BOUNDS ON THE $a$-INVARIANT AND REDUCTION NUMBERS OF IDEALS

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Abstract. Let $R$ be a $d$-dimensional standard graded ring over an Artinian local ring. Let $\mathfrak{M}$ be the unique maximal homogeneous ideal of $R$. Let $h^i_n(R)$ denote the length of the $n$th graded component of the local cohomology module $H^i_\mathfrak{M}(R)$. Define the Eisenbud-Goto invariant $EG(R)$ of $R$ to be the number $\sum_{q=0}^{d-1} \binom{d-1}{q} h^q_\mathfrak{M}(R)_{1-q}$. We prove that the $a$-invariant $a(R)$ of the top local cohomology module $H^d_\mathfrak{M}(R)$ satisfies the inequality: $a(R) \leq \ell(R) - \ell(R_1) + (d-1)(\ell(R_0) - 1) + EG(R)$. This bound is used to get upper bounds for the reduction number of an $\mathfrak{m}$-primary ideal $I$ of a Cohen-Macaulay local ring $(R, \mathfrak{m})$, when the associated graded ring of $I$ has depth at least $d-1$.

1. Introduction

Let $R = \mathop{\bigoplus}_{n=0}^{\infty} R_n$ be a $d$-dimensional standard graded ring over an Artinian local ring $R_0$. Let $\mathfrak{M}$ be the maximal homogeneous ideal of $R$. Let $H^i_\mathfrak{M}(R)$ denote the $i$-th local cohomology module of $R$ with respect to $\mathfrak{M}$. For a graded module $M$, we use $[M]_n$ or $M_n$ to denote the $n$th graded component of $M$. The $a$-invariant of $R$, introduced in [GW], is defined as

$$a(R) = \max\{n \mid [H^d_\mathfrak{M}(R)]_n \neq 0\}.$$  

The objective of this paper is to give a bound for the $a$-invariant of $R$ in terms of lengths of graded components of local cohomology modules and use it to get bounds for reduction numbers of ideals. Let $\ell(M)$ denote length of a module $M$. We set $\ell\left([H^q_\mathfrak{M}(R)]_{1-q}\right) = h^q(R)_{1-q}$ for all $q \geq 0$. To state our bound for the $a$-invariant we define the Eisenbud-Goto invariant $EG(R)$ of $R$ to be the number

$$EG(R) = \sum_{q=0}^{d-1} \binom{d-1}{q} h^q(R)_{1-q}.$$  

Keywords: $a$-invariant, reduction number, Eisenbud-Goto invariant, local cohomology.

AMS 2000 classification: 13D45, 13A02.

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The main result of the paper is:

**Theorem 1.1.** Let \( R = \bigoplus_{n=0}^{\infty} R_n \) be a \( d \)-dimensional standard graded algebra over an artinian local ring \( R_0 \) with multiplicity \( e(R) \). Then

\[
a(R) \leq e(R) - \ell(R_1) + (d - 1)(\ell(R_0) - 1) + \text{EG}(R).
\]

Eisenbud and Goto [EG] showed that if \( R_0 \) is a field then

\[
e(R) \geq 1 + \text{codim}(R) - \text{EG}(R).
\]

They showed that if equality holds in the above inequality then \( R/H^0_{\mathfrak{m}}(R) \) has linear resolution. To state our bounds for reduction numbers we recall some basic concepts about reductions of ideals. Let \((R, \mathfrak{m})\) be a local ring. Let \( J \subset I \) be ideals of \( R \). The ideal \( J \) is called a reduction of \( I \) if there exists an \( n \in \mathbb{N} \) such that \( JI^n = I^{n+1} \) [NR]. Among the reductions of \( I \), the smallest ones with respect to inclusion are called minimal reductions of \( I \). If \( R/\mathfrak{m} \) is infinite then any minimal reduction of \( I \) is minimally generated by as many elements as the Krull dimension of the fiber cone \( F(I) := \bigoplus_{n=0}^{\infty} I^n/\mathfrak{m}I^n \). The reduction number, \( r_J(I) \), of \( I \) with respect to a minimal reduction \( J \) is the least integer \( n \) for which \( JI^n = I^{n+1} \). When \( R/\mathfrak{m} \) is infinite, the reduction number of \( I \) is defined as the minimum of the reduction numbers \( r_J(I) \) where \( J \) varies over all the minimal reductions of \( I \). Let \( G(I) := \bigoplus_{n \geq 0} I^n/I^{n+1} \) be the associated graded ring of an ideal \( I \). Let \( \gamma(I) \) denote the depth of the irrelevant ideal \( G_+ \) of \( G(I) \). If \((R, \mathfrak{m})\) is a Cohen-Macaulay local ring, \( I \) is an \( \mathfrak{m} \)-primary ideal and \( \gamma(I) \geq d - 1 \), then \( r(I) = a(G(I)) + d \) [M]. The Ratliff-Rush closure of an ideal \( I \), \( \tilde{I} \), is the stable value of the sequence of the ideals \( \{I^{n+1} : I^n\} \). We will obtain the following bounds for \( r(I) \) as an application of the main theorem:

