Hardness result for the total rainbow $k$-connection of graphs

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Abstract

A path in a total-colored graph is called total rainbow if its edges and internal vertices have distinct colors. For an $\ell$-connected graph $G$ and an integer $k$ with $1 \leq k \leq \ell$, the total rainbow $k$-connection number of $G$, denoted by $trc_k(G)$, is the minimum number of colors used in a total coloring of $G$ to make $G$ total rainbow $k$-connected, that is, any two vertices of $G$ are connected by $k$ internally vertex-disjoint total rainbow paths. In this paper, we study the computational complexity of total rainbow $k$-connection number of graphs. We show that it is NP-complete to decide whether $trc_k(G) = 3$.

Keywords: total rainbow $k$-connection number, computational complexity.
AMS subject classification 2010: 05C15, 05C40, 68Q17, 68Q25, 68R10.

1 Introduction

All graphs considered in this paper are simple, finite, undirected and connected. We follow the terminology and notation of Bondy and Murty\cite{2} for those not defined here. A set of internally vertex-disjoint paths are called disjoint. Let $G$ be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{0, 1, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored the same. A path in $G$ is called a rainbow path if no two edges of the path are colored the same. The graph $G$ is called rainbow connected if for any two vertices of $G$, there is a rainbow path connecting them. The rainbow connection

*Supported by NSFC No.11371205 and 11531011.
number of $G$, denoted by $rc(G)$, is defined as the minimum number of colors that are needed to make $G$ rainbow connected. If $G$ is an $\ell$-connected graph with $\ell \geq 1$, then for any integer $1 \leq k \leq \ell$, $G$ is called rainbow $k$-connected if any two vertices of $G$ are connected by $k$ disjoint rainbow paths. The rainbow $k$-connection number of $G$, denoted by $rc_k(G)$, is the minimum number of colors that are required to make $G$ rainbow $k$-connected. The concepts of rainbow connection and rainbow $k$-connection of graphs were introduced by Chartrand et al. in [6, 5], and have been well-studied since then. For further details, we refer the readers to the book [15].

Let $G$ be a nontrivial connected graph with a vertex-coloring $c : V(G) \to \{0, 1, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent vertices may be colored the same. A path in $G$ is called a vertex-rainbow path if no interval vertices of the path are colored the same. The graph $G$ is rainbow vertex-connected if for any two vertices of $G$, there is a vertex-rainbow path connecting them. The rainbow vertex-connection number of $G$, denoted by $rvc(G)$, is the minimum number of colors used in a vertex-coloring of $G$ to make $G$ rainbow vertex-connected. If $G$ is an $\ell$-connected graph with $\ell \geq 1$, then for any integer $1 \leq k \leq \ell$, the graph $G$ is rainbow vertex $k$-connected if any two vertices of $G$ are connected by $k$ disjoint vertex-rainbow paths. The rainbow vertex $k$-connection number of $G$, denoted by $rvc_k(G)$, is the minimum number of colors that are required to make $G$ rainbow vertex $k$-connected. These concepts of rainbow vertex connection and rainbow vertex $k$-connection of graphs were proposed by Krivelevich and Yuster [11] and Liu et al. [16], respectively.

Liu et al. [17] introduced the analogous concepts of total rainbow $k$-connection of graphs. Let $G$ be a nontrivial $\ell$-connected graph with a total-coloring $c : E(G) \cup V(G) \to \{0, 1, \ldots, t\}$, $t \in \mathbb{N}$, where $\ell \geq 1$. A path in $G$ is called a total-rainbow path if its edges and interval vertices have distinct colors. For any integer $1 \leq k \leq \ell$, the graph $G$ is called total rainbow $k$-connected if any two vertices of $G$ are connected by $k$ disjoint total-rainbow paths. The total rainbow $k$-connection number of $G$, denoted by $trc_k(G)$, is the minimum number of colors that are needed to make $G$ total rainbow $k$-connected.
When $k = 1$, we simply write $trc(G)$, just like $rc(G)$ and $rvc(G)$. From Liu et al.\cite{17}, we have that $trc(G) = 1$ if and only if $G$ is a complete graph, and $trc(G) \geq 3$ if $G$ is not complete. If $G$ is an $\ell$-connected graph with $\ell \geq 1$, then $trc_k(G) \geq 3$ if $2 \leq k \leq \ell$, and $trc_k(G) \geq 2diam(G) - 1$ for $1 \leq k \leq \ell$, where $diam(G)$ denotes the diameter of $G$. In relation to $rc_k(G)$ and $rvc_k(G)$, they have $trc_k(G) \geq \max(rc_k(G), rvc_k(G))$. Also, if $rc_k(G) = 2$, then $trc_k(G) = 3$. If $rvc_k(G) \geq 2$, then $trc_k(G) \geq 5$.

