Well-posedness for the Navier-Stokes equations with datum in Sobolev-Fourier-Lorentz spaces

D. Q. Khai, N. M. Tri

Institute of Mathematics, Vietnam Academy of Science and Technology
18 Hoang Quoc Viet, 10307 Cau Giay, Hanoi, Vietnam

Abstract: In this note, for \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), we introduce and study Sobolev-Fourier-Lorentz spaces \( \dot{H}^s_{L^p r}(\mathbb{R}^d) \). In the family spaces \( \dot{H}^s_{L^p r}(\mathbb{R}^d) \), the critical invariant spaces for the Navier-Stokes equations correspond to the value \( s = \frac{d}{p} - 1 \). When the initial datum belongs to the critical spaces \( \dot{H}^{\frac{d}{p} - 1}_{L^p r}(\mathbb{R}^d) \) with \( d \geq 2 \), \( 1 \leq p < \infty \), and \( 1 \leq r < \infty \), we establish the existence of local mild solutions to the Cauchy problem for the Navier-Stokes equations in spaces \( L^\infty([0, T]; \dot{H}^{\frac{d}{p} - 1}_{L^p r}(\mathbb{R}^d)) \) with arbitrary initial value, and existence of global mild solutions in spaces \( L^\infty([0, \infty); \dot{H}^{\frac{d}{p} - 1}_{L^p r}(\mathbb{R}^d)) \) when the norm of the initial value in the Besov spaces \( \dot{B}^{\frac{d}{p} - 1, \infty}_{\tilde{p}, \infty}(\mathbb{R}^d) \) is small enough, where \( \tilde{p} \) may take some suitable values.

§1. INTRODUCTION

We consider the Navier-Stokes equations (NSE) in \( d \) dimensions in special setting of a viscous, homogeneous, incompressible fluid which fills the entire space and is not submitted to external forces. Thus, the equations we consider are the system:

\[
\begin{align*}
\partial_t u &= \Delta u - \nabla.(u \otimes u) - \nabla p, \\
\text{div}(u) &= 0, \\
u(0, x) &= u_0,
\end{align*}
\]

which is a condensed writing for

\[
\begin{align*}
1 \leq k \leq d, \quad \partial_k u_k &= \Delta u_k - \sum_{i=1}^d \partial_i(u_k u_i) - \partial_k p, \\
\sum_{i=1}^d \partial_i u_i &= 0, \\
1 \leq k \leq d, \quad u_k(0, x) &= u_{0k}.
\end{align*}
\]

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3e-mail address: triminh@math.ac.vn
The unknown quantities are the velocity \( u(t, x) = (u_1(t, x), \ldots, u_d(t, x)) \) of the fluid element at time \( t \) and position \( x \) and the pressure \( p(t, x) \).

A translation invariant Banach space of tempered distributions \( \mathcal{E} \) is called a critical space for NSE if its norm is invariant under the action of the scaling \( f(\lambda \cdot) \mapsto \lambda f(\lambda \cdot) \). One can take, for example, \( \mathcal{E} = L^d(\mathbb{R}^d) \) or the smaller space \( \mathcal{E} = \dot{H}^{\frac{d}{2}} - 1(\mathbb{R}^d) \). In fact, one has the chain of critical spaces given by the continuous imbedding

\[
\dot{H}^{\frac{d}{2}} - 1(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d) \hookrightarrow \dot{B}^{d - 1}_{p, \infty}(\mathbb{R}^d) \hookrightarrow BMO^{-1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^d). \tag{1}
\]

It is remarkable feature that the NSE are well-posed in the sense of Hadamard (existence, uniqueness and continuous dependence on data) when the initial datum is divergence-free and belongs to the critical function spaces (except \( \dot{B}^{-1}_{\infty, \infty} \)) listed in (1) (see [4] for \( \dot{H}^{d - 1}(\mathbb{R}^d), L^d(\mathbb{R}^d), \) and \( \dot{B}^{d - 1}_{1, \infty}(\mathbb{R}^d) \), see [23] for \( BMO^{-1}(\mathbb{R}^d) \), and the recent ill-posedness result [3] for \( \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^d) \)).

In the 1960s, mild solutions were first constructed by Kato and Fujita ([17], [18]) that are continuous in time and take values in the Sobolev space \( H^s(\mathbb{R}^d) \). In 1981, Weissler [29] gave the first existence result of mild solutions in the \( L^3(\mathbb{R}^3) \). In 1984, Kato [20] obtained, by means of a purely analytical tool (involving only Hölder and Young inequalities and without using any estimate of fractional powers of the Stokes operator), an existence theorem in the whole space \( L^3(\mathbb{R}^3) \). In [3], [5], [6], Cannone showed how to simplify Kato’s proof. The idea is to take advantage of the structure of the bilinear operator in its scalar form. In particular, the divergence \( \nabla \) and heat \( e^{t\Delta} \) operators can be treated as a single convolution operator. In 1994, Kato and Ponce [22] showed that the NSE are well-posed when the initial datum belongs to homogeneous Sobolev spaces \( \dot{H}^{\frac{d}{2} - 1}_{p, \infty}(\mathbb{R}^d), (d \leq p < \infty) \). Recently, the authors of this article have
Fourier-Lorentz spaces ˙

Sobolev-Fourier-Lorentz spaces ˙

Stokes equations are well-posed when the initial datum belongs to the Sobolev spaces ˙

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perturbed by the critical Sobolev-Fourier-Lebesgue spaces ˙

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showed that NSE are well-posed when the initial datum belongs to Sobolev

spaces ˙

s ≤ 0). In [13], we showed that the bilinear operator

B(u, v)(t) = \int_0^t e^{(t-\tau)\Lambda} \nabla \cdot (u(\tau) \otimes v(\tau)) \, d\tau

(2)

is bicontinuous in \( L^\infty([0, T]; \dot{H}_p^d(\mathbb{R}^d)) \) with super-critical and non-negative-regular indexes \( 0 \leq d, p > 1, \) and \( \frac{4}{d} < \frac{1}{p} < \frac{d+1}{d} \), and we established the inequality

\[ \|B(u, v)\|_{L^\infty([0, T]; \dot{H}_p^d)} \leq C_{s, p, d} T^{\frac{1}{2} + s - \frac{1}{p}} \|u\|_{L^\infty([0, T]; \dot{H}_p^s)} \|v\|_{L^\infty([0, T]; \dot{H}_p^s)}. \]

In this case existence and uniqueness theorems of local mild solutions can therefore be easily deduced. In [16] we prove that NSE are well-posed when the initial datum belongs to the Sobolev spaces ˙

H_p^d(\mathbb{R}^d) with \( 0 < d, p \leq d \).

In this paper, for \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), we first recall the notion of the Fourier-Lebesgue spaces \( \dot{L}^p(\mathbb{R}^d) \), introduced and investigated in [12]; then we introduce and study Sobolev-Fourier-Lebesgue spaces \( \dot{H}_p^s(\mathbb{R}^d) \), and Sobolev-Fourier-Lorentz spaces \( \dot{H}_p^s(\mathbb{R}^d) \). After that we show that the Navier-Stokes equations are well-posed when the initial datum belongs to the critical Sobolev-Fourier-Lorentz spaces \( \dot{H}_p^s(\mathbb{R}^d) \) with \( d \geq 2, 1 \leq p < \infty \), and \( 1 < r < \infty \). The spaces \( \dot{H}_p^{d+1}(\mathbb{R}^d) \) are more general than the spaces \( \dot{H}_p^d(\mathbb{R}^d) \). In particular, \( \dot{H}_p^{d+1}(\mathbb{R}^d) = \dot{H}_p^{d+1}(\mathbb{R}^d) \) when \( \frac{1}{d} + \frac{1}{p} = 1 \).

In 1997, Le Jan and Sznitman [26] considered a very simple space convenient to the study of NSE, which is the space \( E \) of tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^d) \) so that \( \hat{f}(\xi) \) is a locally integrable function on \( \mathbb{R}^d \) and \( \sup_{\xi} |\xi|^{d-1} |\hat{f}(\xi)| < \infty \), with ‘’ standing for the Fourier transform. This space may be defined as a Besov space based on the spaces \( PM \) of pseudomeasures \( (PM) \) is the space of the image of the Fourier transforms of essentially bounded functions: \( PM = \mathcal{F} L^\infty \). More precisely, \( E = \dot{B}_{PM}^{d-1, \infty}(\mathbb{R}^d). \) They showed that the bilinear operator \( B \) is bicontinuous in \( L^\infty([0, T]; \dot{B}_{PM}^{d-1, \infty}(\mathbb{R}^d) \) for all \( 0 < T \leq \infty \). Therefore they can easily deduce the existence of global mild solutions in spaces \( L^\infty([0, \infty); \dot{B}_{PM}^{d-1, \infty}(\mathbb{R}^d) \) when norm of the initial value in the spaces \( \dot{B}_{PM}^{d-1, \infty}(\mathbb{R}^d) \) is small enough. From Definitions 1 and 2 in Section 2, we have

\[ PM = \mathcal{L}^1, \dot{B}_{PM}^{d-1, \infty}(\mathbb{R}^d) = \dot{H}_p^{d-1}(\mathbb{R}^d). \]
In 2011, Lei and Lin \cite{25} showed that NSE are well-posed when the initial datum belongs to the spaces $\mathcal{X}^{-1}(\mathbb{R}^d)$, which is defined by

$$f \in \mathcal{X}^{-1}(\mathbb{R}^d) \text{ if and only if } \|(-\Delta)^{-\frac{1}{2}}f\|_{\mathcal{X}} < \infty, \text{where } \|f\|_{\mathcal{X}} = \|\hat{f}\|_{L^1}.$$ 

They established the existence of global mild solutions in the space $L^\infty([0,\infty); \mathcal{X}^{-1})$ when norm of the initial value in the spaces $\mathcal{X}^{-1}(\mathbb{R}^d)$ is small enough. From Definitions 1 and 2 in Section 2, we see that

$$\mathcal{X}^{-1}(\mathbb{R}^d) = \dot{H}^{-\frac{1}{p}}_{L^\infty}(\mathbb{R}^d).$$

Thus, the spaces $\dot{B}^{d-1,\infty}_{p,p}$ and $\mathcal{X}^{-1}$, studied in \cite{26} and \cite{25}, are particular cases of the critical Sobolev-Fourier-Lebesgue spaces $\dot{H}^{d}_{L^p}$ with $p = 1$ and $p = \infty$, respectively. Note that estimates in the Lorentz spaces were also studied in \cite{1}, \cite{19} (see also the references therein). Very recently, ill-posedness of NSE in critical Besov spaces $\dot{B}^{-1}_{\infty,q}$ was investigated in \cite{28}.

The paper is organized as follows. In Section 2 we introduce and investigate the Sobolev-Fourier-Lorentz spaces and some auxiliary lemmas. In Section 3 we present the main results of the paper. Due to some technical difficulties we will consider three cases $1 < p \leq d, d \leq q < \infty$, and $p = 1$ separately. In subsection 3.1 we treat the case $1 < p \leq d$. In subsection 3.2 we consider the case $d \leq q < \infty$. Finally, in subsection 3.3 we study the case $p = 1$. In the sequence, for a space of functions defined on $\mathbb{R}^d$, say $E(\mathbb{R}^d)$, we will abbreviate it as $E$. Throughout the paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$ with a uniform constant $C$. The notation $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$.

\section{SOBOLEV-FOURIER-LORENTZ SPACES}

\textbf{Definition 1.} (Fourier-Lebesgue spaces). (See \cite{12}.)

For $1 \leq p \leq \infty$, the Fourier-Lebesgue spaces $\mathcal{L}^p(\mathbb{R}^d)$ is defined as the space $\mathcal{F}^{-1}(L^p(\mathbb{R}^d)), (\frac{1}{p} + \frac{1}{p'} = 1)$, equipped with the norm

$$\|f\|_{\mathcal{L}^p(\mathbb{R}^d)} := \|\mathcal{F}(f)\|_{L^{p'}(\mathbb{R}^d)},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse.

\textbf{Definition 2.} (Sobolev-Fourier-Lebesgue spaces).

For $s \in \mathbb{R}$, and $1 \leq p \leq \infty$, the Sobolev-Fourier-Lebesgue spaces $\dot{H}^s_{L^p}(\mathbb{R}^d)$ is defined as the space $\dot{\Lambda}^{-s}\mathcal{L}^p(\mathbb{R}^d)$, equipped with the norm

$$\|u\|_{\dot{H}^s_{L^p}} := \|\dot{\Lambda}^s u\|_{\mathcal{L}^p}.$$
where \( \dot{\Lambda} = \sqrt{-\Delta} \) is the homogeneous Calderon pseudo-differential operator defined as
\[
\hat{\dot{\Lambda}} g(\xi) = |\xi| \hat{g}(\xi).
\]

**Definition 3.** (Lorentz spaces). (See [2].) For \( 1 \leq p, r \leq \infty \), the Lorentz space \( L^{p,r}(\mathbb{R}^d) \) is defined as follows. A measurable function \( f \in L^{p,r}(\mathbb{R}^d) \) if and only if
\[
\|f\|_{L^{p,r}(\mathbb{R}^d)}\ := \left( \int_{0}^{\infty} (t^{\frac{p}{r}} f^*(t))^{\frac{r}{p}} \, dt \right)^{\frac{r}{p}} < \infty \quad \text{for } 1 \leq r < \infty,
\]
\[
\|f\|_{L^{p,\infty}(\mathbb{R}^d)} := \sup_{t > 0} t^{\frac{1}{p}} f^*(t) < \infty \quad \text{for } r = \infty,
\]
where \( f^*(t) = \inf \{ \tau : M^d(\{ x : |f(x)| > \tau \}) \leq t \} \), with \( M^d \) being the Lebesgue measure in \( \mathbb{R}^d \).

