Generalized, Master and Nonlocal Symmetries of Certain Deformed Nonlinear Partial Differential Equations

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It is shown that the deformed Nonlinear Schrödinger (NLS), Hirota and AKNS equations with
(1 + 1) dimension admit infinitely many generalized (nonpoint) symmetries and polynomial con-
served quantities, master symmetries and recursion operator ensuring their complete integrability.
Also shown that each of them admits infinitely many nonlocal symmetries. The nature of the
deformed equation whether bi-Hamiltonian or not is briefly analyzed.

Keywords: Integrable equations; nonlinear partial differential equations; soliton equations; deformed
equations.

1. Introduction

The investigation of completely integrable higher order nonlinear partial differential equa-
tions (PDEs) with (1 + 1) dimension admitting solitons has drawn considerable interest in
recent years [13, 15–18, 23, 25, 29–31, 33, 35, 36]. The question of integrability of nonlinear
PDEs can be investigated through several approaches [2, 4, 20, 32]. If a nonlinear PDE,
written in Hamiltonian description, admits infinitely many generalized symmetries, then it
is expected to be integrable in the sense of Liouville [26, 27, 37]. If the generalized sym-
metries of a given nonlinear PDE are explicitly known, then it is possible to construct the
so called recursion operator. The mathematical characterization of the recursion operator
is that it maps a symmetry to another symmetry of a given equation. The existence of a
recursion operator guarantees that the PDE has infinitely many higher order symmetries,
which is a key feature of complete integrability [6, 10, 12, 28].

Another interesting class of symmetries admitted by nonlinear PDEs possessing solitons
is master symmetries which involves both dependent and independent variables and are
related with generalized symmetries [8–10]. A master symmetry (of degree n) for a non-
linear PDE, is a derivation in the Lie algebra of vector fields having the property that n
fold applications leaves the commutator of the flow under consideration invariant. When a
nonlinear PDE admits a master symmetry it usually admits infinitely many such symmetries where the successive elements involve independent variables, dependent variables as well as spatial derivatives of the dependent variables. A remarkable feature of master symmetries for nonlinear PDEs is that they constitute a centerless Virasoro algebra. The existence of a sequence of master symmetries is one of the characteristics of completely integrable nonlinear PDEs. If the infinitesimal symmetries of the given PDE involves local variables (that is independent variables and dependent variables and its partial derivatives only) then it is called local symmetries otherwise known as nonlocal symmetries [3, 5, 14, 34]. One of the reasons to introduce the concept of a nonlocal symmetry, involving nonlocal variables, is that the generating function of local symmetries are solution spaces of the equations under consideration. These generating functions depend on independent and dependent variables as well as on their derivatives of higher order. It is appropriate to mention here that the theory of coverings over differential equations provides an interesting tool to describe various nonlocal phenomena: nonlocal symmetries and conservation laws, Bäcklund transformations, prolongation structures, etc.

If a nonlinear PDE admitting solitons gets perturbed or deformed it is of interest to investigate whether it preserves the integrability properties of un-deformed counterpart. In this article, we report that the deformed NLS, Hirota and AKNS equations with (1 + 1) dimension, respectively, given by

\begin{align}
iv - u_{xx} - 2u^2u^* &= g,  \\
g_x &= -2ub,  \\
b_x &= i(ug^* - u^*g),  \\
u_i &= i(u_{xx} + 6|u|^2u_x) + \frac{u_{xxx}}{2} + |u|^2u = g,  \\
g_x &= -2ub,  \\
b_x &= i(ug^* - u^*g) \quad \text{a - parameter},  \\
u_x &= u_{xx} + 2u^2v + \tilde{g},  \\
v_x &= v_{xx} - 2v^2u + h,  \\
\tilde{g}_x &= 2ub,  \\
h_x &= 2vb,  \\
b_x &= uh + v\tilde{g},
\end{align}

where * denotes complex conjugate preserve the integrability properties of their undeformed counterpart. Note that on eliminating \(g(x, t), \tilde{g}(x, t), h(x, t), \) and \(b(x, t)\) in the above coupled equations one can obtain higher order nonlinear PDEs.

We would like to mention that the deformed equations (1.1)–(1.3) arise from the compatibility condition of a system of linear equations. More precisely the deformed equations (1.1)–(1.3) admit Lax pair satisfying the Lax equation [1, 21]

\[ L_t - M_x + [L, M] = 0 \quad \text{or} \quad L_t - M_x + LM - ML = 0. \]
The explicit form of the Lax matrices $L$ and $M$ are given below:

(i) Lax pair of deformed NLS equation, (1.1)

$$L = \begin{pmatrix} i\lambda & vu \\ vu^* & -i\lambda \end{pmatrix}, \quad M = \begin{pmatrix} 2i\lambda^2 - i\nu u^* + \frac{ib}{2\lambda} & 2i\lambda u + ig \\ 2i\lambda u^* - u_x^* + \frac{ig^*}{2\lambda} & -2i\lambda^2 + i\nu u^* - \frac{ib}{2\lambda} \end{pmatrix}. $$

(ii) Lax pair of deformed Hirota equation, (1.2)

$$L = \begin{pmatrix} i\lambda & vu \\ vu^* & -i\lambda \end{pmatrix}, \quad M = \begin{pmatrix} 4i\alpha\lambda^3 - i\lambda^2 + \frac{i\nu u^*}{2} - 2i\alpha\lambda u^* \\ 4i\alpha\lambda^3 + 2i\alpha u_x - i\lambda u - i\lambda u_{xx} \end{pmatrix} + a(uu_x^* - u^* u_x) + \frac{ib}{2\lambda} \begin{pmatrix} 4i\alpha\lambda^2 u + 2i\alpha u_x - i\lambda u - i\lambda u_{xx} \\ -2i\alpha u_x^* + \frac{ig}{2\lambda} \end{pmatrix} + \frac{u^*}{2} - i\alpha uu_{xx}^* - 2i\alpha u^* - \frac{ig^*}{2\lambda} + i\lambda^2 - \frac{i\nu u^*}{2} + 2i\alpha uu^* \\ + \frac{u^*}{2} - i\alpha uu_{xx}^* - 2i\alpha u^* - \frac{ig^*}{2\lambda} - a(uu_x^* - u^* u_x) - \frac{ib}{2\lambda} \end{pmatrix}. $$

(iii) Lax pair of deformed AKNS equation, (1.3)

$$L = \begin{pmatrix} i\lambda & vu \\ -iv & -i\lambda \end{pmatrix}, \quad M = \begin{pmatrix} 2\lambda^2 + uv - \frac{ib}{2\lambda} & 2\lambda u - i\nu u^* - \frac{ig}{2\lambda} \\ -2\lambda v - iv_e - \frac{ib}{2\lambda} - 2\lambda^2 - uv + \frac{ib}{2\lambda} \end{pmatrix} + a(uu_x^* - u^* u_x) + \frac{ib}{2\lambda} \begin{pmatrix} 2\lambda^2 + uv - \frac{ib}{2\lambda} \\ -2\lambda v - iv_e - \frac{ib}{2\lambda} - 2\lambda^2 - uv + \frac{ib}{2\lambda} \end{pmatrix}.$$

where $\lambda$ is a spectral parameter. The derivation of the Lax matrices $L$ and $M$ associated with the deformed equations are given in [31].

