LARGE-\(N\) LIMIT AND CONTACT TERMS IN UNBROKEN \(YM_4\)

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ABSTRACT

I characterize the structure of the master field for \(F^{0}_{\bar{z}z}\) in \(SU(\infty)-YM_4\) on a product of two Riemann surfaces \(Z \times W\) in the gauge \(F^{ch}_{\bar{z}z} = 0\) as the sum of a ‘bulk’ constant term and of delta-like ‘contact’ terms. The contact terms may occur because the localization of the functional integral at \(N = \infty\) on a master orbit of a constant connection under the action of singular gauge transformations is still compatible with the large-\(N\) factorization and translational invariance. In addition I argue that if the gauge group is unbroken and there is a mass gap, that is if the theory confines, the functional measure at \(N = \infty\), in the gauge \(F^{ch}_{\bar{z}z} = 0\), must be localized on the moduli space of flat connections with punctures on \(Z \times W\).

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1 Contact terms and the master orbit

Many years ago, it was suggested in [1] that because of the factorization of correlation functions at large-$N$ and translational invariance, the functional measure for $YM$ theories at $N = \infty$ should be localized on the gauge orbit of some constant connection $A_\mu$, the master orbit:

$$A_\mu^g = gA_\mu g^{-1} - ig^{-1} \partial_\mu g.$$  \hspace{1cm} (1)

However, in this paper I show that if the gauge transformation $g$ is not smooth, some gauge-invariant local operators do not need to be constant when computed on the master orbit, but in addition to the constant part get the contribution of some ‘contact terms’.

It turns out that the most general structure of the contact terms for gauge-invariant local operators at $N = \infty$, compatible with translational invariance and factorization, is that they are ultralocal distributions (linear combinations of delta functions and their derivatives) localized at submanifolds that depend on moduli that contain some copies of the translations. For example I show in this paper that, in the gauge $F^{ch}_{zz} = 0$ (the superscript $^{ch}$ means the charged part with respect to the diagonal $U(1)^{N-1}$), in $SU(N)$-$YM_4$ on a product of two Riemann surfaces $Z \times W$, the structure of the master field for the neutral part of $F_{zz}$, that is the eigenvalues of $F_{zz}$ (that determine the correlation functions of all the traces of $F_{zz}$), may be in general the sum of a constant ‘bulk’ contribution and some ‘contact’ terms that are delta-like distributions.

This situation is reminiscent of topological field theories [2],[3],[4], in which contact terms arise because of the topological non-triviality of the gauge orbit [5].

From now on I will restrict my argument to $SU(N)$-$YM_4$ on a product of two Riemann surfaces $Z \times W$.

I will assume that Eq.\((1)\) holds for some not-necessarily smooth gauge transformation $g$. Let us suppose that the gauge $F^{ch}_{zz} = 0$ can be reached. Should the gauge transformation $g$ in Eq.\((1)\) be smooth on the entire orbit, $F^{0}_{zz}$ would simply be a constant. However, under a singular gauge transformation $F^{g}_{zz}$ transforms as:

$$F^{g}_{zz} = gF_{zz}g^{-1} + F_{zz}(g^{-1} \partial_z g, g^{-1} \partial_{\bar{z}} g).$$  \hspace{1cm} (2)
where the second term represents the field strength of a connection that is locally a pure gauge. In the gauge $F_{zz}^{ch} = 0$, that allows residual $U(1)^{N-1}$ transformations, Eq.(2) reduces to:

$$F_{zz}^{0\omega} = F_{zz}^{0} + \partial_{z}\partial_{\bar{z}}\omega^{0} - \partial_{\bar{z}}\partial_{z}\omega^{0}.$$  \hspace{1cm} (3)

The second term is zero for smooth $\omega^{0}$, but it is proportional to a singular delta function if the gauge transformation $\omega^{0}$ carries a non-trivial $\pi_{1}$, that is if there is a magnetic vortex in the theory:

$$\omega^{0} = h^{0} \text{Imlog}(z - z_{1}).$$ \hspace{1cm} (4)

Only $\pi_{1}$ needs to be considered here since the $w$-coordinates appear as a parameter in Eq.(3).

