Whispering gallery modes in a microfiber coil with an \( n \)-fold helical symmetry: classical dynamics, stochasticity, long period gratings, and wave parametric resonance

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Abstract: Propagation of whispering gallery modes in a microfiber coil having an \( n \)-fold helical symmetry is considered. The \( n \)-fold helically symmetric deformations of a coiled microfiber is similar to a long period grating introduced by periodic microfiber bending with the azimuthal angle period \( 2\pi/n \). The considered modes are localized near a geodesic situated at the peripheral part of the microfiber surface. Using the perturbation theory, a simple condition for the stability of this geodesic is found. Violation of this condition happens in a small region localized near a point determined by the phase matching condition for the introduced long period grating. In the classical limit, the unstable region corresponds to the parametric resonance and stochastization of whispering gallery rays. Generally, this region corresponds to resonance intermode coupling, which may result either in the periodic transmission of radiation power between two modes or in the aperiodic wave parametric resonance, which causes simultaneous coupling and power transfer between numerous whispering gallery modes. The obtained results are important for engineering of miniature optical fiber coils and analysis of their propagation loss.

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1. Introduction

There has been an increasing interest in development of theory, fabrication methods, and applications of optical fiber and microfiber coils [1–15]. Coiling an optical fiber results in significant reduction of its dimensions, which is important for applications. Not less importantly, a coiled waveguide exhibits remarkable physical properties. Initially, special interest in these devices was caused by a class of problems related to the polarization of light propagating along a fiber coil, e.g., to the geometric rotation of polarization [1–3,6]. Later, a new type of miniature optical resonator composed of tightly coiled optical microfiber with coupling turns was considered [4,8–12]. Recently, a miniature multimode fiber coil (MFC) with uncoupled turns illustrated in Fig. 1(a) was suggested to be used as a compact optical delay line having small losses, small dimensions, and, at the same time, broadband and smooth transmission characteristics [14]. The performance of this delay line is based on the fundamental whispering gallery mode (WGM) propagating along the periphery of the coiled microfiber. Besides, a similar device can be used as a very perceptive evanescent optical sensor based on the single fundamental WGM propagation [13] and multi-WGM interference [15]. The advantage of this type of optical sensor is in its miniature dimensions, which are significantly smaller than the length of the microfiber waveguide.

In practice, the shape of an MFC is never perfectly uniform. The MFC nonuniformities are introduced in the process of microfiber and central rod fabrication as well as in the process of winding of an MFC. Special MFC nonuniformities can be introduced on purpose in order to provide certain device functionalities. Understanding of the effect of MFC nonuniformities is
critical both for the experimental realization of a low-loss MFC and for engineering of advanced nonuniform MFC devices.

Consider an MFC created of a microfiber having a diameter of a few tens of microns. This MFC supports a series of WGMs propagating near the peripheral part of the coiled microfiber. Ideal MFC, which consists of a uniform microfiber coiled on an axially symmetric and uniform rod, possesses the infinite helical symmetry and does not exhibit intermode transitions. Deviation from this symmetry causes coupling between modes. This paper addresses the problem of propagation of light along the deformed MFC having an \( n \)-fold helical symmetry, which can be fabricated by coiling a microfiber on a central rod having an \( n \)-fold axial symmetry as illustrated in Fig. 1(b) for \( n = 5 \). The effect of the introduced periodic bending of the microfiber is similar to the effect of a long period grating (LPG) in a straight optical waveguide [16–18]. Due to the special properties of the WGMs (e.g., the quasi-equal spacing between propagation constants) the introduced perturbation is responsible for a quite complex dynamics of WGMs investigated below. Beside the general interest in engineering of the MFCs, the obtained results are useful for understanding and evaluation of the MFC transmission loss.

In Section 2, the structure and analytical expressions of the WGMs propagating along the nonuniform MFC are described. In Section 3, the classical perturbation theory for the whispering gallery rays propagating along the peripheral part of the coiled microfiber is developed and the condition of stability of these rays is found. In Section 4, it is shown that breaking of the condition of stability coincides with the condition of classical parametric resonance and, simultaneously, with the phase matching condition of the introduced LPG. In Section 5, a numerical example of the classical ray stochasticization in the neighborhood of parametric resonance is given. In Section 6, the behavior of WGMs in the neighborhood of parametric resonance is demonstrated numerically using the beam propagation method and analyzed using the coupled wave equations. It is shown that, in certain cases, there exists a periodic power transfer between two modes, while, in other cases, the wave parametric resonance causes complex simultaneous and aperiodic multi-mode coupling. Finally, in Section 7, the results of the paper are discussed and summarized.