**Theorem 1.2.** Let \((R, \mathfrak{m})\) be a \( d \)-dimensional Cohen-Macaulay local ring with infinite residue field. Let \( I \) be an \( \mathfrak{m} \)-primary ideal with \( \gamma(I) \geq d - 1 \). Let \( J \) be any minimal reduction of \( I \).

1. Let \( d = 1 \). Then \( r(I) \leq e(I) - \left( \ell(I/(I \cap \tilde{I}^2)) - 1 \right) \leq e(I) \).
2. Let \( d = 2 \). Put \( X = \text{Proj}(G(I)) \). Then \( r(I) \leq 1 + e(I) - \ell(I/I^2) + \ell(H^0(X, \mathcal{O}_X)) \).
3. Let \( d \geq 3 \). Then \( r(I) \leq 1 + \ell(I^2/JI) + h^{d-1}(G(I))_{2-d} \).

We will show by an example that our bounds for the \( a \)-invariant and reduction number are sharp.
2. A BOUND ON THE $a$-INARIANT OF STANDARD GRADED ALGEBRAS

In this section we prove our bound on the $a$-invariant of a standard graded algebra $R$ over an Artinian local ring $R_0$.

**Theorem 2.1.** Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a $d$-dimensional standard graded algebra over an Artinian local ring $R_0$. Then

\begin{equation}
  a(R) \leq e(R) - \ell(R_1) + (d - 1)(\ell(R_0) - 1) + EG(R).
\end{equation}

**Proof.** We may assume without loss of generality that the residue field of $R_0$ is infinite. We prove the theorem by induction on $d$. Let $d = 0$. Then

\[ e(R) = \ell(R_0) + \ell(R_1) + \cdots + \ell(R_m), \]

where $m = a_0(R)$. Thus

\[ e(R) - \ell(R_1) - \ell(R_0) + 1 = 1 + \ell(R_2) + \cdots + \ell(R_m) \geq m. \]

Let $R$ be Cohen-Macaulay and pick a degree one nonzerodivisor $x$ to see that

\[ a(R) = a(R/xR) - 1 \]

\[ \leq e(R/xR) - \ell([R/xR]_1) + (d - 2)(\ell(R_0) - 1) - 1 \]

\[ = e(R) - \ell(R_1) + \ell(R_0) + (d - 2)(\ell(R_0) - 1) - 1 \]

\[ = e(R) - \ell(R_1) + (d - 1)(\ell(R_0) - 1). \]

Now let $d = 1$. If $R$ is Cohen-Macaulay, we are done by the above argument. So let depth$(R) = 0$. Then $S := R/H^{0}_{m}(R)$ is Cohen-Macaulay, $e(S) = e(R)$ and $a(R) = a(S)$. Hence

\[ a(R) = a(S) \leq e(S) - \ell(S_1) = e(R) - \ell(R_1) + h^{0}(R)_1. \]

Suppose $d \geq 2$. Let $x \in R_1$ be a superficial element. We first prove that for a degree one superficial element in $R$,

\[ EG(R/xR) \leq EG(R). \]

Since $x$ is superficial of degree one,

\[ H^{i}_{m}(0 : R x) = \begin{cases} 
(0 : R x) & \text{if } i = 0 \\
0 & \text{if otherwise.} 
\end{cases} \]
Hence from the short exact sequence

\[ 0 \rightarrow (0 :_R x) \rightarrow R \rightarrow \frac{R}{(0 :_R x)} \rightarrow 0 \]

we get \( H^i_{2n}(R/(0 :_R x)) = H^i_{2n}(R) \) for all \( i \geq 1 \). From the exact sequence

\[ 0 \rightarrow \frac{R}{(0 :_R x)}(-1) \rightarrow R \rightarrow \frac{R}{xR} \rightarrow 0 \]

we get the long exact sequence

\[ \cdots \rightarrow [H^i_{2n}(R)]_n \rightarrow [H^i_{2n}(R/xR)]_n \rightarrow [H^{i+1}_{2n}(R)]_{n-1} \rightarrow \cdots \]

Hence for all \( i \geq 0 \),

\[ h^i(R/xR)_n \leq h^i(R)_n + h^{i+1}(R)_{n-1}. \]

