ON THE ALPERIN-MCKAY CONJECTURE FOR SIMPLE GROUPS OF TYPE A

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Abstract. In this paper characters of the normaliser of $d$-split Levi subgroups in $SL_n(q)$ and $SU_n(q)$ are parametrized with a particular focus on the Clifford theory between the Levi subgroup and its normalizer. These results are applied to verify the Alperin-McKay conjecture for primes $\ell$ with $\ell \nmid 6(q^2 - 1)$ and the Alperin weight conjecture for $\ell$-blocks of those quasi-simple groups with abelian defect. The inductive Alperin-McKay condition and inductive Alperin weight condition by the second author are verified for certain blocks of $SL_n(q)$ and $SU_n(q)$.

Keywords. Alperin-McKay conjecture, inductive Alperin-McKay condition, inductive blockwise Alperin weight condition, special linear group, special unitary group

1. Introduction

The McKay conjecture and its blockwise version, the Alperin-McKay conjecture, are two fundamental conjectures in representation theory of finite groups.

For a finite group $G$, a prime $\ell$ and $N$ the normalizer of a Sylow $\ell$-subgroup of $G$, the McKay conjecture is concerned with the cardinality of $\text{Irr}_{\ell'}(X)$ where $\text{Irr}_{\ell'}(X) = \{ \chi \in \text{Irr}(X) | \chi(1)_{\ell} = 1 \}$ and states that

$$| \text{Irr}_{\ell'}(G) | = | \text{Irr}_{\ell'}(N) |.$$

For an $\ell$-block $B$ of $G$ with defect group $D$ and $b$ the Brauer correspondent of $B$, an $\ell$-block of the normalizer in $G$ of $D$, the Alperin-McKay conjecture claims that

$$| \text{Irr}_0(B) | = | \text{Irr}_0(b) |,$$

where $\text{Irr}_0(C)$ is the set of height 0 characters in a block $C$. By summing over blocks with maximal defect, the Alperin-McKay conjecture implies the McKay conjecture.

Alperin introduced with the Alperin Weight Conjecture a similar global-local conjecture that claimed that the number of irreducible Brauer characters of a given $\ell$-block coincides with the number of weights of this $\ell$-block. If the block $B$ has abelian defect group, the conjecture is equivalent to

$$| \text{IBr}(B) | = | \text{IBr}(b) |,$$
where $b$ is as before the Brauer correspondent of $B$. A more structural explanation of both conjectures was suggested by Broué, namely that in case of abelian defect the blocks $B$ and $b$ are derived equivalent.

In [IMN07, Spå13a] the McKay and the Alperin-McKay conjecture have been reduced to finite simple groups. It remains to check the corresponding so-called inductive condition for all finite simple groups and primes $\ell$. Substantial progress has been made towards proving the McKay conjecture along that line. In particular, using the inductive McKay condition, Malle and the second author established the McKay conjecture for $\ell = 2$ [MS16]. Furthermore for odd primes the inductive McKay condition has been verified in multiple cases of simple groups [Spå12, CS13, CS17b, CS17a]. Less cases have been checked for the inductive Alperin-McKay (AM) condition. [Spå13a] verified the inductive AM condition for simple groups of Lie type, when $\ell$ is the defining characteristic, and for alternating groups, when the prime $\ell$ is odd. Additionally [Mal14, SF14] have dealt with simple groups of types $2^d B_2, 2^d G_2$ and $2^d F_4$. Further results have been established in [CS15, KS16b, KS16a] by considering particular structures of the defect group of the block.

The present paper is concerned with the Alperin-McKay condition for quasi-simple groups of type $A$ and primes $\ell$ different from the defining characteristic with $\ell \geq 5$. Note that the inductive Alperin-McKay (AM) and the blockwise Alperin weight (BAW) conditions hold for most blocks in the defining characteristic according to [Spå13a] and [Spå13b]. In order to verify the inductive Alperin-McKay condition a new criterion is introduced in Theorem 2.4, which will have applications to other series of simple groups, see [CSFS]. It complements the criterion given in [CS15]. This leads to the following statement, where we write $\text{SL}_n(-q)$ for $\text{SU}_n(q)$ and $\text{GL}_n(-q)$ for $\text{GU}_n(q)$.

**Theorem 1.1.** Let $\ell$ be a prime, $q$ a prime power and $\epsilon \in \{\pm 1\}$ with $\ell \nmid 3q(q - \epsilon)$, $G := \text{SL}_n(\mathbb{F}_q)$, $G := \text{SL}_n(q\epsilon)$, $B_0$ an $\ell$-block of $G$ with defect group $D$, and $B$ the $\text{GL}_n(q\epsilon)$-orbit containing $B_0$. Assume that $\text{PSL}_n(q\epsilon)$ is simple, $G$ is its universal covering group and the stabilizer $\text{Out}(G)_B$ is abelian.

(a) The inductive AM condition from Definition 7.2 of [Spå13a] holds for $B_0$.
(b) Let $d$ be the order of $q$ in $(\mathbb{Z}/\ell\mathbb{Z})^\times$. If $D$ is abelian and $C_G(D)$ is a $d$-split Levi subgroup of $G$, then the inductive BAW condition from [Spå13b] holds for $B_0$.

In Corollary 6.1, the inductive AM condition is also proven for blocks that are $\text{GL}_n(q\epsilon)$-stable and satisfy another similar assumption.

For unipotent blocks Feng has established in [Fen18] the inductive BAW condition for unipotent blocks and Li-Zhang have treated in [LZ18] other blocks under additional assumptions on the outer automorphism group. Also Li constructed in [Li18] an equivariant bijection for the inductive BAW condition in symplectic groups under some assumption on the $\ell$-modular decomposition matrix. More particular cases of simple groups of small rank were checked in [FLL17, Sch16, SF14].

In the proof of Theorem 1.1 a main step is to parametrize the characters of the normalizers of $d$-split Levi subgroups. These normalizers serve as local subgroups in the inductive AM condition, see Theorem 5.1. We investigate the action of automorphisms on those characters in terms of their parameters. Essential is to understand the Clifford theory of irreducible characters of a $d$-split Levi subgroup $L$ in $N_G(L)$. Furthermore, we consider the action of the stabilizer $\text{Aut}(G)_B(L)$ on the irreducible characters and verify that the corresponding inertia groups are of a particular structure.
Theorem 1.2. Let $G := \text{SL}_n(\mathbb{F}_q)$, $\overline{G} := \text{GL}_n(\mathbb{F}_q)$, $F : G \rightarrow \overline{G}$ a Frobenius endomorphism defining an $\mathbb{F}_q$-structure, $L$ a $d$-split Levi subgroup of $(G, F)$, $N_0 := N_{G^F}(L)$, and $\tilde{N}_0 := N_{G^F}(L)$.

(a) Every $\lambda \in \text{Irr}(L^F)$ extends to its inertia group in $N_0$.
(b) Let $E_0 \leq \text{Aut}(G^F)$ be the image of $E$ defined in 3.4 and let $\psi \in \text{Irr}(N_0)$. Then there exists a $\tilde{N}_0$-conjugate $\psi_0$ of $\psi$ such that

(i) $O_0 = (\tilde{G}^F \cap O_0) \rtimes (E_0 \cap O_0)$ for $O_0 := G^F(\tilde{G}^F \rtimes E_0)^L_{\psi_0}$, and
(ii) $\psi_0$ extends to $(G^F \rtimes E_0)^L_{\psi_0}$.

For groups of Lie type with abelian Sylow $\ell$-subgroup, bijections implying the Alperin-McKay conjecture and blockwise Alperin weight were constructed in [Mal14, Theorem 2.9] assuming the above 1.2(b)(a) for analogous local subgroups. As a consequence of our proof of Theorem 1.2, Alperin-McKay conjecture holds via [Mal14, Theorem 2.9 and Corollary 3.7] for special linear and unitary groups with abelian Sylow $\ell$-subgroup. Thanks to Theorem 5.2 we are able to generalize Malle’s approach from [Mal07], where he constructed a bijection for the inductive McKay condition. By considerations inspired by [Spå09, §10] we deduce from this the Alperin-McKay conjecture for all blocks, using results of Puig and Zhou on the so-called inertial blocks. Note that for $\ell \mid q$ the Alperin-McKay conjecture was proven in [Spå13a] based on earlier work by Green-Lehrer-Lusztig while for $\ell \mid (q - \epsilon)$ results of Puig in [Pui94, §5] imply the conjecture for most $\ell$-blocks of $\text{SL}_n(\mathbb{F}_q)$ with abelian defect.

Theorem 1.3. Let $G = \text{SL}_n(\mathbb{F}_q)$. Let $\ell$ be a prime with $\ell \nmid 3q(q - \epsilon)$.

(a) The Alperin-McKay Conjecture holds for all $\ell$-blocks of $G$.
(b) The Alperin weight Conjecture holds for all $\ell$-blocks of $G$ with abelian defect.

This paper is organised in the following way: in Section 2 we provide a criterion for the inductive AM condition and give some helpful statements using inertial blocks. Then in Section 3 we construct explicitly the $d$-split Levi subgroups, their normalizers and we highlight some important properties of the irreducible characters. The Clifford theory between $d$-split Levi subgroups and their normalizers is studied in Section 4 in order to prove Theorem 1.2. The final sections deduce from this the main results using $d$-Harish-Chandra theory and Jordan decomposition.

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2. THE INDUCTIVE AM CONDITION

In this section we recall the inductive AM condition and give a criterion for proving it in our situation. The proof requires some arguments relying on Dade’s ramification group. We also give a short lemma that will later be used to verify the Alperin-McKay conjecture.

For characters of finite groups and blocks we will freely make use of the notation from [Isa94] and [Nav98]. Our blocks are considered with regard to a prime $\ell$. For an $\ell$-block $c$ of a subgroup of a finite group $H$ we denote the induced block by $c^H$. For a generalized character $\chi$ of $H$ we denote by $\text{Irr}(\chi) \subseteq \text{Irr}(H)$ its set of irreducible constituents and for an irreducible character $\chi$ we denote by $\text{bl}(\chi)$ the $\ell$-block that it belongs to. We denote
by \( \chi^o \) the restriction of \( \chi \) to the \( \ell' \)-elements of \( H \). Recall that whenever the group \( A \) acts on a set \( M \) we denote by \( A_m \) the stabilizer of an element \( m \) of \( M \) in \( A \).

### 2.A. The inductive AM condition for a set of blocks.

In [Spä13a, Definition 7.1] the inductive AM condition for a simple group and a prime \( \ell \) were introduced. An alternative version, relative to a radical \( \ell \)-subgroup was given in [CS13, Section 7.1]. In addition [KS16b, Def. 3.2] provided one related to a single block.

First recall the following standard notations. For \( D \) an \( \ell \)-subgroup of a finite group \( X \) we denote by \( \text{Bl}(X / D) \) the set of \( \ell \)-blocks of \( X \) which have \( D \) as a defect group. Recall the notation \( \text{Irr}(0)(B) \) for the set of height zero characters in a block \( B \). For any subset \( B \subseteq \text{Bl}(X) \) let the sets \( \text{Irr}(B) \) and \( \text{Irr}(0)(B) \) be analogously defined.

**Definition 2.1.** Let \( S \) be a finite non-abelian simple group and \( \ell \) a prime. Let \( G \) be the universal covering group of \( S \) and \( B \in \text{Bl}(G) \) with defect group \( D \). Then we say that the inductive AM condition holds for \( B \) if

1. there exists some \( \text{Aut}(G)_{B,D} \)-stable subgroup \( M \) such that \( N_G(D) \leq M \leq G \).
2. For \( B' \in \text{Bl}(M) \) with \( B'^G = B \) there exists an \( \text{Aut}(G)_{B,D} \)-equivariant bijection
   
   \[ \Omega_B : \text{Irr}(0)(B) \rightarrow \text{Irr}(0)(B') \]
   
   such that
   - \( \Omega_B(\text{Irr}(0)(B) \cap \text{Irr}(G \mid \nu)) \subseteq \text{Irr}(M \mid \nu) \) for every \( \nu \in \text{Irr}(Z(G)) \),
   - \( \text{bl}(\chi) = \text{bl}(\Omega_B(\chi))^G \) for every \( \chi \in \text{Irr}(0)(B) \).
3. For every character \( \chi \in \text{Irr}(0)(B) \) there exists a group \( A \) and characters \( \tilde{\chi} \) and \( \tilde{\chi}' \) such that
   - (1) For \( Z := \ker(\chi) \cap Z(G) \) and \( \overline{G} := G/Z \) the group \( A \) satisfies \( \overline{G} \triangleleft A \), \( C_A(\overline{G}) = Z(A) \) and \( A/Z(A) \cong \text{Aut}(\overline{G})^{\chi} \),
   - (2) \( \tilde{\chi} \in \text{Irr}(A) \) is an extension of \( \chi \), where \( \chi \in \text{Irr}(\overline{G}) \) lifts to \( \chi \),
   - (3) for \( \overline{M} := M/Z \) and \( \overline{D} := D/Z \) the character \( \tilde{\chi}' \in \text{Irr}(\overline{M}N_A(\overline{D})) \) is an extension of \( \chi' := \Omega_B(\chi) \in \text{Irr}(\overline{M}J) \),
   - (4) \( \text{bl}(\text{Res}_{\overline{M}J}(\chi')) = \text{bl}(\text{Res}_{\overline{M}N_A(\overline{D})}(\tilde{\chi}))^J \) for every \( J \) with \( \overline{G} \leq J \leq A \),
   - (5) \( \text{Irr}(\text{Res}_{\overline{M}J}(\chi')) = \text{Irr}(\text{Res}_{\overline{M}N_A(\overline{D})}(\tilde{\chi})) \).

In particular, [CS15, Proposition 2.2] shows that if the inductive AM condition holds for any \( B \in \text{Bl}(G) \), then \( S \) satisfies the inductive AM condition [Spä13a, Definition 7.2] with respect to \( \ell \). Furthermore, [CS15, Theorem 4.1], an alternative version of the above condition was given in the case the radical subgroup is a Sylow \( \ell \)-subgroup. In other words, a modified version to consider blocks with maximal defect. The aim is to provide an analogous statement which focuses on blocks with a “nice” stabilizer subgroup in the automorphism group of \( G \).

### 2.B. Some block theory.

To prove Theorem 2.4 below, Dade’s ramification group provides a fundamental tool. It was introduced in [Dad73] and then reformulated in [Mur13] by Murai. We use it to study blocks and to define a bijection required for Definition 2.1.

For each block \( b \) we denote by \( \lambda_b \) the associated map defined in [Nav98, §3]. For a group \( G \) and \( x \in G \) we denote by \( \text{Cl}_G(x)^+ \) the sum of the \( G \)-conjugacy class containing \( x \) in the group algebra \( Z[G] \).

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4. **ON THE ALPERIN-MCKAY CONJECTURE FOR SIMPLE GROUPS OF TYPE A**
Definition 2.2. Let $X \lhd \tilde{X}$ and $B \in \text{Bl}(X)$, then the group $\tilde{X}[b]$ is defined by

$$\tilde{X}[B] := \left\{ x \in \tilde{X}_B \mid \lambda_{b(x)}(\text{Cl}(x,y)) \neq 0 \text{ for some } y \in xX \right\},$$

where $b(x)$ denotes an arbitrary block of $\langle X, x \rangle$ which covers $B$. (This definition of $\tilde{X}[B]$ is independent of the choice of $b(x)$, see [Mur13, 3.3].)

In the above situation let $\chi \in \text{Irr}(B)$. Then $\tilde{X}[B] \leq \tilde{X}_x \leq \tilde{X}_B$ (see [KS15, Lemma 3.2]).

For the proof of Theorem 2.4 we use the following technical lemma. Recall that for $X \lhd \tilde{X}$ and $B \in \text{Bl}(X)$ we denote by $\text{Bl}(\tilde{X} \mid B)$ the set of all blocks of $\tilde{X}$ covering $B$.

Lemma 2.3. Let $X \lhd \tilde{X}$ be finite groups with abelian quotient $\tilde{X} / X$, and $B \in \text{Bl}(X)$ with defect group $D$. Let $M \leq X$ with $M \geq N_X(D)$, $\tilde{M} \leq \tilde{X}$ with $\tilde{M} \cap X = M$ and $\tilde{M} \geq MN_{\tilde{X}}(D)$, $b \in \text{Bl}(M)$ with $b^X = B$ and $\tilde{b} \in \text{Bl}(\tilde{M} \mid b)$.

Let $\tilde{M} \leq M_b$, $\tilde{X} := X M$ and $\tilde{B} := \tilde{b}^{\tilde{X}}$.

(a) If $\tilde{b} \in \text{Bl}(\tilde{M} \mid b)$ is covered by $\tilde{b}$, $\tilde{B} = \tilde{b}^{\tilde{M} X}$ covers $B$ and is covered by $\tilde{B}$.

(b) If $\tilde{B} \in \text{Bl}(\tilde{X} \mid B)$ is covered by $\tilde{B}$, then $\tilde{b}$ covers some $\tilde{b} \in \text{Bl}(\tilde{M} \mid b)$ with $\tilde{b}^{\tilde{X}} = \tilde{B}$.

Proof. Note that since $\tilde{X} / X$ is abelian, $\tilde{M} \lhd \tilde{M}$. By [Nav98, 9.28], $\tilde{B}$ is defined, covers $B = b^X$, and is covered by $\tilde{b}^{\tilde{X}} = \tilde{B}$.

Via [Nav98, 9.28] we see that $\tilde{c} \mapsto \tilde{c}^{\tilde{X}}$ defines a bijection $\text{Bl}(\tilde{M} \mid b) \to \text{Bl}(\tilde{X} \mid B)$. Hence there exists a block $\tilde{b}$ with $\tilde{b}^{\tilde{X}} = \tilde{B}$. The set $\text{Bl}(\tilde{M} \mid \tilde{b})$ is contained in $\text{Bl}(\tilde{M} \mid b)$ and is, via block induction, in bijection with $\text{Bl}(\tilde{X} \mid \tilde{B})$. Clearly $\tilde{B} \in \text{Bl}(\tilde{X} \mid \tilde{B})$ and hence there exists some $\tilde{b}' \in \text{Bl}(\tilde{M} \mid \tilde{b})$ with $\tilde{b}'^{\tilde{X}} = \tilde{B}$. Since $\tilde{b}^{\tilde{X}} = \tilde{B}$, this implies $\tilde{b} = \tilde{b}'$ and hence $\tilde{b}$ covers $\tilde{b}$. \hfill \Box

2.C. Alternative inductive condition. In this section a criterion for the inductive AM condition adapted to simple groups of Lie type is given. The condition is closely related to the one in [CS15, Theorem 4.1]. In fact conditions (i) – (iv) are the same, and condition (v) which considers the structure of stabilizers of a block is altered. Naturally, this will limit which blocks these conditions can be considered for.