2. Whispering gallery modes (WGMs) in a microfiber coil (MFC)

Propagation of WGMs in a uniform MFC was considered in Ref [14]. Here it is assumed that the nonuniformities of the MFC are relatively smooth having a characteristic length much greater than the radiation wavelength. In addition, the mode behavior near a thin layer surrounding the MFC surface is approximated by the zero boundary condition at the surface. Then, the propagation of a WGM localized near a bent surface can be described through the parameters of classical geodesics on this surface [19]. The surface geodesics serve as “classical bones” near which a WGM “wave flesh” can be constructed [20]. The geodesic of our interest is the helical trajectory \( s \) shown in Fig. 1(c). Introduce the local orthogonal coordinate system near this geodesic \((s, p, q)\) where \(s\) is the coordinate along the geodesic and \(p\) and \(q\) are, respectively, the coordinates along the normal and binormal vector of this geodesic at point \(s\). The asymptotic expression for the modes (or their linear combinations) localized near geodesic \(s\) has the form [19]:

\[
E_{nl}(s, p, q) \approx \rho^{-\frac{1}{16}}(s)\sigma^{1/2}(s)\exp\left[i\int_{s_0}^{s} \beta_{nl}(s)\, ds + \frac{\beta_0}{2\sigma(s)}\left(i\frac{d\sigma}{ds} - \frac{1}{\sigma(s)}\right)q^2\right]\times 
H_n\left(\frac{\beta_0\rho^2 q}{\sigma^2(s)}\right)Ai\left(-\left(\frac{2\beta_0^2}{\rho^2(s)}\right)^{1/3}p - \sigma_i\right), \quad \beta_0 = \frac{2\pi n_f}{\lambda}.
\]

Here, \(\beta_{nl}(s)\) is the mode propagation constant, which is determined as
In Eq. (1) and (2), \( \rho(s) \) is the local radius of curvature of the geodesic, \( n_f \) is the microfiber refractive index, \( \lambda \) is the radiation wavelength, \( H_n(x) \) is the Hermite polynomial, \( Ai(x) \) is the Airy function, \( \varsigma_i \) is the root of Airy function \( (\varsigma_0 = 2.338, \varsigma_1 = 4.088, \varsigma_2 = 5.52, \ldots) \) and \( \beta \) is the local radius of curvature of the geodesic, \( \beta_0 = \sqrt{\frac{\rho(s)}{2}} \), \( \beta_1 = \frac{1}{\sigma^2(s)} \), \( m = \frac{1}{2} \), \( \sigma(s) = \sqrt{\sum_{j=1}^{2} a_j \theta_j(s) \theta_j(s)} \), where functions \( \theta_j(s) \) satisfy the linear differential equation:

\[
\frac{d^2 \theta_j(s)}{ds^2} + K(s) \theta_j(s) = 0.
\]

As a particular case, consider a uniform MCR with microfiber radius \( r_0 \) and central rod radius \( R_0 \). The radius of curvature of the geodesic in this case is constant:

\[
\rho(s) = R_0 = R_0 + 2r_0,
\]

and solutions of Eq. (4) are \( \theta_{1,2}(s) = \exp(\pm iK_0^{1/2} s) \) where the Gaussian curvature

\[
K(s) = K_0 = \frac{1}{R_0 r_0}.
\]

Generally, Eqs. (1) and (2) describe a linear combination of WGMs. Individual WGMs can be obtained by setting \( a_{11} = a_{22} = 0 \), so that, from Eq. (5), \( a_{12} = a_{21} = 1/(2K_0^{1/2}) \) and, from Eq. (3), \( \sigma^2(s) = 1/K_0^{1/2} \). Substitution of these values into Eqs. (1) and (2) yields the expression for the WGMs considered in Ref [14]:

\[
E_m(s, p, q) = \exp \left[ \frac{i \beta_0 q}{2} - \frac{1}{2} \left( \frac{\beta_0^2}{R_0 r_0} \right)^{1/2} q^2 \right] H_m \left[ \frac{\beta_0^2}{R_0 r_0} \right] Ai \left[ - \frac{2 \beta_0^2 q}{R_0} \right] p - \varsigma_i.
\]

