The thermodynamics of self-gravitating systems in equilibrium is holographic

Ntina Savvidou and Charis Anastopoulos

Department of Physics, University of Patras, Patras, 26500, Greece
E-mail: ksavvidou@physics.upatras.gr and anastop@physics.upatras.gr

Received 19 September 2013, revised 30 December 2013
Accepted for publication 15 January 2014
Published 11 February 2014

Abstract
We show that, when we study the coexistence of the general relativity with thermodynamics, some physical properties that are usually thought of as holographic and lying in the domain of quantum gravity can actually be accessed even at the classical level. In particular, we demonstrate that the thermodynamics of the gravitating systems in equilibrium is fully specified by the variables defined on the system’s boundary, namely, the boundary’s geometry and extrinsic curvature. Hence, information is non-trivially incorporated in boundary variables because of the structure (the symmetries) of the classical gravity theory, without any input from the quantum theory (such as black hole entropy).

Keywords: self-gravitating systems, entropy, holographic principle, axiomatic thermodynamics
PACS numbers: 04.20.Cv, 04.40.-b, 05.70.—a

1. Introduction
The association of entropy to black holes by Bekenstein and Hawking strongly suggests a fundamental relationship between gravity and thermodynamics. It has led to the conjecture that the fundamental theory of gravity should satisfy the holographic principle [1], namely, the statement that the full information about a gravitational system is contained in the degrees of freedom of the system’s boundary.

The holographic principle refers to a mapping of bulk degrees of freedom into boundary degrees of freedom at the Planck scale. In this work, we designate the equilibrium thermodynamics of self-gravitating systems as ‘holographic’ because the thermodynamic state space consists solely of geometric variables defined at the boundary; furthermore, measurements at the boundary suffice for establishing the thermodynamic properties of the system. This is a restricted sense of holography, but it is highly non-trivial and it appears at the level of the coarse-grained quasiclassical description of an underlying quantum gravity
theory—like, for example, in the proposals of [7–9]. The most important feature of the holographic properties presented here is that they require no input from the quantum theory (such as black-hole entropy), let alone quantum gravity effects. They originate from the structure of the classical gravity theories and the basic principles of thermodynamics.

Our result follows from an important relation between Einstein’s equations for static spacetimes and the principle of maximum entropy. Einstein’s equations for a general static spacetime follows from the maximization of matter entropy subject to the continuity equation for the stress–energy tensor and the initial value constraints—for an earlier proof of this statement, see [2, 3] for previous work and [4–6] for special cases. Here, we show that, for solutions to Einstein’s equations, only boundary terms contribute to the total entropy of the system and that all thermodynamic properties are encoded into geometric variables on the boundary.

The structure of the paper is as follows. In section 2, we briefly describe the proof that the maximum entropy principle leads to Einstein’s equations for static spacetimes and we compute explicitly the boundary term associated with the variations of the appropriate thermodynamic potential. These results are employed in section 3, in order to construct the thermodynamic state space of the self-gravitating systems, explain the precise sense in which it is holographic and set out the basic thermodynamic properties for self-gravitating systems. In section 4, we summarize our results and their implications.

2. The maximum entropy principle in self-gravitating systems

In this section, we have shown that the principle of maximum entropy for matter in the bulk implies that, for solutions to Einstein’s equations, all thermodynamic variations can be expressed in terms of geometric properties of the boundary.

2.1. Notation and definitions

We consider a static globally hyperbolic spacetime $M = R \times \Sigma$ with 4-metric

$$d\ell^2 = -L^2(x) dt^2 + h_{ij}(x) dx^i dx^j,$$

expressed in terms of the spatial coordinates $x^i$ and the time coordinate $t$. The time-like vector field $\xi^\mu = (\partial/\partial t)^\mu$ is a Killing vector of the metric equation (1). $L$ is the lapse function, and $h_{ij}$ is a $t$-independent Riemannian 3-metric on the surfaces $\Sigma_t$ of constant $t$. The time-like unit normal on $\Sigma_t$ is $n_\mu = L \partial_\mu t$ and the extrinsic curvature tensor on $\Sigma_t$ vanishes.

Let $C \subset \Sigma$ be a compact spatial region, with boundary $B = \partial C$. $C$ contains an isotropic fluid in thermal and dynamical equilibrium, described by the stress–energy tensor

$$T_{\mu\nu} = \rho(x)n_\mu n_\nu + P(x)(g_{\mu\nu} + n_\mu n_\nu),$$

where $\rho(x)$ and $P(x)$ are the energy density and the pressure, respectively. The stress–energy tensor satisfies the continuity equation $\nabla_\mu T^{\mu\nu} = 0$ which implies that

$$\frac{\nabla P}{\rho + P} = -\frac{\nabla L}{L}. \quad (3)$$

Equation (2) is the standard form of the stress–energy tensor for ideal fluids. Since dissipative processes are absent in equilibrium configurations, equation (2) also applies to non-ideal fluids.

We assume that the fluid consists of $q$ different particle species. The associated particle-number densities $n_a(x)$, $a = 1, \ldots, q$, together with the energy density $\rho(x)$, define the
thermodynamic state space. We assume that all local thermodynamic properties of the fluid are encoded in the entropy-density functional \( s(\rho, n_a) \). The first law of thermodynamics for the fluid takes the form

\[
Td\sigma = d\rho - \sum a \mu_a d n_a,
\]

(4)

where \( \mu_a = -T \frac{\partial s}{\partial n_a} \) is the chemical potential associated with the particle species \( a \) and \( T = \left( \frac{\partial s}{\partial \rho} \right)^{-1} \) is the local temperature of the fluid. The pressure \( P \) is defined through the Euler equation

\[
\rho + P - Ts - \sum a \mu_a n_a = 0.
\]

(5)

Combining equations (5) and (4), we derive the Gibbs–Duhem relation, \( dP = sdT + \sum a n_a d\mu_a \).

