A SPACE OF WEIGHT ONE MODULAR FORMS ATTACHED TO TOTALLY REAL CUBIC NUMBER FIELDS

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Abstract. Let $K$ be a totally real cubic number field with fundamental discriminant. In this note we construct a weight one modular form $f_K$ with level and nebentypus depending only on the discriminant of $K$. We show that, up to isomorphism class, the assignment $K \mapsto f_K$ is injective. Furthermore, if $d$ is a positive fundamental discriminant we show that \{$f_K : K \in \mathcal{C}_d$\} is a linearly independent subset of $\mathcal{M}_1(\Gamma_0(Nd), \epsilon_d)$, where $\mathcal{C}_d$ denotes the set of isomorphism classes of cubic number fields with discriminant $d$.

1. Introduction

An interesting and useful problem in number theory is the evaluation of the dimension of the space of modular forms. The case of modular forms of weight larger than one is a well understood problem, see for instance [D-I]. In contrast the weight one case is, as Duke calls it (see [Duke]), mysterious. In the present work we described what we believe is an interesting subspace of weight one modular forms. We can construct an explicit basis for this subspace, hence obtaining some bounds on the space of weight one modular forms. We should remark that the bounds we get are no better than previously known results, however the subspace we have found seems quite interesting, and as far as we know it has not been studied yet.

To be more explicit the goal of this paper is to exhibit a canonical subspace of a space of weight one modular forms that depends only in a set of conjugacy classes of cubic fields of a fixed discriminant. Let $\mathcal{M}_k(\Gamma_0(N), \epsilon)$ be the space of weight $k$ modular forms of level $N$ and nebentypus $\epsilon$, and let $\left(\frac{\cdot}{d}\right)$ be Kronecker symbol. If $d$ is a negative fundamental discriminant\footnote{recall that an integer $d$ is called fundamental whenever $d$ is the discriminant of a quadratic field.} we construct a canonical subspace $V_d$ of $\mathcal{M}_1\left(\Gamma_0(|d|), \left(\frac{d}{\cdot}\right)\right)$ that depends only on the unramified $\mathbb{Z}/3\mathbb{Z}$-extensions of $\mathbb{Q}(\sqrt{-3d})$. By class field theory such extensions are determined by cubic fields of fundamental discriminant, and our previous studies of the integral trace form over such fields (see [Man]) allowed us to construct an explicit basis for the space $V_d$.

2. Main construction:

Let $K$ be a number field and let $O_K^0 \subset K$ be the set of integral elements with zero trace. The map
\[
t^0_K : O_K^0 \xrightarrow{x} \mathbb{Z} \xrightarrow{tr_{K/Q}(x^2)}
\]
defines a quadratic form with corresponding bilinear form

\[ T_K(x, y) = \text{tr}_{K/Q}(xy) \bigg|_{O_K^0}. \]

**Proposition 2.1.** Let \( K \) be a cubic field with fundamental discriminant \( d \). Then

\[ t_K := \frac{t_K^0}{2 \gcd(3, d)} \]

is a primitive, integral, binary quadratic form of discriminant \(-\frac{3d}{\gcd(3, d)^2}\).

**Proof.** If \( 3 \nmid d \) this is precisely [Man, Corollary 5.4]. Otherwise, the result follows from [Man, Theorem 6.4]. \( \square \)

**Definition 2.2.** Let \( K \) be a totally real cubic field with fundamental discriminant \( d \). Let

\[ f_K(z) := \sum_{x \in O_K^0} q^{t_K}(x^2) \]

where \( q = e^{2\pi i z} \), and \( z \) lies in the upper half plane.

We have associated to \( K \) a theta series given by the quadratic form \( q_K \). The following classic result of Schoeneberg [Sch] gives us some information about \( f_K \).

**Theorem 2.3** (Schoenberg). Let \( Q \) be a positive definite, integral quadratic form of dimension \( 2k \), and discriminant \( \Delta \). Let

\[ \theta(Q, z) := \sum_{x \in \mathbb{Z}^{2k}} q^Q(x) \]

be the theta series associated to \( Q \). Then,

\[ \theta(Q, z) \in \mathcal{M}_k \left( \Gamma_0(|\Delta|), \left( \frac{\Delta}{\cdot} \right) \right). \]

A proof of Schoenberg’s Theorem can be found in [Ogg, Theorem 20-20].

**Definition 2.4.** Let \( d \) be a non-zero integer. The 3-reflection \( d_3 \) of \( d \) is the integer defined by \( d_3 := -\frac{3d}{\gcd(3, d)^2} \).

**Remark 2.5.** We have chosen this terminology in light of the Scholz reflection principle. Notice that if \( d \) is a fundamental discriminant so is \( d_3 \). In fact, the two statements are equivalent since \((d_3)_3 = d\).

