{E},$$

$$0 \rightarrow \mathbb{Z}^r \overset{Q}{\rightarrow} \mathbb{Z}^{d+r} \rightarrow 0 = \hat{F},$$

where $Q$ is an integer matrix. Then there is a map $\mathbb{Z}^n \rightarrow \mathbb{Z}^{d+r}$ defined by a matrix $B$ which gives the map between $\hat{E}$ and $\hat{F}$. The mapping cone $\text{Cone}(\beta)$ is given by the following complex:

$$0 \rightarrow \mathbb{Z}^{n+r} \overset{[B,Q]}{\rightarrow} \mathbb{Z}^{d+r} \rightarrow 0 = \text{Cone}(\beta).$$

(2.1) is then obtained by applying the snake lemma to the following diagram

$$\begin{array}{cccccc}
0 & \overset{}{\longrightarrow} & 0 & \overset{}{\longrightarrow} & (\mathbb{Z}^{d+r})^* & \overset{}{\longrightarrow} & (\mathbb{Z}^{d+r})^* & \overset{}{\longrightarrow} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \overset{}{\longrightarrow} & (\mathbb{Z}^n)^* & \overset{}{\longrightarrow} & (\mathbb{Z}^{n+r})^* & \overset{}{\longrightarrow} & (\mathbb{Z}^r)^* & \overset{}{\longrightarrow} & 0
\end{array}$$

Then $N^\vee = (\mathbb{Z}^{n+r})^*/\text{Im}([B,Q]^*)$ and $\beta^\vee$ is the composite map of the inclusion $(\mathbb{Z}^n)^* \hookrightarrow (\mathbb{Z}^{n+r})^*$ and the quotient map $(\mathbb{Z}^{n+r})^* \rightarrow (\mathbb{Z}^{n+r})^*/\text{Im}([B,Q]^*)$.

**Remark 2.2.** If $N$ is free, i.e. there is no torsion part in the group $N$. Then by (2.2), the Gale dual $\beta^\vee$ is the quotient map $(\mathbb{Z}^n)^* \rightarrow (\mathbb{Z}^n)^*/\text{Im}([B]^*)$ and we have an exact sequence

$$N^* \xrightarrow{\beta^\vee} (\mathbb{Z}^n)^* \xrightarrow{\beta^\vee} H^1(\text{Cone}(\beta)^*) \rightarrow 0.$$
(2.5) \[ 1 \rightarrow \mu \rightarrow G \xrightarrow{\alpha} (\mathbb{C}^*)^n \rightarrow T \rightarrow 1, \]

where \( \mu = \text{Hom}_\mathbb{Z}(\text{Coker}(\beta^\vee), \mathbb{C}^*) \) is finite, \( G = \text{Hom}_\mathbb{Z}(\mathcal{N}^\vee, \mathbb{Z}^*) \) and \( T \) is the \( d \)-dimensional torus \((\mathbb{C}^*)^d\).

Let \( \mathbb{C}[z_1, \ldots, z_n] \) be the coordinate ring of the affine variety \( \mathbb{A}^n \). Associated to the simplicial fan \( \Sigma \), there is an irrelevant ideal \( J_\Sigma \) generated by the elements:

\[
(2.6) \quad \left\langle \prod_{\rho \not\in \sigma} z_i : \sigma \in \Sigma \right\rangle.
\]

Let \( Z := \mathbb{A}^n \setminus V(J_\Sigma) \). Then \( Z \) is a quasi-affine variety. The torus \((\mathbb{C}^*)^n\) acts on \( Z \) naturally since \( Z \) is the complement of coordinate subspaces. The algebraic group \( G \) acts on the variety \( Z \) through the map \( \alpha \) in (2.5). Then we have an action groupoid \( Z \times G \rightrightarrows Z \).

**Definition 2.4** ([BCS]). The toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) \) associated to the stacky fan \( \Sigma \) is defined to be the quotient stack \([Z/G]\).

Since \( N \) is a finitely generated abelian group of rank \( d \), we may write

\[ N = \mathbb{Z}^d \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}. \]

Then \( \overline{N} = \mathbb{Z}^d \), and

\[ \overline{\beta} : \mathbb{Z}^n \rightarrow \overline{N} \]

is given by \( \{\overline{b}_1, \ldots, \overline{b}_n\} \). So \( \Sigma_{\text{red}} := (\overline{N}, \Sigma, \overline{\beta}) \) is a stacky fan. In the exact sequence (2.5), let \( G = \text{Im}(\alpha) \), then we have an exact sequence of abelian groups

\[
1 \rightarrow \mu \rightarrow G \rightarrow \overline{G} \rightarrow 1.
\]

This is a central extension. By [DP], the quotient stack \([Z/G]\) is the \( \mu \)-gerbe over the quotient stack \([Z/\overline{G}] \) identified by this central extension.

**Remark 2.5.** The stack \( \mathcal{X}(\Sigma_{\text{red}}) \) can be constructed as follows. Consider the following exact sequences

\[
0 \rightarrow (\overline{N'})^* \rightarrow \mathbb{Z}^n \xrightarrow{\overline{\beta}} \overline{N} \rightarrow 0;\]

where \( \overline{\beta} \) is given by the vectors \( \{\overline{b}_1, \ldots, \overline{b}_n\} \), and

\[
(2.7) \quad 0 \rightarrow N^* \rightarrow (\mathbb{Z}^n)^* \xrightarrow{\overline{\beta'}} \overline{N'} \rightarrow 0.
\]

So \( A_{d-1}(\mathcal{X}(\Sigma)) = \overline{N'} \) and from the construction of Cox [Cox], \( \mathcal{X}(\Sigma_{\text{red}}) = [Z/\overline{G}] \), where \( \overline{G} = \text{Hom}_\mathbb{Z}(\overline{N'}, \mathbb{C}^*). \)
2.2. Construction of toric Deligne-Mumford stacks. It is known that every toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is a $\mu$-gerbe over the underlying toric orbifold for a finite abelian group $\mu$ and some finite abelian gerbes over $\mathcal{X}(\Sigma)$ are again toric Deligne-Mumford stacks, see [Jiang2]. In this section we classify trivial and nontrivial gerbes over toric orbifolds.

**Lemma 2.6.** Let $\mathbb{Z}^s$ and $\mathbb{Z}^t$ be two free abelian groups of ranks $s$ and $t$ respectively. Suppose that there is a map $\beta : \mathbb{Z}^s \to \mathbb{Z}^t$ which is given by an integral $t \times s$ matrix $A$. Then the dual map $\beta^* : (\mathbb{Z}^t)^* \to (\mathbb{Z}^s)^*$ is given by the transpose $A^t$ and

$$\text{coker}(\beta^*) \cong \ker(\beta) \oplus \text{coker}\beta.$$  

**Proof.** For simplicity, we assume that $\beta$ has finite cokernel and $s \geq t$. Since the matrix $A$ is an integer matrix, there exist invertible $s \times s$ integer matrix $P$ and an $t \times t$ integer matrix $P'$ such that $P'AP$ is the matrix

$$\begin{bmatrix}
    a_1 & 0 & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & a_t \\
\end{bmatrix},$$

with $a_1|\cdots|a_t$. This is the Smith normal form (see e.g. [Pra]). It follows that $\text{coker}(\beta) \cong \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_t}$. After taking dual we get that the map $\beta^* : (\mathbb{Z}^t)^* \to (\mathbb{Z}^s)^*$ is given by the matrix

$$\begin{bmatrix}
    a_1 & 0 \\
    \vdots & \ddots \\
    0 & a_t \\
    0 & \cdots & 0 \\
\end{bmatrix}.$$  

So it is easy to see that $\text{coker}(\beta^*) = \mathbb{Z}^{s-t} \oplus \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_t}$. Since the kernel of $\beta$ is isomorphic to $\mathbb{Z}^{s-t}$, we complete the proof. $\square$

**Lemma 2.7.** Let $\Sigma$ be a stacky fan. If the vectors $\{b_1, \cdots, b_n\}$ generate the torsion part $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}$ of $N$, then $r \leq n - d$ and $N^\vee \cong N'^\vee$.

**Proof.** From the following diagram

\[
\begin{array}{c}
\mathbb{Z}^d \\
\downarrow \beta \\
\mathbb{Z}^n \\
\downarrow \beta \\\n0
\end{array}
\]

It is easy to see that if $\beta$ generate the torsion part of $N$, then $r \leq n - d$.

