IMPLEMENTING UNITARY OPERATORS WITH DECOMPOSITION INTO DIAGONAL MATRICES OF TRANSFORM DOMAINS

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Abstract: We have proposed and demonstrated a general and scalable scheme for programmable unitary gates. Our method is based on matrix decomposition into diagonal and Fourier factors. Thus, we are able to construct arbitrary matrix operators only by diagonal matrices alternately acting on two photonic encoding bases that belong to a Fourier transform pair. Thus, the technical difficulties to implement arbitrary unitary operators are significantly reduced. As examples, two protocols for optical OAM and path domain are considered to verify our proposal and evaluate the performance. For OAM domain, unitary matrices with dimensionality up to $6 \times 6$ are achieved and discussed with the numerical simulations. For path domain, an average fidelity of 0.98 is evaluated through 80 experimental results with dimensionality of $3 \times 3$. Furthermore, our proposal is also potential to provide a quantum operation protocol for any other photonic domain, if there exists the corresponding transformed domain.

1. Introduction

Unitary operators are powerful tools in quantum information processing for both investigating non-classical phenomena and exploring quantum computational resources [1]. Great efforts have been made to apply universal operation protocols to high-dimensional quantum bit, or qudit, for extending information capacity and enhancing computing versatility [2–4]. As a promising platform, photonics provides various degrees of freedoms (DOFs) for quantum encoding including optical path [5,6], frequency [7], temporal-mode [8,9] and transverse-spatial-mode [10–12]. Any lossless optical setup can be described by a unitary operator. However, for the inverse problem, i.e., how to design an on-demand lossless implementation for unitary operation, which is feasible for both linear operations and nonlinear operations with the KLM computing scheme [13], has not been well-solved yet.

Implementing discrete unitary operators of path-domain has been proposed and demonstrated by Reck et.al [5]. But it is hard to directly apply Reck’s scheme to other photonic DOFs. When dealing with path-encoded qudits, the dimensionality of the investigated Hilbert space is finite and determined by the physical structure of multi-port interferometers [14]. For most of other DOFs such as optical frequency or optical orbital angular momentum (OAM), a proper cutoff would be necessary to obtain a finite subset from the natural encoding basis, which is actually an infinite Hilbert space. As a concrete example, the requirement of closed operation has made it tough to demonstrate unitary operations for OAM encoded qudits, especially with the demand for unity success probability at the same time. Recently, there are some reports of OAM-domain unitary gates with the techniques of OAM parity sorters [15,16], multi-plane light conversion (MPLC) [17] and spin-orbital coupling [18,19]. However, none of them could achieve universality, scalability and unity success probability simultaneously, even theoretically. For the frequency-domain, linear operation protocols have been reported with electro-optic modulators (EOM) and pulse shapers [20,21], but the universality as well as the efficiency are temporarily limited by the speed of EOMs.
In this work, inspired by the approach of frequency-domain, we propose and demonstrate a general and scalable scheme for programmable unitary gates that could be applied on multiple DOFs. Our method is based on matrix decomposition into diagonal and Fourier factors. Thus, we are able to construct arbitrary matrix operators only by diagonal matrices alternately acting on two photonic encoding bases that belong to a Fourier transform pair. The OAM-domain and path-domain unitary gates are selected as two examples to present our method with simulation and experiments, respectively. For OAM domain, the numerical simulation of unitary matrices with dimensionality up to $6 \times 6$ is given. For path domain, an average fidelity value of 0.98 is evaluated through 80 experimental results with dimensionality of $3 \times 3$. It will be further discussed that the electro-optic frequency-domain unitary gates [20] can be considered as another application of our method. Besides implementing quantum operations on OAM-encoded and path-encoded qudits, our proposal is potential to provide a quantum operation protocol when exploring another new photonic DOF, if there exists the corresponding Fourier transform pair.

