REFINED CYCLIC SIEVING ON WORDS
FOR THE MAJOR INDEX STATISTIC

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Abstract. Reiner-Stanton-White [RSW04] defined the cyclic sieving phenomenon (CSP) associated to a finite cyclic group action and a polynomial. A key example arises from the length generating function for minimal length coset representatives of a parabolic quotient of a finite Coxeter group. In type $A$, this result can be phrased in terms of the natural cyclic action on words of fixed content.

There is a natural notion of refinement for many CSP’s. We formulate and prove a refinement, with respect to the major index statistic, of this CSP on words of fixed content by also fixing the cyclic descent type. The argument presented is completely different from Reiner-Stanton-White’s representation-theoretic approach. It is combinatorial and largely, though not entirely, bijective in a sense we make precise with a “universal” sieving statistic on words, flex.

A building block of our argument involves cyclic sieving for shifted subset sums, which also appeared in Reiner-Stanton-White. We give an alternate, largely bijective proof of a refinement of this result by extending some ideas of Wagon-Wilf [WW94].

1. Introduction

Since Reiner, Stanton, and White introduced the cyclic sieving phenomenon (CSP) in 2004 [RSW04], the cyclic sieving phenomenon has become an important companion to any cyclic action on a finite set. Some remarkable examples of the CSP involve the action of a Springer regular element on Coxeter groups [RSW04, Theorem 1.6], the action of Schützenberger’s promotion on Young tableaux of fixed rectangular shape [Rho10], and the creation of new CSPs from old using multisets and plethysms with homogeneous symmetric functions [BER11, Proposition 8]. See [Sag11] for Sagan’s thorough introduction to the cyclic sieving phenomenon. More recent work on the CSP includes, for instance, [ARR15, Pec14, PSV16]. An earlier “extended abstract” for the present work appeared in [AS17]. We assume some familiarity with the CSP, though we recall certain key statements.

Definition 1.1. Suppose $C_n$ is a cyclic group of order $n$ generated by $\sigma_n$, $W$ is a finite set on which $C_n$ acts, and $f(q) \in \mathbb{N}[q]$. We say the triple $(W,C_n,f(q))$ exhibits the cyclic sieving phenomenon (CSP) if for all $k \in \mathbb{Z}$,

\[
\#W^{\sigma_n^k} := \# \{ w \in W : \sigma_n^k \cdot w = w \} = f(\omega_n^k),
\]

where $\omega_n$ is any fixed primitive $n$-th root of unity.

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Representation theoretically, evaluations of $f$ at $n$-th roots of unity yield the characters of the $C_n$-action on $W$.

In many instances of cyclic sieving, and all of those considered here, $f(q)$ is the generating function for some statistic on $W$. Given a statistic $\text{stat}: W \to \mathbb{N}$, let

$$W^{\text{stat}}(q) := \sum_{w \in W} q^{\text{stat} w} \in \mathbb{N}[q].$$

(2)

We say two statistics $\text{stat}, \text{stat}': W \to \mathbb{N}$ are equidistributed on $W$ if $W^{\text{stat}}(q) = W^{\text{stat}'}(q)$, and we say they are equidistributed modulo $n$ on $W$ if $W^{\text{stat}}(q) \equiv W^{\text{stat}'}(q) \pmod{q^n - 1}$.

Our main result is a refinement of a CSP triple first observed by Reiner-Stanton-White, which we now summarize; see Section 2 for missing definitions. Consider words in the alphabet $\mathcal{P} := \{1, 2, \ldots\}$. Given a word $w = w_1 \cdots w_n$ of length $n$, let $\text{cont}(w)$ denote the content of $w$ and write

$$W_\alpha := \{\text{words } w : \text{cont}(w) = \alpha\}$$

for the set of words with content $\alpha$. Write $\text{maj}(w)$ for the major index of $w$. The cyclic group $C_n$ acts on words of length $n$ by rotation. The following expresses an interesting result of Reiner, Stanton, and White in our notation.

**Theorem 1.2.** [RSW04, Proposition 4.4]. Let $\alpha \vdash n$. The triple $(W_\alpha, C_n, W_\alpha^{\text{maj}}(q))$ exhibits the CSP.

Reiner, Stanton, and White deduced Theorem 1.2 from the following more general result about Coxeter systems.

**Theorem 1.3.** [RSW04, Theorem 1.6]. Let $(W, S)$ be a finite Coxeter system and $J \subseteq S$. Let $W_J$ be the corresponding parabolic subgroup, $W^J$ the set of minimal length representatives for left cosets $X := W/W_J$, and $X^\ell(q) := \sum_{w \in W^J} q^{\ell(w)}$. Let $C$ be a cyclic subgroup of $W$ generated by a Springer regular element. Then $(X, C, X^\ell(q))$ exhibits the cyclic sieving phenomenon.

Theorem 1.2 follows from Theorem 1.3 when $W = S_n$ by identifying $W/W_J$ with words of fixed content $\alpha$, where $\alpha$ is the composition recording the lengths of consecutive subsequences of $J$, and $C$ is generated by an $n$-cycle. One must also use the classical result of MacMahon that $\text{maj}$ is equidistributed with the inversion statistic on words, from which it follows that $W_\alpha^{\text{maj}}(q) = X^\ell(q)$ [Mac, §1].

**Definition 1.4.** A refinement of a CSP triple $(W, C_n, W^{\text{stat}}(q))$ is a CSP triple $(V, C_n, V^{\text{stat}}(q))$ where $V \subset W$ has the restricted $C_n$-action.

If $(V, C_n, V^{\text{stat}}(q))$ refines $(W, C_n, W^{\text{stat}}(q))$, then so does $(U, C_n, U^{\text{stat}}(q))$ where $U := W - V$. Thus, a CSP refinement partitions $W$ into smaller CSPs with the same statistic. If $W$ is an orbit, its only refinements are $W$ and $\emptyset$. In Section 8, we define a statistic on words, $\text{flex}$, which is universal in the sense that it refines to all $C_n$-orbits. Such universal statistics are essentially equivalent to the choice of a total ordering for each orbit $O$ of $W$. 
We partition words of fixed content into fixed cyclic descent type (CDT). One computes CDT($w$) by building up $w$ by adding all 1’s, 2’s, ..., and counting the number of cyclic descents introduced at each step. For precise details, see Definition 4.1 and Example 4.2. We write the set of words with fixed content and cyclic descent type as

$$W_{\alpha, \delta} := \{ \text{words } w : \text{cont}(w) = \alpha, \text{CDT}(w) = \delta \}. \quad (4)$$

Our main result is the following.

**Theorem 1.5.** Let $\alpha \models n$ and $\delta$ be any composition. The triple

$$(W_{\alpha, \delta}, C_n, W_{\alpha, \delta}^{\text{maj}}(q))$$

refines the CSP triple $(W_{\alpha}, C_n, W_{\alpha}^{\text{maj}}(q))$.

Indeed, we derive an explicit product formula for $W_{\alpha, \delta}^{\text{maj}}(q)$ mod $(q^n - 1)$ involving $q$-binomial coefficients, Theorem 5.19. The formula results in a $q$-identity similar to the Vandermonde convolution identity; see Corollary 5.20. The argument involves constructing $W_{\alpha, \delta}$ algorithmically by recursively building a certain tree.

The two-letter case of Theorem 1.5 can be rephrased as follows. Fix $n \in \mathbb{Z}_{\geq 1}$ and $k, b \in \mathbb{Z}_{\geq 0}$. Let $S_{k,b}$ denote the set of subsets $\Delta$ of $\mathbb{Z}/n$ of size $k$ where $\# \{ i \in \Delta : i + 1 \not\in \Delta \} = b$. Define the statistic $\text{mbs}: S_{k,b} \to \mathbb{N}$ by identifying $\mathbb{Z}/n$ with $\{1, \ldots, n\}$ and setting $\text{mbs}(\Delta) := \sum_{i \in \Delta : i + 1 \not\in \Delta} i$, which sums the maximum of the cyclic blocks of $\Delta$.

**Corollary 1.6.** The triple

$$(S_{k,b}, C_n, S_{k,b}^{\text{mbs}}(q))$$

exhibits the CSP.

**Example 1.7.** When $n = 5, k = 3, b = 2$, $S_{3,2} = \{ \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 1\}, \{4, 5, 2\}, \{5, 1, 3\} \}$, which have mbs statistic 6, 8, 5, 7, 4, respectively, so $S_{3,2}^{\text{mbs}}(q) = q^4 + q^5 + q^6 + q^7 + q^8$. We then have $S_{3,2}^{\text{mbs}}(\omega_5) = 0, S_{3,2}^{\text{mbs}}(1) = 5$, in agreement with (1).

Theorem 8.3 in [RSW04] and hence Theorem 1.2 builds on a representation-theoretic result due to Springer [Spr74, Proposition 4.5]. Our argument is highly combinatorial, but it is not entirely bijective. Finding an explicit bijection would be quite interesting. See Section 8 for more details.

A key building block of our proof of Theorem 1.5 involves cyclic sieving on multisubsets and subsets, which was also first stated in [RSW04]. We describe refinements of these results as well, Theorem 7.4 and Theorem 7.11, restricting to certain gcd requirements in the subset case. We present a completely different inductive proof of our subset refinement in the spirit of our proof of Theorem 1.5. Both our proof of Theorem 7.11 and Theorem 1.5 use an extension lemma, Lemma 3.3, which allows us to extend CSPs from smaller cyclic groups to larger ones.

The rest of the paper is organized as follows. In Section 2, we recall combinatorial background. In Section 3, we introduce the concept of modular periodicity and
prove our extension lemma, Lemma 3.3. In Section 4, we define cyclic descent type. In Section 5, we decompose words with fixed content and cyclic descent type and prove a product formula for \( W_{\alpha, \delta}(q) \mod q^n - 1 \), Theorem 5.19. Section 6 uses the results of Section 5 to prove our main result, Theorem 1.5. Section 7 refines cyclic sieving on multisubsets and subsets with respect to shifted sum statistics. In Section 8, we introduce the flex statistic and use it to reinterpret Theorem 1.5.

2. Combinatorial Background

In this section, we briefly recall or introduce combinatorial notions on words and fix our notation. We use the alphabet of positive integers \( \mathbb{P} := \{1, 2, \ldots\} \) throughout unless otherwise noted. We also write \( #S \) or \( |S| \) for the cardinality of a set \( S \). For \( \text{stat}: W \to \mathbb{N} \), recall the notation

\[
W^{\text{stat}}(q) := \sum_{w \in W} q^{\text{stat}(w)}.
\]

A word \( w \) of length \( n \) is a sequence \( w = w_1w_2 \cdots w_n \) of letters \( w_i \in \mathbb{P} \). Let \( |w| \) denote the length of a word \( w \). Let \( W_n \) denote the set of all words of length \( n \). The descent set of \( w \) is \( \text{Des}(w) := \{1 \leq i < n : w_i > w_{i+1}\} \), and the number of descents is \( \text{des}(w) := \# \text{Des}(w) \). The major index of \( w \) is \( \text{maj}(w) := \sum_{i \in \text{Des}(w)} i \).

The cyclic descent set of \( w \) is \( \text{CDes}(w) := \{1 \leq i \leq n : w_i > w_{i+1}\} \), where now the subscripts are taken mod \( n \), and we write \( \text{cdes}(w) := \# \text{CDes}(w) \) for the number of cyclic descents. Any position \( 1 \leq i \leq n \) that is not a cyclic descent is a cyclic weak ascent. Cyclic descents were introduced by Cellini in an algebraic context; see [Cel98]. Since then, cyclic descents have been used by Lam and Postnikov in studying alcoved polytopes [LP12] and by Petersen in studying \( P \)-partitions [Pet05].

The inversion number of \( w \) is \( \text{inv}(w) := \# \{(i, j) : 1 \leq i < j \leq n \text{ and } w_i > w_j\} \). We use lower dots between letters to indicate cyclic descents and upper dots to indicate cyclic weak ascents throughout the paper as in the following example.

