Integrable, molecular-type solutions of the extended Skyrme-Faddeev model

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I. INTRODUCTION

The Skyrme-Faddeev model was introduced in the seventies [1] as a clever generalization to (3 + 1) dimensions of the O(3) non-linear sigma model in (2 + 1) dimensions [2]. The Skyrme term, quartic in derivatives of the field, balances the quadratic kinetic term and according to Derrick’s theorem, allows the existence of stable solutions with non-trivial Hopf topological charges. Due to the highly non-linear character of the model and the lack of symmetries, the first soliton solutions were only constructed in the late nineties using numerical methods [3–6]. Several physical applications based on the model have been extensively studied in many areas due mainly to the knotted character of the solutions [7]. The numerical efforts in the construction of the solutions have improved our understanding of the properties of the model [8] and even the scattering of knotted solitons has been investigated [9]. One of the aspects of the model that has attracted considerable attention has been its connection with gauge theories. Faddeev and Niemi have conjectured that it might describe the low energy limit of the pure SU(2) Yang-Mills theory [10]. They based their argument on a decomposition of the physical degrees of freedom of the SU(2) connection, proposed in the eighties by Cho [11], and involving a triplet of scalar fields ̃n taking values on the sphere S^2 (̃n^2 = 1). Gies [12] has calculated the Wilsonian one loop effective action for the pure SU(2) Yang-Mills theory assuming Cho’s decomposition, and found that the Skyrme-Faddeev action is indeed part of it, but additional quartic terms in the derivatives of the triplet ̃n are unavoidable.

The extended version of such Skyrme-Faddeev (ESF) model

\[ \mathcal{L} = M^2 \partial_\mu \vec{n} \cdot \partial^\mu \vec{n} - \frac{1}{\varepsilon^2} (\partial_\mu \vec{n} \times \partial^\mu \vec{n})^2 + \frac{\beta}{2} (\partial_\mu \vec{n} \cdot \partial^\mu \vec{n})^2 - V(n_1, n_2, n_3) \]  

has already been studied: The static energy density \( \langle \mathcal{H}_{\text{static}} = -\mathcal{L} \rangle \) associated to (1) is positive definite if \( V > 0 \), \( M^2 > 0 \), \( \varepsilon^2 > 0 \) and \( \beta < 0 \). That is the sector explored in [3] and where Hopf soliton solutions were first constructed (for \( V = 0 \)). In addition, that is also the sector explored in [13] but with additional terms involving second derivatives of the ̃n field, and where Hopf soliton were also constructed. The static energy density of (1) is also positive definite for \( V > 0 \) if

\[ M^2 > 0; \quad \varepsilon^2 < 0; \quad \beta < 0; \quad \beta \varepsilon^2 \geq 1. \]  

That is the sector that agrees with the signature of the terms in the one loop effective action calculated in [12] and it is the sector that we will consider in this paper. Static Hopf solitons with axial symmetry were constructed in [14, 15] for the sector (2) (with \( V = 0 \)) and their quantum excitations, including comparison with glue ball spectrum, were considered in [16]. An interesting feature of the Hopf solitons constructed in [14] is that they shrink in size and then disappear as \( \beta \varepsilon^2 \to 1 \), which is exactly the point where the vortex solution of the class (1) exists. Full numerical simulation was followed for the existence of such knotted solutions in [17].

The action (1) also possesses the vortex solutions. The first exact vortex solutions for the theory were constructed in [18], and by exploring the integrability properties of a submodel of (1). In order to describe those exact vortex solutions it is better to perform the stereographic projection of the target space \( S^2 \) onto the plane parameterized by the complex scalar field \( u \) and related to ̃n by

\[ \vec{n} = (u + u^*, -i (u - u^*), |u|^2 - 1) / (1 + |u|^2). \]  

