A mean-field limit for the Vlasov-Poisson system

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Abstract
We present a probabilistic proof of the mean-field limit and propagation of chaos for an N-particle Coulomb system in 3-dimensional with potentials scaling like $\frac{1}{N}$ and N-dependent cut-off of order $N^{-\frac{1}{3}+\epsilon}$. In particular, we show convergence of the empirical distribution to solutions of the Vlasov-Poisson equation for typical initial conditions.

1 Introduction
We are interested in a microscopic derivation of the non-relativistic Vlasov-Poisson system. This equations describes a plasma of identical charged particles with Coulomb interactions

$$\partial_t f + p \cdot \nabla_q f + (k \ast \rho_t) \cdot \nabla_p f = 0,$$

where $k$ is the Coulomb kernel

$$k(q) := \frac{q}{|q|^3},$$

and

$$\rho_t(q) = \rho[f_t](q) = \int d^3p \, f(t, q, p)$$

is the charge-density induced by the distribution $f(t, p, q)$.

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Here, units are chosen such that all constants, in particular the mass and charge of the particles, are equal to 1. The distribution function \( f(t,q,p) \geq 0 \) describes the microscopic density of particles with position \( q \in \mathbb{R}^3 \) and momentum \( p \in \mathbb{R}^3 \).

Our discussion applies equally to the gravitational case, i.e. for the Vlasov-Newton system, which differs from the above equations only by the sign of \( k \).

Kinetic equations of the Vlasov-type are usually conceived as mean-field equations, that is, effective descriptions of large many-particle systems in which the N-particle interactions are well approximated by an “average effect” determining the time-evolution of \( f \).

In the literature, there are two different (but related) approaches to deriving kinetic equations as the mean-field limit of a particular microscopic model.

1. The first starts directly with the empirical density of an N-particle system. If for initial conditions \( X = (q_1,p_1; q_2,p_2; \ldots; q_N,p_N) \) the empirical distribution \( \mu^N[X] = \frac{1}{N} \sum_{i=1}^{N} \delta_{q_i} \delta_{p_i} \) is well approximated by a continuous density \( f_0(q,p) \), one wants to show that at \( t > 0 \), the time-evolved distribution \( \mu^N[X_t] = \mu^N[X_t] = \frac{1}{N} \sum_{i=1}^{N} \delta_{q_i(t)} \delta_{p_i(t)} \) is well approximated by \( f_t(p,q) \), where \( f_t \) is a solution of a Vlasov-equation of the form (1). “Well approximated” is understood in terms of the weak topology on the space of probability measures, metrized (for instance) by an appropriate Wasserstein distance.

Note that although in this problem we are formally concerned with probability measures, the (desired) result is actually deterministic.
The second point of view is concerned with random initial conditions, rather than deterministic ones and focuses on probability distributions on the $N$-particle phase space. The goal is still to prove convergence of the empirical density, however only for typical initial conditions. To this end, one usually aims at a result of the following type: If at $t = 0$ the particles are independently distributed according to the law $f_0$, i.e., if we consider an ensemble of systems with initial configurations $X$ distributed according to

$$F_N^0(X) = \prod_{i=1}^{N} f_0(q_i, p_i)$$

on $\mathbb{R}^{6N}$, then for $t > 0$, the particles are still “approximately independent” such that

$$F_N^t(X) = \Psi_t \# F_N^0[X] \approx \prod_{i=1}^{N} f_t(p_i, q_i),$$

where $\Psi_t(X)$ is the Hamiltonian flow generated by the microscopic equations of motion. Here, the approximation is understood in terms of the convergence of marginals, that is, writing $x_i = (q_i, p_i)$:

$$(k) F_t^N(x_1, ..., x_k) := \int F_t^N(X) \, dx_{k+1}...dx_N \rightarrow \prod_{i=1}^{k} f_t(x_i).$$

(4)

for $N \rightarrow \infty$. This is also known as molecular chaos or Kac’s chaos. By a well-known result of probability theory, molecular chaos is equivalent to the convergence in law of the empirical measures $\mu_t^N[X] = \mu^N[\Psi_t X]$ against the constant variable $f_t$, i.e. to

$$\lim_{N \rightarrow \infty} \mathbb{P}_0(\mu_t^N[X] \rightarrow f_t) = 1.$$ 

(5)

(E.g. Kac, 1956 [12], Grünbaum, 1971 [8], Sznitman, 1991, Prop.2.2 [21], see Mischler and Hauray, 2014 [17] for recent quantitative results).

Classical results, concerning simplified models with Lipschitz-continuous forces, are generally of the first (deterministic) type (Braun and Hepp, 1977 [3], Dobrushin, 1979 [4], Neunzert, 1984 [18]). However, generalization of this technique to more realistic systems with singular forces proved to be difficult even when suitable (N-dependent) cut-offs are employed. The understanding that has grown in recent years is that such deterministic results are in fact too strong in these cases, the reason being that there exist “bad” initial conditions leading to the clustering of particles and hence to significant deviations from the typical mean-field behavior.

The most ambitious mean-field results for singular potentials to date thus impose additional constraints on the initial configurations, subsequently showing that the set of “good” initial conditions (for which these constraints are satisfied) approaches measure 1 in the limit $N \rightarrow \infty$ (Hauray and Jabin, 2013 [10]). With this method, the authors prove propagation of chaos for singular forces behaving near zero like
\( \sim \frac{1}{|q|^\alpha} \) with \( \alpha < 2 \) (in \( d = 3 \) dimensions) and an N-dependent cut-off decreasing as \( N^{-\frac{1}{\alpha-1}}+\epsilon \).

In contrast, Kiessling, 2014 \[13\] proves a (non-quantitative) approximation result including the Coulomb singularity under the assumption of an additional (uniform in \( N \)) a priori bound on the microscopic forces, but wasn’t yet able to show that this bound is typical.

Recently, Boers and Pickl \[2\] proposed a novel method for deriving mean-field equations which is designed for stochastic initial conditions, thus aiming directly at a typicality result. In their paper, they too prove molecular chaos for singular potentials up to - but not including - the Coulomb case with a microscopic cut-off width of order \( N^{-1/3} \).

The aim of the present paper is to generalize this result to include the Coulomb singularity, thus providing a microscopic derivation of the Vlasov-Poisson system which (to our knowledge) has been an open problem, so far.

### 2 The microscopic model

For the microscopic model, we consider Coulomb pair interactions with an N-dependent cut-off. For \( N \in \mathbb{N} \) and \( \delta \geq 0 \), let

\[
k_N^\delta(q) := \begin{cases} \frac{q}{|q|^{\alpha}} , & \text{if } |q| \geq N^{-\delta} \\ qN^{3\delta} , & \text{else.} \end{cases}
\]

(6)

For \( N \to \infty \) and any \( \delta > 0 \) this converges point-wise to the Coulomb kernel on \( \mathbb{R}^3 \setminus \{0\} \), which justifies the notation \( k^\infty(q) := k(q) = \frac{q}{|q|^\alpha} \).

Moreover, we note that \( |k_N^\delta(q)| \leq N^{3\delta} \) and \( k_N^\delta(0) = 0 \).

In the so-called mean-field scaling, the equations of motion for the (regularized) N-particle system are thus given by

\[
\begin{aligned}
\dot{q}_i(t) &= p_i(t) \\
\dot{p}_i(t) &= \frac{1}{N} \sum_{j=1}^N k_N^\delta(q_j - q_i),
\end{aligned}
\]

(7)

for \( i \in 1, ..., N \). Since the vector field is globally Lipschitz for fixed \( \delta, N \), we have global existence and uniqueness of solutions and hence an N-particle Hamiltonian flow, which we denote by \( N\Psi_{t,s}(X) = (N\Psi_{t,s}^1(X), N\Psi_{t,s}^2(X)) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N} \). Introducing the N-particle force field \( K : \mathbb{R}^{3N} \to \mathbb{R}^{3N} \) given by \( (K(q_1, ..., q_N))_i := \frac{1}{N} \sum_{j=1}^N k_N^\delta(q_j - q_i) \),

we can also characterize \( \Psi_{t,s} \) as the solution of \( \frac{d}{dt}(\Psi_{t,s}^1(X), \Psi_{t,s}^2(X)) = K(\Psi_{t,s}^1(X), \Psi_{t,s}^2(X)) \)
\((\Psi_{t,s}^2(X), K(\Psi_{t,s}^1(X)))\) with \(\Psi_{s,s}(X) = X\).

For any \(\delta > 0\) and \(N \in \mathbb{N} \cup \{\infty\}\), we also consider the corresponding mean-field equation

\[
\partial_t f + p \cdot \nabla_q f + \left(k_N^\delta \ast \int d^3 p f\right) \cdot \nabla_p f = 0. \tag{8}
\]

For (formally) \(N = \infty\), this reduces to the Vlasov-Poisson equation (1).