Example 2.1. If \( w = 155.3.155.3. = 15531553 \), then \( |w| = 8 \), \( \text{Des}(w) = \{3, 4, 7\} \), \( \text{des}(w) = 3 \), \( \text{CDes}(w) = \{3, 4, 7, 8\} \), \( \text{cdes}(w) = 4 \), \( \text{maj}(w) = 14 \), and \( \text{inv}(w) = 9 \).

A composition or weak composition of \( n \) is a sequence \( \alpha = (\alpha_1, \ldots, \alpha_m) \) of non-negative integers summing to \( n \), typically denoted \( \alpha \vdash n \). A composition is strong if \( \alpha_i > 0 \) for all \( i \). The content of a word \( w \), denoted \( \text{cont}(w) \), is the sequence \( \alpha \) whose \( j \)-th part is the number of \( j \)'s in \( w \). For \( w \in W_n \), \( \text{cont}(w) \) is a weak composition of \( n \). We write

\[
W_\alpha := \{ w \in W_n : \text{cont}(w) = \alpha \}.
\]

The cyclic group \( C_n := \langle \sigma_n \rangle \) of order \( n \) acts on \( W_n \) by rotation as

\[
\sigma_n \cdot w_1 \cdots w_{n-1} w_n := w_n w_1 \cdots w_{n-1}.
\]

Typically we consider \( \sigma_n \) to be the long cycle \( (12 \cdots n) \in S_n \).

The set of all words in \( \mathbb{P} \) is a monoid under concatenation. A word is primitive if it is not a power of a smaller word. Any non-empty word \( w \) may be written uniquely as \( w = v^f \) for \( f \geq 1 \) with \( v \) primitive. We call \( |v| \) the period of \( w \), written \( \text{period}(w) \), and
f the frequency of w, written \text{freq}(w). An orbit of \( W_n \) under rotation is a necklace, usually denoted \([w]\). We have period\((w) = \#[w] \) and freq\((w) \cdot \text{period}(w) = |w|\).

Content, primitivity, period, frequency, and cdes are all constant on necklaces.

**Example 2.2.** The necklace of \( w = 15531553 = (1553)^2 \) is

\[ [w] := \{15531553, 55315531, 53155315, 31553155\} \subset W_{(2,0,2,0,4)} \subset W_8, \]

which has period 4, frequency 2, and cdes 4.

Reiner-Stanton-White gave several equivalent conditions for a triple \((W,C_n,f(q))\) to exhibit the CSP. In place of (1) in Definition 1.1, we may instead require

\[ f(q) \equiv \sum_{\text{orbits } O \subset W} \frac{q^n - 1}{q^n/|O| - 1} \pmod{q^n - 1}, \]

where the sum is over all orbits \( O \) under the action of \( C_n \) on \( W \). Note that for \( d \mid n \),

\[ \frac{q^n - 1}{q^d - 1} = \sum_{i=0}^{n/d-1} q^{di} \not\equiv 0 \pmod{q^n - 1}. \]

This means every \( C_n \)-action on a finite set \( W \) gives rise to a CSP \((W,C_n,f(q))\), where \( f(q) \) is the right hand side of (5). We refer the interested reader to [RSW04, Proposition 2.1] for the proof of the equivalence of (1) and (5).

**Remark 2.3.** If \((V,C_n,f(q))\) exhibits the CSP, then so do both \((V,C_g,f(q))\) and \((V,C_n,f(q^{-1}))\) when \( g \mid n \) by (1). In the latter case we have relaxed the constraint \( f(q) \in \mathbb{N}[q] \) to \( f(q) \in \mathbb{N}[q,q^{-1}] \), which does no harm since (1) involves evaluations at roots of unity. Further, if \((V,C_n,f(q))\) and \((W,C_n,h(q))\) exhibit the CSP, then \((V \coprod W,C_n,f(q) + g(q))\) and \((V \times W,C_n,f(q)h(q))\) exhibit the CSP, where \( C_n \) acts on \( V \times W \) by \( \tau \cdot (v,w) := (\tau \cdot v, \tau \cdot w) \) [BER11, Prop. 2.2].

For a set \( S \), write

\[ \binom{S}{k} := \{ \text{all } k \text{-element subsets of } S \}, \]

\[ \binom{S}{k}^q := \{ \text{all } k \text{-element multisubsets of } S \}. \]

Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \vdash n \). We use the following standard \( q \)-analogues:

\[ [n]_q := 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}, \]

\[ [n]_q! := [n]_q[n-1]_q \cdots [1]_q, \]

\[ \binom{n}{\alpha}_q := \frac{[n]_q!}{[\alpha_1]_q! \cdots [\alpha_m]_q!} \in \mathbb{N}[q], \]

\[ \binom{n}{k}_q := \binom{n + k - 1}{k}_q := \binom{n + k - 1}{k,n - 1}_q. \]
We write \([a, b] := \{i \in \mathbb{Z} : a \leq i \leq b\}\). Observe that the cyclic group \(C_n = \langle \sigma_n \rangle\) of order \(n\) acts on \([0, n-1]\) by \(\sigma_n(i) := i + 1 \pmod{n}\). This induces actions of \(C_n\) on \(\binom{[0, n-1]}{k}\) and \(\binom{[0, n-1]}{n}\) by acting on values in each subset or multiset. For example, \(\sigma_4 \cdot \{0, 0, 0, 2, 2, 3\} = \{0, 1, 1, 1, 3, 3\}\). These actions, in slightly more generality, appear in one of the original, foundational CSP results as follows.

**Theorem 2.4.** [RSW04, Thm. 1.1]. In the notation above, the triples \((\binom{[0, n-1]}{k}, C_n, \binom{n}{k})\) and \((\binom{[0, n-1]}{n}, C_n, \binom{n}{k})\) exhibit the CSP.

We will also have use of the following principal specializations (see [Mac95, Example I.2.2] or [Sta99, Proposition 7.8.3]):

\[
\begin{align*}
\binom{[0, n-1]}{k} \text{sum} (q) &= e_k(1, q, q^2, \ldots, q^{n-1}) = q^{\binom{k}{2}} \binom{n}{k}, \\
\binom{[0, n-1]}{k} \text{sum} (q) &= h_k(1, q, q^2, \ldots, q^{n-1}) = \binom{n}{k}. 
\end{align*}
\]

(8) (9)

Here the sum statistic denotes the sum of the elements of a subset or submultiset of \(\mathbb{Z}\).

Recall that the length function \(\ell\) on \(S_n\) coincides with the inversion statistic defined above on words of content \((1, 1, \ldots, 1)\). More generally, minimal length coset representatives of parabolic quotients \(S_n/S_J\) also have length given by the inversion statistic on the corresponding words \(W_\alpha\). The following classical result is due to MacMahon.

**Theorem 2.5.** [Mac, §1]. For each \(\alpha \vDash n\), \(maj\) and \(inv\) are equidistributed on \(W_\alpha\) with

\[
W_\alpha^{maj}(q) = \binom{n}{\alpha}_q = W_\alpha^{inv}(q).
\]

(10)

Despite (10), \(maj\) and \(inv\) are not equidistributed even modulo \(n\) on \(W_{\alpha, \delta}\) in general, so \((W_{\alpha, \delta}, C_n, W_{\alpha, \delta}^{maj}(q))\) does not generally exhibit the CSP. As an explicit example, once we have defined the cyclic descent type, an easy computation will show that \(W_{(2,2),(0,2)} = \{1212, 2121\}\). The corresponding major index generating function is \(q^2 + q^4\), while the inversion generating function is \(q^1 + q^3\), which are not even congruent modulo \(q^4 - 1\).

3. **Modular Periodicity and an Extension Lemma**

We now introduce the concept of modular periodicity and use it to give an extension lemma, Lemma 3.3, which allows us to extend CSP’s from certain subgroups to larger groups. We will verify the hypotheses of Lemma 3.3 in the subsequent sections to deduce Theorem 1.5.
Definition 3.1. We say a statistic $\text{stat}: W \rightarrow \mathbb{Z}$ has period $a$ modulo $b$ on $W$ if for all $i \in \mathbb{Z}$,

$$\# \{w \in W : \text{stat}(w) \equiv_b i \} = \# \{w \in W : \text{stat}(w) \equiv_b i + a \}.$$ 

Similarly, we say a Laurent polynomial $f(q) \in \mathbb{C}[q,q^{-1}]$ has period $a$ modulo $b$ if

$$q^a \cdot f(q) \equiv f(q) \pmod{q^b - 1},$$

or equivalently if $(q^b - 1) | (q^a - 1) f(q)$.

For example, $1 + 5q + q^2 + 5q^3 + q^4 + 5q^5$ has period 2 modulo 6. Note that stat has period $a$ modulo $b$ on $W$ if and only if $W^{\text{stat}(q)}$ has period $a$ modulo $b$. The following basic properties of modular periodicity will be useful throughout the paper.

Lemma 3.2. Let $f(q) \in \mathbb{C}[q,q^{-1}]$ and $a, b, c \in \mathbb{Z}$.

(i) If $f(q)$ has period $a$ modulo $c$ and period $b$ modulo $c$, then $f(q)$ has period $ua + vb$ modulo $c$ for any $u, v \in \mathbb{Z}$. In particular, $f(q)$ has period $\gcd(a, b)$ modulo $c$.

(ii) If $f(q)$ has period $a$ modulo $b$ and period $b$ modulo $c$, then $f(q)$ has period $a$ modulo $c$.

(iii) If $f(q)$ has period $a$ modulo $c$ and $b | c$, then $f(q)$ has period $a$ modulo $b$.

(iv) If $f(q)$ has period $a$ modulo $b$, then so does $f(q)h(q)$ for any Laurent polynomial $h(q)$.

(v) If $f(q)$ has period $a$ modulo $b$ and $a \mid b$, then

$$f(q) \equiv \frac{a}{b} \left( \frac{q^b - 1}{q^a - 1} \right) f(q) \pmod{q^b - 1}.$$

Proof. (i), (iii), (iv), and (v) are straightforward. For (ii), suppose

$$(q^b - 1) \mid (q^a - 1)f(q), \quad (q^c - 1) \mid (q^b - 1)f(q).$$

Write $q^c - 1 = \prod_{k=1}^{c} (q - \omega_k^c)$. If $q - \omega_k^c$ does not divide $f(q)$, then it must divide $q^b - 1$ and hence $q^a - 1$. It follows that

$$(q^c - 1) \mid (q^a - 1)f(q).$$

Lemma 3.3. Suppose $C_n = \langle \sigma_n \rangle$ acts on $W$. Let $g \mid n$ and $C_g := \langle \sigma_n^{|g|} \rangle \subset C_n$. If

(i) $(W, C_g, f(q))$ exhibits the CSP,

(ii) $f(q)$ has period $g$ modulo $n$, and

(iii) for all $C_n$-orbits $O \subset W$, we have $\frac{n}{|O|} \mid g$,

then $(W, C_n, f(q))$ exhibits the CSP.

Proof. Let

$$F(q) := \sum_{C_n \text{-orbits } O \subset W} \frac{q^g - 1}{q^{|O|} - 1}.$$
By (5), \((W, C_a, F(q))\) exhibits the CSP, so \((W, C_g, F(q))\) also exhibits the CSP by Remark 2.3. Thus, by (5) and condition (i),
\[
f(q) = F(q) + p(q)(q^g - 1)
\]
for some \(p(q) \in \mathbb{C}[q]\). Each summand of \(F(q)\) has period \(g\) modulo \(n\) since
\[
(q^n - 1) | (q^g - 1) \frac{q^n - 1}{q^{n/|\mathcal{O}|} - 1},
\]
by condition (iii). Putting this together with condition (ii), \(f(q)\) and \(F(q)\) have period \(g\) modulo \(n\). Using Lemma 3.2(v) twice along with (11) now gives
\[
f(q) \equiv g \frac{q^n - 1}{n q^g - 1} F(q) \equiv F(q) \mod (q^n - 1).
\]
\[
\square
\]

4. Cyclic Descent Type

In this section, we introduce the cyclic descent type of a word. We also verify hypothesis (iii) of Lemma 3.3 for \(W_{\alpha, \delta}\) for a particular \(g\); see Lemma 4.3.