It was shown in [18] that the field configurations of the...
form
\[ u \equiv u(z, y), \quad u^* \equiv u^*(z^*, y) \quad \text{for} \quad \beta e^2 = 1, \quad V = 0 \]

are exact solutions of (1), where \( z = x^1 + i \varepsilon_1 x^2 \) and \( y = x^3 - \varepsilon_2 x^0 \), with \( \varepsilon_\alpha = \pm 1 \), \( \alpha = 1, 2 \), and \( x^\mu, \mu = 0, 1, 2, 3 \), are the Cartesian coordinates of the Minkowski space-time. Despite the fact that (1) constitutes a very large class of solutions, no finite energy solutions were found within it. If the dependence of the \( u \) field upon the variable \( y \) is in the form of phases like \( e^{i k y} \), then one finds solutions with finite energy per unit of length along the \( x^3 \)-axis. The simplest solution is of the form \( u = z^n e^{i k y} \), with \( n \) to be integer, and it corresponds to a vortex parallel to the \( x^3 \)-axis and with waves traveling along it with the speed of light. More general solutions of the class (1) were constructed in [15], including multivortices separated from each other and all parallel to the \( x^3 \)-axis.

The vortex solutions for the model continue to exist when the condition \( \beta e^2 = 1 \) is relaxed by introducing a potential \( V \) [20]. The potential is essentially introduced to stabilize the vortex solutions. It is well known that there are some variations for the potential when it is a functional of the third component \( n_3 \) of the triplet \( \vec{n} \), such as so-called old-baby potential (one-vacuum type) [21] and new-baby potential (two-vacuum type) [22]. The baby-Skyrme model is a 2+1 mimic of the Skyrme model and has the static planar solution called baby-skyrmions. The present model has close relation with the baby-Skyrme model when we restrict our analysis only in the static planar solution. In the old-baby potential, the rotational symmetry of the baby-skyrmions is spontaneously broken while in the new-baby, no such transition of the structure occurs [22]. Similar behavior has also been observed in our vortex solution [24]. By using the old-baby potential, the deformation grows as \( \beta e^2 \) increases. Note that if the potential is a functional of the third component \( n_3 \) it breaks the \( O(3) \) symmetry of the original Skyrme-Faddeev down to \( O(2) \), the group of rotations on the plane \( n_1 - n_2 \), and so eliminating two of the three Goldstone boson degrees of freedom.

If one extends the form of the potential which contains all components \( n_1, n_2 \) and \( n_3 \), there must be several possibilities for choice of the potential. In the easy plane potential \( V = \frac{1}{2} m^2 n_3^2 \), the baby-skyrmions possess the dihedral symmetry \( D_2 \) [22]. Originally, such a symmetrical solution was found by the choice of \( V = \frac{1}{2} m^2 (1 - n_3^2)(1 - n_1^2) \) [20]. A sophisticated form of the potential \( V = m^2 [1 - (n_1 + i n_2) N] \), \( (N \geq 2) \) is essentially same as the old-baby but exhibits the \( D_N \) symmetry [23]. The baby-skyrmions or the fractional vortex states for a variant of these potentials are extensively investigated in [23]. In a rough speculation, if one wants to get solutions with platonic (dihedral) symmetries, he/she should employ some potentials which contain terms with \( n_1 \) or (and) \( n_2 \) components. In the studies, they share the procedure to get their new solutions: first they employ special potentials for one’s which are motivated by a physics or a mathematics, and then they numerically solve the Euler-Lagrange equations or the hamiltonian.

The method examined in [23] is unique and independent from others. First the authors assume an existence of a static and exact \( N \)-centered vortex solutions. Next, they determine the form of the potential in order that the solution satisfies the Euler-Lagrange equation. They succeeded to get the analytical \( N = 2 \) solution but failed to find \( N > 2 \) ones Thus they solved the problem numerically. Although any guiding principle to find the solution and the potential is absent in their discussion, the idea seems promising. In this paper, we try to construct the exact \( N \)-centered vortex solutions and the corresponding potentials. First, we introduce the \( N \)-centered ansatz which is essentially similar with one proposed in [23]. Main difference is that our ansatz describes the time-dependent, traveling wave vortex solutions and these are exactly solution of the corresponding submodel equation. Plugging them into the Euler-Lagrange equation of the model and we are able to construct the potential in order that the solution of the submodel becomes one of the equation itself. As a result, we can get the analytical vortex solution of the model which possesses an infinite number of the conserved quantities. The method is straightforwardly applicable to the other related soliton models such as the baby-Skyrme model, the Skyrme model, so on.

