Abstract

A standard fact about two incompressible surfaces in an irreducible 3-manifold is that one can move one of them by isotopy so that their intersection becomes \( \pi_1 \)-injective. By extending it on the maps of some 3-dimensional \( \mathbb{Z}_n \)-manifolds into 4-manifolds, we prove that any homotopy equivalence of 4-dimensional graph-manifolds with reduced graph-structures is homotopic to a diffeomorphism preserving the structures.

Keywords: graph-manifold, \( \pi_1 \)-injective \( \mathbb{Z}_n \)-submanifold.

1 \( \mathbb{Z}_n \)-submanifolds of 4-manifolds

A \( \mathbb{Z}_n \)-manifold of dimension \( k \) is an object \( Z \) obtained from a compact oriented \( k \)-manifold with boundary by identifying (with respect of the orientation) \( n \) isomorphic parts of boundary in such a way that if a point of the boundary component participate in the identification, then all the points of this component do. Every point of \( Z \) has a neighborhood isomorphic either to a \( k \)-cell, or to a \( k \)-semi-cell, or to a \( n \)-page book of \( k \)-cells. The identified parts form the singular set of \( Z \) denoted \( Z_s \), and the closure of their complement is its regular set denoted \( Z' \). Non-identified boundary components form the boundary \( \partial Z \).

For example, by identifying boundaries of three oriented surfaces each one having one boundary component, one obtain a 2-dimensional \( \mathbb{Z}_3 \)-manifold without boundary. Mapping cylinder of \( n \)-fold covering \( S^1 \to S^1 \) gives an example of 2-dimensional \( \mathbb{Z}_n \)-manifold with boundary; it appears in 3-dimensional Seifert manifolds as the preimage of an arc in the base orbifold going from the projection of a singular fiber to the boundary.

A standard fact about two incompressible surfaces in an irreducible 3-manifold is that one can move one of them by isotopy so that their intersection becomes \( \pi_1 \)-injective. Under natural homotopic assumption, this remains true for the map of 3-manifold and 3-dimensional submanifold of 4-manifold (Proposition 2.B.2 of [5] and Proposition 1 of [4]). The following lemma extends it on the maps of some 3-dimensional \( \mathbb{Z}_n \)-manifolds into 4-manifolds.

To fix the notations, let \( c : S^1 \to S^1 \) be \( n \)-fold covering. Its mapping cylinder is a 2-dimensional \( \mathbb{Z}_n \)-manifolds \( \text{Map}^n(S^1, S^1) = (S^1 \times [0, 1])/\{S^1 \times \{1\}\} = c(S^1 \times \{1\}) \). Let \( Z = S^1 \times \text{Map}^n(S^1, S^1) \), it is a 3-dimensional \( \mathbb{Z}_n \)-manifold with boundary \( \partial Z = S^1 \times \partial\text{Map}^n(S^1, S^1) \), singular set \( Z_s = S^1 \times \text{Map}^n(S^1, S^1)_s \) and regular set \( Z' = S^1 \times (S^1 \times [0; 1]) \).

Lemma 1. Let \( W \) be a compact smooth oriented 4-manifold with \( \pi_2(W) = \pi_3(W) = 0 \) and \( M \) be a compact oriented \( \pi_1 \)-injective 3-submanifold with \( \pi_2(M) = 0 \). Let \( f : (Z, \partial Z) \to (W, W \setminus M) \) be a \( \pi_1 \)-injective map.

Then one can move \( f \) by homotopy (that is constant on \( \partial Z \)) so that \( f(Z_s) \cap M = \emptyset \) and each connected component of \( F = f^{-1}(M) \) is a \( \pi_1 \)-injective torus in \( M \).
Proof. Move $f$ by a small homotopy to make it transverse to $M$. Then $F = f - 1(M)$ is a 2-dimensional $\mathbb{Z}_n$-manifold whose embedding $(F,F_s) \to (Z,(\partial Z,Z_s))$ induces an embedding of regular sets $F' \to Z'$. As $M$ is closed and $Z'$ is compact, $F'$ has a finite number of connected components ([1], corollary 17.2(IV)).

