Minimax optimal goodness-of-fit testing for densities under a local differential privacy constraint

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Abstract

Finding anonymization mechanisms to protect personal data is at the heart of machine learning research. Here we consider the consequences of local differential privacy constraints on goodness-of-fit testing, i.e. the statistical problem assessing whether sample points are generated from a fixed density $f_0$, or not. The observations are hidden and replaced by a stochastic transformation satisfying the local differential privacy constraint. In this setting, we propose a new testing procedure which is based on an estimation of the quadratic distance between the density $f$ of the unobserved sample and $f_0$. We establish minimax separation rates for our test over Besov balls. We also provide a lower bound, proving the optimality of our result. To the best of our knowledge, we provide the first minimax optimal test and associated private transformation under a local differential privacy constraint, quantifying the price to pay for data privacy.

1 Literature

Ensuring user privacy is at the core of the development of Artificial Intelligence. In particular, someone with access to the training set or the outcome of the algorithm should not be able to retrieve the original dataset. However, classical anonymization and cryptographic approaches fail to prevent the disclosure of sensitive information in the context of learning. Hence differential privacy mechanisms were developed to cope with such issues.

Such considerations can be traced back to a few major papers. In particular, [War65] presents the first privacy mechanism which is now a baseline method for binary data: Randomized response. Another important result is presented in the works of [DL86, DL89, FS98], where they expose a trade-off between statistical utility, or in other terms performance, and privacy.

Differential privacy as expressed in [DMNS06, DKM+06] is the most common formalization of the problem of privacy. It can be summed up as the following condition: altering a single data point of the training set only affects the probability of an outcome to a limited degree. One main advantage of such a definition of the privacy is that it can be parametrized by some $\alpha$, where low values of $\alpha$ correspond to a more restrictive privacy condition. Such a definition is global to the private dataset.

Now we consider a stronger privacy condition, where the analyst himself is not trusted with the data: local differential privacy. This has been extensively studied through the concept of local algorithms, especially in the context of privacy-preserving data mining [War65, AS00, AA01, vdHvdH02, EGS03, AH05, MS06, JSW08, KLN+11]. Recent results detailed in [DJW13a, DWJ13, DJW13b] give information processing inequalities where $\alpha$ appears. Those can be used to obtain Fano or Le Cam-type inequalities in order to obtain a minimax lower bound for estimation or testing problems.

Testing problems have appeared as crucial tools in machine learning since they enable to assess whether a model fits the observations, hence enabling anomalies or novelties to be detected. In particular, goodness-of-fit measures the discrepancy between observed values and a known density provided by the expected model for the behavior of the data. This motivates our study of goodness-of-fit testing under a local differential privacy constraint. Goodness-of-fit testing is a classical hypothesis testing problem in statistics. It consists in testing whether the density $f$ of $n$ independent and identically distributed (i.i.d.) observations equals a specified distribution $f_0$ or not. Assuming that $f$ and $f_0$ belong to $L_2([0,1]) = \{ f : [0,1] \to \mathbb{R}, \|f\|_2^2 = \int_0^1 f^2(x)dx < \infty \}$, it is natural to propose a test based on an estimation of the squared $L_2$-distance $\|f - f_0\|_2^2$ between
We consider the uniform separation rate as defined in [Bar02]. Let \( s > \) be a wide range of regular classes with smoothness parameter \( \rho \). The infimum is taken over all tests of level \( \alpha \). We want to design our tests so that they can reject the null hypothesis \( H_0 : f = f_0 \) if the data was not actually generated from the given model with a given confidence level. Additionally, we want to find the limitations of the test by determining how close the two hypotheses can get while remaining separated by the testing procedure. This classical problem has been studied under the lens of minimax optimality in the seminal work by [Ing87, Ing93]. Non-asymptotic performances and an extension to composite null hypotheses are provided in the paper by [FL06]. In order to introduce the notion of minimax optimality for a testing procedure, let us recall some definitions. We consider the uniform separation rate as defined in [Bar02]. Let \( \Delta_0 \) be a \( \gamma \)-level test with values in \( \{0, 1\} \), where \( \Delta_0 = 1 \) corresponds to the decision of rejecting the null hypothesis \( f = f_0 \). The uniform separation rate \( \rho_n (\Delta_0, B_s, \beta) \) of the test \( \Delta_0 \), over a class \( B_s \) of alternatives \( f \) such that \( f - f_0 \) satisfies smoothness assumptions, with respect to the \( L_2 \)-norm, is defined for all \( \beta \) in \( (0, 1) \) by

\[
\rho_n (\Delta_0, B_s, \beta) = \inf \left\{ \rho > 0 ; \sup_{f \in B_s \| f - f_0 \|_2 > \rho} P_f (\Delta_0 (X_1, \ldots, X_n) = 0) \leq \beta \right\},
\]

where \( P_f \) denotes the distribution of the i.i.d. sample \( (X_1, \ldots, X_n) \) with common density \( f \).

The uniform separation rate is then the smallest value in the sense of the \( L_2 \)-norm of \( f - f_0 \) for which the second kind error of the test is uniformly controlled by \( \beta \) over \( B_s \). This definition extends the notion of critical radius introduced in [Ing93] to the non-asymptotic framework. Note that in general, minimax separation rates are different from minimax estimation rates since testing and estimation are problems of different kinds.

A test of level \( \gamma \) having the optimal performances should then have the smallest possible uniform separation rate (up to a multiplicative constant) over \( B_s \). To quantify this, [Bar02] introduces the non-asymptotic minimax rate of testing defined by

\[
\rho_n^* (B_s, \gamma, \beta) = \inf_{\Delta_0} \rho_n (\Delta_0, B_s, \beta),
\]

where the infimum is taken over all tests of level \( \gamma \). A test is optimal in the minimax sense over the class \( B_s \) if its uniform separation rate is upper-bounded, up to some constant, by the non-asymptotic minimax rate of testing.

Assuming that \( B_s \) is some Hölder class with smoothness parameter \( s > 0 \), [Ing93] establishes the asymptotic minimax rate of testing \( n^{-2s/(4s+1)} \). The test proposed in his paper is not adaptive since it makes use of a known smoothness parameter \( s \). Adaptive goodness-of-fit tests are provided in [Ing00] and [FL06]. These tests achieve the separation rate \( (n/\sqrt{\log \log (n)})^{-2s/(4s+1)} \) over a wide range of regular classes with smoothness parameter \( s > 0 \), the \( \log \log (n) \) term being the price to pay for adaptation to the unknown parameter \( s > 0 \).

A few problems have already been tackled in order to obtain minimax rates under local privacy constraint. The main question is whether the minimax rates are affected by the local privacy constraint and to quantify the degradation of the rate in that case. For a few problems, a degradation of the effective sample size by a multiplicative constant is found. In [DW13], they obtain minimax estimation rates for multinomial distributions with \( d \) dimensions and find a sample degradation of \( \alpha^2/d \). That is, if \( n \) is the necessary and sufficient number of samples in order to solve the classical problem, the \( \alpha \)-local differential private problem is solved with \( n \alpha^2/d \) samples, where \( d \) is the number of dimensions. In [DJW15], they also find a multiplicative sample degradation of \( \alpha^2/d \) for generalized linear models, and \( \alpha^2 \) for median estimation. However, in other problems, a polynomial degradation is noted. For one-dimensional mean estimation, the usual minimax rate is \( n^{-(1/\sqrt{2-2/k})} \), whereas the private rate is \( (n \alpha^2)^{-(\alpha^2/(1-1/k))} \) for random variables \( X \) such that \( E(X) \in [-1, 1] \) and \( E(|X|^k) \). As for the problem of nonparametric density estimation, the rate goes from \( n^{-2s/(2s+1)} \) to \( (n \alpha^2)^{-2s/(2s+2)} \) over an elliptical Sobolev space. This result was extended in [BDK19] over Besov ellipsoids. The classical minimax mean squared errors were presented in [Vu97, YB99, Tsy04].

We list out our contributions:
• We provide the first minimax lower bound for the problem of goodness-of-fit test under local privacy constraint over Besov balls.

• We present the first minimax optimal test with the associated local differentially private channel in this setting.

• The test is made adaptive to the smoothness up to a logarithmic term.

In a setting very similar to ours, [GLRV16] tackles the problems of independence testing and identity testing. More precisely, they test whether sample points were drawn from a known multinomial distribution. However, we consider densities instead. Besides they work under differential privacy constraints, whereas we enforce local privacy. Note that they apply Laplace perturbation to the frequencies, whereas we apply the perturbation onto the coefficients of a wavelet basis, and the choice of the basis is crucial in obtaining the optimal rate. Finally and most importantly, they do not provide guarantees on the convergence rates. [ADKR19] tackle two-sample equivalence testing with unequal sized samples and independence testing under a global differential privacy constraint. In particular, their novel privatization method maintains sample efficiency of the testing method presented in [DK16].

The rest of the paper is articulated as follows. In Section 2, we detail our setting and sum up our results. We introduce a test and a privacy mechanism in Section 3. This will lead to an upper bound on the minimax separation distance for identity testing. However, the proposed test depends on the smoothness parameter which is unknown in general. That is the reason why we present a version of the test in Section 4 that is adaptive to $s$. A lower bound that matches the upper bound is introduced in Section 5. Afterwards, we conclude the paper with a final discussion in Section 6. Finally, in the Supplementary Material, the proofs of all the results presented in this paper are contained in Section A and discussions on possible alternatives for the proof of the lower bound in Section B.

All along the paper, $C$ will denote some absolute constant, $c(a, b, \ldots), C(a, b, \ldots)$ will be constants depending only on their arguments. The constants may vary from line to line.

2 Setting

2.1 Local differential privacy

Let $n$ be some positive integer. Consider unobserved random variables $X_1, \ldots, X_n$, which are independent and identically distributed (i.i.d.) with density $f$ with respect to Lebesgue measure on $[0, 1]$.

Let $\alpha > 0$. Observe $Z_1, \ldots, Z_n$ which are $\alpha$-local differentially private views of $X_1, \ldots, X_n$. That is, there exist $Q_1, \ldots, Q_n$ such that for all $0 \leq i \leq n$, $Z_i$ is a stochastic transformation of $X_i$ by the channel $Q_i$, and

$$\sup_{S, z_j, x, x'} \frac{Q_i(Z_i \in S | X_i = x, Z_j = z_j, j \neq i)}{Q_i(Z_i \in S | X_i = x', Z_j = z_j, j \neq i)} \leq \exp(\alpha).$$

(3)

Equation (3) represents the $\alpha$-local differential privacy assumption in the general interactive case. The stronger assumption corresponding to the non-interactive case (see [War65] and [EGS03]) is expressed, for all $1 \leq i \leq n$, as

$$\sup_{S, x, x'} \frac{Q_i(Z_i \in S | X_i = x)}{Q_i(Z_i \in S | X_i = x')} \leq \exp(\alpha).$$

(4)

Our results focus on the non-interactive case. Let $Q_\alpha$ be the set of channels satisfying the condition in Equation (4).

2.2 Separation rates over Besov balls

The aim of the paper is to provide optimal separation rates for goodness-of-fit tests over Besov balls under privacy constraints. We first recall the definition of Besov balls and we define the uniform separation rates of testing in the private setting.
We denote for all \( j \in \mathbb{N}, \)
\[
\{ \varphi_{j,k}(\cdot) = 2^{j/2} \varphi(2^j \cdot - k), k \in \Lambda(j) \} \cup \{ \psi_{j,k}(\cdot) = 2^{j/2} \psi(2^j \cdot - k), j \geq j, k \in \Lambda(j) \}
\]
is an orthonormal basis of \( L_2([0,1]) \). For the sake of simplicity, we consider the Haar basis where \( \varphi = \mathbb{I}_{[0,1)} \) and \( \psi = \mathbb{I}_{[0,1/2)} - \mathbb{I}_{[1/2,1)} \). In this case, for all \( j \in \mathbb{N}, \Lambda(j) = \{0,1, \ldots 2^j - 1 \} \).

We denote for all \( j \geq 0, k \in \Lambda(j) \), \( \alpha_{j,k} = \int f \varphi_{j,k} \) and \( \beta_{j,k} = \int f \psi_{j,k} \).

Let \( R > 0 \) and \( s > 0 \). The Besov ball \( B_{s,2,\infty}(R) \) with radius \( R \) associated with the Haar basis is defined as
\[
B_{s,2,\infty}(R) = \left\{ f \in L_2([0,1]), \forall j \geq 0, \sum_{k \in \Lambda(j)} \beta_{j,k}^2 \leq R^{2-2js} \right\}.
\]

Fix a density \( f_0 \in L_2([0,1]) \). We want to test the hypotheses
\[
H_0 : f = f_0, \quad \text{versus} \quad H_1 : f \neq f_0.
\]

The twist on classical goodness-of-fit testing is in the fact that the samples from \( f \) are unobserved, we only observe their private views.

