Neutron star (NS) is regarded as the natural laboratory for nuclear physics. The equation of state (EoS) extracted in flat spacetime is used chronically as an input to the Tolman–Oppenheimer–Volkoff (TOV) equation to constrain the structure of NS. However, using such EoS to characterize the NS with obvious gravitational effect seems controversial. In our work, we demonstrate the EoS of the same nuclear matter, either on earth or inside NS, ought to be in the same form due to the relativity principle. Gravity only enhances the temperature and the chemical potential, known as Tolman’s law and Klein’s law. We also clarify the self-consistency of the TOV equation, i.e., the equilibrium thermodynamics and gravity are included uniformly. The reason for conclusions in JCAP 02, 026 (2021) and Phys. Rev. D 104, 123005 (2021) is that the equilibrium thermodynamic relations protected by the equivalence principle in local spacetime are not taken into account.

I. INTRODUCTION

In our universe, the NS is one of the densest stars, whose mass is usually estimated between 1.4 $M_\odot$ (known as the Chandrasekhar limit) and 2.16 $M_\odot$ [1–3]. In 2019, the astronomer observed a binary system, named PSR J0740+6620, consisting of an NS and a white dwarf [4]. In this binary system, the NS is deemed to be $2.08 \pm 0.07 M_\odot$, nearing the mass upper limit [5]. This is the most massive NS ever observed so far. Usually around the nuclear saturation densities, the NS is thought to consist of nucleon matter combined with leptons. One also argues that quark matter will emerge in the core [6]. Recently, an amazing star has been observed as a candidate with quark components [7]. With the stellar density increasing further, the strangeness matter may appear in NSs as well, like hyperons components and even $s$–quark matter [8, 9]. From these, the broad definition of NS can be subdivided into hyperon stars, hybrid stars and even quark stars. Abundant nuclear configurations prompt NSs as natural laboratories for studying nuclear physics. Except for interesting nuclear structures, the binary merging events are the important processes to study both astronomy and nuclear physics [10, 11]. NSs construct the bridge from microcosmic to macroscopic, from quantum to gravity, and from nuclear physics to astrophysics. There exist numerous observations, however, plenty of unclear issues still remain. For example, the emergence of hyperon matter in dense enough NSs seems inevitable. But this will soften the EoS and induce the largest mass predicted by theory lighter than the Chandrasekhar limit. This problem is known as the “hyperon puzzle” [12]. Although several effective explanations are proposed, the root may originate from the obscurity of interactions among the many-body hyperons system [13, 14]. It’s still doubtful if there is quark matter in the NS and the crossover to the hadronic phase is uncertain [2, 15]. Understanding the origin of the strong magnetic field of NS is also a puzzle [16]. To study these issues, one of the top priorities is to depict the configurations of NSs correctly. In theory, plenty of phenomenological nuclear EoS are used to do that. Some successful potential models, like TNI2, TNI3, TNI2u, TNI3u [17, 18], AV18 + TBF, Paris+TBF [19], CD Bonn [20], etc., are proposed to describe the potential between hadrons accurately and applied to describe the nuclear matter in the NS. At the same time, effective models based on quantum field theory also play essential roles to depict relativistic nuclear matter in the NS [13, 21–23]. While these effective EoS are either determined by quantum theory in global flat spacetime or extracted from the nuclear data in labs on earth, in both cases the gravity effect is not taken into account at all. Whereas, the TOV equation, in charge of determining the configurations of spherical star based on EoS, is derived from the ideal fluid staying in equilibrium based on Einstein’s equation [24–26]. It seems discordant to put such EoS into the TOV equation to determine the structure of NSs. A question comes into being: Can we use these EoS to reproduce the nuclear matter in NS? Recently, some works claim that this manipulation is not self-consistent and the gravity correction omitted in previous works will enhance the upper limit of mass dramatically [27, 28].

In this paper, we want to clarify the validity of applying the EoS deriving in flat spacetime into the TOV equation, and the constancy of thermal variables in the TOV equation. Our work is organized as follows: the correct version of EoS entering the TOV equation is explained in Sec. II. The constancy of the thermal variables, i.e. temperature and chemical potential, in the TOV equation is illustrated in Sec. III. To visualize, a model step is established in Sec. IV and the corresponding numerical results are shown in Sec. V. Finally, a summary is provided in Sec. VI. Some supplements are appended in Appendix A, Appendix B and Appendix C.

The natural unit is applied throughout. Meanwhile, $G$ represents the gravitational constant.
II. EOS IN THE TOV EQUATION

A. A brief review of the TOV equation

Firstly, we would briefly review the derivation of the TOV equation, based on the process given in Ref. [26]. To a stable, isolated and isotropically spherical NS, the line element is commonly written as

\[ ds^2 = e^{2\Phi(r)}dr^2 - e^{2\nu(r)}d\tau^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2. \]  

The corresponding metric is defined as

\[ g_{\mu\nu} = \text{diag} \left( e^{2\Phi(r)}, -e^{2\nu(r)}, -r^2, -r^2 \sin^2 \theta \right) , \]
\[ g^{\mu\nu} = \text{diag} \left( e^{-2\Phi(r)}, -e^{-2\nu(r)}, -r^2, -r^2 \sin^2 \theta \right). \]  

Here, \( \Phi \) and \( \nu \) are arbitrary functions dependent on the radial coordinate \( r \). Since we seek the static solutions, we can take the four-velocity \( U_\mu \) pointing the timelike direction. For normalization \( U^2 = -1 \), it becomes \( U_\mu = \left( e^{\Phi(r)}, 0, 0, 0 \right) \). As an ideal model, the NS is traditionally regarded as consisting of perfect fluid. In terms of the energy-momentum tensor of perfect fluid

\[ T_{\mu\nu} = (P + \epsilon) U_\mu U_\nu + P g_{\mu\nu}, \]  

the components are

\[ T_{tt} = e^{2\Phi(r)} \epsilon(r) , \quad T_{rr} = e^{2\nu(r)} P(r) , \quad T_{\theta\theta} = r^2 P(r) , \quad T_{\phi\phi} = r^2 \sin^2 \theta P(r). \]  

where \( P \) and \( \epsilon \) are pressure and energy density at \( r \), respectively. According to the Einstein equation \( G_{\mu\nu} = 8\pi G T_{\mu\nu} \), three independent equations are derived as

\[
\begin{cases}
\text{the } tt \text{ component:} & 8\pi G \epsilon(r) = \frac{e^{-2\nu(r)}}{r^2} \left[ 2r \nu'(r) - 1 + e^{2\nu(r)} \right], \\
\text{the } rr \text{ component:} & 8\pi G P(r) = \frac{e^{-2\nu(r)}}{r^2} \left[ 2r \Phi'(r) + 1 - e^{2\nu(r)} \right], \\
\text{the } \theta\theta \text{ component:} & 8\pi G P(r) = e^{-2\nu(r)} \left[ \Phi''(r) + \Phi'^2(r) - \Phi'(r) \nu'(r) + \frac{\Phi'(r) - \nu'(r)}{r} \right].
\end{cases}
\]  

The prime means derivative on \( r \). The \( \phi\phi \) component is proportional to the \( \theta\theta \) component. With a helpful function defined as

\[ m(r) = \frac{1}{2G} \left( r - re^{-2\nu} \right), \]

the \( tt \) component becomes

\[ \frac{dm(r)}{dr} = 4\pi r^2 \epsilon(r). \]  

