Stochastic Nicholson’s blowflies delay differential equation with regime switching

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Abstract In this paper, we investigate the global existence of almost surely positive solution to a stochastic Nicholson’s blowflies delay differential equation with regime switching, and give the estimation of the path. The results presented in this paper extend some corresponding results in Wang et al. stochastic Nicholson’s blowflies delayed differential equations, Appl. Math. Lett. 87 (2019) 20-26.

Keywords Nicholson’s blowflies model, Regime switching, Global solution

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1 Introduction

The following delay differential equation, known as the famous Nicholson’s blowflies model

\[ X'(t) = -\delta X(t) + p X(t - \tau) e^{-aX(t-\tau)} \]

was presented by Gurney et al. (1990)\textsuperscript{1} to model the population of Lucilia cuprina (the Australian sheep-blowfly). The biological meaning of the parameters in this model is that: $\delta$ denotes the adult death rate of per capita daily, $p$ denotes the maximum egg production rate of per capita daily, $\tau$ denotes the generation time, and $1/a$ is the size that the Lucilia cuprina reproduces at its maximum rate. Since then, the dynamics of this model was investigated by many researchers, see Berezansky et al. (2010)\textsuperscript{2} for an overview.

Considering that the per capita daily adult death rate $\delta$ is affected by the white noise of the environment, one often uses $\delta - \sigma dB(t)$ in lieu of $\delta$, where $\sigma$ denotes the intensity of the white noise and $B(t)$ is a one-dimensional standard Brown motion defined on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, which is a complete probability space. Wang et al. (2019)\textsuperscript{3} studied the following stochastic Nicholson’s blowflies delay differential equation:

\[ dX(t) = [-\delta X(t) + p X(t - \tau) e^{-aX(t-\tau)}]dt + \sigma X(t)dB(t), \tag{1.1} \]

with initial conditions $X(s) = \phi(s)$, for $s \in [-\tau, 0]$, $\phi \in C([-\tau, 0], [0, +\infty))$, $\phi(0) > 0$. Under the condition $\delta > \sigma^2/2$, they prove that equation (1.1) has a unique global solution on $[-\tau, \infty)$, which is positive a.s. on $[0, \infty)$, and give the estimations of $\limsup_{t \to \infty} E[X(t)]$ and $\limsup_{t \to \infty} \frac{1}{t} \int_0^t E[X(s)]ds$, respectively.

But we find from numerical simulation that the condition $\delta > \sigma^2/2$ is too strict to the nonexplosion of the solution to equation (1.1). For example, let $\delta = 1$, $p = 5$, $\tau = 1$, $a = 1$, $\sigma = 2$, then the numerical simulation in Figure 1(left) shows that the solution $X(t)$ is globally existent on $[0, +\infty)$, and oscillating about $N^* = \log p/\delta$, which is the positive equilibrium of the corresponding deterministic equation of (1.1), but $\delta < \sigma^2/2$.

Meanwhile, as one know that besides the white noises there is a so called telegraph noise in the real ecosystem. A switching process between two or more regimes can be used to illustrate this type of noise. One can model

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the regime switching by a continuous time Markov chain \((r_t)_{t \geq 0}\), which takes values in a finite state spaces \(S = \{1, 2, ..., m\}\) and satisfies

\[
P(r_{t+\Delta} = j | r_t = i) = \begin{cases} 
q_{ij} \Delta + o(\Delta), & \text{for } i \neq j, \\
1 + q_{ij} \Delta + o(\Delta), & \text{for } i = j,
\end{cases} \quad \text{as } \Delta \to 0^+,
\]

where \(Q = (q_{ij}) \in R^{m \times m}\) is the infinitesimal generator, and \(q_{ij} > 0\) denotes the transition rate from \(i\) to \(j\) for \(i \neq j\), and \(\sum_{j \in S} q_{ij} = 0, \forall i \in S\). In consideration of that the white noise and the telegraph noise are deferent types of noise, we assume that the Markov chain \(r_t\) is independent of the Brown motion \(B(t)\). For more details on the theorem of regime switching system, one can refer to Mao et al. (2006) [4], Khasminskii et al. (2007) [5] and Yin et al. (2009) [6].

