FUNNEL FLOWS FROM DISKS TO MAGNETIZED STARS

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ABSTRACT

This work considers flows from an accretion disk corotating with the aligned dipole magnetic field of a rotating star. Ideal magnetohydrodynamics (MHD) is assumed with the pressure and density related as \( p \propto \rho^\gamma \) and with \( \rho v^2 \ll B^2/4\pi \), where \( \rho \) is the flow velocity. Transonic flows, which go from subsonic motion near the disk to supersonic inflow near the star, are shown to be possible only for a range of \( R_d \sim r_c \equiv (GM/\Omega^2)^{1/3} \), where \( R_d \) is the radius at which the dipole field line intersects the disk, \( r_c \) is the corotation radius, \( M \) is the mass of the star, and \( \Omega \) is its angular rotation rate. Over a larger range of \( R_d \lesssim r_c \), subsonic flows from the disk to the star are possible. The transonic flows have very different behaviors for \( \gamma > 7/5 \) and \( \gamma < 7/5 \). In both cases, the plasma flow velocity \( v \) (which is parallel to \( B \)) increases with decreasing distance \( R \) from the star. However, for \( \gamma > 7/5 \), the Mach number \( \mathcal{M} \equiv |v|/c_s \) (with \( c_s \) the sound speed) initially increases to values \( \succeq 1 \) with \( R \) decreasing from \( R_d \), but for \( R \) decreasing from \( \approx 0.22R_d \) (for \( \gamma = 5/3 \)) the Mach number decreases while still being \( \succeq 1 \). In the other limit, \( \gamma < 7/5 \), \( \mathcal{M} \) increases monotonically with decreasing \( R \). Application of these results is made to funnel flows to magnetized neutron stars and young stellar objects. We argue that the rotation needs to be included in the calculation of emission-line profiles of young stellar objects.

Key words: accretion, accretion disks — hydrodynamics — plasmas — pulsars: general — stars: formation — stars: magnetic fields

1. INTRODUCTION

Models of magnetohydrodynamic (MHD) disk accretion to a rotating star with an aligned dipole magnetic field have been discussed by a number of authors (e.g., Ghosh & Lamb 1979; Lovelace, Romanova, & Bisnovatyi-Kogan 1995, 1999). Early theoretical work on accretion onto magnetized T Tauri stars was done by Camenzind (1990) and Königl (1991). More recently funnel flows have been considered by Ostriker & Shu (1995) and Li & Wilson (1999).

Definite predictions can be made about funnel flows from disks to magnetized stars assuming a dipole magnetic field. This strong field limit is relevant to disk accretion to neutron stars where the typical surface magnetic field (Shapiro & Teukolsky 1983) is strong in the sense discussed below (\( \S 9 \)). Much less is known regarding the magnitudes of the surface magnetic fields of T Tauri stars, but typical values are thought to also be strong, \( \sim 1–10 \) kG (Johns-Krull, Valenti, & Koresko 1999). These flows, shown in Figure 1, naturally explain the large infall velocities observed in many T Tauri stars (Hartmann, Hewett, & Calvet 1994; Martin 1996; Najita et al. 2000). Such flows may also occur in disk accretion to rotating magnetized neutron stars (Li & Wilson 1999). Here a study is made of stationary, axisymmetric ideal MHD flows from an accretion disk to a rotating star with an aligned dipole magnetic field. The MHD equations lead to a Bernoulli equation that we analyze in the general case in which pressure and density of a fluid particle are related as \( p \propto \rho^\gamma \) at constant entropy. A Bernoulli equation for the isothermal case \( p \propto \rho \) was discussed earlier by Li & Wilson (1999). Detailed calculations of emission-line shapes were made by Hartmann et al. (1994), Martin (1996), and Muzerolle, Calvet, & Hartmann (2001) for a model where plasma free-falls along dipole field lines, but these authors do not analyze the Bernoulli equation and do not consider the sonic point of the flow. Ostriker & Shu (1995) give calculated field configurations for funnel flows, but they do not consider the variation of fluid variables along field lines and do not discuss the sonic point of the flow.

We use inertial, cylindrical \((r, \phi, z)\), and spherical \((R, \theta, \phi)\) coordinate systems. As in the case of matter outflow along open magnetic field lines threading a disk (Ustyugova et al. 1999), there are a number of integrals of the ideal MHD funnel flow that are conserved along magnetic field lines labeled by the flux function \( \Psi(r, z) = rA_\phi = \text{const} \), where \( A_\phi \) is the vector potential. The constant \( K(\Psi) \) arises from the conservation of mass, \( \Omega(\Psi) \) from conservation of helicity, \( \Lambda(\Psi) \) from conservation of angular momentum, \( S(\Psi) \) from conservation of specific entropy, and \( E(\Psi) \) from conservation of energy (Bernoulli’s constant) (see, e.g., Ustyugova et al. 1999):

\[
\frac{v_p}{4\pi \rho} \propto B, \quad \Omega(\Psi) = \frac{\omega - KB_\phi}{4\pi \rho r}, \quad \Lambda(\Psi) = \omega r^2 - \frac{rB_\phi}{\psi}, \quad S = S(\Psi), \quad E(\Psi) = \frac{v_p^2}{2} + \frac{1}{2} (\omega - \Omega)^2 + w + \Phi_g - \frac{\Omega^2 r^2}{2},
\]

where the \( p \) subscript indicates the poloidal \((r, z)\) or \((R, \theta)\) part of the vector, \( \omega = v_\phi / r \) is the angular velocity of the matter, \( w \) is the specific enthalpy, and \( \Phi_g = -GM/R \) is the gravitational potential of the star. (The mass of the disk is...
assumed negligible.) An additional equation, the Grad-Shafranov equation, follows from the equilibrium of forces across magnetic field lines (Lovelace et al. 1986).

