Abstract. Let $f$ traverse a sequence of classical holomorphic newforms of fixed weight and increasing squarefree level $q \to \infty$. We prove that the pushforward of the mass of $f$ to the modular curve of level 1 equidistributes with respect to the Poincaré measure.

Our result answers affirmatively the squarefree level case of a conjecture spelled out in 2002 by Kowalski, Michel, and VanderKam [20] in the spirit of a conjecture of Rudnick and Sarnak [30] made in 1994.

Our proof follows the strategy of Holowinsky and Soundarajan [15] who showed in 2008 that newforms of level 1 and large weight have equidistributed mass. The new ingredients required to treat forms of fixed weight and large level are an adaptation of Holowinsky’s reduction of the problem to one of bounding shifted sums of Fourier coefficients, a refinement of his bounds for shifted sums, an evaluation of the $p$-adic integral needed to extend Watson’s formula to the case of three newforms where the level of one divides but need not equal the common squarefree level of the other two, and some additional technical work in the problematic case that the level has many small prime factors.

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1. Introduction

1.1. Statement of result. A basic problem in modern number theory and the analytic theory of modular forms is to understand the limiting behavior of modular forms in families. Let $f : \mathbb{H} \to \mathbb{C}$ be a classical holomorphic newform of weight $k$ and level $q$. The mass of $f$ is the finite measure $d\nu_f = |f(z)|^2 y^{k-2} \, dx \, dy$ ($z = x + iy$) on the modular curve $Y_0(q) = \Gamma_0(q) \backslash \mathbb{H}$. In a recent breakthrough, Holowinsky and Soundararajan [15] proved that newforms of large weight $k$ and fixed level $q = 1$ have equidistributed mass, answering affirmatively a natural variant of the quantum unique ergodicity conjecture of Rudnick and Sarnak [30].

**Theorem 1.1** (Mass equidistribution for $\text{SL}(2,\mathbb{Z})$ in the weight aspect). Let $f$ traverse a sequence of newforms of increasing weight $k \to \infty$ and fixed level $q = 1$. Then the mass $\nu_f$ equidistributes with respect to the Poincaré measure $d\mu = y^{-2} \, dx \, dy$ on the modular curve $Y_0(q)$.

Kowalski, Michel, and VanderKam [20, Conj 1.5] formulated an analogue of the Rudnick-Sarnak conjecture in which the roles of the parameters $k$ and $q$ are reversed: they conjectured that the masses of newforms of fixed weight and large level $q$ are equidistributed amongst the fibers of the canonical projection $\pi_q : Y_0(q) \to Y_0(1)$ in the following sense.

**Conjecture 1.2** (Mass equidistribution for $\text{SL}(2,\mathbb{Z})$ in the level aspect). Let $f$ traverse a sequence of newforms of fixed weight and increasing level $q \to \infty$. Then the pushforward $\mu_f := \pi_{q*}(\nu_f)$ of the mass of $f$ to $Y_0(1)$ equidistributes with respect to $\mu$.

Kowalski, Michel, and VanderKam remark that Conjecture 1.2 follows in the special case of dihedral forms from their subconvex bounds for Rankin-Selberg $L$-functions modulo an unestablished extension of Watson’s formula [11], which is now known by Theorem 4.1 of this paper. Recently Koyama [21], following the method of Luo and Sarnak [23], proved the analogue of Conjecture 1.2 for unitary Eisenstein series of increasing prime level by reducing the problem to known subconvex bounds for automorphic $L$-functions of degree two.

Our aim in this paper is to establish the squarefree level case of Conjecture 1.2. Our result is the first of its kind for nondihedral cusp forms.

**Theorem 1.3** (Mass equidistribution for $\text{SL}(2,\mathbb{Z})$ in the squarefree level aspect). Let $f$ traverse a sequence of newforms of fixed weight and increasing squarefree level $q \to \infty$. Then $\mu_f$ equidistributes with respect to $\mu$.

**Remark 1.4.** Our extension (Theorem 4.1 of Watson’s formula [11]) shows that Theorem 1.3 would follow from subconvex bounds $L(f \times f \times \phi, 1/2) \ll_q q^{1-\delta}$ ($\delta > 0$) for the central $L$-values of the triple product $L$-functions attached to $f$ as above and each Maass cusp form or unitary Eisenstein series $\phi$ on $Y_0(1)$. Such bounds are known to follow from the generalized Lindelöf hypothesis, which itself follows from the generalized Riemann hypothesis, so one can view Theorem 1.3 as an unconditionally proven consequence of a central unresolved conjecture.

**Remark 1.5.** One cannot relax entirely the restriction of Theorem 1.3 to newforms, for instance a cusp form of level 1 may be regarded as an oldform of arbitrary level $q > 1$.

**Remark 1.6.** Rudnick [29] showed that Theorem 1.1 implies that the zeros of newforms of level 1 and weight $k \to \infty$ equidistribute on $Y_0(1)$. At the 2010 Arizona Winter School, Soundararajan asked whether there is an analogue of Rudnick’s result for newforms of large level. We do not know whether such an analogue

---

1. As spelled out by Luo and Sarnak [24], we refer to Sarnak [31, 32] and the references in [15] for further discussion.
2. We say that a sequence of finite Radon measures $\mu_0$ on a locally compact Hausdorff space $X$ equidistributes with respect to some fixed finite Radon measure $\mu$ if for each function $\phi \in C_c(X)$ we have $\mu_j(\phi)/\mu_j(1) \to \mu(\phi)/\mu(1)$ as $j \to \infty$, here and always identifying a measure $\mu$ with the corresponding linear functional $\phi \mapsto \mu(\phi) := \int_X \phi \, d\mu$ on the space $C_c(X)$ and writing 1 for the constant function.
exists and highlight here one of the difficulties in adapting Rudnick’s method. Let \( f \) be a newform of weight \( k \) and level \( q \), let \( \mathcal{Z} \) be the left \( \Gamma_0(q) \)-multiset of zeros of \( f \) in \( \mathbb{H} \) and let \( \mathcal{Z}_1 \) be the left \( \Gamma \)-multiset \((\Gamma = \text{PSL}(2, \mathbb{Z})) \) obtained by summing the images of \( \mathcal{Z} \) under coset representatives for \( \Gamma(1)/\Gamma_0(q) \). We ask: does \( \Gamma \backslash \mathcal{Z}_1 \) equidistribute on \( Y_0(1) \) as \( q \to \infty \)? Following Rudnick, one may show for \( \phi \in C_c^\infty(\mathbb{H}) \) and \( \Phi(z) = \sum_{\gamma \in \Gamma} \phi(\gamma z) \) that

\[
\frac{12}{k \psi(q)} \sum_{z \in \Gamma \backslash \mathcal{Z}_1} \frac{\Phi(z)}{\# \text{Stab}_\Gamma(z)} = \int_{\Gamma \backslash \mathbb{H}} \Phi \, dV + \int_{\Gamma \backslash \mathbb{H}} \frac{\pi_{q^k}(\log \nu_f)}{k \psi(q)} \Delta \Phi \, dV,
\]

where \( \psi(q) = [\Gamma(1) : \Gamma_0(q)] \), \( \Delta = y^2(\partial^2_x + \partial^2_y) \) is the hyperbolic Laplacian, and \( dV \) is the hyperbolic probability measure on \( \Gamma \backslash \mathbb{H} \); the formula (1) follows by some elementary manipulations of the identity \( \int_{\mathbb{H}} \log |z - z_0| \Delta \phi(z) y^{-2} \, dx \, dy = 2\pi \phi(z_0) \), which holds for any \( z_0 \in \mathbb{H} \) and follows from Green’s identities. Since the total number of inequivalent zeros is \#(\Gamma \backslash \mathcal{Z}_1 = \# \Gamma_0(q) \backslash \mathcal{Z} \sim k \psi(q)/12 \) \cite{36} §2, the first term on the right-hand side of (1) may be regarded as a main term, the second as an error term that one would like to show tends to 0. An important step toward adapting Rudnick’s method would be to rule out the possibility that \( \pi_{q^k}(\log \nu_f) / k \psi(q) \) tends to \(-\infty\) uniformly on compact subsets as \( q \to \infty \). The difficulty in doing so is that Theorem 1.3 does not seem to preclude the masses \( \nu_f \) from being very small somewhere within each fiber of the projection \( Y_0(q) \to Y_0(1) \); stated another way, the sum of the values taken by \( y^k |f|^2 \) in a fiber of \( Y_0(q) \to Y_0(1) \) are controlled (in an average sense as the fiber varies) by Theorem 1.3 but their product could still conceivably be quite small. There are further difficulties in adapting Rudnick’s method that we shall not mention here.

Remark 1.7. Lindenstrauss \cite{22} and Soundararajan \cite{38} proved that Maass eigenforms of fixed level \( q \) and large Laplace eigenvalue \( \lambda \to \infty \) have equidistributed mass. We ask: do Maass newforms of large level \( q \to \infty \) (with \( \lambda \) taken to lie in a fixed subinterval of \([1/4, +\infty]\), say) satisfy the natural analogue of Conjecture \ref{conj:equidist}? An affirmative answer to this question would answer to the generalized Riemann hypothesis (at least for \( q \) squarefree, as in remark \ref{rem:conj}), but appears beyond the reach of our methods because the Ramanujan conjecture is not known for Maass forms (compare with \cite{15} p.2).

Remark 1.8. We shall actually establish the following stronger hybrid equidistribution result: for a newform \( f \) (possibly varying) weight \( k \) and squarefree level \( q \), the measures \( \mu_f = \pi_{q^k}(\nu_f) \) equidistribute as \( qk \to \infty \). The novelty in our argument concerns only the variation of \( q \), so we encourage the reader to regard \( k \) as fixed.

Remark 1.9. With minor modifications our arguments should extend to the general case of not necessarily squarefree levels \( q \) as soon as an appropriate extension of Watson’s formula is worked out. However, we shall invoke the assumption that the level \( q \) is squarefree whenever doing so simplifies the exposition. The parts of our argument that require modification to treat the general case are Lemmas 3.4 and 3.15 and 4.4. One should be able to generalize Lemmas 3.4 and 3.15 using that for any level \( q \) the cusps of \( \Gamma_0(q) \) fall into classes indexed by the divisors \( d \) of \( q \) consisting of \( \phi(\gcd(d, q/d)) \) cusps of width \( d/\gcd(d, q/d) \). To generalize 4.4 one must compute (or sharply bound) a \( p \)-adic integral involving matrix coefficients of supercuspidal representations of \( \text{GL}(2, \mathbb{Q}_p) \). We plan to consider this generalization in future work.

1.2. A very brief review of the motivating work of Holowinsky-Soundararajan. Our proof of Theorem 1.3 is an adaptation of the Holowinsky-Soundararajan proof \cite{11} of Theorem 1.1 which in turn synthesizes the independent arguments of Holowinsky \cite{14} and Soundararajan \cite{39}. Here we briefly recall their independent arguments and refer to the survey \cite{32} and the original papers for further background.

Holowinsky \cite{14} employs a clever unfolding trick and an asymptotic analysis of certain archimedean integrals in the limit \( k \to \infty \) to reduce the study of the periods \( \mu_f(\phi) \) to the problem of bounding sums.
roughly of the form

\[ \sum_{n \leq k} \lambda_f(n)\lambda_f(n + l), \]

where \( l \) is an essentially bounded nonzero integer and \( \lambda_f(n) \) the \( n \)th Fourier coefficient of the newform \( f \) of weight \( k \to \infty \) and level 1, normalized so that the Deligne bound reads \( |\lambda_f(p)| \leq 2 \). He reduces bounds for the sums (2) to those for the mean values \( \sum_{n \leq k} |\lambda_f(n)| \) by a sieving technique that quantifies, using the Deligne bound for \( \lambda_f \), the “independence” of the sums \( n \mapsto \lambda_f(n) \) and \( n \mapsto \lambda_f(n + l) \) for \( l \neq 0 \).

Soundararajan’s method \[39\] takes as input a precise identity (given in this case by Watson \[41\]) relating Soundarajan’s argument succeeds unless this is so.

1.3. What’s new in this paper. The synthetic part of the Holowinsky-Soundararajan argument works just as well in the level aspect as in the weight aspect (see §5), so we highlight here four of the more substantial difficulties encountered in adapting the independent arguments of Holowinsky and Soundararajan to the level aspect.