### 2.2. Entropy maximization

Next, we maximize the total entropy of matter \( S = \int_C d^3x \sqrt{h} s(\rho, n_a) \) for fixed values of the total particle numbers in \( C \), \( N_a = \int_C d^3x \sqrt{h} n_a(\rho, n_a) \) and for fixed values of the fields on the boundary.

Entropy maximization is subject to the continuity equation (3) for the fluid, and to the first-class constraints of general relativity. For static spacetimes, the Hamiltonian constraint reads

\[
\mathcal{H}(x) := 16\pi \rho(x) - R(x) = 0,
\]

(6)

where \( R \) is the Ricci scalar associated with the 3-metric \( h_{ij} \). The momentum constraint has been implemented in the choice of coordinates corresponding to the metric equation (1), namely, the gauge-fixing condition that the shift vector vanishes.

The Hamiltonian constraint expresses the energy density \( \rho \) as a function of the metric \( h_{ij} \) and the continuity equation expresses the lapse \( L \) in terms of the energy density \( \rho \). It follows that the entropy \( S \) and the particle numbers \( N_a \) are functionals of the 3-metric \( h_{ij} \) and the particle-number densities \( n_a \). Entropy maximization for fixed values of \( N_a \) requires the variation of \( S + \sum a b_a N_a \) with respect to \( n_a \) and \( h_{ij} \) for some Lagrange multipliers \( b_a \).

Variation of \( S + \sum a b_a N_a \) with respect to \( n_a \) yields,

\[
\delta S = \sum a \int_C d^3x \sqrt{h} \left( \frac{\partial s}{\partial n_a} + b_a \right) \delta n_a = \sum a \int_C d^3x \sqrt{h} \left( -\frac{\mu_a}{T} + b_a \right) \delta n_a = 0,
\]

(7)

leading to \( b_a = \frac{\mu_a}{T} \). Hence, for equilibrium configurations, the thermodynamic variables \( \frac{\mu_a}{T} \) are constant in \( V \).

We introduce the function \( \omega \) (a Massieu function [11]) as the Legendre transform of the entropy density \( s \) with respect to \( n_a \)

\[
\omega(\rho, b_a) := s - \sum a \frac{\partial s}{\partial n_a} n_a = s + \sum a b_a n_a = \frac{\rho + P}{T}.
\]

(8)

Expressed in terms of \( \omega \), equation (4) takes the form

\[
d\omega = \frac{d\rho}{T} - \sum a n_a db_a.
\]

(9)

It follows that \( T^{-1} = \frac{d\omega}{d\rho} \) and \( n_a = -\frac{\partial \omega}{\partial b_a} \). The Gibbs–Duhem relation becomes \( dP = \omega dT + T \sum_a n_a db_a \).
Since the variables $b_a$ are constant for entropy-maximizing configurations, $dP/dT = \omega = (P + \rho)/T$. Combining with equation (3), we obtain
\[ \frac{\nabla_i T}{T} = -\frac{\nabla_i L}{L}, \]
which leads to the Tolman’s relation between local temperature and lapse function [10]
\[ LT = T^*, \]
where $T^*$ is a constant. In an asymptotically flat spacetime, $L = 1$ at spacelike infinity. Then, $T^*$ is identified with the temperature as seen by an observer at infinity.

2.3. The thermodynamic boundary term

Next, we show that for solutions to Einstein’s equation, $\delta\Omega$ becomes a boundary term.

By employing the definition (8), entropy maximization is equivalent to the maximization of the functional
\[ \Omega[h_{ij}, b_a] := \int_C d^3x \sqrt{h} \omega(\rho, b_a). \]
Since the density $\rho$ depends on the metric $h_{ij}$ through the Hamiltonian constraint (6) and $b_a$ are constant, $\Omega$ is a function on the space $\text{Riem}(C)$ of all Riemannian metrics $h_{ij}$ on $C$. Variation with respect to $h_{ij}$ yields
\[ \delta \Omega = \int_C d^3x \sqrt{h} \left[ \frac{\omega}{2} h^{ij} \delta h_{ij} + \frac{\partial \omega}{\partial \rho} \delta \rho \right] = \int_C d^3x \sqrt{h} \left( \frac{\rho + P}{2T} h^{ij} \delta h_{ij} + \frac{\delta R}{16\pi T} \right). \]
Using the equation $\delta R = -R^{ij} \delta h_{ij} + \nabla^i (\nabla^j \delta h_{ij} - h^{kl} \nabla^i \delta h_{kl})$ for the variation of the Ricci scalar $R$, together with equation (10), we find
\[ \delta \Omega = -\int_C d^3x \sqrt{h} \left[ S^{ij} \delta h_{ij} + \frac{1}{16\pi} \int_C d^3x \sqrt{h} \left[ \nabla^i (T^{-1} \nabla^j \delta h_{ij} - T^{-1} h^{kl} \nabla^i \delta h_{kl}) - \nabla^j \left( \frac{N}{LT} \delta h_{ij} - \frac{\nabla_j N}{LT} h^{ij} \delta h_{kl} \right) \right] \right], \]
where
\[ S^{ij} := R^{ij} - \frac{1}{2} h^{ij} R - \frac{1}{L} (\nabla^i \nabla^j L - h^{ij} \nabla_k L) - 8\pi h^{ij} P. \]

The condition $S^{ij} = 0$ coincides the spatial components of Einstein’s equations for the static spacetime metric, equation (1). For fixed values of the metric $h_{ij}$ and of its first derivatives on the boundary $B$ of $C$, $\delta \Omega = -\int_C d^3x \sqrt{h} S^{ij} \delta h_{ij}$. Thus, the principle of maximum entropy $\delta \Omega = 0$ leads to the full set of Einstein’s equations [2] for static spacetimes.

