This article discusses the usage of a partition based Fubini calculus for Poisson processes. The approach is an amplification of Bayesian techniques developed in Lo and Wong for gamma/Dirichlet processes. Applications to models are considered which all fall within an inhomogeneous spatial extension of the size biased framework used in Perman, Pitman and Yor. Among some of the results; an explicit partition based calculus is then developed for such models, which also includes a series of important exponential change of measure formula. These results are then applied to solve the mostly unknown calculus for spatial Lévy-Cox moving average models. The analysis then proceeds to exploit a structural feature of a scaling operation which arises in Brownian excursion theory. From this a series of new mixture representations and posterior characterizations for large classes of random measures, including probability measures, are given. These results are applied to yield new results/identities related to the large class of two-parameter Poisson-Dirichlet models. The results also yields easily perhaps the most general and certainly quite informative characterizations of extensions of the Markov-Krein correspondence exhibited by the linear functionals of Dirichlet processes. This article then defines a natural extension of Doksum’s Neutral to the Right priors (NTR) to a spatial setting. NTR models are practically synonymous with exponential functions of subordinators and arise in Bayesian non-parametric survival models. It is shown that manipulation of the exponential formulae makes what has been otherwise formidable analysis transparent. Additional interesting results related to the Dirichlet process and other measures are developed. Based on practical considerations, computational procedures which are extensions of the Chinese restaurant process are also developed.
1 Introduction

This paper discusses the active usage of a Poisson process partition based Fubini calculus to solve a variety of problems. That is, a method which will be used to solve problems associated with large classes of random partitions $p := \{C_1, \ldots, C_n(p)\}$ of the integers $\{1, \ldots, n\}$. The method is based on the formal statement of two results, which are known in various levels of generality, concerning a Laplace functional change of measure and a partition based Fubini representation. In terms of technique, this is an amplification of the methods discussed in Lo and Weng (1989) [see also Lo (1984)] for a class of Bayesian nonparametric weighted gamma process mixture models. The idea to choose a Poisson process framework was based on suggestions from Jim Pitman. The utility of the approach is demonstrated by its application to a suite of problems which are within the general size-biased framework of Pitman, Perman and Yor (1992, Section 4), with now a spatial inhomogeneous component. Methodologically this article may be viewed as a treatment of combinatorial stochastic processes from a Bayesian (infinite-dimensional calculus) technical viewpoint.

A key role in the works of Lo (1984) and Lo and Weng (1989), is played by a partition distribution on the integers $\{1, \ldots, n\}$ which is a variant of Ewens sampling formula [see Ewens (1972) and Antoniak (1974)] associated with the Poisson-Dirichlet partition distribution. One particular feature is that posterior quantities written with respect to a Blackwell and MacQueen (1973) urn scheme can be further simplified to calculations which amount to sums over partitions $p$ of $\{1, \ldots, n\}$. This makes $p$ what I term a separating class. The methodology discussed there amounts to a partition based Fubini calculus for Dirichlet [see Ferguson
(1973, 1974), Freedman (1963) and Fabius (1963)] and gamma processes. Pitman (1996), extends the description of the Blackwell-MacQueen sampling from the Dirichlet process to a large class of species sampling random measures. Importantly, he develops ideas surrounding the two-parameter Poisson-Dirichlet family of distributions in a Bayesian context. This provides a nice bridge to related work, where the two-parameter family appears, in for instance Pitman (1995a, b, 1997a, 1999), Pitman, Perman and Yor (1992), Pitman and Yor (1992, 1997, 2001). Those works are non-Bayesian and center around topics such as Brownian excursion theory and Kingman’s theory of partition structures as developed in Aldous (1985) and Pitman (1995a). Returning to a Bayesian setting, Ishwaran and James (2001a) recently develop the calculus for a class of species sampling mixture models analogous to and extending the model of Lo (1984), based on the work of Pitman (1995a, 1996).

The interest here is to extend these ideas to other random measures, not necessarily mixture models, via more general partition structures. The question of how indeed to obtain information via as yet possibly unknown partition structures related to classes of random measures suggests that one may need considerable expertise in combinatorial calculations. Here, this issue is circumvented by usage of what is referred to as a Poisson Process Partition Calculus. Note that Poisson Palm calculus is employed in Pitman, Perman and Yor (1992) and Pitman and Yor (1992). [See also Fitzsimmons, Pitman and Yor (1992)]. The intersection with those results manifests itself in section 5.

One may infer from Lo (1984) and Lo and Weng (1989), that Bayesian infinite-dimensional calculus is a calculus based on the disintegration of joint product structures on abstract spaces which exploits properties of partitions p or some other separating class. Examples of other separating classes, which will not be discussed, are the s-path models found in Brunner and Lo (1989) and the classification based methods for generalised Dirichlet stick-breaking models discussed in Ishwaran and James (2001b). Specifically these ideas are applied to classes of boundedly finite random measures, say μ, on complete and separable spaces[see Daley and Vere-Jones (1986)] which are linked to Poisson random measures. I will mention early that this is not synonymous with the notion of stochastic integration which suggests simply a marginalization over the infinite-dimensional component; although these types of ideas will play a role. Here the primary interest is in the derivation and various characterizations of the joint structure in terms of an infinite-dimensional posterior law of μ and its marginal components. A key aspect of disintegration of measures on Polish spaces is the availability of a well defined Fubini’s theorem. The notion of Bayesian models and disintegrations is made quite clear in Le Cam (1986, Chapter 12). The term Bayesian is meant primarily in terms of technique. The treatment of problems is more in line with a broader point of view such as in Kingman (1975) rather than Ferguson (1973). Of course I shall cover quite thoroughly models which arise in Bayesian nonparametrics. Readily accessible general disussions on disintegrations of measures may be found in Pollard (2001) and Kallenberg (1997). See also Blackwell and Maitra (1984), Dellacherie and Meyer (1978), and Pachl (1978). Additionally, Daley and Vere-Jones (1988), Matthes, Kerstan and Mecke (1978) and Kallenberg (1986) provide details about Fubini’s theorem for random measures cast within the language of Palm calculus.

1.1 Basic principles and motivation

For motivation the typical but rich mixture model setup is described. Let K denote a non-negative integrable kernel on a complete and separable space \( \mathcal{X} \times \mathcal{Y} \) and let \( \mu \) denote a boundedly finite discrete random measure on \( \mathcal{Y} \). A mixture model is defined as follows

\[
    f(X|\mu) := \int_{\mathcal{Y}} K(X|Y)\mu(dY).
\]


for $X$ varying in $\mathcal{X}$. When $\mu$ is a Dirichlet process, a two-parameter Poisson-Dirichlet process or a weighted gamma process then we are in the setting described earlier. When $K$ is chosen to be a density such as a normal kernel and $\mu$ is a probability measure then $f$ is a random density. This presents one way to describe random measures over spaces of densities and is analogous to the idea behind classical density estimation where one convolves an empirical distribution function with a kernel. A similar construct holds for random hazard rates which might be useful in models in a multiplicative intensity setting. However, the fact that the kernel may be specified rather arbitrarily leads to the description of a large body of models which appear in nonparametric statistics, spatial modelling and general inverse problems. Letting $K(A_{ij}, Y) := I\{Y \in A_{i,j}\}$ corresponds to partially observed models such as interval censoring, right censoring, double censoring described, from a frequentist viewpoint, in Turnbull (1976) and Groeneboom and Wellner (1992). See Lindsay (1995) and Groeneboom (1996) for much more general models. As discussed in Lo (1984) and Lo and Weng (1989) one can induce specific shapes such as the class of monotone densities or hazards via a uniform kernel or completely monotone models via mixtures of exponential kernels. In statistical terms, when $\mu$ is a probability measure $P$, the general mixture model has an interpretation where $X \mid Y, P$ is $K(X \mid Y)$ and $Y \mid P$ is missing information with distribution $P$. In spatial statistics, Wolpert and Ickstadt (1998b) propose to specify $\mu$ as a Lévy random field where \( \int K(x - y)\mu(dy) \) or more specifically

$$\int_0^t e^{(s-t)}\mu(ds)$$

which is reminiscent of a stationary Ornstein-Uhlenbeck process. Barndorff-Nielsen and Shepard (2001) propose the usage of non-Gaussian Ornstein-Uhlenbeck processes which also can be viewed as a mixture representation where $\mu$ is a Lévy process. See also Le Cam (1961) for an early mathematical discussion of random measures and shot-noise models. While such models are indeed rich in terms of flexibility and diversity in terms of applications, many questions remain open about their properties. Suppose that now one has the joint product measure of $\{X_1, \ldots, X_n, \mu\}$

$$\mathcal{P}(d\mu) \prod_{i=1}^n \int_{\mathcal{Y}} K(X_i \mid Y_i)\mu(dY_i)$$

which arises in a variety of contexts. The quantity above represents one possible disintegration of the joint structure $\{X_1, \ldots, X_n, \mu\}$. Most often direct evaluation of $\{X_1, \ldots, X_n, \mu\}$ is not simple or practically implementable. Stripping away the kernel $K$ one is left with the joint product structure, $\{Y_1, \ldots, Y_n; \mu\}$ which due to versatility of an available Fubini’s theorem becomes the main object of interest. That is knowledge of this structure reduces the problem of the mixture model above to a special case of a cadre of possibilities. Hence the goal is to find the following disintegration

$$\mathcal{P}(d\mu) \prod_{i=1}^n \mu(dY_i) := \mathcal{P}(d\mu \mid \mathcal{Y}) M_{\mu}(dY_1, \ldots, dY_n)$$

where $M_{\mu}$ is a possibly sigma-finite joint moment measure of $\mathcal{Y}$ and $\mathcal{P}(d\mu \mid \mathcal{Y})$ can be thought of as the posterior distribution of $\mu \mid \mathcal{Y}$. However, the result is best understood and its utility is revealed by an equivalent
statement via Fubini’s theorem,

\[
\int g(Y, \mu) \prod_{i=1}^{n} \mu(dY_i) \mathcal{P}(d\mu) := \int g(Y, \mu) \mathcal{P}(d\mu|Y) M_\mu(dY_1, \ldots, dY_n)
\]

for \( g \) an integrable or positive function. As is certainly known it is sufficient to check for \( g \) specified to be an indicator of appropriate cylinder sets or other characterizing function. The structure \( M_\mu(dY_1, \ldots, dY_n) \) is an urn-type structure that can be generated sequentially via conditional moment measures. If \( \mu \) is a probability measure the notion of conditional moment measures is synonymous with the notion of Bayesian prediction rules. However the exchangeable urn structure becomes a bit un-wieldy and one seeks a further disintegration of \( Y \). A natural one is based on the often quite informative decomposition \( Y := (Y^*, p) \) where \( Y^* = \{Y_1, \ldots, Y_{n(p)}\} \) are the unique random values given a random partition \( p \) of the integers \( \{1, \ldots, n\} \).

In other words the joint (sigma-finite) measure admits a disintegration in terms of a conditional measure on its unique values and a measure on \( p \). Neither of which need be a proper probability measure.

The main point boils down to the following basic principles; suppose that a random measure \( \mu^* \) is some function of \( \mu \), i.e. \( \mu^* = g(\mu) \). Then via (5) its posterior law, marginal law and partition structure can all be derived from those corresponding aspects of \( \mu \). This is of course provided that one has explicit information about \( \mu \). The utility of such a procedure for \( \mu \) is then amplified by its richness. That is, a measure of how many interesting processes \( \mu^* \) can it capture. The interesting aspect of this is that the measure of richness of \( \mu \) must correspond to the simplicity of its posterior laws, marginal moment and partition structures, while still being informative. The structures must indeed act in a way like canonical basis functions. In other words complex structures can be derived from simple ones. The Poisson random measure, \( N \), emerges as a natural candidate given its prevalence in various theories of random measures and its basic connections (via the Poisson random variable and Bell’s number) to random partitions of the integers. Albeit there is a duality to an approach using combinatorial arguments, an exploitation of the Poisson random measure analogue of (3), with a further partition disintegration, allows one to proceed in a pure framework of disintegration of measures to directly derive many aspects of large classes of measures \( \mu \).

The notation \( \sum_p \) will be used to denote the sum over all partitions of the integers \( \{1, \ldots, n\} \). As is well known[see Rota (1964)], this sum is equivalent to Bell’s number. For papers which discuss the natural relationships of the Poisson process/random variable to partitions, see for instance Constantine and Savits (1994), Pitman (1997b), Constantine (1999) and Di Nardo and Senato (2001). The papers by Constantine and Savits (1994) and Constantine (1999), and references therein, are certainly related to this one. Constantine and Savits (1994) discuss methods to evaluate identity/moment formulae for compound Poisson processes via Faa Di Bruno’s formula. Certainly one can infer from Theorem 2.1 of Constantine and Savits (1994) that many of the formulae here, expressed in terms of \( \sum_p \), can be re-expressed in terms of infinite-sum notation related to Dobinski’s formula or more obviously cycle notation.

1.2 Notation and preliminaries

Again let \( p = \{C_1, \ldots, C_{n(p)}\} \) denote a partition of size \( n(p) \) of the integers \( \{1, \ldots, n\} \), let \( e_{j,n} \) denote the cardinality of each cell \( C_j \) for \( j = 1, \ldots, n(p) \). This partition structure is related to a description of general analogues of a Chinese restaurant scheme to generate partitions described in terms of a sequential seating of customers[see Aldous (1985), Pitman (1996) and Kerov (1998)]. The results will be closely connected to such a structure generated from the exchangeable partition probability function (EPPF) (partition distribution). See Pitman (1995a,b, 1996) for a thorough description of the EPPF concept. Additionally, for \( r > 1 \), let
\( p_r = \{C_{1,r}, \ldots, C_{n(p_r),r}\} \) denote a partition of \( \{1, 2 \ldots, r\} \), where \( C_{i,r} \) denotes the current configuration of table \( i \) after \( r \) customers have been seated and \( e_{i,r} \) denotes the number of customers seated at \( C_{i,r} \). The partition \( p_{r+1} \) then denotes the (updated) one step larger partition on \( \{1, 2, \ldots, r+1\} \).

Now specific notation is given for models which shall be looked at in some detail. That is, for the two-parameter Poisson-Dirichlet models and closely connected generalised gamma family of random measures. In addition notation is given for a general spatial variation of the Beta process of Hjort (1990). First, we briefly describe the two-parameter Poisson-Dirichlet class of models. See Pitman and Yor (1997) and Pitman (1996) for more details. Let \((Z_i)\) denote a collection of iid random variables whose distribution is a diffuse probability measure \(H\) and independently of \((Z_i)\), let \((P_i)\) denote a collection of ranked probabilities which sum to one and have a two-parameter Poisson-Dirichlet distribution denoted as \(PD(\alpha, \theta)\) with parameter values \(0 \leq \alpha < 1\) and \(\theta > -\alpha\). The corresponding random probability measure has a representation,

\[
P_{\alpha,\theta}(\cdot) := \sum_{i=1}^{\infty} P_i \delta_{Z_i}(\cdot).
\]

The law of \(P_{\alpha,\theta}\), denoted \(PD_{\alpha,\theta}(dP|H)\) is uniquely associated with its prediction rule and exchangeable partition probability function (EPPF) given as,

\[
P\{Y_{n+1} \in \cdot | Y_1, \ldots, Y_n\} = \frac{\theta + n(p)\alpha}{\theta + n} H(\cdot) + \sum_{j=1}^{n(p)} \left(\frac{e_{j,n} - \alpha}{\theta + n}\right) \delta_{Y_j}(\cdot),
\]

and,

\[
PD(p|\alpha, \theta) = \frac{\left(\prod_{j=1}^{n(p)}(\theta + \alpha j)^{\frac{1}{\Gamma(\alpha)}} \left(\prod_{j=1}^{n(p)} \Gamma(e_j - \alpha)\right)\right)^{-1}}{\prod_{j=1}^{n-1}(\theta + j)}.
\]

The extreme cases are the normalised stable law process \(P_{\alpha,0}\) and the Dirichlet process, \(P_{0,\theta}\) with shape parameter \(\theta H\). The laws of the \((P_i)\) and Ewens sampling EPPF formula for the Dirichlet process are denoted as \(PD(\theta)\) and \(PD(p|\theta)\). Similarly for the stable process write \(PD(\alpha)\) and \(PD(p|\alpha)\). Now for \(b \geq 0\) the rich family of generalized gamma random measures [see Brix (1999)] is generated by the Lévy measure

\[
\rho_{\alpha,b}(ds) = \frac{1}{\Gamma(1-\alpha)} s^{-\alpha-1} e^{-bs} ds,
\]

which includes the stable law subordinator, \(b = 0\), gamma processes subordinator \(\alpha = 0\), and the inverse-Gaussian law, \(\alpha = .5\), \(b > 0\), among others. The notation \(\rho_{\alpha} := \rho_{\alpha,0}\) will be reserved for the stable law and the choice \(\theta \rho_{0,1}\) will be used to generate the Dirichlet process family of models. The general subordinator has increments which have a distribution belonging to an exponential family of distributions with a power variance function, introduced by Tweedie (1984) and further discussed in Hougaard (1986), Bar-Lev and Enis (1996), and Jørgensen (1997). See Küchler and Sorensen (1997). Classes of compound Poisson process models based on this distribution are discussed in Aalen (1992), Lee and Whitmore (1993) and Hougaard, Lee and Whitmore (1997). Lastly, a spatial version of Hjort’s (1990) (two-parameter) Beta process corresponds to the Lévy process generated by the inhomogeneous Lévy measure,

\[
u^{-1}(1-u)^{c(s)-1} du A_0(ds, dx)
\]

for \((u, s, x) \in (0, 1] \times (0, \infty) \times \mathcal{X}\). The quantity \(c(s)\) is a decreasing function on \((0, \infty)\) and \(A_0\) is a hazard measure. The symbols \(G(a,b), \beta(a,b)\) will be used to denote gamma and beta random variables respectively. \(G(dx|a,b)\) will denote a gamma density.
Some other references connected to the Poisson-Dirichlet family, not mentioned later, include McCloskey (1965), Engen (1978), Carlton (1999), Donnelly and Tavaré (1987), Gyllenberg and Koski (2001). See the article by Ewens and Tavaré (1997) for a discussion of the wide applicability of the two-parameter model. See also Pitman (1995b), which will be referenced later.

2 Poisson Process Partition Calculus

Let \( N \) denote an inhomogeneous Poisson process (measure) on a complete and separable space \( X \) with (diffuse) mean measure \( \nu(\cdot) \). That is, the Laplace functional of \( N \) is of the form

\[
L_N(f|\nu) = \int_M \exp \left\{ -\int_X f(x)N(dx) \right\} P(dN|\nu)
\]

\[
= \exp \left( -\int_X \left( 1 - e^{-f(x)} \right) \nu(dx) \right)
\]

for non-negative functions \( f \in BM(X) \) on \( (X, B(X)) \) where \( BM(X) \) denotes the collection of measurable functions of bounded support on \( X \). See Daley and Vere-Jones (1988) for a description of these concepts. For brevity we use the shorthand notation of the type,

\[
e^{-N(f)} = \exp \left\{ -\int_X f(x)N(dx) \right\}.
\]

The exposition of this paper centers around the utilization of disintegration results related to the joint measure

\[
\mathcal{P}(dN|\nu) \prod_{i=1}^n N(dX_i),
\]

where \([\Box]\) represents a disintegration of the joint product measure of \( \{X_1, \ldots, X_n, N\} \). Moreover, the collection \( X = \{X_1, \ldots, X_n\} \) can be considered as conditionally independent given \( N \). However importantly once integration is done over \( N \) the collection \( X \) will usually consist of tied values. It follows that one can always represent \( X = (X^*, p) \) where \( X^* = \{X^*_1, \ldots, X^*_n(p)\} \) denotes the unique values and \( p \) dictates which variables are equal according to the relationship \( X_i = X^*_j \) if and only if \( i \in C_j \). The main purpose of this section is to describe two results concerning the Poisson process which are fashioned as tools to be tailor-made to solve a variety of problems in an expeditious manner.

2.1 Basic tools

First an (exponential) change of measure or disintegration formulae based on Laplace functionals is given below. Such an operation is commonly called exponential tilting.

Lemma 2.1 For non-negative functions \( f \in BM(X) \) on \( (X, B(X)) \) and \( g \) on \( (M, B(M)) \)

\[
\int_M g(N)e^{-N(f)}P(dN|\nu) = L_N(f|\nu) \int_M g(N)P(dN|e^{-f}\nu),
\]

where \( P(dN|e^{-f}\nu) \) is the law of a Poisson Process with intensity

\[
e^{-f(x)}\nu(dx).
\]

In other words the following absolute continuity result holds,

\[
e^{-N(f)}P(dN|\nu) = L_N(f|\nu)P(dN|e^{-f}\nu).
\]
**Proof.** By the unicity of of Laplace functionals for random measures on \( \mathcal{X} \) it suffices to check this result for the case \( g(N) = e^{-N(h)} \). Thus it follows that,

\[
\int_{\mathcal{M}} e^{-N(f+h)}\mathcal{P}(dN|\nu) = \mathcal{L}_N(f|\nu) \int_{\mathcal{M}} e^{-N(h)}\mathcal{P}_f(dN)
\]

where for the time being \( \mathcal{P}_f \) denotes some law on \( N \). Simple algebra shows that

\[
\int_{\mathcal{M}} e^{-N(h)}\mathcal{P}_f(dN) = \frac{\mathcal{L}_N(f+h|\nu)}{\mathcal{L}_N(f|\nu)}
\]

and hence \( \mathcal{P}_f(dN) := \mathcal{P}(dN|e^{-f}\nu) \) which concludes the result. \( \blacksquare \)

**Remark 1.** Lemma 2.1 is a simple functional extension, mod the Gaussian and drift component, of an analogous result for Lévy processes on \( \mathcal{R} \) or more generally \( \mathcal{R}^d \) which may be found in Küchler and Sorensen (1997) Proposition 2.1.3. The utility of Lemma 2.1 will be demonstrated throughout. Poisson processes with laws described by \( \mathcal{P}(dN|e^{-f}\nu) \) can be found in Pitman and Yor (1992)[See Section 5 of this manuscript].

**Remark 2.** Naturally Lemma 2.1 extends to the following somewhat more vague generalisation; Suppose that \( \mu \) is a random measure with Laplace functional \( \mathcal{L}_{\mu} \), then given the setup in Lemma 2.1 with \( \mu \) in place of \( N \), \( e^{-\mu(f)}\mathcal{P}(d\mu) = \mathcal{L}_{\mu}(f)\mathcal{P}_f(d\mu) \) where \( \mathcal{P}_f \) is characterized by its Laplace functional

\[
\int_{\mathcal{M}} e^{-\mu(h)}\mathcal{P}_f(d\mu) = \frac{\mathcal{L}_\mu(f+h)}{\mathcal{L}_\mu(f)}.
\]

One can replace the argument with characteristic functionals.

Results which identify the disintegration of \((11)\) in terms of the posterior distribution of the Poisson process and the marginal joint measure

\[
(13) \quad M(dX_1, \ldots, dX_n) = \int_{\mathcal{M}} \left[ \prod_{i=1}^{n} N(dX_i) \right] \mathcal{P}(dN|\nu)
\]

are well known in the literature via Palm calculus for Poisson processes. The quantity \( M \) in \((13)\) is also known as the joint moment measure. These existing results are customized in Lemma 2.2 below where emphasis is placed on the partition structure.

**Lemma 2.2** Let \( g \) be a non-negative or integrable function on \( \mathcal{X}^n \times \mathcal{M} \), then for each \( n \geq 1 \),

\[
(14) \quad \int_{\mathcal{M} \times \mathcal{X}^n} g(x, N) \prod_{i=1}^{n} N(dx_i) \mathcal{P}(dN|\nu) = \sum_{p} \int_{\mathcal{X}^n(p)} \left[ \int_{\mathcal{M}} g(\mathbf{X}^*, p, N + \sum_{j=1}^{n(p)} \delta_{X_j}^*) \mathcal{P}(dN|\nu) \right] \prod_{j=1}^{n(p)} \nu(dX_j^*),
\]

with

\[
(15) \quad \int_{\mathcal{M}} g(\mathbf{X}^*, p, N + \sum_{j=1}^{n(p)} \delta_{X_j}^*) \mathcal{P}(dN|\nu) = \int_{\mathcal{M}} g(\mathbf{X}^*, p, N) \mathcal{P}(dN|\nu, \mathbf{X}).
\]

The moment measure \( M \) is also expressible via the conditional moment measures as,

\[
(16) \quad M(d\mathbf{X}) = \nu(dX_1) \prod_{i=2}^{n} \left[ \nu(dX_i) + \sum_{j=1}^{n(p_{i-1})} \delta_{X_j}^*(dX_i) \right]
\]
REMARK 3. It is of course true that Lemma 2.2 is not entirely novel. However, the partition representation that is used is certainly not readily seen in the literature. Moreover, it has been tailor made to assume its present purpose as a general tool. One way to deduce the partition representation is to examine carefully Daley and Vere-Jones (1988), [equation (5.517), Lemma 5.2.VI, and the discussion on page 192]. A simple minded but informative approach is to simply refer back to the case of Poisson random variables. For clarity and also to showcase what is believed to be interesting side results involving partly Lemma 2.1 we prove this result in its entirety in the next section using alternate means.

