STRICTLY OUTER ACTIONS OF GROUPS AND QUANTUM GROUPS

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Abstract. An action of a locally compact group or quantum group on a factor is said to be strictly outer when the relative commutant of the factor in the crossed product is trivial. We show that all locally compact quantum groups can act strictly outerly on a free Araki-Woods factor and that all locally compact groups can act strictly outerly on the hyperfinite $\text{II}_1$ factor. We define a kind of Connes’ $T$ invariant for locally compact quantum groups and prove a link with the possibility of acting strictly outerly on a factor with a given $T$ invariant. Necessary and sufficient conditions for the existence of strictly outer actions of compact Kac algebras on the hyperfinite $\text{II}_1$ factor are given.

1. Introduction

The most natural appearance of a group is as a symmetry group of a space. Several types of groups are simply defined by their action on a space. More generally, actions of locally compact (l.c.) groups on quantum spaces, i.e. von Neumann algebras, attracted a lot of attention. Therefore, it is a very natural idea to consider also actions of l.c. quantum groups on von Neumann algebras.

The history of the theory of l.c. quantum groups dates back to the 1960’s, with the important work of Kac, Kac & Vainerman and Enock & Schwartz, see [12] for an overview. Building on important contributions of Baaj & Skandalis [1], Woronowicz [14, 15] and Van Daele [11], the general theory of l.c. quantum groups was developed by Kustermans and the author [22, 23]. The general theory of actions of l.c. quantum groups on von Neumann algebras was developed in [37].

It has been shown by Enock and Nest [10, 11] that quantum group symmetries appear in the study of irreducible inclusions of factors, of depth 2 and infinite index. If $N_0 \subset N_1$ is such an inclusion and satisfies a technical regularity condition, they consider the Jones tower $N_0 \subset N_1 \subset N_2 \subset \cdots$. On the relative commutant $N_2 \cap N_0'$, Enock and Nest construct a l.c. quantum group structure. They construct a strictly outer action of this l.c. quantum group on $N_1$ such that $N_0$ is the fixed point algebra. This link between quantum groups and irreducible inclusions of depth 2 goes back to Ocneanu and was also generalized to reducible inclusions, yielding quantum groupoids, see e.g. [25].

It is now an obvious question to study which l.c. quantum groups can appear in Enock and Nest’s construction? Equivalently (see [37]), the question is which l.c. quantum groups can act strictly outerly on a factor, i.e. such that the relative commutant of the factor in the crossed product is trivial.

Even for l.c. groups, this question is not so easy. Blattner [7] constructed for any l.c. group $G$, an action $(\alpha_g)$ of $G$ on the hyperfinite $\text{II}_1$ factor $\mathcal{R}$ and showed that none of the $\alpha_g$ is an inner automorphism (for $g \neq e$). This is a necessary condition in order to get a trivial relative commutant of the factor in the crossed product, but far from being a sufficient condition.

For compact Lie groups, the question is more easy to answer: if $g \mapsto u_g$ is a finite-dimensional representation, say on $\mathbb{C}^n$, such that $g \mapsto \text{Ad} u_g$ is faithful, Wassermann (34, page 212) considered the diagonal action $\text{Ad} u_g$ on the infinite tensor product $\mathcal{R} = \bigotimes_{n=1}^\infty M_n(\mathbb{C})$ and showed that
this action is minimal. This means that the relative commutant of the fixed point algebra in \( R \) is trivial and implies, in particular, that the action is strictly outer.

For certain actions of \( R \) on the hyperfinite II\(_1\) factor, Kawahigashi proved in Proposition 3.2 of [18] the strict outerness. It might be possible to use Kawahigashi’s techniques to prove that Blattner’s action is strictly outer for every infinite abelian l.c. group.

Our contribution to the case of l.c. groups consists in proving the following result: if \((\alpha_g)\) is an action of a l.c. group \( G \) on a factor \( N \) leaving invariant a faithful normal state and such that \( g \mapsto \alpha_g \) is faithful, than the diagonal action of \( G \) on the infinite tensor product of copies of \( N \) (w.r.t. \( \omega \)) is strictly outer. As a corollary, we obtain that all l.c. groups can act strictly outerly on the hyperfinite II\(_1\) factor. In fact, a variant of Blattner’s action, taking infinitely many copies of the regular representation, is of this form and hence, strictly outer. Further, we give a more geometrical construction of a strictly outer action of any linear group on the hyperfinite II\(_1\) factor.

We turn now to the quantum setting. For finite quantum groups, Yamanouchi [47] constructed a minimal (and hence, strictly outer) action on the hyperfinite II\(_1\) factor. Even to obtain strictly outer actions of compact quantum groups is not so easy. For instance, it was shown recently by Izumi [17] that an infinite tensor product action in the style of Wasserman, fails to be strictly outer for the quantum \( SU_q(2) \) group. Strictly outer actions of arbitrary compact quantum groups were constructed by Ueda [35], by taking the free product of the action of the quantum group on itself by translation, and a trivial action. Of course, using the free product, the obtained actions are no longer on hyperfinite factors.

Below, we prove that there exists a III\(_1\) factor – it is a free Araki-Woods factor in the sense of Shlyakhtenko [30] – on which every l.c. quantum group can act strictly outerly. In fact, the key point in the argument of Ueda is the following: because the Haar measure of a compact quantum group is a state and invariant under the comultiplication, it is possible to make a free product w.r.t. this invariant state. In the non-compact case, we can no longer use the Haar measure, because it is infinite and the free product construction w.r.t. weights does not work. So, we have to produce first, in a different way, an action of an arbitrary l.c. quantum group on a von Neumann algebra leaving invariant a faithful state. Afterwards, we can apply the idea of Ueda and take the free product with a trivial action.

Having strictly outer actions of an arbitrary l.c. quantum group on a factor, it is a natural idea to ask when such a strictly outer action can be found on certain types of factors. The general construction gives a strictly outer action on a type III\(_1\) factor. Of course, a strictly outer action cannot exist on a factor of type I. Below, we give necessary and sufficient conditions for a l.c. quantum group being able to act strictly outerly on a factor of type II\(_1\), II\(_\infty\), III\(_1\) for \( 0 < \lambda < 1 \) or certain III\(_{11}\) factors. In fact, we introduce a kind of Connes \( T \) invariant \( T(M, \Delta) \) for a l.c. quantum group \((M, \Delta)\) (which is not the \( T \) invariant of the von Neumann algebra \( M \)). We prove that if a l.c. quantum group \((M, \Delta)\) acts strictly outerly on a factor \( N \), then \( T(N) \subset T(M, \Delta) \).

As we already remarked, the free Araki-Woods factors on which we find strictly outer actions of arbitrary l.c. quantum groups, are not injective. Since every l.c. group can act strictly outerly on an injective factor, it is a natural question to study which l.c. quantum groups can act strictly outerly on an injective factor. We prove that a necessary condition is the co-amenability of the l.c. quantum group. The converse is however far from clear (and probably, false), but we prove that a compact Kac algebra can act strictly outerly on an injective factor if and only if it is co-amenable. We further show that the bicrossed product l.c. quantum groups [38, 2] (which need not be Kac algebras) can act strictly outerly on an injective factor if and only if they are co-amenable. Using results of Ueda [36], we show that if the compact quantum group \( SU_q(2) \) acts strictly outerly on an injective factor, then there exists an irreducible subfactor of the hyperfinite II\(_1\) factor with index \((q + q^{-1})^2\). Since most important unpublished work of Popa makes this last statement highly improbable, we are convinced that not all the \( SU_q(2) \) can act strictly outerly on an injective factor.

Observe that Banica ([3, Section 4]) has shown that a discrete Kac algebra with a faithful finite dimensional corepresentation (see Definition 2.8) can act strictly outerly on the hyperfinite II\(_1\)
We study actions of locally compact (l.c.) groups and quantum groups on von Neumann algebras. These assumptions are not essential but give us lighter statements of our results. We have separable predual. So, we only work with second countable locally compact quantum groups. For simplicity, we will assume that all Hilbert spaces are separable and all von Neumann algebras have their essential contribution to the results of this paper.

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2. Preliminaries

For simplicity, we will assume that all Hilbert spaces are separable and all von Neumann algebras have separable predual. So, we only work with second countable locally compact quantum groups. These assumptions are not essential but give us lighter statements of our results.

We study actions of locally compact (l.c.) groups and quantum groups on von Neumann algebras. An action of $G$ on a von Neumann algebra is a morphism $G \to \text{Aut} G : g \mapsto \alpha_g$ such that $g \mapsto \alpha_g(x)$ is strongly* continuous for every $x \in N$. We then define the crossed product

$$G \rtimes N := (\alpha(N) \cup \mathcal{L}(G) \otimes 1)'' \subset B(L^2(G)) \otimes N,$$

where $\alpha : N \to L^\infty(G) \otimes N : (\alpha(x))(g) = \alpha_g^{-1}(x)$ and $\mathcal{L}(G)$ is the group von Neumann algebra generated by the left regular representation on $L^2(G)$ for the left Haar measure, defined by $(\lambda_g f)(h) = \xi(f(g^{-1} h)).$

**Definition 2.1.** An action $(\alpha_g)$ of a l.c. group $G$ on a factor $N$ is called **strictly outer** if the relative commutant of $N$ in the crossed product is trivial, i.e.

$$G_{\alpha} \rtimes N \cap \alpha(N)' = C.$$ 

It is obvious that a strictly outer action is outer, i.e. if $g \neq e$, then $\alpha_g$ is not an inner automorphism. But, if $G$ is not a discrete group, not every outer action is strictly outer. For instance, if $N$ is a $\text{III}_0$ factor with trivial $T$ invariant, the modular automorphism group $(\sigma_t)$ of an n.s.f. weight $\theta$ on $N$ is outer ($\sigma_t^\theta$ is not inner for $t \neq 0$), but not strictly outer: the crossed product is even not a factor, but its center is the space of the flow of weights.

We generalize these concepts to the world of l.c. quantum groups. The general theory of l.c. quantum groups was developed by Kustermans and the author in [22, 23]. We recall some of the basic definitions and properties.

**Definition 2.2.** A pair $(M, \Delta)$ is called a (von Neumann algebraic) l.c. quantum group when

- $M$ is a von Neumann algebra and $\Delta : M \to M \otimes M$ is a normal and unital $*$-homomorphism satisfying the coassociativity relation: $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.
- There exist normal semi-finite faithful (n.s.f.) weights $\varphi$ and $\psi$ on $M$ such that
  
  - $\varphi$ is left invariant in the sense that $\varphi((\omega \otimes \iota)\Delta(x)) = \varphi(x)\omega(1)$ for all $x \in M^+_\varphi$ and $\omega \in M^+_\varphi$,
  
  - $\psi$ is right invariant in the sense that $\psi((\iota \otimes \omega)\Delta(x)) = \psi(x)\omega(1)$ for all $x \in M^+_{\psi}$ and $\omega \in M^+_{\psi}$.

Ordinary l.c. groups appear in this theory in the form of $M = L^\infty(G)$ and $(\Delta(f))(p,q) = f(pq)$ for $f \in L^\infty(G)$. All l.c. quantum groups whose von Neumann algebra $M$ is commutative are of this form.

Fix a l.c. quantum group $(M, \Delta)$.

We first define the analogue of the left regular representation. As usual, we write $N_\varphi = \{x \in M \mid \varphi(x^* x) < \infty\}$. Represent $M$ in the GNS-construction of $\varphi$ with GNS-map $\Lambda : N_\varphi \to H$. We define a unitary $W$ on $H \otimes H$ by

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)) \quad \text{for all } a, b \in N_\varphi.$$

Here, $\Lambda \otimes \Lambda$ denotes the canonical GNS-map for the tensor product weight $\varphi \otimes \varphi$. One proves that $W$ satisfies the pentagonal equation: $W_{12}W_{13}W_{23} = W_{23}W_{12}$, and we say that $W$ is a...
multiplicative unitary. It is the left regular corepresentation. The von Neumann algebra $M$ is the strong closure of the algebra $\{(i \otimes \omega)(W) \mid \omega \in B(H)_*\}$ and $\Delta(x) = W^*(1 \otimes x)W$, for all $x \in M$. The von Neumann algebraic quantum group $(M, \Delta)$ has an underlying C*-algebraic quantum group $(A, \Delta)$, where $A$ is the norm closure of $\{(i \otimes \omega)(W) \mid \omega \in B(H)_*\}$. If $M = L^\infty(G)$, we have $A = C_0(G)$, the continuous functions on $G$ vanishing at infinity.

Next, the l.c. quantum group $(M, \Delta)$ has an antipode $S$, which is the unique $\sigma$-strong* closed linear map from $M$ to $M$ satisfying $(i \otimes \omega)(W) \in D(S)$ for all $\omega \in B(H)_*$ and

$$S\{(i \otimes \omega)(W)\} = (i \otimes \omega)(W^*)$$

and such that the elements $(i \otimes \omega)(W)$ form a $\sigma$-strong* corepresentation of $S$. The antipode $S$ has a polar decomposition $S = R\tau - i/2$ where $R$ is an anti-automorphism of $M$ and $(\tau)$ is a strongly continuous one-parameter group of automorphisms of $M$. We call $R$ the unitary antipode and $(\tau_t)$ the scaling group of $(M, \Delta)$.

The dual l.c. quantum group $(\hat{M}, \hat{\Delta})$ is defined in [22], Section 8. Its von Neumann algebra $\hat{M}$ is the strong closure of the algebra $\{(\omega \otimes i)(W) \mid \omega \in B(H)_*\}$ and the comultiplication is given by $\hat{\Delta}(x) = \Sigma W(x \otimes 1)W^*\Sigma$ for all $x \in \hat{M}$, where $\Sigma$ is the flip map on $H \otimes H$. On $\hat{M}$ there exists a canonical left invariant weight $\hat{\varphi}$ and the associated multiplicative unitary is given by $\Sigma W^*\Sigma$. We have again an underlying C*-algebraic quantum group $(\hat{A}, \hat{\Delta})$ where $\hat{A}$ is the norm closure of $\{(\omega \otimes i)(W) \mid \omega \in B(H)_*\}$.

Since $(\hat{M}, \hat{\Delta})$ is again a l.c. quantum group, we can introduce the antipode $\hat{S}$, the unitary antipode $\hat{R}$ and the scaling group $(\hat{\tau}_t)$ exactly as we did it for $(M, \Delta)$. Observe that

$$\hat{S}((\omega \otimes i)(W^*)) = (\omega \otimes i)(W).$$

The modular conjugations of the weights $\varphi$ and $\hat{\varphi}$ will be denoted by $J$ and $\hat{J}$ respectively. Then it is worthwhile to mention that

$$R(x) = \hat{J} x^* \hat{J} \quad \text{for all } x \in M \quad \text{and} \quad \hat{R}(\tilde{y}) = Jy^*J \quad \text{for all } \tilde{y} \in \hat{M}.$$  

The modular operators of the weights $\varphi$ and $\hat{\varphi}$ are denoted by $\nabla$ and $\hat{\nabla}$. We mention that

$$\tau_t(x) = \nabla^t x \nabla^{-it} \quad \text{for all } x \in M \quad \text{and} \quad \hat{\tau}_t(x) = \nabla^t y \nabla^{-it} \quad \text{for all } \tilde{y} \in \hat{M}.$$  

**Definition 2.3.** A unitary $U \in M \otimes B(K)$ is called a corepresentation of a l.c. quantum group $(M, \Delta)$ on the Hilbert space $K$, if $(\Delta \otimes i)(U) = U_1 U_2$.

We say that $U$ is faithful if

$$\{(i \otimes \mu)(U) \mid \mu \in B(K)_*\}'' = M.$$  

We observe that a corepresentation $U$ is, in a sense, automatically continuous: we have $U \in M(A \otimes K(K))$, where $K(K)$ denotes the C*-algebra of compact operators.

Taking into account all unitary corepresentations, we can define a universal C*-algebraic dual $(\hat{A}_u, \hat{\Delta}_u)$. The non-degenerate *-representations of the C*-algebra $\hat{A}_u$, $\rho : \hat{A}_u \to B(K)$ are in one-to-one correspondence with the corepresentations of $U$ of $(M, \Delta)$ through the formula $U = (i \otimes \rho)(W)$, where $W \in M(A \otimes \hat{A}_u)$ is the so-called universal corepresentation. See [21] for details. If $M = L^\infty(G)$, the universal C*-algebraic dual is $C^*(G)$, the full group C*-algebra.

Of course, since $(M, \Delta)$ has a universal dual $(\hat{A}_u, \hat{\Delta}_u)$, also $(\hat{M}, \hat{\Delta})$ has a universal dual $(\hat{A}_u, \hat{\Delta}_u)$ and we have a universal corepresentation $\hat{W} \in M(\hat{A} \otimes \hat{A}_u)$.

We use [37] as a reference for actions of l.c. quantum groups, but we recall the necessary elements of the theory. Let $(M, \Delta)$ be a l.c. quantum group and $N$ a von Neumann algebra. A faithful, normal *-homomorphism $\alpha : N \to M \otimes N$ is called a (left) action of $(M, \Delta)$ on $N$ if

$$(\Delta \otimes i)\alpha = (i \otimes \alpha)\alpha.$$  

One can define the crossed product as

$$M \rtimes^\alpha N = (\alpha(N) \cup \hat{M} \otimes 1)'' \subset B(H) \otimes N.$$
If $\alpha : N \to M \otimes N$ is an action, the fixed point algebra is denoted by $N^\alpha$ and defined as the von Neumann subalgebra of elements $x \in N$ satisfying $\alpha(x) = 1 \otimes x$.

We repeat Definition 6.1 of [37].

