A strong version of implicit function theorem

Genrich Belitskii and Dmitry Kerner

Abstract. Consider the system of equations \( F(x, y) = 0 \). The classical Implicit Function Theorem starts from the assumption: "the derivative \( F'_x(0) \) is nondegenerate/right-invertible". This condition is far from necessary. It has been weakened by J.C. Tougeron (Tougeron’s implicit function theorem) and then further by B. Fischer. Another direction was to extend the theorem to a more general class of rings, beyond the rings of analytic/smooth functions.

We obtain the "weakest possible" condition that ensures the solvability of \( F(x, y) = 0 \). The condition is in terms of some suitable filtration, it is necessary and sufficient for the existence of "good" (differentiable) solutions. In the simplest cases this filtration condition reproduces Tougeron’s/Fisher’s theorems. For more delicate filtrations we get significant strengthenings.

This gives a version of the strong Implicit Function Theorem in the category of filtered groups. In particular, we get the solvability for a broad class of (commutative, associative) rings.

Finally, we prove the Artin-type approximation theorem: if a system of \( C^\infty \) equations has a formal solution and the derivative \( F'_x(0) \) satisfies a Lojasiewicz-type condition then the system has a \( C^\infty \)-solution.

1. Introduction

All the rings in this paper are commutative and associative. We use the multivariable notation, \( \underline{x} = (x_1, \ldots, x_m) \), \( \underline{y} = (y_1, \ldots, y_n) \).

Consider a system of (analytic/formal/\( C^\infty /C^k \)) equations \( F(x, y) = 0 \). The classical Implicit Function Theorem reads: if the matrix of derivatives \( \partial_i F(0, 0) = F'_x(0, 0) \) is right invertible (i.e. is of the full rank) then \( F(\underline{x}, \underline{y}) = 0 \), has a (analytic/formal/etc.) solution. The condition "\( F'_x(0, 0) \) is right invertible" is quite restrictive. For example, the theorem does not ensure the solution of the equation \( xy = 0 \) (in the vicinity of \((0, 0)\)) or \( y^2 = 0 \) (at any point).

Various strengthenings/generalizations of the theorem have been proved (including Hensel lemma). For example, Tougeron’s implicit function theorem:

**Theorem 1.1.** [Tougeron-book, pg.56],[Tougeron1968] Let \( R = k[[\underline{x}, \underline{y}]] \) or \( k\{\underline{x}, \underline{y}\} \) (for \( k\)-normed) or \( C^\infty(\mathbb{R}^m \times \mathbb{R}^n, 0) \). Let \( F(\underline{x}, \underline{y}) \in R^{\oplus p} \), with \( p \leq n \), let \( I \subset R \) be a proper ideal. If \( F(\underline{x}, 0) \in I(I_{\text{max}} F'_x(\underline{x}, 0))^2 R^{\oplus p} \) then there exists a solution, \( F(\underline{x}, y(\underline{x})) \equiv 0 \), satisfying: \( y(\underline{x}) \in IR^{\oplus n} \).

Here \( I_{\text{max}}(\underline{x}) \) is the ideal of the maximal minors of the matrix.

While this theorem ensures the solution of \( yx = 0 \) and \( y^2 = 0 \), it fails to ensure the solution of the system

\[
\begin{align*}
\left\{ \begin{array}{l}
y_1^2 + y_1x = x^3 \\
y_2^2 + y_2x = x^3
\end{array} \right.
\end{align*}
\]

Here \( F(x, 0) \equiv x^3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( I_{\text{max}} \left( F'_x(\underline{x}, 0) \right) = (x^2) \), thus \( F(\underline{x}, 0) \notin I_{\text{max}} \left( F'_x(\underline{x}, 0) \right)^2 \).

Tougeron himself realized in [Tougeron1966] that one can replace in the condition \( F(\underline{x}, 0) \in I(I_{\text{max}} F'_x(\underline{x}, 0))^2 R^{\oplus p} \) the ideal \( I_{\text{max}} F'_x(\underline{x}, 0) \) by the bigger ideal \( A F'_x(\underline{x}, 0) := \text{Ann} \left( \text{Coker}(F'_x(\underline{x}, 0)) \right) \), the annihilator of the cokernel of the map \( R^{\oplus n} \rightarrow F'_x(\underline{x}, 0) R^{\oplus p} \). Some properties of this ideal are given in \$2.3$. Here we just mention that for \( p = 1 \), i.e. the case of one equation, the two ideals coincide: \( A F'_x(\underline{x}, 0) = I_{\text{max}} F'_x(\underline{x}, 0) \).

The statement was further strengthened by B. Fisher, by replacing one of the factors in \( \text{Ann} \left( \text{Coker}(F'_x(\underline{x}, 0)) \right)^2 \) by the image \( \text{Im}(F'_x(\underline{x}, 0)) \subset R^{\oplus p} \). (The initial version was for \( p \)-adic rings, we give a more general version relevant for our context.)

**Theorem 1.2.** [Fisher1997] Let \( (R, \mathfrak{m}) \) be a local Henselian ring over a field of zero characteristic. Let \( F_1, \ldots, F_p \in R[[y_1, \ldots, y_p]] \). Suppose \( F(\underline{x}, 0) \in \mathfrak{m} A F'_x \text{Im}(F'_x(\underline{x}, 0)) \). Then there exists a solution, \( F(\underline{x}, y(\underline{x})) \equiv 0 \), such that \( y(\underline{x}) \in \mathfrak{m} \text{Ann}(\text{Coker}(F'_x(\underline{x}, 0))) R^{\oplus p} \).
In the case of one equation, \( p = 1 \), this coincides with Tougeron’s result. For \( p > 1 \) Fisher’s result is stronger. (Note that \( A_p = \text{Im}(F_p') \supseteq A_p^{\omega} \otimes \mathbb{R}^p \), and for \( p > 1 \) the inclusion is almost always proper.)

Though this version solves the examples above, it cannot cope with the slightly more complicated example:

\[
\begin{align*}
y^2 - y_1^2 - 2y_1y_3 + y_2^2 + y_3 + y_4 + p(x_1, x_2) &= 0, \quad \text{where } p(x_1, x_2) \in \mathbb{R}^{k+1}, \quad m = (x_1, x_2) \in R = k[[x_1, x_2]].
\end{align*}
\]

Here \( m A_p^\omega \text{Im}(F_p'(x_1, 0)) = m (x_1, x_2) \), thus in general \( F_p'(x_1, 0) = p(x_1, x_2) \notin m A_p^\omega \text{Im}(F_p'(x_1, 0)) \).

