TOPS OF DYADIC GRIDS AND T1 THEOREMS

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Abstract. We extend the notion of a dyadic grid of cubes in \( \mathbb{R}^n \) to include infinite dyadic cubes. These 'tops' of a dyadic grid form a tiling of \( \mathbb{R}^n \) which is subject to the constraints similar to those arising in tiling Euclidean space by (finite) unit cubes. These tops arise in the theory of two weight norm inequalities through weighted Haar and Alpert wavelets.

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1. Introduction

Let \( \mu \) be a positive locally finite Borel measure on \( \mathbb{R}^n \), and let \( \mathcal{D} \) be a dyadic grid on \( \mathbb{R}^n \). As pioneered by Nazarov, Treil and Volberg in [NTV], [NTV4], etc., the weighted Haar decomposition associated with a function \( f \in L^2(\mu) \) and an integer \( N \in \mathbb{N} \) is given by

\[
\sum_{Q \in \mathcal{D}, \ell(Q) \leq 2^N} \Delta_Q^\mu f + \sum_{Q \in \mathcal{D}, \ell(Q) = 2^N} E_Q^\mu f,
\]

and converges to \( f \) both in \( L^2(\mu) \) and \( \mu \)-almost everywhere. See below for definitions of the Haar projections \( \Delta_Q^\mu \) and \( E_Q^\mu \). It is this form of Haar decomposition that is used to prove virtually all \( T_1 \) theorems in the literature. However, in order to estimate the bilinear form

\[
\langle T(f \sigma), g \rangle_\omega = \left\langle T \left( \sum_{I \in \mathcal{D}, \ell(I) < 2^N} \Delta_I^\sigma f \right), \sum_{J \in \mathcal{D}, \ell(J) < 2^N} \Delta_J^\omega g \right\rangle_\omega + \left\langle T \left( \sum_{I \in \mathcal{D}, \ell(I) = 2^N} E_I^\sigma f \right), \sum_{J \in \mathcal{D}, \ell(J) = 2^N} E_J^\omega g \right\rangle_\omega + \left\langle T \left( \sum_{I \in \mathcal{D}, \ell(I) = 2^N} E_I^\sigma f \right), \sum_{J \in \mathcal{D}, \ell(J) < 2^N} \Delta_J^\omega g \right\rangle_\omega + \left\langle T \left( \sum_{I \in \mathcal{D}, \ell(I) < 2^N} \Delta_I^\sigma f \right), \sum_{J \in \mathcal{D}, \ell(J) = 2^N} E_J^\omega g \right\rangle_\omega,
\]

associated with a truncation of a singular integral operator \( T \), one must begin by estimating the final three inner products on the right hand side above. Since the cubes \( I, J \in \mathcal{D} \) in the tiling with \( \ell(I) = \ell(J) = 2^N \), are infinite in number, one must exercise special care in summing up estimates over these cubes \( I \) and \( J \).

In this paper, we push the terms \( \sum_{I \in \mathcal{D}, \ell(I) = 2^N} E_I^\sigma f \) and \( \sum_{J \in \mathcal{D}, \ell(J) = 2^N} E_J^\omega g \) to "infinity", where the corresponding sums reduce to finite sums over 'tops' associated to the dyadic grid \( \mathcal{D} \), and where these tops form a tiling of \( \mathbb{R}^n \) by at most \( 2^n \) infinite cubes. Since the sums are now finite, no special care is needed in their estimation.

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1.1. Tops of dyadic grids. We say that a cube $Q$ in $\mathbb{R}^n$ is a PSA cube if its sides are parallel to the coordinate axes. Any union of an increasing sequence $\{Q_k\}_{k=1}^{\infty}$ of PSA cubes $Q_k$ is either a PSA cube in $\mathbb{R}^n$ or a set of infinite diameter, that we refer to as an infinite PSA cube. We say that a set $Q$ is a supercube if it is either a PSA cube or an infinite PSA cube.

