C*-Algebra valued fuzzy normed spaces with application of Hyers–Ulam stability of a random integral equation

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Abstract
In this paper, we consider C*-algebra valued fuzzy normed spaces. We study the random integral equation \( \frac{1}{2} \int_{x-cd}^{x+cd} u(y, \tau, d_0) d\tau = u(y, x, d) \) which is related to the stochastic wave equation. In addition, using a C*-algebra valued fuzzy controller function, we consider its C*-algebra valued fuzzy Hyers–Ulam stability.

MSC: 54E50; 39B52; 39B62; 46L05; 47H10; 39B82

Keywords: Nonlinear random integral equation; Stochastic wave equation; Iterative method; Random operator; Fuzzy normed space; C*-algebra valued fuzzy set

1 Introduction
The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for additive groups in Banach spaces. Hyers’ theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6–11]).

Let \( A \) be a C*-algebra and \( x \) be a self-adjoint element in \( A \). Then if \( x \) is of the form \( yy^* \) for some \( y \in A \), then \( x \) is called a positive element. Denote by \( A^+ \) the cone of positive elements of \( A \). We will denote \( z \leq w \) when \( w - z \in A^+ \) (see [12]).

Using random normed spaces introduced by Šerstnev [13] and studied by Muşări [14] and Radu [15], Cheng and Mordeson [16] defined fuzzy normed spaces.

In this paper, we generalize a recent paper of Saadati [17] using C*-algebra valued fuzzy sets and applying \( t \)-norms on C*-algebras (see [18, 19]).

2 C*-Algebra valued fuzzy normed spaces
In this section, we discuss C*-algebra. For more details, we refer the reader to [20–22].
Definition 1 Let $\mathcal{A}$ be an order commutative $C^*$-algebra and $\mathcal{A}^+$ be the positive section of $\mathcal{A}$. Let $U \neq \emptyset$. A $C^*$-algebra valued fuzzy set $C$ on $U$ is a function $C: U \rightarrow \mathcal{A}^+$. For each $u$ in $U$, $C(u)$ represents the degree (in $\mathcal{A}^+$) to which $u$ satisfies $\mathcal{A}^+$.

We put $0 = \inf \mathcal{A}^+$ and $1 = \sup \mathcal{A}^+$. Now, we define the triangular norm ($t$-norm) on $\mathcal{A}^+$.

Definition 2 A function $T: \mathcal{A}^+ \times \mathcal{A}^+ \rightarrow \mathcal{A}^+$ which satisfies

(i) $(\forall u \in \mathcal{A}^+) \ (T(u, 1) = u)$; (boundary condition)
(ii) $(\forall (u, v) \in \mathcal{A}^+ \times \mathcal{A}^+) \ (T(u, v) = T(v, u))$; (commutativity)
(iii) $(\forall (u, v, w) \in \mathcal{A}^+ \times \mathcal{A}^+ \times \mathcal{A}^+) \ (T(u, T(v, w)) = T(T(u, v), w))$; (associativity)
(iv) $(\forall (u, u', v, v') \in \mathcal{A}^+ \times \mathcal{A}^+ \times \mathcal{A}^+ \times \mathcal{A}^+) \ (u \leq u' \text{ and } v \leq v' \Rightarrow T(u, v) \leq T(u', v'))$,

(monotonicity)

is called a $t$-norm.

If, for every $u, v \in \mathcal{A}^+$ and sequences $\{u_n\}$ and $\{v_n\}$ converging to $u$ and $v$, we have

$$\lim_n T(u_n, v_n) = T(u, v),$$

then we say $T$ on $\mathcal{A}^+$ is continuous (in short, a $ct$-norm).

Definition 3 Assume that $\mathcal{F}: \mathcal{A}^+ \rightarrow \mathcal{A}^+$ satisfies $\mathcal{F}(0) = 1$ and $\mathcal{F}(1) = 0$ and is decreasing. Then $\mathcal{F}$ is called a negation on $\mathcal{A}^+$.