### 2.1 Method of Characteristics

It is useful to consider the characteristics of the system, exploiting the fact that Vlasov-equations are transport equations, in which an initial distribution \(f_0\) is transported along an effective one-particle flow. For \(N \in \mathbb{N}, \delta > 0\) and \(\rho \in L^1(\mathbb{R}^3)\), we define \(\hat{K}_N^\delta(q,p;\rho) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3\) by

\[
\hat{K}_N^\delta(q,p;\rho) := (p, k_N^\delta \ast \rho(q)) \tag{9}
\]

Then, the (regularized) Vlasov-Poisson equation (8) with initial \(f_0\) is equivalent to the following system of integro-differential equations:

\[
\begin{align*}
\frac{d}{dt} \varphi_{t,s}^N(z; f_0) &= \hat{K}_N^\delta(\varphi_{t,s}^N(z; f_0); \rho_t) \\
\rho_t(q) &= \int d^3 p f(t,q,p) \\
f(t,\cdot) &= \varphi_{t,s}^N(\cdot; f_0) \# f_s \\
\varphi_{s,s}^N(z; f_0) &= z
\end{align*} \tag{10}
\]

Here, \(\varphi(\cdot) \# f\) denotes the image-measure of \(f\) under \(\varphi\), defined by \(\varphi \# f(A) = f(\varphi^{-1}(A))\) for any Borel set \(A \subseteq \mathbb{R}^6\).

In other words, we have a non-linear dynamics in which \(\varphi_{t,s}^N(\cdot; f_0)\) is the one-particle flow induced by the mean-field dynamics with initial distribution \(f_0\), while, in turn, \(f_0\) is transported with the flow \(\varphi_{t,s}^N = \varphi_{t,s}^N\). Due to the semi-group property \(\varphi_{t,s}^N \circ \varphi_{s',s}^N = \varphi_{t,s}^N\) it usually suffices to consider the initial time \(s = 0\).

In the following, we shall also require the lift of \(\varphi_{t,s}^N(\cdot)\) to the \(N\)-particle phase-space, which we denote by \(N \Phi_{t,s}\). That is, for \(f_0 \in L^1(\mathbb{R}^6)\) and \(X = (Q,P) = (q_1, q_2, ..., q_N; p_1, p_2, ..., p_N)\), we define

\[
N \Phi_{t,s}(X; f_0) := (\varphi_{t,s}^N(q_1, p_1; f_0), ..., \varphi_{t,s}^N(q_N, p_N; f_0)) \tag{11}
\]

(modulo rearrangement of the position and momentum variables). We shall often omit the index \(N\) and the initial datum \(f_0\), unless necessary.
As before, $\Phi_1^t(X)$ and $\Phi_2^t(X)$ will denote the projection on the position and momentum variables, respectively. By $\mathcal{K} : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ we denote the lift of the mean-field force to the $N$-particle phase-space, i.e. $(\mathcal{K}(Z))_i := k_N^i \ast \rho_t^N(z_i)$, for $Z = (z_1, ..., z_N)$.

### 2.2 Existence of Solutions for Vlasov-Poisson

For the regularized Vlasov-Poisson equation \cite{[5]}, all forces are Lipschitz and the solution theory is fairly standard \cite{[5],[6]}. In the Coulomb case, the issue is more subtle. Fortunately, we can rely on various results establishing global existence and uniqueness of (strong) solutions under fairly mild conditions on the initial configuration $f_0$ (Pfaefelmoser, 1990 \cite{[19]}, Schaeffer, 1991 \cite{[20]}, Lions and Perthame, 1991 \cite{[15]}, Horst, 1993 \cite{[11]}). For our purposes, the following existence result due to Lions and Perthame will prove to be very useful:

**Theorem 2.1** (Lions and Perthame).

Let $f_0 \geq 0$, $f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ satisfy

$$
\int |p|^m f_0(q,p) \, dq \, dp < +\infty, \tag{12}
$$

for all $m < m_0$ and some $m_0 > 3$.

a) Then, the Vlasov-Poisson system defined by equations (1–3) has a continuous, bounded solution $f(t, \cdot, \cdot) \in C(\mathbb{R}^+; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ for $1 \leq p < \infty$ satisfying

$$
\sup_{t \in [0,T]} \int |p|^m f(t,q,p) \, dq \, dp < +\infty, \tag{13}
$$

for all $T < \infty, m < m_0$.

b) If, in fact, $m_0 > 6$ and we assume that $f_0$ satisfies

$$
\text{supess}\{f_0(q' + pt,p') : |q - q'| \leq Rt^2, |p - p'| < Rt\} \\
\in L^\infty((0,T) \times \mathbb{R}_q^3; L^1(\mathbb{R}_p^3)) \tag{14}
$$

for all $R > 0$ and $T > 0$, then

$$
\sup_{t \in [0,T]} \|\rho_t(q)\|_\infty < +\infty, \ \forall T \in (0, +\infty). \tag{15}
$$

Given a bounded charge-density, a very strong uniqueness result was obtained by Loeper, 2006 \cite{[16]}.
The result of Lions and Perthame actually yields bounds that are uniform in $N$ as we consider the sequence of regularized Vlasov-Poisson equations (8) with $N$-dependent cut-off. This is expressed in the following Lemma:

**Lemma 2.2.** Let $f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and $f^N_t$ be the solution of the regularized Vlasov-Poisson equation (8) (with corresponding cut-off) and initial datum $f^N(t,\cdot,\cdot) = f_0$. If $f_0$ satisfies assumption (14) of the theorem above, there exists a $C_\rho > 0$ such that

$$\|\rho^N_t\|_\infty < C_\rho, \forall N \in \mathbb{N} \cup \{\infty\}, \forall t > 0,$$

where formally $\rho^\infty_t = \rho[f_t]$.

**Proof.** By following the proof in [15], one can readily see that the bound for the propagation of moments of $f_0$ is uniform in $N$, if the transport equation is regularized as (e.g.) in (8). But the control on $\sup_{t \in [0,T]} \int |p|^m f(t,q,p)$ is all that is required for part b) of Thm. 2.1, so that the bound on $\|\rho_t\|_\infty$ obtained from (14) is actually an upper bound for the sequence $(\rho^N_t)_{N \in \mathbb{N}}$ corresponding to the regularized equations.

Since Assumption (14) is rather formal, we want to state a more intuitive sufficient condition.

**Lemma 2.3.** Let $f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, $f \geq 0$. Suppose there exist a $S > 0$ s.t. for all $|p| > S$:

$$f_0(q,p) = \rho(q)\vartheta(|p|),$$

with $\rho \in L^\infty(\mathbb{R}^3)$ and $\vartheta(|p|) \in L^1(\mathbb{R}^3)$ monotonously decreasing. Then $f_0$ satisfies assumption (14). Special cases:

- $f_0$ has compact support in the $p$-variables.
- $f_0$ is a thermal state of the form $\rho(q) e^{-\beta p^2}$ with $\|\rho\|_\infty < \infty, \beta > 0$.

**Proof.** For given $R, t > 0$ we have to consider the function

$$\tilde{f}(t,q,p) := \text{supess}\{f_0(q'+pt,p') : |q - q'| \leq R t^2, |p - p'| < R t\}.$$

Choosing $R' > S + RT$, we have

$$\int_{\mathbb{R}^3} \tilde{f}(t,q,p) d^3p = \int_{|p| \leq R'} + \int_{|p| > R'} \tilde{f}(t,q,p) d^3p$$

$$\leq \frac{4}{3}\pi R'^3 \|\tilde{f}(t,\cdot,\cdot)\|_\infty + \|\rho\|_\infty \int_{|p-p'| < R t} \vartheta(|p'|) d^3p$$

$$\leq \frac{4}{3}\pi R'^3 \|f_0\|_\infty + \|\rho\|_\infty \int \vartheta(|p| - R t) d^3p$$

$$\leq C \|f_0\|_\infty + \|\rho\|_\infty \|\vartheta\|_1 < \infty,$$
where in the second to last line we used the monotonicity of $\vartheta(|p|)$ and the fact that $\|\hat{f}\|_\infty = \|f_0\|_\infty$.

One simple consequence of the bounded density is that the mean-field force remains bounded, as well.

**Lemma 2.4.** Let $k$ be the Coulomb kernel, and $\rho \in L^1(\mathbb{R}^3; \mathbb{R}^+)$. Then:

$$\|k * \rho\|_\infty \leq 4\pi\|\rho\|^{2/3}_\infty.$$  \hspace{1cm} (17)

**Proof.** For $R > 0$, we compute:

$$\|k * \rho\|_\infty \leq \left\| \int_{|y|<R} k(y)\rho(x-y) \, d^3y \right\|_\infty + \left\| \int_{|y|>R} k(y)\rho(x-y) \, d^3y \right\|_\infty \leq \|\rho\|_\infty \int_{|y|<R} \frac{1}{|y|^2} \, d^3y + R^{-2}\|\rho\|_1 = 4\pi R\|\rho\|_\infty + R^{-2}.$$  

This is optimized by setting $R = \|\rho\|^{-1/3}_\infty$, yielding $\|k * \rho\|_\infty \leq 4\pi\|\rho\|^{2/3}_\infty$. \hspace{1cm} \(\Box\)

### 3 Statement of the Results

In the following, all probabilities and expectation values are meant with respect to the product measure given at a certain time. That is, for any random variable $H : \mathbb{R}^{6N} \to \mathbb{R}$ and any element $A$ of the Borel-algebra

$$\mathbb{P}_t(H \in A) = \int_{H^{-1}(A)} \prod_{j=1}^N f_t^N(x_j) \, dX$$

$$\mathbb{E}_t(H) = \int_{\mathbb{R}^{6N}} H(X) \prod_{j=1}^N f_t^N(x_j) \, dX.$$  \hspace{1cm} (18)

Note that since $^N\Phi_{t,s}$ leaves the measure invariant,

$$\mathbb{E}_s(H \circ \Phi_{t,s}) = \int_{\mathbb{R}^{6N}} H(\Phi_{t,s}^N(X)) \prod_{j=1}^N f_s^N(x_j) \, dX$$

$$= \int_{\mathbb{R}^{6N}} H(X) \prod_{j=1}^N f_s^N(\varphi_{s,t}^N(x_j)) \, dX$$

$$= \int_{\mathbb{R}^{6N}} H(X) \prod_{j=1}^N f_t^N(x_j) \, dX = \mathbb{E}_t(H).$$
In particular:
\[ \mathbb{P}_t(X \in A) = \mathbb{P}_0(\Phi_t,0(X) \in A). \]

To metrize convergence of probability measures, we introduce the Wasserstein distances, that are now extensively studied in the field of optimal transportation and have been introduced to the theory of kinetic equations by Dobrushin [6]. For details and proofs, we refer the reader to [22, Section 6].