2.2 Supporting results

Note that Lemma 2.1 implies the following result for each bounded set \( B \),

\[
\int_M N(B) \exp \left\{ -\int_X f(x) N(dx) \right\} P(dN|\nu) = \mathcal{L}_N(f|\nu) \int_B e^{-f(x)} \nu(dx).
\]

This is reminiscent of the expression which appears in Lemma 10.6 in Kallenberg (1986) and perhaps more clearly in Proposition 12.1.V in Daley and Vere-Jones (1998). That is, the expression (17) identifies the conditional Laplace functional of the Poisson process given one observation [see Daley and Vere-Jones (1988), p. 458]. A point to note is that in contrast to Daley and Vere-Jones (1988) Proposition 12.I.V, this result is not obtained by taking derivatives. This suggests that Lemma 2.1 can be used repeatedly to obtain the conditional Laplace functional given \( n \) observations. Thus providing an alternative to an argument using repeatedly say Lemma 12.1.V. The general dual of Lemma 12.1.V. can be deduced from Remark 2 as follows; Suppose that \( \mu \) is an random measure (as in Remark 2) with finite 1st moment measure, say \( m_\mu \), then

\[
\int_M \mu(B) e^{-\mu(f)} P(d\mu) = \mathcal{L}_\mu(f) \int_B \mu(dx) P_f(d\mu) := \mathcal{L}_\mu(f) \int_B r(f|x) m_\mu(dx)
\]

for some function \( r \) determined by the second expression. That is, the evaluation of \( \int_B \int_M \mu(dx) P_f(d\mu) \). Hence the conditional Laplace functional of \( \mu|x \) is

\[
\mathcal{L}_\mu(f|x) := \mathcal{L}_\mu(f) r(f|x).
\]

This general form can be applied repeatedly to (conditional) random measures \( \mu|x_1, \ldots, x_i \) etc, where the requirement is the existence of a finite conditional measure \( m_\mu(\cdot|x_1, \ldots, x_i) \). All such results can be deduced from an argument similar to what is used in Proposition 2.1 below.

REMARK 4. A result for general \( \mu \) is quite applicable. As an example consider finite random mixtures of infinitely divisible random variables. That is,

\[
\sum_{k=1}^m W_{k,m} \delta_{Z_k}
\]

where \( W_{k,m} \) are iid infinitely divisible random variables and \( Z_k \) are iid random variables. Such models can be used as approximations to many of the models discussed here. The emphasis on the change of measure interpretation should also prove useful. In fact, for such an infinitely divisible class the results in Section 3 apply with small modification.
REMARK 5. James (2001a, b) using the analogue of Lemma 2.1 for weighted gamma and generalised weighted gamma process obtained their posterior characterizations in this manner without any specific mention of Poisson processes. The idea for this approach is based on an extension of the arguments in Lo and Weng (1989). In Section 3 it is shown that Lemma 2.1 actually implies these analogues.

**Proposition 2.1** Lemma 2.1 implies that the conditional Laplace functional of $N|X_1,\ldots,X_n$ based on the model is,

$$L_N(f|\nu) \left[ \prod_{j=1}^{n(p)} e^{-f(X_j^\ast)} \right]$$

**Proof.** The result proceeds by induction. Let $s, f \in BM(\mathcal{X})$ and choose $g(N) = \int_{\mathcal{X}} s(\nu)N(d\nu)$. The case for $n = 1$ follows from (17). For general $n = r$ it follows from Lemma 2.1 that the conditional Laplace functional of $N$ given $(X_r, X_{r+1})$ is determined by the expression

$$L_N(f|\nu) \left[ \prod_{j=1}^{n(p_r)} e^{-f(X_j^\ast)} \right] \left[ \int_{\mathcal{X}} s(x_{r+1})e^{-f(X_{r+1})}\nu(dX_{r+1}) + \sum_{j=1}^{n(p_r)} s(X_j^\ast) \right]$$

Now define a function $t(x_{r+1})$ to be $e^{-f(x_{r+1})}$ if $X_{r+1}$ is not equal to $\{X_1^\ast,\ldots,X_n^\ast\}$ and is set to be one otherwise. Then,

$$\int_{\mathcal{X}} s(x_{r+1})t(x_{r+1}) \left[ \nu(dX_{r+1}) + \sum_{j=1}^{n(p_r)} \delta_{X_j^\ast}(dX_{r+1}) \right] = \int_{\mathcal{X}} s(x_{r+1})e^{-f(x_{r+1})}\nu(dX_{r+1}) + \sum_{j=1}^{n(p_r)} s(X_j^\ast).$$

Hence, the conditional Laplace functional is,

$$L_N(f|\nu) \left[ \prod_{j=1}^{n(p_r)} e^{-f(X_j^\ast)} \right] t(X_{r+1}) = L_N(f|\nu) \left[ \prod_{j=1}^{n(p_{r+1})} e^{-f(X_j^\ast)} \right]$$

as desired. □

**Remark 6.** The proof of Proposition 2.2 below follows closely an unpublished proof by Albert Y. Lo for the case of gamma processes. That is, it is an alternate proof for Lemma 2 in Lo (1984) which yields the appropriate partition representation for integrals with respect to a Blackwell-MacQueen urn distribution derived from a Dirichlet process. The style of proof exploits properties of partitions similar to those stated in Pitman (1995a, Proposition 10). In particular see (24) below. See also the proof of Lemma 5 in Hansen and Pitman (2001) for general species sampling models. Details in the proof of Proposition 2.2 translate into generalizations of a weighted Chinese restaurant algorithm given in the next section. Proposition 2.1 and 2.2 combine to yield Lemma 2.2.

**Proposition 2.2** For $i = 1,\ldots,n$, let $g_i$ be non-negative functions in $BM(\mathcal{X})$ then,

$$\int_{\mathcal{X}} \prod_{i=1}^{n(p)} \prod_{i=1}^{n(p_i)} g_i(x_i)N(dx_i) P(dN|\nu) = \sum_{p} \prod_{j=1}^{n(p)} \prod_{i=1}^{n(p_i)} g_i(x_j^\ast) \nu(dx_j^\ast).$$

Equivalently, $M(dX) = \prod_{j=1}^{n(p)} \nu(dX_j^\ast)$. 

**Proof.**
Proof. The proof of (23) proceeds by induction. Case $n = 1$ is obvious. Now suppose it is true for $n = r$. Let $p_{r+1}$ denote a partition of $\{1, \ldots, r+1\}$, and define for each $r > 0$,

$$
\phi_g(p_r) = \prod_{j=1}^{n(p_r)} \int_{X} \left[ \prod_{i \in C_{j,r}} g_i(x_j^r) \right] \nu(dx_j^r)
$$

It follows that $\phi_g(p_{r+1})$ is

$$
\phi_g(p_r) \int_{Y} g_{r+1}(v) \nu(dv)
$$

if $n(p_{r+1}) = n(p_r) + 1$, otherwise if the index $r+1$ is in an existing cell/table $C_{i,r}$ then it is equivalent to

$$
\phi_g(p_r) \int_{Y} g_{r+1}(v) \pi_g(dv|C_{i,r})
$$

where

$$
\pi_g(dv|C_{i,r}) = \frac{\prod_{l \in C_{i,r}} g_l(v)}{\int_{Y} \prod_{l \in C_{i,r}} g_l(v) \nu(dv)}
$$

for $i = 1, \ldots, n(p_r)$. Note that this implies that,

$$
\sum_{p_{r+1}} \phi_g(p_{r+1}) = \sum_{p_r} \phi_g(p_r) \left[ \int_{X} g_{r+1}(v) \nu(dv) + \sum_{i=1}^{n(p_r)} \int_{X} g_{r+1}(v) \pi_g(dv|C_{i,r}) \right].
$$

Now by (simple algebra) and the induction hypothesis on $r$ it follows that,

$$
\sum_{p_{r+1}} \phi_g(p_{r+1}) = \int_{X^n} \left[ \int_{X} g_{r+1}(v) \nu(dv) + \sum_{j=1}^{n(p_r)} g_{r+1}(X_j^r) \right] \left[ \prod_{i=1}^{r} g_i(X_i) \right] M(dX_r).
$$

Now utilizing the fact that, $M(dX_{r+1}) = \left[ \nu(dX_{r+1}) + \sum_{j=1}^{n(p_r)} \delta_{X_j^r}(dX_{r+1}) \right] M(dX_r)$, concludes the proof. $\blacksquare$

Remark 7. Of course (22) in its most basic form leads to well-known results for moments and cumulants of a Poisson random variable. For instance, setting $g_i$ to be indicators of a bounded set $A$ yields,

$$
E(N(A)^n) = \sum_{P} \nu(A)^{n(P)}
$$

Where $N(A)$ is a Poisson random variable with mean measure $\nu(A)$.

2.3 Chinese restaurant like approximation methods

In this section a new algorithm for approximating complex integrals and in fact posterior distributions is described. This algorithm works by sequentially sampling from a partition distribution and structurally behaves similar to the Chinese restaurant process seating algorithm discussed in Aldous (1985), Pitman (1996) and Kerov (1998). In particular, the proposed scheme is influenced by the weighted Chinese restaurant (WCR) procedure developed in Lo, Brunner and Chan (1996) [see also Brunner, Chan, James and Lo (2001)] for mixtures of Dirichlet process and weighted gamma process posterior models. Ishwaran and James (2001a) subsequently generalise the WCR to include the class of species sampling mixture models. The essence of all these algorithms will be revisited in this section.
Notice that the left hand side of (22).

\[
\sum_{\mathbf{p}} \prod_{j=1}^{n(\mathbf{p})} \int_{\mathcal{X}} \prod_{i \in C_j} g_i(x_j^*) \nu(dx_j^*),
\]

is a function of partitions of the form, \( \sum_{\mathbf{p}} t(\mathbf{p}) \). This result is analogous to Lo (1984) where he points out that complex multiple integrals with respect to a Blackwell-MacQueen urn are equivalent to considerably more manageable sums over partitions. However, it is known that the complexity of the number of partitions behaves like Bell’s number as \( n \) increases and hence one needs some method to approximate such quantities. In order to further illustrate a connection to a Chinese restaurant we introduce the following expressions;

\[
\sum_{\mathbf{p}} \theta^n(\mathbf{p}) \prod_{j=1}^{n(\mathbf{p})} \Gamma(e_j - \alpha) \prod_{j=1}^{Y_{r+1}} \int_{\mathcal{X}} \prod_{i \in C_j} K(X_i|Y_j^*) \eta(dy_j^*),
\]

and

\[
\sum_{\mathbf{p}} PD(\mathbf{p}|\alpha, \theta) \prod_{j=1}^{n(\mathbf{p})} \prod_{i \in C_j} K(X_i|Y_j^*) H(dy_j^*).
\]

The expressions above may arise respectively from a generalized gamma mixture model and a two-parameter Poisson-Dirichlet process mixture model. The first expression appears in James (2001b) and reduces to an expression for the gamma process in Lo and Weng (1989) and James (2001a). The latter expression appears in Ishwaran and James (2001a) and extends the analogous result for the the Dirichlet process in Lo (1984). That is, the latter corresponds to the marginal likelihood of \( X|Y \) when \( Y_1, \ldots, Y_n | P \) are iid \( P \), the law of \( P \) is \( P_{\alpha, \theta}(dP|H) \). Allowing for more flexibility in the interpretation of \( g_i \) and \( \nu \) these terms can be written as special cases of (25). Consequently, an understanding of the mechanism behind the WCR algorithm as outlined in Brunner, Chan and Lo (1996) translates into a general algorithm which is now described. From (24) set

\[
l(r) = \int_{\mathcal{X}} g_{r+1}(v) \nu(dx) + \sum_{i=1}^{n(\mathbf{p}_r)} \int_{\mathcal{X}} g_{r+1}(x) \pi_g(dx|C_i, r).
\]

The procedure relies on a method to generate partitions \( \mathbf{p} \) based on the following rule, described in terms of customers entering a restaurant,

**Algorithm 1**  \( r+1 \): Seat the first customer to a table with probability \( l(0)/l(0) = 1 \).

**Step (r+1):** Given \( \mathbf{p}_r \), customer \( r+1 \) sits at table \( C_{j,r} \) with probability

\[
P_r(\mathbf{p}_{r+1} | \mathbf{p}_r) = l(r)^{-1} \int_{\mathcal{X}} g_{r+1}(x) \pi_g(dx|C_{j,r}),
\]

where \( \mathbf{p}_{r+1} = \mathbf{p}_r \cup \{r+1 \in C_{i,r} \} \) for \( i = 1, \ldots, n(\mathbf{p}_r) \). Otherwise, customer \( r+1 \) sits at a new table with probability

\[
P_r(\mathbf{p}_{r+1} | \mathbf{p}_r) = l(r)^{-1} \int_{\mathcal{X}} g_{r+1}(x) \nu(dx).
\]

The completion of Step \( n \) produces a \( \mathbf{p} = \{C_1, \ldots, C_{n(\mathbf{p})}\} = \mathbf{p}_n \), where now \( \mathbf{p} \) is drawn from a density \( q(\mathbf{p}|\mathbf{g}) \) whose form is described in the Lemma 2.3.
Lemma 2.3 The $n$-step seating algorithm results in a partition $\mathbf{p}$ drawn from a density/distribution given by $q(\mathbf{p}|\mathbf{g})$ that satisfies,

$$
\mathcal{I}(\mathbf{p}|\mathbf{g})q(\mathbf{p}|\mathbf{g}) = \prod_{j=1}^{n(\mathbf{p})} \int_{X} \left\{ \prod_{i \in C_j} g_i(X_j^*) \right\} \nu(dX_j^*),
$$

where $\mathcal{I}(\mathbf{p}|\mathbf{g}) = \prod_{r=1}^{n} l(r-1)$.

**Proof.** As in the proof of Proposition 2.2, define $\phi_g(\mathbf{p}_r)$ from (23). Now note carefully that,

$$
\phi_g(\mathbf{p}_r) \pi_g(dX_j^*|C_{j,r}) = \nu(dX_j^*) \prod_{i \in C_{j,r}} g_i(X_j^*) \times \prod_{l \neq j} \int_{i \in C_{l,r}} g_l(u) \nu(du).
$$

Hence it follows that if $\mathbf{p}_{r+1} = \mathbf{p}_r \cup \{r+1 \in C_{j,r}\}$, then

$$
\int g_{r+1}(u) \pi_g(du|C_{j,r}) = \frac{\phi_g(\mathbf{p}_{r+1})}{\phi_g(\mathbf{p}_r)} \text{ and } \mathbb{P}(\mathbf{p}_{r+1}|\mathbf{p}_r) = \frac{\phi_g(\mathbf{p}_{r+1})}{l(r) \phi_g(\mathbf{p}_r)}.
$$

A similar argument for $\mathbf{p}_{r+1}$ forming a new table shows that (29) holds in general. Now notice that since $\mathbf{p}_{r+1}$ contains all the information in $\mathbf{p}_r$, the product rule of probability gives

$$
q(\mathbf{p}_n|\mathbf{g}) = \mathbb{P}(\mathbf{p}_1) \prod_{r=1}^{n-1} \mathbb{P}(\mathbf{p}_{r+1}|\mathbf{p}_r) = \frac{\phi_g(\mathbf{p}_n)}{\mathcal{I}(\mathbf{p}|\mathbf{g})},
$$

where $\mathbb{P}(\mathbf{p}_1) = \phi_g(\mathbf{p}_1)/l(0) = 1$. Now setting $\mathbf{p} = \mathbf{p}_n$ yields the desired result. \[\blacksquare\]

Now to approximate terms such as $\mathcal{I}(\mathbf{p}|\mathbf{g})$, draw an iid sample, say $\mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(B)}$, of size $B$ from $q(\mathbf{p}|\mathbf{g})$ and use $B^{-1} \sum_{b=1}^{B} \mathcal{I}(\mathbf{p}^{(b)}|\mathbf{g})$. The Chinese restaurant algorithm to generate a draw from $PD(\mathbf{p}|\alpha, \theta)$ is recovered by setting $l(r) = \theta + r$, $\int_{X} g_{r+1}(x) \nu(dx) := \theta + n(\mathbf{p}_r) \alpha$, and $\int_{X} g_{r+1}(x) \pi_g(dx|C_{i,r}) := e_{i,r} - \alpha$. In other words under this specification $q(\mathbf{p}|\mathbf{g}) = PD(\mathbf{p}|\alpha, \theta)$.

**Remark 8.** The algorithm above is an example of a sequential importance sampling procedure. The efficient Dirichlet process algorithm discussed in MachEachern, Clyde and Liu (1999) can be seen as a special case of the WCR when using a binomial kernel. The Chinese restaurant process structure however is not emphasized in that work. There are also analogous MCMC methods which can now readily be deduced from the descriptions given in Brunner, Chan, and Lo (1996) or Ishwaran and James (2001a). See Ishwaran and James (2001b) and Ishwaran, James and Sun (2001) for applications and ideas for other algorithms.

## 3 Size-biased generalizations of completely random measures

In this section it is shown how specific applications of Lemma 2.1 and 2.2 yield explicit disintegration results for a class of random measures which includes completely random measures. One feature of the analysis reveals that cumulants assume in many respects the role played by the EPPF in posterior calculus for random probability measures. The present construction is influenced by section 4 of Pitman, Perman and Yor (1992). The results will be applied throughout.
Let $N$ denote a Poisson process on an arbitrary Polish space $\mathcal{X} = \mathcal{S} \times \mathcal{Y}$ with intensity $\nu(ds, dy) = \rho(ds|y)\eta(dy)$ for $\rho$ a Lévy measure on the Polish space $\mathcal{S}$ depending on $y$ in a fairly arbitrary way and $\eta$ a sigma-finite (non-atomic) measure on $\mathcal{Y}$. Denote the law of $N$ as $\mathcal{P}(dN|\rho, \eta)$. As in Pitman, Perman and Yor (1992, section 4) let $h$ denote an arbitrary strictly positive function on $\mathcal{S}$. Furthermore it is assumed that $h, \rho, \eta$ are selected such that for each bounded set $B$ in $\mathcal{Y}$,

$$\int_B \int_\mathcal{S} \min(h(s), 1)\rho(ds|y)\eta(dy) < \infty. \tag{30}$$

Now define a random measure $\mu$ on $\mathcal{Y}$ such that it may be represented in a distributional sense as,

$$\mu(dy) = \int_\mathcal{S} h(s)N(ds, dy). \tag{31}$$

The law of $\mu$ is denoted as $\mathcal{P}(d\mu|\rho, \eta)$. When $\rho$ does not depend on $y$, then write $\rho(ds|y) := \rho(ds)$. Similar to Tsilevich, Vershik and Yor (2001) the term homogeneous will sometimes be applied to this special case of $\rho$ and $\mu$. In the case that $\mathcal{S} = (0, \infty)$ and $h(s) := s$ then following Kingman (1967, 1993) $\mu$ is a completely random measure without a deterministic component.

**Remark 9.** Related to completely random measures, Ferguson and Klass (1972) discuss constructions for the class of Lévy processes on $(0, \infty)$ without a Gaussian component but allowing for fixed points of discontinuity. See also Wolpert and Ickstadt (1998a,b), Brix (1999) for recent applications of completely random measures to spatial statistics. The condition (30) guarantees that $\mu$ in (31) is a *boundedly finite* measure in the language of Daley and Vere-Jones (1988, Definition 6.1.I.). Theorem 6.3.VIII of that work discusses the representation of completely random measures. Kallenberg (1997, chapter 10) describes conditions under which Poisson functionals are finite. Certain aspects of the presentation below are of course implicit in Kallenberg (1986).

The Laplace functional for $\mu$ can be represented as follows

$$\mathcal{L}_\mu(g) = \exp \left( -\int_\mathcal{Y} \int_\mathcal{S} (1 - e^{-g(y)h(s)})\rho(ds|y)\eta(dy) \right). \tag{32}$$

As in Pitman, Perman and Yor (1992) the notation $T := \mu(\mathcal{Y})$ will be used to denote an almost surely finite total mass.

### 3.1 Disintegrations and posterior distributions

Define, the following moments with respect to the measure $\rho(s|y)$ as for each fixed $n$ and $y$,

$$\kappa_n(\rho|y) = \int h(s)^n\rho(ds|y) \quad \text{and} \quad \kappa_n(\rho) = \int h(s)^n\rho(ds).$$

Note that, $m_n(dv|\rho, \eta) = \left[\int \mathcal{S} h(s)\rho(ds|v)\right] \eta(dv)$ denotes the first moment measure of $\mu$.

Now similar to the case of Poisson processes let $\mathcal{P}(d\mu|\rho, \eta)\prod_{i=1}^n \mu(dY_i)$ represent a disintegration of the joint product measure of $\{Y_1, \ldots, Y_n, \mu\}$; where further $\{Y_1, \ldots, Y_n\}$ can be viewed as conditionally independent given $\mu$. The techniques in Section 2 will now be applied to identify the disintegration which describes the posterior distribution of $\mu$ given $\{Y_1, \ldots, Y_n\}$, and the marginal joint measure of $\{Y_1, \ldots, Y_n\}$ and its corresponding disintegration $(Y^*, \mathbf{p})$. A description of the posterior law of $\mu$ will now be given. The result will then be justified in Theorem 3.1.
Now for each \( n \), let \( \mathcal{P}(d\mu|\rho, \eta, \mathbf{Y}) \) denote the conditional law of \( \mu \) corresponding to the random measure,

\[
\mu(\cdot) + \sum_{j=1}^{n(p)} h(J_{j,n})
\]

where \( J_{j,n} \) are independent random variables each with (conditional) distribution depending on \( Y_j^* \),

\[
\mathbb{P}(J_{j,n} \in ds|\rho, Y_j^*) = \frac{h(s)^{\sum_j n}\rho(ds|Y_j^*)}{\int_0^{\infty} h(u)^{\sum_j n}\rho(du|Y_j^*)}
\]

and chosen independently of \( \mu \) which is \( \mathcal{P}(d\mu|\rho, \eta) \).