By the construction of Gale duality in (2.2), $N^\vee \cong (\mathbb{Z}^{n-r})^*/\text{Im}([B, Q]^*).$ Since $\overline{N}$ is free, we have $\overline{N}^\vee \cong (\mathbb{Z}^n)^*/\text{Im}(\bar{[B, Q]}^*)$. The map $[B, Q] : \mathbb{Z}^{n+r} \to \mathbb{Z}^{d+r}$ in the mapping cone $\text{Cone}(\beta)$ has the same cokernel as the map $\beta$. Since the map $\beta$ generate the torsion part of $N$, the map $\beta$ has the same cokernel as the map $\beta$.  

\[\text{coker}(\beta^*) \cong \ker(\beta) \oplus \text{coker}\beta.\]
Thus the map $[B, Q]$ has the same cokernel as the map $[\overline{B}] : \mathbb{Z}^n \to \mathbb{Z}^d$. Also, $\ker[B, Q] \simeq \mathbb{Z}^{n-d} \simeq \ker[\overline{B}]$. By Lemma 2.6 we have

$$N^\vee = \text{coker}[B, Q]^* \simeq \text{coker}[\overline{B}]^* = \overline{N}^\vee.$$ 

\[ \square \]

Let $\Sigma$ be the stacky fan and $\Sigma_{\text{red}}$ the corresponding reduced stacky fan. Consider the following diagram

$$\begin{array}{c}
\mathbb{Z}^n \xrightarrow{\beta} N \\
\downarrow \quad \downarrow \\
\mathbb{Z}^n \xrightarrow{\overline{\beta}} \overline{N}.
\end{array}$$

Taking Gale dual yields

$$\begin{array}{c}
0 \longrightarrow \overline{N}^* \longrightarrow \mathbb{Z}^n \longrightarrow \overline{N}^\vee \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow N^* \longrightarrow \mathbb{Z}^n \longrightarrow N^\vee \longrightarrow \text{coker}(\overline{\beta}^\vee) \longrightarrow 0.
\end{array} \tag{2.8}$$

**Lemma 2.8.** Let $\Sigma = (N, \Sigma, \beta)$ be a stacky fan. If the vectors $\{b_1, \ldots, b_n\}$ in the map $\beta$ generate the torsion part $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}$ of $N$, then the map $\varphi$ in (2.8) is diagonalizable over integers.

**Proof.** First in the reduced stacky fan $\Sigma_{\text{red}}$, the map $\overline{\beta} : \mathbb{Z}^n \to \mathbb{Z}^d$ is given by $\{\overline{b}_1, \ldots, \overline{b}_n\}$. Consider the following diagram

$$\begin{array}{c}
0 \longrightarrow \mathbb{Z}^n \xrightarrow{id} \mathbb{Z}^n \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r} \longrightarrow N \longrightarrow \overline{N} \longrightarrow 0.
\end{array} \tag{2.9}$$

From the definition of Gale dual in Section 2.1, we have the morphisms of mapping cones $\text{Cone}(\overline{\beta})$, $\text{Cone}(\beta)$ and $\text{Cone}(\overline{\beta})$:

$$\begin{array}{c}
0 \longrightarrow \mathbb{Z}^r \longrightarrow \mathbb{Z}^r \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow \mathbb{Z}^{n+r} \longrightarrow \mathbb{Z}^{d+r} \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^d \longrightarrow 0.
\end{array} \tag{2.10}$$
where $\overline{Q}$ is the diagonal matrix in $Q$. Dualizing gives the following diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (Z^d)^* & \xrightarrow{\overline{p}} & (Z^n)^* & \xrightarrow{\overline{p}'} & N^\vee & \rightarrow & 0 \\
\downarrow & & \downarrow i & & \downarrow & & \downarrow \varphi & & \\
0 & \rightarrow & (Z^{d+r})^* & \xrightarrow{[B,Q]^*} & (Z^{n+r})^* & \xrightarrow{\pi} & N^\vee & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (Z^r)^* & \xrightarrow{\overline{Q}} & (Z^r)^* & \rightarrow & \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r} & \rightarrow & 0,
\end{array}
\]

where $\pi \circ i = \beta^\vee$. Since the map $\varphi$ is induced from the map $i$ in (2.11), it is given by an integer matrix $A$. Then from the general fact in the finitely generated group theory there exist integer matrices $P, P'$ such that $PAP'$ is a diagonal matrix with entries $n_1, \ldots, n_s, 0, \ldots, 0$ which satisfy the condition $n_1 | \cdots | n_s$. This is again the Smith normal form. From the diagram (2.11) the third column is exact and the cokernel is $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}$, so the diagonal matrix given by $\varphi$ is of the form

\[
\begin{bmatrix}
1 \\
\vdots \\
1 \\
\vdots \\
m_1 \\
\vdots \\
m_r
\end{bmatrix}.
\]

\[\square\]

**Proposition 2.9.** Let $\Sigma = (N, \Sigma, \beta)$ be a stacky fan and $\mu = \mu_{m_1} \times \cdots \times \mu_{m_r}$. If the vectors $\{b_1, \ldots, b_n\}$ in the map $\beta$ generate the torsion part $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}$ of $N$, then $X(\Sigma)$ is a nontrivial $\mu$-gerbe over the toric orbifold $X(\Sigma_{\text{red}})$.

**Proof.** Applying Hom$_\mathbb{Z}(-, \mathbb{C}^*)$ to the diagram (2.8) yields

\[
\begin{array}{ccccccccc}
1 & \rightarrow & \mu & \rightarrow & G & \xrightarrow{\alpha} & (\mathbb{C}^*)^n & \rightarrow & T & \rightarrow & 1 \\
\downarrow & & \downarrow \alpha(\varphi) & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & 1 & \rightarrow & G & \xrightarrow{\pi} & (\mathbb{C}^*)^n & \rightarrow & T & \rightarrow & 1,
\end{array}
\]
where the map $\alpha(\varphi)$ is given by the diagonal matrix:

$$
\begin{pmatrix}
(\cdot)^1 \\
\vdots \\
(\cdot)^1 \\
(\cdot)^{m_1} \\
\vdots \\
(\cdot)^{m_r}
\end{pmatrix},
$$

since by Lemma 2.8 the map $\varphi$ in (2.8) is diagonalizable. By Lemma 2.7, $N^\vee \cong \overline{N^\vee}$ and so $G \cong \overline{G}$. So we get an exact sequence

$$
1 \rightarrow \mu \xrightarrow{\varphi} G \rightarrow \overline{G} \rightarrow 1
$$

which is a central extension, where $\mu = \text{Hom}_{\mathbb{Z}}(\text{cok}(\beta^\vee), \mathbb{C}^*) = \mu_{m_1} \times \cdots \times \mu_{m_r}$. We can decompose the left side of the diagram (2.12) according to (2.13) to get the following diagram for each $\mu_{m_i}$:

$$
\begin{array}{cccccc}
1 & \rightarrow & \mu_{m_i} & \rightarrow & \mathbb{C}^* & \rightarrow & (\mathbb{C}^*)^n & \rightarrow & T & \rightarrow & 1 \\
\downarrow & & \downarrow & & (\cdot)^{m_i} & & \downarrow & & \downarrow & & \\
1 & \rightarrow & 1 & \rightarrow & \mathbb{C}^* & \rightarrow & (\mathbb{C}^*)^n & \rightarrow & T & \rightarrow & 1.
\end{array}
$$

The corresponding component $\mathbb{C}^* \subset G$ determines a line bundle $M_i$ over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$. The central extension (2.14) is nontrivial, because $\overline{G} \cong G$ and the map $\varphi$ in (2.14) is nontrivial on each component. So from the definition of gerbes, the quotient stack $\mathcal{X}(\Sigma) = [\mathbb{Z}/G]$ is a nontrivial $\mu$-gerbe over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}}) = [\mathbb{Z}/\overline{G}]$ coming from the direct sum of line bundles $\bigoplus_i M_i$.

**Remark 2.10.** Recall that the stack of roots of a line bundle can be constructed as follows. More details can be found in Appendix B of [AGV2], or [Ca]. Let $L$ be a line bundle over a variety (or an Artin stack) $X$. Let $m$ be a positive integer and consider the Kummer exact sequence:

$$
1 \rightarrow \mu_m \rightarrow \mathbb{C}^* \rightarrow \mathbb{C}^* \rightarrow 1.
$$

Then we have the quotient stack $[\mathbb{C}^*/\mathbb{C}^*] \cong B\mu_m$, where the action is given by $\lambda \cdot x = \lambda^m x$. Let $L^*$ be the complement of the zero section in the total space of $L$. The following twist

$$
\sqrt[m]{L} := [L^*/\mathbb{C}^*] = [L^* \times_{\mathbb{C}^*} \mathbb{C}^*/\mathbb{C}^*],
$$

is the $\mu_m$-gerbe over $X$ coming from the line bundle $L$, which is called the stack of $m$-th root of $L$. $\sqrt[m]{L}$ may be viewed as a toric stack bundle in the sense of [Jiang1] since the stack $[\mathbb{C}^*/\mathbb{C}^*]$ is a toric Deligne-Mumford stack.

In the proof of Proposition 2.9, the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is obtained by applying a sequence of root constructions to line bundles $M_i$ over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$, i.e.

$$
\mathcal{X}(\Sigma) \cong \sqrt[m_1]{M_1} \times_{\mathcal{X}(\Sigma_{\text{red}})} \cdots \times_{\mathcal{X}(\Sigma_{\text{red}})} \sqrt[m_r]{M_r}.
$$
Similar descriptions have also been obtained independently by F. Perroni [Perroni].

