INVERSE-CLOSEDNESS OF THE SET
OF INTEGRAL OPERATORS
WITH $L_1$-CONTINUOUSLY VARYING KERNE LTS

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Abstract. Let $N$ be an integral operator of the form

$$(Nu)(x) = \int_{\mathbb{R}^c} n(x, x-y) u(y) \, dy$$

acting in $L_p(\mathbb{R}^c)$ with a measurable kernel $n$ satisfying the estimate

$$|n(x, y)| \leq \beta(y),$$

where $\beta \in L_1$. It is proved that if the function $t \mapsto n(t, \cdot)$ is continuous in the $L_1$-norm and the operator $1 + N$ has an inverse, then $(1 + N)^{-1} = 1 + M$, where $M$ is an integral operator possessing the same properties.

1. Introduction

A class $\mathcal{A}$ of linear operators is called inverse-closed if the inverse to any operator from $\mathcal{A}$ also belongs to $\mathcal{A}$. Usually, an inverse-closed class forms a subalgebra of the algebra of all bounded operators. The investigation of inverse-closed subalgebras (full subalgebras) had its origin in Wiener's theorem on absolutely convergent Fourier series [1, 2, 3]. Wiener's theorem implies that if the operator $1 + N$, where $N$ is an operator of convolution with a summable function, is invertible, then $(1 + N)^{-1} = 1 + M$, where $M$ is also an operator of convolution with a summable function. For more recent results on inverse-closed classes, see [5]–[26] and references therein.

This paper deals with the integral operator in $L_p(\mathbb{R}^c, \mathbb{E})$, $1 \leq p \leq \infty$, of the form

$$(Nu)(x) = \int_{\mathbb{R}^c} n(x, x-y) u(y) \, dy.$$ 

It is assumed that $\mathbb{E}$ is a finite-dimensional Banach space, the values $n(x, y)$ of the kernel $n$ are bounded linear operators acting in $\mathbb{E}$, and the kernel $n$ is measurable and satisfies the estimate

$$(*) \quad \|n(x, y)\| \leq \beta(y),$$

where $\beta \in L_1$. The main result of the paper (Theorem 5.3) states that if the function $t \mapsto n(t, \cdot)$ is continuous in the $L_1$-norm and the operator $1 + N$ has an inverse, then $(1 + N)^{-1} = 1 + M$, where $M$ is an integral operator possessing the same properties.

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Proposition 2.2. (Lusin’s theorem [27, ch. 4, § 11, 20, 21]) states that if estimate (⋆) holds for the kernel $m$ of the operator $N$ and $1 + N$ is invertible in $L_p(\mathbb{R}^c, E)$, then $(1 + N)^{-1} = 1 + M$, where $M$ is an integral operator with a kernel $m$ satisfying estimate (⋆) as well. This easily implies that the fact of the invertibility of $1 + N$ and the kernel $m$ do not depend on $p$. On the other hand, it was known [20] that (for a wide class of operators $T$) if ‘coefficients’ of $T : L_p \to L_p$ vary continuously in an abstract sense, then $T^{-1}$ has the same property (provided $T^{-1}$ exists). Here the abstract continuity means that $S_hTS_{-h}$, where $(S_hu)(x) = u(x-h)$, continuously depends upon $h$ on subspaces of compactly supported (in a uniform sense) functions. If $p = \infty$ such abstract continuity of the integral operator $N$ exactly means that the function $t \mapsto n(t, \cdot)$ is continuous in the $L_1$-norm. Technical difficulties of the proof mostly consist of the correct usage of the Lebesgue integral. As a generalization of the main result, we show (Theorem 7.3) that an inverse to a difference-integral operator with ‘continuous’ coefficients also has ‘continuous’ coefficients.

The idea of the proof consists of a combination of two results. The first result [8, 11, 20, 21] states that if estimate (⋆) holds for the kernel $n$ of the operator $N$ and $1 + N$ is invertible in $L_p(\mathbb{R}^c, E)$, then $(1 + N)^{-1} = 1 + M$, where $M$ is an integral operator with a kernel $m$ satisfying estimate (⋆) as well. This easily implies that the fact of the invertibility of $1 + N$ and the kernel $m$ do not depend on $p$. On the other hand, it was known [20] that (for a wide class of operators $T$) if ‘coefficients’ of $T : L_p \to L_p$ vary continuously in an abstract sense, then $T^{-1}$ has the same property (provided $T^{-1}$ exists). Here the abstract continuity means that $S_hTS_{-h}$, where $(S_hu)(x) = u(x-h)$, continuously depends upon $h$ on subspaces of compactly supported (in a uniform sense) functions. If $p = \infty$ such abstract continuity of the integral operator $N$ exactly means that the function $t \mapsto n(t, \cdot)$ is continuous in the $L_1$-norm. Technical difficulties of the proof mostly consist of the correct usage of the Lebesgue integral. As a generalization of the main result, we show (Theorem 7.3) that an inverse to a difference-integral operator with ‘continuous’ coefficients also has ‘continuous’ coefficients.

The paper is organized as follows. General facts concerning the Lebesgue integral are recalled in Section 2. In Section 3, we describe some properties of the class of integral operators majorized by a convolution with a function $\beta \in L_1$. In Section 4, we discuss operators whose coefficients (kernels) vary continuously in an abstract sense. In Section 5, we prove Theorem 5.3 which is the main result of this paper. In Section 6, we show that the integral operator $N$ considered possesses the property of local compactness. This fact allows us to generalize Theorem 5.3 to a class of difference-integral operators (Section 7).

2. General notation and the Lebesgue integral

Let $X$ and $Y$ be Banach spaces. We denote by $B(X, Y)$ the space of all bounded linear operators acting on $X$ to $Y$. If $X = Y$ we use the brief notation $B(X)$. We denote by $1 \in B(X)$ the identity operator.

As usual, $Z$ is the set of all integers and $\mathbb{N}$ is the set of all positive integers.

Let $c \in \mathbb{N}$. Unless otherwise is explicitly stated, the linear space $\mathbb{R}^c$ is considered with the Euclidian norm $| \cdot |$. For $x \in \mathbb{R}^c$ and $r > 0$, we denote by $B(x, r)$ the open ball $\{ y \in \mathbb{R}^c : |y - x| < r \}$ with centre at $x$ and radius $r$.

We denote by $\mu$ the Lebesgue measure on $\mathbb{R}^c$. We accept that a measurable function [27, 28, 29] may be undefined on a set of measure zero. We say that $E \subseteq \mathbb{R}^c$ is a set of full measure if its complement has measure zero.

Let $E$ be a measurable subset of $\mathbb{R}^c$. The point $x \in E$ is called [30, ch. 1, § 2] a point of density of $E$ if

$$\lim_{r \to +0} \frac{\mu(E \cap B(x, r))}{\mu(B(x, r))} = 1.$$ 

Proposition 2.1 (Lebesgue’s density theorem [30, ch. 1, § 2, Proposition 1]). Almost every point of a measurable set $E \subseteq \mathbb{R}^c$ is a point of density of $E$.