**Definition 4.** (Fourier-Lorentz spaces). For \( 1 \leq p, r \leq \infty \), the Fourier-Lorentz spaces \( L^{p,r}(\mathbb{R}^d) \) is defined as the space \( \mathcal{F}^{-1}(L^{p',r}(\mathbb{R}^d)) \), equipped with the norm
\[
\|f\|_{L^{p,r}(\mathbb{R}^d)} := \|\mathcal{F}(f)\|_{L^{p',r}(\mathbb{R}^d)}.
\]

**Definition 5.** (Sobolev-Fourier-Lorentz spaces). For \( s \in \mathbb{R} \) and \( 1 \leq r, p \leq \infty \), the Sobolev-Fourier-Lorentz spaces \( \dot{H}^s_{p,r}(\mathbb{R}^d) \) is defined as the space \( \dot{\Lambda}^{-s} L^{p,r}(\mathbb{R}^d) \), equipped with the norm
\[
\|u\|_{\dot{H}^s_{p,r}(\mathbb{R}^d)} := \|\dot{\Lambda}^s u\|_{L^{p,r}(\mathbb{R}^d)}.
\]

**Theorem 1.** (Holder’s inequality in Fourier-Lorentz spaces). Let \( 1 < r, q, \tilde{q} < \infty \) and \( 1 \leq h, \tilde{h}, \hat{h} \leq +\infty \) satisfy the relations
\[
\frac{1}{r} = \frac{1}{q} + \frac{1}{\tilde{q}} \quad \text{and} \quad \frac{1}{h} = \frac{1}{\tilde{h}} + \frac{1}{\hat{h}}.
\]
Suppose that \( u \in L^{q,\tilde{h}} \) and \( v \in L^{\tilde{q},\hat{h}} \). Then \( uv \in L^{r,h} \) and we have the inequality
\[
\|uv\|_{L^{r,h}} \lesssim \|u\|_{L^{q,\tilde{h}}} \|v\|_{L^{\tilde{q},\hat{h}}}. \tag{3}
\]

**Proof.** Let \( r', q', \) and \( \tilde{q}' \) be such that
\[
\frac{1}{r} + \frac{1}{r'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad \text{and} \quad \frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1.
\]
It is easily checked that the following conditions are satisfied
\[
1 < r', q', \tilde{q}' < +\infty \quad \text{and} \quad \frac{1}{r'} + 1 = \frac{1}{q'} + \frac{1}{\tilde{q}'}.
\]
We have
\[ \|uv\|_{L^{r,h}} = \|\hat{u}\hat{v}\|_{L^{r',h}} = \frac{1}{(2\pi)^{d/2}} \|\hat{u} \ast \hat{v}\|_{L^{r',h}}. \] (4)

Applying Proposition 2.4 (c) in ([24], p. 20), we have
\[ \|\hat{u} \ast \hat{v}\|_{L^{r',h}} \lesssim \|\hat{u}\|_{L^{q',\tilde{h}}} \|\hat{v}\|_{L^{q',\hat{h}}} = \|u\|_{L^{q,\tilde{h}}} \|v\|_{L^{q,\hat{h}}}. \] (5)

Now, the estimate (3) follows from the equality (4) and the inequality (5).

**Theorem 2.** (Young’s inequality for convolution in Fourier-Lorentz spaces). Let \( 1 < r, q, \tilde{q} < \infty \), and \( 1 \leq h, \tilde{h}, \hat{h} \leq \infty \) satisfy the relations
\[ \frac{1}{r} + 1 = \frac{1}{q} + \frac{1}{\tilde{q}} \quad \text{and} \quad \frac{1}{h} = \frac{1}{\tilde{h}} + \frac{1}{\hat{h}}. \]

Suppose that \( u \in L^{q,\tilde{h}} \) and \( v \in L^{\tilde{q},\hat{h}} \). Then \( u \ast v \in L^{r,h} \) and the following inequality holds
\[ \|u \ast v\|_{L^{r,h}} \lesssim \|u\|_{L^{q,\tilde{h}}} \|v\|_{L^{\tilde{q},\hat{h}}}. \] (6)

**Proof.** Let \( r', q', \) and \( \tilde{q}' \) be such that
\[ \frac{1}{r'} + \frac{1}{q'} = 1, \frac{1}{q} + \frac{1}{\tilde{q}'} = 1, \quad \text{and} \quad \frac{1}{\tilde{q}} + \frac{1}{q'} = 1. \]

By definition
\[ \|u \ast v\|_{L^{r,h}} = \|\hat{u}\ast\hat{v}\|_{L^{r',h}} = (2\pi)^{d/2} \|\hat{u}\hat{v}\|_{L^{r',h}}. \] (7)

We can check that the following conditions are satisfied
\[ 1 < r', q', \tilde{q}' < +\infty \quad \text{and} \quad \frac{1}{r'} = \frac{1}{q} + \frac{1}{\tilde{q}'}.
\]

Applying Proposition 2.3 (c) in ([24], p. 19), we have
\[ \|\hat{u}\hat{v}\|_{L^{r',h}} \lesssim \|\hat{u}\|_{L^{q',\tilde{h}}} \|\hat{v}\|_{L^{q',\hat{h}}} = \|u\|_{L^{q,\tilde{h}}} \|v\|_{L^{q,\hat{h}}}. \] (8)

Now, the estimate (3) follows from the equality (7) and the inequality (8).

**Theorem 3.** (Sobolev inequality for Sobolev-Fourier-Lorentz spaces). Let \( 1 < q \leq \tilde{q} < \infty, s, \tilde{s} \in \mathbb{R}, s - \frac{d}{q} = \tilde{s} - \frac{d}{\tilde{q}} \), and \( 1 \leq r \leq \infty \). Then
\[ \|u\|_{H^{s,\tilde{q},r}_x} \lesssim \|u\|_{H^{\tilde{s},\tilde{q},r}_x}, \forall u \in H^s_{\tilde{q},r}. \] (9)
Proof. We have
\[ \|u\|_{\dot{H}^s_L^{q,r}} = \|\dot{\Lambda}^{s-s}u\|_{L^{q,r}} = \|\xi^{s-s}\hat{u}(\xi)\|_{L^{q',r}}, \]  
where
\[ \frac{1}{q} + \frac{1}{q'} = 1. \]

Note that \(|\xi|^{-r} \in L^\infty_\mathbb{R}^d\) for all \(r\) satisfying \(0 < r \leq d\).

Applying Proposition 2.3 (c) in ([24], p. 19), we have
\[ \|\xi^{s-s}\hat{u}(\xi)\|_{L^{q',r}} \lesssim \|\xi^{s-s}\|_{L^{\infty,q',r}} \|\dot{\Lambda}^{s-s}u(\xi)\|_{L^{q',r}} \approx \|u\|_{\dot{H}^s_{L^{q,r}}}. \]  

The estimate (9) follows from the equality (10) and the inequality (11).

Lemma 1. Let \(s \in \mathbb{R}, 1 \leq p \leq \infty, \) and \(1 \leq r \leq \tilde{r} \leq \infty\).

(a) We have the following imbedding maps
\[ L^{p,1} \hookrightarrow L^{p,r} \hookrightarrow L^{p,\tilde{r}} \hookrightarrow L^{p,\infty}, \]
\[ \dot{H}^s_{L^{p,1}} \hookrightarrow \dot{H}^s_{L^{p,r}} \hookrightarrow \dot{H}^s_{L^{p,\tilde{r}}} \hookrightarrow \dot{H}^s_{L^{p,\infty}}. \]

(b) \(\dot{H}^s_{L^p} = \dot{H}^s_{L^p,q'}\) (equality of the norm), where \(\frac{1}{p} + \frac{1}{p'} = 1\).

Proof. It is easily deduced from the properties of the standard Lorentz spaces.

Lemma 2. Let \(s \in \mathbb{R}\) and \(1 < p < \infty\). We have

(a) If \(1 < q \leq 2\) then \(\dot{H}^s_q \hookrightarrow \dot{H}^s_{L^p}\).

(b) If \(2 \leq q < \infty\) then \(\dot{H}^s_{L^q} \hookrightarrow \dot{H}^s_q\).

Proof. It is deduced from Theorem 1.2.1 ([2], p. 6).

Lemma 3. Assume that \(1 \leq r, p \leq \infty\) and \(k \in \mathbb{N}\), then the two quantities
\[ \|u\|_{\dot{H}^k_{L^p,r}}\text{ and } \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^{p,r}} \]
are equivalent.

Proof. First, we prove that
\[ \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^{p,r}} \lesssim \|u\|_{\dot{H}^k_{L^p,r}}. \]
We have
\[ \sum_{|\alpha|=k} \| \partial^\alpha u \|_{L^p,r} = \sum_{|\alpha|=k} \| i^k \xi^\alpha \hat{u}(\xi) \|_{L^{p',r}} = \sum_{|\alpha|=k} \| \frac{\xi^\alpha}{|\xi|^k} |\xi|^k \hat{u}(\xi) \|_{L^{p',r}} \]
\[ \leq \sum_{|\alpha|=k} \| |\xi|^k \hat{u}(\xi) \|_{L^{p',r}} \leq \| \hat{\Lambda}^k u(\xi) \|_{L^{p',r}} = \| u \|_{\dot{H}^k_{p,r}}. \]

Next, we prove that
\[ \| u \|_{\dot{H}^k_{p,r}} \lesssim \sum_{|\alpha|=k} \| \partial^\alpha u \|_{L^p,r}. \]

It is easy to see that for all \( \xi \in \mathbb{R}^d \), we have
\[ |\xi|^k \leq d^\frac{k}{d} \sum_{|\alpha|=k} |\xi|^\alpha. \]

This gives the desired result
\[ \| u \|_{\dot{H}^k_{p,r}} = \| |\xi|^k \hat{u}(\xi) \|_{L^{p',r}} \leq d^\frac{k}{d} \sum_{|\alpha|=k} |\xi|^\alpha \hat{u}(\xi) \|_{L^{p',r}} \]
\[ \leq d^\frac{k}{d} \sum_{|\alpha|=k} \| \xi^\alpha \hat{u}(\xi) \|_{L^{p',r}} = d^\frac{k}{d} \sum_{|\alpha|=k} \| \partial^\alpha u \|_{L^p,r}. \]

\[ \text{Lemma 4.} \]
Let \( k \in \mathbb{N}, p \in \mathbb{R}, \) and \( r \in \mathbb{R} \) be such that
\[ 0 \leq k \leq d - 1, \frac{k}{d} < \frac{1}{p} < \frac{1}{2} + \frac{k}{2d}, \] and \( 1 \leq r \leq \infty. \)

Then the following inequality holds
\[ \| uv \|_{\dot{H}^k_{p,q}} \lesssim \| u \|_{\dot{H}^k_{p,p}} \| v \|_{\dot{H}^k_{p,r}}, \forall u, v \in \dot{H}^k_{p,r}, \]
where
\[ \frac{1}{q} = \frac{2}{p} - \frac{k}{d}. \]

\[ \text{Proof.} \]
First, we estimate \( \| \partial^\alpha (uv) \|_{L^q,r}, \) where
\[ \alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{N}^d, \ |\alpha| = \sum_{i=1}^d \alpha_i = k. \]

By the general Leibniz rule, we have
\[ \partial^\alpha (uv) = \sum_{\gamma+\beta=\alpha} \binom{\alpha}{\gamma} (\partial^\gamma u)(\partial^\beta v). \]
\[ \frac{1}{q_1} = \frac{1}{p} - \frac{k - |\gamma|}{d}, \quad \frac{1}{q_2} = \frac{1}{p} - \frac{k - |\beta|}{d}. \]

Therefore applying Theorems 1, 3 and Lemma 1 (a) in order to obtain
\[
\| (\partial^\gamma u)(\partial^\beta v) \|_{\mathcal{L}_{p,r}} \lesssim \| \partial^\gamma u \|_{\mathcal{L}_{p,1}} \| \partial^\beta v \|_{\mathcal{L}_{p,\infty}} \lesssim \| \partial^\gamma u \|_{\mathcal{H}_{p,r}^{k-|\gamma|}} \| \partial^\beta v \|_{\mathcal{H}_{p,r}^{k-|\beta|}}.
\]

Thus, for all \( \alpha \in \mathbb{N}^d \) with \( |\alpha| = k \), we have
\[
\| \partial^\alpha (uv) \|_{\mathcal{L}_{p,r}} \lesssim \| u \|_{\mathcal{H}_{p,r}^{k}} \| v \|_{\mathcal{H}_{p,r}^{k}}.
\]

Applying Lemma 3, we have
\[
\| uv \|_{\mathcal{H}_{p,r}^{k}} \lesssim \| u \|_{\mathcal{H}_{p,r}^{k}} \| v \|_{\mathcal{H}_{p,r}^{k}}, \quad \forall u, v \in \dot{H}_{p,r}^{k}.
\]

**Lemma 5.** Assume that \( 1 \leq p, r \leq \infty \) and \( s \in \mathbb{R} \). If \( u_0 \in \dot{H}_{p,r}^{s} \) then \( e^{t\Delta}u_0 \in L^\infty([0, \infty); \dot{H}_{p,r}^{s}) \) and
\[
\| e^{t\Delta}u_0 \|_{L^\infty([0, \infty); \dot{H}_{p,r}^{s})} \leq \| u_0 \|_{\dot{H}_{p,r}^{s}}.
\]

**Proof.** For \( t \geq 0 \), we have
\[
\| e^{t\Delta}u_0 \|_{\dot{H}_{p,r}^{s}} = \| e^{t\Delta}\dot{\lambda}^s u_0 \|_{\mathcal{L}_{p,r}} = \| e^{-t|\xi|^2}|\xi|^s \hat{u}_0 \|_{\mathcal{L}_{p,r}} \leq \| \| \xi \|^s \hat{u}_0 \|_{\mathcal{L}_{p,r}} = \| \dot{\lambda}^s u_0(\xi) \|_{\mathcal{L}_{p,r}} = \| \dot{\lambda}^s u_0(\xi) \|_{\mathcal{L}_{p,r}} = \| u_0 \|_{\dot{H}_{p,r}^{s}}.
\]

Finally, let us recall the following result on solutions of a quadratic equation in Banach spaces (Theorem 22.4, [24], p. 227).

