WHITTAKER MODULES FOR $\hat{\mathfrak{g}}l$ AND $W_{1+\infty}$–MODULES WHICH ARE NOT TENSOR PRODUCTS

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Abstract. We consider the Whittaker modules $M_1(\lambda, \mu)$ for the Weyl vertex algebra $M$ (also called $\beta\gamma$ vertex algebra), constructed in [4], where it was proved that these modules are irreducible for each finite cyclic orbifold $M^\mathbb{Z}_n$. In this paper, we consider the modules $M_1(\lambda, \mu)$ as modules for the $\mathbb{Z}$–orbifold of $M$, denoted by $M_0$. $M_0$ is isomorphic to the vertex algebra $W_{1+\infty},c=-1=M(2)\otimes M(1)$ which is the tensor product of the Heisenberg vertex algebra $M(1)$ and the singlet algebra $M(2)$ (cf. [2], [14], [19]). Furthermore, these modules are also modules of the Lie algebra $\hat{\mathfrak{g}}l$ with central charge $c = -1$. We prove that they are reducible as $\hat{\mathfrak{g}}l$–modules (and therefore also as $M_0$–modules), and we completely describe their irreducible quotients $L(d, \lambda, \mu)$. We show that $L(d, \lambda, \mu)$ in most cases are not tensor product modules for the vertex algebra $M(2)\otimes M(1)$. Moreover, we show that all constructed modules are typical in the sense that they are irreducible for the Heisenberg-Virasoro vertex subalgebra of $W_{1+\infty},c=-1$.

Contents

1. Introduction 2
Free field realization of $L(d, \lambda, \mu)$. 3
Typical $W_{1+\infty},c$–modules 4
A connection with Whittaker $\mathfrak{g}l(2\ell)$–modules. 4
2. Preliminaries 5
2.1. Classical Whittaker modules 5
2.2. Generalized Whittaker modules 5
2.3. Example: $\mathfrak{g} = \mathfrak{g}l(2\ell, \mathbb{C})$ 6
2.4. Whittaker modules for $\mathfrak{g} = \hat{\mathfrak{g}}l$ 6
3. Weyl vertex algebra and its Whittaker modules 7
4. $W_{1+\infty}$-algebra at central charge $c = -1$ and its Whittaker modules 10
4.1. The $W_{1+\infty}$-algebra approach 10
4.2. Approach using the Lie algebra $\hat{\mathfrak{g}}l$ 10
4.3. Connection between $\hat{\mathfrak{g}}l$–modules and $W_{1+\infty}$–modules 11
5. $M_1(\lambda, \mu)$ as a Whittaker $\hat{\mathfrak{g}}l$–module 11
6. The structure of the Whittaker module $M_1(\lambda, \mu)$ as a $\hat{\mathfrak{g}}l$–module 14
7. Whittaker vectors in $M_1(\lambda, \mu)$ 16
8. Bosonic realisations of $M_1(\lambda, \mu)$ and $L(d, \lambda, \mu)$ 18
8.1. Realisation of $M_1(\lambda, \mu)$ from $\Pi(0)$–modules 18

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1. Introduction

The representation theory of Weyl algebras provides an important tool in the construction of modules for finite and infinite-dimensional Lie algebras, and vertex algebras. Let $\hat{A}$ denote the Weyl algebra, and let $M$ denote the associated Weyl vertex algebra ($\beta\gamma$–system) of rank one. In [4], we have proved that irreducible Whittaker modules $M_1(\lambda, \mu)$ for the Weyl vertex algebra $M$ remain irreducible when we restrict them to the $\mathbb{Z}_n$–invariant subalgebra $M^{\mathbb{Z}_n}$. In this paper, we consider a limit case and restrict these modules to a $\mathbb{Z}$–invariant subalgebra $M^0$. Surprisingly, we prove that the irreducible Weyl modules of Whittaker type are always reducible as $M^0$–modules.

However, the analysis of these reducible modules leads to many interesting observations and new constructions.

First, we recall that the $\mathbb{Z}$–invariant subalgebra $M^0$ of the rank one Weyl vertex algebra $M$ is isomorphic to the simple vertex algebra $W_{1+\infty, c}$ at central charge $c = -1$ (cf. [14], [19]). Thus every Whittaker module for the Weyl vertex algebra is a module for $W_{1+\infty, c}$. On the other hand, the representation theory of the vertex algebra $W_{1+\infty, c}$ is deeply connected with the representation theory of the Lie algebra $\hat{D}$, which is a central extension of the Lie algebra of infinite matrices (cf. [13], [14]). More precisely, the universal vertex algebra $W_{1+\infty}^c$ is the vacuum module of central charge $c$ for the Lie algebra $\hat{D}$, which is the central extension of the Lie algebra of complex regular differential operators on $\mathbb{C}^\ast$. Moreover, there exists a homomorphism $\Phi_0 : \hat{D} \rightarrow \hat{gl}$, defined by (4.2); hence each $\hat{gl}$–module of central charge $c$ is a $\hat{D}$–module of the same central charge. If the resulting module is restricted, we get a $W_{1+\infty}^c$–module. Using the Weyl (vertex) algebras, one can construct $\hat{gl}$–modules which become modules for the simple vertex algebra $W_{1+\infty, c}$. Given the Lie homomorphism $\hat{gl} \rightarrow \hat{A}$ defined by

$$E_{i,j} \mapsto a(-i)a^\ast(j); \quad C \mapsto -1,$$

every restricted $\hat{A}$–module has the structure of a $\hat{gl}$–module, and one shows that they are also $W_{1+\infty, c} = -1$–modules.

Similarly as in the case of the quasi-finite modules from [14], we show in Theorem 5.2 that the structure of the module $M_1(\lambda, \mu)$ as a $W_{1+\infty, c} = -1$–module is equivalent to the structure of the same module as a $\hat{gl}$–module.
As far as we have noticed, Whittaker modules for the Lie algebra $\hat{\mathfrak{gl}}$ and for the vertex algebra $W_{1+\infty}$ haven’t been investigated in the literature yet.

Hence our main task is to analyze $M_1(\lambda, \mu)$ as a $\hat{\mathfrak{gl}}$-module. This will directly imply the structure of $M_1(\lambda, \mu)$ as a $W_{1+\infty}$-module. We prove:

- $M_1(\lambda, \mu)$ is a cyclic $\hat{\mathfrak{gl}}$-module of Whittaker type for the Whittaker pair $(\hat{\mathfrak{gl}}, \mathfrak{p})$, where $\mathfrak{p}$ is a certain commutative subalgebra of upper triangular matrices of $\hat{\mathfrak{gl}}$. Thus, $M_1(\lambda, \mu)$ is not a classical Whittaker module for $\hat{\mathfrak{gl}}$, but a generalized Whittaker module in the sense of [7].
- In order to see that $M_1(\lambda, \mu)$ is not irreducible, we use the Casimir element $I = \sum_i E_{i,i}$ of $U(\hat{\mathfrak{gl}})$. We show that $I$ acts non-trivially on $M_1(\lambda, \mu)$.
- We prove that $\mathbb{C}[I]w_{\lambda, \mu}$ provides all Whittaker vectors in $M_1(\lambda, \mu)$ and that for every $d \in \mathbb{C}$, the quotient
  \[ L(d, \lambda, \mu) = \frac{M_1(\lambda, \mu)}{(I-d)\mathbb{C}[I]w_{\lambda, \mu}} \]
  is simple both as a $\hat{\mathfrak{gl}}$-module and as $W_{1+\infty, c = -1}$-module.

**Free field realization of $L(d, \lambda, \mu)$**. Note that modules $L(d, \lambda, \mu)$ are constructed as quotients of Whittaker Weyl modules. A natural question is to provide an explicit realization of these simple modules.

There exist another realization of Whittaker $W_{1+\infty, c}$-modules which are based on the fact that $W_{1+\infty, c}$ can be embedded into Heisenberg vertex algebra for integral central charges. Let $M_n(1)$ denotes the Heisenberg vertex algebra generated by $n$ Heisenberg fields.

- $W_{1+\infty, c = 1}$ is isomorphic to $M_1(1)$.
- $W_{1+\infty, c = n}$ is isomorphic to the principal $\mathfrak{w}$-algebra $W_1(\mathfrak{gl}(n), f_{\text{princ}})$ which is a vertex subalgebra of $M_n(1)$. (cf. [10]).
- $W_{1+\infty, c = -1}$ is isomorphic to the tensor product $\mathcal{M}(2) \otimes M_1(1)$, where $\mathcal{M}(2)$ is the singlet vertex algebra at central charge $-2$ (isomorphic to the simple $W(2,3)$-algebra at $c = -2$) and $M_1(1)$ is rank one Heisenberg vertex algebra (cf. [19]). In particular, $W_{1+\infty, c = -1}$ is embedded into $M_2(1)$.
- In general, $W_{1+\infty, c = -n}$ is embedded into $M_{2n}(1)$ (cf. [1]).

To the best of our knowledge, the following problem is still unsolved:

**Problem 1.1.** For embedding $W_{1+\infty, c} \hookrightarrow M_n(1)$ mentioned above, identify Whittaker $M_n(1)$-modules as Whittaker $W_{1+\infty, c}$-module.

We will be focused on the case $c = -1$. Then $W_{1+\infty, c = -1}$ is a subalgebra of $M_2(1)$ and every Whittaker $M_2(1)$-module is naturally Whittaker $W_{1+\infty, c = -1}$-module. Then by using the isomorphism $W_{1+\infty, c = -1} \cong \mathcal{M}(2) \otimes M_1(1)$ one can realize a family of Whittaker modules as the tensor product modules

\[(1.1) \quad Z_1 \otimes Z_2 \]

where $Z_1$ is an irreducible Whittaker $\mathcal{M}(2)$-module and $Z_2$ is an irreducible Whittaker $M_1(1)$-module. K. Tanabe showed in [18] that every Whittaker $\mathcal{M}(p)$-module is obtained from Whittaker modules for the Heisenberg vertex algebra. Therefore, we have a family
of irreducible $W_{1+\infty,c=-1}$-modules obtained using free-field realization from the Whittaker $M_2(1)$–modules.

**Question 1.2.** Can we realize $L(d,\lambda,\mu)$ as irreducible Whittaker $M_2(1)$–modules? This is equivalent to the question can we represent $L(d,\lambda,\mu)$ as the tensor product modules (1.1)?

In Section 8 we find a direct realization of $L(d,\lambda,\mu)$ as $M_2(1)$–modules in the special cases when $M_1(\lambda,\mu)$ has the structure of a module for the vertex algebra $\Pi(0)$ (cf. Proposition 8.2). Then the Casimir element $I$ can be seen as an element of the vertex algebra $\Pi(0)$.