Theorem 2.4. Let $S$ be a finite non-abelian simple group and $\ell$ a prime dividing $|S|$. Let $G$ be the universal covering group of $S$, $D$ a radical $\ell$-subgroup of $G$ and $B \subseteq \text{Bl}(G \mid D)$ a $\tilde{G}_D$-stable subset with $(\tilde{G}E)_B \leq (\tilde{G}E)_B$. Assume we have a semi-direct product $\tilde{G} \rtimes E$, a $\text{Aut}(G)_B$-stable subgroup $M$ with $N_G(D) \leq M \leq G$ and a group $\tilde{M} \leq \tilde{G}$ with $\tilde{M} \geq M N_{\tilde{G}}(D)$ and $M = \tilde{M} \cap G$ such that the following conditions hold:

(i) $G = [\tilde{G}, \tilde{G}]$ and $E$ is abelian,

$C_{\tilde{G} \rtimes E}(G) = Z(\tilde{G})$ and $\tilde{G}E / Z(\tilde{G}) \cong \text{Im}(G) \text{Aut}(G)_D$ by the natural map,

any element of $\text{Irr}_0(B)$ extends to its stabiliser in $\tilde{G}$,

any element of $\text{Irr}_0(B')$ extends to its stabiliser in $\tilde{M}$.

(ii) Let $B' \subseteq \text{Bl}(M)$ be the set of all Brauer correspondents of the blocks in $B$. For $\tilde{G} := \text{Irr}(\tilde{G} \mid \text{Irr}_0(B))$ and $\tilde{M} := \text{Irr}(\tilde{M} \mid \text{Irr}_0(B'))$ there exists an $N_{\tilde{G}E}(D)_B$-equivariant bijection

$$\tilde{\Omega} : \tilde{G} \to \tilde{M}$$

with
\[
(1) \tilde{\Omega}(\mathcal{G} \cap \text{Irr}(\tilde{G} \mid \tilde{\nu})) = \mathcal{M} \cap \text{Irr}(\tilde{M} \mid \tilde{\nu}) \text{ for all } \tilde{\nu} \in \text{Irr}(\tilde{Z}(\tilde{G})) ,
\]
\[
(2) \text{bl} \left( \tilde{\Omega}(\chi) \right) = \text{bl}(\chi) \text{ for all } \chi \in \mathcal{G} , \text{ and}
\]
\[
(3) \tilde{\Omega}(\mu) = \tilde{\Omega}(\chi) \text{ Res}_{\tilde{M}}^\tilde{G}(\tilde{\mu}) \text{ for every } \tilde{\mu} \in \text{Irr}(\tilde{G} \mid 1_G) \text{ and every } \tilde{\chi} \in \mathcal{G} .
\]

(iii) For every \( \tilde{\chi} \in \mathcal{G} \) there exists some \( \chi_0 \in \text{Irr}(G \mid \tilde{\chi}) \) such that

- \( (\tilde{G} \times E)_{\chi_0} = \tilde{G}_{\tilde{\chi}_0} \times E_{\chi_0} \), and
- \( \chi_0 \text{ extends to } G \times E_{\chi_0} \).

(iv) For every \( \tilde{\psi} \in \mathcal{M} \) there exists some \( \psi_0 \in \text{Irr}(M \mid \tilde{\psi}) \) such that

- \( O = (\tilde{G} \cap O) \times (E \cap O) \) for \( O := G(\tilde{G} \times E)_{D,\psi_0} \), and
- \( \psi_0 \text{ extends to } M(G \times E)_{D,\psi_0} \).

(v) For any \( \tilde{G} \)-orbit \( B \) in \( \mathcal{B} \) the group \( \text{Out}(G)_B \) is abelian.

Then the inductive AM condition holds for all \( \ell \)-blocks in \( \mathcal{B} \).

Note that assumption 2.4(v) implies that the stabilizer subgroup in \( \text{Out}(G) \) of characters of \( G \) is stable under \( \tilde{G} \)-conjugation. The following statement highlights a situation, based upon the underlying central character, for which this condition of Theorem 2.4 holds.

The above criterion is tailored to simple groups \( S \) of Lie type and there are canonical candidates for the groups \( \tilde{G} \), \( G \) and \( E \). If \( S \) is of Lie type \( B \), \( C \) or \( E_7 \), then condition 2.4(v) holds with the usual choices of the groups.

Let \( \nu \in \text{Irr}(Z(G)) \) with \( Z(G)_{\ell} \leq \text{ker}(\nu) \). If \( Z(G) \times E_{\nu} \) is abelian, then \( \text{Out}(G)_b \) is abelian for every \( \tilde{G} \)-orbit \( b \) in \( \text{Bl}(G \mid \text{bl}(\nu)) \). If the character \( \nu \) is faithful, condition 2.4(v) holds for every \( \tilde{G} \)-orbit \( b \) in \( \text{Bl}(G \mid \text{bl}(\nu)) \).

As said before, Theorem 2.4 and [CS15, Theorem 4.1] coincide in all but the last assumption. Hence the proof of Theorem 2.4 requires altering the previous proof in all situations where the last assumption is used. We now construct a bijection \( \Omega_B : \text{Irr}_0(\mathcal{B}) \to \text{Irr}_0(\mathcal{B}') \). Note that the proof [CS15, Theorem 4.1] gives in all cases a bijection \( \Omega_B : \text{Irr}_0(\mathcal{B}) \to \text{Irr}_0(\mathcal{B}') \) but it is not clear if the other requirements are satisfied.

2.2.4. A bijection \( \Omega_B \) for Definition 2.1(ii): We first choose in \( \mathcal{B} \) a \( (\tilde{G}E)_{D,\psi} \)-transversal \( \mathcal{B}_0 \). The idea is to define \( \Omega_B \) on a \( (\tilde{G}E)_{D,\psi} \)-transversal \( \mathcal{G}_0 \) in \( \text{Irr}_0(\mathcal{B}) \).

Let \( \chi_0 \in \mathcal{G}_0 \). Note that \( (\tilde{G}E)_{\chi_0} = \tilde{G}_{\chi_0}E_{\chi_0} \) by assumption (iii) and (v). Let \( \chi \in \text{Irr}(\tilde{G} \mid \chi_0) \) and set \( \phi := \tilde{\Omega}(\chi) \), \( B := \text{bl}(\chi_0) \), \( \tilde{B} := \text{bl}(\chi) \in \text{Bl}(\tilde{G} \mid B) \) and \( \tilde{b} = \text{bl}(\phi) \). Then \( b^{\tilde{G}} = \tilde{B} \) by condition 2.4(ii),2. By [Nav98, 9.28] the block \( b \in \text{Bl}(M) \) with \( b^{\tilde{G}} = B \) is covered by \( \tilde{b} \).

Let \( \tilde{G} \) to be the group with \( G \leq \tilde{G} \leq \tilde{G}_B[B] \) such that \( \tilde{G}/G \) is a \( \ell ' \)-Hall subgroup of \( \tilde{G}_B[B]/G \). As every character of \( G \) extends to its stabilizer in \( \tilde{G} \), it follows that \( \chi_0 \) extends to \( \tilde{G} \leq \tilde{G}_B[B] \leq \tilde{G}_{\chi_0} \). Moreover by Clifford theory \( \text{Irr}(\text{Res}_{\tilde{G}}^{\tilde{G}}(\chi)) \cap \text{Irr}(\tilde{G} \mid \chi_0) = \{ \tilde{\chi}_0 \} \), so set \( \tilde{B} := \text{bl}(\tilde{\chi}_0) \).

According to Lemma 2.3 there exists \( \tilde{b} \in \text{Bl}(\tilde{M} \mid b) \) for \( \tilde{M} := \tilde{M} \cap \tilde{G} \) with \( \tilde{b}^{\tilde{G}} = \tilde{B} \) that is covered by \( \tilde{b} \). Since \( \text{bl}(\phi) \) covers \( \tilde{b} \) there exists some \( \tilde{\phi}_0 \in \text{Irr}(\text{Res}_{\tilde{M}}^{\tilde{G}}(\phi)) \cap \text{Irr}(\tilde{b}) \). As \( \chi_0 \) extends to \( \tilde{G} \), restriction gives a bijection between \( \text{Irr}(\tilde{b}) \) and \( \text{Irr}(b) \), see [KS15, Lemma 3.7]. Thus \( \phi_0 := \text{Res}_{\tilde{M}}^{\tilde{G}}(\tilde{\phi}_0) \in \text{Irr}(b) \) and we set \( \Omega_B(\chi_0) = \phi_0 \).
We first ensure that $\Omega_{B}$ satisfies the properties from 2.1(ii): In order to prove

$$\Omega_{B}(\text{Irr}_0(B) \cap \text{Irr}(G \mid \nu)) \subseteq \text{Irr}(M \mid \nu)$$

for every $\nu \in \text{Irr}(Z(G))$ it is sufficient to check that for $\chi_0$ as above $\Omega_{B}(\chi_0) \in \text{Irr}(M \mid \nu)$ where $\nu \in \text{Irr}(\text{Res}_{G}^{Z(G)} \chi_0)$. Note that $\bar{\Omega}(\chi) \in \text{Irr}(\bar{M} \mid \bar{\nu})$ for some $\bar{\nu} \in \text{Irr}(Z(\bar{G}) \mid \nu)$, where $\chi \in \text{Irr}(\bar{G} \mid \nu)$. This implies $\Omega_{B}(\chi_0) = \phi_0 \in \text{Irr}(M \mid \nu)$.

To prove the second half of 2.1(ii) it is sufficient to check $\text{bl}(\chi) = \text{bl}(\Omega_{B}(\chi))^{G}$. By the construction of $\Omega_{B}$, the block $\text{bl}(\chi_0)$ is covered by $\bar{B}$ and $\text{bl}(\phi_0)$ is covered by $\hat{b}$, where $\phi_0 = \Omega_{B}(\chi_0)$ and $\hat{G} = \bar{B}$. Let $b \in \text{Bl}(M)$ with $b^{G} = B$. Then by construction, both $b$ and $\text{bl}(\phi_0)$ are covered by $\hat{b}$. By definition of $\bar{M}$ this implies $b = \text{bl}(\phi_0)$, since $\bar{M} \leq M[b]$ as $\hat{G}[B] = G\bar{M}[b]$.

For the proof of Theorem 2.4, it remains to show that any $\chi_0 \in \text{Irr}_0(B)$ satisfies the condition given by Definition 2.1(iii). By assumption 2.4(v), we can focus on $\chi_0 \in G_0$. Furthermore, by the proof of [CS15, Theorem 4.1], it is enough to verify the following proposition, which is an analogue of [CS15, Proposition 4.2].

**Proposition 2.5.** Let $\chi \in G_0$ and $\Omega_{B}$ be the bijection constructed above. Then there exists a group $A$ and characters $\bar{\chi}$ and $\phi$ such that

(i) for $Z := \ker(\chi) \cap Z(G)$ and $\bar{G} := G/Z$, then $\bar{G} \triangleleft A$, $C_{A}(\bar{G}) = Z(A)$ and $A/Z(A) \cong \text{Aut}(G)\chi$,

(ii) $\bar{\chi} \in \text{Irr}(A)$ is an extension of the character $\chi \in \text{Irr}(G)$ that lifts to $\chi$,

(iii) $\phi \in \text{Irr}(\bar{M}N_{A}(\bar{D}))$ is an extension of the character $\bar{\phi} \in \text{Irr}(\bar{M})$ that lifts to $\phi := \Omega(\chi)$ where $\bar{M} := M/Z$ and $\bar{D} := DZ/Z$,

(iv) there exists a group $J$ with $\bar{G}Z(A) \leq J < A$ with abelian quotients $A/J$ and $J/\bar{G}$, such that

$$\text{bl} \left( \text{Res}_{J_2}(\bar{\chi}) \right) = \text{bl} \left( \text{Res}_{\bar{M}N_{J_2}(\bar{D})}^{\bar{M}N_{A}(\bar{D})}(\bar{\phi}) \right)^{J_2} \text{ for every } J_2 \text{ with } G \leq J_2 \leq J$$

(v) $\text{Irr}(\text{Res}_{Z(A)}^{A}(\bar{\chi})) = \text{Irr}(\text{Res}_{\bar{M}N_{A}(\bar{D})}^{\bar{M}N_{J_2}(\bar{D})}(\bar{\phi}))$.

**Proof.** Let $\chi_1 \in \text{Irr}(\hat{G}(\chi_0 \mid \chi_0)$ and $\phi_1 \in \text{Irr}(\hat{M}_{\phi_0} \mid \phi_0)$ be the Clifford correspondents to $\chi$ and $\phi$ respectively. They satisfy $\text{Res}_{G}^{\hat{G}}(\chi_1) = \hat{\chi}_0$ and $\text{Res}_{\hat{M}}^{\hat{M}_{\phi_0}}(\phi_1) = \hat{\phi}_0$ and hence $\text{bl}(\text{Res}_{G}^{\hat{G}}(\chi_1)) = \text{bl}(\text{Res}_{\hat{M}}^{\hat{M}_{\phi_0}}(\phi_1))^{\hat{G}}$. In addition $\hat{G}[B] = \hat{G}\bar{M}[b]$ by [Mur13] and thus $\hat{G} = \hat{G} \cap G\bar{M}[b] = G(\hat{G} \cap M[b]) = G\hat{M}$. Note that $M = G \cap \hat{M}$ and $\ell \mid |\hat{G} : G|$. Hence

$$\text{bl}(\text{Res}_{\hat{G}}^{\hat{G}}(\hat{\chi}_0)) = \text{bl}(\text{Res}_{\hat{M}N_{x}(\hat{\phi})}^{\hat{M}N_{J_2}(\hat{D})}(\hat{\phi}))^{(\hat{G} \cap M[x])} \text{ for every } x \in \hat{M}$, see [KS15, Lemma 2.4].

Let $G \leq J_1 \leq \hat{G}[B]^{J_1}$, then $J_1 = G(J_1 \cap \hat{M}[b])$. Since $(J_1 \cap \hat{M}[b])_{\phi_0} \leq \hat{M}$ we see $\text{bl}(\text{Res}_{J_1}^{\hat{G}}(\chi_1)) = \text{bl}(\text{Res}_{J_1 \cap \hat{M}[b]}^{\hat{M}_{\phi_0}}(\phi_1))^{J_1}$ according to [KS15, Lemma 2.5].

Let $G \leq J_2 \leq \hat{G}[\chi_0]$ and $J_1 := J_2 \cap \hat{G}[B] = J_2[B]$. Then $\text{bl}(\text{Res}_{J_1}^{\hat{G}}(\chi_1)) = \text{bl}(\text{Res}_{J_1 \cap \hat{M}[b]}^{\hat{M}_{\phi_0}}(\phi_1))^{J_1}$. Furthermore, as $B$ is $J_2$-stable, it follows that

$$\text{bl}(\text{Res}_{J_2}^{\hat{G}}(\chi_1)) = \text{bl}(\text{Res}_{J_1}^{\hat{G}}(\chi_1))^{J_2}.$$
and so $\text{bl}(\text{Res}_{J_2 \cap \bar{M}_{\phi_0}}^M(\phi_1))$ is covered by $\text{bl}(\text{Res}_{J_2 \cap \bar{M}_{\phi_0}}^M(\phi_1))$. Since $\text{bl}(\text{Res}_{J_2 \cap \bar{M}_{\phi_0}}^M(\chi_1))$ is the unique block of $J_2$ covering $\text{bl}(\text{Res}_{J_2 \cap \bar{M}_{\phi_0}}^M(\phi_1))J_1$ and $\text{bl}(\text{Res}_{J_2 \cap \bar{M}_{\phi_0}}^M(\phi_1))$ is uniquely covered by $\text{bl}(\text{Res}_{J_2 \cap \bar{M}_{\phi_0}}^M(\phi_1))$, we see $\text{bl}(\text{Res}_{J_2 \cap \bar{M}_{\phi_0}}^M(\chi_1)) = \text{bl}(\text{Res}_{J_2 \cap \bar{M}_{\phi_0}}^M(\phi_1))J_2$.

Now combining this equality with the proof of [CS15, Proposition 4.2] verifies our statement.

For the proof of the statements in 1.1(b) and 1.3(b) about the blockwise Alperin weight condition we point out the following property of $\Omega$ which is clear from its construction.

**Corollary 2.6.** Assume that in the situation of Theorem 2.1. If $B \subset \text{Irr}(B)$ is a union of $\bar{G}$-orbits and $B' \subset \text{Irr}(B')$ is a union of $\bar{M}$-orbits with $\tilde{\Omega}(\text{Irr}(G | B)) = \text{Irr}(B') \cap \text{Irr}(M | B')$. Then $S_B(B) = B'$.

**2.D. Application of inertial blocks.** For Puig’s notion of inertial blocks we refer to [Pui99, 1.5]. Here are two propositions showing an instance where the McKa y conjecture for a quotient group implies the Alperin-Mckay conjecture for a certain block. It will later be used in the proof of Theorem 1.3(a). The idea, inspired by [Spä09, §10], is to use Clifford-theoretic properties of the Brauer correspondence for inertial blocks in order to verify that a block satisfies the Alperin-Mckay Conjecture. Recall that an extension map $\Lambda$ with respect to $X < Y$ for $M \subseteq \text{Irr}(X)$ is a map such that for every $\chi \in M$, $\Lambda(\chi)$ is an extension of $\chi$ to $Y$. For $L < N$ and $L \subset \text{Irr}(L)$ recall $\text{Irr}(N | L) := \bigcup_{\chi \in L} \text{Irr}(N | \chi)$.