**Theorem 4.** Let \( E \) be a Banach space, and \( B : E \times E \to E \) be a continuous bilinear form such that there exists \( \eta > 0 \) so that
\[
\| B(x, y) \| \leq \eta \| x \| \| y \|,
\]
for all \( x \) and \( y \) in \( E \). Then for any fixed \( y \in E \) such that \( \| y \| \leq \frac{1}{2\eta} \), the equation \( x = y - B(x, x) \) has a unique solution \( \overline{x} \in E \) satisfying \( \| \overline{x} \| \leq \frac{1}{2\eta} \).


§3. MAIN RESULTS

For $T > 0$, we say that $u$ is a mild solution of NSE on $[0, T]$ corresponding to a divergence-free initial data $u_0$ when $u$ satisfies the integral equation

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}P\nabla.(u(\tau) \otimes u(\tau))d\tau.$$

Above we have used the following notation: For a tensor $F = (F_{ij})$ we define the vector $\nabla F$ by $(\nabla F)_i = \sum_{d=1}^d \partial_j F_{ij}$ and for vectors $u$ and $v$, we define their tensor product $(u \otimes v)_{ij} = u_i v_j$. The operator $P$ is the Leray projection onto the divergence-free fields

$$(Pf)_j = f_j + \sum_{1 \leq k \leq d} R_j R_k f_k,$$

where $R_j$ is the Riesz transforms defined as

$$R_j = \frac{\partial_i}{\Lambda}, \text{ i.e. } \hat{R}_j g(\xi) = \frac{i\xi_j}{|\xi|} \hat{g}(\xi).$$

The heat kernel $e^{t\Delta}$ is defined as

$$e^{t\Delta}u(x) = ((4\pi t)^{-d/2}e^{-|\cdot|^2/4t} * u)(x).$$

If $X$ is a normed space and $u = (u_1, u_2, ..., u_d), u_i \in X, 1 \leq i \leq d$, then we write

$$u \in X, \|u\|_X = \left(\sum_{i=1}^d \|u_i\|^2_X\right)^{1/2}.$$

In this main section we investigate mild solutions to NSE when the initial datum belongs to critical spaces $\dot{H}^{d}_{p,r,1}(\mathbb{R}^d)$ with $1 \leq p < \infty$ and $1 \leq r < \infty$. We consider three cases $1 < p \leq d, d \leq q < \infty$, and $p = 1$ separately.

3.1. Solutions to the Navier-Stokes equations with the initial value in the critical spaces $\dot{H}^{d}_{p,r,1}(\mathbb{R}^d)$ with $1 < p \leq d$ and $1 \leq r < \infty$.

We define an auxiliary space $K^p_{p,r,T}$ which is made up by the functions $u(t, x)$ such that

$$\|u\|_{K^p_{p,r,T}} := \sup_{0 < t < T} t^{\frac{1}{p'}} \left\|u(t, x)\right\|_{\dot{H}^{d}_{p,r,1}} < \infty,$$
and
\[ \lim_{t \to 0} \alpha_{t, \beta} = 0, \tag{12} \]
with
\[ 1 < p \leq \tilde{p} < \infty, \quad \frac{1}{p} - \frac{1}{d} < \frac{1}{\tilde{p}}, \quad 1 \leq r \leq \infty, \quad T > 0, \]
and
\[ \alpha = \alpha(p, \tilde{p}) = d\left(\frac{1}{p} - \frac{1}{\tilde{p}}\right). \]
In the case \( \tilde{p} = p \), it is also convenient to define the space \( \mathcal{K}^{\tilde{p}}_{p,s} \) as the natural space \( L^\infty([0, T]; \dot{H}_{L^p}^{d-1}) \) with the additional condition that its elements \( u(t, x) \) satisfy
\[ \lim_{t \to 0} \| u(t, x) \|_{\dot{H}_{L^p}^{d-1}} = 0. \tag{13} \]

Lemma 6. Let \( 1 \leq r \leq \tilde{r} \leq \infty \). Then we have the following imbedding
\[ \mathcal{K}^{\tilde{p}}_{p,1,T} \hookrightarrow \mathcal{K}^{\tilde{p}}_{p,r,T} \hookrightarrow \mathcal{K}^{\tilde{p}}_{p,\tilde{r},T} \hookrightarrow \mathcal{K}^{\tilde{p}}_{p,\infty,T}. \]

Proof. It is easily deduced from Lemma 1 (a) and the definition of \( \mathcal{K}^{\tilde{p}}_{p,r,T} \). [11]

Lemma 7. Suppose that \( u_0 \in L^q_r(\mathbb{R}^d) \) with \( 1 \leq q \leq \infty \) and \( 1 \leq r < \infty \), then \( e^{t\Delta}u_0 \in \mathcal{K}^{\tilde{p}}_{p,1,\infty} \) with \( \frac{1}{p} - \frac{1}{d} < \frac{1}{\tilde{p}} < \frac{1}{p} \).

Proof. Before proving this lemma, we need to prove the following lemma.

Lemma 8. Suppose that \( u_0 \in L^q_r(\mathbb{R}^d) \) with \( 1 \leq q \leq \infty \) and \( 1 \leq r < \infty \). Then \( \lim_{n \to \infty} \| 1_{B_n^c}u_0 \|_{L^q_r} = 0 \), where \( n \in \mathbb{N}, B_n = \{ x \in \mathbb{R}^d : |x| < n \}, B_n^c = \mathbb{R}^d \setminus B_n, \) and \( 1_{B_n^c} \) is the indicator function of the set \( B_n^c \) on \( \mathbb{R}^d : 1_{B_n^c}(x) = 1 \) for \( x \in B_n^c \) and \( 1_{B_n^c}(x) = 0 \) otherwise.

Proof. With \( \delta > 0 \) being fixed, we have
\[ \{ x : |1_{B_n^c}u_0(x)| > \delta \} \supseteq \{ x : |1_{B_{n+1}^c}u_0(x)| > \delta \}, \tag{14} \]
and
\[ \bigcap_{n=0}^{\infty} \{ x : |1_{B_n^c}u_0(x)| > \delta \} = \emptyset. \tag{15} \]
Note that
\[ \mathcal{M}^d(\{ x : |1_{B_n^c}u_0(x)| > \delta \}) = \mathcal{M}^d(\{ x : |u_0(x)| > \delta \}). \]
We prove that
\[ M^d \left( \{ x : |u_0(x)| > \delta \} \right) < \infty, \]  
(16)
assuming on the contrary
\[ M^d \left( \{ x : |u_0(x)| > \delta \} \right) = \infty. \]

Set
\[ u_0^*(t) = \inf \{ \tau : M^d \left( \{ x : |u_0(x)| > \tau \} \right) \leq t \}. \]

We have \[ u_0^*(t) = \delta \] for all \( t > 0 \), from the definition of the Lorentz space, we get
\[ \left\| u_0 \right\|_{L^{q,r}} = \left( \int_0^\infty (t^{\frac{1}{q}} u_0^*(t))^{\frac{r}{t}} \frac{dt}{t} \right)^{\frac{q}{r}} \geq \delta \left( \int_0^\infty t^{\frac{1}{q} - 1} dt \right)^{\frac{q}{r}} = \infty, \]
a contradiction.

From (14), (15), and (16), we have
\[ \lim_{n \to \infty} M^d \left( \{ x : |1_{B^n} u_0(x)| > \delta \} \right) = 0. \]  
(17)

Set
\[ u_n^*(t) = \inf \{ \tau : M^d \left( \{ x : |1_{B^n} u_0(x)| > \tau \} \right) \leq t \}. \]

We have
\[ u_n^*(t) \geq u_{n+1}^*(t). \]  
(18)

Fixed \( t > 0 \). For any \( \epsilon > 0 \), from (17) it follows that there exist \( n_0 = n_0(t, \epsilon) \) is large enough such that
\[ M^d \left( \{ x : |1_{B^n} u_0(x)| > \epsilon \} \right) \leq t, \forall n \geq n_0. \]

From this we deduce that
\[ u_n^*(t) \leq \epsilon, \forall n \geq n_0, \]
therefore
\[ \lim_{n \to \infty} u_n^*(t) = 0. \]  
(19)

From (18) and (19), we apply Lebesgue’s monotone convergence theorem to get
\[ \lim_{n \to \infty} \left\| 1_{B^n} u_0 \right\|_{L^{q,r}} = \lim_{n \to \infty} \left( \int_0^\infty (t^{\frac{1}{q}} u_n^*(t))^{\frac{r}{t}} \frac{dt}{t} \right)^{\frac{q}{r}} = 0. \]

Now we return to prove Lemma 7. We prove that
\[ \sup_{0 < t < \infty} \left\| e^{(\Delta) u_0} \right\|_{H^{\frac{d}{p} - 1}, \infty} \lesssim \left\| u_0 \right\|_{H^{\frac{d}{p} - 1}, L^{p,r}}. \]  
(20)
Let \( p' \) and \( \tilde{p}' \) be such that
\[
\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{and} \quad \frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1.
\]

We have
\[
\left\| e^{t \Delta} u_0 \right\|_{\mathcal{H}_{\tilde{p}', \tilde{p}}} = \left\| e^{-t|\xi|^2} \xi^{\frac{d}{	ilde{p}} - 1} \tilde{u}_0(\xi) \right\|_{L^{p'}}, \tag{21}
\]
Applying Holder’s inequality in the Lorentz spaces (see Proposition 2.3 (c) in [24], p. 19), we have
\[
\| e^{-t|\xi|^2} \xi^{\frac{d}{\tilde{p}} - 1} \tilde{u}_0(\xi) \|_{L^{p', \infty}} = t^{-\frac{\tilde{p}}{\tilde{p}'}(\frac{1}{\tilde{p}} - \frac{1}{\tilde{p}'})}\| e^{-t|\xi|^2} \|| \xi \xi^{\frac{d}{\tilde{p}} - 1} \tilde{u}_0(\xi) ||_{L^{p', \infty}} = t^{-\frac{\tilde{p}}{\tilde{p}'}(\frac{1}{\tilde{p}} - \frac{1}{\tilde{p}'})}\| e^{-t|\xi|^2} \|_{L^{\tilde{p}', \tilde{p}}}, \tag{22}
\]
The estimate (20) follows from the equality (21) and the estimate (22).
We claim now that
\[
\lim_{t \to 0} t^{\frac{\tilde{p}}{\tilde{p}'}} \| e^{t \Delta} u_0 \|_{\mathcal{H}_{\tilde{p}', \tilde{p}}} = 0. \tag{23}
\]
From the equality (21), we have
\[
t^{\frac{\tilde{p}}{\tilde{p}'}(\frac{1}{\tilde{p}} - \frac{1}{\tilde{p}'})}\| e^{-t|\xi|^2} \xi^{\frac{d}{\tilde{p}} - 1} \tilde{u}_0(\xi) ||_{L^{p', \infty}} = C \| e^{-t|\xi|^2} \xi^{\frac{d}{\tilde{p}} - 1} \tilde{u}_0(\xi) ||_{L^{p', \infty}} \tag{24}
\]
for any \( \epsilon > 0 \). Fixed one of such \( n \) and applying Holder’s inequality in the Lorentz spaces, we have
\[
\left\| e^{-t|\xi|^2} \xi^{\frac{d}{\tilde{p}} - 1} \tilde{u}_0(\xi) \right\|_{L^{p', \infty}} \leq C \left\| e^{-t|\xi|^2} \xi^{\frac{d}{\tilde{p}} - 1} \tilde{u}_0(\xi) \right\|_{L^{p', \infty}} \leq C \left\| e^{-t|\xi|^2} \xi^{\frac{d}{\tilde{p}} - 1} \tilde{u}_0(\xi) \right\|_{L^{p', \infty}} < \epsilon \quad \text{for large enough} \ n.
\]


for small enough $t = t(n) > 0$. From estimates (24) and (25), we have,

$$ t^2 \left\| e^{t\Delta} u_0 \right\|_{H^{\frac{d}{2}}_{L^p}} \leq C' \left\| 1_{B^n_c} |\xi|^{\frac{d-1}{2}} \hat{u}_0(\xi) \right\|_{L^{p',r}} + C''(n) t^2 \left\| u_0 \right\|_{H^{\frac{d}{2}}_{L^p}} < \epsilon. \quad \square $$

In the following lemmas a particular attention will be devoted to the study of the bilinear operator $B(u,v)(t)$ defined by (2).

In the following lemmas, denote by $[x]$ the integer part of $x$ and by $\{x\}$ the fraction part of $x$.

**Lemma 9.** Let $1 < p \leq d$. Then for all $\tilde{p}$ be such that

$$ \frac{1}{2p} + \frac{[\tilde{p}] - 1}{2d} < \frac{1}{\tilde{p}} < \min \left\{ \frac{[\tilde{p}]}{d}, \frac{1}{2d} \right\}, $$

(26)

the bilinear operator $B(u,v)(t)$ is continuous from $K^0_{d,(\frac{\tilde{p}}{p}),T} \times K^0_{d,(\frac{\tilde{p}}{p}),T}$ into $K^p_{p,1,T}$ and the following inequality holds

$$ \left\| B(u,v) \right\|_{K^p_{p,1,T}} \leq C \left\| u \right\|_{K^\tilde{p}_{(\frac{\tilde{p}}{p}),T}} \left\| v \right\|_{K^\tilde{p}_{(\frac{\tilde{p}}{p}),T}}, $$

(27)

where $C$ is a positive constant and independent of $T$.

