LATTICES AND PARAMETER REDUCTION IN DIVISION ALGEBRAS

M. LORENZ AND Z. REICHSTEIN

Abstract. Let $k$ be an algebraically closed field of characteristic 0 and let $D$ be a division algebra whose center $F$ contains $k$. We shall say that $D$ can be reduced to $r$ parameters if we can write $D \simeq D_0 \otimes_{F_0} F$, where $D_0$ is a division algebra, the center $F_0$ of $D_0$ contains $k$ and trdeg$_{k}(F_0) = r$.

We show that every division algebra of odd degree $n \geq 5$ can be reduced to $\leq \frac{1}{2}(n-1)(n-2)$ parameters. Moreover, every crossed product division algebra of degree $n \geq 4$ can be reduced to $\leq (\lfloor \log_2(n) \rfloor - 1)n + 1$ parameters. Our proofs of these results rely on lattice-theoretic techniques.

Contents

1. Introduction 2
2. Preliminaries 3
  2.1. $G$-varieties 3
  2.2. $(G, H)$-sections and compressions 4
  2.3. $G$-lattices 5
  2.4. The symmetric and exterior squares 5
3. Groups of the form $T_{n-1} \rtimes G$ and lattices 5
  3.1. Notations 5
  3.2. $T_{n-1} \rtimes G$-varieties and $G$-lattices 5
  3.3. Compressions and $G$-lattices 6
  3.4. Linearization and essential dimension 7
4. Proof of Theorem 1.1 7
  4.1. Reduction to a lattice-theoretic problem 7
  4.2. Solution of the lattice-theoretic problem 9
5. Proof of Theorem 1.2 11
  5.1. General observations 11
  5.2. Conclusion of the proof 14
6. Algebras of degree four 15
References 18

1991 Mathematics Subject Classification. 16K20, 14L30, 20C10, 20J06.

Key words and phrases. division algebra, integral representation, lattice, algebraic transformation group, essential dimension, crossed product, central simple algebra.

M. Lorenz was supported in part by NSF grant DMS-9618521.
Z. Reichstein was supported in part by NSF grant DMS-9801675.
1. Introduction

Throughout this paper, $k$ denotes a (fixed) algebraically closed base field of characteristic zero. Let $K$ be a field containing $k$ and let $A$ be a finite-dimensional $K$-algebra. We would like to write $A$ as $A = A_0 \otimes_{K_0} K$ for some $K_0$-algebra $A_0$ over an intermediate field $k \subseteq K_0 \subseteq K$ with $\text{trdeg}_K(K_0)$ as low as possible; the minimal value of $\text{trdeg}_K(K_0)$ will be denoted by $\tau(A)$. Note that if $\text{trdeg}_K(K_0) < \text{trdeg}_K(K)$ then passing from $A$ to $A_0$ may be viewed as “parameter reduction” in $A$.

We shall be particularly interested in the case where $A = \text{UD}(n)$ is the universal division algebra of degree $n$ and $K$ is the center of $\text{UD}(n)$ which we shall denote by $Z(n)$. Recall that $\text{UD}(n)$ is the subalgebra of $M_n(k(x_{ij}, y_{ij}))$ generated (as a division algebra) by two generic $n \times n$-matrices $X = (x_{ij})$ and $Y = (y_{ij})$, where $x_{ij}$ and $y_{ij}$ are $2n^2$ independent variables over $k$; see, e.g., [Pr, Section II.1] or [Row, Section 3.2]. We will denote $\tau(\text{UD}(n))$ by $d(n)$. It is easy to show that $d(n) \geq \tau(A)$ for any central simple algebra $A$ of degree $n$ whose center contains $k$ (see, e.g., [Re, Lemma 9.2]); in other words, every central simple algebra of degree $n$ can be “reduced to at most $d(n)$ parameters”. In the language of [Re2], $d(n) = \text{ed}(\text{PGL}_n)$, where $\text{ed}$ denotes the essential dimension; see [Re2, Lemma 9.2].

To the best of our knowledge, the earliest attempt to determine the value of $d(n)$ is due to Procesi, who showed that $d(n) \leq n^2$; see [Pr, Thm. 2.1]. Note that if $\text{UD}(n)$ is cyclic then $d(n) = 2$, because we can take $A_0$ to be a symbol algebra; cf. [Re2, Lemma 9.4]. This is known to be the case for $n = 2, 3$ and 6. For other $n$ the exact value of $d(n)$ is not known. However, the following inequalities hold:

\begin{align*}
d(n) &\leq n^2 - 2n \quad (\text{Re2, Proposition 4.5}), \\ d(n) &\leq d(nm) \leq d(n) + d(m) \quad \text{if } (n, m) = 1 \quad (\text{Re2, Section 9.4}), \\ d(n^r) &\geq 2r \quad (\text{Re1, Theorem 16.1}), \\ d(n) &\leq \frac{1}{2}(n-1)(n-2) + n \quad \text{if } n \text{ is odd} \quad (\text{Row2; cf. Re2, Section 9.3}).
\end{align*}

The last inequality is due to Rowen. In this paper we will sharpen it by showing that, in fact, $d(n) \leq \frac{1}{2}(n-1)(n-2)$ for every odd $n \geq 5$. Moreover, in $\text{UD}(n)$, reduction to this number of parameters can be arranged in a particularly nice fashion:

**Theorem 1.1.** Let $n \geq 5$ be an odd integer, $\text{UD}(n)$ be the universal division algebra of degree $n$ and $Z(n)$ be its center. Then there exists a subfield $F$ of $Z(n)$ and a division algebra $D$ of degree $n$ with center $F$ such that

(a) $\text{UD}(n) = D \otimes_F Z(n)$,
(b) $\text{trdeg}_F(F) = \frac{1}{2}(n-1)(n-2)$ and
(c) $Z(n)$ is a rational extension of $F$.

In particular, $d(n) = \text{ed}(\text{PGL}_n) \leq \frac{1}{2}(n-1)(n-2)$.

In the course of the proof of Theorem [1], we will obtain an explicit description of the center $F$ of $D$: $F \simeq k(\wedge^2 A_{n-1})^{S_n}$, where $S_n$ denotes the symmetric group on $n$ symbols and

\begin{equation}
A_{n-1} = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid a_1 + \cdots + a_n = 0\},
\end{equation}
with the natural $S_n$-action. Our argument relies on the results of [LL], where the symmetric square $\text{Sym}^2 A_{n-1}$ is shown to be stably permutation for $n$ odd; see Proposition 4.3 below.

For our next result, recall that if $A$ is a central simple algebra of degree $n$ with center $F$ and $L$ is a subfield of $A$ then $L$ is called strictly maximal if $F \subset L$ and $[L:F] = n$.

**Theorem 1.2.** Let $A$ be a finite-dimensional central simple algebra of degree $n$ with center $F$, $L$ be a strictly maximal subfield of $A$, $L^{\text{norm}}$ be the normal closure of $L$ over $F$, and $\mathcal{G} = \text{Gal}(L^{\text{norm}}/F)$. Suppose $\mathcal{G}$ is generated by $r$ elements together with $\mathcal{H} = \text{Gal}(L^{\text{norm}}/L)$. If either $r \geq 2$ or $\mathcal{H} \neq \{1\}$ then $\tau(A) \leq r|\mathcal{G}| - n + 1$.

If $A$ be a central simple algebra of degree $n$ then the upper bounds we have for $\tau(A)$ (or, equivalently, for $d(n)$), are all quadratic in $n$; see (1.1), (1.4) and Theorem 1.1. However, if we assume that $A$ is a crossed product, Theorem 1.2 yields an asymptotically better bound:

**Corollary 1.3.** Suppose a group $\mathcal{G}$ of order $n$ can be generated by $r \geq 2$ elements. Then $\tau(A) \leq (r - 1)n + 1$ for any $\mathcal{G}$-crossed product central simple algebra $A$. In particular, $\tau(A) \leq (\lfloor \log_2(n) \rfloor - 1)n + 1$, for any crossed product central simple algebra of degree $n \geq 4$. Here, as usual, $\lfloor x \rfloor$ denotes the largest integer $\leq x$.

Recall that $A$ is called a $\mathcal{G}$-crossed product if it contains a strictly maximal subfield $L$, such that $L/F$ is a Galois extension and $\text{Gal}(L/F) = \mathcal{G}$; cf. [Row], Definition 3.1.23]. Thus the first assertion of the corollary is an immediate consequence of Theorem 1.2.

The second assertion follows from the first, because any group of order $n$ can be generated by $r \leq \log_2(n)$ elements. (Indeed, $|\langle G_0, g \rangle| \geq 2|G_0|$ for any subgroup $G_0$ of $\mathcal{G}$ and any $g \in G \setminus G_0$.) Note also that $\lfloor \log_2(n) \rfloor \geq 2$ for any $n \geq 4$.