2. Theory

Without loss of generality, our target is to implement an arbitrary unitary gate $U$ acting on photonic qudit, which could be expressed as,

$$|\beta\rangle = U|\alpha\rangle,$$

where $|\alpha\rangle$ and $|\beta\rangle$ denote the input and output qudit states. For most practical applications, $U$ is a finite-dimensional operator and can be mathematically expressed as a unitary matrix with respect to the utilized encoding basis. It has been presented [22] that an arbitrary matrix of dimensionality $N$ can be mathematically decomposed into the production of diagonal matrices and cyclic matrices alternately, where the total number of matrices is $M$, or equivalently speaking, $M$ diagonal matrices $D$ spaced by discrete Fourier transformation (DFT) matrix $F$ and its inverse matrix $F^T$,

$$U_{\text{decom}} = F D_5 F^T \cdots F D_1 F^T D_1 F D_1 F^T = (F D_5 F^T) \cdots (F D_1 F^T) D_1 (F D_1 F^T)$$

where $M \leq 2N - 1$ [23] denotes the number of diagonal matrices required for the decomposition and it would vary according to the target matrices. Specifically, to establish a unitary operator, the diagonal matrices $D$ in Eq. (2) would also be unitary. To clearly present our idea, the right-hand term in the first line of Eq. (2) is modified as the form of the second one. Particularly, the term in bracket is a diagonal matrix sandwiched by DFT and invers DFT matrices. Actually, it is also a diagonal matrix but acting on a transformed domain, which is determined by the DFT. Thus, Eq. (2) could be understood as that a unitary operator can be decomposed into a series of diagonal matrices between two domains, which are linked with Fourier transformation. It should be mentioned that a diagonal matrix would not lead to cross-mode effects and is also referred as a spectral shaper. Actually, the required cross-components for unitary operations are achieved by DFT matrices in Eq. (2). Thus, if we could find two photonic DOFs that are Fourier transform pair, any unitary matrix can be implemented by performing diagonal matrix operations on these two DOFs in succession for the investigated photonic qudits. There are some Fourier transform pairs that have been well investigated in photonic domain, including OAM and azimuthal angle [24], frequency and time-bin, Bessel modes and perfect OAM modes [25], linear basis ($x$-basis) and transverse wave vector basis ($k$-basis), etc. For most photonic DOFs, it is simple and flexible to implement diagonal matrix operations (or spectral shaping). Thus, the technical difficulties to implement arbitrary unitary operators are significantly reduced. In this work, two protocols for optical OAM and path domain are considered to verify our proposal and evaluate the performance. Moreover, our
proposed approach is intrinsically lossless so that it is potential to achieve reversible operations, which are particularly essential for quantum computing [26].

3. OAM-domain Unitary Operation Protocol

With well-defined OAM basis, an arbitrary state vector $|\alpha\rangle$ can be expanded as,

$$|\alpha\rangle = \sum_{l} c_{l} |l\rangle,$$

where $\{c_{l}\}$ represent OAM spectrum coefficients and $|l\rangle$ denotes the $l$th eigenstate with OAM of $l\hbar$ carried per photon. It has been investigated that transverse OAM basis $|l\rangle\langle l|$ and angle basis $|\varphi\rangle\langle \varphi|$ are connected via Fourier transformation [24]:

$$|l\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \exp(-il\varphi) |\varphi\rangle d\varphi$$

$$|\varphi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} \exp(il\varphi) |l\rangle$$

Thus, it is feasible to implement well-designed diagonal matrices within OAM-domain and angle-domain alternately to establish any target OAM-domain matrix operator with decomposition form shown in Eq. (2). The question is how to implement the diagonal matrices for both OAM and angle domains. Fig. 1 shows an example of OAM-domain unitary operator with three-layer scheme corresponding to the case of $M = 3$ in Eq. (2). There are two layers to implement diagonal matrices in angle domain and one layer for OAM domain. As shown later, this scheme could perform most unitary operations with dimensionality of $3 \times 3$, and some particular operations of $4 \times 4$ and $6 \times 6$.