In $Z = S^1 \times \text{Map}^n(S^1,S^1)$ consider the subspace $G = \{0\} \times \text{Map}^n(S^1,S^1)$ where $0 \in S^1$. We have $(G,G_s) \subset (Z,Z_s)$ and $G' \subset Z'$. So $F'$ (which is union of surfaces with boundaries) and $G'$ (which is an annulus) are embedded in $Z' = S^1 \times S^1 \times I$. Denote the boundary component of $G'$ lying $\partial Z$ by $\partial_0 G'$ and the other one by $\partial_1 G'$.

![Figure 1: Regular parts, the covering $\partial_1 Z' \to Z_s$, and $Z_s \cup G \cup F$](image1)

**Step 1: elimination of trivial circles of $\partial F'$.** Denote the boundary component of $Z'$ giving $\partial Z$ by $\partial_0 Z'$ and the other one (participating in the identification) by $\partial_1 Z'$. Suppose that there is a circle of $\partial F'$ that is trivial in $\partial_1 Z'$. Then its projection on $Z_s$ is trivial, too. As its projection is embedded in $Z_s \simeq T^2$ (because $\partial F'$ is the preimage of $F_s$ which is embedded), it bounds an embedded disk there. Now we can proceed as in Proposition 1 of [4]: take a map of 2-disk in $M$ bounding the same loop. As $\pi_2(W) = 0$, the union of these two disks bounds a map of 3-disk $\alpha : D^3 \to W$, which can be separated from $M$. Denote by $N$ a small book neighborhood of $D^2 \subset Z_s$ in $Z$. The map $f$ will not be changed on $Z \setminus N$; on $D^2$ the homotopy of $f$ will be the pushing across $\alpha(D^3)$, and $N \setminus D$ will serve to rely the new map on $D^2$ and the old map on $Z \setminus N$, doing it separately on each leaf of the book.

![Figure 2: Changing of $f$ on $D^2 \subset Z_s$.](image2)

This homotopy of $f$ will eliminate the trivial circle from $F'$:

![Figure 3: Surgery on a trivial loop of $\partial F'$.](image3)
Step 2: Reduction of $F$ to a union of closed surfaces and annuli. Now all the circles $\partial F' \subset \partial_1 Z' \simeq T^2$ are non-trivial and embedded, hence, parallel. They are either parallel to $\partial_1 G'$, or not, which corresponds respectively to $\partial F' \cap \partial_1 G' = \emptyset$ or $\partial F' \cap \partial_1 G' \neq \emptyset$. Take one connected component of $F'$, denote it still by $F'$.

We will separate two cases: $F' \cap G' = \emptyset$ or $F' \cap G' \neq \emptyset$.

Case 1: $F' \cap G' = \emptyset$. It implies $F' \subset (Z' \setminus G') \simeq S^1 \times I \times I$. Note that $(Z' \setminus G') \subset Z'$ is $\pi_1$-injective but $F' \subset Z'$ is not, so neither is $F' \subset (Z' \setminus G')$.

