A NUMERICAL APPROACH FOR THE FILTERED GENERALIZED ČECH COMPLEX

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Abstract. In this paper, we present an algorithm to compute the filtered generalized Čech complex for a finite collection of disks in the plane, which don’t necessarily have the same radius.

The key step behind the algorithm is to calculate the minimum scale factor needed to ensure rescaled disks have a nonempty intersection, through a numerical approach, whose convergence is guaranteed by a generalization of the well-known Vietoris-Rips Lemma, which we also prove in an alternative way, using elementary geometric arguments.

We present two applications of our main results. We give an algorithm for computing the 2-dimensional filtered generalized Čech complex of a finite collection of \( d \)-dimensional disks in \( \mathbb{R}^d \). In addition, we show how the algorithm yields the minimal enclosing ball for a finite set of points in the plane.

1. INTRODUCTION

Recently, in the study of data point clouds from a topological approach (cf. \cite{4, 5, 11, 15, 23}), the need to develop algorithms to construct different simplicial structures has arisen, such as the Vietoris-Rips complex, the Čech complex, the PL-lower star complex, etc. (cf. \cite{8, 22}).

Of particular interest to us is the generalized Čech complex structure: whereas the standard Čech complex is induced by the intersection of a collection of disks with fixed radius, the generalized version admits different radii (see \cite{14}); when radiuses are rescaled, using the same scale factor each time, the corresponding simplicial complexes forms the filtered generalized Čech complex.

There exist efficient algorithms to calculate the standard Čech complex (e.g. \cite{6}), and software currently available to obtain the associated filtration (cf. \cite{16, 17}); also, in \cite{13} the authors propose an algorithm to approximate the Čech filtration. On the other hand, we can find algorithms to calculate the generalized Čech complex (e.g. \cite{14}), however as far as we know, there are neither algorithms nor software to provide the filtered generalized Čech complex. In the present work we show an algorithm to compute the filtered generalized Čech complex for a finite collection of disks, specifically, in the plane. Actually, we also show an algorithm to build up to the 2-dimensional filtered generalized structure (or 2-skeleton), for higher dimensional disk systems, which many applications only require, as we can see in \cite{2, 7, 12, 18, 20}.

The key step behind these proposed algorithms, is to calculate the minimum scale factor (called Čech scale) needed to ensure that the rescaled disks have a nonempty intersection; the generalized Vietoris-Rips Lemma over multiple radii will allow us to calculate these scales numerically.

We must emphasize that, our main algorithm (Algorithm 5) is only generalizable to higher dimensional disk systems to obtain the 2-dimensional filtered generalized Čech structure, as we show as an application. Additionally, we show how our algorithm yields the minimal enclosing ball for a finite set of points in the plane.

Date: September 24, 2018.

Key words and phrases. Disk system; generalized Čech complex; Čech scale; generalized Vietoris-Rips Lemma; miniball problem.
This paper is organized as follows. In Section 2 we introduce basic notions and notation which will be used throughout the paper. We define the Vietoris-Rips system and the Čech system, associated to a finite collection of closed \(d\)-disks (or \(d\)-disk system) in terms of their intersection. We also introduce the fundamental notions of Vietoris-Rips scale and Čech scale for a \(d\)-disk system, as the infimum over all rescaling factors such that the \(d\)-disk system becomes a Vietoris-Rips system or a Čech system respectively. In Lemma 5 we state and prove a generalization, over multiple radii, of the well-known Vietoris-Rips Lemma [7, Th. 2.5] using elementary geometric arguments. In [1] can be found a proof in the generalized case, following the ideas in [7].

In Section 3 we describe the generalized versions of standard Vietoris-Rips and Čech simplicial complex structures, to the case of \(d\)-disk systems with different radiuses. We explain how their respective filtrations are induced by weight functions, and we propose an algorithm to obtain the Čech-weight function of a given \(d\)-disk system, associating to each Čech simplex its corresponding Čech scale.

Section 4 focuses on studying the intersection properties of collections of disks in the plane. We define a real-valuated function associated to each 2-disk system, such that, if it turns out to be nonnegative, then its Čech scale agrees with its Vietoris-Rips scale, being then easy to compute; otherwise, the Čech scale will correspond to a root of such function, and we propose a numerical approach to obtain this Čech scale (Section 5), supported on the generalized Vietoris-Rips Lemma which provides appropriated bounds.

Section 5 contains our main result, the Čech-scale algorithm, whose input is a 2-disk system, and the output is the corresponding Čech scale, as well as the unique intersection point of the rescaled disk system at its Čech scale (see Remark 3).

Finally, we conclude the paper illustrating two applications in Section 6. First, we recall the miniball problem, to show how our Čech-scale algorithm yields the minimal enclosing ball for a finite point cloud in the plane. Secondly, we present an algorithm for computing the Čech filtration of the 2-skeleton of the generalized Čech complex structure for a \(d\)-dimensional disk systems in an arbitrary euclidean space \(\mathbb{R}^d\).

2. Vietoris-Rips and Čech systems

Throughout this paper, a finite collection of \(d\)-disks in euclidean space \(\mathbb{R}^d\) with positive radius, \[\text{(1)}\]

\[M = \{D_i(c_i; r_i) \subset \mathbb{R}^d \mid r_i > 0, 1 \leq i \leq m\}\]

will be called \(d\)-disk system. In this section we introduce and analyze two fundamental subclasses of \(d\)-disk systems, namely, the Vietoris-Rips systems and the Čech systems. We study the infimum of those scales that turn a \(d\)-disk system into a Vietoris-Rips or Čech system. We conclude this section presenting a generalized version of the Vietoris-Rips Lemma, extended to \(d\)-disk systems.

Definition 1. Let \(M = \{D_1, D_2, \ldots, D_m\}\) be a \(d\)-disk system. We say \(M\) is a Vietoris-Rips system if \(D_i \cap D_j \neq \emptyset\) for each pair \(i, j \in \{1, 2, \ldots, m\}\). Moreover, if the \(d\)-disk system \(M\) has the nonempty intersection property \([D_{i \in M} D_i \neq \emptyset]\), then \(M\) is called a Čech system.

For each \(\lambda \geq 0\), and \(d\)-disk system \(M\) as in (1), we define the collection \[M_\lambda := \{D_i(c_i; \lambda r_i) \subset \mathbb{R}^d \mid D_i \in M\}\]

and say that \(\lambda\) is a scale. Geometrically, the set \(M_\lambda\) consists of \(d\)-disks with the same centers than those in \(M\), but with rescaled radii by \(\lambda\). Clearly, only when \(\lambda > 0\) the set \(M_\lambda\) will be again a \(d\)-disk system. Note that \(M_1 = M\), and \(M_0\) is the set consisting of the centers of the \(d\)-disks in \(M\).
**Definition 2.** Let $M$ be a $d$-disk system. The Vietoris-Rips scale of $M$ is defined by,

$$\nu_M := \inf \{ \lambda \in \mathbb{R} \mid M_\lambda \text{ is a Vietoris-Rips system} \}.$$ 

Analogously, the Čech scale of $M$ is defined by,

$$\mu_M := \inf \{ \lambda \in \mathbb{R} \mid M_\lambda \text{ is a Čech system} \}.$$ 

A straightforward calculation shows the following characterizations: $M$ is a Vietoris-Rips system if and only if, $\nu_M \leq 1$ (in particular, $\nu_{M_{\nu_M}} = 1$); similarly, $M$ is a Čech system if and only if, $\mu_M \leq 1$.

