Retrieving Quality Factors for Reflection & Transmission Measurements

Juliang Li∗ and P. Barry, C. Change
High Energy Physics Division, Argonne National Laboratory

This article presents pedagogical explanation of retrieving the resonance parameters $Q_L$, $Q_o$ and $Q_c$ from both reflection and transmission measurement of microwave resonator. Here $Q_L$ stands for the total or loaded quality factor (Q), $Q_o$ is the internal Q and $Q_c$ is the coupling or external Q. Matlab Code based on the methods is available for download for direct calculation of the Qs. [1]

I. MODEL OF REFLECTION MEASUREMENT

For reflective type resonator as shown in figure [1] (a) the transmitted signal measured by the network analyzer is given as: [2, 3]

$$\Gamma_i = ae^{-2\pi f/\tau} \Gamma_d \left[ 1 - \frac{2\kappa e^{i\phi}}{1 + \kappa + j2Q_o \delta_L} \right]$$

(1)

with

$$\delta_L = \delta_i - \frac{X_e R_o}{2Q_o (Z_o^2 + X_e^2)}$$

(2)

$$\delta_i = \frac{\omega - \omega_o}{\omega_o}$$

(3)

Due to the coupling reactance $X_e$ original resonant frequency $\omega_o$ is shifted down by $X_e R_o \omega_o / [2Q_o (Z_o^2 + X_e^2)]$. $\alpha$ accounts for the overall loss and amplification gain of the line. $\tau$ is the cable delay and it tilts the phase of reflected signal by a slope $2\pi \tau$. Without any cable delay the resonance traces out a circle in the complex plane but the cable delay deforms this circle to a loop like curve. $\phi$ stands for the asymmetry in the resonance due to impedance mismatch. Removing this effect is further elaborated in part II D. $\Gamma_d$, defined as in Eq. 4, is the decoupled reflection coefficient away from resonance. Together with cable phase delay $\tau$ it traces out a larger circle with wide enough measurement bandwidth. $\kappa$ (Eq. 5) is the coupling coefficient which tells the strength the resonator is coupled to the external circuit. Both $\gamma_d$ and $\kappa$ are function of $X_e$ and figure [2] shows its effect on the resonance circle in phase space. It rotates the resonance circle away from the real axis by angle $\alpha$ as well as shrinks the resonance circle at the same time. Larger $X_e$ will rotate the circle further away from the real axis and shrink it to a smaller circle.

$$\Gamma_d = \left( \frac{jX_e - Z_o}{jX_e + Z_o} \right) = e^{-2j \\text{atan}(X_e/Z_o)} = e^{-j\alpha}$$

(4)

$$\kappa = \frac{Q_o}{Q_c}$$

(5)

Definition of other parameters are illustrated in figure [1].

*A. Removing cable delay*}

The cable delay deforms the resonance circle to a loop shape as illustrated in figure [3] (a). In phase format the delay was indicated as a sloped phase away from the resonance...
point as shown in figure 3 (b). The cable delay is equal to the slope divided by \( 2\pi \) and can be retrieved by linear fit to the line segment either before or after the resonance. The two slopes from either side of the resonance are generally not equal due to impedance mismatch in the microwave line which could be retrieved in step II D. Preliminary solution is taking the average of the two slopes or linear fit the two line segments together. Once the impedance mismatch is determined at step II D and corrected a linear fit to the phase plot could be fitted to the mismatched corrected data to tune up the results. After removing cable delay the over all phase is flat as shown in figure 4 (a).

B. Circle fit

The resonance circle is rotated away from the real axis with angle \( \alpha \) due to the coupling reactance \( X_c \) (Eq. 2). A circle is fitted to the data to extract the radius and center of the resonance circle. The resonance circle is shifted to the center of the phase space (figure 5 a) and (if impedance mismatch is not present) the resonance point is aligned with the real axis (figure 5 b). Angle \( \alpha \) is calculated with Eq. 6 with \( C_x \) and \( C_y \) being the coordinates of the fitted circle centers.

\[
\alpha = \arctan \left( \frac{C_y}{C_x} \right) \quad (6)
\]

