On the Optimal Recovery Threshold of Coded Matrix Multiplication

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Abstract

We provide novel coded computation strategies for distributed matrix-matrix products that outperform the recent “Polynomial code” constructions in recovery threshold, \textit{i.e.}, the required number of successful workers. When \(m\)-th fraction of each matrix can be stored in each worker node, polynomial codes require \(m^2\) successful workers, while our MatDot codes only require \(2m - 1\) successful workers, albeit at a higher communication cost from each worker to the fusion node. Further, we propose “PolyDot” coding that interpolates between Polynomial codes and MatDot codes. Finally, we demonstrate an application of MatDot codes to multiplying multiple (> 2) matrices.

I. INTRODUCTION

As the era of Big Data advances, massive parallelization has emerged as a natural approach to overcome limitations imposed by saturation of Moore’s law (and thereby of single processor compute speeds). However, massive parallelization leads to computational bottlenecks due to straggling or faulty nodes, which has been a topic of active recent interest in the emerging area of “coded computation” (\textit{e.g.} \cite{1}-\cite{23}), which not only advances on coding approaches in classical works on Algorithm-Based Fault Tolerance (ABFT), but also provides novel analyses of required computation time (\textit{e.g.} expected time \cite{1} and deadline exponents \cite{24}). Perhaps most importantly, it brings an information-theoretic lens to the problem by examining fundamental limits and comparing them with existing strategies. A broader survey of results and techniques is provided in \cite{25}.

This paper focuses on the coded matrix multiplication problem. Matrix multiplication is central to many modern computing applications, including machine learning and scientific computing. There is a lot of interest in classical ABFT literature (starting from \cite{26}) and more recently in coded computation literature (\textit{e.g.} \cite{4}, \cite{27}) to make matrix multiplications resilient to faults and delays. In particular, Yu, Maddah-Ali, and Avestimehr \cite{4} provide novel coded matrix-multiplication constructions called \textit{Polynomial codes} that outperform classical work from ABFT literature in terms of the recovery threshold, the minimum number of successful (non-delayed, non-faulty) processing nodes required for completing the overall matrix multiplication.

Our work advances on those constructions in scaling sense. Concretely, when \(m\)-th fraction of each matrix can be stored in each worker node, Polynomial codes require \(m^2\) successful workers, while MatDot codes only require \(2m - 1\) successful workers. However, as we note in Section \ref{motivation}, this comes at an increased per-worker communication cost. We begin with some motivating examples that show our constructions and a summary of results.

A. Motivating examples and summary of results

Consider the setting where each processing worker node stores \(N^2/m\) linear combinations of the entries of \(\textbf{A}\) and \(N^2/m\) linear combinations of the entries of \(\textbf{B}\). Assume that \(P\) worker nodes perform the computation, we evaluate the straggler tolerance of a technique by its recovery threshold \(k\). In the following, we demonstrate our construction for \(m = 3\). We describe (i) ABFT matrix multiplication \cite{26} (also called product-coded matrices in \cite{27}), (ii) Polynomial codes \cite{4} and then (iii) our proposed construction, MatDot codes, each with successively improving, \textit{i.e.}, smaller, recovery threshold. We begin by describing ABFT matrix multiplication.

Example 1. [ABFT codes \cite{26}, Fig. 1] \(k = 6\) Consider two \(N \times N\) matrices

\[
\textbf{A} = \begin{bmatrix} \hat{\textbf{A}}_1 \\ \hat{\textbf{A}}_2 \end{bmatrix}, \textbf{B} = \begin{bmatrix} \hat{\textbf{B}}_1 & \hat{\textbf{B}}_2 \end{bmatrix}.
\]

Using ABFT, it is possible to compute \(\textbf{AB}\) over \(P\) nodes such that, (i) each node uses one linear combination of \(\textbf{A}\) and one linear combination of \(\textbf{B}\) and (ii) the overall computation is tolerant to \(P - 6\) stragglers, \textit{i.e.}, 6 nodes suffice to recover \(\textbf{AB}\)? ABFT codes use the strategy as per Fig. 1 where 4 of the 9 worker nodes compute \(\hat{\textbf{A}}_i, \hat{\textbf{B}}_j, i, j \in \{1, 2\}\) and the remaining worker nodes compute \(\hat{\textbf{A}}_1(\hat{\textbf{B}}_1 + \hat{\textbf{B}}_2), (\hat{\textbf{A}}_1 + \hat{\textbf{A}}_2)\hat{\textbf{B}}_1, (\hat{\textbf{A}}_1 + \hat{\textbf{A}}_2)(\hat{\textbf{B}}_1 + \hat{\textbf{B}}_2)\) for \(i = 1, 2\), respectively. The general principle of ABFT is to encode the rows of \(\textbf{A}\) and the columns of \(\textbf{B}\) using systematic MDS codes of dimension \(m\).

A superior recovery threshold was obtained in \cite{4}, which gave the following polynomial code construction.

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Fig. 1. ABFT matrix multiplication \[26\] for \( P = 9 \) worker nodes with \( m = 2 \), where \( A = \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} \) and \( B = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \). The recovery threshold is 6.

Example 2. [Polynomial codes \[4\], Fig. 2, \( k = 4 \)] Consider two \( N \times N \) matrices

\[
A = \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix}, \quad B = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}.
\]

Polynomial codes computes \( AB \) over \( P \) nodes such that, (i) each node uses \( N^2/2 \) linear combination of the entries of \( A \) and \( N^2/2 \) linear combination of the entries of \( B \) and (ii) the overall computation is tolerant to \( P - 4 \) straggler, i.e., 4 nodes suffice to recover \( AB \). Polynomial codes use the following strategy: Node \( i \) computes \( (\tilde{A}_1 B_1 + \tilde{A}_2 B_2 x + \tilde{A}_1 B_2 x^2 + \tilde{A}_2 B_2 x^3) \) can be interpolated. Having interpolated the polynomial, the coefficient (matrices) can be used to evaluate \( AB \).

Fig. 2. Polynomial Codes \[4\] with \( m = 2 \), where \( A = \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} \) and \( B = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \). The recovery threshold is 4.

Our novel MatDot construction obtains a smaller recovery threshold as compared with polynomial codes. Unlike ABFT and polynomial codes, MatDot divides matrix \( A \) column-wise and matrix \( B \) row-wise.

Example 3. [MatDot codes, Fig. 3, \( k = 3 \)] MatDot codes compute \( AB \) over \( P \) nodes such that, (i) each node uses \( N^2/2 \) linear combination of the entries of \( A \) and \( N^2/2 \) linear combination of the entries of \( B \) and (ii) the overall computation is tolerant to \( P - 3 \) straggler, i.e., 3 nodes suffice to recover \( AB \). The proposed MatDot codes use the following strategy: Matrix \( A \) is split into submatrices \( A_0, A_1 \) vertically each of dimension \( N \times N/2 \) and matrix \( B \) is split into submatrices \( B_0, B_1 \) horizontally each of dimension \( N/2 \times N \) as follows:

\[
A = \begin{bmatrix} A_0 & A_1 \end{bmatrix}, \quad B = \begin{bmatrix} B_0 \\ B_1 \end{bmatrix}.
\]

(1)

Now, we define the encoding functions \( p_A(x) \) and \( p_B(x) \) as \( p_A(x) = A_0 + A_1 x \) and \( p_B(x) = B_0 x + B_1 \). Let \( x_0, x_1, \ldots, x_{P-1} \) be distinct elements of field \( F \), the master node sends \( p_A(x_{l-1}) \) and \( p_B(x_{l-1}) \) to the \( l \)-th worker node where the \( l \)-th worker node performs the multiplication \( p_A(x_{l-1})p_B(x_{l-1}) \) and sends the output to the fusion node. The exact computations at each worker node are depicted in Fig. 3. We can observe that the fusion node can obtain the product \( AB \) using the output of any
three successful workers as follows: Let the worker nodes 1, 2, and 3 be the first three successful worker nodes, then the fusion node owns the following three matrices:

\[
p_A(x_0)p_B(x_0) = A_0B_1 + (A_0B_0 + A_1B_1)x_0 + A_1B_0x_0^2,
\]

\[
p_A(x_1)p_B(x_1) = A_0B_1 + (A_0B_0 + A_1B_1)x_1 + A_1B_0x_1^2,
\]

\[
p_A(x_2)p_B(x_2) = A_0B_1 + (A_0B_0 + A_1B_1)x_2 + A_1B_0x_2^2.
\]

Since these three matrices can be seen as three evaluations of the matrix polynomial \(p_A(x)p_B(x)\) of degree two at three distinct evaluation points \(x_0, x_1, x_2\), the coefficients of powers of \(x\) in \(p_A(x)p_B(x)\) can be obtained using polynomial interpolation. This includes the coefficient of \(x\), \(A_0B_0 + A_1B_1\), i.e., \(AB\).

Since the worker nodes require to communicate \(N \times N\) matrices to the fusion node, MatDot incurs higher communication cost as compared to polynomial codes.

Our main contributions are as follows:

- We present our system model in the next section, and describe MatDot codes in Section III. While polynomial codes have a recovery threshold of \(\Theta(m^2)\), MatDot codes have a recovery threshold of \(\Theta(m)\).
- We present a systematic version of MatDot codes - where the first operations of the first \(m\) worker nodes may be viewed as multiplication in uncoded form - in Section IV.
- In Section V we propose “PolyDot codes”, a unified view of MatDot and Polynomial codes that leads to a trade-off between recovery threshold and communication costs.
- In Section VI we apply the constructions of Section III to study coded computation for multiplying more than two matrices.

We note that recent works of Yu, Maddah-Ali, and Avestimehr [28] and Dutta, Bai, Grover, and Low [29] obtain constructions that outperform PolyDot codes in tradeoffs between communication and recovery threshold (although MatDot codes continue to have the smallest recovery threshold for given storage constraints). Importantly, Yu et al. [28] also provide interesting impossibility (converse) results demonstrating approximate optimality of these improved strategies.

II. SYSTEM MODEL AND PROBLEM STATEMENT

A. System model

The system, illustrated in Fig. 3, consists of a master node, multiple worker nodes, and a fusion node that combines outputs from successful worker nodes. These are defined more formally below.

Definition 1. [Computational system] A computational system consists of (i) a master node that receives computational inputs and processes them, (ii) memory-constrained worker nodes that perform pre-determined computations on their respective inputs in parallel, and (iii) a fusion node that receives messages from successful workers and performs decoding operations on them.