Let \(w^{(i)}\) denote the subsequence of \(w\) with all letters larger than \(i\) removed. We have a “filtration”
\[
\emptyset \preceq w^{(1)} \preceq w^{(2)} \preceq \cdots \preceq w^{(m-1)} \preceq w^{(m)} = w,
\]
where \(u \preceq v\) means that \(u\) is a subsequence of \(v\). We think of this filtration as building up \(w\) by recursively adding all of the copies of the next largest letter “where they fit.” The cyclic descent type of a word \(w\), denoted CDT\((w)\), is the sequence which tracks the number of new cyclic descents at each stage of the filtration. Precisely, we have the following.

**Definition 4.1.** The cyclic descent type (CDT) of a word \(w\) is the weak composition of \(\text{cdes}(w)\) given by
\[
\text{CDT}(w) := (\text{cdes}(w^{(1)}), \text{cdes}(w^{(2)}) - \text{cdes}(w^{(1)}), \ldots, \text{cdes}(w^{(m)}) - \text{cdes}(w^{(m-1)})).
\]

Note that CDT is constant on necklaces since rotating \(w\) rotates each \(w^{(i)}\) and \(\text{cdes}\) is constant under rotations. Furthermore, \(\text{cdes}(w^{(1)}) = 0\) always, so CDT\((w)\) always begins with 0.
Example 4.2. Suppose $w = 143124114223$, so

\begin{align*}
  w^{(1)} &= 1111 & \text{cdes}(w^{(1)}) &= 0, \\
  w^{(2)} &= 112.1122. & \text{cdes}(w^{(2)}) &= 2, \\
  w^{(3)} &= 13.12.11223. & \text{cdes}(w^{(3)}) &= 3, \\
  w^{(4)} &= 14.3.124.114.223. & \text{cdes}(w^{(4)}) &= 5.
\end{align*}

Hence, $\text{CDT}(143124114223) = (0,2-0,3-2,5-3) = (0,2,1,2)$.

Recall from (4) that

$$W_{\alpha,\delta} := \{ w \in W_n : \text{cont}(w) = \alpha, \text{CDT}(w) = \delta \}.$$  

We could define $W_{\alpha,\delta}$ more “symmetrically” by replacing cont with “cyclic weak ascent type,” which would be the point-wise difference of cont and CDT. However, content is ubiquitous in the literature, so we use it.

Lemma 4.3. If $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\delta = (\delta_1, \ldots, \delta_m)$, $N \subset W_{\alpha,\delta}$ is a necklace, and $g := \text{gcd}(\alpha_1, \ldots, \alpha_m, \delta_1, \ldots \delta_m)$, then $\frac{n}{|N|} \mid g$.

Proof. We can write $N = [w]$ with $w = u \frac{n}{|N|}$ since $\text{freq}(w) = \frac{n}{|N|}$. Hence, using pointwise multiplication,

$$\text{cont}(w) = \frac{n}{|N|} \cdot \text{cont}(u), \quad \text{CDT}(w) = \frac{n}{|N|} \cdot \text{CDT}(u).$$

In particular, $\frac{n}{|N|}$ divides $\alpha_1, \ldots, \alpha_m, \delta_1, \ldots \delta_m$, so $\frac{n}{|N|} \mid g$.  

\[\square\]

5. Runs and Falls

In this section, we give a method to algorithmically construct $W_{\alpha,\delta}$ and use it to prove a product formula for $W_{\alpha,\delta}^{\text{maj}}(q)$, Theorem 5.19. We conclude the section by using this formula to verify hypothesis (ii) of Lemma 3.3 for $W_{\alpha,\delta}$; see Proposition 5.21.

5.1. A Tree Decomposition for $W_{\alpha,\delta}$. We now describe a way to create words with a fixed content and CDT in terms of insertions into runs and falls. This procedure is organized into a tree, Definition 5.11, whose edges are labeled with sets and multisets. Lemma 5.8 describes changes in the major index upon traversing an edge of this tree.

Definition 5.1. Write $w = w_1 \cdots w_n \in W_n$. A fall in $w$ is a maximal set of distinct consecutive indices $i, i+1, \ldots, j-1, j$ such that $w_i > w_{i+1} > \cdots > w_j$, where we take indices modulo $n$. A run in a non-constant word $w$ is a maximal set of distinct consecutive indices $i, i+1, \ldots, j$ such that $w_i \leq w_{i+1} \leq \cdots \leq w_j$, where we take indices modulo $n$. The constant word $w = \ell^n$ by convention has no runs, and it has $n$ falls.
Note that each letter in \( w \) is part of a unique fall and a unique run, except when \( w = \ell^n \) is constant. Index falls from 0 from left to right starting at the fall containing the first letter of \( w \), and do the same with runs. It is easy to see that \( w \) has \( n - \operatorname{cdes}(w) \) falls and \( \operatorname{cdes}(w) \) runs, since they are separated by cyclic weak ascents and cyclic descents, respectively.

**Definition 5.2.** We write
\[
F(w) := [0, |w| - \operatorname{cdes}(w) - 1] \quad \text{and} \quad R(w) := [0, \operatorname{cdes}(w) - 1]
\]
for the indices of the falls and runs of \( w \), respectively.

**Example 5.3.** Let \( w = 26534611 = 2\dot{6}53\dot{4}\dot{6}1\dot{1} = 26.5.346.11 \in W_8 \), where upper dots indicate cyclic weak ascents and lower dots indicate cyclic descents. Since \( \operatorname{cdes}(w) = 3 \), we have \( F(w) = [0, 4] \) and \( R(w) = [0, 2] \). The 5 falls of \( w \) are \( 2, 653, 4, 61, 1 \), with respective indices \( 0, 1, 2, 3, 4 \). The 3 runs of \( w \) are \( 1126, 5, 346 \), with respective indices \( 0, 1, 2 \).

**Definition 5.4.** Fix a letter \( \ell \) and pick a subset \( F \) of the falls \( F(w) \). Assume \( \ell \) does not appear in any of the falls in \( F \). We **insert \( \ell \) into falls** \( F \) by successively inserting \( \ell \) into each fall \( w_i > w_{i+1} > \cdots > w_j \) in \( F \) so that \( w_i \cdots \ell \cdots w_j \) is still decreasing.

Similarly, we may fix a letter \( \ell \) and pick a multisubset \( R \) of \( R(w) \) (this time \( \ell \) may already appear in a run in \( R \)). We **insert \( \ell \) into runs** \( R \) by successively inserting \( \ell \) into each run \( w_i \leq w_{i+1} \leq \cdots \leq w_j \) in \( R \) so that \( w_i \cdots \ell \cdots w_j \) is still weakly increasing.

When inserting \( \ell \) into a run already containing \( \ell \), the resulting word is independent of precisely which of the possible positions is used. This is the reason we insert into runs and falls instead of positions.

Note that there is a slight ambiguity in our description of insertion into falls and runs, since it may be possible to insert either at the beginning or at the end of \( w \) while still satisfying the relevant inequalities. Given the choice, we always insert at the beginning of \( w \).

**Example 5.5.** Let \( w = 2\dot{6}534611’ \). Insert 7 into falls of \( w \) with indices 0 and 3 to successively obtain \( \overline{2\dot{6}534611}’ \) and then \( w’ := 72\dot{6}534\overline{7}611’ \). Note that \( w’ = 7.26.5.347.6.11 \) has two more runs (or cyclic descents) than \( w \). Now insert 7 into the runs of \( w’ \) with multiset of indices \( \{0, 2, 3, 3\} \) to successively obtain \( 77.26.5.347.6.11, 77.26.57.347.6.11, 77.26.57.34\overline{7}7.6.11, \) and \( w'' := 77.26.57.34\overline{7}77.6.11 \).

Let
\[
\widetilde{W}_n = \{ w \in W_n : w \text{ ends in a } 1 \},
\]
\[
\widetilde{W}_{\alpha, \delta} = \{ w \in W_{\alpha, \delta} : w \text{ ends in a } 1 \}.
\]

We restrict to \( \widetilde{W}_n \) and \( \widetilde{W}_{\alpha, \delta} \) since the major index generating function is easier to find and extends to \( W_{\alpha, \delta}^{\text{maj}}(q) \pmod{q^n - 1} \).
Definition 5.6. Fix $w \in \tilde{W}_n$, a letter $\ell$ not in $w$, and 

$$F \subseteq F(w) = [0, |w| - \text{cdes}(w) - 1] \quad \text{and} \quad R \subseteq [0, \text{cdes}(w) + |F| - 1]$$

where $\subseteq$ denotes a multisubset. Let $w'$ be obtained by inserting $\ell$ into falls $F$ of $w$. Note that $[0, \text{cdes}(w) + |F| - 1] = R(w')$ indexes the runs of $w'$. Now let $w''$ be obtained by inserting $\ell$ into runs $R$ of $w'$. We say $w''$ is obtained by inserting the triple $(\ell, F, R)$ into $w$. Observe that $[0, \text{cdes}(w'') = cdes(w'') = cdes(w) + |F|$ and $w'' \in \tilde{W}_{n+|F|}|R].$

First we define the cyclic descent type $\delta = \text{CDT}(w)$ of any $w \in W_\alpha$. Then we give a product formula for $W_{\alpha, \delta}^\text{maj}(q)$ modulo $q^n - 1$, Theorem 5.19. The $q = 1$ specialization gives a formula for $\#W_{\alpha, \delta}$, Proposition 5.16. Along the way, we describe how to build words in $W_{\alpha, \delta}$ by walking along a tree whose edges are labeled by sets and multisets. We describe a fixed point lemma arising from the tree which will play a key role in Section 6. We also introduce the notion of modular periodicity in order to transfer certain results between different cyclic groups or moduli.

We next describe the effect of inserting a single letter on maj. We restrict to $\tilde{W}_n$ so we preserve a cyclic weak ascent at the end and never add a letter to the end. The fact that the increments in major index from inserting a new letter into all possible positions form a permutation was first observed by Gupta [Gup78]. Lemma 5.7 tells us exactly the increment in major index based on which run or fall the newly inserted letter fits into.

