Joint Invariants of Linear Symplectic Actions

Fredrik Andreassen and Boris Kruglikov *

Institute of Mathematics and Statistics, UiT the Arctic University of Norway, 90-37 Tromsø, Norway; fredrik.andreassen@gmail.com
* Correspondence: boris.kruglikov@uit.no

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Abstract: We review computations of joint invariants on a linear symplectic space, discuss variations for an extension of group and space and relate this to other equivalence problems and approaches, most importantly to differential invariants.

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1. Introduction

The classical invariant theory [1–3] investigates polynomial invariants of linear actions of a Lie group $G$ on a vector space $V$, i.e., describes the algebra $(S V^*)^G$. For instance, the case of binary forms corresponds to $G = \text{SL}(2, \mathbb{C})$ and $V = \mathbb{C}^2$; equivalently for $G = \text{GL}(2, \mathbb{C})$ one studies instead the algebra of relative invariants. The covariants correspond to invariants in the tensor product $V \otimes W$ for another representation $W$. Changing to the Cartesian product $V \times W$ leads to joint invariants of $G$.

In this paper, we discuss joint invariants corresponding to the (diagonal) action of $G$ on the iterated Cartesian product $V \times^m$ for increasing number of copies $m \in \mathbb{N}$. We will focus on the case $G = \text{Sp}(2n, \mathbb{R})$, $V = \mathbb{R}^{2n}$ and discuss the conformal $G = \text{CSp}(2n, \mathbb{R}) = \text{Sp}(2n, \mathbb{R}) \times \mathbb{R}_+$ and affine $G = \text{ASp}(2n, \mathbb{R}) = \text{Sp}(2n, \mathbb{R}) \rtimes \mathbb{R}^{2n}$ versions later.

This corresponds to invariants of $m$-tuples of points in $V$, i.e., finite ordered subsets. By the Hilbert-Mumford [1] and Rosenlicht [4] theorems, the algebra of polynomial invariants (for the semi-simple $G$) or the field of rational invariants (in all other cases considered) can be interpreted as the space of functions on the quotient space $V \times^m / G$.

For $G = \text{Sp}(2n, \mathbb{C})$ the algebra of invariants is known [5]. Generators and relations (syzygies) are described in the first and the second fundamental theorems, respectively. We review this in Theorem 1 (real version), and complement by explicit examples of free resolutions of the algebra. In addition, we describe the field of rational invariants.

We also discuss invariants with respect to the group $G = \text{Sp}(2n, \mathbb{R}) \times S_m$, in which case considerably less is known. Another generalization we consider is the field of invariants for the conformal symplectic Lie group $G = \text{CSp}(2n, \mathbb{R})$ on the contact space.

When approaching invariants of infinite sets, like curves or domains with smooth boundary, the theory of joint invariants is not directly applicable and the equivalence problem is solved via differential invariants [6]. In the case of a group $G$ and a space $V$ as above this problem was solved in [7]. We claim that the differential invariants from this reference can be obtained in a proper limit of joint invariants, i.e., via a certain discretization and quasiclassical limit, and demonstrate it explicitly in several cases.

In this paper, we focus on discussion of various interrelations of joint invariants. In particular, at the conclusion we note that joint invariants can be applied to the equivalence problem of binary
forms. Since these have been studied also via differential invariants \[2,8\] a further link to the above symplectic discretization is possible.

The relation to binary forms mentioned above is based on the Sylvester theorem \[9\], which in turn can be extended to more general Waring decompositions, important in algebraic geometry \[10\]. Our computations should carry over to the general case. This note is partially based on the results of \[11\], generalized and elaborated in several respects.

2. Recollection: Invariants

We briefly recall the basics of invariant theory, referring to \[3,12\] for more details.

Let \( G \) be a Lie group acting on a manifold \( V \). A point \( x \in V \) is regular if a neighborhood of the orbit \( G \cdot x \) is fibred by \( G \)-orbits. A point \( x \in V \) is weakly regular, if its (not necessary \( G \)-invariant) neighborhood is foliated by the orbits of the Lie algebra \( g = \text{Lie}(G) \). In general, the action can lack regular points, but a generic point is weakly regular. For algebraic actions a Zariski open set of points is regular.

2.1. Smooth Invariants

If \( G \) and \( V \) are only smooth (and non-compact), there is little one can do to guarantee regularity a priori. An alternative is to look for local invariants, i.e., functions \( I = I(x) \) in a neighborhood \( U \subset V \) such that \( I(x) = I(g \cdot x) \) as long as \( x \in U \) and \( g \in G \) satisfy \( g \cdot x \in U \).

The standard method to search for such \( I \) is by elimination of group parameters, namely by computing quasi-transversals \[3\] or using normalization and moving frame \[2\]. Another way is to solve the linear PDE system \( L_g(I) = 0 \) for \( \xi \in g = \text{Lie}(G) \).

Given the space of invariants \( \{I\} \) one can extend \( U \subset V \) and address regularity. In our case the invariants are easy to compute and we do not rely on any of these methods; however instead we describe the algebra and the field of invariants depending on specification of the type of functions \( I \).

2.2. Polynomial Invariants

If \( G \) is semi-simple and \( V \) is linear, then by the Hilbert-Mumford theorem generic orbits can be separated by polynomial invariants \( I \in (SV^*)^G \), where \( SV^* = \bigoplus_{k=0}^{\infty} S^kV^* \) is the algebra of homogeneous polynomials on \( V \). With a choice of linear coordinates \( x = (x_1, ..., x_n) \) on \( V \) we identify \( SV^* = \mathbb{R}[x] \).

Moreover, by the Hilbert basis theorem, the algebra of polynomial invariants \( A_G = (SV^*)^G \) is Noetherian, i.e., finitely generated by some \( a = (a_1, ..., a_s) \), \( a_j = a_j(x) \in A_G \).

Denote by \( R = \mathbb{R}[a] \) the free commutative \( \mathbb{R} \)-algebra generated by \( a \). It forms a free module \( F_0 \) over itself. \( A_G \) is also an \( R \)-module with surjective \( R \)-homomorphism \( \phi_0 : F_0 \rightarrow A_G, \phi_0(a_j) = a_j(x) \).

The first syzygy module \( S_1 = \text{Ker}(\phi_0) \) fits the exact sequence

\[ 0 \rightarrow S_1 \rightarrow F_0 \rightarrow A_G \rightarrow 0. \]

A syzygy is an element of \( S_1 \), i.e., a relation \( r = r(a) \) between the generators of \( A_G \) of the form \( \sum_{p=1}^k r_p a_{p_j} = 0 \), \( r_p \in \mathbb{R} \).

The module \( S_1 \) is Noetherian, i.e., finitely generated by some \( b = (b_1, ..., b_l) \). Denote the free \( R \)-module generated by \( b \) by \( F_1 = R[b] \). The natural homomorphism \( \phi_1 : F_1 \rightarrow S_1 \subset F_0 \), \( \phi_1(b_j) = b_j(a) \), defines the second syzygy module \( S_2 = \text{Ker}(\phi_1) \), and we can continue obtaining \( S_2 \subset F_2 = R[e] \), etc. This yields the exact sequence of \( R \)-modules:

\[ \ldots \rightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} A_G \rightarrow 0. \]

The Hilbert syzygy theorem states that \( q \)-th module of syzygies \( S_q \) is free for \( q \geq s = \#a \).