For practical utility, it is important to have the amount of processing that the worker nodes perform to be much smaller than the processing at the master and the fusion node.
Before we prove Theorem 1, we first describe the general MatDot code construction. In this section, we provide a code construction, named MatDot code, for the problem stated in Section II-B that obtains a recovery threshold of $m$ symbols from the master node, where each symbol is an element of $\mathbb{F}$. For the simplicity, we assume that $m$ divides $N$. The computational complexities of the master and fusion nodes, in terms of the matrix parameter $N$, are strictly less than the computational complexity at any worker node. The goal is to perform this matrix product utilizing faulty/delay prone workers with as low recovery threshold as possible.

### III. MatDot Codes

In this section, we provide a code construction, named MatDot code, for the problem stated in Section II-B that obtains a recovery threshold of $2m - 1$ where $m \geq 2$ is any positive integer that divides $N$.

**Theorem 1.** For the matrix multiplication problem specified in Section II-B computed on the system defined in Definition 1, a recovery threshold of $2m - 1$ is achievable where $m \geq 2$ is any positive integer that divides $N$.

Before we prove Theorem 1, we first describe the general MatDot code construction.

#### Construction 1. [MatDot Codes]

**Splitting of input matrices:** The matrix $A$ is split “vertically” and $B$ is split “horizontally” into $m$ equally sized pieces (of $N^2/m$ symbols each) as follows:

$$A = [A_0 | A_1 | \ldots | A_{m-1}], \; \; B = \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{m-1} \end{bmatrix},$$  \hspace{1cm} (2)

where, for $i \in \{0, \ldots, m-1\}$, and $A_i, B_i$ are $N \times N/m$ and $N/m \times N$ dimensional matrices, respectively.

**Master node (encoding):** Let $x_1, x_2, \ldots, x_P$ be distinct elements of $\mathbb{F}$. Let $p_A(x) = \sum_{i=0}^{m-1} A_i x^i$ and $p_B(x) = \sum_{j=0}^{m-1} B_j x^{m-1-j}$. The master node sends to the $l$-th worker evaluations of $p_A(x), p_B(x)$ at $x = x_l$, that is, it sends $p_A(x_l), p_B(x_l)$ to the $l$-th worker.

**Worker nodes:** For $l \in \{1, 2, \ldots, P\}$, the $l$-th worker node computes the matrix product $C(x_l) = p_A(x_l)p_B(x_l)$ and sends it to the fusion node on successful completion.

If the fusion node is allowed to have higher computational complexity, the workers can simply store $A, B$ using maximum distance separable codes to get a recovery threshold of $m$; the fusion node simply recovers $A, B$ and then multiplies them, essentially performing the whole operation.
Fusion node (decoding): The fusion node uses outputs of any $2m - 1$ successful workers to compute the coefficient of $x^{m-1}$ in the product $C(x) = p_A(x)p_B(x)$ (the feasibility of this step will be shown later in the proof of Theorem 7). If the number of successful workers is smaller than $2m - 1$, the fusion node declares a failure.

Notice that in MatDot codes, we have

$$AB = \sum_{i=0}^{m-1} A_iB_i,$$

where, for $i \in \{0, \cdots, m - 1\}$, $A_i$ and $B_i$ are as defined in (2). The simple observation of (3) leads to a different way of computing the matrix product as compared with Polynomial codes based computation. In particular, to compute the product, we only require, for each $i$, the product of $A_i$ and $B_i$. That is, we do not require products of the form $A_iB_j$ for $i \neq j$ unlike Polynomial codes, where, after splitting the matrices $A$, $B$ in to $m$ parts, all $m^2$ cross-products are required to evaluate the overall matrix product. This leads to a significantly smaller recovery threshold for our construction.

Proof of Theorem 7: To prove the theorem, it suffices to show that in the MatDot code construction described above, the fusion node is able to construct $C$ from any $2m - 1$ worker nodes. Observe that the coefficient of $x^{m-1}$ in:

$$C(x) = p_A(x)p_B(x) = \left( \sum_{i=0}^{m-1} A_i x^i \right) \left( \sum_{j=0}^{m-1} B_j x^{m-1-j} \right)$$

is $AB = \sum_{i=0}^{m-1} A_iB_i$ (from (3)), which is the desired matrix-matrix product. Thus it is sufficient to compute this coefficient at the fusion node as the computation output for successful computation. Now, because the polynomial $C(x)$ has degree $2m - 2$, evaluation of the polynomial at any $2m - 1$ distinct points is sufficient to compute all of the coefficients of powers of $x$ in $p_A(x)p_B(x)$ using polynomial interpolation. This includes $AB = \sum_{i=0}^{m-1} A_iB_i$, the coefficient of $x^{m-1}$. The next section has a complexity analysis that shows that master and fusion nodes have a lower computational complexity as compared with the workers.

A. Complexity analyses of MatDot codes

Encoding/decoding complexity: Decoding requires interpolating a $2m - 2$ degree polynomial for $N^2$ elements. Using polynomial interpolation algorithms of complexity $O(k \log^2 k)$ [30], where $k = 2m - 1$, the decoding complexity per matrix element is $O(m \log^2 m)$. Thus, for $N^2$ elements, the decoding complexity is $O(N^2 m \log^2 m)$.

Encoding for each worker requires performing two additions, each adding $m$ scaled matrices of size $N^2/m$, for an overall encoding complexity for each worker of $O(mN^2/m) = O(N^2)$. Thus, the overall computational complexity of encoding for $P$ workers is $O(N^2P)$.

Each worker’s computational cost: Each worker multiplies two matrices of dimensions $N \times N/m$ and $N/m \times N$, requiring $N^3/m$ operations (using straightforward multiplication algorithms). Hence, the computational complexity for each worker is $O(N^3/m)$. Thus, as long as $P \ll N$ (and hence $m \ll N$), encoding and decoding complexity is much smaller than per-worker complexity.

Communication cost: The master node communicates $O(PN^2/m)$ symbols, and the fusion node receives $O(mN^2)$ symbols from the successful worker nodes. While the master node communication is identical to that in Polynomial codes, the fusion node there only receives $O(mN^2/m^2) = O(N^2)$ symbols.

B. Why do MatDot exceeds the fundamental limits in [4]

The fundamental limit in [4] concludes that the recovery threshold is $\Omega(m^2)$, whereas our recovery threshold is lower: $2m - 1$. To understand why this is possible, one needs to carefully examine the derivation of the fundamental limit in [4], which uses a cut-set argument to count the number of bits/symbols required for computing the product $AB$. In doing so, the authors make the assumption that the number of symbols communicated by each worker to the fusion node is $N^2/m^2$, which is a fallout of a horizontal division of matrix $A$, and a vertical division of matrix $B$ (the opposite of the division used here).

The bound does not apply to our construction because each worker now communicates $N^2$ symbols to the fusion node. Note that while the amount of information in each worker’s transmissions is less, $O(N^2/m)$ (because the $N \times N$ matrices communicated by the workers have rank $N^2/m$), this is still significantly larger than $N^2/m^2$ assumption made in the fundamental limits in [4].

From a communication viewpoint, MatDot requires communicating a total of $(2m - 1)N^2$ symbols, which is larger than the $N^2$ symbols in the product $AB$. This is suggestive of a trade-off between minimal number of workers and minimal (sum-rate)

[31] More sophisticated algorithms also require super-quadratic complexity in $N$, so the conclusion will hold if those algorithms are used at workers as well.
communication from non-straggling workers. Section VII describes a unified view of MatDot and Polynomial codes, which describes the trade-off between worker-fusion communication cost and recovery threshold achieved by our construction.

In practice, whether this increased worker-fusion node communication cost using MatDot codes is worth paying for will depend on the computational fabric and system implementation choices. Even in systems where communication costs may by significant, it is possible that more communication from fewer successful workers is less expensive than requiring more successful workers needed in Polynomial codes. Also note that if $P = \Omega(m^2)$ (e.g. when the system is highly fault prone or the deadline [24] is very short), communication complexity at the master node will dominate, and hence MatDot codes may not impose a substantial computing overhead.

IV. SYSTEMATIC CODE CONSTRUCTIONS

In this section, we provide a systematic code construction for the problem stated in Section II-B that obtains a recovery threshold of $2m - 1$ where $m \geq 2$ is any positive integer that divides $N$. First, we need to note that a systematic code in the context of the distributed matrix-matrix multiplication problem is a linear code designed for the problem stated in Section II-B computed on the system defined in Definition 1 such that the matrices $A$ and $B$ are split as in (2), and the output of the $l$-th worker node is the product $A_{l-1}B_{l-1}$, for all $l \in \{1, \cdots, m\}$. Notice that, in systematic codes, the product $AB$ is the summation of the output of the first $m$ worker nodes. This follows from noting that the output of the $l$-th worker node is the product $A_{l-1}B_{l-1}$, for all $l \in \{1, \cdots, m\}$, in the systematic codes along with the fact that $AB = \sum_{l=0}^{m-1} A_{l}B_{l}$. The presented systematic code, named systematic MatDot code, is advantageous over MatDot codes in three aspects. First, although both MatDot and systematic MatDot codes have the same recovery threshold, unlike MatDot codes where for every single instance of the problem $2m - 1$ workers are required to finish before obtaining the result of the computation problem, in many instances of the systematic MatDot code, the outputs of only $m$ successful workers suffice. For e.g., if the first $m$ worker nodes finish, then the fusion node can recover $AB$. Secondly, for the instances of systematic MatDot code in which the outputs of only $m$ successful workers suffice, the decoding complexity is $O(mN^2)$, which is less than the decoding complexity of MatDot codes $O(mN^2 \log^2 m)$. Another advantage for systematic MatDot codes over MatDot codes is that the systematic MatDot approach may be useful for backward-compatibility current-day practice. That is, for systems that are already established and operating with no straggler tolerance, but do $m$-way parallelization, it’s easier to apply the systematic approach as the infrastructure could be appended to additional worker nodes without modifying what the first $m$ nodes are doing.

**Theorem 2.** For the matrix multiplication problem specified in Section II-B computed on the system defined in Definition 1 there exists a systematic code, where the product $AB$ is the summation of the output of the first $m$ worker nodes, that solves this problem with a recovery threshold of $2m - 1$, where $m \geq 2$ is any positive integer that divides $N$.

Before we describe the general systematic MatDot code construction that will be used to prove Theorem 2 we first present the following simple systematic MatDot code example.

