Dynamics and multiqubit entanglement in distant resonators

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Abstract
We consider the dynamics of the photon states in distant resonators coupled to a common bus resonator at different positions. The frequencies of distant resonators from a common bus resonator are equally detuned. These frequency detunings are kept larger than the coupling strengths of each resonator to the common bus resonator to satisfy the dispersive interaction regime. In the dispersive regime, we show that the time dynamics of the system evolve to an arbitrary W-type state in a single step at various interaction times. Our results show that a one-step generation of arbitrary W-type states can be achieved with high fidelity in a system of superconducting resonators.

Keywords: time dynamics, entanglement, distant resonators, dispersive interaction, superconducting resonators, W-state, photon storage

(Some figures may appear in colour only in the online journal)

1. Introduction

The coherent interaction of matter with quantized fields within high reflective resonators or Cavity known as cavity Quantum Electrodynamics (QED) provides a powerful platform for quantum information processing [1–3] and quantum networks [4]. Attentions have been focused on the dynamics of coupled cavities for encoding the quantum information [5–12]. For example, the dynamics of the coupled-cavity array system with each cavity containing single two-level atoms [11] or ensembles of atoms [9] revealed evidence for the formation of W-state.

In an alternative scenario of circuit-QED [13], there are some interesting studies, for example, the realization of quantum logic gates [14–21], quantum algorithms [22], microwave pulse or photon storage [23–26], and entanglements [27–33]. In some recent studies, the generation of W-state entanglement by encoding the quantum information in artificial qubits coupled to a microwave transmission line resonator was proposed [34–39]. In a system of artificial qubits coupled to a superconducting resonator, three-qubit W-state was generated through a dispersive collective quantum non-demolition measurement [34] and with resonant interaction [35, 37].

In the present paper, we theoretically examine the time dynamics and realization of W-type states in distant quantum resonators coupled to a common bus resonator at different positions with dispersive interaction. In the dispersive regime, we consider equal frequency detuning of each distant resonator from a common bus resonator, which is larger than their coupling strengths to common bus resonator. We show that the time dynamics of the photon states |0⟩ and |1⟩ (which serves as a qubit) in distant resonators lead to an arbitrary n-qubit W-state [40, 41]

\[ |W⟩_n = C_1|100...0⟩ + C_2|010...0⟩ + ... + C_n|000...1⟩ \] (1)

in a single step at various interaction times. Here, \(|C_1|^2 = |C_2|^2 = ... = |C_n|^2 = 1/n\). We also discuss the experimental feasibility of our scheme by considering the effect of the finite lifetime of the photon in the resonator, inhomogeneous coupling of distant resonators to common bus resonator, direct resonator-resonator coupling, and mixed initial state. Our results show that a one-step generation of arbitrary W-state can be achieved with high fidelity.
The W-type entangled state presented here may have potential applications in quantum sensing networks [42, 43] and optimal processing of quantum information [40]. Our scheme of generating a W-type entangled state is general and can be applied to a different kind of coupled quantum system that follows the ladder algebra of quantum harmonic oscillators such as optical cavities and superconducting resonators. Furthermore, encoding quantum information tasks in artificial atoms coupled to resonators is not required unlike some earlier studies [33, 35, 37], which may make the realization simpler.

The paper is organized as follows. In section 2, we discuss the physical model and derive its effective Hamiltonian in the dispersive interaction regime. In section 3, we study the time dynamics of the photon states population in distant resonators and show the evidence of W-state appearance. In section 4, we discuss the experimental feasibility in superconducting resonators and show the evidence of W-state appearance. Finally, we conclude our results in section 5.

2. Physical model

Consider a one dimensional common resonator $R_0$ of frequency $\Omega_0$ with Hamiltonian $H_{R_0} = \hbar \omega_0 a a^\dagger$. Here, $a^\dagger$ and $a$ are the creation and annihilation operators of the common resonator $R_0$ holding the commutator property $[a, a^\dagger] = 1$. Next, we consider one dimensional distant resonators $R_j$ $(j = 1, 2, 3, \ldots)$ coupled to a common resonator $R_0$ at different positions having Hamiltonian $H_{R_j} = \hbar \omega_j b_j^\dagger b_j$. Here, $\omega_j$ is the frequency of $j^{th}$ distant resonator and $b_j^\dagger$ ($b_j$) is the creation (annihilation) operators, which follows the property of $[b_j, b_k^\dagger] = \delta_{jk}$. It is assumed that the spacing between each distant resonator $R_j$ is large enough to avoid interaction among them. Our proposed physical system using superconducting resonators is shown in figure 1. The coupling strength $g_j$ between the $j^{th}$ resonator $R_j$ to common resonator $R_0$ can be tuned by external flux through the Superconducting Ring Couplers [44, 45]. The catch and release of microwave photon states $|0\rangle_j$ and $|1\rangle_j$ from Transmission Line (TL) into superconducting resonator $R_j$ can be performed by turning on the capacitance $C$ [45]. The dotted line box in figure 1 shows the subsystem composed of superconducting resonators $R_j$ and $R_0$, which is used to generate the entanglement on the microwave photon states.

We first consider four-resonator subsystem composed of $R_j$ $(j = 1, 2, 3)$ and $R_0$ in dispersive regime. In the interaction picture, the Hamiltonian of four-resonator subsystem under rotating wave approximation can be written as (here, we assume $\hbar = 1$) [44]

$$H = \sum_{j=1}^{3} g_j^j a^\dagger b_j e^{-i \Delta_j t} + a b_j^\dagger e^{i \Delta_j t},$$

where, $g_j^j$ is the coupling strength of resonator $R_j$ to common resonator $R_0$ at different positions. The detuning of the each resonator $R_j$ from common resonator $R_0$ is given by $\Delta_j = \Omega_0 - \omega_j$.

