Fundamental Group of Locally Symmetric Varieties

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The geometry of moduli spaces of complex abelian varieties and of compactifications of those moduli spaces has been the object of much study in the last few years. Given such a compactified moduli space it is natural to ask about the fundamental group of a resolution of singularities. This problem has been studied in some special cases, for instance in [K], [HK] and [HS]. It follows from the results of [K] and the well-known fact that every arithmetic subgroup of \( \text{Sp}(2g, \mathbb{Q}) \) is a congruence subgroup of some level that the fundamental group must be finite except when \( g = 1 \), i.e., except in the case of modular curves. The method in all these cases, and here, is to use the toroidal compactification of [SC], considering the moduli spaces as quotients of the Siegel upper half-space by arithmetic subgroups of the symplectic group.

In this paper we treat the subject in greater generality. In many cases we are able to identify the fundamental group explicitly as a quotient of the arithmetic group in question. For this purpose we do not need to restrict ourselves to the symplectic group but instead may consider any locally symmetric variety. Later we return to the case of moduli of abelian varieties (specifically, to Siegel modular varieties) and calculate the fundamental group in some interesting special cases. In many cases, including those studied in the papers mentioned above, the fundamental group is trivial, but we give examples to show that this need not be true in general.

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1. Unipotent elements in parabolic subgroups

We shall study locally symmetric varieties and their compactifications as described in [SC]. We therefore let $D$ be a bounded symmetric domain with $\text{Aut}(D)^0 = G$, a simple real Lie group defined over $\mathbb{Q}$. We fix an arithmetic subgroup $\Gamma$ of $G$. We put $X = D/\Gamma$ and use the methods of [SC] to construct a compactification $\bar{X}$ of $X$. Since we have not required $\Gamma$ to be a neat subgroup of $G$ we cannot guarantee that we can choose $\bar{X}$ to be smooth, but we can take a resolution of singularities $\tilde{X} \to \bar{X}$ if we wish. The object of interest to us is the topological fundamental group $\pi_1(\tilde{X})$. Some arbitrary choices have to be made in constructing $\bar{X}$ and $\tilde{X}$ but $\pi_1(\tilde{X})$ does not depend on the choices.

We begin as in [HS] with some topological lemmas, repeated here for ease of reference.

**Lemma 1.1.** Let $M$ be a connected, simply-connected real manifold and $G$ a group acting discontinuously on $M$. Take a base point $x \in M$. Then the quotient map $\phi : M \to M/G$ induces a surjective homomorphism $\phi_* : G \to \pi_1(M/G, \phi(x))$.

*Proof:* [G], Satz 5. The quotient map $\phi_*$ is easy to describe and this is done in [HS].

**Lemma 1.2.** Let $M$ be a connected complex manifold and $M_0$ an analytic subvariety. Let $x \in M \setminus M_0$. Then the inclusion $M \setminus M_0 \hookrightarrow M$ induces a map $\pi_1(M \setminus M_0, x) \to \pi_1(M, x)$, which is surjective if $\text{codim}_\mathbb{C} M_0 \geq 1$ and an isomorphism if $\text{codim}_\mathbb{C} M_0 \geq 2$.

*Proof:* See [HK].

**Lemma 1.3.** If $X_1$ and $X_2$ are bimeromorphically equivalent connected complex manifolds, then $\pi_1(X_1) \cong \pi_1(X_2)$.

*Proof:* Also in [HK].

Now let $\phi : D \to D/\Gamma = X$ be the quotient map and let $X_{\text{reg}}$ be the nonsingular locus of $X$. Put $D' = \phi^{-1}(X_{\text{reg}})$. The complement $D \setminus D'$ consists of countably many analytic subspaces of $D$ of complex codimension at least 2; recall that $D$ has naturally the structure of a complex manifold. So $D'$ is simply-connected and we get a surjection $\Gamma \to \pi_1(X_{\text{reg}})$. Composing this with the surjection $\pi_1(X_{\text{reg}}) \to \pi_1(\bar{X})$ induced by the inclusion $X_{\text{reg}} \hookrightarrow \bar{X}$ we get a surjection $\psi : \Gamma \to \pi_1(\bar{X})$. 
Theorem 1.4. If $P$ is a parabolic subgroup of $G$ and $U_P$ is the centre of the unipotent radical of $P$ then $U_P \cap \Gamma \subseteq \ker \psi$.

Proof: We can assume that $P \cap \Gamma \neq 1$. By [SC], III.3.2, Proposition 2, $P$ corresponds to a boundary component $F_P = F$. Since $P \cap \Gamma \neq 1$, $F$ is a rational boundary component and we may adopt the notation of [SC] and write $U_P \cap \Gamma = U(F)_Z$.