1. It is not a priori clear how best to extend Holowinsky’s unfolding trick in the presence of multiple (possibly unboundedly many) cusps, nor what should take the place of his asymptotic analysis of archimedean integrals in studying the fixed weight, large level limit; several fundamentally different approaches are possible, one of which we shall present in §3.1.

When \( q \) is squarefree, the problem then becomes to bound sums roughly of the form\[ 3\]

\[ \sum_{d|q} \sum_{n \leq dk} \lambda_f(n)\lambda_f(n + dl), \]

where again \( l \neq 0 \) is essentially bounded. As we now explain, the sums (3) differ from those (2) studied by Holowinsky in two important ways.

2. The shifts \( dl \) are now nearly as large as the length of the interval \( \approx dk \) over which we are summing\[ 4\]

Much of the existing work on bounds for such sums (see remark 3.11) applies only when the shift is substantially smaller than the summation interval. Holowinsky’s treatment of (2) does allow shifts as large as the summation interval, but gives a bound for \( \sum_{n \leq qk} \lambda_f(n)\lambda_f(n + ql) \) that involves an extraneous factor of \( \tau(ql) \), which is prohibitively large (e.g., \( \gg \log(q)^A \) for any \( A \)) if \( q \) has many small prime factors. In Theorem 3.10 we refine Holowinsky’s method to allows shifts as large as the summation interval with \textit{full uniformity} in the size of the shift, e.g., without the factor \( \tau(ql) \). This refinement may be of independent interest.

3. Let \( \omega(q) \) denote the number of prime divisors of the squarefree integer \( q \). Then the number of shifted sums in (3) is \( 2^{\omega(q)} \), which can be quite large\[ 5\]. In the crucial case\[ 6\] that \( \lambda_f(p) \) is typically small for primes \( p \ll qk \), our refinement of Holowinsky’s method saves nearly two logarithmic powers of \( dk \) over the trivial bound \( \ll dk \) for the shifted sum in (3) of length \( \approx dk \). Thus we save very little over the trivial bound if \( d \) is a small divisor of \( q \), and it is not immediately clear whether such savings are enough to produce a sufficient saving in the sum over all \( d \). One needs here an inequality of the

3Here one should think of a divisor \( d \) of \( q \) as indexing the unique cusp of \( \Gamma_0(q) \) of width \( d \), where as usual the \textit{width} of a cusp is its ramification index over the cusp \( \infty \) for \( \Gamma_0(1) \).

4This difficulty corresponds the fact that cusps for \( \Gamma_0(q) \) may have \textit{large} width.

5This difficulty corresponds to the fact that \( \Gamma_0(q) \) may have \textit{many} cusps.

6Soundararajan’s argument succeeds unless this is so.
shape

\[
\sum_{d|q} \frac{dk}{\log(dk)^{2-\varepsilon}} \ll \frac{qk}{\log(qk)^{2-\varepsilon}} \log(\varepsilon q),
\]

which one can interpret as saying that the divisors of any squarefree integer are well distributed in a certain sense. Indeed, if hypothetically \( q \) were to have “too many” large divisors, then the LHS of (4) might be large enough to swamp the small logarithmic savings, while if \( q \) were to have “too many” small divisors, then the savings for each term on the LHS might be too small to produce an overall savings. A convexity argument and a (weak form of the) prime number theorem are sufficient to establish (4); see Lemma 3.1.6

(4) The identity relating \( \mu_f(\phi) \) to \( L(\phi \times f \times f, \frac{1}{2}) \) that Soundararajan’s method takes as input is given by Watson [41] when \( f \) and \( \phi \) are newforms of the same (squarefree) level. In the level aspect, the relevant Weyl periods are those for which \( f \) has large level and \( \phi \) has fixed level, so Watson’s formula does not apply. We extend Watson’s result in Theorem 4.1 by computing (Lemma 4.4) a \( p \)-adic integral arising in Ichino’s general formula [16], specifically

\[
\int_{g \in \operatorname{PGL}_2(\mathbb{Q}_p)} \frac{\langle g : \phi_p, \phi_p \rangle}{\langle \phi_p, \phi_p \rangle} \frac{\langle g : f_p, f_p \rangle}{\langle f_p, f_p \rangle} \frac{\langle g : f_p, f_p \rangle}{\langle f_p, f_p \rangle} \, dg,
\]

where \( \phi_p \) (resp. \( f_p \)) is the newvector at \( p \) for the adelization of \( \phi \) (resp. \( f \)) and \( \langle \cdot, \cdot \rangle \) denotes a \( \operatorname{PGL}_2(\mathbb{Q}_p) \)-invariant Hermitian pairing on the appropriate representation space. The crucial case for us is when \( p \) divides the squarefree level \( q \) of the newform \( f \), so that \( \phi_p \) lives in a spherical representation of \( \operatorname{PGL}_2(\mathbb{Q}_p) \) and \( f_p \) in a special representation. As we discuss in Remark 4.2, our evaluation of (5) leads to a precise formula relating \( \int \psi_1 \psi_2 \psi_3 \) to \( L(\frac{1}{2}, \psi_1 \times \psi_2 \times \psi_3) \) for any three newforms of squarefree level (and trivial central character); such an identity should be of general use in future work that exploits the connection between periods and \( L \)-values.

1.4. Plan for the paper. Our paper is organized as follows. In [2] we recall some standard properties of our basic objects of study: holomorphic newforms, Maass eigenforms, unitary Eisenstein series and incomplete Eisenstein series. In [3] we prove the level aspect analogue of Holowinsky’s main result [14, Corollary 3], as described above; we emphasize the aspects of his argument that do not immediately generalize to the level aspect and refer to his paper for the details of arguments that do. In [4] we extend Watson’s formula to cover the additional case that we need. In [5] we complete the proof of Theorem 1.3 using the main results of [3] and [4]. Sections 3 and 4 are independent of each other, but both depend upon the definitions, notation and facts recalled in [2].

1.5. Notation and conventions. Recall the standard notation for the upper half-plane \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \), the modular group \( \Gamma = \operatorname{SL}(2, \mathbb{Z}) \cap \mathbb{H} \) acting by fractional linear transformations, its congruence subgroup \( \Gamma_0(q) \) consisting of those elements with lower-left entry divisible by \( q \), the modular curve \( Y_0(q) = \Gamma_0(q) \backslash \mathbb{H} \), the natural projection \( \pi_q : Y_0(q) \to Y_0(1) \), the Poincaré measure \( d\mu = y^{-2} \, dx \, dy \), and the stabilizer \( \Gamma_\infty = \{ \pm \left( \begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) : n \in \mathbb{Z} \} \) in \( \Gamma \) of \( \infty \in \mathbb{P}^1(\mathbb{R}) \). We denote a typical element of \( \mathbb{H} \) as \( z = x + iy \) with \( x, y \in \mathbb{R} \).

There is a natural inclusion \( C_c(Y_0(1)) \hookrightarrow C_c(Y_0(q)) \) obtained by pulling back under the projection \( \pi_q \); here \( C_c \) denotes the space of compactly supported continuous functions. For a newform \( f \) of weight \( k \) on \( \Gamma_0(q) \) the pushforward measure \( d\mu_f := \pi_q_* ([|f|^2 y^k \, d\mu]) \) on the modular curve \( Y_0(1) \) corresponds, by definition, to the linear functional

\[
\mu_f(\phi) = \int_{\Gamma_0(q) \backslash \mathbb{H}} \phi(z) |f|^2(z) y^k \frac{dx \, dy}{y^2} \quad \text{for } \phi \in C_c(Y_0(1)) \hookrightarrow C_c(Y_0(q)).
\]
We let $\mu$ denote the standard measure on $Y_0(1)$, so that

$$\mu(\phi) = \int_{Y_0(1)} \phi(z) \frac{dx \, dy}{y^2} \quad \text{for } \phi \in C_c(Y_0(1)).$$

Since $\mu$ and $\mu_f$ are finite, they extend to the space of bounded continuous functions on $Y_0(1)$, where we shall denote also by $\mu$ and $\mu_f$ their extensions. In particular, $\mu(1)$ denotes the volume of $Y_0(1)$ and $\mu_f(1)$ the Petersson norm of $f$.

As is customary, we let $\varepsilon > 0$ denote a sufficiently small positive number whose precise value may change from line to line. We use the asymptotic notation $f(x, y, z) \ll_{x,y} g(x, y, z)$ to indicate that there exists a positive real $C(x, y)$, possibly depending upon $x$ and $y$ but not upon $z$, such that $|f(x, y, z)| \leq C(x, y)|g(x, y, z)|$ for all $x$, $y$, and $z$ under consideration. We write $f(x, y, z) = O_{x,y}(g(x, y, z))$ synonymously for $f(x, y, z) \ll_{x,y} g(x, y, z)$ and write $f(x, y, z) \sim_{x,y} g(x, y, z)$ synonymously for $f(x, y, z) \ll_{x,y} g(x, y, z) \ll_{x,y} f(x, y, z)$.

1.6. Weyl’s criterion. The following standard lemma provides essential motivation for what follows.

**Lemma 1.10.** Suppose that for each fixed Maass eigencuspform or incomplete Eisenstein series $\phi$, we have

$$\frac{\mu_f(\phi)}{\mu_f(1)} \to \frac{\mu(\phi)}{\mu(1)}$$

as $qk \to \infty$ for $q$ squarefree and $f$ a holomorphic newform of weight $k$ and level $q$; the convergence need not be uniform in $\phi$. Then Theorem 1.3 follows if we can show that $\mu_f(\phi)/\mu_f(1) \to \mu(\phi)/\mu(1)$ as $q \to \infty$ for a set of bounded functions $\phi$ the uniform closure of whose span contains $C_c(Y_0(1))$; such a set is furnished by the Maass eigencuspforms and incomplete Eisenstein series as defined in §2.

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2. Background on automorphic forms

We collect here some standard properties of classical automorphic forms. We refer to Serre [35], Shimura [36], Iwaniec [18, 19] and Atkin-Lehner [1] for complete definitions and proofs.

2.1. Holomorphic newforms. Let $k$ be a positive even integer, and let $\alpha$ be an element of $GL(2, \mathbb{R})$ with positive determinant; the element $\alpha$ acts on $\mathbb{H}$ by fractional linear transformations in the usual way. Given a function $f : \mathbb{H} \to \mathbb{C}$, we denote by $f|_k \alpha$ the function $z \mapsto \det(\alpha)^{k/2}j(\alpha, z)^{-k}f(\alpha z)$, where $j\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z = cz + d$.

A holomorphic cusp form on $\Gamma_0(q)$ of weight $k$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ that satisfies $f|_k \gamma = f$ for all $\gamma \in \Gamma_0(q)$ and vanishes at the cusps of $\Gamma_0(q)$. A holomorphic newform is a cusp form that is an eigenform of the algebra of Hecke operators and orthogonal with respect to the Petersson inner product to the oldforms.

7The terms we leave undefined are standard and their precise definitions, which may be found in the references mentioned above, are not necessary for our purposes.
\(\lambda_f(1) = 1\) in the Fourier expansion

\[
y^{k/2}f(z) = \sum_{n \in \mathbb{N}} \lambda_f(n) \sqrt{n} \kappa_f(ny)e(nx),
\]

where \(\kappa_f(y) = y^{k/2}e^{-2\pi y}\) and \(e(x) = e^{2\pi ix}\); in that case the Fourier coefficients \(\lambda_f(n)\) are real, multiplicative, and satisfy the Deligne bound \(|\lambda_f(n)| \leq \tau(n)\), where \(\tau(n)\) denotes the number of positive divisors of \(n\). If \(\gamma \in \Gamma_0(q)\) and \(z' = \gamma z = x' + iy'\), then \(y^{k/2}f(z') = (j(\gamma, z)/|j(\gamma, z)|)^k y^{k/2}f(z)\), so that in particular \(z \mapsto y^k|f(z)|^2\) is \(\Gamma_0(q)\)-invariant and our definition of \(\mu_f\) given in Section 1.5 makes sense.

