FEDOSOV SUPERMANIFOLDS: BASIC PROPERTIES
AND THE DIFFERENCE IN EVEN AND ODD CASES

B. GEYER $^{a)1}$, and P.M. LAVROV $^{a),b)2}$

$^a)$ Center of Theoretical Studies, Leipzig University,
Augustusplatz 10/11, D-04109 Leipzig, Germany

$^b)$ Tomsk State Pedagogical University, 634041 Tomsk, Russia

We study basic properties of supermanifolds endowed with an even (odd) symplectic structure and a connection respecting this symplectic structure. Such supermanifolds can be considered as generalization of Fedosov manifolds to the supersymmetric case. Choosing an appropriate definition of inverse (second-rank) tensor fields on supermanifolds we consider the symmetry behavior of tensor fields as well as the properties of the symplectic curvature and of the Ricci tensor on even (odd) Fedosov supermanifolds. We show that for odd Fedosov supermanifolds the scalar curvature, in general, is non-trivial while for even Fedosov supermanifolds it necessarily vanishes.

1. INTRODUCTION

The methods of modern differential geometry have become an universal tool of theoretical physics starting from the recognition of their crucial role in general relativity. In general relativity the basic objects are Riemannian manifolds equipped with a symmetric connection respecting a metric tensor field. The formulation of classical mechanics involves symplectic manifolds, i.e., manifolds endowed with a non-degenerate closed 2-form used for the construction of the Poisson bracket. It has also been realized that the so-called deformation quantization [1] can be formulated in terms of symplectic manifolds equipped with a symmetric connection respecting the symplectic structure (Fedosov manifolds) [2].

The discovery of supersymmetric particle theories as well as of supergravity has introduced into modern quantum field theory a number of applications of differential geometry which are based on the notion of supermanifolds, first proposed and analyzed by Berezin [3] (see also [4]). In these cases, one has to equip the supermanifolds with a suitable connection. The general consideration of the well-known Batalin-Vilkovisky quantization method [5] involves an odd symplectic supermanifold, i.e., a supermanifold endowed with an odd symplectic structure [6]. In some specific considerations of modern gauge field theory (see, e.g., [7, 8]) one introduced also even symplectic supermanifolds equipped with a flat connection respecting a given symplectic structure.

The goal of the present paper is to study the symmetry properties of tensor fields on supermanifolds, as well as the properties of the curvature tensor for arbitrary even (odd) symplectic supermanifolds endowed with a symmetric connection respecting a given symplectic structure.

The paper is organised as follows. In Sect. 2, we remind the definition of tensor fields on supermanifolds, as well as the properties of the curvature tensor for arbitrary even (odd) symplectic supermanifolds endowed with a symmetric connection respecting a given symplectic structure.

In Sect. 3, we consider affine connections on a supermanifold and their curvature tensors. In Sect. 4, we discuss relations between even (odd) symplectic supermanifolds and even (odd) Poisson supermanifolds. In Sect. 5, we present the notion of even (odd) Fedosov supermanifolds and of even (odd) symplectic curvature tensors. In Sect. 6, we study the Ricci tensor constructed from the symplectic curvature and the scalar curvature which is non-trivial for odd Fedosov supermanifolds. In Sect. 7, we give a short summary.

1E-mail: geyer@itp.uni-leipzig.de
2E-mail: lavrov@tspu.edu.ru; lavrov@itp.uni-leipzig.de
We use the condensed notation suggested by DeWitt. Derivatives with respect to the coordinates \( x^i \) are understood as acting from the left and for them the notation \( \partial_i A = \partial A / \partial x^i \) is used. Right derivatives with respect to \( x^i \) are labelled by the subscript "r" or the notation \( A_{,i} = \partial_r A / \partial x^i \) is used. The Grassmann parity of any quantity \( A \) is denoted by \( \epsilon(A) \).

## 2. Tensor Fields on Supermanifolds

In this Chapter we review explicitly some basic definitions and simple relations of tensor analysis on supermanifolds which are useful in order to avoid elementary pitfalls in the course of the computations. Thereby, we adopt the conventions of DeWitt [4].

Let the variables \( x^i, \epsilon(x^i) = \epsilon_i \) be local coordinates of a supermanifold \( M, \dim M = N \), in the vicinity of a point \( P \). Let the sets \( \{\epsilon_i\} \) and \( \{\epsilon^i\} \) be coordinate bases in the tangent space \( T_P M \) and the cotangent space \( T_P^* M \), respectively. If one goes over to another set \( \bar{x}^i = \bar{x}^i(x) \) of local coordinates the basis vectors in \( T_P M \) and \( T_P^* M \) transform as follows:

\[
\bar{e}_i = e_j \frac{\partial x^j}{\partial \bar{x}^i}, \quad \bar{e}^i = e^j \frac{\partial \bar{x}^i}{\partial x^j}.
\]

For the transformation matrices the following relations hold:

\[
\frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^j} = \delta^i_j, \quad \frac{\partial x^k}{\partial \bar{x}^j} \frac{\partial \bar{x}^i}{\partial x^k} = \delta^i_j, \quad \frac{\partial x^k}{\partial \bar{x}^j} \frac{\partial \bar{x}^i}{\partial x^k} = \delta^i_j.
\]