The case of central simple algebras of degree 4 is of special interest since, by a theorem of Albert, every such algebra is a $\mathbb{Z}/2 \mathbb{Z} \times \mathbb{Z}/2 \mathbb{Z}$-crossed product; see e.g., [Row], Theorem 3.2.28]. Thus Corollary 1.3 says that $d(4) = \tau(\text{UD}(4)) \leq 5$. On the other hand, $d(4) \geq 4$ by (1.3). This proves:

**Corollary 1.4.** $d(4) = \text{cd}(\text{PGL}_4) = 4$ or 5.

The rest of this paper is structured as follows. In Section 2 we discuss preliminary material from invariant theory and the theory of $\mathcal{G}$-lattices. In Section 3 we explain how $\mathcal{G}$-lattices can be used to give an upper bound on essential dimensions of certain groups. We prove Theorem 1.1 in Section 4 and Theorem 1.2 in Section 5. In Section 6 we show that the methods of this paper cannot be used to decide whether the exact value of $d(4)$ equals 4 or 5.

2. Preliminaries

2.1. $G$-varieties. A $G$-variety $X$ is an algebraic variety with a (regular) action of an algebraic group $G$. If $G$ acts freely (i.e., with trivial stabilizers) on a dense open subset of $X$, then $X$ is called a generically free $G$-variety.

A dominant rational map $\pi: X \dashrightarrow Y$ is called the rational quotient map if $k(Y) = k(X)^G$ and $\pi^*: k(Y) \hookrightarrow k(X)$ is the natural inclusion $k(X)^G \hookrightarrow k(X)$. We will usually
denote the rational quotient $Y$ by $X/G$; note that $X/G$ is only defined up to birational equivalence. By a theorem of Rosenlicht a rational quotient map separates points in general position in $X$; see [Ros1, Theorem 2] and [Ros2] (also cf. [PV, Theorem 2.3]). In other words, there exists a dense open subset $U$ of $X$ such that $f$ is regular on $U$ and $x, y \in U$ lie in the same $G$-orbit iff $f(x) = f(y)$.

We will not generally assume that $X$ is irreducible; however, we will always require $X$ to be primitive. This means that $G$ transitively permutes the irreducible components of $X$; equivalently, $X/G$ is irreducible, i.e., $k(X)^G$ is a field; cf. [Re2, Section 2.2]. Note that an irreducible $G$-variety is always primitive, and, if $G$ is connected, a primitive variety is necessarily irreducible. Thus the notion of a primitive variety is only of interest if the group $G$ is disconnected.

If $N$ is a normal subgroup of $G$ then the $G$-action on $X$ induces a (rational) $G/N$-action on $X/N$; moreover, one can choose a model $Y$ of $X/N$ such that the $G/N$-action on $Y$ is regular; see [PV, Proposition 2.6], [Re2, Remark 2.6].

**Lemma 2.1.** Let $N$ be a normal subgroup of $G$ and let $X$ be a $G$-variety. Then $X$ is generically free as a $G$-variety if and only if

(a) $X$ is generically free as an $N$-variety and
(b) $X/N$ is generically free as a $G/N$-variety.

**Proof.** Assume (a) and (b) hold. Choose $x$ in $X$ in general position and suppose $g \in \text{Stab}(x)$. Then (b) implies that $g \in N$ and (a) says that $g = 1$. This shows that the $G$-action on $X$ is generically free. The converse is proved in a similar manner. □

2.2. $(G,H)$-sections and compressions. Let $X$ be a $G$-variety and let $\pi : X \rightarrow X/G$ be the rational quotient map. Furthermore, let $H$ be a closed subgroup of $G$. An $H$-invariant subvariety $S$ of $X$ is called a $(G,H)$-section if the following conditions are satisfied.

(i) $\pi(S)$ contains an open dense subset of $X/G$ and
(ii) if $x$ is a point in general position in $S$ then $gx \in S$ if and only if $g \in H$.

Recall that a $G$-compression $X \rightarrow Y$ is a dominant rational map of generically free $G$-varieties.

**Lemma 2.2.** Suppose $S$ is a $(G,H)$-section of $X$. Then

(a) $k(S)^H = k(X)^G$ and
(b) any $H$-compression $S \rightarrow S'$ lifts to a $G$-compression $X \rightarrow X'$, where $S'$ is a $(G,H)$-section of $X'$.

**Proof.** (a) See [PV, Section 2.8], [Po, 1.7.2] or [Re2, Lemma 2.11].

(b) Note that $X$ and $G \ast_H S$ are birationally equivalent as $G$-varieties, where $G \ast_H S$ is defined as $G \times S/H$ for the $H$-action given by $h(g,s) = (gh^{-1}, hs)$; see [Po, Theorem 1.7.5] or [Re2, Lemma 2.14]. Now we set $X' = G \ast_H S'$ and extend $f$ to a rational map $X \rightarrow X'$ by $f(g,s) \rightarrow (g, f(s))$. It is easy to see that this map has the desired properties; cf. the proof of [Re2, Lemma 4.1]. □
2.3. \(G\)-lattices. Let \(G\) be a finite group. A \(G\)-lattice is a (left) module over the integral group ring \(\mathbb{Z}[G]\) that is free of finite rank as a \(\mathbb{Z}\)-module. A \(G\)-lattice \(M\) is called

- faithful if the structure map \(G \to \text{Aut}_\mathbb{Z}(M)\) is injective, and
- a permutation lattice if \(M\) has a \(\mathbb{Z}\)-basis that is permuted by \(G\).

For any \(G\)-lattice \(M\), the \(G\)-action on \(M\) extends canonically to actions of \(G\) on the group algebra \(k[M]\) and on the field of fractions \(k(M)\) of \(k[M]\). The operative fact concerning the \(G\)-fields (i.e., fields with \(G\)-action) of the form \(k(M)\) for our purposes is the following result of Masuda [M]; cf. also [L, proof of Prop. (1.5)].

**Proposition 2.3.** Let \(0 \to M \to E \to P \to 0\) be an exact sequence of \(G\)-lattices with \(M\) faithful and \(P\) permutation. Then, as \(G\)-fields, \(k(E) \simeq k(M)(t_1, \ldots, t_r)\), where the elements \(t_i\) are \(G\)-invariant and transcendental over \(k(M)\) and \(r = \text{rank} P\).

2.4. The symmetric and exterior squares. Let \(M\) be a \(G\)-lattice. By definition, the symmetric square \(\text{Sym}^2 M\) is the quotient of \(M \otimes M\), modulo the subgroup generated by the elements \(m \otimes m' - m' \otimes m\) for \(m, m' \in M\). Similarly, the exterior square \(\Lambda^2 M\) is the quotient of \(M \otimes M\) modulo the subgroup generated by the elements \(m \otimes m\), as \(m\) ranges over \(M\). The action of \(G\) on \(M \otimes M\) restricts down to \(\text{Sym}^2 M\) and \(\Lambda^2 M\), making each a \(G\)-lattice. The \(G\)-lattice \(\Lambda^2 M\) can be identified with the sublattice of antisymmetric tensors in \(M \otimes M\), that is,

\[
\Lambda^2 M \simeq A_2^G(M) = \{x \in M \otimes M \mid x^\tau = -x\}
\]

where \(\tau : M \otimes M \to M \otimes M\) is the switch \((m \otimes m')^\tau = m' \otimes m\); see [Bou, Exerc. 8 on p. A III.190]. Furthermore, \(A_2^G(M)\) is exactly the kernel of the canonical map \(M \otimes M \to \text{Sym}^2 M\). Hence, we have an exact sequence of \(G\)-lattices

\[
0 \to \Lambda^2 M \to M \otimes M \to \text{Sym}^2 M \to 0.
\] (2.1)

3. Groups of the form \(T_{n-1} \rtimes G\) and lattices

3.1. Notations. In this section we shall focus on the following situation. Let \(T_{n-1} = (\mathbb{G}_m)^n / \Delta\) be the (diagonal) maximal torus of \(\text{PGL}_n\); here \(\Delta \simeq \mathbb{G}_m\) diagonally embedded in \((\mathbb{G}_m)^n\). Recall that \(S_n\) acts on \(T_{n-1}\) by permuting the \(n\) copies of \(\mathbb{G}_m\) and that the normalizer \(N(T_{n-1})\) of \(T_{n-1}\) in \(\text{PGL}_n\) is isomorphic to \(T_{n-1} \rtimes S_n\). We shall be interested in subgroups of \(N(T_{n-1})\) of the form \(T_{n-1} \rtimes G\), where \(G\) is a subgroup of \(S_n\). These groups have two properties that will be important to us in the sequel: (i) \(T_{n-1} \rtimes G\)-varieties and their compressions can be constructed from \(G\)-lattices and (ii) certain \((\text{PGL}_n, T_{n-1} \rtimes G)\)-sections will naturally come up in the proofs of Theorems [L1] and [L2]. In this section we will focus on the relationship between \(T_{n-1} \rtimes G\)-varieties and \(G\)-lattices.

3.2. \(T_{n-1} \rtimes G\)-varieties and \(G\)-lattices. Suppose we are given a morphism

\[f : M \to A_{n-1}\]
of $G$-lattices, where $A_{n-1}$ is the root lattice defined in \([1,3]\). Note that $A_{n-1} \cong X_s(T_{n-1})$ as an $S_n$-lattice (and hence as $G$-lattice), where $X_s(T_{n-1})$ is the lattice of characters of $T_{n-1}$. We will always identify $A_{n-1}$ with $X_s(T_{n-1})$.