In §2 we discuss simplifications that occur in the strong field limit, and in §3 we analyze the Bernoulli equation (3). In §4 dimensionless variables are introduced. In §5 we discuss the nature of the transonic solutions that go from subsonic flow near the disk to supersonic flow near the star and show that the behavior of the flows is different for $\gamma < 7/5$ and $\gamma > 7/5$. In §6 we describe the nature of the possible subsonic outflows from the disk to the star. In §7 we contrast the considered flows with spherical Bondi accretion. In §8 we discuss conditions for the strong field approximation to hold. In §9 we apply the theory to accretion to a neutron star and to a young stellar object. We argue that the rotation needs to be included in the calculation of emission-line profiles of young stellar objects. In §10 we summarize the work.

2. STRONG FIELD APPROXIMATION

The poloidal magnetic field in the region of the funnel flow sketched in Figure 1 is assumed to be the star’s dipole field $B_p = 3R(\mu \cdot R)/R^3 - \mu/R^3$, where the magnetic moment $\mu$ coincides with the rotation axis of the star and that of the disk. This approximation is applicable under conditions where the magnetic field energy density is larger than the matter kinetic energies,

$$\frac{B_p^2}{8\pi} \gg \left( \rho, \rho v^2 \right).$$

Hence, the matter flow does not disturb the dipole magnetic field significantly, and the toroidal component of the magnetic field is small, $|B_p| \ll |B_d|$. In §8 we give necessary conditions for the strong field approximation to hold.

Plasma in the disk penetrates inward, across the dipole magnetic field to reach the starting region of the funnel flow at $\sim R_d$ as shown in Figure 1. We let $\tau_B$ denote the timescale for radial diffusion/drift of disk plasma across the magnetic field at $\sim R_B$, and note that $\tau_B \geq \tau_{B0} = R_B^2/\eta_B$, where the inequality reflects the dependence of $\tau_B$ on the geometry of the ordered field and $\eta_B$ is the magnetic diffusivity (see Lovelace, Romanova, & Newman 1994). At the same time, plasma drifts inward on a timescale $\tau_{vis} = R_p^2/\nu$, where $\nu = \alpha_{SS} c_s h$ is the Shakura & Sunyaev (1973) formula for the turbulent viscosity in the disk with sound speed $c_s$, half-thickness $h$, and $\alpha_{SS} \leq 1$ a dimensionless constant. For $\tau_{B0} \gg \tau_{vis}$, the disk behaves almost as a perfect conductor: it may extend inward to distances much less than the corotation radius by distorting the dipole field into a cusplike shape (see, e.g., Scharlemann 1978; Aly 1980; Lipunov 1987). Here we consider essentially the opposite limit where $\tau_B \sim \tau_{B0} \sim \tau_{vis}$ and where the dipole field is changed by only a small fractional amount.

In spherical coordinates, the flux function of the dipole magnetic field is $\Psi = \mu \sin^2 \theta/R$. Thus, the equation for the dipole field lines is

$$R = R_d(\Psi) \sin^2 \theta,$$

where $R_d(\Psi) = \mu/\Psi$ is the radius at which the magnetic field line $\Psi = \text{const}$ passes through the disk ($z = 0$). Note that $B_r = (R^2 \sin \theta)^{-1} \partial \Psi / \partial \theta = 2\mu \cos \theta / R^3$ and $B_\theta = -\left( \partial R \sin \theta \right)^{-1} \partial \Psi / \partial R = \mu \sin \theta / R^3$. Figure 1 shows the envisioned geometry.

Equations (1) and (2) can be solved for $\omega$ and $B_p$ to give

$$\omega = \Omega \frac{1 - (\rho_A/\rho)(h/r^2)}{1 - \rho_A/\rho}, \quad B_p = r\Omega \sqrt{4\pi \rho_A} \frac{1 - h/r^2}{1 - \rho_A/\rho},$$

where $\rho_A \equiv K^2/(4\pi)$ and $h \equiv \Lambda/\Omega$. Note that $\rho_A/\rho = \nu_A^2/\nu_{\text{vis}}^2$, the poloidal Alfvén Mach number squared calculated using the Alfven velocity $\nu_A \equiv |B_p|/\sqrt{4\pi \rho}$. Owing to our assumptions, $\rho_A/\rho \ll 1$ everywhere in the considered region. Consequently,

$$\omega - \Omega = \Omega \frac{\rho_A}{\rho} \left( 1 - \frac{h}{r^2} \right), \quad B_p = \Omega r \sqrt{4\pi \rho_A} \left( 1 - \frac{h}{r^2} \right).$$