A tensor field of type \((n, m)\) with rank \( n + m \) is defined as a geometric object which, in each local coordinate system \((x) = (x^1, ..., x^N)\), is given by a set of functions with \( n \) upper and \( m \) lower indices obeying definite transformation rules. For example, let that set of functions be given either as \( T^{i_1 ... i_n}_{j_1 ... j_m}(x) \) or as \( T_{j_1 ... j_m}^{i_1 ... i_n}(x) \) with Grassmann parity \( \epsilon(T^{i_1 ... i_n}_{j_1 ... j_m}) = \epsilon(T_{j_1 ... j_m}^{i_1 ... i_n}) = \epsilon(T) + \epsilon_{i_1} + \cdots + \epsilon_{i_n} + \epsilon_{j_1} + \cdots + \epsilon_{j_m} \). Then the transformation rules under a change of coordinates, \((x) \rightarrow (\bar{x})\), are given as follows (assuming the usual convention on the ordering of transformation matrices):

\[
\bar{T}^{i_1 ... i_n}_{j_1 ... j_m} = T^{k_1 ... k_m}_{l_1 ... l_n} \frac{\partial x^{k_1}}{\partial \bar{x}^{i_1}} ... \frac{\partial x^{k_m}}{\partial \bar{x}^{i_n}} \frac{\partial \bar{x}^{j_1}}{\partial x^{l_1}} ... \frac{\partial \bar{x}^{j_m}}{\partial x^{l_n}} \times (-1)^{\sum_{s \in 1}^n \sum_{p \in +1} \epsilon_{i_p}^{s} (\epsilon_{j_s} + \epsilon_{l_s}) + \sum_{s \in 1}^n \sum_{p \in +1} \epsilon_{i_s}^{p} (\epsilon_{j_p} + \epsilon_{l_p}) + \sum_{s \in 1}^n \sum_{p \in +1} \epsilon_{j_s} (\epsilon_{k_p} + \epsilon_{l_p}) + \sum_{s \in 1}^n \sum_{p \in +1} \epsilon_{l_s} (\epsilon_{k_p} + \epsilon_{j_p})}
\]

or

\[
\bar{T}_{j_1 ... j_m}^{i_1 ... i_n} = T_{k_1 ... k_m}^{l_1 ... l_n} \frac{\partial x^{i_1}}{\partial x^{l_1}} ... \frac{\partial x^{i_n}}{\partial x^{l_n}} \frac{\partial \bar{x}^{j_1}}{\partial \bar{x}^{k_1}} ... \frac{\partial \bar{x}^{j_m}}{\partial \bar{x}^{k_m}} \times (-1)^{\sum_{s \in 1}^n \sum_{p \in +1} \epsilon_{i_p}^{s} (\epsilon_{i_s} + \epsilon_{l_s}) + \sum_{s \in 1}^n \sum_{p \in +1} \epsilon_{i_s} (\epsilon_{i_p} + \epsilon_{l_p}) + \sum_{s \in 1}^n \sum_{p \in +1} \epsilon_{j_s} (\epsilon_{k_p} + \epsilon_{l_p}) + \sum_{s \in 1}^n \sum_{p \in +1} \epsilon_{l_s} (\epsilon_{k_p} + \epsilon_{j_p})}
\]

For vector fields, \( T^i \), and covector fields, \( T_i \), the transformation rule is obvious:

\[
\bar{T}^i = T^n \frac{\partial \bar{x}^i}{\partial x^n}, \quad \bar{T}_i = T^n \frac{\partial x^n}{\partial \bar{x}^i}.
\]

For second-rank tensor fields of different type, \([3]\) and \([4]\) imply the transformation rules

\[
\bar{T}^{ij} = T^{mn} \frac{\partial \bar{x}^j}{\partial x^m} \frac{\partial \bar{x}^i}{\partial x^n} (-1)^{\epsilon_j \epsilon_i + \epsilon_m \epsilon_n}, \quad \bar{T}_{ij} = T^{mn} \frac{\partial x^n}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^i} (-1)^{\epsilon_j \epsilon_i + \epsilon_m \epsilon_n},
\]

\[
\bar{T}^{i} = T^{mn} \frac{\partial \bar{x}^i}{\partial x^n} (-1)^{\epsilon_i \epsilon + \epsilon_m \epsilon_n}, \quad \bar{T}_i = T_m^n \frac{\partial x^n}{\partial \bar{x}^i} (-1)^{\epsilon_i \epsilon + \epsilon_m \epsilon_n}.
\]
Note that the unit matrix $\delta^i_j$ is connected with unit tensor fields $\delta^i_j$ and $\delta_j^i$, transforming according to (5) and (9), as follows

$$\delta^i_j = \delta^j_i = (-1)^{\epsilon_i} \delta^i_j = (-1)^{\epsilon_j} \delta_j^i. \tag{10}$$

From a tensor field of type $(n, m)$ with rank $n + m$, where $n \neq 0$, $m \neq 0$, one can construct a tensor field of type $(n - 1, m - 1)$ with rank $n + m - 2$ by the contraction of an upper and a lower index by the rules

$$T^{i_1\ldots i_{n-1}}_{i n} j_1\ldots j_{m-1} j_m = (-1)^{\epsilon_i} (\epsilon_{i_1} + \epsilon_{i_n} + \epsilon_{j_1} + \ldots + \epsilon_{j_{m-1}} + 1), \tag{11}$$

$$T_{j_1\ldots j_{n-1} i j_n}^{i_1\ldots i_{m-1} i m} = (-1)^{\epsilon_i} (\epsilon_{i_1} + \epsilon_{i_n} + \epsilon_{j_1} + \ldots + \epsilon_{j_{n-1}} + 1). \tag{12}$$