Lemma 5.7. Suppose $w' \in \tilde{W}_{n+1}$ is obtained by adding a letter $\ell$ to $w \in \tilde{W}_n$ in any position. Then $w'$ is obtained by inserting $\ell$ into some run or fall of $w$, and

$$\text{maj}(w') - \text{maj}(w) = \begin{cases} 
\text{cdes}(w) - r & \text{if } \ell \text{ is inserted into run } r \text{ of } w \\
\text{cdes}(w) + 1 + f & \text{if } \ell \text{ is inserted into fall } f \text{ of } w.
\end{cases} \tag{15}$$

Proof. If $\text{cdes}(w') = \text{cdes}(w)$, then $\ell$ is inserted into some run of $w$, and otherwise $\text{cdes}(w') = \text{cdes}(w) + 1$ and $\ell$ is inserted into some fall of $w$. Inserting $\ell$ into run $r$ of $w$ will increment the position of $\text{cdes}(w) - r$ descents by 1 each, so

$$\text{maj}(w') - \text{maj}(w) = \text{cdes}(w) - r.$$

Let $\text{comaj}(w) := 1 + 2 + \cdots + (|w| - 1) - \text{maj}(w)$, which is the sum of $i \in [|w| - 1]$ where $w_i \leq w_{i+1}$. Inserting $\ell$ into fall $f$ of $w$ will increment the position of $(|w| - 1) - \text{cdes}(w) - f$ weak ascents by 1 each, so

$$\text{comaj}(w') - \text{comaj}(w) = (|w| - 1) - \text{cdes}(w) - f,$$

from which it follows that

$$\text{maj}(w') - \text{maj}(w) = \text{cdes}(w) + 1 + f.$$
Lemma 5.8. Suppose \( w'' \) is obtained by inserting the triple \((\ell, F, R)\) into \( w \in \tilde{W}_n \). Then
\[
\text{maj}(w'') - \text{maj}(w) = \left(\frac{|F| + 1}{2}\right) + (\text{cdes}(w))|F| + |F||R| + \sum_{f \in F} f - \sum_{r \in R} r.
\]

Proof. Let \( w' \) be obtained by inserting \( \ell \) into falls \( F \) of \( w \). It suffices to show
\[
\text{maj}(w') - \text{maj}(w) = \left(\frac{|F| + 1}{2}\right) + (\text{cdes}(w))|F| + \sum_{f \in F} f
\]
and
\[
\text{maj}(w'') - \text{maj}(w') = (\text{cdes}(w'))|R| - \sum_{r \in R} r
\]
since \( \text{cdes}(w') = \text{cdes}(w'') = \text{cdes}(w) + |F| \). Both (17) and (18) follow from iterating Lemma 5.7 and recalling \( \text{cdes} \) is incremented by 1 each time we insert into a fall. □

Notation 5.9. For the rest of this section, fix a strong composition \( \alpha = (\alpha_1, \ldots, \alpha_m) \) of \( n \geq 1 \) and \( \delta = (\delta_1, \ldots, \delta_m) \vdash k \) with \( \delta_1 = 0 \). We emphasize that \( \alpha \) and \( \delta \) have the same number, \( m \), of parts. For \( \ell = 1, \ldots, m \), let
\[
n_\ell := \alpha_1 + \cdots + \alpha_\ell,
\]
\[
k_\ell := \delta_1 + \cdots + \delta_\ell.
\]
For \( w \in W_{\alpha,\delta} \), we have the defining conditions \( |w^{(\ell)}| = n_\ell \) and \( \text{cdes}(w^{(\ell)}) = k_\ell \). Furthermore, let
\[
S_\ell := \left( [0, n_{\ell-1} - k_{\ell-1} - 1], \delta_\ell \right), \quad M_\ell := \left( [0, k_\ell - 1], \alpha_\ell - \delta_\ell \right)
\]
and
\[
g := \gcd(\alpha_1, \alpha_2, \ldots, \alpha_m, \delta_1, \delta_2, \ldots, \delta_m).
\]
If \( w \in \tilde{W}_{\alpha,\delta} \), then the set \( S_\ell \) consists of all subsets of the falls \( F(w^{(\ell-1)}) \) which, when \( \ell \) is inserted into those falls of \( w^{(\ell-1)} \), result in a word \( w' \) with \( k_\ell \) cyclic descents. The multiset \( M_\ell \) similarly consists of all choices of runs \( R(w') \) which, when \( \ell \) is inserted into those runs, result in a word with length \( n_\ell \).

Remark 5.10. We restrict to strong compositions \( \alpha \) for notational simplicity, though the results in this section may easily be generalized to arbitrary weak compositions by “flattening” weak compositions to strong ones by removing zeros.

Definition 5.11. Construct a rooted, vertex-labeled and edge-labeled tree \( T_{\alpha,\delta} \) recursively as follows. Begin with a tree \( T^{(1)} \) containing only a root labeled by the word \( 1^{\alpha_1} \). For \( \ell = 2, \ldots, m \), to obtain \( T^{(\ell)} \), do the following. For each leaf \( w \) of \( T^{(\ell-1)} \) and for each triple \((\ell, F, R)\) with
\[
F \in S_\ell \quad \text{and} \quad R \in M_\ell,
\]
add an edge labeled by $(F,R)$ to $T^{(\ell-1)}$ from $w$ to $w''$ where $w''$ is obtained by inserting $(\ell,F,R)$ into $w$. Define $T_{\alpha,\delta} := T^{(m)}$.

**Example 5.12.** Let $\alpha = (4,2,3)$ and $\delta = (0,2,1)$. Figure 1 is the subgraph of $T_{\alpha,\delta}$ consisting of paths from the root to leaves that are rotations of 112113323.

![Figure 1](image_url)

**Figure 1.** Subgraph of tree $T_{\alpha,\delta}$ with $\alpha = (4,2,3)$, $\delta = (0,2,1)$.

For this full $T_{\alpha,\delta}$, the root has $\binom{4}{3} = 6$ children since 1111 has 4 falls. Each child of the root itself has $\binom{4}{1} \binom{3}{0} = 24$ children. Hence, $T_{\alpha,\delta}$ has 144 leaves. Notice that the cyclic rotations of 31121332 appearing as leaves in Figure 1 are precisely those ending in 1. It will shortly become apparent that in this example, $\#W_{\alpha,\delta} = \frac{n}{4} \cdot 144 = 324$.

**Lemma 5.13.** The vertices of $T_{\alpha,\delta}$ which are $\ell < m$ edges away from the root are precisely the elements of $\{w^{(\ell+1)} : w \in \tilde{W}_{\alpha,\delta}\}$, each occurring once. In particular, the leaves of $T_{\alpha,\delta}$ are precisely the elements of $\tilde{W}_{\alpha,\delta}$, each occurring once.

**Proof.** By definition of $S_\ell$ and $M_\ell$, any leaf of $T^{(\ell)}$ has content $(\alpha_1, \ldots, \alpha_\ell)$, cyclic descent type $(\delta_1, \ldots, \delta_\ell)$, and ends in a 1, so is in $\{w^{(\ell)} : w \in \tilde{W}_{\alpha,\delta}\}$. Conversely, given any $w \in \tilde{W}_{\alpha,\delta}$, the word $w^{(\ell)}$ is obtained by inserting a unique triple $(\ell,F,R)$ into $w^{(\ell-1)}$ by repeated applications of Lemma 5.7. \qed

**Definition 5.14.** By Lemma 5.13, the tree $T_{\alpha,\delta}$ encodes a bijection

$$\Phi : \tilde{W}_{\alpha,\delta} \sim \prod_{\ell=2}^{m} S_\ell \times M_\ell$$

given by reading the edge labels from the root to $w$. We suppress the dependence of $\Phi$ on $\alpha$ and $\delta$ from the notation since they can be computed from the input $w$.

**Lemma 5.15.** For any $w \in W_{\alpha,\delta}$,

$$(21) \quad \# [w] = \frac{n}{\alpha_1} \cdot \# ([w] \cap \tilde{W}_{\alpha,\delta}).$$

Consequently,

$$(22) \quad \# W_{\alpha,\delta} = \frac{n}{\alpha_1} \# \tilde{W}_{\alpha,\delta}.$$
Proof. Each \( w \in W_{\alpha,\delta} \) has period \((w) = n/\text{freq}(w)\) distinct cyclic rotations, of which \(\alpha_1/\text{freq}(w)\) end in 1. \(\Box\)

**Proposition 5.16.** Using Notation 5.9, we have

\[
\#W_{\alpha,\delta} = \frac{n}{\alpha_1} \prod_{\ell=2}^{m} \left( \frac{n_{\ell-1} - k_{\ell-1}}{\delta_{\ell}} \right) \binom{k_{\ell}}{\alpha_{\ell} - \delta_{\ell} \mod q}.\tag{23}
\]

In particular, \(W_{\alpha,\delta} \neq \emptyset\) if and only if

\[
0 \leq \delta_{\ell} \leq \alpha_{\ell} \quad \text{for all } 1 \leq \ell \leq m, \text{ and }
\delta_1 + \cdots + \delta_{\ell+1} \leq \alpha_1 + \cdots + \alpha_{\ell} \quad \text{for all } 1 \leq \ell < m.\tag{24}
\]

Proof. The product in (23) is \(\# \prod_{\ell=2}^{m} S_{\ell} \times M_{\ell}\), which is \(\#\tilde{W}_{\alpha,\delta}\) by the bijection \(\Phi\). Now (23) follows from (22), and (24) follows from (23). \(\Box\)

5.2. Major Index Generating Functions. We next use the bijection \(\Phi\) and Lemma 5.8 to give a product formula for \(\tilde{W}_{\alpha,\delta}^{\text{maj}}(q)\), Theorem 5.17. We then use modular periodicity to obtain an analogous expression for \(W_{\alpha,\delta}^{\text{maj}}(q)\) modulo \(q^n - 1\), Theorem 5.19.

**Theorem 5.17.** Using Notation 5.9, we have

\[
\tilde{W}_{\alpha,\delta}^{\text{maj}}(q) = \prod_{\ell=2}^{m} q^{k_{\ell} \alpha_{\ell}} \left( \frac{n_{\ell-1} - k_{\ell-1}}{\delta_{\ell}} \right)_q \binom{k_{\ell}}{\alpha_{\ell} - \delta_{\ell} \mod q}^{-1}.\tag{25}
\]

\[
= q^{\eta(\alpha,\delta)} \prod_{\ell=2}^{m} \left( \frac{n_{\ell-1} - k_{\ell-1}}{\delta_{\ell}} \right)_q \binom{k_{\ell}}{\alpha_{\ell} - \delta_{\ell} \mod q}^{-1}.\tag{26}
\]

where

\[
\eta(\alpha, \delta) := n - \alpha_1 + \binom{k}{2} + \sum_{\ell=2}^{m} \binom{\delta_{\ell}}{2}.\tag{27}
\]

Proof. Combining \(\Phi\) with Lemma 5.8 shows that

\[
\tilde{W}_{\alpha,\delta}^{\text{maj}}(q) = \prod_{\ell=2}^{m} \sum_{F \in S_{\ell}} \sum_{R \in M_{\ell}} q^{\epsilon(\ell,F,R)},\tag{27}
\]

where \(\epsilon(\ell,F,R) := \left( \frac{\delta_{\ell} + 1}{2} \right) + k_{\ell-1} \alpha_{\ell} + \delta_{\ell} (\alpha_{\ell} - \delta_{\ell}) + \text{sum}(F) - \text{sum}(R)\).