But in general $L(d,\lambda,\mu)$ does not have form (1.1). The reason is that $L(d,\lambda,\mu)$ contains vectors which are not locally finite for the action the element $J^0(k)$, for $k > 0$, which is a necessary condition for a $W_{1+\infty,c=-1}$–module to have the form (1.1). Therefore modules $L(d,\lambda,\mu)$ are irreducible modules of a completely new type.

We can slightly generalize these examples by applying the spectral-flow automorphism $\rho_s$ of the Weyl algebras, which induces the spectral flow automorphism $\tilde{\rho}_s$ of the Lie algebra $\hat{g}_l$.

Let us summerize:

**Theorem 1.3.** The vertex algebra $W_{1+\infty,c=-1}$ has the following two families of irreducible modules of Whittaker types:

1. Tensor product modules $Z_1 \otimes Z_2$, where $Z_1$ is a Whittaker modules for the singlet algebra $M(2)$ (cf. [18]) and $Z_2$ is a Whittaker module for the Heisenberg vertex algebra $M_1(1)$.
2. Modules $\tilde{\rho}_s(L(d,\lambda,\mu))$ which are irreducible quotients of the Weyl Whittaker modules $\rho_s(M_1(\lambda,\mu))$.

Moreover, the following statements are equivalent (cf. Propositions 8.2, 8.3):

(a) Module in (2) is of type (1);
(b) $M_1(\lambda,\mu)$ has the structure of a $\Pi(0)$–module;
(c) $\lambda = (\lambda_0,0,\cdots)$, $\lambda_0 \neq 0$.

**Typical $W_{1+\infty,c}$-modules.** Weak modules for the vertex operator algebra $W_{1+\infty,c}$ can be naturally divided in two classes: typical and atypical. We say that a weak $W_{1+\infty,c}$–module is typical if it is irreducible as a module for the Heisenberg-Virasoro subalgebra of $W_{1+\infty,c}$, which we denote by $L^{HVir}_c$. In the case $c = -1$, one can construct typical/atypical highest weight $W_{1+\infty,c}$–modules by using typical/atypical modules for the singlet vertex algebra (cf. [2, 6, 19, 8]).

Tensor product modules in Theorem 1.3(1) are typical since Whittaker $M(2)$–modules are realized on irreducible, Whittaker Virasoro modules (cf. [18]). In Theorem 9.4 in Appendix we prove typicality of our new series of modules:

**Theorem 1.4.** Modules $L(d,\lambda,\mu)$ are typical $W_{1+\infty,c=-1}$–modules.

**A connection with Whittaker $gl(2\ell)$-modules.** The study of Whittaker modules $M_1(\lambda,\mu)$ as $\hat{g}_l$–modules is motivated by [4] and the orbifold theory of vertex operator algebras. As a result we get a nice family of Whittaker modules for the Lie algebra $\hat{g}_l$ and describe a complete set of Whittaker vectors. Since one can embed finite-dimensional Lie algebra $gl(n)$ into $\hat{g}_l$, it is a natural question what is $gl(n)$–version of the Whittaker modules $M_1(\lambda,\mu)$. It
turns out that the theoretical background for these modules was given in [7] in the context of generalized Whittaker modules. But the paper [7] only presented a construction of universal generalized \(\mathfrak{gl}(n)\)-modules, and it seems the corresponding simple quotients haven’t been investigated before our paper. Because of simplicity we consider here the \(\mathfrak{gl}(2\ell)\)-modules \(W(\alpha, \beta)\), where \(\alpha, \beta \in \mathbb{C}^\ell\), which are quotients of universal generalized Whittaker modules from [7]. We present this construction in Section 9. We describe the structure of \(W(\alpha, \beta)\) and obtained a complete family of Whittaker vectors. We prove that if \(\alpha, \beta \neq 0\), then
\[
L(d, \alpha, \beta) = \frac{W(\alpha, \beta)}{(I - d)\mathbb{C}[I\mathfrak{w}_{\alpha, \beta}]}
\]
is a simple generalized Whittaker \(\mathfrak{gl}(2\ell)\)-module.

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2. Preliminaries

2.1. Classical Whittaker modules. For the structural theory of Whittaker modules for Whittaker pairs see [7].

Assume that \(\mathfrak{g}\) is a Lie algebra with triangular decomposition \(\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+\).

Let \(\mathfrak{w}\) be a \(\mathfrak{g}\)-module. Vector \(v \in \mathfrak{w}\) is called a Whittaker vector if there is a Lie functional \(\lambda : \mathfrak{n}_+ \to \mathbb{C}\) such that \(xv = \lambda(x)v\) for all \(x \in \mathfrak{n}_+\).

Let \(\lambda : \mathfrak{n}_+ \to \mathbb{C}\) be a Lie functional. Let \(\mathbb{C}v_\lambda\) be a 1-dimensional \(\mathfrak{n}_+\)-module such that \(xv_\lambda = \lambda(x)v_\lambda\) for all \(x \in \mathfrak{n}_+\). Then
\[
M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{n}_+)} \mathbb{C}v_\lambda,
\]
is a \(U(\mathfrak{g})\)-module, which is called the universal (or standard) Whittaker module.

2.2. Generalized Whittaker modules. Let \(\mathfrak{g}\) be a Lie algebra, and \(\mathfrak{n}\) any nilpotent subalgebra of \(\mathfrak{g}\). Whittaker modules with respect to \(\mathfrak{n}\) are sometimes called generalized Whittaker modules, or Whittaker modules with respect to the pair \((\mathfrak{g}, \mathfrak{n})\). For example, we can take \(\mathfrak{n}\) to be any commutative subalgebra of \(\mathfrak{n}_+\).

This makes this situation more general to the previous one where the nilpotent subalgebra is exactly \(\mathfrak{n}_+\).

Let \(\mathfrak{w}\) be a \(\mathfrak{g}\)-module. Vector \(v \in \mathfrak{w}\) is called Whittaker vector for the pair \((\mathfrak{g}, \mathfrak{n})\) if there is a Lie functional \(\lambda : \mathfrak{n} \to \mathbb{C}\) such that \(xv = \lambda(x)v\) for all \(x \in \mathfrak{n}\).

Let \(\lambda : \mathfrak{n} \to \mathbb{C}\) be a Lie functional. Let \(\mathbb{C}v_\lambda\) be a 1-dimensional \(\mathfrak{n}\)-module such that \(xv_\lambda = \lambda(x)v_\lambda\) for all \(x \in \mathfrak{n}\). Then
\[
M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}v_\lambda,
\]
is a \(U(\mathfrak{g})\)-module, which is called the universal Whittaker module for the pair \((\mathfrak{g}, \mathfrak{n})\).
Such Whittaker modules and corresponding Whittaker vectors are sometimes called the generalized Whittaker modules/Whittaker vectors.

2.3. Example: \( g = gl(2\ell, \mathbb{C}) \). In this Subsection we take \( g = gl(2\ell, \mathbb{C}) \). Take the usual basis \( e_{i,j}, i, j = 1, \ldots, n \) of \( g \).

Recall the triangular decomposition:

\[
g = n^- \oplus h \oplus n^+.
\]

Here

\[
n^+ = \text{span}_\mathbb{C}\{e_{i,j} \mid i < j\},
\]

\[
n^- = \text{span}_\mathbb{C}\{e_{i,j} \mid i > j\},
\]

\[
h = \text{span}_\mathbb{C}\{e_{i,i} \mid i = 1, \ldots, 2\ell\}.
\]

Note that \( n^+ \) is not commutative. For example \([e_{1,2}, e_{2,3}] = e_{1,3}\).

Whittaker modules with respect to this triangular decomposition are called classical Whittaker modules.

But we can take the following subalgebra \( n \subset n^+ \).

\[
n = \text{span}_\mathbb{C}\{e_{i,j} + \ell \mid i, j = 1, \ldots, \ell\}.
\]

Then \( n \) is a commutative Lie algebra.

Whittaker modules with respect to the pair \((g, n)\) are called the generalized Whittaker modules. Let us describe these modules. Take \( \lambda \in n^* \). Consider the universal Whittaker module

\[
M_\lambda = U(g) \otimes_{U(n)} \mathbb{C}v_\lambda.
\]

Now we have two natural questions:

- Is \( M_\lambda \) irreducible?
- If \( M_\lambda \) is reducible, describe simple quotients of \( M_\lambda \).

These modules appeared in [7, Example 23]. Quite surprisingly, the irreducibility analysis has not been presented neither in [7] nor at any other recent work. We shall see later that \( M_\lambda \) is never irreducible, and describe its simple quotients.

2.4. Whittaker modules for \( g = \hat{gl} \). The basis is given by (cf. [13, 14]):

\[
C, E_{i,j}, \quad i, j \in \mathbb{Z}.
\]

The triangular decomposition:

\[
g = n^- \oplus h \oplus n^+.
\]

Here

\[
n^+ = \text{span}_\mathbb{C}\{E_{i,j} \mid i < j\},
\]

\[
n^- = \text{span}_\mathbb{C}\{E_{i,j} \mid i > j\},
\]

\[
h = \mathbb{C}C \oplus \text{span}_\mathbb{C}\{E_{i,i} \mid i \in \mathbb{Z}\}.
\]

Let

\[
p = \text{span}_\mathbb{C}\{E_{-i-1,j} \mid i, j \in \mathbb{Z}_{\geq 0}\}.
\]

Then \( p \) is a commutative subalgebra of \( n^+ \).

As before we call the Whittaker modules for the Whittaker pair \((g, n^+)\) the classical Whittaker \( \hat{gl} \)-modules, and Whittaker modules for the pair \((g, p)\) the generalized Whittaker \( \hat{gl} \)-modules.

In what follows we shall describe a family of simple, generalized Whittaker \( \hat{gl} \)-modules.
3. Weyl vertex algebra and its Whittaker modules

We now recall the notion of Weyl vertex algebra as this will be our main object of study. To define this vertex algebra, first we remind the reader of the notion of Weyl algebra \( \hat{A} \)

Let \( L \) be the infinite-dimensional Lie algebra with generators

\[
K, a(n), a^*(n), \quad n \in \mathbb{Z},
\]

such that \( K \) is in the center where and the only nontrivial relations are

\[
[a(n), a^*(m)] = \delta_{n+m,0}K, \quad n, m \in \mathbb{Z}.
\]

The Weyl algebra \( \hat{A} \) is:

\[
\hat{A} = U(L)_{\langle K-1 \rangle},
\]

where \( \langle K-1 \rangle \) is the two sided ideal generated by \( K-1 \). So, in \( \hat{A} \) we have \( K = 1 \).

To construct the Weyl vertex algebra, we need a vector space, so we choose the simple Weyl algebra module \( M \) generated by a cyclic vector \( 1 \) such that

\[
a(n)1 = a^*(n+1)1 = 0 \quad (n \geq 0),
\]

that is, \( M = \mathbb{C}[a(-n), a^*(-m) \mid n > 0, m \geq 0] \).