The corresponding (conditional) moment measure and Laplace functional for (34) are given by

\[
m_\mu(dv|\rho, \eta, \mathbf{Y}) = m_\mu(dv|\rho, \eta) + \sum_{j=1}^{n(p)} E[h(J_{j,n})|Y_j^*] \delta_{Y_j^*}(dv)
\]

and

\[
\mathcal{L}_\mu(g|\rho, \eta, \mathbf{Y}) = \mathcal{L}_\mu(g|\rho, \eta) \prod_{j=1}^{n(p)} \int_0^{\infty} e^{-g(Y_j^*)h(s)} \mathbb{P}(J_{j,n} \in ds|\rho, Y_j^*)
\]

The joint marginal measure of \( \mathbf{Y} \) is expressible as

\[
M_\mu(d\mathbf{Y}|\rho, \eta) = m_\mu(dY_1|\rho, \eta) \prod_{i=2}^n m_\mu(dY_i|\rho, \eta, Y_1, \ldots, Y_{i-1}).
\]

**Theorem 3.1** Suppose that \( \mu \) is a random measure defined by (32) and assume that \( \kappa_n(\rho|y) < \infty \) for each fixed \( y \). Let \( g \) be a non-negative or integrable function on \( \mathcal{Y}^n \times \mathcal{M} \), then for each \( n \geq 1 \),

\[
\int_{\mathcal{M} \times \mathcal{Y}^n} g(\mathbf{Y}, \mu) \prod_{i=1}^n \mu(dY_i) \mathcal{P}(d\mu|\rho, \eta) = \int_{\mathcal{M} \times \mathcal{Y}^n} g(\mathbf{Y}, \mu) \mathcal{P}(d\mu|\rho, \eta, \mathbf{Y}) M_\mu(d\mathbf{Y}|\rho, \eta).
\]

The expressions in (37) are equivalent to,

\[
\sum_{\mathbf{p}} \int_{\mathcal{Y}} \prod_{i=1}^{n(p)} \int_{\mathcal{M}} g(\mathbf{Y}^*, \mathbf{p}, \mu) \mathcal{P}(d\mu|\rho, \eta, \mathbf{Y}) \prod_{j=1}^{n(p)} \kappa_{e_j,n}(\rho|Y_j^*) \eta(dy_j^*)
\]

and \( M_\mu(d\mathbf{Y}|\rho, \eta) := \prod_{j=1}^{n(p)} \kappa_{e_j,n}(\rho|Y_j^*) \eta(dy_j^*) \)

**Proof.** First by definition, \( M_\mu \) is completely determined by

\[
\int_{\mathcal{Y}^n} \left[ \prod_{i=1}^n \tilde{g}_i(Y_i) \right] M_\mu(d\mathbf{Y}|\rho, \eta) = \int_{\mathcal{M} \times \mathcal{Y}^n} \left[ \prod_{i=1}^n \tilde{g}_i(Y_i) \right] \prod_{i=1}^n \mu(dY_i) \mathcal{P}(d\mu|\rho, \eta)
\]

for \( \tilde{g}_i \) in \( \mathcal{B}(\mathcal{Y}) \). But this is equivalent to

\[
\int_{\mathcal{M} \times \mathcal{Y}^n} \left[ \prod_{i=1}^n \tilde{g}_i(Y_i) \right] \left[ \prod_{i=1}^n \int_0^{\infty} h(s_i)N(ds_i, dY_i) \right] \mathcal{P}(dN|\rho, \eta).
\]
A direct application of Lemma 2.2, or Proposition 2.2, yields all the desired forms of $M_\mu$. This is seen immediately by setting $g_i(x_i) = h(s_i)g_i(y_i)$ and replacing $\nu(dx)$ with $\rho(ds|y)\eta(dy)$. Now it simply remains to show that the conditional Laplace functional of $\mu$ is $\mathbb{L}$. Since the form of the marginal measure $M_\mu$ is established, Lemma 2.2 now shows that the conditional Laplace functional is obtained by using the fact that $\mu(g) := \int_\mathbb{Y} \int_\mathcal{S} h(s)g(y)N(ds, dy)$ and replacing $f(x^*)$ in Proposition 2.1 with $g(y^*)h(s_j)$. Hence, the conditional Laplace functional of $\mu$ is equivalent to,

$$\int_{\mathbb{S}^{n(p)}} \left( \int_{\mathcal{M}} e^{-\mu(h)} \mathcal{P}(d\mu|\rho, \eta, \mathbf{y}^*) \right)^{\frac{n(p)}{j=1}} \mathcal{P}(\mathcal{J}_{j,n} \in ds|\rho, \eta, \mathbf{y}^*_j).$$

\[\blacksquare\]

### 3.2 Cumulants and moment representations

The joint marginal measure $M_\mu$ disintegrates into

$$M_\mu(d\mathbf{y}|\rho, \eta) = \left[ \prod_{j=1}^{n(p)} \kappa_{e_{j,n}}(\rho|\mathbf{y}_j^*) \right] \prod_{j=1}^{n(p)} \eta(d\mathbf{y}_j^*),$$

which shows that the possibly sigma-finite measure of $\mathbf{Y}$, disintegrates into an appropriate joint measure of $(\mathbf{Y}^*, \mathbf{p})$. Such structures will play a fundamental role throughout. As a simple application the joint structure can be used to obtain expressions for the moments of the corresponding random variable $\mu(B)$, for $B$ finite, as follows,

$$E[\mu(B)^n|\rho, \eta] := \sum_{\mathbf{p}} \prod_{j=1}^{n(p)} \int_B \kappa_{e_{j,n}}(\rho|\mathbf{y}_j^*) \eta(d\mathbf{y}_j^*)$$

which corresponds to the classical relationship between moments and cumulants. Additionally for integrable linear functionals $\mu(f_1)$, one might be interested in calculating the joint moments,

$$E\left[ \prod_{i=1}^{q} (\mu(f))^{\eta} \right] = \int_{\mathcal{M}} \left[ \prod_{i=1}^{q} \mathcal{P}(d\mu|\rho, \eta) \right] f_{1}(y_{i,n})\mu(dy_{i,n})$$

for integers $n_i$ such that without loss of generality $n = \sum_{i=1}^{q} n_i$. An application of Theorem 3.1 easily yields,

$$E\left[ \prod_{i=1}^{q} (\mu(f_1))^{e_{i,n}} \right] = \sum_{\mathbf{p}} \prod_{j=1}^{n(p)} \int_{\mathcal{Y}} \kappa_{e_{j,n}}(\rho|u) \left[ \prod_{j=1}^{q} f_{1}^{e_{j,n}}(u) \right] \eta(du)$$

where $e_{j,n}^i$, satisfying $e_{j,n} := \sum_{i=1}^{q} e_{j,n}^i$, denotes the number of indices associated with $f_i$ in $C_j$. Suppose that $E[T^\eta|\rho, \eta] < \infty$, then an important case of (41) is,

$$E[T^\eta|\rho, \eta] := \sum_{\mathbf{p}} \prod_{i=1}^{n(p)} \kappa_{e_{i,n}}(\Omega) = \sum_{\mathbf{p}} \prod_{j=1}^{n(p)} \int_{\mathcal{Y}} \kappa_{e_{j,n}}(\rho|\mathbf{y}_j^*) \eta(d\mathbf{y}_j^*).$$

The consequences of this representation will play a major role in Section 5.

**Example** [1]. As an important example consider the generalised gamma Levy measure $\rho_{\alpha,b}$ for $b > 0$. In this case the $J_{j,n}$ are $\mathcal{G}(e_{j,n} - \alpha, b)$,

$$\prod_{i=1}^{n(p)} \kappa_{e_{i,n}}(\rho_{\alpha,b}) := \prod_{j=1}^{n(p)} \Gamma(e_{j,n} - \alpha) b^{-(e_{j,n} - \alpha)} := b^{-(n-n(p))} \prod_{j=1}^{n(p)} \Gamma(e_{j,n} - \alpha),$$

$$\prod_{i=1}^{n(p)} \kappa_{e_{i,n}}(\rho_{\alpha,b}) := \prod_{j=1}^{n(p)} \Gamma(e_{j,n} - \alpha) b^{-(e_{j,n} - \alpha)} := b^{-(n-n(p))} \prod_{j=1}^{n(p)} \Gamma(e_{j,n} - \alpha),$$
and hence
\[ E[\mu(B)^n|\rho, b, \eta] := b^{-n} \sum_{\mathbf{p}} b^{n(p)} \eta(B)^{n(p)} \prod_{j=1}^{n(p)} \Gamma(e_{j,n} - \alpha) \]

When \( \alpha := 0 \) and \( b = 1 \) this expression combined with (33) corresponds to the gamma process with shape measure \( \eta \). In this case, where \( \eta(Y) = \theta \) is finite, it follows that normalising the expression (33) by \( E[T^n|\rho, b, \eta] \) yields the EPPF, \( PD(\eta|\theta) \) of the Dirichlet process \( PD_{0,0}(d\eta|H) \). Otherwise in the un-normalised case one obtains expressions for the generalised gamma random measure in Brix (1999). In the weighted version of this model, discussed in James (2001b), set \( b := \delta(Y^*) \), which reduces to expressions for the weighted gamma process when \( \alpha = 0 \) in Lo and Weng (1989). Note that although dividing by \( E[T^n|\rho, b, \eta] \) yields a proper distribution for \( \mathbf{p} \) it is not an EPPF except for the case of the Dirichlet process.

**Example [2].** For the Beta process, with parameters \( c, A \), the \( J_{j,n}|Y^*_j \) are \( B(c_{j,n}, c(Y^*_j)) \). Hence,
\[ \kappa_{e_{j,n}}(\rho|Y^*_j) := \frac{\Gamma(e_{j,n})\Gamma(c(Y^*_j))}{\Gamma(e_{j,n} + c(Y^*_j))}, \quad \text{and} \quad E[J_{j,n}|Y^*_j] := \frac{\kappa_{1+e_{j,n}}(\rho|Y^*_j)}{\kappa_{e_{j,n}}(\rho|Y^*_j)} := \frac{e_{j,n}}{e_{j,n} + c(Y^*_j)} \]

These types of integrable operations identify joint distributional structures for \( (Y^*, \mathbf{p}) \) which have product form. This fact is summarized in the next result. The result is important for applications of mixture models where \( Y \) again are missing values and not observables. The result will also play a significant role in Section 5.

**Corollary 3.1** Let \( \prod_{i=1}^{n} \int_{\mathcal{Y}} \tilde{g}_i(Y_i) \mu(dY_i) \) be an integrable function of \( \mathcal{P}(d\mu|\rho, \eta) \), then there exists a conditional distribution of \( Y|\mathbf{p} \) such that the unique values \( \{Y^*_1, \ldots, Y^*_n(\mathbf{p})\} \) are independent with distribution
\[ \mathbb{P}(dY^*_j|\rho, \eta, \tilde{g}) \propto \left[ \prod_{i \in C_j} \tilde{g}_i(Y^*_j) \right] \kappa_{e_{j,n}}(\rho|Y^*_j) \eta(dY^*_j). \]

and \( \mathbf{p} \) has a distribution proportional to \( \prod_{j=1}^{n(\mathbf{p})} \int_{\mathcal{Y}} \left[ \prod_{i \in C_j} \tilde{g}_i(Y^*_j) \right] \kappa_{e_{j,n}}(\rho|Y^*_j) \eta(dY^*_j) \). If the integrability condition still holds when the \( (\tilde{g}_i) \) are equal to one, then
\[ \mathbb{P}(dY^*_j|\rho, \eta) \propto \kappa_{e_{j,n}}(\rho|Y^*_j) \eta(dY^*_j), \]

the distribution of \( \mathbf{p} \), is proportional to \( \prod_{j=1}^{n(\mathbf{p})} \kappa_{e_{j,n}}(\Omega) \). In the homogeneous case the integrability condition holds only if \( \eta \) is a finite measure. When \( \eta \) is a finite measure it follows also that the unique values \( Y^*_1, \ldots, Y^*_n(\mathbf{p}) \) are iid \( \eta(\cdot)/\eta(\mathcal{Y}) := H(\cdot) \).

Hereafter, based on (36) and Theorem 3.1, denote a joint law (conditional on \( \mathbf{p} \)) of \( \{(J_{j,n}, Y^*_j)\} \) as
\[ \mathbb{P}(d\mathbf{J}, dY^*|\rho, \eta) := \prod_{j=1}^{n(\mathbf{p})} \mathbb{P}(dJ_{j,n}|\rho, Y^*_j) \mathbb{P}(dY^*_j|\rho, \eta). \]

It follows that there exists joint laws of \( N, \mathbf{J}, Y^*|\mathbf{p} \) denoted as
\[ \mathcal{P}(dN, d\mathbf{J}, dY^*|\rho, \eta) := \mathcal{P}(dN|\rho, \eta, \mathbf{J}, Y^*) \mathcal{P}(d\mathbf{J}, dY^*|\rho, \eta) := \mathcal{P}(dN, d\mathbf{J}|\rho, \eta, Y^*) \mathcal{P}(dY^*|\rho, \eta) \]
and also a joint law of $\mu, Y$ given $p$ denoted as

$$\mathcal{P}(d\mu, dY^*|\rho, \eta) := \mathcal{P}(d\mu|\rho, Y) \prod_{j=1}^{n(p)} \mathbb{P}(dY_j^*|\rho, \eta).$$

### 3.3 Updating and moment formulae

This section presents a series of important exponential based updating (change of measure) formulae which will be used throughout.

**Proposition 3.1** (Updating and moment formulae I) Let $N$ denote a Poisson process with law $\mathcal{P}(dN|\rho, \eta)$. In addition, let $f$ denote a positive function on $S \times Y$. Suppose that $w(\mu)$ is a positive integrable function of $\mu$, such that it is representable as $w(\mu) = e^{-N(f)}$. Then,

(i) $$w(\mu)\mathcal{P}(d\mu|\rho, \eta) = \mathcal{P}(d\mu|e^{-f} \rho, \eta) E[w(\mu)|\rho, \eta] := \mathcal{P}(d\mu|e^{-f} \rho, \eta) \mathcal{L}_N(f|\rho, \eta).$$

(ii) If $\kappa_n(e^{-f} \rho|y) < \infty$ then for each $n$, $$w(\mu) \prod_{i=1}^{n} \mu(dY_i)\mathcal{P}(d\mu|\rho, \eta) = \mathcal{P}(d\mu|e^{-f} \rho, \eta, Y) \mathcal{L}_N(f|\rho, \eta) M_\mu(dY|e^{-f} \rho, \eta)$$

(iii) If $\kappa_n(\rho|y) < \infty$ holds then, $$w(\mu)\mathcal{P}(d\mu|\rho, \eta, Y) = \mathcal{P}(d\mu|e^{-f} \rho, \eta, Y) E[w(\mu)|\rho, \eta, Y]$$

and

$$\mathcal{L}_N(f|\rho, \eta) M_\mu(dY|e^{-f} \rho, \eta) := E[w(\mu)|\rho, \eta, Y] M_\mu(dY|\rho, \eta),$$

where $E[w(\mu)|\rho, \eta, Y] := \mathcal{L}_N(f|\rho, \eta) \prod_{j=1}^{n(p)} \left[ \int_S e^{-f(s, Y^*_j)} \mathbb{P}(\tilde{J}_{j,n} \in ds|Y^*_j, \rho) \right].$

**Proof.** As in Lemma 2.1, it suffices to show that the two sides in statement (i) have the same Laplace functional. An application of Lemma 2.1, combined with the fact that $\mu$ is a functional of $N$ implies that

$$\int_{\mathcal{M}} e^{-\mu(g)} \mathcal{P}(d\mu|\rho, \eta) = \int_{\mathcal{M}} e^{-\mu(g)} e^{-N(f)} \mathcal{P}(dN|\rho, \eta) := \int_{\mathcal{M}} e^{-\mu(g)} \mathcal{P}(d\mu|e^{-f} \rho, \eta) E[w(\mu)|\rho, \eta]$$

which yields statement (i). The result in statement (ii) is obtained by replacing $e^{-\mu(g)}$ in (39) with $e^{-\mu(g)} e^{-N(f)}$ and applying Lemma 2.1 to the inner integral. 

**Proposition 3.2** (Updating and moment formulae II) Suppose that $f$ in Proposition 3.1 is replaced by $f_n = \sum_{i=1}^{n} f_i$, for positive integrable functions $f_i$, this implies that $w(\mu) := \prod_{i=1}^{n} w_i(\mu)$ where $w_i(\mu) = e^{-N(f_i)}$. Now with respect to the joint model $\mathcal{P}(d\mu|\rho, \eta) \prod_{i=1}^{n} w_i(\mu)\mathcal{P}(dY_i)$, the following additional formulae are given; (all expressions are assumed to be finite)

(i) $$\mathcal{L}_N(f_n|\rho, \eta) := \mathcal{L}_N(f_1|\rho, \eta) \prod_{i=2}^{n} \mathcal{L}_N(f_i|e^{-f_{i-1}} \rho, \eta) := E[\prod_{i=1}^{n} w_i(\mu)|\rho, \eta],$$

where $f_i := \sum_{j=1}^{i} f_j$. 
(ii) The expressions below are equivalent.

\[ E[w_i(\mu)|e^{-f_{i-1}}\rho, \eta, Y_{i-1}]]m_\mu(dY_i|e^{-f_i}\rho, \eta, Y_{i-1}). \]

\[ E[w_i(\mu)|e^{-f_i-1}\rho, \eta, Y_i]]m_\mu(dY_i|e^{-f_i-1}\rho, \eta, Y_{i-1}). \]

(iii) The above statements coupled with Proposition 3.1 imply that the marginal calculation,

\[ \int_{M} \prod_{i=1}^{n} w_i(\mu)\mu(dY_i) \mathcal{P}(d\mu|\rho, \eta) \]

is equivalent to the following formulae,

\[ \mathcal{L}_N(f_\eta|\rho, \eta)M_\mu(dY|e^{-f_\eta}\rho, \eta), \]

\[ E[w(\mu)|\rho, \eta, Y]M_\mu(dY|\rho, \eta), \]

\[ E[w(\mu)|\rho, \eta, Y_1]m_\mu(dY_1|e^{-f_\eta}\rho, \eta) \prod_{i=2}^{n} E[w_i(\mu)|e^{-f_{i-1}}\rho, \eta, Y_{i-1}]]m_\mu(dY_i|e^{-f_i}\rho, \eta, Y_{i-1}), \]

\[ E[w_1(\mu)|\rho, \eta, Y_1]m_\mu(dY_1|\rho, \eta) \prod_{i=2}^{n} E[w_i(\mu)|e^{-f_{i-1}}\rho, \eta, Y_i]]m_\mu(dY_i|e^{-f_i-1}\rho, \eta, Y_{i-1}). \]

**Remark 10.**

A notable special case of Proposition 3.1 (i) was established in Lo and Weng (1989)[see also Lo(1982) and James (2001a)] for the weighted gamma process. In Lo and Weng (1989, Proposition 3.1), the statement proceeds as follows

**Proposition 3.1, Lo and Weng (1989) 1** Let \( G_{\eta, \beta} \) denote the law of a weighted gamma process with shape \( \eta \) and weight \( \beta(\cdot) \), then for each positive \( g \)

\[ \int_{M} g(\mu)e^{-\mu(f)}G_{\eta, \beta}(d\mu) = L_{G_{\eta, \beta}}(f) \int_{M} g(\mu)G_{\eta, \beta}^*(d\mu), \]

where \( \beta^* = \beta/(1 + \beta f) \).

This result establishes the absolute continuity of weighted gamma processes and identifies the specific densities. In other words the result of Lo and Weng (1989, Proposition 3.1) includes the quasi-invariance result for the gamma process recently established independently in Tsilevich, Vershik and Yor (2001, Theorem 3.1). The applications considered by Tsilevich, Vershik and Yor (2001) are vastly different from Lo and Weng (1989) and it is not surprising that this result emerges in another context. The development of the exponential formulae used here are directly inspired by Lo and Weng (1989, Proposition 3.1).
3.4 A simple proof for the almost sure discreteness of size-biased measures

In this section a Fubini argument is used to establish the almost sure discreteness of $\mu$ with law $\mathcal{P}(d\mu|\rho,\eta)$. This will include a simple alternative proof for the class of completely random measures as discussed in Kingman (1993, Chapter 10). Kingman’s result is based on a modification of Blackwell’s (1973) argument for the Dirichlet process. The present technique is based on the approach of Berk and Savage (1979) and Lo and Weng (1989) for the Dirichlet process and weighted gamma process respectively. The only requirement I will need is that $\mu$ admits a disintegration, has a 1st moment measure, or is absolutely continuous with respect to the law of another measure $\mu^*$ which has one. The latter case of course will yield the result for the stable law. Measurability issues vanish on Polish spaces. The idea is to apply a 1-step disintegration of $\mu(dx)\mathcal{P}(d\mu|\rho,\eta)$. For the arguments below it suffices to show that the result holds over all bounded sets $B$, i.e. sets such that $\eta(B) < \infty$ so without loss of generality we can assume that $\eta$ is a finite measure.

Proposition 3.3 (Almost sure discreteness of measures) Suppose that $\mu$ is $\mathcal{P}(d\mu|\rho,\eta)$ such that $\mu$ has a 1st moment measure $m_\mu(\cdot|\rho,\eta)$. Otherwise suppose that there exists a measure $\mu^*$ which admits a 1st moment measure and satisfies the absolute continuity relationship,

$$\mathcal{P}(d\mu|\rho,\eta) := g(\mu^*)\mathcal{P}(d\mu^*|\rho^*,\eta),$$

for some positive integrable function $g$. Then $\mu$ is almost surely discrete. That is,

$$\int_\mathcal{M} \mu(\{x : \mu(\{x\}) = 0\})\mathcal{P}(d\mu|\rho,\eta) := 0$$

Proof. Using a disintegration argument,

$$\int_\mathcal{M} \mu(\{x : \mu(\{x\}) = 0\})\mathcal{P}(d\mu|\rho,\eta) := \int \left[ \int_\mathcal{M} I\{x : \mu(\{x\}) = 0\}\mathcal{P}(d\mu|\rho,\eta) \right] m_\mu(dx|\rho,\eta)$$

From Theorem 3.1 the law of $\mu$ taken with respect to the inner term(that is given $x$) is the same as

$$\mu(\cdot) + h(J)\delta_x(\cdot)$$

where $h(J)$ is strictly positive and $J$ has law $\propto h(s)\rho(ds|x)$ for almost all $x$. But since $\mu$ is not negative it follows that

$$\mu(\{x\}) + h(J)\delta_x(\{x\}) \geq h(J) > 0.$$ 

for almost all $x$ with respect to $m_\mu$. Hence the inner term is zero which concludes the result. If again $\mu$ does not admit such a disintegration, apply the result to the random measure $\mu^*$, with for instance Lévy measure $e^{-sb}\rho$, or some other operation, and use the absolute continuity of measures. $\blacksquare$

Remark 11. This method also implies the almost sure discreteness of the measures in Section 5. Beyond the mild restriction to Polish spaces, I believe this is the most general result of this type. The absolute continuity in (52) coupled with the existence of such $\mu^*$, for instance via Proposition 3.1, seems to exhaust the possibilities.
4 Intensity rate mixture models, Lévy moving averages and shot-noise processes

In this section analysis for the class of mixture of hazards models as discussed in the introduction, otherwise known as Lévy-Cox moving average models, is given. Very little is known about the posterior structure of such models with a notable exception being the case of mixtures of weighted gamma processes which is discussed in various degrees of generality in Dykstra and Laud (1981), Lo (1982), Lo and Weng (1989). Wolpert and Ickstadt (1998a) and James (2001a) consider semiparametric extensions of this model. Full partition based posterior analysis in a general multiplicative setting is given in Lo and Weng (1989) and James (2001a). One consequence of the absence of a general analysis of this model is the unavailability of computational procedures which sample from the updated or posterior based models. As mentioned in the introduction such models are currently used in Spatial statistical applications and survival analysis. An interesting class of models which fits into this framework is the generalised gamma process proposed in Brix (1999). Wolpert and Ickstadt (1998b) propose models based on arbitrary Lévy processes. The analysis here includes these models as well as mixture models based on the general size-biased random measures described in section 3.

Consider the random hazard or intensity rate,

\[ \lambda(t|\mu) = \int Y K(t|v) \mu(dv), \]  

where \( K \) denotes a known \((\tau, \eta)\)-integrable kernel on a Polish space \( X \times Y \) and \( \mu \) is modelled as a random measure with law \( P(d\mu|\rho, \eta) \). The representation in (53) defines a large class of random measures \( \lambda \) which are not independent increment processes.

In this section explicit posterior characteristics of \( \mu \) and hence \( \lambda \) based on the multiplicative intensity likelihood,

\[ L(X|\mu)P(d\mu|\rho, \eta) = \prod_{i=1}^{n} \int Y K(X_i|Y_i) \mu(dY_i) e^{-\mu(f_K)} P(d\mu|\rho, \eta) \]

are derived where, \( f_K(v) = \left[ \int S Y(s) K(s|v) \tau(ds) \right] \).

Here \( X_i \) are observations in a (Polish space) region \( X \) and \( Y_i \) can be viewed as missing observations on \( Y \). \( Y(s) \) is a non-negative predictable function which for many applications in event history analysis denotes the number of observed individuals still at risk just before time \( s \). An important point is that the structural form of \( L(X|\mu) \) remains the same under right censoring and left filtering [see Jacod (1975) and Andersen, Borgan, Gill and Keiding (1993)]. If \( Y(s) = 1 \) then the model may correspond to the likelihood of an inhomogeneous Poisson process with intensity rate \( \lambda(t|\mu) \). One purpose of the development of Lemma 2.1 is to handle the exponential term in (43). Thus mimicking the application of Lo and Weng (1989, Proposition 3.1). Proposition 3.1 and Theorem 3.1 in this paper readily yield the desired results. Moreover the appearance of the exponential term combined with Proposition 3.1 show that analysis of this model only requires the weaker condition,

\[ \kappa_n(e^{-f_K} \rho|y) = \int_0^\infty e^{-f_K(y)s} h(s)^n \rho(ds|y) < \infty. \]

The condition (55) will be assumed throughout this section and now for instance admits analysis for the stable law \( \rho_{n,0} \), via Proposition 3.1.
An application of Proposition 3.1 combined with Corollary 3.1 show that the marginal likelihood is,

$$
\int_M L(X|\mu)\mathcal{P}(d\mu|\rho, \eta) = \mathcal{L}_\mu(f_K|\rho, \eta) \sum_p \prod_{j=1}^{n(p)} \prod_{i \in C_j} K(X_i|Y_j^*) \kappa_{\varepsilon_j,n}(e^{-f_K \rho}|Y_j^*) \eta(dY_j^*).
$$

4.1 Posterior characterizations

Explicit posterior characterizations are now given which follows immediately from the Theorem 3.1, Corollary 3.1. (Note that it is assumed throughout that all relevant integrals are finite).