We consider the dispersive regime such that $g_j / \Delta_j \ll 1$. This regime can be studied using a second-order perturbation expansion in $g_j / \Delta_j$. Under this condition, it is convenient to rewrite the Hamiltonian $H$ of the four-resonator subsystem in Schrödinger’s picture such that

$$H = H_{0,s} + H_{t,s},$$

with

$$H_{0,s} = \Omega_0 a^\dagger a + \sum_{j=1}^{3} \omega_j b_j^\dagger b_j$$

and

$$H_{t,s} = \sum_{j=1}^{3} g_j (a^\dagger b_j + a b_j^\dagger).$$

The interaction part of the Hamiltonian given by equation (5) describes the coupling of three resonators to a common bus resonator $R_0$. More insight into the dispersive regime can be gained by adiabatically eliminating the degree of freedom of common resonator $R_0$, which results in effective interaction among the distant resonators $R_j$. For this purpose, we apply the unitary transformation $[13], U H U^\dagger$, with

$$U = \exp \left[ \sum_{j=1}^{3} \frac{g_j^j}{\Delta_j} (a b_j^\dagger - a^\dagger b_j) \right].$$

Expanding the above transformation to the second order in the small parameter $g_j / \Delta_j$, we can get the dispersive Hamiltonian given by

$$H' = H_0 + H_t,$$
where,
\[ H_0 = \Omega_0^a a^\dagger a + \sum_{j=1}^3 \omega_j b_j^\dagger b_j, \tag{8a} \]
\[ H_1 = \chi_{12}(b_1 b_2^\dagger + b_2 b_1^\dagger) + \chi_{13}(b_1 b_3^\dagger + b_3 b_1^\dagger) + \chi_{23}(b_2 b_3^\dagger + b_3 b_2^\dagger). \tag{8b} \]

The frequencies of the resonators \( R_j \) are Lamb-shifted to \( \omega_j' = \omega_j + g_j^2/\Delta_j \), where \( j = 1, 2, 3 \). Similarly, the frequency of resonator bus \( R_0 \) is Lamb-shifted to \( \Omega_0' = \Omega_0 - \sum_{j=1}^3 g_j^2/\Delta_j \). For completeness, we give detailed derivation of equation (8) in appendix. The dispersive couplings between the resonator pairs of \( R_j \) are given by
\[ \chi_{12} = \frac{g_1 g_2}{2 \Delta_1 \Delta_2}, \tag{9a} \]
\[ \chi_{13} = \frac{g_1 g_3}{2 \Delta_1 \Delta_3}, \tag{9b} \]
\[ \chi_{23} = \frac{g_2 g_3}{2 \Delta_2 \Delta_3}. \tag{9c} \]

The Hamiltonian \( H' \) in the interaction picture, can be obtained using \( H_{\text{eff}} = e^{iH_0 t} H e^{-iH_0 t} \), and is given by
\[ H_{\text{eff}} = \chi_{12} b_1 b_2^\dagger e^{i\delta t} + \chi_{13} b_1 b_3^\dagger e^{i\delta t} + \chi_{23} b_2 b_3^\dagger e^{i\delta t} + H. \tag{10} \]
where, \( \delta_t = \omega_1' - \omega_2' \), \( \delta_{13} = \omega_1' - \omega_3' \) and \( \delta_{23} = \omega_2' - \omega_3' \). Equation (10) describes the controllable interaction between the resonators \( R_1, R_2, \) and \( R_3 \). In what follows, we study the time dynamics of the photon population in each resonator \( R_j \) using the effective Hamiltonian given by equation (10), which produces W-entangled state of photon states \( |0\rangle \) and \( |1\rangle \).

### 3. Time dynamics and multiqubit entanglement

In order to study the time dynamics of photon state population, we consider the subspace composed of photon states \( |1,0,0,0\rangle, |0,1,0,0\rangle \) and \( |0,0,1,0\rangle \). In this subspace, the state of the four-resonator subsystem is a linear combination of three states
\[ |\psi(t)\rangle = C_1(t)|\psi_1\rangle + C_2(t)|\psi_2\rangle + C_3(t)|\psi_3\rangle. \tag{11} \]
Here, \( |\psi_1\rangle = |1,0,0,0\rangle |0_0\rangle \), \( |\psi_2\rangle = |0,1,0,0\rangle |0_0\rangle \), and \( |\psi_3\rangle = |0,0,1,0\rangle |0_0\rangle \). The fourth qubit with subscript 0 represents the vacuum photonic state of common resonator \( R_0 \).

The time evolution of the state \( |\psi(t)\rangle \) of the four-resonator subsystem under effective Hamiltonian given by equation (10) can be obtained by solving the Schrödinger’s equation
\[ \frac{\partial |\psi(t)\rangle}{\partial t} = -iH_{\text{eff}} |\psi(t)\rangle. \tag{12} \]

Using \( |\psi(t)\rangle \) in equation (12), the probability amplitudes can be expressed in the form of following coupled differential equations:
\[ \dot{C}_1 = -i[\chi_{12} e^{-i\delta t} C_2 + \chi_{13} e^{i\delta t} C_3], \tag{13a} \]
\[ \dot{C}_2 = -i[\chi_{12} e^{i\delta t} C_1 + \chi_{23} e^{-i\delta t} C_3], \tag{13b} \]
\[ \dot{C}_3 = -i[\chi_{13} e^{-i\delta t} C_1 + \chi_{23} e^{i\delta t} C_2]. \tag{13c} \]

The initial state of the four-resonator subsystem is assumed to be \( |1,0,0,0\rangle |0_0\rangle \), which shows that there is a single photon in resonator \( R_1 \). When resonators \( R_1, R_2, \) and \( R_3 \) are resonant to each other with \( \omega_i = \omega_1 = \omega_2 = \omega_3 \), than each resonator will be equally detuned from common resonator with detuning \( \Delta = \Delta_1 = \Delta_2 = \Delta_3 = \Omega_0 - \omega_r \). Assuming identical couplings i.e., \( g_1 = g_2 = g_3 \) to a common resonator, we have zero detuning values \( \delta_{12} = \delta_{13} = \delta_{23} = 0 \) and identical dispersive couplings between the resonators pairs \( R_j \) as \( \chi_{12} = \chi_{13} = \chi_{23} = \chi \) (see equation (9)). Under this condition, the state of four-resonator subsystem evolves at time \( t \) with coefficients
\[ C_1(t) = \frac{1}{3}[2 \cos \chi t + \cos 2\chi t], \tag{14a} \]
\[ C_2(t) = C_3(t) = \frac{1}{3}[-\cos \chi t + \cos 2\chi t], \tag{14b} \]