The paper is organized as follows. In the next section we briefly describe the extended Skyrme-Faddeev model. The equations of motion are also introduced in Sec. II. The method how to get the solutions of the integrable sector of the present model is discussed in Sec. III. Sec. IV is devoted for the zero-curvature conditions and the conservation of the currents. In Sec. V, we show the numerical solutions. A brief summary is presented in Sec. VI.

II. THE MODEL

The Lagrangian density of the extended Skyrme-Faddeev model reads [11], where unit vector \( \vec{n} \), i.e. \( \vec{n} \cdot \vec{n} = 1 \), is a triplet of real scalar fields taking values on the sphere \( S^2 \). The coupling constant \( \mathcal{M} \) is dimensional whereas \( e^2 \) and \( \beta \) are some dimensionless coupling constants. The potential \( V \) depends on all the components of the triplet \( \vec{n} \). One can introduce the complex fields \( u \) and \( u^* \) using a stereographic projection [9] which leads to the following expression for the Lagrangian

\[
\mathcal{L} = 4 \mathcal{M}^2 \frac{\partial_\mu u \partial_\mu u^*}{(1 + |u|^2)^2} + \frac{8}{e^2} \left[ \frac{\partial_\mu u^2}{(1 + |u|^2)^2} \right] - V(u, u^*). \tag{5}
\]

If one set \( \beta e^2 = 1 \), then the model possesses some lump shaped analytical solutions in absence of the potential \( V \).
For the case of $\beta e^2 \neq 1$, a potential is needed to stabilize the solution \cite{15,20}. In this paper, we try to find an exact form of potential for some solutions with $\beta e^2 \neq 1$. The Euler-Lagrange equations corresponding to (3) read

$$(1 + |u|^2)\partial^2 K_\mu - 2 u^* K_\mu \partial^2 u = \frac{1}{4}(1 + |u|^2)^3 \frac{\partial V}{\partial u^*}, \tag{6}$$

together with the complex conjugated equation. The symbol $K_\mu$ stands for the expression

$$K_\mu \equiv M^2 \partial_\mu u + \frac{4}{c^2} \left[ \frac{(\partial_\mu u \partial^\mu u^*) \partial^2 u^*}{(1 + |u|^2)^2} \right. + \left. \frac{(\beta e^2 - 1)(\partial_\mu u \partial^\mu u^*) \partial^2 u^*}{(1 + |u|^2)^2} \right]. \tag{7}$$

We use the dimensionless polar coordinates $(t, \rho, \varphi, z)$ defined as follows

$$x_0 = ct, \quad x_1 = \rho \cos \varphi, \quad x_2 = \rho \sin \varphi, \quad x_3 = z.$$

here we choose $c = 1$. The metric is of the form

$$(ds)^2 = \eta_{\mu\nu} dx^\mu dx^\nu = (dt)^2 - (d\rho)^2 - \rho^2 (d\varphi)^2 - (dz)^2,$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \det[\eta] = -\rho^2.$$

III. CONSTRUCTION OF THE EXACT SOLUTIONS

A. The general formula

We would like to examine solutions which satisfy so-called zero curvature condition \cite{30}. The zero curvature condition of the ESF model reads

$$\partial_\mu u \partial^\mu u = 0. \tag{8}$$

As we shall describe at Sec. IV in detail, the solution possesses an infinite set of conserved currents. Within the condition \cite{38}, the equation of motion \cite{40} can be written as

$$\frac{\partial V}{\partial u^*} = -4 \frac{\partial^2 K_\mu}{(1 + |u|^2)^2}. \tag{9}$$