**Proof.** We have

$$ \left\| B(u,v)(t) \right\|_{H^{\frac{d}{2}}_{L^p}} \leq \int_0^t \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla_i (u(\tau) \otimes v(\tau)) \right\|_{H^{\frac{d}{2}}_{L^p}} d\tau $$

$$ = \int_0^t \left\| \Lambda_{\frac{d}{2}}^{-1} e^{(t-\tau)\Delta} \mathbb{P} \nabla_i (u(\tau) \otimes v(\tau)) \right\|_{L^p} d\tau. $$

(28)

Note that

$$ \left( \Lambda_{\frac{d}{2}}^{-1} e^{(t-\tau)\Delta} \mathbb{P} \nabla_i (u(\tau) \otimes v(\tau)) \right)^j(\xi) $$

$$ = \left( \Lambda_{\frac{d}{2}} \right)^j e^{(t-\tau)\Delta} \mathbb{P} \nabla_i \Lambda_{\frac{d}{2}}^{-1} (u(\tau) \otimes v(\tau)) \right)^j(\xi) $$

$$ = |\xi|^j \tilde{p} e^{- (t-\tau)|\xi|^2} \sum_{l,k=1}^d \left( \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (i\xi_l) \left( \Lambda_{\frac{d}{2}}^{-1}(u_l v_k(\tau)) \right)^j(\xi). $$

Thus

$$ \left( \Lambda_{\frac{d}{2}}^{-1} e^{(t-\tau)\Delta} \mathbb{P} \nabla_i (u(\tau) \otimes v(\tau)) \right)^j $$

$$ = \frac{1}{(t-\tau)^{\frac{d}{2}}+d+1} \sum_{l,k=1}^d K_{l,k,j} \left( \frac{\cdot \delta_{jk}}{\sqrt{t-\tau}} \right) * \left( \Lambda_{\frac{d}{2}}^{-1}(u_l v_k(\tau)) \right), $$

(29)
where

\[
\hat{K}_{l,j}(\xi) = \frac{1}{(2\pi)^{d/2}} |\xi|^{\frac{d}{2}} e^{-|\xi|^2} \left( \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (i\xi_l). \tag{30}
\]

Setting the tensor \( K(x) = \{ K_{l,k,j}(x) \} \), we can rewrite the equality (29) in the tensor form

\[
\hat{\Lambda}^{\frac{d}{2} - 1} e^{(t-\tau)\Delta} \mathbb{F} \nabla \cdot (u(\tau) \otimes v(\tau)) = \frac{1}{(t - \tau)^{\frac{d}{2} + \frac{d+1}{2}}} K \left( \frac{\cdot}{\sqrt{t - \tau}} \right) * \left( \hat{\Lambda}^{\frac{d}{2} - 1} (u(\tau) \otimes v(\tau)) \right).
\]

Applying Theorem 2 for convolution in the Fourier-Lorentz spaces, we have

\[
\left\| \hat{\Lambda}^{\frac{d}{2} - 1} e^{(t-\tau)\Delta} \mathbb{F} \nabla \cdot (u(\tau) \otimes v(\tau)) \right\|_L^p, \lesssim \frac{1}{(t - \tau)^{\frac{d}{2} + \frac{d+1}{2}}} \left\| K \left( \frac{\cdot}{\sqrt{t - \tau}} \right) \right\|_L^r \left\| \hat{\Lambda}^{\frac{d}{2} - 1} (u(\tau) \otimes v(\tau)) \right\|_L^{q, \infty}, \tag{31}
\]

where

\[
\frac{1}{q} = \frac{2}{p} - \frac{[\frac{d}{p}] - 1}{d} \quad \text{and} \quad \frac{1}{r} = 1 + \frac{1}{p} - \frac{2}{p} + \frac{[\frac{d}{p}] - 1}{d}. \tag{32}
\]

Note that from the inequality (26), we can check that \( r \) and \( q \) satisfy the relations

\[
1 < r, q < \infty, \quad \frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}.
\]

Applying Lemma 4, we have

\[
\left\| \hat{\Lambda}^{\frac{d}{2} - 1} (u(\tau) \otimes v(\tau)) \right\|_L^{q, \infty} \lesssim \left\| u(\tau) \right\|_{H^{[\frac{d}{p}] - 1}_{L^{p, \infty}}} \left\| v(\tau) \right\|_{H^{[\frac{d}{p}] - 1}_{L^{p, \infty}}}. \tag{33}
\]

From the equalities (30) and (32), we obtain

\[
\left\| K \left( \frac{\cdot}{\sqrt{t - \tau}} \right) \right\|_L^r = (t - \tau)^{\frac{d}{2}} \left\| \hat{K} \left( \frac{\cdot}{\sqrt{t - \tau}} \right) \right\|_{L^{r, 1}} = (t - \tau)^{\frac{d}{2} - \frac{d+1}{2}} \left\| \hat{K} \right\|_{L^{1, \infty}} \simeq (t - \tau)^{\frac{d}{2}} \left( 1 + \frac{1}{p} - \frac{2}{p} + \frac{[\frac{d}{p}] - 1}{d} \right). \tag{34}
\]

From the estimates (31), (33), and (34), we deduce that

\[
\left\| \hat{\Lambda}^{\frac{d}{2} - 1} e^{(t-\tau)\Delta} \mathbb{F} \nabla \cdot (u(\tau) \otimes v(\tau)) \right\|_L^p \lesssim (t - \tau)^{\frac{d}{2} - \frac{d+1}{2}} \left\| u(\tau) \right\|_{H^{[\frac{d}{p}] - 1}_{L^{p, \infty}}} \left\| v(\tau) \right\|_{H^{[\frac{d}{p}] - 1}_{L^{p, \infty}}} = (t - \tau)^{\alpha - 1} \left\| u(\tau) \right\|_{H^{[\frac{d}{p}] - 1}_{L^{p, \infty}}} \left\| v(\tau) \right\|_{H^{[\frac{d}{p}] - 1}_{L^{p, \infty}}},
\]

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where
\[
\alpha = \alpha \left( \frac{d}{\tilde{d}}, \tilde{p} \right) = \left[ \frac{d}{\tilde{p}} \right] - \frac{d}{\tilde{p}},
\]
this gives the desired result
\[
\left\| B(u, v)(t) \right\|_{\dot{H}^{\frac{d}{\tilde{p}} - 1}_{L^p}} \lesssim \int_0^t (t - \tau)^{\alpha - 1} \left\| u(\tau) \right\|_{\dot{H}^{\frac{d}{\tilde{p}} - 1}_{L^p}} \left\| v(\tau) \right\|_{\dot{H}^{\frac{d}{\tilde{p}} - 1}_{L^p}} d\tau
\]
\[
\lesssim \int_0^t (t - \tau)^{\alpha - 1} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \left\| u(\eta) \right\|_{\dot{H}^{\frac{d}{\tilde{p}} - 1}_{L^p}} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \left\| v(\eta) \right\|_{\dot{H}^{\frac{d}{\tilde{p}} - 1}_{L^p}} d\tau
\]
\[
= \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \left\| u(\eta) \right\|_{\dot{H}^{\frac{d}{\tilde{p}} - 1}_{L^p}} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \left\| v(\eta) \right\|_{\dot{H}^{\frac{d}{\tilde{p}} - 1}_{L^p}} \int_0^t (t - \tau)^{\alpha - 1} \tau^{-\alpha} d\tau
\]
\[
\simeq \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \left\| u(\eta) \right\|_{\dot{H}^{\frac{d}{\tilde{p}} - 1}_{L^p}} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \left\| v(\eta) \right\|_{\dot{H}^{\frac{d}{\tilde{p}} - 1}_{L^p}},
\]
(35)

Let us now check the validity of the condition (13) for the bilinear term \( B(u, v)(t) \). Indeed, from (35)
\[
\lim_{t \to 0} \left\| B(u, v)(t) \right\|_{\dot{H}^{\frac{d}{\tilde{p}} - 1}_{L^p}} = 0,
\]
whenever
\[
\lim_{t \to 0} t^{\frac{\alpha}{2}} \left\| u(t) \right\|_{\dot{H}^{\frac{d}{\tilde{p}} - 1}_{L^p}} = \lim_{t \to 0} t^{\frac{\alpha}{2}} \left\| v(t) \right\|_{\dot{H}^{\frac{d}{\tilde{p}} - 1}_{L^p}} = 0.
\]
The estimate (27) is deduced from the inequality (35).

\[\square\]

**Lemma 10.** Let \( 1 < p \leq d \). Then for all \( \tilde{p} \) be such that
\[
\left\lfloor \frac{d}{\tilde{p}} \right\rfloor - 1 < \frac{1}{p} < \min\left\{ \frac{\left\lfloor \frac{d}{\tilde{p}} \right\rfloor - 1}{d}, \frac{1}{2} + \frac{\left\lfloor \frac{d}{\tilde{p}} \right\rfloor - 1}{2d} \right\}
\]
(36)
the bilinear operator \( B(u, v)(t) \) is continuous from \( \mathcal{K}_{d, \infty, T}^{\tilde{p}} \times \mathcal{K}_{d, \infty, T}^{\tilde{p}} \) into \( \mathcal{K}_{d, \frac{1}{\tilde{p}}, 1, T}^{\tilde{p}} \) and the following inequality holds
\[
\left\| B(u, v) \right\|_{\mathcal{K}_{d, \frac{1}{\tilde{p}}, 1, T}^{\tilde{p}}} \leq C \left\| u \right\|_{\mathcal{K}_{d, \frac{1}{\tilde{p}}, 1, T}^{\tilde{p}}} \left\| v \right\|_{\mathcal{K}_{d, \frac{1}{\tilde{p}}, \infty, T}^{\tilde{p}}},
\]
(37)
where \( C \) is a positive constant and independent of \( T \).
Proof. First, arguing as in Lemma [9], we derive
\[
\hat{\Lambda}^{[\frac{d}{p}]-1}e^{(t-\tau)\Delta P}\nabla. (u(\tau) \otimes v(\tau)) = \frac{1}{(t-\tau)^{d+2}}K\left(\frac{\cdot}{\sqrt{t-\tau}}\right) * \left(\hat{\Lambda}^{[\frac{d}{p}]-1}(u(\tau) \otimes v(\tau))\right),
\]
where
\[
\hat{K}_{l,k,j}(\xi) = \frac{1}{(2\pi)^d/2}e^{-|\xi|^2/2}\left(\delta_{jk} - \frac{\xi_j k_l}{|\xi|^2}\right)(i\xi_l). \tag{38}
\]
Applying Theorem [2] for the convolution in the Fourier-Lorentz spaces, we have
\[
\left\|\hat{\Lambda}^{[\frac{d}{p}]-1}e^{(t-\tau)\Delta P}\nabla. (u(\tau) \otimes v(\tau))\right\|_{L^{p,1}} \lesssim \frac{1}{(t-\tau)^{d+2}}\left\|K\left(\frac{\cdot}{\sqrt{t-\tau}}\right)\right\|_{L^{r,1}}\left\|\hat{\Lambda}^{[\frac{d}{p}]-1}(u(\tau) \otimes v(\tau))\right\|_{L^{q,\infty}}, \tag{39}
\]
where
\[
\frac{1}{q} = \frac{2}{p} - \frac{[d]}{d-1} \quad \text{and} \quad \frac{1}{r} = 1 - \frac{1}{p} + \frac{[d]}{d-1}. \tag{40}
\]
Note that from the inequality (36), we can check that \(r\) and \(q\) satisfy the relations
\[
1 < r, q < \infty, \quad \frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}.
\]
Applying Lemma [4] we have
\[
\left\|\hat{\Lambda}^{[\frac{d}{p}]-1}(u(\tau) \otimes v(\tau))\right\|_{L^{q,\infty}} \lesssim \left\|u(\tau)\right\|_{H^{[\frac{d}{p}]-1}}\left\|v(\tau)\right\|_{H^{[\frac{d}{p}]-1}}. \tag{41}
\]
From the equalities (38) and (40), we obtain
\[
\left\|K\left(\frac{\cdot}{\sqrt{t-\tau}}\right)\right\|_{L^{r,1}} = (t-\tau)^{\frac{d}{2}}\left\|\hat{K}\right\|_{L^{r,1}} \simeq (t-\tau)^{\frac{d}{2}}\left(1 - \frac{1}{p} + \frac{[d]}{d} - 1\right). \tag{42}
\]
From the estimates (39), (41), and (42), we deduce that
\[
\left\|\hat{\Lambda}^{[\frac{d}{p}]-1}e^{(t-\tau)\Delta P}\nabla. (u(\tau) \otimes v(\tau))\right\|_{L^{p,1}} \lesssim (t-\tau)^{\frac{d}{2}}\left(\frac{[d]}{p} - 1\right)\left\|u(\tau)\right\|_{H^{[\frac{d}{p}]-1}}\left\|v(\tau)\right\|_{H^{[\frac{d}{p}]-1}}
\]
\[
\quad = (t-\tau)^{\frac{d}{2} - 1}\left\|u(\tau)\right\|_{H^{[\frac{d}{p}]-1}}\left\|v(\tau)\right\|_{H^{[\frac{d}{p}]-1}}.
\]

where
\[ \alpha = \alpha \left( \frac{d}{p}, \tilde{p} \right) = \left[ \frac{d}{p} \right] - \frac{d}{\tilde{p}} \]
this gives the desired result
\[
\left\| \mathcal{B}(u, v)(t) \right\|_{\dot{H}^{\frac{d}{r} - 1}_{p, \infty}} \leq \int_{0}^{t} (t - \tau)^{\frac{\alpha}{2} - 1} \left\| u(\tau) \right\|_{\dot{H}^{\frac{d}{r} - 1}_{p, \infty}} \left\| v(\tau) \right\|_{\dot{H}^{\frac{d}{r} - 1}_{p, \infty}} d\tau
\]
\[
\leq \int_{0}^{t} (t - \tau)^{\frac{\alpha}{2} - 1} \tau^{-\alpha} \sup_{0 < \eta < t} \eta^{\frac{d}{2}} \left\| u(\eta) \right\|_{H^{\frac{d}{r} - 1}_{p, \infty}} \sup_{0 < \eta < t} \eta^{\frac{d}{2}} \left\| v(\eta) \right\|_{H^{\frac{d}{r} - 1}_{p, \infty}} d\tau
\]
\[
= \sup_{0 < \eta < t} \eta^{\frac{d}{2}} \left\| u(\eta) \right\|_{H^{\frac{d}{r} - 1}_{p, \infty}} \sup_{0 < \eta < t} \eta^{\frac{d}{2}} \left\| v(\eta) \right\|_{H^{\frac{d}{r} - 1}_{p, \infty}} \int_{0}^{t} (t - \tau)^{\frac{\alpha}{2} - 1} \tau^{-\alpha} d\tau
\]
\[
\approx t^{-\frac{\alpha}{2}} \sup_{0 < \eta < t} \eta^{\frac{d}{2}} \left\| u(\eta) \right\|_{H^{\frac{d}{r} - 1}_{p, \infty}} \sup_{0 < \eta < t} \eta^{\frac{d}{2}} \left\| v(\eta) \right\|_{H^{\frac{d}{r} - 1}_{p, \infty}}
\]
(43)