Proposition 2.2 (Lusin’s theorem [27, ch. 4, § 5 Proposition 1]). Let $X$ be a Banach space. A function $f : \mathbb{R}^c \to X$ is measurable if and only if for any compact set $K \subseteq \mathbb{R}^c$ and any $\varepsilon > 0$ there exists a compact set $K_1 \subseteq K$ such that $\mu(K \setminus K_1) < \varepsilon$ and the restriction of $f$ to $K_1$ is continuous.

Let $E$ be a fixed finite-dimensional Banach space with the norm $| \cdot |$. We denote by $L_p = L_p(\mathbb{R}^c, E)$, $1 \leq p < \infty$, the space of all measurable functions $u : \mathbb{R}^c \to E$
bounded by the semi-norm
\[ \|u\| = \|u\|_{L_p} = \left( \int_{\mathbb{R}^c} |u(x)|^p \, dx \right)^{1/p}, \]
and we denote by \( L^\infty = L^\infty(\mathbb{R}^c, \mathbb{E}) \) the space of all measurable essentially bounded functions \( u : \mathbb{R}^c \to \mathbb{E} \) with the semi-norm
\[ \|u\| = \|u\|_{L^\infty} = \text{ess sup} |u(x)|. \]
Finally, we denote by \( L_p = L_p(\mathbb{R}^c, \mathbb{E}) \), \( 1 \leq p \leq \infty \), the Banach space of all classes of functions \( u \in L_p \) with the identification almost everywhere. For more details, see [27, 28, 29]. Usually they do not distinguish the spaces \( L_p \) and \( L_p \). For our purposes it is not always convenient. Both the semi-norm on \( L_p \) and the induced norm on \( L_p \) are called \( L_p \)-norms.

**Proposition 2.3** (Lebesgue’s theorem [27, ch. 4, § 3, 7, Theorem 6]). Let \( p < \infty \), \( u_i \in L_p(\mathbb{R}^c, \mathbb{E}) \) converges almost everywhere to a function \( u \), and there exists a nonnegative function \( g \in L_p(\mathbb{R}^c, \mathbb{R}) \) such that \( |u_i(x)| \leq g(x) \) for almost all \( x \) and all \( i \). Then \( u \in L_p \) and \( u_i \) converges to \( u \) in \( L_p \)-norm.

**Proposition 2.4.** Let \( 1 \leq p \leq \infty \), \( u_i \in L_p(\mathbb{R}^c, \mathbb{E}) \), and the series \( \sum_{i=1}^{\infty} u_i \) converges absolutely, i.e. \( \sum_{i=1}^{\infty} \|u_i\|_{L_p} < \infty \). Then the series \( \sum_{i=1}^{\infty} u_i(x) \) converges absolutely at almost all \( x \), the function \( s(x) = \sum_{i=1}^{\infty} u_i(x) \) belongs to \( L_p \), and the series \( \sum_{i=1}^{\infty} u_i \) converges to \( s \) in \( L_p \)-norm.

**Proof.** For the case \( p < \infty \), e.g. see [27, ch. 4, § 3, 3, Proposition 6]. The case \( p = \infty \) is evident. \( \square \)

**Proposition 2.5** (Fubini’s theorem, [27, ch. 5, § 8, 4]). Let \( X \) be an arbitrary Banach space.

If \( n \in L_1(\mathbb{R}^c \times \mathbb{R}^c, X) \), then for almost all \( x \in \mathbb{R}^c \) the function
\[ y \mapsto n(x, y) \]
is defined for almost all \( y \in \mathbb{R}^c \) and belongs to \( L_1(\mathbb{R}^c, X) \); the function
\[ x \mapsto \int_{\mathbb{R}^c} n(x, y) \, dy \]
is defined for almost all \( x \in \mathbb{R}^c \) and belongs to \( L_1(\mathbb{R}^c, X) \); and
\[ \int_{\mathbb{R}^c} \left( \int_{\mathbb{R}^c} n(x, y) \, dy \right) \, dx = \int_{\mathbb{R}^c} \left( \int_{\mathbb{R}^c} n(x, y) \, dx \right) \, dy. \]

If \( n : \mathbb{R}^c \times \mathbb{R}^c \to X \) is measurable and
\[ \int_{\mathbb{R}^c} \left( \int_{\mathbb{R}^c} \|n(x, y)\| \, dy \right) \, dx < \infty, \]
then \( n \in L_1(\mathbb{R}^c \times \mathbb{R}^c, X) \).

**Corollary 2.6.** A measurable subset \( E \subset \mathbb{R}^c \times \mathbb{R}^c \) has measure zero if and only if for almost all \( x \in \mathbb{R}^c \) the set \( E_x = \{ y \in \mathbb{R}^c : (x, y) \in E \} \) has measure zero.

**Proposition 2.7.** Let \( X \) be a Banach space and a sequence \( u_i \in L_1(\mathbb{R}^c, X) \) converge to \( u_0 \in L_1(\mathbb{R}^c, X) \) in norm. Then there exists a subsequence \( u_{i_k} \) that converges to \( u_0 \) almost everywhere.

**Proof.** This is a consequence of [27, ch. 4, § 3, 4, Theorem 3] and Proposition 2.4. \( \square \)
3. The class $\mathbf{N}_1$

We denote by $\mathbf{N}_1 = \mathbf{N}_1(\mathbb{R}^c, \mathcal{E})$ the set of all measurable functions $n : \mathbb{R}^c \times \mathbb{R}^c \to \mathcal{B}(\mathcal{E})$ satisfying the property: there exists a function $\beta \in L_1(\mathbb{R}^c, \mathbb{R})$ such that for almost all $(x, y) \in \mathbb{R}^c \times \mathbb{R}^c$

$$\|n(x, y)\| \leq \beta(y).$$

(3.1)

For convenience (without loss of generality), we assume that $\beta$ is defined everywhere. Kernels of the class $\mathbf{N}_1$ and the operators induced by them were considered in [8, 11], [20, § 5.4], and [21]. In order to show that the notation used in [20, § 5.4] and [21] is equivalent to the notation used in the present paper, we note the following Proposition.

**Proposition 3.1.** The function $n : \mathbb{R}^c \times \mathbb{R}^c \to \mathcal{B}(\mathcal{E})$ is measurable if and only if the functions $n_1(x, y) = n(x, x - y)$ is measurable.

**Proof.** The proof can be obtained as a word to word repetition of the proof of [20, Lemma 4.1.5]. □

**Proposition 3.2** ([20, Proposition 5.4.3]). For any $n \in \mathbf{N}_1(\mathbb{R}^c, \mathcal{E})$, the operator

$$\begin{equation}
(NU)(x) = \int_{\mathbb{R}^c} n(x, x - y) u(y) \, dy
\end{equation}
$$

(3.2)

acts in $L_p(\mathbb{R}^c, \mathcal{E})$ for all $1 \leq p \leq \infty$. More precisely, for any $u \in L_p(\mathbb{R}^c, \mathcal{E})$ the function $y \mapsto n(x, x - y) u(y)$ is integrable for almost all $x$, and the function $Nu$ belongs to $L_p(\mathbb{R}^c, \mathcal{E})$; if $u_1$ and $u_2$ coincide almost everywhere, then $Nu_1$ and $Nu_2$ also coincide almost everywhere. Besides,

$$\|N : L_p \to L_p\| \leq \|\beta\|_{L_1}.$$

(3.3)

We denote the set of all operators $N \in \mathcal{B}(L_p)$, $1 \leq p \leq \infty$, of the form (3.2) by $\mathbf{N}_1 = \mathbf{N}_1(L_p)$.