Now, by the generating fields theorem, there is a unique vertex algebra \((M, Y, 1)\) where the vertex operator is given by

\[
Y : M \to \text{End}(M)[[z, z^{-1}]]
\]

such that

\[
Y(a(-1)1, z) = a(z), \quad Y(a^*(0)1, z) = a^*(z),
\]

\[
a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}, \quad a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n)z^{-n}.
\]

We choose the following conformal vector of central charge \( c = -1 \) (cf. [14]):

\[
\omega = \frac{1}{2}(a(-1)a^*(-1) - a(-2)a^*(0))1.
\]

Then \((M, Y, 1, \omega)\) has the structure of a \( \frac{1}{2}\mathbb{Z}_{\geq 0} \)-graded vertex operator algebra. Weak and ordinary modules for \((M, Y, 1, \omega)\) can be defined as in the case of \( \mathbb{Z} \)-graded vertex operator algebras.

Following [4], we define the Whittaker module for \( \hat{A} \) to be the quotient

\[
M_1(\lambda, \mu) = \hat{A}/I,
\]

where \( \lambda = (\lambda_0, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_m) \) and \( I \) is the left ideal

\[
I = \langle a(0) - \lambda_0, \ldots, a(n) - \lambda_n, a^*(1) - \mu_1, \ldots, a^*(m) - \mu_m, a(n+1), \ldots \rangle.
\]

Let \( \mathfrak{n} \) be the subalgebra of \( L \) generated by \( a(n), a^*(n+1), n \in \mathbb{Z}_{\geq 0} \). Then \( \mathfrak{n} \) is commutative, and therefore nilpotent subalgebra of \( L \).

**Proposition 3.1.** [4] We have:
Then $g$ with the Whittaker function $\Lambda = \lambda \mu : n \to \mathbb{C}$:
\[
\Lambda(a(i)) = \lambda_i \quad (i = 0, \ldots, n), \quad \Lambda(a(i)) = 0 \quad (i > n)
\]
\[
\Lambda(a^*(i)) = \mu_i \quad (i = 1, \ldots, m), \quad \Lambda(a^*(i)) = 0 \quad (i > m).
\]

(2) $M_1(\lambda, \mu)$ is an irreducible $\hat{A}$-module.

(3) $M_1(\lambda, \mu)$ is an irreducible weak module for the Weyl vertex operator algebra $M$.

The Whittaker vector will be denoted by $w_{\lambda, \mu}$. For each $s \in \mathbb{Z}$, the Weyl algebra $\hat{A}$ contains the following automorphism $\rho_s$:
\[
\rho_s(a(n)) = a(n + s), \quad \rho_s(a^*(n)) = a^*(n - s), \quad (n \in \mathbb{Z}).
\]

For any $\hat{A}$-module $N$, let $\rho_s(N)$ denote the $\hat{A}$-module $\rho_s(N)$ such that $\rho_s(N) = N$ as a vector space and
\[
x.v = \rho_s(x)v, \quad x \in \hat{A}, \quad v \in N.
\]

Clearly, $N$ is irreducible $\hat{A}$-module if and only if $\rho_s(N)$ is irreducible $\hat{A}$-module.

Let $n^{(s)}$ denote the commutative subalgebra of $L$ generated by $a(n - s), a^*(n + 1 + s)$, $n \in \mathbb{Z}_{\geq 0}$.

**Proposition 3.2.** For each $s \in \mathbb{Z}$ we have:
\begin{enumerate}
  \item $\rho_s(M_1(\lambda, \mu))$ is a standard Whittaker module for the Whittaker pair $(\mathcal{L}, n^{(s)})$ of level $K = 1$ with the Whittaker function $\Lambda^{(s)} : n \to \mathbb{C}$:
\[
\Lambda^{(s)}(a(i - s)) = \lambda_i \quad (i = 0, \ldots, n), \quad \Lambda^{(s)}(a(i - s)) = 0 \quad (i > n)
\]
\[
\Lambda^{(s)}(a^*(i + s)) = \mu_i \quad (i = 1, \ldots, m), \quad \Lambda^{(s)}(a^*(i + s)) = 0 \quad (i > m).
\]
  \item $\rho_s(M_1(\lambda, \mu))$ is an irreducible $\hat{A}$-module.
  \item $\rho_s(M_1(\lambda, \mu))$ is an irreducible weak module for the Weyl vertex operator algebra $M$.
\end{enumerate}

Let $\zeta_p = e^{2\pi i / p}$ be $p$-th root of unity. Let $g_p$ be the automorphism of the vertex operator algebra $M$ which is uniquely determined by the following automorphism of the Weyl algebra $\hat{A}$:
\[
a(n) \mapsto \zeta_p a(n), \quad a^*(n) \mapsto \zeta_p^{-1} a^*(n) \quad (n \in \mathbb{Z}).
\]

Then $g_p$ is the automorphism of $M$ of order $p$.

The following result was proved in [4] in the case $s = 0$ and the proof for arbitrary $s \in \mathbb{Z}$ is completely analogous.

**Theorem 3.3.** [4] Assume that $\Lambda = (\lambda, \mu) \neq 0$ and $s \in \mathbb{Z}$. Then $\rho_s(M_1(\lambda, \mu))$ is an irreducible weak module for the orbifold subalgebra $M^{g_p} = M^{(g_p)}$, for each $p \geq 1$.

Let us define an operator $J^0 := aa^*$: Then the components of the field
\[
Y(J^0, z) = \sum_{n \in \mathbb{Z}} J^0(n) z^{-n-1}
\]
satisfies the commutation relations for the Heisenberg algebra at level $-1$. We have the following subalgebra of $M$:
\[
M^0 = \text{Ker}_M J^0(0).
\]
Let \( \zeta \in \mathbb{C}, |\zeta| = 1 \), which is not a root of unity. Let \( g \) be the automorphism of the vertex operator algebra \( M \) uniquely determined by the following automorphism of the Weyl algebra \( \hat{A} \):

\[
a(n) \mapsto \zeta a(n), \quad a^*(n) \mapsto \zeta^{-1} a^*(n) \quad (n \in \mathbb{Z}).
\]

Then \( g \) is the automorphism of \( M \) of infinite order, and the group \( G = \langle g \rangle \) is isomorphic to \( \mathbb{Z} \). Clearly, we have

\[
M^0 \cong M^G.
\]

Note also that

\[
M^0 = \bigcap_{p=1}^{\infty} M^{\mathbb{Z}_p}, \quad M \supset M^{\mathbb{Z}_2} \supset \cdots M^{\mathbb{Z}_p} \supset \cdots \supset M^0.
\]

Since \( M_1(\lambda, \mu) \) is an irreducible \( M^{\mathbb{Z}_p} \)-module for each \( p \geq 1 \), it is natural to ask if \( M_1(\lambda, \mu) \) is irreducible also as an \( M^0 \)-module. However, in this paper we prove that \( M_1(\lambda, \mu) \) is a reducible and indecomposable \( M^0 \)-module.

Let us finish this section by describing the connections between \( \rho_s(M_1(\lambda, \mu)) \) and \( M_1(\lambda, \mu) \). As \( M \)-modules these modules are constructed by applying the automorphism \( \rho_s \) which has a nice description in terms of generators of \( \hat{A} \). But, the automorphism \( \rho_s \) does also have sense as automorphism of vertex \( J^0(0) \)-invariant subalgebras, which can be explain by using H. Li’s \( \Delta \)-operator.

First we notice that as \( M \)-modules (see [5] for a proof):

\[
(\rho_s(M_1(\lambda, \mu)), Y_s(v, z)) = (M_1(\lambda, \mu), Y(\Delta(-sJ^0(z)v, z)), z))
\]

where

\[
\Delta(h, z) = z^{h(0)} \exp \left( \sum_{n=1}^{\infty} h(n) \frac{z(-z)^{-n}}{-n} \right).
\]

In particular, the action of \( J^0(0) \) on \( \rho_s(M_1(\lambda, \mu)) \) is given by

\[
J^0_s(z) = J^0(z) + \frac{(-z)^{-1}}{-1} Y(-sJ^0(1), J^0, z) = J^0(z) + sz^{-1}\text{Id}.
\]

By restricting this realization on vertex subalgebras of \( M \) which are \( J^0(0) \)-invariant we get:

**Proposition 3.4.** Let \( U \) be one of the subalgebras \( M^{\mathbb{Z}_p} \) or \( M^0 \). As an \( U \)-module \( \rho_s(M_1(\lambda, \mu)) \) is obtained from \( M_1(\lambda, \mu) \) as follows:

\[
(\rho_s(M_1(\lambda, \mu)), Y_s(v, z)) = (M_1(\lambda, \mu), Y(\Delta(-sJ^0(z)v, z)), z))
\]

where \( v \in U \).

**Remark 1.** In the case \( U = M^0 \), the previous proposition shows that if we describe the structure of \( M_1(\lambda, \mu) \) as a \( U \)-module (i.e., description of submodules, irreducible quotients) we will automatically describe the structure of \( \rho_s(M_1(\lambda, \mu)) \) as \( U \)-module.
4. \( \mathcal{W}_{1+\infty} \)-algebra at central charge \( c = -1 \) and its Whittaker modules

A peculiarity of the orbifold \( M^0 \) is that it has two additional important realizations which we plan to explore in our paper:

- \( M^0 \) is isomorphic to the vertex algebra \( \mathcal{W}_{1+\infty} \)-algebra at central charge \( c = -1 \).
- \( M^0 \) is isomorphic to the simple module for the Lie algebra \( \hat{\mathfrak{gl}} \), which is the central extension of the Lie algebra of infinite matrices.

4.1. The \( \mathcal{W}_{1+\infty} \)-algebra approach. The universal vertex algebra \( \mathcal{W}_{1+\infty}^c \) is generated by the fields

\[
J^k(z) = \sum_{n \in \mathbb{Z}} J^k(n) z^{-n-k-1} \quad (k \in \mathbb{Z}_{\geq 0}),
\]

whose components satisfy the commutation relations for the Lie algebra \( \hat{\mathfrak{D}} \) at central charge \( c \), which is a central extension of the Lie algebra of complex regular differential operators on \( \mathbb{C}^* \) (cf. [10], [14]). It has a simple quotient, which we denote by \( \mathcal{W}_{1+\infty,c} \).