To discover the meaning of \( m \), the integral version of Eq. (7)

\[ m(r) = 4\pi \int_0^r \epsilon(r') r'^2 dr' \]  

is illuminating. It seems \( m(r) \) is the integral of energy density over the stellar interior, which can be interpreted as the mass within range \( r \). Hence, the Schwarzschild (gravitational) mass of a star with a radius \( R \) is \( M = m(R) \). In the view of general relativity, the equivalence principle tells an equivalence between the gravitational and the inertial mass [29]. The energy-momentum conservation \( \nabla_\mu T_{\mu\nu} = 0 \) always makes sense, where \( \nabla_\mu T_{\mu\nu} = 0 \) combined with Eq. (5) gives

\[ \frac{dP(r)}{dr} = - \left[ P(r) + \epsilon(r) \right] \frac{d\Phi(r)}{dr}. \]
Then, with the help of Eq. (7), the \( rr \) component is simplified to

\[
\frac{d \Phi (r)}{dr} = \frac{Gm (r) + 4 \pi G r^3 P (r)}{r [r - 2 Gm (r)]}.
\]  

Eq. (7), Eq. (9) and Eq. (10) are collectively known as the TOV equation:

\[
\begin{align*}
\frac{d P (r)}{dr} &= - [P (r) + \epsilon (r)] \frac{d \Phi (r)}{dr}, \\
\frac{d \Phi (r)}{dr} &= \frac{Gm (r) + 4 \pi G r^3 P (r)}{r [r - 2 Gm (r)]}, \\
\frac{dm (r)}{dr} &= 4 \pi r^2 \epsilon (r),
\end{align*}
\] 

As Eq. (7) and Eq. (9) show that the TOV equation links the EoS in local spacetime to the global property of the NS step by step. Eq. (10) states the gravity configuration is related to the local matter property. And the energy-momentum conservation of each fluid element is satisfied self-consistently.

### B. EoS in local spacetime inside NS

We can imagine that the EoS is actually locally effectual on the plane tangent to the geodesic on a manifold. Hence, to a sphere, the EoS is tenable in a local spacetime, either on earth or inside the NS. It’s a rigorous fact: “small enough regions of spacetime look like flat (Minkowski) space” [26]. On earth, the gravity effect is unconspicuous. Hence, any local spacetimes are equal inertial frames. Taking the orthogonal Cartesian coordinate \( V^\mu = (t, x) = (t, X, Y, Z) \) for instance, the according metric is \( \eta^{\mu \nu} = \text{diag} (1, -1, -1, -1) \). It’s obvious to satisfy

\[
d_\sigma \eta^{\mu \nu} = 0,
\] 

i.e. in the view of general relativity, such coordinate is known as \textit{locally inertial coordinate}. \( V^\mu \) is the basis vector constitute, known as the \textit{local Lorentz frame}. Einstein’s relativity principle assumes that all inertial frames are totally equivalent to the performance of all physical experiments, and any identical local observations within the same conditions cannot distinguish. Compared with \( r \), \( r_1 \) seems a rescale of the radial length, therefore they have approximately the properties of the radius. Just Eq. (14) is more helpful in our following derivations. The most general line element in a static and asymptotically flat spacetime in spherical polar coordinates appears to

\[
ds^2 = e^{\Phi_1 (r_1)} dr^2 - e^{\nu_1 (r_1)} \left( dr_1^2 + r_1^2 d\theta^2 + r_1^2 \sin^2 \theta d\phi^2 \right),
\] 

Comparing \( r, r_1 \) seems a rescale of the radial length, therefore they have approximately the properties of the radius. Just Eq. (14) is more helpful in our following derivations. The most general line element in a static and asymptotically flat spacetime in spherical polar coordinates appears to

\[
ds^2 = U (r) dr^2 - V (r) dr^2 - W (r) \left( r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right),
\] 

where \( U, V, W \) are arbitrary functions of \( r \) [32]. Eq. (14) is in the case of isotropy which claims \( W (r) = V (r) \).

For a nuclear matter fluid element, the length scale ought to be in the unit of nuclei radius \( r_N = 6.10 \text{ GeV}^{-1} \). Compared with the NS radius, such an element is small enough to act as a candidate to be considered in a locally flat spacetime. Based on the TOV equation, the change of \( \Phi \) in this volume element is nearly negligible. More numerical detail is shown in Appendix A. In other words, to such a nuclear element, \( \Phi \), or \( \Phi_1 \) at present, is nearly a constant. According to the covariance of spacetime, \( \nu_1 \)
normal coordinates are not unique, \( V \) the thermodynamic limit, the factor \( \exp (\nu_1) \) in the spatial component will be omitted during the statistical physics steps, so that \( x^* \) has no difference from \( x \). But the gravity effect remains in \( \exp (\Phi) \) in \( r^* \). This is known as the time dilation effect. Although normal coordinates are not unique, \( V^\mu \) and \( V^\mu_* \) have the most analogous information of spacetime. Taken together, a general inertial coordinate has been found to be used to project the EoS derived on earth to construct the NS. The local EoS of NS ought to be in the same form as on earth.

Universally, the EoS is a function of thermal variables, i.e. temperature and chemical potential. The general equivalence principle said in small enough regions of spacetime, the laws of physics reduce to those of special relativity and it is impossible to detect the existence of a gravitational field by means of local experiments [26, 35]. That means if a fluid element sufficiently is small, its EoS ought to be described as a function of local thermodynamic variables in its local rest frame [33]. Like the free-fall experiment, the falling process seems obviously different after moving the same object from the earth to the moon because of the difference in gravity. But Newton’s second law is not violated. As an analogy, the EoS of nuclear matter in NS is like Newtonian mechanics in the falling experiment, while the local thermal variable is like gravitational acceleration. Then, we will ask if the thermal variables are \( (T^i, \mu^i) \) in totally flat spacetime, e.g on the earth, how will gravity change them in the local spacetime inside NS? This doubt has been answered in a recent work Ref. [34] within the framework of equilibrium thermodynamics.

The local thermal variables ought to respect the fundamental laws of thermodynamics owing to the general equivalent principle. With the boundary condition of asymptotically flat, based on the Gibbs-Duhem relation and the first law of thermodynamics, they found that the local thermal variables \( (T^*, \mu^*) \) are related to \( (T^i, \mu^i) \) via

\[
T^* = T^i e^{-\Phi}, \quad \mu^* = \mu^i e^{-\Phi}.
\] (19)

Actually, Eq. (19) is known as Tolman’s law and Klein’s law (TK law), respectively [36–38]. \( (T^*, \mu^*) \) is the real thermal variables determining the EoS inside the NS, while \( (T^i, \mu^i) \) are the variables existing on the zero gravitational potential hypersurface and redshifted by the gravity [39]. In Appendix B, we illustrate the TK law ought to be caused by the time dilation effect via a relativistic free fermion gas model. Their formalism clarifies the TK law is self-consistent in the framework of equilibrium thermodynamics, without any approximations. Moreover, Ref. [34] also proved that the global thermodynamic potentials obey the conventional first law of thermodynamics by assuming the additivity of extensive thermodynamic quantities. Such self-consistency is guaranteed by the equivalence principle in general relativity. More basically, Ref. [40] emphasized that Tolman’s law is general for thermal equilibrium in curved spacetime with spherical symmetry, which is protected by the energy-momentum conservation of the stress tensor. Then, based on the grand canonical ensemble, we can define pressure \( P \), conserved charge number density \( n \), entropy density \( s \) and energy density \( \epsilon \) in local spacetime naturally

\[
P = -\Omega, \quad n = -\frac{\partial \Omega (T^*, \mu^*)}{\partial T^*}, \quad s = -\frac{\partial \Omega (T^*, \mu^*)}{\partial \mu^*}, \quad \epsilon = -P + T^* s + \mu^* n. \] (20)

Actually, the neglection of Eq. (20) in Refs. [27, 28] induces their remarkable enhancement of mass upper limit.

