JACOBIAN FIBRATIONS ON THE SINGULAR K3 SURFACE OF DISCRIMINANT 3

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Abstract. In this paper we give the Weierstrass equations and the generators of Mordell-Weil groups for Jacobian fibrations on the singular K3 surface of discriminant 3.

1. Introduction

A K3 surface defined over the complex number field whose Picard number equals to maximum possible number 20 is called a singular K3 surface. Shioda and Inose [10] showed that the map a singular K3 surface $X$ corresponds to its transcendental lattice $T_X$ is a bijective correspondence from the set of singular K3 surfaces onto the set of equivalence classes of positive-definite even integral lattice of rank two with respect to $SL_2(\mathbb{Z})$. The discriminant of a singular K3 surface $X$ is the determinant of the Gram matrix of the transcendental lattice $T_X$.

In this paper we study Jacobian fibrations, i.e., elliptic fibrations with a section, on the singular K3 surface $X_3$ of discriminant 3, which corresponds to the lattice defined by

$\begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix}$

and is uniquely determined up to isomorphism. Jacobian fibrations on $X_3$ were classified by Nishiyama [8]. He classified all configurations of singular fibers of Jacobian fibrations on $X_3$ into 6 classes and determined their Mordell-Weil groups. Then, we give for each fibration a Weierstrass model. More precisely, we state our main theorem.

Theorem 1. Let $X_3$ be the singular K3 surface of discriminant 3. For each Jacobian fibration in Nishiyama’s list [8, Table 1.1], an elliptic parameter $u_i$, a Weierstrass equation and the generators of the Mordell-Weil group are given by Table 1.

We explain about Table 1. The first column shows the name of each Jacobian fibrations following Nishiyama’s notaion. The second column shows the configuration of singular fibers. Here, for example, by $2\Pi^* + 4$ means that the surface has two singular fibers of type $\Pi^*$ and a singular fiber of $\Pi$ type $IV$ (Kodaira’s notation [4]). The third column shows the Mordell-Weil group (MWG) of the fibration. The fourth column shows an elliptic parameter $u_i$ of the fibration under the singular affine model (2.6) of $X_3$. The index $i$ is the name of the fibration. The last column shows a Weierstrass equation and rational points corresponding to Mordell-Weil generator of the fibration, where $O$ is the rational point corresponding to the zero of MWG.

Recently, Braun, Kimura and Watari [2] showed that Nishiyama’s list also gives the classification of Jacobian fibrations on $X_3$ modulo isomorphism. Thus, our and their results answer completely a question of Kuwata and Shioda [7].

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2. Notation

The singular K3 surface $X_3$ is known as a generalized Kummer surface constructed by the following. Let $C_\omega$ be the complex elliptic curve with the fundamental periods 1 and $\omega = e^{2\pi \sqrt{-1}/3}$. Let $\sigma$ be an automorphism of $C_\omega \times C_\omega$ defined by $\sigma(z_1, z_2) \mapsto (\omega z_1, \omega^2 z_2)$. Then the minimal resolution of the quotient $C_\omega \times C_\omega / \langle \sigma \rangle$ is isomorphic to the singular K3 surface $X_3$ (see [10, Lemma 5.1]). The automorphism $\sigma$ has the 9 fixed points $(v_i, v_j)$ ($1 \leq i, j \leq 3$), where $\{v_i\}$ are the fixed points of the automorphism $\sigma_1$ of $C_\omega$ defined by $\sigma_1(z) = \omega z$. These 9 points $(v_i, v_j)$ correspond to the singular points $p_{ij}$ of the quotient $C_\omega \times C_\omega / \langle \sigma \rangle$. The minimal resolution $X_3$ of $C_\omega \times C_\omega / \langle \sigma \rangle$ is obtained by replacing each $p_{ij}$ by 2 non-singular rational curves $E_{i,j}$ and $E'_{i,j}$ with $E_{i,j} : E'_{i,j} = 1$. Moreover, $X_3$ contains 6 non-singular rational curves, i.e. the image $F_i$ (or $G_j$) of $\{v_i\} \times C_\omega$ (or $C_\omega \times \{v_j\}$) in $X_3$. We have the following intersection numbers.

\begin{align}
F_i^2 = G_i^2 = E_{i,j}^2 = E'_{i,j}^2 = -2, & \quad F_i \cdot E_{j,k} = G_i \cdot E'_{j,k} = F_i \cdot G_j = 0,
E_{i,j} \cdot E'_{i,j} = \delta_{i,k} \cdot \delta_{j,l}, & \quad F_i \cdot E'_{j,k} = G_i \cdot E_{k,j} = \delta_{i,j}.
\end{align}

These 24 curves on $X_3$ form the configuration of Figure 1.