**Definition 2.4.** An action $\alpha$ of $(M, \Delta)$ on $N$ is called faithful when
\[
\{(t \otimes \omega)\alpha(x) \mid \omega \in N^\Lambda, x \in N\}'' = M.
\]
The action is called minimal when it is faithful and $N \cap (N^\alpha)' = C$.

We finally define strictly outer actions of l.c. quantum groups.

**Definition 2.5.** An action of a l.c. quantum group $(M, \Delta)$ on a factor $N$ is called strictly outer if the relative commutant of this factor in the crossed product is trivial, i.e.
\[
M_{\alpha^G} N \cap \alpha(N)' = C.
\]

From Proposition 6.2 in [37], we know that every minimal action is strictly outer. An integrable strictly outer action is minimal. In particular, for compact quantum groups the notions of minimal and strictly outer actions coincide.

Observe also that an action of an ordinary l.c. group $G$ on a von Neumann algebra $N$ is faithful in the sense of Definition 2.4 if and only if the morphism $g \mapsto \alpha_g$ is faithful.

An important technical tool in the study of actions of l.c. quantum groups on von Neumann algebras is the Radon-Nikodym derivative of an n.s.f. weight under the action, as introduced by Yamanouchi [38] (see the appendix of [3] for an easy approach). If $\alpha : N \to M \otimes N$ is an action of $(M, \Delta)$ on $N$ and if $\theta$ is an n.s.f. weight on $N$, we can consider unitaries $D_t := [D\theta \circ \alpha : D\theta]_t \in M \otimes N$, satisfying the following properties:
\[
(\Delta \otimes i)(D_t) = (t \otimes \alpha)(D_t)(1 \otimes D_t) \quad \text{and} \quad D_{t+s} = D_t (\tau_t \otimes \sigma_\theta)(D_s).
\]

If we represent the crossed product $M_{\alpha^G} N$ as above, as a subalgebra of $B(H) \otimes N$, we can define a dual n.s.f. weight $\theta$ on $M_{\alpha^G} N$ with a modular operator $\tilde{\nabla}$ such that $D_t = \tilde{\nabla}^\mu t (\nabla_{\tilde{\nabla}}^{-it} \otimes \nabla_{\tilde{\nabla}}^{-it})$, where $\nabla$ is the modular operator of the left invariant weight on the dual $(\hat{M}, \hat{\Delta})$ and $\nabla_\theta$ is the modular operator of $\theta$.

**Definition 2.6.** Let $\alpha : N \to M \otimes N$ be an action of $(M, \Delta)$ on $N$ and let $\theta$ be an n.s.f. weight on $N$. We say that $\theta$ is $\rho$-invariant, when $\rho$ is a strictly positive, self-adjoint operator affiliated with $M$, satisfying
\[
\Delta(\rho) = \rho \otimes \rho \quad \text{and} \quad \theta((\omega \otimes \xi) \alpha(x)) = \|\rho^{1/2} \xi\|^2 \theta(x) \quad \text{for} \quad x \in M_\theta^+, \xi \in D(\rho^{1/2}).
\]

We say that a state $\omega$ on $N$ is invariant, when $(\omega \otimes \omega)\alpha(x) = \omega(x) 1$ for all $x \in N$.

We finish this section of preliminaries with the following alternative characterization of strict outerness.

**Proposition 2.7.** Let $\alpha : N \to M \otimes N$ be an action of a l.c. quantum group on a factor $N$. Then, $\alpha$ is strictly outer if and only if
\[
B(H) \otimes N \cap \alpha(N)' = M' \otimes 1.
\]

In particular, a strictly outer action is faithful.

**Proof.** Suppose first that $B(H) \otimes N \cap \alpha(N)' = M' \otimes 1$. In order to prove that $\alpha$ is strictly outer, it suffices to show that $M_{\alpha^G} N \cap M' \otimes 1 = C1$. If $V \in M' \otimes M$ denotes the right regular corepresentation of $(M, \Delta)$ (see [24]), it is easy to check that $(\iota \otimes \alpha)(z) = V_{12} \alpha_{13} V_{12}^*$ for all $z \in M_{\alpha^G} N$. So, if $a \in M'$ and $a \otimes 1 \in M_{\alpha^G} N$, we get $a \in M' \cap M = C1$ and we are done.

Suppose conversely that $\alpha$ is strictly outer and let $z \in B(H) \otimes N \cap \alpha(N)'$. Then,
\[
W_{12}(\iota \otimes \alpha)(z)W_{12}^* \in B(H) \otimes (M_{\alpha^G} N) \cap (1 \otimes \alpha(N))' = B(H) \otimes 1 \otimes 1.
\]
So, we can take $a \in B(H)$ such that $(\iota \otimes \alpha)(z) = W^*(a \otimes 1)W \otimes 1$. The left hand side belongs to $B(H) \otimes M \otimes N$, while the right hand side belongs to $B(H) \otimes M \otimes 1$. Since $M \cap M = C$, the right hand side belongs to $B(H) \otimes 1 \otimes 1$ and we can take $b \in B(H)$ such that $z = b \otimes 1 = W^*(a \otimes 1)W$. If we apply $\iota \otimes \Delta$, we conclude that

$$b \otimes 1 \otimes 1 = (\iota \otimes \hat{\Delta})(W^*(a \otimes 1)W) = W_{12}W_{13}W_{12} = W_{12}^*(b \otimes 1 \otimes 1)W_{12}.$$  

It follows that $b \in M'$ and hence, $z \in M' \otimes 1$. This proves the first part of the proposition.

If $a$ commutes with $(\iota \otimes \mu)(x)$ for all $\mu \in N$, and $x \in N$, it is clear that $a \otimes 1 \in B(H) \otimes N \cap (N')'$. Hence, $a \in M'$. The faithfulness of $\alpha$ follows immediately. 

3. Strictly outer actions of locally compact quantum groups

We shall prove, in Theorem 3.1(a) below, that there exists a type $\text{III}_1$ free Araki-Woods factor (in the sense of Shlyakhtenko [30]) on which every l.c. quantum group acts strictly outerly. We introduce a $T$ invariant for l.c. quantum groups (Definition 3.4) and determine when a given l.c. quantum group can act strictly outerly on a factor of a given type (Theorems 3.5 and 3.6).

Let us recall Shlyakhtenko’s construction of these free Araki-Woods factors. Let $T$ be an involution on a Hilbert space $K$. This means that $T$ is a densely defined, closed, injective, anti-linear operator on $K$ satisfying $T^{-1} = T$. If $T = \mathcal{J}Q^{1/2}$ is its polar decomposition, $\mathcal{J}$ is an anti-unitary operator and $Q$ is a strictly positive, self-adjoint operator satisfying $\mathcal{J}Q\mathcal{J} = Q^{-1}$. We also have $\mathcal{J}Q^t\mathcal{J} = Q^{-t}$. Define

$$K_\mathcal{J} := \{\xi \in H \mid \mathcal{J}\xi = \xi\}.$$  

Then, $K_\mathcal{J}$ is a real Hilbert space and the restriction of $Q^t$ gives a one-parameter group of orthogonal transformations of $K_\mathcal{J}$.

Consider the full Fock space

$$\mathcal{F}(K) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} K^{\otimes (n)}.$$  

Shlyakhtenko’s free Araki-Woods von Neumann algebra [30] can be defined as

$$\Gamma(K_\mathcal{J}, Q^t)^t = \{s(\xi) \mid \xi \in D(T)^t \subset B(\mathcal{F}(K)) \text{ with } s(\xi) = \ell(\xi) + \ell(T\xi)^* \},$$  

where $\ell(\xi)$ denotes the left creation operator in $B(\mathcal{F}(K))$ whenever $\xi \in K$. The vector state $\omega_K$ is faithful on $\Gamma(K_\mathcal{J}, Q^t)^t$ and is called the free quasi-free state.

Instead of starting with an involution $T$, we can of course start with a real Hilbert space with a one-parameter group of orthogonal operators and complexify to obtain $T$.

We show now that any l.c. quantum group can act in a natural way on a free Araki-Woods von Neumann algebra, whenever we have a corepresentation of $(M, \Delta)$ on a Hilbert space $K$ and a compatible involution $T$ on $K$.

Proposition 3.1. Let $U \in M \otimes B(K)$ be a corepresentation of a l.c. quantum group $(M, \Delta)$ on a Hilbert space $K$. Suppose that there exists an involution $T$ on $K$ satisfying

$$(\mu \otimes 1)(U^*)T \subset T(\overline{\mu} \otimes 1)(U^*)$$  

for all $\mu \in M_*$. Let $T = \mathcal{J}Q^{1/2}$ be the polar decomposition. Denote by $\mathcal{F}(U)$ the amplified corepresentation on the full Fock space $\mathcal{F}(K)$ defined by

$$\mathcal{F}(U) = \bigoplus_{n=0}^{\infty} U^{(n)},$$  

where $U^{(0)} = 1 \in M \otimes B(\mathbb{C}\Omega)$ and $U^{(n)} = U_{1,n+1} \ldots U_{12} \in M \otimes B(K^{\otimes (n)})$ for $n \geq 1$. Let $N := \Gamma(K_\mathcal{J}, Q^t)^t$ be the free Araki-Woods factor corresponding to $T$. Defining

$$\alpha : N \rightarrow M \otimes N : \alpha(z) = \mathcal{F}(U)^*(1 \otimes z)\mathcal{F}(U),$$  

we get a well-defined action of $(M, \Delta)$ on $N$ leaving invariant the free quasi-free state.

If the corepresentation $U$ is faithful, the action $\alpha$ is faithful.
Proof. It is obvious that \( \mathcal{F}(U) \) is a corepresentation of \((M, \Delta)\) on the full Fock space \( \mathcal{F}(K) \). We observe that
\[
(\mu \otimes \iota)(\mathcal{F}(U)^*(1 \otimes \ell(\xi))\mathcal{F}(U)) = \ell((\mu \otimes \iota)(U^*)\xi).
\]
Using Equation \( \text{(3.1)} \), we get that
\[
(\mu \otimes \iota)(\mathcal{F}(U)^*(1 \otimes s(\xi))\mathcal{F}(U)) = s((\mu \otimes \iota)(U^*)\xi),
\]
for all \( \mu \in M_* \) and \( \xi \in D(T) \). Hence, the definition of \( \alpha \) in the statement of the proposition yields an action of \((M, \Delta)\) on the von Neumann algebra \( \Gamma(K_{\mathcal{F}}, Q^{it})'' \). It is obvious that the free quasi-free state is invariant under the action \( \alpha \).

Further, we observe that \((\iota \otimes \omega_{t,\eta})(s(\xi)) = (\iota \otimes \omega_{t,\eta})(U^*)\) for \( \xi \in D(T) \) and \( \eta \in H \mapsto \mathcal{F}(H) \). So, the action \( \alpha \) is faithful if the corepresentation \( U \) is faithful.

\[\square\]

**Remark 3.2.** The assumption on the existence of an involution \( T \) satisfying Equation \( \text{(3.1)} \) is crucial for the construction of the action \( \alpha \). It has a very natural interpretation, as follows. Let \( \rho : \hat{A}_u \to \mathcal{B}(K) \) be the representation of the universal \( C^* \)-algebraic dual \((\hat{A}_u, \Delta_u)\) corresponding to \( U \) through the formula \((\iota \otimes \rho)(W) = U \) (see Preliminaries). From \( \text{(21)} \), we know that \((\hat{A}_u, \Delta_u)\) has an antipode \( \hat{S}_u \) satisfying \( \hat{S}_u((\mu \otimes \iota)(W^*)) = (\mu \otimes \iota)(W) \) for all \( \mu \in A^* \). This antipode has a polar decomposition \( \hat{S}_u = \hat{R}_u\hat{\tau}_u^{\frac{1}{2}} \). It is not hard to check that Equation \( \text{(3.1)} \) is then equivalent with the equations
\[
\rho(\hat{R}_u(x)) = J\rho(x)^*J \quad \text{and} \quad \rho(\hat{\tau}_u^{it}(x)) = Q^{it}\rho(x)Q^{-it}
\]
for all \( x \in \hat{A}_u \) and \( t \in \mathbb{R} \).

If we define the real \( C^* \)-algebra
\[
\hat{A}_u^{\text{real}} := \{ x \in \hat{A}_u \mid \hat{R}_u(x) = x^* \}
\]
and the real Hilbert space \( K_{\mathcal{F}} \) as before, we see that \( \rho(\hat{A}_u^{\text{real}}) \) leaves \( K_{\mathcal{F}} \) invariant. Further, the one-parameter group \((\hat{\tau}_u^{it})\) restricts to a one-parameter group of \( \hat{A}_u^{\text{real}} \). So, a compatible pair of a corepresentation \( U \) of \((M, \Delta)\) on a Hilbert space \( K \) and an involution \( T = JQ^{1/2} \) is the same thing as a representation \( \rho \) of the real \( C^* \)-algebra \( \hat{A}_u^{\text{real}} \) on the real Hilbert space \( K_{\mathcal{F}} \) satisfying
\[
\rho(\hat{\tau}_u^{it}(x)) = Q^{it}\rho(x)Q^{-it}
\]
for all \( x \in \hat{A}_u^{\text{real}} \) and \( t \in \mathbb{R} \).

Finally, the involution \( T \) exists in an important case. If we take the left regular corepresentation \( W \in \mathcal{B}(H \otimes H) \), we know that Equation \( \text{(3.1)} \) holds for \( W \) with \( T = J\nabla^{1/2} \). This is the case because \( J \) implements the unitary antipode \( R \) of \((M, \Delta)\), while \( \nabla^{it} \) implements the scaling group of \((M, \Delta)\) and \( \hat{S}((\omega \otimes \iota)(W^*)) = (\omega \otimes \iota)(W) \). We refer to the section of preliminaries for the definition of \( W, J \) and \( \nabla \).

**Corollary 3.3.** Every l.c. quantum group can act faithfully on \( \Gamma(L^2(\mathbb{R}, K_{\mathcal{F}}), \lambda_t)'' \) leaving invariant the free quasi-free state. Here, \( K_{\mathcal{F}} \) is the separable, infinite-dimensional real Hilbert space and \( \lambda_t \) acts by left translations on \( \mathbb{R} \).

Every l.c. quantum group with trivial scaling group \( \tau_t = \iota \) can act faithfully on the free group factor \( \mathcal{L}(\mathbb{R}_\infty) \) leaving invariant the trace.

*Proof.* Consider the Hilbert space \( L^2(\mathbb{R}, H) \) and define \( \mathcal{J}(\xi)(x) = J\xi(x), (Q^{it}\xi)(x) = \nabla^{it}\xi(x-t) \) for \( \xi \in L^2(\mathbb{R}, H) \). Then, \( T := JQ^{1/2} \) is an involution. Identifying \( L^2(\mathbb{R}, H) \) with \( L^2(\mathbb{R}) \otimes H \), we define a corepresentation \( U := W_{13} \) of \((M, \Delta)\) on \( L^2(\mathbb{R}, H) \). We already remarked above that Equation \( \text{(3.1)} \) is satisfied for \( W \) and \( J\nabla^{1/2} \). Hence, Equation \( \text{(3.1)} \) is satisfied for our \( U \) and \( T \). From Proposition \( \text{(21)} \), we get a faithful action of \((M, \Delta)\) on the free Araki-Woods factor \( \Gamma(L^2(\mathbb{R}, H), Q^{it})'' \) leaving invariant the free quasi-free state.

Using the unitary \((Z\xi)(x) = \nabla^{-ix}\xi(x) \) on \( L^2(\mathbb{R}, H) \), it is clear that
\[
(L^2(\mathbb{R}, H), Q^{it}) \cong (L^2(\mathbb{R}, H_{\mathcal{F}}), \lambda_t).
\]
So, the first part of the corollary is proven. If \( M \) happens to be finite dimensional, we take above a direct sum of infinitely many copies of \( W \) and assure as such that we can always assume \( K_{\mathcal{F}} \) to be infinite dimensional.
To prove the second part, suppose that \( \tau_t = \iota \) for all \( t \). Then, \( \hat{\tau}_t = \iota \) as well and \( (\mu \otimes \iota)(W^*) J = J(\pi \otimes \iota)(W^*) \) for \( \mu \in M_* \). By proposition 2.4, we get an action of \( (M, \Delta) \) on \( \Gamma(H_J, \iota)^n \) leaving invariant the trace. As above, we can assume that \( H \) is infinite dimensional and then, \( \Gamma(H_J, \iota)^n \cong \mathcal{L}(\mathbb{F}_\infty) \).

The following observation is due to Y. Ueda [35]. If \( \alpha \) is a faithful action of \( (M, \Delta) \) on a von Neumann algebra \( N_0 \) leaving invariant a faithful state \( \omega_0 \) and if \( N_1 \) is an arbitrary von Neumann algebra with faithful state \( \omega_1 \), it is easy to extend \( \alpha \) to an action \( \beta \) on the free product (see [12]) \( (N, \omega) := (N_0, \omega_0) \star (N_1, \omega_1) \) acting trivially on \( N_1 \). If now, \( N \cap N_1' = \mathbb{C} \[, \) the action \( \beta \) is minimal and hence, strictly outer. It is easy to give examples such that \( N \cap N_1' = \mathbb{C} \). Indeed, we can take \( (N_1, \omega_1) = (A, \eta) \star (B, \mu) \) such that the centralizer \( A^\theta \) contains a non-trivial group of orthogonal unitaries and such that the centralizer \( B^\theta \) contains a group of orthogonal unitaries with at least three elements. It follows from a result of Barnett (Theorem 11 in [9]) that

\[
N \cap N_1' = (N \star A) \star B \cap (A \star B)' = \mathbb{C}.
\]

Moreover, \( N \) is a full factor.