We will now associate to $f$ a $T_{n-1} \rtimes G$-variety $X_f$ as follows. Let $G$ act on $k[M]$ as usual and define a $T_{n-1}$-action on $k[M]$ by putting

$$t(m) = f(m)(t) \cdot m \quad (t \in T_{n-1}, \ m \in M).$$

One easily checks that, by $k$-linear extension of this rule, one obtains a well-defined action of $T_{n-1}$ by automorphism on $k[M]$. Moreover, for $t \in T_{n-1}$, $g \in G$ and $m \in M$, one calculates

$$t(gm) = f(gm)(t) \cdot gm = [gf(m)](t) \cdot gm = f(m)(t^g) \cdot gm = [gt^g](m);$$

so the actions of $G$ and $T_{n-1}$ combine to yield a locally finite action of $T_{n-1} \rtimes G$ on $k[M]$ and thus an algebraic action on $X_f = \text{Spec} \ k[M]$.

**Lemma 3.1.** The $T_{n-1} \rtimes G$-variety $X_f$ is a generically free if and only if

(a) $f$ is surjective and  
(b) $\ker(f)$ is a faithful $G$-lattice.

**Proof.** Condition (a) is equivalent to saying that the $T_{n-1}$-action on $X_f$ is generically free; cf., e.g., [OV, Theorem 3.2.5]. To interpret condition (b) geometrically, note that $k(X_f) = k(M)$ and $k(X_f/T_{n-1}) = k(M)^{T_{n-1}} = k(\ker(f))$. Thus condition (b) holds iff $G$ acts faithfully on $X_f/T_{n-1}$ or, equivalently, iff the $G$-action on $X_f/T_{n-1}$ is and generically free (the two notions coincide for finite groups). The desired conclusion now follows from Lemma [27].

3.3. **Compressions and $G$-lattices.** In the sequel we will only be interested in generically free $T_{n-1} \rtimes G$-varieties. In particular, we will assume that $f$ is surjective and expand it into an exact sequence of $G$-lattices:

$$0 \longrightarrow K = \ker(f) \longrightarrow M \xrightarrow{f} A_{n-1} \longrightarrow 0,$$

where $K$ is a faithful $G$-lattice. For future reference, we extract the following equality from the proof of Lemma 3.1:

$$k(X_f/T_{n-1} \rtimes G) = k(X_f)^{T_{n-1} \rtimes G} = [k(M)^{T_{n-1}}] G = [k(K)] G. \quad (3.2)$$

We can now obtain information about $T_{n-1} \rtimes G$-compressions of $X_f$ by studying this sequence more closely.

**Lemma 3.2.** Suppose that the exact sequence (3.1) extends to a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & M & \xrightarrow{f} & A_{n-1} & \longrightarrow & 0 \\
 & \uparrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_0 & \longrightarrow & M_0 & \xrightarrow{f_0} & A_{n-1} & \longrightarrow & 0 \\
\end{array}$$

of $G$-lattices, where $K_0$ is faithful, and the vertical map $M_0 \longrightarrow M$ is injective. Then there exists a $T_{n-1} \rtimes G$-compression $X_f \longrightarrow X_{f_0}$. 
Proof. Since \( k(M_0) = k(X_f) \) is an \( T_{n-1} \times \mathcal{G} \)-invariant subfield of \( k(M) = k(X_f) \), it defines a dominant \( T_{n-1} \times \mathcal{G} \)-equivariant map \( X_f \to X_{f_0} \). Furthermore, by Lemma 3.1, the \( T_{n-1} \times \mathcal{G} \)-action on both \( X_f \) and \( X_{f_0} \) is generically free. Thus, the rational map \( X_f \to X_{f_0} \) we have constructed is a \( T_{n-1} \times \mathcal{G} \)-compression. \( \square \)

3.4. Linearization and essential dimension. If the \( \mathcal{G} \)-lattice \( M \) in Section 3.2 is a permutation lattice then the \( T_{n-1} \times \mathcal{G} \)-variety \( X_f \) is birationally linearizable. To see this, fix a \( \mathbb{Z} \)-basis \( m_1, \ldots, m_r \) of \( M \) that is permuted by \( \mathcal{G} \). Clearly, \( k(X_f) = k(M) = k(m_1, \ldots, m_r) \). Thus, putting \( V_f = \sum_i km_i \) we obtain a \( T_{n-1} \times \mathcal{G} \)-invariant \( k \)-subspace of \( k(X_f) \) with \( k(X_f) = k(V_f) \).

A similar argument goes through if \( M \) to be permutation projective, i.e., \( M \) is a direct summand of a permutation \( \mathcal{G} \)-lattice.

Lemma 3.3. If \( M \) is permutation projective in (3.1) then there is a \( T_{n-1} \times \mathcal{G} \)-compression \( V \to X_f \) with \( V \) a generically free linear \( T_{n-1} \times \mathcal{G} \)-variety. In particular, \( \text{ed}(T_{n-1} \times \mathcal{G}) \leq \text{rank } K \).

Proof. Suppose \( M \oplus N = P \), where \( P \) is a permutation \( \mathcal{G} \)-lattice. Then the sequence (3.1) embeds in the obvious fashion in an exact sequence \( 0 \to K \oplus N \to P = M \oplus N \to A_{n-1} \to 0 \). In view of the foregoing and Lemma 3.2, this proves the first assertion.

To complete the proof, recall that the essential dimension of an algebraic group \( G \) is defined as the smallest possible value \( \dim(X/G) \), where \( X \) is a generically free \( G \)-variety so that there is a \( G \)-compression \( V \to X \) with \( V \) a generically free linear \( G \)-variety; see [Re2, Definition 3.5]. For \( G = T_{n-1} \times \mathcal{G} \) and \( X = X_f \), we have \( \dim(X/G) = \text{rank } K \) by (3.2). \( \square \)

4. Proof of Theorem 1.1

4.1. Reduction to a lattice-theoretic problem. The universal division algebra \( \text{UD}(n) \) is represented by a class \( c \in H^1(Z(n), \text{PGL}_n) \). We can write \( \text{UD}(n) = D \otimes_{Z(D)} Z(n) \) if and only if \( c \) lies in the image of the natural map

\[
H^1(Z(D), \text{PGL}_n) \to H^1(Z(n), \text{PGL}_n)
\] (4.1)

Recall that for any finitely generated field extension \( L/k \), an element of \( \alpha \in H^1(L, \text{PGL}_n) \) may also be interpreted as a \( \text{PGL}_n \)-torsor, i.e., a generically free \( \text{PGL}_n \)-variety \( X_\alpha \) such that \( k(X_\alpha)^{\text{PGL}_n} = L \). Moreover, \( X_\alpha \) is uniquely determined (up to birational isomorphism of \( \text{PGL}_n \)-varieties), and the central simple algebra corresponding to \( \alpha \) can be recovered as the algebra of \( \text{PGL}_n \)-equivariant rational maps \( X_\alpha \to M_n \).

In particular, \( X_c = M_n \times M_n \), where \( \text{PGL}_n \) acts on \( M_n \times M_n \) by simultaneous conjugation; to say that \( c \) lies in the image of the map (L.1) is equivalent to saying that there exists a \( \text{PGL}_n \)-compression \( M_n \times M_n \to X' \) such that \( k(X')^{\text{PGL}_n} = Z(D) \).

For a more detailed discussion of these facts and further references, see [RY, Section 3].

Denote the linear subspace of \( M_n \) consisting of diagonal matrices by \( D_n \). It is easy to see that \( D_n \times M_n \) is a \( (\text{PGL}_n, N(T_{n-1})) \)-section of \( M_n \times M_n \), where \( N(T_{n-1}) = T_{n-1} \rtimes S_n \) is the normalizer of the maximal torus \( T_{n-1} = (G_m)^n/\Delta \) in \( \text{PGL}_n \), as in the previous section. (See Lemma 3.2 below for a more general fact.) Thus, in view of Lemma 2.3, we have the following.
Reduction 4.1. In order to prove Theorem 1.1 it is enough to show that there exists an $N(T_{n-1})$-compression

$$D_n \times M_n \rightarrow X$$

such that

(i) $\dim(X) = \frac{1}{2}(n-1)(n-2) + n - 1$ or equivalently, $\dim(X/N(T_{n-1})) = \text{trdeg}_k k(X)^{N(T_{n-1})} = \frac{1}{2}(n-1)(n-2)$.

(ii) $Z(n) = k(M_n \times M_n)^{\text{PGL}_n} = k(D_n \times M_n)^{N(T_{n-1})}$ is purely transcendental over $k(X)^{N(T_{n-1})}$.

