Mass dimension one fermions from flag dipole spinors

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**Abstract**

According to the Lounesto classification, there are six classes of spinors. The Dirac and Weyl spinors belong to the first three and the sixth classes respectively. The remaining fourth and fifth classes are known as the flag dipole and flag pole spinors respectively. In this letter, a mass dimension one fermionic field with flag dipole spinors as expansion coefficients is constructed. These spinors are shown to be related to Elko (flag pole spinors) by a matrix transformation. It shows that the flag dipole spinors are generalizations of Elko. To construct a Lorentz-covariant quantum field, an infinitesimal deformation is applied to the spinor dual. Subsequently, we show that the fermionic fields constructed from Elko and flag dipole spinors are physically equivalent.

**Keywords:** Mass dimension one fermions, Lounesto classification, Elko, Flag pole spinors, Flag dipole spinors

1. **Introduction**

In the Standard Model (SM) of particle physics, the Dirac and Weyl spinors and their quantum field operators have both played important roles in describing the dynamics of the fermions as well as elucidating the entire structure of the theory.

Despite the success of the SM, there remains many outstanding questions in particle physics which led to the common consensus that the theory is incomplete. In this letter, we propose an approach to investigate the physics.
beyond the SM that has thus far not received much attention. This approach is based on the fact that according to the Lounesto classification, the Dirac and Weyl spinors are not the only spin-half representations of the Lorentz group. Instead, there exists six classes of spinors, each uniquely defined by their bilinear covariants (Lounesto, 2001, Chap. 12)

Dirac spinors
1. \( \Omega_1 \neq 0, \Omega_2 \neq 0. \)
2. \( \Omega_1 \neq 0, \Omega_2 = 0. \)
3. \( \Omega_1 = 0, \Omega_2 \neq 0. \)

Singular spinors
4. \( \Omega_1 = 0, \Omega_2 = 0, K^\mu \neq 0, S^{\mu\nu} \neq 0. \)
5. \( \Omega_1 = 0, \Omega_2 = 0, K^\mu = 0, S^{\mu\nu} \neq 0. \)
6. \( \Omega_1 = 0, \Omega_2 = 0, K^\mu \neq 0, S^{\mu\nu} = 0. \)

Given a spinor \( \psi \) that transforms under the \( (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) representation of the Lorentz group, the bilinear covariants are defined as

\[
\begin{align*}
\Omega_1 &= \bar{\psi}\psi, \\
K^\mu &= \bar{\psi}\gamma^5\gamma^\mu\psi, \\
S^{\mu\nu} &= \bar{\psi}\gamma^\mu\gamma^\nu\psi
\end{align*}
\]

(1)

where \( \bar{\psi} = \psi^\dagger \gamma^0 \) is the Dirac dual and the \( \gamma^\mu \) matrices are chosen to be

\[
\gamma^0 = \begin{pmatrix} O & I \\ I & O \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} O & -\sigma^i \\ \sigma^i & O \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}.
\]

(2)

The Dirac spinors belong to the first three classes and the Weyl spinors reside in the sixth. The remaining fourth and fifth class are known as the flag dipole and flag pole spinors respectively.

A natural question arises from the above classification — Do the flag dipole and flag pole spinors and their quantum field operators have any relevance in particle physics? This question was partially answered in a series of publications on the theory of mass dimension one fermionic fields with Elko (flag pole spinors) as expansion coefficients Ahluwalia and Grumiller (2005a,b); da Rocha and Rodrigues Jr. (2006); da Rocha and Hoff da Silva.
(2007, 2009, 2010); Hoff da Silva and da Rocha (2009); Ahluwalia et al. (2010, 2011); Fabbri (2010a,b, 2011); da Rocha et al. (2011a,b); Bernardini and da Rocha (2012); Lee (2015, 2016a,b); Lee and Dias (2016); Ahluwalia (2013, 2017a,b); Cavalcanti et al. (2014); Nikitin (2014). These fields have two fascinating properties. They are of mass dimension one and satisfy the Klein-Gordon but not the Dirac equation. The mass dimension one fermions have been studied as a dark matter candidate. Their signatures at the LHC and in cosmology have been investigated Dias et al. (2012); Alves et al. (2015); Agarwal et al. (2015). Their gravitational interactions have also received much attention Boehmer (2007a,b, 2008); Boehmer and Mota (2008); Boehmer and Burnett (2008, 2010a,b); Shankaranarayanan (2009, 2010); Boehmer et al. (2010); Chee (2010); Gredat and Shankaranarayanan (2010); Wei (2011); Basak et al. (2013); da Rocha et al. (2013); da Silva and Pereira (2014). In these works, due to the endowed the Klein-Gordon kinematics, Elko was proposed as an inflaton candidate as well as a source of dark energy. da Rocha and Hoff da Silva (2014) have also investigated the dynamics of Elko in the presence of black holes.