**Theorem 4.1** The posterior distribution of $Y, \mu|X$ based on the model (54) is representable as,

$$
\pi(dY, d\mu|X) \propto \mathcal{P}(d\mu|e^{-f_K \rho}, \eta, Y) \prod_{i=1}^{n(p)} \mathbb{P}(dY_i^*|e^{-f_K \rho}, \eta, K) \pi(p|X),
$$

where $\pi(p|X) \propto \prod_{j=1}^{n(p)} \int_Y \prod_{i \in C_j} K(X_i|Y_j^*) \kappa_{\varepsilon_j,n}(e^{-f_K \rho}|Y_j^*) \eta(dY_j^*)$ is a (posterior) distribution of $p|X$.

Similar to Lo and Weng (1989, Theorem 4.2), Theorem 4.1 implies for instance that the posterior expectation of the intensity, $\lambda|Y$, is

$$
E[\lambda(t|\mu)|Y] = \int_Y K(t|v) \kappa_1(e^{-f_K \rho}|v) \eta(dv) + \sum_{j=1}^{n(p)} K(t|Y_j^*) \frac{\kappa_{1+\varepsilon_j,n}(e^{-f_K \rho}|Y_j^*)}{\kappa_{\varepsilon_j,n}(e^{-f_K \rho}|Y_j^*)}
$$

and hence the posterior expectation given $X$ is,

$$
E[\lambda(t|\mu)|X] = \sum_p \left( \int_Y K(t|v) \kappa_1(\rho_{f_K}|v) \eta(dv) + \sum_{j=1}^{n(p)} \int_Y K(t|v) \frac{\kappa_{1+\varepsilon_j,n}(e^{-f_K \rho}|v)}{\kappa_{\varepsilon_j,n}(e^{-f_K \rho}|v)} \pi(dv|C_j) \right) \pi(p|X).
$$

**Example** [Generalised gamma process]. A brief description of the results related to the usage of a generalised gamma process with intensity $\rho_{\alpha,b}(ds)\eta(dy)$ as in Brix (1999) are given. Note in particular that the result holds for the stable case $b = 0$. The posterior distribution of $\mu$ given $Y$ is denoted as $\mathcal{P}(d\mu|e^{-f_K \rho}, \alpha, b, Y)$. That is, $J_{\varepsilon_j,n}$ given $Y_j^*$ are independent $G(e_j, n - \alpha, b + f_K(Y_j^*))$, and $\mu$ is now a weighted generalised gamma process with Laplace function

$$
\exp \left\{ -\frac{1}{\alpha} \int_Y [(b + f_K(v) + g(v))^\alpha - (b + f_K(v))^\alpha] \eta(dv) \right\}
$$

The joint moment measure of $Y$ can be expressed as,

$$
M_\mu(dY|e^{-f_K \rho}, \alpha, b, \eta) = \prod_{j=1}^{n(p)} \Gamma(e_j, n - \alpha) \prod_{j=1}^{n(p)} (b + f_K(Y_j^*))^{-e_j,n-\alpha} \eta(dY_j^*)
$$

which generalizes an expression for the weighted gamma process, see Lo and Weng (1989) and James (2001a). See James (2001b) for more details related to the generalised gamma model.
REMARK 12. Note that from a practical point of view the distribution of $J_{j,n}$ based on $P(d\mu | e^{-fK}\rho, \eta, Y)$ may not always be easy to simulate. If however the moment condition in Theorem 3.1 holds then one can use an alternative characterization of the posterior based on $P(d\mu | \rho, \eta, Y)$. In that case one does not marginalize over the exponential term but instead works with the measure,

$$e^{-\mu(fK)} P(d\mu | \rho, \eta, Y)$$

REMARK 13. James (2001a) gives results for semi-parametric weighted Gamma process mixture models under more complex multiplicative intensity structures. That is for cases where the kernel $K$ depends on a Euclidean parameter $\psi$ and where for instance there may be several independent Poisson processes. A careful examination of that work, coupled with the results given here provides an obvious way to obtain the corresponding result for the general processes. A notable wrinkle is that the Laplace functionals will depend on $\psi$. A discussion of this is omitted for brevity.

4.2 Simulating the posterior

Here the algorithm discussed in Section 2.3 is applied to this setting to demonstrate a possible approach to approximate posterior quantities. Again MCMC based methods can also be deduced from the algorithm below. First set,

$$l(r|K) = \int_{Y^*} K_{r+1}(X_{r+1}|v) \kappa_1(e^{-fK}\rho|v) \eta(dv) + \sum_{j=1}^{n(p_r)} \int_{Y} K_{r+1}(X_{r+1}|v) \frac{\kappa_1 + \epsilon_{j,r}}{\kappa_{e_{j,r}}(e^{-fK}\rho|v)} \pi(dv|C_{j,r}),$$

where in particular, $l(0|K) = \int_{Y} K_{r+1}(X_{r+1}|v) \kappa_1(e^{-fK}\rho|v) \eta(dv)$.

One can now use the variant of the WCR described in Section 2.3 with the seating rule: Given $p_r$, customer $r+1$ sits at table $C_{j,r}$ with probability

$$\mathbb{P}(p_{r+1}|p_r) = l(r|K)^{-1} \int_{Y^*} K_{r+1}(X_{r+1}|v) \frac{\kappa_1 + \epsilon_{j,r}}{\kappa_{e_{j,r}}(e^{-fK}\rho|v)} \pi(dv|C_{j,r}),$$

where $p_{r+1} = p_r \cup \{r+1 \in C_{i,r}\}$ for $i = 1, \ldots, n(p_r)$. Otherwise, customer $r+1$ sits at a new table with probability

$$\mathbb{P}(p_{r+1}|p_r) = l(r|K)^{-1} \int_{Y} K(X_{r+1}|v) \kappa_1(e^{-fK}\rho|v) \eta(dv).$$

The completion of Step $n$ produces a $p = \{C_{1}, \ldots, C_{n(p)}\} = p_n$, where now, from Lemma 2.3, $p$ is drawn from the density $q(p|K)$ which satisfies,

$$\mathcal{I}(p|K)q(p|K) = \prod_{j=1}^{n(p)} \int_{Y^*} \left[ \prod_{i \in C_j} K(X_i|Y_i^*) \right] \frac{1}{\kappa_{e_{j,n}}(e^{-fK}\rho|Y_i^*)} \eta(dY_i^*),$$

This fact, together with Theorem 4.1, implies that for any integrable function $t(p)$,

$$\sum_{p} t(p) \pi(p|X) = \frac{\sum_{p} t(p) \mathcal{I}(p|K)q(p|K)}{\sum_{p} \mathcal{I}(p|K)q(p|K)}.$$

The expression (60) and Theorem 4.1 now suggest a method to approximate the posterior law $P(d\mu|X)$.
1. Using the seating algorithm above, draw $B$ iid random partitions $p = \{C_1, \ldots, C_{n(p)}\}$ from $q(p|K)$.

2. Use the value of $p$ to draw $Y_j^*$ independently from $\pi(dY_j^*|C_j)$ for $j = 1, \ldots, n(p)$. This yields $Y^* = (Y_1^*, \ldots, Y_n^*(p))$.

3. Using the current value of $(Y^*, p)$, approximate a draw from the random measure

$$
\mu_{f_k}(\cdot) + \sum_{j=1}^{n(p)} h(J_{j,n}) \delta_{Y_j^*}(\cdot),
$$

which is distributed as $P(\mu|e^{-f_k}\rho, \eta, Y)$.

4. To approximate the posterior law of a functional $g(\mu)$, run the previous steps $B$ times independently obtaining values $\mu^{(b)}$ with importance weights $I(p^{(b)})$, for $b = 1, \ldots, B$. Approximate the law, $P(g(\mu) \in \cdot|X)$, with

$$
\frac{\sum_{b=1}^{B} I\{g(\mu^{(b)}) \in \cdot\} I(p^{(b)}|K)}{\sum_{b=1}^{B} I(p^{(b)}|K)}.
$$

If one only needs to approximate moments, or an integration which yields a $t(p)$ in closed form, for instance the likelihood, then steps 2 and 3 can be eliminated and one can replace (62) with

$$
\frac{\sum_{b=1}^{B} t(p^{(b)}) I(p^{(b)}|K)}{\sum_{b=1}^{B} I(p^{(b)}|K)}.
$$

REMARK 14. Note that in (61) the main difficulty is to approximate a draw from $\mu_{f_k}$. Brix (1999) discusses methods on how to approximate a generalized gamma process $P(d\mu|\rho, \eta)$. It should be straightforward to extend this to a $P(d\mu|e^{-f_k}\rho, \eta)$. See also Wolpert and Ickstadt (1998b) for some possible ideas in the general setting. I believe that the mixture representations given in the next section may also be useful in this regard.

5 Analysis of a Scaling operation which arises in Brownian Excursion theory

The previous section describes applications of various exponential change of measure operations. As shown from Lemma 2.1 this operation results in a change of measure from a Poisson process with intensity $\nu$ to another Poisson process law with intensity $e^{-f}\nu$. An important aspect of that is one can still apply directly Lemma 2.2 to the transformed Poisson law, which yields the various results in the previous section. In particular this operation transforms processes $\mu$ which do not admit moment measures to ones which do. An important example is the stable law which is transformed to a form of weighted generalised gamma process. Another important operation, besides the exponential change of measure, is a type of scaling, which arises for instance in Brownian excursion theory[see for instance Pitman and Yor (1992, 1997, 2001)]. This operation no longer preserves the Poisson nature of $N$ or similarly the structure of the biased models $\mu$. This in itself does not present a major obstacle as one could still apply Lemma 2.2 to the Poisson law first. The law resulting from the scaling operation may not have an obviously understandable form. However, indeed hidden in a scaling operation is an exponential form via a gamma integral identity. This identity has been used frequently in various contexts. Here, a slightly different variation will be used, where the exponential
change of measure idea will be applied internally leading to a variety of interesting consequences. As an important special case we look at the \(\text{PD}(\alpha, \theta)\) model. In general, analysis of the simple scaling structure leads to results quite related to Pitman and Yor (1992, 1997, 2001). [See also Perman, Pitman and Yor (1992), Section 4]. In particular see Pitman and Yor (1992) Section 3.

**REMARK 15.** It will become quite clear to the experts, on excursion theory and such matters, that the overlap with Pitman and Yor (1992, Section 3) is hardly coincidental. Although this was not my initial motivation. Some of the results given below amplify on Pitman and Yor (1992, Theorem 3.1 and especially Remarks 3.3 and 3.4). This section may be viewed as extensions of their Section 3. Given the new results I obtain for the \(\text{PD}(\alpha, \theta)\), among other things, the present exposition should clearly provide new insights, for the experts, into matters of which I myself have no expertise. Again applications of Lemma 2.1 and Proposition 3.1 play a fundamental role.

**REMARK 16.** In this section it is assumed that \(\eta\) satisfies the integrability condition in Corollary 3.1 when \(\tilde{g} := 1\). That is when \(\rho\) is homogeneous then \(\eta := cH\) for some scalar \(c\). For the Dirichlet process \(\eta := \theta H\) but is not to be confused with \(\theta\) used below to mimic the scaling operation associated with \(\text{PD}(\alpha, \theta)\) for \(\alpha > 0\).

The present analysis yields new mixture representations of random measures \(N\) and \(\mu\). What is most important is how they arise within the context of the scaling operation. [The reader should again note Pitman and Yor (1992, Remark 3.4)]. This leads to an analogous representations of random probability measures defined as,

\[
P(\cdot) = \frac{\mu(\cdot)}{\mu(Y)} = \frac{\int_S h(s)N(ds, \cdot)}{\int_S h(u)N(du, Y)} := \frac{\mu(\cdot)}{T}
\]

whose law is determined by an appropriate law on \(N\). That is either a Poison law or a scaled Poison law to be described below. In addition various characterizations of the the posterior distributions of \(N, \mu,\) and \(P\) are obtained. These results are applied to obtain general identities and representations for the two-parameter family with parameters \(0 < \alpha < 1, \theta > -\alpha\). This extends an identity given for the case \(\text{PD}(\alpha, \alpha)\) in Pitman and Yor (2001).

It is known that the general \(\text{PD}_{\alpha, \theta}(d\mu|H)\) cannot directly be defined via normalization of an independent increment process. The exceptions are the Dirichlet process and the Stable law process. Pitman and Yor (1997),[see also Tsilevich, Vershik and Yor (2000)] establish the following relationship. Let \(PD_{\alpha, \theta}(d\mu|\eta)\) denote a law of \(\mu\) such that its normalisation results in a Poisson-Dirichlet random probability measure, with law denoted as \(\text{PD}_{\alpha, \theta}(P|H)\). In particular \(\text{PD}_{\alpha, \theta}(d\mu|\eta) := \mathcal{P}(d\mu|\rho_\alpha, \eta)\) is the Stable process. From Pitman and Yor (1997), [see in particular Tsilevich, Vershik and Yor (2000) where this form is taken], it follows that for \(0 < \alpha < 1, \theta > -\alpha\),

\[
\text{PD}_{\alpha, \theta}(d\mu|\eta) = c_{\alpha, \theta}T^{-\theta}\text{PD}_{\alpha, 0}(d\mu|\eta),
\]

where \(c_{\alpha, \theta}\) is a normalizing constant. Pitman and Yor (1997) describe various results concerning the \(PD(\alpha, \theta)\) law of the \((P_t)\), related to this fact. Notably, Proposition 21, Proposition 22, and Proposition 33. In particular they show that if the sequence \((P_t)\) is \(PD(\alpha, \theta)\) then

\[
PD(\alpha, \theta) = \int_0^\infty PD(\alpha|t)\gamma_{\alpha, \theta}(dt)
\]
where $PD(\alpha|t)$ denotes the conditional Poisson-Kingman[see section 8] law of $PD(\alpha)$ conditioned on $T$, 
$\gamma_{\alpha,0}(dt) = c_{\alpha,1}^{-1}t^{-\theta}f_\alpha(t)$, $c_{\alpha,\theta} = E[T^{-\theta}] = \int_0^\infty t^{-\theta}f_\alpha(t)$ is the normalising constant and $f_\alpha(t)$ denotes the density of a stable law random variable which is the distribution of $T$. Tsilevich, Verhisk and Yor (2000) using (23) in combination with the following gamma identity for $\theta > 0$,

\begin{equation}
T^{-\theta} := \frac{1}{\Gamma(\theta)} \int_0^\infty v^{\theta-1}e^{-vT}dv,
\end{equation}

establish quite remarkably and simply a two-parameter extension of the Markov-Krein correspondence via the Laplace functional of a stable law.[Their result will be extended in a general fashion in Section 6]. This identity is also used in Peman, Pitman and Yor (1992) and Pitman and Yor (1997) among other places. The results discussed above are used primarily to deduce properties related to the normalized process $\mu(\cdot)/T$. That is, equivalently the $PD(\alpha, \theta)$ family. The interest here however is in another characterization of the law $P_{\alpha,\theta}(d\mu|\eta)$ which allows more direct usage of it and of course implies results for the normalized process and synonymously $PD(\alpha, \theta)$. The method relies on using the identity (67) conditioning on various transformed densities for $V$ rather than $T$ which will lead to a variety of interesting results. This analysis will be applied to general processes, $\mu$ and $N$, subject to the same type of scaling operation. For all $\theta > -\alpha$, in particular the case $-\alpha < \theta \leq 0$, the same identity (67) will be used for each fixed $n$,

\begin{equation}
\frac{1}{\Gamma(n)} \int_0^\infty T^nv^{n-1}e^{-vT}dv := 1,
\end{equation}

where the main point now is to work with $T^n$ rather than its reciprocal, and otherwise use the fact that the left hand-side of (68) is one. Now assuming that

\begin{equation}
E[T^{-\theta}|\rho, \eta] := \int_{\mathcal{M}} T^{-\theta}(dN|\rho, \eta) := \int_{\mathcal{M}} T^{-\theta}P(d\mu|\rho, \eta) < \infty
\end{equation}

consider the equality of laws,

\begin{equation}
\hat{P}(dN|\rho, \eta, \theta) := \frac{T^{-\theta}P(dN|\rho, \eta)}{E[T^{-\theta}|\rho, \eta]}, \text{and, } \hat{P}(d\mu|\rho, \eta, \theta) := \frac{T^{-\theta}P(d\mu|\rho, \eta)}{E[T^{-\theta}|\rho, \eta]}
\end{equation}

For each fixed $v$, let $E[e^{-vT}|\rho, \eta] := \int_{\mathcal{M}} e^{-vT}P(dN|\rho, \eta)$ denote the Laplace transform of $T$ taken relative to $P(dN|\rho, \eta)$. The following relationship will prove useful and allows analysis for the case $-\alpha < \theta \leq 0$. Suppose that for $n \geq 1$,

\begin{equation}
E[T^n|\rho, \eta, \theta + n] := \int_{\mathcal{M}} T^n\hat{P}(dN|\rho, \eta, \theta + n) = \frac{E[T^{-\theta}|\rho, \eta]}{E[T^{-(\theta+n)}|\rho, \eta]} < \infty
\end{equation}

then

\begin{equation}
\hat{P}(dN|\rho, \eta, \theta) := \frac{T^n\hat{P}(dN|\rho, \eta, \theta + n)}{E[T^n|\rho, \eta, \theta + n]}, \text{and } \hat{P}(d\mu|\rho, \eta, \theta) := \frac{T^n\hat{P}(d\mu|\rho, \eta, \theta + n)}{E[T^n|\rho, \eta, \theta + n]}.
\end{equation}

Additionally, denote the laws of $P$ taken relative to $P(\cdot|\rho, \eta)$ and $\hat{P}(dN|\rho, \eta, \theta)$ as $P(dP|\rho, \eta)$ and $\hat{P}(dP|\rho, \eta, \theta)$ respectively. When $\rho$ does not depend on $y$ and $\eta := cH$, then

\begin{equation}
P := \sum_{i=1}^{\infty} P_i \delta z_i
\end{equation}
where the sequence $(P_i)$ is independent of $(Z_i)$ which are iid $H$. In that case the analysis of $P$ is in principle equivalent to the analysis of $(P_i)$. That is the $P$ models are special cases of species sampling models [See Pitman (1996), Hansen and Pitman (2001) and Ishwaran and James (2001a)]. Hence, for instance, the results of Perman, Pitman and Yor (1992) and Pitman (1995b), concerning the EPPF etc.; can be applied to this setting to obtain information about $P$. When $\rho$ depends on $y$, then one can still represent $P$ in the form (73). However the independence property between $(P_i)$ and $(Z_i)$ no longer holds. It will become clear that if interest is simply the marginal distributional properties of the $(P_i)$ then the results of Pitman, Perman and Yor (1992) and Pitman (1995b) can be applied using $\Omega$ rather than $\rho$. At any rate, a different analysis will be used here which is based primarily on information contained in the random measures $N$ and $\mu$ which will yield relevant information about the $(P_i)$ etc. This is useful even in the species sampling case where the EPPF may be intractable or does not easily convey information about $P$.

**Remark 17.** Although the emphasis seems to be on the scaled laws $\tilde{P}(\cdot|\rho, \eta, \theta+n)$ this is only partially the case. In particular it should be clear that the negative moment conditions (69) and (71) do not hold in general. This is the case for a gamma process, which could be denoted as $P_{0,\theta}(d\mu|\eta)$, and hence it is not a proper $\tilde{P}(d\mu|\rho, \eta, \theta)$. The forthcoming discussion investigates properties of general $N$ and $\mu$ based on manipulation of both (67) and (68). The results below related to $\tilde{P}(\cdot|\rho, \eta, \theta+n)$ hold provided (63) and (64) are true.

### 5.1 Mixture representations for general processes

The results follow from the proof of Theorem 5.1 below.

**Proposition 5.1** The following identities hold which have various implications.

(i) For $n > 0$,

$$
\Gamma(n) := \sum_{p} \int_{0}^{\infty} \left[ \prod_{j=1}^{n(p)} \kappa_{j,n} (e^{-v\Omega}) \right] v^{n-1} E[e^{-vT}|\rho, \eta] dv,
$$

which is equivalent to, $\int_{0}^{\infty} E[T^n|e^{-v\Omega}, \eta] v^{n-1} E[e^{-vT}|\rho, \eta] dv$.

(ii) Statement (i) implies that there exist a joint distribution of $V, p$ given by,

$$
\pi_{n,0}(dv, p|\rho, \eta) := \frac{\prod_{j=1}^{n(p)} \kappa_{j,n} (e^{-v\Omega}) v^{n-1} E[e^{-vT}|\rho, \eta] dv}{\Gamma(n)}.
$$

(iii) Suppose that (63) holds for $\theta+n > 0$ then, $\Gamma(\theta+n)E[T^{-(\theta+n)}|\rho, \eta] := \int_{0}^{\infty} v^{\theta+n-1} E[e^{-vT}|\rho, \eta] dv$, which identifies a random variable $V$ on $(0, \infty)$ with density,

$$
\pi_{\theta+n}(dv|\rho, \eta) := \frac{v^{\theta+n-1} E[e^{-vT}|\rho, \eta] dv}{\Gamma(\theta+n)E[T^{-(\theta+n)}|\rho, \eta]}
$$

Additionally, $E[T^n|\rho, \eta, \theta+n] := \int_{0}^{\infty} E[T^n|e^{-v\Omega}, \eta] \pi_{\theta+n}(dv)$. Hence there exists a random variable $V$ such that $V, p$ has joint density,

$$
\pi_{\theta+n,0}(dv, p|\rho, \eta) := \frac{E[T^{-(\theta+n)}|\rho, \eta]}{E[T^{-\theta}|\rho, \eta]} \left[ \prod_{j=1}^{n(p)} \kappa_{j,n} (e^{-v\Omega}) \right] \pi_{\theta+n}(dv)
$$
(iv) The marginal distribution of $p$ in (73) and (77) is given by,

$$
\pi_{\theta+n,\theta}(p|\rho,\eta) := \frac{E[T^{-(\theta+n)}|\rho,\eta]}{E[T^{-\theta}|\rho,\eta]} \int_0^\infty \left[ \prod_{j=1}^{n(p)} \kappa_{\epsilon_j,n}(e^{-v\theta}) \right] \pi_{\theta+n}(dv)
$$

REMARK 18. The formula $\pi_{n,0}(p)$ derived from (78) corresponds to the EPPF formula given in Pitman (1995b, Corollary 6 formula (32)), when $h(s) := s \in (0,\infty)$ and $\rho$ is homogeneous. Pitman (1995b) uses the identity (73) applied to $T^{-n}$. Indeed it will be stated formally that (78) are EPPF formulas which can be seen as an extension of Pitman’s result.

The next result identifies $\tilde{P}(dN|\rho,\eta,\theta)$ and in fact arbitrary Poisson laws $\mathcal{P}(dN|\rho,\eta)$ as a mixture relative to Poisson random measures $N^*$ with conditional (given $V$) laws $\mathcal{P}(dN|e^{-v\rho,\eta})$. Moreover since the conditional law on $N^*$ has moments the result also contains mixture representations of $\mu$ of the type $N^* + \sum_{j=1}^{n(p)} \delta_{J_j,n,Y_j^*}$. The marginal distribution of $N^*$ actually depends on $n$.

