Dispersion of Elastic Waves Propagating in a Fluid-Saturated Porous Medium with Cavities

V I Erofeev\textsuperscript{a}, A V Leonteva\textsuperscript{b}

Mechanical Engineering Research Institute of the Russian Academy of Sciences – Branch of Federal Research Center “Institute of Applied Physics of the RAS”, Nizhny Novgorod, Russian Federation

E-mail: \textsuperscript{a}erof.vi@yandex.ru, \textsuperscript{b}aleonav@mail.ru

Abstract. The propagation of plane longitudinal waves in a fluid-saturated porous medium with cavities is considered in linear elastic approximation. It has been shown that as distinct from the classical porous medium (Biot's medium) where two longitudinal dispersionless waves can propagate (a fast wave and a slow wave), in the porous-cavity medium three dispersive longitudinal waves propagate and one of them is solely induced due to cavities in the material. The study of waves is carried out by obtaining and analyzing the dispersion equation, phase velocity and group velocity. The frequency spectrum density is considered, which enables to identify the degree of dispersion expressiveness. At certain values of the system parameters some regions of strong and weak dispersion, regions of normal and anomalous dispersion have been identified.

1. Introduction
Mathematical models of deformable porous media are widely used for studying processes in geophysics, mechanics of natural and man-made composites. It is conventional to use in calculations both classical models going back to the works of M.A. Bio \cite{1, 2}, Ya.I. Frenkel \cite{3}, F. Gassman \cite{4}, L.Ya. Kosachevsky \cite{5} and their later modifications \cite{6-9}.

The fundamentals of mechanics and physics of porous materials are stated in monographs \cite{6-11}.

The study of wave processes in porous materials is dealt with, in particular, in papers \cite{8, 9, 12-16}.

International multidisciplinary conferences on mechanics of porous materials (“Poromechanics”) have been regularly held since \cite{17-22}.

It has been previously noted \cite{23} that when testing materials numerous fluid-filled cavities are frequently occurred therein in addition to porosity. It determines the relevance of studying porous-cavity solid media.

2. Mathematical Model
The equations describing the dynamics of a porous fluid-saturated medium with fluid-filled cavities in Lagrangian coordinates are shown in \cite{23}. It is implied that pores are cylindrically shaped and cavities are spherical. Such a medium is a continuum with inner degrees of freedom, since under the elastic wave action the cavities will oscillate exerting in turn an effect on the wave propagation.

If it is assumed that there is no viscosity in this system, all types of nonlinearity (geometric, physical, cavity one) are neglected and a plane longitudinal wave propagating along the $x_3$ axis is further considered, the three-dimensional system of equations is transformed into one-dimensional and takes the form:
\[
\begin{align*}
\rho_{11} \frac{\partial^2 u_1}{\partial t^2} + \rho_{12} \frac{\partial^2 v_1}{\partial t^2} - (\lambda_2 + 2\mu) \frac{\partial^2 u_1}{\partial x_3^2} - Q \frac{\partial^2 v_1}{\partial x_3^2} + N(\lambda_2 + 2\mu) \frac{\partial v_1}{\partial x_3} &= 0 \\
\rho_{22} \frac{\partial^2 v_1}{\partial t^2} + \rho_{12} \frac{\partial^2 u_1}{\partial t^2} - Q \frac{\partial^2 u_1}{\partial x_3^2} - R \frac{\partial^2 v_1}{\partial x_3^2} &= 0 \\
\frac{\partial^2 v}{\partial t^2} + \omega^2 v - \frac{4\pi\rho_2}{\rho_0} \left[ (\lambda_2 + 2\mu) \left( \frac{\partial u_1}{\partial x_3} - N v \right) + \frac{\partial}{\partial x_3} (Q v_1 + Ru_1) \right] &= 0
\end{align*}
\]