From Eq. (1), the characteristic widths of the WGMs along axes \( x \) and \( y \) are \( \rho(s)^{1/3} \beta_0^{-1/3} \) and \( \sigma(s)^{1/2} \beta_0^{-1/2} \), respectively. For the uniform MFC, these widths are \( R_0^{1/3} \beta_0^{-1/3} \) and \( R_0^{-1/2} \beta_0^{-1/2} \). WGMs which are localized near the geodesic for arbitrarily large \( s \) and experience only small changes for a weakly perturbed MFC are called the stable WGMs. In other words, we call a WGM stable if (a) its characteristic widths are bounded for any \( s \) and (b) if any small \( s \)-dependent perturbation of the MFC parameters does not cause unlimited
growth of these widths. The condition of stability along the \( n \)-axis is simple: it holds if the radius of curvature \( \rho(s) \) is bounded. The condition of stability along the \( r \)-axis is less trivial: it coincides with the condition of stability of the geodesic \( s \) at the MFC surface, which is determined by the boundedness of the solution of Eq. (4). This differential equation is the Jacobi equation, which determines the geodesics at the MFC surface adjacent to the geodesic \( s \) [22].

Equations (1)-(5) define the evolution of WGMs along the MFC with varying parameters. It allows to obtain simple analytical expressions for the WGMs and for the transmission amplitudes between WGMs provided that the shape of the geodesic \( s \), its curvature, \( \rho(s) \), and the Gaussian curvature of the MFC surface at this geodesic, \( K(s) \), are expressed through the MFC parameters. A model of a perturbed MFC is considered in the next Section.

3. Stable geodesic and the Gaussian curvature

In this section, an MFC surface is modeled in the following simplified form:

\[
\begin{align*}
  x &= \left[ R(\phi) + r(\phi)(1 + \cos(\theta)) \right] \cos(\phi) \\
  y &= \left[ R(\phi) + r(\phi)(1 + \cos(\theta)) \right] \sin(\phi) \\
  z &= r(\phi) \sin(\theta) + \tan(\gamma) \int_0^\phi [R(\phi) + r(\phi)] d\phi
\end{align*}
\]

where \( R(\phi) \) is the local radius of the central rod and \( r(\phi) \) is the local radius of the coiled microfiber. For simplicity, the local asymmetry of microfiber, i.e., the dependence of \( r \) on \( \theta \) is ignored. For a uniform MFC, \( R(\phi) \) and \( r(\phi) \) are constants and the coil pitch is \( p = 2\pi \tan(\gamma) R \). Samples of MFCs depicted with Eq. (9) are shown in Fig. 1.

The shape of an MFC is assumed to be close to a uniform MFC, so that

\[
R(\phi) = R_0 + \Delta R(\phi), \quad r(\phi) = r_0 + \Delta r(\phi), \quad \Delta R(\phi) \ll R_0, \quad \Delta r(\phi) \ll r_0. \quad (10)
\]

The WGMs of our interest are localized in the neighborhood of the geodesic \( s \) located at the peripheral part of the MFC surface as shown in Fig. 1(c). For a uniform MFC, \( s \) is a helix defined by the equations:

\[
\begin{align*}
  x &= R_0 \cos(\phi), \quad R_0 = R_0 + 2r_0 \\
  y &= R_0 \sin(\phi) \\
  z &= (R_0 - r_0) \tan(\gamma) \phi
\end{align*}
\]

For the relatively weak dependence of \( R(\phi) \) and \( r(\phi) \) on azimuthal angle \( \phi \) determined by Eq. (10), the shape of the trajectory \( s \) is obtained from Eq. (9) by substitution \( \theta = \theta(\phi) \) where \( \theta(\phi) \ll 1 \) is a solution of the nonuniform Jacobi equation:

\[
\theta_\omega + K_0 \theta = \frac{\tan(\gamma) K_0}{\Lambda} (R_0 \Delta r_0 - r_0 \Delta R_0), \quad K_0 = \frac{1}{r_0 R_0}, \quad \Lambda = \left[ R_0^2 + (R_0 - r_0)^2 \tan^2(\gamma) \right]^{1/2}.
\]

(12)

Here \( s \) is the coordinate along the unperturbed helical trajectory, \( s = \lambda \phi \), so that \( \Delta R_\phi = \lambda \Delta R_0 \) and \( \Delta r_\phi = \lambda \Delta r_0 \).

