Twisted logarithmic modules of vertex algebras

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A vertex algebra is a vector space $V$ with $1 \in V$ and bilinear products $a_{(n)}b \in V$ ($a, b \in V$, $n \in \mathbb{Z}$) satisfying:

$$
\begin{align*}
    a_{(n)}1 &= \delta_{n,-1} a \quad (n \geq -1) \\
    1_{(n)}b &= \delta_{n,-1} b \quad (n \in \mathbb{Z}) \\
    a_{(n)}b &= 0 \quad (n \gg 0)
\end{align*}
$$

and the Borcherds identity

$$
\sum_{i=0}^{\infty} (-1)^i \binom{n}{i} \left( a_{(m+n-i)}(b_{(k+i)}c) - (-1)^n b_{(k+n-i)}(a_{(m+i)}c) \right)
= \sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)}c
$$
A **logarithmic quantum field** on $W$ is a linear map $a(ζ, z)$ from $W$ to the space of series of the form

$$
\sum_{k ∈ S} \sum_{j=0}^{∞} w_{j,k}(ζ) z^{j+k}, \quad w_{j,k}(ζ) ∈ W[ζ]
$$

for some finite $S ⊂ \mathbb{C}$.

$\text{LFie}(W) =$ space of logarithmic fields on $W$

**Notation:**

$$
D_z = \partial_z + z^{-1} \partial_ζ, \quad D_ζ = z \partial_z + \partial_ζ \quad (ζ = \log z)
$$
Locality and $n$-th products

A pair of logarithmic fields $a, b$ is **local** if

$$z_{12}^N a(\zeta_1, z_1)b(\zeta_2, z_2) = z_{12}^N b(\zeta_2, z_2)a(\zeta_1, z_1), \quad z_{12} = z_1 - z_2,$$

for some integer $N \geq 0$.

The **$n$-th product** $a(n)b$ of two local logarithmic fields:

$$(a(n)b)(\zeta, z)v = D_{z_1}^{(N-1-n)} \left( z_{12}^N a(\zeta_1, z_1)b(\zeta_2, z_2)v \right) \bigg|_{z_1 = z_2 = z}^{\zeta_1 = \zeta_2 = \zeta}$$

for $n \leq N - 1$. For $n \geq N$, let $a(n)b = 0$. 
Definition of twisted modules

$V$ – vertex algebra, $\varphi \in \text{Aut}(V)$

A $\varphi$-twisted $V$-module is a vector space $W$ with a linear map $Y: V \rightarrow \text{LFie}(W)$ such that $Y(V)$ is local, $Y(1) = I$,

$$Y(\varphi a, z) = e^{2\pi i D_\zeta} Y(a, z),$$

and

$$Y(a_{(n)} b, z) = Y(a, z)_{(n)} Y(b, z)$$

for $a, b \in V$, $n \in \mathbb{Z}$. 
Modes in a twisted module

\( \varphi \in \text{Aut}(V) – \) locally finite \( \Rightarrow \varphi = \sigma e^{-2\pi i N} \)

with \( \sigma \in \text{Aut}(V) – \) semisimple, \( N \in \text{Der}(V) – \) locally nilpotent

\( \Rightarrow \quad Y(\sigma a, z) = e^{2\pi i z \partial_z} Y(a, z), \quad Y(N a, z) = -\partial_\zeta Y(a, z) \)

\( Y(a, z) \big|_{\zeta=0} = Y(e^{\zeta N} a, z) = \sum_{m \in \alpha} a_{(m+N)} z^{-m-1}, \quad a_{(m+N)} \in \text{End}(W) \)

\( Y(a, z) = \sum_{m \in \alpha} (z^{-m-1} N a)_{(m+N)} \)

where \( \sigma a = e^{-2\pi i \alpha} a, \quad \alpha \in \mathbb{C}/\mathbb{Z}, \quad z^{-N} = e^{-\zeta N} \)
Borcherds identity for twisted modules

**Theorem.** If \( \varphi = \sigma e^{-2\pi i N} \) is locally finite, in every \( \varphi \)-twisted \( V \)-module \( W \) we have

\[
\sum_{i=0}^{\infty} (-1)^i \binom{n}{i} a_{(m+n-i+N)}(b_{(k+i+N)} v)
- \sum_{i=0}^{\infty} (-1)^{n+i} \binom{n}{i} b_{(k+n-i+N)}(a_{(m+i+N)} v)
= \sum_{j=0}^{\infty} \left( \left( \binom{m+N}{j} a \right)_{(n+j)} b \right)_{(m+k-j+N)} v
\]

where \( a, b \in V, \ v \in W, \ \sigma a = e^{-2\pi i m} a, \ \sigma b = e^{-2\pi i k} b, \ n \in \mathbb{Z}. \)
Commutator formulas

\[
\left[ a_{(m+N)}, b_{(k+N)} \right] = \sum_{j=0}^{\infty} \left( \left( \binom{m+N}{j} a \right)_{(j)} b \right)_{(m+k-j+N)}
\]