**Example 2.11.** Let \( N = \mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \), and \( \beta : \mathbb{Z}^4 \to N \) be given by the vectors \\
\[ \{ b_1 = (1,0,1,0), b_2 = (0,1,0,0), b_3 = (-1,2,0,0), b_4 = (0,-1,0,1) \} \]
Then \( \overline{N} = \mathbb{Z}^2 \) and let \( \Sigma \) be the complete fan in \( \mathbb{R}^2 \) generated by \( \{ \overline{b}_1, \overline{b}_2, \overline{b}_3, \overline{b}_4 \} \). The fan \( \Sigma \) is the fan of Hirzebruch surface \( \mathbb{F}_2 \). Then \( \Sigma = (N, \Sigma, \beta) \) is a stacky fan. We compute that \( \beta^\vee : \mathbb{Z}^4 \to N^\vee = \mathbb{Z}^2 \) is given by the matrix
\[
\begin{bmatrix}
2 & -4 & 2 & 0 \\
0 & 4 & 0 & 4
\end{bmatrix}
\]
The reduced stacky fan is \( \Sigma_{\text{red}} = (\mathbb{Z}^2, \Sigma, \overline{\beta}) \), where \( \overline{\beta} : \mathbb{Z}^4 \to \mathbb{Z}^2 \) is given by \( \{ \overline{b}_1, \overline{b}_2, \overline{b}_3, \overline{b}_4 \} \). The Gale dual \( \overline{\beta}^\vee : \mathbb{Z}^4 \to N^\vee = \mathbb{Z}^2 \) is given by the matrix
\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix}
\]
So in this example the diagram (2.16) is:
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^4 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \downarrow \ast & & \overline{\beta}^\vee & & \downarrow \varphi & & \downarrow \ast & & \downarrow \ast \\
0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^4 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_4 & \longrightarrow & 0,
\end{array}
\]
where \( \varphi \) is the diagonal matrix
\[
\begin{bmatrix}
2 & 0 \\
0 & 4
\end{bmatrix}
\]
So from (2.16) we get the following exact sequence:
\[
\begin{array}{cccccc}
1 & \longrightarrow & \mu_2 \times \mu_4 & \longrightarrow & (\mathbb{C}^*)^2 & \longrightarrow & (\mathbb{C}^*)^4 & \longrightarrow & (\mathbb{C}^*)^2 & \longrightarrow & 1 \\
& & \downarrow \alpha(\varphi) & & \downarrow \ast & & \downarrow \ast & & \downarrow \ast & & \downarrow \ast \\
1 & \longrightarrow & 1 & \longrightarrow & (\mathbb{C}^*)^2 & \longrightarrow & (\mathbb{C}^*)^4 & \longrightarrow & (\mathbb{C}^*)^2 & \longrightarrow & 1.
\end{array}
\]
The toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) = [\mathbb{C}^4 - V(J_\Sigma)]/(\mathbb{C}^*)^2 \), where the action is given by the transpose of the matrix \( \beta^\vee \). Let \( L_1, L_2 \) be the two line bundles over \( \mathbb{F}_2 \) which are the two canonical generators in the Picard group \( \mathbb{Z}^2 \) of \( \mathbb{F}_2 \). The toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) \) is a \( \mu_2 \times \mu_4 \)-gerbe over the Hirzebruch surface \( \mathcal{X}(\Sigma_{\text{red}}) = \mathbb{F}_2 \) coming from the direct sum of line bundles \( L_1 \oplus L_2 \). \( \mathcal{X}(\Sigma) \) can be constructed by taking square root and quartic root of line bundles \( L_1 \) and \( L_2 \) respectively: \( \mathcal{X}(\Sigma) \cong \sqrt{L_1} \times_{\mathbb{F}_2} \sqrt{L_2} \).

**Remark 2.12.** In the map \( \beta : \mathbb{Z}^n \to N \), if the components of the torsion part of \( b_i \) are zero, then it is easy to check that \( N^\vee \cong \overline{N} \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r} \) and \( \mathcal{X}(\Sigma) = \mathcal{X}(\Sigma_{\text{red}}) \times \mathcal{B}_\mu \) with \( \mu = \mu_{m_1} \times \cdots \times \mu_{m_r} \).

**Proposition 2.13.** Every toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) \) has a decomposition:
\[
\mathcal{X}(\Sigma) \cong \mathcal{X}(\Sigma') \times \mathcal{B}_\mu,
\]
where \( \mathcal{X}(\Sigma') \) is a nontrivial gerbe over the toric orbifold \( \mathcal{X}(\Sigma_{\text{red}}) \) and \( B\mu \) is the classifying stack for a finite abelian group \( \mu \).

**Proof.** Consider the map \( \beta : \mathbb{Z}^n \to N \) given by integral vectors \( \{b_1, \ldots, b_n\} \subseteq N \). Let \( N = \mathbb{Z}^d \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r} \) and \( N_{tor} = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r} \). Then there exists a subgroup \( N'_{tor} \subseteq N_{tor} \) such that that \( \{b_1, \ldots, b_n\} \) generate \( N'_{tor} \). Let \( N' = \mathbb{Z}^d \oplus N'_{tor} \) and let \( \beta' : \mathbb{Z}^n \to N' \) be the map given by \( \{b_1, \ldots, b_n\} \). Then \( \Sigma' = (N', \Sigma, \beta') \) is a stacky fan such that \( \{b_1, \ldots, b_n\} \) generate the torsion part of \( N' \).

Let \( \Sigma_{\text{red}} \) be the corresponding reduced stacky fan. Then by Lemma 2.7 in the Gale dual \( (\beta')^\vee : \mathbb{Z}^n \to (N')^\vee \) and \( \beta^\vee : \mathbb{Z}^n \to (N)^\vee \), \( (N')^\vee \cong (N)^\vee \). Since the cokernel of the map \( \beta \) is

\[
cok(\beta') \oplus N_{tor}/N'_{tor} \cong cok(\beta') \oplus N_{tor}/N'_{tor},
\]

by Lemma 2.6 the Gale dual map of \( \beta \) is

\[
\beta^\vee : \mathbb{Z}^n \to (N')^\vee \oplus N_{tor}/N'_{tor}.
\]

Let \( \mu = N_{tor}/N'_{tor} \). By Lemma 2.9 and Remark 2.12 the proposition follows. \( \square \)

### 3. Line Bundles over Toric Deligne-Mumford Stacks

In this section we prove that the Picard group of a toric Deligne-Mumford stacks is isomorphic to \( N^\vee \) in the Gale dual \( \beta^\vee : \mathbb{Z}^n \to N^\vee \) of \( \beta : \mathbb{Z}^n \to N \).

Let \( \mathcal{X}(\Sigma) = [Z/G] \) be a toric Deligne-Mumford stack associated to the stacky fan \( \Sigma = (N, \Sigma, \beta) \). In the case of a quotient stack, a line bundle \( L \) on \( \mathcal{X}(\Sigma) \) is a \( G \)-equivariant line bundle \( L \) on \( Z \). A \( G \)-equivariant line bundle on \( Z \) is a line bundle \( L \) on \( Z \) together with an isomorphism \( \varphi : pr^*L \to \mu^*L \), where \( pr : G \times Z \to Z \) is the projection and \( \mu : G \times Z \to Z \) is the action. Let \( Pic(\mathcal{X}(\Sigma)) \) denote the Picard group of \( \mathcal{X}(\Sigma) \).

**Proposition 3.1.** \( Pic(\mathcal{X}(\Sigma)) \cong N^\vee \).

**Proof.** Since \( G = \text{Hom}_{\mathbb{Z}}(N^\vee, \mathbb{C}^*) \), Pontryagin duality implies that the character group \( G^\vee \) of \( G \) is isomorphic to \( N^\vee \).

As discussed above, \( \mathcal{X}(\Sigma) = [Z/G] \) where \( Z = \mathbb{A}^n \setminus \mathbb{V}(J_\Sigma) \) with codim \( \mathbb{V}(J_\Sigma) \geq 2 \). This implies that \( Pic(Z) \cong Pic(\mathbb{A}^n) \cong \mathbb{Z} \), generated by the trivial line bundle.

Let \( \mathcal{L} \in Pic(\mathcal{X}(\Sigma)) \), then \( \mathcal{L} \) corresponds to a \( G \)-equivariant line bundle \( L \to Z \), which is trivial but not necessarily \( G \)-equivariantly trivial. The \( G \)-action on \( L \) is given by a character \( \chi_L : G \to \mathbb{C}^* \). Thus there is a map

\[
Pic(\mathcal{X}(\Sigma)) \to G^\vee, \quad \mathcal{L} \to \chi_L.
\]

On the other hand, any character \( \chi : G \to \mathbb{C}^* \) gives a \( G \)-equivariant line bundle \( L_\chi \) over \( Z \), hence a line bundle on \( \mathcal{X}(\Sigma) \). The result follows. \( \square \)

From the results in [EG], [Kr] that the Picard group of a quotient stack is isomorphic to its first Chow group, we have:

**Corollary 3.2.** There is an isomorphism \( A^1(\mathcal{X}(\Sigma), \mathbb{Z}) \cong N^\vee \).
For each ray \( \rho_i \), we define a line bundle \( L_i \) over \( \mathcal{X}(\Sigma) \) to be the trivial line bundle \( \mathbb{C} \) with the \( G \) action on \( \mathbb{C} \) given by the \( i \)-th component of the map \( \alpha : G \to (\mathbb{C}^*)^n \). We can similarly define a line bundle \( L_i \) over the toric orbifold \( \mathcal{X}(\Sigma_{\text{red}}) \). This line bundle \( L_i \) is the trivial line bundle \( \mathbb{C} \) over \( \mathbb{Z} \) on which the action of \( G \) is through the \( i \)-th component of the map \( \alpha : G \to (\mathbb{C}^*)^n \), where \( \alpha \) is obtained by taking \( \text{Hom}_\mathbb{Z}(-, \mathbb{C}^*) \) to the map \( \beta^\vee \) in (2.7).