**Proposition 3.3.** If two functions $n, n_1 \in \mathbf{N}_1(\mathbb{R}^c, \mathcal{E})$ coincide almost everywhere on $\mathbb{R}^c \times \mathbb{R}^c$, then they induce the same operator (3.2).

**Proof.** Indeed, for any $u \in L_p$, by Proposition 3.2 and Corollary 2.6, for almost all $x \in \mathbb{R}^c$ the functions $v(y) = n(x, x - y) u(y)$ and $v_1(y) = n_1(x, x - y) u(y)$ coincide almost everywhere. Therefore $Nu$ and $N_1 u$ coincide almost everywhere. □

**Theorem 3.4** ([20, Theorem 5.4.7]). Let $N \in \mathbf{N}_1(L_p)$, $1 \leq p \leq \infty$. If the operator $1 + N$ is invertible, then $(1 + N)^{-1} = 1 + M$, where $M \in \mathbf{N}_1(L_p)$.

A version of Theorem 3.4 for the case of infinite-dimensional $\mathcal{E}$ can be found in [21].

**Corollary 3.5** ([20, Corollary 5.4.8]). Let $n \in \mathbf{N}_1$, and the operator $N$ be defined by (3.2). If the operator $1 + N$ is invertible in $L_p$ for some $1 \leq p \leq \infty$, then it is invertible in $L_p$ for all $1 \leq p \leq \infty$. Moreover, the kernel $m$ of the operator $M$, where $(1 + N)^{-1} = 1 + M$, does not depend on $p$.

For any $n \in \mathbf{N}_1(\mathbb{R}^c, \mathcal{E})$, we denote by $\tilde{n}$ the function that assigns to each $x \in \mathbb{R}^c$ the function $\tilde{n}(x) : \mathbb{R}^c \to \mathcal{B}(\mathcal{E})$ defined by the rule

$$\tilde{n}(x)(y) = n(x, x - y).$$

(3.4)
Proposition 3.6. Let \( n \in \mathbf{N}(\mathbb{R}^c, \mathbb{E}) \). Then formula (3.4) defines a measurable function \( \bar{n} \) with values in \( L_1(\mathbb{R}^c, \mathcal{B}(\mathbb{E})) \) for almost all \( x \in \mathbb{R}^c \). The function \( \bar{n} : \mathbb{R}^c \to L_1(\mathbb{R}^c, \mathcal{B}(\mathbb{E})) \) is essentially bounded, i.e. \( \bar{n} \in L_\infty(\mathbb{R}^c, L_1(\mathbb{R}^c, \mathcal{B}(\mathbb{E}))) \).

Proof. We take an arbitrary \( \alpha \in \mathbb{N} \). We redefine \( n \) by the formula \( n(x, y) = 0 \) for \( x \notin [-\alpha, \alpha]^c \). By estimate (3.1), the redefinition of the function \( n \) is summable. Hence, by Proposition 2.5, for almost all \( x \in [-\alpha, \alpha]^c \), the values \( \bar{n}(x)(y) = n(x, x-y) \) are defined for almost all \( y \), \( \bar{n}(x) \) belongs to \( L_1 \), and \( \| \bar{n}(x) \| = \int_{\mathbb{R}^c} \| n(x, y) \| \, dy \leq \| \| \beta \|_{L_1} \| \). \( \square \)

Proposition 3.7. For any \( n \in \mathbf{N}(\mathbb{R}^c, \mathbb{E}) \), the norm of the operator \( N : L_\infty \to L_\infty \) defined by formula (3.2) satisfies the estimate

\[
\text{ess sup}_x \| \bar{n}(x) \|_{L_1} \leq C \| N : L_\infty \to L_\infty \|,
\]

where \( C \) depends only on the norm on \( \mathbb{E} \).

Proof. We set

\[
M = \text{ess sup}_x \| n(x, \cdot) \|_{L_1}.
\]

We take an arbitrary \( \varepsilon > 0 \). By assumption, there exists a measurable set \( K \subset \mathbb{R}^c \) such that \( \mu(K) \neq 0 \) and

\[
\| n(x, \cdot) \|_{L_1} > M - \varepsilon, \quad x \in K.
\]

Without loss of generality, we may assume that \( K \subseteq [-\alpha, \alpha]^c \) for some \( \alpha \in \mathbb{N} \).

By Proposition 3.6, the function \( \bar{n} \) and its restriction to \( [-\alpha, \alpha]^c \) are measurable. Consequently, by Proposition 2.2, for any \( \varepsilon_1 > 0 \) there exists a compact set \( K_1 \subseteq [-\alpha, \alpha]^c \) such that the restriction of \( \bar{n} \) to \( K_1 \) is continuous and \( \mu([[-\alpha, \alpha]^c \setminus K_1] < \varepsilon_1 \).

If \( \varepsilon_1 \) is small enough, \( \mu(K \cap K_1) \neq 0 \).

Suppose that \( \dim \mathbb{E} = 1 \). Let \( x_0 \in K \cap K_1 \) be a point of density of the set \( K \cap K_1 \). Since \( L_\infty \) is the conjugate space of \( L_1 \), there exists \( u \in L_\infty \), \( \| u \| \leq 1 \), such that

\[
\int_{\mathbb{R}^c} n(x_0, x_0-y)u(y) \, dy \geq M - 2\varepsilon.
\]

By continuity, for \( x \in K \cap K_1 \) close enough to \( x_0 \) we have

\[
\left| \int_{\mathbb{R}^c} n(x, x-y)u(y) \, dy \right| \geq M - 3\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, it follows that \( \| N : L_\infty \to L_\infty \| \geq M \).

Now we suppose that \( \dim \mathbb{E} \) is arbitrary (but finite). We identify elements of \( \mathcal{B}(\mathbb{E}) \) with matrices \( \{a_{ij}\} \) and consider in \( \mathcal{B}(\mathbb{E}) \) another norm \( \| \{a_{ij}\} \|_\bullet = \max_{ij} |a_{ij}| \). It is equivalent to the initial norm on \( \mathcal{B}(\mathbb{E}) \), because all norms on a finite-dimensional space are equivalent [31, ch. 1, § 2, 3, Theorem 2]. Repeating the reasoning from the above paragraph, we obtain

\[
\| N : L_\infty \to L_\infty \| \geq \text{ess sup}_x \int_{\mathbb{R}^c} \| n(x, y) \|_\bullet \, dy,
\]

which completes the proof. \( \square \)

For all \( r > 0 \), we consider the function

\[
\bar{n}_r(x) = \frac{1}{\mu(B(0, r))} \int_{B(0, r)} \bar{n}(x-y) \, dy.
\]
By Proposition 3.6, the function $\bar{n}$ is essentially bounded. Therefore, the functions $\bar{n}_r$ are defined everywhere and continuous.

