Boosts in an arbitrary direction and maximal causal velocities in a deformed Minkowski space

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October 30, 2018

Abstract

We discuss boosts in a deformed Minkowski space, i.e. a four-dimensional space-time with metric coefficients depending on non-metric coordinates (in particular on the energy). The general form of a boost in an arbitrary direction is derived in the case of space anisotropy. Two maximal 3-vector velocities are mathematically possible, an isotropic and an anisotropic one. However, only the anisotropic
velocity has physical meaning, being invariant indeed under deformed boosts.

1 Introduction

In the last years, two of the present authors (F.C. and R.M.) proposed a generalization of Standard Special Relativity (SR) based on a “deformation” of space-time, assumed to be endowed with a metric whose coefficients depend on the energy of the process considered [1]. Such a formalism (Deformed Special Relativity, DSR) applies in principle to all four interactions (electromagnetic, weak, strong and gravitational) - at least as far as their nonlocal behavior and nonpotential part is concerned - and provides a metric representation of them (at least for the process and in the energy range considered) ([1]-[5]). Moreover, it was shown that such a formalism is actually a five-dimensional one, in the sense that the deformed Minkowski space is embedded in a larger Riemannian manifold, with energy as fifth dimension [6].

In particular, the explicit expression of the Lorentz boost in the deformed Minkowski space $\tilde{M}_4$ for velocity along one of the coordinate axes was derived in Ref.s [1] in the simpler case of an isotropic space. In this paper, following the line of mathematical-formal research started with [7], we want to derive the form of the deformed boost for velocity in an arbitrary direction, and in the general case of spatial anisotropy. Such a problem is far from being trivial, on account of the intrinsic anisotropic nature of $\tilde{M}_4$. Moreover, as we shall see, this will allow us to further clarify the meaning of the maximal causal velocities in the deformed space-time.

The organization of the paper is as follows. In Subsect. 2.1 we briefly introduce the concept of deformed Minkowski space. Subsect. 2.2 deals with the problem of the maximal velocity in $\tilde{M}_4$, namely the equivalent of the light speed in SR. It is shown that two different mathematical procedures lead to two different maximal velocities, an isotropic and an anisotropic one.
Subsect.s 3.1 and 3.2 deal, respectively, with a boost along one of the coordinate axis and in an arbitrary direction, in the general spatially anisotropic case. The expression of the deformed boosts symmetric in space and time coordinates is given in Subsect. 2.3. We derive in Subsect. 2.4 the generalized velocity composition law, which allows us to state that only the anisotropic velocity is actually invariant under deformed boosts, and has therefore a true physical meaning.

2 Deformed Special Relativity in four dimensions (DSR4)

2.1 Deformed Minkowski space-time

The generalized (“deformed”) Minkowski space \( \tilde{M}_4 \) (DMS4) is defined as a space with the same local coordinates \( x \) of \( M_4 \) (the four-vectors of the usual Minkowski space), but with metric given by the metric tensor

\[
\eta_{\mu\nu}(x^5) = \text{diag}(b_0^2(x^5), -b_1^2(x^5), -b_2^2(x^5), -b_3^2(x^5)) =
\]

where the \( \{b_\mu^2(x^5)\} \) are dimensionless, real, positive functions of the independent, non-metrical (n.m.) variable \( x^5 \). The generalized interval in \( \tilde{M}_4 \) is

\[
\text{ESC off} \delta_{\mu\nu} \left[ b_0^2(x^5)\delta_{\mu0} - b_1^2(x^5)\delta_{\mu1} - b_2^2(x^5)\delta_{\mu2} - b_3^2(x^5)\delta_{\mu3} \right]
\]

(1)

where the \( \{b_\mu^2(x^5)\} \) are dimensionless, real, positive functions of the independent, non-metrical (n.m.) variable \( x^5 \). The generalized interval in \( \tilde{M}_4 \) is

\[1\]In the following, we shall employ the notation “ESC on” ("ESC off") to mean that the Einstein sum convention on repeated indices is (is not) used.

\[2\]Such a coordinate is to be interpreted as the energy \( E \) (see Ref.s [1]-[5]). However, since the metric coefficients \( b_\mu^2(x^5) \) are dimensionless, their dependence on \( E \) is actually of the type

\[ b_\mu^2 \left( \frac{E}{E_0} \right), \]

where \( E_0 \) plays the role of a threshold energy, characteristic of the interaction considered (as before, see Ref.s [1]-[5]). Moreover, the index 5 explicitly refers to the above-mentioned fact that the deformed Minkowski space can be "naturally" embedded in a five-dimensional (Riemannian) space [6]. In this last framework, it is worth to give \( x^5 \) the dimension of a length.
therefore given by \((x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)\), with \(c\) being the usual light speed in vacuum\) (ESC on)

\[
ds^2 = b_0^2 c^2 (dt)^2 - [b_1^2 (dx)^2 + b_2^2 (dy)^2 + b_3^2 (dz)^2] = \eta_{\mu\nu}dx^\mu dx^\nu = dx * dx.
\]

(2)

The last step in (2) defines the scalar product \(*\) in the deformed Minkowski space \(\tilde{M}_4\). In order to emphasize the dependence of DMS4 on the variable \(x^5\), we shall sometimes use the notation \(\tilde{M}_4(x^5)\). It follows immediately that it can be regarded as a particular case of a Riemann space with null curvature.

Let us stress that metric (1) is supposed to hold at a local (and not global) scale. We shall therefore refer to it as a “topical” deformed metric, because it is supposed to be valid not everywhere, but only in a suitable (local) space-time region (characteristic of both the system and the interaction considered).

The two basic postulates of DSR4 (which generalize those of standard SR) are [1]:

1- Space-time properties: Space-time is homogeneous, but space is not necessarily isotropic; a reference frame in which space-time is endowed with such properties is called a ”topical” inertial reference frame (TIRF). Two TIRF’s are in general moving uniformly with respect to each other (i.e., as in SR, they are connected by a ”inertiality” relation, which defines an equivalence class of \(\infty^3\) TIRF);

2- Generalized Principle of Relativity (or Principle of Metric Invariance): All physical measurements within each TIRF must be carried out via the same metric.