In order to find the explicit form of $V$, it is not straightforward task that one directly solve the partial differential equation \cite{49}. Instead, we employ a method to obtain a possible candidate of $V$ in terms of the following a few of steps:

(i) we construct a $N$-centered solution $u_N$ which satisfies the zero curvature condition \cite{38},

(ii) substituting the solution into \cite{9}, we obtain the derivative of potential

$$\frac{\partial V}{\partial u^*} \equiv -4 \frac{\partial^2 K_\mu}{(1 + |u|^2)^2} \bigg|_{u = u_N}, \tag{10}$$

(iii) we assume a candidate of $V_N$ for $u_N$ and compute its derivative $\frac{\partial V_N}{\partial u^*}$,

(iv) comparing the results of $\frac{\partial V_N}{\partial u^*}$ and $\frac{\partial V}{\partial u^*}$, we find a form of $V_N$.

As a simple example, we consider a solution

$$u = \frac{1}{a^N}(z^N - \xi_1^N). \tag{11}$$

If we set ‘the centers’ of the solution at $\xi_k \equiv ce^{i\alpha + \frac{2\pi(k-1)}{N}}$, where $\alpha$ and $c$ are arbitrary constants, the solution \cite{10} can straightforwardly be contracted in the form

$$\partial V \frac{\partial V}{\partial u^*} = \frac{128 N^3 (\beta e^2 - 1)}{c^2 (1 + |u|^2)^5} \left[ (u + \xi_1^N)^2 - 2/N (u^* + \xi_1^N)^{-1/2} \right. \left. \times (1 + |u|^2 + N (-1 + 2\xi_1^N u + |u|^2)) \right]. \tag{12}$$

For a candidate satisfying \cite{12}, we introduce the following expression

$$V_N = \frac{\lambda}{(1 + |u|^2)^2} (u + \xi_1^N)^2 (u^* + \xi_1^N)^b, \tag{13}$$

where $\lambda, a,$ and $b$ are arbitrary constants. Comparing $\partial V_N$ and $\partial V$ \cite{49}, we can fix the constants and obtain the form of the potential

$$V_N = \frac{\lambda}{(1 + |u|^2)^2} (u + \xi_1^N)^2 - 2/N (u^* + \xi_1^N)^2 - 2/N. \tag{14}$$

Here $\lambda$ is determined as

$$\lambda = \frac{-32 (\beta e^2 - 1)}{a^N c^2}. \tag{15}$$

Although it remains some possibilities that the form \cite{13} is not unique, our method is quite useful to find the solutions. In the following subsection, we shall examine for the several solutions in terms of this procedure and see how it works.
B. Examples of the solutions and the corresponding potential

In this subsection, we get several solutions/potentials in terms of the procedure. The solution of the form \[11\] is already known as so-called lump solution and is extensively studied in \[20\], \[29\]. Thus, in this paper we mainly focus on several multi-centered holomorphic solutions and obtain the corresponding potentials.

1. The lump solution and the corresponding potential

First we examine the case of the lump solution and check the validity of our procedure. The holomorphic lump-shaped vortex solution with the winding number \(n = 1\) is written as \[18\]

\[u_1 = \left(\frac{\rho}{a_1}\right) e^{i[\varphi + k(t + z)]},\]  

where \(a_1\) is an arbitrary scale parameter. Note that \(u_1\) satisfies the zero curvature condition \[8\]. One can show that the following formulas hold

\[\partial_\mu u_1 \partial^\mu u_1^* = -\frac{2}{\rho^2} |u_1|^2, \quad \partial_\mu^\mu \partial_\mu u_1 = 0.\]  

Substituting (16) into (9), we get

\[\frac{\partial \bar{V}}{\partial u^*} = 128 \frac{\beta e^2 - 1}{e^{2a_1^2}} \frac{u}{(1 + |u|^2)^3}.\]  

On the other hand, plugging (16) into (13), we obtain

\[V_1 = 4\rho^2 \frac{1}{a_1^2} \frac{1}{(1 + |u|^2)^4}\]  

and differentiating this, we obtain

\[\frac{\partial V_1}{\partial u^*} = 128 \frac{\beta e^2 - 1}{e^{2a_1^2}} \frac{u}{(1 + |u|^2)^3}.\]  

Both expressions is obviously equal if we choose \(s = -8\).

