MONOTONICITY FORMULAS OF EIGENVALUES AND ENERGY FUNCTIONALS ALONG THE RESCALED LIST’S EXTENDED RICCI FLOW

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Abstract. In this paper, we study monotonicity formulas of eigenvalues and entropies along the rescaled List’s extended Ricci flow. We derive some monotonicity formulas of eigenvalues of Laplacian which generalize those of Li in [8] and Cao-Hou-Ling in [3]. Moreover, we also consider monotonicity formulas of $F_k$-functional which can be seen as a generalized $F$-functional corresponding with steady Ricci breathers, and $W_k$-functional which generalizes $W$-functional corresponding with expanding Ricci breathers.

1. Introduction

Let $(M^n, g(t))$ be a compact Riemannian manifold, $g(t)$ be a solution to the following List’s extended Ricci flow which was introduced by B. List:

\[ \begin{cases} \frac{\partial}{\partial t}g = -2\text{Ric} + 2\alpha d\varphi \otimes d\varphi, \\ \varphi_t = \Delta \varphi, \end{cases} \tag{1.1} \]

where $\alpha > 0$ is a real constant, $\varphi = \varphi(t)$ is a smooth scalar function defined on $M^n$ and $\Delta$ denotes the Laplacian given by $g(t)$. When $\alpha = 2$ and $\alpha = \frac{n-1}{n-2}$, the extended Ricci flow (1.1) have been studied by List in [6] and [7], respectively. Denote by $S_{ij} = R_{ij} - \alpha \varphi_i \varphi_j$ a symmetric two-tensor. Then (1.1) becomes

\[ \begin{cases} \frac{\partial}{\partial t}g_{ij} = -2S_{ij}, \\ \varphi_t = \Delta \varphi. \end{cases} \tag{1.2} \]

In this paper, we consider the following rescaled List’s extended Ricci flow

\[ \begin{cases} \frac{\partial}{\partial t}g_{ij} = -2(S_{ij} - \frac{r}{n} g_{ij}), \\ \varphi_t = \Delta \varphi, \end{cases} \tag{1.3} \]

where $r = r(t)$ is a function depending only on $t$. In particular, (1.2) can be seen as a special case of (1.3) when $r = 0$. On the other hand, if $r(t) = (\int_M S \, dv)/(\int_M dv)$ (that is, $r$ is the average value of $S$), then (1.3) can be seen as the extended Hamilton normalized flow under the List’s extended Ricci flow. Here

\[ S = g^{ij}S_{ij} = R - \alpha |\nabla \varphi|^2 \tag{1.4} \]
is the trace of the two-tensor $S_{ij}$. For a constant $k \geq 1$, we defined the $\mathcal{F}_k$-functional as follows

$$\mathcal{F}_k = \int_M (|\nabla f|^2 + kS) e^{-f} dv. \quad (1.5)$$

If we define

$$\lambda(g) = \inf_f \mathcal{F}_k, \quad (1.6)$$

where the infimum is taken over all smooth function $f$ which satisfies

$$\int_M e^{-f} dv = 1, \quad (1.7)$$

then the nondecreasing of the $\mathcal{F}_k$-functional implies the nondecreasing of $\lambda(g)$. In particular, $\lambda(g)$ defined in (1.6) is the lowest eigenvalue of the operator $-4\Delta + kS$. In the first part, we consider eigenvalues of the operator

$$-\Delta + bS \quad (1.8)$$

with $b$ a constant. For $b = 0$, we first derive the following evolution equation of eigenvalues on Laplacian under the rescaled List’s extended Ricci flow (1.3). That is, we obtain

**Theorem 1.1.** Let $\lambda^{-\Delta}(t)$ be the eigenvalue of the operator $-\Delta$ corresponding to the normalized eigenfunction $u$, that is,

$$-\Delta u = \lambda^{-\Delta} u, \quad \int_M u^2 dv = 1. \quad (1.9)$$

Then under the rescaled List’s extended Ricci flow (1.3),

$$\frac{d}{dt} \lambda^{-\Delta} = -\frac{2r}{n} \lambda^{-\Delta} + \int_M \left( \lambda^{-\Delta} S u^2 - S|\nabla u|^2 + 2S^{ij} u_i u_j \right) dv. \quad (1.9)$$

**Theorem 1.2.** Let $\lambda^{-\Delta+\frac{1}{2}S}(t)$ be the eigenvalue of the operator $-\Delta + \frac{1}{2}S$ corresponding to the normalized eigenfunction $u$, that is,

$$(-\Delta + \frac{1}{2}S) u = \lambda^{-\Delta+\frac{1}{2}S} u, \quad \int_M u^2 dv = 1. \quad (1.10)$$

Then under the rescaled List’s extended Ricci flow (1.3),

$$\frac{d}{dt} \lambda^{-\Delta+\frac{1}{2}S} = -\frac{2r}{n} \lambda^{-\Delta+\frac{1}{2}S} + \int_M \left[ |S_{ij}|^2 u^2 + 2S^{ij} u_i u_j + \alpha(\Delta \varphi)^2 u^2 \right] dv. \quad (1.10)$$
Moreover, if \( S_{ij}(t) \geq 0 \) for all \( t \), the eigenvalues of the operator \(-\Delta + \frac{1}{2} S\) satisfy

\[
\frac{d}{dt} \left( \lambda - \Delta e^{\frac{1}{2} \int_0^t r(s) \, ds} \right) = e^{\frac{1}{2} \int_0^t r(s) \, ds} \left\{ \int_M \left[ |S_{ij}|^2 u^2 + 2 S_{ij} u_i u_j + \alpha (\Delta \varphi)^2 u^2 \right] dv \right\} \geq 0
\]

(1.11)

and \( \lambda - \Delta e^{\frac{1}{2} \int_0^t r(s) \, ds} \) is nondecreasing under the rescaled List's extended Ricci flow (1.3). Furthermore, the monotonicity is strict unless the metric is Ricci flat.

It is well-known that, under the List’s extended Ricci flow (1.2), the nonnegativity of \( S \) is preserved. In this paper, we will prove that the nonnegativity of \( S \) is also preserved under the rescaled List’s extended Ricci flow (1.3) for all \( r(t) \). That is,

**Theorem 1.3.** The nonnegativity of \( S \) is preserved under the rescaled List’s extended Ricci flow (1.3).

In order to state the following results on eigenvalues, we first introduce the following definition:

\[ S_{\min}(0) = \min_{x \in M} S(x, 0). \]

By virtue of Theorem 1.3, we prove the following

**Theorem 1.4.** Let \((g(t), \varphi(t))\) be a solution to the rescaled List’s extended Ricci flow (1.3) with \( S_{ij}(t) \geq \theta g_{ij}(t) \) holding for some \( \theta \geq \frac{1}{2} \). Let \( \lambda - \Delta(t) \) be the eigenvalue of the operator \(-\Delta\).

1. If \( S_{\min}(0) \geq 0 \), then \( \lambda - \Delta e^{\frac{1}{2} \int_0^t r(s) \, ds} \) is nondecreasing along the rescaled List’s extended Ricci flow (1.3).