![Fig.1. The three-layer scheme of OAM-domain unitary operation.](image)

In the angle-domain, a diagonal matrix can be readily achieved by a single wave front modulation element such as a spatial light modulator (SLM) [27] or metasurfaces [28]. Here, the SLM is considered and noted as angle modulator in Fig. 1. A typical angle modulation function $f(\varphi)$ can be achieved by the following pattern $\Phi_{ANG}(r, \varphi)$ settles on SLM:

$$\Phi_{ANG}(r, \varphi) = f(\varphi),$$

where $r$ and $\varphi$ indicate the radius and azimuthal angle under polar coordinates in the transverse plane according to beam propagation direction, respectively. Thus, the only question is how to realize the required diagonal matrices in OAM-domain, which is also referred as an OAM spectral shaper [29].
Here, a straightforward way is considered to achieve a programmable and arbitrary OAM spectral shaper. With the help of OAM mode-sorter based on Cartesian to log-polar coordinate transformation [30], each OAM eigenstate can be mapped to different position in the focal plane. As shown in Fig. 1, two static optical elements $\Phi_1(x, y)$ and $\Phi_2(u, v)$ are employed to perform the mode sorter [30, 31]:

$$\Phi_1(x, y) = \frac{2\pi a}{\lambda f} \left[ y \arctan \left( \frac{y}{x} \right) - x \ln \left( \frac{x^2 + y^2}{b} \right) + x \right],$$

$$\Phi_2(u, v) = -\frac{2\pi ab}{\lambda f} \exp \left( -\frac{u}{a} \right) \cos \left( \frac{v}{a} \right),$$

where $a$ and $b$ are adjustable parameters, which can be designed according to the beam width of the exact transverse optical field. After passing through these two static optical elements together with an auxiliary lens of focal length $f$, the $l$th OAM eigenstate is sorted to a single spot whose center coordinate $x$ in the focal plane reads,

$$C_l = \frac{f \lambda}{2\pi a},$$

while the center coordinate $y$ is only determined by parameter $b$ in Eq. (6) and irrelevant to $l$. As indicated in Fig. 1, the desired OAM spectral shaper noted as the shaping function $g(l)$ can be achieved by the following wave front modulation function on another SLM placed at the focal plane of the second optical elements $\Phi_2$:

$$\Phi_{OAM}(x, y) = g \left( \frac{2\pi ax}{f \lambda} \right),$$

where $[\cdot]$ rounds the value to the nearest integer. After OAM spectral shaping, another two static optical elements are required to recover the OAM modes, i.e. to transform the Cartesian coordinate back to log-polar coordinate. The mode-sorter has been proved to be reversible [32] and the static optical elements to perform the reverse mode-sorting are exactly the same as those shown in Eq. (6). Since the static elements for OAM mode-sorter can be replaced by customized spherical lens and cylindrical lens [31], only one programmable SLM performing $\Phi_{OAM}(x, y)$ indicated in Eq. (8) would be enough for any OAM spectral shaping task.

We have mathematically modeled the scheme shown in Fig. 1 and simulated the field evolution according to Huygens-Fresnel principle under paraxial approximation. According to the characteristics of SLMs (Holoeye Pluto) employed in the experiment, the simulation area is set to $1080 \times 1080$ pixels with a pixel size of $8 \times 8$. First, the performance of our proposed OAM spectral shaper is examined. As shown in Fig. 2(a), L1 and L2 are lenses of focal length $f$. A superposed OAM state of 11 dimensions is shown in Fig. 2(c) as the input state. The mode interval between adjacent OAM encoding channels is chosen as 3 to compress the mode crossing induced by mode-sorter [33]. From left to right, Fig. 2(b) shows field evolution of an initial superposed OAM state passing through the spectral shaper step-by-step. Brightness and color indicate amplitude and phase distributions, respectively. After OAM mode-sorter, the initial angular momentum state is mapped to linear one. With the help of lens L1, linear momentum is transformed to linear coordinate in the focal plane. Then OAM spectral shaping is performed by the SLM settled on the focal plane. After that, the optical vortex is rolled back again by another reversely operating mode-sorter.