To a newform \(f\) one attaches the finite part of the adjoint \(L\)-function \(L(ad f, s) = \prod_p L_p(ad f, s)\) and its completion \(\Lambda(ad f, s) = L_\infty(ad f, s)L(ad f, s) = \prod_v L_v(ad f, s)\), where \(p\) traverses the set of primes and \(v\) the set of places of \(\mathbb{Q}\); the local factors \(L_v(ad f, s)\) are as in [41, §3.1.1]. The Rankin-Selberg method [28, 33] and a standard calculation [41, §3.2.1] show that

\[
\mu_f(1) := \int_{\Gamma_0(q) \backslash \mathbb{H}} |f|^2(z)y^k \frac{dx dy}{y^2} = \frac{\Gamma(k-1)}{4\pi^k} \frac{k-1}{2\pi^2} L(ad f, 1).
\]

As in the analogous weight aspect [13, p.7], the work of Gelbart-Jacquet [8] (following Shimura [37]) and the theorem of Hoffstein-Lockhart [13, Theorem 0.1] (with appendix by Goldfeld-Hoffstein-Lieman) imply that

\[
L(ad f, 1)^{-1} \ll \log(qk).
\]

Let \(\sigma\) traverse a set of representatives for the double coset space \(\Gamma_\infty \backslash \Gamma/\Gamma_0(q)\). Then the points \(a_\sigma := \sigma^{-1} \in \mathbb{P}^1(\mathbb{Q})\) traverse a set of inequivalent cusps of \(\Gamma_0(q)\). The integer \(d_\sigma := [\Gamma_\infty : \Gamma_\infty \cap \sigma\Gamma_0(q)\sigma^{-1}]\) is the width of the cusp \(a_\sigma\), while

\[
\omega_\sigma := \sigma^{-1} \left( \frac{d_\sigma}{1} \right)
\]

is the scaling matrix for \(a_\sigma\), which means that \(z \mapsto z_\sigma := \omega_\sigma z\) is a proper isometry of \(\mathbb{H}\) under which \(z_\sigma \mapsto z_\sigma + 1\) corresponds to the action on \(z\) by a generator for the \(\Gamma_0(q)\)-stabilizer of \(a_\sigma\).

If the bottom row of \(\sigma^{-1}\) is \((c, d)\), then \(d_\sigma = q/(c, d^2)\); moreover, as \(\sigma\) varies, the multiset of widths \(\{d_\sigma\}\) is the set \(\{d : d|q\}\) of positive divisors of \(q\) [19, §2.4]. In particular, \(c\) and \(d_\sigma\) are coprime, so we may and shall assume (after multiplying \(\sigma\) on the left by an element of \(\Gamma_\infty\) if necessary) that \(d_\sigma\) divides \(d\). Since \(q\) is squarefree, the numbers \(d_\sigma\) and \(q/d_\sigma\) are coprime, so that \(\omega_\sigma\) is an Atkin-Lehner operator \(W_Q\)” in the sense of [1] p.138. Thus by applying [1] Thm 3) to the newform \(f\), we obtain

\[
f_{|k}\omega_\sigma = \pm f.
\]

Since \(f\) is \(\Gamma_0(q)\)-invariant, the property (ii) does not depend upon the choice of cusp representative \(\sigma\).

2.2. Maass eigencuspforms. A Maass cusp form \((\text{of level } 1)\) is a \(\Gamma\)-invariant eigenfunction of the hyperbolic Laplacian \(\Delta := y^{-2}(\partial_x^2 + \partial_y^2)\) on \(\mathbb{H}\) that decays rapidly at the cusp of \(\Gamma\). By Selberg’s “\(\lambda_1 \geq 1/4\)” theorem [44] there exists a real number \(r \in \mathbb{R}\) such that \((\Delta + 1/4 + r^2)\phi = 0\); our arguments use only that \(r \in \mathbb{R} \cup (i(-1/2, 1/2)\), and so apply verbatim in contexts where “\(\lambda_1 \geq 1/4\)” is not known.

A Maass eigencuspform is a Maass cusp form that is an eigenfunction of the (non-archimedean) Hecke operators and the involution \(T_{-1} : \phi \mapsto [z \mapsto \phi(-\bar{z})]\), which commute one another as well as with \(\Delta\). A Maass eigencuspform \(\phi\) has a Fourier expansion

\[
\phi(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{\lambda_\phi(n)}{\sqrt{|n|}} \kappa_\phi(ny)e(nx)
\]

where \(\kappa_\phi(y) = 2|y|^{1/2}K_{\phi}(2\pi |y|))sgn(y)^{1/4}\) with \(K_{\phi}\) the standard \(K\)-Bessel function, \(\text{sgn}(y) = 1\) or \(-1\) according as \(y\) is positive or negative, and \(\phi \in \{1\text{ or } \phi'\}\) the \(T_{\phi}\)-eigenvalue of \(\phi\). We have \(\kappa_\phi(y) \leq 1\) for all \(s \in i\mathbb{R} \cup (-1/2, 1/2)\) and all \(y \in \mathbb{R}_{\geq 0}\). A normalized Maass eigencuspform further satisfies \(\lambda_\phi(1) = 1\); in that
Because $f(-z) = \bar{f}(z)$ for any normalized holomorphic newform $f$, we have $\mu_f(\phi) = 0$ whenever $T_{-1}\phi = \delta\phi$ with $\delta = -1$. Thus we shall assume throughout this paper that $\delta = 1$, i.e., that $\phi$ is an even Maass form.

2.3. Eisenstein series. Let $s \in \mathbb{C}$ and $z \in \mathbb{H}$. The real-analytic Eisenstein series $E(s, z) = \sum_{\gamma \in \Gamma \backslash \Gamma} \Im(\gamma z)^s$ converges normally for $\Re(s) > 1$ and continues meromorphically to the half-plane $\Re(s) \geq 1/2$ where the map $s \mapsto E(s, z)$ is holomorphic with the exception of a unique simple pole at $s = 1$ of constant residue $\Res_{s=1} E(s, z) = \mu(1)^{-1}$. The Eisenstein series satisfies the invariance $E(s, \gamma z) = E(s, z)$ for all $\gamma \in \Gamma$ and admits the Fourier expansion

$$E(s, z) = y^s + M(s) y^{1-s} + \frac{1}{\xi(2s)} \sum_{n \in \mathbb{Z} \setminus 0} \frac{\lambda_{s-1/2}(n)}{\sqrt{|n|}} \kappa_{s-1/2}(ny) e(nx),$$

where $\lambda_s(n) = \sum_{a,b} (a/b)^s$, $\kappa_s(y) = 2|y|^{1/2} K_s(2\pi|y|)$, $M(s) = \xi(2s-1)/\xi(2s)$, $\xi(s) = \Gamma(s) \zeta(s)$, $\Gamma_R(s) = \pi^{-s/2} \Gamma(s/2)$, and $\zeta(s) = \sum_{n \in \mathbb{N}} n^{-s}$ (for $\Re(s) > 1$) is the Riemann zeta function. The identity $|M(s)| = 1$ for $\Re(s) = 1/2$ follows from (for instance) the functional equation for the zeta function and the prime number theorem. When $\Re(s) = 1/2$ we call $E(s, z)$ a unitary Eisenstein series.

2.4. Incomplete Eisenstein series. Let $\Psi \in C_c^\infty(\mathbb{R}_+^*)$ be a nonnegative-valued test function with Mellin transform $\Psi^\wedge(s) = \int_{\mathbb{R}} \Psi(y) y^{-s-1} \, dy$. Repeated partial integration shows that $|\Psi^\wedge(s)| \ll_{\Psi, A} (1 + |s|)^A$ for each positive integer $A$, uniformly for $s$ in vertical strips. The Mellin inversion formula asserts that $\Psi(y) = \int_{(2)} \Psi^\wedge(s) y^s \frac{ds}{2\pi i}$, where $\int_{(\sigma)}$ denotes the integral taken over the vertical contour from $\sigma-i\infty$ to $\sigma+i\infty$. To such $\Psi$ we attach the incomplete Eisenstein series

$$E(\Psi, z) = \sum_{\gamma \in \Gamma \backslash \mathbb{H}} \Psi(\Im(\gamma z)).$$

The sum has a uniformly bounded finite number of nonzero terms for $z$ in a fixed compact subset of $\mathbb{H}$. By Mellin inversion, the rapid decay of $\Psi^\wedge$ and Cauchy’s theorem, we have

$$E(\Psi, z) = \int_{(2)} \Psi^\wedge(s) E(s, z) \frac{ds}{2\pi i} = \frac{\Psi^\wedge(1)}{\vol(\Gamma \backslash \mathbb{H})} + \int_{(1/2)} \Psi^\wedge(s) E(s, z) \frac{ds}{2\pi i}.$$

Let $\phi = E(\Psi, \cdot)$ be an incomplete Eisenstein series. Note that $\mu(\phi) = \Psi^\wedge(1)$. By comparing (14) and (12), we may express the Fourier coefficients $\phi_n(y)$ in the Fourier series $\phi(z) = \sum_{n \in \mathbb{Z}} \phi_n(y) e(nx)$ as

$$\phi_n(y) = \int_{(1/2)} \Psi^\wedge(s) \frac{\lambda_{s-1/2}(n)}{\xi(2s)} \sqrt{|n|} \kappa_{s-1/2}(ny) \frac{ds}{2\pi i},$$

$$\phi_0(y) = \frac{\mu(\phi)}{\mu(1)} + \int_{(1/2)} \Psi^\wedge(s) (y^s + M(s) y^{1-s}) \frac{ds}{2\pi i}.$$
Theorem 3.1. Let $f$ be a normalized holomorphic newform of weight $k$ and squarefree level $q$. If $\phi$ is a Maass eigencuspform, then

$$\frac{\mu_f(\phi)}{\mu_f(1)} \ll_{\phi, \varepsilon} \log(qk)^c M_f(qk)^{1/2}.$$  

If $\phi$ is an incomplete Eisenstein series, then

$$\frac{\mu_f(\phi)}{\mu_f(1)} - \frac{\mu(\phi)}{\mu(1)} \ll_{\phi, \varepsilon} \log(qk)^c M_f(qk)^{1/2} \left(1 + R_f(qk)\right).$$

In this section $k$ is a positive even integer, $f$ is a normalized holomorphic newform of weight $k$ and squarefree level $q$, and $\phi$ is a Maass eigencuspform or incomplete Eisenstein series. In [3.1] we reduce Theorem 3.1 to a problem of estimating shifted sums (see Definition 3.2). In [3.2] we apply a refinement of [14, Theorem 2] to bound such shifted sums. In [3.3] we complete the proof of Theorem 3.1.

3.1. Reduction to shifted sums. Fix once and for all an everywhere nonnegative test function $h \in C_c^\infty(\mathbb{R}^*_+)$ with Mellin transform $h^\wedge(s) = \int_0^\infty h(y)y^{s-1}\,dy$ such that $h^\wedge(1) = \mu(1)$. In what follows, all implied constants may depend upon $h$ without mention.

Definition 3.2. To the parameters $s \in \mathbb{C}$, $l \in \mathbb{Z}_{\neq 0}$ and $x \geq 1$ we associate the shifted sums

$$S_s(l, x) = \sum_{m \in \mathbb{N}} \frac{\lambda_f(m)}{\sqrt{m}} \frac{\lambda_f(n)}{\sqrt{n}} I_s(l, n, x),$$

where $I_s(l, n, x)$ is an integral depending upon our fixed test function $h$:

$$I_s(l, n, x) = \int_0^\infty h(xy)\kappa_s(ly)\kappa_f(my)\kappa_f(ny)y^{\varepsilon-1}\,\frac{dy}{y}, \quad m := n + l.$$

Our aim in this section is to reduce Theorem 3.1 to the problem of bounding such shifted sums. We shall subsequently refer to the statement below of Proposition 3.3 but not the details of its proof.

