ON A CLASSICAL CORRESPONDENCE  
BETWEEN K3 SURFACES  

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To 80-th Birthday of I.R. Shafarevich

ABSTRACT. Let $X$ be a K3 surface which is intersection of three (i.e. a net $P^2$) of quadrics in $P^5$. The curve of degenerate quadrics has degree 6 and defines a natural double covering $Y$ of $P^2$ ramified in this curve which is again a K3. This is a classical example of a correspondence between K3 surfaces which is related with moduli of sheaves on K3’s studied by Mukai. When general (for fixed Picard lattices) $X$ and $Y$ are isomorphic? We give necessary and sufficient conditions in terms of Picard lattices of $X$ and $Y$.

E.g. for Picard number 2 the Picard lattice of $X$ and $Y$ is defined by its determinant $-d$ where $d > 0$, $d \equiv 1 \mod 8$, and one of equations $a^2 - db^2 = 8$ or $a^2 - db^2 = -8$ has an integral solution $(a, b)$. Clearly, the set of these $d$ is infinite: $d \in \{(a^2 \pm 8)/b^2\}$ where $a$ and $b$ are odd integers. This gives all possible divisorial conditions on the 19-dimensional moduli of intersections of three quadrics $X$ in $P^5$ which imply $Y \cong X$. One of them, when $X$ has a line is classical and corresponds to $d = 17$.

Similar considerations can be applied to a realization of an isomorphism $(T(X) \otimes \mathbb{Q}, H^{2,0}(X)) \cong (T(Y) \otimes \mathbb{Q}, H^{2,0}(Y))$ of transcendental periods over $\mathbb{Q}$ of two K3 surfaces $X$ and $Y$ by a fixed sequence of types of Mukai vectors.

0. Introduction

In this paper we study a classical correspondence between algebraic K3 surfaces over $\mathbb{C}$.

Let a K3 surface $X$ be an intersection of three quadrics in $P^5$ (more generally, $X$ is a K3 surface with a primitive polarization $H$ of degree 8). The three quadrics define the projective plane $P^2$ (the net) of quadrics. Let $C \subset P^2$ be the curve of degenerate quadrics. The curve $C$ has degree 6 and defines another K3 surface $Y$ which is the minimal resolution of singularities of the double covering of $P^2$ ramified in $C$. It has the natural linear system $|h|$ with $h^2 = 2$ which is preimage of lines in $P^2$. This is a classical and a very beautiful example of a correspondence between K3 surfaces. It is defined by a 2-dimensional algebraic cycle $Z \subset X \times Y$.

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This example is related with the moduli of sheaves on K3 surfaces studied by Mukai [5], [6]. It is well-known that the K3 surface \( Y \) is the moduli of sheaves \( \mathcal{E} \) on \( X \) with the rank \( r = 2 \), the first Chern class \( c_1(\mathcal{E}) = H \) and the Euler characteristic \( \chi = \chi(\mathcal{E}) = 4 \). We apply this construction to study the following questions.

**Question 1.** When \( Y \) is isomorphic to \( X \)?

We want to answer this question in terms of the Picard lattices \( N(X) \) and \( N(Y) \) of \( X \) and \( Y \). Then our question is as follows:

**Question 2.** Assume that \( N \) is a hyperbolic lattice, \( \tilde{H} \in N \) a primitive element with square 8. What are conditions on \( N \) and \( \tilde{H} \) such that for any K3 surface \( X \) with the Picard lattice \( N(X) \) and a primitive polarization \( H \in N(X) \) of degree 8 the corresponding K3 surface \( Y \) is isomorphic to \( X \) if the the pairs of lattices \( (N(X), H) \) and \( (N, \tilde{H}) \) are isomorphic as abstract lattices with fixed elements?

In other words, what are conditions on \( (N(X), H) \) as an abstract lattice with a primitive vector \( H \) with \( H^2 = 8 \) which are sufficient for \( Y \) to be isomorphic to \( X \) and are necessary if \( X \) is a general K3 surfaces with the Picard lattice \( N(X) \)?

We give an answer to the questions 2 in Theorems 2.2.3 and 2.2.4 and also Propositions 2.2.1 and 2.2.2. In particular, if the Picard number \( \rho(X) = \text{rk} N(X) \geq 12 \), the result is very simple: \( X \cong Y \), if and only if there exists \( x \in N(X) \) such that \( x \cdot H \equiv 1 \mod 2 \). This follows from results of Mukai [6] and also [7], [8].

K3 surfaces \( X \) and \( Y \) with \( \rho(X) = \rho(Y) = 2 \) are especially interesting. Really, it is well-known that the moduli space of K3 surfaces \( X \), which are intersections of three quadrics, is 19-dimensional. If \( X \) is general, i.e. \( \rho(X) = 1 \), then the surface \( Y \) cannot be isomorphic to \( X \) because \( N(X) = \mathbb{Z}H \) where \( H^2 = 8 \) and \( N(X) \) does not have elements with square 2 which is necessary if \( Y \cong X \). Thus, if \( Y \cong X \), then \( \rho(X) \geq 2 \), and \( X \) belongs to a codimension 1 submoduli space of K3 surfaces which is a divisor in the 19-dimensional moduli space of intersections of three quadrics in \( \mathbb{P}^5 \) (up to codimension 2). To describe connected components of this divisor, it is equivalent to describe Picard lattices \( N(X) \cong N(Y) \) of the surfaces \( X \cong Y \) above with \( \rho(X) = \rho(Y) = 2 \) such that general K3 surfaces \( X \) and \( Y \) with these Picard lattices have \( X \cong Y \). We show that the Picard lattice \( N(X) \cong N(Y) \) of these \( X \cong Y \) is defined by its determinant which is equal to \(-d\) where \( d > 0 \) and \( d \equiv 1 \mod 8 \). We show that the set \( \mathcal{D} \) of these numbers \( d \) is exactly the set of \( d \in \mathbb{N} \) such that \( d \equiv 1 \mod 8 \) and one of the equations

\[
a^2 - db^2 = 8
\]

or

\[
a^2 - db^2 = -8
\]

has an integral solution. It is easy to see that solutions \((a, b)\) of these equations are odd. It follows that \( \mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_- \) where

\[
\mathcal{D}_+ = \left\{ \frac{a^2 - 8}{b^2} \in \mathbb{N} \mid a, b \in \mathbb{N} \text{ are odd} \right\}
\]
and
\[ D_- = \left\{ \frac{a^2 + 8}{b^2} \in \mathbb{N} \mid a, b \in \mathbb{N} \text{ are odd} \right\}. \tag{0.4} \]

Both sets \( D_+ \) and \( D_- \) are infinite. The set \( D_+ \) contains the infinite sequence \( a^2 - 8 \) where \( a \in \mathbb{N} \) is odd. The set \( D_- \) contains the infinite sequence \( a^2 + 8 \) where \( a \in \mathbb{N} \) is odd. Similarly, we show that the sets \( D_- \cap D_+ \) and \( D_- \cap D_+ \) are infinite (see considerations after Theorem 3.1.7). In contrary, we don’t know if the set \( D_- \cap D_- \) is infinite. In Sect. 3.2, we give an algorithm and a program for calculation of \( D \) and describe geometry of the surfaces \( Y \cong X \) for \( d \in D \). See Theorems 3.2.1 — 3.2.5 and also results of calculations (for small \( d \)) after Theorem 3.2.5. Calculations using this algorithm give the list of first numbers from \( D_+ \): 1, 9, 17, 33, 41, 57, 73, 89, 97, 113, ..., 2009. See Theorem 3.2.5.

The set \( D \) labels connected components of the divisor, where \( Y \cong X \), in 19-dimensional moduli of intersections of three quadrics \( X \). Each \( d \in D \), gives a connected 18-dimensional moduli space of K3 surfaces with the Picard lattice \( N(X) = N(Y) \) of rank 2 and determinant \( -d \) where \( X \cong Y \). E.g. it is well-known that \( Y \cong X \) if \( X \) has a line. This is a divisorial condition on moduli of \( X \). This component is labeled by \( d = 17 \in D \). Thus, our results show that: “There exists an infinite number of different divisorial conditions on moduli of intersections of three quadrics \( X \in \mathbb{P}^5 \) such that each of them implies \( Y \cong X \). They are labeled by elements of the infinite set \( D \) which was described above. The case \( d = 17 \) corresponds to the classical example above of \( X \) containing a line.”

We mention that solutions \((a, b)\) of the equations (0.1) and (0.2) can be interpreted as elements of Picard lattices of \( X \) and \( Y \). E.g. for general \( X \) and \( Y \) with \( \rho(X) = \rho(Y) = 2 \), we have \( Y \cong X \) if and only if \( \det N(X) \) is odd and there exists \( h_1 \in N(X) \) such that \( (h_1)^2 = \pm 4 \) (for one of signs) and \( h_1 \cdot H \equiv 0 \mod 2 \). Similar condition in terms of \( Y \) is: \( \det N(X) \) is odd and there exists \( h_1 \in N(Y) \) such that \( h_1^2 = \pm 4 \) (for one of signs). It follows the following very simple sufficient condition when \( Y \cong X \) (see Corollary 3.1.9) which we want to formulate exactly.

**Theorem.** Let \( X \) be a K3 surface. Assume that \( X \) is an intersection of three quadrics (more generally, \( X \) has a primitive polarization \( H \) of degree 8). Let \( Y \) be a K3 surface which is the double covering of the net \( \mathbb{P}^2 \) of the quadrics defining \( X \) ramified along the curve of degenerate quadrics (more generally, \( Y \) is the moduli space of sheaves on \( X \) with the Mukai vector \( v = (2, H, 2) \)).

Then \( Y \cong X \), if there exists \( h_1 \in N(X) \) such that the primitive sublattice \([H, h_1]_{pr}\) in \( N(X) \) generated by \( H \) and \( h_1 \) has odd determinant, and

\[ (h_1)^2 = \pm 4 \text{ and } h_1 \cdot H \equiv 0 \mod 2. \]

These conditions are necessary if either \( \rho(X) = 1 \), or \( \rho(X) = 2 \) and \( X \) is a general K3 surface with its Picard lattice.

From our point of view, this statement is very interesting because elements \( h_1 \) of the Picard lattice \( N(X) \) with negative square \( (h_1)^2 = -4 \) get some geometrical
meaning. For K3 surfaces it is well-known only for elements $\delta$ of the Picard lattice $N(X)$ with negative square $\delta^2 = -2$: then $\delta$ or $-\delta$ is effective.

It seems, many known examples of $Y \cong X$ (e. g. see [3], [15]) follow from the theorem.

Similar methods can be developed for much more general situation. Let $X$ and $Y$ are K3 surfaces,

$$\phi : (T(X) \otimes \mathbb{Q}, H^{2,0}(X)) \cong (T(Y) \otimes \mathbb{Q}, H^{2,0}(Y)) \quad (0.5)$$

an isomorphism of their transcendental periods over $\mathbb{Q}$, and

$$(a_1, H_1, b_1)^\pm, \ldots, (a_k, H_k, b_k)^\pm, \quad (0.6)$$

a sequence of types of isotropic Mukai vectors of moduli of sheaves on K3 where $\pm$ shows the direction of the correspondence.

Similar methods and calculations can be applied to study the following question: When there exists a correspondence between $X$ and $Y$ which is given by the sequence $(0.6)$ of Mukai vectors, and which gives the isomorphism $(0.5)$ between their transcendental periods?

In [6] and [9] sufficient and necessary conditions on $(0.5)$ were given when there exists at least one such a sequence $(0.6)$ with coprime Mukai vectors $(a_i, H_i, b_i)$.

The fundamental tool to get the results above is the Global Torelli Theorem for K3 surfaces proved by I.I. Piatetskii-Shapiro and I.R. Shafarevich in [10]: Using results of Mukai [5], [6], we can calculate periods of $Y$ using periods of $X$; by the Global Torelli Theorem [10], we can find out if $Y$ is isomorphic to $X$.

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1. Preliminary notations and results about lattices and K3 surfaces

1.1. Some notations about lattices. We use notations and terminology from [8] about lattices, their discriminant groups and forms. A lattice $L$ is a non-degenerate integral symmetric bilinear form. I. e. $L$ is a free $\mathbb{Z}$-module equipped with a symmetric pairing $x \cdot y \in \mathbb{Z}$ for $x, y \in L$, and this pairing should be non-degenerate. We denote $x^2 = x \cdot x$. The signature of $L$ is the signature of the corresponding real form $L \otimes \mathbb{R}$. The lattice $L$ is called even if $x^2$ is even for any $x \in L$. Otherwise, $L$ is called odd. The determinant of $L$ is defined to be $\det L = \det(e_i \cdot e_j)$ where $\{e_i\}$ is some basis of $L$. The lattice $L$ is unimodular if $\det L = \pm 1$.

The dual lattice of $L$ is $L^* = \text{Hom}(L, \mathbb{Z}) \subset L \otimes \mathbb{Q}$. The discriminant group of $L$ is $A_L = L^*/L$. It has the order $|\det L|$. The group $A_L$ is equipped with the discriminant bilinear form $b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ and the discriminant quadratic form $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$ if $L$ is even. To get this forms, one should extend the form of $L$ to the form on the dual lattice $L^*$ with values in $\mathbb{Q}$. 
For \( x \in L \), we shall consider the invariant \( \gamma(x) \geq 0 \) where

\[
x \cdot L = \gamma(x)\mathbb{Z}.
\]  

(1.1.1)

Clearly, \( \gamma(x) \mid x^2 \) if \( x \neq 0 \).

We denote by \( L(k) \) the lattice obtained from a lattice \( L \) by multiplication of the form of \( L \) by \( k \in \mathbb{Q} \).

The orthogonal sum of lattices \( L_1 \) and \( L_2 \) is denoted by \( L_1 \oplus L_2 \).

For a symmetric integral matrix \( A \), we denote by \( \langle A \rangle \) a lattice which is given by the matrix \( A \) in some bases. We denote

\[
U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]  

(1.1.2)

Any even unimodular lattice of the signature \((1, 1)\) is isomorphic to \( U \).

An embedding \( L_1 \subset L_2 \) of lattices is called \emph{primitive} if \( L_2/L_1 \) has no torsion.

We denote by \( O(L) \), \( O(bL) \) and \( O(qL) \) the automorphism groups of the corresponding forms. Any \( \delta \in L \) with \( \delta^2 = -2 \) defines a reflection \( s_\delta \in O(L) \) which is given by the formula \( x \to x + (x \cdot \delta)\delta \), \( x \in L \). All such reflections generate the \( 2 \)-\emph{reflection group} \( W^{(-2)}(L) \subset O(L) \).

1.2. Some notations about K3 surfaces. Here we remind some basic notions and results about K3 surfaces, e. g. see [10], [11], [12]. A K3 surface \( S \) is a projective algebraic surface over \( \mathbb{C} \) such that its canonical class \( K_S \) is zero and the irregularity \( q_S = 0 \). We denote by \( N(S) \) the \emph{Picard lattice} of \( S \) which is a hyperbolic lattice with the intersection pairing \( x \cdot y \) for \( x, y \in N(S) \). Since the canonical class \( K_S = 0 \), the space \( H^{2,0}(S) \) of 2-dimensional holomorphic differential forms on \( S \) has dimension one over \( \mathbb{C} \), and

\[
N(S) = \{ x \in H^2(S, \mathbb{Z}) \mid x \cdot H^{2,0}(S) = 0 \}
\]  

where \( H^2(S, \mathbb{Z}) \) with the intersection pairing is a 22-dimensional even unimodular lattice of signature \((3, 19)\). The orthogonal lattice \( T(S) \) to \( N(S) \) in \( H^2(S, \mathbb{Z}) \) is called the \emph{transcendental lattice} of \( S \). We have \( H^{2,0}(S) \subset T(S) \otimes \mathbb{C} \). The pair \( (T(S), H^{2,0}(S)) \) is called the \emph{transcendental periods of} \( S \). The \emph{Picard number} of \( S \) is \( \rho(S) = \text{rk} \ N(S) \). An non-zero element \( x \in N(S) \otimes \mathbb{R} \) is called \emph{nef} if \( x \neq 0 \) and \( x \cdot C \geq 0 \) for any effective curve \( C \subset S \). It is known that an element \( x \in N(S) \) is ample if \( x^2 > 0 \), \( x \) is \emph{nef}, and the orthogonal complement \( x^\perp \) to \( x \) in \( N(S) \) has no elements with square \(-2\). For any element \( x \in N(S) \) with \( x^2 \geq 0 \), there exists a reflection \( w \in W^{(-2)}(N(S)) \) such that the element \( \pm w(x) \) is nef; it then is ample, if \( x^2 > 0 \) and \( x^\perp \) had no elements with square \(-2\) in \( N(S) \).

We denote by \( V^+(S) \) the light cone of \( S \), which is the half-cone of

\[
V(S) = \{ x \in N(S) \otimes \mathbb{R} \mid x^2 > 0 \}
\]  

(1.2.2)
containing a polarization of $X$. In particular, all nef elements $x$ of $S$ belong to $\overline{V^+(S)}$: one has $x \cdot V^+(S) > 0$ for them.

The reflection group $W^{(-2)}(N(S))$ acts in $V^+(S)$ discretely, and its fundamental chamber is the closure $\overline{\mathcal{K}(S)}$ of the Kähler cone $\mathcal{K}(S)$ of $S$. It is the same as the set of all nef elements of $S$. Its faces are orthogonal to the set $\text{Exc}(S)$ of all exceptional curves $r$ on $S$ which are non-singular rational curves $r$ on $S$ with $r^2 = -2$. Thus, we have

$$\overline{\mathcal{K}(S)} = \{0 \neq x \in V^+(S) \mid x \cdot \text{Exc}(S) \geq 0\}.$$  \hspace{1cm} (1.2.3)

1.3. K3 surfaces with polarizations of degree 8 and 2. The following results are well-known (see Mayer [4], Saint-Donat [11] and Shokurov [13]).

Let $X$ be a K3 surface with a primitive polarization $H \in N(X)$ of degree $H^2 = 8$. Here primitive means that the sublattice $\mathbb{Z}H \subset N(X)$ is primitive.

**Proposition 1.3.1.** The linear system $|H|$ has dimension 5, and there are the following and the only following cases for the linear system $|H|$:  

(a) $|H \cdot E| > 2$ for any elliptic curve $E$ on $X$ (it has $E^2 = 0$). Then the linear system $|H|$ gives an embedding of $X$ to $\mathbb{P}^5$ as intersection of three quadrics.

(b) $|H \cdot E| \geq 2$ for any elliptic curve $E$ on $X$, and there exists an elliptic curve $E$ on $X$ such that $H \cdot E = 2$. Then the linear system $|H|$ is hyperelliptic and gives a double covering of a rational scroll. In this case, $H$ and $E$ generate a primitive sublattice of $N(X)$ which is isomorphic to $U(2)$.

(c) $|H \cdot E| \geq 1$ for any elliptic curve $E$ on $X$, and there exists an elliptic curve $E$ on $X$ such that $H \cdot E = 1$. Then the linear system $|H|$ has a fixed component $D$ which is a non-singular rational curve on $X$ (it has $D^2 = -2$), and $|H| = 5|E| + D$ where $|E|$ is an elliptic pencil on $X$ and $E \cdot D = 1$. In this case, $H$ and $E$ generate a primitive sublattice of $N(X)$ which is isomorphic to $U$.

Since $H^2 = 8$, we then have $\gamma(H) = 1, 2, 4$ or 8 (for $H \in N(X)$).

For the case (c), $\rho(X) \geq 2$ and $\gamma(H) = 1$.  

For the case (b), $\rho(X) \geq 2$ and $\gamma(H) = 1$ or 2. If $\rho(X) = 2$, then $\gamma(H) = 2$.  