Example 4 Let

$$\text{diag} M_n([0,1]) = \left\{ \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} : \text{diag}[u_1, \ldots, u_n], u_1, \ldots, u_n \in [0,1] \right\}.$$ 

We say $\text{diag}[u_1, \ldots, u_n] \leq \text{diag}[b_1, \ldots, b_n]$ if and only if $a_i \leq b_i$ for all $i = 1, \ldots, n$ and also $1 = \text{diag}[1, \ldots, 1]$ and $0 = \text{diag}[0, \ldots, 0]$. Now, we see that if $\mathcal{A} = \text{diag} M_n([0,1])$, then $\text{diag} M_n([0,1]) = \mathcal{A}^+$. Let $T_F : \text{diag} M_n([0,1]) \times \text{diag} M_n([0,1]) \rightarrow \text{diag} M_n([0,1])$ be

$$T_F(\text{diag}[u_1, \ldots, u_n], \text{diag}[v_1, \ldots, v_n]) = \text{diag}[u_1 \cdot v_1, \ldots, u_n \cdot v_n].$$

Then $T_F$ is a $t$-norm (product $t$-norm). Note that this $t$-norm is continuous.

Example 5 Let $\text{diag} M_n([0,1]) = \mathcal{A}^+$ and $T_M : \text{diag} M_n([0,1]) \times \text{diag} M_n([0,1]) \rightarrow \text{diag} M_n([0,1])$ be

$$T_M(\text{diag}[u_1, \ldots, u_n], \text{diag}[v_1, \ldots, v_n]) = \text{diag}[\min(u_1, v_1), \ldots, \min(u_n, v_n)].$$

Then $T_M$ is a $t$-norm (minimum $t$-norm). Note that this $t$-norm is continuous.

Definition 6 The triple $(S, \eta, T)$ is called a $C^*$-algebra valued fuzzy normed space (in short, $C^*$-AVFN-space) if $S \neq \emptyset$, $T$ is a $ct$-norm on $\mathcal{A}^+$ and $\eta$ is a $C^*$-algebra valued fuzzy set on $S^2 \times ]0, +\infty[$ such that, for each $t, s, p \in T$ and $\tau, \varsigma$ in $]0, +\infty[$, we have
(a) \( \eta(s, \tau) > 0 \);
(b) \( \eta(s, \tau) = 1 \) for all \( \tau > 0 \) if and only if \( s = 0 \);
(c) \( \eta(as, \tau) = \eta(s, \frac{a}{|a|}) \) for all \( s \in S \) and \( a \in \mathbb{R} \) with \( a \neq 0 \);
(d) \( \eta(t + s, \tau + \zeta) \geq \tau(\eta(t, \tau), \eta(s, \zeta)) \) for all \( t, s \in S \) and \( \tau, \zeta \geq 0 \);
(e) \( \eta(s, \cdot) : (0, \infty) \to \mathcal{A}^+ \setminus \{0\} \) is left continuous;
(f) \( \lim_{t \to \infty} \eta(s, \tau) = 1 \).

Also, \( \eta \) is a \( C^* \)-algebra valued fuzzy norm.

Let \((S, \eta, \mathcal{T})\) be a \( C^* \)-AVFN-space. For \( \tau > 0 \), define the open ball \( B(t, \varrho, \tau) \) as

\[
B(s, \varrho, \tau) = \{ t \in S : \eta(t - s, \tau) \geq \mathcal{F}(\varrho) \},
\]

in which \( s \in S \) is the center and \( \varrho \in \mathcal{A}^+ \setminus \{0, 1\} \) is the radius. We say that \( A \subseteq S \) is open if, for each \( s \in A \), there exist \( \tau > 0 \) and \( \varrho \in \mathcal{A}^+ \setminus \{0, 1\} \) such that \( B(s, \varrho, \tau) \subseteq A \). We denote the family of all open subsets of \( S \) by \( \tau_\eta \), and so \( \tau_\eta \) is the \( C^* \)-fuzzy topology induced by the \( C^* \)-algebra valued fuzzy norm \( \eta \).

**Example 7** Consider the linear normed space \((S, \| \cdot \|)\). Let \( \mathcal{T} = \mathcal{T}_M \) and define the fuzzy set \( \eta \) on \( S^2 \times (0, \infty) \) as follows:

\[
\eta(s, \tau) = \text{diag} \left[ \frac{\tau}{\tau + \| s \|}, \exp \left( -\frac{\| s \|}{\tau} \right) \right]
\]

for all \( \tau \in \mathbb{R}^+ \). Then \((S, \eta, \mathcal{T}_M)\) is a \( C^* \)-AVFN-space.