The components of the fields \( J^0(z), J^1(z) \) satisfy the commutation relation for the Heisenberg Virasoro Lie algebra \( \mathcal{H} \) generated by \( J^0(r), J^1(r), C_1, C_2 \) with the commutation relations (\( r, s \in \mathbb{Z} \)):

\[
\begin{align*}
[J^0(r), J^0(s)] &= r \delta_{r+s,0} C_2, \\
[J^1(r), J^0(s)] &= -s J^1(r + s), \\
[J^1(r), J^1(s)] &= (r-s) J^1(r + s) + \frac{r^3 - r}{12} \delta_{r+s,0} C_1,
\end{align*}
\]

with central elements \( C_1 \) and \( C_2 \). On each weak \( \mathcal{W}_{1+\infty,c} \)-module the actions of the central elements are given by \( C_1 = -2c, C_2 = c \). Let \( \mathcal{L}^{HVir}_c \) be the Heisenberg-Virasoro vertex subalgebra of \( \mathcal{W}_{1+\infty,c} \) generated by \( J^0(z), J^1(z) \).

**Definition 4.1.** An irreducible weak \( \mathcal{W}_{1+\infty,c} \)-module \( M \) is called typical (resp. atypical) if it is irreducible (resp. reducible) as a \( \mathcal{L}^{HVir}_c \)-module.

It was proved by V. Kac and A. Radul in [14] that \( M^0 \cong \mathcal{W}_{1+\infty,c} \) for \( c = -1 \). As a consequence, we have that \( M^0 \) is generated by the fields

\[
J^k(z) = Y(a^*(-k)a, z) =: \left( \partial_z^k a^*(z) \right) a(z) := \sum_{n \in \mathbb{Z}} J^k(n) z^{-n-k-1} \quad (k \in \mathbb{Z}_{\geq 0}).
\]

**Remark 2.** One can show that modules obtained in the decomposition of \( M \) as a \( M^0 \)-module are atypical (cf. [19]), but the modules constructed from the irreducible relaxed modules in [5] are typical highest weight \( \mathcal{W}_{1+\infty,c=-1} \)-modules.

Our Whittaker modules for the Weyl vertex algebra are automatically Whittaker weak modules for \( \mathcal{W}_{1+\infty,c=-1} \).

4.2. Approach using the Lie algebra \( \hat{\mathfrak{gl}} \). Define the generating function

\[
E(z, w) =: a(z) a^*(w) := \sum_{i,j \in \mathbb{Z}} E_{ij} z^{i-1} w^{-j}.
\]
In other words, the operators $E_{i,j}$ are defined as

$$E_{i,j} = a(-i)a^*(j):$$

These operators endow $M$ with the structure of a $\hat{\mathfrak{gl}}$–module at central charge $K = -1$ (see formula (2.7) in [14] for commutation relations for $\hat{\mathfrak{gl}}$).

We have (cf. [14]):

$$[E_{i,j}, a(-m)] = \delta_{j,m}a(-i), \quad [E_{i,j}, a^*(m)] = -\delta_{i,m}a^*(j).$$

4.3. **Connection between $\hat{\mathfrak{gl}}$–modules and $\mathcal{W}_{1+\infty}$–modules.** In [10], [14] the authors demonstrated how one can construct $\mathcal{W}_{1+\infty}$–modules from $\hat{\mathfrak{gl}}$–modules. Strictly speaking, their approach was used only for quasi-finite modules, which are necessarily weight modules. Therefore, they formulated their results for a category of modules which does not include Whittaker modules. However, we extend their approach to a broader category which also includes Whittaker modules.

Following [10], we have a homomorphism of Lie algebras $\Phi_0 : \hat{\mathcal{D}} \to \hat{\mathfrak{gl}}$, determined by

$$\Phi_0(t^m f(D)) = \sum_{j \in \mathbb{Z}} f(-j)E_{j-m,j}, \quad \Phi_0(C) = K,$$

where $f \in \mathbb{C}[x]$, $D = t\partial_t$. This homomorphism enables us to consider $\hat{\mathfrak{gl}}$–modules as $\hat{\mathcal{D}}$–modules, and therefore as modules for the vertex algebra $\mathcal{W}_{1+\infty}^c$.

It seems that the representation theory of $\hat{\mathfrak{gl}}$ is easier for analysis than the representation theory of $\mathcal{W}_{1+\infty}^c$. Nonetheless, a problem is in the fact that $\Phi_0$ is not surjective homomorphism. Hence the restriction of irreducible $\hat{\mathfrak{gl}}$–module to $\mathcal{D}$ (and therefore to $\mathcal{W}_{1+\infty}^c$) need not be irreducible. However, for a family of quasi-finite modules, V. Kac and A. Radul [13] showed that the image of $\Phi_0$ is dense in a certain way which we explain latter (cf. [13, Proposition 4.3]). So for quasi-finite modules, irreducible $\hat{\mathfrak{gl}}$–modules are also irreducible $\mathcal{W}_{1+\infty}^c$–modules. Our goal is to show that we can replace quasi-finite modules with certain Whittaker modules, so that irreducible $\hat{\mathfrak{gl}}$–modules will become irreducible for $\mathcal{W}_{1+\infty}^c$.

5. **$M_1(\lambda, \mu)$ as a Whittaker $\hat{\mathfrak{gl}}$–module**

**Proposition 5.1.** Assume that $\lambda \neq 0$ and $\mu \neq 0$. Then $M_1(\lambda, \mu)$ is a Whittaker $\hat{\mathfrak{gl}}$–module for the pair $p \subset \hat{\mathfrak{gl}}$ at central charge $K = -1$, generated by the Whittaker vector $w_{\lambda, \mu}$.

**Proof.** Let us prove that $w_{\lambda, \mu}$ is indeed cyclic. As a vector space, $M_1(\lambda, \mu) \cong M$, so its basis elements are of the form

$$w = a(-n_1) \cdots a(-n_r)a^*(-m_1) \cdots a^*(-m_s)w_{\lambda, \mu}.$$

We will now use double induction on the length of a vector in $M_1(\lambda, \mu)$. First inductive hypothesis is the following:

Every element of the form $w = a(-n_1) \cdots a(-n_r)w_{\lambda, \mu}$ is an element of $U(\hat{\mathfrak{gl}})w_{\lambda, \mu}$.
Let us first prove the initial case. Let \( a(-n)w_{\lambda, \mu} \) be an element of \( M_1(\lambda, \mu) \). Since \( \mu \neq 0 \), there is an index \( j_0 \) such that \( \mu_{j_0} \neq 0 \), and \( a^*(j_0)w_{\lambda, \mu} = \mu_jw_{\lambda, \mu} \). Therefore, we have

\[
\frac{1}{\mu_{j_0}} E_{i, j_0}w_{\lambda, \mu} = a(-n)w_{\lambda, \mu}.
\]

Now, let us assume that for any vector of length at most \( r \), the inductive hypothesis holds. Let us take a vector of the form

\[
v = a(-n_1)a(-n_2)\cdots a(-n_{r+1})w_{\lambda, \mu}.
\]

We have that

\[
\frac{1}{\mu_{j_0}^{r+1}} E_{n_1, j_0}E_{n_2, j_0} \cdots E_{n_{r+1}, j_0}w_{\lambda, \mu}
\]

\[
= \frac{1}{\mu_{j_0}^{r+1}} : a(-n_1)a^*(j_0) : a(-n_2)a^*(j_0) : \cdots : a(-n_{r+1})a^*(j_0) : w_{\lambda, \mu}
\]

\[
= \frac{1}{\mu_{j_0}^{r+1}} a(-n_1)a(-n_2) \cdots a(-n_{r+1})\underbrace{a^*(j_0)a^*(j_0) \cdots a^*(j_0)}_{r+1 \text{ factors}} w_{\lambda, \mu}
\]

\[
+ C_1 \frac{1}{\mu_{j_0}^{r+1}} a(-n_1) \cdots a(-n_k) \underbrace{a^*(j_0) \cdots a^*(j_0)}_{k \text{ factors}} w_{\lambda, \mu}
\]

\[
+ \cdots + C_{r+1} w_{\lambda, \mu},
\]

where \( C_i \) are constants that can be zero, \( a(-n) \) means that this element is left out, and \( K \) is a (possibly zero) sum of vectors of length at most \( r \). Therefore, we have that

\[
v = \frac{1}{\mu_{j_0}^{r+1}} E_{n_1, j_0}E_{n_2, j_0} \cdots E_{n_{r+1}, j_0}w_{\lambda, \mu} - K,
\]

and by the induction hypothesis, we have that \( v \in U(\widehat{\mathfrak{gl}})w_{\lambda, \mu} \).

The second induction hypothesis is that every vector of the form

\[
a^*(-m_1) \cdots a^*(-m_s)a(-n_1) \cdots a(-n_r)w_{\lambda, \mu}
\]

is in \( U(\widehat{\mathfrak{gl}})w_{\lambda, \mu} \).

Let us first prove the initial case. Let \( v = a^*(-m)a(-n_1)a(-n_2) \cdots a(-n_r)w_{\lambda, \mu} \), where \( r \) is any non-negative integer. Since \( \lambda \neq 0 \), there is an index \( \lambda_{j_0} \neq 0 \). Therefore,

\[
\frac{1}{\lambda_{i_0}} E_{-i_0, -m}a(-n_1) \cdots a(-n_r)w_{\lambda, \mu}
\]

\[
= \frac{1}{\lambda_{i_0}} : a(i_0)a^*(-m) : a(n_1) \cdots a(-n_r)w_{\lambda, \mu}
\]

\[
= v.
\]
Now, let us assume that for any vector with at most $s$ factors $a^s()$ the inductive hypothesis holds. Let us take a vector of the form
\[ v = a^s(-m_1) \cdots a^s(-m_{s+1})a(-n_1)a(-n_2) \cdots a(-n_r)w_{\lambda, \mu}. \]

We have that
\[
\frac{1}{\lambda^s_{i_0}} E_{-i_0, -m_1} E_{-i_0, -m_2} \cdots E_{-i_0, -m_{s+1}} a(-n_1)a(-n_2) \cdots a(-n_r)w_{\lambda, \mu}
\]
\[
= \frac{1}{\lambda^s_{i_0+1}} :a(i_0)a^s(-m_1) :a(i_0)a^s(-m_2) : \cdots :a(i_0)a^s(-m_{s+1}):
\]
\[
= \frac{1}{\lambda^s_{i_0+1}} a^s(-m_1)a^s(-m_2) \cdots a^s(-m_{s+1})a(-n_1) \cdots a(-n_r) a(i_0)a(i_0) \cdots a(i_0) w_{\lambda, \mu}
\]
\[
+ D_1 \frac{1}{\lambda^s_{i_0}} a^s(-m_1) \cdots a^s(-m_k) \cdots a^s(-m_{s+1})a(-n_1) \cdots a(-n_r) a(i_0) \cdots a(i_0) w_{\lambda, \mu}
\]
\[
+ \cdots + D_s+1 w_{\lambda, \mu},
\]
\[
= \frac{1}{\lambda^s_{i_0+1}} \lambda^s_{i_0+1} + L,
\]

where $D_i$ are constants that can be zero, $a^s(-m)$ means that this element is left out, and $L$ is a (possibly zero) sum of vectors of length at most $s$. Therefore, we have that
\[
v = \frac{1}{\lambda^s_{i_0}} E_{-i_0, -m_1} E_{-i_0, -m_2} \cdots E_{-i_0, -m_{s+1}} a(-n_1)a(-n_2) \cdots a(-n_r)w_{\lambda, \mu} - L,
\]

and by the induction hypothesis, we have that $v \in U(\hat{gl}).w_{\lambda, \mu}$.