We assume further that $|\rho_A/\rho| \ll 1$ and $h/r^2 \ll 1$. From equation (7) note that $B_p/|B_p| \ll \Omega \ll \Omega$. Thus, a given fluid particle moves along the poloidal magnetic field $v_p \propto B_p$, and its angular velocity $\Omega(\Psi)$ is a constant. The value $\Omega(\Psi)$ is the angular velocity of rotation of the footpoint of the magnetic field line $\Psi = \text{const}$, which we suppose is frozen into the star. Thus, $\Omega(\Psi) = \Omega_{\text{star}} = \text{const}$ is the angular rotation rate of the star. We omit the subscript on $\Omega$ in the following.

3. BERNOULLI’S CONSTANT

Consider now the matter flow along a specific magnetic field line $R = R_d(\Psi) \sin^2 \theta$. The effective potential along this
line in a reference frame rotating with rate $\Omega$ is

$$\Phi(R) = \Phi_0 + \Phi_c = -\frac{GM}{R} \frac{\Omega^2 R^3}{2} \sin^2 \theta$$

$$= -\frac{GM \Omega^2 R^3}{2R_d}.$$  \hspace{1cm} (8)

The magnitude of the magnetic field along this field line is

$$B_p(R) = \frac{\mu}{R^3} (4 - 3 \sin^2 \theta)^{1/2} = \frac{\mu}{R^3} \left( 4 - 3 \frac{R}{R_d} \right)^{1/2}.$$ \hspace{1cm} (9)

Thus, the Bernoulli constant along this field line is

$$E = F(R, \rho) = \frac{K^2 \mu^2}{32\pi^2 \rho R^6} \left( 4 - 3 \frac{R}{R_d} \right) + w + \Phi(R),$$ \hspace{1cm} (10)

where $w = S \rho^{-1}/(\gamma - 1) = c_s^2/(\gamma - 1)$, with $c_s$ the sound speed and $S \equiv \gamma \rho \rho_\star^2$ the entropy.

We consider flows that are subsonic near the disk, $|v_p| \ll c_s$ for $R \lesssim R_d$. There are then three possible cases: (1) the flow is transonic, going from subsonic to supersonic at some distances $R^* < R_d$; (2) the flow remains subsonic between the disk and the star; or (3) no stationary solution to the Bernoulli equation exists.

For the transonic flows, note that the $\rho$ derivative of equation (10) is

$$\frac{\partial F}{\partial \rho} = \frac{c_s^2 - v_p^2}{\rho},$$ \hspace{1cm} (11)

in that $(\partial v/\partial \rho)_\rho = c_s^2/\rho$. In general, $F(R, \rho)$ has a minimum as a function of $\rho$ at, say, $\rho_\star(R)$, and $E = F(R, \rho)$ has two solutions for $\rho$. The larger $\rho$ solution has $\partial F/\partial \rho > 0$ and corresponds to subsonic flow. The smaller $\rho$ solution has $\partial F/\partial \rho < 0$ and corresponds to supersonic flow.

For the transonic flows, the conditions for a smooth transition through the sonic point are

$$\left. \frac{\partial F}{\partial R} \right|_{R^*} = 0, \quad \left. \frac{\partial F}{\partial \rho} \right|_{R^*} = 0.$$ \hspace{1cm} (12)

In order to pass from subsonic to supersonic flow, this extremum of $F(R, \rho)$ must be a saddle point that corresponds to $\left( \partial^2 F / \partial \rho^2 \right)_{R^*} \left( \partial^2 F / \partial R \partial \rho \right)_{R^*} < 0$. The conditions for this are given in §5.

4. DIMENSIONLESS VARIABLES

With $R$ measured in units of $R_d$, we can write the effective potential as

$$\Phi = -\Omega^2 c_s^2 \left[ \frac{\alpha}{R/R_d} \frac{(R/R_d)^{3}}{2 \alpha^2} \right],$$

where $\alpha = r_c/R_d$, and the characteristic length

$$r_c \equiv \left( \frac{GM}{\Omega^2} \right)^{1/3} \approx 1.5 \times 10^8 \text{ cm} \left( \frac{M}{M_\odot} \right)^{1/2} \left( \frac{P}{1 \text{ s}} \right)^{2/3}.$$ \hspace{1cm} (13)

is the corotation radius and $P = 2\pi/\Omega$ is the star’s rotation period. For a young star of solar mass, $r_c \approx 2.9 \times 10^{11} \text{ cm}(P/1 \text{ day})^{2/3}$.

Useful dimensionless variables are

$$\hat{R} = \frac{R}{R_d}, \quad \hat{\rho} = \frac{4\pi \alpha \Omega R_d^4 \rho}{K\mu},$$

$$\hat{E} = \frac{E}{\Omega^2 c_s^2}, \quad \hat{S} = \frac{S}{\Omega^2 c_s^2} \left( \frac{K\mu}{4\pi \alpha \Omega R_d^4} \right)^{\gamma-1}.$$ \hspace{1cm} (14)

and $F = F/(\Omega r_c^2)$, $(\hat{E})^2 = S\hat{\rho}^{\gamma-1} = c_s^2/\Omega r_c^2$, $\hat{v}_p = v_p/\Omega r_c$ = $(4 - 3R/R_d^2)/(\hat{\rho} R_d)$). Note that at $\hat{R} = 1$, $\hat{\rho}\hat{v}_p = 1$. For simplicity of notation we now drop the hats. We then have $R_{\star\star}/R_d < R \leq 1$, where $R_{\star\star}$ is the star’s radius.

