EXAMPLES OF DEFORMED G_2-INSTANTONS/DONALDSON–THOMAS
CONNECTIONS

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Abstract. In this note, we provide the first non-trivial examples of deformed G_2-instantons, originally called deformed Donaldson–Thomas connections. As a consequence, we see how deformed G_2-instantons can be used to distinguish between nearly parallel G_2-structures and isometric G_2-structures on 3-Sasakian 7-manifolds. Our examples give non-trivial deformed G_2-instantons with obstructed deformation theory and situations where the moduli space of deformed G_2-instantons has components of different dimensions. We finally study the relation between our examples and a Chern–Simons type functional which has deformed G_2-instantons as critical points.

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1. Introduction

Gauge theory and calibrated geometry are central topics of study in the context of G_2 geometry, and they are intimately related. Based on ideas stemming from Mirror Symmetry, particularly SYZ fibrations, and the real Fourier–Mukai transform, the authors of [LL09] introduced the following gauge-theoretic equation (in the context of complex line bundles over G_2-manifolds) as a proposed mirror to certain calibrated cycles in G_2-manifolds.

Definition 1.1. Let (X^7, ϕ) be a 7-manifold with a coclosed G_2 structure ϕ, let ψ = *_ϕϕ be the dual of ϕ, and let L be a Hermitian complex line bundle on X. A unitary connection A on L is a deformed G_2-instanton if its curvature F_A satisfies

\[
\frac{1}{6} F_A^3 + F_A \wedge ψ = 0.
\]

The definition can obviously be extended to higher rank vector bundles and principal bundles but, based on [LL09], one is primarily interested in the case of complex line bundles. When (X, ϕ) is additionally a G_2-manifold, deformed G_2-instantons are, in a certain sense, “mirror” to (co)associative cycles.

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Remark 1.2. In [LL09], deformed $G_2$-instantons are called deformed Donaldson–Thomas connections. However, since $A$ is a $G_2$-instanton on $(X^7, \varphi)$ if and only if
\begin{equation}
F_A \wedge \psi = 0,
\end{equation}
the authors feel it is more appropriate that solutions of (1.1) are called deformed $G_2$-instantons. Moreover, there is a natural relationship between deformed $G_2$-instantons and deformed Hermitian-Yang–Mills connections, which is parallel to the relationship between $G_2$-instantons and Hermitian-Yang–Mills connections, that gives further rationale for the nomenclature.

1.1. Main results. The main results of this article are the first constructions of non-trivial solutions to the deformed $G_2$-instanton equation (1.1). Here, by non-trivial, we mean deformed $G_2$-instantons that are not flat and do not arise via pullback from lower-dimensional constructions.

As an application of our construction we have the following result (Corollary 4.5).

Theorem 1.3. Let $X^7$ be a 3-Sasakian 7-manifold and let $L_0$ be the trivial complex line bundle on $X$. Let $\varphi^{ts}$ be the standard nearly parallel $G_2$-structure on $X$ inducing the 3-Sasakian Einstein metric $g^{ts}$, and let $\varphi^{np}$ be the second (strictly) nearly parallel $G_2$-structure on $X$ inducing the “squashed” Einstein metric $g^{np}$.

- There is a circle of non-trivial deformed $G_2$-instantons on $L_0$ for $\varphi^{ts}$.
- There is a 2-sphere of non-trivial deformed $G_2$-instantons on $L_0$ for $\varphi^{np}$.

Remark 1.4. There are infinitely many compact 3-Sasakian 7-manifolds $X^7$ [BGM94, GWZ08]. Key examples are giving by the 7-sphere $S^7$ and the Aloff–Wallach space $SU(3)/U(1)_{1,1}$ (c.f. Example 3.8), which is the SO(3)-frame bundle of $\Lambda^2 \mathbb{CP}^2$.

Remark 1.5. We recall that, if $X^7$ is a compact 3-Sasakian 7-manifold, then the metric cone on $(X^7, g^{ts})$ is hyperkähler, and has holonomy $\text{Sp}(2)$ if it is not flat, whilst the metric cone on $(X^7, g^{np})$ has holonomy $\text{Spin}(7)$.

Remark 1.6. Theorem 1.3 shows how non-trivial deformed $G_2$-instantons distinguish between the nearly parallel $G_2$-structures $\varphi^{np}$ and $\varphi^{ts}$. We shall also see that non-trivial deformed $G_2$-instantons can discriminate between two coclosed $G_2$-structures inducing the same metric, including the Einstein metrics $g^{ts}$ and $g^{np}$ (c.f. Corollary 4.7).

We can also apply our results to non-trivial complex line bundles when $X^7$ is the 3-Sasakian Aloff–Wallach space (Corollary 4.12).

Theorem 1.7. Let $\pi : X \to \mathbb{CP}^2$ be the SO(3)-frame bundle of $\Lambda^2 \mathbb{CP}^2$ and let $k \in \mathbb{Z}$.

- There is a circle of non-trivial deformed $G_2$-instantons on $\pi^* \mathcal{O}(k)$ for $\varphi^{ts}$.
- There is a 2-sphere of non-trivial deformed $G_2$-instantons on $\pi^* \mathcal{O}(k)$ for $\varphi^{np}$.

More generally, we give examples of deformed $G_2$-instantons for families of coclosed $G_2$-structures $\varphi_{t, \varepsilon}$ on a 3-Sasakian 7-manifold $X$ depending on two parameters: $t > 0$ and $\varepsilon \in \{\pm 1\}$. Our ansatz depends on $a_1, a_2, a_3 \in \mathbb{R}$, where $a_1 = a_2 = a_3 = 0$ yields the trivial flat connection in the case of the trivial complex line bundle $L_0$. Hence, if we let $r = \sqrt{a_1^2 + a_2^2 + a_3^2}$, which can be viewed as the distance to the trivial connection on $L_0$, we can represent our main result (Proposition 4.4) in Figure 1 below. The overall picture for the non-trivial line bundles on the 3-Sasakian Aloff–Wallach space (Proposition 4.10) is the same.
Remark 1.8. In particular, we observe in Figure 1 how deformed $G_2$-instantons die as $t$ varies from 0 to $1/\sqrt{2}$ in both the cases $\varepsilon = \pm 1$, but when $\varepsilon = -1$ we have a circle that lives for all $t > 0$. For context, we mention that $\varphi^{np}$ corresponds to $(t, \varepsilon) = (1/\sqrt{5}, +1)$ and $\varphi^{ts}$ corresponds to $(t, \varepsilon) = (1, -1)$. We also note that $\varphi_{t,\varepsilon}$ induce the same metric for a fixed value of $t$; it is an interesting feature resulting from our work that the space of deformed $G_2$-instantons differs for these isometric $G_2$-structures.

We may also relate our examples to the recently developed moduli space theory for deformed $G_2$-instantons in [KY20]. We refer the reader to Definition 4.14 for the formal definition for a deformed $G_2$-instanton to be obstructed in the sense of deformation theory.

Theorem 1.9. Let $X^7$ be a 3-Sasakian 7-manifold and let $L_0$ be the trivial complex line bundle on $X$. The non-trivial deformed $G_2$-instantons given by Theorem 1.3 are obstructed. Moreover, the moduli spaces of deformed $G_2$-instantons on $L_0$ for the nearly parallel $G_2$-structures $\varphi^{np}$ and $\varphi^{ts}$ both contain at least two components of different dimensions.

A Chern–Simons type functional was introduced in [KL09] whose critical points are deformed $G_2$-instantons. We study this functional in our setting and, in particular, deduce the following (c.f. Lemma 4.19 and Corollary 5.7), which reflects the fact that non-trivial deformed $G_2$-instantons coalesce with the trivial connection at $t = 1/\sqrt{2}$.

Theorem 1.10. Let $X^7$ be a 3-Sasakian 7-manifold, let $L_0$ be the trivial complex line bundle, and let $A_0$ be the trivial flat connection on $L_0$. Then $A_0$ is unobstructed (and hence rigid and locally isolated) as a deformed $G_2$-instanton with respect to $\varphi^{np} = \varphi_{1/\sqrt{5},+1}$ and $\varphi^{ts} = \varphi_{1,-1}$, but obstructed as a deformed $G_2$-instanton with respect to $\varphi_{1/\sqrt{2},\varepsilon}$ for $\varepsilon = \pm 1$.

1.2. Summary. This article is organized as follows. Section 2 introduces some background on 3-Sasakian geometry which will later be of use in constructing the examples of deformed $G_2$-instantons. In section 3 we give some simple examples of deformed $G_2$-instantons which arise from pulling back connections in 6 and 4 dimensions. The non-trivial examples which constitute the main contribution of this article are constructed and presented in section 4. Finally, in section 5, this article concludes with a discussion of the Chern–Simons type functional mentioned above; in particular, the functional is explicitly computed and analyzed in some of the cases developed in this article.

Remark 1.11. We shall use summation convention over repeated indices throughout the article.
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2. 3-SASAKIAN GEOMETRY

We shall now give a short introduction to some aspects of 3-Sasakian geometry and its relation to $G_2$ geometry. We refer the reader to the survey article [BG01] and references therein for more information on 3-Sasakian geometry.