**Example 4.** [Systematic MatDot code, $m = 2, k = 3$]

Matrix $A$ is split vertically into two submatrices $A_0$ and $A_1$, each of dimension $N \times \frac{N}{2}$ and matrix $B$ is split horizontally into two submatrices $B_0$ and $B_1$, each of dimension $\frac{N}{2} \times N$ as follows:

$$A = \begin{bmatrix} A_0 & A_1 \end{bmatrix}, \quad B = \begin{bmatrix} B_0 \\ B_1 \end{bmatrix}. \quad (5)$$

Now, we define the encoding functions $p_A(x)$ and $p_B(x)$ as $p_A(x) = A_0 \frac{x}{x_0 - x_1} + A_1 \frac{x}{x_1 - x_0}$ and $p_B(x) = B_0 \frac{x}{x_0 - x_1} + B_1 \frac{x}{x_1 - x_0}$, for distinct $x_0, x_1 \in \mathbb{F}$. Let $x_2, \cdots, x_{p-1}$ be elements of $\mathbb{F}$ such that $x_0, x_1, x_2, \cdots, x_{p-1}$ are distinct, the master node sends $p_A(x_{l-1})$ and $p_B(x_{l-1})$ to the $l$-th worker node, $l \in \{1, \cdots, P\}$, where the $l$-th worker node performs the multiplication $p_A(x_{l-1})p_B(x_{l-1})$ and sends the output to the fusion node. The exact computations at each worker node are depicted in Fig. 5.

We can observe that the outputs of the worker nodes 1, 2 are $A_0B_0, A_1B_1$, respectively, i.e., this code is systematic. In addition, assuming, for some instance, that these are the two worker nodes, worker nodes 1, 2, send their computation outputs first, the fusion node can directly obtain the product $AB$ by performing the summation $A_0B_0 + A_1B_1$, since $AB = A_0B_0 + A_1B_1$. Otherwise, assume, for some other instance, that the first three worker nodes to finish do not include worker node 1 or worker node 2. That is, for example, let worker nodes 1, 3, 4 be the first three worker nodes to send their computation outputs to the fusion node, then the fusion node owns the following three matrices: $p_A(x_0)p_B(x_0), p_A(x_2)p_B(x_2)$, and $p_A(x_3)p_B(x_3)$. Since these three matrices can be seen as three evaluations of the matrix polynomial $p_A(x)p_B(x)$ of degree two at three distinct evaluation points $x_0, x_2, x_3$, the matrix polynomial $p_A(x)p_B(x)$ can be obtained using polynomial interpolation. Finally, the product $AB$ is obtained by summing the evaluations of the interpolated matrix polynomial $p_A(x)p_B(x)$ at $x_0$ and $x_1$. That is,

$$p_A(x_0)p_B(x_0) + p_A(x_1)p_B(x_1) = A_0B_0 + A_1B_1 = AB.$$
The following describes the general construction of the systematic codes for matrix-matrix multiplication.

**Construction 2.** [Systematic MatDot codes]

**Splitting of input matrices:** \( \mathbf{A} \) and \( \mathbf{B} \) are split as in (2).

**Master node (encoding):** Let \( x_0, x_1, \ldots, x_{P-1} \) be arbitrary distinct elements of \( \mathbb{F} \). Let \( p_{\mathbf{A}}(x) = \sum_{i=0}^{m-1} \mathbf{A}_i L_i(x) \) and \( p_{\mathbf{B}}(x) = \sum_{i=0}^{m-1} \mathbf{B}_i L_i(x) \), such that

\[
L_i(x) = \prod_{j \in \{0, \ldots, m-1\} - \{i\}} \frac{x - x_j}{x_i - x_j},
\]

where \( i \in \{0, \ldots, m-1\} \).

The master node sends to the \( l \)-th worker evaluations of \( p_{\mathbf{A}}(x) \), \( p_{\mathbf{B}}(x) \) at \( x = x_{l-1} \), that is, it sends \( p_{\mathbf{A}}(x_{l-1}), p_{\mathbf{B}}(x_{l-1}) \) to the \( l \)-th worker, \( l \in \{1, 2, \ldots, P\} \).

**Worker nodes:** For \( l \in \{1, 2, \ldots, P\} \), the \( l \)-th worker node computes the matrix product \( \mathbf{C}(x_{l-1}) = p_{\mathbf{A}}(x_{l-1}) p_{\mathbf{B}}(x_{l-1}) \) and sends it to the fusion node on successful completion.

**Fusion node (decoding):** For any \( k \) such that \( m \leq k \leq 2m - 1 \), whenever the outputs of the first \( k \) workers to finish contain the outputs of the worker nodes \( 1, \ldots, m \), i.e., \( \{\mathbf{C}(x_{l-1})\}_{l \in \{1, \ldots, m\}} \) is contained in the set of the first \( k \) evaluations received by the fusion node, the fusion node performs the summation \( \sum_{i=0}^{m-1} \mathbf{C}(x_i) \). Otherwise, if \( \{\mathbf{C}(x_{l-1})\}_{l \in \{1, \ldots, m\}} \) is not contained in the set of the first \( 2m - 1 \) evaluations received by the fusion node, the fusion node performs the following steps: (i) interpolates the polynomial \( \mathbf{C}(x) = p_{\mathbf{A}}(x) p_{\mathbf{B}}(x) \) (the feasibility of this step will be shown later in the proof of Theorem 1), (ii) evaluates \( \mathbf{C}(x) \) at \( x_0, \ldots, x_{m-1} \), (iii) performs the summation \( \sum_{i=0}^{m-1} \mathbf{C}(x_i) \).

If the number of successful worker nodes is smaller than \( 2m - 1 \) and the first \( m \) worker nodes are not included in the successful worker nodes, the fusion node declares a failure.

**Remark 1.** The polynomials \( L_i(x) \) with \( i \in \{0, \ldots, m-1\} \) defined in (6) have the following evaluations at \( x_0, \ldots, x_{m-1} \),

\[
L_i(x_j) = \begin{cases} 
1 & \text{if } j = i, \\
0 & \text{if } j \in \{0, \ldots, m-1\} - \{i\},
\end{cases}
\]

**Lemma 1.** For Construction 2, the output of the \( l \)-th worker node, for \( l \in \{1, \ldots, m\} \), is the product \( \mathbf{A}_l \mathbf{B}_{l-1} \). That is, Construction 2 is a systematic code for matrix-matrix multiplication.

**Proof of Lemma 7** The lemma follows from noting that the output of the \( l \)-th worker, for \( l \in \{1, \ldots, m\} \), can be written as

\[
\mathbf{C}(x_{l-1}) = p_{\mathbf{A}}(x_{l-1}) p_{\mathbf{B}}(x_{l-1}) \\
= \sum_{i=0}^{m-1} \mathbf{A}_i L_i(x_{l-1}) \sum_{i=0}^{m-1} \mathbf{B}_i L_i(x_{l-1}) \\
= \mathbf{A}_l \mathbf{B}_{l-1},
\]

where the last equality follows from Remark 1. □
Now, we proceed with the proof of Theorem 2.

**Proof of Theorem 2.** Since Construction 2 is a systematic code for matrix-matrix multiplication (Lemma 1), in order to prove the theorem, it suffices to show that Construction 2 is a valid construction with a recovery threshold $k = 2m - 1$.

For the first case in which $\{C(x_{i-1})\} \subseteq \{1, \ldots, m\}$ is contained in the set of the first $k$ evaluations received by the fusion node, for some $k \in \{m, \ldots, 2m - 1\}$, since Construction 2 is systematic (Lemma 1), the product $AB$ is the summation of the outputs of the first $m$ workers.

For the other case in which $\{C(x_{i-1})\} \subseteq \{1, \ldots, m\}$ is not contained in the set of the first $2m - 1$ evaluations received by the fusion node, we need to argue that the fusion node is able to construct $AB$. From [6], observe that the polynomials $L_i(x)$, $i \in \{0, \ldots, m - 1\}$, have degrees $m - 1$ each. Therefore, each of $p_A(x) = \sum_{i=0}^{m-1} A_i L_i(x)$ and $p_B(x) = \sum_{i=0}^{m-1} B_i L_i(x)$ has a degree of $m - 1$ as well. Consequently, $C(x) = p_A(x) \cdot p_B(x)$ has a degree of $2m - 2$. Now, because the polynomial $C(x)$ has degree $2m - 2$, evaluation of the polynomial at any $2m - 1$ distinct points is sufficient to interpolate $C(x)$ using polynomial interpolation algorithm. Now, since Construction 2 is systematic (Lemma 1), the product $AB$ is the summation of the outputs of the first $m$ workers, i.e., $AB = \sum_{i=0}^{m-1} C(x_i)$. Therefore, after the fusion node interpolates $C(x)$, evaluating $C(x)$ at $x_0, \ldots, x_{m-1}$, and performing the summation $\sum_{i=0}^{m-1} C(x_i)$ yields the product $AB$.

The next section has a complexity analysis that shows that master and fusion nodes have a lower computational complexity as compared with the workers.

### A. Complexity analyses of the systematic codes

Apart from the encoding/decoding complexity, the complexity analyses of Construction 2 are the same as their MatDot codes counterparts. In the following, we investigate the encoding/decoding complexity of Construction 2.

**Encoding/decoding Complexity:** For decoding, first, for the interpolation step, we interpolate a $2m - 2$ degree polynomial for $N^2$ elements. Using polynomial interpolation algorithms of complexity $O(k \log^2 k)$ [30], where $k = 2m - 1$, the interpolation complexity per matrix element is $O(m \log^2 m)$. Thus, for $N^2$ elements, the interpolation complexity is $O(N^2 m \log^2 m)$. For the evaluation of $C(x)$ at $x_0, \ldots, x_{m-1}$, each evaluation involves adding $2m - 1$ scaled matrices of size $N^2$ with a complexity of $O(mN^2)$. Hence, for all $m$ evaluations the complexity is $O(mN^2N^2)$. Finally, the complexity of the final addition of $m$ matrices of size $N^2$ is $O(mN^2)$. Hence, the overall decoding complexity is $O(m^2N^2)$.

Encoding for each worker requires first performing evaluations of polynomials $L_i(x)$ for all $i \in \{0, \ldots, m - 1\}$, with each evaluation requires $O(m)$ operations. This gives $O(m^2)$ operations for all polynomial evaluations. Afterwards, two additions are performed, each adding $m$ scaled matrices of size $N^2/m$, with complexity $O(mN^2/m) = O(N^2)$. Therefore, the overall encoding complexity for each worker is $O(\max(N^2, m^2)) = O(N^2)$. Thus, the overall computational complexity of encoding for $P$ workers is $O(N^2P)$.

Since the encoding complexity of systematic MatDot codes is $O(N^2P)$, and that the computational cost of each worker in MatDot codes, and hence in systematic MatDot codes, is $O(N^3/m)$, we conclude that as long as $P \ll N$ (and hence $m \ll N$), encoding and decoding complexity is much smaller than per-worker complexity for Construction 2.