This completes the proof that the Whittaker vector $w_{\lambda, \mu}$ is cyclic. \qed

Next, using a connection between $\hat{gl}$-modules and $W_{1+\infty}$-modules we have:

**Theorem 5.2.** Assume that $\lambda \neq 0$ and $\mu \neq 0$. Then $M_1(\lambda, \mu)$ is a Whittaker $W_{1+\infty,c=-1}$-module generated by the cyclic vector $w_{\lambda, \mu}$. In particular, $M_1(\lambda, \mu)$ is a cyclic module for the orbifold vertex algebra $M^0$.

**Proof.** By Proposition 5.1 we have that $M_1(\lambda, \mu)$ is a cyclic $\hat{gl}$-module and we want to prove it is a cyclic $\hat{D}$-module. Take a homomorphism $\Phi_0 : \hat{D} \to \hat{gl}$. Although $M_1(\lambda, \mu)$ is not a quasi-finite weight module, we will show that one can still apply [13, Proposition 4.3]. We use the following arguments:

- Let $M = M_1(\lambda, \mu)$. Recall that $M \cong M$ as a vector spaces.
- Take the Virasoro vector $\omega = \frac{1}{2}(a(-1)a^s(-1) - a(-2)a^s(0))1$ of central charge $c = -1$ and define $L(n) = \omega_{n+1}$. Then $L(0)$ defines a $\frac{1}{2}\mathbb{Z}_{\geq 0}$ gradation on $M$:
  \[
  M = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}_{\geq 0}} M_m
  \]
  such that $v \in M_m \iff L(0)v = mv$.
• Define $\mathcal{M}(m) = \bigcup_{k \leq m} M_k$.

• Since $\dim M_k < \infty$ for all $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, we have that $\dim \mathcal{M}(m) < \infty$ for all $m \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

• Then by applying the homomorphism $\Phi_0$ on $M_1(\lambda, \mu)$ we get:

\[
\Phi_0(t^m f(D)) = \sum_{j \in \mathbb{Z}} f(-j)E_{j-m,j}, \quad \Phi_0(C) = -1
\]

\[
= \sum_{j \in \mathbb{Z}} f(-j) : a(m-j)a^*(j) :
\]

• For $v \in \mathcal{M}(m)$, we have that $\Phi_0(t^m f(D))v \in \mathcal{M}(m+N)$, where $N$ is determined by Whittaker function, has the property

\[\lambda_i = \mu_i = 0 \quad \forall i > N,\]

and can be chosen independently on $m$.

• Therefore $\Phi_0(t^m f(D))$ defines a homomorphism

\[
(*) \quad \mathbb{C}[x] \rightarrow \text{Hom}(\mathcal{M}(m), \mathcal{M}(m+N)).
\]

• Since $\dim \mathcal{M}(m) < \infty$ and $\dim \mathcal{M}(m+N) < \infty$, the proof of [13, Proposition 4.3] gives that $(*)$ is a continuous map. Therefore, it can be uniquely extended to a homomorphism

\[
(*) \quad \mathcal{O} \rightarrow \text{Hom}(\mathcal{M}(m), \mathcal{M}(m+N)),
\]

where $\mathcal{O}$ denotes the algebra of holomorphic functions on $\mathbb{C}$.

• Since for each $j \in \mathbb{Z}$ there is a holomorphic function $g$ such that $g(\ell) = \delta_{\ell,j}$, $\ell \in \mathbb{Z}$, we conclude that each $E_{j,j}$ is in the image of map (5.1).

• Since $\mathcal{M}$ is a Whittaker $\widehat{\mathfrak{g}l}$–module generated by $w_{\lambda,\mu}$, $\mathcal{M}$ is also generated by $w_{\lambda,\mu}$ as a $\widehat{D}$–module.

The proof follows. $\square$

6. THE STRUCTURE OF THE WHITTAKER MODULE $M_1(\lambda, \mu)$ AS A $\widehat{\mathfrak{g}l}$–MODULE

We have proved in Proposition 5.1 that $M_1(\lambda, \mu)$ is a cyclic $\widehat{\mathfrak{g}l}$–module. Let us prove that it is not irreducible. We will use the following Casimir element.

Define

\[
I := \sum_{j \in \mathbb{Z}} E_{j,j}.
\]

As far we can see, $I$ is introduced in [12] for a slightly different category of modules. However, it is well defined on the family of Whittaker $\widehat{\mathfrak{g}l}$–modules which we considered.

Lemma 6.1. On any weak $M$–module $\mathcal{M}$ we have that $I = J^0(0)$. In particular:

1. $I \in \text{End}(M_1(\lambda, \mu))$.
2. The action of $I$ commutes with the action of $\widehat{\mathfrak{g}l}$ on $M_1(\lambda, \mu)$.
Proof. On each weak $M$–module the operator $J^0(n) = \sum_{j \in \mathbb{Z}}: a(j+n)a(-j) :$ is well defined. In particular, we have:

$$J^0(0) = \sum_{j \in \mathbb{Z}}: a(-j)a^*(j) := I.$$ 

In particular this implies that $I$ is well defined operator acting on $M_1(\lambda, \mu)$. Thus (1) holds.

For the proof of assertion (2) we use commutation relations for $\hat{\mathfrak{gl}}$:

$$[E_{i,j}, E_{s,t}] = \delta_{j,s}E_{i,t} - \delta_{i,t}E_{s,j} - C\Phi(E_{i,j}, E_{s,t}).$$

We calculate:

$$[E_{i,j}, I]v = \left(\sum_{k \in \mathbb{Z}}[E_{i,j}, E_{k,k}]\right)v = (E_{i,j} - E_{i,j})v = 0.$$ 

Therefore, our second assertion also holds.

Lemma 6.2.  

(1) For every $n \in \mathbb{Z}_{\geq 1}$, $I^n w_{\lambda, \mu}$ is a non-trivial Whittaker vector in $M_1(\lambda, \mu)$.

(2) For any $S \subset \mathbb{C}[I]w_{\lambda, \mu}$, let $\langle S \rangle$ be the submodule generated by Whittaker vectors from $S$. Then $\langle (I - d)\mathbb{C}[I]w_{\lambda, \mu} \rangle$ is a proper submodule of $M_1(\lambda, \mu)$ for each $d \in \mathbb{C}$.

We have proved the following result.

Theorem 6.3. $M_1(\lambda, \mu)$ is a reducible, cyclic $\hat{\mathfrak{gl}}$–module.

The automorphism $\rho_s$ of $\hat{\mathfrak{A}}$ induces the following automorphism of $\hat{\mathfrak{gl}}$:

$$\tilde{\rho}_s : E_{i,j} =: a(-i)a^*(j) \rightarrow E_{i-s,j-s} =: a(-i+s)a^*(j-s) : .$$

So this means that as $\hat{\mathfrak{gl}}$–module $\rho_s(M_1(\lambda, \mu))$ is obtained from $M_1(\lambda, \mu)$ by applying the automorphism $\tilde{\rho}_s$. As a consequence we get

Corollary 6.4. We have: 

(1) $\rho_s(M_1(\lambda, \mu))$ is a reducible, cyclic $\hat{\mathfrak{gl}}$–module.

(2) Let $L$ be any irreducible quotient of $M_1(\lambda, \mu)$. Then $\tilde{\rho}_s(L)$ is a irreducible quotient of $\rho_s(M_1(\lambda, \mu))$.

Let $L$ be any irreducible quotient of $M_1(\lambda, \mu)$. (We will see below that it is not unique). Two very important problems arise:

(A) Find an explicit realization of $L$ if possible.

(B) Determine the complete set of Whittaker vectors in $M_1(\lambda, \mu)$ which generate the maximal submodule of $M_1(\lambda, \mu)$.

In what follows we will solve both problems.
7. Whittaker vectors in $M_1(\lambda, \mu)$

In this section we describe the complete set of Whittaker vectors in $M_1(\lambda, \mu)$ and determine its simple quotients. In particular, we completely solve the problem (B).

Let $M = M_1(\lambda, \mu)$. As before, we choose $N \in \mathbb{Z}_{\geq 0}$ such that $a(n)w_{\lambda, \mu} = a^*(n)w_{\lambda, \mu} = 0$ for $n \geq N$. Then

$$v = Iw_{\lambda, \mu} = (a(0)a^*(0) : \cdots + a(N)a^*(-N): w_{\lambda, \mu} + (a(-1)a^*(1) : \cdots + a(-N)a^*(N): w_{\lambda, \mu}$$

(7.1)

$$= \sum_{k=0}^{N} \lambda_k a^*(-k)w_{\lambda, \mu} + \sum_{k=1}^{N} \mu_k a(-k)w_{\lambda, \mu}$$

is a new Whittaker vector. Now we choose $i_0$ and $j_0$ such that

$$\lambda_{i_0} \cdot \mu_{j_0} \neq 0.$$

**Lemma 7.1.** For $n, k \geq 0$ we have

$$a(n)I^k w_{\lambda, \mu} = \lambda_n(I + 1)^k w_{\lambda, \mu}.$$  

$$a^*(n+1)I^k w_{\lambda, \mu} = \mu_{n+1}(I - 1)^k w_{\lambda, \mu}.$$  

**Proof.** By direct calculation we have

$$[a(n), I] = \sum_{j \in \mathbb{Z}} [a(n), E_{j,j}] = [a(n), a^*(-n)a(n)] = a(n).$$

This implies that

$$a(n)Iw_{\lambda, \mu} = \lambda_n Iw_{\lambda, \mu} + [a(n), I]w_{\lambda, \mu} = \lambda_n(Iw_{\lambda, \mu} + w_{\lambda, \mu}) = \lambda_n(I + 1)w_{\lambda, \mu}.$$  

$$a(n)I^2 w_{\lambda, \mu} = \lambda_n(I^2 + 2I + 1)w_{\lambda, \mu} = \lambda_n(I + 1)^2 w_{\lambda, \mu}.$$  

Assume that $a(n)I^k w_{\lambda, \mu} \in \mathbb{C}[I]w_{\lambda, \mu}$. Then

$$a(n)I^{k+1} w_{\lambda, \mu} = I(a(n)I^k w_{\lambda, \mu}) + a(n)I^k w_{\lambda, \mu} = \lambda_n(I(I + 1)^k + (I + 1)^k)w_{\lambda, \mu}.$$  

Now first claim holds by the induction. The proof of the second claim is analogous. \qed