In this case $(F', \partial F') \to (S^1 \times I \times I, \partial(S^1 \times I \times I))$, $\partial F'$ going to a curve parallel to the generator of $\pi_1(S^1 \times I \times I) = H_1(S^1 \times I \times I)$. We need to find an embedded disk in $S^1 \times I \times I$ for the trivializing loop. As $\partial F'$ bounds $F' \subset S^1 \times I \times I$, $\partial F' \sim 0$ in $H_1(S^1 \times I \times I, \mathbb{Z}_2)$. As all the curves of $\partial F'$ are $\neq 0$ and parallel in $\partial(S^1 \times I \times I)$, they are in even number: say $m$ with one orientation, $m$ with the opposite one. $(F', \partial F') \to (S^1 \times I \times I, \partial(S^1 \times I \times I))$, so each $\partial F'$ is sent to a generator of $\pi_1(S^1 \times I \times I)$. Take an embedded disk $D = \{0\} \times I \times I \subset (S^1 \times I \times I)$. We have then $(D, \partial D) \to (S^1 \times I \times I, \partial(S^1 \times I \times I))$ and $D \cap F'$ is the union of circles and annuli. Take the innermost circle of $D \cap F'$; if it is $\sim 0$ in $F'$, we have a disk for the surgery, otherwise move $D$ by isotopy to eliminate it, and so on. After treating all the circles either we have a disk for the surgery, or there are only arcs in $D \cap F'$. Take two components of $\partial F$ corresponding to the arc whose ends are neighbouring in $\partial D$, denote them by $a$ and $a^{-1}$. Then $F'$ can be presented as $F' = A \cup F''$ where $A$ is an annulus with a hole, $\partial A = a \cup a^{-1} \cup \gamma$ and $F''$ is the remaining part.

![Figure 4: $F'$ as union of an annulus with a hole and $F''$.](image)

Denote by $A'$ the annulus in $\partial(S^1 \times I \times I)$ lying between $a$ and $a^{-1}$. Note that $\gamma \sim aa^{-1} \sim 0$ in $\pi_1(S^1 \times I \times I)$ (but $\gamma \sim 0$ in $F'$ as $F' \neq S^1 \times I \times I$). The arcs $A \cap D$ and $A' \cap D$ bound a disk in $D$, whose interior does not intersect $F'$.

![Figure 5: $D \cap F'$ in $D$ and $D$ in $S^1 \times I \times I$.](image)

Push $A \cup A'$ along its normal bundle toward the inside of this disk, living $\gamma$ unchanged, and do the surgery of the pushed part (which is a torus with a hole) on the interior disk. We obtain a disk $\tilde{D}$ embedded in $S^1 \times I \times I$ such that $\partial \tilde{D} = \gamma$ and $\tilde{D} \cap F' = \gamma$. Hence we can move $f$ by
Then \( \alpha \frac{\partial F}{f} \) allows to change \( F \) (by homotopy in order to do the surgery of \( F \) such that \( \alpha(\partial D) \)).

Case 2: \( F' \cap G' \neq \emptyset \) is a 1-dim manifold (embedded in both \( F' \) and \( G' \)); recall that \( G' \) was chosen such that \( (F'_s \cap G'_s = \emptyset \leftrightarrow \partial F' \cap \partial_1 G' = \emptyset) \).

Case 2.1: \( F' \cap G' \neq \emptyset \) but \( F' \cap \partial_1 G' = \emptyset \). Then \( \partial F' \) is a union of circles parallel to \( \partial_1 G' \); and \( F' \cap G' \) is a closed 1-manifold. Circles of \( F' \cap G' \) that are trivial in \( \pi_1(F') \), can be eliminated by moving \( G' \) (without moving \( \partial_1 G' \)). So, if all the circles of \( F' \cap G' \) are trivial in \( \pi_1(F') \), we are back to the Case 1. If there is a circle of \( F' \cap G' \) that is non-trivial in \( \pi_1(F') \), it is either parallel to a component of \( \partial F' \) or not.

• If there exists a circle not parallel to a component of \( \partial F' \), then it is embedded in \( G' \) and not parallel to its boundary, so it bounds a disk in \( G' \). We can do the surgery on this disk (choosing the innermost one).

• If all circles of \( F' \cap G' \) are parallel to a component of \( \partial F' \), then cutting \( Z' \) along \( G' \), we'll have some number of \( \pi_1 \)-injective annuli embedded in \( S^1 \times I \times I \), boundary going to boundary, hence isotopic to annuli in \( \partial(S^1 \times I \times I) \) by an isotopy that is trivial on the boundary. Use these isotopies in \( Z' \) to move \( G' \) in order to separate it from \( F' \), and we are again in the settings of the Case 1.

