Mach’s Principle: A Response to Mashhoon and Wesson’s Paper arXiv: 1106.6036

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Abstract. In their recent “Mach’s principle and higher-dimensional dynamics”, Mashhoon and Wesson argue that Mach’s principle is not properly incorporated into general relativity and that in Einstein’s theory “the origin of inertia remains essentially the same as in Newtonian physics.” While it is true that the motion of a single test particle in a Newtonian inertial frame of reference appears essentially the same as in an Einsteinian local inertial frame, this misses the point. The issue is not what motion looks like in an inertial frame of reference but what is the origin of the inertial frame. Unlike Newtonian dynamics, general relativity does implement Mach’s principle when considered from this correctly formulated point of view.

1 Introduction

In arXiv: 1106.6036[1], Mashhoon and Wesson argue that Mach’s principle is not properly incorporated into general relativity (GR) because in it “the origin of inertia remains essentially the same as in Newtonian physics”: the motion of a single test particle in a Newtonian inertial frame of reference appears essentially the same as in an Einsteinian local inertial frame. But Mach’s concern was not the description of the motion of a single particle in an inertial frame of reference but the origin of the inertial frame. I shall show in this note that in Newtonian theory there is no mechanism that determines inertial frames whereas there is in GR.

The key to establishing this difference between the two theories is the identification of a precise defect in Newtonian theory (Sec. 2). In Sec. 3, I show how this defect can be eliminated in a simple Machian model of nonrelativistic particles by a process called best matching. This is already sufficient to show how Mach’s principle should be implemented, and I therefore merely indicate briefly in Sec 4 how the same mechanism of best matching represents the core of GR when treated as a dynamical theory. This shows that GR is Machian in a way that Mach would have approved. In Sec. 5, I briefly comment on a response to my note by Mashhoon and Wesson.

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2 The Defect of Newtonian Dynamics

Mach [2] argued that in dynamics position must be defined by observable distances from other particles and not by absolute space or an inertial frame of reference. However, Mach was an intuitive thinker and, apart from making it clear that the complete universe must be involved in the determination of inertial motion, did not give a precise mathematical criterion that would allow one to say that a dynamical theory is satisfactory from his point of view. This lack of a definition of a Machian theory was compounded by the fact that Einstein made no attempt to implement Mach’s ideas directly in GR. In fact, in 1902 Poincaré [4, 5] had made an insightful analysis of Newtonian dynamics, from which a clear defect of it emerged. The implementation of Mach’s principle requires a formulation of dynamics that eliminates this defect. In this section, I shall give the essence of Poincaré’s analysis in a form suitable for the further discussion. A fuller account can be found in [6].

Consider the dynamics of $N, N \geq 3$, point particles in Euclidean space that form a closed dynamical system taken to model the universe. Only the distances $r_{ab}, a, b = 1, 2, ..., N$, between them are observable. Suppose the $r_{ab}$ are determined at two instants 1 and 2; the two resulting sets of $r_{ab}$ define two relative configurations. In Cartesian frames of reference, the particles will have coordinates $x^1_a$ and $x^2_a$. Newton’s problem in laying the foundations of dynamics was this: how can one define the displacements and hence velocities of the individual particles between the two instants? The difficulty is that the Cartesian frames chosen for the two relative configurations are unrelated. There is nothing intrinsic to the observable $r^1_{ab}$ and $r^2_{ab}$ that establishes a connection between the frame-dependent $x^1_a$ and $x^2_a$. This indeterminacy and its consequences for 3-body Newtonian dynamics are illustrated in Fig. 1.

Newton analyzed the problem of defining individual particle displacements given only relative positions in an unpublished paper called *De gravitatione* (see [7], p. 609ff). The associated difficulties led him to introduce the notions of absolute space and time. However, this had the consequence that Newtonian dynamics is not as predictive as one might expect, for it allows many different evolutions of the observable $r_{ab}$ to arise from identical observable $r_{ab}, \dot{r}_{ab}$. The change in the initial displacements associated with translations (the difference between 1 and t in Fig. 1) has no effect on the evolution of the $r_{ab}$ because of Galilean relativity. However, rotations

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2A typical comment is: “When, accordingly, we say that a body preserves unchanged its direction in space, our assertion is nothing more or less than an abbreviated reference to the entire universe” ([3], p. 286, Mach’s emphasis.)
Figure 1. The observable separations $r_{ab}^1$ and $r_{ab}^2 = r_{ab}^1 + \delta r_{ab}$ define two slightly different (shaded and dashed) triangles and, in the limit $\delta t \to 0$, observable initial data $r_{ab}, \dot{r}_{ab}$. If, with Newton, one assumes that position in space has physical meaning, the triangles can be placed separately anywhere in space (in this representation, on the paper or screen carrying the images). This is equivalent to choosing arbitrary Cartesian frames of reference for each of them. For all such choices, $r_{ab}, \dot{r}_{ab}$ are the same (invariant), but the Newtonian displacements in space are different. Most importantly, the initial velocities $\dot{x}_a$, obtained in the limit $\delta t \to 0$ from $\delta x_a/\delta t$, can be changed from the possibility 1 by the mere group operations of translating (t) and rotating (d) one triangle relative to the other. They generate different Newtonian initial velocities $\dot{x}_a$, but the $r_{ab}$ are invariant. This has the consequence that different Newtonian evolutions of the observable $r_{ab}$ can arise from observationally identical initial $r_{ab}, \dot{r}_{ab}$. Poincaré identified this defect of Newtonian dynamics. Mach’s intuition about dynamics will be realized if this defect is eliminated: the observable initial data must determine the observable evolution.