Definition 2.1. A Riemannian 7-manifold $(X^7, g^{ts})$ is 3-Sasakian if it has three orthonormal Killing vector fields $\{\xi_i\}_{i=1}^3$ which satisfy the relations $[\xi_i, \xi_j] = 2\varepsilon_{ijk}\xi_k$, for $\varepsilon_{ijk}$ the sign of the permutation taking $(1,2,3)$ to $(i,j,k)$.

2.1. SU(2) leaf space. For a 3-Sasakian manifold $(X^7, g^{ts})$ as in Definition 2.1, the Killing vector fields $\{\xi_i\}_{i=1}^3$ generate a locally free action of SU(2). The leaf space of this SU(2) action, denoted $Z^4$, can be endowed with a metric $g_Z$ so that the canonical projection

$$\pi : X \to Z$$

is an orbifold Riemannian submersion. This metric is anti-self-dual and Einstein of positive scalar curvature $s > 0$. For convenience, we will scale the metric $g^{ts}$ so that $s = 48$: this fits with the canonical example of $S^7$ with its constant curvature 1 metric. In the particular case when $Z$ is spin, we shall regard (2.1) as the lift to SU(2) = Spin(3) of an SO(3)-(orbi)bundle of frames of $\Lambda^2_+ Z$, the bundle of self-dual 2-forms on $Z$.

The Levi-Civita connection of $Z$ induces a connection $\eta$ on the bundle (2.1) which, seen as a 1-form on $X$ with values in $\mathfrak{su}(2)$, can be written as

$$\eta = \eta_i \otimes T_i,$$

where the $T_i$ are a standard basis of $\mathfrak{su}(2)$ satisfying $[T_i, T_j] = 2\varepsilon_{ijk}T_k$, and the $\eta_i$ are 1-forms on $X$. The horizontal space of the connection is $H = \ker(\eta)$. Knowing that $Z$ is anti-self-dual Einstein means that the curvature of the connection $\eta$ in (2.2) is given by

$$F_\eta = d\eta + \frac{1}{2}[\eta \wedge \eta] = -2\omega_i \otimes T_i,$$

with the 2-forms $\omega_1, \omega_2, \omega_3$ forming an orthogonal basis of $(\Lambda^2_+ H, g^{ts}|_H)$ with $|\omega_i| = \sqrt{2}$. Notice that we have the relations

$$\omega_i \wedge \omega_j = 2\delta_{ij}\pi^* \text{vol}_Z,$$

where $\text{vol}_Z$ is the Riemannian volume form on $(Z, g_Z)$.

2.2. Metrics and $G_2$-structures. The 3-Sasakian metric $g^{ts}$ can be written

$$g^{ts} = \eta_i \otimes \eta_i + \pi^* g_Z$$

and it is well-known to be Einstein. In fact, there is a second Einstein metric on $X$ for which $\pi$ is a Riemannian submersion. This can be obtained from $g^{ts}$ by squashing the fibers of $\pi$ by a factor which reduces the length of any $\xi_i$-orbit by $\sqrt{5}$. This yields the metric

$$g^{np} = \frac{1}{5} \eta_i \otimes \eta_i + \pi^* g_Z.$$
The metrics $g^{ts}$ in (2.5) and $g^{np}$ in (2.6) are induced by natural distinguished $G_2$-structures $\varphi^{ts}$ and $\varphi^{np}$ on $X^7$. (For details on $G_2$-structures we refer the reader to [KLL20], for example.) Using (2.2) and (2.3), it is convenient to consider two 1-parameter families, depending on $t \in \mathbb{R}^+$ and $\varepsilon \in \{\pm 1\}$, of $G_2$-structures on $X^7$ determined by the 3-forms

$$\varphi_{t,\varepsilon} = \varepsilon t^3 \eta_{123} - t(\eta_1 \wedge \omega_1 + \eta_2 \wedge \omega_2 + \varepsilon \eta_3 \wedge \omega_3),$$

where we have used the notation $\eta_{123} = \eta_1 \wedge \eta_2 \wedge \eta_3$ for brevity. Each $\varphi_{t,\varepsilon}$ determines the metric

$$g_t = t^2 \eta_i \otimes \eta_i + \pi^* g_Z,$$

which is independent of $\varepsilon$, and the associated 4-forms $\psi_{t,\varepsilon} = \pi^* \varphi_{t,\varepsilon}$ are

$$\psi_{t,\varepsilon} = \pi^* \text{vol}_Z - t^2 (\varepsilon \eta_{23} \wedge \omega_1 + \varepsilon \eta_{31} \wedge \omega_2 + \eta_{12} \wedge \omega_3),$$

where we write $\eta_{ij} = \eta_i \wedge \eta_j$ for short. Note that if we set

$$\varphi^{ts} = \varphi_{1/\sqrt{5},-1} \quad \text{and} \quad \varphi^{np} = \varphi_{1/\sqrt{5},+1},$$

then (2.8) shows that $\varphi^{ts}$ induces $g^{ts}$ in (2.5) and $\varphi^{np}$ induces $g^{np}$ in (2.6).

**Remark 2.2.** Changing the sign of the parameter $\varepsilon$ corresponds to the change $\eta_3 \mapsto -\eta_3$, which gives a change of orientation on the vertical space in the projection (2.1). However, since we have fixed the structure equations on $SU(2)$, we are not free to change $\eta_3 \mapsto -\eta_3$, and so $\varepsilon$ represents a genuine parameter.

We now recall a notable class of $G_2$-structures in this setting.

**Definition 2.3.** A $G_2$-structure determined by a 3-form $\varphi$, with dual 4-form $\psi$, is nearly parallel if $d\varphi = \lambda \psi$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. By possibly changing orientation, we can always ensure that $\lambda > 0$.

Using the equation (2.3) for the curvature of $\eta$, we find the following structure equations:

$$d\eta_i = -2\omega_i - 2\eta_j \wedge \eta_k,$$

$$d\omega_i = 2\omega_j \wedge \eta_k - 2\eta_j \wedge \omega_k,$$

for $(i,j,k)$ a cyclic permutation of $(1,2,3)$. Equations (2.11)-(2.12) immediately show that

$$d\psi_{t,\varepsilon} = 0$$

for all $t, \varepsilon$. Using (2.11)–(2.12) we compute from (2.7) and (2.9) that the equation

$$d\varphi_{t,\varepsilon} = \lambda \psi_{t,\varepsilon}$$

only has solutions when

$$(t, \varepsilon, \lambda) = \left( \frac{1}{\sqrt{5}}, +1, \frac{12}{\sqrt{5}} \right) \quad \text{and} \quad (t, \varepsilon, \lambda) = (1, -1, 4).$$

By Definition 2.3, $(t, \varepsilon) = (1/\sqrt{5},+1)$ and $(t, \varepsilon) = (1,-1)$ are the only values for which $\varphi_{t,\varepsilon}$ is nearly parallel [FKMS97]. We see from (2.5), (2.6) and (2.8) that $g_{1/\sqrt{5}} = g^{np}$ and $g_1 = g^{ts}$, and (2.10), (2.14) and (2.15) show that $\varphi^{np}$ and $\varphi^{ts}$ are nearly parallel.

We conclude this section with a familiar concrete example.
Example 2.4. The standard 7-sphere $S^7$ with its round constant curvature 1 metric $g_{S^7}$ is 3-Sasakian, and its SU(2) leaf space is $Z = S^4$, naturally endowed with its round constant curvature 4 metric $g_Z$. It is well-known that $S^7$ admits two homogeneous Einstein metrics: $g_{S^7}$ (which is $g^7$ in (2.5)) and its “squashed” metric (which is $g^{np}$ in (2.6)). Each of these Einstein metrics is induced by a homogeneous nearly parallel $G_2$-structure, given in (2.10) by $\varphi^{ts}$ and $\varphi^{np}$ respectively.

3. Deformed Hermitian-Yang–Mills connections and ASD instantons

In this section we provide examples of deformed $G_2$-instantons arising from lower-dimensional geometries: specifically, deformed Hermitian-Yang–Mills connections on Calabi–Yau 3-folds and anti-self-dual instantons on anti-self-dual Einstein 4-orbifolds and hypersymplectic 4-manifolds.

3.1. Deformed Hermitian-Yang–Mills connections. We recall the following definition, which originated in [LYZ00, MMMS00].

Definition 3.1. Let $(Y, \omega)$ be a Kähler $n$-fold, where $\omega$ is the Kähler form, and let $L$ be a Hermitian complex line bundle on $Y$. A unitary connection $A$ on $L$ is a deformed Hermitian-Yang–Mills connection (with phase $e^{i\alpha}$) if

$$F_A^{(0,2)} = 0 \quad \text{and} \quad \text{Im}(e^{-i\alpha}(\omega + F_A)^n) = 0.$$  

When $Y$ is a Calabi–Yau manifold, deformed Hermitian-Yang–Mills connections are, in a sense, “mirror” to special Lagrangian $n$-folds.