2. For all \( t \), we have

\[
\frac{d}{dt} \ln \left( \lambda - \Delta e^{\frac{1}{2} \int_0^t r(s) \, ds} \right) \geq 2 \theta x(t).
\]

(1.12)

Moreover, \( \lambda(t) \) has the lower bound

\[
\lambda - \Delta(t) e^{\frac{1}{2} \int_0^t r(s) \, ds} \geq \lambda(0) e^{2 \theta \int_0^t x(s) \, ds}
\]

(1.13)

depending only on \( t \), where

\[
x(t) = \frac{S_{\min}(0) e^{\frac{1}{2} \int_0^t r(s) \, ds}}{1 - \frac{2}{n} S_{\min}(0) \int_0^t \left( e^{-\frac{1}{2} \int_0^s r(\tilde{s}) \, d\tilde{s}} \right) ds}.
\]

In particular, for compact Riemannian surfaces, we obtain the following consequences from Theorem 1.4.
Corollary 1.5. Let \((g(t), \varphi(t))\) be a solution to the rescaled List’s extended Ricci flow \((1.3)\) on \(M^2\). Let \(\lambda^{-\Delta}(t)\) be the eigenvalue of the operator \(-\Delta\).

(i) If \(R_{ij} \leq \epsilon u_i u_j\),

where \(\epsilon \leq \frac{2\alpha(\theta - 1)}{2\theta - 1}\) with \(\theta > \frac{1}{2}\), then the following holds:

1. \(\lambda^{-\Delta} e^{\int_0^t r(s) ds} \) is nondecreasing along the rescaled List’s extended Ricci flow \((1.3)\).

Moreover, \(\lambda(t)\) has the lower bound

\[\lambda^{-\Delta}(t) e^{\int_0^t r(s) ds} \geq \lambda(0) e^{2\theta \int_0^t x(s) ds},\] \hspace{1cm} (1.15)

depending only on \(t\), where

\[x(t) = \frac{S_{\min}(0) e^{\int_0^t r(s) ds}}{1 - S_{\min}(0) \int_0^t \left( e^{-r(s) ds} \right) ds}.\]

(ii) If \(|\nabla \varphi|^2 g_{ij} \geq 2\varphi_i \varphi_j\), then the following holds:

1. \(\lambda^{-\Delta} e^{\int_0^t r(s) ds} \) is nondecreasing along the rescaled List’s extended Ricci flow \((1.3)\).

Moreover, \(\lambda(t)\) has the lower bound

\[\lambda^{-\Delta}(t) e^{\int_0^t r(s) ds} \geq \lambda(0) e^{\int_0^t x(s) ds},\] \hspace{1cm} (1.16)

depending only on \(t\), where

\[x(t) = \frac{S_{\min}(0) e^{-\int_0^t r(s) ds}}{1 - S_{\min}(0) \int_0^t \left( e^{-r(s) ds} \right) ds}.\]

Remark 1.1. It should be pointed out that for \(r = 0\) and \(\alpha = 2\), our above results on eigenvalues reduce to the corresponding results of Li in \([8]\). In particular, our Theorem \([1.3]\) is new.

Remark 1.2. Some related results for monotonicity formulas of eigenvalues on Laplacian along the Ricci flow, we refer to \([1,2,10]\) and among others \([5]\) for later development.

Next, we study monotonicity formulas of eigenvalues on Laplacian on Riemannian surfaces. We obtain the following results:
Theorem 1.6. Let $\lambda^{-\Delta + bS}(t)$ be the eigenvalue of the operator $-\Delta + bS$ with normalized eigenfunction $u$ on $M^2$ with $S(t) \geq 0$ holding for all $t$, that is,

$$(-\Delta + bS)u = \lambda^{-\Delta + bS} u, \quad \int_M u^2 \, dv = 1.$$ 

Then if $|\nabla \varphi|^2 g_{ij} \geq 2\varphi_i \varphi_j$ and $0 < b \leq \frac{1}{2}$, we have

$$\frac{d}{dt} \left( \lambda^{-\Delta + bS} e^{\int_0^t r(s) \, ds} \right) \geq e^{\int_0^t r(s) \, ds} \left\{ \int_M \left\{ 2b^2 S^2 u^2 + (1 - 2b)\lambda Su^2 + 2bS|\nabla u|^2 + ba^2 |\nabla \varphi|^4 u^2 + 2ba(\Delta \varphi)^2 u^2 \right\} \, dv \right\} \geq 0.$$ 

(1.18)

We also obtain the following bounds for eigenvalues of the operator $-\Delta + bS$ on compact Riemannian surfaces.

Theorem 1.7. Let $\lambda^{-\Delta + bS}(t)$ be the eigenvalue of the operator $-\Delta + bS$ on $M^2$. If $|\nabla \varphi|^2 g_{ij} \geq 2\varphi_i \varphi_j$ and $0 < b \leq \frac{1}{2}$, we have

$$[1 - tS_{\min}(0)] \lambda - \frac{b^2 S_{\min}^2(0)}{2} \ln[1 - tS_{\min}(0)]$$

is nondecreasing under the List’s extended Ricci flow [1.2]. Moreover, $\lambda(t)$ has the lower bound

$$\lambda(t) \geq \frac{1}{1 - tS_{\min}(0)} \lambda(0) + \frac{b^2 S_{\min}^2(0)}{2[1 - tS_{\min}(0)]} \ln[1 - tS_{\min}(0)]$$

(1.20)

depending only on $t$.

Theorem 1.8. Let $\lambda^{-\Delta + bS}(t)$ be the eigenvalue of the operator $-\Delta + bS$ on $M^2$ with $r > 0$ and $S_{\min}(0) > 0$. If $|\nabla \varphi|^2 g_{ij} \geq 2\varphi_i \varphi_j$ and $0 < b \leq \frac{1}{2}$, we have

$$\frac{d}{dt} (\ln \lambda) \geq \frac{S_{\min}(0)e^{-\int_0^t r(s) \, ds}}{1 - S_{\min}(0) \int_0^t \left( e^{-\int_0^s r(\tilde{s}) \, d\tilde{s}} \right) ds} - r$$

(1.21)

under the rescaled List’s extended Ricci flow [1.3]. Moreover, $\lambda(t)$ has the lower bound

$$\lambda(t) \geq \lambda(0)e^{\int_0^t \tilde{x}(s) \, ds},$$

(1.22)

depending only on $t$, where

$$\tilde{x}(t) = \frac{S_{\min}(0)e^{-\int_0^t r(s) \, ds}}{1 - S_{\min}(0) \int_0^t \left( e^{-\int_0^s r(\tilde{s}) \, d\tilde{s}} \right) ds} - r.$$
Remark 1.3. When \( r = 0 \), Theorem 1.6 becomes Theorem 1.6 of Cao-Hou-Ling in \([3]\). When \( \varphi = 0 \) and \( r(t) = (\int_M R \, dv) / (\int_M dv) \), our Theorems 1.7, 1.8 reduce to Theorem 3.4 and Theorem 3.3 of Cao-Hou-Ling in \([3]\), respectively.

