Minimal surfaces with genus zero

M. Magdalena Rodríguez

ABSTRACT. A very interesting problem in the classical theory of minimal surfaces consists of the classification of such surfaces under some geometrical and topological constraints. In this short paper, we give a brief summary of the known classification results for properly embedded minimal surfaces with genus zero in $\mathbb{R}^3$ or quotients of $\mathbb{R}^3$ by one or two independent translations. This does not intend to be an exhaustive review of the tools or proofs in the field, but a simple explanation of the currently known results.

1 Introduction

The classical theory of minimal surfaces, whose roots go back to Euler and Lagrange in the 18th century, still remains extremely active nowadays. New examples discovered in an explosion of activity in the eighties have gradually focused the subject on the problem of classification. Recently, important advances have been made towards the goal of characterizing the minimal surfaces under some topological and/or geometrical constraints, and new families of unexpected examples have been found.

In this short paper, we briefly expose the history and some known classification results for minimal surfaces whose genus is zero (i.e. those which are topologically a punctured sphere). For more detailed reviews, see [1, 2, 9] and the references therein.

2 Genus zero minimal surfaces in $\mathbb{R}^3$

In 1740, Euler constructed the first nonplanar minimal surface: the catenoid (see Figure 1 center). He also showed that this is the only nonplanar, complete minimal surface of revolution in $\mathbb{R}^3$. The following theorem provides us another important characterization of the catenoid.
Figure 1: The helicoid (left), the catenoid (center) and a Riemann minimal example (right).

**Theorem 1 (López & Ros, [8])** The only nonplanar properly embedded minimal surface in $\mathbb{R}^3$ with finite total curvature and genus zero is the catenoid.

Thirty years later the discovery of the catenoid, Meusnier found another minimal example: the helicoid (see Figure 1 left), that was classified in 1842 by Catalan as the only nonplanar ruled minimal surface in $\mathbb{R}^3$. Recently, the helicoid has also been classified, in a beautiful application of the outstanding theory developed by Colding and Minicozzi, as the only properly embedded minimal surface with genus zero and one end.

**Theorem 2 (Meeks & Rosenberg, [13])** A properly embedded simply-connected minimal surface in $\mathbb{R}^3$ is either the plane or the helicoid.

A theorem by Collin [3] assures that, if $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with at least two ends, then $M$ has finite topology if and only if it has finite total curvature. Hence, we obtain from Theorems 1 and 2 that the only properly embedded minimal surfaces in $\mathbb{R}^3$ with genus zero and finite topology (i.e. with finitely many ends) are the plane, the helicoid and the catenoid. In particular, there are no properly embedded minimal surfaces with genus zero and $n$ ends, for any $n \in \mathbb{N}, n \geq 3$. However, there exist properly embedded minimal surfaces with genus zero and infinitely many ends: the Riemann minimal examples. In a posthumously published paper, Riemann constructed a 1-parameter family $\mathcal{R} = \{R_t\}_{t>0}$ of minimal surfaces in $\mathbb{R}^3$, and classified them together with the plane, the helicoid and the catenoid as the only minimal surfaces of $\mathbb{R}^3$ foliated by straight lines and/or circles. Each $R_t$ has genus zero with infinitely many ends in $\mathbb{R}^3$, but

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1Let $M \subset \mathbb{R}^3$ be a minimal surface, and $K$ its Gaussian curvature. We define the total curvature of $M$ as $C(M) = \int_M K$. 

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its geometry simplifies considerably modulo its symmetries: \( R_t \) is invariant
by the translation by \((t, 0, 2)\) and has two planar ends and genus one in the
quotient by \((t, 0, 2)\). Viewed in \( \mathbb{R}^3 \), \( R_t \) has two limit ends\(^1\) one top and
one bottom limit end (see Figure 1, right). These examples have also been
classified as follows.

**Theorem 3 (Meeks, Pérez & Ros, [10])** The Riemann minimal examples are the unique periodic non simply-connected properly embedded
minimal surfaces in \( \mathbb{R}^3 \) with genus zero.

The extension of the above characterization by eliminating the hypoth-
esis on the periodicity constitutes the following conjecture, by the same
authors.

**Conjecture 1 (Genus zero conjecture, [11])** If \( M \subset \mathbb{R}^3 \) is a properly
embedded minimal surface with genus zero and infinitely many ends, then
\( M \) is a Riemann minimal example.

Meeks, Pérez and Ros have proven some strong partial results in this
direction; for example, they prove in [12] that a surface \( M \) in the hypothesis
of Conjecture 1 cannot have exactly one limit end.