I conclude that the most general structure for the master field of $F_{zz}^{0}$ is a constant part plus a vortex condensate:

$$iF_{zz}^{0}(z, w) = H_{0}^{0} + \sum_{i} h_{i}^{0}(z, w)\delta^{(2)}(z - z_{i}(w)) \hspace{1cm} (5)$$

or, in a singular $U(1)^{N-1}$ gauge in which the phase of the charged connections is multivalued, a condensate of strings. The factor of $i$ has been introduced into Eq.(5) to make the constant $H_{0}^{0}$ real.

The occurrence of these ‘contact’ terms is obviously compatible with translational invariance since the functional measure will contain the integration over the moduli (positions) of the ‘contacts’. In the next section I show that it is also compatible with the large-$N$ factorization of the gauge invariant correlations.

2 \hspace{1cm} Contact terms and factorization

I start presenting a slightly different argument that does not make explicit use of the assumption that Eq.(1) holds. In the gauge $F_{zz}^{ch} = 0$, the effective action for $F_{zz}^{0}$, $\Gamma$,

$$Z = \int \exp[-S_{YM}]\delta(F_{zz}^{ch})\delta(F_{zz}^{0} - H_{0})\Delta_{FP}DADH^{0}$$

$$= \int \exp[-\Gamma(H_{0})]DH^{0}, \hspace{1cm} (6)$$

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defined integrating out all the other fields but $F_{zz}$, though a priori unknown is of order $N^2$ for $N$ large, while the integration measure grows as $N$. Therefore, for the functional integral in which $\Gamma$ occurs, the saddle-point method applies for large $N$. Because of translational invariance the minima or the saddles of $\Gamma$ must be either constant or non-constant configurations containing the translations among their moduli, that is:

$$H^0(x) = H^0_0 + H^0(x; [x_i])$$ (7)

Because of large-$N$ clustering, for the non-constant part of $H^0(x)$ I may assume:

$$H^0(x; [x_i]) = \sum_i H^0(x - x_i)$$ (8)

where for simplicity I made the unnecessary assumption that the irreducible constituents of the master field with respect to translations $H^0_0(x - x_i)$ are all of the same ‘type’. Now I compute the two-point correlation function making the ‘ansatz’ of Eqs.(7)-(8):

$$\langle H^i(x)H^j(y) \rangle = H^iH^j + H^i \frac{n}{V} \int H^j(y - x_1)dx_1 +$$

$$+ H^j \frac{n}{V} \int H^i(x - x_1)dx_1 +$$

$$+ \frac{1}{V} \int \sum_k H^i(x - x_k) \sum_{k'} H^j(y - x_{k'}) \Pi dx_i =$$

$$H^iH^j + H^i \frac{n}{V} \int H^j(y - x_1)dx_1 +$$

$$+ H^j \frac{n}{V} \int H^i(x - x_1)dx_1 +$$

$$+ \frac{1}{V} \sum_k \int H^i(x - x_k)H^j(y - x_k)dx_k +$$

$$+ \sum_{k \neq k'} \frac{1}{V} \int H^i(x - x_k)dx_k \frac{1}{V} \int H^j(y - x_{k'})dx_{k'} =$$

$$= H^iH^j + H^i \frac{n}{V} \int H^j(y - x_1)dx_1 +$$

$$+ H^j \frac{n}{V} \int H^i(x - x_1)dx_1 +$$

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\( + \frac{n}{V} \int H^i(x-x_1)H^j(y-x_1)dx_1 + \\
+ \frac{n^2-n}{V^2} [\int H^i(x-x_1)dx_1][\int H^j(y-x_1)dx_1] \) \hspace{1cm} (9)

This correlation function should be compared with the disconnected product:

\[ <H^i(x)> <H^j(y)> = \left[ H^i + \frac{n}{V} \int H^i(x-x_1)dx_1 \right] \times \]
\[ \times \left[ H^j + \frac{n}{V} \int H^j(y-x_1)dx_1 \right] \] \hspace{1cm} (10)

where \( n \) is the number of irreducible constituents with respect to translations. I also assume that the limit \( n \to \infty \) is taken keeping constant the number of constituents per unit of ‘volume’. The two-point correlation function factorizes only if the constituents of the master field are either a constant or an ultralocal distribution, otherwise the non-trivial overlap between different irreducible constituents that appears after the last equality in Eq.(9) would imply a non-vanishing two-point connected function. Quite analogous formulae hold for all the other correlation functions of \( F^0_{\bar{z}z} \).