\[
\left[ Y(a, z_1), Y(b, z_2) \right] = \sum_{j=0}^{\infty} Y \left( \left( D_{z_2}^{(j)} \delta_{\alpha+N}(z_1, z_2) a \right)_{(j)} b, z_2 \right)
\]

where

\[
\sigma a = e^{-2\pi i m} a, \quad \alpha = m + \mathbb{Z} \in \mathbb{C}/\mathbb{Z}, \quad \sigma b = e^{-2\pi i k} b
\]

and

\[
\delta_{\alpha+N}(z_1, z_2) = \sum_{m \in \alpha} z_1^{-m-1-N} z_2^{m+N}
\]
Normally ordered product and $(-1)$-st product

\[ Y(a, z) = \underbrace{Y(a, z)_+}_{z^\gamma, \, \text{Re} \, \gamma \geq 0} + \underbrace{Y(a, z)_-}_{z^\gamma, \, \text{Re} \, \gamma < 0} \]

\[ :Y(a, z)Y(b, z): = Y(a, z)_+ Y(b, z) + Y(b, z)Y(a, z)_- \]

\[ = \sum_{j=-1}^{\infty} z^{-j-1} Y(\left( \begin{pmatrix} \alpha_0 + N \\ j + 1 \end{pmatrix} a \right)_{(j)} b, z) \]

where \( \sigma a = e^{-2\pi i \alpha_0} a, \quad \alpha_0 \in \mathbb{C}, \quad -1 < \text{Re} \, \alpha_0 \leq 0 \)
Affine Lie algebras

\( g \) – Lie algebra with a symmetric invariant bilinear form \((\cdot|\cdot)\).

\[ \varphi \in \text{Aut}(g), \quad (\varphi a|\varphi b) = (a|b) \]

\[ \varphi = \sigma e^{-2\pi i N}, \quad \sigma \in \text{Aut}(g), \quad N \in \text{Der}(g) \]

\[ (\sigma a|\sigma b) = (a|b), \quad (N a|b) + (a|N b) = 0 \]

**\( \varphi \)-twisted affinization** \( \hat{g}_\varphi \):

\[ [a_{(m+N)}, b_{(k+N)}] = [a, b]_{(m+k+N)} + \delta_{m,-k}( (m+N) a|b ) K \]

where \( \sigma a = e^{-2\pi i m} a, \quad \sigma b = e^{-2\pi i k} b \)
Twisted Heisenberg algebras $\hat{h}_\varphi$

$$[a_{(m+N)}, b_{(k+N)}] = \delta_{m,-k}((m+N)a|b)K$$

Case $\dim \hat{h} = 2\ell$: \quad $(v_i|v_j) = \delta_{i+j,2\ell+1}, \quad \lambda \in \mathbb{C}$

$$\sigma v_i = \lambda v_i \quad (1 \leq i \leq \ell), \quad \sigma v_i = \lambda^{-1} v_i \quad (\ell + 1 \leq i \leq 2\ell),$$

$$\mathcal{N}: \quad v_1 \leftrightarrow v_2 \leftrightarrow v_3 \leftrightarrow \cdots \leftrightarrow v_\ell \leftrightarrow 0$$
$$-\mathcal{N}: \quad v_{\ell+1} \leftrightarrow v_{\ell+2} \leftrightarrow v_{\ell+3} \leftrightarrow \cdots \leftrightarrow v_{2\ell} \rightarrow 0$$

Case $\dim \hat{h} = 2\ell - 1$: \quad $(v_i|v_j) = \delta_{i+j,2\ell}, \quad \lambda = \pm 1, \quad \sigma v_i = \lambda v_i$

$$\mathcal{N}: \quad v_1 \leftrightarrow v_2 \leftrightarrow -v_3 \leftrightarrow -v_4 \leftrightarrow v_5 \leftrightarrow \cdots \leftrightarrow (-1)^{\ell-1}v_{2\ell-1} \leftrightarrow 0$$
Example: $A_1^{(1)}$

$$(\alpha_1|\alpha_1) = 2, \quad (\delta|\Lambda_0) = (\Lambda_0|\delta) = 1$$

$$\varphi = t_{\alpha_1} : \quad \alpha_1 \mapsto \alpha_1 - 2\delta, \quad \delta \mapsto \delta, \quad \Lambda_0 \mapsto \Lambda_0 + \alpha_1 - \delta$$

$$v_1 = -\frac{2\pi i}{\sqrt{2}} \Lambda_0, \quad v_2 = \frac{\alpha_1}{\sqrt{2}}, \quad v_3 = -\frac{\sqrt{2}}{2\pi i} \delta$$

$$\varphi = e^{-2\pi i \mathcal{N}} \quad \text{where} \quad \mathcal{N} : v_1 \leftrightarrow v_2 \leftrightarrow -v_3 \leftrightarrow 0$$

$$(v_i|v_j) = \delta_{i+j,4}$$
Extended Toda hierarchy
[Milanov 2007]

Vertex operators

\[ \Gamma_{\pm}^{\alpha/2}(z) = \exp\left( \pm \sum_{n=0}^{\infty} \frac{z^{n+1}}{2(n+1)!} q_{1,n} + \frac{z^n}{n!} (\log z - c_n) q_{0,n} \right) \]

\[ \times \exp\left( \frac{1}{2} \partial q_{0,0} + \sum_{n=0}^{\infty} \frac{n!}{z^{n+1}} \partial q_{1,n} \right) \]

where \( c_0 = 0, \quad c_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \)