**Proposition 2.7.** Let $L < N$ and $C \in \text{Bl}(N)$ with a defect group $D$ satisfying $C_N(D) \leq L$. Let $C_0 \in \text{Bl}(L)$, such that some $C_0 \in \text{Bl}(NC_0 | C_0) \cap \text{Bl}(NC_0 | D)$ satisfies $C_0^N = C$. (Note that $C_0$ exists by [Nav98, 9.14].) Let $B := \left( \bigcup_{\kappa \in \text{Irr}(B)} \text{Irr}(\text{Res}_{\kappa}^N(\kappa)) \right) \cap \text{Irr}(C_0)$. Assume that every $\xi \in B$ extends to $N_\xi$ and $N_\xi/L$ satisfies the McKay Conjecture for $\ell$. Assume that $C_0$ is inertial. Then the Alperin-Mckay Conjecture holds for $C$.

**Proof.** Let $C'$ be the Brauer correspondent in $N_X(D)$ of $C$. According to [Nav98, 9.14] induction defines a bijection between $\text{Irr}_0(C_0)$ and $\text{Irr}_0(C)$. Let $\Lambda$ be an extension map with respect to $L < N$ for $\left( \bigcup_{\kappa \in \text{Irr}(B)} \text{Irr}(\text{Res}_{\kappa}^N(\kappa)) \right) \cap \text{Irr}(C_0)$. The characters in $\text{Irr}_0(C_0)$ are of the form $\text{Ind}_{N_\xi}^{N_\xi}(\Lambda(\xi)\eta)$ where $\xi \in B$ with $D \leq N_\xi$ and $\eta \in \text{Irr}(N_\xi/L)$ with $\ell \nmid \eta(1)$. Since $C_N(D) \leq L$, all characters of this form belong to $C_0$. Since $C_0$ is the only block of $NC_0$ that covers $C_0$ we have $\ell \nmid |NC_0 : LD|$ according to [Nav98, 9.17].

Then $\{ \kappa \in \text{Irr}(N_\xi | \xi) \mid \kappa(1)\ell = \xi(1)\ell = 1 \}$ corresponds to $\{ \eta \in \text{Irr}(N_\xi/L) \mid \eta(1)\ell = 1 \} = \text{Irr}_{\ell'}(N_\xi/L)$. Let $M := N_\xi C_0(\ell')$. Now the McKay conjecture for $N_\xi/L$ implies $|\text{Irr}_{\ell'}(N_\xi/L)| = |\text{Irr}_{\ell'}(M_\xi)|$. The later set corresponds to $\{ \kappa \in \text{Irr}(M \xi | \xi) \mid \kappa(1)\ell = \xi(1)\ell \}$. Let $\tilde{C}_0$ be the block of $M$ with $\tilde{C}_0^{N_\xi} = C_0$. The above implies $|\text{Irr}_0(C_0)| = |\text{Irr}_0(C)|$.

Recall that by assumption the block $C_1 \in \text{Bl}(LD | C_0)$ is inertial in the sense of [Pui11, 1.5]. Because of $\ell \nmid |NC_0 : LD|$ we see that $LD \leq M$ with $\ell'$-index. Then $\tilde{C}_0$ covers $C_1$ and is inertial according to [Zho16, Corollary]. Accordingly $\tilde{C}_0$ is basic Morita equivalent in the sense of [Pui99, §7] to the corresponding block $\tilde{C}_0$ of its stabilizer subgroup $N_M(D, c_0)$, where $c_0 \in \text{Bl}(C_M(D))$ that is covered by the Brauer correspondent $\tilde{C}_0 \in \text{Bl}(N_M(D))$ of $\tilde{C}_0$. Note
that the stabilizer subgroup from [Pui11, 1.5] is defined in terms of local pointed groups, while here we use the equivalent description of this group in terms of a maximal Brauer pairs, see [Thé95, 40.13(d)]. A basic Morita equivalence implies \(|\text{Irr}_0(\widetilde{C}_0)| = |\text{Irr}_0(\widetilde{c}_0)|\). By definition, \(\widetilde{c}_0\) is the Fong-Reynolds correspondent of \(C'\) with respect to \(c_0\), i.e., \(\widetilde{c}_0^{N_x(D)} = C'\) and \(|\text{Irr}_0(\widetilde{c}_0)| = |\text{Irr}_0(C')|\). This implies the statement.

For proving the blockwise Alperin weight conjecture we can prove the following analogue of the above.

**Proposition 2.8.** Assume that \(D\) is abelian in the situation of Proposition 2.7.

(a) Then \(C\) satisfies the blockwise Alperin weight conjecture.

(b) If \(\mathbb{B}' \subset \left( \bigcup_{\kappa \in \text{Irr}(B)} \text{Irr}(\text{Res}_L^N(\kappa)) \right) \cap \text{Irr}(C_0)\) forms an \(N_{C_0}\)-stable basic set of \(C_0\) in the sense of [CE04, 14.3] whose corresponding \(\ell\)-modular decomposition matrix is unitriangular, then \(|\text{Irr}(N | \mathbb{B}')| = |\text{IBr}(C)|\).

**Proof.** Since \(D\) is abelian, \(D \leq C_N(D) \leq L\), hence \(LD = L\) and \(C_0\) is inertial. Recall \(\ell \nmid \left| N_{C_0} : L \right|\) by the proof of Proposition 2.7, hence \(M := N_{NC_0}(LD) = N_{C_0}\). By the above proof the Fong-Reynolds correspondent \(\widetilde{C}_0 \in \text{Bl}(M)\) of \(C_0\) is inertial, as well.

We see that \(|\text{IBr}(C)| = |\text{IBr}(\widetilde{C}_0)|\). Let \(\widetilde{c}_0 \in \text{Bl}(N_M(D))\) be the Brauer correspondent of \(\widetilde{C}_0\). Now \(\widetilde{C}_0\) and \(\widetilde{c}_0\) are basic Morita equivalent and hence satisfy \(|\text{IBr}(\widetilde{C}_0)| = |\text{IBr}(\widetilde{c}_0)|\).

Since \(\mathbb{B}'\) is an \(N_{C_0}\)-stable set and the \(\ell\)-modular decomposition matrix of \(\text{Irr}(C_0)\) is unitriangular with respect to \(\mathbb{B}'\) there exists a \(N_{C_0}\)-equivariant bijection \(\Upsilon : \mathbb{B}' \to \text{IBr}(C_0)\) such that \(\Upsilon(\xi)\) has multiplicity one in \(\xi^o\), where \(\xi^o\) denotes the restriction of \(\xi\) to \(\ell\)-regular elements. Let \(\xi \in \mathbb{B}'\). Then \(N_\xi = N_{\Upsilon(\xi)}\) because of the equivariance of \(\Upsilon\). Some extension of \(\Upsilon(\xi)\) is a constituent of \(\Lambda(\xi)^o\), since \(\Lambda(\xi)^o\) has a constituent \(\phi\) in \(\text{IBr}(N_\xi \uparrow \Upsilon(\xi))\) and \(\text{Res}_{L_\xi}(\phi)\) is a summand of \(\text{Res}_{L_\xi}(\Lambda(\xi)^o) = \xi^o\). Let \(\Lambda'\) be the \(N_{C_0}\)-equivariant extension map with respect to \(L < N_{C_0}\) for \(\mathbb{B}'\) where for \(\xi \in \mathbb{B}'\), \(\Lambda'(\Upsilon(\xi))\) is defined as the unique extension of \(\Upsilon(\xi)\) that is a constituent of \(\Lambda(\xi)^o\). Recall \(\ell \nmid \left| N_{C_0} : L \right|\). Then via \(\Lambda'\) there is a correspondence between \(\text{Irr}(N | \mathbb{B}')\) and \(\text{IBr}(C)\). □

3. \(d\)-Split Levi Subgroups in Type \(\mathfrak{A}_{n-1}\)

The conditions presented in Theorem 2.4 are aimed specifically at groups of Lie type. We will apply it for simple groups \(S\) such that \(G = \text{SL}_n(q)\) or \(\text{SU}_n(q)\) for \(q\) a power of a prime \(\neq \ell\).

For that purpose it is important to study the Clifford theory arising from \(d\)-split Levi subgroups and their normalizers. The case of minimal \(d\)-split Levi subgroups is the subject of [CS17a, §5]. Here we treat the general case.

3.A. Notation for type \(\mathfrak{A}_{n-1}\). This section is used to fix some of the standard notation that will be used throughout.

Fix a prime \(p\) and \(q = p^m\) for some positive integer \(m\). Let \(n \geq 2\). We denote

\[ G := \text{SL}_n(\mathbb{F}_p) \leq \widetilde{G} := \text{GL}_n(\mathbb{F}_p). \]

Let \(\mathbb{T}\) the maximal torus of diagonal matrices in \(G\) and \(T := \mathbb{T} \cap G\). The corresponding root system \(\Phi\) identifies with the subset \(\{e_i - e_j \mid 1 \leq i, j \leq n, i \neq j\}\) of \(\mathbb{R}^n\), where \((e_i)_{1 \leq i \leq n}\) denotes the standard orthonormal basis [GLS98, 1.8.8]. For every root \(\alpha \in \Phi\), let \(x_\alpha(t)\) the
matrix such that \( x_{e_i-e_j}(t) - \text{Id}_n \) is the elementary matrix with entry \( t \) at position \((i, j)\). Set \( n_a := x_{a}(1)x_{-a}(-1)x_{a}(1) \). For \( 1 \leq i \leq n-1 \), denote \( n_{e_{i+1}-e_i} \in N_{G}(T) \) by \( n_i \).

Set \( \gamma_0 : G \to \tilde{G} \) to be the automorphism defined by \( g \mapsto (g^{tr})^{-1} \), where \( g^{tr} \) denotes the transpose matrix of \( g \). Let \( v_0 \in \tilde{G} \) be the matrix with the entry \((-1)^{k+1}\) at position \((k, n+1-k)\) and 0 elsewhere [GLS98, 2.7]. Then
\[
\gamma : \tilde{G} \to \tilde{G}, \quad g \mapsto v_0 \gamma_0(g) v_0^{-1},
\]
is a graph automorphism of \( \tilde{G} \) with \( \gamma(x_{a}(t)) = x_{\gamma(a)}((-1)^{|a|+1}t) \) where \( |e_i - e_j| := |i - j| \) and \( \gamma(e_i - e_j) = e_{n+1-i} - e_{n-i} \). Also denote by
\[
F_q : \tilde{G} \to \tilde{G}
\]
the raising of matrix entries to the \( q \)-th power for \( q = p^m \) \((m \geq 1)\), so that \( F_q = F_p^m \).

For \( \epsilon \in \{\pm1\} \) set \( F = \gamma^{\frac{1}{2}} \circ F_q, \tilde{G} := \tilde{G}^F = \text{GL}_n(eq) \) and \( G := G^F = \text{SL}_n(eq) \). As \( F_q \) commutes with \( v_0 \), it follows that \( G^F \) is \( \tilde{G} \)-conjugate to \( G^{\gamma_0} \circ F_q \), in particular they are isomorphic.

For each root \( \alpha \), it is clear that \( n_{\alpha} \) is fixed by \( F_p \). Moreover, by considering the action of \( \gamma_0 \) on \( n_{\alpha} \), it can be seen that \( \gamma_0 \) also fixes each such element.

Let \( V := \langle n_{\alpha} \mid \alpha \in \Gamma \rangle \), \( H := T \cap V \). Let \( e := 2|V| \) and define
\[
E := C_{em} \times C_2
\]
which acts on \( \tilde{G}^{F_p} \) such that the generating element of the first factor acts by \( F_p \) and the second factor acts by \( \gamma \). Denote by \( \hat{F} \) the element \( \gamma \hat{F}^{em} \) of \( E \) inducing the automorphism \( F \) on \( G^{F_p} \). Observe that \( F^e = F^{em} \) and thus \( G \leq G^{F_p} \), an \( E \)-stable subgroup. For any \( F \)-stable subgroup \( H \leq G \) the normalizer in \( G^F \times E \) is well-defined and will be denoted by \( (G^F \times E)_H \).

3.B. Construction of \( d \)-split Levi subgroups. The aim is to study how \( \text{Aut}(G^F)^L \) acts on \( \text{Irr}(N_G(L)^F) \) for \( d \)-split Levi subgroups \( L \) of \( G \). The groups are studied by considering the corresponding subgroups of \( G^{\varphi F} \) for some well chosen element \( \varphi \in G^{F_p} \).

Let \( V \) be the \( \mathbb{R} \)-span of \( \Gamma \) in \( \mathbb{R}^n \) and fix \( 1 \leq d \leq n \) and \( d_0 \geq 1 \) the integer such that \( \Phi_{d_0}(-x) = \pm \Phi_d(x) \), where \( \Phi_d(x) \) denotes the \( d^\text{th} \)-cyclotomic polynomial. Set \( a := \left\lfloor \frac{n}{d_0} \right\rfloor \) and
\[
v := n_1 \ldots n_{d_0-1} n_{d_0+1} \ldots n_{ad_0-1}
\]
that is the product \( n_1 \ldots n_{ad_0} \) where the \( n_{ad_0} \)'s for \( 1 \leq i \leq a \) are removed. Then \( w := \rho(v) \in N_G(T)/T \cong \mathfrak{S}_a \) is a product of \( a \) disjoint \( d_0 \)-cycles and \( v \) acts on \( \Gamma \) by \( v(e_i - e_j) = e_{w(i)} - e_{w(j)} \). Let \( \zeta \) be a primitive \( d_0 \)-root of unity in \( \mathbb{C} \) and set \( V(w, \zeta) \) to be the eigenspace of \( w \) on \( V \) with eigenvalue \( \zeta \). Let \( \Gamma' \) be a \( w \)-stable parabolic root subsystem of \( \Gamma \) such that
\[
\left(V(w, \zeta) \cap \Gamma' \right) \cap \Gamma = \Gamma'.
\]

Then the corresponding \( d \)-split Levi subgroup of \( G \) is given by
\[
\tilde{L}_{\Gamma'} := \langle \tilde{T}, X_n \mid \alpha \in \Gamma' \rangle = C_{\tilde{G}}(\tilde{S}),
\]
where \( \tilde{S} \) is a \( vF \)-stable \( \Phi_d \)-torus of \( \tilde{G} \). In addition, taking \( S \) and \( L_{\Gamma'} \) to be the kernel of the determinant map restricted to \( \tilde{S} \) respectively \( \tilde{L}_{\Gamma'} \), yields the corresponding \( d \)-split Levi subgroup of \( G \).
3.C. Identification by partitions. A parabolic root subsystem $\Gamma' \subseteq \Gamma$ yields an equivalence relation on $\underline{n} := \{1, \ldots, n\}$ by saying $i \sim j$ if and only if $e_i - e_j \in \Gamma' \cup \{0\}$ and the equivalence classes provide a partition of $\underline{n}$. Conversely, a partition $\lambda = (\lambda_1, \ldots, \lambda_0) \vdash \underline{n}$ yields a parabolic root subsystem

$$\Gamma_{\lambda} := \{e_i - e_j \mid \{i, j\} \subseteq \lambda_k \text{ for some } k\}.$$ 

Lemma 3.1. There is a bijection $\gamma$ between the set of $\sigma$-stable parabolic root subsystems of $\Gamma$ and the set of $w$-stable partitions of $\underline{n}$.

Proposition 3.2. The following are equivalent for a $w$-stable partition $\lambda$:

(i) $\Gamma_{\lambda}$ satisfies Equation (3.2).

(ii) $w$ acts on $\lambda$ with at most one fixed point and all other orbits have length $d_0$.

Proof. Assume $\Gamma'$ satisfies Equation (3.2) and set $\lambda = \gamma(\Gamma')$. Set $X = \mathcal{V}(w, \zeta) \cap (\Gamma')^1$ so that $\Gamma' = X^1 \cap \Gamma'$ and $e_i - e_j \in \Gamma'$ if and only if $v_i = v_j$ for all $v \in X$. Let $\lambda_0 := \{i \mid e_i \in X^1\}$. If $i \in \lambda_0$, and $e_i - e_j \in \Gamma'$, then $j \in \lambda_0$ and thus $\lambda_0 \subseteq \lambda$ and is fixed by $w$. Take $\mu \in \lambda \setminus \{\lambda_0\}$ and $i \in \lambda_k$. If $j = w^k(i) \in \mu$, then $v_i = v_j = \zeta^{-k}v_i$ for all $v \in X$ and so $i \in \lambda_0$, which is a contradiction. This shows that all $w$-orbits on $\lambda \setminus \{\lambda_0\}$ have length $d_0$.

In the converse direction, assume that $w$ acts on $\lambda$ with at most one fixed point and all other orbits have length $d_0$. Let $e_i - e_j \in (\mathcal{V}(w, \zeta) \cap \Gamma^1) \setminus \{\Gamma_{\lambda}\}$. Then there must exist a root of the form $e_i - e_k \cdot w \cdot \epsilon_j \in \Gamma_{\lambda}$, with $k \neq 0$. Thus $v_i = v_j = v_{w^k \cdot \epsilon_j}$, which implies $\{i, j\}$ is contained within the fixed set of $\lambda$ giving a contradiction. Thus $\Gamma_{\lambda}$ satisfies Equation (3.2). \[\square\]

3.D. Some notation associated to partitions. Assume $\lambda \vdash \underline{n}$ satisfies condition (ii) of Proposition 3.2. If $\lambda$ has a set fixed by $w$, then we denote this set by $\lambda_0$ and $\lambda' := \lambda \setminus \{\lambda_0\}$, otherwise $\lambda'$ coincides with $\lambda$. Let $\overline{\mu}$ denote the $w$-orbit of $\mu \in \lambda'$ and $\overline{X}$ the set of all $w$-orbits. As all $w$-orbits on $\lambda'$ have length $d_0$, for each $f \in \{1, \ldots, n\}$ define $\lambda_f := \{\mu \in \lambda' \mid |\mu| = f\}$ and

$$t_f := \frac{\lambda_f}{d_0},$$

so that $t := \sum_{f=1}^n t_f$ equals the number of $w$-orbits on $\lambda'$. In addition, for each $w$-orbit $\overline{\mu}$ of $\lambda'$ define

$$I_{\overline{\mu}} := \{i \mid \text{there exists } \mu' \in \overline{\mu} \text{ such that } i \in \mu'\}.$$ 

Take $\mu, \mu' \in \lambda'$ such that $|\mu| = |\mu'| = f$ but $\overline{\mu} \neq \overline{\mu'}$. Set $I_{\overline{\mu}} = \{I_{\overline{\mu}}(1), \ldots, I_{\overline{\mu}}(d_0 f)\}$ with $I_{\overline{\mu}}(i) < I_{\overline{\mu}}(i')$ if and only if $i < i'$. This labelling yields an order preserving bijection

$$\tau_{\overline{\mu}, \overline{\nu}} : I_{\overline{\mu}} \rightarrow I_{\overline{\nu}}, \quad I_{\overline{\mu}}(i) \mapsto I_{\overline{\nu}}(i).$$

Furthermore, up to conjugation of $\lambda$ by an element of the centraliser in the symmetric group $\mathfrak{S}_n$ of $w$, it can be assumed that $\tau_{\overline{\mu}, \overline{\nu}}(w^k(\mu)) \in \overline{\nu}$.