Our construction of the compression (1.2) will be based on Lemma 3.2. In order to apply this lemma, we need to write the linear $N(T_{n-1})$-variety $D_n \times M_n$ birationally in the form $X_f$, where $f$ is as in (3.1). Let $x_i$ and $y_{rs}$ be the standard coordinates on $D_n$ and $M_n$ respectively. In this coordinate system, the $S_n$-action on $D_n \times M_n$ is given by

$$\sigma(x_i) = x_{\sigma(i)} \quad \text{and} \quad \sigma(y_{rs}) = y_{\sigma(r)\sigma(s)}.$$ 

Thus, monomials in these coordinates and their inverses form an $S_n$-lattice isomorphic to $M = U_n \oplus U_n^{\otimes 2}$, where $U_n = \mathbb{Z}^n$ be the standard permutation $S_n$-lattice. Moreover, an element $t = (t_1, \ldots, t_n)$ of $T_{n-1} = (\mathbb{G}_m)^n/\Delta$ acts on monomials in $x_i$, $y_{rs}$ by characters determined (multiplicatively) by

$$t(x_i) = x_i \quad \text{and} \quad t(y_{rs}) = t_eq^{-1}y_{rs}.$$ 

Denoting the standard basis of $U_n$ by $b_1, \ldots, b_n$ and defining $f: M = U_n \oplus U_n^{\otimes 2} \rightarrow A_{n-1}$ by $f(b_i, b_i \otimes b_s) = b_s - b_s$, the above formulas give exactly the action of $N(T_{n-1}) = T_{n-1} \triangleleft S_n$ on $X_f$ as described in (3.2). The exact sequence (3.1) for this $f$ is the Formanek – Procesi exact sequence

$$0 \rightarrow K = \text{Ker}(f) \rightarrow U_n \oplus U_n^{\otimes 2} \rightarrow A_{n-1} \rightarrow 0;$$

see (3.1). Note that

$$K \cong U_n \oplus U_n \oplus A_{n-1}^{\otimes 2}.$$ 

Here, the first copy of $U_n$ is mapped identically onto the first summand of $U_n \oplus U_n^{\otimes 2}$, the second $U_n$ corresponds to the sublattice of $U_n^{\otimes 2}$ consisting of the monomials in $y_{ii}$, and $A_{n-1}^{\otimes 2}$ describes the kernel of $f$, restricted to the sublattice $\langle y_{rs} | r \neq s \rangle \cong U_n \otimes A_{n-1} \subset U_n^{\otimes 2}$; cf. [3], p. 3573.

We now want to apply Lemma 3.2 to the above sequence, with $K_0 = \wedge^2 A_{n-1}$. Recall that $\wedge^2 A_{n-1}$ may be viewed as a sublattice $A_{n-1}^{\otimes 2}$; see (2.1). Let

$$\varphi: \wedge^2 A_{n-1} \hookrightarrow U_n \oplus U_n \oplus A_{n-1}^{\otimes 2}$$

be the natural embedding of $\wedge^2 A_{n-1}$ into the third component of $U_n \oplus U_n \oplus A_{n-1}^{\otimes 2}$. We remark that for $n > 3$, $S_n$ acts faithfully on $\wedge^2 A_{n-1}$; in fact, $\wedge^2 A_{n-1} \otimes \mathbb{Q}$ is the irreducible $S_n$-representation corresponding to the partition $(n-2, 1^2)$ of $n$; see, e.g., [7], Exercise 4.6.

Combining Reduction 4.1 with Lemma 3.2 we obtain:
Reduction 4.2. Theorem \ref{t1} follows from Propositions \ref{p3} and \ref{p4} stated below.

**Proposition 4.3.** For odd \( n \geq 5 \), \( k(U_n \oplus U_n \oplus A_{n-1}^{\otimes 2}) \cong k(\bigwedge^2 A_{n-1})(y_1, \ldots, y_n) \) as \( S_n \)-fields, where the elements \( y_i \) are \( S_n \)-invariant and transcendental over \( k(\bigwedge^2 A_{n-1}) \). In particular, \( k(U_n \oplus U_n \oplus A_{n-1}^{\otimes 2})^{S_n} \) is rational over \( k(\bigwedge^2 A_{n-1})^{S_n} \).

**Proposition 4.4.** For odd \( n \), there exists a commutative diagram of \( S_n \)-lattices:

\[
\begin{array}{cccccc}
0 & \longrightarrow & U_n \oplus U_n \oplus A_{n-1}^{\otimes 2} & \longrightarrow & U_n \oplus U_n^{\otimes 2} & \longrightarrow & A_{n-1} \longrightarrow 0 \\
& \uparrow \phi & & \uparrow & f & \downarrow & \\
0 & \longrightarrow & \bigwedge^2 A_{n-1} & \longrightarrow & L & \longrightarrow & A_{n-1} \longrightarrow 0 \\
\end{array}
\]

Here the first row is the Formanek – Procesi sequence \((1.3)\).

Indeed, Proposition \ref{p4} in conjunction with Lemma \ref{le2} yields an \( N(T_{n-1}) \)-compression

\[
D_n \times M_n \xrightarrow{\sim} X_f \longrightarrow X = X_{f_0},
\]

and formula \((1.2)\) further implies that \( k(D_n \times M_n)^{N(T_{n-1})} = k(X_f)^{N(T_{n-1})} = k(U_n \oplus U_n \oplus A_{n-1}^{\otimes 2})^{S_n} \) and \( k(X)^{N(T_{n-1})} = k(\bigwedge^2 A_{n-1})^{S_n} \). Thus, condition (i) in Reduction \ref{r4} is clearly satisfied and Proposition \ref{p4} ensures that (ii) holds as well.

4.2. Solution of the lattice-theoretic problem. Our proofs of Propositions \ref{p3} and \ref{p4} will be based on the following result from \cite[Section 3.5]{1}. If \( \mathcal{G} \) is a finite group, \( \mathcal{H} \) is a subgroup of \( \mathcal{G} \) and \( M \) a \( \mathbb{Z}[\mathcal{H}] \)-module then \( M^\mathcal{G}_\mathcal{H} = \mathbb{Z}[\mathcal{G}] \otimes_{\mathbb{Z}[\mathcal{H}]} M \) will denote the induced \( \mathbb{Z}[\mathcal{G}] \)-module.

**Proposition 4.5.** For odd \( n \), there is an isomorphism of \( S_n \)-lattices

\[
\text{Sym}^2 A_{n-1} \oplus U_n \oplus \mathbb{Z} \cong \mathbb{Z}[S_n]_{S_n \times S_2} \oplus U_n \oplus \mathbb{Z} ,
\]

where \( \mathbb{Z} \) has the trivial \( S_n \)-action. In particular, \( \text{Sym}^2 A_{n-1} \oplus U_n \oplus \mathbb{Z} \) is a permutation lattice.

**Proof of Proposition 4.5.** First, applying Proposition \ref{p3} to the obvious sequence \( 0 \longrightarrow U_n \oplus A_{n-1}^{\otimes 2} \longrightarrow U_n \oplus U_n \oplus A_{n-1}^{\otimes 2} \longrightarrow U_n \longrightarrow 0 \), we see that

\[
k(U_n \oplus U_n \oplus A_{n-1}^{\otimes 2}) \cong k(U_n \oplus A_{n-1}^{\otimes 2})(t_1, \ldots, t_n)
\]
as \( S_n \)-fields.

Next, sequence \((2.1)\) for \( M = A_{n-1} \) combined with Proposition \ref{p5} gives rise to an exact sequence of \( S_n \)-lattices

\[
0 \longrightarrow \bigwedge^2 A_{n-1} \longrightarrow A_{n-1}^{\otimes 2} \oplus U_n \oplus \mathbb{Z} \longrightarrow P \longrightarrow 0 ,
\]

where \( P = \text{Sym}^2 A_{n-1} \oplus U_n \oplus \mathbb{Z} \) is permutation. Applying Proposition \ref{p3} to this sequence, we deduce that

\[
k(A_{n-1}^{\otimes 2} \oplus U_n \oplus \mathbb{Z}) \cong k(\bigwedge^2 A_{n-1})(x_1, \ldots, x_m)
\]
as $S_n$-fields.

Finally,

$$k(U_n \oplus U_n \oplus A_{n-1}^{\otimes 2}) \simeq k(U_n \oplus A_{n-1}^{\otimes 2})(t_1, \ldots, t_n) = k(A_{n-1}^{\otimes 2} \oplus U_n \oplus \mathbb{Z})(t_1, \ldots, t_{n-1}) \simeq k(\bigwedge A_{n-1}(x_1, \ldots, x_m, t_1, \ldots, t_{n-1})$$

as $S_n$-fields, which proves the first assertion of Proposition 4.3. The second assertion is an immediate consequence of the first.

Proof of Proposition 4.4. Recall that the embedding $\varphi$ of (4.3) is defined as the composition

$$\varphi: \bigwedge^2 A_{n-1} \hookrightarrow A_{n-1} \hookrightarrow U_n \oplus U_n \oplus A_{n-1}^{\otimes 2},$$

where $\psi$ is the injection from (2.3) (with $M = A_{n-1}$) and the second map identifies $A_{n-1}^{\otimes 2}$ with the third component of $U_n \oplus U_n \oplus A_{n-1}^{\otimes 2}$. We aim to show that $\varphi$ together with sequence (4.3) will give rise to a commutative diagram as in the statement of Proposition 4.4. In other words, our goal is to show that the class in $\text{Ext}_{\mathbb{Z}[S_n]}(A_{n-1}, U_n \oplus U_n \oplus A_{n-1}^{\otimes 2})$ corresponding to the extension (4.3) belongs to the image of the map

$$\varphi_*: \text{Ext}_{\mathbb{Z}[S_n]}(A_{n-1}, \bigwedge^2 A_{n-1}) \to \text{Ext}_{\mathbb{Z}[S_n]}(A_{n-1}, U_n \oplus U_n \oplus A_{n-1}^{\otimes 2}).$$

In fact, we will prove:

**Lemma 4.6.** For odd $n$, the map $\varphi_*$ is surjective.

**Proof.** We will tacitly use the following standard facts from homological algebra, valid for any finite group $G$:

- If $V$ is a $G$-module and $M$ an $H$-module for some subgroup $H \leq G$ then $\text{Ext}^*_{\mathbb{Z}[G]}(V, M^H) \simeq \text{Ext}^*_{\mathbb{Z}[H]}(V|M_H)$; see [BS, Prop. IV.12.3] and [Br, Prop. III.5.9]. For $V = \mathbb{Z}$, the trivial $G$-module, this isomorphism is the “Shapiro isomorphism” $H^*(G, M^G_H) \simeq H^*(H, M)$. In case, $M$ is actually a $G$-module, the restriction map $\text{Res}^G_H: H^*(G, M) \to H^*(H, M)$ factors through the Shapiro isomorphism:

$$\text{Res}^G_H: H^*(G, M) \xrightarrow{\mu} H^*(G, M^G_H) \xrightarrow{\sim} H^*(H, M),$$

where $\mu: M \to M^G_H$ sends $m \mapsto \sum_{g \in G/H} g \otimes g^{-1}m$; see [Br, p. 81].

