The infinitesimal form of induced representation of the $\kappa$--Poincaré group

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Abstract
The infinitesimal form of the induced representation of the $\kappa$--Poincaré group is constructed. The infinitesimal action of the $\kappa$--Poincaré group on the $\kappa$--Minkowski space is described. The actions of these two infinitesimal forms on the solution of Klein-Gordon equation are compared.

1 Introduction
Recently, considerable interest has been paid to the deformations of group and algebras of space-time symmetries [7]. An interesting deformation of the Poincaré algebra [8], [6] as well as group [1] has been introduced which depend on the dimensional deformation parameter $\kappa$; the relevant objects are called $\kappa$--Poincaré algebra and $\kappa$--Poincaré group, respectively. Their structure was studied in some detail and many of their properties are now well understood. In particular, the induced representations of the deformed group were found [3] and the duality between $\kappa$--Poincaré group and $\kappa$--Poincaré algebra was also given [4]. Having the representations of the $\kappa$--Poincaré group and duality relations one can consider the infinitesimal form of the representation; in order to check whether we obtain the representation of the $\kappa$--Poincaré algebra. This is nontrivial as it is known from the construction of the induced representations the action of the Lorentz group on $q$--space is standard. Therefore the support spaces are described by the classical equation $q^2 = \text{const}$. On the other hand $q^2$ is not the Casimir operator of $\kappa$--Poincaré algebra.

In section 2 we describe the general definition of the induced representation of quantum group, than we find the infinitesimal form of the induced representation of $\kappa$--Poincaré group constructed by [3]. We show that in the massive case the infinitesimal form of the induced representation is the representation of the

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\(\kappa\)-Poincaré algebra. Next in section 3 we consider the infinitesimal action of \(\mathcal{P}\kappa\) on \(\kappa\)-Minkowski space \(\mathcal{M}\kappa\), and describe the Klein-Gordon equation. And on the end we compare actions of our two infinitesimal forms over the solution K-G equation.

In that paper we assume that \(g_{\mu\nu}\) is diagonal \((+,-,-,-)\) tensor matrix.

Let us remind the definition of \(\kappa\)-Poincaré group. The \(\kappa\)-Poincaré group \(\mathcal{P}\kappa\) is the Hopf *-algebra generated by selfadjoint elements \(\Lambda_{\mu\nu}\), \(v^\mu\) subject to the following relations:

\[
\begin{align*}
[\Lambda_{\alpha\beta}, v^\kappa] &= -\frac{i}{\kappa}((\Lambda_{\alpha0} - \delta_{\alpha0})\Lambda^\kappa_{\beta} + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\kappa}), \\
[v^\kappa, v^\sigma] &= \frac{i}{\kappa}(\delta^\kappa_0 v^\sigma - \delta^\sigma_0 v^\kappa), \\
[\Lambda_{\alpha\beta}, \Lambda_{\mu\nu}] &= 0.
\end{align*}
\]

The comultiplication, antipode and counit are defined as follows:

\[
\begin{align*}
\Delta \Lambda_{\mu\nu} &= \Lambda_{\mu\alpha} \otimes \Lambda^\alpha_{\nu}, \\
\Delta v^\mu &= \Lambda_{\mu\nu} \otimes v^\nu + v^\mu \otimes I, \\
S(\Lambda_{\mu\nu}) &= \Lambda^\nu_{\mu}, \\
S(v^\mu) &= -\Lambda_{\mu\nu} v^\nu, \\
\epsilon(\Lambda_{\mu\nu}) &= \delta_{\mu\nu}, \\
\epsilon(v^\mu) &= 0.
\end{align*}
\]

Its dual structure, the \(\kappa\)-Poincaré algebra \(\mathcal{\bar{P}}\kappa\) (in the Majid and Ruegg basis \([9]\)) is a quantized universal enveloping algebra in the sense of Drinfeld \([10]\) described by the following relations:

\[
\begin{align*}
[M_{ij}, P_0] &= 0, \\
[M_{ij}, P_k] &= i(g_{jk}P_i - g_{ik}P_j), \\
[M_{i0}, P_0] &= iP_i, \\
[M_{i0}, P_k] &= -\frac{i\kappa}{2}g_{ik}(1 - e^{2\kappa P_0}) + \frac{i}{2\kappa}g_{ik}P_rP_r - \frac{i}{\kappa}P_i P_k, \\
[P_{\mu\nu}, P_0] &= 0, \\
[M_{ij}, M_{rs}] &= i(g_isM_{jr} - g_{js}M_{ir} + g_{jr}M_{is} - g_{ir}M_{js}), \\
[M_{i0}, M_{rs}] &= -i(g_isM_{0r} - g_{ir}M_{0s}), \\
[M_{00}, M_{j0}] &= -iM_{ij}.
\end{align*}
\] (1.1)