For the case (a), $\rho(X) \geq 1$ and $\gamma(H) = 1, 2, 4$ or 8. If $\rho(X) = 1$, then $\gamma(H) = 8$.

A general K3 surface $X$ has $\rho(X) = 1$, then one has the case (a) and $X$ is an intersection of three quadrics.

Now let $Y$ be a K3 surface with a nef element $h \in N(Y)$ of degree $h^2 = 2$. Obviously, $h$ is primitive.

**Proposition 1.3.2.** The linear system $|h|$ has dimension 2, and there are the following and the only following cases:

(a) $|h \cdot E| \geq 2$ for any elliptic curve $E$ on $Y$. Then the linear system $|h|$ gives a double covering of $\mathbb{P}^2$ ramified along a curve of degree 6 with at most double singularities.

(b) $|h \cdot E| \geq 1$ for any elliptic curve $E$ on $Y$, and there exists an elliptic curve $E$ on $Y$ such that $h \cdot E = 1$. Then the linear system $|h|$ has a fixed component $D$ which is a non-singular rational curve (it has $D^2 = -2$), and $|h| = 2|E| + D$ where
$|E|$ is an elliptic pencil and $E \cdot D = 1$. In this case, $h$ and $E$ generate a primitive sublattice of $N(X)$ which is isomorphic to $U$.

Since $h^2 = 2$, we have $\gamma(h) = 1$ or 2.
For the case (b), $\rho(Y) \geq 2$ and $\gamma(h) = 1$.
For the case (a), $\rho(Y) \leq 1$ and $\gamma(h) = 1$ or 2. If $\rho(Y) = 1$, then $\gamma(h) = 2$.
A general K3 surface $Y$ has $\rho(Y) = 1$, then $\gamma(h) = 2$, and one has the case (a).

2. General results on the classical correspondence between K3 surfaces with primitive polarizations of degree 8 and 2 which gives isomorphic K3's

2.1. The correspondence. Let a K3 surface $X$ be an intersection of three quadrics in $\mathbb{P}^5$ (more generally, $X$ is a K3 surface with a primitive polarization $H$ of degree 8). See general results about intersections of quadrics in [14]. The three quadrics define the projective plane $\mathbb{P}^2$ (the net) of quadrics. Let $C \subset \mathbb{P}^2$ be the curve of degenerate quadrics. The curve $C$ has degree 6 and defines another K3 surface $Y$ which is the minimal resolution of singularities of the double covering of $\mathbb{P}^2$ ramified in $C$. It has the natural linear system $|h|$ with $h^2 = 2$ which is preimage of lines on $\mathbb{P}^2$. This is a classical and a very beautiful example of a correspondence between K3 surfaces. It is defined by a 2-dimensional algebraic cycle $Z \subset X \times Y$.

This example is related with the moduli of sheaves on K3 surfaces studied by Mukai [5], [6]. It is well-known that the K3 surface $Y$ is the moduli of sheaves $\mathcal{E}$ on $X$ with rank $r = 2$, first Chern class $c_1(\mathcal{E}) = H$ and Euler characteristic $\chi = \chi(\mathcal{E}) = 4$.

Let

$$H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$$

be the full cohomology lattice of $X$ with the Mukai product

$$(u, v) = -(u_0 \cdot v_2 + u_2 \cdot v_0) + u_1 \cdot v_1$$

for $u_0, v_0 \in H^0(X, \mathbb{Z})$, $u_1, v_1 \in H^2(X, \mathbb{Z})$, $u_2, v_2 \in H^4(X, \mathbb{Z})$. We naturally identify $H^0(X, \mathbb{Z})$ and $H^4(X, \mathbb{Z})$ with $\mathbb{Z}$. Then the Mukai product is

$$(u, v) = -(u_0 v_2 + u_2 v_0) + u_1 \cdot v_1.$$ (2.1.3)

The element

$$v = (2, H, 2) = (r, H, \chi - r) \in H^*(X, \mathbb{Z})$$

is called the Mukai vector of the net of quadrics or (more generally) of sheaves on $X$ for which $Y$ is the moduli space. It is isotropic, i.e. $v^2 = 0$. Mukai [5], [6] showed that one has the natural identification

$$H^2(Y, \mathbb{Z}) = (v^\perp / \mathbb{Z}v)$$

which also gives the isomorphism of the Hodge structures of $X$ and $Y$. The element $h = (-1, 0, 1) \in v^\perp$ has square $h^2 = 2$, $h \mod \mathbb{Z}v$ belongs to the Picard lattice $N(Y)$ of $Y$, and the linear system $|h|$ defines the structure of double plane $Y$ for the net of quadrics defining $X$.

We apply this construction to study the following questions.
Question 2.1.1. When Y is isomorphic to X?

We want to answer this question in terms of Picard lattices of X and Y. Then the exact formulation of our question is as follows:

Question 2.1.2. Assume that N is a hyperbolic lattice, \( \tilde{H} \in N \) a primitive element with square 8. What are conditions on \( N \) and \( \tilde{H} \) such that for any K3 surface \( X \) with the Picard lattice \( N(X) \) and a primitive polarization \( H \in N(X) \) of degree 8 the corresponding K3 surface \( Y \) is isomorphic to \( X \) if the the pairs of lattices \( (N(X), H) \) and \( (N, \tilde{H}) \) are isomorphic as abstract lattices with fixed elements?

In other words, what are conditions on \((N(X), H)\) as an abstract lattice with a primitive vector \( H \) with \( H^2 = 8 \) which are sufficient for \( Y \) to be isomorphic to \( X \) and are necessary if \( X \) is a general K3 surface with the Picard lattice \( N(X) \)?

We formulate the results below.

2.2. Formulation of general results. Below \( X \) is a K3 surface with a primitive polarization \( H \) of degree \( H^2 = 8 \), and \( Y \) the corresponding K3 surface \( Y \) with the nef element \( h \) of degree \( h^2 = 2 \) defined in Sect. 2.1.

The following statement follows from the Mukai identification (2.1.5) and results from [8].

Proposition 2.2.1. If \( Y \) is isomorphic to \( X \), then the invariant \( \gamma(H) \) for \( H \) in \( N(X) \) (see (1.1.1)) is equal to 1.

Assume that \( \gamma(H) = 1 \) for \( H \) in \( N(X) \). Then the Mukai identification (2.1.5) canonically identifies the transcendental periods \( (T(X), H^{2,0}(X)) \) and \( (T(Y), H^{2,0}(Y)) \). It follows that the Picard lattices \( N(Y) \) and \( N(X) \) have the same genus. In particular, \( N(Y) \) is isomorphic to \( N(X) \) if the genus of \( N(X) \) contains only one class. If the genus of \( N(X) \) contains only one class, then \( Y \) is isomorphic to \( X \), if additionally the canonical homomorphism \( O(N(X)) \to O(q_{N(X)}) \) is epimorphic. Both these conditions are valid (in particular, \( Y \cong X \)), if either \( \rho(X) \geq 12 \) or \( N(X) \) contains a copy of \( U \) (see (1.1.2)).

From now on we can assume that \( \gamma(H) = 1 \) in \( N(X) \) since only for this case we may have \( Y \cong X \).

Calculations below are valid for an arbitrary K3 surface \( X \) and a primitive vector \( H \in N(X) \) with \( H^2 = 8 \) and \( \gamma(H) = 1 \). Let \( K(H) = H^+_{N(X)} \) be the orthogonal complement to \( H \) in \( N(X) \). Let \( H^* = H/8 \). Then any element \( x \in N(X) \) can be written as

\[ x = aH^* + k^* \]  \hspace{1cm} (2.2.1)

where \( a \in \mathbb{Z} \) and \( k^* \in K(H)^* \), because \( \mathbb{Z}H \oplus K(H) \subset N(X) \subset N(X)^* \subset \mathbb{Z}H^* \oplus K(H)^* \). Since \( \gamma(H) = 1 \), the map \( aH^* + [H] \mapsto k^* + K(H) \) gives an isomorphism of the groups \( \mathbb{Z}/8 \cong [H^*]/[H] \cong [u^* + K(H)]/K(H) \) where \( u^* + K(H) \) has order 8 in \( A_{K(H)} = K(H)^*/K(H) \). It follows,

\[ N(X) = [\mathbb{Z}H, K(H), H^* + u^*]. \]  \hspace{1cm} (2.2.2)
The element $u^*$ is defined canonically mod $K(H)$. Since $H^* + u^*$ belongs to the even lattice $N(X)$, it follows

$$(H^* + u^*)^2 = \frac{1}{8} + u^* \equiv 0 \mod 2. \quad (2.2.3)$$

Let $\overline{T} = H^* \mod [H] \in [H^*]/[H] \cong \mathbb{Z}/8$ and $\overline{k}^* = k^* \mod K(H) \in A_{K(H)} = K(H)^*/K(H)$. We then have

$$N(X)/[H, K(H)] = (\mathbb{Z}/8)(\overline{H}^* + \overline{u}^*) \subset (\mathbb{Z}/8)\overline{H}^* + K(H)^*/K(H). \quad (2.2.4)$$

Also $N(X)^* \subset ZH^* + K(H)^*$ since $H + K(H) \subset N(X)$, and for $a \in \mathbb{Z}$, $k^* \in K(H)^*$ we have $x = aH^* + k^* \in N(X)$ if and only if $(aH^* + k^*) \cdot (H^* + u^*) = \frac{a}{8} + k^* \cdot u^* \in \mathbb{Z}$.

It follows,

$$N(X)^* = \{aH^* + k^* \mid a \in \mathbb{Z}, k^* \in K(H)^*, a \equiv -8k^* \cdot u^* \mod 8\} \subset ZH^* + K(H)^*, \quad (2.2.5)$$

and

$$N(X)^*/[H, K(H)] = \{(-8\overline{k}^* \cdot \overline{u}^*)\overline{H}^* + \overline{k}^* \mid \overline{k}^* \in A_{K(H)}\} \subset (\mathbb{Z}/8)\overline{H}^* + A_{K(H)}. \quad (2.2.6)$$

We introduce the characteristic map of the polarization $H$

$$\kappa(H) : K(H)^* \to A_{K(H)}/(\mathbb{Z}/8)(u^* + K(H)) \to A_{N(X)} \quad (2.2.7)$$

where for $k^* \in K(H)^*$ we have

$$\kappa(H)(k^*) = (-8k^* \cdot u^*)H^* + k^* + N(X) \in A_{N(X)}. \quad (2.2.8)$$

It is epimorphic, its kernel is $(\mathbb{Z}/8)(u^* + K(H))$, and it gives the canonical isomorphism

$$\overline{\kappa(H)} : A_{K(H)}/(\mathbb{Z}/8)(u^* + K(H)) \cong A_{N(X)}. \quad (2.2.9)$$

For the corresponding discriminant forms we have

$$\kappa(k^*)^2 \mod 2 = (k^*)^2 + 8(k^* \cdot u^*)^2 \mod 2. \quad (2.2.10)$$

Similar results we have for the polarization $h$ of $Y$. We denote by $K(h)$ the orthogonal complement to $h$ in $N(Y)$. 
Proposition 2.2.2. If $Y$ is isomorphic to $X$, then the invariant $\gamma(H)$ of $H$ in $N(X)$ (see (1.1.1)) is equal to 1. If $\gamma(H) = 1$ for $H \in N(X)$, then $\gamma(h) = 1$ for $h \in N(Y)$, and $\det K(h) = \det K(H)/4$.

Assume that $\gamma(H) = 1$ for $H \in N(X)$. Then the Mukai identification (2.1.5) canonically identifies the transcendental periods $(T(X), H^{2,0}(X))$ and $(T(Y), H^{2,0}(Y))$. It follows that the Picard lattices $N(Y)$ and $N(X)$ have the same genus. In particular, $N(X)$ is isomorphic to $N(Y)$, if the genus of $N(Y)$ contains only one class. If the genus of $N(Y)$ contains only one class, then $Y$ is isomorphic to $X$, if additionally the canonical homomorphism $O(N(Y)) \to O(q_{N(Y)})$ is epimorphic. Both these conditions are valid, if either $\rho(Y) \geq 12$ or $N(Y)$ contains a copy of $U$ (see (1.1.2)).

Thus, $Y \cong X$, if $\gamma(H) = 1$ for $H \in N(X)$ and either $\rho(Y) \geq 12$ or $N(Y)$ contains a copy of $U$.

Calculations below are valid for an arbitrary K3 surface $Y$ and a primitive vector $h \in N(Y)$ with $h^2 = 2$ and $\gamma(h) = 1$. Let $K(h)$ be the orthogonal complement to $h$ in $N(Y)$. Let $h^* = h/2$. Then any element $x \in N(Y)$ can be written as

$$x = ah^* + k^* \quad (2.2.11)$$

where $a \in \mathbb{Z}$ and $k^* \in K(h)^*$. Since $\gamma(h) = 1$, the map $ah^* + [h] \mapsto k^* + K(h)$ gives the isomorphism of the groups $\mathbb{Z}/2 = [h^*]/[h] \cong [w^* + K(h)]/K(h)$ where $w^* + K(h)$ has order 2 in $K(h)^*/K(h) = A_{K(h)}$. It follows,

$$N(Y) = [Zh, K(h), h^* + w^*] \quad (2.2.12)$$

where $w^* + K(h)$ is an element of order 2 in $A_{K(h)}$. The element $w^*$ is defined canonically mod $K(h)$. The element $h^* + w^*$ belongs to the even lattice $N(Y)$, it follows

$$(h^* + w^*)^2 = \frac{1}{2} + w^*^2 \equiv 0 \mod 2. \quad (2.2.13)$$

Let $\overline{h^*} = h^* \mod [h] \in [h^*]/[h] \cong \mathbb{Z}/2$ and $\overline{k^*} = k^* \mod K(h) \in A_{K(h)}$. We then have

$$N(Y)/[h, K(h)] = (\mathbb{Z}/2)(\overline{h^*} + \overline{w^*}) \subset (\mathbb{Z}/2)\overline{h^*} + A_{K(h)} \quad (2.2.14)$$

Since $N(Y)^* \subset Zh^* + K(h)^*$, for $a \in \mathbb{Z}, k^* \in K(h)^*$ we have $x = ah^* + k^* \in N(Y)$ if and only if $(ah^* + k^*) \cdot (h^* + w^*) = \frac{a}{2} + k^* \cdot w^* \in \mathbb{Z}$. It follows,

$$N(Y)^* = \{ah^* + k^* \mid a \in \mathbb{Z}, k^* \in K(h)^*, a \equiv -2k^* \cdot w^* \mod 2\} \subset Zh^* + K(h)^* \quad (2.2.15)$$

and

$$N(Y)^*/[h, K(h)] = \{(-2k^* \cdot w^*)\overline{h^*} + \overline{k^*} \mid \overline{k^*} \in K(h)^*/K(h)\} \subset (\mathbb{Z}/2)\overline{h^*} + A_{K(h)} \quad (2.2.16)$$
We introduce the characteristic map of the polarization $h$

$$\kappa(h) : K(h)^* \to A_K(h)/(\mathbb{Z}/2)(w^* + K(h)) \to A_N(Y)$$  \hspace{1cm} (2.2.17)

where for $k^* \in K(h)^*$ we have

$$\kappa(h)(k^*) = (-2k^* \cdot w^*)h^* + k^* + N(Y) \in A_N(Y).$$  \hspace{1cm} (2.2.18)

It is epimorphic, its kernel is $(\mathbb{Z}/2)(w^* + K(h))$, and it gives the canonical isomorphism

$$\overline{\kappa(h)} : A_K(h)/(\mathbb{Z}/2)(w^* + K(h)) = A_N(Y).$$  \hspace{1cm} (2.2.19)

For the corresponding discriminant forms we have

$$\kappa(k^*)^2 \mod 2 = (k^*)^2 + 2(k^* \cdot w^*)^2 \mod 2.$$  \hspace{1cm} (2.2.20)

Now we can formulate our main result:

**Theorem 2.2.3.** The surface $Y$ is isomorphic to $X$ if the following conditions (a), (b), (c) are valid:

(a) $\gamma(H) = 1$ for $H \in N(X)$;

(b) there exists $\tilde{h} \in N(X)$ with $\tilde{h}^2 = 2$, $\gamma(\tilde{h}) = 1$ and such that there exists an embedding $f : K(H) \to K(\tilde{h})$ of negative definite lattices such that $K(\tilde{h}) = [f(K(H)), 4f(u^*)], w^* + K(\tilde{h}) = 2f(u^*) + K(\tilde{h})$;

(c) the dual to $f$ embedding $f^* : K(\tilde{h})^* \to K(H)^*$ commutes (up to multiplication by $\pm 1$) with the characteristic maps $\kappa(H)$ and $\kappa(\tilde{h})$, i.e.

$$\kappa(\tilde{h})(k^*) = \pm \kappa(H)(f^*(k^*))$$  \hspace{1cm} (2.2.21)

for any $k^* \in K(\tilde{h})^*$.

The conditions (a), (b) and (c) are necessary if $\text{rk } N(X) \leq 19$ and $X$ is a general K3 surface with the Picard lattice $N(X)$ in the following sense: the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ is $\pm 1$. (Remind that, by Proposition 2.2.1, $Y \cong X$ if $\text{rk } N(X) = 20$.)

We can also formulate similar result using the surface $Y$.

**Theorem 2.2.4.** The surface $Y$ is isomorphic to $X$ if the following conditions (a), (b) and (c) are valid:

(a) $\gamma(H) = 1$ for $H \in N(X)$, then $\gamma(h) = 1$ for $h \in N(Y)$;

(b) there exists a primitive $\tilde{H} \in N(Y)$ with $\tilde{H}^2 = 8$, $\gamma(\tilde{H}) = 1$ and such that there exists an embedding $f : K(\tilde{H}) \to K(h)$ of negative definite lattices such that $K(h) = [f(K(\tilde{H})), 4f(u^*)], w^* + K(h) = 2f(u^*) + K(h)$;
(c) the dual to $f$ embedding $f^* : K(h)^* \to K(\tilde{H})^*$ commutes (up to multiplication by ±1) with the characteristic maps $\kappa(\tilde{H})$ and $\kappa(h)$, i. e.

$$\kappa(h)(k^*) = \pm \kappa(\tilde{H})(f^*(k^*)) \quad (2.2.22)$$

for any $k^* \in K(h)^*$.

The conditions (a), (b) and (c) are necessary if $\text{rk} \, N(Y) \leq 19$ and $Y$ is a general $K3$ surface with the Picard lattice $N(Y)$ in the following sense: the automorphism group of the transcendental periods $(T(Y), H^{2,0}(Y))$ is ±1. (Remind that, by Proposition 2.2.2, $Y \cong X$ if $\gamma(H) = 1$ and $\text{rk} \, N(Y) = 20$.)