For the stacky fan \( \Sigma \) and its reduced stacky fan \( \Sigma_{\text{red}} \), we consider again the diagram (2.8). Since we have that \( \text{cok}(\beta^\vee) = \mathbb{Z} m_1 \oplus \cdots \oplus \mathbb{Z} m_{n-d} \) by Lemma 2.8, the map \( \varphi \) is given by the diagonal matrix

\[
\begin{pmatrix}
m_1 \\
\vdots \\
m_{n-d}
\end{pmatrix},
\]

where some \( m_i \)'s are 1 since \( r \leq n-d \).

Let \( t := \{ t_1, \ldots, t_{n-d} \} \) be the generators of \( \overline{\mathbb{N}}^\vee \) so that the map \( \varphi \) is diagonalized. Let \( \{ e_1, \ldots, e_n \} \) be the standard basis of \( \mathbb{Z}^n \). Let \( x = (x_i) \) and \( t = (t_i) \) be column vectors, then there exist a matrix \( A \) such that \( x = At \). Then under these bases the map \( \overline{\beta}^\vee \) is given by the matrix \( A^t \). Suppose that the map \( \beta^\vee \) is given by a matrix \( B \), then we have \( B = MA^t \). Since \( \overline{\beta}^\vee \) is surjective, there exists an integral matrix \( C \) such that \( t = Cx \), where \( x := (x_1, \ldots, x_n)^t \) with \( x_j := Ae_j \). Let \( \tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_n)^t \) be defined by

\[
(3.1) \quad \tilde{x} = AMt = AMCx.
\]

Then every \( \tilde{x}_i \) is an integral linear combination of \( x_i \)'s by the above formula. We call the formula (3.1) the associated formula of the stacky fan \( \Sigma \).

Let \( \pi : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma_{\text{red}}) \) be the rigidification map. For the line bundle \( L_j \) over \( \mathcal{X}(\Sigma_{\text{red}}) \) corresponding to ray \( \rho_j \), we have the pullback \( \pi^*(L_j) = L_j \) over \( \mathcal{X}(\Sigma) \). From Lemma 2.8 we have the morphism \( \varphi : \overline{\mathbb{N}}^\vee \to \mathbb{N}^\vee \) after choosing the basis of \( \overline{\mathbb{N}}^\vee \cong \mathbb{N}^\vee \) which is diagonalizable. Let \( id : \overline{\mathbb{N}}^\vee \to \mathbb{N}^\vee \) be the identity morphism under the basis, dualizing we an isomorphism

\[
(3.1) \quad id : G \cong \overline{\mathbb{G}}.
\]

Under this isomorphism, we can take \( L_j \) as a line bundle over \( \mathcal{X}(\Sigma) \) using the character \( \overline{\alpha}_j : \overline{\mathbb{G}} \to \mathbb{C} \) in the \( j \)-th component of the map \( \overline{\alpha} \). Suppose that in the formula (3.1),

\[
\tilde{x}_i = \sum_{j=1}^n a_{j,i} x_j,
\]

where \( a_{1,i}, \ldots, a_{n,i} \) are integers, then we have the following proposition.

**Proposition 3.3.** \( \mathcal{L}_i = \bigotimes_j L_j^{a_{j,i}} \).

**Proof.** From the construction of the line bundle \( \mathcal{L}_i \) on the toric Deligne-Mumford stack the proposition can be easily proved from the diagram (2.8) and the formula (3.1). \( \square \)
4. Integral Chow Ring of Toric Deligne-Mumford Stacks

In this section we study the integral Chow ring of toric Deligne-Mumford stacks. For references of integral Chow ring of stacks, see [EG] and [Kr]. In this section, every Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is semi-projective satisfying the condition of Lemma 2.9. We use the results in [Iwa1], [Iwa2] concerning the integral Chow ring of a simplicial toric orbifold.

4.1. The Chow ring of stack of roots of a line bundle. In this section we compute the Chow ring with integer coefficients of the stack $\sqrt[m]{L}$ of $m$-th root of a line bundle $L$.

Let $\mathbb{C}^*$ act on the total space of $L$ by acting trivially on the base $X$ and acting by $\lambda \cdot x := \lambda^m x$ on the fiber. As recalled in Remark 2.10, $\sqrt[m]{L}$ is the stack quotient $[L^*/\mathbb{C}^*]$ with respect to this $\mathbb{C}^*$-action.

Let $i : X \hookrightarrow L$ be the inclusion of the zero section, and $j : L \setminus X \hookrightarrow L$ the inclusion of its complement. Then we have an exact sequence on the $\mathbb{C}^*$-equivariant Chow groups:

$$A_*(X)_{\mathbb{C}^*} \xrightarrow{i_*} A_*(L)_{\mathbb{C}^*} \xrightarrow{j^*} A_*(L \setminus X)_{\mathbb{C}^*} \rightarrow 0,$$

where $i_*$ is the pushforward and $j^*$ is the flat pullback. By [EG],

$$A^*[L^*/\mathbb{C}^*]] = A^*_{\mathbb{C}^*}(L \setminus X).$$

Let $\pi : L \rightarrow X$ be the structure map of the line bundle. Then we have

$$\pi^* : A_*(X)_{\mathbb{C}^*} \xrightarrow{\cong} A_{*-1}(L)_{\mathbb{C}^*}.$$

Let

$$b := (\pi^*)^{-1} \circ i_* : A_*(X)_{\mathbb{C}^*} \rightarrow A_{*-1}(X)_{\mathbb{C}^*}.$$  

We may rewrite the sequence (4.1) as

$$A_*(X)_{\mathbb{C}^*} \xrightarrow{b} A_{*-1}(X)_{\mathbb{C}^*} \xrightarrow{j^*} A_*(L \setminus X)_{\mathbb{C}^*} \rightarrow 0.$$  

This implies that

$$A_*([L^*/\mathbb{C}^*]) \cong A_{*-1}(X)_{\mathbb{C}^*}/\text{Im}(b).$$

Clearly the map $b$ is given by the $\mathbb{C}^*$-equivariant first Chern class of the line bundle $L$, which is $c_1(L) - mt$, where $c_1(L)$ is the non-equivariant first Chern class of $L$, and $t$ is the equivariant parameter. Thus we have the following proposition:

**Proposition 4.1.** The Chow ring $A^*([L^*/\mathbb{C}^*])$ is isomorphic to the quotient ring $A^*(X)[t]/(c_1(L) - mt)$.

**Proof.** Since $\mathbb{C}^*$ acts trivially on $X$, we have $A_*(X)_{\mathbb{C}^*} = A_*(X)[t]$. Since $c_1(L) - mt$ is not a zero divisor, (4.3) is exact on the left, and the image of $b$ is the ideal generated by $c_1(L) - mt$. □
4.2. Integral Chow ring of toric Deligne-Mumford stacks. Let \( \Sigma \) be a stacky fan such that the map \( \beta \) generate the torsion part of the group \( N \) and \( \mathcal{X}(\Sigma) \) the associated toric Deligne-Mumford stack. Let \( \mathcal{X}_{\text{orb}}(\Sigma) \) be the underlying toric orbifold given by the simplicial fan \( \Sigma \). Let

\[
\mathbb{Z}[x_i : \rho_i \in \Sigma(1)] \quad \frac{(I_\Sigma + \text{Cir}(\Sigma))}{(I_\Sigma + \text{Cir}(\Sigma))}
\]

be the Stanley-Reisner ring of the fan \( \Sigma \), where \( x_i \) corresponds to the torus invariant divisor \( D_{\rho_i} \), \( \text{Cir}(\Sigma) \) is the ideal generated by the linear relations:

\[
\left( \sum_{\rho_i \in \Sigma} \theta(v_i) x_i \right)_{\theta \in \mathbb{N}^*},
\]

where \( v_i \) is the first lattice point of the ray \( \rho_i \), and \( I_\Sigma \) is the ideal generated by square-free monomials

\[
\{ x_{i_1} \cdots x_{i_k} : \rho_{i_1} + \cdots + \rho_{i_k} \text{ is not a cone } \sigma \in \Sigma \}
\]
in (1.2).

Proposition 4.2 ([Iwa1]). The integral Chow ring \( A^*(\mathcal{X}_{\text{orb}}(\Sigma), \mathbb{Z}) \) is isomorphic to (4.5).

Consider \( \Sigma_{\text{red}} = (N, \Sigma, \overline{\beta}) \), where \( \overline{\beta} : \mathbb{Z}^n \rightarrow \overline{N} \) is given by the vectors \( \{ \overline{b}_1, \cdots, \overline{b}_n \} \), and the toric Deligne-Mumford stack \( \mathcal{X}(\Sigma_{\text{red}}) = [\mathbb{Z}/G] \). Let \( C(\Sigma_{\text{red}}) \) is the ideal generated by the linear relations:

\[
\left( \sum_{\rho_i \in \Sigma} \theta(b_i) x_i \right)_{\theta \in \mathbb{N}^*}
\]
in (1.3).