- For any $G$-lattices $V$ and $W$, $\text{Ext}^*_{\mathbb{Z}[G]}(V, W) \simeq H^*(G, V^* \otimes W)$, where $\otimes = \otimes_{\mathbb{Z}}$ and $V^* = \text{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$ is the dual $G$-lattice; see [Br, Prop. III.2.2].

- If $V$ and $W$ are both permutation $G$-lattices then $\text{Ext}^*_{\mathbb{Z}[G]}(V, W) = 0$; cf. [L, Propositions 1.1, 1.2].
Armed with these facts, we proceed as follows. First, \( \text{Ext}_{\mathbb{Z}[S_n]}(A_{n-1}, U_n) = 0 \), because \( U_n \simeq \mathbb{Z}^\ast_{\mathcal{G}} \) and \( A_{n-1}|_{\mathcal{G}} \simeq U_{n-1} \). Therefore, it suffices to show that the map

\[
\psi_2 : \text{Ext}_{\mathbb{Z}[S_n]}(A_{n-1}, \bigwedge A_{n-1}) \rightarrow \text{Ext}_{\mathbb{Z}[S_n]}(A_{n-1}, A_{n-1}^2)
\]

is surjective. But the extension (1.1) (for \( M = A_{n-1} \)) gives rise to an exact sequence

\[
\text{Ext}_{\mathbb{Z}[S_n]}(A_{n-1}, \bigwedge A_{n-1}) \xrightarrow{\psi_2} \text{Ext}_{\mathbb{Z}[S_n]}(A_{n-1}, A_{n-1}^2) \rightarrow \text{Ext}_{\mathbb{Z}[S_n]}(A_{n-1}, \text{Sym}^2 A_{n-1}) .
\]

Therefore, it suffices to prove:

\[
\text{Ext}_{\mathbb{Z}[S_n]}(A_{n-1}, \text{Sym}^2 A_{n-1}) = 0 \quad \text{for odd } n.
\]

For this, we use the isomorphism \( \text{Sym}^2 A_{n-1} \oplus U_n \oplus \mathbb{Z} \simeq \mathbb{Z}^\ast_{\mathcal{G}} \oplus U_n \oplus \mathbb{Z} \) of Proposition 4.3, where we have put \( \mathcal{G} = S_{n-2} \times S_2 \) for simplicity. This isomorphism entails

\[
\text{Ext}_{\mathbb{Z}[S_n]}(A_{n-1}, \text{Sym}^2 A_{n-1}) \simeq \text{Ext}_{\mathbb{Z}[S_n]}(A_{n-1}, \mathbb{Z}^\ast_{\mathcal{G}})
\]

\[
\simeq \text{Ext}_{\mathcal{G}}(A_{n-1}|_{\mathcal{G}}, \mathbb{Z})
\]

\[
\simeq H^1(\mathcal{G}, A_{n-1}^\ast).
\]

Dualizing the augmentation sequence \( 0 \rightarrow A_{n-1} \rightarrow U_n \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0 \) we obtain an exact sequence \( 0 \rightarrow \mathbb{Z} \xrightarrow{\epsilon^* U_n} A_{n-1}^\ast \rightarrow 0 \), where \( \epsilon^*(1) = \sum_i e_i \), the sum of the natural basis elements of \( U_n \). This sequence, viewed as exact sequence of \( \mathcal{G} \)-lattices, in turn yields an exact sequence

\[
H^1(\mathcal{G}, U_n) = 0 \rightarrow H^1(\mathcal{G}, A_{n-1}^\ast) \rightarrow H^2(\mathcal{G}, \mathbb{Z}) \rightarrow H^2(\mathcal{G}, U_n) .
\]

Thus, it suffices to show that \( H^2(\mathcal{G}, \mathbb{Z}) \rightarrow H^2(\mathcal{G}, U_n) \) is injective. As a \( \mathcal{G} \)-module, \( U_n = V \oplus W \), where \( V = \bigoplus_{j=1}^{n-2} \mathbb{Z} e_j \simeq \mathbb{Z}^\ast_{S_{n-3} \times S_2} \) and \( W = \mathbb{Z} e_{n-1} \oplus \mathbb{Z} e_n \simeq \mathbb{Z}^\ast_{S_{n-2}} \). Therefore, the Shapiro isomorphism gives

\[
H^2(\mathcal{G}, U_n) \simeq H^2(S_{n-3} \times S_2, \mathbb{Z}) \oplus H^2(S_{n-2}, \mathbb{Z})
\]

and the map \( H^2(\mathcal{G}, \mathbb{Z}) \rightarrow H^2(\mathcal{G}, U_n) \) becomes the restriction map

\[
\text{Res}_{S_{n-3} \times S_2}^\mathcal{G} \times \text{Res}_{S_{n-2}}^\mathcal{G} : H^2(\mathcal{G}, \mathbb{Z}) \rightarrow H^2(S_{n-3} \times S_2, \mathbb{Z}) \oplus H^2(S_{n-2}, \mathbb{Z}) .
\]

This map is indeed injective, as is easily seen by identifying \( H^2(\mathcal{G}, \mathbb{Z}) \) in the usual fashion with \( \text{Hom}(\mathcal{G}, \mathbb{Q}/\mathbb{Z}) \), and similarly for the subgroups \( S_{n-3} \times S_2 \) and \( S_{n-2} \). This finishes the proof of the Lemma, and hence, of Proposition 1.3 and of Theorem 1.1. \( \square \)

5. Proof of Theorem 1.2

5.1. General observations. The following notations will be used throughout this section:
\begin{itemize}
  \item $F$ will be a field containing $k$;
  \item $A$ will be a finite-dimensional central simple algebra with center $F$;
  \item $L$ will be a strictly maximal commutative subfield of $A$;
  \item $n$ denotes the degree of $A$, so $\dim_F A = n^2$ and $[L:F] = n$;
  \item $\mathcal{G}$ is the Galois group of the normal closure $L^{\text{norm}}$ of $L$ over $F$;
  \item $T_{n-1}$ denotes the diagonal maximal torus of $\text{PGL}_n$.
\end{itemize}

**Reduction 5.1.** In the course of proving Theorem 1.2 we may assume without loss of generality that $F$ is a finitely generated field extension of $k$.

**Proof.** Indeed, choose a primitive element $x$ for the extension $L/F$ and complete $1,x,\ldots,x^{n-1}$ to an $F$-basis $e_1,\ldots,e_{n^2}$ of $A$, where $e_i = x^{i-1}$ for $i = 1,\ldots,n$. Let \( c^i_{rs} \) be the structure constants for $A$ in this basis, i.e.,
\[
e_r e_s = \sum_{t=1}^{n^2} c^i_{rs} e_t \tag{5.1}\]
for every $r,s = 1,\ldots,n^2$. Then $A = A_0 \otimes_{F_0} F$, where $F_0 = k(c^i_{rs})$ and $A_0$ is the $n^2$-dimensional $F_0$-algebra spanned by $e_1,\ldots,e_{n^2}$ with multiplication given by (5.1). Moreover, $A_0$ is a central simple algebra of degree $n$ with center $F_0$ and $A_0$ contains the strictly maximal subfield $L_0 = F_0(x)$ whose normal closure $L_0^{\text{norm}}$ has Galois group $\mathcal{G}$ over $F_0$. Clearly, $\tau(A) \leq \tau(A_0)$. Thus, after replacing $A$ by $A_0$ we may assume that $F$ is finitely generated extension of $k$. \(\square\)

Next we pass from central simple algebras to generically free $\text{PGL}_n$-varieties, as we did at the beginning of Section 4. Recall that a central simple algebra $A$ of degree $n$ with center $F$ defines a class $H^1(F,\text{PGL}_n)$. This class, in turn, gives rise to a $\text{PGL}_n$-torsor $X_A$ over $F$. Since we are assuming $F$ is a finitely generated extension of $k$, $X_A$ is a generically free $\text{PGL}_n$-variety; moreover, $F = k(X_A/\text{PGL}_n)$ and $A$ can be recovered from $X_A$ as the algebra of $\text{PGL}_n$-equivariant rational maps $X_A \to M_n$; cf. [RY, Section 3].