Below, only MFCs with a relatively small pitch \( p \ll R \) will be considered. It is seen from Eqs. (10) and (12) that, in the approximation linear in \( \Delta R / R_0, \Delta r / r_0 \), and \( \gamma \sim p / R \), the shape of the trajectory \( s \) is not changed.
As it is explained in Section 2, the propagating WGMs are localized only near a geodesic, which is stable on the MFC surface. To check for stability, it is sufficient to determine how the adjacent geodesics spread away from $s$. The deviation of these geodesics is determined by the Jacobi Eq. (4) where, in the linear in $\Delta R/ R_0$, $\Delta r/r_0$, and $\gamma$ approximation, the Gaussian curvature is reduced to

$$K(s) = K_0 - \frac{\Delta R_0 + \Delta r}{r_0}, \quad s = R_0 \phi. \quad (13)$$

### 4. Long period grating: mode coupling and the classical parametric resonance

Assume that the microfiber radius is uniform, $r(\phi) = r_0$, while the central rod is axially asymmetric having the $n$-fold rotational symmetry:

$$R(\phi) = R_0 + \Delta R_0 \sin(n \phi), \quad n = 1, 2, .... \quad (14)$$

In this case, the surface of the coiled microfiber, which supports the geodesic $s$, experiences periodic bending with the period $P_0 = 2\pi R_0/n$. This bending is a particular case of a long period grating explicitly investigated in fiber optics [16–18]. It is known that the long period grating, which period matches the difference between two propagation constants of two modes $\Delta \beta$ so that $|\Delta \beta| = 2\pi / P_0$, causes resonant coupling between these modes. For a uniform MFC, the propagation constants are approximately uniformly spaced in quantum number $m$ (see Eq. (8)) and the matching condition for these propagation constants causes multiple coupling between modes. From Eqs. (8) and (14), due to the symmetry of perturbation, only coupling between the adjacent modes with the same parity is not zero. For these modes, the expression for the propagation constant in Eq. (8) gives $\Delta \beta = 2(R_{\chi} r_0)^{1/2}$ and the phase matching condition reads:

$$r_0 = \frac{4R_0}{n^2}. \quad (15)$$

In classical mechanics, the latter effect is the well known parametric resonance, which, in our case, is described by the Jacobi Eq. (4). In fact, with Eq. (13), Eq. (4) takes the form of the Mathieu equation:

$$\theta_{\phi \phi} + \left( \frac{R_0}{r_0} + n^2 \frac{\Delta R_0}{r_0} \sin(n \phi) \right) \theta = 0. \quad (16)$$

In the cylindrically symmetric case, $\Delta R_0 = 0$, Eq. (16) shows that the geodesics adjacent to the helical trajectory $s$ experience oscillations with period $\phi_0 = 2\pi (r_0 / R_0)^{1/2}$ as a function of $\phi$ (or with period $2\pi / K_0^{1/2}$ as a function of $s$). For $\Delta R_0 \neq 0$, the classical parametric resonance [23] occurs when the period of harmonic perturbation, $\phi_0 = 2\pi / k$, is close to a half of period $\phi_0$, i.e., $\phi_0 = (2 + \delta)\phi_0$, $\delta << 1$. Substitution of the values of periods into the latter equation yields the condition of the classical parametric resonance for a MFC:

$$r_0 = r_{\chi \phi}(1 + \delta), \quad r_{\chi \phi} = \frac{4R_0}{n^2}, \quad \delta << 1, \quad n = 3, 4, 5, .... \quad (17)$$

For $\delta = 0$, this equation coincides with the LPG phase matching condition, Eq. (15) that will be further considered in Section 6. Clearly, the rotational orders $n = 1, 2$ are excluded from...
this equation because \( r_0 < R_0 = R_0 + 2r_0 \). More detailed calculation \([24]\) shows that the solutions of Eq. (16) are stable for

\[
\frac{n^2 \Delta R_0}{2R_0} < \delta \quad \text{(18)}
\]

and are unstable otherwise. In particular, for the microfiber radius close to the parametric resonance value, \( r_0 = r_{pr} + \Delta r_0, \Delta r_0 << r_{pr} \), Eqs. (17) - (19) yield the following condition of stability:

\[
|\Delta R_0| < \Delta r_0 \left| \frac{R_0 + 2r_{pr}}{2R_0} \right. \quad \text{(19)}
\]

Usually, \( r_0 << R_0 \) so that \( R_0 + 2r_{pr} \approx R_0 \) and Eq. (19) is reduced to \( |\Delta R_0| < |\Delta r_0|/2 \).