### V. Unifying MatDot and Polynomial Codes: Trade-off between communication cost and recovery threshold

**A. PolyDot Codes: A trade-off between communication cost and recovery threshold**

In this section, we present a code construction, named PolyDot, that provides a trade-off between communication cost and recovery threshold. While Polynomial codes [4] have a higher recovery threshold of $m^2$, and a lower communication cost of $O(N^2/m^2)$ per worker node, MatDot codes have a lower recovery threshold of $2m - 1$, but a higher communication cost of $O(N^2)$ per worker node. Here, we present PolyDot codes, which may be viewed as an interpolation of MatDot codes. For the proposed codes, we consider the matrix computation problem specified in Section II-B computed on the system defined in Definition 4. The next theorem shows that, for any positive integers $s, t$ such that $st = m$ and both $s$ and $t$ divide $N$, the product $AB$ can be recovered using outputs from any $t^2(2s - 1)$ worker nodes with a communication cost from each worker node to the fusion node of $O(N^2/t^2)$.

**Theorem 3.** For the matrix multiplication problem specified in Section II-B computed on the system defined in Definition 4 there exist codes with a recovery threshold of $t^2(2s - 1)$ and a communication cost from each worker node to the fusion node bounded by $O(N^2/t^2)$ for any positive integers $s, t$ such that $st = m$ and both $s$ and $t$ divide $N$.

Before describing the general PolyDot code construction that will be used to prove Theorem 3, we first introduce the following simple PolyDot code example.

**Example 5.** [PolyDot code, $m = 4, s = 2, k = 12$]
Matrix $A$ is split into submatrices $A_{0,0}, A_{0,1}, A_{1,0}, A_{1,1}$ each of dimension $N/2 \times N/2$. Similarly, matrix $B$ is split into submatrices $B_{0,0}, B_{0,1}, B_{1,0}, B_{1,1}$ each of dimension $N/2 \times N/2$ as follows:

$$A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix}, \quad B = \begin{bmatrix} B_{0,0} & B_{0,1} \\ B_{1,0} & B_{1,1} \end{bmatrix}.$$  \hspace{1cm} (9)

Notice that, from (9), the product $AB$ can be written as

$$AB = \left[ \begin{array}{c} \sum_{i=0}^{1} A_{0,i} B_{i,0} \\ \sum_{i=0}^{1} A_{1,i} B_{i,0} \\ \sum_{i=0}^{1} A_{0,i} B_{i,1} \\ \sum_{i=0}^{1} A_{1,i} B_{i,1} \end{array} \right].$$ \hspace{1cm} (10)

Now, we define the encoding functions $p_A(x)$ and $p_B(x)$ as

$$p_A(x) = A_{0,0} x + A_{1,0} x^1 + A_{0,1} x^2 + A_{1,1} x^3,$$

$$p_B(x) = B_{0,0} x^2 + B_{1,0} x^3 + B_{0,1} x^4 + B_{1,1} x^5.$$

Observe the following:

(i) the coefficient of $x^2$ in $p_A(x)p_B(x)$ is $\sum_{i=0}^{1} A_{0,i} B_{i,0},$

(ii) the coefficient of $x^8$ in $p_A(x)p_B(x)$ is $\sum_{i=0}^{1} A_{0,i} B_{i,1},$

(iii) the coefficient of $x^3$ in $p_A(x)p_B(x)$ is $\sum_{i=0}^{1} A_{1,i} B_{i,0},$ and

(iv) the coefficient of $x^9$ in $p_A(x)p_B(x)$ is $\sum_{i=0}^{1} A_{1,i} B_{i,1}.$

Let $x_0, \cdots, x_{P-1}$ be distinct elements of $\mathbb{F}$, the master node sends $p_A(x_{l-1})$ and $p_B(x_{l-1})$ to the $l$-th worker node, $l \in \{1, \cdots, P\}$, where the $l$-th worker node performs the multiplication $p_A(x_{l-1})p_B(x_{l-1})$ and sends the output to the fusion node.

Let worker nodes $1, \cdots, 12$ be the first 12 worker nodes to send their computation outputs to the fusion node, then the fusion node owns the matrices $p_A(x_{l-1})p_B(x_{l-1})$ for all $l \in \{1, \cdots, 12\}$. Since these 12 matrices can be seen as 12 evaluations of the matrix polynomial $p_A(x)p_B(x)$ of degree 11 at 12 distinct evaluation points $x_0, \cdots, x_{11}$, the coefficients of the matrix polynomial $p_A(x)p_B(x)$ can be obtained using polynomial interpolation. This includes the coefficients of $x^{r+2+6c}$ for all $r, c \in \{0, 1\}$, i.e., $\sum_{i=0}^{1} A_{r,i} B_{i,c}$ for all $r, c \in \{0, 1\}$. Once the matrices $\sum_{i=0}^{1} A_{r,i} B_{i,c}$ for all $r, c \in \{0, 1\}$ are obtained, the product $AB$ is obtained by (10).

Notice that although only the parameters $m$ and $s$ are sufficient to characterize a PolyDot code, we use parameter $t$ in addition to $m$ and $s$ to characterize such codes for readability. The following describes the general construction of PolyDot($m, s, t$) codes.

**Construction 3. [PolyDot($m, s, t$) codes]**

**Splitting of input matrices: $A$ and $B$ are split both horizontally and vertically:**

$$A = \begin{bmatrix} A_{0,0} & \cdots & A_{0,s-1} \\ \vdots & \ddots & \vdots \\ A_{t-1,0} & \cdots & A_{t-1,s-1} \end{bmatrix},$$

$$B = \begin{bmatrix} B_{0,0} & \cdots & B_{0,t-1} \\ \vdots & \ddots & \vdots \\ B_{s-1,0} & \cdots & B_{s-1,t-1} \end{bmatrix},$$ \hspace{1cm} (11)

where, for $i = 0, \cdots, s-1, j = 0, \cdots, t-1$, $A_{i,j}$'s are $N/t \times N/s$ matrices and $B_{i,j}$'s are $N/s \times N/t$ matrices. We choose $s$ and $t$ such that both $s$ and $t$ divide $N$ and $st = m$.

**Master node (encoding):** Define the encoding polynomials as

$$p_A(x, y) = \sum_{s=0}^{t-1} \sum_{j=0}^{s-1} A_{i,j} x^i y^j,$$

$$p_B(y, z) = \sum_{k=0}^{s-1} \sum_{l=0}^{t-1} B_{k,l} y^{s-k} z^l.$$ \hspace{1cm} (12)

The master node sends to the $r$-th worker evaluations of $p_A(x, y), p_B(y, z)$ at $x = x_r, y = x_r, z = x_r^{t(2s-1)}$. $x_r$'s are all distinct for $r \in \{1, 2, \cdots, P\}$. 

Worker nodes: For \( r \in \{1, 2, \ldots, P\} \), the \( r \)-th worker node computes the matrix product \( C(x_r, y_r, z_r) = p_A(x_r, y_r)p_B(y_r, z_r) \) and sends it to the fusion node on successful completion.

Fusion node (decoding): The fusion node uses outputs of any \( t^2(2s - 1) \) successful workers to compute the coefficient of \( x^{i-1}y^{s-1}z^{l-1} \) in \( C(x, y, z) = p_A(x, y)p_B(y, z) \). That is, it computes the coefficient of \( x^{i-1+((s-1)t+2s-1)t(l-1)} \) of the transformed single-variable polynomial. The proof of Theorem 3 shows that this is indeed possible. If the number of successful workers is smaller than \( t^2(2s - 1) \), the fusion node declares a failure.

Remark 2. For PolyDot codes, we transform polynomials in 3 variables \( x, y, z \) to a polynomial in one variable \( x \) by choosing \( y = x^t \) and \( z = x^{(2s-1)} \). In fact, we show later in this section that this transformation is one-to-one.

Before we prove the theorem, we discuss the utility of PolyDot codes. By choosing different \( s \) and \( t \), we can trade off communication cost and recovery threshold. For \( s = m \) and \( t = 1 \), PolyDot\((m, s = m, t = 1)\) code is a MatDot code which has a low recovery threshold but high communication cost. At the other extreme, for \( s = 1 \) and \( t = m \), PolyDot\((m, s = 1, t = m)\) code is a Polynomial code.

Now let us consider a code with intermediate \( s \) and \( t \) values such as \( s = \sqrt{m} \) and \( t = \sqrt{m} \). A Polydot\((m, s = \sqrt{m}, t = \sqrt{m})\) code has a recovery threshold of \( m(2\sqrt{m} - 1) = \Theta(m^{1.5}) \), and the total number of symbols to be communicated to the fusion node is \( \Theta((N/\sqrt{m})^2 \cdot m^{1.5}) = \Theta(\sqrt{m}N^2) \) which is smaller than \( \Theta(mN^2) \) required by MatDot codes, and larger than \( \Theta(N^2) \) required by Polynomial codes. This tradeoff is illustrated in Fig. 6 for \( m = 36 \).

To prove Theorem 3, we need the following lemma.

**Lemma 2.** The following function
\[
f: \{0, \ldots, t - 1\} \times \{0, \ldots, 2s - 2\} \times \{0, \ldots, t - 1\} \\
\rightarrow \{0, \ldots, t^2(2s - 1) - 1\}
\]
\[
(\alpha, \beta, \gamma) \mapsto \alpha + t\beta + t(2s - 1)\gamma \tag{13}
\]
is a bijection.

**Proof.** Let us assume, for the sake of contradiction, that for some \((\alpha', \beta', \gamma') \neq (\alpha, \beta, \gamma)\), \( f(\alpha', \beta', \gamma') = f(\alpha, \beta, \gamma) \). Then \( f(\alpha, \beta, \gamma) \mod t = \alpha = f(\alpha', \beta', \gamma') \mod t = \alpha' \) and hence \( \alpha = \alpha' \). Similarly, \( f(\alpha, \beta, \gamma) \mod t(2s - 1) = f(\alpha', \beta', \gamma') \mod t(2s - 1) \) gives \( \alpha + t\beta = \alpha' + t\beta' \), and thus \( \beta = \beta' \) (because \( \alpha = \alpha' \)). Now, because \( \alpha = \alpha' \) and \( \beta = \beta' \), as we just established, using \( f(\alpha, \beta, \gamma) = f(\alpha', \beta', \gamma') \), it follows that \( \gamma = \gamma' \).
The product of $p_A(x, y)$ and $p_B(y, z)$ can be written as follows:

$$C(x, y, z) = p_A(x, y)p_B(y, z)$$

$$= \left( \sum_{i=0}^{t-1} \sum_{j=0}^{s-1} A_{i,j} x^i y^j \right) \left( \sum_{k=0}^{t-1} \sum_{l=0}^{s-1} B_{k,l} y^{s-1-k} z^l \right)$$

$$= \sum_{i,j,k,l} A_{i,j} B_{k,l} x^i y^{s-1-j} z^l. \quad (14)$$

Note that the coefficient of $x^{i-1}y^{s-1}z^{l-1}$ in $C(x, y, z)$ is equal to $C_{i,l} = \sum_{k=0}^{s-1} A_{i,k} B_{k,l}$. By our choice of $y = x^i$ and $z = x^{(2s-1)}$, we can further simplify $C(x, x^i, x^{(2s-1)})$:

$$C(x, y, z) = C(x)$$

$$= \sum_{i,j,k,l} A_{i,j} B_{k,l} x^{i+t(s-1+j-k)+t(2s-1)l}. \quad (15)$$

The maximum degree of this polynomial is when $i = t - 1, j - k = s - 1$ and $l = t - 1$, which is $(t - 1) + (2s - 2)t + t(2s - 1)(t - 1) = t^2(2s - 1) - 1$. Furthermore, if we let $\alpha = i, \beta = s - 1 + j - k, \gamma = l$, the function $f(\alpha, \beta, \gamma)$ in Lemma 2 is the degree of $x$ in (15). This implies that for different pairs of $(i, j, k, l)$, we get different powers of $x$. When $j - k = 0$, we obtain ($\sum_{k=0}^{s-1} A_{i,k} B_{k,l}) x^{i+t(s-1)} + t(2s-1)l = C_{i,l} x^{i+t(s-1)+t(2s-1)l}$ which is the desired product we want to recover.