**Proposition 3.8.** Let $\bar{n} \in L_\infty (\mathbb{R}^n, L_1(\mathbb{R}^n, B(E)))$. Then there exists a sequence $r_i \to 0$ such that the functions $\bar{n}_{r_i}$ converges almost everywhere to $\bar{n}$.

**Remark 1.** For a locally summable function $\bar{n}$ taking its values in $\mathbb{R}$, it is known [30, ch. 1, Corollary 1 of Theorem 1] that the whole family $\bar{n}_r$ converges almost everywhere to $\bar{n}$ as $r \to +0$ (Lebesgue’s differentiation theorem). For our aims, the weaker assertion formulated above (which has an essentially easier proof) is enough.

**Proof.** Without loss of generality we may assume that $\bar{n}$ has a compact support, and thus $\bar{n} \in L_1(\mathbb{R}^n, L_1(\mathbb{R}^n, B(E)))$.

We consider the convolution operator

$$
(T_r \bar{n})(x) = \frac{1}{\mu(B(0, r))} \int_{B(0, r)} \bar{n}(x-y) \, dy.
$$

It is known (see, e.g. [20, Theorem 4.4.4(a)]) that

$$
\|T_r : L_1(\mathbb{R}^n, L_1(\mathbb{R}^n, B(E))) \to L_1(\mathbb{R}^n, L_1(\mathbb{R}^n, B(E)))\| \leq 1.
$$

We recall that we consider the case $\bar{n} \in L_1(\mathbb{R}^n, L_1(\mathbb{R}^n, B(E)))$. We take an arbitrary $\varepsilon > 0$ and a continuous function $\bar{k}$ with a compact support such that $\|\bar{k} - \bar{n}\| < \varepsilon$ (the latter is possible by the definition [27, ch. 4, § 3, Definition 2] of $L_1$). We have

$$
\|T_r \bar{n} - \bar{n}\|_{L_1} \leq \|T_r (\bar{n} - \bar{k})\|_{L_1} + \|T_r \bar{k} - \bar{k}\|_{L_1} + \|\bar{k} - \bar{n}\|_{L_1} \leq 2\varepsilon + \|T_r \bar{k} - \bar{k}\|_{L_1}.
$$

Since $\bar{k}$ is uniformly continuous, $T_r \bar{k}$ converges uniformly to $\bar{k}$ and hence in $L_1$-norm. Thus, if $r$ is small enough, $\|T_r \bar{k} - \bar{k}\|_{L_1} < \varepsilon$. Hence, if $r$ is small enough, we have

$$
\|\bar{n}_r - \bar{n}\|_{L_1} = \|T_r \bar{n} - \bar{n}\|_{L_1} \leq 3\varepsilon.
$$

Thus $\bar{n}_r$ converges to $\bar{n}$ in $L_1$-norm. By Proposition 2.7, we can choose a sequence $r_i \to 0$ such that $\bar{n}_{r_i}$ converges to $\bar{n}$ almost everywhere. $\square$

## 4. The class $\mathbf{C}$

For any $\alpha \in \mathbb{N}$ and $1 \leq p < \infty$, we denote by $L_p^\alpha$ the subspace of $L_p$ that consists of all $u \in L_p$ such that

$$
u(x) = 0 \quad \text{for} \ x \notin [-\alpha, \alpha]^c,
$$

and we denote by $L_p^{\alpha} \setminus \mathbb{N}$ the subspace of $L_p$ that consists of all $u \in L_p$ such that

$$
u(x) = 0 \quad \text{for} \ x \in [-\alpha, \alpha]^c.
$$

We denote by $t_f = t_f(L_p)$ the set of all operators $T \in B(L_p)$ possessing the property: for any $\alpha \in \mathbb{N}$ there exists $\gamma \in \mathbb{N}$ such that

$$
TL_p^\alpha \subseteq L_p^\gamma,
$$

$TL_p^{\gamma} \subseteq L_p^{\alpha}$.

We denote by $t = t(L_p)$ the closure of $t_f(L_p)$ in norm. The classes $t_f$ and $t$ were considered in [19] and [20, § 5.5].

**Proposition 4.1.** Let $1 \leq p \leq \infty$. Then the class $N_1(L_p)$ is included into $t(L_p)$. 

Proof. For any \( \delta \in \mathbb{N} \), we consider the operator
\[
(N_\delta u)(x) = \int_{x+[-\delta,\delta]} n(x, x-y) u(y) dy.
\]
Clearly, \( N_\delta \in \mathbb{N}_1 \). From estimate (3.3) we have
\[
\|N_\delta - N\| \leq \int_{\mathbb{R}^c \setminus [-\delta,\delta]} \beta(y) dy,
\]
which implies \( N_\delta \to N \) as \( \delta \to \infty \). It remains to observe that
\[
N_\delta L_p^\alpha \subseteq L_p^{\alpha+\delta},
N_\delta L_p^{(\alpha+\delta)} \subseteq L_p^\alpha.
\]
Thus, \( N_\delta \in \mathfrak{t}_f \). \( \square \)

Clearly, the operator
\[
(S_h u)(x) = u(x-h),
\]
where \( h \in \mathbb{R}^c \), acts in \( L_p \) for all \( 1 \leq p \leq \infty \) and \( \|S_h\| = 1 \).

**Proposition 4.2.** Let \( n \in \mathbb{N}_1 \), and the operator \( N \) be defined by formula (3.2). Then for any \( h \in \mathbb{R}^c \) the operator \( S_h NS_{-h} \) is defined by the formula
\[
(S_h NS_{-h} u)(x) = \int_{\mathbb{R}^c} n(x-h, x-y) u(y) dy.
\]

Proof. One has
\[
(NS_{-h} u)(x) = \int_{\mathbb{R}^c} n(x, x-y) u(y+h) dy,
\]
\[
(S_h NS_{-h} u)(x) = \int_{\mathbb{R}^c} n(x-h, x-h-y) u(y+h) dy
= \int_{\mathbb{R}^c} n(x-h, x-y) u(y) dy. \quad \square
\]

We denote by \( \mathbf{C}_u = \mathbf{C}_u(L_p) \) the set of all operators \( T \in B(L_p) \) such that the function
\[
h \mapsto S_h TS_{-h}, \quad h \in \mathbb{R}^c,
\]
is continuous in norm. Clearly, if function (4.1) is continuous at zero, it is continuous everywhere. The class \( \mathbf{C}_u \) was considered in [20, § 5.6]; see also [22], where the case of the strongly differentiable function (4.1) was considered.

We denote \( \mathbf{C} = \mathbf{C}(L_p) \) the set of all operators \( T \in \mathfrak{t}(L_p) \) such that the restriction of function (4.1) to \( L_p^\alpha \) is continuous in norm for all \( \alpha \in \mathbb{N} \). The class \( \mathbf{C} \) was discussed in [18] and [20, § 5.6].

**Theorem 4.3** ([20, Proposition 5.6.1]). Let \( 1 \leq p \leq \infty \). If an operator \( T \in \mathbf{C}_u(L_p) \) is invertible, then \( T^{-1} \in \mathbf{C}_u(L_p) \).