With the theoretical discovery of Elko and its fermionic fields, to answer our question, the remaining task is to study the properties of the flag dipole spinors. In this letter, using the results obtained by Cavalcanti (2014); Ahluwalia (2017b), we show that there exists a mass dimension one fermionic field with flag dipole spinors as expansion coefficients. The flag dipole spinors are shown to be related to Elko by a one-parameter matrix transformation.

The spin-sums for the flag dipole spinors computed using the dual given in Ahluwalia et al. (2010, 2011) are not Lorentz-covariant. The non-covariance can removed using the new dual introduced in Ahluwalia (2017b). Consequently, we show that the mass dimension one fermionic fields constructed from Elko and flag dipole spinors are physically equivalent.

2. Flag dipole spinors

Cavalcanti (2014) showed that the flag dipole spinors have two solutions, one with two non-zero components and another with four non-zero compo-
ments. They are given by

$$
\psi_2^{(1)} = \begin{pmatrix} a \\ 0 \\ 0 \\ b \end{pmatrix}, \quad \psi_2^{(2)} = \begin{pmatrix} 0 \\ c \\ d \\ 0 \end{pmatrix}, \quad |a|^2 \neq |b|^2, \quad |c|^2 \neq |d|^2
$$

$$
\psi_4 = \begin{pmatrix} -fhg^*/|g|^2 \\ f \\ g \\ h \end{pmatrix}, \quad |f|^2 \neq |g|^2
$$

where \( a, \cdots, h \) are arbitrary complex numbers. The reason for the existence of two different types of solutions is that \( \psi_2^{(i)} \) and \( \psi_4 \) are related by a boost in the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation

$$
\psi_4^{(1,2)}(p) = \kappa(p)\psi_2^{(1,2)}(0)
$$

where \( \psi_4^{(1,2)}(p) \) satisfy the condition of \( \psi_4 \) given in eq. (3) and

$$
\kappa(p) = \sqrt{E + m} \left( I + \frac{\sigma \cdot p}{E + m} \right)
$$

is the boost operator with \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) being the Pauli matrices.

The above solutions for the flag dipole spinors do not necessarily guarantee the existence of a physical quantum field operator. By physical, we mean that these fields must be local, furnish fermionic statistics and have positive-definite free Hamiltonian. After imposing these conditions, up to a global phase, we find that the flag dipole spinors which give rise to a physical quantum field operator, is related to Elko given in Ahluwalia (2017b) by a matrix transformation

$$
\lambda_S(\epsilon, \alpha) = Z(z)\lambda_S(\epsilon, \alpha), \quad \lambda_A(\epsilon, \alpha) = Z(z)\lambda_A(\epsilon, \alpha)
$$

where \( Z(z) \) is an one-parameter matrix defined as

$$
Z(z) = \begin{pmatrix} z^{-1}I & O \\ O & zI \end{pmatrix}
$$
and \( z \) is a complex number satisfying the condition \(|z|^2 \neq 1\). The spinors in eqs. (3) and (4) are in the polarization basis. The spinors in eq. (3) are constructed in the helicity basis where \( \epsilon \) is defined as \( \epsilon = \lim_{p \to 0} \hat{p} \). Since \( \mathcal{Z}(z) \) is comprised of the identity matrix, it commutes with all the Lorentz generators of the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation. Therefore, eq. (3) is valid for all momentum

\[
\begin{align*}
\lambda^S_z(p, \alpha) &= \mathcal{Z}(z) \lambda^S(p, \alpha), \\
\lambda^A_z(p, \alpha) &= \mathcal{Z}(z) \lambda^A(p, \alpha).
\end{align*}
\]