This implies that if we have $t^2(2s - 1)$ successful worker nodes, we can compute all the coefficients in (15) by polynomial interpolation. Hence, we can recover all $C_{i,l}$'s for $i, l = 0, \cdots, t - 1$. $lacksquare$

### B. Complexity analyses of PolyDot codes

**Encoding/decoding complexity**: Decoding requires interpolating a polynomial of degree $t^2(2s - 1) - 1$ for $N^2$ elements. Using interpolation polynomial algorithms of complexity $O(k \log^2 k)$ [30], [32] where $k = t^2(2s - 1)$, the decoding complexity per matrix element is $O(t^2(2s - 1) \log^2 t^2)$. Thus, the overall decoding complexity is $O(N^2 t^2(2s - 1) \log^2 t^2)$

Encoding for each worker requires performing two additions, each adding $m$ scaled matrices of size $N^2/m$, for an overall encoding complexity for each worker of $O(m N^2/m) = O(N^2)$. Thus, the overall computational complexity of encoding for $P$ worker nodes is $O(N^2 P)$.

**Each worker’s computational complexity**: Multiplication of matrices of size $N/t \times N/s$ and $N/s \times N/t$ requires $O(\frac{N^4}{st^2})$ computations.

**Communication complexity**: Master node communicates $O(N^2/ts) = O(N^2/m)$ symbols to each worker, hence total outgoing symbols from the master node will be $O(PN^2/M)$. For decoding, each node sends $O(N^2/t^2)$ symbols to the fusion node and recovery threshold is $O(t^2(2s - 1))$. Total number of symbols communicated to the fusion node is $O((2s - 1)N^2)$.

### VI. MULTIPLYING MORE THAN TWO MATRICES

We now present our coding technique for multiplying $n$ matrices. Before introducing our result for multiplying more than two matrices, we first provide a formal description of the problem.

**A. Problem Statement**

Compute the product $C = \prod_{i=1}^{n} A_i$ of $N \times N$ square matrices $\tilde{A}_1, \cdots, \tilde{A}_n$ in the computational system specified in Section II-A. For the ease of readability of this section, we define

$$\tilde{A}_i = \begin{cases} A_{i/2} & \text{if } i \text{ is odd,} \\ B_{i/2} & \text{if } i \text{ is even,} \end{cases} \quad (16)$$

for all $i \in \{1, \cdots, n\}$. Using (16), $C$ can be written as

$$C = \begin{cases} \prod_{i=1}^{n} A_i & \text{if } n \text{ is even,} \\ \left( \prod_{i=1}^{[n/2]} A_i B_i \right) A_{[n/2]} & \text{if } n \text{ is odd.} \end{cases} \quad (17)$$

Each worker can receive at most $nN^2/m$ symbols from the master node, where each symbol is an element of $\mathbb{F}$. Similar to Section II-B, the computational complexities of the master and fusion nodes, in terms of the matrix parameter $N$, are required to be strictly less than the computational complexity at any worker node. The goal is to perform this matrix product utilizing faulty/delay prone workers with as low recovery threshold as possible. Again, in the sequel, we will assume that $|\mathbb{F}| > P$. 

B. Codes for n−matrix multiplication

Theorem 4 (Recovery threshold for multiple matrix multiplications). For the matrix multiplication problem specified in Section VI-A computed on the system defined in Definition 1 there exists a code with a recovery threshold of

$$k(n, m) = \begin{cases} 2m^{n/2} - 1 & \text{if } n \text{ is even}, \\ (m + 1)m\left(\frac{n}{2}\right) - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Before we describe the general code construction that will be used to prove Theorem 4 we first present the following simple examples.

Example 6. [Multiplying 4 matrices $m = 2, k = 7$]

Here, we give an example of multiplying 4 matrices and show that a recovery threshold of 7 is achievable. For $i \in \{1, 2\}$, matrix $A_i$ is split vertically into submatrices $A_{i1}, A_{i2}$ each of dimension $N \times \frac{N}{2}$ as follows: $A_i = [A_{i1}A_{i2}]$, while, for $i \in \{1, 2\}$, matrix $B_i$ is split horizontally into submatrices $B_{i1}, B_{i2}$ each of dimension $\frac{N}{2} \times N$ as follows:

$$B_i = \begin{bmatrix} B_{i1} \\ B_{i2} \end{bmatrix}.$$ 

Notice that the product $\prod_{i=1}^{2} A_i B_i$ can now be written as

$$\prod_{i=1}^{2} A_i B_i = (A_1 B_1) (A_2 B_2) = (A_{11} B_{11} + A_{12} B_{12}) (A_{21} B_{21} + A_{22} B_{22}).$$

Now, we define the encoding polynomials $p_{A_i}(x), p_{B_j}(x), i \in \{1, 2\}$ as follows:

- $p_{A_1}(x) = A_{11} + A_{12}x$,
- $p_{B_1}(x) = B_{11}x + B_{12}$,
- $p_{A_2}(x) = A_{21} + A_{22}x$,
- $p_{B_2}(x) = B_{21}x + B_{22}$.

From (21), we have

$$p_{A_1}(x)p_{B_1}(x) = (A_{11} B_{11} + A_{12} B_{12}) x + A_{12} B_{11} x^2,$$
$$p_{A_2}(x)p_{B_2}(x) = (A_{21} B_{21} + A_{22} B_{22}) x^2 + A_{22} B_{21} x^4.$$

From (20) along with (22), we can observe the following:
(i) the coefficient of $x$ in $p_{A_1}(x)p_{B_1}(x)$ is $A_{11} B_{11} + A_{12} B_{12} = A_1 B_1$,
(ii) the coefficient of $x^2$ in $p_{A_2}(x)p_{B_2}(x^2)$ is the product $A_{21} B_{21} + A_{22} B_{22} = A_2 B_2$, and
(iii) the coefficient of $x^3$ in $p_{A_1}(x)p_{B_1}(x)p_{A_2}(x^2)p_{B_2}(x^2)$ is the product $\prod_{i=1}^{2} A_i B_i$.

Let $x_0, \ldots, x_{P-1}$ be distinct elements of $F$, the master node sends $p_{A_i}(x^l_{i-1})$ and $p_{B_j}(x^l_{j-1})$, for all $i \in \{1, 2\}$, to the $l$-th worker node, $l \in \{1, \ldots, P\}$, where the $l$-th worker node performs the multiplication $\prod_{i=1}^{2} p_{A_i}(x^l_{i-1}) p_{B_j}(x^l_{j-1})$ and sends the output to the fusion node.

Let worker nodes $1, \ldots, 7$ be the first 7 worker nodes to send their computation outputs to the fusion node, then the fusion node receives the matrices $\prod_{i=1}^{2} p_{A_i}(x^l_{i-1}) p_{B_j}(x^l_{j-1})$ for all $l \in \{1, \ldots, 7\}$. Since these 7 matrices can be seen as 7 evaluations of the matrix polynomial $\prod_{i=1}^{2} p_{A_i}(x^l) p_{B_j}(x^l)$ of degree 6 at 7 distinct evaluation points $x_0, \ldots, x_6$, the coefficients of the matrix polynomial $\prod_{i=1}^{2} p_{A_i}(x^l) p_{B_j}(x^l)$ can be obtained using polynomial interpolation. This includes the coefficient of $x^3$, i.e., $\prod_{i=1}^{2} A_i B_i$.

Example 7. [Multiplying 3 matrices, $m = 2, k = 5$]

Here, we give an example of multiplying 3 matrices and show that a recovery threshold of 5 is achievable. In this example, we have three input matrices $A_1, B_1, A_2$, each of dimension $N \times N$ and need to compute the product $A_1 B_1 A_2$. First, the three input matrices are split in the same way as in Example 6. The product $A_1 B_1 A_2$ can now be written as

$$A_1 B_1 A_2 = [A_1 B_1 A_{21} A_1 B_1 A_{22}],$$

where $A_1 B_1 = A_{11} B_{11} + A_{12} B_{12}$.

$$A_1 B_1 A_2 = [A_1 B_1 A_{21} A_1 B_1 A_{22}],$$

where $A_1 B_1 = A_{11} B_{11} + A_{12} B_{12}$.
Now, we define the encoding polynomials \( p_{A_1}(x), p_{B_1}(x), p_{A_2}(x) \) as follows:

\[
\begin{align*}
p_{A_1}(x) &= A_{11} + A_{12}x, \\
p_{B_1}(x) &= B_{11}x + B_{12}, \\
p_{A_2}(x) &= A_{21} + A_{22}x.
\end{align*}
\]

From (24), we have

\[
\begin{align*}
p_{A_1}(x)p_{B_1}(x)p_{A_2}(x^2) &= A_{11}B_{12}A_{21} + (A_{11}B_{11} + A_{12}B_{12})A_{21}x \\
&+ (A_{12}B_{11}A_{21} + A_{11}B_{12}A_{22})x^2 \\
&+ (A_{11}B_{11} + A_{12}B_{12})A_{22}x^3 + A_{12}B_{11}A_{22}x^4.
\end{align*}
\]

From (25), we can observe the following:

(i) the coefficient of \( x \) in \( p_{A_1}(x)p_{B_1}(x)p_{A_2}(x^2) \) is the product \( A_{11}B_{12}A_{21} \), and

(ii) the coefficient of \( x^3 \) in \( p_{A_1}(x)p_{B_1}(x)p_{A_2}(x^2) \) is the product \( A_{11}B_{12}A_{22} \).