**Theorem 4.4** ([20, Theorem 5.6.3]). Let \( 1 \leq p \leq \infty \). If an operator \( T \in \mathbf{C}(L_p) \) is invertible, then \( T^{-1} \in \mathbf{C}(L_p) \).
5. The class $\mathbf{CN}_1$

We denote by $\mathbf{C}_n\mathbf{N}_1 = \mathbf{C}_n\mathbf{N}_1(\mathbb{R}^c, E)$ the class of kernels $n \in \mathbf{N}_1$ such that the function $n$ can be redefined on a set of measure zero so that it becomes defined everywhere, estimate (3.1) holds for all $x$ and $y$, and the corresponding function $x \mapsto \bar{n}(x)$ becomes uniformly continuous in the norm of $L_1(\mathbb{R}^c, \mathcal{B}(E))$.

We denote by $\mathbf{CN}_1 = \mathbf{CN}_1(\mathbb{R}^c, E)$ the class of kernels $n \in \mathbf{N}_1$ such that the function $n$ can be redefined on a set of measure zero so that it becomes defined everywhere, estimate (3.1) holds for all $x$ and $y$, and the corresponding function $x \mapsto \bar{n}(x)$ becomes continuous in the norm of $L_1(\mathbb{R}^c, \mathcal{B}(E))$.

**Theorem 5.1.** Let $n \in \mathbf{N}_1(\mathbb{R}^c, E)$, and the operator $N$ be defined by formula (3.2). If the operator $N$ belongs to $\mathbf{C}_n(L_\infty)$, then $n \in \mathbf{C}_n\mathbf{N}_1$.

**Proof.** From Propositions 4.2 and 3.7 it follows that

$$\text{ess sup}_x \|\bar{n}(x-h) - \bar{n}(x)\|_{L_1} \leq C\|S_hNS_{-h} - N : L_\infty \to L_\infty\|.$$  

We recall that the assumption $N \in \mathbf{C}_n(L_\infty)$ means that

$$\forall \varepsilon > 0 \ \exists R > 0 \ \forall (h : |h| < R) \ \|S_hNS_{-h} - N : L_\infty \to L_\infty\| < \varepsilon,$$

which implies

$$\forall \varepsilon > 0 \ \exists R > 0 \ \forall (h : |h| < R) \ \text{ess sup}_{x \in \mathbb{R}^c} \|\bar{n}(x-h) - \bar{n}(x)\|_{L_1} < C\varepsilon.$$

Next from the estimate (for the sake of definiteness we assume that $s < r$)

\[
\|\bar{n}_r(x) - \bar{n}_s(x)\| = \left\| \frac{1}{\mu(B(0,r))} \int_{B(0,r)} \bar{n}(x-y) \, dy - \frac{1}{\mu(B(0,s))} \int_{B(0,s)} \bar{n}(x-z) \, dz \right\| \\
= \left\| \frac{1}{\mu(B(0,r))} \int_{B(0,r)} \bar{n}(x-y) \, dy - \left( \frac{s}{r} \right)^c \frac{1}{\mu(B(0,s))} \int_{B(0,r)} \bar{n}(x-z) \, dz \right\| \\
= \left\| \frac{1}{\mu(B(0,r))} \int_{B(0,r)} \bar{n}(x-y) \, dy - \frac{1}{\mu(B(0,r))} \int_{B(0,r)} \bar{n}(x-y) \, dy \right\| \\
\leq \text{ess sup}_{x \in \mathbb{R}^c} \|\bar{n}(x-h) - \bar{n}(x)\|,
\]

where $h = (1 - \frac{s}{r})y$ (clearly, $|h| = |(1 - \frac{s}{r})y| \leq r$ since $0 < s < r$), it follows that

$$\forall \varepsilon > 0 \ \exists R > 0 \ \forall (r,s : |r|, |s| < R) \ \forall x \in \mathbb{R}^c \ \|\bar{n}_r(x) - \bar{n}_s(x)\| < C\varepsilon.$$

Thus $\bar{n}_r$ converges uniformly to a function $\bar{n}_* : \mathbb{R}^c \to L_1(\mathbb{R}^c, \mathcal{B}(E))$ as $r \to 0$. Since the functions $\bar{n}_r$ are continuous, the limit function $\bar{n}_*$ is also continuous. On the other hand, by Proposition 3.8, there exists a sequence $r_i \to 0$ such that the functions $\bar{n}_{r_i}$ converges to $\bar{n}$ almost everywhere. Consequently, $\bar{n}$ coincides with a continuous function $\bar{n}_*$ on a set $F_1$ of full measure.

Now we describe the desired redefinition $n_0 : \mathbb{R}^c \times \mathbb{R}^c \to \mathcal{B}(E)$ of the function $n$.

First we consider the set $E$ of all points $(x,y) \in \mathbb{R}^c \times \mathbb{R}^c$ such that estimate (3.1) does not hold. By assumption, $E$ is a set of measure zero. We denote by $F_2$ the set of all $x \in \mathbb{R}^c$ such that the set $E_x = \{ y \in \mathbb{R}^c : (x,y) \in E \}$ has measure zero. By Corollary 2.6, the set $F_2$ is a set of full measure. For $x \in F_1 \cap F_2$ and
y ∈ E_x, we redefine n by the rule n(x, y) = 0. So, estimate (3.1) holds for all y when x ∈ F_1 ∩ F_2 (we assume that β is defined everywhere).

We set n_0(x, y) = n(x, y) for x ∈ F_1 ∩ F_2 and all y ∈ R^c (for x ∉ F_1 ∩ F_2 the value n_0(x, y) is yet undefined). By Corollary 2.6, n and n_0 coincide on a set of full measure. If we define n_0(x, y) for x ∉ F_1 ∩ F_2 in an arbitrary way, n_0 and n remain to be equivalent functions. The problem is to make n_0 continuous and to ensure estimate (3.1). We note that for any x ∈ F_1 ∩ F_2 the functions n(x, y) = n(x, y) and n_0(x, y) = n_0(x, y) coincide.

Next we define n_0(x, y) for x ∉ F_1 ∩ F_2. Since F_1 ∩ F_2 is a set of full measure, for any x ∉ F_1 ∩ F_2 there exists a sequence x_k ∈ F_1 ∩ F_2 that converges to x. By the continuity of n_*, it follows that n_*(x_k) = n_*(x_k, ·) converges to n_*(x) in L_1-norm. By Proposition 2.7, this implies that there exists a subsequence n_*(x_{k_i}) that converges to n_*(x) (not only in L_1-norm, but also) almost everywhere. So, we set n_0(x, y) = n_*(x)(y) for y’s such that lim_{i→∞} n_*(x_{k_i})(y) = n_*(x)(y) (we recall that n_*(x_{k_i}) = n(x_{k_i}) since x_{k_i} ∈ F_1), and n_0(x, y) = 0 otherwise. By the definition of n_0, we have ∥ n_0(x_k, y) ∥ ≤ β(y) for all y. Therefore, ∥ n_0(x, y) ∥ ≤ β(y) for almost all y. Finally, we redefine n_0(x, ·) on a set of measure zero so that the estimate ∥ n_0(x, y) ∥ ≤ β(y) holds for all y.