For \( \alpha > 0 \) and \( \gamma \in (0,1) \), we construct an \( \alpha \)-local differentially private channel \( Q \in \mathcal{Q}_\alpha \) and a \( \gamma \)-level test \( \Delta_{\gamma,Q} \) such that
\[
\mathbb{P}_{Q_0^\gamma} (\Delta_{\gamma,Q}(Z_1, \ldots, Z_n) = 1) \leq \gamma,
\]
where
\[
\mathbb{P}_{Q_0^\gamma} (Z_1, \ldots, Z_n) \in \prod_{i=1}^n A_i = \int \prod_i Q(Z_i \in A_i | X_i = x_i) f_0(x_i) dx_i
\]
is the joint marginal channel such that Equation (1) holds (we assume here that \( Q_i = Q \) for all \( i \)).

We then define the uniform separation rate of the test \( \Delta_{\gamma,Q} \) over the class \( \mathcal{B}_s \) as
\[
\rho_n (\Delta_{\gamma,Q}, \mathcal{B}_s, \beta) = \inf \left\{ \rho > 0 : \sup_{f \in \mathcal{B}_s, \|f-f_0\|_2 > \rho} \mathbb{P}_{Q_0^\gamma} (\Delta_{\gamma,Q}(Z_1, \ldots, Z_n) = 0) \leq \beta \right\}.
\]

A good channel \( Q \) and a good test \( \Delta_{\gamma,Q} \) are characterized by a small uniform separation rate. This leads us to the definition of the \( \alpha \)-private minimax separation rate over the class \( \mathcal{B}_s \)
\[
\rho_n^*(\mathcal{B}_s, \alpha, \gamma, \beta) = \inf_{Q \in \mathcal{Q}_\alpha} \inf_{\Delta_{\gamma,Q}} \rho_n (\Delta_{\gamma,Q}, \mathcal{B}_s, \beta),
\]
where the infimum is taken over every possible \( \alpha \)-private channel \( Q \) and all \( \gamma \)-level test \( \Delta_{\gamma,Q} \) based on the private observations \( Z_1, \ldots, Z_n \).

### 2.3 Overview of the results

We introduce the following classes of alternatives: for any \( s > 0 \), \( R, R' > 0 \), we define the set \( \mathcal{B}_{s,2,\infty}(R, R') \) as follows
\[
\mathcal{B}_{s,2,\infty}(R, R') = \{ f, f - f_0 \in \mathcal{B}_{s,2,\infty}(R), \|f\|_\infty \leq R' \}.
\]

We also assume that \( \|f_0\|_\infty \leq R' \). Note that the class \( \mathcal{B}_{s,2,\infty}(R, R') \) depends on \( f_0 \) since only regularity for the difference \( f - f_0 \) is required to establish the separation rates. Nevertheless, for the sake of simplicity we omit \( f_0 \) in the notation of this set.

The results presented in Theorems 3.3 and 5.2 can be condensed into the following conclusion that holds for any \( n \geq (\log n)^{1+3/(4s)} \), \( s > 0 \), \( R, R' > 0 \), \( \alpha \geq 1/\sqrt{n} \), \( \gamma, \beta \in (0,1)^2 \) such that \( 2\gamma + \beta < 1 \),
\[
c(\gamma, \beta, R, R') \left[ (n^{-2s/(4s+3)} e^{-\alpha}) \lor n^{-2s/(4s+1)} \right]
\]
\[
\leq \rho_n^* (\mathcal{B}_{s,2,\infty}(R, R'), \alpha, \gamma, \beta)
\]
\[
\leq C(\gamma, \beta, R, R') \left[ (n\alpha^2)^{-2s/(4s+3)} \lor n^{-2s/(4s+1)} \right].
\]

(8)
Remark 2.1. Since we obtain matching bounds up to a log term in Equation (8), we can deduce the minimax separation rate for goodness-of-fit testing under a local privacy constraint. It can be decomposed into two different regimes. When \( \alpha \) is larger than \( n^{1/(4s+1)} \), then the rates of our upper and lower bounds match exactly. Then the minimax rate is of order \( n^{-2s/(4s+1)} \), which coincides with the rate obtained in the non-private case in \([Ing87]\). However, when \( \alpha \) is smaller than \( \frac{4 \log n}{(2s+1)(4s+3)} \), the rates of our upper and lower bounds only match in \( n \). The minimax rate is then of order \( n^{-2s/(4s+3)} \) and so we show a polynomial degradation in the rate due to the privacy constraints. Such a degradation has also been discovered in the related problem of second moment estimation and mean estimation, as well as for the density estimation in \([BDKS19]\). Our bounds do not match in \( \alpha \) however and this leads to untight bounds when \( \alpha \in \left(\frac{4 \log n}{(2s+1)(4s+3)}, n^{1/(4s+1)}\right) \). This is not an issue in practice, since \( \alpha \) will be taken small in order to guarantee privacy.

3 Definition of a test and a privacy mechanism

We will firstly define a testing procedure coupled with a privacy mechanism for which an upper bound on the uniform separation rate matches the right-hand side in Equation (8).

Let \( X_1, \ldots, X_n \) be i.i.d. with common density \( f \). We assume that \( f \) and \( f_0 \) are supported on \([0, 1]\) and belong to \( L_2([0, 1]) \). We want to test

\[
H_0 : f = f_0, \quad \text{versus} \quad H_1 : f \neq f_0, \tag{9}
\]

from \( \alpha \)-local differentially private views of \( X_1, \ldots, X_n \). Let us first propose a transformation of the data, satisfying the differentially privacy constraints.

3.1 Privacy mechanism

We consider the privacy mechanism introduced in \([BDKS19]\). Let us fix some integer \( J > 0 \). We consider for all \( k \in \Lambda(J) = \{0, 1, \ldots, 2^J - 1\} \) the functions \( \varphi_{i,k} \) introduced in Section 2.2. We define, for all \( i \in \{1, \ldots, n\} \) the vector \( Z_i = (Z_{i,J,k})_{k \in \Lambda(J)} \), by

\[
\forall k \in \Lambda(J), \quad Z_{i,J,k} = \varphi_{i,k}(X_i) + \sigma_J W_{i,J,k}, \tag{10}
\]

where \((W_{i,J,k})_{1 \leq i \leq n, k \in \Lambda(J)} \) are i.i.d. Laplace distributed random variables with parameter 1 and

\[
\sigma_J = \frac{2||\varphi||_\infty}{\alpha} 2^{J/2}.
\]

Lemma 3.1. To each random variable \( X_i \) of the sample set \((X_1, \ldots, X_n)\), we associate the vector \( Z_i = (Z_{i,J,k})_{k \in \Lambda(J)} \). The random vectors \((Z_1, \ldots, Z_n)\) are non-interactive \( \alpha \)-local differentially private views of the samples \((X_1, \ldots, X_n)\). Namely, they satisfy the condition in Equation (4).

The proof is due to \([BDKS19]\). We recall here the main arguments.

Proof. The random vectors \((Z_{i,J,k})_{1 \leq i \leq n}\) are i.i.d. by definition. Let us denote by \( Q^{Z_{i,J,k} = \varphi_{i,k}(x_i)} \) the density of the vector \( Z_i \), conditionally to \( X_i = x_i \). For any \( x_i, x_i' \) in \([0, 1]\),

\[
\frac{Q^{Z_{i,J,k} = \varphi_{i,k}(x_i)}}{Q^{Z_{i,J,k} = \varphi_{i,k}(x_i')}} = \prod_{k \in \Lambda(J)} \exp \left[ \frac{|z_{i,J,k} - \varphi_{i,k}(x_i')| - |z_{i,J,k} - \varphi_{i,k}(x_i)|}{\sigma_J} \right] \leq \exp \left[ \frac{1}{\sigma_J} \sum_{k \in \Lambda(J)} (|\varphi_{i,k}(x_i')| + |\varphi_{i,k}(x_i)|) \right].
\]

Since \( |\Lambda(J)| = 2^J \) and since \( \varphi_{i,k}(x_i) \neq 0 \) for a single value of \( k \in \Lambda(J) \), we get

\[
\frac{Q^{Z_{i,J,k} = \varphi_{i,k}(x_i)}}{Q^{Z_{i,J,k} = \varphi_{i,k}(x_i')}} \leq \exp \left[ \frac{2||\varphi||_\infty 2^{J/2}}{\sigma_J} \right] \leq \exp(\alpha),
\]

by definition of \( \sigma_J \), which concludes the proof by application of Lemma 3.2.
Lemma 3.2. Denote $Q^{Z/X=x}(.)$ the density of the vector $Z$, conditionally to $X = x$, with respect to the measure $\mu$. Then $Q \in Q_\alpha$, if and only if there exists a measurable set $Z$ with $\mu(Z \in Z) = 1$ such that $\frac{Q^{Z/X=x}(z)}{\int Z^{x}} \leq \alpha$ for any $z \in Z$.

Proof. Assume there exists $Z$ with $Q(Z \in Z) = 1$ such that $\frac{Q^{Z/X=x}(z)}{\int Z^{x}} \leq \alpha$ for any $z \in Z$.

Let $\tilde{S}$ be some measurable subset of the support of $Z$. Let $S = \tilde{S} \cap Z$.

$$\frac{Q(Z \in S | X = x)}{Q(Z \in S | X = x')} = \frac{Q(Z \in S | X = x)}{Q(Z \in S | X = x')}.$$

Then

$$\frac{Q(Z \in S | X = x)}{Q(Z \in S | X = x')} = \frac{\int_S Q^{Z/X=x}(z)d\mu(z)}{\int_S Q^{Z/X=x'}(z)d\mu(z)} \leq \frac{\int_S Q^{Z/X=x'}(z)e^\alpha d\mu(z)}{\int_S Q^{Z/X=x'}(z)e^{-\alpha} d\mu(z)} = \frac{Q(Z \in S | X = x')}{Q(Z \in S | X = x)} e^{2\alpha}.$$

So

$$\frac{Q(Z \in \tilde{S} | X = x)}{Q(Z \in S | X = x')} \leq e^{\alpha}.$$

Assume that $Q \in Q_\alpha$. Then for any measurable $S$, we have $Q(Z \in S | X = x) \leq e^\alpha Q(Z \in S | X = x')$. That is, for any $S$,

$$\int_S (e^\alpha Q^{Z/X=x'}(z)- Q^{Z/X=x}(z))d\mu(z) \geq 0.$$

So there exists $Z$ with $Q(Z \in Z) = 1$ such that $\frac{Q^{Z/X=x}(z)}{\int Z^{x}} \leq \alpha$ for any $z \in Z$.

3.2 Definition of the test

Our aim is now to define a testing procedure for the testing problem defined in Equation (1) from the observation of the vectors $(Z_1, \ldots, Z_n)$. Our test statistic $T_J$ is defined as

$$\frac{1}{n(n-1)} \sum_{i \neq j = 1}^n \sum_{k \in \Lambda(J)} (Z_{i,j,k} - \alpha_{j,k}^0) (Z_{i,j,k} - \alpha_{j,k}^0),$$  \hspace{1cm} (11)

where $\alpha_{j,k}^0 = \int_0^1 \varphi_{J,k}(x)f_0(x)dx$.

We consider the test function

$$\Delta_{J, \gamma, Q}(Z_1, \ldots, Z_n) = 1_{T_J > t_{\alpha}^0(1-\gamma)}.$$  \hspace{1cm} (12)

where $t_{\alpha}^0(1-\gamma)$ denotes the $(1-\gamma)$-quantile of $T_J$ under $H_0$. Note that this quantile can be estimated by simulations, under the hypothesis $f = f_0$. We can indeed simulate the vector $(Z_1, \ldots, Z_n)$ if the density of $(X_1, \ldots, X_n)$ is assumed to be $f_0$. Hence the test rejects the null hypothesis $H_0$ if

$$T_J > t_{\alpha}^0(1-\gamma).$$

The test is obviously of level $\gamma$ by definition of the threshold.

Comments:

In a similar way as in [FL06], the test is based on an estimation of the quantity $\|f - f_0\|^2_2$. Note indeed that $T_J$ is an unbiased estimator of $\|\Pi_{S,J} (f - f_0)\|^2$, where $\Pi_{S,J}$ denotes the orthogonal projection in $L_2([0,1])$ onto the space generated by the functions $(\varphi_{J,k}, k \in \Lambda(J))$.

In the next section, we provide non-asymptotic theoretical results for the power of this test.
3.3 Upper bound on the minimax separation rate

We provide an upper bound on the uniform separation rate for our test and privacy channel over Besov balls in Theorem 3.3. It also constitutes an upper bound on the minimax separation rate.

**Theorem 3.3.** Let \((X_1, \ldots, X_n)\) be i.i.d. with common density \(f\) on \([0, 1]\). Let \(f_0\) be some given density on \([0, 1]\). From the observation of the random vectors \((Z_1, \ldots, Z_n)\) defined by Equation (10), for a given \(\alpha > 0\), we want to test the hypotheses
\[
H_0 : f = f_0, \quad \text{versus} \quad H_1 : f \neq f_0.
\]

We assume that \(f_0\) is uniformly bounded by \(R' > 0\) and that \(n\alpha^2 \geq 1\).