3.E. The structure of $d$-split Levi subgroups. Let $\lambda \vdash \underline{n}$ be a partition satisfying condition (ii) of Proposition 3.2. We will make use of the notation from Section 3.D. For each $\mu \in \lambda$ define

$$\overline{T}_\mu := \{\text{diag}(z_1, \ldots, z_n) \in \mathbb{T} \mid z_j = 1 \text{ if } j \not\in \mu\}$$

and

$$\overline{L}_\mu := \langle \overline{T}_\mu, X_\alpha \mid \alpha \in \Gamma_\mu \rangle.$$
Then
\[ \tilde{L}_\lambda := \langle \tilde{T}, X_\alpha \mid \alpha \in \Gamma_\lambda \rangle = \tilde{L}_{\lambda_0} \times \prod_{\mu \in \lambda'} \tilde{L}_\mu \cong \GL_{|\lambda_0|}(\overline{\mathbb{F}}_p) \times \prod_{j=1}^n \GL_f(\overline{\mathbb{F}}_p)^{d_0 t_f}. \]

Furthermore, by construction \( F_\mu(\tilde{L}_\mu) = \tilde{L}_\mu \) and \( v(\tilde{L}_\mu) = \tilde{L}_{w(\mu)} \). Therefore
\[ \tilde{L}_\lambda := \tilde{L}_\lambda^{vF} \cong \GL_{|\lambda_0|}(\epsilon q) \times \prod_{f=1}^n \GL_f(\epsilon q^{d_0} t_f). \]

In addition \( L_\lambda \) and \( L_\lambda \) are the kernel of the determinant maps on \( \tilde{L}_\lambda \) and \( \tilde{L}_\lambda \) respectively.

3.F. **The structure of** \( N_\lambda/L_\lambda \). Given \( G \) and an \( F \)-stable Levi subgroup \( L \), let \( W_G \) and \( W_L \) denote the Weyl groups of \( G \) and \( L \) respectively. In addition, set \( N := N_G(L) \) so that \( N = N^vF \). The relative Weyl group of \( L \) in \( G \) (see [BM98, Section 4]) is defined to be
\[ W_G(L) := N/L \cong N_{W_G(W_L)}/W_L. \]

If \( L \) is \( vF \)-stable, it follows that \( N \) is also \( vF \)-stable, and therefore \( vF \) induces an automorphism \( \sigma \) on \( W_G(L) \). Furthermore, by the Lang-Steinberg Theorem
\[ N/L \cong C_{W_G(L)}(\sigma). \]

For a partition \( \lambda \vdash n \) set \( W_\lambda = \bigcap_{\mu \in \lambda} W_\mu \) the intersection of the setwise stabilisers of \( \mu \in \lambda \) in \( W \cong \mathfrak{S}_n \). If \( \lambda \) is a partition satisfying condition (ii) of Proposition 3.2 and \( L := L_\lambda \), then
\[ N/L \cong \{ x \in \mathfrak{S}_n \mid \lambda^x = \lambda \text{ and } w^x w^{-1} \in W_\lambda \}/W_\lambda. \]

If \( x \in \mathfrak{S}_n \) fixes \( \lambda \) and \( w^x w^{-1} \in W_\lambda \), then \( w^x w^{-1} \) fixes \( \lambda_0 \) and so \( (\lambda_0)^x \) is fixed by \( w \). Therefore \( (\lambda_0)^x = \lambda_0 \). Furthermore, as \( |\mu^x| = |\mu| \) for all \( \mu \in \lambda' \), it follows that
\[ C_{W_G(L_\lambda)}(w) = \prod_{e=1}^n C_{\Sym(\lambda_e)}(w), \]
where we recall that \( \lambda_f := \{ \mu \in \lambda \mid |\mu| = f \} \).

**Proposition 3.3.** Let \( \lambda \) be a \( v \)-stable partition of \( n \) satisfying condition (ii) of Proposition 3.2. Then for \( L = L_\lambda^{vF} \) and \( N = N_G(L_\lambda)^vF \),
\[ N/L \cong \prod_{e=1}^n C_{d_0} \times \mathfrak{S}_{t_f}. \]

3.G. **The structure of** \( N_\lambda \). Let \( \lambda \vdash n \) satisfying Proposition 3.2 and recall that \( \lambda_f = \{ \mu \in \lambda' \mid |\mu| = f \} \) and set \( \overline{\lambda}_f \) to be the set of \( w \)-orbits on \( \lambda_f \), so \( t_f = |\overline{\lambda}_f| \). Then for \( \Sym(\overline{\lambda})_{\{\overline{\lambda_1}, \ldots, \overline{\lambda_n}\}} \), the intersection of the setwise stabilisers \( \Sym(\overline{\lambda})_{\overline{\lambda}_f} \), Proposition 3.3 shows that
\[ N/L \cong (C_{d_0})^t \times \Sym(\overline{\lambda})_{\{\overline{\lambda_1}, \ldots, \overline{\lambda_n}\}}. \]

We construct subgroups \( V_0, S \leq V \cap N \) such that \( \rho(V_0) = C_{d_0}^t \) and \( \rho(S) = \Sym(\overline{\lambda})_{\{\overline{\lambda_1}, \ldots, \overline{\lambda_n}\}}. \)
3.G.1. The construction of $V_0$. Let $\underline{\mu} \in \bar{\lambda}$ and recall the elements $n_i$ as defined in Section 3.B. Then define

$$v_\underline{\mu} := \prod_{i \in \underline{\mu}} n_i,$$

where the product is taken with respect to the natural ordering ”$\leq$” on $\mathbb{N}$. Then by construction $[v_\underline{\mu}, v] = 1$ and the two elements $v_\underline{\mu}$ and $v$ induce the same action on $\bar{L}_\underline{\mu}$ while $[v_\underline{\mu}, \bar{L}_\underline{\mu}] = 1$ whenever $\underline{\mu} \neq \underline{\mu}$. Set

$$V_f := \langle v_\underline{\mu} \mid |\mu| = f \rangle$$

and $V_0 := \langle v_\underline{\mu} \mid \underline{\mu} \in \bar{\lambda} \rangle$.

Then $\rho(V_f) = C_0^{d_f}$ and $\rho(V_0) = C_0^{d_0}$.

3.G.2. The construction of $S$. Let $\underline{\mu} \neq \underline{\mu}' \in \bar{\lambda}_f$ and $\tau_{\underline{\mu}, \underline{\mu}'}$ the bijection between $I_{\underline{\mu}}$ and $I_{\underline{\mu}'}$ from Equation (3.3). Define an element

$$s_{\underline{\mu}, \underline{\mu}'} := \prod_{i \in I_{\underline{\mu}}} n_i^{-e_{\tau_{\underline{\mu}, \underline{\mu}'(i)}}(1)}.$$

Then $s_{\underline{\mu}, \underline{\mu}'}$ acts on $\tilde{L}$ by permuting $\tilde{L}_{\underline{\mu}}$ and $\tilde{L}_{\underline{\mu}'}$, while $[s_{\underline{\mu}, \underline{\mu}'}, \tilde{L}_{\underline{\mu}'}] = 1$ for every $\mu'' \neq \underline{\mu}$ or $\underline{\mu}'$. Let

$$S_f := \langle s_{\underline{\mu}, \underline{\mu}'} \mid \underline{\mu}, \underline{\mu}' \in \bar{\lambda}_f \rangle$$

and $S = \langle S_f \mid 1 \leq f \leq n \rangle$.

Then by construction $\rho(S_f) = \text{Sym}(\bar{\lambda}_f)$, and $\rho(S) = \text{Sym}(\bar{\lambda})(\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$. Furthermore, $v_{s_{\underline{\mu}, \underline{\mu}'}} = v_{\underline{\mu}}$ so $S$ is in the normalizer of $V_0$ and $[s_{\underline{\mu}, \underline{\mu}'}, v] = 1$.

**Proposition 3.4.** Let $\lambda$ be a $w$-stable partition satisfying condition (ii) of Proposition 3.2. Let $L = L_\lambda^F$ and $N = N_{G,F}(L_\lambda)$. Then there exist subgroups $V_0, S \leq V \cap N$ such that

$$N = LV_0S$$

where $\rho(V_0) = C_0^{d_0}$ and $\rho(S) = \rho(S) = \text{Sym}(\bar{\lambda})(\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$.

3.G.3. Constructing an extension map with respect to $L \cap S \triangleleft S$. Let $f \in \{1, \ldots, n\}$ and write $\bar{\lambda}_f = \{\underline{\mu}_1, \ldots, \underline{\mu}_t\}$. Set $V^{(f)} := \{n_{i_1}^{-e_{j_1}}(1) \mid 1 \leq i, j \leq t_f\} \leq \text{SL}_{t_f}(p)$. There is a natural isomorphism from $S_f$ to $V^{(f)}$ given by $s_{\underline{\mu}_i, \underline{\mu}_j} \mapsto n_{i_1}^{-e_{j_1}}(1)$. Moreover, for $T_f$ the diagonal maximal torus of $\text{SL}_{t_f}(p)$, it follows that the image of $L \cap S_f$ is $T_f \cap V^{(f)}$.

**Corollary 3.5.** There exists an extension map with respect to $L \cap S \triangleleft S$.

**Proof.** This follows from observing that

$$L \cap S = \langle s_{\underline{\mu}, \underline{\mu}'} \mid \underline{\mu}, \underline{\mu}' \in \bar{\lambda}_f \text{ for some } f \rangle = \prod_{f=1}^n L \cap S_f$$

and that there is an extension map with respect to $T_f \cap V^{(f)} \triangleleft V^{(f)}$ by [CS17a, Proposition 5.5].
3.3. Characters of $d$-split Levi subgroups. Define $L_0 := [L, L] = [L, L], L_f := [L_f, L_f]$ and $\bar{L}_\pi := [\bar{L}_\pi, \bar{L}_\pi]$ so that
\[
L_0 = L_{\lambda_0} \times \prod_{f=1}^{n} L_f = L_{\lambda_0} \times \prod_{f=1}^{n} \left( \prod_{\pi \in \pi_f} L_\pi \right) \cong \text{SL}_{|\lambda_0|}(eq) \times \prod_{f=1}^{n} \text{SL}_f(eq)^{d_f}.
\]

Lemma 3.6. Let $\chi \in \text{Irr}(L)$ and $\chi_0 \in \text{Irr}(\text{Res}_{L_0}^L(\chi))$, then the Clifford correspondent of $\chi$ over $\chi_0$ restricts irreducibly to $\chi_0$.

Proof. Write
\[
\chi_0 = \chi_{\lambda_0} \times \prod_{\pi \in \pi_0} \chi_\pi
\]
with $\chi_\pi \in \text{Irr}(\bar{L}_\pi)$ and $\chi_{\lambda_0} \in \text{Irr}(L_{\lambda_0})$. Then
\[
\bar{L}_{\chi_0} = (\bar{L}_{\lambda_0})_{\chi_{\lambda_0}} \times \prod_{\pi \in \pi_0} (\bar{L}_\pi)_{\chi_\pi}
\]
Each factor $\bar{L}_\pi/L_\pi$ and $\bar{L}_{\lambda_0}/L_{\lambda_0}$ is cyclic, therefore $\chi_0$ extends from $L_0$ to $\bar{L}_{\chi_0}$ and hence to $L_{\chi_0}$. Moreover as $L_{\chi_0}/L_0$ is abelian, it follows from Gallagher’s Theorem that $\text{Res}_{L_0}^L(\chi) = \chi_0$.

Lemma 3.7. Let $\chi \in \text{Irr}(L)$ and $\chi_0 \in \text{Irr}(\text{Res}_{L_0}^L(\chi))$. Then $\bar{L}_\chi = L\bar{L}_{\chi_0}$.

Proof. As $\bar{L}_\chi \leq L\bar{L}_{\chi_0}$, it suffices to show that $\chi$ extends to $L\bar{L}_{\chi_0}$. Let $\psi \in \text{Irr}(L\bar{L}_{\chi_0} | \chi)$. Then $\chi_0 \in \text{Irr}(\text{Res}_{L_0}^{L\bar{L}_{\chi_0}}(\psi))$ and hence there exists a unique $\tilde{\psi} \in \text{Irr}(\bar{L}_{\chi_0} | \chi_0)$ such that $\text{Ind}_{L_{\chi_0}}^{\bar{L}_{\chi_0}}(\tilde{\psi}) = \psi$. However, $\tilde{\psi}$ restricts to an irreducible character of $L_{\chi_0}$ as $\tilde{\psi}$ restricts to $\chi_0$ by Lemma 3.6. Therefore $\text{Ind}_{L_{\chi_0}}^{\bar{L}_{\chi_0}}(\text{Res}_{L_{\chi_0}}^{\bar{L}_{\chi_0}}(\tilde{\psi}))$ is irreducible and thus $\text{Res}_{L}^{L_{\chi_0}}(\psi) = \chi$.

Let us recall the fundamental property of stabilizers of characters in type A.

Theorem 3.8 (Cabanes-Späth,[CS17a, Thm 4.1]). Let $E$ be defined as in Equation (3.1). If $\psi \in \text{Irr}(G^F)$, then there exists a $\psi_0 \in \text{Irr}(G^F | \psi)$ such that
\[
(G^F \rtimes E)\psi_0 = (G^F)\psi_0 \rtimes E \psi_0
\]
and $\psi_0$ extends to a character of $G^F \rtimes E$.

We apply this result to the Levi subgroups $L_\pi$. That is, set $T_\pi \subseteq \text{Irr}(L_\pi)$ to be an $(E(\nu_\pi))$-transversal on $\text{Irr}(L_\pi)$ satisfying Theorem 3.8 with respect to $(L_\pi)$ and $(\bar{L}_\pi)$. Furthermore, each $T_\pi$ is chosen so that if $\theta \in T_\pi$ and $\bar{\theta} \bar{\pi} \pi \in S_f$ then $\theta \bar{\pi} \pi \in T_f$. Then we can define $T \subseteq \text{Irr}(L_0)$ by
\[
T := \left\{ \chi_{\lambda_0} \times \prod_{\pi \in \pi_0} \chi_\pi \mid \chi_{\lambda_0} \in T_{\lambda_0} \text{ and } \chi_\pi \in T_\pi \right\}.
\]

Lemma 3.9. Let $\chi_0 \in T \subseteq \text{Irr}(L_0)$ from Equation (3.4) and $V_0$, $S$ as in Proposition 3.4. Then
\[
(LV_0\bar{E}S)\chi_0 = \bar{L}_{\chi_0}(V_0 E)\chi_0 S_{\chi_0}.
\]
In particular, for \( \hat{N} := NE \) it follows that
\[
\hat{N}_x = L_x V_{0,x} S_{x_0} \quad \text{and} \quad \hat{N}_x = L_x (V_0 E) x_0 S_{x_0}.
\]

**Proof.** Take \( x \in \hat{L} V_0 E \) and \( s \in S \). If \( x s \in N_x \) and \( \overline{m} \in \overline{\Lambda} \), then \( \chi^s \) and \( \chi^{\rho(s)}(\overline{m}) \) give the same character on \( L^{\rho(s)}(\overline{m}) \). Thus by the choice of characters in \( T \), it follows that \( \chi^s = \chi^{\rho(s)}(\overline{m}) \). Therefore \( s \in S_{x_0} \) and
\[
(\hat{L} V_0 E s) x_0 = (\hat{L} V_0 E) x_0 S_{x_0}.
\]
Furthermore, the choice of \( T \) implies that \( (\hat{L} V_0 E) x_0 = \hat{L} x_0 (V_0 E) x_0 \).

**Corollary 3.10.** Let \( \chi \in \operatorname{Irr}(L) \). Then \( \hat{N} \chi = \hat{L} \chi N_x \). Moreover, there exists \( T \) an \( \hat{L} \)-transversal on \( \operatorname{Irr}(L) \) such that
\[
(\hat{N} \hat{N}) \chi = \hat{N} \chi \hat{N} \chi
\]
for each \( \chi \in T \).

**Proof.** It suffices to prove these properties hold for an \( L \hat{N} \)-transversal of \( \operatorname{Irr}(L) \). In particular, it can be assumed that there exists a \( \chi_0 \in \operatorname{Irr}(\operatorname{Res}_{\chi_0}^L(\chi)) \cap T \). Let \( \tilde{\chi}_0 \in \operatorname{Irr}(\hat{L} x_0 | \chi_0) \) be the Clifford correspondent inducing to \( \chi \). Hence by Lemma 3.9 and Lemma 3.7
\[
\hat{N} \chi = L(\hat{N}) \chi_0 = L(\hat{L} x_0 V_{0,x} S_{x_0}) \chi_0 = \hat{L}_\chi (V_{0,x} S_{x_0}) \chi_0 = \hat{L} \chi N_x.
\]
This proves the first statement. Similarly by Lemma 3.9
\[
(\tilde{N} \chi_0) \chi_0 (\hat{N}) \chi_0 = \left( L (V_{0,x} S_{x_0}) \chi_0 \right) \left( L (V_{0,x} S_{x_0}) \chi_0 \right) = \hat{L}_\chi (V_{0,x} S_{x_0}) \chi_0 = (\hat{N} \hat{N}) \chi_0.
\]
Thus the second statement follows.

4. The Clifford Theory for d-split Levi subgroups

In [CS17b] the authors provide a criterion to check property 2.2(iv) of [Spä12] which arises in the inductive McKay condition. In this section we consider the generalisation of this criterion to the case of a d-split Levi subgroup, which will be used to prove Theorem 1.2.

4.A. Clifford-theoretic tools. There are two technical statements that will be applied in our context but have appeared in the context of the McKay Conjecture.