In hand with Eq. (20), let’s revisit Eq. (9), then we will find another way to explain our point above. According to \( \epsilon = -P + T^* s + \mu^* n \), Eq. (9) becomes

\[
\frac{dP (r)}{dr} = -\left[ T^* s (T^*, \mu^*) + \mu^* n (T^*, \mu^*) \right] \frac{d\Phi (r)}{dr} = -\left[ T^* \frac{\partial P (T^*, \mu^*)}{\partial T^*} + \mu^* \frac{\partial P (T^*, \mu^*)}{\partial \mu^*} \right] \frac{d\Phi (r)}{dr}. \] (21)

Comparing with the total differential of \( P \)

\[
\frac{dP (r)}{dr} = \left( \frac{\partial P}{\partial \Phi} + \frac{\partial P}{\partial T^*} \frac{dT^*}{d\Phi} + \frac{\partial P}{\partial \mu^*} \frac{d\mu^*}{d\Phi} \right) \frac{d\Phi (r)}{dr}. \] (22)
we find that
\[ \frac{\partial P}{\partial \Phi} \equiv 0, \tag{23} \]
and
\[ \frac{dT^*}{d\Phi} = -T^*, \quad \frac{d\mu^*}{d\Phi} = -\mu^*. \tag{24} \]

Eq. (23) and Eq. (24) indicate that the grand canonical EoS depends on \( \Phi \) only through the temperature and chemical potential, i.e., the gravity changes the EoS only through the redshift of local thermal variables \( T^* \) and \( \mu^* \). Hence, the form of EoS is unchanged. At the same time, the solutions of Eq. (24) are
\[ T^* (r) \propto e^{-\Phi(r)}, \quad \mu^* (r) \propto e^{-\Phi(r)}. \tag{25} \]

Taking the boundary condition of asymptotically flat into account, the TK law is recovered again. Unlike the proof based on the relativity principle, the equivalent principle and energy-momentum conservation are combined to arrive at the same conclusions.

In a nutshell, the fundamental principles of relativity make sure the EoS in the local frame of NS is in the same form in flat spacetime. The gravity effect only enters EoS through thermal variables which are linked to the variables in flat spacetime via the TK law.

### III. Constancy of Thermal Variables in the TOV Equation

In terms of the TK law, thermal variables in gravity field are composed of the redshifted variables as well as the gravity enhancement. Via the chain rule, the l.h.s of Eq. (9) yields to
\[ \frac{dP (r)}{dr} = \frac{dP}{dT^*} \frac{dT^*}{dr} + \frac{dP}{d\mu^*} \frac{d\mu^*}{dr} = e^{-\Phi(r)} \left[ s \frac{dT^I (r)}{dr} + n \frac{d\mu^I (r)}{dr} \right] - (T^* s + \mu^* n) \frac{d\Phi (r)}{dr}. \tag{26} \]
Then, Eq. (9) becomes
\[ e^{-\Phi(r)} \left[ s \frac{dT^I (r)}{dr} + n \frac{d\mu^I (r)}{dr} \right] = - \left[ P + \epsilon - T^* s - \mu^* n \right] \frac{d\Phi (r)}{dr} \rightarrow s \frac{dT^I (r)}{dr} + n \frac{d\mu^I (r)}{dr} = 0, \tag{27} \]
where the definitions of thermal quantities in Eq. (20) are inserted. Similarly, Eq. (27) has been derived in Ref. [40] (Notice that \( \sigma = s/n \) in Eq. (15) in their work). As pointed out in Ref. [40], there is no specific reason to chop off the relevance between the temperature and chemical potential. While in Klein’s primal work Ref. [38], he demanded Tolman’s law always tenable, as in the zero chemical potential case. His shortcut approach equals to inserting another constrain, i.e. \( T^*/\mu^* = \) constant. Actually, over different kinds of matter, Tolman’s law is general because of the energy-momentum conservation, but the universality of Klein’s law does not always hold [40, 41]. Fortunately, as the logarithm of fugacity, \( \mu^*/T^* \) ought to be position independent in NSs. Since a constant fugacity ensures the heat flow and diffusion vanish in an equilibrium system [42]. In fact, \( (\mu^*/T^*)' = 0 \) is also the prerequisite to arriving at the TK law in Ref. [34]. Assuming \( \mu^*/T^* = \mu^I/T^I = C \) with \( C \) for a position independent constant, we have
\[ \frac{s}{C} \frac{dT^I (r)}{dr} + n \frac{d\mu^I (r)}{dr} = \left\{ \begin{aligned} \frac{(s + C n)}{C} \frac{d\mu^I (r)}{dr} = 0 & \rightarrow \frac{d\mu^I (r)}{dr} = 0, \\ \frac{(s)}{C} \frac{dT^I (r)}{dr} = 0 & \rightarrow \frac{dT^I (r)}{dr} = 0. \end{aligned} \tag{28} \]

To the NS in equilibrium, the redshifted thermal variables observed at spatial infinity are always robust. The matter distribution varied in the radial direction is totally caused by gravity. A schematic diagram is shown in FIG. 1 to help understand this conclusion:
FIG. 1: A schematic diagram of the gravity effect induced by an NS. The red line in the upper half is the local thermal variables. The blue mesh in the lower half is the gravity potential of the NS.

IV. MODEL STEP

To present our conclusions above and visualize the evolution of the gravity effect in the NS, a framework based on an effective model is established in this section.

After a huge number of neutrinos emitting, much of the energy is carried out. After that, the temperature at the surface of an isolated and stable NS is around $10^6$ K $\approx 86.17$ eV [43]. Hence, a zero temperature approximation is reasonable. Some helpful thermal quantities with explicit expressions are presented in Appendix C. It’s convenient to solve

$$
\frac{d\mu^e(r)}{dr} = -\mu^e(r) \frac{d\Phi}{dr},
$$

$$
\Phi(r) = \frac{Gm(r) + 4\pi Gr^3 P(r)}{r[r - 2Gm(r)]},
$$

$$
\frac{dm(r)}{dr} = 4\pi r^2 \epsilon(r),
$$

(29)

at zero temperature based on the effective EoS, instead of Eq. (11).