We conclude that every l.c. quantum group can act strictly outerly on a full factor.

In Theorem 3.6 (a) below, we want to construct such a strictly outer action on the canonical free Araki-Woods factor \( \Gamma(L^2(\mathbb{R}, K_\mathbb{R}), \lambda_t)^n \) and for this, we will need a generalization of Barnett’s result, see Lemma 3.1.

As we know now that every l.c. quantum group can act strictly outerly on a factor, it is a natural question to study the possibility of acting strictly outerly on factors with certain given invariants.

We introduce some kind of Connes’ \( T \) invariant of a l.c. quantum group and show how it is related to the possibility of acting strictly outerly on a factor with a certain \( T \) invariant.

**Definition 3.4.** Let \( (M, \Delta) \) be a l.c. quantum group and \( (\tau_t) \) its scaling group. We define

\[
T(M, \Delta) := \{ t \in \mathbb{R} \mid \text{There exists a unitary } u \in M \text{ satisfying } \tau_t = \text{Ad } u \text{ and } \Delta(u) = u \otimes u \}.
\]

Remark that, exactly as in the case of the usual \( T \) invariant of von Neumann algebras defined by Connes in [4], we have that \( T(M, \Delta) \) is a subgroup of \( \mathbb{R} \).

Since \( \Delta \tau_t = (\tau_t \otimes \iota) \Delta \), it is natural to consider, in the definition of \( T(M, \Delta) \) that \( \tau_t \) is implemented by a unitary satisfying \( \Delta(u) = u \otimes u \).

We denote by \( T(N) \) the \( T \) invariant of a von Neumann algebra \( N \).

In the following result, we prove that a small \( T \) invariant \( T(M, \Delta) \) makes it impossible to act on certain factors. More precisely, we have the following.

**Theorem 3.5.** Let \( (M, \Delta) \) be a l.c. quantum group.

a) If \( (M, \Delta) \) acts strictly outerly on a factor \( N \), then \( T(N) \subset T(M, \Delta) \).

b) If \( (M, \Delta) \) acts strictly outerly on a \( II_\infty \) factor, there exists a one-parameter group \( \rho^t \in M \) of unitaries such that \( \Delta(\rho^t) = \rho^t \otimes \rho^t \) and \( \tau_t = \text{Ad } \rho^t \) for all \( t \in \mathbb{R} \).

c) If \( (M, \Delta) \) acts strictly outerly on a \( II_1 \) factor, the scaling group \( (\tau_t) \) is trivial.

**Proof.** Let \( \alpha \) be a strictly outer action of \((M, \Delta)\) on a factor \( N \) and \( t_0 \in T(N) \). Take an n.s.f. weight \( \theta \) on \( N \) such that \( \sigma^\theta_{t_0} = \iota \). Then, we have a dual n.s.f. weight \( \tilde{\theta} \) on the crossed product \( M \times \alpha \times \mathbb{N} \), with modular operator \( \nabla \). Denote by \( D_t := [D \theta \circ \alpha : D \theta^t] \) the Radon-Nikodym derivative and by \( \nabla_{\tilde{\theta}} \) the modular operator of \( \theta \). As above, \( \nabla \) denotes the modular operator of the left invariant weight on the dual \( (M, \Delta) \). We know that \( \nabla^{u^t} = D_t(\nabla^{u^t} \otimes \nabla^{u^t}) \). Take \( x \in N \). Because \( D_t \in M \times \mathbb{N} \), we have \( D_t(\nabla^{u^t} \otimes 1) = B(H) \otimes N \). Further, because \( \nabla^{u^t_0} = 1 \) and \( \sigma^\theta_{t_0} = \alpha \sigma^\theta_{t_0} \), we have

\[
D_{t_0}(\nabla^{u^t_0} \otimes 1) \alpha(x) = \nabla^{u^t_0} \alpha(x) = \alpha(x) \nabla^{u^t_0} = \alpha(x) D_{t_0}(\nabla^{u^t_0} \otimes 1).
\]

From Proposition 2.7 and the strict outerness of \( \alpha \), it follows that \( B(H) \otimes N \cap \alpha(N)' = M' \otimes 1 \). So, because \( D_{t_0} \in M \otimes N \), there exists a unitary \( u \in M \) such that \( D_{t_0} = u^* \otimes 1 \) and \( u^* \nabla^{u^t_0} \in \)
M'. Because \( \hat{V}^{\Delta t_0} \) implements \( \tau_{t_0} \), we get \( \tau_{t_0} = \text{Ad} u \). Finally, we know that \( (\Delta \otimes \iota)(D_{t_0}) = (\iota \otimes \alpha)(D_{t_0})(1 \otimes D_{t_0}) \) and so, \( \Delta(u) = u \otimes u \). We conclude that \( t_0 \in T(M, \Delta) \).

In the reasoning above, we can take \( \theta = \text{Tr} \), a trace on \( N \). We find that \( D_t = u_t^* \otimes 1 \) for \( u_t \in M \) and \( t \in \mathbb{R} \), such that \( \Delta(u_t) = u_t + u_t \) and \( \tau_t = \text{Ad} u_t \). Because \( D_{t+s} = D_t(\tau_t \otimes \iota)(D_s) \) and because any group-like unitary is invariant under the scaling group, we find that \( (u_t) \) is a one-parameter family of unitaries. So, we can write \( u_t = \rho^{it} \).

Assume that, moreover, \( \text{Tr} \) is finite. Because \( D_t = \rho^{-it} \otimes 1 \), it follows that \( \text{Tr} \) is \( \rho \)-invariant. This implies that \( \rho \text{ Tr}(1) = (\iota \otimes \text{Tr})\alpha(1) = 1 \text{ Tr}(1) \) and hence, \( \rho = 1 \) and the scaling group is trivial. \( \square \)

From the constructive point of view, we prove the following. Recall from [30], Theorem 6.4 that for \( 0 < \lambda < 1 \), there exists a unique free Araki-Woods factor of type III\( \lambda \) given by \( \Gamma(K, S, Q^{it})'' \) whenever the subgroup of \( \mathbb{R}_+^2 \) generated by the spectrum of \( Q \) is \( \lambda \mathbb{Z} \).

**Theorem 3.6.** Let \( (M, \Delta) \) be a l.c. quantum group.

a) Every l.c. quantum group can act strictly outerly on \( \Gamma(L^2(\mathbb{R}, K_R), \lambda_0)'' \) leaving invariant the free quasi-free state.

b) If there exists a one-parameter group \( \rho^{it} \in M \) of unitaries such that \( \Delta(\rho^{it}) = \rho^{it} \otimes \rho^{it} \) and \( \tau_t = \text{Ad} \rho^{it} \) for all \( t \in \mathbb{R} \), \( (M, \Delta) \) can act strictly outerly on \( L(\mathcal{F}_\infty) \otimes B(K) \) leaving \( \rho \)-invariant the trace.

c) If \( t_0 \in T(M, \Delta) \), \( (M, \Delta) \) can act strictly outerly on the unique free Araki-Woods factor \( T_\lambda \) of type III\( \lambda \) with 0 < \( \lambda < 1 \) and \( |t_0| = \frac{2\pi}{\log \lambda} \).

d) If \( (\tau_t) \) is trivial, \( (M, \Delta) \) can act strictly outerly on \( L(\mathcal{F}_\infty) \) leaving invariant the trace.

e) Let \( \Gamma \) be a countable dense subgroup of \( \mathbb{R} \). If \( \Gamma \subset T(M, \Delta) \), \( (M, \Delta) \) can act strictly outerly on a III\( \lambda_0 \) factor whose flow of weights is the canonical action of \( \mathbb{R} \) on the dual compact group \( \hat{\Gamma} \).

**Proof.** From Corollary 3.3, we can take a faithful action \( \beta \) of \( (M, \Delta) \) on the free Araki-Woods factor \( N_0 := \Gamma(L^2(\mathbb{R}, K_R), \lambda_0)'' \) leaving invariant the free quasi-free state \( \omega_0 \). Define \( N_1 := \Gamma(L^2(\mathbb{R}, K_R^3), \lambda_0)'' \), with free quasi-free state \( \omega_1 \). Put \( (N, \omega) = (N_0, \omega_0) \ast (N_1, \omega_1) \). Then, \( (N, \omega) \cong (N_0, \omega_0) \) by [30], Theorem 2.11. Because \( \omega_0 \) is invariant under \( \beta \), it is easy to extend \( \beta \) to an action \( \alpha \) of \( (M, \Delta) \) on \( N \), acting trivially on \( N_1 \). It is clear that \( \alpha \) is still faithful. Moreover, if we write \( A = \Gamma(L^2(\mathbb{R}, K_R), \lambda_0)'' \), we have \( N = (N_0 \ast A) \ast A \ast A \) and \( A \ast A \ast A \subset N^\alpha \) (all the free products are with respect to the free quasi-free states). Combining Lemmas 3.1 and 3.2 below, we get that \( N \cap (N^\alpha)' = C \). So, \( \alpha \) is a minimal action and hence, a strictly outer action.

To start with, take the above action \( \alpha \) of \( (M, \Delta) \) on \( N \) with invariant free quasi-free-state \( \omega_0 \). Define \( C = N \rtimes (\sigma^\omega) \mathbb{R} \) to be the continuous core of \( N \). We realize \( C \) as the subalgebra of \( N \otimes B(L^2(\mathbb{R})) \) generated by \( \sigma^\omega(N) \) and \( 1 \otimes \rho(\mathbb{R}) \), where \( \sigma^\omega(x) \in N \otimes \mathcal{L}^\infty(\mathbb{R}) \) is defined by \( \sigma^\omega(x)(t) = \sigma^\omega_t(x) \) and where \( \rho_t \) denotes the right translation by \( t \).

Because the state \( \omega \) is invariant, we know from [37], proof of Proposition 4.3, that \( \beta \sigma^\omega = (\tau_t \otimes \sigma^\omega_t)\beta \). Defining the unitary \( \mathcal{V} \in M \otimes 1 \otimes \mathcal{L}^\infty(\mathbb{R}) \) by \( \mathcal{V}(s) = \rho^{s} \otimes 1 \), we can define the action
\[
\gamma : C \to M \otimes C : \gamma(z) = \mathcal{V}^*(\alpha \otimes \iota)(z)\mathcal{V} \quad \text{satisfying} \quad \\
\gamma \sigma^\omega = (\iota \otimes \sigma^\omega)(\alpha) , \quad \gamma(1 \otimes \rho_t) = (\rho^{-it} \otimes 1 \otimes \rho_t) .
\]

It is clear that the dual weight of \( \omega \) is invariant under \( \gamma \) and hence, the trace of \( C \) will be \( \rho \)-invariant. Further, \( \sigma^\omega(N^\alpha) \subset C^\gamma \) and \( \gamma \) remains a faithful action. We claim that \( C \sigma^\omega(N^\alpha)' = C \) and then, it follows that \( \gamma \) is a minimal and hence, strictly outer action. From Theorem 5.2 in [32], we know that \( C \cong \mathcal{L}(\mathcal{F}_\infty) \otimes B(K) \). To conclude the proof of item d), it suffices hence to prove our claim. Write again \( N = (N_0 \ast A) \ast A \ast A \). Take a sequence \( \alpha_n \) of unitaries in the first copy of \( A \) satisfying the conclusions of Lemma 3.2. Take the same sequence in the second and third copy of \( A \) and denote them by \( b_n \) and \( c_n \). Let \( x \in C \cap \sigma^\omega(N^\alpha)' \) and take \( \mu \in B(L^2(\mathbb{R}))_z \). Let \( L \) be a compact subset of \( \mathbb{R} \) and write \( p_L \) for the characteristic function of
L. Write $x_L = (1 \otimes p_L)x(1 \otimes p_L)$. Because $a_n \in N^\alpha$, $x$ commutes with $\sigma^\omega(a_n)$. But, $\sigma^\omega(a_n)$ commutes with $\theta$ as well. Because $\|\sigma^\omega(a_n) - a_n\| \to 0$ uniformly on compact subsets of $\mathbb{R}$, it follows that $\|(\iota \otimes \mu)(x_L), a_n\| \to 0$. The same holds for $b_n$ and $c_n$. From Lemma 4.4 we get that $(\iota \otimes \mu)(x_L) \in \mathbb{C}$. This holds for all $\mu \in B(L^2(\mathbb{R}))$, and we conclude that $x_L \in 1 \otimes B(L^2(\mathbb{R}))$ for all compact subsets $L \subset \mathbb{R}$. We finally get that $x \in 1 \otimes B(L^2(\mathbb{R}))$ as well. Because $x \in C$, it follows that $x = 1 \otimes y$ with $y \in \rho(\mathbb{R})''$. Since, with the notations of the proof of item a), $(\mu \otimes \iota)(\sigma^\omega(N_1)) | \mu \in N_1'' = L^\infty(\mathbb{R})$, we get $y \in \rho(\mathbb{R})'' \cap L^\infty(\mathbb{R})' = \mathbb{C}$. This proves our claim.

\[\] Suppose that $t_0 \in T(M, \Delta)$. Exactly as in item c), we can start with the strictly outer action $\alpha$ of $(M, \Delta)$ on $N$ obtained in item a) and next, extend this action, to an action $\gamma$ of $(M, \Delta)$ on $C_{t_0} := N \rtimes_{\alpha(\sigma^\omega_{t_0})} \mathbb{Z}$.

We can consider $C_{t_0}$ as an intermediate subalgebra: $\sigma^\omega(N) \subset C_{t_0} \subset C$, where $C$ denotes the continuous core of $N$. Because we have seen already that $C \cap \sigma^\omega(N')' = \mathbb{C}$, it follows that $C_{t_0} \cap \sigma^\omega(N')' = \mathbb{C}$. Hence, $\gamma$ will be a strictly outer action. To conclude the proof of item, we have to show that $C_{t_0} \cong T_\lambda$, where $|t_0| = \frac{2\pi}{|\log \lambda|}$.

As mentioned above, Shlyakhtenko has shown in Theorem 5.2 of [22] that the continuous core of $N := \Gamma(L^2(\mathbb{R}, K_\lambda), \lambda_\mu''$ is isomorphic with $\mathcal{L}(\mathcal{F}_\infty) \otimes B(K)$. More precisely, he has given a model for the continuous core of $N$ that corresponds to the model for $\mathcal{L}(\mathcal{F}_\infty) \otimes B(K)$ given by Rădulescu [26] and such that the dual action $(\theta_\lambda)$ on the continuous core is precisely the trace-scaling action of Rădulescu.

Moreover, as explained just before Proposition 6.9 in [30] (using the result of Rădulescu proven in [27]), the discrete core of $T_\lambda$ with its dual action $(\mu_\rho) \subset \mathbb{C}$ can be described by the same model of $\mathcal{L}(\mathcal{F}_\infty) \otimes B(K)$ and $\mu_\rho$ corresponds to $\theta_\lambda \rho_{\log \lambda}$. Because $T_\lambda$ is isomorphic with the fixed point algebra of the dual action on the discrete core of $T_\lambda$, we get

$T_\lambda \cong \{ x \in N \rtimes (\sigma^\omega) \mathbb{R} | \theta_{\log \lambda}(x) = x \}$.

It is clear that this last algebra is isomorphic with our $C_{t_0}$.

\[\] From 3.3, we get a faithful action of $(M, \Delta)$ on $\mathcal{L}(\mathcal{F}_\infty)$ leaving invariant the trace. Taking the free product with a trivial action on $\mathcal{L}(\mathcal{F}_2)$, we immediately get a strictly outer action of $(M, \Delta)$ on $\mathcal{L}(\mathcal{F}_\infty)$ leaving invariant the trace.

\[\] We start once again with the strictly outer action $\alpha$ of $(M, \Delta)$ on $N$ obtained in item a). As in the proof of item c), we can extend $\alpha$ to a strictly outer action $\gamma$ of $(M, \Delta)$ on $N \rtimes (\sigma^\omega) \Gamma$. Because $N$ is a type III$_1$ factor (30, Theorem 6.10), $N \rtimes (\sigma^\omega) \Gamma$ is a type III$_1$ factor whose flow of weights is the action of $\mathbb{R} \subset \Gamma$ on the compact group $\Gamma$ by translations.

We recall that a l.c. quantum group is said to be unimodular if the left invariant weight is also right invariant.

**Corollary 3.7.** If the dual of a l.c. quantum group $(M, \Delta)$ is unimodular, then $(M, \Delta)$ can act strictly outerly on $\mathcal{L}(\mathcal{F}_\infty) \otimes B(K)$. This holds in particular for discrete quantum groups.

If a compact quantum group acts strictly outerly on a II$_\infty$ factor, then $\tau_t = t$ for all $t \in \mathbb{R}$.

**Proof.** Let $(M, \Delta)$ be a l.c. quantum group. If the dual of $(M, \Delta)$ is unimodular, it follows from Proposition 2.4 in 39 that $\tau_t = \text{Ad} \delta^{-it}/2$, where $\delta$ is the modular element of $(M, \Delta)$, i.e. the Radon-Nikodym derivative between the left and right invariant weight. Hence, $(M, \Delta)$ can act on $\mathcal{L}(\mathcal{F}_\infty) \otimes B(K)$, because $\Delta(\delta^it) = \delta^it \otimes \delta^it$. It is well known that a compact quantum group is unimodular and so, our result can be applied to discrete quantum groups.