□

Theorem 5.2. Let n ∈ N_1(R^c, E), and the operator N be defined by formula (3.2). The operator N belongs to C(L_∞) if and only if n ∈ CN_1.

Proof. For any α ∈ N, we consider the operator

\[ (N_α u)(x) = \int \chi_{[-α, α]^c}(x) u(y) dy = \int_\mathbb{R} n(x, x - y) \chi_{[-α, α]^c}(y) u(y) dy, \]

where \( \chi_{[-α, α]^c} \) is the characteristic function of the set \([-α, α]^c\). Since \( N_α \) coincides with \( N \) on \( L^p_0 \), we have

\[ \| S_h N_α S_{-h} - N_α \|_{L^p_0} = \| S_h N_α S_{-h} - N_α \|_{L^p_0} \]

\[ = \| S_h N S_{-h} - N \|_{L^p_0}, \]

which together with \( N \in C \) implies that \( N_α \in C_a \). Therefore, by Theorem 5.1, the restriction

\[ n_α(x)(y) = n(x, x - y) \chi_{[-α, α]^c}(y) \]

of the function \( n_α \) coincides with a continuous function almost everywhere. By (3.1), for almost all \( x \) we have the estimate

\[ \| n_α(x) \|_{L_1} \leq \int \beta(x - y) dy = \int \chi_{[-α, α]^c}(y) \beta(y) dy. \]

Let us take a sequence \( α_i \in N \) such that \( α_{i+1} - α_i > 2 \) for all \( i \). We set

\[ n_{α_{i+1} \setminus α_i}(x)(y) = n(x, x - y) \chi_{[-α_{i+1}, α_{i+1}]\setminus[-α_i, α_i]^c}(y). \]

Clearly, \( n_{α_{i+1} \setminus α_i} = n_{α_{i+1}} - n_{α_i} \). Obviously, the function \( n_{α_{i+1} \setminus α_i} \) coincides with a continuous one almost everywhere. We replace the functions \( n_{α_{i+1} \setminus α_i} \) by the corresponding continuous functions. From (3.1) for almost all \( x \), it follows the estimate

\[ \| n_{α_{i+1} \setminus α_i}(x) \|_{L_1} \leq \int \beta(x - y) dy \]

\[ = \int \chi_{[-α_{i+1}, α_{i+1}]\setminus[-α_i, α_i]^c}(y) \beta(y) dy. \]
Next we take an arbitrary \( x_0 \in \mathbb{R}^c \) and show that the function \( \bar{n} \) can be redefined on a set of measure zero so that it becomes continuous on \( x_0 + (-1, 1)^c \). We note that \( x \in x_0 + (-1, 1)^c \) and \( y \in x + [-\alpha_{i+1}, \alpha_{i+1}]^c \setminus [-\alpha_i, \alpha_i]^c \) imply that \( y \in x_0 + [-\alpha_{i+1} - 1, \alpha_{i+1} + 1]^c \setminus [-\alpha_i + 1, \alpha_i - 1]^c \). Therefore for \( x_0 + (-1, 1)^c \) we have

\[
\|\bar{n}_{\alpha_{i+1}}(x)\|_{L_1} \leq \int_{x + [-\alpha_{i+1}, \alpha_{i+1}]^c \setminus [-\alpha_i, \alpha_i]^c} \beta(y) \, dy \\
\leq \int_{x_0 + [-\alpha_{i+1} - 1, \alpha_{i+1} + 1]^c \setminus [-\alpha_i + 1, \alpha_i - 1]^c} \beta(y) \, dy.
\]

Clearly,

\[
\sum_{i=1}^{\infty} \int_{x_0 + [-\alpha_{i+1} - 1, \alpha_{i+1} + 1]^c \setminus [-\alpha_i + 1, \alpha_i - 1]^c} \beta(y) \, dy \leq 2 \int_{\mathbb{R}^c} \beta(y) \, dy < \infty.
\]

Hence the series \( \sum_{i=1}^{\infty} \bar{n}_{\alpha_{i+1}}(x) \) (consisting of continuous functions) converges uniformly on \( x_0 + (-1, 1)^c \). Therefore, its sum is a continuous function. Obviously, its sum coincides with \( \bar{n} \) on \( x_0 + (-1, 1)^c \) almost everywhere.

Since \( x_0 \) is arbitrary, \( \bar{n} \) coincides with a continuous function \( \bar{n}_* \) on a set of full measure. Now the proof of the possibility of a redefinition of \( n \) repeats the corresponding part of the proof of Theorem 5.1.

Let us prove the converse statement. By Propositions 4.2 and estimate (3.3), for any \( \alpha \in \mathbb{N} \) we have

\[
\|S_hNS_h - N : L^\alpha_\infty \to L^\infty_\alpha\| \leq \text{ess sup}_{x \in \mathbb{R}^c} \int_{[-\alpha, \alpha]^c} \|n(x-h, x-y) - n(x, x-y)\| \, dy.
\]

We take a large \( \gamma \in \mathbb{N} \). For \( x \notin [-\gamma, \gamma]^c \), we have the estimate

\[
\int_{[-\alpha, \alpha]^c} \|n(x-h, x-y) - n(x, x-y)\| \, dy \\
\leq \int_{[-\alpha, \alpha]^c} \|n(x-h, x-y)\| \, dy + \int_{[-\alpha, \alpha]^c} \|n(x, x-y)\| \, dy \\
\leq 2 \int_{[-\alpha, \alpha]^c} \beta(x-y) \, dy = 2 \int_{x+[-\alpha, \alpha]^c} \beta(y) \, dy,
\]

which is small provided \( \gamma \) is large enough. For \( x \in [-\gamma, \gamma]^c \), we have the estimate

\[
\int_{[-\alpha, \alpha]^c} \|n(x-h, x-y) - n(x, x-y)\| \, dy \leq \int_{\mathbb{R}^c} \|n(x-h, x-y) - n(x, x-y)\| \, dy,
\]

which is small provided \( h \) is small, by continuity of \( \bar{n} \). \( \square \)

**Theorem 5.3.** Let \( n \in \mathcal{CN}_1 \), and the operator \( N \in \mathcal{B}(L_p) \), \( 1 \leq p \leq \infty \), be defined by formula (3.2). If the operator \( 1 + N \) is invertible, then \( (1 + N)^{-1} = 1 + M \), where

\[
(Mu)(x) = \int_{\mathbb{R}^c} m(x, x-y) u(y) \, dy
\]

with \( m \in \mathcal{CN}_1 \).
Proof. Let the operator $1 + N : L_p \to L_p$ be invertible. Then, by Corollary 3.5, it is invertible in $L_\infty$, and by Theorem 3.4, $(1 + N)^{-1} = 1 + M$, where $M \in \mathbf{N}_1(L_\infty)$. On the other hand, by Theorem 4.4, $(1 + N)^{-1} \in \mathcal{C}(L_\infty)$. Therefore by Theorem 5.2, $m \in \mathbf{CN}_1$. \hfill \square

6. The class $\mathbf{h}$

For every $k \in \mathbb{Z}^c$, we consider the operator

$$(P_ku)(x) = \chi_{k+(0,1)^c}(x)u(x),$$

where $\chi_{k+(0,1)^c}$ is the characteristic function of the set $k + (0,1]^c \subset \mathbb{R}^c$. We call (see [20, Proposition 6.1.1]) an operator $K \in \mathfrak{t}(L_p)$, $1 \leq p \leq \infty$, locally compact if for all $k, m \in \mathbb{Z}^c$, the operator $P_m TP_k$ is compact. We denote the set of all locally compact operators $K \in \mathfrak{t}(L_p)$ by $\mathbf{h}(L_p)$. Clearly, the class $\mathbf{h}(L_p)$ is closed in norm.