Proposition 3.3. Let $Y \geq 1$. If $\phi$ is a Maass eigencuspform of eigenvalue $1/4 + r^2$, then

$$\frac{\mu_f(\phi)}{\mu_f(1)} = \frac{1}{Y \mu_f(1)} \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{\lambda_\phi(l)}{\sqrt{|l|}} \sum_{d | q} S_{\sigma}(dl, dY) + O_{\phi, \varepsilon}(Y^{-1/2}).$$

If $\phi = E(\Psi, \cdot)$ is an incomplete Eisenstein series, then

$$\frac{\mu_f(\phi)}{\mu_f(1)} - \frac{\mu(\phi)}{\mu(1)} = \frac{1}{Y \mu_f(1)} \int_{\mathbb{R}} \frac{\Psi^\wedge\left(\frac{1}{2} + it\right)}{\xi(1 + 2it)} \left(\sum_{l \in \mathbb{Z}_{\geq 0}} \frac{\lambda_\phi(l)}{\sqrt{|l|}} \sum_{d | q} S_{\sigma}(dl, dY)\right) \frac{dt}{2\pi} + \frac{1 + R_f(qk)}{Y^{1/2}} + O_{\phi, \varepsilon}(Y^{-1/2}).$$

Our proof follows a sequence of lemmas. Let $k, f, q, Y, \phi, h$ be as above and let $h_Y$ be the function $y \mapsto h(Yy)$. To $h_Y$ we attach the incomplete Eisenstein series $E(h_Y, z)$ by the usual recipe (15).

Lemma 3.4. We have the following approximate formula for the quantity $\mu_f(\phi)$:

$$Y \mu_f(\phi) = \sum_{d | q} \int_{y=0}^\infty h_Y(dy) \int_{x=0}^1 \phi(dz) |f(z)|^2 y^k \frac{dx\,dy}{y^2} + O_{\phi}(Y^{1/2} \mu_f(1)).$$
Proof. By Mellin inversion and Cauchy’s theorem as in \([14]\), we have

\[
Y \mu_f(\phi) = \mu_f(E(h_Y,\cdot)\phi) - \int_{(1/2)} h^{*}(s)Y^{s}\mu_f(E(s,\cdot)\phi) \frac{ds}{2\pi i}.
\]

The argument of \([14]\) Proof of Lemma 3.1a shows without modification that

\[
\int_{(1/2)} h^{*}(s)Y^{s}\mu_f(E(s,\cdot)\phi) \frac{ds}{2\pi i} \ll_{\phi} Y^{1/2}\mu_f(1);
\]

since the proof is short, we sketch it here. By the Fourier expansion for \(E(s, z)\) and the rapid decay of \(\phi(z)\) as \(y \to \infty\), we have \(E(s, z)\phi(z) \ll_{\phi} |s|^{O(1)}\) for \(\text{Re}(s) = 1/2\) and \(z\) in the Siegel domain \(\{z : x \in [0, 1], y > 1/2\}\) for \(\Gamma \setminus \mathbb{H}\). By the rapid decay of \(h^{*}\) we have \(h^{*}(s)Y^{s}E(s, z)\phi(z) \ll_{\phi} Y^{1/2}|s|^{-2}\) for \(s, z\) as above; the estimate \([13]\) follows by integrating in \(z\) against \(\mu_f\) and then integrating in \(s\).

Having established that \(Y \mu_f(\phi) = \mu_f(E(h_Y,\cdot)\phi) + O_{\phi}(Y^{1/2}\mu_f(1))\), it remains now only to evaluate \(\mu_f(E(h_Y,\cdot)\phi)\). Let \(\Gamma_{\infty} \setminus \Gamma/\Gamma_0(q) = \{\sigma\}\) be a set of double-coset representatives as in \([21]\) and set

\[
d_{\sigma} = [\Gamma_{\infty} : \Gamma_{\infty} \cap \sigma \Gamma_0(q) \sigma^{-1}].
\]

By decomposing the transitive right \(\Gamma\)-set \(\Gamma_{\infty} \setminus \Gamma\) into \(\Gamma_0(q)\)-orbits

\[
\Gamma_{\infty} \setminus \Gamma = \sqcup \Gamma_{\infty} \setminus \Gamma_{\infty} \sigma \Gamma_0(q) = \sqcup \sigma^{-1} \Gamma_{\infty} \sigma \cap \Gamma_0(q) \setminus \Gamma_0(q),
\]

we obtain

\[
E(h_Y, z) = \sum_{\sigma \in \Gamma_{\infty} \setminus \Gamma/\Gamma_0(q)} \sum_{\gamma \in \sigma^{-1} \Gamma_{\infty} \sigma \cap \Gamma_0(q) \setminus \Gamma_0(q)} h_Y(\text{Im}(\sigma \gamma z)).
\]

By invoking the \(\Gamma_0(q)\)-invariance of \(z \mapsto \phi(z)|f|^2(z)y^k \frac{dx dy}{y^2}\) and unfolding the sum over \(\gamma \in \sigma^{-1} \Gamma_{\infty} \sigma \cap \Gamma_0(q) \setminus \Gamma_0(q)\) with the integral over \(z \in \Gamma_0(q) \setminus \mathbb{H}\), we get

\[
\mu_f(E(h_Y,\cdot)\phi) = \sum_{\sigma \in \Gamma_{\infty} \setminus \Gamma/\Gamma_0(q)} \int \sigma^{-1} \Gamma_{\infty} \sigma \cap \Gamma_0(q) \setminus \mathbb{H} h_Y(\text{Im}(\sigma z))\phi(z)|f|^2(z)y^k \frac{dx dy}{y^2}.
\]

The change of variables \(z \mapsto \sigma^{-1}z\) transforms the integral above into

\[
\int_{\Gamma_{\infty} \cap \sigma \Gamma_0(q) \sigma^{-1} \setminus \mathbb{H}} h_Y(y)\phi(z)|f|^2(\sigma^{-1}z)\text{Im}(\sigma^{-1}z)^k \frac{dx dy}{y^2}.
\]

Integrating over a fundamental domain for \(\Gamma_{\infty} \cap \sigma \Gamma_0(q) \sigma^{-1} = \{\pm \{1, d_{\sigma} n\} : n \in \mathbb{Z}\}\) acting on \(\mathbb{H}\), we get

\[
\int_{y=0}^{\infty} \int_{x=0}^{d_{\sigma}} h_Y(y)\phi(z)|f|^2(\sigma^{-1}z)\text{Im}(\sigma^{-1}z)^k \frac{dx dy}{y^2}.
\]

Applying now the change of variables \(z \mapsto d_{\sigma}z\) gives

\[
\int_{y=0}^{\infty} \int_{x=0}^{1} \phi(d_{\sigma} z)|f| (d_{\sigma} \sigma^{-1}(d_{\sigma} - 1))^2(z)y^k \frac{dx dy}{y^2}.
\]

Since \(f| (d_{\sigma} \sigma^{-1}(d_{\sigma} - 1)) = \pm f\) by the consequence \([9]\) of Atkin-Lehner theory (using here that \(q\) is squarefree), we find that

\[
\mu_f(E(h_Y,\cdot)\phi) = \sum_{\sigma \in \Gamma_{\infty} \setminus \Gamma/\Gamma_0(q)} \int_{y=0}^{\infty} h_Y(d_{\sigma} y)\int_{x=0}^{1} \phi(d_{\sigma} z)|f|^2(z)y^k \frac{dx dy}{y^2}.
\]

Since \(\{d_{\sigma}\} = \{d : d\mid q\}\), we obtain the claimed formula. \(\square\)
In the expression for $Y \mu_f(\phi)$ given by Lemma 3.4, we expand $\phi$ in a Fourier series $\phi(z) = \sum_{l \in \mathbb{Z}} \phi_l(y)e(lx)$ and consider separately the contributions from $l$ in various ranges; specifically, we set

\[ S_0 = \sum_{d_1} \int_{y=0}^{\infty} h_Y(dy) \int_{x=0}^{1} \phi_0(dy)|f|^2(z)y^k \frac{dx\,dy}{y^2}, \]

\[ S_{(0,Y^{1+\varepsilon})} = \sum_{d_1} \int_{y=0}^{\infty} h_Y(dy) \int_{x=0}^{1} \sum_{0 \leq |l| \leq Y^{1+\varepsilon}} \phi_l(dy)|f|^2(z)y^k \frac{dx\,dy}{y^2}, \]

\[ S_{\geq Y^{1+\varepsilon}} = \sum_{d_1} \int_{y=0}^{\infty} h_Y(dy) \int_{x=0}^{1} \sum_{|l| \geq Y^{1+\varepsilon}} \phi_l(dy)|f|^2(z)y^k \frac{dx\,dy}{y^2}, \]

so that

\[ \sum_{d_1} \int_{y=0}^{\infty} h_Y(dy) \int_{x=0}^{1} \phi(dz)|f|^2(z)y^k \frac{dx\,dy}{y^2} = S_0 + S_{(0,Y^{1+\varepsilon})} + S_{\geq Y^{1+\varepsilon}}. \]

We treat these contributions in Lemmas 3.6, 3.7 and 3.8 respectively; in doing so we shall repeatedly use the following technical result.

**Lemma 3.5.** The quantity $\mu_f(E(h_Y, \cdot))$ satisfies the formulas and estimates

\[ \mu_f(E(h_Y, \cdot)) = \sum_{d_1} \int_{y=0}^{\infty} h_Y(dy) \int_{x=0}^{1} |f|^2(z)y^k \frac{dx\,dy}{y^2} \]

\[ = Y \mu_f(1)(1 + E_f(qY)) \]

\[ = Y \mu_f(1) \left(1 + O\left(Y^{-1/2}R_f(qk)\right)\right), \]

where

\[ E_f(x) := \frac{2\pi^2}{x} \int_{(1/2)} h^\wedge(s) \left(\frac{x}{4\pi}\right)^s \frac{\Gamma(s+k-1)}{\Gamma(k)} \frac{\zeta(s)}{\zeta(2s)} \frac{L(adf,s)}{L(adf,1)} \frac{ds}{2\pi i}. \]

More, $\mu_f(E(h_Y, \cdot)) \ll Y \mu_f(1)$.

**Proof.** The first equality follows from the same argument as in the proof of Lemma 3.4, the second from the Mellin formula and the unfolding method by a direct computation, the third from the bounds $|\Gamma(k-1/2 + it)| \leq \Gamma(k)^{-1/2} \Gamma(k)$, $\zeta(1/2 + it) \ll (1 + |t|)^{1/4}$ and $\zeta(1 + 2it) \gg 1/\log(1 + |t|)$ as in [39, p.7]. Finally, because the quantity $\mu_f(E(h_Y, \cdot))$ is majorized by the integral of the $\Gamma$-invariant measure $\mu_f$ over the region on which the function $\Gamma^\wedge \not\in \mathbb{H}$ and the region intersects $\ll Y$ fundamental domains for $\Gamma \backslash \mathbb{H}$, we have $\mu_f(E(h_Y, \cdot)) \ll Y \mu_f(1)$.

**Lemma 3.6** (The main term $S_0$). If $\phi$ is a Maass eigencuspform, then $\phi_0(y) = 0$ and $S_0 = 0$. If $\phi$ is an incomplete Eisenstein series, then

\[ S_0 = Y \mu_f(1) \left(\frac{\mu(\phi)}{\mu(1)} + O_{\phi}\left(\frac{1 + R_f(qk)}{Y^{1/2}}\right)\right). \]

**Proof.** If $\phi$ is a Maass eigencuspform then $\phi_0(y) = 0$ holds by definition, hence $S_0 = 0$. Suppose otherwise that $\phi$ is an incomplete Eisenstein series. It follows from [10] that for every $y \in \mathbb{R}_+^*$ such that $h_Y(y) \neq 0$,
we have \( \phi_0(y) = \mu(\phi)/\mu(1) + O_\phi(Y^{-1/2}) \). Thus two applications of Lemma 3.5 show that
\[
S_0 = \mu_f(E(h_{Y, \cdot})) \left( \frac{\mu(\phi)}{\mu(1)} + O_\phi(Y^{-1/2}) \right) = Y \mu_f(1) \left( 1 + O \left( \frac{R_f(qk)}{Y^{1/2}} \right) \right) \left( \frac{\mu(\phi)}{\mu(1)} + O_\phi(Y^{-1/2}) \right) = Y \mu_f(1) \left( \mu(\phi) / \mu(1) + O_\phi \left( \frac{1 + R_f(qk)}{Y^{1/2}} \right) \right).
\]

\[ \square \]

**Lemma 3.7** (The essential error term \( S_{(0,Y^{1+\varepsilon})} \)). If \( \phi \) is a Maass eigencuspform, then
\[
S_{(0,Y^{1+\varepsilon})} = \sum_{0 < |l| < Y^{1+\varepsilon}} \frac{\lambda_\phi(l)}{\sqrt{|l|}} \sum_{d \mid q} S_d(l, dY).
\]

If \( \phi \) is an incomplete Eisenstein series, then
\[
S_{(0,Y^{1+\varepsilon})} = \int_\mathbb{R} \frac{\Psi^{\wedge}(\frac{1}{2} + it)}{\xi(1 + 2it)} \sum_{0 < |l| < Y^{1+\varepsilon}} \frac{\lambda_\phi(l)}{\sqrt{|l|}} \sum_{d \mid q} S_d(l, Yd) \, \frac{dt}{2\pi}.
\]

**Proof.** Follows by integrating the Fourier expansion (5) of a newform, the Fourier expansion (10) of a Maass cusp form, and the formula (15) for the non-constant Fourier coefficients of an Eisenstein series. \( \square \)

**Lemma 3.8** (The trivial error term \( S_{\geq Y^{1+\varepsilon}} \)). We have \( S_{\geq Y^{1+\varepsilon}} \ll_{\phi, \varepsilon} Y^{-10} \mu_f(1) \).