The OAM spectral shaper under test is expressed as $g(l) = \pi l/4$. Since only lossless scheme and unitary matrices are concerned, the required diagonal operators in matrix decomposition in
Eq. (2) are phase-only and thus referred as \( D(g(l)) = \text{diag}(\exp(ig(l))) \) for simplicity. In the following sections, \( f(\phi) \) and \( g(l) \) are adopted instead of \( \exp(if(\phi)) \) and \( \exp(ig(l)) \) for angular phase modulation and OAM phase spectral shaping, respectively. In this case, \( g(l) = \pi l/4 \) means 0 and \( \pi \) phase modulation for adjacent OAM channels. Actually, it is the toughest case since modulation function is “sharpest” and the phase crosstalk is maximum. Such extreme example is considered to examine the crosstalk tolerance of our proposed OAM spectral shaper. Besides, \( g(l) = \pi l/4 \) is a typical clock matrix and could also be achieved by field rotation of \( \pi/4 \) rad with a Dove prism [34]. Thus, the OAM spectral shaping effect can be observed intuitively through the rotation between output and initial field in Fig. 2(b). We also provided the OAM spectrum after spectral shaper in Fig. 2(d). The complex OAM spectrum is calculated with mode-matching method [35]. From these simulations, it can be seen that the OAM spectral shaper is valid for arbitrary shaping of high-dimensional input state and the unitary operations could be achieved in succession.

Secondly, the OAM spectral shaping function \( g(l) \) and two required angular modulation functions of \( f_1(\phi) \) and \( f_2(\phi) \) for a given target matrix are calculated through an optimization algorithm. We maximum \( \text{trace}(VV^T) \) subject to \( \text{fidelity}(V, U) \geq 0.999 \), while \( U \) and \( V \) denote target unitary matrix and implemented matrix, respectively. The success probability of implemented matrix after optimization is calculated by \( \text{trace}(VV^T) \). The fidelity is defined as [14]:

\[
\text{fidelity}(U, V) = \left| \frac{\text{Tr}(U^T V)}{\sqrt{\text{Tr}(U^T U) \cdot \text{Tr}(V^T V)}} \right|^2
\]  

(9)

The optimized matrix \( V \) in Eq. (9) is calculated by OAM spectral shaping function \( g(l) \) and two angular modulation functions \( f_1(\phi) \) and \( f_2(\phi) \). Here, we give an example with dimension of \( 2 \times 2 \):
\[
\begin{pmatrix}
\ddots \\
V_{2 \times 2}^{21} & V_{2 \times 2}^{12} \\
V_{2 \times 2}^{21} & V_{2 \times 2}^{22}
\end{pmatrix} = F^T D(f_1(\varphi))F \times D(g(l)) \times F^T D(f_2(\varphi))F \quad (10)
\]

The number \(K\) in Eq. (10) is the dimension of numerical optimization. The central \(2 \times 2\) region in the left side of Eq. (10) is the optimized matrix \(V\). We note that \(K = N + 2d\), where \(N\) is the dimension of target matrix (\(N = 2\) for the case in Eq. (10)) and \(d\) is the number of guard bands. The guard bands are necessary to eliminate cut-off effects, which arises from the differences between the finite-dimension discrete Fourier transform required in Eq. (2) and the infinite-dimension nature of the Fourier relationship of OAM eigenstates and angle eigenstates expressed in Eq. (4). According to our calculation as well as the results of frequency-domain unitary operation \([20]\), the condition of \(d \geq N - 1\) would be enough for most practical cases.