Recall the procedure of partial compactification in the direction of $F$ as it is described in [SC]: we can write

$$D = \text{Im}^{-1}C(F) \times \mathbb{C}^k \times F \subseteq D(F) = U(F)_C \times \mathbb{C}^k \times F,$$

where $U(F) = U_P$, $U(F)_C$ is the complexification of $U(F)$, $C(F)$ is an open convex cone in $U(F)$ and $\text{Im} : U(F)_C \rightarrow U(F)$ is the imaginary part. $U(F)_Z$ acts by translation in the real direction in $U(F)_C$ and hence preserves $C(F)$: it acts trivially on $\mathbb{C}^k \times F$ and there is an embedding

$$D/U(F)_Z \hookrightarrow T(F) \times \mathbb{C}^k \times F$$

where $T(F) = U(F)_C/U(F)_Z$ is an algebraic torus over $\mathbb{C}$. By choosing an appropriate fan $\{\sigma_\alpha\}$ subdividing $C(F)$ we construct a partial compactification $T(F)_{\{\sigma_\alpha\}}$ of $T(F)$ and hence

$$\overline{D/U(F)_Z} = \left(D/U(F)_Z\right)_{\{\sigma_\alpha\}} \subseteq T(F)_{\{\sigma_\alpha\}} \times \mathbb{C}^k \times F.$$

We can even do this in such a way as to make $\overline{D/U(F)_Z}$ smooth and keep everything $\Gamma$-equivariant.

$T(F)_{\{\sigma_\alpha\}}$ is simply-connected: see for instance [F]. If $T(F) = \text{Hom}(M, \mathbb{C}^*)$, where $M$ is a lattice, and if $N$ is the dual lattice, then $\pi_1(T(F))$ consists of the classes of the loops $s \mapsto \exp\{2\pi i s < , n >\}$ for $n \in N$. If $n \in N \cap \sigma_\alpha$ then this loop is killed in $\pi_1(T(F)_{\{\sigma_\alpha\}})$ by a retraction given by

$$R_n(s,t) = t \exp\{2\pi i s < , n >\} \quad \text{for } (s,t) \in [0,1]^2, t \neq 0,$$

extended by putting $R_n(s,0) = \lim_{t \to 0} R_n(s,t)$, which exists in $T(F)_{\{\sigma_\alpha\}}$ (and is independent of $s$). Since $N \cap \sigma_\alpha$ generates $N$ if $\sigma_\alpha$ is of maximal dimension, $T(F)_{\{\sigma_\alpha\}}$ is simply-connected.
We can identify $T(F)$ with $N \otimes \mathbb{C}^*$, so that $R_n(s, t) = n \otimes te^{2\pi is}$. We can consistently choose a logarithm for $t \in (0, 1]$, so that $R_n$ comes from a map $\hat{R}_n : [0, 1] \times (0, 1] \to N \otimes \mathbb{C}$ given by $\hat{R}_n(s, t) = n \otimes (\log t + 2\pi is)$. Note also that $\text{Im} \hat{R}_n(s, t) \in C(F)$ if $n \in C(F)$. From this it follows that $\hat{R}_n$ comes from a map $\hat{\hat{R}}_n : [0, 1] \times (0, 1] \to N \otimes \mathbb{Z}$ given by $\hat{\hat{R}}_n(s, t) = n \otimes (\log t + 2\pi is)$. Note also that $\text{Im} \hat{\hat{R}}_n(s, t) \in C(F)$ if $n \in C(F)$. From this it follows that $D/U(F)Z$ is also simply-connected, since $F$ and $C^k$ are both simply-connected themselves. As $D/U(F)Z$ maps under the action of $\Gamma$ onto a Zariski-open set in $\bar{X}$, and as $U(F)Z$ acts trivially on $D/U(F)Z$, we see that $U(F)Z$ is in the kernel of $\Gamma \to \pi_1(\bar{X})$.