**Lemma 8** (\([23]\)) Let \((S, \eta, \mathcal{T})\) be a \( C^* \)-AVFN-space. Then \( \eta(s, \tau) \) is nondecreasing with respect to \( \tau \) for all \( s \in S \).

**Definition 9** Let \( \{s_n\}_{n \in \mathbb{N}} \) be a sequence in a \( C^* \)-AVFN-space \((S, \eta, \mathcal{T})\). If, for all \( \varepsilon \in \mathcal{A}^+ \setminus \{0\} \) and \( \tau > 0 \), there exists \( n_0 \in \mathbb{N} \) such that, for all \( m \geq n \geq n_0 \),

\[
\eta(s_m - s_n, \tau) \geq \mathcal{F}(\varepsilon),
\]

then \( \{s_n\}_{n \in \mathbb{N}} \) is said to be Cauchy.

Also \( \{s_n\}_{n \in \mathbb{N}} \) is said to be convergent to \( s \in S \) (\( s_n \xrightarrow{\eta} s \)) if \( \eta(s_n - s, \tau) = \eta(s - s_n, \tau) \to 1 \) as \( n \to +\infty \) for every \( \tau > 0 \). If every Cauchy sequence is convergent in a \( C^* \)-AVFN-space, then the space is said to be complete. A complete \( C^* \)-AVFN-space is called a \( C^* \)-algebra valued fuzzy Banach space (in short, a \( C^* \)-AVFB-space).

### 3 Random operators in \( C^* \)-AVFB-spaces

Let \((\Gamma, \Sigma, \xi)\) be a probability measure space. Assume that \((T, \mathcal{B}_T)\) and \((S, \mathcal{B}_S)\) are Borel measurable spaces, in which \( T \) and \( S \) are \( C^* \)-AVFB-spaces. A mapping \( F : \Gamma \times T \to S \) is said to be a random operator if \( \{ \gamma : F(\gamma, t) \in B \} \in \Sigma \) for all \( t \in T \) and \( B \in \mathcal{B}_S \). Also, \( F \) is a random operator if \( F(\gamma, t) = \xi(\gamma) \) is an \( S \)-valued random variable for every \( t \) in \( T \). A random operator \( F : \Gamma \times T \to S \) is called linear if \( F(\gamma, at_1 + bt_2) = aF(\gamma, t_1) + bF(\gamma, t_2) \) for almost every \( \gamma \) for each \( t_1, t_2 \) in \( T \) and scalars \( a, b \) and **bounded** if there exists a nonnegative
real-valued random variable $M(\gamma)$ such that

$$\eta(F(\gamma, t_1) - F(\gamma, t_2), M(\gamma) \tau) \geq \eta(t_1 - t_2, \tau),$$

almost every $\gamma$ for each $t_1, t_2$ in $T$ and $\tau > 0$.

Recently, some authors discussed the approximation of functional equations in several spaces by using a direct technique and a fixed point technique; for fuzzy Menger normed algebras, see [24]; for fuzzy metric spaces, see [25, 26]; for FN spaces, see [27]; for non-Archimedean random Lie $C^*$-algebras, see [28]; for non-Archimedean random normed spaces, see [29]; for random multi-normed space, see [30]; and we also refer the reader to [31–34].

Note that a $[0, \infty]$-valued metric is called a generalized metric.

**Theorem 10** ([35, 36]) Consider a complete generalized metric space $(T, \delta)$ and a strictly contractive function $\Lambda: T \rightarrow T$ with Lipschitz constant $L < 1$. For every given element $t \in T$, either

$$\delta(A^n t, A^{n+1} t) = \infty$$

for each $n \in \mathbb{N}$ or there is $n_0 \in \mathbb{N}$ such that

1. $\delta(A^n t, A^{n+1} t) < \infty, \forall n \geq n_0$;
2. the fixed point $s^*$ of $\Lambda$ is the convergent point of sequence $\{A^n t\}$;
3. in the set $V = \{s \in T | \delta(A^n_0 t, s) < \infty\}$, $s^*$ is the unique fixed point of $\Lambda$;
4. $(1 - L)\delta(s, s^*) \leq \delta(s, As)$ for every $s \in V$.