Our work has been based on the observation that Fisher’s condition can be further weakened: instead of \( F_p'(x_1, 0) \in m A_p^\omega \text{Im}(F_p'(x_1, 0)) \) it is enough to ask for \( F_p'(x_1, 0) \in m J \text{Im}(F_p'(x_1, 0)) \), where \( J \subset R \) is the maximal possible ideal satisfying \( J^2 = J A_p^\omega \). (See example 3.8 and §4.1 for more detail.) This gives the further strengthening of Fisher’s/Tougeron’s statements. Still, this strengthening does not help to address the very simple system

\[
\begin{align*}
y^1 &+ y_1 x_1 = x_1^{100} \\
y^2 &+ y_2 x_2 = x_2^{100} 
\end{align*}
\]

where \( A_p^\omega = (x_1 x_2) \) thus \( J = (x_1 x_2) \), but \( F_p'(x_1, 0) \notin J \text{Im}(F_p'(x_1, 0)) \).

The goals of this note are:

• To weaken the condition further (to the "weakest possible" condition of "iff" type) so that we get a Strong Implicit Function Theorem.
• To extend the result to a broader category. It is natural to extend from \( k[[x, y]] \), \( k\{x, y\}, C_p(\mathbb{R}^n, \mathbb{R}^m, 0) \) to the local Henselian rings (not necessarily regular or Noetherian) over a field. In fact even the ring structure is not necessary, our main result is for the filtered (not necessarily abelian) groups, cf. theorems 3.1 and 3.2.

In the subsequent works we hope to apply this strong form to various problems of Algebra and Geometry (finite determinacy [du Plessis-Wall], tactile maps [Brusche-Hauser] etc.).

Remark 1.3. 1. The classical approach to construct a solution is the order-by-order approximation: first solve the linear part (modulo quadratic terms), then quadratic, cubic et c. Accordingly we always present the equation in the form \( u + L y + H(y) = 0 \). Here:

\[
\begin{align*}
u &= F_p'(x_1, 0) \in V \text{ is an element of an } R\text{-module (or just of an abelian group);} \\
V &\overset{v}{\longrightarrow} W \text{ is a homomorphism of } R\text{-modules (or of abelian groups);} \\
H(y) &= \text{the remaining } "\text{higher order terms}".
\end{align*}
\]

Further, as we always start from a solution of the linear part, \( u + L y = 0 \), we assume \( u \in L(V) \), i.e. \( u = -L v \), for some \( v \in V \).

2. If the equations \( F_p(x, y) = 0 \) are linear in \( y \), i.e. \( F_p(x, y) = F_p'(x_1, 0) + F_p'(x_1, 0) y_1 \), then the obvious sufficient condition for solvability is: the entries of \( F_p'(x_1, 0) \) lie in \( A_p\text{Im}(F_p'(x_1, 0)) \). While the (tautological) necessary and sufficient condition is: \( F_p'(x_1, 0) \subseteq \text{Im}(F_p'(x_1, 0)) \). This condition is much weaker than those of Tougeron/Fisher. Therefore as landmarks for the criteria one should consider equations that are non-linear in \( y \).

3. In practice one usually needs not just a solution. One needs a statement of the type

\[
\begin{align*}
\text{there exists a subgroup/submodule } V_1 \subset V \text{ such that } \forall v \in V_1 \text{ the equation } L(y - v) + H(y) = 0 \text{ has a solution, } y, \in V_1 \text{ which is "close" to } v \text{ and depends on } v \text{ "smoothly".}
\end{align*}
\]

We call this a smooth solution, the precise formulation is in §2.1.

4. In our approach we expand \( F_p(x, y) = 0 \) in powers of \( y \) (i.e. at \( y = 0 \)), hence the criteria are formulated in terms of \( F_p'(x_1, 0), F_p'(x_1, 0) y_1 \) etc. One could expand at some \( y = y^{(0)}(x) \), then the criteria are written in terms of \( F_p'(x_1, y^{(0)}(x)), F_p'(x_1, y^{(0)}(x)) y_1, \ldots \). (For example, theorem 1.2 is stated in [Fisher1997] in such a form.) Such an expansion at \( y^{(0)}(x) \) is helpful if one has a good initial approximation for the solution. As the two approaches are obviously equivalent, e.g. by changing the variable \( y \rightarrow y - y^{(0)}(x) \), we prefer to expand at \( y = 0 \), to avoid cumbersome formulas.

5. Usually the main problem is to establish the order-by-order solution procedure. Thus many of our results are of the form "If (…) then there exists a Cauchy sequence \( (y^{(n)}) \), such that \( L(y^{(n)} - v) + H(y^{(n)}) \rightarrow 0 \)." The topology here comes from filtration, e.g. the \( m\)-adic topology on \( R \).

Once such a result is established one has an honest solution in the completion/henselization of \( V \). Then (if \( V \) is non-complete) one uses the Artin-type approximation theorems [K.P.R.M.] to establish a solution in \( V \). Over some rings we can directly ensure a solution, see §3.2.

6. The geometric meaning. The equations \( F_p(x, y) = 0 \) over \( R = R_X \otimes R_Y \) define the (germ of the) locus \( V(F) \subset \text{Spec}(R_X) \times \text{Spec}(R_Y) \) with the natural projections into \( \text{Spec}(R_X), \text{Spec}(R_Y) \). The solvability of the equation means that this locus has a smooth component that projects isomorphically onto the \( \text{Spec}(R) \times \{0\} \subset \text{Spec}(R) \).
Today the factorization of ideals into prime ideals is realized by many computer packages. But this is done only in Noetherian rings (e.g. \( \mathbb{k}[x, y]/\mathbb{k}\langle x, y \rangle \)). So, this does not help for the questions in e.g. \( C^\infty, C^k \) rings.

7. A reformulation in terms of commutative algebra. Suppose a ring \( R \) is "built" from two ingredients, \( R_X \) and \( R_Y \), e.g. \( R = R_X \otimes R_Y \), with a prescribed inclusion \( R_X \hookrightarrow R \). Given an ideal \( F = (F_1, \ldots, F_p) \subset R \), a solution of

\[
F(x, y) = 0 \text{ is a projection } R^{\frac{y-\gamma(x)}{\gamma}} \rightarrow R_X \text{ whose kernel is precisely } F.
\]

2. Definitions, Notations, Preliminaries

2.1. Filtered groups. A decreasing filtration \( V_n \) of a group \( V \) is the sequence of normal subgroups \( V \supset V_1 \supset V_2 \supset \cdots \). The filtration is called faithful if \( V_n \) if \( \cap_{i \geq 1} V_i = \{0\} \). The filtration induces Krull topology, the fundamental system of neighborhoods of \( v \in V \) is \( \{vV_j\}_{j \geq 1} \).

Example 2.1. 1. Let \((R, m)\) be a local ring, consider the group of invertible matrices over \( R: GL(n, R) \). Corresponding to the filtration \( R \supset m \supset m^2 \supset \cdots \) we have the filtration by the normal subgroups \( V_j := \{I + A| A \in Mat(n, m)\} \).