Let \( x_0, \ldots, x_{p-1} \) be distinct elements of \( F \), the master node sends \( p_{A_i}(x_i^{j-1}) \), for all \( i \in \{1, 2\} \), and \( p_{B_1}(x_i^{j-1}) \) to the \( l \)-th worker node, \( l \in \{1, \ldots, P\} \), where the \( l \)-th worker node performs the multiplication \( p_{A_1}(x_i^{j-1})p_{B_1}(x_i^{j-1})p_{A_2}(x_i^{j-2}) \) and sends the output to the fusion node.

Let worker nodes 1, \ldots, 5 be the first 5 worker nodes to send their computation outputs to the fusion node, then the fusion node receives the matrices \( p_{A_1}(x_i^{j-1})p_{B_1}(x_i^{j-1})p_{A_2}(x_i^{j-2}) \) for all \( l \in \{1, \ldots, 5\} \). Since these 5 matrices can be seen as 5 evaluations of the matrix polynomial \( p_{A_1}(x)p_{B_1}(x)p_{A_2}(x^2) \) of degree 4 at 5 distinct evaluation points \( x_0, \ldots, x_4 \), the coefficients of the matrix polynomial \( p_{A_1}(x)p_{B_1}(x)p_{A_2}(x^{j-1}) \) can be obtained using polynomial interpolation. This includes the coefficients of \( x \) and \( x^3 \), i.e., \( A_{11}B_{12}A_{21} \) and \( A_{11}B_{12}A_{22} \).

In the following, we present the general code construction for \( n \)-matrix multiplication.

**Construction 4. [An \( n \)-matrix multiplication code]**

**Splitting of input matrices:** for every \( i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \) and \( j \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \), \( A_i \) and \( B_j \) are split as follows

\[
A_i = [A_{i1} | A_{i2} | \ldots | A_{im}], \quad B_j = \begin{bmatrix} B_{j1} \\ B_{j2} \\ \vdots \\ B_{jm} \end{bmatrix},
\]

where, for \( k \in \{1, \ldots, m\} \), \( A_{ik}, B_{jk} \) are \( N \times N/m \) and \( N/m \times N \) dimensional matrices, respectively.

**Master node (encoding):** Let \( x_0, x_1, \ldots, x_{p-1} \) be arbitrary distinct elements of \( F \). For \( i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \), define \( p_{A_i}(x) = \sum_{j=1}^{m} A_{ij}x^{j-1} \), and, for \( i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \), define \( p_{B_i}(x) = \sum_{j=1}^{m} B_{ij}x^{j-1} \). For \( l \in \{1, 2, \ldots, P\} \), the master node sends to the \( l \)-th worker evaluations of \( p_{A_i}(x_{l-1}^{j-1}) \) and \( p_{B_i}(x_{l-1}^{j-1}) \) for all \( i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \) and \( j \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \).

**Worker nodes:** For \( i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \), define

\[
p_{C_i}(x) = \begin{cases} p_{A_i}(x)p_{B_i}(x) & \text{if } i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \}, \\
p_{A_i}(x) & \text{if } i \text{ is odd and } i = \lfloor \frac{n}{2} \rfloor. \end{cases}
\]

For \( l \in \{1, 2, \ldots, P\} \), the \( l \)-th worker node computes the matrix product \( \Pi_{i=1}^\lfloor \frac{n}{2} \rfloor p_{C_i}(x_{l-1}^{m-1}) \) and sends it to the fusion node on successful completion.

**Fusion node (decoding):** If \( n \) is even, the fusion node uses outputs of any \( 2m \lfloor \frac{n}{2} \rfloor - 1 \) successful workers to compute the coefficient of \( x^{m\lfloor \frac{n}{2} \rfloor - 1} \) in the matrix polynomial \( \Pi_{i=1}^\lfloor \frac{n}{2} \rfloor p_{C_i}(x^{m-1}) \), and if \( n \) is odd, the fusion node uses outputs of any \( m\lfloor \frac{n}{2} \rfloor (m+1) - 1 \) successful workers to compute the coefficients of \( x^{j(m\lfloor \frac{n}{2} \rfloor - 1)} \), for all \( j \in \{1, \ldots, m\} \), in the matrix polynomial \( \Pi_{i=1}^\lfloor \frac{n}{2} \rfloor p_{C_i}(x^{m-1}) \) (the feasibility of this step will be shown later in the proof of Theorem 4).

If the number of successful workers is smaller than \( 2m \lfloor \frac{n}{2} \rfloor - 1 \) for even \( n \) or smaller than \( m\lfloor \frac{n}{2} \rfloor (m+1) - 1 \) for odd \( n \), the fusion node declares a failure.

**Remark 3.** The coefficient of \( x^{m-1} \) in \( p_{C_i}(x^{m-1}) \), for any \( i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \), is \( \sum_{j=1}^{m} A_{ij}B_{ij} = A_iB_i \).

In the following, we show Lemmas 3 and 4, which state properties of coefficients of products of polynomials. Using Lemmas 3 and 4, we show Claims 5 and 6, which demonstrate that the product \( C \) is contained in a set of coefficients of the matrix.
polynomial $\prod_{i=1}^{\frac{n}{2}} p_{C_i}(x^{m_i-1})$, where $p_{C_i}(x)$ is as defined in (27), for $i \in \{1, \cdots, [n/2]\}$. Finally, we provide a proof of Theorem 3 using Claims 5 and 6.

**Lemma 3.** If $p(x) = \sum_{j=0}^{2d^{i-1} - 2} p_j x^j$ is a polynomial with degree $2d^{i-1} - 2$ for some $i \geq 2$, and $q(x) = \sum_{j=0}^{2d^{i-2}} q_j x^j$ is any other polynomial with degree $2d - 2$, then $p_{d^{i-1}} q_{d^{i-1}}$ is the coefficient of $x^{d^{i-1}}$ in $p(x)q(x^{d^{i-1}})$.

**Proof.** We first expand out $p(x)$ and $q(x)$ as following:

$$p(x) = \sum_{j=0}^{d^{i-1}-2} p_j x^j + p_{d^{i-1}-1} x^{d^{i-1}-1} + \sum_{j=d^{i}}^{2d^{i-1}-2} p_j x^j \tag{28}$$

$$q(x) = \sum_{j=0}^{d^{i-2}} q_j x^j + q_{d^{i-1}-1} x^{d^{i-1}-1} + \sum_{j=d^{i}}^{2d^{i-2}} q_j x^j \tag{29}$$

We show that the term of degree $d^i - 1$ in $p(x)q(x^{d^{i-1}})$ is only generated by multiplication of the term of degree $d^i - 1$ in $p(x)$ and the term of degree $d^{i-1}(d - 1)$ in $q(x^{d^{i-1}})$. For this purpose, we consider following terms:

1. Consider the multiplication of two lowest degree terms in $\tilde{p}_1(x)$ and $\tilde{q}_2(x^{d^{i-1}})$ of equations (28) and (29). That is, $q_0 x^d p_0 = p_0 q_0 x^d$ which has higher degree in comparison to $x^{d^i - 1}$. Consequently, the degree of any term in the multiplication of $\tilde{p}_1(x)$ and $\tilde{q}_2(x^{d^{i-1}})$ will be strictly greater than $d^i - 1$.

2. Consider the multiplication of two highest degree terms of $\tilde{p}_2(x)$ and $\tilde{q}_1(x^{d^{i-1}})$ of equations (28) and (29).

$$q_{d-2} x^{d^{i-1}(d-2)} p_{d^{i-1}-2} x^{2d^{i-1}-2} = q_{d-2} p_{d^{i-1}-1} x^{2d^{i-1}-2}$$

is less than $d^i - 1$. Consequently, the degree of any term in the multiplication of $\tilde{p}_2(x)$ and $\tilde{q}_1(x^{d^{i-1}})$ will be strictly less than $d^i - 1$.

3. Since the degree of any term in the multiplication of $\tilde{p}_1(x)$ and $\tilde{q}_1(x^{d^{i-1}})$ is strictly less than $d^i - 1$, and any term in $\tilde{p}_1(x)$ has degree less than the degree of any term in $\tilde{p}_2(x)$, we conclude that any term in the multiplication of $\tilde{p}_1(x)$ and $\tilde{q}_1(x^{d^{i-1}})$ has degree strictly less than $d^i - 1$.

4. Since the degree of any term in the multiplication of $\tilde{p}_1(x)$ and $\tilde{q}_2(x^{d^{i-1}})$ is strictly greater than $d^i - 1$, and any term in $\tilde{p}_2(x)$ has degree larger than the degree of any term in $\tilde{p}_1(x)$, we conclude that any term in $\tilde{p}_2(x)\tilde{q}_2(x^{d^{i-1}})$ has degree strictly greater than $d^i - 1$ which completes the proof.

**Lemma 4.** If $p(x) = \sum_{j=0}^{2d^{i-1} - 2} p_j x^j$ is a polynomial with degree $2d^{i-1} - 2$ for some $i \geq 2$, and $q(x) = \sum_{j=0}^{d^{i-1}} q_j x^j$ is any other polynomial with degree $d - 1$, then, for $0 \leq j \leq d - 1$, $p_{d^{i-1}-1} q_j$ are the coefficients of $x^{(j+1)d^{i-1}-1}$ in $p(x)q(x^{d^{i-1}})$.

**Proof.** First, we expand out $p(x)$ as in (28), and expand $q(x)$ as follows:

$$q(x) = \sum_{j=0}^{d^{i-1}} q_j x^j + \sum_{j=d^{i}}^{2d^{i-1}-2} q_j x^j \tag{30}$$

In order to prove Lemma 4, we show that the term of degree $x^{(j+1)d^{i-1}-1}$ in $p(x)q(x^{d^{i-1}})$ is produced by the multiplication of only the term $p_{d^{i-1}-1} x^{d^{i-1}-1}$ in $p(x)$ with the term $q_j x^{j(d^{i-1})}$ in $q(x^{d^{i-1}})$. First, it is clear that the product of the term $p_{d^{i-1}-1} x^{d^{i-1}-1}$ in $p(x)$ with the term $q_j x^{j(d^{i-1})}$ in $q(x^{d^{i-1}})$ has degree $(j+1)d^{i-1} - 1$. Thus, to complete the proof, we show that no other terms in $p(x)$ produce a degree of $x^{(j+1)d^{i-1}-1}$ when multiplied with any term in $q(x^{d^{i-1}})$. To do so, we consider the following terms:

1. Consider the multiplication of two lowest degree terms in $\tilde{p}_1(x)$ and $\tilde{q}_1(x^{d^{i-1}})$ of equations (28) and (30). That is, $p_0 q_{j+1} x^{(j+1)d^{i-1}}$ which has higher degree in comparison to $x^{(j+1)d^{i-1}-1}$. Consequently, the degree of any term in the multiplication of $\tilde{p}_1(x)$ and $\tilde{q}_1(x^{d^{i-1}})$ will be strictly greater than $(j+1)d^{i-1} - 1$.