The algebras $A$ we are concerned with in the context Theorem 1.2 are of a special form: they have a maximal subfield $L/F$ such that $\text{Gal}(L^{\text{norm}}/F)$ is the given group $\mathcal{G}$. We would like to know how this extra structure is reflected in the geometry of the $\text{PGL}_n$-variety $X_A$. The following lemma gives a partial answer. We continue to let $T_{n-1}$ denote the diagonal maximal torus of $\text{PGL}_n$, as in Section 3.1.

**Lemma 5.2.** $X_A$ has a ($\text{PGL}_n,T_{n-1} \rtimes \mathcal{G}$)-section.

Note that the group $\mathcal{G}$ comes with a natural permutation representation of $\mathcal{G}$ on the $n$ embeddings $L \hookrightarrow L^{\text{norm}}$ over $F$. This permutation representation gives an embedding $\alpha: \mathcal{G} \hookrightarrow S_n$ that we use to define the semidirect product $T_{n-1} \rtimes \mathcal{G}$. Note that $\alpha$ is only defined up to an inner automorphism of $S_n$, since the $n$ embedding $L \hookrightarrow L^{\text{norm}}$ are not naturally in a 1-1 correspondence with \{1,\ldots,n\}. Relabeling these embeddings (or, equivalently, reordering the roots of a defining polynomial for $L/F$) will cause $T_{n-1} \rtimes \mathcal{G}$ to be replaced by a conjugate subgroup of $\text{PGL}_n$; the corresponding section will be translates of each other by elements of $S_n \subset \text{PGL}_n$.\]
Thus it is sufficient to verify that such a section exists for a particular numbering of the embeddings $L \hookrightarrow L^{\text{norm}}$.

Proof. Suppose $L = F(r)$ for some $r \in L$ and $r = r_1, r_2, \ldots, r_n$ are the conjugates of $r$ in $L^{\text{norm}}$. As we remarked above, the order of the roots is not intrinsic; however, we choose it at this point, and it will not be changed in the sequel. The group $G$ permutes $r_1, \ldots, r_n$ transitively via a permutation representation $\alpha: G \rightarrow S_n$.

Now let $x_1, \ldots, x_n$ be commuting independent variables over $k$. The symmetric group acts on the polynomial ring $k[x_1, \ldots, x_n]$ by permuting these variables; composing this action with $\alpha$, we obtain a (permutation) action of $G$ on $k[x_1, \ldots, x_n]$. Note that for every $f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]^G$, we have $f(r_1, \ldots, r_n) \in F$; we shall write this element of $F$ as $f_r$.

Recall that $A$ is the algebra of $\text{PGL}_n$-equivariant rational maps $X_A \rightarrow M_n$. We claim that

$$S = \left\{ x \in X_A : \begin{array}{c} r(x) = \text{diag}(\lambda_1, \ldots, \lambda_n) \text{ is a diagonal matrix} \\ f(\lambda_1, \ldots, \lambda_n) = f_r(x) \text{ for every } f \in k[x_1, \ldots, x_n]^G \end{array} \right\}$$

is an $(\text{PGL}_n, T_{n-1} \rtimes G)$-section $S$ of $X_A$. Note that $S$ is a $T_{n-1} \rtimes G$-invariant subvariety of $X_A$: indeed, $T_{n-1}$ acts trivially on the set of diagonal matrices, and $f_r \in F$ is a $\text{PGL}_n$-invariant rational function on $X_A$. Thus we need to show that

(i) $\text{PGL}_n x$ intersects $S$ for $x$ in general position in $X_A$ and

(ii) $g s \in S \implies g \in T_{n-1} \rtimes G$ for $s$ in general position in $S$;

cf. Section 2.2.

Let $\pi: X_A \rightarrow X_A / \text{PGL}_n$ be the rational quotient map. Recall that $k(X_A / \text{PGL}_n) = k(X_A)^{\text{PGL}_n} = F$. Suppose $p(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ is the minimal polynomial of $r \in L$ over $F$. Note $a_1, \ldots, a_n \in F$ are $\text{PGL}_n$-invariant rational functions on $X_A$; in particular, for $x \in X_A$ in general position, the matrix $r(x) \in M_n$ satisfies the polynomial $p_x(t) = t^n + a_1(x) t^{n-1} + \cdots + a_n(x) \in k[t]$. Since $p(t)$ is an irreducible polynomial over $F$, its discriminant $\delta$ is a non-zero element of $F$, i.e., a non-zero $\text{PGL}_n$-invariant rational function $X_A$. This means that for $x \in X_A$ in general position (i.e., away from the zero locus of $\delta$ and the indeterminacy locus of $r$), the $n \times n$-matrix $r(x)$ has distinct eigenvalues. We conclude that in this case $p_x(t)$ is the characteristic polynomial for $r(x)$; in particular, the eigenvalues of $r(x)$ are precisely the roots of $p_x(t)$.

To see what these eigenvalues are more explicitly, let $Y \rightarrow X_A / \text{PGL}_n$ be a rational map of varieties induced by the field extension $L^{\text{norm}} / F$. Then $r_1, \ldots, r_n \in L^{\text{norm}}$ are
rational functions on $Y$. Thus we have the following diagram of rational maps

$$
\begin{array}{c}
X_A \xrightarrow{r} \pi \xrightarrow{\pi} Y \xrightarrow{r_i} k \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{array}
$$

Suppose $x$ be a point of $X_A$ in general position and $y$ is a point of $Y$ lying above $\pi(x)$. Since $r_1, \ldots, r_n$ are distinct elements of $L^{\text{norm}} = k(Y)$, $\lambda_1 = r_1(y), \ldots, \lambda_n = r_n(y)$ are the $n$ distinct roots of $p_x(t)$, i.e., the $n$ distinct eigenvalues of the $n \times n$-matrix $r(x)$.

Proof of (i): In view of the above discussion, we may assume without loss of generality that $r(x)$ is diagonalizable. In other words, the $\text{PGL}_n$-orbit of $r(x)$ in $M_n$ contains the diagonal matrix $\text{diag}(\lambda_1, \ldots, \lambda_n)$ or equivalently, $r(x') = \text{diag}(\lambda_1, \ldots, \lambda_n)$ for some $x' \in \text{PGL}_n x$. It remains to show that $x' \in S$. Indeed, for any $f \in k[x_1, \ldots, x_n]^G$, we have

$$
f_r(x') = f(r_1(y), \ldots, r_n(y)) = f(\lambda_1, \ldots, \lambda_n),
$$

as desired. This completes the proof of (i).

Proof of (ii): Let $x$ be a point of $S$ in general position. We may assume without loss of generality that $r(x)$ is diagonalizable. In other words, the $\text{PGL}_n$-orbit of $r(x)$ in $M_n$ contains the diagonal matrix $\text{diag}(\lambda_1, \ldots, \lambda_n)$ or equivalently, $r(x') = \text{diag}(\lambda_1, \ldots, \lambda_n)$ for some $x' \in \text{PGL}_n x$. It remains to show that $x' \in S$. Indeed, for any $f \in k[x_1, \ldots, x_n]^G$, we have

$$
f_r(x') = f(r_1(y), \ldots, r_n(y)) = f(\lambda_1, \ldots, \lambda_n),
$$

as desired. This completes the proof of (i).

Remark 5.3. One can show that the converse of Lemma 5.2 is also true: if $X_A$ has a $(\text{PGL}_n, T_{n-1} \rtimes \mathcal{G})$-section then $A$ contains a strictly maximal subfield $L$ such that $\text{Gal}(L^{\text{norm}}/F) = \mathcal{G}$. Since this result is not needed in the sequel, we omit the proof.

5.2. Conclusion of the proof. We are now ready to finish the proof of Theorem 1.3. Let $A$ be a central simple algebra of degree $n$ and let $X_A$ denote the $\text{PGL}_n$-variety associated to $A$. Recall that $\tau(A) = \text{ed}(X_A, \text{PGL}_n)$; see [Rea] Theorem 8.8 and Lemma 9.1. Moreover, if $X$ has a $(\text{PGL}_n, H)$-section $S$ then $\text{ed}(X, \text{PGL}_n) \leq \text{ed}(S, H) \leq \text{ed}(H)$; see [Rea] Lemma 4.1 and Definition 3.5. Applying these inequalities to the
situation described by Lemma 5.2 with \( H = T_{n-1} \rtimes \mathcal{G} \), we see that
\[ \tau(A) \leq \text{ed}(T_{n-1} \rtimes \mathcal{G}). \] (5.2)
Thus Theorem 1.2 is a consequence of the following:

**Lemma 5.4.** Suppose \( \mathcal{G} \) is a transitive subgroup of \( \mathcal{S}_n \) generated by the subgroup \( \mathcal{H} = \mathcal{G} \cap \mathcal{S}_{n-1} \) together with elements \( g_1, \ldots, g_r \). Assume that either \( r \geq 2 \) or \( \mathcal{H} \neq \{1\} \). Then \( \text{ed}(T_{n-1} \rtimes \mathcal{G}) \leq r|\mathcal{G}| - n + 1 \).