If a compact quantum group acts strictly outerly on a II$_\infty$ factor, we get a strictly positive, self-adjoint operator $\rho$ affiliated with $M$ such that $\Delta(\rho^it) = \rho^it \otimes \rho^it$ for all $t \in \mathbb{R}$ and $\tau_t = \text{Ad} \rho^it$. Because $\Delta(\rho) = \rho \otimes \rho$, it follows with the same proof as in Proposition 7.10 of [22] that $\rho$ is affiliated (in the C*-algebraic sense) with the C*-algebra of $(M, \Delta)$. This C*-algebra is unital and it follows that $\rho$ is bounded. Because $\Delta(\rho) = \rho \otimes \rho$, we get $\rho = 1$. So, $\tau_t = t$ for all $t \in \mathbb{R}$. □
We give some more information on the $T$ invariant $T(M, \Delta)$ and calculate it for the bicrossed product quantum groups of $[38]$.  

The following result is not true for the usual $T$ invariant of von Neumann algebras.

**Proposition 3.8.** Let $(M, \Delta)$ be a l.c. quantum group (and we still assume that $M_1$ is separable). Then $T(M, \Delta)$ is either a countable subgroup of $\mathbb{R}$, either the whole of $\mathbb{R}$. In the latter case, there exists a one-parameter family $\rho^t$ of unitaries in $M$ satisfying $\Delta(\rho^t) = \rho^t \otimes \rho^t$ and $\tau_t = \text{Ad} \rho^t$ for all $t \in \mathbb{R}$.

*Proof.* We define the intrinsic group of $(M, \Delta)$.

$$G_M := \{ u \in M \mid u \text{ is unitary and } \Delta(u) = u \otimes u \}.$$  

Equipped with the strong* topology, $G_M$ is a l.c. group. We know that for $u \in G_M$, $\tau_s(u) = u$ for all $s \in \mathbb{R}$. So, if $v \in G_M$ and $\text{Ad} v = \tau_s$ for some $s \in \mathbb{R}$, then $v \in \mathcal{Z}(G_M)$, the center of $G_M$.  

We define the l.c. abelian group $K$ as

$$K := \{(u, t) \in \mathcal{Z}(G_M) \times \mathbb{R} \mid \text{Ad } u = \tau_t \}$$

and the continuous homomorphism $\eta : K \to \mathbb{R} : \eta(u, t) = t$. By definition and the remark above, $T(M, \Delta) = \eta(K)$.

Define $K_0 = \text{Ker } \eta$ and consider the dual, continuous homomorphism $\hat{\eta} : \mathbb{R} \to \widehat{K/K_0}$ with dense range. So, $\widehat{K/K_0}$ is connected and hence, isomorphic to $\mathbb{R}^n \times L$ for $n \in \mathbb{N} \cup \{0\}$. In the latter case, there is a compact open subgroup containing a compact open subgroup. Suppose that the restriction of $\eta$ to $\mathbb{R}^n$ is not surjective. Then, $\eta(\mathbb{R}^n) = \{0\}$ and the restriction of $\eta$ to $L$ is surjective. The image of the compact open subgroup of $L$ is a compact subgroup of $\mathbb{R}$ and hence, trivial. So, the image of $L$ is at most countable, contradicting the surjectivity of $\eta$. Hence, the restriction of $\eta$ to $\mathbb{R}^n$ is surjective and we find a continuous homomorphism $\mu : \mathbb{R} \to K$ such that $\eta \mu = \iota$. So, we are done.  

We now compute $T(M, \Delta)$ for bicrossed product l.c. quantum groups. We use the conventions of $[2]$. So, we are given a (second countable) l.c. group $G$ with two closed subgroups $G_1, G_2$ such that $G_1 \cap G_2 = \{e\}$ and $G_1 G_2$ has a complement of Haar measure zero in $G$. We say that the pair $G_1, G_2$ is a matched pair. We define for almost all $x \in G$, $p_i(x) \in G_i$ such that $x = p_1(x)p_2(x)$. In this way, we identify (as measure spaces) $G_1$ and $G/G_2$. We define the von Neumann algebra $M = G_2 \rtimes L^\infty(G/G_2)$. From $[38]$, we know that $M$ is the underlying von Neumann algebra of a l.c. quantum group $(M, \Delta)$, the bicrossed product of $G_1$ and $G_2$. The underlying von Neumann algebra of the dual $(M, \Delta)$ is given by $L^\infty(G_1^\prime G) \rtimes G_1$. We have the following result.

**Proposition 3.9.** If $(M, \Delta)$ is the above bicrossed product l.c. quantum group, then

$$T(M, \Delta) = \{ t \in \mathbb{R} \mid \text{The character } \delta^\text{it}_2 \text{ on } G_2 \text{ can be extended to a continuous character of } G \},$$

where $\delta_2$ is the modular function of the l.c. group $G_2$.

*Proof.* We represent $M$ as $\left( L(G_2) \otimes 1 \cup \alpha(L^\infty(G_1)) \right)^\prime$ on $L^2(G_2 \times G_1)$, where $L(G_2)$ is the group von Neumann algebra generated by the left translations and $\alpha(F)$ is the multiplication operator with the function $\alpha(F)(s, g) = F(p_1(sg))$ for $g \in G_1, s \in G_2$, whenever $F \in L^\infty(G_1)$. From Proposition 4.16 in $[38]$, we know that $\tau_t$ is implemented by the multiplication operator

$$P^\text{it}(s, g) = \delta^\text{it}(g^{-1} p_1(sg)) \delta_1^\text{it}(gp_1(sg)^{-1}) \delta_2^\text{it}(s^{-1} p_2(sg)),$$
where $\delta_1, \delta_2, \delta$ denote the modular functions of $G_1, G_2, G$. Given our representation of $M$, it is clear that $1 \otimes L^\infty(G_1) \subset M'$. Hence, $\tau_t$ is implemented by the multiplication operator with the function

$$(s, g) \mapsto \alpha(\delta_t, \delta_1^{-it})(s, g) \delta^{it}_2(s^{-1}p_2(sg)) .$$

We also know from Proposition 4.1 in [33] that the group-like unitaries of $(M, \Delta)$ are precisely of the form $\alpha(\omega)(\lambda_\tau \otimes 1)$, where $\omega$ is a character on $G_1$ and $s \in G_2$ is such that $p_2(sg) = s$ for all $g \in G_1$. Then, it is not difficult to conclude that $t \in T(M, \Delta)$ if and only if there exists a character $\omega$ on $G_1$ such that the multiplication operator

$$K(s, g) = \omega(p_1(sg)) \delta^{it}_2(s^{-1}p_2(sg))$$

commutes with $\mathcal{L}(G_2) \otimes 1$. This holds if and only if

$$\omega(p_1(sg)) \delta^{it}_2(p_2(sg)) = \omega(g) \delta^{it}_2(s) \quad \text{almost everywhere},$$

i.e. if and only if there exists a (measurable, hence continuous) character $\mu$ on $G$ such that $\mu(gs) = \omega(g) \delta^{it}_2(s)$. So, we are done.

In Example 5.4 of [33], there is given a matched pair $G_1, G_2 \subset G$, such that $G \cong \text{PSL}_2(\mathbb{R})$ and $G_1$ is the $(ax + b)$-group. Interchanging $G_1$ and $G_2$, the bicrossed product $(M, \Delta)$ satisfies $T(M, \Delta) = \{0\}$.

We also remark that, for an arbitrary l.c. quantum group $(M, \Delta)$, the scaling group $(\tau_t)$ satisfies $\Delta \tau_t = (\tau_t \otimes \tau_t)\Delta$. Hence, $\mathbb{R}$ acts by quantum group automorphisms on $(M, \Delta)$. If $\Gamma$ is any subgroup of $\mathbb{R}$, that we equip with the discrete topology, the crossed product $\Gamma \rtimes M$ carries a natural l.c. quantum group structure: the comultiplication on $M$ is extended by putting $\Delta(\lambda_\tau) = \lambda_\tau \otimes \lambda_\tau$. It is now obvious, that if $T(M, \Delta) = \{0\}$, then $T(\Gamma \rtimes M, \Delta) = \Gamma$. So, we can obtain any subgroup of $\mathbb{R}$ as the $T$ invarant of a l.c. quantum group and any countable subgroup as the $T$ invariant of a l.c. quantum group with separable predual.

Another case where the invariant $T(M, \Delta)$ can be easily calculated, is for certain compact quantum groups. We state the result for Woronowicz’ quantum $SU_q(n)$ group.

**Proposition 3.10.** Consider the compact quantum group $SU_q(n)$, $0 < q < 1$ and $n \geq 2$ [40]. Then,

$$T(SU_q(n)) = \mathbb{Z} \frac{\pi}{\log q} .$$

In particular, $SU_q(n)$ acts strictly outerly on the free Araki-Woods factor $T_q^{\infty}$ of type III$_{q^2}$.

**Proof.** Write $(M, \Delta) = SU_q(n)$. Since a group-like unitary $u \in M$, is a one-dimensional corepresentation and since we know from [40] the corepresentations of $(M, \Delta)$, we conclude that $(M, \Delta)$ has no non-trivial group-like unitaries. Hence, $T(M, \Delta)$ consists of those $t \in \mathbb{R}$ with $\tau_t = \iota$. But, this is the same set as the set of $t \in \mathbb{R}$ with $\tau_t = \iota$. We know that the scaling group on the dual is implemented by the $F$-matrices. Hence, $\tau_t = \iota$ if and only if $q^{2\iota t} = 1$.

In the course of the proof of the Theorem [33] we needed the extension of an action $\alpha$ of a l.c. quantum group $(M, \Delta)$ on a von Neumann algebra $N$ to an action on the crossed product $N \rtimes (\sigma^\theta_f) \mathbb{R}$, where $\theta$ is an n.s.f. weight on $N$. Such extensions were considered by Yamanouchi in [18] under the assumption that $\nabla = \delta^{-1}$, which implies in particular that $\tau_t = \text{Ad} \delta^{-it}$, i.e. the scaling group is implemented by the modular element, which is group-like.

It is more natural to consider these extensions in the more general setting where the scaling group is implemented by an arbitrary group-like operator. In fact, we show that one cannot do better.

**Proposition 3.11.** Let $(M, \Delta)$ be a l.c. quantum group. The following are equivalent.

- Every action $\alpha : N \rightarrow M \otimes N$ of $(M, \Delta)$ on a von Neumann algebra $N$ can be extended to an action of $(M, \Delta)$ on the crossed product $N \rtimes (\sigma^\theta_f) \mathbb{R}$, whenever $\theta$ is an n.s.f. weight on $N$. 

Proof. Suppose first that every action \( \alpha \) can be extended to an action on the core. Consider the strictly outer action \( \alpha \) of \((M, \Delta)\) on \((N, \omega) = (N_0, \omega_0) \ast (N_1, \omega_1)\) constructed in the proof of Theorem 3.6, using the notation introduced there. By assumption, there exists an action \( \beta : C \to M \otimes C \) extending \( \alpha \), where \( C = N \rtimes \sigma_\tau^\omega \). Since \( \beta \) extends \( \alpha \), \( \beta \) is a faithful action. Moreover, from the proof of Theorem 3.6, we get that \( C \cap \sigma_\tau^\omega(N^\omega)' = C \). Hence, \( C \cap (\sigma_\tau^\omega)' = C \). So, \( \beta \) is strictly outer. Because \( C \) is a II\(_\infty\) factor, the second condition of the proposition follows from Theorem 3.5.

Suppose, conversely, that the second condition is fulfilled. Let \( \alpha : N \to M \otimes N \) be an action of \((M, \Delta)\) on a von Neumann algebra \( N \) with n.s.f. weight \( \theta \). Define \( \mathcal{U} \in M \otimes N \otimes L^\infty(\mathbb{R}) \) by \( \mathcal{U}(t) = (\rho^{-it} \otimes 1)D_t^\rho \). If we define, for \( z \in N \rtimes \sigma_\tau^\omega \mathbb{R} \subset N \otimes \mathbb{B}(L^2(\mathbb{R})) \), \( \beta(z) = \mathcal{U}(\alpha \otimes i)(z)\mathcal{U}^* \). One verifies immediately that

\[
\beta \sigma^\theta = (i \otimes \sigma^\theta)\alpha \quad \text{and} \quad \beta(1 \otimes \rho_t) = (i \otimes \sigma^\theta)(D_t) (\rho^{it} \otimes 1 \otimes \rho_t) ,
\]

where

\[
\sigma^\theta : N \to N \otimes L^\infty(\mathbb{R}) : \sigma^\theta(x)(t) = \sigma^\theta(x) , \quad (\rho_t \xi)(s) = \xi(s + t) \quad \text{for} \quad \xi \in L^2(\mathbb{R}) \quad \text{and}
\]

\[
N \rtimes \sigma_\tau^\omega \mathbb{R} = (\sigma^\theta(N) \uplus 1 \otimes \rho_t)^\omega \subset N \otimes \mathbb{B}(L^2(\mathbb{R})) ,
\]

\[
D_t = [D \theta \circ \alpha : D \theta]_{1 \in M \otimes N} \text{ is the Radon-Nikodym derivative.}
\]

4. A TECHNICAL LEMMA

We now include two lemmas that were needed to establish the trivial relative commutant properties \( N \cap (N^\omega)' = C \) and \( C \cap \sigma_\tau^\omega(N^\omega)' = C \) in the proof of Theorem 3.6. Lemma 4.1 below is a generalization of Theorem 11 in [6]. We do not need the full strength of the lemma, but it could be interesting to state its most general form.

The beautiful idea of the proof of Lemma 4.1 is due to G. Skandalis.

In Theorem 11 in [6], one considers a free product \((N, \omega) = (N_1, \omega_1) \ast (N_2, \omega_2)\) (see [12]), a unitary \( a \in N_1 \) in the centralizer of \( \omega_1 \) satisfying \( \omega_1(a) = 0 \) and unitaries \( b, c \in N_2 \) in the centralizer of \( \omega_2 \) satisfying \( \omega_2(b) = \omega_2(c) = \omega_2(cb^*) = 0 \), to obtain the inequality

\[
\|x - \omega(x)1\|_2 \leq 14 \max\{\|x, a\|_2, \|x, b\|_2, \|x, c\|_2\} , \quad \text{for all} \quad x \in N .
\]

Here, and below, we use the \( L^2 \)-norm \( \|x\|_2 = \omega(x^* x) \).

We will prove a same kind of inequality for general elements \( a, b, c \) keeping track of their non-unitarity and non-invariance under the modular group. This will allow to conclude the relative commutant properties needed above (in cases where the centralizer is trivial).

Lemma 4.1. Let \( N_{1,2} \) be von Neumann algebras with faithful normal states \( \omega_{1,2} \). Let \((N, \omega) = (N_1, \omega_1) \ast (N_2, \omega_2)\). Let \( a \in N_1 \) and \( b, c \in N_2 \). Suppose that \( a, b, c \in D(\sigma_\tau^{\omega_{1,2}}) \). Then, for all \( x \in N \),

\[
\|x - \omega(x)1\|_2 \leq D(a, b, c) \max\{\|x, a\|_2, \|x, b\|_2, \|x, c\|_2\} + E(a, b, c) \|x\|_2 ,
\]

where

\[
D(a, b, c) = 3\|a\|^2 + 2\|b\|^3 + 2\|c\|^3 \quad \text{and} \quad E(a, b, c) = 3C(a) + 2C(b) + 2C(c) + 6|\omega(cb^*)| \|cb^*\|
\]

with

\[
C(a) = \|a\|^3 \|\sigma_\tau^{\omega_{1,2}}(a) - a\| + \|a\|^2 \|a^* a - 1\| + 2(1 + \|a\|^2) \|aa^* - 1\| + 3|\omega(a)| \|a\| .
\]
Hence, if there exists a sequence \( a_n \) in \( N_1 \) and sequences \( b_n, c_n \) in \( N_2 \) such that \( a_n, b_n, c_n \) satisfy
\[
\|a_n a_n^* - 1\| \to 0 , \quad \|a_n^* a_n - 1\| \to 0 , \quad a_n \in D(\sigma_{i/2}^n) \quad \text{and} \quad \|\sigma_{i/2}^n(a_n) - a_n\| \to 0 ,
\]
\[
\omega(a_n) \to 0 .
\]
and such that \( \omega(c_n b_n^*) \to 0 \), then
\[
N \cap \{a_n, b_n, c_n \mid n \in \mathbb{N}\}' = \mathbb{C} .
\]
Moreover, every central sequence in \( N \) is trivial.

**Proof.** Take \( a \in N_1 \) and \( b, c \in N_2 \) such that \( a, b, c \in D(\sigma_{i/2}^n) \). Choose \( x \in N \). Replacing \( x \) by \( x - \omega(x)1 \) (which reduces the \( L^2 \)-norm), we may suppose that \( \omega(x) = 0 \). Represent \( N_1 \), resp. \( N_2 \), on its GNS-space \((H_1, \xi_1)\), resp. \((H_2, \xi_2)\). Let \((H, \xi) = (H_1, \xi_1) \star (H_2, \xi_2)\). Recall that
\[
H = C\xi \oplus (\hat{H}_1 \otimes H(2, l)) \oplus (\hat{H}_2 \otimes H(1, l)) ,
\]
with
\[
H(2, l) = C\xi \oplus \hat{H}_2 \oplus (\hat{H}_2 \oplus \hat{H}_1) \oplus (\hat{H}_2 \oplus \hat{H}_1) \oplus \cdots \quad \text{and}
\]
\[
H(1, l) = C\xi \oplus \hat{H}_1 \oplus (\hat{H}_1 \oplus \hat{H}_2) \oplus (\hat{H}_1 \oplus \hat{H}_2) \oplus \cdots ,
\]
where \( \hat{H}_1 = H_1 \otimes C\xi_1 \).