The computational complexity of the rainbow connectivity and vertex-connectivity has been attracted much attention. In \cite{4}, Chakraborty et al. proved that deciding whether $rc(G) = 2$ is NP-Complete. Analogously, Chen et al.\cite{8} showed that it is NP-complete to decide whether $rvc(G) = 2$. Motivated by \cite{4, 8}, we consider the computational complexity of computing the total rainbow $k$-connectivity $trc_k(G)$ of a graph $G$. For $k = 1$, Chen et al. recently gave reductions to prove that it is NP-complete to decide whether $trc(G) = 3$ in \cite{7}. In this paper, we prove that for any fixed $k \geq 1$ it is NP-complete to decide whether $trc_k(G) = 3$. The reduction of our proof is different from that in \cite{7}.

\section{Main results}

In the following, we will show that deciding whether $trc_k(G) = 3$ is NP-complete for fixed $k \geq 1$.

\textbf{Theorem 2.1.} Given a graph $G$, deciding whether $trc_k(G) = 3$ is NP-Complete for fixed $k \geq 1$.

We first define the following three problems.

\textbf{Problem 1.} The total rainbow connection number $3$.

Given: Graph $G = (V, E)$.

Decide: Whether there is a total coloring of $G$ with 3 colors such that all the pairs $\{u, v\} \in (V \times V)$ are total rainbow $k$-connected?
Problem 2. The subset total rainbow $k$-connection number 3.

Given: Graph $G = (V, E)$ and a set of pairs $P \subseteq (V \times V)$, where $P$ contains nonadjacent vertex pairs.

Decide: Whether there is a total-coloring of $G$ with 3 colors such that all the pairs $\{u, v\} \in P$ are total rainbow $k$-connected?

Problem 3. The subset partial edge-coloring.

Given: Graph $G = (V, E)$ with a set of pairs $Q \subseteq V \times V$ where $Q$ contains nonadjacent vertex pairs, and a partial 2-edge-coloring $\hat{\chi}$ for $\hat{E} \subseteq E$.

Decide: Whether $\hat{\chi}$ can be extended to a 3-total-coloring $\chi$ of $G$ that makes all the pairs in $Q$ total rainbow $k$-connected and $\chi(e) \notin \{\chi(u), \chi(v)\}$ for all $e = uv \in \hat{E}$?

In the following, we first reduce Problem 2 to Problem 1, and then reduce Problem 3 to Problem 2. Finally, Theorem 2.1 is completed by reducing 3-SAT to Problem 3.

Before proving Theorem 2.1, we need an useful result shown in [6].

Lemma 2.2. [6] For every $k \geq 2$, $rc_k(K_{(k+1)^2}) = 2$. Furtherly, the following 2-edge coloring can make $G$ rainbow $k$-connected. Let $G_1, G_2, \ldots, G_{k+1}$ be mutually vertex-disjoint graphs, where $V(G_i) = V_i$, such that $G_i = K_{k+1}$ for $1 \leq i \leq k + 1$. Let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,k+1}\}$ for $1 \leq i \leq k+1$. Let $G$ be the join of the graphs $G_1, G_2, \ldots, G_{k+1}$. Thus $G = K_{(k+1)^2}$ and $V(G) = \bigcup_{i=1}^{k+1} V_i$. We assign the edge $uv$ of $G$ the color 0 if either $uv \in E(G_i)$ for some $i(1 \leq i \leq k + 1)$ or if $uv = v_{i,j}v_{j,l}$ for some $i,j,l$ with $1 \leq i,j,l \leq k + 1$ and $i \neq j$. All other edges of $G$ are assigned the color 1.

For $k = 1$, since $rc_1(K_{(k+1)^2}) = 1$, the above coloring surly makes $G$ rainbow 1-connected.

Note that from the above coloring, for every vertex $v \in V(G)$, we have $d(v) = k^2 + 2k$, $2k$ edges incident with $v$ colored with 0, and $k^2$ edges incident with $v$ colored with 1.

Lemma 2.3. $\text{Problem 2} \preceq \text{Problem 1}$. 