2.3. Proofs. Let us denote by $e_1$ the canonical generator of $H^0(X, \mathbb{Z})$ and by $e_2$ the canonical generator of $H^4(X, \mathbb{Z})$. They generate the sublattice $U$ in $H^*(X, \mathbb{Z})$ with the Gram matrix $U$. Consider Mukai vector $v = (2e_1 + 2e_2 + H)$. We have

$$N(Y) = v_\perp \mathbb{Z} / \mathbb{Z}v. \quad (2.3.1)$$

Let us calculate $N(Y)$. Let $K(H) = (H)_\perp^{1 \mathbb{N}(X)}$. Then we have embedding of lattices of finite index

$$\mathbb{Z}H \oplus K(H) \subset N(X) \subset N(X)^* \subset \mathbb{Z}H^* \oplus K(H)^* \quad (2.3.2)$$

where $H^* = H/2$. We have the orthogonal decomposition up to finite index

$$U \oplus \mathbb{Z}H \oplus K(H) \subset U \oplus N(X) \subset U \oplus \mathbb{Z}H^* \oplus K(H)^*. \quad (2.3.3)$$

Let $s = x_1e_1 + x_2e_2 + yH^* + z^* \in v_\perp \mathbb{Z}\mathbb{N}(X)$, $z^* \in K(H)^*$. Then $-2x_1 - 2x_2 + y = 0$ since $s \in v_\perp$ and hence $(s, v) = 0$. Thus, $y = 2x_1 + 2x_2$ and

$$s = x_1e_1 + x_2e_2 + 2(x_1 + x_2)H^* + z^*. \quad (2.3.4)$$

Here $s \in U \oplus N(X)$ if and only if $x_1, x_2 \in \mathbb{Z}$ and $2(x_1 + x_2)H^* + z^* \in N(X)$. This orthogonal complement contains

$$[\mathbb{Z}v, K(H), \mathbb{Z}h] \quad (2.3.5)$$

where $h = -e_1 + e_2$, and this is a sublattice of finite index in $(v_\perp)_{U \oplus \mathbb{N}(X)}$. The generators $v$, generators of $K(H)$ and $h$ are free, and we can rewrite $s$ above using these generators with rational coefficients as follows:

$$s = -\frac{x_1 + x_2}{2}h + \frac{x_1 + x_2}{4}v + z^*, \quad (2.3.6)$$

where $2(x_1 + x_2)H^* + z^* \in N(X)$. Equivalently,

$$s = ah^* + b\frac{v}{4} + z^*, \quad (2.3.7)$$
where \( a, b \in \mathbb{Z} \), \( z^* \in K(H)^* \), \( a \equiv b \mod 2 \), and \( 2bH^* + z^* \in N(X) \).

Thus, we get the following cases:

Assume that \( \gamma(H) = 8 \). Then \( 2b \equiv 0 \mod 8 \), or \( b \equiv 0 \mod 4 \). Then \( z^* \in K(H) \), \( a \equiv b \equiv 0 \mod 2 \), \( s \in [h, K(H)] \mod \mathbb{Z}v \). It follows,

\[
N(X) = [H, K(H)] \quad (2.3.8)
\]

and

\[
N(Y) = [h, K(h) = K(H)]. \quad (2.3.9)
\]

We have \( \det N(Y) = \det N(X)/4 \).

Assume that \( \gamma(H) = 4 \). Then either \( 2b \equiv 0 \mod 8 \) or \( 2b \equiv 4 \mod 8 \). Equivalently, \( b \equiv 0 \), \( 2 \mod 4 \). If \( b \equiv 0 \mod 4 \), we get for \( N(Y) \) the same elements as above. If \( b \equiv 2 \mod 4 \), we get an additional element \( \mod \mathbb{Z}v \) which is equal to \( z^* \) where \( z^* \in K(H)^* \) is defined by the condition that \( 4H^* + z^* \in N(X) \). Here \( z^* + K(H) \) has order two in \( A_{K(H)} \). Thus, for this case

\[
N(X) = [H, K(H), \frac{H}{2} + z^*], \quad (2.3.10)
\]

\[
N(Y) = [h, K(h) = [K(H), z^*]] \quad (2.3.11)
\]

where \( z^* + K(H) \) has order two in \( K(H)^*/K(H) \). We have \( \det N(X) = 2 \det K(H) \) and \( \det N(Y) = 2 \det K(H)/4 = \det N(X)/4 \).

Assume that \( \gamma(H) = 2 \). Then \( 2b \equiv 0, 4, \pm 2 \mod 8 \). Or \( b \equiv 0, 2, \pm 1 \mod 4 \). If \( b \equiv 0, 2 \mod 4 \), we get the same elements as for \( \gamma(H) = 4 \). If \( b \equiv \pm 1 \mod 4 \), we get additional elements \( \pm (\frac{b}{2} + z_1^*) \) where \( z_1^* + K(H) \) has order 4 in \( A_{K(H)} \). Finally we get (changing notations) that

\[
N(X) = [H, K(H), \frac{H}{4} + z^*], \quad (2.3.12)
\]

\[
N(Y) = [h, K(h) = [K(H), 2z^*], \frac{h}{2} + z^*], \quad (2.3.13)
\]

where \( z^* + K(H) \) has order 4 in \( K(H)^*/K(H) \). We have \( \det N(X) = \det K(H)/2 \), \( \det N(Y) = \det K(H)/8 \), and \( \det N(Y) = \det N(X)/4 \).

Assume that \( \gamma(H) = 1 \). Then \( 2b \equiv 0, 4, \pm 2 \mod 8 \), and we get the same lattice \( N(Y) \) as above. Thus,

\[
N(X) = [H, K(H), \frac{H}{8} + u^*], \quad (2.3.14)
\]

\[
N(Y) = [h, K(h) = [K(H), 4u^*], \frac{h}{2} + 2u^*] = [h, K(h), \frac{h}{2} + w^*], \quad (2.3.15)
\]

where \( u^* + K(H) \) has order 8 in \( A_{K(H)} \), \( w^* = 2u^* \), \( K(h) = [K(H), 2w^* = 4u^*] \). Here we agreed notations with Sect. 2.2. We have \( \det N(X) = \det K(H)/8 \) and
\[ \det N(Y) = \det K(H)/8. \] Thus, \( \det N(X) = \det N(Y) \) for this case. We can formally put here \( h = \frac{H}{2} \) since \( h^2 = \left( \frac{H}{2} \right)^2 = 2. \) Then

\[ N(X) \cap N(Y) = [H, K(H), \frac{H}{4} + 2u^*]. \quad (2.3.16) \]

We have

\[ [N(X) : N(X) \cap N(Y)] = [N(Y) : N(X) \cap N(Y)] = 2. \quad (2.3.17) \]

From these calculations, we have:

**Lemma 2.3.1.** For Mukai identification (2.1.5), the sublattice \( T(X) \subset T(Y) \) has index 2, if \( \gamma(H) = 2, 4, 8, \) and \( T(X) = T(Y) \), if the \( \gamma(H) = 1 \) for \( H \in N(X) \).

Also \( T(X) \subset T(Y) \) has index 2 if either \( \gamma(h) = 2 \) for \( h \in N(Y) \) or \( \det K(h) \neq \det K(H)/4 \). If \( \gamma(h) = 1 \) and \( \det K(h) = \det K(H)/4 \), then \( \gamma(H) = 2 \) or 1, and \( T(X) \subset T(Y) \) has index 2, if \( \gamma(H) = 2, \) and \( T(X) = T(Y) \), if \( \gamma(H) = 1 \) (we cannot get \( T(X) = T(Y) \) using only the Picard lattice \( N(Y) \) of \( Y \)).

**Proof.** Really, since \( H \in N(X), T(X) \perp N(X) \) and \( T(X) \cap \mathbb{Z}v = \{0\} \), the Mukai identification (2.1.5) gives an embedding \( T(X) \subset T(Y) \). We then have \( \det T(Y) = \det T(X)/[T(Y) : T(X)]^2 \). Moreover, \( |\det T(X)| = |\det N(X)| \) and \( |\det T(Y)| = |\det N(Y)| \) because the transcendental and the Picard lattice are orthogonal complements to each other in a unimodular lattice \( H^2(\ast, \mathbb{Z}) \). By calculations above, we get the statement.

We remark that the first statement of Lemma 2.3.1 is a particular case of the general statement by Mukai [6] that \( [T(Y) : T(X)] = q \) where

\[ q = \min |v \cdot x| \quad (2.3.18) \]

for all \( x \in H^0(X, \mathbb{Z}) \oplus N(X) \oplus H^4(X, \mathbb{Z}) \) such that \( v \cdot x \neq 0 \). For our Mukai vector \( v = (2, H, 2) \), it is easy to see that \( q = 2, \) if \( \gamma(H) = 2, 4, 8, \) and \( q = 1, \) if \( \gamma(H) = 1. \)

Now we can prove Propositions 2.2.1 and 2.2.2. If \( X \cong Y \), then \( T(X) \cong T(Y) \). Then \( \det T(X) = \det T(Y) \), and \( [T(Y) : T(X)] = 1 \) for the Mukai identification. Then, \( T(X) = T(Y) \) for the Mukai identification (2.1.5). By Lemma 2.3.1, we then get first statements of Propositions 2.2.1 and 2.2.2.

Assume that \( \gamma(H) = 1 \) as for Propositions 2.2.1 and 2.2.2. Then \( T(X) = T(Y) \) for the Mukai identification (2.1.5). By the discriminant forms technique (see [8]), then the discriminant quadratic forms \( q_{N(X)} = -q_{T(X)} \) and \( q_{N(Y)} = -q_{T(Y)} \) are isomorphic. Thus, lattices \( N(X) \) and \( N(Y) \) have the same signatures and discriminant quadratic forms. It follows (see [8]) that they have the same genus: \( N(X) \otimes \mathbb{Z}_p \cong N(Y) \otimes \mathbb{Z}_p \) for any prime \( p \) and the ring of \( p \)-adic integers \( \mathbb{Z}_p \). Additionally, assume that either the genus of \( N(X) \) or the genus of \( N(Y) \) contains only one class. Then \( N(X) \) and \( N(Y) \) are isomorphic.
If additionally the canonical homomorphism \( O(N(X)) \to O(q_{N(X)}) \) (equivalently, \( O(N(Y)) \to O(q_{N(Y)}) \)) is epimorphic, then the Mukai identification \( T(X) = T(Y) \) can be extended to give an isomorphism \( \phi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z}) \) of cohomology lattices. The Mukai identification is identical on \( H^{2,0}(X) = H^{2,0}(Y) \). Multiplying \( \phi \) by \( \pm 1 \) and by elements of the reflection group \( W^{(-2)}(N(X)) \), if necessary, we can assume that \( \phi(H^{2,0}(X)) = H^{2,0}(Y) \) and \( \phi \) maps the Kähler cone of \( X \) to the Kähler cone of \( Y \). By global Torelli Theorem for K3 surfaces proved by Piatetskii-Shapiro and Shafarevich [10], \( \phi \) is then defined by an isomorphism of K3 surfaces \( X \) and \( Y \).

If \( \rho(X) \geq 12 \), by [8, Theorem 1.14.4], the primitive embedding of \( T(X) = T(Y) \) into the cohomology lattice \( H^2(X, \mathbb{Z}) \) of K3 surfaces is unique up to automorphisms of the lattice \( H^2(X, \mathbb{Z}) \). Like above, it then follows that \( X \) is isomorphic to \( Y \).

Let us prove Theorems 2.2.3 and 2.2.4.

Assume that \( \gamma(H) = 1 \). The Mukai identification then gives the canonical identification

\[
T(X) = T(Y). \tag{2.3.19}
\]

Thus, it gives the canonical identifications

\[
A_{N(X)} = N(X)^*/N(X) = (U \oplus N(X))^*/(U \oplus N(X)) = T(X)^*/T(X) = A_{T(X)} \\
= A_{T(Y)} = T(Y)^*/T(Y) = N(Y)^*/N(Y) = A_{N(Y)}. \tag{2.3.20}
\]

Here \( A_{N(X)} = N(X)^*/N(X) = (U \oplus N(X))^*/(U \oplus N(X)) \) because \( U \) is unimodular, \( (U \oplus N(X))^*/(U \oplus N(X)) = T(X)^*/T(X) = A_{T(X)} \) because \( U \oplus N(X) \) and \( T(X) \) are orthogonal complements to each other in the unimodular lattice \( H^*(X, \mathbb{Z}) \). Here \( A_{T(Y)} = T(Y)^*/T(Y) = N(Y)^*/N(Y) = A_{N(Y)} \) because \( T(Y) \) and \( N(Y) \) are orthogonal complements to each other in the unimodular lattice \( H^2(Y, \mathbb{Z}) \). E.g. the identification \( (U \oplus N(X))^*/(U \oplus N(X)) = T(X)^*/T(X) = A_{T(X)} \) is given by the canonical correspondence

\[
x^* + (U \oplus N(X)) \to t^* + T(X) \tag{2.3.21}
\]

if \( x^* \in (U \oplus N(X))^* \), \( t^* \in T(X)^* \) and \( x^* + t^* \in H^*(X, \mathbb{Z}) \).

By (2.3.15), we also have the canonical embedding of lattices

\[
K(H) \subset K(h) = [K(H), 4u^*]. \tag{2.3.22}
\]

We have the key statement:

**Lemma 2.3.2.** The canonical embedding (2.3.22) (it is given by (2.3.15)) \( K(H) \subset K(h) \) of lattices, and the canonical identification \( A_{N(X)} = A_{N(Y)} \) (given by (2.3.20)) agree with the characteristic homomorphisms \( \kappa(H) : K(H)^* \to A_{N(X)} \) and \( \kappa(h) : K(h)^* \to A_{N(Y)} \), i.e. \( \kappa(h)(k^*) = \kappa(H)(k^*) \) for any \( \kappa^* \in K(h)^* \subset K(H)^* \) (this embedding is dual to (2.3.22)).
Proof. From definitions of the identifications (2.3.20), the identification \((U \oplus N(X))^*/(U \oplus N(X)) = A_{N(Y)}\) is given by the canonical embeddings

\[
(U \oplus N(X))^* \supset (v^\perp)_0^* = (v^\perp/Zv)^*
\]  

(2.3.23)

where \((v^\perp)_0^* = \{s^* \in (U \oplus N(X))^* \mid s^* \cdot v = 0\}\).

By (2.2.5), \(s^* = x_1e_1 + x_2e_2 + yH^* + k^* \in (U \oplus N(X))^*\) if and only if \(x_1, x_2, y \in \mathbb{Z}, k^* \in K(H)^*\), and \(y \equiv -8k^* \cdot u^* \mod 8\). Here \(\kappa(k^*) = s^* + (U \oplus N(X)).\) We have \(s^* \cdot v = -2x_1 - 2x_2 + y\), and \(s^* \in (v^\perp)_0^*\), if additionally \(y = 2x_1 + 2x_2\). Thus, we have \(2(x_1 + x_2) \equiv -8k^* \cdot u^* \mod 8\). It follows, \(k^* \cdot (4u^*) \in \mathbb{Z}\), and \(k^* \in K(h)^* = [K(h), 4u^*].\) Moreover, \(x_1 + x_2 \equiv -2k^* \cdot w^* \mod 4\) and \(-x_1 + x_2 \equiv x_1 + x_2 \equiv -2k^* \cdot w^* \mod 2\) where \(w^* = 2u^*\). Like in (2.3.7), we then have that \(s^* \in (v^\perp)_0^*\) if and only if \(s^* = (-x_1 + x_2)h^* + \frac{x_1 + x_2}{4}v + k^*\) where \(x_1, x_2, k^*\) satisfy the conditions above. Finally, we have \(s^* \in (v^\perp)_0^*\), if and only if

\[
s^* = ah^* + b\frac{v}{4} + k^*
\]  

(2.3.24)

where \(a, b \in \mathbb{Z}, k^* \in K(h)^*, a \equiv -2k^* \cdot w^* \mod 2\) and \(b \equiv -2k^* \cdot w^* \mod 4\). Here \(s^* \) gives \(ah^* + k^* \in N(Y)^*\), and \(\kappa(h)(k^*) = ah^* + k^* + N(Y) = s^* + U \oplus N(X) = \kappa(H)(k^*)\) under the identification (2.3.20). It proves the statement.

Proof of Theorem 2.2.3. We have the Mukai identification (it is defined by (2.1.5)) of the transcendental periods

\[
(T(X), H^{2,0}(X)) = (T(Y), H^{2,0}(Y)).
\]  

(2.3.25)

For general \(X\) with the Picard lattice \(N(X)\), it is the unique isomorphism of the transcendental periods up to multiplication by \(\pm 1\). If \(X \cong Y\), this (up to \(\pm 1\)) isomorphism can be extended to \(\phi : H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z}).\) The restriction of \(\phi\) on \(N(X)\) gives then isomorphism \(\phi_1 : N(X) \cong N(Y)\) which is \(\pm 1\) on \(A_{N(X)} = A_{N(Y)}\) under the identification (2.3.20). The element \(\tilde{h} = (\phi_1)^{-1}(h)\) and \(f = \phi^{-1}\) satisfy Theorem 2.2.3 by Lemma 2.3.2.

The other way round, under conditions of Theorem 2.2.3, by Lemma 2.3.2, one can construct an isomorphism \(\phi_1 : N(X) \cong N(Y)\) which is \(\pm 1\) on \(A_{N(X)} = A_{N(Y)}\). It can be extended to be \(\pm 1\) on the transcendental periods under the Mukai identification (2.3.25). Then it is defined by the isomorphism \(\phi : H^2(X, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z}).\) Multiplying \(\phi\) by \(\pm 1\) and by reflections from \(W^{(-2)}(N(X))\), if necessary (the group \(W^{(-2)}(N(X))\) acts identically on the discriminant group \(N(X)^*/N(X)\)), we can assume that \(\phi\) maps the Kähler cone of \(X\) to the Kähler cone of \(Y\). By global Torelli Theorem [10], it is then defined by an isomorphism of \(X\) and \(Y\).

To prove Theorem 2.2.4, one should argue similarly.
3. The case of Picard number 2

3.1. General results. Here we want to apply results of Sect. 2 to $X$ and $Y$ with Picard number 2.

We start with some preliminary considerations on K3 surfaces with Picard number 2 and primitive polarizations of degree 8 and 2.

Assume that $\text{rk} \ N(X) = 2$. Let $H \in N(X)$ be a primitive polarization with $H^2 = 8$. Assume that $\gamma(H) = 1$ for $H \in N(X)$ (we have this condition, if $Y \cong X$). Let $K(H) = H^\perp_{N(X)} = \mathbb{Z}\delta$ and $\delta^2 = -t$ where $t > 0$ is even. It then follows that $N(X) = [\mathbb{Z}H, \mathbb{Z}\delta, H^\ast + \mu \delta^2]$ where $H^\ast = H/8$ and $\mu = \pm 1, \pm 3$. Since $(H^\ast + \mu \delta^2)^2 = 1/8 - \mu^2 t/64 \equiv 0 \pmod{2}$, it follows $t = 8d$ where $d \in \mathbb{N}$ and $1 - \mu^2 d \equiv 0 \pmod{16}$. Then $d$ is odd and $\mu^2 d \equiv 1 \pmod{16}$. It follows that $d \equiv 1 \pmod{8}$ and $\mu^2 \equiv d \pmod{16}$. Changing $\delta$ by $-\delta$ if necessary, we can always assume that $\mu = 1$ or $\mu = 3$ where $\mu = 1$ if $d \equiv 1 \pmod{16}$, and $\mu = 3$ if $d \equiv 9 \pmod{16}$. These simple calculations show that the lattice $N(X)$ and $H$ are defined uniquely, up to isomorphisms, by $d$ where $-d = \det N(X)$. One can even replace $H$ by any primitive element of $N(X)$ with square 8. Thus, we have

**Proposition 3.1.1.** Let $X$ be a K3 surface with the Picard number $\rho = 2$. Assume that $X$ has a primitive polarization $H$ of degree $H^2 = 8$, and $\gamma(H) = 1$ for $H \in N(X)$. Then the lattice $N(X)$ is defined by its determinant $\det N(X) = -d$ where $d \in \mathbb{N}$ and $d \equiv 1 \pmod{8}$. There exists a unique choice of a primitive orthogonal vector $\delta \in K(H) = H^\perp_{N(X)}$ such that

\[
N(X) = [H, \delta, (H + \mu \delta)/8] \quad (3.1.1)
\]

where $\delta^2 = -8d$,

\[
\mu = \begin{cases} 
1, & \text{if } d \equiv 1 \pmod{16}, \\
3, & \text{if } d \equiv 9 \pmod{16}. 
\end{cases} \quad (3.1.2)
\]

We have

\[
N(X) = \{z = (xH + y\delta)/8 \mid x, y \in \mathbb{Z} \text{ and } \mu x \equiv y \pmod{8}\}. \quad (3.1.3)
\]

For any primitive element $H' \in N(X)$ with $(H')^2 = H^2 = 8$, there exists a unique automorphism $\phi \in O(N(X))$ such that $\phi(H) = H'$.