We consider the test \(\Delta_{J^*, \gamma, \alpha, Q}\) defined by Equation (12) with \(J = J^*\), where \(J^*\) is the smallest integer \(J\) such that
\[
2^J \geq (n\alpha^2)^{2/(4s+3)} \cdot n^{2/(4s+1)}.
\]

The uniform separation rate, defined by Equation (3), of the test \(\Delta_{J^*, \gamma, \alpha, Q}\) over \(B_{s,2,\infty}(R, R')\) defined by Equation (7) satisfies for all \(n \in \mathbb{N}^*\), \(R, R' > 0\), \(\alpha \geq 1/\sqrt{n}\), \(\gamma, \beta \in (0, 1)^2\) such that \(\gamma + \beta < 1\)
\[
\rho_n(\Delta_{J^*, \gamma, \alpha, Q}, B_{s,2,\infty}(R, R'), \beta) \leq C(R, R', \gamma, \beta) \left[ (n\alpha^2)^{-2s/(4s+3)} \lor n^{-2s/(4s+1)} \right].
\]

The proof of this result is in Section A.1 of the Supplementary Material.

**Comments:**

When the sample \((X_1, \ldots, X_n)\) is directly observed, [FL06] propose a testing procedure with uniform separation rate over the set \(B_{s,2,\infty}(R, R')\) controlled by
\[
C(R, R', \gamma, \beta)n^{-2s/(4s+1)},
\]
which is an optimal result. Hence we obtain here a loss in the uniform separation rate, due to the fact that we only observe \(\alpha\)-differentially private views of the original sample. This loss occurs when \(\alpha \leq n^{1/(4s+1)}\), otherwise, we get the same rate as when the original sample is observed. We will see in Section 5 that this result is optimal.

Finally, having \(\alpha < 1/\sqrt{n}\) represents an extreme case, where the sample size is really low in conjunction with a very strict privacy condition. In such a range of \(\alpha\), \(J\) is taken equal to 0, but this does not lead to optimal rates.

The test proposed in Theorem 3.3 depends (via the parameter \(J^*\)) on the smoothness parameter \(s\) of the Besov ball \(B_{s,2,\infty}(R)\). In a second step, we will propose a test adaptive to \(s\). In Section 4 we construct an aggregated testing procedure, which is independent of the smoothness parameter and achieves the minimax separation rates established in Equation (8) over a wide range of Besov balls simultaneously, up to a logarithmic term.

4 Adaptive tests

In Section 2.2 we have defined a testing procedure which depends on the parameter \(J\). The performances of the test depend on this parameter. We have optimized the choice of \(J\) to obtain the smallest possible upper bound for separation rate over the set \(B_{s,2,\infty}(R, R')\). Nevertheless, the test is not adaptive since this optimal choice of \(J\) depends on the smoothness parameter \(s\).

In order to obtain adaptive procedure, we propose, as in [FL06], to aggregate a collection of tests. For this, we introduce the set
\[
\mathcal{J} = \{ J \in \mathbb{N}, 2^J \leq n^2 \}.
\]

For a given level \(\gamma \in (0, 1)\), the aggregated testing procedure rejects the hypothesis \(H_0 : f = f_0\) if
\[
\exists J \in \mathcal{J}, \quad \hat{T}_J > t^0_J(1 - u_\gamma),
\]
where \(u_\gamma\) is defined by
\[
u_\gamma = \sup \left\{ u \in (0, 1), \mathbb{P}_{\mathbb{Q}_0} \left( \sup_{J \in \mathcal{J}} \left( \hat{T}_J - t^0_J(1 - u_\gamma) \right) > 0 \right) \leq \gamma \right\}.
\]
Hence $u_\gamma$ is the least conservative choice leading to a $\gamma$-level test. We easily notice that $u_\gamma \geq \gamma/|J|$. Indeed,

\[
\Pr_{Q_{f_0}} \left( \sup_{J \in J} \left( \hat{T}_J - t_J^0(1 - \gamma/|J|) \right) > 0 \right) \\
\leq \sum_{J \in J} \Pr_{Q_{f_0}}^{\gamma} \left( \hat{T}_J > t_J^0(1 - \gamma/|J|) \right) \\
\leq \sum_{J \in J} \gamma/|J| \leq \gamma.
\]

Let us now consider the second kind error for the aggregated test, which is the probability to accept the null hypothesis $H_0$ incorrectly. This quantity is upper bounded by the smallest second kind error of the tests of the collection, at the price that $\gamma$ has been replaced by $u_\gamma$. Indeed,

\[
\Pr_{Q_{f_1}} \left( \sup_{J \in J} \left( \hat{T}_J - t_J^0(1 - u_\gamma) \right) \leq 0 \right) \\
= \Pr_{Q_{f_1}} \left( \cap_{J \in J} \left( \hat{T}_J \leq t_J^0(1 - u_\gamma) \right) \right) \\
\leq \inf_{J \in J} \Pr_{Q_{f_1}}^{\gamma} \left( \hat{T}_J \leq t_J^0(1 - u_\gamma) \right).
\]

We obtain the following theorem for the aggregated procedure.

**Theorem 4.1.** Let $(X_1, \ldots, X_n)$ be i.i.d. with common density $f$ in $L_2([0,1])$. Let $f_0$ be some given density in $L_2([0,1])$. From the observation of the random vectors $(Z_1, \ldots, Z_n)$ defined by Equation (10), for a given $\alpha > 0$, we want to test the hypotheses

\[ H_0 : f = f_0, \quad \text{versus} \quad H_1 : f \neq f_0. \]

We assume that $f_0$ is uniformly bounded by $R' > 0$ and we assume that $na^2/\sqrt{\log(n)} \geq 1$.

We consider the set $\mathcal{J} = \{ J \in \mathbb{N}, 2^J \leq n^2 \}$ and the aggregated test

\[ \Delta_{\gamma,Q}^J = \mathbf{1}_{\sup_{J \in J} (\hat{T}_J - t_J^0(1 - u_\gamma)) > 0} \]

where $u_\gamma$ is defined by Equation (13). The uniform separation rate, defined by Equation (15), of the test $\Delta_{\gamma,Q}^J$ over the set $\mathcal{B}_{s,2}\infty(R, R')$ defined by Equation (7) satisfies for all $n \in \mathbb{N}^*$, $s > 0$, $R, R' > 0$, $\alpha > 0$, $\gamma, \beta \in (0,1)^2$ such that $\gamma + \beta < 1$,

\[ \rho_n \left( \Delta_{\gamma,Q}^J, \mathcal{B}_{s,2}\infty(R, R'), \beta \right) \]

\[ \leq C(R, R', \gamma, \beta) \left( na^2/\sqrt{\log(n)} \right)^{-2s/(4s+3)} \vee \left( n/\sqrt{\log(n)} \right)^{-2s/(4s+1)}. \]

The proof of this result is in Section A.2 of the Supplementary Material.

**Comments:** Comparing this result with the rates obtained in Theorem 3.3 which will be proved to be optimal in the next section, we have here a logarithmic loss due to the adaptation. We recall the separation rates in the non-private setting obtained by [Ing00] and [FL06] for adaptive procedures over Besov balls. They were of order $\left( n/\sqrt{\log(n)} \right)^{-2s/(4s+1)}$. We do not know if the logarithmic term that we obtain here is optimal or not.

## 5 Lower bound on the minimax separation rate

We consider for any $s > 0$, $R, R' > 0$, the classes of alternatives $\mathcal{B}_{s,2}\infty(R, R')$ defined by Equation (7).

This section will focus on the presentation of a lower bound on the minimax separation rate over Besov balls defined in Equation (9) for the problem of identity testing under a local differential privacy constraint. The test and privacy mechanism showcased in Section 3 will turn out to be minimax optimal since the lower bound will match the upper bound obtained in Theorem 3.3.

Let us apply a Bayesian approach, where we will define a prior distribution which corresponds to a mixture of densities satisfying $H_1$. Such a proof technique is classical for lower bounds in
We note the total variation distance between two probability measures are distinguishable. Another natural idea to prove Theorem 5.2 is to bound the total variation distance between arbitrary distributions with respective supports in $H_0$ and $H_1$. It turns out that the closer the distributions from $H_0$ and $H_1$ can be, the higher the second kind error. So if we are able to provide distributions from $H_0$ and $H_1$ which are close from one another, we can guarantee that the second kind error of any test will have to be high.

**Lemma 5.1.** Let $\gamma, \beta \in (0,1)^2$ and $\delta \in [0,1]$ such that $\gamma + \beta + \delta < 1$. Let $f_0 \in L_2([0,1])$ and $\rho > 0$. We define

$$F_\rho(B_{s,2,\infty}(R,R')) = \{ f \in B_{s,2,\infty}(R,R'), \| f - f_0 \|_2 \geq \rho \}.$$  

Let $\alpha > 0$ and let $Q \in Q_\alpha$ be some $\alpha$-private channel. Let $\nu_\rho$ be some probability measure such that $\nu_\rho(F_\rho(B_{s,2,\infty}(R,R'))) \geq 1 - \delta$ and let $Q_{\alpha}^n$ be defined, for all measurable set $A$ by

$$P_{Q_{\alpha}^n}(\{Z_1, \ldots, Z_n\} \in A) = \int P_{Q_\alpha^n}(\{Z_1, \ldots, Z_n\} \in A) d\nu_\rho(f).$$

We note the total variation distance between two probability measures $P_1$ and $P_2$ as $\|P_1 - P_2\|_{TV} = \sup_A |P_1(A) - P_2(A)|$.

Then if

$$\|P_{Q_{\alpha}^n} - P_{Q_{\alpha}^n}\|_{TV} < 1 - \gamma - \beta - \delta,$$

we have

$$\inf_{\Delta_{\gamma,Q}} \rho_n(\Delta_{\gamma,Q}, B_{s,2,\infty}(R,R'), \beta) \geq \rho,$$

where the infimum is taken over all possible $\gamma$-level test, hence satisfying

$$P_{Q_{\alpha}^n}(\Delta_{\gamma,Q}(Z_1, \ldots, Z_n) = 1) \leq \gamma.$$

The idea is to establish the connection between the second kind error and the total variation distance between arbitrary distributions with respective supports in $H_0$ and $H_1$. It turns out that the closer the distributions from $H_0$ and $H_1$ can be, the higher the second kind error. So if we are able to provide distributions from $H_0$ and $H_1$ which are close from one another, we can guarantee that the second kind error of any test will have to be high.

**Theorem 5.2.** Let $\gamma, \beta \in (0,1)^2$ such that $2\gamma + \beta < 1$. Let $\alpha > 0$, $R > 0$, $s > 0$.

We obtain the following lower bound for the $\alpha$-private minimax separation rate defined by Equation (6) for non-interactive channels in $Q_\alpha$ over the class of alternatives $B_{s,2,\infty}(R,2)$

$$\rho_\alpha^n(B_{s,2,\infty}(R,2), \alpha, \gamma, \beta) \geq c(\gamma, \beta, R) \left[ (n^{-2s/(4s+3)} \wedge (\log n)^{-1/2} e^{-\alpha}) \vee n^{-2s/(4s+1)} \right].$$

Note that the result only holds for non-interactive channels. The details of the proof can be found in Section A.4 of the Supplementary Material. We outline the intuition behind the main arguments in the following sketch.

**Sketch of proof.** We want to find the largest $L_2$-distance between the initial density $f_0$ under the null hypothesis and the density in the alternative hypothesis such that their transformed counterparts by an $\alpha$-private channel $Q$ cannot be discriminated by a test. We will rely on the singular vectors of $Q$ in order to define densities and their private counterparts with ease. We define a mixture of densities in the private space such that they have a fixed $L_2$-distance to $f_0$, which is the private transformation of $f_0$ by $Q$. We obtain a sufficient condition for the total variation distance between the mixture and $f_0$ to be small enough for both hypotheses to be indistinguishable. Then we ensure that the densities that we have considered in the private set are associated with densities for the original sample that belong to the regularity class $B_{s,2,\infty}(R,2)$. Employing bounds on the singular values of $Q$, we obtain sufficient conditions for the original densities to have the right regularity. Collecting all these elements, the conclusion relies on Lemma 5.1.

**Remark 5.3.** The total variation distance is a good criterion in order to determine whether two distributions are distinguishable. Another natural idea to prove Theorem 5.2 is to bound the total variation distance between two private densities by the total variation distance between the densities of the original samples, up to some constants depending on the privacy constraints. Following this intuitive approach, we can provide a lower bound using Theorem 1 in [DJW13b] combined with Pinsker’s inequality. However, the resulting lower bound does not match the upper bound for the separation rates of goodness-of-fit testing presented in our Section 3. Details on this approach are provided in Section B.3 of the Supplementary Material.
6 Discussion

We provided the first minimax optimal test and local differentially private channel for the problem of goodness-of-fit testing over Besov balls. Besides the test and channel remain optimal up to a log factor even if the smoothness parameter is unknown. Among our technical contributions, it is to note that we used a proof technique in the lower bound that does not involve Theorem 1 from [DJW13b]. The minimax separation rate turns out to suffer from a polynomial degradation in the private case. However, we point out an elbow effect, where the rate is the same as the usual case up to some constant if $\alpha$ is large enough. Future possible works could extend our results to larger Besov classes and to the discrete case. Besides, a lower bound including the study of interactive channels is open for further research.