Define \( \eta := x\xi \). Because \( \langle \eta, \xi \rangle = 0 \), we write \( \eta = \mu + \gamma \) with \( \mu \in \hat{H}_1 \otimes H(2, l) \) and \( \gamma \in \hat{H}_2 \otimes H(1, l) \). Define, for \( \zeta \in H \) and \( y \in \mathbb{N} \), \( \zeta \cdot y := Jy^* J\zeta \) and observe that \( (z\xi) \cdot \sigma_{i/2}^n(y) = zy\xi \) for \( y \in D(\sigma_{i/2}^n) \).

Define \( \tilde{\eta} = a^* \cdot \eta \cdot a \) for all values of \( a \). We observe that
\[
\|\tilde{\eta} - \eta\| \leq \|a\| \cdot \|\sigma_{i/2}^n(a) - a\| \cdot \|x\|_2 + \|a\| \cdot \|\|x, a||_2 \| + \|a^* a - 1\| \cdot \|x\|_2 .
\]
Writing \( \mu = a^* \cdot \mu \cdot a \) and \( \tilde{\gamma} = a^* \cdot \gamma \cdot a \), we have \( \tilde{a} = \mu + \tilde{\gamma} \). It is obvious that
\[
\|\mu\|_2^2 \geq (1 + \|a\|^2) \cdot \|aa^* - 1\| \cdot \|x\|_2^2 \quad \text{and} \quad \|\tilde{\gamma}\|_2^2 \geq (1 + \|a\|^2) \cdot \|aa^* - 1\| \cdot \|x\|_2^2 .
\]
If \( P_2 \) denotes the projection onto \( \hat{H}_2 \otimes H(1, l) \) and because \( a \in N_1 \), we know that
\[
\|P_2 a^* \cdot a \cdot J_\gamma\| \leq \|\omega(a)\| \cdot \|J_\gamma\| \leq \|\omega(a)\| \cdot \|a\| \cdot \|x\|_2 .
\]
So, we conclude that \( \|\tilde{\gamma}, \gamma\| \leq \|\omega(a)\| \cdot \|a\| \cdot \|x\|_2^2 .
\]
Write \( \tilde{\zeta} = \eta - \gamma - \tilde{\gamma} \). With the inequalities obtained so far, we can estimate all the inner products between \( \tilde{\zeta}, \gamma \) and \( \tilde{\gamma} \). Since we also estimate the difference between \( \|\tilde{\gamma}\|_2^2 \) and \( \|\gamma\|_2^2 \), we arrive at
\[
\|\eta\|_2^2 = \|\tilde{\zeta} + \gamma + \tilde{\gamma}\|_2^2 \geq \|\tilde{\zeta}\|_2^2 + 2\|\gamma\|_2^2 + \|a\|_2^3 \cdot \|x, a\|_2 - C(a) \cdot \|x\|_2^2 ,
\]
where \( C(a) \) is as in the statement of the lemma. Since \( \|\eta\|_2^2 = \|\mu\|_2^2 + \|\tilde{\gamma}\|_2^2 \), we arrive at
\[
\|\mu\|_2^2 \geq \|\gamma\|_2^2 - \|a\|_2^3 \cdot \|x, a\|_2 - C(a) \cdot \|x\|_2^2 .
\]
Next, we use quite analogously \( b \) and \( c \). Define \( \eta' = b^* \cdot b \cdot \eta \), \( \eta'' = c^* \cdot \eta \cdot c \) and analogously \( \mu', \mu'', \gamma' \) and \( \gamma'' \). Write \( \zeta' = \eta - \mu - \mu' - \mu'' \). We can estimate in exactly the same way as above, all the inner products between \( \zeta', \mu, \mu' \) and \( \mu'' \). It will not be surprising that we arrive in this way at
\[
\|\gamma\|_2^2 \geq 2\|\mu\|_2^2 + \|b\|_2^3 \cdot \|x, b\|_2 - \|c\|_2^3 \cdot \|x, c\|_2 - (C(b) + C(c)) \cdot \|x\|_2^2 .
\]
Combining Inequalities (1.1) and (1.2) and using that \( \|x\|_2 = (\|\mu\|_2 + \|\gamma\|_2^2) /\|x\|_2 \), we precisely arrive at the inequality stated in the lemma.

Suppose now that we have sequences \( a_n, b_n, c_n \) as stated in the lemma. It is clear that \( D(a_n, b_n, c_n) \) will remain bounded, while \( E(a_n, b_n, c_n) \) converges to zero. So, we have a constant \( D \geq 0 \) and a sequence \( \kappa_n \geq 0 \) converging to zero, such that
\[
\|x - \omega(x)1\|_2 \leq D \max\{\|x, a_n\|_2, \|x, b_n\|_2, \|x, c_n\|_2\} + \kappa_n \cdot \|x\|_2 ,
\]
for all $x \in N$ and $n \in \mathbb{N}$.

If $x \in N \cap \{a_n, b_n, c_n \mid n \in \mathbb{N}\}'$, we immediately get that $x = \omega(x)1 \in \mathbb{C}$.

Let $x_n$ be a central sequence in $N$ (i.e. a bounded sequence in $N$ such that $\| [x_n, a] \|_2, \| [x_n^*, a] \|_2 \to 0$ for all $a \in N$). Choose $\varepsilon > 0$. Take $n$ such that $\kappa_n \| x_n \|_2 < \varepsilon/2$ for all $m$. Next, take $m_0$ such that for all $m \geq m_0$, $\text{Dmax} \{ \| [x_m, a_n] \|_2, \| [x_m, b_n] \|_2, \| [x_m, c_n] \|_2 \} < \varepsilon/2$. It follows that $\| x_m - \omega(x_m)1 \|_2 < \varepsilon$ for all $m \geq m_0$. We can do the same thing with $x_n^*$ and conclude that the sequence $x_n$ is trivial. □

The following lemma is very easy to prove.

**Lemma 4.2.** The free Araki-Woods factor $N = \Gamma(L^2(\mathbb{R}, \mathbb{R}), \lambda_\lambda)'$ with free quasi-free state $\omega$ contains a sequence of unitaries $u_n$ such that $u_n$ is analytic w.r.t. $(\sigma_\lambda^*)$, $\| \sigma_\lambda^*(u_n) - u_n \| \to 0$ uniformly on compact subsets of $\mathbb{C}$ and $\omega(u_n) \to 0$.

**Proof.** By Fourier transformation, we consider rather $K_\lambda = \{ \xi \in L^2(\mathbb{R}) \mid \xi(-x) = \overline{\xi(x)} \}$, with orthogonal transformations $(U_\lambda \xi)(x) = \exp(itx)\xi(x)$. This corresponds to the involution $(T \xi)(x) = \exp(-x/2)\xi(-x)$ on $L^2(\mathbb{R})$.

Take unit vectors $\mu_n$ in $K_\lambda$, bounded and with support in $[-1/n, 1/n]$. Recall that generators for the free Araki-Woods factor $N$ can be written as $s(\xi) = (\ell(\xi) + \ell(T \xi)^*)/2$. Also, if $\xi$ is a bounded, compactly supported function in $L^2(\mathbb{R})$, then $s(\xi)$ is analytic w.r.t. $(\sigma_\lambda^*)$ and $\sigma_\lambda^* s(\xi) = s(\xi)$, where $\xi(z) = \exp(izx)\xi(x)$.

Put $x_n := (s(\mu_n) + s(\mu_n^*))/2 \in N$. Put also $y_n = (\ell(\mu_n) + \ell(\mu_n^*))/2 \in B(L^2(\mathbb{R}))$. We know that $y_n$ has the semi-circular distribution with respect to the vacuum state. Take a real number $k$ such that $\int_1^2 \exp(ikt)\sqrt{1-t^2}dt = 0$. Put $u_n = \exp(ikx_n)$. It is clear that $u_n$ are unitaries in $N$, analytic w.r.t. $(\sigma_\lambda^*)$ and $\| \sigma_\lambda^*(u_n) - u_n \| \to 0$ uniformly on compact subsets of $\mathbb{C}$. Since $\| x_n - y_n \| \to 0$, we get $\| u_n - \exp(iky_n) \| \to 0$, which yields $\omega(u_n) \to 0$ by our choice of $k \in \mathbb{R}$. □

5. Strictly outer actions of locally compact groups on the hyperfinite $\text{II}_1$ factor

In this section, we construct strictly outer actions of ordinary l.c. groups on the hyperfinite II$_1$ factor. We first provide a general construction procedure, based on a general strict outerness result. Next, we give a more geometrical construction for linear groups.

**Theorem 5.1.** Let $N$ be a factor with a faithful state $\omega$ and suppose that a l.c. group $G$ acts faithfully on $N$ by automorphisms $(a_g)$ leaving invariant $\omega$, i.e. such that $a_g = e$ and $\omega a_g = \omega$.

Denote $(N_\infty, \omega_\infty) = \bigotimes_{n=1}^\infty (N, \omega)$, which is a factor equipped with the faithful state $\omega_\infty$. Let $G$ act diagonally on $N_\infty$.

Then, the action of $G$ on $N_\infty$ is strictly outer.

**Proof.** Write $N_k = \bigotimes_{n=1}^k (N, \omega)$.

For every $n \in \mathbb{N}$, it is easy to define an automorphism $i_n$ of $N_\infty$ such that

$i_n(x_1 \otimes \cdots \otimes x_n \otimes \cdots) = x_n \otimes x_1 \otimes \cdots \otimes x_{n-1} \otimes x_{n+1} \otimes \cdots$.

One defines $i_n$ on all $N_k$, $k \geq n$, and one can extend to $N_\infty$ because $i_n$ preserves $\omega_\infty$.

We have an obvious isomorphism $\Psi : N \otimes N_\infty \to N_\infty$ given by $\Psi(x_0 \otimes (x_1 \otimes x_2 \otimes \cdots)) = x_0 \otimes x_1 \otimes \cdots$. We write $j(z) = \Psi(1 \otimes z)$ and claim that $i_n(z) \to j(z)$ strongly$^*$ for all $z \in N_\infty$.

To prove our claim, let $(H_\infty, \xi_\infty) = \bigotimes_{n=1}^\infty (H, \xi_n)$ be a GNS-construction for $(N_\infty, \omega_\infty)$ with cyclic and separating vector $\xi_\infty$. We define unitaries $U_n$ on $H_\infty$ by the formula $U_n x \xi_\infty = i_n(x) \xi_\infty$. We also define an isometry $V$ on $H_\infty$ given by $V x \xi_\infty = j(x) \xi_\infty$. By definition, it is clear that
$U_n x \xi_\infty \to V x \xi_\infty$ whenever $x \in N_k$ for some $k \in \mathbb{N}$. Because the sequence $U_n$ is bounded, it follows that $U_n \to V$ strongly. Hence, for $z \in N_\infty$ and $y \in N_\infty'$, we get

$$i_n(z) y \xi_\infty = y i_n(z) \xi_\infty = y U_n z \xi_\infty \to y V z \xi_\infty = y j(z) \xi_\infty = j(z) y \xi_\infty.$$  

Because the sequence $i_n(z)$ is bounded, we conclude that $i_n(z) \to j(z)$ strongly for all $z \in N_\infty$. Because $i_n$ and $j$ are *-homomorphisms, the convergence is strong*. This proves our claim.

We realize $G \rtimes N_\infty = (\alpha(N_\infty) \cup L(G) \otimes 1)'' \subset B(L^2(G)) \otimes N_\infty$, where $\alpha : N_\infty \to L^\infty(G) \otimes N_\infty : \alpha(x)(g) = \alpha_{g-1}(x)$.

Let now $a \in G \rtimes N_\infty \cap \alpha(N_\infty)'$. Using the realization of $G \rtimes N_\infty \subset B(L^2(G)) \otimes N_\infty$ and the fact that $i_n$ and $j$ commute with the action of $G$, we find that

$$(i \circ i_n)(a) \in G \rtimes N_\infty \cap \alpha(N_\infty)'$$

for all $n$ and hence,

$$(i \circ j)(a) \in G \rtimes N_\infty \cap \alpha(N_\infty)'$$

Applying the isomorphism $i \otimes \Psi^{-1}$, we conclude that $\alpha_{13}$ and $\alpha(x) \otimes 1$ commute in $B(L^2(G)) \otimes N \otimes N_\infty$ for all $x \in N$. So, $a$ and $(i \otimes \Psi)\alpha(x) \otimes 1$ commute in $B(L^2(G)) \otimes N_\infty$ for all $x \in N, \mu \in N_*$. The self-adjoint family of functions

$$\{g \mapsto \mu (\alpha_{g^{-1}}(x)) \mid \mu \in N_*, x \in N\}$$

separates the points of $G$, which yields

$$\{g \mapsto \mu (\alpha_{g^{-1}}(x)) \mid \mu \in N_*, x \in N\}' = L^\infty(G).$$

It follows that $a \in L^\infty(G) \otimes N_\infty$. From the duality for crossed products, we get that

$$a \in G \rtimes N_\infty \cap L^\infty(G) \otimes N_\infty = \alpha(N_\infty).$$

But, $a \in \alpha(N_\infty)'$, which gives $a \in \mathbb{C}1$ and we are done.

In [7], Blattner constructed a canonical action of any l.c. group on the hyperfinite II$_1$ factor. His construction is as follows. Let $H_\mathbb{R}$ be a real Hilbert space and consider its Clifford algebra $\text{Cl}(H_\mathbb{R})$: $\text{Cl}(H_\mathbb{R})$ is the *-algebra generated by the self-adjoint elements $c(\xi)$, $\xi \in H_\mathbb{R}$ with relations $c(\xi)c(\eta) + c(\eta)c(\xi) = 2\langle \xi, \eta \rangle$. For every orthogonal transformation $u$ on $H_\mathbb{R}$, we have an automorphism $\alpha_u$ of $\text{Cl}(H_\mathbb{R})$ such that $\alpha_{u}(c(\xi)) = c(u\xi)$. In particular, taking $u\xi = -\xi$ for all $\xi \in H_\mathbb{R}$, we obtain a $\mathbb{Z}/2\mathbb{Z}$-grading of $\text{Cl}(H_\mathbb{R})$. On $\text{Cl}(H_\mathbb{R})$ there exists a unique trace $\tau$ such that $\tau(1) = 1$ and $\tau(a) = 0$ if $a$ has odd degree. Using a GNS-representation for this trace, we define a von Neumann algebra, which happens to be the hyperfinite II$_1$ factor $\mathcal{R}$ if $H_\mathbb{R}$ is of separable infinite dimension. So, whenever such a real Hilbert space $H_\mathbb{R}$ is fixed, we consider

$$\mathcal{R} = \{c(\xi) \mid \xi \in H_\mathbb{R}\}''.$$  

It is clear that, for any orthogonal transformation $u$ on $H_\mathbb{R}$, $\alpha_u$ extends to an automorphism of $\mathcal{R}$, still denoted by $\alpha_u$.

Let now $G$ be a l.c. group and let $(u_g)$ be a continuous representation of $G$ by orthogonal transformations of a real Hilbert space $H_\mathbb{R}$ of infinite separable dimension. Defining $\alpha_g := \alpha_{u_g}$, we get an action of $G$ by automorphisms of $\mathcal{R}$. If the representation $(u_g)$ is faithful, the automorphism group $(\alpha_g)$ is clearly faithful as well.

Also, observe that taking a direct sum of a family of orthogonal transformations yields an action of $\mathcal{R}$, which is isomorphic to the diagonal action on the tensor product of the family of copies of $\mathcal{R}$ associated to the given family of real Hilbert spaces.

Hence, we obtain the following corollary of Theorem [11].

**Corollary 5.2.** Every l.c. group $G$ can act strictly outerly on the hyperfinite II$_1$ factor. In particular, Blattner’s action of $G$ on the hyperfinite II$_1$ factor associated as above with an infinite direct sum of copies of the regular representation, is a strictly outer action.
Finally, we explain how strictly outer actions of linear groups on the hyperfinite II$_1$ factor can be obtained in an alternative, more intuitive and geometric way.

Let $n \in \mathbb{N}$. Define $G = SL(n + 2, \mathbb{C})$ and consider the subgroups

$$H = \left\{ \begin{pmatrix} 1_{n,n} & 0_{n,2} \\ 0_{2,n} & 1_k \\ 0 \end{pmatrix} \left| k \in \mathbb{Z} \right. \right\}, \quad K = \left\{ \begin{pmatrix} SL(n, \mathbb{C}) & 0_{n,2} \\ 0_{2,n} & 1_z \\ 0 \end{pmatrix} \left| z \in \mathbb{C} \right. \right\} \quad \text{and}

$$

$$L = \left\{ \begin{pmatrix} SL(n, \mathbb{C}) & 0_{n,2} \\ 0_{2,n} & 1_k \\ 0 \end{pmatrix} \left| k \in \mathbb{Z} \right. \right\}.$$

Let $\Gamma$ be a lattice in $G$ (i.e. a discrete subgroup with finite covolume). Write $X = G/\Gamma$, with its finite measure invariant under the action of $G$.

We first claim that the action of $K$ on $X$ is (measure theoretically) free, in the sense that for almost all $x \in X$, the stabilizer $S_x \subset K$ is trivial. So, we have to prove that

$$\bigcup_{\gamma \in \Gamma, \gamma \neq e} \{ x \in G \mid x\gamma x^{-1} \in K \}$$

has measure zero in $G$. Because $\Gamma$ is countable, it is enough to show that, for every $\gamma \in \Gamma \setminus \{e\}$, $\{ x \in G \mid x\gamma x^{-1} \in K \}$ has measure zero. This clearly is an algebraic subvariety of $G$, which is not the whole of $G$, because the center of $G$ intersects $K$ trivially. Hence, it is a set of measure zero.

We conclude a fortiori that the action of $L$ on $X$ is free in the same measure theoretic sense.