We denote the above (unique up to isomorphisms) hyperbolic lattice $N(X)$ by $N_d^8$ where $d \in \mathbb{N}$ and $d \equiv 1 \pmod{8}$.

**Proof.** Let $H'$ be a primitive element of $N(X) \cong N_d^8$ with square 8. It is easy to see that, if $\gamma(H') \neq 1$, then $\det N(X)$ is even which is impossible. The rest calculations are elementary.

Now let $Y$ be a K3 surface with $\text{rk} \ N(Y) = 2$. Let $h \in N(Y)$ be a nef element of degree $h^2 = 2$. Assume that $\gamma(h) = 1$ (this condition is necessary to have $Y \cong X$). Let $K(h) = h^\perp_{N(Y)} = \mathbb{Z}\alpha$ and $\alpha^2 = -t$ where $t > 0$ is even. It then follows that $N(Y) = [Zh, \mathbb{Z}\alpha, h^\ast + \frac{\delta}{8}]$ where $h^\ast = h/2$. We have $(h^\ast + \frac{\delta}{8})^2 = 1/2 - t/4 \equiv 0 \pmod{2}$, and hence $t = 2d$ where $d \in \mathbb{N}$ and $d \equiv 1 \pmod{4}$. Like for $X$ above, we get
Proposition 3.1.2. Let $Y$ be a K3 surface with the Picard number $\rho = 2$. Assume that $Y$ has a nef element $h \in N(Y)$ of degree $h^2 = 2$, and $\gamma(h) = 1$ for $h \in N(Y)$. Then the lattice $N(Y)$ is defined by its determinant $\det N(Y) = -d$ where $d \in \mathbb{N}$ and $d \equiv 1 \mod 4$. For a primitive orthogonal vector $\alpha \in K(h) = h_{N(Y)}^\perp$ (it is unique up to changing by $-\alpha$), we have

$$N(Y) = [h, \alpha, (h + \alpha)/2]$$

where $\alpha^2 = -2d$. We have

$$N(Y) = \{ z = (xh + y\alpha)/2 \mid x, y \in \mathbb{Z} \text{ and } x \equiv y \mod 2 \}.$$  \hfill (3.1.4)

For any element $h' \in N(Y)$ with $(h')^2 = h^2 = 2$, there exists an automorphism $\phi \in O(N(Y))$ such that $\phi(h) = h'$. It is unique up to changing to $\phi s_\alpha$ where $s_\alpha(h) = h$, $s_\alpha(\alpha) = -\alpha$.

We denote the above (unique up to isomorphisms) hyperbolic lattice $N(Y)$ of the determinant $-d$ by $N_d^2$ where $d \in \mathbb{N}$ and $d \equiv 1 \mod 4$.

Proof. It is trivial.

From Propositions 3.1.1 and 3.1.2, we then get

Proposition 3.1.3. Under conditions and notations of Propositions 3.1.1 and 3.1.2, all elements $h' = (xH + y\delta)/8 \in N(X)$ with $(h')^2 = 2$ are in one to one correspondence with solutions $(x, y)$ of the equation

$$x^2 - dy^2 = 16$$

with odd $x$, $y$ and $\mu x \equiv y \mod 8$. Changing an odd solution $(x, y)$ of the equation to $(x, -y)$, if necessary, one can always satisfy the last congruence.

The Picard lattices of $X$ and $Y$ are isomorphic, $N(X) \cong N(Y)$, if and only if $\det N(X) = \det N(Y) = -d$ (it follows that $d \equiv 1 \mod 8$), and there exists an element $h' \in N(X)$ with $(h')^2 = 2$ as above. Equivalently, the equation $x^2 - dy^2 = 16$ has a solution with odd $x$ and $y$.

Proof. Assume, $h' = (xH + y\delta)/8 \in N(X)$ and $(h')^2 = 2$. This is equivalent to $x, y \in \mathbb{Z}$, $\mu x \equiv y \mod 8$ and $2 = (x^2 - dy^2)/8$. Thus, the elements $h'$ are in one to one correspondence with integral solutions $(x, y)$ of the equation $x^2 - dy^2 = 16$ which satisfy the condition $x\mu \equiv y \mod 8$.

Let $(x, y)$ be an integral solution of the equation. Clearly, then $x \equiv y \mod 2$. Let $(x, y)$ be even. Then $(x, y) = 2(x_1, y_1)$ where $(x_1, y_1)$ is an integral solution of the equation $x_1^2 - dy_1^2 = 4$. Again $x_1 \equiv y_1 \mod 2$. If $x_1 \equiv y_1 \equiv 1 \mod 2$, we get a contradiction with $d \equiv 1 \mod 8$. If $x_1 \equiv y_1 \equiv 0 \mod 2$, we get that $(x, y) = 4(x_2, y_2)$ where $(x_2, y_2)$ are integral solutions of the equation $x_2^2 - dy_2^2 = 1$. It follows that $x_2 \mod 2$ and $y_2 \mod 2$ are different. Then the congruence $\mu x \equiv y \mod 8$ is not satisfied, and the solution $(x, y)$ does not give an element of $N(X)$.
Assume that $(x, y)$ is a solution of $x^2 - dy^2 = 16$ with odd $x$, $y$. We have $d \equiv \mu^2 \mod 16$ and $(x - y\mu)(x + y\mu) \equiv 0 \mod 16$. If $x - y\mu \equiv 0 \mod 4$ and $x + y\mu \equiv 0 \mod 4$, then $2x \equiv 0 \mod 4$ which is impossible for odd $x$. Thus, we have that only one of congruences: $x - y\mu \equiv 0 \mod 8$ or $x + y\mu \equiv 0 \mod 8$ is valid. It follows that exactly one of solutions $(x, y)$ or $(x, -y)$ gives an element of $N(X)$. It proves first statement.

If $N(Y) \cong N(X)$, the lattices have the same determinant. By Proposition 3.1.2, $N(X) \cong N(Y)$, if and only if $N(X)$ has an element $h'$ with square 2. This finishes the proof.

We have a similar statement in terms of $Y$.

**Proposition 3.1.4.** Under conditions and notations of Propositions 3.1.1 and 3.1.2, all primitive elements $H' = (xh + ya)/2 \in N(Y)$ with square $(H')^2 = 8$ are in one to one correspondence with solutions $(x, y)$ of the equation

$$x^2 - dy^2 = 16$$

with odd $x$, $y$.

The Picard lattices of $X$ and $Y$ are isomorphic, $N(X) \cong N(Y)$, if and only if det $N(X) = \det N(Y) = -d$ (it follows that $d \equiv 1 \mod 8$), and there exists an element $H' \in N(Y)$ with $(H')^2 = 8$ as above. Equivalently, the equation $x^2 - dy^2 = 16$ has a solution with odd $x$ and $y$. It follows that $d \equiv 1 \mod 8$.

**Proof.** Let $H' = (xh + ya)/2$ where $x, y \in \mathbb{Z}$ and $x \equiv y \mod 2$. Then $2 = (x^2 - dy^2)/8$, hence $x^2 - dy^2 = 16$. The element $H'$ is not primitive if and only if $x/2, y/2 \in \mathbb{Z}$ and $x/2 \equiv y/2 \mod 2$. This is equivalent that $x \equiv y \equiv 0 \mod 2$ and $x \equiv y \mod 4$. Thus, the elements $H'$ are in one to one correspondence with integral solutions $(x, y)$ of the equation $x^2 - dy^2 = 16$ which satisfy the condition that either they are both odd or both even, but $x \mod 4$ and $y \mod 4$ are different.

Assume that $(x, y)$ is an integral solution of the equation $x^2 - dy^2 = 16$. Clearly $x \equiv y \mod 2$ since $d \equiv 1 \mod 8$. If $x \equiv y \equiv 1 \mod 2$, then $(x, y)$ gives a primitive element $H'$ by previous considerations. Let $(x, y)$ be even. Then $(x, y) = 2(x_1, y_1)$ where $(x_1, y_1)$ is an integral solution of the equation $x_1^2 - dy_1^2 = 4$. Again $x_1 \equiv y_1 \mod 2$. It then follows that $x \equiv y \mod 4$, and the solution $(x, y)$ does not give a primitive element $H'$ of $N(Y)$.

If $N(Y) \cong N(X)$, the lattices have the same determinant. By Proposition 3.1.1, $N(X) \cong N(Y)$, if and only if the lattice $N(Y)$ has a primitive element $H'$ with $(H')^2 = 8$. It finishes the proof.

Now we can apply Theorems 2.2.3 and 2.2.4 to find out when $Y \cong X$.

**Theorem 3.1.5.** Let $X$ be a K3 surface and $\rho(X) = 2$. Assume that $X$ is an intersection of three quadrics (more generally, $X$ has a primitive polarization $H$ of degree 8). Let $Y$ be a K3 surface which is the double covering of the net $\mathbb{P}^2$ of the quadrics defining $X$, ramified along the curve of degenerate quadrics and $h$ be the
preimage of a line in $\mathbb{P}^2$ (more generally, $Y$ is the moduli space of sheaves on $X$ with the Mukai vector $v = (2, H, 2)$ and the canonical nef element $h = (-1, 0, 1)$ mod $\mathbb{Z}v$).

If $Y \cong X$, then

$$\gamma(H) = 1, \ det N(X) = -d \ where \ d \equiv 1 \ mod \ 8. \quad (3.1.7)$$

In the case (3.1.7), with notations of Propositions 3.1.1 and 3.1.3 for $X$ and $H$, we have that all elements $\tilde{h} = (xH + y\delta)/8 \in N(X)$ with square $\tilde{h}^2 = 2$ satisfying Theorem 2.2.3 are in one to one correspondence with integral solutions $(x, y)$ of the equation

$$x^2 - dy^2 = 16 \quad (3.1.8)$$

with odd $x$, $y$, and $x \equiv \pm 4 \ mod \ d$, $\mu x \equiv y \ mod \ 8$ (one can always satisfy the last congruence changing $y$ to $-y$ if necessary).

In particular (by Theorem 2.2.3), for a general K3 surface $X$ with $\rho(X) = 2$ and det $N(X) = -d$, $d \equiv 1 \ mod \ 8$, we have: $Y \cong X$ if and only if the equation

$$x^2 - dy^2 = 16$$

has an integral solution $(x, y)$ with odd $(x, y)$ and $x \equiv \pm 4 \ mod \ d$. Moreover, a nef element $h = (xH + y\delta)/8$ of $X$ with $h^2 = 2$ defines the structure of a double plane on $X$ which is isomorphic to the double plane $Y$ if and only if $x \equiv \pm 4 \ mod \ d$.

Proof. If $Y \cong X$, then $\gamma(H) = 1$ by Proposition 2.2.1. By Proposition 3.1.1, $N(X) \cong N_d^8$ where $d \in \mathbb{N}$, $d \equiv 1 \ mod \ 8$ and det $N(X) = -d$.

Assume that $Y \cong X$ for a general $X$ with the Picard lattice $N_d^8$. Let $\tilde{h} \in N(X)$ satisfies conditions of Theorem 2.2.3.

By Proposition 3.1.3, all primitive

$$\tilde{h} = (xH + y\delta)/8 \quad (3.1.9)$$

with $(\tilde{h})^2 = 2$ are in one to one correspondence with odd $(x, y)$ which satisfy the equation $x^2 - dy^2 = 16$ and $y \equiv \mu x \ mod \ 8$, and any integral solution of the equation $x^2 - dy^2 = 16$ with odd $x$, $y$ gives such $\tilde{h}$ after replacing $y$ to $-y$ if necessary, which does not matter for the statement of Theorem.

Let $k = aH + b\delta \in \tilde{h}^\perp = \mathbb{Z}\alpha$. Then $(k, h) = ax - byd = 0$ and $(a, b) = \lambda(yd, x)$. Hence, we have $(\lambda(ydH + x\delta))^2 = \lambda^2(8y^2d^2 - 8dx^2) = 8\lambda^2d(y^2d - x^2) = -2^7\lambda^2d$.

Since $\alpha^2 = -2d$, we get $\lambda = 2^{-3}$ and $\alpha = (ydH + x\delta)/8$. There exists a unique (up to $\pm 1$) embedding $f : K(H) = \mathbb{Z}\delta \to K(\tilde{h}) = \mathbb{Z}\alpha$ of one-dimensional lattices. It is given by $f(\delta) = 2\alpha$ up to $\pm 1$. Thus, its dual is defined by $f^*(\alpha^*) = 2\delta^*$ where $\alpha^* = \alpha/2d$ and $\delta^* = \delta/8d$. To satisfy conditions of Theorem 2.2.3, we should have

$$\kappa(\tilde{h})(\alpha^*) = \pm 2\kappa(H)(\delta^*). \quad (3.1.10)$$

We have $u^* = \mu d\delta^*$, $w^* = 2f(u^*) = \mu d\alpha^* = \mu \alpha/2$, and

$$\kappa(\tilde{h})(\alpha^*) = (-2\alpha^* \cdot w^*)\tilde{h}^* + \alpha^* + N(X) \quad (3.1.11)$$
by (2.2.18). Here \( \tilde{h}^* = \tilde{h}/2 \). We then have \( \alpha^* \cdot w^* = \mu(\alpha/2d) \cdot (\alpha/2) = -\mu/2 \), and 
\[
\kappa(\tilde{h})(\alpha^*) = \mu\tilde{h}^* + \alpha^* + N(X) = \mu(x(H/16) + y(\delta/16)) + y(H/16) + x\delta/16d.
\]
It follows,
\[
\kappa(\tilde{h})(\alpha^*) = \frac{\mu x + y}{16} H + \frac{x + \mu y d}{16d} \delta + N(X) = \frac{\mu x + y}{2} H^* + \frac{x + \mu y d}{2} \delta^* + N(X)
\]  
(3.1.12)
where \( H^* = H/8 \). We have \( u^* = \mu d\delta^* = \mu\delta/8 \). By (2.2.8), \( \kappa(H)(\delta^*) = (-8\delta^* \cdot u^*)H^* + \delta^* + N(X) \). We have \( \delta^* \cdot u^* = -\mu/8 \). It follows,
\[
\kappa(H)(\delta^*) = \mu H^* + \delta^*.
\]  
(3.1.13)
By (3.1.12) and (3.1.13), we then get that \( \kappa(H)(2\delta^*) = \pm \kappa(\tilde{h})(\alpha^*) \) is equivalent to 
\[
(x + \mu yd)/2 \equiv \pm2 \mod d \text{ or } x + \mu yd \equiv \pm4 \mod d
\]
since the group \( N(X)^*/N(X) \) is cyclic of order \( d \) and it is generated by \( \mu H^* + \delta^* + N(X) \). Thus, finally we get 
\[ x \equiv \pm4 \mod d. \]

The condition to have an odd solution \((x, y)\) of (3.1.18) with \( x \equiv \pm4 \mod d \) does not depend on the choice of \( X \) and the polarization \( H \). If this condition is satisfied, it is valid for any \( X \) with the Picard lattice \( N(X) \) and any of its primitive polarization \( H \) of degree 8. By Theorem 2.2.3, \( Y \cong X \) for all of them. This finishes the proof.

We have a similar statement in terms of \( Y \).

**Theorem 3.1.6.** Let \( X \) be a K3 surface and \( \rho(X) = 2 \). Assume that \( X \) is an intersection of three quadrics (more generally, \( X \) has a primitive polarization \( H \) of degree 8). Let \( Y \) be a K3 surface which is the double covering of the net \( \mathbb{P}^2 \) of the quadrics ramified along the curve of degenerate quadrics and \( h \in N(Y) \) be the preimage of line in \( \mathbb{P}^2 \) (more generally, \( Y \) is the moduli space of sheaves on \( X \) with the Mukai vector \( v = (2, H, 2) \) and the canonical nef element \( h = (-1, 0, 1) \mod \mathbb{Z}v \)).

If \( Y \cong X \), then
\[
\gamma(H) = 1, \quad \gamma(h) = 1, \quad \det N(Y) = -d \text{ where } d \equiv 1 \mod 8.
\]  
(3.1.14)

In the case (3.1.14) with notations of Propositions 3.1.2 and 3.1.4 for \( Y \) and \( h \), then we have that all elements \( \tilde{H} = (xh + y\alpha)/2 \in N(Y) \) with square \( \tilde{H}^2 = 8 \) satisfying Theorem 2.2.4 are in one to one correspondence with integral solutions \((x, y)\) of the equation
\[
x^2 - dy^2 = 16
\]  
(3.1.15)
with odd \( x, y \), and \( x \equiv \pm4 \mod d \).

In particular (by Theorem 2.2.4), for a general K3 surface \( Y \) with \( \rho(Y) = 2 \) and \( \det N(Y) = -d, \ d \equiv 1 \mod 8 \), we have: \( Y \cong X \) if and only if the equation \( x^2 - dy^2 = 16 \) has an integral solution \((x, y)\) with odd \((x, y)\) and \( x \equiv \pm4 \mod d \).
Moreover, a primitive polarization \( H = (xh + y\alpha)/2 \) of \( Y \) with \( H^2 = 8 \) defines a structure of intersection of three quadrics on \( Y \) which is isomorphic to the structure of intersection of three quadrics of \( X \) if and only if \( x \equiv \pm 4 \mod d \).

**Proof.** If \( Y \cong X \), then \( \gamma(H) = 1 \) and \( \gamma(h) = 1 \), by Proposition 2.2.2. By Propositions 3.1.2 and 3.1.4, \( N(Y) \cong N_d^2 \) where \( d \equiv 1 \mod 8 \) and \( \det N(Y) = -d \).

Assume that \( Y \cong X \) for a general \( Y \) with the Picard lattice \( N_d^2 \). Let \( \tilde{H} \in N(Y) \) satisfies conditions of Theorem 2.2.4.

By Proposition 3.1.4, all primitive \( \tilde{H} = (xh + y\alpha)/2 \) (3.1.16) with \( \tilde{H}^2 = 8 \) are in one to one correspondence with odd \((x, y)\) which satisfy the equation \( x^2 - dy^2 = 16 \).

Let \( k = ah + b\alpha \in \tilde{H}_N(Y) = \mathbb{Z}\delta \). Then \( ax - byd = 0 \) and \((a, b) = \lambda(yd, x)\). We have \( (\lambda(ydh + x\alpha))^2 = \lambda^2(2y^2d^2 - 2dx^2) = 2\lambda^2d(y^2d - x^2) = -2^5\lambda^2d \). Since \( \delta^2 = -8d \), we get \( \lambda = 2^{-1} \), and
\[
\delta = (ydh + x\alpha)/2. \quad (3.1.17)
\]

There exists a unique (up to \( \pm 1 \)) embedding \( f : K(\tilde{H}) = \mathbb{Z}\delta \to K(h) = \mathbb{Z}\alpha \) of one-dimensional lattices. It is given by \( f(\delta) = 2\alpha \) up to \( \pm 1 \). Thus, its dual is defined by \( f^*(\alpha^*) = 2\delta^* \) where \( \alpha^* = \alpha/2d \) and \( \delta^* = \delta/8d \). To satisfy conditions of Theorem 2.2.4, we should have
\[
\kappa(h)(\alpha^*) = \pm 2\kappa(\tilde{H})(\delta^*). \quad (3.1.18)
\]

Like for the proof above, we have
\[
\kappa(h)(\alpha^*) = \mu h^* + \alpha^* + N(Y) \quad (3.1.19)
\]
where \( h^* = h/2 \), and
\[
\kappa(\tilde{H})(\delta^*) = \mu \tilde{H}^* + \delta^* \quad (3.1.20)
\]
where \( \tilde{H}^* = \tilde{H}/8 \). By (3.1.16) and (3.1.17), we then have
\[
\kappa(\tilde{H})(\delta^*) = \mu(xh + y\alpha)/16 + (ydh + x\alpha)/16d + N(Y). \quad (3.1.21)
\]

It follows,
\[
\kappa(\tilde{H})(\delta^*) = \frac{\mu x + y}{8} h^* + \frac{x + \mu dy}{8} \alpha^* + N(Y). \quad (3.1.22)
\]

Thus, \( \kappa(\tilde{H})(2\delta^*) = \pm \kappa(h)(\alpha^*) \) is equivalent to \( (x + \mu y d)/4 \equiv \pm 1 \mod d \) or \( x + \mu y d \equiv \pm 4 \mod d \) since the group \( N(Y)^*/N(Y) \) is cyclic of the order \( d \), and it is generated by \( \mu h^* + \alpha^* + N(Y) \). Thus, finally we get \( x \equiv \pm 4 \mod d \).
The condition to have an odd solution \((x, y)\) of (3.1.15) with \(x \equiv \pm 4 \mod d\) does not depend on the choice of \(Y\) and the polarization \(h\). If this condition is satisfied, it is valid for any \(Y\) with the Picard lattice \(N(Y)\) and any its polarization \(h\) of degree 2. By Theorem 2.2.4, \(Y \cong X\) for all of them. This finishes the proof.