In particular, the minimal free resolution exists and has length \( s \), see \[13\].
To emphasize the generating sets, we depict free resolutions as follows:

\[ \mathbb{R}[x] \supset A_G \leftarrow \mathbb{R}[a] \leftarrow \mathbb{R}[b] \leftarrow \mathbb{R}[c] \leftarrow \cdots \leftarrow 0. \]

### 2.3. Rational Invariants

If \( G \) is algebraic, in particular reductive, then by the Rosenlicht theorem \[4\] generic orbits can be separated by rational invariants \( I \in \mathcal{F}_G \). Here \( \mathbb{R}(x) \) is the field of rational functions on \( V \) and \( \mathcal{F}_G = \mathbb{R}(x)^G \).

Let \( d \) be the transcendence degree of \( \mathcal{F}_G \). This means that there exist \( (a_1, \ldots, a_d) = \bar{a}, a_j \in I \), such that \( \mathcal{F}_G \) is an algebraic extension of \( \mathbb{R}(\bar{a}) \). Then either \( \mathcal{F}_G = \mathbb{R}(\bar{a}) \) for \( a = \bar{a} \) or \( \mathcal{F}_G \) is generated by \( a \supset \bar{a} \), which by the primitive element theorem can be assumed of cardinality \( s = \# a = d + 1 \), i.e., \( a = (a_1, \ldots, a_d, a_{d+1}) \). In the latter case there is one algebraic relation on \( a \). Please note that \( d \leq n \) because \( \mathbb{R}(\bar{a}) \subset \mathbb{R}(x) \).

We adopt the following convention for depicting this:

\[ \mathbb{R}(x) \supset \mathcal{F}_G \overset{\text{alg}}{\supset} \mathbb{R}(a) \overset{d}{\supset} \mathbb{R}. \]

### 2.4. Our Setup

If the Lie group \( G \) acts effectively on \( V \), then for some \( q \) it acts freely on \( V^{\times q} \), and hence on all \( V^{\times m} \) for \( m \geq q \). The number of rational invariants separating a generic orbit in \( V^{\times m} \) is equal to the codimension of the orbit.

It turns out that knowing all those invariants \( I \) on \( V^{\times q} \) is enough to generate the invariants on \( V^{\times m} \) for \( m > q \). Indeed, let \( \pi_{i_1, \ldots, i_q} : V^{\times m} \rightarrow V^{\times q} \) be the projection to the factors \( (i_1, \ldots, i_q) \). Then the union of \( \pi_{i_1, \ldots, i_q} \) \( I \) for \( I \) from the field \( \mathcal{F}_G(V^{\times q}) \) gives the generating set of the field \( \mathcal{F}_G(V^{\times m}) \), and similarly for the algebra of invariants.

Below we denote \( A_G^m = A_G(V^{\times m}) \) and \( \mathcal{F}_G^m = \mathcal{F}_G(V^{\times m}) \).

### 2.5. The Equivalence Problem

For a semi-simple Lie group \( G \) the field \( \mathcal{F}_G \) is obtained from the ring \( A_G \) by localization (field of fractions): \( \mathcal{F}_G = F(A_G) \). Hence we discuss a solution to the equivalence problem through rational invariants.

Let \( I_1, \ldots, I_s \) be a generating set of invariants of the action of \( G \) on \( V^{\times q} \). If \( s = d + 1 \), this set of generators is subject to an algebraic condition, which constrains the generators to an algebraic set \( \Sigma \subset \mathbb{R}^d \). If \( s = d \) then \( \Sigma = \mathbb{R}^d \). This \( \Sigma \) is the signature space, cf. \[14\].

Now the \( q \)-tuple of points \( X = (x_1, \ldots, x_q) \) is mapped to \( I_1(X), \ldots, I_s(X) \in \Sigma \). Denote this map by \( \Psi \). Two generic configurations of points \( X', X'' \in V^{\times q} \) are \( G \)-equivalent iff their signatures coincide \( \Psi(X') = \Psi(X'') \).

### 3. Invariants on Symplectic Vector Spaces

Let \( V = \mathbb{R}^{2n}(x_1^1, \ldots, x_n^1, y_1^1, \ldots, y_n^1) \) be equipped with the standard symplectic form \( \omega = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n \). The group \( G = \text{Sp}(2n, \mathbb{R}) \) acts almost transitively on \( V \), preserving the origin \( O \). Thus, there are no continuous invariants of the action, \( \mathcal{F}_G^1 = \mathbb{R} \). The first invariant occurs already for two copies of \( V \). Namely for a pair of points \( A_i, A_j \in V \) the double symplectic area of the triangle \( OA_iA_j \) is

\[ a_{ij} = \omega(OA_i, OA_j) = x_i y_j - x_j y_i = \sum_{k=1}^n x_i^k y_j^k - x_j^k y_i^k. \]
3.1. The Case \(n = 1\)

Consider at first the case of dimension 2, where \(V = \mathbb{R}^2(x, y), \omega = dx \wedge dy\). The invariant \(a_{12} = x_1y_2 - x_2y_1\) on \(V \times V\) generates pairwise invariants \(a_{ij}\) on \(V^\times m\) for \(m \geq 2\) induced through the pull-back of the projection \(\pi_{ij} : V^\times m \to V \times V\) to the corresponding factors. Below we describe minimal free resolutions of \(A^m_G\) for \(m \geq 2\).

3.1.1. \(V \times V\)

Here the algebra is generated by one element, whence the resolution:

\[
\mathbb{R}[x_1, x_2, y_1, y_2] \supset A^2_G \leftarrow \mathbb{R}[a_{12}] \leftarrow 0
\]

In other words, \(A^2_G \simeq R := \mathbb{R}[a_{12}]\). Please note that \(F^2_G = \mathbb{R}(a_{12})\).

3.1.2. \(V^\times 3 = V \times V \times V\)

Here the action is free on the level of \(m = 3\) copies of \(V\) and we get \(3 = \dim V^\times 3 - \dim G\) independent invariants \(a_{12}, a_{13}, a_{23}\). They generate the entire algebra, and we get the following minimal free resolution:

\[
\mathbb{R}[x_1, x_2, x_3, y_1, y_2, y_3] \supset A^3_G \leftarrow \mathbb{R}[a_{12}, a_{13}, a_{23}] \leftarrow 0
\]

Once again, \(A^3_G \simeq R := \mathbb{R}[a_{12}, a_{13}, a_{23}]\). Also \(F^3_G = \mathbb{R}(a_{12}, a_{13}, a_{23})\).

