NERON-SEVERI GROUPS UNDER SPECIALIZATION

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Abstract. We prove that given a smooth proper family \( \mathcal{X} \to B \) of varieties over an algebraically closed field \( k \) of characteristic 0, there exists a closed fiber having the same Picard number as the geometric generic fiber, even if \( k \) is countable. In fact, we give two proofs, and they show that the locus on the base where the Picard number jumps is “small” in two different senses. The first proof uses Hodge theory and the actions of geometric monodromy groups and Galois groups to show that the locus is small in a sense related to Hilbert irreducibility. The second proof uses the “\( p \)-adic Lefschetz (1,1) theorem” of Berthelot and Ogus to show that in a family of varieties with good reduction at \( p \), the locus is nowhere \( p \)-adically dense. Finally, we prove analogous statements for cycles of higher codimension, under the assumption of the variational Hodge conjecture or a \( p \)-adic analogue conjectured by M. Emerton.

1. Introduction

1.1. The jumping locus. For a smooth proper variety \( X \) over an algebraically closed field, let \( \text{NS} \, X \) be its Néron-Severi group, and let \( \rho(X) \) be the rank of \( \text{NS} \, X \). (See Sections 2 and 3 for definitions and basic facts.)

Now suppose that we have a smooth proper family \( \mathcal{X} \to B \), where \( B \) is an irreducible variety over an algebraically closed field \( k \) of characteristic 0. If \( b \in B(k) \), then one can choose an injection of the Néron-Severi group \( \text{NS} \, \mathcal{X}_b \) of the geometric generic fiber into the Néron-Severi group \( \text{NS} \, \mathcal{X}_b \) of the fiber above \( b \), so \( \rho(\mathcal{X}_b) \geq \rho(\mathcal{X}_b) \). The jumping locus

\[ B(k)_{\text{jumping}} := \{ b \in B(k) : \rho(\mathcal{X}_b) > \rho(\mathcal{X}_b) \} \]

is a countable union of lower-dimensional subvarieties of \( B \). If \( k \) is uncountable, it follows that \( B(k)_{\text{jumping}} \neq B(k) \). Our main result is that this holds even when \( k \) is countable:

**Theorem 1.1.** Let \( k \) be an algebraically closed field of characteristic 0. Let \( B \) be an irreducible variety over \( k \). Let \( \mathcal{X} \to B \) be a smooth proper morphism. Then there exists \( b \in B(k) \) such that \( \rho(\mathcal{X}_b) = \rho(\mathcal{X}_b) \).

**Remark 1.2.** The condition \( \rho(\mathcal{X}_b) = \rho(\mathcal{X}_b) \) is equivalent to the condition that \( \text{NS} \, \mathcal{X}_b \to \text{NS} \, \mathcal{X}_b \) is an isomorphism: see Proposition 3.6.

In fact, we will give two very different proofs of Theorem 1.1. The two approaches, which prove that the jumping locus is small in two different senses, are outlined in Sections 1.3 and 1.4.

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1.2. **Relationship to previous and current work.** Work on Theorem 1.1 and related questions goes back at least 25 years. The literature contains an existence proof or construction of non-jumping Picard numbers in some special cases:

- The question of the existence of smooth surfaces in $\mathbb{P}^3$ defined over $\mathbb{Q}$ having the generic Picard number was addressed by T. Shioda [Shi81]. His construction is explicit and involves surfaces with large automorphism group.
- Fix $d_1, \ldots, d_n \geq 1$, and consider the universal family $\mathcal{X} \to B$ of smooth $2r$-dimensional complete intersections in a fixed $\mathbb{P}^{2r+n}$ cut out by homogeneous polynomials of degrees $d_1, \ldots, d_n$. Assume that $\rho^r(\mathcal{X}_b)$ (as defined in Section 10) equals 1: this holds for most tuples $(d_1, \ldots, d_n)$. Then T. Terasoma [Ter85] proves the existence of a complete intersection $\mathcal{X}_b$ over $\mathbb{Q}$ in this family such that $\rho^r(\mathcal{X}_b) = 1$. The proof proceeds by using the monodromy action on $H^{2r}_{\text{et}}(\mathcal{X}_b, \mathbb{Z}_\ell(r))$ to bound the Tate classes of the closed fibers.

- The article [Ell04] uses a similar method to prove the existence of a degree-$d$ polarized $K3$ surface $X$ over $\mathbb{Q}$ with $\rho(X) = 1$ for each even positive integer $d$.
- A similar method is adapted in [vL07] to construct an explicit $K3$ surface $X$ over $\mathbb{Q}$ with $\rho(X) = 1$.
- Terasoma’s method is applied in [AV08] to study the variety of lines on a cubic 4-fold. On the other hand, E. Amerik [Ame09] uses our Theorem 1.1 to construct a $K3$ surface $S$ over a number field with $\rho(S) = 2$ and $\text{Aut } S$ finite such that rational points on $\text{Hilb}^2 S$ are potentially dense, even though there is no abelian fibration to explain this.

1.3. **The complex approach.** Our first approach involves complex Hodge theory. To state the more precise information about the jumping locus that this approach yields, we introduce the notion of “sparse”, which is a version for algebraically closed fields of the notion “thin” defined in [Ser97, §9.1] in the context of Hilbert irreducibility.

**Definition 1.3.** Let $k$ be an algebraic closure of a field $k_f$ that is finitely generated over $\mathbb{Q}$. Let $B$ be an irreducible $k_f$-variety. Call a subset $S$ of $B(k)$ **sparse** (relative to $k_f$) if there exists a dominant and generically finite morphism $\pi : B' \to B$ of irreducible $k_f$-varieties such that for each closed point $s \in B$ associated to a point of $S$, the fiber $\pi^{-1}(s)$ is not a single closed point.

We refer to the end of this introduction for a discussion of this notion.

The final output of the complex approach is the following:

**Theorem 1.4.** Suppose that $k$ is an algebraic closure of a field $k_f$ that is finitely generated over $\mathbb{Q}$. Let $B$ be an irreducible $k_f$-variety, and let $f : \mathcal{X} \to B$ be a smooth proper morphism. Then $B(k)_{\text{jumping}}$ is sparse.

Although Theorem 1.4 has a restriction on $k$ (to make sense of the notion of sparse), an easy argument to be given in Section 8 shows that it still implies the general case of Theorem 1.1.

The strategy of the proof of Theorem 1.4 is to use Deligne’s global invariant cycle theorem ([Del71] or [Voi03 4.3.3]) and the semisimplicity of the category of polarized Hodge structures to decompose the Hodge structure on the Betti cohomology $H^2(\mathcal{X}^\text{sm}, \mathbb{Q})$ as $H \oplus H^\perp$, where $H$ consists of the classes invariant under some finite-index subgroup of the geometric monodromy group. The space $H$ carries a Hodge structure independent of $b$; after an étale
Then the set \( A \) is dense in \( B \).

Assume Setup 1.5. For Theorem 1.6, \( f \) is nowhere dense in \( B \). Let \( Q \) and \( [1.1] \) be a smooth proper morphism.

**Remark 1.8**

For an elliptic curve \( E \), CM points in the jumping locus (cf. [Voi03, 5.3.4]): In the setting of Theorem 1.1 over an algebraically closed field, the jumping locus is nowhere \( p \)-adically dense. For example, let \( X \to B \) be a smooth proper morphism.

**Theorem 1.6.** Assume Setup 1.5. For \( b \in B(\mathcal{O}_C) \subseteq B(C) \), let \( \mathcal{X}_b \) be the \( C \)-variety above \( b \). Then the set

\[
B(\mathcal{O}_C)_{\text{jumping}} := \{ b \in B(\mathcal{O}_C) : \rho(\mathcal{X}_b) > \rho(\mathcal{X}_q) \}
\]

is nowhere dense in \( B(\mathcal{O}_C) \) for the analytic topology.

To prove Theorem 1.6, we apply a “\( p \)-adic Lefschetz \((1,1)\) theorem” of P. Berthelot and A. Ogus [BOS3] to obtain a down-to-earth local analytic description (Lemma 5.2) of the jumping locus in \( B(\mathcal{O}_C) \). This eventually reduces the problem to a peculiar statement (Proposition 6.1) about linear independence of values of linearly independent \( p \)-adic power series.

**Remark 1.7.** It is well known (cf. [BLR90, p. 235]) that the archimedean analogue of Theorem 1.6 is false. For example, let \( B \) be an irreducible \( \mathbb{C} \)-variety, let \( \mathcal{E} \to B \) be a family of elliptic curves such that the \( j \)-invariant map \( j : B \to \mathbb{A}^1 \) is dominant, and let \( \mathcal{X} = \mathcal{E} \times_B \mathcal{E} \). For an elliptic curve \( E \) over an algebraically closed field, \( \rho(E \times E) = 2 + \text{rk} \text{End} E \) (cf. the Rosati involution comment in the proof of Proposition 1.11). So \( B(\mathbb{C})_{\text{jumping}} \) is the set of CM points in \( B(\mathbb{C}) \). In the \( j \)-line, the set of CM points is the image of \( \{ z \in \mathbb{C} : \text{im}(z) > 0 \} \) and \( [\mathbb{Q}(z) : \mathbb{Q}] = 2 \) under the usual analytic uniformization by the upper half plane. This image is dense in \( \mathbb{A}^1(\mathbb{C}) \), so its preimage under \( j \) is dense in \( B(\mathbb{C}) \).

**Remark 1.8.** Remark 1.7 is a particular case of a general topological density theorem for the jumping locus (cf. [Voit03, 5.3.4]): In the setting of Theorem 1.1 over \( k = \mathbb{C} \), if the jumping locus in \( B \) has a component that is reduced and of the “expected codimension” \( h^{2,0} \), then the jumping locus is topologically dense in \( B(\mathbb{C}) \). This fact ultimately relies on the topological density of \( \mathbb{Q} \) inside \( \mathbb{R} \).

**Remark 1.9.** We can give a heuristic explanation of the difference between \( \mathbb{C} \) and a field like \( C = \mathbb{C}_p \). Namely, \( [\mathbb{C} : \mathbb{R}] = 2 \), but the analogous \( p \)-adic quantity \( [\mathbb{C}_p : \mathbb{Q}_p] \) is infinite (in fact, equal to \( 2^{\aleph_0} \) [Lam86]). So a subvariety in \( B(\mathbb{C}_p) \) of positive codimension can be thought of as having infinite \( \mathbb{Q}_p \)-codimension. This makes it less surprising that a countable union of such subvarieties could be nowhere \( p \)-adically dense.

**Remark 1.10.** If \( \mathcal{X} \to B \) is as in Remark 1.7, but over an algebraic closure \( k \) of a finite field \( \mathbb{F}_p \), then again we have \( \rho(\mathcal{X}_q) = 3 \), but now \( \rho(\mathcal{X}_b) \geq 4 \) for all \( b \in B(k) \) since every elliptic curve \( E \) over \( k \) has endomorphism ring larger than \( \mathbb{Z} \). Thus the characteristic \( p \) analogue
of Theorem 1.1 fails. On the other hand, it seems likely that it holds for any algebraically closed field \( k \) that is not algebraic over a finite field.

1.5. Applications to abelian varieties. J.-P. Serre [Ser00, pp. 1–17] and R. Noot [Noo95, Corollary 1.5] used something like Terasoma’s method, combined with G. Faltings’ proof of the Tate conjecture for homomorphisms between abelian varieties, to prove that in a family of abelian varieties over a finitely generated field of characteristic 0, there exists a geometric closed fiber whose endomorphism ring equals that of the geometric generic fiber; in fact, their argument proves a sparseness statement analogous to Theorem 1.4. Our Theorem 1.1 reproves the existence result without Faltings’ work, and Theorem 1.6 strengthens this by showing that in the \( p \)-adic setting, the corresponding jumping locus is nowhere \( p \)-adically dense in the good reduction locus:

**Proposition 1.11.** Assume Setup 1.5, and assume moreover that \( \mathcal{X} \to B \) is an abelian scheme. Then

\[
\{ b \in B(\mathcal{O}_C) : \text{End} \mathcal{X}_\eta \to \text{End} \mathcal{X}_b \text{ is not an isomorphism} \}
\]

is nowhere dense in \( B(\mathcal{O}_C) \) for the analytic topology.

**Proof.** Choose a polarization on \( \mathcal{X}_\eta \), and replace \( B \) by a dense open subvariety to assume that it extends to a polarization of \( \mathcal{X} \to B \). For a polarized abelian variety \( A \), the group \((\text{NS} A)_Q\) is isomorphic to the subspace of \((\text{End} A)_Q\) fixed by the Rosati involution: see [Mum70, p. 190], for instance. (The subscript \( Q \) denotes \( \otimes \mathbb{Q} \).) Thus the \((\text{End} A)_Q\) jumping locus for \( \mathcal{X} \to B \) equals the Picard number jumping locus for \( \mathcal{X} \times_B \mathcal{X} \to B \). Apply Theorem 1.6 with \( X := \mathcal{X} \times_B \mathcal{X} \). Finally, if \((\text{End} X_\eta)_Q \to (\text{End} X_b)_Q\) is an isomorphism, then \( \text{End} X_\eta \to \text{End} X_b \) is an isomorphism, as one sees by considering the action on torsion points (this uses characteristic 0). \( \square \)

**Remark 1.12.** Theorem 1.7 of [Noo95] states that for any algebraic group \( G \) arising as a Mumford-Tate group of a complex abelian variety, there exists an abelian variety \( A \) over a number field \( F \) such that the Mumford-Tate group of \( A \) equals \( G \) and such that moreover the Mumford-Tate conjecture holds; i.e., the action of \( \text{Gal}(\overline{F}/F) \) on a Tate module \( T_\ell A' \) gives an open subgroup in \( G(\mathbb{Q}_\ell) \).

It would be natural to conjecture a “nowhere dense” analogue, i.e., that the locus in a family of abelian varieties where the Mumford-Tate group jumps (downward) is nowhere \( p \)-adically dense in the good reduction locus. But we know how to prove this only if we assume Conjecture 10.6 from Section 10.

A proof similar to that of Proposition 1.11 yields another application of Theorem 1.1.

**Proposition 1.13.** Let \( k \) be an algebraically closed field of characteristic 0. Let \( A \) be an abelian variety defined over \( k \). Let \( B \) be an irreducible \( k \)-variety. Let \( \mathcal{X} \to B \) be an abelian scheme that is not locally isotrivial for the étale topology. Then there exists \( b \in B(k) \) such that \( \mathcal{X}_b \) is not isogenous to \( A \).

**Remark 1.14.** Under the appropriate hypotheses on \( k \) and \( \mathcal{X} \to B \), Theorems 1.4 and 1.6 prove the analogous strengthenings of Proposition 1.13.
1.6. **Sparse versus nowhere \( p \)-adically dense.** The purpose of this subsection is to develop basic properties of sparse sets, and to discuss the difference between the two notions of smallness appearing in Theorems 1.4 and 1.6.

**Proposition 1.15.**

(a) For any closed subscheme \( Z \subseteq B \), the subset \( Z(k) \) is sparse.

(b) A finite union of sparse sets is sparse.

(c) A dominant and generically finite morphism of irreducible varieties maps sparse sets to sparse sets.

(d) If \( S \) is a sparse subset of \( \mathbb{A}^n(k) \), then \( S \cap \mathbb{A}^n(k_f) \) is a thin set.

(e) The complement of a sparse subset of \( B(k) \) is Zariski dense in \( B \).