In the rest of this paper, we consider monotonicity formulas of \( F_k \)-functional which can be seen as a generalized \( F \)-functional corresponding with steady Ricci breathers, and \( W_k \)-functional which can be seen as a generalized \( W \)-functional corresponding with expanding Ricci breathers. Under the following evolution equation

\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij} &= -2(S_{ij} - \frac{r}{n} g_{ij}), \\
\varphi_t &= \Delta \varphi, \\
f_t &= -\Delta f + |\nabla f|^2 - S + r,
\end{align*}
\]

we proved the following results:

**Theorem 1.9.** Under the system \((1.23)\), we have

\[
\begin{align*}
\frac{d}{dt} F_k &= -\frac{2r}{n} F_k + 2(k - 1) \int_M (|S_{ij}|^2 + \alpha(\Delta \varphi)^2) e^{-f} \, dv \\
&\quad + 2 \int_M (|S_{ij} + f_{ij}|^2 + \alpha|\Delta \varphi - \langle \nabla f, \nabla \varphi \rangle|^2) e^{-f} \, dv,
\end{align*}
\]

or equivalently,

\[
\begin{align*}
\frac{d}{dt} F_k &= 2r \frac{(F_k - k r)}{n} + 2(k - 1) \int_M (|S_{ij} - \frac{r}{n} g_{ij}|^2 + \alpha(\Delta \varphi)^2) e^{-f} \, dv \\
&\quad + 2 \int_M (|S_{ij} + f_{ij} - \frac{r}{n} g_{ij}|^2 + \alpha|\Delta \varphi - \langle \nabla f, \nabla \varphi \rangle|^2) e^{-f} \, dv.
\end{align*}
\]

Remark 1.4. It was pointed out that for \( \alpha = 2 \) and \( k = 1 \), List \([6]\) and Müller \([11]\) studied the monotonicity of \( F_k \)-functional under the List’s extended Ricci flow \((1.2)\). Later, In \([8]\), Li studied the monotonicity of \( F_k \)-functional for \( \alpha = 2 \) and all \( k \geq 1 \) under \((1.2)\). In particular, when \( \varphi = 0 \) and \( k = 1 \), the \( F_k \)-functional \((1.5)\) becomes the Perelman’s \( F \)-functional.

Applying \((1.24)\) in Theorem 1.9, we can also obtain the following result:

**Theorem 1.10.** Let \((M^n, g(t))\) be a compact Riemannian manifold with \( g(t) \) satisfying the rescaled List’s extended Ricci flow \((1.3)\). We let \( \lambda(t) \) be the lowest eigenvalue of the operator \(-4\Delta + kS\) with \( k \geq 1 \). If the average value of \( S \) is nonnegative for all \( t \), then \( \lambda(t) \) is nondecreasing along \((1.3)\). Moreover, the monotonicity is strict unless the metric is Einstein.

As a direct application of \((1.24)\) in Theorem 1.9 we can obtain the following results:
Corollary 1.11. Under the system \((1.23)\) and \(k \geq 1\), we have

\[
\frac{d}{dt} \left( \mathcal{F}_k e^{\frac{2}{n} \int_0^t r(s) ds} \right) = e^{\frac{2}{n} \int_0^t r(s) ds} \left\{ 2(k-1) \int_M \left( |S_{ij}|^2 + \alpha (\Delta \varphi)^2 \right) e^{-f} dv \\
+ 2 \int_M \left( |S_{ij} + f_{ij}|^2 + \alpha |\Delta \varphi - \langle \nabla f, \nabla \varphi \rangle|^2 \right) e^{-f} dv \right\} \\
\geq 0,
\]

which shows that \(\mathcal{F}_k e^{\frac{2}{n} \int_0^t r(s) ds}\) is nondecreasing along \((1.23)\). Moreover, the monotonicity is strict unless the metric is Ricci flat.

Remark 1.5. Choosing \(r = 0\) and \(\alpha = 2\), then \((1.24)\) reduces the formula (1-6) in Theorem 1.1 of Li [8]. On the other hand, under the normalized Ricci flow of Hamilton, Li in [9] also obtained a similar result as Theorem 1.10.

As in [8], for a constant \(k \geq 1\), we define the following \(W_k\)-functional:

\[
W_k = \tau^2 \int_M \left[ k(S + \frac{n}{2\tau}) + |\nabla f|^2 \right] e^{-f} dv.
\]

Under the following coupled system

\[
\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= -2(S_{ij} - \frac{\tau}{n} g_{ij}), \\
\varphi_t &= \Delta \varphi, \\
f_t &= -\Delta f + |\nabla f|^2 - S + r \\
\tau_t &= 1,
\end{aligned}
\]

we obtain the following results:

Theorem 1.12. Under the system \((1.28)\) and \(k \geq 1\), we have

\[
\frac{d}{dt} W_k = 2\tau^2 \left\{ -\frac{\tau}{n} \mathcal{F}_k + (k-1) \int_M \left( |S_{ij} + \frac{1}{2\tau} g_{ij}|^2 + \alpha (\Delta \varphi)^2 \right) e^{-f} dv \\
+ \int_M \left( |S_{ij} + f_{ij} + \frac{1}{2\tau} g_{ij}|^2 + \alpha |\Delta \varphi - \langle \nabla f, \nabla \varphi \rangle|^2 \right) e^{-f} dv \right\},
\]
or equivalently,
\[
\frac{d}{dt} W_k = 2\tau^2 \left\{ \frac{r}{n} (F_k - kr) + \frac{kr}{\tau} \right. \\
+ (k - 1) \int_M \left( |S_{ij} + \frac{1}{2\tau} g_{ij} - \frac{r}{n} g_{ij}|^2 + \alpha (\Delta \phi)^2 \right) e^{-f} \, dv \\
\left. + \int_M \left( |S_{ij} + f_{ij} + \frac{1}{2\tau} g_{ij} - \frac{r}{n} g_{ij}|^2 + \alpha |\Delta \phi - \langle \nabla f, \nabla \phi \rangle|^2 \right) e^{-f} \, dv \right\}.
\]

(1.30)

As a direct application of the formula (1.29), we obtain

**Corollary 1.13.** Under the system (1.28) and \( k \geq 1 \), we have

\[
\frac{d}{dt} W_k + 2\tau^2 F_k = 2\tau^2 \left\{ (k - 1) \int_M \left( |S_{ij} + \frac{1}{2\tau} g_{ij}|^2 + \alpha (\Delta \phi)^2 \right) e^{-f} \, dv \\
+ \int_M \left( |S_{ij} + f_{ij} + \frac{1}{2\tau} g_{ij}|^2 + \alpha |\Delta \phi - \langle \nabla f, \nabla \phi \rangle|^2 \right) e^{-f} \, dv \right\} \geq 0.
\]

(1.31)

Moreover, the equality in (1.31) holds if and only if the metric is Einstein.