Different from the discrete OAM spectral shaping function \(g(l)\), the angular modulation functions \(f_1(\varphi)\) and \(f_2(\varphi)\) are continuous. The angular modulation functions are represented by sine expansion:

\[
f(\varphi) = \sum_{n=1}^{p} A_n \sin(n\varphi + \theta_n) \quad (11)
\]

The parameters \(\{A_n\} \in [0, 2\pi]\) and \(\{\theta_n\} \in [0, 2\pi]\) are to be determined. The number \(p\) in Eq. (11) is determined by the dimension \(N\) of target matrix since higher orders of sine components in angular modulation functions allow interactions between OAM eigenstates with larger distance. \(p = 3\) is chosen in our simulations. The entire optimization is done with the MATLAB “fmincon” function. To further explain the optimization procedure and provide an executable example, the functions \(g(l), f_1(\varphi)\) and \(f_2(\varphi)\) required for \(2 \times 2\) and \(3 \times 3\) Hadamard matrices \(H_2\) and \(H_3\) as well as modulation functions settled on SLMs are shown in Fig. 3.

\[
H_2 = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}, H_3 = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
e^{i2\pi/3} & e^{i4\pi/3} & e^{i4\pi/3}
\end{pmatrix} \quad (12)
\]

The modelled setup for OAM-domain matrix transformation is shown in Fig. 3(a). Three SLMs labelled as SLM1, SLM2 and SLM3 perform the first angular modulator, OAM spectral shaper and the second angular modulator, respectively. The colored phase modulation patterns corresponding to each SLM are also shown. For Hadamard matrix \(H_2\), the optimized angular modulation functions are shown under polar coordinates in Fig. 3(b), where red and blue solid lines indicate \(f_1(\varphi)\) and \(f_2(\varphi)\), respectively. The unit of polar axis in Fig. 3(b) is rad. The optimized OAM spectral shaper noted as \(g(l)\) is shown in Fig. 3(c), where OAM eigenstates of \(l = 0\) and \(l = 1\) are chosen as encoding channel for \(H_2\). In Fig. 3(c), 10 guard bands are plotted. Those optimization results for Hadamard matrix \(H_3\) are shown in Fig. 3(d) and (e), where OAM eigenstates \(l = -1, l = 0\) and \(l = 1\) are chosen as encoding channel. According to the optimization, the fidelity above 0.999 and success probability above 0.970 are achieved simultaneously for \(H_2\) and \(H_3\).
After the phase patterns settled on SLMs are obtained, the field evolution of Hadamard matrix $H_2$ has been calculated according to Huygens-Fresnel principle. The OAM eigenstates of $l = -3$ and $l = 3$ are chosen to be the encoding channels. Each row in Fig. 4 represents an independent field evolution of different initial OAM state. From different columns in Fig. 4, it can be seen that the amplitude (brightness) and phase (color) distributions of the optical field at six observable transverse planes shown in Fig. 3(a). Fig. 4 shows the initial field, field after SLM1, field after mode-sorter, field before SLM2, field before L2 and output field (from left to right). It should be mentioned that there is no need for post-selection to perform OAM-domain closed transformation with our proposal and thus the unity success probability is preserved. Thus, our results about implementing OAM-domain Hadamard matrix (or OAM beam splitter) may inspire some new applications such as observation of OAM-domain Hong-Ou-Mandel (HOM) effect [36] and OAM-encoded measurement-device-independent quantum key distribution (MDI-QKD) [37].
Additionally, another important feature of our proposal is parallel computing, which is considered as an advantage of optical information processing. With our scheme, the same matrix could be applied on different OAM channels simultaneously without any extra hardware cost. To further explain this, the optimized OAM spectral shaping function \( g(l) \) is shown in Fig. 5(a) for two \( H_2 \) gates acting on two groups of OAM channels with \( l = -3 & -2 \) and \( l = 4 & 5 \). It could be found that the two \( H_2 \) gates work independently with no crosstalk if the interval of utilized OAM channels are larger than two times of guard bands. For SLM modulation functions, the only change is to replace the OAM spectral shaping function \( g(l) \) by

\[
\text{conv}(g(l), \delta(l-l_1) + \delta(l-l_2) + \cdots),
\]