**Definition 1.** A dyadic supergrid $\mathcal{D}$ in $\mathbb{R}^n$ is a collection of supercubes satisfying

1. The set of all cubes in $\mathcal{D}$ form a dyadic grid.
2. The supercubes in $\mathcal{D}$ are nested in the sense that given $Q, Q' \in \mathcal{D}$, then either $Q \cap Q' = \infty$, $Q \subset Q'$ or $Q' \subset Q$.

We shall now investigate the nature of the supercubes of infinite diameter in a dyadic supergrid. In particular we will show that they are uniquely determined by the underlying dyadic grid, and that they form a very special type of tiling of $\mathbb{R}^n$.

In order to motivate this investigation, we now recall the construction of weighted Alpert wavelets in [RaSaWi], and correct a small oversight there. Let $\mu$ be a locally finite positive Borel measure on $\mathbb{R}^n$, and fix $\kappa \in \mathbb{N}$. For each cube $Q$, denote by $L^2_{Q;\kappa}(\mu)$ the finite dimensional subspace of $L^2(\mu)$ that consists of linear combinations of the indicators of the children $\mathcal{C}(Q)$ of $Q$ multiplied by polynomials of degree less than $\kappa$, and such that the linear combinations have vanishing $\mu$-moments on the cube $Q$ up to order $\kappa - 1$:

$$L^2_{Q;\kappa}(\mu) = \left\{ f = \sum_{Q' \in \mathcal{C}(Q)} 1_{Q'} p_{Q',\kappa}(x) : \int_Q f(x) x^\beta d\mu(x) = 0, \quad \text{for } 0 \leq |\beta| < \kappa \right\},$$

where $p_{Q',\kappa}(x) = \sum_{\beta \in \mathbb{Z}^n : |\beta| \leq \kappa - 1} a_{Q',\beta} x^\beta$ is a polynomial in $\mathbb{R}^n$ of degree less than $\kappa$. Here $x^\beta = x_1^{\beta_1} x_2^{\beta_2} ... x_n^{\beta_n}$.

Let $d_{\kappa,n} = \dim L^2_{Q;\kappa}(\mu)$ be the dimension of the finite dimensional linear space $L^2_{Q;\kappa}(\mu)$.

Consider an arbitrary dyadic grid $\mathcal{D}$. For $Q \in \mathcal{D}$, let $\triangle_{Q;\kappa}$ denote orthogonal projection onto the finite dimensional subspace $L^2_{Q;\kappa}(\mu)$, and let $\mathcal{P}_{Q;\kappa}^{\mu}$ denote orthogonal projection onto the finite dimensional subspace

$$\mathcal{P}_{Q;\kappa}^{\mu} = \text{Span}\{1_Q x^\beta : 0 \leq |\beta| < \kappa\}.$$

To obtain a complete set of orthonormal projections, we use the following definition and lemma.

**Definition 2.** Define a $\mathcal{D}$-tower $\Gamma \subset \mathcal{D}$ to be an infinite sequence $\{I_m\}_{m=1}^{\infty}$ of nested dyadic cubes $I_m \subset I_{m+1}$ with side lengths $\ell(I_m) = 2^m$, and define top $\Gamma = \bigcup_{m=1}^{\infty} I_m$, which we refer to as the top of the tower $\Gamma$.

**Definition 3.** We define an equivalence relation $\sim$ on $\mathcal{D}$-towers $\Gamma_1$ and $\Gamma_2$ by $\Gamma_1 \sim \Gamma_2$ if $\Gamma_1 \cap \Gamma_2 = \emptyset$.

Note that the tops of two towers intersect if and only if the towers are equivalent, and so we can associate to each equivalence class the top of any representative.

**Lemma 4.** For every dyadic grid $\mathcal{D}$, there are at most $2^n$ equivalence classes of $\mathcal{D}$-towers $\{\Gamma_1, ..., \Gamma_T\}$, $1 \leq T \leq 2^n$. Moreover, $\mathbb{R}^n$ is the disjoint union of the tops associated with these equivalence classes.

