NUMERICAL PERFORMANCE OF ABS CODES
FOR NONLINEAR SYSTEMS OF EQUATIONS

E. Bodon 1, A. Del Popolo 2, L. Lukšan 3 and E. Spedicato 5

1. Introduction

The nonlinear ABS algorithms solve a nonlinear system of algebraic equations \( F(x) = 0 \) where \( F : R^n \rightarrow R^n \) in component form \( f_1(x) = 0, f_2(x) = 0, \ldots, f_n(x) = 0, x \in R^n \). The ABS algorithms can be viewed as a modification of the Newton method. Newon method constructs an iterative sequence of points \( x_i, i = 1, 2, \ldots \) by the relation \( x_{i+1} = x_i + t_i \), where \( t_i \) is the solution of the linear system \( J(x_i)t_i = -F(x_i) \) or in component form \( a^T_i(x_i)t_i = -f_k(x_i), k = 1, \ldots, n \), where \( J(x_i) \) indicates the Jacobian matrix of \( F \) evaluated at \( x_i \), while \( a_k(x_i) \) denotes the \( k \)-th row of \( J \) evaluated at \( x_i \). The set of linear equations \( a^T_k(x_i)t_i = -f_k(x_i), k = 1, \ldots, n \) is solved by the ABS method for linear systems after modifying the definition of \( J \) and \( F \). The motivation is based on the fact that at the \( k \)-th step the ABS algorithms generate the next approximation to the solution \( t_i \) using the \( k \)-th equation and the vector \( t_i \), which is a solution of the system formed by the first \( k - 1 \) equations. Since the new approximate point \( x_i + t_i \) is expected to be closer to the solution \( x^* \) of \( F(x) = 0 \), it can be used to evaluate the \( k \)-th row of \( J \) and the \( k \)-th component of \( F \). The procedure is the following.

Assing an initial vector \( x_0 \), set \( m = 1 \),

\[
(\ast) \text{ set } y_1 = x_{m-1}, H_1 = I, \text{ for } k = 1 \text{ to } m \text{ compute:}
\]

\[
p_k = H_k^T z_k \text{ where } z_k \in R^n \text{ is a parameter which gives the type of ABS method,}
\]

\[
\beta_k = f_k(y_k)/\delta_k, \quad \delta_k = p_k^T a_k(y_k), \quad y_{k+1} = y_k - \beta_k p_k, \quad H_{k+1} = H_k - H_k a_k(y_k) p_k^T / \delta_k,
\]

let \( x_m = y_{n+1} \),

if the stopping criterion is satisfied, stop; otherwise increment the index \( m \) by 1 and return to \((\ast)\).

2. The implemented codes

The ABS codes that we have implemented based on the above algorithm use three choices of the available parameter \( z_k \) that define three well-known ABS methods for the linear case.

The first code, routine nl-huang1, and the second code, routine nl-huang2, use the Huang method defined by \( z_k = a_k(y_k) \), and the modified Huang method defined by \( z_k = H_k^T a_k(y_k) \) and \( \delta_k = p_k^T p_k \).

In the routine nl-huang1 the matrix \( H_k \) is updated by formula \( H_{k+1} = H_k - H_k a_k(y_k) p_k^T / \delta_k \).

In one major iteration (consisting of a sequence of \( n \) minor iterates such that system \( J(x)t_i = -F(x) \) is solved) the number of multiplications is \( 3/2n^3 \) for the Huang algorithm,

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2Department of Mathematics, University of Bergamo, Bergamo 24129, Italy (bodon@unibg.it)
3Department of Mathematics, University of Bergamo, Bergamo 24129, Italy (delpopolo@unibg.it)
4Institute of Computer Science, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 2, 182 07 Prague 8, Czech Republic (luksan@uivt.cas.cz)
5Department of Mathematics, University of Bergamo, Bergamo 24129, Italy (emilio@unibg.it)
5/2n³ for the modified Huang algorithm and symmetry of $H_k$ is forced by computing only the elements on a triangular part of the matrix. The memory requirement is $n²/2$.