2. Consider the multiplication of two highest degree terms of $\tilde{p}_2(x)$ and $\tilde{q}_1(x^{d^{i-1}})$ of equations (28) and (30). The product

$$q_{j-1} x^{(j-1)d^{i-1}} p_{2d^{i-1}-2} x^{2d^{i-1}-2} = q_{j-1} p_{2d^{i-1}-1} x^{2d^{i-1}-2}$$
has degree less than \((j + 1)d^{-1} - 1\). Consequently, the degree of any term in the multiplication of \(\tilde{p}_2(x)\) and \(\tilde{q}_1(x^{d_{1}})\) will be strictly less than \((j + 1)d^{-1} - 1\).

3. Since the degree of any term in the multiplication of \(\tilde{p}_2(x)\) and \(\tilde{q}_1(x^{d_{1}})\) is strictly less than \((j + 1)d^{-1} - 1\), and the degree of any term in \(\tilde{p}_1(x)\) is less than the degree of any term in \(\tilde{p}_2(x)\), we conclude that any term in the multiplication of \(\tilde{p}_1(x)\) and \(\tilde{q}_1(x^{d_{1}})\) has degree strictly less than \((j + 1)d^{-1} - 1\).

4. Since the degree of any term in the multiplication of \(\tilde{p}_1(x)\) and \(\tilde{q}_2(x^{d_{1}})\) is strictly greater than \((j + 1)d^{-1} - 1\), and the degree of any term in \(\tilde{p}_2(x)\) is larger than the degree of any term in \(\tilde{p}_1(x)\), we conclude that any term in \(\tilde{p}_2(x)\tilde{q}_2(x^{d_{1}})\) has degree strictly greater than \((j + 1)d^{-1} - 1\) which completes the proof.

Now, we are able to state the following claims.

**Claim 5.** The coefficient of \(x^{m^{\lceil \frac{n}{2} \rceil} - 1}\) in \(\prod_{i=1}^{\lceil \frac{n}{2} \rceil} p_{C_i}(x^{m^{i-1}})\) is \(\prod_{i=1}^{\lceil \frac{n}{2} \rceil} A_iB_i\), where, for \(i \in \{1, \cdots, \lceil \frac{n}{2} \rceil\}\), \(p_{C_i}(x)\) is as defined in (27).

**Proof.** We prove the claim iteratively. Since \(p_{C_1}(x)\) has degree \(2m^{i-1} - 2\) with \(i = 2\), and \(p_{C_2}(x)\) has degree \(2m - 2\), we have, by Lemma 3, that the coefficient of \(x^{m^{2} - 1}\) in \(p_{C_1}(x)p_{C_2}(x^{m})\) is the product of the coefficient of \(x^{m^{1} - 1}\) in \(p_{C_1}(x)\) and the coefficient of \(x^{2m^{-1} - m}\) in \(p_{C_2}(x^{m})\). However, from Remark 3, we already know that \(A_1B_1\) is the coefficient of \(x^{2m^{-1}}\) in \(p_{C_1}(x)\) and that \(A_2B_2\) is the coefficient of \(x^{2m^{-1} - m}\) in \(p_{C_2}(x^{m})\). Therefore, \(A_1B_1A_2B_2\) is the coefficient of \(x^{m^{1} - 1}\) in \(p_{C_1}(x)p_{C_2}(x^{m})\).

Similarly, consider the two polynomials \(p'(x) = p_{C_1}(x)p_{C_2}(x^{m})\) and \(p_{C_3}(x)\). Notice that \(p'(x)\) has degree \(2m^{i-1} - 2\) with \(i = 3\), and \(p_{C_3}(x)\) has degree \(2m - 2\), therefore, from Lemma 3, the coefficient of \(x^{m^{3} - 1}\) in \(p'(x)p_{C_3}(x^{2m})\) is the product of the coefficient of \(x^{m^{2} - 1}\) in \(p'(x)\) and the coefficient of \(x^{3m^{-1} - 2m}\) in \(p_{C_3}(x^{2m})\). However, from the previous step, we already know that \(A_1B_1A_2B_2\) is the coefficient of \(x^{m^{2} - 1}\) in \(p'(x)p_{C_3}(x^{m})\). In addition, from Remark 3, we already know that \(A_3B_3\) is the coefficient of \(x^{m^{3} - 2m}\) in \(p_{C_3}(x^{m})\). Therefore, \(A_1B_1A_2B_2A_3B_3\) is the coefficient of \(x^{m^{1} - 1}\) in \(p'(x)p_{C_3}(x^{m}) = p_{C_1}(x)p_{C_2}(x^{m})p_{C_3}(x^{m})\).

Repeating the same procedure, we conclude that \(\prod_{i=1}^{\lceil \frac{n}{2} \rceil} A_iB_i\) is the coefficient of \(x^{m^{\lceil \frac{n}{2} \rceil} - 1}\) in \(\prod_{i=1}^{\lceil \frac{n}{2} \rceil} p_{C_i}(x^{m^{i-1}})\).

**Claim 6.** If \(n \geq 3\) and odd, then, for any \(j \in \{1, \cdots, m\}\), \((\prod_{i=1}^{\lceil \frac{n}{2} \rceil} A_iB_i)A_{\lceil \frac{n}{2} \rceil}j\) is the coefficient of \(x^{j^{m^{\lceil \frac{n}{2} \rceil}} - 1}\) in \(\prod_{i=1}^{\lceil \frac{n}{2} \rceil} p_{C_i}(x^{m^{i-1}})\), where, for \(i \in \{1, \cdots, \lceil \frac{n}{2} \rceil\}\), \(p_{C_i}(x)\) is as defined in (27).

**Proof.** First, notice that since the degree of \(p_{C_1}(x)\) is \(2m - 2\) for all \(i \in \{1, \cdots, \lceil \frac{n}{2} \rceil\}\), the degree of \(\prod_{i=1}^{\lceil \frac{n}{2} \rceil} p_{C_i}(x^{m^{i-1}})\) is \((2m - 2)\sum_{i=1}^{\lceil \frac{n}{2} \rceil} m^{i-1} = 2m^{\lceil \frac{n}{2} \rceil} - 2\). In addition, the matrix polynomial \(p_{C_{\lceil \frac{n}{2} \rceil}}(x)\) has degree \(m - 1\). Therefore, from Lemma 4, for \(1 \leq j \leq m\), the product of the coefficient of \(x^{m^{\lceil \frac{n}{2} \rceil}} - 1\) in \(\prod_{i=1}^{\lceil \frac{n}{2} \rceil} p_{C_i}(x^{m^{i-1}})\) and the coefficient of \(x^{j^{m^{\lceil \frac{n}{2} \rceil}} - 1}\) in \(p_{C_{\lceil \frac{n}{2} \rceil}}(x^{m^{i-1}})\) is \(\prod_{i=1}^{\lceil \frac{n}{2} \rceil} A_iB_i\), also, by definition, the coefficient of \(x^{j^{m^{i-1}} - 1}\) in \(p_{C_{\lceil \frac{n}{2} \rceil}}(x^{m^{i-1}})\) is \(A_{\lceil \frac{n}{2} \rceil}j\). Thus, \((\prod_{i=1}^{\lceil \frac{n}{2} \rceil} A_iB_i)A_{\lceil \frac{n}{2} \rceil}j\) is the coefficient of \(x^{j^{m^{\lceil \frac{n}{2} \rceil}} - 1}\) in \(\prod_{i=1}^{\lceil \frac{n}{2} \rceil} p_{C_i}(x^{m^{i-1}})\).

Now, we prove Theorem 4.

**Proof of Theorem 4.** To prove the theorem, it suffices to show that for Construction 3 the fusion node is able to construct \(C\) from any \(2m^{n/2} - 1\) worker nodes if \(n\) is even or from any \((m + 1)m^{\lceil \frac{n}{2} \rceil} - 1\) if \(n\) is odd.

First, for the case in which \(n\) is even, we need to compute \(C = \prod_{i=1}^{\lceil \frac{n}{2} \rceil} A_iB_i\). Notice, from Claim 5, that the desired matrix product \(C\) is the coefficient of \(x^{m^{\lceil \frac{n}{2} \rceil} - 1}\) in \(\prod_{i=1}^{\lceil \frac{n}{2} \rceil} p_{C_i}(x^{m^{i-1}})\). Thus, it is sufficient to compute this coefficient at the fusion node as the computation output for successful computation. Now, because the polynomial \(\prod_{i=1}^{\lceil \frac{n}{2} \rceil} p_{C_i}(x^{m^{i-1}})\) has degree \(2m^{n/2} - 2\), evaluation of the polynomial at any \(2m^{n/2} - 1\) distinct points is sufficient to compute all of the coefficients of powers of \(x\) in \(\prod_{i=1}^{\lceil \frac{n}{2} \rceil} p_{C_i}(x^{m^{i-1}})\) using polynomial interpolation. This includes \(C\), the coefficient of \(x^{m^{\lceil \frac{n}{2} \rceil} - 1}\).
Now, for the case in which \( n \) is odd, we need to compute \( C = \left( \prod_{i=1}^{\frac{m}{2}} A_i B_i \right) A \left[ \frac{m}{2} \right] \). First, notice that \( C \) is a concatenation of the matrices \( \left( \prod_{i=1}^{\frac{m}{2}} A_i B_i \right) A \left[ \frac{m}{2} \right] \), \( j \in \{1, \ldots, m\} \) as follows:

\[
C = \left( \prod_{i=1}^{\frac{m}{2}} A_i B_i \right) A \left[ \frac{m}{2} \right] = \\
\left[ \prod_{i=1}^{\frac{m}{2}} A_i B_i \right] A \left[ \frac{m}{2} \right]_1 \cdots \left[ \prod_{i=1}^{\frac{m}{2}} A_i B_i \right] A \left[ \frac{m}{2} \right]_m . \tag{31}
\]

From Claim 6 for all \( j \in \{1, \ldots, m\} \), the product \( \left( \prod_{i=1}^{\frac{m}{2}} A_i B_i \right) A \left[ \frac{m}{2} \right] \) is the coefficient of \( x^{m \left( \frac{m}{2} \right) - 1} \) in \( \prod_{i=1}^{\frac{m}{2}} pC_i(x^{m-1}) \). Thus, it is sufficient to compute these coefficients, for all \( j \in \{1, \ldots, m\} \), at the fusion node as the computation output for successful computation. Now, because the polynomial \( \prod_{i=1}^{\frac{m}{2}} pC_i(x^{m-1}) \) has degree \( m \left( \frac{m}{2} \right) (m + 1) - 2 \), evaluation of the polynomial at any \( m \left( \frac{m}{2} \right) (m + 1) - 1 \) distinct points is sufficient to compute all of the coefficients of powers in \( \prod_{i=1}^{\frac{m}{2}} pC_i(x^{m-1}) \) using polynomial interpolation. This includes the coefficients of \( x^{m \left( \frac{m}{2} \right) - 1} \), i.e. \( \left( \prod_{i=1}^{\frac{m}{2}} A_i B_i \right) A \left[ \frac{m}{2} \right] \), for all \( j \in \{1, \ldots, m\} \).