It’s known that the single pion exchange process determines long-range interaction between nucleons [44]. While the exchange of multi-pions as well as single heavy meson, like $\sigma$, $\omega$, $\rho$, characterize mid- and short-range interaction [45]. In the single-meson exchange model, single heavy meson exchange processes can effectively parameterize the multi pions process [46]. In the hadron matter crust of NSs, mid- and short-range interactions are dominant. Hence, in this work, we apply the well-known Walecka model to characterize the nucleon matter in the NS [47]. Without the requirement of renormalization in low-energy effective theories, the cubic and quartic interaction terms are added phenomenologically to describe the bound nuclear matter in the non-relativistic potential approach [21]. The Lagrangian density of this modified Walecka (mW) model is written as

$$
L_{\text{NS}} = \sum_{N=n,p} \bar{\psi}_N \left[ i\gamma^\mu \left( \partial_\mu - i\delta^\mu_\nu \mu_N \right) \right] - m_N + g_{\sigma N} \sigma - g_{\omega N} \omega_\mu \psi_N + \sum_{l=e,\mu,\nu} \bar{\psi}_l \left[ i\gamma^\mu \left( \partial_\mu - i\delta^\mu_\nu \mu_l \right) - m_l \right] \psi_l
$$

+ $\frac{1}{2} \left( \partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2 \right) - \frac{1}{3} bm_N \left( g_{\sigma N} \sigma \right)^3 - \frac{1}{4} c (g_{\sigma N} \sigma)^4 - \frac{1}{4} \omega_\mu^\nu \omega_\nu^\rho + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\rho,$

(30)

where $\omega_\mu^\nu = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$. In this model, nucleons $N$, i.e., protons $p$ and neutrons $n$, are included. Free electrons $e^-$ and muons $\mu^-$ are appended to keep $\beta$-equilibrium. The isospin symmetry between nucleons is considered, i.e. $g_{MP} = g_{MN}$, where $g_{MN}$ are the interaction coupling constants between mesons $M$ and nucleons $N$. $m_N$, $m_l$, $m_\mu$, and $m_\nu$ are the particle masses of nucleons, leptons, the $\sigma$ meson and the $\omega$ meson, respectively. Because of the existence of leptons, the conserved charges are baryon number $B$ and electric charge number $Q$. The chemical potentials of particles are defined as

$$
\mu_n = \mu_B, \quad \mu_p = \mu_B + \mu_Q, \quad \mu_{e^-} = \mu_{e^-} = -\mu_Q.
$$

(31)

Accordingly, the particle densities of conserved charges are

$$
n_B = n_p + n_n, \quad n_Q = n_p - n_e - n_{\mu^-}.
$$

(32)
The electrically neutral condition of NSs asks for \( n_\Omega = 0 \).

We adopt the relativistic mean-field (RMF) approximation \[48\]. The meson fields are allowed to receive density-dependent expectations: \( \bar{\sigma}, \bar{n}_\Omega \). Based on RMF, one can define meson-dressed nucleon mass \( \bar{m}_N \) and corresponding effective nucleon chemical potential \( \bar{\mu}_N \):

\[
\bar{m}_N = m_N - g_{\sigma N} \bar{\sigma}, \quad \bar{\mu}_N = \mu_N - g_{\omega N} \bar{\omega}_0.
\] (33)

Notice that the local gravity field has no influence on interaction terms so that the mesonic correction in \( \bar{\mu}_N \) is unchanged. In other words, \( \exp(\Phi) \) only dress \( \mu_N \) to \( \bar{\mu}_N \), i.e.

\[
\bar{\mu}_N = \mu^*_N - g_{\omega N} \bar{\omega}_0.
\] (34)

With the expressions in Appendix C, the local pressure in the NS is

\[
P_{\text{NS}} (\mu^*_B) = \sum_{N=\nu,\bar{\nu}} P_{\text{FG}} (\bar{\mu}_N, \bar{m}_N) + P_{\text{FG}} (\mu^*_c, m_c) + P_{\text{FG}} (\mu^*_\nu, m_\nu) - \frac{1}{2} m_\sigma^2 \bar{\sigma}^2 - \frac{1}{3} b m_N (g_{\sigma N} \bar{\sigma})^3 - \frac{1}{4} c (g_{\sigma N} \bar{\sigma})^4 + \frac{1}{2} m_\omega^2 \bar{\omega}_0^2.
\] (35)

To a system in equilibrium, the pressure ought to be maximum. To maximize the pressure, we have extremum conditions

\[
\frac{dP_{\text{NS}}}{d\bar{\omega}_0} = \frac{dP_{\text{NS}}}{d\bar{\sigma}} = 0.
\] (36)

For \( \omega \) meson, the gap equations yields to

\[
\frac{dP_{\text{NS}}}{d\bar{\omega}_0} = \frac{dP_{\text{NS}}}{d\bar{\sigma}} + \sum_{N=\nu,\bar{\nu}} \frac{\partial P_{\text{NS}}}{\partial \bar{\mu}_N} \frac{d\bar{\mu}_N}{d\bar{\omega}_0} = m_\omega \bar{\omega}_0 - \sum_{N=\nu,\bar{\nu}} g_{\omega N} n_N = 0.
\] (37)

And for \( \sigma \) meson, we have

\[
\frac{dP_{\text{NS}}}{d\bar{\sigma}} = \frac{\partial P_{\text{NS}}}{\partial \bar{\sigma}} + \sum_{N=\nu,\bar{\nu}} \frac{\partial P_{\text{NS}}}{\partial \bar{\mu}_N} \frac{d\bar{\mu}_N}{d\bar{\sigma}} = - \left( m_\sigma^2 \bar{\sigma} + b m_N g_{\sigma N} \bar{\sigma}^2 + c g_{\sigma N} \bar{\sigma}^3 \right) + \sum_{N} g_{\sigma N} n^*_N.
\] (38)

where \( n^*_N \) is the scalar condensate number density shown as Eq. (C7). Then, we have gap equations of mW model in RMF:

\[
\begin{align*}
 & \left\{ m_\omega^2 \bar{\omega}_0 - \sum_{N} g_{\omega N} n_N (\bar{\mu}_N, \bar{m}_N) = 0, \\
 & m_\sigma^2 \bar{\sigma} + b m_N g_{\sigma N} \bar{\sigma}^2 + c g_{\sigma N} \bar{\sigma}^3 - \sum_{N} g_{\sigma N} n^*_N (\bar{\mu}_N, \bar{m}_N) = 0.
\end{align*}
\] (39)

With the help of gap equations, the corresponding energy density is simplified to

\[
e_{\text{NS}} (\mu^*_B) = \sum_{N} e_{\text{FG}} (\bar{\mu}_N, \bar{m}_N) + e_{\text{FG}} (\mu^*_c, m_c) + e_{\text{FG}} (\mu^*_\nu, m_\nu) + \frac{1}{2} m_\sigma^2 \bar{\sigma}^2 + \frac{1}{3} b m_N (g_{\sigma N} \bar{\sigma})^3 + \frac{1}{4} c (g_{\sigma N} \bar{\sigma})^4 + \frac{1}{2} m_\omega^2 \bar{\omega}_0^2.
\] (40)

In a phenomenological way, we have set the divergent vacuum energy to zero.