**Proof.** Each top $\Gamma$ is a product of $n$ infinite intervals, top $\Gamma = \prod_{k=1}^{n} F_k$, where $F_k = (-\infty, a_k)_n$ or $(-\infty, \infty)_n$. Consider the set of $2^n$ sequences $\Omega_n = \{-\infty, a_k, 0, \infty\}_n$. We say that top $\Gamma$ is associated with a sequence $\theta = (\theta_1, ..., \theta_n) \in \Omega_n$ if $\theta_k$ is an endpoint of $F_k$. Since any two tops associated with the same $\theta$ have nonempty intersection, they must coincide. Thus the number of tops is at most the number of $\theta$’s in $\Omega_n$. Finally, every $x \in \mathbb{R}^n$ is in some unit cube and hence in some top. This completes the proof of the lemma.

**Remark 5.** These tops are precisely the supercubes of infinite diameter in the unique dyadic supergrid containing the dyadic grid $\mathcal{D}$.

For the standard dyadic grid, the sets top $\Gamma_t$ are precisely the quadrants in dimension 2, the octants in dimension 3, etc... Now define

$$\mathcal{F}^\kappa_{\top \Gamma_t}(\mu) = \left\{ \beta \in \mathbb{Z}^n_+ : |\beta| \leq \kappa - 1 : x^\beta \in L^2(1_{\top \Gamma_t}\mu) \right\},$$

and

$$\mathcal{P}^\kappa_{\top \Gamma_t}(\mu) = \text{Span}\{x^\beta : \beta \in \mathcal{F}^\kappa_{\top \Gamma_t}\}.$$
The following theorem was proved for the standard dyadic grid in [RaSaWi], which establishes the existence of Alpert wavelets, for $L^2(\mu)$ in all dimensions, having the three important properties of orthogonality, telescoping and moment vanishing.

**Theorem 6 (Weighted Alpert Bases).** Let $\mu$ be a locally finite positive Borel measure on $\mathbb{R}^n$, fix $\kappa \in \mathbb{N}$, and fix a dyadic grid $\mathcal{D}$ in $\mathbb{R}^n$.

1. Then $\left\{E_{\text{top } \Gamma; \kappa}^\mu \right\}_{t=1}^T \cup \left\{\Delta_{Q; \kappa}^\mu \right\}_{Q \in \mathcal{D}}$ is a complete set of orthogonal projections in $L^2(\mu)$ and

\begin{equation}
    f = \sum_{t=1}^T E_{\text{top } \Gamma; \kappa}^\mu f + \sum_{Q \in \mathcal{D}} \Delta_{Q; \kappa}^\mu f, \quad f \in L^2(\mu),
\end{equation}

where convergence in the first line holds both in $L^2(\mu)$ norm and pointwise $\mu$-almost everywhere.

2. Moreover we have the telescoping identities

\begin{equation}
    1_Q \sum_{\Gamma; Q \subseteq \sigma \subsetneq P} \Delta_{\Gamma; \kappa}^\mu = E_{Q; \kappa}^\mu - 1_Q E_{P; \kappa}^\mu \quad \text{for } P, Q \in \mathcal{D} \text{ with } Q \subseteq P,
\end{equation}

3. and the moment vanishing conditions

\begin{equation}
    \int_{\mathbb{R}^n} \Delta_{Q; \kappa}^\mu f(x) x^2 \mu(x) = 0, \quad \text{for } Q \in \mathcal{D}, \quad \beta \in \mathbb{Z}^n, \quad 0 \leq |\beta| < \kappa.
\end{equation}