The explicit form of the Lax matrices $L$ and $M$ are given below:

(i) Lax pair of deformed NLS equation, (1.1)

$$L = \begin{pmatrix} i\lambda & vu \\ vu^* & -i\lambda \end{pmatrix}, \quad M = \begin{pmatrix} 2i\lambda^2 - i\nu u^* + \frac{ib}{2\lambda} & 2i\lambda u + ig \\ 2i\lambda u^* - u_x^* + \frac{ig^*}{2\lambda} & -2i\lambda^2 + i\nu u^* - \frac{ib}{2\lambda} \end{pmatrix}. $$

(ii) Lax pair of deformed Hirota equation, (1.2)

$$L = \begin{pmatrix} i\lambda & vu \\ vu^* & -i\lambda \end{pmatrix}, \quad M = \begin{pmatrix} 4i\alpha\lambda^3 - i\lambda^2 + \frac{i\nu u^*}{2} - 2i\alpha\lambda u^* \\ 4i\alpha\lambda^3 + 2i\alpha u_x - i\lambda u - i\lambda u_{xx} \end{pmatrix} + a(uu_x^* - u^* u_x) + \frac{ib}{2\lambda} \begin{pmatrix} 4i\alpha\lambda^2 u + 2i\alpha u_x - i\lambda u - i\lambda u_{xx} \\ -2i\alpha u_x^* + \frac{ig}{2\lambda} \end{pmatrix} + \frac{u^*}{2} - i\alpha uu_{xx}^* - 2i\alpha u^* + \frac{ig^*}{2\lambda} - a(uu_x^* - u^* u_x) - \frac{ib}{2\lambda} \end{pmatrix}. $$

(iii) Lax pair of deformed AKNS equation, (1.3)

$$L = \begin{pmatrix} i\lambda & vu \\ -iv & -i\lambda \end{pmatrix}, \quad M = \begin{pmatrix} 2\lambda^2 + uv - \frac{ib}{2\lambda} & 2\lambda u - i\nu u^* - \frac{ig}{2\lambda} \\ -2\lambda v - iv_e - \frac{ib}{2\lambda} - 2\lambda^2 - uv + \frac{ib}{2\lambda} \end{pmatrix} + a(uu_x^* - u^* u_x) + \frac{ib}{2\lambda} \begin{pmatrix} 2\lambda^2 + uv - \frac{ib}{2\lambda} \\ -2\lambda v - iv_e - \frac{ib}{2\lambda} - 2\lambda^2 - uv + \frac{ib}{2\lambda} \end{pmatrix}.$$
2. Generalized Symmetries, Recursion Operator and Polynomial Conserved Quantities of Deformed NLS Equation

It is easy to check that the deformed NLS equation and its complex conjugate
\[ iu_t - u_{xx} - 2u^2u^* = g \] (2.1a)
\[ iu^*_t + u^*_{xx} + 2u^2u = -g^* \] (2.1b)
\[ g_x = -2iub \] (2.1c)
\[ g^*_x = 2iu^*b \] (2.1d)
\[ b_x = i(ug^* - u^*g) \] (2.1e)

are invariant under the scaling symmetry
\[ (t, x, u, u^*, g, g^*, b) \rightarrow (s^{-2}t, s^{-1}x, s^1u, s^1u^*, s^3g, s^3g^*, s^3b), \]
where \( s \) is an arbitrary parameter which suggests that \( u \) corresponds to one derivative with respect to \( x \), \( g \) and \( b \) corresponds to three derivatives with respect to \( x \).

2.1. Generalized symmetries

Let us assume that the deformed NLS equation (2.1) is invariant under a one parameter nonpoint transformations
\[ \tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = u + \epsilon G^1_n + O(\epsilon^2), \quad \tilde{u}^* = u^* + \epsilon G^2_n + O(\epsilon^2), \]
\[ \tilde{g} = g + \epsilon H^1_n + O(\epsilon^2), \quad \tilde{g}^* = g^* + \epsilon H^2_n + O(\epsilon^2), \quad \tilde{b} = b + \epsilon B_n + O(\epsilon^2), \] (2.2)
where
\[ K_n = (G^1_n, G^2_n, H^1_n, H^2_n, B_n)^T \]
are functions of \((u, u^*, g, g^*, b, u_x, u^*_x, u_{xx}, u^*_{xx}, \ldots, g_x, g^*_x, b_x, \ldots)\), provided \( u, u^*, g, g^* \) and \( b \) satisfy Eq. (2.1). Consequently we obtain the following invariant equations
\[ \frac{DG^1_n}{Dt} - \frac{D^2G^1_n}{Dx^2} - 2u^2G^1_n - 4uu^*G^2_n - H^2_n = 0, \] (2.3)
\[ \frac{DG^2_n}{Dt} + \frac{D^2G^2_n}{Dx^2} + 2u^2G^1_n + 4uu^*G^2_n + H^2_n = 0, \] (2.4)
\[ \frac{DH^1_n}{Dx} + 2ub + 2bG^1_n = 0, \] (2.5)
\[ \frac{DH^2_n}{Dx} - 2uu^*b - 2bG^2_n = 0, \] (2.6)
\[ \frac{DB_n}{Dx} - i(g^*G^1_n + uH^2_n - gG^2_n - u^*H^1_n) = 0. \] (2.7)
where

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u^2} + u_3 \frac{\partial}{\partial u^3} + u_4 \frac{\partial}{\partial u^4} + u_5 \frac{\partial}{\partial u^5} + u_6 \frac{\partial}{\partial u^6} + \ldots
\]

\[
\frac{D}{Dx} = \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial v} + v_2 \frac{\partial}{\partial v^2} + v_3 \frac{\partial}{\partial v^3} + v_4 \frac{\partial}{\partial v^4} + v_5 \frac{\partial}{\partial v^5} + v_6 \frac{\partial}{\partial v^6} + \ldots
\]

Note that the invariant equations (2.3)–(2.7) can be solved in more than one way [11, 27]. However in this article we follow the algorithmic method devised by Hereman [11] to derive generalized symmetries, conserved quantities and recursion operators for nonlinear partial differential and differential-difference equations [11]. Hereman’s algorithm is based on the concept of weights and ranks. The weight \( W \) of a variable is defined as the exponent in the scaling parameter \( s \) which multiplies the variable. Weights of the dependent variables are nonnegative and rational. An expression is said to be uniform in rank if all its terms have the same rank. Setting \( W(u) = 1, W(g) = 3, W(b) = 3 \) and \( W(\partial/\partial t) = 2 \) and hence Eqs. (2.1) are of rank \( 3, 3, 4, 4, 4, 4 \). This property is called uniformity in rank. The rank of a monomial is defined as the total weight of the monomial, again in terms of derivatives with respect to \( x \). From Eqs. (2.3)–(2.7), it is easy to check that

\[
K_0 = \begin{pmatrix} G_1^0 & G_0^1 \\ H_1^0 & H_0^1 \\ B_0 & \end{pmatrix}, \quad K_1 = \begin{pmatrix} G_1^1 & G_1^0 \\ H_1^1 & H_1^0 \\ B_1 & \end{pmatrix}, \quad K_2 = \begin{pmatrix} G_2^1 & G_2^0 \\ H_2^1 & H_2^0 \\ B_2 & \end{pmatrix}
\]

are trivial symmetries with rank \( 2, 2, 4, 4, 4 \) and \( 3, 3, 5, 5, 5 \) respectively. Obviously the next generalized symmetry \( K_2 \) must have rank \( 4, 4, 6, 6, 6 \). With this in mind we first form monomials in \( u, u^*, g, g^* \) and \( b \) of rank \( 4, 4, 6, 6, 6 \). Thus the most general form of \( K_2 \) will be

