The Minimum Stretch Spanning Tree Problem for Typical Graphs

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Abstract With applications in communication networks, the minimum stretch spanning tree problem is to find a spanning tree \( T \) of a graph \( G \) such that the maximum distance in \( T \) between two adjacent vertices is minimized. The problem has been proved NP-hard and fixed-parameter polynomial algorithms have been obtained for some special families of graphs. In this paper, we concentrate on the optimality characterizations for typical classes of graphs. We determine the exact formulae for the complete \( k \)-partite graphs, split graphs, generalized convex graphs, and several planar grids, including rectangular grids, triangular grids, and triangulated-rectangular grids.

Keywords communication network; spanning tree optimization; tree spanner; max-stretch; congestion

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1 Introduction

Since Peleg et al. [22] in 1989, a series of tree spanner problems arise in connection with applications in distribution systems and communication networks (see survey [16]). A basic decision version of the tree spanner problems for a graph \( G \) is as follows: For a given integer \( k \), is there a spanning tree \( T \) of \( G \) (called a tree \( k \)-spanner) such that the distance in \( T \) between every pair of vertices is at most \( k \) times their distance in \( G \)? The corresponding optimization version of the problem is to find the minimum \( k \) such that there exists a tree \( k \)-spanner of \( G \). This spanning tree optimization problem is referred to as the minimum stretch spanning tree problem and MSST for short [4–6, 10, 11, 18].

We formulate this optimization problem formally. Let \( G \) be a simple connected graph with vertex set \( V(G) \) and edge set \( E(G) \). Given a spanning tree \( T \) of \( G \), for \( uv \in E(G) \), let \( d_T(u,v) \) denote the distance between \( u \) and \( v \) in \( T \), that is the length of the unique \( u-v \)-path in \( T \). Then the stretch of a spanning tree \( T \) is defined by

\[
\sigma_T(G,T) := \max_{uv \in E(G)} d_T(u,v).
\]  

Furthermore, the minimum stretch spanning tree problem is to determine

\[
\sigma_T(G) := \min \{ \sigma_T(G,T) : T \text{ is a spanning tree of } G \}.
\]

This gives rise to a graph invariant \( \sigma_T(G) \), called the tree-stretch of \( G \). Here, we follow the notation \( \sigma_T(G) \) in [10].

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For an edge $e = uv$ not in $T$, the unique cycle in $T + e$ is called the fundamental cycle with respect to $e$. So, the above problem is equivalent to finding a spanning tree such that the length of a maximum fundamental cycle is minimized, where the tree-stretch $\sigma_T(G)$ is one less than the length of this cycle. This is precisely the shortest maximal fundamental cycle problem proposed by Galbiati[12]. As is well known, all fundamental cycles with respect to a spanning tree $T$ constitute a basis of the cycle space of $G$[11]. Thus we have an optimal basis problem in the cycle space.

In the duality point of view, for each $e \in T$, the edge-cut between two components of $T - e$ is a fundamental edge-cut (cocycle). Let $X_e$ and $Y_e$ be the vertex sets of these components of $T - e$. Write $\partial(X_e) := \{uv \in E(G) : u \in X_e, v \in Y_e\}$. Then $\partial(X_e)$ is the fundamental edge-cut with respect to $e$, and $|\partial(X_e)|$ is called the congestion of edge $e$. The minimum congestion spanning tree problem, proposed by Ostrovskii [20] in 2004, is to determine

$$c_T(G) := \min_{e \in T} \max_{e \in T} |\partial(X_e)| : T \text{ is a spanning tree of } G.$$

This graph invariant $c_T(G)$ is called the tree-congestion of $G$.

Admittedly, the tree-congestion $c_T(G)$ is a variant of the cutwidth $c(G)$ of $G$ and the tree-stretch $\sigma_T(G)$ is a variant of the bandwidth $B(G)$ of $G$ (see surveys [8, 9]). In the circuit layout of VLSI designs and network communication, the quality of an embedding is usually evaluated by two parameters, namely, the dilation and the congestion. The dilation motivates the bandwidth problem and the congestion leads to the cutwidth problem.

So far the main concern of the tree spanner problems is in the algorithmic aspects, including the NP-hardness [4-6, 10, 12], the fixed-parameter polynomial algorithms [4, 5, 10, 11] and the approximability [12]. Moreover, for the characterization problem, it is known that determining $\sigma_T(G) \leq 2$ is polynomially solvable [6], while determining $\sigma_T \leq k$ for $k \geq 4$ is NP-complete. A long-standing open problem is to characterize $\sigma_T(G) = 3$. In this respect, it is significant to determine exact value of $\sigma_T(G)$ for typical classes of graphs.

The minimum congestion spanning tree problem has been studied extensively in the literature. On the complexity aspect, the NP-hardness even for chain graphs or split graphs was shown in [19]. Linear time algorithms for fixed parameter $k$ and for planar graphs, bounded-degree graphs and treewidth bounded graphs were presented in [3]. Additionally, determining the exact values of $c_T(G)$ for special graphs has found an increasing interest during the last decade, for example:

- The complete graphs $K_n$, the complete bipartite graphs $K_{m,n}$, and the planar grids $P_m \times P_n$ [7, 14].
- The complete $k$-partite graphs $K_{n_1,n_2,\ldots,n_k}$ and the torus grids $C_m \times C_n$ [7, 15].
- The triangular grids $T_n$ [21].
- The $k$-outerplanar graphs [2].

Motivated by the above results on $c_T(G)$, our goal is to investigate the dual invariant $\sigma_T(G)$ for some basic families of graphs. The main results are parallel to those for $c_T(G)$.