**Theorem 4.1.** Let \( d \) be a positive integer and \( \hat{L} \) a vF-stable d-split Levi subgroup of \( \hat{G} \). Assume the groups \( N := N_G(L)^{vF}, \hat{N} := N_G(L)^{vF} \) and \( \hat{N} := (C_{G^0}^{vF}(v \hat{F}))_L \) satisfy the following conditions:

(i) There exists some set \( T \subseteq \operatorname{Irr}(L) \), such that
\[
\hat{N} \chi = L \chi \hat{N} \chi \quad \text{for every} \quad \chi \in T,
\]
\[
(\hat{N} \hat{N}) \chi = \hat{N} \chi \hat{N} \chi \quad \text{for every} \quad \chi \in T,
\]
\[
\hat{N} \chi \text{ contains some} \hat{N} \text{-stable} L \text{-transversal of} \operatorname{Irr}(L).
\]

(ii) There exists an extension map \( \Lambda \) with respect to \( L \triangleleft N \) such that
\[
\Lambda \text{ is} \hat{N} \text{-equivariant.}
\]
\[
\text{Every character} \quad \chi \in T \quad \text{has an extension} \quad \tilde{\chi} \in \operatorname{Irr}(\hat{N}_\chi) \quad \text{with} \operatorname{Res}_{\hat{N}_\chi}^\hat{N}_\chi(\tilde{\chi}) = \Lambda(\chi) \text{ and} \quad v \hat{F} \in \ker(\tilde{\chi}).
\]
(iii) Let $W_d := N/L$ and $\tilde{W}_d := \tilde{N}/L$. For $\xi \in \Irr(L)$ and $\tilde{\xi} \in \Irr(\tilde{L}_\xi | \xi)$ let $W_\tilde{\xi} := N_\tilde{\xi}/L$, $W_\xi := N_\xi/L$, $K := N_{\tilde{W}_d}(W_\xi,W_{\tilde{\xi}})$ and $\tilde{K} := N_{\tilde{W}_d}(W_\xi,W_{\tilde{\xi}})$. Then there exists for every $\eta_0 \in \Irr(W_\tilde{\xi})$ some $\eta \in \Irr(W_\xi | \eta_0)$ such that
(1) $\eta$ is $\tilde{K}_{\eta_0}$-invariant.
(2) If $D$ is non-cyclic, $\eta$ extends to some $\tilde{\eta} \in \Irr(\tilde{K}_\eta)$ with $v\tilde{F} \in \ker(\tilde{\eta})$.

Then:

(a) For every $\chi \in \Irr(\tilde{N})$ there exists some $\chi_0 \in \Irr(N | \chi)$ such that
(1) $(\tilde{N}\tilde{N})_{\chi_0} = \tilde{N}_{\chi_0}\tilde{N}_{\chi_0}$, and
(2) $\chi_0$ has an extension $\tilde{\chi}_0$ to $\tilde{N}_{\chi_0}$ with $v\tilde{F} \in \ker(\tilde{\chi})$.

(b) Moreover, there exists some $\tilde{N}$-equivariant extension map with respect to $\tilde{L} \triangleleft \tilde{N}$ that is compatible with $\Irr(\tilde{N}/N)$.

Proof. The proof of part (a) is the same as the proof of [CS17b, Theorem 4.3]. Part (b) uses the assumption that $\mathcal{T}$ is $\tilde{N}$-stable. The proof follows from the considerations as in the proof of Theorem 4.2 of [CS18].

The following result helps to construct extensions in our context. Recall that for a finite group $X$ and $\xi \in \Irr(X)$ there is an associated subgroup $Z(\xi) := \{z \in X \mid |\xi(z)| = |\xi(1)|\}$, see [Isa94, 2.26].

Proposition 4.2 ([Spä19, Prop. 2.6]). Let $K \triangleleft M$, $K_0 \triangleleft M$ with $K_0 \leq K$ and $M$ a finite group. Let $\xi \in \Irr(K)$ a character with $\xi_0 := \Res^K_{K_0}(\xi) \in \Irr(K_0)$. Assume

(i) $K = Z(\xi)K_0$;
(ii) assume there exists some group $V \leq M$ such that
(1) $M = KV$ and $H := V \cap K \leq C_K(K_0)$; and
(2) there is a $\zeta \in \Irr(\Res^K_{K_0}(\xi))$ which extends to some $\tilde{\zeta} \in \Irr(V_\zeta)$;
(iii) $\xi_0$ extends to $K_0 \rtimes \epsilon(V_{\xi_0})$, where $\epsilon : V \rightarrow V/H$ is the canonical epimorphism.

Then there exists an extension of $\xi$ to $M_\xi$ that is afforded by a representation $\tilde{D}$ satisfying

\begin{equation}
\tilde{D}(kv) = \tilde{\zeta}(v)D'(\rho(v))D(k) \text{ for every } k \in K \text{ and } v \in V_{\lambda},
\end{equation}

where $D$ is a representation of $K$ affording $\xi$, and $D'$ is a representation of $K_0 \rtimes \epsilon(V_{\xi_0})$ extending $D_{K_0}$.

4.B. An extension map with respect to $L \triangleleft N$. Let $\lambda \vdash n$ satisfying condition (ii) of Proposition 3.2. Set $\tilde{L} := \tilde{L}_\lambda^{rf}$, $L := \tilde{L} \cap G^{rf}$ and $N = N_{G^{rf}}(L)$. All the notations from Section 3 will be used without further reference. The aim in this section is to prove Theorem 1.3 via the following stronger statement.

Theorem 4.3. Let $\tilde{N} := NE$. Then there exists an extension map $\Lambda$ with respect to $L \triangleleft N$ such that

(i) $\Lambda$ is $\tilde{N}$-equivariant,
(ii) there exists a set $\mathcal{T} \subseteq \Irr(L)$ which contains an $\tilde{L}$-transversal of $\Irr(L)$ such that if $D$ is non-cyclic, then every character $\chi \in \mathcal{T}$ has an extension $\tilde{\chi} \in \Irr(\tilde{N}_\chi)$ with $\Res^{\tilde{N}}_{N_\chi}(\tilde{\chi}) = \Lambda(\chi)$ and $v\tilde{F} \in \ker(\tilde{\chi})$. 

Clearly $N$ is normalised by $\tilde{L}$ and therefore it suffices to produce an extension for characters in a $\tilde{L}$-transversal of $\text{Irr}(L)$. Furthermore, to prove Theorem 4.3 it suffices to define $\Lambda$ on a $\tilde{N}$-transversal of $\text{Irr}(L)$ and the remaining values of $\Lambda$ can be constructed using $\tilde{N}$-conjugation. In particular for the remainder of this section we assume that $\chi \in \text{Irr}(L)$ and $\chi_0 \in \text{Irr}(\text{Res}_{\lambda_0}^L(\chi)) \cap \mathcal{T}$, where $\mathcal{T}$ is taken from Equation (3.4). Let $\tilde{\chi}_0 \in \text{Irr}(L \chi_0 \mid \chi_0)$ with $\text{Ind}_{L \chi_0}^L(\tilde{\chi}_0) = \chi$. Then Lemma 3.6 shows that $\tilde{\chi}_0$ restricts to $\chi_0$.

Consider the structure of $\tilde{N}_\chi$. As $L \chi_0 \leq L$ and $\tilde{N}_\chi$ permutes the $L$-conjugates of $\chi_0$, it follows that $L \chi_0 \leq \tilde{N}_\chi$. Hence $\tilde{N}_\chi \leq L(\tilde{N}_\chi \chi_0)$. However $\tilde{\chi}_0$ induces to $\chi$ and $L \chi_0 \leq \tilde{N}_\chi \chi_0$, therefore $(\tilde{N}_\chi \chi_0) \tilde{\chi}_0 \leq N_\chi$. Thus

$$\tilde{N}_\chi = L(\tilde{N}_\chi \chi_0).$$

Furthermore for $\tilde{V}_0 := V_0 E$, Lemma 3.9 shows that

$$(\tilde{N}_\chi \chi_0) \chi_0 = L \chi_0 \left( (\tilde{V}_0 \chi_0 S \chi_0) \right) \chi_0.$$

4.B.1. A useful subgroup of $L \chi_0$ containing $L_0$. As $L_0 \cap S$ can be a proper subgroup of $L \cap S$, we need to consider a subgroup $L_0 \leq K \leq L \chi_0$ such that $K \cap S = L \cap S$. Define $-\text{Id}_{\tilde{L}_\mathcal{P}}$ to be the unique central element of order two in $\tilde{L}_\mathcal{P}$ and set

$$Z_2 := \langle -\text{Id}_{\tilde{L}_\mathcal{P}} \mid \mathcal{P} \in \mathcal{L} \rangle.$$

In addition define $K_\mathcal{P} := \langle L_{\mathcal{P}}, L_{\mathcal{P}} \cap Z_2 \rangle$ and

$$\tilde{K} := \langle L_0, Z_2 \rangle = K \chi_0 \times \prod_{\mathcal{P} \in \mathcal{L}} K_{\mathcal{P}}.$$

Then for $K := L \cap \tilde{K} = \ker(\text{det}_{\tilde{K}})$, it follows that $K \cap S = L \cap S$. Set $\psi := \text{Res}_{L \chi_0}^L(\tilde{\chi}_0) \in \text{Irr}(K)$ and let $\tilde{\psi} \in \text{Irr}(\tilde{K})$ be an extension of $\psi$. The restriction of $\tilde{\psi}$ to $Z_2$ determines the extension of $\chi_0$ to $\tilde{K}$. However $[Z_2, \tilde{V}_0] = 1$ and therefore $\tilde{V}_0$ fixes the restriction of $\tilde{\psi}$ to $Z_2$. Thus $(\tilde{V}_0)_{\tilde{\psi}} = (\tilde{V}_0)_{\chi_0}$ and hence

$$(\tilde{V}_0)_{\psi} = (\tilde{V}_0)_{\chi_0}.$$

Lemma 4.4. Let $M := K\tilde{V}_0 S$. If $\psi$ extends to $M_\psi$, then $\chi$ extends to $\tilde{N}_\chi$.

Proof. By construction $\tilde{N}_\chi = L(\tilde{L}_\chi \chi_0) \chi_0$. Moreover $L \chi_0 \cap (M \chi_0) \chi_0 = K$ and $(M \chi_0) \chi_0 \leq M_\psi$. Thus an extension of $\psi$ to $M_\psi$ gives an extension $\phi$ of $\tilde{\chi}_0$ to $(\tilde{L}_\chi \chi_0) \chi_0$ by [Spä10b, Lemma 4.1]. Moreover, by the Mackey formula,

$$\text{Res}_{\tilde{L}}^{\tilde{N}_\chi} \left( \text{Ind}_{L \chi_0 M \chi_0}^{\tilde{N}_\chi}(\phi) \right) = \text{Ind}_{L \chi_0}^{\tilde{L}_\chi}(\tilde{\chi}) = \chi.$$

4.B.2. Extending $\psi$ to $M_\psi$. First we observe that $M_\psi \leq M \chi_0$ and so

$$M_\psi = K(\tilde{V}_0)_{\psi} S_\psi.$$

For each $\mathcal{P} \in \mathcal{L}$, define a group

$$E_\mathcal{P} := \langle \tilde{E}_\mathcal{P}, \gamma_0 \mathcal{P} \rangle$$
where \( \hat{F} \) and \( \gamma_0, \pi \) act as \( \hat{F} \) and \( \gamma_0 \) on \( \tilde{L} \), while \( [E, \tilde{L}] = 1 \) whenever \( \pi \neq \tilde{\pi} \). Then the group

\[
E_0 := E_{\lambda_0} \times \prod_{\pi \in \chi} E_{\pi}
\]

contains \( E \) as a diagonally embedded subgroup. Set \( \hat{\nu}_F = V_0 \hat{F} \). Then by Theorem 3.8 and the choice of \( \chi_0 \in T \), each factor \( \chi_\pi \) of \( \chi_0 \) extends to \( L_\pi(V_0 \hat{F})_{\chi_\pi}/(\nu_\pi \hat{F}) \). Hence \( \chi_\pi \) extends to some \( \phi_\pi \in \text{Irr} \left( L_\pi(V_\pi F)_{\chi_\pi} \right) \) such that \( \nu_\pi \hat{F} \) is contained in \( \ker(\phi_\pi) \). Furthermore, these extensions can be taken so that, \( \phi_{\tilde{\pi} \pi} := \phi_\pi \) whenever \( \chi_{\tilde{\pi} \pi} = \chi_\pi \). Therefore

\[
\phi_0 := \phi_{\lambda_0} \times \prod_{\pi \in \chi} \phi_\pi
\]

provides an extension of \( \chi_0 \) to \( L_0(V_0 E_0)_{\chi_0} \) with \( v \hat{F} \) in \( \ker(\phi_0) \) and \( S_{\chi_0} = S_{\phi_0} \).

As \( \hat{K} \) is the central product of \( L_0 \) and \( Z_2 \), there is a character \( \tau \in \text{Irr}(Z_2) \) such that \( \tilde{\tau}(lz) = \chi_0(l) \tau(z) \). Furthermore, \( \hat{K}(V_0 E_0 )_{\tilde{\tau}} \) is the central product of \( Z_2 \) and \( L_0(V_0 E_0)_{\chi_0} \), and therefore \( \phi := \phi_0 \tau \) defines an extension of \( \tilde{\tau} \). Set \( \phi := \text{Res}_{K(V_0 E_0)_{\tilde{\tau}}}^K(\phi) \) which then forms an extension of \( \tilde{\tau} \). Thus \( \eta := \text{Res}_{K(V_0 E_0)_{\tilde{\tau}}}^K(\phi) \) provides an extension of \( \tilde{\tau} \) to \( K(V_0 E_0)_{\tilde{\tau}} \) and by construction contains \( v \hat{F} \) in its kernel.

It is clear that \( S_{\phi} \leq S_{\psi} \). Therefore let \( s \in S_{\psi} \) and consider \( \phi^s \). As \( \tilde{\psi}^s \) is also an extension of \( \psi \), it follows that \( \psi^s = \chi_0 \tau \) or \( \chi_0(\tau \cdot \text{det} Z_2) \). However \( S_{\psi} \leq S_{\chi_0} \) and therefore \( \tilde{\psi}^s = \chi_0 \tau^s \) and \( \tau^s = \tau \) or \( \tau \cdot \text{det} Z_2 \). Hence \( \phi^s = \phi_0 \tau \) or \( \phi_0(\tau \cdot \text{det} Z_2) \) and thus

\[
\phi^s = \text{Res}_{K(V_0 E_0)_{\tilde{\tau}}}^K(\phi^s) = \text{Res}_{K(V_0 E_0)_{\tilde{\tau}}}^K(\phi) \cdot \phi = \phi.
\]

In particular, \( S_{\phi} = S_{\psi} = S_{\gamma} \).

By construction, \( K = L_0 Z(K) \) with \( S \cap K \leq Z(K) \). Therefore \( \text{Res}_{S \cap K}^K(\psi) = m \zeta \) for some \( m \in \mathbb{N} \) and \( \zeta \in \text{Irr} \left( \text{Res}_{S \cap K}^K(\psi) \right) \). Furthermore, \( \zeta \) extends to a character \( \tilde{\zeta} \in \text{Irr}(S_{\zeta}) \) by Corollary 3.5. As

\[
L_0 \ltimes \rho(S) = L_{\lambda_0} \times \prod_{f=1}^n (L_f \ltimes \rho(S_f)),
\]

it follows that \( \chi_0 \) extends to \( L_0 \ltimes \rho(S_{\chi_0}) \) by [Nav18, Corollary 10.2]. Hence by Proposition 4.2, \( \psi \) extends to \( K S_{\psi} \).

Observe that \( L \hat{\nu}_0 \cap S \leq K \) and therefore \( K \hat{\nu}_0 \cap KS = K(\hat{\nu}_0 \cap S) = K \). As \( \psi \) extends to \( K S_{\psi} \), [Spä10b, Lemma 4.1] implies that \( \eta \) extends to \( (K \hat{\nu})_{\psi} \). Hence \( \psi \) extends to \( M_{\psi} \) with \( v \hat{F} \) contained in its kernel. This completes the proof of Theorem 4.3.

4.C. **Characters of the quotient** \( N/L \). Fix \( \lambda \vdash n \) satisfying condition (ii) of Proposition 3.2, \( \tilde{L} := L_{\psi} F, L := \tilde{L} \cap G_{\psi} F \) and \( N = N_{G_{\psi} F}(L_\lambda) \). All the notations from Section 3 will be used without further reference. The aim in this section is to study characters of certain subgroups of the relative Weyl group \( W := N/L \).
Let $\chi \in \Irr(L)$ and $\overline{\chi} \in \Irr(\overline{L}_\chi | \chi)$. The quotient $\overline{L}/L$ is cyclic and therefore $\Res_{\overline{L}}^L(\overline{\chi}) = \chi$. Moreover, $\overline{\chi} := \Ind_{\overline{L}}^L(\overline{\chi}) \in \Irr(\overline{L})$. Because $\overline{L}_\chi \triangleleft N$, we see that $N_{\overline{\chi}} \leq N_{\overline{\chi}} \leq L \overline{N}_{\overline{\chi}}$. However all elements in $\overline{N}$ have determinant one and therefore $N_{\overline{\chi}} = L \overline{N}_{\overline{\chi}} = N_{\overline{\chi}}$. Furthermore, if $x \in N_{\overline{\chi}}$, then $\overline{\chi}^x = \overline{\chi}\overline{\beta}$ for some $\beta \in \Irr(\overline{L}_\chi/L)$. However, $\beta$ is a power of the determinant homomorphism and therefore $N_{\overline{\chi}} < N_{\overline{\chi}}$.

**Proposition 4.5.** Let $\chi \in \Irr(L)$, $\overline{\chi} \in \Irr(\overline{L}_\chi | \chi)$, $W_\chi := N_{\chi}/L$, $W_\overline{\chi} := N_{\overline{\chi}}/L$, $W_{\chi} := N_{\chi}/L$, $K := N_W(W_\chi, W_\overline{\chi})$ and $\eta_0 \in \Irr(W_{\overline{\chi}})$. Then there exists a character $\eta \in \Irr(W_\chi | \eta_0)$ such that

(i) $\{\eta^w \mid w \in K\} \cap \Irr(W_\chi | \eta_0) = \{\eta\}$,

(ii) $\eta$ extends to $K_\eta$,

(iii) $\eta$ has an extension $\tilde{\eta} \in \Irr(K_\eta \times E)$ with $\tilde{\eta} \in \ker(\eta)$.