5. Ray dynamics: stability and the classical parametric resonance

It is interesting to investigate the evolution of optical rays, which propagate inside the MFC near the geodesic \( s \) and possess finite but small angle of reflection from the MFC surface. The problem of 3D evolution of an optical ray is solved as follows. Parameter \( \theta \) is eliminated from Eq. (9) of the MFC surface and the latter is represented in the cylindrical coordinates

\[
\rho = \rho_{MFC}(z, \phi) = R(\phi) + \left[ x^2(\phi) - (\tan(\gamma))^2 \right]^{1/2} R(\phi) d\phi - z^2 \right]^{1/2}.
\]

The equations of a straight optical ray are represented parametrically as a function of \( \phi \) in the form

\[
\begin{align*}
x &= x_0 + t_x f(\phi), \\
y &= y_0 + t_y f(\phi), \\
z &= z_0 + t_z f(\phi),
\end{align*}
\]

where \( f(\phi) = (y_0 - x_0 \tan(\phi)) / (t_x \sin(\phi) - t_y) \) and \( x_0, y_0, z_0, t_x, t_y, t_z \) are constants. Then, the point of intersection between the MFC surface and a
ray is found by numerical solution of equation $\rho_{\text{MFC}}^2(\varphi, \varphi) = x^2(\varphi) + y^2(\varphi)$ This approach allows us to perform much faster numerical calculations compared to a more general method, in which determination of the intersection between a straight line and a surface requests solution of two equations with two unknowns (in our case, $\theta$ and $\varphi$).

Here, the model of an MFC having the $n$-fold rotational symmetry with $n = 10$ is considered. It is assumed that the parameters of an MFC are close to the parametric resonance.

Fig. 3. (a)-(g) – Fragments of the Poincaré surfaces of sections (left hand side figures) and the relative deviation of the reflection points (right hand side figures) versus the number of turns for the launched ray shown in Fig. 2. (h) and (i) – The sine of incidence angle at the reflection point and the relative deviation of the same point, respectively, after completion of 200 turns as a function of the microfiber radius $r_0$. 

$R_0 = 2.30 \mu m$
The radius of the central rod is set to \( R_0 = 230 \mu\text{m} \) (the length units considered in this Section are not important and can be rescaled). Then, from Eq. (17), we have \( r_{PR} = 10 \mu\text{m} \). The MFC pitch is put equal to \( h = 4r_{PR} \). The MFC coil with \( R_0 = 230 \mu\text{m}, r_0 = 10 \mu\text{m}, \) and \( h = 40 \mu\text{m} \) is shown in Fig. 2. The amplitude of the central rod radius perturbation in Eq. (14) is set to \( \Delta R_0 = 0.01r_0 \). The point plots in Figs. 3(a)-(g) show fragments of the Poincaré surfaces of sections, i.e., the sine of the reflection angle at reflection points, (left hand side figures) and the relative deviation of the reflection points (right hand side figures) versus the number of turns. The fragments are calculated for the ray (red line in the inset of Fig. 2) launched parallel to the geodesic \( s \) from the point at axis \( n \), which is spaced by \( 0.07r_0 \) from \( s \). The spacing \( 0.07r_0 \) is chosen to arrive at the unstable and stochastic behavior of the ray for \( r_0 = 10.3 \mu\text{m} \) shown in Fig. 3(d). In our numerical simulations, the evolution of the similarly defined rays was investigated for different microfiber radii \( r_0 \). Comparison of this behavior for the MFC with microfiber radii \( r_0 = 10.2797 \mu\text{m}, r_0 = 10.27978 \mu\text{m}, r_0 = 10.28 \mu\text{m}, \) and \( r_0 = 10.3 \mu\text{m} \) plotted in Fig. 3(a)-(d) shows that there is a sharp lower threshold of instability, so that for \( r_0 < 10.2797 \mu\text{m} \) the ray behavior is stable (for the case of Fig. 3(a), simulation beyond the plotted data up to 2000 turns did not show any instability). Alternatively, Figs. 3(d)-(g) show that the unstable behavior smoothly vanishes with increasing \( r_0 \) and the evolution of the ray becomes stable for \( r_0 > 10.75 \mu\text{m} \). Figure 3(h) and (i) plot the sine of incidence angle and the relative deviation for a single reflection point, which immediately follows completion of the 200th turn, as a function of the microfiber radius. The latter plots are useful for comparison with the wave calculations of Section 6. The size of the region of instability found from Fig. 3 is \( \approx 0.45 \mu\text{m} \) in reasonable agreement with Eq. (19), which gives for the region of parametric resonance \( 2\Delta R_0 = 4\Delta R_0 \approx 0.4 \mu\text{m} \).