The metric (1) is just a possible realization of the above postulates. We refer the reader to Ref.s [1]-[5] for the explicit expressions of the phenomenological energy-dependent metrics for the four fundamental interactions.

In \(\tilde{M}_4\), it is possible a priori consider two scalar products between trivectors \(\tilde{v}_1, \tilde{v}_2\): the standard, Euclidean one \(\cdot\), defined by means of the metric tensor \(g_{ik} = \delta_{ik}\) (here and in the following small Latin indices will range in \(\{1, 2, 3\}\)), and the deformed one, induced by the deformed scalar product \(*\) in \(\tilde{M}_4\), and defined by means of the metric tensor \(-\eta_{ik}(x^5)\) \(\equiv\) \(b_i^2(x^5)\delta_{ik}\) (where the sign \(\cdot\) is obviously introduced in order to get a positive trivector
norm) as follows:

\[ v_1 \ast v_2 \equiv -\sum_{i=1}^{3} \eta_{ij}(x^5)(v_1)^i(v_2)^j = \sum_{i=1}^{3} b_i^2(x^5) \delta_{ij}(v_1)^i(v_2)^j = \]

\[ = b_1^2(x^5)(v_1)^1(v_2)^1 + b_2^2(x^5)(v_1)^2(v_2)^2 + b_3^2(x^5)(v_1)^3(v_2)^3 \quad (3) \]

Notice that, in the case of space isotropy \(( b_i(x^5) = b(x^5), \forall i = 1, 2, 3 )\), the metric tensor reads:

\[ \eta_{\mu \nu,iso}(x^5) = \text{diag}(b_0^2(x^5), -b_1^2(x^5), -b_2^2(x^5), -b_3^2(x^5)) = \]

\[ = \delta_{\mu \nu} [\delta_{\mu 0} b_0^2(x^5) - (\delta_{\mu 1} + \delta_{\mu 2} + \delta_{\mu 3}) b_i^2(x^5)] \quad (4) \]

It is easily seen that the two (space) scalar products \( \cdot \) and \( \ast_{iso} \) are proportional:

\[ a \cdot d = \sum_{i=1}^{3} a^i d^i = \frac{1}{b^2(x^5)} \sum_{i=1}^{3} b^2(x^5) a^i d^i = \frac{1}{b^2(x^5)} a \ast_{iso} d \quad (5) \]

### 2.2 The trivector ”maximal causal velocity”(m.c.v.) in SR and in DSR4

As is well known, the maximal causal speed in \( M_4 \) is obtained by putting \( ds^2 = 0 \), whence

\[ ds^2 = 0 \Leftrightarrow c^2 dt^2 - dx^2 - dy^2 - dz^2 = 0 \Leftrightarrow \frac{dx^2 + dy^2 + dz^2}{dt^2} = c^2 \quad (6) \]

Then one interprets \( c \) as the maximal causal speed along any direction of the (Euclidean) space \( R^3 \) (embedded in the pseudoeuclidean Minkowski space-time \( M_4 \)). Such an interpretation is obviously based on the physical fact that \( c \) coincides with the light speed in vacuum, and on the isotropy of
$R^3$. Therefore $c$ represents the value of any of the three components of the vector m.c.v. of SR, $\mathbf{u}_{SR}$, namely:

$$\mathbf{u}_{SR} = c(\vec{x}, \vec{y}, \vec{z}) \quad (7)$$

Then, $c^2$ is not, in general, a square modulus, but the square of any component of $\mathbf{u}_{SR}$, whose square modulus (obviously with respect to the Euclidean scalar product $\cdot$), is instead:

$$|\mathbf{u}_{SR}|^2 \equiv \sum_{i=1}^{3} (u_{SR}^i)^2 = 3c^2 \quad (8)$$

so that

$$u_{SR}^i = \frac{1}{\sqrt{3}} |\mathbf{u}_{SR}| \quad \forall i = 1, 2, 3 \quad (9)$$

The above procedure must be suitably modified in the DSR4 case, due to the space anisotropy of $\widehat{M}_4$.

Actually, in order to sort out a single component of the 3-vector m.c.v., in a general 4-d special-relativistic theory (characterized by a diagonal metric tensor $g_{\mu\nu}(\{x\}_{n.m.})$, where $\{x\}_{n.m.}$ is a set of non-metrical variables), one has to exploit a "directional separation" (or "dimensional separation") method, which consists of the following three-step recipe (ESC off throughout):

1- Set $ds^2$ equal to zero:

$$ds^2 = 0 \Leftrightarrow g_{00}(\{x\}_{n.m.})c^2 dt^2 + \sum_{i=1}^{3} g_{ii}(\{x\}_{n.m.})(dx^i)^2 = 0 \quad (10)$$

$\Rightarrow$ Actually, it can be shown that the existence of an invariant, real quantity, having the dimensions of the square of a speed follows from the Principle of Relativity and the properties of homogeneity and isotropy of space-time. The value of such a speed must be experimentally determined in the framework of the total class $C_T$ of the physical phenomena considered (for instance, it is obviously $\infty$ for Galilei’s relativity, when only mechanics is considered, and $c$ for Einstein’s relativity, when also electromagnetism is taken into account). See Ref.s [1], and references therein.
2- In order to find the i-th component \( u^i(\{x\}_{n.m.}) \) of the m.c.v., put \( dx^j = 0 \) (\( j \neq i \)), thus getting

\[
g_{00}(\{x\}_{n.m.})c^2 dt^2 + g_{ii}(\{x\}_{n.m.})(dx^i)^2 = 0
\]

(11)

3- Evidence on the lhs of (11) a quantity with physical dimensions \([\text{space}] = [\text{velocity}]\); at this point, we have two different subcases:

I) One carries to the lhs of (11) \( \frac{dx^i}{dt} \) (which amounts to consider the 3-d Euclidean product \( \cdot \)), thus getting an anisotropic m.c.v.:

\[
u^i(\{x\}_{n.m.}) \equiv \frac{dx^i}{dt} = \left( \frac{g_{00}(\{x\}_{n.m.})}{(-g_{ii}(\{x\}_{n.m.}))} \right)^{1/2} c \quad \forall i = 1, 2, 3.
\]