**Proof.** For each $f \in \{1, \ldots, n\}$ set $W_f := LV_fS_f/L$, $\hat{Z}_f = LV_f/L$ and $\hat{S}_f = LS_f/L$. Then $\hat{Z}_f \cong (C_{d_0})^{t_f}$, $\hat{S}_f \cong \mathfrak{S}_{t_f}$ and $W_f = \hat{Z}_f \times \hat{S}_f$. Moreover $W = \prod_{f=1}^n W_f$ and for $\overline{\chi}_f := \Res_{\hat{S}_f}^L(\overline{\chi})$ it follows that

$$W_{\overline{\chi}} = \prod_{f=1}^n (W_f)_{\overline{\chi}_f}.$$ 

Therefore $\eta_0 \in \Irr(W_{\overline{\chi}})$ can be written as $\eta_0 = \prod_{f=1}^n \eta_{0,f}$, where $\eta_{0,f} := \Res_{\overline{W}_f}^W(\overline{\chi}_f)(\eta_0)$.

After suitable $V_f$-conjugation, the character $\overline{\chi}_f$ has a stabiliser in $W_{\chi}$, with the following description: there exists positive integers $r_f, d_{f,j}, a_{f,j}$ with $1 \leq j \leq r_f$ and a partition $M_{f,1}, \ldots, M_{f,r_f}$ of $\{1, \ldots, t_f\}$ such that $|M_{f,j}| = a_{f,j}$ and

$$W_f, \overline{\chi}_f = \left\{((\zeta_1, \ldots, \zeta_{t_f}), \sigma) \in C_{d_0} \chi \mathfrak{S}_{t_f} \mid \sigma(M_{i,j}) = M_{i,j} \text{ for all } 1 \leq j \leq r_i \text{ and } \zeta_k^{d_{f,j}} = 1 \text{ for all } k \in M_{i,j}\right\}.$$ 

The group $W_f, \overline{\chi}_f$ is isomorphic to a group as considered in [CS17a, Proposition 5.12]. Therefore the following constructions from [CS17a, Proposition 5.12] are taken: For $\nu_f \in \Irr\left(\Res_{\overline{Z}_f, \overline{\chi}_f}^{W_f, \overline{\chi}_f}(\eta_{0,f})\right)$, there exists an extension $\psi_f \in \Irr\left(\hat{Z}_f(W_f, \overline{\chi}_f)_{\nu_f}\right)$ with $(S_f, \overline{\chi}_f)_{\nu_f} \in \ker(\psi_f)$. Furthermore there exists another character $\tilde{\nu}_f \in \Irr\left(\hat{Z}_f(W_f, \overline{\chi}_f)_{\nu_f}\right)$ with $\tilde{\nu}_f \in \ker(\tilde{\nu}_f)$ so that the character

$$\tilde{\eta}_{0,f} := \Ind_{\overline{Z}_f(W_f, \overline{\chi}_f)_{\nu_f}}^{\hat{Z}_f(W_f, \overline{\chi}_f)_{\nu_f}}(\psi_f \tilde{\nu}_f)$$

satisfies

$$\Res_{W_f, \overline{\chi}_f}(\tilde{\eta}_{0,f}) = \eta_{0,f}.$$ 

Moreover $N_{W_f}(W_f, \overline{\chi}_f)_{\eta_{0,f}} \leq N_{W_f}(\hat{Z}_f(W_f, \overline{\chi}_f)_{\eta_{0,f}}$ and $\eta_{0,f}$ has an extension $\phi_f \in \Irr\left(N_{W_f}(\hat{Z}_f(W_f, \overline{\chi}_f)_{\eta_{0,f}})\right)$.

Define $\tilde{\eta}_0 := \prod_{f=1}^n \tilde{\eta}_{0,f}$ and $\tilde{Z} := \prod_{f=1}^n \hat{Z}_f$ so that

$$\Res_{W_{\overline{\chi}}}(\tilde{\eta}_0) = \eta_0.$$
Then
\[ N_W(W_\chi)_{\eta_0} = \prod_{f=1}^{n} N_{W_f}(W_{f,\eta_0})_{\eta_0} \leq \prod_{f=1}^{n} N_{W_f}(\tilde{Z}W_\chi)_{\tilde{\eta}_0} = N_W(\tilde{Z}W_\chi)_{\tilde{\eta}_0} \]
and so \( \phi := \prod_{f=1}^{n} \phi_f \in \text{Irr} \left( N_W(\tilde{Z}W_\chi)_{\tilde{\eta}_0} \right) \) is an extension of \( \tilde{\eta}_0 \). Hence as \( W_\chi \leq N_W(W_\chi) \), the character
\[ \tilde{\eta}_0 := \text{Res}^{N_W(\tilde{Z}W_\chi)}_{\eta_0} (\phi) \]
is an extension of \( \eta_0 \) and
\[ \eta := \text{Ind}^{W_\chi}_{(W_\chi)_{\eta_0}} (\tilde{\eta}_0) \in \text{Irr}(W_\chi | \eta_0) \).

If \( w \in K \) and \( \eta^w \in \text{Irr}(W_\chi | \eta_0) \) then \( w \in W_\chi K_{\eta_0} \) and hence it can be assumed that \( w \in K_{\eta_0} \). Furthermore, \( K_{\eta_0} \leq N_W(\tilde{Z}W_\chi)_{\tilde{\eta}_0} \) and so \( \phi^w = \phi \). However this implies that \( \tilde{\eta}_0^w = \tilde{\eta}_0 \) as \( \tilde{\eta}_0 \) is the restriction of \( \phi \), and thus \( \eta^w = \eta \) as \( \eta \) arises as the induced character which is fixed by elements of \( K_{\eta_0} \), proving the first statement. Moreover, the observation \( K_{\eta} = W_\chi K_{\eta_0} \) yields that
\[ \tilde{\eta} := \text{Ind}^{K_{\eta_0}}_{K_{\eta_0}} \left( \text{Res}^{N_W(\tilde{Z}W_\chi)}_{\eta_0} (\phi) \right) \in \text{Irr}(K_\eta) \]
is an extension of \( \eta \) proving the second statement. The final property is the same as the property given in [CS17a, Proposition 5.12] and the proof is the same.

The explicit description allows us to see that the relative Weyl groups \( W_\chi \) satisfy the McKay Conjecture. This is applied in the proof of Theorem 1.3(a).

**Proposition 4.6.** The McKay Conjecture holds for \( W_\chi \) from 4.5, whenever \( \chi \in \text{Irr}(L) \).

**Proof.** Recall that \( W = \tilde{Z} \rtimes \tilde{S} \) and \( W_\chi = Z \rtimes S \) for an extension \( \tilde{\chi} \) of \( \chi \) to \( \tilde{L}_\chi \). By construction \( W_\chi / W_\tilde{\chi} \) is isomorphic to a subgroup of \( \tilde{L}_\chi / L \), hence cyclic. The group \( Z \lhd W_\chi \) is abelian and by the constructions given in the previous proof, every character of \( Z \) extends to its stabilizer in \( W_\chi \). Let \( \mu \in \text{Irr}(Z) \). Then \( S_\mu \) is a direct product of symmetric groups and \( (W_\chi)_\mu / (ZS_\mu) \) is cyclic. According to [Spä10a, Lemma 12.4], \( (W_\chi)_\mu / Z \) satisfies the McKay conjecture. This allows us to apply [Spä10a, Lemma 12.1] and obtain that the McKay conjecture holds for \( W_\chi \).

**Corollary 4.7.** (a) For every \( \chi \in \text{Irr}(\tilde{N}) \) there exists some \( \chi_0 \in \text{Irr}(N \times \chi) \) such that

(i) \( (\tilde{\pi} \tilde{\chi} \chi_0)_{\chi_0} = \tilde{\pi} \chi \chi_0 \), and

(ii) \( \chi_0 \) has an extension \( \tilde{\chi}_0 \) to \( \tilde{N}_{\chi_0} \) with \( v \tilde{F} \in \ker(\tilde{\chi}) \).

(b) There exists some \( \tilde{N} \)-equivariant extension map \( \tilde{\Lambda} \) with respect to \( \tilde{L} \lhd \tilde{N} \) that is compatible with \( \text{Irr}(\tilde{N}/N) \).

**Proof.** This follows by combining Theorem 4.1 with the verification of the required conditions in Theorem 4.3, Proposition 4.5 and Corollary 3.10.

We are finally able to verify Theorem 1.2.
Proof of Theorem 1.2. Using the proof of [CS17a, Proposition 5.3] there exists an isomorphism \( \epsilon : \tilde{G}^F \times E_0 \rightarrow \mathbf{C}_{\tilde{G}^F \circ \Phi}(v \tilde{F})/(v \tilde{F}) \). Let \( L' \), \( N' \) and \( \tilde{N}' \) be the images of \( L_0 \), \( N_0 \) and \( \tilde{N}_0 \) respectively under this isomorphism. By the proof of [CS17a, Proposition 5.3], Corollary 4.7 implies the statement. \( \square \)

5. Block-theoretic considerations

The aim in this section is to study the \( \ell \)-blocks of \( \text{SL}_n(\mathbb{F}_q) \) via relating the normalizer of a defect subgroup with normalizers of \( d \)-split Levi subgroups (\( d \) is the multiplicative order of \( q \) mod \( \ell \)) and constructing a bijection between height zero characters.

Here we rely on Broué-Malle-Michel’s theory of generic Sylow \( d \)-tori and generic blocks for finite reductive groups from [BMM93]. Using this language, Cabanes-Enguehard parametrized the \( \ell \)-blocks of finite reductive groups \( G^F \) with \( d \)-cuspidal pairs \( (L, \zeta) \) where \( L \) is a \( d \)-split Levi subgroup and \( \zeta \) a so-called \( d \)-cuspidal character of \( L^F \), see [CE99a, 4.1]. The \( \ell \)-block \( B \) of finite reductive group \( (G, F) \) associated with \( (L, \zeta) \), denoted by \( b_{G^F}(L, \zeta) \), is the \( \ell \)-block that contains all irreducible constituents of \( R_{L^F}^G(\zeta) \).

In order to prove Theorem 1.1, via an application of the criterion introduced in Theorem 2.4, it is necessary to prove that the normalizers of the \( d \)-split Levi subgroups from Theorem 1.2 can be chosen as the group denoted by \( M \) in Theorem 2.4. We prove this in the case where \( \ell \mid 3q(q - \epsilon) \).

In a second step we construct a bijection \( \tilde{\Omega}_B \) with the properties required in 2.4(ii). This bijection has obvious similarities with the one constructed for the inductive McKay condition, which was first developed in [Mal07] and then later in [CS17a]. As main ingredients it uses the so-called \( d \)-Harish-Chandra theory, the Jordan decomposition of characters and extension maps for \( d \)-split Levi normalizers. This bijection has to be transferred to the new context where also blocks and the height of the characters have to be taken into account.

5.A. Normalizers of defect groups and height zero characters. Recall that for a odd prime \( \ell \) and a prime power \( q \) with \( \ell \nmid q \) we denote by \( d_s(q) \) the order of \( q \) in \( (\mathbb{Z}/\ell \mathbb{Z})^\times \).

For \( (H, F) \) a reductive group defined over a finite field, denote by \( \mathcal{E}_F(H^F) \) the union of Lusztig series associated to semi-simple \( \ell \)-elements of \( H^F \).

Theorem 5.1. Let \( \tilde{G} = \text{GL}_n(\mathbb{F}_q) \), \( F : \tilde{G} \rightarrow \tilde{G} \) a Frobenius endomorphism defining an \( \mathbb{F}_q \)-structure, \( \epsilon \in \{ \pm 1 \} \) with \( \tilde{G}^F = \text{GL}_n(\mathbb{F}_q) \) and \( \ell \) a prime with \( \ell \nmid 3q(q - \epsilon) \). Let \( \tilde{B} \in \text{Bl}(\tilde{G}^F) \) and set \( d : = d_s(q) \). Let \( (L, \zeta) \) be a \( d \)-cuspidal pair of \( (\tilde{G}, F) \) associated to \( \tilde{B} \) as in [CE99a, 4.1]. Let \( S \) be the Sylow \( \Phi_d \)-subtorus of \( Z^\circ(L) \). Then there exists some defect group \( D \) of \( \tilde{B} \) such that \( \mathbf{N}_{[\tilde{G}, \tilde{G}]}^F(S) \) is \( \text{Aut}([\tilde{G}, \tilde{G}])_{\tilde{B}, D} \)-stable and \( \mathbf{N}_{\tilde{G}^F}^F(D) \subseteq \mathbf{N}_{\tilde{G}^F}^F(S) \). Moreover \( C_{\tilde{G}^F}(D) \leq C_{\tilde{G}^F}(S) = L^F \).

Proof. We have \( \tilde{B} = b_{\tilde{G}^F}(L, \zeta) \) in the notation of [CE99a, 2.6], i.e., \( L \) is a \( d \)-split Levi subgroup, \( \zeta \in \mathcal{E}_F(L^F) \) is \( d \)-cuspidal and all constituents of \( R_L^G(\zeta) \) are contained in \( \tilde{B} \).

We observe that by our assumption on \( \ell \) the groups \( \tilde{G}_a \) and \( \tilde{G}_b \) defined in the paragraph after Proposition 3.3 of [CE99a] satisfy \( \tilde{G}_a = Z(\tilde{G}) \) and \( \tilde{G}_b = \tilde{G} \cong \text{SL}_n(\mathbb{F}_q) \). Further note that \( L \) coincides with the Levi subgroup \( K \) defined in [CE99a, 3.2 and 3.4].

Let \( M \) be the group defined in the paragraph before [CE99a, 4.4]. Then \( S \) is also the Sylow \( \Phi_d \)-subtorus of \( Z^\circ(M) \) thanks to [CE99a, 4.4.(iii)]. Denote \( Z := Z(M)^F \). Then \( M = C_{\tilde{G}}^0(Z) \) according to [CE99a, Lemma 4.8].
By [CE99a, Lemma 4.16], $Z$ is a characteristic subgroup of a defect group $D$ of $\tilde{B}$ and hence $N_{G_F}(D) \subseteq N_{G_F}(Z)$. We also have $N_{G_F}(Z) = N_{G_F}(M) = N_{G_F}(Z^\circ(M)) \leq N_{G_F}(S)$.

Denote $G = [G, G]$. We must ensure that every $\phi_0 \in \text{Aut}(G^F)_{\tilde{B}, D}$ stabilizes $N_{G_F}(L)$. Such $\phi_0$ is induced by a bijective endomorphism $\phi$ of $\tilde{G}$ commuting with $F$. The equality $M = C_{G_F}(Z)$ implies $\phi(M) = M$. Similarly $\phi$ stabilizes $L$ since $L \cap G_b$ is the smallest $d$-split Levi subgroup containing $M$, see [CE99a, proof of Lemma 4.4].

Since the centralizer of any semi-simple element is connected in $G$, $C_{G}(Z)$ is connected and we may write $C_G(D)^F \leq C_G(Z)^F = C_{G_F}(Z)^F = MF \leq LF = C_G(S)^F$ by the definition of $S$.

To construct the bijection required by Theorem 2.4 the following description of height 0 characters of unipotent blocks is needed. The general proof below was communicated to us by Marc Cabanes. The particular case of type A could be treated with a simpler proof.

**Theorem 5.2** (Height zero and series). Let $(G, F)$ be a reductive group defined over a field of cardinality $q$, $\ell$ an odd prime good for $G$, not dividing $q$, and $\neq 3$ if $G^F$ is of type $^3D_4$. $d$ is the multiplicative order of $q$ mod $\ell$. Let $B$ be a unipotent $\ell$-block of $G^F$ defined by a $d$-cuspidal pair $(K, \zeta)$ as in [CE94, 4.4]. Then

$$\text{Irr}_0(B) \subseteq \cup_t \mathcal{E}(G^F, t),$$

where $t$ ranges over $Z(K^*)^F$. Each such $t \in Z(K^*)^F$ satisfies $\ell \nmid |N_G(K)^F : N_{G(t)}(K)^F|$, where $G(t)$ denotes a Levi subgroup of $G$ dual to $C_{G^t}(t)$ with $K \leq G(t)$.

**Proof.** A character in $\text{Irr}_0(B)$ is of the form $\chi := \pm R_{G(t)}^G(\hat{t}\mu_t)$ where $t \in G^*_t, G(t)$ is an $F$-stable Levi subgroup of $G$ in duality with the $F$-stable Levi subgroup $C_{G^t}(t)$, $\hat{t}$ is a linear character of $G(t)^F$ defined by duality and $\mu_t \in \mathcal{E}(G(t)^F, 1)$ is a component of $R_{G(t)}^G(\zeta)$ where $(K_t, \zeta_t)$ is a unipotent $d$-cuspidal pair in $G(t)$ (see [CE94, 4.4(iii)] and $[\mathbf{K}, \mathbf{K}] = [K_t, K_t]$). By [CE94, 4.4(ii)] any Sylow $\ell$-subgroup of $C_{G^t}([K, K])^F$ is a defect group of $B$. Similarly $\mu_t$ belongs to an $\ell$-block of $G(t)^F$ such that any Sylow $\ell$-subgroup of $C_{G(t)}([K, K])^F$ is a defect group of this block.

If $\chi$ has height zero, then $\chi(1) = |G^F : G(t)^F| \ell \cdot \mu_t(1) = |G^F : C_{G^t}([K, K])^F| \ell$. On the other hand $\mu_t(1) = \ell^{h(\mu_t)} |G(t)^F : C_{G(t)}([K, K])^F|, 0 \leq h(\mu_t)$ denotes the height of $\mu_t$, by what has been said about $b(\mu_t) \in \mathcal{B}(G(t)^F)$. Thus $\ell^{h(\mu_t)} |G^F : C_{G^t}([K, K])^F| \ell = |G^F : C_{G}(([K, K])^F| \ell$ and therefore

$$h(\mu_t) = 0 \text{ and } |C_{G^t}([K, K])^F : C_{G(t)}([K, K])^F| = 1.$$

We use the second equality. As duality preserves the order of groups of rational points, one has

$$|C_{G^t}([K^*, K^*])^F : C_{G^t}([K^*, K^*], t)^F| = 1$$

with $K^*$ in duality with $K$ and $t \in C_{G^t}([K^*, K^*])^F$. This means that $t$ centralizes a Sylow $\ell$-subgroup of $C := C_{G^t}([K^*, K^*])^F$. So it has a $C$-conjugate that centralizes $Z^\circ(K^*)^F$. But $C_{G^t}(Z^\circ(K^*)^F) = K^F$ by [CE94, 3.3(ii)] and therefore $C_G(Z^\circ(K^*)^F) \leq C_{K^*}([K^*, K^*])^F = Z(K^*)^F$. This gives that $t$ is actually conjugate to an element of $Z(K^*)^F$ in $G^F$.