2. Let \((X, 0)\) be the germ of a space (algebraic/formal/analytic etc). Consider its automorphisms, \( Aut(X, 0) \). The natural filtration is by the subgroups of automorphisms that are identity up to \( j\)th order.

3. The classical case is when \( V \) is abelian, a module over a ring, with filtration defined by some ideal, \( V_j = PV \).

Given two groups, \( V, W \), we consider the implicit function equation, \( L(y)H(y) = L(v) \), where

\[
* L \text{ is a homomorphism, } V \xrightarrow{L} W;
* H : V \rightarrow W \text{ is a "higher order" map, usually not a homomorphism, satisfying } H(1_v) = 1_w. \text{ We will specify the precise conditions of being of higher order later.}

Given a decreasing filtration \( V \) on \( V \) we define the filtration \( V_j := L(V_j) \) on \( W \). An order-by-order solution of the equation \( L(y)H(y) = L(v) \) is a Cauchy sequence, \( \{y^{(n)}\}_{n \geq 1} \) for \( V \), i.e. \( y^{(n)}(y^{(n+1)})^{-1} \in V_n \), such that \( L(y^{(n)}H(y^{(n)})L(v)^{-1})_{n \geq 1} \) is a Cauchy sequence for \( L(V) \).

Let \( V \) be a decreasing filtration of \( V \) by normal subgroups. We say that the equation \( L(y)H(y) = L(v) \) admits a smooth solution on \( V_1 \) if there exists a map \( V_1 \xrightarrow{L} V \) satisfying:

\[
* L(y_v)H(y_v) = L(v) \text{ for any } v \in V_1;
* y_1v = 1_w
\]

* the map \( y_v \) is "differentiable" at any \( v \in V_1 \). Namely for any \( \Delta_j \in V_j: y_v\Delta_jy_v^{-1}\Delta_j^{-1} \in V_{j+1} \). By the normality, \( V_{j+1} \supset V_j \), this condition can be written in the form \( y_v^{-1}\Delta_j^{-1}y_v\Delta_j \in V_{j+1} \). (If in the last condition we put \( v = 1_v \) then, using the second condition, we get that the map \( y_v \) is "close" to identity: for \( \Delta \in V_1, y_v\Delta\Delta^{-1} \in V_2 \).

Combining these notions we get the notion of smooth order-by-order solution: \( V_1 \xrightarrow{y^{(n)}} V \) with the properties as above.

If \( V, W \) are abelian groups then all the notions simplify accordingly. An order-by-order smooth solution means a Cauchy sequence:

\[
y^{(n)}_v - y^{(n+1)}_v \in V_n; \quad \forall j \geq 1, \forall \Delta_j \in V_j: y^{(n)}_v + \Delta_j - y^{(n)}_v - \Delta_j \in V_{j+1}; \quad L(y^{(n)}_v - v) + H(y^{(n)}_v) \in L(V_n).
\]

2.2. Rings. All the rings in this paper are commutative, associative with unit element. If \( V, W \) are modules over a ring \( R \), then the implicit function equation is \( L(y)H(y) = L(v) \), where \( L \) is a homomorphism of \( R \)-modules, while the higher order \( H(y) \) satisfies: for any ideal \( J \subset R: H(JV) \subset J^2W \). We say that the map \( V \xrightarrow{L} W \) is of order \( \geq k \) if for any ideal \( J \subset R: H(JV) \subset J^kW \). In particular, \( H \) is of higher order iff its order is at least two.

Fix some ideal \( J \subset R \). The pair \((R, J)\) is said to satisfy the (classical) implicit function theorem, denote this by \( IFT_J \), if for any surjective morphism of free \( R \)-modules of finite rank, \( V \xrightarrow{L} W \), any \( v \in JV \) and any "higher order term" \( V \xrightarrow{L} W \) the equation \( L(y) - L(v) + H(y) = 0 \) has a solution. Note that if \( R \) satisfies \( IFT_J \) then for any ideal \( J \subset R \) the ring \( J \) satisfies \( IFT_J \) as well.

Example 2.2. Let \((R, m)\) be any local Henselian ring over a field \( k \). For example, \( R = k[[x_1, \ldots, x_m]]/I \), \( R = k[x_1, \ldots, x_m]/\langle I \rangle \) (for \( k \)-normed), \( R = C^\infty(R^m, 0)/I \), or \( R = C^p(R^m, 0)/I \). Then \((R, m)\) satisfies the \( I.F.T. \).

The rings \( k[x], k[x]/I \) do not satisfy \( I.F.T. \), e.g. the equation \( y^2 + y = x^2 \) is not solvable over these rings.

We say that \((R, J)\) satisfies implicit function theorem with unit linear part, denote this by \( IFT_{J,1} \) if for the system of equations, \( y - v + H(y) = 0 \) has a solution over \( R \) (for any \( v \in JV \)).

This system is a particular case of the classical implicit function equations. Therefore the Henselian rings of the example above satisfy \( IFT_{J,1} \). Note that the condition \( IFT_{J,1} \) is weaker than \( IFT_J \). For example, \( IFT_{J,1} \) is satisfied by \( \mathbb{Z}[\mathbb{Z}] / \langle I \rangle \), \( \mathbb{Z}[\mathbb{Z}] / \langle J \rangle \), for \( J = (x) \).
2.3. Annihilator of cokernel. Consider a homomorphism of finitely generated $R$-modules, $V \xrightarrow{L} W$. Its image, $\text{Im}(L)$, is an $R$-submodule of $W$. Its cokernel, $W/\text{Im}(L)$, is an $R$-module as well. The annihilator-of-cokernel is defined as:

\[(4) \quad A_L := \text{ann}(\text{coker}(L)) := \{f \in R : fW/\text{Im}(L) = \{0\}, \text{ i.e. } fW \subseteq \text{Im}(L)\} \]

Recall the classical relation [Eisenbud, Proposition 20.7]: for $L \in \text{Mat}(m, n; R)$ with $m \leq n$, $A_L \supseteq I_m(L) \supseteq (A_L)^m$. (Here the maximal minors come from $m \times m$ blocks, i.e. $I_{\text{max}}(L) = I_m(L)$.) In particular, for $m = 1$: $A_L = I_1(L)$, the ideal generated by all the entries of $L$.

By definition $A_L W \subseteq L(V)$. In many cases one has the stronger property: $A_L W \subseteq mL(V)$.

**Lemma 2.3.** If $L \in \text{Mat}(m, n; m)$, $1 < n \leq m$ and $I_{\text{max}}(L)$ is radical then $A_L W \subseteq m^{m-1}L(V)$.