In the routine nl-huang2 the vector $p_k$ is computed by the equivalent formula $p_k = a_k(y_k) - P_{k-1}D_{k-1}^{-1}P_{k-1}^T a_k(y_k)$ in Huang method and $p_k = s_k - P_{k-1}D_{k-1}^{-1}P_{k-1}^T s_k$, $s_k = a_k(y_k) - P_{k-1}D_{k-1}^{-1}P_{k-1}^T a_k(y_k)$ in the modified Huang method, where $P_{k-1} = (p₁,..,p_{k-1})$, $D_{k-1} = (δ₁,..,δ_{k-1})$. In one major iteration the number of multlications is $n³$ for the Huang algorithm, $2n³$ for the modified Huang algorithm, the test for linear dependence is $n²$. This code requires less operations but more memory than nl-huang1.

In the third code, routine nl-ilu, the parameter choices are the same of the implicit LU algorithm with column pivoting, that is $z_k = e_j$ where $j$ is such that $|e_j^T H_k a_k(y_k)| = max\{|e_l^T H_k a_k(y_k)|, l = 1,..,n\}$. A major iteration requires $n³/3$ multiplications and $n²/2$ memory (memory requirement can be reduced to $n²/4$ by using a different implementation).

In a major iteration the number of component function evaluations is $n$ and the number of element jacobian evaluation is $n²$ for all the codes.

The jacobian is singular if at least one row $k$ is linearly dependent, in which case in all the algorithms the iteration continues setting $y_{k+1} = y_k$. In the theoretical ABS method the test for linearly dependence is $H_k a_k = 0$, in the present codes the test is done on the value of $δ_k$ and the zero is substituted by a tolerance $t$ or $t * \|a_k\|$ where $t$ is given by the user.

The programs have four stopping criteria. The first criterion depends on a measure of the residual $\|F(x_i)\|_∞ ≤ eps$, the second criterion is based on a measure of the relative distance between consecutive iterates $\|x_{i+1} - x_i\|_∞ ≤ tol\|x_i\|_∞$, the third criterion terminates the iterations if no progress occurs in decreasing the function after a fixed number $n_s$ of steps. The last criterion terminates the iterations after a fixed number $itmax$ of steps. The parameters $t$, $eps$, $tol$, $n_s$, $itmax$ are given by the user.

Optionally, if the norm of $F$ is increasing at an iteration $i$, the implemented algorithms can use a line search technique. Setting $\bar{x} = x_i$, the line search technique reduces the width of the interval between the points $x_{i-1}$ and $\bar{x}$ by 1/2 (the new point is $\bar{x} = (x_{i-1} + \bar{x})/2$) and so on, until a point $\bar{x}$ is found where the norm of $F(\bar{x})$ decreases, or a maximum number $nhalf$ of halvings is achieved, where $nhalf$ is given by the user.

The codes are written in Fortran 77 in single and in double precision.

3. The numerical experiments

Some numerical results are given in the following tables. The symbols mod.huang1, mod.huang2, implicit lu, refer to the codes nl-huang1, nl-huang2, nl-ilu, the symbols m.hua1 line search, m.hua2 line search, imp.lu line search, mean that the line search technique was applied in the algorithms nl-huang1, nl-huang2, nl-ilu, where the function norm grows at some iterations. The symbol $\|F\|_∞$ means the minimum value found by the algorithm, $it\|F\|_∞$ the iteration where $\|F\|_∞$ was found, $it$ the number of performed iterations, time the execution time. The symbols in the brackets have the following meanings: (x) the second stopping criterion is verified, (div) divergence, stop because $\|F\|_∞$ increases during a fixed number of steps, (o) oscillations, stop because there is no improvement in the last $n_s$ steps. Lack of marks means that the first convergence test is satisfied.