\[
K_2 = \begin{pmatrix} G_2^1 & G_2^0 \\ H_2^1 & H_2^0 \\ B_2 & \end{pmatrix} = \begin{pmatrix} l_1 u_{xx} + l_2 u_{x} u_x^* + l_3 u_{x} u_{x}^* + l_4 u_{x} u_{x}^* + l_5 g_{xx} + l_6 g_{x}^* \\ m_1 u_{xx}^* + m_2 u_{x} u_{x}^{*2} + m_3 u_{x} u_{x}^{*2} + m_4 u_{x} u_{x}^{*2} + m_5 g_{xx}^* + m_6 g_{x}^{*2} \\ p_1 g_{xx} + p_2 u_{x} u_{x}^* g + p_3 u_{x} u_{x}^* g + p_4 u_{x} u_{x}^* g + p_5 g_{xx}^* + p_6 g_{x}^{*2} \\ q_1 g_{xx} + q_2 u_{x} u_{x}^* g + q_3 u_{x} u_{x}^* g + q_4 u_{x} u_{x}^* g + q_5 g_{xx}^* + q_6 g_{x}^{*2} \\ r_1 b_{xx} + r_2 u_{x} u_{x}^* b + r_3 u_{x} u_{x}^* b + r_4 u_{x} u_{x}^* b + r_5 u_{x} u_{x}^* b + r_6 u_{x} u_{x}^* b \end{pmatrix}
\]

where \( l_j, m_j, p_j, q_j, \) and \( r_j, j = 1, 2, \ldots, 6 \) are arbitrary constants to be determined. Hereafter we denote \( u_{x}, u_{xx}, g, g_{xx}, \) etc. by \( u_{ab}, g_{ab}, \) etc. We now substitute \( G_2^1, G_2^0, H_2^1, H_2^0 \) and \( B_2 \) in the invariant equations (2.3)–(2.7), with \( n = 2 \) and solving them by using (2.1) we obtain
the following nontrivial generalized symmetry

\[
\begin{pmatrix}
G_1^2 \\
G_2^2 \\
H_1^2 \\
H_2^2 \\
B_2
\end{pmatrix} =
\begin{pmatrix}
u u_x + 6u u_x^* \\
6u u_x^* + 6u^* u_x \\
g_{2x} + 6u^* u_x + 6u u_x g - 6u u_x g^* \\
b_x + 6u u_x b + 6u^* u_x b
\end{pmatrix},
\]

(2.10)

with rank \((4, 4, 6, 6)\). In a similar manner we obtain the next generalized symmetry \(K_3\) for (2.1) with rank \((5, 5, 7, 7)\). They are

\[
\begin{pmatrix}
G_1^3 \\
G_2^3 \\
H_1^3 \\
H_2^3 \\
B_3
\end{pmatrix} =
\begin{pmatrix}
u u_x + 2u^2 u_x^* + 8u u_x u_{xx} + 4u u_x u_x^* + 6u^* u_x^2 + 6u^3 u^2 \\
g_{2x} - 2u^2 g_x^* - 6g g_x^2 - 12u^3 u x^* + 4u g u_{xx} + 8u^2 g u_x - 8u g^2 u_{xx} \\
6u u_x g_x + 4u u_x^* - 12u^2 u g^2_x + 12u u_x^* g_x + 12u^2 u^* g_x + 18u^2 u^3 g \\
b_x + 8u^2 b u_{xx} + 12u u_x u^2 + 20u^2 u b_{xx} + 64u^2 u^2 b + 8u b u_{xx} + 2g u_{xx} \\
10u^2 g^2 u_x^* - 10u^2 g u_{xx} + 12u^2 u^3 g_x
\end{pmatrix},
\]

(2.11)

Proceeding as above, we find that the deformed NLS equation (2.1) admits a sequence of generalized symmetries \(K_n\) with rank \((n+2, n+2, n+4, n+4)\). Since each entry of the generalized symmetry \(K_n, n \geq 4\) involves a lengthy expression we refrain from presenting them here. We have also checked that the commutator

\[
[K_i, K_j] = K'_j[K_i] - K'_i[K_j] = 0 \quad \forall \ i, j
\]

(2.12)

showing that the obtained generalized symmetries are in commute [7]. Here the Fréchet derivative of \(K\) is defined as

\[
K'(u)[v] = \frac{\partial}{\partial \epsilon} K(u + \epsilon v)|_{\epsilon=0}.
\]

2.2. Recursion operator

An operator valued function \(R\) is said to be a recursion operator of a scalar nonlinear PDE with two independent variables if it satisfies

\[
\hat{K} = RK,
\]

where \(\hat{K}\) and \(K\) are successive generalized symmetries. For the deformed NLS equation (2.1) the recursion operator \(R\) will be \((5 \times 5)\) matrix and so the above equation can be written as

\[
K_{m+1} = RK_m.
\]
where $K_m$ and $K_{m+1}$ are successive generalized symmetries and $R_{ij}$, $i, j = 1, 2, 3, 4, 5$ are functions of dependent variable and their differential and integral operators. We below explain how the recursion operator $R$ for the deformed NLS equation can be constructed. For $m = 2$, Eq. (2.13) becomes

$$
\begin{bmatrix}
G^1_m \\
G^2_m \\
H^1_m \\
H^2_m \\
B_{m+1}
\end{bmatrix} =
\begin{bmatrix}
R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\
R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\
R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\
R_{41} & R_{42} & R_{43} & R_{44} & R_{45} \\
R_{51} & R_{52} & R_{53} & R_{54} & R_{55}
\end{bmatrix}
\begin{bmatrix}
G^1_m \\
G^2_m \\
H^1_m \\
H^2_m \\
B_{m+1}
\end{bmatrix},
$$

(2.13)

where $K_4$ and $K_5$ are successive generalized symmetries given in (2.10) and (2.11) with ranks $(4, 4, 6, 6)$ and $(5, 5, 7, 7)$, respectively. From Eq. (2.14) it is clear that

\begin{align}
\text{rank } G^1_4 &= \text{rank } R_{11} + \text{rank } G^1_4 = \text{rank } R_{12} + \text{rank } G^2_4 = \text{rank } R_{13} + \text{rank } H^1_2 \\
&= \text{rank } R_{14} + \text{rank } H^2_1 = \text{rank } R_{15} + \text{rank } B_2,
\end{align}

(2.15)

\begin{align}
\text{rank } G^2_4 &= \text{rank } R_{21} + \text{rank } G^2_4 = \text{rank } R_{22} + \text{rank } G^2_4 = \text{rank } R_{23} + \text{rank } H^1_2 \\
&= \text{rank } R_{24} + \text{rank } H^2_1 = \text{rank } R_{25} + \text{rank } B_2,
\end{align}

(2.16)

\begin{align}
\text{rank } H^1_2 &= \text{rank } R_{31} + \text{rank } G^1_4 = \text{rank } R_{32} + \text{rank } G^2_4 = \text{rank } R_{33} + \text{rank } H^1_2 \\
&= \text{rank } R_{34} + \text{rank } H^2_1 = \text{rank } R_{35} + \text{rank } B_1,
\end{align}

(2.17)

\begin{align}
\text{rank } H^2_1 &= \text{rank } R_{41} + \text{rank } G^1_4 = \text{rank } R_{42} + \text{rank } G^2_4 = \text{rank } R_{43} + \text{rank } H^1_2 \\
&= \text{rank } R_{44} + \text{rank } H^2_1 = \text{rank } R_{45} + \text{rank } B_1,
\end{align}

(2.18)

\begin{align}
\text{rank } B_4 &= \text{rank } R_{41} + \text{rank } G^1_4 = \text{rank } R_{42} + \text{rank } G^2_4 = \text{rank } R_{43} + \text{rank } H^1_2 \\
&= \text{rank } R_{44} + \text{rank } H^2_1 = \text{rank } R_{45} + \text{rank } B_2.
\end{align}

(2.19)