In particular, the action of $H$ on $X$ is free and it is ergodic by Moore’s ergodicity theorem, see e.g. [19]. So, we can define $\mathcal{R} = H \ltimes L^\infty(X)$, which is the hyperfinite II$_1$ factor by amenability of $\mathbb{Z}$, ergodicity and freeness.

We next have a natural action of $SL(n, \mathbb{C})$ on $\mathcal{R}$ such that

$$SL(n, \mathbb{C}) \ltimes \mathcal{R} = L \ltimes L^\infty(X).$$

It follows that $SL(n, \mathbb{C}) \ltimes \mathcal{R} \cap \mathcal{R}' \subset L \ltimes L^\infty(X) \cap L^\infty(X)' = L^\infty(X)$ because of the freeness of the action of $L$ on $X$, see [29]. But then, $SL(n, \mathbb{C}) \ltimes \mathcal{R} \cap \mathcal{R}' \subset L^\infty(X) \cap \mathcal{R}' = \mathbb{C}$ because of the ergodicity of the action of $H$ on $X$.

Hence, we have found a strictly outer action of $SL(n, \mathbb{C})$ on the hyperfinite II$_1$ factor $\mathcal{R}$ for any $n \in \mathbb{N}$. Because the restriction of a strictly outer action to a closed subgroup is a strictly outer action, we have found a strictly outer action on $\mathcal{R}$ of any linear group.

6. **Strictly outer actions of locally compact quantum groups on injective factors**

First, we have shown that every l.c. quantum group can act strictly outerly on a free Araki-Woods factor. These factors are far from being injective. Secondly, we have seen that every l.c. group can act strictly outerly on the hyperfinite II$_1$ factor. From Theorem [30] we know that not all l.c. quantum groups can act strictly outerly on a II$_1$ factor. Nevertheless, we ask the natural question when it is possible to act strictly outerly on an injective factor of arbitrary type.

In this elementary section, we prove a general result giving a necessary condition for the possibility to act strictly outerly on an injective factor. We prove that for bicrossed product quantum groups, this condition is sufficient as well.

Given a corepresentation of a l.c. quantum group in a factor $N$, we consider the infinite tensor product of the associated inner action on $N$. We give a sufficient condition for its strict outerness that will allow us, in the next two sections, to deal with strictly outer actions of compact and discrete quantum groups on injective factors.

We finally show that the possibility of acting strictly outerly on an injective factor is stable under cocycle deformation of the quantum group.
**Definition 6.1.** Let \((M, \Delta)\) be a l.c. quantum group. A (not necessarily normal) state \(m : M \to \mathbb{C}\) is called a left invariant mean, if \(m(\omega \otimes (\Delta(x))) = \omega(1) m(x)\) for all \(x \in M\) and \(\omega \in M_*\). We analogously define a right invariant mean. We call \(m\) an invariant mean, if \(m\) is a left and right invariant mean.

A l.c. quantum group has a left invariant mean if and only if it has an invariant mean, see e.g. Proposition 3 in [3].

**Proposition 6.2.** Let \(\alpha : N \to M \otimes N\) be a strictly outer action of a l.c. quantum group \((M, \Delta)\) on \(N\). If the crossed product \(M \rtimes N\) is injective, \((M, \Delta)\) has an invariant mean. If \(N\) is injective, the dual l.c. quantum group \((\hat{M}, \hat{\Delta})\) has an invariant mean.

**Proof.** It suffices to prove the first statement: the second follows by considering the dual action on the crossed product and using the fact that the double crossed product is \(B(\hat{H}) \otimes N\), which is injective if \(N\) is injective (see [2] for the notion of dual action and double crossed product).

So, suppose that \(M \rtimes N\) is injective. Let \(P : B(H) \otimes N \to M \rtimes N\) be a norm one projection. Take \(z \in M'\). Then, \(P(z \otimes 1) \in M \rtimes N\) and for all \(x \in N\), we have
\[
P(z \otimes 1) \alpha(x) = P((z \otimes 1) \alpha(x)) = P(\alpha(x)(z \otimes 1)) = \alpha(x) P(z \otimes 1).
\]

By strict outerness of \(\alpha\), we get that \(P(z \otimes 1) \in \mathbb{C}\). Hence, we can define a state \(\mu\) on \(M'\) such that \(P(z \otimes 1) = \mu(z) I\) for all \(z \in M'\).

Define \(\rho : M' \to M \otimes M' : \rho(z) = W(1 \otimes z) W^*\). Here, \(W\) is the multiplicative unitary associated with \((M, \Delta)\) and we know that \((\hat{J} \otimes J) W(\hat{J} \otimes J) = W^*\) (see Preliminaries). Hence, we get that \(\rho(z) = (\hat{J} \otimes J) \Delta(J z J)(\hat{J} \otimes J)\) for all \(z \in M'\). Let \(Q\) be the norm one projection from \(B(H) \otimes B(H) \otimes N\) onto \(B(H) \otimes (M \rtimes N)\) such that \((\omega \otimes \iota \otimes \iota) Q(z) = P((\omega \otimes \iota \otimes \iota)(z))\) for all \(z \in B(H) \otimes B(H) \otimes N\) and \(\omega \in B(H)_*\). We observe that, for \(z \in M'\) and \(\omega \in B(H)_*\),
\[
\mu((\omega \otimes \iota)(\rho(z)))(1 \otimes 1) = P((\omega \otimes \iota) \rho(z) \otimes 1) = (\omega \otimes \iota \otimes \iota) Q(W_{12}(1 \otimes z \otimes 1) W_{12}^*)
\]
\[
= (\omega \otimes \iota \otimes \iota)(W_{12} Q(1 \otimes z \otimes 1) W_{12}^*) = \omega(1) \mu(z) (1 \otimes 1).
\]

We immediately conclude that \(m(z) = \mu(J z^* J)\) defines a left invariant mean on \((M, \Delta)\). \(\square\)

Combining several results, we obtain the following.

**Proposition 6.3.** Let \((M, \Delta)\) be a bicrossed product locally compact quantum group, with \(M = G_2 \ltimes L^\infty(G/G_2)\) as explained just before Proposition 3.8. Then, \((M, \Delta)\) can act strictly outerly on an injective factor if and only if \((\hat{M}, \hat{\Delta})\) has an invariant mean.

**Proof.** One implication follows from Proposition 6.2. So, suppose that \((\hat{M}, \hat{\Delta})\) has an invariant mean. Since \((L^\infty(G_2), \Delta_2)\) is a closed quantum subgroup of \((\hat{M}, \hat{\Delta})\), the restriction of the invariant mean on \((\hat{M}, \hat{\Delta})\) gives an invariant mean on \(L^\infty(G_2)\). So, \(G_2\) is an amenable l.c. group. Using Corollary 6.2 we can take a strictly outer action of \(G\) on the hyperfinite \(\Pi_1\) factor \(\mathcal{R}\). From Proposition 6.1 in [3], we get that \((M, \Delta)\) can act strictly outerly on \(G_2 \ltimes \mathcal{R}\), where \(G_2\) acts by the restriction of the action of \(G\) on \(\mathcal{R}\). Since \(G_2\) is amenable, \(G_2 \ltimes \mathcal{R}\) is injective. \(\square\)

Taking into account Theorem 6.1 and since the tensor product preserves injectiveness, it is a natural idea to consider an infinite tensor product action to obtain strictly outer actions on injective factors. But, due to the non-commutativity of the algebra \(M\), we cannot perform the tensor product of two arbitrary actions. However, we can make the tensor product of two inner actions because we can make the tensor product of corepresentations.

Let \((M, \Delta)\) be a l.c. quantum group and \(U \in M \otimes N\) a corepresentation of \((M, \Delta)\) in the factor \(N\). Suppose that \(\alpha\) is a faithful normal state on \(N\) that is invariant under the inner action
\[
\beta : N \to M \otimes N : \beta(z) = U^* (1 \otimes z) U,
\]
\[ (\iota \otimes \omega)\beta(z) = \omega(z)1 \text{ for all } z \in N. \]

Writing \( X_n \) for the \( n \)-fold tensor product, \( X_n := U_{1,n+1} \ldots U_{12} \in M \otimes N^\otimes_n \), we define the inner action

\[ \beta_n : N^\otimes_n \to M \otimes N^\otimes : \beta_n(z) = X_n^*(1 \otimes z)X_n. \]

It is clear that the tensor product state \( \omega^\otimes \) is invariant under \( \beta_n \) and that the actions \( \beta_n \) and \( \beta_{n+1} \) are compatible with the inclusion \( N^\otimes_n \hookrightarrow N^\otimes_{n+1} : z \mapsto z \otimes 1 \). So, we can take easily the direct limit \( (N^\otimes, \omega^\otimes) = \bigotimes_n^\infty (N, \omega) \) with the limit action \( \alpha : N^\infty \to M \otimes N^\infty \) that we call the infinite tensor product action.

We will give a sufficient condition for \( \alpha \) to be a strictly outer action.

Recall that the l.c. quantum group \( (M, \Delta) \) has a universal \( C^* \)-algebraic dual \( (\hat{A}_u, \hat{\Delta}_u) \), such that the \( * \)-representations of \( A_u \) are in one-to-one correspondence with the unitary corepresentations of \( (M, \Delta) \). We denote by \( W \in M(\hat{A} \otimes \hat{A}_u) \) the universal corepresentation. Here \( A \) is the reduced \( C^* \)-algebra of \( (M, \Delta) \), which can be defined as the norm closure of \( \{ (\iota \otimes \omega)(W) \mid \omega \in B(H)_* \} \). See [21] for details.

**Proposition 6.4.** Let \( (M, \Delta) \) be a locally compact quantum group. Let \( N \) be a factor with a faithful state \( \omega \) and let \( U \in M \otimes N \) be a unitary corepresentation of \( (M, \Delta) \) in \( N \). Suppose that the following two conditions hold.

a) The state \( \omega \) is invariant under the inner action \( \beta : N \to M \otimes N : \beta(z) = U^*(1 \otimes z)U \).

b) Let \( \rho : \hat{A}_u \to N \) be the \( * \)-representation satisfying \((\iota \otimes \rho)(W) = U \). Then, every bounded sequence \((a_n) \) in \( M(\hat{A}_u) \) satisfying \((\iota \otimes \rho)(\Delta_u(a_{n+1})) = a_n \) for all \( n \), is a constant scalar sequence.

Then, the infinite tensor product action \( \alpha : N^\infty \to M \otimes N^\infty \) is strictly outer.

**Proof.** Denote by \( E_n : N^\infty \to N^\otimes_n \) the natural conditional expectations. We have \((\iota \otimes E_n)\alpha(z) = \beta_n(E_n(z)) \) for all \( z \in N^\infty \).

Since \( \beta_n \) is an inner action of \( (M, \Delta) \) on \( N^\otimes_n \) (i.e. cocycle equivalent with the trivial action), we obtain that \( \text{Ad} X_n : M_{\beta_n^\otimes N^\otimes_n} \to M \otimes N^\otimes_n \) is an isomorphism, sending \( \beta_n(N^\otimes_n) \) to \( 1 \otimes N^\otimes_n \). So, we conclude that

\[ M_{\beta_n^\otimes N^\otimes_n} \cap \beta_n(N^\otimes_n)' = X_n^*(\hat{M} \otimes 1)X_n. \]

Let \( z \in M_{\alpha^\otimes N^\infty} \cap \alpha(N^\infty)' \). Considering \( M_{\alpha^\otimes N^\infty} \) as a subalgebra of \( B(H) \otimes N^\infty \), we can write \( z_n = (\iota \otimes E_n)(z) \). Then, for all \( n \), \( z_n \in M_{\beta_n^\otimes N^\otimes_n} \cap \beta_n(N^\otimes_n)' \). So, we can take \( y_n \in M \), with \( \|y_n\| \leq \|z\| \), such that

\[ (\iota \otimes E_n)(z) = X_n^*(y_n \otimes 1)X_n. \]

Since \( E_n \circ E_{n+1} = E_n \), we get that \( y_n = (\iota \otimes \omega)(U^*(y_{n+1} \otimes 1)U) \) for all \( n \).

Define the (right) \( C^* \)-algebraic action \( \eta : K(H) \to M(K(H) \otimes \hat{A}_u) \) : \( \eta(y) = W^*(y \otimes 1)W \). Fix \( \mu \in B(H)_* \) and define \( a_n = (\mu \otimes \iota)\eta(y_n) \). Clearly, \( \|a_n\| \leq \|\mu\| \|z\| \) for all \( n \) and

\[ (\iota \otimes \omega)(\Delta_u(a_{n+1})) = (\mu \otimes \iota \otimes \omega)(\eta(y_{n+1})) = (\mu \otimes \iota)(\eta(\iota \otimes \omega)(\eta(y_{n+1}))) = (\mu \otimes \iota)\eta(y_n) = a_n. \]

By assumption, the sequence \((a_n) \) is a constant scalar sequence. Since this holds for all \( \mu \in B(H)_* \), we conclude that \( \eta(y_n) \in B(H) \otimes 1 \). But then, \( \eta(y_n) = y_n \otimes 1 \). This implies that \( y_n \in M' \). We already know that \( y_n \in M \) and conclude that \( y_n \in \mathbb{C}1 \) for all \( n \). Hence, \( z \in \mathbb{C}1 \) and we are done.

In certain cases, the conditions in Proposition 6.4 can be weakened:

**Proposition 6.5.** Suppose that we are in the setting of Proposition 6.4.

- If the scaling group \( (\tau_t) \) of \( (M, \Delta) \) is trivial and \( \omega \) is a trace, then condition 23 is automatically fulfilled.
• If there exists a state $\omega_1$ on $\hat{A}_u$ and a number $0 < t < 1$ such that $\omega \rho = (1 - t)\omega_1 + t \tilde{\epsilon}$, where $\tilde{\epsilon}$ is the co-unit of $(\hat{A}_u, \hat{\Delta}_u)$, then condition [4] can be weakened to the condition: every $a \in M(\hat{A}_u)$ satisfying $(\iota \otimes \omega \rho)\hat{\Delta}_u(a) = a$ is scalar.

Proof. First, suppose that the scaling group of $(M, \Delta)$ is trivial and that $\omega$ is a trace. Then, also the scaling group of $(\hat{A}_u, \hat{\Delta}_u)$ is trivial and we get a $^*$-anti-automorphism $R_\mu$ of $\hat{A}_u$ such that

$$R_\mu((\mu \otimes \iota)(W)) = (\mu \otimes \iota)(W^*)$$

for all $\mu \in B(H)_+$. Let $f, g \in H$ and let $(e_n)$ be an orthonormal basis for $H$. Let $z \in N$. We make the following computation, writing $\omega_{f,g} \in B(H)_+$, defined by $\omega_{f,g}(x) = (xf, g)$.

$$\omega_{f,g}((U^* (1 \otimes z) U) = \omega((\omega_{f,g} \otimes \iota)(U^* (1 \otimes z) U)) = \sum_{n=1}^\infty \omega((\omega_{e_n,g} \otimes \iota)(U^* z(\omega_{e_n,g} \otimes \iota)(U))) = \sum_{n=1}^\infty \omega(z \rho((\omega_{e_n,g} \otimes \iota)(W)(\omega_{e_n,g} \otimes \iota)(W^*)))) .$$

We now use that, with strict convergence,

$$\sum_{n=1}^\infty (\omega_{f,e_n} \otimes \iota)(W)(\omega_{e_n,g} \otimes \iota)(W^*) = R_u\left(\sum_{n=1}^\infty (\omega_{e_n,g} \otimes \iota)(W)(\omega_{e_n,g} \otimes \iota)(W^*)\right)$$

$$= R_u((\omega_{f,g} \otimes \iota)(WW^*)) = \omega_{f,g}(1)1 .$$

We conclude that

$$(\omega_{f,g} \otimes \omega)(U^* (1 \otimes z) U) = \omega_{f,g}(1)\omega(z)$$

which yields a proof of our first statement.

Suppose next that $\omega \rho = (1 - t)\omega_1 + t \tilde{\epsilon}$ Let $(a_n)$ be a bounded sequence in $M(\hat{A}_u)$ satisfying $(\iota \otimes \omega \rho)\hat{\Delta}_u(a_{n+1}) = a_n$ for all $n$.

Define the probability measure $\gamma$ on $\mathbb{Z}$ by $\gamma(\{0\}) = t$ and $\gamma(\{1\}) = 1 - t$. If we define the $n$-fold convolution $\omega_n = \omega_1 \ast \cdots \ast \omega_1$ ($n$ times), we get

$$(\omega \rho)^n = \sum_{k=0}^\infty \gamma^\ast k(\{k\})\omega_k ,$$

where $\gamma^\ast k$ is the $n$-fold convolution of the measure $\gamma$ on $\mathbb{Z}$. By Corollary 2 in [13], we know that $\|\gamma^\ast n - \gamma^\ast (n+1)\|_1 \rightarrow 0$ if $n \rightarrow \infty$. So, we find that $\|\omega(\rho)^n - (\omega \rho)^{n+1}\| \rightarrow 0$ if $n \rightarrow \infty$. Then, we have, for all $n$ and $k$,

$$\|a_{k+1} - a_k\| = \|(\iota \otimes \omega(\rho)^n)\hat{\Delta}_u(a_{n+k+1}) - (\iota \otimes \omega(\rho)^{n+1})\hat{\Delta}_u(a_{n+k+1})\|$$

$$\leq \|(\omega(\rho)^n - (\omega \rho)^{n+1})\| \|a_{n+k+1}\| .$$

We let $n \rightarrow \infty$ and use that the sequence $(a_n)$ is bounded, to conclude that $a_{k+1} = a_k$ for all $k$. Hence, there exists an $a \in M(\hat{A}_u)$ such that $a_k = a$ for all $k$. Then, $(\iota \otimes \omega(\rho)\hat{\Delta}_u(a) = a$ and our weakened condition yields that $a \in C1$. Hence, $(a_n)$ is a constant scalar sequence. □

As a final general result, we prove that the possibility of acting strictly outerly on an injective factor is stable under cocycle perturbation. More precisely, let $(M, \Delta)$ be a l.c. quantum group. A 2-cocycle $\Omega$ on $M$ is a unitary operator $\Omega \in \mathfrak{M} \otimes \mathfrak{M}$ satisfying

$$(1 \otimes \Omega)(\iota \otimes \Delta)(\Omega) = (\Omega \otimes 1)((\Delta \otimes \iota)(\Omega) .$$

Let as in the Preliminaries, $W$ denote the left regular corepresentation of $(M, \Delta)$, which is a multiplicative unitary on $H \otimes H$. Recall that we introduced as well the notations $J$ and $\hat{J}$ for the modular conjugations of the left invariant weights on $(M, \Delta)$ and $(\hat{M}, \hat{\Delta})$, respectively.