**Definition 3.1.** Let \( \mathcal{P}(\mathbb{R}^n) \) be the set of probability measures on \( \mathbb{R}^n \). For given \( \mu, \nu \in \mathcal{P}(\mathbb{R}^n) \), let \( \Pi(\mu, \nu) \) be the set of all probability measures \( \mathbb{R}^n \times \mathbb{R}^n \) with marginal \( \mu \) and \( \nu \), respectively. Then, for \( p \in [1, +\infty) \), the \( p \)'th Wasserstein distance on \( \mathcal{P}(\mathbb{R}^n) \) is defined by

\[
W_p(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^p \, d\pi(x, y) \right)^{1/p}.
\] (20)

Convergence in \( W_p \) implies convergence in the weak topology of \( \mathcal{P}(\mathbb{R}^n) \), as well as convergence of the first \( p \) moments. A particularly useful case is the first Wasserstein distance, for which we have the Kantorovich-Rubinstein duality:

\[
W_1(\mu, \nu) = \sup_{\|g\|_{Lip} \leq 1} \left\{ \int g(x) \, d\mu(x) - \int g(x) \, d\nu(x) \right\},
\] (21)

where \( \|g\|_{Lip} := \sup_{x,y} \frac{|g(x) - g(y)|}{|x - y|} \), for \( g : \mathbb{R}^n \to \mathbb{R} \).

**Theorem 3.2.**

Let \( f_0 \geq 0, f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \) satisfy the assumptions of Theorem 2.1 a) and b) and let \( f_t \) be the unique solution of the Vlasov-Poisson equation with \( f(0, p, q) = f_0(p, q) \). Assume in addition that there exists a \( m > 2 \) such that \( \int |q|^m f_0(q, p) \, dq \, dp < +\infty \).

For \( N \in \mathbb{N} \) and \( \delta > 0 \) let \( N \Psi_{t,s} \) be the \( N \)-particle flow corresponding to the regularized Coulomb-system (7) with cut-off width \( N^{-\delta} \). Then, the empirical density \( \mu_t^N[X] = \rho_0^N[\Psi_t,0(X)] \) converges to the solution of the Vlasov-Poisson equation in the following sense:

For \( \delta < \frac{1}{3}, \lambda < \min\{\frac{1}{4}, \delta\} \), and \( T > 0 \) there exist constants \( C_0(f_0), C_1(\lambda, m) \), and an \( N_0 \in \mathbb{N} \) such that

\[
\mathbb{P}_0 \left( \exists t \in [0, T] : W_1(\mu_t^N[X], f_t) > N^{-\lambda} e^{tC_0\sqrt{\ln N}} \right) \\
\leq 2 \left( \frac{C_1 N^{-1+2\lambda} + N^{-1+3\delta} e^{T C_0 \sqrt{\ln N}}}{N} \right)
\] (22)

for all \( N \geq N_0 \).
Remark 3.3.

1. Note that since $\exp\left[\sqrt{\ln(N)}\right] = \exp\left[\ln(N)/\sqrt{\ln(N)}\right] = N^{\sqrt{\ln N}}$, we have $e^{\sqrt{\ln N}} = o(N^{-\epsilon})$ for arbitrary small $\epsilon > 0$, so that the right-hand-side of (22) is actually of order $o(N^{-1+3\delta+\epsilon})$.

2. The restriction to $N \geq N_0$ is only for convenience. Bounds valid for all $N \geq 1$ can be read off of the proof in section 7.1.

3. Without the additional assumption of spatial moments, molecular chaos still holds, albeit without the bounds in Wasserstein-distance stated in the theorem (see Appendix A1).

4. Our result allows us to choose the order of the cut-off width arbitrary close to $N^{-1/3}$, which is physically distinguished as the scale of the mean inter-particle distance.

The strategy of the proof, following Boers and Pickl [2], is to control the deviation of the microscopic time-evolution from the mean-field time evolution in terms of the following $N$-dependent quantity:

**Definition 3.4.**

Let $J(t)$ be the stochastic process given by

$$J_t^N(X) := \min\left\{1, \sqrt{\ln(N)} A^6 \sup_{0 \leq s \leq t} |N \Psi_s^1(X) - N \Phi_s^1(X)|_\infty + N^6 \sup_{0 \leq s \leq t} |N \Psi_s^2(X) - N \Phi_s^2(X)|_\infty\right\}. \quad (23)$$

Here, $|X|_\infty$ denotes the maximum norm on $\mathbb{R}^{6N}$. Note that the small but crucial innovation with respect to [2] is that distances in spatial coordinates and momentum coordinates are weighted differently, exploiting the second-order nature of the dynamics.

The relevance of $J(t)$ for part 1) of the theorem is obvious from $P_0\left(\sup_{0 \leq s \leq t} |N \Psi_s - N \Phi_s|_\infty \geq N^{-\delta}\right) \leq P_0(J_t^N(X) = 1) \leq E_0(J_t^N)$, while the relevance for part 2) is grounded in the following observation:

**Proposition 3.5.** For all $\epsilon \geq 2N^{-\delta}$ and $T > 0$ we have

$$P_0\left(\exists t \in [0, T] : W_1(\mu_t^N, f_t^N) > \epsilon\right) \leq E_0(J_T) + P_0\left(\sup_{0 \leq t \leq T} W_1(\varphi_t^N \# \mu_0^N, f_t^N) > \epsilon/2\right), \quad (24)$$

where $\varphi_t^N$ is the characteristic flow of $f_t^N$ defined in (10).
Note that the second term on the r.h.s. of (21) depends only on the mean-field dynamics, while fluctuations of the microscopic dynamics are controlled by $\mathbb{E}_0(J_t)$. We will provide a bound for that second term in Section 6.

**Proof.** Let $\mathcal{L} := \{ g \in L^1(\mathbb{R}^6) : \|g\|_{Lip} = 1 \}$ and consider

$$
P_0 \left( \exists t \in [0,T] : \sup_{g \in \mathcal{L}} \left| \int g \, d\mu_t^N - \int g f_t^N \right| > \epsilon \right)
$$

$$= P_0 \left( \exists t \in [0,T] : \sup_{g \in \mathcal{L}} \frac{1}{N} \sum_{i=1}^N g(\Psi_t(X)_i) - \int g f_t^N \right) > \epsilon$$

$$\leq P_0 \left( \exists t \in [0,T], g \in \mathcal{L} : \frac{1}{N} \sum_{i=1}^N g(\Psi_t(X)_i) - \sum_{i=1}^N g(\Phi_t(X)_i) \right) > \epsilon/2 \right) \tag{25}
$$

$$+ P_0 \left( \exists t \in [0,T], g \in \mathcal{L} : \left| \frac{1}{N} \sum_{i=1}^N g(\Phi_t(X)_i) - \int g f_t^N \right| > \epsilon/2 \right) \tag{26}
$$

Concerning (25), we recall that $J_T(X) < 1 \Rightarrow \sup_{0 \leq t \leq T} |\Psi_t(X) - \Phi_t(X)|_\infty < N^{-\delta}$ and hence, using $\|g\|_{Lip} = 1$, we see that for $\epsilon \geq 2N^{-\delta}$:

$$\leq P_0 \left( \exists t \in [0,T], g \in \mathcal{L} : \left| \frac{1}{N} \sum_{i=1}^N g(\Psi_t(X)_i) - \Phi_t(X)_i \right| > \epsilon/2 \right) \tag{27}
$$

$$\leq P_0 \left( \exists t \in [0,T] : |\Psi_t(X) - \Phi_t(X)|_\infty > \epsilon/2 \right) \tag{28}
$$

$$\leq P_0 (J_T = 1) \leq \mathbb{E}_0(J_T) \tag{29}
$$

For the second term, observe that $\frac{1}{N} \sum_{i=1}^N g(\Phi_t(X)_i) = \int g(x) \mu_t^N \circ \varphi_{0,t}^N \, d\mathbb{P} x$ and hence

$$\left| \frac{1}{N} \sum_{i=1}^N g(\Phi_t(X)_i) - \int g(x) f_t^N(x) \right| = \left| \int g(x) \mu_t^N (\varphi_{0,t}^N(x)) - \int g(x) f_t (\varphi_{0,t}^N(x)) \right| \leq W_1(\varphi_t^N \# \mu_t^N, \varphi_t^N \# f_0).$$

This means that

$$P_0 \left( \exists t \in [0,T], g \in \mathcal{L} : \left| \frac{1}{N} \sum_{i=1}^N g(\Phi_t(X)_i) - \int g f_t \right| > \epsilon/2 \right)
$$

$$= P_0 \left( \sup_{0 \leq t \leq T} W_1(\varphi_t^N \# \mu_t^N, \varphi_t^N \# f_0) > \epsilon/2 \right).$$

$\Box$
4 Local Lipschitz Bound

If all forces were Lipschitz continuous with a Lipschitz constant that remains bounded uniformly in $N$, the difference between the “true” microscopic time-evolution and the mean-field time-evolution would be straightforwardly controlled by $|\Psi_t(X) - \Phi_t(X)|_\infty$. The desired convergence for $\mathbb{E}_0(J_t)$ would then immediately follow by a simple application of Gronwall’s Lemma.