Since \( F^0_{\bar{z}z} \) has scaling dimension two, it is not restrictive to assume that \( F^0_{\bar{z}z} \) is a linear combination with dimensionless coefficients of delta-two (anomalous dimensions are expected to appear only as \( \frac{1}{N} \) corrections). All the other ultralocal distributions can be obtained as limits of linear combinations of delta, because Dirac measures are dense in the distributions \( \mathbb{D} \). Now I show explicitly that the ansatz in Eq.(5) is compatible with factorization, provided the v.e.v.’s of \( h^0(z, w) \) factorize and are translational invariant (for example \( h^0(z, w) \) are constant):

\[ i^2 < F^i_{\bar{z}z}(z_1, w_1)F^j_{\bar{z}z}(z_2, w_2)> = H^i \delta^j + H^j \delta^i + \frac{\sum_k h^i_k(z_k, w_2)}{A} + \\
+ \frac{1}{A^2} \left[ \int \sum_k h^i_k(z_1, w_1)\delta^{(2)}(z_1 - z_k(w_1)) \prod dz_k(w_1) \right] \times \\
\times \left[ \int \sum_k h^j_k(z_2, w_2)\delta^{(2)}(z_2 - z_k'(w_2)) \prod dz_k'(w_2) \right] . \] \hspace{1cm} (11)
For \( w_1 \neq w_2 \) the two-point function should be compared with the disconnected product:

\[
i^2 < F_{zz}^{i}(z_1, w_1) > < F_{zz}^{j}(z_2, w_2) > = \left[H_0^i + \frac{< \sum_k h_k^{i}(z_k, w_1) >}{A}\right] \times \\
\times \left[H_0^j + \frac{< \sum_{k'} h_{k'}^{j}(z_{k'}, w_2) >}{A}\right]
\]

In this case factorization and translational invariance follow from the assumed factorization and translational invariance of \( < h_k^{i}(z, w) > \). When \( w_1 = w_2 \) the factorization follows from the preceding assumptions about the v.e.v. of \( h_k^{i}(z, w) \) and from the computation in Eq.(9)-Eq.(10).

The occurrence of delta-two can be interpreted as vortices, as I did in the first section. The vortex charge is quantized according to the topological class defined by \( \pi_1(SU(N)/\mathbb{Z}_2) \). Alternatively the gauge fixing \( F_{zz}^{ch} \) leaves a residual \( U(1)^{N-1} \), and solutions of the system:

\[
F_{zz}^{ch}(z, w) = 0 \\
iF_{zz}^{0}(z, w) = H_0^0 + H^0(z, w; [z_i])
\]

(13)
can be classified by \( \pi_1(U(1)^{N-1}) \). If we allow \( k \)-fold covers of the original surface \( Z \), rational holonomies and vortex charges are allowed at large \( N \). This completes the classification of the contact terms that may occur at large \( N \). In the next section I present a physical interpretation of the structure of the master field for \( F_{zz}^{0} \).

3 Confinement and the master field

According to [7], if there is a mass gap, the phase of pure \( SU(N) \)-gauge theories can be classified either as the Higgs or the confining one, depending whether either the electric or the magnetic fluxes condense, respectively. If there is no mass gap and the gauge symmetry is unbroken, the theory is in the Coulomb phase.

It is quite obvious that if pure \( SU(N)-YM_4 \) were in the Higgs phase, free vortices could not occur in the master field, since the magnetic charge is confined in this phase. Indeed if \( F_{zz}^{0} \) is a non-zero constant, the \( SU(N) \)
gauge group must be spontaneously broken to the isotropy subgroup of \( H^0 \), since the gauge orbit under global gauge transformations is non-trivial. This case is analogous to the one in which the eigenvalues of a scalar field in the adjoint representation condense in the vacuum. This theory looks like a superconductor of type one.