C. Translation to the origin

\[\Gamma_i = a \left( 1 - \frac{2\kappa}{1 + \kappa + j2Q_L\delta L} \right) \quad (7)\]
\[= a \left( 1 - \frac{2\kappa}{1 + j2Q_L\delta L} \right) \quad (8)\]

The equation can also be seen as the sum of vectors in a complex plane

\[
\overrightarrow{\Gamma_i'} = \overrightarrow{\Gamma_i} - \overrightarrow{\Gamma_i'}
\]

(9)

With \( \overrightarrow{\Gamma_i} \) represents the unit vector on the real axis as in figure 6, \( \overrightarrow{\Gamma_i'} \) is the resonance vector that traces out the resonance circle and \( \overrightarrow{\Gamma_i} \) is the vector corresponding to

\[
\Gamma_i = a \left( \frac{2\kappa}{1 + j2Q_L\delta L} \right)
\]

(10)

which will traces out the same resonance circle as if the off resonance point is the origin of the polar coordinate and the phase of the vector is \( \theta_i/2 = \arctan(-j2Q_L\delta L) \). The same circle will be achieved with vector \( \overrightarrow{\Gamma_i} \) as if the origin of the polar coordinate system is moved to the center of the circle which will has the phase \( \theta_i \). When the circle is transferred to the origin and rotated back to align with the real axis, the reflection coefficient is defined by the vector \( \overrightarrow{\Gamma_i} \) which defines the circle in figure 3 b. The diameter \( d \) and angle \( \theta \) of the circle is related to the resonance circle as:

\[
d = a \left( \frac{2\kappa}{1 + \kappa} \right)
\]

(11)
\[
\theta = 2\arctan(-2Q_L\delta L)
\]

(12)

When one selects two frequencies \( f_3 \) and \( f_4 \) where \( \theta_{3,4} = \pm 90^\circ \) (figure 3 b). \( Q_L \) can be calculated as

\[
Q_L = \frac{f_L}{f_3 - f_4}
\]

(13)
The reflection model (1) ignores the coupling loss. When coupling loss present the resonance circle will be distorted and
\[ \theta \approx 2\operatorname{atan}(-2Q_L\delta)L \] (14)
Least square regression should be applied instead to retrieve \( Q_L \) with higher accuracy[2]. After minimizing the quantity:
\[ E = \sum_i \left[ \theta_i - 2\operatorname{atan}\left( -\frac{2Q_L}{f_L}(f_i - f_L) \right) \right]^2 \] (15)
\( Q_L \), \( f_L \) and \( \theta_o \) are retrieved together.

D. Resonance asymmetry

If impedance mismatch is present in the circuit the resonance will be asymmetric in amplitude plot as shown in figure 8. In polar plot the center of the resonance circle is rotated away from the axis connecting the origin of the polar coordinate to the off resonance point by angle \( \phi \) (figure 9). To correct this effect the straight forward method is rotating the circle back after recovering the angle \( \phi \). This method turns to over estimate the quality factor as the diameter to be used to retrieve the Qs should be the segment along the real axis instead of the diameter of the fitted circle as indicated in figure 9[4, 5].

FIG. 6: (a) Phase space illustration of vectors in Eq. 9. \( \vec{r} \) is the vector representing the resonance circle. It could also be represented by vector \( \vec{r}_i \) as if the origin of the polar system is at the off resonance point. For the simplicity of calculating the phase the circle will be represented by vector \( \vec{r}_c \) by shifting the origin of the phase space to the center of the circle. The phase will be twice of the phase of vector \( \vec{r}_i \).

FIG. 7: a. fit of Eq. 15 for retrieving the \( Q_L \) and \( f_L \) b. plot of \( \theta_i \) vs. 2atan \( -\frac{2Q_L}{f_L}(f_i - f_L) \) with the retrieved \( Q_L \) and \( f_L \) to evaluate the fitting quality. A straight line should be achieved with good fitting quality. The upper and lower end of the line is off from the straight line due to impedance mismatch.

FIG. 8: Illustration of resonance with impedance mismatch. a. polar plot of the resonance circle with \( \phi = -0.12\pi \), b. amplitude plot of the resonance dip at the same angle. c. polar plot of the resonance circle with \( \phi = -0.12\pi \), d. amplitude plot of the resonance dip at the same angle.