(f) Let \( L \) be a finite extension of \( k_f \), let \( B \) be an irreducible \( L \)-variety, and let \( \pi : B' \to B \) be the same scheme viewed as a \( k_f \)-variety. Then the inclusion \( B(k) \hookrightarrow k_f B(k) \) maps any sparse subset of \( B(k) \) relative to \( L \) to a sparse subset of \( k_f B(k) \) relative to \( k_f \).

(g) Let \( L \) be a finite extension of \( k_f \), let \( B \) be an irreducible \( k_f \)-variety, and let \( C \) be an irreducible component of the base extension \( B_L \). Then the inclusion \( C(k) \hookrightarrow B_L(k) = B(k) \) maps any sparse subset of \( C(k) \) relative to \( L \) to a sparse subset of \( B(k) \) relative to \( k_f \).

**Proof.**

(a) Take \( B' = B - Z \).

(b) Choose a irreducible component dominating \( B \) in the fiber product of the varieties \( B' \) over \( B \).

(c) Trivial.

(d) By definition.

(e) Suppose not. Then there is a nonempty open subscheme \( U \) of \( B \) such that \( U(k) \) is sparse. After shrinking \( U \), there is a finite surjective morphism from \( U \) to a nonempty open subscheme \( V \) of some \( \mathbb{A}^n \). Then \( V(k) \) is sparse by [1], so \( V(k_f) \) is a thin set by [1]. This contradicts the fact that \( k_f \) is Hilbertian.

(f) Given a dominant and generically finite morphism of irreducible \( L \)-varieties \( B' \to B \), consider the \( k_f \)-morphism \( k_f B' \to k_f B \).

(g) The \( k_f \)-morphism \( k_f B_L \to B \) induces a dominant and generically finite \( k_f \)-morphism \( k_f C \to B \). Apply (d) and (e). \( \square \)

**Warning 1.16.** Unfortunately, the notion of sparse is sensitive to the ground field within its algebraic closure. For example, if \( B = \mathbb{A}^1_{k_f} \) and \( L \) is a nontrivial finite extension of \( k_f \), then the set of points in \( B(k) = k \) whose field of definition contains \( L \) is sparse relative to \( k_f \) (use \( B' := \mathbb{A}^1_L \to B \)), but not sparse relative to \( L \) (see Proposition 1.15(d)).

The following proposition, which is not needed elsewhere in this article, shows that neither of the properties of being sparse or nowhere \( p \)-adically dense implies the other.

**Proposition 1.17.**

(i) Let \( k_f \) be a number field. Fix embeddings \( k_f \subseteq \overline{\mathbb{Q}} \subseteq \mathbb{C}_p \). Let \( \pi : B' \to B \) be any dominant and generically finite morphism of irreducible \( k_f \)-varieties such that \( \deg \pi > 1 \). Let \( S \subseteq B(\overline{\mathbb{Q}}) \) be the sparse subset defined by \( \pi \). Then \( S \) is dense in \( B(\mathbb{C}_p) \).

(ii) There exists a subset \( S \) of \( \overline{\mathbb{Q}} \) that is nowhere \( p \)-adically dense but not sparse.
Proof.

(i) It suffices to prove that given a generically finite morphism $B' \to B$ over $k_f$, any nonempty $p$-adic open subset $U$ of $B'(\mathbb{C}_p)$ contains a $\mathbb{Q}$-point $u$ whose field of definition $k_f(u)$ equals $k_f(\pi(u))$. This will be shown by a variant of the argument in [Poo01]. By replacing $B'$ and $B$ by dense open subschemes and composing with a generically finite morphism $B \to \mathbb{A}^n$, we reduce to the case $B = \mathbb{A}^n$. We may also assume that $n \geq 1$, that $B'$ is étale over $B$, and that $B'$ is a locally closed subscheme in $\mathbb{A}^{n+1}$, with $\pi$ being the projection onto the first $n$ coordinates. Choose $u_0 \in U \cap B'(\mathbb{Q})$, and let $F = k_f(u_0)$. Using weak approximation, we find a line $L$ in $\mathbb{A}^n_F$ passing $p$-adically close to $u_0$ such that the Galois conjugates of $\pi(L)$ other than $\pi(L)$ do not pass near $\pi(u_0)$. A general such $L$ is not parallel to any coordinate hyperplane, and intersects $B'$ in a point $u \in B'(\mathbb{Q}) \cap U$ $p$-adically near $u_0$. The assumptions on $\pi(L)$ imply that $F \subseteq k_f(\pi(u)) \subseteq k_f(u)$. Since each of the $n+1$ coordinates of $u$ is a linear polynomial in any other coordinate, each generates the same field extension of $F$, so $k_f(\pi(u)) = k_f(u)$.

(ii) The set $\mathbb{Q}$ is nowhere $p$-adically dense in $\mathbb{C}_p$. On the other hand, $\mathbb{Q}$ is not thin in $\mathbb{Q}$, so Proposition 1.15(d) proves that $\mathbb{Q}$ is not sparse.

\[\square\]

Remark 1.18. In the setting of Theorem 1.1 but over a field $k$ of characteristic 0 that need not be algebraically closed, either Theorem 1.4 or Theorem 1.6 can be used to show that there exists $d \geq 1$ such that

$$\{b \in B \mid \kappa(b) : k \leq d \text{ and } \rho(\mathcal{X}_b) = \rho(\mathcal{X}_0)\}$$

is Zariski dense in $B$. In fact, Theorem 1.4 and Proposition 1.15(c,d) show that the degree of any generically finite rational map $B \to \mathbb{P}^n$ can serve as $d$.

1.7. Outline of the article. After introducing some notation in Section 2, we review some standard facts about Néron-Severi groups and specialization maps in Section 3. Section 4, which uses only étale and Betti cohomology, and some Hodge theory, is devoted to the proof of Theorem 1.4. The next three sections prove Theorem 1.6. Section 5 discusses some basic properties of crystalline cohomology and convergent isocrystals, and applies them to give a local description of the jumping locus; Section 6 proves the key $p$-adic power series proposition to be applied to understand this local description; and Section 7 completes the proof. Section 8 shows that either of Theorems 1.4 or 1.6 can be used to prove Theorem 1.1.

Section 9 gives an application of Theorem 1.1: if all closed fibers in a smooth proper family are projective, then there exists a dense open subvariety of the base over which the family is projective, assuming that the base is a variety in characteristic 0.

Finally, Section 10 explains conditional generalizations of our results to cycles of higher codimension. The generalization of Theorem 1.4 is proved assuming the variational Hodge conjecture, and the generalization of Theorem 1.6 is proved assuming M. Emerton’s $p$-adic version of the variational Hodge conjecture (Conjecture 10.6). Either of these would yield the generalization of Theorem 1.1. We also prove that the $p$-adic variational Hodge conjecture implies the classical variational Hodge conjecture.
2. Notation

If $A$ is a commutative domain, let $\text{Frac}(A)$ denote its fraction field. If $A \to B$ is a ring homomorphism, and $M$ is an $A$-module, let $M_B$ denote the $B$-module $M \otimes_A B$. If $k$ is a field, then $\overline{k}$ denotes an algebraic closure, chosen consistently whenever possible. Given a prime number $p$, let $\mathbb{Z}_p$ be the ring of $p$-adic integers, let $\mathbb{Q}_p = \text{Frac}(\mathbb{Z}_p)$, choose algebraic closures $\overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}_p$, and let $\mathbb{C}_p$ denote the completion of $\overline{\mathbb{Q}}_p$.

For any $S$-schemes $X$ and $T$, let $X_T$ be the $T$-scheme $X \times_S T$. For a commutative ring $R$, we may write $R$ as an abbreviation for $\text{Spec} R$. A variety is a separated scheme of finite type over a field. If $B$ is an irreducible scheme, let $\eta$ denote its generic point. If $b \in B$, let $\kappa(b)$ be its residue field and let $\overline{b} = \text{Spec} \kappa(b)$. For example, if $\mathcal{X} \to B$ is a morphism, then $\mathcal{X}_\overline{b}$ is called the geometric generic fiber. If $X$ is a variety over a field equipped with an embedding in $\mathbb{C}$, then $X^{an}$ denotes the associated complex analytic space.

If $\mathcal{X}$ is a complex analytic space and $i$ is a nonnegative integer, then we have the Betti cohomology $H^i(\mathcal{X}, F)$ for any field $F$. If $X$ is a variety over a field $k$, and $i$ and $j$ are integers with $i \geq 0$, and $\ell$ is a prime not divisible by the characteristic of $k$, then we have the étale cohomology $H^i_{\text{et}}(X_k, \mathbb{Q}_\ell(j))$, which is equipped with a $\text{Gal}(\overline{k}/k)$-action (replace $\overline{k}$ by a separable closure if $k$ is not perfect).

3. Basic facts on Néron-Severi groups

3.1. Picard groups and Néron-Severi groups. For a scheme or formal scheme $X$, let $\text{Pic} X$ be its Picard group. If $X$ is a smooth proper variety over an algebraically closed field, let $\text{Pic}^0 X$ be the subgroup consisting of isomorphism classes of line bundles algebraically equivalent to 0 (i.e., to $\mathcal{O}_X$), and define the Néron-Severi group $\text{NS} X := \text{Pic} X / \text{Pic}^0 X$. The abelian group $\text{NS} X$ is finitely generated \cite{n2} p. 145, Théorème 2] (see \cite{sga6} XIII.5.1 for another proof), and its rank is called the Picard number $\rho(X)$.

**Proposition 3.1.** If $k \subseteq k'$ are algebraically closed fields, and $X$ is a smooth proper $k$-variety, then the natural homomorphism $\text{NS} X \to \text{NS} X_{k'}$ is an isomorphism.

**Proof.** The Picard scheme $\text{Pic}_{X/k}$ exists as a group scheme that is locally of finite type over $k$ (this holds more generally for any proper scheme over a field: see \cite{m} II.15], which uses \cite{o62}). Then $\text{Pic}^0 X$ is the set of $k$-points of the identity component of $\text{Pic}_{X/k}$ \cite{k05} 9.5.10]. So $\text{NS} X$ is the group of components of $\text{Pic}_{X/k}$. Thus $\text{NS} X$ is unchanged by algebraically closed base extension. \hfill $\square$

**Remark 3.2.** The Nakai-Moishezon criterion \cite{h77} A.5.1] implies that ampleness of a line bundle $\mathcal{L}$ on a variety $X$ over any field $K$ depends only on its class in $\text{NS} \times X_L$ for any algebraically closed field $L$ containing $K$.

3.2. Specialization of Néron-Severi groups.

**Proposition 3.3** (cf. \cite{sga6} X App 7]). Let $R$ be a discrete valuation ring with fraction field $K$ and residue field $k$. Fix an algebraic closure $\overline{K}$ of $K$. Choose a nonzero prime ideal $p$ of the integral closure $\overline{R}$ of $R$ in $\overline{K}$, so $\overline{k} := \overline{R}/p$ is an algebraic closure of $k$. Let $X$ be a smooth proper $R$-scheme. Then there is a natural homomorphism

$$\text{sp}_{\overline{K}, k} : \text{NS} X_{\overline{K}} \to \text{NS} X_{\overline{k}}.$$
In all cases, if Proposition 3.1.

Proof. As in [SGA 6, X App 7.8], we have

\[(3.4) \quad \text{Pic} \, X_K \cong \text{Pic} \, X \to \text{Pic} \, X_k.\]

If \( L \) is a line bundle on \( X_K \) whose image in \( \text{Pic} \, X_k \) is ample, then the corresponding line bundle on \( X \) is ample relative to \( \text{Spec} \, R \) by [EGA III.1. 4.7.1], so \( L \) is ample too.

For each finite extension \( L \) of \( K \) in \( K \), the integral closure \( R_L \) of \( R \) in \( L \) is a Dedekind ring by the Krull-Akizuki theorem [Bou98, VII.2.§5, Proposition 5], and localizing at \( p \cap R_L \) gives a discrete valuation ring \( R'_L \). Take the direct limit over \( L \) of the analogue of \((3.4)\) for \( R'_L \) to get \( \text{Pic} \, X_{\overline{K}} \to \text{Pic} \, X_{\overline{K}} \) (cf. [SGA 6, X App 7.13.3]). This induces \( \text{NS} \, X_{\overline{K}} \to \text{NS} \, X_{\overline{K}} \) (cf. [SGA 6, X App 7.12.1]). The ampleness claim follows from Remark 3.2 and the statement for \( \text{Pic} \) already discussed.

Remark 3.5. In Proposition 3.3 if \( R \) is complete, or more generally henselian, then there is only one choice of \( p \).

Proposition 3.6. Let \( B \) be a noetherian scheme. Let \( s, t \in B \) be such that \( s \) is a specialization of \( t \) (i.e., \( s \) is in the closure of \( \{ t \} \)). Let \( p = \text{char} \, \kappa(s) \). Let \( \mathcal{X} \to B \) be a smooth proper morphism. Then it is possible to choose a homomorphism

\( \text{sp}_{t,s} : \text{NS} \, \mathcal{X}_t \to \text{NS} \, \mathcal{X}_s \)

with the following properties:

(a) If \( p = 0 \), then \( \text{sp}_{t,s} \) is injective.
(b) If \( p = 0 \), then \( \text{coker} \, \text{sp}_{t,s} \) is torsion-free.
(c) If \( p > 0 \), then \( \text{sp}_{t,s} \) and \( \text{sp}_{t,s} \) hold after tensoring with \( \mathbb{Z}[1/p] \).
(d) In all cases, \( \rho(\mathcal{X}_s) \geq \rho(\mathcal{X}_t) \).
(e) If \( \text{sp}_{t,s} \) maps a class \([ \mathcal{L} ]\) to an ample class, then \( \mathcal{L} \) is ample.

Proof. A construction of \( \text{sp}_{t,s} \) is explained at the beginning of [SGA 6, X App 7.17]: the idea is to choose a discrete valuation ring \( R \) with a morphism \( \text{Spec} \, R = \{ s', t' \} \to B \) mapping \( s' \) to \( s \) and \( t' \) to \( t \), to obtain

\[ \text{NS} \, \mathcal{X}_t \cong \text{NS} \, \mathcal{X}_p^{\text{sp}_{t,s}} \text{NS} \, \mathcal{X}_{s'} \cong \text{NS} \, \mathcal{X}_s, \]

with the outer isomorphisms coming from Proposition 3.1.

For any prime \( \ell \neq p \), there is a commutative diagram

\[(3.7) \quad \begin{array}{ccc}
\text{NS} \, \mathcal{X}_t \otimes \mathbb{Z}_\ell & \overset{\text{sp}_{t,s}}{\longrightarrow} & H^2_{et}(\mathcal{X}_t, \mathbb{Z}_\ell(1)) \\
\downarrow & & \downarrow \\
\text{NS} \, \mathcal{X}_s \otimes \mathbb{Z}_\ell & \overset{\text{sp}_{t,s}}{\longrightarrow} & H^2_{et}(\mathcal{X}_s, \mathbb{Z}_\ell(1))
\end{array}
\]

(cf. [SGA 6, 7.13.10]: there everything is tensored with \( \mathbb{Q} \), but the explanation shows that in our setting we need only tensor with \( \mathbb{Z}[1/(i-1)!] \) with \( i = 1 \)). This proves \( \text{(m)} \) and its analogue for \( p > 0 \). By \( \text{sp}_{t,s} \otimes \mathbb{Z}_\ell \) is contained in \( \text{coker} \, (\text{NS} \, \mathcal{X}_t \otimes \mathbb{Z}_\ell \to H^2_{et}(\mathcal{X}_t, \mathbb{Z}_\ell(1))) \).