The remaining of the paper is organized as follows. In Section 2, we present a basic lower bound by using the girth and derive the exact results for $K_n$, $C_n$, $K_{m,n}$, etc. In Section 3, we characterize $K_{n_1,n_2,\ldots,n_k}$, split graphs and generalized convex graphs. Section 4 is devoted to the exact representations for a class of plane graphs, including rectangular grids $P_m \times P_n$, triangular grids $T_n$, and triangulated-rectangular grids $T_{m,n}$.

2 Preliminaries

We shall follow the graph-theoretic terminology and notation of [1]. Let $G$ be a simple connected graph on $n$ vertices with vertex set $V(G)$ and edge set $E(G)$. For a subset $S \subseteq V(G)$, the neighbor set of $S$ is defined by $N_G(S) := \{v \in V(G) \mid S : u \in S, uv \in E(G)\}$. We abbreviate
for the complete graphs $K_n$ ($n \geq 3$), the wheels $W_n = C_{n-1} \lor K_1$ ($n \geq 4$), and the diamonds $D_n = K_2 \lor K_{n-2}$ ($n \geq 4$).

(2) If a bipartite graph $G$ has an edge $uv$ that $\{u, v\}$ is a dominating set, then $\sigma_T(G) = g(G) - 1 = 3$. So $\sigma_T(G) = 3$ for the complete bipartite graphs $K_{m,n}$ ($m, n \geq 2$) ([18]).

(3) $\sigma_T(G) = 3$ for the special planar grids $P_3 \times P_n$ ($n \geq 2$).

(4) $\sigma_T(C_n) = n - 1$ for the cycles $C_n$ ($n \geq 3$).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\caption{Examples in Proposition 2.3}
\end{figure}
Proof. (1) If $G$ has a dominating vertex $v$ (namely, $v$ is adjacent to all other vertices), then the star $K_{1,n-1}$ with center $v$ is an optimal tree, for which every fundamental cycle is a triangle. An example is shown in Figure 1(a).

(2) Suppose that $G$ is a bipartite graph with an edge $uv$ that $\{u, v\}$ is a dominating set. We can construct a spanning tree $T$ by the edge $uv$ and joining each other vertex to either $u$ or $v$ (which is called a double star with diameter three, see Figure 1(b)). Then every fundamental cycle with respect to $T$ has length 4, and thus $T$ is optimal.

(3) For the planar grid $P_3 \times P_n$, the girth is 4 and the ‘caterpillar’ with leaves on the boundary of outer face is an optimal tree (see Figure 1(c)).

(4) The cycle $C_n$ $(n \geq 3)$ has the unique fundamental cycle itself, which has length $n - 1$. This completes the proof.

It is interesting to characterize the graphs satisfying $\sigma_T(G) = g(G) - 1$, namely, those graphs having a spanning tree that every fundamental cycle is a shortest cycle. We shall see more examples in the next section.

3 Characterization of Low Stretch Graphs

This section is intended to connect with the open problem of characterizing $\sigma_T(G) = 3$. Madan-\mbox{e}-\mbox{l} et al.\cite{18} showed that $\sigma_T(G) \leq 3$ for all interval and permutation graphs, and that a regular bipartite graph $G$ has $\sigma_T(G) \leq 3$ if and only if it is complete. Moreover, Brandstät\mbox{d} et al.\cite{5} showed $\sigma_T(G) = 3$ for bipartite ATE-free graphs and convex graphs. Here, an ATE (asteroidal triple of edges) in a graph $G$ is a set $A$ of three edges that for any two edges $e_1, e_2 \in A$, there is a path from $e_1$ to $e_2$ that avoids the neighborhood of the third edge $e_3$ (the neighborhood of $uv$ is $N_G(u) \cup N_G(v)$). An ATE-free (asteroidal-triple-edge-free) graph is one which does not contain any ATE. The bipartite convex graphs form a special class of bipartite ATE-free graphs. A bipartite graph $G$ with bipartition $(X, Y)$ is said to be convex if $Y$ can be ordered as $Y = \{y_1, y_2, \ldots, y_n\}$ such that the neighbor set $N_G(x_i)$ is a consecutive sequence in $Y$ for each $x_i \in X$. We present more results in this context.

3.1 Complete $k$-partite Graphs

Let $\{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$ with $n_i = |V_i|$ $(1 \leq i \leq k)$. The complete $k$-partite graph $K_{n_1, n_2, \ldots, n_k}$ with $k \geq 2$ is a graph such that $uv \in E(G)$ if and only if $u \in V_i$ and $v \in V_j$ for $i \neq j$.

Theorem 3.1. Suppose that $n_1 \leq n_2 \leq \cdots \leq n_k$ and $k \geq 3$. Then

$$\sigma_T(K_{n_1, n_2, \ldots, n_k}) = \begin{cases} 2, & \text{if } n_1 = 1, \\ 3, & \text{otherwise.} \end{cases}$$

Proof. Let $G = K_{n_1, n_2, \ldots, n_k}$ $(k \geq 3)$. Obviously, the girth of $G$ is 3. When $n_1 = 1$, we have $\sigma_T(G) = 2$ by (1) of Proposition 2.3. When $n_1 \geq 2$, we will show that for any spanning tree $T$, $\sigma_T(G, T) \geq 3$. By letting $X = V_2 \cup \cdots \cup V_k$, we have a complete bipartite graph $G'$ of bipartition $(V_1, X)$. There are two cases to consider.