2. The two-centered solution and the potential

We introduce a form of the two centered solution such as \[20\]

\[u_2 = \left(\frac{\rho}{a_2}\right)^2 \left(\frac{c}{a_2}\right)^2 e^{i[2\varphi + k(t + z)]} - \left(\frac{c}{a_2}\right)^2 e^{i[2\alpha + k(t + z)]},\]  

where \(a_2\) is an arbitrary scale parameter. Roughly speaking, \(c\) and \(\alpha\) are parameters that characterize positions of center of each soliton. (We shall discuss it later more precisely.) This type of solution was already studied as a planar soliton solution of the baby-Skyrme model \[29\]. Our model may be identified as a kind of generalization of the baby-Skyrme model, then the form (21) should be applicable in the case of our vortex system. Note that (21) satisfies the zero curvature condition. Substituting (21) into (9), and also introducing the suitable form of the potential, we finally obtain form of the potential for the two centered vortex solution

\[V_2 = \frac{\mu^2}{4a_2^2} \left\{ n_1 + in_2 + \left(\frac{c}{a_2}\right)^2 e^{i[2\alpha + k(t + z)]} (1 - n_3) \right\} \times \left\{ n_1 - in_2 + \left(\frac{c}{a_2}\right)^2 e^{i[2\alpha + k(t + z)]} (1 - n_3^2) \right\} (1 - n_3)^2.\]  

For the case of static vortex \(k = 0\) (which coincides with the planar baby-skyrmion), the similar potential was obtained in \[29\].

3. The three-centered solution and the potential

Next we move to a three-centered form of the vortex solution. As the beginning, we construct a holomorphic solution with three independent centers. Each center is located on the tip of a triangle (see Fig.1) and the cores of the vortices are located at \(P_1, P_2\) and \(P_3\). We describes the point \(P_1\) by the polar coordinates \((c, \alpha)\). It is convenient to define the fixed point \(P\). From the point \(P\), the center of each vortex has coordinates \((\rho_1, \varphi_1)\), \((\rho_2, \varphi_2)\) and \((\rho_3, \varphi_3)\), where the \(\rho_i\) are distances between \(P\) and \(P_i\) and \(\varphi_i\) are angles measured from the horizontal axis.
Each components of such coordinates are defined as
\[
\rho_1 := a_3 \sqrt{\rho^2 - 2\rho c \cos(\varphi - \alpha) + c^2},
\]
\[
\varphi_1 := \arctan \left( \frac{\rho \sin \varphi - c \sin \alpha}{\rho \cos \varphi - c \cos \alpha} \right);
\]
\[
\rho_2 := a_3 \sqrt{\rho^2 + \rho c \cos(\varphi - \alpha) - \sqrt{3} \rho c \sin(\varphi - \alpha) + c^2},
\]
\[
\varphi_2 := \arctan \left( \frac{2\rho \sin \varphi - \sqrt{3} \rho \cos \alpha + c \sin \alpha}{2\rho \cos \varphi + c \cos \alpha + \sqrt{3} \rho \sin \alpha} \right);
\]
\[
\rho_3 := a_3 \sqrt{\rho^2 + \rho c \cos(\varphi - \alpha) + \sqrt{3} \rho c \sin(\varphi - \alpha) + c^2},
\]
\[
\varphi_3 := \arctan \left( \frac{2\rho \sin \varphi + \sqrt{3} \rho \cos \alpha + c \sin \alpha}{2\rho \cos \varphi + c \cos \alpha - \sqrt{3} \rho \sin \alpha} \right),
\]
where \(a_3\) is an arbitrary scale parameter.

