EXACT SELF-SIMILAR AND TWO-PHASE SOLUTIONS OF SYSTEMS OF SEMILINEAR PARABOLIC EQUATIONS

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Abstract

Interest to the tasks by bound with systems of the quasilinear equations recently has revived. Therefore we have decided to expose in electronic sort our operation published in Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 101, No. 2, pp. 189-199, November, 1994. Original article submitted October 26, 1993.

Exact single-wave and two-wave solutions of systems of equations of the Newell-Whitehead type are presented. The Painleve test and calculations in the spirit of Hirota are used to construct these solutions.

INTRODUCTION

In [1-4], methods were developed for constructing the explicit expressions that describe nonlinear waves (kinks) that are solutions of the semilinear parabolic equations

\[ U_t - U_{xx} = F(U), \]  

(0.1)

where \( F(U) \) are quadratic and cubic polynomials. The solutions of Eq.(0.1) considered in the cited studies satisfy boundary conditions that are determined by the roots of the nonlinearity, i.e., solutions propagating in strips were constructed. Besides bounded solutions, solutions having a discontinuity of the second kind (so-called monsters) were considered. In addition,
expression were obtained for solutions that describe the interaction of kinks and also kinks in discontinuous solutions. At the present time, only two types of solution of Eq.(0.1) represented by explicit expressions are known - so-called self-similar (single-phase, single-wave) solutions and solutions describing the interaction of self-similar waves (two-phase solutions) [1,4]. The self-similar solutions have the form (for the notation, see below)

\[ \chi(\tau) = U\left(\frac{\varphi \exp(\lambda \tau)}{\psi \exp(\lambda \tau)}\right), \]

\[ \chi(\tau) = U\left(\frac{\varphi(\lambda \tau)}{\psi(\lambda \tau)}\right), \]

(0.2)

For the nonself-similar solutions describing the interaction of single-phase waves, there is a representation of the form

\[ \chi(\tau_1, \tau_2) = U\left(\frac{\varphi(\exp(\lambda_1 \tau_1), \exp(\lambda_2 \tau_2), \tau_1, \tau_2)}{\psi(\exp(\lambda_1 \tau_1), \exp(\lambda_2 \tau_2), \tau_1, \tau_2)}\right), \]

(0.3)

Here, \( U(z) \) is a function that is entire in some region \( \Omega \in C^1 \); \( \varphi \) and \( \psi \) are polynomials; \( \tau = x + pt + c \) is the self-similar variable; \( \lambda \) and \( p \) are phase (related) constants; and \( c \) is an arbitrary constant. Similarly, \( \tau_i = x + p_i + c_i, i = 1, 2 \), where \( \lambda_i \) and \( p_i \) - are phase constants, \( c_i \) is an arbitrary constant, and the phases \( \tau_i \) are independent, i.e., a linear combination of them with integer coefficients does not vanish identically. We note that self-similar solutions of Eq.(0.1) are determined by the solution of the ordinary differential equation

\[ \lambda p \chi' - \lambda^2 \chi'' - F(\chi) = 0. \]

(0.4)

It is shown in [3,4] that for equations of the form (0.4) the store of solutions of the form (0.2) and, hence, the set of constants \( \lambda \) and \( p \) is very restricted. Similar results have also been obtained by means of the Painlevé test [5-7], the idea of which is to represent the solution of an equation of the form (1) by a Laurent series in the neighborhood of a moving pole, i.e., to expand the required function in powers of \( \tau \), the self-similar variable. The Painlevé test, applied to various equations of the form (0.1), showed that the set of possible velocities for self-similar solutions represented by a Laurent series is, as a rule, small and can be uniquely determined. In addition, for each
concrete value of the velocity $p$ obtained by means of the Painlevé test the 
Laurent series can uniquely, up to a shift constant, determine a function of 
the form (0.2). Therefore, in the case of a positive Painlevé test, depending 
on its results, it is natural to assume the existence for equations of the form 
(0.1) of solutions in the class of functions defined by the expressions (0.2)-(0.3). In this case, one can substitute in the original equation the general 
form of the assumed solution (the ansatz), where $\varphi$ and $\psi$ are polynomials 
with undetermined coefficients. Equating the coefficients of the same powers 
of the exponentials or (and) equal powers of the self-similar variables, we 
obtain are overdetermined, in general, system of nonlinear algebraic equa-
tions for the unknown coefficients of the polynomials $\varphi$ and $\psi$ and the phase 
constants. The solution of this system (if it exists) is necessary and suffi-
cient for constructing an explicit expression for the solution of the original 
equation. We note that it was precisely an assumption about the form of 
the solutions of equations of the form (0.1) that made possible the successful 
use of constructive methods of solution finding, for example, modification of 
Hirota’s method ([2], p. 184,[2’],p.68).