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A Proofs

In the following, \(\text{Im}(\cdot)\) denotes the image function and \(\text{dim}(\cdot)\) the dimension function.

A.1 Upper bound: proof of Theorem 3.3

We want to establish a condition on \(\|f - f_0\|^2\), under which the second kind of error of the test is controlled by \(\beta\), namely

\[
\mathbb{P}_{Q^n_j} \left( \hat{T}_J \leq t_J^0(1 - \gamma) \right) \leq \beta. 
\]

(15)

Denoting by \(t_J(\beta)\) the \(\beta\)-quantile of \(\hat{T}_J\) under \(\mathbb{P}_{Q^n_j}\), the condition in Equation (15) holds as soon as \(t_J^0(1 - \gamma) \leq t_J(\beta)\). Hence, we provide an upper bound for \(t_J^0(1 - \gamma)\) and a lower bound for \(t_J(\beta)\).

A.1.1 Upper bound for \(t_J^0(1 - \gamma)\)

Since \(\mathbb{E}_{Q^n_j}[\hat{T}_J] = 0\), we have by Markov’s inequality, for all \(t > 0\),

\[
\mathbb{P}_{Q^n_j} \left( \left| \hat{T}_J \right| > t \right) \leq \frac{\text{Var}_{Q^n_j}(\hat{T}_J)}{t^2}.
\]

So, considering the \((1 - \gamma)\)-quantile of \(\hat{T}_J\) under \(H_0\), we have

\[
1 - \gamma = 1 - \mathbb{P}_{Q^n_j} \left( \left| \hat{T}_J \right| > t_J^0(1 - \gamma) \right) \geq 1 - \frac{\text{Var}_{Q^n_j}(\hat{T}_J)}{(t_J^0(1 - \gamma))^2}.
\]

So

\[
t_J^0(1 - \gamma) \leq \sqrt{\text{Var}_{Q^n_j}(\hat{T}_J)/\gamma}.
\]

(16)

Note that one can rewrite \(\hat{T}_J\) as

\[
\hat{T}_J = \frac{1}{n(n-1)} \sum_{i \neq j=1}^{n} h_J(Z_i, Z_i),
\]

where

\[
h_J(Z_i, Z_i) = \sum_{k \in \Lambda(J)} \left( Z_{i,k} - \alpha_{0,i,k} \right) \left( Z_{i,k} - \alpha_{0,j,k} \right).
\]

In order to provide an upper bound for the variance \(\text{Var}_{Q^n_j}(\hat{T}_J)\), let us first state a lemma controlling the variance of a \(U\)-statistic of order 2. This result is a particular case of Lemma 8 in [MALM19].

Lemma A.1. Let \(h\) be a symmetric function with 2 inputs, \(Z_1, \ldots, Z_n\) be independent and identically distributed random vectors and \(U_n\) be the \(U\)-statistic of order 2 defined by

\[
U_n = \frac{1}{n(n-1)} \sum_{i \neq j=1}^{n} h(Z_i, Z_i).
\]

The following inequality gives an upper bound on the variance of \(U_n\),

\[
\text{Var}(U_n) \leq C \left( \frac{\sigma^2}{n} + \frac{s^2}{n^2} \right),
\]

where \(\sigma^2 = \text{Var}(\mathbb{E}[h(Z_1, Z_2) \mid Z_1])\) and \(s^2 = \text{Var}(h(Z_1, Z_2))\).

We have that \(\mathbb{E}_{Q^n_j}[h_J(Z_1, Z_2) \mid Z_1] = 0\). Moreover,

\[
h_J(Z_1, Z_2)
= \sum_{k \in \Lambda(J)} (\varphi_{J,k}(X_1) - \alpha_{0,k}) (\varphi_{J,k}(X_2) - \alpha_{0,k}) + \sigma_J^2 \sum_{k \in \Lambda(J)} W_{1,k} W_{2,k}
+ \sigma_J \sum_{k \in \Lambda(J)} W_{1,k} (\varphi_{J,k}(X_2) - \alpha_{0,k}) + \sigma_J \sum_{k \in \Lambda(J)} W_{2,k} (\varphi_{J,k}(X_1) - \alpha_{0,k}).
\]
So, since $\mathbb{E}_{Q^o_0} \left( \varphi_{j,k}(X_i) - \alpha_{j,k}^0 \right) = 0$ and $\mathbb{E}_{Q^o_0} W_{i,j,k} = 0$ for any $i$, we have

$$\text{Var}_{Q^o_0} (h_j(Z_1, Z_2))$$

$$= \text{Var}_{Q^o_0} \left[ \sum_{k \in \Lambda(J)} (\varphi_{j,k}(X_1) - \alpha_{j,k}^0) (\varphi_{j,k}(X_2) - \alpha_{j,k}^0) \right] + \text{Var}_{Q^o_0} \left[ \sigma_j^2 \sum_{k \in \Lambda(J)} W_{1,j,k} W_{2,j,k} \right]$$

$$+ 2\text{Var}_{Q^o_0} \left[ \sigma_j \sum_{k \in \Lambda(J)} W_{1,j,k} (\varphi_{j,k}(X_2) - \alpha_{j,k}^0) \right].$$

Now, by independence of $X_1$ and $X_2$,

$$\text{Var}_{Q^o_0} \left[ \sum_{k \in \Lambda(J)} (\varphi_{j,k}(X_1) - \alpha_{j,k}^0) (\varphi_{j,k}(X_2) - \alpha_{j,k}^0) \right]$$

$$= \sum_{k, l \in \Lambda(J)} \mathbb{E} \left[ (\varphi_{j,k}(X_1) - \alpha_{j,k}^0) (\varphi_{j,l}(X_1) - \alpha_{j,l}^0) \right] \mathbb{E} \left[ (\varphi_{j,k}(X_2) - \alpha_{j,k}^0) (\varphi_{j,l}(X_2) - \alpha_{j,l}^0) \right]$$

$$= \sum_{k, l \in \Lambda(J)} \left[ \int \varphi_{j,k} \varphi_{j,l} f_0 - \alpha_{j,k}^0 \alpha_{j,l}^0 \right]^2.$$

So

$$\text{Var}_{Q^o_0} \left[ \sum_{k \in \Lambda(J)} (\varphi_{j,k}(X_1) - \alpha_{j,k}^0) (\varphi_{j,k}(X_2) - \alpha_{j,k}^0) \right]$$

$$= \int \int \left( \sum_{k \in \Lambda(J)} \varphi_{j,k}(x) \varphi_{j,k}(y) \right)^2 f_0(x) f_0(y) dx dy - 2 \int \left( \sum_{k \in \Lambda(J)} \alpha_{j,k}^0 \varphi_{j,k}(x) \right)^2 f_0(x) dx$$

$$+ \left( \sum_{k \in \Lambda(J)} (\alpha_{j,k}^0)^2 \right)^2$$

$$\leq \|f_0\|_\infty^2 \int \int \sum_{k, l \in \Lambda(J)} \varphi_{j,k}(x) \varphi_{j,l}(y) \varphi_{j,l}(x) \varphi_{j,l}(y) dx dy + \left( \int f_0^2 \right)$$

$$\leq C(\|f_0\|_\infty) |\Lambda(J)| \leq C(\|f_0\|_\infty) 2^J \sigma_j^2$$

by orthogonality and since $|\Lambda(J)| \leq 2^J$.

By independence of the variables $(W_{i,j,k})$,

$$\text{Var} \left( \sigma_j^2 \sum_{k \in \Lambda(J)} W_{1,j,k} W_{2,j,k} \right) = \sigma_j^4 \sum_{k \in \Lambda(J)} \text{Var}(W_{1,j,k} W_{2,j,k}) = C2^J \sigma_j^4,$$

since $|\Lambda(J)| \leq 2^J$.

Finally, using again the independence of the variables $(W_{1,j,k})_{k \in \Lambda(J)}$, and their independence with $X_2$,

$$\text{Var}_{Q^o_0} \left[ \sigma_j \sum_{k \in \Lambda(J)} W_{1,j,k} (\varphi_{j,k}(X_2) - \alpha_{j,k}^0) \right]$$

$$= \sigma_j^2 \mathbb{E}_{Q^o_0} \sum_{k, k' \in \Lambda(J)} W_{1,j,k} W_{1,j,k'} (\varphi_{j,k}(X_2) - \alpha_{j,k}^0) (\varphi_{j,k'}(X_2) - \alpha_{j,k'}^0)$$

$$= \sigma_j^2 \sum_{k \in \Lambda(J)} \mathbb{E}(W_{1,j,k}^2) \mathbb{E}_{Q^o_0} \left[ (\varphi_{j,k}(X_2) - \alpha_{j,k}^0)^2 \right]$$

$$\leq C(\|f_0\|_\infty) 2^J \sigma_j^2 \sigma_j^2.$$
This leads to the following upper bound for $\text{Var}_{Q_{h_j}}(h_j(Z_1, Z_2))$,

$$\text{Var}_{Q_{h_j}}(h_j(Z_1, Z_2)) \leq C(\|f_0\|_\infty)(1 + \sigma_j^2 + \sigma_j^4)2^j,$$

from which, by application of Lemma A.1, we deduce that

$$\text{Var}_{Q_{h_j}}(\hat{T}_j) \leq C(\|f_0\|_\infty)(1 + \sigma_j^2)^{2^j}.$$

Then, plugging the last inequality into Equation (16), we have

$$\theta_j^0(1 - \gamma) \leq C(\|f_0\|_\infty)(1 + \sigma_j^2)^{2^j/n} \sqrt{\gamma}.$$  (17)

A.1.2 Lower bound for $t_J(\beta)$

We recall that $t_J(\beta)$ denotes the $\beta$-quantile of $\hat{T}_j$ under $P_{Q_j}$. Let us define

$$\hat{U}_j = \frac{1}{n(n-1)} \sum_{i \neq l=1}^n \sum_{k \in \Lambda(J)} (Z_{i,J,k} - \alpha_{J,k}) (Z_{l,J,k} - \alpha_{J,k}),$$

$$\hat{V}_j = 2 \sum_{k \in \Lambda(J)} (\alpha_{J,k} - \alpha_{0,J,k}^0) \frac{1}{n} \sum_{i=1}^n (Z_{i,J,k} - \alpha_{J,k}).$$

We have

$$\hat{T}_j = \hat{U}_j + \hat{V}_j + \|\Pi_{S_j}(f - f_0)\|^2.$$

Note that $\text{Var}_{Q_j}(\hat{U}_j)$ can be upper bounded in the same way as $\text{Var}_{Q_{h_j}}(\hat{T}_j)$. So, following the derivations from the previous section, we have

$$\text{Var}_{Q_j}(\hat{U}_j) \leq C(\|f\|_\infty)(1 + \sigma_j^4)^{2^j/n^2}.$$

Let us now compute $\text{Var}_{Q_j}(\hat{V}_j)$. Since $\hat{V}_j$ is centered,

$$\text{Var}_{Q_j}(\hat{V}_j) = \mathbb{E}_{Q_j}(\hat{V}_j^2) = \frac{4}{n^2} \sum_{k, k' \in \Lambda(J)} (\alpha_{J,k} - \alpha_{0,J,k}^0) (\alpha_{J,k'} - \alpha_{0,J,k'}^0) \sum_{i,l=1}^n \mathbb{E}_{Q_j} [(Z_{i,J,k} - \alpha_{J,k})(Z_{l,J,k'} - \alpha_{J,k'})].$$

Note that, if $i \neq l$,

$$\mathbb{E}_{Q_j} [(Z_{i,J,k} - \alpha_{J,k})(Z_{l,J,k'} - \alpha_{J,k'})] = 0.$$