Case 2.2: \( F' \cap G' \neq \emptyset \) and \( \partial F' \cap \partial_1 G' \neq \emptyset \). If the components of \( \partial F' \) are homotopic to \( \partial_1 G' \), move \( G' \) by isotopy to separate \( \partial_1 G' \) from \( \partial F' \), and we are in the previous case. If the curves of \( \partial F' \) are not homotopic to \( \partial_1 G' \), choose an embedded curve \( \alpha \subset \partial_1 Z' \), parallel to the curves \( \partial F' \) (different from them) and make \( \alpha \times I \subset Z' \) be the new \( G' \). Then, we are back to Case 2.1.

Step 3: the image of the singular set \( Z_s \) can be separated from \( M \). Now every connected component of \( F' \) is a torus or an annulus. Let us show that all the annuli can be eliminated. To simplify the notations, \( F \) will stand for \( F \) without tori-components.

Fix a decomposition of \( Map^n(S^1, S^1) \) by a wedge of intervals as follows:

![Figure 6: Decomposition of Map^n(S^1, S^1) on sheets.](image)

After multiplying by \( S^1 \), it gives a wedge of annuli \( \bigvee S_i \) (identified along one boundary component) embedded in \( Z \) and decomposing \( Z \) into sheets \( I \times I \times S^1 \), in which the corresponding parts of \( F \) are manifolds. One generator of \( \pi_1(Z_s) \) is given by the singular circle of \( Map^n(S^1, S^1) \). The loop \( (\bigvee S_i) \cap Z_s \) corresponds to the second generator, and can be chosen not to be parallel to \( F_s \). Then every connected component of \( F \) intersects a decomposing annulus \( S_i \) by arcs. Take an arc coming from a component \( F_1 \subset F' \), and suppose it is the innermost one in \( S_i \) (i.e. its union with an arc from \( \partial S_i \) bounds a disk in \( S_i \) that does not contain other arcs from \( F \cap S_i \)); denote it by \( \alpha \subset F_1 \cap S_1 \). Then the arcs \( F_1 \cap S_j \) are the innermost ones in \( S_j \forall j \neq i \). The product of \( I \) with the union of the corresponding disks in \( S_1 \) and \( S_2 \) (see Fig. 7 below)

![Figure 7: Disks in the sheets.](image)

allows to change \( f \) by homotopy so that the new image of the singular set \( Z_s \) coincide with the image of \( I \times (F_1 \cap S_1) \). Hence by pushing off along the normal bundle of \( M \), \( \alpha \times I \) can be
As in dim 3, on the fundamental group level the singular fibers create roots of the regular fiber.

2 Application to 4-dim graph-manifolds

2.1 Seifert manifolds

Following [6, 7], we say that an orientable 4-manifold $S$ with boundary is a Seifert manifold if it has the structure of a fibered orbifold $\pi : S \to B$ over a 2-orbifold with generic fiber $T^2$, and $S$ and $\partial S$ are non-singular as orbifolds. Note that $B$ as orbifold has no boundary, but the underlying surface of $B$ does.

Local picture.

Any point $b \in B$ has a neighborhood of type $D = D^2/G$ where $G$ is a finite subgroup of $\text{O}(2)$ corresponding to the stabilizer of $p$. Then $\pi^{-1}(D) = (T^2 \times D^2)/G$ where the action of $G$ is free and is a lifting of an action of $G$ on $D^2$ (that is $\pi|_{\pi^{-1}(D)} : (T^2 \times D^2)/G \to D^2/G$ is induced by the canonical projection $T^2 \times D^2 \to D^2$).

Case 0: $G = \{1\}$.

In that case $p$ is non-singular and $\pi^{-1}(p) = T^2$ is a regular fiber. The preimage of an arc in $B$ that joins $p$ with $\partial B$ is $T^2 \times I$.