Consequently, we obtain

\[
\tilde{f}_z(x) = \mathcal{Z}(z) f(x),
\]

\[
\tilde{f}_z(x) = \mathcal{Z}(z) f(x),
\]

where \( f(x) \) is the mass dimension one fermionic field given in (Ahluwalia, 2017b). The matrix \( \mathcal{Z}(z) \) is dimensionless so \( \tilde{f}_z(x) \) is still of mass dimension one. The dual for the flag dipole spinors are related to their Elko counterpart by

\[
\begin{align*}
\tilde{\lambda}^S_z(p, \alpha) &= \tilde{\lambda}^S(p, \alpha) \mathcal{Z}^{-1}(z), \\
\tilde{\lambda}^A_z(p, \alpha) &= \tilde{\lambda}^A(p, \alpha) \mathcal{Z}^{-1}(z).
\end{align*}
\]

The spin-sums for the flag dipole spinors are given by

\[
\begin{align*}
\sum_{\alpha} \lambda^S_z(p, \alpha) \tilde{\lambda}^S_z(p, \alpha) &= m[\mathcal{G}_z(\phi) + I], \\
\sum_{\alpha} \lambda^A_z(p, \alpha) \tilde{\lambda}^A_z(p, \alpha) &= m[\mathcal{G}_z(\phi) - I]
\end{align*}
\]

where \( \mathcal{G}_z(\phi) \) is defined as

\[
\mathcal{G}_z(\phi) = \begin{pmatrix}
0 & 0 & 0 & -ie^{-i\phi}|z|^2 \\
0 & 0 & ie^{i\phi}|z|^2 & 0 \\
0 & -ie^{-i\phi}|z|^2 & 0 & 0 \\
ie^{i\phi}|z|^2 & 0 & 0 & 0
\end{pmatrix}.
\]

The non-covariance of the spin-sums due to \( \mathcal{G}_z(\phi) \) can be removed by defining a new dual with an infinitesimal deformation (Ahluwalia, 2017b)

\[
\tilde{\lambda}^S(p, \sigma) \equiv \tilde{\lambda}^S(p, \sigma) A_z, \quad \tilde{\lambda}^A(p, \sigma) \equiv \tilde{\lambda}^A(p, \sigma) B_z
\]
where
\[ A_z = \left[ I - \tau G_z(\phi) \right] \]
\[ B_z = \left[ I + \tau G_z(\phi) \right]. \]  
(15)

In the limit \( \tau \to 1 \), the spin-sums become \( z \)-independent
\[ \sum_{\sigma} \lambda^S(p, \sigma) \bar{\lambda}^S(p, \sigma) = m_\lambda I, \quad \sum_{\sigma} \lambda^A(p, \sigma) \bar{\lambda}^A(p, \sigma) = -m_\lambda I. \]  
(16)

Therefore, in the limit \( \tau \to 1 \), the propagator for \( f_z(x) \) is given by
\[ S_{f_z}(x, y) = \left\langle \left| T[f_z(x)\bar{f}_z(y)] \right| \right\rangle \]
\[ = \frac{i}{(2\pi)^4} \int d^4p e^{-ip\cdot(x-y)} \frac{I}{p^2 - m_\lambda^2 + i\epsilon} \]  
(17)

and the Lagrangian is
\[ \mathcal{L}_{f_z} = \partial^\mu \bar{f}_z \partial_\mu f_z - m_\lambda^2 \bar{f}_z f_z. \]  
(18)

Since the norms and the spin-sums are \( z \)-independent, it shows that under the new dual, the mass dimension one field constructed from the flag-dipole spinors and Elko are physically equivalent.

3. Conclusions

The Lounesto classification provides a new and intriguing possibility to study the physics beyond the SM. Four of the six classes of spinors, namely Dirac and Weyl spinors have already found important applications in particle physics. From a theoretical perspective, it is natural to ask whether the remaining spinors have any applications in particle physics. The works of Ahluwalia and Grumiller (2005a, b) and da Rocha and Rodrigues Jr. (2006) showed that a mass dimension one fermionic field naturally emerges from Elko (flag pole spinors). Following their works, the results obtained in this letter bring us a step closer to a complete understanding on the relationships between Lounesto classification and quantum field theory.

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