**Proof.** Lemma 3.8 follows from Lemma 3.5 and the following claim: for all \( y \in \mathbb{R}^+ \) such that \( h_Y(y) \neq 0 \), we have \( \sum_{|l| \geq Y^{1+\varepsilon}} |\phi_l(y)| \ll_{\phi, \varepsilon} Y^{-11} \). The claim is proved in [14] §3.2, as follows. When \( \phi \) is a cusp form of eigenvalue \( 1/4 + r^2 \), so that \( \phi_l(y) = y^{-1/2} \lambda_\phi(l) \kappa_{\varepsilon}(ly) \), the claim follows from the exponential decay of \( l \mapsto \kappa_{\varepsilon}(ly) \) for \( l \geq Y^{1+\varepsilon} \) and \( y \ll Y^{-1} \) together with the polynomial growth of \( l \mapsto \lambda_\phi(l) \). When \( \phi \) is an incomplete Eisenstein series, the integral formula (15) and standard bounds for the \( K \)-Bessel function show that for each positive integer \( A \), we have \( \phi_l(y) \ll_{\phi, \varepsilon, A} Y^{A-1/2} |l|^{-A(1 + |y|/|l|)^{\varepsilon}} \); the claim then follows by summing over \( |l| \geq Y^{1+\varepsilon} \). \( \square \)

**Proof of Proposition 3.3.** By Lemma 3.4 and equation (19), we have
\[
\frac{\mu_f(\phi)}{\mu_f(1)} = \frac{1}{Y \mu_f(1)} (S_0 + S_{(0,Y^{1+\varepsilon})} + S_{\geq Y^{1+\varepsilon}}) + O_{\phi, \varepsilon}(Y^{-1/2}).
\]

Proposition 3.3 follows by combining the results of Lemma 3.6, Lemma 3.8, and Lemma 3.7. \( \square \)

### 3.2. Bounds for individual shifted sums

We bound the individual shifted sums appearing in Definition [3.2] in subsequent sections we shall need only our main result, Corollary 3.14. We first recall a special case of Holowinsky’s bound [14] Theorem 2).

**Theorem 3.9** (Holowinsky). Let \( \varepsilon \in (0, 1) \). Then for \( x \geq 1 \) and \( l \in \mathbb{Z}_{\neq 0} \), we have
\[
\sum_{m,n \in \mathbb{N}, m \equiv n + l \in \mathbb{N}, \max(m,n) \leq x} |\lambda_f(m)\lambda_f(n)| \ll_{\varepsilon} \tau(l) \frac{x \prod_{p \leq x} (1 + 2|\lambda_f(p)|/p)}{\log(ex)^{2-\varepsilon}}.
\]

Unfortunately, Theorem 3.9 is insufficient for our purposes because \( \tau(ql) \) can be quite large, even larger asymptotically than every power of \( \log(eq) \), when \( q \) has many small prime factors. The following refinement will suffice.
Theorem 3.10. With conditions as in the statement of Theorem 3.7, we have

\[
\sum_{n \leq x} |\lambda_f(m)\lambda_f(n)| \ll_{\varepsilon} \frac{x \prod_{p \leq z} (1 + 2|\lambda_f(p)|/p)}{\log(ex)^{k-\varepsilon}}
\]

where all implied constants are absolute.

Proof. In [26, Thm 3.1], we generalized Holowinsky’s bound [14, Thm 2] to totally real number fields \( \mathbb{F} \). Along the way we proved a pair of results [26, Thm 4.10] and [26, Thm 7.2] either of which imply Theorem 3.10. For completeness, we shall give the argument here in the special case \( \mathbb{F} = \mathbb{Q} \), which borrows heavily from that of Holowinsky; up to [24] we essentially recall his argument, and after that introduce our refinement.

Let \( \lambda(n) = |\lambda_f(n)| \), so that

\[
\lambda \text{ is a nonnegative multiplicative function satisfying } \lambda(n) \leq \tau(n).
\]

We may assume that \( 1 \leq l \leq x \). Fix \( \alpha \in (0, 1/2) \) (to be chosen sufficiently small at the end of the proof) and set

\[
y = x^\alpha, \quad s = \alpha \log \log(x), \quad z = x^{1/\alpha}.
\]

For \( x \gg_\alpha 1 \) we have \( 10 \leq z \leq y \leq x \), as we shall henceforth assume. For each \( n \in \mathbb{N} \), write \( m = n + l \in \mathbb{N} \). Define the \( z \)-part of a positive integer to be the greatest divisor of that integer supported on primes \( p \leq z \). There exist unique positive integers \( a, b, c \) such that \( \gcd(a, b) = 1 \) and \( ac \) (resp. \( bc \)) is the \( z \)-part of \( m \) (resp. \( n \)); such triples \( a, b, c \) satisfy

\[
p | abc \Rightarrow p \leq z, \quad c | l, \quad \text{and } \gcd(a, b) = 1.
\]

Write \( \mathbb{N} = \sqcup_{a, b, c} \mathbb{N}_{abc} \) for the fibers of \( n \mapsto (a, b, c) \). The assumption (24) implies \( \lambda(m)\lambda(n) \leq 4^s \lambda(ac)\lambda(bc) \), so that

\[
\sum_{n \in \mathbb{N}\cap[1,x]} \lambda(m)\lambda(n) \leq \sum_{a, b, c} \lambda(ac)\lambda(bc) \cdot \#(\mathbb{N}_{abc} \cap [1, x]).
\]

Holowinsky asserts that Rankin’s trick implies that the contribution to the above from \( a, b, c \) for which \( |ac| > y \) or \( |bc| > y \) is \( \ll_{\alpha,A} x \log(x)^{-A} \) for any \( A \); we spell out an alternate proof of this assertion in [26, Lemma 7.3]. Now, an integer belongs to \( \mathbb{N}_{abc} \) only if it satisfies some congruence conditions modulo each prime \( p \leq z \) (see [14, p.14], or [26, Lemma 7.3] for a detailed discussion); as in [14] or [26, Corollary 7.8], an application of the large sieve (or Selberg’s sieve) shows that if \( |ac| \leq y, |bc| \leq y \) and \( x \gg y^2 \), then

\[
\#(\mathbb{N}_{abc} \cap [1, x]) \ll \frac{x + (yz)^2}{\log(z)^2} \frac{l}{c^2 \phi(abc^{-1} l)}.
\]

Since \( (yz)^2 \ll x, \log(z)^2 \ll \alpha \log(x)^2 \log(x)^2 \), \( 4^s \ll \varepsilon \log(x)^\varepsilon \) (for \( \alpha \ll \varepsilon 1 \)), and \( \phi(abc^{-1} l) \geq \phi(c^{-1} l)\phi(a)\phi(b) \), we see that Theorem 3.10 follows from the bound

\[
\sum_{c | l} \frac{1}{c} \frac{l/c \phi(l/c)}{\phi(abc^{-1} l)} \sum_{|ac| \leq y, |bc| \leq y} \lambda(ac)\lambda(bc) \phi(a)\phi(b) \ll \log \log(x)^{O(1)} \prod_{p \leq z} \left(1 + \frac{2\lambda(p)}{p}\right),
\]

This bound is slightly poorer than that obtained by Holowinsky because we have been more precise in our calculation of the residue classes sieved out by prime divisors of \( c^{-1} l \); the discrepancy here does not matter in the end.

\[\text{(24)}\]

\[\text{(23)}\]
which we now establish. Note first that

\[(25)\]

\[
\sum_{|ac| \leq y \atop p | ab \Rightarrow p \leq z} \frac{\lambda(ac)\lambda(bc)}{\phi(a)\phi(b)} \leq \left( \prod_{p \leq z} \frac{\lambda(p^{k+\nu}(c))}{\phi(p^{k})} \right)^2.
\]

Using that \(\lambda(p^k) \leq k + 1\) and \(p \geq 2\) and summing some geometric series as in [20] Lemma 7.4 gives

\[
\sum_{k \geq 0} \lambda(p^{k+\nu}) \leq \nu + 1 + \sum_{k \geq 1} \frac{\nu + k + 1}{p^{k-1}(p-1)} \leq 3\nu + 3
\]

for each \(\nu \geq 1\), while for \(\nu = 0\)

\[
\sum_{k \geq 0} \lambda(p^k) \phi(p^k) = \left(1 + \frac{\lambda(p)}{p}\right) \left(1 + \frac{\lambda(p)\left(\frac{1}{p^2} - \frac{1}{p}\right) + \sum_{k \geq 2} \frac{\lambda(p^{k+1})}{p^{k+1}}}{1 + \frac{\lambda(p)}{p}}\right)
\]

\[
\leq \left(1 + \frac{\lambda(p)}{p}\right) \left(1 + \frac{20}{p}\right).
\]

Thus the LHS of (24) is bounded by \(\zeta(2)^4\psi(l)\prod_{p \leq x}(1 + \lambda(p)p^{-1})^2\), where \(\psi\) is the multiplicative function

\[(26)\]

\[
\psi(l) = \sum_{c | l} \frac{l/c}{\phi(l/c)} \prod_{p | l} (3\nu + 3)^2.
\]

By direct calculation and the inequality \(p \geq 2\), we have

\[
\psi(p^a) = \frac{1}{1 - p^{-1}} + \frac{9}{p^2} \left(a + 1\right)^2 + \frac{1}{1 - p^{-1}} \left(1 - \frac{a^2}{p^2} \sum_{i=1}^{a-1} \frac{(i+1)^2}{p^i}\right) \leq 1 + C p^{-1}
\]

for some constant \(C \leq 10^6\), so that \(\psi(l) \leq \prod_{p | l}(1 + C p^{-1}) \ll \log \log(x)^C\) for \(1 \leq l \leq x\). This estimate for \(\psi(l)\) establishes the claimed bound (24).

\[\square\]

Remark 3.11. A bound of the form (20) but with an unspecified dependence on the parameter \(l\) may be derived from the work of Nair [25]. We have attempted to quantify this dependence by working through the details of Nair’s arguments, and have shown that they imply

\[(27)\]

\[
\sum_{n \in \mathbb{N} \atop m = n + l \in \mathbb{N} \atop \max(m,n) \leq x} |\lambda_f(m)\lambda_f(n)| \ll_{\varepsilon} \tau_m(l) \frac{x \prod_{p \leq x}(1 + 2|\lambda_f(p)|/p)}{\log(ex)^{2-\varepsilon}}
\]

for some \(m \geq 2\) (probably \(m = 2\)) and all \(0 \neq |l| \leq x^{1/16-\varepsilon}\); in deducing this we have used the Ramanujan bound \(|\lambda_f(p)| \leq 2\). This strength and uniformity falls far short of what is needed in treating the level aspect of QUE.

A mild strengthening of (20) subject to the additional constraint \(4l^2 \leq x\) appears in the recent book of Iwaniec-Friendlander [8] Thm 15.6], which was released after we completed the work of this paper. The condition \(4l^2 \leq x\) makes their result inapplicable in our treatment of the level aspect of QUE, where \(l\) can be nearly as large as \(x\). However, it seems to us that one can remove this condition by a suitable modification of their arguments.