The experiments are obtained setting the tolerance parameters $t = 1.e - 6$, $eps = 1.e - 6$, $tol = 1.e - 10$ for the single precision case and $t = 1.e - 15$, $eps = 1.e - 15$, $tol = 1.e - 18$ for the double precision case.
We have used the following test functions:

- **Rosenbrock function** \((n = 2)\) and the extended Rosenbrock function, where, for 
  \[ i = 1, 2, \ldots, n/2 \]
  \[ f_{2i-1} = 1 - x_{2i-1} \]
  \[ f_{2i} = 10(x_{2i} - x_{2i-1}^2) \]
  \[ x_{0_{2i-1}} = -1.2, \quad x_{0_{2i}} = 1.0 \]

- **Powell singular function** with \(n = 4\), where
  \[ f_1 = x_1 + 10x_2 \]
  \[ f_2 = 5^{1/2}(x_3 - x_4) \]
  \[ f_3 = (x_2 - 2x_3)^2 \]
  \[ f_4 = 10^{1/2}(x_1 - x_4)^2 \]
  \[ x_0 = (3, -1, 0, 1) \]

- **Brown almost linear function**
  \[ f_i = x_i + \sum_{j=1}^{n} x_j - (n + 1) \]
  \[ f_n = (\prod_{j=1}^{n} x_j) - 1 \]
  \[ x_0 = 1/2, \quad i = 1, 2, \ldots, n \]

- **Schubert Broyden function**
  \[ f_1 = (3 - x_1)x_1 + 1 - 2x_2 \]
  \[ f_i = (3 - x_i)x_i + 1 - x_{i-1} - 2x_{i+1} \quad i = 2, \ldots, n - 1 \]
  \[ f_n = (3 - x_n)x_n + 1 - x_{n-1} \]
  \[ x_0 = -1, \quad i = 1, 2, \ldots, n \]

4. References
| function         | method           | $\|F\|_\infty$ | it | $\|F\|_\infty$ | it | time |
|------------------|------------------|----------------|----|----------------|----|------|
| Brown almost linear n=4  
$x_0$ | mod.huang1 | 0.60e-6 | 3 | 3 | 0.0 |
| Brown almost linear n=4  
$1.1x_0$ | mod.huang1 | 0.12e-6 | 3 | 3 | 0.0 |
| Brown almost linear n=4  
$10x_0$ | mod.huang1 | 0.60e-6 | 4 | 4 | 0.0 |
| Brown almost linear n=4  
100$x_0$ | mod.huang1 | 0.42e-6 | 14 | 14 | 0.0 |
| Schubert Broyden n=10  
$x_0$ | mod.huang1 | 0.42e-6 | 4 | 4 | 0.0 |
| Schubert Broyden n=10  
$10x_0$ | mod.huang1 | 0.48e-6 | 8 | 8 | 0.5 |
| Schubert Broyden n=50  
$x_0$ | mod.huang1 | 0.24e-6 | 4 | 4 | 2.03 |
| Schubert Broyden n=50  
$10x_0$ | mod.huang1 | 0.30e-6 | 9 | 9 | 4.73 |
| Schubert Broyden n=100  
$x_0$ | mod.huang1 | 0.24e-6 | 4 | 4 | 16.32 |
| Schubert Broyden n=100  
$10x_0$ | mod.huang1 | 0.30e-6 | 9 | 9 | 38.56 |
| Schubert Broyden n=100  
$100x_0$ | mod.huang1 | 0.83e-6 | 11 | 11 | 47.62 |