(i) The spanning tree $T$ contains no edges between vertices in $X$. Then $T$ is a spanning tree of $G'$ and a fundamental cycle with respect to $T$ in $G'$ is one in $G$. As $G'$ is bipartite, a fundamental cycle in $G'$ has length at least 4, whence $\sigma_T(G, T) \geq 3$.

(ii) The spanning tree $T$ contains some edges between vertices in $X$. Suppose that $xy \in T$ with $x \in V_i$ and $y \in V_j$ $(2 \leq i < j \leq k)$. Let $u \in V_i$ be such that $d_T(u, x) < d_T(u, y)$. If $d_T(u, x) \geq 2$, then $d_T(u, y) \geq 3$, thus $\sigma_T(G, T) \geq 3$. Otherwise $ux \in T$. Take $z \in V_i, z \neq x$. 

Then \(d_T(x, z) \geq 2\). If the path \(P_{x,z}\) in \(T\) contains \(u\), then \(d_T(y, z) \geq 3\). Otherwise \(d_T(u, z) \geq 3\), whence \(\sigma_T(G, T) \geq 3\).

On the other hand, we can construct a spanning tree \(T\) in the complete bipartite graph \(G'\) as a double star (as in Proposition 2.3(2)). Then for an edge between the vertices of \(V_1\) and \(X\), the fundamental cycle has length four, while for an edge between the vertices of \(X\), the fundamental cycle has length three. Thus \(\sigma_T(G, T) = 3\). This completes the proof.

\[\square\]

### 3.2 Split Graphs

A graph \(G\) is a **split graph** if its vertex set \(V(G)\) can be partitioned into a clique \(X\) of \(G\) and an independent set \(Y\) of \(G\) (see [13]). For split graphs, [19] showed that the spanning tree congestion problem is NP-complete. However, the dual problem is easy. It has been known in [4, 23] that \(\sigma_T(G) \leq 3\) for split graphs \(G\). Here we describe a precise characterization as follows.

**Theorem 3.2.** For a split graph \(G\) (apart from a tree), \(\sigma_T(G) = 2\) if and only if there exists a vertex \(x_0 \in X\) such that every vertex \(y \in Y \setminus N_G(x_0)\) is a pendant vertex (of degree one). Otherwise \(\sigma_T(G) = 3\).

**Proof.** If there exists a vertex \(x_0 \in X\) such that every vertex \(y \in Y \setminus N_G(x_0)\) is pendant, then we can construct a spanning tree \(T^*\) by the star with edges from \(x_0\) to \(N_G(x_0)\), and by joining each remaining vertex \(y \in Y \setminus N_G(x_0)\) to its unique neighbor in \(X\). Then for any \(x, x' \in X\), we have \(d_{T^*}(x, x') \leq 2\). For any edge \(xy \in E(G)\) with \(x \in X\) and \(y \in N_G(x_0)\), the path between \(x\) and \(y\) in \(T^*\) is either \(x_0y\) or \(xxy_0y\), thus \(d_{T^*}(x, y) \leq 2\). For any edge \(xy \in E(G)\) with \(x \in X\) and \(y \in Y \setminus N_G(x_0)\), we have \(x \neq x_0\). Then \(x\) is the unique neighbor of \(y\), thus \(d_{T^*}(x, y) = 1\). Therefore \(\sigma_T(G, T^*) = 2\) and so \(\sigma_T(G) = 2\).

Conversely, if \(\sigma_T(G) = 2\), then there is a spanning tree \(T\) such that \(\sigma_T(G, T) = 2\). This spanning tree \(T\) restricted in \(G[X]\) must be a star with center \(x_0\). For otherwise there would be \(x, x' \in X\) such that \(d_T(x, x') \geq 3\). If a vertex \(y \in Y \setminus N_G(x_0)\) is adjacent to two vertices \(x_1, x_2 \in X\) (where \(xy_1 \in T\)), then the fundamental cycle \(yx_1x_0x_2\) has length greater than three, which contradicts \(\sigma_T(G, T) = 2\).

Furthermore, we show that \(\sigma_T(G) \leq 3\) in any case. To this end, we construct a spanning tree \(T\) as follows. We choose a vertex \(x_0 \in X\) arbitrarily and take the star from \(x_0\) to \(N_G(x_0)\), and join each vertex \(y \in Y \setminus N_G(x_0)\) to a neighbor in \(X\). For any \(x, x' \in X\), we have \(d_T(x, x') \leq 2\). For any edge \(xy \in \overline{T}\) with \(x \in X\) and \(y \in N_G(x_0)\), the path between \(x\) and \(y\) in \(T\) is \(xx_0y\), thus \(d_T(x, y) = 2\). If there is an edge \(xy \in \overline{T}\) with \(x \in X\) and \(y \in Y \setminus N_G(x_0)\), and \(yx' \in T\), then the path between \(x\) and \(y\) in \(T\) is \(x_0x'y\). Thus \(d_T(x, y) = 3\). Therefore, \(\sigma_T(G, T) \leq 3\) and so \(\sigma_T(G) \leq 3\). This completes the proof.

\[\square\]

### 3.3 Generalized Convex Graphs

A bipartite graph \(G\) with bipartition \((X, Y)\) is a **chain graph** if there is an order \(x_1, x_2, \ldots, x_m\) in \(X\) such that \(N_G(x_1) \subseteq N_G(x_2) \subseteq \cdots \subseteq N_G(x_m)\) in \(Y\). Previously, [19] showed that the minimum congestion spanning tree problem is NP-hard even for chain graphs. However, the counterpart in the tree-stretch problem is quite easy, since a chain graph is a special convex graph and \(\sigma_T(G) \leq 3\) is known in [5].