If a constant field and a vortex condensate occur at the same time in the master field for the same eigenvalues, the theory resembles a superconductor of type two, which can be penetrated by magnetic flux vortices (they would form lines in 3d and sheets in 4d).

One way of seeing this is to look at the equations (for \( U(N) \)):

\[
F_{iz}^{ih} = \partial_i \lambda \lambda^{ih} + i (A^i - A^h)[\lambda \lambda^{ih}] + i \sum_j A^{ij} A^{j} = 0
\]

\[
iF_{iz}^{ij} = i (\partial_i \lambda A^{i} + i \sum_j A^{ij} A^{j}) = -H^i
\]

that appear as a constraint at \( N = \infty \) in the functional integral. Using the ansatz (reduction):

\[
A^{ih} = A^{ih} = 0 \quad |i - h| > 1
\]

\[
A_{i+1} = 0 \quad \text{vortex}
\]

\[
A_{i+1} = 0 \quad \text{anti-vortex },
\]

the following Toda equations corresponding to vortices or antivortices are obtained:

\[
(A^i - A^{i+1})_z = -i \partial_z \log A^{i+1}_z
\]

\[
-\partial_z \partial \log |A^{i+1}_z|^2 - 2 |A^{i+1}_z|^2 + |A^{i-1}_z|^2 + |A^{i+1+2}_z|^2 = H^{i+1} - H^i
\]

\[
i \sum_i \partial_i \lambda^{ih} = - \sum_i H^i
\]

and

\[
(A^i - A^{i+1})_z = i \partial_z \log A^{i+1}_z
\]

\[
\partial_z \partial \log |A^{i+1}_z|^2 + 2 |A^{i+1}_z|^2 - |A^{i-1}_z|^2 - |A^{i+1+2}_z|^2 = H^{i+1} - H^i
\]

\[
i \sum_i \partial_i \lambda^{ih} = - \sum_i H^i.
\]

Toda equations are a \( SU(N) \) generalization of the Liouville equation (\( SU(2) \)) involving only \( 3N - 1 \) generators, the Cartan generators and the immediately
off-diagonal charged generators.

Liouville and Toda equations are known to possess vortex solutions. In fact they are the paradigm of vortex equations [8].

To be more precise, a vortex with magnetic charge:

\[
\frac{2\pi n_{i+1}^{i+1}}{k}
\]  

arises wherever the charged field \(A_{z_{i+1}}^{i+1}\) has a zero of the form:

\[
h(z, \bar{z})^{i+1}(\bar{z} - \bar{z}_1)^{n_{i+1}^{i+1}}.
\]

This corresponds to a pole singularity of the vector potential in the Cartan subalgebra and a \(\delta\)-like singularity in the Abelian field strength. An anti-vortex corresponds to a zero involving the complex conjugate variable and component of the connection on the \(Z\) surface. The ‘order parameter’ \(A_{z_{i+1}}^{i+1}\) for vortices of type \(i, i+1\) approaches exponentially, with exponent of order \(|H^i - H^{i+1}|^{\frac{1}{2}}\), its asymptotic value \(H^{i+1} - H^i\) from the vanishing value in the centre of the vortex. Hence in 3d the magnetic flux would be squeezed into long flux tubes of transverse width of the order \(|H^i - H^{i+1}|^{-\frac{1}{2}}\). In the case where vortices and a non-vanishing zero mode occur in the master field, the v.e.v. of \(F_{z\bar{z}}^0\) is in general still different from zero and the symmetry is broken, unless there is enough magnetic flux to compensate the constant part and the v.e.v. of \(F_{z\bar{z}}^0\) is zero. In this last case the symmetry is unbroken, and the superconductor is at its transition point with the Coulomb phase.

There is only one possibility left. The constant part of the master field vanishes and the magnetic vortices condense in the vacuum: this is the confining phase of \(YM_4\). Hence at large \(N\), in the gauge \(F_{z\bar{z}}^{ch} = 0\), the \(SU(N)\) functional integral must be localized on the moduli space of flat connections with punctures, if the theory confines the electric charge.

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