(12)

II) One carries to the lhs of (11) \( (-g_{ii}(\{x\}_{n.m.}))^{1/2} \frac{dx^i}{dt} \) (which amounts to consider the 3-d deformed product \( \ast \) defined by \( -g_{ij}(\{x\}_{n.m.}) = \delta_{ij} |g_{ii}(\{x\}_{n.m.})| \)), thus getting an isotropic m.c.v.:

\[
u^i(\{x\}_{n.m.}) \equiv (-g_{ii}(\{x\}_{n.m.}))^{1/2} \frac{dx^i}{dt} = (g_{00}(\{x\}_{n.m.}))^{1/2} c \quad \forall i = 1, 2, 3.
\]

(13)

The two subcases I and II differ essentially by the different way of implementing the space anisotropy. In the former case, the anisotropy is embedded in the definition of m.c.v.; in the latter one, in the scalar product\(^4\)\(^5\).

\(^4\) Of course, the procedure of “directional separation” gives (in either subcase) the same standard result when applied to SR. In fact:

\[
u^i_{SR} = (-g_{ii})^{1/2} \frac{dx^i}{dt} = (g_{00})^{1/2} c = \frac{dx^i}{dt} = \frac{(g_{00})^{1/2}}{(-g_{ii})^{1/2}} c = c \quad \forall i = 1, 2, 3
\]

\(^5\) Let us notice that the directionally separating procedure can be consistently applied only to (special- or general-relativistic) metrics which are fully diagonal. This is obviously due to the mixings between different space directions which arise in the case of non-diagonal metrics.
Specializing the above equations to the DSR4 framework, we get therefore, in the two subcases:

I)

\[ u_{DSR4,I}^i(x^5) \equiv u^i(x^5) = c \frac{b_0(x^5)}{b_i(x^5)}, \tag{14} \]

\[ |u_{DSR4,I}(x^5)| = \left( \sum_{i=1}^{3} (u_{DSR4,I}^i(x^5))^2 \right)^{1/2} = \]

\[ = cb_0(x^5) \left( \frac{1}{b_1^2(x^5)} + \frac{1}{b_2^2(x^5)} + \frac{1}{b_3^2(x^5)} \right)^{1/2} \tag{15} \]

The vector \( u \) is the (spatially) anisotropic generalization of the maximal causal speed introduced in the (spatially) isotropic case [1];

II)

\[ u_{DSR4,II}^i(x^5) \equiv w^i(x^5) = cb_0(x^5), \tag{16} \]

\[ |u_{DSR4,II}(x^5)| = \left( \sum_{i=1}^{3} b_i^2(x^5) (u_{DSR4,II}^i(x^5))^2 \right)^{1/2} = cb_0(x^5) \left( b_1^2(x^5) + b_2^2(x^5) + b_3^2(x^5) \right)^{1/2} \tag{17} \]

whence

\[ u_{DSR4,II}^i(x^5) = \left( b_1^2(x^5) + b_2^2(x^5) + b_3^2(x^5) \right)^{-1/2} \left| u_{DSR4,II}(x^5) \right|, \tag{18} \]

i.e in this subcase (unlike the previous one, see Eqs (14),(15)) one can state a proportionality relation by an overall factor (even if dependent on the metric coefficients) between \( u_{DSR4,II}^i(x^5) \) and \( |u_{DSR4,II}(x^5)| \).

We have therefore shown that the two different procedures of directional separation lead to two different mathematical definitions of maximal velocity, an isotropic (\( w \)) and an anisotropic (\( u \)) one. The choice between them must be done on a physical basis (see Subsect. 3.4).
3 Boosts in DSR4

3.1 Boost direction along $\hat{x}^i$ ($i = 1, 2, 3$)

It follows from the principles 1), 2) of DSR4 that the transformations among TIRF’s, called ”Deformed Lorentz transformations” (DLT), are those (homogeneous) coordinate transformations which leave the metric tensor $\eta_{\mu\nu}$ invariant. Since the deformed Minkowski space is a special case of a Riemann space, we can state that the DLT’s are the space-time rotational component of the (maximal) Killing group of $\tilde{M}_4(x^5)$. Then, physical laws are to be covariant with respect to such generalized transformations.

By following a procedure analogous to that used in SR to derive LT’s (see Appendix A), we get, for a boost along $\hat{x}^i$, $\forall i = 1, 2, 3$ (ESC off throughout):

\[
\begin{align*}
(x')^i &= (x^i)', \quad \gamma(x^i - v^i t) = \gamma \left( x^i - \tilde{\beta} b_0 (x^5) \right) \frac{ct}{b_i (x^5)} \\
(x')^{k \neq i} &= (x^{k \neq i})', \quad t' = \gamma \left( t - \frac{v^i b_i (x^5)}{c^2 b_0 (x^5)} x^i \right) = \tilde{\gamma} \left( t - \tilde{\beta}^2 \frac{v^i}{v^i} x^i \right)
\end{align*}
\]

(19)

where:

\[
\tilde{\beta} = \tilde{\beta}^i = \frac{v^i b_i (x^5)}{c b_0 (x^5)} ; \quad (20)
\]

\[
\tilde{\gamma} = \left[ 1 - (\tilde{\beta}^i)^2 \right]^{-1/2} = \left[ 1 - \left( \frac{v^i b_i (x^5)}{c b_0 (x^5)} \right)^2 \right]^{-1/2} \quad (21)
\]

Of course, in the non-relativistic limit $\lim_{c \to \infty}$, $\tilde{\beta} \to 0^+$ and $\tilde{\gamma} \to 1^+$, so that the deformed boosts reduce to the Galilean transformations.
3.2 Boost in a generic direction

In this case, the relative velocity is $\mathbf{v} = v^1\hat{x} + v^2\hat{y} + v^3\hat{z}$, and we have to suitably generalize definitions (20), (21) as follows:\(^6\)

$\tilde{\beta} \equiv \left( \frac{\mathbf{v}}{\mathbf{u}} \right) = \left( \frac{v^1b_1(x^5)}{cb_0(x^5)}, \frac{v^2b_2(x^5)}{cb_0(x^5)}, \frac{v^3b_3(x^5)}{cb_0(x^5)} \right) \quad (22)$