Hirota quadratic equations

\[ \text{Res}_{z} \ z^n \left( \Gamma^{\delta} \otimes \Gamma^{\delta} \right) \left( \Gamma^{\alpha/2} \otimes \Gamma^{-\alpha/2} - \Gamma^{-\alpha/2} \otimes \Gamma^{\alpha/2} \right) \tau \otimes \tau = 0, \quad n \geq 0 \]
Symmetries of the extended Toda hierarchy

[Dubrovin–Zhang 2004], [B.–Wheeless 2015]

Virasoro algebra

\[
\mathcal{L}_{-1} = \sum_{n=1}^{\infty} \left( 2x_n \partial_{t_{n-1}} + nt_n \partial_{t_{n-1}} + nx_n \partial_{x_{n-1}} \right) + t_1 x_0 + 2x_0 x_1
\]

\[
\mathcal{L}_0 = \sum_{n=1}^{\infty} \left( 2x_n \partial_{t_n} + nt_n \partial_{t_n} + nx_n \partial_{x_n} \right) + x_0^2
\]

\[
\mathcal{L}_p = \sum_{n=1}^{\infty} \left( 2x_n \partial_{t_{n+p}} + nt_n \partial_{t_{n+p}} + nx_n \partial_{x_{n+p}} \right) + 2x_0 \partial_{t_p} + \sum_{n=1}^{p-1} \partial_{t_n} \partial_{t_{p-n}}
\]

\[(p \geq 1, \quad \partial_{t_0} := 0)\]
Example: $A_1^{(1)}$

**Fock space** $\mathbb{C}[x_0, x_1, m, x_2, m, x_3, m]|m=1,2,3,...$

$\nu_1(\mathcal{N}) = x_0$, $\nu_2(\mathcal{N}) = -\partial x_0$, $\nu_3(\mathcal{N}) = a = \text{const}$

$\nu_i(-m+\mathcal{N}) = x_{i,m}$ $(i = 1, 2, 3, \ m = 1, 2, 3,...)$

$\nu_i(m+\mathcal{N}) = m\partial x_{4-i,m} + (-1)^{i+1}\partial x_{3-i,m}$ $(\partial x_{0,m} := 0)$

**Virasoro algebra**

$L_p = \frac{1}{2} \sum_{i=1}^{3} \sum_{m \in \mathbb{Z}} \nu_i(m+p+\mathcal{N}) \nu_{4-i}(-m+\mathcal{N})$, $p \neq 0$

$L_0 = \sum_{i=1}^{3} \sum_{m=1}^{\infty} x_{i,m}(m\partial x_{i,m} + (-1)^{i-1}\partial x_{i-1,m}) + \frac{1}{2} \partial^2 x_0 + ax_0$
Symplectic fermions

[Kausch 1995], [B.–Sullivan 2015]

\[ \{ a_{(m)}, b_{(k)} \} = m\delta_{m,-k}(a|b)K \]

\( (\cdot|\cdot) \) – skewsymmetric, \( a, b \in \text{span}(u_1, u_2) \)

\( (u_1|u_2) = -(u_2|u_1) = 1, \quad (u_i|u_i) = 0 \)

Conformal vector \( \omega = -u_1(-1)u_2 \Rightarrow \text{Virasoro algebra with } c = -2 \)

Twisted modules

\[ \mathcal{N}: u_1 \mapsto u_2 \mapsto 0, \quad \sigma u_i = \lambda u_i \quad (\lambda = \pm 1) \]

\[ \{ a_{(m+\mathcal{N})}, b_{(k+\mathcal{N})} \} = \delta_{m,-k}((m+\mathcal{N})a|b)K \]

\( m, k \in \mathbb{Z} \) for \( \lambda = 1 \) and \( m, k \in \frac{1}{2} + \mathbb{Z} \) for \( \lambda = -1 \)
Symplectic fermions ($\lambda = 1$)

Fock space $\bigwedge (\xi_1, m, \xi_2, m)_{m=0,1,2,\ldots}$ where $(\xi_1, 0)^2 = -\frac{1}{2}$

$$u_i(-m+N) = \xi_i, m \quad (i = 1, 2, \ m = 0, 1, 2, \ldots)$$

$$u_1(m+N) = m\partial\xi_2, m - \partial\xi_1, m \quad (m = 1, 2, 3, \ldots)$$

$$u_2(m+N) = -m\partial\xi_1, m \quad (m = 1, 2, 3, \ldots)$$

Virasoro algebra

$$L_p = -\sum_{m\in\mathbb{Z}} u_1(m+p+N) u_2(-m+N), \quad p \neq 0$$

$$L_0 = \sum_{m=1}^{\infty} \left( m\xi_1, m\partial\xi_1, m + m\xi_2, m\partial\xi_2, m - \xi_2, m\partial\xi_1, m \right) - \xi_1, 0\xi_2, 0$$