In this paper, we describe a modification of the Painlevé test for sys-
tems of semilinear parabolic equations. It turns on that the set of possible 
velocities of self-similar solutions of systems is, as in the case of individual 
equations, restricted and can be uniquely determined. The analogy with the 
scalar case made it possible to assume that the systems have solutions of 
the definite form, for the finding of which the method of undetermined co-
efficients described above was used. It is characteristic that in the case of 
the systems considered in this paper, as in the one-dimensional case, it is, 
as a rule, possible to find all similar solutions whose velocity is determined 
by the Painlevé test by making a restriction to functions of the form (0.2). 
By means of the same method we have succeeded in constructing two-phase 
solutions of the form (0.3) of some of the consider systems.

We note however that the mechanism of interaction of waves that are 
solutions of systems of semilinear parabolic equation has been found to be 
very specific. Thus, in [4] (p. 44,[2’]p.70) there is an asymptotic description 
of the process of wave creation from a finite perturbation for the Kolmogorov - 
Petrovskii - Piskunov (KPP) - Fisher equation, from which there follows 
the impossibility of constructing an expression that describes the interaction 
of the waves in the form dictated by Hirota’s method. It turned out that the 
situation is quite different for systems of semilinear parabolic equations. In 
the paper, we consider example of a system for which explicit expressions for
a solution describing the annihilation of kinks are constructed (In this text
the figures are not reduced. Look figures in the original of a paper in TMF
. see Fig. 2 below).

The subject of the investigation in the present paper is systems of the
form

\[
U_t - U_{xx} = U^l(1 - U^n - \theta^n),
\]

\[
\theta_t - \theta_{xx} = -BU^k\theta^n,
\]

where \(U, \theta\) are the unknown functions, \(B\) is a nonvanishing constant, and
\(m, n, k, q, l\) are natural numbers. The paper consist of two sections. In Sec.
1, we give self-similar solutions of a system that is one of the forms of the
Belousov - Zhabotinski reaction model [8,9] and has, in addition, a biophys-
ical application [10]. In Sec. 2, we give self-similar and two-phase solutions
of various systems describing processes of the type of nonlinear kinetics.

1 SELF-SIMILAR SOLUTIONS
IN THE BELOUSOV–ZHABOTINSKII
REACTION MODEL

Suppose that in (0.5) \(m = n = k - q = l = l, B \neq 0, B \neq 1/2\), i.e., the system
has the form

\[
U_t - U_{xx} = U^l(1 - U^n - \theta^n),
\]

\[
\theta_t - \theta_{xx} = -BU^k\theta^n,
\]

Then the following theorem holds.

**Theorem 1.1** All self-similar solutions of the system (1.1) that can be rep-
resented by a Laurent series in a punctured neighborhood of the moving pole
\(x = -pt + c\), where \(p\) is the velocity and \(c\) is an arbitrary constant, have the
form

\[
U = \frac{1}{(1 + \exp \tau)^2}, \quad \theta = \frac{(1 - B)(2 \exp \tau + \exp 2\tau)}{(1 + \exp \tau)^2},
\]

\[
\tau = ax + bt, \quad a^2 = B/6, \quad b = -5B/6;
\]

(1.2)
\[ U = \frac{1 - 2 \exp \tau}{(1 + \exp \tau)^2}, \quad \theta = \frac{(1 - B) \exp 2\tau}{(1 + \exp \tau)^2}, \]
\[ \tau = ax + bt, \quad a^2 = -B/6, \quad b = -5B/6, \quad p = b/a; \quad (1.3) \]

**Proof.**

The fact that the given functions solve the system can be verified by substitution. We consider in more detail question of the uniqueness of the presented self-similar solutions associated with modification of the Painlevé test for system. We shall seek a solution in the form of Laurent series in powers of \( \tau \).