6. Propagation of WGMs: mode coupling and the wave parametric resonance

Let us investigate the effect of the introduced periodic perturbation on the propagation of WGMs along the MFC. As previously, it is assumed that the coiled microfiber has the uniform radius \( r_0 \), while the central rod is cylindrically asymmetric and possesses the \( n \)-fold rotational symmetry defined by Eq. (14). Similar to Ref [14], calculation is performed with the beam propagation model (BPM) where the local microfiber radius of curvature \( R_{loc}(\varphi) \) along the geodesic \( s \) is introduced by replacing of the refractive index \( n_r \) with the effective index \( n_r(1 + x/R_{loc}(\varphi)) \). In this approximation, the pitch \( p \) is set to zero, i.e., the torsion effect is ignored. Therefore, the essentially vector effects (polarization dependence, Berry phase [1–3], etc.) are neglected. The local radius of curvature in the latter equation is expressed through \( R(\varphi) = R(\varphi) + 2r(\varphi) \) as \( R_{loc} = (\rho^2 + \rho_\varphi^2)^{3/2} / (\rho^2 + 2\rho_\varphi^2 - \rho \rho_\varphi \varphi) \). For \( r(\varphi) = r_0 \) and \( R(\varphi) \) defined from Eq. (14) with \( \Delta R_0 \ll R_0 \), we have:

\[
R_{loc}(\varphi) = R_0 - \Delta R_{loc} \sin(n\varphi), \quad \Delta R_{loc} = (n^2 + 2\Delta R_0).
\]  

Close to the MFC parameters considered section 5, here we set \( n = 10 \) and \( R_0 = R_0 + 2r_{PR} = 230 \mu\text{m} + 2 \cdot 10 \mu\text{m} = 250 \mu\text{m} \). In addition, we set \( \Delta R_{loc} = 0.05R_0 \approx 12.5 \mu\text{m} \), which, as follows from Eq. (20), approximately corresponds to \( \Delta R_0 \approx 0.1 \mu\text{m} \) of Section 5. The length of the coiled microfiber, \( L_{MF} \), is set to 10 turns, which is equal to 100 grating periods or \( L_{MF} = 100 - 2\pi R_0 = 15.71 \text{mm} \).

The fundamental WGM is launched along the defined MFC. The transmission power of this mode, \( P(z) \), is found by calculation of the squared overlap integral between the field...
distribution at position \( z \) along the microfiber and the initial distribution of the WGM. Generally, the behavior of \( P(z) \) depends on MFC parameters \( R_0, r_0, \Delta R_m \), and radiation wavelength \( \lambda \).

Figure 4(a) plots the fundamental WGM power \( P(L_{MF}) \) at the end of the MFC as a function of microfiber radius \( r_0 \) for the wavelength \( \lambda = 1.5 \mu m \). It is seen that the transmission power has a resonant behavior in the region between 11 and 13 \( \mu m \). Shifting of the resonant region from \( r_m = 10 \mu m \) defined in Section 5 and from the unstable region in Fig. 3(h) and (i) towards larger microfiber radius values is due to the smaller effective radius of the WGM trajectory compared to \( R_0 = R_0 + 2r_0 \).