The coproducts, counit and antipode:

\[
\begin{align*}
\Delta P_0 &= I \otimes P_0 + P_0 \otimes I, \\
\Delta P_k &= P_k \otimes e^{-\frac{P_0}{\kappa}} + I \otimes P_k.
\end{align*}
\]
\[\Delta M_{ij} = M_{ij} \otimes I + I \otimes M_{ij},\]
\[\Delta M_{i0} = I \otimes M_{i0} + M_{i0} \otimes e^{-\frac{P_0}{\kappa}} + \frac{1}{\kappa} M_{ij} \otimes P_j,\]
\[\varepsilon(M_{\mu\nu}) = 0; \quad \varepsilon(P_\nu) = 0,\]
\[S(P_0) = -P_0,\]
\[S(P_i) = -e^{\frac{P_i}{\kappa}} P_i,\]
\[S(M_{ij}) = -M_{ij},\]
\[S(M_{i0}) = -e^{\frac{P_0}{\kappa}} (M_{i0} - \frac{1}{\kappa} M_{ij} P_j),\]

where \(i, j, k, r, s = 1, 2, 3.\)

Fact that the \(\kappa\)-Poincarè algebra is dual to the \(\kappa\)-Poincarè group was proved by [4]. The fundamental duality relations read:

\[< P_\mu, f(v) > = i \left. \frac{\partial}{\partial \nu} f(v) \right|_{v=0},\]
\[< M_{\mu\nu}, f(\Lambda) > = i \left. \left( \frac{\partial}{\partial \Lambda^{\mu\nu}} - \frac{\partial}{\partial \Lambda^{\nu\mu}} \right) f(\Lambda) \right|_{\Lambda=I} \]  

(1.2)

2 The infinitesimal form of the induced representation

Let us recall the definition of the representation of a quantum group \(A(G),\) acting in the linear space \(V.\) It is simply a map

\[\varrho : V \rightarrow V \otimes A(G)\]

satisfying

\[(I \otimes \Delta) \otimes \varrho = (\varrho \otimes I) \otimes \varrho\]

The induced representations are defined as follow [3], [11]: given any quantum group \(A(G),\) its quantum subgroup \(A(H)\) and the representation \(\varrho\) of the latter acting in the linear space \(V,\) we consider the subspace of the space \(V \otimes A(G)\) defined by the coequivariance condition:

\[\tilde{V} = \{ F \in V \otimes A(G) : id \otimes (\Pi \otimes id) \circ \Delta_G F = (\varrho \circ id) F \}\]

where \(\Pi : A(G) \rightarrow A(H)\) is epimorphism defining the subgroup \(A(H).\)

The induced representation is defined as a right action:

\[\tilde{\varrho} : \tilde{V} \rightarrow \tilde{V} \otimes A(G)\]
\[\tilde{\varrho} = id \otimes \Delta\]
In the paper [3] Maślanka obtained the following form of the induced representation in the massive case:

\[ \rho_R : f_i(q_\mu) \rightarrow D_{ij}(R(\tilde{q}, \Lambda)) \cdot \exp(-i\kappa \ln(\cosh(\frac{m}{\kappa}) - \frac{q_0}{m} \sinh(\frac{m}{\kappa})) \otimes v^0) \cdot \exp(\frac{i\kappa \sinh(\frac{m}{\kappa}) q_k}{m \cosh(\frac{m}{\kappa}) - q_0 \sinh(\frac{m}{\kappa})} \otimes v^k) f_j(q_\mu \otimes \Lambda^\mu_\nu) \]

where:

\[ q_\mu = m \Lambda^\mu_\mu \]