here, \( \chi = g^2/\Delta \). We first plot the population of photon states in resonator \( R_1, R_2, \) and \( R_3 \) as a function of operation time in dimensionless units \( \chi t/\pi \) as shown in figure 2. The time evolution of photon state population can be obtained using [11].
\[ P_m = \langle \psi_m | \rho(t) | \psi_m \rangle = C_m C_m^*, \] (15)

where \( m = 1, 2, 3 \) and \( \rho(t) = |\psi(t)|^2 |\psi(t)\rangle \). In figure 2, dashed-blue curve denotes the time evolution of the population \( P_1 \) of state \( |1_00_3\rangle |0_0\rangle \) while solid-red curve denote the time evolution of the population \( P_{2,3} \) of state \( |0_11_2\rangle |0_0\rangle \) and \( |0_02_1\rangle |0_0\rangle \). In order to get a comparison of the resulting dynamics obtained from the effective Hamiltonian equation (10), we also show the results of our numerical simulations obtained from the \textit{ab initio} Hamiltonian in equation (3), which are shown by filled circles in figure 2. Here, we choose parameters \( \omega_r = 2\pi \times 5.75 \text{ GHz} \), \( \Omega_0 = 2\pi \times 6.75 \text{ GHz} \), and \( \kappa = 0 \text{ MHz} \). The coupling strength \( g = 2\pi \times 50 \text{ MHz} \) is chosen to satisfy the rotating wave approximation in equation (2).

The dynamics of the system shown in figure 2 clearly reflect the transfer of photon population among the resonators \( R_1, R_2, \) and \( R_3 \). The oscillation period of the photon population depends on the dispersive coupling \( \chi \). When there is a single photon in resonator \( R_1 \), figure 2 shows that the dynamic of the system can evolve to a three-photon W-state at \( \chi t / \pi \approx 0.22, 0.44, 0.88, 1.10 \) with \( |C_1|^2 \approx |C_2|^2 \approx |C_3|^2 \approx 0.33 \), just in a single step. During the entire time dynamics, the photon state population in resonator \( R_2 \) and \( R_3 \) oscillates with an equal magnitude due to the identical coupling strengths and frequencies. It is in-phase with each other while out of phase with the population in resonator \( R_1 \).

It may be mentioned here that for the case of coupled-cavity airy array system \( [9] \), the probabilities of excitation (photon)-transfer was observed as high as 100%. In distant resonator case considered in this study, the photon state population \( P_1 \) is never fully transferred \( (P_1 \neq 0 \text{ throughout the dynamics}) \) to the resonators \( R_2 \) and \( R_3 \) as can be seen in figure 2. For interaction time other than \( \chi t / \pi \approx 0.22, 0.44, 0.88, 1.10 \), the condition \( |C_1|^2 \approx |C_2|^2 \approx |C_3|^2 \) still holds. Therefore, it is clear that three-qubit entangled states with unequal probability amplitudes can still be generated except for time at which \( P_1 \to 1 \) and \( P_{2,3} \to 0 \) as shown in figure 2. For example, choosing the operation time \( \chi t = \pi / 3 \), the final state evolves to

\[ |\psi_3\rangle = \frac{1 + i \sqrt{3}}{6} (|1_02_03\rangle - 2|0_11_20\rangle - 2|0_02_10\rangle). \] (16)

The subscript \( \pi / 3 \) indicates the operation time \( \chi t \) at which time dynamics evolves to an entangled state. Similarly, for \( \chi t = \pi \), the final state is given by

\[ |\psi_3\rangle = \frac{1}{3} (|1_02_03\rangle + 2|0_11_20\rangle + 2|0_02_10\rangle). \] (17)

Both \( |\psi_3\rangle \) and \( |\psi_3\rangle \) are three-qubit entangled state of photons in three distant resonators \( R_1, R_2, \) and \( R_3 \). Note that for both states \( |C_1|^2 = 1/9 \) and \( |C_2|^2 = |C_3|^2 = 4/9 \). The behavior of the system symmetrically changes if a single photon is initially present in resonator \( R_2 \) i.e., \( P_2 \to P_1 \) and \( P_1 = P_3 \), as a result, W-state entanglement still preserved. This situation is in contrast to coupled-cavity array system \( [9] \) where entanglement can disappear when initial excitation changes.

Next, we extend the scheme for time dynamics and generation of entanglement to an arbitrary \( n \)-distant resonators. We consider \( n \) resonators \( R_j \) \( (j = 1, 2, 3, \ldots, n) \) coupled to a common resonator \( R_0 \) in the dispersive regime having identical coupling constant. In this case, following the similar unitary transformation (See equation (6)) up to \( j = n \), the effective Hamiltonian for \( n + 1 \)-resonator subsystem can be written in the following form:

\[ H_{\text{eff}}^n = \sum_{i=j, j} \chi_i (b_i b_i^\dagger e^{i\beta_i t} + H.c. \). \] (18)

Here, we assume that the initial state of \( n + 1 \)-resonator subsystem is \( |1_02_00_0\rangle |0_0\rangle \) and all the distant resonators are resonant to each other i.e., equally detuned from the common resonator \( R_0 \). Under this condition, the state of \( n + 1 \)-resonator subsystem evolves at time \( t \) with coefficients

\[ C_1(t) = \frac{1}{n} [(n - 1) \cos \chi t + \cos (n - 1) \chi t] + i \frac{1}{n} [(n - 1) \sin \chi t - \sin (n - 1) \chi t], \] (19a)

\[ C_2(t) = \ldots = C_n(t) = \frac{1}{n} [- \cos \chi t + \cos (n - 1) \chi t] - i \frac{1}{n} \sin \chi t + \sin (n - 1) \chi t], \] (19b)

which reduce to equation (14) i.e., four-resonator subsystem for \( n = 3 \). As an example, we consider the case of five-resonator subsystem composed of resonators \( R_1, R_2, R_3, \) and \( R_4 \) coupled to \( R_0 \). The initial state of this system is assumed to