We are interested in the case where $(Y, \omega, \Omega)$ is a Calabi–Yau 3-fold, with holomorphic volume form $\Omega$, and $A$ is a deformed Hermitian-Yang–Mills connection with phase 1. In this case (3.1) can be rewritten as

$$F_A \wedge \text{Im} \Omega = 0 \quad \text{and} \quad \frac{1}{6} F_A^3 + F_A \wedge \frac{1}{2} \omega^2 = 0.$$  

The analogy between (1.1) and (3.2) should be clear. We then provide an observation which extends Lemma 5.5 in [KY20].

Lemma 3.2. Let $(Y, \omega, \Omega)$ be a Calabi–Yau 3-fold and let $L$ be a Hermitian complex line bundle on $Y$. Let $\pi : X^7 \to Y$ be an $S^1$-bundle over $Y$ with a connection 1-form $\eta$, which is Hermitian-Yang–Mills–Mills, endowed with the standard $G_2$-structure

$$\varphi = \eta \wedge \pi^* \omega + \pi^* \text{Re} \Omega \quad \text{and} \quad \psi = \frac{1}{2} \pi^* \omega^2 - \eta \wedge \pi^* \text{Im} \Omega.$$  

Note that as $\eta$ is Hermitian-Yang–Mills–Mills we have $d\psi = 0$.

Then $A$ is a deformed Hermitian-Yang–Mills connection with phase 1 on $L$ if and only if $\pi^* A$ is a deformed $G_2$-instanton on $\pi^* L$.

Proof. The proof follows immediately from (1.1), (3.2) and (3.3), just as in the proof of Lemma 5.5 in [KY20].

Remark 3.3. Lemma 3.2 has a well-known analogue where deformed Hermitian-Yang–Mills connections and deformed $G_2$-instantons are replaced by Hermitian-Yang–Mills conceptions and $G_2$-instantons. When $X^7 = S^1 \times Y$ with the product $G_2$-structure, where $\eta = d\theta$ for $\theta$ the coordinate on $S^1$, the only $G_2$-instantons on $\pi^* L$ are (up to gauge transformations) given by $\pi^* A$ for a Hermitian-Yang–Mills connection on $Y$ [Wan18].
There are now many examples of deformed Hermitian-Yang–Mills connections, in particular provided by the recent relationship between existence of such connections and stability proved in [Che20]. We may construct a simple example of a $G_2$-manifold, i.e. $X^7$ with a torsion-free $G_2$-structure $\varphi$ (satisfying $d\varphi = 0$ and $d\psi = 0$), with a non-trivial line bundle admitting a deformed $G_2$-instanton which is not a $G_2$-instanton as follows.

**Example 3.4.** Suppose that $(Y, \omega, \Omega)$ is a Calabi–Yau 3-fold such that $[\sqrt{3}\omega]$ is an integral class in $H^2(Y)$, and so defines a Hermitian complex line bundle $L$ with a unitary connection $A$ such that $F_A = i\sqrt{3}\omega$. Then (3.2) is satisfied and so $A$ is a deformed Hermitian-Yang–Mills connection with phase 1.

If we let $X = S^1 \times Y$ with the product torsion-free $G_2$-structure as in (3.3), where $\pi : X \rightarrow Y$ is the natural projection and $\eta = d\theta$ for $\theta$ the coordinate on $S^1$, then $\pi^*A$ is a deformed $G_2$-instanton on $\pi^*L$ by Lemma 3.2. Notice that $\pi^*F_A^3 \neq 0$ and so $\pi^*A$ is not a $G_2$-instanton on $X = S^1 \times Y$.

**Remark 3.5.** One can perform a similar construction to Example 3.4 for so-called Calabi–Yau links $X^7$ in $S^9$, which are nontrivial $S^1$-bundles over Calabi–Yau 3-(orbi)folds $Y$ arising as hypersurfaces in $\mathbb{CP}^4$, to obtain deformed $G_2$-instantons which are not $G_2$-instantons on $X$. The study of $G_2$-instantons on Calabi–Yau links was initiated in [CARDSE20], using Hermitian-Yang–Mills connections on $Y$.

### 3.2. ASD instantons on anti-self-dual Einstein 4-orbifolds

Let $X^7$ be a 3-Sasakian 7-manifold as in Definition 2.1 and let $(Z^4, g_Z)$ as in (2.1) be the SU(2) leaf space. Recall that $(Z^4, g_Z)$ is an anti-self-dual Einstein 4-orbifold, and recall the $G_2$-structures on $X$ whose 4-forms are given by $\psi_{t,\varepsilon}$ in (2.9). In particular, recall the forms $\omega_i$ in (2.3), which are pullbacks of self-dual 2-forms on $Z$, used to construct $\psi_{t,\varepsilon}$.

We now have the following simple observation concerning anti-self-dual (ASD) instantons on $Z$, i.e. connections $A$ on $Z$ whose curvature satisfies

\begin{equation}
F_A = -* F_A. 
\end{equation}

Since $\pi^* F_A^3 = 0$ automatically for dimension reasons, for any connection $A$ on $Z$, we see from (1.1) that the notions of deformed $G_2$-instanton and $G_2$-instanton coincide for $\pi^*A$. We may thus obtain trivial examples of deformed $G_2$-instantons as follows.

**Lemma 3.6.** Let $X^7$ be a 3-Sasakian 7-manifold and let $Z$ be its SU(2) leaf space as in (2.1). Let $L$ be a Hermitian complex line bundle on $Z$, and let $A$ be a unitary connection on $L$.

Then $\pi^*A$ is a (deformed) $G_2$-instanton on $\pi^*L$ over $X$, with respect to some (and hence all) $\psi_{t,\varepsilon}$ in (2.9) if and only if $A$ is an ASD instanton.

**Proof.** First, let $A$ be an anti-self-dual (ASD) instanton on $Z$. Then, since $F_A$ is anti-self-dual and the forms $\omega_i$ appearing in (2.9) are self-dual on the horizontal space $H$ for the projection (2.1), we see that $\pi^* F_A \wedge \psi_{t,\varepsilon} = 0$, i.e. that $\pi^*A$ is a $G_2$-instanton.

Conversely, if $\pi^* F_A \wedge \psi_{t,\varepsilon} = 0$ for some $t$ and $\varepsilon$, then we must have

\begin{equation}
\pi^* F_A \wedge \omega_i = 0 \quad \text{for all } i.
\end{equation}

Since the $\omega_i$ span the self-dual 2-forms on $H$, (3.5) implies that (3.4) holds, i.e. $A$ is an ASD instanton.

**Remark 3.7.** Lemma 3.6 has some overlap with Proposition 18 in [BO19].

We now give an example of using Lemma 3.12 which will be useful later.
Example 3.8. Let $X^7$ be the Aloff–Wallach space $SU(3)/U(1)_{1,1}$ where
\begin{equation}
U(1)_{1,1} = \left\{ \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix} \in SU(3) : \theta \in \mathbb{R} \right\}.
\end{equation}
Then $X$ is a homogeneous 3-Sasakian 7-manifold whose $SU(2)$ leaf space is $Z = \mathbb{CP}^2$. Let $L = \mathcal{O}(k)$ on $Z$ for $k \in \mathbb{Z}$. The connection $A$ on $L$ with harmonic curvature will be unitary and have the property that $F_A$ is a multiple of the Fubini–Study form, and so will be an ASD instanton (since $\mathbb{CP}^2$ has the opposite orientation). Moreover, $A$ will be non-trivial whenever $k \neq 0$. Lemma 3.6 then gives a deformed $G_2$-instanton (which is a non-trivial $G_2$-instanton for $k \neq 0$) on $\pi^*L$ with respect to both of the homogeneous nearly parallel $G_2$-structures $\varphi^{ts}$ and $\varphi^{hyp}$ in (2.10) on $X$.

Remark 3.9. Gauge theory on $X = SU(3)/U(1)_{1,1}$, in particular $G_2$-instantons with respect to the two homogeneous nearly parallel $G_2$-structures, is studied in some detail in [BO19].

3.3. ASD instantons on hypersymplectic 4-manifolds. We shall now give some simple examples arising from pull-backs of anti-self-dual connections on a hypersymplectic 4-dimensional manifold. As we shall see, this is directly analogous to the construction arising from anti-self-dual Einstein 4-orbifolds considered in the previous subsection.

Definition 3.10. A \textit{hypersymplectic} structure on an oriented 4-manifold $Z^4$ is a triple of closed 2-forms $(\omega_1, \omega_2, \omega_3)$ on $Z$ so that, for a volume form $\text{vol}_Z$ on $Z$,
\begin{equation}
\omega_i \wedge \omega_j = 2Q_{ij} \text{vol}_Z,
\end{equation}
where the matrix $Q_{ij}$ is positive definite. We may choose $\text{vol}_Z$ in (3.7) so that the matrix $Q_{ij}$ has determinant one. In this manner, the hypersymplectic structure determines a Riemannian metric $g_Z$ on $Z$ whose bundle of self-dual two forms is spanned by $\{\omega_1, \omega_2, \omega_3\}$ and whose Riemannian volume form is $\text{vol}_Z$.