Recall from Definition 3.2 that the sums \(S_\varepsilon(l, x)\) involve a certain integral \(I_\varepsilon(l, n, x)\). 

Lemma 3.12. For each positive integer $A$, the integral $I_s(l, n, x)$ satisfies the upper bound

$$I_s(l, n, x) \ll_A \frac{\Gamma(k - 1)}{(4\pi)^{k-1}} \sqrt{mn \cdot \max\left(1, \frac{\max(m, n)}{xk}\right)^{-A}}$$

uniformly for $s \in i\mathbb{R} \cup (-1/2, 1/2)$, $n \in \mathbb{N}$, $l \in \mathbb{Z}_{\neq 0}$, and $x \geq 1$. Here $m := n + l$, as usual.

Proof. Let $s, l, m, n$ be as above, and let $A \geq 0$. Then $|\kappa_s(y)| \leq 1$, so that by the Mellin formula we have

$$I_s(l, n, x) \leq \int_0^\infty h(xy)\kappa_f(my)\kappa_f(ny)y^{-1} \frac{dy}{y}$$

$$= \int (A) h^\wedge(w)x^w \int_{\mathbb{R}^+} y^{w-1} \kappa_f(my)\kappa_f(ny) \frac{dy}{y} \frac{dw}{2\pi i}$$

$$= \frac{(\sqrt{mn})^k}{(4\pi)^{k-1} \pi} \int (A) h^\wedge(w) \left(\frac{x}{\pi (\frac{m+n}{2})}\right)^w \Gamma(w + k - 1) \frac{dw}{2\pi i}$$

$$\ll_A \frac{\Gamma(k - 1)}{(4\pi)^{k-1}} \sqrt{mn \cdot \max\left(1, \frac{\max(m, n)}{xk}\right)^{-A}}.$$

Here we have used the arithmetic mean-geometric mean inequality, the well-known bound \[42, \text{Ch 7, Misc. Ex 44}]

$$\frac{\Gamma(w + k - 1)}{\Gamma(k - 1)} \ll_A (k - 1)^A (1 + k^{-1}(1 + |w|^2)) \ll k^A (1 + |w|^2)$$

for $\Re(w) = A$, and the rapid decay of $h^\wedge$. The case $A = 0$ gives $I_s(l, n, x) \ll_k (4\pi)^{-k+1}\Gamma(k - 1)\sqrt{mn}$, which combined with the case that $A$ is a positive integer yields the assertion of the lemma. \qed

Remark 3.13. See \[26, \text{Lem 4.3} \] and \[26, \text{Cor 4.4} \] for a fairly sharp refinement of Lemma 3.12.

Corollary 3.14. The shifted sums $S_s(l, x)$ satisfy the upper bound

$$(28) \quad S_s(l, x) \ll_\varepsilon \frac{\Gamma(k - 1)}{(4\pi)^{k-1}} \frac{X}{\log(xk)^{1/2 - \varepsilon}} \prod_{p \leq xk} \left(1 + \frac{2|\lambda_f(p)|}{p}\right)^{-A}$$

uniformly for $s \in i\mathbb{R} \cup (-1/2, 1/2)$ and $x \geq 1$.

Proof. Let us set $X = xk$ and temporarily denote by $T_f(x, l, \varepsilon)$ the right-hand side of (28) without the factor $(4\pi)^{-k+1}\Gamma(k - 1)$. By Definition 3.2 and Lemma 3.12 we need only show that

$$\sum_{m \in \mathbb{N}, m = n + l} |\lambda_f(m)| \lambda_f(n) \cdot \max\left(1, \frac{\max(m, n)}{X}\right)^{-A} \ll_\varepsilon T_f(x, l, \varepsilon)$$

for some positive integer $A$. Take $A = 2$. We may assume that $X = xk \geq 10$. By Theorem 3.10 and the Deligne bound $|\lambda_f(p)| \leq 2$, the left hand side of (29) is

$$\ll_\varepsilon T_f(x, l, \varepsilon) \sum_{n=0}^{\infty} 2^{-n\Lambda} 2^n \left(\frac{\log(X)}{\log(2^n X)}\right)^{2-\varepsilon} \prod_{X < p \leq 2^n X} \left(1 + \frac{2|\lambda_f(p)|}{p}\right)$$

$$
\ll T_f(x, l, \varepsilon) \sum_{n=0}^{\infty} 2^{-(A-1)n} \exp\left(4 \log \left(\frac{\log(2^n X)}{\log(X)}\right)\right).
$$

The inner sum converges and is bounded uniformly in $X$, so we obtain the desired estimate (29). \qed
3.3. Bounds for sums of shifted sums. We complete the proof of Theorem 3.1 by bounding the sums of shifted sums that arose in Proposition 3.3.

Lemma 3.15. For each $\varepsilon \in (0, 1)$ and each squarefree number $q$, we have

$$\sum_{d|q} \frac{d}{\log(dk)^{2-\varepsilon}} \ll \frac{q \log \log(e^\varepsilon q)}{\log(qk)^{2-\varepsilon}} \ll \varepsilon \frac{q}{\log(qk)^{2-2\varepsilon}}.$$  \hspace{1cm} \Box$$

Proof. Suppose that $q$ is the product of $r \geq 1$ primes $q_1 < \cdots < q_r$. Let $p_1 < \cdots < p_r$ be the first $r$ primes, so that $p_i \leq q_i$ for $i = 1, \ldots, r$. Define $\beta(x) = x/\log(e^\varepsilon xk)^{2-\varepsilon}$; we have chosen this particular definition so that $\beta$ is increasing on $\mathbb{R}_{\geq 1}$ and $\beta(x) \asymp x/\log(xk)^{2-\varepsilon}$ for $x \in \mathbb{R}_{\geq 1}$. The map

$$\mathbb{R}_{\geq 1} \ni x \mapsto \beta(e^x) = x - (2 - \varepsilon) \log(2 + x)$$

is convex, so that for each $a = (a_1, \ldots, a_r) \in \{0, 1\}^r$ we have

$$\beta(p_1 \cdots p_r) \leq \beta(p_1 a_1 q_1 \cdots a_r q_r) \leq \beta(p_1 a_1 q_1 \cdots a_r q_r)^{e^\mu} \leq \cdots \leq \beta(p_1 \cdots p_r)^{e^\mu}.$$  \hspace{1cm} (30)

The prime number theorem implies that $\log(p_1 \cdots p_r) = \log(r)(1 + o(1))$, where the notation $o(1)$ refers to the limit as $r \to \infty$; we may and shall assume that $r$ is sufficiently large (and at least 100) because the assertion of the lemma holds trivially when $q$ has a bounded number of prime factors. Set $r_0 = [r/10]$. Observe that

$$p_{r-r_0+1} \cdots p_r = \exp\left(r \log(r) - (r - r_0) \log(r - r_0) + o(r \log(r))\right)$$

$$= \exp\left(r_0 \log(r) + (r - r_0) \log\left(\frac{r}{r - r_0}\right) + o(r \log(r))\right)$$

$$= \exp(r_0 \log(r)(1 + o(1)))$$

$$\ll (p_1 \cdots p_r)^{1/3^r},$$  \hspace{1cm} (31)

and

$$\log(p_1 \cdots p_{r_0}) = r_0 \log(r_0)(1 + o(1)) \asymp r \log(r)(1 + o(1)) = \log(p_1 \cdots p_r).$$  \hspace{1cm} (32)

Let $\Omega_0$ denote the set of all $a \in \{0, 1\}^r$ for which $a_1 + \cdots + a_r \leq r_0$ and $\Omega_1$ the set of all $a \in \{0, 1\}^r$ for which $a_1 + \cdots + a_r > r_0$, so that $\{0, 1\}^r = \Omega_0 \cup \Omega_1$. Then by (31) we have

$$\sum_{a \in \Omega_0} \frac{\beta(p_1 a_1 \cdots p_r a_r)}{\beta(p_1 \cdots p_r)} \ll 2^r \frac{\beta(p_1 \cdots p_r)}{\beta(p_1 \cdots p_r)} \ll 2^r (p_1 \cdots p_r)^{-7/8} \leq \sqrt{2}.$$  \hspace{1cm} (33)

If $a \in \Omega_1$, then (32) implies $\beta(p_1 a_1 \cdots p_r a_r)/\beta(p_1 \cdots p_r) \asymp p_1 a_1^{-1} \cdots p_r a_r^{-1}$, so that

$$\sum_{a \in \Omega_1} \frac{\beta(p_1 a_1 \cdots p_r a_r)}{\beta(p_1 \cdots p_r)} \ll \sum_{d|p_1 \cdots p_r} \frac{1}{d} \leq (1 + o(1)) e^\gamma \log(p_1 \cdots p_r) \ll \log(e^\varepsilon q).$$  \hspace{1cm} (34)

Since $\beta(x) \asymp x/\log(e^x)^{2-\varepsilon}$ for $x \in \mathbb{R}_{\geq 1}$, it follows from (30), (33), and (34) that

$$\sum_{d|q} \frac{d}{\log(dk)^{2-\varepsilon}} \ll \sum_{d|q} \frac{\beta(d)}{\beta(q)} = \sum_{a \in \Omega_1} \beta(q_1 a_1 \cdots q_r a_r) \ll \log(e^\varepsilon q),$$

which establishes the lemma. \hspace{1cm} (B)
Lemma 3.17. Let

\[
S_s(dy) = \sum_{d\mid q} \frac{\Gamma(k - 1)}{(4\pi)^{k-1}} Y \prod_{p\leq qk} \left(1 + \frac{2|\lambda_f(p)|}{p}\right),
\]

uniformly for \(s \in i\mathbb{R} \cup (-1/2, 1/2)\) and \(x \geq 1\).

Proof. By Corollary 3.14 we have

\[
\sum_{d\mid q} S_s(dy) \ll_{\varepsilon, c_1, c_2} \frac{\Gamma(k - 1)}{(4\pi)^{k-1}} Y \prod_{p\leq qk} \left(1 + \frac{2|\lambda_f(p)|}{p}\right) \sum_{d\mid q} \frac{dk}{\log(dk)^{2-\varepsilon}}.
\]

By the Deligne bound \(|\lambda_f(p)| \leq 2\), the part of the product in (35) taken over \(qk < p \leq qkY\) is \(\ll \log(eY)^4 \ll_{c_1, c_2} \log(e\varepsilon qk)^4\). The claim now follows from Lemma 3.15.

Lemma 3.17. Let \(\varepsilon > 0\), \(Y \geq 1\). If \(\phi\) is a normalized Maass eigencuspform, then

\[
\sum_{0 < |l| < Y^{1+\varepsilon}} \frac{|\lambda_\phi(l)|}{\sqrt{|l|}} \ll_{\phi, \varepsilon} Y^{1/4 + 2\varepsilon},
\]

where (as indicated) the implied constant may depend upon \(\phi\). On the other hand, if \(t \in \mathbb{R}\), then

\[
\sum_{0 < |l| < Y^{1+\varepsilon}} \frac{|\lambda_t(l)|}{\sqrt{|l|}} \ll_{\varepsilon} Y^{1/2 + 2\varepsilon},
\]

where the implied constant does not depend upon \(t\).

Proof. Follows from the Cauchy-Schwarz inequality, partial summation, the Rankin-Selberg bound for \(|\lambda_\phi|\) and the uniform bound \(|\lambda_t| \leq \tau(l)|\) for \(\lambda_t\).