Theorem 6.1. Let $1 \leq p \leq \infty$. Then the class $\mathbf{CN}_1(L_p)$ is included into $\mathbf{h}(L_p)$.

Proof. It is known (see, e.g. [20, Proposition 6.2.2]) that an integral operator in $L_p[a,b]$ with a continuous kernel $k(\cdot, \cdot)$ is locally compact. Consequently, an operator of convolution with a continuous compactly supported kernel is locally compact. By virtue of estimate (3.3), an operator of convolution with a summable kernel is also locally compact.

Let an operator $N \in \mathbf{CN}_1(L_p)$ has the form (3.2). Since we want to prove the compactness of the operator $P_m TP_k$, without loss of generality we may assume that the functions $\beta$ and $\tilde{n}$ are compactly supported. More precisely, we may assume that $\tilde{n}$ is supported in $[m-1, m+2]^c$ and $\beta$ is supported in $[m-k-1, m-k+2]^c$; moreover, the function $\tilde{n}$ is $L_1$-continuous.

For any $i \in \mathbb{N}$, we consider the function

$$\tilde{n}_i(x) = \tilde{n}(x^*),$$

where $x^* = (x_1^*, \ldots, x_i^*)$ is the nearest to $x = (x_1, \ldots, x_i)$ from the right $(\frac{1}{i})^c$-integer point in the sense that $0 \leq x_i^* - x_i < \frac{1}{i}$ and $x^* \in \mathbb{Z}^c/i$. We consider the integral operators $N_i \in \mathbf{N}_1$ generated by $\tilde{n}_i$. Since $\tilde{n}$ is continuous and compactly supported, $N_i$ converges to $N$ in norm by estimate (3.3).

Clearly, any operator $N_i$ can be represented as a finite sum of the operators $P_{k,i}N_i$, $k \in \mathbb{Z}^c$, where

$$(P_{k,i}u)(x) = \chi_{k/i+(0,1/i)^c}(x)u(x).$$

By what was proved, the operators $P_{k,i}N_i$ are locally compact. Hence the operators $N_i$ and the operator $N$ are locally compact as well. \hfill \square

7. The class $\mathbf{CS}$

Let $X$ be a Banach space. We denote by $C = C(\mathbb{R}^c, X)$ the Banach space of all bounded continuous functions $u : \mathbb{R}^c \to X$ with the norm

$$\|u\| = \sup_{x \in \mathbb{R}^c} \|u(x)\|.$$ 

We denote by $\mathbf{CS} = \mathbf{CS}(L_p)$ the set of all operators of the form

$$(Du)(x) = \sum_{i=1}^{\infty} d_i(x)u(x - h_i),$$

(7.1)
where \( h_i \in \mathbb{R}^c \), \( d_i \in C(\mathbb{R}^c, \mathcal{B}(\mathbb{E})) \), \( \sum_{i=1}^{\infty} \| d_i \|_C < \infty \). Clearly, \( D \) acts in \( L_p \), \( 1 \leq p \leq \infty \), and in \( C(\mathbb{R}^c, \mathbb{E}) \), and in all cases \( \| D \| \leq \sum_{i=1}^{\infty} \| d_i \| \).

**Theorem 7.1** ([20, Corollary 5.6.10]). If an operator \( D \in \mathbf{CS} \) is invertible in \( L_p \), \( 1 \leq p \leq \infty \), then \( D^{-1} \in \mathbf{CS} \) as well. If \( D \in \mathbf{CS} \) is invertible in \( L_p \) for some \( 1 \leq p \leq \infty \), then it is invertible in \( L_p \) for all \( 1 \leq p \leq \infty \).

**Proof.** It is enough to observe that our class \( \mathbf{CS} \) coincides with the class \( \mathbf{S}(C) \) in notation of [20, see 5.2.1 and 5.1.1]. \( \square \)

**Proposition 7.2.** Let \( N \in \mathbf{CN}_1(\mathcal{L}_p) \) and \( D \in \mathbf{CS}(\mathcal{L}_p) \), \( 1 \leq p \leq \infty \). Then \( DN, ND \in \mathbf{CN}_1(\mathcal{L}_p) \).

**Proof.** Let an operator \( N \in \mathbf{CN}_1(\mathcal{L}_p) \) has the form (3.2), and an operator \( D \in \mathbf{CS}(\mathcal{L}_p) \) has the form (7.1). By the definition of composition of operators, we have

\[
(DN) (x) = \sum_{i=1}^{\infty} d_i(x) \int_{\mathbb{R}^c} n(x - h_i, x - y) u(y) dy,
\]

(7.2)

We consider the operators

\[
(N_1 u) (x) = \int_{\mathbb{R}^c} \| n(x, x - y) \| u(y) dy,
\]

\[
(D_1 u) (x) = \sum_{i=1}^{\infty} \| d_i(x) \| \| u(x - h_i) \|
\]

acting in \( L_p (\mathbb{R}^c, \mathbb{R}) \). In the formula

\[
(D_1 N_1 | u) (x) = \sum_{i=1}^{\infty} \| d_i(x) \| \int_{\mathbb{R}^c} \| n(x - h_i, x - y) \| \cdot | u(y) | dy,
\]

(7.3)

each of the functions

\[
v_i (x) = \int_{\mathbb{R}^c} \| n(x - h_i, x - y) \| \cdot | u(y) | dy,
\]

by Proposition 3.2, is defined almost everywhere and belongs to \( L_p \); furthermore,

\[
\| v_i \|_{L_p} \leq \| \beta \|_{L_1} \cdot \| u \|_{L_p}.
\]

Since \( \sum_{i=1}^{\infty} \| d_i \|_C < \infty \), the series \( \sum_{i=1}^{\infty} w_i \), where

\[
w_i(x) = \| d_i(x) \| \cdot v_i(x) = \| d_i(x) \| \int_{\mathbb{R}^c} \| n(x - h_i, x - y - h_i) \| \cdot | u(y) | dy,
\]

converges absolutely in \( L_p \)-norm. We denote by \( F \) the set of all \( x \) such that the series \( \sum_{i=1}^{\infty} w_i(x) \) converges (absolutely). By Proposition 2.4, \( F \) is a set of full measure. Thus for \( x \in F \), formulas (7.3) and (7.2) can be rewritten as

\[
(D_1 N_1 | u) (x) = \int_{\mathbb{R}^c} \sum_{i=1}^{\infty} \| d_i(x) \| \cdot \| n(x - h_i, x - y - h_i) \| \cdot | u(y) | dy,
\]