3 **Genus zero minimal surfaces in \( T^3 \)**

Let us denote by \( T^3 \) a 3-dimensional flat torus obtained as a quotient of
\( \mathbb{R}^3 \) by three independent translations. Since \( T^3 \) is a compact space, every
minimal surface \( M \subset T^3 \) is also compact, so it cannot have any end. In
particular, a genus zero minimal surface in \( T^3 \) is topologically a sphere, and
so it lifts to a minimal sphere in \( \mathbb{R}^3 \), which is impossible. Hence there are
no properly embedded, triply periodic, minimal surfaces in \( \mathbb{R}^3 \) with genus
zero in the quotient.

\(^1\)A limit end \( e \) of a noncompact surface \( M \) is an accumulation point of the set \( E(M) \) of
ends of \( M \); this makes sense since \( E(M) \) can be endowed with a natural topology which
makes it a compact, totally disconnected subspace of the real interval \([0, 1] \), see [9].

\(^3\)A surface in \( \mathbb{R}^3 \) is said to be *singly, doubly or triply periodic* when it is invariant by a
discrete infinite group \( G \) of isometries of \( \mathbb{R}^3 \) of rank one, two or three, respectively, that
acts properly and discontinuously. After passing to a finitely sheeted covering, we can
assume that the flat 3-manifold \( \mathbb{R}^3/G \) is either \( S^1 \times \mathbb{R}^2, \mathbb{R}^3/S_\theta, T^2 \times \mathbb{R} \) or \( T^3 \), where \( S_\theta \)
denotes a screw motion symmetry and \( T^k \) is a \( k \)-dimensional flat torus, \( k = 2, 3 \).
4 Genus zero minimal surfaces in $\mathbb{T}^2 \times \mathbb{R}$

In 1835, Scherk showed a properly embedded minimal surface $S$ in $\mathbb{R}^3$ invariant by two independent translations. The surface $S$, which is obtained from a minimal graph over the unit square with boundary data $\pm \infty$ disposed alternately, is invariant by $(2,0,0), (0,2,0)$. This surface $S$ was generalized later on to a 1-parameter family of properly embedded minimal surfaces $S_\theta$ invariant by translations of vectors $(2,0,0), (2 \cos \theta, 2 \sin \theta, 0)$, for $\theta \in (0, \frac{\pi}{2}]$ (in particular, $S = S_{\frac{\pi}{2}}$). These surfaces are known as doubly periodic Scherk minimal examples and have genus zero and four ends asymptotic to flat vertical annuli in the quotient by such translations. Annular ends of this kind are usually called Scherk-type ends.

Theorem 4 (Lazard-Holly & Meeks, [7]) The only nonplanar properly embedded minimal surfaces in $\mathbb{T}^2 \times \mathbb{R}$ with genus zero are the doubly periodic Scherk minimal examples.

5 Genus zero minimal surfaces in $S^1 \times \mathbb{R}^2$

The first known properly embedded minimal surfaces of $\mathbb{R}^3$ invariant by only one independent translation were the conjugate surfaces $S^*_\theta$ of the doubly periodic Scherk minimal examples, $\theta \in (0, \frac{\pi}{2}]$, called singly periodic Scherk minimal examples. They may be viewed as the desingularization of two vertical planes meeting at angle $\theta$, and are invariant by the translation by $T = (0,0,2)$. In the quotient by $T$, denoted as $S^1 \times \mathbb{R}^2$, each $S_\theta$ has genus zero and four Scherk-type ends.

H. Karcher [5] generalized the previous Scherk examples by constructing, for each natural $n \geq 2$, a $(2n - 3)$-parameter family of properly embedded minimal surfaces in $\mathbb{R}^3$, invariant by $T$, and with genus zero and $2n$ Scherk-type ends in the quotient. These surfaces are called saddle towers. Let us now recall their construction: Consider any convex polygonal domain $\Omega_n$ whose boundary consists of $2n$ edges of length one, with $n \geq 2$, and mark its edges alternately by $\pm \infty$. Assume $\Omega_n$ is non-special (see definition [1] below). By a theorem of Jenkins and Serrin [4], there exists a minimal graph defined on $\Omega_n$ which diverges to $\pm \infty$, as indicated by the marking, when we approach to the edges of $\Omega_n$. The boundary of this minimal graph consists of $2n$ vertical lines above the vertices of $\Omega_n$. Hence the conjugate minimal surface of this graph is bounded by $2n$ horizontal symmetry curves, lying between two horizontal planes separated by distance one. By reflecting
about one of the two symmetry planes, we obtain a fundamental domain for a saddle tower with $2n$ Scherk-type ends.