have a significant effect because they change the angular momentum in the system. We have the undesirable consequence that different evolutions can arise from identical observable initial data. One can regard the different relative placings in Fig. 1 as gauge transformations that leave the observable initial data $r_{ab}, \dot{r}_{ab}$ invariant but not the evolution of the $r_{ab}$.

Poincaré found this state of affairs ‘repugnant’ but faced with the manifest presence of angular momentum in the solar system concluded regretfully that there was no alternative but to accept the active participation of an invisible agent in local dynamics. Curiously, he did not consider Mach’s suggestion [2] (cf. footnote 2) that the universe as a whole might determine the local inertial frames in which nearby bodies are observed to evolve.

I shall now explain the simple way to construct theories that are free
from the defect Poincaré found in Newtonian dynamics. As I have written extensively about the definition of Mach’s principle in [6] and its implementation in [8], I will merely give the basic idea and an indication of why it applies universally, including the theory of dynamical geometry and thus general relativity (GR). This will show that Newtonian dynamics and GR are very different precisely because the latter implements Mach’s principle in a formulation that I feel confident Mach would have accepted. Since absolute time is as invisible as absolute space, I shall replace the derivatives $\dot{r}_{ab}$ wrt time by derivatives $r'_{ab}$ wrt an arbitrary parameter $\lambda$.

### 3 Best Matching Implements Mach’s Principle

The defect that Poincaré identified in Newtonian dynamics has a purely group-theoretical origin: the generators of Euclidean rotations change the Newtonian initial data without changing the observable initial data. The best-matching implementation of Mach’s principle employs the very same generators to solve the problem in Machian dynamics that they create in Newtonian dynamics. The first thing is to change the aim of dynamics: not to find a law that each body in the universe satisfies separately in space (as in Newton’s law of inertia) but one that governs the changes of the observable separations $r_{ab}$ within the universe, treated as a single closed system. Crucially, an observable initial state $r_{ab}, r'_{ab}$ of the universe must uniquely determine its observable evolution.

The idea, in its simplest form, is to define a metric on the space $\mathcal{R}$ of possible relative configurations of the universe. If this can be done, then geodesics with respect to this metric will define Machian evolutions in $\mathcal{R}$. This is because a point and direction, which correspond to the observable initial data $r_{ab}, r'_{ab}$ in $\mathcal{R}$, define a geodesic uniquely. Poincaré’s requirement will be met. Figure 2 illustrates the mechanism that implements this by creating a metric on $\mathcal{R}$ for the case of the 3-body problem.

The mechanism is very simple. To define a metric on $\mathcal{R}$, we need to define a ‘difference’ between any two nearly identical relative configurations of the system. These are represented by triangles in the 3-body problem. To measure their difference, we represent one – either will do – in an arbitrary Cartesian frame. The particles then acquire coordinates $x_a, a = 1, 2, 3$. We then put the other triangle in an initial arbitrary trial placing somewhere near the first triangle. This corresponds to choosing some Cartesian frame.

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3 The extension to scale-invariant particle dynamics and dynamical geometry together with issues related to time and the expansion of the universe are discussed in [6, 8].
Figure 2. a) An arbitrary placing of the dashed triangle relative to the undashed triangle; b) the best-matched placing reached by translational and rotational minimization of (1). Best matching minimizes the expression (1), brings the centres of mass to coincidence and reduces the net rotation to zero. Note that in this toy ‘island-universe’ model best matching automatically forces us to take into account the relative changes within the complete universe.

for the second triangle. Its particles then have coordinates $x_a + \delta x_a$. In this trial position, we calculate

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\delta s_{\text{trial}} := \sqrt{(E - V) \sum_a m_a \delta x_a \cdot \delta x_a},
$$

(1)

where $E$ is a constant and $V$ is a function on $\mathbb{R}$, i.e., $V = V(r_{ab})$. It is clear that in a Machian approach (1) can have no physical significance since the trial placing is arbitrary. However, we can use the generators of translations and rotations to move the second triangle into the unique position in which (1) is minimized. This is the best-matched position; it is determined by the triangles alone. The corresponding extremal value of (1) defines the ‘difference’, or ‘distance’, between the two relative configurations. This is all that we need to implement Mach’s principle. Note that the best-matched displacements $\delta x_a^{bm}$ are not defined relative to any space but relative to the triangle taken to represent the initial configuration. The role of space as a frame of reference is eliminated. We now consider the consequences.