In general $A_L W \not\subseteq mL(V)$. For example, let $L = \begin{pmatrix} f & 0 \\ 0 & L_1 \end{pmatrix}$, where $\det(L_1) = f$. Then $A_L = (f)$ and $(f)W \not\subseteq mL(V)$.

2.4. Ideals that satisfy $J^n \subseteq J A_L$. Fix some $n \in \mathbb{N}$ and consider the set $\mathfrak{J}_n$ of all the ideals satisfying $J^n \subseteq J A_L$. This is an inductive set, i.e. for any increasing sequence, $J_1 \subseteq J_2 \subseteq \cdots$ the union $\bigcup_k J_k$ is an ideal that satisfies $J^n \subseteq J A_L$. (If $f, g \subseteq \bigcup_k J_k$ then $f, g \subseteq J_k$ for some $k < \infty$, thus $fg \subseteq J_k$.) Therefore in $\mathfrak{J}_n$ there exist(s) ideal(s) that are/is maximal by inclusion.

**Lemma 2.4.** Let $n = 2$ and let $J \subset R$ be such a (maximal by inclusion) ideal.

1. If $J$ is finitely generated then $J \subseteq \overline{A_L}$. (Here $\overline{A_L}$ is the integral closure.)

2. If $A_L$ is radical then $J = A_L$. If $R$ is integrally closed and $A_L$ is principal, generated by a non-zero divisor, then $J = A_L$.

3. In many cases $A_L \subseteq J \subseteq \overline{A_L}$ and a maximal by inclusion ideal $J$ is non-unique.

**Proof.**

1. If $J^2 \subseteq JA_L$, then obviously the inclusion is satisfied by $J + A_L$ as well. Thus $A_L \subseteq J$.

For the second part, note that $A_L$ is a reduction of $J$, see [Huneke-Swanson, Definition 1.2.1], thus $J \subseteq \overline{A_L}$ by [Huneke-Swanson, Corollary 1.2.5].

2. If $J^2 \subseteq JA_L$ then in particular $J^2 \subseteq A_L$. Then, $A_L$ being radical, we get $J \subseteq A_L$.

The second part follows from [Huneke-Swanson, Proposition 1.5.2]; in our case $\overline{A_L} = A_L$.

3. Let $R = \mathbb{K}[x, y, z]$ and $L = (x^p, y^p, z^p)$ in $\text{Mat}(1, 3; R)$. Then $A_L = (x^p, y^p, z^p)$ while $\overline{A_L} = \mathbb{K}^p$. Define: $J_1 = ((x, y)^p, z^p)$, $J_y = ((x, z)^p, y^p)$, $J_z = ((y, z)^p, x^p)$. By direct check, each of them satisfies $J^2 = J A_L$. But there is no bigger ideal $J$ that contains say $J_x + J_y$ and satisfies $J^2 = J A_L$. Indeed, suppose $y^{p-1}z \in J$ and $x^{p-1}z^2 \in J$, for some $i, j$ satisfying $i + j < p$. Then $J A_L = J^2 \ni x^{p-j}y^{p-i}z^{i+j}$. Contradicting $x^{p-j}y^{p-i}z^{i+j} \not\in A_L$, as $i + j < p$. Thus, in this case there are several biggest ideals. ■

3. The results

3.1. Criteria for order-by-order solvability. Our main contribution is the following general criterion of order-by-order solvability of the equation for arbitrary filtered groups.

Consider a homomorphism of (arbitrary) groups $V \xrightarrow{L} W$. Let $V \supseteq V_1 \supseteq V_2 \supseteq \cdots$ be a sequence of normal sub-groups. Consider the equation $(L y)H(y) = L(v)$, here $H : V \to W$ is a map representing "higher order terms".

**Theorem 3.1.** Assume: $H(1) = 1$ and $(H(y))^{-1}H(yV_j) \subseteq L(V_{j+1})$, for $j = 1, 2, \ldots$ and all $y \in V_1$.

1. For every $v \in V_1$ there exists a smooth order-by-order solution (see §2.1).

2. If $V$ is complete with respect to $V_1$ then there exists a smooth solution of $(L y)H(y) = L(v)$.

**Proof.**

1. The proof is by induction. Choose $y^{(1)} = 1$ and $y^{(2)} = v$. Suppose $y^{(2)}, \ldots, y^{(j)}$ have been constructed for some $j \geq 2$. Present $y^{(j+1)} = z y^{(j)}$, so we should find the necessary $z \in V_j$. Note:

\[(5) \quad L(y^{(j+1)})H(y^{(j+1)})L(v^{-1}) = L(z)L(y^{(j)})H(y^{(j)})H(y^{(j)})^{-1}H(z y^{(j)})L(v^{-1}) =
\]

\[= \left( L(z)L(y^{(j)})H(y^{(j)})L(v^{-1}) \right) \left( L(v)H(y^{(j)})^{-1}H(z y^{(j)})L(v^{-1}) \right) \]

As $w = L(y^{(j)})H(y^{(j)})L(v^{-1}) \in L(V_j)$ there exists $z \in V_j$ satisfying $L(z) = w^{-1}$. Then equation (5) reads

\[(6) \quad L(y^{(j+1)})H(y^{(j+1)})L(v^{-1}) = L(v)H(y^{(j)})^{-1}H(z y^{(j)})L(v^{-1}) \in L(V_{j+1}). \]

Here we use the normality $V_{j+1} \triangleleft V_j$. By construction this (order-by-order) solution is smooth.
2. Given the Cauchy sequence \( y^{(n)} \) from the main theorem take the limit \( y = \lim_{n \to \infty} y^{(n)} \). Then one has
\[
\lim_{n \to \infty} L(y^{(n)})H(y^{(n)}) \in \cap L(V_j).
\]

Usually one needs results of such type for abelian groups, i.e. one solves the equation \( L(y-v) + H(y) = 0 \). We state the corresponding criterion separately.

**Corollary 3.2.** 1. Given abelian groups \( V, W \) and a homomorphism \( V \xrightarrow{f} W \). Suppose there exists a decreasing filtration \( V_{\cdot} \) of \( V \) satisfying: \( \forall y \in V_1, \forall j \geq 1: H(y + V_j) - H(y) \in L(V_{j+1}). \) Then for any \( v \in V_1 \) there exists a smooth order-by-order solution.
2. If \( V \) is complete with respect to \( V_{\cdot} \), then the conditions \( H(y + V_j) - H(y) \in L(V_{j+1}) \) imply a smooth solution of the equation \( L(y-v) + H(y) = 0 \).

The natural question is whether this theorem/corollary can be further improved, i.e. whether the assumptions on \( L, H \) can be weakened. We prove that the assumptions are the "weakest possible".