In order to properly interpret this result, we note that the equation $S^{ij} = 0$ corresponds to Hamilton’s equations for the momentum $\pi^{ij}$, conjugate to $h_{ij}$ [12]. In the Hamiltonian formulation of general relativity, the equations of motion incorporate the symplectic structure (Poisson bracket) and the constraints of the gravitational state space. The Hamiltonian constraint is employed in the derivation of equation (19); and the momentum constraint is implemented in the specification of the metric equation (1) and the gauge-choice of vanishing shift vector. It follows that the thermodynamic description of the gravitating systems contains implicitly the information about the symplectic structure of the theory [13].

1 It was shown that the symplectic form of general relativity incorporates the notion of logical temporal ordering of the theory. The expectation that this temporal ordering may be associated with the temporal ordering due to thermodynamic irreversibility has been a major motivation for the present work. This idea is a part of the work in preparation by one of us (NS).
However, the relation between the maximum entropy principle and Hamilton’s equations, as incorporated in equation (14), is not specific to general relativity. It only requires the existence of the first-class constraints that relate the metric $h_{ij}$ and the energy density $\rho$ (for static configurations). Thus, it applies to a larger class of the parameterized systems (i.e., systems whose Hamiltonian vanishes by virtue of first-class constraints). Consider, for example, a parameterized theory described by the metric $h_{ij}$ and its conjugate momentum $\pi_{ij}$, and subject to a Hamiltonian constraint of the form

$$H = T(\pi, h) + V(h) - \rho,$$

where $T$ is a ‘kinetic-energy’ term and $V$ a ‘potential’ term that may involve arbitrarily high derivatives of the 3-metric $h_{ij}$. Assume that $T$ and $\delta T/\delta \pi_{ij}$ vanish for $\pi_{ij} = 0$. For this system, Hamilton’s equations for static solutions also follow from the maximum entropy principle.

The total divergence term in equation (14) determines the thermodynamic properties of the solutions to Einstein’s equations. We evaluate this term by performing a $2 + 1$ decomposition of the metric $h_{ij}$ at the boundary $B$.

Let $B$ be described locally by the condition $f(x) = 0$. The unit normal to $B$ is $m_i = \alpha \nabla_i f$, where $\alpha = 1/\sqrt{h_{ij} \nabla_i f \nabla_j f}$. The induced metric on $B$ is

$$\sigma_{ij} = h_{ij} - m_im_j,$$  

(17)

The tensor $\sigma^l_j = \delta^l_j - m^l m_j$ projects onto the 2-surface $B$. Then, we define the extrinsic curvature tensor of $B$ as

$$\kappa_{ij} := \sigma^k_i \sigma^l_j \nabla_k m_l.$$  

(18)

The metric and extrinsic curvature satisfy $\sigma_{ij} m^l = \kappa_{ij} m^l = 0$. Hence, they are tensors on the 2-surface $B$.

Then, for solutions to Einstein’s equations, equation (14) yields

$$\delta \Omega = \frac{1}{16\pi T_\ast} \oint_B \frac{d^2 y \sqrt{\sigma}}{\sqrt{\sigma}} (m^l \nabla_i \Lambda^i \delta \sigma_{ij} - 2\Lambda^{ij} \delta \kappa_{ij} + \Lambda_{ij} \delta \sigma_{ij}),$$  

(19)

where the coordinates on $B$ are denoted as $y$.

A remarkable property of $\delta \Omega$, equation (19), is that it depends only on the variations $\delta \sigma_{ij}$ of the 2-metric and $\delta \kappa_{ij}$ of the extrinsic curvature $\kappa_{ij}$ of the boundary. The variations of the remaining components of the 3-metric $h_{ij}$ and of its first derivatives do not contribute to $\delta \Omega$. This is a specific property of general relativity: the boundary term (19) depends crucially on the form of the Hamiltonian constraint. The independence of $\delta \Omega$ from variations other than $\delta \sigma_{ij}$ and $\delta \kappa_{ij}$ does not follow from a generic first-class constraint, such as equation (16).

3. Thermodynamics of self-gravitating systems

3.1. The thermodynamic state space

In what follows, we argue that equation (19) leads to holographic thermodynamics for self-gravitating systems in equilibrium. We employ the term ‘holographic thermodynamics’ in the sense that the state space consists only of boundary variables and that all thermal properties of the system can be determined from local measurements at the boundary.

Thermodynamic systems are defined in terms of a physical boundary that distinguishes the system from the rest of the universe. Equilibrium configurations are characterized by a number of quantities that are conserved in absence of external intervention; these quantities include the internal energy $U$ and the particle-species numbers $N_a$. In a flat background spacetime, the geometry of the boundary fully determines the enclosed volume $V$. Ordinary
thermodynamic systems are extensive, i.e., their thermodynamic variables scale linearly with
the volume \( V \), so properties of the boundary other than the enclosed volume do not contribute
to thermodynamics. In such systems, the state space \( \Gamma \) of a thermodynamic system consists
of the variables \( (V, U, N_\nu) \), and the thermodynamic description requires the definition of an
entropy functional on \( \Gamma \).