**Proof.** By Lemma 3.3, it suffices to construct an exact sequence (3.1) with \( M \) permutation projective and \( K \) faithful having rank \( K = r|\mathcal{G}| - n + 1 \). To this end, note that \( U_n \simeq \mathbb{Z}[\mathcal{G}/\mathcal{H}] \) as \( \mathcal{G} \)-lattices. Let \( \Theta : \mathbb{Z}[\mathcal{G}] \to \mathbb{Z}[\mathcal{G}/\mathcal{H}] = U_n \) denote the canonical epimorphism; the kernel of \( \Theta \) is \( \mathbb{Z}[\mathcal{G}]/\omega \mathcal{H} \), where \( \omega \mathcal{H} \) denotes the augmentation ideal of \( \mathbb{Z}[\mathcal{G}] \); cf. [Pa]. Then \( \sum_i \mathbb{Z}[\mathcal{G}] (g_i - 1) = A_{n-1} \); see [Pa, Lemma 3.1.1]. Therefore, we obtain an epimorphism of \( \mathcal{G} \)-lattices
\[ f : M = \mathbb{Z}[\mathcal{G}]^r \to A_{n-1}, \quad (\alpha_1, \ldots, \alpha_r) \mapsto \sum_i \alpha_i(g_i - 1). \]

Put \( K = \text{Ker} f \); so \( K \) certainly has the required rank. For faithfulness, we may consider \( K \otimes \mathbb{Q} \) instead of \( K \) and work over the semisimple algebra \( \mathbb{Q}[\mathcal{G}] \). Since \( f \otimes \mathbb{Q} \) and \( \Theta \otimes \mathbb{Q} \) are split, we have \( \mathbb{Q}[\mathcal{G}] \)-isomorphisms \( (A_{n-1} \otimes \mathbb{Q}) \oplus (K \otimes \mathbb{Q}) \simeq \mathbb{Q}[\mathcal{G}]^r \) and \( (A_{n-1} \otimes \mathbb{Q}) \oplus \mathbb{Q} \otimes \mathbb{Q}[\mathcal{G}] \mathcal{H} \simeq \mathbb{Q}[\mathcal{G}] \). Therefore,
\[ K \otimes \mathbb{Q} \simeq \mathbb{Q}[\mathcal{G}]^{r-1} \oplus \mathbb{Q} \otimes \mathbb{Q}[\mathcal{G}] \mathcal{H}. \]

If \( r \geq 2 \) then \( \mathbb{Q}[\mathcal{G}]^{r-1} \) is \( \mathcal{G} \)-faithful, and if \( \mathcal{H} \neq \{1\} \) then \( \omega \mathcal{H} \otimes \mathbb{Q} \) is \( \mathcal{H} \)-faithful and so \( \mathbb{Q}[\mathcal{G}] \mathcal{H} \simeq (\omega \mathcal{H} \otimes \mathbb{Q})^1 \mathcal{H} \) is \( \mathcal{G} \)-faithful. In either case, \( K \otimes \mathbb{Q} \) is faithful, and hence so is \( K \), as desired. \( \square \)

6. Algebras of degree four

Recall that Corollary 1.4 asserts that \( d(4) \) equals 4 or 5. Whether the true value of \( d(4) \) is four or five is an open question. The purpose of this section is to show that this question cannot be resolved by the methods of this paper.

For the rest of this section we will identify the Klein 4-group \( \mathcal{V} = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \) with the subgroup of \( \mathcal{S}_4 \) generated by (12)(34) and (13)(24). Let \( A_3 \) be the augmentation (or root) lattice of \( \mathcal{S}_4 \), restricted to \( \mathcal{V} \); see (1.3). In other words,
\[ A_3 \simeq \omega \mathcal{V}, \] (6.1)
where \( \omega \mathcal{V} \) is the augmentation ideal of the group ring \( \mathbb{Z}[\mathcal{V}] \).

We now briefly recall how we arrived at the bound \( d(4) \leq 5 \). First of all, since \( \text{UD}(4) \) is a \( \mathcal{V} \)-crossed product, \( d(4) = \tau(\text{UD}(4)) \leq \text{ed}(T_3 \rtimes \mathcal{V}) \); see (5.2). Secondly, Lemma 8.3 tells us that \( \text{ed}(T_3 \rtimes \mathcal{V}) \leq \text{rank}(K_0) \), for any commutative diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & M & \overset{f}{\longrightarrow} & A_3 & \longrightarrow & 0 \\
\phi \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0 & \longrightarrow & M_0 & \overset{f_0}{\longrightarrow} & A_3 & \longrightarrow & 0
\end{array}
\] (6.2)
of \( \mathcal{V} \)-lattices with \( M \) permutation projective, \( K_0 \) faithful, and \( \varphi \) is injective. Finally, in the course of the proof of Lemma 5.4 (with \( \mathcal{G} = \mathcal{V} \) and \( r = 2 \)) we constructed a particular diagram \((6.2)\) with \( M = M_0 = \mathbb{Z}[V]^2 \) and \( \text{rank}(K_0) = 5 \). This gave us the bound \( d(4) \leq 5 \). The question we will now address is whether or not one can sharpen this bound by choosing a different diagram \((6.2)\). The following proposition shows that the answer is “no”.

**Proposition 6.1.** Let \((6.2)\) be a commutative diagram of \( \mathcal{V} = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})\)-lattices with \( M \) permutation projective. Then \( K_0 \) is faithful and \( \text{rank} K_0 \geq 5 \).

Note that in the setting of Lemma 3.2 we assumed that (i) \( K_0 \) is faithful and (ii) \( \varphi \) is injective. Here we see that (i) is automatic and (ii) is irrelevant for the rank estimate.

**Proof.** Since \( M \) is permutation projective, we have \( H^1(\mathcal{H}, M) = 0 = H^{-1}(\mathcal{H}, M) \) for every subgroup \( \mathcal{H} \) of \( \mathcal{V} \). (This condition is actually equivalent to \( M \) being permutation projective; see [C-TS Proposition 4].) Consequently, \((6.2)\) yields a commutative diagram

\[
\begin{array}{ccc}
0 = H^1(\mathcal{H}, M) & \longrightarrow & H^1(\mathcal{H}, A_3) \xrightarrow{\delta} H^2(\mathcal{H}, K) \\
& & \downarrow \varphi \\
& & H^1(\mathcal{H}, A_3) \xrightarrow{\delta_0} H^2(\mathcal{H}, K_0)
\end{array}
\]

Thus, \( \delta_0 \) is mono. Since \( H^1(\mathcal{V}, A_3) \cong \mathbb{Z}/4\mathbb{Z} \) and \( H^1(\mathcal{H}, A_3) \cong \mathbb{Z}/2\mathbb{Z} \) for any nonidentity cyclic subgroup \( \mathcal{H} \) of \( \mathcal{V} \) (see [LL, Lemma 4.3]), we obtain

\[
\mathbb{Z}/4\mathbb{Z} \hookrightarrow H^2(\mathcal{V}, K_0) \quad \text{and} \quad H^2(\mathcal{H}, K_0) \neq 0 \text{ for all } 1 \neq \mathcal{H} \leq \mathcal{V}. \tag{6.3}
\]

Similarly, \( 0 = H^{-1}(\mathcal{H}, M) \) implies \( H^{-1}(\mathcal{H}, A_3) \hookrightarrow \tilde{H}^0(\mathcal{H}, K_0) \). Using the identification \((6.1)\), we have \( H^{-1}(\mathcal{V}, A_3) \cong \omega \mathcal{V}/(\omega \mathcal{V})^2 \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). Thus:

\[
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \hookrightarrow \tilde{H}^0(\mathcal{H}, K_0). \tag{6.4}
\]

We will show that \((6.3)\) forces \( K_0 \) to be faithful, and \((6.3)\) and \((6.4)\) together imply that \( \text{rank} K_0 \geq 5 \). The discussion below could be shortened somewhat by a reference to \((5.3)\); however, for the sake of completeness, we will give a self-contained argument.

**Lemma 6.2.** Let \( L \) be a \( \mathcal{V} \)-lattice, \( 1 \neq x \in \mathcal{V} \) and \( L_\pm = \{ l \in L \mid x l = \pm l \} \). If \( L\mid_{(x)} = L_+ \oplus L_- \) then \( 2 \cdot H^2(\mathcal{V}, L) = 0 \).

**Proof.** Since \( L_+ \) and \( L_- \) are \( \mathcal{V} \)-sublattices of \( L \), we may assume \( L = L_+ \) or \( L = L_- \). Write \( \mathcal{V} = \langle x, y \rangle \). Then \( \langle y \rangle \)-sublattices of \( L \) are stable under \( \mathcal{V} \). Therefore, we may assume that \( L \) is indecomposable as a \( \langle y \rangle \)-lattice. This leaves the following possibilities for \( L \): \( \mathbb{Z} \uparrow \langle x \rangle \) or \( \mathbb{Z} \lambda \) for some \( \lambda \in \text{Hom}(\mathcal{V}, \mathbb{Z}) \). In each case, \( 2 \cdot H^2(\mathcal{V}, L) = 0 \) is easy to verify.

Lemma 6.2 in combination with the first condition in \((6.3)\), implies that \( K_0 \) is faithful. Consequently, \( K_0^V = K_0/K_0^V \) is faithful as well, and so \( \text{rank} K_0^V \geq 2 \). In addition, we know by \((6.4)\) that \( \tilde{H}^0(\mathcal{V}, K_0) = K_0^V/(\sum v v)K_0 \) is not cyclic. Hence, neither is \( K_0^V \), which forces \( \text{rank} K_0^V \geq 2 \) and thus \( \text{rank} K_0 \geq 4 \). Suppose, by way of
contradicting (6.3).