Noting that

\[
\left( \frac{\delta_{\ell} + 1}{2} \right) + k_{\ell-1} \alpha_{\ell} + \delta_{\ell} (\alpha_{\ell} - \delta_{\ell}) = k_{\ell} \alpha_{\ell} - \left( \frac{\delta_{\ell}}{2} \right),
\]

simplifying (27) gives

\[
\tilde{W}_{\alpha,\delta}^{\text{maj}}(q) = \prod_{\ell=2}^{m} q^{k_{\ell} \alpha_{\ell} - \left( \frac{\delta_{\ell}}{2} \right)} S_{\ell}^{\text{sum}}(q) M_{\ell}^{\text{sum}}(q^{-1}).\tag{28}
\]
Equation (25) now follows from (8), (9), and the definition of $S_\ell$ and $M_\ell$. As for (26), consider the reversal bijection $r: M_\ell \rightarrow M_\ell$ induced by

$$x \mapsto k_\ell - 1 - x$$
on [0, k_\ell - 1]. This bijection satisfies $\text{sum}(r(A)) = (k_\ell - 1)(\alpha_\ell - \delta_\ell) - \text{sum}(A)$, so

(29) $$M_\ell^{\text{sum}}(q^{-1}) = q^{-(k_\ell - 1)(\alpha_\ell - \delta_\ell)}M_\ell^{\text{sum}}(q).$$

Plugging (29) into (28) and noting that

$$
\sum_{\ell=2}^{m} \left( k_\ell \alpha_\ell - \frac{\delta_\ell}{2} - (k_\ell - 1)(\alpha_\ell - \delta_\ell) \right) = \sum_{\ell=2}^{m} \left( \alpha_\ell - \frac{\delta_\ell}{2} - \frac{\delta_\ell^2}{2} + k_\ell \delta_\ell \right) \\
= n - \alpha_1 - \frac{k}{2} + \sum_{\ell=2}^{m} \left( -\frac{\delta_\ell^2}{2} + \sum_{j=2}^{\ell} \delta_j \delta_\ell \right) \\
= n - \alpha_1 - \frac{k}{2} + \frac{1}{2} \sum_{\ell=2}^{m} \sum_{j=2}^{\ell} \delta_j \delta_\ell \\
= n - \alpha_1 - \frac{k}{2} + \frac{k^2}{2}
$$
gives

$$
\tilde{W}_{\alpha,\delta}^{\text{maj}}(q) = q^{n - \alpha_1 + \left(\frac{k}{2}\right)} \prod_{\ell=2}^{m} S_\ell^{\text{sum}}(q) M_\ell^{\text{sum}}(q).
$$

Using (8) and (9) now yields (26). □

**Lemma 5.18.** Let $\alpha \vdash n$, $\delta \vdash k$. The statistic $\text{maj}$ has period $k$ modulo $n$ on $W_{\alpha,\delta}$. Moreover, $\text{maj}$ is constant modulo $d := \gcd(n, k)$ on necklaces in $W_{\alpha,\delta}$, and

(30) $$W_{\alpha,\delta}^{\text{maj}}(q) \equiv \frac{n}{\alpha_1} \tilde{W}_{\alpha,\delta}^{\text{maj}}(q) \pmod{q^d - 1}.$$

**Proof.** Since cyclically rotating $w \in W_{\alpha,\delta}$ increments each cyclic descent by 1 modulo $n$, we have

(31) $$\text{maj}(\sigma_n \cdot w) \equiv_n \text{maj}(w) + k.$$

In particular, maj has period $k$ modulo $n$ on necklaces in $W_{\alpha,\delta}$. Furthermore, maj is constant on necklaces in $W_{\alpha,\delta}$ modulo $d$. Now (30) follows from (21). □

**Theorem 5.19.** Using Notation 5.9, let $d := \gcd(n, k)$. Then, modulo $q^n - 1$,

(32) $$
W_{\alpha,\delta}^{\text{maj}}(q) \equiv d \left( \frac{q^n - 1}{q^d - 1} \right) \prod_{\ell=2}^{m} q^{k_\ell \alpha_\ell \left( \frac{n_\ell - 1 - k_\ell - 1}{\delta_\ell} \right)} q^{\left( \frac{k_\ell}{\alpha_\ell - \delta_\ell} \right)} q^{-1} \\
\equiv d \left( \frac{q^n - 1}{q^d - 1} \right) q^\left( \frac{k}{2} \right) + \sum_{\ell=2}^{m} \left( \frac{k_\ell}{2} \right) - \alpha_1 \prod_{\ell=2}^{m} \left( \frac{n_\ell - 1 - k_\ell - 1}{\delta_\ell} \right) q^{\left( \frac{k_\ell}{\alpha_\ell - \delta_\ell} \right)} q^{-1}.
$$
Proof. By Lemma 5.18, maj has period $k$ modulo $n$ on $W_{\alpha, \delta}$. Hence by Lemma 3.2(i), maj has period $d$ modulo $n$ on $W_{\alpha, \delta}$. Using Lemma 3.2(v) and (30) gives

$$W_{\alpha, \delta}^{\text{maj}}(q) \equiv \frac{d}{\alpha_1} \left( \frac{q^n - 1}{q^d - 1} \right) W_{\alpha, \delta}^{\text{maj}}(q)$$

$$\equiv \frac{d}{\alpha_1} \left( \frac{q^n - 1}{q^d - 1} \right) \left( \frac{n}{\alpha_1} \widetilde{W}_{\alpha, \delta}^{\text{maj}}(q) + p(q)(q^d - 1) \right)$$

$$\equiv \frac{d}{\alpha_1} \left( \frac{q^n - 1}{q^d - 1} \right) \widetilde{W}_{\alpha, \delta}^{\text{maj}} \pmod{q^n - 1},$$

where $p(q) \in \mathbb{C}[q]$. Theorem 5.19 now follows from Theorem 5.17. \qed

Corollary 5.20. Using Notation 5.9, let $d := \gcd(n, k)$. Then, modulo $q^n - 1$,

$$W_{\alpha}^{\text{maj}}(q) = \left( \frac{n}{\alpha} \right) \equiv \sum_{\delta} \frac{d}{\alpha_1} \left( \frac{q^n - 1}{q^d - 1} \right) \prod_{\ell=2}^{m} q^{k_{\ell} \delta_{\ell}} \left( \frac{n_{\ell-1} - k_{\ell-1}}{\delta_{\ell}} \right) q^{\left( \frac{k_{\ell}}{\alpha_{\ell} - \delta_{\ell}} \right) q^{-1}}$$

where the sum is over weak compositions $\delta$ of $k$ satisfying (24). In particular,

$$\#W_{\alpha} = \left( \frac{n}{\alpha} \right) = \sum_{\delta} \frac{n}{\alpha_1} \prod_{\ell=2}^{m} \left( \frac{n_{\ell-1} - k_{\ell-1}}{\delta_{\ell}} \right) \left( \frac{k_{\ell}}{\alpha_{\ell} - \delta_{\ell}} \right) q^{-1}.$$

Note that the two-letter case of (34) is a special case of the classical Vandermonde convolution identity [Sta12, Ex. 1.1.17].

5.3. Verifying Hypothesis (ii) of Lemma 3.3 for $W_{\alpha, \delta}$.

Proposition 5.21. Using Notation 5.9, $W_{\alpha, \delta}^{\text{maj}}(q)$ has period $g$ modulo $n$.

Proof. Let $d = \gcd(n, k)$. By Theorem 5.19,

$$W_{\alpha, \delta}^{\text{maj}}(q) \equiv \frac{d}{\alpha_1} \left( \frac{q^n - 1}{q^d - 1} \right) \prod_{\ell=2}^{m} q^{k_{\ell} \delta_{\ell}} \left( \frac{n_{\ell-1} - k_{\ell-1}}{\delta_{\ell}} \right) q^{\left( \frac{k_{\ell}}{\alpha_{\ell} - \delta_{\ell}} \right) q^{-1}}$$

modulo $q^n - 1$. The action of rotation on elements of $S_{\ell} = \left\{ [0, n_{\ell-1} - k_{\ell-1}]_{\delta_{\ell}} \right\}$ increases their sum by $\delta_{\ell}$ modulo $n_{\ell-1} - k_{\ell-1}$. Thus by (8), $\left( \frac{n_{\ell-1} - k_{\ell-1}}{\delta_{\ell}} \right)$ has period $\delta_{\ell}$ modulo $n_{\ell-1} - k_{\ell-1}$. Similarly by (9), $\left( \frac{k_{\ell}}{\alpha_{\ell} - \delta_{\ell}} \right) q^{-1}$ has period $\alpha_{\ell} - \delta_{\ell}$ modulo $k_{\ell}$. For $\ell = 2, \ldots, m$, by Lemma 3.2(iv) we then have

$$W_{\alpha, \delta}^{\text{maj}}(q) \text{ has period } \delta_{\ell} \text{ modulo } n_{\ell-1} - k_{\ell-1}, \text{ and}$$

$$W_{\alpha, \delta}^{\text{maj}}(q) \text{ has period } \alpha_{\ell} - \delta_{\ell} \text{ modulo } k_{\ell}.$$

We show $W_{\alpha, \delta}^{\text{maj}}(q)$ has period $\alpha_{\ell}$ and $\delta_{\ell}$ modulo $n$ by downward induction on $\ell$, for $m \geq \ell \geq 2$. Note that the base case $\ell = m$ is accounted for by our argument as well.
Suppose $W_{\alpha,\delta}^{\text{maj}}(q)$ has period $\alpha_j$ and $\delta_j$ modulo $n$ for all $j > \ell$. By Lemma 5.18, $W_{\alpha,\delta}^{\text{maj}}(q)$ has period $k$ modulo $n$. By Lemma 3.2(i), $W_{\alpha,\delta}^{\text{maj}}(q)$ thus has period
\[ k_\ell = k - (\delta_m + \cdots + \delta_{\ell+1}) \]
modulo $n$. Since $W_{\alpha,\delta}^{\text{maj}}(q)$ has period $\alpha_\ell - \delta_\ell$ modulo $k_\ell$, $W_{\alpha,\delta}^{\text{maj}}(q)$ has period $\alpha_\ell - \delta_\ell$ modulo $n$ by Lemma 3.2(ii).

As noted, $W_{\alpha,\delta}^{\text{maj}}(q)$ has period $\delta_\ell$ modulo $n_{\ell-1} - k_{\ell-1}$. By Lemma 3.2(i), $W_{\alpha,\delta}^{\text{maj}}(q)$ also has period
\[ n_{\ell-1} - k_{\ell-1} = n - (\alpha_m + \cdots + \alpha_{\ell+1}) - k + (\delta_m + \cdots + \delta_{\ell+1}) - (\alpha_\ell - \delta_\ell) \]
modulo $n$. Hence, as $W_{\alpha,\delta}^{\text{maj}}(q)$ has period $\delta_\ell$ modulo $n_{\ell-1} - k_{\ell-1}$, $W_{\alpha,\delta}^{\text{maj}}(q)$ has period $\delta_\ell$ modulo $n$ by Lemma 3.2(ii). By another application of Lemma 3.2(i), $W_{\alpha,\delta}^{\text{maj}}(q)$ has period $\alpha_\ell$ modulo $n$ as well, completing the induction.

Indeed, $W_{\alpha,\delta}^{\text{maj}}(q)$ has period $\delta_1 = 0$ modulo $n$ trivially, and $W_{\alpha,\delta}^{\text{maj}}(q)$ has period $\alpha_1 = n - (\alpha_m + \cdots + \alpha_2)$ modulo $n$ by Lemma 3.2(i). Putting everything together, $W_{\alpha,\delta}^{\text{maj}}(q)$ has periods $\alpha_1, \ldots, \alpha_m, \delta_1, \ldots, \delta_n$ modulo $n$, so by one more application of Lemma 3.2(i), $W_{\alpha,\delta}^{\text{maj}}(q)$ has period $g$ modulo $n$. \(\square\)

6. Refining the CSP to Fixed Content and Cyclic Descent Type

In this section, we verify the final hypothesis (i) of Lemma 3.3 for $W_{\alpha,\delta}$ and deduce Theorem 1.5. Throughout this section we continue to follow Notation 5.9. We recall in particular that
\[ S_\ell := \left( [0,n_{\ell-1} - k_{\ell-1} - 1] \right), \quad M_\ell := \left( [0,k_{\ell} - 1] \right) \]
and
\[ g := \gcd(\alpha_1, \ldots, \alpha_m, \delta_1, \ldots, \delta_m). \]

6.1. A Fixed Point Lemma. To prove our main result, Theorem 1.5, one approach would be to find a $C_n$-equivariant isomorphism between a known CSP triple and $(W_{\alpha,\delta}, C_n, W_{\alpha,\delta}^{\text{maj}}(q))$. Such a triple is hinted at by (25) and the bijection $\Phi$ using products of CSP’s coming from Theorem 2.4, though the approach encounters immediate difficulties. For instance, $\tilde{W}_{\alpha,\delta}$ is not generally closed under the $C_n$-action. In this section, we instead give a fixed point lemma, Lemma 6.5, which is intuitively a weakened version of the equivariant isomorphism approach.

**Definition 6.1.** Since $g \mid n_{\ell-1} - k_{\ell-1}$, $g \mid k_\ell$, and $g \mid n$, $C_g$ acts on each of $S_\ell$, $M_\ell$, and $W_{\alpha,\delta}$ by restricting the actions of $C_{n_{\ell-1} - k_{\ell-1}}$, $C_{k_\ell}$, and $C_n$ to their unique subgroups of size $g$. For instance, the action of $C_g$ on $W_{\alpha,\delta}$ is generated by rotation by $n/g$.