Note that it is not hard to calculate the Vietoris-Rips scale $\nu_M$ for a given $d$-disk system $M = \{D_1, D_2, \ldots, D_m\}$: if $r_i$ denotes the radius of $D_i$ and $\|c_i - c_j\|$ represents the distance between the center of $D_i$ and $D_j$, then $\nu_M = \max_{i<j} \{ \|c_i - c_j\|/(r_i + r_j) \}$.

For a $d$-disk system $M$ with just one disk, its Vietoris-Rips scale is $\nu_M = 0$; if $M = \{D_1, D_2\}$ has two disks, then $\nu_M = \|c_1 - c_2\|/(r_1 + r_2)$. Actually, in both cases the Vietoris-Rips scale agrees with the Čech scale.

On the other hand, calculating the Čech scale is a more complicated issue if the $d$-disk system has at least three disks.

Concerning Čech scales, we have the following remark, which will become important for our implementation.

**Remark 3.** If $\mu_M$ is the Čech scale for $M$, then the $\mu_M$-rescaled system $M_{\mu_M}$, has only one point in the intersection $\bigcap_{D_i \in M} D_i(c_i; \mu_M r_i)$. Such point will be denoted by $c_M$.

Indeed, suppose this were not true. Then there would be $p_1, p_2 \in \bigcap_{D_i \in M} D_i(c_i; \mu_M r_i)$ such that $p_1 \neq p_2$. By convexity of the $d$-disks, it follows that the middle point $\bar{p} = \frac{1}{2}(p_1 + p_2)$ must belong to every $d$-disk $D_i(c_i; \mu_M r_i)$. On the other hand, $\|\bar{p} - c_i\| < \max\{\|p_1 - c_i\|, \|p_2 - c_i\|\}$ for any center $c_i$ in the disk system. Let $\mu_i < \mu_M$ be a scale such that $\bar{p} \in D_i(c_i; \mu_i r_i)$ for every $d$-disk in $M$. It follows that $\bar{\mu} = \max\{\mu_i\} < \mu_M$ and $\bar{\mu} \in \bigcap_{D_i \in M} D_i(c_i; \bar{\mu} r_i)$ which contradicts the minimality of the Čech scale $\mu_M$.

**Example 4.** Figure 1 shows (left picture) the following 2-disk system in $\mathbb{R}^2$,

$$M = \{D_1((-3, 4); 4), D_2((1, 3); 3), D_3((2, -1); 2)\}.$$

![Figure 1](image-url)
which is not a Čech system. For such disk system, the \texttt{Cech.scale} algorithm in Section 5 yields to $\mu_M = 0.9188$ and the Čech system $M_{\mu_M}$ is shown in the right picture.

There exists a close relationship between Vietoris-Rips systems and Čech systems. Obviously, every Čech system is also a Vietoris-Rips system, but the opposite statement does not hold in general (as we saw in the example above). However, we have the following result which extends the standard Vietoris-Rips Lemma [7, Th. 2.5], to any Vietoris-Rips system (compare with [1, Th. 3.2] and [13, Th. 4]).

Lemma 5. Let $M = \{D_i(c_i; r_i) \subset \mathbb{R}^d \mid r_i > 0\}$ be a finite set of closed $d$-disks in $\mathbb{R}^d$. If $D_i(c_i; r_i) \cap D_j(c_j; r_j) \neq \emptyset$ for every pair of disks in $M$, then

$$\bigcap_{D_i \in M} D_i(c_i; \sqrt{2d/(d+1)}r_i) \neq \emptyset.$$  

In other words, for every Vietoris-Rips system $M$ in $\mathbb{R}^d$, the $\sqrt{2d/(d+1)}$-rescaled $d$-disk system $M_{\sqrt{2d/(d+1)}}$ is a Čech system.

Proof. First, we prove the result for the case where $M$ has at most $d + 1$ disks, say $M = \{D_1, \ldots, D_{d'}\}$, $d' \leq d + 1$. Note that we need to prove that $\mu_M \leq \sqrt{\frac{2d}{d+1}}$.

Let $\{c_M\} = \bigcap_i D_i(c_i; \mu_M r_i)$ be the unique intersection point of the $\mu_M$-rescaled $d$-disk system (see Remark 3). Suppose in addition, without loss of generality, that $\|c_i - c_M\| = \mu_M r_i$ for $c_1, \ldots, c_m$ and $m \leq d'$.

Then $c_M$ belongs to the convex hull of the set $\{c_1, \ldots, c_m\}$, for if this were not true, there would exist an hyperplane across $c_M$ such that the set $\{c_1, \ldots, c_m\}$ is completely contained at one side, let $v$ be the normal vector for such hyperplane in the opposite direction, then $\langle v, c_i - c_M \rangle > 0$ for all $i = 1, \ldots, m$, hence

$$\|c_i - c_M\|^2 = \|c_i - (c_M + tv)\|^2 + 2t\langle v, c_i - c_M \rangle - t^2\|v\|^2$$

for every $t \in I_v := (0, 2\langle v, c_i - c_M \rangle/\|v\|^2)$; this implies that $c_M + tv \in D_i(c_i; \mu_M r_i)$ for any $i = 1, \ldots, m$ and $t \in I_v$, which is a contradiction since $c_M$ is the only point in the intersection $\bigcap_i D_i(c_i; \mu_M r_i)$. Therefore, $c_M$ is in the convex hull.

Now, define $\hat{c}_i := c_i - c_M$ and let $\theta_{ij}$ denote the angle between vectors $\hat{c}_i$ and $\hat{c}_j$. Since $c_M$ is in the convex hull of $\{c_1, \ldots, c_m\}$, then the vector $0 \in \mathbb{R}^d$ can be written as a convex combination $\sum_{j=1}^m a_j \hat{c}_j = 0$. Thus $\langle \sum_{j=1}^m a_j \hat{c}_j, \hat{c}_i \rangle = 0$ for any $i = 1, \ldots, m$, and

$$\sum_{j=1}^m \langle a_j \hat{c}_j, \hat{c}_i \rangle = \sum_{j=1}^m a_j \|\hat{c}_j\| \|\hat{c}_i\| \cos \theta_{ji} = 0.$$  

Taking out the common factor $\|\hat{c}_i\|$, we have $\sum_{j=1}^m a_j \|\hat{c}_j\| \cos \theta_{ji} = 0$. Now, taking the sum over $i$, we deduce that

$$\sum_{i=1}^m \sum_{j=1}^m a_j \|\hat{c}_j\| \cos \theta_{ji} = \sum_{j=1}^m a_j \|\hat{c}_j\| \sum_{i=1}^m \cos \theta_{ji} = 0.$$  

Note that $\cos \theta_{ii} = 1$. On the other hand, if were $\cos \theta_{ji} > -\frac{1}{m-1}$ for all $1 \leq i, j \leq m$, $i \neq j$, then, for each $j$ we should have $\sum_{i=1}^m \cos \theta_{ji} = 1 + \sum_{j=1,j \neq i}^m \cos \theta_{ji} > 0$. But, this contradicts (2).
because $\sum_{j=1}^{m} a_j = 1$ and $a_j \geq 0$. Therefore, there must exist $i \neq j$, say $i = 1$ and $j = 2$, such that $\cos \theta_{12} \leq -\frac{1}{m-1} \leq -\frac{1}{d}$, then

\[ 0 \leq \frac{d}{d-1} (1 + \cos(\theta_{12})) \leq 1. \]