Example 3.11. If $(Z^4, g_Z)$ is a K3 surface with $(\omega_1, \omega_2, \omega_3)$ a hyperkähler triple, then (3.7) is satisfied with $Q_{ij} = \delta_{ij}$ and $g_Z$ in Definition 3.10 is the associated Ricci-flat Kähler metric. See [Don06] for more about hypersymplectic structures and their relation to hyperkähler geometry.

Consider a 3-torus bundle $\pi : X \to Z$ over such a hypersymplectic 4-manifold $Z$ with an anti-self-dual connection $\eta$. We regard the connection $\eta = (\eta_1, \eta_2, \eta_3) \in \Omega^1(X, \mathbb{R}^3)$ as three 1-forms on the total space whose curvatures $d\eta_i$ are the pull-back of anti-self-dual 2-forms on $(Z, g_Z)$. Then, we consider $G_2$-structures on $X$ whose corresponding 4-forms are
\begin{equation}
\psi_t = \pi^*\text{vol}_Z - t^2 (\eta_{23} \wedge \pi^* \omega_1 + \eta_{31} \wedge \pi^* \omega_2 + \eta_{12} \wedge \pi^* \omega_3),
\end{equation}
for some constant $t > 0$. (For more details on the relation between $G_2$-structures and hypersymplectic structures see [FY18].) We see that $d\psi_t = 0$ as the $d\eta_i$ are assumed to be the pull-back of anti-self-dual 2-forms and thus $d\psi_i \wedge \omega_j = 0$ for all $i, j$. This is the only point for which the condition that $\eta$ is anti-self-dual is used.

We now construct some simple deformed $G_2$-instantons with respect to $\psi_t$ in (3.8). We regard these as being trivial examples since, as in Lemma 3.6 above, they are $G_2$-instantons for which the cubic term in the curvature vanishes. The proof is almost identical to Lemma 3.6 so we omit it.

Lemma 3.12. Let $Z$ be a hypersymplectic 4-manifold, let $L$ be a Hermitian complex line bundle on $Z$ and let $A$ be a unitary connection on $L$. Let $\pi : X \to Z$ be a $T^3$-bundle over $Z$ with anti-self-dual connection $\eta$ and let $\psi_t$ be as in (3.8).
Then $\pi^*A$ is a (deformed) $G_2$-instanton on $\pi^*L$ with respect to some (and hence all) $\psi_t$ if and only if $A$ is an ASD instanton.

**Example 3.13.** Let $Z$ be a K3 surface and $\omega_1, \omega_2, \omega_3$ a hyperkähler triple, and let $X = T^3 \times Z$ so that $\eta$ is the trivial flat connection on $\pi : X \to Z$. Note that in this case $X$ is a $G_2$-manifold with the product $G_2$-structure whose 4-form is $\psi = \psi_1$ in (3.8).

Denote by $\mathcal{H}_+ = (\omega_1, \omega_2, \omega_3)$ the space of self-dual harmonic 2-forms and by $\mathcal{H}_-$ that of anti-self-dual ones. Then $H^2(X, \mathbb{R}) \cong \mathcal{H}_+ \oplus \mathcal{H}_-$ and suppose that $\mathcal{H}_- \cap H^2(X, \mathbb{Z}) \neq 0$. Then, for any complex line bundle so that $c_1(L) \in \mathcal{H}_- \cap H^2(X, \mathbb{Z})$, Lemma 3.12 applies and we obtain a connection $A$ which is both a $G_2$-instanton and a deformed $G_2$-instanton for $\psi$ on the $G_2$-manifold $X = T^3 \times Z$.

**Remark 3.14.** Similar constructions of $G_2$-instantons to Example 3.13, also performed in the higher rank case, can be found in [CGFT20].

4. Examples

We now turn to the main goal of this article, which is to construct the first non-trivial examples of deformed $G_2$-instantons. At the end of the section, we shall also discuss implications of this construction for the deformation theory of deformed $G_2$-instantons.

In this section, unless we state otherwise, we let $X^7$ be a 3-Sasakian 7-manifold as in Definition 2.1, and we shall use the notation introduced in section 2. In particular we let $Z^4$ be the leaf space of the SU(2) action on $X$ given in (2.1). We also let $L_0$ denote the trivial complex line bundle over $X$.

4.1. $G_2$-instantons. We consider connections $A$ on the trivial complex line bundle $L_0$ over $X$, given by

\begin{equation}
A = i(a_1\eta_1 + a_2\eta_2 + a_3\eta_3),
\end{equation}

for $a_1, a_2, a_3 \in \mathbb{R}$, where the $\eta_i$ are given in (2.2). (Here, we identify the Lie algebra $\mathfrak{u}(1)$ with $i\mathbb{R}$.) The curvature of $A$ is given by

\begin{equation}
F_A = -2ia_1(\omega_1 + \eta_3) - 2ia_2(\omega_2 + \eta_31) - 2ia_3(\omega_3 + \eta_12),
\end{equation}

where the $\omega_i$ are given in (2.3), and we recall that $\eta_{ij} = \eta_i \wedge \eta_j$. Using (2.4), (2.9) and (4.2), we compute

\begin{equation}
F_A \wedge \psi_{t,\varepsilon} = -2i((1 - 2\varepsilon t^2)(a_1\eta_{23} + a_2\eta_{31}) + (1 - 2i^2)a_3\eta_{12}) \wedge \pi^*\text{vol}_Z.
\end{equation}

From (1.2) and (4.3) we are led to the following conclusion.

**Proposition 4.1.** Suppose that $A$ in (4.1) is a $G_2$-instanton with respect to $\psi_{t,\varepsilon}$ in (2.9). Then

- $a_1 = a_2 = a_3 = 0$, in which case $A$ is the trivial flat connection, or
- $t = 1/\sqrt{2}$, $\varepsilon = -1$, $a_1 = a_2 = 0$ and $a_3 \neq 0$, so $A = ia_3\eta_3$ is a $G_2$-instanton with respect to $\psi_{1/\sqrt{2}, -1}$, or
- $t = 1/\sqrt{2}$ and $\varepsilon = +1$, in which case all $A$ in (4.1) are $G_2$-instantons with respect to $\psi_{1/\sqrt{2}, +1}$.

**Remark 4.2.** The $G_2$-structures given by $(t, \varepsilon) = (1/\sqrt{2}, +1)$ are special, as they support a real 3-parameter family of $G_2$-instantons on the trivial complex line bundle $L_0$ on $X$ by Proposition 4.1. This extends observations made in Proposition 17 of [BO19]. Moreover, the $G_2$-structures given by $(t, \varepsilon) = (1/\sqrt{2}, -1)$ admit a real 1-parameter family of $G_2$-instantons on $L_0$ on $X$. 


Remark 4.3. There are examples of $G_2$-instantons on higher rank bundles on $X$, such as Example 2 in [CO20], which describes an irreducible $G_2$-instanton with gauge group $SO(3)$ in this setting.

4.2. Deformed $G_2$-instantons. We now analyze solutions to the deformed $G_2$-instanton (or, equivalently, deformed Donaldson–Thomas connection) equation with respect to $\psi_{t,\varepsilon}$:

$$\frac{1}{6} F_A^3 + F_A \wedge \psi_{t,\varepsilon} = 0,$$

for $A$ a connection on the trivial complex line bundle $L_0$ on $X$. For $A$ as in (4.1) we compute

$$\frac{1}{6} F_A^3 = 8i (a_1^2 + a_2^2 + a_3^2) (a_1 \eta_{23} + a_2 \eta_{31} + a_3 \eta_{12}) \wedge \pi^* \text{vol}_Z,$$

Using (4.3) and (4.5), we see that $A$ solves (4.4) if and only if

$$(4(a_1^2 + a_2^2 + a_3^2) - (1 - 2\varepsilon t^2))(a_1 \eta_{23} + a_2 \eta_{31} + (4(a_1^2 + a_2^2 + a_3^2) - (1 - 2t^2))a_3 \eta_{12} = 0.$$

We see that (4.6) always has the solution $a_1 = a_2 = a_3 = 0$, which corresponds to the flat connection. Otherwise, if $\varepsilon = +1$ then we must have

$$a_1^2 + a_2^2 + a_3^2 = \frac{1}{4} (1 - 2t^2).$$

We immediately see that (4.7) can be solved for a non-flat connection $A$ if and only if $2t^2 < 1$, in which case there is a whole 2-sphere of solutions. If, instead, $\varepsilon = -1$ then if $a_1^2 + a_2^2 \neq 0$ we must have

$$a_1^2 + a_2^2 = \frac{1}{4} (1 + 2t^2) \text{ and } a_3 = 0,$$

and if $a_3 \neq 0$ we must have

$$a_1^2 + a_2^2 = 0 \text{ and } a_3^2 = \frac{1}{4} (1 - 2t^2).$$

Clearly, (4.8) gives a circle of solutions for any $t > 0$, whereas (4.9) gives non-trivial solutions if and only if $2t^2 < 1$, in which case there are two solutions.

We state these findings as follows.