Proof of Theorem 3.17. Suppose that \(\phi\) is a normalized Maass eigencuspform of eigenvalue \(\frac{1}{4} + r^2\). By Proposition 3.3 we have

\[
\frac{\mu_f(\phi)}{\mu_f(1)} = \frac{1}{Y \mu_f(1)} \sum_{0 < |l| < Y^{1+\varepsilon} \sum_{d\mid q} S_{\nu}(dl, dy) + O_{\phi, \varepsilon}(Y^{-1/2}).
\]

Recall from (17) that

\[
\mu_f(1) \approx q \frac{\Gamma(k - 1)}{(4\pi)^{k-1}} L(ad f, 1)
\]

and recall the definition (17) of \(M_f(qk)\). We shall ultimately choose \(Y \ll \log(qk)^{O(1)}\), so Corollary 3.16 gives the bound

\[
\frac{1}{Y \mu_f(1)} \sum_{d\mid q} S_{\nu}(dl, dy) \ll_{\varepsilon} \log(qk)^{\varepsilon} M_f(qk).
\]

By (37) and Lemma 3.17 applied to (36), we find that

\[
\frac{\mu_f(\phi)}{\mu_f(1)} \ll_{\phi, \varepsilon} \log(qk)^{\varepsilon} M_f(qk) \sum_{0 < |l| < Y^{1+\varepsilon}} \frac{|\lambda_\phi(l)|}{\sqrt{|l|}} + Y^{-1/2}
\]

\[
\ll_{\phi, \varepsilon} Y^{1/2 + 2\varepsilon} \log(qk)^{\varepsilon} M_f(qk) + Y^{-1/2}.
\]

Choosing \(Y = \max(1, M_f(qk)^{-1}) \ll \log(qk)^{O(1)}\) gives the cuspidal case of the theorem.
Suppose now that $\phi = E(\Psi, \cdot)$ is an incomplete Eisenstein series. Proposition 3.3, Corollary 3.16 and Lemma 4.17 show, as in the cuspidal case, that

$$\mu_f(\phi) - \mu_f(1) \leq \phi_{\varphi_e} Y^{1/2 + 2\varepsilon} \log(qk)^e M_f(qk) \int_{\mathbb{R}} \Psi^\wedge(\frac{1}{2} + it) dt + \frac{1 + R_f(qk)}{Y^{1/2}}.$$ 

The same choice of $Y$ as above completes the proof. \qed

4. An extension of Watson’s formula

Watson \cite{Watson}, building on earlier work of Garrett \cite{Garrett}, Piatetski-Shapiro and Rallis \cite{Piatetski-Rallis}, Harris and Kudla \cite{Harris-Kudla}, and Gross and Kudla \cite{Gross-Kudla}, proved a beautiful formula relating the integral of the product of three modular forms to the central value of their triple product $L$-function. Unfortunately, Watson’s formula applies only to triples of newforms having the same squarefree level. In \cite{Nelson} we shall refer only to the statement of the following extension of Watson’s formula to the case of interest, not the details of its proof.

**Theorem 4.1.** Let $\phi$ be a Maass eigencuspform of level 1 and $f$ a holomorphic newform of squarefree level $q$, as in \cite{Nelson}. Then

$$\int_{\Gamma_0(q) \backslash \mathbb{H}} \phi(z) |f|^2(z) g^k \frac{dz\,dy}{y^2} \left( \int_{\Gamma_0(q) \backslash \mathbb{H}} |f|^2(z) g^k \frac{dz\,dy}{y^2} \right)^2 = \frac{1}{8q} \Lambda(\phi \times f \times f, \frac{1}{2}).$$

The $L$-functions $L(\cdot \cdot \cdot) = \prod_p L_p(\cdot \cdot \cdot)$ and their completions $\Lambda(\cdot \cdot \cdot) = L_\infty(\cdot \cdot \cdot) L(\cdot \cdot \cdot) = \prod_v L_v(\cdot \cdot \cdot)$ are as in \cite{Watson} §3.

**Remark 4.2.** For simplicity, we have stated Theorem 4.1 only in the special case that we need it, but our calculations (Lemma 4.4) lead to a more general formula. Let $\psi_j$ ($j = 1, 2, 3$) be newforms of weight $k_j$ and level $q_j$. We allow the possibility $k_j = 0$, in which case we require that $\psi_j$ be an even or odd Maass eigencuspform. If $k_1 + k_2 + k_3 \neq 0$ or some prime $p$ divides exactly one of the $q_j$, then it is straightforward to see that $\int \psi_1 \psi_2 \psi_3 = 0$. Otherwise $k_1 + k_2 + k_3 = 0$ and each prime divides the $q_j$ either 0, 2 or 3 times, so one can read off from Watson \cite{Watson} Theorem 3, Ichino \cite{Ichino} and Lemma 4.4 the identity

$$(38) \int_X \psi_1 \psi_2 \psi_3 = \frac{1}{8} \prod \Lambda(1, \varphi_j) \prod c_v$$

where $X = \lim X_0(q) \backslash \mathbb{H}$ with $\text{vol}(X) := \text{vol}(X_0(1) \backslash \mathbb{H}) = \pi / 3$, $c_\infty$ is $Q_\infty \in \{0, 1, 2\}$ from \cite{Watson} Theorem 3, $c_p = 1$ if $p$ divides none of the $q_j$, $c_p = p^{-1}$ if $p$ divides exactly two of the $q_j$, and $c_p = p^{-1}(1 + p^{-1})(1 + \varepsilon_p)$ if $p$ divides all of the $q_j$ with $-\varepsilon_p$ the product of the Atkin-Lehner eigenvalues for the $\psi_j$ at $p$ as in \cite{Watson} Theorem 3.

Watson proved his formula only for three forms of the same squarefree level because Gross and Kudla \cite{Gross-Kudla} evaluated the $p$-adic zeta integrals of Harris and Kudla \cite{Harris-Kudla} only when (the factorizable automorphic representations generated by) the three forms are special at $p$; Harris and Kudla had already considered the case that all three forms are spherical at $p$. Ichino \cite{Ichino} showed that the local zeta integrals of Harris and Kudla are equal to simpler integrals over the group $\text{PGL}(2, \mathbb{Q}_p)$. Ichino and Ikeda \cite{Ichino-Ikeda} §7, §12] computed these simpler integrals when all three forms are special at $p$. Since we are interested in the integral of $\phi |f|^2$ when $\phi$ has level 1 and $f$ has squarefree level $q$, we must consider the case that two representations are special and one is spherical. We remark in passing that Böcherer and Schulze-Pillot \cite{Bocherer-Schulze-Pillot} considered similar problems for modular forms on definite rational quaternion algebras in the classical language, but their results are not directly applicable here.
To state (a special case of) Ichino’s result, we introduce some notation. In what follows, \( v \) denotes a place of \( \mathbb{Q} \) and \( p \) a prime number. Let \( G = \text{PGL}(2)/\mathbb{Q}, G_v = G(\mathbb{Q}_v), K_\infty = \text{SO}(2)/\{ \pm 1 \}, K_p = G(\mathbb{Z}_p), \) and \( G_\mathbb{A} = G(\mathbb{A}) = \prod_v G_v, \) where \( \mathbb{A} = \prod_v \mathbb{Q}_v \) is the adele ring of \( \mathbb{Q} \). Regard \( \phi \) and \( f \) as pure tensors \( \phi = \bigotimes \phi_v \) and \( f = \bigotimes f_v \) in (factorizable) cuspidal automorphic representations \( \pi_{\phi} = \bigotimes \pi_{\phi,v} \) and \( \pi_f = \bigotimes \pi_{f,v} \) of \( G_\mathbb{A} = \prod_v G_v \). Set \( f_v = (-1) \cdot f_v \) and \( \bar{f} = \bigotimes f_v \). Then \( f_p = \bar{f}_p \) for all (finite) primes \( p \).

Although the vectors \( \phi_v \) and \( f_v \) are defined only up to a nonzero scalar multiple, the matrix coefficients

\[
\Phi_{\phi,v}(g_v) = \frac{\langle g_v \cdot \phi_v, \phi_v \rangle}{\langle \phi_v, \phi_v \rangle}, \quad \Phi_{f,v}(g_v) = \frac{\langle g_v \cdot f_v, f_v \rangle}{\langle f_v, f_v \rangle}, \quad \Phi_{\bar{f},v}(g_v) = \frac{\langle g_v \cdot \bar{f}_v, \bar{f}_v \rangle}{\langle f_v, f_v \rangle}
\]

are well-defined; here \( g_v \) belongs to \( G_v \) and \( \langle \cdot, \cdot \rangle_v \) denotes the (unique up to a scalar) \( G_v \)-invariant Hermitian pairings on the irreducible admissible self-contragredient representations \( \pi_{\phi,v} \) and \( \pi_{f,v} \). Let \( dg_v \) denote the Haar measure on the group \( G_v \) with respect to which \( \text{vol}(K_v) = 1 \). Define the local integrals

\[
I_v = \int_{G_v} \Phi_{\phi,v}(g_v) \Phi_{f,v}(g_v) \Phi_{\bar{f},v}(g_v) \, dg_v
\]

and the normalized local integrals

\[
\tilde{I}_v = \left( \frac{\zeta_v(2) \zeta_v(3)}{\zeta_v(2) \zeta_v(1, \text{ad} \phi) \zeta_v(1, \text{ad} f)} \right)^{-1} I_v.
\]

**Theorem 4.3 (Ichino).** We have \( \tilde{I}_v = 1 \) for all but finitely many places \( v \), and

\[
\frac{\left| \int_{G_v \backslash \mathbb{H}} \phi f |f|^2 y^k \frac{dx \, dy}{y^2} \right|^2}{\int_{G_v \backslash \mathbb{H}} |\phi|^2 \frac{dx \, dy}{y^2} \left( \int_{G_v \backslash \mathbb{H}} |f|^2 y^k \frac{dx \, dy}{y^2} \right)^2} = \frac{1}{8} \frac{\Lambda(\frac{1}{2}, \phi \times f \times f)}{\Lambda(1, \text{ad} \phi) \Lambda(1, \text{ad} f)} \prod_v \tilde{I}_v.
\]

**Proof.** See [10] Theorem 1.1, Remark 1.3. We have taken into account the relation between classical modular forms and automorphic forms on the adele group \( G_\mathbb{A} \) (see Gelbart [9]) and the comparison (see for instance Vignéras [40, §III.2]) between the Poincaré measure on the upper half-plane and the Tamagawa measure on \( G_\mathbb{A} \).

We know by work of Harris and Kudla [12], Gross and Kudla [11], Watson [41], Ichino and Ikeda [17], and Ichino that \( \tilde{I}_\infty = 1 \) and \( \tilde{I}_p = 1 \) for all primes \( p \) that do not divide the level \( q \). We contribute the following computation, with which we deduce Theorem 4.1 from Theorem 4.3.

**Lemma 4.4.** Let \( p \) be a prime divisor of the squarefree level \( q \). Then \( \tilde{I}_p = 1/p \).

Before embarking on the proof, let us introduce some notation and recall formulas for the matrix coefficients \( \Phi_{\phi,p} \) and \( \Phi_{f,p} \). Let \( G_p = \text{PGL}_2(\mathbb{Q}_p), \) let \( K_p = \text{PGL}_2(\mathbb{Z}_p), \) and let \( A_p \) be the subgroup of diagonal matrices in \( G_p \). Recall the Cartan decomposition \( G_p = K_p A_p K_p \). For \( y \in \mathbb{Q}_p^* \) we write \( a(y) = (y^{-1} 1) \in A_p \).