In order to show that if $\eta \in U(F)Z$ then $\psi(\eta) = 1 \in \pi_1(\bar{X})$ we have to do a little more: we must avoid the singularities of $\bar{X}$. Suppose $H : [0, 1]^2 \to D/U(F)Z$ is a null homotopy for a loop in $D/U(F)Z$ coming from $\eta \in U(F)Z$, and let $\bar{H}$ be the corresponding null homotopy in $\bar{X}$. We may assume that $H$ and $\bar{H}$ are smooth maps. The singularities of $\bar{X}$ are quotient singularities arising from the action of $\Gamma$ on $D/U(F)Z$, since we have chosen $\{\sigma_\alpha\}$ so as not to lead to singularities in $D/U(F)Z$. We may assume, by the description of $R$ above, that there is a unique point $x_0$ (which we can take to be $(\frac{1}{2}, \frac{1}{2})$) in $[0, 1]^2$ such that $H(x_0) \notin D/U(F)Z$, and that away from this point $H$ lifts from a map into $D/U(F)Z$ to a map

$$\hat{H} : [0, 1]^2 \setminus \{(\frac{1}{2}, \frac{1}{2})\} \to D.$$  

Denote by $Z$ the preimage in $D/U(F)Z$ of $\text{Sing} \bar{X}$. Then $Z$ is the union of countably many analytic submanifolds of codimension $\geq 2$. The same is true of $\hat{Z}$, the preimage of $Z \cap (D/U(F)Z)$ in $D$. By choosing a path from $x$ to $\eta(x)$ in $D \setminus \hat{Z}$ we may assume that $\hat{H}(0, t)$ misses $\hat{Z}$ and that the loop $H(0, t)$ misses $Z$. The idea now is to move $H$ so that its image misses $Z$, without changing $H$ on $\{0, 1\} \times \{0, 1\}$: this will produce a null homotopy of the path $\bar{H}(0, t)$ corresponding to $\eta$ that takes place entirely in $\bar{X} \setminus \text{Sing} \bar{X}$ and therefore lifts to $\bar{X}$.

To move the part of $H$ that lies inside $D/U(F)Z$, we choose a neighbourhood $N_1$ of $1 \in G$ and a diffeomorphism

$$\theta_1 : N_1 \to \Delta_1 = \{x \mid |x| < 1\} \subseteq g = \text{Lie} G.$$
Then we define $\hat{H}_1 : [0, 1]^2 \times \Delta_1 \to D$ by

$$\hat{H}_1((a, b), x) = \theta_1^{-1}(\lambda_1(a, b)) \hat{H}(a, b),$$

where $\lambda_1 : \mathbb{R}^2 \to \mathbb{R}$ is a $C^\infty$ function satisfying $0 \leq \lambda(a, b) < 1$ and $\lambda_1(a, b) = 0$ if and only if $a = b = \frac{1}{2}$ or $(a, b) \notin [0, 1]^2$. Let $\hat{H}_1 : [0, 1]^2 \times \Delta_1 \to D/U(F)_Z$ be the composition with the quotient map.

To move $H$ near the boundary, choose a neighbourhood $N_\infty$ of $H((\frac{1}{2}, \frac{1}{2}))$ and a diffeomorphism

$$\theta_\infty : N_\infty \to \Delta_\infty = \{x \mid |x| < 1\} \subseteq \mathbb{R}^{2 \dim C_D}.$$

Let $\Delta$ be an open disc centred at $(\frac{1}{2}, \frac{1}{2})$ such that $H(\Delta) \subseteq N_\infty$ and $\theta_\infty H(\Delta) \subseteq \frac{1}{2}\Delta_\infty$. We translate $H$ near $(\frac{1}{2}, \frac{1}{2})$, defining $\mathcal{H}_\infty : [0, 1]^2 \times \Delta_\infty \to D/U(F)_Z$ by

$$\mathcal{H}_\infty((a, b), x) = \theta_\infty^{-1}(\theta_\infty H(a, b) + \lambda_\infty(a, b)x),$$

where $\lambda_\infty : \mathbb{R}^2 \to \mathbb{R}$ is a $C^\infty$ function satisfying $0 \leq \lambda_\infty(a, b) < \frac{1}{2}$ and $\lambda_\infty(a, b) = 0$ if and only if $(a, b) \notin \Delta$.

Now we compose $\mathcal{H}_1$ and $\mathcal{H}_\infty$ so as to get a map $\mathcal{H} : [0, 1]^2 \times \Delta_1 \times \Delta_\infty \to D/U(F)_Z$ given by

$$\mathcal{H}((a, b), x, y) = \theta_\infty^{-1}\{\theta_\infty \mathcal{H}_1((a, b), x) + \lambda_\infty(a, b)y\}.$$

Since the real tangent space to $D/U(F)_Z$ at any point is a quotient of $g$ as a real vector space, $\mathcal{H}$ is a submersion for $(a, b) \in (0, 1)^2 \setminus \{(\frac{1}{2}, \frac{1}{2})\}$. By the choice of $\mathcal{H}_\infty$ it is a submersion at $(\frac{1}{2}, \frac{1}{2})$ as well. So by the Thom transversality theorem (see [BG]) there exist $x_0 \in \Delta_1$ and $y_0 \in \Delta_\infty$ such that $\mathcal{H}((s, t), x_0, y_0)$ is a null homotopy transversal to each component of $Z$. As $Z$ has real codimension 4 this means the homotopy misses $Z$, and we are done. 