single precision
| function                  | method    | $\|F\|_\infty$ | $it\|F\|_\infty$ | $it$ | time |
|--------------------------|-----------|----------------|------------------|-----|-----|
| Rosenbrock n=2 $x_0$    | mod.huang1 | 0.0            | 1                | 1   | 0.0 |
|                         | mod.huang2 | 0.0            | 1                | 1   | 0.0 |
|                         | implicit lu | 0.0            | 1                | 1   | 0.0 |
| Rosenbrock n=2 $1.1x_0$ | mod.huang1 | 0.12e-6        | 1                | 1   | 0.0 |
|                         | mod.huang2 | 0.12e-6        | 1                | 1   | 0.0 |
|                         | implicit lu | 0.12e-6        | 1                | 1   | 0.0 |
| Rosenbrock n=2 $10x_0$  | mod.huang1 | 0.0            | 2                | 2   | 0.0 |
|                         | mod.huang2 | 0.0            | 2                | 2   | 0.0 |
|                         | implicit lu | 0.0            | 1                | 1   | 0.0 |
| Rosenbrock n=2 $100x_0$ | mod.huang1 | 0.0            | 2                | 2   | 0.0 |
|                         | mod.huang2 | 0.0            | 2                | 2   | 0.0 |
|                         | implicit lu | 0.0            | 1                | 1   | 0.0 |
| Extended Rosenbrock n=10 | mod.huang1 | 0.0            | 1                | 1   | 0.0 |
|                         | mod.huang2 | 0.0            | 1                | 1   | 0.0 |
|                         | implicit lu | 0.0            | 1                | 1   | 0.0 |
| Extended Rosenbrock n=10 | mod.huang1 | 0.12e-6        | 1                | 1   | 0.0 |
|                         | mod.huang2 | 0.12e-6        | 1                | 1   | 0.0 |
|                         | implicit lu | 0.12e-6        | 1                | 1   | 0.0 |
| Extended Rosenbrock n=10 | mod.huang1 | 0.0            | 2                | 2   | 0.0 |
|                         | mod.huang2 | 0.0            | 2                | 2   | 0.0 |
|                         | implicit lu | 0.0            | 1                | 1   | 0.0 |
| Extended Rosenbrock n=10 | mod.huang1 | 0.0            | 2                | 2   | 0.0 |
|                         | mod.huang2 | 0.0            | 2                | 2   | 0.0 |
|                         | implicit lu | 0.0            | 1                | 1   | 0.0 |
| Extended Rosenbrock n=100 | mod.huang1 | 0.0            | 1                | 1   | 4.01 |
|                         | mod.huang2 | 0.0            | 1                | 1   | 2.96 |
|                         | implicit lu | 0.0            | 1                | 1   | 0.33 |
| Extended Rosenbrock n=100 | mod.huang1 | 0.12e-6        | 1                | 1   | 4.01 |
|                         | mod.huang2 | 0.12e-6        | 1                | 1   | 2.96 |
|                         | implicit lu | 0.12e-6        | 1                | 1   | 0.33 |
| Extended Rosenbrock n=100 | mod.huang1 | 0.0            | 2                | 2   | 8.02 |
|                         | mod.huang2 | 0.0            | 2                | 2   | 5.94 |
|                         | implicit lu | 0.0            | 1                | 1   | 0.33 |
| Extended Rosenbrock n=100 | mod.huang1 | 0.0            | 2                | 2   | 8.01 |
|                         | mod.huang2 | 0.0            | 2                | 2   | 5.93 |