**Corollary 2.6.** Let \( K \) be a totally real cubic field with fundamental discriminant \( d \). Then,

\[ f_K \in \mathcal{M}_1 \left( \Gamma_0(|d_3|), \left( \frac{d_3}{\cdot} \right) \right) \]

**Proof.** Since \( q_K \) is a binary form of discriminant \( d_3 \), the result is a particular case of Schoenberg’s Theorem. \( \square \)

Let \( d \) be a positive fundamental discriminant, and let us denote by \( C_d \) denotes the set of isomorphism classes of cubic number fields with discriminant \( d \). The following are the two main results in the paper. Intermediate steps used in the proofs of the Theorems can be found in section 3.
Theorem 2.7. The map
\[ \Theta : C_d \to M_1 \left( \Gamma_0(|d_3|), \left( \frac{d_3}{d} \right) \right) \]
\[ K \mapsto f_K \]
is injective.

Proof. Suppose that \( \Theta(K) = \Theta(L) \). Then, by Lemma 3.1 the forms \( t_K \) and \( t_L \) equivalent, hence the forms \( t_K^0 \) and \( t_L^0 \) are equivalent. The later equivalence implies that \( K \) and \( L \) are isomorphic (see \[ Man \] Theorem 6.5 ). □

Theorem 2.8. Let \( d \) be a positive fundamental discriminant. Then \( \Theta(C_d) \) is a linearly independent subset of \( M_1 \left( \Gamma_0(|d_3|), \left( \frac{d_3}{d} \right) \right) \).

Proof. We may assume that \( \Theta(C_d) \) is non-empty. Let \( K_1, \ldots, K_n \) be a set of representatives of \( C_d \), and let \( f_i := \Theta(K_i) \) for \( i = 1, \ldots, n \). Suppose there are complex numbers \( \lambda_1, \ldots, \lambda_n \) such that
\[ \lambda_1 f_1 + \ldots + \lambda_n f_n = 0. \]
Let \( p \) be a prime represented by \( t_{K_1} \), which exists by Corollary 3.5. Since \( \Theta \) is injective it follows from Corollary 3.3 that \( p \) is not represented by \( t_{K_i} \) for any \( i = 2, \ldots, n \). In particular, the coefficient of \( q^p \) in \( f_i \) is equal to zero for all \( i = 2, \ldots, n \). Since the order of the \( \lambda \)'s is irrelevant we deduce that \( \lambda_i = 0 \) for all \( i = 1, \ldots, n \). □

Let \( d \) be a negative fundamental discriminant. We have exhibited a natural subspace of
\[ M_1 \left( \Gamma_0(|d|), \left( \frac{d}{d} \right) \right) \],
that arises from totally real cubic fields of discriminant \( d_3 \). We called this space \( V_d := \text{Span}_C \Theta(C_{d_3}) \) the \( d \)-cubic space.

Corollary 2.9. Let \( d \) be as above and let \( \text{Cl}_{\mathbb{Q}(\sqrt{d_3})} \) be the ideal class group of \( \mathbb{Q}(\sqrt{d_3}) \). Then
\[ \frac{3^{r_3(d_3)} - 1}{2} \leq \dim_{\mathbb{C}} M_1 \left( \Gamma_0(|d|), \left( \frac{d}{d} \right) \right) \]
where \( r_3(d_3) = \dim_{\mathbb{F}_3}(\text{Cl}_{\mathbb{Q}(\sqrt{d_3})} \otimes_{\mathbb{Z}} \mathbb{F}_3) \).

Proof. First notice that \( d = (d_3)_3 \). Thus by Theorem 2.8 \( \dim_{\mathbb{C}}(V_d) = \# \Theta(C_{d_3}) \), and by Theorem 2.7 \( \dim_{\mathbb{C}}(V_d) = \#C_{d_3} \). On the other hand \( \#C_{d_3} \), the number of isomorphism classes of cubic fields of discriminant \( d_3 \), is equal to \( \frac{3^{r_3(d_3)} - 1}{2} \) (See III). □

3. Auxiliary results

The following are standard results we needed for our Theorems. We include proofs for them here to make the present paper self-contained.

Lemma 3.1. Let \( Q \) and \( Q_1 \) be two primitive, positive definite, integral binary quadratic forms of the same discriminant. The first two non-zero terms of \( \theta(Q, z) \) and \( \theta(Q_1, z) \) coincide if and only if \( Q \) and \( Q_1 \) are equivalent.
Proof. Recall that \( Q \) is \( \text{GL}_2(\mathbb{Z}) \) equivalent to a unique form \( ax^2 + bxy + cy^2 \) of the same discriminant such that \( 0 \leq b \leq a \leq c \), i.e., a reduced form. The values \( a, c \) and \( a - b + c \) are the successive minima of the form \( Q \). It is not difficult to see that if the first two non-zero terms of \( \theta(Q, z) \) and \( \theta(Q_1, z) \) coincide then \( Q \) and \( Q_1 \) have the same successive minima, and in particular they are equivalent to the same reduced form so they are equivalent. \( \square \)

**Lemma 3.2.** Let \( Q \) be integral binary quadratic form, and suppose \( p \) is a prime represented by \( Q \). Then \( Q \) is equivalent to a form \( px^2 + bxy + cy^2 \) such that \( 0 \leq b \leq p \).