Using the Kummer sequence, one shows [Mil80, V.3.29(d)] that the latter is \( T_1 \text{Br} \mathcal{X} := \lim_{\leftarrow n} (\text{Br} \mathcal{X})[\ell^n] \), which is automatically torsion-free; this proves \( \text{(m)} \) and its analogue for \( p > 0 \).
Finally, (b) follows from the earlier parts, and (c) follows from the corresponding part of Proposition 3.3.

**Corollary 3.8.** Let \( B \) be an irreducible variety over an algebraically closed field, let \( \mathcal{X} \to B \) be a smooth proper morphism, and let \( b \in B(k) \). If every element of \( \text{NS} \mathcal{X}_b \) has a positive integer multiple that lies in the image of \( \text{Pic} \mathcal{X} \to \text{Pic} \mathcal{X}_b \to \text{NS} \mathcal{X}_b \), then \( \rho(\mathcal{X}_b) = \rho(\mathcal{X}_b') \).

**Proof.** The construction of \( \text{sp}_{\eta,b} \) shows that \( \text{Pic} \mathcal{X} \to \text{NS} \mathcal{X}_b \) factors as

\[
\text{Pic} \mathcal{X} \to \text{Pic} \mathcal{X}_b \to \text{Pic} \mathcal{X}_b \to \text{NS} \mathcal{X}_b \text{sp}_{\eta,b} \to \text{NS} \mathcal{X}_b,
\]

so the hypothesis implies that \( \text{sp}_{\eta,b} : (\text{NS} \mathcal{X}_b)_Q \to (\text{NS} \mathcal{X}_b)_Q \) is surjective. Thus \( \rho(\mathcal{X}_b) \geq \rho(\mathcal{X}_b') \). Proposition 3.6(c) supplies the opposite inequality. □

**Proposition 3.9.** Let \( B \) be a noetherian scheme. For a smooth proper morphism \( \mathcal{X} \to B \) and a nonnegative integer \( n \), the set

\[
B_{\geq n} := \{ b \in B : \rho(\mathcal{X}_b) \geq n \}
\]

is a countable union of Zariski closed subsets of \( B \).

**Proof.** Proposition 3.6(d) says that \( B_{\geq n} \) contains the closure of any point in \( B_{\geq n} \). So if \( B = \text{Spec} \ A \) for some finitely generated \( \mathbb{Z} \)-algebra \( A \), then \( B_{\geq n} \) is the (countable) union over \( b \in B_{\geq n} \) of the closure of \( \{ b \} \). If the statement is true for \( \mathcal{X} \to B \), then it is true for its base extension by any morphism \( \iota : B' \to B \) of noetherian schemes, because Proposition 3.1 implies \( B_{\geq n} = \iota^{-1}(B_{\geq n}) \). Combining the previous two sentences proves the result for any noetherian affine scheme. Finally, if \( B \) is any noetherian scheme, write \( B = \bigcup_{i=1}^{n} B_i \) with \( B_i \) affine, let \( C_i \) be the union of the closures in \( B \) of the generic points of all the irreducible components of the closed subsets of \( B_i \) appearing in the countable union for \( (B_i)_{\geq n} \), and let \( C = \bigcup_{i=1}^{n} C_i \). Then \( B_{\geq n} = \bigcup_{i=1}^{n} (B_i)_{\geq n} \subseteq C \) and the opposite inclusion follows from the first sentence of this proof. □

**Corollary 3.10.** Let \( k \subseteq k' \) be algebraically closed fields. Let \( B \) be an irreducible \( k \)-variety. For a smooth proper morphism \( \mathcal{X} \to B \), the jumping locus

\[
B(k')_{\text{jumping}} := \{ b \in B(k') : \rho(\mathcal{X}_b) > \rho(\mathcal{X}_b') \}
\]

is the union of \( Z(k') \) where \( Z \) ranges over a countable collection of closed \( k \)-subvarieties of \( B \).

**Proof.** Proposition 3.9 yields subvarieties \( Z \) for the case \( k' = k \). The same subvarieties work for larger \( k' \) by Proposition 3.11, applied as in the proof of Proposition 3.9 to the base change morphism \( \iota : B_{k'} \to B \).

3.3 Pathological behavior in positive characteristic. The material in this section is not needed elsewhere in this article. Let \( R \) be a discrete valuation ring, and define \( K, k, \overline{K}, \overline{k} \) as in Section 3.2. The two examples below show that \( \text{sp}_{\overline{K}, k} \) is not always injective.

**Example 3.11.** There exists \( R \) of equal characteristic 2 and \( X \to \text{Spec} \ R \) such that \( X_{\overline{K}} \) and \( X_k \) are Enriques surfaces of type \( \mathbb{Z}/2\mathbb{Z} \) and \( \alpha_2 \), respectively [BM76, p. 222]. (The type refers to the structure of the scheme \( \text{Pic}^r \) parameterizing line bundles numerically equivalent to 0.) In this case \( \text{NS} X_{\overline{K}} \to \text{NS} X_k \) has a nontrivial kernel, generated by the canonical class of \( X_{\overline{K}} \), an element of order 2.
Example 3.12. There exists $R$ of mixed characteristic $(0,2)$ and $X \to \text{Spec } R$ such that $X_K$ and $X_{\overline{k}}$ are Enriques surfaces of type $\mathbb{Z}/2\mathbb{Z}$ and $\mu_2$, respectively [Lan83, Theorem 1.3], so again we have a nontrivial kernel.

Next, we give an example showing that $\text{coker}(sp_{\mathbb{K}}^{\mathbb{K}})$ is not always torsion-free.

Example 3.13. Let $\mathcal{O}$ be the maximal order of an imaginary quadratic field in which $p$ splits. Let $\mathcal{O}'$ be the order of conductor $p$ in $\mathcal{O}$. Over a finite extension $R$ of $\mathbb{Z}_p$, there exists a $p$-isogeny $\psi: E \to E'$ between elliptic curves over $R$ such that $\text{End } E_{\mathbb{K}} \simeq \mathcal{O}$ and $\text{End } E'_{\mathbb{K}} \simeq \mathcal{O}'$. Since $p$ splits, $E$ has good ordinary reduction and $\text{End } E_{\mathbb{K}} \simeq \mathcal{O}$. But $\psi$ must reduce to either Frobenius or Verschiebung, so $\text{End } E'_{\mathbb{K}} \simeq \mathcal{O}'$ too. Using that $\text{coker}(\text{End } E_{\mathbb{K}} \to \text{End } E'_{\mathbb{K}})$ is of order $p$, one can show that the cokernel of $\text{NS}((E' \times E')_{\mathbb{K}}) \to \text{NS}((E' \times E')_{\mathbb{K}})$ contains nonzero elements of order $p$.

4. THE COMPLEX POINT OF VIEW AND AN APPLICATION OF TERASOMA’S LEMMA

Our goal is to prove Theorem 1.4. After recalling a few standard definitions in Section 4.1 we state in Section 4.2 an auxiliary result (Theorem 4.2) about restrictions of Tate classes and Hodge-Tate classes. Sections 4.3 through 4.7 use Deligne’s global invariant cycle theorem, semisimplicity of the category of polarized Hodge structures, and Terasoma’s lemma to prove Theorem 4.2. Finally, Theorem 1.4 is deduced from Theorem 4.2 in Sections 4.8 and 4.9.

4.1. Tate classes and Hodge-Tate classes. Let $Y$ be a smooth projective variety over a finitely generated subfield $k_f$ of $\mathbb{C}$, and let $k$ be the algebraic closure of $k_f$ in $\mathbb{C}$.

A degree $2r$ Tate class on $Y_k$ is a cohomology class in $H^{2r}_\text{ét}(Y_k, \mathbb{Q}_\ell(r))$ whose orbit under $\text{Gal}(k/k_f)$ is finite. Given a codimension-$r$ cycle $Z$ on $Y_k$, its cohomology class (see Definition 10.1) is a Tate class, because $Z$ is definable over a finite extension of $k_f$. The Tate conjecture asserts the converse, that Tate classes are classes of algebraic cycles.

A (rational) Hodge class on $Y^{\text{an}}$ is an element of $H^{2r}(Y^{\text{an}}, \mathbb{Q}) \cap \text{Fil}^r H^{2r}(Y^{\text{an}}, \mathbb{C})$, where $\text{Fil}^r$ denotes the $r$th level of the Hodge filtration. The class in $H^{2r}(Y^{\text{an}}, \mathbb{Q})$ of any codimension-$r$ cycle on $Y^{\text{an}}$ is a Hodge class. The Hodge conjecture asserts the converse, that Hodge classes are classes of algebraic cycles.

A Hodge-Tate class is a Hodge class whose image under the comparison isomorphism $H^{2r}(Y^{\text{an}}, \mathbb{Q}) \otimes \mathbb{Q}_\ell \to H^{2r}_\text{ét}(Y_k, \mathbb{Q}_\ell(r))$ is a Tate class. If the Hodge conjecture is true, then Hodge classes are Hodge-Tate. In particular, if $r = 1$, where the Hodge conjecture is known as the Lefschetz theorem on $(1,1)$-classes, Hodge classes are Hodge-Tate.

4.2. Tate classes of closed geometric fibers. For the rest of Section 4 except Section 4.9 we work under the following assumptions.

Setup 4.1. Let $k_f$ be a finitely generated subfield of $\mathbb{C}$, and let $k$ be the algebraic closure of $k_f$ in $\mathbb{C}$. Let $B$ be a smooth quasi-projective geometrically irreducible $k_f$-variety, and let $f: \mathcal{X} \to B$ be a smooth projective morphism.

Section 4.9 will show how to reduce the general case of Theorem 1.4 to this setting.

Theorem 4.2. Assume Setup 4.1. After possibly replacing $k_f$ with a finite extension and $\mathcal{X} \to B$ by its base extension by a finite étale morphism $\overline{B} \to B$ of geometrically irreducible
extension by $\mathbf{Q}_\ell$ outside a sparse subset, the restriction map
\begin{equation}
\tag{4.3}
f^*_{\mathrm{et}} : H^2_{\mathrm{et}}(\overline{X}, \mathbf{Q}_\ell(r)) \to H^2_{\mathrm{et}}(X_b, \mathbf{Q}_\ell(r))
\end{equation}
induced by the inclusion $j : X_b \hookrightarrow \overline{X}$ restricts to a surjection between the sets of Tate classes on both sides, and between the sets of Hodge-Tate classes on both sides.

4.3. Essentially invariant classes. Let $b \in B(\mathbb{C})$. The geometric monodromy group is the analytic fundamental group $\pi_1(B_{\mathrm{an}}, b)$. Say that a class $\alpha \in H^2(X_b^{\mathrm{an}}, \mathbf{Q})$ is essentially invariant if its orbit under $\pi_1(B_{\mathrm{an}}, b)$ is finite.

Let $H$ be the subspace of essentially invariant classes. Since $\dim H < \infty$, there is a finite-index normal subgroup $\Gamma \subset \pi_1(B_{\mathrm{an}}, b)$ such that $H$ is the set of $\Gamma$-invariant elements in $H^2(X_b^{\mathrm{an}}, \mathbf{Q})$. The finite unramified analytic cover corresponding to $\Gamma$ is the analytization of some finite etale cover $\tilde{B} \to B$ equipped with a base-point $\tilde{b}$ mapping to $b$, all defined over some finite extension of $k_f$. Replace $k_f$ by this finite extension, replace $X \to B$ by its base extension by $\tilde{B} \to B$, and replace $b$ by $\tilde{b}$ to reduce to the case that $\pi_1(B_{\mathrm{an}}, b)$ acts trivially on $H$. Let $\overline{X}$ be a smooth projective completion of the new $X$.

As $b$ varies, the $H$ form a locally constant subsystem of $\mathcal{R}_{f_{\mathrm{an}}, \mathbf{Q}}$, and the geometric monodromy action on each fiber of this subsystem is now trivial.

4.4. Application of Deligne’s global invariant cycle theorem. This section is devoted to the Hodge-theoretic ingredients needed for the proof of Theorem 4.2. Fix an ample line bundle $\mathcal{L}$ on $\overline{X}$.

Proposition 4.4. Let $b \in B(\mathbb{C})$. Let $H$ be the set of essentially invariant elements of $H^2(X_b^{\mathrm{an}}, \mathbf{Q})$. As in Section 4.3, assume that $\pi_1(B_{\mathrm{an}}, b)$ acts trivially on $H$. Then

(i) The set $H$ is a Hodge substructure of $H^2(X_b^{\mathrm{an}}, \mathbf{Q})$. There is a natural (and in particular, monodromy-invariant) decomposition of Hodge structures
\[ H^2(X_b^{\mathrm{an}}, \mathbf{Q}) = H \oplus H^\perp. \]

(ii) There is a decomposition of Hodge structures
\[ H^2(\overline{X}^{\mathrm{an}}, \mathbf{Q}) = K \oplus K^\perp \]
such that the restriction map
\[ j^* : H^2(\overline{X}^{\mathrm{an}}, \mathbf{Q}) \to H^2(X_b^{\mathrm{an}}, \mathbf{Q}) \]
maps $K$ to 0 and induces an isomorphism of Hodge structures $K^\perp \to H$.

(iii) Any nonzero $\alpha \in H^\perp$ has an infinite orbit under $\pi_1(B_{\mathrm{an}}, b)$.

Proof. This is an immediate consequence of Deligne’s global invariant cycle theorem (or theorem of the fixed part) and the semisimplicity of the category of polarized Hodge structures (cf. [Del71]). For the convenience of the reader, we construct this decomposition in more detail, because we want to make clear that it is also Galois-invariant.

Under our hypothesis on $H$, Deligne’s global invariant cycle theorem (cf. [Del71] or [Voi03]) says that $H = \operatorname{im} j^*$. Since $j$ is a morphism of projective varieties, $j^*$ is a morphism of Hodge structures, so $H$ is a Hodge substructure of $H^2(X_b^{\mathrm{an}}, \mathbf{Q})$.

The Hodge structure $H^2(\overline{X}^{\mathrm{an}}, \mathbf{Q})$ is polarized using $\mathcal{L}$, and similarly $H^2(X_b^{\mathrm{an}}, \mathbf{Q})$ is polarized using $\mathcal{L}_b := \mathcal{L}|_{X_b}$. Namely, the rational cohomology class $c_1(\mathcal{L}_b)$ yields a nondegenerate
pairing on \( H^{2r}(\mathcal{X}_b^{an}, \mathbb{Q}) \) and yields the Lefschetz decomposition of this Hodge structure into a direct orthogonal sum of "primitive components" which are rational Hodge substructures. On each primitive component, the restriction of the pairing above gives (up to sign) a polarization of the corresponding Hodge substructure (see [Vo02] 6.3.2 for more detail). Changing the signs of the above pairings where needed on some pieces of this Lefschetz decomposition, we get a polarized Hodge structure on \( H^{2r}(\mathcal{X}_b^{an}, \mathbb{Q}) \). Similarly we get a polarized Hodge structure on \( H^{2r}(\overline{\mathcal{X}}^{an}, \mathbb{Q}) \).

The importance of the polarizations lies in the fact that for a polarized Hodge structure, the polarization remains nondegenerate on any Hodge substructure. This yields (i) with \( 4.5 \).

**4.5. The corresponding decomposition of étale cohomology.** In this section we prove an analogue of Proposition 4.4 for étale cohomology. We continue to use the notation of Setup 4.1 and Proposition 4.4, but assume moreover that \( b \) is defined over a finite extension \( k_f(b) \) of \( k \) contained in \( k \).