where \( u_1, v_3 \) are the displacement of the solid and liquid phases, respectively, along the direction of the \( x_3 \) axis; \( v \) is the volume of the cavity disturbed by the wave; \( \rho_{11}, \rho_{22} \) are effective initial mass densities of the solid and liquid phases, respectively; \( \rho_{11} = \rho_1 - \rho_{12}, \rho_{22} = \rho_2 - \rho_{12}, \rho_{12} < 0 \), \( \rho_1, \rho_2 \) are masses of the solid and liquid phases, respectively; \( \rho_0 \) is the added mass density; \( \lambda_2, \mu \) are Lame’s coefficients; \( Q, R \) are known coefficients characterizing a fluid-saturated medium, their values are given in [10]; \( N \) is the number of cavities per unit of volume; \( \omega_0 \) is resonant frequency, \( \omega^2 = (3\rho_0 + 4\mu) / (r_0^2 \rho_0) \); \( r_2 \) is the radius of cavities; \( \rho_0 \) is the solid phase density; \( p_0 \) is the initial pressure in the intra-cavity fluid; \( \gamma \) is the adiabatic exponent of the fluid.

3. Plane Longitudinal Wave Dispersion

We try the solution for the system of linear differential equations (1) in the form of monochromatic waves

\[
u_3 = u_3^0 \exp(i\omega - ikx), \quad v_3 = v_3^0 \exp(i\omega - ikx), \quad v = v^0 \exp(i\omega - ikx)
\]

where \( u_3^0, v_3^0, v^0 \), are complex amplitudes.

The insertion of the said exponents into (1) enables us to obtain the dispersion relation

\[
-\rho_0 \rho_0 \omega^6 + \left( \rho_1 \rho_0 \omega_0^6 + n \right) + \rho_0 Q \left( \mu k^2 + \left( -\rho_0 \omega^2 q_1 + n q_0 \right) k^2 + \rho_0 q_0 k^4 \right) + \rho_0 Q \left( -\rho_0 \omega^2 q + n R^2 \right) k^4 = 0
\]

binding the frequency (\( \omega \)) and wave number (\( k \)) of a plane longitudinal wave and having the sixth order for \( \omega \) and the fourth order for \( k \).

The following designations are introduced in order to shorten the notation (2)

\[
\rho = \rho_{11} \rho_{22} - \rho_{12}^2, \quad \kappa = \lambda_2 + 2\mu, \quad q_1 = \rho_{11} R + \rho_{22} \kappa - 2\rho_{12} Q, \quad q_2 = \rho_{12} Q - (\rho_{11} - \rho_{12}) R, \quad q = Q^2 - \kappa R, \quad n = 4\pi \rho_2 \kappa \kappa.
\]

We obtain for dimensionless frequency of \( \bar{\omega} = \frac{\omega}{\omega_s} \) and wave number \( \bar{k} = \frac{k}{\omega_s / c} \) (having chosen the resonant frequency and the ratio of the resonance frequency to the propagation speed of an elastic longitudinal wave in water as characteristic values of frequency and wave number) we obtain the dispersion relation (2) in dimensionless form (the line is omitted):

\[
\omega^6 - \left( 1 + a_1 + a_2 k^2 \right) \omega^4 + \left( (a_2 - a_1 a_4) k^2 - a_2 k^4 \right) \omega^2 + (a_3 + a_4 a_5) k^4 = 0,
\]

here \( a_i (i = 1...5) \) are dimensionless coefficients

\[
a_1 = \frac{n}{\rho_0 \omega^2}, \quad a_2 = \frac{q_1}{\rho c^2}, \quad a_3 = \frac{q}{\rho c^4}, \quad a_4 = \frac{q_2}{\rho c^2}, \quad a_5 = \frac{R^2}{\rho c^4}.
\]

The analysis of equation (3) has shown that the dispersion curves described thereby have three branches. Two of them come from the origin of coordinates and one from the points \( \left( 0, \pm \sqrt{1 + a_i} \right) \). The curves have two horizontal asymptotes \( \omega = \pm \sqrt{1 + a_i a_3} / a_3 \) and four oblique asymptotes passing
through the origin of coordinates, oblique asymptotes $\omega = \pm b_2 k$. Here $b_2 = \frac{\sqrt{2}}{2} \sqrt{a_2 \pm \sqrt{a_2^2 + 4a_3}}$. The curves have no vertical asymptotes.