Now we check the validity of condition (12) for the bilinear term \( \mathcal{B}(u, v)(t) \).
From (43) we infer that
\[
\lim_{t \to 0^+} t^{\frac{\alpha}{2}} \left\| \mathcal{B}(u, v)(t) \right\|_{\dot{H}^{\frac{d}{r} - 1}_{p, 1}} = 0,
\]
whenever
\[
\lim_{t \to 0^+} \left\| u(t) \right\|_{\dot{H}^{\frac{d}{r} - 1}_{p, \infty}} = \lim_{t \to 0^+} t^{\frac{\alpha}{2}} \left\| v(t) \right\|_{\dot{H}^{\frac{d}{r} - 1}_{p, \infty}} = 0.
\]
Finally, the estimate (37) can be deduced from the inequality (43).

**Theorem 5.** Let \( 1 < p \leq d \) and \( 1 \leq r < \infty \). Then for all \( \tilde{p} \) be such that
\[
\frac{1}{2p} + \frac{[\frac{d}{p}] - 1}{2d} < \frac{1}{\tilde{p}} < \min \left\{ \frac{[\frac{d}{p}]}{d}, \frac{1}{2}, \frac{[\frac{d}{p}]}{d} + \frac{[\frac{d}{p}] - 1}{2d} \right\},
\]
there exists a positive constant \( \delta_{p, \tilde{p}, d} \) such that for all \( T > 0 \) and for all \( u_0 \in \dot{H}^{\frac{d}{r} - 1}_{p, r} (\mathbb{R}^d) \) with \( \text{div}(u_0) = 0 \) satisfying
\[
\sup_{0 < t < T} t^{\frac{1}{2}([\frac{d}{p}] - 1)} \left\| e^{t \Delta} u_0 \right\|_{\dot{H}^{\frac{d}{r} - 1}_{p, \infty}} \leq \delta_{p, \tilde{p}, d},
\]
(44)
\[\text{NSE}\] has a unique mild solution \( u \in K^{\frac{d}{r} - 1}_{p, 1, T} \cap L^{\infty} ([0, T]; \dot{H}^{\frac{d}{r} - 1}_{p, r}) \).

In particular, the inequality (14) holds for arbitrary \( u_0 \in \dot{H}^{\frac{d}{r} - 1}_{p, r} (\mathbb{R}^d) \) when \( T(u_0) \) is small enough, and there exists a positive constant \( \sigma_{p, \tilde{p}, d} \) such that we can take \( T = \infty \) whenever \( \left\| u_0 \right\|_{\dot{B}^{\frac{d}{r} - 1}_{p, \infty}} \leq \sigma_{p, \tilde{p}, d} \).
exists a positive constant $C_{p,\tilde{p},d}$ is positive constant independent of $T$. From Theorem 4 and the above inequality, we deduce that for any $u_0 \in H_{\tilde{L}^{p,r}}^{\tilde{d}-1}$ such that
\[
\|e^{t}\Delta u_0\|_{\mathcal{K}_{\tilde{d}}^p,\infty,T} = \sup_{0<t<T} t^{\frac{1}{2}((\tilde{d})^{-1}-p)} \|e^{t}\Delta u_0\|_{H_{\tilde{L}^{p,r}}^{\tilde{d}-1}} \leq \frac{1}{4C_{p,\tilde{p},d}},
\]
the Navier-Stokes equations has a solution $u$ on the interval $(0, T)$ so that
\[
u \in \mathcal{K}_{\tilde{d}}^p,\infty,T.
\]
From Lemmas 6 and 10 and (45), we have
\[
B(u, u) \in \mathcal{K}_{p,1,T}^p \subseteq \mathcal{K}_{p,r,T}^p \subseteq L^\infty([0, T]; H_{\tilde{L}^{p,r}}^{\tilde{d}-1}).
\]
From Lemma 5 we also have $e^{t}\Delta u_0 \in L^\infty([0, T]; H_{\tilde{L}^{p,r}}^{\tilde{d}-1})$. Therefore
\[
u = e^{t}\Delta u_0 - B(u, u) \in L^\infty([0, T]; H_{\tilde{L}^{p,r}}^{\tilde{d}-1}).
\]
For all $u_0 \in H_{\tilde{L}^{p,r}}^{\tilde{d}-1}$, applying Theorem 3 we deduce that
\[
u_0 \in H_{\tilde{L}^{d/\tilde{p}},r}^{\tilde{d}-1}.
\]
From (46), applying Lemma 7 we get $e^{t}\Delta u_0 \in \mathcal{K}_{d/\tilde{p},1,T}^p$. From the definition of $\mathcal{K}_{p,r,T}$, we deduce that the left-hand side of the inequality (44) converges to 0 when $T$ tends to 0. Therefore the inequality (44) holds for arbitrary $u_0 \in H_{\tilde{L}^{p,r}}^{\tilde{d}-1}$ when $T(u_0)$ is small enough. Applying Lemmas 7 and 10 we conclude that $u \in \mathcal{K}_{d/\tilde{p},1,T}^p$.

Next, applying Theorem 5.4 ([24], p. 45), we deduce that the two quantities $\|u_0\|_{B_{\tilde{L}^{d/\tilde{p}},\infty}^{\tilde{d}-1,\infty}}$ and $\sup_{0<t<T} t^{\frac{1}{2}((\tilde{d})^{-1}-p)} \|e^{t}\Delta u_0\|_{H_{\tilde{L}^{d/\tilde{p}},\infty}^{\tilde{d}-1}}$ are equivalent, then there exists a positive constant $\sigma_{p,\tilde{p},d}$ such that $T = \infty$ and (44) holds whenever
\[
\|u_0\|_{B_{\tilde{L}^{d/\tilde{p}},\infty}^{\tilde{d}-1,\infty}} \leq \sigma_{p,\tilde{p},d}.
\]

Proof. From Lemmas 6 and 10 the bilinear operator $B(u, v)(t)$ is continuous from $\mathcal{K}_{\tilde{d}}^p,\infty,T \times \mathcal{K}_{\tilde{d}}^p,\infty,T$ into $\mathcal{K}_{\tilde{d}}^p,1,T$ and we have the inequality
\[
\|B(u, v)\|_{\mathcal{K}_{\tilde{d}}^p,\infty,T} \leq \|B(u, v)\|_{\mathcal{K}_{\tilde{d}}^p,1,T} \leq C_{p,\tilde{p},d} \|u\|_{\mathcal{K}_{d/\tilde{p},1,T}^p} \|v\|_{\mathcal{K}_{d/\tilde{p},1,T}^p},
\]
where $C_{p,\tilde{p},d}$ is positive constant independent of $T$. From Lemmas 6 and 9, and (45), we have $\|u\|_{\mathcal{K}_{d/\tilde{p},1,T}^p} \leq C_{p,\tilde{p},d} \|\Delta u_0\|_{\mathcal{K}_{d/\tilde{p},1,T}^p}$. Therefore the inequality (44) holds for arbitrary $u_0 \in H_{\tilde{L}^{p,r}}^{\tilde{d}-1}$, we deduce that the left-hand side of the inequality (44) converges to 0 when $T$ tends to 0. Therefore the inequality (44) holds for arbitrary $u_0 \in H_{\tilde{L}^{p,r}}^{\tilde{d}-1}$ when $T(u_0)$ is small enough. Applying Lemmas 7 and 10 we conclude that $u \in \mathcal{K}_{d/\tilde{p},1,T}^p$.
Remark 1. From Theorem \(3\) and the proof of Lemma \(7\) and Theorem 5.4 ([24], p. 45), we have the following imbedding maps

\[
\dot{H}^\frac{d}{p-1} L_{p,r}^d (\mathbb{R}^d) \hookrightarrow H^\frac{d}{p-1} L_{p,r}^d (\mathbb{R}^d) \hookrightarrow \dot{B}^\frac{d}{p-1} B_{p,1}^{\mu} (\mathbb{R}^d) \hookrightarrow \dot{B}^\frac{d}{p-1} B_{p,\infty} (\mathbb{R}^d).
\]

On the other hand, a function in \(\dot{H}^\frac{d}{p-1} L_{p,r}^d (\mathbb{R}^d) \) can be arbitrarily large in the \(\dot{H}^\frac{d}{p-1} L_{p,r}^d (\mathbb{R}^d) \) norm but small in the \(\dot{B}^\frac{d}{p-1} B_{p,\infty} (\mathbb{R}^d) \) norm.

3.2. Solutions to the Navier-Stokes equations with the initial value in the critical spaces \(\dot{H}^\frac{d}{p-1} L_{p,r}^d (\mathbb{R}^d) \) with \(d \leq p < \infty\) and \(1 \leq r < \infty\).

Lemma 11. Suppose that \(u_0 \in \dot{H}^\frac{d}{p-1} L_{p,r}^d (\mathbb{R}^d) \) with \(d \leq p < \infty\) and \(1 \leq r < \infty\). Then \(e^{\Delta t} u_0 \in K_{d,1,\infty}^p \) for all \(p > p\).

Proof. We prove that

\[
\sup_{0 < t < \infty} t^{\frac{\alpha}{p}} \| e^{\Delta t} u_0 \|_{L^{p,1}} \lesssim \| u_0 \|_{\dot{H}^\frac{d}{p-1} L_{p,r}^d},
\]

where

\[
\alpha = \alpha(d,p) = 1 - \frac{d}{p}.
\]

Let \(p'\) and \(p'\) be such that

\[
\frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{p} + \frac{1}{p'} = 1.
\]

We have

\[
\| e^{\Delta t} u_0 \|_{L^{p,1}} = \| e^{-t|\xi|^2} \hat{u}_0(\xi) \|_{L^{p',1}} = \| e^{-t|\xi|^2} |\xi|^{1 - \frac{d}{p'}} |\xi|^{\frac{d}{p'} - 1} \hat{u}_0(\xi) \|_{L^{p',1}}.
\]

Applying Holder’s inequality in the Lorentz spaces to obtain

\[
\| e^{-t|\xi|^2} |\xi|^{1 - \frac{d}{p'}} |\xi|^{\frac{d}{p'} - 1} \hat{u}_0(\xi) \|_{L^{p',1}} = t^{-\frac{1}{2}(1 - \frac{d}{p})} \| e^{-t|\xi|^2} |\xi|^{1 - \frac{d}{p'}} \hat{u}_0(\xi) \|_{L^{p',p}} \| |\xi|^{\frac{d}{p'} - 1} \hat{u}_0(\xi) \|_{L^{p',\infty}}
\]

\[
\approx t^{-\frac{d}{p}} \| |\xi|^{\frac{d}{p'} - 1} \hat{u}_0(\xi) \|_{L^{p',\infty}} \lesssim t^{-\frac{d}{p}} \| |\xi|^{\frac{d}{p'} - 1} \hat{u}_0(\xi) \|_{L^{p',r}} = t^{-\frac{d}{p}} \| u_0 \|_{\dot{H}^\frac{d}{p-1} L_{p,r}^d}.
\]

Therefore this gives the desired result

\[
\| e^{\Delta t} u_0 \|_{L^{p,1}} \lesssim t^{-\frac{d}{p}} \| u_0 \|_{\dot{H}^\frac{d}{p-1} L_{p,r}^d}.
\]
We claim now that 
\[ \lim_{t \to 0} t^{\frac{d}{2}} \| e^{t\Delta} u_0 \|_{L^{p,1}} = 0. \]
For any \( \epsilon > 0 \). Applying Lemma \[ \text{and from the above proof we deduce that} \]
\[ t^{\frac{d}{2}} \| e^{t\Delta} u_0 \|_{L^{p,1}} \leq \]
\[ t^{\frac{d}{2}} \| e^{-t|\xi|^2} |\xi|^{1-\frac{d}{p}} 1_{B_n} |\xi|^{\frac{d}{p}} \hat{u}_0(\xi) \|_{L^{p',1}} + t^{\frac{d}{2}} \| e^{-t|\xi|^2} |\xi|^{1-\frac{d}{p}} 1_{B_n} |\xi|^{\frac{d}{p}} \hat{u}_0(\xi) \|_{L^{p',1}} \leq \]
\[ C_1 \| e^{-|\xi|^2} |\xi|^{1-\frac{d}{p}} \|_{L^{p',1}} 1_{B_n} |\xi|^{\frac{d}{p}} \hat{u}_0(\xi) \|_{L^{p',\infty}} + C_2 t^{\frac{d}{2}} 1_{B_n} |\xi|^{1-\frac{d}{p}} \|_{L^{p',1}} \| |\xi|^{\frac{d}{p}} \hat{u}_0(\xi) \|_{L^{p',\infty}} \]
\[ \leq C_3 1_{B_n} |\xi|^{\frac{d}{p}} \hat{u}_0(\xi) \|_{L^{p',r}} + C_4 (n) t^{\frac{d}{2}} \| |\xi|^{\frac{d}{p}} \hat{u}_0(\xi) \|_{L^{p',r}} < \epsilon \]
for large enough \( n \) and small enough \( t = t(n) > 0. \)

\[ \text{Lemma 12. Let} \]
\[ p \geq d \text{ and } d < \tilde{p} < 2p. \]