Considering both the effects of white noise and telegraph noise of environment, we present the following stochastic Nicholson’s blowflies delay differential equation with regime switching

\[
dX(t) = \left[ -\delta_i X(t) + p_{rt} X(t - \tau_r) e^{-\alpha_r X(t - \tau_r)} \right] dt + \sigma_r X(t) dB(t)
\]

with initial conditions:

\[
X(s) = \phi(s), \quad \text{for } s \in [-\tau, 0], \phi \in C([-\tau, 0], \mathbb{R}^+),
\]

where \(\tau = \max_{i \in S} \{\tau_i\}, \mathbb{R}^+ = (0, +\infty)\). The contribution of this paper is as follows:

- Model (1.2) considered in this paper includes regime switching, which is more general than model (1.1);
- We show the global existence of almost surely positive solution to equation (1.2) without the restricted condition \(\delta_i > \sigma_i^2/2, i \in S\), which extends the one in [3].
- We give the estimations of \(\limsup_{t \to \infty} E[X_\theta(t)], \limsup_{t \to \infty} \frac{1}{t} \int_0^t E[X_\theta(s)] ds, \liminf_{t \to \infty} X(t)\) and the sample Lyapunov exponent of \(X(t)\), which are more general than the ones in literature.

For simplicity, in the following and next sections we use the following notations:

\[
\hat{\gamma} = \min_{i \in S} \{g_i\}, \ \check{\gamma} = \max_{i \in S} \{g_i\}, \ \mathbb{R}_0^+ = [0, \infty), \ \beta_r = \theta \left[ \delta_i + (1 - \theta)\sigma_i^2/2 \right], \ \gamma_r = p_r (a_r c)^{-1}, \ \rho_r = \delta_i / \sigma_i^2, \ \mu_r = 2 \delta_i - p_r - \sigma_i^2, \ \text{and} \ C_i = \begin{cases} 
\frac{\sigma_i^2 - 2 \delta_i + \sqrt{\sigma_i^4 - 2 \sigma_i^2 \delta_i + 4 \rho_i^2}}{2}, & \text{if } \delta_i \leq \sigma_i^2/2, \\
\frac{\sigma_i^2}{2 \rho_i^{1/2}}, & \text{if } \delta_i > \sigma_i^2/2,
\end{cases} \quad i \in S.
\]

**Assumption (A):** The Markov chain \((r_t)_{t \geq 0}\) is irreducible with an invariant distribution \(\pi = (\pi_i, i \in S)\).

Before the main results, we first give two lemmas and omit the proof here for saving layouts.

**Lemma 1.1.** Let \(f(x) = xe^{-ax}, a \in \mathbb{R}^+, x \in \mathbb{R}_0^+, \) then \(f(x) \leq (ae)^{-1}\).

**Lemma 1.2.** For \(a \in \mathbb{R}, b \in \mathbb{R}^+, \) we have \(\frac{ax^2 + bx}{x^2 + 4} \leq C(a)\) for \(x \in \mathbb{R}, \) where \(C(a) = \begin{cases} 
(a + \sqrt{a^2 + b^2})/2, & a \geq 0; \\
-b^2/4a, & a < 0.
\end{cases}\)

2 **Main Results**

In this section, we will show the global existence of almost surely positive solution of equation (1.2) with initial value (1.3), and give some estimations of the solution \(X(t)\).

**Theorem 2.1.** For any given initial value (1.3), equation (1.2) has a unique solution \(X(t)\) on \([-\tau, \infty)\), which is positive on \(\mathbb{R}_0^+\) a.s.

**Proof.** Since all the coefficients of equation (1.2) are locally Lipschitz continuous on \(\mathbb{R}_0^+\), then for any given initial value (1.3), there is a unique maximal local solution \(X(t)\) on \([-\tau, \Lambda_e)\), where \(\Lambda_e\) is the explosion time. Assume that the subsystems of (1.2) switch along the Markov chain \(r_0, r_1, r_2, ..., r_m\), and the times on each state are \([0, t_0], [t_0, t_1], [t_1, t_2], ..., [t_{m-1}, \Lambda_e]\), respectively.
Step 1. We first show that $X(t)$ is positive almost surely on $[0, \Lambda_\varepsilon)$. System (1.2) switching along the Markov chain $r_0, r_1, r_2, \ldots, r_n$ gives

$$
dX(t) = [\delta_{r_0}X(t) + G(X(t - \tau_{r_0}), r_0)] dt + \sigma_{r_0}X(t) dB(t), \quad t \in [0, t_0],
$$

$$
dX(t) = [\delta_{r_1}X(t) + G(X(t - \tau_{r_1}), r_1)] dt + \sigma_{r_1}X(t) dB(t), \quad t \in [t_0, t_1],
$$

$$
\vdots
$$

$$
dX(t) = [\delta_{r_n}X(t) + G(X(t - \tau_{r_n}), r_n)] dt + \sigma_{r_n}X(t) dB(t), \quad t \in [t_{n-1}, \Lambda_\varepsilon),
$$

where $G(X(t - \tau_{r_j}), r_j) = p_{r_j}X(t - \tau_{r_j})e^{-\alpha_{r_j}X(t-\tau_{r_j})}$. Obviously, each equation is a linear stochastic differential equation, which can be solved step by step, see Mao (1997, p. 98-99) for more details.