Proposition 4.3 ([Iwa2]). There is an isomorphism of rings:

\[
A^*(\mathcal{X}(\Sigma_{\text{red}}), \mathbb{Z}) \simeq \frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{(I_\Sigma + \text{C}(\Sigma_{\text{red}}))}.
\]

Remark 4.4. Let \( \Sigma = (N, \Sigma, \beta) \) be a stacky fan. For the simplicial fan \( \Sigma \), the toric orbifold \( \mathcal{X}_{\text{orb}}(\Sigma) \) associated to \( \Sigma \) has stack structures in codimension at least 2. The toric orbifold \( \mathcal{X}(\Sigma_{\text{red}}) \) has stack structures in codimension at least 1. The toric orbifold \( \mathcal{X}(\Sigma_{\text{red}}) \) can be obtained from \( \mathcal{X}_{\text{orb}}(\Sigma) \) by taking roots of divisors. Since we don’t need this result here, we omit the details.

In view of Proposition [2.9], the idea of proving Theorem 1.1 is to compute the integral Chow ring of \( \mathcal{X}(\Sigma) \) by combining Propositions 4.3 and 4.1.

Proof of Theorem 1.1. As in the proof of Proposition 2.9, let \( M_i \rightarrow \mathcal{X}(\Sigma_{\text{red}}) \) be the line bundle over the toric orbifold given by one generator in \( \overline{N}^\lor \) and Let \( m_i \) be the corresponding positive integer. The toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) \) is a nontrivial \( \mu_{m_1} \times \cdots \times \mu_{m_r} \)-gerbe over the toric orbifold \( \mathcal{X}(\Sigma_{\text{red}}) \) obtained
from a sequence of root gerbe constructions determined by the line bundles $M_i$ for $1 \leq i \leq n - d$.

Since the integral Chow ring $A^*(\mathcal{X}(\Sigma_{\text{red}}), \mathbb{Z})$ is generated by the Picard group $N^\vee$ and $\frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{C(\Sigma_{\text{red}})} \cong N^\vee$, so by Proposition 4.3

$$A^*(\mathcal{X}(\Sigma_{\text{red}}); \mathbb{Z}) \cong \frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{(I_{\Sigma_{\text{red}}} + C(\Sigma_{\text{red}}))} \cong \frac{\mathbb{Z}[t_1, \ldots, t_{n-d}]}{I_{\Sigma_{\text{red}}}},$$

where $\{t_1, \ldots, t_{n-d}\}$ is a basis of $N^\vee$, the ideal $I_{\Sigma_{\text{red}}}$ is obtained from $I_{\Sigma_{\text{red}}}$ by expressing $x_i$'s in terms of $t_i$'s. By Proposition 1.1 the Chow ring $A^*(\mathcal{X}(\Sigma), \mathbb{Z})$ is isomorphic to the ring obtained from $\frac{\mathbb{Z}[t_1, \ldots, t_{n-d}]}{I_{\Sigma_{\text{red}}}}$ by replacing the canonical generators $\{t_1, \ldots, t_{n-d}\}$ by $\{m_1, \ldots, m_{n-d}\}$. In view of (3.1), this is isomorphic to the Stanley-Reisner ring $SR(\Sigma)$ of the stacky fan $\Sigma$ since in the ring $SR(\Sigma)$, the ideal $I_{\Sigma}$ is obtained from the ideal $I_{\Sigma_{\text{red}}}$ in (1.5) replacing $x_i$ by $\tilde{x}_i$ for each ray $\rho_i$. $\square$

By Proposition 2.13 every toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ has a decomposition: $\mathcal{X}(\Sigma) \cong \mathcal{X}(\Sigma') \times B\mu$, where $\mathcal{X}(\Sigma')$ is a nontrivial gerbe over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$ and $B\mu$ is the classifying stack of a finite abelian group $\mu = \mu_{r_1} \times \cdots \times \mu_{r_s}$. It is known that

$$A^*(B\mu, \mathbb{Z}) \cong \mathbb{Z}[t_1, \ldots, t_s]/(r_1 t_1, \ldots, r_s t_s).$$

Thus we have

**Proposition 4.5.** The integral Chow ring of $\mathcal{X}(\Sigma)$ is given by

$$A^*(\mathcal{X}(\Sigma), \mathbb{Z}) \cong A^*(\mathcal{X}(\Sigma'), \mathbb{Z})[t_1, \ldots, t_s]/(r_1 t_1, \ldots, r_s t_s).$$

5. The Integral Orbifold Chow Ring

In this section we compute the integral orbifold Chow ring of toric Deligne-Mumford stacks.

5.1. The inertia stack. Let $\mathcal{X}(\Sigma)$ be a toric Deligne-Mumford stack associated to the stacky fan $\Sigma = (N, \Sigma, \beta)$. For a cone $\sigma$ in the simplicial fan $\Sigma$, let $\text{link}(\sigma) = \{b_i : \rho_i + \sigma \text{ is a cone in } \Sigma\}$. Then we have a quotient stacky fan $\Sigma/\sigma = (N(\sigma), \Sigma/\sigma, \beta(\sigma))$, where

$$\beta(\sigma) : \mathbb{Z}^l \rightarrow N(\sigma)$$

is given by the images of $\{b_i\}$'s in $\text{link}(\sigma)$. Let $m := |\sigma|$, then $\dim(N_{\sigma}) = m$ since $\sigma$ is simplicial. Consider the commutative diagrams

$$
\begin{array}{c}
0 \rightarrow \mathbb{Z}^{l+m} \rightarrow \mathbb{Z}^n \rightarrow \mathbb{Z}^{n-l-m} \rightarrow 0 \\
\downarrow \beta \quad \downarrow \beta \quad \downarrow \beta \\
0 \rightarrow N \rightarrow N \rightarrow 0 \rightarrow 0,
\end{array}
$$

(5.1)
Let $Z$ and fan $\Sigma/\sigma$ and $X_{\substack{\text{of} \\ \text{Proof.}}}$

\begin{align*}
0 & \longrightarrow \mathbb{Z}^m \longrightarrow \mathbb{Z}^{l+m} \longrightarrow \mathbb{Z}^l \longrightarrow 0 \\
(5.2) & \quad \downarrow \beta \quad \downarrow \bar{\beta} \quad \downarrow \beta(\sigma) \\
0 & \longrightarrow N_{\sigma} \longrightarrow N \longrightarrow N(\sigma) \longrightarrow 0.
\end{align*}

Applying the Gale dual yields

\begin{align*}
0 & \longrightarrow \mathbb{Z}^{n-l-m} \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^{l+m} \longrightarrow 0 \\
(5.3) & \quad \downarrow \cong \quad \downarrow \beta^\vee \quad \downarrow \bar{\beta}^\vee \\
0 & \longrightarrow \mathbb{Z}^{n-l-m} \longrightarrow N^\vee \xrightarrow{\phi_1} \bar{N}^\vee \longrightarrow 0,
\end{align*}

and

\begin{align*}
0 & \longrightarrow \mathbb{Z}^l \longrightarrow \mathbb{Z}^{l+m} \longrightarrow \mathbb{Z}^m \longrightarrow 0 \\
(5.4) & \quad \downarrow \beta(\sigma)^\vee \quad \downarrow \bar{\beta}^\vee \quad \downarrow \beta_{\sigma}^\vee \\
0 & \longrightarrow N(\sigma)^\vee \xrightarrow{\phi_2} \bar{N}^\vee \longrightarrow N_{\sigma}^\vee \longrightarrow 0.
\end{align*}

Since $\mathbb{Z}^m \cong N_{\sigma}$, we have $N_{\sigma}^\vee = 0$. Applying $\text{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ to (5.3), (5.4) yields

\begin{align*}
1 & \longrightarrow \tilde{G} \xrightarrow{\varphi_1} G \longrightarrow (\mathbb{C}^*)^{n-l-m} \longrightarrow 1 \\
(5.5) & \quad \downarrow \tilde{\alpha} \quad \downarrow \alpha \quad \downarrow \cong \\
1 & \longrightarrow (\mathbb{C}^*)^{l+m} \longrightarrow (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^{n-l-m} \longrightarrow 1,
\end{align*}

and

\begin{align*}
1 & \longrightarrow 1 \longrightarrow \tilde{G} \xrightarrow{\cong} G(\sigma) \longrightarrow 1 \\
(5.6) & \quad \downarrow \tilde{\alpha} \quad \downarrow \alpha(\sigma) \\
1 & \longrightarrow (\mathbb{C}^*)^m \longrightarrow (\mathbb{C}^*)^{l+m} \longrightarrow (\mathbb{C}^*)^l \longrightarrow 1.
\end{align*}

Let $Z(\sigma) = \mathcal{X}/\vee(J_{\Sigma/\sigma})$, where $J_{\Sigma/\sigma}$ is the irrelevant ideal of the quotient simplicial fan $\Sigma/\sigma$. By the definition of toric Deligne-Mumford stack, we have $\mathcal{X}(\Sigma/\sigma) = [Z(\sigma)/G(\sigma)]$, where the action of $G(\sigma)$ is via the map $\alpha(\sigma)$ in (5.6).

**Proposition 5.1.** If $\sigma$ is a cone in the simplicial fan $\Sigma$, then $\mathcal{X}(\Sigma/\sigma)$ is a closed substack of $\mathcal{X}(\Sigma)$.