3.1.3. \(V^\times 4\)

Here \(\dim V^\times 4 = 8, \dim G = 3\) and we have 6 invariants \(a = \{a_{ij} : 1 \leq i < j \leq 4\}\). To obtain a relation, we try eliminating the variables \(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\), but this fails with the standard MAPLE command. Yet, using the transitivity of the \(G\)-action we fix \(A_1\) at \((1, 0)\) and \(A_2\) at \((0, a_{12})\), and then obtain the only relation

\[
b_{1234} := a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0
\]

that we identify as the Plücker relation. Thus, the first syzygy is a module over \(R := \mathbb{R}[a]\) with one generator, hence the minimal free resolution is:

\[
\mathbb{R}[x, y] \supset A^4_G \leftarrow \mathbb{R}[a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}] \leftarrow \mathbb{R}[b_{1234}] \leftarrow 0.
\]

For the field of rational invariants one of the generators is superfluous, for instance we can resolve the relation \(b_{1234} = 0\) for \(a_{34} = (a_{13}a_{24} - a_{14}a_{23})/a_{12}\), and get

\[
\mathbb{R}(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \supset F^4_G \simeq \mathbb{R}(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}) \supset \mathbb{R}
\]

3.1.4. \(V^\times 5\)

The algebra of invariants \(A^5_G\) is generated by \(a = \{a_{ij} : 1 \leq i < j \leq 5\}\). This time the number of generators is 10, while codimension of the orbit is \(10 - 3 = 7\). Using the same method we obtain that the first syzygy module is generated by the Plücker relations

\[
b_{ijkl} := a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk} = 0.
\]

We have 5 of those: \(b = \{b_{ijkl} : 1 \leq i < j < k < l \leq 5\}\). Thus, there should be relations among relations, or equivalently second syzygies. If \(F_0 = \mathbb{R}[a] =: R\) and \(F_1 = R[b]\) then this module is
S_2 = \text{Ker}(\phi_1 : F_1 \to S_1 \subset F_0)$. Using elimination of parameters, we find that $S_2$ is generated by $c = \{c_i : 1 \leq i \leq 5\}$ with

$$c_i := \sum_{j=1}^{5} (-1)^{j}a_{ij}b_{1...j...5}.$$  

For instance, $c_1 = a_{12}b_{1345} - a_{13}b_{1245} + a_{14}b_{1235} - a_{15}b_{1234}$. Then we look for relations between the generators $c$ of $S_2$, defining the third syzygy module $S_3$. It is generated by one element

$$d := (a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34})c_1 + (-a_{13}a_{45} + a_{14}a_{35} - a_{15}a_{34})c_2$$

$$+ (a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24})c_3 + (-a_{12}a_{35} + a_{13}a_{25} - a_{15}a_{23})c_4$$

$$+ (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})c_5 = 0.$$  

Thus, the minimal free resolution of $\mathcal{A}_G^5$ is (note that here, as well as in our other examples, the length of the resolution is smaller than what the Hilbert theorem predicts):

$$\mathbb{R}[x,y] \supset \mathcal{A}_G^5 \leftarrow \mathbb{R}[a] \leftarrow \mathbb{R}[b] \leftarrow \mathbb{R}[c] \leftarrow \mathbb{R}[d] \leftarrow 0.$$  

As before, to generate the field of rational invariants, we express superfluous generators in terms of the others using the first syzygies. Specifically, we express $a_{34}, a_{35}, a_{45}$ from the relations $b_{1234}, b_{1325}, b_{1245}$; the other 2 syzygies follow from the higher syzygies. Removing these generators, we obtain a set of 7 independent generators $\bar{a} = a \setminus \{a_{34}, a_{35}, a_{45}\}$ whence

$$\mathbb{R}(x,y) \supset \mathcal{F}_G^5 \simeq \mathbb{R}(\bar{a}) \supset \mathbb{R}.$$  

3.1.5. General $V \times m$

The previous arguments generalize straightforwardly to conclude that $\mathcal{A}_G^m$ is generated by $a = \{a_{ij} : 1 \leq i < j \leq m\}$. The first syzygy module is generated by the Plücker relations $b = \{b_{ijkl} : 1 \leq i < j < k < l \leq m\}$. In other words we have:

$$\mathcal{A}_G^m = \langle a \mid b \rangle.$$  

Similarly, the field of rational invariants is generated by $a$, yet all of them except for $a_{1j}, a_{2j}$ can be expressed (rationally) through the rest via the Plücker relations $b_{12kl}$. Denote $\bar{a} := \{a_{12}, a_{13}, \ldots, a_{1m}, a_{23}, \ldots, a_{2m}\}, \#\bar{a} = 2m - 3$. Then we get for $m \geq 2$:

$$\mathbb{R}(x,p) \supset \mathcal{F}_G^m \simeq \mathbb{R}(\bar{a}) \supset \mathbb{R}.$$  

3.2. The General Case: Algebra of Polynomial Invariants

Minimal free resolutions can be computed in many examples for $n \geq 1$. However, in what follows we restrict our attention to describing generators/relations of $\mathcal{A}_G^m$.

Let us count the number of local smooth invariants. The action of $G$ on $V$ is almost transitive, so the stabilizer of a nonzero point $A_1$ has dim $G_{A_1} = \binom{2n+1}{2} - 2n = \binom{2n}{2}$. For a generic $A_2$ there is only one invariant $a_{12}$ (the orbit has codimension 1) and the stabilizer of $A_2$ in $G_{A_1}$ has dim $G_{A_1,A_2} = \binom{2n}{2} - (2n - 1) = \binom{2n-1}{2}$. For a generic $A_3$ there are two more new invariants $a_{13}, a_{23}$ (the orbit has codimension 2 + 1 = 3) and the stabilizer of $A_3$ in $G_{A_1,A_2}$ has dim $G_{A_1,A_2,A_3} = \binom{2n-2}{2} - (2n - 2) = \binom{2n-2}{2}$. By the same reason for $k \leq 2n$ the stabilizer of a generic $k$-tuple of points $A_1, \ldots, A_k$ has dim $G_{A_1,\ldots,A_k} = \binom{2n-k+1}{2}$. Finally, for $k = 2n$ the stabilizer of generic $A_1, \ldots, A_{2n}$ is trivial.
Thus, we get the expected number of invariants \( a_{ij} \). For \( m \leq 2n + 1 \) there are no relations between them, and the first comes at \( m = 2n + 2 \). These can be obtained by successively studying cases of increasing \( n \) resulting in the Pfaffian relation:

\[
b_{1123 \ldots i_{2n+1} i_{2n+2}} := \text{Pf}(a_{ij})_{1 \leq p,q \leq 2n+2} = 0.
\]

Recall that the Pfaffian of a skew-symmetric operator \( S \) on \( V \) with respect to \( \omega \) is \( \text{Pf}(S) = \text{vol}_\omega(S e_1, \ldots, S e_{2n}) \) for any symplectic basis \( e_i \) of \( V \). The properties of the Pfaffian are:

\[
\text{Pf}(S)^2 = \text{det}(S), \text{Pf}(TST^t) = \text{det}(T) \text{Pf}(S).
\]

For \( n = 1 \) we get

\[
b_{1234} = \text{Pf} \begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
-a_{12} & 0 & a_{23} & a_{24} \\
a_{13} & -a_{23} & 0 & a_{34} \\
a_{14} & -a_{24} & -a_{34} & 0
\end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.
\]

Similarly, for \( n = 2 \) we get

\[
b_{123456} = a_{12}a_{34}a_{56} - a_{12}a_{35}a_{46} + a_{12}a_{36}a_{45} - a_{13}a_{24}a_{56} + a_{13}a_{25}a_{46} - a_{13}a_{26}a_{45} + a_{14}a_{23}a_{56} - a_{14}a_{25}a_{36} + a_{14}a_{26}a_{35} - a_{15}a_{23}a_{46} + a_{15}a_{24}a_{36} - a_{15}a_{26}a_{34} + a_{16}a_{23}a_{45} - a_{16}a_{24}a_{35} + a_{16}a_{25}a_{34} = 0.
\]

Denote \( b = \{ b_{112 \ldots i_{2n+1} i_{2n+2}} : 1 \leq i_1 < i_2 < \cdots < i_{2n+1} < i_{2n+2} \leq m \} \).