The three vortex solution has of the form
\[
u_3 = \frac{\rho_1}{a_3} \frac{\rho_2}{a_3} \frac{\rho_3}{a_3} \left( \frac{\rho}{a_3} \right)^3 e^{i(3\varphi + k(t+z))} - \left( \frac{c}{a_N} \right)^3 e^{i[3\alpha + k(t+z)]},
\]
where \(\psi \equiv \left( \frac{\rho}{a_3} \right)^3 e^{3i\varphi} e^{ik(t+z)}\). Substituting [23] into [11] and introducing suitable form of the potential, we finally obtain form of the potential of three-centered solution as
\[
V_3 = \frac{\mu^2}{4a_3^3} \left\{ n_1 + in_2 + \left( \frac{c}{a_3} \right)^3 e^{i[3\alpha + k(t+z)]}(1 - n_3) \right\}^{\frac{1}{2}}
\]
\times \left\{ n_1 - in_2 + \left( \frac{c}{a_3} \right)^3 e^{-i[3\alpha + k(t+z)]}(1 - n_3) \right\}^{\frac{1}{2}} (1 - n_3)^{\frac{1}{2}}.
\]

4. The \(N\)-centered solution and the potential

We can straightforwardly generalize the procedure for an arbitrary \(N\)-centered vortex solutions. The \(N\)-centered solution can be written as the form
\[
u_N = \left( \frac{\rho}{a_N} \right)^N e^{i[N\varphi + k(t+z)]} - \left( \frac{c}{a_N} \right)^N e^{i[N\alpha + k(t+z)]},
\]
where \(a_N\) is an arbitrary scale parameter. Here \(c\) and \(\alpha\) are parameters which give information of the positions of each vortex. The each center is located on the top of an \(N\)-positive square. Note that the solution [26] satisfies the the zero curvature condition [8]. Again substituting [26] into the generalized potential [9], we finally obtain form of the potential as
\[
V_N = \frac{\mu^2}{4a_N^3} \left\{ n_1 + in_2 + \left( \frac{c}{a_N} \right)^N e^{i[N\alpha + k(t+z)]}(1 - n_3) \right\}^{2 - \frac{4}{5}}
\]
\times \left\{ n_1 - in_2 + \left( \frac{c}{a_N} \right)^N e^{-i[N\alpha + k(t+z)]}(1 - n_3) \right\}^{2 - \frac{4}{5}}
\times (1 - n_3)^{\frac{4}{5}}.
\]

In the case of \(c = 0\) the potential [27] becomes \((1 + n_3)^{2-2/N}(1 - n_3)^{2+2/N}\), which exactly coincides with the result proposed in [20].

IV. THE ZERO CURVATURE CONDITION

There is quite a interesting feature of our potential; the potential [13] can generally be rewritten as of the form
\[
V \to \mu^2 \left( \frac{\partial_\mu u \partial^\mu u^*}{1 + |u|^2} \right)^2,
\]
where the coefficient \(\mu^2\) is
\[
\mu^2 \equiv 8s \frac{\beta^2 - 1}{c^2}
\]
in the context of the \(CP^1\) model for integrable theories in any dimension \([30]\), and then applied to many models with target space being the sphere \(S_2\), or \(CP^1\). It leads to an infinite number of local conserved currents. Indeed, the equation of motion \([10]\) implies the conserved currents given by
\[
J^G_\mu \equiv \tilde{K}_\mu \frac{\delta G}{\delta u} - \tilde{K}^*\mu \frac{\delta G}{\delta u^*},
\]
where \(G\) is assumed to be any functional of \(u, u^*\). \(\tilde{K}_\mu\) can be defined as follows
\[
\tilde{K}_\mu = M^2 \partial_\mu u + \frac{4}{c^2} \left( \partial_\mu u \partial^\mu u^* \right) \left( \frac{1}{1 + |u|^2} \right)^2
\]
+ \(\beta c^2 - 1 - s) \left( \partial_\mu u \partial^\mu u^* \right) \partial_\mu u \]
by using above replacement formula (28).