\[ \tilde{q}_0 = \frac{mq_0 \cosh(\frac{m}{\kappa}) - m^2 \sinh(\frac{m}{\kappa})}{m \cosh(\frac{m}{\kappa}) - q_0 \sinh(\frac{m}{\kappa})} \]

\[ \tilde{q}_k = \frac{mq_k}{m \cosh(\frac{m}{\kappa}) - q_0 \sinh(\frac{m}{\kappa})} \]

and \( f(q) \) are the square integrable functions defined on the hyperboloid \( q^2 = m^2 \), \( q_0 = \sqrt{q_i q_i + m^2} \) and taking values in the vector space carrying the unitary representation of the rotation group, the matrices \( D_{ij} \) are constructed in the same way as the matrices of the representation of classical orthogonal group and \( R(\tilde{q}, \Lambda) \) is a classical Wigner rotation corresponding to the momentum \( \tilde{q} \) and transformation \( \Lambda \), [3]. Of course, the right-hand side of eq.(2.1) is to be understood here as an element of the tensor product of the algebra of functions on the hyperboloid \( q^2 = m^2 \) and the group \( \mathcal{P}_\kappa \). Following Woronowicz [12] we define the infinitesimal form of the induced representation. For any element \( X \) of enveloping algebra \( \mathcal{P}_\kappa \) and for any \( f \in \mathcal{P}_\kappa \) if

\[ X(f) = \langle X, f \rangle \]

and

\[ \rho_R : \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{P}_\kappa \]

we define

\[ \tilde{X} : \mathcal{V} \rightarrow \mathcal{V} \]

by

\[ \tilde{X} = (I \otimes X) \circ \rho_R \]

Using the duality relations [1,2] after some calculi we arrive at the following formulas describing the infinitesimal form of our representation:

\[ \tilde{M}_{ij} = i(q_i \frac{\partial}{\partial q^j} - q_j \frac{\partial}{\partial q^i}) + \varepsilon_{ijk} s_k \]

\[ \tilde{M}_{i0} = -iq_0 \frac{\partial}{\partial q^i} + \varepsilon_{ijk} q_j s_k \]

\[ \tilde{M}_{0i} = -iq_0 \frac{\partial}{\partial q^i} + \varepsilon_{ijk} q_j s_k \]

\[ \tilde{M}_{00} = -q_0 \frac{\partial}{\partial q^0} + \varepsilon_{ijk} q_j s_k \]
\[ \tilde{P}_0 = p_0 = \kappa \ln(\cosh(m/\kappa) - q_0/m \sinh(m/\kappa)) \]
\[ \tilde{P}_j = p_j = -\kappa \frac{\sinh(m/\kappa) q_j}{m \cosh(m/\kappa) - q_0 \sinh(m/\kappa)} \]  

(2.2)

where \( s_k, \quad (k = 1, 2, 3) \) are the infinitesimal forms of representation \( D_{ij} \) (The representation of SU(2) algebra).

It is easy to check that our operators satisfy the relations of the \( \kappa \)-Poincaré algebra (1.1), so the infinitesimal form is the representation of the \( \kappa \)-Poincaré algebra.

3 The \( \kappa \)-Minkowski space, the infinitesimal action, K-G equation

The \( \kappa \)-Minkowski space ([1], [2]) \( M_\kappa \) is a universal \(*\)-algebra with unity generated by four selfadjoint elements \( x^\mu \) subject to the following conditions:

\[ [x^\mu, x^\nu] = \frac{i}{\kappa} (\delta^\mu_0 x^\nu - \delta^\nu_0 x^\mu) . \]

Equipped with the standard coproduct:

\[ \Delta x^\mu = x^\mu \otimes I + I \otimes x^\mu, \]

antipode \( S(x^\mu) = -x^\mu \) and counit \( \varepsilon(x^\mu) = 0 \) it becomes a quantum group.

The product of generators \( x^\mu \) will be called normally ordered if all \( x^0 \) factors stand leftmost. This definition can be used to ascribe a unique element \( :f(x) : \) of \( M_\kappa \) to any polynomial function of four variables \( f \). Formally, it can be extended to any analytic function \( f \).