$\tilde{\gamma} \equiv \left( 1 - \left| \tilde{\beta} \right|^2 \right)^{-1/2} \quad (23)$

where (cfr. Eq.(14))

$\mathbf{u} = \left( \frac{b_0(x^5)}{b_1(x^5)}, \frac{b_0(x^5)}{b_2(x^5)}, \frac{b_0(x^5)}{b_3(x^5)} \right) \quad (24)$

In order to derive the expression of the deformed boost in a generic direction, it is possible to use the same method of the previous case (see Appendix A). However, it is simpler to consider the notion of parallelism between 3-vectors in $\tilde{M}_4(x^5)$ \(^7\) and decompose the space vector $\mathbf{x}$ in two components, $x_\parallel$ and $x_\perp$, parallel and orthogonal, respectively, to the boost direction $\hat{v}$

$x = x_\parallel + x_\perp. \quad (25)$

\(^6\)Notice that $\tilde{\beta} := \left( \frac{\mathbf{v}}{\mathbf{u}} \right) \neq \frac{\mathbf{v}}{\mathbf{u}}$. This follows from the anisotropy of the 3-vector $\mathbf{u}$, and it is to be compared with the SR case, where $\beta := \left( \frac{\mathbf{v}}{\mathbf{c}} \right) = \frac{\mathbf{v}}{\mathbf{c}}$. In general, it is possible to state that

$\left( \frac{m}{n} \right) = \frac{1}{m} \iff n = n(\hat{x}, \hat{y}, \hat{z})$

i.e iff $\mathbf{n}$ is a spatially isotropic trivector.

\(^7\)The definitions of parallelism and orthogonality are to be meant in the sense of the deformed 3-d. scalar product $\ast$ (see Eq. (3)).
\[ x_\parallel \equiv \hat{\bar{v}} (\hat{\bar{v}} \ast x) = \frac{\| v \|}{\| v \|} (v \ast x) = \frac{\| v \|}{v \ast v} (v \ast x) = \]
\[ = \sum_{i=1}^{3} b_i^2 (x^5) v_i x_i \frac{v}{v \ast v} \quad \text{in DSR4:} \quad \beta \equiv (\hat{\bar{v}}) \neq \hat{\bar{v}} (\hat{\bar{v}} \ast x) = \]
\[ = \frac{\beta}{\| \beta \|} (\hat{\bar{v}} \ast x) = \frac{\beta}{\| \beta \|} (\hat{\bar{v}} \ast x) = \frac{\sum_{i=1}^{3} b_i^2 (x^5) \beta_i x_i}{\sum_{i=1}^{3} b_i^2 (x^5) (\beta_i^2)} \tilde{\beta}, \quad (26) \]

(with \(\|\), denoting the absolute value of a trivector with respect to the deformed scalar product \(\ast\), whereas the notation \(\|\) will be used for the trivector norm with respect to the standard product \(\cdot\).)

\[ x^i_\parallel \equiv \frac{\sum_{k=1}^{3} b_k^2 (x^5) v^k x^k}{\sum_{k=1}^{3} b_k^2 (x^5) (v^k)^2} v^i \quad \text{in DSR4:} \quad \beta \equiv (\hat{\bar{v}}) \neq \hat{\bar{v}} \sum_{i=1}^{3} b_i^2 (x^5) \beta_i x_i \frac{v}{v \ast v} \sum_{i=1}^{3} b_i^2 (x^5) (\beta_i^2) \tilde{\beta}, \quad (27) \]

\[ x_\perp \equiv x - x_\parallel = x - \frac{\sum_{i=1}^{3} b_i^2 (x^5) v_i x_i}{\sum_{i=1}^{3} b_i^2 (x^5) (v_i)^2} \neq \]

in DSR4: \(\beta \equiv (\hat{\bar{v}}) \neq \hat{\bar{v}} \sum_{i=1}^{3} b_i^2 (x^5) \beta_i x_i \frac{v}{v \ast v} \sum_{i=1}^{3} b_i^2 (x^5) (\beta_i^2) \tilde{\beta}, \quad (28) \]

\[ x^i_\perp \equiv x^i - \frac{\sum_{k=1}^{3} b_k^2 (x^5) v^k x^k}{\sum_{k=1}^{3} b_k^2 (x^5) (v^k)^2} v^i \neq \]

in DSR4: \(\beta \equiv (\hat{\bar{v}}) \neq \hat{\bar{v}} \sum_{i=1}^{3} b_i^2 (x^5) \beta_i x_i \frac{v}{v \ast v} \sum_{i=1}^{3} b_i^2 (x^5) (\beta_i^2) \tilde{\beta}, \quad (29) \]
It is easily checked that indeed

\[ x \ast \nu = \sum_{i=1}^{3} b_i^2(x^5)x^i \nu^i = \frac{\sum_{i=1}^{3} b_i^2(x^5)x^i \nu^i}{\sum_{i=1}^{3} b_i^2(x^5)(\nu^i)^2} \sum_{k=1}^{3} b_k^2(x^5)(\nu^k)^2 = \]

\[ = \frac{\sum_{i=1}^{3} b_i^2(x^5)x^i \nu^i}{\sum_{i=1}^{3} b_i^2(x^5)(\nu^i)^2} \nu \ast \nu = x \parallel \ast \nu = |x|_2 |\nu|_2, \quad (30) \]

\[ x_\perp \ast \nu = x \ast \nu - x \parallel \ast \nu = 0. \quad (31) \]