We have to show now that $|N_G(K)^F : N_{G(t)}(K)^F| = 1$. Letting $g \in N_G(K)^F$ we must show that $g \in G(t)$. By [CE99b, 6], we have $g \in (K C_{G^t}([K, K]))^F$. Thanks to [CE94, 3.3(ii)] telling us that $Z(K)^F = Z^\circ(K)^F$ and Lang-Steinberg’s theorem, we have
(KC^G_G([K, K]))^F = K^F C^G_G([K, K])^F. But we have seen before that C^G_G([K, K])^F \leq G(t),
so indeed g \in KG(t) = G(t). \hfill \Box

Note that the above proof simplifies when G is a general linear group, as in that case
G = G^* and all centralizers are connected.

5. Bijections between certain characters of \( \tilde{G}^F \) and \( N_{\tilde{G}^F}(S) \). Recall that for a
given character \( \chi \) we use \( \text{bl}(\chi) \) to denote the \( \ell \)-block \( \chi \) belongs to.

In the next step we show that a bijection as required in 2.4(ii) exists for a block \( \tilde{B} \in \text{Bl}(GL_q(eq)) \). Note that although Broué’s conjecture is known for blocks with abelian defect
of \( GL_q \), the Alperin-McKay conjecture hasn’t been proven for \( SL_q \).

Let \( G, F : \tilde{G} \to \tilde{G} \) such that \( \tilde{G}^F \cong GL_q(eq) \). Note that \( \text{Irr}(\tilde{G}^F/[\tilde{G}, \tilde{G}]^F) \) acts on
\( \text{Irr}(\tilde{G}^F) \) by multiplication and thereby induces an action on \( \text{Bl}(\tilde{G}^F) \). Note that any
\( \text{Irr}(\tilde{G}^F/[\tilde{G}, \tilde{G}]^F) \)-orbit coincides with \( \text{Bl}(\tilde{G}^F | B) \), the set of blocks of \( \tilde{G}^F \) covering \( B \) for some \( B \in \text{Bl}([G, \tilde{G}]^F) \).

5. B.1. A parameter set and two character sets associated to a \( \Phi_d \)-torus. Let \( d \geq 1 \) and \( S \)
a \( \Phi_d \)-torus of \( (\tilde{G}, F) \). Denote \( L := C_{\tilde{G}}(S) \).
For any \( F \)-stable torus \( T \) of \( \tilde{G} \) we denote by
\( T_{\Phi_d} \) the Sylow \( \Phi_d \)-torus of \( (T, F) \).

For \( s \in (\tilde{G}^*)^F \) semi-simple, let \( \tilde{G}(s) \) be an \( F \)-stable Levi subgroup of \( \tilde{G} \) dual to \( C_{\tilde{G}^*}(s) \)
and \( \tilde{s} \) the linear character of \( \tilde{G}(s)^F \) associated to \( s \) by duality. If \( L' \) is an \( F \)-stable Levi
subgroup of \( \tilde{G}, K \) an \( F \)-stable Levi subgroup of \( L' \) and \( \kappa \in \text{Irr}(K^F) \), then \( W_{L'}(K, \kappa)^F \) is
defined as \( N_{L^F}(K)^\kappa/K^F \). If additionally \( (K, \kappa) \) is a unipotent \( d \)-cuspidal pair of \( (L', F) \)
we denote by \( \mathcal{E}(L^F, (K, \kappa)) \) the set of constituents of \( R^F_K(\kappa) \) and there is a bijection
\[ \text{Irr}(W_{L'}(K, \kappa)^F) \to \mathcal{E}(L^F, (K, \kappa)) \]
according to [BMM93, 3.2(2)] which we denote by
\[ \eta \mapsto R^F_K(\kappa) \eta. \]

For any element \( x \) of a finite group we denote by \( x^e \) an element of \langle x \rangle \) whose order is
not divisible by \( \ell \) and for which \( xx^{-1} = \ell \)-element. We define \( P_S \) as the set of triples
\( (s, \kappa, \eta) \) where
\begin{itemize}
  \item \( s \in \tilde{G}^* \) is a semi-simple element with \( S \leq \tilde{G}(s) \) and \( C_{\tilde{G}(s)}(e) \leq \tilde{G}(s) \),
  \item \( \kappa \) is a unipotent \( d \)-cuspidal character of \( K^F \), where \( K := C_{\tilde{G}(s)}(S) \) with \( S = Z(K)_{\Phi_d} \), and
  \item \( \eta \in \text{Irr}(W_{\tilde{G}(s)}(K, \kappa)^F) \).
\end{itemize}
The group \( \tilde{N} := N_{\tilde{G}^F}(S) \) acts via conjugation on \( P_S \) and we write \( \overline{P}_S \) for the set of \( \tilde{N} \)-orbits in \( P_S \).

Let \( \Upsilon^\circ : P_S \to \text{Irr}(\tilde{G}^F) \) be given by \( (s, \kappa, \eta) \mapsto \epsilon_{\tilde{G}(s)} \epsilon_{\tilde{G}(s)} R^F_{\tilde{G}(s)}(s R^F_k(\kappa) \eta) \), where \( \epsilon_{\tilde{G}(s)} \)
\( \epsilon_{\tilde{G}(s)} \) are signs (see [CE04, 8.27]). Denote
\[ G_S := \Upsilon^\circ(P_S). \]

**Lemma 5.3.** Let \( (s, \kappa, \eta) \in P_S \). Then \( \Upsilon^\circ(s, \kappa, \eta)(1)_\ell = |G^F : N_{\tilde{G}(s)^F}(K)|_\ell \eta(1)_\ell \kappa(1)_\ell. \)
Proof. In this case Jordan decomposition coincides with Deligne-Lusztig induction and 
\[ R^\mathbb{G}_{\mathbb{G}(s)}(\mathbb{G}(\mathbb{G}(s)) (\kappa_\eta)(1) = \xi_{\mathbb{G}(s)}(\kappa_\eta)|_{\mathbb{G}(s)} \mathbb{G}(s) : \mathbb{K}^F |_{\ell} \kappa(1) \eta(1) \ell. \] The degree of \( R^\mathbb{G}_{\mathbb{G}(s)}(\kappa_\eta)|_{\mathbb{G}(s)} \mathbb{G}(s) : \mathbb{K}^F |_{\ell} \kappa(1) \eta(1) \ell. \) is given in Theorem 4.2 of [Mal07]. Let \( D_\eta \in \mathbb{Q}(X) \) be as in [Mal07, 4.2] a rational function with zeros and poles only at roots of unity and 0 such that \( D_\eta(\zeta_d) = \eta(1)/|W_{\mathbb{G}(s)}(\mathbb{K})| \) for a certain primitive \( d \)th root \( \zeta_d \) of unity and 
\[ R^\mathbb{G}_{\mathbb{K}}(\kappa_\eta)|_{\mathbb{G}(s)} \mathbb{G}(s) : \mathbb{K}^F |_{\ell} \kappa(1) \eta(1) \ell. \] This implies the stated formula in the general case.

We see from the definition \( \Upsilon \) of \( \mathbb{G} \) and \( \mathbb{K} \).

Proof. The statement is proven by first defining an equivalence relation on \( \mathcal{P}_S \) and proving that the maps \( \Upsilon^\circ \) and \( \Upsilon^\circ \) induce well-defined injective maps on the set \( \mathcal{P}_S \) of equivalence classes in \( \mathcal{P}_S \). In a second step we then see that the bijection obtained has the required properties.
Recall $\tilde{N}$ acts by conjugation on $\mathcal{P}_S$ inducing an equivalence relation. We denote by $\overline{\mathcal{P}}_S$ the set of equivalence classes in $\mathcal{P}_S$.

Via $d$-Harish-Chandra theory and Deligne-Lusztig induction, we see that triples lying in the same $\tilde{N}$-orbit correspond to the same character of $\tilde{G}$ by the equivariance properties from [CS13, 3.1 and 3.4]. (Note that Theorem 3.4 of [CS13] states the equivariance of the $d$-Harish-Chandra theory only in the case of minimal $d$-split Levi subgroups, but the proof applies also in the general case.) Hence $\Upsilon^\circ$ induces a well-defined map

$$\Upsilon : \overline{\mathcal{P}}_S \rightarrow \mathcal{G}_S.$$ 

Assume for a given character $\chi \in \mathcal{G}_S$ that $\Upsilon^\circ(s,\kappa,\eta) = \Upsilon^\circ(s',\kappa',\eta') = \chi$. We see that $s$ and $s'$ have to be $(G^*)^F$-conjugate by the disjointness of Lusztig series. Let $g \in (G^*)^F$ with $s' = s^g$. According to [BMM93, 3.2(1)] the characters $\kappa^g \in \text{Irr}(C_{\tilde{G}}(s)^F(S))$ and $\kappa' \in \text{Irr}(C_{\tilde{G}}(s')^F(S))$ are $\tilde{G}(s)^F$-conjugate, i.e. $\kappa^g h = \kappa'$ for some $h \in \tilde{G}(s)^F$ and $\eta^gh = \eta'$. Since $Z(C_{\tilde{G}}(s')^F(S))_{\mathfrak{g}_d} = S$, we see $gh \in \tilde{N}$. Hence $\Upsilon$ is injective and hence bijective.

Recall that $\Lambda$ is $\tilde{N}$-equivariant. Therefore $\Upsilon^\circ$ induces a bijection

$$\Upsilon' : \overline{\mathcal{P}}_S \rightarrow \mathcal{N}_S$$

by Clifford theory and the equivariance of Deligne-Lusztig induction.

We now study the bijection

$$\Omega_S : \mathcal{G}_S \rightarrow \mathcal{N}_S$$

given by $\Upsilon' \circ \Upsilon^{-1}$.

The considerations in [CS17a, §6] prove that $\Omega_S$ is $(\tilde{G}^F E)_S$-equivariant and satisfies

- $\Omega_S(\mathcal{G}_S \cap \text{Irr}(\tilde{G} \mid \nu)) \subseteq \text{Irr}(\tilde{N} \mid \nu)$ for every $\nu \in \text{Irr}(Z(\tilde{G})^F)$
- $\Omega_S$ is compatible with the action of $\text{Irr}(\tilde{G}^F/G^F)$ by multiplication, i.e.

$$\Omega_S(\chi \mu) = \Omega_S(\chi) \text{Res}^{\tilde{G}^F}_{\tilde{N}}(\mu) \text{ for every } \chi \in \mathcal{G}_S \text{ and } \mu \in \text{Irr}(\tilde{G}^F/G^F).$$

It remains to prove $\text{bl}(\chi) = \text{bl}(\Omega_S(\chi))^\tilde{G}$ for every $\chi \in \Upsilon(\overline{\mathcal{P}}_S)$. Let $(s,\kappa,\eta) \in \mathcal{P}_S$ and $\chi = \Upsilon^\circ(s,\kappa,\eta)$. Then we see that $\chi$ is a constituent of $R^\tilde{G}_{G(s)}(\tilde{G}^{G(s)}(\kappa))$. The constituents of $R^\tilde{G}_{G(s)}(\tilde{G}^{G(s)}(\kappa))$ lie all in the same block and similarly the constituents of $R^\tilde{G}_{G(s)}(\tilde{S}\tilde{R}^\tilde{G}_{K}(\kappa))$ belong to the same block where $\tilde{S}\tilde{R}^\tilde{G}_{K}(\kappa)$. We see that $\Omega_S$ is $\text{Irr}(\tilde{G}^F/G^F)$-equivariant.

In order to compute the block $\text{bl}(\Upsilon^\circ(s,\kappa,\eta))^\tilde{G}$ we recall that $\mathbf{S}^\ell$ is a normal $\ell$-subgroup of $\tilde{N}$ and hence the defect group of the block $\text{bl}(\Upsilon^\circ(s,\kappa,\eta))$ contains $\mathbf{S}^\ell$. By Theorem 5.1, $L^F \succeq C_{\tilde{G}^F}(\mathbf{S}^\ell)$. Then

$$\text{bl}(\text{Ind}_{\tilde{N}^\mathfrak{g}_{K}(\tilde{S}\kappa)}^{\tilde{N}^\mathfrak{g}_{K}(\tilde{S}\kappa)}) = \text{bl}(R^L_{K}(\tilde{S}\kappa))^\tilde{N}$$

according to [Nav98, 9.8]. By the definition of $s_0$ we see that $\text{bl}(R^L_{K}(\tilde{S}\kappa)) = \text{bl}(R^L_{K}(\tilde{S}\kappa))$. This altogether implies that

$$\text{bl}(\Upsilon'(s,\kappa,\eta))^\tilde{G} = \text{bl}(R^L_{K}(\tilde{S}\kappa))^\tilde{G}.$$ 

According to [CE99a, 2.5] one knows $\text{bl}(R^L_{K}(\tilde{S}\kappa))^\tilde{G} = \text{bl}(R^L_{K}(\tilde{S}\kappa))$. This shows $\text{bl}(\chi) = \text{bl}(\Omega_S(\chi))^\tilde{G}$. 

$\blacksquare$
In the next step we obtain a bijection between the height zero characters of Brauer corresponding blocks satisfying the assumption 2.4(ii).

**Theorem 5.6.** Assume the situation of Proposition 5.5. Let \( L := C_G(S) \), \( \zeta \in \mathcal{E}_\ell'(L^F) \) be d-cuspidal, \( \widetilde{B}_0 = b_{G_F}(L, \zeta) \) the \( \ell \)-block containing all components of \( R^G_L \zeta \), and \( \widetilde{B} \) be the sum of \( \text{Irr}((G^F/F^F) \text{-orbit in } \text{Bl}(G^F) \text{ containing } \widetilde{B}_0). \) Let \( \widetilde{b} \) be the sum of \( c \in \text{Bl}(N) \) with the property that \( \phi^c \) is in \( \widetilde{B}_0 \).

Then there exists a \((G^F E)\mathcal{B}, \tilde{N}\)-equivariant bijection
\[ \tilde{\Omega}_\mathcal{B} : \text{Irr}_0(\widetilde{B}) \rightarrow \text{Irr}_0(\widetilde{b}) \]
with
- \( \tilde{\Omega}_\mathcal{B}(\text{Irr}_0(\widetilde{B}) \cap \text{Irr}(\tilde{G}^F | \nu)) \subseteq \text{Irr}(\tilde{N} | \nu) \) for every \( \nu \in \text{Irr}(\mathbb{Z}(\tilde{G}^F)), \)
- \( \tilde{\Omega}_\mathcal{B} \) is compatible with the action of \( \text{Irr}(\tilde{G}^F/F^F) \) by multiplication, i.e.,
\[ \tilde{\Omega}_\mathcal{B}(\chi\mu) = \tilde{\Omega}_\mathcal{B}(\chi) \text{Res}_{\tilde{N}}(\chi) \text{Res}_{\tilde{N}}(\mu) \text{ for every } \mu \in \text{Irr}(\tilde{G}^F/F^F) \text{ and } \chi \in \text{Irr}(\tilde{B}). \]

**Proof.** We have to show \( \text{Irr}_0(\widetilde{B}) \subseteq \mathcal{G}_S \) and \( \tilde{\Omega}_S(\text{Irr}_0(\widetilde{B})) = \text{Irr}_0(\tilde{b}) \) for the bijection \( \tilde{\Omega}_S \) from Proposition 5.5.

Let \( \chi_0 \in \text{Irr}(\widetilde{B}) \cap \mathcal{E}_\ell'(\tilde{G}) \), where \( \mathcal{E}_\ell'(\tilde{G}) \) is the union of Lusztig series of \( \tilde{G}^F \) associated to semi-simple \( \ell \)-elements, \( \chi_0 = Y^\circ(s_0, \kappa, \eta) \) and \( K \) the Levi subgroup of \( G \) such that \( \kappa \) is a character of \( K^F \). For any \( \chi \in \text{Irr}(\widetilde{B}) \) there exists some semi-simple \( s \in \tilde{G}^* \) with \( s_\ell = s_0 \) and \( \chi \in \mathcal{E}(\tilde{G}^F, s) \) \([\text{CE04}, 9.12.(i)]\). Furthermore \( R^G_{G(s_0)}(\chi)s_0^{-1} \) lies in a unipotent block. Thanks to Theorem 5.2 we can assume that \( K \leq G(s) \). The proof of \([\text{CE04}, 23.4]\) can be applied and proves that \( R^G_{G(s_0)}(\chi)s_0^{-1} \) lies in the \( d \)-Harish-Chandra series of \((K, \kappa)\), hence \( R^G_{G(s_0)}(\chi)s_0^{-1} \in \mathcal{E}(\tilde{G}(s)^F, (K, \kappa)) \). This proves \( \text{Irr}_0(\widetilde{B}) \subseteq \mathcal{G}_S \).

Let \( \mathcal{P}_\widetilde{B} := \mathcal{Y}^{-1}(\text{Irr}_0(\widetilde{B})). \) We see that \( \mathcal{Y}(\mathcal{P}_\widetilde{B}) \subseteq \text{Irr}(\tilde{b}) \), since \( \text{bl}(\chi) = \text{bl}(\mathcal{Y}(\chi)) \) for every \( \chi \in \mathcal{G}_S \) and \( \tilde{b} \) is the sum of all blocks \( c \) of \( \tilde{N} \) such that \( c^\tilde{G} \) is in \( \widetilde{B} \).

It remains to prove that \( \text{Irr}(\tilde{b}) \subseteq \mathcal{N}_S \). The blocks of \( \tilde{b} \) cover blocks of \( L^F \). Hence, via Jordan decomposition, every height zero character of \( c \) corresponds to a character in the \( d \)-Harish-Chandra series of \((K, \kappa)\) with \( S = \mathbb{Z}(K')_{\Phi_\eta} \). This implies \( \text{Irr}_0(\tilde{b}) \subseteq \mathcal{N}_S \).

Accordingly the restriction of \( \tilde{\Omega}_S \) to \( \text{Irr}_0(\widetilde{B}) \) yields an injective map
\[ \tilde{\Omega}_\mathcal{B} : \text{Irr}_0(\widetilde{B}) \rightarrow \text{Irr}(\tilde{b}) \]
with the required equivariance properties.