The comparison theorem for étale and Betti cohomology yields an isomorphism of \( \mathbb{Q}_\ell \)-vector spaces

\[
H_{et}^{2r}(\mathcal{X}_b, \mathbb{Q}_\ell(r)) \simeq H_{et}^{2r}(\mathcal{X}_b^{an}, \mathbb{Q}) \otimes \mathbb{Q}_\ell.
\]

Let \( \pi_1(\overline{B^{an}}, b) \) denote the profinite completion of \( \pi_1(B^{an}, b) \). Since \( \pi_1(B^{an}, b) \) acts on \( H^{2r}(\mathcal{X}_b^{an}, \mathbb{Z}) \), there is an induced action of \( \pi_1(\overline{B^{an}}, b) \) on \( H^{2r}(\mathcal{X}_b^{an}, \mathbb{Q}) \otimes \mathbb{Q}_\ell \).

**Proposition 4.6.**

(i) We have \( \mathbb{Q}_\ell \)-vector space decompositions

\[
H_{et}^{2r}(\mathcal{X}_b, \mathbb{Q}_\ell(r)) = H_{Q_\ell} \oplus (H^\perp)_{Q_\ell} \\
H_{et}^{2r}(\overline{\mathcal{X}}, \mathbb{Q}_\ell(r)) = K_{Q_\ell} \oplus (K^\perp)_{Q_\ell}.
\]

All four summands are preserved by the action of \( \text{Gal}(k/k_f(b)) \) on the left hand sides. The summands \( H_{Q_\ell} \) and \( (H^\perp)_{Q_\ell} \) are preserved also by \( \pi_1(\overline{B^{an}}, b) \).

(ii) The restriction map \( j^*_\ell \) in (4.3) maps \( K_{Q_\ell} \) to 0 and induces a \( \text{Gal}(k/k_f(b)) \)-equivariant isomorphism \( (K^\perp)_{Q_\ell} \to H_{Q_\ell} \).

(iii) Any nonzero \( \alpha \in (H^\perp)_{Q_\ell} \) has an infinite orbit under \( \pi_1(\overline{B^{an}}, b) \).

**Proof.** We start with Proposition 4.4(iii), tensor with \( \mathbb{Q}_\ell \), and apply the comparison isomorphisms. This yields the decompositions, and transforms \( j^* \) into \( j^*_\ell \), and the two summands \( H_{Q_\ell} \) and \( (H^\perp)_{Q_\ell} \) are invariant under \( \pi_1(B^{an}, b) \) and hence also under \( \pi_1(\overline{B^{an}}, b) \), but we must check that all four summands are preserved by \( \text{Gal}(k/k_f(b)) \).

Since \( j^*_\ell \) arises from a \( k_f(b) \)-morphism, it is Galois-equivariant, so its kernel \( K_{Q_\ell} \) and image \( H_{Q_\ell} \) are Galois-invariant. On the other hand, the intersection pairing we used on \( H_{et}^{2r}(\mathcal{X}_b, \mathbb{Q}_\ell(r)) = H_{et}^{2r}(\mathcal{X}_b^{an}, \mathbb{Q}) \otimes \mathbb{Q}_\ell \) to construct the orthogonal complement \( (H^\perp)_{Q_\ell} \) is Galois-invariant, because it is deduced from the natural intersection pairing induced by the hard Lefschetz isomorphism satisfied by the Galois-invariant class \( c_1(Z_b) \), by changing signs of this pairing on certain pieces of the Lefschetz decomposition relative to \( c_1(Z_b) \). This proves (i) and (iii).
We turn to \( \text{(iii)} \). Suppose that the class \( \alpha \in (H^1)^\perp_{\mathbb{Q}_\ell} \) has finite orbit under \( \pi_1(B^{\text{an}}, b) \). Then there is a finite-index subgroup \( \Gamma \leq \pi_1(B^{\text{an}}, b) \) fixing \( \alpha \). Now
\[
\alpha \in ((H^1)^\perp_{\mathbb{Q}_\ell})^\Gamma = (H^1)^\Gamma \otimes \mathbb{Q}_\ell = 0
\]
by Proposition 4.4\((\text{iii})\). \( \square \)

4.6. **Tate classes in the essentially varying part: Terasoma’s lemma.** The goal of this section is to prove the following.

**Proposition 4.7.** For all \( b \in B(k) \) outside a sparse subset, the summand \( (H^1)^\perp_{\mathbb{Q}_\ell} \) of \( H^2_{\text{et}}(\mathcal{X}_b, \mathbb{Q}_\ell(r)) \) does not contain any nonzero Tate classes.

**Proof.** Choices as in Section 3.2 let us define a specialization isomorphism between
\[
(4.8) \quad H^2_{\text{et}}(\mathcal{X}_\tilde{b}, \mathbb{Q}_\ell(r)) \quad \text{and} \quad H^2_{\text{et}}(\mathcal{X}_b, \mathbb{Q}_\ell(r)) \quad \text{comparison} \quad H^2_{\text{et}}(\mathcal{X}_b^{\text{an}}, \mathbb{Q}) \otimes \mathbb{Q}_\ell
\]
and an isomorphism of étale fundamental groups \( \pi_1^{\text{alg}}(B_k, \tilde{b}) \simeq \pi_1^{\text{alg}}(B_k, \tilde{b}) \) making the respective actions of the groups
\[
(4.9) \quad \text{Gal}(k_f(B)/k_f(B)) \supseteq \text{Gal}(k(B)/k(B)) \to \pi_1^{\text{alg}}(B_k, \tilde{b}) \quad \text{and} \quad \pi_1^{\text{alg}}(B_k, \tilde{b}) \quad \text{comparison} \quad \pi_1(B^{\text{an}}, b)
\]
on the spaces in \( (4.8) \) agree. Identify the three spaces in \( (4.8) \) so that all five groups in \( (4.9) \) act. Then Proposition 4.6\((\text{iii})\) implies that each nonzero \( \alpha \in (H^1)^\perp_{\mathbb{Q}_\ell} \) has an infinite orbit under \( \text{Gal}(k_f(B)/k_f(B)) \). The Tate classes are those with finite orbit under a sixth group \( \text{Gal}(k/k_f(b)) \) acting on the middle in \( (4.8) \), so Proposition 4.7 now follows from Lemma 4.10 below. \( \square \)

**Lemma 4.10** (Terasoma). For all \( b \in B(k) \) outside a sparse subset, the image of \( \text{Gal}(k/k_f(b)) \to \text{Aut} H^2_{\text{et}}(\mathcal{X}_b, \mathbb{Q}_\ell(r)) \) contains the image of \( \text{Gal}(k_f(B)/k_f(B)) \to \text{Aut} H^2_{\text{et}}(\mathcal{X}_b, \mathbb{Q}_\ell(r)) \).

**Proof.** See [Ter85], and also [GGP04, Lemma 8], which states the result in the slightly more general context we are using. \( \square \)

4.7. **Proof of Theorem 4.2.** As explained in Section 1.3 we may assume that \( \pi_1(B^{\text{an}}, b) \) acts trivially on \( H \). Let \( \alpha_b \in H^2_{\text{et}}(\mathcal{X}_b, \mathbb{Q}_\ell(r)) \) be a Tate class. If \( b \) is outside the sparse set of Proposition 4.7, then \( \alpha_b \) lies in the part \( H^2_{\mathbb{Q}_r} \) of the decomposition in Proposition 4.6\((\text{iii})\). By Proposition 4.6\((\text{iii})\), \( j_b^* \) restricts to a \( \text{Gal}(k/k_f(b)) \)-equivariant isomorphism \( (K^1)^\perp_{\mathbb{Q}_\ell} \to H^2_{\text{et}}(\mathcal{X}^\text{an}, \mathbb{Q}_\ell(r)) \) mapped by \( j_b^* \) to \( \alpha_b \), and \( \alpha \) is a Tate class on \( \mathcal{X} \).

If moreover \( \alpha_b \) is Hodge-Tate, then the Hodge structure isomorphism \( K^1 \to H \) in Proposition 4.4\((\text{iii})\) maps \( \alpha \) to \( \alpha_b \), so \( \alpha \) is Hodge-Tate too.

4.8. **Proof of Theorem 1.4** in the projective case. In this section we prove Theorem 1.4 under the assumptions in Setup 1.1.

We may replace \( k_f \) with a field extension and replace \( B \) by a finite étale cover \( \tilde{B} \), as required by Theorem 4.2, without affecting the hypotheses or conclusion of Theorem 1.4 because of Proposition 1.15\((\text{gr})\). Let \( S \) be the sparse subset promised by Theorem 4.2.

We need to prove that \( \rho(\mathcal{X}_b) = \rho(\mathcal{X}_\tilde{b}) \) for all \( b \in B(k) \) outside \( S \). Fix \( b \), and suppose that \( [H] \in \text{NS} \mathcal{X}_b \), where \( H \in \text{Pic} \mathcal{X}_b \). The Hodge class \( c_1(H) \) on \( \mathcal{X}_b \) is Hodge-Tate, so by Theorem 4.2 it comes from some Hodge-Tate class \( \beta \) on \( \mathcal{X}^\text{an} \). By the Lefschetz (1,1) theorem,
there is a nonzero integer \( n \) such that \( n\beta = c_1(H) \) for some \( H \in \text{Pic} X \). Corollary 3.8 applied to \( X_k \to B_k \) and \( b \) shows that \( \rho(X_b) = \rho(X_{\eta}) \).

4.9. Reductions. We now complete the proof of Theorem 1.4 by showing how to eliminate the extra hypotheses made in Setup 4.1. First, we may base-extend by \( B_{\text{red}} \to B \) to assume that \( B \) is reduced. By Proposition 1.15(a), we may replace \( B \) by any nonempty open subscheme \( U \) and \( X \) by \( f^{-1}(U) \). In particular, we may assume that \( B \) is smooth and quasi-projective. We may base extend to a suitable finite extension of \( k_f \) and select an irreducible component, because of Proposition 1.15(g); this lets us assume moreover that \( B \) is geometrically irreducible.

It remains to replace “proper” by “projective”. If \( B \) is fixed, then the property that a given \( X \to U \) over a given dense open subscheme \( U \) of \( B \) satisfies Theorem 1.4 depends only on the generic fiber \( X_{\eta} \), by Proposition 1.15(a) again. If \( X' \to U' \) is another instance such that \( X'_{\eta} \) is the blow-up of \( X_{\eta} \) along a smooth center, then Theorem 1.4 for \( X' \) is equivalent to Theorem 1.4 for \( X \), because \( \rho(X'_b) = \rho(X'_\eta) - \rho(X_\eta) \) for all \( k \)-points \( b \) in a sufficiently small dense open subscheme of \( B \) (cf. Remark 10.5). The same holds more generally if \( X'_{\eta} \) and \( X_{\eta} \) are birational, because birational maps in characteristic 0 admit a weak factorization: see the first sentence of Remark 2 following Theorem 0.3.1 in [AKMW02]. Chow’s lemma implies that the generic fiber \( X_{\eta} \) of our proper \( B \)-scheme \( X \) is birational to some smooth projective \( \kappa(\eta) \)-variety \( Y_{\eta} \), which can be spread out over a dense open subscheme of \( B \), so the projective case now implies the general case.

5. Convergent isocrystals and de Rham cohomology

We now begin work toward the \( p \)-adic proof of Theorem 1.1.

5.1. Goal of this section.

Definition 5.1. Assume Setup 1.5. Let \( d = \dim B_K \). Let \( b \) be a smooth \( \overline{K} \)-point on \( B_K \). If \( B \) is a closed subscheme of \( \mathbb{A}^n \), a polydisk neighborhood of \( b \) is a neighborhood \( U \) of \( b \) in \( B(\overline{K}) \) in the analytic topology equipped with, for some \( \epsilon > 0 \), a bijection

\[
\{(z_1, \ldots, z_d) \in \overline{K}^d : |z_i| \leq \epsilon \} \to U
\]

defined by an \( n \)-tuple of power series in \( z_1, \ldots, z_d \) with coefficients in some finite extension of \( K \). (Such neighborhoods exist by the implicit function theorem.) A polydisk neighborhood of \( b \) in an arbitrary \( B \) is a polydisk neighborhood of \( b \) in some affine open subscheme of \( B \). Let \( \mathcal{H}(U) \) be the subring of \( \overline{K}[[z_1, \ldots, z_d]] \) consisting of power series \( g \) with coefficients in some finite extension of \( K \) such that \( g \) converges on the polydisk.

The goal of Section 5 is to prove the following:

Lemma 5.2. Assume Setup 1.5. Let \( b_0 \in B(\mathcal{O}_K) \subseteq B(\overline{K}) \) be such that \( B_{\overline{K}} \) is smooth at \( b_0 \). Then there exists a polydisk neighborhood \( U \) of \( b_0 \) contained in \( B(\mathcal{O}_K) \) and a finitely generated \( \mathbb{Z} \)-submodule \( \Lambda \subseteq \mathcal{H}(U)^n \) for some \( n \) such that

\[
\{ b \in U : \rho(X_b) > \rho(X_{\eta}) \} = \bigcup_{\lambda \in \Lambda, \lambda \neq 0} (\text{zeros of } \lambda \text{ in } U).
\]
Remark 5.3. The analogue over \( \mathbb{C} \) is a well-known consequence of the Lefschetz (1, 1) theorem \([\text{Voï} 03] \) §5.3. But the union will often be dense in \( B(\mathbb{C}) \), so this analogue is not useful for our purposes.

5.2. Coherent sheaves on \( p \)-adic formal schemes. In the next two subsections, we recall some key notions of \([\text{Ogu}84] \), specialized slightly to the case we need. Assume Setup 5.5.

Definition 5.4 (cf. \([\text{Ogu}84] \) §1]). A \( p \)-adic formal scheme over \( \mathcal{O}_K \) is a noetherian formal scheme \( T \) of finite type over \( \text{Spf} \mathcal{O}_K \). (See \([\text{EGA I}] \) §10 for basic definitions regarding formal schemes.)

Example 5.5. Let \( X \) be a smooth proper scheme over \( \mathcal{O}_K \). Its completion with respect to the ideal sheaf \( p \mathcal{O}_X \) is a \( p \)-adic formal scheme \( \hat{X} \). “Formal GAGA” states:

(a) The functor \( \text{Coh}(X) \to \text{Coh}(\hat{X}) \) mapping \( \mathcal{F} \) to its \( p \)-adic completion \( \hat{\mathcal{F}} \) is an equivalence between the categories of coherent sheaves \([\text{EGA III}] \) Corollaire 5.1.6.

(b) Under this equivalence, line bundles on \( X \) correspond to line bundles on \( \hat{X} \).

(c) For any \( \mathcal{F} \in \text{Coh}(X) \) and \( q \in \mathbb{Z}_{\geq 0} \), there is an isomorphism of \( \mathcal{O}_K \)-modules \( H^q(X, \mathcal{F}) \simeq H^q(\hat{X}, \hat{\mathcal{F}}) \) \([\text{EGA III}] \) Corollaire 4.1.7.

We write \( K \otimes \cdot \) as an abbreviation for \( K \otimes_{\mathcal{O}_K} \cdot \). Similarly, \( K \otimes \cdot \) means \( K \otimes_{\mathcal{O}_K} \cdot \).

Definition 5.6 (\([\text{Ogu}84] \) Definition 1.1]). For any \( p \)-adic formal scheme \( T \), let \( \text{Coh}(K \otimes \mathcal{O}_T) \) denote the full subcategory of \((K \otimes \mathcal{O}_T)\)-modules isomorphic to \( K \otimes \mathcal{F} \) for some coherent \( \mathcal{O}_T \)-module \( \mathcal{F} \).