Then the bilinear operator \( B(u, v)(t) \) is continuous from \( K_{d,\infty,T}^{\tilde{p}} \times K_{d,\infty,T}^{\tilde{p}} \) into \( K_{p,1,T}^{p} \), and we have the inequality
\[ \| B(u, v) \|_{K_{p,1,T}^{p}} \leq C \| u \|_{K_{d,\infty,T}^{\tilde{p}}} \| v \|_{K_{d,\infty,T}^{\tilde{p}}}, \]
where \( C \) is a positive constant and independent of \( T \).

\[ \text{Proof. First, arguing as in Lemma 9, we derive} \]
\[ \hat{A}^{\frac{d}{p}-1} e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau) \otimes v(\tau)) = \frac{1}{(t-\tau)\frac{d}{2}(\frac{d}{2}+1)} K \left( \frac{\xi}{\sqrt{t-\tau}} \right) (u(\tau) \otimes v(\tau)), \]
where the tensor \( K(x) = \{ K_{i,k,j}(x) \} \) is given by the formula
\[ \hat{K}_{i,k,j}(\xi) = \frac{1}{(2\pi)^d} |\xi|^{\frac{d}{2}-1} e^{-|\xi|^2} \left( \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (i\xi). \]

Applying Theorem 2 for the convolution in the Fourier-Lorentz spaces, we have
\[ \left\| \hat{A}^{\frac{d}{p}-1} e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau) \otimes v(\tau)) \right\|_{L^{p,1}} \]
\[ \lesssim \frac{1}{(t-\tau)\frac{d}{2}(\frac{d}{2}+1)} \left\| K \left( \frac{\xi}{\sqrt{t-\tau}} \right) \right\|_{L^{p,1}} \left\| (u(\tau) \otimes v(\tau)) \right\|_{L^{\frac{d}{p},\infty}}, \]
where
\[
\frac{1}{r} = 1 + \frac{1}{p} - \frac{2}{\hat{p}}. \tag{51}
\]

Note that from the inequality (47), we can check that \(1 < r < \infty\). Applying Theorem 1, we have
\[
\|u(\tau) \otimes v(\tau)\|_{L^\hat{p}p,\infty} \lesssim \|u(\tau)\|_{L^\hat{p},\infty} \|v(\tau)\|_{L^\hat{p},\infty}. \tag{52}
\]
From the equalities (49) and (51) it follows that
\[
\left\|K\left(\frac{\sqrt{t - \tau}}{\sqrt{t - \tau}}\right)\right\|_{L^{r,1}} = (t - \tau)^{\frac{d}{2}} \left\| \hat{K} \right\|_{L^{r,1}} \simeq (t - \tau)^{\frac{d}{2}(1 + \frac{1}{p} - \frac{2}{\hat{p}})}. \tag{53}
\]
From the estimates (50), (52), and (53), we deduce that
\[
\left\|e^{(t - \tau)A}F \nabla (u(\tau) \otimes v(\tau))\right\|_{H^{\frac{d}{2},p-1}} \lesssim (t - \tau)^{-\frac{d}{2}} \left\|u(\tau)\right\|_{L^\hat{p},\infty} \left\|v(\tau)\right\|_{L^\hat{p},\infty} = (t - \tau)^{a-1} \left\|u(\tau)\right\|_{L^\hat{p},\infty} \left\|v(\tau)\right\|_{L^\hat{p},\infty},
\]
where
\[
\alpha = \alpha(d, \hat{p}) = 1 - \frac{d}{\hat{p}}.
\]
This gives the desired result
\[
\left\|B(u, v)(t)\right\|_{H^{\frac{d}{2},p-1}} \lesssim \int_0^t (t - \tau)^{a-1} \left\|u(\tau)\right\|_{L^\hat{p},\infty} \left\|v(\tau)\right\|_{L^\hat{p},\infty} d\tau
\]
\[
\leq \int_0^t (t - \tau)^{a-1} \tau^{-\alpha} \sup_0<\eta<t \eta^{\frac{d}{2}} \left\|u(\eta)\right\|_{L^\hat{p},\infty} \sup_0<\eta<t \eta^{\frac{d}{2}} \left\|v(\eta)\right\|_{L^\hat{p},\infty} d\tau
\]
\[
= \sup_0<\eta<t \eta^{\frac{d}{2}} \left\|u(\eta)\right\|_{L^\hat{p},\infty} \sup_0<\eta<t \eta^{\frac{d}{2}} \left\|v(\eta)\right\|_{L^\hat{p},\infty} \int_0^t (t - \tau)^{a-1} \tau^{-\alpha} d\tau
\]
\[
\simeq \sup_0<\eta<t \eta^{\frac{d}{2}} \left\|u(\eta)\right\|_{L^\hat{p},\infty} \sup_0<\eta<t \eta^{\frac{d}{2}} \left\|v(\eta)\right\|_{L^\hat{p},\infty}. \tag{54}
\]
From (54) it follows the validity of (13) since
\[
\lim_{t \to 0} \left\|B(u, v)(t)\right\|_{H^{\frac{d}{2},p-1}} = 0,
\]
whenever
\[
\lim_{t \to 0} t^{\frac{d}{2}} \left\|u(t)\right\|_{L^\hat{p},\infty} = \lim_{t \to 0} t^{\frac{d}{2}} \left\|v(t)\right\|_{L^\hat{p},\infty} = 0.
\]
The estimate (48) can be deduced from the inequality (54). \qed
Lemma 13. Let \( \tilde{p} > d \), then the bilinear operator \( B(u, v)(t) \) is continuous from \( K_{d, \infty}^{\tilde{p}} \times K_{d, \infty}^{\tilde{p}} \) into \( K_{d, 1}^{\tilde{p}}, \) and we have the inequality

\[
\|B(u, v)\|_{K_{d, 1}^{\tilde{p}}} \leq C \|u\|_{K_{d, \infty}^{\tilde{p}}} \|v\|_{K_{d, \infty}^{\tilde{p}}},
\]

where \( C \) is a positive constant and independent of \( T \).

Proof. First, arguing as in Lemma 9, we derive

\[
e^{(t-\tau)\Delta} \nabla.(u(\tau) \otimes v(\tau)) = \frac{1}{(t-\tau)^{d+1}} K \left( \frac{x}{\sqrt{t-\tau}} \right) \ast (u(\tau) \otimes v(\tau)),
\]

where the tensor \( K(x) = \{K_{l,k,j}(x)\} \) is given by the formula

\[
\hat{K}_{l,k,j}(\xi) = \frac{1}{(2\pi)^{d/2}} e^{-|\xi|^2} \left( \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (i\xi)_l.
\]

Applying Theorem 2 for the convolution in the Fourier-Lorentz spaces, we have

\[
\|e^{(t-\tau)\Delta} \nabla.(u(\tau) \otimes v(\tau))\|_{L^{\tilde{p}, 1}} \lesssim \frac{1}{(t-\tau)^{d+1}} \|K \left( \frac{x}{\sqrt{t-\tau}} \right)\|_{L^{r, 1}} \|u(\tau) \otimes v(\tau)\|_{L^{1/\tilde{p}, \infty}},
\]

where

\[
\frac{1}{r} = 1 - \frac{1}{\tilde{p}}.
\]

Applying Theorem 1, we have

\[
\|u(\tau) \otimes v(\tau)\|_{L^{1/\tilde{p}, \infty}} \lesssim \|u(\tau)\|_{L^{1/\tilde{p}, \infty}} \|v(\tau)\|_{L^{1/\tilde{p}, \infty}}.
\]

From the equalities (56) and (58) it follows that

\[
\|K \left( \frac{x}{\sqrt{t-\tau}} \right)\|_{L^{r, 1}} = (t-\tau)^{-\frac{d}{2r}} \|K\|_{L^{r', 1}} \simeq (t-\tau)^{\frac{d}{2} - \frac{1}{r}}.
\]

From the estimates (57), (59), and (60), we deduce that

\[
\|e^{(t-\tau)\Delta} \nabla.(u(\tau) \otimes v(\tau))\|_{L^{\tilde{p}, 1}} \lesssim (t-\tau)^{-\frac{d}{2} - \frac{1}{r}} \|u(\tau)\|_{L^{1/\tilde{p}, \infty}} \|v(\tau)\|_{L^{1/\tilde{p}, \infty}}
\]

\[
= (t-\tau)^{\frac{d}{2} - 1} \|u(\tau)\|_{L^{1/\tilde{p}, \infty}} \|v(\tau)\|_{L^{1/\tilde{p}, \infty}},
\]

\[
23.
\]
\[ \alpha = \alpha(d, \tilde{p}) = 1 - \frac{d}{\tilde{p}}. \]

This gives the desired result

\[
\|B(u, v)(t)\|_{L^{\tilde{p}_1}} \leq \int_0^t (t - \tau)^{\frac{\tilde{p}_1}{2} - 1} \|u(\tau)\|_{L^{\tilde{p}_2}} \|v(\tau)\|_{L^{\tilde{p}_2}} d\tau
\]

\[
\leq \int_0^t (t - \tau)^{\frac{\tilde{p}_1}{2} - 1} \tau^{-\alpha} \sup_{0 < \eta < t} \eta^{\frac{\tilde{p}_1}{2}} \|u(\eta)\|_{L^{\tilde{p}_2}} \sup_{0 < \eta < t} \eta^{\frac{\tilde{p}_1}{2}} \|v(\eta)\|_{L^{\tilde{p}_2}} d\tau
\]

\[
= \sup_{0 < \eta < t} \eta^{\frac{\tilde{p}_1}{2}} \|u(\eta)\|_{L^{\tilde{p}_2}} \sup_{0 < \eta < t} \eta^{\frac{\tilde{p}_1}{2}} \|v(\eta)\|_{L^{\tilde{p}_2}} \int_0^t (t - \tau)^{\frac{\tilde{p}_1}{2} - 1} \tau^{-\alpha} d\tau
\]

\[
\simeq t^{-\frac{\tilde{p}_1}{2}} \sup_{0 < \eta < t} \eta^{\frac{\tilde{p}_1}{2}} \|u(\eta)\|_{L^{\tilde{p}_2}} \sup_{0 < \eta < t} \eta^{\frac{\tilde{p}_1}{2}} \|v(\eta)\|_{L^{\tilde{p}_2}}. \tag{61}
\]

From (61) it follows the validity of (12) since

\[
\lim_{t \to 0} t^{\frac{\tilde{p}_1}{2}} \|B(u, v)(t)\|_{L^{\tilde{p}_1}} = 0,
\]

whenever

\[
\lim_{t \to 0} t^{\frac{\tilde{p}_1}{2}} \|u(t)\|_{L^{\tilde{p}_2}} = \lim_{t \to 0} t^{\frac{\tilde{p}_1}{2}} \|v(t)\|_{L^{\tilde{p}_2}} = 0.
\]

Finally, the estimate (55) can be deduced from the inequality (61). The following lemma is a generalization of Lemma 13.

**Lemma 14.** Let \( d < \tilde{p}_1 < \infty \) and \( d \leq \tilde{p}_2 < \infty \) be such that one of the following conditions is satisfied

\[ d < \tilde{p}_1 < 2d, d \leq \tilde{p}_2 < \frac{d\tilde{p}_1}{2d - \tilde{p}_1}, \]

or

\[ \tilde{p}_1 = 2d, d \leq \tilde{p}_2 < \infty, \]

or

\[ 2d < \tilde{p}_1 < \infty, \frac{\tilde{p}_1}{2} < \tilde{p}_2 < \infty. \]

Then the bilinear operator \( B(u, v)(t) \) is continuous from \( K^{\tilde{p}_1}_{d, \infty, T} \times K^{\tilde{p}_1}_{d, \infty, T} \) into \( K^{\tilde{p}_2}_{d, 1, T} \), and we have the inequality

\[
\|B(u, v)\|_{K^{\tilde{p}_2}_{d, 1, T}} \leq C \|u\|_{K^{\tilde{p}_1}_{d, \infty, T}} \|v\|_{K^{\tilde{p}_1}_{d, \infty, T}},
\]

where \( C \) is a positive constant and independent of \( T \).
Theorem 6. Let \( p \geq d \) and \( 1 \leq r < \infty \). Then for any \( \tilde{p} \) such that
\[
\tilde{p} > p,
\]
there exists a positive constant \( \delta_{\tilde{p},d} \) such that for all \( T > 0 \) and for all \( u_0 \in \dot{H}^{\frac{d}{p}-1}_{L^r}(\mathbb{R}^d) \), with \( \text{div}(u_0) = 0 \) satisfying
\[
\sup_{0 < t < T} t^{\frac{1}{2}(1 - \frac{d}{p})} \| e^{t\Delta} u_0 \|_{L^r,\infty} \leq \delta_{\tilde{p},d},
\]
NSE has a unique mild solution \( u \in \cap_{q>p} K_{d,1,T}^q \cap L^\infty([0,T]; \dot{H}^{\frac{d}{p}-1}_{L^r}) \).
In particular, the inequality \((63)\) holds for arbitrary \( u_0 \in \dot{H}^{\frac{d}{p}-1}_{L^r}(\mathbb{R}^d) \) with \( T(u_0) \) is small enough, and there exists a positive constant \( \sigma_{\tilde{p},d} \) such that we can take \( T = \infty \) whenever \( \| u_0 \|_{B^{\frac{d}{p}-1}_{L^\infty}} \leq \sigma_{\tilde{p},d} \).