It follows from inequality above and from the AM-GM inequality, that

\[ \sqrt{\frac{d}{d-1}} (1 + \cos(\theta_{12})) \cdot \sqrt{r_1 r_2} \leq \sqrt{r_1 r_2} \leq \frac{r_1 + r_2}{2}. \]

A straightforward calculation on (4) leads us to the following inequality

\[ (d + 1)(r_1 + r_2)^2 \leq 2d(r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_{12})) \]

so,

\[ (d + 1)\mu_M^2 (r_1 + r_2)^2 \leq 2d(\mu_M^2 r_1^2 + \mu_M^2 r_2^2 - 2\mu_M^2 r_1 r_2 \cos(\theta_{12})) \]

\[ = 2d(\|\hat{c}_1\|^2 + \|\hat{c}_2\|^2 - 2\|\hat{c}_1\|\|\hat{c}_2\| \cos(\theta_{12})) \]

\[ = 2d(\|\hat{c}_1\|^2 + \|\hat{c}_2\|^2 - 2\langle \hat{c}_1, \hat{c}_2 \rangle) \]

\[ = 2d\|\hat{c}_1 - \hat{c}_2\|^2 \]

which implies,

\[ \mu_M^2 \leq \frac{2d}{d+1} \cdot \|\hat{c}_1 - \hat{c}_2\|^2 = \frac{2d}{d+1} \cdot \|c_1 - c_2\|^2 \leq \frac{2d}{d+1} \]

(the last inequality holds because $M$ is a Vietoris-Rips system) or equivalently $\mu_M \leq \sqrt{\frac{2d}{d+1}}$.

For a collection with more than $d + 1$ disks, the claim is a consequence of the Helly’s Theorem (see [3, Problem 29]) which establishes that for any finite collection of convex subsets of $\mathbb{R}^d$ greater than $d$, if the intersection of every $d + 1$ of such sets is nonempty, then the whole collection has a nonempty intersection. This concludes the proof.

The upper bound $\sqrt{2d/(d+1)}$ in Lemma 5 is optimal: it suffices to take a $d$-disk system with $d + 1$ disks of equal radii and pairwise tangents (cf. [8, Sec. III.2]).

In the Example 4 we can see what the Vietoris-Rips Lemma claim for the 2-disk system $M$: $\mu_M = 0.9188 < 0.9812 = \sqrt{4/3} \nu_M$.

To conclude this section, notice that as for a Čech system $\mu_M \leq 1$, then the Vietoris-Rips Lemma implies the following result.

**Corollary 6.** If $M$ is an arbitrary $d$-disk system and $\nu_M$ is its Vietoris-Rips scale, then its Čech scale satisfies that $\mu_M \leq [\nu_M, \sqrt{2d/(d+1)} \nu_M]$. In consequence, for every $d$-disk system $M$, the rescaled disk system $M \sqrt{2d/(d+1)} \nu_M$ is always a Čech system.

In particular, if $\sqrt{2d/(d+1)} \nu_M \leq 1$ then $M \nu_M$ is a Čech system.

3. Filtered generalized simplicial structures for $d$-disk systems

In this section we introduce two simplicial structures associated with a $d$-disk system $M$, as well as the filtration induced by rescaling the system $M$. The importance of these notions lies in their relation to the topological analysis through persistent homology of filtered simplicial structures, induced by point clouds with non-homogeneous neighborhoods.

Let $M$ be a $d$-disk system. Denote by $\mathcal{VR}(M)$ the family of all Vietoris-Rips subsystems of $M$, this is,

\[ \mathcal{VR}(M) = \{ \sigma \subset M \mid D_i \cap D_j \neq \emptyset, D_i, D_j \in \sigma \}. \]
the following relation is fulfilled:

\[ \Delta(\bigcap_{i \in D} D_i) \neq \emptyset. \]

Analogously, denote by \( \mathcal{C}(M) \) the set of all Čech subsystems,

\[ \mathcal{C}(M) = \{ \sigma \subset M \mid \bigcap_{i \in \sigma} D_i \neq \emptyset \}. \]

On the other hand, recall that a simplicial structure on a (finite) set \( V \) is defined as a family \( \Delta(V) \subset 2^V \) of subsets of \( V \) such that: if \( \sigma \in \Delta(V) \) and \( \tau \subset \sigma \), then \( \tau \in \Delta(V) \). It is immediate that for any \( \sigma \in \mathcal{VR}(M) \), every \( d \)-disk subsystem \( \tau \subset \sigma \) is also in the family \( \tau \in \mathcal{VR}(M) \). The same property is valid for the family \( \mathcal{C}(M) \). These properties imply that \( \mathcal{VR}(M) \) and \( \mathcal{C}(M) \) are simplicial complexes.

We refer to \( \mathcal{VR}(M) \) as the generalized Vietoris-Rips complex associated to the \( d \)-disk system \( M \), and to \( \mathcal{C}(M) \) as the generalized Čech complex of \( M \).

The above construction allows us to study the topology of a data cloud through the persistent homology of the generalized Vietoris-Rips or Čech complexes, i.e., to perform a topological data analysis. However, to perform such analysis it is necessary to construct a filtered simplicial structure. We will define a filtration through weight functions.

Let \( \Delta \) be a simplicial complex and let \( \omega : \Delta \to \mathbb{R} \) be a function. We call \( \omega \) a weight function over the simplicial complex \( \Delta \) if \( \tau, \sigma \in \Delta \) and \( \tau \subset \sigma \), implies \( \omega(\tau) \leq \omega(\sigma) \).

For example, to the generalized Čech complex \( \mathcal{C}(M) \) of the \( d \)-disk system \( M \), the function \( \omega : \mathcal{C}(M) \to \mathbb{R} \), \( \sigma \mapsto \mu_{\sigma} \) which assigns the Čech scale to any Čech subsystem \( \sigma \subset M \), is a weight function, called the Čech-weight function. The analogous property holds for the Vietoris-Rips complex and the Vietoris-Rips scale (see [1] for the construction of the filtered generalized Čech complex using weighted point clouds).

Moreover, from the definition we have for the Čech-weight function and to every nonnegative scale \( \lambda \geq 0 \), that

\[ \omega^{-1}((\infty, \lambda]) = \mathcal{C}(M_\lambda). \]

We will denote by \( \mathcal{C}_M(\lambda) \) the family \( \mathcal{C}(M_\lambda) \) for \( \lambda \geq 0 \), i.e., the family of all Čech subsystems of the \( \lambda \)-rescaled disk system \( M_\lambda \), in order to make the dependence explicit with respect to the parameter. We establish the analogous definition for \( \mathcal{VR}_M(\lambda) \), for any \( \lambda \geq 0 \).

It is important to note that there is no restriction on the scale \( \lambda \geq 0 \), additional to the non-negativity, i.e., we allow greater values of \( \lambda \) than 1, in the interest of studying the generalized Čech complex of rescaled disk systems beyond the original.