Now we consider a generalization of convex graphs. A subset family \(\mathcal{F}\) is called **laminar** (or **nested**) if for any two sets \(A, B \in \mathcal{F}\), at least one of \(A \setminus B, B \setminus A, A \cap B\) is empty, that is, \(A \cap B \neq \emptyset \Rightarrow A \subseteq B\) or \(B \subseteq A\).

**Definition 3.3.** A bipartite graph \(G\) with bipartition \((X, Y)\) is a **generalized convex graph** if there exists a tree \(\tau(Y)\) on the vertex set \(Y\) such that for each \(x_i \in X\), the neighbor set
\( Y_i = N_G(x_i) \) induces a subpath in \( \tau(Y) \) and the subset family \( \Sigma = \{Y_i : x_i \in X\} \) satisfies the following

**Laminar Property.** For each maximal subset \( Y_0 \in \Sigma \) (there exists no \( Y_i \in \Sigma \) such that \( Y_0 \subset Y_i \)) the subset family \( \{Y_i \setminus Y_0 : Y_i \cap Y_0 \neq \emptyset, Y_i \in \Sigma\} \) is laminar.

For a convex graph \( G \), \( \tau(Y) \) is itself a path and the subset family \( \Sigma = \{Y_i : x_i \in X\} \) can be regarded as a set of intervals on the line of \( \tau(Y) \). For each maximal interval \( Y_0 \in \Sigma \), if \( Y_i \cap Y_0 \neq \emptyset, Y_j \cap Y_0 \neq \emptyset \), then \( Y_i \setminus Y_0 \) and \( Y_j \setminus Y_0 \) are either disjoint or included one another. Hence the subset family \( \{Y_i \setminus Y_0 : Y_i \cap Y_0 \neq \emptyset, Y_i \in \Sigma\} \) is laminar. Thus the above definition is indeed a generalization of that of convex graphs. Moreover, a generalized convex graph is not necessarily an ATE-free graph. For example, when \( \tau(Y) \) is not a path, let \( y_1, y_2, y_3 \) be three leaves (pendant vertices) of \( \tau(Y) \) in different branches such that there is a path from \( y_i \) to \( y_j \) that avoids the neighborhood of \( y_k \) (for \( \{i, j, k\} = \{1, 2, 3\} \)). Then the three edges \( e_1, e_2, e_3 \) incident with \( y_1, y_2, y_3 \), respectively, in \( G \) constitute an ATE.

We are going to show that \( \sigma_T(G) = 3 \) for generalized convex graphs. Since a bipartite graph (apart from a tree) has girth \( g(G) = 4 \), we have \( \sigma_T(G) \geq 3 \). It suffices to construct an optimal spanning tree with three levels.

Let \( \Sigma = \{Y_1, Y_2, \ldots, Y_m\} \) be the family of neighbor sets, where \( Y_i = N_G(x_i) \) for \( x_i \in X \) \((1 \leq i \leq m)\). By assumption, we are given a tree \( \tau(Y) \) on \( Y \) that each \( Y_i \) induces a subpath of the tree \((1 \leq i \leq m)\). Suppose that \( Y_1 \) contains a leaf (pendant vertex) of \( \tau(Y) \) and it is maximal in \( \Sigma \) in the sense of inclusion. We consider this leaf as the root of the tree. Starting with \( Y_1 \), we define the level sets \( L_k \) in \( \Sigma \) by the following procedure:

1. Define \( L_1 := \{Y_1\} \). Set \( \Sigma := \Sigma \setminus L_1 \) and \( k := 1 \).
2. For each \( Y_i \in L_k \), if \( Y_j \in \Sigma \) satisfies that \( Y_i \cap Y_j \neq \emptyset, Y_j \setminus Y_i \neq \emptyset \), and \( Y_i \cup Y_j \) is maximal (i.e., there is no other \( Y_i \) such that \( Y_i \cap Y_i \neq \emptyset, Y_i \setminus Y_i \neq \emptyset \), and \( Y_i \cup Y_j \subset Y_i \cup Y_i \)), then \( Y_j \) is called a successor of \( Y_i \) (and \( Y_i \) is the predecessor of \( Y_j \)). Let \( L_{k+1} \) be the set of successors of \( Y_i \) for all \( Y_i \in L_k \).
3. Set \( \Sigma := \Sigma \setminus L_{k+1} \) and \( k := k + 1 \). If \( \bigcup_{Y_i \in L_{1} \cup L_{2} \cup \ldots \cup L_k} Y_i = Y \), then let \( h := k \) and stop, else go to (ii).

By this procedure, we construct the level sets \( L_1, L_2, \ldots, L_h \). Let \( \Sigma^* := \bigcup_{1 \leq k \leq h} L_k \), which is a subfamily of \( \Sigma \). For all neighbor sets \( Y_i \) in \( \Sigma^* \), no one is contained in another, and they constitute a cover of \( Y \). Also, they can be regarded as a directed tree rooted at \( Y_1 \) and running down level by level. If \( Y_j \) and \( Y_i \) are successors of \( Y_i \) in this directed tree, then by the laminar property, we see that \( (Y_j \setminus Y_i) \cap (Y_i \setminus Y_i) = \emptyset \). Also, for \( Y_i \in L_{k-1} \) and \( Y_j \in L_{k+1} \), we have \( Y_i \cap Y_j = \emptyset \).

Note that there may be some neighbor sets \( Y_q \in \Sigma \setminus \Sigma^* \) which are discarded in the above procedure. For each \( Y_q \in \Sigma \setminus \Sigma^* \), there must be a \( Y_i \in L_k \) and its successor \( Y_j \in L_{k+1} \) such that \( Y_i \cap Y_q \neq \emptyset \), and \( Y_i \cup Y_q \subseteq Y_i \cup Y_j \). For otherwise we may choose \( Y_q \) in the above procedure.