\[ U = \sum_{k=-q_1}^{\infty} a_{k+q_1}(x + pt + c_1)^k, \]
\[ \theta = \sum_{k=-q_2}^{\infty} B_{k+q_2}(x + pt + c_1)^k, \quad (1.4) \]

where \( q_1 \) and \( q_2 \) are certain natural numbers that determine the order of the poles of the functions \( U \) and \( \theta \), respectively, and in what follows we shall, without loss of generality, assume that the constants \( c_1 \) and \( c_2 \) are equal to zero. In the general case, the possible values of \( q_1 \) and \( q_2 \) are determined by means of Newton’s polygon method; however, in the present case it can be done by using elementary arguments and equating the exponents of the definitely highest negative powers of the second derivatives and the quadratic terms of the polynomials. It is easy to see that in the given case \( q_1 = q_2 = 2 \), i.e., both functions \( U \) and \( \theta \) have a pole of second order. Substituting the obtained series in the system and equating the coefficients of equal powers of \( \tau \), we obtain a system of nonlinear algebraic equations for the determination of the coefficients of the series:

\[ a_0(a_0 + b_0 - 6) = 0, \quad b_0(a_0B - 6) = 0, \]
\[ 2a_1a_0 + a_1b_0 - 2a_1 + a_0b_1 - 2a_0p = 0, \]
\[ a_1b_0B + a_0b_1B - 2b_1 - 2b_0p = 0, \]
\[ 2a_2a_0 + a_2b_0 + a_1^2 + a_1b_1 - a_1p + a_0b_2 - a_0 = 0, \]
\[ a_2b_0B + a_1b_1B + a_0b_2B - b_1p - 0, \]
\[ 2a_3a_0 + a_3b_0 + 2a_2a_1 + a_2b_1 + a_1b_2 - a_1 + a_0b_3 = 0, \]
\[ B(a_3b_0 + a_2b_1 + a_1b_2 + a_0b_3) = 0, \]
\[2a_4a_0 + a_4b_0 - 2a_4 + 2a_3a_1 + a_3b_1 + a_3p + a_2^2 + a_2b_2 - a_2 + a_1b_3 + a_0b_4 = 0,\]
\[a_4b_0B + a_3b_1B + a_2b_2B + a_1b_3B + a_0b_4B - 2b_4 + b_3p = 0,\]
\[2a_2a_0 + a_5b_0 - 6a_5 + 2a_4a_1 + a_4b_1 + 2a_4p + 2a_3b_2 + a_3b_2 - a_3 + a_2b_3 + a_1b_4 + a_0b_5 = 0;\]
\[a_5b_0B + a_4b_1B + a_3b_2B + a_2b_3B + a_1b_4B + a_0b_5B - 6b_5 + 2b_4p = 0,\]
\[2a_6a_0 + a_6b_0 - 12a_6 + 2a_5a_1 + a_5b_1 + 3a_5p + 2a_4a_2 + a_4b_2 - a_4 + a_3^2 + a_3b_3 + a_2b_4 + a_1b_5 + a_0b_6 = 0,\]
\[a_6b_0B + a_5b_1B + a_4b_2B + a_3b_3B + a_2b_4B + a_1b_5B + a_0b_6B - 12b_6 + 3b_5p = 0.\]

(1.5)

Solving successively this system for the unknown coefficients \(a_k\) and \(b_k\), we obtain
\[a_0 = \frac{6}{B}, \quad b_0 = \frac{6(B - 1)}{B}, \quad a_1 = \frac{6p}{5B}, \quad b_1 = \frac{6(B - 1)p}{5B},\]
\[a_2 = \frac{25B - p^2}{50B}, \quad b_2 = -\frac{(1 - B)(p^2 + 25B)}{50B},\]
\[a_3 = \frac{p^3}{250B^3}, \quad b_3 = \frac{(B - 1)p^3}{250B^3},\]
\[a_4 = \frac{125B^2 - 7p^4}{5000B}, \quad b_4 = \frac{(B - 1)(125B^2 - 7p^4)}{5000B},\]
\[a_5 = -\frac{1375B^2 - 79p^4}{75000B}, \quad b_5 = \frac{p(1 - B)(1375B^2 - 79p^4)}{75000B},\]
\[a_6 = \frac{37500b_6 - 625B^2p^2 + 36p^4}{37500(B - 1)}.\]

(1.6)