**Remark 1.6.** When \( r = 0 \) and \( \alpha = 2 \), then (1.29) reduces the formula (1.8) in Theorem 1.3 of Li [8].

This paper is organized as follows: In Section two, we study monotonicity formulas for eigenvalues of the operator \(-\Delta + bS\). We mainly consider monotonicity formulas for eigenvalues of the Laplacian under the rescaled List’s extended Ricci flow (1.3) and give some dependent lower bounds of eigenvalues. Moreover, Theorem 1.1-Theorem 1.4 have been proved in this part. In Section three, we deal with monotonicity formulas for eigenvalues of the operator \(-\Delta + bS\) with \( 0 < b \leq \frac{1}{2} \) on compact Riemannian surfaces and obtain some interesting results. In this part, we give proofs of Theorem 1.6-Theorem 1.8. In the last Section four, we study the first variations of \( F_k \)-functional and \( W_k \)-functional. These functionals are very important to study entropies corresponding to the List’s extended Ricci flow (1.2). In this part, Theorem 1.9-Theorem 1.12 have been proved.

2. **Proof of Theorem 1.1-Theorem 1.4**

We first give a Lemma which will be used later.
Lemma 2.1. Under the rescaled List’s extended Ricci flow (1.3), we have
\[ \frac{\partial}{\partial t} g^{ij} = 2 g^{ik} g^{jl} (S_{kl} - \frac{r}{n} g_{kl}), \]
in particular, when \( r(t) = \frac{\int_M S \, dv}{\int_M dv} \) (that is, \( r \) is the average value of \( S \)), then (2.2) shows that the volume of \((M, g(t))\) is a constant for all \( t \).

Proof. It has been shown in [6] (see Lemma 1.4 in [6]) that if the metric \( g(t) \) satisfies
\[ \frac{\partial}{\partial t} g^{ij} = v_{ij}, \]
where \( v_{ij} \) is a symmetric two tensor, then
\[ \frac{\partial}{\partial t} g^{ij} = -v^{ij}, \]
with \( v \) denoting the trace of \( v_{ij} \) with respect to \( g \), and
\[ R_t = -\Delta (\text{tr}_g v) + g^{k j} g^{i l} v_{ij,kl} - g^{k j} g^{i l} R_{kl} v_{ij}. \]
Hence, inserting \( v_{ij} = -2(S_{ij} - \frac{r}{n} g_{ij}) \), \( \text{tr}_g v = -2(S - r) \) into (2.4) and (2.5), we obtain (2.1) and (2.2), respectively.

Next, we prove (2.3). In fact, from (2.7), we have
\[ R_t = 2\Delta S - 2g^{k j} g^{i l} S_{ij,kl} + 2g^{k j} g^{i l} R_{kl} (S_{ij} - \frac{r}{n} g_{ij}) \]
\[ = 2\Delta S - 2g^{k j} g^{i l} S_{ij,kl} + 2g^{k j} g^{i l} (S_{kl} + \alpha \varphi_k \varphi_l) (S_{ij} - \frac{r}{n} g_{ij}) \]
\[ = 2\Delta S - 2g^{k j} g^{i l} S_{ij,kl} + 2|S_{ij}|^2 - \frac{2r}{n} S + 2\alpha S_{ij} \varphi^i \varphi^j - \alpha \frac{2r}{n} |\nabla \varphi|^2. \]

By the definition of \( S_{ij} \) and the contracted Bianchi identity, we also have
\[ g^{k j} S_{ij,k} = g^{k j} R_{ij,k} - g^{k j} \alpha (\varphi_i \varphi_j)_k \]
\[ = \frac{1}{2}(R - \alpha |\nabla \varphi|^2)_i - \alpha (\Delta \varphi) \varphi_i \]
\[ = \frac{1}{2} S_{i} - \alpha (\Delta \varphi) \varphi_i. \]

Thus, (2.8) becomes
\[ R_t = 2\Delta S + 2|S_{ij}|^2 - \frac{2r}{n} S + 2\alpha S_{ij} \varphi^i \varphi^j - \alpha \frac{2r}{n} |\nabla \varphi|^2 + 2\alpha (\Delta \varphi)^2 \]
\[ + 2\alpha (\nabla \Delta \varphi, \nabla \varphi). \]

It follows that
\[ S_t = (R - \alpha |\nabla \varphi|^2)_t \]
\[ = 2\Delta S + 2|S_{ij}|^2 - \frac{2r}{n} S + 2\alpha (\Delta \varphi)^2 \]
and the desired (2.3) follows. We complete the proof of Lemma 2.1.

Let \( u \) be the eigenfunction corresponding to eigenvalue \( \lambda \) of the operator \(-\Delta + bS\), that is,

\[
(-\Delta + bS)u = \lambda u. \tag{2.12}
\]

Multiplying both sides of (2.12) with \( u \) and integrating on \( M \), we have

\[
\lambda = \int_M (|\nabla u|^2 + bS u^2) \, dv. \tag{2.13}
\]

Using Lemma 2.1 we have

\[
\frac{d}{dt}\lambda = \int_M (2u_i u_j \frac{\partial}{\partial t} g^{ij} + 2(u_t)^i u_i + bS_t u^2 + 2bS u u_t) \, dv \\
+ \int_M (|\nabla u|^2 + bS u^2)(-S + r) \, dv \\
= \int_M (2S^{ij} u_i u_j - \frac{2r}{n} |\nabla u|^2 - 2u_t \Delta u + bS_t u^2 + 2bS u u_t) \, dv \\
+ \int_M (|\nabla u|^2 + bS u^2)(-S + r) \, dv. \tag{2.14}
\]

From (2.9) and the Stokes formula, we have

\[
2 \int_M S^{ij} u_i u_j \, dv = \int_M (-S_{,i}u^i - 2S^{ij} u_{ij} + 2\alpha(\Delta \varphi)(\nabla u, \nabla \varphi) u) \, dv. \tag{2.15}
\]

On the other hand,

\[
-\int_M |\nabla u|^2 S \, dv = \int_M (S\Delta u + S_{,i}u^i) u \, dv. \tag{2.16}
\]
Inserting (2.15) and (2.16) into (2.14) yields
\[
\frac{d}{dt} \lambda = \int_M \left\{ -\frac{2r}{n} |\nabla u|^2 + bS|u|^2 - 2S^{ij}u_{ij}u + 2\alpha(\Delta \varphi)\langle \nabla u, \nabla \varphi \rangle u \\
+ 2u_t(-\Delta u + bSu) - Su(-\Delta u + bSu) + r(|\nabla u|^2 + bS|u|^2) \right\} dv
\]
\[
= \int_M \left\{ -\frac{2r}{n} |\nabla u|^2 + bS|u|^2 - 2S^{ij}u_{ij}u + 2\alpha(\Delta \varphi)\langle \nabla u, \nabla \varphi \rangle u \\
+ \lambda 2u_t u + \lambda (-S + r)|u|^2 \right\} dv
\]
\[
= \int_M \left\{ -\frac{2r}{n} |\nabla u|^2 + bS|u|^2 - 2S^{ij}u_{ij}u + 2\alpha(\Delta \varphi)\langle \nabla u, \nabla \varphi \rangle u \\
+ \lambda \left( \int_M u^2 dv \right)_t
\right\} dv
\]
\[
= \int_M \left\{ -\frac{2r}{n} |\nabla u|^2 + bS|u|^2 - 2S^{ij}u_{ij}u + 2\alpha(\Delta \varphi)\langle \nabla u, \nabla \varphi \rangle u \right\} dv.
\] \hspace{1cm} (2.17)