Making use of ranks of generalized symmetries we obtain nonzero ranks for the following $R_{ij}$'s:

\begin{align}
\text{rank } R_{11} &= \text{rank } R_{12} = \text{rank } R_{21} = \text{rank } R_{22} = \text{rank } R_{33} = 1, \\
\text{rank } R_{44} &= \text{rank } R_{45} = \text{rank } R_{42} = \text{rank } R_{44} = \text{rank } R_{45} = 1, \\
\text{rank } R_{53} &= \text{rank } R_{54} = \text{rank } R_{55} = 1, \\
\text{rank } R_{11} &= \text{rank } R_{42} = \text{rank } R_{44} = \text{rank } R_{42} = \text{rank } R_{51} = \text{rank } R_{52} = 3.
\end{align}

(2.20a-d)
respectively are as follows: We have also verified that the defining equation (2.13) holds for \( x, t \)

A local conservation law of a nonlinear PDE with two independent variables \((x, t)\) is defined by

\[
\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0
\]

which is satisfied on all solutions. The function \(\rho(x, t)\) is usually called local conserved density and \(J(x, t)\) is the associated flux also known as current density. Extending the method devised by Hereman et al. [11], we find that the deformed NLS admits a sequence of polynomial conserved quantities \(\{\rho^{(n)}, J^{(n)}\}\). First three of them with ranks \(2, 3, 4, 5\) respectively are as follows:

\[
\begin{align*}
\rho^{(1)} &= uu^*,\quad J^{(1)} = -b - i(\omega u_y^* - u^* u_x) \\
\rho^{(2)} &= u^* u_x,\quad J^{(2)} = i(u^2 u_x^* + u^* u_{xx} - u_y u_x^* - u^* u_y^*) \\
\rho^{(3)} &= u^2 u_x^2 - u_y u_x^* \quad J^{(3)} = i(2 uu_{xx}^* u_x - 2 u^2 u_x u_y^* + u_{xx} u_{xx}^* - u_{xy} u_y^*) - 2 uu^* b
\end{align*}
\]

equations (2.22) in addition with (2.1). From the above analysis we observe that the effect of deformation changes the structure of the local current densities \(J^{(n)}\), \(n = 1, 2, \ldots\), which contain the deforming functions \(g, b\), but not the densities \(\rho^{(n)}\), which generate the conserved quantities.
3. Master Symmetries of the Deformed NLS Equation

A function $\tau = \tau(x, t, u, u_x, \ldots, \partial_x^{-1}u, \ldots)$ is said to be a master symmetry of a PDE with two independent variables $(x, t)$ if

$$[\cdot, [\cdot, \cdot]] = 0 \quad \text{and} \quad \tau, \cdot \neq 0,$$

where the commutator relation $[\cdot, \cdot]$ is defined as

$$[F, G] = G'F - F'G.$$

Here the Frechet derivative of $F$ is defined as

$$F'(u)[v] = \frac{\partial}{\partial \epsilon} F(u + \epsilon v) |_{\epsilon = 0}.$$

We show below how to derive a sequence of master symmetries for (2.1). Obviously (2.1) is invariant under the dilation symmetry

$$(t, x, u, u^*, g, g^*, b) \rightarrow (s^{-2}t, s^{-1}x, s^1u, s^1u^*, s^3g, s^3g^*, s^3b),$$

where $s$ is an arbitrary parameter. As mentioned earlier, we have $W(\partial/\partial x) = 1, W(u) = 1, W(u^*) = 1, W(g) = 3, W(g^*) = 3, W(b) = 3, W(\partial/\partial t) = 2$, and hence (2.1) have ranks $(3, 3, 5, 5, 5)$. It is known that soliton equations with $(1+1)$ dimension admits infinitely many master symmetries $\{\tau_j\}_{j=0}^{\infty}$ satisfying the following relations [7]:

$$[\tau_j, \tau_l] = (l - j)\tau_{j+l},$$

$$[K_j, \tau_l] = d_j K_{j+l},$$

$$[K_j, K_l] = 0,$$

where $K_j, j = 0, 1, 2, \ldots$ are generalized symmetries. It appears that the relations (3.2) also hold good for deformed NLS equation (2.1) if master symmetries exist. Since the generalized symmetries of deformed NLS equation are commutable, Eq. (3.2c) readily holds from Eq. (2.12). For $j = 0, l = 0$, Eq. (3.2b) becomes

$$[K_0, \tau_0] = d_0 K_0.$$

Now the rank of the generalized symmetry $K_0 = (u_x, u^*_x, g, g^*_x, b)$ is $(2, 2, 4, 4, 4)$ and so the rank of the master symmetry $\tau_0 = (\tau_1^0, \tau_2^0, \tau_3^0, \tau_4^0, \tau_5^0)^T$ satisfying

$$[G^1_0, \tau_0^1] = d_0 G^1_0, \quad [G^2_0, \tau_0^2] = d_0 G^2_0, \quad [H^1_0, \tau_0^1] = d_0 H^1_0, \quad [H^2_0, \tau_0^2] = d_0 H^2_0, \quad [B_0, \tau_0] = d_0 B_0,$$
where \( \phi \) satisfy the relations showing that they constitute a symmetry algebra of Virasoro type.

Proceeding as above, we find that the deformed NLS equation (2.1) admits a sequence of master symmetries satisfying (3.2a,b). The first two (nontrivial) members of the sequence of master symmetries with ranks (2, 2) are:

\[
\begin{align*}
\tau_0 &= \begin{pmatrix} \tau_1^0 \\ \tau_2^0 \\ \tau_3^0 \\ \tau_4^0 \end{pmatrix} = \begin{pmatrix} (a_1 u_u + a_2 u) \\ a_3 u_u^2 + a_4 u^* \\ a_5 x g_1 + a_6 g \\ a_7 x g_2^* + a_8 g^* \\ a_9 x b_x + a_{10} b \end{pmatrix}, \\
\tau_1 &= \begin{pmatrix} \tau_1^1 \\ \tau_2^1 \\ \tau_3^1 \\ \tau_4^1 \end{pmatrix} = \begin{pmatrix} x(u_{xx} + 2 u u^*) + 2 u u + 2 x \\ -x(u_{xx} + 2 u^2 u) + 2 u u^* + 2 x \\ x(g_{xx} + 4 u^2 g - 2 u^2 g^*) + 2 g + 2 g_x + 2 u b \\ -x(g_{xx}^* + 4 u^2 u - 2 u^2 g^*) + 2 g^* + 2 g_x^* - 2 u^* b \end{pmatrix}, \\
\tau_2 &= \begin{pmatrix} \tau_1^2 \\ \tau_2^2 \\ \tau_3^2 \\ \tau_4^2 \end{pmatrix} = \begin{pmatrix} x G_{x}^2 + 3 u_{xx} + 2 u_{x} \phi + 4 u v + 2 u^2 u^* \\ x G_{x}^2 + 3 u_{xx}^* + 2 u^* \phi - 4 u^* v + 6 u^2 u \\ x H_{x}^2 + 3 g_{xx} + 2 u_{x} \phi + 4 \psi + 4 u u^* g - 6 u^2 g^* + 2 i b_{xx} \\ x H_{x}^2 + 3 g_{xx}^* + 2 u^* \phi - 4 \psi^* + 8 u u^* g^* - 6 u^2 g^* - 2 i b_{xx}^* \end{pmatrix}.
\end{align*}
\]

where \( \phi = u^* \) and \( \psi = u^* u \), and \( K_2 = (G_2, G_2^*, H_{x}^2, H_{x}^2, B_{x})^T \) is the generalized symmetry of deformed NLS equation. We have checked that the obtained master symmetries also satisfy the relations

\[
[K_j, \tau_j] = -(j + 1)K_{i+j}, \quad \forall j, l
\]

showing that they constitute a symmetry algebra of Virasoro type.
4. Nonlocal Symmetries of Deformed NLS Equation