|                         | implicit lu | 0.0            | 1                | 1   | 0.33 |
| Powell singular n=4 $x_0$ | mod.huang1 | 0.64e-7        | 19               | 19  | 0.0 |
|                         | mod.huang2 | 0.81e-10       | 19               | 19  | 0.0 |
|                         | implicit lu | 0.38e+2        | (div)            | 1   | 100 0.11 |
|                         | imp.lu line search | 0.40e-6        | 86               | 86  | 0.06 |
| Powell singular n=4 $1.1x_0$ | mod.huang1 | 0.44e-7        | 25               | 25  | 0.0 |
|                         | mod.huang2 | 0.56e-10       | 25               | 25  | 0.0 |
|                         | implicit lu | 0.42e+2        | (div)            | 1   | 100 0.06 |
|                         | imp.lu line search | 0.61e-6        | 60               | 60  | 0.05 |
| Powell singular n=4 $10x_0$ | mod.huang1 | 0.34e-7        | 20               | 20  | 0.0 |
|                         | mod.huang2 | 0.43e-10       | 20               | 20  | 0.0 |
|                         | implicit lu | 0.38e+3        | (div)            | 1   | 100 0.06 |
|                         | imp.lu line search | 0.69e-6        | 69               | 69  | 1.1 |
| Powell singular n=4 $100x_0$ | mod.huang1 | 0.70e-7        | 30               | 30  | 0.0 |
|                         | mod.huang2 | 0.88e-10       | 30               | 30  | 0.0 |
|                         | implicit lu | 0.93e+3        | (div)            | 9   | 100 0.05 |
|                         | imp.lu line search | 0.69e-6        | 73               | 73  | 0.05 |
| function          | method             | $\|F\|_\infty$ | it | $\|F\|_\infty$ | time |
|-------------------|--------------------|---------------|----|---------------|------|
| Brown almost linear n=4  | mod.huang1       | 0.22d-15     | 5  | 5  | 0.0  |
| x₀                 | mod.huang2       | 0.44d-15     | 5  | 5  | 0.0  |
|                   | implicit lu      | 0.11d-15     | 6  | 6  | 0.0  |
| Brown almost linear n=4  | mod.huang1       | 0.0          | 5  | 5  | 0.0  |
| 1.1x₀              | mod.huang2       | 0.0          | 5  | 5  | 0.0  |
|                   | implicit lu      | 0.0          | 6  | 6  | 0.0  |
| Brown almost linear n=4  | mod.huang1       | 0.67d-15     | 8  | 8  | 0.0  |
| 10x₀               | mod.huang2       | 0.22d-15     | 5  | 5  | 0.0  |
|                   | implicit lu      | 0.10d+1      | 1  | 2  | 0.0  |
| Brown almost linear n=4  | mod.huang1       | 0.0          | 16 | 16 | 0.0  |
| 100x₀              | m.hual line search | 0.78d-15 | 11 | 11 | 0.0  |
|                   | mod.huang2       | 0.11d-15     | 16 | 16 | 0.0  |
|                   | m.hu2 line search | 0.33d-15     | 11 | 11 | 0.0  |
|                   | implicit lu      | 0.44d-15     | 18 | 18 | 0.0  |
| Brown almost linear n=20 | mod.huang1      | 0.56d-14     | (o) | 5  | 9  | 0.44 |
| x₀                 | m.hu1 line search | 0.44d-15   | 21 | 21 | 0.93 |
|                   | mod.huang2       | 0.24d-14     | (o) | 15 | 22 | 1.05 |
|                   | m.hu2 line search | 0.89d-15   | 22 | 22 | 1.05 |
|                   | implicit lu      | 0.29d-14     | (o) | 12 | 17 | 0.22 |
|                   | imp.lu line search | 0.89d-15 | 14 | 14 | 0.16 |
| Brown almost linear n=20 | mod.huang1      | 0.29d-14     | (o) | 16 | 21 | 0.88 |
| 1.1x₀              | m.hu1 line search | 0.89d-15   | 22 | 22 | 0.99 |