It is readily seen that in the final equation of this system the coefficient of the unknown \(b_6\) is equal to zero and the obtained equation is an expression from which the possible values of the velocity can be determined. This expression is usually called the dispersion relation. It has the form
\[(2B - l)(36p^4 - 625B^2) = 0.\]

(1.7)

By virtue of the conditions of the theorem, the solution of Eq.(1.6!!) for \(p\) gives two (up to the sign) possible values of the
\[p_{1,2} = \pm \frac{5\sqrt{B}}{\sqrt{6}}, \quad p_{3,4} = \pm \frac{5i\sqrt{B}}{\sqrt{6}}.\]

(1.8)
For each particular value of the velocity $p$, the coefficients $a_k$ and $b_k$ of the series for $U$ and $\theta$ are uniquely determined; therefore, there exist only two different self-similar solutions of the system that can be represented by a Laurent series with second-order pole. It is obvious that to each of these solutions there corresponds one of the two possible values of the velocity and that for a definite value of the parameter $B$ with fixed sign these values are different. By virtue of the invariance of the substitution $x^- > -x$ in the system (1.5), reversal of the sign of the phase constant $a$ corresponds to reflection of the traveling wave with respect to the ordinate with change of the direction of its motion, i.e., reversal of the velocity. Thus, the above solutions completely exhaust the store of velocities obtained by means of the Painlevé test. □

**Remark 1.1** It is clear that for any $B \neq 0$ only one of the two solutions (1.2) and (1.3) is real. The boundary condition that real solutions of the system (1.1) satisfy have the form

$$U_{x \to \infty} \to 0, \quad \theta_{x \to \infty} \to 1 - B, \quad U_{x \to -\infty} \to 1, \quad \theta_{x \to -\infty} \to 0. \quad (1.9)$$

**Remark 1.2** It follows from Eq. (1.6) that for $B = l/2$ there may also exist other solutions in addition to (1.2) and (1.3) that are given by explicit expressions.

Note that as $B \to 1$ in each of the solutions (1.2) and (1.3) $\theta \to 0$, and at the same time the solution of the system it continuously transformed into the solution of the scalar equation

$$U_t - U_{xx} - U(1 - U) = 0. \quad (1.10)$$

which is known as the KPP - Fisher equation. The Painlevé test for Eq. (1.8) gives, up to the sign, two possible values of that velocity for solutions that can be represented by a Laurent series, namely

$$p_{1,2} = \pm \frac{5}{\sqrt{6}}, \quad p_{3,4} = \pm \frac{5i}{\sqrt{6}}. \quad (1.11)$$

The solutions with such values of speed were earlier calculated in [2] p.184.
It is therefore obvious that there do not exist any other solutions of Eq. (1.8) that can be expanded in a Laurent series apart from those that are obtained from the expressions (1.2) and (1.3) for $B=1$. In addition, it is clear that in the given case the velocity (1.7) are continuous functions of the parameter $B$ of the system on the complete domain of definition, i.e., for any $B \neq 0$ will be shown below that this is not always the case.

2 SELF-SIMILAR AND TWO-PHASE SOLUTIONS IN MODELS OF NONLINEAR KINETICS

Suppose that in (0.5) $m = n = k = 2, q = l = 1$. We obtain the system

$$U_t - U_{xx} = U(1 - U^2 - \theta^2),$$
$$\theta_t - \theta_{xx} = -BU^2\theta,$$

(2.1)

As in the case of the system (1.1), the order of the poles of the functions $U$ and $\theta$ in the system (2.1) is determined. In the given case, $ql = q2 = 1$ in (1.4) and (1.5). The unique (up to sign) possible expression for the velocity that is obtained by me; of the Painlevé test has the form

$$p_{1,2} = \frac{\pm 3\sqrt{B}}{\sqrt{2(2B - 1)}},$$

(2.2)

It proved to be possible to construct solutions of the system (2.1) for $B = 2$.