We find from the dispersion equation (3) the dependence of the phase velocity ($v_{ph} = \frac{v}{k}$) on the wave number

$$v_{ph}^6 - \left( a_2 + \frac{1 + a_3}{k^2} \right) v_{ph}^4 - \left( a_3 - a_2 a_4 - a_2 \right) v_{ph}^2 + \frac{a_2 + a_4 a_3}{k^2} = 0. \quad (4)$$

It is seen from (4) that at $k \to 0$ one of the values of the phase velocity tends to infinity (i.e., the process ceases to be a wave process) and the other two values are determined from the relation

$$v_{ph} = \sqrt{\frac{a_2 - a_4 a_3}{2(1 + a_1)}} + \sqrt{\frac{a_2 - a_4 a_3}{2(1 + a_1)}} \frac{a_4 + 2a_3}{1 + a_1}.$$

The curves describing the phase velocity determined by dependence (4) have six horizontal asymptotes, including two zero ones, and two vertical zero asymptotes: $v_{ph} = \pm b_2$, $v_{ph} = 0$, $k = 0$. There are no oblique asymptotes.

Group velocity ($v_{gr} = \frac{d\omega}{dk}$) is assigned by the ratio:

$$v_{gr} = -\frac{k(\omega^4 + 2(2\omega^2 + a_1 a_4 - a_2)\omega^2 - 2(a_3 + a_4 a_3 - a_2)^2)}{\omega(-3\omega^4 + 2(a_2 k^2 + a_1 + 1)\omega^2 + a_3 k^2 - (a_1 a_4 - a_2)^2)}.$$

where $\omega$ is determined from the dispersion equation (3). It is seen from (5) that at $k \to 0$ one of the values of the group velocity tends to zero (i.e., the energy is not transferred, the process ceases to be a wave process) and the other two values take corresponding values of the phase velocities. The curves determined by dependence (5), which describe the group velocity, have six horizontal asymptotes, two of them are zero: $v_{gr} = \pm b_2$, $v_{gr} = 0$. There are no oblique and vertical asymptotes.

For the analysis convenience we represent the dependences $\omega(k)$, $v_{ph}(k)$, $v_{gr}(k)$ in the same axes (Fig. 1). The symmetry of the graphs for these functions relative to the coordinate axes herein makes it sufficient to display them only in the first quarter. The solid line shows in the Figure the dependence $\omega(k)$, the long dotted line indicates $v_{ph}(k)$, the dash-dotted line denotes $v_{gr}(k)$, the asymptotes of the curves are marked with the short dotted line.

It can be seen from the Figure that the first branch of the dispersion curve is originated at the frequency $\omega = \omega_0 = 0$. At small wave numbers ($k < 1$), i.e. for long waves ($\lambda >> 1$) the frequency tends to the value $\omega = b_2 k$, at infinity the frequency tends to the value $\omega = b_2 k$. The phase and group velocities of the respective branches tend to the value $v = b_2$ from different sides. The second branch of the dispersion curve originates at the frequency $\omega = \omega_0 \neq 0$. At infinity the frequency tends to the value $\omega = b_2 k$. The respective branches of the curves of the phase and group velocities tend to the common asymptote $v_{ph} = v_{gr} = b_2$ from different sides. The third branch of the dispersion curve, like the first branch, is originated at the frequency $\omega = \omega_0 = 0$. At infinity the frequency tends to the value $\omega = (1 + a_3 a_5)/a_j$, the values of the phase and group velocities tend to zero. The group velocity in the interval $(0, +\infty)$ is less than the phase velocity; it proves the normal dispersion at selected values of the system parameters.
An important characteristic of dispersive waves is the spectral distribution density calculated by the following formula [24]

$$\rho(\omega) = \text{const} \cdot \frac{dk}{d\omega}.$$ 

Density represents up to a multiplier the value inverse to the group velocity. In this case the constant in the last formula can be chosen equal to one, since it will not affect the quality of the curve. For each branch of the dispersion curve determined from (3) the spectral distribution density is shown in Fig. 2.

