We calculate frequencies and damping rates of the lowest collective modes of a dilute Bose gas confined in an anisotropic trapping potential above the Bose-Einstein transition temperature. From the Boltzmann equation with a simplified collision integral we derive a general dispersion relation that interpolates between the collisionless and hydrodynamic regimes. In the case of axially symmetric traps we obtain explicit expressions for the frequencies and damping rates of the lowest modes in terms of a phenomenological collision time. Our results are compared with microscopic calculations and experiments.

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I. INTRODUCTION

Above the Bose-Einstein transition temperature both collisionless and hydrodynamic modes can exist in a trapped gas. In the collisionless regime the frequency of the mode is large compared with the interatomic collision frequency, and the mean free path is larger than the wavelength of the mode and the dimension of the cloud. In the hydrodynamic regime the opposite is true. Previous theoretical studies have focused on the collisionless and hydrodynamic regimes. Analysis of recent measurements of frequencies and damping rates of collective modes in trapped Bose gases above the Bose-Einstein transition temperature indicates that the experiments were performed under conditions intermediate between the collisionless and hydrodynamic regimes, and in Ref. a simple interpolation formula for the intermediate regime was proposed. The aim of this paper is to calculate frequencies and damping rates of modes in the intermediate regime, and to examine the validity of the interpolation formula.

The difficulty in this problem arises from the inhomogeneity of the system. Unlike the case of homogeneous systems conditions may be hydrodynamic in the center of the cloud and collisionless near its surface. In a trapped gas the collisionless limit is always well-defined in the sense that the frequency may exceed the collision rate everywhere, but hydrodynamic conditions can only be achieved in a limited region of space, not in the outer parts of the cloud.
Therefore one would expect a single relaxation-time approximation to work less well than for a homogeneous system.

At temperatures above the transition temperature the gas is dilute in the sense that the mean interatomic spacing is much larger than the interatomic scattering length. Mean-field effects can be neglected since the potential energy due to interactions, $nU_0$, is much less than the thermal energy $k_BT$. Here $n$ is the number density, $U_0$ is the effective two-body interaction (which is given in terms of the scattering length $a$ as $U_0 = 4\pi ah^2/m$, where $m$ is the atomic mass), $k_B$ is the Boltzmann constant, and $T$ is the temperature. Then the atoms can be described by a distribution function that obeys the Boltzmann equation. We solve the Boltzmann equation with a trial solution appropriate to the collective modes being studied, using a relaxation time approximation to the collision integral.

In the next section we derive a general relation for mode frequencies that interpolates between the collisionless and hydrodynamic regimes in an anisotropic trap. In section III we consider the experimentally relevant case of axially symmetric traps and calculate frequencies and damping rates of the lowest modes. In section IV we discuss the validity of our approach by comparing our results with microscopic calculations and with experiment. Section V contains a summary of our conclusions.

II. THE BOLTZMANN EQUATION AND COLLECTIVE MODES

In this section we derive the dispersion relation of the lowest modes of a Bose gas confined by an anisotropic harmonic potential given by

$$V(r) = \frac{1}{2} m (\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2),$$

(1)

where $\omega_1$, $\omega_2$, and $\omega_3$ are the characteristic frequencies of the trap in the three directions.

The dynamics of the dilute Bose gas above the transition temperature can be described by a semi-classical distribution function $f(p, r, t)$, where $p$ is the particle momentum. The distribution function satisfies the Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{1}{m} p \cdot \nabla_r f - \nabla_r V \cdot \nabla_p f = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}.$$

(2)

Here $\left( \frac{\partial f}{\partial t} \right)_{\text{coll}}$ is the contribution of collisions to the rate of change of $f$. For two-body scattering it is given by

$$\left( \frac{\partial f(p, r, t)}{\partial t} \right)_{\text{coll}} = -\frac{1}{(2\pi\hbar)^3} \int dp_1 \int d\sigma |\mathbf{v} - \mathbf{v}_1| [f(p, r)f(p_1, r)(1 + f(p, r))(1 + f(p_1, r)) - f(p, r)f(p_1, r)(1 + f(p, r))(1 + f(p_1, r))].$$

(3)
The collision integral involves scattering processes with two incoming particles of momenta \( p \) and \( p_1 \) (and velocities \( v \) and \( v_1 \)) and two outgoing particles of momenta \( p' \) and \( p'_1 \), respectively. The differential cross section is denoted by \( d\sigma \).