**Proof.** Let $W(\sigma)$ be the subvariety of $Z$ defined by the ideal $J(\sigma) := \langle z_i : \rho_i \subset \sigma \rangle$. Then $W(\sigma)$ contains the $\mathbb{C}$-points $z \in \mathbb{C}^n$ such that the cone spanned by $\{\rho_i : z_i = 0\}$ containing $\sigma$ belongs to $\Sigma$. Then the $\mathbb{C}$-point $z$ in $W(\sigma)$ such that $\rho_i \not\subset \sigma \cup \text{link}(\sigma)$ implies that $z_i \neq 0$. This implies that $W(\sigma) \cong Z(\sigma) \times (\mathbb{C}^*)^{n-l-m}$. It is clear that $W(\sigma)$ is invariant under the $G$-action.

Let $\varphi_0 : Z(\sigma) \rightarrow W(\sigma)$ be the inclusion given by $z \mapsto (z, 1)$. Then we have a morphism of groupoids $\varphi_0 \times \varphi_1 : Z(\sigma) \times G(\sigma) \rightarrow W(\sigma) \times G$ which induces a morphism of stacks $\varphi : [Z(\sigma)/G(\sigma)] \rightarrow [W(\sigma)/G]$. To prove that it is an
isomorphism, we first prove that the following diagram is cartesian:

$$Z(\sigma) \times G(\sigma) \xrightarrow{\varphi_0 \times \varphi_1} W(\sigma) \times G$$

\[ Z(\sigma) \times Z(\sigma) \xrightarrow{\varphi_0 \times \varphi_0} W(\sigma) \times W(\sigma). \]

This is easy to prove. Given an element \((z_1, z_2) \in Z(\sigma) \times Z(\sigma)\), under the map \(\varphi_0 \times \varphi_0\), we get \(((z_1, 1), (z_2, 1)) \in W(\sigma) \times W(\sigma)\). If there is an element \(g \in G\) such that \(g(z_1, 1) = (z_2, 1)\), then from the exact sequence in the first row of (5.5), there is an element \(g(\sigma) \in G(\sigma)\) such that \(g(\sigma)z_1 = z_2\). Thus we have an element \((z_1, g(\sigma)) \in Z(\sigma) \times G(\sigma)\). So the morphism \(\varphi : [Z(\sigma)/G(\sigma)] \to [W(\sigma)/G]\) is injective. Let \((z, 1)\) be an element in \(W(\sigma)\), then there exists an element \(g \in \mathbb{C}^{n-l-m}\) such that \(g(z, 1) = (z, 1)\). By (5.5), \(g\) determines an element in \(G\), so \(\varphi\) is surjective and \(\varphi\) is an isomorphism. Clearly the stack \([W(\sigma)/G]\) is a closed substack of \(X(\Sigma)\), so \(X(\Sigma) = [Z(\sigma)/G(\sigma)]\) is also a closed substack of \(X(\Sigma)\).

**Remark 5.2.** Proposition 5.1 is Proposition 4.2 of [BCS]. However the proof given there has a gap: some incorrect exact sequences were used. We choose to give a new proof here. One can prove this result using extended stacky fan defined in [Jian] since the quotient stacky fan is naturally an extended stacky fan.

Following [BCS], for each top dimensional cone \(\sigma\) in \(\Sigma\), denote by \(\text{Box}(\sigma)\) the set of elements \(v \in N\) such that \(\nu = \sum_{\rho_i \subseteq \sigma} a_i e_i\) for some \(0 \leq a_i < 1\). The elements in \(\text{Box}(\sigma)\) are in one-to-one correspondence with the elements in the finite group \(N(\sigma) = N/N_\sigma\), where \(N(\sigma)\) is a local isotropy group of the stack \(X(\Sigma)\). If \(\tau \subseteq \sigma\) is a subcone, we define \(\text{Box}(\tau)\) to be the set of elements in \(v \in N\) such that \(\nu = \sum_{\rho_i \subseteq \tau} a_i e_i\), where \(0 \leq a_i < 1\). It is easy to see that \(\text{Box}(\tau) \subset \text{Box}(\sigma)\). In fact the elements in \(\text{Box}(\tau)\) generate a subgroup of the local group \(N(\sigma)\). Let \(\text{Box}(\Sigma)\) be the union of \(\text{Box}(\sigma)\) for all \(d\)-dimensional cones \(\sigma \in \Sigma\). For \(v_1, \ldots, v_n \in N\), let \(\sigma(\overline{v}_1, \ldots, \overline{v}_n)\) be the unique minimal cone in \(\Sigma\) containing \(\overline{v}_1, \ldots, \overline{v}_n\).

**Proposition 5.3 (BCS).** The \(r\)-th inertia stack of the stack \(X(\Sigma)\) is

\[ I_r(X(\Sigma)) = \coprod_{(v_1, \ldots, v_r) \in \text{Box}(\Sigma)^r} X(\Sigma/\sigma(\overline{v}_1, \ldots, \overline{v}_r)). \]

We are interested in the cases \(r = 1\) or \(2\). When \(r = 1\),

\[ I(X(\Sigma)) = \coprod_{v \in \text{Box}(\Sigma)} X(\Sigma/\sigma(\overline{v})) \]

is the inertia stack. The orbifold Chow ring is the Chow ring of the inertia stack as \(\mathbb{Z}\)-modules.

When \(r = 2\), for any pair \((v_1, v_2)\) in \(\text{Box}(\Sigma)\), there is a unique \(v_3 \in \text{Box}(\Sigma)\) such that \(v_1 + v_2 + v_3 \equiv 0 \mod N\). We have:

\[ I_2(X(\Sigma)) = \coprod_{(v_1, v_2, v_3); v_1 + v_2 + v_3 \equiv 0 \mod N} X(\Sigma/\sigma(\overline{v}_1, \overline{v}_2, \overline{v}_3)). \]

The components are called 3-twisted sectors in [CR1].
5.2. **The integral orbifold Chow ring.** Let \( X(\Sigma) \) be a toric Deligne-Mumford stack with stacky fan \( \Sigma \) and \( A^*_\text{orb}(X(\Sigma), \mathbb{Z}) \) its integral orbifold Chow ring. We first study the \( A^*_\text{orb}(X(\Sigma), \mathbb{Z}) \)-module structure of \( A^*_\text{orb}(X(\Sigma), \mathbb{Z}) \). Because \( \Sigma \) is a simplicial fan, we have the following two lemmas in [Jiang1]:

**Lemma 5.4.** For any \( c \in \mathbb{N} \), let \( \sigma \) be the minimal cone in \( \Sigma \) containing \( \overline{c} \), then there exists a unique expression

\[
c = v + \sum_{\rho_i \subset \sigma} m_i b_i
\]

where \( m_i \in \mathbb{Z}_{\geq 0} \), and \( v \in \text{Box}(\sigma) \). □

**Lemma 5.5.** Let \( \tau \) be a cone in the complete simplicial fan \( \Sigma \) and \( \{\rho_1, \ldots, \rho_s\} \subset \text{link}(\tau) \). Suppose \( \rho_1, \ldots, \rho_s \) are contained in a cone \( \sigma \subset \Sigma \). Then \( \sigma \cup \tau \) is contained in a cone of \( \Sigma \). □

Let \( v \in \text{Box}(\Sigma) \) and \( \sigma := \sigma(\overline{v}) \) the minimal cone containing \( \overline{v} \). Then we have the quotient stacky fan \( \Sigma/\sigma \) and \( \Sigma_{\text{red}}/\sigma \). From the diagrams (5.1) and (5.2) we consider the following diagrams:

(5.8)

Taking Gale dual yields

(5.9)

For the quotient stacky fan \( \Sigma/\sigma \), if in the map \( \beta : \mathbb{Z}^n \rightarrow N \), the vectors \( \{b_1, \cdots, b_n\} \) generate the torsion part of \( N \), then from (5.8) and (5.9), the vectors \( \{\tilde{b}_1, \cdots, \tilde{b}_l\} \) in the map \( \beta(\sigma) : \mathbb{Z}^l \rightarrow N(\sigma) \) generate the torsion part of \( N(\sigma) \). So we have:
Proposition 5.6. Given a toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ associated to the stacky fan $\Sigma$. Suppose that the map $\beta$ generate the torsion part of $N$, then for a cone $\sigma \subset \Sigma$, the closed substack $\mathcal{X}(\Sigma/\sigma)$ is a nontrivial gerbe over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}}/\sigma)$. \hfill \Box

By Lemma 2.8 the map $\varphi$ in (5.9) is diagonalizable, so the map $\tilde{\varphi}$ in (5.9) is diagonalizable. From (5.3) and (5.4), $N(\sigma)^{\vee} \cong \tilde{N}^{\vee}$, and $N(\sigma)^{\vee} \cong \tilde{N}^{\vee}$, so the map $\varphi(\sigma)$ in (5.9) is diagonalizable. We assume that $\varphi(\sigma)$ is given by the diagonal integral matrix $M(\sigma)$. Let $\tilde{x}_i = (\tilde{x}_1, \ldots, \tilde{x}_i)$ and $x_i = (x_1, \ldots, x_i)$ be column vectors. Then using the same analysis of the formula (3.1), we have the following lemma:

Lemma 5.7. The formula in (3.1) induces a formula

$$\tilde{x}_i = A(\sigma)M(\sigma)C(\sigma)x_i$$

for the quotient stacky fan, where $A(\sigma)$ and $C(\sigma)$ are integral matrices. When we take $x_j$ as first Chern class of the line bundle $L_j$, the definition of $\tilde{x}_i$ are compatible with restrictions to components of the inertia stack. \hfill \Box