In particular, for the tensor fields of type $(1, 1)$ the contraction leads to the supertraces,

$$T^i_j (-1)^{\epsilon_i} \quad \text{and} \quad T_i^j. \tag{13}$$

Having in mind the index shifting rules of DeWitt [4],

$$T^i = i T (-1)^{\epsilon(T)i} \quad \text{and} \quad T_i = i T (-1)^{\epsilon(T)i}, \tag{14}$$

and similarly for tensors of higher rank, let us construct from two tensor fields $U^{i_1\ldots i_{n+k}}_{j_1\ldots j_m}$ and $V^{j_1\ldots j_m}_{i_1\ldots i_{n+k}}$ new tensor fields by

$$(-1)^{\epsilon(V)} (\epsilon_{i_1} + \ldots + \epsilon_{i_n} + \epsilon_k) U^{i_1\ldots i_{n+k}}_{j_1\ldots j_m} V^{j_1\ldots j_m}_{k_{j_1}\ldots k_{j_m}} \equiv T^{i_1\ldots i_{n+k}}_{j_1\ldots j_m}, \tag{15}$$

$$(-1)^{\epsilon(U)} (\epsilon_{j_1} + \ldots + \epsilon_{j_m} + \epsilon_k) V^{j_1\ldots j_m}_{i_1\ldots i_{n+k}} U^{i_1\ldots i_{n+k}}_{k_{j_1}\ldots k_{j_m}} \equiv \hat{T}^{i_1\ldots i_{n+k}}_{j_1\ldots j_m}, \tag{16}$$

which really transform according to (6) and (10), respectively. For example, from vector and covector fields, $U^i$ and $V_i$, one gets a scalar field, i.e., a tensor field of rank zero,

$$(-1)^{\epsilon(V) + 1} U^i V_i = (-1)^{\epsilon(U) + \epsilon(V)} V_i U^i, \tag{17}$$

being an invariant, and from two second rank tensor fields, $U^{ij}$ and $V_{ij}$, we obtain

$$(-1)^{(\epsilon_i + \epsilon_k) \epsilon(V) + \epsilon_k} U^{ij} V_{ij} \quad \text{and} \quad (-1)^{(\epsilon_i + \epsilon_k) \epsilon(U)} V_{ij} U^{ij}, \tag{18}$$

transforming according to (6) and (10), respectively, but after a further contraction, on gets the scalar

$$(-1)^{(\epsilon_i + \epsilon_k) \epsilon(V) + 1} U^{ij} V_{kj} = (-1)^{\epsilon(U) \epsilon(V) + (\epsilon_i + \epsilon_k) \epsilon(U)} V_{kj} U^{kj}. \tag{19}$$

Furthermore, taking into account (15) and (16), the unique inverse of a (non-degenerate) second rank tensor field of type $(2,0)$ will be defined as follows:

$$(-1)^{(\epsilon_i + \epsilon_k) \epsilon(T) + \epsilon_k} T^{ik}_{kj} (T^{-1})_{kj} = \delta^i_j, \tag{20}$$

$$(-1)^{(\epsilon_j + \epsilon_k) \epsilon(T)} (T^{-1})_{jk} T^{kj} = \delta^i_j, \tag{21}$$

and correspondingly for tensor fields of type $(0,2)$.

Let us remark that the inclusion of the correct sign factors into the definitions of contractions, (11) and (12), and of the inverse tensors, (20) and (21), is essential. Namely, let us consider a second rank tensor field of type $(2,0)$ obeying the property of generalized (anti)symmetry,

$$T^{ij}_{\pm} = \pm (-1)^{\epsilon_i \epsilon_j} T^{ij}_{\pm}. \tag{22}$$
Obviously, that property is in agreement with the transformation law (9),
\[
T^i_{\pm} = T^i_{\pm} \frac{\partial \bar{x}^j}{\partial x^n} \frac{\partial x^n}{\partial x^m} (-1)^{\epsilon_j (\epsilon_i + \epsilon_m)} = \pm T^i_{\pm} \frac{\partial \bar{x}^j}{\partial x^n} \frac{\partial x^n}{\partial x^m} (-1)^{\epsilon_i} = \pm (-1)^{\epsilon_i} T^i_{\pm}.
\]

Thus, the notion of generalized (anti)symmetry of a tensor field of type (2,0) is invariantly defined in any coordinate system.

Now, suppose that \( T^i_{\pm} \) is non-degenerate, thus allowing for the introduction of the corresponding inverse tensor fields of type (0,2) according to (20) and (21). From (22) one gets
\[
(T_{\pm}^{-1})^i_j = \pm (-1)^{\epsilon_i} T_{\pm}^{i} \epsilon(T_{\pm}^{-1})_{ji},
\]
and, as it should be, also this generalized (anti)symmetry is invariantly defined.

However, if the inverse tensor field had been defined naively according to
\[
T^i_{\pm} (-T_{\pm}^{-1})_{kj} = \delta^i_j \quad (T_{\pm}^{-1})_{ik} T^k_{\pm} = \delta^i_j,
\]
then, instead of Eq. (23) one obtains
\[
(-T_{\pm}^{-1})_{ij} = \mp (1)^{\epsilon_i} (\epsilon_j + 1) \epsilon(T_{\pm}^{-1})_{ij}, \quad \epsilon((T_{\pm}^{-1})_{ij}) = \epsilon(T_{\pm}) + \epsilon_i + \epsilon_j.
\]
However, that symmetry property of \((-T_{\pm}^{-1})_{ij}\) has no invariant meaning. Namely, the transformed field \((-T^{-1}_{\pm})_{ij}\), being determined according to the rule (7), does not have the symmetry (25). The reason is, that (24) does not introduce \((-T_{\pm}^{-1})_{ij}\) as a tensor field on a supermanifold.