Applying (2.3) in Lemma 2.1 into (2.17) yields
\[
\frac{d}{dt} \lambda = \int_M \left\{ -\frac{2r}{n} (|\nabla u|^2 + bS|u|^2 + 2b|S_{ij}|^2u^2 - 2S^{ij}u_{ij}u + bu^2(\Delta S) \\
+ 2b\alpha(\Delta \varphi)^2u^2 + 2\alpha(\Delta \varphi)\langle \nabla u, \nabla \varphi \rangle u \right\} dv
\]
\[
= -\frac{2r}{n} \lambda + \int_M \left\{ 2b|S_{ij}|^2u^2 - 2S^{ij}u_{ij}u + bu^2 \Delta S \\
+ 2b\alpha(\Delta \varphi)^2u^2 + 2\alpha(\Delta \varphi)\langle \nabla u, \nabla \varphi \rangle u \right\} dv.
\] \hspace{1cm} (2.18)

Using the Stokes formula again, we have
\[
-2 \int_M S^{ij}u_{ij}u dv = \int_M (2S^{ij}u_{ij}u dv + 2S^{ij}u_{ij}u) dv
\]
\[
= \int_M [S_{ij}u_{ij} + 2S^{ij}u_{ij} - 2\alpha(\Delta \varphi)\langle \nabla u, \nabla \varphi \rangle u] dv
\]
\[
= \int_M \left[ -\frac{1}{2}u^2 \Delta S + 2S^{ij}u_{ij} - 2\alpha(\Delta \varphi)\langle \nabla u, \nabla \varphi \rangle u \right] dv.
\] \hspace{1cm} (2.19)
Therefore, (2.18) becomes
\[
\frac{d}{dt} \lambda = -\frac{2r}{n} \lambda + \int_{M} \left\{ 2b|S_{ij}|^2 u^2 + 2S_{ij} u_i u_j + (b - \frac{1}{2})u^2 \Delta S + 2b\alpha(\Delta \varphi)^2 u^2 \right\} dv
\]
\[
= -\frac{2r}{n} \lambda + \int_{M} \left\{ 2b|S_{ij}|^2 u^2 + 2S_{ij} u_i u_j + 2b\alpha(\Delta \varphi)^2 u^2 
\right. \\
+ (2b - 1)S[(bS - \lambda)u^2 + |\nabla u|^2] \right\} dv.
\] (2.20)

Hence, the following consequence follows:

**Proposition 2.2.** Let \(\lambda^{-\Delta + bS}(t)\) be the eigenvalue of the operator \(-\Delta + bS\) corresponding to the normalized eigenfunction \(u\), that is,
\[
(-\Delta + bS)u = \lambda^{-\Delta + bS}u, \quad \int_{M} u^2 dv = 1.
\]

Then
\[
\frac{d}{dt} \lambda^{-\Delta + bS} = -\frac{2r}{n} \lambda^{-\Delta + bS} + \int_{M} \left\{ 2b|S_{ij}|^2 u^2 + 2S_{ij} u_i u_j + 2b\alpha(\Delta \varphi)^2 u^2 
\right. \\
+ (2b - 1)S[(bS - \lambda)u^2 + |\nabla u|^2] \right\} dv.
\] (2.21)

**Proof of Theorems 1.1 and 1.2.** From Proposition 2.2 it is easy to derive Theorem 1.1 by letting \(b = 0\) and Theorem 1.2 by letting \(b = \frac{1}{2}\), respectively.

Next, we give the proof of Theorem 1.3 by the Lemma 2.12 in [4] (see Page 99 in [4]).

**Proof of Theorem 1.3.** Using the Cauchy inequality
\[
|S_{ij}|^2 \geq \frac{1}{n}S^2,
\]
we obtain from (2.23)
\[
S_t \geq \Delta S + \frac{2}{n}S^2 - \frac{2r}{n}S.
\] (2.22)
Comparing \(S\) with the corresponding solution of ODE
\[
\frac{d}{dt} x = \frac{2}{n}x^2 - \frac{2r}{n}x, \quad x(0) = S_{\text{min}}(0)
\] (2.23)
gives
\[
S(x, t) \geq x(t) := \frac{S_{\text{min}}(0)e^{-\frac{2}{n}\int_{0}^{t} r(s)ds}}{1 - \frac{2}{n}S_{\text{min}}(0) \int_{0}^{t} \left( e^{-\frac{2}{n}\int_{0}^{s} r(\tilde{s})ds} \right) ds},
\] (2.24)
where \( x(t) \) is the solution of (2.23). In particular, when \( S_{\min}(0) = 0 \), then \( S(0) \geq 0 \) and (2.24) gives
\[
S(x, t) \geq 0 \quad (2.25)
\]
for all \( t \). The desired Theorem 1.3 is attained.

**Proof of Theorem 1.4.** From (2.21), we have
\[
\frac{d}{dt} \left( \lambda - \Delta e^{\frac{2}{n}} \int_0^t r(s) \, ds \right) = e^{\frac{2}{n}} \int_0^t r(s) \, ds \left\{ \int_M \left( \lambda - \Delta S + \theta - 1 \right) \int_M S \, dv + (2\theta - 1) \int_M S |\nabla u|^2 \, dv \right\},
\]
which shows (1) holds. On the other hand, applying (2.24) into (2.26), we achieve
\[
\frac{d}{dt} \left( \lambda - \Delta e^{\frac{2}{n}} \int_0^t r(s) \, ds \right) \geq e^{\frac{2}{n}} \int_0^t r(s) \, ds \left\{ \lambda - \Delta x(t) + (2\theta - 1) x(t) \lambda - \Delta \right\}
\]
which shows (2) holds.

**Proof of Corollary 1.5.** As in [8] of Li, using the fact \( R_{ij} = \frac{R}{2} g_{ij} \), we can compute
\[
S_{ij} V^i V^j = \left( \frac{R}{2} g_{ij} - \alpha \varphi_i \varphi_j \right) V^i V^j
\]
\[
\geq \frac{R}{2} |V|^2 - \alpha |\nabla \varphi|^2 |V|^2
\]
\[
\geq \left( \frac{R}{2} - \alpha |\nabla \varphi|^2 \right) |V|^2,
\]
where \( V = (V^i) \). Since \( R_{ij} \leq \epsilon u_i u_j \) and \( \epsilon \leq \frac{2a(\theta - 1)}{2\theta - 1} \) with \( \theta > \frac{1}{2} \), we have
\[
\left( \frac{1}{2} - \theta \right) R + (\theta - 1) \alpha |\nabla \varphi|^2 \geq 0
\]
which is equivalent to
\[
\left( \frac{R}{2} - \alpha |\nabla \varphi|^2 \right) |V|^2 \geq \theta S |V|^2.
\]
Therefore, we have \( S_{ij} \geq \theta S g_{ij} \) from (2.28) and the consequence (i) follows.
On the other hand, we can check that if $|\nabla \phi|^2 g_{ij} \geq 2 \phi_i \phi_j$, then

$$S_{ij} V^i V^j = \frac{R}{2} |V|^2 - \frac{\alpha}{2} |\nabla \phi|^2 |V|^2$$

$$\geq \frac{R}{2} |V|^2 - \frac{\alpha}{2} |\nabla \phi|^2 |V|^2\tag{2.29}$$

which shows that $S_{ij} \geq \frac{1}{2} S g_{ij}$ and the consequence (ii) follows. We complete the proof of Corollary 1.5.