Case 1: $G = \mathbb{Z}_m = \{ g \mid g^m = 1 \}$ and the action is given by $g(x, y, z) = (x-a/m, y-b/m, ze^{2\pi i/m})$, where $(x, y) \in T^2$, $z \in D^2 = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$, and $a, b$ being mutually relatively prime. Then $p$ is an arc point of angle $2\pi/m$ and $\pi^{-1}(p) = T^2$. The covering (regular fiber) $\to$ (singular fiber) corresponds to the stabilizer of $p$ with boundary is a manifold $S_1 \times (S_1 \times D^2)$ beeing trivial because $(T^2 \times D^2)/G$ is orientable). That means that $(T^2 \times D^2)/G = S_1 \times (3$-dimensional model). So in this case, the preimage of an arc in $B$ that joins $p$ with $\partial B$ is the $\mathbb{Z}_m$-manifold $S_1 \times Map^n(S_1, S_1)$.

Case 2: $G = \mathbb{Z}_2 = \{ \tau \mid \tau^2 = 1 \}$ and the action is given by $\tau \cdot (x, y, z) = (x + 1/2, -y, \bar{z})$. Then $\pi^{-1}(p) = K^2$, $p$ lies on a reflector circle and $\pi^{-1}(D)$ is a twisted $D^2$-bundle over $K^2$. The preimage of an arc in $B$ that joins the point $p$ with $\partial B$ is in this case a manifold $K^2 \times I$.

Case 3: $G = D_{2m} = \{ \tau, g \mid \tau^2 = g^m = 1, \ \tau g \tau^{-1} = g^{-1} \}$ and the action is given by $\tau \cdot (x, y, z) = (x + 1/2, -y, \bar{z})$ and $g \cdot (x, y, z) = (x, y-b/m, ze^{2\pi i/m})$. Then $p$ is a corner reflector of angle $\pi/m$ and $\pi^{-1}(p) = K^2$ which is $m$-fold covered by $K^2$ over a regular point of the reflector circle.

The preimage of an arc in $B$ that joins $p$ with $\partial B$ is the mapping cylinder of a covering $T^2 \to K^2$ which is a 2-fold covering over the loop reversing the orientation of $K^2$ and a $m$-fold covering over the other loop. This is not a $\mathbb{Z}_k$-manifold, but one can join $p$ with $\partial B$ by 2 consecutive arcs $\alpha$ and $\beta$, the first one lying on the reflector circle and not containing another corner reflector, the second one joining the other end of $\alpha$ with $\partial B$. Then $\pi^{-1}(\alpha) = S_1 \times Map^n(S_1, S_1)$ and $\pi^{-1}(\beta) = I \times K^2$.

As in dim 3, on the fundamental group level the singular fibers create roots of the regular fiber.

$S_1$ and $S_2$ can be pushed away from $S_3$.) Taking the product of the union of disks in $S_k$ and $S_{k+1}$ by $I$ and doing the same homotopy for the sheet lying between $S_k$ and $S_{k+1}$, one will eliminate all the annuli of $F$. \hfill \box
2.2 4-dimensional graph-manifolds

A block is a Seifert bundle (with boundary) over a hyperbolic 2-orbifold. A graph-manifold structure on a compact closed oriented 4-manifold is a decomposition as a union of blocks, glued by diffeomorphisms of the boundary. Note that the boundary of a block has the structure of a $T^2$-bundle over a circle. A graph-manifold structure is reduced if none of the glueing maps are isotopic to fiber-preserving maps of $T^2$-bundles. Any graph-structure give rise to a reduced one by forming blocks glued by bundle maps into larger blocks. Like in the non-singular case, 4-dimensional graph-manifolds are aspherical, their Euler characteristic is 0 (because the blocks are finitely covered by $T^2$-bundles over hyperbolic surfaces, hence have $\chi = 0$, and the glueings are made along 3-manifolds), and can be smoothed.