The physical boundary \( B \) of a self-gravitating system in equilibrium may correspond to
a bounding box (as in ordinary thermodynamics) or to a stellar surface. \( B \) separates between
a region in which the metric \( h_{ij} \) is given by a solution of Einstein’s equation with matter (the
interior) and a region where \( h_{ij} \) is a solution of the vacuum Einstein equations (the exterior).
For stellar-surface boundary conditions, the pressure \( P \) vanishes at \( B \). When \( B \) corresponds to a
bounding box, the description of the spacetime geometry requires the use of junction conditions
on \( B \). According to the thin-shell formalism [12, 14], the variables \( \sigma_{ij} \) and \( \kappa_{ij} \) are continuous
across \( B \), while derivatives of the 3-metric \( h_{ij} \) normal to \( B \) exhibit discontinuities proportional
to the stress–energy tensor of the box. It is, therefore, significant that the only variables whose
variations contribute to \( \delta \Omega \) in equation (19) are the ones that remain continuous across \( B \). If the
variation of other variables, discontinuous across \( B \), contributed to \( \delta \Omega \), the thermodynamics
would be strongly dependent on the physical properties of the box, as encoded in its boundary
stress–energy tensor.

An equilibrium configuration corresponds to a specific solution \( h_{ij} \) of Einstein’s equations
in the interior of \( B \). According to equation (19), only variations of the 2-metric \( \sigma_{ij} \) and of the
extrinsic curvature \( \kappa_{ij} \) have thermodynamic significance, i.e., their variations contribute to
\( \delta \Omega \). By equation (18), the extrinsic curvature can be viewed as a tangent vector on the space
of Riemannian metric on \( \text{Riem}(B) \); hence, the tensors \( \sigma_{ij} \) and \( \kappa_{ij} \) span the tangent bundle
\( T\text{Riem}(B) \).

Viewed as boundary data for Einstein’s equations \( S^{ij} = 0 \) in the interior, \( \sigma_{ij} \) and \( \kappa_{ij} \) do
not suffice for finding a unique solution. Hence, in principle, many different solutions
to Einstein’s equations correspond to the same values of the thermodynamic variables on the
boundary. Moreover, the integration of the Einstein’s equation from the boundary inwards may
lead to singularities or other forms of internal boundaries—we have given such examples in
section 3.2. In such a case, the variation of \( \delta \Omega \), equation (19) has to incorporate a contribution
from the internal boundary which is, in general, inaccessible to an external observer.

Here, we restrict to the case of regular 3-metrics \( h_{ij} \), i.e., to 3-metrics that are everywhere
locally Minkowskian in the interior of the physical boundary \( B \). In this case, equation (19)
involves integration only over the physical boundary of the system and no contributions from
the internal boundaries. Hence, all variations in equation (19) can be attributed to interventions
by an external observer, as is necessary for the consistent thermodynamic description of the
system.

The thermodynamic properties of the system enclosed by \( B \) are fully specified by the
function \( \Omega \), which depends on the surface fields \( \sigma_{ij} \) and \( \kappa_{ij} \) and on the variables \( h_{\nu} \). The
latter are constant in the interior, and thus their value is also determined at the boundary.
It follows that the thermodynamic state space for a self-gravitating system in equilibrium
\( \Gamma = T\text{Riem}(B) \times \mathbb{R} \) consists only of variables that are accessible at the system’s physical
boundary.

As in ordinary thermodynamics, we define the thermodynamic conjugate variables to \( \sigma_{ij} \)
and \( \kappa_{ij} \) as

\[
A^{ij} := \frac{\delta \Omega}{\delta \sigma_{ij}} = \frac{1}{16\pi T_\nu} \sqrt{\sigma} \left[ \left( m' \nabla_i L - L \sigma^{ij} \sigma_{kj} \right) \right], \quad B^{ij} := \frac{\delta \Omega}{\delta \kappa_{ij}} = -\frac{1}{8\pi T_\nu} L \sqrt{\sigma} \sigma^{ij}.
\]

The conjugate variables above are tensor fields defined on the physical boundary \( B \). The
particle numbers \( N_\nu = -\frac{\delta \Omega}{\delta v_\nu} \) are the other thermodynamical conjugates and these are bulk
variables. However, the particle numbers are conserved in a closed system, and their value can be ascertained from information about the process through which the system was assembled. An external observer with no knowledge about the system other than the conserved numbers \( N_i \), can fully reconstruct the equations of state (the functional relation between state-space variables and their conjugates) solely by local measurements at the boundary. Thus, the thermodynamics of self-gravitating systems exhibit striking manifestation of the holographic principle; all hydrodynamic bulk variables (density, pressure) are encoded into geometric properties of the boundary.

**Boundary temperature and pressure.** In ordinary thermodynamic systems, the conjugate variables on the thermodynamic state space correspond to the temperature and the pressure. In order to interpret the conjugate variables \( A^j \) and \( B^j \), equation (20), in terms of temperature and pressure, we consider the projection of the equations of motion along the normal \( m^j \) to the boundary \( B \), \( S_j m^j m^l = 0 \). We obtain

\[
m^j \nabla_i L = \frac{L}{\kappa} \left[ 8\pi P + \frac{1}{L} \sigma^{ij} \nabla_i \nabla_j L + \frac{1}{2} (\kappa_{ij} k^{ij} - \kappa^2 + 2R) \right]
\]

where \( 2R \) is the curvature scalar on \( B \), \( \kappa = \kappa^i \sigma_{ij} \), and \( 2 \nabla_i \) the covariant derivative associated with \( \sigma_{ij} \).