**Theorem 2.1** All self-similar solutions of the system (2.1), $B = 2$, represented by a Laurent series in the punctured neighborhood of the moving pole $x = -pt + c$, where $p$ is the velocity, and $c$ is an arbitrary constant, have the form

$$U = \frac{\pm 1}{1 + \exp \tau}, \quad \theta = \frac{\pm \exp \tau}{1 + \exp \tau},$$
$$\tau = ax + bt, \quad a = \pm 1, \quad b = -1, \quad p = b/a = \pm 1;$$

(2.3)

$$U = \frac{-a}{a - \tau}, \quad \theta = \frac{\pm \exp \tau}{1 + \exp \tau},$$
$$\tau = ax + bt, \quad b = 2a, \quad p = b/a = 2;$$

(2.4)
where \( c \) is an arbitrary constant. \textit{Proof.} The fact that functions (2.3) and (2.4) satisfy the system can be verified by substitution. The Painlevé test for the considered system gives two possible values of the velocity for solutions that can be represented by a Laurent series: \( p_1 = 2, p_2 = \pm 1 \). It is with such velocities that the above solutions move. From this the statement of the theorem follows, and the theorem is proved. \( \square \)

\textbf{Remark 2.1} The value of the expression (2.2) for \( B = 2(p = \sqrt{3}) \) is different from the values of the velocity given in Theorem 2 for the solutions of the system (2.1) for the same value of the parameter \( B \). Thus, the expression (2.2) is not a continuous function of the parameter \( B \) of the system (2.1).

\textbf{Proposition 1.} The system (2.1), \( B = 2 \), has a two-phase (nonself-similar) solution of the form

\[
\begin{align*}
U &= \frac{a_1 + \exp \tau_2}{a_1 + \tau_1 + \exp \tau_2}, \\
\theta &= \frac{\pm \tau_1}{a_1 + \tau_1 + \exp \tau_2}, \\
\tau_1 &= a_1 x + b_1 t, \\
\tau_2 &= a_2 x + b_2 t, \\
b_1 &= -2a_1, \\
b_2 &= 1, \\
a_2 &= 1;
\end{align*}
\]

(2.5)

where \( a_1 \) is an arbitrary constant. The proposition is verified by substitution. The interaction of the self-similar waves (2.3) and (2.4) described by (2.5) is analogous to the interaction of a kink with a discontinuous solution, which can be described by explicit expressions for individual equations with cubic nonlinearity [4] (p. 39-42), [2] p. 51. Figure 1 gives the graph of the function \( U \) for \( a_1 = 1 \). It can be seen that on the collision of the singularity with the front of the kink the order of the latter increases. For \( a_1 = -1 \), the interaction of the discontinuous solution with the kink described by the function \( U \) leads to annihilation of the singularity and the formation of one kink. A situation analogous to the considered cases also holds for the function \( \theta \). We turn to the next example. Suppose that in (5) \( m = n = q = l, l = k = 2 \). The resulting system has the form

\[
\begin{align*}
U_t - U_{xx} &= U^2(1 - U - \theta), \\
\theta_t - \theta_{xx} &= -BU^2 \theta,
\end{align*}
\]

(2.6)

In this case, it proved possible to construct both self-similar and two-phase solutions containing the parameter \( B \) of the original system (2.6) as a free
The Painlevé test for the system (2.6) gives two possible values of the velocity for self-similar solutions that can be represented by a Laurent series, namely

\[ p_1 = \pm \sqrt{\frac{B}{2}}, \quad p_2 = \pm \sqrt{2B}. \quad (2.7) \]

Note that this is true for any \( B \neq 1/3 \), since for \( B = 1/3 \) the dispersion relation vanishes identically. A consequence of this result is the following theorem, the proof of which is completely analogous to that of Theorems 1 and 2.

**Theorem 2.2** All self-similar solutions of the system (2.6), \( B \neq 1/3 \), that can be represented by a Laurent series in the punctured neighborhood of the moving pole \( x = -pt + c \), where \( p \) is the velocity and \( c \) is an arbitrary constant, have the form

\[ U = \frac{c_1}{a_1 + a_2 \exp \tau}, \quad \theta = \frac{(1-B)a_2 \exp \tau}{c_1 + a_2 \exp \tau}, \]

\[ \tau = ax + bt, \quad a^2 = B/2, \quad b = -B/2, \quad p = b/a = \pm \frac{\sqrt{B}}{\sqrt{2}}, \quad (2.8) \]

\( c_1, c_2 \) are arbitrary constants;

\[ U = \frac{a \sqrt{2}}{\tau \sqrt{B} + a \sqrt{2}}, \quad \theta = \frac{\sqrt{B}(1-B)\tau}{\tau \sqrt{B} + a \sqrt{2}}, \]

\[ \tau = ax + bt, \quad a^2 = B/2, \quad b = -a \sqrt{2B}, \quad p = -\sqrt{2B}, \quad (2.9) \]

\( a \) is an arbitrary constant.