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Finally, the remaining uncolored edges are colored with 0. Now we show that rainbow paths in $G$ are total rainbow $k$-connected. We now extend it to a total rainbow $k$-connection coloring $\chi' : V' \cup E' \to \{0, 1, 2\}$, $\chi'(x) = 2$ for all $x \in V' \setminus V$; $\chi'(v, x_{(v,i)}) = 1$ for all $v \in V$ and $x_{(v,i)} \in V_v$; $\chi'(u, x_{(u,v,i)}) = 0, \chi'(v, x_{(u,v,i)}) = 1$ for all $\{u, v\} \in (V \times V) \setminus P$ and all $x_{(u,v,i)} \in V_{(u,v)}$. The edges in $G'[V_v]$ or $G'[V_{(u,v)}]$ are colored with $0, 1$ as Lemma 2.2 for all $v \in V$ and all $\{u, v\} \in (V \times V) \setminus P$. Finally, the remaining uncolored edges are colored with 0. Now we show that $G'$ is total rainbow $k$-connected under this coloring. For $\{u, v\} \in P$, the $k$ disjoint total rainbow paths in $G$ connecting $u$ and $v$ are also $k$-disjoint total rainbow paths in $G'$. For $\{u, v\} \in (V \times V) \setminus P$, $\{ux_{(u,v,1)}v, ux_{(u,v,2)}v, \ldots, ux_{(u,v,k)}v\}$ are $k$-disjoint total
Lemma 2.4. Problem 3 ≤ Problem 2.

Proof. Since the identity of the colors does not matter, it is more convenient that instead of a partial 2-edge coloring \( \hat{\chi} \) we consider the corresponding partition \( \pi_{\hat{\chi}} = (\hat{E}_1, \hat{E}_2) \). For the sake of convenience, let \( e = e^1 e^2 \) for \( e \in (\hat{E}_1 \cup \hat{E}_2) \). Note that the ends of \( e \) may be labeled by different signs for \( e \in (\hat{E}_1 \cup \hat{E}_2) \). Given such a partial 2-edge coloring \( \hat{\chi} \) and a set of pairs \( Q \subseteq (V \times V) \) where \( Q \) contains nonadjacent vertex pairs. Now we construct a graph \( G' = (V', E') \) and define a set of pairs \( P \subseteq (V' \times V') \) as follows. We first add the vertices

\[
\{c, b_1, b_2\} \cup \left\{\{c^j, d^j_{e^j}, f^j_{e^j}\} : j \in \{1, 2\}, e \in (\hat{E}_1 \cup \hat{E}_2)\right\}
\]

and add the edges

\[
\{b_1 c, b_2 c\} \cup \left\{c c^j_{e^j} : j \in \{1, 2\}, e \in (\hat{E}_1 \cup \hat{E}_2)\right\} \cup \left\{c^j f^j_{e^j}, c^j e^j, d^j_{e^j} : e \in (\hat{E}_1 \cup \hat{E}_2)\right\}.
\]

Now we define the set of pairs \( P \).

\[
P =Q \cup \{b_1, b_2\} \cup \left\{b_i, c_i^j : e \in \hat{E}_i, i, j \in \{1, 2\}\right\} \cup \left\{f^j_{e^j}, c^j, \{d^j_{e^j}, c^j, d^j_{e^j}, e^{(3-j)}\} : j \in \{1, 2\}, e \in (\hat{E}_1 \cup \hat{E}_2)\right\}.
\]
Given a 3CNF formula

**Proof.**

Then we secondly add the new vertices

\[ \{g(u,v,2), g(u,v,3), \ldots, g(u,v,k)\} : \{u,v\} \in P \setminus Q \]

and add the new edges

\[ \{ug(u,v,2)v, ug(u,v,3)v, \ldots, ug(u,v,k)v\} : \{u,v\} \in P \setminus Q\].

On one hand, if there is a 3-total-coloring of \( \chi \) of \( G \) that makes all the pairs in \( Q \) total rainbow \( k \)-connected which extends \( \pi_\chi = (\hat{E}_1, \hat{E}_2) \) and \( \chi(e) \notin \{\chi(e^1), \chi(e^2)\} \) for all \( e = e^1e^2 \in \hat{E}, \) then we give a total-coloring \( \chi' \) of \( G' \) as follows. Suppose w.l.o.g that \( \hat{E}_1 \) are colored with 0, and \( \hat{E}_2 \) are colored with 1. \( \chi'(v) = \chi(v), \) and \( \chi'(e) = \chi(e) \) for all \( v \in V, e \in E; \chi'(v) = 2 \) for all \( v \in V' \setminus V; \chi'(b_1c) = 1, \) and \( \chi'(b_2c) = 0; \chi'(c_1e^j) = \chi'(c_2e^j) = 0, \) and \( \chi'(d_i^je^j) = \{1,2\} \setminus \chi(e^j) \) for all \( e \in \hat{E}_1; \chi'(c_1e^j) = \chi'(c_2e^j) = 1, \) and \( \chi'(d_i^je^j) = \{0,2\} \setminus \chi(e^j) \) for all \( e \in \hat{E}_2; \chi'(ug(u,v,t)) = 0, \) and \( \chi'(g(u,v,t)v) = 1 \) for all \( 2 \leq t \leq k \) and all \( \{u,v\} \in P \setminus Q. \) One can verify that this coloring indeed makes all the pairs in \( P \) total rainbow \( k \)-connected.