We further let $C_g$ act diagonally on the products $S_\ell \times M_\ell$ and $\prod_{\ell=2}^m S_\ell \times M_\ell$. We emphasize that despite having $C_g$-actions on $W_{\alpha,\delta}$ and $\prod_{\ell=2}^m S_\ell \times M_\ell$, the bijection $\Phi: \tilde{W}_{\alpha,\delta} \cong \prod_{\ell=2}^m S_\ell \times M_\ell$ is not in general equivariant since $\tilde{W}_{\alpha,\delta}$ is not closed under the $C_g$ action on $W_{\alpha,\delta}$. 
Definition 6.2. Given a multisubset of some set \([0,a]\), we may encode it as a multiplicity word \(w_0w_1\ldots w_a\) where \(w_i\) is the multiplicity of \(i\). In particular, we may consider the bijection \(\Phi: \tilde{W}_{\alpha,\delta} \sim \prod_{\ell=2}^n S_\ell \times M_\ell\) as mapping words to sequences of pairs of certain words.

Example 6.3. Consider the leaf \(w = 211332311\) in Figure 1 from Example 5.12. Reading edge labels gives \(\Phi(w) = (((0,2),\emptyset), (\{2\}, \{1,2\}))\). Recalling that \(S_2\) consists of subsets of \([0,4-1]\), \(M_2\) consists of multisubsets of \(\emptyset\), \(S_3\) consists of subsets of \([0,4-1]\), and \(M_3\) consists of multisubsets of \([0,3-1]\), the corresponding sequence of words is \(((1010\epsilon), (0010, 011))\), where \(\epsilon\) denotes the empty word. Table 1 summarizes several similar translations.

| \(w\)         | \(\Phi(w)\)                      | sequence of pairs of words |
|---------------|----------------------------------|-----------------------------|
| 211332311     | (((0,2),\emptyset), (\{2\}, \{1,2\})) | ((1010\epsilon), (0010, 011)) |
| 1211332311    | (((1,3),\emptyset), (\{3\}, \{1,2\})) | ((0101\epsilon), (0001, 011)) |
| (211332311)\(^2\) | (((0,2, 4, 6),\emptyset), (\{2, 6\}, \{1, 2, 4, 5\})) | ((1010^2\epsilon), (0010^2, 011^2)) |
| 2221123311    | (((0,2), \{0, 0\}), (\emptyset, \{1, 1\})) | ((1010, 20), (000000, 02)) |

Table 1. Examples of words, corresponding sequences of edge labels in \(T_{\alpha,\delta}\), and corresponding sequences of words. Note that the second word is a cyclic rotation of the first.

Lemma 6.4. Suppose \(w = u^k\) for some word \(u\). If \(\Phi(u) = ((x_2, y_2), \ldots, (x_m, y_m))\) encoded as multiplicity words as in Definition 6.2, then \(\Phi(w) = ((x_2^k, y_2^k), \ldots, (x_m^k, y_m^k))\).

Proof. The insertion triples needed to build \(w\) are the sequences of \(k\) shifted copies of the insertion triples needed to build \(u\). \(\square\)

Lemma 6.5. An element \(\tau \in C_g\) fixes \(w \in \tilde{W}_{\alpha,\delta}\) if and only if \(\tau\) fixes \(\Phi(w)\).

Proof. For \(\tau \in C_n\), let \(o(\tau)\) denote the order of \(\tau\). It is easy to see that \(\tau \in C_n\) fixes \(w \in W_n\) if and only if there is some word \(u\) such that \(w = u^{o(\tau)}\).

Suppose \(\tau \in C_g\) fixes \(w\), so that \(w = u^{o(\tau)}\). By Lemma 6.4,

\[
\Phi(w) = ((x_2^{o(\tau)}, y_2^{o(\tau)}), \ldots, (x_m^{o(\tau)}, y_m^{o(\tau)}))
\]

Each of the words \(x_i^{o(\tau)}\) and \(y_3^{o(\tau)}\) is fixed by \(\tau\), so \(\Phi(w)\) is fixed by \(\tau\). The reverse implication follows analogously using the fact that \(\Phi\) is a bijection. \(\square\)

6.2. Verifying Hypothesis (i) of Lemma 3.3 for \(W_{\alpha,\delta}\).

Theorem 6.6. Using Notation 5.9, \((W_{\alpha,\delta}, C_g, W_{\alpha,\delta}^{maj}(q))\) exhibits the CSP.

Proof. We use the notation and actions in Definition 6.1. Recall that

\[
S_\ell := \left(\left[0, n_{\ell-1} - k_{\ell-1} - 1\right], \frac{\left[0, k_{\ell-1} - 1\right]}{\delta_{\ell}}\right), \quad M_\ell := \left(\left[0, k_{\ell} - 1\right]\right) \left(\left[0, m_{\ell} - 1\right]\right).
\]
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From Theorem 2.4, for each \(2 \leq \ell \leq m\), \((S_\ell, C_g, (\binom{n_\ell-\ell}{\delta_\ell}^q)\) and \((M_\ell, C_g, (\binom{k_\ell}{\alpha_\ell-\delta_\ell}^q)_{q^{-1}})\) exhibit the CSP. Taking products,
\[
\prod_{\ell=2}^{m} S_\ell \times M_\ell, C_g, \prod_{\ell=2}^{m} \left( \binom{n_\ell-\ell}{\delta_\ell}^q \right) \left( \binom{k_\ell}{\alpha_\ell-\delta_\ell}^q \right)_{q^{-1}}
\]
exhibits the CSP. Comparing this to Theorem 5.17, we have
\[
\tilde{W}_{\alpha,\delta}^{\text{maj}} \equiv \prod_{\ell=2}^{m} \left( \binom{n_\ell-\ell}{\delta_\ell}^q \right) \left( \binom{k_\ell}{\alpha_\ell-\delta_\ell}^q \right)_{q^{-1}} \pmod{q^g - 1},
\]
modulo \(q^g - 1\), as \(\sum_{\ell=2}^{m} k_\ell \alpha_\ell \equiv 0 \pmod{g}\) because \(g | \alpha_\ell\) for all \(\ell\). Thus,
\[
\prod_{\ell=2}^{m} S_\ell \times M_\ell, C_g, \tilde{W}_{\alpha,\delta}^{\text{maj}}(q)
\]
exhibits the CSP.

By Lemma 5.15, for any \(w \in W_{\alpha,\delta}\),
\[
\#|w| = \frac{n}{\alpha_1} \cdot \#\left([w] \cap \tilde{W}_{\alpha,\delta}\right).
\]
Since \([w]\) is an orbit under \(C_n\), an element \(\tau \in C_n\) fixes \(w\) if and only if \(\tau\) fixes \([w]\) pointwise. Thus, for any \(\tau \in C_n\),
\[
\#W_{\alpha,\delta}^\tau = \frac{n}{\alpha_1} \cdot \#\tilde{W}_{\alpha,\delta}^\tau.
\]
Combining (38) and Lemma 6.5 now shows that for any \(\tau \in C_g\),
\[
\#W_{\alpha,\delta}^\tau = \frac{n}{\alpha_1} \cdot \#\left(\prod_{\ell=2}^{m} S_\ell \times M_\ell\right)^\tau.
\]
Hence, by (39), the CSP in (37), and (1),
\[
\left( W_{\alpha,\delta}, C_g, \frac{n}{\alpha_1} \tilde{W}_{\alpha,\delta}^{\text{maj}}(q) \right)
\]
exhibits the CSP. By (30), \(\frac{n}{\alpha_1} \tilde{W}_{\alpha,\delta}(q) \equiv W_{\alpha,\delta}^{\text{maj}}(q) \pmod{q^d - 1}\), hence also modulo \(q^g - 1\) since \(g \mid d\), completing the proof.

We have now finished the verification of the conditions in Lemma 3.3 for \(W_{\alpha,\delta}\). Condition (i) is Theorem 6.6, Condition (ii) is Proposition 5.21, and Condition (iii) is Lemma 4.3. This completes the proof of Theorem 1.5.

7. Refinements of Binomial CSP’s

A key step in the proof of Theorem 6.6 was Theorem 2.4 due to Reiner-Stanton-White, which says that the triples
\[
\left( \binom{[0,n-1]}{k}, C_n, \binom{n}{k}_q \right) \quad \text{and} \quad \left( \binom{[0,n-1]}{k}, C_n, \binom{n}{k}_q \right)
\]
exhibit the CSP. Indeed, [RSW04] contains two proofs, one via representation theory [RSW04, §3] and another by direct calculation [RSW04, §4]. In this section, we give two refinements of related CSP’s involving an action of \( C_d \) on sets of subsets (Theorem 7.11) and multisubsets (Theorem 7.4) for all \( d \mid n \), using shifted sum statistics. Our proof of the subset refinement, Theorem 7.11, does not use Theorem 2.4, so it can be used as an alternative proof of the subset case of Theorem 2.4. Our method is inspired by the rotation of subintervals used by Wagon and Wilf in [WW94, §3].

7.1. Cyclic Actions and Notation. We define two different cyclic actions of the cyclic group of order \( d \) on \([0, n - 1]\) and induce these actions to \( \binom{[0,n-1]}{k} \) and \( \binom{[0,n-1]}{k} \).

Notation 7.1. Fix \( n \in \mathbb{Z}_{\geq 1}, k \in \mathbb{Z}_{\geq 0}, \) and \( d \mid n \). Let

\[
S = \binom{[0,n-1]}{k}, \quad M = \binom{[0,n-1]}{k}
\]

For all \( j \in [1, \frac{n}{d}] \), let

\[
I_d^j := [(j - 1)d, jd - 1],
\]

which we call a \( d \)-interval. For any composition \( \alpha = (\alpha_1, \ldots, \alpha_{n/d}) \models k \) with \( n/d \) parts, let

\[
S_\alpha := \{ A \in S : \#(A \cap I_d^j) = \alpha_j \text{ for all } j \},
\]

\[
M_\alpha := \{ A \in M : \#(A \cap I_d^j) = \alpha_j \text{ for all } j \},
\]

where the intersection in (41) preserves the multiplicity of \( A \). We also fix cyclic groups \( C_d, C'_{d} \) of order \( d \) whose actions are described below.

Let \( C_d \) act on \([0, n - 1]\) by simultaneous rotation of \( d \)-intervals, which is generated by the permutation

\[
\sigma_d := (0 \ 1 \ \ldots \ (d - 1)) \ldots ((n - d) \ (n - d + 1) \ \ldots \ (n - 1))
\]

in cycle notation. On the other hand, \( C_n \) has a unique subgroup \( C'_{d} \) of order \( d \) which also acts on \([0, n - 1]\) and is generated by the permutation

\[
\sigma_{n/d}^n = \left( 0 \ \frac{n}{d} \ \ldots \ \frac{n}{d} \right) \ldots \left( \frac{n}{d} - 1 \ \frac{2n}{d} - 1 \ \ldots \ (n - 1) \right).
\]

Induce these actions of \( C_d \) and \( C'_{d} \) up to \( S \) and \( M \) by

\[
g \cdot \{ a_1, \ldots, a_k \} := \{ g \cdot a_1, \ldots, g \cdot a_k \}.
\]

Notice that the action of \( C_d \) restricts to \( S_\alpha \) and \( M_\alpha \) for any \( \alpha = (\alpha_1, \ldots, \alpha_{n/d}) \models k \).