However, the forces we consider here become singular in the limit $N \rightarrow \infty$ and hence do not satisfy a uniform Lipschitz bound. Nevertheless, we observe that for the mean-field force $k^N \ast \rho_t$, the global Lipschitz constant $\|k^N \ast \rho_t\|_{\text{Lip}}$ diverges only logarithmically as the cutoff is lifted with increasing $N$. Together with the particular scaling of $J^N$ - “trading” part of this divergence for a tighter control on spatial fluctuations - this will suffice to establish the desired convergence of $\mathbb{E}_0(J_t)$ by virtue of $\mathbb{E}_0(J_t + \Delta t) \sim \sqrt{\ln(N)} \mathbb{E}_0(J_t) \Delta t + o(\Delta t)$.

Our first goal is to show that for typical initial conditions, the fluctuations in the microscopic forces can be bound in a similar fashion, as long as $\Psi_t(X)$ and $\Phi_t(X)$ are close. Hence, following [2], we introduce a function controlling the difference $|k(q) - k(q + \xi)|$, for $|\xi| < 2N^{-\delta}$.

**Definition 4.1.** Let

$$l^N_\delta(q) := \begin{cases} \frac{54}{|q|^\delta} N^{3\delta}, & \text{if } |q| \geq 3N^{-\delta} \\ N^{3\delta}, & \text{else} \end{cases}$$

(30)

and $L : \mathbb{R}^{6N} \rightarrow \mathbb{R}^N$ be defined by $(L(X))_i := \frac{1}{N} \sum_{j \neq i} l(q_j - q_i)$. Furthermore, for a given density $f_t$, we define $\overline{L}_t(X)$ by $(\overline{L}_t(X))_i := l \ast \rho_t(q_i)$.

**Lemma 4.2.** For any $\xi \in \mathbb{R}^3$ with $|\xi|_\infty < 2N^{-\delta}$, we have

$$|k^N_\delta(q) - k^N_\delta(q + \xi)|_\infty \leq l^N_\delta(q)|\xi|_\infty$$

(31)

**Proof:** First, note that the derivative of $k^N$ is bounded by $N^{3\delta}$, so that (31) holds for $|q| < 3N^{-\delta}$. For $|q| \geq 3N^{-\delta}$, the difference between $k^N(q)$ and $k^N(q + \xi)$ is largest if $\xi$ is antiparallel to $q$. Between $q$ and $q + \xi$, the derivative of $k$ takes its greatest value at the point closest to the center. Hence, we have

$$\frac{|k^N_\delta(q) - k^N_\delta(q + \xi)|_\infty}{|\xi|_\infty} \leq \left| \frac{d}{dr} r^{-2} \right|_{|q| - |\xi|} = 2(|q| - |\xi|)^{-3}.$$  

(32)

Since $|q| \geq 3N^{-\delta}$ and $|\xi| < 2N^{-\delta}$, it follows that $|q| - |\xi| \geq \frac{1}{3}|q|$. Hence, as claimed, $|k(q) - k(q + \xi)|_\infty |\xi|_\infty \leq 2\left( \frac{3}{|q|} \right)^3 |\xi|_\infty \leq \frac{54}{|q|^\delta} |\xi|_\infty \quad \Box$
5 The law of large numbers

Our proof of the mean-field limit is based on controlling the difference between the mean-field dynamics and the “true” microscopic dynamics. As an intermediate step, we require that the mean-field force is a good approximation to the microscopic forces for configurations $N\Phi_t(X)$ which evolve according to the mean-field dynamics. The key observation here is that if an $N$-particle configuration evolves according to the mean-field flow $N\Phi_t$, the individual particles remain statistically independent for all $t$, thus giving rise to a law-of-large-numbers result.

Definition 5.1.
For any $t > 0$ and fixed $\delta < \frac{1}{3}$, we consider the following (in fact time-dependent) sets $A, B, C$ defined by

\[
X \in A \iff |J_t(X)| < 1
\]

\[
X \in B \iff |K(\Phi_{t,0}(X)) - \overline{K}(\Phi_{t,0}(X))|_\infty < N^{-1+2\delta}
\]

\[
X \in C \iff |L(\Phi_{t,0}(X)) - \overline{L}(\Phi_{t,0}(X))|_\infty < 1
\]

We want to show that (for any $t$), initial conditions in $B \cap C$ are typical with respect to the product measure $F_0(X) := \prod_{j=1}^N f_0(x_j)$ on $\mathbb{R}^{6N}$.

Departing from the rest of our presentation, we will formulate the next result for the general case, allowing direct adaptation to singularities milder than the Coulomb case.

Proposition 5.2. Let $\rho_t \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ as before. Let $h : \mathbb{R}^3 \to \mathbb{R}$ and suppose that for given $\delta > 0$ and $N \in \mathbb{N}$ there exists $c > 0$ and an exponent $\kappa > 0$ such that $|h(x)| \leq c \min\{N^{\kappa\delta}, |x|^{-\kappa}\}$, $\forall x \in \mathbb{R}^3$.

Assume furthermore that

\[
\delta < \min\left\{\frac{1 - 2\beta}{2\kappa - 3}, \frac{1 - \beta}{\kappa}\right\}.
\]  

Then, for all $\gamma > 0$ there exists a $C_\gamma > 0$ such that

\[
\mathbb{P}_t\left(\sup_{1 \leq i \leq N} \frac{1}{N} \sum_{j \neq i} h(q_j - q_i) - h * \rho_t(q_i) \geq N^{-\beta}\right) \leq \frac{C_\gamma}{N^\gamma}.
\]  

Proof. Let

\[
D_i := \left\{ X \in \mathbb{R}^6 : \left| \frac{1}{N} \sum_{j \neq i} h(q_j - q_i) - h * \rho_t(q_i) \right| \geq N^{-\beta} \right\}
\]
and $D := \bigcup_{i=1}^{N} D_i$. Then $\mathbb{P}(D) \leq \sum_{i=1}^{N} \mathbb{P}(D_i) = N\mathbb{P}(D_1)$.

By Markov’s inequality, we have for every $M \in \mathbb{N}$:

$$\mathbb{P}_i(D_1) \leq \mathbb{E}_i \left( \frac{1}{N} \sum_{j=1}^{N} |h(q_j - q_1) - h \ast \rho_\ell(q_1)|^{2M} \right)$$

Let $M := \{ \alpha \in \mathbb{N}_0^N \mid |\alpha| = 2M \}$ the set of multiindices $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ with $\sum_{k=1}^{N} \alpha_k = 2M$. Let

$$G^\alpha := \prod_{j=1}^{N} (h(q_j - q_1) - h \ast \rho_\ell(q_1))^\alpha_j.$$ 

Then:

$$\mathbb{E}_i \left[ \left( \sum_{j=1}^{N} (h(q_j - q_1) - h \ast \rho_\ell(q_1)) \right)^{2M} \right] = \sum_{\alpha \in M} \binom{2M}{\alpha} \mathbb{E}_i(G^\alpha)$$

Now we note that $\mathbb{E}_i(G^\alpha) = 0$ whenever there exists a $1 \leq j \leq N$ such that $\alpha_j = 1$. This can be seen by integrating the $j$th variable first.

For the remaining terms, we first note that for any $1 \leq m \leq M$:

$$\int |h(q_i - q_j)|^m f_i(q_j, p_j) \, d^3p_j \, d^3q_j = \int |h|^m(q_i - q_j) \rho_\ell(q_j) \, d^3q_j$$

Now, for $\kappa m \leq 2$, we estimate

$$\int |h|^m(q_i - q_j) \rho_\ell(q_j) \, d^3q_j \leq c \int_{|q_i|<1} |q_i|^{-\kappa m} \rho_\ell(q_i - q_j) \, d^3q_j + c \int_{|q_i|\geq1} |q_i|^{-\kappa m} \rho_\ell(q_i - q_j) \, d^3q_j$$

$$\leq c(4\pi\rho_\ell_\infty + \|\rho_\ell\|_1)$$

While for $\kappa m > 2$, we find

$$\int |h|^m(q_i - q_j) \rho_\ell(q_j) \, d^3q_j = \int |h|^m(q_j) \rho_\ell(q_i - q_j) \, d^3q_j$$

$$\leq \int_{|q_j|<N^{-\delta}} |h|^m(q_j) \rho_\ell(q_i - q_j) \, d^3q_j + \int_{|q_j|\geq N^{-\delta}} |h|^m(q_j) \rho_\ell(q_i - q_j) \, d^3q_j$$

$$\leq c\|\rho_\ell\|_\infty \left(4\pi N^{-3\delta} N^{\kappa m} + \int_{|q_j|\geq N^{-\delta}} \frac{1}{|y|^m} \, d^3q_j \right) \leq 8\pi c\|\rho_\ell\|_\infty N^{(\kappa m - 3)\delta}$$