For \( \lambda' \leq \lambda \) we have the families contention: \( \mathcal{VR}_M(\lambda') \subset \mathcal{VR}_M(\lambda) \) and \( \mathcal{C}_M(\lambda') \subset \mathcal{C}_M(\lambda) \). In general, given a simplicial complex \( \Delta \) and a weight function \( \omega : \Delta \to \mathbb{R} \), any increasing sequence \( \lambda_1 < \cdots < \lambda_s \) of real numbers induces a simplicial filtration: \( \Delta_1 \subset \cdots \subset \Delta_s \) for \( \Delta_i := \omega^{-1}((-\infty, \lambda_i]) \). Thus, for any \( d \)-disk system \( M \), the generalized Čech complex \( \mathcal{C}(M) \) has a filtered simplicial complex structure,

\[ \mathcal{C}_M(0) \subset \mathcal{C}_M(\lambda_1) \subset \cdots \subset \mathcal{C}_M(\lambda_s). \]

Of course, even when we can vary the scale \( \lambda \) on an interval, the length of the above filtration is finite. Now the importance of the Čech scale in the construction of the filtration of the generalized Čech complex is evident. The goal of the next sections is the construction of algorithms to numerically estimate the Čech scale of every Čech subsystem of \( M \).

The filtered generalized Čech complex can be approximated by the Vietoris-Rips structure, in the following sense. The inclusion \( \mathcal{C}(M) \subset \mathcal{VR}(M) \) holds clearly, in consequence: \( \mathcal{C}_M(\lambda) \subset \mathcal{VR}_M(\lambda) \) for any scale \( \lambda \geq 0 \), then by Lemma 5 any Vietoris-Rips \( d \)-system \( \sigma \in \mathcal{VR}(M) \) rescaled by a factor of \( \sqrt{2d/(d+1)} \) is also a Čech \( d \)-system: \( \sigma \sqrt{2d/(d+1)} \in \mathcal{C}(M) \). Therefore, for any \( d \)-disk system \( M \) the following relation is fulfilled:

\[ \mathcal{VR}_M(\lambda') \subset \mathcal{C}_M(\lambda) \subset \mathcal{VR}_M(\lambda), \]

where \( \sqrt{2d/(d+1)} \cdot \lambda' \leq \lambda \).
To any $d$-disk system $M$, the simplicial substructure $\mathcal{C}(M)^{(1)}$ given by the 1-skeleton of the generalized Čech complex of $M$ is a basic combinatorial structure (actually, a graph) that can be easily defined, it just takes the relationship into account if every two vertices are neighbors: the set of vertices is $M$, and there exists an edge $\{D_i, D_j\}$ whenever $D_i \cap D_j \neq \emptyset$. The Čech-weight function restricted to $\mathcal{C}(M)^{(1)}$ is, in fact: $\omega(\{D_i\}) = 0$ to every vertice, and $\omega(\{D_i, D_j\}) = \|c_i - c_j\|/(r_i + r_j)$ to any edge.

In Algorithm 1 we calculate the Čech-weight function $\omega: \mathcal{C}(M)(\lambda)^{(dim)} \to \mathbb{R}$, for the $dim$-skeleton of a $\lambda$-rescaled $d$-disk system $M$. To do this, we assume an arbitrary linear order in the $d$-disk system $M$, and for every disk $D \in M$ we consider the following set (cf. [22]):

$$\lambda\text{-LowerNbrs}(D) = \{ \tilde{D} \in M \mid \tilde{D} < D, \omega(\{D, \tilde{D}\}) \leq \lambda \}.$$

**Algorithm 1:** Čech-weight function of a $d$-disk system.

**Input:** A $d$-disk system $M$, a nonnegative parameter $\lambda$ and an integer $dim \geq 2$.

**Output:** The Čech-weight function $\omega: \mathcal{C}(M)(\lambda)^{(dim)} \to \mathbb{R}$.

1. for $n \leftarrow 0$ to $dim - 1$ do
2.  for $\sigma \in \mathcal{C}(M)(n) \setminus \mathcal{C}(M)(n-1)$ do
3.    $LN_\sigma \leftarrow \bigcap_{D \in \sigma} \lambda\text{-LowerNbrs}(D)$;
4.    for $D \in LN_\sigma$ do
5.      $N \leftarrow \{D\} \cup \sigma$;
6.      if $n \leq d - 1$ then
7.        Calculate $\omega(N) \leftarrow \mu_N$;
8.      else
9.        $\omega(N) \leftarrow \max\{\omega(\tau) \mid \tau \subseteq N\}$;
10.     end
11.    if $\omega(N) \leq \lambda$ then
12.       Update $\mathcal{C}(M)(\lambda)^{(dim)} \leftarrow \mathcal{C}(M)(\lambda)^{(dim)} \cup \{N\}$;
13.    end
14.  end
15. end
16. end
17. return $(\mathcal{C}(M)(\lambda)^{(dim)}, \omega: \mathcal{C}(M)(\lambda)^{(dim)} \to \mathbb{R})$

We conclude the section with an application of Algorithm 1 to a 2-disk system.

**Example 7.** Let $M$ be the following 2-disk system,

$$M = \{D_1((2.99, 0.56); 1.5), D_2((0.99, 0.11); 1.0),$$
$$D_3((1.69, 1.30); 0.6), D_4((1.07, 1.93); 0.4),$$
$$D_5((1.96, 2.64); 0.8)\}.$$

The output of Algorithm 1 applied to $M$, with $d = 2$, $\lambda = 1$ and $dim = 2$ gives the Čech scales indicated next to every edge and in the triangle, in Figure 2. The Čech scale of the 2-disk system $\{D_1, D_2, D_3\}$ was calculated with the Čech.scale script from Algorithm 5.
4. INTERSECTION PROPERTIES OF DISK SYSTEMS

In this section we focus on studying 2-disk systems in the plane, called simply disk systems. As we have seen in the last section, the study of the Čech scale is a key aspect to the construction and study of filtered generalized Čech complex. In this section, we establish several intersection properties of disk systems, which will lead us to be able to calculate the Čech scale.

Let \( \partial D_i(c_i; r_i) := \{ x \in \mathbb{R}^2 \mid \| x - c_i \| = r_i \} \) be the boundary of the disk \( D_i(c_i, r_i) \).

Let \( D_i \) and \( D_j \) be two disks such that \( D_i \cap D_j \neq \emptyset \). We define \( D_i \cap D_j \) to be the unitary set \( \{ d_{ij} \} \) constructed as follows:

1. If \( \partial D_i \cap \partial D_j \neq \emptyset \), then \( d_{ij} \in \partial D_i \cap \partial D_j \) is the only one point with the property \( \langle d_{ij} - c_i, n_{ij} \rangle \geq 0 \), where \( n_{ij} = (-b, a) \) is the normal vector to \( c_j - c_i = (a, b) \).
2. If \( \partial D_i \cap \partial D_j = \emptyset \), we define \( d_{ij} \) as the unique intersection point in \( \partial D_i(c_i; \lambda r_i) \cap \partial D_j(c_j; \lambda r_j) \), for \( \lambda \) given as the minimal scale such that \( D_i(c_i; \lambda r_i) \subset D_j(c_j; \lambda r_j) \), i.e. \( \lambda = \| c_i - c_j \| / |r_i - r_j| \).

In particular, when \( D_i \) and \( D_j \) are concentric, then \( D_i \cap D_j = \{ c_i \} = \{ c_j \} \). On the other hand, if the disks \( D_i \) and \( D_j \) are internally or externally tangent, then \( d_{ij} = d_{ji} \). We can think about \( d_{ij} \), when \( \partial D_i \cap \partial D_j \) is not empty, as the intersection point of the boundaries at the left of the vector from \( c_i \) to \( c_j \). Figure 3 shows the above construction.