**Theorem 1.** The algebra of \( G \)-invariants is generated by \( a \) with syzygies \( b \):

\[
\mathcal{A}^m_G = \langle a \mid b \rangle.
\]

**Proof.** Let us first prove that the invariants \( a_{ij} \) generate the field \( \mathcal{F}^m_G \) of rational invariants for \( m = 2n \). We use the symplectic analog of Gram-Schmidt normalization: given points \( A_1, \ldots, A_{2n} \) in general position, we normalize them using \( G = \text{Sp}(2n, \mathbb{R}) \) as follows.

Let \( e_1, \ldots, e_{2n} \) be a symplectic basis of \( V \), i.e., \( \omega(e_{2k-1}, e_{2k}) = 1 \) and \( \omega(e_i, e_j) = 0 \) else. At first \( A_1 \) can be mapped to the vector \( e_1 \). The point \( A_2 \) can be mapped to the line \( \mathbb{R}e_2 \), and because of \( \omega(OA_1, OA_2) = a_{12} \) it is mapped to the vector \( a_{12}e_2 \). Next in mapping \( A_3 \) we have two constraints \( \omega(OA_1, OA_3) = a_{13}, \omega(OA_2, OA_3) = a_{23} \), and the point can be mapped to the space spanned by \( e_1, e_2, e_3 \) satisfying those constraints. Continuing like this, we arrive to the following matrix with columns \( OA_i \):

\[
\begin{pmatrix}
1 & 0 & -a_{23} & -a_{24} & \cdots & -a_{2n-1} & -a_{2n} \\
0 & a_{12} & a_{13} & a_{14} & \cdots & a_{1,2n-1} & a_{1,2n} \\
0 & 0 & 1 & 0 & \cdots & * & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & b_{1234} & \cdots & * & * \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & a_{2n-1,2n}
\end{pmatrix}
\]

where \( b_{1234} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} \) (this does not vanish in general if \( n > 1 \)) and by * we denote some rational expressions in \( a_{ij} \) that do not fit the table.

If \( m < 2n \) then only the first \( m \) columns of this matrix have to be kept. If \( m > 2n \) then the remaining points \( A_{2n+1}, \ldots, A_m \) have all their coordinates invariant as the stabilizer of the first \( 2n \) points is trivial. Thus, the invariants are expressed rationally in \( a_{ij} \).

To obtain polynomial invariants one clears the denominators in these rational expressions, and so \( \mathcal{A}^m_G \) is generated by \( a \) as well.
Now the Pfaffian of the skew-symmetric matrix \((a_{ij})_{2k \times 2k}\) is the square root of the determinant of the Gram matrix of the vectors \(OA_i, 1 \leq i \leq k\), with respect to \(\omega\). If we take \(k = n + 1\) then the vectors are linearly dependent and therefore the Pfaffian vanishes. Thus, \(b\) are syzygies among the generators \(a\). That they form a complete set follows from the same normalization procedure as above. \(\square\)

**Remark 1.** Theorem 1 is basically known: H. Weyl described the generators \(a\) as the first fundamental theorem; his second fundamental theorem gives not only the syzygy denoted above by \(b\), but also several different Pfaffians of larger sizes. Namely he lists in ([5], VI.1) the syzygies \(b_{1,\ldots,2n+2} := \text{Pf}(a_{i_{\ell}i_{\ell}})_{1 \leq \ell, \ell \leq 2n+2} = 0, 1 \leq k \leq n.\) Those however are abundant, and should be excluded. For instance, for \(n = 2\)

\[
b_{12345678} = a_{12}b_{345678} - a_{13}b_{245678} - a_{14}b_{235678} + a_{15}b_{234678} + a_{16}b_{234578} - a_{17}b_{234568} + a_{18}b_{234567}.
\]

In general, the larger Pfaffians can be expressed via the smallest through the expansion by minors [15] (this fact was also noticed in [16]). Here is the corresponding Pfaffian identity (below we denote \(S_{2n+1} = \{\sigma \in S_{2n+2} : \sigma(1) = 1\}\))

\[
b_{12,\ldots,2n+1,2n+2} = \frac{1}{n!} \sum_{\ell \in S_{2n+1}} (-1)^{\text{sgn}(\ell)} a_{12}b_{\ell(2)}b_{\ell(3)}\ldots b_{\ell(2n+2)}.
\]

In (3), §9.5 another set of syzygies was added: \(q_{i_1,\ldots,i_{2n+1}} = \det(a_{i_1,i_{2n+1}})_{1 \leq j \leq 1} = 0.\) These are also abundant, and should be excluded. For instance, for \(n = 1\) we get

\[
q_{123456} = a_{12}b_{3456} - a_{34}b_{1256} + a_{35}b_{1246} - a_{36}b_{1245}.
\]

### 3.3. The General Case: Field of Rational Invariants

Since \(G\) is simple, the field of rational invariants is the field of fractions of the algebra of polynomial invariants: \(\mathcal{F}_G^m = F(A_G^m).\) To obtain its basis one can use the syzygies \(b_{1,\ldots,2n+2} = 0\) to express all invariants through \(a = \{a_{ij} : 1 \leq i \leq 2n; i < j \leq m\}.\)

This can be done rationally (with \(b_{1,\ldots,2n} \not\equiv 0\) in the denominator), for instance for \(n = 2\) we can express \(a_{56}\) from the syzygy \(b_{123456} = 0\) as follows:

\[
a_{56} = (a_{12}a_{35}a_{46} - a_{12}a_{36}a_{45} - a_{13}a_{25}a_{46} + a_{13}a_{26}a_{45} + a_{14}a_{25}a_{36} - a_{14}a_{26}a_{35} + a_{15}a_{23}a_{46} - a_{15}a_{24}a_{36} + a_{15}a_{26}a_{34} - a_{16}a_{23}a_{45} + a_{16}a_{24}a_{35} - a_{16}a_{25}a_{34})/(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}).
\]