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Note that one of these two estimates also provides the bound for \( \|h * \rho_t\|_\infty \), which corresponds to the case \( m = 1 \). In total, we can conclude that there exists a constant \( C \) depending on \( \|\rho_t\|_\infty \), such that for all \( m \geq 2 \):

\[
|h(q_j - q_i) - h(q_i)|^m \leq C^m \max\{N^{(\kappa m - 3)\delta}, 1\}. \tag{39}
\]

Now, for \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \in \mathcal{M} \), let \( \#\alpha \) denote the number of \( \alpha_j \) with \( \kappa \alpha_j \geq 3 \). Note that if \( \#\alpha > M \), we must have \( \alpha_j = 1 \) for at least one \( 1 \leq j \leq N \), so that \( \mathbb{E}_t(G^\alpha) = 0 \). For the other multiindices, we get (using that the particles are statistically independent):

\[
\mathbb{E}_t(G^\alpha) = \mathbb{E}_t \left[ \prod_{j=1}^N (k_\delta(q_j - q_i) - k * \rho_t(q_i))^{\alpha_j} \right] \\
\leq \prod_{j=1}^N \mathbb{E}_t \left[ (|h(q_j - q_i)| + |h * \rho_t(q_i)|)^{\alpha_j} \right] \\
\leq \prod_{j=1}^N C^{\alpha_j} \max\{N^{(\kappa \alpha_j - 3)\delta}, 1\} \\
\leq C^{2M} N^{2M\kappa\delta} N^{-3\delta \#\alpha}. \tag{40}
\]

Finally, we observe that for any \( k \geq 1 \), the number of multiindices \( \alpha \in \mathcal{M} \) with \( \#\alpha = k \) is certainly bounded by

\[
\sum_{\#\alpha = k} 1 \leq \binom{N}{k} (2M)^k \leq (2M)^k N^k
\]

Thus:

\[
\mathbb{P}_t(D_1) \leq \frac{1}{N^{2M(1-\beta)}} \sum_{\alpha \in \mathcal{M}} \binom{2M}{\alpha} \mathbb{E}_t(G^\alpha) \\
\leq C_M \frac{N^{2M\kappa\delta}}{N^{2M(1-\beta)}} \sum_{k=1}^M N^{(1-3\delta)k} \\
\leq M C_M N^{2M(\kappa + \beta - 1)} \max\{N^{M(1-3\delta)}, 1\} \\
\leq M C_M N^{-\epsilon M}
\]

where \( C_M \) is a constant depending on \( M \) and

\[
\epsilon := \begin{cases} 
1 - 2\beta + \delta(3 - 2\kappa) & \text{if } 3\delta < 1 \\
2(1 - \beta - \kappa\delta) & \text{if } 3\delta \geq 1.
\end{cases} \tag{41}
\]

\( \epsilon \geq 0 \) according to (33). We conclude the proof by noting that

\[
\mathbb{P}_t(D) \leq \mathbb{P}_t(D_1) \leq C'_M N^{1-\epsilon M}, \tag{42}
\]

which goes arbitrarily fast to 0 as \( N \to \infty \) for sufficiently large \( M \). \( \square \)
Corollary 5.3. Applying the previous result once for \(k^N_\delta\) with \(\kappa = 2\) and \(\beta = 1 - 2\delta\) and once for \(l^N_\delta\) with \(\kappa = 3\) and \(\beta = 0\), we get for any \(\gamma > 0\) a \(C_\gamma < \infty\) such that

\[
P_t\left( \sup_{1 \leq i \leq N} \left| \frac{1}{N} \sum_{j \neq i} k^N_\delta(q_j - q_i) - k^N_\delta * \rho_t(q_i) \right| \geq N^{-1 + 2\delta} \right) \leq \frac{C_\gamma}{N^\gamma}, \quad (43)
\]

\[
P_t\left( \sup_{1 \leq i \leq N} \left| \frac{1}{N} \sum_{j \neq i} l^N_\delta(q_j - q_i) - l^N_\delta * \rho_t(q_i) \right| \geq 1 \right) \leq \frac{C_\gamma}{N^\gamma}, \quad (44)
\]

for all \(N \in \mathbb{N}\). Observing that \(P_0(B) = P_t(\Phi_t(B))\) and \(P_0(C) = P_t(\Phi_t(C))\), this implies

\[
P_0(B) \geq 1 - \frac{C_\gamma}{N^\gamma},
\]

\[
P_0(C) \geq 1 - \frac{C_\gamma}{N^\gamma}.
\]

In other words, for any fixed \(t\), initial conditions in \(B_t \cap C_t\) are typical with the measure of “bad” initial conditions decreasing exponentially fast as \(N \to \infty\).

6 Gronwall-estimate

We conclude our proof of the second part of the Theorem by establishing a Gronwall-type estimate for \(E_0(J_t)\).

Lemma 6.1. There exists a constant \(C_1 > 0\) such that for any \(N > 1\):

\[
\|l^N_\delta * \rho_t(x)\|_\infty \leq C_1 \ln(N) \left( \|\rho_t\|_1 + \|\rho_t\|_\infty \right) \quad (45)
\]

Since \(|\nabla k^N_\delta(q)| < l^N_\delta(q)|\), this also implies

\[
\|\nabla k^N_\delta * \rho_t(x)\|_\infty \leq C_1 \ln(N) \left( \|\rho_t\|_1 + \|\rho_t\|_\infty \right). \quad (46)
\]

Proof:

\[
\|l * \rho_t(x)\|_\infty = \| \int l(x - y) \rho_t(y) \, d^3y \|_\infty
\]

\[
\leq \left\| \int_{|x - y| < N^{-\delta}} l(x - y) \rho_t(y) \, d^3y \right\|_\infty + \left\| \int_{N^{-\delta} < |x - y| < 1} l(x - y) \rho_t(y) \, d^3y \right\|_\infty
\]

\[
+ \left\| \int_{|x - y| > 1} l(x - y) \rho_t(y) \, d^3y \right\|_\infty
\]

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The first term can be estimated by

$$\left\| \int_{|x-y|<N^{-\delta}} t(x-y)\rho_t(y) \, d^3 y \right\|_\infty \leq \|\rho_t\|_\infty N^{-3\delta} |B(N^{-\delta})| \leq \frac{4}{3} \pi \|\rho_t\|_\infty$$

The last term is bounded by

$$\left\| \int_{|x-y|>1} t(x-y)\rho_t(y) \, d^3 y \right\|_\infty \leq 54 \|\rho_t\|_1$$

Finally, the second term yields

$$\left\| \int_{N^{-\delta} <|x-y|<1} g(x-y)\rho_t(y) \, d^3 y \right\|_\infty \leq \|\rho_t\|_\infty \int 1(N^{-\delta} <|y|<1) \frac{54}{|y|^3} \, d^3 y$$

$$\leq \frac{4}{3} \pi \|\rho_t\|_\infty 54 \ln(N^\delta) = 72 \pi \delta \|\rho_t\|_\infty \ln(N)$$

Now we can prove part 2) of our main Theorem 3.2 by establishing the following proposition.

**Proposition 6.2.** Under the assumptions of Theorem 3.2, we have for all $t \geq 0$ and $\delta < \frac{1}{3}$:

$$E_0(J_t^N) \leq (N^{-1} + N^{-1+3\delta}) \exp[C_1 \sqrt{\ln(N)} \int_0^t (\|\rho_s^N\|_\infty + 1) \, ds] \quad (47)$$

**Proof.** Recall that

$$J_t^N(X) := \min \left\{ 1, \sqrt{\ln(N)} N^\delta \sup_{0 \leq s \leq t} |N^{\Psi^1_t}(X) - N^{\Phi^1_t}(X)|_\infty + N^\delta \sup_{0 \leq s \leq t} |N^{\Psi^2_t}(X) - N^{\Phi^2_t}(X)|_\infty \right\}.$$  

We split the expectation $E_0(J_t)$ in the following way:

$$E_0(J_t) = E_0(J_t \mid A^c) + E_0(J_t \mid A \setminus (B \cap C)) + E_0(J_t \mid (A \cap B \cap C)) \quad (48)$$

1) Clearly, on $A^c$, we have $\frac{d}{dt} J_t \leq 0$ (since $J_t$ is already maximal) and thus also

$$\frac{d}{dt} E_t(J_t \mid A^c) \leq 0.$$  

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2) For $X \in \mathcal{A}$, we have to consider

$$
\frac{d}{dt} \sup_{0 \leq s \leq t} |\Psi_s^1(X) - \Phi_s^1(X)|_{\infty} \leq \frac{d}{dt} |\Psi_t^1(X) - \Phi_t^1(X)|_{\infty}
$$

and

$$
\frac{d}{dt} \sup_{0 \leq s \leq t} |\Psi_s^2(X) - \Phi_s^2(X)|_{\infty} \leq \sup_{0 \leq s \leq t} |\Psi_s^2(X) - \Phi_s^2(X)|_{\infty}
$$

(50)

and

$$
\frac{d}{dt} \sup_{0 \leq s \leq t} |\Psi_s^1(X) - \Phi_s^1(X)|_{\infty} \leq \frac{d}{dt} |\Psi_t^2(X) - \Phi_t^2(X)|_{\infty}
$$

and

$$
\frac{d}{dt} \sup_{0 \leq s \leq t} |\Psi_s^2(X) - \Phi_s^2(X)|_{\infty} \leq \sup_{0 \leq s \leq t} |\Psi_s^2(X) - \Phi_s^2(X)|_{\infty}
$$

(51)

We begin by controlling the contribution of the “bad” initial conditions not contained in $\mathcal{B}$ and $\mathcal{C}$. Since the two-particle force is bounded by $N^{2\delta-1}$, the total force acting on each particle is bounded as $|K(X)|_{\infty} \leq N^{2\delta}$. The mean-field force $K$ is of order 1, according to Lemma 2.4. Finally, since $X \in \mathcal{A}$, $N^{\delta}|\Psi_t^1(X) - \Phi_t^1(X)|_{\infty} \leq 1$.