**Remark.** The technical difficulties above arose almost entirely from the possibility that $\bar{X}$ might have singularities at or near the boundary. If $\Gamma$ is neat, so that $\bar{X}$ is smooth, or if, as in [HS], we need only consider boundary components far from $\text{Sing} \bar{X}$, the situation is very simple. Even if we cannot easily avoid $\text{Sing} \bar{X}$, it is often the case that the resolution $\tilde{X} \to \bar{X}$ can be chosen to have simply-connected fibres (for instance if $\bar{X}$ has only cyclic
quotient singularities), and then $\pi_1(\tilde{X}) \cong \pi_1(X)$ anyway. Perhaps this can always be done, even when $\tilde{X}$ has unknown finite quotient singularities.

For practical purposes, the following easy corollary, or something like it, is often useful.

**Corollary 1.5.** The fundamental group $\pi_1(\tilde{X})$ is a quotient of $\Pi(\Gamma) = \Gamma/\Upsilon$, where $\Upsilon$ is the normal subgroup of $\Gamma$ generated by all elements in the centre of the unipotent radical of some parabolic subgroup of $G$.

**Proof:** $\Pi(\Gamma)$ is the largest quotient of $\Gamma$ satisfying the conditions of Theorem 1.4. □

In fact if we let $\tilde{\Gamma}$ be the normaliser of $\Gamma$ in $G(\mathbb{Q})$ then $\tilde{\Gamma}/\Gamma$ acts on $X$ and preserves the set of rational boundary components; the action therefore extends to $\tilde{X}$ if we choose the compactification correctly. The resolution $\tilde{X} \to \bar{X}$ may also be chosen to be $\tilde{\Gamma}/\Gamma$-equivariant, and therefore $\tilde{\Gamma}/\Gamma$ acts on $\pi_1(\tilde{X})$. So the kernel of $\psi : \Gamma \to \pi_1(\tilde{X})$ must be a normal subgroup of $\tilde{\Gamma}$. But in fact $\Pi(\Gamma)$ satisfies this condition as well. For if $g \in \tilde{\Gamma}$ and $P$ is a parabolic subgroup of $G$ defined over $\mathbb{Q}$, then so is $P^g$ and the unipotent radical $R_u$ satisfies $R_u(P^g) = R_u(P)^g$. So $U_{P^g} = U_P^g$. This shows also that $\Upsilon$ is actually the subgroup of $\Gamma$ generated by all the $U_P$s, which is automatically normal. But for purposes of calculation the statement in Corollary 1.5 is preferable, because rather than work with all parabolic subgroups it is easier to take a representative from each conjugacy class. Geometrically, this means that one works with the boundary components of $\tilde{X}$ (i.e., with the Tits building) rather than with boundary components of $D$. It is not even necessary to look at all boundary components: in view of the fact ([SC], III.4.4, Theorem 3) that $U(F) \subseteq U(F')$ if $F'$ is a boundary component of $F$ it is enough to look at the minimal ones. But it is usually easiest to write down $U(F)$ when $F$ is a maximal proper boundary component, corresponding to a divisor in the toroidal compactification. This is often sufficient because it happens that such $U(F)$ already generate $U(F')$ for all $F'$.

Finally, the condition that $G$ is simple is also stronger than we need. If we allow $G$ to be semisimple then nothing changes except that we have (cf. [SC], p.208) $D = \prod D_i$ (for instance $D = \mathbb{H}^2$ and $X$ is a Hilbert modular surface) and $G = \prod \text{Aut}(D_i)^0 = \prod G_i$ (as a set), and we must allow $P = \prod P_i$, where each $P_i$ is either parabolic or else equal to $G_i$.

We summarise our results as follows.
Corollary 1.6. Let $D$ be a bounded symmetric domain with $G = \text{Aut}(D)^0$ a semisimple real Lie group defined over $\mathbb{Q}$. Let $\Gamma$ be an arithmetic subgroup of $G$. Then $\pi_1(\tilde{X})$ is a quotient of $\Pi(\Gamma) = \Gamma/\Upsilon$, where $\Upsilon$ is the subgroup generated by $U(F) \cap \Gamma$ for all rational boundary components $F$ of $D$. Equivalently, $\Upsilon$ is the normal subgroup of $\Gamma$ generated by $U(F) \cap \Gamma$ as $F$ runs through a set of representatives for all boundary components of $\tilde{X}$ (or all minimal boundary components of $\tilde{X}$).