By Theorems 3.1.5 and 3.1.6, general \(X\) and \(Y\) with \(\rho = 2\) and \(X \cong Y\) are labeled by \(d \in \mathbb{N}\) such that the conditions

\[
x^2 - dy^2 = 16, \ x \equiv y \equiv 1 \mod 2, \ x \equiv \pm 4 \mod d
\]

are satisfied for some integers \(x, y\). Since \(x\) and \(y\) are odd, it follows that \(d \equiv 1 \mod 8\), which we had known. Since \(x \equiv \pm 4 \mod d\) and \(x\) is odd, we can write down all these \(x\) as

\[
x = \pm 4 + kd, \ k \in \mathbb{Z}, \ k \equiv 1 \mod 2.
\]

(3.1.24)

The equation (3.1.23) gives then \(16 \pm 8kd + k^2d^2 - dy^2 = 16\), i.e. \(\pm 8k + k^2d - y^2 = 0\). It follows

\[
d = \frac{y^2 + 8k}{k^2}
\]

(3.1.25)

where \(k\) is any odd integer. Let \(p\) be an odd prime. Assume that \(p^{2t+2} \nmid k\) for some \(t \geq 0\). Since \(k \mid (y^2 + 8k)\), it follows \(k \mid y^2\) and \(p^{t+1} \mid y\). Then \(p^{2t+2} \mid y^2\). Since \(p^{4t+2} \mid k^2\), it then follows that \(p^{2t+2} \mid y^2 + 8k\) and \(p^{2t+2} \mid y^2\). Thus, \(p^{2t+2} \mid k\). We get a contradiction. It shows that an odd prime \(p\) may divide \(k\) only in an even maximal power. Thus \(k = \pm b^2\) for some \(b \in \mathbb{N}\). Then \(b \mid y\) (since \(k \mid y^2\)), and \(y = ab\) for some integer \(a\). Thus, finally we get that

\[
d = \frac{a^2 \mp 8}{b^2}
\]

(3.1.26)

for some odd integers \(a\) and \(b\). Then \((a, b)\) is an odd solution of one of equations

\[
a^2 - db^2 = 8,
\]

(3.1.27)

or

\[
a^2 - db^2 = -8.
\]

(3.1.28)

If one of these equations has a solution with odd \(a\) and \(b\), then \(d \equiv 1 \mod 8\). Further we assume that: \(d \equiv 1 \mod 8\). It is easy to see that then any solution \((a, b)\) of the equations (3.1.27) and (3.1.28) has odd \(a\) and \(b\). Our considerations show that solutions \((x, y)\) of (3.1.23) are in one to one correspondence with solutions \((a, b)\) of the equations (3.1.27) and (3.1.28). More exactly, we get solutions

\[
(x, y) = \pm \left(a^2 - 4, \ ab\right), \ \text{if} \ a^2 - db^2 = 8,
\]

(3.1.29)

and

\[
(x, y) = \pm \left(a^2 + 4, \ ab\right), \ \text{if} \ a^2 - db^2 = -8
\]

(3.1.30)

of (3.1.23), and any solution of (3.1.23) can be written in this form. We call solutions (3.1.29) and (3.1.30) of (3.1.23) as associated with solutions \((a, b)\) of the equations (3.1.27) and (3.1.28) respectively.

Thus, we get the final result
Theorem 3.1.7. Let $X$ be a K3 surface and $\rho(X) = 2$. Assume that $X$ is an intersection of three quadrics (more generally, $X$ has a primitive polarization $H$ of degree 8). Let $Y$ be a K3 surface which is the double covering of the net $\mathbb{P}^2$ of the quadrics defining $X$ ramified along the curve of degenerate quadrics and $h$ be preimage of a line in $\mathbb{P}^2$ (more generally, $Y$ is the moduli space of sheaves on $X$ with the Mukai vector $v = (2, H, 2)$ and the canonical nef element $h = (-1, 0, 1)$ mod $\mathbb{Z}v$).

Then $Y \cong X$ for a general $X$ with $\rho(X) = 2$, if and only if $\det(N(X)) = -d$ where $d \equiv 1 \mod 8$, and one of equations

\[ a^2 - db^2 = 8, \tag{3.1.31} \]

or

\[ a^2 - db^2 = -8 \tag{3.1.32} \]

has an integral solutions. All solutions of these equations have odd $a$ and $b$, and the set of possible $d$ is union of the two infinite sets

\[ D_+ = \{ \frac{a^2 - 8}{b^2} \in \mathbb{N} \mid a, b \in \mathbb{N} \text{ are odd} \} \tag{3.1.33} \]

for the equation (3.1.31), and

\[ D_- = \{ \frac{a^2 + 8}{b^2} \in \mathbb{N} \mid a, b \in \mathbb{N} \text{ are odd} \} \tag{3.1.34} \]

for the equation (3.1.32). Here $D_+$ is infinite because it contains the infinite subset \{ $a^2 - 8 | a \in \mathbb{N} \text{ is odd}$\}, and $D_-$ is infinite because it contains the infinite subset \{ $a^2 + 8 | a \in \mathbb{N} \text{ is odd}$\}.

Solutions of (3.1.31) and (3.1.32) give all solutions of (3.1.23) as associated solutions (3.1.29) and (3.1.30), and all primitive elements $\tilde{h} \in N(X)$ with $\tilde{H}^2 = 2$ of Theorem 3.1.5, and $\tilde{H} \in N(Y)$ with $\tilde{H}^2 = 8$ of Theorem 3.1.6.

We also mention the following: The sets $D_+ - D_+ \cap D_- \text{ and } D_- - D_+ \cap D_-$ are infinite. The infinite sequence \{(1 + 14k)^2 - 8 | k \in \mathbb{N}\} is in $D_+ - D_+ \cap D_-$. Its elements $d$ are not in $D_-$ because $7|d$ and $-2$ is not a square mod 7. Similarly, the infinite sequence \{(1 + 6k)^2 + 8 | k \in \mathbb{N}\} is in $D_- - D_+ \cap D_-$. We don’t know: If the set $D_+ \cap D_-$ is infinite?

It is interesting to interpret solutions of (3.1.31) and (3.1.32) as appropriate elements of $N(X)$ and $N(Y)$.

We have

Theorem 3.1.8. Under conditions and notations of Proposition 3.1.1, the elements

\[ h_1 = \frac{(2aH + 2b\delta)}{8} \in N(X), \tag{3.1.35} \]
where \((a, b)\) is any integral solution of \(a^2 - db^2 = \pm 8\) satisfying the congruence \(\mu a \equiv b \mod 4\) (one can always satisfy the congruence changing \(b\) to \(-b\) if necessary) are all elements \(h_1 \in N(X)\) with

\[(h_1)^2 = \pm 4 \text{ and } h_1 \cdot H \equiv 0 \mod 2. \tag{3.1.36}\]

In particular, if \((a, b)\) is a solution of \(a^2 - db^2 = 8\), the surface \(X\) has a nef element \(h_1\) with square 4 and the structure of a quartic, if the \(h_1\) is very ample.

The existence of an element \(h_1 \in N(X)\) satisfying (3.1.36) (for one of signs \(+\) or \(-\)) is equivalent to \(Y \cong X\) for a general \(X\) with \(\rho(X) = 2\). In particular, \(Y \cong X\) if \(X\) has a structure of quartic with the linear system \(|h_1|\) of planes of even degree with respect to the hyperplane \(H\) of the intersection of quadrics (i. e. \(H \cdot h_1 \equiv 0 \mod 2\)). It is even sufficient to have an element \(\tilde{h}_1 \in W(\cdot ^2)(N(X))(h_1)\) with \(\tilde{h}_1 \cdot H \equiv 0 \mod 2\).

**Proof.** Let \((a, b)\) be a solution of \(a^2 - db^2 = \pm 8\). It follows that \(a, b\) are odd. We have \((a - b\mu)(a + b\mu) \equiv 0 \mod 8\). It follows that either \(a - b\mu \equiv 0 \mod 4\) or \(a + b\mu \equiv 0 \mod 4\). Changing \(b\) to \(-b\) if necessary we can assume that \(a \equiv b\mu \mod 4\). Then \(2a \equiv 2b\mu \mod 8\) and \(h_1 \in N(X)\). We have \((h_1)^2 = 4(a^2 - db^2)/8 = \pm 8/2 = \pm 4\). Vice versa, if \(h_1 \in N(X)\) satisfies (3.1.36), it can be written in the form (3.1.35), where \((a, b)\) satisfies \(a^2 - db^2 = \pm 8\).

Additionally applying Theorem 2.2.3, we get the following simple sufficient condition when \(Y \cong X\) which is valid for \(X\) with any \(\rho(X)\):

**Corollary 3.1.9.** Let \(X\) be a K3 surface. Assume that \(X\) is an intersection of three quadrics (more generally, \(X\) has a primitive polarization \(H\) of degree 8). Let \(Y\) be a K3 surface which is the double covering of the net \(\mathbb{P}^2\) of the quadrics defining \(X\) ramified along the curve of degenerate quadrics (more generally, \(Y\) is the moduli space of sheaves on \(X\) with the Mukai vector \(v = (2, H, 2)\)).

Then \(Y \cong X\) if there exists \(h_1 \in N(X)\) such that the primitive sublattice \([H, h_1]_{\text{pr}}\) in \(N(X)\) generated by \(H\) and \(h_1\) has odd determinant and

\[(h_1)^2 = \pm 4 \text{ and } h_1 \cdot H \equiv 0 \mod 2. \tag{3.1.37}\]

This condition is necessary to have \(Y \cong X\) if either \(\rho(X) = 1\), or \(\rho(X) = 2\) and \(X\) is a general K3 surface with its Picard lattice (i. e. the automorphism group of the transcendental periods \((T(X), H^{2,0}(X))\) is \(\pm 1\)).

**Proof.** The cases \(\rho(X) \leq 2\) had been considered. We can assume that \(\rho(X) > 2\). Let \(N = [H, h_1]_{\text{pr}}\). All considerations above for \(N(X)\) of \(\text{rk } N(X) = 2\) will be valid for \(N\). We can construct an associated with \(h_1\) solution \(\tilde{h} \in N\) with \(\tilde{h}^2 = 2\) such that \(H\) and \(\tilde{h}\) satisfy conditions of Theorem 2.2.3 for \(N(X)\) replaced by \(N\). It is easy to see that the conditions (b) and (c) will be still satisfied if we extend \(f\) in \((b)\) \(\pm\) identically on the orthogonal complement \(N_{N(X)}^\perp\). It finishes the proof.

It seems, many known examples of \(Y \cong X\) (e. g. [3], [15]) follow from the corollary.

A similar (to Theorem 3.1.8) statement for \(Y\) is simpler.
Theorem 3.1.10. Under notations of Proposition 3.1.2, assume that \( d \equiv 1 \mod 8 \). Then elements
\[
h_1 = (ah + b\delta)/2 \in N(Y),
\]
where \((a, b)\) is any integral solution of \( a^2 - db^2 = \pm 8 \) are all elements \( h_1 \in N(Y) \) with
\[
(h_1)^2 = \pm 4.
\]
In particular, if \((a, b)\) is a solution of \( a^2 - db^2 = 8 \), the surface \( X \) has a nef element \( h_1 \) with square \( 4 \), and a structure of quartic, if the \( h_1 \) is very ample.

The equality \( \gamma(H) = 1 \) (equivalently, \( d = -\det N(X) \) is odd) and the existence of an elements \( h_1 \in N(Y) \) satisfying (3.1.38) are equivalent to \( Y \cong X \) for a general \( Y \) with \( \rho(Y) = 2 \). In particular, \( Y \cong X \), if \( \gamma(H) = 1 \) and \( Y \) has a structure of quartic.

Proof. It is similar and simpler.

We remark that if \( N(X) \) of Theorem 3.1.8 has elements \( h \) with \( h^2 = 2 \), then \( N(X) \cong N(Y) \) by Proposition 3.1.3. By Theorem 3.1.10, then \( Y \cong X \) if \( N(X) \) has an element \( h_1 \) with \( h_1^2 = 4 \) (one does not need the congruence \( h_1 \cdot H \equiv 0 \mod 2 \)).

From our point of view, the statements 3.1.8 — 3.1.10 are very interesting also because they give some geometric meaning of elements \( h_1 \) of the Picard lattice with the negative square \((h_1)^2 = -4\). This is well-known only for elements \( \delta \) of the Picard lattice with \( \delta^2 = -2 \); then \( \delta \) or \(-\delta\) is effective.

3.2. Geometry of \( X \cong Y \), and calculation of the sets \( \mathcal{D}_+ \) and \( \mathcal{D}_- \). Here we study geometry of surfaces \( X \) and \( Y \) with \( \rho = 2 \) when general \( X \cong Y \). Equivalently, \( N(X) \cong N(Y) \cong N^2_d \cong N^2_d \) where \( d \in \mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_- \). Moreover, we give an algorithm to calculate the sets \( \mathcal{D}_+ \) and \( \mathcal{D}_- \). Using this algorithm, we find the first elements of these sets.

It is more convenient to work with the surface \( Y \) and its nef element \( h \) with \( h^2 = 2 \) which defines the structure of double plane, and its Picard lattice \( N(Y) \cong N^2_d \). To define \( X \), we should show a polarization \( H \in N(Y) \) with \( H^2 = 8 \) which defines the structure of intersection of three quadrics. They should be related by the correspondence described in Sect. 3.1. To find \( H \), we show the corresponding \( h_1 \) in \( N(X) \) with \( h_1^2 = 4 \) such that \( H \) is the associated with \( h_1 \).

We use notations of Proposition 3.1.2. Thus,
\[
N(Y) = [h, \alpha, (h + \alpha)/2], \ h^2 = 2, \ \alpha^2 = -2d, \ h \perp \alpha.
\]
Here \( d = -\det(N(Y)) > 0 \) and \( d \equiv 1 \mod 8 \), if \( d \in \mathcal{D} \). We assume that \( h \) is nef.

Proposition 3.2.1. Given \( d \in \mathcal{D} \), the lattice \( N(X) \cong N(Y) \) has non-zero elements with zero square, if and only if \( d \) is a square. It happens only for \( d = 1 \) or \( d = 9 \).
If $d = 1$ we have $1 \in \mathcal{D}_+ \cap \mathcal{D}_-$. The set $\text{Exc}(Y) = \{ \alpha \}$ (one should change $\alpha$ to $-\alpha$, if necessary), and the Kähler cone $\overline{\mathcal{K}(Y)}$ is generated by $h$ and by the elliptic curve $E = (h - \alpha)/2$ (up to the linear equivalence). This is the case (b) of Proposition 1.3.2. The surface $Y$ has only one nef primitive element $H$ with $H^2 = 8$. It defines $X$. The element $H = (5h - 3\alpha)/2 = 5E + \alpha$ which corresponds to the case (c) of Proposition 1.3.1. The $H$ is associated with the ample element $h_1 = (3h - \alpha)/2 = \mathcal{E} + \alpha \in N(Y)$ with $(h_1)^2 = 4$. It is also associated with the element $h_1 = (h - 3\alpha)/2$ with $(h_1)^2 = -4$.

If $d = 9$, $9 \in \mathcal{D}_- - \mathcal{D}_+ \cap \mathcal{D}_-$. For this case, $\text{Exc}(Y) = \emptyset$, the Kähler cone is generated by the elliptic curves $E_1 = (3h - \alpha)/2$ and $E_2 = (3h + \alpha)/2$, the surface $Y$ has only one nef element with square 2 which is $h$, and it has exactly two nef (and then ample) primitive elements $H = (5h \pm \alpha)/2$ with $H^2 = 8$. They both define $X$ as intersection of three quadrics and are associated with two elements $h_1 = (h \pm \alpha)/2$ with $h_1^2 = -4$.

**Proof.** Let $N(Y)$ has an element $c = (xh + y\alpha)/2$ with $c^2 = (x^2 - dy^2)/2 = 0$ where $x, y$ are not both zero. Then $d$ is a square in $\mathbb{Q}$, and then $d = (d_1)^2$ for $d_1 \in \mathbb{N}$. If $d \in \mathcal{D}$, then one of equations $(a - d_1b)(a + d_1b) = 8$ or $(a - d_1b)(a + d_1b) = -8$ has an integral solution $(a, b)$. Simple calculations show that then $d = d_1 = 1$, if the first equation has solution; and $d_1 = 1$ or $d_1 = 3$, if the second equation has a solution. Respectively, $d = 1$ or $d = 9$.

Assume that $d = 1$. All elements $r = (ah + b\alpha)/2 \in N(Y)$ with $r^2 = -2$ correspond to solutions of the equation $a^2 - b^2 = -4$ with $a \equiv b \mod 2$. It follows that $(a, b) = (0, \pm 2)$, and the set $\text{Exc}(Y) = \{ \alpha \}$. Thus, nef elements $z = (ah + b\alpha)/2 \in N(Y)$ are defined by the conditions: $z^2 = (a^2 - b^2)/2 \geq 0$, $a = h \cdot z > 0$ (these conditions give that $z \in \mathcal{V}^+(Y)$), and $-b = z \cdot \alpha \geq 0$. It follows that $z = gh + pE$ with $q \geq 0$ and $p \geq 0$ where $E = (h - \alpha)/2$. Thus $h$ and $E$ generate the $\overline{\mathcal{K}(Y)}$. The nef element $E$ has $E^2 = 0$ and is primitive. It follows [10] that $E$ is an elliptic curve on $X$ up to the linear equivalence: $[E]$ is an elliptic pencil on $Y$.

The equation $a^2 - b^2 = 8$ has the only solutions $(a, b) = (\pm 3, \pm 1)$. The equation $a^2 - b^2 = -8$ has the only solutions $(a, b) = (\pm 1, \pm 3)$. The corresponding nef elements $h_1 \in N(Y)$ with $(h_1)^2 = 4$ are the only $h_1 = (3h - \alpha)/2$. The corresponding associated nef elements $H \in N(Y)$ are the only $H = (5h - \alpha)/2 = 5E + \alpha$. It follows that $H$ is ample since its orthogonal complement is generated by $\delta$ with $\delta^2 = -8d = -8$.

Let $\tilde{h} = (xh + y\alpha)/2 \in N(Y)$ has $\tilde{h}^2 = 2$. Like above, we then get that $\tilde{h} = \pm h$. Thus, $h$ is the only nef element of $Y$ with square 2.