To check the constancy of the TOV equation, i.e. \( d\mu^*/dr = 0 \), we would like to calculate \( dP/d\Phi \) in the framework of mW model:

\[
\frac{dP_{\text{NS}}}{d\Phi} = \frac{\partial P_{\text{NS}}}{\partial \Phi} + \frac{\partial P_{\text{NS}}}{\partial \bar{\mu}_Q} \frac{d\bar{\mu}_Q}{d\Phi} + \frac{\partial P_{\text{NS}}}{\partial \bar{\sigma}} \frac{d\bar{\sigma}}{d\Phi} + \frac{\partial P_{\text{NS}}}{\partial \bar{\omega}_0} \frac{d\bar{\omega}_0}{d\Phi}
\]

\[
= \sum_{j=NJ} \frac{\partial P_{\text{NS}}}{\partial \bar{\mu}^*_j} \frac{d\bar{\mu}^*_j}{d\Phi} + \frac{\partial P_{\text{NS}}}{\partial \bar{\mu}^*_Q} \frac{d\bar{\mu}^*_Q}{d\Phi}
\]

\[
= - \sum_{j=NJ} (\bar{\mu}^*_j \delta_{Nj} + Q_j \bar{\mu}^*_j) n_j (\bar{\mu}^*_j) - \bar{\mu}^*_Q n_Q (\bar{\mu}^*_Q)
\]

\[
= - \bar{\mu}^*_Q n_Q - m_\omega^2 \bar{\omega}_0^2,
\]

where we have inserted both extremum condition Eq. (36) and the electrically neutral condition. Based on Eq. (35) and Eq. (40),
we find that \( P_{\text{NS}} + \epsilon_{\text{NS}} = \tilde{\mu}_B n_B + m_e^2 \tilde{\omega}_0^2 \). Actually, with the help of gap equations Eq. (39) and Eq. (34), it turns to \( dP_{\text{NS}}/d\Phi = -\mu_B n_B \). Hence, as shown, the constancy of TOV equation is exhibited within mW model.

V. NUMERICAL RESULTS AND DISCUSSIONS

A. Rescale the TOV equation

As mentioned in Appendix A, \( G \) is a very small constant in the natural unit. To put the TOV equation into practice numerically, a rescale scheme is needed. We select a helpful scheme mentioned in Ref. [49]. The rescale unit are presented in TABLE I.

| Physical quantity | Pressure | Energy density | Distance | Mass |
|-------------------|----------|---------------|----------|------|
| Dimensionless quantity | \( \hat{P} \) | \( \hat{\epsilon} \) | \( \hat{x} \) | \( \hat{m} \) |
| Unit | \( P_0 = m_n^4/ \left(8\pi^2\right) \) | \( \epsilon_0 = m_n^4/ \left(8\pi^2\right) \) | \( R_0 = \sqrt{3\pi/\alpha_G \lambda_N} \) | \( M_0 = (R_0/R_0^n) M_0 \) |
| Value | 1.282 GeV/fm\(^3\) | 1.282 GeV/fm\(^3\) | 8.378 km | 2.837 \( M_\odot \) |

TABLE I: \( \alpha_G = Gm_n^2 \) is the dimensionless strength of the gravitational coupling between two neutrons. \( \lambda_N = m_n^{-1} \) is the Compton wavelength of the neutron. \( M_\odot \) is the mass of sun. \( R_0^n = 2GM_0 \) is the Schwarzschild radius of the sun.

In this way, we can rescale the too large or the too small constants to \( 2GM_0/R_0 = 4\pi R_0^3 \epsilon_0 / (3M_0) = 1 \). Then, the original TOV equation reduces to

\[
\begin{align*}
\frac{d\hat{P}(x)}{dx} &= - \left[ \hat{P}(x) + \hat{\epsilon}(x) \right] \frac{d\Phi(x)}{dx}, \\
\frac{d\Phi(x)}{dx} &= \frac{\hat{m}(x) + 3\hat{\chi} \hat{P}(x)}{2x \left[ x - \hat{m}(x) \right]}, \\
\frac{d\hat{m}(x)}{dx} &= 3x^2 \hat{\epsilon}(x).
\end{align*}
\]

Especially, at zero temperature, the first equation can be simplified to

\[
\frac{d\mu_B(x)}{dx} = -\mu_B(x) \frac{d\Phi(x)}{dx},
\]

as in Eq. (29). The radius of neutron star is \( R_{\text{NS}} = \hat{R}_{\text{NS}} R_0 \) and the total mass is \( M_{\text{NS}} = \hat{M}_{\text{NS}} M_0 \). At the surface of NS, \( \hat{P}(x_{\text{NS}}) \) ought to be zero. Thus, the gravity factor at the surface \( \Phi_s \) can be solved

\[
\Phi_s = \frac{1}{2} \ln \left( 1 - \frac{2GM_{\text{NS}}}{R_{\text{NS}}} \right) = \frac{1}{2} \ln \left( 1 - \frac{\hat{M}_{\text{NS}}}{\hat{R}_{\text{NS}}} \right).
\]

B. The choice of parameter

We list the masses of each particle adopted in this work in TABLE II [50]. According to the scalar–isoscalar resonance in \( \pi - \pi \) scattering, \( m_\sigma \) is chosen 550 MeV in our work.

| Particle | \( p \) | \( n \) | \( \sigma \) | \( \omega \) | \( e^- \) | \( \mu^- \) |
|----------|-----|-----|-----|-----|-----|-----|
| Mass (MeV) | 939 | 939 | 550 | 783 | 0.511 | 105.66 |

TABLE II: Masses of different particles.

With different corrections taken into account, there exist several adjustable schemes of parameters. Here, we adopt the conventional one mentioned in Ref. [21]. In RMF, the coupling constants can be determined by fitting empirically known
properties of nuclear: the saturation density $n_0 = 0.153 \text{ fm}^{-3}$, the binding energy $E_b = -16.3 \text{ MeV}$, the Landau mass $m_L = 0.83m_N$ and the compressibility $K = 250 \text{ MeV}$ [51, 52]. Of these, the compressibility has the greatest uncertainty, approximately $\pm 30 \text{ MeV}$. Then, the parameters are fitted to [21]

| $g_{\sigma N}$ | $g_{\rho N}$ | $b \times 10^3$ | $c \times 10^4$ |
|---------------|-------------|----------------|----------------|
| 8.685         | 8.646       | 7.950          | 6.952          |

TABLE III: Interaction parameters.

C. Numerical results

In this subsection, the local chemical potential at the core $\mu_B^c$ is chosen as $\mu_B^c \in [0.95 \text{ GeV, 5 GeV}]$. The mass varying with the radius of NS as well as the baryon density at the core $n_B^c$ is shown in FIG 2.

![FIG 2: The mass-radius relation (left panel) and the mass as a function of the baryon density at core (right panel). From top to bottom, two red dashed lines are the upper (2.16 $M_\odot$) and lower limit (1.4 $M_\odot$) of the NS’s mass, respectively. The neutron star in PSR J0740+6620 is depicted in the orange band.](image)

To reveal the evolution of the gravity effect, i.e. $\Phi$, we design the following algorithm shown in FIG. 3.
FIG. 3: The algorithm to distinguish the chemical potential and $\Phi$ self-consistently.

In this algorithm, firstly with a given local chemical potential $\mu^c_B$ at the core, the pressure decreases to zero gradually with the evolution of the TOV equation within Eq. (29). Then, both the gravity factor $\Phi_s$ and the chemical potential $\mu^s_B$ at the surface are known. Based on the constancy of the TOV equation, i.e. $\mu^f_B$ is always a constant, $\mu^f_B$ can be determined at the surface via $\mu^s_B = \mu^f_B \exp (-\Phi_s)$. Dividing $\mu^c_B$ with $\mu^f_B$ at each distance, we can get the evolution of $\Phi$. With this algorithm, the self-consistent evolution of $\Phi$ in the interior of NS is shown in FIG. 4.