2. Neat arithmetic groups.

Throughout this section we assume that $\Gamma$ is neat. In particular this implies that $\Gamma$ is torsion-free. In this special (but not very special) case we can identify $\pi_1(\tilde{X})$ precisely. Note that now $\tilde{X} = \tilde{X}$.

Theorem 2.1. Let $D$ be a bounded symmetric domain and $G = \text{Aut}(D)^0$ a semisimple Lie group defined over $\mathbb{Q}$. Let $\Gamma$ be a neat arithmetic subgroup of $G$. Then $\pi_1(\tilde{X}) \cong \Pi(\Gamma)$.

Proof: Let $\Upsilon$ be as in Corollary 1.6 above. We should like to proceed by putting $X_u = D/\Upsilon$ and constructing a toroidal compactification $\tilde{X}_u$ (which could be assumed to be smooth because $\Upsilon$, being a subgroup of $\Gamma$, is neat). Then we should expect $\Gamma/\Upsilon$ to act freely on $\tilde{X}_u$ with quotient $\tilde{X}$. Unfortunately $\Upsilon$ need not have finite index in $\Gamma$ so $\Upsilon$ will not, in general, be an arithmetic group. But this is not a serious difficulty: the only consequence is that $\tilde{X}_u$ need not be compact, which does not matter to us.

We construct $\tilde{X}_u$ as in [SC], III.5. For each rational boundary component $F$ we have $\Gamma \cap U(F) = \Upsilon \cap U(F)$, by the definition of $\Upsilon$, so we can call this group $U(F)_\mathbb{Z}$ without ambiguity. We take the same fans $\{\sigma_\alpha\}$ as before, requiring them to be admissible for the action of $\Gamma$, not just of $\Upsilon$, and we define $\tilde{X}_u$ as the quotient of $\bigcup_F D/U(F)_\mathbb{Z}$ by the closure of the equivalence relation defined by the action of $\Upsilon$. Thus (cf. [SC], p.255) if $x_i \in \overline{D/U(F_i)_\mathbb{Z}}$, $i = 1$, 2, then $x_1 \sim x_2$ if and only if there are a rational boundary component $F$, an element $\eta \in \Upsilon$ and a point $x \in \overline{D/U(F)_\mathbb{Z}}$ such that $F_1$ and $\eta F_2$ are boundary components of $F$ and $x$ projects to $x_1$ and to $\eta x_2$ under the projection maps $\overline{D/U(F)_\mathbb{Z}} \to \overline{D/U(F_1)_\mathbb{Z}}$ and $\overline{D/U(F)_\mathbb{Z}} \to \overline{D/U(\eta F_2)_\mathbb{Z}}$. This equivalence relation is closed, so $\tilde{X}_u$ is Hausdorff: indeed, $\pi_{F_u}^{(u)} : \overline{D/U(F)_\mathbb{Z}} \to \tilde{X}_u$ is biholomorphic onto its image. But
the map $π_F : \overline{D/U(F)} \to \bar{X}$ factors as $π_F = qπ_F^{(u)}$, where $q : \bar{X}_u \to \bar{X}$ is the quotient map under the action of $Γ/Υ$, and $π_F$ is étale since $Γ$ is neat. Therefore $q$ is étale and so $Γ/Υ \hookrightarrow π_1(X)$.

3. Examples.

If $X$ is a curve then $π_1(\bar{X})$ will not be finite unless $\bar{X} = \mathbb{P}^1$. In this case (for instance when $G = SL(2, \mathbb{R})$ and $Γ$ is the principal congruence subgroup of some level $l \geq 5$) $Υ$ is of infinite index in $Γ$ and in particular is not an arithmetic group. In higher-dimensional cases (other than products with a factor of this type) $Γ/Υ$ seems to be finite, but I am not aware of any definite general result of that nature. In many cases $Γ/Υ$ is trivial and thus $\bar{X}$ is simply-connected: this is shown for various Siegel modular varieties in [K], [HK] and [HS]. The most frequently considered Hilbert modular surfaces are also simply-connected but some others are not: see [vdG]. Likewise, Siegel modular threefolds are not simply-connected in general, and the fundamental group can even be quite big, as the following examples show.