Consider now some examples

$$E_{-i_0, n_1} a(-n_1)I^k w_{\lambda, \mu} = [E_{-i_0, n_1}, a(-n_1)]I^k w_{\lambda, \mu} + a(-n_1)E_{-i_0, n_1}I^k w_{\lambda, \mu}$$

$$= -a(i_0)I^k w_{\lambda, \mu} + a(-n_1)I^k E_{-i_0, n_1} w_{\lambda, \mu}$$

$$= -\lambda_{i_0}(I + 1)^k w_{\lambda, \mu} + \lambda_{i_0} \mu_{n_1} a(-n_1)I^k w_{\lambda, \mu}.$$  

$$E_{m_1, j_0} a^*(-m_1)I^k w_{\lambda, \mu} = [E_{m_1, j_0}, a^*(-m_1)]I^k w_{\lambda, \mu} + a^*(-m_1)E_{m_1, j_0}I^k w_{\lambda, \mu}$$

$$= a^*(j_0)I^k w_{\lambda, \mu} + a^*(-m_1)I^k E_{m_1, j_0} w_{\lambda, \mu}$$

$$= \mu_{j_0}(I - 1)^k w_{\lambda, \mu} + \lambda_{m_1} \mu_{j_0} a^*(-m_1)I^k w_{\lambda, \mu}.$$
This implies:
\[ \hat{E}_{m_1,j_0} a^*(-m_1) I^k w_{\lambda,\mu} = \frac{1}{\mu_{j_0}} (E_{m_1,j_0} - \lambda_{m_1} \mu_{j_0}) a^*(-n_1) I^k w_{\lambda,\mu} = (I - 1)^k w_{\lambda,\mu}, \]
and
In this way we get:

**Lemma 7.2.** Let \( p \in \mathbb{Z}_{\geq 0} \).

1. Let \( \Phi \in \mathbb{C}[a(-n - 1), a^*(-m) \mid n, m \in \mathbb{Z}_{\geq 0}, m \neq i_0} \].

   Then
   \[ \hat{E}_{-i_0,p} \Phi I^k w_{\lambda,\mu} = \left( \frac{\partial}{\partial a(-p)} \Phi \right) (I + 1)^k w_{\lambda,\mu}. \]

2. Let \( \Phi^* \in \mathbb{C}[a^*(-m) \mid m \in \mathbb{Z}_{\geq 0}, m \neq i_0} \].

   Then
   \[ \hat{E}_{p,j_0} \Phi^* I^k w_{\lambda,\mu} = \left( \frac{\partial}{\partial a^*(-p)} \Phi \right) (I - 1)^k w_{\lambda,\mu}. \]

**Proposition 7.3.** Assume that \( v \) is a Whittaker vector in \( \mathcal{M} \). Then \( v \in \mathbb{C}[I]w_{\lambda,\mu} \).

**Proof.** First we see that as a vector space:
\[ \mathcal{M} \cong \mathbb{C}[a(-n - 1), a^*(-m) \mid n, m \in \mathbb{Z}_{\geq 0}, m \neq i_0} \] \( \otimes \mathbb{C}[I], \)
i.e., the basis of \( \mathcal{M} \) consists of the vectors:
\[ (\ast) \ a(-n_1) \cdots a(-n_r) a^*(-m_1) \cdots a^*(-m_s) I^k w_{\lambda,\mu}, \ n_i \geq 1, m_i \geq 0, m_i \neq i_0, k \geq 0. \]

So we can assume that we have a Whittaker vector of the form
\[ (7.2) \ v = \sum_{j=0}^{k} \Phi_j I^j w_{\lambda,\mu} \]
for certain polynomials \( \Phi_j \in \mathbb{C}[a(-n - 1), a^*(-m) \mid n, m \in \mathbb{Z}_{\geq 0}, m \neq i_0} \]. By using Lemma 7.2 (1) we get
\[ \hat{E}_{-i_0,p} v = \sum_{j=0}^{k} \left( \frac{\partial}{\partial a(-p)} \Phi_j \right) (I + 1)^j w_{\lambda,\mu}. \]

Since \( v \) is a Whittaker vector \( \hat{E}_{-i_0,p} \) must act trivially on it. Since vectors (\( \ast \)) are linearly independent we conclude that
\[ \frac{\partial}{\partial a(-p)} \Phi_j = 0 \ \ \forall p \in \mathbb{Z}_{> 1}, \ j = 0, \ldots, k. \]
This implies that \( \Phi_j = \Phi_j^* \) for certain \( \Phi_j^* \in \mathbb{C}[a^*(m) \mid m \in \mathbb{Z}_{\geq 0}, m \neq i_0} \) and
\[ v = \sum_{j=0}^{k} \Phi_j^* I^j w_{\lambda,\mu} \]
Now using Lemma 7.2 (2) we get
\[
\hat{E}_{p,j_0} v = \sum_{j=0}^{k} \left( \frac{\partial}{\partial a^* (-p)} \Phi^*_j \right) (I-1)^j w_{\lambda,\mu}.
\]
Since \(v\) is a Whittaker vector, we have \(\hat{E}_{p,j_0} v = 0\), implying that \(\frac{\partial}{\partial a^* (-p)} \Phi^*_j = 0\) \(\forall p \geq 0\). This implies that \(\Phi^*_j\) is a constant, and therefore \(v \in \mathbb{C}[I]w_{\lambda,\mu}\). The Claim holds. \(\square\)

As a consequence, we get description of irreducible Whittaker modules.

**Theorem 7.4.** For each \(d \in \mathbb{C}\):
\[
L(d, \lambda, \mu) = \frac{M_1(\lambda, \mu)}{(I-d)\mathbb{C}[I]w_{\lambda,\mu}}
\]
is irreducible.

**Proof.** It suffices to prove that each vector \(v\) of the form (7.2), whose projection to \(L(d, \lambda, \mu)\) is non-zero, is cyclic. By using exactly the same arguments as in the proof of Proposition 7.3 we see that \(w_{\lambda,\mu} \in U(\hat{\mathfrak{gl}}).v\). Since \(w_{\lambda,\mu}\) is cyclic vector in \(M_1(\lambda, \mu)\), we conclude that \(v\) is cyclic. The proof follows. \(\square\)

By Applying Corollary 6.4 on the irreducible \(\hat{\mathfrak{gl}}\)-module \(L(d, \lambda, \mu)\) we get:

**Corollary 7.5.** For any \(s \in \mathbb{Z}\), \(\tilde{\rho}_s(L(d, \lambda, \mu))\) is an irreducible quotient of \(\rho_s(M_1(\lambda, \mu))\).

8. Bosonic realisations of \(M_1(\lambda, \mu)\) and \(L(d, \lambda, \mu)\)

K. Tanabe in [18] showed that the Whittaker modules for the rank one Heisenberg vertex algebra remain irreducible when we restrict them to the singlet vertex algebra introduced in [2]. Since \(\mathcal{W}_{1+\infty}\)-algebra at central charge \(c = -1\) is isomorphic to the tensor product \(\mathcal{W}_{3,-2} \otimes M_1(1)\), where \(\mathcal{W}_{3,-2}\) is the singlet vertex algebra at central charge \(-2\) and \(M_1(1)\) is rank one Heisenberg vertex algebra (cf. [19]), \(\mathcal{W}_{1+\infty,c=-1}\) can be realized as a subalgebra of the rank two Heisenberg vertex algebra \(M_2(1)\).

Using Tanabe’s result, one shows that irreducible Whittaker \(M_2(1)\)-modules remain irreducible when we restrict them to the vertex algebra \(\mathcal{W}_{1+\infty,c=-1}\). But it is a hard problem to identify them with our Whittaker modules \(L(d, \lambda, \mu)\) realised as quotients of Whittaker modules for the Weyl vertex algebra. In this article, we identify modules \(L(d, \lambda, \mu)\) as \(M_2(1)\)-modules in the cases when \(M_1(\lambda, \mu)\) has the structure of a \(\Pi(0)\)-module which will be described Subsection 8.1. But in general, our Whittaker modules are different to those obtained from Tanabe paper.

8.1. Realisation of \(M_1(\lambda, \mu)\) from \(\Pi(0)\)-modules. Let \(V_L = M_{\gamma,\delta}(1) \otimes \mathbb{C}[L]\) be the lattice vertex algebra associated to the lattice \(L = \mathbb{Z}\gamma + \mathbb{Z}\delta\) with products
\[
\langle \gamma, \gamma \rangle = \langle \delta, \delta \rangle = 0, \quad \langle \gamma, \delta \rangle = 2,
\]
where \(M_{\gamma,\delta}(1)\) is the Heisenberg vertex algebra generated by \(\gamma(z), \delta(z)\), and \(\mathbb{C}[L]\) the group algebra of the lattice \(L\).
Consider the following subalgebra of $V_L$:

$$\Pi(0) = M_{\gamma, \delta}(1) \otimes \mathbb{C}[Z\gamma].$$

Let $M_{\varphi}(0)$ be the commutative vertex algebra generated by the commutative even field $\varphi(z) = \sum_{i \in \mathbb{Z}} \varphi(i)z^{-i-1}$, so

$$M_{\varphi}(0) = \mathbb{C}[\varphi(-1), \cdots].$$

For $\chi = \chi(z) = \sum_{n \in \mathbb{Z}} \chi_i z^{-i} \in \mathbb{C}[[z]]$, let $L_{\varphi}(\chi) = \mathbb{C}1_{\chi}$ be the 1–dimensional $M_{\varphi}(0)$–module such that $\varphi(i) \equiv \chi_i \text{Id}$ on $L_{\varphi}(\chi)$.

Note that $\gamma, \delta = \delta - 2\varphi, e^{i\gamma}$ generate a subalgebra $\tilde{\Pi}(0)$ of $M_{\varphi}(0) \otimes \Pi(0)$ and that $\tilde{\Pi}(0) \cong \Pi(0)$.

There is an embedding of vertex algebras $\Phi : M \hookrightarrow \tilde{\Pi}(0) \hookrightarrow M_{\varphi}(0) \otimes \Pi(0)$ given by

$$\Phi(a) = e^\gamma, \Phi(a^*) = -\frac{1}{2}(\gamma(-1) + \delta(-1) - 2\varphi(-1))e^{-\gamma}.$$

Let $a^{-1} = e^{-\gamma}$. Define $a(n) = e^n, a^{-1}(n) = e^{-n}z$. For $\lambda \in \mathbb{C}, \lambda \neq 0$, let $\Pi_{\lambda}$ be the $\mathbb{Z}_{\geq 0}$–graded irreducible Whittaker $\Pi(0)$–module (cf. [3]), generated by the Whittaker vector $v_\lambda$ such that

$$a(n)v_\lambda = \lambda \delta_{n,0}v_\lambda, \quad a^{-1}(n)v_\lambda = \frac{1}{\lambda} \delta_{n,0}v_\lambda \quad (n \in \mathbb{Z}_{\geq 0}).$$

As vector spaces:

$$\Pi_{\lambda} \cong \mathbb{C}[\gamma(-n), \delta(-n+1) \mid n \in \mathbb{Z}_{>0}], \quad (\Pi_{\lambda})_{\text{top}} = \mathbb{C}[\delta(0)]v_\lambda \cong \mathbb{C}[\delta(0)].$$

Now consider module $L_{\varphi}(\chi) \otimes \Pi_{\lambda}$. Define $w_{\lambda, \chi} = 1_{\chi} \otimes v_\lambda$.