Let us assume that the deformed NLS equation (2.1) is invariant under a nonpoint continuous transformations,

\[ \tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = u + \epsilon S^1_n + O(\epsilon^2), \quad \tilde{u}^* = u^* + \epsilon S^2_n + O(\epsilon^2), \]

\[ \tilde{g} = g + \epsilon T^1_n + O(\epsilon^2), \quad \tilde{g}^* = g^* + \epsilon T^2_n + O(\epsilon^2), \quad \tilde{b} = b + \epsilon A_n + O(\epsilon^2) \]  

(4.1)

where \( P_n = (S^1_n, S^2_n, T^1_n, T^2_n, A_n)^T \) are functions of \((x, t, u, u^*, g, g^*, u_x, u_x^*, u_{xx}, u_{xx}^*, g_x, g_x^*, g_{xx}, g_{xx}^*, b, b_x, \ldots)\), provided \( u, u^*, g, g^* \) and \( b \) satisfy Eq. (2.1). Then the invariant equations are

\[ \frac{D S^1_n}{D x} - \frac{D S^2_n}{D x^2} - 2u^* S^1_n - 4u u^* S^1_n - T^1_n = 0, \]  

(4.2)

\[ \frac{D S^2_n}{D x} + \frac{D S^1_n}{D x^2} + 2u^2 S^2_n + 4u u^* S^2_n + T^2_n = 0, \]  

(4.3)

\[ \frac{D T^2_n}{D x} - 2u^* A_n - 2b S^2_n = 0, \]  

(4.4)

\[ \frac{D A_n}{D x} - i(\epsilon T^1_n + u T^2_n - g S^2_n - u^* T^2_n) = 0. \]  

(4.6)

Following the procedure of Hereman et al. described in Sec. 2.1, we find that the deformed NLS equation (2.1) admits a sequence of nonlocal symmetries \( P_n \). The first three members of the sequence with ranks \((1, 1, 3, 3, 3), (2, 2, 4, 4, 4)\) and \((3, 3, 5, 5, 5)\) are:

\[ P_1 = \begin{pmatrix} S^1_1 \\ S^2_1 \\ T^1_1 \\ T^2_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} -2it(u_{xx} + 2u^* u) + xu_x + u \\ 2it(u_{xx} + 2u^2 u) + xu_x^* + u^* \\ -2it(g_{xx} + 4uu^* g - 2u^2 g) + xg_x + g \\ 2it(g_{xx} + 4uu^* g - 2u^2 g) + xg_x^* + g^* \\ -2it(4u^* b + 2u^* u_x) + xb_x + b \end{pmatrix} \]  

(4.7)

\[ P_2 = \begin{pmatrix} S^1_2 \\ S^2_2 \\ T^1_2 \\ T^2_2 \\ A_2 \end{pmatrix} = \begin{pmatrix} -2igtG^1_2 + xG^1_2 + 2u^2 + 2u \\ -2igtG^2_2 + xG^2_2 - 2u^2 \phi - 2u^* \phi \\ -2itH^1_x + xH^1_x + 2g \phi + 2u^* b + 2u b \\ -2itH^2_x + xH^2_x - 2g^* \phi - 2u^* \phi - 2u^* b \\ -2itB_2 + xB_1 + b + 2u^* g \end{pmatrix} \]  

(4.8)

\[ P_3 = \begin{pmatrix} S^1_3 \\ S^2_3 \\ T^1_3 \\ T^2_3 \\ A_3 \end{pmatrix} = \begin{pmatrix} -2igtG^1_3 + xG^1_3 + 3u_{xx} + 2u^* \phi + 4u \phi + 2u^2 u^* \\ -2igtG^2_3 + xG^2_3 + 3u_{xx}^* + 2u_{xx}^* \phi - 4u^* \psi + 6u^2 u^* \\ -2itH^1_x + xH^1_x + 3g_{xx} + 2g \phi + 4g \psi + 4uu^* g - 6u^2 g - 6u^2 g^* + 2u_x b \\ -2itH^2_x + xH^2_x + 3g_{xx}^* + 2g^* \phi - 4g^* \psi + 8u^2 g^* - 6u^2 g^* - 6u^2 g - 2u_x b \\ -2itB_3 + xB_2 + 2b_{xx} + 2u_x g - 2u_x g^* \end{pmatrix} \]  

(4.9)
where $\phi$, and $\psi$ are nonlocal variables satisfying

$$
\phi_x = uu^*, \quad \phi_t = b + i(uu^* - u^*u), \quad \psi_x = u^*u, \\
\psi_t = i(uu^* + uu - u^2u^2 - u^*u).
$$

(4.10)

Here $K_2 = (G^1_2, G^2_2, H_1^2, H_2^2, B)^T$ and $K_3 = (G^1_3, G^2_3, H_1^3, H_2^3, B)^T$ are generalized symmetries of the deformed NLS equation given in Eqs. (2.10)–(2.11). Note that the nonlocal variables $\phi$ and $\psi$ can be connected with conservation laws, that is,

$$
\frac{\partial}{\partial t}(uu^*) + \frac{\partial}{\partial x}(-b - i(uu^* - u^*u)) = 0,
$$

$$
\frac{\partial}{\partial t}(u^*u) + \frac{\partial}{\partial x}(-i(uu^* + uu - u^2u^2 - u^*u)) = 0.
$$

5. Bi-Hamiltonian Representation of Deformed NLS Equation

We would like to mention that the KdV given by

$$
(\partial^3_x + 8uu_x \partial_x + 4u_{xxx})(u_1 + u_{xxx} + 6u^2_x) = 0,
$$

(5.1)
can be written as

$$
v_1 + v_{xxx} + 12uv_x - w_x = 0, \quad w_{xxx} + 8uw_x + 4vw_x = 0
$$

(5.2)

where $v = u_x$, $w = u_t + u_{xxx} + 6u^2_x$. Recently, Kupershmidt [15] has shown that the KdV can be written in the following Hamiltonian description

$$
u_1 = \theta_1 \left( \frac{\delta H_1}{\delta u} \right) - \theta_1(w) = \theta_2 \left( \frac{\delta H_2}{\delta w} \right) - \theta_2(w) = 0,
$$

(5.3)

where

$$
\theta_1 = \partial_x, \quad \theta_2 = \partial^3_x + 2(u_1 + \partial_x u)
$$

are Hamiltonian operators of the KdV equation $u_t - 6uu_x - u_{xxx} = 0$, and

$$
H_1 = u, \quad H_2 = \frac{u^2}{2} \cdots
$$

are conserved densities. Kersten et al. [19] have demonstrated that the Kupershmidt deformation of a bi-Hamiltonian system is itself bi-Hamiltonian. It is straightforward to check that the deformed NLS equation can also be written in the Hamiltonian description

$$
\begin{pmatrix}
\frac{\delta H_1}{\delta u} \\
\frac{\delta H_1}{\delta u^*}
\end{pmatrix}
= \theta_1 \left( \frac{\delta H_1}{\delta u} \right) + \theta_1 \left( g^* \right) = \theta_2 \left( \frac{\delta H_2}{\delta u^*} \right) + \theta_1 \left( g^* \right)
$$

(5.4)
and

$$\theta_2\left(\frac{g^*}{g}\right) = 0.$$  \hfill (5.5)

Here

$$\theta_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \theta_2 = i \begin{pmatrix} 2u\partial_x^{-1}u & -\partial_x - 2u\partial_x^{-1}u^* \\ -\partial_x - 2u^*\partial_x^{-1}u & 2u^*\partial_x^{-1}u^* \end{pmatrix} \hfill (5.6)$$

are Hamiltonian operators of NLS equation

$$iu_t - u_{xx} - 2u^2u^* = 0 \hfill (5.6)$$

and

$$H_2 = u^*u_x, \quad H_3 = u^2u'^2 - u_xu'^* \hfill (5.7)$$

are conserved densities of the NLS equation. Hence the deformed NLS equation (2.1) is a bi-Hamiltonian system.