![Figure 3. Illustration for the time evolution of the photon population in five-resonator subsystem, which is assumed to be in the initial state |1_02_03_0\rangle |0_0\rangle. Dashed-blue curve represents the population of photons P_1 in resonator R_1 while solid red curve denotes the population of photons P_{2,3,4} in resonator R_2, R_3, and R_4. The horizontal dashed-dotted line marks the point of interactions at which the dynamics of the system can nearly produce four-qubit W-state on photon states. The parameters adopted here are similar to figure 2.](image-url)
be $|1\rangle_2 |0\rangle_4 |0\rangle_5$. In figure 3, dashed-blue curve shows the
time evolution of the population $P_1$ of state $|1\rangle_2 |0\rangle_4 |0\rangle_5$
while solid-red curve denotes the time evolution of the population
$P_{2,3,4,5}$ of state $|0\rangle_1 |1\rangle_2 |0\rangle_4 |0\rangle_5$, $|0\rangle_1 |0\rangle_2 |1\rangle_4 |0\rangle_5$, and
$|0\rangle_1 |0\rangle_2 |0\rangle_4 |1\rangle_5$. The parameters taken here are similar to the
case considered in figure 2. The time evolution of the photon
state clearly produces four-qubit W-state at interaction times
$\chi t/\pi = 1/4, 3/4, 5/4$. By setting the operation time $\chi t =
\pi/4$, one can get the following four-qubit entangled state:

$$|\psi_4\rangle_{e/4} = \frac{1 + i}{2\sqrt{2}} (|1\rangle_2 |0\rangle_3 |0\rangle_4
- |0\rangle_1 |1\rangle_2 |0\rangle_4
- |0\rangle_1 |0\rangle_2 |1\rangle_4
- |0\rangle_1 |0\rangle_2 |0\rangle_4 |1\rangle_5),$$

which is similar to the prototype four-qubit W-state.

Next, we consider the case of six-resonator subsystem
composed of resonators $R_1$, $R_2$, $R_3$, $R_4$, and $R_5$ coupled to $R_6$
with initial state $|1\rangle_2 |0\rangle_3 |0\rangle_4 |0\rangle_5 |0\rangle_6$. In figure 4(a), we plot time
evolution of the population $P_1$ of state $|1\rangle_2 |0\rangle_3 |0\rangle_4 |0\rangle_5 |0\rangle_6$
(dashed-blue curve) and $P_{2,3,4,5}$ of states $|0\rangle_1 |1\rangle_2 |0\rangle_4 |0\rangle_5
|0\rangle_6$, $|0\rangle_1 |0\rangle_2 |1\rangle_4 |0\rangle_5 |0\rangle_6$, $|0\rangle_1 |0\rangle_2 |0\rangle_4 |1\rangle_5 |0\rangle_6$
(denoted by solid-red curves). The period of oscillation for the
photone state population gets shorter as compared to the four
and five-resonator-qubit subsystem. Furthermore, the peak
amplitude of the oscillation for the photon states also
decreases as the number of resonators increase. As a result, a
at appears in the photon state population $P_1$ and $P_{2,3,5}$,
which is clearly visible in figure 4(a). In order to obtain the
prototype five-qubit W-state with equal probability amplitudes
of $1/5$, one needs crossing point of the photon state
population. For this purpose, we numerically solve the den-
sity matrix equation $d\rho /dt = -i[H, \rho]$ by considering full Hamiltonian
given by equation (2) with $g_1$ different than $g = g_2 = \ldots = g_5$. The results of our simulations are shown
in figure 4(b) with $g = 2\pi \times 62.5$ MHz while rest of the
parameters are the same as considered in figure 2. Figure 4(b)
clearly reveals that the dynamics of the system can nearly
evolve to a five-qubit W-state at four different interaction
times $\chi t/\pi$ as indicated by the dashed-dotted horizontal line.

4. Possible experimental implementation

In general, our scheme for generating W-state is possible
using different kind of quantum system that follows the ladder
algebra of quantum harmonic oscillator i.e., $[a, a^\dagger] = 1,$
and dispersive interaction regime. Such as photon states inside
the optical cavities or superconducting resonators [45, 46].
However, considering two-level neutral or artificial atoms
(Trasmon or Cooper Pair Box etc) as a qubit coupled to
common resonator [34, 35, 37] may not work. Because, in
case of a two-level atom, ladder algebra follows $[\sigma^+, \sigma^-] = 1,$
which may not result in the adiabatic elimination of the
common resonators with dispersive interaction.

Recently, quantum information processing using micro-
wave photons in a one-dimensional superconducting resonator
has been focused due to high-quality factor and large zero-point
electric fields. Some interesting outcomes have been presented
such as catching and releasing of microwave photon states [45]
and quantum superposition of a single microwave photon [47].
Therefore, we consider the experimental feasibility of our
results on microwave photon states in a system of super-
conducting resonators as shown in figure 1.

Imperfections in the implementation can affect the time
dynamics and hence the realization of the W-state entangle-
ments. The possible sources of imperfections could be the
finite lifetime of the photons in the superconducting resona-
tors, inhomogeneous coupling strength $g_j$, direct resonator-
resonator interactions due to mutual inductance or capaci-
tance, and noisy initial state preparation. We first consider the
effects of photon lifetime $\kappa$ in the four-resonator subsystem.
The time evolution of the four-resonator subsystem under
the influence of photon decay rates can be described by the