Theorem 1. Any homotopy equivalence of closed oriented 4-manifolds with reduced graph-structures is homotopic to a diffeomorphism preserving the structures.

We will say that a $\pi_1$-injective map between the blocks $f : S \to S'$ is fiber covering if in the fundamental groups it sends the (normal) fiber subgroup of $\pi_1(S)$ into the (normal) fiber subgroup of $\pi_1(S')$. To extend the proof of the non-singular case ([4]), one has to show that any $\pi_1$-injective map $f : W = \cup W_i \to W' = \cup W'_i$ of graph-manifolds with reduced graph-structures is homotopic to $\bigcup f_i$, where each $f_i : (W_i, \partial W_i) \to (W'_j, \partial W'_j)$ is a fiber covering map. The missing step is the following.

Proposition 1. Let $S$ be a block and $W$ be a 4-dim graph-manifold with reduced graph-structure. Then any $\pi_1$-injective map $f : S \to W$ is homotopic to a map into one block of $W$.

Proof. In the base of $\pi : S \to B$, take the wedge of circle on which $B$ retracts. Join its core by arcs with every cone point and every reflector circle. As $S$ retracts on

$$S' = \pi^{-1}(\text{wedge of circles + wedge of arcs + reflector circles}),$$

we have to show that $f|_{S'}$ is homotopic to a map into one bloc of $W$.

Step 1: $f|_{\pi^{-1}(\text{wedge of circles})}$ is homotopic to a map in one block of $W$. To make the reasonnings as in the singular case, one only needs to show that up to homotopy rel boundary, any $\pi_1$-injective map $g : (T^2 \times I, \partial(T^2 \times I)) \to (S, \partial S)$ is either fiber-covering or is a map into $\partial S$. Note that $S$ is finitely covered by $\tilde{S}$ which is a $T^2$-bundle over a hyperbolic surface (with boundary). As $g$ sends boundary to boundary, $\text{Im } g_* \subset \pi_1(\tilde{S}) \subset \pi_1(S)$, hence $g$ lifts to a map $\tilde{g} : (T^2 \times I, \partial) \to (\tilde{S}, \partial \tilde{S})$, to which Lemma 2 ([4]) applies.

Step 2: $f|_{\pi^{-1}(\text{wedge of arcs + reflector circles})}$ is homotopic to a map in one block of $W$. Denote by $M$ the block of $W$ in which $f(\pi^{-1}(\text{wedge of circles}))$ lies and denote $p$ the point of $B$ which is the common core of wedge of circles and wedge of arcs. Then $f|_{\pi^{-1}(p)}$ is a covering of the regular fiber of $M$.

Step 2.1: arcs to the cone points. The preimage of such an arc is $Z = S^1 \times Map^p(S^1, S^1)$, and $f|Z$ sends $\partial Z$ into $M$, hence $f(\partial Z)$ does not intersect the decomposing submanifolds of $W$. 

$S = M \cup M_S$, where $M$ is a $T^2$-bundle over a surface and $M_S$ is its Seifert-part. Note that $0 \to \pi_1(M) \to \pi_1(S)$. Theorem ([9], [8], [6], [7].) Let $S, S'$ be 4-dimensional closed orientable Seifert bundles with hyperbolic bases. Then $\pi_1(S) = \pi_1(S')$ if and only if there is a fiber-preserving diffeomorphism between $S$ and $S'$.