We can now apply the boost (19) to \( x_\parallel \) and \( x_\perp \), thus getting

\[
\begin{align*}
x_\parallel' &= \tilde{\gamma}(x_\parallel - \nu t) \\
x_\perp' &= \frac{x_\perp}{\tilde{\gamma}} \\
t' &= \tilde{\gamma} \left( t - \sum_{i=1}^{3} \frac{v^i b_i^2(x^5)}{c^2 b_0^2(x^5)} x^i \right) = \tilde{\gamma} (t - \vec{B} \cdot x) = \tilde{\gamma} (t - \tilde{\beta}^{(\ast)} \ast x) \quad (32)
\end{align*}
\]

where we put (cfr. Eq.s (22)-(24))

\[
\tilde{\gamma} \equiv (1 - \tilde{\beta} \cdot \tilde{\beta})^{-1/2} = (1 - \tilde{\beta}^{(\ast)} \ast \tilde{\beta}^{(\ast)})^{-1/2} = \]

\[
= \left[ 1 - \left( \frac{v^1 b_1(x^5)}{c b_0(x^5)} \right)^2 - \left( \frac{v^2 b_2(x^5)}{c b_0(x^5)} \right)^2 - \left( \frac{v^3 b_3(x^5)}{c b_0(x^5)} \right)^2 \right]^{-1/2}, \quad (33)
\]

\[
\tilde{\beta}^{(\ast)} \equiv \left( \frac{\nu}{|\nu|} \right) = \left( \frac{v^1}{c b_0(x^5)} \tilde{x}, \frac{v^2}{c b_0(x^5)} \tilde{y}, \frac{v^3}{c b_0(x^5)} \tilde{z} \right) = \frac{1}{c b_0(x^5)} \nu, \quad (34)
\]

\[
w \equiv c b_0(x^5)(\tilde{x}, \tilde{y}, \tilde{z}), \quad (35)
\]

\[
\vec{B} \equiv \left( \frac{\nu}{|\nu|^2} \right) = \left( \frac{v^1 b_1^2(x^5)}{c^2 b_0^2(x^5)} \tilde{x}, \frac{v^2 b_2^2(x^5)}{c^2 b_0^2(x^5)} \tilde{y}, \frac{v^3 b_3^2(x^5)}{c^2 b_0^2(x^5)} \tilde{z} \right), \quad (36)
\]
\[ \widetilde{B}^{(*)} \equiv \left( \frac{v}{w^2} \right) = \frac{1}{c^2 b_0(x^5)} v. \] (37)

We can therefore state that the deformed boosts admit a double treatment, either:

I) In terms of the Euclidean scalar product \( \cdot \), of the (anisotropic) m.c.v. \( \bar{u} \) and of the related "rapidities" \( \tilde{\beta} \) and \( \widetilde{B} \), or

II) in terms of the deformed product \( * \), of the (isotropic) m.c.v. \( w \) and of the related quantities \( \tilde{\beta}^{(*)} \) and \( \widetilde{B}^{(*)} \).

Then, the space vector transforms as:

\[ x' = \bar{x} = \widetilde{x} + \tilde{\gamma} (x - v t) + x = \]

\[ = \bar{x} + (\tilde{\gamma} - 1) \hat{\gamma}(\bar{u} \ast \bar{x}) - \tilde{\gamma} v t = \bar{x} + (\tilde{\gamma} - 1) \frac{v}{|v|_*} (v \ast \bar{x}) - \tilde{\gamma} v t \] (38)

and we eventually find the expression of the deformed boost in a generic direction:

\[
\begin{align*}
\left\{ \begin{array}{l}
x' = x + (\tilde{\gamma} - 1) \frac{v}{|v|_*} (v \ast \bar{x}) - \tilde{\gamma} v t \\
t' = \tilde{\gamma} (t - \widetilde{B} \cdot \bar{x}) = \tilde{\gamma} (t - \widetilde{B}^{(*)} \ast \bar{x})
\end{array} \right.
\end{align*}
\] (39)

### 3.3 Symmetrization of deformed boosts

As in the case of standard SR, it is possible to symmetrize the expression of boosts in DSR4 by introducing suitable time coordinates.

Let us first consider a deformed boost along \( \hat{\gamma} \) (\( i = 1, 2, 3 \)); the symmetrization transformation (a "dimensionally homogenizing dilato-contraction") of \( t \) is given by

---

8It is possible to show that, in this case, more equivalent forms of the deformed boost (32) exist. As is easily seen, this is due to the fact that, in general, \( \tilde{\beta} \neq \bar{v} \) and \( \widetilde{B} \neq \bar{v} \), whereas \( \tilde{\beta}^{(*)} = \bar{v} = \widetilde{B}^{(*)} \) (cfr. Eq.s (22), (34), (36) and (37)).
\[ x^0 \equiv u^t = c b_0(x^5) t; \quad (x')^i = (x^i)' \equiv x^i \quad (40) \]

The deformed metric tensor in the new "primed" coordinates, \( \{(x')^\mu = (x^\mu)'\} = \{x^0, x, y, z\} \), reads:

\[ \eta'_{\mu\nu}(x^5) \equiv \eta_{\alpha\beta}(x^5) \frac{\partial x^\alpha}{\partial x'\mu} \frac{\partial x^\beta}{\partial x'\nu} = \text{diag}(b_1^2(x^5), -b_1^2(x^5), -b_2^2(x^5), -b_3^2(x^5)) \]

Eq. (19) takes therefore the symmetric form in \( x^i e x^0 \) (ESC off):

\[ (x')^i = (x^i)' = \tilde{\gamma}(x^i - \tilde{\beta}^i x^0) \]
\[ (x')^k \neq i = (x^k \neq i)' = x^k \neq i \]
\[ (x')^0 = (x^0)' = \tilde{\gamma}(x^0 - \tilde{\beta}^i x^i) \quad (42) \]

Transformation (40) does not symmetrize the deformed boost in a generic direction (unlike the case of SR, where the same transformation \( x^0 = ct \) symmetrizes both boosts). In this case, the symmetrization is possible only if the treatment II (based on the deformed scalar product *) is used.