4 Random integral equation related to the stochastic wave equation

Let $(\Gamma, \Sigma, \xi)$ be a probability space and $(S, \eta, T_\mathcal{M})$ be a $C^*$ AVFB-space. Assume that the real numbers $c > 0$ and $d_0$ are fixed, and suppose that $\gamma \in \Gamma$. Consider the stochastic wave equation

$$u_{dd}(\gamma, x, d) = c^2 u_{xx}(\gamma, x, d). \quad (4.1)$$

Since

$$u_d(\gamma, x, d) = \frac{1}{2c} \frac{\partial}{\partial d} \int_{x-cd}^{x+cd} H(\gamma, \tau, d_0) d\tau$$

$$= \frac{1}{2} H(\gamma, x + cd, d_0) + \frac{1}{2} H(\gamma, x - cd, d_0),$$

$$u_{dd}(\gamma, x, d) = \frac{c}{2} H_d(\gamma, x + cd, d_0) - \frac{c}{2} H_d(\gamma, x - cd, d_0), \quad (4.2)$$

we have that

$$u(\gamma, x, d) := \frac{1}{2c} \int_{x-cd}^{x+cd} H(\gamma, \tau, d_0) d\tau \quad (4.3)$$
is a solution of (4.1) for any random differentiable \( S \)-valued function \( H \) on \( \Gamma \times \mathbb{R} \).

On the other hand, Jung [37] showed that if the \( S \)-valued functions \( F \) and \( G \) on \( \Gamma \times \mathbb{R}^2 \) are twice differentiable, then the \( S \)-valued solution \( u \) on \( \Gamma \times \mathbb{R}^2 \) of (4.1) has a representation of the form

\[
u(\gamma, x, d) = F(\gamma, x + cd) + G(\gamma, x - cd),
\]

in which

\[
\frac{1}{2c} \int_{x-cd}^{x+cd} F(\gamma, \tau) \, d\tau = F(\gamma, x + cd), \tag{4.5}
\]

\[
\frac{1}{2c} \int_{x-cd}^{x+cd} G(\gamma, \tau) \, d\tau = G(\gamma, x - cd).
\]

Consider the random integral equation

\[
\frac{1}{2c} \int_{x-cd}^{x+cd} u(\gamma, \tau, d_0) \, d\tau = u(\gamma, x, d), \tag{4.6}
\]

which is controlled by the continuous fuzzy set \( \varphi(x, d, t) \) as

\[
\eta \left( \frac{1}{2c} \int_{x-cd}^{x+cd} u(\gamma, \tau, d_0) \, d\tau - u(\gamma, x, d), t \right) \geq \varphi(x, d, t).
\]

We say that the random integral equation (4.6) has fuzzy Hyers–Ulam stability if there are \( u_0(\gamma, x, d) \) and \( \lambda > 0 \) such that

\[
\frac{1}{2c} \int_{x-cd}^{x+cd} u_0(\gamma, \tau, d_0) \, d\tau = u_0(\gamma, x, d),
\]

\[
\eta \left( u(\gamma, x, d) - u_0(\gamma, x, d), t \right) \geq \varphi \left( x, d, \frac{t}{\lambda} \right).
\]

5 \( C^* \)-Algebra-valued fuzzy Hyers–Ulam stability

Let \( c > 0, d_0 > 0, \) and \( a + cd_0 < b - cd_0 \). Let \( (\Gamma, \Sigma, \xi) \) be a probability measure space, \( (S, \eta, T_M) \) be a \( C^* \)-AVFB-space, \( \alpha := [a, b], \beta := (0, d_0], \) and \( \alpha_0 := [a + cd_0, b - cd_0]. \) Let \( M > 0 \) and \( 0 < L < 1. \) Consider a continuous \( C^* \)-algebra-valued fuzzy set \( \varphi : \alpha \times \beta \times (0, \infty) \rightarrow J \) which is increasing in the second and third components and satisfies

\[
\inf_{\tau \in [x-cd, x+cd]} \varphi \left( \tau, d, \frac{t}{d} \right) \geq \varphi \left( x, d, \frac{t}{L} \right), \tag{5.1}
\]

for all \( x \in \alpha_0, d \in \beta, \) and \( t > 0. \)

The set \( T \) consists of all random operators \( F : \Gamma \times \alpha \times \beta \rightarrow S \) which satisfy the following:

(a) \( F(\gamma, x, d) \) is continuous for each \( x \in \alpha_0, d \in \beta, \) and \( \gamma \in \Gamma; \)

(b) \( F(\gamma, x, d) = 0_S \) for all \( x \in \alpha \setminus \alpha_0, d \in \beta, \) and \( \gamma \in \Gamma; \)

(c) \( \eta(F(\gamma, x, d), t) \geq \varphi(x, d, \frac{t}{M}) \) for all \( x \in \alpha_0, d \in \beta, t > 0, \) and \( \gamma \in \Gamma. \)
**Theorem 11** Suppose that a random operator $u \in T$ satisfies the random integral inequality

$$\eta\left(\frac{1}{2c}\int_{x-cd}^{x+cd} u(\gamma, \tau, d\tau) d\tau - u(\gamma, x, d), t\right) \geq \varphi(x, d, t)$$

(5.2)

for all $x \in \alpha_0$, $d \in \beta$, $t > 0$, and $\gamma \in \Gamma$. Then there is a unique random operator $u_0 \in T$ which satisfies

$$\frac{1}{2c}\int_{x-cd}^{x+cd} u_0(\gamma, \tau, d\tau) d\tau = u_0(\gamma, x, d),$$

(5.3)

$$\eta(u(\gamma, x, d) - u_0(\gamma, x, d)) \geq \varphi(x, d, (1-L)t)$$

(5.4)

for all $x \in \alpha_0$, $d \in \beta$, $t > 0$, and $\gamma \in \Gamma$.

**Proof** We consider the $[0, \infty]$-valued metric $\delta$ on $T$ defined by

$$\delta(F, G) = \inf\left\{ \lambda \in [0, \infty] \middle| \eta(F(\gamma, x, d) - G(\gamma, x, d), t) \geq \varphi(x, d, t/\lambda) \right\} \forall x \in \alpha_0, d \in \beta, \gamma \in \Gamma, t > 0$$

(5.5)

In [38], Miheţ and Radu proved that $(B, \delta)$ is complete (see also [39]).

Consider the operator $A : T \rightarrow T$ given by

$$(AH)\gamma, x, d = \begin{cases} \frac{1}{2c} \int_{x-cd}^{x+cd} H(\gamma, \tau, d\tau) d\tau, & (x \in \alpha_0, d \in \beta, \gamma \in \Gamma), \\ 0, & \text{(otherwise).} \end{cases}$$

(5.6)

It is easy to show that $AH$ is continuous on $\Gamma \times \alpha_0 \times \beta$. Let $x - cd = \xi_1 < \xi_2 < \cdots < \xi_k = x + cd$, $\Delta s_i = \xi_i - \xi_{i-1}, i = 1, 2, \ldots, k$. Using (5.1), (c), and (5.6), we obtain

$$\eta(AH)(\gamma, x, d, t) = \eta\left(\frac{1}{2c}\int_{x-cd}^{x+cd} H(\gamma, \tau, d\tau) d\tau, t\right)$$

$$= \eta\left(\frac{1}{2c} \lim_{|\Delta| \rightarrow 0} \sum_{i=1}^{k} H(\gamma, \xi_i, d\tau) \Delta s_i, t\right)$$

$$= \eta\left(\lim_{|\Delta| \rightarrow 0} \sum_{i=1}^{k} H(\gamma, \xi_i, d\tau) \Delta s_i, 2ct\right)$$

$$= \lim_{|\Delta| \rightarrow 0} \eta\left(\sum_{i=1}^{k} H(\gamma, \xi_i, d\tau) \Delta s_i, 2ct\right)$$

$$\geq \lim_{|\Delta| \rightarrow 0} T_M \eta\left(H(\gamma, \xi_i, d\tau) \Delta s_i, \frac{2ct}{k}\right)$$

$$\geq \inf_{\tau \in [x-cd, x+cd]} \eta\left(H(\gamma, \xi_i, d\tau), \frac{2ct}{|\Delta s_i|k}\right)$$
for any given $x \in \alpha_0$, $d \in \beta$, $t > 0$, and $\gamma \in \Gamma$, and then $\Lambda H \in T$. Let $F, G \in T$ and $\lambda_{FG} \in [0, \infty]$ such that $\delta(F, G) \leq \lambda_{FG}$. Then we have

$$
\eta(F(\gamma, x, d) - G(\gamma, x, d), t) \geq \varphi(x, d, \frac{t}{\lambda_{FG}})
$$