We shall linearize the Boltzmann equation in small deviations of the distribution function from the equilibrium one. This can be achieved simply by inserting the following expression

\[
f(p, r) = f^{(0)}(p, r) + f^{(0)}(p, r)(1 + f^{(0)}(p, r))\psi(p, r)
\]

in Eqs. (2) and (3), where \( \psi(p, r) \) describes the deviation from equilibrium, and \( f^{(0)} \) is the equilibrium distribution function

\[
f^{(0)}(p, r) = \left\{ \exp \left[ \frac{p^2}{2m} + V(r) - \mu \right]/k_B T \right\}^{-1}.
\]

To first order in \( \psi \), the Boltzmann equation assumes the following form

\[
\left( \frac{\partial}{\partial t} + \frac{1}{m} p \cdot \nabla r - \nabla r \cdot \nabla p \right) \psi(p, r) = -I[\psi(p, r)],
\]

where \( I \) is a linear operator given by

\[
I[\psi] = \frac{1}{(2\pi\hbar)^3} \int dp_1 \int d\sigma \left| v - v_1 \right| f^{(0)}(p_1, r)(1 + f^{(0)}(p, r))(1 + f^{(0)}(p'_1, r))(1 + f^{(0)}(p'_1, r)) \times
\]

\[
\left[ \psi(p, r) + \psi(p_1, r) - \psi(p'_1, r) - \psi(p'_1, r) \right].
\]

Since the scattering processes considered here conserve the number of particles, momentum, and energy [12], the zeroth, first, and second moments of the collision integral, calculated by multiplying it by \( 1, p, \) and \( p^2 \), respectively, and then integrating over \( p \), must vanish. The quantities \( 1, p, \) and \( p^2 \) are called the collision invariants.

### A. Dispersion Relation for Low-Lying Modes

The lowest collective mode corresponds to the center of mass motion (the dipole mode). The frequency of this mode equals \( \omega_i \) \( (i = 1, 2, 3) \) and does not depend on interatomic collisions. Therefore this mode will not be considered further in this paper. We shall instead focus on the higher modes which have a frequency \( 2\omega_i \) in the collisionless limit. In the hydrodynamic limit one can show that the fluid velocity \( u \) associated with these modes is given generally by \( u = (ax, by, cz) \), where \( a, b, \) and \( c \) are constants [3–7]. This expression is general in the sense that it does not depend on temperature or the anisotropy of the trap. In spherical traps these modes correspond to the monopole mode (or breathing mode) characterized by \( a = b = c \), and the quadrupole modes for which \( a + b + c = 0 \).
Let us attempt to find a solution $\psi$ that corresponds to the lowest modes of the system described above. The deviation function $\psi$ corresponding to a flow with drift velocity $\mathbf{u}$ is proportional to $\mathbf{u} \cdot \mathbf{p}$. Acting on $\mathbf{u} \cdot \mathbf{p}$ with the left hand side of Eq. (6) will thus introduce terms like $x^2$, $p_x^2$ etc., and therefore we adopt as our trial function the general form

$$\psi = e^{-i\omega t}[a_1 x^2 + b_1 x p_x + c_1 p_x^2 + a_2 y^2 + b_2 y p_y + c_2 p_y^2 + a_3 z^2 + b_3 z p_z + c_3 p_z^2].$$

Here the terms $x^2$, $y^2$, and $z^2$ correspond to changes in the number of particles, and $x p_x$, $y p_y$, and $z p_z$ correspond to changes in the local momentum density, and hence all these terms are collision invariants. On the other hand the terms $p_x^2$, $p_y^2$, and $p_z^2$ are not collision invariants since only their sum $p^2 = p_x^2 + p_y^2 + p_z^2$, which is proportional to the kinetic energy, is. Mode damping therefore arises from only these latter terms.