Remark 5.7. Let \( B \) be a separated finite-type \( \mathcal{O}_K \)-scheme. The rigid generic fiber of the formal scheme \( \hat{B} \) is open in the rigid analytification \((B_K)^{\text{an}} \) of \( B_K \). So, given a coherent sheaf \( \mathcal{F} \) on \( B \) and \( b_0 \in B(\mathcal{O}_K) \subseteq B(K) \) such that \( B_K \) is smooth at \( b_0 \), we have two routes to construct the “restriction” of \( \mathcal{F} \) to a sheaf on a sufficiently small rigid analytic neighborhood of \( b_0 \); one route goes through \( \hat{\mathcal{F}} \), and the other goes through \( \mathcal{F}_K \) on \( B_K \). In particular, if \( \mathcal{F}_K \) is locally free of rank \( n \) on \( B_K \), and \( U \) is a sufficiently small polydisk neighborhood of \( b_0 \), then a choice of local trivialization of \( \mathcal{F}_K \) lets us “restrict” global sections of \( \overline{K} \otimes \hat{\mathcal{F}} \) to obtain elements of \( \mathcal{H}(U)^n \). (Although we have used some language of rigid geometry in this remark, it is not needed anywhere else in this article.)

5.3. Definition of convergent isocrystal. Given a \( p \)-adic formal \( \mathcal{O}_K \)-scheme \( T \), let \( T_1 \) be the closed subscheme defined by the ideal sheaf \( p \mathcal{O}_T \), and let \( T_0 \) be the associated reduced subscheme \((T_1)_{\text{red}} \).

Definition 5.8 (cf. \([\text{Ogu}84] \) Definition 2.1]). An enlargement of \( B_k \) is a \( p \)-adic formal \( \mathcal{O}_K \)-scheme \( T \) equipped with a \( k \)-morphism \( z: T_0 \to B_k \). A morphism of enlargements \((T', z') \to (T, z) \) is an \( \mathcal{O}_K \)-morphism \( T' \to T \) such that the induced \( k \)-morphism \( T_0' \to T_0 \) followed by \( z \) equals \( z' \).

Example 5.9. Given \( s \in B(k) \), let \([s] \) denote the enlargement \((\text{Spf} \mathcal{O}_K, \text{Spec} k \to B_k) \) of \( B_k \).

Example 5.10. If \( \gamma: B' \to B \) is a morphism of \( \mathcal{O}_K \)-schemes of finite type, then we view \( B' \) as an enlargement of \( B_k \) by equipping it with the \( k \)-morphism \( (B'_k)_0 \to B_k \) induced by \( \gamma \).
Definition 5.11 (cf. [Ogu84, Definition 2.7]). A convergent isocrystal on $B_k$ consists of the following data:

(a) For every enlargement $(T, z)$ of $B_k$, a sheaf $E_T \in \text{Coh}(K \otimes \mathcal{O}_T)$.

(b) For every morphism of enlargements $g: (T', z') \to (T, z)$, an isomorphism $\theta_g: g^*E_T \to E_{T'}$ in $\text{Coh}(K \otimes \mathcal{O}_{T'})$. If $h: (T'', z'') \to (T', z')$ is another, the cocycle condition $\theta_h \circ h^*g = \theta_{gh}$ is required, and $\theta_{id} = id$.

5.4. Crystalline and de Rham cohomology.

Definition 5.12 (cf. [Gro68, §7] and [Ber74, III.1.1]). Let $K$ and $k$ be as in Setup 1.5 and let $W$ be the Witt ring of $k$, so $\mathcal{O}_K$ is finite as a $W$-module. Given a smooth proper $k$-variety $X$ and $q \in \mathbb{Z}_{\geq 0}$, we have the crystalline cohomology $H^q_{\text{cris}}(X/W)$, which is a finite $W$-module. Define

$$H^q_{\text{cris}}(X/K) := K \otimes_W H^q_{\text{cris}}(X/W).$$

There is a Chern class homomorphism $[\text{Gro68, §7.4}]
\[ c^\text{cris}_1: \text{Pic} X \to H^2_{\text{cris}}(X/K). \]

Remark 5.13. The fact that crystalline cohomology is a Weil cohomology [GM87] implies [Kle68, 1.2.1] that $c^\text{cris}_1(\mathcal{L})$ depends only on the image of $\mathcal{L}$ in $\text{NS}_X$.

Definition 5.14. If $X \xrightarrow{q} \text{Spf} \mathcal{O}_K$ is a smooth proper $p$-adic formal scheme and $q \in \mathbb{Z}_{\geq 0}$, its de Rham cohomology over $K$ is defined in terms of hypercohomology as follows (cf. [Gro66]):

$$H^q_{\text{dR}}(X/K) := K \otimes \mathbb{R}^q g_* \Omega^\bullet_{X/\mathcal{O}_K}.$$

This is a finite-dimensional $K$-vector space equipped with the Hodge filtration $\text{Fil}^\bullet H^q_{\text{dR}}(X/K)$. There is a Chern class homomorphism (cf. [Har75, II.7.7])

$$c^\text{dR}_1: \text{Pic} X \to H^2_{\text{dR}}(X/K).$$

5.5. Crystalline and de Rham cohomology in families. If instead of a single $k$-variety or $p$-adic formal $\mathcal{O}_K$-scheme we have a family, then the $K$-vector spaces $H^2_{\text{dR}}(X/K)$ and $H^q_{\text{cris}}(X/K)$ are replaced by a coherent sheaf, or a compatible system of sheaves (namely, a convergent isocrystal).

The statement for crystalline cohomology takes the following form:

Theorem 5.15 (cf. [Ogu84, Theorems 3.1 and 3.7]). Assume Setup 1.3. For each $q \in \mathbb{Z}_{\geq 0}$, there exists a convergent isocrystal $E := R^q_{\text{cris}} f_* \mathcal{O}_{\mathcal{Y}/K}$ on $B_k$ with an isomorphism of $K$-vector spaces $E_{[s]} \simeq H^q_{\text{cris}}(\mathcal{X}_{[s]}/K)$ for each $s \in B(k)$.

The de Rham version can be stated as follows:

Theorem 5.16. Let $g: \mathcal{Y} \to T$ be a smooth proper morphism of $p$-adic formal schemes and let $q \in \mathbb{Z}_{\geq 0}$. The sheaf $K \otimes \mathbb{R}^q g_* \Omega^\bullet_{\mathcal{Y}/T} \in \text{Coh}(K \otimes \mathcal{O}_T)$ has a Hodge filtration in $\text{Coh}(K \otimes \mathcal{O}_T)$. Pulling back $g$ along a morphism of $p$-adic formal schemes $t: \text{Spf} \mathcal{O}_K \to T$, yields a $p$-adic formal scheme $\mathcal{Y}_t \to \text{Spf} \mathcal{O}_K$, and pulling back the sheaf yields the filtered $K$-vector space $H^q_{\text{dR}}(\mathcal{Y}_t/K)$.

Definition 5.17. For a smooth proper morphism of $p$-adic formal schemes $g: \mathcal{Y} \to T$, define

$$H^q_{\text{dR}}(\mathcal{Y}/K) := \left( K \otimes \mathbb{R}^2 g_* \Omega^\bullet_{\mathcal{Y}/T} \right) / \text{Fil}^1 \left( K \otimes \mathbb{R}^2 g_* \Omega^\bullet_{\mathcal{Y}/T} \right).$$
Remark 5.18. If \( \mathcal{Y} \rightarrow T \) arises as the \( p \)-adic completion of a smooth proper morphism \( f : \mathcal{X} \rightarrow B \) of \( \mathcal{O}_K \)-schemes, then \( H^{02}(\mathcal{Y}/K) \simeq K \otimes R^2f_*\mathcal{O}_\mathcal{X} \) in \( \text{Coh}(K \otimes \mathcal{O}_T) \). Moreover, \( (R^2f_*\mathcal{O}_\mathcal{X})|_{B_K} \) is a locally free \( \mathcal{O}_{B_K} \)-module \([\text{Del68 5.5(i)}]\).

5.6. Comparison and the \( p \)-adic Lefschetz (1,1) theorem. The following result identifies crystalline and de Rham cohomologies, even in the family setting.

Theorem 5.19. Assume Setup 5.2. Let \( (T, z) \) be an enlargement of \( B_k \). Let \( f_0 : \mathcal{X}_0 \rightarrow T_0 \) be obtained from \( f : \mathcal{X} \rightarrow B \) by base change along \( z : T_0 \rightarrow B_k \leftarrow B \). Let \( g : \mathcal{Y} \rightarrow T \) be a smooth proper lifting of \( f_0 \). Then for each \( q \in \mathbb{Z}_{\geq 0} \) there is a canonical isomorphism (cf. \([\text{Ogu84 Theorem 3.8.2}]\))

\[
\sigma_{\text{cris},T} : K \otimes \mathbb{R}^q g_* \Omega^\bullet_{\mathcal{Y}/T} \rightarrow (R^q_{\text{cris}}f_* \mathcal{O}_{\mathcal{X}/K})_T.
\]

Moreover, if \( t \in T(\mathcal{O}_K) \) and \( s = z(t(\text{Spec } k)) \in B(k) \), then the isomorphism \( \sigma_{\text{cris}, T} \) induced by \( \sigma_{\text{cris}, T} \) on the fibers above \( t \) fits in a commutative diagram (cf. \([\text{B70 2.3}], \text{BO83 3.4}]\))

\[
\begin{array}{ccc}
\text{Pic} \mathcal{X}_t & \longrightarrow & \text{Pic} \mathcal{X}_s \\
\downarrow \sigma_{\text{cris}, T} & & \downarrow \sigma_{\text{cris}, T} \\
H^2_{\text{dR}}(\mathcal{X}_t/K) & \longrightarrow & H^2_{\text{cris}}(\mathcal{X}_s/K).
\end{array}
\]

Finally, we have what one might call a \( p \)-adic analogue of the Lefschetz (1,1) theorem:

Theorem 5.20 (cf. \([\text{BO83 Theorem 3.8}]\)). Let \( X \xrightarrow{\phi} \text{Spf } \mathcal{O}_K \) be a smooth proper \( p \)-adic formal scheme. Let \( \mathcal{L}_k \) be a line bundle on \( X_k \). Then the following are equivalent:

(a) There exists \( m \) such that \( \mathcal{L}_k \otimes \mathcal{O}_k^m \) lifts to a line bundle on \( X \).

(b) The element of \( H^2_{\text{dR}}(X/K) := K \otimes \mathbb{R}^q g_* \Omega^\bullet_{X/\mathcal{O}_K} \) corresponding under \( \sigma_{\text{cris}, \mathcal{O}_K} \) to \( c_1(\mathcal{L}_k) \in H^2_{\text{cris}}(X_k/K) \) maps to 0 in the quotient \( H^{02}(X/K) \).

All the above definitions and results are compatible with respect to base change from \( \mathcal{O}_K \) to \( \mathcal{O}_L \) for a finite extension \( L \) of \( K \) \([\text{Ogu84 3.6, 3.9.1, 3.11.1}]\).

5.7. Proof of Lemma 5.2. There is a unique irreducible component of \( B_{\overline{K}} \) containing the point of \( B(\overline{K}) \) corresponding to \( b_0 \). Replace \( K \) by a finite extension so that \( b_0 \) and this component are defined over \( K \), and replace \( B \) by the closure of this component. Then replace \( B \) by an open subscheme to assume that \( B \) is a closed subscheme of \( \mathbb{A}^r = \text{Spec } \mathcal{O}_K[x_1, \ldots, x_r] \) for some \( r \). Translate so that \( b_0 \) is the origin in \( \mathbb{A}^r \). Let \( s \in B(k) \) be the reduction of \( b_0 \), the origin in \( \mathbb{A}^r(k) \).

Let \( \varphi : \mathbb{A}^r \rightarrow \mathbb{A}^r \) be the morphism induced by the \( \mathcal{O}_K \)-algebra homomorphism mapping each variable \( x_i \) to \( px_i \). Let \( B' = \varphi^{-1}(B) \). Let \( b'_0 \in B'(\mathcal{O}_K) \) be the origin, so \( \varphi(b'_0) = b_0 \). Let \( T = B' \). Pulling back \( f : \mathcal{X} \rightarrow B \) yields a morphism of \( \mathcal{O}_K \)-schemes \( \mathcal{X}' \rightarrow B' \). Completing yields a morphism of \( p \)-adic formal \( \mathcal{O}_K \)-schemes \( \mathcal{X}'_T \rightarrow T \). We write \( f \) for any of these.

Let \( E \) be the convergent isocrystal \( R^2\sigma_{\text{cris}}f_* \mathcal{O}_{\mathcal{X}/K} \) on \( B_k \) given by Theorem 5.15. Because the special fiber of \( T \) maps to \( s \in B(k) \), we have a morphism of enlargements \( T \rightarrow [s] \), so the definition of convergent isocrystal gives an identification

\[
E_T \simeq E_{[s]} \otimes \mathcal{O}_K \mathcal{O}_T \simeq H^2_{\text{cris}}(\mathcal{X}_s/K) \otimes \mathcal{O}_K \mathcal{O}_T,
\]

and the latter is a trivialized sheaf in \( \text{Coh}(K \otimes \mathcal{O}_T) \).
Let \( \mathcal{L}_k \) be a line bundle on \( \mathcal{X}_s \). Then \( c_1^\text{cris}(\mathcal{L}_k) \in H^2_{\text{cris}}(\mathcal{X}_s/K) \) gives rise to a constant section \( \gamma_{\text{cris}}(\mathcal{L}_k) \) of \( H^2_{\text{cris}}(\mathcal{X}_s/K) \otimes_{\mathcal{O}_K} \mathcal{O}_T \simeq E_T \). Applying \( \sigma_{\text{cris,T}}^{-1} \) yields a section \( \gamma_{\text{dR}}(\mathcal{L}_k) \) of \( K \otimes \mathbb{R}^2 \mathfrak{f}_s \mathcal{O}_{\mathcal{X}_T/T} \), which can be mapped to a section \( \gamma_{02}(\mathcal{L}_k) \) of the quotient sheaf \( H^{02}(\mathcal{X}_T/K) \).

Let \( b' \in B'(\mathcal{O}_K) \). Let \( \mathcal{X}'_b \) be the \( \mathcal{O}_K \)-scheme obtained by pulling back \( \mathcal{X} \to B \) by the composition \( \text{Spec} \mathcal{O}_K \to b' \to B \). Let \( \mathcal{X}'_b \times K = \mathcal{X}'_b \times K \). We can “evaluate” \( \gamma_{\text{cris}}(\mathcal{L}_k) \), \( \gamma_{\text{dR}}(\mathcal{L}_k) \), and \( \gamma_{02}(\mathcal{L}_k) \) at \( b' \) by pulling back via \( \text{Spf} \mathcal{O}_K \to b' \to T \) to obtain values in \( K \)-vector spaces

\[
\gamma_{\text{cris}}(\mathcal{L}_k, b') \in H^2_{\text{cris}}(\mathcal{X}_s/K),
\gamma_{\text{dR}}(\mathcal{L}_k, b') \in H^2_{\text{dR}}(\overline{\mathcal{X}}_b/K),
\gamma_{02}(\mathcal{L}_k, b') \in H^{02}(\overline{\mathcal{X}}_b/K).
\]

Because the composition of enlargements \( \hat{b} \to T \to [s] \) is the identity, the cocycle condition in Definition 5.11 yields \( \gamma_{\text{cris}}(\mathcal{L}_k, b') = c_1^\text{cris}(\mathcal{L}_k) \).