(7.4)

\[
(DN) u (x) = \int_{\mathbb{R}^c} \sum_{i=1}^{\infty} d_i(x) n(x - h_i, x - y - h_i) u(y) dy
\]

The later representation is obviously equivalent to

\[
(DN) u (x) = \int_{\mathbb{R}^c} \left( \sum_{i=1}^{\infty} d_i(x) n(x - h_i, x - y - h_i) \right) u(y) dy.
\]
Thus $DN$ is an integral operator with the kernel

$$n_1(x, y) = \sum_{i=1}^{\infty} d_i(x) n(x - h_i, y - h_i).$$

For almost all $(x, y)$, this kernel satisfies the estimate

$$\|n_1(x, y)\| \leq \sum_{i=1}^{\infty} \|d_i(x)\| \beta(y - h_i) \leq \sum_{i=1}^{\infty} \|d_i\|_{L_\infty} \beta(y - h_i).$$

Clearly, $\beta_1(y) = \sum_{i=1}^{\infty} \|d_i\|_{L_\infty} \beta(y - h_i)$ is a summable function. Thus $n_1 \in N_1$. It remains to note that the series

$$\overline{n}_1(x, \cdot) = \sum_{i=1}^{\infty} d_i(x) \overline{n}(x - h_i, \cdot - h_i)$$

converges to a continuous function, because it consists of continuous functions and converges uniformly.

Next we discuss the composition $ND$. By the definition of composition of operators, we have

$$(NDu)(x) = \int_{\mathbb{R}^c} n(x, x - y) \left( \sum_{i=1}^{\infty} d_i(y) u(y - h_i) \right) dy.$$ 

Let us consider the function

$$w(y) \mapsto \sum_{i=1}^{\infty} ||d_i(y)|| \cdot |u(y - h_i)|.$$ 

Clearly, this series converges absolutely in $L_\rho$-norm. Hence by Proposition 2.4 it converges absolutely almost everywhere (say, on $F_1$) to the function $w$. Let $F_2$ be the set of all $x$ such that $\int_{\mathbb{R}^c} ||n(x, x - y)|| w(y) dy < \infty$. By Proposition 3.2, $F_2$ is a set of full measure.

Let $x \in F_2$ be fixed. Since $n \in CN_1$, we may assume that $n(x, x - y)$ is defined for all $y$. Therefore for all $y \in F_1$

$$n(x, x - y) \sum_{i=1}^{\infty} d_i(y) u(y - h_i) = \sum_{i=1}^{\infty} n(x, x - y) d_i(y) u(y - h_i).$$

Thus

$$(NDu)(x) = \int_{\mathbb{R}^c} n(x, x - y) \left( \sum_{i=1}^{\infty} d_i(y) u(y - h_i) \right) dy$$

$$= \int_{\mathbb{R}^c} \left( \sum_{i=1}^{\infty} n(x, x - y) d_i(y) u(y - h_i) \right) dy.$$ 

Since $x \in F_2$, the function

$$y \mapsto ||n(x, x - y)|| w(y)$$

$$= ||n(x, x - y)|| \sum_{i=1}^{\infty} ||d_i(y)|| \cdot |u(y - h_i)|$$

$$= \sum_{i=1}^{\infty} ||n(x, x - y)|| \cdot ||d_i(y)|| \cdot |u(y - h_i)|$$
is integrable. Therefore, by Proposition 2.3, the series

\[ y \rightarrow \sum_{i=1}^{\infty} n(x, x - y) d_i(y) u(y - h_i) \]

is absolutely convergent in \( L_{11} \)-norm. Hence

\[ (NDu)(x) = \int_{\mathbb{R}^c} \left( \sum_{i=1}^{\infty} n(x, x - y) d_i(y) u(y - h_i) \right) dy \]

\[ = \sum_{i=1}^{\infty} \int_{\mathbb{R}^c} n(x, x - y) d_i(y) u(y - h_i) dy. \]

Performing a simple change of variables and using the absolute convergence of the series in \( L_{11} \)-norm, we arrive at

\[ (NDu)(x) = \sum_{i=1}^{\infty} \int_{\mathbb{R}^c} n(x, x - y) d_i(y) u(y - h_i) dy \]

\[ = \sum_{i=1}^{\infty} \int_{\mathbb{R}^c} n(x, x - y - h_i) d_i(y + h_i) u(y) dy \]

\[ = \int_{\mathbb{R}^c} \left( \sum_{i=1}^{\infty} n(x, x - y - h_i) d_i(y + h_i) \right) u(y) dy. \]

Thus \( ND \) is an integral operator with the kernel

\[ n_1(x, y) = \sum_{i=1}^{\infty} n(x, x - y - h_i) d_i(y + h_i). \]

For almost all \((x, y)\), the kernel satisfies the estimate

\[ \|n_1(x, y)\| \leq \sum_{i=1}^{\infty} \beta(y + h_i) \|d_i(y + h_i)\| \leq \sum_{i=1}^{\infty} \|d_i\|_{L_\infty} \cdot \beta(y + h_i). \]

We note again that

\[ \beta_1(y) = \sum_{i=1}^{\infty} \|d_i\|_{L_\infty} \beta(y + h_i) \]

is a summable function. Thus \( n_1 \in \mathbf{N}_1 \). It remains to observe that the series

\[ \tilde{n}_1(x, \cdot) = \sum_{i=1}^{\infty} \tilde{n}(x, x - h_i - \cdot) d_i(\cdot + h_i) \]

consists of continuous functions and converges uniformly. \( \square \)

**Theorem 7.3.** Let \( n \in \mathbf{CN}_1 \), \( 1 \leq p < \infty \), the operator \( N \in \mathbf{B}(L_p) \) be defined by formula (3.2), and \( D \in \mathbf{CS}(L_p) \). If the operator \( D + N \) is invertible in \( L_p \), then \( (D+N)^{-1} = A + M \), where \( A \in \mathbf{CS}(L_p) \) and \( M \) has the form (5.1) with \( m \in \mathbf{CN}_1 \).

**Proof.** Let \( D + N \) be invertible. Then by [20, Theorem 6.2.1], the operator \( D \) is invertible. Therefore the operator \( (D + N)^{-1} \) can be represented in the form

\[ (D + N)^{-1} = (1 + D^{-1}N)^{-1} D^{-1}. \]

By Theorem 7.1, \( D^{-1} \in \mathbf{CS} \). By Proposition 7.2, \( D^{-1}N \in \mathbf{CN}_1(L_p) \). By Theorem 5.3, \( (1 + D^{-1}N)^{-1} \) has the form \( 1 + K \), where \( K \in \mathbf{CN}_1(L_p) \). Thus
\[(D + N)^{-1} = (1 + D^{-1}N)^{-1}D^{-1} = (1 + K)D^{-1} = D^{-1} + KD^{-1}.\] Finally, again by Proposition 7.2, \(M = KD^{-1} \in \mathbb{CN}_1(L_p)\).

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