E. Other Qs

The coupling \( Q_c \) is calculated through
\[ Q_c = \frac{|z_c| + r}{r} Q_L \] (16)
Here \( |z_c| + r \) is a in Eq[1] that accounts for the attenuation and gain in the measurement line. If the circle is rotated due to impedance mismatch, the real \( r \) should be the \( r' \) illustrated in figure 9. Clearly \( r' = r \) if no rotation is present. \( r' \) could be calculated through:
\[ 2r' = 2r \cos \phi \] (17)
Finally the internal \( Q_o \) is calculated through the equation:
\[ \frac{1}{Q_L} = \frac{1}{Q_c} + \frac{1}{Q_o} \] (18)

III. MODEL OF TRANSMISSION MEASUREMENT

For transmission or hanger type resonator, the reflected signal measured by the network analyzer is given as: [4][7]
FIG. 9: illustration of the three angles $\alpha$, $\phi$ and $\theta_i$. $\alpha$ is the rotation of the center of the resonance circle. $\phi$ is the rotation of the circle center away from the segment from the off resonance center to the coordinate origin, which is result of impedance mismatch. With perfect impedance match $\phi = 0$ and the circle center align with the off resonance point and the origin. $\theta_i$ is the rotation of the resonance point away from the segment between the circle center and the origin.

\[\Gamma_i = ae^{-2\pi i f/\tau} \Gamma_d \left[ 1 - \frac{\kappa e^{i\phi}}{1 + \kappa + j2Q_o \delta_i} \right] \quad (19)\]

\[d = \frac{\kappa}{1 + \kappa} \quad (20)\]
\[\tan \theta = -2Q_i \delta_i \quad (21)\]

A. Other Qs

\[Q_e = \frac{|z_e| + r}{2r} Q_i \quad (22)\]

$Q_i$ can be calculated through the same relationship as Eq. 18

Appendix A: Calculation of $\omega_o$ shift $\Delta \omega$ by $X_e$

introduce $\Delta \omega$ into $\omega_o$

\[\delta_i = \frac{\omega}{\omega_o(1 + \frac{\Delta \omega}{\omega_o})} - 1 \quad (A1)\]
\[= \frac{\omega}{\omega_o} - \frac{\Delta \omega}{\omega_o} - 1 \quad (A2)\]
\[= \frac{\omega}{\omega_o} - \frac{\Delta \omega}{\omega_o} - 1 \quad (A3)\]

Compare to Eq. [2]

\[\frac{\Delta \omega}{\omega_o} = \frac{X_e R_o}{2Q_o Z_o^2 + X_e^2} \quad (A4)\]

[1] https://github.com/julesli/Qs_Reflection_Transmission.git
[2] S. Shahid, J. A. R. Ball, C. G. Wells, and P. Wen, Reflection type q-factor measurement using standard least squares methods, IET Microwaves, Antennas Propagation 5, 426 (2011)
[3] D. Kajfez and E. J. Hwan, Q-factor measurement with network analyzer, IEEE Transactions on Microwave Theory and Techniques 32, 666 (1984)
[4] M. S. Khalil, M. J. A. Stoutimore, F. C. Wellstood, and K. D. Osborn, An analysis method for asymmetric resonator transmission applied to superconducting devices, Journal of Applied Physics 111, 054510 (2012), https://doi.org/10.1063/1.3692073
[5] S. Probst, F. B. Song, P. A. Bushev, A. V. Ustinov, and M. Whites, Efficient and robust analysis of complex scattering data under noise in microwave resonators, Review of Scientific Instruments 86, 024706 (2015) https://doi.org/10.1063/1.4907935
[6] J. Gao, The Physics of Superconducting Microwave Resonators, Ph.d. diss., Pasadena CA (2008).
[7] A. Megrant, C. Neill, R. Barends, B. Chiaro, Y. Chen, L. Feigl, J. Kelly, E. Lucero, M. Marriott, P. J. J. O’Malley, D. Sank, A. Vainsencher, J. Wenner, T. C. White, Y. Yin, J. Zhao, C. J. Palmstrøm, J. M. Martinis, and A. N. Cleland, Planar superconducting resonators with internal quality factors above one million, Applied Physics Letters 100, 113510 (2012) https://doi.org/10.1063/1.3693409