A natural thermodynamic assumption is that the local temperature is constant at the boundary \( B \), \( 2 \nabla_i T = 0 \), so that there is no boundary heat flow. This implies that \( 2 \nabla_i L = 0 \) in equation (21). Denoting the temperature at the boundary as \( T_B \), equation (20) leads to the equations of state for boundary pressure and momentum in a self-gravitating system

\[
\frac{1}{T_B} = -\frac{8\pi}{\sqrt{\sigma}} \frac{\delta \Omega}{\delta \kappa_{ij}} \frac{1}{\sigma_{ij} \kappa}
\]

\[
P = \frac{\kappa}{\sqrt{\sigma}} \frac{\delta \Omega}{\delta \sigma_{ij}} \frac{1}{32\pi T_B} (\kappa_{ij} k^{ij} + \kappa^2 + 2R)
\]

**Boundary diffeomorphisms.** From equations (22) and (23) we observe that there are fewer functionally independent conjugate field variables (only the pressure \( P \)) than variables of \( \Gamma \). This is problematic for defining different thermodynamic representations of the system (such as the enthalpy or the free energy representations) using the Legendre transform. There are not enough conjugate variables to capture the information contained in the variables of \( \Gamma \).

This issue arises because we have not taken into account the diffeomorphism symmetry of the boundary, namely the fact that the thermodynamic description ought to be independent of the choice of the coordinate system on the boundary \( B \). Let us assume for concreteness that the boundary is topologically a 2-sphere: \( B = S^2 \). For any 2-metric \( \sigma \), there exists a coordinate system \( \gamma^m \) on \( S^2 \), s.t. \( \sigma_{nm} = e^\phi \tilde{\sigma}_{nm} \), where \( \phi \) is a scalar field and \( \tilde{\sigma}_{nm} \) the homogeneous sphere metric of unit scalar curvature. Then, equation (19) becomes

\[
\delta \Omega = \frac{1}{16\pi T_o} \oint_B d^2y \sqrt{\tilde{\sigma}} e^\phi [2m^j \nabla_j L - L \kappa] \delta \phi - 2 L \delta \kappa \]

The independent variations in equation (24) correspond to the conformal factor \( \phi \) of the 2-metric and the trace of the extrinsic curvature \( \kappa \). This implies that the reduced state space \( \Gamma_{red} \) of true thermodynamical degrees of freedom consists of \( (\phi(y), \kappa(y), b_o) \) and it can be expressed as \( \Gamma_{red} = T(\text{Riem}(B)/\text{Diff}(B)) \times \mathbb{R}^l \). Let us denote by \( \pi : \Gamma \to \Gamma_{red} \) the projection map \( \pi(\sigma_{ij}, k_{ij}, b_o) = (\phi, \kappa, b_o) \). Then the thermodynamic function \( \Omega \) on \( \Gamma \) should be of the form \( \Omega = \pi^* \tilde{\Omega} \), where \( \tilde{\Omega} \) is a function on \( \Gamma_{red} \).
Equations (22) and (23) imply that there exist two functionally independent fields on \( B \) among the conjugate variables \( \delta \Omega / \delta \phi (\tilde{y}) \) and \( \delta \Omega / \delta \kappa (\tilde{y}) \). These functionally independent fields may be chosen as the pressure \( P \) and the scalar \( \kappa_{ab} \kappa^{ab} \). Hence, there is same number of independent thermodynamically conjugate fields with the number of fundamental thermodynamic field \( \phi \) and \( \kappa \) on \( \Gamma_{\text{red}} \). There is no problem of principle in defining alternative thermodynamic representations through the Legendre transform of the function \( \Omega \) on \( \Gamma_{\text{red}} \).

### 3.2. Non-regular solutions

Next, we elaborate the physical interpretation of the non-regular solutions to Einstein’s equations, and conjecture that their thermodynamic properties could lead to a much stronger version of the holographic thermodynamics than the one derived here.

Our analysis so far is restricted to thermodynamic systems that correspond to the regular solutions of Einstein’s equations in the interior region. As explained earlier, the thermodynamic variables on the boundary under-determine the solution to Einstein’s equations. Hence, it is in principle possible, that the same values of the thermodynamic variables correspond to solutions of Einstein’s equations in the interior region. As explained earlier, the thermodynamic version of the holographic thermodynamics than the one derived here.

Next, we elaborate the physical interpretation of the non-regular solutions to Einstein’s equations clarifies this point. The interior solution corresponds to a 3-metric

\[
d s_3^2 = \frac{d r^2}{1 - \frac{2M}{r}} + r^2 (d \theta^2 + \sin^2 \theta \; d \phi^2),
\]

in terms of the standard coordinates \((r, \theta, \phi)\); \( m(r) \) is the mass function related to the density \( \rho \), as \( dm/dr = 4\pi r^2 \rho \). The boundary \( B \) corresponds to constant ‘radius’ \( r = r_0 \). For \( r > r_0 \), the system is described by the Schwarzschild metric with mass \( M \). Then, the 2-metric and extrinsic curvature on \( B \) are

\[
d r^2 = r_0^2 (d \theta^2 + \sin^2 \theta \; d \phi^2), \quad \kappa_{ij} = \frac{\sqrt{1 - \frac{2M}{r_0}}}{r_0} \sigma_{ij},
\]

i.e., they only depend on the two parameters \( M \) and \( r_0 \), subject to the condition \( r_0 > 2M \). It follows that the thermodynamic state space \( \Gamma \) of a spherically symmetric system is \((q + 2)\)-dimensional.

The interior metric \( h_{ij} \) is a solution to the Oppenheimer–Volkoff (OV) equation. A unique solution to the OV equation requires the determination of \( r_0 \), \( m(r_0) = M \) and of the pressure \( P(r_0) \) at the boundary, together with the variables \( b_s \). Thus, the space \( \Gamma_0 \) of all solutions is \((q + 3)\)-dimensional, and \( \Gamma \) is a submanifold of \( \Gamma_0 \), of codimension one.