Everything so far has been compatible with base extension from \( \mathcal{O}_K \) to \( \mathcal{O}_L \) for a finite extension \( L \) of \( K \), and we may take direct limits to adapt the definitions and results above to \( \mathcal{O}_L \).

Proposition 5.6 gives an injective homomorphism

\[ sp_{b, s} : (\text{NS} \mathcal{X}_s, K) \to (\text{NS} \mathcal{X}_s). \]

**Claim 5.21.** The class \([\mathcal{L}_k] \in (\text{NS} \mathcal{X}_s) \) is in the image of \( sp_{b, s} \) if and only if \( \gamma_{02}(\mathcal{L}_k, b') = 0 \).

**Proof.** Suppose that \([\mathcal{L}_k] \) is in the image of \( sp_{b, s} \). After multiplying by a power of \( p \), replacing \( K \) by a finite extension, and tensoring \( \mathcal{L}_k \) with a line bundle algebraically equivalent to 0 (which, by Remark 5.13, does not change any of the sections and values \( \gamma(\cdot) \)), we may assume that the isomorphism class of \( \mathcal{L}_k \) in \( \text{Pic} \mathcal{X}_s \) is the specialization of the isomorphism class of some line bundle \( \mathcal{L} \) on \( \mathcal{X}_s \). Lift \( \mathcal{L} \) to a line bundle \( \mathcal{L}_k \) on \( \mathcal{X}_s \). Completing yields \( \mathcal{L}_k \in \text{Pic} \overline{\mathcal{X}}_b \). The commutative diagram in Theorem 5.19 shows that the element \( c_1^\text{dR}(\mathcal{L}_k) \in H^2_{\text{dR}}(\overline{\mathcal{X}}_b/K) \) is mapped by \( \sigma_{\text{cris,b}} \) to \( c_1^\text{cris}(\mathcal{L}_k) \). Then Theorem 5.20 applied to \( \overline{\mathcal{X}}_b \) shows that \( \gamma_{02}(\mathcal{L}_k, b') = 0 \).

Conversely, suppose that \( \gamma_{02}(\mathcal{L}_k, b') = 0 \). Then Theorem 5.20 applied to \( \overline{\mathcal{X}}_b \) shows that after raising \( \mathcal{L}_k \) to a power of \( p \), we have that \( \mathcal{L}_k \) comes from some \( \mathcal{L} \) on \( \mathcal{X}_s \). By Example 5.5(b), \( \mathcal{L} \) comes from some \( \mathcal{L} \) on \( \mathcal{X}_s \). Then \([\mathcal{L}_k] = sp_{b, s}([K \otimes \mathcal{L}]) \). This completes the proof of Claim 5.21.

Because of Remark 5.13, \( \gamma_{02}(\mathcal{L}_k) \) defines a homomorphism from \( \text{NS} \mathcal{X}_s \) to the space of sections of the sheaf \( K \otimes_{K} H^{02}(\mathcal{X}_T/K) \). Let \( \Lambda_T \) be the image, so \( \Lambda_T \) is a finitely generated \( \mathbb{Z} \)-module. For any \( b' \in B'(\mathcal{O}_K) \), evaluation at \( b' \) defines a homomorphism \( ev_{b'} \) from \( \Lambda_T \) (or \((\Lambda_T)_Q\)) to \( H^{02}(\overline{\mathcal{X}}_b/K) \).

Applying Claim 5.21 over all finite extensions of \( \mathcal{O}_K \) yields

**Corollary 5.22.**
(a) For any \( b' \in B'(O_K) \), \( \rho(\mathcal{X}_{b'}) \) is the rank of the kernel of the composition

\[
\begin{array}{c}
\text{NS } \chi_{b'} \mapsto (\Lambda_T)_{\mathcal{O}} \xrightarrow{ev_{b'}} H^0(\widetilde{\mathcal{X}}_{b'}/K).
\end{array}
\]

(b) In particular,

\[
\rho(\mathcal{X}_{b'}) \geq \text{rk ker } \gamma_{02},
\]

with equality if and only if \( \text{ev}_{b'}: \Lambda_T \to H^0(\widetilde{\mathcal{X}}_{b'}/K) \) is injective.

Remark \ref{rmk5.18} lets us apply Remark \ref{rmk5.17} to \( \mathcal{F} := \mathcal{R}^2 f_* \mathcal{O}_X \) on \( B' \) to obtain a polydisk neighborhood \( U' \) of \( b'_0 \) in \( B'(O_K) \) such that the subgroup \( \Lambda_T \) of global sections of \( K \otimes H^0(\mathcal{X}_T/K) \) is expressed on \( U' \) as a subgroup \( \Lambda' \) of \( \mathcal{H}(U')^n \): in fact, if \( K \) is enlarged so that all elements of \( \text{NS}(\mathcal{X}_b) \) are defined over the residue field \( k \), then the coefficients of the elements in \( \Lambda' \) lie in \( K \). For \( b' \in U' \), we may interpret \( \text{ev}_{b'} \) concretely in terms of evaluation of power series in \( \Lambda' \).

We will prove

\[
\rho(\mathcal{X}_{b'}) = \rho(\mathcal{X}_{b'})
\]

by comparing both with \( \rho(\mathcal{X}_{b'}) \) for a “very general” \( \beta \in B'(O_K) \).

Lemma 5.25. Let \( Z \) be a finite-type \( K \)-scheme that is smooth of dimension \( n \). Let \( z_0 \in Z(K) \). Then no countable union of subschemes of \( Z \) of dimension less than \( n \) can contain a neighborhood of \( z_0 \) in \( Z(K) \).

Proof. Shrink \( Z \) so that there is an étale morphism \( \pi: Z \to \mathbb{A}^n \). By the implicit function theorem, \( \pi \) maps any sufficiently small neighborhood of \( z_0 \) in \( Z(K) \) bijectively to a neighborhood of \( \pi(z_0) \) in \( \mathbb{A}^n(K) \), so we reduce to the case \( Z = \mathbb{A}^n \). This follows by induction on \( n \) by projecting onto \( \mathbb{A}^{n-1} \) and using the uncountability of \( K \).

Corollary \ref{cor3.10} and Lemma \ref{lem5.25} show that there exists \( \beta \in B'(O_K) \) near \( b'_0 \) such that \( \rho(\mathcal{X}_{\beta}) = \rho(\mathcal{X}_{b'}) \). For any \( b' \in B'(O_K) \), we have

\[
\rho(\mathcal{X}_{b'}) \leq \rho(\mathcal{X}_{\beta}) \leq \rho(\mathcal{X}_{b'}),
\]

by \ref{eq:5.23}, the choice of \( \beta \), and Proposition \ref{prop3.6}(d), respectively.

Claim 5.27. For some \( b' \in B'(O_K) \), we have \( \text{rk ker } \gamma_{02} = \rho(\mathcal{X}_{b'}) \).

Proof. Applying Proposition \ref{prop6.23} to \( \Lambda' \) gives a nonempty open subset \( V \) of the polydisk in \( C^d \) such that \( \text{ev}_u \) is injective for \( u \in V \). Since \( K \) is dense in \( C \), the set \( V \) contains a point in \( K^d \), and we let \( b' \) be its image in \( B'(O_K) \), so \( \text{ev}_{b'} \) is injective. Apply Corollary \ref{cor5.22} to \( b' \).

Applying \ref{eq:5.23} with \( b' \) as in Claim \ref{claim5.27} proves \ref{eq:5.24}.

Next, \( \varphi \) maps \( U' \) isomorphically to a polydisk neighborhood \( U \) of \( b_0 \) in \( B(O_K) \), and \( \Lambda' \) corresponds to some \( \Lambda \subseteq \mathcal{H}(U)^n \). Substituting \ref{eq:5.24} into Corollary \ref{cor5.22}, expressed on \( U \) in terms of \( \Lambda \), shows that for \( b \in U \),

\[
\rho(\mathcal{X}_{b}) \geq \rho(\mathcal{X}_{b'}),
\]

\footnote{The proofs in Section 5 do not rely on any results in this section.}
with equality if and only if \( \lambda(b) \neq 0 \) for every nonzero \( \lambda \in \Lambda \). This completes the proof of Lemma 5.2.

**Remark 5.28.** We conjecture that Lemma 5.2 holds with \( B(\mathcal{O}_K) \) replaced by \( B(K) \), but crystalline methods do not suffice to prove this. This would imply that Theorem 1.6 holds with \( B(C) \) in place of \( B(\mathcal{O}_C) \).

6. **Unions of zero loci of power series**

To understand the nature of the following statement, the reader is urged to consider the \( r = 1 \) case first.

**Proposition 6.1.** Let \( C \) be as in Setup 1.5. Let \( D = \{(z_1, \ldots, z_d) \in C^d : v(z_i) \leq \epsilon \text{ for all } i\} \) for some \( \epsilon > 0 \). Let \( R \) be the subring of \( C[[z_1, \ldots, z_d]] \) consisting of power series that converge on \( D \). Let \( r \) be a nonnegative integer. Let \( \Lambda \) be a finite-dimensional \( \mathbb{Q}_p \)-subspace of \( R \). Then there exists a nonempty analytic open subset \( U \) of \( D \) such that for all \( u \in U \), the evaluation-at-\( u \) map

\[
ev_u : \Lambda \to C^r \\
(f_1, \ldots, f_r) \mapsto (f_1(u), \ldots, f_r(u))
\]
is injective.

**Remark 6.2.** The archimedean analogue of Proposition 6.1 is false. For example, if \( \Lambda \) is the \( \mathbb{R} \)-span of \( 1, x, x^2 \), then there is no \( u \in \mathbb{C} \) such that the evaluation-at-\( u \) map \( \Lambda \to \mathbb{C} \) is injective. Even if we consider only the \( \mathbb{Z} \)-span of \( 1, x, x^2 \), the evaluation-at-\( u \) map fails to be injective on a dense subset of \( \mathbb{C} \).

The rest of this section is devoted to the proof of Proposition 6.1. The proof is by induction on \( r \). Because the base case \( r = 1 \) is rather involved, we begin by explaining the inductive step.

Suppose that \( r > 1 \), and that Proposition 6.1 is known for \( r' < r \). Let \( \pi : R^r \to R^{r-1} \) be the projection to the first \( r - 1 \) coordinates. Let \( \Lambda^{(r)} \) and \( \Lambda_{r-1} \) be the kernel and image of \( \pi|_\Lambda \). View \( \Lambda^{(r)} \) as a subgroup of \( R \). For any \( u \in D \), we have a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \to & \Lambda^{(r)} & \to & \Lambda & \to & \Lambda_{r-1} & \to & 0 \\
\downarrow \ev_u & & \downarrow \ev_u & & \downarrow \ev_u & & \\
0 & \to & C & \to & C^r & \to & C^{r-1} & \to & 0.
\end{array}
\]

The inductive hypothesis applied to \( \Lambda_{r-1} \subseteq R^{r-1} \) gives a closed polydisk \( D' \subseteq D \) such that the right vertical map in (6.3) is injective for all \( u \in D' \). The inductive hypothesis applied to \( \Lambda^{(r)} \subseteq R \) gives a closed polydisk \( D'' \subseteq D' \) such that the left vertical map in (6.3) is injective for all \( u \in D'' \). Then for \( u \in D'' \), the middle vertical map in (6.3) is injective. This completes the proof of the inductive step.

Before discussing the base case \( r = 1 \), we prove another lemma. Let \( v : C \to \mathbb{Q} \cup \{+\infty\} \) be the valuation on \( C \), normalized by \( v(p) = 1 \). If \( \vec{t} = (t_1, \ldots, t_s) \in C^s \) for some \( s \), define \( v(\vec{t}) := \min_j v(t_j) \).

**Lemma 6.4.**
(a) If $\vec{t}_1, \ldots, \vec{t}_n \in C^*$ are $\mathbb{Q}_p$-independent, then \( \{ v\left( \sum a_i \vec{t}_i \right) : (a_1, \ldots, a_n) \in (\mathbb{Z}_p)^n - (p\mathbb{Z}_p)^n \} \) is finite.

(b) If $t_1, \ldots, t_n \in C$, then \( \{ v(t) : t = \sum a_i t_i \neq 0 \text{ for some } a_i \in \mathbb{Q}_p \} \) has finite image in $\mathbb{Q}/\mathbb{Z}$. 

Proof.

(a) The function \( (\mathbb{Z}_p)^n - (p\mathbb{Z}_p)^n \to \mathbb{Q} \)
\[
(a_1, \ldots, a_n) \mapsto v\left( \sum a_i \vec{t}_i \right)
\]
is continuous for the $p$-adic topology on the left and the discrete topology on the right, so by compactness its image is finite.

(b) By replacing the $t_i$ with a basis for their $\mathbb{Q}_p$-span, we reduce to the $s = 1$ case of (a).

From now on, we assume $r = 1$; i.e., $\Lambda \subseteq R$. Choose a $\mathbb{Q}_p$-basis $f_1, \ldots, f_n$ of $\Lambda$. We may assume that $D$ is the unit polydisk, so the coefficients of each $f_i$ tend to 0. Multiply all the $f_i$ by a single power of $p$ to assume that $f_i \in \mathcal{O}_C[[z_1, \ldots, z_d]]$. For some $m$, the images of $f_i$ in $C[[z_1, \ldots, z_d]]/(z_1, \ldots, z_d)^m$ are $\mathbb{Q}_p$-independent, because a descending sequence of vector spaces in $\Lambda$ with zero intersection must be eventually zero. Fix such an $m$.

Let $M$ be the set of monomials $\mu$ in the $z_i$ whose total degree $\deg \mu$ is less than $m$. For each $\mu$, let $c_\mu \in C$ be the coefficient of $\mu$ in $f_i$. For each $\mu \in M$, apply Lemma 6.4(b) to $c_\mu$ to obtain a finite subset $S_\mu$ of $\mathbb{Q}/\mathbb{Z}$. Let $S = \bigcup_{\mu \in M} S_\mu$. Let $q_1, \ldots, q_d$ be distinct primes greater than $m$ that do not appear in the denominators of rational numbers representing elements of $S$. For $i = 1, \ldots, n$, let $\vec{t}_i \in C^{#M}$ be the vector whose coordinates are the $c_\mu$ for $\mu \in M$. By choice of $m$, the $\vec{t}_i$ are $\mathbb{Q}_p$-independent. By Lemma 6.4(a), the set
\[
\{ v\left( \sum a_i \vec{t}_i \right) : (a_1, \ldots, a_n) \in (\mathbb{Z}_p)^n - (p\mathbb{Z}_p)^n \}
\]
is finite; choose $A \in \mathbb{Q}$ larger than all its elements. Choose a positive integer $N$ such that
\[
(6.5) \quad mN > (m - 1)(N + 1/q_i) + A
\]
for all $i$. Let
\[
U := \{ (z_1, \ldots, z_n) \in C^n : v(z_i) = N + 1/q_i \text{ for all } i \},
\]
so $U$ is open in $D$.

Consider an arbitrary nonzero element $f = \sum_{\mu} c_\mu \mu$ of $\Lambda$. So $f = \sum_{i=1}^n a_i f_i$ for some $(a_1, \ldots, a_n) \in \mathbb{Q}_p^n - \{0\}$. Let $u = (u_1, \ldots, u_d) \in U$. It remains to show that $f(u) \neq 0$.