This observations have to be taken into account later on when considering Poisson and symplectic supermanifolds. Furthermore, among supermatrices with all possible generalized symmetry properties (see, e.g., [9]) only supermatrices with the properties (22) have an invariant meaning. Therefore, differential geometry on supermanifolds should be developed solely on the basis of tensor fields that possess the properties of the generalized (anti)symmetry.

### 3. AFFINE CONNECTIONS ON SUPERMANIFOLDS AND CURVATURE

As in the case of tensor analysis on manifolds, on a supermanifold \( M \) one can introduce the covariant derivation (or affine connection) as a mapping \( \nabla \) (with components \( \nabla_i, \epsilon(\nabla_i) = \epsilon_i \)) from the set of tensor fields on \( M \) to itself by the requirement that it should be a tensor operation acting from the right and adding one more lower index and, when it is possible locally to introduce Cartesian coordinates on \( M \), that it should reduce to the usual (right–)differentiation.

Let us first discuss the latter case, with \((x)\) being a Cartesian coordinate system and \((\bar{x})\) an arbitrary one. Let us consider a vector field \( T^i \). Then, in the system \((\bar{x})\), we have
\[
T^i \nabla_j = \bar{T}_{i,j},
\]
and in the coordinate system \((\bar{x})\), by virtue of (8) and (9), we obtain
\[
\bar{T}_{i,j} = (T^m_{,n}) \frac{\partial r x^n}{\partial \bar{x}^i} \frac{\partial \bar{x}^i}{\partial x^m} (-1)^{\epsilon_j (\epsilon_i + \epsilon_m)} = \bar{T}_{i,j} + \bar{T}^k \Gamma^i_{kj} (-1)^{\epsilon_k (\epsilon_i + 1)};
\]

here, \( \Gamma^i_{jk} \) are the affine connection components (or Christoffel symbols),
\[
\Gamma^i_{jk} = \frac{\partial r x^k}{\partial x^m} \frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^i}.
\]

By definition they possess the property of generalized symmetry w.r.t. the lower indices,
\[
\Gamma^i_{jk} = (-1)^{\epsilon_j \epsilon_k} \Gamma^i_{kj}.
\]

Similarly, the action of the covariant derivative on covector fields \( T_i \) is given by
\[
\bar{T}i \nabla_j = \bar{T}_{i,j} + \bar{T}^k \bar{\Gamma}^i_{kj} \quad \text{with} \quad \bar{\Gamma}^i_{kj} = (-1)^{\epsilon_j (\epsilon_i + 1)} \frac{\partial^2 x^k}{\partial x^l \partial x^m} \frac{\partial x^m}{\partial \bar{x}^j} \frac{\partial x^l}{\partial \bar{x}^i}.
\]
Using the first of relations (2), one establishes straightforwardly that
\[ \bar{\Gamma}^{k}_{ij} = -\Gamma^{k}_{ij}, \]
and therefore
\[ T_{i} \nabla_{j} = T_{i,j} - T_{k} \Gamma^{k}_{ij}. \]

In general, on arbitrary supermanifolds \( M \), the Christoffel symbols are not necessarily given by partial derivatives with respect to the coordinates. However, also \( M \), for arbitrary supermanifolds the covariant derivative \( \nabla \) (or connection \( \Gamma \)) is defined through the (right–) differentiation and the separate contraction of upper and lower indices with the connection components analogous to the case of vector and co-vector fields. More explicitly, they are given as local operations acting on scalar, vector and co-vector fields by the rules
\[
T \nabla_{i} = T_{i}, \tag{28}
T^{i} \nabla_{j} = T^{i,j} + T^{k} \Gamma^{i}_{kj} (1)^{\epsilon_{j}(\epsilon_{i}+1)}, \tag{29}
T_{i} \nabla_{j} = T_{i,j} - T^{k} \Gamma^{k}_{ij}, \tag{30}
\]
and on second-rank tensor fields of type \((2,0), (0,2)\) and \((1,1)\) by the rules
\[
T^{ij} \nabla_{k} = T^{ij}_{k} + T^{il} \Gamma_{lk} (1)^{\epsilon_{l}(\epsilon_{i}+1)} + T^{kl} \Gamma_{ij} (1)^{\epsilon_{l}(\epsilon_{i}+\epsilon_{j}+1)}, \tag{31}
T_{ij} \nabla_{k} = T_{ij,k} - T_{kl} \Gamma_{jk} - T_{lj} \Gamma_{ik} (1)^{\epsilon_{l}(\epsilon_{i}+\epsilon_{j})}, \tag{32}
T^{i}_{j} \nabla_{k} = T^{i}_{jk} - T^{l}_{ik} \Gamma^{l}_{jk} + T^{l}_{ij} \Gamma^{l}_{ik} (1)^{\epsilon_{l}(\epsilon_{i}+\epsilon_{j}+1)}. \tag{33}
\]

Similarly, the action of the covariant derivative on a tensor field of any rank and type is given in terms of their tensor components, their ordinary derivatives and the connection components.