3. **Proof of Theorem 1.6** **Theorem 1.8**

We first prove Theorem 1.6.

**Proof of Theorem 1.6** When $n = 2$, we have $R_{ij} = \frac{R}{2} g_{ij}$ and

$$S_{ij} = \frac{R}{2} g_{ij} - \alpha \phi_i \phi_j = \frac{1}{2} (S + \alpha |\nabla \phi|^2) g_{ij} - \alpha \phi_i \phi_j.\tag{3.1}$$

Hence,

$$|S_{ij}|^2 = \frac{R^2}{2} + \alpha^2 |\nabla \phi|^4 - \alpha R |\nabla \phi|^2$$

$$= \frac{1}{2} S^2 + \frac{1}{2} \alpha^2 |\nabla \phi|^4\tag{3.2}$$

and (2.20) becomes

$$\frac{d}{dt} \lambda = - r \lambda + \int_M \left\{ 2b |S_{ij}|^2 u^2 + 25^{ij} u_i u_j + 2b \alpha (\Delta \phi)^2 u^2$$

$$+ (2b - 1) b S^2 u^2 - (2b - 1) \lambda S u^2 + (2b - 1) S |\nabla u|^2 \right\} dv$$

$$= - r \lambda + \int_M \left\{ b S^2 u^2 + b \alpha^2 |\nabla \phi|^4 u^2 + S |\nabla u|^2 + \alpha |\nabla \phi|^2 |\nabla u|^2 - 2 \alpha \langle \nabla u, \nabla \phi \rangle^2$$

$$+ 2 b \alpha (\Delta \phi)^2 u^2 + (2b - 1) b S^2 u^2 - (2b - 1) \lambda S u^2 + (2b - 1) S |\nabla u|^2 \right\} dv$$

$$= - r \lambda + \int_M \left\{ 2b^2 S^2 u^2 - (2b - 1) \lambda S u^2 + 2b |\nabla u|^2$$

$$+ b \alpha^2 |\nabla \phi|^4 u^2 + \alpha |\nabla \phi|^2 |\nabla u|^2 - 2 \alpha \langle \nabla u, \nabla \phi \rangle^2 + 2 b \alpha (\Delta \phi)^2 u^2 \right\} dv.$$\tag{3.3}

Therefore, we obtain Theorem 1.6.

**Proof of Theorem 1.7** In particular, when $r = 0$, the rescaled List’s extended Ricci flow (1.3) becomes the List’s extended Ricci flow (1.2) and
(3.3) becomes
\[
\frac{d}{dt} \lambda = \int_M \left\{ 2b^2 S^2 u^2 + (1 - 2b) \lambda S u^2 + 2b S |\nabla u|^2 \\
+ b \alpha^2 |\nabla \varphi|^4 u^2 + \alpha |\nabla \varphi|^2 |\nabla u|^2 - 2\alpha \langle \nabla u, \nabla \varphi \rangle^2 + 2b \alpha (\Delta \varphi)^2 u^2 \right\} dv,
\]
respectively. In particular, for \( r = 0 \), we have from (3.24)
\[
S(x, t) \geq S_{\min}(0).
\]
Therefore, (3.4) gives
\[
\frac{d}{dt} \lambda \geq \int_M \left\{ 2b^2 S^2 u^2 + (1 - 2b) \lambda S u^2 + 2b S |\nabla u|^2 \right\} dv \geq \int_M \left\{ 2b^2 S_{\min}(0) \frac{1}{1 - t S_{\min}(0)} |\nabla u|^2 \right\} dv.
\]
Using the inequality \( y^2 - cy \geq -\frac{1}{4} c^2 \) into (3.3), we derive
\[
\frac{d}{dt} \lambda \geq \frac{S_{\min}(0)}{1 - t S_{\min}(0)} \lambda - \frac{b^2 S_{\min}(0)}{2[1 - t S_{\min}(0)]^2},
\]
which shows
\[
\frac{d}{dt} \left\{ [1 - t S_{\min}(0)] \lambda - \frac{b^2 S_{\min}(0)}{2} \ln[1 - t S_{\min}(0)] \right\} \geq 0.
\]
Integrating both sides of (3.8) on \( t \), we have
\[
\lambda(t) \geq \frac{1}{1 - t S_{\min}(0)} \lambda(0) + \frac{b^2 S_{\min}(0)}{2[1 - t S_{\min}(0)]} \ln[1 - t S_{\min}(0)]
\]
and Theorem 1.7 follows.

**Proof of Theorem 1.8.** Note that (3.3) becomes
\[
\frac{d}{dt} \lambda = -r \lambda + \int_M \left\{ 2b^2 S^2 u^2 + (1 - 2b) \lambda S u^2 + 2b S |\nabla u|^2 \\
+ b \alpha^2 |\nabla \varphi|^4 u^2 + \alpha |\nabla \varphi|^2 |\nabla u|^2 - 2\alpha \langle \nabla u, \nabla \varphi \rangle^2 + 2b \alpha (\Delta \varphi)^2 u^2 \right\} dv.
\]

In particular, for \( n = 2 \), we have from (2.24)
\[
S(x, t) \geq x(t) := S_{\min}(0) e^{-\int_0^t r(s) ds} \frac{S_{\min}(0)}{1 - S_{\min}(0) \int_0^t \left(e^{-\int_0^\sigma r(\tau) d\tau}\right) d\sigma}.
\] (3.11)

Hence, (3.10) yields
\[
\frac{d}{dt} \lambda \geq -r \lambda + \int_M \left\{2b^2 S^2 u^2 + (1 - 2b) \lambda x(t) + 2bx(t) \int M |\nabla u|^2\right\} dv
\]
\[
\geq -r \lambda + 2b^2 \int_M S^2 u^2 dv + (1 - 2b) \lambda x(t) + 2bx(t) \left\{\lambda - b \int_M S^2 dv\right\}
\]
\[
= (x(t) - r) \lambda + 2b^2 \int_M (S - x(t)) S u^2 dv \
\geq (x(t) - r) \lambda.
\] (3.12)

Note that \( \lambda(t) > 0 \). We obtain from (3.12)
\[
\frac{d}{dt} (\ln \lambda) \geq x(t) - r.
\] (3.13)

We complete the proof of Theorem 1.8.