6. Summary and Concluding Remarks

In this article we have shown that the deformed Nonlinear Schrödinger (NLS), Hirota and AKNS equations with (1+1) dimension admit infinitely many generalized (nonpoint) symmetries and polynomial conserved quantities, master symmetries and a recursion operator ensuring their complete integrability. Also shown that each of them admits infinitely many nonlocal symmetries. The nature of the deformed equation whether bi-Hamiltonian or not is also analyzed.

From the analysis of the deformed NLS we observe that the conserved densities for the deformed and un-deformed remain the same while the current densities (fluxes) explicitly contain the deforming functions. This shows that the nonholonomic deformations can appear only at the equation level, while the conserved integrals of motion remain the same under deformation. Also we observe that the obtained sequence of nonlocal symmetries and master symmetries of deformed NLS equation satisfy

$$P_{n+1} = R P_n, \quad \tau_{n+1} = R \tau_n, \quad \forall n$$  \hfill (6.1)

where $R$ is a recursion operator given in Eq. (2.21) Furthermore, the sequence of master symmetries, nonlocal symmetries and generalized symmetries satisfy the following relation:

$$P_{i+1} = -2iK_{i+1} + \tau_i, \quad i = 0, 1, \ldots.$$  \hfill (6.2)

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Appendix: A Brief Details of Generalized, Master and Nonlocal Symmetries of Deformed Hirota and AKNS Equations

A. Deformed Hirota Equation (1.2)

Proceeding in a similar manner explained in Secs. 2–4 we find that the deformed Hirota equation admits infinitely many generalized symmetries, master symmetries and nonlocal symmetries and a recursion operator. The first few members of the sequence of generalized symmetries, conserved quantities, master symmetries and nonlocal symmetries are given below:

First member of generalized symmetries \( K_0 = (G_1^0, G_2^0, H_1^0, H_2^0, B_0)^T \)

\[
G_1^0 = \begin{pmatrix} u_x + a(u_{xx} + 2u^2u) \\ u_x^* - a(u_{xx} + 2u^2u) \\ g_x + a(g_{xx} + 4u^*u - 2u^2g) \\ g_x^* - a(g_{xx} + 4u^*u - 2u^2g) \\ b_x + a(b_{xx} + 4u^*b + 2igu^*g) \end{pmatrix}
\]

Second member of generalized symmetries \( K_1 = (G_1^1, G_2^1, H_1^1, H_2^1, B_1)^T \)

\[
G_1^1 = \begin{pmatrix} u_{xx} + 2u^2u + a(u_{xx} + 6u^*u_x) \\ -u_{xx} - 2u^2u + a(u_{xx} + 6u^*u_x) \\ g_{xx} + 4u^*u - 2u^2g + a(g_{xx} + 6u^*u_x - 6u^*u_x - 6u^*u_x) \\ -g_{xx} - 4u^*u + 2u^2g + a(g_{xx} + 6u^*u_x + 6u^*u_x - 6u^*u_x) \\ b_{xx} + 4u^*b + 2igu^* + a(b_{xx} + 6u^*b_x + 6u^*b_x + 6u^*b_x) \end{pmatrix}
\]

Third member of generalized symmetries \( K_2 = (G_1^2, G_2^2, H_1^2, H_2^2, B_2)^T \)

\[
G_1^2 = \begin{pmatrix} u_{xx} + 6u^*u_x + a(u_{xx} + 2u^2u_x + 8u^*u_{xx} + 4u^*u_x + 6u^*u_x + 6u^*u_x) \\ -u_{xx} - 2u^2u + a(u_{xx} + 6u^*u_x) + 2u^2g + 6u^*u_x - 6u^*u_x - 12u^1u^1 \\ g_{xx} + 6u^*g_x + 6u^*g_x - 6u^*g_x + 6u^*g_x - 6u^*g_x + 6u^*g_x \\ g_{xx} + 6u^*g_x + 6u^*g_x - 6u^*g_x - a(g_{xx} - 2u^2g_{xx} - 6u^*u_x - 12u^1u^1) \\ b_{xx} + 6u^*b_x + 6u^*b_x + 6u^*b_x + 6u^*b_x + 6u^*b_x + 6u^*b_x \end{pmatrix}
\]

etc.
First member of conserved quantities ($\rho^{(1)}, J^{(1)}$)

$$\rho^{(1)} = uu - 2ia^2ux,$$

$$J^{(1)} = 2a^2(6uu^2ux + uaa_{xx} - u_xu_{xx} + u_x^2u_{xx} + 2u^2u_{xx})$$

$$- 2u_xu_{xx} + 4u^2u_{xx} + 2ug^*) - \frac{4}{7}(a^*u_x - uu^*) - b$$

Second member of conserved quantities ($\rho^{(2)}, J^{(2)}$)

$$\rho^{(2)} = uu + 2a(u^2u_{xx} - uu^*),$$

$$J^{(2)} = 2a^2(-u_u^2ux - u_x^2u_{xx} + 2u^3u_{xx} - 10uu^2u_x - u_x^3u - u^2u_{xx} - u_x^2u^2$$

$$+ 2u_{xx}^2u_{xx} + 2u^3u_{xx}) + a(8uu^2u_x + 2u_xu_{xx} + uu^*x - 6u_{xx}u_x - 2u^3u_{xx}$$

$$- 4uu^*b) - \frac{4}{7}(u^2u_{xx} + uu^* - uu^* - 2ug^*)$$

Third member of conserved quantities ($\rho^{(3)}, J^{(3)}$)

$$\rho^{(3)} = (u^2u_{xx} - uu^*),$$

$$J^{(3)} = 2a^2(-12u^3u_{xx} + 3u^2u_{xx} + 6uu^2u_x - uz^2u_{xx} - u_x^2u_{xx} + 6u^3u_{xx}$$

$$+ 10uu^2u_x - 2u_x^3u + uu^*_{xx}u_{xx}) + a(-2u^2u + 2u^3u_x - 2u^2u^2$$

$$+ 4u^2u^2 - 5uu^2u_{xx} + 6u^3u^2 - 4iu^2u_x - 16uu^2u - 2u_x^2u_{xx} + 2uu_xu_{xx}$$

$$- uz^2u_x + 5iu^2u_{xx} - iu^2u_{xx} - 2uu^*b - \frac{1}{2}u_xu_{xx} + \frac{1}{2}u_xu_{xx}$$

etc.

First member of master symmetries $\tau_0 = (\tau_0^1, \tau_0^2, \tau_0^3, \tau_0^4, \tau_0^5)^T$

$$\begin{align*}
\tau_0^1 &= (x + a)ux + u \\
\tau_0^2 &= (x + a)ux + u_x \\
\tau_0^3 &= (x + a)ux + u_x \\
\tau_0^4 &= (x + a)ux + u_x \\
\tau_0^5 &= (x + a)ux + u_x
\end{align*}$$

Second member of master symmetries $\tau_1 = (\tau_1^1, \tau_1^2, \tau_1^3, \tau_1^4, \tau_1^5)^T$

$$\begin{align*}
\tau_1^1 &= (x + a)(ux + 2u^2u) + 2u^2 + 2ux \\
\tau_1^2 &= -((x + a)(ux + 2u^2u) + 2u^2 + 2ux) \\
\tau_1^3 &= (x + a)(ux + 2u^2u) + 2u^2 + 2ux \\
\tau_1^4 &= -((x + a)(ux + 2u^2u) + 2u^2 + 2ux) \\
\tau_1^5 &= (x + a)(ux + 2u^2u) + 2u^2 + 2ux
\end{align*}$$
Third member of nonlocal symmetries $\tau_2 = (\tau_2^1, \tau_2^2, \tau_2^3, \tau_2^4)^T$

$$
\begin{align*}
\tau_2^1 = & \begin{pmatrix} 
(x + a)(ua_x + 6u^a u_x) + 3u_{xx} + 2u_x \phi + 4u \psi + 2u^a u_x^2 \\
(x + a)(ua_x + 6u^a u_x^2) + 3u_{xx} + 2u_x \phi - 4u \psi + 6u^2 u_x^2 \\
(x + a)(g_{xx} + 6u^a u_x + 6u^a u_x^2 - 6u^g_{xx} u_x) + 3g_{xx} + 2g_x \phi \\
\end{pmatrix} \\
\tau_2^2 = & \begin{pmatrix} 
+ 4u \psi + 4u^a u_x - 6u^2 g_x^2 + 2b_{ux} \\
- 4u \psi + 8u^a u_x - 6u^2 g_x^2 - 2b_{ux} \\
(x + a)(b_{xx} + 6u^a b_x + 6u^a b_{xx} + 6u_b u_x) + 2b_{ux} + 2g^x u_x - 2g u_x^2 \\
\end{pmatrix}
\end{align*}
$$

etc.

where $\phi_x = uu^x$ and $\psi_x = u^a u_x$.