Proposition 4.4. Let $a_1, a_2, a_3 \in \mathbb{R}$ and let $A = ia_j \eta_j$ as in (4.1) be a connection on the trivial complex line bundle $L_0$ on $X$. Then $A$ is a deformed $G_2$-instanton with respect to $\psi_{t,\varepsilon}$ in (2.9) if and only if either $a_1 = a_2 = a_3 = 0$, so $A$ is the trivial flat connection, or one of the following holds:

- $t \in (0, 1/\sqrt{2})$, $\varepsilon = +1$ and $a_1, a_2, a_3$ satisfy (4.7) (so there is a 2-sphere of solutions);
- $t \in (0, 1/\sqrt{2})$, $\varepsilon = -1$ and $a_1, a_2, a_3$ satisfy (4.8) or (4.9) (so the solutions consist of a circle and two points);
- $t \geq 1/\sqrt{2}$, $\varepsilon = -1$, and $a_1, a_2, a_3$ satisfy (4.8) (so there is a circle of solutions).

By (2.10), Proposition 4.4 immediately gives the following two results.

Corollary 4.5. Recall the two nearly parallel $G_2$-structures $\varphi^{op}$ and $\varphi^{ts}$ on $X$ given in (2.10).

- There is a 2-sphere of non-trivial deformed $G_2$-instantons on $L_0$ over $X$ with respect to $\varphi^{op}$ arising from (4.1).
- There is a circle of non-trivial deformed $G_2$-instantons on $L_0$ over $X$ with respect to $\varphi^{ts}$ arising from (4.1).
Remark 4.6. Corollary 4.5 demonstrates how Proposition 4.4 can be used to show that deformed $G_2$-instantons can discriminate between $G_2$-structures on $X$; in particular, between the two natural nearly parallel $G_2$-structures on $X$. We also see that the family of deformed $G_2$-structures for these two nearly parallel $G_2$-structures has different dimensions, and there is no obvious relation between them.

Corollary 4.7. Recall the two Einstein metrics $g^{ts}$ and $g^{np}$ on $X$ given in (2.5) and (2.6), and recall that $\varphi_1,\varepsilon$ induces $g^{ts}$ and $\varphi_1/\sqrt{5},\varepsilon$ induces $g^{np}$ for $\varepsilon \in \{\pm 1\}$.

- Using the ansatz (4.1), there is a circle of non-trivial deformed $G_2$-instantons with respect to $\varphi_{1,-1}$, whereas there are no non-trivial deformed $G_2$-instantons with respect to $\varphi_{1,+1}$.
- Using the ansatz (4.1), there is a circle plus two isolated examples of non-trivial deformed $G_2$-instantons with respect to $\varphi_{1/\sqrt{5},-1}$, whereas there is a 2-sphere of non-trivial deformed $G_2$-instantons with respect to $\varphi_{1/\sqrt{5},+1}$.

Remark 4.8. Corollary 4.7 indicates how Proposition 4.4 shows that deformed $G_2$-instantons can be used to distinguish between isometric $G_2$-structures on $X$; in particular, between the two natural Einstein metrics on $X$. However, we observe that for these two Einstein metrics, whilst the spaces of deformed $G_2$-instantons are very different for the two isometric $G_2$-structures, their Euler characteristics are the same. This is pertinent since one might hope to use the Euler characteristic of the moduli space as a possible enumerative invariant for deformed $G_2$-instantons.

We give a concrete example of the construction.

Example 4.9. Take the 7-sphere $S^7$ as in Example 2.4 and let $L_0$ be the trivial complex line bundle over $S^7$. Corollary 4.5 gives a 2-sphere of deformed $G_2$-instantons on $L_0$ over $(S^7,\varphi^{np})$, and a circle of deformed $G_2$-instanton on $L_0$ over $(S^7,\varphi^{ts})$. On the other hand, Corollary 4.7 shows that we have a family of deformed $G_2$-instantons on $L_0$ consisting of a circle plus two further points for another $G_2$-structure inducing the squashed metric $g^{np}$, and we have no known non-trivial deformed $G_2$-instantons on $L_0$ for another $G_2$-structure inducing the round metric $g^{ts}$.

In this way, deformed $G_2$-instantons on the trivial complex line bundle can be used to distinguish between the two homogeneous nearly parallel $G_2$-structures on $S^7$, and between isometric $G_2$-structures for the two homogeneous Einstein metrics on $S^7$.

4.3. Non-trivial line bundles. One may ask whether there are non-trivial examples of deformed $G_2$-instantons on non-trivial line bundles. The natural approach is to use a non-trivial bundle $L$ over $Z$ equipped with an anti-self-dual connection $A_0$, using the ideas in subsection 3.2. We shall give a particular example, which may clearly be generalized to other 3-Sasakian 7-manifolds, of a existence result for non-trivial deformed $G_2$-instantons on non-trivial line bundles.

Consider the setting of Example 3.8, where $X$ is the Aloff–Wallach space $\text{SU}(3)/\text{U}(1)_{1,1}$, where $\text{U}(1)_{1,1}$ is given in (3.6), and recall that $Z = \mathbb{CP}$. Let $L = \mathcal{O}_{\mathbb{CP}^2}(k)$ for some $k \in \mathbb{Z}$, and let $A_0$ be the connection on $L$ with harmonic curvature (which must be an ASD instanton as observed in Example 3.8).

Recalling the $\eta_i$ in (2.2), we may consider a connection $A$ on $\pi^*L$ over $X$ given by

\begin{equation}
A = \pi^*A_0 + ia, \quad \text{for} \quad a = a_j\eta_j,
\end{equation}

for $i = 1, \ldots, 7$.
where \(a_1, a_2, a_3 \in \mathbb{R}\). The curvature of \(A\) is \(F_A = F_{A_0} + i \alpha a\) and as \(A_0\) is anti-self-dual we have \(\pi^* F_{A_0} \wedge \psi_{t, \varepsilon} = 0\) by Lemma 3.6. Hence,

\[(4.11) \quad F_A \wedge \psi_{t, \varepsilon} = i \alpha a \wedge \psi_{t, \varepsilon} = -2i ((1 - 2 \varepsilon t^2) \left( a_1 \eta_{23} + a_2 \eta_{31} \right) + (1 - 2t^2) a_3 \eta_{12}) \wedge \pi^* \text{vol}_Z,\]

by (4.3). As for the cubic term in the curvature we find that

\[(4.12) \quad F_A^3 = 3i \pi^* F_{A_0}^2 \wedge da - 3 \pi^* F_{A_0} \wedge (da)^2 - i (da)^3\]
as \(F_{A_0}^3 = 0\) for dimensional reasons. By inspection, we see that \((da)^2 = \beta_i \wedge \omega_i\) for some 2-forms \(\beta_i\). As the \(\omega_i\) are self-dual and \(F_{A_0}\) is anti-self-dual we find that \(F_{A_0} \wedge (da)^2 = 0\). Hence, (4.12) becomes

\[(4.13) \quad F_A^3 = -i (da)^3 + 3i \pi^* F_{A_0}^2 \wedge da.\]
The first term in (4.13) is given by the right-hand side of (4.5) (multiplied by 6). As for the second term, we find that \(F_{A_0}^2 = |F_{A_0}|^2 \text{vol}_Z\), with \(|\cdot|\) denoting the norm with respect to \(g_Z\), since \(F_{A_0}\) is anti-self-dual (recalling that \(F_{A_0}\) is imaginary-valued). Thus, using (4.2), we see that

\[(4.14) \quad 3i \pi^* F_{A_0}^2 \wedge da = -6i \pi^* |F_{A_0}|^2 (a_1 \eta_{23} + a_2 \eta_{31} + a_3 \eta_{12}) \wedge \pi^* \text{vol}_Z.\]

Inserting (4.5) and (4.14) in (4.13) shows that the deformed G_2-instanton equation (4.4) for \(\psi_{t, \varepsilon}\) is equivalent to

\[(4.15) \quad (8(a_1^2 + a_2^2 + a_3^2) - \pi^* |F_{A_0}|^2 - 2(1 - 2 \varepsilon t^2)(a_1 \eta_{23} + a_2 \eta_{31}) + (8(a_1^2 + a_2^2 + a_3^2) - \pi^* |F_{A_0}|^2 - 2(1 - 2t^2)) a_3 \eta_{12} = 0.\]

At this point, we use the fact that \(F_{A_0} = ik\omega\), where \(\omega\) is the Fubini–Study form on \(\mathbb{CP}^2\), and thus \(|F_{A_0}|^2 = 2k^2\). (Note that we need \(|F_{A_0}|^2\) to be constant in order for (4.15) to have a solution for constant \(a_1, a_2, a_3\) which are not all zero.) Inserting this into (4.15) for \(\varepsilon = +1\) gives that non-trivial solutions must satisfy

\[(4.16) \quad a_1^2 + a_2^2 + a_3^2 = \frac{1}{4} (1 - 2t^2 + k^2).\]

We see that (4.16) has non-trivial solutions for \(a_1, a_2, a_3\) if and only if \(2t^2 < 1 + k^2\). We therefore obtain deformed G_2-instantons with respect to \(\psi_{t, +1}\) on \(\pi^* \mathcal{O}(k)\) for \(k \neq 0\) arising from the ansatz (4.10) which are not given by the pullback of the ASD instanton on \(\mathcal{O}(k)\) (which is a deformed G_2-instanton by Lemma 3.6) for these values of \(t\).