(13)

where \( \text{conv}(\cdot) \) denotes convolution and \( \delta(l-l_2) \) denotes the Dirac delta function. The OAM charge \( l_1, l_2 \ldots \) are the central OAM channels utilized by each identical matrix. It is unnecessary to change the angular modulation functions \( f_1(q) \) and \( f_2(q) \) for such parallel processing. Two \( H_2 \) gates acting on OAM channels of \( l = -14 & -10 \) and \( l = 10 & 14 \) are simulated and the entire \( 4 \times 4 \) matrix operator is evaluated and shown in Fig. 5(b). Furthermore, the simulated results of dual \( H_3 \) gates are shown in Fig. 5(c). The utilized OAM channels are \( l = -18 & -14 & -10 \) and \( l = 10 & 14 & 18 \). The \( H_3 \) gate is also noted as \( 3 \times 3 \) DFT. In Fig. 5(c), the \( H_3 \) gate is tested with input superposed OAM states of \( |\omega_1\rangle, |\omega_2\rangle \) and \( |\omega_3\rangle \) one-by-one, while \( |\omega_n\rangle \) denotes the \( n \)th column of \( H_3^T \). With this test, the phase accuracy of the unitary matrix can be evaluated. The colored bars and empty bars in Fig. 5(c) indicate input complex OAM and output OAM spectrum, respectively. It should be mentioned that the simulation in Fig. 5 is made under the same three-layer model shown in Fig. 3(a). It has been observed that the OAM-domain parallel gates work well without any extra hardware cost.
Fig. 5. The OAM-domain parallel $H_2$ gates and $H_3$ gates achieved by the three-layer structure. (a) OAM spectral shaping function for dual $H_2$ gates. (b) The entire $4 \times 4$ matrix operator made up of two $H_2$ gates. (c) Dual $H_3$ gates tested with input superposed OAM states of $|\omega_1\rangle$, $|\omega_2\rangle$ and $|\omega_3\rangle$ one-by-one.

Actually, the only limitation of the number of parallel matrices is the number of available OAM channels. Here, the mode interval of adjacent OAM channels is settled as 3 to avoid mode crosstalk induced by OAM mode-sorters. The OAM channel resources would be utilized more efficiently if some efforts are made to reduce the crosstalk [38]. Finally, it should be mentioned that our method is not constrained by the way of OAM spectral shaping. The system arrangement shown in Fig. 1 would be largely simplified with compact OAM spectral shaper. For some specific target matrices, the recently reported single-step OAM shaper [29,39] may be employed.

4. Path-domain Unitary Operation Protocol
Our unitary operation protocol can also be applied on path domain. According to the experimental results, our scheme is totally different to Reck’s scheme but keeps the same features of programmability, scalability and intrinsically lossless architecture.

To utilize the matrix decomposition in Eq. (2), we notice that there is a one-to-one mapping from transverse wave vector in the object plane to transverse coordinate in the focal plane. To encode qudits, the path basis can be defined by a series of transverse coordinates lying on $y -$ axis in the focal plane. Thus, the optical field $\hat{E}_o$ in the object plane is linked to the aforementioned path-encoded state vector through the Fourier transformation and expressed as:

$$\hat{E}_o(y) = \sum_{n=-\infty}^{\infty} c_n \exp(ink_y), \quad (14)$$

where $c_n$ denote the complex amplitude of optical field in the $n$th path-encoding channel. The distance between adjacent paths is noted as $L$, while the distance between adjacent SLMs is set to $2f$. This leads to a corresponding transverse wave vector of

$$k_y = \frac{L}{2f} \frac{\pi L}{\lambda f} \quad (15)$$

The experimental setup is designed as Fig. 6(a). The modulation functions on SLM1 and SLM3 are settled as following to introduce the required transverse wave vector components.