Let \( (G, X) \) be a pair where \( G \) is a group acting on a set \( X \). A morphism of group actions \( (G, X) \to (G', X') \) is a pair \((\phi, \psi)\) where \( \phi : G \to G' \) is a group homomorphism and \( \psi : X \to X' \) is a map of sets which satisfy

\[
\psi(g \cdot x) = \phi(g) \cdot \psi(x) \text{ for all } g \in G, x \in X.
\]
Remark 7.2. The actions of $C_d$ and $C_d'$ on $[0, n-1]$ are isomorphic since $\sigma_d$ and $\sigma_{n/d}$ have the same cycle type. This isomorphism explicitly arises from $\phi: \sigma_d \mapsto \sigma_{n/d}$ with $\psi: 0 \mapsto 0$, $1 \mapsto \frac{n}{d}$, etc. Thus the actions of $C_d$ and $C_d'$ on $S$ and $M$ are isomorphic as well.

Recall the sum statistic sums the elements of a set of multiset. We also use the following shifted sum statistic. For $A \in S$, let

$$\text{sum}'(A) := \sum_{a \in A} a - \sum_{i=0}^{k-1} i = \text{sum}(A) - \binom{k}{2}.\tag{44}$$

Recall from (8) and (9) that

$$S^\text{sum}'(q) = \binom{n}{k}_q, \quad M^{\text{sum}}(q) = \binom{n}{k}_q.\tag{45}$$

Using (45), we may restate Theorem 2.4 as saying that $(S, C_n, S^{\text{sum}}(q))$ and $(M, C_n, M^{\text{sum}}(q))$ exhibit the CSP. Moreover, under the restricted action of $C'_d \subset C_n$ on $M$ and $S$,

$(S, C'_d, S^{\text{sum}}(q))$ and $(M, C'_d, M^{\text{sum}}(q))$ exhibit the CSP by Remark 2.3. By Remark 7.2,

$(S, C_d, S^{\text{sum}}(q))$ and $(M, C_d, M^{\text{sum}}(q))$ also exhibit the CSP.

Example 7.3. Let $n = 8$, $k = 4$, and $d = 4$. Abbreviating $\{0, 4, 5, 6\}$ as 0456, etc., gives

$$S_{(1,3)} = \{0456, 0457, 0467, 1456, 1457, 1467, 1567,
2456, 2457, 2467, 2567, 3456, 3457, 3467, 3567\}.$$

Here, $C_4$ acts on $[0, 8-1]$ by the permutation $(0123)(4567)$, and $C'_4$ acts by $(0246)(1357)$. $M_{(1,3)}$ contains $S_{(1,3)}$ in addition to, for instance, 0444.

7.2. A Multisubset Refinement. We next prove a refinement of the CSP triple $(M, C_d, M^{\text{sum}}(q))$ in (46) by fixing sizes of intersections with the $d$-intervals.

Theorem 7.4. Recall Notation 7.1, and fix a composition $\alpha = (\alpha_1, \ldots, \alpha_{n/d}) \vdash k$. Then, $(M_\alpha, C_d, M_\alpha^{\text{sum}}(q))$ refines the CSP triple $(M, C_d, M^{\text{sum}}(q))$.

Proof. Separating the $d$-intervals into different multisubsets gives

$$M_\alpha \cong \left(\binom{[0, d-1]}{\alpha_1}\right) \times \cdots \times \left(\binom{[0, d-1]}{\alpha_{n/d}}\right),\tag{47}$$

which preserves the natural $C_d$-action and sum statistic modulo $d$. Since

$$\left(\binom{[0, d-1]}{\alpha_j}, C_d, \binom{[0, d-1]}{\alpha_j}^{\text{sum}}(q)\right)$$

exhibits the CSP for all $j$, the result follows from Remark 2.3. \(\square\)

The following analogous result holds for subsets.
Proposition 7.5. Recall Notation 7.1, and fix a composition $\alpha = (\alpha_1, \ldots, \alpha_{n/d}) \vdash k$. Then $(S_\alpha, C_d, S_\alpha^{\text{sum}^*}(q))$ exhibits the CSP, where

$$\text{sum}^*(A) := \text{sum}(A) - \sum_{j=1}^{n/d} \frac{\alpha_j}{2}.$$  

(48)

Proof. Separating the $d$-intervals into different subsets gives

$$S_\alpha \cong \left(\frac{[0, d-1]}{\alpha_1}\right) \times \cdots \times \left(\frac{[0, d-1]}{\alpha_{n/d}}\right),$$

which preserves the $C_d$-action and sum statistic modulo $d$. Since

$$\left(\left(\frac{[0, d-1]}{\alpha_j}\right), C_d, \left(\frac{[0, d-1]}{\alpha_j}\right) \text{sum}^* - \frac{\alpha_j}{2}\right)(q)$$

exhibits the CSP for all $j$, $(S_\alpha, C_d, S_\alpha^{\text{sum}^*}(q))$ exhibits the CSP by Remark 2.3.

□

Remark 7.6. Since we must shift the sum statistic by different amounts depending on $\alpha$, Proposition 7.5 is not a CSP refinement, in contrast to Theorem 7.4.

7.3. A Subset Refinement. We next prove an honest refinement of the CSP triple $(S, C_d, S^{\text{sum}^*}(q))$ in (46). To do so, we restrict to certain subsets of $S$ for each divisibility chain ending in $n$. Our proof again inductively extends CSP’s up from cyclic subgroups of $C_d$ using Lemma 3.3. In this subsection we first define our restricted subsets and give some examples. We then present a series of lemmata verifying the conditions of Lemma 3.3 before proving our refinement, Theorem 7.11.

Definition 7.7. Suppose $e \mid d \mid n$. Let

$$G_{d,e} := \{A \in S : \gcd(d, \#(A \cap I_1^1), \#(A \cap I_2^2), \ldots, \#(A \cap I_{n/d}^n)) = e\}.$$  

(50)

We have $G_{n,\gcd(n,k)} = S$ and $G_{n,e} = \emptyset$ for all other $e$.

Remark 7.8. Note that $G_{d,e}$ decomposes as the disjoint union

$$G_{d,e} = \coprod S_\alpha,$$

ranging over all $\alpha = (\alpha_1, \ldots, \alpha_{n/d}) \vdash k$ satisfying

$$\gcd(d, \alpha_1, \ldots, \alpha_{n/d}) = e.$$  

Example 7.9. If $n = 4, k = 2$, then abbreviating $\{0, 2\}$ as 02, etc., gives

$$G_{1,1} = \{01, 02, 03, 12, 13, 23\} = S,$$

$$G_{2,1} = \{02, 03, 12, 13\}, \quad G_{2,2} = \{01, 23\},$$

$$G_{4,1} = \emptyset, \quad G_{4,2} = \{01, 02, 03, 12, 13, 23\} = S, \quad G_{4,4} = \emptyset.$$  

Consequently, $G_{4,2} \cap G_{2,1} = \{02, 03, 12, 13\}$ and $G_{4,2} \cap G_{2,2} = \{01, 23\}$. 


Definition 7.10. Suppose $D$ is a totally ordered chain in the divisibility lattice ending with $\gcd(n, k) \mid n$, i.e. $D = d_p \mid d_{p-1} \mid \cdots \mid d_0 \mid n$ where $d_0 := \gcd(n, k)$. Write
\[ G_D := G_{n, d_0} \cap G_{d_0, d_1} \cap \cdots \cap G_{d_{p-1}, d_p} \subset S. \]

We may now state our subset refinement. The proof is postponed to the end of this subsection.

Theorem 7.11. Using Notation 7.1, let $D$ be a totally ordered chain in the divisibility lattice ending with $\gcd(n, k) \mid n$ and starting with $e \mid d$. Then, $(G_D, C_d, G_{d, \text{sum}'}(q))$ refines the CSP triple $(S, C_d, S_{\text{sum}'}(q))$.

Example 7.12. If $n = 4$, $k = 2$, and $D = 1 \mid 2 \mid 4$, then $G = G_{4,2} \cap G_{2,1}$ has $C_2$ orbits $\{02, 13\}$ and $\{03, 12\}$. Moreover,
\[ G_{\text{sum}'}(q) = q^1 + 2q^2 + q^3 \equiv 2(q^0 + q^1) \pmod{q^2 - 1}, \]
so $(G, C_2, G_{\text{sum}'}(q))$ exhibits the CSP by (5).

In fact, the subset case of Theorem 2.4 is the special case $D = \gcd(n, k) \mid n$ of Theorem 7.11, so the proof below of Theorem 7.11 yields an alternative proof of the subset case of Theorem 2.4.

Corollary 7.13. $(S, C_n, S_{\text{sum}'}(q))$ exhibits the CSP.

Lemma 7.14. Let $D$ be a totally ordered chain in the divisibility lattice ending with $\gcd(n, k) \mid n$ and beginning with $e \mid d$. Suppose $C'_e$ is the unique subgroup of $C_d$ of order $e$.

(i) $G_D = \coprod S_\alpha$, where the disjoint union is over some set of $\alpha$ satisfying $\alpha = \alpha_1, \ldots, \alpha_{n/d} \vdash k$ and $\gcd(d, \alpha_1, \ldots, \alpha_{n/d}) = e$.

(ii) $G_D$ is closed under the $C_d$ and $C_e$-actions on $S$.

(iii) The $C_e$ and $C'_e$-actions on $G_D$ are isomorphic.

(iv) For any $C_d$-orbit $O$ of $G_D$, we have $\frac{d}{|O|} \mid e$.

(v) The sum’ statistic has period $e$ modulo $d$ on $G_D$.

Proof. For (i), by (51) we have $G_{d,e} = \coprod S_\alpha$ where $\alpha$ ranges over all compositions satisfying the constraints in (i). Similarly, if $e \mid b$, then $G_{b,e} = \coprod S_\beta$ where in particular $\beta = (\beta_1, \ldots, \beta_{n/b}) \vdash k$. Now if $d \mid c$, we may break up each $b$-interval into $b/d$ $d$-intervals. It is then easy to see that
\[ S_\alpha \cap S_\beta = \emptyset \text{ or } S_\alpha. \]

Now (i) follows inductively.

For (ii), by (i) it suffices to show that each $S_\alpha$ is closed under the $C_d$ and $C'_e$-actions. Since $\sigma_d$ rotates $d$-intervals, it preserves the size of each $d$-interval, so $\sigma_d$ indeed maps $S_\alpha$ to itself. The same argument applies with $\sigma_e$ in place of $\sigma_d$.

For (iii), by (i), it suffices to show the $C_e$ and $C'_e$-actions on $S_\alpha$ are isomorphic. Recalling (49), we have
\[ S_\alpha \cong \left( \left[0, d-1\right] \atop \alpha_1 \right) \times \cdots \times \left[0, d-1\right] \atop \alpha_{n/d}. \]
By Remark 7.2, the actions of $C_e$ and $C'_e$ on $\left(\frac{[0,d-1]}{\alpha_j}\right)$ are isomorphic for each $j$, so their actions on $S_\alpha$ are isomorphic as well.

For (iv), pick $A \in \mathcal{O}$ with $A \in S_\alpha$ for $\alpha$ as in (i). Let $A_j := A \cap I^j_d$, which has $\alpha_j$ elements. Viewing $A_j$ as a multiplicity word $w_j$ as in Definition 6.2, we see that $A_j$ has $d - \alpha_j$ zeros and $\alpha_j$ ones. For all $j$, $w_j$ is some word repeated $\frac{d}{\|O\|}$ times. Using the two-letter case of Lemma 4.3, we have $\frac{d}{\|O\|} | \text{freq}(w_j) | \alpha_j$. Thus $\frac{d}{\|O\|} | \gcd(d, \alpha_1, \ldots, \alpha_{n/d}) = e$.