We will denote by \( d_{ij}(\lambda) \), instead of simply \( d_{ij} \), for the intersection point of the \( \lambda \)-rescaled disks \( D_i(c_i; \lambda r_i) \) and \( D_j(c_j; \lambda r_j) \).

In order to study Čech systems, we give the following characterization, according the intersection points \( d_{ij} \).

**Lemma 8.** Let \( M = \{ D_1, D_2, \ldots, D_m \} \) be a disk system. Then \( M \) is a Čech system if and only if, there exist \( d_{ij} \in D_i \cap D_j \) such that \( d_{ij} \in D_k \) for all \( k \neq i, j \).

**Proof.** Suppose \( M \) is a Čech system. Define \( A := \bigcap_{1 \leq i \leq m} D_i \neq \emptyset \). Then \( A \) has only one of the following geometries:

1. \( A = \{ c_M \} \),
2. \( A \) is the region, bounded by more than one circumference arc,
3. \( A = D_i \) for some \( i \in \{1, \ldots, m\} \).

**Figure 2.** The Čech-weight function of the 2-disk system \( M \).
In the first case, necessarily $c_M \in \partial D_i \cap \partial D_j$ for some $i, j$ and, it follows $d_{ij} = c_M \in D_k$ for all $k$.

For the second case, if $a \in \partial A \subset A$ belongs to the boundary and is in the intersection of two arcs, say $\partial D_i$ and $\partial D_j$, then $a = d_{ij}$ and it satisfy $d_{ij} \in D_k$ for every $k \neq i, j$.

For the last case, if $A = D_i$ for some $i$, then for each $j \neq i$ we have $d_{ij} \in A \subset D_k$ for all $k \neq i, j$.

Therefore, in any case there exists such point $d_{ij}$.

The converse is clear by definition of a Čech system. □

This criterion was presented in [14, Sec. III] for a disk system.

Next, we define the map $\rho$, which in a certain sense quantifies the intersection of a disk system. This map will allow us to discern the minimal scale in which a disk system has the nonempty intersection property.

**Definition 9.** Let $M = \{D_1, D_2, \ldots, D_m\}$ be a Vietoris-Rips system, with $m \geq 3$. We define $\rho(M) := \max_{1 \leq i, j \leq m} \{\min_{k \neq i, j} \{r_k - \|d_{ij} - c_k\|\}\}$. If $\nu_M$ is the Vietoris-Rips scale of $M$, we define $\rho_M : [\nu_M, \infty) \to \mathbb{R}$, $\lambda \mapsto \rho_M(\lambda) = \rho(M, \lambda)$.

It follows, by construction, that $\rho_M$ is continuous throughout the interval $[\nu_M, \infty)$. Indeed, given three disks $D_i(c_i; r_i)$, $D_j(c_j; r_j)$ and $D_k(c_k; r_k)$, denote by $\Lambda^k_{ij} : [\nu_M, \infty) \to \mathbb{R}$ the map $\lambda \mapsto \lambda r_k - \|d_{ij}(\lambda) - c_k\|$, where $d_{ij}(\lambda)$ is the element in $D_i(c_i; \lambda r_i) \cap D_j(c_j; \lambda r_j)$.

If $r_i \neq r_j$, then for each $k \neq i, j$ the map $\Lambda^k_{ij}$ is continuous in the range $\|c_i - c_j\|/(r_i + r_j) \leq \lambda \leq \|c_i - c_j\|/|r_i - r_j|$, and vary linearly in the range $[\|c_i - c_j\|/|r_i - r_j|, \infty)$ because for $\lambda \geq \|c_i - c_j\|/|r_i - r_j|$, the term $\|d_{ij}(\lambda) - c_k\|$ remains constant. The left picture in Figure 4 shows in bold red color the geometric place of $\{d_{ij}(\lambda), d_{ji}(\lambda)\}$ which vary continuously respect to the parameter $\lambda$ and also the distance from it to the fix point $c_k$.

On the other hand, for $r_i = r_j$ the points $\{d_{ij}(\lambda), d_{ji}(\lambda)\}$ vary continuously respect to $\lambda$ on the line showed in the right picture of Figure 4. Therefore, $\lambda r_k - \|d_{ij}(\lambda) - c_k\|$ also depend continuously of $\lambda$. 

![Figure 3. Intersection point $d_{ij}$.](image-url)
The continuity of $\rho_M$ is a consequence of the continuity of each map $\Lambda^k_{i,j}$ and the continuity of the min-max functions:

$$\rho_M(\lambda) = \max_{1 \leq i, j \leq m} \left\{ \min_{k \neq i, j} \Lambda^k_{i,j}(\lambda) \right\}.$$ 

The map $\rho_M$ plays a key role in the rest of this work. The Algorithm 2 allows to systematically reach the value $\rho_M(\lambda)$. In addition, we present the next characterization of Čech systems in terms of $\rho_M$.

**Algorithm 2**: Evaluation of the map $\rho_M$.

**Input**: Disk system $M$, and scale $\lambda$.

**Output**: Value $\rho_M(\lambda)$.

1. $\rho \leftarrow -\infty$;
2. for $(i,j \leftarrow 1$ to $|M|) \& (i \neq j)$ do
3. 
4. Calculate $d_{ij}(\lambda)$;
5. $\rho_{\text{max}} \leftarrow \text{TRUE}$;
6. for $k \neq i,j$ do
7. 
8. Calculate $\Lambda^k_{i,j}(\lambda)$;
9. if $\Lambda^k_{i,j}(\lambda) \leq \rho$ then
10. 
11. $\rho_{\text{max}} \leftarrow \text{FALSE}$;
12. break for
13. end
14. if $\rho_{\text{max}} \leftarrow \text{TRUE}$ then
15. 
16. Update $\rho \leftarrow \min_k \{\Lambda^k_{i,j}(\lambda)\}$;
17. end
18. end
19. return $\rho$

**Lemma 10.** Let $M$ be a disk system. Then $M_\lambda$ is a Čech system if and only if, $\rho_M(\lambda) \geq 0$. In particular $\rho_M(\sqrt{4/3 \nu_M}) \geq 0$.

**Proof.** By Lemma 8, $M_\lambda$ is a Čech system if and only if, there exists $d_{ij}(\lambda)$ such that $d_{ij}(\lambda) \in D_k(c_k; \lambda r_k)$ for every $k \neq i, j$, i.e. $\Lambda^k_{i,j}(\lambda) \geq 0$ for every $k \neq i, j$, which is equivalent to $\rho_M(\lambda) \geq 0$.

On the other hand, from Corollary 6 and taking $d = 2$, the rescaled system $M_{\sqrt{4/3 \nu_M}}$ is a Čech system, then by the first assertion, $\rho_M(\sqrt{4/3 \nu_M}) \geq 0$. □
5. The Čech scale algorithm

Our main algorithm (Algorithm 5) computes the Čech scale of a given disk system $M$. The key aspect on which this algorithm is based, is precisely the function $\rho_M$. Before we describe the algorithm, we need to analyze additional properties of $\rho_M$.