In the relaxation time approximation the collision operator is associated with a mean relaxation time $\tau(\mathbf{r})$, which depends on the position $\mathbf{r}$. One is restricted, however, in that for any approximate expression for the collision operator $I$, the above-mentioned conservation laws should be satisfied. This can be ensured by requiring the collision invariants to be eigenfunctions of the collision operator with eigenvalue zero. However, for terms such as $p_x^2$, which are not collision invariants and in momentum space have the symmetry of $Y^m_2$, where $Y^m_l$ are the spherical harmonics, we have

$$I[p_x^2] = -\frac{1}{3 \tau(\mathbf{r})} (p_x^2 - p^2/3),$$

and similarly for $p_y^2$ and $p_z^2$. The term $p^2/3$ in Eq. (10) ensures that $I[p_x^2 + p_y^2 + p_z^2] = 0$.

Since the scattering rate depends on the local number density $\rho$, the mode damping involves a spatial average of the relaxation time. (The simplest functional dependence one can assume for the collision rate is that of the equilibrium distribution function.) To obtain collective modes, we insert the trial function (9) in the Boltzmann equation (6) and then take moments of this equation by multiplying it by $f(0)(1 + f(0))$ and by each of the terms in $\psi$ and then integrating over $\mathbf{p}$ and $\mathbf{r}$ taking into account Eq. (10). For each moment we get a separate equation and thus obtain nine coupled equations. The condition for the existence of nontrivial solutions results in the following dispersion relation for the frequency of the collective modes:

$$\left(\omega^6 - 4\omega_0^2\omega^4 + 16\omega_0^4\omega^2 - 64\omega_0^6\right)$$

$$+ \frac{2}{3} \left(\omega^6 - 10\omega_0^2\omega^4 + 32\omega_0^4\omega^2 - 96\omega_0^6\right)/(\omega \tau)$$

$$- \frac{1}{4} \left(\omega^6 - 8\omega_0^2\omega^4 + 20\omega_0^4\omega^2 - 48\omega_0^6\right)/(\omega \tau)^2 = 0,$$

where

$$\omega_0^2 = \frac{p^2}{3\tau(\mathbf{r})}.$$
\[ \omega_a^2 = \omega_1^2 + \omega_2^2 + \omega_3^2, \]
\[ \omega_b^4 = \omega_1^2 \omega_2^2 + \omega_1^2 \omega_3^2 + \omega_2^2 \omega_3^2, \]
and
\[ \omega_c^6 = (\omega_1 \omega_2 \omega_3)^2, \]
with
\[ \frac{1}{\tau} = \frac{\int dr \tau^{-1}(r) \int dp f^{(0)}(1 + f^{(0)}) \varepsilon^2}{\int dr \int dp f^{(0)}(1 + f^{(0)}) \varepsilon^2}, \tag{12} \]
where \( \varepsilon = p^2/2m \) is the kinetic energy. Equation (11) is our main result, which describes the frequency and attenuation of collective modes in the collisionless and hydrodynamic regimes, as well as in the intermediate regime.

III. MODE FREQUENCIES AND DAMPING RATES FOR A TRAP WITH AN AXIS OF SYMMETRY

In traps with an axis of symmetry the oscillations in the axial and radial directions are decoupled and therefore the projection of angular momentum \( m \) about the symmetry axis of the trap is a good quantum number and can be used to label modes. In this case the modes described for a general trap in subsection II A have the following three forms:

i) The oscillations in the radial and axial directions are in phase (In the spherically symmetric case this corresponds to the breathing mode, \( a = b = c \)).

ii) The oscillations in the two radial directions are out-of-phase with those in the axial direction (\( a = b = -c/2 \) for the case of spherical symmetry).

iii) The oscillations in the two perpendicular radial directions are out-of-phase with respect to each other and there is no oscillation in the axial direction (\( a = -b, c = 0 \)). Both of the first and second modes have \( m = 0 \), while for the last one \( m = 2 \). For the rest of this paper we denote the first mode as the \( 0^+ \)-mode, the second one as the \( 0^- \)-mode, and the third one as the \( 2 \)-mode.