$$f(y) = \sum_{n=0}^{p} A_n \sin(nk_y + \theta_n) \quad (16)$$

Eq. (16) is very similar to the azimuthal phase modulation function in Eq. (11) for OAM-domain matrix transformation. The parameters $\{A_n\} \in [0,2\pi]$ and $\{\theta_n\} \in [0,2\pi]$ are to be determined by optimizations and possess the same meanings as shown in Eq. (11). The only difference is that the azimuthal coordinate $\varphi$ is replaced by linear coordinate $y$. The modulation function on SLM2 is settled as

$$\Phi(x,y) = g\left[\frac{x}{L}\right], \quad (17)$$

which is also a direct mapping of Eq. (8). The function of $g(n)$ represents the phase modulation applied to the $n$th path channel. Practically, $g(n)$, $f_1(y)$, $f_2(y)$ are optimized with the same method as $g(l)$, $f_1(\varphi)$, $f_2(\varphi)$ in OAM-domain unitary transformation. The $H_3$ gate is shown as an example to clarify it. The optimized $g(n)$ is shown in Fig. 6(b), where path channels $n = -1&0&1$ are chosen for encoding. The $f_1(y)$ and $f_2(y)$ functions are shown in Fig. 6(c) as phase-only gratings programmed on SLM1 and SLM3, respectively. Here, only one period of grating is plotted for simplicity and clarity. A lens with focal length of $f$ is added on SLM2 to maintain the Fourier relation between optical field on adjacent SLM planes. Two extra lenses noted as L1 and L2 in Fig. 6(a) are employed to compensate for transverse wave vector so that the setup can be scalable. As indicated in Eq. (14), we are only interested in the field variations along $y -$ axis in the transverse plane. The field along $x -$ axis is simply set as Gaussian shape and some ancillary lenses are programmed on SLMs to compensate the natural expansion of Gaussian beams.
80 different unitary transformations have been performed with the experimental setup of Fig. 6(a), while 75 of them are randomly generated. For each different target matrices, a particular set of modulation functions has to be optimized. The number of guard bands are set as d=2. A laser operating at 1550 nm (RIO Orion) is incident to the optical system through a collimator. Three SLMs (Holoeye Pluto) operating with reflective mode are utilized and the focal length mentioned before is set as 40 cm. The distance L between adjacent paths is designed as 2 mm considering the reflective area of SLMs. Specially, the input state is also changed by SLM1 at the same time with a superposed SLM pattern performing the weighted beam splitting [40].

The experimental results are summarized in Fig. 7(a) and (b). The fidelity distribution and success probability distribution of implemented matrices are shown as solid bars. Here, the fidelity is evaluated. Since the target matrix U is unitary, the value of fidelity(V, U) between U and implemented matrix V equals to fidelity(VV^T, UU^T) according to Eq. (9). Thus, the fidelity can be estimated by comparing VV^T and identical matrix. Actually, it is a phase test. During the test, the input states of |ψ_1⟩, |ψ_2⟩ and |ψ_3⟩ are settled as the columns of target matrix U^T. The results would also form an identical matrix if the implemented matrix V is sufficiently accurate, as shown in Fig. 7 (c–h). The success probability is calculated by the output optical power from the three path-encoding channels normalized by the total output power received by the charge coupled device (CCD) camera.
The red solid line in Fig. 7(a) is Gaussian fit of experimental results for calculating fidelity. The average value of fidelity is larger than 0.98 among 80 different tests. The success probability is shown in Fig. 7(b). The main reason of deteriorated probability is that part of optical energy entered guard bands and failed to vanish through interference. The experimental results of $H_1$ gate and $H_2$ gate (acting on the first and third channel) are shown in Fig. 7(c) and (d), respectively. The original data captured by CCD camera are also shown. As concrete examples, the data of another four typical unitary matrices are shown in Fig. 7 (e ~ h), while the fidelity ranges from 0.96 to 0.99.