The affine connection components do not transform as mixed tensor fields, instead they obtain an additional inhomogeneous term:
\[ \bar{\Gamma}^{i}_{jk} = (1)^{\epsilon_{i}(\epsilon_{m}+\epsilon_{j})} \frac{\partial \tilde{x}^{i}}{\partial x^{m}} \Gamma^{i}_{lk} \frac{\partial x^{m}}{\partial x^{k}} + \frac{\partial \tilde{x}^{i}}{\partial x^{m}} \frac{\partial^{2} x^{m}}{\partial x^{j} \partial \tilde{x}^{k}}. \tag{34} \]

In general, the connection components \( \Gamma^{i}_{jk} \) do not have the property of (generalized) symmetry w.r.t. the lower indices. The deviation from this symmetry is the torsion,
\[ T^{i}_{jk} := \Gamma^{i}_{jk} - (1)^{\epsilon_{j}(\epsilon_{k}+\epsilon_{i})} \Gamma^{i}_{kj}, \tag{35} \]
which transforms as a tensor field. If the supermanifold \( M \) is torsionless, i.e., if the Christoffel symbols obey the relation (27), then one says that a symmetric connection is defined on \( M \). Here, with the aim of studying Fedosov supermanifolds, we consider only symmetric connections.

The Riemannian tensor field \( R^{i}_{mj,k} \), according to Ref. [4], is defined in a coordinate basis by the action of the commutator of covariant derivatives, \([\nabla_{i}, \nabla_{j}] = \nabla_{i} \nabla_{j} - (1)^{\epsilon_{i}(\epsilon_{j})} \nabla_{j} \nabla_{i}, \) on a vector field \( T^{i} \) as follows:
\[ T^{i} [\nabla_{j}, \nabla_{k}] = (1)^{\epsilon_{m}(\epsilon_{i}+1)} T^{m} R^{i}_{mj,k}. \tag{36} \]
A straightforward calculation yields
\[ R^{i}_{mj,k} = -\Gamma^{i}_{mj,k} + \Gamma^{i}_{mk,j} (1)^{\epsilon_{j}(\epsilon_{k})} + \Gamma^{i}_{jn} \Gamma^{n}_{mk} (1)^{\epsilon_{j}(\epsilon_{m})} - \Gamma^{i}_{kn} \Gamma^{n}_{mj} (1)^{\epsilon_{k}(\epsilon_{m}+\epsilon_{j})}. \tag{37} \]
The Riemannian tensor field possesses the following generalized antisymmetry property,
\[ R^{i}_{mj,k} = - (1)^{\epsilon_{j}(\epsilon_{k})} R^{i}_{mk,j}; \tag{38} \]

furthermore, it obeys the (super) Jacobi identity,
\[ (1)^{\epsilon_{m}(\epsilon_{k})} R^{i}_{mj,k} + (1)^{\epsilon_{j}(\epsilon_{m})} R^{i}_{jm,k} + (1)^{\epsilon_{k}(\epsilon_{j})} R^{i}_{km,j} \equiv 0. \tag{39} \]
Using the (super) Jacobi identity for the covariant derivatives,
\[
[\nabla_i, [\nabla_j, \nabla_k]](-1)^{\varepsilon_i \varepsilon_k} + [\nabla_k, [\nabla_i, \nabla_j]](-1)^{\varepsilon_k \varepsilon_j} + [\nabla_j, [\nabla_k, \nabla_i]](-1)^{\varepsilon_j \varepsilon_i} \equiv 0, \tag{40}
\]
one obtains the (super) Bianchi identity,
\[
(-1)^{\varepsilon_i \varepsilon_j} R^m_{mjk;i} + (-1)^{\varepsilon_k \varepsilon_j} R^m_{mij,k} + (-1)^{\varepsilon_j \varepsilon_k} R^m_{mk;i,j} \equiv 0, \tag{41}
\]
with the notation \( R^m_{mjk;i} := R^m_{mjk} \nabla_i \).

4. SYMPLECTIC AND POISSON SUPERMANIFOLDS

Suppose now that we are given a supermanifold \( M \) of an even dimension, \( \dim M = 2n \). Let \( \omega \) be an even, \( \epsilon(\omega) = 0 \), (resp. odd, \( \epsilon(\omega) = 1 \)) non-degenerate exterior 2-form on \( M \). Then, the pair \((M, \omega)\) is called an even (resp. odd) almost symplectic supermanifold; it is called an even (resp. odd) symplectic supermanifold if \( \omega \) is closed, \( d\omega = 0 \).

In an arbitrary coordinate basis on \( M \), \( \omega \) can be written as
\[
\omega = \omega_{ij} \, dx^j \wedge dx^i \tag{42}
\]
is also invariant under a change of local coordinates, \( \overline{d\omega} = d\bar{\omega} \). The requirement of closure, \( d\omega = 0 \), leads to the following identity for \( \omega_{ij} \):
\[
\omega_{ij,k}(-1)^{\varepsilon_i \varepsilon_k} + \text{cycle}(i, j, k) \equiv 0 \iff \partial_i \omega_{jk}(-1)^{\varepsilon_i \varepsilon_j} + \text{cycle}(i, j, k) \equiv 0. \tag{45}
\]