Moreover,

$$\mathbb{E}_{Q_j} [(Z_{i,J,k} - \alpha_{J,k})(Z_{l,J,k'} - \alpha_{J,k'})] = \mathbb{E}[(\varphi_{J,k}(X_i) - \alpha_{J,k})(\varphi_{J,k'}(X_l) - \alpha_{J,k'}) + \sigma_j^2 \mathbb{E}_{Q_j}(W_{i,J,k}W_{l,J,k'})] = \int \varphi_{J,k} \varphi_{J,k'} f - \alpha_{J,k} \alpha_{J,k'} + 2\sigma_j^2 \mathbb{I}_{k=k'}.$$
Hence,

\[ \text{Var}_{Q_J} \left( \tilde{V}_J \right) = \frac{4}{n} \sum_{k, k' \in \Lambda(J)} \left( \alpha_{J,k} - \alpha_{J,k'}^0 \right) \left( \alpha_{J,k} - \alpha_{J,k'}^0 \right) \left( \int \varphi_{J,k} \varphi_{J,k'} f - \alpha_{J,k} \alpha_{J,k'} + 2\sigma_J^2 I_{k=k'} \right) \]

\[ = \frac{4}{n} \int \left( \sum_{k \in \Lambda(J)} (\alpha_{J,k} - \alpha_{J,k}^0) \varphi_{J,k} \right)^2 \frac{1}{n} \left( \sum_{k' \in \Lambda(J)} \alpha_{J,k} (\alpha_{J,k} - \alpha_{J,k}^0) \right)^2 \]

\[ + \frac{8}{n} \sigma_J^2 \sum_{k \in \Lambda(J)} (\alpha_{J,k} - \alpha_{J,k}^0)^2 \]

\[ \leq \frac{4}{n} \|f\|_\infty \sum_{k \in \Lambda(J)} (\alpha_{J,k} - \alpha_{J,k}^0)^2 + \frac{8}{n} \sigma_J^2 \sum_{k \in \Lambda(J)} (\alpha_{J,k} - \alpha_{J,k}^0)^2 \]

\[ = \frac{1}{n} \left( 4\|f\|_\infty + 8\sigma_J^2 \right) \|\Pi_{S,j}(f - f_0)\|^2. \]

We finally obtain,

\[ \text{Var}_{Q_J} \left( \tilde{V}_J \right) \leq C(\|f\|_\infty) \left( \frac{1 + \sigma_J^2}{n} \right) \|\Pi_{S,j}(f - f_0)\|^2. \]

By Chebyshev’s inequality,

\[ P_{Q_J} \left( \hat{T}_J \leq E_{Q_J}(\hat{T}_J) - \sqrt{\text{Var}_{f} \left( \hat{T}_J \right) / \beta} \right) \leq \beta. \]

Now \( E_{Q_J}(\hat{T}_J) = \|\Pi_{S,j}(f - f_0)\|^2 \), hence we obtain the following lower bound on the \( \beta \)-quantile,

\[ \|\Pi_{S,j}(f - f_0)\|^2 \leq \sqrt{\text{Var}_{f} \left( \hat{T}_J \right) / \beta} \leq t_J(\beta). \]

Moreover,

\[ \sqrt{\text{Var}_{f} \left( \hat{T}_J \right) / \beta} \leq C(\|f\|_\infty, \beta) \left[ \|\Pi_{S,j}(f - f_0)\|^2 \left( \frac{1 + \sigma_J}{\sqrt{n}} \right) + \frac{(1 + \sigma_J^2)2^{J/2}}{n} \right] \]

\[ \leq \frac{1}{2} \|\Pi_{S,j}(f - f_0)\|^2 + C(\|f\|_\infty, \beta) \frac{(1 + \sigma_J^2)2^{J/2}}{n}, \]

where we have used the inequality \( ab \leq a^2/2 + b^2/2 \). This leads us to the following lower bound for \( t_J(\beta) \)

\[ t_J(\beta) \geq \frac{1}{2} \|\Pi_{S,j}(f - f_0)\|^2 - C(\|f\|_\infty, \beta) \frac{(\sigma_J^2 + 1)2^{J/2}}{n}. \]  

(18)

Recalling that the condition in Equation (15) holds as soon as \( t_J^0(1 - \gamma) \leq t_J(\beta) \), we obtain the following sufficient condition by combining Equations (17) and (18),

\[ \|\Pi_{S,j}(f - f_0)\|^2 \geq C(\|f_0\|_\infty, \|f\|_\infty, \gamma, \beta) \left( \frac{(\sigma_J^2 + 1)2^{J/2}}{n} \right). \]

That is,

\[ \|f - f_0\|^2 \geq \|f - f_0 - \Pi_{S,j}(f - f_0)\|^2 + C(\|f_0\|_\infty, \|f\|_\infty, \gamma, \beta) \frac{(\sigma_J^2 + 1)2^{J/2}}{n}. \]

Since \( f - f_0 \in B_{s,2,\infty}(R) \), we have

\[ \|f - f_0 - \Pi_{S,j}(f - f_0)\|^2 \leq R^22^{-2J}, \]

which leads to the sufficient condition

\[ \|f - f_0\|^2 \geq R^22^{-2J} + C(\|f\|_\infty, \|f_0\|_\infty, \gamma, \beta) \frac{(\sigma_J^2 + 1)2^{J/2}}{n}. \]
We recall that $\sigma_J = C2^{J/2}/\alpha$. That is, the sufficient condition turns out to be:

$$
\|f - f_0\|^2 \geq C(\|f\|_\infty, \|f_0\|_\infty, R, \gamma, \beta) \left(2^{-2Js} + \frac{\alpha^{3/2}}{\alpha^2 n} + \frac{2^{J/2}}{n}\right).
$$

We consider two cases.

- If $1/\sqrt{n} \leq \alpha \leq n^{1/(4s+1)}$, then $(n\alpha^2)^{2/(4s+3)} \leq n^{2/(4s+1)}$ and the right hand term of the inequality in Equation (19) for $J = J^*$ is upper bounded by

  $$
  C(\|f\|_\infty, \|f_0\|_\infty, R, \gamma, \beta)(n\alpha^2)^{-4s/(4s+3)}.
  $$

- If $\alpha > n^{1/(4s+1)}$, then $(n\alpha^2)^{2/(4s+3)} > n^{2/(4s+1)}$ and the right hand term of the inequality in Equation (19) for $J = J^*$ is upper bounded by

  $$
  C(\|f\|_\infty, \|f_0\|_\infty, R, \gamma, \beta)n^{-4s/(4s+1)}.
  $$

Hence, the separation rate of our test over the set $B_{s,2,\infty}(R, R')$ is controlled by

$$
C(R, R', \gamma, \beta) \left[(n\alpha^2)^{-2s/(4s+3)} \wedge n^{-2s/(4s+1)}\right],
$$

which concludes the proof of Theorem 4.3.

### A.2 Adaptivity: proof of Theorem 4.1

Using Inequality (14), and the fact that $\gamma \geq \gamma/|J|$, we obtain that

$$
P_{Q_f^{\gamma}}(\Delta_{\gamma, Q_f} = 0) \leq \beta
$$

as soon as

$$
\exists J \in J, t^0_J(1 - \gamma/|J|) \leq t_J(\beta).
$$

Using Equations (17) and (18), and the fact that $|J| \leq C \log(n)$, we get that Equation (20) holds as soon as there exists $J \in J$ such that

$$
\|\Pi_{S,J}(f - f_0)\|^2 \geq C(\|f_0\|_\infty, \|f\|_\infty, \gamma, \beta) \left(\frac{(\sigma_J^2 + 1)2^{J/2} \sqrt{\log(n)}}{n}\right),
$$

or equivalently

$$
\|f - f_0\|^2 \geq \inf_{J \in J} \left[\|f - f_0 - \Pi_{S,J}(f - f_0)\|^2 + C(\|f_0\|_\infty, \|f\|_\infty, \gamma, \beta)\left(\frac{(\sigma_J^2 + 1)2^{J/2} \sqrt{\log(n)}}{n}\right)\right].
$$

Assuming that $f - f_0 \in B_{s,2,\infty}(R)$, for some $s > 0$ and $R > 0$, we get that Equation (20) holds if

$$
\|f - f_0\|^2 \geq \inf_{J \in J} \left[R^22^{-2Js} + C(\|f_0\|_\infty, \|f\|_\infty, \gamma, \beta)\left(2^{J/2} + \frac{2^{3J/2}}{\alpha^2} \sqrt{\log(n)}\right)\right].
$$

Choosing $J \in J$ as the smallest integer in $J$ such that

$$
2^J \geq (n^2 \alpha^4 / \log(n))^{1/(4s+3)} \wedge (n^2 / \log(n))^{1/(4s+1)},
$$

we obtain the sufficient condition

$$
\|f - f_0\|^2 \geq C(\|f\|_\infty, \|f_0\|_\infty, R, \gamma, \beta) \left((n\alpha^2 / \sqrt{\log(n)})^{-4s/(4s+3)} \wedge (n/ \sqrt{\log(n)})^{-4s/(4s+1)}\right).
$$

Hence, for all $s > 0$, $R, R' > 0$, the separation rate of the aggregated test over the set $B_{s,2,\infty}(R, R')$ is controlled by

$$
C(R, R', \gamma, \beta) \left[(n\alpha^2 / \sqrt{\log(n)})^{-2s/(4s+3)} \wedge (n/ \sqrt{\log(n)})^{-2s/(4s+1)}\right],
$$

which concludes the proof of Theorem 4.1.
A.3 Lower bound: proof of Lemma 5.1

Since \( \nu_p(\mathcal{F}_p(B_{\alpha,2}^\infty(R, R'))) \geq 1 - \delta \), we first notice that

\[
\inf_{\Delta, Q} \sup_{f \in \mathcal{F}_p(B_{\alpha,2}^\infty(R, R'))} \mathbb{P}_{Q_f}^p(\Delta, Q(Z_1, \ldots, Z_n) = 0) \\
\geq \inf_{\Delta, Q} \mathbb{P}_{Q_f}^p(\Delta, Q(Z_1, \ldots, Z_n) = 0) - \delta \\
= \inf_{\Delta, Q} (\mathbb{P}_{Q_f}^p(\Delta, Q(Z_1, \ldots, Z_n) = 0) + \mathbb{P}_{Q_f}^p(\Delta, Q(Z_1, \ldots, Z_n) = 0) \\
- \mathbb{P}_{Q_f}^p(\Delta, Q(Z_1, \ldots, Z_n) = 0)) - \delta \\
\geq 1 - \gamma - \sup_{\Delta, Q} \left| \mathbb{P}_{Q_f}^p(\Delta, Q(Z_1, \ldots, Z_n) = 0) - \mathbb{P}_{Q_f}^p(\Delta, Q(Z_1, \ldots, Z_n) = 0) \right| - \delta
\]

by definition of \( \Delta, Q(Z_1, \ldots, Z_n) \). Finally, by definition of the total variation distance,

\[
\inf_{\Delta, Q} \sup_{f \in \mathcal{F}_p(B_{\alpha,2}^\infty(R, R'))} \mathbb{P}_{Q_f}^p(\Delta, Q(Z_1, \ldots, Z_n) = 0) \geq 1 - \gamma - \delta - \| \mathbb{P}_{Q_f}^p - \mathbb{P}_{Q_f}^p \|_{TV}.
\]

So we have

\[
\inf_{\Delta, Q} \sup_{f \in \mathcal{F}_p(B_{\alpha,2}^\infty(R, R'))} \mathbb{P}_{Q_f}^p(\Delta, Q(Z_1, \ldots, Z_n) = 0) > \beta,
\]

provided that

\[
\| \mathbb{P}_{Q_f}^p - \mathbb{P}_{Q_f}^p \|_{TV} < 1 - \gamma - \beta - \delta.
\]

A.4 Lower bound: proof of Theorem 5.2

A.4.1 Preliminary results

The following lemma sheds light on the equivalence between the local differential privacy condition and a similar condition on the density of the channel.

**Lemma A.2.** Let \( Q \in Q_\alpha \) be an \( \alpha \)-private channel. Let \( X \) be a random variable with distribution \( P \). Then there exists a probability measure with respect to which \( Q(\cdot|x) \) is absolutely continuous for any \( x \).

**Proof.** Let \( \mu = \int Q(\cdot|x)dP(x) \). Let \( S \) be a measurable set such that \( \mu(S) = 0 \). Then since \( Q(S|x) \geq 0 \) for any \( x \), there exists \( x \) such that \( Q(S|x) = 0 \). Now by \( \alpha \)-local differential privacy, \( Q(S|x) = 0 \) for any \( x \).

For the sake of completeness, we prove the following classical inequality between the total variation distance and the chi-squared distance. It will be used in order to reduce the study of the distance between the distributions to that of an expected squared likelihood ratio.