In order to finish our proof it remains to show that \( \tilde{\Omega}_\mathcal{B}(\text{Irr}_0(\widetilde{B})) = \text{Irr}_0(\tilde{b}) \). Recall that \( \chi_0 = Y^\circ(s_0, \kappa, \eta) \in \text{Irr}(\widetilde{B}) \). Lemma 5.3 shows that the \( \ell \)-part of the degree of \( Y^\circ(s_0, \kappa, 1) \) is minimal amongst \( \text{Irr}(\widetilde{B}) \). Accordingly this character has height 0. We see that the block has defect \( |\tilde{G}(s)^F : K^F|_{\ell} |W_{\tilde{G}(s)}(K, \kappa)|_{\ell} \). According to the description of the defect group this proves that \( |\tilde{G}(s)^F : K^F|_{\ell} = |W_{\tilde{G}(s)}(K, \kappa)|_{\ell} \). This proves that \( \text{Irr}_0(\widetilde{B}) = \mathcal{Y}(\mathcal{P}_\widetilde{B}) \), where \( \mathcal{P}_\widetilde{B} \) is the set of triples \((s', \kappa', \eta') \in \mathcal{P}_S \) with the following properties:
- \( (s')_\ell \in \mathbb{Z}((\tilde{G}^*)^F)_{s_0} \), \( \kappa' = \kappa \), and
- \( \eta' \in \text{Irr}_\ell(W_{\tilde{G}(s')}(K, \kappa)^F) \), and
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5.6

BMM93

Spä 5.6

assuming

allow also an application of

2.1

1.2(b)

2.4(ii)

has the properties required in

and Theorem

1.3

2.4(v)

5.7

1.2(b)

\[ W_{G_{(\kappa')}}(K, \kappa)^F : W_{\tilde{G}_{(\kappa')}}(K, \kappa)^F \].

Note that \( W_{\tilde{G}_{(\kappa')}}(K, \kappa)^F = W_{\tilde{G}_{(\kappa')}}(K)^F \) by the description of unipotent d-cuspidal pairs given in [CE04, 21.6] and [BMM93, proof of 3.3].

On the other hand considering the description of the defect group from [CE99a, Lemma 4.16] and the degrees given in Lemma 5.4 we see \( \text{Irr}_0(\tilde{b}) = \Upsilon^\kappa(\mathcal{P}_B) \). This implies that \( \tilde{\Omega}_B \) defines a bijection with the required properties. \( \square \)

For later we point out the following equality.

**Corollary 5.7.** Let \( B \in \text{Bl}(\text{SL}_n(eq)) \) be covered by \( \tilde{B}_0 \) in the situation of Theorem 5.6. Then \( |\text{Irr}_0(B)| = |\text{Irr}_0(b)| \), where \( N := \tilde{N} \cap \text{SL}_n(eq) \) and \( b \in \text{Bl}(N) \) with \( b^G = B \).

**Proof.** The map \( \tilde{\Omega}_B \) is \( \text{Irr}(\tilde{G}/G) \)-compatible. Accordingly the number of constituents of \( \text{Res}^\tilde{G}_{G}(\chi) \) and \( \text{Res}^\tilde{N}_{N}(\Omega(\chi)) \) coincide for all \( \chi \in \text{Irr}_0(\tilde{B}) \). Also every character of \( \text{Irr}_0(B) \) and \( \text{Irr}_0(b) \) is the constituent of the the restriction of some character of \( \text{Irr}_0(\tilde{B}) \) and \( \text{Irr}_0(\tilde{b}) \), respectively, since \( \ell | [\tilde{G} : G] \). Furthermore we see that all blocks of \( G \) covered by \( B \) have the same number of height zero characters. Accordingly

\[
\bigcup_{\chi \in \text{Irr}_0(\tilde{B})} \text{Irr}(\text{Res}^\tilde{G}_{G}(\chi)) = \bigcup_{b \in \text{Bl}(\tilde{G}/\tilde{B})} \text{Irr}_0(B).
\]

This implies the stated equality via analogous local considerations. \( \square \)

6. PROOFS OF THE MAIN STATEMENTS

In this section we combine the results of the previous sections to provide a proof of Theorem 1.1 and Theorem 1.3. We are first concerned with the consequences towards the Alperin-McKay Conjecture and the inductive AM condition. At the end we turn our focus towards the blockwise Alperin weight conjecture and the blockwise Alperin weight condition.

**Proof of Theorem 1.1(a).** Let \( \ell \) be a prime \( \ell \not| q(q - \epsilon) \), \( B_0 \) be an \( \ell \)-block of \( \text{SL}_n(eq) \) and \( B \) the sum of \( \text{GL}_n(eq) \)-conjugates of \( B_0 \). Assume that \( \text{Out}(G)_B \) is abelian. We prove the inductive AM condition as given in [Spä13a, 7.2] for \( B \) via an application of Theorem 2.1. While clearly Assumption (i) holds, Assumption (iii) follows from 3.8. The bijection from Corollary 5.6 has the properties required in 2.4(ii). Assumption 2.4(iv) is known from Theorem 1.2(b). Recall that \( \text{Out}(G)_B \) is assumed to be abelian. This implies that the inductive AM condition from [Spä13a, 7.2] holds for \( B_0 \). \( \square \)

In the above we apply the criterion from Theorem 2.4 assuming 2.4(v). Corollary 5.7 and Theorem 1.2(b) allow also an application of [CS15, Theorem 4.1] towards the verification of the inductive AM condition for more \( \ell \)-blocks.

**Corollary 6.1.** Let \( \ell \) be a prime, \( q \) a prime power and \( \epsilon \in \{ \pm 1 \} \) with \( \ell \not| 3q(q - \epsilon) \), let \( b \) be an \( \ell \)-block of \( G := \text{SL}_n(eq) \). If for every \( J \) with \( G \leq J \leq \text{GL}_n(eq) \) every \( c \in \text{Bl}(J \mid b) \) is \( \text{GL}(eq) \)-stable then the inductive AM condition from Definition 7.2 of [Spä13a] holds for \( b \).

In [CS15, §5] the condition on \( b \) is studied using the \( d \)-cuspidal pair \( (L, \zeta) \) associated with \( b \). For every \( J \) with \( G \leq J \leq \text{GL}_n(eq) \) every \( c \in \text{Bl}(J \mid b) \) is \( \text{GL}_n(eq) \)-stable if \( Z(L) \) is connected according to the considerations given there.
Proof. The criterion [CS15, Theorem 4.1] can also be applied to a $\widetilde{G}$-orbit of blocks of $\mathrm{SL}_n(eq)$. The assumptions 4.1(i)-(iv) are ensured as in the proof of 1.1(a). The remaining condition (v) is satisfied by assumption.

The equality in Corollary 5.7 is close to the Alperin-McKay conjecture for those blocks, just the block of the normalizer of the defect group is replaced by the induced block of the normalizer of the Sylow $\Phi_d$-torus. We use the result of 2.7 to relate the blocks of those two groups.

Proof of Theorem 1.3(a). Assume $G = \mathrm{SL}_n(eq)$ writes as $G^F$ as before. Let $B \in \mathrm{Bl}(\mathrm{SL}_n(eq))$, $N$ and $b \in \mathrm{Bl}(N)$ be corresponding to $B$ as in Corollary 5.7. Then $|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(b)|$ according to Corollary 5.7. Denote by $(L, \zeta)$ the $d$-cuspidal pair defining $B$ as in [CE99a, 4.1]. Note that $N = N_G(L)$.

Some defect group $D$ of $B$ is also a defect group of any covering block of $\mathrm{GL}_n(eq)$ and satisfies $C_G(D) \leq L^F$ by Theorem 5.1 and [CE99a, §5.3]. Note that $D$ is also a defect group of $b$. All characters of $L := L^F$ extend to their stabilizers in $N$ by Theorem 1.2. For $\xi \in \mathrm{Irr}(L)$ the group $N_\xi/L$ satisfies the McKay Conjecture for $\ell$ by Proposition 4.6. Now every block $b$ of $\widetilde{L} := (Z(\widetilde{G})L)^F$ contains at least one character of $\mathrm{Irr}(\widetilde{L} \mid \zeta)$ and those are $d$-cuspidals according to [CE94, proof of 1.10(i)]. The block $\tilde{b}$ is splendid Rickard equivalent to a unipotent block via a Jordan decomposition, see [BDR17]. This block has a $d$-cuspidal unipotent character. According to [CE04, 22.9] this block has central defect and is hence nilpotent. This implies that $\tilde{b}$ is nilpotent, since the fusion systems of the blocks are preserved by splendid Rickard equivalence according to [Pui99, 19.7] and nilpotency of a block can be read off from its fusion system. The $\ell$-group $D$ acts on the set $\mathrm{Bl}(\widetilde{L} \mid b)$ since $b$ is $D$-stable. Observe that $\ell \nmid |\mathrm{Bl}(\widetilde{L} \mid b)|$ since multiplication with characters of $\mathrm{Irr}(\widetilde{L}/L)$ defines an action on $\mathrm{Bl}(\widetilde{L} \mid b)$.

Accordingly there exists some block $\tilde{b} \in \mathrm{Bl}(\widetilde{L} \mid b)$ that is $D$-stable. Since $\mathrm{Bl}(\widetilde{L} \mid b)$ forms an $\mathrm{Irr}(\widetilde{L}/L)$-orbit, all blocks in $\mathrm{Bl}(\widetilde{L} \mid b)$ are $D$-stable. Then the block $\tilde{c} \in \mathrm{Bl}(\widetilde{L}D)$ covering $\tilde{b}$ is nilpotent as well by [Cab87, Theorem 2]. Let $\tilde{b} \in \mathrm{Bl}(LD \mid b)$ be covered by $\tilde{c}$. This block is inertial by [Pui11, Theorem 3.13]. Hence Theorem 2.7 can be applied and proves the statement. 

While the above proves that all $\ell$-blocks of $\mathrm{SL}_n(eq)$ satisfy the Alperin-McKay conjecture for primes $\ell$ with $\ell \nmid 3q(q - e)$, we finally deduce the consequences of our considerations towards the Alperin weight Conjecture.

Proof of Theorem 1.1(b). Let $B_0 \in \mathrm{Irr}(\mathrm{SL}_n(eq))$ with defect group $D$, and $(L, \zeta)$ and $N$ be defined as in Corollary 5.7. Let $c \in \mathrm{Bl}(N)$ be the block with $c^{\mathrm{SL}_n(eq)}$ corresponding to $B_0$ and $c_0 = \mathrm{bl}(\zeta) \in \mathrm{Bl}(L^F)$. By assumption $C_G(D)$ is $d$-split and therefore it coincides with $L$, since $L$ is the minimal $d$-split Levi containing $C_G^0(D)$ according to [CE99a, 4.4(iii)]. The defect group hence satisfies $D = Z(L)^F$. This implies $N = N_{G^F}(D)$.

The block $c_0 := \mathrm{bl}(\zeta)$ contains a character of $E_{\ell}^c(L^F)$ and the characters of $E_{\ell}^c(L^F)$ are trivial on $Z(L)_{\ell}$ according to [CE99a, 1.2(v)]. On the other hand $c_0$ as block with central defect has a unique character that is trivial on $D$ and that corresponds to a defect zero character of $L/D$. For $\mathbb{E}_0 := \mathrm{Irr}(c_0) \cap E_{\ell}^c(L^F)$ this implies $\mathbb{E}_0 = \mathrm{Irr}(c_0) \cap \mathrm{Irr}(L/D)$ and hence $\mathrm{Irr}(N \mid \mathbb{E}_0) = \mathrm{Irr}(c) \cap \mathrm{Irr}(N/D)$. As $c$ has normal defect,

$$|\mathrm{IBr}(c)| = |\mathrm{Irr}(c) \cap \mathrm{Irr}(N/D)| = |\mathrm{Irr}(N \mid \mathbb{E}_0)|$$
Let \( \tilde{G}^F = \text{GL}_n(\mathbb{F}_q) \) be defined as in as in 5.1, \( \tilde{L} := C_{\tilde{G}^F}(\mathbb{Z}(L)) \) and \( \tilde{N} := \text{N}_{\tilde{G}^F}(L) \). We denote by \( \tilde{B} \) the sum of blocks of \( \tilde{G}^F \) covering \( B_0 \) and by \( \tilde{b} \) the sum of corresponding blocks of \( \tilde{N} \). Denote by \( b_0 \) the sum of blocks of \( L \) covered by one of \( \tilde{b} \). The construction of \( \tilde{\Omega}_B : \text{Irr}_0(\tilde{B}) \rightarrow \text{Irr}_0(\tilde{b}) \) in 5.6 implies

\[
\tilde{\Omega}_B(\tilde{B}) = \text{Irr}(\tilde{N} | \tilde{B}'),
\]

where \( \tilde{B} := \text{Irr}_0(\tilde{B}) \cap \mathcal{E}_\mathfrak{p}(\tilde{G}^F) \) and \( \tilde{B}' := \text{Irr}(\tilde{b}_0) \cap \mathcal{E}_\mathfrak{p}(\tilde{L}) \).

Let \( B \) be the sum of blocks in \( \text{Bl}(G^F) \) covered by a block of \( \tilde{B} \) and \( b \) be the sum of blocks in \( \text{Bl}(N) \) covered by a blocks of \( \tilde{b} \). By the construction in Section 2 we obtain a bijection \( \Omega_B : \text{Irr}_0(B) \rightarrow \text{Irr}_0(b) \). For \( \mathbb{B} := \bigcup \chi \in \mathbb{B} \text{Irr}(\text{Res}_{\tilde{G}^F}(\chi)) \) and \( \mathbb{B}' := \bigcup \psi \in \mathbb{B}' \text{Irr}(\text{Res}_L(\psi)) \) Corollary 2.6 implies \( \Omega_B(\mathbb{B}) = \text{Irr}(b) \cap \text{Irr}(N/D) \).

Note that \( \mathbb{B} \subseteq \mathcal{E}_\mathfrak{p}(G^F) \). Since the \( \ell \)-modular decomposition matrix of \( B_0 \) is unitriangular with respect to the \( \text{Aut}(G^F)_{B}-\text{stable set}, \) see \([Gec93, \text{Theorem C and Proposition 2.6(iii)}]\), this implies the inductive blockwise Alperin weight condition, see \([KS16b, \text{Theorem 1.2}]\).

We conclude by proving the Alperin weight conjecture for blocks with abelian defect in our situation.

**Proof of 1.3(b).** Let \( B \) be the sum of a \( \tilde{G} \)-orbit in \( \text{Bl}(\text{SL}_n(\mathbb{F}_q)) \). Let \( G := \text{SL}_n(\mathbb{F}_q) \) and \( F : G \rightarrow G \) be a Frobenius endomorphism such that \( G^F = \text{SL}_n(\mathbb{F}_q) \). Let \( (\mathbb{L}, \zeta) \) be the \( d \)-cuspidal pair of a block \( B_0 \) from the blocks in \( B \), \( L := F^F \text{and} N := \text{N}_{G^F}(L) \). By Theorem 5.1, it follows that \( \text{N}_{G^F}(L) \geq \text{N}_{G^F}(D) \) for some defect group \( D \) of \( B_0 \) and hence by Brauer correspondence \([Nav98, \text{4.12}]\) the blocks in \( B \) correspond to a \( \tilde{N} \)-orbit in \( \text{Bl}(N) \) of the same length. We denote the sum of those blocks by \( b \).

Let \( \Omega_B : \text{Irr}_0(B) \rightarrow \text{Irr}_0(b) \) be a bijection that is derived from the bijection of Theorem 5.6 as in Section 2. The arguments in the proof of Theorem 1.1(b) imply that

\[
\Omega_B(\text{Irr}_0(B) \cap \mathcal{E}_\mathfrak{p}(G^F)) = \text{Irr}_0(b) \cap \text{Irr}(N \mid \mathcal{E}_\mathfrak{p}(L^F)).
\]

Let \( c_0 := \text{bl}(\zeta) \in \text{Bl}(L^F) \) and denote by \( c \) the block of \( N \) covering \( c_0 \). Let \( r \) be the length of the \( \tilde{G} \)-orbit of \( G \) containing \( B_0 \). Recall that \( \mathcal{E}_\mathfrak{p}(G^F) \) is a basic set of \( G^F \) by \([Gec93, \text{Theorem C and Proposition 2.6(iii)}]\). Then \( b \) has \( r \) summands, one is \( c \), and

\[
|\text{IBr}(B_0)| = \frac{|\text{Irr}_0(B) \cap \mathcal{E}_\mathfrak{p}(G^F)|}{r} = \frac{|\text{Irr}_0(b) \cap \text{Irr}(N \mid \mathcal{E}_\mathfrak{p}(L^F))|}{r} = |\text{Irr}_0(c) \cap \text{Irr}(N \mid \mathbb{B}')|,
\]

where \( \mathbb{B}' := \text{Irr}(c_0) \cap \mathcal{E}_\mathfrak{p}(L^F) \). Then by \([BDR17]\) \( c_0 \) is basic Morita equivalent to a block \( c_0' \) of \( C_{L^*}(s)^F \) above a unipotent block \( c_0'' \in \text{Bl}(C_{L^*}(s)^F) \) with central defect. The unipotent character of \( c_0'' \) is trivial on the central defect, hence restricts to an irreducible Brauer character and hence forms a basic set. By Clifford theory the unipotent characters of \( c_0' \) form a basic set with a diagonal \( \ell \)-modular decomposition matrix as well, since \( C_{L^*}(s)^F / C_{L^*}(s)^F \) is an \( \ell \)-group. The Morita equivalence from \([BDR17]\) maps the unipotent characters to \( \mathbb{B}' := \text{Irr}(c_0) \cap \mathcal{E}_\mathfrak{p}(L^F) \) and hence \( \text{Irr}(c_0) \cap \mathcal{E}_\mathfrak{p}(L^F) \) is a \( N_{\text{sp}} \)-stable basic set with a unitriangular \( \ell \)-decomposition matrix. Recall that by the proof of Theorem 1.3(a) the assumptions of Proposition 2.7 are satisfied. Then the assumption 2.8(b) applies and we obtain

\[
|\text{IBr}(c)| = |\text{Irr}(N \mid \mathbb{B}')|.
\]
Since $c$ satisfies the Alperin weight Conjecture by Proposition 2.8(a), this implies the Alperin weight conjecture for $B_0$ and all blocks of $B$. □

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