In general, we have \(#a = 2nm - n(2n + 1)\) for \(m \geq 2n,\) in summary:

\[
\mathbb{R}(x, y) \supset \mathcal{F}_G^m = \mathbb{R}(a) \overset{d(m,n)}{\supset} \mathbb{R},
\]

where

\[
d(m,n) = \begin{cases} 
2nm - n(2n + 1) & \text{for } m \geq 2n \\
\binom{m}{2} & \text{for } m \leq 2n.
\end{cases}
\]

### 4. Variation on the Group and Space

Let us consider inclusion of symmetrization, scaling and translations to the transformation group \(G.\) We also discuss contactization of the action.
4.1. Symmetric Joint Invariants

Invariants of the extended group $\tilde{G} = \text{Sp}(2m, \mathbb{R}) \times S_m$ on $V \times m$ are equivalent to $G$-invariants on configurations of unordered sets of points $V \times m / S_m$ (which is an orbifold). Denote the algebra of polynomial $\tilde{G}$-invariants on $V \times m$ by $S^m_{\tilde{G}} \subset A^m_G$. The projection $\pi : A^m_G \rightarrow S^m_{\tilde{G}}$ is given by

$$\pi(f) = \frac{1}{m!} \sum_{\sigma \in S_m} \sigma \cdot f.$$

As a Noetherian algebra $S^m_{\tilde{G}}$ is finitely generated, yet it is not easy to establish its generating set explicitly. All linear terms average to zero, $\pi(a_{ij}) = 0$, but there are several invariant quadratic terms in terms of the homogeneous decomposition $A^m_G = \oplus_{k=0}^\infty A^m_k$.

For example, for $n = 1, m = 4$ we have $A^4_0 = \mathbb{R}$, $A^4_1 = \mathbb{R}^6 = (a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})$, $A^4_2 = \mathbb{R}^{20}$ (21 monomials $a_{ij}a_{kl}$ modulo 1 Plücker relation), etc. Then $\pi(A^4_0) = \mathbb{R}$, $\pi(A^4_1) = 0$, and $\pi(A^4_2) = \mathbb{R}^2$ has generators

$$6\pi(a_{12}^2) = a_{12}^2 + a_{13}^2 + a_{14}^2 + a_{23}^2 + a_{24}^2 + a_{34}^2,$$
$$12\pi(a_{12}a_{13}) = a_{12}a_{13} + a_{12}a_{14} + a_{13}a_{14} - a_{12}a_{23} - a_{12}a_{24} + a_{23}a_{24} + a_{13}a_{23} - a_{13}a_{34} - a_{23}a_{34} + a_{14}a_{24} + a_{14}a_{34} + a_{24}a_{34}.$$

**Theorem 2.** The field of symmetric rational invariants $S^m_{\tilde{G}} = \pi(F^m_G)$ is the field of fractions $S^m_{\tilde{G}} = F(S^m_{\tilde{G}})$ and its transcendence degree is $d(m,n)$.

**Proof.** This follows from general theorems ([17], §2.5) and discussion in Section 2. □

The last statement can be made more constructive: Let $\ell$ numerate indices $(ij)$ of the basis $\mathbf{a}$ of $F^m_G$ as in Section 3.3, $1 \leq \ell \leq d = d(m,n)$. One can check that $q_k = \pi(\prod_{\ell \leq k} a_{ij}^\ell)$ are algebraically independent. Thus, denoting $q = (q_1, \ldots, q_d)$ we obtain the presentation

$$\mathbb{R}(x, y) \supset S^m_{\tilde{G}} \supset \mathbb{R}(q)^{d(m,n)} \supset \mathbb{R}.$$

Here is an algorithm to obtain generators of $S^m_{\tilde{G}}$.

**Proposition 1.** Fix an order on generators $a_{ij} \in A^m_G$, and induce the total lexicographic order on monomials $a^\sigma \in \mathcal{R} = \mathbb{R}[a]$. Let $\Sigma$ be the Gröbner basis of the $\mathcal{R}$-ideal generated by $\pi(a^\sigma)$. Then elements $\pi(a^\sigma)$, contributing to $\Sigma$, generate $S^m_{\tilde{G}} = \pi(A^m_G)$.

**Proof.** Please note that the algorithm proceeds in total degree of $a^\sigma$ until the Gröbner basis stabilizes. That the involved $\pi(a^\sigma)$ generate $S^m_{\tilde{G}}$ as an algebra (initially they generate the ideal $\mathcal{R} \cdot \pi(A^m_G) \subset A^m_G$) follows from the same argument as in the proof of Hilbert’s theorem on invariants [1]. (The above $\pi$ is the Reynolds operator used there.) □

Let us illustrate how this works in the first nontrivial case $m = 3$, for any $n$.

In this case, the graded components of $S^3_{\tilde{G}} = \pi(A^3_G)$ have the following dimensions: $\dim S^3_0 = 1$, $\dim S^3_1 = 0$, $\dim S^3_2 = 2$, $\dim S^3_3 = 1$, $\dim S^3_4 = 4$, $\dim S^3_5 = 2$, $\dim S^3_6 = 7$, etc., encoded into the Poincaré series

$$P^3_S(z) = 1 + 2z^2 + z^3 + 4z^4 + 2z^5 + 7z^6 + 4z^7 + 10z^8 + 7z^9 + \ldots = \frac{1 + z^4}{(1 - z^2)^2(1 - z^3)}.$$
For the monomial order \( a_{12} > a_{13} > a_{23} \) the invariants

\[
I_{2a} = 3\pi (a_{12}^2) = a_{12}^2 + a_{13}^2 + a_{23}^2, \quad I_{2b} = 3\pi (a_{12}a_{13}) = a_{12}a_{13} - a_{12}a_{23} + a_{13}a_{23}, \\
I_3 = 6\pi (a_{12}^2a_{13}) = a_{12}^2(a_{13} + a_{23}) - a_{23}^2(a_{12} + a_{13}) + a_{13}^2(a_{12} - a_{23}), \\
I_4 = 3\pi (a_{12}a_{13}^3) = a_{12}^2a_{13}^2 + a_{12}^2a_{23} + a_{13}^2a_{23}
\]

generate a \( \text{Gröbner} \) basis of the ideal \( \mathcal{R} \cdot \pi (A_n^m) \) with the leading monomials of the corresponding \( \text{Gröbner} \) basis equal: \( a_{12}^2, a_{12}a_{13}, a_{13}^2, a_{12}a_{23}, a_{12}a_{23}, a_{13}a_{23}, a_{13}^2, a_{23}^2 \).