If we instead look at (4.15) for \(\varepsilon = -1\) then for non-trivial solutions we either have

\[(4.17) \quad a_1^2 + a_2^2 = \frac{1}{4} (1 + 2t^2 + k^2) \quad \text{and} \quad a_3 = 0, \quad \text{or} \]

\[(4.18) \quad a_1^2 + a_2^2 = 0 \quad \text{and} \quad a_3^2 = \frac{1}{4} (1 - 2t^2 + k^2).\]

We see that (4.17) admits non-trivial solutions for all \(t\), whereas (4.18) admits non-trivial solutions if and only if \(2t^2 < 1 + k^2\), just as for (4.16).

Overall, we have the following proposition, which generalizes Proposition 4.4 for the case of the Aloff–Wallach space \(SU(3)/U(1)^{1,1}\).

**Proposition 4.10.** Let \(X\) be the Aloff–Wallach space \(SU(3)/U(1)^{1,1}\) as in Example 3.8, with projection \(\pi : X \to Z = \mathbb{CP}^2\). Let \(L = \mathcal{O}(k)\) on \(Z\) for \(k \in \mathbb{Z}\), let \(A_0\) be the unitary ASD instanton on \(L\) and let \(A\) be a connection on \(\pi^* L\) as in (4.10) which is a deformed G_2-instanton
with respect to $\psi_{t,\varepsilon}$ given in (2.9). Then either $a_1 = a_2 = a_3 = 0$, and so $A = \pi^* A_0$ (and thus is a $G_2$-instanton), or one of the following holds:

- $t \in (0, (1 + k^2)/2)$, $\varepsilon = +1$ and $a_1, a_2, a_3$ satisfy (4.16);
- $t \in (0, (1 + k^2)/2)$, $\varepsilon = -1$ and $a_1, a_2, a_3$ satisfy (4.17) or (4.18);
- $t \geq (1 + k^2)/2$, $\varepsilon = -1$ and $a_1, a_2, a_3$ satisfy (4.17).

Remark 4.11. We see that (4.18) has non-trivial solutions for $t = 1$ if and only if $|k| > 2$.

In particular, Proposition 4.10 gives non-trivial deformed $G_2$-instantons on $\pi^* \mathcal{O}(k)$ over $X$ with respect to the $G_2$-structure $\varphi_{1,+1}$, which induces the 3-Sasakian metric $g^{ts}$, if and only if $|k| > 2$.

Moreover, when $|k| > 2$, Proposition 4.10 gives a 2-sphere of non-trivial deformed $G_2$-instantons on $\pi^* \mathcal{O}(k)$ with respect to $\varphi_{1,+1}$, whereas it gives a family of deformed $G_2$-instantons consisting of a circle and two points with respect to $\varphi^{ts}$.

Proposition 4.10 has the following immediate corollary.

Corollary 4.12. Let $X$ be the Aloff–Wallach space $SU(3)/U(1)_{1,1}$ as in Example 3.8, with projection $\pi : X \to \mathbb{CP}^2$, and recall the nearly parallel $G_2$-structures $\varphi^{np}$ and $\varphi^{ts}$ on $X$ given in (2.10).

- For every $k \in \mathbb{Z}$, there is a 2-sphere of non-trivial deformed $G_2$-instantons on $\pi^* \mathcal{O}(k)$ with respect to $\varphi^{np}$.
- If $k \in \{0, \pm 1\}$, there is a circle of non-trivial deformed $G_2$-instantons on $\pi^* \mathcal{O}(k)$ with respect to $\varphi^{ts}$, and if $k \in \mathbb{Z} \setminus \{0, \pm 1\}$ there is a circle and two further examples of non-trivial deformed $G_2$-instantons on $\pi^* \mathcal{O}(k)$ with respect to $\varphi^{ts}$.

Remark 4.13. Proposition 4.10 continues to demonstrate how deformed $G_2$-instantons can distinguish between nearly parallel $G_2$-structures and isometric $G_2$-structures. Moreover, the observed equality between the Euler characteristics of the families of deformed $G_2$-structures for isometric $G_2$ structures continues to hold in this setting.

4.4. Moduli spaces. We now make some observations on the families of deformed $G_2$-instantons we have constructed, and their relation to the deformation theory of deformed $G_2$-instantons developed in [KY20]. To state the deformation theory result we recall the following definition.

Definition 4.14. For a 7-manifold $X$ with a coclosed $G_2$-structure $\varphi$ and a Hermitian complex line bundle $L$ on $X$, we let $\mathcal{M}(X, \varphi, L)$ denote the moduli space of deformed $G_2$-instantons on $L$ with respect to $\varphi$. Let $A \in \mathcal{M}(X, \varphi, L)$ and consider the complex

$$
0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{(\frac{1}{2} F_A^2 + * \varphi) \wedge d} \Omega^5(X) \longrightarrow 0.
$$

Then $A$ is unobstructed if $H^2 = 0$ for (4.19), i.e. if the linearisation of the deformed $G_2$-instanton condition $(\frac{1}{2} F_A^2 + * \varphi) \wedge d : \Omega^1(X) \to \Omega^5(X)$ is surjective; otherwise $A$ is obstructed.

We now state a deformation theory result that follows from [KY20], which motivates the definition of unobstructed in Definition 4.14.

Theorem 4.15. Let $(X^7, \varphi)$ be a compact 7-manifold with a coclosed $G_2$-structure, and let $L$ be a Hermitian complex line bundle on $X$. Then $\mathcal{M}(X, \varphi, L)$ has expected dimension $b^1(X)$.

Therefore, if $A \in \mathcal{M}(X, \varphi, L)$ is unobstructed, then $\mathcal{M}(X, \varphi, L)$ is a smooth manifold near $A$ of dimension $b^1(X)$.

Moreover, if $A \in \mathcal{M}(X, \varphi, L)$ and

- $d \varphi = 0$, or
• $F_A^3 \neq 0$ everywhere,
then for generic coclosed $G_2$-structures $\varphi'$ sufficiently near $\varphi$ so that $[\ast \varphi' \varphi'] = [\ast \varphi \varphi] \in H^4(X)$, the subset of $\mathcal{M}(X, \varphi', L)$ of connections sufficiently near $A$ is a smooth manifold of dimension $b^1(X)$ (if it is non-empty).

Theorem 4.15 has the following consequence.

**Corollary 4.16.** Let $(X^7, \varphi)$ be a compact 7-manifold with a coclosed $G_2$-structure, and suppose that $X^7$ admits a nearly parallel $G_2$-structure. Let $L$ be a Hermitian complex line bundle on $X$. Then the expected dimension of $\mathcal{M}(X, \varphi, L)$ is 0. In particular, if $A \in \mathcal{M}(X, \varphi, L)$ is unobstructed then $A$ is rigid and locally isolated in $\mathcal{M}(X, \varphi, L)$.

Moreover, given $A \in \mathcal{M}(X, \varphi, L)$ such that $F_A^3 \neq 0$ everywhere, for generic coclosed $G_2$-structures $\varphi'$ near $\varphi$ with $[\ast \varphi' \varphi'] = [\ast \varphi \varphi]$, the subset of $\mathcal{M}(X, \varphi', L)$ of connections sufficiently near $A$ is a discrete collection of points (if it is non-empty).

**Proof.** The result follows from Theorem 4.15 and the fact that $X^7$ must have finite fundamental group by Myers theorem, since the induced metric from a nearly parallel $G_2$-structure is Einstein with positive scalar curvature. \qed

Since the non-trivial deformed $G_2$-instantons given by Corollary 4.5 are not rigid for the nearly parallel $G_2$-structures $\varphi^{np}$ and $\varphi^{ts}$ on the $3$-Sasakian $X^7$, Corollary 4.16 yields the following result.

**Proposition 4.17.** All of the non-trivial deformed $G_2$-instantons from Propositions 4.4 and 4.10 that exist in positive-dimensional families are obstructed.

In particular, the non-trivial deformed $G_2$-instantons on $(X^7, \varphi^{ts})$ and $(X^7, \varphi^{np})$ on the trivial line bundle $L_0$ from Proposition 4.4 are obstructed.

**Remark 4.18.** We see from (4.5) and Propositions 4.4 and 4.10 that all of the non-trivial deformed $G_2$-instantons $A$ with respect to $\varphi_{t,\varepsilon}$ we have constructed have the property that $F_A^3 \neq 0$ everywhere, and thus Corollary 4.16 applies. Moreover, when $\varepsilon = +1$ we see that $[\hat{\varphi}_{t+1}] = [\hat{\varphi}_{t'+1}]$ for all $t, t'$. However, $\mathcal{M}(X, \varphi_{t'+1}, L_0)$ still contains an $S^2$ family of deformed $G_2$-instantons near $A$ for all $t'$ near $t$. We conclude that the family $\varphi_{t+1}$ is not sufficiently generic to enable us to peturb $A$ to become locally isolated.

We now make an elementary observation, which follows from Remark 5.12 in [KY20].