According to proposition 5.2, the probability of initial conditions in $\mathcal{B}^c \cup \mathcal{C}^c$ decreases faster than any power of $N$, as $N \to \infty$. Hence, we can find for any $\gamma > 0$ a constant $C$, such that

$$
\frac{d}{dt} \mathbb{E}_t(J_t | \mathcal{A} \setminus (\mathcal{B} \cap \mathcal{C})) \leq C N^{3\delta} \mathbb{P}_0((\mathcal{A} \cap \mathcal{B})^c) \leq \frac{C}{N^{\gamma}}.
$$

(52)

3) It remains to control the change of $J_t$ for typical initial conditions, i.e. $X \in \mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$. To this end, we consider:

$$
|K(\Psi_t^1(X)) - K_t(\Phi_t^1(X))|_{\infty} \leq |K(\Psi_t^1(X)) - K(\Phi_t^1(X))|_{\infty} + |K(\Phi_t^1(X)) - K_t(\Phi_t^1(X))|_{\infty}
$$

(53)

(54)

Since $X \in \mathcal{B}$, it follows that

$$
|K(\Phi_t^1(X)) - K_t(\Phi_t^1(X))|_{\infty} < N^{-1+2\delta},
$$

(55)

which controls (54).

By the triangle inequality, we get for any $1 \leq i \leq N$:

$$
\left| (K(\Psi_t^1(X)) - K(\Phi_t^1(X)))_i \right|_{\infty} \leq \sum_{j=1}^{N} k(\Psi_j^1 - \Phi_j^1) - k(\Phi_j^1 - \Phi_t^1)
$$

$$
\leq \sum_{j=1}^{N} |k(\Psi_j^1 - \Phi_j^1) - k(\Phi_j^1 - \Phi_t^1)|_{\infty}
$$

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Thus, Lemma 4.2 implies:

\[
\|k(\Psi_j - \Psi_i^1) - k(\Phi_j - \Phi_i^1)\|_\infty \leq l(\Phi_j - \Phi_i^1)|\Psi_j - \Psi_i - \Phi_j^1|_\infty \\
\leq 2l(\Phi_j - \Phi_i^1)\|\Psi - \Phi_i^1\|_\infty
\]

Since \(X \in C\), it follows that

\[
\sum_{j=1}^{N} l(\Phi_j^1 - \Phi_i^1) = (L(\Phi_t(X)))_i \leq \|t^N * \rho_t(q)\|_\infty + 1 \leq C_l \ln(N)(1 + \|\rho_t\|_\infty)
\]

Hence, we have found that for \(X \in A \cap B \cap C\):

\[
\frac{d}{dt}|\Psi_t^2(X) - \Phi_t^2(X)|_\infty \leq C_l \ln(N)(1 + \|\rho_t\|_\infty) \|\Psi_t^1(X) - \Phi_t^1(X)\|_\infty
\]

Together with (50), this yields:

\[
\frac{d}{dt} J_t \big|_{A \cap B \cap C} \leq \sqrt{\ln N} N^\delta \frac{d}{dt}|\Psi_t^1(X) - \Phi_t^1(X)|_\infty + N^\delta \frac{d}{dt}|\Psi_t^2(X) - \Phi_t^2(X)|_\infty \\
\leq \sqrt{\ln N} N^\delta |\Psi_t^2(X) - \Phi_t^2(X)|_\infty \\
+ N^\delta \left[C_l \ln N \left(1 + \|\rho_t\|_\infty\right) |\Psi_t^1(X) - \Phi_t^1(X)|_\infty + N^{-1+2\delta}\right] \\
\leq C_l \left(1 + \|\rho_t\|_\infty\right) \sqrt{\ln N} J_t(X) + N^{-1+3\delta}.
\]

Together with the above results and observing that \(E_0(J_0) = 0\), we have for every \(t \geq 0\) and some \(\gamma > 1\):

\[
E_0(J_t) \leq \int_0^t \left(C N^{-\gamma} + C_l \left(1 + \|\rho_s\|_\infty\right) \sqrt{\ln N} E_0(J_s) + N^{-1+3\delta}\right) ds
\]

With Gronwall’s Lemma and choosing \(\gamma\) large enough, we conclude

\[
E_0(J_t) \leq \left(N^{-1} + N^{1-3\delta}\right) \exp\left[C_l \sqrt{\ln(N)} \int_0^t \left(\|\rho_s\|_\infty + 1\right) ds\right],
\]

as claimed. \(\square\)
Controlling the mean-field dynamics

The result for $E_0(J_t)$ proves part 1) of Thm. 3.2 and provides all the necessary bounds on microscopic fluctuations, as long as a suitable $N$-dependent cut-off is employed. To complete the proof of part 2) of the theorem and show that the empirical density converges in law to solutions of the Vlasov-Poisson dynamics, we still require certain bounds on the mean-field dynamics themselves. First, we have to show that the solutions $f_t^N$ of the regularized Vlasov-Poisson equation converge to a solution of the proper Vlasov-Poisson equation as the cut-off is lifted with $N \to \infty$. Second we have to provide a uniform-in-time bound for the approximation of a continuous Vlasov-density by a discrete sample, appearing in Proposition 3.5.

First, the proof of $f_t^N \rightharpoonup f_t$ is based on the following stability result by Loeper [16, Thm. 2.9], which is proven by methods from the theory of optimal transportation.

Proposition 7.1 (Loeper).
Let $k(q)$ be the Coulomb-kernel and $\rho_1, \rho_2 \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ two (probability) densities. Then we have the stability result

$$\|k * \rho_1 - k * \rho_2\|_{L^2(\mathbb{R}^3)} \leq \left( \max \{\|\rho_1\|_\infty, \|\rho_2\|_\infty\} \right)^{1/2} W_2(\rho_1, \rho_2).$$

(57)

From this we derive the following approximation result:

Proposition 7.2. Let $f_0$ satisfy the assumptions of Theorem 3.2. Let $f_t^N$ and $f_t$ be the solution of the regularized, respectively the unregularized Vlasov-Poisson equation with initial datum $f_0$. Then we have for any $p \in [1, \infty)$:

$$W_p(f_t^N, f_t) \leq 4\pi C_\rho N^{-\delta} e^{(C_\rho + C_0 \sqrt{\ln N})},$$

(58)

where $C_0 := C_1(1 + C_\rho)$ depends on $\sup_{t,N} \{\|\rho_t^N\|_\infty, \|\rho_t\|_\infty\} < +\infty$.

Proof. Let $\rho_t^N := \rho[f_t^N]$ and $\rho_t^f := \rho[f_t]$ denote the charge-density induced by $f_t^N$ and $f_t$, respectively. Let $\varphi_t^N = (Q_t^N, P_t^N)$ be the flow defined by the characteristic equation (10) for $f_t^N$. For the (unregularized) Vlasov-Poisson equation, the corresponding vector-field is not Lipschitz continuous. However, as we assume the existence of a solution $f_t$ with bounded density $\rho_t$, the mean-field force $k * \rho_t$ does satisfy a Log-Lip bound of the form $|k * \rho_t(x) - k * \rho_t(y)| \leq C|x - y| \ln^-(|x - y|)$, where $\ln^-(x) = \max\{0, -\ln(x)\}$. This is sufficient to ensure the existence of a characteristic flow $\psi_{t,s}^f = (Q_{t,s}^f, P_{t,s}^f)$ such that $f_t = \psi_{t,s}^f f_s$. 

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Now we consider \( \pi_0(x, y) := f_0(x)\delta(x-y) \in \Pi(f_0, f_0) \), which is already the minimizer yielding \( W_p(f_0^N, f_t)|_{t=0} = W_p(f_0, f_0) = 0 \) and define \( \pi_t = (\varphi_t^N, \psi_t) \# \pi_0 \). Then \( \pi_t \in \Pi(f_0^N, f_t), \forall t \in [0, T] \). For \( p \in [2, \infty) \), set

\[
D_p(t) := \left[ \int_{\mathbb{R}^6 \times \mathbb{R}^6} \left( \sqrt{\ln N} |x^1 - y^1| + |x^2 - y^2| \right)^p \, d\pi_t(x, y) \right]^{1/p}
\]

\[
= \left[ \int_{\mathbb{R}^6 \times \mathbb{R}^6} \left( \sqrt{\ln N} |Q^N(t, x) - Q^f(t, y)| + |P^N(t, x) - P^f(t, y)| \right)^p \, d\pi_0(x, y) \right]^{1/p}
\]

(59)

Note that \( W_p(f_0^N, f_t) < D_p(t) \) for any \( \pi_0 \in \Pi(f_0, f_0) \).