Hence

\[
EG(R/xR) = \sum_{q=0}^{d-2} \binom{d-2}{q} h^q(R/xR)_{1-q} \\
\leq \sum_{q=0}^{d-2} \binom{d-2}{q} [h^q(R)_{1-q} + h^{q+1}(R)_{-q}] \\
= \sum_{q=0}^{d-2} \binom{d-2}{q} h^q(R)_{1-q} + \sum_{q=1}^{d-1} \binom{d-2}{q-1} h^q(R)_{1-q} \\
= \sum_{q=0}^{d-1} \binom{d-1}{q} h^q(R)_{1-q} \\
= EG(R).
\]

Therefore

\[
a(R) \leq a(R/xR) - 1 \text{ by [1]}
\leq e(R/xR) - \ell(R/xR)_1 + (d - 2) \left( \ell ([R/xR]_0) - 1 \right) + EG(R) - 1 \\
= e(R) - \ell(R_1) + \ell(R_0) - \ell ((0 : x)_{R_0}) + (d - 2) \left( \ell(R_0) - 1 \right) \\
+ EG(R) - 1 \\
\leq e(R) - \ell(R_1) + (d - 1) (\ell(R_0) - 1) + EG(R).
\]

\[ \square \]

We now demonstrate that the bound in Theorem 2.1 is sharp.
Example 2.2. Let $k$ be a field and $x,y,a,b,c,d$ be indeterminates. Consider the ideal $I = (x^3, x^2y^4, xy^5, y^7)$ in the polynomial ring $S = k[x,y]$. Using Hilbert series we show that $F(I) \simeq k[a,b,c,d]/(bd,bc,b^2,c^3)$. Consider the ring homomorphism $\phi : R = k[a,b,c,d] \longrightarrow F(I)$ defined by

$$
\phi(a) = x^3, \quad \phi(b) = x^2y^4, \quad \phi(c) = xy^5, \quad \text{and} \quad \phi(d) = y^7.
$$

Here the overbar indicates the image in the first graded component of $F(I)$. Let $L = \ker \phi$. The equations

$$
(x^2y^4)(y^7) = (xy^5)^2 y, \quad (x^2y^4)(xy^5) = (x^3y^7)y^2, \quad (x^2y^4)^2 = (x^3)(xy^5)y^3, \quad (xy^5)^3 = x^3(y^7)^2 y,
$$

show that $N = (bd,bc,b^2,c^3) \subset L$. To show that $N = L$, we show that $R/N$ and $R/L$ have same Hilbert series. We denote the Hilbert series of a graded algebra $G$ by $H(G, \lambda)$. By the propositions 2.3 and 2.6 of [H] we find that $\mu(I^n) = 3n + 1$ for all $n \geq 0$. Here $\mu$ denotes the minimum number of generators. This shows that $H(F(I), \lambda) = (1 + 2\lambda)/(1 - \lambda)^2$. By the well known ”divide and conquer strategy” for finding Hilbert series of quotients of polynomial rings by monomial ideals we get, $H(R/N, \lambda) = H(F(I), \lambda)$. Thus $F(I) \cong R/N$. Therefore $F(I)$ is a two - dimensional ring with depth one. Notice that $N = (b,c^3) \cap (c,d,b^2)$. Put $J = (b,c^3)$ and $K = (c,d,b^2)$. In order to get the desired information about local cohomology of $F(I)$, consider the exact sequence :

$$
0 \longrightarrow F(I) \longrightarrow R/J \bigoplus R/K \longrightarrow R/(J + K) \longrightarrow 0.
$$

Hence we get the following long exact sequence of local cohomology modules with respect to the maximal homogeneous ideal $m = (a,b,c,d) :

$$
0 \longrightarrow H^1_m(F(I)) \longrightarrow H^1_m(R/K) \longrightarrow H^1_m(R/(J + K)) \longrightarrow H^2_m(F(I)) \longrightarrow H^2_m(R/J) \longrightarrow 0.
$$

We now show that $a(F(I)) = 0$ and $h^1(F(I))_0 = 1$. Since

$$
R/J \simeq k[a,c,d]/(c^3), \quad R/(J + K) \simeq k[a] \quad \text{and} \quad R/K \simeq k[a,b]/(b^2),
$$

by using that fact that $a(R/(f)) = a(R) + \deg(f)$ for a homogeneous regular element $f$ of a graded algebra $R$, we conclude that $a(R/J) = 0, \quad a(R/(J + K)) = \cdots$
In the next result we will need the formula: $H_\gamma$. Let Proposition 3.1.

$n \geq i$ be an $I$ since $H$. Proof. By $[T]$ and $[M]$, we know that Hence $r(I) = a(G(I)) + d$ where $(R, m)$ is a Cohen-Macaulay local ring of dimension $d$ and $\gamma(I) \geq d - 1$. We will consider the cases where $d = 1$, $d = 2$ and $d \geq 3$ separately. In the next result we will need the formula: $[H_0^G, (G(I))](I) = (I^n \cap \tilde{I}^{n+1})/I^{n+1}$ for all $n \geq 0$ $[HJLS]$.