Suppose now that the tensor field \( \omega_{ij} \) is non-degenerate. Taking into account the relations [10] and the definitions of the inverse, Eqs. [20] and [21], the tensor field \( \omega^{ij} \), being the (unique) inverse of \( \omega_{ij} \), is given by
\[
\omega^{ik} \omega_{kj}(-1)^{\varepsilon_k + \epsilon(\omega)(\varepsilon_i + \varepsilon_k)} = \delta^i_j, \quad (-1)^{\varepsilon_i + \varepsilon(\omega)(\varepsilon_i + \varepsilon_k)} \omega_{ik} \omega^{kj} = \delta_i^j. \tag{46}
\]
The inverse tensor field \( \omega^{ij} \) has the following symmetry property:
\[
\omega^{ij} = -(-1)^{\varepsilon_i \varepsilon_j} + \epsilon(\omega) \omega^{ji}, \quad \epsilon(\omega^{ij}) = \epsilon(\omega) + \varepsilon_i + \varepsilon_j. \tag{47}
\]
In terms of the inverse tensor field \( \omega^{ij} \), the relations [45] can be rewritten as follows:
\[
\omega^{i} \partial_{n} \omega^{jk}(-1)^{\varepsilon_i + \epsilon(\omega)(\varepsilon_k + \varepsilon(\omega)) + \text{cycle}(i, j, k) \equiv 0. \tag{48}
\]

With the help of \( \omega^{ij} = \omega^{ij}(\bar{x}) \) one can introduce the even (resp. odd) Poisson bracket,
\[
(A, B) = \frac{\partial_x A}{\partial x^i} (-1)^{\varepsilon_i \varepsilon_x} \omega^{ij} \frac{\partial B}{\partial x^j}, \quad \epsilon((A, B)) = \epsilon(A) + \epsilon(B) + \epsilon(\omega), \tag{49}
\]
which is invariant under transformations \( (x) \to (\bar{x}) \) of the coordinates, \( (\tilde{A}, \tilde{B}) = (A, B) \). One easily verifies that all the properties which are required for a bilinear form to be a Poisson bracket are fulfilled. In particular, the relation [45] is just the (super) Jacobi identity for the Poisson bracket.

Supermanifolds equipped with a tensor field obeying the properties [47] and [48] are called even (odd) Poisson supermanifolds. From the above considerations it follows that, as in the case of ordinary differential geometry, there exists an one-to-one correspondence between even (odd) non-degenerate Poisson supermanifolds and even (odd) symplectic supermanifolds.
5. FEDOSOV SUPERMANIFOLDS

Suppose now we are given an even (odd) symplectic supermanifold, \((M, \omega)\). Let \(\nabla\) (or \(\Gamma\)) be a covariant derivative (connection) on \(M\) which preserves the 2-form \(\omega\), \(\omega \nabla = 0\). In a coordinate basis this requirement reads
\[
\omega_{ij,k} - \omega_{im} \Gamma^m_{jk} + \omega_{jm} \Gamma^m_{ik} (-1)^{i+e_j} = 0. \tag{50}
\]
If, in addition, \(\Gamma\) is symmetric then we have an even (odd) symplectic connection (or symplectic covariant derivative) on \(M\). Now, an even (odd) Fedosov supermanifold \((M, \omega, \Gamma)\) is defined as an even (odd) symplectic supermanifold with a given even (odd) symplectic connection.