The \( \text{Gröbner} \) basis also gives the following syzygy \( R_8 \):

\[
(4I_{2a}^2 + 4I_{2a}I_{2b} + 3I_{2b}^2)I_{2b} - (8I_{2a}^2 + 4I_{2a}I_{2b} + 14I_{2b}^2)I_4 + 4(I_{2a} - 2I_{2b})I_3^2 + 27I_4^2 = 0.
\]

In other words, \( S_G^3 = \langle I_{2a}, I_{2b}, I_3, I_4 \mid R_8 \rangle \). We also derive a presentation of the field of rational invariants (\( 2 : 1 \) means quadratic extension)

\[
\mathbb{R}(x, y) \supset \delta_3^1 \supset \mathbb{R}(I_{2a}, I_{2b}, I_3) \supset \mathbb{R}.
\]

### 4.2. Conformal and Affine Symplectic Groups

For the group \( G_1 = \text{CSp}(2n, \mathbb{R}) = \text{Sp}(2n, \mathbb{R}) \times \mathbb{R}_+ \) the scaling makes the invariants \( a_{ij} \) relative, yet of the same weight, so their ratios \( a_{12} : a_{13} : \cdots : a_{m-1,m} \) or simply the invariants \( I_{ij} = \frac{a_{ij}}{a_{12}} \) are absolute invariants. These generate the field of invariants of transcendence degree \( d(m,n) - 1 \).

For the group \( G_2 = \text{ASp}(2n, \mathbb{R}) = \text{Sp}(2n, \mathbb{R}) \rtimes \mathbb{R}^{2n} \) the translations do not preserve the origin \( O \) and this makes \( a_{ij} \) non-invariant. However due to the formula \( 2\omega(A_1A_2A_3) = a_{12} + a_{23} - a_{13} \) (or more symmetrically: \( a_{12} + a_{23} + a_{31} \)), with the proper orientation of the triangle \( A_1A_2A_3 \), we easily recover the absolute invariants \( a_{ij} + a_{ik} + a_{ki} \).

Alternatively, using the translational freedom, we can move the point \( A_1 \) to the origin \( O \). Then its stabilizer in \( G_2 \) is \( G = \text{Sp}(2n, \mathbb{R}) \) and we compute the invariants of \((m-1)\) tuples of points \( A_2, \ldots, A_m \) as before. In particular they generate the field of invariants of transcendence degree \( d(m-1,n) \).

### 4.3. Invariants in the Contact Space

Infinitesimal symmetries of the contact structure \( \Pi = \ker(\alpha) \), \( \alpha = du - y \, dx \) in the contact space \( M = \mathbb{R}^{2n+1}(x, y, u) \), where \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \), are given by the contact vector field \( X_H \) with the generating function \( H = H(x, y, u) \). Taking quadratic functions \( H \) with weights \( w(x) = 1, w(y) = 1, w(u) = 2 \) results in the conformally symplectic Lie algebra, which integrates to the conformally symplectic group \( G_1 = \text{CSp}(2n, \mathbb{R}) \) (taking \( H \) of degree \( \leq 2 \) results in the affine extension of it by the Heisenberg group).

Alternatively, one considers the natural lift of the linear action of \( G = \text{Sp}(2n, \mathbb{R}) \) on \( V = \mathbb{R}^{2n} \) to the contactization \( M \) and makes a central extension of it. We will discuss the invariants of this action. Please note that this action is no longer linear, so the invariants cannot be taken to be polynomial, but can be assumed rational.

#### 4.3.1. The Case \( n = 1 \)

In the 3-dimensional case the group \( G_1 = \text{GL}(2, \mathbb{R}) \) acts on \( M = \mathbb{R}^3(x, y, u) \) as follows:

\[
G_1 \ni g = \begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix} : (x, y, u) \mapsto (ax + \beta y, \gamma x + \delta y, f(x, y, u)),
\]

where \( f(x, y, u) = (ax + \beta y)(\gamma x + \delta y) \neq 0 \cdot \frac{xy}{2} + \frac{(ax + \beta y)(\gamma x + \delta y)}{2} \).
This action is almost transitive (no invariants); however there are singular orbits and a relative invariant \( R = xy - 2u \). Extending the action to multiple copies of \( M \), i.e., considering the diagonal action of \( G_1 \) on \( M^{\times m} \), results in \( m \) copies of this relative invariant, but also in the lifted invariants from various \( V^x \):

\[
R_k = x_k y_k - 2u_k \quad (1 \leq k \leq m), \quad R_{ij} = x_i y_j - x_j y_i \quad (1 \leq i < j \leq m).
\]

These are all relative invariants of the same weight, therefore their ratios are absolute invariants:

\[
T_k = \frac{R_k}{R_{m}} \quad (1 \leq k < m), \quad T_{ij} = \frac{R_{ij}}{R_{m}} \quad (1 \leq i < j \leq m).
\]

Since \( u_k \) enter only \( R_k \) there are no relations involving those, and the relations on \( T_{ij} \) are the same as for \( a_{ij} \), namely they are Plücker relations (since those are homogeneous, they are satisfied by both \( R_{ij} \) and \( T_{ij} \)). As previously, we can use them to eliminate all invariants except for \( \tilde{T} = \{ T_k, T_{1i}, T_{2j} \} \):

\[
T_{kl} = \frac{T_{1k} T_{2l} - T_{2k} T_{1l}}{T_{12}}, \quad 3 \leq k < l \leq m.
\]

The field of rational invariants for \( m > 1 \) is then described as follows:

\[
\mathbb{R}(x, y, u) \supset \mathcal{F}_{G_1}^m \simeq \mathbb{R}(\tilde{T})^{3m-4} \supset \mathbb{R}.
\]

4.3.2. The General Case

In general, we also have no invariants on \( M \) and the following relative invariants on \( M^{\times m} \)

\[
R_k = x_k y_k - 2u_k \quad (1 \leq k \leq m), \quad R_{ij} = x_i y_j - x_j y_i \quad (1 \leq i < j \leq m)
\]

resulting in absolute invariants \( T_k, T_{ij} \) given by the same formulae. Again, using the Pfaffian relations we can rationally eliminate superfluous generators, and denote the resulting set by \( \tilde{T} = \{ T_k, T_{ij} : 1 \leq k < m, i < j \leq m, 1 \leq i \leq 2n \} \). This set is independent and contains \( \tilde{d}(m, n) \) elements, where

\[
\tilde{d}(m, n) = \begin{cases} 
(2n + 1) m - n(2n + 1) - 1 & \text{for } m \geq 2n \\
\binom{m}{2} + m - 1 = \binom{m+1}{2} - 1 & \text{for } m \leq 2n.
\end{cases}
\]

This \( \tilde{d}(m, n) \) is thus the transcendence degree of the field of rational invariants:

\[
\mathbb{R}(x, y, u) \supset \mathcal{F}_{G_1}^m \simeq \mathbb{R}(\tilde{T})^{\tilde{d}(m, n)} \supset \mathbb{R}.
\]

5. From Joint to Differential Invariants

When we pass from finite to continuous objects the equivalence problem is solved through differential invariants. In [7] this was done for submanifolds and functions with respect to our groups \( G \). After briefly recalling the results, we will demonstrate how to perform the discretization in several different cases.