It is well known that the elliptic curve $C_\omega$ has the following Weierstrass form

\begin{equation}
C_\omega : y^2 = x^3 + 1.
\end{equation}

We denote each factor of $C_\omega \times C_\omega$ by

\begin{align}
C^1_\omega : y_1^2 = x_1^3 + 1, & \quad C^2_\omega : y_2^2 = x_2^3 + 1.
\end{align}

Then the automorphism $\sigma$ is written by

\begin{align}
\sigma : C^1_\omega \times C^2_\omega \rightarrow C^1_\omega \times C^2_\omega & \quad (x_1, y_1, x_2, y_2) \mapsto (\omega x_1, y_1, \omega^2 x_2, y_2).
\end{align}
The function field $\mathbb{C}(X_3)$ is equal to the invariant subfield of the function field $\mathbb{C}(C_\omega^1 \times C_\omega^2) = \mathbb{C}(x_1, x_2, y_1, y_2)$ under the automorphism $\sigma$. Then we have

\begin{equation}
\mathbb{C}(X_3) = \mathbb{C}(y_1, y_2, t), \quad t = x_1 x_2,
\end{equation}

where $y_1, y_2,$ and $t$ are naturally regarded as functions on $X_3$ with the relation

\begin{equation}
t^3 = (y_1^2 - 1)(y_2^2 - 1).
\end{equation}

This gives a singular affine model of $X_3$. We start from the equation to obtain a Weierstrass form for each Jacobian fibration on $X_3$. Under the above notation, we
see that the divisor of typical functions are as follows.

(2.7)

\[(y_1 - 1) = 3F_2 + 2(E'_{2,1} + E'_{2,2} + E'_{2,3}) + E_{2,1} + E_{2,2} + E_{2,3}
- (3F_1 + 2 (E'_{1,1} + E'_{1,2} + E'_{1,3}) + E_{1,1} + E_{1,2} + E_{1,3})\]

\[(y_1 + 1) = 3F_3 + 2(E'_{3,1} + E'_{3,2} + E'_{3,3}) + E_{3,1} + E_{3,2} + E_{3,3}
- (3F_1 + 2 (E'_{1,1} + E'_{1,2} + E'_{1,3}) + E_{1,1} + E_{1,2} + E_{1,3})\]

\[(y_2 - 1) = 3G_2 + 2(E_{1,2} + E_{2,2} + E_{3,2}) + E'_{1,2} + E'_{2,2} + E'_{3,2}
- (3G_1 + 2 (E_{1,1} + E_{2,1} + E'_{1,1}) + E'_{1,1} + E'_{2,1} + E'_{3,1})\]

\[(y_2 + 1) = 3G_3 + 2(E_{1,3} + E_{2,3} + E_{3,3}) + E'_{1,3} + E'_{2,3} + E'_{3,3}
- (3G_1 + 2 (E_{1,1} + E_{2,1} + E_{3,1}) + E'_{1,1} + E'_{2,1} + E'_{3,1})\]

\[(t) = F_2 + E'_{2,3} + E_{2,3} + G_3 + E_{3,3} + E'_{3,3} + F_3 + E'_{3,2} + E_{3,2} + G_2 + E_{2,2} + E'_{2,2}
- (E_{2,1} + E_{3,1} + 2 (G_{1} + E_{1,1} + E'_{1,1} + F_{1}) + E'_{1,2} + E'_{1,3})\]
3. Fibration 1

An elliptic parameter for Fibration 1 is given by

\[(3.1) \quad u_1 = \frac{2(y_1 + 1)}{(y_1 - 1)^2}.\]