**Proposition 3.3.** Given a filtration \( V_{\cdot} \) of \( V \), suppose there exists a smooth order-by-order solution \( V_1 \xrightarrow{y^{(n)}} V_1 \) of \( L(y)V(y) = L(v) \). Suppose either \( V \) is complete or the maps \( y^{(n)} \) have open images for \( n \gg 1 \). Then for any \( v \in V_1 \) and any \( n > j > 1 \):
\[
H(y^{(n)})^{-1}H(y^{(n)}V_j) \in L(V_{j+1}).
\]

**Proof.** First we prove that for any \( v \in V_1, \Delta_j \in V_j: \Delta_j y^{(n)} = y^{(n)}_{\Delta_j v} \) for some \( \Delta_j \in V_j \). We construct \( v\tilde{\Delta}_j \) by approximations. Note that \( y^{(n)}_{\Delta_j v}(y^{(n)})^{-1} \Delta_j^{-1} = \Delta_j^{-1} + V_{j+1} \). Then \( y^{(n)}_{\Delta_j v}(y^{(n)})^{-1} \Delta_j^{-1} = \Delta_j^{-1} + V_{j+2} \) and so on. If \( V_1 \) is complete then the Cauchy sequence \( y^{(n)}_{\Delta_j v} \) converges to some \( y^{(n)}_{\Delta_j v} \) with the needed property.

If the image of \( y^{(n)} \) is open then for \( n \gg 1 \) we get \( \Delta_j y^{(n)} = y^{(n)}_{\Delta_j v} = y^{(n)}_{\Delta_j v} \) as well.

Now we use the property that \( y^{(n)}_{\Delta_j v} \) is an order-by-order solution:
\[
(7) \quad H(y^{(n)})^{-1}H(\Delta_j y^{(n)}_{\Delta_j v}) = H(y^{(n)})^{-1}H(y^{(n)}_{\Delta_j v}) = L((y^{(n)}_{\Delta_j v})^{-1}v^{-1})L((y^{(n)}_{\Delta_j v}^{-1})^{-1}v)(V_n) = L(v^{-1}y^{(n)}_{\Delta_j v}^{-1}v)(V_n)
\]

As \( y \) is smooth, \( y^{(n)}_{\Delta_j v} = \Delta_j v \Delta_j y^{(n)}_{\Delta_j v} \). Substitute this to get:
\[
(8) \quad H(y^{(n)}_{\Delta_j v})H(y^{(n)}_{\Delta_j v})^{-1}L(v^{-1}y^{(n)}_{\Delta_j v}^{-1}v)(V_n) \subset L(V_{j+1}).
\]

Here in the last equality we use the normality of \( V_{j+1} \).

In general it is not easy to find the appropriate filtration \( V_{\cdot} \). Our criterion simplifies for modules over a ring.

**Corollary 3.4.** Let \( R \) be a (commutative, associative) ring. Fix some ideal \( I \subset R \) and a submodule \( V_1 \subset V \).
1. If \( H(y + \Delta_j) - H(y) \in JL(V_1) \) for \( y \in V_1 \) and \( \Delta_j \in I^jV_1 \), then for any \( v \in V_1 \) there exists a smooth order-by-order solution of \( L(y) - H(y) = 0 \).
2. Suppose \( V \) is complete with respect to the filtration \( \{V_j = I^{j-1}V_1 \} \). If for any \( v \in V_1 \) there exists a smooth order-by-order solution of \( L(y) + H(y) = 0 \) then \( (y + \Delta_j) - H(y) \in J^jL(V_1) \).

**Example 3.5.** The most important case is when the ring \( R \) is over a base ring \( k \), which has no zero divisors, e.g. \( k \) is a field. Suppose further that the term \( H(y) \) admits the "linear approximation" with the remainder in the form of Lagrange:
\[
(9) \quad H(y + \Delta) = H(y) + H_1(y)(\Delta) + H_2(y, \Delta)(\Delta, \Delta),
\]

here \( H_1(y)(z) \) is linear in \( z \) while \( H_2(y, \Delta)(z, z) \) is quadratic in \( z \). Suppose \( H(V_1) \subset JL(V_1) \), then we get \( tH_1(y)(\Delta) + t^2H_2(y, t\Delta)(\Delta, \Delta) \in JL(V_1) \), for \( t \in k \). Divide by \( t \) and put \( t = 0 \) in the remaining expression to get \( H_1(y)(\Delta) \in JL(V_1) \). Then \( H_2(y, t\Delta)(\Delta, \Delta) \in JL(V_1) \). Thus \( H_1(y)(\Delta, \Delta) \in J^{k+1}L(V_1) \) and \( H_2(y, t\Delta)(\Delta, \Delta) \in J^{k+2}L(V_1) \), i.e. Thus \( H(y + \Delta) - H(y) \in J^kL(V_1) \) is implied by \( H(V_1) \subset JL(V_1) \). Thus, in this case the condition \( H(V_1) \subset JL(V_1) \) ensures the solvability. Vice versa, the existence of a smooth solution implies \( H(y + \Delta) - H(y) \in J^kL(V_1) \) and hence \( H(V_1) \subset JL(V_1) \).

Thus, for modules over the rings the question is reduced to the search for an appropriate submodule \( V_1 \subset V \). The simplest type of submodule is \( V_1 = JV \), for some ideal \( J \subset R \). More generally, suppose \( I_1V \supseteq V_1 \supseteq I_2V \), where \( I_1 \) is the minimal possible, \( I_2 \) is the maximal possible.

**Corollary 3.6.** Let \( (R, m, k) \) be a local ring over a field \( k \) of zero characteristic. Suppose \( H(y) \) as in the last example, in particular for \( J \subset R: H(JV) \subseteq J^2W \).
1. If \( I_1^2 W \subseteq m I_2 L(V) \) then there exists a smooth order-by-order solution.
2. If \( V_1 = JV \) and for all \( v \in V_1 \) there exists a smooth order-by-order solution then \( J^2 W \supseteq m J L(V) \).

**Proof.**
1. We should check \( H(y + \Delta) - H(y) \in L(m V_1) \). Indeed, for \( y, \Delta \in J_1 V \) one has: \( H(y + \Delta), H(y) \in I_1^2 W \subseteq m I_2 L(V) \subseteq m L(V_1) \).
2. By the assumption there exists a solution for any choice of \( H \). In particular, choose \( H(y) = q(y) \xi \), where \( q(y) \) is a quadratic polynomial in \( y \), while \( \xi \in W \) is a generic element. By proposition 3.3 we have:

\[
L(m V_1) \ni H(y + t \Delta) - H(y) = t \xi (q(y + t \Delta) - q(y)).
\]

As this holds for any \( t \in k \) we get: \( (y \Delta_i) \xi \in J L(V), \) for any \( i, j \). By choosing \( y_i, \Delta_j \) as all the generators of \( J \) and \( \xi \) as all the generators of \( W \) we get: \( J^2 W \subseteq L(m J V) \).