It follows immediately, from Lemma 10 that $\rho_M(\lambda) \geq 0$ for every $\lambda \geq \mu_M$. Also, if at the Vietoris-Rips scale it holds that $\rho_M(\nu_M) \geq 0$, then $\mu_M = \nu_M$ by the minimality of the Čech scale. We conclude that in this case (this is, $\rho_M(\nu_M) \geq 0$), the Čech scale is easily computable.

On the other hand, if $\rho_M(\nu_M) < 0$ then the Čech scale satisfies $\mu_M \in (\nu_M, \sqrt{4/3}\nu_M]$ and moreover $\rho_M(\mu_M) = 0$. This is a consequence of the continuity of $\rho_M$, and the fact that $\rho_M(\lambda) < 0$ for every $\mu_M \leq \lambda < \mu_M$ and $\rho_M(\lambda) \geq 0$ for $\mu_M \leq \lambda$. Thus, to find the Čech scale of a disk system $M$ for which $\rho_M(\nu_M) < 0$, we need to solve the equation $\rho_M(\lambda) = 0$.

On the practical side, we conclude that a priori validation of the sign of the evaluated function $\rho_M(\nu_M)$ is a useful step in an efficient procedure that computes Čech scales. For this purpose, we present the Algorithm 3 which has as output the logical TRUE if $\rho(\lambda) \geq 0$, for given disk system $M$ and scale $\lambda$.

**Algorithm 3**: Nonnegative verification of the map $\rho$.

```
Input : Disk system $M$, and scale $\lambda$.
Output: The logical TRUE if $\rho_M(\lambda) \geq 0$ and the logical FALSE in otherwise.
1 for $(i, j \leftarrow 1$ to $|M|) \& (i \neq j)$ do
2     rho.nonnegative $\leftarrow$ TRUE;
3     for $k \neq i, j$ do
4         Calculate $\Lambda^k(i,j)(\lambda)$;
5         if $\Lambda^k(i,j)(\lambda) < 0$ then
6             rho.nonnegative $\leftarrow$ FALSE;
7             break for
8         end
9     end
10 if rho.nonnegative $=$ TRUE then
11     return (TRUE)
12 end
13 end
14 return (FALSE)
```

Notice that some radii configurations yields high probability of getting the same Vietoris-Rips and Čech scale: in the case of a Vietoris-Rips system $M$ with 3 elements and equal radii, the probability that $M$ is also a Čech system is $p_3 := P(\mu_M = \nu_M) = 3\pi/(8\pi - 3\sqrt{3}) \approx 0.47$; for a Vietoris-Rips system with $m$-elements and equal radii, the probability to be a Čech system is $p_3^{m-2}$.

We propose a numerical approach, in order to solve the equation $\rho_M(\lambda) = 0$, to calculate the Čech scale under the hypothesis $\rho_M(\nu_M) < 0$, since in this case, we actually know that $\mu_M \in (\nu_M, \sqrt{4/3}\nu_M]$ (see Section 2) as consequence of the generalized Vietoris-Rips Lemma. We have chosen the bisection method for this purpose. We will denote the implementation of bisection method for the map $\rho_M$ through the interval $[a,b]$, by bisection($\rho_M, a, b$). The output of bisection($\rho_M, a, b$) is a real number $\lambda \in [a, b]$ such that $\rho_M(\lambda) = 0$. For the numerical method we are working with a precision of $10^{-12}$.

It is important to mention that the numerical method *regula falsi* was also used instead of the numerical method of bisection, in order to calculating the Čech scale. However, in our context the
efficiency of the program using the *regula falsi* numerical method is not better than if the numerical method of bisection is used.

The Algorithm 4 (below) has as input a disk system $M$, and produces as output the Čech scale $\mu_M$ as well as the intersection point $\{c_M\} = \bigcap_{D_i \in M} D_i(c_i; \mu_M r_i)$. This algorithm takes a naive approach to calculate the Čech scale, and is established to completeness and to be a reference for the principal algorithm (Algorithm 5).

**Algorithm 4:** The Čech scale calculation for disk system.

**Input:** A disk system $M$.

**Output:** The Čech scale $\mu_M$ and the intersection point $c_M$.

1. Calculate $\nu_M$;
2. $\mu^* \leftarrow \nu_M$;
3. if $\rho_M(\mu^*) \geq 0$ then
   4. return $(\mu_M = \mu^*, c_M)$
4. else
   6. Update $\mu^* \leftarrow \sqrt{4/3} \nu_M$;
7. end
8. Update $\mu^* \leftarrow \text{bisection}(\rho_M, \nu_M, \mu^*)$;
9. Calculate $\sqcap M_{\mu^*} \leftarrow \{d_{ij}(\mu^*) \in D_i \cap D_j \mid D_i, D_j \in M_{\mu^*}\}$;
10. if $|\sqcap M_{\mu^*}| > 1$ then
11. Find $\mu' \in (\nu_M, \mu^*)$ such that $\rho_M(\mu') > 0$;
12. Update $\mu^* \leftarrow \mu'$;
13. go to (8);
14. end
15. return $(\mu_M = \mu^*, c_M)$

The following lemma claims that the Algorithm 4 is consistent.

**Lemma 11.** For any disk system $M$, the Algorithm 4 has as output the Čech scale $\mu_M$ of $M$, and the unique intersection point $\{c_M\} = \bigcap_{D_i \in M} D_i(c_i; \mu_M r_i)$.

**Proof.** In the case $\rho_M(\nu_M) \geq 0$, it is clear that the algorithm has generated asseverated data (steps [3]-[4]). In otherwise, for the case $\rho_M(\nu_M) < 0$, we assign $\mu^* := \sqrt{4/3} \nu_M$.

Then, $\rho_M(\nu_M) \cdot \rho_M(\mu^*) \leq 0$ and lets call again $\mu^*$ the output root in step (8). To check if $\mu^*$ is the Čech scale we are looking for, we calculate the set of pairwise intersection points of the $\mu^*$-rescaled system $\sqcap M_{\mu^*}$, in step (9).

If the set $\sqcap M_{\mu^*}$ is unitary, then steps (10)-(14) are omitted and the algorithm returns the Čech scale as well as the intersection point $\sqcap M_{\mu^*}$ at step (15); in otherwise, the root $\mu^*$ is not the Čech scale (see Figure 5), and then we should find another scale $\mu' \in (\nu_M, \mu^*)$ such that $\rho_M(\mu') > 0$, and repeat from step (8). It is possible, for some configurations of the disk system, that the map $\rho_M$ has a behavior as in Figure 5.
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The last iterative part is a finite process because \( \rho_M \) is algebraic over \( \mathbb{Q} \), then eventually the set \( \cap M_{\mu^*} \) will be unitary and the Čech scale will be calculated.

The step (9) in Algorithm 4 is necessary, as show the following example, in which the map \( \rho_M \) has another root along side the Čech scale in the interval \( [\nu_M, \sqrt{4/3} \nu_M] \).

**Example 12.** Let \( M = \{D_1, D_2, D_3\} \) be the disk system in Figure 6. A direct calculation, yields that \( \nu_M = \mu_M = 0.8947 \). On the other hand, we also have that \( \rho_M(\nu_M) = \rho_M(\mu_M) = \rho_M(\lambda) = 0 \) for \( \lambda = 1 \). Therefore, the map \( \rho_M \) has more than one root on the interval \( [\nu_M, \sqrt{4/3} \nu_M] = [0.8947, 1.0331] \).