Assume that $\bar{H} \in N(Y)$ is a primitive ample element with $\bar{H}^2 = 8$. The pair $(Y, \bar{H})$ defines then a K3 surface $X$ with the polarization of degree 8. Since $d \in \mathcal{D}$, we then know, by Theorem 3.1.7, that the corresponding surface $Y$ with the polarization of degree two is $(Y, h)$ (since $Y$ has the only one nef element $h$ with $h^2 = 2$). Thus, $\bar{H}$ should be one of the found above associated nef solutions, which
are all equal to \( H \).

Let \( d = 9 \). Here we argue similarly to \( d = 1 \). Like above (considering the equation \( a^2 - 9b^2 = -4 \)), we can see that \( \text{Exc}(Y) = \emptyset \). Then \( \overline{K}(Y) = \overline{V} + (Y) \) coincides with the set of all nef elements of \( Y \) which are then \( (xh + y\alpha)/2 \) where \( x^2 - 9y^2 \geq 0 \) and \( x > 0 \). It follows that \( \overline{K}(Y) \) is generated by the elliptic curves \( E_1 \) and \( E_2 \). Like above (considering the equation \( a^2 - 9b^2 = 4 \)), we can see that \( h \) is the only nef element of \( Y \) with \( h^2 = 2 \). Like above, we can see that the equation \( a^2 - 9b^2 = -8 \) has the only solutions \((a, b) = (\pm 1, \pm 1)\). They giveind the two only two associated nef elements \( H = (5h \pm \alpha)/2 \) with \( H^2 = 8 \). Like above, the elements \( H \) are the only primitive nef elements (they are then ample) of \( Y \) with \( H^2 = 8 \). The found above elliptic curves \( E_1 \) and \( E_2 \) are the only elliptic curves (up to linear equivalence) on \( Y \). We have \( H \cdot E_1 \geq 3 \) and \( H \cdot E_2 \geq 3 \). By Proposition 1.3.1, \( X \) is then intersection of three quadrics.

It proves the statement.

We now assume that \( d \in D \) and \( d > 9 \). Then \( d \) is not a square, \( N(X) \) has no non-zero elements with square 0, and \( X \) is intersection of three quadrics by Proposition 1.3.1.

We remind [1] that the lattice \( N(Y) \cong N^2_d \) (for \( d \in \mathbb{N} \), \( d \equiv 1 \mod 4 \) and \( d \) which is not a square) with the fixed elements \( h \) and \( \alpha \) can be considered as an order \( N(Y) \subset \mathbb{K} = \mathbb{Q}(\sqrt{d}) \) of the real quadratic field \( \mathbb{K} = \mathbb{Q}(\sqrt{d}) \). One should correspond to \( z = (xh + y\alpha)/2 \in N(Y) \), the element \( z = (x + y\sqrt{d})/2 \in \mathbb{K} \). Thus, \( 1 = h \) and \( \sqrt{d} = \alpha \). Then \( N(Y) \) becomes a full \( \mathbb{Z} \)-submodule and a subring of \( \mathbb{K} = \mathbb{Q}(\sqrt{d}) \). The \( d \) is equal to the discriminant of the order \( N(Y) \). The order \( N(Y) \) is the ring of all integers in \( \mathbb{K} \) if \( d \) is square-free.

The intersection pairing in \( N(Y) \) then corresponds to the norm \( N \) in \( \mathbb{K} \):

\[
z^2/2 = N(z) = z\overline{z} = (x^2 - dy^2)/4
\]  

(3.2.2)

where \( \overline{z} = (x - y\sqrt{d})/2 \). An element \( \epsilon \in N(Y) \) is called a unit if \( N(\epsilon) = \pm 1 \). A unit \( \epsilon \) is positive (respectively negative) if \( N(\epsilon) = 1 \) (respectively, \( N(\epsilon) = -1 \)). By (3.2.2), positive units are in one to one correspondence with \( h \in N(Y) \) with \( \overline{h}^2 = 2 \), and negative units are in one to one correspondence with roots \( r \in N(Y) \) with \( r^2 = -2 \). Each unit \( \epsilon \) defines an automorphism \( z \to \epsilon z \) of \( N(Y) \) as a module, which preserves the intersection pairing (i.e. \( \epsilon \in O(N(Y)) \)) and is proper (i.e. \( \epsilon \in SO(N(Y)) \)), if \( N(\epsilon) = 1 \). If \( N(\epsilon) = -1 \), then \( \epsilon \) multiplies the intersection pairing by \(-1 \) (we shall call it then as an anti-automorphism of the lattice \( N(Y) \), \( \epsilon \in AO(N(Y)) \)), and it is non-proper, \( \det(\epsilon) = -1 \), i.e. \( \epsilon \in AO^-(N(Y)) \). The group \( SO(N(Y)) \cup AO^-(N(Y)) \) is then identified with the group of units. The group of units is \( \pm 1_{\mathbb{Z}} \) where

\[
\epsilon_0 = (s + t\sqrt{d})/2 = (sh + t\alpha)/2, \ s, t > 0,
\]  

(3.2.3)

is the fundamental unit. It is defined uniquely. To get the full group \( O(N(Y)) \cup AO(N(Y)) \) of automorphisms and anti-automorphisms of the lattice \( N(Y) \), one
needs to add the \textit{involution} \( \phi_h \in O(N(Y)) \) of the \textit{double plane structure} of \( Y \): \( \phi_h(h) = h \) and \( \phi_h(\alpha) = -\alpha \). It preserves the intersection pairing and is non-proper. We remind (e.g. see [1]) that there exists a very effective algorithm for finding the fundamental unit which uses continuous fractions.

Depending on the norm of the fundamental unit \( \epsilon_0 \), one has a different geometry of \( Y \) and \( X \).

Assume that \( N(\epsilon_0) = -1 \). Then \( r = \epsilon_0 \) has \( r^2 = -2 \), and (by elementary considerations)

\[
\text{Exc}(Y) = \{ r = (sh + t\alpha)/2, \ r = \phi_h(r) = (sh - t\alpha)/2 \}. \tag{3.2.4}
\]

Thus, the \textit{nef} cone \( K(Y) \) of \( Y \) is the set of \( z = (xh + y\alpha)/2 \in N(Y) \otimes \mathbb{R} \) such that \( z \cdot h = x > 0 \), \( z^2 = (x^2 - dy^2)/2 > 0 \), \( r \cdot z = (xs - ytd)/2 \geq 0 \) and \( \tilde{r} \cdot z = (xs + ytd)/2 \geq 0 \). The inequalities \( x > 0 \) and \( z^2 = (x^2 - dy^2)/2 > 0 \) follow from the last two (we assume that \( z \neq 0 \)). Thus,

\[
K(Y) = \{ z = (xh + y\alpha)/2 \in N(Y) \otimes \mathbb{R} \mid xs \geq ytd \geq -xs \}. \tag{3.2.5}
\]

Any automorphism of \( Y \) should preserve the set \( \text{Exc}(Y) \). It follows that the action of \( \text{Aut}(Y) \) on \( N(Y) \) is then generated by the involution \( \phi_h \), which generates the full automorphism group \( \text{Aut}(Y) \) of a general \( Y \) with \( N(Y) \). Thus, up to action of the group generated by \( \pm 1 \), \( W(-2)(N(Y)) \) and \( \text{Aut}(Y) \), any \( z \in N(Y) \otimes \mathbb{R} \) with \( z^2 > 0 \) has a unique representative \( z_0 \) in the \textit{fundamental domain}

\[
K(Y)^+ = \{ (xh + y\alpha)/2 \in N(Y) \otimes \mathbb{R} \mid xs \geq ytd \geq 0 \}. \tag{3.2.6}
\]

Similarly, any element \( \omega \in N(Y) \otimes \mathbb{R} \) with \( \omega^2 < 0 \) has a unique representative \( \omega_0 \) such that the orthogonal line to \( \omega_0 \) intersects the angle \( K(Y)^+ \), and \( -\alpha \cdot \omega_0 \geq 0 \). Let \( \omega = (xh + y\alpha)/2 \in N(Y) \) where \( \omega^2 = (x^2 - dy^2)/2 < 0 \) and \( x \geq 0 \), \( y > 0 \). Then the orthogonal complement to \( \omega \) is \( \mathbb{R}\omega^\perp \) where

\[
\omega^\perp = (dyh + x\alpha)/2, \quad (\omega^\perp)^2 = -d\omega^2. \tag{3.2.7}
\]

The above conditions mean that \( \omega^\perp = (dyh + x\alpha)/2 \in K(Y)^+ \), equivalently

\[
ys \geq xt \geq 0. \tag{3.2.8}
\]

Elements \( \omega_0 \in N(Y) \) with \( \omega_0^2 < 0 \) satisfying (3.2.8) are called \textit{reduced}. E.g. \( r = \epsilon_0 \) is reduced, and

\[
\epsilon^\perp_0 = (dth + s\alpha)/2, \quad (\epsilon^\perp_0)^2 = 2d. \tag{3.2.9}
\]

We shall use the hyperbolic distance in the hyperbolic space (it is a line for our case)

\[
\mathcal{L}((N(Y)) = V^+(N(Y))/\mathbb{R}^+ \tag{3.2.10}
\]
where \( \mathbb{R}_{++} \) is the set of positive real numbers. The distance is given by
\[
\cosh(\rho(\mathbb{R}_{++}z_1, \mathbb{R}_{++}z_2)) = \frac{z_1 \cdot z_2}{\sqrt{z_1^2 z_2^2}} \tag{3.2.11}
\]
for \( z_1, z_2 \in V^+(N(Y)) \). The \( \mathcal{K}(Y)^+/\mathbb{R}_{++} \) is a closed interval of the hyperbolic line with terminals \( \mathbb{R}_{++}h \) and \( \mathbb{R}_{++}e_0^- \). It follows that points \( \mathbb{R}_{++}z \) where \( z = (xh + y\alpha)/2 \) and \( x^2 - dy^2 > 0 \), of this interval are characterized by the properties that
\[
x > 0, \text{ and } 1 = \frac{h^2}{\sqrt{h^2 h^2}} \leq \frac{h \cdot z}{\sqrt{2z^2}} \leq \frac{h \cdot e_0^-}{\sqrt{h^2 (e_0^-)^2}}. \tag{3.2.12}
\]
Equivalently, we have
\[
1 \leq \frac{x}{\sqrt{x^2 - dy^2}} \leq \frac{t\sqrt{d}}{2}. \tag{3.2.13}
\]
The interval has only a finite number of elements of \( N(Y) \) with the fixed square, i.e. when \( x^2 - dy^2 > 0 \) is fixed. Similar condition for \( \omega = (xh + y\alpha)/2 \) with \( \omega^2 < 0 \) to be reduced is then
\[
\frac{1}{\sqrt{d}} \leq \frac{y}{\sqrt{-(x^2 - dy^2)}} \leq \frac{t}{2}. \tag{3.2.14}
\]

Assume that \( \tilde{h} \in \mathcal{K}(Y) \) is a nef element with \( \tilde{h}^2 = 2 \). Then there exists an involution \( \phi \) of \( N(Y) \) such that \( \phi(\tilde{h}) = \tilde{h} \) and \( \phi \) is \(-1\) on the \( \tilde{h}^\perp \) which is generated by the element \( \tilde{\alpha} \) with \( \tilde{\alpha}^2 = -2d < -2 \). The automorphism \( \phi \) then preserves \( \mathcal{K}(Y) \), the set \( \text{Exc}(Y) \), and \( \phi(\alpha) = \tilde{\alpha} \). It follows that \( \phi = \phi_h \) and \( \tilde{h} = h \). Thus, the surface \( Y \) has the only one structure of double plane. It is defined by the \( h \).

Assume that \( H \in \mathcal{K}(Y) \) is a nef (and then ample) primitive element with \( H^2 = 8 \). Assume that \( H' \in \mathcal{K}(Y) \) is another such an element. By Proposition 3.1.1, there exists an automorphism \( \phi \in O(N(Y)) \) such that \( \phi(H) = H' \). Then \( \phi \) preserves the fundamental chamber \( \mathcal{K}(Y) \) for \( W(-2)(N(Y)) \) and preserves \( \text{Exc}(Y) = \{r, \tilde{r}\} \). It follows that either \( \phi \) is identical or \( \phi = \phi_h \) is defined by the involution of the double plane of \( Y \). It follows that
\[
H = \frac{(xh + y\alpha)/2}{\phi_h(H) = H' = (xh - y\alpha)/2} \tag{3.2.15}
\]
are the only primitive nef elements of \( Y \) with square 8. They define equivalent structures \( X \cong (Y, H) \cong (Y, H') \) of intersection of three quadrics. Since we assume that \( d \in D \), the corresponding double plane is \( (Y, h) \) since \( h \) is the only nef element of \( Y \) with \( h^2 = 2 \). By (3.2.13), the element \( H \) (and similarly \( H' \)) is characterized by the conditions (where \( x, y \) are integers):
\[
H = (xh + y\alpha)/2, \quad x^2 - dy^2 = 16, \quad x \equiv y \equiv 1 \mod 2,
\]
x > 0, y > 0, 4 < x < 2t\sqrt{d}. \tag{3.2.16}
If \( z = (a + b\sqrt{d})/2 \) gives a solution \((a, b)\) of (3.1.31), then \( \epsilon_0 z \) gives a solution of (3.1.32), since \( N(\epsilon_0) = -1 \). It follows that both equations (3.1.31) and (3.1.32) have solutions. Thus, \( d \in D_+ \cap D_- \). The opposite statement is also valid. Really, assume that \( h_1 = (ah + b\alpha)/2 \) gives a solution of (3.1.31) and \( h'_1 = (a'h + b'\alpha)/2 \) gives a solution of (3.1.32). Since \( h_1^2 = 4 \) and \( (h'_1)^2 = -4 \), like for the propositions 3.1.1 and 3.1.2, one can prove that there exists an anti-automorphism \( \phi \in AO(N(Y)) \) such that \( \phi(h_1) = h'_1 \). If \( \det(\phi) = 1 \) the anti-automorphism \( \phi' = \phi h \phi \) has \( \det(\phi') = -1 \). It then corresponds to a unit \( \epsilon \) with \( N(\epsilon) = -1 \). It follows that \( N(\epsilon_0) = -1 \) for the fundamental unit \( \epsilon_0 \). Thus, conditions \( N(\epsilon_0) = -1 \) and \( d \in D_+ \cap D_- \) are equivalent, if \( d \in D \).

By Theorem 3.1.7, \( H \) is the associated solution (3.1.29) or (3.1.30) with a solution of the equation (3.1.31) or (3.1.32) respectively.

Assume that \( H \) is associated with a solution \( h_1 = (ah + b\alpha)/2 \) of the equation (3.1.31), thus \( (h_1)^2 = 4 \). Then \( x = a^2 - 4, \ y = ab \). By (3.2.16), we then have (changing signs of \( a, b \) if necessary): \( a > 0, b > 0 \) and \( 4 < a^2 - 4 \leq 2t\sqrt{d} \). It follows that \( h_1 \in K(Y)^+ \) is nef and then ample element with square 4 since for elements \( h_1 \) with \( (h_1)^2 = 4 \) the condition (3.2.13) is

\[
2\sqrt{2} \leq a \leq t\sqrt{2d}. \tag{3.2.17}
\]

Thus, if \( H \) is associated with \( h_1 = (ah + b\alpha)/2 \) where \( a > 0 \) and \( b > 0 \) and \( (h_1)^2 = 4 \), then \( h_1 \) is ample and

\[
8 < a^2 \leq 2t\sqrt{d} + 4. \tag{3.2.18}
\]

The condition (3.2.18) is much stronger than (3.2.17).

Now assume that \( H \) is associated with \( h_1 = (ah + b\alpha)/2 \) having \( h_1^2 = -4 \). Then \( x = a^2 + 4, \ y = ab \). By (3.2.16), we then have (changing signs of \( a \) and \( b \), if necessary) \( a > 0, b > 0 \) and \( 4 < a^2 + 4 \leq 2t\sqrt{d} \). Thus,

\[
0 < a^2 \leq 2t\sqrt{d} - 4. \tag{3.2.19}
\]

By (3.2.14), \( h_1^\perp \in K(Y)^+ \) (or \( h_1 \) is reduced) if and only if \( 2\sqrt{2} \leq b \leq \sqrt{2t} \). Since \( a^2 - db^2 = -8 \), this is equivalent to

\[
0 < a^2 \leq 2dt^2 - 8. \tag{3.2.20}
\]

The condition (3.2.19) is much stronger than (3.2.20).

Thus, if \( d \in D \), then either equation (3.1.31) has a solution \((a, b)\) satisfying (3.2.18), or equation (3.1.32) has a solution \((a, b)\) satisfying (3.2.19). This gives an effective algorithm to find out if \( d \in D \). Then \( d \in D_+ \cap D_- \).

If \( d \in D \), then there exist nef (and then ample) elements \( \tilde{h}_1 \in N(Y) \) with \( (\tilde{h}_1)^2 = 4 \). Let us find them. Like for \( h^2 = 2 \) or \( H^2 = 8 \) above, there exist only two such elements \( \tilde{h}_1 \). One of them is \( \tilde{h}_1 = (ph + qa)/2 \) where \( p > 0, q > 0 \), and
another one is \( \phi_h(h_1) = (ph - qa)/2 \). If there exists a solution \((a, b)\) of (3.1.31), satisfying (3.2.18), then \((p, q) = (a, b)\) and \(h_1 = h_1 = (ah + ba)/2\), since the \(h_1\) is nef. If there exists a solution \((a, b)\), \(a > 0, b > 0\) of (3.1.32), satisfying (3.2.19), then \(\tilde{h}_1 = -\epsilon_0(a - b\sqrt{d})/2\) has \((\tilde{h}_1)^2 = 4, \tilde{h}_1 \in \mathcal{K}(Y)^+\), and the \(\tilde{h}_1\) is then nef and ample.

Finally we have proved

**Theorem 3.2.2.** Let \(d \in \mathbb{N}, d \equiv 1 \mod 8\) and \(d > 9\) is not a square. Further we follow conditions and notations of Proposition 3.1.2. Let \(\epsilon_0 = (s + t\sqrt{d})/2\), where \(s > 0, t > 0\), be the fundamental unit of the order \(N(Y)\) (where \(1 = h\) and \(\sqrt{d} = \alpha\)). Then \(d \in \mathcal{D}_+ \cap \mathcal{D}_-\), if and only if the norm \(N(\epsilon_0) = -1\) and either the equation \(a^2 - db^2 = 8\) has a solution \((a, b)\) satisfying (3.2.18) or the equation \(a^2 - db^2 = -8\) has a solution \((a, b)\) satisfying (3.2.19). This gives an effective algorithm of \(d \in \mathcal{D}_+ \cap \mathcal{D}_-\) (see the Program in Sect. 5). For \(d \in \mathcal{D}_+ \cap \mathcal{D}_-\), let \(h_1 = (ah + ba)/2\) corresponds to such a solution \((a, b)\) with \(a > 0\) and \(b > 0\). We have \(h_1^2 = 4\), for the first case, and \(h_1^2 = -4\) for the second.

Further we assume that \(d \in \mathcal{D}_+ \cap \mathcal{D}_-\). Then the surface \(Y\) has a unique nef (and then ample) element of degree two; it is equal to \(h\) and defines the structure of a double plane for \(Y\) and the involution \(\phi_h\) of the double plane such that \(\phi_h^*(h) = h\), \(\phi_h^*(\alpha) = -\alpha\). The set \(\text{Exc}(Y) = \{r = \epsilon_0 = (sh + ta)/2, \tilde{r} = \phi_h^*(r) = (sh - ta)/2\}\).

For a general \(Y\) with \(N(Y)\) the involution \(\phi_h\) generates \(\text{Aut}(Y)\).