**Theorem 5.1 (Poisson Mixture representations)** (i) For $\theta > 0$, the law $\tilde{P}(dN|\rho,\theta,\eta)$ defined in (74) is expressible as the following mixture,

$$
\tilde{P}(dN|\rho,\rho,\eta) := \int_0^\infty \mathcal{P}(dN|e^{\rho,\eta}) \pi_\theta(dv|\rho,\eta).
$$

This indicates that $N|V = v$ has a Poisson law $\mathcal{P}(dN|e^{\rho,\eta})$ with intensity $e^\rho\eta(ds|y)\eta(dy)$ and $V$ is a random variable on $(0,\infty)$ with density $\pi_\theta(dv|\rho,\eta)$. Equivalently the Laplace function of $N$ is,

$$
\mathcal{L}_N(f|\rho,\eta,\theta) := \int_0^\infty \mathcal{L}_N(f|e^{\rho,\eta}) \pi_\theta(dv|\rho,\eta)
$$

[Compare statement (i) with Pitman and Yor (1992, Remarks 3.3, 3.4)]

(ii) For each $\alpha > 0$, a Poisson law $\mathcal{P}(dN|\rho,\eta)$ can be represented as,

$$
\mathcal{P}(dN|\rho,\eta) := \sum_p \int_{\mathcal{S}^{\alpha}(p) \times \mathcal{Y}^{\alpha}(p) \times \mathbb{R}^+} \mathcal{P}(dN,dJ,dY^*|e^{\rho,\eta}) \pi_{\rho,\theta}(dv,p|\rho,\eta)
$$

That is from (78), the random measure $N$ can be represented as

$$
N^* + \sum_{j=1}^{n(p)} \delta_{J_j,n,Y_j^*},
$$

where given $V,p$ with law $\pi_{\rho,\theta}(dv,p|\rho,\eta)$, $N^*$ is conditionally independent of $J,Y^*$, with distribution $\mathcal{P}(dN|e^{\rho,\eta},\rho,\eta)$. The random variables $(J,Y^*)$ given $V,p$ have distribution $\mathcal{P}(dJ,dY^*|e^{\rho,\eta})$ as in (44).

(iii) For $\theta + n > 0$, this includes $-\alpha < \theta \leq 0$, the law $\tilde{P}(dN|\rho,\eta,\theta)$ is equivalent to,

$$
\sum_p \int_{\mathcal{S}^{\alpha}(p) \times \mathcal{Y}^{\alpha}(p) \times \mathbb{R}^+} \mathcal{P}(dN,dJ,dY^*|e^{\rho,\eta}) \pi_{\theta+n,\theta}(dv,p|\rho,\eta).
$$

That is, $N$ can be represented as (73) except that the distribution of $V,p$ is $\pi_{\theta+n,\theta}(dv,p|\rho,\eta)$. When $\theta := 0$ statement (iii) reduces to statement (ii).
**Proof.** For the result in (i), it suffices to evaluate the Laplace functional of $\hat{\mathcal{P}}$ defined in (70). That is,
\[
E[T^{-\theta}\rho,\eta] \int_{\mathcal{M}} e^{-N(f)}\hat{\mathcal{P}}(dN|\rho,\eta,\theta) := \int_{\mathcal{M}} e^{-N(f)}T^{-\theta}\mathcal{P}(dN|\rho,\eta).
\]
Now apply the identity (67) to see that the right hand side is equal to
\[
\frac{1}{\Gamma(\theta)} \int_{0}^{\infty} v^{\theta-1}E[e^{-N(f)}e^{-vT}\rho,\eta]dv.
\]
An application of Lemma 2.1 and dividing through by $E[T^{-\theta}\rho,\eta]$ now show that the Laplace functional of $\hat{\mathcal{P}}(dN|\rho,\eta,\theta)$ is,
\[
\int_{0}^{\infty} E[e^{-N(f)}e^{-v\rho,\eta}] v^{\theta-1}E[e^{-vT}\rho,\eta]dv \frac{1}{\Gamma(\theta)} E[T^{-\theta}\rho,\eta],
\]
as desired. For (ii), use (88) as follows
\[
\Gamma(n)\mathcal{L}_{N}(f|\rho,\eta) := \int_{0}^{\infty} E[T^n e^{-N(f)}e^{-vT}\rho,\eta]v^{n-1}dv.
\]
Noting that $T^n := \int_{Y_1}^{\infty} \prod_{i=1}^{n} \mu(dY_i)$, the result follows immediately by an application of (ii) in Proposition 3.1 and Corollary 3.1. Statement (iii) follows by first applying (i) to $\hat{\mathcal{P}}(dN|\rho,\eta,\theta) + n$ in (72) and then applying (ii) to $\mathcal{P}(dN|e^{-v\rho,\eta})$.

**Remark 19.** A clear distinction between a Poisson law $\mathcal{P}(dN|\rho,\eta)$ and the law on $N$ defined as $\hat{\mathcal{P}}(dN|\rho,\eta,\theta)$ was used above. This distinction was made because the conditions and techniques used to derive the results were slightly different. Hereafter, for some brevity, the notation $\theta$ will be used for all objects. When $\theta$ is set equal to zero (when applicable) this will correspond to results for the Poisson based laws of $N$, $\mu$, and $\mathcal{P}$ and their corresponding mixing laws.

**Corollary 5.1 (Mixture representations for $\mu$)** The results (i), (ii), (iii) in Theorem 5.1 imply analogous results for $\hat{\mathcal{P}}(d\mu|\rho,\eta,\theta)$ and $\mathcal{P}(d\mu|\rho,\eta)$.

(i) For $\theta > 0$,
\[
\hat{\mathcal{P}}(d\mu|\rho,\eta,\theta) := \int_{0}^{\infty} \mathcal{P}(d\mu|e^{-v\rho,\eta})\pi_{\theta}(dv|\rho,\eta)
\]
(ii) For $n \geq 1$ and $\theta + n > 0$, including $-\alpha < \theta \leq 0$, Statement (ii) and (iii) in Theorem 5.1 implies that, for each $n > 0$ the random measures $\mu$ with distribution $\mathcal{P}(d\mu|\rho,\eta)$ or $\hat{\mathcal{P}}(d\mu|\rho,\eta,\theta)$ can be represented as
\[
\mu^* + \sum_{j=1}^{n(p)} h(J_{j,n})\delta_{Y_j^*},
\]
where $\mu^*|V, p$ is $\mathcal{P}(d\mu|e^{-v\rho,\eta})$, and $(J_{j,n},Y_j^*)|V,p$ are conditionally independent of $\mu^*$ with distribution $\mathcal{P}(dJ,dY^*|e^{-v\rho,\eta})$. The distribution of $V,p$ is $\pi_{\theta+n}(dv,dv|\rho,\eta)$.

(iii) Equivalently these results yield the following expressions for the respective Laplace functionals. For $\theta > 0$,
\[
\mathcal{L}_{\mu}(g|\rho,\eta,\theta) := \int_{0}^{\infty} \mathcal{L}_{\mu}(g|e^{-v\rho,\eta})\pi_{\theta}(dv|\rho,\eta),
\]
and for $\theta + n > 0, n \geq 1$,

\begin{equation}
L_\mu(g|\rho, \eta, \theta) := \sum_p \int_0^\infty L_\mu(g|e^{-vh}\rho, \eta, p) \pi_{\theta+n}(dv, p|\rho, \eta),
\end{equation}

where

\begin{equation}
L_\mu(g|e^{-vh}\rho, \eta, p) := \int_{Y_{n(p)}} L_\mu(g|e^{-vh}\rho, \eta, Y) \prod_{j=1}^{n(p)} P(dY_j|e^{-vh}\rho, \eta)
\end{equation}

Setting $\theta = 0$ in (89) yields an identity for the Laplace functional of $P(d\mu|\rho, \eta)$.

Now mixture representations for $P$ are given. First set,

\begin{equation}
p^*_n := \frac{\mu^*(Y)}{\mu^*(Y) + \sum_{j=1}^{n(p)} h(J_{j,n})} = \frac{T^*}{T^* + \sum_{j=1}^{n(p)} h(J_{j,n})}
\end{equation}

and define a random probability measure as

\begin{equation}
P^*(\cdot) := \frac{\mu^*(\cdot)}{\mu^*(Y)} := \frac{T^*}{T^*}.
\end{equation}

**Proposition 5.2** (Mixture representations for random probability measures)

(i) If $\theta > 0$ then the random probability measure,

\begin{equation}
\tilde{P}(dP|\rho, \eta, \theta) := \int_0^\infty P(dP|e^{-vh}\rho, \eta) \pi_{\theta}(dv, \rho, \eta)
\end{equation}

(ii) If $P$ is $\tilde{P}(dP|\rho, \eta, \theta)$ or $P(dP|\rho, \eta)$ then for each $n \geq 1$ and $\theta + n > 0$, it is equivalent in distribution to the random probability measure

\begin{equation}
p^*_n P^*(\cdot) + (1 - p^*_n) \frac{\sum_{j=1}^{n(p)} h(J_{j,n}) \delta_{Y^*_j}(\cdot)}{\sum_{j=1}^{n(p)} h(J_{j,n})}
\end{equation}

with distribution specified by statement (ii) Corollary 5.1.

### 5.2 Duality of mixture representations and posterior distributions

In this section $Y_1, \ldots, Y_n|P$ are iid random variables with distribution $P$. That is, this implies the joint model for $Y|P$, is $\prod_{i=1}^n P(dY_i)$. The law of $P$ is either $P(dP|\rho, \eta)$ or $\tilde{P}(dP|\rho, \eta, \theta)$. The interest is in obtaining posterior distributions of $N, \mu$ and hence $P$ and relevant information about the marginal structure of $Y$.

Define a conditional distribution of $V|Y$ as,

\begin{equation}
\pi_{\theta+n}(dv|p, Y^*) \propto \prod_{j=1}^{n(p)} \kappa_{e_{j,n}}(e^{-vh}\rho|Y^*_j) \pi_{\theta+n}(dv)
\end{equation}

when $\rho$ does not depend on $y$, then the distribution of $V|Y$ only depends on $p$ and (92) reduces to

\begin{equation}
\pi_{\theta+n}(dv|p, Y^*) \propto \prod_{j=1}^{n(p)} \kappa_{e_{j,n}}(e^{-vh}\rho) \pi_{\theta+n}(dv).
\end{equation}
Theorem 5.2 (i) The marginal distribution of \( Y \) is

\[
\pi(dY) := \int_0^\infty \left[ \prod_{j=1}^{n(p)} \mathcal{P}(dY_j^*|e^{-\nu_h} \rho, \eta) \right] \pi_{\theta+n,\varrho}(d\rho, d\eta).
\]

This implies that the quantity \( \pi_{\theta+n,\varrho}(p|\rho, \eta) \) is an EPPF. When \( \rho \) does not depend on \( y \) then the marginal distribution of \( Y \) is expressible as

\[
\pi(dY) := \pi_{\theta+n,\varrho}(p|\rho, \eta) \prod_{j=1}^{n(p)} H(dY_j^*).
\]

(ii) Statement (i) combined with Theorem 5.1 imply that the posterior distribution of \( N, \mu \) given \( Y \) is identical to the mixtures,

\[
\int_{\mathbb{R}^+ \times S^{n(p)}} \mathcal{P}(dN, dJ|e^{-\nu_h} \rho, \nu, Y^*) \pi_{\theta+n}(d\rho, \nu, Y^*); \int_{\mathbb{R}^+} \mathcal{P}(d\mu|e^{-\nu_h} \rho, \nu, Y^*) \pi_{\theta+n}(d\rho, \nu, Y^*)
\]

respectively.

(iii) Statement (ii) implies that the posterior distribution of \( P \) is determined by either of the laws in \( (95) \).

Combined with the mixture representations this implies that the distribution of \( P \) given \( Y \) is equivalent to the distribution of the random measure

\[
p_n^* P^*(\cdot) + (1 - p_n^*) \sum_{j=1}^{n(p)} h(J_{j,n}) \delta_{Y_j^*}(\cdot)
\]

where the distributions of \( \mu^* \) and \( (J_{j,n}) \) given \( V, Y \) is specified by \( \mathcal{P}(d\mu|e^{-\nu_h} \rho, \nu, Y^*) \) and \( V|Y \) is \( \pi_{\theta+n}(d\rho, d\nu, d\nu, Y^*) \)

**Proof.** Given the mixture representations in Theorem 5.1, the result follows by an appeal to Fubini’s theorem which identifies the posterior laws of \( N, \mu, P \) as mixtures relative to the marginal distribution of \( Y \). That is, for instance \( \mathcal{P}(dN) := \int_\mathbb{N} \mathcal{P}(dN|Y) \pi(dY) \). (More formally one could evaluate the Laplace of \( (95) \).)

The conclusion that \( \pi_{\theta+n,\varrho}(p|\rho, \eta) \) is an EPPF perhaps requires further discussion as the technique I used may be a bit unfamiliar. Essentially from the theory of exchangeability if \( Y_1, \ldots, Y_n|P \) are iid \( P \) then the marginal distribution of \( Y = (Y^*, p) \) is exchangeable. Hence once the unique values \( Y^* \) are exposed an integration with respect to \( \eta \) leaves only a marginal distribution of \( p \) which must be an EPPF regardless of whether or not the unique \( \{Y_1^*, \ldots, Y_n^*(p)\} \) are iid \( H \). That is regardless of whether or not \( P \) is a species sampling model as described in Pitman (1996). What is lost is the 1-1 correspondence between \( (p, H) \) and the random probability measure \( P \). For instance, it is conceivable that the EPPF \( PD(p|\alpha, \theta) \) could be embedded in a model \( P \) which is not \( \mathcal{P}_{\alpha,\varrho}(dP|H) \). In that case the \( Y_1^*, \ldots, Y_n^*(p) \) cannot be iid \( H \). In other words an EPPF can always be found by working with a \( P \) model, finding the joint marginal distribution of \( Y \), and then marginalizing over the unique values. This is obvious when \( P \) is a species sampling model. However, it is a simple matter to verify the addition rules given in Pitman (1995a,b, 1996) for an EPPF by applying the Bayesian idea of a prediction rule. In particular for each \( n \) evaluate,

\[
E[P(Y)|Y] := E[p_n^*|Y] + E[1 - p_n^*]|Y| := 1.
\]

Proper manipulation of the middle expression will yield the obvious rules.
5.3 Results for $PD(\alpha, \theta)$

A description of the $PD_{\alpha, \theta}(d\mu|\eta)$ laws is now given. Throughout this section the notation $\mu_L$, $T_L$ etc will be used to denote the (conditional law) of $\mu$, $T$ depending on a random variable $L$.

5.3.1 Distributional properties of $PD_{\alpha, \theta}(d\mu|\eta)$

Corollary 5.2 (i) If $N$ is $P(dN|\rho, \eta)$ and $h(s) := s$, then for $\theta > 0$, $\tilde{P}(dN|\rho, \eta, \theta)$ is such that $N|V$ is a Poisson random measure with intensity,

$$
\rho_{\alpha,v}(ds)\eta(dy) := e^{-vs}\rho_{\alpha}(ds)\eta(dy)
$$

corresponding to the Lévy measure of a generalised gamma process. The density of $V$ is

$$
\tau_\theta(du|\rho) := \frac{1}{c_{\alpha,\theta}^v}e^{\theta-1}e^{-Kv^\alpha}dv.
$$

By a change of variable the distribution of $L := V^\alpha$ has a gamma distribution with parameters $(\frac{\theta}{\alpha}, K)$. That is, $L$ is $G(\frac{\theta}{\alpha}, K)$. The factor $K$ is determined in part by the total mass of $\eta$. It can be dispensed with by re-scaling.

Remark 20. Now the connection to Pitman and Yor (1992, section 3, p. 335-336) should be more transparent. [See also Pitman and Yor (2001, Theorem 3)]. The exponential law arises by setting $K = 1$ (or by rescaling $L$) and the choice of $\theta = \alpha$.

Proposition 5.3 Corollary 5.2 implies that the distribution of $PD_{\alpha, \theta}(d\mu|\eta)$ with respect to the mixing distribution of $L$ is,

$$
PD_{\alpha, \theta}(d\mu|\eta) := \int_0^\infty \mathcal{P}(d\mu|\rho_{\alpha,L^1/\alpha}, \eta)\mathcal{G}(dL|\theta/\alpha, K)
$$

In other words $\mu|L$ is a generalized gamma random measure with Lévy measure $\rho_{\alpha,L^1/\alpha}(ds)\eta(dy)$ and $L$ is a gamma random variable with parameters $(\frac{\theta}{\alpha}, K)$. When $K = 1$ and $\theta = \alpha$, $L$ is a standard exponential random variable. The Laplace functional of $PD_{\alpha, \theta}(d\mu|\eta)$ can be expressed as,

$$
\int_M e^{-\mu(g)}PD_{\alpha, \theta}(d\mu|\eta) := \int_0^\infty \mathcal{L}_\mu(g|L, \alpha)\mathcal{G}(dL|\theta/\alpha, K)
$$

where

$$
\mathcal{L}_\mu(g|L, \alpha) := e^{-\frac{1}{\alpha} \int_y (g(y) + L^\frac{1}{\alpha})^\alpha \eta(dy)} e^{KL}
$$

is the conditional Laplace functional of $\mu$ given $L$.

(ii) Furthermore, suppose that conditioned on $L$, $\mu$ is multiplied by $L^{1/\alpha}$. Then the unconditional law of $L^{1/\alpha}\mu_L$ is given by its Laplace functional,

$$
E[e^{-L^{1/\alpha}\mu_L(g)}] := \int_0^\infty \mathcal{L}_\mu(L^{1/\alpha}g|L, \alpha)\mathcal{G}(dL|\theta/\alpha, K) := \left[ \int_\gamma (g(y) + 1)^\alpha H(dy) \right]^{-\theta/\alpha}
$$

(iii) Statement (ii) implies that the distribution of $L^{1/\alpha}T_L$ is $\mathcal{G}(\theta)$. 

REMARK 21. Statement (iii) should be compared with $T$ of Pitman and Yor (1997, Proposition 21). Note that the explicit expression for the Laplace functional in (ii) is obtained via Brix’s (1999) expression for the generalised gamma measure.

Now noting that,  

\begin{equation}
\prod_{j=1}^{n(p)} \kappa_{e_j,n}(e^{-\nu h}) := e^{-(n-n(p)\alpha)\eta(Y)}\prod_{j=1}^{n(p)} \Gamma(e_j,n - \alpha),
\end{equation}

the results below are easily deduced.

**Proposition 5.4**

(i) For all $\theta > -\alpha$ and $n \geq 1$, the $PD_{\alpha,\theta}(d\mu|\eta)$ law is representable as the random measure, 

\begin{equation}
\mu^*(\cdot) + \sum_{j=1}^{n(p)} J_{j,n} \delta_{Y_j^*}(\cdot)
\end{equation}

where $\mu^*$ is given a random variable $L$ is a generalised gamma random measure with intensity $\rho_{\alpha,L^1/\alpha}(ds)\eta(dy)$. The $(J_{j,n})$ given $(L, Y^*)$ are independent of $\mu^*$ with respective Gamma distributions $\mathcal{G}(e_{j,n} - \alpha, L^1/\alpha)$. $L$ given $p$ is $\mathcal{G}(n(p) + \theta/\alpha, K)$ for some constant $K$. In particular by cancellation one can set $K = \alpha$ or $K = 1$. Conditionally independent of $L$, \{Y_1^*, \ldots, Y_{n(p)}^*\} are iid $\eta(\cdot)/\eta(Y) := H(\cdot)$.

(ii) The distribution of $p$ is the EPPF, $PD(\alpha, \theta)$. In addition the marginal law of $\mu^*|p$ is $PD_{\alpha,\theta+n(p)\alpha}(d\mu|\eta)$, which follows since similar to $(103)$ its Laplace functional is

\begin{equation}
\int_0^\infty \mathcal{L}_\mu(g|L, \alpha)\mathcal{G}(dL|n(p) + \theta/\alpha, K)
\end{equation}

(iii) Hence the random probability measure

$$\frac{\mu^*(\cdot)}{\mu^*(Y)}$$

given $p$ is $PD_{\alpha,\theta+n(p)\alpha}(d\mu|H)$. Denote the random probability measure with this law as $P_{\alpha,\theta+n(p)}\alpha$.

(iv) Given $L, p$, the random variables $G_{j,n} := L^{1/\alpha}J_{j,n}$ are independent $\mathcal{G}(e_{j,n} - \alpha)$ independent of $L$ and $\mu^*$. Moreover given $p$ the distribution of $L^{1/\alpha}\mu_L^*|P$ is given by the Laplace functional

$$\left[\int_Y (g(y) + 1)^{\alpha}H(dy)\right]^{-(\theta+n(p)\alpha)/\alpha}$$

and $L^{1/\alpha}T_L^*$ given $p$ has distribution $\mathcal{G}(\theta + n(p)\alpha)$

REMARK 22. Suppose that $K = 1$, and $n = 1$, then in particular for the stable case $PD_{\alpha,0}(d\mu|\eta)$, it follows that $L$ is exponential (1). For the $PD_{\alpha,\alpha}(d\mu|\eta)$, $L$ is $\mathcal{G}(2,1)$.

5.3.2 Identities for $PD(\alpha, \theta)$

The propositions above are now applied to derive an alternate representation for the distribution of the general $PD(\alpha, \theta)$ and related models. This, in particular generalizes the right-hand side of the construction of a $PD(\alpha, \alpha)$ model given in Pitman and Yor (2001, Example 8, eq. (33)) to previously unknown ones for the general $PD(\alpha, \theta)$ model.
Notice that using the change of variable \( u = vs \), \( e^{-vs} \rho_\alpha(ds) \) transforms to the \( \mathbb{L} \) measure

\[
v^\alpha e^{-u} \rho_\alpha(du)
\]

which yields the equivalence of the sets

\[
\{ y : \int_y^\infty e^{-vs} \rho_\alpha(ds) \leq x \} := \{ u : \int_u^\infty e^{-s} \rho_\alpha(ds) \leq x/v^\alpha \}.
\]

Define,

\[
\Lambda^{-1}(x) := \inf\{ u : \int_u^\infty e^{-s} \rho_\alpha(ds) \leq x \},
\]

and set \( \Gamma_j := \sum_{i=1}^j E_i \) for \( (E_i) \) a collection of independent standard exponential random variables. In addition define for all \( \theta > 0 \)

\[
\Sigma_{\theta/\alpha} := \sum_{j=1}^\infty \Lambda^{-1}(\Gamma_j/L)
\]

where \( L \) is a \( \mathcal{G}(\theta/\alpha) \) random variable, independent of \( (E_i) \). Now using the change of variable \( L = V^\alpha \), and Proposition 5.3, 5.4, it is easy to see from an application of Khintchine’s (1937) Inverse \( \mathbb{L} \) method, [see also Ferguson and Klass (1972), Wolpert and Ickstadt (1998b), Sato (1999), Rosinski (2001), and Banjevic, Ishwaran and Zarepour (2002)], that the following identities hold;

**Proposition 5.5 (Distributional representations for \( PD(\alpha, \theta) \))** Choose \( 0 < \alpha < 1 \),

(i) then for \( \theta > 0 \), the distribution of the sequence

\[
(\Lambda^{-1}(\Gamma_j/L)/\Sigma_{\theta/\alpha}; \ j = 1, 2, \ldots),
\]

is \( PD(\alpha, \theta) \).

(ii) Let \( (Z_j) \) denote an iid sequence with distribution \( H \) chosen independently of \( L \) and \( (E_j) \). If \( \theta > 0 \), then equivalent to (i), the random probability measure,

\[
P_{\alpha, \theta}(\cdot) := \sum_{j=1}^\infty \Lambda^{-1}(\Gamma_j/L)/\Sigma_{\theta/\alpha} \delta_{Z_j}(\cdot)
\]

is \( PD_{\alpha, \theta}(dP|H) \).

(iii) For all \( \theta > -\alpha \) and \( n \geq 1 \), a \( PD_{\alpha, \theta}(dP|H) \) random probability measure is representable as,

\[
P_{\alpha, \theta}(\cdot) := \frac{\Sigma_{n(p)+\theta/\alpha}}{\Sigma_{n(p)+\theta/\alpha} + \sum_{j=1}^n J_{j,n}} P_{\alpha,\theta+n(p)|\alpha}(\cdot) + \frac{\sum_{j=1}^n J_{j,n} \delta_{Y_j^*}(\cdot)}{\Sigma_{n(p)+\theta/\alpha} + \sum_{j=1}^n J_{j,n}}
\]

where, for clarity, \( \Sigma_{n(p)+\theta/\alpha} := \sum_{j=1}^\infty \Lambda^{-1}(\Gamma_j/L) \). \( L|p \) is gamma distributed with parameters \( (n(p)+\theta/\alpha, 1) \), \( (J_{j,n})|L, p \) are respectively independent \( \mathcal{G}(\epsilon_{j,n} - \alpha, L^\alpha) \) and \( (Y_j^*)|p \) are iid \( H \). Note that \( (J_{j,n}) \) are not independent of \( \Sigma_{n(p)+\theta/\alpha} \).