Let us introduce the curvature tensor of an even (odd) symplectic connection,
\[
R_{ijkl} = \omega_{lm} R^n_{m,jkl}, \quad \epsilon(R_{ijkl}) = \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l, \tag{51}
\]
where \(R^n_{m,jkl}\) is given by (37). This leads to the following representation,
\[
R_{imjk} = -\omega_{ln} \Gamma^n_{m,j,k} + \omega_{ln} \Gamma^n_{m,k,j} (-1)^{\epsilon_j \epsilon_k} + \Gamma_{ijm} \Gamma^n_{mk} (-1)^{\epsilon_j \epsilon_m} - \Gamma_{ikn} \Gamma^n_{mj} (-1)^{\epsilon_k (\epsilon_m + \epsilon_j)}, \tag{52}
\]
where we used the notation
\[
\Gamma_{ijk} = \omega_{ln} \Gamma^n_{jlk}, \quad \epsilon(\Gamma_{ijk}) = \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k . \tag{53}
\]
Using this, the relation (50) reads
\[
\omega_{ij,k} = \Gamma_{ijk} + (-1)^{\epsilon_i \epsilon_j} R_{ijlk} . \tag{54}
\]
Furthermore, from Eq. (37) it is obvious that
\[
R_{ijkl} = -(-1)^{\epsilon_k \epsilon_l} R_{ijlk}, \tag{55}
\]
and, using (51) and (39), one deduces the (super) Jacobi identity for \(R_{ijkl}\),
\[
(-1)^{\epsilon_i \epsilon_l} R_{ijlk} + (-1)^{\epsilon_i \epsilon_k} R_{iljk} + (-1)^{\epsilon_k \epsilon_l} R_{iklj} = 0 . \tag{56}
\]
In addition, the curvature tensor \(R_{ijkl}\) is (generalized) symmetric w.r.t. the first two indices,
\[
R_{ijkl} = (-1)^{\epsilon_i \epsilon_j} R_{jilk}. \tag{57}
\]
In order to prove this, let us consider
\[
\omega_{ij,kl} = \Gamma_{ijk,l} - \Gamma_{jik,l} (-1)^{\epsilon_i \epsilon_j} . \tag{58}
\]
Then, using the relations
\[
\Gamma_{ijk,l} = \omega_{lm} \Gamma^n_{jkl} + \omega_{lm} \Gamma^n_{jk,l} (-1)^{\epsilon_n + \epsilon_j + \epsilon_k} \epsilon_l \tag{59}
\]
and the definitions (52) and (53), we get
\[
0 = \omega_{ij,kl} - (-1)^{\epsilon_k \epsilon_l} \omega_{ij,lk} = \Gamma_{ijk,l} - \Gamma_{jik,l} (-1)^{\epsilon_i \epsilon_j} - \Gamma_{ijl,k} (-1)^{\epsilon_k \epsilon_l} + \Gamma_{jil,k} (-1)^{\epsilon_i + \epsilon_j + \epsilon_k} \epsilon_l = -R_{ijkl} + (-1)^{\epsilon_i \epsilon_j} R_{jikl}. \tag{60}
\]
For any even (odd) symplectic connection there holds the identity
\[
(-1)^{\epsilon_i \epsilon_l} R_{ijlk} + (-1)^{\epsilon_i \epsilon_k} R_{iljk} + (-1)^{\epsilon_k \epsilon_l} R_{iklj} + (-1)^{\epsilon_i \epsilon_j + \epsilon_k \epsilon_l} R_{klji} + (-1)^{\epsilon_i \epsilon_j + \epsilon_k \epsilon_l} R_{jkl} = 0. \tag{61}
\]
This is proved by using the Jacobi identity (56) together with a cyclic change of the indices:
\[
(-1)^{\epsilon_i \epsilon_l} R_{ijlk} + (-1)^{\epsilon_i \epsilon_k} R_{iljk} + (-1)^{\epsilon_k \epsilon_l} R_{iklj} = 0, \tag{62}
\]
\[
(-1)^{\epsilon_i \epsilon_k} R_{ljik} + (-1)^{\epsilon_{j} \epsilon_{k}} R_{klij} + (-1)^{\epsilon_{j} \epsilon_{i}} R_{iklj} = 0, \tag{63}
\]
\[
(-1)^{\epsilon_{i} \epsilon_{l}} R_{klij} + (-1)^{\epsilon_{i} \epsilon_{j}} R_{kjli} + (-1)^{\epsilon_{i} \epsilon_{k}} R_{kijl} = 0, \tag{64}
\]
\[
(-1)^{\epsilon_{i} \epsilon_{k}} R_{jkl} + (-1)^{\epsilon_{i} \epsilon_{l}} R_{jikl} + (-1)^{\epsilon_{i} \epsilon_{j}} R_{jkl} = 0. \tag{65}
\]
Now, multiplying Eq. (62) by the factor \((-1)^{\epsilon_i\epsilon_j}\) and Eq. (64) by the factor \((-1)^{\epsilon_i\epsilon_k+\epsilon_j\epsilon_l}\) and summing the obtained results, one gets the identity (61). The same is obtained by multiplying Eq. (63) by the factor \((-1)^{\epsilon_i\epsilon_k+\epsilon_j\epsilon_l}\) and Eq. (65) by the factor \((-1)^{\epsilon_i\epsilon_j+\epsilon_k\epsilon_l}\) and then summing the results. Moreover, any other combination of Eqs. (62)–(65) containing four components of the symplectic curvature with a cyclic permutation of all the indices are reduced to the identities (61). In the case of ordinary Fedosov manifolds, i.e., when all the variables \(x^i\) are even \((\epsilon_i = 0)\), Eq. (61) obtains the symmetric form [2],

\[
R_{ijkl} + R_{ijlk} + R_{klij} + R_{jkli} = 0.
\] (66)

In the identity (61) the components of the symplectic curvature tensor occur with cyclic permutations of all the indices. However, the pre-factors depending on the Grassmann parities of indices are not obtained by cyclic permutation. One may consider this as an unexpected result, but the Jacobi identity [50] obeys this property only w.r.t. the last three indices and, in addition, also the (anti) symmetry properties [55] and [57] do not fulfill this requirement.

6. RICCI TENSOR

Having the curvature tensor, \(R_{ijkl}\), and the tensor field \(\omega^{ij}\), with allowance made for the symmetry properties of these tensors, [47], [55] and [57], one can define the following three different tensor fields of type \((0,2)\),

\[
R_{ij} = \omega^{kn}R_{mklj}(-1)^{(\epsilon(\omega)+1)(\epsilon_k+\epsilon_n)} = R^k_{ij}(-1)^{\epsilon_k},
\] (67)
\[
K_{ij} = \omega^{kn}R_{mnikj}(-1)^{(\epsilon_i\epsilon_j+\epsilon(\omega)+1)(\epsilon_k+\epsilon_n)} = R^k_{ikj}(-1)^{\epsilon_k(\epsilon_i+1)},
\] (68)
\[
Q_{ij} = \omega^{kn}R_{ijmk}(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_n)+(\epsilon(\omega)+1)(\epsilon_k+\epsilon_n),
\] (69)
\[
\epsilon(R_{ij}) = \epsilon(K_{ij}) = \epsilon(Q_{ij}) = \epsilon_i + \epsilon_j,
\]

where, obviously, \(R_{ij}\) and \(K_{ij}\) do not depend on \(\omega\).

From the definitions (67), (69) and the symmetry properties of \(R_{ijkl}\), it follows immediately that for any symplectic connection one has

\[
R_{ij} = -(-1)^{\epsilon_i\epsilon_j}R_{ji},
\] (70)
\[
Q_{ij} = (-1)^{\epsilon_i\epsilon_j}Q_{ji}.
\] (71)

Therefore, on any even Fedosov supermanifold the tensor \(R_{ij}\) \((Q_{ij})\) equals to zero (is non-trivial), while on any odd Fedosov supermanifold this tensor is non-trivial (equals to zero). Indeed, using the symmetry properties of the tensor fields \(\omega^{ij}\) and \(R_{ijkl}\), one obtains the relations

\[
[1 + (-1)^{\epsilon(\omega)}]R_{ij} = 0,
\] (72)
\[
[1 - (-1)^{\epsilon(\omega)}]Q_{ij} = 0.
\] (73)