The current is conserved because in the derivative of the current

\[ \partial^\mu J^G_\mu = \frac{\delta^2 G}{\delta u^2} \partial^\mu u \tilde{K}_\mu + \frac{\delta G}{\delta u} \partial^\mu \tilde{K}_\mu + \frac{\delta^2 G}{\delta u \delta u^*} \partial^\mu u^* \tilde{K}_\mu - \frac{\delta^2 G}{\delta u^2} \partial^\mu u^* \tilde{K}_\mu - \frac{\delta G}{\delta u^*} \partial^\mu \tilde{K}_\mu - \frac{\delta^2 G}{\delta u \delta u^*} \partial^\mu u \tilde{K}_\mu \]

the first and fourth terms vanish due to the constraint [3], and the third and sixth terms cancel out and, the second and fifth terms vanish because now the Euler-Lagrange equation has the form

\[ \partial^\mu \tilde{K}_\mu = 0, \quad \partial^\mu \tilde{K}_\mu^* = 0. \] (33)

Thus the current \( J^G_\mu \) is conserved and it leads to an infinite number of conserved quantities.

**V. THE NUMERICAL STUDY**

In the previous sections(7,3),(994,990), assuming the form of the solutions which satisfy the zero curvature condition, we have obtained a general form of potential \([27]\) for \( \beta e^2 \neq 1 \). In this section, conversely, we numerically check whether the potentials have the desired solutions. The simulated annealing method \([31]\) is a Hamiltonian minimization scheme which successfully finds the solution without any assumption. The Hamiltonian density has the form

\[ \mathcal{H} = 4M^2 \left[ \frac{|\dot{u}|^2 + \nabla u \cdot \nabla u^*}{(1 + |u|^2)^2} - \frac{24}{c^2} (\nabla u^* \nabla u)^2 \left( \frac{2}{3} \right)^2 - F^2 \right. \]

\[ - \frac{24(\beta e^2 - 1)}{c^2} \frac{|\dot{u}|^2 + \frac{1}{2} \nabla u \cdot \nabla u^* |\nabla u \cdot \nabla u^* - |u|^2|}{(1 + |u|^2)^4} + V(u, u^*), \]

where \( \dot{u} \) denotes the \( \dot{u} \)-derivative of \( u \), and \( \nabla u \) its spatial gradient, and where we have denoted

\[ \frac{|\dot{u}|^2}{(\nabla u)^2} = \frac{1}{3} + F \exp[i \Phi]. \] (35)

If we choose \( u \) such that it satisfies the zero curvature condition, \( F \) becomes \( \frac{2}{3} \) (see [13]). Here we employ \([27]\) as the potential \( V \). The results of the energy density for \( N = 2, \ldots, 5 \) are given in Fig. 4. These constitute \( N \)th peaks of which each have unit topological charge. Note that for larger \( a_N \), the radius of the solution grows but never split into \( N \) independent constituents. In Fig. 4 we plot the total energy for \( N = 2, \ldots, 5 \) with the analytical results for the one-centered solutions previously found in \([20]\)

\[ E_{\text{tori}} = 4M^2 \left[ 2\pi N + \frac{2\pi}{3} \frac{1}{a_N^3} (\beta e^2 - 1)(N^2 - 1)I(N) \right. \]

\[ + k^2 \left\{ 2\pi a_N^3 I(N) + \frac{2\pi}{3} (\beta e^2 - 1)N \right\}. \] (36)

where \( I(N) = \frac{1}{\Gamma(1 + \frac{1}{N})} \Gamma(1 - \frac{1}{N}) \) and \( k^2 = 1 \). The energies multi-centered solution is higher than that of one-centered. It means that the energy grows as we employ larger value of \( c \). However, it does not mean that the solution is unstable under the scale changes. Fig. 4 shows a time process of our simulation. We start with the toroidal (one-centered) initial condition, and finally we arrive lower energy state with multi-centered configuration. So, at least it is numerically stable and is the solution for the choice of \( c \) (see Fig. 4).