A metric equation (25) is regular if \( m(0) = 0 \); \( m(0) \) may take no positive values for solutions to the OV equation [15]. For \( m(0) = -M_0 \), \( (M_0 > 0) \), the metric near \( r = 0 \) is approximated by

\[
d s_3^2 = \frac{r dr^2}{2M_0} + r^2 (d \theta^2 + \sin^2 \theta \; d \phi^2).
\]

The proper radius coordinate corresponding to the metric equation (27) is \( x = \frac{r^3}{M_0} / \sqrt{2M_0} \). A 2-sphere of proper radius \( x \) around \( r = 0 \) has area equal to \( 4\pi (\frac{r}{M_0})^{2/3} \). This implies that the spacetime is not locally Minkowskian around \( r = 0 \) (since in that case the area should be \( 4\pi x^2 \)), and that the point \( r = 0 \) is a conical singularity. This singularity does not give rise to inextensible causal geodesics, and it corresponds to an interior boundary.
of spacetime. Hence, the space of the regular solutions $\Gamma_{\text{reg}}$ is also a submanifold of $\Gamma_0$ of codimension one.

However, $\Gamma_{\text{reg}}$ does not coincide with the thermodynamic state space $\Gamma$. For some values of $(M, r_0, b_a)$ there exist more than one elements of $\Gamma_{\text{reg}}$, and for some other there exist none. It would seem that the thermodynamic state space $\Gamma$ should be restricted to the values of $(M, r_0)$ that correspond to a unique element of $\Gamma_{\text{reg}}$; otherwise the pressure at the boundary cannot be uniquely determined from the knowledge of $(M, r_0, b_a)$ and the equations of state are ambiguous.

In [16], we argued that the restrictions described above are problematic for the conceptual consistency of gravitational thermodynamics. We proposed a resolution to this problem: a unique solution of Einstein’s equation will be assigned to each element of $\Gamma$ through the principle of maximum entropy. Of all solutions to Einstein’s equations characterized by the same boundary conditions (corresponding to an element of $\Gamma$), the physical solution will be selected by the requirement that it maximizes the entropy functional. The implementation of this idea requires the assignment of an entropy term to the internal boundaries of the non-regular solutions, which we interpreted as a form of gravitational entropy to be added to the entropy of matter. We applied this idea successfully to a model system of spherically symmetric self-gravitating radiation in a bounding box [16]. We found that the maximum entropy principle always selects a (unique) regular solution as an equilibrium configuration, provided a regular solution exists for given values of $(M, r_0)$; if a regular solution does not exist, then a non-regular solution is chosen uniquely. The generalization of this result to a larger class of spacetimes, and eventually to any self-gravitating system in equilibrium, is currently under investigation. This would lead to a much stronger version of the holographic principle than the one presented here, namely that the thermodynamic variables at the boundary fully specify the geometry of the interior.

3.3. Weak gravity limit

At the weak gravity limit (i.e., for low densities and small system size), we expect that the thermodynamics of self-gravitating system reduces to ordinary thermodynamics expressed in terms of extensive variables. In extensive thermodynamics, all variables are homogeneously distributed inside a bounding box $B$. Hence, at the weak gravity limit, the matter density $\rho$ is approximately constant on the interior of the box. By equation (6), the Ricci scalar $R$ will also be constant. Hence, the geometry inside the box corresponds to a region of a homogeneous 3-sphere of radius $\sqrt{6}/R$.

The thermodynamic properties of an extensive system are not affected by the shape of the bounding box; they depend only on the enclosed volume. Therefore, it suffices to consider a spherical boundary box $r = r_0$. It is convenient to employ the spherical coordinates $(r, \theta, \phi)$ for the interior metric.

$$\text{d}s^2 = \frac{\text{d}r^2}{1 - \frac{r^2}{6}} + r^2(\text{d}\theta^2 + \sin^2 \theta \, \text{d}\phi^2).$$

At the weak gravity limit, $R$ is close to zero, so equation (28) is close to the Minkowskian metric.

For a spherical bounding box $B$, the 2-metric $\sigma_{ij}$ and the extrinsic curvature $\kappa_{ij}$ are given by equation (26). Denoting by $\kappa_{ij}^{(0)}$, the extrinsic curvature of the surface $r = r_0$ when embedded in Minkowski spacetime, we find

$$\Delta \kappa := \kappa - \kappa_{ij}^{(0)} \simeq -\frac{R r_0^2}{12} \kappa_{ij}^{(0)}$$

$$\kappa_{ij}^{(0)} = \frac{1}{r^2} \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \right) \kappa_{ij}^{(0)}.$$
to leading order in $R$, with $\kappa^{(0)} = -2/r_0$. Thus, the energy density $\rho$ of the ordinary, extensive thermodynamics is proportional to the deviation of the extrinsic curvature from its Minkowskian value, while the box’s geometry is kept constant. To leading approximation, the volume enclosed by the box is $V_0 := \frac{4\pi}{3} r_0^3$ and the internal energy $U = \rho V_0$ of the system is

$$U = -\frac{r_0}{12} \frac{\Delta \kappa}{\kappa^{(0)}},$$

(30)

The function $\Omega(U, V_0, b_0) \propto \omega(\rho, b_0)V_0$ is expressed solely in terms of the geometric variables $r_0$ and $\Delta \kappa$ of the boundary. Hence, the theory of the ordinary extensive thermodynamics can be reformulated in terms of the Riemannian geometry of the 3-sphere.

Finally, we note that an alternative characterization for $U$ can be made in terms of the actual volume $V$ enclosed by the spherical boundary, whence

$$U = \frac{3r_0}{5} \Delta V,$$

(31)

where $\Delta V = V - V_0$.