By means of the level structure \( \{L_1, L_2, \ldots, L_h\} \), we construct the spanning tree \( T \) by the following algorithm.

**Construction Algorithm**

1. For \( L_1 = \{Y_1\} \), construct a star \( T_1 \) with center \( x_1 \) and all leaves \( y \in Y_1 \). Set \( T := T_1 \) and \( k := 1 \).
2. For each neighbor set \( Y_i \in L_k \), consider a successor \( Y_j \in L_{k+1} \), and construct a star \( T_j \) with center \( x_j \) and all leaves \( y \in \{Y_j \setminus Y_i\} \cup \{\bar{y}\} \), where \( \bar{y} \) is the last vertex in \( Y_i \cap Y_j \) (according to the order of the path of \( Y_j \)). Set \( T := T \cup T_j \). Repeat this step for all successors of \( Y_i \) and all \( Y_i \in L_k \).
3. Set \( k := k + 1 \). If \( k < h \), then go to (2).
For each neighbor set $Y_q \in \Sigma \setminus \Sigma^*$, suppose that $Y_i \cap Y_q \neq \emptyset$ and $Y_q \subseteq Y_i \cup Y_j$ for some $Y_i \in L_k$ and its successor $Y_j \in L_{k+1}$. Let $\bar{y}$ be the last vertex in $Y_i \cap Y_q$. Then set $T := T \cup \{x_q \bar{y}\}$.

We claim that the output $T$ of the above algorithm is indeed a spanning tree of $G$. In fact, we first construct a star $T_i$ with center $x_1$ for level $L_1$. When $Y_i \in L_k$ has been considered, we have a star $T_i$ with center $x_i$. Then we consider a successor $Y_j \in L_{k+1}$ of $Y_i$ and add a star $T_j$. Since the stars $T_i$ and $T_j$ have only one leaf in common, $T_i \cup T_j$ is connected and contains no cycles, and thus is a tree. If $Y_i$ has another successor $Y_j$, then by the laminar property, we have $(Y_j \setminus Y_i) \cap (Y_i \setminus Y_j) = \emptyset$. Then $T_i$ and $T_j$ have one leaf in common, $T_i$ and $T_j$ have at most one leaf in common (if the $\bar{y} \in Y_i$ is the same for $T_i$ and $T_j$). Hence $T_i \cup T_j \cup T_l$ is also a tree. In this way, we construct a set of stars in which any two stars have at most one leaf in common. So we obtain a tree $T$ in Steps (1)-(3). In Step (4), we add more pendant edges (with new leaves $x_q$) to $T$. Additionally, all vertices of $G$ are considered when the algorithm terminates. Therefore $T$ is finally a spanning tree.

**Theorem 3.4.** For a generalized convex graph $G$ (apart from a tree), it holds that $\sigma_T(G) = 3$.

**Proof.** We proceed to show that the spanning tree $T$ constructed by the above algorithm has stretch three. For each cotree-edge $e \in T$, there are two cases to consider:

**Case 1.** $e = x_jy$ with $Y_j \in L_k$ for some $L_k \in \Sigma^*$. Let $Y_i \in L_{k-1}$ be the predecessor of $Y_j$. Then $y \in Y_i \cap Y_j$. Thus $x_i y, x_j \bar{y}$ and $x_i \bar{y}$ are contained in $T$ (where $\bar{y}$ is the last vertex in $Y_i \cap Y_j$). Hence $e = x_jy$ and these three edges in $T$ constitute the fundamental cycle with respect to $e$, which has length four.

**Case 2.** $e = x_qy$ with $Y_q \in \Sigma \setminus \Sigma^*$. Then there is some $Y_i \in L_k$ and its successor $Y_j \in L_{k+1}$ such that $Y_i \cap Y_q \neq \emptyset$ and $Y_q \subseteq Y_i \cup Y_j$. If $y \in Y_i$ (say $Y_q \subseteq Y_i$), then $x_i y, x_j \bar{y}, x_q \bar{y} \in T$ (where $\bar{y}$ is the last vertex in $Y_i \cap Y_q$). Thus $e = x_qy$ and these three edges in $T$ constitute the fundamental cycle with respect to $e$, which has length four. If $y \in Y_j \setminus Y_i$, then by the laminar property, $Y_q \setminus Y_i \subseteq Y_j \setminus Y_i$. Let $\bar{y}$ be the last vertex in $Y_i \cap Y_j$. Then $\bar{y} \in Y_q$ and $x_j y, x_j \bar{y}, x_q \bar{y} \in T$. Thus these three edges in $T$ and $e = x_q y \notin T$ also yield a length four fundamental cycle.

To summarize, for every cotree-edge $e \in T$, the fundamental cycle with respect to $e$ has length four. Therefore, $\sigma_T(G, T) = 3$ and the theorem is proved.

## 4 Planar Grids

It is known that the minimum stretch spanning tree problem is NP-hard for planar graphs in general\[^{10}\]. We investigate some planar grids in this section.