FIG. 4: $\Phi$ in the interior of NS. In both panels, $\mu^c_B = 1.14$ GeV produces the lightest NS, i.e. $M_{NS} = 1.4 \ M_\odot$, in mW model framework. (a) $\Phi$ at both surface ($\Phi_s$, the red solid line) and core ($\Phi_c$, the blue solid line) are depicted as functions of baryon number density at the stellar core $n^c_B$. The colored dashed lines are labeled by different $\mu^c_B$ with arrows pointing from the core to the surface. (b) The thin dashed lines are the evolution of $\Phi$ inside the NS, which are functions of the distance away from the core. Different colors, as in (a), are used to label different $\mu^c_B$. The thick boundary (dark gray solid line) wrapping the dashed lines is $\Phi_s$.

Finally, we would like to have an insight into how gravity enhances the redshifted chemical potential. The results are shown in FIG. 5.
Based on FIG. 4 and FIG. 5, we find that the gravity effect is of essence to the NS. With the growth of matter density at the core, $\Phi_c$ decreases dramatically. In FIG. 4 (b), although different densities of matter at the core cause distinct differences of $\Phi_c$, to the NS with large core densities, $\Phi_c$ tends to be nearly identical in mW model. To produce a denser NS core, the local chemical potential ought to increase, but the redshifted chemical potential is decreasing slightly and also seems identical in the dense region, as shown in FIG. 5 (b). In fact, the property of nucleon matter characterized by mW model is mainly dominated by free relativistic fermion gas. In the dense core region where $\mu_B^* \gg m_N \gg \mu_Q^*$, the pressure as well as the energy density, are proportional to $\mu_B^*$ \footnote{Based on the TOV equation Eq. (29), large evolution gradients of $\mu_B^*$ and $\Phi$ are presented as well. Hence, this competitive relation between large $\mu_B^*$ and the large decreasing evolution gradient induces the non-monotonic tendency of the mass-radius relation as shown in FIG. 2. That is to say, even if the matter density of the core is much denser, the NS may also be configured in a lighter mass and a smaller radius. This conclusion is model-dependent.}. Based on the TOV equation Eq. (29), large evolution gradients of $\mu_B^*$ and $\Phi$ are presented as well. Hence, this competitive relation between large $\mu_B^*$ and the large decreasing evolution gradient induces the non-monotonic tendency of the mass-radius relation as shown in FIG. 2. That is to say, even if the matter density of the core is much denser, the NS may also be configured in a lighter mass and a smaller radius. This conclusion is model-dependent.

VI. SUMMARY AND OUTLOOK

The EoS derived in flat spacetime is usually used to characterize NS. In this paper, we explain the rationality of this manipulation. The fundamental principles of relativity ensure the EoS used to construct the nuclear structure inside NS ought to be in the same form as the one derived on earth. Owing to the equivalent principle, the equilibrium thermodynamics should also be tenable in local inertial spacetime. The gravity effect only acts as a background to enhance the local thermal variables, known as the TK law. Furthermore, we also illustrate the constancy of thermal variables in the TOV equation. To a stable and isolated NS, the redshifted thermal variables remain constants in global spacetime. Temperature and chemical potential varying inside NS are caused by gravity. All the gravity effects are included self-consistently in the TOV equation. Solving the TOV equation equals evolving the gravity field of the NS. Based on our conclusion, the striking enhancement of the mass upper limit in Refs. [27, 28] caused by the so-called gravity effect is questionable. Because the proper definitions of thermodynamic quantities in local spacetime protected by the general equivalent principle are not considered in their works.

The constancy of the thermal variables equation as in Eq. (28) can be used to explore the thermal environment in the interior of NSs if the mass, radius as well as some properties of the surface are known. The corrections from gravity to thermal variables, like the TK law, should be checked in simulations of binary merging. And a complete illustration in the language of quantum field theory is looking forward.

VII. ACKNOWLEDGMENT

In this work, J. L., T. G. and L. H. are supported by the National Natural Science Foundation of China under Grants No. 11890712, and J. Z. is supported by No. 12047535.
Appendix A: Estimation of the magnitude of $\Delta \Phi$ in a fluid element

In the natural unit, the gravitational constant $G$ is $6.71 \times 10^{-39}$ GeV$^{-2}$. Usually, the effective EoS of nuclear matter depicts the system in the size of nuclei radius $r_N \sim 6.10$ GeV$^{-1}$. Based on the TOV equation, the change of $\Phi$ in a fluid element inside an NS is estimated to

$$\Delta \Phi \sim \frac{Gm + 4\pi Gr^3P}{r(r - 2Gm)}r_N.$$  \hspace{1cm} (A1)

If the fluid element is near the core, i.e. $r \sim 0$ and $m \sim 4\pi r^3\epsilon/3$, we have

$$\Delta \Phi \sim \frac{G4\pi r^3\epsilon/3 + 4\pi Gr^3P}{r^2 - 2G4\pi r^4\epsilon/3}r_N \sim rG(\epsilon + P)r_N \sim GM m_B r r_N,$$  \hspace{1cm} (A2)

In the vicinity of the core, the baryon density $n_B$ is a little amount as well and $\mu_B \sim 1$ GeV. The changing of $\Phi$ is in the same magnitude as $G$, i.e. $\Delta \Phi \sim 10^{-39}$.

Then, before turning to the element far from the core, we first list the unit transform in TABLE IV.

| Physical quantity | SI units ↔ natural units |
|-------------------|--------------------------|
| Mass              | 1 kg = $5.56 \times 10^{26}$ GeV |
| Length            | 1 km = $5.08 \times 10^{18}$ GeV$^{-1}$ |
| Time              | 1 s = $1.52 \times 10^{24}$ GeV$^{-1}$ |
| Pressure          | 1 Pa = $1$ kg$^{-1}$m$^{-1}$s$^{-2}$ = $4.74 \times 10^{-38}$ GeV$^4$ |

TABLE IV: The transformation of different units.

Some properties of NS with magnitude are also presented here [1, 53]:

$$m \sim M_\odot = 1.99 \times 10^{30} \text{ kg} = 1.11 \times 10^{57} \text{ GeV},$$

$$r \sim (10^0 \text{ to } 10^1) \text{ km} = (5.08 \times 10^{18} \text{ to } 5.08 \times 10^{19}) \text{ GeV}^{-1},$$

$$P \sim (3.2 \times 10^{31} \text{ to } 1.6 \times 10^{34}) \text{ Pa} = (1.52 \times 10^{-6} \text{ to } 7.58 \times 10^{-4}) \text{ GeV}^4.$$  \hspace{1cm} (A3)

Then, the magnitudes of elements in the TOV equation are

$$GM \sim 7.45 \times 10^{18} \text{ GeV}^{-1},$$

$$4\pi Gr^3P \sim (1.67 \times 10^{13} \text{ to } 8.36 \times 10^{13}) \text{ GeV}^{-1},$$

$$r(r - 2Gm) \sim (-4.99 \times 10^{37} \text{ to } 1.82 \times 10^{39}) \text{ GeV}^{-2}.$$  \hspace{1cm} (A4)

From this, the magnitude of $\Delta \Phi$ is estimated to be $10^{-20}$ and $10^{-18}$. Hence, we have $\exp(\Phi + \Delta \Phi) \approx \exp(\Phi)$, so that $\Phi$ can be regarded as a constant in such fluid elements.

In fact, gravity is a macroscopic effect. $G$ is too little to make the gravity effect obvious in such a microscopic size of nuclear element inside the NS.