In addition, for the tensor fields [67] – [69] there exists a relation which follows from the identity (61). Indeed, multiplying the equation (61) by the factor \((-1)^{\epsilon_i\epsilon_j+(\epsilon(\omega)+1)(\epsilon_i+\epsilon_j)}\) and by the tensor \(\omega^{ij}\), with allowance made for summation over indices \(i, j\) and for the definitions [67] – [69], we obtain

\[
R_{ij} + Q_{ij} + (-1)^{\epsilon_i\epsilon_j}K_{ji} + (-1)^{\epsilon(\omega)}K_{ij} = 0.
\] (74)

A second independent relation can be derived from (61) by multiplying with the tensor \(\omega^{kl}\) and the factor \((-1)^{\epsilon_i\epsilon_j+\epsilon_k\epsilon_l(\epsilon(\omega)+1)(\epsilon_i+\epsilon_j)}\); after subsequent summation over the indices \(i, k\) this leads to the following result:

\[
[1 + (-1)^{\epsilon(\omega)}] (K_{ij} - (-1)^{\epsilon_i\epsilon_j}K_{ji}) = 0.
\] (75)

From the relations (72), (73) and (75) one concludes: For any even symplectic connection we obtain

\[
K_{ij} = -(-1)^{\epsilon_i\epsilon_j}K_{ji}, \quad R_{ij} = 0, \quad Q_{ij} = -2K_{ij}.
\] (76)
while for any odd symplectic connection we have

\[ Q_{ij} = 0, \quad R_{ij} = K_{ij} - (-1)^{\epsilon_i \epsilon_j} K_{ji}. \] (77)

Therefore, the tensor field \( K_{ij} \) should be considered as the only independent second-rank tensor which can be constructed from the symplectic curvature. We refer to \( K_{ij} \) as the Ricci tensor of an even (odd) Fedosov supermanifold.

Let us define the scalar curvature \( K \) by the formula

\[ K = \omega^{ji} K_{ij} (-1)^{\epsilon_i + \epsilon_j} = \omega^{ji} \omega^{kn} R_{nikj} (-1)^{\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_n}(\epsilon(\omega)+1). \] (78)

From the symmetry properties of \( R_{ijkl} \) and \( \omega^{ij} \), it follows that on any Fedosov supermanifold one has

\[ [1 + (-1)^{\epsilon(\omega)}] K = 0. \] (79)

Therefore, as is the case for ordinary Fedosov manifolds \([2]\), for any even symplectic connection the scalar curvature necessarily vanishes. But the situation becomes different for odd Fedosov supermanifolds where no restriction on the scalar curvature occurs. Therefore, in contrast to both the usual Fedosov manifolds and the even Fedosov supermanifolds, any odd Fedosov supermanifolds can be characterized by the scalar curvature as additional geometrical structure.

7. SUMMARY

We have considered some properties of tensor fields defined on supermanifolds \( M \). It was shown that only the generalized (anti)symmetry of tensor fields has an invariant meaning, and that differential geometry on supermanifolds should be constructed in terms of such tensor fields.

Any supermanifold \( M \) can be equipped with a symmetric connection \( \Gamma \) (covariant derivative \( \nabla \)). The Riemannian tensor \( R_{ijkl} \) corresponding to this symmetric connection \( \Gamma \) satisfies both the Jacobi identity and the Bianchi identity.

Any supermanifold \( M \) of an even dimension can be endowed with an even (odd) 2-form \( \omega \). If this 2-form is non-degenerate and closed, the pair \((M, \omega)\) defines an even (odd) symplectic supermanifold. In a coordinate basis on the supermanifold \( M \), the 2-form \( \omega \) is described by a second-rank tensor field \( \omega_{ij} \) obeying the property of generalized antisymmetry in both the even and odd cases. In its turn, the tensor field \( \omega^{ij} \), being inverse to \( \omega_{ij} \), obeys the property of generalized antisymmetry in the even case, while in the odd case it has the property of generalized symmetry. The tensor \( \omega^{ij} \) defines the even (odd) Poisson bracket on a supermanifold \( M \). The Jacobi identity for the even (odd) Poisson bracket follows from the closure of the 2-form \( \omega \). Supermanifolds equipped with an even (odd) non-degenerate Poisson structure \( \omega^{ij} \) are called even (odd) Poisson supermanifolds. Therefore, there exists an one-to-one correspondence between an even (odd) symplectic supermanifold and the corresponding even (odd) Poisson supermanifold.

Any even (odd) symplectic supermanifold can be equipped with a symmetric connection respecting the given symplectic structure. Such a symmetric connection is called a symplectic connection. The triplet \((M, \omega, \Gamma)\) is called an even (odd) Fedosov supermanifold. The curvature tensor \( R_{ijkl} \) of a symplectic connection obeys the property of generalized symmetry with respect to the first two indices, and the property of generalized antisymmetry with respect to the last two indices. The tensor \( R_{ijkl} \) satisfies the Jacobi identity and the specific (for the symplectic geometry) identity (see (61)) containing the sum of components of this tensor with a cyclic permutation of all the indices, which, however, does not (!) contain cyclic permuted factors depending on the Grassmann parities of the indices.

On any even (odd) Fedosov manifold, the Ricci tensor \( K_{ij} \) can be defined. In the even case, the Ricci tensor obeys the property of generalized symmetry and gives a trivial result for the scalar curvature. On the contrary, in the odd case the scalar curvature, in general, is nontrivial.

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