In fact, by using the proportionality (already stressed in footnote 8) among \( \tilde{\beta}^{(s)} \), \( \tilde{R}^{(s)} \) and \( v \), the following transformation on \( t \)

\[ x^0 \equiv c b_0(x^5) t = w^k t \quad (\forall k = 1, 2, 3) ; \quad (x')^i = (x^i)' \equiv x^i \quad (\forall i = 1, 2, 3) \quad (43) \]
does symmetrize Eq. (32) in $x_\parallel$ and $x^0$:

$$
\begin{align*}
  x_\parallel' &= (1 - \bar{\beta}^{(*)} * \bar{\beta}^{(*)})^{-1/2} (x_\parallel - \bar{\beta}^{(*)} x^0) \\
  x_\perp' &= x_\perp \\
  (x')^0 &= (x^0)' = (1 - \bar{\beta}^{(*)} * \bar{\beta}^{(*)})^{-1/2} (x^0 - \bar{\beta}^{(*)} * x) = (1 - \bar{\beta}^{(*)} * \bar{\beta}^{(*)})^{-1/2} (x^0 - \bar{\beta}^{(*)} * x_\parallel)
\end{align*}
$$

Under transformation (43), the metric tensor becomes:

$$
\eta_{\mu\nu}'(x^5) = \eta_{\alpha\beta}(x^5) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} = \text{diag}(1, -b_1^2(x^5), -b_2^2(x^5), -b_3^2(x^5))
$$

Therefore the symmetrization of the deformed boost in a generic direction makes the 4-d. metric isochronous, since $\eta_{00}' = 1$ so that $\tau = t$ (namely proper time coincides with coordinate time).

Let us finally notice that, like in the SR case, the boost in generic direction expressed in terms of $x_\parallel$ and $t$ (Eq. (39)) cannot in general be symmetrized.

### 3.4 Velocity composition law in $\tilde{M}_4$ and the invariant maximal speed

We have seen in Subsect. 2.2 that the directionally separating approach (mandatory in the deformed case) yields two different mathematical definitions $\underline{u}$ (Eq. (14)) and $\overline{u}$ (Eq. (16)) of maximal causal velocity in DSR4. The choice between them must be done on a physical basis, by checking their actual invariance under deformed boosts.

To this aim, we have to derive the generalized velocity composition law valid in $\tilde{M}_4$. For a deformed boost in the direction $\hat{\mathbf{x}}^i$, we have, by differentiating the inverse of Eq. (19) (on account of the fact that $dx^5 = 0$ in DSR4)
\[
\begin{align*}
\frac{dx^i}{dt} &= \tilde{\gamma} \left[ (dx^i)' + v^i (dt)' \right] \\
\frac{dx^{k \neq i}}{dt} &= (dx^{k \neq i})' \\
\frac{dt}{dt} &= \tilde{\gamma} \left[ (dt)' + \frac{v^ib^2_5(x^5)}{c^2 b^2_0(x^5)} (dx^i)' \right]
\end{align*}
\]

(46)

with \( \tilde{\gamma} \) given by (21). Since

\[
\begin{align*}
\frac{dx^i}{dt} &= v^i, \\
\frac{(dx^i)'}{(dt)'} &= \frac{(dx^i)'}{(dt)'} = (v')^i = (v^i)', \\
\frac{dx^{k \neq i}}{dt} &= v^{k \neq i}, \\
\frac{(dx^{k \neq i})'}{(dt)'} &= \frac{(dx^{k \neq i})'}{(dt)'} = (v^{k \neq i})' = (v^{k \neq i})',
\end{align*}
\]

we get the deformed velocity composition law (in compact notation, ESC off)

\[
v^k = \frac{(v^k)'}{1 + \left( \frac{b_1(x^5)}{b_0(x^5)} \right)^2 v^i (v^i)'} \left\{ \tilde{\gamma}(x^5) + \delta_{ik} \left[ 1 - \tilde{\gamma}(x^5) \right] \right\}
\]

(48)

This relation can be expressed in terms of the standard 3-d. scalar product \( \cdot \) (and therefore of the anisotropic maximal velocity \( \mathbf{u} \)) (approach I) as

\[
v^k = \frac{(v^k)'}{1 + \frac{\mathbf{u} \cdot v^i}{(u^i(x^5))^2} \left\{ \tilde{\gamma}(x^5) + \delta_{ik} \left[ 1 - \tilde{\gamma}(x^5) \right] \right\}}
\]

\[
= \frac{(v^k)'}{1 + \frac{\tilde{\beta} \cdot v^i}{\mathbf{u}^i(x^5)} \left\{ \tilde{\gamma}(x^5) + \delta_{ik} \left[ 1 - \tilde{\gamma}(x^5) \right] \right\}}
\]

(49)

16
where (cfr. Eq.s (22), (23))

\[
\tilde{\beta}^i(x^5) = \frac{v^i}{w^i(x^5)} ; \quad \tilde{\gamma}(x^5) = \left(1 - \frac{\tilde{\beta}(x^5) \cdot \tilde{\beta}(x^5)}{w^i(x^5)}\right)^{-1/2}
\] (50)

Alternatively, we can use approach II, based on the deformed scalar product ∗ (and therefore the isotropic maximal velocity \(w\)) and write Eq.(48) as

\[
v^k = \frac{(v^k)' + \delta_{ik}v^i}{1 + \frac{\bar{\beta}(x^5) \cdot v'}{w^i(x^5)}} \begin{cases} 
\tilde{\gamma}(x^5) + \delta_{ik} [1 - \tilde{\gamma}(x^5)] 
\end{cases} = \\
= \frac{(v^k)' + \delta_{ik}v^i}{1 + \frac{\tilde{\beta}(x^5) \cdot v'}{w^i(x^5)}} \begin{cases} 
\tilde{\gamma}(x^5) + \delta_{ik} [1 - \tilde{\gamma}(x^5)] 
\end{cases}
\] (51)

with (cfr. Eq.s (34), (33))

\[
\tilde{\beta}(x^5) = \frac{v^i}{w^i(x^5)} ; \quad \tilde{\gamma}(x^5) = \left(1 - \frac{\tilde{\beta}(x^5) \cdot \tilde{\beta}(x^5)}{w^i(x^5)}\right)^{-1/2}
\] (52)

It is now an easy task to check the truly maximal character of the two velocities. Indeed, if \((v^i)' = u^i(x^5)\), one gets, from Eq.(49)

\[
v_i = \frac{u^i(x^5) + v^i}{1 + \frac{v^i}{w^i(x^5)}} = u^i(x^5)
\] (53)

whereas, for \((v^i)' = w^i(x^5)\), Eq. (51) yields

\[
v_i = \frac{w^i(x^5) + v^i}{1 + \frac{(b_1(x^5))^2 v^i}{w^i(x^5)}} \neq w^i(x^5)
\] (54)

We can therefore conclude, on a physical basis, that \(u\) is the maximal, invariant causal velocity in DSR4, and it plays in the deformed Minkowski
space $\tilde{M}_4$ the role of the light speed in standard SR\(^9\).