Proof. Applying Lemma \[ ] and Theorem \[ ] we deduce that there exists a positive constant \( \delta_{\tilde{p},d} \) such that for all \( T > 0 \) and for all \( u_0 \in \dot{H}^{\frac{d}{p}-1}_{L^r}(\mathbb{R}^d) \), with \( \text{div}(u_0) = 0 \) satisfying the inequality \((63)\) then NSE has a unique mild solution \( u \in K_{d,1,T}^p \). Next, we prove that \( u \in \cap_{q>p} K_{d,1,T}^q \).
Consider two cases \( d < \tilde{p} < 2d \) and \( 2d \leq \tilde{p} < \infty \) separately.
First, we consider the case \( d < \tilde{p} < 2d \). We consider two possibilities \( \tilde{p} > \frac{4d}{3} \) and \( \tilde{p} \leq \frac{4d}{3} \). In the case \( \tilde{p} > \frac{4d}{3} \), we apply Lemmas \[ ] and \[ ] to obtained \( u \in K_{d,1,T}^{\tilde{p}} \) for all \( q \) satisfying \( p < q < \tilde{p}_1 \) where \( \tilde{p}_1 = \frac{dp}{2d-\tilde{p}} > 2d \). Thus, \( u \in K_{d,1,T}^{\tilde{p}_1} \). Applying again Lemmas \[ ] and \[ ], we deduce that \( u \in K_{d,1,T}^{q} \) for all \( q > p \). In the case \( \tilde{p} \leq \frac{4d}{3} \), we set up the following series of numbers \( \{\tilde{p}_i\}_{0 \leq i \leq N} \) by inductive. Set \( \tilde{p}_0 = \tilde{p} \) and \( \tilde{p}_1 = \frac{dp}{2d-\tilde{p}_0} \). We have \( \tilde{p}_1 > \tilde{p}_0 \). If \( \tilde{p}_1 > \frac{4d}{3} \), then set \( N = 1 \) and stop here. In the case \( \tilde{p}_1 \leq \frac{4d}{3} \), set \( \tilde{p}_2 = \frac{dp}{2d-\tilde{p}_1} \). We have \( \tilde{p}_2 > \tilde{p}_1 \). If \( \tilde{p}_2 > \frac{4d}{3} \), then set \( N = 2 \) and stop here. In the case \( \tilde{p}_2 \leq \frac{4d}{3} \), set \( \tilde{p}_3 = \frac{dp}{2d-\tilde{p}_2} \). We have \( \tilde{p}_3 > \tilde{p}_2 \), and so on, there exists \( k \geq 0 \) such that \( \tilde{p}_k \leq \frac{4d}{3} \). \( \tilde{p}_{k+1} = \frac{dp}{2d-\tilde{p}_k} > \frac{4d}{3} \). We set \( N = k + 1 \) and stop here, and we have
\[
\tilde{p}_0 = \tilde{p}, \tilde{p}_i = \frac{dp_{i-1}}{2d - \tilde{p}_{i-1}}, \tilde{p}_i > \tilde{p}_{i-1} \text{ for } i = 1, 2, 3, ..., N,
\]
\[
2d \geq \tilde{p}_N > \frac{4d}{3} \geq \tilde{p}_{N-1}.
\]
From \( u \in K_{d,1,T}^{\tilde{p}_0} \), applying Lemmas \[ ] and \[ ] to obtained \( u \in K_{d,1,T}^{q} \) for all \( q \) satisfying \( p < q < \tilde{p}_1 \). Then applying again Lemmas \[ ] and \[ ] to
obtained $u \in K_{d,1,T}^q$ for all $q$ satisfying $p < q < \hat{p}_2$, and so on, finishing we have $u \in K_{d,1,T}^q$ for all $q$ satisfying $p < q < \hat{p}_N$. Therefore $u \in K_{d,1,T}^q$ for all $q$ satisfying $\frac{4d}{3} < q < \hat{p}_N$. From the proof of the case $\hat{p}_N > \frac{4d}{3}$, we have $u \in K_{d,1,T}^q$ for all $q > p$.

Next, we consider the case $2d \leq \hat{p} < \infty$. Let $i \in \mathbb{N}$ be such that

$$\frac{\hat{p} - 1}{2i} \geq \max\{2d, p\} > \frac{\hat{p}}{2i}.$$ 

From $\hat{p} \geq \max\{2d, p\}$, we have $i \geq 1$. Applying the Lemmas 11 and 14 to obtained $u \in K_{d,1,T}^q$ for all $q > \frac{\hat{p}}{2}$. Applying again Lemmas 11 and 14 to obtained $u \in K_{d,1,T}^q$ for all $q > \frac{\hat{p}}{2}$, and so on, finishing we have $u \in K_{d,1,T}^q$ for all $q > \frac{\hat{p}}{2}$. Applying again Lemmas 11 and 14 to obtained $u \in K_{d,1,T}^q$ for all $q > \max\{p, \frac{\hat{p}}{2}\}$. If $p \geq \frac{\hat{p}}{2}$ then we have $u \in K_{d,1,T}^q$ for all $q > p$. If $p < \frac{\hat{p}}{2}$ then $2d > \frac{\hat{p}}{2}$. Thus $u \in K_{d,1,T}^q$ for all $q$ satisfying $\frac{\hat{p}}{2} < q < 2d$. Therefore, from the proof of the case $d < \hat{p} < 2d$, we have $u \in K_{d,1,T}^q$ for all $q > p$.

The fact that $u \in L^\infty([0, T]; H_{L,p,r}^{\frac{d}{2} - 1})$ can be deduced from Lemmas 5 and 12. Applying Lemma 11, we get $e^{t\Delta}u_0 \in K_{d,\infty,T}^1$. From the definition of $K_{p,r,T}^1$, we deduce that the left-hand side of the inequality (63) converges to 0 when $T$ tends to 0. Therefore the inequality (63) holds for arbitrary $u_0 \in H_{L,p,r}^{\frac{d}{2} - 1}$ when $T(u_0)$ is small enough.

Next, applying Theorem 5.4 ([24], p. 45), we deduce that the two quantities $\|u_0\|_{H_{L,p,r}^{\frac{d}{2} - 1}}$ and $\sup_{0 < t < \infty} t^{\frac{d}{2}(1 - \frac{d}{r})}\|e^{t\Delta}u_0\|_{L^{p,d}}$ are equivalent, then there exists a positive constant $\sigma_{p,d}$ such that $T = \infty$ and (63) holds whenever

$$\|u_0\|_{H_{L,p,r}^{\frac{d}{2} - 1}} \leq \sigma_{p,d}.$$ 

**Remark 2.** From the proof of Lemma 11 and Theorem 5.4 ([24], p. 45), we have the following imbedding maps

$$H_{L,p,r}^{\frac{d}{2} - 1}(\mathbb{R}^d) \hookrightarrow B_{L,p,r}^{\frac{d}{2} - 1,\infty}(\mathbb{R}^d) \hookrightarrow B_{L,p,r}^{\frac{d}{2} - 1,\infty}(\mathbb{R}^d).$$

On the other hand, a function in $H_{L,p,r}^{\frac{d}{2} - 1}(\mathbb{R}^d)$ can be arbitrarily large in the $H_{L,p,r}^{\frac{d}{2} - 1}(\mathbb{R}^d)$ norm but small in the $B_{L,p,r}^{\frac{d}{2} - 1,\infty}(\mathbb{R}^d)$ norm.

### 3.3. Solutions to the Navier-Stokes equations with initial value in the critical spaces $H_{L,r}^{d-1}(\mathbb{R}^d)$ with $1 \leq r < \infty$. 

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We define an auxiliary space $K_{s,r,T}$ which is made up by the functions $u(t,x)$ such that
\[ \|u\|_{K_{s,r,T}} := \sup_{0 < t < T} t^\frac{\alpha}{2} \|u(t,x)\|_{H^s_{L^1,r}} < \infty, \]
and
\[ \lim_{t \to 0} t^\frac{\alpha}{2} \|u(t,x)\|_{H^s_{L^1,r}} = 0, \quad (64) \]
with
\[ d - 1 \leq s < d, \quad 1 \leq r \leq \infty, \quad T > 0, \]
and
\[ \alpha = \alpha(s) = s + 1 - d. \]
In the case $s = d - 1$, it is also convenient to define the space $K_{d-1,r,T}$ as
the natural space $L^\infty([0,T]; \dot{H}^{d-1}_{L^1,r})$ with the additional condition that its
elements $u(t,x)$ satisfy
\[ \lim_{t \to 0} \|u(t,x)\|_{\dot{H}^{d-1}_{L^1,r}} = 0. \quad (65) \]

**Lemma 15.** Let $1 \leq r \leq \tilde{r} \leq \infty$. Then we have the following imbedding

\[ K_{s,1,T} \hookrightarrow K_{s,r,T} \hookrightarrow K_{s,\tilde{r},T} \hookrightarrow K_{s,\infty,T}. \]

**Proof.** It is deduced from Lemma 1 (a) and the definition of $K_{s,r,T}$. \qed

**Lemma 16.** Suppose that $u_0 \in \dot{H}^{d-1}_{L^1,r}(\mathbb{R}^d)$ with $1 \leq r < \infty$, then $e^{t\Delta}u_0 \in K_{s,r,\infty}$ with $d - 1 < s < d$.

**Proof.** We prove that
\[ \sup_{0 < t < \infty} t^\frac{\alpha}{2} \|e^{t\Delta}u_0\|_{H^s_{L^1,r}} \lesssim \|u_0\|_{\dot{H}^{d-1}_{L^1,r}} \text{ for } 1 \leq r \leq \infty. \quad (66) \]
We have
\[ \|e^{t\Delta}u_0\|_{H^s_{L^1,r}} = \|e^{-t|\xi|^2} |\xi|^s \hat{u}_0(\xi)\|_{L^\infty} = \||\xi|^{s+1-d}e^{-t|\xi|^2} |\xi|^{d-1} \hat{u}_0(\xi)\|_{L^\infty} \]
\[ \leq t^{-\frac{s+1-d}{2}} \|\xi|^{s+1-d}e^{-t|\xi|^2}\|_{L^\infty} \|\xi|^{d-1} \hat{u}_0(\xi)\|_{L^\infty} \approx t^{-\frac{\alpha}{2}} \|u_0\|_{\dot{H}^{d-1}_{L^1,r}}. \quad (67) \]
We claim now that
\[ \lim_{t \to 0} t^\frac{\alpha}{2} \|e^{t\Delta}u_0\|_{H^s_{L^1,r}} = 0 \text{ for } 1 \leq r < \infty. \]
From the inequality (67), we have
\[
 t^{\frac{2}{d}} \left\| e^{t\Delta} u_0 \right\|_{H^{s+1}_t L^1_x} \leq 
 t^{\frac{2}{d}} \left\| |\xi|^{s+1} \hat{e}^{-t|\xi|^2} \chi_{B_n} |\xi|^{d-1} \hat{u}_0(\xi) \right\|_{L_\xi^\infty} + t^{\frac{2}{d}} \left\| |\xi|^{s+1} \hat{e}^{-t|\xi|^2} \chi_{B_n} |\xi|^{d-1} \hat{u}_0(\xi) \right\|_{L_\xi^\infty}.
\]
For any \( \epsilon > 0 \), applying Lemma 8, we have
\[
 t^{\frac{2}{d}} \left\| |\xi|^{s+1} \hat{e}^{-t|\xi|^2} \chi_{B_n} |\xi|^{d-1} \hat{u}_0(\xi) \right\|_{L_\xi^\infty} \leq \left\| |\xi|^{s+1} \hat{e}^{-t|\xi|^2} \right\|_{L_\xi^\infty} \left\| \chi_{B_n} |\xi|^{d-1} \hat{u}_0(\xi) \right\|_{L_\xi^\infty} = C \left\| \chi_{B_n} |\xi|^{d-1} \hat{u}_0(\xi) \right\|_{L_\xi^\infty} < \frac{\epsilon}{2},
\]
for large enough \( n \). Fixed one of such \( n \), we have the following estimates
\[
 t^{\frac{2}{d}} \left\| |\xi|^{s+1} \hat{e}^{-t|\xi|^2} \chi_{B_n} |\xi|^{d-1} \hat{u}_0(\xi) \right\|_{L_\xi^\infty} \leq t^{\frac{2}{d}} \left\| \chi_{B_n} |\xi|^{s+1} \hat{e}^{-t|\xi|^2} \right\|_{L_\xi^\infty} \left\| |\xi|^{d-1} \hat{u}_0(\xi) \right\|_{L_\xi^\infty} \leq t^{\frac{2}{d}} \left\| \chi_{B_n} |\xi|^{s+1} \hat{e}^{-t|\xi|^2} \right\|_{L_\xi^\infty} \left\| |\xi|^{d-1} \hat{u}_0(\xi) \right\|_{L_\xi^\infty} = t^{\frac{2}{d}} n^{s+1-d} \left\| |\xi|^{d-1} \hat{u}_0(\xi) \right\|_{L_\xi^\infty} \leq \frac{\epsilon}{2}
\]
for small enough \( t = t(n) > 0 \). From the estimates (68) and (69), we have,
\[
 t^{\frac{2}{d}} \left\| \hat{e}^{t\Delta} u_0 \right\|_{H^s_{t,x}} \leq C \left\| \chi_{B_n} |\xi|^{d-1} \hat{u}_0(\xi) \right\|_{L_\xi^\infty} + t^{\frac{2}{d}} n^{s+1-d} \left\| u_0 \right\|_{H^{d-1}_x} < \epsilon. \quad \Box
\]

Lemma 17. Let \( d - 1 < s < d \). Then the bilinear operator \( B(u,v)(t) \) is continuous from \( K_{s,\infty,T} \times K_{s,\infty,T} \) into \( K_{s,1,T} \) and we have the inequality
\[
 \| B(u,v) \|_{K_{s,1,T}} \leq C \left\| u \right\|_{K_{s,\infty,T}} \left\| v \right\|_{K_{s,\infty,T}},
\]
where \( C \) is a positive constant and independent of \( T \).