Suppose that $\Omega$ is a 2-cocycle on $(M, \Delta)$. Following [40], we define

$$W_\Omega := \hat{\Omega}W\Omega^* \quad \text{with} \quad \hat{\Omega} = (1 \otimes J\hat{J})\Sigma \Omega \Sigma(1 \otimes J\hat{J}) .$$
We also define $\Delta_{\Omega}(x) = \Omega \Delta(x) \Omega^*$. It is easy to verify that $W_{\Omega}$ is a multiplicative unitary on $H \otimes H$ and that $\Delta_{\Omega}$ is a co-associative comultiplication on $(M, \Delta)$. Up to now, a general theory giving necessary and sufficient conditions for $(M, \Delta_{\Omega})$ to be a l.c. quantum group is not available. See [10] for results in this direction. Nevertheless, we have the following result.

**Proposition 6.6.** Let $(M, \Delta)$ be a l.c. quantum group and $\alpha : N \to M \otimes N$ a strictly outer action on the injective factor $N$. If $\Omega$ is a 2-cocycle on $M$ such that $(M, \Delta_{\Omega})$ is again a l.c. quantum group with left corepresentation $W_{\Omega}$, then

$$\beta : N \otimes B(H) \to M \otimes N \otimes B(H) : \beta(z) = (\Omega W^*)_{13} (\alpha \otimes \iota)(z) (W \Omega^*)_1$$

defines a strictly outer action of $(M, \Delta_{\Omega})$ on the injective factor $N \otimes B(H)$.

**Proof.** It is obvious that $\alpha : N \to M \otimes N$ is a cocycle action (in the sense of [38], Definition 1.1) of $(M, \Delta_{\Omega})$ on $N$ with cocycle $\Omega^* \otimes 1$. Using Definition 1.3 in [38], we can define the cocycle crossed product

$$(M, \Delta_{\Omega})_{(\alpha, \Omega^* \otimes 1)} \ltimes N := \{ (\alpha(N) \cup \{ (\omega \otimes \iota)(\hat{\Omega} W) \otimes 1 | \omega \in B(H)_\alpha \}) \}'' .$$

If we consider the cocycle action $\alpha \otimes \iota$ of $(M, \Delta_{\Omega})$ on $N \otimes B(H)$, with cocycle $\Omega^* \otimes 1 \otimes 1$, the cocycle formula [10] yields that this cocycle action is stabilizable with the unitary $(\Omega W^*)_{13} \in M \otimes N \otimes B(H)$, in the sense of [38], Definition 1.7. This means that, defining $\beta$ as in the statement of the proposition, we indeed get an action of $(M, \Delta_{\Omega})$ on $N \otimes B(H)$. We claim that this action is strictly outer. Using Proposition 1.8 of [38], we have to prove that

$$(M, \Delta_{\Omega})_{(\alpha, \Omega^* \otimes 1)} \ltimes N \cap (\alpha(N))' = M' \otimes 1$$

by the strict outerness of $\alpha$ and Proposition 7.4. As an intermediate step, we show that every $z \in (M, \Delta_{\Omega})_{(\alpha, \Omega^* \otimes 1)} \ltimes N$ satisfies

(6.2) $(\iota \otimes \alpha)(z) = (V (JJ^*1) \Sigma_{\Omega^*} \Sigma(JJ^*1)_{12}) z_{13} (V (JJ^*1) \Sigma_{\Omega^*} \Sigma(JJ^*1)_{12})^*$,

where $V \in M' \otimes M$ denotes the right regular corepresentation of $(M, \Delta)$. Recall that

$$V = (JJ^*1) \Sigma W \Sigma(JJ^*1)$$

and $\Delta(x) = V(x1)V^*$ for all $x \in M$. Since, for $x \in N$, $(\iota \otimes \alpha)(x) = (\Delta \otimes \iota)(x) = V_{12} \alpha(x)_{13} V_{12}^*$ and since $(JJ^*1) \Sigma_{\Omega^*} \Sigma(JJ^*1)$ belongs to $M' \otimes M$, Equation (6.2) is clear for $z = \alpha(x)$. To obtain Equation (6.2) for $z = (\omega \otimes \iota)(\hat{\Omega} W) \otimes 1$ and $\omega \in B(H)_\alpha$, we have to show that

$$V (JJ^*1) \Sigma_{\Omega^*} \Sigma(JJ^*1)$$

and $(\hat{\Omega} W)_{12}$ commute. Using the equality for $V$ recalled above, we have to show that $(\Omega W^*)_1$ and $(\Omega V)_{23}$ commute. Observe that

$$(\Omega W^*)_{12} (\Omega V)_{23} (W \Omega^*)_1 = \Omega_{12} (\Delta \otimes \iota)(\Omega) V_{23} \Omega_{12}^* = \Omega_{23} (\iota \otimes \Delta)(\Omega) V_{23} \Omega_{12}^* = (\Omega V)_{23} ,$$

where we used the cocycle equation (6.1). Hence, we have proven Equation (6.2).

Let now $z \in (M, \Delta_{\Omega})_{(\alpha, \Omega^* \otimes 1)} \ltimes N \cap (\alpha(N))'$. Then, $z = JJ^*1$ for $a \in M$. Since $z$ satisfies Equation (6.2), we conclude that $a \otimes 1$ and $\Sigma W \Omega^* \Sigma$ commute. Hence, $\Delta_{\Omega}(a) = 1 \otimes a$. So, $a \in \mathbb{C}$ and we are done. \(\square\)

**7. The case of compact quantum $\Delta$ groups**

As another partial converse to Proposition 6.2, we show that every compact Kac algebra whose dual has an invariant mean, can act strictly outerly on the hyperfinite II$_1$ factor. In [15], Theorem 8.6, the same kind of result is stated, but we were unable to understand their complicated approach. Below we present a fairly easy construction.

At the end of this section, we show that it is highly improbable that there exists, for all $0 < q < 1$, a strictly outer action of the compact quantum group $SU_q(2)$ on an injective factor.
Recall that we still continue to assume that our quantum groups are second countable (i.e. all Hilbert spaces and preduals of von Neumann algebras are separable).

In our von Neumann algebraic setting, a l.c. quantum group \((M, \Delta)\) is compact if and only if its Haar measure is finite. We refer to [24] for a nicely written overview of the theory of compact quantum groups and their duals: discrete quantum groups. Suppose that \((M, \Delta)\) is a compact quantum group. There exists a unique left invariant state \(h \in M\), which is called the Haar state. The state \(h\) is right invariant as well (a compact quantum group is unimodular). Denote by \((\hat{M}, \hat{\Delta})\) the dual l.c. quantum group, constructed as in the preliminaries out of the multiplicative unitary \(W \in M \otimes \hat{M}\). We know that \(\hat{M} = \bigoplus_{n \in \mathbb{Z}} M_n\), where \(M_n\) are finite-dimensional full matrix algebras. Denote by \(e_n\) the minimal central projections of \(\hat{M}\). Then, \(U_n := W(1 \otimes e_n)\) provide exactly the irreducible corepresentations of \((M, \Delta)\). A compact quantum group is a compact Kac algebra if and only if the scaling group \((\tau_i)\) is trivial. Equivalently, the Woronowicz characters on the Hopf subalgebra \(\mathcal{A} \subset M\) (of matrix coefficients of finite dimensional corepresentations) are trivial.

The following lemma is a quantum version of Theorem 4.3 in [19]. The result only works in the non-Kac case. Even a normal \(q\)-trace \(\phi\) which satisfies the same conclusion does not exist in the non-Kac case.

Recall that we define the convolution product \(\omega * \mu := (\omega \otimes \mu)\hat{\Delta}\) for \(\omega, \mu \in \hat{M}_\ast\).

**Lemma 7.1.** Let \((\hat{M}, \hat{\Delta})\) be a discrete Kac algebra with an invariant mean. Then, there exists a normal tracial state \(\phi\) on \(\hat{M}\) such that
\[
\|\mu * \phi_n - \mu(1) \phi_n\| \to 0 \quad \text{for all} \quad \mu \in \hat{M}_\ast \quad \text{where} \quad \phi_n = \phi \ast \cdots \ast \phi \quad \text{\(n\) times}.
\]

**Proof.** As a first step, we prove that there exists a sequence \(\omega_n\) of normal tracial states on \(\hat{M}\) such that \(\|\mu * \omega_n - \mu(1) \omega_n\| \to 0\) for all \(\mu \in \hat{M}_\ast\). Let \(m_0\) be an invariant mean on \((\hat{M}, \hat{\Delta})\). Denote by \(W \in M \otimes \hat{M}\) the multiplicative unitary of \((M, \Delta)\). Define
\[
\Phi : \hat{M} \to M \otimes \hat{M} : \Phi(x) = W^\ast(1 \otimes x)W.
\]

Then, \(\Phi\) is an action of \((M, \Delta)\) on \(\hat{M}\) (the adjoint action). Let \(h\) be the Haar state on the compact quantum group \((M, \Delta)\) and define a (non-normal) state \(m\) on \(\hat{M}\) by the formula
\[
m(x) = m_0((h \otimes \iota)\Phi(x)) \quad \text{for} \quad x \in \hat{M}.
\]

We claim that \(m\) is a left invariant mean on \((\hat{M}, \hat{\Delta})\). Let \(x \in \hat{M}\) and \(\mu \in \hat{M}_\ast\). Then,
\[
m((\mu \otimes x)\hat{\Delta}^{-1}(x)^\ast) = m_0((h \otimes \iota \mu)(W_{12}^\ast W_{23}(1 \otimes x \otimes 1)W_{23}^\ast W_{12}^\ast))
\]
\[
= m_0((h \otimes \iota \mu)(W_{13}W_{23}(\Phi(x) \otimes 1)W_{23}^\ast W_{13}^\ast))
\]
\[
= m_0(((h \otimes \mu) \circ \text{Ad}W) \otimes \iota)(\iota \otimes \hat{\Delta})(\Phi(x))
\]
\[
= m_0(((h \otimes \mu) \circ \text{Ad}W) \otimes \iota)(\Phi(x)_{13}^\ast).
\]

It is easy to check that, for all \(z \in M\), \((h \otimes \iota)(W(z \otimes 1)W^\ast) = h(z)1\), see e.g. the proof of Corollary 3.9 in [17]. Here, we have used in a crucial way that we are in the Kac case. So, we can continue the computation above and get
\[
m((\mu \otimes x)\hat{\Delta}^{-1}(x)^\ast) = \mu(1) m_0((h \otimes \iota)\Phi(x)) = \mu(1) m(x).
\]

Hence, \(m\) is indeed a left invariant mean on \((\hat{M}, \hat{\Delta})\).

Take a sequence \(\eta_n\) of normal states on \(\hat{M}\) such that \(\eta_n(x) \to m_0(x)\) for all \(x \in \hat{M}\). Define \(\gamma_n := (h \otimes \eta_n)\Phi\). Then, \(\gamma_n\) is a sequence of normal states on \(\hat{M}\) such that \(\gamma_n(x) \to m(x)\) for all \(x \in \hat{M}\). By invariance of \(h\), we also have that, for all \(n\), \(\gamma_n\) is invariant under the action \(\Phi\):
\[
(\iota \otimes \gamma_n)^\Phi(x) = \gamma_n(x) \quad \text{for all} \quad x \in \hat{M}.
\]

We claim that a normal state \(\omega\) on \(M\) is a trace if and only if \(\omega\) is invariant under \(\Phi\). We know that \(\hat{M}\) is a direct sum of matrix algebras \(M_n\). It is also clear that \(\Phi\) leaves invariant all the
matrix algebras $\hat{M}_\alpha$, yielding actions $\Phi_\alpha$ of $(M, \Delta)$ on $\hat{M}_\alpha$. Every of these restrictions $\Phi_\alpha$ is ergodic (i.e. has a trivial fixed point algebra). Let $\tau_\alpha$ be the normalized trace on $\hat{M}_\alpha$. It is easy to verify that $\tau_\alpha$ is invariant under $\Phi_\alpha$ (see e.g. Lemma 2.1 in [17]). If now $\omega$ is a normal state on $\hat{M}$ which is invariant under $\Phi$, we get that $\omega = \sum \tau_\alpha(K_\alpha)$, for certain positive matrices $K_\alpha$, satisfying $\Phi_\alpha(K_\alpha) = 1 \otimes K_\alpha$. Hence, all the $K_\alpha$ are scalar and $\omega$ is a trace.

We conclude that $\gamma_n$ is a sequence of normal tracial states on $M$. Since $\gamma_n \to m$ pointwise and since $m$ is left invariant, a classical technique allows to find a sequence of normal states $\omega_n$ that are convex combinations of the $\gamma_n$ and that satisfy $\|\mu \ast \omega_n - \mu(1)\omega_n\| \to 0$ for all $\mu \in M$. Since a convex combination of traces is still a trace, the first step of the proof is done.

For the second step of the proof, we can follow almost literally the proof of Theorem 4.3 in [19], yielding a sequence $t_n$ of positive real numbers, satisfying $\sum t_n = 1$ and such that

$$\phi := \sum_n t_n \omega_n$$

is the normal tracial state that we are looking for. □

This allows us to prove the announced result.

**Theorem 7.2.** Let $(M, \Delta)$ be a compact Kac algebra. If the dual discrete Kac algebra $(\hat{M}, \hat{\Delta})$ has an invariant mean, then $(M, \Delta)$ can act strictly outerly on the hyperfinite II$_1$ factor.

**Proof.** Take a normal tracial state $\phi$ satisfying the conclusion of Lemma 7.1. Write $\phi = \sum \tau_\alpha$, where $\tau_\alpha$ is the normalized trace on $\hat{M}_\alpha$. Take the subset $I_0$ of $\alpha \in I$ satisfying $t_\alpha > 0$. Denote by $\mathcal{R}$ the hyperfinite II$_1$ factor and denote by $\tau$ its tracial state. Take a family of orthogonal projections $\{e_\alpha \in \mathcal{R} \mid \alpha \in I_0\}$ such that $\tau(e_\alpha) = t_\alpha$. Choosing isomorphisms $e_\alpha \mathcal{R} e_\alpha \cong M_\alpha \otimes \mathcal{R}$, we get embeddings $M_\alpha \to e_\alpha \mathcal{R} e_\alpha$ and hence, a normal *-homomorphism $\pi : M \to \mathcal{R}$. Define $U := (i \otimes \pi)(W) \in M \otimes \mathcal{R}$ and $\beta : \mathcal{R} \to M \otimes \mathcal{R} : \beta(x) = U^* (1 \otimes x)U$. From Proposition 6.5, we know that the trace $\tau$ is invariant under $\beta$. So, as explained before Proposition 6.4, we can make an infinite tensor product action $\alpha$ of $(M, \Delta)$ on $\mathcal{R} \cong \bigotimes_1^{\infty} \mathcal{R}$. In order to prove that $\alpha$ is strictly outer, we have to verify condition [17] in Proposition 6.4. Observe that, since $(M, \Delta)$ is compact, $M(\hat{A}_\alpha) = \hat{M}$. From Lemma 7.1, we know that $\|\phi_n - \phi_{n+1}\| \to 0$. Hence, as in the proof of the second item of Proposition 6.4, it suffices to check that an element $a \in M$ satisfying $(i \otimes \phi)\Delta(a) = a$ is scalar. But, such an element satisfies $\mu(a) = (\mu \ast \phi_n)(a)$ for all $n$ and hence, $|\mu(a) - \mu(1)\phi_n(a)| \to 0$ if $n \to \infty$. This means that $a - \phi_n(a)1 \to 0$ weakly and hence, $a$ is scalar. □

We now prove, combining results of Ueda [36] and a classical trick, that a strictly outer action of $SU_q(2)$ on an injective factor gives rise to an irreducible subfactor of the hyperfinite II$_1$ factor with index $(q + q^{-1})^2$.

We will make use of the extension of Jones’ index theory to factors which are not necessarily type II$_1$, see [20]. In this theory, an index $\text{Index} E$ is associated to an inclusion $N \subset M$ and a faithful normal conditional expectation $E : M \to N$. It coincides with the Jones index, if $N$ and $M$ are II$_1$ factors and $E$ is the unique conditional expectation satisfying $\tau_N \circ E = \tau_M$ (where $\tau_N$ and $\tau_M$ are the normalized traces on $N$ and $M$, resp.).

The following result is probably well known, but we include a proof for the convenience of the reader. It uses essentially a non-compact version of Wassermann’s invariance principle [43].

**Lemma 7.3.** Let $N \subset M$ be an inclusion of factors and $E : M \to N$ a faithful normal conditional expectation with $\text{Index} E = \lambda < \infty$. Let $N \subset M \subset M_1 \subset \cdots$ be the Jones tower with associated conditional expectations $E_n : M_n \to M_{n-1}$.