It is also easy to see why - although approach II) looks at first sight more rigorous mathematically, because it permits to connect the peculiar features of spatial anisotropy of DSR4 to the deformed product $\ast$, "naturally induced" from the metric of $\tilde{M}_4(x^5)$ - actually it is approach I) which yields the physically relevant result. Indeed, the velocity $\underline{u}$ is just defined as $\frac{dx}{dt}$, and it therefore represents the physically measured velocity, for a particle moving in the usual, physical Euclidean 3-d space. On the other hand, this result clearly shows that the space anisotropy introduced by the deformed metric is not a mere mathematical artifact, but it reflects itself in the physical properties (imposed by the interaction involved) of the phenomenon described by the deformed space-time.

The comparison of the deformed boost expressions (Eq.s (19), (32)) with the corresponding ones of the standard Lorentz boosts shows clearly that the transition from SR (based on $M_4$) to DSR4 (based on $\tilde{M}_4$) is simply carried out by letting

$$u_{SR} = c(\tilde{x}, \tilde{y}, \tilde{z}) \longrightarrow u_{DSR4}(x^5) = cb_0(x^5) \left( \frac{1}{b_1(x^5)} \tilde{x}, \frac{1}{b_2(x^5)} \tilde{y}, \frac{1}{b_3(x^5)} \tilde{z} \right) \tag{55}$$

In other words, the difference between $M_4$ and $\tilde{M}_4(x^5)$ (at least as far as

Of course, in the case of space isotropy (cfr. Eq.(4)), we recover the result of Ref. [1], namely an isotropic maximal causal velocity given by

$$u^i_{iso}(x^5) = u^i_{DSR4,II}(x^5)|_{b_i(x^5)=b(x^5)} = c \frac{b_0(x^5)}{b(x^5)} \quad \forall i = 1, 2, 3,$$

$$|u_{iso}(x^5)| = \left( \sum_{i=1}^3 (u^i_{iso}(x^5))^2 \right)^{1/2} = \sqrt{3}c \frac{b_0(x^5)}{b(x^5)}$$

Moreover, notice that it can be

$$u^i >_c c \quad \text{according to} \quad \frac{b_0}{b^i} >_c 1.$$

In other words, there may be maximal causal speeds superluminal, depending on the interaction considered (without any violation of causality).
the finite coordinate transformations are concerned) is completely embodied in the trivector m.c.v. $\mathbf{u}$.

### 3.5 Choosing the boost direction in DSR4

We want now to remark a difficulty arising in the context of DSR4, due to the space anisotropy.

Indeed, the space anisotropy (reflected in the physical anisotropic m.c.v. $\mathbf{u}$) produces a triple indetermination in the process of identifying the motion axis with any of the space coordinate axes, since now - unlike the SR case - the space dimensions are no longer equivalent.

However, this indeterminacy can be removed (at least in principle) by means of the following *Gedankenexperiment*. Consider three particles (ruled by one and the same interaction) in general different but able to move at the maximal causal velocity $u^i(x^5)$. Suppose they are moving in the 3-d Euclidean space along mutually independent (orthogonal) spatial directions. Assigning an arbitrary labelling to the particle motion directions, we can fix an orthogonal, left-handed fame of axes. Since by assumption we know the interaction which the particles are subjected to, we know the deformed metric and therefore the metric coefficients as functions of the energy, $b_\mu^2(E)$. Then, a measurement of the particle velocities allows us to determine the right labelling of the spatial frame (cfr. Eq. (24)).

This implies that in the context of DSR4, too, it is always possible, at physical level, to let one of the three space axes to coincide with the direction of motion of a physical object, and therefore apply the suitable deformed boost.

### Appendix A

Consider two TIRF’s, $K$ and $K'$; by definition, the DLT’s are metric isometries, i.e. leave *invariant* the deformed metric interval (2), whence:

$$b_0^2(x^5)c^2 t^2 - b_1^2(x^5)x^2 - b_2^2(x^5)y^2 - b_3^2(x^5)z^2 =$$

$$= b_0^2(x^5)c^2 (t')^2 - b_1^2(x^5)(x')^2 - b_2^2(x^5)(y')^2 - b_3^2(x^5)(z')^2$$

(A.1)
Moreover, without loss of generality, we can assume that the frames $K$ and $K'$ are in *standard configuration* (i.e. their spatial frames coincide at $t = t' = 0$). By choosing the boost direction along $\hat{x} = \hat{x}_1$, we have therefore $y' = y$, $z' = z$, and Eq. (A.1) reduces to

$$b_0^2(x^5)c^2t^2 - b_1^2(x^5)x^2 = b_0^2(x^5)c^2(t')^2 - b_1^2(x^5)(x')^2$$

(A.2)

From *space-time homogeneity* it follows that the functional relations between the two sets of coordinates $\{x, y, z, t\}$ and $\{x', y', z', t'\}$ must be linear. Then, in general, the deformed coordinate transformations are to be searched in the form

$$\begin{align*}
x' &= A_{11}x + A_{14}t \\
y' &= y \\
z' &= z \\
t' &= A_{41}x + A_{44}t
\end{align*}$$

(A.3)

where the coefficients $A_{11}, A_{14}, A_{41}, A_{44}$ depend *a priori* in general on $v$ and $\hat{x}$ (and, parametrically, on $x^5$).