Appendix B: In the view of imaginary-time path integral

As mentioned in Sec. 1, some effective EoS are derived based on field theory models. The imaginary-path integral is another version of equilibrium thermodynamics based on quantum field theory [21]. Remember that as we analysis in Sec. A, $\Phi$ nearly invariant in a fluid element. Therefore, we treat $\Phi$ as a constant in our following derivations. Notice that here, the global spacetime coordinates are labeled by Greek alphabets and the local Lorentz spacetime, or local laboratory, coordinates are labeled by Latin scripts.

Since fermions are mainly compositions of NS, we would take the free Dirac field for instance. It’s well-known that the action
of free Dirac field with thermal variable \((T^i, \mu^i)\) in flat spacetime is written as

\[
S_{\text{FG}} = \int_0^{\pi} dt \int d^3x \sqrt{-g_{\mu\nu}} \left( i\gamma^\mu \partial_\mu - m + \gamma^0 \mu^i \right) \psi. \tag{B1}
\]

where \(\gamma^\mu\) is the Dirac matrix satisfying Dirac algebra. To illustrate the gravity effect remains in the local spacetime inside NS, the tetrad formalism is needed here. The vielbein, \(e^a_\mu\), is defined to be the matrix for linking the global spacetime coordinates to the local spacetime coordinates. Considering the thermodynamic limit, we have \(V^\mu = (t', x)\). The metric related to local coordinates is \(\eta^{ab}\), while the metric related to global coordinates ought to be

\[
g^{\mu\nu} = \text{diag} \left( e^{-2\Phi}, -1, -1, -1 \right). \tag{B2}
\]

In terms of \(g^{\mu\nu} = e^a_\mu e^b_\nu \eta^{ab}\), the vielbein yields to

\[
e^a_\mu = \text{diag} \left( e^{-\Phi}, 1, 1, 1 \right). \tag{B3}
\]

Moreover, the Christoffel symbol in local spacetime is

\[
\Gamma^t_{\mu\nu} = \frac{1}{2} g^{kr} \left( \partial_\mu g_{nr} + \partial_n g_{\mu r} - \partial_r g_{\mu n} \right) = 0. \tag{B4}
\]

The spin connection is defined as

\[
\omega^{ab}_\mu = e^a_\nu \Gamma^\nu_{\sigma\mu} e^{\sigma b} + e^a_\nu \partial_\mu e^{\nu b} = e^a_\nu \Gamma^\nu_{\sigma\mu} e^{\sigma b} - e^{\nu b} \partial_\mu e^a_\nu,
\]

so that the covariant derivative is:

\[
D_\mu \psi = \partial_\mu \psi - \frac{1}{2} \omega^{ab}_{\mu\sigma} \left[ \gamma^\sigma, \gamma^b \right] = \partial_\mu \psi. \tag{B6}
\]

Based on these quantities, the action in local spacetime inside NS written in the global spacetime coordinates appears to

\[
S_{\text{FG}} = \int_0^{\pi} dt \int d^3x \sqrt{-g_{\mu\nu}} \left( i\gamma^\mu e^a_\mu D_\mu - m + \gamma^0 \mu^i \right) \psi = \int_0^{\pi} dt \int d^3x \sqrt{-g_{\mu\nu}} \left[ i\gamma^0 \left( \frac{\partial}{\partial t} + \mu^i \right) + e^0 \left( i\gamma \cdot \nabla + m \right) \right] \psi \tag{B7}
\]

It’s interesting to find \(\sqrt{-g_{\mu\nu}}\) rescales \(t\) to \(t'\) and \(e^0_0\) changes \(\partial_t\) to \(\partial_{t'}\). From this, the grand potential \(\Omega_{\text{FG}}\) is derived as

\[
\Omega_{\text{FG}} = 2 \int \frac{d^3k}{(2\pi)^3} \left[ E_k + T^* \ln \left( 1 + e^{-\frac{E_k}{T^*}} \right) + T^* \ln \left( 1 + e^{-\frac{E_k}{T^*}} \right) \right], \tag{B8}
\]

with fermion energy \(E_k = \sqrt{k^2 + m^2}\). Apparently, Eq. (B8) is in the same form with the flat version. The TK law recovers automatically. Looking back to Eq. (17), in a local inertial frame, the locally thermal equilibrium condition is satisfied coherently. Although an observer in local spacetime in gravity field cannot discover the gravity or curvature distinctly works as a background field. A trick in mathematics is presented in Eq. (B7). In general, the invariant action is defined as \(S = \int d^4x \sqrt{-g_{\mu\nu}} \mathcal{L} \left[ \partial^a_\mu \right]\), where the Lagrangian density can be rearranged by the orders of deriviative operators. With the preview in Eq. (B7), we can find that \(\sqrt{-g_{\mu\nu}}\) and \(e^a_0\) will help to transform the time component in the integral measure and \(\partial_0\) at each order from \(t\) to \(t'\), respectively. The time dilation transforming \(t\) to \(t'\) precisely induce the local thermal variables enhanced by gravity via the redshift.
Appendix C: Low temperature approximation

The thermal quantities of free relativistic Fermi gas are presented here:

\[ P_{FG}(T^*, \mu^*) = 2T^* \int \frac{d^3p}{(2\pi)^3} \left[ \ln \left( 1 + e^{-\beta^* (E - \mu^*)} \right) + \ln \left( 1 + e^{-\beta^* (E + \mu^*)} \right) \right], \]

\[ n_{FG}(T^*, \mu^*) = \left( \frac{dP_{FG}}{d\mu^*} \right)_T = 2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\beta^* (E - \mu^*)} + 1} - \frac{1}{e^{\beta^* (E + \mu^*)} + 1}, \]

\[ s_{FG}(T^*, \mu^*) = \left( \frac{dP_{FG}}{dT^*} \right)_{\mu^*} = 2 \sum_{\eta = \pm 1} \int d^3p \left\{ \frac{(E + \eta \mu^*)/T^*}{e^{(E + \eta \mu^*)/T^*}} + \frac{1}{e^{(E + \eta \mu^*)/T^*}} + 1 \right\}, \]

\[ \epsilon_{FG}(T^*, \mu^*) = -P_{FG}(T^*, \mu^*) + T^* s_{FG}(T^*, \mu^*) + \mu^* n_{FG}(T^*, \mu^*), \]

where \( \beta^* = 1/T^* \). The vacuum energy in grand potential is omitted phenomenologically.