The representation \( \pi_{\phi,p} \) is unramified principal series with Satake parameters \( \alpha_{\phi}(p) \) and \( \beta_{\phi}(p) \); for clarity we write simply \( \alpha = \alpha_{\phi}(p) \) and \( \beta = \beta_{\phi}(p) \). The vector \( \phi_p \) lies on the unique \( K_p \)-fixed line in \( \pi_{\phi,p} \). The matrix coefficient \( \Phi_{f,p} \) is bi-\( K_p \)-invariant, so by the Cartan decomposition we need only specify \( \Phi_{\phi,p}(a(p^m)) \) for \( m \geq 0 \), which is given by the Macdonald formula [3] Theorem 4.6.6

\[
\Phi_{\phi,p}(a(p^m)) = \frac{1}{1 + p^{-1}p^{-m/2}} \left[ \alpha^m \frac{1 - p^{-1/2}}{1 - \frac{p^{-1/2}}{\alpha}} + \beta^m \frac{1 - p^{-1/2}}{1 - \frac{p^{-1/2}}{\beta}} \right].
\]

The representation \( \pi_{f,p} \) is an unramified quadratic twist of the Steinberg representation of \( G_p \). The vector \( f_p \) lies on the unique \( I_p \)-fixed line in \( \pi_{f,p} \), where \( I_p \) is the Iwahori subgroup of \( K_p \) consisting of matrices that are upper-triangular mod \( p \). Thus to determine \( \Phi_{f,p} \), we need only specify the values it takes on
representatives for the double coset space $I_p \backslash G_p / I_p$, whose structure we now recall following [10, §7] (see also [17, §7] for a similar discussion). Define the elements

$$w_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} p & p^{-1} \\ 0 & 1 \end{pmatrix}, \quad \omega = \begin{pmatrix} p & 1 \\ 0 & 1 \end{pmatrix}$$

of $G_p$. Note that since $G_p = \text{PGL}_2(\mathbb{Q}_p)$, we have $w_1^2 = w_2^2 = \omega^2 = 1$. For $w$ in the group $W_a = \langle w_1, w_2 \rangle$ generated by $w_1$ and $w_2$, let $\lambda(w)$ be the length of the shortest string expressing $w$ in the alphabet $\{w_1, w_2\}$, so that $\lambda(w_1) = \lambda(w_2) = 1$. Extend $\lambda$ to the group $\hat{W} = \langle w_1, w_2, \omega \rangle$, which is the semidirect product of $W_a$ by the group of order 2 generated by $\omega$, via the formula $\lambda(\omega^j w) = \lambda(w)$ when $w \in W_a$, so that in particular $\lambda(\omega) = 0$. We have a Bruhat decomposition $G_p = \bigsqcup_{w \in \hat{W}} I_p w I_p$; unwinding the definitions, this reads more concretely as

$$G_p = \left( \bigsqcup_{n \in \mathbb{Z}} I_p \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} I_p \right) \bigsqcup \left( \bigsqcup_{n \in \mathbb{Z}} I_p w_1 \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} I_p \right),$$

but we shall not adopt this perspective. With our normalization of measures we have $\text{vol}(I_p w I_p) = (p + 1)^{-1} p^{\lambda(w)}$. Suppose temporarily that $\pi_{f,p}$ is (the trivial twist of) the Steinberg representation. The matrix coefficient $\Phi_{f,p}$ is bi-$I_p$-invariant and given by

$$\Phi_{f,p}(\omega^j w) = (-1)^j (-p^{-1})^{\lambda(w)}$$

for all $j \in \{0, 1\}$ and $w \in W_a$. In particular

$$\Phi_{f,p}(\omega^j w)^2 = p^{-2\lambda(w)}.$$

In the general case that $\pi_{f,p}$ is a possibly nontrivial unramified quadratic twist of Steinberg, the formula (44) for the squared matrix coefficient still holds.

**Proof of Lemma** Having recalled the formulas above, we see that

$$I_p = \int_{G_p} \Phi_{f,p}(g) \Phi_{f,p}(g)^2 \, dg = \sum_{w \in \hat{W}} \text{vol}(I_p w I_p) \Phi_{f,p}(w) p^{-2\lambda(w)}$$

$$= (p + 1)^{-1} \sum_{w \in \hat{W}} \Phi_{f,p}(w) p^{-\lambda(w)},$$

where $\Phi_{f,p}$ is given by (40). The evaluation of the Poincaré series

$$\sum_{w \in \hat{W}} t^{\lambda(w)} = 2^{1 + t} \frac{1 + t}{1 - t},$$

where $t$ is an indeterminate, is asserted and used in [17, §7], but we need a finer result here. For $w \in \hat{W}$ let us write $\mu(w)$ for the unique nonnegative integer with the property that $K_p w K_p = K_p a(p^{\mu(w)}) K_p$. Then we claim that for indeterminates $x, t$ we have the relation of formal power series

$$\sum_{w \in \hat{W}} x^{\mu(w)} t^{\lambda(w)} = \frac{(1 + x)(1 + t)}{1 - xt}.$$

Note that we recover (43) upon taking $x = 1$. To prove (44), observe that since $\omega^1 w_1 = w_2 \omega$ and $\omega^2 = 1$, every element $w$ of $\hat{W}$ is of the form $u_{abn} = \omega^n (w_1 w_2)^n w_2^b$ or $v_{abn} = \omega^n (w_2 w_1)^n w_1^b$ for some $a \in \{0, 1\}$, $b \in \{0, 1\}$, and $n \in \mathbb{Z}_{\geq 0}$. Computing $u_{abn}$ and $v_{abn}$ explicitly to be

$$u_{00n} = \begin{pmatrix} p^n & 0 \\ 0 & p^{-n} \end{pmatrix}, \quad u_{01n} = \begin{pmatrix} p^{-n} & p^n \\ 0 & 1 \end{pmatrix}, \quad u_{11n} = \begin{pmatrix} p^{-n} & 0 \\ 0 & p^n \end{pmatrix}, \quad u_{10n} = \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}, \quad v_{00n} = \begin{pmatrix} p^n & 0 \\ 0 & p^{-n} \end{pmatrix}, \quad v_{01n} = \begin{pmatrix} p^{-n} & p^n \\ 0 & 1 \end{pmatrix}, \quad v_{11n} = \begin{pmatrix} p^{-n} & 0 \\ 0 & p^n \end{pmatrix}, \quad v_{10n} = \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}.$$
we see that this parametrization of $\tilde{\varphi}$ and $\tilde{L}$ is unique except that $u_{a00} = v_{a00}$ for each $a \in \{0, 1\}$; furthermore, we can read off that $\mu(u_{abn}) = 2n + a$, that $\mu(v_{abn}) = 2(n + b) - a$, and that $\lambda(u_{abn}) = \lambda(v_{abn}) = 2n + b$. Thus

$$\sum_{w \in W} \mu(w) \lambda(w) = (1 + x) + \sum_{b=0,1} \sum_{n \geq 0} \sum_{a=0,1} t_{2n+b} \sum_{a=0,1} \left(x^{2n+a} + x^{2(n+b)-a}\right)$$

$$= (1 + x) + \sum_{b=0,1} \sum_{n \geq 0} \sum_{a=0,1} t_{2n+b} x^{2n+b-1} \sum_{a=0,1} \left(x^{1+a-b} + x^{1+b-a}\right)$$

$$= (1 + x) + (1 + x)^2 \sum_{m>0} t^m x^{m-1},$$

from which (44) follows upon summing the geometric series. We now combine (40), (42) and (44), noting that the series converge because $|\alpha| < p^{1/2}$ and $|\beta| < p^{1/2}$; the contributions to the formula (42) for $I_p$ of the two terms in the formula (40) for $\Phi_{\phi,p}$ are respectively

$$(p + 1)^{-1}(1 + p^{-1})^{-1} \frac{1 - p^{-1/2} \beta}{1 - \alpha} \frac{(1 + p^{-1/2} \alpha)(1 + p^{-1})}{1 - p^{-3/2} \alpha},$$

and

$$(p + 1)^{-1}(1 + p^{-1})^{-1} \frac{1 - p^{-1/2} \alpha}{1 - \beta} \frac{(1 + p^{-1/2} \beta)(1 + p^{-1})}{1 - p^{-3/2} \beta},$$

Summing these fractions by cross-multiplication and then simplifying, we obtain

$$I_p = p^{-1}(1 - p^{-1}) \frac{(1 + \alpha p^{-1/2})(1 + \beta p^{-1/2})}{(1 - \alpha p^{-3/2})(1 - \beta p^{-3/2})},$$

Recall the definition (39) of $\tilde{I}_p$. The local $L$-factors are given by (see [41, §3.1])

$$L_p(1, \text{ad } f) = \zeta_p(2), \quad L_p(1, \text{ad } \phi) = [(1 - \alpha^2 p^{-1})(1 - p^{-1})(1 - \beta^2 p^{-1})]^{-1},$$

$$L_p(\frac{1}{2}, \phi \times f \times f) = [(1 - \alpha p^{-1/2})(1 - \beta p^{-1/2})(1 - \alpha p^{-3/2})(1 - \beta p^{-3/2})]^{-1},$$

thus the normalized local integral $\tilde{I}_p$ is

$$\tilde{I}_p = p^{-1}(1 - p^{-1}) \frac{(1 + \alpha p^{-1/2})(1 - \beta p^{-1/2})(1 + \alpha p^{-1/2})(1 + \beta p^{-1/2})}{(1 - \alpha^2 p^{-1})(1 - p^{-1})(1 - \beta^2 p^{-1})} = p^{-1},$$

as asserted. □
5. Proof of Theorem 1.3

We combine Theorem 4.1 and Theorem 4.4 with Soundararajan’s weak subconvex bounds 39 to complete the proof of Theorem 1.3. Fix a positive even integer $k$. Let $f$ be a newform of weight $k$ and squarefree level $q$. Fix a Maass eigencuspform or incomplete Eisenstein series $\phi$. We will show that the “discrepancy”

$$D_f(\phi) := \frac{\mu_f(\phi)}{\mu_f(1)} - \frac{\mu(\phi)}{\mu(1)}$$

tends to 0 as $qk \to \infty$, thereby fulfilling the criterion of Lemma 1.10 by combining the complementary estimates for $D_f(\phi)$ provided below by Proposition 5.2 and Proposition 5.3.

Lemma 5.1. The quantities $M_f(x)$ and $R_f(x)$ 17 appearing in the statement of Theorem 3.1 satisfy the estimates

$$M_f(qk) \ll \epsilon \log(qk)^{1/6 + \epsilon} L(ad f, 1)^{1/2}, \quad R_f(qk) \ll \epsilon \frac{\log(qk)^{-1 + \epsilon}}{L(ad f, 1)} \ll \log(qk)\epsilon^2.$$

Proof. The bound for $M_f(qk)$ follows from the proof of [15, Lemma 3] with “$k$” replaced by “$qk$,” noting that $\lambda_f(p)^2 \leq 1 + \lambda_f(p^2)$ for all primes $p$. The bound for $R_f(qk)$ follows from the arguments of [39, Example 1], [15, Lemma 1] with “$k$” replaced by “$qk$” and the lower bound [8] for $L(ad f, 1)$. 

Proposition 5.2. We have $D_f(\phi) \ll_{\phi, \epsilon} \log(qk)^{1/12 + \epsilon} L(ad f, 1)^{1/4}$.

Proof. Follows immediately from Theorem 5.1 and Lemma 5.2. 

Proposition 5.3. We have $D_f(\phi) \ll_{\phi, \epsilon} \log(qk)^{-\delta + \epsilon} L(ad f, 1)^{-1}$, where $\delta = 1/2$ if $\phi$ is a Maass eigencusp-form and $\delta = 1$ if $\phi$ is an incomplete Eisenstein series.

Proof. If $\phi$ is a Maass eigencuspform, then the analytic conductor of $\phi \times f \times f$ is $\asymp (qk)^4$, so Theorem 4.4 and the arguments of Soundararajan [39, Example 2] with “$k$” replaced by “$qk$” show that

$$\left| \frac{\mu_f(\phi)}{\mu_f(1)} \right|^2 \ll \epsilon \frac{L(\phi \times f \times f, \frac{1}{2})}{qk \cdot L(ad f, 1)^2} \ll \epsilon \frac{1}{\log(qk)^{1-\epsilon} L(ad f, 1)^2}.$$

If $\phi = E(\Psi, \cdot)$ is an incomplete Eisenstein series, then the unfolding method as in Lemma 3.3 and the bound for $R_f(q)$ given by Lemma 5.1 show that

$$\frac{\mu_f(\phi)}{\mu_f(1)} - \frac{\mu(\phi)}{\mu(1)} = 2\pi^2 \int_{(1/2)} \Psi^\wedge(s) \left( \frac{s}{4\pi} \right)^k \Gamma(s + k - 1) \zeta(s) L(ad f, s) \frac{ds}{\Gamma(k)} \zeta(2s) L(ad f, 1) 2\pi i \ll \epsilon \frac{\log(qk)^{-1+\epsilon}}{L(ad f, 1)}.$$

Proof of Theorem 1.3. By Propositions 5.2 and 5.3 there exists $\delta \in \{1/2, 1\}$ such that

$$D_f(\phi) \ll_{\phi, \epsilon} \min \left( \log(qk)^{-\delta + \epsilon} L(ad f, 1)^{-1}, \log(qk)^{1/12 + \epsilon} L(ad f, 1)^{1/4} \right);$$

it follows by the argument of [15, §3] with “$k$” replaced by “$qk$” that $D_f(\phi) \to 0$ as $qk \to \infty$. 

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