By multiplying $f$ by a power of $p$, we may assume that $(a_1, \ldots, a_n) \in (\mathbb{Z}_p)^n - (p\mathbb{Z}_p)^n$. If $\mu = z_1^{e_1} \cdots z_d^{e_d}$ and $\deg \mu \geq m$, then
\[
(6.6) \quad v(\mu(u)) \geq 0 + e_1 N + \cdots + e_d N \geq mN
\]
On the other hand, the definition of $A$ yields $\xi \in M$ such that $v(c^{\xi}) < A$, so that
\[
(6.7) \quad v(c^{\xi} \xi(u)) < A + \sum_{i=1}^d e_i (N + 1/q_i) \leq mN
\]
by (6.5).

To show that $f(u) \neq 0$, it remains to show that the valuations $v(c_\mu \mu(u))$ for $\mu \in M$ are distinct, since then the minimum of these equals $v(f(u))$, by (6.6) and (6.7). In fact, if
\[ \mu = z_1^{e_1} \cdots z_d^{e_d} \text{ and } \deg \mu < m, \] then for each \( i \) the choice of the \( q_i \) implies \( v(c^\mu_i) \in \mathbb{Z}_{(q_i)} \) (i.e., \( q_i \) does not divide the denominator of \( v(c^\mu_i) \)), so
\[ v(c^\mu_i(u)) \in \frac{e_i}{q_i} + \mathbb{Z}_{(q_i)}; \]
moreover \( e_i \leq \deg \mu < m < q_i \), so \( e_i \) is determined by \( v(c^\mu_i(u)) \). This completes the proof of Proposition 6.1.

7. Proof of Theorem 1.6

Lemma 7.1. Let \( k \subseteq k' \) be an extension of algebraically closed valued fields such that \( k \) is dense in \( k' \). Then for any \( k \)-variety \( B \), the set \( B(k) \) is dense in \( B(k') \) with respect to the analytic topology.

Proof. Let \( b' \in B(k') \) be a point to be approximated by \( k \)-points. We may replace \( B \) by the Zariski closure of the image of \( b' \) under \( B_{k'} \rightarrow B \). Then \( b' \) is a smooth point. We may shrink \( B \) to assume that \( B \) is étale over \( \mathbb{A}^n \) for some \( n \). Since \( k \) is algebraically closed, we may reduce to the case that \( B \) is open in \( \mathbb{A}^n \). This case follows from \( k \) being dense in \( k' \). \( \square \)

Proof of Theorem 1.6. Let \( b_0 \in B(\mathcal{O}_C) \). We need to show that any neighborhood \( U_0 \) of \( b_0 \) in \( B(\mathcal{O}_C) \) contains a nonempty open set \( V \) that does not meet \( B(O(\mathcal{O}_C))_{\text{jumping}} \). By Lemma 7.1, \( b_0 \) can be replaced by a smaller neighborhood \( U_1 \) of \( b_0 \) in \( B(\mathcal{O}_C) \), and Proposition 6.1 gives a nonempty open subset \( V \) of \( U_1 \) such that \( \rho(X_0) = \rho(X_\eta) \) for all \( b \in V \cap B(\mathcal{O}_C) \).

Suppose that \( b \in V \cap B(\mathcal{O}_C)_{\text{jumping}} \). By Corollary 3.10, \( b \) is contained in a \( \overline{K} \)-variety \( Z \) such that \( Z(C) \subseteq B(C)_{\text{jumping}} \). Lemma 7.1 gives \( b' \in V \cap Z(\overline{K}) \subseteq B(\mathcal{O}(\overline{K}))_{\text{jumping}} \), which contradicts the definition of \( V \). \( \square \)

8. Two proofs of Theorem 1.1

We show that the output of either the complex approach or the \( p \)-adic approach can be used to prove Theorem 1.1.

Proof that Theorem 1.4 implies Theorem 1.1. Let \( k \) and \( X \rightarrow B \) be as in Theorem 1.1. Let \( k_f \) be the subfield of \( k \) generated by all coefficients involved in the description of \( X \rightarrow B \). Let \( \overline{k_f} \) be the algebraic closure of \( k_f \) in \( k \). Theorem 1.4 shows that there exists \( b \in B(\overline{k_f}) \) outside \( B(\overline{k_f})_{\text{jumping}} \). Proposition 3.1 shows that Picard numbers are unchanged by base extension from one algebraically closed field to another, so \( b \notin B(k)_{\text{jumping}} \) too. \( \square \)

Proof that Theorem 1.6 implies Theorem 1.1. Assume that \( k \) and \( X \rightarrow B \) are as in Theorem 1.1. Replacing \( B \) by a dense open subscheme, we may assume that \( B \) is smooth over \( k \). Choose a finitely generated \( \mathbb{Z} \)-algebra \( A \) in \( k \) such that \( X \rightarrow B \) is the base extension of a morphism \( X_A \rightarrow B_A \) of \( A \)-schemes. By localizing \( A \), we may assume that \( X_A \rightarrow B_A \) is a smooth proper morphism, and that \( B_A \rightarrow \text{Spec } A \) is smooth with geometrically irreducible fibers. Because of Proposition 3.1, we may replace \( k \) by the algebraic closure of \( \text{Frac}(A) \) in \( k \).

By [Cas86, Chapter 5, Theorem 1.1], there exists an embedding \( A \hookrightarrow \mathbb{Z}_p \) for some prime \( p \). Extend it to an embedding \( k \hookrightarrow \mathbb{C}_p \). Base extend by \( A \hookrightarrow \mathbb{Z}_p \) to obtain \( X_{\mathbb{Z}_p} \rightarrow B_{\mathbb{Z}_p} \). Since \( B_{\mathbb{Z}_p} \rightarrow \text{Spec } \mathbb{Z}_p \) is smooth with geometrically irreducible special fiber, the set \( B_{\mathbb{Z}_p}(\mathcal{O}_{\mathbb{C}_p}) \) is nonempty. Apply Theorem 1.6 to \( X_{\mathbb{Z}_p} \rightarrow B_{\mathbb{Z}_p} \) to find a nonempty open subset \( U \) of
The field \( k \) is dense in \( \mathbb{C} \), since even \( \mathbb{Q} \) is dense in \( \mathbb{C} \), so Lemma 7.1 shows that \( U \) contains a point \( b \) of \( B(k) \); this \( b \) is as required in Theorem 1.1, because of Proposition 3.1.

9. Projective vs. proper

**Theorem 9.1.** Let \( B \) be a variety over a field \( k \) of characteristic 0. Let \( X \to B \) be a smooth proper morphism such that all closed fibers are projective. Then there exists a Zariski dense open subvariety \( U \) of \( B \) such that \( X_U \to U \) is projective.

**Proof.** We may assume that \( B \) is irreducible, say with generic point \( \eta \). Theorem 1.1 yields \( b \in B(k) \) such that a specialization map \( \text{NS}_X \to \text{NS}_B \) is an isomorphism. By assumption, \( X_b \) is projective, so we may choose an ample line bundle \( L_b \) on \( X_b \). By construction of \( b \), the class \( [L_b] \in \text{NS}_B \) comes from the class in \( 
abla \text{NS}_X \) of some line bundle \( L \) on \( X_\eta \). By Proposition 3.6, \( L \) is ample. Then \( L \) comes from a line bundle on \( X_L \) for some finite extension \( L \) of \( \kappa(B) \), so \( X_L \) is projective, so \( \eta \) is projective [EGA II, 6.6.5]. Finally, spreading out shows that \( X_U \to U \) is projective for some dense open subscheme \( U \) of \( B \) [EGA IV.III, 8.10.5(xiii)].

**Remark 9.2.** Under the hypotheses of Theorem 9.1, we may also deduce that every fiber of \( X \to B \) is projective: just apply Theorem 9.1 to each irreducible subvariety of \( B \). But we cannot deduce that \( X \to B \) is projective, or even that \( B \) can be covered by open sets \( U \) such that \( X_U \to U \) is projective, as the following example of Raynaud shows.

**Example 9.3** ([Ray70, XII.2]). Let \( A \) be a nonzero abelian variety over a field \( k \). Let \( \sigma \) be the automorphism \((x,y) \mapsto (x, x+y)\) of \( A \times A \). Let \( B \) be a rational curve whose singular locus is a single node, and let \( \tilde{B} \) be its normalization. Then one can construct an abelian scheme \( \mathcal{A} \to B \) whose base extension to \( \tilde{B} \) is simply \( (A \times A) \times \tilde{B} \) with the fibers above the two preimages of the node identified via \( \sigma \). Moreover, for any neighborhood \( U \) of the node, \( \mathcal{A}_U \to U \) is not projective.

**Question 9.4.** Can one construct a counterexample to Theorem 9.1 over \( k = \mathbb{F}_p \)?

**Example 9.5.** With notation as in Example 9.3, one can show that the first projection \( A \times A \times \tilde{B} \to A \) factors through a morphism \( \mathcal{A} \to A \) whose fiber above a given \( a \in A(k) \) is projective if and only if \( a \) is a torsion point. This provides a counterexample over \( \mathbb{F}_p \) except for the fact that \( \mathcal{A} \to A \) is not smooth.

10. Cycles of higher codimension

Many of our arguments apply to specialization of cycles of codimension greater than 1, but unfortunately there is a missing ingredient in each of our two approaches. In the complex approach, we would need a higher-codimension analogue of the Lefschetz (1, 1) theorem, i.e., either the Hodge conjecture or a weak version of it such as the variational Hodge conjecture. In the \( p \)-adic approach, we would need a higher-codimension analogue of the \( p \)-adic Lefschetz (1, 1) theorem (Theorem 5.20). Emerton has conjectured such an analogue (Conjecture 10.6).
10.1. **Cycle class maps.** For a smooth proper variety $X$ over a field, let $\mathcal{Z}^r(X)\mathbb{Q}$ be the vector space of codimension-$r$ cycles with $\mathbb{Q}$ coefficients.

**Definition 10.1.** Let $X$ be a smooth proper variety over an algebraically closed field $k$ of characteristic 0. Fix a prime $\ell$. Define $\rho^r(X)$ as the dimension of the $\mathbb{Q}_\ell$-span of the image of the $\ell$-adic cycle class map

$$\text{cl}_{\text{et}} : \mathcal{Z}^r(X)\mathbb{Q} \to H^{2r}_{\text{et}}(X, \mathbb{Q}_\ell(r)).$$

The homomorphism $\text{cl}_{\text{et}}$ factors through the vector space $\mathcal{Z}^r(X)\mathbb{Q}/\text{alg}$ of cycles modulo algebraic equivalence. Suppose that $k'$ is an algebraically closed field extension of $k$. Then every element of $\mathcal{Z}^r(X_{k'})\mathbb{Q}/\text{alg}$ comes from a cycle over $\kappa(V) \subseteq k'$ for some $k$-variety $V$, and can be spread out over some dense open subvariety in $V$, and hence is algebraically equivalent to the $k'$-base extension of the resulting cycle above any $k$-point of $V$. So $\mathcal{Z}^r(X)\mathbb{Q}/\text{alg} \to \mathcal{Z}^r(X_{k'})\mathbb{Q}/\text{alg}$ is surjective, and $\rho^r(X) = \rho^r(X_{k'})$. This, together with standard comparison theorems, shows that $\rho^r(X)$ equals its analogue defined using the Betti cycle class map

$$\text{cl}_{\text{Betti}} : \mathcal{Z}^r(X)\mathbb{Q} \to H^{2r}(X^{\text{an}}, \mathbb{Q})$$

when $k = \mathbb{C}$, and also its analogue defined using de Rham cohomology. The comparison with Betti cohomology shows that $\rho^r(X)$ equals $\dim_{\mathbb{Q}} \text{cl}_{\text{Betti}}(\mathcal{Z}^r(X)\mathbb{Q})$, which is independent of $\ell$.

As follows from [SGA 6, X App 7, especially §7.14], all the facts in Section 3.2 (other than facts not needed for the proofs of Theorem 1.1, namely the claims about ampleness and about the cokernel of specialization being torsion-free) have analogues with $\text{NS}X$ replaced by $\text{cl}_{\text{et}}(\mathcal{Z}^r(X)\mathbb{Q})$. In particular, in a smooth proper family, $\rho^r(X)$ can only increase under specialization, and the jumping locus for $\rho^r$ is a countable union of Zariski closed subsets.

10.2. **The complex approach.** The following will suffice for the higher-codimension analogue of Theorem 1.4.

**Conjecture 10.2 (Variational Hodge conjecture).** Let $f : \mathcal{X} \to B$ be a smooth projective morphism between smooth quasi-projective complex varieties with $B$ irreducible, and let $r$ be a nonnegative integer. Let $b \in B(\mathbb{C})$ and let $\alpha_b \in \text{cl}_{\text{Betti}}(\mathcal{Z}^r(\mathcal{X}_b)\mathbb{Q})$. If $\alpha_b$ is the restriction of a class $\alpha \in H^{2r}(\mathcal{X}^{\text{an}}, \mathbb{Q})$ (or equivalently, $\alpha_b$ is invariant under the monodromy action of $\pi_1(B^{\text{an}}, b)$), then there is a cycle class $\alpha' \in \text{cl}_{\text{Betti}}(\mathcal{Z}^r(\mathcal{X})\mathbb{Q})$ such that $\alpha_b$ is the restriction of $\alpha'$.

**Remark 10.3.** The variational Hodge conjecture is a consequence of the Hodge conjecture because, as in Proposition 1.4, the class $\alpha_b$ above is the restriction of a Hodge class $\overline{\alpha}$ on some smooth completion of $\mathcal{X}$. On the other hand, it is a priori a much weaker statement, since the Hodge class $\overline{\alpha}$ is an “absolute Hodge class” in the sense of [DMOS82, p. 28].

Assuming Conjecture 10.2, Theorem 1.4 has the following extension, whose proof is as in Sections 2.8 and 2.9 modified in the nonprojective case according to Remark 10.3.

**Theorem 10.4.** Assume that $k$ is an algebraic closure of a field $k_f$ that is finitely generated over $\mathbb{Q}$. Let $B$ be an irreducible $k_f$-variety. Let $\mathcal{X} \to B$ be a smooth proper morphism. Let $r$ be a nonnegative integer. If Conjecture 10.2 holds for $r$, then the set of $b \in B(k)$ such that $\rho^r(\mathcal{X}_b) > \rho^r(\mathcal{X}_0)$ is sparse.
Remark 10.5. In the blowing-up step in Section 4.9, we can no longer assert that blowing-up increases $\rho^r$ by an amount that is constant for all fibers above a Zariski open subset of $B$. Instead we use the following formula (cf. [Voi03, Theorem 9.27]) for the blow-up $X'$ of a smooth proper variety $X$ along a smooth center $Z$ of codimension $c$:

$$\rho^r(X') = \rho^r(X) + \sum_{0 \leq k \leq c-2} \rho^{r-k-1}(Z),$$

where $\rho^s(Z) := 0$ if $s < 0$. This shows that under specialization, the values of $\rho^r(X')$ and $\rho^r(X)$ jump by the same amount, provided that the values $\rho^{r-k-1}(Z)$ do not jump, which can be guaranteed outside a sparse subset of $B(k)$, by induction on $r$ and Proposition 1.15(b).