We can fix an orthonormal basis $\left\{h_{Q; \kappa}^{a, a} \right\}_{a \in \Gamma_{Q, n, \kappa}}$ of $L^2(\mu)$ where $\Gamma_{Q, n, \kappa}$ is a convenient finite index set. Then

$$\left\{h_{Q; \kappa}^{a, a} \right\}_{a \in \Gamma_{Q, n, \kappa}} \text{ and } Q \in \mathcal{D}$$

is an orthonormal basis for $L^2(\mu)$, with the understanding that we add an orthonormal basis of each space $P^n_{\text{top } \Gamma; \kappa}(\mu)$ if it is nontrivial. In particular we have from the theorem above that when $P^n_{\text{top } \Gamma; \kappa}(\mu) = \{0\}$ for all $1 \leq t \leq T$, which is the case for the doubling measures $\mu$ considered in this paper, then

$$\|f\|^2_{L^2(\mu)} = \sum_{Q \in \mathcal{D}} \left\|\Delta_{Q; \kappa}^\mu f\right\|^2_{L^2(\mu)} = \sum_{Q \in \mathcal{D}} \left|\hat{f}(Q)\right|^2,$$

where $\left|\hat{f}(Q)\right|^2 \equiv \sum_{a \in \Gamma_{Q, n, \kappa}} \left\langle f, h_{Q; \kappa}^{a, a} \right\rangle_{\mu}^2$.

**Remark 7.** A dyadic grid $\mathcal{D}$ on the real line is a translate of the standard grid $\mathcal{D}_0$ if and only if there are exactly two tops, and if the tops are $(-\infty, a)$ and $(a, \infty)$, then $\mathcal{D} = \mathcal{D}_0 + a$.

In the case $\kappa = 1$, the construction in Theorem 6 reduces to the familiar Haar wavelets, where we have the following bound for the Alpert projections $E_{I; \kappa}$ (see (4.7) on page 14):

\begin{equation}
    \left\|E_{I; \kappa}^\mu f\right\|_{L^2(\mu)} \lesssim E_{I; \kappa}^\mu |f| \leq \frac{1}{|I_{\mu}|} \int_I |f|^2 d\mu, \quad \text{for all } f \in L^2_{\text{loc}}(\mu).
\end{equation}

In terms of the Alpert coefficient vectors $\hat{f}(I) \equiv \left\{\left\langle f, h_{Q; \kappa}^{a, a} \right\rangle \right\}_{a \in \Gamma_{Q, n, \kappa}}$, we thus have

\begin{equation}
    \left|\hat{f}(I)\right| = \left\|\Delta_{I; \kappa}^\mu f\right\|_{L^2(\sigma)} \lesssim \left\|\Delta_{I; \kappa}^\mu f\right\|_{L^2(\sigma)} \sqrt{|I_{\sigma}|} \leq C \left\|\Delta_{I; \kappa}^\mu f\right\|_{L^2(\sigma)} = C \left|\hat{f}(I)\right|.
\end{equation}
2. A Generalization

Let \( \mu \) be a locally finite positive Borel measure on \( \mathbb{R}^n \), and let \( \mathcal{P} = \operatorname{Span} \{ \varphi_i \}_{i=1}^d \) denote a \( d \)-dimensional subspace of locally \( L^2(\mu) \) complex-valued functions, \( \varphi_i \in L^2_{\text{loc}}(\mu) \), which contains the constant function \( 1 \) on \( \mathbb{R}^n \). For example, \( \mathcal{P} \) could be the real finite dimensional linear space of complex polynomials of degree less than \( \kappa \) used in the construction of weighted Alpert wavelets above. More generally, \( \mathcal{P} \) could consist of complex polynomials of \( z \) (or instead \( \overline{z} \)) of degree less than \( \kappa \). In any of these cases, the finite dimensional space \( \mathcal{P} \) is invariant under translations, dilations and rotations, making them well suited to analysis involving Taylor’s subspace of locally \( L^2(\mu) \).