4. PROOF OF THEOREM 1.9-1.12

In order to derive our results, we first prove the following three lemmas:

**Lemma 4.1.** Under the evolution equation (1.23), we have
\[
(e^{-f} dv)_t = (-f_t - S + r)e^{-f} dv = -(\Delta e^{-f}) dv,
\] (4.1)
\[
(|\nabla f|^2)_t = 2S_{ij} f^i f^j - \frac{2r}{n} |\nabla f|^2 + 2\nabla f \nabla (\Delta f + |\nabla f|^2 - S + r).
\] (4.2)

**Proof.** The formula (4.1) is a direct conclusion of (2.2) in Lemma 2.1 and \( f_t = -\Delta f + |\nabla f|^2 - S + r \). Similarly, we have
\[
(|\nabla f|^2)_t = (g^{ij} f_i f_j)_t
\]
\[
= f_i f_j \frac{\partial}{\partial t}g^{ij} + 2g^{ij}(f_i)_t f_j
\]
\[
= 2S^{ij} f_i f_j - \frac{2r}{n} |\nabla f|^2 + 2\nabla f \nabla (\Delta f + |\nabla f|^2 - S + r)
\] (4.3)

and (4.2) is achieved. We complete the proof of Lemma 4.1.
Lemma 4.2. With the help of Lemma 4.1, we have

\[
\left( \int_M \mathcal{S}_e^{-f} dv \right)_t = \int_M \left( 2|S_{ij}|^2 - \frac{2r}{n} S + 2\alpha(\Delta\phi)^2 \right) e^{-f} dv, 
\]

\[
(4.4)
\]

\[
\left( \int_M |\nabla f|^2 e^{-f} dv \right)_t = \int_M \left( 2S_{ij} f^i f^j - \frac{2r}{n} |\nabla f|^2 - 2\Delta^2 f - 2\Delta S + \Delta|\nabla f|^2 \right) e^{-f} dv,
\]

\[
(4.5)
\]

\[
\left( \int_M (|\nabla f|^2 + S) e^{-f} dv \right)_t = \int_M \left( 2|S_{ij}|^2 - \frac{2r}{n} (S + |\nabla f|^2) + 2S_{ij} f^i f^j - 2\Delta^2 f - 2\Delta S + \Delta|\nabla f|^2 + 2\alpha(\Delta\phi)^2 \right) e^{-f} dv.
\]

\[
(4.6)
\]

Proof. By the Stoke formula, we can obtain (4.4) from (2.3) in Lemma 2.1. It is easy to see that (4.5) holds from (4.2). Thus, the formula (4.6) follows by combining (4.4) with (4.5).

Lemma 4.3. Under the rescaled List's extended Ricci flow (1.3), we have

\[
\int_M |f_{ij}|^2 e^{-f} dv = \int_M \left( \frac{1}{2} \Delta|\nabla f|^2 - \Delta^2 f - S_{ij} f^i f^j - \alpha(\nabla f, \nabla \phi)^2 \right) e^{-f} dv,
\]

\[
(4.7)
\]

\[
2 \int_M S_{ij} f_{ij} e^{-f} dv = \int_M \left[ 2S_{ij} f^i f^j - \Delta S + 2\alpha(\Delta\phi)(\nabla f, \nabla \phi) \right] e^{-f} dv,
\]

\[
(4.8)
\]

\[
\int_M |S_{ij} + f_{ij}|^2 e^{-f} dv = \int_M \left[ |S_{ij}|^2 + S_{ij} f^i f^j + \frac{1}{2} \Delta|\nabla f|^2 - \Delta^2 f - \Delta S - \alpha(\nabla f, \nabla \phi)^2 + 2\alpha(\Delta\phi)(\nabla f, \nabla \phi) \right] e^{-f} dv.
\]

\[
(4.9)
\]

Proof. Multiplying both sides of the following well-known Bochner formula with $e^{-f}$

\[
\frac{1}{2} \Delta|\nabla f|^2 = |f_{ij}|^2 + 2\nabla f \nabla \Delta f + R_{ij} f^i f^j
\]

\[
(4.10)
\]

and integrating on it, we have

\[
\int_M |f_{ij}|^2 e^{-f} dv = \int_M \left( \frac{1}{2} \Delta|\nabla f|^2 - \Delta^2 f - R_{ij} f^i f^j \right) e^{-f} dv
\]

\[
= \int_M \left( \frac{1}{2} \Delta|\nabla f|^2 - \Delta^2 f - S_{ij} f^i f^j - \alpha(\nabla f, \nabla \phi)^2 \right) e^{-f} dv
\]

\[
(4.11)
\]
and (4.7) follows. On the other hand,
\[
2 \int_M S_{ij} f_i f_j e^{-f} dv = 2 \int_M (S_{ij} f_i)_j e^{-f} dv
\]
\[
= 2 \int_M (S_{ij} f_i + S_{ij} f_{ij}) e^{-f} dv.
\]  (4.12)

Applying (4.7) into (4.12) gives
\[
2 \int_M S_{ij} f_i f_j e^{-f} dv = 2 \int_M \left( \frac{1}{2} S_{ij} f_i - \alpha (\Delta \varphi) \varphi_i f_i + S_{ij} f_{ij} \right) e^{-f} dv
\]
\[
= \int_M \left[ \Delta S - 2 \alpha (\Delta \varphi) \langle \nabla f, \nabla \varphi \rangle + 2 S_{ij} f_{ij} \right] e^{-f} dv,
\]  (4.13)

which gives the desired formula (4.8).

By combining (4.7) with (4.8), we derive (4.9) finally.

**Proof of Theorem 1.9.** Applying (4.9) into (4.6), we obtain
\[
\frac{d}{dt} \mathcal{F} = \frac{d}{dt} \int_M (|\nabla f|^2 + S) e^{-f} dv
\]
\[
= \int_M \left( \frac{2}{n} |S_{ij}|^2 - \frac{2r}{n} (S + |\nabla f|^2) + 2 S_{ij} f_{ij} \right.
\]
\[
- 2 \Delta^2 f - 2 \Delta S + \Delta |\nabla f|^2 + 2 \alpha (\Delta \varphi)^2 \right) e^{-f} dv
\]
\[
= \int_M \left( - \frac{2r}{n} (|\nabla f|^2 + S) + 2|S_{ij} + f_{ij}|^2 \right.
\]
\[
+ 2 \alpha (\Delta \varphi)^2 + 2 \alpha (\Delta \varphi) \langle \nabla f, \nabla \varphi \rangle - 4 \alpha (\Delta \varphi) \langle \nabla f, \nabla \varphi \rangle \right) e^{-f} dv
\]
\[
= - \frac{2r}{n} \mathcal{F} + 2 \int_M \left( |S_{ij} + f_{ij}|^2 + \alpha |\Delta \varphi - \langle \nabla f, \nabla \varphi \rangle|^2 \right) e^{-f} dv.
\]  (4.14)

Therefore, we obtain
\[
\frac{d}{dt} \mathcal{F}_k = \frac{d}{dt} \mathcal{F} + (k - 1) \frac{d}{dt} \int_M S e^{-f} dv
\]
\[
= - \frac{2r}{n} \mathcal{F}_k + 2(k - 1) \int_M \left( |S_{ij}|^2 + \alpha (\Delta \varphi)^2 \right) e^{-f} dv
\]
\[
+ 2 \int_M \left( |S_{ij} + f_{ij}|^2 + \alpha |\Delta \varphi - \langle \nabla f, \nabla \varphi \rangle|^2 \right) e^{-f} dv
\]  (4.15)

and the desired formula (1.24) is achieved.
The formula (1.25) can be achieved by a direct computation.