\[ H = \sum_{j=1}^{6} \omega_j a_j^\dagger a_j. \]
master equation [48]
\[
\frac{d\hat{\rho}}{dt} = -i[H, \hat{\rho}] + \kappa_0 L[a] + \kappa_f L[b_f] \tag{21}
\]
where, \(j = 1, 2, 3, \kappa_0\) is decay rate of superconducting resonator bus and \(\kappa_f\) is the decay rate of the \(j\)th superconducting resonator. The Liouvillian of an operator \(\hat{\xi} \in (a, b_f)\) is \(L[\hat{\xi}] = \xi \rho \hat{\xi}^\dagger - \hat{\xi}^\dagger \xi \rho/2 + \rho \hat{\xi}^\dagger \xi/2\), which describes the photon decay through the resonator. In equation (21), \(\tilde{\rho}(t)\) is the final density operator of the system when the operation is performed under photon leakage from the superconducting resonator. We numerically obtained the density operator \(\tilde{\rho}(t)\) [49] with initial state \(|\psi_i\rangle = |1, 0, 0\rangle|0_0\rangle\) and for Hamiltonian \(H\) given by equation (2). We first numerically simulate the population of microwave photons \(P_1\), and \(P_{2,3}\) as a function of \(\chi t/\pi\) as shown in figure 2 by dotted curves. Here, we adopted \(\kappa = \kappa_0 = \kappa_1 = \kappa_2 = \kappa_3 = 0.5\) MHz (lifetime of 2\(\mu s\)) while the rest of the parameters are the same as used in figure 2. The damping occurs in the oscillation of \(P_1\) and \(P_{2,3}\) due to the photon decay through the resonator. Its effect becomes more prominent at longer operation times as clearly visible by dotted curves in figure 2. Therefore, choosing shorter operation time (first crossing point) is more feasible to obtain a W-state.

Next, we present the results of our numerical simulations for the fidelity \(F_3\) of the entangled state \(|\psi_3\rangle\) generated from the initial state \(|\psi_1\rangle\) as a function of operation time using the definition
\[
F_3 = \langle \psi_3 | \tilde{\rho}(t) | \psi_3 \rangle. \tag{22}
\]
Here, \(|\psi_3\rangle\) is the output state of an ideal system and \(\tilde{\rho}(t)\) is the final density operator under photon leakage from the superconducting resonator. Figure 5(a) shows the result of simulation obtained from equation (22) for different photon decay rates [19]. Here, solid black, dashed-red, and dotted-blue curves correspond to \(\kappa = 0\) MHz, \(\kappa = 0.25\) MHz (life time \(\kappa^{-1} = 4\) \(\mu s\)), and \(\kappa = 0.5\) MHz (life time \(\kappa^{-1} = 2\) \(\mu s\)), respectively. The rest of the parameters are the same as used in figure 2. Figure 5(a) clearly shows that the fidelity varies with operation time for different decay rates of the resonators. Here, fidelity is maximum for \(\kappa = 0\) MHz (ideal case), 98.7% with \(\kappa = 0.25\) MHz, and 97.7% with \(\kappa = 0.5\) MHz for optimum operation time \(\chi t/\pi \approx 0.22\).

Next, we numerically simulate the fidelity \(F_4\) of the W-state generated in five-resonator subsystem using the expression
\[
F_4 = \langle \psi_4 | \tilde{\rho}(t) | \psi_4 \rangle. \tag{23}
\]
Here, \(|\psi_4\rangle\) is the W-state given by equation (20) under ideal condition and \(\tilde{\rho}(t)\) is the final density operator under photon leakage from the superconducting resonator. The results of our numerical simulations are shown in figure 5(b). Again fidelity \(F_4\) varies with time at different decay rates and the maximum value of the fidelity is obtained around operation time \(\chi t/\pi \approx 1/4\). Here, maximum fidelity can approximately reach 100% with \(\kappa = 0\) MHz (ideal case), 98.4% with \(\kappa = 0.25\) MHz, and 97.4% with \(\kappa = 0.5\) MHz. It is evident from the results of figures 5(a) and (b) that W-state can be generated in a single step with high fidelity. The small oscillations in figures 5(a) and (b) arises due the interactions between \(R_j\) and \(R_0\). These oscillations can be further minimized by increasing the detuning \(\Delta\) or reducing the coupling strengths \(g_j\).

Furthermore, so far, we have assumed identical coupling constant \(g_j\) between each distant resonator \(R_j\) and common superconducting resonator \(R_0\) in our analysis. Needless to say, this assumption would not be perfectly met in an experimental setup. The coupling inaccuracies can induce small difference in the value of \(g_j\) for each resonator. In order to get further insight, next, we consider the fidelity of W-state against the inhomogeneous coupling strength. For simplicity, we consider four-resonator subsystem in which we vary \(g_2\) relative to the other couplings i.e., \(g = g_1 = g_3\). In figure 6, we show a contour plot of fidelity for three-qubit W-state as function of \(\chi t/\pi\) and \(g_2/g\) for photon decay rate of \(\kappa = 0.10\) MHz (life time \(\kappa^{-1} = 10\) \(\mu s\)). The small dark-shaded region of the
contour in figure 6 shows that the fidelity can reach up to approximately 99% when \( g_2 \approx g \) for operation time around \( \chi t/\pi \approx 0.22 \).

In our scheme, \( R_j \) resonators are coupled through resonator-bus-mediated mechanism instead of direct resonator-resonator coupling. This mediated mechanism may allow neglecting any parasitic mutual inductance or capacitance. Further more, the spacing between each superconducting resonators \( R_j \) can be kept large enough to avoid interaction among them due to mutual capacitance and mutual inductance. However, in a realistic situation, a direct resonator-resonator coupling my still appear. In order to study the effects of this direct interaction in the generation of W-state, we added the additional interaction term [9, 11]

\[
H_{\text{int}} = G_M (b_1 b_2^* + b_2 b_3^* + H.c.)
\]

in equation (5) for four-resonator subsystem. Here, \( G_M \) is the direct resonator-resonator coupling parameter, which includes the effects of mutual inductance or capacitance. Equation (24) shows that the first resonator gets decoupled from the third resonator if we only consider the direct resonator-resonator interaction (\( G_M = 0 \)) instead of resonator-bus-mediated mechanism i.e., \( g_j = 0 \) in equation (5). As a result time dynamics will change significantly. Now applying the unitary transformation equation (6) breaks down the Schrieffe-Wolff Transformation [50] (See appendix). As a result, calculations become extremely tedious to obtain an effective Hamiltonian with renormalization of parameters \( \chi 's \) in simplified form. Therefore, we numerically obtained the fidelity \( F_3 \) as a function of coupling ratio \( g/G_M \) as shown in figure 7. The results in figure 7 clearly show that high fidelity can be obtained when direct resonator-resonator coupling \( G_M \) is approximately 100 time weaker as compared with the resonator-bus coupling \( g \).