For (v), it suffices to show that sum' has period $e$ modulo $d$ on $S_\alpha$ for $\alpha$ as in (i). By the gcd condition, there exist $c_1, \ldots, c_{n/d} \in \mathbb{Z}$ such that

$$c_1 \alpha_1 + \cdots + c_{n/d} \alpha_{n/d} \equiv e \pmod{d}.$$ 

For some particular $A \in S_\alpha$, consider cyclically rotating the elements of $A \cap I^j_d$ forward by $c_j$ in $I^j_d$ for all $j$. The result is a bijection $\phi: S_\alpha \rightarrow S_\alpha$ such that for all $A \in S_\alpha$,

$$\text{sum}'(\phi(A)) \equiv \text{sum}'(A) + e \pmod{d}.$$ 

Hence sum' indeed has period $e$ modulo $d$ on $S_\alpha$. \hfill \Box

Example 7.15. Let $n = 12, k = 8$, and $D = 1 \mid 2 \mid 4 \mid 12$. Then

$$G_D = G_{12,4} \cap G_{4,2} \cap G_{2,1} = G_{4,2} \cap G_{2,1}.$$ 

We have $G_{2,1} = \coprod S_\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_6) \vdash 8$ and $\gcd(2, \alpha_1, \ldots, \alpha_6) = 1$. Similarly $G_{4,2} = \coprod S_\beta$ where $\beta = (\beta_1, \beta_2, \beta_3) \vdash 8$ and $\gcd(4, \beta_1, \beta_2, \beta_3) = 2$. In fact,

$$\emptyset \subsetneq G_D \subsetneq G_{2,1}$$ 

since, for instance, $S_\alpha \subset G_D$ when $\alpha = (4, 0, 1, 1, 1, 1)$ while $S_\alpha \subset G_{2,1} - G_D$ when $\alpha = (2, 0, 2, 1, 2, 1)$.

Lemma 7.16. Let $C_e$ act on $S$ by simultaneous rotations of $e$-intervals, and $C'_e$ be unique subgroup of $C_d$ of size $e$. Then the actions of $C_e$ and $C'_e$ on $G_D$ are isomorphic.

Proof. In fact, we will show $C_e$ and $C'_e$ have isomorphic actions on $S_\alpha$ for any $\alpha = (\alpha_1, \ldots, \alpha_{n/d}) \vdash k$, which gives the result by (51). Recalling (49), we have

$$S_\alpha \cong \left(\frac{[0,d-1]}{\alpha_1}\right) \times \cdots \times \left(\frac{[0,d-1]}{\alpha_{n/d}}\right).$$ 

By Remark 7.2, the actions of $C_e$ and $C'_e$ on $\left(\frac{[0,d-1]}{\alpha_j}\right)$ are isomorphic for each $j$, so their actions on $S_\alpha$ are isomorphic as well. \hfill \Box

Lemma 7.17. Let $D$ be a totally ordered chain in the divisibility lattice ending with $\gcd(n,k) \mid n$ and beginning with $e \mid d$. Then $G_D$ is closed under the $C_d$-action on $S$. Moreover, for any orbit $O$ of $G_D$, we have $\frac{d}{\|O\|} \mid e$.

Proof. For the stability of $G_D$ under the $C_d$-action, pick $A \in G_D$. Since $\sigma_d$ rotates $d$-intervals, $\sigma_d$ preserves the size of each $c$-interval of $A$ for all $d \mid c$. Hence $C_d$ preserves all gcd conditions defining $G_D$. 

\hfill \Box
In particular, for each orbit $O$ of $G_D$, we have $O \subset S_\alpha$ for some $\alpha = (\alpha_1, \ldots, \alpha_{n/d})$ satisfying $\gcd(d, \alpha_1, \ldots, \alpha_{n/d}) = e$. Pick some $A \in O$. Let $A_j := A \cap I_d^j$, which has $a_j$ elements. Viewing $A_j$ as a multiplicity word $w_{A_j}$ as in Definition 6.2, we see that $A_j$ has $d - \alpha_j$ zeros and $\alpha_j$ ones. For all $j$, $w_{A_j}$ is some word repeated $\frac{d}{|O|}$ times, so $\frac{d}{|O|} \mid \text{freq}(w_{A_j}) \mid \alpha_j$, using the two-letter case of Lemma 4.3. Hence, $\frac{d}{|O|} \mid \gcd(d, \alpha_1, \ldots, \alpha_{n/d}) = e$. \hfill \Box

**Lemma 7.18.** Let $e \mid d \mid n$. Then the sum$'$ statistic has period $e$ modulo $d$ on $S_\alpha$ for all $\alpha = (\alpha_1, \ldots, \alpha_{n/d})$ satisfying $\gcd(d, \alpha_1, \ldots, \alpha_{n/d}) = e$. Furthermore, if $D$ is a totally ordered chain beginning with $e \mid d$ and ending with $\gcd(n, k) \mid n$, then sum$'$ has period $e$ modulo $d$ on $G_D$.

**Proof.** For the first claim, by the gcd condition, there exist $c_1, \ldots, c_{n/d} \in \mathbb{Z}$ such that

$$c_1\alpha_1 + \cdots + c_{n/d}\alpha_{n/d} \equiv e \pmod{d}.$$ 

For some particular $A \in S_\alpha$, consider cyclically rotating the elements of $A \cap I_d^j$ forward by $c_j$ in $I_d^j$ for all $j$. The result is a bijection $\phi : S_\alpha \to S_\alpha$ such that for all $A \in S_\alpha$,

$$\text{sum}'(\phi(A)) \equiv \text{sum}'(A) + e \pmod{d}.$$ 

It follows that sum$'$ has period $e$ modulo $d$ on $S_\alpha$. Since $\phi$ preserves the size of each $d$-interval, $\phi$ also preserves the size of each $c$-interval for $d \mid c$. The second claim now follows from (51). \hfill \Box

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**Lemma 7.19.** Let $d \mid n$. The $C_d$ action on $G_{d,d}$ is trivial and $(G_{d,d}, C_d, G_{d,d}^{\text{sum}'}(q))$ exhibits the CSP.

**Proof.** All subsets in $G_{d,d}$ have each $d$-interval either full or empty, so $C_d$ fixes every $A \in G_{d,d}$. By (5), $(G_{d,d}, C_d, G_{d,d}^{\text{sum}'}(q))$ thus exhibits the CSP if and only if $G_{d,d}^{\text{sum}'}(q) \equiv \#G_{d,d} \bmod (q^d - 1)$. If $G_{d,d} = \emptyset$ the result is trivial, so take $G_{d,d} \neq \emptyset$. For any $A \in G_{d,d}$, since each $d$-interval is full or empty, we have $d \mid k$ and

$$\text{sum}'(A) \equiv \frac{k(d)}{2} - \frac{k(d-k)}{2} \equiv 0 \pmod{d}.$$ 

\hfill \Box

We may now prove Theorem 7.11.

**Proof of Theorem 7.11.** We induct on $d$. If $d = 1$, then $(G_D, C_1, G_D^{\text{sum}'}(q))$ exhibits the CSP trivially. For the induction step, we first claim that $(G_D, C_e, G_D^{\text{sum}'}(q))$ exhibits the CSP. If $e = d$, then $G_D \subset G_{d,d}$, so by Lemma 7.19 the $C_e$ action is trivial. It is easy to see that CSP’s with trivial actions refine to arbitrary subsets,
so \((G_D, C_e, G^\text{sum'}_D(q))\) exhibits the CSP in this case. If \(e < d\), by conditioning on the sizes of the intersections of the \(e\)-intervals, we can write
\[
G_D = \prod_{f \mid e} G_{f|D}
\]
where \(f \mid D\) denotes the chain with \(f\) prepended to \(D\). Hence \((G_{f|D}, C_e, G^\text{sum'}_{f|D}(q))\) exhibits the CSP by induction for each \(f \mid e\), since \(f \mid D\) begins with \(f \mid e\). Thus \((G_D, C_e, G^\text{sum'}_{D}(q))\) exhibits the CSP by (54), proving the claim.

In order to realize the \((G_D, C_d, G^\text{sum'}_{D}(q))\) CSP triple from the \((G_D, C_e, G^\text{sum'}_{D}(q))\) CSP triple, we verify the conditions of Lemma 3.3. From Lemma 7.14(ii), the restriction of the \(C_d\)-action on \(G_D\) to the subgroup \(C'_e \subset C_d\) of size \(e\) is isomorphic to the \(C_e\)-action on \(G_D\), giving Condition (i). Condition (ii) is Lemma 7.14(v), and Condition (iii) is Lemma 7.14(iv). Thus \((G_D, C_d, G^\text{sum'}_{D}(q))\) exhibits the CSP by Lemma 3.3.

\section{The Flex Statistic}

We conclude by formalizing the notion of universal sieving statistics and giving an example, flex, in the context of words. We end with an open problem.

**Definition 8.1.** Given a finite set \(W\) with a \(C_n\)-action, we say \(\text{stat}: W \to \mathbb{Z}_{\geq 0}\) is a universal CSP statistic for \((W, C_n)\) if \((O, C_n, \text{stat}^O(q))\) exhibits the CSP for all \(C_n\)-orbits \(O\) of \(W\).

**Definition 8.2.** Let \(\text{lex}(w)\) denote the index at which \(w\) appears when lexicographically ordering the necklace \([w]\), starting from 0. Let \(\text{flex}\) be the product
\[
\text{flex}(w) := \text{freq}(w) \cdot \text{lex}(w).
\]
For example, the necklace in Example 2.2 has lex statistics 0, 3, 2, 1, respectively, so that \(\text{lex}(55315531) = 3\) and \(\text{flex}(55315531) = 2 \cdot 3 = 6\).

**Lemma 8.3.** The function \(\text{flex}\) is a universal CSP statistic for \((W_n, C_n)\).

**Proof.** Let \(N\) be any necklace of length \(n\) words. Since \(\text{freq}(N) = \frac{n}{|N|}\) and \(\text{lex}(N) = \{0, 1, \ldots, |N| - 1\}\), we have
\[
N^{\text{flex}}(q) = \sum_{j=0}^{|N|-1} q^j |N|^{-1} = \frac{q^n - 1}{q^{n/|N|} - 1},
\]
so \((N, C_n, N^{\text{flex}}(q))\) exhibits the CSP by (5). \(\square\)

Given a universal sieving statistic \(\text{stat}\) on some set \(W\), \(\text{stat}\) takes on the values \(\{0, n/d, \ldots, n - n/d\}\) modulo \(n\) on any orbit of size \(d\). The converse holds as well. In this sense, up to shifting values by \(n\), universal sieving statistics are equivalent to total orderings on each orbit \(O\) of \(W\).

Standing in contrast to Lemma 8.3, \((N, C_n, N^{\text{maj}}(q))\) does not exhibit the CSP when \(N = [123123]\), so \(\text{maj}\) is not a universal CSP statistic on \((W_n, C_n)\). However, \(\text{maj}\) trivially refines to the orbit \(N = \{1^n\}\) for any \(n\). Since refinement is not

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[This text likely contains a mathematical or computational problem, with references to lemmas and definitions that are part of a larger body of work. The context suggests it is from a research paper in mathematics or computer science, possibly related to computational complexity or symmetry groups.]
generally closed under intersections, it is not clear if there is any useful sense in which \( \text{maj} \) on words can be “maximally refined.”

It follows from Lemma 8.3 and (5) that Theorem 1.5 is equivalent to the following.

**Theorem 8.4.** The statistics \( \text{flex} \) and \( \text{maj} \) are equidistributed modulo \( n \) on \( W_{\alpha,\delta} \).

Indeed, we were originally led to Theorem 1.5 through an exploration of the irreducible multiplicities of the so-called higher Lie modules (see e.g. [Sch03]), which uncovered the fact that \( \text{flex} \) and \( \text{maj} \) are equidistributed modulo \( n \) on \( W_\alpha \). Data exploration led us to conjecture this equidistribution refined to fixed cyclic descent type as in Theorem 8.4. These connections will be explained in a future publication. They also naturally suggest the problem of finding explicit bijections proving Theorem 8.4, which we leave as an open problem.

**Open Problem 8.5.** For \( \alpha \models n \) and \( \delta \) any weak composition, find a bijection \( \varphi : W_{\alpha,\delta} \rightarrow W_{\alpha,\delta} \) satisfying

\[
\text{maj}(\varphi(w)) \equiv \text{flex}(w) \pmod{n}.
\]

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