Then we consider the low temperature approximation. For a degenerate system, at low temperatures, the contribution from anti-particle is suppressed. As mentioned, an isolated NS is colder than 1 MeV after cooling. The Fermi-Dirac distribution recovers a Heaviside step function \( \Theta \)

\[ \frac{1}{e^{\beta^* (E - \mu^*)} + 1} \approx \Theta (\mu^* - E). \]

From this, we have

\[ \ln \left( 1 + e^{-\beta^* (E - \mu^*)} \right) \approx \beta^* (\mu^* - E) \Theta (\mu^* - E). \]

The pressure at zero temperature is

\[ P_{FG}(\mu^*, m) = \frac{1}{24\pi^2} \left[ \mu^* \left( 2\mu^* - 5m^2 \right) \sqrt{\mu^* - m^2} + 3m^4 \arccosh \left( \frac{\mu^*}{m} \right) \right]. \]

Based on Eq. (C4), the particle density at zero temperature is

\[ n_{FG}(\mu^*, m) = \frac{dP_{FG}}{d\mu^*} = \frac{k_F^3}{3\pi^2}, \]

where the Fermi momentum is defined as \( k_F = \sqrt{\mu^* - m^2} \). The energy density for free fermion at zero temperature is

\[ \epsilon_{FG}(\mu^*, m) = \int_0^\infty E \Theta (\mu^* - E) dE = \frac{1}{8\pi^2} \left[ \mu^* \sqrt{\mu^* - m^2} \left( 2\mu^* - 2m^2 \right) - m^4 \arccosh \left( \frac{\mu^*}{m} \right) \right]. \]

For the convenience of calculations in mW model, one introduces an auxiliary function \( n_{FG}^{\prime} \) working as the scalar condensate number density of free fermion. As shown in Eq. (33), \( d\tilde{n}/d\tilde{r} \) only contributes a coupling constant in mW model, while \( dP_{FG}/d\tilde{n} \) is a more general expression. Hence, it is defined as

\[ n_{FG}^{\prime}(\mu^*, m) = \frac{d\Omega_{FG}}{d\mu^*} = \frac{\tilde{m}}{2\pi^2} \left[ \mu^* \sqrt{\mu^* - \tilde{m}^2} - \tilde{m}^2 \arccosh \left( \frac{\mu^*}{\tilde{m}} \right) \right]. \]

[1] K. N. Glendenning, *Compact stars: Nuclear physics, particle physics and general relativity* (Springer Science & Business Media, 2012).
[2] P. A. Mazzali, F. K. Ropke, S. Benetti and W. Hillebrandt, Science 315, 825 (2007).
[3] L. Rezzolla, E. R. Most and L. R. Weih, Astrophys. J. Lett. 852, no.2, L25 (2018).
[4] H. T. Cromartie et al. [NANOGrav], Nature Astron. 4, no.1, 72-76 (2019).
[5] E. Fonseca, H. T. Cromartie, T. T. Pennucci, P. S. Ray, A. Y. Kirichenko, S. M. Ransom, P. B. Demorest, I. H. Stairs, Z. Arzoumanian and L. Guillot et al. [Atel], Astrophys. J. Lett. 915, no.1, L12 (2021).
[6] E. Annala, T. Gorda, A. Kurkela, J. Nättilä and A. Vuorinen, Nature Phys. 16, no.9, 907-910 (2020).
[7] I. Bombaci, A. Drago, D. Logoteta, G. Pagliara and I. Vidaña, Phys. Rev. Lett. 126, no.16, 162702 (2021).
[8] J. R. Ellis, J. I. Kapusta and K. A. Olive, Nucl. Phys. B 348, 345-372 (1991).
[9] K. Masuda, T. Hatsuda and T. Takatsuka, Astrophys. J. 764, 12 (2013).
[10] B. P. Abbott et al. [LIGO Scientific and Virgo], Phys. Rev. Lett. 119, no.16, 161101 (2017).
[11] E. R. Most, A. Motornenko, J. Steinheimer, V. Dexheimer, M. Hanauske, L. Rezzolla and H. Stoecker, [arXiv:2201.13150 [nucl-th]].
[12] I. Bombaci, JPS Conf. Proc. 17, 101002 (2017).
[13] K. Masuda, T. Hatsuda and T. Takatsuka, Eur. Phys. J. A 52, no.3, 65 (2016).
[14] W. Z. Jiang, B. A. Li and L. W. Chen, Astrophys. J. 756, 56 (2012).
[15] R. Somasundaram, I. Tews and J. Margueron, [arXiv:2112.08157 [nucl-th]].
[16] A. Reisenegger, in International Workshop on Strong Magnetic Fields and Neutron Star (2003) pp. 33–49.
[17] S. Nishizaki, T. Takatsuka and Y. Yamamoto, Prog. Theor. Phys. 105, 607-626 (2001).
[18] S. Nishizaki, T. Takatsuka and Y. Yamamoto, Prog. Theor. Phys. 108, 703-718 (2002).
[19] M. Baldo, G. F. Burgio and H. J. Schulze, Phys. Rev. C 61, 055801 (2000).
[20] R. Machleidt, Phys. Rev. C 63, 024001 (2001).
[21] J. I. Kapusta and C. Gale, Finite-temperature field theory: Principles and applications, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2011).
[22] K. Masuda, T. Hatsuda and T. Takatsuka, PTEP 2013, no.7, 073D01 (2013).
[23] A. Li, Z. Y. Zhu, E. P. Zhou, J. M. Dong, J. N. Hu and C. J. Xia, JHEAp 28, 19-46 (2020).
[24] J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. 55, 374-381 (1939).
[25] R. C. Tolman, Phys. Rev. 55, 364-373 (1939).
[26] S. M. Carroll, Spacetime and Geometry (Cambridge University Press, 2019).
[27] G. M. Hossain and S. Mandal, JCAP 02, 026 (2021).
[28] G. M. Hossain and S. Mandal, Phys. Rev. D 104, no.12, 123005 (2021).
[29] A. Einstein, The meaning of relativity (Routledge, 2003).
[30] A. Einstein and F. A. David, The principle of relativity (Courier Corporation, 2013).
[31] H. Poincaré, The principles of mathematical physics, Vol. 15 (1905) pp. 1-24.
[32] A. S. Eddington, The mathematical theory of relativity (The University Press, 1923).
[33] K. S. Thorne, C. W. Misner, and J. A. Wheeler, Gravitation (Freeman San Francisco, 2000).
[34] U. Aydemir and J. Ren, [arXiv:2201.00025 [gr-qc]].
[35] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (John Wiley and Sons, New York, 1972).
[36] R. C. Tolman, Phys. Rev. 35, 904-924 (1930).
[37] R. Tolman and P. Ehrenfest, Phys. Rev. 36, no.12, 1791-1798 (1930).
[38] O. Klein, Rev. Mod. Phys. 21, 531-533 (1949).
[39] J. Santiago and M. Visser, Eur. J. Phys. 40, no.2. 025604 (2019).
[40] J. A. Lima, A. Del Popolo and A. R. Plastino, Phys. Rev. D 100, no.10, 104042 (2019).
[41] H. C. Kim and Y. Lee, Phys. Rev. D 105, no.8, L081501 (2022).
[42] W. Israel, Annals Phys. 100, 310-331 (1976).
[43] J. M. Lattimer, AIP Conf. Proc. 1645, no.1, 61-78 (2015).
[44] H. Yukawa, Proc. Phys. Math. Soc. Jap. 17, 48-57 (1935).
[45] M. Taketani, S. Nakamura, and M. Sasaki, Progress of Theoretical Physics 6, 581-586 (1951).
[46] R. Machleidt, The meson theory of nuclear forces and nuclear structure, in Advances in Nuclear Physics, edited by J. W. Negele and E. Vogt (Springer US, Boston, MA, 1989) pp. 189–376.
[47] J. D. Walecka, Annals Phys. 83, 491-529 (1974).
[48] T. Niksic, D. Vretenar and P. Ring, Prog. Part. Nucl. Phys. 66, 519-548 (2011).
[49] J. Piekarewicz, Neutron star matter equation of state, in Handbook of Supernovae, edited by A. W. Alsabti and P. Murdin (Springer International Publishing, Cham, 2017) pp. 1075–1094.