Let $G$ be a simple connected planar graph. Suppose that we have a planar embedding of $G$ on the plane so that it is a *plane graph*. For a face $f$ of $G$, the degree of $f$, denoted by $d(f)$, is the number of edges in its boundary. Our approach is based on the spanning trees of the dual graph. The dual graph $G^*$ of $G$ is defined as follows. Each face $f$ of $G$ (including the outer face) corresponds to a vertex $f^*$ in $G^*$, and each edge $e$ of $G$ corresponds to an edge $e^*$ of $G^*$ in such a way that two vertices $f^*$ and $g^*$ are joined by an edge $e^*$ in $G^*$ if and only if their corresponding faces $f$ and $g$ are separated by the edge $e$ in $G$. We may place each vertex $f^*$ in the face $f$ of $G$ and draw each edge $e^*$ to cross the edge $e$ of $G$ exactly once. This dual graph $G^*$ is also a plane graph.

A prominent property of duality is: A cycle $C$ of $G$ corresponds an edge-cut (cocycle) $C^*$ of $G^*$, and an edge-cut $B$ of $G$ corresponds a cycle $B^*$ of $G^*$. In particular, for a spanning tree $T$ of $G$, the cotree $T$ corresponds to a spanning tree $T^*$ of $G^*$. A fundamental cycle with respect to $T$ in $G$ corresponds to a fundamental edge-cut with respect to $T^*$ in $G^*$ (see [1] for
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For example, the cube $Q_3$ is shown in Figure 2(a) and a spanning tree $T$ with solid lines in Figure 2(b). Meanwhile, the spanning tree $T^*$ with dotted lines of the dual graph $G^*$ is also drawn in Figure 2(b), in which the vertices of faces are represented by small circles and the vertex of outer face is denoted by $O$.

![Figure 2. The cube $Q_3$ and its spanning trees](image)

For a face $f$ of plane graph $G$ (a vertex of $G^*$), we define the level of $f$, denoted by $\lambda(f)$, to be the length of a shortest path from the vertex $f$ to the vertex $O$ of outer face in $G^*$. We denote by $L_i$ the set of faces having level $i$ ($i = 0, 1, \cdots$). Then the levels can be determined by the following procedure:

(i) Let $\lambda(O) = 0$ and $L_0 = \{O\}$.

(ii) If $L_i$ has been defined, then for any face $f$ whose level $\lambda(f)$ is not defined and it is adjacent to a face $g \in L_i$, set $\lambda(f) = i + 1$.

For example, the levels of the faces in $Q_3$ are shown in Figure 2(a) by the number in each face (except $O$ with level 0), where $|L_1| = 4, |L_2| = 1$. Here, we first consider the outer vertex $O$ as the root. Then, all vertices in $L_1$ have the same predecessor $O$. In general, when $\lambda(f) = i + 1$ is defined in terms of an adjacent vertex $g \in L_i$, $g$ is the predecessor of $f$. Thus, a rooted tree (called search tree) is obtained level by level. In this respect, we define the maximum level of $G$ by

$$\lambda_{\text{max}}(G) := \max_{f \in F} \lambda(f),$$

where $F$ is the set of the faces of $G$. This is the height of the search tree.

4.1 Rectangular Grids

First, we consider the rectangular grids $G = P_m \times P_n$ ($2 \leq m \leq n$) on the plane. Let $V(G) := \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$ denote the vertex set of $G$, and $(i, j)$ is adjacent to $(i', j')$ if $|i - i'| + |j - j'| = 1$ (see Figure 3(a)). Similar to the notation of matrices, we may call $R_i := \{(i, j) : 1 \leq j \leq n\}$ the $i$-th row, and $Q_j := \{(i, j) : 1 \leq i \leq m\}$ the $j$-th column. The edges in the rows are called horizontal edges. The edges in the columns are called vertical edges.

Hruska[14] proved the tree-congestion as follows ($m \leq n$):

$$c_T(P_m \times P_n) = \begin{cases} m, & \text{if } m = n \text{ or } m \text{ odd}, \\ m + 1, & \text{otherwise}. \end{cases}$$

In the following we derive a similar formula for the tree-stretch:
Theorem 4.1. For the rectangular grids $P_m \times P_n$ with $2 \leq m \leq n$, we have

$$\sigma_T(P_m \times P_n) = 2 \left\lceil \frac{m}{2} \right\rceil + 1.$$ 

Proof. Let $G = P_m \times P_n (2 \leq m \leq n)$. We first show that

$$\lambda_{\text{max}}(G) = \left\lceil \frac{m}{2} \right\rceil.$$ 

By induction on $m$. When $m = 2, 3$, all faces have level 1, so $\lambda_{\text{max}}(G) = 1$ and the assertion holds. Assume that $m \geq 4$ and the assertion holds for smaller $m$. We delete the boundary of the outer faces from $G$ (the vertices and the edges on this boundary are deleted). Then the remaining graph is $G' = P_{m-2} \times P_{n-2}$. In this transformation, all faces with level 1 are removed. Therefore $\lambda_{\text{max}}(G) = \lambda_{\text{max}}(G') + 1$. By induction hypothesis, $\lambda_{\text{max}}(G') = \left\lfloor (m-2)/2 \right\rfloor$. Hence

$$\lambda_{\text{max}}(G) = \left\lceil \frac{m-2}{2} \right\rceil + 1 = \left\lceil \frac{m}{2} \right\rceil.$$ 

For example, the levels of $P_4 \times P_5$ are shown in Figure 3(a) and $\lambda_{\text{max}}(G) = \lfloor m/2 \rfloor = 2$.