We must emphasize here that, the spacetime curvature persists even at the weak gravity limit and it is essential for the thermodynamic description in terms of the boundary variables. The holographic properties presented here are not accessible in the Newtonian theory of gravity.

3.4. Thermodynamic properties

Next, we formulate the basic laws of thermodynamics in light of the previous results, we examine possible candidates for the notion of internal energy in self-gravitating systems and we derive non-trivial thermodynamic inequalities.

Zero-th law of thermodynamics. The generalization of the zero-th law of thermodynamics to self-gravitating systems is provided by Tolman’s relation between lapse function and local temperature, equation (11). Indeed, for an asymptotically flat spacetime, equation (11) is equivalent to the statement that the temperature of a self-gravitating body in equilibrium is everywhere constant when transformed to the frame of an observer at infinity.

From the analysis of section 2, we note that the only assumptions required for the derivation of the equation (11) are (i) that the total entropy is expressed in terms of an entropy density $s$, and (ii) that the Hamiltonian constraint does not involve the particle densities $n_a$. Tolman’s relation is essentially kinematical, in the sense that its derivation does not depend on the specific form of Einstein’s equations—it holds for any theory described by a constraint of the form equation (16). In effect, Tolman’s law follows from Galileo’s principle that gravity is insensitive to the physical composition of the material bodies.

We also note the inter-relation between the zero-th law of thermodynamics and the continuity equation, equation (3). Either of the two can be assumed as a priori given, and the other will be derived as a consequence of the maximum entropy principle.

First law of thermodynamics. Including the contribution from variations to $b_a$ in equation (19) and using equation (21), we derive the first law of thermodynamics for a self-gravitating system,

$$\delta \Omega = \frac{1}{T_*} \int_B \frac{d^3y LP}{\kappa} \delta \left(\sqrt{\sigma}\right) + \sum_a N_a \delta b_a$$

$$+ \frac{1}{16\pi T_*} \int_B d^3y \sqrt{\sigma} \left( \kappa^{ij} \delta \sigma_{ij} - 2 \sigma^{ij} \delta \kappa_{ij} + \frac{\kappa_{ij} \kappa^{ij} - \kappa^2 + 2^2}{2\kappa} \sigma^{ij} \delta \sigma_{ij} \right),$$

(32)
or, using the entropy functional $S$ instead of $\Omega$

$$\Theta = T_0 dS - \oint_B d^2y \frac{LP}{\kappa} \delta(\sqrt{\sigma}) + \sum_a (L\mu_a) y \delta N_a,$$

(33)

where we denoted the second line of equation (32) by $\Theta/T_0$; $\Theta$ is a 1-form on the thermodynamic state space $\Gamma$. The terms in the right-hand side of equation (33) have an obvious thermodynamic significance, as heat, mechanical work and ‘chemical’ work. However, $\Theta$ is not an exact form: $\delta \Theta \neq 0$. So, $\Theta$ cannot be written as $\delta U$, where $U$ is a scalar function on $\Gamma$, the system’s internal energy. $\Theta$ also contains work terms.

There is no obvious definition of internal energy for a general self-gravitating system. This is due to the fact that the maximum entropy principle leading to equation (19) does not require the specification of constant internal energy [12]. Candidates for the internal energy such as the Komar mass (see below) are boundary terms; hence, they do not affect the derivation of Einstein’s equations from the maximum entropy principle. The internal energy is not an independent variable in the thermodynamic state space of self-gravitating systems—unlike ordinary thermodynamics—but a function of the more fundamental geometric variables $\sigma_{ij}$ and $\kappa_{ij}$.

For a spherically symmetric system, the variables $\sigma_{ij}$ and $\kappa_{ij}$ are given by equation (26), and $L = \sqrt{1 - 2M/\rho_0}$. The 1-form $\Theta$ becomes $\Theta = \delta M$, and the first law of thermodynamics simplifies

$$\delta M = T_0 dS - P(\rho_0) \left(4\pi \rho_0^2 \delta \rho_0 \right) + \sqrt{1 - 2M/\rho_0} \sum_a \mu_a(\rho_0) \delta N_a.$$  

(34)

Hence, in spherically symmetric systems, internal energy is identified with the Arnowitt–Deser–Misner mass $M$, and the 1-form $\Theta$ contains no work terms. This suggests that, in the general case, the work terms in $\Theta$ are associated to inhomogeneities of the boundary.

**The Komar mass.** The leading candidate for the internal energy is the Komar mass $M$, defined in terms of the surface integral [17]

$$M = \frac{1}{4\pi} \oint_{B'} d^2y \sqrt{\sigma} m^i \nabla_i L$$

(35)

for any closed surface $B'$ enclosing $B$, since the energy density and pressure vanishes in the exterior of $B$. The Komar mass is the conserved quantity associated with the symmetry of the time translations in a static spacetime via Noether’s theorem, so it is natural to identify it with the internal energy. However, there are two problems to such an identification. First, there is no obvious physical interpretation of $\Theta - \delta M$ as a work term. Second, for a system in a bounding box, the variable $m^i \nabla_i L$ appearing in equation (35) is discontinuous across the boundary. Hence, the Komar mass may turn out to depend on the stress–energy tensor of the bounding box, and not only on the properties of the enclosed fluid. This is unacceptable for the internal energy of a thermodynamic system.