|                   | mod.huang2       | 0.40d-14     | (o) | 9  | 14 | 0.66 |
|                   | m.hu2 line search | 0.78d-15   | 12 | 12 | 0.60 |
|                   | implicit lu      | 0.32d-14     | (o) | 8  | 12 | 0.11 |
|                   | imp.lu line search | 0.44d-15 | 16 | 16 | 0.17 |
| Schubert Broyden n=10 | mod.huang1      | 0.99d-15     | 5  | 5  | 0.0  |
| x₀                 | mod.huang2       | 0.99d-15     | 5  | 5  | 0.0  |
|                   | implicit lu      | 0.88d-15     | 5  | 5  | 0.0  |
| Schubert Broyden n=10 | mod.huang1      | 0.89d-15     | 9  | 9  | 0.5  |
| 10x₀               | mod.huang2       | 0.89d-15     | 9  | 9  | 0.5  |
|                   | implicit lu      | 0.89d-15     | 8  | 8  | 0.0  |
| Schubert Broyden n=50 | mod.huang1      | 0.78d-15     | 5  | 5  | 2.80 |
| x₀                 | mod.huang2       | 0.78d-15     | 5  | 5  | 2.63 |
|                   | implicit lu      | 0.89d-15     | 5  | 5  | 0.33 |
| Schubert Broyden n=50 | mod.huang1      | 0.67d-15     | 10 | 10 | 5.55 |
| 10x₀               | mod.huang2       | 0.99d-15     | 9  | 9  | 4.78 |
|                   | implicit lu      | 0.67d-15     | 9  | 9  | 0.60 |
| Schubert Broyden n=50 | mod.huang1      | 0.78d-15     | 13 | 13 | 7.25 |
| 100x₀              | mod.huang2       | 0.78d-15     | 13 | 13 | 6.92 |
|                   | implicit lu      | 0.89d-15     | 12 | 12 | 0.77 |
| Schubert Broyden n=100 | mod.huang1     | 0.89d-15     | 5  | 5  | 21.37 |
| x₀                 | mod.huang2       | 0.89d-15     | 5  | 5  | 20.71 |
|                   | implicit lu      | 0.89d-15     | 5  | 5  | 1.70 |
| Schubert Broyden n=100 | mod.huang1     | 0.89d-15     | 10 | 10 | 42.74 |
| 10x₀               | mod.huang2       | 0.89d-15     | 10 | 10 | 41.57 |
|                   | implicit lu      | 0.89d-15     | 9  | 9  | 3.08 |
| Schubert Broyden n=100 | mod.huang1     | 0.89d-15     | 13 | 13 | 55.64 |
| 100x₀              | mod.huang2       | 0.78d-15     | 13 | 13 | 54.10 |
|                   | implicit lu      | 0.89d-15     | 12 | 12 | 4.12 |
| function         | method           | $\|F\|_\infty$ | $it_{\|F\|_\infty}$ | $it$ | time |
|------------------|------------------|----------------|----------------------|------|------|
| Rosenbrock n=2  | mod.huang1       | 0.22d-15       | 1                     | 1    | 0.0  |
| $x_0$            | mod.huang2       | 0.22d-15       | 1                     | 1    | 0.0  |
|                  | implicit lu      | 0.22d-15       | 1                     | 1    | 0.0  |
| Rosenbrock n=2  | mod.huang1       | 0.22d-15       | 1                     | 1    | 0.0  |
| 1.1$x_0$         | mod.huang2       | 0.22d-15       | 1                     | 1    | 0.0  |
|                  | implicit lu      | 0.22d-15       | 1                     | 1    | 0.0  |
| Rosenbrock n=2  | mod.huang1       | 0.0            | 1                     | 1    | 0.0  |
| 10$x_0$          | mod.huang2       | 0.0            | 1                     | 1    | 0.0  |
|                  | implicit lu      | 0.0            | 1                     | 1    | 0.0  |
| Rosenbrock n=2  | mod.huang1       | 0.0            | 1                     | 1    | 0.0  |
| 100$x_0$         | mod.huang2       | 0.0            | 1                     | 1    | 0.0  |
|                  | implicit lu      | 0.0            | 1                     | 1    | 0.0  |