Now we compute:

\[
\frac{d}{dt} D_p^p(t) \leq p \int \left( \sqrt{\ln N} |Q^N(t, x) - Q^f(t, y)| + |P^N(t, x) - P^f(t, y)| \right)^{p-1} \\
\left( \sqrt{\ln N} |P^N(t, x) - P^f(t, y)| + |k_0^N \ast \rho^N_t(Q^N_t(x)) - k \ast \rho^f_t(Q^f_t(y))| \right) \, d\pi_0(x, y)
\]

(60)

The interesting term to control is the interaction term

\[
\int |k_0^N \ast \rho^N_t(Q^N_t(x)) - k \ast \rho^f_t(Q^f_t(y))| \, d\pi_0(x, y)
\]

\[
\leq \int |k_0^N \ast \rho^N_t(Q^N_t(x)) - k_0^N \ast \rho_t^N(Q^N_t(y))| \, d\pi_0(x, y)
\]

(61)

\[
+ \int |k_0^N \ast \rho_t^N(Q_t^N(y)) - k \ast \rho^f_t(Q^f_t(y))| \, d\pi_0(x, y)
\]

(62)

We begin with (61) and find with lemma 6.1

\[
\int |k_0^N \ast \rho^N_t(Q^N_t(x)) - k_0^N \ast \rho_t^N(Q^N_t(y))| \, d\pi_0(x, y)
\]

\[
\leq C_1 \ln(N) (\|\rho_t^N\|_\infty + 1) \int |Q^N_t(x) - Q^N_t(y)| \, d\pi_0(x, y),
\]

(63)

For the second term, we estimate:

\[
\leq \frac{1}{2} \|\rho_t^f\|^{1/2} \left\| k_0^N \ast \rho^N_t - k \ast \rho^f_t \right\|_2 \leq C_1^{1/2} \|k_0^N \ast \rho^N_t - k \ast \rho^f_t\|_2
\]

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Finally:
\[ \| k_N^\ast \rho - k \ast \rho_f^N \|_2 \leq \| k \ast \rho_f^N - k \ast \rho_f^t \|_2 + \| k_N^\ast \rho - k \ast \rho_f^N \|_2 \]

According to Proposition 7.1, the first summand is bounded by
\[ \| k \ast \rho_f^N - k \ast \rho_f^t \|_2 \leq C \rho W_2(\rho_f^N, \rho_f^t) \leq C_1/2 \rho W_2(\rho_f^N, \rho_f^t) \leq C_1/2 \rho W_p(t) \]

For the second term, we get with Young’s inequality:
\[ \| (k - k_N^\ast)^\ast \rho_f^N \|_2 \leq \| \rho_f^N \|_2 \| k - k_N^\ast \|_1 \leq (\| \rho_f^N \|_{\infty} \| \rho_f^t \|_1)^{1/2} \| k - k_N^\ast \|_1 \leq C_1/2 \int_{|q|<N^{-\delta}} \frac{1}{|q|^2} dq \leq 4\pi C_1/2 N^{-\delta} \]

Putting everything together, we have with \( C_0 := C_1 (1 + C_\rho) \)
\[
\frac{d}{dt} D_p(\rho_f^N) \leq p C_0 \sqrt{\ln N} D_p(\rho_f^N) + p \rho W_p(\rho_f^N, \rho_f^t) \\
+ p 4\pi C_\rho N^{-\delta} \int (\sqrt{\ln N} |Q^N(t, x) - Q_f^t(t, y)| + |P^N(t, x) - P_f^t(t, y)|)^{p-1} \\
\leq p \left( (C_0 \sqrt{\ln N} + C_\rho) D_p(\rho_f^N) + 4\pi C_\rho N^{-\delta} (D_p(\rho_f^N))^{p-1} \right)
\]
or
\[
D_p(t + \Delta t) - D_p(t) \leq \left( (C_0 \sqrt{\ln N} + C_\rho) D_p(t) + C_\rho 4\pi N^{-\delta} \right) \Delta t + o(\Delta t).
\]

Using Gronwall’s inequality and the fact that \( D_p(0) = 0 \), we conclude
\[ W_p(\rho_f^N, \rho_f^t) \leq D_p(t) \leq 4\pi C_\rho N^{-\delta} e^{(C_0 + C_\rho \sqrt{\ln N})}. \tag{64} \]

For a more detailed presentation of this method (and an alternative approach to the mean-field limit for Vlasov-Poisson), see also [14].

A similar, but simpler Gronwall estimate yields the following bound:

**Proposition 7.3.** For fixed \( f_0 \) and \( N \in \mathbb{N} \), let \( \varphi^N_t = (Q(t, \cdot), P(t, \cdot)) \) the characteristic flow of \( f_t^N \) defined by (10). Then we have:
\[ W_1(\varphi^N_t \# \mu_0^N, \varphi^N_t \# f_0) \leq \sqrt{\ln N} W_1(\mu_0^N, f_0) e^{(C_0 + C_\rho \sqrt{\ln N})}. \tag{65} \]
Proof. For $X \in \mathbb{R}^{6N}$ take some $\pi_0(x,y) \in \Pi(\mu_0^N, f_0)$ and define $\pi_t = (\varphi_t^N, \varphi^N_t) * \pi_0 \in \Pi(\varphi_t^N * \mu_0^N [X], \varphi_t^N * f_t)$. Set
\[
D(t) := \int_{\mathbb{R}^6 \times \mathbb{R}^6} \sqrt{\ln N} |x^1 - y^1| + |x^2 - y^2| \, d\pi_t(x,y)
= \int_{\mathbb{R}^6 \times \mathbb{R}^6} \sqrt{\ln N} |Q(t,x) - Q(t,y)| + |P(t,x) - P(t,y)| \, d\pi_0(x,y).
\]
Using again the Lipschitz bound as in equation (63), we derive
\[
D(t) \leq D(0) + C_0 \sqrt{\ln N} \int D(s) \, ds
\]
and hence by Gronwall’s inequality:
\[
W_1(\varphi_t^N * \mu_0^N, \varphi_t^N * f_t) \leq D(t) \leq D(0)e^{C_0 \sqrt{\ln N}}
\]
Taking on the right-hand side the infimum over all $\pi_0(x,y) \in \Pi(\mu_0^N, f_0)$,
\[
W_1(\varphi_t^N * \mu_0^N, \varphi_t^N * f_t) \leq \sqrt{\ln N} W_1(\mu_0^N, f_0)e^{C_0 \sqrt{\ln N}},
\]
from which the statement follows. \qed

Note that the previous two results were deterministic. To complete our
proof of the main theorem, we still require a probabilistic “concentration
estimate” for $\mathbb{P}_0(W_1(\mu_0^N [X], f_0) > \epsilon)$. Fortunately, we can rely
on various recent works, providing quantitative results about the conver-
gence of typical samples under various regularity assumptions on the
law $f_0$ (e.g. [4], [3], [1]). We will cite here the following (particu-
larly strong) result from Fournier and Guillin, 2014 [7].

**Theorem 7.4 (Fournier and Guillin).**

Let $f$ be a probability measure on $\mathbb{R}^d$ such that $\exists q > 2$:
\[
M_q(f) := \int |x|^q \, df(x) < +\infty
\]
Let $(X_i)_{i=1,\ldots,N}$ be a sample of independent variables, distributed
according to the law $f$ and $\mu^N[X] := \sum_{i=1}^N \delta_{X_i}$. Then, for any $\epsilon > 0$
there exist constants $c, C$ depending only on $q, M_q(f), \epsilon$ such that for
all $N \geq 1$ and $\xi > 0$:
\[
\mathbb{P}(W_1(\mu_0^N, f) > \xi) \leq C_1 \xi \exp(-cN\xi^d) + CN(\xi)^{-(q-\epsilon)}.
\]
7.1 Proof of Main Theorem

Proof of Theorem 3.2 part 2). Let $\gamma \in (0, 1)$ and choose $\gamma' > \gamma$ in the same interval. Applying the previous Theorem with $\delta = N^{-\frac{2}{3}}$ and $\epsilon = q - 2$, we get $C, c > 0$ such that

$$\mathbb{P}
\left(W_1(\mu^N, f) > N^{-\frac{\gamma'}{6}}\right) \leq C \exp(-cN^{1-\gamma'}) + CN^{-1+\frac{2}{3}}.$$ 

Proposition 7.3 then yields:

$$\mathbb{P}
\left(\exists t \in [0, T] : W_1(\phi^N_t \# \mu^N_0, \phi^N_t \# f^N_t) \geq \sqrt{\ln(N)} N^{-\frac{\gamma'}{6}} e^{tC_0 \sqrt{\ln N}}\right) < C \exp(-cN^{1-\gamma'}) + CN^{-1+\frac{2}{3}},$$

or, for sufficiently large $N$:

$$\mathbb{P}
\left(\exists t \in [0, T] : W_1(\phi^N_t \# \mu^N_0, \phi^N_t \# f^N_t) \geq \frac{1}{4} N^{-\frac{\gamma}{6}} e^{tC_0 \sqrt{\ln N}}\right) < 2CN^{-1+\frac{2}{3}},$$

Now we use this with Proposition 3.3 to conclude

$$\mathbb{P}
\left(\exists t \in [0, T] : W_1(\mu^N_t, f^N_t) > \frac{1}{2} N^{-\frac{\gamma}{6}} e^{tC_0 \sqrt{\ln N}}\right) \leq \mathbb{E}_0(J_T)+2CN^{-1+\frac{2}{3}}.$$ 

Finally, by Proposition 7.2

$$W_1(\mu^N_t, f_t) \leq W_1(\mu^N_t, f^N_t) + W_1(f^N_t, f_t)$$

$$\leq W_1(\mu^N_t, f^N_t) + 4\pi C_\rho N^{-\delta} e^{tC_0 \sqrt{\ln N}+C_\rho},$$

or, if $N$ is large enough,

$$W_1(\mu^N_t, f_t) \leq W_1(\mu^N_t, f^N_t) + \frac{1}{2} N^{-\lambda} e^{tC_0 \sqrt{\ln N}},$$

with $\lambda = \gamma \min\{\frac{1}{6}, \delta\}$. Hence, we get

$$\mathbb{P}
\left(\exists t \in [0, T] : W_1(\mu^N_t, f_t) > N^{-\lambda} e^{tC_0 \sqrt{\ln N}}\right) \leq \mathbb{E}_0(J_T)+2CN^{-1+2\lambda}.$$ 

Recalling Prop. 6.2 and the fact that $\mathbb{E}_0(J_T) < (N^{-1}+N^{-1+3\delta}) e^{tC_0 \sqrt{\ln N}}$, the theorem follows. \hfill \Box
8 Weaker Singularities, Open Questions

For better comparison with known results, we state here without further proof the application of our method to more general forces with singularities milder than Coulomb. Generalization to arbitrary dimensions would be straightforward, as well.