In terms of these variables, Bernoulli’s equation becomes

$$E = F(R, \rho) = \frac{4 - 3R}{2\rho^2 R_d^2} + \frac{S\rho^{-1}}{\gamma - 1} - \left( \frac{\alpha}{R} + \frac{R^3}{2\alpha^2} \right).$$ \hspace{1cm} (15)

We suppose that the disk matter is at a relatively low temperature, that is, $w(R_d) = c_s^2(R_d)/(\gamma - 1) \ll -\Phi(R_d)$, and that the outflow speed from the disk is small, $\hat{v}_p^2(R_d) \ll -\Phi(R_d)$. Under these conditions the dimensionless quantities at the starting point of the flow ($R = 1$ are the Bernoulli constant, $E \approx -\alpha - 1/2\alpha^2$; the sound speed, $c_s(1) \ll 1$; and the density, $\rho(1) \gg 1$. Because $S = c_s^2/\rho^{-1}$, we also have $S = S(1) \ll 1$.

5. TRANSONIC FLOWS

The mentioned conditions for a smooth transition through the sonic point give

$$0 = \frac{\partial F}{\partial \rho} = \frac{3(8 - 5R^*_\star)}{2\rho^*_\star R^*_\star} + \frac{3R^*_\star}{2\alpha^2},$$ \hspace{1cm} (16)

$$0 = \frac{\partial F}{\partial \rho} = -\frac{4 - 3R^*_\star}{\rho^*_\star R^*_\star} + S\rho^*_\star^{-2}.$$ \hspace{1cm} (17)

In general, $F_{\rho\rho^*_\star} > 0$. In order for this critical point to be a saddle point of $F(R, \rho)$, we must have $\left( \partial^2 F / \partial \rho^2 \right)_{R^*_\star} < \left( \partial^2 F / \partial R \partial \rho \right)_{R^*_\star}$. This requirement is satisfied if $R^*_\star > (N/D)^{1/4} / (2\alpha^3/3)^{1/4}$, where $N = 30\gamma R^*_\star^2 - 45R^*_\star^3 + 140\gamma R^*_\star - 100\gamma R^*_\star + 80\gamma - 112$ and $D = 60\gamma R^*_\star^2 - 15R^*_\star^3 + 52\gamma R^*_\star - 188\gamma + 144\gamma - 48$. This condition on $R^*_\star$ is always satisfied if $\gamma \leq 7/4$. For $\gamma = 5/3$, the condition is satisfied for $R^*_\star > 0.581(2\alpha^3/3)^{1/4}$. The nature of the function $F(R, \rho)$ can be understood from its contour plot shown in Figure 2a for a sample case.

The term in equation (16) involving $\rho^*_\star$ can be rewritten in terms of $S$ using equation (17). This term is then seen to be negligible compared with the terms involving $\alpha$ because $S \ll 1$. Equation (16) is then seen to correspond to a balance of the gravitational attraction and centrifugal force along the field line. Under this condition, equation (16) can be satisfied only for

$$\alpha \leq \alpha_0 \equiv \left( \frac{3}{2} \right)^{1/3} \approx 1.145.$$ \hspace{1cm} (18)

Consequently,

$$R^*_\star \approx \left( \frac{2\alpha^3}{3} \right)^{1/4}, \quad \rho^*_\star \approx S^{1/(\gamma + 1)} \left( \frac{4 - 3R^*_\star}{R^*_\star} \right)^{1/(\gamma + 1)},$$

$$c_s^* \approx \left( S\rho^*_\star^{-1} \right)^{1/2} \approx S^{1/(\gamma + 1)} \left( \frac{4 - 3R^*_\star}{R^*_\star} \right)^{1/2(\gamma + 1)}. \hspace{1cm} (19)$$
Thus, the mass flux density is \( \rho_s v_s = \rho_s c_s \) where \( \rho_s \) is the density at the saddle point and \( v_s \) and \( c_s \) are the speed of sound and the sound speed, respectively.

A possible subsonic flow would be defined by \( S < 0.863 \), then as mentioned below the saddle point given by equation (17) disappears (by becoming a minimum). The saddle point given by equation (19) is the only saddle point with \( R^* \leq 1 \).

Figure 3 shows the dependencies of \( R_s \) and \( c_s \) on \( \alpha = r_p / R_p \) for the cases of \( \gamma = 5/3 \) and 1.3 and dimensionless entropy \( S = 0.005 \). The dashed line is for the approximation of eq. (19). For the \( \gamma = 5/3 \) case, the saddle point of \( F(R, \rho) \) changes into a minimum for \( \alpha < 0.218 \) so that these values are not of interest.