| Extended Rosenbrock n=10 | mod.huang1 | 0.22d-15 | 1 | 1 | 0.0 |
| $x_0$            | mod.huang2       | 0.22d-15       | 1                     | 1    | 0.0  |
|                  | implicit lu      | 0.22d-15       | 1                     | 1    | 0.0  |
| Extended Rosenbrock n=10 | mod.huang1 | 0.22d-15 | 1 | 1 | 0.0 |
| 1.1$x_0$         | mod.huang2       | 0.22d-15       | 1                     | 1    | 0.0  |
|                  | implicit lu      | 0.22d-15       | 1                     | 1    | 0.0  |
| Extended Rosenbrock n=10 | mod.huang1 | 0.0 | 1 | 1 | 0.0 |
| 10$x_0$          | mod.huang2       | 0.0            | 1                     | 1    | 0.0  |
|                  | implicit lu      | 0.0            | 1                     | 1    | 0.0  |
| Extended Rosenbrock n=10 | mod.huang1 | 0.0 | 1 | 1 | 0.0 |
| 100$x_0$         | mod.huang2       | 0.0            | 1                     | 1    | 0.0  |
|                  | implicit lu      | 0.0            | 1                     | 1    | 0.0  |
| Extended Rosenbrock n=100 | mod.huang1 | 0.22d-15 | 1 | 1 | 4.23 |
| $x_0$            | mod.huang2       | 0.22d-15       | 1                     | 1    | 3.18 |
|                  | implicit lu      | 0.22d-15       | 1                     | 1    | 0.33 |
| Extended Rosenbrock n=100 | mod.huang1 | 0.23d-15 | 1 | 1 | 4.23 |
| 1.1$x_0$         | mod.huang2       | 0.23d-15       | 1                     | 1    | 3.13 |
|                  | implicit lu      | 0.23d-15       | 1                     | 1    | 0.33 |
| Extended Rosenbrock n=100 | mod.huang1 | 0.0 | 1 | 1 | 4.29 |
| 10$x_0$          | mod.huang2       | 0.0            | 1                     | 1    | 3.13 |
|                  | implicit lu      | 0.0            | 1                     | 1    | 0.28 |
| Extended Rosenbrock n=100 | mod.huang1 | 0.0 | 1 | 1 | 4.23 |
| 100$x_0$         | mod.huang2       | 0.0            | 1                     | 1    | 3.14 |
|                  | implicit lu      | 0.0            | 1                     | 1    | 0.38 |
| Powell singular n=4 | mod.huang1 | 0.62d-16 | 44 | 44 | 0.0 |
| $x_0$            | mod.huang2       | 0.78d-19       | 44                     | 44   | 0.0  |
|                  | implicit lu      | 0.38d+2        | (div)                  | 1     | 100  |
|                  | imp.lu line search | 0.56d-15 | 186 | 186 | 0.11 |
| Powell singular n=4 | mod.huang1 | 0.33d-16 | 45 | 45 | 0.0 |
| 1.1$x_0$         | mod.huang2       | 0.41d-19       | 45                     | 45   | 0.0  |
|                  | implicit lu      | 0.42d+2        | (div)                  | 1     | 100  |
|                  | imp.lu line search | 0.85d-15 | 126 | 126 | 0.06 |
| Powell singular n=4 | mod.huang1 | 0.43d-16 | 50 | 50 | 0.0 |
| 10$x_0$          | mod.huang2       | 0.54d-19       | 50                     | 50   | 0.0  |
|                  | implicit lu      | 0.38d+3        | (div)                  | 1     | 100  |
|                  | imp.lu line search | 0.56d-15 | 135 | 135 | 0.11 |
| Powell singular n=4 | mod.huang1 | 0.79d-16 | 55 | 55 | 0.0 |
| 100$x_0$         | mod.huang2       | 0.84d-19       | 55                     | 55   | 0.0  |
|                  | implicit lu      | 0.93d+4        | (div)                  | 9     | 100  |
|                  | imp.lu line search | 0.39d-15 | 140 | 140 | 0.05 |