It is seen from the Figure that the spectrum becomes denser and continuous at $\omega \to \sqrt{1 + a_1 a_2 / a_3} - 0$. The natural frequency spectrum is $\omega \in \left[0, \sqrt{1 + a_1 a_2 / a_3}\right]$. It means that for a slow wave (the third branch in Fig. 1) dispersion occurs at high frequencies. It is also seen that the frequency spectrum becomes continuous at $\omega \to \sqrt{1 + a_1} + 0$. The natural frequency spectrum is within the range $\left(\sqrt{1 + a_1}, +\infty\right)$. So, in a fast wave caused by the availability of cavities in the medium, the dispersion occurs at low frequencies. A slight increase in the spectral distribution density for the first branch of the dispersion curve corresponding to a fast wave resulting from the porosity of the medium corresponds to an insignificant dispersion. The natural frequency spectrum is $\omega \in \left[0, +\infty\right)$.

4. Parametric Analysis of Dispersion Characteristics

The dynamics of dispersion curves of a plane longitudinal wave at a change in the parameter $a_2$ is shown in Fig. 3.

It is seen that the inner branches losing their smoothness tend to zero. The clear separation of the dispersion curves into three pairs occurs at the origin of coordinates. One of them begins to stretch along the axis $x$, the other changes its shape taking the form of a broken line touches the third one. The discontinuity of the dispersion curve occurs here at the point of contact. The further increase in the parameter results in the disappearance of one branch from the real plane, while the other two branches having separated into two common parts continue to move away from each other.
Fig. 3. Dynamics of the dispersion curve upon changing the parameter $a_5$. 
With the parameter value $a_5 = a_5^*$, $a_5^* = a_3 + (a_2 + a_4) b_5^2$ the dispersion curve discontinues. The coordinates of the discontinuity point $(k, \omega)$ are found from the condition of the intersection of curve (3) with its asymptote or from the condition of the intersection of two branches of the dispersion curve. The dispersion curve after discontinuity is shown in Fig. 4.

The dispersion curve still has three branches but their form has qualitatively changed. A branch with two generation frequencies and a "free" branch emerged. The curve retains its linear asymptotics at infinity. The sections of the first and second branches of the dispersion curve "overjumped" the asymptote $\omega = b_1 k$. The group velocity curve acquires asymptotes, the analytical form thereof cannot be found. The dispersion became anomalous on those sections of the dispersion curve which have changed their position relative to the asymptote $\omega = b_1 k$.

The frequency spectrum (Fig. 5) becomes continuous when the frequency tends to four critical frequencies: $\omega \rightarrow \sqrt{1 + a_1} = 0$, $\omega \rightarrow \sqrt{1 + a_1} + 0$ and frequencies near thereof the dispersion curves acquire peaks.

**Conclusion**

Thus, the paper states that:

1) Three longitudinal waves propagate in the porous medium with cavities; two of them (a fast wave and a slow wave) are generated due to the fluid-saturated porosity and the third one due to cavities available in the material. That is the principal distinction of the considered medium from the porous medium (Biot’s medium) including two longitudinal waves (a fast wave and a slow wave) and the classical elastic medium with one longitudinal wave.

2) The process of wave propagation in porous medium with cavities is dispersing. Its feature is the dispersion of fast waves in the low-frequency range and the dispersion of slow waves in the high-frequency range.

3) There were identified the parameters of the system, where the dispersion of longitudinal waves is anomalous.
The work was supported by the Russian Science Foundation (Grant No 20-19-00613).