Let us now define the infinitesimal action of \( P_\kappa \) on \( M_\kappa \). The \( \kappa \)-Minkowski space carries a left-covariant action of \( \kappa \)-Poincaré group \( P_\kappa \), \( \varrho_L : M_\kappa \to P_\kappa \otimes M_\kappa \), given by

\[ \varrho_L(x^\mu) = \Lambda^\mu_\nu \otimes x^\nu + v^\mu \otimes I. \]  

(3.1)

Let \( X \) be any element of the Hopf algebra dual to \( P_\kappa \) – the \( \kappa \)-Poincaré algebra \( P_\kappa \). The corresponding infinitesimal action:

\[ \tilde{X} : M_\kappa \to M_\kappa \]

is defined as follows: for any \( f \in M_\kappa \),

\[ \tilde{X} f = (X \otimes I) \circ \varrho_L(f). \]

The following forms of generators were obtained in [3]:

\[ \tilde{P}_\mu : f : = i \frac{\partial f}{\partial x^\mu}. \]
\[ \hat{M}_{ij} : f : = -i(x_i \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^i}) f : \]

\[ \hat{M}_{i0} : f : = : \left[ ix^i_0 \frac{\partial}{\partial x^i} - x_i \left( \frac{\kappa}{2} \left( 1 - e^{-\frac{2\kappa}{\sigma} q^a} \right) - \frac{1}{2\kappa} \Delta \right) + \frac{1}{\kappa} x^k \frac{\partial^2}{\partial x^k \partial x^i} \right] f : \]

In that case these operators are not satisfying relation of \( \kappa \)-Poincaré algebra (1.1), because the action of \( \kappa \)-Poincaré algebra (group) on the \( \kappa \)-Minkowski space is antirepresentation (not representation). It is clear from the following equation:

\[ < AB f(P), \varphi(x) > = < f(P), \hat{B} \hat{A} \varphi(x) > \]

for \( A, B, P \in \tilde{\mathcal{P}}_\kappa \).

The deformed Klein-Gordon equation we write in the two equivalent forms:

\[ \left( \partial + \frac{m^2}{8} \right) f = 0, \]

or

\[ \left[ \partial_0^2 - \partial_i^2 + m^2 \left( 1 + \frac{m^2}{4\kappa^2} \right) \right] f = 0, \]

where operators \( \partial_0, \partial_i, \partial \) are defined:

\[ \partial_0 : f : = : \left( \kappa \sin \left( \frac{1}{\kappa} \frac{\partial}{\partial x^0} \right) + \frac{i}{2\kappa} e^{\frac{i}{\kappa} \frac{\partial}{\partial x^0}} \Delta \right) f : \]

\[ \partial_i : f : = : \left( e^{\frac{i}{\kappa} \frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^i} \right) f : \]

\[ \partial : f : = : \left( \frac{\kappa^2}{4} \left( 1 - \cos \left( \frac{1}{\kappa} \frac{\partial}{\partial x^0} \right) \right) - \frac{1}{8} e^{\frac{i}{\kappa} \frac{\partial}{\partial x^0}} \Delta \right) f : . \]

We can write the solution of K-G equation in the following wave function [5]:

\[ \Phi(x^\mu) = : \int \frac{d^3q}{q^0} a(\vec{q}) e^{-ip_\mu(q)x^\mu} : \]

where \( p_\mu \) is deformed of \( q_\mu \) defined in eq.(2.2).

For \( \hat{X} = \hat{M}_{i0}, \hat{M}_{ij}, \hat{P}_\mu \), following the paper [4], let us define the operators \( \mathcal{X}(q) \) by the following relation:

\[ \hat{X}(q_\nu) \Phi(x^\mu) = : \int \frac{d^3q}{q^0} \{ \mathcal{X}(q_\nu) a(\vec{q}) \} e^{-ip_\mu(q)x^\mu} : \]

It is easy to see that:

\[ \mathcal{M}_{ij}(q) = -i(q_i \frac{\partial}{\partial q^j} - q_j \frac{\partial}{\partial q^i}) = -\hat{M}_{ij}(q) \]
\[ \mathcal{M}_{0\theta}(q) = i q_0 \frac{\partial}{\partial q^\theta} = -\tilde{M}_{0\theta}(q) \]
\[ \mathcal{P}_\mu(q) = p_\mu(q) = \tilde{P}_\mu(q) \]

The last relations compare generators defined in [5] from with our found from induced representation.

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