**Proof.** Let \( x_0 \) and \( y_0 \) be integers such that \( Q(x_0, y_0) = p \). Since \( \gcd(x_0, y_0) = 1 \) (any factor of \( \gcd(x_0, y_0) \) is a square factor of \( p \)) there are \( r, s \) integers such that \( x_0r - y_0s = 1 \). Let

\[
S = \begin{bmatrix} x_0 & s \\ y_0 & r \end{bmatrix} \in \text{GL}_2(\mathbb{Z}).
\]

By acting on \( Q \) with \( S \) we obtain an equivalent form such that the coefficient of \( x^2 \) is \( p \). Let

\[
T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \text{GL}_2(\mathbb{Z}).
\]

Since \( T^n \) takes the form \( px^2 + bxy + cy^2 \) to a form \( px^2 + (b + 2pn)x_0 + c'y^2 \) we can pick \( n \in \mathbb{Z} \) such that \( Q \) is equivalent to a form \( px^2 + bxy + cy^2 \) with \( -p \leq b < p \). By using

\[
M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \text{GL}_2(\mathbb{Z})
\]

we can make sure that \( 0 \leq b \leq p \). \( \square \)

**Corollary 3.3.** Let \( Q \) and \( Q_1 \) be two integral binary quadratic forms of the same discriminant. If there exists a prime \( p \) which is represented by \( Q \) and \( Q_1 \), then the forms are equivalent.

**Proof.** We may assume that \( Q(x, y) = px^2 + bxy + cy^2 \), and that \( Q_1(x, y) = px^2 + b_1xy + c_1y^2 \) where \( 0 \leq b, b_1 \leq p \). Since \( b^2 - 4pc = b_1^2 - 4pc_1 \) we have that \( 2p | (b + eb_1) \) where \( e \in \{1, -1\} \). If \( b + eb_1 \neq 0 \) then \( 2p \leq |b + eb_1| \leq b + b_1 \leq p + p = 2p \), hence \( b = p = b_1 \) and \( c = c_1 \). If \( b + eb_1 = 0 \) then \( b = \pm b_1 \), and since they are both non-negative \( b = b_1 \) and again \( c = c_1 \). In any case \( Q = Q_1 \). \( \square \)

**Proposition 3.4.** Let \( K \) be a Galois number field and \( I \subseteq O_K \) a non-zero ideal. Then the class of \( I \), in the ideal class group of \( K \), contains infinitely many ideals of prime norm.

**Proof.** Let \( [I] \) be the class of \( I \) in the ideal class group of and let

\[
\zeta(s, [I]) := \sum_{\substack{J \in [I] \\ J \subseteq O_K}} \frac{1}{||J||^s}
\]

be the partial zeta function of \( [I] \) defined for \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \). Recall that \( \zeta(s, [I]) \) can be analytically continued to the region \( \text{Re}(s) > 1 - \frac{1}{[K:Q]} \) except for the simple pole at \( s = 1 \), which leads to the analytic class number formula. In an analog way to Euler’s proof of infinitude of rational primes, actually more closely to Dirichlet’s theorem on arithmetic progressions, it is possible to deduce that

\[
F_1(s) := \sum_{\substack{P \in [I] \\ P \subseteq \text{Integral primes}}} \frac{1}{||P||^s}
\]
has a pole at \( s = 1 \) (See [La] VIII §4 for details). If there were only finitely many ideals of prime norm in \([I]\), say \( N \) of them, then
\[
|F_I(1)| \leq \frac{N}{2} + [K : \mathbb{Q}] \sum_{p \in \mathbb{Z} \text{ prime}} \frac{1}{p^s} = O(1)
\]
contradicting that \( F_I(s) \) has a pole at 1.

\[\square\]

Corollary 3.5. Let \( Q \) be a primitive, positive definite, integral binary quadratic form of discriminant \( \Delta \). Then, there are primes represented by \( Q \).

Proof. Let \( Cl(\Delta) \) be the set of \( SL_2(\mathbb{Z}) \) classes of primitive, positive definite, integral binary quadratic forms of discriminant \( \Delta \), and let \( Cl(\mathbb{Q}(\sqrt{-\Delta})) \) be ideal class group of \( \mathbb{Q}(\sqrt{-\Delta}) \). Recall that there is a bijection \( \Gamma_{\Delta} : Cl(\Delta) \rightarrow Cl(\mathbb{Q}(\sqrt{-\Delta})) \), which in fact is a group isomorphism for the group structure on \( Cl(\Delta) \) given by Gauss’s composition (See [Bu] for details). From the definition of \( \Gamma_{\Delta} \), and since conjugation acts as taking inverses on \( Cl(\mathbb{Q}(\sqrt{-\Delta})) \), it follows that a prime \( p \) is represented by \( Q \) if and only if \( p \) is the norm of a prime ideal \( P \in \Gamma_{\Delta}([Q]) \). The result follows by the above proposition with \( K = \mathbb{Q}(\sqrt{-\Delta}) \) and \( [I] = \Gamma_{\Delta}([Q]) \). \[\square\]

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