C. Complexity analyses of Construction 2

**Encoding/decoding complexity:** Decoding requires interpolating a \( 2m n^2 / 2 - 2 \) degree polynomial if \( n \) is even or a \( m \left( \frac{m}{2} \right) (m + 1) - 2 \) degree polynomial if \( n \) is odd for \( N^2 \) elements. Using polynomial interpolation algorithms of complexity \( O(q \log^2 q) \) [30], where \( q = k(n, m) \), defined in [18], complexity per matrix element is \( O(m \left( \frac{m}{2} \right) \log^2 m \left( \frac{m}{2} \right)) \). Thus, for \( N^2 \) elements, the decoding complexity is \( O(N^2 m \left( \frac{m}{2} \right) \log^2 m \left( \frac{m}{2} \right)) \).

Encoding for each worker requires performing \( n \) additions, each adding \( m \) scaled matrices of size \( N^2 / m \), for an overall encoding complexity for each worker of \( O(nN^2 / m) = O(nN^2) \). Thus, the overall computational complexity of encoding for \( P \) workers is \( O(nN^2 P) \).

**Each worker’s computational cost:** Each worker multiplies \( n \) matrices of dimensions \( N \times N / m \) and \( N / m \times N \). For any worker \( l \) with \( l \in \{1, \ldots, P\} \), the multiplication can be performed as follows:

**Case 1:** \( n \) is even

In this case, worker \( l \) wishes to compute \( pA_1(x_l-1)pB_1(x_l-1)pA_2(x_l-1)pB_2(x_l-1) \cdots pA_n/2(x_l-n_{/2}-1)pB_n/2(x_l-n_{/2}-1) \).

**Worker \( l \) does this multiplication in the following order:**
1. Compute \( pB_1(x_l-1)pA_{i-1}(x_l-1) \) for all \( i \in \{1, \ldots, n/2 - 1\} \) with a total complexity of \( O(nN^2 / m^2) \).
2. Compute the product of the output matrices of the previous step with a total complexity of \( O(nN^3 / m^3) \). Call this product matrix \( D \). Notice that \( D \) has a dimension of \( N / m \times N / m \).
3. Compute \( pA_1(x_l-1)D \) with complexity \( O(N^2 / m) \).

Hence, the overall computational complexity per worker for even \( n \) is

\[
O\left(\max(nN^3 / m^2, nN^3 / m^3, N^3 / m^2, N^3 / m)\right) = O\left(\max(nN^3 / m^2, N^3 / m)\right).
\]

**Case 2:** \( n \) is odd

In this case, worker \( l \) wishes to compute

\[
pA_1(x_l-1)pB_1(x_l-1)pA_2(x_l-1)pB_2(x_l-1) \cdots pA_{n-1/2}(x_l-n_{/2})pB_{n-1/2}(x_l-n_{/2})pA_{n+1/2}(x_l-n_{/2}^{-1}).
\]

**Worker \( l \) does this multiplication in the following order:**
1. Compute \( pB_1(x_l-1)pA_{i-1}(x_l-1) \) for all \( i \in \{1, \ldots, (n - 1)/2\} \) with a total complexity of \( O(nN^3 / m^2) \).
2. Compute the product of the output matrices of the previous step with a total complexity of \( O(nN^3 / m^3) \). Call this product matrix \( D \). Notice that \( D \) has a dimension of \( N / m \times N / m \).
3. Compute \( pA_1(x_l-1)D \) with complexity \( O(N^3 / m^2) \).

Hence, the overall computational complexity per worker for odd \( n \) is

\[
O\left(\max(nN^3 / m^2, nN^3 / m^3, N^3 / m^2)\right) = O(nN^3 / m^2).
\]

In conclusion, the computational complexity per worker is \( O\left(\max(nN^3 / m^2, N^3 / m)\right) \) if \( n \) is even, and \( O(nN^3 / m^2) \) if \( n \) is odd.

**Communication cost:** The master node communicates \( O(nPN^2 / m) \) symbols, and the fusion node receives \( O(m \left( \frac{m}{2} \right) / N^2) \) symbols from the successful worker nodes.
VII. CONCLUSION

We provide the MatDot construction for coded matrix multiplication, which has a recovery threshold of $2m - 1$. We also present the systematic MatDot construction providing the same recovery threshold of MatDot construction. Since a natural lower bound for the recovery threshold is $m$, an open question is whether the threshold we obtain is optimal. Another open question is whether the worker-fusion communication cost can be reduced without increasing the recovery threshold of MatDot. In addition, we provide the PolyDot construction which allows a trade-off between communication cost and recovery threshold. Finally, we provide code constructions for multiplying more than two matrices.

Open problem: Tensor product multiplication

We conclude with a discussion on the possibility of extending our ideas to computing tensor products. Consider the problem, where the goal is to compute $A \otimes B$ of two $N \times N$ square matrices in the distributed system specified in system model section over $P$ workers with the highest possible recovery threshold. For this problem, an obvious application of Polynomial codes [1] yields a recovery threshold of $m^2$.

To see this, let

$$A = [A_0 \ A_2 \ \ldots \ A_{m-1}], \quad B = [B_0 \ B_2 \ \ldots \ B_{m-1}].$$

Note that $A \otimes B = [A_0 \otimes B_0 \ \ldots \ A_{m-1} \otimes B_{m-1}]$.

While the above approach obtains a recovery threshold of $m^2$, note workers do not necessarily carry a greater burden of computation as compared with the fusion node. Specifically the computational complexity of the fusion node is $\Theta(p)$, which has the same complexity order as the entire tensor product $A \otimes B$, thus in an order sense, the strategy is not gaining in terms of computational complexity from distributed computing. An open question motivated by our work is the above approach has any interesting applications.

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REFERENCES

[1] K. Lee, M. Lam, R. Pedarsani, D. Papailiopoulos, and K. Ramchandran. Speeding up distributed machine learning using codes. In IEEE International Symposium on Information Theory, pages 1143–1147, 2016.
[2] S. Li, M. A. Maddah-Ali, and A. S. Avestimehr. Coded mapreduce. In Communication, Control, and Computing (Allerton), 2015 53rd Annual Allerton Conference on, pages 964–971. IEEE, 2015.
[3] R. Tandon, Q. Lei, A. G. Dimakis, and N. Karampatziakis. Gradient coding. In Machine Learning Systems Workshop, Advances in Neural Information Processing Systems (NIPS), 2016.
[4] Qian Yu, Mohammad Ali Maddah-Ali, and Amir Salman Avestimehr. Coded distributed computing: Straggling servers and multistage dataflows. In Proceedings of the 34th International Conference on Machine Learning (ICML), pages 3368–3376, 2017.
[16] Mehmet Fatih Aktas, Pei Peng, and Emina Soljanin. Effective straggler mitigation: Which clones should attack and when? arXiv preprint arXiv:1710.00748, 2017.

[17] Mehmet Fatih Aktas, Pei Peng, and Emina Soljanin. Straggler mitigation by delayed relaunch of tasks. arXiv preprint arXiv:1710.00414, 2017.

[18] Malihe Aliasgari, Jörg Kliewer, and Osvaldo Simeone. Coded computation against straggling decoders for network function virtualization. arXiv preprint arXiv:1709.01031, 2017.

[19] Amirhossein Reisizadehmoarakeh, Saurav Prakash, Ramtin Pedarsani, and Salman Avestimehr. Coded computation over heterogeneous clusters. arXiv preprint arXiv:1701.05973, 2017.

[20] Wael Halbawi, Navid Azizan-Ruhi, Fariborz Salehi, and Babak Hassibi. Improving distributed gradient descent using reed-solomon codes. arXiv preprint arXiv:1706.05436, 2017.

[21] Can Karakus, Yifan Sun, and Suhas Diggavi. Encoded distributed optimization. In Information Theory (ISIT), 2017 IEEE International Symposium on, pages 2890–2894. IEEE, 2017.

[22] Yaoqing Yang, Pulkit Grover, and Soummya Kar. Coding method for parallel iterative linear solver. NIPS 2017, to appear, arXiv:1706.00163, 2017.

[23] Amirhossein Reisizadeh and Ramtin Pedarsani. Latency analysis of coded computation schemes over wireless networks. arXiv preprint arXiv:1707.00040, 2017.

[24] Sanghamitra Dutta, Viveck Cadambe, and Pulkit Grover. Coded convolution for parallel and distributed computing within a deadline. In IEEE International Symposium on Information Theory (ISIT), July 2017.

[25] Viveck Cadambe and Pulkit Grover. Codes for distributed computing: A tutorial. IEEE Information Theory Society Newsletter, 67(4):3, December 2017.

[26] K.-H. Huang and J.A. Abraham. Algorithm-based fault tolerance for matrix operations. IEEE Transactions on Computers, C-33(6):518–528, June 1984.

[27] Kangwook Lee, Changho Suh, and Kannan Ramchandran. High-dimensional coded matrix multiplication. In Information Theory (ISIT), 2017 IEEE International Symposium on, pages 2418–2422, 2017.

[28] Qian Yu, Mohammad Ali Maddah-Ali, and A Salman Avestimehr. Straggler mitigation in distributed matrix multiplication: Fundamental limits and optimal coding. arXiv preprint:1801.07487, 2018.

[29] Sanghamitra Dutta, Ziqian Bai, Pulkit Grover, and Tze Meng Low. Coded training of model parallel deep neural networks under soft-errors. Submitted to ISIT 2018, 2018.

[30] H T Kung. Fast evaluation and interpolation. Technical report, Carnegie Mellon University, 1973.

[31] Volker Strassen. Gaussian elimination is not optimal. Numerische Mathematik, 13(4):354–356, 1969.

[32] Lei Li. On the arithmetic operational complexity for solving vandermonde linear equations. Japan journal of industrial and applied mathematics, 17(1):15–18, 2000.