First member of nonlocal symmetries $P_1 = (S_1^1, S_2^1, T_1^1, T_2^1, A_1)^T$

$$
\begin{align*}
S_1^1 = & \begin{pmatrix} 
\begin{pmatrix} 
(\phi(u_{xx} + 2u^2 u_x) - 3a(u_{xx} + 6u^a u_x) + (x + a)u_x + u \\
(\phi(-u_{xx}^2 + 2u^2 u_x) - 3a(u_{xx}^2 + 6u^a u_x^2) + (x + a)u_x + u^x \\
(\phi(g_{xx} + 4u^a u_x - 2u^2 g_x^2) - 3a(g_{xx} + 6u^a u_x + 6u^a g_x \\
- 6u^g_{xx} u_x) + (x + a)g_x + g \\
(\phi(g_{xx} - 4u^a g_x^2 + 2u^2 g_x^2) - 3a(g_{xx} + 6u^a u_x + 6u^a g_x \\
- 6u^g_{xx} u_x) + (x + a)g_x^2 + g^x \\
(\phi(b_{xx} + 4u^a b_x + 2g u_x^2) - 3a(b_{xx} + 6u^a b_x + 6u^a b_{xx, x} \\
+ 6u_{b u_x}) + (x + a)b_x + b
\end{pmatrix}
\end{pmatrix} \\
\end{align*}
$$

Second member of nonlocal symmetries $P_2 = (S_1^2, S_2^2, T_1^2, T_2^2, A_2)^T$, where

$$
\begin{align*}
S_1^2 = & \begin{pmatrix} 
\begin{pmatrix} 
(\phi(u_{xx} + 6u^a u_x) - 3a(u_{xx} + 6u^a u_x) + 4u_{xx} u_x^2 + 8u^a u_{xx} + 2u_{xx} u_x^2 + 6u^3 u_x^2) \\
(\phi(\phi(u_{xx} + 2u^2 u_x) - 3a(u_{xx} + 6u^a u_x) + (x + a)u_x + u \\
(\phi(-u_{xx}^2 + 2u^2 u_x) - 3a(u_{xx}^2 + 6u^a u_x^2) + (x + a)u_x + u^x \\
(\phi(g_{xx} + 4u^a u_x - 2u^2 g_x^2) - 3a(g_{xx} + 6u^a u_x + 6u^a g_x \\
- 6u^g_{xx} u_x) + (x + a)g_x + g \\
(\phi(g_{xx} - 4u^a g_x^2 + 2u^2 g_x^2) - 3a(g_{xx} + 6u^a u_x + 6u^a g_x \\
- 6u^g_{xx} u_x) + (x + a)g_x^2 + g^x \\
(\phi(b_{xx} + 4u^a b_x + 2g u_x^2) - 3a(b_{xx} + 6u^a b_x + 6u^a b_{xx, x} \\
+ 6u_{b u_x}) + (x + a)b_x + b
\end{pmatrix}
\end{pmatrix} \\
\end{align*}
$$

$$
\begin{align*}
T_1^2 = & \begin{pmatrix} 
\begin{pmatrix} 
(\phi(g_{xx} + 6u^a u_x g + 6u^a g_x - 6u_{g x} g) - 3a(g_{xx} - 2u^2 g_x^2 - 6u^2 g^2 - 12u^a u_x g^2 \\
+ 4u_{xx, x} g + 8u^a u_{xx} g - 8u^a u_{xx, x} g + 8u^a u_{xx} g + 4u^a u_{xx, x} g + 4u^a u_{xx, x} g - 4u^a u_{xx, x} g \\
+ 12u_{xx, x} g + 18u^a u_{xx, x} g + (x + a)(g_{xx} + 4u^a u_x - 2u^2 g_x) + 2g x + 2g x + 2u_{b, b}
\end{pmatrix}
\end{pmatrix} \\
\end{align*}
$$

$$
\begin{align*}
T_2^2 = & \begin{pmatrix} 
\begin{pmatrix} 
(\phi(g_{xx} + 6u^a u_x g + 6u^a u_x g - 6u^a u_x g) - 3a(-g_{xx} + 2u^2 g_x^2 + 6u^2 g^2 + 12u^a u_x g \\
- 4u^a u_{xx, x} g - 8u^a u_{xx, x} g + 8u^a u_{xx, x} g + 4u^a u_{xx, x} g + 4u^a u_{xx, x} g - 4u^a u_{xx, x} g \\
- 12u_{xx, x} g + 18u^a u_{xx, x} g + (x + a)(-g_{xx} - 2u^2 g_x^2 + 2u^2 g^2) + 2g x - 2g x + 2u_{b, b}
\end{pmatrix}
\end{pmatrix} \\
\end{align*}
$$
A_2 = it(b_{xx} + 6u^*b_x + 6u^*u_xb + 6uu^*b) - 3at(b_{xu} + 8u_xu^*b + 12u_xu^*b + 20uu^*b_{xx})
+ 6at^2b + 8uu_xu^*b + 2uu_xg + 10at^2u^*g^2 - 10b_{xx}u^*g + 12uu^*u_x^g + (x + a)(b_{xx} + 4uu^*b + 2uu_x^g) + b_x + 2iu^*g,

etc.

where \( \phi \) is a nonlocal variable defined by

\[
\phi_x = uu^*, \quad \phi_t = b - \frac{i}{2}(uu_x^* - u^*u_x) - 3uu^*u^2 - a(u_{xx} + u_x^u - u_xu_x^*),
\]

(A.1)

The recursion operator of the deformed Hirota equation (1.2) is same as for the deformed NLS equation as in (2.21). Also we observe that the obtained sequence of nonlocal symmetries and master symmetries of deformed Hirota equation satisfy (6.1). Furthermore, it is observed that master symmetries \( \tau_i \) and nonlocal symmetries \( P_i \) and generalized symmetries \( K_i \) satisfy the following relation:

\[
P_{i+1} = itK_{i+1} - 3atK_{i+2} + \tau_i, \quad \forall i.
\]

Bi-Hamiltonian representation:

The deformed Hirota equation can also be written in the Hamiltonian description

\[
\begin{pmatrix}
  u \\
  u^* 
\end{pmatrix}
\begin{pmatrix}
  \delta H_1 \\
  \delta H_3 \\
  \delta H_5 \\
  \delta u \\
  \delta u^* 
\end{pmatrix}
= \theta_1
\begin{pmatrix}
  -g^2 \\
  -g 
\end{pmatrix}
= \theta_2
\begin{pmatrix}
  \delta H_2 \\
  \delta H_4 \\
  \delta H_6 \\
  \delta u \\
  \delta u^* 
\end{pmatrix}
+ \theta_1
\begin{pmatrix}
  -g^2 \\
  -g 
\end{pmatrix}
\]

and

\[
\theta_2
\begin{pmatrix}
  -g^2 \\
  -g 
\end{pmatrix}
= 0.
\]