A. Frequencies and Damping Rates

The roots of Eq. (11) can be obtained analytically as functions of \( \tau \) and the characteristic frequencies \( \omega_a, \omega_b, \) and \( \omega_c \), but they have a complicated functional dependence on the parameters. However, for experimental traps with axial symmetry characterized by \( \omega_1 = \omega_2 = \omega_0 \) and \( \omega_3 = \lambda \omega_0 \), where \( \lambda \) is the anisotropy ratio, the roots simplify considerably. In this case the dispersion relation (11) takes the form

\[ \omega \left[ (\omega^2 - 2\omega_0^2) - i\omega \tau (\omega^2 - 4\omega_0^2) \right] \left[ (\omega^2 - r_- \omega_0^2) (\omega^2 - r_+ \omega_0^2) - i\omega \tau (\omega^2 - 4\omega_0^2) (\omega^2 - 4\lambda^2 \omega_0^2) \right] = 0, \tag{13} \]
where

$$r_{\pm} = \frac{5 + 4\lambda^2}{3} \pm \frac{1}{3} \sqrt{25 - 32\lambda^2 + 16\lambda^4}. \quad (14)$$

Note that the dispersion relation in this case factorizes into two terms (apart from the $\omega$ factor), one of which is independent of $\lambda$. This factorization can be understood by finding the eigenmodes that correspond to the roots of (13). As we shall discuss in more detail below, the $\lambda$-independent term corresponds to the 2-mode, in which there is no oscillation in the $z$-direction. The other two modes, corresponding to the $\lambda$-dependent factor, are the $0^+$- and $0^-$-modes. There are three modes of frequency $\omega = 0$ in the collisionless limit which correspond to thermal expansion.

The first corresponds to constant temperature change in all directions. This mode is associated with the over-all factor $\omega$ in Eq. (13). The second mode has $m = 0$ and frequency $\omega = -i/2\tau$ in the collisionless limit ($\omega_0\tau \gg 1$), and $\omega = -i/\tau$ in the hydrodynamic limit ($\omega_0\tau \ll 1$). The third mode has $m = 2$ and the same values of frequency in the collisionless and hydrodynamic limits as for the previous mode.

We start the analysis of (13) by looking at mode frequencies in the collisionless and hydrodynamic limits. In the collisionless limit, $\omega\tau \gg 1$, we get:

$$\omega \equiv \omega_C = 2\omega_0, 2\omega_0, 2\lambda\omega_0, \quad (15)$$

while in the hydrodynamic limit, $\omega\tau \ll 1$:

$$\omega \equiv \omega_H = \sqrt{2}\omega_0, \sqrt{r^-\omega_0}, \sqrt{r^+\omega_0}, \quad (16)$$

where $\omega_C$ denotes the frequency in the collisionless limit and $\omega_H$ is the corresponding frequency in the hydrodynamic limit. Thus we obtain the simple picture that the three collisionless modes of frequencies $2\omega_0$, $2\omega_0$ and $2\lambda\omega_0$ correspond to the three hydrodynamic modes of frequencies $\sqrt{2}\omega_0$, $\sqrt{r^-\omega_0}$ and $\sqrt{r^+\omega_0}$, respectively. (This is true only for $\lambda > 1$. When $\lambda < 1$ the mode of frequency $2\lambda\omega_0$ corresponds to the mode of frequency $\sqrt{r^-\omega_0}$ instead of that of frequency $\sqrt{r^+\omega_0}$.) We note here that our result (16) agrees with previous works based on either the conservation laws [3], or taking moments of the kinetic equation [5].

To obtain frequencies of modes and their attenuation in the intermediate regime, we insert $\omega = \omega_r - i\omega_i$ in Eq. (13), and thus obtain two coupled equations for the real and imaginary parts. In Figs. 1 and 2 we plot the frequency and the damping rate of the $0^-$-mode for values of $\lambda$ that satisfy the experimentally relevant condition $\lambda \ll 1$.