The surface \(Y\) has exactly two nef (and then ample) elements \(h_1\) and \(\phi_h(h_1)\) with square 4 which define two isomorphic structures of quartic on \(Y\). Here \(h_1 = h_1\) if \(h_1^2 = 4\), and \(\tilde{h}_1 = -\epsilon_0(a - b\sqrt{d})/2\) if \(h_1^2 = -4\) and \(h_1 = (ah + ba)/2\).

The surface \(Y\) has exactly two nef (and then ample) primitive elements \(H = (xh \pm y\alpha)/2\) with square 8. They are associated with the element \(h_1\) above:

\[(x, y) = (a^2 - 4, ab), \text{ if } a^2 - db^2 = 8,\]

and

\[(x, y) = (a^2 + 4, ab), \text{ if } a^2 - db^2 = -8.\]

These ample elements \(H\) define isomorphic structures \(X = (Y, H)\) of intersection of three quadrics on \(Y\) such that the corresponding double plane is isomorphic to \((Y, h)\).

Now we assume that \(d \in D\) and \(N(\epsilon_0) = 1\). Then all units of the order \(N(Y)\) have the norm 1. It follows that \(N(Y)\) has no roots \(r\) with \(r^2 = -2\), and the nef cone is \(\mathcal{K}(Y) = \mathcal{V}^+(\mathcal{N}(Y))\). It follows that any \(z = (xh + y\alpha)/2 \in N(Y)\) with \(z^2 > 0\) and \(x > 0\) is nef and ample. All nef elements \(h \in N(Y)\) with \(h^2 = 2\) are in one to one correspondence with units \(\epsilon \in \mathbb{Z}_{\mathbb{N}}\) (we call them nef too). For this correspondence, the nef element \(\epsilon(h)\) with square 2 corresponds to the nef unit \(\epsilon\). Here we identify units with automorphisms of the order \(N(Y)\).

Each nef element \(\epsilon(h)\) (or nef element with square 2) defines a structure of double plane for \(Y\) and the involution \(\phi_{\epsilon(h)}\) of \(Y\) such that \(\phi_{\epsilon(h)}(\epsilon(h)) = \epsilon(h)\) and \(\phi_{\epsilon(h)}\) is \(-1\).
on the orthogonal complement to $\epsilon(h)$ in $N(Y)$. We have $\phi_{\epsilon_1(h)}\phi_{\epsilon_2(h)} = \epsilon_1^2\epsilon_2^{-2}$.

For general $Y$ with $N(Y)$, the involutions $\phi_{\epsilon(h)}$ generate the full automorphism group $\text{Aut}(Y)$ (we shall see that soon).

For the hyperbolic distance (3.2.11), the nef element $h' = \epsilon_0(h)$ with square 2 is the nearest to $h$ comparing with other nef elements with square 2. It follows that $\mathbb{R}_{++}h$ and $\mathbb{R}_{++}h'$ are the terminals of the closed interval

$$I = [\mathbb{R}_{++}h, \mathbb{R}_{++}h'] \subset V^+(N(Y))/\mathbb{R}_{++} \text{ where } h' = \epsilon_0(h)$$

of the hyperbolic line $V^+(N(Y))/\mathbb{R}_{++}$. For general $Y$, the interval $I$ gives the fundamental domain for the action of $\text{Aut}(Y)$ on the line, and the action of $\text{Aut}(Y)$ is generated by reflections $\phi_h$ and $\phi_{h'}$ in the terminals $\mathbb{R}_{++}h$ and $\mathbb{R}_{++}h'$ of $I$. Really, otherwise, there exists an automorphism $\phi$ of $Y$ such that $\phi^*(h) = h'$ and $\phi^*(h') = h$. Then $\phi$ is the symmetry in the center of the interval $I$, and $\phi^*(h + h') = h + h'$ (here $(h + h')^2 > 0$), and $\phi$ is equal to $-1$ on the orthogonal complement $\mathbb{Z}\beta$ to $h + h'$. Thus, $\phi^*$ of $N(Y)$ is generated by the reflection $s_\beta$ in the root $\beta \in N(Y)$. Since $Y$ is general, the action of $s_\beta$ should be $\pm 1$ on the discriminant group of $N(Y)$ (to be continued as $\pm 1$ to the transcendental lattice $T(Y)$). This is only possible if either $\beta^2 = -2$ or the orthogonal complement to $\beta$ in $N(Y)$ is generated by nef elements with square two. The first case is impossible since $N(Y)$ has no elements with square $-2$. The second one is impossible because $\mathbb{R}_{++}e = \mathbb{R}_{++}(h + h')$ is the center of the interval $I$ which has smaller distance from the terminal $\mathbb{R}_{++}h$ than $\mathbb{R}_{++}h'$. It also proves that the automorphism group of $Y$ is generated by the involutions $\phi_{\epsilon(h)}$.

From this description of the action of $\text{Aut}(Y)$ on $N(Y)$, we get that the nef elements $h$ and $h' = \epsilon_0(h)$ are not conjugate by $\text{Aut}(Y)$, but their orbits give all nef elements with square 2 of $Y$. There are two classes $\epsilon_0^{2\mathbb{Z}}(h)$ and $\epsilon_0^{2\mathbb{Z}}(h') = \epsilon_0^{1+2\mathbb{Z}}(h)$ of nef elements with square 2 of $Y$, up to automorphisms of $Y$.

On the other hand, there exists an involution $\xi = \epsilon_0\phi_h \in O^+(N(S))$ which acts as a symmetry in the center of the interval $I$. Then $\xi(h) = h'$, $\xi(h') = h$. Clearly, the involutions $\phi_{\epsilon(h)}$ and $\xi$ generate the automorphism group $O^+(N(Y))$ where $O^+(N(Y))$ is the subgroup of $O(N(Y))$ of index two, which preserves the angle $V^+(N(Y))$. Really, this group is transitive on the set of all nef elements in $N(Y)$ with square 2, and its stabilizer subgroup has the order two.

Further we use the interval $I$ or the angle $\mathbb{R}_{++}I$ over $I$ instead of $K(Y)^+$ for the previous case.

Up to the action of $\pm 1$ and $\text{Aut}(Y)$, any $z \in N(Y) \otimes \mathbb{R}$ with $z^2 > 0$ has a unique representative $z_0$ in $\mathbb{R}_{++}I$. Like for (3.2.12), $z = (xh + y\alpha)/2 \in \mathbb{R}_{++}I$, if and only if $z^2 > 0$, $x > 0$ and

$$1 = \frac{h^2}{\sqrt{h^2h^2}} \leq \frac{h \cdot z}{\sqrt{2z^2}} \leq \frac{h \cdot \epsilon_0(h)}{\sqrt{h^2(\epsilon_0(h))^2}},$$

equivalently,

$$1 \leq \frac{x}{\sqrt{x^2 - dy^2}} \leq \frac{s}{2}.$$
Simple calculations show that $\mathbb{R}_{++}z$ belongs to the left half of $I$ (containing $\mathbb{R}_{++}h$), if and only if
\begin{equation}
1 \leq \frac{x}{\sqrt{x^2 - dy^2}} \leq \frac{\sqrt{s+2}}{2}.
\end{equation}

Similarly, an element $\omega \in N(Y) \otimes \mathbb{R}$ with $\omega^2 < 0$ has a unique representative $\omega_0$ such that the orthogonal line to $\omega_0$ intersects the angle $\mathbb{R}_{++}I$, and $-\alpha \cdot \omega_0 \geq 0$. This $\omega_0$ is called reduced. Like for (3.2.14), we have that $\omega = (xh + y\alpha)/2 \in N(Y) \otimes \mathbb{R}$ where $\omega^2 = (x^2 - dy^2)/2 < 0$ and $x \geq 0, y > 0$, is reduced (or $\omega^\perp = (dyh + x\alpha)/2 \in \mathbb{R}_{++}I$), if and only if
\begin{equation}
\frac{1}{\sqrt{d}} \leq \frac{y}{\sqrt{-(x^2 - dy^2)}} \leq \frac{s}{2\sqrt{d}}.
\end{equation}

Also $\omega^\perp = (dyh + x\alpha)/2$ belongs to the left half of $I$ if and only if
\begin{equation}
\frac{1}{\sqrt{d}} \leq \frac{y}{\sqrt{-(x^2 - dy^2)}} \leq \frac{\sqrt{s+2}}{2\sqrt{d}}.
\end{equation}

By Proposition 3.1.1, the group $O^+(N(Y))$ is transitive on the set of all primitive ample elements $H \in N(Y)$ with $H^2 = 8$. It follows that there exist exactly two such elements $H$ and $H' = \xi(H)$ which give rays $\mathbb{R}_{++}H$ and $\mathbb{R}_{++}H'$ of the interval $I$. If $H = H'$, then all ample primitive elements of $Y$ with square $8$ are $\text{Aut}(Y)$-equivalent and define equivalent structures of double plane for $Y$. Moreover, by Theorem 3.1.6, any structure of double plane of $Y$ can be obtained by this way. On the other hand, $h$ and $h'$ are not $\text{Aut}(Y)$-equivalent, and define then different structures of double plane for $Y$. It proves that $H'$ is different from $H$. We further assume that $H$ defines the structure of double plane equivalent to $h$, and then $H'$ defines the structure of double plane equivalent to $h'$. Assume that $H = (xh + y\alpha)/2$. Since $\mathbb{R}_{++}H \in I$, then $x > 0, y > 0$, and by (3.2.23)
\begin{equation}
4 < x < 2s.
\end{equation}

Let $d \in \mathcal{D}_+$. Then $H = (xh + y\alpha)/2$ is associated with a solution $(a, b)$ of $a^2 - db^2 = 8$. Then $h_1 = (ah + b\alpha)/2 \in N(Y)$ has $h_1^2 = 4$, and $x = a^2 - 4, y = ab$. We can suppose that $a > 0, b > 0$. By (3.2.27),
\begin{equation}
8 < a^2 < 2s + 4
\end{equation}
and
\begin{equation}
2\sqrt{2} < a < \sqrt{2s + 4}.
\end{equation}

By (3.2.24), this is equivalent to the condition that $\mathbb{R}_{++}h_1$ belongs to the left half of $I$. Like for nef elements with square $2$ or $8$ above, we can prove that there exist exactly two nef elements of $Y$ with square $4$ which belong to the interval $I$. They are conjugate by the involution $\xi$ and give two non-isomorphic structures of quartic
on $Y$ (for a general $Y$ with $N(Y)$). The element $h_1$ belongs to the left half of $I$. The element $\xi(h_1)$ belongs to the right half of $I$ and gives another non-isomorphic structure of quartic on $Y$. It gives the associated element $\phi_{h'}(H)$ with square 8 in the image of $I$ by the symmetry in its right terminal $\mathbb{R}_{++}h'$.

Now assume that $d \in D_-. Then $H = (xh + y\alpha)/2$ is associated with a solution $(a, b)$ of $a^2 - db^2 = -8$, $h_1 = (ah + bo)/2 \in N(Y)$ has $h_1^2 = -4$ and $x = a^2 + 4$, $y = ab$. We can suppose that $a > 0, b > 0$. By (3.2.27),

$$0 < a^2 < 2s - 4.$$

This is equivalent (use $a^2 - db^2 = -8$) to

$$\frac{1}{\sqrt{d}} \leq \frac{b}{\sqrt{8}} < \frac{\sqrt{s + 2}}{2\sqrt{d}}$$

which (by (3.2.26)) means that $\mathbb{R}_{++}h_1 \perp$ belongs to the left half of $I$. Like for nef elements with square 2 or 8 above, we can prove that there exist exactly two elements of $N(Y)$ with square $-4$ such that the corresponding orthogonal elements belong to the interval $I$ (or there are exactly two reduced elements with square $-4$). They are conjugate by the involution $\xi$ and are not $\text{Aut}(Y)$-equivalent (for a general $Y$ with $N(Y)$). The element $h_1 \perp$ belongs to the left half of $I$. The element $\xi(h_1 \perp)$ belongs to the right half of $I$. The element $\xi(h_1)$ gives the associated element $\phi_{h'}(H)$ with square 8 in the image of $I$ by the symmetry in its right terminal $\mathbb{R}_{++}h'$.

Thus, we have proved

**Theorem 3.2.3.** Let $d \in \mathbb{N}, d \equiv 1 \mod 8$ and $d > 9$ is not a square. Further we follow conditions and notations of Proposition 3.1.2. Let $\epsilon_0 = (s + t\sqrt{d})/2$, $s > 0$, $t > 0$ be the fundamental unit of the order $N(Y)$ (where $1 = h$ and $\sqrt{d} = \alpha$). Then $d \in \mathcal{D}_- \cap \mathcal{D}_+$ if and only if the norm $N(\epsilon_0) = 1$ and the equation $a^2 - db^2 = \pm 8$ has a solution $(a, b)$ satisfying (3.2.28) for $a^2 - db^2 = 8$, and satisfying (3.2.30) for the equation $a^2 - db^2 = -8$. This gives an effective algorithm of $d \in \mathcal{D}_- \cap \mathcal{D}_+$ (see the Program in Sect. 5). For $d \in \mathcal{D}_- \cap \mathcal{D}_+$, we denote $h_1 = (ah + bo)/2 \in N(Y)$ where $(a, b)$ is such solution with $a > 0$ and $b > 0$. We have $h_1^2 = \pm 4$.

Further we assume that $d \in \mathcal{D}_- \cap \mathcal{D}_+ \cap \mathcal{D}_-$. The surface $Y$ has two nef (and then ample) elements of degree two, $h$ and $h' = \epsilon_0(h)$, which define two double plane structures of $Y$ and the involutions $\phi_h$ and $\phi_{h'}$ of the double planes. The set $\text{Exc}(Y) = \emptyset$. For a general $Y$ with $N(Y)$ these two double plane structures of $Y$ are not isomorphic (i.e. $h$ and $h'$ are not $\text{Aut}(Y)$-equivalent), but any double plane structure of $Y$ is isomorphic to one of these two; the involutions $\phi_h$ and $\phi_{h'}$ generate $\text{Aut}(Y)$. There exists an involution $\xi \in O^+(N(Y)) = \epsilon_0 \phi_h^*$ such that $\xi(h) = h'$; the involution $\xi$ and $\text{Aut}(Y)$ generate the group $O^+(N(Y))$.

The surface $Y$ has exactly two nef (and then ample) primitive elements $H$ and $H'$ with square 8 such that the rays $\mathbb{R}_{++}H$ and $\mathbb{R}_{++}H'$ are in between the rays $\mathbb{R}_{++}h$ and $\mathbb{R}_{++}h'$. For general $Y$ with $N(Y)$, the elements $H$ and $H'$ give two
non-isomorphic structures of intersection of three quadrics of $Y$. Any structure of intersection of three quadrics of $Y$ is isomorphic to one of them. One of these elements, say $H = (xh + b\alpha)/2$, is associated with the element $h_1$ above:

$$(x, y) = (a^2 - 4, ab), \text{ if } d \in \mathcal{D}_+ - \mathcal{D}_+ \cap \mathcal{D}_-, \quad (3.2.32)$$

and

$$(x, y) = (a^2 + 4, ab), \text{ if } d \in \mathcal{D}_- - \mathcal{D}_+ \cap \mathcal{D}_-. \quad (3.2.33)$$

The corresponding to $H$ double plane structure of $Y$ is isomorphic to the one defined by $h$. The corresponding to $H'$ double plane structure of $Y$ is isomorphic to the one defined by $h'$.

The ray $\mathbb{R}^+ h_1$, if $d \in \mathcal{D}_+ - \mathcal{D}_+ \cap \mathcal{D}_-$ (respectively, the ray $\mathbb{R}^+ h_1^\perp$, if $d \in \mathcal{D}_- - \mathcal{D}_+ \cap \mathcal{D}_-$) belongs to the left half, containing $\mathbb{R}^+ h$, of the interval $I = [\mathbb{R}^+ h, \mathbb{R}^+ h']$ of the hyperbolic line $V^+(N(Y))/\mathbb{R}^+$. For general $Y$ with $N(Y)$, the elements $h_1$ and $\xi(h_1)$ are not $\text{Aut}(Y)$-equivalent; any element $\tilde{h}_1 \in N(Y)$ with $\tilde{h}_1^2 = \pm 4$ is $\text{Aut}(Y)$-equivalent to $\pm h_1$ or $\pm \xi(h_1)$. In particular, $Y$ has exactly two non-isomorphic structures of quartic (defined by $h_1$ and $\xi(h_1)$) if $d \in \mathcal{D}_+ - \mathcal{D}_+ \cap \mathcal{D}_-$.

Now we consider conditions when equations (3.1.31) and (3.1.32), i.e. $a^2 - db^2 = \pm 8$, have solutions locally. Equivalently, when these equations have solutions in the ring $\mathbb{Z}_p$ of $p$-adic integers for any prime $p$ (obviously, they have solutions in $\mathbb{R}$). We assume that $d \equiv 1 \mod 8$. By Theorem 3.1.8, existence of a solution is equivalent to existence of an element $h_1 \in N^2_d$ with $h_1^2 = \pm 4$. Since the lattice $N^2_d$ is even, $h_1$ is primitive in $N^2_d$. Very often the genus of the lattice $N^2_d$ has only one class. Then these local conditions are sufficient for existence of solutions of the equations (3.1.31) and (3.1.32). We have

**Proposition 3.2.4.** Assume that $d \in \mathbb{N}$ and $d \equiv 1 \mod 8$.

Then equation $a^2 - db^2 = \pm 8$ has a solution in $\mathbb{Z}_p$ (equivalently, the lattice $N^2_d \otimes \mathbb{Z}_p$ has an element $h_1$ with $h_1^2 = \pm 4$) for any prime $p$, if and only if

$$\left(\frac{\pm 2}{p}\right) = 1 \quad (3.2.34)$$

for any odd prime $p|d$. We remind that

$$\left(\frac{2}{p}\right) = (-1)^{\omega(p)} \text{ where } \omega(p) = \frac{p^2 - 1}{8},$$

and

$$\left(\frac{-2}{p}\right) = (-1)^{\omega(p) + \varepsilon(p)} \text{ where } \varepsilon(p) = \frac{p - 1}{2}.$$  

In particular, if the genus of the lattice $N^2_d$ has only one class, $\text{cl}(N^2_d) = 1$, then the equation $a^2 - db^2 = \pm 8$ has an integral solution, if and only if (3.2.34) is valid for any odd prime $p|d$. 
Proof. If \((a, b)\) is a solution of \(a^2 - db^2 = \pm 8\), then \((a/2)^2 \equiv \pm 2 \mod d\). It follows (3.2.34).

Now we assume that \(d \equiv 1 \mod 8\) and (3.2.34) is valid. We denote \(S_p = N_d^2 \otimes \mathbb{Z}_p\).

Since \(\det S = -d\), the lattice \(S_p\) is unimodular if \(p \nmid d\).

An unimodular \(p\)-adic lattice, which is even for \(p = 2\), is defined by its rank and determinant. It follows that \(S_2 \cong U \otimes \mathbb{Z}_2\) (see (1.1.2)). The lattice \(U \otimes \mathbb{Z}_2\) has primitive elements with any even square. In particular, it has primitive elements with the square \(\pm 4\). For odd prime \(p \nmid d\) the lattice \(S_p \cong (2) \oplus (-2d) \cong \langle \pm 4 \rangle \oplus \langle \pm 4(-d) \rangle\). Thus, \(S_p\) has elements with square \(\pm 4\).

Assume that odd \(p|d\). Then \(S_p \cong (2) \oplus (-2d) \cong ((\pm 2) \cdot 2) \oplus (-2d)\) because \(\pm 2 \in (\mathbb{Z}_p)^2\) if (3.2.34) is valid. It follows that \(S_p\) has elements with square \(\pm 4\).

This proves the statement.