Proposition 5.8. Let $\mathcal{X}(\Sigma)$ be a toric Deligne-Mumford stack associated to the stacky fan $\Sigma$, then we have an isomorphism of $A^*(\mathcal{X}(\Sigma), \mathbb{Z})$-modules:

$$\bigoplus_{v \in \text{Box}(\Sigma)} A^*(\mathcal{X}(\Sigma/\sigma(v)), \mathbb{Z}) \left[\text{deg}(y^v)\right] \cong \frac{\mathbb{Z}[\Sigma]}{\text{Cir}(\Sigma)}.$$

PROOF. We use a method similar to that in Proposition 4.7 of [Jiang1]. Let

$$S_\Sigma := \mathbb{Z}[y^{b_i} : \rho_i \in \Sigma(1)]/I_\Sigma.$$

Then $S_\Sigma/\text{Cir}(\Sigma) \cong A^*(\mathcal{X}(\Sigma), \mathbb{Z})$ given by $y^{b_i} \mapsto x_i$. By the definition of $\mathbb{Z}[\Sigma]$ and Lemma 5.4, we see that $\mathbb{Z}[\Sigma] = \bigoplus_{v \in \text{Box}(\Sigma)} y^v \cdot S_\Sigma$. And we obtain an isomorphism of $A^*(\mathcal{X}(\Sigma), \mathbb{Z})$-modules:

$$\mathbb{Z}[\Sigma]/\text{Cir}(\Sigma) \cong \bigoplus_{v \in \text{Box}(\Sigma)} \mathbb{Z} \cdot y^v \cdot S_\Sigma.$$

For any $v \in \text{Box}(\Sigma)$, let $\sigma(\tau)$ be the minimal cone in $\Sigma$ containing $\tau$. Let $\rho_1, \ldots, \rho_l \in \text{link}(\sigma(\tau))$, and $\tilde{\rho}_i$ be the image of $\rho_i$ under the natural map $\pi : N \rightarrow N(\sigma(\tau)) = N/N_\sigma(\tau)$. Then $S_{\Sigma/\sigma(\tau)} \subset \mathbb{Z}[\Sigma/\sigma(\tau)]$ is the subring generated by: $\tilde{y}^{\tilde{b}_i}$ for $\rho_i \in \text{link}(\sigma(\tau))$. Let $\tilde{a}$ be the order of the torsion subgroup of $N(\sigma(\tau))$. Then let $a = s\tilde{a}$, and conversely we have $\tilde{a} = \frac{1}{s}a$. By Lemmas 5.5 and 5.7 it is easy to check that the ideal $I_{\Sigma/\sigma(\tau)}$ goes to the ideal $I_{\Sigma}$ and we have a morphism $\tilde{\Psi}_v : S_{\Sigma/\sigma(\tau)}[\text{deg}(y^v)] \rightarrow y^v \cdot S_\Sigma$ given by: $y^{b_i} \mapsto y^v \cdot sy^{b_i}$. If $\sum_{i=1}^{l} \theta(\tilde{b}_i)ay^{\tilde{b}_i}$ belongs to the ideal $\text{Cir}(\Sigma/\sigma(\tau))$, then

$$\tilde{\Psi}_v \left( \sum_{i=1}^{l} \theta(\tilde{b}_i)ay^{\tilde{b}_i} \right) = y^v \cdot \left( \sum_{i=1}^{n} \theta(b_i)s\tilde{a}y^{b_i} \right) = y^v \cdot \left( \sum_{i=1}^{n} \theta(b_i)ay^{b_i} \right).$$
where $\theta = \tilde{\theta} \circ \pi$ and $\theta(b_i) = \tilde{\theta}(b_i)$. So we obtain that $\tilde{\Psi}_v(\sum_{i=1}^l \tilde{\theta}(b_i) ay^{b_i}) \in y^v \cdot Cir(\Sigma)$. So $\tilde{\Psi}_v$ induce a morphism $\Psi_v : \frac{S_{\Sigma/\sigma(\overline{\tau})}}{Cir(\Sigma/\sigma(\overline{\tau}))}[deg(y^v)] \to \frac{y^v \cdot S_{\Sigma}}{y^v \cdot Cir(\Sigma)}$ such that $\Psi_v([y^{b_i}]) = [y^v \cdot sy^{b_i}]$.

Conversely, for such $v \in Box(\Sigma)$ and $\rho_i \subset \sigma(\overline{\tau})$, choose $\theta_i \in Hom_{\mathbb{Z}}(N, \mathbb{Q})$ such that $\theta_i(b_i) = 1$ and $\theta_i(b_i') = 0$ for $b_i' \neq b_i \in \sigma(\overline{\tau})$. Again by Lemmas 5.3 and 5.7 we consider the following morphism $\tilde{\Phi}_v : y^v \cdot S_{\Sigma} \to S_{\Sigma/\sigma(\overline{\tau})}[deg(y^v)]$ given by:

$$y^b_i \mapsto \begin{cases} \frac{1}{n} y^{b_i} & \text{if } \rho_i \subset link(\sigma(\overline{\tau})), \\ -\sum_{j=1}^l \theta_i(b_j)y^{b_j} & \text{if } \rho_i \subset \sigma(\overline{\tau}), \\ 0 & \text{if } \rho_i \notin \sigma(\overline{\tau}) \cup link(\sigma(\overline{\tau})). \end{cases}$$

Let $y^v \cdot (\sum_{i=1}^n \theta(b_i) ay^{b_i})$ belong to the ideal $y^v \cdot Cir(\Sigma)$. For $\theta \in M$, we have $\theta = \theta_v + \theta'_v$, where $\theta_v \in N(\sigma(\overline{\tau}))^* = M \cap (\sigma(\overline{\tau}))^\perp$ and $\theta'_v$ belongs to the orthogonal complement of the subspace $\sigma(\overline{\tau})^\perp$ in $M$. We have

$$\tilde{\Phi}_v \left( y^v \cdot \left( \sum_{i=1}^n \theta(b_i) ay^{b_i} \right) \right) = \sum_{i=1}^l \theta_v(b_i) ay^{b_i} + \sum_{\rho_i \subset \sigma(\overline{\tau})} \theta'_v(b_i) \left( -\sum_{j=1}^l \theta_i(b_j)y^{b_j} \right) + \sum_{i=1}^l \theta'_v(b_i)y^{b_i}.$$

Note that $\sum_{i=1}^l \theta_v(b_i) ay^{b_i} \in Cir(\Sigma/\sigma(\overline{\tau}))$. Now let $\theta'_v = \sum_{\rho_i \subset \sigma(\overline{\tau})} a_i \theta_i$, where $a_i \in \mathbb{Q}$, then $\sum_{\rho_i \subset \sigma(\overline{\tau})} \theta'_v(b_i) = \sum_{\rho_i \subset \sigma(\overline{\tau})} a_i \theta_i(b_i)$. We have:

$$\sum_{\rho_i \subset \sigma(\overline{\tau})} a_i \theta_i(b_i) \left( -\sum_{j=1}^l \theta_i(b_j)y^{b_j} \right) + \sum_{\rho_i \subset \sigma(\overline{\tau})} \sum_{j=1}^l a_i \theta_i(b_j)y^{b_j} = 0,$$

so we have $\tilde{\Phi}_v \left( y^v \cdot \left( \sum_{i=1}^n \theta(b_i) ay^{b_i} \right) \right) \in Cir(\Sigma/\sigma(\overline{\tau}))$. So $\tilde{\Phi}_v$ induces a morphism

$$\Phi : \frac{y^v \cdot S_{\Sigma}}{y^v \cdot Cir(\Sigma)} \to \frac{S_{\Sigma/\sigma(\overline{\tau})}}{Cir(\Sigma/\sigma(\overline{\tau}))}[deg(y^v)].$$

We check that $\Phi_v \Psi_v = 1$ and $\Psi_v \Phi_v = 1$. So $\Phi_v$ is an isomorphism. Note that both sides of (5.10) are $S_{\Sigma/\sigma(\overline{\tau})}$-modules, we complete the proof. □

Now we compute the ring structure. The key part of the orbifold cup product is the orbifold obstruction bundle. For the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$, the obstruction bundle over the 3-twisted sectors in (5.7) is given by:

**Proposition 5.9.** Let $\mathcal{X}(\Sigma/\sigma(\overline{\tau}_1, \overline{\tau}_2, \overline{\tau}_3))$ be a 3-twisted sector of the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$. Let $v_1 + v_2 + v_3 = \sum_{\rho_i \subset \sigma(\overline{\tau}_1, \overline{\tau}_2, \overline{\tau}_3)} a_i b_i$, $a_i = 1, 2$, then the Euler class of the obstruction bundle $O_{(v_1, v_2, v_3)}$ over $\mathcal{X}(\Sigma)(v_1, v_2, v_3)$ is:

$$\prod_{a_i=2} c_1(L_i)|\mathcal{X}(\Sigma/\sigma(\overline{\tau}_1, \overline{\tau}_2, \overline{\tau}_3)),$$

where $\mathcal{L}_i$ is the line bundle over $\mathcal{X}(\Sigma)$ corresponding to the ray $\rho_i$. 
Proof of Theorem 1.2. Proposition 5.8 gives an isomorphism between $A^*(\mathcal{X}(\Sigma), \mathbb{Z})$-modules:

$$A^*_{orb}(\mathcal{X}(\Sigma), \mathbb{Z}) \simeq \bigoplus_{v \in Box(\Sigma)} A^*(\mathcal{X}(\Sigma/\sigma(\overline{v})), \mathbb{Z}) [\deg(y^v)] \cong \frac{Z[\Sigma]}{Cir(\Sigma)}.$$ 

All we need is to show that the orbifold cup product defined in [AGV] coincides with the product in ring $Z[\Sigma]/Cir(\Sigma)$. From the above isomorphisms, it suffices to consider the canonical generators $y^v$, $y^w$ where $v \in Box(\Sigma)$.