Irrespective of its status as a candidate for internal energy, the Komar mass is closely related to the thermodynamic function $\Omega$. For a gravitating system bound by a box, the pressure is discontinuous at $B$. It is non-zero inside the box, but vanishes outside. The metric components and the derivatives corresponding to the thermodynamical variables $\sigma_{ij}$ and $\kappa_{ij}$ are continuous across the boundary, but the derivative $m^i \nabla_i L$ exhibits a jump due to the discontinuity of the pressure. We denote the value of $m^i \nabla_i L$ in the outside of $B$ with the suffix $+$ and its value inside $B$ by the suffix $-$. From equation (21), we obtain

$$(m^i \nabla_i L)_+ - (m^i \nabla_i L)_- = -8\pi \frac{LP}{\kappa}.$$  

(36)
Taking the limit of $B' \rightarrow B$ from the outside, equation (35) becomes
\[ M = -2 \oint_B d^2y \sqrt{\sigma} \frac{LP}{\kappa} + T_* \int_C d^3x \sqrt{h} \frac{\rho + 3P}{T}. \] (37)

In deriving equation (37), we employed the equation
\[ 4\pi (\rho + 3P)L = \nabla_i \nabla^i L, \] (38)
that follows from $S_{ij} h^{ij} = 0$, and Tolman’s law, equation (11). Using equation (12), we find
\[ 2 \int_C d^3x \sqrt{h} \frac{\rho}{T} = \frac{3}{\Omega_1} - M + 2 \oint_B d^2y \sqrt{\sigma} LP/\kappa \frac{T}{T_*}. \] (39)

Equation (39) is the result relating the Komar mass $M$ and the thermodynamic function $\Omega_1$, because the bulk term in the left-hand side of equation (39) can often be explicitly related to $\Omega_1$. Consider, for example, a fluid with a linear equation of state $P = \gamma \rho$, where $\gamma$ is a constant. For such a fluid
\[ \Omega = (1 + \gamma) \int_C d^3x \sqrt{h} \rho / T, \] (40)
and, equation (39) becomes
\[ \Omega = \frac{1 + \gamma}{1 + 3\gamma^2} \frac{M + 2 \oint_B d^2y \sqrt{\sigma} LP/\kappa}{T_*}. \] (41)

Equation (41) demonstrates explicitly how $\Omega$ is expressed solely in terms of variables that are defined on the boundary $B$. For spherically symmetric systems, equation (41) reproduces the results of [4].

**Thermodynamics inequalities.** An important motivation for formulating gravitational thermodynamic in terms of boundary variables is the possibility of formulating the entropy bounds suggested by black hole thermodynamics in an invariant geometric language [18] which can also be used for studying their implications to ordinary thermodynamics. While a proof of such entropy bounds requires a significant extension of the present formalism, equation (39) does lead to several non-trivial thermodynamic inequalities.

For a fluid that satisfies the weak and dominant energy conditions ($0 \leq P \leq \rho$),
\[ \frac{1}{2} \Omega \leq \int_C d^3x \sqrt{h} \rho / T \leq \Omega. \] (42)

Hence, by equation (39)
\[ \frac{1}{2} \frac{\Omega T_*}{M + 2 \oint_B d^2y \sqrt{\sigma} LP/\kappa} \leq 1. \] (43)

For stellar-surface boundary conditions, $P = 0$, and equation (43) simplifies
\[ \frac{M}{2T_*} \leq \Omega \leq \frac{M}{T_*}. \] (44)

Interestingly, the lower bound to $\Omega$ in equation (44) is saturated by a black hole. Strictly speaking the present formalism does not apply to black holes. However, in a heuristic sense, we can model a Schwartzshild black hole as a thermodynamical system with vanishing chemical potential (because the particle numbers are not preserved due to the no-hair theorem) that is characterized by the Bekenstein–Hawking entropy $S = 4\pi M^2 / \hbar$ and by the Hawking temperature $T_* = \hbar / (8\pi M)$. Then, we readily verify that $S = \Omega = \frac{M}{2T_*}$. 


A different bound to $\Omega$ follows by noting that in a stellar solution, the temperature $T$ in the interior is larger than the surface temperature and consequently $T \leq T_*$ (this property does not hold for the non-regular solutions). Equation (39) then becomes

$$2M_P \leq 3\Omega T_* - M,$$

(45)

where $M_P = \int_C \sqrt{h} \rho$ is the proper mass of the fluid in the interior. The difference between $M$ and $M_P$ is interpreted as the gravitational binding energy $E_B$ of the configuration

$$E_B = M_P - M.$$

(46)

Then, equation (45) becomes

$$M - \Omega T_* \leq -\frac{2}{3} E_B,$$

(47)

i.e., the binding energy defines an upper bound to the generalized free energy $M - \Omega T_*$.  

4. Conclusions

The main result of this paper is the demonstration that the thermodynamics of gravitating systems in equilibrium is holographic at the level of classical general relativity, in a very precise sense: (i) the thermodynamic state space consists of variables defined on the boundary, and (ii) the thermodynamic properties of the system are ascertained by local measurements at the boundary. We believe that these results can be extended toward a stronger sense of holography. To this end, we showed in [16] that the full geometry in the bulk follows from the knowledge of thermodynamic variables at the boundary in a specific self-gravitating system, and we believe that this result can be generalized.

We showed that the definition of thermodynamics in terms of the boundary variables is a property of parameterized theories, in general, not of general relativity in particular. However, general relativity is distinguished because it leads to a geometrically natural state space, that consists of the boundary 2-metric and its extrinsic curvature. Thus, the equilibrium thermodynamics of gravitating systems can be described in a geometrical language, and this property persists at the limit of ordinary extensive thermodynamics.

We view these results as an important stepping stone toward the formulation of a general axiomatic theory of equilibrium thermodynamics in gravitational systems that will also include black holes. At the moment, the major open issues are: (i) the thermodynamic properties of the interior boundaries that characterize the non-regular solutions, and (ii) a thermodynamically natural definition of internal energy.

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