Figures 4 and 5 show sample radial profiles of the fluid variables for \( \gamma = 5/3 \) and 1.3, respectively. A value of \( \gamma \) smaller than the ideal monotonic gas value is of interest for high-temperature plasmas where the electron heat conduction can be modeled by smaller \( \gamma \) (as is the case for the solar wind). The difference in behavior of the cases shown in Figures 4 and 5 can be understood by considering the Bernoulli equations for the cases of \( \gamma = 5/3 \) and 1.3, which is the freefall speed, \( \rho \approx \left(2/\alpha \right)^{1/2} R^{-3/2} \), and \( c_s \approx \left(2/\alpha \right)^{3/4} R^{3/2} S^{-1/4} \). Hence, the Mach number varies as \( M = v_p / c_s \approx 2(\alpha/2)^{1/4} S^{-1/2} R^{(5\gamma - 7)/4} \). Thus, for \( \gamma > 7/5 \), the Mach number initially increases with decreasing \( R \), but for small enough \( R \), \( M \) decreases with decreasing \( R \).

The radius of the maximum of \( M \) is found to be \( R_{\text{max}} \approx (5\gamma - 7)/{6(\gamma - 1)\left[1 + 1/(2\alpha^3)\right]} \). The maximum value of \( M \) is therefore \( S^{-1/2} \). On the other hand, for \( \gamma < 7/5 \), \( M \) increases monotonically with decreasing \( R \). Therefore, the changeover in behavior between Figures 4 and 5 occurs at \( \gamma = 7/5 \). The radial variation of both \( v_p \) and \( M \) is clearly important for calculations of the emission-line profiles (Hartmann et al. 1994).

For \( \gamma > 7/5 \), the Mach number does not decrease through \( M = 1 \). For \( R \leq 1 \), Bernoulli’s equation is \( 2(\rho^2 R^3) + S \rho^{-1} (\gamma - 1) \approx \alpha \). Substituting \( \rho = \eta / R^\theta \) with \( \theta = \gamma / (\gamma + 1) \) gives \( 2/\eta^2 + S \eta^{-1} / (\gamma - 1) \) \( \equiv L(\eta) \approx \alpha R^{(5\gamma - 7)/(\gamma - 1)} \). Clearly, \( L(\eta) \) has a minimum value of \( L_{\text{min}} = [2 + 4/(\gamma - 1)S]^{1/(\gamma - 1)} \) at \( \eta_{\text{min}} = (4/S) \), which corresponds to \( M = 1 \). Thus, for \( \gamma > 7/5 \) there is a minimum radius where a stationary flow exists, \( R_{\text{min}} = [(2/\alpha)(\gamma + 1)/(\gamma - 1)]^{1/(5\gamma - 7)} \). For most conditions this radius is expected to be less than the radius of the star.

Near the star the flow is expected to have a standing shock where \( M \) discontinuously decreases to a value less than...
unity. For $\gamma < 7/5$ there also must be a shock. Note that the ratio of the temperatures across this shock for an ideal gas is $T_2/T_1 = 1 + 2(\gamma - 1)(\mathcal{M}^2 + 1)(\mathcal{M}^2 - 1)/[(\gamma + 1)^2 \mathcal{M}^2]$ [$\approx 2\gamma(\gamma - 1)/[\mathcal{M}^2]$ for $\mathcal{M} \gg 1$], where $T_1$ is the upstream temperature and $T_2$ is downstream (closer to the star). Calculation of the location of this shock is beyond the scope of this paper.

The value of entropy $S$ for transonic flows $S_{ts}$ is determined by conditions at the origin of the flow at the disk where $c_{sd}^2 \ll c_{ad}^2$, where $c_{sd}$ is the sound speed at the disk. At the disk

$$E = \frac{c_{ad}^2}{\gamma - 1} - \left(\alpha + \frac{1}{2\alpha^2}\right),$$

while at the sonic point

$$E = \frac{1}{2\gamma - 1} c_{s*}^2 - \left(\alpha + \frac{R_s^3}{R_{*}^3}\right).$$

Note that $c_{s*}^2 = S \rho_{s*}^{-1}$ and $\rho_{s*} = (4 - 3R_s)^{1/2}/(c_{s*} R_{*}^4)$. Therefore, in order to have a transonic flow, we need

$$S = S_{ts} \equiv \left(\frac{2}{\gamma + 1}\right)^{(\gamma+1)/2} \left(\frac{R_s^6}{4 - 3R_s}\right)^{(\gamma-1)/2} \times \left[c_{ad}^2 - C^2(\alpha)\right]^{(\gamma+1)/2},$$

where

$$C^2(\alpha) = (\gamma - 1)\left(\alpha + \frac{1}{2\alpha^2} - c_1 \alpha^{1/4}\right)$$

is obtained by using equation (19) and $c_1 = (3/2)^{1/4} + (2/3)^{1/4}/2 \approx 1.476$. Figure 6 shows $C(\alpha)$. Clearly, for a given value of $\alpha$ it is necessary to have sufficiently high disk temperature or disk sound speed, $c_{sd} \geq C(\alpha)$. This inequality and equation (18) for transonic flow can be satisfied only for

$$\alpha_{\min} \leq \alpha \leq \alpha_0,$$

where $\alpha_{\min}$ is such that $c_{sd} = C(\alpha_{\min})$. Equation (21) corresponds to a limited range of $R_{d}, r_c/\alpha_0 < R_d < r_c/\alpha_{\min}$. For
example, for constant \( c_{\text{sd}} = 0.1 \) and \( \gamma = 5/3 \), we have \( c_{\text{sd}} \geq C(\alpha) \) for \( 1.03 \leq \alpha \leq 1.145 \) and \( S \) varies between 0 and 0.00147; for \( c_{\text{sd}} = 0.2 \), the allowed range is \( 0.928 \leq \alpha \leq 1.145 \) and \( S \) varies between 0 and 0.00933.