Now to obtain another representation of \( P_{\alpha, \theta} \), which also serves to directly recover Pitman’s (1996) description of the posterior distribution. Notice from Proposition 5.3 and 5.4 that given \( p \) the following equivalence in distribution holds for each \( n \geq 1 \);

\[
\frac{L^{1/\alpha} L_{1/\alpha}}{L_{1/\alpha} + L_{1/\alpha} \sum_{j=1}^n J_{j,n}} := \frac{G_{\theta+n(p)\alpha}}{G_{\theta+n(p)\alpha} + \sum_{j=1}^n G_{j,n}}.
\]
This is also true for \( n=0 \) by Proposition 5.3. The key point is that the quantity above is independent of the mixing distribution on \( L \) for all \( n \). Moreover, \( L^{1/\alpha} T_L := G_{\theta+n(p)\alpha} \) is \( \mathcal{G}(\theta + n(p)\alpha) \) and independent of the gamma random variables \( (G_{j,n}) \) as defined in Proposition 5.4. Hence the following result,

**Proposition 5.6** For all \( \theta > -\alpha \),

\[
P_{\alpha,\theta}() := p_n P_{\alpha,\theta+n(p)\alpha}() + (1 - p_n) \frac{\sum_{j=1}^{n(p)} G_{j,n}\delta_{Y^*_j}()}{\sum_{j=1}^{n(p)} G_{j,n}}
\]

where

\[
p_n := \frac{G_{\theta+n(p)\alpha}}{G_{\theta+n(p)\alpha} + \sum_{j=1}^{n(p)} G_{j,n}}.
\]

Given \( p \) the quantity above does not depend on the mixing distribution \( L \). Hence given \( Y = (Y^*, p) \) the posterior distribution of a \( P \) which is \( PD_{\alpha,\theta}(dP|H) \) is immediately seen to be equivalent to the random measure on the right of (114) when \( (Y^*_j) \) and \( p \) are fixed. This corresponds exactly with the posterior distribution described in Pitman (1996).

**Remark 23.** It is certainly obvious that one could use Pitman’s (1996) posterior characterization to obtain the simple mixture representation in proposition 5.6. The main point however is really how the previous descriptions (which are less obvious) led up to this result. Moreover, none of the arguments appealed to the stick-breaking representation of \( PD_{\alpha,\theta}(dP|H) \).

**Remark 24.** The analysis of the \( PD(\alpha,\theta) \) models revealed various independence structures via a simple transformation. In general this will not be the case but there are certainly many instances where an appropriate transformation of the \( (J_{j,n}) \) will render them independent of \( \mu^* \) and the mixing distribution on \( V \). Such an operation should simplify the analysis. One might try this with the intensities described in Pitman and Yor (2001).

**Remark 25.** I wonder what if any interpretation does an adjustment to the left hand-side of Pitman and Yor (2001, Theorem 3, eq. (19)) have when \( \epsilon_0 \) is replaced by what one might guess from Proposition 5.3 to be a \( \mathcal{G}(\theta/\alpha) \) random variable.

### 5.4 Results for the Dirichlet Process and generalised gamma process

Results for the gamma process with shape \( \eta(\cdot) := \theta H(\cdot) \), that is \( PD_{0,\theta}(d\mu|\eta) \), follow by using \( \rho_{0,1} \) in place of \( \rho_\alpha \). In this case \( T \) is \( \mathcal{G}(\theta) \), and

\[
\prod_{j=1}^{n(p)} \kappa_{e^j,n}(e^{-\epsilon_{h}\Omega}) := (1 + \epsilon)^{-n} \theta^{n(p)} \prod_{j=1}^{n(p)} \Gamma(e_{j,n}).
\]

Which yields readily,

**Proposition 5.7**

\[
\pi_{n,0}(dv, p) := PD(p|\theta) \tau_{\theta, n}(dv)
\]
where
\[
\tau_{\theta,n}(dv) := \frac{\Gamma(\theta+n)(1+v)^{-(n+\theta)}v^{n-1}dv}{\Gamma(\theta)}
\]
which implies that \( V \) and \( p \) are independent. Note the density \( \tau_{\theta,n}(dv) \) is well defined provided \( \theta > 0 \).

(ii) The \( (J_{j,n})|V,p \) are independent \( \mathcal{G}(e_{j,n},1+V) \) and the distribution of \( \mu^*|V,p \) is a (simple) weighted gamma process, i.e. has intensity \( \rho_{0,1+V}\theta H \), and is independent of \( p \).

(iii) Hence it follows that given \( V \) and \( p \)
\[
(V + 1) \left[ \mu^*(\cdot) + \sum_{j=1}^{n(p)} J_{j,n} \delta_{Y^*_j} \right]
\]
is a mixture of gamma processes independent of \( V \). That is additionally given \( Y^* \), the measure in (118) is a gamma process with shape \( \theta H(\cdot) + \sum_{j=1}^{n(p)} e_{j,n} \delta_{Y^*_j}(\cdot) \). This fact serves to recover the well-known result of Ferguson (1973) for the posterior distribution of the Dirichlet process.

REMARK 26. It is not so surprising that the gamma/Dirichlet model is such that the mixing distribution \( V \) and \( p \) are independent. It is also perhaps true, given the properties of \( PD(\theta) \), that this is the only species sampling model with this property.

The arguments above may be applied to the generalised gamma model with intensity \( \rho_{\alpha,b} \). In this case, from section 3, it follows that,
\[
\prod_{j=1}^{n(p)} \kappa_{e_{j,n}}(e^{-v\theta}) := (v + b)^{-(n-n(p)\alpha)} \beta^{n(p)} \prod_{j=1}^{n(p)} \Gamma(e_{j,n} - \alpha)
\]
where it is assumed that \( \eta(Y) := \theta \). This leads to a joint density of \( V,p \) specified as
\[
\beta^{n(p)} \left[ \prod_{j=1}^{n(p)} \Gamma(e_{j,n} - \alpha) \right] \frac{(v + b)^{-(n+n(p)\alpha)}v^{n-1}e^{-[(v+b)^{\alpha}-b^{\alpha}]K}dv}{\Gamma(n)}
\]
In addition the \( (J_{j,n})|V,p \) are \( \mathcal{G}(e_{j,n} - \alpha,b + V) \) and \( \mu^*|V,p \) is a generalised gamma process with Lévy measure \( \rho_{\alpha,b+V} \). Hence \( (b+V)J_{j,n} \) are \( \mathcal{G}(e_{j,n} - \alpha) \). The Laplace functional of \( (V + b)\mu^*|V,p \) is
\[
e^{-\frac{1}{\theta}(v+b)^{\alpha}} \int_{\mathcal{S}} [(\varphi(y)+1)^{\alpha} - 1] \varphi(dy)
\]

REMARK 27. In order to incorporate larger classes of models for \( P \) one could use a weighted Poisson law
\[
Q(dN|\rho,\eta) := \frac{w(N)\mathcal{P}(dN|\rho,\eta)}{E[w(N)]}
\]
for an arbitrary integrable function \( w \). This will be used in Section 8.
6 Distributions of joint linear functionals of $P$; variations of the Markov-Moment problem

This section is a continuation of the previous one. Here, it is shown that the joint Cauchy-Stieltjes transform of linear functionals of $P$, which are $\hat{P}(dP|\rho, \eta, \theta)$ and $P(dP|\rho, \eta)$, is equivalent to expressions involving the Laplace functional of random measures $V_{\mu V}$. The precise meaning of $V_{\mu V}$ will be clear from the context below. The method of proof, given the results in the previous section, is an easy extension of the beautiful approach used by Tsilevich, Vershik and Yor (2000) for the Dirichlet process and the general $P_{\alpha, \theta}(dP|H)$ family. The results given here represents the most general ones that I know of. More importantly the explicit relationship between $P$ and $V_{\mu V}$ is a new insight. See Kerov (1998) for many implications of this type of result.

Let $f_l$ denote real-valued functions on $Y$ and define $P_{f_l} = \int_Y f_l(y)P(dy)$ for $l = 1, \ldots, q$. In addition let $z_l$ for $l = 1, \ldots, q$ denote non-negative scalars. In this section the calculation of the following transform (in relation to Laplace functionals of $V_{\mu V}$) is discussed;

\begin{equation}
\int_{\mathcal{M}^*} \frac{1}{1 + \sum_{l=1}^{q} z_l P_{f_l}} \hat{P}(dP|\rho, \eta, \theta)
\end{equation}

for all $\theta > -\alpha$. Again when $\theta = 0$ this coincides with the $P(dP|\rho, \eta)$ laws. The quantity characterizes the joint distribution of $(P_{f_1}, \ldots, P_{f_q})$. Kerov and Tsilevich (1998) in the case of $P_{\alpha, \theta}(dP|H)$ used combinatorial arguments to obtain the moment expressions in the case of the Dirichlet and two-parameter models to yield extensions of the Markov-Krein identity for $(P_{f_1}, \ldots, P_{f_q})$. Their results extend the work of Cifarelli and Regazzini (1990) for the case of the distribution of the mean functional, $\int yP(dy)$, when $P$ has a Dirichlet process law. The mean case is also discussed in Diaconis and Kemperman (1996) where in addition the result for the joint distribution of functionals like $(P_{f_1}, \ldots, P_{f_q})$ was proposed as an open problem. Tsilevich (1997) establishes the case for the mean with respect to the general two-parameter processes. These results used hard analytic techniques which would not be easily extendable to a general scenario. However, recently Tsilevich, Vershik and Yor (2000) devise a beautiful simple proof of the corresponding result in the case of the Dirichlet process and the general two-parameter extension via Laplace functionals. Given the results in section 5.1 it is a simple matter to extend their result to the general class of probability measure $\hat{P}(dP|\rho, \eta, \theta)$. That is, following closely their approach, relationships between and the Laplace functional of $V_{\mu V}$ are established. The results below follow by rewriting

\begin{equation}
\frac{1}{1 + \sum_{l=1}^{q} z_l P_{f_l}} := \frac{T}{T + \sum_{l=1}^{q} z_l \mu(f_l)}
\end{equation}

and applying Corollary 5.1.

6.1 Joint Cauchy-Stieltjes transforms and Laplace functionals

Proposition 6.1 For $\theta > 0$,

\begin{equation}
\int_{\mathcal{M}^*} \left(1 + \sum_{l=1}^{q} z_l P_{f_l}\right)^{-\theta} \hat{P}(dP|\rho, \eta, \theta) := \mathcal{L}_{\nu \mu V}(\sum_{l=1}^{q} z_l f_l|\rho, \eta, \theta)
\end{equation}

where

\begin{equation}
\mathcal{L}_{\nu \mu V}(\sum_{l=1}^{q} z_l f_l|\rho, \eta, \theta) := \int_{0}^{\infty} \mathcal{L}_{\nu}(\sum_{l=1}^{q} z_l f_l|e^{-\nu} \rho, \eta) \pi_\theta(d\nu|\rho, \eta)
\end{equation}
Proposition 6.2 For $\theta + n > 0$ and $n \geq 1$,

(i) \[ \int_{\mathcal{M}^*} \left( 1 + \sum_{l=1}^{q} z_l P f_l \right)^{-(\theta + n)} \, \tilde{P}(dP|\rho, \eta, \theta) := \sum_{p} \int_{0}^{\infty} L_{\mu}(v) \left( \sum_{l=1}^{q} z_l f_l | e^{-v\rho}, \eta, p \right) \pi_{\theta + n, \theta}(dv|\rho, \eta) \]

(ii) The expressions in (i) are equal to:

\[ \sum_{p} \int_{Y_{\rho}(p)} \left[ \int_{0}^{\infty} L_{\mu}(v) \left( \sum_{l=1}^{q} z_l f_l | e^{-v\rho}, \eta, p, Y^* \right) \pi_{\theta + n, \theta}(dv|\rho, \eta, p, Y^*) \right] \pi(dY) \]

which indicates that the posterior Cauchy-Stieltjes transform,

\[ \int_{\mathcal{M}^*} \left( 1 + \sum_{l=1}^{q} z_l P f_l \right)^{-(\theta + n)} \, \tilde{P}(dP|\rho, \eta, \theta, Y) \]

is equal to,

\[ \int_{0}^{\infty} L_{\mu}(v) \left( \sum_{j=1}^{q} z_l f_l | e^{-v\rho}, \eta, \rho, Y^* \right) \pi_{\theta + n, \theta}(dv|\rho, \eta, p, Y^*) \]

(125)

Note again the various relationships to the (posterior) laws of $V_{\mu_Y}$.

Now setting $g(y) := \sum_{i=1}^{q} z_l f_l(y)$ in the Laplace transform of $L^{1/\alpha}_{\mu_L}$ in statement (ii) of Proposition (5.3) yields,

\[ \left[ \int_{Y} (\sum_{i=1}^{q} z_l f_l(y) + 1) \right]^{-\alpha/\theta} H(dy) \]

Hence the result of Tsilevich, Vershik and Yor (2000) for the $P_{\alpha,\theta}(dP|H)$ family is recovered. However the relationship to $L^{1/\alpha}_{\mu_L}$ is not noted in their work.

6.2 A remark on moment calculations

As mentioned previously, Kerov and Tsilevich (1998) used nontrivial combinatorial arguments to calculate the joint moments of $(P f_1, \ldots, P f_q)$ in the case of $P_{\alpha,\theta}(dP|H)$. Here similar to Ishwaran and James (2001a) for species sampling models it is demonstrated that one can easily obtain the relevant moment calculations by using Theorem 5.2. This calculation will only be presented for the case where $\rho$ does not depend on $y$.

The task is to calculate

\[ E \left[ \prod_{l=1}^{q} (P f_l)^{n_l} \right] = \int_{\mathcal{M}} \left[ \prod_{l=1}^{q} \prod_{i=1}^{n_l} \int_{Y} f_l(y, i) P(dy, i) \right] \tilde{P}(dP|\rho, \eta, \theta) \]

Now analogous to (111) an application of Theorem 5.2 yields the result

(126) \[ E \left[ \prod_{l=1}^{q} (P f_l)^{n_l} \right] = \sum_{p} \pi_{\theta + n, \theta}(p|\rho, \eta) \prod_{j=1}^{n(p)} \int_{Y} \left[ \prod_{l=1}^{q} f_l^{j,n} (u) \right] H(du) \]
7 Posterior Calculus for Extended Neutral to the Right processes

In this section I focus on the concept of neutral to the right processes (NTR) originally proposed in Doksum (1974). The Dirichlet process is the most notable member of this class. Here, a new natural extension of the NTR concept to more abstract spaces is given. It is then shown how Proposition 3.1 can be used to yield the most transparent and simplest posterior analysis of such models. This includes the case of survival data models subject to right censoring when a NTR prior is used or synonymously when Lévy process priors are used to model the cumulative hazard. Additionally, using Proposition 3.1 a change of measure formula is established which relates Beta/Dirichlet processes to their more complex Beta/Beta-Neutral (Stacy) generalizations.

**Remark 28.** Doksum (1974, Theorem 3.1) establishes essentially a 1-1 correspondence between NTR processes and exponential functions of subordinators. See below for explicit details. This fact seems not to be widely noticed by probabilists investigating problems where models under the latter description arise. One consequence is that the calculus that is described below for NTR’s can be exploited in other areas besides Bayesian nonparametrics. Here I will omit the drift component. It is a simple matter to make adjustments starting from the obvious modification of Lemma 2.1. (see Remark 2). Doksum (1974, Corollary 3.2), establishes the almost sure discreteness of NTR’s under the condition that the drift component is zero.

**Remark 29.** The notation \( \Lambda \) will be used in this section to denote a random cumulative hazard measure. The dependence of quantities \( F_0, A_0 \) on \( \rho, \eta \) will be suppressed. The arguments \( (s),(t) \) will be used to denote time as is usual in survival analysis. The argument \( (u) \) plays the role of \( (s) \) in the previous sections.

First the original definition proposed by Doksum is given

**Definition 1.** (Doksum (1974)) A random distribution function \( F \) on the positive real line is said to be **neutral to the right** if for each \( k > 1 \) \( t_1 < t_2 \ldots < t_k \), there exists non-negative independent random variables \( V_1, \ldots, V_k \) such that the vectors satisfy,

\[
\mathcal{L}\{(F(t_1), F(t_2) - F(t_1), F(t_k) - F(t_{k-1}))\} = \mathcal{L}\{(V_1, V_2(1-V_1), \ldots, V_k \prod_{j=1}^{k-1}(1-V_j))\},
\]

where \( \mathcal{L} \) denotes the law.

In the special case where \( F \) is a Dirichlet process with shape \( \theta F_0(\cdot) \) then each increment \( F(t_k) - F(t_{k-1}) \) is \( B(\theta F_0([t_{k-1}, t_k])); \theta[1 - F_0([t_{k-1}, t_k])] \). Doksum discusses various equivalences and implications of this definition. From Theorem 3.1 of Doksum it follows that \( F \) is a NTR process if and only if for \( t \geq 0 \),

\[
S(t) = 1 - F(t) = e^{-Z(t)},
\]

where \( Z \) is an increasing Levy process satisfying \( Z(0) = 0 \) and \( \lim_{n \to \infty} Z(t) = \infty \). The analysis here will consider subordinators \( Z \) without a drift component. In other words \( Z \), is a completely random measure on \( (0, \infty) \) with associated intensity \( \rho_z(du|y)\eta(dy) \) for \( (u, y) \in (0, \infty) \times (0, \infty) \). Ferguson (1974) shows that a Dirichlet process with finite shape measure, \( \theta F_0(\cdot) \), results if

\[
\rho_z(du|y)\eta(dy) = \frac{1}{1 - e^{-\theta}} e^{-u \theta F_0(y, \infty)} du \theta F_0(dy)
\]
It follows from the theory of product integration that an NTR process can also be represented as

\[ S(t) = \prod_{u \leq t} (1 - \Lambda(du)) \]

where \( \Lambda \) denotes a cumulative hazard which is further modelled as a completely random measure with intensity \( \rho\Lambda(u|y)ds\eta(dy) \) for \( (u, y) \in (0, 1] \times (0, \infty) \). The symbol \( \prod \), denotes the product integral which has played a primary role in survival analysis. In particular the Kaplan-Meier estimator for \( S \), Kaplan and Meier (1958), is obtained by replacing \( \Lambda \) by its empirical counterpart, the Nelson-Aalen estimator. See the text by Andersen, Borgan, Gill and Keiding (1993) for further elaboration. Gill and Johansen (1990) discuss in detail the properties of the product integral. The product integral is also expressible as,

\[ \prod_{[0,t]} (1 - \Lambda(dv, \mathcal{Y})) = \exp(-\Lambda^c(t)) \prod_{[0,t]} (1 - \Lambda_d(v)) , \]

where \( \Lambda^c \) denotes the continuous part of \( \Lambda \). Suppressing the dependence on \( \rho, \eta \), \( E[\Lambda(t)] = A_0(t) \) where \( A_0 \) denotes a prior cumulative hazard specification. It follows that,

\[ E[S(t)] := 1 - F_0(t) = \prod_{u \leq t} (1 - E[\Lambda(du)]) := e^{-A_0(t)}. \]

The restriction of the jumps of the process to \([0,1]\) ensures that \( \Lambda \) is an element in the space of cumulative hazards and hence \( S \) is a proper survival function. Hjort (1990) first proposed working directly with Lévy priors on the space of cumulative hazards which is more in line with the frequentist counting process analysis of event-history models [See Aalen(1975, 1978) and Andersen, Borgan, Gill and Keiding (1993)]. Hjort (1990) shows that if \( \Lambda \) is specified to be a Beta process then it is a conjugate model with respect to right censoring. Hjort (1990, section 7A), under a Beta process specification for \( \Lambda \) in (130), also defines a class of generalised Dirichlet processes on \( \mathbb{R}^+ \). He shows that the Dirichlet process is a special case of this model by setting \( c(s) = \theta F_0([s, \infty)) \). In summary, Bayesian nonparametric methods for the simple survival setting subject to censoring have been discussed following the framework of Ferguson’s (1973, 1974) (see also Friedman (1963) and Fabius (1964)) Dirichlet process in the works of Doksum (1974), Susarla and van Ryzin (1976), Blum and Susarla (1977), Ferguson and Phadia (1979), Lo (1993), Doss (1994), and Walker and Muliere (1997) among others. The methods discussed above operate by placing a Dirichlet or more general NTR prior on the unknown survival or distribution function. An alternative but essentially equivalent approach involves working with priors on the cumulative hazard measure discussed in Hjort (1990), Lo (1993) and most recently Kim (1999). However, unlike the simplicity of the Dirichlet process discussed in Ferguson (1973, 1974) for complete data models, the technical aspects of these models appear to be formidable. Moreover, the technical arguments used do not easily extend to more complex settings. In particular, the prior to posterior characterizations given in Ferguson and Phadia (1979)(see also Doksum (1974)), are only developed for distribution functions on the real line. In addition very little is known about the marginal and partition based structures. It will be shown that an alternate representation makes the calculus for NTR processes indeed straightforward.

Note importantly that there is a 1-1 correspondence between each \( Z \) and \( \Lambda \). Formally, the Lévy measure of \( Z \) arises as the image of \( \rho\Lambda(du|y)\eta(dy) \) via the map \( (u, y) \) to \((-\log(1 - u), y)\). For further discussion see Dey (1999) and Dey, Erickson and Ramamoorthi (2000). An important consequence, which has not
been exploited in this context, is the following identity, which holds in distribution for each \( n \) where \( N \) is \( \mathcal{P}(dN|\rho_N, \eta) \). Define for \( v > 0 \),

\[
\tilde{f}_{v-}(u, y) := -I\{v > y\} \log(1 - u)
\]

then,

\[
S(v-) := e^{Z(v-)} := e^{-N(\tilde{f}_{v-})},
\]

where the law of \( N \) is \( \mathcal{P}(dN|\rho_N, \eta) \).

### 7.1 Definition of Extended NTR processes

Suppose that \((T, X)\) denotes a marked pair of random variables on \( \mathcal{R}^+ \times \mathcal{X} \) with distribution \( F(ds, dx) \). In this section an answer is provided to the open question of how to extend an NTR process from \( \mathcal{R}^+ \) to more general marked Polish spaces. This provides for instance a new class of Bayesian models for multivariate survival models. While indeed it is easy to extend \( Z \) or \( \Lambda \) to more abstract spaces, the representation in (133) or (134) do not immediately suggest an obvious extension for \( F \). The Dirichlet process which is defined over arbitrary spaces is a notable exception. However, James and Kwon (2000) recently propose a method which extends the Beta-Neutral prior of Lo (1993), and by virtue of the equivalences, the Beta-Stacy process in Muliere and Walker (1997) and Beta distribution function discussed in Hjort (1990, section 7A), to a spatial setting. A general definition can be deduced from elements of their construction. A definition for \( F \) on \( \mathcal{R}^+ \times \mathcal{X} \) can be facilitated by the usage of its associated hazard measure on \( \mathcal{R}^+ \times \mathcal{X} \). From Last and Brandt (1995, A5.3), it follows that such a measure always exists and is defined by,

\[
\Lambda(ds, dx) := I\{t > 0\} \frac{F(ds, dx)}{S(s-)}.
\]

In particular, \( \Lambda(ds, \mathcal{X}) := \Lambda(ds) \) and hence

\[
S(s-) := \prod_{u<s} (1 - \Lambda(du, \mathcal{X})).
\]

An extended NTR is defined as,

**Definition 2.** (Extended Neutral to the Right Process) Let \( \Lambda \) denote a completely random measure with intensity \( \rho_\Lambda(du|s)\eta(ds, dx) \) for \((u, s, x) \in (0,1] \times (0, \infty) \times \mathcal{X} \). Furthermore, the intensity measures are chosen such that

\[
A_0(ds, dx) := E[\Lambda(ds, dx)|\rho_\Lambda, \eta] := \left[ \int_0^1 u\rho_\Lambda(du|s) \right] \eta(ds, dx)
\]

is a hazard measure. [Denote the marginal cumulative hazard \( A_0(ds, \mathcal{X}) := A_0(ds) \)]. Then an Extended Neutral to the Right process on \( \mathcal{R}^+ \times \mathcal{X} \) is defined for \( t > 0 \) and each \( B, \) an arbitrary measurable set in \( \mathcal{X} \),

\[
F(t, B) := \int_0^t S(s-)\Lambda(ds, B) := \int_0^t \prod_{u<s} (1 - \Lambda(du)) \Lambda(ds, B)
\]

In particular, \( F(ds, dx) := S(s-)\Lambda(ds, dx) \). The law of \( F \) is denoted \( \mathcal{P}(dF|\rho_\Lambda, \eta) \). The random quantities \( S(s-) \) and \( \Lambda(ds, dx) \) are independent for each \( s \) and arbitrary \( x \) and

\[
E[F(ds, dx)] := E[S(s-)]E[\Lambda(ds, dx)] := e^{-A_0(s)} A_0(ds, dx) := F_0(ds, dx).
\]
REMARK 30. The definition of an extended neutral to the right process yields, as a special case, a class of random probability measures on arbitrary spaces $\mathcal{X}$, defined as

$$(139) \quad F(dx) := \int_0^\infty S(s-)\Lambda(ds, dx).$$

For instance this expression offers another identity for a Dirichlet process on $\mathcal{X}$.