Assume that \(d \in \mathbb{N}\) and \(d \equiv 1 \mod 8\). In Sect. 5: Appendix, we give Program for GP/PARI calculator (version 1.38) which uses Proposition 3.2.4 and Theorems 3.2.2 and 3.2.3 to check if \(d \in D^\pm\) where \(d > 9\). The program first checks that \(d\) is not a square. Then it checks that the local condition (3.2.34) is satisfied for one of signs \(\pm\). If that is true, it calculates the fundamental unit \(\epsilon_0 = (s + t\sqrt{d})/2\) and its norm \(N_\epsilon = N(\epsilon_0)\). If \(N_\epsilon = -1\), we apply Theorem 3.2.2. We find all odd positive \((a, b)\) satisfying \(a^2 - db^2 = 8\) and (3.2.18), and \(a^2 - db^2 = -8\) and (3.2.19). To make the algorithm more efficient, we also use that \(a^2 \equiv \pm 8 \mod d\). If such \((a, b)\) don’t exist, then \(d \notin \mathcal{D}\). If such \((a, b)\) exist, then \(d \in \mathcal{D}_+ \cap \mathcal{D}_-\), and we get the element \(h_1 = (ah + ba)/2\) with \(h_1^2 = \pm 4\). If \(N_\epsilon = 1\), we similarly apply Theorem 3.2.3 to check that \(d \in \mathcal{D}_+ \cap \mathcal{D}_-\) and to find \(h_1 = (ah + ba)/2\) with \(h_1^2 = \pm 4\), if \(d \in \mathcal{D}_+ \cap \mathcal{D}_-\).

If \(d \in \mathcal{D}_+ \cap \mathcal{D}_-\), we also find (according to Theorem 3.2.2) classes \(r = \epsilon_0(h)\) with \(r^2 = -2\), \(\tilde{h}_1\) with \((\tilde{h}_1)^2 = 4\) and \(H = (xh + y\alpha)/2\) with \(H^2 = 8\).

If \(d \in \mathcal{D}_+ \cap \mathcal{D}_-\), we also find (according to Theorem 3.2.3) classes \(h' = \epsilon_0(h)\) with \((h')^2 = 2\) and \(H = (xh + ba)/2\) with \(H^2 = 8\).

The program additionally calculates the class number \(cl = cl(d)\) of the lattice \(N_d^2\). If \(cl = 1\), then the local condition (3.2.34) is sufficient for \(d \in \mathcal{D}_\pm\). This can be used to check that calculations are correct.

Using this program, we get
Theorem 3.2.5. The first elements (up to 2009) of $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$ are

$$1(\pm), 9(-), 17(\pm), 33(-), 41(\pm), 57(-), 73(\pm), 89(\pm), 97(\pm), 113(\pm), 129(-), 137(\pm), 153(-), 161(+), 177(-), 193(\pm), 201(-), 209(-), 217(+), 233(\pm), 241(\pm), 249(-), 281(\pm), 297(-), 313(\pm), 329(+), 337(\pm), 353(\pm), 369(-), 393(-), 409(\pm), 417(-), 433(\pm), 449(\pm), 457(\pm), 489(-), 497(+), 513(-), 521(\pm), 537(-), 553(+), 561(-), 569(\pm), 593(\pm), 601(\pm), 617(\pm), 633(-), 641(\pm), 649(-), 657(-), 673(\pm), 681(-), 713(+), 721(+), 737(-), 753(-), 769, (\pm), 801(-), 809(\pm), 833(+), 849(-), 857(\pm), 873(-), 881(\pm), 889(+), 913(-), 921(-), 929(\pm), 937(\pm), 953(\pm), 969(-), 977(\pm), 1017(-), 1033(\pm), 1041(-), 1049(\pm), 1057(+), 1081(+), 1097(\pm), 1121(-), 1137(-), 1153(\pm), 1161(-), 1169(+), 1177(-), 1193(\pm), 1201(\pm), 1217(\pm), 1233(-), 1241(\pm), 1249(\pm), 1273(-), 1289(\pm), 1321(\pm), 1329(-), 1337(+), 1353(-), 1361(\pm), 1377(-), 1401(-), 1409(\pm), 1433(\pm), 1441(-), 1457(+), 1473(-), 1481(\pm), 1497(-), 1513(+), 1529(-), 1553(\pm), 1561(+), 1569(-), 1577(-), 1609(\pm), 1633(+), 1649(\pm), 1657(\pm), 1673(+), 1689(-), 1697(\pm), 1713(-), 1721(\pm), 1737(-), 1753(\pm), 1777(\pm), 1793(-), 1801(\pm), 1809(-), 1817(+), 1841(+), 1857(-), 1873(\pm), 1881(-), 1889(\pm), 1913(\pm), 1921(\pm), 1969(-), 1977(-), 1993(\pm), 2009(+)

where we mark $d \in \mathcal{D}$ by $+$ (respectively $-$) if $d \in \mathcal{D}_+ - \mathcal{D}_+ \cap \mathcal{D}_-$ (respectively $d \in \mathcal{D}_- - \mathcal{D}_+ \cap \mathcal{D}_-$), and by $\pm$, if $d \in \mathcal{D}_+ \cap \mathcal{D}_-$.

Calculations using the Program give for the first 10 non-square elements of $\mathcal{D}$ the following (where $\omega = (1 + \sqrt{d})/2$):

- $d = 17 \in \mathcal{D}_+ \cap \mathcal{D}_-: cl(d) = 1, \epsilon_0 = 3 + 2\omega, N(\epsilon_0) = -1, h_1 = (3h + \alpha)/2, r = (8h + 2\alpha)/2, h_1 = (5h + \alpha)/2, H = (13h + 3\alpha)/2$.
- $d = 33 \in \mathcal{D}_- - \mathcal{D}_+ \cap \mathcal{D}_-: cl(d) = 1, \epsilon_0 = 19 + 8\omega, N(\epsilon_0) = 1, h_1 = (5h + \alpha)/2, h' = (46h + 8\alpha)/2, H = (29h + 5\alpha)/2$.
- $d = 41 \in \mathcal{D}_+ \cap \mathcal{D}_-: cl(d) = 1, \epsilon_0 = 27 + 10\omega, N(\epsilon_0) = -1, h_1 = (7h + \alpha)/2, r = (64h + 10\alpha)/2, h_1 = h_1, H = (45h + 7\alpha)/2$.
- $d = 57 \in \mathcal{D}_- - \mathcal{D}_+ \cap \mathcal{D}_-: cl(d) = 1, \epsilon_0 = 131 + 40\omega, N(\epsilon_0) = 1, h_1 = (7h + \alpha)/2, h' = (302h + 40\alpha)/2, H = (53h + 7\alpha)/2$.
- $d = 73 \in \mathcal{D}_+ \cap \mathcal{D}_-: cl(d) = 1, \epsilon_0 = 943 + 250\omega, N(\epsilon_0) = -1, h_1 = (9h + \alpha)/2, r = (2136h + 250\alpha)/2, h_1 = h_1, H = (77h + 9\alpha)/2$.
- $d = 89 \in \mathcal{D}_+ \cap \mathcal{D}_-: cl(d) = 1, \epsilon_0 = 447 + 106\omega, N(\epsilon_0) = -1, h_1 = (9h + \alpha)/2, r = (1000h + 106\alpha)/2, h_1 = (217h + 23\alpha)/2, H = (85h + 9\alpha)/2$.
\[d = 97 \in D_+ \cap D_-: cl(d) = 1, \epsilon_0 = 5035 + 1138\omega, N(\epsilon_0) = -1, h_1 = (69h + 7\alpha)/2,\]
\[r = (11208h + 1138\alpha)/2, \tilde{h}_1 = h_1, H = (4757h + 483\alpha)/2;\]
\[d = 113 \in D_+ \cap D_-: cl(d) = 1, \epsilon_0 = 703 + 146\omega, N(\epsilon_0) = -1, h_1 = (11h + \alpha)/2,\]
\[r = (1552h + 146\alpha)/2, \tilde{h}_1 = h_1, H = (117h + 11\alpha)/2;\]
\[d = 129 \in D_- - D_+ \cap D_-: cl(d) = 1, \epsilon_0 = 15371 + 2968\omega, N(\epsilon_0) = -1, h_1 = (11h + \alpha)/2,\]
\[r = (33710h + 2968\alpha)/2, H = (125h + 11\alpha)/2;\]
\[d = 137 \in D_+ \cap D_-: cl(d) = 1, \epsilon_0 = 1595 + 298\omega, N(\epsilon_0) = -1, h_1 = (35h + 3\alpha)/2,\]
\[r = (3488h + 298\alpha)/2, h_1 = (199h + 17\alpha)/2, H = (1229h + 105\alpha)/2.\]

3.3. An application to moduli of \(X\) and \(Y\). The results above can be interpreted from the point of view of moduli of intersections of three quadrics in \(\mathbb{P}^5\).

By period map and local or global Torelli Theorem, the moduli space of K3 surfaces \(X\) which are intersections of three quadrics in \(\mathbb{P}^5\) (more generally, K3 surfaces \(X\) with a primitive polarization \(H\) of degree \(H^2 = 8\)) are 19-dimensional. Moreover, surfaces \(X\) with \(\rho(X) = \rho\) belong to a 20 – \(\rho\)-dimensional submoduli space. If \(X\) is general, then \(\rho(X) = 1\), and the surface \(Y\) cannot be isomorphic to \(X\) because \(N(X) = \mathbb{Z}H\) where \(H^2 = 8\), and \(N(X)\) does not have elements with square 2 which is necessary if \(Y \cong X\). Thus, if \(Y \cong X\), then \(\rho(X) \geq 2\), and \(X\) belongs to a codimension 1 submoduli space of K3 surfaces which is a divisor in the moduli space (up to codimension 2). The set \(D\) labels connected components of the divisor. Each \(d \in D\), gives an irreducible and connected codimension one moduli subspace of K3 surfaces \(X\) with the Picard lattice \(N(X) = N^8_d \cong N^2_d\) (more generally, \(N^8_d \subset N(X)\), but the polarization \(H \in N^8_d\), see Corollary 3.1.9); \(Y \cong X\) for all \(X\) from this subspace. See [7], [8] on corresponding results about connected components of moduli of K3 surfaces with condition on Picard lattice. See also [2].

E.g. it is well-known that \(Y \cong X\) if \(X\) has a line. This is a divisorial condition on moduli of \(X\). The corresponding component is labeled by \(d = 17 \in D\). Really, let \(l \in N(X)\) be the class of line. Then the intersection matrix of \(H\) and \(l\) is

\[
\begin{pmatrix}
H^2 & H \cdot l \\
l \cdot H & l^2
\end{pmatrix} = \begin{pmatrix} 8 & 1 \\ 1 & -2 \end{pmatrix}
\]

which has the determinant \(-17\). Since \(17 \in D\), then \(Y \cong X\). The projection from the line \(l\) gives an embedding of \(X\) to \(\mathbb{P}^3\) as a quartic. The corresponding hyperplane section is \(h_1 = H - l\), it has \(h_1^2 = 4\). We have \(H \cdot h_1 = 7\) which is odd. The reflection of \(h_1\) in \(l\) gives \(\tilde{h}_1 = h_1 + (h_1 \cdot l)l = h_1 + 3l = H + 2l\). We have \((\tilde{h}_1)^2 = 4\) and \(\tilde{h}_1 \cdot H = 10 \equiv 0 \mod 2\). Then \(Y \cong X\) by Corollary 3.1.9. Of course, there exists a classical direct geometric isomorphism between \(X\) and \(Y\) in this case.

Our results in Sect. 2 imply: There exists an infinite number of different divisorial conditions on moduli of intersections \(X\) of three quadrics in \(\mathbb{P}^5\) such that each of them implies \(Y \cong X\). All these divisorial conditions are labeled by elements of the infinite set \(D \subset \mathbb{N}\). The number \(d = 17 \in D\) corresponds to the classical example of \(X\) containing a line.
It seems, a direct geometric construction of the isomorphism between \( X \) and \( Y \) is known only for the first \( d = 1, 9 \) and 17. For all other \( d \in \mathcal{D} \) we were managed to prove that \( Y \cong X \) only using the fundamental Global Torelli Theorem for K3 surfaces proved by I.I. Piatetski-Shapiro and I.R. Shafarevich in [10].

It would be interesting to find all possible codimension two (or bigger) conditions on moduli of intersections \( X \) of three quadrics in \( \mathbb{P}^5 \) which imply \( Y \cong X \) and which don’t follow from the divisorial conditions on moduli which were described above.

4. A general perspective

Similar methods and calculations can be developed in the following general situation.

Let \( X \) and \( Y \) are K3 surfaces,

\[
\phi : (T(X) \otimes \mathbb{Q}, H^{2,0}(X)) \cong (T(Y) \otimes \mathbb{Q}, H^{2,0}(Y))
\]

(4.1)
an isomorphism of their transcendental periods over \( \mathbb{Q} \), and

\[
(a_1, H_1, b_1) \pm \ldots (a_k, H_k, b_k) \pm,
\]

(4.2)
a sequence of types of isotropic Mukai vectors of sheaves on K3, and \( \pm \) shows the direction of the correspondence.

Similar methods and calculations can be applied to answer the following question:

When there exists a correspondence between \( X \) and \( Y \) given by the sequence (4.2) of types of Mukai vectors, which gives the isomorphism (4.1) between their transcendental periods?

In [6] and [9] sufficient and necessary conditions on (4.1) are given when there exists at least one such a sequence (4.2) with coprime Mukai vectors \( (a_i, H_i, b_i) \).

In this paper, we had considered the case when \( Y = X \), \( \phi = \pm \text{id} \), the sequence (4.2) consists of one primitive Mukai vector \((2, H, 2)^+\) with \( H^2 = 8 \) where \( + \) means that \( Y \) is the moduli space of sheaves on \( X \) with the Mukai vector \((2, H, 2)^+\).

5. Appendix: A program for GP/PARI calculator

\[
\text{\texttt{for d which is N, d\ equiv 1 mod 8 and d>9 it finds out}}
\]
\[
\text{\texttt{if d\ in \Da, and it finds basic polarizations}}
\]
\[
\text{\texttt{default(compatible,3)}}
\]
\[
\text{\texttt{pprint("$d=$","d","$"});}}
\]
\[
\text{\texttt{if(issquare(d)==1,pprint("$d=$","d","$" is square and it is not in \Da"),}}
\]
\[
\text{\texttt{u=factor(d);uu=u[1] ;u=uu;kill(uu);k1=matsize(u)[2];}}
\]
\[
\text{\texttt{dplus=1;dminus=1;for(k=1,k1,if(kro(2,u[k])=-1,dplus=-1,);}}
\]
\[
\text{\texttt{if(kro(-2,u[k])==-1,dminus=-1,));}}
\]
\[
\text{\texttt{if(dplus==-1,pprint("$d=$","d," is not in \Da+ by local conditions"),);}}
\]
\[
\text{\texttt{if(dminus==-1,pprint("$d=$","d," is not in \Da- by local conditions"),);}}
\]
\[
\text{\texttt{if(dplus==-1&&dminus==-1,pprint("$d=$","d," is not in \Da by local conditions"),)}}\]
cl=classno(d);pprint("cl(d)="",cl);\ 
if(dplus==1,aa0=vector(d,k,0);f=1;\ 
for(k=1,d-1,if(mod(k^2,d)==mod(8,d),aa0[f]=k;f=f+1,));\ 
apl0=vector(f-1,k,aa0[k]);kill(aa0);kill(f););\ 
if(dminus==1,aa0=vector(d,k,0);f=1;\ 
for(k=1,d-1,if(mod(k^2,d)==mod(-8,d),aa0[f]=k;f=f+1,));\ 
amin0=vector(f-1,k,aa0[k]);kill(aa0);kill(f),);\ 
eps=unit(d);neps=norm(eps);\ 
s=2*real(eps)+imag(eps);t=imag(eps);\ 
pprint("fund unit=",eps);\ 
pprint("norm fund unit=",neps);\ 
if(neps==-1,\ 
gampl=0;gammin=0;f=matsize(apl0)[2];lim2=sqrt(4+2*t*sqrt(d));\ 
a1=-d;for(n=0,lim2/d+1,a1=a1+d;for(m=1,f,a=a1+apl0[m];\ 
if(type(a/2)==1,a2=(a^2-8)/d;\ 
if(issquare(a2)!=1,b=isqrt(a2);if(a^2-4<s*2,\ 
gampl=1;apl=a;bpl=b;pprint("h_1=","apl","h+","bpl","alpha")/2$"),))));\ 
if(gampl==0,f=matsize(amin0)[2];lim2=sqrt(2*t*sqrt(d)-4);\ 
a1=-d;for(n=0,lim2/d+1,a1=a1+d;for(m=1,f,a=a1+amin0[m];\ 
if(type(a/2)==1,a2=(a^2+8)/d;\ 
if(issquare(a2)!=1,b=isqrt(a2);if(a^2+4<s*2,\ 
gammin=1;amin=a;bmin=b;\ 
pprint("h_1=","amin","h+","bmin","alpha")/2$"),))));\ 
if(gampl==0&gammin==0,pprint("d="",d," is not in \(\text{Da}\)"),\ 
pprint("d="",d," is in \(\text{Da_+}\)\cap \(\text{Da_-}\)"),);\ 
pprint("S=(","s","h+","t","alpha")/2$"));\ 
if(gampl==1,x=apl^2-4;y=apl*bpl;\ 
pprint("$\widetilde{\alpha}_1=h_1$"),\ 
pprint("$H=","x","h+","y","alpha")/2$"),\ 
pp=-(s*amin-t*bmin+d)/2;qq=-(t*amin-s*bmin)/2;\ 
x=amin^2+4;y=amin*bmin;\ 
pprint("$\widetilde{\alpha}_1=","pp","h+","qq","alpha")/2$"),;\ 
pprint("$H=","x","h+","y","alpha")/2$"),;\ 
gampl=0;gammin=0;\ 
if(dplus==1,\ 
f=matsize(apl0)[2];lim2=sqrt(2*s+4);\ 
a1=-d;for(n=0,lim2/d+1,a1=a1+d;for(m=1,f,a=a1+apl0[m];\ 
if(type(a/2)==1,a2=(a^2-8)/d;\ 
if(issquare(a2)!=1,b=isqrt(a2);if(a^2-4<=s*2,\ 
gampl=1;apl=a;bpl=b;\ 
pprint("$\widetilde{\alpha}_1=","apl","h+","bpl","alpha")/2$"),))))),;\ 
if(dminus==1,\ 
f=matsize(amin0)[2];lim2=sqrt(2*s-4);\ 
}
a1 = -d; for (n = 0; lim2/d + 1, a1 = a1 + d; for (m = 1, f, a = a1 + amin0[m];
if (type(a/2) == 1, a2 = (a^2 + 8)/d;
if (issquare(a2) != 1, b = isqrt(a2);
if (a^2 + 4 < s * 2, 
\gamma_{\text{min}} = 1; a_{\text{min}} = a; b_{\text{min}} = b;
pprint("h_{1}=("a_{\text{min}}","h_{+}",b_{\text{min}}","\alpha")/2$",)))));
if (gamlp == 0 & gammin == 0, pprint("d=", d, " is not in $D_{a}^{+}$");
if (gamlp == 1, pprint("d=", d, "$ is in $D_{a}^{+} - D_{a}^{-} + \cap D_{a}^{-}\$"));
pprint("h_{1}\prime=("s,"h_{+}",t,"\alpha")/2$"), x = apl^2 - 4; y = apl*bpl;
pprint("H=("x,"h_{+}",y,"\alpha")/2$"),
pprint("h_{1}\prime=("s,"h_{+}",t,"\alpha")/2$"),
\gamma_{\text{d}} = "d," "$ is in $D_{a}^{+} - D_{a}^{-} + \cap D_{a}^{-}\$";
x = amin^2 + 4; y = amin*bmin; pprint("H=("x,"h_{+}",y,"\alpha")/2$")));

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