References
[1] Biot M A 1941 General theory of three-dimentional consolidation. J. Appl. Phys. 12, No. 1, pp. 155-164
[2] Biot M A 1962 Mechanics of deformation and acoustical propagation in porous media. J. Appl. Phys. 33, No. 10, pp.1482-1498
[3] Frenkel' Ia I 1944 K teorii seismicheskikh i seismoelektricheskikh iavlenii vo vlazhnoi pochve [To the theory of the seismic and seismoelectric phenomena in the damp soil]. Izvestiia AN SSSR. Seriia geograficheskaia i geofizicheskaia 8, No. 4, pp. 133-149
[4] Gassmann F 1951 Uber die elastizatat poroser medien. Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich 96, pp.1-23
[5] Kosachevskii L Y 1959 O rasprostranenii uprugikh voln v dvukhkomponentnykh sredakh [On the propagation of elastic waves in two-component media]. PMM 23(6) pp.1115-1123
[6] Nikolaevskii V N, Basniev A T, Gorbunov A T and Zotov G A 1970 Mekhanika poristykh nasyshchenykh sred [Mechanics of porous saturated environments]. Moscow, Nedra p 335
[7] Nigmatulin R I 1978 Osnovy mekhaniki heterogenykh sred [Bases of mechanics of heterogeneous environments]. Moscow, Nauka p 336
[8] Bourbie T, Coussy O and Zinzner B 1987 Acoustics of Porous Media. Houston, Gulf Publishing Co p 334
[9] Nikolaevskiy V N 1996 Geomechanics and Fluidodynamics. Kluwer Academic Publishers, Netherlands p 477
[10] Coussy O 2004 Poromechanics. Wiley p 312
[11] Coussy O 2010 Mechanics and Physics of Porous Solids. Wiley p 282
[12] Leclario F, Cohen-Tenou djy and Aguirre Puente Y 1994 Extension of Boit’s theory of waves propagation to frozen porous media. J. Acoust. Soc. Amer. 96, No. 6, pp. 3753-3768
[13] Schanz M 2001 Wave Propagation in Viscoelastic and Poroelastic Continua: A Boundary Element Approach. Springer, Berlin p 170
[14] Markov M G 2005 Propagation of longitudinal elastic waves in a fluid-saturated porous medium with spherical inclusions. Acoustical Physics. 51(1, Supplement), pp. S115-S121
[15] Markov M G 2006 Rayleigh wave propagation along the boundary of a non-Newtonian fluid-saturated porous medium. Acoustical Physics 52, No. 4, pp. 429-434
[16] Markov M G, Markova I A, Jarillo G F R and Pervago E V 2020 Propagation of elastic compressional waves in a porous-fractured medium saturated with immiscible fluids. Izvestiya. Physics of the Solid Earth 56, No 3, pp. 357-363
[17] Poromechanics – A Tribute to Maurice A Biot, Proceedings, Biot Conference on Poromechanics / Thimus, J-F, Abousleiman, Y, Cheng, A H-D, Coussy, O, and Detournay, E (eds.). A.A. Balkema, Rotterdam, Brookfield, 1998 p 648
[18] Poromechanics II / Auriault J-L, Geindrean C, Royer P, Bloch J-F, Boutin C, Lewandowska L (eds.). A.A. Balkema, Rotterdam, Brookfield, 2002 p 955
[19] Poromechanics III / Abousleiman Y N, Cheng A H-D, Ulm F-J (eds.). A.A. Balkema, Leidon, London, New York, Phyledelphia, Singapore, 2005 p 828
[20] Poromechanics IV / Ling H I, Smyth A, Betti R (eds.). DEStech Publications. Inc., PA, USA, 2009 p 1151
[21] Poromechanics V / Hellmich C, Pichler B, Adam D (eds.). ASCE, 2013 (CD)
[22] Poromechanics VI / Vandamme M, Dangla P, Pereira J-M, Grabezloo S (eds.). ASCE, 2017 (CD)
[23] Bagdoev A G, Erofeyev V I and Shekoyan A V 2016 Wave Dynamics of Generalized Continua. Springer-Verlag, Berlin Heidelberg p 274
[24] Rabinovich M I and Trubetskoy D I 1984 Vvedenie v teoriu kolebanii i voln [Introduction to the theory of fluctuations and waves]. Moscow, Nauka p 432