**Proposition 3.1.** Let $(R, m)$ be a one-dimensional Cohen-Macaulay local ring. Let $I$ be an $m$-primary ideal. Then

$$r(I) \leq e(I) - \left[ \ell(I/(I \cap \tilde{I}^2) - 1 \right] \leq e(I).$$

**Proof.** Since $d = 1$,

$$a(G(I)) \leq e(I) - \ell(I/I^2) + h_1^0(G) = e(I) - \ell(I/I^2) + \ell((I \cap \tilde{I}^2)/I^2) = e(I) - \ell(I/(I \cap \tilde{I}^2)).$$

Hence $r(I) = a(G(I)) + 1 \leq 1 + e(I) - \ell(I/(I \cap \tilde{I}^2))$. If $\ell(I/(I \cap \tilde{I}^2) = 0$ then $I \subseteq \tilde{I}^2$. But $\tilde{I}^2 = I^{n+2} : I^n$ for large $n$. Hence $I^{n+1} = I^{n+2}$. This is a contradiction. Hence $\ell(I/(I \cap \tilde{I}^2)) \geq 1$. Thus we obtain the classical bound $r(I) \leq e(I)$.

**Example 3.2.** Let $k$ be a field and $t$ be an indeterminate. Put $R = k[[t^4, t^5, t^6, t^7]]$ and $I = (t^4, t^5, t^6)$. Let $m$ denote the unique maximal ideal of $R$. Let $G$ denote the associated graded ring $G(I)$ of $I$. Then $G$ is not Cohen-Macaulay since $t^7 I \subset I^2$. To find the associated Ratliff-Rush ideal of $I$ notice that $I^2 = m^2$. Since $r(m) = 1$, the associated graded ring $G(m)$ is Cohen-Macaulay by $[S]$. Therefore all powers of $m$ are Ratliff-Rush. Hence, $\tilde{I}^2 = \tilde{m}^2 = m^2 = I^2$. Hence $(I \cap \tilde{I}^2/I^2) = 0$. Therefore $r(I) \leq 1 + e(I) - \ell(I/I^2) = 2$. It can be checked that $r_{(t^4)}(I) = 2$. Therefore the bound in the above result is sharp.
Proposition 3.3. Let $I$ be an $m$-primary ideal of a two dimensional Cohen-Macaulay local ring with $\gamma(I) \geq 1$. Let $X = \text{Proj } G(I)$. Then

$$r(I) \leq 1 + e(I) - \ell(I/I^2) + \ell(H^0(X, \mathcal{O}_X)).$$

Proof. Since $\gamma(I) \geq 1$,

$$r(I) \leq 1 + e(I) - \ell(I/I^2) + \ell(R/I) + h^1(G)_0.$$

By the exact sequence

$$0 \rightarrow H^0_{G_+}(G) \rightarrow G \rightarrow \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n)) \rightarrow H^1_{G_+}(G) \rightarrow 0$$

we get, by taking the 0th component of all the modules in the above exact sequence:

$$\ell(H^0(X, \mathcal{O}_X)) - \ell(R/I) = h^1(G)_0.$$

Putting this in the above bound for $r(I)$ we get the desired upper bound. \(\square\)

Proposition 3.4. Let $(R, m)$ be a Cohen-Macaulay local ring of dimension $d \geq 3$ with $R/m$ infinite. Let $I$ be an $m$-primary ideal with a minimal reduction $J$ and $\gamma(I) \geq d - 1$. Then

$$r(I) \leq 1 + \ell(I^2/JI) + h^{d-1}(G)_{2-d}.$$

Proof. Since $\gamma(I) \geq d - 1$, by the Theorem 2.1

$$r(I) \leq e(I) - \ell(I/I^2) + (d-1)(\ell(R/I) - 1) + h^{d-1}(G)_{2-d} + d$$

$$= \ell(R/J) - \ell(R/I^2) + \ell(R/I) + (d-1)\ell(R/I)$$

$$- (d-1) + d + h^{d-1}(G)_{2-d}$$

$$= \ell(R/J) - \ell(R/I^2) + \ell(J/II) + 1 + h^{d-1}(G)_{2-d}$$

$$= 1 + \ell(I^2/II) + h^{d-1}(G)_{2-d}$$

\(\square\)

Acknowledgements: The first author wishes to acknowledge the hospitality of the Indian Institute of Technology Bombay, where part of the paper was done. The last named author wishes to thank the Institute of Mathematical Sciences where this work was initiated.
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