By the well-known classification of finite subgroups of $\text{GL}_2(\mathbb{Z})$, the action of $\mathcal{V}$ on $\overline{K_0}$ is given by either

\[ \text{diag:} \text{ the matrices } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \text{ so } \overline{K_0} \simeq \mathbb{Z}_{+,-} \oplus \mathbb{Z}_{-,+}, \text{ or} \]

\[ \text{non-diag:} \text{ the matrices } \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \text{ so } \overline{K_0} \simeq \mathbb{Z}_{-} \uparrow_{\mathcal{H}} \mathcal{V} \text{ for some cyclic } \mathcal{H} \leq \mathcal{V}. \]

In case non-diag, $H^2(\mathcal{V}, \overline{K_0}) \simeq H^2(\mathcal{H}, \mathbb{Z}_{-}) \simeq \widehat{H}^0(\mathcal{H}, \mathbb{Z}_{-}) = 0$, and hence $H^2(\mathcal{V}, K_0^\mathcal{V})$ maps onto $H^2(\mathcal{V}, K_0)$. But $H^2(\mathcal{V}, K_0^\mathcal{V}) \simeq H^2(\mathcal{V}, \mathbb{Z})^2 \simeq \text{Hom}(\mathcal{V}, \mathbb{Q}/\mathbb{Z})^2 \simeq (\mathbb{Z}/2\mathbb{Z})^4$. Thus $H^2(\mathcal{V}, K_0^\mathcal{V})$ is annihilated by 2, and hence so is $H^2(\mathcal{V}, K_0)$, contradicting (6.3).

Therefore, diag must hold:

\[ \overline{K_0} = K_0/K_0^\mathcal{V} \simeq \mathbb{Z}_{+,-} \oplus \mathbb{Z}_{-,+}. \]

The action of $\mathcal{V}$ on $K_0$ is given by matrices

\[ c = \begin{pmatrix} 1_{2 \times 2} & 0 & \gamma \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 1_{2 \times 2} & \delta & 0 \\ -1 & 1 \end{pmatrix} \]

with $\gamma, \delta \in M_{2 \times 1}(\mathbb{Z})$ and $0 = (0 \ 0)$. By Lemma 5.2, $\gamma \neq 0$ and $\delta \neq 0$. Conjugating by a suitable matrix of the form \( \begin{pmatrix} 1_{2 \times 2} & 0 & 0 \\ 0 & 1_{2 \times 2} \end{pmatrix} \) we can ensure that the entries of $\gamma$ are 0 or 1, and similarly for $\delta$. If $\gamma = (\frac{1}{1})$ or $(\frac{0}{1})$ then conjugating, respectively, by \( \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \) or \( \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \), we can replace $\gamma$ by $\gamma = (\frac{0}{1})$. Thus we may assume that $\gamma = (\frac{0}{1})$. If $\delta = (\frac{1}{1})$, then conjugating by \( \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \), we replace $\delta$ by $(\frac{0}{1})$ without changing $c$.

This leaves us with two cases to consider:

\[ \underline{\delta = (\frac{0}{1})}: \text{ Then } c = (1 \ c) \text{ and } d = (1 \ d) \text{ with } c' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } d' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \]

Therefore, $K_0 \simeq \mathbb{Z} \oplus (A_3 \otimes_{\mathbb{Z}} \mathbb{Z})$, where $\lambda : \mathcal{V} \rightarrow \mathbb{Z}$ is the map sending the elements of $\mathcal{V}$ acting via $c$ and $d$ both to $-1$.

Tensoring the augmentation sequence $0 \rightarrow A_3 = \omega \mathcal{V} \rightarrow \mathbb{Z}[\mathcal{V}] \rightarrow \mathbb{Z} \rightarrow 0$ with $\mathbb{Z}_\lambda$ we obtain an exact sequence $0 \rightarrow A_3 \otimes \mathbb{Z}_\lambda \rightarrow \mathbb{Z}[\mathcal{V}] \otimes \mathbb{Z}_\lambda = \mathbb{Z}[\mathcal{V}] \rightarrow \mathbb{Z}_\lambda \rightarrow 0$. This sequence in turn implies that $H^2(\mathcal{V}, A_3 \otimes \mathbb{Z}_\lambda) \simeq H^1(\mathcal{V}, \mathbb{Z}_\lambda)$, and the inflation-restriction sequence easily gives $H^1(\mathcal{V}, \mathbb{Z}_\lambda) = \mathbb{Z}/2\mathbb{Z}$. Thus, $H^2(\mathcal{V}, K_0) = H^2(\mathcal{V}, \mathbb{Z}) \oplus H^2(\mathcal{V}, A_3 \otimes \mathbb{Z}_\lambda) \simeq (\mathbb{Z}/2\mathbb{Z})^3$, contradicting (6.3).

\[ \underline{\delta = (\frac{1}{1})}: \text{ In this case, } cd = \begin{pmatrix} 1_{2 \times 2} & 1_{2 \times 2} \\ 1_{2 \times 2} & -1_{2 \times 2} \end{pmatrix}. \text{ Letting } \mathcal{H} \text{ denote the cyclic subgroup of } \mathcal{V} \text{ acting via } cd, \text{ we have } K_0|_\mathcal{H} \simeq \mathbb{Z}[\mathcal{H}]^2. \text{ Thus, } H^2(\mathcal{H}, K_0) = 0, \text{ contradicting (6.3).} \]

This completes the proof of the proposition. \qed
Acknowledgment. The work on this article was started while the authors were attending the Noncommutative Algebra program at MSRI in the Fall of 1999. The authors would like to thank the organizers of this program and MSRI staff for their hospitality and support.

REFERENCES

[B] E. Beneish, Induction theorems on the stable rationality of the center of the ring of generic matrices, Trans. Amer. Math. Soc. 350 (1998), 3571 – 3585.
[Bou] N. Bourbaki, Algèbre, chap. 1–3, Hermann, Paris, 1970.
[Br] K. S. Brown, Cohomology of Groups, Springer-Verlag, New York, 1982.
[C-TS] J.-L. Colliot-Thélène and J.-J. Sansuc, La R-équivalence sur les tores, Ann. scient. Éc. Norm. Sup. 10 (1977), 175–230.
[F] E. Formanek, The center of the ring of 3×3 generic matrices, Linear and Multilinear Algebra 7 (1979), 203–212.
[FH] W. Fulton and J. Harris, Representation Theory, A First Course, Springer-Verlag, New York, 1991.
[HS] P. J. Hilton and U. Stammbach, A Course in Homological Algebra, 2nd ed., Springer-Verlag, New York, 1997.
[LL] N. Lemire, M. Lorenz, On certain lattices associated with generic division algebras, preprint, 1999. Available at http://www.math.temple.edu/~lorenz/papers/lattices.html
[L] H. W. Lenstra, Rational functions invariant under a finite abelian group, Invent. math. 25 (1974), 299–325.
[M] K. Masuda, On a problem of Chevalley, Nagoya Math. J. 8 (1955), 59–63.
[N] L. A. Nazarova, Integral representations of Klein’s four-group, Soviet Math. Dokl. 2 (1961), 1304–1307.
[OV] A. L. Onishchik, E. B. Vinberg, Lie Groups and Algebraic Groups, Springer-Verlag, 1990.
[Pa] D. S. Passman, The Algebraic Structure of Group Rings, John Wiley & Sons, New York, 1977.
[Po] V. V. Popov, Sections in Invariant Theory, Proceedings of the Sophus Lie Memorial Conference, Scandinavian University Press, 1994, 315–362.
[PV] V. L. Popov, E. B. Vinberg, Invariant Theory, in Encyclopedia of Math. Sciences 55, Algebraic Geometry IV, edited by A. N. Parshin and I. R. Shafarevich, Springer-Verlag, 1994.
[Pr] C. Procesi, Non-commutative affine rings, Atti Acc. Naz. Lincei, S. VIII, v. VIII, fo. 6 (1967), 239–255.
[Re1] Z. Reichstein, On a theorem of Hermite and Joubert, Canadian Journal of Math., 51 (1) (1999), 69–95.
[Re2] Z. Reichstein, On the notion of essential dimension for algebraic groups, Transformation groups, to appear. Available at http://ucs.orst.edu/~reichstz/pub.html.
[RY] Z. Reichstein, B. Youssin, Splitting fields of G-varieties, preprint. Available at http://ucs.orst.edu/~reichstz/pub.html.
[Ros1] M. Rosenlicht, Some basic theorems on algebraic groups, American Journal of Math., 78 (1956), 401–443.
[Ros2] M. Rosenlicht, A remark on quotient spaces, Anais da Academia Brasileira de Ciências 35 (1963), 487–489.
[Row1] L. H. Rowen, Polynomial Identities in Ring Theory, Academic Press, 1980.
[Row2] L. H. Rowen, Brauer factor sets and simple algebras, Trans. Amer. Math. Soc., 282, no. 2 (1984), 765–772.
Department of Mathematics, Temple University, Philadelphia, PA 19122-6094
E-mail address: lorenz@math.temple.edu

Department of Mathematics, Oregon State University, Corvallis, OR 97331
E-mail address: zinovy@math.orst.edu