Assume that $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$

$$\chi_i = \lambda \mu_i, \quad i = 1, \ldots, n, \chi_j = 0 \quad \forall j > n.$$

We have:

$$a^*(i)w_{\lambda, \chi} = \mu_i w_{\lambda, \chi}, \quad i = 1, \ldots, n, \quad a^*(i)w_{\lambda, \chi} = 0, \quad \text{for } i > n.$$

**Proposition 8.1.** Assume that $\lambda \neq 0$, $\mu_n \neq 0$, $\lambda = (\lambda, 0, 0, \ldots)$, $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ and

$$\chi(z) = \sum_{i \in \mathbb{Z}_{<0}} \chi_i z^{-i-1} + \frac{1}{\lambda} \sum_{i=1}^n \mu_i z^{-i-1},$$

Then we have:

- $M_1(\lambda, \mu) \cong L_{\varphi}(\chi) \otimes \Pi_{\lambda}$ as modules for the Weyl vertex algebra $M$.

- $M_1(\lambda, \mu)$ has the structure of an irreducible $\Pi(0)$–module.

**Remark 3.** Note that if $\lambda$ has not of the form $(\lambda, 0, 0, \ldots)$, then the $M$–module structure on $M_1(\lambda, \mu)$ can not be extended to a structure of $\Pi(0)$–module. We simply can not define the vertex operator

$$Y_{M_1(\lambda, \mu)}(e^c, z)$$

in the usual sense. But we believe that the certain generalised vertex operator of type (8.1) can be constructed using irregular vertex operators as in the Gaiotto paper [11].
8.2. Free field realisation of $L(d,\lambda,\mu)$: the case $M_2(\lambda,\mu)$ is a $\Pi(0)$–module. We start with realisation of $M_2(\lambda,\mu)$ as a $\Pi(0)$–module. Note that the operator $\delta(0) \in \text{End}(M_2(\lambda,\mu))$ commutes with the action of $M_\varphi(0) \otimes M_{\eta,\delta}(1)$, and since $M^0 = W_{1+\infty,-1} \subset M_\varphi(0) \otimes M_{\eta,\delta}(1)$, we get that $\delta(0)$ commutes with the action of $W_{1+\infty,-1}$.

By construction we have that $M_2(\lambda,\mu)$ is a module for the Heisenberg vertex algebra $M_{\gamma,\delta}(1)$, generated by $\gamma(z)$ and $\delta(z) = \delta(0) - 2\varphi(z)$, with the free action of $\delta(0) = \delta(0) - 2\chi_0$.

For each $d \in \mathbb{C}$, we get the following irreducible $W_{1+\infty,-1}$–module

$$L(d,\lambda,\mu) = \frac{M_1(\lambda,\mu)}{J_d},$$

where $J_d = W_{1+\infty,-1}(\delta(0) - d)w_{\lambda,\chi}$. This implies that $L(d,\lambda,\mu)$ is an $M_{\gamma,\delta}(1)$–module such that $\delta(0)$ acts as $d \cdot \text{Id}$. Therefore we have the following:

**Proposition 8.2.** Identify $W_{1+\infty,-1}$ as a subalgebra of the Heisenberg vertex algebra $M_{\gamma,\delta}(1)$. Assume that

$$\lambda = (\lambda,0,0,\ldots), \quad \mu = (\mu_1,\ldots,\mu_n).$$

Then the Whittaker $W_{1+\infty,-1}$–module $L(d,\lambda,\mu)$ has the structure of an irreducible Whittaker module for the Heisenberg vertex algebra $M_{\gamma,\delta}(1)$, generated by the Whittaker vector $w_{\lambda,\mu}$ such that $\gamma(0) \equiv -\text{Id}$, $\delta(0) \equiv d \cdot \text{Id}$ and

$$\gamma(i)w_{\lambda,\mu} = 0, \quad \delta(n)w_{\lambda,\mu} = -\frac{2H_i}{\lambda}w_{\lambda,\mu} \quad (n \in \mathbb{Z}_{>0}).$$

8.3. The general case and non-tensor product modules. Proposition 8.2 shows that if $\lambda = (\lambda,0,0,\ldots)$, then $L(d,\lambda,\mu)$ has the structure of a module for the Heisenberg vertex algebra $M_2(1)$. One can ask if $L(d,\lambda,\mu)$ in general are isomorphic to a Whittaker module for $M_2(1)$.

It is important note that $M^0 = W_{1+\infty,-c=-1} \cong M(2) \otimes M_1(1)$, and therefore Whittaker modules for $M^0$ can be obtained by using recent Tanabe paper [18] on Whittaker modules for the singlet algebra $M(p)$, $p \geq 2$. He proved in [18, Theorem 1.1] that every irreducible $M(p)$–module which is generated by the Virasoro Whittaker vector can be realized as a Whittaker module for the Heisenberg vertex algebra $M_1(1)$.

Next result shows that in general $L(d,\lambda,\mu)$ is not realized as an irreducible, Whittaker $M_2(1)$–module.

**Proposition 8.3.** Assume that $\lambda = (\lambda_0,\ldots,\lambda_n)$, $\mu = (\mu_1,\ldots,\mu_m)$, $n > 0$ and $\lambda_n,\mu_m \neq 0$. Then $L(d,\lambda,\mu)$ does not have the form (1.1), i.e, it is not realized as a Whittaker $M_2(1)$–module.

**Proof.** For any $v \in M_1(\lambda,\mu)$, denote by $[v]$ the projection of $v$ to $L(d,\lambda,\mu)$. Then $[w_{\lambda,\mu}]$ is the Whittaker vector $L(d,\lambda,\mu)$ for the Whittaker pair $(\hat{\mathfrak{h}},\mathfrak{p})$ (cf. Subsection 2.4), which is unique, up to a scalar factor.

Assume that $L(d,\lambda,\mu) \cong Z_1 \otimes Z_2$ where $Z_1$ is an irreducible Whittaker $M(2)$–module, and $Z_2$ is an irreducible Whittaker $M_{j_0}(1)$–module. An irreducible Whittaker $M_{j_0}(1)$–module is generated by the Whittaker vector $w_{\text{heis}}$ such that for $k \geq 0$:

$$J^0(k)w_{\text{heis}} = \chi_k w_{\text{heis}}, \quad \chi_k \in \mathbb{C},$$
and one easily sees that each vector in the Whittaker module must be locally finite for $J^0(k), k \geq 0$. This implies that arbitrary vector $w \in L(d, \lambda, \mu)$ is locally finite for $J^0(k_0)$ for all $k_0 > 0$, meaning that

\begin{equation}
\dim \text{span}_C \{ J^0(k_0)^i w | i \geq 0 \} < \infty.
\end{equation}

We shall prove that it is not possible. We shall find $k_0 > 0$ and $w$ such that (8.2) does not hold. We take $w = [w_{\lambda, \mu}]$.

Now we consider $J^0(k_0)^i w_{\lambda, \mu}, i \geq 1$. For simplicity we consider the case $n \geq m$, when $k_0 = n$. We have

\begin{equation}
\text{span}_C \{ J^0(k_0)^i [w_{\lambda, \mu}] | i \geq 0 \} = \text{span}_C \{ [a^*(0)^i w_{\lambda, \mu}] | i \geq 0 \}.
\end{equation}

Assume that there are constants $C_0, \ldots, C_m, C_m \neq 0$ such that

\begin{equation}
C_0[w_{\lambda, \mu}] + \cdots + C_m[a^*(0)^m w_{\lambda, \mu}] = 0.
\end{equation}

Set

$$u = C_0 w_{\lambda, \mu} + \cdots + C_m a^*(0)^m w_{\lambda, \mu} \in M_1(\lambda, \mu).$$

Since

$$\frac{1}{\lambda_n} (E_{0,m} - \lambda_0 \mu_m)^m u = \nu w_{\lambda, \mu} \quad (\nu \neq 0),$$

we conclude that $u$ is not in the ideal generated by $(I - d)C[F]$, implying that $[u] \neq 0$. This contradicts relation (8.3). Therefore $\text{span}_C \{ J^0(k_0)^i [w_{\lambda, \mu}] | i \geq 0 \}$ is infinite-dimensional, which implies that $L(d, \lambda, \mu)$ does not have the form (1.1). The proof follows.

\begin{remark}
In Appendix we shall present a different proof of Proposition 8.3 by proving that $L(d, \lambda, \mu)$ is isomorphic an irreducible module for the Heisenberg-Virasoro algebra which is not a tensor product module.
\end{remark}

9. Generalized Whittaker modules for $g = gl(2, \mathbb{C})$

In this Section we take $g = gl(2, \mathbb{C})$. Recall from Example 2.3 that $g$ has triangular decomposition $g = n_- \oplus h \oplus n_+$. We take the subalgebra $n \subset n_+$.

$$n = \text{span}_C \{ e_{i,j}+\ell | i,j = 1, \ldots, \ell \}. $$

Now we are interested in Whittaker modules with respect to the pair $(g,n)$, which we called generalized Whittaker modules. In particular we are interested in a construction of simple quotients of the universal Whittaker module

$$M_\lambda = U(g) \otimes_{U(n)} C\nu_\lambda.$$

9.1. Weyl algebra $A_\ell$ and its Whittaker modules. Here we recall the definition of the Weyl algebra $A_\ell$. It is an complex associative algebra with generators

$$a_i, a^*_i, \quad i = 1, \ldots, 2\ell$$

and relations

$$[a_i, a_j] = [a^*_i, a^*_j] = 0, \quad [a_i, a^*_j] = \delta_{i,j}, \quad (i,j = 1, \ldots, 2\ell).$$
Define the normal ordering on $A$ with

$$: xy := \frac{1}{2} (xy + yx).$$

In particular, we have

$$: a_i a_i^* := a_i a_i^* - \frac{1}{2} = a_i^* a_i + \frac{1}{2}.$$

For $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, $\beta = (\beta_1, \ldots, \beta_\ell) \in \mathbb{C}^\ell$, we define the Whittaker module $W(\alpha, \beta)$ to be the quotient

$$W(\alpha, \beta) = A^\ell I_\ell(\alpha, \beta),$$

where and $I_\ell(\alpha, \beta)$ is the left ideal

$$I_\ell(\alpha, \beta) = \langle a_1 - \alpha_1, \ldots, a_\ell - \alpha_\ell, a_{\ell+1}^* - \beta_1, \ldots, a_{2\ell}^* - \beta_\ell \rangle.$$

One shows that

Lemma 9.1. $W(\alpha, \beta)$ is an irreducible $A^\ell$–module. It is generated by the Whittaker vector $w_{\alpha, \beta} = 1 + I_\ell(\alpha, \beta)$ such that

$$a_i w_{\alpha, \beta} = \alpha_i w_{\alpha, \beta}, \quad a_{\ell+1}^* w_{\alpha, \beta} = \beta_1 w_{\alpha, \beta}, \quad i = 1, \ldots, \ell.$$

As a vector space

$$W(\alpha, \beta) \cong \mathbb{C}[a_1^*, \ldots, a_\ell^*, a_{\ell+1}, \ldots, a_{2\ell}].$$

9.2. Embedding of $g$ into $A^\ell$. It is well known that there is a Lie algebra homomorphism

$$\Phi : g \rightarrow (A^\ell)_{\text{Lie}}$$

uniquely determined by

$$e_{i,j} \mapsto a_i a_j^*.$$

This implies that each $A^\ell$–module can be treated as a $g$–module.