On the other hand, any 3-total-coloring of \( G' \) that makes all the pairs in \( P \) total rainbow \( k \)-connected indeed makes all the pairs in \( Q \) total rainbow \( k \)-connected in \( G, \) because \( G' \) contains no path of length 2 between any pair in \( Q \) that is not contained in \( G. \)

Note that there exactly exist \( k \) disjoint total rainbow paths between any pair in \( P \setminus Q. \) For any \( e \in \hat{E}_i, i \in \{1,2\}, \) from the set of pairs \( \{\{b_1,b_2\}, \{b_i,c_i\}, \{f_j^i,e^j\}, \{d_j^i,c_i\}, \{d_j^i(3-j)\}: j \in \{1,2\}\}, \) we have \( \chi'(b_1c) \neq \chi'(b_2c), \chi'(e) = \chi'(c_1e^j) = \chi'(c_2e^j) = \chi'(b_{(3-j)}c) \) and \( \chi'(e) \notin \{\chi'(e^1), \chi'(e^2)\} \) for \( j \in \{1,2\}. \) Hence the coloring \( \chi' \) of \( G' \) not only provides a 3-total-coloring \( \chi \) of \( G \) that makes all the pairs in \( Q \) are total rainbow \( k \)-connected, but it also make sure that \( \chi \) extends the original partial coloring \( \pi_\chi = (\hat{E}_1, \hat{E}_2) \) and \( \chi(e) \notin \{\chi(e^1), \chi(e^2)\} \) for all \( e = e^1e^2 \in \hat{E}. \)

\[ \square \]

**Lemma 2.5.** 3-SAT \( \leq \) Problem 3.

**Proof.** Given a 3CNF formula \( \phi = \bigwedge_{i=1}^m c_i \) over variables \( \{x_1, x_2, \ldots, x_n\}, \) we construct a graph \( G = (V, E), \) a partial 2-edge coloring suppose w.l.o.g that \( \chi : \hat{E} \to \{0,1\}, \) and
a set of pairs $Q \subseteq (V \times V)$ where $Q$ contains nonadjacent vertex pairs such that there is an extension $\chi$ of $\hat{\chi}$ that makes all the pairs in $Q$ total rainbow $k$-connected and $\chi(e) \notin \{\chi(u), \chi(v)\}$ for all $e = uv \in \hat{E}$ if and only if $\phi$ is satisfiable. We define $G$ as follows:

$$V(G) = \{c_t : t \in [m]\} \cup \{c^j_t, t \in [m], 2 \leq j \leq k\} \cup \{x_i : i \in [n]\} \cup \{s\}$$

and

$$E(G) = \{c_t x_i : x_i \in c_t \text{ in } \phi\} \cup \{sx_i : i \in [n]\} \cup \{sc^j_t c_t : t \in [m], 2 \leq j \leq k\}.$$ 

Now we define the set of pairs $Q$ as follows:

$$Q = \\{\{s, c_t\} : t \in [m]\}.$$ 

Finally we define the partial 2-edge coloring $\hat{\chi}$ as follows:

$$\hat{\chi}(c_t x_i) = \begin{cases} 
0 & \text{if } x_i \text{ is positive in } c_t, \\
1 & \text{if } x_i \text{ is negative in } c_t.
\end{cases}$$

On one hand, if $\phi$ is satisfiable with a truth assignment over $\{x_1, x_2, \ldots, x_n\}$, we extend $\hat{\chi}$ to $\chi$ as follows: $\chi(v) = 2$ for all $v \in V$; $\chi(sc^j_t) = 0$, and $\chi(c^j_t c_t) = 1$ for all $t \in [m]$ and all $2 \leq j \leq k$; $\chi(sx_i) = x_i$ for all $i \in [n]$. One can verify $\chi$ is as desired. On the other hand, suppose that $\chi$ is as desired as above. Note that for any $c_t$, there must exist a total rainbow path $sx_i c_t$ by some vertex $x_i$. Set such $x_i = \{\chi(sx_i), \chi(x_i)\} \cap \{0, 1\}$ which can make $c_t$ true. One can verify $\phi$ is satisfiable.

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