![Figure 6. Fidelity of three-qubit W-state versus \( g_2/g \) and operation time \( \chi t/\pi \) for \( \kappa = 0.10 \) MHz in four-resonator subsystem. The rest of the parameters are the same as used in figure 5. The small dark shaded contour corresponds to maximum fidelity.](image)

![Figure 7. Fidelity of W-state versus coupling ratio \( g/G_M \) for \( \kappa = 0.0 \) MHz (solid curve) and \( \kappa = 0.5 \) MHz (dashed curve) in four-resonator subsystem.](image)

Furthermore, initial state \( |\psi_i\rangle \) can be realized with high fidelity by catching the photons from the transmission lines by turning on the coupling capacitance [45]. However, real situation is that the initial state will be a complicated mixed state. Next we are interested to study how the fidelity of the W-state preparation is affected if one includes noise in the initial state preparation. For simplicity, we consider the initial state which is similar to noisy Werner’s mixed State [51–53] in four resonator subsystem given by

\[
\rho_i = p |\phi_i\rangle \langle \phi_i | + \frac{1 - p}{4} I_4.
\]

Here, \( 0 \leq p \leq 1 \) is a noise parameter which includes imperfection due to time controlled capture and release of microwave photons and unwanted hoping of microwave photons among resonators. An arbitrary pure state is given by \( |\phi_i\rangle = (\cos \theta |0_1 0_2\rangle + i \sin \theta |0_1 1_2\rangle) |0_3 0_4\rangle \) and \( I_4 \) is an identity operator. The parameters \( p \) and \( \theta \) measure the overlap of the mixed state equation (25) with an ideal initial pure state \( |\psi_i\rangle \). In figure 8, we show the fidelity \( F_3 \) as a function of noise parameter \( p \) for different values of angles \( \theta \). As expected, when \( p = 1 \) and \( \theta = n \times \pi \) (such that \( n = 0, 1, 2, \ldots \)), the initial state becomes \( \rho_i = |\psi_i\rangle \langle \psi_i | \) and fidelity approaches to unity.

The influence of superconducting ring couplers may also arise in time dynamics and entanglement. However, it can be neglected because their frequencies are designed at large detuning with those of superconducting resonators [44]. Finally, the quality factor \( Q \) for a resonator describes the life time of photons inside the resonator. Higher \( Q \) value indicate a lower rate of photon loss from the resonator. It may be worth mentioning that the quality factor for a one-dimensional superconducting resonator which has been achieved in some recent experiment is of the order of \( Q \sim 2 \times 10^6 \) [54, 55]. Using this quality factor, the estimated lifetime of the photons in a superconducting resonator is \( \kappa^{-1} = 1/(2Q) \approx 55 \mu s \), which is sufficiently longer than the operation time in our scheme. The analysis here implies that the high fidelity implementation of
multiqubit W-State is feasible in distant resonators in a single-step.

5. Conclusion

In conclusion, we have studied the time dynamics of the photon state population in distant resonators. Each resonator $R_j$ is coupled to a common quantum bus resonator $R_b$. In our scheme, all resonator $R_j$ have identical frequency detuning from a common bus resonator $R_b$. In order to satisfy the dispersive regime, the frequency detuning is kept larger than the coupling strengths to a common bus resonator. In the dispersive regime, the time dynamics of three and four resonators coupled to a common bus evolve to three and four-qubit W-state, respectively. A population gap appears when five or higher number of resonators are coupled to a common bus resonator. We show that the dynamics of the system can still produce multiqubit W-state by increasing the coupling strength of the first resonator to a larger optimum value as compared to the rest of the coupling strengths. Our numerical simulations in a system of superconducting microwave resonators show the high fidelity generation of single-step W-state against the possible sources of imperfections in a realistic situation. Since it is possible to achieve high Q-value resonators, catching and releasing a photon in resonator from a transmission line in the experiment, therefore, our results may stimulate experimental activities in the near future.

Appendix. Derivation of dispersive Hamiltonian

Here we list some useful commutator relations, which we have used to diagonalize the Hamiltonian given by the equation (3)

$$[ab^+_j - a^j b^+_j, a^j a] = -(a^j b^+_j + a^j b_j^+)$$ \hspace{1cm} (A.1a)

$$[ab^+_j - a^j b^+_j, b^+_j b_j] = (a^j b_j + a^j b_j^+)\delta_{jj'}$$ \hspace{1cm} (A.1b)

$$[ab^+_j - a^j b^+_j, a^j a] = (b^+_j b_j + b_j b_j^+) - (a^j a + a^j a)\delta_{jj'}.$$ \hspace{1cm} (A.1c)

In order to diagonalize the Hamiltonian in equation (3), we consider the unitary transformation of equation (6) as $U = e^{\hat{S}}$, with

$$S = \sum_{j'=1}^{3} \frac{g_j}{\Delta_j} (ab^+_j - a^j b^+_j).$$ \hspace{1cm} (A.2)

By considering $H_i = H_{0,i} + H_{t,i}$, the unitary transformation $U = UH_iU^+$ yields

$$H' = H_{0,i} + H_{t,i} + \frac{1}{2}[S, [S, H_{0,i}]] + \frac{1}{2}[S, [S, H_{t,i}]] + ..., \quad (A.3)$$

where,

$$[S, H_{0,i}] = \sum_{j'=1}^{3} \frac{g_j}{\Delta_j} [(ab^+_j - a^j b^+_j), a^j a]$$

$$+ \sum_{j''=1}^{3} \frac{g_j}{\Delta_j} [(ab^+_j - a^j b^+_j), b^+_j b_j].$$ \hspace{1cm} (A.4)

On using equation (A.1), we obtain

$$[S, H_{0,i}] = \sum_{j'=1}^{3} \frac{g_j}{\Delta_j} [(ab^+_j - a^j b^+_j, a^j a)]$$

$$+ \sum_{j''=1}^{3} \frac{g_j}{\Delta_j} [(ab^+_j - a^j b^+_j), b^+_j b_j].$$ \hspace{1cm} (A.5)