In many question it is useful to reformulate the conditions \( J^2 W \subseteq m J L(V), I_1^2 W \subseteq m I_2 L(V) \) in terms of the annihilator of cokernel, \( A_L \). Thus we get:

**Corollary 3.7.**
1. If \( I_1^2 \subseteq m I_2 A \) then there is an order-by-order solution.
2. Suppose \( A_L W \subseteq m L(E) \). If \( I_1^2 \subseteq A_L I_2 \) then there is an order-by-order solution.

The last corollaries give the optimal criterion for submodules of type \( V_1 = JV \), in the case of arbitrary higher order terms. With some restrictions on \( H(y) \) the conditions on \( J \) can be weakened:

**Example 3.8.** Suppose \( \text{ord}(H(y)) \geq k \) i.e. \( \forall I \subset R: H(J V) \subset I^k W \). In most cases \( k \geq 2 \). If in addition \( J^k W \subseteq m J L(V) \) then for any \( v \in J V \) there exists an order-by-order solution. (Indeed: \( H(J V) \subset J^k W \subset m J L(V) \), now use example 3.5.)

In the lowest order case, \( k = 2 \), we get a sufficient condition for order-by-order solvability: \( J^2 W \subset m J L(V) \). It is weaker than Tougeron’s/Fisher’s conditions.

**Remark 3.9.** In the classical case of the equation \( F(\underline{x}, y) = 0 \) one asks that the map \( F_y'(\underline{x}, 0) : V \to W \) is right invertible, i.e. surjective. Our criterion asks that \( F_y'(\underline{x}, 0)(m V_1) \) contains the image of \( V_1 \) under the higher order terms \( F(\underline{x}, y) - F(\underline{x}, 0) - F_y'(\underline{x}, 0) y \).

### 3.2. **Criterion for exact solutions.** Recall the Artin approximation property: if a finite system of polynomial equations over \( R \) has a solution over \( \hat{R} \) then it has a solution over \( R \). Many rings have this approximation property, for example excellent Henselian rings (in particular complete rings, analytic rings).

In our case we have more general rings and more general class of equations. Thus we give a criterion for exact (and not just order-by-order) solution.

**Proposition 3.10.** Let \( R \) be a ring with some ideal \( m \subset R \), suppose \( IFT_{m, 1} \) holds for \( (R, m) \). Let \( V, W \) be some \( R \)-modules of finite ranks, let \( V_1 \subset V \) be a submodule. If \( H(V_1) \subset m L(V_1) \) then for any \( v \in V_1 \) the equation \( L(y - v) + H(y) = 0 \) has a solution (over \( R \)) which is smooth in \( v \).

**Proof.** Let \( v = \sum v_i \xi_i \), where \( \xi_i \in V_1, v_i \in R \). We look for a solution in the form \( y = \sum y_i \xi_i \). As \( H(y) \) represents the higher order terms and \( H(V_1) \subset m L(V_1) \) one has: \( H(\sum y_i \xi_i) = \sum h_i(y)L(\xi_i) \), where \( h_i(y) \) again represents some higher order terms. Thus we get \( \sum (y_i - v_i + h_i(y))L(\xi_i) = 0 \), i.e. we should solve the very particular system of equations: \( \{ y_i - v_i + h_i(y) = 0 \} \). By the assumption \( IFT_{m, 1} \) holds for \( (R, m) \), hence the statement.

**Corollary 3.11.** Suppose \( R \) satisfies \( IFT_{m, 1} \), consider the equation \( L(y - v) + H(y) = 0 \).
1. If \( J^2 W \subseteq m J L(V) \) then for any \( v \in J V \) there exists a (smooth) solution.
2. If \( J^2 \subseteq J A_L \) then for any \( v \in m J V \) there exists a (smooth) solution.
3. If \( J^2 \subseteq J A_L \) and \( A_L W \subseteq m L(V) \) then for any \( v \in J V \) there exists a (smooth) solution.

**Example 3.12.** Let \( (R, m) \) be a local Henselian ring over a field. Take \( J = A_L \), then the corollary implies Tougeron’s/Fisher’s theorems. As mentioned in the introduction, if one takes \( J \) the maximal possible that satisfies \( J^2 = J A_L \) then one gets the strengthening of Tougeron’s/Fisher’s theorems.

But the corollary is useful for more general rings, e.g. if in equation (2) the term \( p(x_1, x_2) \) has integral coefficients then we get a solution over \( \mathbb{Z}[[x_1, x_2]] \).

### 4. Remarks and Examples

#### 4.1. **Comparison to Fisher’s and Tougeron’s theorems.** The condition \( J^2 \subseteq J A_{F'_1(\xi, 0)} \) is a weakening of the condition \( J \subseteq A_{F'_1(\xi, 0)} \).
Example 4.1. Let $R = k[[x_1, x_2]]$, take $m = (x_1, x_2)$. Consider the equation $H(y, z) + y_1x_1^k + y_2x_2^k + p(z) = 0$, compare to equation (2). Here $H(y, z)$ presents the higher order terms, it is at least quadratic in $y_1, y_2$, while $p(z) \in k[[z]]$. In this case: $L = F^0(y, 0) = (x_1^k, x_2^k) \in Mat(1, 2; R)$ and $I_{max}(L) = A_L = (x_1^k, x_2^k) \subset R$. Thus $(A_L)^2 = (x_1^{2k}, x_1^kx_2^k, x_2^{2k})$. To apply Tougeron’s/Fisher’s theorems we have to assume: $p(x) \in m(a_1^k, x_1x_2, x_1^{2k}, x_2^{2k})$.

The other hand, by direct check, the ideal $m^k = (x_1, x_2)^k$ satisfies: $(m^k)^2 = (m^k)(x_1^k, x_2^k) = (m^k)A_L$. Therefore corollary 3.11 gives:

* if $k$ is algebraically closed and $p(x) \in m^{2k}$ then the equation has a solution;

* if $k$ is arbitrary and $p(x) \in m^{2k+1}$ then the equation has a solution.

Note that to write down an explicit solution is not a trivial task even in the particular case of equation (2).

Therefore, even in the case of just one equation, the condition $J^2 = JA_L$ significantly strengthens the versions of Tougeron/Fisher.

4.2. Comparison of the condition $J^2 = JA_{F^0(\xi, 0)}$ to $H(V_1) \subset mL(V_1)$. It is simpler to check the ideals, $J^2 = JA_{F^0(\xi, 0)}$, than to look for a submodule satisfying the needed property. But the “ideal-type” criterion is in general weaker than the criterion via $V_1$.