**Lemma A.3.**

\[
\| \mathbb{P}_{Q_f}^p - \mathbb{P}_{Q_f}^p \|_{TV} \leq \frac{1}{2} \left( \mathbb{E}_{Q_f}^p \left[ L_{Q_f}^p(Z_1, \ldots, Z_n) - 1 \right] \right)^{1/2}.
\]

**Proof.** We have

\[
\| \mathbb{P}_{Q_f}^p - \mathbb{P}_{Q_f}^p \|_{TV} = \frac{1}{2} \int \left| L_{Q_f}^p - 1 \right| d\mathbb{P}_{Q_f}^p
\]

\[
= \frac{1}{2} \mathbb{E}_{Q_f}^p \left[ L_{Q_f}^p(Z_1, \ldots, Z_n) - 1 \right]
\]

\[
\leq \frac{1}{2} \left( \mathbb{E}_{Q_f}^p \left[ L_{Q_f}^p(Z_1, \ldots, Z_n) - 1 \right] \right)^{1/2},
\]

by Cauchy-Schwarz inequality and since \( \mathbb{E}_{Q_f}^p \left( L_{Q_f}^p(Z_1, \ldots, Z_n) \right) = 1 \).
Lemma A.4. Let \( P_f, P_g \) be probability measures over the sample space \( \Omega \) with respective densities \( f \) and \( g \) with respect to a measure \( \mu \). Let \( Q \) be a stochastic channel. Then

\[
\|P_f - P_g\|_{TV} \geq \|P_{Q_z} - P_{Q_{z'}}\|_{TV}.
\]

Proof. For any measurable set \( S \),

\[
\int_{\Omega} Q(S|x)(f(x) - g(x))d\mu(x) = \int_{\Omega} Q(S|x)(f(x) - g(x))1\{f - g \geq 0\}(x)d\mu(x) + \int_{\Omega} Q(S|x)(f(x) - g(x))1\{f - g < 0\}(x)d\mu(x).
\]

Now, since \( 0 \leq Q(S|x) \leq 1 \) for any measurable set \( S \) and \( x \in \Omega \),

\[
0 \leq \int_{\Omega} Q(S|x)(f(x) - g(x))1\{f - g \geq 0\}(x)d\mu(x) \leq \int_{\Omega} (f(x) - g(x))1\{f - g > 0\}(x)d\mu(x).
\]

and

\[
0 \geq \int_{\Omega} Q(S|x)(f(x) - g(x))1\{f - g < 0\}(x)d\mu(x) \geq \int_{\Omega} (f(x) - g(x))1\{f - g < 0\}(x)d\mu(x).
\]

So for any measurable set \( S \),

\[
\int_{\Omega} (f(x) - g(x))1\{f - g < 0\}(x)d\mu(x) \leq \int_{\Omega} Q(S|x)(f(x) - g(x))d\mu(x) \leq \int_{\Omega} (f(x) - g(x))1\{f - g > 0\}(x)d\mu(x).
\]

That is, for any measurable set \( S \)

\[
\left| \int_{\Omega} Q(S|x)(f(x) - g(x))d\mu(x) \right| \leq \sup_A \left| \int_A (f(x) - g(x))d\mu(x) \right| = \|P_f - P_g\|_{TV}.
\]

The following lemma ensures that the transformation we will consider in the coming lemmas preserves the \( \alpha \)-local differential privacy condition.

Lemma A.5. Let \( Q \in Q_\alpha \) be a non-interactive \( \alpha \)-private channel. Let \( X \) be a random variable and \( Z \) be its associated private view through \( Q \). Now, let \( \tilde{Z} \) be independent from \( X \) conditionally to \( Z \). Then \( \tilde{Z} \) is an \( \alpha \)-private view of \( X \).

Proof. Let \( S \) be a measurable set.

\[
\frac{Q(\tilde{Z} \in S|X = x)}{Q(Z \in S|X = x')} = \frac{\int Q(\tilde{Z} \in S|Z = z, X = x)Q(Z = z|X = x)d\mu(z)}{\int Q(Z \in S|Z = z, X = x')Q(Z = z|X = x')d\mu(z)} \leq e^\alpha \frac{\int Q(\tilde{Z} \in S|Z = z)Q(Z = z|X = x')d\mu(z)}{\int Q(Z \in S|Z = z)Q(Z = z|X = x')d\mu(z)} = e^\alpha,
\]

where the inequality is due to \( \alpha \)-local privacy and the conditional independence of \( \tilde{Z} \) and \( X \).
We now provide the following two lemmas to show that we can consider a channel \( Q \in \mathcal{Q}_\alpha \) such that \( \int Q(Z \in \cdot |x)f_0(x)dx \) is uniform over \([0,1]\), without loss of generality.

**Lemma A.6.** There exists an injective measurable transformation from \( \mathbb{R}^d \) to \((0,1)^d\). 

**Proof.** Firstly, applying arctan elementwise is an injective measurable transformation from \( \mathbb{R}^d \) to \((0,1)^d\). Now, let us show that there exists an injective measurable function from \((0,1)^2\) to \((0,1)\). This will ensure the existence for any \( d \geq 2 \). Let \( x \in (0,1)^2 \). Then consider its decimal representation, \( x = \left( \sum_{i=1}^{\infty} x_{i}^010^{-i} \right) \sum_{i=1}^{\infty} x_{i}^110^{-i} \) with the following conditions. Let \( \eta \in \{1,2\} \). For any \( i, x_i^\eta \) is an integer between 0 and 9 and there exists \( j \geq i \) such that \( x_i^\eta \neq 9 \). Then there exists a unique decimal representation of \( x \). Now we exhibit the following injective measurable transformation from \((0,1)^2\) to \((0,1)\),

\[
x \rightarrow \sum_{i=1}^{\infty} (x_{i}^010^{-2i} + x_{i}^110^{1-2i}).
\]

\[\square\]

**Lemma A.7.** Let \( \mathcal{U} = (U_i)_{i \in \mathbb{N}} \) such that the \( U_i \)'s are independent random variables which are uniform over \([0,1]\). Let \( Y \) be a real random variable with support \( \mathcal{Y} \) and independent from \( \mathcal{U} \). There exists a measurable function \( G_Y : \mathbb{R} \times [0,1]^\mathcal{U} \rightarrow [0,1] \) such that \( G_Y(Y;\mathcal{U}) \) is uniform over \((0,1)\) and the following holds. For any random variables \( X, Z \) which are independent from \( \mathcal{U} \) and such that their respective supports are subsets of \( \mathcal{Y} \),

\[
\|\mathbb{P}(X \in \cdot) - \mathbb{P}(Z \in \cdot)\|_{TV} = \|\mathbb{P}(G_Y(X;\mathcal{U}) \in \cdot) - \mathbb{P}(G_Y(Z;\mathcal{U}) \in \cdot)\|_{TV}.
\]

**Proof.** Let \( F_Y : t \rightarrow \mathbb{P}(Y \leq t) \) be the cumulative distribution function of \( Y \). Its number of discontinuities is at most countable. Let \( I \subset \mathbb{N} \). Denote the points where \( F_Y \) is discontinuous as \( \omega_i \), for any \( i \in I \) and let \( \Delta_i = P(Y = \omega_i) \) be the respective sizes of the discontinuities. Let \( \mathcal{Y} \) be the support of \( Y \).

We define \( G_Y(y;\mathcal{U}) = F_Y(y)(1 - 1\{y \in \{\omega_i : i \in I\}\}) + \sum_{i \in I} F_Y(\omega_i) - \Delta_i, U_i1\{y = \omega_i\} \), where \( \mathcal{U} = (U_i)_{i \in \mathbb{N}} \). Firstly, \( G_Y(Y;\mathcal{U}) \) is uniform over \((0,1)\) by construction, since for any \( i \),

\[
\lim_{t \rightarrow 0, t > 0} F_Y(\omega_i - t) = F_Y(\omega_i) - \Delta_i.
\]

We now verify that \( G_Y(\cdot;\mathcal{U}) \) preserves the total variation distance. We exhibit the inverse of \( G_Y(\cdot;\mathcal{U}) \) restricted to the support \( \mathcal{Y} \) of \( Y \). For any \( \tilde{y} \in \text{Im}(G_Y(\cdot;\mathcal{U})) \),

\[
G_Y^{-1}(\tilde{y}) = F_Y^{-1}(\tilde{y})1\{\tilde{y} \notin \bigcup_{i \in I}[F_Y(\omega_i) - \Delta_i, F_Y(\omega_i)]\} + \sum_{i \in I} \omega_i 1\{\tilde{y} \in [F_Y(\omega_i) - \Delta_i, F_Y(\omega_i)]\},
\]

where \( F_Y^{-1} \) is the generalized inverse distribution function.

Then we can apply the data processing inequality in Lemma A.4 on the channel defined for any measurable set \( S \) conditionally on any point \( x \) such as: \( Q(S|x) = \mathbb{P}(G_Y(x;\mathcal{U}) \in S) \). So the total variation distance is contracted by \( G_Y \). That is,

\[
\sup_{A \in \mathcal{A}} |\mathbb{P}(X \in A) - \mathbb{P}(Z \in A)| \geq \sup_{A \in \mathcal{A}} |\mathbb{P}(G_Y(X;\mathcal{U}) \in G_Y(A;[0,1]^\mathcal{U})) - \mathbb{P}(G_Y(Z;\mathcal{U}) \in G_Y(A;[0,1]^\mathcal{U}))|.
\]

Let \( A \) be an element of the \( \sigma \)-Algebra \( \mathcal{A} \).

\[
|\mathbb{P}(X \in A) - \mathbb{P}(Z \in A)| = |\mathbb{P}(X \in A \cap \mathcal{Y}) - \mathbb{P}(Z \in A \cap \mathcal{Y})| = |\mathbb{P}(G_Y(X;\mathcal{U}) \in G_Y(A \cap \mathcal{Y};[0,1]^\mathcal{U})) - \mathbb{P}(G_Y(Z;\mathcal{U}) \in G_Y(A \cap \mathcal{Y};[0,1]^\mathcal{U}))|,
\]

since \( G_Y(\cdot;\mathcal{U}) \) is invertible over \( \mathcal{Y} \). So defining \( \mathcal{A}_\mathcal{Y} \) as the restriction of \( \mathcal{A} \) to \( \mathcal{Y} \),

\[
\sup_{A \in \mathcal{A}} |\mathbb{P}(X \in A) - \mathbb{P}(Z \in A)| = \sup_{A \in \mathcal{A}_\mathcal{Y}} |\mathbb{P}(G_Y(X;\mathcal{U}) \in G_Y(A;[0,1]^\mathcal{Y})) - \mathbb{P}(G_Y(Z;\mathcal{U}) \in G_Y(A;[0,1]^\mathcal{Y}))| \leq \sup_{A \in \mathcal{A}} |\mathbb{P}(G_Y(X;\mathcal{U}) \in G_Y(A;[0,1]^\mathcal{Y})) - \mathbb{P}(G_Y(Z;\mathcal{U}) \in G_Y(A;[0,1]^\mathcal{Y}))|.
\]

Hence the equality holds and the total variation distance is preserved by the transformation \( G_Y(\cdot;\mathcal{U}) \). 

\[\square\]
A.4.2 Definition of prior distributions

Let $Q \in Q_0$ be a non-interactive $\alpha$-private channel. We assume that $f_0$ is the uniform density on $[0, 1]$. We define the function $\psi \in L^2([0, 1])$ by $\psi(x) = \mathbf{1}_{[0,\frac{1}{2}]} - \mathbf{1}_{[\frac{1}{2}, 1]}$, and for some given $J \in \mathbb{N}$, that will be specified later, we define, for all $k \in \Lambda(J) = \{0, 1, \ldots 2^J - 1\}$, $\psi_{J,k}(x) = 2^{J/2}\psi(2^J x - k)$. We denote by $V$ the linear subspace of $L^2([0, 1])$ generated by the functions $(f_0, \psi_{J,k}, k \in \Lambda(J))$.

Then by application of Lemma A.6 there exists a probability measure $\mu$ with respect to which $Q(Z \in \cdot | x)$ is absolutely continuous for any $x$. Denote the density of $Q(Z \in \cdot | x)$ by $q(\cdot | x)$. Let $Q$ be the operator such that for any $x \in V$, $Q(x) = \int q(\cdot | x)\xi(x)dx$. Over $V$, $Q$ is a linear operator and $\text{dim}(\text{Im}(Q)) \leq K = 2^J + 1$.

Assume that $\int q(z|x)f_0(x)dx = \tilde{f}_0(z)$, where $\tilde{f}_0$ is the uniform density over $[0, 1]$. This assumption is made without loss of generality. Indeed, by application of Lemma A.6 we can transform any random variable into a one-dimensional random variable injectively. So such a transformation simply amounts to a simple relabeling. Now let $Z_{f_0}$ and $Z_f$ be linear one-dimensional random variables which are privatized samples from $f_0$ and $f$ respectively. Then the support of $Z_f$ is included in the support of $Z_{f_0}$ since $f_0$ is uniform over $[0, 1]$. So by Lemma A.7 $G_{Z_{f_0}}(Z_{f_0}, U)$ is uniformly distributed over $(0, 1)$ and the total variation distance between $Z_{f_0}$ and $Z_f$ is preserved. It is key in obtaining the lower bound, by Lemma 5.1. Finally, we show that such transformations preserve the privacy condition by Lemma A.8. Besides, $Q(Z \in \cdot | x)$ is absolutely continuous with respect to the Lebesgue measure for any $x$ since $f_0$ is the uniform density over $[0, 1]$.

Then we complete $(f_0)$ and $(\tilde{f}_0)$ into respective orthogonal bases $(f_0, u_i)_{2 \leq i \leq K}$ and $(\tilde{f}_0, g_i)_{2 \leq i \leq K'}$ of $V$ and $\text{Im}(Q)$ using the SVD method on $Q$. Note that $Q(u_i) = \int q(\cdot | x)u_i(x)dx = \lambda_i g_i$ for any $2 \leq i \leq K$. Besides, each $u_i$ and $g_i$ are normed.