6. SUBSONIC FLOWS

For \( \alpha_{\text{min}} \leq \alpha \leq \alpha_0 \) (with \( \alpha_{\text{min}} \) given above), it is easy to show that there are outflows from the disk to the star that remain subsonic. For \( R \ll 1 \), equation (15) gives \( c_s \approx |\alpha(\gamma - 1)/R|^{1/2} \) and \( \rho \approx |\alpha(\gamma - 1)/(SR)|^{1/(\gamma - 1)} \). Generally, the densities of these flows are much larger than for the transonic flows. The subsonic flows are not relevant to the line formation in young stellar objects.

For \( \alpha < \alpha_{\text{min}} \), there are no stationary solutions to the Bernoulli equation (15). Equivalently, there are no stationary outflows for \( R_D > r_f/\alpha_{\text{min}} \). For \( \alpha > \alpha_0 \) or \( R_D < r_f/\alpha_0 \), there are also subsonic outflows from the disk to the star.

7. CONTRAST WITH BONDI ACCRETION

The Bernoulli equation (10) differs from that for spherical Bondi (1952) accretion in that here the plasma flows along the magnetic field lines, the matter is rotating, and the flow starts from a finite radius of a disk. For the Bondi flow, mass conservation gives \( 4\pi R^2 \rho v_R = \dot{M} = \text{const} \), so that the Bernoulli equation is

\[
E = \frac{(\dot{M}/4\pi)^2}{2\rho^2 R^2} + \frac{S\rho^{\gamma - 1}}{\gamma - 1} - \frac{GM}{R},
\]

where \( S = c_{\text{ex}}^2 / \rho_{\infty}^{\gamma - 1} \), with \( c_{\text{ex}} \) and \( \rho_{\infty} \) the sound speed and density at a large distance from the star. In contrast, the first term of equation (10) is \( \propto 1/(\rho^2 R^6) \) owing to the “focusing” of the centrifugal potential, and further, equation (10) includes the centrifugal potential, and equation (10) for small \( R \), the Bondi flow has \( v_R \approx (2GM/R)^{1/2} \) as in the funnel flow; however, \( \rho \sim 1/R^{3/2} \) and \( c_s \sim 1/R^{(3\gamma - 1)/4} \), whereas for the funnel flow \( \rho \sim 1/R^{5/2} \) and \( c_s \sim 1/R^{(5\gamma - 1)/4} \). Thus, the Mach number for the Bondi flow \( M_c \sim 1/R^{(5\gamma - 3)/4} \) always increases (or is constant) with decreasing \( R \) for \( \gamma \leq 5/3 \).

8. VALIDITY OF STRONG FIELD APPROXIMATION

In the strong field approximation of § 2, the disk plasma is assumed to corotate with the star. Equilibrium of this part of the disk requires an additional outward radial force that arises naturally from the slight bending of the field lines passing through the disk. This radial magnetic force (per unit area of the disk) is simply

\[
F_{\text{mag}} = (B_f B_c)/2\pi > 0,
\]

where the \( h \) subscript indicates evaluation at the disk surface \( z = h \) (eq. [5] of Lovelace et al. 1995). Because the region of the disk has \( r_e - r_l < 1 \), the radial magnetic field is \( (B_f/B_c)_1 \approx 6\pi\Omega^2(r_e - r_l)/B_c^2 \). Introducing the nominal Alfvén radius \( r_A = [\mu^4/(2GM^2)]^{1/7} \) (Shapiro & Teukolsky 1983) and the viscosity parameter \( \alpha_{\text{SS}} \) (Shakura & Sunyaev 1973) so that \( \dot{M} = 2\pi \sigma v_r |r_v| \) with \( |r_v| = \alpha_{\text{SS}} \sigma c_s R/h \) allows us to write

\[
\frac{|B_z|}{B_c} \approx \frac{3}{\sqrt{2}} \left( \frac{r}{r_A} \right)^{7/2} \Omega^2 r_e - r_l - \frac{c_s}{\alpha_{\text{SS}}^2 c_s^2}
\]

assuming \( h/r \approx c_s/(\Omega r) \). The validity of the strong field limit assumed here requires \( |B_z|/B_c |r_v| < 1 \), which corresponds to

\[
\frac{(r_c/r_A)^{7/2}}{\alpha_{\text{SS}} \left( \frac{c_s}{\Omega r_A} \right)^2} \frac{c_s}{|r_v| - r_l}.
\]

With representative values, \( \alpha_{\text{SS}} = 0.1, c_s/(\Omega r_A) = 0.1 \), and \( r_c/|r_v| < 10 \), it is clear that we need \( (r_c/r_A)^{7/2} < 0.01 \) or \( r_c/r_A < 0.027 \). The very strong dependence of equation (23) on the ratio \( r_c/r_A \) means that \( r_A \) needs to be only somewhat larger than \( r_c \) for the strong field approximation to apply.