Instead of the Coulomb kernel \( (2) \), we consider an interaction-force
\[
k(q) = \frac{q}{|q|^{\alpha+1}}
\]
with \( \alpha < 2 \) and a regularization near the origin satisfying
\[
\left\{\begin{array}{ll}
k_N(x) = \frac{q}{|q|^{\alpha+1}} & \text{for } |x| \geq N^{-\delta} \\
|k_N(x)| \leq N^{-\delta \alpha} & \text{for } |x| < N^{-\delta}.
\end{array}\right.
\]
Then we have the following result:

**Theorem 8.1.** Assume that \( k_N \) satisfies \((71)\) for \( \alpha < 2 \) and
\[
d < \frac{1}{1+\alpha}
\]
Assume (for simplicity) that \( f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \) has compact support and let \( f_t, f_t^N \) the solution of the Vlasov-equation with force-kernel \( k \) and \( k_N \), respectively. Then, we have for all \( T \geq 0 \):
\[
P_0\left( \sup_{0 \leq t \leq T} N^{\Psi_t} - N^{\Phi_t}|_{\infty} \geq N^{-1+\alpha \delta} \right) \leq e^{TC_2 N^{-1+(\alpha+1)\delta}}
\]
And molecular chaos in the sense that for \( \lambda = \min\{\frac{1}{\alpha}, \delta\} \)
\[
P_0(\exists t \in [0, T] : W_1(\mu_t^N[X], f_t) > e^{TC_3 N^{-\lambda}}) \leq 2e^{TC_2 N^{-1+(\alpha+1)\delta}},
\]
for sufficiently large \( N \), with constants \( C_1, C_2 \) depending on \( f_0 \) and \( \alpha \).

This can be compared to the results in Hauray and Jabin, 2013 \[10\], where a statement similar to \((74)\) is derived for the case \( 1 \leq \alpha < 2 \) with a cut-off of order
\[
d < \frac{1}{6} \min\left\{\frac{1}{\alpha-1}, \frac{5}{\alpha}\right\},
\]
by providing an explicit control on the minimal particle distance (in phase-space, strictly speaking). For \( \alpha \in [1, 2) \), the upper bound in \( d \) given by \((75)\) ranges between \( \frac{5}{2} \) and \( \frac{1}{6} \), while in \((72)\) it ranges between \( \frac{1}{2} \) and \( \frac{1}{3} \). In particular, it is interesting to note that the cut-off required in \[10\] is smaller than ours for \( \alpha < \frac{7}{5} \) but larger for \( \frac{7}{5} < \alpha < 2 \). This suggests that the purely probabilistic estimates presented here fare better for strong singularities (in the sense of admitting a significantly
smaller cut-off albeit for the price of a significantly slower rate of convergence), while the method proposed in \cite{10} provides better controls for mild singularities, notably allowing to treat the cases $0 < \alpha < 1$ with no cut-off at all. (C.f. also their previous results in \cite{9}).

As it stands, the methods presented in the present paper require a cut-off even for the weakest of singularities in the interaction-force, the reason being that they provide no further control on the “bad” initial configurations but “neglect” them on the basis that they have (arbitrarily) small measure.

However, we conjecture that this stochastic method can be extended to treat the case $0 < \alpha < 1$ with no cut-off, by a) integrating the dynamics over small time-intervals (similar to \cite{10}) thus improving the estimates such that $\delta$ can be chosen greater than 1 and then b) proving that for typical initial conditions, no two particles will ever come closer than $\sim N^{-1}$ over any finite time-interval. We leave this problem to be treated in a future publication.
A Appendix

A.1 Convergence of Marginals

We show here that without further assumptions on \( f_0 \), \( \lim_{N \to \infty} E_0(J^N_t) = 0 \) implies molecular chaos, albeit without the strong quantitative bounds stated in Theorem 3.2.

Definition A.1 (Bounded Lipschitz distance). Let \( \mathcal{L}' \) be the space of functions \( g : \mathbb{R}^n \to \mathbb{R} \) satisfying

\[
\|g\|_\infty := \sup_x |g(x)| = 1, \quad \|g\|_{\text{Lip}} := \sup_{x,y} \frac{|g(x) - g(y)|}{|x - y|} = 1.
\]

(76)

For two probability densities \( \mu, \nu \) on \( \mathbb{R}^k \), the \textit{bounded Lipschitz distance} is defined by

\[
d_{BL}(\mu, \nu) := \sup_{g \in \mathcal{L}'} \left| \int g(x) \, d\mu(x) - \int g(x) \, d\nu(x) \right|.
\]

This metrizes weak convergence in \( \mathcal{P}(\mathbb{R}^n) \).

Proposition A.2. Suppose that \( \lim_{N \to \infty} E_0(J^N_t) = 0 \). Then, the reduced \( k \)-particle marginal given by

\[
(k) F_t(x_1, ..., x_k) := \int f_t(X) \, d^3x_{k+1}...d^3x_N
\]

(77)

converges weakly to

\[
\prod_{i=1}^k f_t(x_1, ..., x_k) = f_t(x_1)f_t(x_2)\cdot \ldots \cdot f_t(x_k),
\]

as \( N \to \infty \) for all \( k \in \mathbb{N} \). More precisely, we have:

\[
d_{BL}((k) F^N_s, \otimes^k f^N_s) \leq E_0(J_t) + N^{-\delta}, \forall s \leq t.
\]

(78)

Proof. Let \( g : \mathbb{R}^{6k} \to \mathbb{R} \) be a test-function with \( \|g\|_{\text{Lip}} = \|g\|_\infty = 1 \). Let \( \mathcal{A} \subset \mathbb{R}^{6N} \) be given by \( X \in \mathcal{A} \iff J_t(X) < 1 \). Then \( X \in \mathcal{A} \) implies in particular \( |\Psi_{s,0}(X) - \Phi_{s,0}(X)|_\infty \leq N^{-\delta}, \forall s \in [0, t] \), while \( \lim_{N \to \infty} E_0(J_t) = 0 \) implies \( \lim_{N \to \infty} \mathbb{P}_0(\mathcal{A}^c) = 0 \). Thus, we find for all \( s \leq t \):
\[ d_{BL}(^k F_s^N, \otimes_k f_s^N) \]

\[ = \sup_{g \in L} \left| \int (^k F_s^N - \otimes_k f_s^N) g(x_1, ..., x_k) d^3 x_1 ... d^3 x_k \right| \]

\[ = \sup_{g \in L} \left| \int (F_s^N(X) - \otimes^N f_s^N(X)) g(x_1, ..., x_k) d^3 x_1 ... d^3 x_k \right| \]

\[ = \sup_{g \in L} \left| \int (F_0(\Psi_{s,0}(X)) - F_0(\Phi_{s,0}(X))) g(x_1, ..., x_k) d^6 N x \right| \]

\[ = \sup_{g \in L} \left| \int F_0(X) \left( g(P_k \Psi_{s,0}(X)) - g(P_k \Phi_{s,0}(X)) \right) d^6 N x \right| \]

\[ = \sup_{g \in L} \left| \int A F_0(X) \left( g(P_k \Psi_{s,0}(X)) - g(P_k \Phi_{s,0}(X)) \right) d^6 N x \right| + \sup_{g \in L} \left| \int A c F_0(X) \left( g(P_k \Psi_{s,0}(X)) - g(P_k \Phi_{s,0}(X)) \right) d^6 N x \right| \]

\[ (79) \]

\[ (80) \]

where \( P_k : \mathbb{R}^N \rightarrow \mathbb{R}^k, (x_1, ..., x_N) \mapsto (x_1, ..., x_k) \) is the projection onto the first \( k \) coordinates. Since \( g \) and \( F_0 \) are bounded by 1, we have \( (80) \leq P_0(A^c) \leq E_0(J_t) \).

Using that \( \| g \|_{Lip} = 1 \), we obtain

\[ \sup_{X \in A} |g(P_k \Psi_{s,0}(X)) - g(P_k \Phi_{s,0}(X))| \leq |\Psi_{s,0} - \Phi_{s,0}|_{\infty} \leq N^{-\delta} \quad (81) \]

Hence, also \( (79) \leq N^{-\delta} \) and the proposition follows. \( \square \)

**Remark A.3.** In the context of these results, \( d_{BL} \) could be replaced by the stronger metric \( W_1 \) if one could establish a (uniform in \( N \)) bound on the maximal deviation \( |\Phi_t(X) - \Psi_t(X)|_{\infty} \) for all initial configurations \( X \), rather than just typical ones.
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