By taking the doubles, the previous theorem implies that if two Seifert manifolds with boundaries $(S, \partial S)$ and $(S', \partial S')$ over hyperbolic 2-orbifolds are homotopy equivalent rel boundary then $S$ and $S'$ are diffeomorphic.
Then \( f \) is \( \pi_1 \)-injectively embedded in \( Z = S^1 \times S^1 \times I \). Hence \( (f|Z)^{-1}(\bigcup M_{\varphi_i}) \) is a torus \( \pi_1 \)-injectively embedded in \( Z \) that are parallel to \( \partial Z \) and hence are sent by \( f \) on the covering of regular fiber of \( M \). Take the first \( T_2 \) from \( \bigcup T^2 \) counting from \( \partial Z \), say it comes from \( M_\varphi \subset M \cap M' \). If it exists, denote the next torus from \( \bigcup T^2 \) by \( T_2' \). The restriction of \( f \) on \( T^2 \times I \subset Z \) lying between \( T_1^2 \) and \( T_2^2 \) is a \( \pi_1 \)-injective map \( f_1 : (T^2 \times I, \partial) \rightarrow (M', \partial M') \). As \( f(T_2^2) \) is a covering of the regular fiber of \( M \) and the graph-structure is reduced, \( f(T_2^2) \) is not a covering of the regular fiber of the neighbouring block \( M' \), hence \( f(T_2^2) \subset M_\varphi \) and \( f_1 \) is homotopic rel boundary to a map into \( M_\varphi \). Continuing like this, we change \( f|Z \) so that \( (f|Z)^{-1}(\bigcup M_{\varphi_i}) \) contains at most one manifold component. If there are none, we are done. If there is one such component, we have a \( \pi_1 \)-injective map \( f : (Z, \partial Z, Z_a) \rightarrow (M \cup M', M, M') \).

\[ \begin{array}{cc}
\gamma^n \beta & \gamma^n \\
\gamma \beta & \gamma \end{array} \]

Figure 8: \( f(S^1 \times Map^n(S^1, S^1)) \) in the blocs \( M \) and \( M' \)

As the torus \( (f|Z)^{-1}(\bigcup M_{\varphi_i}) \) is parallel to \( \partial Z \), we have a \( \pi_1 \)-injective map \( f' : (Z, \partial Z) \rightarrow (M', \partial M') \), such that \( f'(\partial Z) \) is not fiber-covering.

**Case 1:** \( f' \) lifts to a map into \( M' \). If the torus \( f'(\partial Z) \) corresponds to a \( (Z \oplus Z) \)-subgroup \( \langle \alpha^n, \beta \rangle \) in \( \pi_1(W) \), then \( f'(Z_a) \) corresponds to the \( (Z \oplus Z) \)-subgroup \( \langle \alpha, \beta \rangle \). Denote \( p' \) the projection of \( M' \). If \( p'_a(\alpha^n) = 0 \) then \( p'_a(\alpha^n) = 0 \), too because \( \pi_1(B') \) is free. Hence \( \alpha \) is homotopic to a loop in the fiber of \( M' \). If \( p'_a(\alpha^n) \neq 0 \) then \( p'_a(\alpha) \neq 0 \), and we can apply

**Fact 1.** Let \( B \) be a surface with boundary, \( S \) a component of \( \partial B \). Then the image \( \pi_1(S) \rightarrow \pi_1(B) \) is root closed if and only if \( B \) is not a Möbius band and roots are squared.

**Proof.** First note that by \([3]\) roots are unique in free groups. If \( \partial B \) has more than one component, then \( S \) corresponds to a primitive element of \( \pi_1(B) \), and the statement comes from \([3]\) \( (a^k = s^k \) implies that \( a, s \) are powers of a common element, and as \( a \) is primitive, \( s \) is a power of \( a \)).