For any $v_1, v_2 \in Box(\Sigma)$, let $v_3 \in Box(\Sigma)$ be the unique box element such that $v_1 + v_2 + v_3 \equiv 0 \pmod{N}$. Then $\mathcal{X}(\Sigma/\sigma(\overline{v}_1, \overline{v}_2, \overline{v}_3))$ is a 3-twisted sector. Let $e_i : \mathcal{X}(\Sigma/\sigma(\overline{v}_1, \overline{v}_2, \overline{v}_3)) \to \mathcal{X}(\Sigma/\sigma(\overline{v}_i))$ be the evaluation map for $1 \leq i \leq 3$. Let $\tilde{e}_3$ be the inverse of $v_3$ in the local group, and $I : \mathcal{X}(\Sigma/\sigma(\overline{v}_3)) \to \mathcal{X}(\Sigma/\sigma(\overline{v}_3))$ be the map given by $(x, v_3) \mapsto (x, \tilde{e}_3)$. Let $\tilde{e}_3 = I \circ e_3$. Then the orbifold cup product is defined by:

$$y^{v_1} \cup_{orb} y^{v_2} = \tilde{e}_3, *(e_1^* y^{v_1} \cup e_2^* y^{v_2} \cup e(O_{(v_1, v_2, v_3)})),$$

where $O_{(v_1, v_2, v_3)}$ is the obstruction bundle in Proposition 5.9. Since $e_1, e_2, \tilde{e}_3$ are all inclusion, so are representable as morphisms of Deligne-Mumford stacks. By [K], the pullback and pushforward are well-defined for integral Chow classes. Let $\pi : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma_{red})$ be the natural morphism of rigidification. The first Chern class of the line bundle $L_i$ is $y^{b_i}$, so by Proposition 5.3, the first Chern class of $L_i$ is $\sum_{j=1}^n a_{j,i} y^{b_j}$ which is $\tilde{y}^{b_j}$. This class represents an integral Chow class of $\mathcal{X}(\Sigma/\sigma(\overline{v}_1, \overline{v}_2, \overline{v}_3))$. We have that $\tilde{e}_3, * (y^{b_j}) = y^{b_j} \in A^*(\mathcal{X}(\Sigma/\sigma(\overline{v}_3)), \mathbb{Z})$. So by the definition of orbifold cup product we have

$$y^{v_1} \cdot y^{v_2} = y^{v_3} \prod_{i \in I} \tilde{y}^{b_i} \cdot \prod_{i \in J} y^{b_i}.$$

6. Examples

In this section we compute some examples of the integral Chow ring and integral orbifold Chow rings.

Example 6.1 (The moduli stack of 1-pointed elliptic curves). Let $\Sigma$ be the complete fan of the projective line, $N = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $\beta : \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ be given by the vectors $\{b_1 = (2, 1), b_2 = (-3, 0)\}$. Then $\Sigma = (N, \Sigma, \beta)$ is a stacky fan. We compute that $(\beta)^\vee : \mathbb{Z}^2 \to N^\vee = \mathbb{Z}$ is given by the matrix $[6,4]$. So we get the following exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \xrightarrow{\beta} \mathbb{Z}^2 \oplus \mathbb{Z}_2 \to 0,$$

and

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \xrightarrow{\beta^\vee} \mathbb{Z} \to \mathbb{Z}_2 \to 0,$$

and

$$(6.1) \quad 1 \to \mu_2 \to \mathbb{C}^* \xrightarrow{[6,4]^t} (\mathbb{C}^*)^2 \to \mathbb{C}^* \to 1.$$
The toric Deligne-Mumford stack $\mathcal{X}(\Sigma) = [\mathbb{C}^2 - \{0\}/\mathbb{C}^*] =: \mathbb{P}(6,4)$, where the action is given by $\lambda(x,y) = (\lambda^6 x, \lambda^4 y)$, may be identified with the moduli stack $\mathcal{M}_{1,1}$ of 1-pointed elliptic curves. The stack $\mathcal{X}(\Sigma)$ is the nontrivial $\mu_2$-gerbe over $\mathbb{P}(3,2)$ coming from the canonical line bundle over $\mathbb{P}(3,2)$. Since $N = \mathbb{Z} \oplus \mathbb{Z}_2$, we have $m_i = 2$. By Theorem 1.1, we have
\[
A^*(\mathcal{X}(\Sigma), \mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, x_2]}{(2x_1 - 3x_2, 2x_1^2x_2)} \cong \mathbb{Z}[t]/(24t^2),
\]
which is the same as the result in [EG1]. We compute the integral orbifold Chow ring. There are 7 box elements: $v = (1,1), w_1 = (-1,0), w_2 = (-2,0), u = (0,1), \rho_1 = (1,0), \rho_2 = (-1,1)$ and $\rho_3 = (-2,1)$ corresponding to 7 twisted sectors. The three box elements $v, w_1, u$ generate the others. So by Theorem 1.2 we have
\[
A^*_\text{orb}(\mathcal{X}(\Sigma), \mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, x_2, v, w_1, u]}{(2x_1 - 3x_2, 2x_1^2x_2, v^2 - 2x_1u, w_1^3 - 2x_2u, vw_1, v_2x_2, w_12x_1, u^2 - 1)} \cong \frac{\mathbb{Z}[t, v, w_1, u]}{(24t^2, v^2 - 6tu, w_1^3 - 4tu, vw_1, 4vt, 6w_1t, u^2 - 1)},
\]
which is the same as the result in [AGV].

Example 6.2. In this example we discuss the relation between integral orbifold Chow ring and the integral Chow ring of crepant resolutions. Let $N = \mathbb{Z}^2$, and $\beta : \mathbb{Z}^3 \to \mathbb{Z}^2$ be given by the vectors $\{b_1 = (1,0), b_2 = (0,1), b_3 = (-1,-2)\}$. Let $\Sigma$ be the complete fan in $\mathbb{R}^2$ generated by $\{b_1, b_2, b_3\}$. Then $\Sigma = (N, \Sigma, \beta)$ is a stacky fan. We compute that $(\beta)^{\vee} : \mathbb{Z}^3 \to N^{\vee} = \mathbb{Z}$ is given by the matrix $[1,1,2]$. So we get the following exact sequence:
\[
1 \to \mathbb{C}^* \xrightarrow{[1,1,2]^t} (\mathbb{C}^*)^3 \to (\mathbb{C}^*)^2 \to 1.
\]
The toric Deligne-Mumford stack $\mathcal{X}(\Sigma) = [\mathbb{C}^3 - \{0\}/\mathbb{C}^*]$, where the action is given by $\lambda(x,y,z) = (\lambda^3 x, \lambda^3 y, \lambda^2 z)$, is the weighted projective stack $\mathbb{P}(1,1,2)$ which is a toric orbifold. We compute the integral orbifold Chow ring. There is one box element: $v = \frac{1}{2}b_1 + \frac{1}{2}b_3 = (0,1)$ corresponding to one twisted sector. So by Theorem 1.2 we have
\[
A^*_\text{orb}(\mathcal{X}(\Sigma), \mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, x_2, x_3, v]}{(x_1x_2x_3, x_1 - x_3, x_2 - 2x_3, v^2 - x_1x_3, vx_1, vx_2, vx_3)} \cong \frac{\mathbb{Z}[x_3, v]}{(2x_3^2, 2vx_3, v^2 - x_3^2)}.
\]
Let $\rho_4$ be a ray generated by $v = b_4$. Then the complete fan $\Sigma' = \{b_1, b_2, b_3, b_4\}$ generated by the rays $\{\rho_1, \rho_2, \rho_3, \rho_4\}$ is the fan of Hirzebruch surface $\mathbb{F}_2$. It is well-known that
\[
A^*(\mathbb{F}_2, \mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, x_2, x_3, x_4]}{(x_1x_2x_3, x_1 - x_3, x_2 - 2x_3 - x_4, x_2x_4, x_1x_3)} \cong \frac{\mathbb{Z}[x_3, x_4]}{(2x_3^3 + x_3^2x_4, x_3^2, 2x_3x_4 + x_4^2)}.
\]
It is easy to see that these two rings are not isomorphic. So under integer coefficients, \( A^*_{\text{orb}}(\mathcal{X}(\Sigma), \mathbb{Z}) \not\cong A^*(\mathbb{F}_2, \mathbb{Z}) \). In [BMP], the authors proved that
\[
A^*_{\text{orb}}(\mathcal{X}(\Sigma), \mathbb{C}) \cong A^*_Q(\mathbb{F}_2, \mathbb{C})
\]
where \( A^*_Q(\mathbb{F}_2, \mathbb{Z}) \) is the quantum corrected cohomology of \( \mathbb{F}_2 \) under complex number coefficients, thus verifying the cohomological crepant resolution conjecture [R].

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