9.3. $W(\alpha, \beta)$ as a $g$–modules. Now we consider the generalized Whittaker modules $W(\alpha, \beta)$ as $g$–modules.

Proposition 9.2. Assume that $\alpha \neq 0$ and $\beta \neq 0$. $W(\alpha, \beta)$ is a Whittaker module for the Whittaker pair $(g, n)$. In a particular, $w_{\alpha, \beta}$ is a Whittaker vector such that

$$e_{i,\ell+j} w_{\alpha, \beta} = \alpha_i \beta_j w_{\alpha, \beta} \quad \forall i, j \in \{1, \ldots, \ell\}$$

and $W(\alpha, \beta) = U(g) w_{\alpha, \beta}$.

In particular, $W(\alpha, \beta)$ is a quotient of the universal Whittaker module $M_\lambda$ with Whittaker function:

$$\lambda = \lambda_{\alpha, \beta} : n \rightarrow \mathbb{C}, \quad \lambda(e_{i,\ell+j}) = \alpha_i \beta_j, \quad \forall i, j \in \{1, \ldots, \ell\}.$$

Proof. The proof is completely analogous to that presented in Proposition 5.1 below in the case $\hat{g}l$. \qed
Note that the central element \( \sum_{i=1}^{2\ell} c_i t_i \in \mathfrak{g} \) acts on \( W(\alpha, \beta) \) as

\[
I = \sum_{i=1}^{2\ell} : a_i a_i^* :.
\]

Since

\[
I w_{\alpha, \beta} = \sum_{i=1}^{\ell} (a_i a_i^* + \beta_i a_{i+\ell}) w_{\alpha, \beta},
\]

which is not proportional to \( w_{\alpha, \beta} \), we conclude that \( W(\alpha, \beta) \) is reducible \( \mathfrak{g} \)-module.

**Theorem 9.3.** For each \( d \in \mathbb{C}, \alpha, \beta \in \mathbb{C}^\ell, \alpha, \beta \neq 0 \), we have

\[
L(d, \alpha, \beta) = \frac{W(\alpha, \beta)}{\langle (I - d) \mathbb{C}[I] w_{\alpha, \beta} \rangle}
\]

is an irreducible \( \mathfrak{g} \)-module.

**Proof.** Proof is analogous to that of Theorem 7.4. \( \square \)

**APPENDIX: \( M_1(\lambda, \mu) \) AS A MODULE FOR THE HEISENBERG-VIRASORO ALGEBRA**

In this section we prove a stronger statement that \( M_1(\lambda, \mu) \) is a cyclic module for the Heisenberg-Virasoro vertex subalgebra of \( \mathfrak{W}_{1+\infty,-1} \). Our main argument will be that the action of singlet \( \mathcal{M}(2) \) on the Whittaker vector \( w_{\lambda, \mu} \) can be replaced by the action of the singlet Virasoro subalgebra. Next we identify \( M_1(\lambda, \mu) \) and \( L(d, \lambda, \mu) \) as restricted modules for the Heisenberg-Virasoro algebra recently classified in [20].

Recall that

\[
J^k(z) = Y((a^*(-k)a, z) = \sum_{s \in \mathbb{Z}} J^k(s) z^{-k-s-1}.
\]

The singlet algebra \( \mathcal{M}(2) \) (cf. [2], [19]) is generated by the Virasoro vector \( L = J^1 + \frac{1}{2} : (J^0)^2 : + \frac{1}{2} DJ^0 \) and by the primary field \( H \) of conformal weight 3. Recall that the \( \mathbb{Z}_2 \)-orbifold of \( M \) is isomorphic to the simple affine vertex algebra \( L_{-1/2}(\mathfrak{sl}_2) \) generated by

\[
e = \frac{1}{2} : a^2 : , h = -J^0, f = -\frac{1}{2} : (a^*)^2 : ,
\]

the singlet algebra is isomorphic to the parafermion vertex algebra \( N_{-1/2}(\mathfrak{sl}_2) \) and \( H \) is a scalar multiple of the parafermion generator \( W^3 \) in [9, Section 2]:

\[
W^3 = k^2 h(-3) + 3kh(-2)h(-1) + 2h(-1)^3 - 6kh(-1) c(-1) f(-1) - 3k^2 e(-2) f(-1) - 3k^2 f(-2) c(-1) (k = -1/2),
\]

(see also [16, Section 4]).

We have that \( \mathcal{L}^{HV\text{ir}}_{c=1} = \mathcal{L}^{Vir}_{c=-2} \otimes M_1(1) \), where \( \mathcal{L}^{Vir}_{c=-2} \) denotes the simple Virasoro vertex algebra of central charge \( c = -2 \) generated by \( L \).

**Theorem 9.4.** We have:

(1) \( M_1(\lambda, \mu) \) is an cyclic \( \mathcal{L}^{HV\text{ir}}_{c=-1} \)-module.

(2) \( L(d, \lambda, \mu) \) is an irreducible \( \mathcal{L}^{HV\text{ir}}_{c=-1} \)-module.
By a direct calculation we get for
\[ J(9.3) \]

Proof. The field \( H \) can be expressed using \( J^0, J^1 \) and it contains the non-trivial summand \( J^0(−1)^3 \). Set \( H(i) := H_{i+2} \), so we have
\[ H(z) = Y(H, z) = \sum_{i \in \mathbb{Z}} H(i)z^{-i-3}. \]

By a direct calculation we get for \( s \in \mathbb{Z}_{\geq 0} \):
\[ H(3n + 3m + s)w_{\lambda, \mu} = \delta_{s,0}q^w w_{\lambda, \mu} \]
for certain \( q \in \mathbb{C}, q \neq 0 \).

Using the following relation in \( M(2) \)
\[ \frac{3}{4}H(-6)1 - L(-2)H(-4)1 + \frac{3}{2}L(-3)H = 0, \]
and using the arguments as in [18, Lemma 3.1] we get that
\[ M(2), w_{\lambda, \mu} = \langle L \rangle w_{\lambda, \mu}, \]
so \( M(2), w_{\lambda, \mu} \) is isomorphic to the Virasoro submodule obtained by the action of the Virasoro generator \( L \) on the Whittaker vector \( w_{\lambda, \mu} \). Since by Theorem 5.2 \( M_1(\lambda, \mu) \) is a cyclic \( \mathcal{M}(2) \otimes M_1(1) \)-module, we conclude that \( M_1(\lambda, \mu) \) is a cyclic \( \mathcal{L}_{c=-1} \)-module. This proves assertion (1). The assertion (2) is a consequence of (1).

Let us prove assertion (3). For simplicity we consider the case \( n + 1 \geq m \). Other cases can be treat similarly.

By a direct calculation we get for \( s \in \mathbb{Z}_{\geq 0} \):
\[ J^0(n + m + s)w_{\lambda, \mu} = J^1(n + m + s)w_{\lambda, \mu} = 0, \]
\[ J^0(n + m)w_{\lambda, \mu} = a_0w_{\lambda, \mu}, \ldots, J^0(n + 1)w_{\lambda, \mu} = a_1w_{\lambda, \mu}, \]
\[ J^1(n + m)w_{\lambda, \mu} = b_0w_{\lambda, \mu}, \ldots, J^1(n)w_{\lambda, \mu} = b_1w_{\lambda, \mu}, \]
where \( a_i, b_i \in \mathbb{C}, a_m, b_{m+1} \neq 0 \), and \( J^0(n)w_{\lambda, \mu}, J^1(n-1)w_{\lambda, \mu} \) are not proportional to \( w_{\lambda, \mu} \).

Consider the nilpotent subalgebra \( \mathcal{P}(\mathcal{D}) \) of \( \mathcal{H} \) spanned by \( J^0(r + 1), J^1(r), r \geq n \) and \( C_1, C_2 \). Then \( \mathcal{C}w_{\lambda, \mu} \) is the 1-dimensional \( \mathcal{P}(\mathcal{D}) \)-module satisfying relations (9.1)-(9.3) and \( C_1w_{\lambda, \mu} = 2w_{\lambda, \mu}, C_2w_{\lambda, \mu} = -w_{\lambda, \mu} \). Then \( M_1(\lambda, \mu) \) and \( L(d, \lambda, \mu) \) are quotients of the universal \( \mathcal{H} \)-module:
\[ \text{Ind}_{\mathcal{P}(\mathcal{D})}^{\mathcal{H}} \mathcal{C}w_{\lambda, \mu}. \]
These universal modules and their simple quotients appeared in a slightly general form in [20, Section 7], where it was proved that they are not tensor product modules (see [20, Example 7.5, Remark A.4]). In particular, it was shown that for each \( d \in \mathbb{C} \):
\[ \frac{\text{Ind}_{\mathcal{P}(\mathcal{D})}^{\mathcal{H}} \mathcal{C}w_{\lambda, \mu}}{U(\mathcal{H})(J^0(0) - d)w_{\lambda, \mu}} \]
is a simple, restricted \( \mathcal{H} \)-module. This easily implies that as \( \mathcal{H} \)-modules:
\[ M_1(\lambda, \mu) = \text{Ind}_{\mathcal{P}(\mathcal{D})}^{\mathcal{H}} \mathcal{C}w_{\lambda, \mu}, \quad L(d, \lambda, \mu) = \frac{M_1(\lambda, \mu)}{U(\mathcal{H})(J^0(0) - d)w_{\lambda, \mu}}. \]
The proof follows. □

Corollary 9.5. $W_{1+\infty,c=-1}$-modules $\tilde{\rho}_s(L(d,\lambda,\mu))$ are typical.

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