Proof. Using the Fourier transform we get
\[
 \mathcal{F}(B(u,v)(t)) (\xi) = \frac{1}{(2\pi)^{d/2}} \int_0^t \hat{e}^{-\langle u-\tau \rangle |\xi|^2} \sum_{l,k=1}^d \left( \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (i \xi_l) \left( \hat{u_l}(\tau) * \hat{v_k}(\tau) \right)(\xi) d\tau.
\]
Thus
\[
 \left\| |\xi|^s \mathcal{F}(B(u,v)(t)) (\xi) \right\| \lesssim \int_0^t \left\| |\xi|^s e^{-\langle u-\tau \rangle |\xi|^2} |\xi| \left( |\hat{u}(\tau)| + |\hat{v}(\tau)| \right) \right\| d\tau.
\]
We have

\[
|\xi|^s |\widehat{u}(\tau)(\xi)| \leq \sup_{\xi \in \mathbb{R}^d} |\xi|^s |\widehat{v}(\tau)(\xi)| = \|u(\tau)\|_{\dot{H}^s}^s \text{ and } |\xi|^s |\widehat{v}(\tau)(\xi)| \leq \|v(\tau)\|_{\dot{H}^s}^s,
\]

therefore

\[
|\widehat{u}(\tau)(\xi)| \leq \frac{\|u(\tau)\|_{\dot{H}^s}^s}{|\xi|^s}, \quad |\widehat{v}(\tau)(\xi)| \leq \frac{\|v(\tau)\|_{\dot{H}^s}^s}{|\xi|^s}.
\]

A standard argument shows that

\[
\frac{1}{|\xi|^s} \cdot \frac{1}{|\xi|^s} = \frac{C}{|\xi|^{2s-d}}.
\]

From the above estimates and Lemma \[\text{(b)}\] (b), we have

\[
\left| \left( \|u(\tau)\|_{\dot{H}^s}^s \right) \right| \left( \|v(\tau)\|_{\dot{H}^s}^s \right) \mid \xi \mid \leq \frac{\|u(\tau)\|_{\dot{H}^s}^s}{|\xi|^s} \frac{\|v(\tau)\|_{\dot{H}^s}^s}{|\xi|^s} \sim \frac{\|u(\tau)\|_{\dot{H}^s}^s \cdot \|v(\tau)\|_{\dot{H}^s}^s}{|\xi|^{2s-d}}.
\]

This gives the desired result

\[
\int_0^t |\xi|^s e^{-r} |\xi|^2 \left( \|\hat{u}(\tau)\| \cdot \|\hat{v}(\tau)\| \right) d\tau \leq \int_0^t |\xi|^d e^{-r} |\xi|^2 \left( \|u(\tau)\|_{\dot{H}^s} \cdot \|v(\tau)\|_{\dot{H}^s} \right) d\tau.
\]

Thus

\[
\left\| \left\| |\xi|^s \mathcal{F}(B(u, v)(t)) \right\| \right\|_{L^2} \leq \int_0^t \left\| |\xi|^s e^{-r} |\xi|^2 \left( \|u(\tau)\|_{\dot{H}^s} \cdot \|v(\tau)\|_{\dot{H}^s} \right) \right\|_{L^2} d\tau
\]

\[
= \int_0^t \left( t - s \right)^{\frac{s-d-1}{2}} \left\| |\xi|^s e^{-r} |\xi|^2 \left( \|u(\tau)\|_{\dot{H}^s} \cdot \|v(\tau)\|_{\dot{H}^s} \right) \right\|_{L^2} d\tau
\]

\[
\leq \int_0^t \left( t - s \right)^{\frac{s-d-1}{2}} \left\| \left\| u(\eta) \right\|_{\dot{H}^s} \cdot \left\| v(\eta) \right\|_{\dot{H}^s} \right\|_{L^\infty,1} d\tau
\]

\[
= \sup_{0 < \eta < t} \left\| u(\eta) \right\|_{\dot{H}^s} \cdot \sup_{0 < \eta < t} \left\| v(\eta) \right\|_{\dot{H}^s} \int_0^t \left( t - s \right)^{\frac{s-d-1}{2}} \left\| u(\eta) \right\|_{\dot{H}^s} \cdot \left\| v(\eta) \right\|_{\dot{H}^s} d\tau
\]

\[
\simeq \left\| t^{-\frac{s}{2}} \sup_{0 < \eta < t} \left\| u(\eta) \right\|_{\dot{H}^s} \cdot \sup_{0 < \eta < t} \left\| v(\eta) \right\|_{\dot{H}^s} \right\|_{L^2}.
\]
Let us now check the validity of the condition \((64)\) for the bilinear term \(B(u, v)(t)\). Indeed, from \((71)\)

\[
\lim_{t \to 0} t^{\frac{d}{2}} \| B(u, v)(t) \|_{H^{s}_{L^2}} = \lim_{t \to 0} t^{\frac{d}{2}} \| \xi^d \mathcal{F}(B(u, v)(t))(\xi) \|_{L^1} = 0,
\]

whenever

\[
\lim_{t \to 0} t^{\frac{d}{2}} \| u(t) \|_{H^{s}_{L^2}} = \lim_{t \to 0} t^{\frac{d}{2}} \| v(t) \|_{H^{s}_{L^2}} = 0.
\]

The estimate \((70)\) is deduced from the inequality \((71)\).

**Lemma 18.** Let \(d - 1 < s < d\). Then the bilinear operator \(B(u, v)(t)\) is continuous from \(K_{s, \infty, T} \times K_{s, \infty, T}\) into \(K_{d-1, T}\) and we have the inequality

\[
\| B(u, v) \|_{K_{d-1, T}} \leq C \| u \|_{K_{s, \infty, T}} \| v \|_{K_{s, \infty, T}},
\]

where \(C\) is a positive constant and independent of \(T\).

**Proof.** First, arguing as in Lemma \([18]\), we have the following estimates

\[
\| \xi^d \mathcal{F}(B(u, v)(t))(\xi) \|_{L^1} \\
\lesssim \int_0^t |\xi|^{d-1} e^{-(t-\tau)|\xi|^2} |\xi||\xi|(\|u(\tau)\| \ast |v(\tau)|)(\xi)d\tau \\
\lesssim \int_0^t |\xi|^{2d-2} e^{-(t-\tau)|\xi|^2} \| u(\tau) \|_{H^{s}_{L^2}} \| v(\tau) \|_{H^{s}_{L^2}} d\tau,
\]

this gives the desired result

\[
\| \xi^d \mathcal{F}(B(u, v)(t))(\xi) \|_{L^1} \\
\lesssim \int_0^t \| \xi |^{d-2} e^{-(t-\tau)|\xi|^2} \|_{L^1} \| u(\tau) \|_{H^{s}_{L^2}} \| v(\tau) \|_{H^{s}_{L^2}} d\tau \]

\[
= \int_0^t (t-s)^{a-d} \| \xi |^{d-2} e^{-(t-\tau)|\xi|^2} \|_{L^1} \| u(\tau) \|_{H^{s}_{L^2}} \| v(\tau) \|_{H^{s}_{L^2}} d\tau \\
\lesssim \int_0^t (t-\tau)^{a-1} \sup_{0<\eta<\xi} \eta^{\frac{d}{2}} \| u(\eta) \|_{H^{s}_{L^2}} \| v(\eta) \|_{H^{s}_{L^2}} d\tau \\
= \sup_{0<\eta<\xi} \eta^{\frac{d}{2}} \| u(\eta) \|_{H^{s}_{L^2}} \| v(\eta) \|_{H^{s}_{L^2}} \int_0^t (t-\tau)^{-a} d\tau \\
\simeq \sup_{0<\eta<\xi} \eta^{\frac{d}{2}} \| u(\eta) \|_{H^{s}_{L^2}} \| v(\eta) \|_{H^{s}_{L^2}} .
\]

(73)
From (73) it follows (65) since
\[
\lim_{t \to 0} \left\| B(u, v)(t) \right\|_{H^{d-1}_{L^1}} = \lim_{t \to 0} \left\| \xi^{d-1} \mathcal{F} \left( B(u, v)(t) \right)(\xi) \right\|_{L^\infty} = 0,
\]
whenever
\[
\lim_{t \to 0} t^s \left\| u(t) \right\|_{H^s_{L^1, \infty}} = \lim_{t \to 0} t^s \left\| v(t) \right\|_{H^s_{L^1, \infty}} = 0.
\]
The estimate (72) can be deduced from the inequality (73).

**Theorem 7.** Let \(d - 1 < s < d\) and \(1 \leq r < \infty\). Then there exists a positive constant \(\delta_{s,d}\) such that for all \(T > 0\) and for all \(u_0 \in \tilde{H}^{d-1}_{L^1, r}(\mathbb{R}^d)\), with \(\text{div}(u_0) = 0\) satisfying
\[
\sup_{0 < t < T} t^\frac{1}{2}(s+1-d) \left\| e^{t \Delta} u_0 \right\|_{H^s_{L^1}} \leq \delta_{s,d},
\]
\(\text{NSE}\) has a unique mild solution \(u \in \mathcal{K}_{s,r,T} \cap L^\infty([0, T]; \tilde{H}^{d-1}_{L^1, r})\).

In particular, the inequality (74) holds for arbitrary \(u_0 \in \tilde{H}^{d-1}_{L^1, r}(\mathbb{R}^d)\) when \(T(u_0)\) is small enough, and there exists a positive constant \(\sigma_{s,d}\) such that we can take \(T = \infty\) whenever \(\left\| u_0 \right\|_{\tilde{H}^{d-1}_{L^1}} \leq \sigma_{s,d}\).

**Proof.** The proof of Theorem 7 is similar to that of Theorem 3. Applying Lemma 17 and Theorem 4 we deduce that there exists a positive constant \(\delta_{s,d}\) such that for any \(u_0 \in \tilde{H}^{d-1}_{L^1, r}(\mathbb{R}^d)\) with \(\text{div}(u_0) = 0\) such that
\[
\sup_{0 < t < T} t^\frac{1}{2}(s+1-d) \left\| e^{t \Delta} u_0 \right\|_{H^s_{L^1, \infty}} = \sup_{0 < t < T} t^\frac{1}{2}(s+1-d) \left\| e^{t \Delta} u_0 \right\|_{H^s_{L^1}} \leq \delta_{s,d},
\]
the Navier-Stokes equations has a solution \(u \in \mathcal{K}_{s,r,T}\). Applying Lemmas 5 and 18 we deduce that \(u \in L^\infty([0, T]; \tilde{H}^{d-1}_{L^1, r})\). Applying Lemma 16 we get \(e^{t \Delta} u_0 \in \mathcal{K}_{s,r,T}\). From the definition of \(\mathcal{K}_{s,r,T}\), we deduce that the left-hand side of the inequality (74) converges to 0 when \(T\) tends to 0. Therefore the inequality (74) holds for arbitrary \(u_0 \in \tilde{H}^{d-1}_{L^1, r}(\mathbb{R}^d)\) when \(T(u_0)\) is small enough.

Next, from the inequality (66) with \(r = \infty\), we deduce that
\[
\sup_{0 < t < \infty} t^\frac{1}{2}(s+1-d) \left\| e^{t \Delta} u_0 \right\|_{H^s_{L^1}} \lesssim \left\| u_0 \right\|_{\tilde{H}^{d-1}_{L^1}},
\]
then there exists a positive constant \(\sigma_{s,d}\) such that \(T = \infty\) and (74) holds whenever \(\left\| u_0 \right\|_{\tilde{H}^{d-1}_{L^1}} \leq \sigma_{s,d}\). \(\square\)
Remark 3. The case $r = \infty$ was studied by Le Jan and Sznitman in [26]. They showed that NSE are well-posed when the initial datum belongs to the space $\dot{H}^{d-1}_{L^1,\infty}$. For $1 \leq r < \infty$ we have the following imbedding map

$$\dot{H}^{d-1}_{L^1,r}(\mathbb{R}^d) \hookrightarrow \dot{H}^{d-1}_{L^1,\infty}(\mathbb{R}^d) = \dot{H}^{d-1}_{L^1}(\mathbb{R}^d).$$

However, note that for $1 \leq r < \infty$ a function in $\dot{H}^{d-1}_{L^1,r}(\mathbb{R}^d)$ can be arbitrarily large in the $\dot{H}^{d-1}_{L^1,\infty}(\mathbb{R}^d)$ norm but small in the $\dot{H}^{d-1}_{L^1}(\mathbb{R}^d)$ norm. Theorem 7 shows the existence of global mild solutions in the spaces $L^\infty([0, \infty); \dot{H}^{d-1}_{L^1,r}(\mathbb{R}^d))$ (with $1 \leq r < \infty$) when the norm of the initial value in the spaces $\dot{H}^{d-1}_{L^1}(\mathbb{R}^d)$ is small enough.

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