![Grid $P_4 \times P_5$ and spanning trees](image)

**Figure 3.** Grid $P_4 \times P_5$ and spanning trees

We next show the lower bound

$$\sigma_T(G) \geq 2\lambda_{\text{max}}(G) + 1. \quad (4.1)$$ 

In fact, let $T$ be any given spanning tree of $G$. Then the cotree $\overline{T}$ determines a spanning tree $\overline{T}^*$ in $G^*$. Suppose that $f_0$ is a face with the maximum level $\lambda_{\text{max}}(G)$. For brevity, we still denote its vertex in $G^*$ by $f_0$ and write $\lambda = \lambda_{\text{max}}(G)$. Then the distance between $f_0$ and $O$ in $\overline{T}^*$ is at least $\lambda$. Let $P^*$ be the path from $f_0$ to $O$ in $\overline{T}^*$ with the last edge $e_0^*$ incident with $O$. The tree-edge $e_0^*$ in the spanning tree $\overline{T}^*$ determines a fundamental edge-cut $C^* = \partial(X_{e_0})$, where $\overline{T}^* - e_0^*$ has two components and $X_{e_0}$ is the vertex set of the component containing $P^*$. Then this fundamental edge-cut $C^*$ with respect to $\overline{T}^*$ in $G^*$ corresponds to a fundamental cycle $C$ with respect to $T$ in $G$. So, this fundamental cycle $C$ is determined by the cotree edge $e_0$ on the boundary of the outer face that corresponds to the edge $e_0^*$ in $P^*$. Note that all faces in $P^*$ (with levels 1, 2, $\cdots$, $\lambda$) are contained in the region surrounded by $C$. Without loss of generality, assume that $e_0$ is on the row $R_1$. We draw $\lambda$ horizontal straight lines passing through the centers of square faces of $P^*$. Then each of these straight lines intersects $C$ at two vertical edges. Besides, $C$ must have at least two more horizontal edges. Hence $C$ has length.
at least $2\lambda + 2$. Consequently, for any spanning tree $T$, we find a fundamental cycle $C$ with length at least $2\lambda + 2$. By the arbitrariness of $T$, the lower bound (4.1) is proved.

On the other hand, we can construct a spanning tree $T^0$ by taking all columns and the row $R_{\lceil m/2 \rceil}$. Then the maximal fundamental cycles have length $2\lfloor m/2 \rfloor + 2$. Thus the spanning tree $T^0$ is optimal. This completes the proof.

4.2 Triangular Grids

We next consider the triangular grids $T_n$, which is defined as follows. The vertex set can be represented as $\{(x, y) \in \mathbb{Z}^2 : x + y \leq n, x, y \geq 0\}$ on the plane, and two vertices $(x, y)$ and $(x', y')$ are joined by an edge if $|x - x'| + |y - y'| = 1$ or $|x - x'| + |y - y'| = 2$ and $x + y = x' + y'$ (refer to [17]). For example, $T_4$ is shown in Figure 4, and $T_1$, $T_2$, and $T_3$ are shown in Figure 5. In this plane embedding of $T_n$, the straight-lines $\{(x, y) \in \mathbb{R}^2 : y = k\} \ (0 \leq k \leq n-1)$ are called horizontal lines and the edges on them are called horizontal edges. Symmetrically, the straight-lines $\{(x, y) \in \mathbb{R}^2 : x = k\} \ (0 \leq k \leq n-1)$ are called vertical lines and the edges on them are called vertical edges. In addition, there are parallel slant edges.

Ostrovskii [21] developed an approach, called center-tail system, to deal with the spanning tree congestion problem for planar graphs, and obtained the result for triangular grids as follows:

$$c_T(T_n) = \begin{cases} 4k, & \text{if } n = 3k, \\ 4k, & \text{if } n = 3k + 1, \\ 4k + 2, & \text{if } n = 3k + 2, \end{cases}$$

(however, $T_n$ here is our $T_{n-1}$). We obtain the corresponding result for tree-stretch as follows.

**Theorem 4.2.** For the triangular grids $T_n$, we have

$$\sigma_T(T_n) = \left\lfloor \frac{2n}{3} \right\rfloor + 1.$$

**Proof.** For the triangular grids $T_n$, we first show that

$$\lambda_{\text{max}}(T_n) = \left\lfloor \frac{2n}{3} \right\rfloor.$$

We use induction on $n$. When $1 \leq n \leq 3$, the levels of faces for $T_1, T_2, T_3$ are shown in Figure 5, in which $\lambda_{\text{max}}(T_1) = 1, \lambda_{\text{max}}(T_2) = \lambda_{\text{max}}(T_3) = 2$. Hence the assertion holds for $1 \leq n \leq 3$. 

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**Figure 4.** Triangular grid $T_4$ and spanning trees

(a) Triangular grid $T_4$  
(b) Spanning trees of $T_4$ and its dual

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Assume that $n \geq 4$ and the assertion holds for smaller $n$. We delete the boundary of the outer faces from $T_n$ (the vertices and the edges on this boundary are deleted). Then the resulting graph is $T_{n-3}$. In this transformation, all faces with levels 1 and 2 are removed. Therefore $\lambda_{\text{max}}(T_{n}) = \lambda_{\text{max}}(T_{n-3}) + 2$. By induction hypothesis, we have

$$
\lambda_{\text{max}}(T_{n}) = \left\lceil \frac{2(n-3)}{3} \right\rceil + 2 = \left\lceil \frac{2n}{3} \right\rceil.
$$

For example, $\lambda_{\text{max}}(T_{3}) = \lambda_{\text{max}}(T_{1}) + 2 = 3$, as shown in Figure 4(a).