For all $i \in \{2, \ldots, K\}$, $u_i \in \text{Span}(\psi_{J,k}, k \in \Lambda(J))$, hence we write

$$u_i = \sum_{k \in \Lambda(J)} a_{i, k} \psi_{J,k}.$$ 

Case $K' < K$. Then there exists $2 \leq i \leq K$ such that $Q(u_i) = 0$. So let $f_\rho = f_0 + \rho u_i$. Moreover, $f_\rho - f_0 = f_\rho G_{\psi_{J,k}}(R)$ if $\sum_{k \in \Lambda(J)} |a_{i,k}|^2 \leq 2^{-2J} R^2$. That is, if $\rho \leq 2^{-J} R$. And $f_\rho$ is a density if $\rho 2^{J/2} |a_{i,k}| \leq 1$ for any $k$ since $\|\psi_{J,k}\|_\infty = 2^{J/2}$. That is, if $\rho \leq 2^{-J/2}$. So we have the following sufficient condition: $\rho \leq (2^{-J} R) \wedge 2^{-J/2}$. And for any $\rho$, $Q(f_0) = Q(f_\rho)$, hence their total variation distance is 0. So for any $(\gamma, \beta) \in (0, 1)^2$ such that $\gamma + \beta < 1$, if $\mathbb{P}_{Q_{\rho_0}}(\Delta_{\gamma, \beta}(Z_1, \ldots, Z_n) = 1) \leq \gamma$, then $\inf_{\Delta_{\gamma, \beta}} \mathbb{P}_{Q_0}^\rho(\Delta_{\gamma, \beta}(Z_1, \ldots, Z_n) = 0) > \beta$. In particular, taking $J = 0$, we obtain $\inf_{\Delta_{\gamma, \beta} \rho_0} (\Delta_{\gamma, \beta}, B_{s,2,\infty}(R), \beta) \geq R \wedge 1$. Such a trivial lower bound is of course not useful and we will work towards another lower bound when $K' = K$.

From this point on, assume $K' = K$. Let $\varepsilon > 0$, for all $\eta \in \{-1, 1\}^{2^J}$, we define

$$f_{\eta}(x) = f_0(x) + \sum_{i=2}^{K} \varepsilon_i \eta_i g_i(x),$$

where for any $2 \leq i \leq K$,

$$\varepsilon_i = \begin{cases} \varepsilon 2^{J/2} \lambda_i & \text{if } \lambda_i \leq 2^{-J/2}, \\ \varepsilon & \text{otherwise}. \end{cases}$$

Note that one can define the corresponding non-private density

$$f_{\eta}(x) = f_0(x) + \sum_{i=2}^{K} \varepsilon_i \eta_i \lambda_i^{-1} u_i(x).$$

We now define the set

$$\mathcal{F} = \left\{ f_{\eta}, \eta \in \{-1, 1\}^{2^J} \right\}.$$ 

Let $\rho = \|f_\rho - f_0\|_2$ and we introduce the uniform distribution $\nu_\rho$ on $\mathcal{F}$. Let $\gamma, \beta \in (0, 1)$ such that $2\gamma + \beta < 1$. In order to apply Lemma 5.1, we will provide sufficient conditions to ensure that

$$\nu_\rho(\mathcal{F}_\rho(B_{s,2,\infty}(R, 2))) \geq 1 - \gamma,$$

where

$$\mathcal{F}_\rho(B_{s,2,\infty}(R, R')) = \{ f \text{ density}, f \in B_{s,2,\infty}(R, R'), \| f - f_0\|_2 \geq \rho \}.$$
A.4.3 Sufficient conditions for \( f_\eta \in \mathcal{F}_\rho(\mathcal{B}_{s,2,\infty}(R,2)) \)

We first prove the following points.

**Lemma A.8.** \( \forall \eta \in \{-1,1\}^2 \):

a) If

\[
\varepsilon \leq \frac{2^{-J}}{\sqrt{2 \log(2^{J+1}/\gamma)}},
\]

then \( f_\eta \) is a density with probability larger than \( 1 - \gamma \) under the prior \( \nu_\rho \).

b) If

\[
\varepsilon \leq R2^{-J(s+1)},
\]

then \( f_\eta - f_0 \in \mathcal{B}_{s,2,\infty}(R,2) \).

**Proof.** a) Since for all \( i, u_i \) is orthogonal to \( f_0 \), uniform density on \([0,1]\), we have that \( \int_0^1 f_\eta(x)dx = 1 \) and we just have to prove that \( f_\eta \) is nonnegative. We remind the reader that

\[
u_\eta = \sum_{k \in \Lambda(J)} \sum_{i_k} a_{i_k,k} \psi_{i_k,k}.
\]

The bases \((u_2,\ldots,u_K)\) and \((\psi_{i_k,k},k \in \Lambda(J))\) are orthonormal. This implies that the matrix \( A = (a_{i_k,k})_{2 \leq i_k \leq K, k \in \Lambda(J)} \) is orthogonal. So

\[
\forall i, \sum_{k \in \Lambda(J)} a_{i_k,k}^2 = 1, \forall k, \sum_{i_k} a_{i_k,k}^2 = 1.
\]

Hence we have for all \( x \in [0,1] \),

\[
(f_\eta - f_0)(x) = \sum_{k \in \Lambda(J)} \sum_{i_k} \eta_i \epsilon_i \lambda_i^{-1} a_{i_k,k} \psi_{i_k,k}(x).
\]

The functions \((\psi_{i_k,k}, k \in \Lambda(J))\) have disjoint supports and \( \sup_{x \in [0,1]} |\psi_{i,k}(x)| = 2^{1/2} \). Hence \( f_\eta \) is nonnegative if and only if for any \( k \in \Lambda(J) \)

\[
2^{1/2} \left| \sum_{i=2}^K \eta_i \epsilon_i \lambda_i^{-1} a_{i_k,k} \right| \leq 1.
\]

Moreover, the condition ensures that \( \|f_\eta\|_\infty \leq 2 \).

Since \( \nu_\rho \) is a uniform distribution on \( \mathcal{F} \), we have that \( f_\eta \) is a density with probability larger than \( 1 - \gamma \) under the prior \( \nu_\rho \) as soon as Equation (23) holds with probability larger than \( 1 - \gamma \). That is,

\[
P \left( \forall k \in \Lambda(J), 2^{1/2} \left| \sum_{i=2}^K \eta_i \epsilon_i \lambda_i^{-1} a_{i,k} \right| \leq 1 \right) \geq 1 - \gamma,
\]

where \((\eta_2,\ldots,\eta_K)\) are i.i.d. Rademacher random variables. Using Hoeffding’s inequality, we get for all \( x > 0 \), for all \( k \in \Lambda(J) \),

\[
P \left( \sum_{i=2}^K \eta_i \epsilon_i \lambda_i^{-1} a_{i,k} > x \right) \leq 2 \exp \left( \frac{-2x^2}{\sum_{i=2}^K (\epsilon_i \lambda_i^{-1} a_{i,k})^2} \right).
\]

Hence

\[
P \left( \forall k \in \Lambda(J), \left| \sum_{i=2}^K \eta_i \epsilon_i \lambda_i^{-1} a_{i,k} \right| > x \right) \leq 2^{J+1} \exp \left( \frac{-x^2}{2 \sum_{i=2}^K (\epsilon_i \lambda_i^{-1} a_{i,k})^2} \right).
\]
This leads to
\[
P\left( \forall k \in \Lambda(J), \sum_{i=2}^{K} \eta_i \epsilon_i \lambda_i^{-1} a_{i,k} \right) > \sqrt{2 \sum_{i=2}^{K} (\epsilon_i \lambda_i^{-1} a_{i,k})^2 \log(2^{J+1}/\gamma)} \leq \gamma.
\]

Hence, \( f_\eta \) is a density with probability larger than \( 1 - \gamma \) under the prior \( \nu_p \) as soon as for any \( k \in \Lambda(J) \),
\[
2^J \cdot 2 \sum_{i=2}^{K} (\epsilon_i \lambda_i^{-1} a_{i,k})^2 \log(2^{J+1}/\gamma) \leq 1.
\]

Now by definition, \( \epsilon_i \lambda_i^{-1} \leq \epsilon 2^{J/2} \). So we have the sufficient condition:
\[
2^J \cdot 2 \epsilon^2 2^J \log(2^{J+1}/\gamma) \sum_{i=2}^{K} a_{i,k}^2 \leq 1.
\]

And \( \sum_{i=2}^{K} a_{i,k}^2 \leq \sum_{i=2}^{K} a_{i,k}^2 = 1 \) leads to the following sufficient condition:
\[
\epsilon \leq \frac{2^{-J}}{\sqrt{2 \log(2^{J+1}/\gamma)}}.
\]

b) We deduce from Equation (22) that \( \forall j \neq J, \forall k, (f_\eta - f_0, \psi_{j,k}) = 0 \) since \( \langle \psi_{j,i}, \psi_{j,k} \rangle = 0 \) for all \( k \) and \( l \). Moreover, for all \( k \in \Lambda(J) \),
\[
(f_\eta - f_0, \psi_{j,k}) = \sum_{i=2}^{K} \eta_i \epsilon_i \lambda_i^{-1} a_{i,k}.
\]

Hence \( f_\eta - f_0 \in B_{s,2,\infty}(R) \) if and only if
\[
\sum_{k \in \Lambda(J)} \left( \sum_{i=2}^{K} \eta_i \epsilon_i \lambda_i^{-1} a_{i,k} \right)^2 \leq R^2 2^{-2J}.
\]

But we also have that
\[
\sum_{k \in \Lambda(J)} \left( \sum_{i=2}^{K} \eta_i \epsilon_i \lambda_i^{-1} a_{i,k} \right)^2 = \| f_\eta - f_0 \|^2 = \sum_{i=2}^{K} \epsilon_i^2 \lambda_i^{-2} \leq \epsilon^2 2^J
\]
by definition of the \( \epsilon_i \)'s and since \( K - 1 \leq 2^J \). Hence we obtain that Equation (21) is a sufficient condition to ensure that \( f_\eta - f_0 \in B_{s,2,\infty}(R) \).

\[\square\]

A.4.4 Information bound

We consider the expected squared likelihood ratio:

\[
\mathbb{E}_{Q_0} \left[ L_{Q_0}^{\psi_{\nu_p}}(Z_1, \ldots, Z_n) \right]
\]

\[
= \mathbb{E}_{Q_0} \mathbb{E}_{\eta, \eta'} \prod_{i=1}^{n} \left( 1 + \frac{\sum_{j=2}^{K} \epsilon_j \eta_j g_j(Z_i)}{f_0(Z_i)} \right) \left( 1 + \frac{\sum_{j=2}^{K} \epsilon_j \eta'_j g_j(Z_i)}{f_0(Z_i)} \right)
\]

\[
= \mathbb{E}_{Q_0} \mathbb{E}_{\eta, \eta'} \prod_{i=1}^{n} \left( 1 + \frac{\sum_{j=2}^{K} \epsilon_j \eta_j g_j(Z_i)}{f_0(Z_i)} \right) \sum_{j=2}^{K} \frac{\epsilon_j \eta_j g_j(Z_i)}{f_0(Z_i)} \sum_{j=2}^{K} \frac{\epsilon_j \eta'_j g_j(Z_i)}{f_0(Z_i)} \sum_{j=2}^{K} \frac{\epsilon_j \eta_j g_j(Z_i) g_j(Z_i)}{f_0(Z_i)^2}.
\]

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Now, for any $j$, 
\[
\mathbb{E}_{\mathcal{Q}_{\mathcal{F}_0}} \left[ \frac{g_j(Z)}{f_0(Z)} \right] = \int g_j(z) dz = 0,
\]
by orthogonality with uniform vector $f_0$. And for any $j, l$, by orthogonality,
\[
\mathbb{E}_{\mathcal{Q}_{\mathcal{F}_0}} \left[ \frac{g_j(Z)g_l(Z)}{f_0(Z)^2} \right] = \int \frac{g_j(z)g_l(z)}{f_0(z)} dz = \int g_j(z)g_l(z) dz = 1 \{j = l\}.
\]

So, considering i.i.d. $Z_i$, we have:
\[
\mathbb{E}_{\mathcal{Q}_{\mathcal{F}_0}} \left[ \mathcal{L}_{\mathcal{Q}_0}^2 (Z_1, \ldots, Z_n) \right] = \mathbb{E}_{\eta, \eta'} \left[ 1 + \sum_{j=2}^{K} \eta_j \eta_j' \right]^n \leq \mathbb{E}_{\eta, \eta'} \exp(n \sum_{j=2}^{K} \eta_j \eta_j' \epsilon_j^2).
\]

Then
\[
\mathbb{E}_{\mathcal{Q}_{\mathcal{F}_0}} \left[ \mathcal{L}_{\mathcal{Q}_0}^2 (Z_1, \ldots, Z_n) \right] \leq \prod_{j=2}^{K} \cosh(n \epsilon_j^2) \leq \prod_{j=2}^{K} \exp(n^2 \epsilon_j^2).
\]