The WGM propagation in the resonant region exhibits remarkable features. In particular, for certain values of \( r_0 \), the fundamental WGM experiences stable periodic exchange of power with a single WGM, the phenomenon well known from the LPG theory [16]. An example of such behavior is shown in Fig. 5(a) for \( r_0 = 12.1 \mu m \). This figure shows the transmission power of the fundamental WGM, \( P(z) \), as a function of propagation distance \( z \)

![Fig. 4](image)

**Fig. 4.** Transmission power \( P \) of the fundamental WGM: (a) – as a function of \( r_0 \) for \( \lambda = 1.5 \mu m \) (inset – magnified region between \( r_0 = 10 \mu m \) and \( r_0 = 14 \mu m \)); (b) – as a function of radiation wavelength \( \lambda \) for \( R_0 = 11.53 \mu m \) and \( r_0 = 12.1 \mu m \).

(upper plots) and the corresponding distribution of the radiation along the microfiber (lower plots). Inset surface plots are the cross-sectional radiation distribution at the positions indicated by arrows. For other values of \( r_0 \), the wave parametric resonance distributes the fundamental WGM power among multiple neighboring WGMs. This situation is illustrated in Fig. 5(b) for \( r_0 = 11.53 \mu m \). It is seen that now the behavior of transmission power \( P(z) \) is aperiodic and coupling between multiple modes accompanies the WGM evolution. The wavelength dependence on the transmission power \( P \) for \( r_0 = 12.1 \mu m \) and \( r_0 = 11.53 \mu m \) considered in Fig. 5(a) and (b) is shown in Fig. 4 (b). Notice the sharp peak in Fig. 4(b) corresponding to \( \lambda = 0.82 \mu m \) and \( r_0 = 11.53 \mu m \), which is attributed to the resonance transmission to the mode \( E_{101} \) (see Eq. (8)).
The analysis and explanation of the obtained numerical results can be performed with the coupled wave equations. Let us introduce the adiabatic cross-sectional modes of the MFC, $E_{nm}(s,n,t)$, which parametrically depends on the coordinate $s$, and represent the propagating radiation field as:

$$E(s, p, q) = \sum_{m,l} a_{ml}(s)E_{ml}(p, q) \exp \left\{-i \int \beta_{ml}(s) ds \right\}, \quad (21)$$

The coupled wave equations for the coefficients $a_{ml}(s)$ are

$$\frac{da_{ml}}{ds} = \sum_{n,l} \left| E_{ml} \right| \frac{dE_{ml}}{ds} \exp \left\{-i \int (\beta_{ml}(s) - \beta_{nl}(s)) ds \right\} a_{nl},$$

$$\left| E_{ml} \right| \frac{dE_{ml}}{ds} = \int_{p,q} E_{ml}(s, p, q) \frac{dE_{ml}(s, p, q)}{ds} dpdq. \quad (22)$$

Fig. 5. Transmission power of the fundamental WGM as a function of propagation distance (upper plots) and corresponding distribution of the radiation along the microfiber (lower plots). Inset surface plots are the cross-sectional radiation distribution at the positions indicated by arrows.
The asymptotic expressions for the WGMs following from Eqs. (1), (2) and (8) are 
\[ E_{m}(s, p, q) = E_{m}(s, p, q) . \] 
Here we are interested in resonant transitions between the series of modes with quantum numbers \((m, 0)\). For this series, the harmonic dependence of local radius of microfiber curvature defined by Eq. (20) yields 
\[ \left( E_{m} \right) \left( dE_{m} \right) = \frac{n(n^2 + 2)\Delta R_{0}}{8R_{0}^2} \cos(n \pi / R_{0}) \sqrt{(m+1)(m+1), \quad m = m \pm 2, \quad m' \neq m \pm 2} \] 
and equidistant spacing between propagation constants, \( \beta_{m} = 2/(R_{0}r_{0})^{1/2} \). Consider, for example, the situation shown in Fig. 5(a) when only two modes, \( E_{00} \) and \( E_{20} \), are coupling. Substituting Eq. (23) into Eq. (22) and neglecting the fast oscillating terms yield the following coupling wave equations for these modes:
\[ \frac{da_{0}}{ds} = \kappa \exp \left[ -is \left( \frac{2}{(R_{0}r_{0})^{1/2} - n / R_{0}} \right) a_{20} \right] \]
\[ \frac{da_{20}}{ds} = -\kappa \exp \left[ is \left( \frac{2}{(R_{0}r_{0})^{1/2} - n / R_{0}} \right) a_{00} \right] \]
\[ \kappa = \frac{n(n^2 + 2)\Delta R_{0}}{2^{7/2}R_{0}^{2}} \] 
From Eq. (24), the condition of phase matching is \( 2(R_{0}r_{0})^{1/2} = n / R_{0} \), which coincides with the classical parametric resonance condition described in Section 4. At this condition, modes \( E_{00} \) and \( E_{20} \) experience periodic full power transfer depicted in Fig. 5(a). The period of the power transfer determined from Eq. (24) is 
\[ S_{p} = \frac{\pi}{\kappa} = \frac{2^{7/2}\pi R_{0}^{2}}{n(n^2 + 2)\Delta R_{0}} \] 
For the parameters of the MFC considered in this Section, \( S_{p} = 17.8 \) mm. The value of this period in Fig. 5(a), determined numerically, is 13.2 mm, which is ~25% less than the estimate given by Eq. (24). The reason of the discrepancy is in the asymptotic nature of Eqs. (1) and (2). In fact, for the considered MFC parameters, the size of mode \( E_{20} \) along the vertical direction in the insets of Fig. 5(a) is comparable with the microfiber diameter, so that the parabolic approximation for the circular microfiber profile in the vicinity of geodesic \( s \), which is assumed in Eqs. (1) and (2), is not accurate. However, Eq. (25) has a reasonable accuracy and is useful for estimates of the characteristic length along which the intermode power transfer takes place. Similarly, for the situation depicted in Fig. 5(b), the propagation constants of modes \( E_{m,0} \) are not accurately equidistant as follows from Eq. (2). Nevertheless, the considered relatively small harmonic perturbation defined by Eq. (20) causes the multimode wave parametric resonance. Further analysis of the behavior of WGMs based on the coupled wave equations, Eq. (22), is beyond the scope of this paper.