**Definition 1** We say that a convex polygonal domain with $2n$ unitary edges is special if $n \geq 3$ and its boundary is a parallelogram with two sides of length one and two sides of length $n - 1$.

**Theorem 5 (Pérez & Traizet, [15])** If $M \subset S^1 \times \mathbb{R}^2$ is a nonplanar, properly embedded minimal surface with genus zero and finitely many Scherk-type ends, then $M$ is a saddle tower.

Now we look for properly embedded minimal surfaces in $S^1 \times \mathbb{R}^2$ with genus zero and infinitely many ends. In a first step, and following the arguments in Section 2, we try to classify those which are periodic. There are two families in this setting: the singly periodic liftings of the 1-parameter family of doubly periodic Scherk minimal examples and singly periodic liftings of the 3-parametric family of KMR examples [5, 16] (also called toroidal halfplane layers). These later examples have been classified in [14] as the only properly embedded, doubly periodic, minimal surfaces in $\mathbb{R}^3$ with parallel ends and genus one in the quotient. If we consider the quotient of a KMR example only by the period at its ends, we obtain a periodic minimal surface in $S^1 \times \mathbb{R}^2$ with genus zero, infinitely many ends, and two limit ends: one top and one bottom.

**Theorem 6** Let $M \subset S^1 \times \mathbb{R}^2$ be a (periodic) properly embedded minimal surface with genus zero so that its lifting to $\mathbb{R}^3$ is a doubly periodic surface. Then $M$ is either a doubly periodic Scherk minimal example (with one limit end) or a KMR example (with two limit ends).

**Proof.** Let $M \subset S^1 \times \mathbb{R}^2$ be in the hypothesis of Theorem 6. Then, its lifting $\tilde{M}$ to $\mathbb{R}^3$ is a properly embedded minimal surface invariant by two independent translations $T_1, T_2$, which can only have genus zero or one in the quotient $\overline{M}$ by both translations.

Suppose that $\tilde{M}$ has non-parallel ends. Thus $T_1$ and $T_2$ are generated by the period at the ends of $\tilde{M}$. When $\tilde{M}$ has genus zero, it must be a doubly periodic Scherk minimal surface, by Theorem 4. Suppose $\tilde{M}$ has genus one, and let $\{\gamma_1, \gamma_2\}$ be a homology basis of $H_1(M, \mathbb{Z})$. By possibly adding to $\gamma_i$, $i = 1, 2$, a finite number of loops around the ends of $\tilde{M}$, we obtain a homology basis $\{\tilde{\gamma}_1, \tilde{\gamma}_2\}$ of $H_1(M, \mathbb{Z})$ so that the period of $\tilde{M}$ along $\tilde{\gamma}_i$ vanishes, which is not possible. Hence, if $\tilde{M}$ has non-parallel ends, it must be a doubly periodic Scherk minimal surface.
Finally, suppose that $\tilde{M}$ has parallel ends. By Theorem 4, $\tilde{M}$ must have genus one, and so it is a KMR example, [14]. This finishes Theorem 6. $\blacksquare$

It was expected that there were no more properly embedded minimal surfaces in $S^1 \times \mathbb{R}^2$ with genus zero and infinitely many ends other than the periodic ones, as in the case of $\mathbb{R}^3$. But this is not true, as the following theorem shows.

**Theorem 7 (Mazet, — & Traizet, [6])** For each non special unbounded convex polygonal domain $\Omega$ with unitary edges (see Definition 2), there exists a non periodic, properly embedded minimal surface $M_\Omega \subset S^1 \times \mathbb{R}^2$ with genus zero, infinitely many ends and one limit end, whose conjugate surface can be obtained from a minimal graph over $\Omega$ with boundary values $\pm \infty$ disposed alternately.

**Definition 2** An unbounded convex polygonal domain is said to be special when its boundary is made of two parallel half lines and one edge of length one (such a domain may be seen as a limit of special domains with $2n$ edges, when $n \to \infty$).

The surfaces $M_\Omega$ appearing in Theorem 7 are obtained by taking limits of saddle towers $M_n$, each $M_n$ with $4n$ ends. In [17] we have proven that the other possible limits for a such sequence $\{M_n\}_n$ of saddle towers are: the singly periodic Scherk minimal example $S_n^\pi$, every doubly periodic Scherk minimal example, a KMR example $M_{\theta,a,0}$ studied in [16] or, after blowing up, a catenoid.
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