Now suppose \( \partial B \) has one component. Denote \( M = Map^k(S^1, S^1) \). Suppose there exists \( a \in \pi_1(B) \) such that \( a \notin \pi_1(\partial B) \) and \( a^k \in \pi_1(\partial B) \). It gives a \( \pi_1 \)-injective map \( (M, \partial M) \rightarrow (B, \partial B) \) such that \( \pi_1(M) \rightarrow a \) and \( \pi_1(\partial M) \rightarrow a^k \). Attach disks to \( M \) and \( B \) to get a \( \pi_1 \)-injective map \( M \cup_{\partial M} D^2 \rightarrow B \cup_{\partial B} D^2 \). As \( a^k \) is a finite power of a generator of \( \pi_1(B) \), \( \pi_1(M \cup_{\partial M} D^2) \rightarrow \pi_1(B \cup_{\partial B} D^2) \) is of finite index. As \( \pi_1(M \cup_{\partial M} D^2) = \mathbb{Z}_k \) and \( (B \cup_{\partial B} D^2) \) is closed, \( (B \cup_{\partial B} D^2) \sim \mathbb{R}^2 \), because \( \mathbb{R}^2 \) is the only closed surface with finite non-trivial \( \pi_1 \). Hence \( k = 2 \) and \( B \) is the Möbius band.

Hence if \( p'(\alpha^n) \) is homotopic to a loop in \( \partial B' \), then so is \( p'(\alpha) \). Hence \( \alpha \) is homotopic to a loop in \( \partial M' \), and \( f' \) is homotopic to a map into \( \partial M' \).

**Case 2:** \( f' \) does not lift to a map into \( M' \). Then \( \alpha^n \) have roots on the singular fibers of \( M' \), hence \( \alpha^n \) is homotopic to a loop in the regular fiber of \( M' \). Denote the Seifert projection of \( M' \) by \( \pi' \). As the graph-structure is reduced and \( \langle \alpha^n, \beta \rangle \) corresponds to the regular fiber of \( M \), \( \pi'_a(\alpha^n, b) \neq 0 \) hence \( \pi'_a(\beta) \neq 0 \). Which means that \( \alpha \) cannot commute with \( \beta \) because the roots of the regular fiber do not commute with the elements of the base.

**Step 2.2:** the image of a reflector circle without corner reflectors lies in one block.

The preimage of the reflector circle is twisted \( K^2 \)-bundle over \( S^1 \). Applying the previous step to the preimage of the arc between \( \partial B \) and the reflector circle, we see that \( f \) restricted to the fiber of this bundle is homotopic to a map into the block \( M \). Take in \( B \) a loop \( \gamma \) near the reflector.
covers the regular fiber of the block $M$, hence the whole $f(M \pm id)$ can be shrunk by homotopy into $M$, in particular $f(\gamma)$ does. Hence the image of the whole 1-skeleton of $\pi^{-1}(\text{reflector circle})$ can be shrunk by homotopy into $M$, and the asphericity implies the statement.

**Step 2.3 : case of reflector circles having corner reflectors.** The preimage of an arc on a reflector circle between a corner reflector and a neighboring point is $\hat{Z} = S^1 \times \text{Map}^a(S^1, S^1)$. Denote as before $\partial \hat{Z} = S^1 \times \partial(\text{Map}^a(S^1, S^1))$ and $\hat{Z}_s = S^1 \times (\text{Map}^a(S^1, S^1))_s$. The preimage of an arc in $B$ between the reflector circle and the boundary is $I \times K^2$, and by previous its image by $f$ can be shrunk into $M$. Hence the image of the whole 1-skeleton of $\pi^{-1}(\text{reflector circle})$ can be shrunk by homotopy into $M$, and the asphericity implies the statement.

Let us show that $f(\hat{Z})$ such that $f(\partial \hat{Z}) \subset M$ can be shrunk into $M$. The double covering $p : Z \to \hat{Z}$ induces a $\pi_1$-injective map $fp : Z \to W$ which by previous can be shrunk into $M$ (by a homotopy which is constant on $\partial Z$). Let us show that the whole mapping cylinder of $p$, $\text{Map}_p(Z, \hat{Z})$ can be shrunk into $M$. For this, note that the mapping cylinders of the 2-fold coverings $p|_{\partial Z}$ and $p|_Z$ are $(I \times K^2)$’s with boundaries lying in $M$, hence both can be shrunk into $M$. It remains to remark that the union of these mapping cylinders with $Z$ contains a 1-skeleton of $\text{Map}_p(Z, \hat{Z})$.

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