The divisor of \(u_1\) is given by

\[(3.2) \quad (u_1) = E'_{3,3} + 2E_{3,3} + 3G_3 + 4E_{1,3} + 5E'_{1,3} + 6F_1 + 3E'_{1,2} + 2E_{1,2} - (E'_{3,1} + 2E_{3,1} + 3G_1 + 4E_{2,1} + 5E'_{2,1} + 6F_2 + 3E'_{2,3} + 4E'_{2,2} + 2E_{2,2}).\]

The zero divisor \((u_1)_0\) (the bold lines in Figure 2) and the polar divisor \((u_1)_\infty\) (the thin lines in Figure 2) are the singular fibers both of type \(\Pi^*\).

Eliminating the variable \(y_2\) from (2.6) and (3.1), we obtain the following equation

\[(3.3) \quad 4t^3 = u_1(y_1 + 1)(y_1 - 1)^3(u_1y_1^2 - 2u_1y_1 + u_1 - 4),\]

which defines a plane curve over \( \mathbb{C}(u_1) \) with a singularity at \((t, y_1) = (0, 1)\). Blowing up by \(t = v(y_1 - 1)\), we have the following equation

\[(3.4) \quad 4v^3 = u_1(y_1 + 1)(u_1y_1^2 - 2u_1y_1 + u_1 - 4),\]

which defines a nonsingular plane cubic curve over \( \mathbb{C}(u_1) \) with a rational point \((v, y_1) = (0, -1)\). Then we can convert it into a Weierstrass form (see Figure 1 or 3). Since the rational point \((v, y_1) = (0, -1)\) corresponds to the divisor \(F_3\) (the dotted line in Figure 2), choosing it as the zero section of the group structure, we obtain the Weierstrass equation for Fibration 1

\[(3.5) \quad Y^2 = X^3 + u_1^5(u_1 - 1)^2,\]

where the change of variables is given by

\[(3.6) \quad X = \sqrt[3]{4(u_1 - 1)u_1t} , \quad Y = -\frac{u_1^2(u_1 - 1)(u_1y_1 - u_1 + 2)}{y_1 + 1}.\]

Besides the two singular fibers of type \(\Pi^*\) at \(u_1 = 0\) and \(\infty\), there is one singular fiber of type IV at \(u_1 = 1\). It is the divisor \(E_{3,2} + E'_{3,2} + Q_1\) (the long dashed dotted lines in Figure 2), where \(Q_1\) is a \((-2)\)-curve on \(X_3\) arising from a curve on \(\mathbb{P}^1 \times \mathbb{P}^1\) below.

Let \(p_j : C^j_{\omega} \to \mathbb{P}^1 (j = 1, 2)\) be the projection given by

\[(3.7) \quad p_j : C^j_{\omega} \quad \to \quad \mathbb{P}^1\]

\[x_j : y_j : z_j \quad \mapsto \quad \left\{ \begin{array}{ll} (y_j : z_j) & \text{if } z_j \neq 0 \\ (1 : 0) & \text{if } z_j = 0. \end{array} \right.\]

Then the map \(p_1 \times p_2 : C^1_{\omega} \times C^2_{\omega} \to \mathbb{P}^1 \times \mathbb{P}^1\) factors through \(\overline{\pi} : C^1_{\omega} \times C^2_{\omega} / \sigma \to \mathbb{P}^1 \times \mathbb{P}^1\). Let \(\overline{\pi}\) be the morphism of degree three from \(X_3\) to \(\mathbb{P}^1 \times \mathbb{P}^1\) that makes the following diagram commutative:

\[
\begin{array}{ccc}
X_3 & \xrightarrow{\pi} & \mathbb{P}^1 \times \mathbb{P}^1 \\
\downarrow & & \\
C^1_{\omega} \times C^2_{\omega} & \xrightarrow{\overline{\pi}} & C^1_{\omega} \times C^2_{\omega} / \sigma
\end{array}
\]
It is easy to verify that the equation \( u_1 = 1 \) means
\[
y_1^2 - 2y_1 - 2y_2 - 1 = 0
\]
from (3.1). This equation defines a curve on \( \mathbb{P}^1 \times \mathbb{P}^1 \). Then it lifts to the \((-2)\)-curve \( Q_1 \) on \( X_3 \) via the map \( \pi \).