5.1. Jets of Curves in Symplectic Vector Spaces

Locally a curve in \( \mathbb{R}^{2n} \) is given as \( u = u(t) \) for \( t = x_1 \) and \( u = (x_2, \ldots, x_n, y_1, \ldots, y_n) \) in the canonical coordinates \( (x_1, x_2, \ldots, x_n, y_1, \ldots, y_n) \), \( \omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n \). The corresponding jet-space \( J^n(V, 1) \) has coordinates \( t, u, u_1, u_2, \ldots \) and \( j^k \) is the truncation of it. For instance, \( j^1(V, 1) = \mathbb{R}^{4n-1}(t, u, u_1) \). Please note that dim \( j^k(V, 1) = 2n + k(2n - 1) \).
In the case of dimension $2n = 2$, the jet-space is $J^k(V, 1) = \mathbb{R}^{k+2}(x, y, y_x, \ldots, y_{x^k})$. Here $G = \text{Sp}(2, \mathbb{R})$ has an open orbit in $J^1(V, 1)$, and the first differential invariant is of order 2:

$$I_2 = \frac{y_{xx}}{(xy_x - y)^3}.$$  

There is also an invariant derivation ($D_x$ is the total derivative with respect to $x$)

$$\nabla = \frac{1}{xy_x - y} D_x.$$  

By differentiation we get new differential invariants $I_3 = \nabla I_2$, $I_4 = \nabla^2 I_2$, etc. The entire algebra of differential invariants is free:

$$\mathcal{A}_G = \langle I_2, \nabla \rangle.$$  

In the general case we denote the canonical coordinates on $V = \mathbb{R}^{2n}$ by $(t, x, y, z)$, where $x$ and $z$ and $(n-1)$-dimensional vectors. $G = \text{Sp}(2n, \mathbb{R})$ acts on $J^\infty(V, 1)$. The invariant derivation is equal to

$$\nabla = \frac{1}{(ty_t - y + xz_t - x_t z)} D_t.$$  

and the first differential invariant of order 2 is

$$I_2 = \frac{x_t z_{tt} - x_{tt} z_t + y_{tt}}{(ty_t - y + xz_t - x_t z)^3}.$$  

There is one invariant $I_3$ of order 3 independent of $I_2$, $\nabla (I_2)$, one invariant $I_4$ of order 4 independent of $I_2$, $\nabla (I_2)$, $I_3$, $\nabla^2 (I_2)$, $\nabla (I_3)$, and so on up to order $2n$. Then the algebra of differential invariants of $G$ is freely generated ([7], §4) so:

$$\mathcal{A}_G = \langle I_2, I_3, \ldots, I_{2n} ; \nabla \rangle.$$  

5.2. Symplectic Discretization

Consider first the case $n = 1$ with coordinates $(x, y)$ on $V = \mathbb{R}^2$. Let $A_i = (x_i, y_i)$, $i = 0, 1, 2$, be three close points lying on the curve $y = y(x)$. We assume $A_i$ is in between $A_0, A_2$ and omit indices for its coordinates, i.e., $A_1 = (x, y)$.

Let $x_0 = x - \delta$ and $x_2 = x + \epsilon$. Denote also $y' = y'(x)$, $y'' = y''(x)$, etc. Then from the Taylor formula we have:

$$y_0 = y - \delta y' + \frac{1}{2} \delta^2 y'' + \frac{1}{6} \delta^3 y'' + o(\delta^3),$$

$$y_2 = y + \epsilon y' + \frac{1}{2} \epsilon^2 y'' + \frac{1}{6} \epsilon^3 y'' + o(\epsilon^3).$$

Therefore, the symplectic invariants $a_{ij} = x_i y_j - x_j y_i$ are:

$$a_{12} = \epsilon (xy' - y) + \frac{1}{2} \epsilon^2 xy'' + \frac{1}{6} \epsilon^3 xy'' + o(\epsilon^3),$$

$$a_{01} = \delta (xy' - y) - \frac{1}{2} \delta^2 xy'' + \frac{1}{6} \delta^3 xy'' + o(\delta^3),$$

$$a_{02} = (\epsilon + \delta)(xy' - y) + \frac{1}{2} (\epsilon^2 - \delta^2) xy''$$

$$+ \frac{1}{6} (\epsilon^3 + \delta^3) xy'' - \frac{1}{2} (\epsilon + \delta) \epsilon \delta y'' + o((|\epsilon| + |\delta|)^3).$$

This implies:

$$\frac{a_{01} - a_{02} + a_{12}}{a_{01} a_{02} a_{12}} = \frac{1}{2} \frac{y''}{(xy' - y)^3} + o(|\delta| + |\epsilon|).$$
Thus, we can extract the invariant exploiting no distance (like $\epsilon = \delta$) but only the topology ($\epsilon, \delta \to 0$) and the symplectic area. This works in any dimension $n$, and using the coordinates from the previous subsection we get

$$\lim_{A_0, A_2 \to A_1} \frac{\text{Area}_\omega(A_0 A_1 A_2)}{\text{Area}_\omega(OA_0 A_1) \text{Area}_\omega(OA_0 A_2) \text{Area}_\omega(OA_1 A_2)} = \frac{2(x_i z_{i0} - x_{i1} z_i + y_{i1})}{(ty_i - y + xz_i - x_i z)^3} = 2I_2.$$ 

Similarly, we obtain the invariant derivation (it uses only two points and hence is of the first order)

$$\lim_{A_0 \to A_1} \frac{\tilde{A}_0 A_1}{\text{Area}_\omega(OA_0 A_1)} = \frac{2D_t}{(ty_i - y + xz_i - x_i z)^2} = 2\nabla.$$ 

The other generators $I_3, I_4, \ldots$ (important for $n > 1$) can be obtained by a higher order discretization, but the formulae become more involved.

### 5.3. Contact Discretization

Now we use joint invariants to obtain differential invariants of curves in contact 3-space $W = \mathbb{R}^3(x, y, u)$ with respect to the group $G = \text{GL}(2, \mathbb{R})$, acting as in §4.3. The curves will be given as $y = y(x), u = u(x)$ and their jet-space is $J^k(W, 1) = \mathbb{R}^{2k+3}(x, y, u, y_s, u_s, \ldots, y_{s_{s_{s_s}}}, u_{s_{s_{s_s}}})$. The differential invariants are generated in the Lie–Tresse sense ([7], §8.1) as

$$A_G = \langle I_1, I_2; \nabla \rangle.$$

where

$$I_1 = \frac{u_s - y}{xy - y}, \quad I_2 = \frac{(xy - 2u)^2}{(xy - y)^3} y_{xx} \quad \nabla = \frac{xy - 2u}{xy - y} D_x.$$ 

Instead of exploiting the absolute rational invariants $T_i, T_{ij}$ we will work with the relative polynomial invariants $R_i, R_{ij}$ from Section 4.3. To get absolute invariants we will then have to pass to weight zero combinations.