In Example 12 the Vietoris-Rips scale \( \nu_M \), of the disk system \( M \), agrees with the Čech scale \( \mu_M \); however is possible to construct more sophisticated disk system \( M \) such that \( \nu_M < \mu_M \) and \( \rho_M(\nu_M) < 0 < \rho_M(\mu_M) \), for which there exists \( \mu^* \in (\nu, \sqrt{4/3} \nu_M) \) with \( \rho_M(\mu^*) = 0 \).

On the other hand, if the disk system \( M \) consists of just three disks and \( \rho_M(\nu_M) < 0 \), then its Čech can be computed with only one application of the numerical method, as we asseverate in the following lemma.

**Lemma 13.** Let \( M = \{D_1, D_2, D_3\} \) be a disk system such that \( \rho_M(\nu_M) < 0 \). Then, there exists a unique root of the map \( \rho_M \) in \([\nu_M, \sqrt{4/3} \nu_M]\). Thus, \( \mu_M \) will be the output of \( \text{bisection}(\rho_M, \nu_M, \sqrt{4/3} \nu_M) \).
Proof. It is straightforward to verify that $\rho_M(\nu_M) \geq 0$ for any configuration with $c_1$, $c_2$, and $c_3$ collinear. Thus, $\{c_1, c_2, c_3\}$ is in general position.

Let $\mu_M$ be the Čech scale of the system $M$ and $c_M$ the intersection point. Note that $c_M \in \partial D_i(c_i; \mu_M r_i)$ for every $i$, since, if this were not true, there would be $k \notin \{i, j\}$ such that $\{c_M\} = \partial D_i(c_i; \mu_M r_i) \cap \partial D_j(c_j; \mu_M r_j)$ and $c_M \in \text{Int}(D_k(c_k; \mu_M r_k))$. Then we would have $\mu_M = \nu_M$ and $\rho_M(\mu_M) = \rho_M(\nu_M) = 0$, contradicting the hypothesis.

On the other hand we may assume, without loss of generality, that $r_1 = \max_i \{r_i\}$ and let $c_M = d_{23}(\mu_M)$ be the intersection point at the Čech scale. By the triangle inequality we have that for any $\lambda > \mu_M$, the following two inequalities holds:

$$\|d_{23}(\lambda) - d_{23}(\mu_M)\| \leq \|d_{23}(\lambda) - c_2\| - \|d_{23}(\mu_M) - c_2\| = (\lambda - \mu_M)r_2$$

$$\|d_{23}(\lambda) - d_{23}(\mu_M)\| \leq \|d_{23}(\lambda) - c_3\| - \|d_{23}(\mu_M) - c_3\| = (\lambda - \mu_M)r_3$$

and at least one of them must be strict. Then

$$\|d_{23}(\lambda) - d_{23}(\mu_M)\| < (\lambda - \mu_M)\frac{r_2 + r_3}{2} \leq (\lambda - \mu_M)r_1$$

so, $d_{23}(\lambda) \in \text{Int}(D_1(c_1; \lambda r_1))$ or equivalently $\lambda r_1 - \|d_{23}(\lambda) - c_1\| > 0$. It follows that $\rho_M(\lambda) > 0$ for any $\lambda > \mu_M$, therefore $\mu_M$ is the only one solution of $\rho_M$. \qed

The following algorithm takes advantage of the unicity property for the root of $\rho_M$, in a disk system with three disks. Essentially, the algorithm consist in iterating the Algorithm 4 systematically over every triplet of disks from $M$.

---

**Algorithm 5: Čech.scale.**

**Input**: Disk system $M$.

**Output**: Čech scale $\mu_M$ and intersection point $c_M$.

1. Calculate $\nu_M$;
2. $\mu^* \leftarrow \nu_M$;
3. if $\rho_M(\mu^*) \geq 0$ then
   4. return $(\mu_M = \mu^*; c_M)$
   5. end
4. for $1 \leq i < j < k \leq |M|$ do
   5. $N \leftarrow \{D_i, D_j, D_k\}$;
   6. Calculate $\nu_N$;
   7. if $\sqrt{4/\sqrt{3}} \nu_N \geq \mu^*$ then
      8. if $\rho_N(\nu_N) \geq 0$ then
         9. $\mu_N \leftarrow \nu_N$;
      10. else
         11. $\mu_N \leftarrow \text{bisection}(\rho_N, \nu_N, \sqrt{4/\sqrt{3}} \nu_N)$
      12. end
      13. if $\mu_N > \mu^*$ then
         14. Update $\mu^* \leftarrow \mu_N$;
      15. end
   16. end
5. end
6. return $(\mu_M = \mu^*; c_M)$
Theorem 14. For any disk system $M$, the Algorithm 5 has as output the Čech scale $\mu_M$ of $M$, and the unique intersection point $\{c_M\} = \bigcap_{D_i \in M} D_i(c_i; \mu_M r_i)$.

Proof. If $\rho_M(\nu_M) \geq 0$, the algorithm returns the right data: steps (1)-(5).

On the other hand, by Helly’s Theorem (cf. [3]) the disk system $M = \{D_1, \ldots, D_m\}$, as a finite family of convex sets in the plane, has a nonempty intersection $\bigcap_{D_i \in M} D_i$ if, and only if, $D_i \cap D_j \cap D_k \neq \emptyset$ for every triplet $1 \leq i < j < k \leq m$. Let $\mu$ be the maximal Čech scale over every triplet in the disk system $M$, i.e.

$$\mu = \max \{\mu_N \mid N = \{D_i, D_j, D_k\} \subset M\}.$$ 

It follows that every $\mu$-rescaled triplet has a nonempty intersection. Hence, the $\mu$-rescaled disk system $M_{\mu}$ also has the nonempty intersection property. Moreover, $\bigcap_{D_i \in M} D_i(c_i; \mu r_i) \subset \bigcap_{D_i \in N \subset M} D_i(c_i; \mu r_i)$ for every triplet $N \subset M$. Therefore, $\mu$ is actually the Čech scale of the disk system, this is, $\mu_M = \mu$.

In steps (6)-(19) the algorithm search the scale $\mu$ systematically, over every triplet $\{D_i, D_j, D_k\} \subset M$, updating the maximal scale found if necessary in steps (15)-(17). By Lemma 13, every Čech scale calculation over any triplet, requires just one application of the bisection method. This implies the correctness of the algorithm.

Additionally, the condition in step (9) avoids calculating unnecessary Čech scales of triplets $N = \{D_i, D_j, D_k\}$. In effect, if $\lambda^*$ is the maximal Čech scale found until the verification of the triplet $N$, and the condition in step (9) does not satisfy, i.e.

$$\mu_N \leq \sqrt{\frac{4}{3}} \nu_M < \lambda^*,$$

then, whatever is the Čech scale of $N$, it would be not greater than $\lambda^*$.

The computational evidence to support the Algorithm 5 is more efficient than Algorithm 4 is given in Figure 7. The graphic shows the average time (in seconds) to computation of both algorithms, with respect to the number of disks in a randomly generated disk system (see Remark 16).

![Figure 7. Average time of the script Čech.scale.](image)

6. Applications

6.1. The miniball problem. The miniball problem or smallest-circle problem in the euclidean space is a classical problem, proposed by James J. Sylvester in 1857.