We next show the lower bound

$$
\sigma_{T}(G) \geq \lambda_{\text{max}}(G) + 1. \tag{4.2}
$$

In fact, let $T$ be any given spanning tree of $G$. Then the cotree $T$ determines a spanning tree $T^*$ in $G^*$. Similar to the previous case, suppose that $f_{0}$ is a face with the maximum level $\lambda = \lambda_{\text{max}}(G)$. Then the distance between $f_{0}$ and $O$ in $T^*$ is at least $\lambda$. Let $P^*$ be the path from $f_{0}$ to $O$ in $T^*$ with the last edge $e^*_0$ incident with $O$. The tree-edge $e^*_0$ in the spanning tree $T^*$ determines a fundamental edge-cut $C^*$, which corresponds to a fundamental cycle $C$ with respect to $T$ in $G$. This fundamental cycle $C$ is determined by the cotree edge $e_0$ on the boundary of the outer face that corresponds to the edge $e^*_0$ in $P^*$. Without loss of generality, assume that $e_0$ is on the horizontal line $R_0 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. Let $P^0$ be the shortest path from $f_{0}$ to $O$ passing though $R_0$ in $G^*$. Suppose that $e^*_0$ is the edge in $R_0$ which corresponds the last edge of $P^0$. Denote by $\partial(P^0)$ the boundary of the region composed of the $\lambda$ faces of $P^0$. Then $\partial(P^0)$ is a $(1 \times \frac{1}{\lambda})$ rectangle (if $\lambda$ is even) or a $(1 \times \frac{3}{2}(\lambda - 1))$ rectangle plus a triangle at $f_{0}$ at the top (if $\lambda$ is odd). It can be seen that the triangle face at the top (or at the bottom) of $\partial(P^0)$ has two boundary edges, and each of the other triangle faces has one boundary edge. Hence the length of $\partial(P^0)$ is $\lambda + 2$. We draw $\left\lceil \frac{1}{\lambda} \right\rceil$ horizontal straight lines passing through the midpoints of the boundary edges in $\partial(P^0)$. Then each of these straight lines intersects the cycle $C$ twice. When $\lambda$ is even, the $\frac{1}{\lambda}$ straight lines intersect the cycle $C$ at $\lambda$ edges. Besides, $C$ must have at least two more horizontal edges (one is $e_0$ and one in $f_0$). Hence the length of $C$ is at least $\lambda + 2$. When $\lambda$ is odd, the $\frac{1}{2}(\lambda + 1)$ straight lines intersect the cycle $C$ at $\lambda + 1$ edges. And $C$ has one more horizontal edge $e_0$. Thus the length of $C$ is at least $\lambda + 2$. Therefore, for any spanning tree $T$, we find a fundamental cycle $C$ with length at least $\lambda + 2$. By the arbitrariness of $T$, the above lower bound (4.2) is proved.

On the other hand, we can construct an optimal spanning tree $T$ as follows:

1. Take a face $f_{0}$ with the maximum level $\lambda_{\text{max}}(G)$.
2. Take the horizontal line $H$ containing the horizontal edge of $f_{0}$, and take the vertical line $V$ containing the vertical edge of $f_{0}$.
3. In the part below $H$, take every vertical line intersecting $H$; In the part above $H$, take every horizontal line intersecting $V$.

![Figure 5](image-url)
(4) In the remaining part of the lower right corner, take all horizontal lines; in the remaining part of the upper left corner, take all vertical lines. An example of $T_4$ can be seen in Figure 4(b). It is easy to check that this spanning tree attain the above lower bound. This completes the proof.

### 4.3 Triangulated-rectangular Grids

Finally, we consider the triangulated-rectangular grids by the same method. A triangulated-rectangular grid $T_{m,n}$ is defined as follows: the vertex set $V(T_{m,n})$ is $\{(x,y) \in \mathbb{Z}^2 : 0 \leq y \leq m-1, 0 \leq x \leq n-1\}$, and two vertices $(x,y)$ and $(x',y')$ are joined by an edge if $|x-x'| + |y-y'| = 1$ or $|x-x'| + |y-y'| = 2$ and $x+y = x'+y'$, as shown in Figure 6. Clearly, $T_{m,n}$ can be obtained from the rectangular grids $P_m \times P_n$ by adding slant edges. 

![Figure 6. Triangulated-rectangle grid $T_{5,6}$](image)

**Theorem 4.3.** For the triangulated-rectangular grids $T_{m,n}$ with $2 \leq m \leq n$, we have

$$\sigma_T(T_{m,n}) = m.$$  

**Proof.** Let $G = T_{m,n}$ ($2 \leq m \leq n$). We first claim that

$$\lambda_{\max}(G) = m - 1.$$  

By induction on $m$. When $m = 2$, all faces have level 1, so $\lambda_{\max}(G) = 1$; When $m = 3$, it is also evident that $\lambda_{\max}(G) = 2$. Assume that $m \geq 4$ and the claim holds for smaller $m$. We delete the boundary of the outer faces from $G$, so that the remaining graph is $G' = T_{m-2,n-2}$. In this transformation, all faces with levels 1 and 2 are removed. Therefore $\lambda_{\max}(G) = \lambda_{\max}(G') + 2$. By induction hypothesis, $\lambda_{\max}(G') = m - 2 - 1 = m - 3$. Hence $\lambda_{\max}(G) = m - 3 + 2 = m - 1$ and the claim follows.

Moreover, by the same method of the previous case we obtain the lower bound

$$\sigma_T(G) \geq \lambda_{\max}(G) + 1. \quad (4.3)$$

On the other hand, we can construct an optimal tree $T$ by taking all columns and the edges from $(\lceil m/2 \rceil, j)$ to $(\lceil m/2 \rceil, j + 1)$ for $1 \leq j \leq n-1$ (see Figure 7). It is easy to check that this spanning tree attain the above lower bound. The proof is complete.
Figure 7. Optimal trees of $T_{m,n}$

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