REMARK 31. The definition is expressed in terms of $\Lambda$ rather than $Z$ extended to $\mathcal{R}^+ \times \mathcal{X}$ due to the natural interpretation of a hazard measure. For instance a description of $F(ds, dx)$ is not easily seen using $Z$. Nonetheless for each $\Lambda$ and $F$ in $\mathcal{R}^+ \times \mathcal{X}$ one can associate a $Z$ on $\mathcal{R}^+ \times \mathcal{X}$ with again the Levy measure of $Z$ arising as the image of $\rho_\Lambda(du|y)\eta(dy, dx)$ via the map $(u, y)$ to $(-\log(1-u), y)$.

REMARK 32. When $B = \mathcal{X}$ it is obvious that $F(\cdot, \mathcal{X})$ is an NTR. In addition, due to the complete randomness properties of $\Lambda$, $F$ satisfies,

$$\mathcal{L}\{(F(t, B), F(t, B) - F(t, B), F(t, B) - F(t, B))\} = \mathcal{L}\{(V_{1,B}, V_{2,B}(1 - V_1), \ldots, V_{k,B} \prod_{j=1}^{k}(1 - V_i))\},$$

where for each $j$, $V_j := V_{j,B} + V_{j,Bc}$ and $V_{i,B}$ is independent of $V_{j,C}$ for $i \neq j$ and $B, C$ arbitrary. A posterior process will be called an extended NTR process if the NTR properties are preserved.

7.2 Posterior distributions and moment formulae

Now suppose that one observes n-iid observations $(T_i, X_i)$ from $\prod_{i=1}^{n} F(dT_i, dX_i)$ and consider the following joint models,

$$(140) \quad \left[ \prod_{i=1}^{n} F(dT_i, dX_i) \right] \mathcal{P}(dF|\rho_\Lambda, \eta) \quad \text{and} \quad \left[ \prod_{i=1}^{n} F(dT_i, dX_i) \right] \mathcal{P}(d\Lambda|\rho_\Lambda, \eta).$$

Here we will work with $\Lambda$, and the equivalent expression

$$(141) \quad \left[ \prod_{i=1}^{n} S(T_i-)\Lambda(dT_i, dX_i) \right].$$

If one assumes the classical univariate right censoring applied to the marked data as in Huang and Louis (1998) then the likelihood under censoring takes the form

$$(142) \quad \left[ \prod_{j=1}^{m} S(C_j-) \right] \left[ \prod_{i=1}^{n} S(T_i-)\Lambda(dT_i, dX_i) \right]$$

where $C_1, \ldots, C_m$ denote $m$ independent censoring times which indicate that there are random variables $T_{n+1}, \ldots, T_{m+n}$ where it is only known that they exceed the respective censored times. Under this assumption no information for the marks associated with the censored points $T_{n+1}, \ldots, T_{m+n}$ is available.

The primary focus will be to deduce the posterior distribution of both $F$ and $\Lambda$ and related characteristics of the marginal distribution of $(T_1, X_1), \ldots, (T_n, X_n)$. This will complete the necessary disintegration which will allow one to apply both the (extended) models for $\Lambda$ and $F$ to a large class of data structures beyond univariate right censoring. In fact it will become clear from the form of (141) that analysis of right censoring data for NTR is really the same affair as analysis of the complete data model.
First, for \( i = 1, \ldots, n \) define \( \tilde{Y}_{T_i-}(s) := I\{T_i > s\} \) and similarly for \( l = 1, \ldots, m \) define \( \tilde{Y}_{C_l-}(s) := I\{C_l > s\} \). In addition for \( i = 1, \ldots, n \) define \( \tilde{f}_{T_i-} \) satisfying,

\[
(1 - u)\tilde{Y}_{T_i-}(s) := e^{-\tilde{f}_{T_i-}(u,s)},
\]

and for \( l = 1, \ldots, m \), let \( \tilde{f}_{C_l-} \) be defined similarly. Then for each \((n, m) \geq 0\)

\[
(1 - u)Y_{n,m}(s) := e^{-f_{n,m}(u,s)}
\]

where \( Y_{n,m}(s) := \sum_{i=1}^{n} \tilde{Y}_{T_i-}(s) + \sum_{l=1}^{m} \tilde{Y}_{C_l-}(s) \). The quantities \( f_{n,m}, \tilde{f} \) are special cases of the functions defined in Proposition 3.1 and 3.2. It follows from the identity in (134) that

\[
S(T_i) := e^{-N(\tilde{f}_{T_i-})} \text{ and } \prod_{i=1}^{n} S(T_i) = e^{-N(f_n)}
\]

where \( N = P(dN | \rho_\Lambda, \eta) \). Now the likelihood (142) can be rewritten as

\[
e^{-N(f_{n,m})} \prod_{i=1}^{n} \Lambda(dT_i, dX_i)
\]

This representation, (144), in combination with Proposition 3.1 and Theorem 3.1 yields the posterior distributions for \( \Lambda, F, Z \) subject to possible right censorship.

**Theorem 7.1**  The posterior distribution of \( \Lambda \) given the model (142) is \( P(d\Lambda | e^{-f_{n,m}}, \rho_\Lambda, \eta, T, X) \). That is, \( \Lambda \) is equivalent to the random measure defined in Proposition 3.1 and 3.2. It follows from the identity in (134) that

\[
\Lambda(\cdot | Y_{n,m}) = e^{-f_{n,m}} P(\cdot | \rho_\Lambda, \eta, T, X)
\]

where the law of \( \Lambda(\cdot | Y_{n,m}) \) is \( P(d\Lambda | e^{-f_{n,m}}, \rho, \eta) \) indicating that its intensity measure is,

\[
(1 - u)Y_{n,m}(s) \rho_\Lambda(du) \eta(ds, dv).
\]

The \( J_{j,n} \) are (conditionally) mutually independent random variables with distribution, for each \( j \), depending on \( T_j^*, Y_{n,m}(T_j^*) \), defined as in (32) as,

\[
P(J_{j,n} \in du | e^{-f_{n,m}}, \rho_\Lambda, T_j^*) := \frac{\kappa_{e_{j,n}}(1 - u)Y_{n,m}(T_j^*) \rho_\Lambda(du | T_j^*)}{\kappa_{e_{j,n}}(e^{-f_{n,m}} \rho_\Lambda | T_j^*)},
\]

and are conditionally independent of \( \Lambda(\cdot | Y_{n,m}) \).

(ii) The posterior distribution of \( F \) is still an extended NTR with distribution \( PN(dF | e^{-f_{n,m}}, \rho_\Lambda, \eta, T, X) \) determined by replacing the random measure \( \Lambda \) with (144).

(iii) A posterior distribution of \( Z \) is equivalent to the law of the random measure

\[
Z(\cdot | Y_{n,m}) + \sum_{j=1}^{n(p)} (1 - e^{-J_{j,n}}) \delta_{T_j^*, X_j^*}(\cdot),
\]

where the Lévy measure of \( Z(\cdot | Y_{n,m}) \) arises as the image of \( (1 - u)Y_{n,m}(s) \rho_\Lambda(du) \eta(ds, dx) \) via the map \((u, s)\) to \((-\log(1 - u), s)\)
When \( m := 0 \), the results correspond to a complete data model.

**Proof.** First set \( \mu := \Lambda \), and \( w(\Lambda) := e^{-\mathcal{N}(f_{n,m})} \). Now apply statement (ii) of Proposition 3.1. \( \blacksquare \)

**Remark 33.** Theorem 7.1 serves to extend the results for the univariate setting to a spatial setting. The previous works for the univariate setting do not use explicitly a partition based representation. More importantly the method of proof, which is new, is quite transparent.

**Remark 34.** Note also that due to the non-atomic nature (continuity) of \( \eta(ds) \) the quantity \( Y_{n,m}(s) \) in (148) can be replaced by

\[
Y_{n,m}^+(s) := \sum_{i=1}^{n} I\{s \leq T_i\} + \sum_{l=1}^{m} I\{s \leq C_l\}.
\]

In other words calculations with respect to \( \Lambda(\cdot|Y_{n,m}) \) should be understood to be equivalent to those with respect to \( \Lambda(\cdot|Y_{n,m}^+) \). This does not apply to the distribution of the jumps \( (J_{j,n}) \).

Little is known in general about the explicit joint moment structure of Neutral to the Right models. The results in Doksum (1974) are rather vague. In recent works expressions for the mean and variance are given. The formulae in Proposition 3.2 can be used to easily obtain various equivalent expressions which goes well beyond a variance calculation. This is seen by setting \( w_i(\Lambda) := e^{-\mathcal{N}(f_{i,-})} \) for \( i = 1, \ldots, n \), and other obvious equivalences. For brevity, I will only present a result which yields the relevant joint structure and EPPF for these models. Such results do not appear in the literature mentioned above.

Define,

\[
\tilde{A}_{n,m}(\infty) := \int_0^\infty \int_0^1 (1 - (1 - u)Y_{n,m}^+(s))\rho_{\Lambda}(du|s)\eta(ds)
\]

In addition for \( i = 1, \ldots, n \) and \( m \geq 0 \) define,

\[
a_{i-1,m}(t) := \int_0^t \int_0^1 u(1 - u)Y_{i-1,m}^+(s)\rho_{\Lambda}(du|s)\eta(ds).
\]

When \( m = 0 \), denote \( a_{i-1,m} \) as \( a_{i-1} \). Now recall that,

\[
\kappa_{e_{j,n}}(e^{-f_{n,m}}\rho_{\Lambda}|T_j) = \int_0^1 u^{e_{j,n}}(1 - u)Y_{n,m}(T_j)\rho_{\Lambda}(du|T_j^*),
\]

**Proposition 7.1** (i) For \( m > 0 \), the (prior) mean calculation for \( \prod_{i=1}^{m} S(C_i) \) is

\[
E[e^{-\mathcal{N}(f_{n,m})}|\rho_{\Lambda}, \eta] = e^{-\tilde{A}_{n,m}(\infty)} = e^{-A_{n}(T_1)} \prod_{l=2}^{n} e^{-A_{l-1}(T_l)} \prod_{l=1}^{m} e^{-A_{n,l-1}(C_l)}
\]

(ii) When \( m = 0 \), the joint marginal distribution of \( (\{T_i, X_i\}) \) can be expressed as

\[
\prod_{j=1}^{n(p)} \left[ \prod_{i \in C_j} e^{-A_{i-1}(T_j)} \right] \kappa_{e_{j,n}}(e^{-f_n}\rho_{\Lambda}|T_j^*)\eta(dT_j^*, dX_j^*)
\]

Adjustments for the censored case are obvious.

(iii) From statement (i) it follows that the corresponding EPPF has the form,

\[
\int_{T^{(p)}} \prod_{j=1}^{n(p)} \left[ \prod_{i \in C_j} e^{-A_{i-1}(T_j)} \right] \kappa_{e_{j,n}}(e^{-f_n}\rho_{\Lambda}|T_j^*)\eta(dT_j^*)
\]
Proposition 7.2 Using an algebraic rearrangement the joint marginal distribution and EPPF can be written respectively as,

\[
\pi(p|\rho, T^*) = \prod_{j=1}^{n(p)} F_0(dT_j^+, dX_j^+) \text{and } \int_{T^n(p)} \pi(p|\rho, T^*) \prod_{j=1}^{n(p)} F_0(dT_j^*).
\]

Where,

\[
\pi(p|\rho, T^*) := \prod_{j=1}^{n(p)} \left[ \prod_{i \in C_j} e^{-A_i - 1(T_j^*)} \right] \kappa_{e_j,n} \left( e^{-f_{\rho} \rho|T^*_j} \right)
\]

\[
\mu_{\tau, \beta}(ds, dx, \{0, 1\}) := \mu_{\tau}(ds, dx, \{0, 1\}) \mu_{\beta}([s, \infty)) + \mu_{\beta}([s, \infty))
\]

Remark 35. Naturally if one were interested in actually generating such partitions etc, then a modification of the algorithm in Section 2.3 can be used. For this one could use expressions for the prediction rule or conditional moment measures which are readily obtainable from an application of Proposition 3.2 combined with Theorem 7.1.

Remark 36. The propositions above combined with Theorem 7.1 yield expressions for the posterior disintegrations. It is now straightforward to obtain posterior characterizations for mixtures of extended NTR models based on kernels \((K_i)\). This framework allows for much more complex structures than right censoring. I have not seen general mixtures of NTR models proposed in the literature.

7.3 Absolute continuity of general Beta,Beta-Neutral/Stacy models to a canonical Beta processes or Dirichlet process

The general construction of the extended NTR models is an extension of (presently unpublished work) James and Sehyug Kwon (2000). In that work the authors extend Lo’s (1993) Beta-Neutral survival and cumulative hazard processes to the spatial setting. Lo (1993) derives these based on the following explicit construction for the hazard

\[
\Lambda_{\tau, \beta}(t) := \int_{0}^{t} \frac{\mu_{\tau}(ds)}{\mu_{\tau}([s, \infty)) + \mu_{\beta}([s, \infty))},
\]

where \(\mu_{\tau, \beta}\) are independent gamma processes with shape measure \(\tau\) and \(\beta\) on \(\mathbb{R}^+\), which as noted in Lo (1993) yields an explicit construction of Hjort’s (1990) Beta cumulative hazard process. James and Kwon (2000) extend this definition by simply extending the gamma processes to a spatial setting. Moreover they show that such models can be always derived from a Dirichlet process on an even larger space. In other words take a two parameter gamma process, say \(\mu_{\tau, \beta}\), on \(\mathbb{R}^+ \times \mathcal{X} \times \{0, 1\}\), such that \(\mu_{\tau, \beta}(ds, dx, \{1\}) := \mu_{\tau}(ds, dx)\) etc. A corresponding Dirichlet process can be defined as

\[
P_{\tau, \beta}(ds, dx, \Delta) := \frac{\mu_{\tau, \beta}(ds, dx, \Delta)}{\mu_{\tau, \beta}(\mathbb{R}^+ \times \mathcal{X} \times \{0, 1\})}.
\]

Then the extension of James and Kwon (2000) can be deduced from the extended Beta-Neutral hazard measure,

\[
\Lambda_{\tau, \beta}(ds, dx) := \frac{\mu_{\tau, \beta}(ds, dx, \{1\})}{\mu_{\tau, \beta}([s, \infty)) \times \mathcal{X} \times \{0, 1\})}
\]
where $\tau$ and $\beta$ are now measures on $\mathcal{R}^+ \times \mathcal{X}$. If $\beta$ is set to zero in (160) then the corresponding (extended) Neutral to the Right process is a Dirichlet process with shape parameter $\tau$ which, without loss of generality, is set to $\theta F_0$. By virtue of the essential equivalence of the Beta-Neutral process to the Beta-Stacy process of Walker and Muliere (1997) and the Beta distribution function of Hjort (1990, Section 7A), the procedure of James and Kwon (2000) includes these models as well. In addition they showed that the (posterior) conjugacy of the Beta-type models on $\mathcal{R}^+$ is preserved under the right censored data spatial model discussed earlier. However their technique relied on very special properties of the Dirichlet process which is quite different than what has been presented in the previous sections. Here a new result is established which shows how one may transform a Dirichlet process or simple Beta process to a more general one via a change of measure. A consequence is that the calculus for such models follows from the calculus for the Dirichlet process plus an application of Proposition 3.1.

Proposition 7.3 Let $\mathcal{P}(d\Lambda|c, A_0)$ denote the law of a Beta process with parameters $c$ and $A_0(ds, dx)$. Let $Z$ denote the corresponding Lévy process defined via the map $(u, y)$ to $(-\log(1-u), y)$ and define a decreasing function $\beta$ on $\mathcal{R}^+$ such that,

$$T_{\beta} := \int_0^\infty \beta(v)Z(dv) < \infty.$$  

Then the following disintegration holds

$$e^{-T_{\beta}}\mathcal{P}(d\Lambda|c, A_0) := \mathcal{P}(d\Lambda|c + \beta, A_0)E[e^{-T_{\beta}}]$$

where $-\log E[e^{-T_{\beta}}]$ is

$$\int_0^\infty \int_0^1 (1 - (1 - u))\beta(s)u^{-1}(1 - u)^{c(s)-1}duA_0(ds).$$

The proof follows by using the alternate representation

$$T_{\beta} := e^{-N(f_{\beta})}$$

where $f_{\beta}(u, s) := -\beta(s)\log(1 - u)$ and $N$ has a Poisson law corresponding to $\mathcal{P}(d\Lambda|c, A_0)$. An interesting feature of this result is that one can obtain quite easily an alternate expression for the EPPF of a Beta-Neutral/Stacy model by first applying the result for a Dirichlet process. That is,

Proposition 7.4 Suppose that $F$ is a NTR process determined by the Beta process with parameters $c^* := \theta F_0 + \beta$, then the EPPF is given by,

$$PD(p|\theta) \prod_{j=1}^{n(p)} \int_0^\infty \left[ \frac{\Gamma(e_{j,n} + \theta F_0([y, \infty)))\Gamma(\theta F_0([y, \infty)) + \beta(y))}{\Gamma(\theta F_0([y, \infty)))\Gamma(e_{j,n} + \theta F_0([y, \infty)) + \beta(y))} \right] F_0(dy).$$

REMARK 37. The change of measure in Proposition 7.3 can of course be extended easily to non-Beta processes. In general, the updated law corresponds to a random hazard measure with Lévy measure $e^{-f_{\beta}\rho}$. Note that much more general choices of $f_{\beta}$ can be used via Proposition 3.1.
The correspondence between the Beta-Stacy process of Walker and Muliere (1997) and Hjort’s (1990, section 7A) process is noted explicitly in Dey (1999) and Dey, Ericson, and Ramamoorthi (2000). The equivalence between the Beta-Neutral process and Hjort’s (1990) process was noted in Lo (1993). Given the gaps in the literature it is apparent that Beta-Neutral processes are not as well known as their equivalent counterparts. This seems to be caused by the title of Lo (1993) which concerns a Censored Data Bayesian Bootstrap.

NTR processes seem to arise naturally in coalescent theory. See in particular Pitman (1999, Proposition 26) which is not a Dirichlet process. Similar types of processes with drift appear in Bertoin (2001).

8 Posterior Distributions of Normalised processes and Poisson-Kingman models

In this section I briefly discuss calculations for probability measures $P$ defined using the weighted Poisson distribution $Q(dN|\rho, \eta)$ which are more in line with the results and methods used in Perman, Pitman and Yor (1992) and Pitman and Yor (1992) and Pitman (1995b). Here descriptions of pertinent quantities will be given in terms of the biased jumps denoted as $\tilde{J}$. One will see that the forms of the results appear quite different than Section 5. For completion the definition of Poisson-Kingman models based on length biased sampling presented in Pitman (1995b) is given,

\[ \text{Definition 8.1 (Pitman (1995b)) Let } P_i = (J_i/T) \text{ be a ranked discrete distribution derived from the ranked points of a Poisson Process with Lévy density } \rho \text{ (not depending on } y) \text{ of random lengths } J_1 \geq J_2 \geq \cdots \geq 0 \text{ by normalizing their lengths by their sum which is } T. \text{ Let } (\tilde{P}_j) \text{ be a size-biased permutation of } (P_i) \text{ and let } \tilde{J}_j = (T\tilde{P}_j) \text{ be the corresponding size-biased permutation of the ranked lengths } (J_i). \text{ The law of the sequence } (P_i) \text{ will be called the Poisson-Kingman distribution with Lévy density } \rho, \text{ and denoted } PK(\rho). \text{ Denote by } PK(\rho|t) \text{ the regular conditional distribution of } (P_i) \text{ given } (T = t) \text{ constructed above. For a probability distribution } \gamma \text{ on } (0, \infty), \text{ let} \]

\[ PK(\rho, \gamma) := \int_0^\infty PK(\rho|t)\gamma(dt) \]

be the distribution on the space of $(P_i)$, [which is the space of decreasing sequences of positive real numbers with sum 1]. Call $PK(\rho, \gamma)$ the Poisson-Kingman distribution with Lévy density $\rho$ and mixing distribution $\gamma$.

Pitman (1995b), points out that knowledge of the conditional law $PK(\rho|t)$ allows one to generate explicit results for distributions $PK(\rho, \gamma)$, in particular the corresponding EPPF, by simply mixing over different candidate densities, $\gamma(dt)$, for $T$. The case of the two parameter Poisson-Dirichlet family is explained in detail in Pitman (1995b) as mentioned in section 5.

The introduction of the measure $Q(dN|\rho, \eta)$ serves to incorporate fully the Poisson-Kingman idea while maintaining the approach of Poisson calculus at the level of $N$. Some properties of this class, which shall become clear in the next section are now described. When $\rho$ is homogeneous and $w(N) := w(N(\cdot, Y))$ then $P$ is a species sampling model. Additionally when $w(N) := g(T)$ and $h(s) := s \in (0, \infty)$ then the random atoms of $P_i$ are $PK(\rho, \gamma)$. In that case

\[ \gamma(dt) := \frac{g(t)E_g(T)|\rho, \eta|}{E[g(T)|\rho, \eta]} \]
where \( g \) is nonnegative and integrable but otherwise arbitrary, \( f_T \) denotes the density of \( T \) with respect to \( \mathcal{P}(dN|\rho, \eta) \). In other words the change of measure at \( N \) via \( \mathcal{Q}(dN|\rho, \eta) \) induces the appropriate change of measure at the level of the PK measure etc. The results below follow from a straightforward application of Lemma 2.2, details are omitted.

### 8.1 Posterior characterizations

In the theorem below it is stated that the posterior law of \( P|Y \), denoted as \( \mathcal{P}(dP|Y) \), corresponds to the law of a random measure defined as,

\[
P^*_n(\cdot) := R_{n(p)}P(\cdot) + (1 - R_{n(p)}) \sum_{j=1}^{n(p)} \frac{h(\tilde{J}_j)}{\sum_{j=1}^{n(p)} h(\tilde{J}_j)} \delta_{Y^*_j}(\cdot),
\]

where, \( R_{n(p)} := T/\left[T + \sum_{j=1}^{n(p)} h(\tilde{J}_j)\right] \).

**Theorem 8.1** Let \( \{Y_1, \ldots, Y_n\} \) be iid \( P \) where the prior law of \( P \) is determined by the weighted Poisson measure \( \mathcal{Q}(dN|\rho, \eta) \). Then, the posterior distribution of \( P|Y \) is equivalent to the distribution of the random measure \( P^*_n \) defined in (167) whose law is now determined by the joint probability measure

\[
\mathcal{Q}(dN, \tilde{J} \in ds|Y, h) \propto w(N + \sum_{j=1}^{n(p)} \delta_{s, Y^*_j})\mathcal{P}(dN|\rho, \eta) \left(T + \sum_{j=1}^{n(p)} h(s_j)\right)^{-n}\prod_{j=1}^{n(p)} h(s_j)^{\rho \cdot \eta} (ds_j|Y^*_j)
\]

The marginal distribution of \( \{Y_1, \ldots, Y_n\} \) corresponds to the un-normalised term on the right hand side of (163), integrated over \( \mathcal{P}(dN|\rho, \eta) \), and multiplied by \( \prod_{j=1}^{n(p)} \eta(dY^*_j) \).

**Remark 40.** Notice that the expression above remains complicated even if \( W(N) \) is replaced by 1. The result is more in line with Pitman, Perman and Yor (1992) and Pitman (1995b). In comparison with section 5 the law of the random measure \( P^*_n \) is a bit more complex as it is described via the biased jumps \( \tilde{J} \) which have a much more complex (non-independent) joint distribution. Under moment conditions other forms of the posterior are easily obtained.