Since, $\Delta_i = \Omega_0 - \omega_j$, therefore, we have

$$[S, H_{0,i}] = -H_{t,i}.$$ \hspace{1cm} (A.6)

Now using equation (A.6) in Baker-Campbell-Hausdorff (BCH) formula given by equation (A.3), we obtain

$$H' = H_{0,i} + \frac{1}{2}[S, H_{t,i}] + ..., \quad (A.7)$$

which is the Schrieffer–Wolff Transformation [50]. One only needs to calculate the commutator

$$[S, H_{t,i}] = \sum_{j'=1}^{3} \frac{g_j}{\Delta_j} (ab^+_j - a^j b^+_j, a^j b_j + a^j b_j^+).$$ \hspace{1cm} (A.8)

By using the commutator relations from equation (A.1), we obtain

$$[S, H_{t,i}] = \sum_{j'=1}^{3} \frac{g_j}{\Delta_j} (b^+_j b_j + b_j b_j^+) - (a^j a + a^j a)\delta_{jj'}$$

$$+ \sum_{j''=1}^{3} \frac{g_j}{\Delta_j} (b^+_j b_j - a^j a)$$

$$+ \sum_{j''=1}^{3} \frac{g_j}{\Delta_j} (b^+_j b_j + b_j b_j^+).$$ \hspace{1cm} (A.9)
By using equation (A.9) in equation (A.7), we obtain

$$ H' = a' \left\{ \Theta_0 - \sum_{j=1}^{3} g^2_j / \Delta_j \right\} + \sum_{j=1}^{3} b^\dagger_j b_j \left( \omega_j + g^2_j / \Delta_j \right) $$

which is diagonal up to $g^2_j / \Delta_j$ and in agreement with equation (8).

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**References**

[1] Bennett C H 1995 Quantum information and computation  *Phys. Today* **48** 24

[2] Raimond J M, Brune M and Haroche S 2001 Manipulating quantum entanglement with atoms and photons in a cavity  *Rev. Mod. Phys.* **73** 565

[3] Aoki T, Dayan B, Wilcut E, Bowen W P, Parkins A S, Kippenberg T J, Vahala K J and Kimble H J 2006 Observation of strong coupling between one atom and a monolithic microresonator  *Nature (London)* **443** 671

[4] Duan L-M, Lukin M D, Cirac J I and Zoller P 2001 Long-distance quantum communication with atomic ensembles and linear optics  *Nature* **414** 413

[5] Škarj M, Borščnik N M, Löffler M and Walther H 1999 Quantum interference and atom-atom entanglement in a two-mode, two-cavity micromaser  *Phys. Rev. A* **60** 3229

[6] Ogden C D, Irish E K and Kim M S 2008 Dynamics in a coupled-cavity array  *Phys. Rev. A* **78** 063805

[7] Almeida G M A, Ciccarello F, Apollaro T J G and Souza A M C 2016 Quantum-state transfer in staggered coupled-cavity arrays  *Phys. Rev. A* **93** 032310

[8] Liu Y-C, Xuan L, Li H-K, Gong Q, Wong C W and Xiao Y-F 2014 Coherent polarization dynamics in coupled highly dissipative cavities  *Phys. Rev. Lett.* **112** 213602

[9] Badshah F, Qamar S and Paternostro M 2014 Dynamics of interacting Dicke model in a coupled-cavity-array  *Phys.Rev. A* **90** 033813

[10] Zhang K and Li Z Y 2010 Transfer behavior of quantum states between atoms in photonic crystal coupled cavities  *Phys. Rev. A* **81** 033843

[11] Zhong Z R, Lin X, Zhang B and Yang Z B 2012 Dynamics of a coupled cavity system composed of three cavities  *Int. J. Quant. Inf.* **10** 1250070

[12] Sampuli E M, Wang Y, Song J and Xia Y 2019 Indirect light-matter interaction in dissipative coupled cavities  *Opt. Express* **27** 22674

[13] Blais A, Huang R-S, Wallraff A, Girvin S M and Schoelkopf R J 2004 Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation  *Phys. Rev. A* **69** 062320

[14] Waseem M, Irfan M and Qamar S 2012 Multiqubit quantum phase gate using four-level superconducting quantum interference devices coupled to superconducting resonator  *Physica C* **477** 24

[15] Hua M, Tao M-J and Deng F-G 2014 universal quantum gates on microwave photons assisted by circuit quantum electrodynamics  *Phys. Rev. A* **90** 012328

[16] Hua M, Tao M-J and Deng F-G 2015 Fast universal quantum gates on microwave photons with all-resonance operations in circuit QED  *Sci. Rep.* **5** 9274

[17] Waseem M, Irfan M and Qamar S 2015 Realization of quantum gates with multiple control qubits or multiple target qubits in a cavity  *Quantum Inf Process* **14** 1869

[18] Liu T, Cao X-Z, Su Q-P, Xiong S-J and Yang C-P 2016 Multi-target-qubit unconventional geometric phase gate in a multi-cavity system  *Sci. Rep.* **6** 21562

[19] Ye B, Zheng Z-F, Zhang Y and Yang C-P 2018 Circuit QED: single-step realization of a multiqubit controlled phase gate with one microwave photonic qubit simultaneously controlling n—1 microwave photonic qubits  *Opt. Express* **26** 30689

[20] Liu T, Guo B-Q, Zhang Y, Yu C-S and Zhang W-N 2018 One-step implementation of a multi-target-qubit controlled phase gate in a multi-resonator circuit QED system  *Quantum Inf Process* **17** 240

[21] Ma L-H, Kang Y-H, Shi Z-C, Huang B-H, Song J and Xia Y 2019 Shortcuts to adiabatic for implementing controlled phase gate with Cooper-pair box qubits in circuit quantum electrodynamics system  *Quantum Inf Process* **18** 65

[22] Waseem M, Ahmed R, Irfan M and Qamar S 2013 Three-qubit Grover’s algorithm using superconducting quantum interference devices in cavity-QED  *Quantum Inf Process* **12** 3649