**Remark 2.2** As in the case (1.1), for \( B = 1/3 \) there can be other solutions in addition to (2.8) and (2.9) that can be represented by explicit expressions.

**Proposition 2.** The system (2.6) has a two-phase solution of the form

\[ U = \frac{a_1 + a_2 \exp \tau_2}{a_1 + a_2 (\tau_1 + \exp \tau_2)}, \quad \theta = \frac{a_2 (1-B) \tau_1}{a_1 + a_2 (\tau_1 + \exp \tau_2)}, \]

\[ \tau_1 = a_1 x + b_1 t, \quad \tau_2 = a_2 x + b_2 t, \]

\[ b_1 = -2a_1 a_2, \quad a_2^2 = B/2, \quad b_2 = B/2, \quad (2.10) \]
is $a_1$ arbitrary constant. The validity of the proposition is proved by substituting the expressions (2.10) in the system (2.6). As in the case of (1.1), the self-similar solutions of the system (2.6), and also the two-phase solution (2.10) describing their interaction are reduced continuously in the limit $B \to 1$, respectively, to the self-similar and two-phase solutions of the one-dimensional problem known as one of the forms of the Zel’dovich equation ([2], p. 73, p. 198): $U_t - U_{xx} - U^2(1 - U) = 0$. Here, as in the system (1.1), the velocities (2.7) are continuous functions of the parameter B on the complete domain of definition. Note that all the given solutions of the system (2.6) are real for $B > 0$. The functions (2.10) describe the interaction of self-similar solutions of the system (2.6). The nature of this interaction is similar to the case considered above (see Fig. 1 in the original of a paper in TMF ). We now give an example of the system (5) in which it proved possible to find not only self-similar solutions but also a bounded two-phase solution that is the result of interaction of waves propagating in strips.

Suppose that in (5) $m = k = q = n = l = l$. Then we obtain the system

$$
U_t - U_{xx} = U(1 - U^2 - \theta),
$$

$$
\theta_t - \theta_{xx} = -BU^2\theta,
$$

(2.11)

**Theorem 2.3** The system (2.11) has nontrivial self-similar solutions that can be represented by a Laurent series for a unique value of the parameter $B$, namely, $B = 1$.

**Proof.** Functions $U$ and $\theta$ that can be represented by a Laurent series in powers of $t$ and satisfy the system (2.11) have a first-order pole, i.e., the coefficients of $a$ and $b_0$ must be nonzero. It follows from the system of equations for these coefficients that this is possible only if $B = 1$:

$$
a_0(a_0^2 - 2) = 0, \quad b_0(a_0^2 B - 2) = 0.
$$

(2.12)

The theorem is proved. □

**Proposition 3.** There exist self-similar solutions of the system (2.11), $B = 1$, having the form

$$
U = \frac{\pm 1}{1 + \exp \tau}, \quad \theta = \frac{\exp \tau_1}{1 + \exp \tau_2},
$$

$$
\tau = ax + bt, \quad b = -1/2, \quad a^2 = 1/2,
$$

(2.13)
Proposition 4. There exists a two-phase solution of the system (2.11), $B = 1$, having the form

$$U = \frac{1 - \exp \tau_1}{1 + \exp \tau_1 + \exp \tau_2}, \quad \theta = \frac{\exp \tau_2}{1 + \exp \tau_1 + \exp \tau_2},$$

$$\tau_1 = a_1 x + b_1 t, \quad \tau_2 = a_2 x + b_2 t,$$

$$b_1 = 0, \quad a_1 = \sqrt{2}, \quad a_2 = 1/\sqrt{2}, \quad b_2 = -1/2,$$

(2.14)

The validity of both propositions can be proved by substituting the given solutions in the system (2.11). The two-phase functions $U$ and $B$ are the result of interaction of the self-similar waves described in Proposition 3. Graphs of the two-phase solution are shown in Fig. 2. It can be clearly seen that the interaction described by the function 6 (see Fig. 2a) represents the annihilation of kinks propagating in the strip between 0 and 1 (for $t > t_0$, the solution is close to zero). In the case of the function $U$ (see Fig. 2b), there is an interaction of the type of merging of waves propagating in different strips [for $t > t_0$, a single wave close to the stationary solution of Semenov’s equation ([2], p. 193) is formed].

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