Example 4.2. Consider the system $\{ y_1^2 + y_1x_1 = x_1^n \quad y_2^2 + y_2x_2 = x_2^n \}$ over $R = [[x_1, x_2]]$. In this case the annihilator of cokernel ideal is principal, $A_{F^0(\xi, 0)} = (x_1x_2)$, thus $J^2 = JA_{F^0(\xi, 0)}$ implies $J = A_{F^0(\xi, 0)}$, see §2.3. And $(x_1x_2)$ does not contain $x_1^n, x_2^n$, regardless of how big are $n$ and $m$.

Of course, the general criterion of proposition 3.10 suffices here. (One starts from $V_1 = (x_1^kR, x_2^kR)$.)

This is a good place to see in a nutshell why no weakening of $J^2 = JA_{F^0(\xi, 0)}$ in the form of some condition on ideals is possible.

Example 4.3. Consider the related system with a modified quadratic part: $\{ y_1^2 + y_1x_1 = x_1^n \quad y_2^2 + y_2x_2 = x_2^n \}$. While the previous system has obvious solutions for $n, m \geq 2$, this system has no solutions in $R$. Indeed, from the second equation it follows that $y_1$ is divisible by $x_2$. Then the left hand side of the first equation must be divisible as well, contradicting the non-divisibility of the right hand side.

Example 4.4. As a further illustration we consider the system $\{ y_1^2 + y_1a_1 = b_1 \quad y_2^2 + y_2a_2 = b_2 \}$, where $a_1, b_1 \in R$, here $R$ is a local Henselian ring. Suppose $gcd(a_1, a_2) = 1$, i.e. $(a_1) \cap (a_2) = (a_1a_2)$. Then $A_L = (a_1a_2)$ is a radical ideal and thus $J^2 = JA_L$ implies $J = A_L$. Thus the approach via $J^2 = JA_L$ gives: if $b_1 \in (a_1)mA_L, b_2 \in (a_2)mA_L$, then the system has a solution.

We check the approach via filtration. To invoke the proposition 3.10 we need $V_1 \subset R^{\oplus 2}$ to satisfy: $\left( \begin{array}{c} v_2 \\ v_1 \end{array} \right) \in (a_1v_1, a_2v_2)$ for any $\left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) \in V_1$. Thus we get $V_1 \subset (a_1a_2R, (a_1)_1R)$ and further substitution gives $V_1 \subset (a_1a_2)^2R^{\oplus 2}$. This ensures $H(V_1) \subset mL(V_1)$, thus the condition on $b_1, b_2$ is the same as above.

Remark 4.5. Suppose the system of equations splits. Then it is natural to choose the split submodule: $V_1 = V_{1,1} \oplus V_{1,2} \subset V$. (Note that the converse does not hold: decomposability of $V_1$ does not imply that the system splits in any sense. For example, all modules of the type $V_1 = JV$ are decomposable if $V$ is of rank $> 1$.) The following questions are important:

* Suppose $L$ is block-diagonal. What are the conditions on $H$ so that we can choose $V_1 = V_{1,1} \oplus V_{1,2}$?

* Formulate some similar statements for $L$ upper-block-triangular vs $V_1$ an appropriate extension.

Remark 4.6. It is not clear how to formulate a condition ensuring the uniqueness of the solution of $L(y - v) + H(y) = 0$. If one seeks for a condition in terms of $v$ and $L$ only, then it is natural to ask: $v$ belongs to a small enough submodule of $V$. For example, $v \in JV$, for some small enough ideal $J \subset R$.

This does not suffice as one sees already in the example of one equation in one variable: $y - x^a(y + x^b) = 0$. Suppose $a < b$, then $A_L = (x^a)$, while $v \in (x^{a+b})$. By taking $b \gg a$ the later ideal can be made arbitrarily small compared to $A_L$. Yet, there is no uniqueness.

5. An approximation theorem

There are several approximation theorems guaranteeing analytical/smooth solutions, provided a formal solution exists. Given the germ of an analytic map at the origin, $F : (R^m, 0) \times (R^n, 0) \to (R^p, 0)$, consider the Implicit
Function Equation

\[ F(\underline{x}, y) = 0 \]

here \( \underline{x} \) is the multi-variable, while \( y \) is an unknown map, \( (\mathbb{R}^m, \underline{0}) \xrightarrow{\delta} (\mathbb{R}^n, \underline{0}) \).

A formal solution of this equation is a formal series \( \hat{y}(\underline{x}) \in \mathbb{R}[[\underline{x}]]^\oplus_m \) satisfying: \( \hat{F}(\underline{x}, \hat{y}(\underline{x})) = 0 \), where \( \hat{F} \) is the formal Taylor series at zero of the map \( F \). In general this solution does not converge off the origin. Two classical results relate it to the "ordinary" solution.

**Theorem 5.1.** Let \( \hat{y}(\underline{x}) \) be a formal solution of the analytic equation \( F(\underline{x}, y) = 0 \).

1. [Artin1968] For every \( r \in \mathbb{N} \) there exists an analytic solution whose \( r \)th jet coincides with the \( r \)th jet of \( \hat{y}(\underline{x}) \).
2. [Tougeron1976] There exists a \( C^\infty \)-solution \( y(\underline{x}) \), whose Taylor series at the origin is precisely \( \hat{y}(\underline{x}) \).

What if the equation \( F(\underline{x}, y) = 0 \) is not analytic but only \( C^\infty \)? Does the existence of a formal solution (for the completion \( \hat{F}(\underline{x}, y) = 0 \)) imply the existence of a smooth solution? The naive generalization of Artin’s/Tougeron’s theorems does not hold.

**Example 5.2.** Let \( m = n = p = 1 \), let \( \tau \) be a flat function, e.g., \( \tau = \begin{cases} e^{-\frac{1}{x}}, & x \neq 0 \\ 0, & x = 0 \end{cases} \). Consider the equation \( \tau^2(x)y(x) = \tau(x) \). The completion of this equation is the identity, \( 0 \equiv 0 \), thus every formal series \( \hat{y} \in \mathbb{R}[[\underline{x}]] \) is a formal solution (of \( \hat{F}(\underline{x}, y) = 0 \)). However, the equation has no local smooth solutions (not even continuous ones). In this example the coefficient of \( y(x) \), i.e. the function \( \tau^2 \), is flat at zero. In other words, the ideal \( \mathcal{A}_{F(\underline{x}, y)} \) is too small and \( \mathcal{A}_{F(\underline{x}, y)} \mathcal{m}^\infty \not\subset \mathcal{m}^\infty \).

The following statement is not a direct consequence of our previous results, but goes in the same spirit. Let \( R = C^\infty(\mathbb{R}^m, 0) \), with the maximal ideal \( \mathcal{m} \subset \mathcal{R} \). Suppose the equation \( F(\underline{x}, y) = 0 \) is a formal solution. By Borel’s lemma, [Rudin-book], we can choose a \( C^\infty \) map \( y_0 \) whose completion is the solution, thus \( F(\underline{x}, y_0) \) is a vector of flat functions.