It is instructive, also, to investigate the functional dependence of the damping rate on $\lambda$ and $\tau$ in the collisionless and hydrodynamic limits. This behavior can be deduced using the property that in both of these limits the imaginary
part \( \omega_i \) is small, and thus one can expand Eq. (13) in \( \omega_i \). Let us first consider the dispersion relation corresponding to the first factor in this equation. In the collisionless limit, \( \omega \tau \gg 1 \), we can expand this factor to first order in \( 1/(\omega \tau) \), and write the solution as

\[
\omega = 2\omega_0 - i\Gamma^c_2, \tag{17}
\]

where \( \Gamma^c_2 \) is the small damping rate associated with the 2-mode. To first order in \( 1/\tau \) we obtain

\[
\Gamma^c_2 = \frac{1}{4} \tau^{-1}. \tag{18}
\]

Similarly, for the other two modes associated with the \( \lambda \)-dependent factor of (13) we obtain

\[
\Gamma^c_0^+ = \frac{1}{12} \tau^{-1}, \tag{19}
\]

and

\[
\Gamma^c_0^- = \frac{1}{6} \tau^{-1}. \tag{20}
\]

In the hydrodynamic limit, \( \omega \tau \ll 1 \), we expand in \( \omega \tau \) and then insert \( \omega = \sqrt{2}\omega_0 - i\Gamma^h_2 \) to get

\[
\Gamma^h_2 = \omega_0^2 \tau, \tag{21}
\]

together with

\[
\Gamma^h_0^+ = \frac{(r_+ - 4\lambda^2)(r_- - 4)}{2(r_- - r_+)} \omega_0^2 \tau, \tag{22}
\]

and

\[
\Gamma^h_0^- = \frac{(r_- - 4\lambda^2)(r_+ - 4)}{2(r_+ - r_-)} \omega_0^2 \tau. \tag{23}
\]

We note that the damping rates in the collisionless limit are proportional to \( 1/\tau \) and independent of \( \lambda \).

Finally, we end this section by discussing some interesting features of the hydrodynamic damping rates \( \Gamma^h_0^+ \) and \( \Gamma^h_0^- \) given by Eqs. (22) and (23). This can be done by plotting these rates as a function of \( \lambda \) as shown in Fig. 3. From this figure, we first note that \( \Gamma^h_0^+ \) vanishes for \( \lambda = 1 \). This is in agreement with the well established fact that in an isotropic harmonic potential the breathing mode does not relax, as was first shown by Boltzmann. \[13\] We also note that \( \Gamma^h_0^- \) has the maximum value \( 1.13\omega_0^2 \tau \) for \( \lambda \approx 1.20 \). In the limit \( \lambda \to 0 \) (cigar-shaped cloud; a quasi one-dimensional system) the damping rate vanishes, while in the limit \( \lambda \to \infty \) (disk-shaped cloud; a quasi two-dimensional system) the damping rate approaches a constant value equal to \( (3/4)\omega_0^2 \tau \).
IV. DISCUSSION

The damping rates found above were all expressed in terms of the phenomenological time \(\tau\), Eq. (12). We shall now identify the damping rates found above with the precise expressions obtained by a microscopic calculation. If we assume that the collision rate is proportional to the local density and take the classical limit, \(f(0) \ll 1\), the spatial average entering Eq. (12) may readily be performed, resulting in \(\tau = 2\sqrt{2}\tau(0)\), where \(\tau(0)\) is the relaxation time at the center of the trap. The damping rate, \(\Gamma_0^+\), for the \(0^+\)-mode, which is given by Eq. (19), therefore becomes \(1/(24\sqrt{2}\tau(0))\). In Ref. [1] this damping rate was calculated from a variational solution to the Boltzmann equation in the collisionless limit, with the result

\[
\Gamma_0^c \leq \frac{1}{24\sqrt{2} \tau_{\text{var}}(0)} \left( 1 + \frac{3}{16} \zeta(3) \left( \frac{T_{\text{BEC}}}{T} \right)^3 \right),
\]