Notice that the origin $O'$ of TIRF $K'$ must move in $K$ with velocity $v = v^1\hat{x}$, and therefore:

$$x' = 0 \Leftrightarrow A_{14} = -vA_{11} \Leftrightarrow x' = A_{11}(x - vt)$$

(A.4)

Replacing (A.3), (A.4) in (A.2) yields

$$b_0^2(x^5)c^2t^2 - b_1^2(x^5)x^2 = b_0^2(x^5)c^2(A_{41}x + A_{44}t)^2 - A_{11}b_1^2(x^5)x^2(x - vt)^2$$

(A.5)

which implies the following $3 \times 3$ quadratic system:

$$\begin{align*}
c^2 &= c^2A_{44}^2 - \left(\frac{b_1(x^5)}{b_0(x^5)}\right)^2A_{11}^2v^2 \\
-1 &= c^2\left(\frac{b_0(x^5)}{b_1(x^5)}\right)^2A_{41}^2 - A_{11}^2 \\
0 &= c^2\left(\frac{b_0(x^5)}{b_1(x^5)}\right)^2A_{41}A_{44} + A_{11}^2v
\end{align*}$$

(A.6)
with general solution

\[ A_{11} = A_{44} = \pm \left( 1 - \left( \frac{vb_1(x^5)}{cb_0(x^5)} \right)^2 \right)^{-1/2} \]  
(A.7)

\[ A_{14} = \mp \left( \frac{vb_1^2(x^5)}{c^2b_0^2(x^5)} \right) \left( 1 - \left( \frac{vb_1(x^5)}{cb_0(x^5)} \right)^2 \right)^{-1/2} = - \left( \frac{vb_1^2(x^5)}{c^2b_0^2(x^5)} \right) A_{11} \]  
(A.7′)

The final result is

\[
\begin{align*}
    x' &= \tilde{\gamma}(x - vt) = \tilde{\gamma} \left( x - \frac{\beta b_0(x^5)}{b_1(x^5)} ct \right) \\
y' &= y \\
z' &= z \\
t' &= \tilde{\gamma} \left( t - \frac{vb_1^2(x^5)}{c^2b_0^2(x^5)} x \right) = \tilde{\gamma} \left( t - \frac{\beta^2}{v} x \right)
\end{align*}
\]  
(A.8)

namely the deformed boost (19) for motion along \( x \).

The same procedure can in principle be followed in deriving the deformed boost in a generic direction. In this case, the coordinate transformations are

\[
\begin{align*}
    x' &= A_{11} x + A_{12} y + A_{13} z + A_{14} t \\
y' &= A_{21} x + A_{22} y + A_{23} z + A_{24} t \\
z' &= A_{31} x + A_{32} y + A_{33} z + A_{34} t \\
t' &= A_{41} x + A_{42} y + A_{43} z + A_{44} t
\end{align*}
\]  
(A.9)

From the physical requirement that the origin \( O' \) of TIRF \( K' \) must move in \( K \) with velocity components \( v^1 \) along \( \hat{x} \), \( v^2 \) along \( \hat{y} \), \( v^3 \) along \( \hat{z} \), one gets:
\[ x' = 0, \quad x = v^1 t \]
\[ y' = 0, \quad y = v^2 t \]
\[ z' = 0, \quad z = v^3 t \]

\[ \iff \begin{cases} 
A_{11} v^1 + A_{12} v^2 + A_{13} v^3 + A_{14} = 0 \\
A_{21} v^1 + A_{22} v^2 + A_{31} v^3 + A_{24} = 0 \\
A_{31} v^1 + A_{32} v^2 + A_{33} v^3 + A_{34} = 0 
\end{cases} \] (A.10)

Eq. (A.9) becomes therefore

\[ \begin{cases} 
x' = A_{11} (x - v^1 t) + A_{12} (y - v^2 t) + A_{13} (z - v^3 t) \\
y' = A_{21} (x - v^1 t) + A_{22} (y - v^2 t) + A_{23} (z - v^3 t) \\
z' = A_{31} (x - v^1 t) + A_{32} (y - v^2 t) + A_{33} (z - v^3 t) \\
t' = A_{41} x + A_{42} y + A_{43} z + A_{44} t 
\end{cases} \] (A.11)

Replacing (A.11) in (A.1) yields

\[
b_0^2(x^5)c^2 t^2 - b_1^2(x^5)x^2 - b_2^2(x^5)y^2 - b_3^2(x^5)z^2 =
\]

\[
= c^2 b_0^2(x^5)(A_{41} x + A_{42} y + A_{43} z + A_{44} t)^2 + 
\]

\[
- b_1^2(x^5)(A_{11} (x - v^1 t) + A_{12} (y - v^2 t) + A_{13} (z - v^3 t))^2 + 
\]

\[
- b_2^2(x^5)(A_{21} (x - v^1 t) + A_{22} (y - v^2 t) + A_{23} (z - v^3 t))^2 + 
\]

\[
- b_3^2(x^5)(A_{31} (x - v^1 t) + A_{32} (y - v^2 t) + A_{33} (z - v^3 t))^2 
\] (A.12)

Equating the coefficients on both sides of (A.12) one gets a system of 10 quadratic equations in the 13 unknown coefficients \( \{ A_{ij}, A_{4i} \} \) \((i,j = 1,2,3)\), namely:
I- from the coefficient of $t^2$:

$$c^2(A_{44}^2 - 1) - \frac{1}{b_0^2(x^5)} \sum_{i,j,l=1}^3 b_j^2(x^5)v^jA_{ij}A_{il} = 0 \quad (A.13)$$

II- from the coefficients of $x^ix^j$ ($i, j = 1, 2, 3$), 6 independent equations:

$$c^2A_{4i}A_{4j} - \frac{1}{b_0^2(x^5)} \sum_{l=1}^3 b_j^2(x^5)(A_{4i}A_{4j} - \delta_{ij}\delta_{il}) = 0 \quad (A.14)$$

III- from the coefficients of $x^it$ ($i = 1, 2, 3$), 3 independent equations:

$$c^2A_{4i}A_{44} + \frac{1}{b_0^2(x^5)} \sum_{j,l=1}^3 b_j^2(x^5)v^jA_{ji}A_{jl} = 0 \quad (A.15)$$

Although the above system in the set $\{A_{ij}, A_{4i}\} (i, j = 1, 2, 3)$ can be exactly solved, the general solution for the boost expressed in the form (A.11) is quite cumbersome. This motivates our choice (adopted in Subsect. 3.2) of deriving the form of the deformed boost in a generic direction by exploiting the notion of ”deformed” parallelism between trivectors. Of course (and it may be checked by explicit calculations), both approaches are completely equivalent.

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