**Lemma 4.19.** Let $(X^7, \varphi)$ be a compact 7-manifold with a nearly parallel $G_2$-structure, and let $L_0$ be the trivial complex line bundle on $X$. Then the trivial flat connection is unobstructed as a deformed $G_2$-instanton, thus rigid and locally isolated in $\mathcal{M}(X, \varphi, L_0)$.

**Lemma 4.19** yields the following interesting result.

**Proposition 4.20.** For any 3-Sasakian 7-manifold $X$, the moduli spaces $\mathcal{M}(X, \varphi^{np}, L_0)$ and $\mathcal{M}(X, \varphi^{ts}, L_0)$ have at least two components of different dimensions.

**Proof.** Since the trivial flat connection $A_0$ lies in $\mathcal{M}(X, \varphi^{np}, L_0)$ and $\mathcal{M}(X, \varphi^{ts}, L_0)$ and is locally isolated, it must define a component of the moduli space in each case. Therefore, in each case the positive-dimensional family of non-trivial deformed $G_2$-instantons from Proposition 4.4 must lie in a different component of the moduli space to the trivial flat connection. \qed

5. **A Chern–Simons type functional**

In this section, we study a functional of Chern–Simons type, introduced in [KL09], which has deformed $G_2$-instantons as critical points, in the setting of our examples.
5.1. **The functional.** Let $X^7$ be a compact 7-manifold with a coclosed $G_2$-structure $\varphi$. (The assumption of compactness is to ensure that integrals of various quantities over $X$ are guaranteed to be well-defined.) Let $L$ be a Hermitian complex line bundle over $X$ and fix a unitary connection $A_0$ on $L$. We let $t$ denote a coordinate on $[0,1]$ and we shall pullback various quantities from $X$ to $X \times [0,1]$ such as $\psi = *_\varphi \varphi$ and $L$; for ease of notation, we shall omit the pullback symbol.

Then, for any other unitary connection $A$ on $L$ we consider a unitary connection $A_t$ on the pullback of $L$ to $[0,1] \times X$, given by

$$A_t = A_0 + t(A - A_0).$$

We let $A_t = A|_{\{t\} \times X}$ and let $F_t$ be the curvature of $A_t$, which is

$$F_t = F_{A_0} + t(F_A - F_{A_0}).$$

Hence, the curvature of $A$ in (5.1) can be written

$$F = dt \wedge (A - A_0) + F_t.$$

With this notation, we can make the following definition.

**Definition 5.1.** Let $(X^7, \varphi)$ be a compact 7-manifold with a coclosed $G_2$-structure and let $L$ be a Hermitian complex line bundle on $X$. Given a unitary connection $A_0$ on $L$ we define the functional $\mathcal{F}$ on unitary connections $A$ on $L$ over $X$ by the formula:

$$\mathcal{F}(A) = \int_{X \times [0,1]} e^{\mathcal{F} + \psi},$$

where $\psi = *_\varphi \varphi$ and $\mathcal{F}$ is given in (5.3). It is shown in [KL09] that the deformed $G_2$-instanton equation (1.1) arises as the critical point equation for the functional $\mathcal{F}$.

Since $X$ is 7-dimensional, it is convenient to expand and rewrite $\mathcal{F}$ in (5.4) as follows:

$$\mathcal{F}(A) = \int_{X \times [0,1]} \frac{1}{3!}(\mathcal{F} + \psi)^3 + \frac{1}{4!}(\mathcal{F} + \psi)^4 = \frac{1}{2} \int_{X \times [0,1]} \mathcal{F}^2 \wedge \psi + \frac{1}{12} \mathcal{F}^4.$$

On the other hand, we have $\mathcal{F}^k = F_{A_0}^k + d(cs_k(A_0, A))$ on $[0,1] \times X$, where $\mathcal{F}$ is given in (5.3) and $cs_k(A_0, A)$ is the $k$th transgression form. This can be explicitly written using the curvature $\mathcal{F}_s$ of the connections $A_s = A_0 + s(A - A_0)$ for $s$ a real valued parameter. Indeed, we have

$$cs_k(A_0, A) = k \int_0^1 (A - A_0) \wedge \mathcal{F}_s^{k-1} \, ds.$$

Thus, from Stokes’ theorem and the fact that $F_{A_0}^2 \wedge \psi = 0 = F_{A_0}^4$ by dimensional reasons, we may write (5.5) as:

$$\mathcal{F}(A) = \frac{1}{2} \int_X cs_2(A_0, A) \wedge \psi + \frac{1}{12} cs_4(A_0, A).$$

In particular, we have the following observation.

**Lemma 5.2.** In the notation of Definition 5.1, let $L = L_0$ be the trivial complex line bundle and let $A_0$ be the trivial flat connection. Then the functional $\mathcal{F}$ in (5.4) is given by

$$\mathcal{F}(A) = \frac{1}{2} \int_X (A - A_0) \wedge (F_A \wedge \psi + \frac{1}{12} F_A^3).$$

**Proof.** The result follows from (5.7) and the observation that $cs_k(A_0, A) = (A - A_0) \wedge F_A^{k-1}$ by (5.6) as $F_{A_0} = 0$. 

□
5.2. Examples. For connections as in (4.1) on 3-Sasakian 7-manifolds we find, after a lengthy but straightforward computation, the following using (5.8).

**Proposition 5.3.** Let \( X^7 \) be a 3-Sasakian 7-manifold with the coclosed \( G_2 \)-structure \( \varphi_{t,\varepsilon} \) on (2.7), and recall the 3-Sasakian metric \( g^{ts} \) in (2.5). For connections \( A = i_{a_j} n_j \) as in (4.1) on the trivial complex line bundle \( L_0 \) on \( X \), we have that the functional \( F \) in (5.4) is given by

\[
F(A) = -c \left[ (x^2 + y^2)(2(x^2 + y^2) - 1) + 2t^2(x^2 + \varepsilon y^2) \right],
\]

where \( x = a_3, y^2 = a_1^2 + a_2^2 \) and

\[
c = 2 \int_X \eta_{123} \wedge \pi^*\text{vol}_Z = 2\text{Vol}(X, g^{ts}) > 0.
\]

**Remark 5.4.** Notice that \( F \) in (5.9) is bounded above and that, as can be seen from (5.10), the 3-Sasakian metric \( g^{ts} \) may be rescaled so that \( c = 1 \), which we will now do for convenience.

**Remark 5.5.** The critical points of the functional \( F \) in (5.9), restricted to connections given by the ansatz (4.1), are given by the vanishing of

\[
\begin{align*}
\frac{\partial F}{\partial x} &= -2x \left( 4(x^2 + y^2) + 2t^2 - 1 \right), \\
\frac{\partial F}{\partial y} &= -2y \left( 4(x^2 + y^2) + 2\varepsilon t^2 - 1 \right).
\end{align*}
\]

We see that we have solutions to (5.11)–(5.12) given by \( x = 0 = y \), corresponding to the flat connection, and

\[
x^2 + y^2 = \frac{1}{4}(1 - 2t^2) \quad \text{when } \varepsilon = 1 \text{ and } 2t^2 < 1,
\]

or \( \varepsilon = -1 \) and either

\[
x = 0 \text{ and } y^2 = \frac{1}{4}(1 + 2t^2), \quad \text{or} \quad y = 0 \text{ and } x^2 = \frac{1}{4}(1 - 2t^2) \quad \text{when } 2t^2 < 1.
\]

Equations (5.13)–(5.14) coincide with the conditions (4.16)–(4.18) we derived earlier that gave our non-trivial deformed \( G_2 \)-instantons in Proposition 4.4.

Using Proposition 5.3 we can examine the relationship between the trivial flat connection and the functional \( F \). In particular, we see that the nature of the trivial connection as a critical point for \( F \) depends on the choice of \( G_2 \)-structure.

**Lemma 5.6.** Recall the notation of Proposition 5.3, in particular the functional \( F \) in (5.9). Let \( A_0 \) be the trivial flat connection on the trivial complex line bundle \( L_0 \) over \( X \). Then the Hessian of \( F \) is nondegenerate at \( A_0 \) if and only if \( t \neq 1/\sqrt{2} \). Moreover:

- if \( t \in (0, 1/\sqrt{2}) \) then \( A_0 \) is a local minimum of \( F \);
- if \( t \geq 1/\sqrt{2} \) and \( \varepsilon = +1 \) then \( A_0 \) is a local maximum of \( F \);
- if \( t \geq 1/\sqrt{2} \) and \( \varepsilon = -1 \) then \( A_0 \) is a saddle point of \( F \).