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Fibration 1}
\end{figure}

4. Fibration 3

An elliptic parameter for Fibration 3 is given by
\[
u_3 = \frac{t}{y_1^2 - 1}.
\]
The divisor of \( u_3 \) is given by
\[
u_3 = G_2 + 2E_{1,2} + 3E_{1,2} + 4F_1 + 3E_{1,1} + 2E_{1,3} + G_3 + 3E_{1,2} - (E_{2,2} + E_{2,3} + 2(F_2 + E_{2,1} + E_{2,1} + G_1 + E_{3,1} + E_{3,1} + F_3) + E_{3,2} + E_{3,3}),
\]
which is indicated in Figure 3. The zero divisor \( (u_3)_0 \) is the singular fiber of type III* (the bold lines) and the polar divisor \( (u_3)_\infty \) is the singular fiber of type I* (the thin lines). The curves \( E_{2,2}, E_{2,3}, E_{3,2} \) and \( E_{3,3} \) (the dotted lines) are all the sections.

Eliminating the variable \( t \) from (2.6) and (4.1), we have the following equation
\[
y_2^2 = u_3^3(y_1^2 - 1)^2 + 1,
\]
which has a rational point \((y_1, y_2) = (1, 1)\) corresponding to the curve \( E_{2,2} \). Thus, choosing \( E_{2,2} \) as the zero section of the group structure, we obtain the Weierstrass equation for Fibration 3
\[
Y^2 = X^3 + 4u_3^3X^2 - 4u_3^3X,
\]
where the change of variables is given by

\[(4.5) \quad X = \frac{2(y_2 + 1)}{(y_1 - 1)^2}, \quad Y = \frac{4(w_3^3(y_1 + 1)(y_1 - 1)^2 + y_2 + 1)}{(y_1 - 1)^3}.
\]

Besides the above two singular fibers of types III∗ and I∗ 6, the fibration has three I1 fibers at \( u_3 = -1, -\omega \) and \(-\omega^2\).

The 2-torsion rational point \((X, Y) = (0, 0)\) corresponds to the curve \(E_{3,3}\). The rational point \((X, Y) = (1, -1)\) corresponds to the curve \(E_{3,2}\) of height \(\langle E_{3,2}, E_{3,2} \rangle = \frac{3}{2}\), which is a generator of the Mordell-Weil lattice of the fibration. The curve \(E_{2,3}\) is another free section corresponding to the rational point \((1, 1)\) with the relation \(E_{2,3} = -E_{3,2}\) in the Mordell-Weil group.

![Figure 3. Fibration 3](image)

5. Fibration 5

An elliptic parameter for Fibration 5 is given by

\[(5.1) \quad u_5 = y_1.
\]

It is clear that this elliptic parameter defines a fibration having three singular fibers all of types IV∗ at \( u_5 = 1, -1 \) and \( \infty \) (the bold lines in Figure 4 from (2.7)). Furthermore the fibration is induced by the composition of the first projection \(C_1^1 \times C_2^1 \to C_1^1\) and the covering map of degree three \(p_1 : C_1^1 \to \mathbb{P}^1\) in (3.7).

The following simple coordinate change

\[(5.2) \quad X = (u_5^2 - 1)t, \quad Y = (u_5^2 - 1)^2y_2
\]

converts the equation (2.6) into the Weierstrass equation for Fibration 5

\[(5.3) \quad Y^2 = X^3 + (u_5^2 - 1)^4.
\]

The curve \(G_1, G_2\) and \(G_3\) correspond to the zero section, 3-torsion rational points \((0, (u_5^2 - 1)^2)\) and \((0, -(u_5^2 - 1)^2)\), respectively (the dotted lines in Figure 4).
6. Fibration 6

An elliptic parameter for Fibration 6 is given by

\[(6.1) \quad u_6 = t.\]