A brief error analysis for experimental results is as follows. First, the conditioned maximum algorithm is utilized to determine the modulation functions of $g(n), f_1(y), f_2(y)$. The fidelity values are conditioned by fidelity $\geq 0.999$. As a tradeoff, the average probability value is lower than the average fidelity value. Second, a five-layer structure is sufficient for arbitrary $3 \times 3$ matrices according to Eq. (2). Though the three-layer structure employed in this work is considered for most cases, the fidelity and probability could not both achieve 1 at the same time for some particular target matrices, hence there is a tradeoff. Third, experimental errors in terms of optical misalignment and nonlinear effects induced by CCD camera would also deteriorate both the fidelity and probability.

It could be concluded that the three-layer structure is valid to achieve both high fidelity and high success probability for most 3-dimensional unitary matrices. Thus, the experimental results of path-encoded unitary transformation are important evidences to prove the versatility of our proposal.

Fig. 7 Experimental results of path-encoded matrix transformation.
5. Discussions

First, the relations and differences between our work and frequency-domain manipulation protocols are discussed. Our method shares the same basic mathematical theorem [22] of matrix decomposition with Lougovski’s work [21]. In their work, this theory is only employed for frequency-encoded qudit. We propose a general scheme by applying this mathematical decomposition on multiple DOFs and provide two representative examples on OAM and path domains. It should be noticed that this protocol can be utilized to any DOFs, if there is a Fourier transform pair. The Fourier pair of frequency and time-bin could be considered as a special case of our proposal. In this work, we are focused on passive setup for unitary operations. Frequency encoding is quite different from other photonic encoding since energy conservation is not satisfied for unitary linear operations such as the Pauli-X gate. Thus, it cannot be achieved by only passive setup. Correspondingly, the unitary operations within both OAM and path domains could be implemented with passive optical elements, which is potential to achieve more compact system arrangement and lower noise.

Next, a brief comparison between our proposal and previous approaches is made for both OAM-domain and path-domain. Different from other OAM-domain approaches, the cut-off effects from infinite OAM-encoding basis to finite subset is largely alleviated, which is due to help of ancillary guard bands. During the unitary operation, part of the input energy would enter the guard bands but such bands could be managed to vanish eventually through interference according to meticulously design. Thus, nearly unity success probability and fidelity can be achieved at the same time. Besides, our proposed OAM-domain scheme can perform parallel unitary operations on different OAM encoding channels simultaneously without increasing system arrangement. The number of required spectral shaper layers grows in $O(N)$ scale according to dimensionality $N$ of unitary operation. The reason is that each spectral shaper could provide $O(N)$ modulation on all the channels in parallel, thus the entire setup can fulfill $N^2$ independent real numbers required by a unitary operation. Compared with the MPLC approach [17], which requires high-resolution SLMs to settle the optimized patterns, the number of tunable pixels needed on spatial modulation elements in our approach is much less.

The path-domain operation protocol with our proposal can be implemented in both free space optics and integrated circuits. The SLMs could be replaced by three-dimensional set of metasurfaces [41] to achieve a compact module for unitary operations. Due to the similar alignment of path channels in our proposal and Reck’s scheme, an on-chip version of our design could also be implemented. The key function of Fourier transformation performed by free space lens could be achieved by refractive Bragg gratings arranged on the Rowland circle according to the recently reported work [42].

In conclusion, we have proposed and demonstrated a general scheme for universal operations in optical path and OAM domains. A three-layer setup is chosen to examine the performance. Field simulation of OAM-domain Hadamard matrices and experimental implementation of randomly generated path-domain unitary matrices have been carried out. The average fidelity value larger than 0.98 is evaluated. Our proposal can successfully reduce the implementation complexity form arbitrary matrices to diagonal matrices. It should be mentioned that our proposal can be extended to any high-dimensional encoding basis, if there exists the corresponding Fourier transform pair.

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