**Theorem 3.1.** Let $l \geq 4$ be an integer and let $p$ be a prime not dividing $2l$. take $Γ(l)$ to be the principal congruence subgroup of level $l$ and let $Γ(l)_p \subseteq Sp(4, \mathbb{F}_p)$ be the image of $Γ(l)$ under the reduction mod $p$ map $red_p : Γ(l) \to Sp(4, \mathbb{F}_p)$. Let $Γ^q(l)_p$ be a Sylow $q$-subgroup of $Γ(l)_p$ for some prime $q \neq p$, and let $Γ = red_p^{-1}(Γ(l)_p)$. Then if $X = D/Γ$ (in this case $D$ is the Siegel upper half-plane), $π_1(\bar{X})$ has a quotient isomorphic to $Γ^q(l)_p$.

**Proof:** We must check first that the groups mentioned exist, by showing that $Γ(l)_p$ is not a $p$-group. The map $red_p : Sp(4, \mathbb{Z}) \to Sp(4, \mathbb{F}_p)$ is surjective, so given $α \in Sp(4, \mathbb{F}_p)$ and $β \in Γ(l)_p$ we can choose $\bar{α} \in Sp(4, \mathbb{Z})$ and $\bar{β} \in Γ(l)$ such that $red_p(\bar{α}) = α$ and $red_p(\bar{β}) = β$. By definition $Γ(l)$ is a normal subgroup of $Sp(4, \mathbb{Z})$, so $\bar{α}^{-1}\bar{β}\bar{α} \in Γ(l)$ and therefore $α^{-1}βα ∈ Γ(l)_p$. So $Γ(l)_p$ is a normal subgroup of $Sp(4, \mathbb{F}_p)$. But the only nontrivial normal subgroup of $Sp(4, \mathbb{F}_p)$ is its centre, which has order 2, and $Γ(l)_p$ contains the element $\begin{pmatrix} I & red_p(l)I \\ 0 & I \end{pmatrix}$, which has order $p \neq 2$. So in fact $Γ(l)_p$ is the whole of $Sp(4, \mathbb{F})$ and the order of this group is not a power of $p$. 

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\(\Gamma(l)\) is neat and so, therefore, is \(\Gamma\). Moreover, \(\Gamma\) is evidently of finite index in \(\Gamma(l)\) and is thus an arithmetic group. So by Theorem 2.1 we can construct a smooth compactification \(\bar{X}\) of \(X\) whose fundamental group is \(\Gamma/\Upsilon\), where \(\Upsilon\) is a group generated by unipotent elements. Suppose \(\tilde{\eta}\) is a unipotent element of \(\Gamma\). Then \(\text{red}_p(\tilde{\eta}) = \eta\) is a unipotent element of \(\text{Sp}(4, \mathbb{F}_p)\) and is therefore conjugate over \(\text{GL}(4, \mathbb{F}_p)\) to an upper-triangular element \(\eta' \in \text{GL}(4, \mathbb{F}_p)\). The order of \(\eta'\) is obviously a power of \(p\), so the order of \(\eta\) is a power of \(p\), but \(\eta \in \Gamma^q(l)_p\) which is a \(q\)-group, so \(\eta\) is the identity. So \(\Upsilon \subseteq \text{Ker} \text{red}_p\) and \(\text{red}_p\) gives a surjective morphism \(\Gamma/\Upsilon \to \Gamma^q(l)_p\). ■

Remarks
i) We can even take \(p = 2\) if we like, since \(\begin{pmatrix} I & \text{red}_2(l)I \\ 0 & I \end{pmatrix}\) is not central.
ii) The varieties exhibited in Theorem 3.1 are all of general type because of the result of Yamazaki ([Y]) that Siegel modular threefolds of level \(l\) are of general type for \(l \geq 4\).

**Corollary 3.2.** The fundamental group of a Siegel modular threefold need not be abelian.

*Proof:* \(\Gamma^q(l)_p\) can be any \(q\)-subgroup, not necessarily a Sylow \(q\)-subgroup. If we take \(l = 5\), \(p = 7\) and \(q = 2\) then we can take \(\Gamma^q(l)_p\) to be non-abelian, since \(\text{Sp}(4, \mathbb{F}_7)\) contains a subgroup isomorphic to \(\text{PSL}(2, \mathbb{F}_7) \cong \text{PSL}(4, \mathbb{F}_3)\), which obviously contains a dihedral group of order 8. ■

In spite of the possibility that \(\pi_1(\bar{X})\) may be a large finite group, it is frequently the case that Siegel modular threefolds arising in geometry turn out to be simply-connected. This is shown for the principal congruence subgroups (in terms of abelian surfaces, the moduli spaces of principally polarised abelian surfaces with level \(l\) structure) in [K] and for the moduli of \((1, p)\)-polarised abelian surfaces with level structure \((p\) an odd prime\) in [HS]. Here we will give two more cases where \(\bar{X}\) is simply-connected and one where it is not.