Now letting

\[
T_{n(p)} := T - \sum_{j=1}^{n(p)} h(s_j)
\]

it is quite clear that one can gain further interpretation by adapting the descriptions given in Perman, Pitman and Yor (1992, section 4) and Pitman (1995b). Note in particular when \( n = 1, \) and \( w(N) := g(T) \) and \( \rho \) is homogeneous an evaluation of the expectation of \( P \) reveals the structural distributions,

\[
P_g(\tilde{J}_1 \in ds, T_1 \in dt_1) = \frac{g(t_1 + s)f_T(t_1)dt_1 h(s)\Omega(s)ds}{(t_1 + s)E[g(T)]} = \frac{g(t_1 + s)E[g(T)]}{E[g(T)]} P(\tilde{J}_1 \in ds, T_1 \in dt_1)
\]

A suitable change of variable yields

\[
P_g(T \in dt, T_1 \in dt_1) = \frac{g(t)}{E[g(T)]} P(T \in dt, T_1 \in dt) = \frac{g(t)f_T(t)}{E[g(T)]} P(T_1 \in dt_1|T = t)
\]
Moreover for every $H$ are now iid for

$$
\mathbb{P}_g \left( p, J \in ds, T_{n(p)} \in dt \right) = \frac{g(t + \sum_{j=1}^{n(p)} h(s_j))}{E[g(T)]} \mathbb{P}_g \left( p, J \in ds, T_{n(p)} \in dt \right).
$$

The joint distributions $\mathbb{P}(\tilde{J}_1 \in ds, T_1 \in dt_1)$ and $\mathbb{P}(p, J \in ds, T_{n(p)} \in dt)$ are given in Pitman (1995b) and Perman, Pitman and Yor (1992). One can for instance obtain formulae for the joint distribution of $(T, T_1, \ldots, T_n)$ via $(g(t)/E[g(T)]) \mathbb{P}(T \in dt, T_1 \in dt_1, \ldots, T_n \in dt_n)$ using Perman, Pitman and Yor (1992, Theorem 2.1). All of these facts correspond with the $PK(\rho, \gamma)$ concept. See additionally Pitman and Yor (1997, Proposition 47).

Combining the description in Pitman (1995b, Lemma 5 and equation (30)) with Theorem 8.1 the next result follows.

**Corollary 8.1** Suppose that $w(N + \sum_{j=1}^{n(p)} \delta_{s_j}, Y^*) := w(N; \sum_{j=1}^{n(p)} \delta_{s_j})$ does not depend on $Y^*$ for all $n$. Then a joint law of $p, \tilde{J}, Y^*$ is defined as:

$$
\frac{1}{E[w(N)]} \left[ \int_\mathcal{M} \frac{w(N; \sum_{j=1}^{n(p)} \delta_{s_j}) \mathcal{P}(dN|\rho, \eta)}{(T + \sum_{j=1}^{n(p)} h(s_j))^n} \prod_{j=1}^{n(p)} h(s_j)^{\delta_{s_j}} \rho(ds_j|Y^*) \eta(dy) \right].
$$

As a consequence, the distribution of $Y^* | \tilde{J}, T_{n(p)}, p$ is conditionally independent of $T_{n(p)}$ such that the sequence $\{Y_1, \ldots, Y_n\}$ consists of $n(p)$ unique values $\{Y^*_1, \ldots, Y^*_n\}$ which are independent with distribution

$$
\mathbb{P}(dY_j^* | \tilde{J}, p) \propto \rho(\tilde{J}_j|Y^*_j) \eta(dy).
$$

for $j = 1, \ldots, n(p)$. Additionally the joint distribution of $\tilde{J}, p$ is

$$
\mathbb{P}(\tilde{J} \in ds, p) = \frac{1}{E[w(N)]} \left[ \int_\mathcal{M} \frac{w(N; \sum_{j=1}^{n(p)} \delta_{s_j}) \mathcal{P}(dN|\rho, \eta)}{(T + \sum_{j=1}^{n(p)} h(s_j))^n} \prod_{j=1}^{n(p)} h(s_j)^{\delta_{s_j}} \int_y \rho(ds_j|y) \eta(dy) \right].
$$

If additionally $\rho$ is homogeneous then

$$
\mathbb{P}(dY) = p_\rho(e_1, \ldots, e_{n(p)}) \prod_{j=1}^{n(p)} H(dY_j^*),
$$

where, the EPPF is

$$
p_\rho(e_1, \ldots, e_{n(p)}) = \frac{1}{E[w(N)]} \int_{\mathcal{M} \times S^{n(p)}} \frac{w(N; \sum_{j=1}^{n(p)} \delta_{s_j}) \mathcal{P}(dN|\rho, \eta) \prod_{j=1}^{n(p)} h(s_j)^{\delta_{s_j}} \rho(ds_j)}{(T + \sum_{j=1}^{n(p)} h(s_j))^n}. \cdot
$$

Moreover for every $n$, the joint distribution of the unique values of $\{Y_1, \ldots, Y_n\}|p$, that is $\{Y^*_1, \ldots, Y^*_n\}$, are now iid $H$. If $W(N) := g(T)$ then the EPPF is:

$$
\int_{\mathbb{R}^+ \times S^{n(p)}} \left[ \frac{g(t + \sum_{j=1}^{n(p)} s_j) f_T(t) \prod_{j=1}^{n(p)} h(s_j)^{\delta_{s_j}} \rho(ds_j)}{\left( t + \sum_{j=1}^{n(p)} h(s_j) \right)^n} \right].
$$

When $h(s) = s \in (0, \infty)$ and $g(T) = 1$, the expression in (177) corresponds exactly with equation (31) of Pitman (1995b).
REMARK 41. One could simply apply a Fubini argument to Theorem 8.1 to deduce the existence of the relevant joint distributions. However, without any interpretation gained via Pitman, Perman and Yor (1992) and Pitman (1995b) the result is somewhat vacuous. Again some of those interpretations may also be of interests to practicing Bayesians statisticians as it certainly goes beyond the usual mean/variance assessment to compare different random probability measures.

REMARK 42. The results above serve also to add to the explicit formulae given in Pitman (1995b) for the length biased case. Note that one could apply (67) to obtain alternate representations for the EPPF as in Pitman (1995b, Corollary 6). An additional interesting feature of Corollary 8.1 is the conditional independence result in (173).

8.2 Mixture models

Theorem 8.1 and its corollary can certainly handle structures of the form

\[ Q(dN|\rho, \eta) \prod_{i=1}^{n} \int_{Y} K(X_i|Y_i) P(dY_i) \]

A description of the relevant posterior laws is presented below.

**Theorem 8.2** The posterior distribution of \( Y, P|X \) based on the model (52) is representable as,

\[
P(dY, dP|X) \propto P(dP|Y) \prod_{i=1}^{n(p)} \mathbb{P}(dY_{j,*}^{+}|\tilde{J}, p, X) \mathbb{P}(\tilde{J} \in ds, p|X),
\]

where a conditional distribution of \( Y|\tilde{J}, p, X \) is given such that the sequence \( \{Y_1, \ldots, Y_n\} \) consists of \( n(p) \) unique values \( Y^* = \{Y_1^*, \ldots, Y_n^*\} \) which are independent with respective distributions,

\[
P(dY_{j,*}^{+}|\tilde{J}, p, X) = \frac{\prod_{i \in C_j} K_i(X_i|Y_{j,*}^+)}{\int_{Y} \prod_{i \in C_j} K_i(X_i|Y_{j,*}^+)} P(dY_{j,*}^{+}|\tilde{J}, p),
\]

for \( j = 1, \ldots, n(p) \). In addition \( \mathbb{P}(\tilde{J} \in ds, p|X) \propto P(\tilde{J} \in ds, p) \prod_{j=1}^{n(p)} \int_{Y} \prod_{i \in C_j} K(X_i|Y_{j,*}^+) P(dY_{j,*}^{+}|\tilde{J}, p) \)

It follows that when \( \rho \) does not depend on \( y \) (179) does not depend on \( \tilde{J}, \mathbf{T} \). Hence in that case posterior characterizations of \( P, Y \) do not differ in form from the results in Lo (1984) and Ishwaran and James (2001a). The general case however is a different matter entirely.

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References

Aalen, O. O. (1975). Statistical inference for a family of counting processes. Ph.D. Thesis. University of California, Berkeley.

Aalen, O. O. (1978). Nonparametric inference for a family of counting processes. Ann. Statist. 6 535-545.

Aalen, O. O. (1992). Modelling heterogeneity in survival analysis by the compound Poisson distribution. Annals of Applied Probability 2 951-972.

Aldous, D. J. (1985). Exchangeability and related topics. In École d’Été de Probabilités de Saint-Flour XII (P. L. Hennequin, editor). Springer Lecture Notes in Mathematics, Vol. 1117.

Andersen, P. K., Borgan, O., Gill, R. D. and Keiding, N. (1993). Statistical Models Based On Counting Processes. Springer-Verlag, New York.

Antoniak, C. E. (1974). Mixtures of Dirichlet processes with applications to Bayesian nonparametric problems. Ann. Statist. 2 1152-1174.

Banjevic, D., Ishwaran, H. and Zarepour, M. (2002). A recursive method for functionals of Poisson processes. To appear in Bernoulli.

Bar-Lev, S.K. and Enis, P. (1986). Reproducibility and natural exponential families with power variance functions. Ann. Statist. 14 1507-1522.

Barndorff-Nielsen, O.E. and Shephard, N. (2001). Normal modified stable processes. Preprint.

Berk, R. and Savage, I. R. (1979). Dirichlet processes produce discrete measures: an elementary proof. Contributions to Statistics, Jaroslav Hajek Memorial Volume. Edited by Jana Jurečková. Reidel, Dordrecht-Boston, Mass.-London. pp. 25-31.

Bertoin, J. (2001). Homogeneous fragmentation processes. Probab. Theory Related Fields 121 301–318.

Blackwell, D. (1973). Discreteness of Ferguson selections. Ann. Statist. 1 356-358.

Blackwell, D. and MacQueen, J. B. (1973). Ferguson distributions via Pólya urn schemes. Ann. Statist. 1 353-355.

Blackwell, D. and Maitra A. (1984). Factorization of probability measures and absolutely measurable sets. Proc. Amer. Math. Soc. 92 251-254.

Blum, J. and Susarla, V. (1977). On the posterior distribution of a Dirichlet process given random right censored data. Stochastic Process. Appl 5 207-211.

Brix, A. (1999). Generalized Gamma measures and shot-noise Cox processes. Adv. in Appl. Probab. 31 929-953.

Brunner, L. J., Chan, A. T., James, L. F. and Lo, A. Y. (2001). Weighted Chinese restaurant processes and Bayesian mixture models. preprint.

Brunner, L. J. and Lo, A. Y. (1989). Bayes methods for a symmetric unimodal density and its mode. Ann. Statist. 17 1550-1566.

Carlton, M.A. (1999). Applications of the Two-Parameter Poisson Dirichlet distribution. Ph.D. Thesis. University of California, Los Angeles. Dept. of Statistics.

Cifarelli, D.M. and Regazzini, E. (1990). Some remarks on the distribution of the means of a Dirichlet process. Ann. Statist. 18 429-442.

Constantine, G. M. (1999). Identities over set partitions. Discrete Mathematics 204 155-162.

Constantine, G. M. and Savits, T. H. (1994). A stochastic process interpretation of partition identities. Siam J. Discrete Math. 7 194-202.
Daley, D. J. AND VERE-JONES, D. (1988). An Introduction to the Theory of Point Processes. Springer-Verlag, New York.

DELLACHERIE, C AND MEYER, P. A. (1978). Probabilities and Potential. Amsterdam

DEY, J. (1999). Some Properties and Characterizations of Neutral-to-the-Right Priors and Beta Processes. Ph.D. Thesis. Michigan State University.

DEY, J, ERICKSON, R.V. AND RAMAMOORTHI, R.V. (2000). Neutral to right priors- A review. Preprint.

DI NARDO, E. AND SENATO, D. (2001). Umbral nature of the Poisson random variables. In Algebraic Combinatorics Computer Science: A Tribute to Gian-Carlo Rota Springer 245-267.

DIACONIS, P. AND KEMPERMAN, J. (1996). Some new tools for Dirichlet priors. Bayesian Statistics 5 (J.M. Bernardo, J.O. Berger, A.P. Dawid and A.F.M. Smith eds.), Oxford University Press, pp. 97-106.

DOKSUM, K. A. (1974). Tailfree and neutral random probabilities and their posterior distributions. Ann. Probab 2 183-201.

DONNELLY, P. AND TAVARÉ, S. (1987). The population genealogy of the infinitely-many neutral alleles model. J. Math. Biol. 25 381-391.

DOSS, H. (1994). Bayesian nonparametric estimation for incomplete data via successive substitution sampling. Ann. Statist. 22 1763-1786.

DYKSTRA, R. L. AND LAUD, P. W. (1981). A Bayesian nonparametric approach to reliability. Ann. Statist. 9 356-367.

ENGEN, S. (1978). Stochastic Abundance Models with Emphasis on Biological Communities and Species Diversity. Chapman and Hall

EWENS, W. J. (1972). The sampling theory of selectively neutral alleles. Theor. Popul. Biol. 3 87-112.

EWENS, W. AND TAVARÉ, S. (1997). Multivariate Ewens distribution. In Discrete Multivariate Distributions (S. Kotz and N. Balakrishnan, eds.). Wiley, New York.

FERGUSON, T. S. (1973). A Bayesian analysis of some nonparametric problems. Ann. Statist. 1 209-230.

FERGUSON, T. S. (1974). Prior distributions on spaces of probability measures. Ann. Statist. 2 615-629.

FERGUSON, T. S. AND KLASS, M. J. (1972). A representation of independent increment processes without Gaussian components. Ann. Math. Statist. 43 1634-1643.

FERGUSON, T. S. AND PHADIA, E. (1979). Bayesian nonparametric estimation based on censored data. Ann. Statist. 7 163-186.

FREEDMAN, D. A. (1963). On the asymptotic behaviour of Bayes estimates in the discrete case. Ann. Math. Statist 34 1386-1403.

FITZSIMMONS, P., PITMAN, J. AND YOR, M. (1992). Markovian bridges: construction, Palm interpretation and splicing, In Seminar on stochastic process 1992 Editors Cinlar, Chung, Sharpe. Birkhauser, Boston. 101-135.

GILL, R. D. AND JOHANSEN, S. (1990). Survey of product-integration with a view towards applications in survival analysis. Ann. Statist. 18 1501-1555.

GROENEBOOM, P. (1996). Lectures on reverse problems. Lectures on probability theory and statistics (Saint-Flour, 1994), 67–164, Lecture Notes in Math., 1648, Springer, Berlin.

GROENEBOOM, P. AND WELLNER, J. A. (1992). Information bounds and nonparametric maximum likelihood estimation. DMV Seminar, 19. Birkhauser Verlag, Basel.

GYLLENBERG, M. AND KOSKI, T. (2001). Probabilistic models for bacterial taxonomy. International Statistical Review 69 249-276.

HANSEN, B. AND PITMAN, J. (2000). Prediction rules for exchangeable sequences related to species sampling. Statist. Prob. Letters 46 251-256.

HJORT, N. L. (1990). Nonparametric Bayes estimators based on Beta processes in models for life history data. Ann. Statist. 18 1259-1294.

HOUGAARD, P. (1986). Survival models for heterogeneous populations derived from stable distributions. Biometrika 73 387-396.

HOUGAARD, P., LEE, M. L. AND WHITMORE, G. (1997). Analysis of overdispersed count data by mixtures of Poisson variables and Poisson processes. Biometrics 53 1225-1238.
Huang, Y. and Louis, T. A. (1998). Nonparametric estimation of the joint distribution of survival time and mark variables. *Biometrika* **85** 785-798.

Ishwaran, H. and James, L. F. (2001a). Generalized weighted Chinese restaurant processes for species sampling models. Manuscript.

Ishwaran, H. and James, L. F. (2001b). Gibbs sampling methods for stick-breaking priors. *J. Amer. Stat. Assoc* 161-173.

Ishwaran, H., James, L. F. and Sun, J. (2001). Bayesian model selection in finite mixtures by marginal density decompositions. *J. Amer. Stat. Assoc* **96** 1316-1332.

Jacod, J. (1975). Multivariate point processes: predictable projection, Radon-Nikodym derivatives, representation of martingales. *Z. Wahrsch. verw. Gebiete* **35** 1-37.

James, L. F. and Kwon, S. (2000). A Bayesian nonparametric approach for the joint distribution of survival time and mark variables under univariate censoring. Unpublished manuscript.

James, L. F. (2001a). Bayesian calculus for Gamma processes with applications to semiparametric models. To appear in *Sankhyā Ser. A*.

James, L. F. (2001b). An analysis of weighted generalised gamma process mixture models. Unpublished notes.

Jorgensen, B. (1997). *The Theory of Dispersion Models. Monographs on Statistics and Applied Probability*, 76. Chapman and Hall, London.

Kallenberg, O. (1986). *Random Measures, 4th Ed*. Akademie-Verlag and Academic Press, Berlin and London.

Kallenberg, O. (1997). *Foundations of modern probability. Probability and its Applications*. Springer-Verlag, New York.

Kaplan, E. L. and Meier, P. (1958). Nonparametric estimation from incomplete observations. *J. Amer. Statist. Assoc.* **53** 457-481.

Kerov, S. (1998). Interlacing measures. *Amer. Math. Soc. Transl.* **181** 35-83.

Kerov, S. and Tsilevich, N. V. (1998). The Markov-Krein correspondence in several dimensions. POMI preprint No. 283, Steklov Institute of Mathematics, St. Petersburg.

Khintchine, A. Ya. (1937). Sur théorie der unbeschränkt teilbaren Verteilungsgesetze. *Mat. Sb.* **44** 79-119.

Kim, Y. (1999). Nonparametric Bayesian estimators for counting processes. *Ann. Statist. Assoc.* **27** 562-588.

Kingman, J. F. C. (1993). *Poisson Processes*. Oxford University Press, Oxford.

Kingman, J. F. C. (1975). Random discrete distributions. *J. Royal Statist. Soc., Series B* **37** 1-22.

Kingman, J. F. C. (1967). Completely random measures. *Pacific J. Math.* **21** 59-78.

Küchler, U. and Sorenson, M. (1997). *Exponential Families of Stochastic Processes*. Springer-Verlag, New York.

Le Cam, L. (1986). *Asymptotic Methods in Statistical Decision Theory*. Springer-Verlag, New York.

Le Cam, L. (1961). A stochastic description of precipitation. In *1961 Proc. 4th Berkeley Sympos. Math. Statist. and Prob.*, Vol. III Univ. California Press, Berkeley Calif. 165-186.

Lee, M.L.T. and Whitmore, G. (1993). Stochastic processes directed by randomized time. *Journal of Applied Probability* **30** 302-314.

Last, G. and Brandt, A. (1995). *Marked Point Processes on the Real Line: The Dynamic Approach*. Springer, New York.

Lindsay, B. (1995). *Mixture models: theory, geometry, and applications*. NSF-CBMS Regional Conference Series in Probability and Statistics, Volume 5. Institute for Mathematical Statistics: Hayward, CA, Hayward, CA.

Lo, A. Y. (1982). Bayesian nonparametric statistical inference for Poisson point processes. *Z. Wahrsch. verw. Gebiete* **59** 55-66.

Lo, A. Y. (1984). On a class of Bayesian nonparametric estimates: I. Density estimates. *Ann. Statist.* **12** 351-357.

Lo, A. Y. (1993). A Bayesian bootstrap for censored data. *Ann. Statist.* **21** 100–123.
Lo, A. Y. and Weng, C. S. (1989). On a class of Bayesian nonparametric estimates: II. Hazard rates estimates. *Ann. Inst. Stat. Math* **41** 227-245.

Lo, A.Y., Brunner, L.J. and Chan, A.T. (1996). Weighted Chinese restaurant processes and Bayesian mixture model. Research Report Hong Kong University of Science and Technology.

MacEachern, S. N., Clyde, M. and Liu, J. S. (1999). Sequential importance sampling for nonparametric Bayes models: the next generation. *Canadian J. Statist.* **27** 251-267.

Matthes, K. Kerstan, J., and Mecke, J. (1978). *Infinitely Divisible Point Processes. English Edition.* Wiley, Chichester.

McCloskey, J. W. (1965). A Model for the Distribution of Individuals by Species in an Environment. Ph.D. Thesis. Michigan State University.

Pachl, J. K. (1978). Disintegration and compact measures. *Math. Scand.* **43** 157-168.

Perman, M., Pitman, J. and Yor, M. (1992). Size-biased sampling of Poisson point processes and excursions. *Probab. Theory Related Fields* **92** 21-39.

Pitman, J. (1995a). Exchangeable and partially exchangeable random partitions. *Probab. Theory Related Fields* **102** 145-158.

Pitman, J. (1995b). Poisson-Kingman partitions. Available at www.stat.berkeley.edu/users/pitman

Pitman, J. (1996). Some developments of the Blackwell-MacQueen urn scheme. In *Statistics, Probability and Game Theory* (T.S. Ferguson, L.S. Shapley and J.B. MacQueen, eds.) 245-267. IMS Lecture Notes-Monograph series, Vol 30.

Pitman, J. (1997a). Partition structures derived from Brownian motion and stable subordinators. *Bernoulli* **3** 79-96.

Pitman, J. (1997b). Some probabilistic aspects of set partitions. *Amer. Math. Monthly*, 201-209.

Pitman, J. (1999). Coalescents with multiple collisions. *Ann. Probab* **27** 1870-1902.

Pitman, J. and Yor, M. (1992). Arcsine laws and interval partitions derived from a stable subordinator. *Proc. London Math. Soc.* **65** 326–356.

Pitman, J. and Yor, M. (1997). The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator. *Ann. Probab.* **25** 855-900.

Pitman, J. and Yor, M. (2001). On the distribution of ranked heights of excursions of a Brownian bridge. *Ann. Probab.* **29** 361-384.

Pollard, D. (2001). *User’s Guide to Measure Theoretic Probability.* Cambridge University Press

Rosinski, J. (2001). Series representations of Lévy processes from the perspective of point processes. In *Lévy Processes: Theory and Applications.* Eds. Brandorff-Nielsen, Mikosch, and Resnick, pp. 401-415. Birkhauser, Boston.,

Rota, G-C. (1964). The number of partitions of a set. *American Mathematical Monthly* **71** 498-504.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible processes.* Cambridge University Press, Cambridge, UK.

Susarla, V. and Van Ryzin, J. (1976). Nonparametric Bayesian estimation of survival curves from incomplete observations. *J. Amer. Statist. Assoc* **71** 897-902.

Tsilevich, N. V. (1997). Distribution of the mean value for certain random measures. POMI preprint No. 240, Steklov Institute of Mathematics, St. Petersburg. English translation in Journal of Mathematical Sciences, vol. 96 (1999), No. 5, pp. 3616-3623.

Tsilevich, N. V., Vershik, A. M., and Yor, M. (2001). An infinite-dimensional analogue of the Lebesgue measure and distinguished properties of the gamma process. *J. Funct. Anal* **185** 274-296.

Tsilevich, N. V., Vershik, A. M., and Yor, M. (2000). Distinguished properties of the gamma process and related topics, Prépublication du Laboratoire de Probabilités et Modèles Aléatoires no. 575, Mars 2000.

Turnbull, B. W. (1976). The empirical distribution function with arbitrarily grouped, censored and truncated data. *J. Roy. Statist. Soc. Ser. B* **38** 290-295.
Tweedie, M.C.K. (1984). An index which distinguishes between some important exponential families. In *Statistics: Applications and New Directions* Eds. J.K. Ghosh and J. Roy, pp. 579-604. Indian Statistical Institute, Calcutta.

Walker, S. and Muliere, P. (1997). Beta-Stacy processes and a generalization of the Polya-urn scheme. *Ann. Statist.* **25** 1762-1780.

Wolpert, R. L. and Ickstadt, K. (1988a). Poisson/Gamma random field models for spatial statistics. *Biometrika* **85** 251-267.

Wolpert, R. L. and Ickstadt, K. (1998b). Simulation of Lévy random fields. In *Practical Nonparametric and Semiparametric Bayesian Statistics* (D. Dey, P. Mueller and D. Sinha, eds.) 227-241. Springer Lecture Notes.

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