Furthermore, in the approximation of § 2 the toroidal magnetic field is neglected. We are interested in the field lines from the star that go through the disk near \( r \), since this is where the funnel flow originates. Field lines that in vacuum would go through the disk at larger radii are expected to be open (Lovelace et al. 1995) as shown in Figure 1. A deviation of the disk rotation rate \( \omega_d \) for \( r \sim r_c \) from the star’s rate \( \Omega \) leads to a toroidal magnetic field at the disk, \( B_0/h = - (h/r \eta)(\omega_d - \Omega) B_z \), where \( \eta \) is the magnetic diffusivity of the disk (eq. [2] of Lovelace et al. 1995). If \( \eta \) is of the order of the viscosity \( \nu \) given by the Shakura & Sunyaev (1973) prescription \( \nu = \alpha_{\text{SS}} c_s h \) (with \( \alpha < 1 \)), then, in order to have \( |(B_0)/B_z| \ll 1 \), we must have \( |(\omega_d - \Omega)| \ll \alpha_{\text{SS}} c_s \) for \( r \sim r_c \).

9. APPLICATIONS

Table 1 gives illustrative parameters for the cases of disk accretion to (1) a rotating magnetized neutron star and (2) a rotating young stellar object. The values for the neutron star are from Shapiro & Teukolsky (1983), and those for the young stellar object are from Johns-Krull et al. (1999). In both cases, the nominal Alfvén radius \( r_A \) (Shapiro & Teukolsky 1983) is larger than \( r_c \), so that the disk plasma is expected to corotate with the star. The disk plasma may move radially across the magnetic field by the interchange instability (Kaisig et al. 1992; Rastätter & Schnell 1999). For the neutron star the surface magnetic field \( (\mu/B_{\text{surf}}^2) \) is assumed to be \( 10^{24} \) G, while for the young star it is \( 10^4 \) G.

For the neutron star case, we assume \( c_{\text{sd}}/(\Omega r_A) = 0.1 \) (with dimensions restored), which corresponds to a surface temperature of the disk \( T_d \approx 4.1 \times 10^7 \) K for a fully ionized hydrogen plasma and \( \gamma = 1.3 \) with mean particle mass \( m_{\text{H}}/\)
The half-thickness of the disk is \( h \sim c_s / \Omega \approx 0.1 r_* \). From Figure 6 with \( \gamma = 1.3 \), we find that transonic outflows from the disk to the star are possible for \( \alpha \) in the range \( \alpha = r_c / r_d \approx 0.95 - 1.145 \), which corresponds to \( R_d \approx (0.873 - 1.053) r_c \). The sonic point location \( r_s / R_d = \sin^2 \theta_s \) has \( \theta_s \) varying between 69° (the lower limit of \( \alpha \)) and 90° where the sonic point is inside the disk. The height of the sonic point \( z_s = R_s \sin^2 \theta_s \cos \theta_s \frac{\Omega}{c_s} \lesssim 0.273 r_c \). Therefore, if all of the disk accretion goes into transonic flow along the field lines, \( M_{\text{acc}} = 2(\pi) R_d^3 \kappa \mu / (4 \pi R_c^2) \), where one of the factors of 2 in the numerator accounts for the two sides of the disk. Thus, the characteristic density normalizing \( \rho \) in equation (14) is

\[
\rho_0 = \frac{K \mu}{4 \pi \alpha \Omega R_d^3} = \frac{M_{\text{acc}}}{4 \pi \alpha \Omega R_d^3 \Delta R_d}.
\]

For the mentioned values and those in Table 1, we find \( \rho_0 \approx 1.7 \times 10^{-8} \) g cm\(^{-3}\). The ratio of the maximum to minimum radii of the flow is \( r_c / R_d = 170 \), and the free-fall speed at the star is \( v_{\text{ff}} = (2GM_*/R_d)^{1/2} \approx 2 \times 10^{10} \) cm s\(^{-1}\).

The angle at which the flow hits the star is \( \theta_s = \sin^{-1} \left( \frac{r_s / R_d}{r_c / R_d} \right) \approx 4.4° \). At this point the flow velocity is an angle \( \approx \theta_s / 2 \) with respect to \( R \).