[23] Waqas M, Ayaz M Q, Waseem M, Qamar S and Qamar S 2018 Gain assisted coherent control of microwave pulse in a one dimensional array of artificial atoms  *Phys. Scr.* **93** 065101

[24] Ayaz M Q, Waqas M, Qamar S and Qamar S 2018 Coherent control and storage of a microwave pulse in a one-dimensional array of artificial atoms using the Butler-Townes effect and electromagnetically induced transparency  *Phys. Rev. A* **97** 022318

[25] Zhou L, Gao Y B, Song Z and Sun C P 2008 Coherent output of photons from coupled superconducting transmission line resonators controlled by charge qubits  *Phys. Rev. A* **77** 013831

[26] Liao J-Q, Huang J-F, Liu Y-X, Kuang L-M and Sun C P 2009 Quantum switch for single-photon transport in a coupled superconducting transmission-line-resonator array  *Phys. Rev. A* **80** 014301

[27] Hu Y and Tian I 2011 Deterministic generation of entangled photons in superconducting resonator arrays  *Phys. Rev. Lett.* **106** 257002

[28] Wang H et al 2011 Deterministic entanglement of photons in two superconducting microwave resonators  *Phys. Rev. Lett.* **106** 060401

[29] Strauch F W 2012 All-resonant control of superconducting resonators  *Phys. Rev. Lett.* **109** 210501

[30] Hua M, Tao M-J, Alzahrani F, Hobiny A, Wei H-R and Deng F-G 2018 One-step entanglements generation on distant superconducting resonators in the dispersive regime  *Quantum Inf Process* **17** 337

[31] Li M, Hua M, Zhang M and Deng F-G 2019 Entangling two high-Q microwave resonators assisted by a resonator terminated with SQUIDs  *New J. Phys.* **21** 073025

[32] Xiong S J, Sun Z, Liu J M, Liu T and Yang C P 2015 Efficient scheme for generation of photonic NOON states in circuit QED  *Opt. Lett.* **40** 2221

[33] Yang C-P, Su Q-P and Han S Y 2012 Generation of Greenberger-Horne-Zeilinger entangled states of photons in multiple cavities via a superconducting qutrit or an atom through resonant interaction  *Phys. Rev. A* **86** 022329

[34] Helmer F and Marquardt F 2009 Measurement-based synthesis of multiqubit entangled states in superconducting cavity  *QED Phys. Rev. A* **79** 052328
[35] -Long G G, -Chang C G, -Sheng H S, -Feng W M and -Quan J N 2012 One-step generation of multi-qubit GHZ and W states in superconducting transmon qubit system Commun. Theor. Phys. 57 205

[36] Huang S-Y, Goan H-S, Li X-Q and Milburn G J 2013 Generation and stabilization of a three-qubit entangled W state in circuit QED via quantum feedback control Phys. Rev. A 88 062311

[37] Wei X and Chen M-F 2015 Generation of N-qubit W state in N separated resonators via resonant interaction Int. J. Theor. Phys. 54 812

[38] Kang Y-H, Chen Y-H, Wu Q-C, Huang B-H, Song J and Xia Y 2016 Fast generation of W states of superconducting qubits with multiple Schrödinger dynamics Sci. Rep. 6 36737

[39] Jones D and Rahmani A 2019 Optimal preparation of the maximally entangled W state of three superconducting qubits arXiv:1909.09289

[40] Ba An N 2004 Optimal processing of quantum information via W-type entangled coherent states Phys. Rev. A 69 022315

[41] Jeong H and Ba An N 2006 Greenberger-Horne-Zeilinger-type and W-type entangled coherent states: Generation and Bell-type inequality tests without photon counting Phys. Rev. A 74 022104

[42] Proctor T J, Knott P A and Dunningham J A 2018 Multiparameter estimation in networked quantum sensors Phys. Rev. Lett. 120 080501

[43] Sidhu J S and Kok P 2020 A geometric perspective on quantum parameter estimation AVS Quantum Science 2 014701

[44] Peropadre B, Zueco D, Wulschler F, Deppe F, Marx A, Gross R and -Ripoll J J G 2013 Tunable coupling engineering between superconducting resonators: from sidebands to effective gauge fields Phys. Rev. B 87 134504

[45] Yin Y et al 2013 Catch and release of microwave photon states Phys. Rev. Lett. 110 107001

[46] Maxwell D, Szwer D J, -Barato D P, Busche H, Pritchard J D, Gauguet A, Weatherill K J, Jones M P A and Adams C S 2013 Storage and control of optical photons using rydberg polaritons Phys. Rev. Lett. 110 103001

[47] Bajjani E Z, Nguyen F, Lee M, Vale L R, Simmonds R W and Aumentado J 2011 Quantum superposition of a single microwave photon in two different 'colour' states Nat. Phys. 7 599

[48] Lambropoulos P and Petrosyan D 2007 Fundamentals of Quantum Optics and Quantum Information (Berlin: Springer)

[49] Johansson J R, Nation P D and Nori F 2013 QuTiP 2: A Python framework for the dynamics of open quantum systems Comput. Phys. Commun. 184 1234

[50] Schrieffer J R and Wolff P A 1966 Relation between the Anderson and Kondo Hamiltonians Phys. Rev. A 74 491

[51] Werner R F 1989 Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model Phys. Rev. A 40 4277

[52] Bennett C H, Brassard G, Popescu S, Schumacher B, Smolin J A and Wootters W K 1996 Purification of noisy entanglement and faithful teleportation via noisy channels Phys. Rev. Lett. 76 722

[53] Wootters W K 1998 Entanglement of formation of an arbitrary state of two qubits Phys. Rev. Lett. 80 2245

[54] Megrant A et al 2012 Planar superconducting resonators with internal quality factors above one million Appl. Phys. Lett. 100 113510

[55] Leek P J, Baur M, Fink J M, Bianchetti R, Steffen L, Filipp S and Wallraff A 2010 Cavity quantum electrodynamics with separate photon storage and qubit readout modes Phys. Rev. Lett. 104 100504