Given a finite point cloud $N \subset \mathbb{R}^d$ the miniball problem consists in finding the center $c \in \mathbb{R}^d$ and minimum radius $r \in \mathbb{R}^+$ of a $d$-disk $D = D(c; r) \subset \mathbb{R}^d$ such that $N \subset D$.

There exist many different approaches to solve this problem, and a variety of algorithms to reach the miniball data (e.g. [9, 21]). In fact, the Čech scale has a close relation with the miniball problem, as we establish in the next lemma.
Lemma 15. Let $N$ be a finite point cloud in $\mathbb{R}^d$, and let $N_1$ be the associated $d$-disk system defined by
\[ N_1 := \{ D_i(c_i; 1) \subset \mathbb{R}^d \mid c_i \in N \}. \]
Then, the Čech scale $\mu_{N_1}$ is the radius of the minimal enclosing ball of $N$, and the intersection point $\{ c_{N_1} \} = \bigcap_{c_i \in N} D_i(c_i; \mu_{N_1})$ its center.

Proof. Let $\mu_{N_1}$ be the Čech scale of the disk system $N_1$ and let $c_{N_1}$ be the intersection point of the $\mu_M$-rescaled disk system. Then the point $c_{N_1}$ belongs to every disk $D_i(c_i; \mu_{N_1} \cdot 1)$, i.e. $\|c_i - c_{N_1}\| \leq \mu_{N_1}$ for any $c_i \in N$, thus $N \subset D(c_{N_1}; \mu_{N_1})$. On the other hand, by definition of Čech scale, $\mu_{N_1}$ is the minimal radius (scale) with such property. Therefore, by uniqueness, $D(c_{N_1}; \mu_{N_1})$ must be the minimal ball enclosing the point cloud $N$. \qed

In particular, for a point cloud $N$ in the plane we can apply our algorithm Čech.scale (Algorithm 5) to the disk system $N_1$, and get the minimal enclosing ball of $N$.

In Figure 8 we show a point cloud $N$ (black dots) and the disk system $N_1$ (blue circles). Applying the Čech.scale script to $N_1$ we get the Čech scale $\mu_{N_1}$ (radio of the red circle) and the point $c_{N_1}$ (red point, center of the red circle).

Figure 8. The miniball of the point cloud $N$.

For the miniball problem there are many efficient algorithms available online, which are easy to find. For example, the C++ script in [10] can compute the miniball for point clouds in any dimension (efficiently up to dimension 10,000).

6.2. The algorithm Čech.scale for higher dimensional disk systems. It is not clear how to generalize the Algorithm 5 to determine the Čech scale of a $d$-disk system in $\mathbb{R}^d$ with $d > 2$. However, it is possible to calculate the Čech scale if the $d$-disk system consists of only three $d$-disks. This makes it possible to calculate the 2-skeleton associated with a $d$-disk system in an arbitrary dimension.

The relevance of this application lies in the possibility of calculating the 2-dimensional filtered simplicial Čech structure of a $d$-disk system immersed in a high-dimensional euclidean space.

The key observation is that
\[ \bigcap_{i=1}^{3} D(c_i; r_i) \neq \emptyset \iff \bigcap_{i=1}^{3} (D(c_i; r_i) \cap P) \neq \emptyset, \]
where $P$ is the affine plane generated by the set $\{ c_1, c_2, c_3 \}$. 
Thus, the problem of determining whether \( \bigcap_{i=1}^{3} (D(c_i; r_i) \cap P) \) is empty or not, can be treated as one in the plane, constructing a disk system in \( \mathbb{R}^2 \) that preserves the affine configuration of the points \{c_1, c_2, c_3\} in the affine space \( P \subset \mathbb{R}^d \). To do this, we set the first center \( c_1 \) as the origin in \( \mathbb{R}^2 \), and “translate” the others centers preserving their original configuration, taking care of moving the second center on the x-axis, as in Figure 9.

**Figure 9.** Affin configuration of the \( d \)-disk system.

More precisely, to any \( d \)-disk system with 3 elements, say \( M = \{B_1(c_1; r_1), B_2(c_2; r_2), B_3(c_3; r_3)\} \), we associate the following disk system, which clearly preserves the affine configuration of the original centers \{c_1, c_2, c_3\},

\[
\text{Aff}(M) = \{D_1(\tilde{c}_1; r_1), D_2(\tilde{c}_2; r_2), D_3(\tilde{c}_3; r_3)\}
\]

where \( \tilde{c}_1 := (0, 0) \), \( \tilde{c}_2 := (\|c_2 - c_1\|, 0) \), \( \tilde{c}_3 := (\|c_3 - c_1\| \cos(\theta), \|c_3 - c_1\| \sin(\theta)) \), where \( \theta \) is the angle between the vectors \( c_2 - c_1 \) and \( c_3 - c_1 \), which satisfies the following relationship:

\[
\cos \theta = \frac{\langle c_2 - c_1, c_3 - c_1 \rangle}{\|c_2 - c_1\| \cdot \|c_3 - c_1\|}.
\]

The next algorithm is a variant of Algorithm [4] taking as input a \( d \)-disk system in \( \mathbb{R}^d \) and a nonnegative parameter \( \lambda \), and as output the Čech weight function of the 2-skeleton of the generalized Čech complex structure. The algorithm first preprocess each triplet of \( d \)-disks as a 2-disk system, then the Čech scale is calculated.
Algorithm 6: 2-skeletal Čech-weight function.

**Input**: A $d$-disk system $M$ and a parameter $\lambda \geq 0$.

**Output**: The Čech-weight function $\omega : \mathcal{C}_M(\lambda) \rightarrow \mathbb{R}$.

1. Calculate $\mathcal{C}_M(\lambda)$;
2. for $\{D_i, D_j\} \in \mathcal{C}_M(\lambda)$ do
   3. $\text{LN}_{ij} \leftarrow \lambda$-LowerNbrs$(D_i) \cap \lambda$-LowerNbrs$(D_j)$;
   4. for $D_k \in \text{LN}_{ij}$ do
      5. $N \leftarrow \text{Aff}([D_i(c_i; r_i), D_j(c_j; r_j), D_k(c_k; r_k)])$;
      6. Calculate $\omega(N) \leftarrow \mu_N$;
      7. if $\omega(N) \leq \lambda$ then
         8. Update $\mathcal{C}_M(\lambda) \leftarrow \mathcal{C}_M(\lambda) \cup \{D_i, D_j, D_k\}$;
      end
   end
3. end
4. return $(\mathcal{C}_M(\lambda), \omega : \mathcal{C}_M(\lambda) \rightarrow \mathbb{R})$

Figure 10 shows the performance of the C/C++ script Čech.scale (available in [19]) vs. the C++ script Miniball (available in [10]).

**Figure 10.** Average time in high dimensions of the Miniball script and Čech.scale script, for a disk systems with three unitary disks.

**Remark 16.** All our timings were done on a 64-bit GNU/Linux machine with two Intel Xeon processors (3.40 GHz), although our script were not threaded and only one core was used per process. We measured all the timings with `clock()` from the Standard C library. The average times in both graphics (Figure 7 and Figure 10) are the mean times for $10^4$ repetitions of each algorithm, for every number of disks multiple of 10 in Figure 7 from 10 to 500, and for every dimension multiple of 200 in Figure 10 from 200 to 10000.

**Acknowledgements**

The first author acknowledges the financial support of PRODEP and of the Universidad de Sonora, as well as the ACARUS (High Performance Computing Area).

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