(24)

where the quantity in the parenthesis results from including degeneracy effects to lowest non-vanishing order. Here \(\tau_{\text{var}}(0)\) is the variational viscous relaxation time given by \(\tau_{\text{var}}^{-1}(0) = (4\sqrt{2}/5)n(0)\sigma\pi\), where \(n(0)\) is the central density, \(\sigma = 8\pi a^2\) is the total cross section, \(\sigma = \sqrt{8k_B T/(\pi m)}\) is the mean thermal velocity, and \(T_{\text{BEC}}\) is the Bose-Einstein transition temperature. The use of an improved trial function changes the number \(1/(24\sqrt{2}\tau(0))\) in Eq. (24) by only \(\sim 1\%\). A comparison between the damping rate Eq. (24) and Eq. (19) thus reveals that

\[
\tau(0) = \left( 1 + \frac{3}{16} \zeta(3) \left( \frac{T_{\text{BEC}}}{T} \right)^3 \right)^{-1} \tau_{\text{var}}(0).
\]

(25)

The same relationship, Eq. (25), applies to the other two modes as well. For the two experiments of Refs. [8,9] our phenomenological time, \(\tau(0)\), in the center of the cloud is therefore given by \(\tau(0) \simeq 0.97\tau_{\text{var}}(0)\) for the experiment performed at \(T \simeq 2T_{\text{BEC}}\), and \(\tau(0) \simeq 0.82\tau_{\text{var}}(0)\) at \(T \simeq T_{\text{BEC}}\).

In the hydrodynamic limit, an accurate microscopic calculation of the collision rate is difficult to obtain due to the failure of the hydrodynamic conditions near the surface of the cloud. This is because the mean free path is too large for hydrodynamics to be valid. To solve this problem a cut-off procedure was introduced in Ref. [4], taking the hydrodynamic description to be valid to a distance from the center of the cloud such that an atom incident from outside the cloud has a probability less than \(1/e\) of not colliding with another atom. This cut-off procedure led to a logarithmic dependence of the damping on the quantity \(\alpha = \sigma n(0)\sqrt{k_B T/(2m\omega_0^2)}\). The leading term of this logarithmic dependence gives a damping rate proportional to \([\ln \alpha]^{3/2}\). The trial function (3) is not capable of describing accurately the hydrodynamic limit, since the very concept of hydrodynamic flow loses its meaning in the surface region. Expressed in other terms, the deviation function has a very different form in the outermost part of the cloud than it does in the center, and this effect is not captured by our simple trial function.
The microscopic calculations performed in Refs. [1,4] accounted for the damping rates only in the collisionless and hydrodynamic limits. To compare with experiments, which are performed under intermediate conditions, a simple phenomenological interpolation formula was suggested [1]. In the following we discuss the validity of this formula. In particular we show that the interpolation formula is identical to our result for the 2-mode. Our results for the $0^+$- and $0^-$-modes cannot in general be expressed in this simple form, but show a close resemblance, as we shall demonstrate below. The interpolation formula suggested in Ref. [1] is

\[ \omega^2 = \omega^2_C + \frac{\omega^2_H - \omega^2_C}{1 - i\omega\tau}. \]  

By construction, the limits $\omega\tau \gg 1$ and $\omega\tau \ll 1$ yield the appropriate frequencies in the collisionless and hydrodynamic limits, respectively. The 2-mode is $\lambda$-independent, thus one can easily put the corresponding dispersion relation (resulting from equating the cubic factor in $\omega$ in Eq. (13) to zero) in the same form as (26). For the $0^-$-mode we consider the two limiting cases $\lambda \ll 1$ and $\lambda \gg 1$, and the corresponding dispersion relation can be written as

\[ \omega^2 = \omega^2_C + \frac{\omega^2_H - \omega^2_C}{1 - i\omega(6\tau/5)}, \quad (m = 0, \ \lambda \ll 1), \]  

\[ \omega^2 = \omega^2_C + \frac{\omega^2_H - \omega^2_C}{1 - i\omega(4\tau/3)}, \quad (m = 0, \ \lambda \gg 1). \]  

The formula corresponding to the $0^+$-mode in these limits takes the same form as (26), but differs slightly from it, when $\lambda$ is comparable to 1. To illustrate how well the simple interpolation formula, Eq. (26), agrees with our results obtained from the solution of Eq. (13) we compare in Fig. 4 our calculated damping rate of the $0^-$-mode, for the case $\lambda = \sqrt{8}$, with the interpolation formula.