Proof of Theorem [1.10] In order to prove Theorem [1.10] we first give the following key proposition:

**Proposition 4.4.** Let \((M^n, g(t))\) be a compact Riemannian manifold with \(g(t)\) satisfying the rescaled List’s extended Ricci flow (1.3). We let \(\lambda(t)\) be the lowest eigenvalue of the operator \(-4\Delta + kS\) with \(k \geq 1\). If there exists a function \(v = v(x, t)\) such that
\[
r(t) = \frac{\int_M (|\nabla v|^2 + kS) e^{-v} dv}{k \int_M e^{-v} dv},
\]
then the lowest eigenvalue \(\lambda(t)\) is nondecreasing under (1.3) provided \(r(t) \leq 0\). The monotonicity is strict unless the metric is Einstein.

**Proof.** Since the lowest eigenvalue of the operator \(-4\Delta + kS\) on a compact Riemannian manifold is given by
\[
\lambda(g(t)) = \inf_f M \int F_k(g, f) e^{-f} dv,
\]
where the infimum is taken over functions satisfying \(\int_M e^{-f} dv = 1\). For the compact Riemannian manifold, the lowest eigenvalue \(\lambda\) can be attained by a smooth function \(u\). Therefore, there exists a smooth function \(u\) such that
\[
\begin{align*}
(-4\Delta + kS)u &= \lambda u \\
\lambda(g(t)) &= \int_M (4|\nabla u|^2 + kSu^2) dv = \int_M (|\nabla \tilde{f}|^2 + kS) e^{-\tilde{f}} dv
\end{align*}
\]
by letting \(-\tilde{f} = 2 \ln u\). Applying (4.16) into (1.25) gives
\[
\frac{d}{dt} \lambda = 2 \frac{r}{n} (\lambda - kr) + 2(k - 1) \int_M \left(|S_{ij} - \frac{r}{n} g_{ij}|^2 + \alpha (\Delta \varphi)^2\right) e^{-\tilde{f}} dv
\]
\[
+ 2 \int M \left(|S_{ij} + \tilde{f}_{ij} - \frac{r}{n} g_{ij}|^2 + \alpha |\Delta \varphi - \langle \nabla \tilde{f}, \nabla \varphi \rangle|^2\right) e^{-\tilde{f}} dv.
\]

By using the assumption (4.16), we get
\[
k r = \int_M (|\nabla v|^2 + kS) e^{-v} dv
\]
\[
\geq \int_M (|\nabla \tilde{f}|^2 + kS) e^{-\tilde{f}} dv
\]
\[
= \lambda
\]
which means that \(\lambda - kr \leq 0\). Therefore, if \(r(t) \leq 0\), then (4.19) shows that \(\lambda(t)\) is nondecreasing under (1.3). The monotonicity is strict unless
\[
S_{ij} - \frac{r}{n} g_{ij} = 0, \quad \Delta \varphi = 0
\]
\[ S_{ij} + \tilde{f}_{ij} - \frac{r}{n} g_{ij} = 0, \quad \Delta \varphi - \langle \nabla \tilde{f}, \nabla \varphi \rangle = 0. \quad (4.22) \]

Notice that the Riemannian manifold is compact. Hence, we have \( \varphi \) is constant from the second equality in (4.21) or the second equality in (4.22). Therefore, the metric \( g \) is Einstein. We complete the proof of Proposition 4.4.

Now, we are in a position to prove Theorem 1.10. We note that for the extended Hamilton normalized flow under the List’s extended Ricci flow, we have

\[ r(t) = \frac{\int_M S \, dv}{\int_M dv}. \]

Choosing \( v = \ln(\text{Vol}(M^n)) \) in (4.16), we derive Theorem 1.10.

**Proof of Theorem 1.12**

Since the \( W \)-functional is related with \( F \) by

\[ W = \tau^2 \int_M \left( S + \frac{n}{2\tau} + |\nabla f|^2 \right) e^{-f} \, dv \]

\[ = \tau^2 F + \frac{n\tau}{2}, \quad (4.23) \]

with the help of (1.24), we obtain

\[ \frac{d}{dt} W = \tau^2 \frac{d}{dt} F + 2\tau F + \frac{n}{2} \]

\[ = \tau^2 \left\{ -2r F + 2 \int_M \left( |S_{ij} + f_{ij}|^2 + \alpha |\Delta \varphi - \langle \nabla f, \nabla \varphi \rangle|^2 \right) e^{-f} \, dv \right\} \]

\[ + 2\tau F + \frac{n}{2}. \quad (4.24) \]

Applying

\[ 2\tau^2 \int_M |S_{ij} + f_{ij} + \frac{1}{2\tau} g_{ij}|^2 e^{-f} \, dv = 2\tau^2 \int_M |S_{ij} + f_{ij}|^2 e^{-f} \, dv + 2\tau F + \frac{n}{2} \]

into (4.24) yields

\[ \frac{d}{dt} W = 2\tau^2 \left\{ -\frac{r}{n} F + \int_M \left( |S_{ij} + f_{ij} + \frac{1}{2\tau} g_{ij}|^2 \right) e^{-f} \, dv \right\}. \quad (4.25) \]

By the definition of \( W_k \), we know

\[ W_k = W + (k-1)\tau^2 \int_M S e^{-f} \, dv + \frac{(k-1)n\tau}{2}. \quad (4.26) \]
Thus, we get from (4.4) and (4.25)

\[
\frac{d}{dt} W_k = \frac{d}{dt} W + (k - 1)n + 2(k - 1)\tau \int_M Se^{-f} dv
\]

\[
+ (k - 1)\tau^2 \left( \int_M Se^{-f} dv \right)_t
\]

\[
= 2\tau^2 \left\{ -\frac{r}{n} F + \int_M \left( |S_{ij} + f_{ij} + \frac{1}{2\tau} g_{ij}|^2 + \alpha|\Delta \varphi - \langle \nabla f, \nabla \varphi \rangle|^2 \right) e^{-f} dv \right\} + \frac{(k - 1)n}{2} + 2(k - 1)\tau \int_M Se^{-f} dv
\]

\[
+ (k - 1)\tau^2 \int_M \left( 2|S_{ij}|^2 - \frac{2r}{n} S + 2\alpha(\Delta \varphi)^2 \right) e^{-f} dv
\]

\[
= 2\tau^2 \left\{ -\frac{r}{n} F_k + (k - 1) \int_M \left( |S_{ij} + \frac{1}{2\tau} g_{ij}|^2 + \alpha(\Delta \varphi)^2 \right) e^{-f} dv
\]

\[
+ \int_M \left( |S_{ij} + f_{ij} + \frac{1}{2\tau} g_{ij}|^2 + \alpha|\Delta \varphi - \langle \nabla f, \nabla \varphi \rangle|^2 \right) e^{-f} dv \right\},
\]

and the desired formula (1.29) is attained.

The formula (1.30) can be achieved by a direct computation.

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