(5.8)

for all $x \in \alpha_0$, $d \in \beta$, $t > 0$, and $\gamma \in \Gamma$, i.e., $\Lambda$ is strictly contractive on $T$. From (5.1), (5.6), and (5.8), we get

$$
\eta((AF)(\gamma, x, d) - (\Lambda G)(\gamma, x, d), t) = \eta\left(\frac{1}{2c} \int_{x-cd}^{x+cd} (F(\gamma, \tau, d_0) - G(\gamma, \tau, d_0)) d\tau, t\right) \\
= \eta\left(\frac{1}{2c} \lim_{|\Delta \tau| \to 0} \sum_{i=1}^{k} (F(\gamma, \xi_i, d_0) - G(\gamma, \xi_i, d_0)) \Delta s_i, t\right) \\
= \eta\left(\lim_{|\Delta \tau| \to 0} \sum_{i=1}^{k} (F(\gamma, \xi_i, d_0) - G(\gamma, \xi_i, d_0)) \Delta s_i, 2ct\right) \\
\geq \lim_{|\Delta \tau| \to 0} \eta\left(F(\gamma, \tau, d_0) - G(\gamma, \tau, d_0), 2ct\right) \\
\geq \inf_{\tau \in [x-cd, x+cd]} \eta\left(\tau, d_0, \frac{t}{d\lambda_{FG}}\right) \\
\geq \varphi\left(x, d_0, \frac{t}{L\lambda_{FG}}\right) \\
\geq \varphi\left(x, d, \frac{t}{L\lambda_{FG}}\right)
$$

(5.9)

for any given $x \in \alpha_0$, $d \in \beta$, $t > 0$, and $\gamma \in \Gamma$, which implies that $\delta(AF, \Lambda G) \leq L\lambda_{FG}$, and so $\delta(AF, \Lambda G) \leq Ld(F, G)$. Suppose $H_0 \in T$. Using (5.2) and (5.5), we get

$$
\eta((\Lambda H_0)(\gamma, x, d) - H_0(\gamma, x, d), t) \geq \eta\left(\frac{1}{2c} \int_{x-cd}^{x+cd} H_0(\gamma, \tau, d_0) d\tau - H_0(\gamma, x, d), t\right) \\
\geq \varphi(x, d, t)
$$

(5.10)
for any \( x \in \alpha_0, \ d \in \beta, \ t > 0, \) and \( \gamma \in \Gamma. \) Thus (5.5) implies that

\[
\delta(\Lambda H_0, H_0) \leq 1 < \infty. \tag{5.11}
\]

Now,

(1) Theorem 10 (2) implies that there is \( u_0 \in T \) such that \( \Lambda^u H_0 \to u_0 \) in \( (T, \delta) \) and \( \Lambda u_0 = u_0. \)

(2) Theorem 10 (3) implies that \( u_0 \) is the unique element of \( T \) which satisfies

\[
(\Lambda u_0)(\gamma, x, d) = u_0(\gamma, x, d)
\]

for any \( x \in \alpha_0, \ d \in \beta, \ t > 0, \) and \( \gamma \in \Gamma. \)

(3) Theorem 10 (3), together with (5.5) and (5.2), implies that

\[
\delta(u, u_0) \leq \frac{1}{1-L} \delta(\Lambda u, u) \leq \frac{1}{1-L}, \tag{5.12}
\]

since (5.2) means that \( \delta(\Lambda u, u) \leq 1. \) In view of (5.5), we can conclude that (5.4) holds for all \( x \in \alpha_0 \) and \( d \in \beta. \)

\[
\square
\]

6 Conclusion

In this paper, we modified and generalized fuzzy normed spaces and introduced the concept of a \( C^* \) AVFN-space. As an application, we studied the Hyers–Ulam stability of a random integral equation related to the stochastic wave equation in \( C^* \) AVFB-spaces.

Acknowledgements

Not applicable.

Funding

This work was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science, and Technology (NRF-2017R1D1A1B04032937).

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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Received: 9 April 2020  Accepted: 17 June 2020  Published online: 02 July 2020

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