**Theorem 3.3.** Let \(X\) be the moduli space of abelian surfaces with a polarisation of type \((1, p)\), \(p\) an odd prime. Then \(\bar{X}\) is simply-connected.

*Proof:* This is a fairly straightforward consequence of the calculations in [HS] (or can be proved directly in the same way, using Theorem 1.4). All we need to do is observe that,
adopting the notations of [HKW] and [HS], the kernel of \( \psi : \Gamma_{1,p}^0 \to \pi_1(\tilde{X}) \) contains not only \( M_0 \) and \( \Gamma(p^2) \) but also the extra element

\[
M'_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & p \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = j_2 \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}.
\]

In [HS] it is shown that a normal subgroup of \( \Gamma_{1,p}^0 \) containing \( M_0 \) and \( \Gamma(p^2) \) must contain \( \Gamma_{1,p}^1 \). The normal subgroup \( \text{Ker} \psi \) is larger than that since \( M'_0 \not\in \Gamma_{1,p}^1 \), and must therefore be equal to \( \Gamma_{1,p}^0 \) since \( \Gamma_{1,p}^0 / \Gamma_{1,p}^1 \) is isomorphic to \( \text{SL}(2, \mathbb{F}_p) \) and hence simple modulo \( \pm I \), which acts trivially. ■

Two Calabi-Yau threefolds occur in the paper [BN] of Barth and Nieto. One, called \( N \) there and also described by Naruki in [Na], is a certain singular quintic hypersurface in \( \mathbb{P}^4 \); the other, called \( \tilde{N} \), is a double cover of \( N \) and is birationally equivalent to the moduli space of abelian surfaces with a polarisation of type (1,3) (or (2,6)) and level-2 structure.

**Theorem 3.4.** If \( Z \) is a desingularisation of \( \tilde{N} \) then \( \pi_1(Z) \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \).

**Proof:** Put \( E = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \) and \( \Lambda = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \). The moduli space of \((1,3)\)-polarised abelian surfaces with level-2 structure is \( \mathbb{H}_2 / \tilde{\Gamma}_{1,3}^0(2) \), where \( \tilde{\Gamma}_{1,3}^0(2) \) is the kernel of reduction mod 2 in \( \tilde{\Gamma}_{1,3}^0 = \text{Sp}(\Lambda, \mathbb{Z}) \) and \( \tilde{\Gamma}_{1,3}^0(2) \) acts on \( \mathbb{H}_2 \) by

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \longrightarrow (AZ + B)(CZ + D)^{-1}E.
\]

One can check, by the same method and with the same notation as in [HS], that \( \tilde{\Gamma}_{1,3}^0(2) \) is generated by \( j_1(\Gamma_1(2)) \), \( j_2(\Gamma_1(2)) \), \( \tilde{M}_1^2 \), \( \tilde{M}_2^2 \), \( \tilde{M}_3^2 \) and \( \tilde{M}_4^2 \). We must describe the normal subgroup generated by the centres of unipotent radicals of parabolic groups in terms of these. \( \tilde{M}_0^2 = j_1 \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \) is in the centre of the unipotent radical of the parabolic subgroup corresponding to the \( \Lambda \)-isotropic line \( \mathbb{Q}(1,0,0,0) \), and the normal subgroup of \( \Gamma_1(2) \) generated by \( \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \) is the whole of \( \Gamma_1(2) \) (this reflects the fact that the modular curve of level 2 is rational). So we get all of \( j_1(\Gamma_1(2)) \), and similarly we get \( j_2(\Gamma_1(2)) \) from \( \tilde{M}_0^2 \) corresponding to \((0,1,0,0)\). The elements \( \tilde{M}_1^2 \) and \( \tilde{M}_2^2 \) are also unipotent and they lie in the centres of the unipotent radicals of the parabolic groups corresponding to the isotropic
planes spanned by \((1,0,0,0)\) and \((0,1,0,0)\) and by \((0,0,1,0)\) and \((0,0,0,1)\) respectively. All other unipotent elements are conjugate to products of these. We can also generate

\[
\tilde{M}_3^4 = j_2 \left( \begin{array}{cc} 1 & 0 \\ -2p & 1 \end{array} \right) \tilde{M}_0^2 \tilde{M}_1^2 \tilde{M}_0^{-2} \tilde{M}_1^{-2},
\]