Since we gave the divisor of \(t\) in (2.7), we know that the zero divisor \((u_6)_0\) is the singular fiber of type \(I_{12}\) (the bold lines in Figure 5) and the polar divisor \((u_6)_\infty\) is the singular fiber of \(I_3^*\) (the thin lines in Figure 5). The curves \(E_{1,2}, E_{1,3}, E'_{2,1}\) and \(E'_{3,1}\) (the dotted lines in Figure 5) are all the sections. Choosing \(E_{1,2}\) as the zero section of the group structure, we obtain the Weierstrass equation for Fibration 6

\[(6.2) \quad Y^2 = X^3 - 2(u_6^3 - 2)X^2 - u_6^6X,\]

where the change of variables is given by

\[(6.3) \quad X = \frac{t^3(y_2 + 1)}{y_2 - 1}, \quad Y = \frac{2t^3y_1(y_2 + 1)}{y_2 - 1}.\]

Besides the two singular fibers of type \(I_{12}\) at \(u_6 = 0\) and of type \(I_3^*\) at \(u_6 = \infty\), there are three \(I_1\) fibers at \(u_6 = 1, \omega\) and \(\omega^2\). The Mordell-Weil group of the fibration is isomorphic to \(\mathbb{Z}/4\mathbb{Z}\). The curve \(E_{1,3}\) corresponds to the rational point \((0,0)\) of order two, and remaining curves \(E'_{2,1}\) and \(E'_{3,1}\) correspond to the rational points \((u_6^3, 2u_6^3), (u_6^3, -2u_6^3)\) of order four, respectively.

7. Fibration 4

To obtain the Weierstrass equation for Fibration 4, we use a 2-neighbor step from Fibration 3. For more detail about 2-neighbor step, we refer to [5, 9, 11].

We compute explicitly the elements of \(\mathcal{O}_{X_3}(F)\) where

\[(7.1) \quad F = E_{2,2} + G_2 + E_{1,2} + E'_{1,2} + F_1 + E'_{1,3} + E_{1,3} + G_3 + E_{3,3} + E'_{3,3} + F_3 + E'_{3,1} + E_{3,1} + G_1 + E_{2,1} + E'_{2,1} + F_2 + E'_{2,2}.\]
Figure 5. Fibration 6

Figure 6. 2-neighbor from Fibration 3 to Fibration 4

is the class of the fiber of type $I_{18}$ we are considering. The linear space $\mathcal{O}_{X_3}(F)$
is 2-dimensional, and the ratio of two linearly independent elements is an elliptic
parameter for $X_3$. Since 1 is an element of $\mathcal{O}_{X_3}$, we may find a non-constant
element of $\mathcal{O}_{X_3}(F)$. Then it will be an elliptic parameter of Fibration 4. Let us
$u'_4 \in \mathcal{O}_{X_3}(F)$ be a non-constant. The function $u'_4$ has a simple pole along $E_{2,2}$ and
$E_{3,3}$, which are the zero section and 2-torsion of Fibration 3. Also, it has a simple
pole along $G_2$, the identity component of the fiber at $u_3 = 0$, a simple pole along
$E'_{3,3}$, the identity component of the fiber at $u_3 = \infty$. Therefore we can put

$$u_4' = \frac{Y}{u_3X},$$

where the variables $u_3, X, Y$ are given by (4.4) and (4.5). Assume $A_1 = 0$, since 1 is an element of $O_{X_3}(F)$. To obtain the coefficients $A_0$ and $A_2$, we look at the order of vanishing along the non-identity components of fibers at $u_3 = \infty$. The function $u_4'$ does not have any pole along $E'_{3,2}$, which intersects with the section $E_{3,2}$ of the fibration at $u_3 = \infty$. Hence $u_4'$ has no pole at $(X, Y, u_3) = (1, -1, \infty)$, and that gives us $A_2 = 0$. Similarly, the component $E'_{2,3}$, which intersects with the section $E_{2,3}$, gives us $A_0 = 0$. Consequently, we have a new elliptic parameter

$$u_4' = \frac{Y}{u_3X},$$

where the variables $u_3, X, Y$ are given by (4.4) and (4.5). Solving for $Y$ and substituting into the Weierstrass equation (4.4), after suitable coordinate changes we have the following

$$y^2 = x^3 + \frac{1}{4}(u_4'^2 x - 16)^2.$$

Although this is a Weierstrass equation for Fibration 4, for latter calculations, we put

$$u_4' = \frac{2}{u_4}, \quad x = \frac{2^3 X}{u_4^6}, \quad y = \frac{2^4 Y}{u_4^6},$$

and obtain another Weierstrass equation for Fibration 4

$$Y^2 = X^3 + (X - u_4^6)^2.$$

The change of variables is given by

$$u_4 = \frac{t}{y_1 + y_2}, \quad X = \frac{(y_1^2 - 1)t^3}{(y_1 + y_2)^3}, \quad Y = \frac{(y_1^2 y_2 + 2y_1 + y_2)t^6}{(y_2 - 1)(y_1 + y_2)^6}.$$