In Fig. 5 we compare the arrow plot of the component \((n_1, n_2)\) of \([23]\) with the corresponding numerical solution using three centered potentials \([25]\). The results almost coincide. Finally, we check the zero curvature condition of our numerical solutions. Fig. 6 indicates that the value is not exactly zero, mainly because of the size effect and the discretization error. However, we think that a zero curvature condition is met, because a value is very small, compared with the net energy, i.e., the ratio is \( \sim 10^{-8} \), which is an order of numerical uncertainty.

**VI. SUMMARY AND OUTLOOK**

In this paper, we mainly discussed how to get analytical, multi-centered vortex solutions and corresponding potentials of the extended version of the Skyrme-Faddeev model. We found forms of the potential \([27]\) for our ansatz of the \( N \)-centered solutions. We confirmed that the forms of the potentials coincide with the previous studies for the one- and two-centered solutions \([21, 29]\). We could extend them for the solution with the arbitrary number of the center.

There are many studies to get such multi-centered solutions introducing some special form of potentials. They are based on the numerical analysis and for the most, the obtained solutions are planar, static solutions. Contrary to the cases, we found the analytical, traveling wave vortex solutions going to the (minus) \( z \) direction. The solutions have the infinite number of conserved quantities. We confirmed how the potentials work by examining the full-field relaxation analysis using the potentials. We got the solutions which coincide with the assumed analytical solutions.

There are similar molecule like solutions and some of them are certainly stable for change of their scale \([27, 28]\). On the contrary, our solutions have always higher energies compared with the corresponding lump shaped solution, then our solutions have no minima for the scale change. Nevertheless, they have several advantages, which are:

(i) they are integrable solutions and then have an infinite number of conserved quantities,

(ii) they are composed of constituents with unit topological charge,

(iii) the method is easily extendable for more complicated configuration such as multi-shell, multi-facet...
forms.

Our scheme is quite general and easily applicable to the related two dimensional solitonic models such as the baby-Skyrme model, the $CP^N$ sigma model and so on. For the physical application, some vortex states in a superconductor may be possible candidates of our solutions. The model has a relationship with the standard electroweak theory, especially when one considers the case of a global $SU(2)$ and a local $U(1)$ breaking into a global $U(1)$, where the model reduces to an Abelian Higgs model with two charged scalar fields. It is interesting to note that the vortices of such model carry the so-called longitudinal electromagnetic currents. Furthermore, the higher winding number solutions exhibit a pipe-like structure.

Our model enjoys a symmetry breaking of the type $O(3)_{\text{global}} \to D_{N\text{global}}$ which is complicated than $SU(2)_{\text{global}} \otimes U(1)_{\text{local}} \to U(1)_{\text{global}}$. It is certainly interesting that we explore the conserved quantities and the nontrivial structures of a type II superconductors by our prediction.

Undoubtedly it is worthwhile to apply our technique to more realistic three dimensional problems. The Skyrme model is a low energy effective model of nuclei and the multi-winding number solutions exhibit the platonic symmetries. The main drawback of such solutions is that the density becomes zero at the vicinity of the center and it is inconsistent with structure of the actual nuclei. There is a certain possibility that our molecule type ansatz may cure of such difficulty and it may describe the basic properties of the observed nuclei, e.g. the charge density.

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FIG. 2: The energy densities with $N = 2 - 5$ for $\beta e^2 = 2.0$ and $\mu = 1.0$.

FIG. 3: The energy for the numerical solutions and one center analytical solutions $N = 2 - 5$ with $\beta e^2 = 2.0$, $\mu = 1.0$ and $a_N = 1$. 
FIG. 4: The energy per charge for the simulation of $N = 3$ with $\beta e^2 = 2.0$ and $\mu = 1.0$.

FIG. 5: The arrow plot $(n_1, n_2)$ of the numerical solution (left) and the analytical solution (right) for $\beta e^2 = 2.0$, $\mu = 1.0$ and $N = 3$. 
FIG. 6: The zero curvature condition in terms of the annealing simulation, for $\beta e^2 = 2.0$, $\mu = 1.0$ and $N = 3$. 