For the case of a young stellar object, we assume \( c_{\text{ad}} / (\Omega r_c) = 0.05 \), which corresponds to a surface temperature of the disk \( T_d \approx 2800 \) K for a slightly ionized hydrogen plasma and \( \gamma = 5/3 \). The half-thickness of the disk is \( h \sim c_s / \Omega \approx 0.05 r_c \). The height of the sonic point is \( z_s \ll 0.174 r_c \). The considerations for this case are similar to those for the neutron star. The range of \( \alpha \) is \( 1.08 - 1.145 \), \( R_d \approx (0.873 - 0.926) r_c \), and \( \Delta R_d \approx 0.053 r_c \). The angle to the sonic point varies from \( \theta_s \approx 78° \) (at the lower limit of \( \alpha \)) to 90°. The characteristic density in this case is \( \rho_0 \approx 2.8 \times 10^{-11} \) g cm\(^{-3}\), assuming that all of the disk accretion goes into transonic flow along the field lines. The ratio \( r_c / R_d = 7.4 \) is much smaller than for the neutron star case, and \( v_{\text{ff}} (R_d) \approx 430 \) km s\(^{-1}\). The angle at which the flow hits the star is \( \theta_s / 2 \approx 21° \). At this point the flow velocity is an angle \( \approx 10^{-7} \)° with respect to \( R \).

The funnel flow has velocity \( v(r) = v_p + \Omega \mathbf{R} \sin(\theta) \phi \), where \( v_p \) is the poloidal velocity along the magnetic field lines. The toroidal velocity for the funnel flow is \( \approx \Omega r \mathbf{R} \sin \theta \) for \( r \lesssim r_c = (GM/R)^{1/3} \). On the other hand, the poloidal velocity is \( v_p \approx (2GM/R)^{1/2} \) close to the star. The ratio of these velocities is \( (R/r_c)^{3/2} (\sin \theta) / \sqrt{2} \).

From Table 1, it is clear that this ratio is much smaller than unity for \( R \sim R_{\text{star}} \) for both young stellar objects and neutron stars. Note, however, that for optically thin emission the contribution to the line profile from a given region depends on the variation of the line-of-sight velocity \( v_p \equiv v \cdot \hat{n} \), where \( \hat{n} \) is the normal in the direction of the observer. The rapid variation of \( v_p \) close to the star tends to reduce the high-velocity shift of the emission. The influence of rotation is to reduce the variation of \( v_p \) over an extended region in the vicinity of the origin of the flow at \( r_c \). For \( \sim r_c \) the influence of rotation on optically thin line shapes is more important. Line calculations by Hartmann et al. (1994) and Martin (1996) neglect the rotation. Sample calculations by Muzerolle et al. (2001) indicate that rotation does not have a large influence on the line shapes, but a systematic analysis has not been done.

### 10. CONCLUSIONS

This work analyzes the stationary ideal MHD funnel flows along the magnetic field lines of a rotating star with an aligned dipole magnetic field. Isentropic flow is assumed so that a given plasma blob maintains \( p / \rho^\gamma = \text{const} \) with \( 1 < \gamma < 5/3 \), and the magnetic field is assumed strong, \( \beta \gg B^2 / 4\pi \). Earlier, Li & Wilson (1999) considered the Bernoulli equation assuming \( p \propto \rho \). Close to the disk the flow is subsonic. There are three possible cases: (1) stationary transonic flow where a sonic point exists along the dipole field line, (2) stationary subsonic flow, and (3) no stationary flow exists. Over a range of \( R_d \ll r_c \), subsonic infall is possible, where \( R_d \) is the radius at which the dipole field line intersects the disk and \( r_c \) is the “corotation radius.” For \( R_d \gg r_c \), there are no stationary flows.

The transonic flows, which become free-fall inflow near the star, are possible only for a narrow range of \( R_d \sim r_c \). At the sonic point these flows have an approximate balance of the centrifugal and gravitational force along the field line. More generally, for cases in which the magnetic field is not a simple dipole the required force balance along the field line at the sonic point can be written as

\[
B_p \cdot \nabla \Phi (r, z) \bigg|_{\text{sonic}} = 0,
\]

where \( \Phi \equiv -GM / (r^2 + z^2)^{1/2} - \Omega^2 r^2 / 2 \) is the effective potential in the rotating reference frame. The flows have different behaviors for \( \gamma > 7/5 \) and \( \gamma < 7/5 \). For \( \gamma > 7/5 \), the Mach number \( \mathcal{M} \equiv |v| / c_s \) initially increases with \( R \) decreasing.
from $R_d$, but for $R$ decreasing from $\approx 0.22 R_d$ (for $\gamma = 5/3$) the Mach number decreases. In the other case, $\gamma < 7/5$, $\mathcal{M}$ increases monotonically with decreasing $R$.

The variation of both $v_p$ and $\mathcal{M}$ is important to the nature of the standing shock near the star and to the determination of emission-line profiles. For the emission-line profiles of young stellar objects we argue that rotation of the funnel flow is important (§ 9). We discuss numerical values for funnel flows to neutron stars and young stellar objects. For the neutron star case we argue that $\gamma$ is likely to be less than $7/5$. This combined with the large value of $r_c/R_{\text{star}}$ implies that the Mach number of the flow approaching the star should be much larger than unity. On the other hand, for flows to a young stellar object where $\gamma > 7/5$ and $r_c/R_{\text{star}}$ is, say, $\lesssim 10$, the Mach numbers are larger than unity but small compared to those in the neutron star case. The flow solutions analyzed in this work are of interest for comparison with MHD simulations of funnel flows (Romanova et al. 2002).

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