Now define

\[
L^2_{\mathcal{P}}(\mu) \equiv \left\{ f = \sum_{Q \in \mathcal{D}(Q)} 1_{Q'} p_{Q';\mathcal{P}}(x) : \int_Q f(x) \varphi_i(x) \, d\mu(x) = 0, \quad \text{for } 1 \leq i \leq d \right\},
\]

where \( p_{Q';\mathcal{P}}(x) = \sum_{i=1}^d a_{Q';i} \varphi_i(x) \) is a linear combination of the functions \( \varphi_i \), \( 1 \leq i \leq d \).

**Theorem 8** (Weighted Alpert Bases). Let \( \mu \) be a locally finite positive Borel measure on \( \mathbb{R}^n \), fix \( \mathcal{P} \) a \( d \)-dimensional subspace of \( L^2_{\text{loc}}(\mu) \) containing the function \( 1 \), and fix a dyadic grid \( \mathcal{D} \) in \( \mathbb{R}^n \).

1. Then \( \left\{ P_{\mathcal{P}}^{\mu} \right\}_{t=1}^T \cup \left\{ \Delta_{Q;\kappa}^{\mu} \right\}_{Q \in \mathcal{D}} \) is a complete set of orthogonal projections in \( L^2_{\mathbb{R}^n}(\mu) \) and

\[
f = \sum_{t=1}^T P_{\mathcal{P}}^{\mu} f + \sum_{Q \in \mathcal{D}} \Delta_{Q;\kappa}^{\mu} f, \quad f \in L^2_{\mathbb{R}^n}(\mu),
\]

\[
\left\langle P_{\mathcal{P}}^{\mu}, \Delta_{Q;\kappa}^{\mu} f, \Delta_{Q;\kappa}^{\mu} f \right\rangle = 0 \quad \text{for } \mathcal{P} \neq Q,
\]

where convergence in the first line holds both in \( L^2_{\mathbb{R}^n}(\mu) \) norm and pointwise \( \mu \)-almost everywhere.

2. Moreover we have the telescoping identities

\[
1_Q \sum_{i : Q \not\subset P} \Delta_{I;\kappa}^{\mu} = E_{Q;\mathcal{P}}^{\mu} - 1_Q E_{P;\mathcal{P}}^{\mu}, \quad \text{for } P, Q \in \mathcal{D} \text{ with } Q \subsetneq P,
\]

3. and the moment vanishing conditions

\[
\int_{\mathbb{R}^n} \Delta_{Q;\kappa}^{\mu} f(x) \varphi(x) \, d\mu(x) = 0, \quad \text{for } Q \in \mathcal{D}, \ \varphi \in \mathcal{P}.
\]

In the case when \( \mathcal{P} \) consists of complex polynomials of \( z \) (or instead \( \overline{z} \)) of degree less than \( \kappa \), we have analogues of (1.4) and (1.5). More generally, if \( N \in \mathbb{N} \) and \( \mathcal{P} = \{ \varphi_i \}_{i=1}^d \) where each of the functions \( \varphi_i \) is uniformly finite type \( \kappa \), then the analogues of (1.4) and (1.5) hold. Here we say \( \varphi \in C^\kappa(\mathbb{R}^n) \) is **uniformly finite type \( \kappa \)** if

\[
\inf_{Q \text{ a cube in } \mathbb{R}^n} \inf_{a \in Q} \sum_{|\alpha| < \kappa} \left| \frac{\partial^{|\alpha|} \varphi(a)}{\partial x^\alpha} \right| |\ell(Q)|^{|\alpha|} > 0.
\]

The inequalities in (1.4) and (1.5) then follow from Taylor’s formula

\[
\varphi(x) = \sum_{\ell=0}^{\kappa-1} \frac{(x-a) \cdot \nabla}{\ell!} \varphi(a) + O(|x-a|^\kappa)
\]

\[
= \sum_{|\alpha| < \kappa} c_{\alpha} \frac{\partial^{|\alpha|} \varphi(a)}{\partial x^\alpha} (x-a)^\alpha + O(|x-a|^\kappa),
\]

together with properties of doubling measures proved in [Saw6]. We leave the straightforward details to the interested reader.
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