7. Discussion and summary

An ideal uniform MFC possesses an infinite helical symmetry. Deformations of an ideal MFC can be accidentally introduced in the process of its fabrication or specially introduced in order to arrive at certain transmission characteristics of this device. Application and further development of the results of this paper can be useful for the analysis of transmission losses caused by MFC deformations as well as for engineering of a specially deformed MFC.

The effect of periodic perturbations on the MFC transmission loss is of special interest. In fact, a conventional drawing station allows to fabricate 20-40 \( \mu \)m diameter silica microfibers, which are uniform enough to ensure the adiabatic suppression of the intermode transitions. In
addition, the translation symmetry of the central rod can be maintained along the distances of a few tens of mm very precisely. However, the accuracy of rotation symmetry of the rod can be hardly made better than a few tenth of a percent. As the result, the light propagating along a coiled microfiber is subject to periodic bending perturbation caused by the axial asymmetry of the central rod which can be expressed by its local radius variation:

\[
R_{\text{loc}}(\varphi) = R_0 + \sum_{n=0}^{\infty} \Delta R_{\text{loc}}^{(n)} \cos(n \varphi + \xi_n),
\]  

(26)

where the terms, which determine the axial asymmetry, are relatively small, \(|\Delta R_{\text{loc}}| << R_0\). For small asymmetry of the central rod, the perturbation theory allows to separate the contribution of harmonics in Eq. (26) and to reduce the problem to the case of a single harmonic with period \(2\pi / n\) considered in this paper. In a more general case, both the microfiber and central rod random nonuniformities should be taken into account. These nonuniformities contain a periodic component, which can be expressed similar to Eq. (26).

For the MFC engineering applications, the partial harmonic amplitude and phase parameters in Eq. (26), \(\Delta R_{\text{loc}}^{(n)}\) and \(\xi_n\), which determine the central rod shape, can be used to arrive at the desired transmission characteristics of the fundamental WGM propagating along the MFC. Here, again, for small \(\Delta R_{\text{loc}}^{(n)}\), the design procedure is reduced to the case of a single harmonic investigated above.

In summary, the classical whispering gallery ray and WGM evolution in an MFC with \(n\)-fold helical symmetry is an interesting new example of the behavior of wave systems that are chaotic in the classical limit [25]. A simple condition of instability is found. For this condition, the classical stochastization, the classical and wave parametric resonance, and the stable WGM coupling, which results in periodic exchange of power, are investigated. Propagation of WGMs along an MFC with more complex nonuniformities can be considered similarly and applied to the evaluation of losses in MFC-based optical delay lines [14] and to the analysis of experimental data for fabricated MFCs [13,15].