10.3. The $p$-adic approach. Now assume that $K, \mathcal{O}_K, k$ are as in Setup 1.5. Let $X \to \text{Spec} \mathcal{O}_K$ be a smooth proper morphism. We have a specialization map $\text{sp}$, cycle class maps $\text{cl}_{\text{dR}}$ and $\text{cl}_{\text{crys}}$ ([Har75, II.7.8] and [GM87], respectively), and a comparison isomorphism $\sigma_{\text{crys}}$ (cf. [Ber74, V.2.3.2]) making the following diagram commute:

$$
\begin{array}{ccc}
\mathcal{Z}^r(X_K)_\mathbb{Q} & \xrightarrow{\text{sp}} & \mathcal{Z}^r(X_k)_\mathbb{Q} \\
\text{cl}_{\text{dR}} & & \Downarrow \text{cl}_{\text{crys}} \\
H^2_{\text{dR}}(X_K/K) & \xrightarrow{\sigma_{\text{crys}}} & H^{2r}_{\text{crys}}(X_k/K).
\end{array}
$$

**Conjecture 10.6** ($p$-adic variational Hodge conjecture; cf. [Eme, Conjecture 2.2]). Let notation be as above. Let $Z_k \in \mathcal{Z}^r(X_k)_\mathbb{Q}$. If $\sigma_{\text{crys}}^{-1}(\text{cl}_{\text{crys}}(Z_k)) \in \text{Fil}^r H^{2r}_{\text{dR}}(X_K/K)$, then $Z_k$ comes from some $Z \in \mathcal{Z}^r(X_K)_\mathbb{Q}$.

**Remark 10.7.** Conjecture 10.6 is slightly more general than [Eme, Conjecture 2.2] in that the latter assumes that $\mathcal{O}_K$ is the Witt ring of its residue field.

**Remark 10.8.** The $r = 1$ case of Conjecture 10.6 is equivalent to Theorem 5.20 (except for tensoring with $\mathbb{Q}$ instead of $\mathbb{Z}[1/p]$): the distinction between $\mathcal{Z}^1$ and Pic is irrelevant since principal divisors on $X_k$ are specializations of principal divisors on $X_K$.

Repeating the $p$-adic proof of Theorem 10.6 yields the following.

**Theorem 10.9.** Assume Conjecture 10.6. Let notation be as in Setup 1.5. Let $r$ be a nonnegative integer. Then the set

$$B(\mathcal{O}_C)_{\text{jumping}} := \{ b \in B(\mathcal{O}_C) : \rho^r(X_b) > \rho^r(X_0) \}$$

is nowhere dense in $B(\mathcal{O}_C)$ for the analytic topology.

10.4. An implication. We introduced above two different variational forms of the Hodge conjecture, in order to extend the results on Néron-Severi groups to higher-degree classes. We now show that Emerton’s conjecture 10.6 is stronger.

**Theorem 10.10.** The $p$-adic variational Hodge conjecture (Conjecture 10.6) implies the variational Hodge conjecture (Conjecture 10.2).

The proof will use the following statement.

**Lemma 10.11.** Let $K$ be any field of characteristic 0. Let $B$ be a $K$-variety. Let $f : X \to B$ be a smooth projective morphism. Let $\alpha$ be a relative de Rham class, i.e., an element of
$\mathbf{H}^0(B, \mathbb{R}^{2r}f_*\Omega^\bullet_{X/B})$. Then there exists a countable set of closed subschemes $V_i$ of $B$ such that for any field extension $L \supseteq K$ and any $b \in B(L)$, the restriction $\alpha_b \in H^r_{\text{dR}}(\mathcal{X}_b/L)$ is algebraic (i.e., in $\text{cl}_{\text{dR}}(\mathcal{Z}^r(\mathcal{X}_b)_{\mathbb{Q}})$) if and only if $b \in \bigcup V_i(L)$.

**Proof.** This is a standard consequence of the fact that the codimension-$r$ subschemes in the fibers of $f$ are parameterized by countably many relative Hilbert schemes $H_j$, each of which is projective over $B$. Let $E$ be $\mathbb{R}^{2r}f_*\Omega^\bullet_{X/B}$ viewed as a geometric vector bundle on $B$. The relative cycle class map is a $B$-morphism $H_j \to \text{Fil}^r E$; its image is projective over $B$, and it parameterizes the classes of effective cycles given by $H_j$. Taking $\mathbb{Q}$-linear combinations of these yields countably many closed subvarieties $W_i$ of $\text{Fil}^r E$ that together parameterize all de Rham cycle classes in fibers of $f$. The desired subschemes $V_i$ are the inverse images of $W_i$ under the section $B \to \text{Fil}^r E$ given by $\alpha$.

**Proof of Theorem 10.14.** Let $X$, $B$, $f$, $b$, $\alpha_b$, and $\alpha$ be as in Conjecture 10.2. Let $Z \in \mathcal{Z}^r(\mathcal{X}_b)_{\mathbb{Q}}$ be such that $\text{cl}_{\text{Betti}}(Z) = \alpha_b$. Let $\overline{X}$ be a smooth projective completion of $X$. As in Remark 10.3 we note that $\alpha_b = \overline{\alpha} |_{\mathcal{X}_b}$, where $\overline{\alpha} \in H^2r(\overline{X}^\text{an}, \mathbb{Q})$ is a Hodge class. In the Leray spectral sequence, $\alpha$ maps to a relative Betti class in $\mathbf{H}^0(B^\text{an}, R^{2r}f_\ast \mathcal{C})$, and the relative Grothendieck comparison isomorphism identifies this with a relative de Rham class $\alpha_{\text{dR}} \in H^0(B, E)$, where $E$ is the sheaf $\mathbb{R}^{2r}f_*\Omega^\bullet_{X/B}$ equipped with the Hodge filtration $\text{Fil}^\bullet$ and Gauss-Manin connection. Since $\overline{\alpha}$ is Hodge and restricts to the same section of $E$ as $\alpha$, $\alpha_{\text{dR}}$ is a horizontal section lying in $H^0(B, \text{Fil}^r E)$.

There is a finitely generated subring $A \subset \mathbb{C}$ such that $X$, $B$, $f$, $b$, $Z$, $E$, and $\alpha_{\text{dR}}$ are obtained via base change from corresponding objects over $A$, which we next base extend by an embedding $A \hookrightarrow \mathbb{Z}_p$ provided by [Cas86, Chapter 5, Theorem 1.1]. To ease notation, from now on we use the notation of Setup 1.5 with $\mathcal{O}_K := \mathbb{Z}_p$, and reuse the symbols above to denote the corresponding objects over $\mathcal{O}_K$. Let $s \in B(k)$ be the reduction of $b \in B(\mathcal{O}_K)$. We may assume that the constituents of $Z$ are flat over $\mathcal{O}_K$, so that the special fiber $Z_s$ is the specialization of $Z_K$.

As in the proof of Lemma 5.2, we use $\text{cl}_{\text{cris}}(Z_s) \in H^r_{\text{cris}}(\mathcal{X}_s/K)$ to construct a horizontal section $\gamma_{\text{dR}}(Z_s)$ of $E|_D$ for some closed polydisk neighborhood $D$ of $b$ inside the residue disk in $B(\mathcal{O}_K)$ corresponding to $s$. (This time we need only the $\mathcal{O}_K$-points, not the $\mathcal{O}_K^*$-points.) Since $\gamma_{\text{dR}}(Z_s)$ and $\alpha_{\text{dR}, K}|_D$ are horizontal sections taking the same value at $b$ (namely, $\text{cl}_{\text{dR}}(Z_k)$), they agree everywhere on $D$ (this is just the fact that a convergent power series whose derivative vanishes identically on a polydisk is constant). In particular, for any $b' \in D \subset B(\mathcal{O}_K)$, the value of $\gamma_{\text{dR}}(Z_s)$ at the generic point $b'_K$ of $b'$ is in $\text{Fil}^r E_{b'_K}$. So Conjecture 10.6 applied to the $\mathcal{O}_K$-morphism $\mathcal{X}_{b'} \to B_{b'}$ implies that $Z_s$ is the specialization of some $Z' \in \mathcal{Z}^r(\mathcal{X}_K)_{\mathbb{Q}}$. Now the classes $\alpha_{\text{dR}, K}|_{b'_K} = \gamma_{\text{dR}}(Z_s)|_{b'_K}$ and $\text{cl}_{\text{dR}}(Z') \in H^2_{\text{dR}}(\mathcal{X}_{b'_K}/K)$ coincide because both are mapped by the isomorphism $\sigma_{\text{cris}}$ to $\text{cl}_{\text{cris}}(Z_s) \in H^r_{\text{cris}}(\mathcal{X}_s/K)$. So the value of $\alpha_{\text{dR}, K}$ on any point in $D$ is an algebraic class.

Applying Lemma 10.11 to $\mathcal{X}_K \to B_K$ and $\alpha_{\text{dR}, K}$ yields a countable set of closed subschemes $V_i$ of $B_K$. The previous paragraph shows that $D \subseteq \bigcup V_i(K)$. By Lemma 5.25 we have $\dim V_i = \dim B_K$ for some $i$. This $V_i$ contains the generic point $\eta$ of $B_K$. So $\alpha_{\text{dR}, K}|_{\eta} = \text{cl}_{\text{dR}}(Y_{\eta})$ for some $Y_{\eta} \in \mathcal{Z}^r(\mathcal{X}_K)_{\mathbb{Q}}$. Taking the closure in $\mathcal{X}_K$ of the constituents of $Y_{\eta}$ defines some $Y_K \in \mathcal{Z}^r(\mathcal{X}_K)_{\mathbb{Q}}$. To $Y_K$ we associate a relative de Rham class $\alpha'_{\text{dR}, K} \in H^0(B_K, \mathbb{R}^{2r}f_*\Omega^\bullet_{X/K}/B_K)$ and a Betti class $\alpha' := \text{cl}_{\text{Betti}}(Y_K) \in H^r(\mathcal{X}^\text{an}, \mathbb{Q})$, where the complex analytic spaces are obtained by fixing an $A$-algebra homomorphism $K \hookrightarrow \mathbb{C}$. The set
\{q \in B_K : \alpha'_{dR,K}|_q = \alpha_{dR,K}|_q\} is closed and contains \eta, so it contains b, which we now view as a K-point. Under the comparison isomorphisms, the equal de Rham classes \(\alpha'_{dR,K}|_b\) and \(\alpha_{dR,K}|_b\) correspond to the Betti classes \(\alpha'_b\) and \(\alpha_b\), so \(\alpha'_b = \alpha_b\).

\[\square\]

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References

[AKMW02] Dan Abramovich, Kalle Karu, Kenji Matsuki, and Jarosław Włodarczyk, Torification and factorization of birational maps, J. Amer. Math. Soc. 15 (2002), no. 3, 531–572 (electronic). MR 1896232 (2003c:14016)

[Ame09] Ekaterina Amerik, On an automorphism of Hilb\(^2\) of certain K3 surfaces, July 20, 2009. Preprint, arXiv:0907.3487.

[AV08] Ekaterina Amerik and Claire Voisin, Potential density of rational points on the variety of lines of a cubic fourfold, Duke Math. J. 145 (2008), no. 2, 379–408. MR 2449551

[Ber74] Pierre Berthelot, Cohomologie cristalline des schémas de caractère \(p > 0\), Lecture Notes in Mathematics, Vol. 407, Springer-Verlag, Berlin, 1974 (French). MR 0384804 (52 #5676)

[BI70] Pierre Berthelot and Luc Illusie, Classes de Chern en cohomologie cristalline, C. R. Acad. Sci. Paris Sér. A-B 270 (1970), A1695-A1697; ibid. 270 (1970), A1750–A1752 (French). MR 0269660 (42 #4555)

[BO83] P. Berthelot and A. Ogus, F-isocrystals and de Rham cohomology. I, Invent. Math. 72 (1983), no. 2, 159–199. MR 700767 (85c:14025)

[BM76] E. Bombieri and D. Mumford, Enriques' classification of surfaces in char. \(p > 0\), Invent. Math. 35 (1976), 197–232. MR 0491720 (58 #10922b)

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, Néron models, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR 1045822 (91i:14034)

[Bou98] Nicolas Bourbaki, Commutative algebra. Chapters 1–7, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998. Translated from the French; Reprint of the 1989 English translation. MR 1727221 (2001g:13001)

[Cas86] J. W. S. Cassels, Local fields, London Mathematical Society Student Texts, vol. 3, Cambridge University Press, Cambridge, 1986. MR 861410 (87i:11172)

[Del68] P. Deligne, Théorème de Lefschetz et critères de dégénérescence de suites spectrales, Inst. Hautes Études Sci. Publ. Math. (1968), no. 35, 259–278 (French). MR 0244265 (39 #5582)

[Del71] Pierre Deligne, Théorie de Hodge. II, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57 (French). MR 0498551 (58 #16653a)

[DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih, Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin, 1982. MR 654325 (84m:14046)

[EGA I] A. Grothendieck, Éléments de géométrie algébrique. I. Le langage des schémas, Inst. Hautes Études Sci. Publ. Math. (1960), no. 4, 228. MR 0217083 (36 #77a)

[EGA II] ———, Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes, Inst. Hautes Études Sci. Publ. Math. (1961), no. 8, 222. MR 0217084 (36 #177b)

[EGA III.I] ———, Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 167. MR 0217085 (36 #177c)
EGA IV.III] Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III, Inst. Hautes Études Sci. Publ. Math. (1966), no. 28, 255. MR 0217086 (36 #178)

[Ell04] Jordan S. Ellenberg, K3 surfaces over number fields with geometric Picard number one, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), Progr. Math., vol. 226, Birkhäuser Boston, Boston, MA, 2004, pp. 135–140. MR 2029866 (2004j:14016)

[Eme] Matthew Emerton, A p-adic variational Hodge conjecture and modular forms with complex multiplication. Preprint, available at http://www.math.northwestern.edu/~emerton/pdffiles/cm.pdf.

[GM87] Henri Gillet and William Messing, Cycle classes and Riemann-Roch for crystalline cohomology, Duke Math. J. 55 (1987), no. 3, 501–538. MR 904940 (89c:14025)

[GGP04] Mark Green, Philip A. Griffiths, and Kapil H. Paranjape, Cycles over fields of transcendence degree 1, Michigan Math. J. 52 (2004), no. 1, 181–187. MR 2043404 (2005f:14019)

[Har75] Robin Hartshorne, On the De Rham cohomology of algebraic varieties, Inst. Hautes Études Sci. Publ. Math. (1975), no. 45, 5–99. MR 0432647 (55 #5633)

[Har77], Algebraic geometry, Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Kle68] S. L. Kleiman, Algebraic cycles and the Weil conjectures, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 359–386. MR 0206011 (34 #5836)

[Kro84] Arthur Ogus, F-isocrystals and de Rham cohomology. II. Convergent isocrystals, Duke Math. J. 51 (1984), no. 4, 765–850. MR 771383 (86j:14012)

[Nér52] André Néron, Problèmes arithmétiques et géométriques rattachés à la notion de rang d’une courbe algébrique dans un corps, Bull. Soc. Math. France 80 (1952), 101–166 (French). MR 0056951 (15,151a)

[Roo95] Rutger Noot, Abelian varieties—Galois representation and properties of ordinary reduction, Compositio Math. 97 (1995), no. 1-2, 161–171. Special issue in honour of Frans Oort. MR 1355123 (97a:11093)

[Ogus84] Arthur Ogus, F-isocrystals and de Rham cohomology. II. Convergent isocrystals, Duke Math. J. 51 (1984), no. 4, 765–850. MR 771383 (86j:14012)

[Oor62] Frans Oort, Sur le schéma de Picard, Bull. Soc. Math. France 90 (1962), 1–14 (French). MR 0138627 (25 #219)

[Ray70] Michel Raynaud, Faisceaux amples sur les schémas en groupes et les espaces homogènes, Lecture Notes in Mathematics, Vol. 119, Springer-Verlag, Berlin, 1970 (French). MR 0260758 (41 #5381)