The fibration has singular fibers of type $1_{18}$ at $u_4 = 0$ and of type $1_1$ at the zeros of $27u_4^6 + 4 = 0$. The zero section corresponds to the divisor $E'_{1,1}$. The 3-torsion rational points $(0, u_4^3)$ and $(0, -u_4^3)$ correspond to the divisors $E_{3,2}$ and $E'_{2,3}$, respectively. The free rational points $(2u_4^3, u_4^3 + 2u_4^3)$ and $(-2u_4^3, u_4^3 - 2u_4^3)$ correspond to the divisors $E_{3,2}$ and $E'_{2,3}$, respectively with the relation $E_{2,3} + E_{3,2} = E'_{2,3}$ in the Mordell-Weil group. Since the height of $E_{2,3}$ is equal to $\frac{3}{2}$, $E_{2,3}$ generates the Mordell-Weil lattice of the fibration.

8. Fibration 2

We obtain the following elliptic parameter $u_4'$ for Fibration 2 by a 2-neighbor step from Fibration 4 (see Figure 8).

$$u_4' = \frac{u_4^6 + X + Y}{u_4^3X}.$$

The variables $u_4, X, Y$ are given by (7.1). Then we get the following Weierstrass equation for Fibration 2.

$$y^2 = x^3 + 2(u_4'^3 - 4)x^2 + 16x.$$
We put

\begin{equation}
  u_2' = \frac{2}{u_2}, \quad x = \frac{2^2 X}{u_2}, \quad y = \frac{2^3 Y}{u_2}.
\end{equation}

and obtain another Weierstrass equation for Fibration 4.

\begin{equation}
  Y^2 = X^3 - 2(u_2^3 - 2)X^2 - u_2^8 X.
\end{equation}
The change of variables is given by

\[
    u_2 = \frac{2t^2}{(y_2 + 1)(y_1^2 + 2y_1 + 2y_2 - 1)},
\]
\[
    X = -\frac{32(y_1 - 1)^2(y_2 - 1)^3 t^2}{(y_2 + 1)^2(y_1^2 + 2y_1 + 2y_2 - 1)^4},
\]
\[
    Y = \frac{128(y_1 - 1)^3(y_2 - 1)^4(y_1 + 1)(y_1 + y_2)}{(y_2 + 1)^2(y_1^2 + 2y_1 + 2y_2 - 1)^5}.
\]

The zero divisor \((u_4)_0\) is the singular fiber of type \(I_{12}^*\) (the bold lines in Figure 9). The polar divisor \((u_4)_\infty = G_3 + E_{2,3} + Q_2\) is the singular fiber of type \(I_3\) (the thin lines in Figure 9), where the divisor \(Q_2\) is the lifting of the curve \(y_2^2 + 2y_1 + 2y_2 - 1 = 0\) on \(\mathbb{P}^1 \times \mathbb{P}^1\) by the map \(\pi\) in \(\S3\). Besides these two singular fibers, there are three \(I_1\) fibers at \(u_2 = 1, \omega\) and \(\omega^2\). The zero section corresponds to the divisor \(E_{1,3}\). The 2-torsion rational point \((0, 0)\) corresponds to the divisor \(E_{3,3}\).

**Figure 9. Fibration 2**

**Remark 2.** We give a Weierstrass equation for Fibration 6 in \(\S6\). Comparing the equations (8.4) and (6.2), we know easily that Fibration 2 is a quadratic twist of Fibration 6. This is the reason why we adopt the equation (8.4) as the Weierstrass equation for Fibration 2 rather than the equation (8.2).

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