and \(\tilde{M}_4^4\) similarly, but not \(\tilde{M}_3^2\) or \(\tilde{M}_4^2\). This is because

\[
\tilde{\Gamma}_{1,3}^0(2) = \left\{ \gamma \in \text{Sp}(\Lambda, \mathbb{Z}) \mid \gamma - I \in \left( \begin{array}{ccc} 2\mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z} \\ 6\mathbb{Z} & 2\mathbb{Z} & 6\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z} \\ 6\mathbb{Z} & 2\mathbb{Z} & 6\mathbb{Z} \end{array} \right) \right\}
\]

contains the normal subgroup

\[
\check{\Gamma}_{1,3}^0(2) = \left\{ \gamma \in \text{Sp}(\Lambda, \mathbb{Z}) \mid \gamma - I \in \left( \begin{array}{ccc} 2\mathbb{Z} & 4\mathbb{Z} & 2\mathbb{Z} \\ 12\mathbb{Z} & 2\mathbb{Z} & 6\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z} \\ 6\mathbb{Z} & 2\mathbb{Z} & 12\mathbb{Z} \end{array} \right) \right\}.
\]

So in this case \(\Pi(\tilde{\Gamma}_{1,3}^0(2)) \cong \tilde{\Gamma}_{1,3}^0(2)/\check{\Gamma}_{1,3}^0(2) \cong \mathbb{Z}/2 \times \mathbb{Z}/2\).

We cannot at once conclude that \(\pi_1(Z) \cong \mathbb{Z}/2 \times \mathbb{Z}/2\) because \(\tilde{\Gamma}_{1,3}^0(2)\) is not neat. The elements that cause it to fail to be neat are the conjugates of \(-I\) (which acts trivially on \(\mathbb{H}_2\)) and of \(\text{diag}(1,-1,1,-1) = I_0\). But \(I_0\) fixes a surface and acts, even at the cusps, as a reflection at its fixed points, so that \(X'\), the compactification of \(\mathbb{H}_2/\tilde{\Gamma}_{1,3}^0(2)\), is smooth even though \(\check{\Gamma}_{1,3}^0(2)\) is not neat. Furthermore, \(\Pi(\tilde{\Gamma}_{1,3}^0(2))\) acts freely on \(X'\) because if the image in \(\Pi(\tilde{\Gamma}_{1,3}^0(2))\) of \(\tilde{M}_3^2\) (or \(\tilde{M}_4^2\)) preserves some boundary component \(F\) and fixes \(x \in \overline{D/U(F)_{\mathbb{Z}}}\), then \(\tilde{M}_3^2\) is in the group generated by \(U(F)_{\mathbb{Z}}\), \(I_0\) and \(-I\), but these are all in \(\check{\Gamma}_{1,3}^0(2)\). But \(X'\) is simply-connected by construction, so \(\pi(Z) \cong \mathbb{Z}/2 \times \mathbb{Z}/2\).

To calculate the fundamental group of a resolution of \(N\) we need a different method, using the projective description of the variety.

**Proposition 3.4.** Any desingularisation \(Y\) of \(N\) is simply-connected.

**Proof:** \(N\) is given by the equations

\[
u_0 + u_1 + u_2 + u_3 + u_4 + u_5 = \sum_{i=0}^{5} \prod_{j \neq i} u_j = 0
\]
in \( \mathbb{P}^5 \). Taking the hyperplanes \( H_0 = (u_0 = 0) \) and \( H_1 = (u_1 = 0) \) and applying the Lefschetz hyperplane theorem for complete intersections (see for instance [D]), we have \( \pi_1(N) \cong \pi(N \cap H_0 \cap H_1) \). But \( N \cap H_0 \cap H_1 \) is given by \( u_0 = u_1 = u_2 + u_3 + u_4 + u_5 = 0 \) and so is simply-connected. \( N \) is singular, but a resolution \( \psi : Y \to N \) is described precisely in [BN], Section 9, especially (9.1) and (9.3). The morphism \( \psi \) contracts only rational varieties (Cayley nodal cubic surfaces, quadrics and rational curves) and in particular has simply-connected fibres, so it does not affect the fundamental group.

Recently Beauville has constructed an example of a Calabi-Yau threefold whose fundamental group is not abelian ([Be]). Perhaps some Siegel modular threefolds also have those properties. Unfortunately it is not easy to tell when a Siegel modular threefold is a Calabi-Yau, and all the examples given in this paper with non-abelian fundamental group are of general type.

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