Here

\[
\theta_1 = \frac{1}{2}
\begin{pmatrix}
  0 & i \\
  -i & 0 
\end{pmatrix}, \quad \theta_2 = \frac{i}{2}
\begin{pmatrix}
  -2u\partial_x^{-1}u & \partial_x + 2u\partial_x^{-1}u^* \\
  \partial_x + 2u\partial_x^{-1}u^* & -2u\partial_x^{-1}u^* 
\end{pmatrix}
\]

are Hamiltonian operators of Hirota equation

\[
iu_x + a(u_{xx} + 6|u|^2u_x) + \frac{u_{xx}}{2} + |u|^2u = 0
\]

and

\[
H_2 = u^*u_x + 2ia(u^2u_x - u_xu^*_2), \quad H_3 = (u^2u_x^2 - u_xu^*_2) + 2ia(3uu^2u_x - u_xu_x^*_2),
\]

are conserved densities of Hirota equation. Hence the deformed Hirota equation (1.2) is a bi-Hamiltonian system.
Recursion operator of deformed AKNS equation

Third member of generalized symmetries $K_3 = (U_2, V_2, G_2, H_2, B_2)^T$

Fourth member of generalized symmetries $K_4 = (U_3, V_3, G_3, H_3, B_3)^T$

First three members of Conserved quantities

Recursion operator of deformed AKNS equation

B. Deformed AKNS Equation (1.3)
First member of master symmetries \( \tau_0 = (\tau_0^1, \tau_0^2, \tau_0^3, \tau_0^4, \tau_0^5)^T \)

\[
\begin{align*}
\tau_0^1 &= (x u_x + u) \\
\tau_0^2 &= (x v_x + v) \\
\tau_0^3 &= x g_x + g \\
\tau_0^4 &= x h_x + h \\
\tau_0^5 &= x b_x + b
\end{align*}
\]

Second member of master symmetries \( \tau_1 = (\tau_1^1, \tau_1^2, \tau_1^3, \tau_1^4, \tau_1^5)^T \)

\[
\begin{align*}
\tau_1^1 &= \begin{pmatrix}
-x u_{xx} + 2uv_x + 2u \phi - 2u_x \\
x (v_{xx} - 2v^2 u) - 2v \phi + 2v_x \\
x (-g_{xx} + 4uv \tilde{g} + 2u^2 h) + 2g \phi - 2g_x + 2vb \\
x (h_{xx} - 4uvh - 2u^2 \tilde{g}) + 2\phi + 2h_x - 2vb \\
x (\tilde{g} v_x - hu_x) + \tilde{g} - uh
\end{pmatrix}
\end{align*}
\]

Third member of master symmetries \( \tau_2 = (\tau_2^1, \tau_2^2, \tau_2^3, \tau_2^4, \tau_2^5)^T \)

\[
\begin{align*}
\tau_2^1 &= \begin{pmatrix}
x (u_{xx} - 6uv u_x + 3u_{xx} - 4u \psi - 2u^2 v - 2u_x \phi) \\
x (v_{xx} - 6uv v_x + 3v_{xx} + 4u \psi - 6u^2 u - 2v_x \phi) \\
x (g_{xx} - 6uv \tilde{g} u_x - 6u h u_x - 6u \tilde{g} x_x) + 3g \tilde{g} x_x \\
x (h_{xx} - 6uvh - 6u h v_x - 6uv h_x) + 3h \tilde{g} x_x \\
x (\tilde{g} v_{xx} - 2u^2 v h - 2u^2 \tilde{g} + \tilde{g} v x_x) + 2u h_x + 2g \tilde{g} x - 2b \phi
\end{pmatrix}
\end{align*}
\]

etc.
where \( \phi = uv \) and \( \psi = vu_x \).

First member of nonlocal symmetries \( P_1 = (S_1, T_1, Q_1, R_1, A_1)^T \)

\[
\begin{align*}
S_1 &= 2(-u_{xx} + 2u^2 v) + xu_x + u \\
T_1 &= 2(-u_{xx} - 2u^2 u) + xv_x + v \\
Q_1 &= 2(-g_{xx} + 4uv \tilde{g} + 2u^2 h) + xg_x + \tilde{g} \\
R_1 &= 2(-h_{xx} - 4uvh - 2u^2 \tilde{g}) + xh_x + h \\
A_1 &= 2(-h_{xx} + 2uvh - 2h u_x) + xb_x + b
\end{align*}
\]
Second member of nonlocal symmetries \( P_2 = (S_2, T_2, Q_2, R_2, A_2) \)^T

\[
\begin{pmatrix}
2t(u_{3x} - 6u_{xx}) + x(-u_{xx} + 2u^2v) + 2u\phi - 2u_s \\
2t(v_{3x} - 6v_{xx}) + x(v_{xx} - 2uv^2) - 2v\phi + 2v_s \\
2t(\tilde{g}_{3x} - 6u\tilde{g}_{xx} - 6u_{xx}h - 6uw\tilde{g}) + x(-\tilde{g}_{xx} + 4uw) \\
2t(h_{3x} - 6v\tilde{g}_{xx} - 6uvv_{xx} - 6uvh_{x}) + x(h_{xx} - 4vch) \\
2t(h_{3x} - 6v\tilde{g}_{xx} - 6uvv_{xx} - 6uvh_{x}) + x(h_{xx} - 4vch) \\
\end{pmatrix},
\]

e etc.

where \( \phi \) is a nonlocal variable defined by

\[
\phi_x = uv, \quad \phi_t = b + uv_{xx} - vu_{xx}.
\]

Also we observe that the obtained sequence of nonlocal symmetries and master symmetries of deformed AKNS equation satisfy (6.1). Furthermore, it is observed that master symmetries \( \tau_i \), and nonlocal symmetries \( P_i \), and generalized symmetries \( K_i \), satisfy the following relation:

\[
P_{i+1} = 2tK_{i+1} + \tau_i, \quad \forall i.
\]

Bi-Hamiltonian representation:

The deformed AKNS equation can also be written in the Hamiltonian description

\[
\begin{pmatrix}
u_t \\
v_x \\
end{pmatrix} = \theta_1 \begin{pmatrix}
\frac{\delta H_1}{\delta u} \\
\frac{\delta H_1}{\delta v} \\
\end{pmatrix} + \theta_2 \begin{pmatrix}
-h \\
-g \\
\end{pmatrix} = \theta_2 \begin{pmatrix}
\frac{\delta H_2}{\delta u} \\
\frac{\delta H_2}{\delta v} \\
\end{pmatrix} + \theta_1 \begin{pmatrix}
-h \\
-g \\
\end{pmatrix}
\]

\[
\theta_2 \begin{pmatrix}
-h \\
-g \\
\end{pmatrix} = 0.
\]

Here

\[
\theta_1 = \begin{pmatrix}
0 & 1 \\
-1 & 0 \\
\end{pmatrix}, \quad \theta_2 = \begin{pmatrix}
-2u\partial_x^{-1}u & -u_x + 2u\partial_x^{-1}v \\
-u_x + 2v\partial_x^{-1}u & -2v\partial_x^{-1}v \\
\end{pmatrix}
\]

are Hamiltonian operator of AKNS equation

\[
\begin{align*}
u_t + u_{xx} - 2u^2v &= 0 \\
v_x - v_{xx} + 2v^2u &= 0
\end{align*}
\]

are conserved densities of deformed AKNS equation. Hence the deformed AKNS equation (1.3) is a bi-Hamiltonian system.
Generalized, Master and Nonlocal Symmetries of Certain Deformed PDEs

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