**Theorem 5.3.** Suppose the equation \( F(\underline{x}, y) = 0 \) has a formal solution and \( \det \left( F_y'(\underline{x}, y_0)(F_y'(\underline{x}, y_0))^T \right) \mathcal{m}^\infty = \mathcal{m}^\infty \).

Then there exists a \( C^\infty \) solution, i.e. \( y \in \mathbb{R}^\oplus_m \) such that \( F(\underline{x}, y) = 0 \).

**Proof.** We seek for the solution in the form \( y = y_0 + \underline{z} \), where the map \( \underline{z} \) is flat. Expand \( F(\underline{x}, y_0 + \underline{z}) \) into the Taylor series with remainder:

\[ F(\underline{x}, y_0 + \underline{z}) = F(\underline{x}, y_0) + F_y'(\underline{x}, y_0)\underline{z} + \left( \int_0^1 (1 - t) \frac{\partial^2 F(\underline{x}, y_0 + tz)}{\partial y^2} dt \right)(\underline{z})^2. \]

Then the equation takes the form

\[ F_y'(\underline{x}, y_0)\underline{z} + G(\underline{x}, \underline{z}) = -F(\underline{x}, y_0) \]

where the map \( F_y(\underline{x}, y_0) \) is flat. Note that the summand \( G(\underline{x}, \lambda \underline{z}) = \lambda^2 h(\underline{x}, \lambda \underline{z}) \) with a \( C^\infty \)-map \( H \) such that \( H_\lambda'(\underline{x}, 0, \lambda) = 0 \). We look for the solution of equation (13) in the form

\[ z = d(\underline{x}) \left( F_y'(\underline{x}, y_0) \right)^T \left( F_y'(\underline{x}, y_0)(F_y'(\underline{x}, y_0))^T \right)^\vee u, \]

where \( d(\underline{x}) := \det \left( (F_y'(\underline{x}, y_0)(F_y'(\underline{x}, y_0))^T \right) \) and \( A^\vee \) denotes the adjugate matrix.

Then we arrive at the equation \( d^2(\underline{x})u + d^2(\underline{x})\hat{G}(\underline{x}, u) = -F(\underline{x}, y_0) \) with the \( C^\infty \)-map \( \hat{G} \) satisfying \( \hat{G}_\lambda'(\underline{x}, 0, \lambda) = 0 \).

Dividing by \( d^2(\underline{x}) \), we obtain the equation

\[ u + \hat{G}(\underline{x}, u) = \tau(\underline{x}), \]

where the map \( \tau \) is flat. By the classical Implicit Function Theorem, the latter equation has a local flat \( C^\infty \)-solution. Hence, the map \( z \) satisfies the equation (13), and \( y = y_0 + \underline{z} \) is the map we need. ■

**Remark 5.4.**

1. The assumption of the theorem can be stated as: "every flat function is divisible by \( \det \left( F_y'(\underline{x}, y_0)(F_y'(\underline{x}, y_0))^T \right) \)". Note that \( y_0 \) is defined up to a flat function, but the assumption does not depend on this choice.

2. Recall that a function \( g(\underline{x}) \) is said to satisfy the Lojasiewicz condition (at the origin) if there exist constants \( C > 0 \) and \( \delta > 0 \) such that for any point \( \underline{x} \in (\mathbb{R}^m, 0) \), \( |g(\underline{x})| \geq Cdist(\underline{x}, 0)^\delta \). As is proved in [Tougeron-book, §V.4]: \( g(\underline{x}) \) satisfies the Lojasiewicz condition at the origin iff \( g(\underline{x})\mathcal{m}^\infty = \mathcal{m}^\infty \). Thus the assumption of the theorem can be stated in the form:

\[ \det \left( F_y'(\underline{x}, y_0)F_y'(\underline{x}, y_0)^T \right) \geq Cdist(\underline{x}, 0)^\delta. \]
REFERENCES

[Artin1968] M.Artin, On the solutions of analytic equations. Invent. Math. 5 1968 277–291
[Artin1969] M.Artin, Algebraic approximation of structures over complete local rings. Inst. Hautes Études Sci. Publ. Math. No. 36 1969 23-58
[Bruschek-Hauser] C.Bruschek, H. Hauser, Arcs, cords, and felssix instances of the linearization principle. Amer. J. Math. 132 (2010), no. 4, 941-986.
[Eisenbud] D.Eisenbud, Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.
[Fisher1997] B.Fisher, A note on Hensel’s lemma in several variables. Proc. Amer. Math. Soc. 125 (1997), no. 11, 3185-3189
[Hauser-Rond] G.Rond, H.Hauser, Artin Approximation, manuscript, http://homepage.univie.ac.at/herwig.hauser/
[Huneke-Swanson] C.Huneke, I.Swanson, Integral closure of ideals, rings, and modules. London Mathematical Society Lecture Note Series, 336. Cambridge University Press, Cambridge, 2006
[K.P.P.R.M.] H.Kurke, G.Pfister, D.Popescu, M.Roczen, T.Mostowski, Die Approximationseigenschaft lokaler Ringe. Lecture Notes in Mathematics, Vol. 634. Springer-Verlag, Berlin-New York, 1978. iv+204 pp.
[Malgrange] B.Malgrange. Ideals of differentiable functions. Tata Institute of Fundamental Research Studies in Mathematics, No. 3 Tata Institute of Fundamental Research, Bombay; Oxford University Press, London 1967 vii+106 pp.
[du Plessis-Wall] A.du Plessis, C.T.C.Wall, The geometry of topological stability. London Mathematical Society Monographs. New Series, 9. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995
[Rudin-book] W.Rudin, Real and complex analysis. Third edition. McGraw-Hill Book Co., New York, 1987. xiv+416 pp
[Tougeron1966] J.C.Tougeron, Une généralisation du théorème des fonctions implicites. C. R. Acad. Sci. Paris Sér. A–B 262 1966 A487-A489
[Tougeron1968] J.C.Tougeron, Idéaux de fonctions différentiables. I. Ann. Inst. Fourier (Grenoble) 18 1968 fasc. 1, 177–240
[Tougeron-book] J.C.Tougeron, Idéaux de fonctions différentiables. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 71. Springer-Verlag, Berlin-New York, 1972
[Tougeron1976] J.C.Tougeron, Solutions d’un système d’équations analytiques réelles et applications. Ann. Inst. Fourier (Grenoble) 26 (1976), no. 3, x, 109-135
[Whitney1934] H.Whitney, Analytic extensions of differentiable functions defined in closed sets. Trans. Amer. Math. Soc. 36 (1934), no. 1, 63-89.

DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE’ER SHEVA 84105, ISRAEL.
E-mail address: genrich@math.bgu.ac.il, dmitry.kerner@gmail.com