**Proof.** By direct computation, one can calculate the Hessian of the functional \( F \) in (5.9). At the flat connection \( A_0 \), when \( (x, y) = (0, 0) \), the Hessian of \( F \) has eigenvalues

\[
2(1 - 2t^2) \quad \text{and} \quad 2(1 - 2\varepsilon t^2).
\]

We see immediately that the Hessian of \( F \) is degenerate if and only if \( 2t^2 = 1 \) as claimed. Moreover, as long as \( 2t^2 \neq 1 \), the critical point is characterised by the signs of the eigenvalues in (5.15). When \( 2t^2 = 1 \) we see that

\[
F(A) = -2(x^2 + y^2)^2 + (1 - \varepsilon)y^2.
\]
When \( \varepsilon = +1 \), \( F \leq 0 \) and equals 0 if and only if \( x = y = 0 \). When \( \varepsilon = -1 \), we instead see that inserting \( x = 0 \) in (5.16) gives a function of \( y \) with a local minimum at \( y = 0 \), and for \( y = 0 \) in (5.16) we obtain a local maximum at \( x = 0 \). The result then follows. \( \square \)

We already observed the significance of the value \( t = 1/\sqrt{2} \) in Proposition 4.4. Lemma 5.6 now leads to the following additional observation concerning this value of \( t \).

**Corollary 5.7.** The trivial flat connection on the trivial complex line bundle over \( X \) is obstructed as a deformed \( G_2 \)-instanton for the \( G_2 \)-structures \( \varphi_{1/\sqrt{2}, \varepsilon} \) in (2.7) for \( \varepsilon = \pm 1 \).

**Proof.** When \( t = 1/\sqrt{2} \), Lemma 5.6 shows that the Hessian of \( F \) in (5.9) is degenerate at the trivial flat connection \( A_0 \). Thus, there exist non-trivial infinitesimal deformations of \( A_0 \) as a deformed \( G_2 \)-instanton within the ansatz (4.1), i.e. \( H^1 \) of the complex (4.19) is non-zero.

However, we know from Proposition 4.4 that \( A_0 \) is locally isolated as a deformed \( G_2 \)-instanton for \( t = 1/\sqrt{2} \) amongst those given by (4.1), and so the non-trivial infinitesimal deformation must be obstructed, i.e. \( H^2 \) of the complex (4.19) must also be non-zero. \( \square \)

**Remark 5.8.** The fact that the flat connection is obstructed at \( t = 1/\sqrt{2} \) was to be expected, as it is at this point that the transition occurs when the 2-sphere or two points consisting of non-flat deformed \( G_2 \)-instantons “shrinks” and merges with the flat connection (see Figure 1). Nevertheless, this observation contrasts with the case of nearly parallel \( G_2 \)-structures for which the flat connection is always unobstructed (see Lemma 4.19).

To conclude, we compare the functional \( F \) in (5.9) for the pairs of isometric \( G_2 \)-structures \( \varphi_{1/\sqrt{5}, \varepsilon} \) and \( \varphi_{1, \varepsilon} \), for \( \varepsilon \in \{ \pm 1 \} \), which induce the Einstein metrics \( g^{np} \) and \( g^{ts} \) respectively.

**Example 5.9.** The coclosed \( G_2 \)-structures \( \varphi_{1/\sqrt{5}, +1} = \varphi^{np} \) and \( \varphi_{1/\sqrt{5}, -1} \) both induce the strictly nearly parallel metric \( g^{np} \) on the 3-Sasakian 7-manifold \( X \), which has the property that the metric cone on \( (X, g^{np}) \) has holonomy \( \text{Spin}(7) \). These \( G_2 \)-structures determine rather different functionals \( F \) by Proposition 4.4. Figure 2 plots the functional \( F \), restricted to connections given by the ansatz (4.1), as in (5.9) for these two \( G_2 \)-structures.

![Figure 2](image-url)

**Figure 2.** The functional \( F \) for \((t, \varepsilon) = (1/\sqrt{5}, +1) \) and \((t, \varepsilon) = (1/\sqrt{5}, -1) \).

One can just discern the local minimum at the origin in the left-hand plot in Figure 2, predicted by Lemma 5.6. We also illustrate the difference between these cases via their levels.
sets in Figure 3. One can see the circle of critical points (which are local maxima) for $\varepsilon = +1$ which gives the 2-sphere of non-trivial deformed $G_2$-instantons in Proposition 4.4, in contrast to the two pairs of critical points for $\varepsilon = -1$ which give the circle (given by local maxima) and two further examples (which are saddle points) of non-trivial deformed $G_2$-instantons in Proposition 4.4.

![Figure 3](image1)

**Figure 3.** Level sets of $F$ for $(t, \varepsilon) = (1/\sqrt{5}, +1)$ and $(t, \varepsilon) = (1/\sqrt{5}, -1)$.

**Example 5.10.** We now focus on the coclosed $G_2$-structures $\varphi_{1,\varepsilon}$, for $\varepsilon = \pm 1$, recalling that $\varphi^{ts} = \varphi_{1,-1}$. These $G_2$-structures induce the 3-Sasakian metric $g^{ts}$ on $X$, which is that whose metric cone is hyperkähler. In this case, we again already know the functionals $F$ for these two $G_2$-structures are very different by Proposition 4.4, and further evidence is provided by the following plots of the functional $F$ in (5.9) in Figure 4.

![Figure 4](image2)

**Figure 4.** The functional $F$ for $(t, \varepsilon) = (1, +1)$ and $(t, \varepsilon) = (1, -1)$.

As for the level sets of the functional $F$, these are plotted in Figure 5 for each case. For $\varepsilon = +1$ we see that the only critical point is the origin, giving the trivial flat connection, and
instead we have a pair of critical points (which are local maxima) for $\varepsilon = -1$ which define a circle of deformed $G_2$-instantons from Proposition 4.4.

Figure 5. Level sets of $\mathcal{F}$ for $(t, \varepsilon) = (1, +1)$ and $(t, \varepsilon) = (1, -1)$.

REFERENCES

[BG01] C. Boyer and K Galicki, 3-Sasaki manifolds, Surveys in Differential Geometry, vol. 6; Essays on Einstein Manifolds, 2001. ↑4
[BGM94] C.P. Boyer, K. Galicki, and B.M. Mann, The geometry and topology of 3-Sasakian manifolds, J. Reine Angew. Math. 455 (1994), 183–220. ↑2
[BO19] G. Ball and G. Oliveira, Gauge theory on Aloff-Wallach spaces, Geom. Topol. 23 (2019), no. 2, 685–743. ↑7, 8, 9
CARDSE20] O. Calvo-Andrade, L.O. Rodríguez Díaz, and H.N. Sá Earp, Gauge theory and $G_2$-geometry on Calabi–Yau links, Rev. Mat. Iberoam. (2020), to appear in print. ↑7
[CGFT20] A. Clarke, M. García-Fernandez, and C. Tipler, T-dual solutions and infinitesimal moduli of the $G_2$-Strominger system, ArXiv e-prints (2020), available at 2005.09977. ↑9
[Che20] G. Chen, Supercritical deformed Hermitian-Yang–Mills equation, ArXiv e-prints (2020), available at 2005.12202. ↑7
[CO20] A. Clarke and G. Oliveira, Spin(7)-instantons from evolution equations, J. Geom. Anal. (2020), to appear in print. ↑10
[Don06] S. K. Donaldson, Two-forms on four-manifolds and elliptic equations, Inspired by S. S. Chern, 2006, pp. 153–172. ↑8
[FKMS97] Th. Friedrich, I. Kath, A. Moroianu, and U. Semmelmann, On nearly parallel $G_2$-structures, J. Geom. Phys. 23 (1997), 259–286. ↑5
[FY18] J. Fine and C. Yao, Hypersymplectic 4-manifolds, the $G_2$-Laplacian flow, and extension assuming bounded scalar curvature, Duke Math. J. 167 (2018), no. 18, 3533–3589. ↑8
[GWZ08] K. Grove, B. Wilking, and W. Ziller, Positively curved cohomogeneity one manifolds and 3-Sasakian geometry, J. Differential Geom. 78 (2008), no. 1, 33–111. ↑2
[KL09] S. Karigiannis and N. C. Leung, Hodge theory for $G_2$-manifolds: intermediate Jacobians and Abel-Jacobi maps, Proc. Lond. Math. Soc. (3) 99 (2009), no. 2, 297–325. ↑3, 14, 15
[KL20] S. Karigiannis, N.C. Leung, and J.D. Lotay (eds.), Lectures and surveys on $G_2$-manifolds and related topics, Springer, 2020. ↑5
[KY20] K. Kawai and H. Yamamoto, Deformation theory of deformed Hermitian Yang–Mills connections and deformed Donaldson–Thomas connections, ArXiv e-prints (2020), available at 2004.00532. ↑3, 6, 13, 14
[LL09] J.-H. Lee and N. C. Leung, Geometric structures on $G_2$ and Spin(7)-manifolds, Adv. Theor. Math. Phys. 13 (2009), no. 1, 1–31. ↑1, 2
[LYZ00] N. C. Leung, S.-T. Yau, and E. Zaslow, From special Lagrangian to Hermitian-Yang–Mills via Fourier–Mukai transform, Adv. Theor. Math. Phys. 4 (2000), no. 6, 1319–1341. ↑6

[MMMS00] M. Mariño, R. Minasian, G. Moore, and A. Strominger, Nonlinear instantons from supersymmetric $p$-branes, J. High Energy Phys. 1 (2000), Paper 5, 32. ↑6

[Wan18] Y. Wang, Moduli spaces of $G_2$ and Spin(7)-instantons on product manifolds, ArXiv e-prints (2018), available at 1803.05899. ↑6

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