As is explained in more detail in [8], the observable separations $r_{ab}$ in a dynamically closed (island) Machian universe evolve exactly as in Newtonian theory with two important differences. First, the total momentum $P$ and angular momentum $L$ of the universe must be exactly zero and its
total energy must be $E$. These restrictions appear as gauge-type constraints on the initial data. The vanishing of $P$ is not significant on account of Galilean relativity, but the constraint $L = 0$ eliminates the defect in Newtonian dynamics identified by Poincaré. Second, if the successive relative configurations are placed relative to the initial configuration in the best-matched position, the evolution unfolds exactly as in a Newtonian inertial frame of reference, but this is not introduced prior to the formulation of the dynamical law. It is emergent. It is important that subsystems of the universe have their inertial frames determined by the best matching applied to the universe but can have nonvanishing angular momentum; the total angular momentum of the universe must however be zero. The manner in which Newtonian time and the energies of individual subsystems emerge out of an initially timeless kinematic framework is described in [8].

4 General Relativity and Gauge Theory

Because best matching is based on transformations generated by continuous groups, it is a universal mechanism that can be employed to implement Machian dynamics whenever the configurations of a dynamical system are invariant under the action of a Lie group. Just as Euclidean transformations leave the interparticle separations invariant in the Newtonian $N$-body problem, three-dimensional diffeomorphisms leave invariant the geodesic distances between points on a 3-manifold on which a Riemannian 3-metric is defined. As was first shown in [9], this makes it possible to implement a refined form of best matching between 3-metrics and recover general relativity as a dynamical theory of the evolution of Riemannian 3-geometry from very basic Machian first principles.

The key result that establishes the Machian nature of GR is that the best matching leads directly to the constraint $\mathcal{P}_{ij, j} = 0$, where $p^{ij}$ is the momentum canonically conjugate to the 3-metric $g_{ij}$, and the semicolon denotes covariant differentiation using the Levi-Civita connection of $g_{ij}$. This constraint is none other than the well-known momentum constraint obtained by Arnowitt, Deser and Misner in their 3+1 dynamical analysis of GR. In fact, best matching explains the presence of constraints linear in the canonical momenta in both GR and gauge theory [8].

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4 Lie groups in the case of finite-dimensional systems, their infinite-dimensional generalizations in the case of gauge theory and dynamical geometry.

5 The Hamiltonian constraints quadratic in the canonical momenta that appear in GR arise because there is no external absolute time in general relativity.
In general dynamical theory, there is a great difference between theories in which constraints are present (gauge theory and GR) and theories in which no constraints are present (Newtonian theory). This difference reflects the implementation of Mach’s principle in GR (and a related form of it in gauge theory). More detailed discussions that take into account the treatment of time, scale invariance implemented through conformal best matching, and a proper constraint analysis can be found in [6, 8, 10].

In summary and to repeat the point made at the start: one cannot judge whether a theory implements Mach’s principle by considering the motion of a single particle in an inertial frame of reference, be it global or local. Instead one must ask: What determines the inertial frame of reference? The answer to this question shows that general relativity is Machian.

5 Comment on Mashhoon and Wesson’s Response

Mashhoon and Wesson have set out a short response “Mach, the Universe, and Foundations of Mechanics” to my comments. They grant that I have given a possible definition and implementation of Mach’s principle in the case of a spatially closed universe but “prefer to rely on the judgement of observational cosmology”. I agree that ultimately observations must judge the viability of a conceptual scheme and that we cannot yet survey the whole universe in the way needed to confirm the correctness of my proposal. However, my concern is with the foundations of mechanics and the ideal Mach intuited. Let me make two points.

1. In their original paper, Mashhoon and Wesson stated as a fact that in GR “the origin of inertia remains essentially the same as in Newtonian physics”. My concern is to counter this claim and to show that Mach’s ideas deal with the most basic foundational question in dynamics: how is motion to be defined? In attempting to answer this question, I do not think the theoretician should be restricted to the analysis of direct observations. Indeed, modern science began when the Greeks introduced theoretical notions into astronomy that went well beyond the observations [7].

2. Mach recognized (footnote 2) that his ideas involved the entire universe. In dynamical geometry, this does indeed require the universe to be spatially closed as Wheeler, following Einstein, advocated (see Isenberg’s

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6 When invoking the universe to explain inertia, Mach was solely concerned with inertia in the sense of Newton’s first law. In accordance with his definition of inertial mass [3], p. 264–271, that is an intrinsic property of each individual body. Einstein introduced much confusion by failing to distinguish the two meanings of inertia.
article in [11]). However, even when boundary conditions are imposed, the
dynamics up to them is best matched and ‘as Machian as it can be’. More-
over, my definition of Mach’s principle and its best-matching implementation
lead directly to the framework that Wheeler merely “let the mathematics
tell us” to adopt (ibid, p. 194).

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