From experiments one obtains independent measurements of the mode frequency $\omega_r$ and its damping rate $\omega_i$. Since our calculated values of these quantities are functions of the collision rate, we can eliminate $\tau$ and obtain a relation between the real and imaginary parts. This allows us to calculate the damping rate of a mode without knowing about the characteristics of the collision processes leading to damping. Substituting $\omega = \omega_r - i\omega_i$ in Eq. (27) and eliminating $\tau$ from the two resulting equations leads to

\[ \omega_i^2 = \frac{1}{2}(\omega_C^2 - 3\omega_H^2) - \omega_r^2 + \frac{1}{2} \sqrt{\omega_C^4 - 10\omega_C^2\omega_H^2 + 9\omega_H^4 + 16\omega_H^2\omega_r^2}. \]  

It is interesting to note that this expression is also what one obtains from the simple interpolation formula (26). In fact the expression Eq. (29) is valid for any number replacing the ratio $6/5$ in Eq. (27). In Fig. 4 we use Eq. (29) to
plot \( \omega_i \) as a function of \( \omega_r \). Two values of \( \lambda \) are used, namely 0.076 and 0.074, corresponding to the two experiments of Refs. [8,9]. It can be clearly seen in this figure that the conditions of these experiments are intermediate between the collisionless and hydrodynamic limits. The deviation of Eq. (29) from the exact solution obtained from Eq. (13) is shown in Fig. 5 where we plot \( \omega_i \) as a function of \( \omega_r \) for \( \lambda = \sqrt{8} \).

The present method of obtaining the frequency and attenuation of collective modes can be extended to higher modes of frequency \( l\omega_i \), where \( l \) is a positive integer, by employing trial functions that differ from the one we have used in Eq. (9), for the \( 2\omega_i \)-modes.

V. CONCLUSION

Starting from the semiclassical Boltzmann equation and using the relaxation time approximation we have derived expressions for the frequencies and damping rates of the low-lying oscillatory modes of a dilute Bose gas above the Bose-Einstein transition temperature. These expressions give frequencies and damping rates as functions of a phenomenological collision time and hence describe modes in the regime intermediate between the collisionless and hydrodynamic regimes. Our results justify the use of a simple interpolation formula for connecting the hydrodynamic with the collisionless regime. The approach used in this work may be readily generalized to higher-frequency modes. After the completion of the present work we received a preprint by Guéry-Odelin et al. [15] who consider the \( 0^+ \)- and \( 0^- \)-modes in the classical limit and obtain conclusions in agreement with ours, and also show that the results are in accord with the results of numerical simulations.

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Captions of Figures

FIG. 1. Frequency of the $0^-$-mode. This plot is obtained by taking the real part of the lower solution ($\sqrt{r_0}\omega_0$) of the dispersion relation resulting from equating the $\lambda$-dependent factor in Eq. (13) to zero in the limit $\lambda \ll 1$. The imaginary part is shown in Fig. 2.

FIG. 2. Damping rate of the $0^-$-mode, in the limit $\lambda \ll 1$.

FIG. 3. The hydrodynamic damping rates $\Gamma^{h}_{0^-}$ and $\Gamma^{h}_{0^+}$ in units of $\omega_0^2\tau$, as a function of $\lambda$. The rate $\Gamma^{h}_{0^-}$ is associated with the $0^+$-mode while $\Gamma^{h}_{0^+}$ corresponds to the $0^+$-mode.

FIG. 4. Damping rate versus mode frequency of the $0^-$-mode for $\lambda = \sqrt{8}$. The solid line represents the exact solution of Eq. (13) and the dashed-dotted line is given by Eq. (29). The dotted line is the difference between the two upper curves (note the change of scale).

FIG. 5. Damping rate versus mode frequency of the $0^-$-mode. The two curves are given by Eq. (29) for different parameters corresponding to the two experiments of Refs. [8,9]. The points indicate experimental results.

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[14] G. M. Kavoulakis, C. J. Pethick, and H. Smith, (in preparation).

[15] D. Guéry-Odelin, F. Zambelli, J. Dalibard, and S. Stringari, cond-mat/9904409.
Fig. 1
Fig. 2
Fig. 4
Fig. 5