Consider three close points $\hat{A}_i = (x_i, y_i, u_i), i = 0, 1, 2$, lying on the curve. We again omit indices for the middle point, so $x_0 = x - \delta, x_1 = x$ and $x_2 = x + \epsilon$. Using the Taylor decomposition as in the preceding subsection, we obtain

$$R_1 = xy - 2u, \quad R_0 - R_1 = \delta(2u' - y - xy') + o(\delta),$$

$$R_{01} = \delta(xy - y) + o(\delta), \quad R_{02} = (\epsilon + \delta)(xy - y) + o(|\epsilon| + |\delta|),$$

$$R_{12} = \epsilon(xy' - y) + o(\epsilon), \quad R_{01} + R_{12} - R_{02} = \frac{1}{2}\epsilon \delta(\epsilon + \delta)y'' + o((|\epsilon| + |\delta|)^3)$$

as well as

$$\tilde{A}_0 A_1 = \delta(\partial_x + y/\partial_y + u/\partial_y) + o(\delta).$$

Passing to jet-notations, we obtain the limit formulae for basic differential invariants:

$$\frac{1}{2} I_1 = \lim_{A_0 \to A_1} \frac{R_0 - R_1}{2R_0} + \frac{1}{2} = \lim_{A_0 \to A_1} \frac{T_0 - 1 + T_{01}}{2T_{01}},$$

$$\frac{1}{2} I_2 = \lim_{A_0, A_2 \to A_1} \frac{R_1^2(R_0 + R_{12} - R_{12})}{R_{01}R_{02}R_{12}} = \lim_{A_0, A_2 \to A_1} \frac{T_{01} + T_{12} - T_{12}}{T_{01}T_{02}T_{12}},$$

$$\nabla = \lim_{A_0 \to A_1} \frac{R_1}{A_0 \tilde{A}_1} = \lim_{A_0 \to A_1} \frac{\tilde{A}_0 A_1}{T_{01}}.$$ 

These formulae straightforwardly generalize to invariants of jets of curves in contact manifolds of dimension $2n + 1, n > 1$, in which case there are also other generators obtained by higher order discretizations.
5.4. Functions and Other Examples

Let us discuss invariants of jets of functions on the symplectic plane. The action of $G = \text{Sp}(2, \mathbb{R})$ on $\mathcal{J}^0 V = V \times \mathbb{R}(u) \simeq \mathbb{R}^3(x, y, u)$, with $J_0 = u$ invariant, prolongs to $\mathcal{J}^\infty (V) = \mathbb{R}^{\infty}(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots)$. Please note that functions can be identified as surfaces in $\mathcal{J}^0 V$ through their graphs.

For any finite set of points $A_k = (x_k, y_k, u_k)$ the values $u_k$ are invariant, and the other invariants $a_{ij}$ are obtained from the projections $A_k = (x_k, y_k)$. In this way we get the basic first order invariant (as before we omit indices $x_1 = x, y_1 = y, u_1 = y$ for the reference point $A_1$ in the right-hand side)

$$I_1 = \lim_{A_0, A_2 \to A_1} \frac{a_{01}(u_1 - u_2) + a_{12}(u_1 - u_0)}{a_{01} - a_{02} + a_{12}} = x u_x + y u_y$$

as well as two invariant derivations

$$\nabla_1 = OA_1 = x D_x + y D_y, \quad \nabla_2 = \lim_{A_0 \to A_1} \frac{I_1}{a_{01}} A_0 A_1 - \frac{u_1 - u_0}{a_{01}} OA_1 = u_x D_y - u_y D_x.$$  

To obtain the second order invariant $I_{2x} = u_x^2 u_{yy} - 2 u_x u_y u_{xy} + u_y^2 u_{xx}$ let $A_0$ belong to the line through $A_1$ in the right-hand side (this constraint reduces the second order formula to depend on only two points), i.e., $A_0 = (x + \epsilon u_y, y - \epsilon u_x), A_1 = (x, y)$. Then $u_0 - u_1 = \epsilon^2 I_{2x} + o(\epsilon^2), a_{01} = \epsilon I_1$ and letting $\epsilon \to 0$ we obtain

$$\lim_{A_0 \to A_1, A_0 A_1 \| \nabla_2} \frac{I_{2x}}{a_{01}^2} = \frac{I_{2x}}{2 I_1^2}.$$  

In the same way we get $I_{2y} = x^2 u_{xx} + 2 x y u_{xy} + y^2 u_{yy}$ and $I_{2h} = x u_x u_{xx} - y u_x u_{yy} + (y u_y - x u_x) u_{xy}$. These however are not required as the algebra of differential invariants applies to other problems treated in [7], §3.1 for some differential syzygies $\mathcal{R}_i$:

$$\mathcal{A}_G = \langle I_0, I_{2x}, \nabla_1, \nabla_2 \mid \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \rangle.$$  

Similarly, one can consider surfaces in the contact 3-space (with the same coordinates $x, y, u$ but different lift of $\text{Sp}(2, \mathbb{R})$ extended to $\text{GL}(2, \mathbb{R})$) and higher-dimensional cases. The idea of discretization of differential invariants applies to other problems treated in [7].

6. Relation to Binary and Higher Order Forms

According to the Sylvester theorem [9] a general binary form $p \in \mathbb{C}[x, y]$ of odd degree $2m - 1$ with complex coefficients can be written as

$$p(x, y) = \sum_{i=1}^{m} (a_i x + b_i y)^{2m-1}.$$  

This decomposition is determined up to permutation of linear factors and independent multiplication of each of them by a $(2m - 1)$-th root of unity.

In other words, we have the branched cover of order $k_m = (2m - 1)^m m!$

$$\times^m (\mathbb{C}^2) \to S_{2m-1} \mathbb{C}^2$$  

and the deck group of this cover is $S_m \times \mathbb{Z}_{2m-1}^m$.

Please note that in the real case, due to uniqueness of the odd root of unity, the corresponding cover over an open subset of the base

$$\times^m (\mathbb{R}^2) \to S_{2m-1} \mathbb{R}^2$$
has the deck group $S_m$.

With this approach the invariants of real binary forms are precisely the joint symmetric invariants studied in this paper, and for complex forms one must additionally quotient by $\mathbb{Z}_{2m-1}$ which is equivalent to passing from $a_{ij}$ to $a_{ij}^{2m-1}$ and other invariant combinations (example for $m = 4$: $a_{12}^3 a_{14}^2 a_{24}^2 a_{34}^3$) and subsequently averaging by the map $\pi$.

Other approaches to classification of binary forms, most importantly through differential invariants [2,8], can be related to this via symplectic discretization.

**Remark 2.** Please note that the standard "root cover" $\mathbb{C}^{2m} \to S^{2m-1}\mathbb{C}^2$:

$$(a_0, a_1, \ldots, a_{2m-1}) \mapsto (p_0, p_1, \ldots, p_{2m-1}), \quad \sum_{i=0}^{2m-1} p_i x^i y^{2m-i-1} = a_0 \prod_{i=1}^{2m-1} (x - a_i y)$$

has order $(2m - 1)! < k_m$. Polynomial $\text{SL}(2, \mathbb{C})$-invariants of binary forms with this approach correspond to functions on the orbifold $\mathbb{C}^{2m}/S_{2m}$.

The above idea extends further to ternary and higher valence forms (see [18] for the differential invariants approach and [19] for an approach using joint differential invariants) with the Waring decompositions [10] as the cover, but here the group $G$ is no longer symplectic. We expect all the ideas of the present paper to generalize to the linear and affine actions of other reductive groups $G$.

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