Since for any $j$, $\epsilon_j \leq \epsilon$, then
\[
\mathbb{E}_{\mathcal{Q}_{\mathcal{F}_0}} \left[ \mathcal{L}_{\mathcal{Q}_0}^2 (Z_1, \ldots, Z_n) \right] \leq \exp(n^2 \epsilon^4 (K - 1)) \leq \exp(n^2 \epsilon^4 2^J).
\]

Then, in order to apply Lemma 5.1 combined with A.3 let us find a sufficient condition for
\[
\mathbb{E}_{\mathcal{Q}_{\mathcal{F}_0}} \left[ \mathcal{L}_{\mathcal{Q}_0}^2 (Z_1, \ldots, Z_n) \right] < 1 + 4(1 - \gamma - \beta - \gamma)^2.
\]

So let us choose $\epsilon$ and $J$ in order to ensure that
\[
\exp(n^2 \epsilon^4 2^J) < 1 + 4(1 - 2\gamma - \beta)^2,
\]
i.e.
\[
2^J \epsilon^4 \leq n^{-2} \log \left[ 1 + 4(1 - 2\gamma - \beta)^2 \right],
\]
i.e.
\[
\epsilon \leq n^{-1/2} \left( \log \left[ 1 + 4(1 - 2\gamma - \beta)^2 \right] / 2^J \right)^{1/4}. \tag{24}
\]

So combining Equation (24) and Lemma A.3 we obtain the following sufficient condition in order to apply Lemma 5.1
\[
\epsilon \leq \left[ n^{-1/2} \left( \log \left[ 1 + 4(1 - 2\gamma - \beta)^2 \right] / 2^J \right)^{1/4} \right] \wedge \left[ R2^{-J(s+1)} \right] \wedge \left[ \frac{2^{-J}}{\sqrt{2 \log(2^{J+1})}} \right].
\]

Now, let us translate this result in terms of distance in the initial space. By definition and orthonormality, for any $\eta \in \{-1, 1\}^{2^J}$
\[
\rho = \|f_\eta - f_0\|_2 = \sqrt{\sum_{i=2}^{K} \lambda_i^2 \|u_i\|_2^2} = \sqrt{\sum_{i=2}^{K} \epsilon_i^2 \lambda_i^2} = \epsilon \sqrt{\sum \lambda_i^2 1\{\lambda_i > 2^{-J/2}\} + 2^J \sum 1\{\lambda_i \leq 2^{-J/2}\}}.
\]

So let us provide guarantees on the singular values in order to determine sufficient conditions for $\rho$ to be a lower bound on $\inf_{\Delta, \mathcal{Q}} \rho_h (\Delta, \mathcal{Q}, \mathcal{B}_{h, \infty}(R), \beta)$. 

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A.4.5 Obtaining the inequalities on the singular values

We have by the privacy constraint and Lemma 3.2 \( \frac{q(z|x)}{q(z|x')} \geq e^{-\alpha} \) for any \( z, x' \). Then for any \( z, x_0 \),

\[
1 = \int_0^1 q(z|x)dx = q(z|x_0) \int_0^1 \frac{q(z|x)}{q(z|x_0)}dx \\
\geq q(z|x_0)e^{-\alpha}.
\]

So for any \( z, x_0 \), we have

\[ q(z|x_0) \leq e^\alpha. \]

This implies that

\[ \|Q\|_F = \sqrt{\int_0^1 \int_0^1 q(z|x)^2dxdz} \leq e^\alpha. \]

We call \( \lambda_i \) the singular values of \( Q \). Then

\[
\sqrt{\sum_{i=2}^{K} \lambda_i^2 \mathbf{1}\{\lambda_i > 2^{-J/2}\}} \leq \|Q\|_F \leq e^\alpha.
\]

Let \( l = \sum_{i=2}^{K} \mathbf{1}\{\lambda_i > 2^{-J/2}\} \). If \( l > 0 \), at least \( l/2 \) singular values are smaller than \( \sqrt{\frac{4}{e}}e^\alpha \) and larger than \( 2^{-J/2} \).

So if \( l \geq 0 \),

\[
\sum_{i=2}^{K} \lambda_i^{-2} \mathbf{1}\{\lambda_i > 2^{-J/2}\} \geq \frac{l^2}{4}e^{-2\alpha}.
\]

Now

\[
\sqrt{\sum_{i=2}^{K} \lambda_i^{-2} \mathbf{1}\{\lambda_i > 2^{-J/2}\} + 2J \sum_{i=2}^{K} \mathbf{1}\{\lambda_i \leq 2^{-J/2}\}} \geq \min_l \sqrt{\frac{l^2}{4}e^{-2\alpha} + 2J(K - 1 - l)} \geq (K - 1)e^{-\alpha}/2.
\]

A.4.6 Conclusion

So if

\[
\rho \leq \left( 2^{3J/4n^{-1/2}} \left( \log \left[ 1 + 4(1 - 2\gamma - \beta)^2 \right] \right)^{1/4} \right) \wedge (2^{-J}s R) \wedge (2 \log(2^{J+1}/\delta))^{-1/2} e^{-\alpha}/2,
\]

then \( \inf_{\Delta, Q} \rho_n(\Delta, Q, B_{s, 2, \infty}(R), \beta) \geq \rho \).

Then, taking \( J \) as the smallest integer such that \( 2^J \geq c(\gamma, \beta, \delta, R) n^{2/(4s+3)} \), we obtain:

\[
\inf_{\Delta, Q} \rho_n(\Delta, Q, B_{s, 2, \infty}(R), \beta) \geq c(\gamma, \beta, R) (n^{-2s/(4s+3)} \wedge (\log n)^{-1/2}) e^{-\alpha}.
\]

And summing up the results over every \( K' \leq K \), we have

\[
\rho_n^*(B_{s, 2, \infty}(R, 2), \alpha, \gamma, \beta) \geq c(\gamma, \beta, R) (n^{-2s/(4s+3)} \wedge (\log n)^{-1/2}) e^{-\alpha}.
\]

So,

\[
\rho_n^*(B_{s, 2, \infty}(R, 2), \alpha, \gamma, \beta) \geq c(\gamma, \beta, R) (n^{-2s/(4s+3)} \wedge (\log n)^{-1/2}) e^{-\alpha}.
\]

Now, we also have

\[
\rho_n^*(B_{s, 2, \infty}(R, 2), +\infty, \gamma, \beta),
\]

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where $\rho^*_{n}(\mathcal{B}_{s,2,\infty}(R,2), +\infty, \gamma, \beta)$ corresponds to the case where there is no local differential privacy condition on $Q$. In particular, taking $Q$ such that $Z = X$ with probability 1 reduces the private problem to the classical testing problem. Now, the data processing inequality in Lemma A.3 justifies that such a $Q$ is optimal by contraction of the total variation distance. And the classical result leads to having $\rho^*_{n}(\mathcal{B}_{s,2,\infty}(R,2), +\infty, \gamma, \beta) = c(\gamma, \beta, R) n^{-2s/(4s+1)}$.

So, we have

$$\rho^*_{n}(\mathcal{B}_{s,2,\infty}(R,2), \alpha, \gamma, \beta) \geq c(\gamma, \beta, R) \left[\left\|n^{-2s/(4s+3)} \wedge (\log n)^{-1/2}\right\| e^{-\alpha} \vee n^{-2s/(4s+1)}\right].$$

**B Naive lower bound**

As explained in Remark 5.3, we provide a lower bound using the main result of [DJW13b], but the resulting rate turns out to be suboptimal.

**Theorem B.1.** Let $\gamma, \beta \in (0,1)$ such that $\gamma + \beta < 1$, let $\alpha > 0, R > 0, s > 0$. We obtain the following lower bound for the $\alpha$-private minimax separation rate defined by Equation (8) for non-interactive channels in $\mathcal{Q}_\alpha$ over the class of alternatives $\mathcal{B}_{s,2,\infty}(R,2)$

$$\rho^*_{n}(\mathcal{B}_{s,2,\infty}(R,2), \alpha, \gamma, \beta) \geq c(\gamma, \beta, R) \left(2^{-J_s} \wedge \frac{1}{(e^\alpha - 1)^{1/4}}\right).$$

The proof will remain concise since some arguments are also presented in the proofs of our main results.

**Proof.** Let us first define the setup similarly to Section A.4.2. Let $Q \in \mathcal{Q}_\alpha$ be a non-interactive $\alpha$-private channel. We assume that $f_0$ is the uniform density on $[0,1]$. We define the function $\psi \in L^2([0,1])$ by $\psi(x) = \mathbf{1}_{[0,\frac{1}{4})} - \mathbf{1}_{(\frac{1}{4},1]}$, and for some given $J \in \mathbb{N}$, that will be specified later, we define, for all $k \in \Lambda(J) = \{0,1,\ldots,2^J - 1\}$, $\psi_{J,i}(x) = 2^J \psi(2^J x - k)$. We denote by $V$ the linear subspace of $L^2([0,1])$ generated by the functions $(f_0, \psi_{J,i}, k \in \Lambda(J))$.

Let

$$f_\eta = f_0 + \rho 2^{-J_s} \sum_{i=1}^{2^J} \eta_i \psi_{J,i},$$

where $\psi_{J,i}$ for every $i$ have disjoint supports, $\int \psi_{J,i} = 0, \int \psi_{J,i}^2 = 1$ and $\|\psi_{J,i}\|_\infty = 2^{J/2}$.

It is possible to show that $f_\eta$ is a density if $\rho \leq 1$ and it is in the Besov set $\mathcal{B}_{s,2,\infty}(R,2)$ if $\rho \leq R^{2^{-J_s}}$.

Note that by orthonormality,

$$\|f_\eta - f_0\|_2^2 = \rho^2.$$

Denote $D_{KL}$ the Kullback-Leibler divergence. Consider Theorem 1 in [DJW13b], for any densities $f, g$ and $Q \in \mathcal{Q}_\alpha$:

$$D_{KL}(\mathbb{P}_Q, \mathbb{P}_Q) + D_{KL}(\mathbb{P}_Q, \mathbb{P}_Q) \leq 4(e^\alpha - 1)^2 \|f - g\|_TV^2. \quad (25)$$

We have

$$D_{KL}(\mathbb{P}_{Q_{f_0}}, \mathbb{P}_{Q_{f_0}}) \leq \frac{1}{2K-1} \sum_{n \in \{-1,1\}^{K-1}} D_{KL}(\mathbb{P}_{Q_{f_0}}, \mathbb{P}_{Q_{f_0}}). \quad (26)$$

And by application of the Kullback-Leibler divergence over products of distributions,

$$D_{KL}(\mathbb{P}_{Q_{f_0}}, \mathbb{P}_{Q_{f_0}}) = nD_{KL}(\mathbb{P}_{Q_{f_0}}, \mathbb{P}_{Q_{f_0}}).$$

So by application of Pinsker’s inequality on one side of the inequality in Equation (26) and using Equation (25) on the other side, this implies

$$2\|\mathbb{P}_{Q_{f_0}} - \mathbb{P}_{Q_{f_0}}\|_TV \leq 4(e^\alpha - 1)^2 \|f_0 - f_0\|_TV^2. \quad (25)$$

So

$$\|\mathbb{P}_{Q_{f_0}} - \mathbb{P}_{Q_{f_0}}\|_TV \leq \sqrt{2}(e^\alpha - 1)\sqrt{n}\|f_0 - f_0\|_TV.$$
And by application of Lemma A.3

\[ \|P_{f_0} - P_{f_\eta}\|_{TV} \leq \frac{1}{2} \left( \mathbb{E}_{f_0} \left[ L^2_{f_\eta}(X_1) - 1 \right] \right)^{1/2}. \]

Now

\[ \mathbb{E}_{f_0} \left[ L^2_{f_\eta}(X_1) \right] = 1 + 2\rho 2^{-J} \sum_{i=1}^{2^J} \eta_i \mathbb{E}_{f_0}(\psi_{J,i}) + \rho^2 2^{-J} \sum_{i=1}^{2^J} \eta_i^2 \mathbb{E}_{f_0}(\psi_{J,i})^2, \]

since \( \psi_{J,i} \) have disjoint supports.

So

\[ \mathbb{E}_{f_0} \left[ L^2_{f_\eta}(X_1) \right] = 1 + \rho^2. \]

Finally,

\[ \|P_{Q_{\nu}} - P_{Q_{\rho}}\|_{TV} \leq \sqrt{\frac{2}{2}(e^\alpha - 1)\sqrt{n}\rho}. \]

Remark B.2. Focusing on the following term from the naive lower bound on the minimax rate

\[ 1/\sqrt{n}, \]

we notice a gap with what we obtain using our proof:

\[ 2^{3/4}/\sqrt{n}. \]

The source of the gap is in the inequality presented in Equation (26). Indeed, on the left-hand side there is a distance describing testing with an alternative hypothesis composed of \( 2^J \) elements. Whereas on the right-hand side, we have the average distance corresponding to testing with only a simple alternative hypothesis. This inequality is nonetheless applied in order to obtain univariate distributions over which Theorem 1 from [DJW13b] is applicable.