Then, there exists an inclusion $\hat{N} \subset \hat{M}$ of II$_1$ factors with $\hat{M} : \hat{N} = \lambda$ and whose Jones tower $\hat{N} \subset \hat{M} \subset \hat{M}_1 \subset \cdots$ satisfies

$$\hat{M}_i \cap \hat{M}_{j-1} = (M_i \cap M_{j-1})^{(\sigma_{E_{ij}})} \text{ where } E_{ij} = E_i \circ E_{i+1} \circ \cdots \circ E_i.$$
If $M$ is injective, $\tilde{N} \cong \tilde{M} \cong \mathcal{R}$, the hyperfinite $\text{II}_1$ factor.

**Proof.** Take a faithful normal state $\eta$ on $N$. Put $\eta_0 := \eta E$ and $\eta_i := \eta_{i-1} E_i$. Let $A$ be the injective factor of type $\text{III}_1$, with faithful normal state $\mu$. Write $\omega_i = \eta_i \otimes \mu$ on the $\text{II}_1$ factor $M_i \otimes \mathcal{A}$ and $\omega = \eta \otimes \mu$ on $N \otimes \mathcal{A}$. Consider the cores $C_i := (M_i \otimes \mathcal{A}) \rtimes_{(\sigma^i_t)} \mathbb{R}$ and $C := (N \otimes \mathcal{A}) \rtimes_{(\sigma^t)} \mathbb{R}$, which we realize as subalgebras of $M_i \otimes \mathcal{A} \otimes B(L^2(\mathbb{R}))$, respectively. Since $M_i \otimes \mathcal{A}$ is a $\text{II}_1$ factor, $C_i$ is a $\text{II}_\infty$ factor for all $i$.

The restriction of $E_i \otimes \iota \otimes \iota$ yields a conditional expectation $F_i : C_i \to C_{i-1}$ and $F : C_0 \to C$. By the characterization of the Jones tower (Theorem 8), $C \subset C_0 \subset C_1 \subset \cdots$ is a Jones tower with compatible conditional expectations $F, F_i$. In particular, $\text{Index} F = \lambda$. Denote by $\tilde{\omega}$ and $\tilde{\omega}_i$ the dual weights on $C$ and $C_i$, resp. It is clear that $\tilde{\omega}_i = \tilde{\omega}_{i-1} F_i$. If we denote by $\rho_i$ the right translation operators in $B(L^2(\mathbb{R}))$, we can consider $1 \otimes 1 \otimes \rho_i \in C_i$. We can define traces $\text{Tr}_i$ on the $\text{II}_\infty$ factors $C_i$ (and Tr on $C$) such that the Connes cocycles w.r.t. $\tilde{\omega}_i$ are given by

$$[\tilde{\omega}_i : \text{Tr}_i]_t = 1 \otimes 1 \otimes \rho_i.$$

It follows that $\text{Tr}_1 = \text{Tr}_{i-1} F_i$. Let $p$ be a projection in $C$ satisfying $\text{Tr}(p) = 1$. Put $\tilde{N} = pCp$, $\tilde{M} = pC_0p$ and $\tilde{M}_i = pC_ip$. It is clear that we get conditional expectations $\tilde{E}_i : \tilde{M}_i \to \tilde{M}_{i-1}$ and $\tilde{E} : M \to \tilde{N}$ such that the towers

$$C \subset C_0 \subset C_1 \subset \cdots \text{ and } \tilde{N} \otimes B(\ell^2(\mathbb{Z})) \subset M \otimes B(\ell^2(\mathbb{Z})) \subset \tilde{M}_1 \otimes B(\ell^2(\mathbb{Z})) \subset \cdots$$

are isomorphic in a way that preserves the conditional expectations. Since $\text{Tr}_i(p) = 1$, it follows that $\tilde{N} \subset \tilde{M}$ is an inclusion of $\text{II}_1$ factors. It is clear that $\text{Index} \tilde{E} = \lambda$. Because $\text{Tr} \circ F = \text{Tr}_0$ and because the restriction of $\text{Tr}$ to $\tilde{N}$ and of $\text{Tr}_0$ to $\tilde{M}$ are the unique tracial states of $\tilde{N}$ and $\tilde{M}$, we get $[\tilde{M} : \tilde{N}] = \lambda$.

Next, we observe that

$$\tilde{M}_i \cap \tilde{M}_{j-1}' \cong (M_i \otimes \mathcal{A}) \rtimes_{(\sigma^i_t)} \mathbb{R} \cap ((M_{j-1} \otimes \mathcal{A}) \rtimes_{(\sigma^t)} \mathbb{R})' = C_i \cap C_{j-1}' .$$

We compute

$$C_i \cap C_{j-1}' \subset M_i \otimes \mathcal{A} \otimes B(L^2(\mathbb{R})) \cap (1 \otimes \sigma^t(A))' = M_i \otimes 1 \otimes L^\infty(\mathbb{R})$$

because $A$ is a $\text{III}_1$ factor. Hence, we get

$$C_i \cap C_{j-1}' \subset \sigma^t(M_i)_{13} \cap \sigma^{t-1}(M_{j-1})_{13}' \cap (1 \otimes \rho(\mathbb{R}))' = (M_i \cap M_{j-1}')_{(\sigma^t_{ij})} \otimes 1 \otimes 1 .$$

Since the converse inclusion $(M_i \cap M_{j-1}')_{(\sigma^t_{i})} \otimes 1 \otimes 1 \subset C_i \cap C_{j-1}'$ is clear, we have proven the formula for $\tilde{M}_i \cap \tilde{M}_{j-1}'$.

To conclude the proof, we see that if $M$ is injective, $\tilde{M}$ and $\tilde{N}$ are injective factors of type $\text{II}_1$. Hence, $\tilde{N} \cong \tilde{M} \cong \mathcal{R}$ in that case.

Combining the previous lemma with results of Ueda [18], we get the following proposition. Let $(M, \Delta)$ be a compact quantum group and $u \in M \otimes M_n(\mathbb{C})$ an irreducible corepresentation on $\mathbb{C}^n$. Associated with such a irreducible corepresentation is a positive invertible $F$-matrix $F_u \in M_n(\mathbb{C})$ satisfying $\text{Tr} F_u = \text{Tr} F_u^{-1}$ (see [14]). The positive real number $\text{Tr} F_u$ is denoted by $\dim u$ and called the quantum dimension of $u$.

**Proposition 7.4.** Let $\alpha : N \to M \otimes N$ be a strictly outer action of a compact quantum group $(M, \Delta)$ on an injective factor $N$. Then, there exist irreducible subfactors of the hyperfinite $\text{II}_1$ factor with index $(\dim u)^2$ for any irreducible corepresentation $u$ of $(M, \Delta)$.

**Proof.** Let $\alpha$ be such an action and $u \in M \otimes M_n(\mathbb{C})$ an irreducible corepresentation. Recall that for compact quantum groups the notions of strictly outer action and minimal action coincide. We can define a new action

$$\gamma : N \otimes M_n(\mathbb{C}) \to M \otimes N \otimes M_n(\mathbb{C}) : \gamma(z) = u_{13} (\alpha \otimes \iota)(z) u_{13} .$$
Consider the inclusion $N^\alpha \otimes 1 \subset (N \otimes M_n(C))^\gamma$. Using the restriction $E_u$ of $\frac{1}{\dim_q u} (\epsilon \otimes \text{Tr}(F_u))$, Ueda proved in [36] that $N^\alpha \otimes 1 \subset (N \otimes M_n(C))^\gamma$ is an irreducible inclusion of factors with Index $E_u = (\dim_q u)^2$. Using Lemma 7.3, we get the existence of an irreducible subfactor of the hyperfinite $\mathbb{II}_1$ factor with index $\dim_q u^2$. □

Remark 7.5. So, if the compact quantum group $SU_q(2)$ would have a strictly outer action on an injective factor for all $0 < q < 1$, we can use its fundamental corepresentation $u$ with $\dim_q u = q + q^{-1}$ and conclude that the hyperfinite $\mathbb{II}_1$ factor would have irreducible subfactors of arbitrary index strictly greater than 4. Since most important unpublished work of Popa states that not all values strictly greater than 4 can be realized as the index of an irreducible subfactor of the hyperfinite $\mathbb{II}_1$ factor, it follows that at least for certain values of $0 < q < 1$, $SU_q(2)$ cannot act strictly outerly on an injective factor. Remark that nevertheless, Banica showed [5] that the dual of $SU_q(2)$ has an invariant mean. This would mean that the converse of Proposition 6.2 does not hold in its full generality.

Remark 7.6. As we explained in the previous remark, there is a strong reason to believe that $SU_q(2)$ cannot act strictly outerly on an injective factor for certain values of $q$. Recently, Szymański [33] has shown that the compact quantum groups $SU_q(2)$ for different values of $q$ are related by a pseudo-2-cocycle. If the pseudo-2-cocycle of Szymański happens to be a 2-cocycle, we can use Proposition 6.6 and can conclude that none of the compact quantum groups $SU_q(2)$, $0 < q < 1$, can act strictly outerly on an injective factor.

Remark 7.7. In the proof of Proposition 7.4, we constructed, following Ueda [36], an irreducible inclusion of factors given a strictly outerly acting compact quantum group $(M, \Delta)$ and an irreducible corepresentation. Ueda computed the Jones tower of this inclusion. The tower of relative commutants only depends on the corepresentation theory of the compact quantum group. The vertices of the principal graph are labeled by the irreducible corepresentations of $(M, \Delta)$ that are subrepresentations of some tensor product $u \otimes u \otimes \cdots$. In particular, whenever the matrix coefficients of $u$ generate a $C^*$-subalgebra of $M$ of infinite dimension, we get an inclusion of infinite depth. This is, of course, almost always the case.

8. The case of discrete quantum groups

Since the dual of a discrete quantum group is compact and since a compact quantum group has an invariant mean (the Haar state), Proposition 6.2 does not exclude the possibility that every discrete quantum group can act strictly outerly on an injective factor.

However, if a discrete quantum group $(\hat{M}, \hat{\Delta})$ with invariant mean acts strictly outerly on an injective factor, the crossed product will be injective as well and the dual action will be a strictly outer action of the compact quantum group $(M, \Delta)$. So, from Remark 7.5, we conclude that we should not expect to find a strictly outer action of the dual of $SU_q(2)$ on an injective factor.

We first prove that a discrete Kac algebra with invariant mean acts strictly outerly on the hyperfinite $\mathbb{II}_1$ factor $\mathcal{R}$. Taking the crossed product and the dual action, we get an alternative proof for Theorem 7.2.

Next, we prove more generally that every discrete Kac algebra with a faithful corepresentation in $\mathcal{R}$ (Definition 2.3), acts strictly outerly on $\mathcal{R}$. This generalizes Banica’s result [11, Section 4] for discrete Kac algebras with a faithful finite-dimensional corepresentation. Note that the discrete quantum groups constructed from vertex models have, by construction, such a faithful finite-dimensional corepresentation.

Recall once again that we assume that our l.c. quantum groups are second countable.

The following is an easy application of Propositions 6.4 and 6.5.

Proposition 8.1. Let $(\hat{M}, \hat{\Delta})$ be a discrete Kac algebra with invariant mean. Then, there exists a strictly outer action of $(\hat{M}, \hat{\Delta})$ on the hyperfinite $\mathbb{II}_1$ factor.
Proof. Let $h$ denote the Haar state on the compact Kac algebra $(M, \Delta)$. Then, $h$ is a faithful tracial state. Moreover, since $(\hat{M}, \hat{\Delta})$ has an invariant mean, it is easy to check that $M$ is an injective von Neumann algebra (see e.g. [28]). Hence, there exists a faithful $^*$-homomorphism $\rho : M \to \mathcal{R}$ such that $\tau \rho = h$, where $\tau$ is the tracial state on $\mathcal{R}$ (this essentially follows from the uniqueness of $\mathcal{R}$, see Corollary 1.23 in [34]).

Consider the multiplicative unitary $\hat{W} = \Sigma W^* \Sigma \in \hat{M} \otimes M$, where $\Sigma$ denotes the flip map. Define $U := (\iota \otimes \rho)(\hat{W})$. We claim that the conditions of Proposition 6.4 are fulfilled. Condition a) is fulfilled because of Proposition 6.3. On the other hand, because $(\hat{\Delta})(\rho(a)) = (a_n, a_n)$, we get that $a_n = h(a_{n+1})1$, because $\tau \rho = h$. So, every $a_n$ is scalar and then, $(a_n)$ is a constant scalar sequence.

The non-trivial point in the proof of the previous proposition is the existence of a faithful, normal $^*$-homomorphism $\rho : M \to \mathcal{R}$. The existence of $\rho$ implies the injectivity of $M$ and this, in turn, implies the existence of an invariant mean on $(\hat{M}, \hat{\Delta})$ (see e.g. [28]). So, we cannot follow the same strategy in the non-amenable case.

We shall consider discrete Kac algebras that have a faithful corepresentation in the hyperfinite $\text{II}_1$ factor $\mathcal{R}$. In the classical case, this corresponds to discrete groups that are subgroups of the unitary group of $\mathcal{R}$. In particular, all residually finite groups belong to this class and they may very well be non-amenable.

**Theorem 8.2.** If a discrete Kac algebra $(\hat{M}, \hat{\Delta})$ has a faithful corepresentation $U \in \hat{M} \otimes \mathcal{R}$ in the hyperfinite $\text{II}_1$ factor $\mathcal{R}$, then $(\hat{M}, \hat{\Delta})$ acts strictly outerly on $\mathcal{R}$.

Proof. Take the $^*$-homomorphism $\rho : A_u \to \mathcal{R}$ such that $(\iota \otimes \rho)(\hat{W}) = U$, where $\hat{W} \in M(\hat{A} \otimes A_u)$ is the universal corepresentation of $(\hat{M}, \hat{\Delta})$. Taking the direct sum with the trivial corepresentation, we may assume that there exists a projection $e_0 \in \mathcal{R} \cap \rho(A_u)'$ satisfying $0 < \tau(e_0) < 1$ and $\rho(a)e_0 = \varepsilon(a)e_0$ for all $a \in A_u$, where $\varepsilon$ denotes the co-unit of $(A_u, \Delta_u)$. From the faithfulness of $U$, we know that

$$^*\text{-alg}\{\iota \otimes \mu(U) \mid \mu \in \mathcal{R}_s\} \text{ is weakly dense in } \hat{M}. \quad (8.1)$$

Taking the tensor product of $U$ and its adjoint corepresentation, we may assume that

$$\text{alg}\{\iota \otimes \mu(U) \mid \mu \in \mathcal{R}_s\} \text{ is weakly dense in } \hat{M}. \quad (8.1)$$

From Proposition 6.3 we know that $\tau$ is invariant under the inner action $\beta : \mathcal{R} \to \hat{M} \otimes \mathcal{R} : \beta(z) = U^*(1 \otimes z)U$. So, we can define the infinite tensor product action $\alpha$. We claim that $\alpha$ is strictly outer.

Write $\omega = \tau \rho$. If we define the state $\omega_1$ on $A_u$ such that $(1 - \tau(e_0))\omega_1(a) = \tau(\rho(a)(1 - e_0))$ and if we put $t = \tau(e_0)$, we observe that $\omega = (1 - t)\omega_1 + t\varepsilon$. We apply Propositions 6.4 and 6.5. So, in order to prove our claim, it suffices to check that the equation

$$(\iota \otimes \omega)\Delta(a) = a, \quad a \in M(A_u)$$

has only scalar solutions.

Let $\Psi$ be a state on $A_u$ that is an accumulation point in the weak$^*$ topology of the sequence $(\Psi_n)$, where

$$\Psi_n := \frac{1}{n} \sum_{k=1}^{n} \omega^{*k}.$$ 

We will prove that $\Psi$ is the Haar state of $(A_u, \Delta_u)$. It is clear that $\omega \ast \Psi = \Psi$. Define, for all $x \in \mathcal{R}$, an element $\mu_x \in A_u^*$ as $\mu_x(a) = \tau(x^* \rho(a)x)$. Because $\mu_x \leq ||x||^2 \omega$, it follows from Lemma 4.3 in [24] that $\mu_x \ast \Psi = \mu_x(1)\Psi$. So, we conclude that $(\mu \rho) \ast \Psi = \mu(1)\Psi$ for all $\mu \in \mathcal{R}_s$. Applying this to the second leg of $\hat{W}$, we get

$$(\iota \otimes \Psi)(\hat{W})(\iota \otimes \mu \rho)(\hat{W}) = \mu(1)(\iota \otimes \Psi)(\hat{W})$$ 

Hence, we know that $\omega$ is the Haar state of $(A_u, \Delta_u)$.
for all $\mu \in \mathbb{R}_+$. Denote by $\hat{\varepsilon} \in \hat{M}$, the co-unit of the discrete Kac algebra $(\hat{M}, \hat{\Delta})$. Using Equation (8.1), we conclude that $(\iota \otimes \Psi)(\hat{W})\alpha = \hat{\varepsilon}(\alpha)(\iota \otimes \Psi)(\hat{W})$ for all $\alpha \in \hat{M}$. From this it follows that $(\iota \otimes \Psi)(\hat{W})$ is the central projection in $\hat{M}$ projecting on the trivial representation. Hence, $\Psi$ is the Haar state.

If $a \in M(A_u)$ and $(\iota \otimes \omega)\Delta(a) = a$, then $(\iota \otimes \Psi_n)\Delta(a) = a$ for all $n$. This implies that $(\iota \otimes \Psi)\Delta(a) = a$. Since $\Psi$ is the Haar state of $(A_u, \Delta_u)$, we get $a = \Psi(a)1$ and we are done. \qed

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