In fact, for $t \in [0, \tau_{r_0}] \subset [0, t_0]$, the solution of the first equation of (2.4) is

$$
X(t) = \Phi_{r_00}(t) \left[ \phi(0) + \int_0^t \Phi_{r_01}^{-1}(s)G(X(s - \tau_{r_0}), r_0) ds \right] > 0 \text{ a.s.,}
$$

where $\Phi_{r_0}(t) = e^{-d_{r_0}(t-\tau_{r_0}) + \sigma_{r_0}[B(t) - B(\tau_{r_0})]}$, and for $t \in [\tau_{r_0}, 2\tau_{r_0}] \subset [\tau_{r_0}, t_0]$ we have

$$
X(t) = \Phi_{r_01}(t) \left[ X(\tau_{r_0}) + \int_{\tau_{r_0}}^t \Phi_{r_01}^{-1}(s)G(X(s - \tau_{r_0}), r_0) ds \right] > 0 \text{ a.s.,}
$$

and for $t \in [j_0\tau_{r_0}, t_0]$, where $j_0 = \lfloor \frac{t_0}{\tau_{r_0}} \rfloor$, we get

$$
X(t) = \Phi_{r_0j_0}(t) \left[ X(j_0\tau_{r_0}) + \int_{j_0\tau_{r_0}}^t \Phi_{r_0j_0}^{-1}(s)G(X(s - \tau_{r_0}), r_0) ds \right] > 0 \text{ a.s.,}
$$

For $t \in [t_0, t_0 + \tau_{r_1}] \subset [t_0, t_1]$, the solution of the second equation of (2.4) is

$$
X(t) = \Phi_{r_10}(t) \left[ X(t_0) + \int_{t_0}^t \Phi_{r_10}^{-1}(s)G(X(s - \tau_{r_1}), r_1) ds \right] > 0 \text{ a.s.,}
$$

where $\Phi_{r_1}(t) = e^{-d_{r_1}[t-(t+\tau_{r_1})] + \sigma_{r_1}[B(t) - B(t+\tau_{r_1})]}$, and, for $t \in [t_0 + \tau_{r_1}, t_0 + 2\tau_{r_1}] \subset [t_0 + \tau_{r_1}, t_1]$, we have

$$
X(t) = \Phi_{r_11}(t) \left[ X(t_0 + \tau_{r_1}) + \int_{t_0+\tau_{r_1}}^t \Phi_{r_11}^{-1}(s)G(X(s - \tau_{r_1}), r_1) ds \right] > 0 \text{ a.s.,}
$$

and for $t \in [t_0 + j_1\tau_{r_1}, t_1]$ with $j_1 = \lfloor \frac{t_1}{\tau_{r_1}} \rfloor$, we get

$$
X(t) = \Phi_{r_1j_1}(t) \left[ X(t_0 + j_1\tau_{r_1}) + \int_{t_0+j_1\tau_{r_1}}^t \Phi_{r_1j_1}^{-1}(s)G(X(s - \tau_{r_1}), r_1) ds \right] > 0 \text{ a.s.}
$$

Repeating this procedure, we can obtain the explicit solution $X(t)$ of (2.4), which is positive on $[0, \Lambda_\varepsilon)$ a.s.

Step 2. Now, we prove that this solution $X(t)$ to equation (1.2) with (1.3) is global existence, that is, the solution will not explode in finite time. We only need to show $\Lambda_\varepsilon = \infty$ a.s. Let $k_0 > 0$ be sufficiently large that $\max_{t \in [0, k_0]} \phi(t) < k_0$, and for each integer $k \geq k_0$ define the stopping time $\Lambda_k = \inf \{ t \in [0, \Lambda_\varepsilon) : X(t) \geq k \}$. It is clear that $\Lambda_k$ is increasing as $k \to \infty$. Let $\Lambda_\infty = \lim_{k \to \infty} \Lambda_k$, whence $\Lambda_\infty \leq \Lambda_\varepsilon$ a.s. In order to show $\Lambda_\infty = \infty$, we only need to prove that $\Lambda_\infty = \infty$ a.s. If it is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that $P\{ \Lambda_\infty \leq T \} > \epsilon$, which yields that there exists an integer $k_1 \geq k_0$ such that $P\{ \Lambda_k \leq T \} > \epsilon$ for all $k \geq k_1$.

Let $V(x, i) = x^\theta$ with $\theta \in [1, 1 + 2\rho)$, then by Itô’s formula, we have

$$
dV(X(t), r_i) = [-\beta_{r_i}X^\theta(t) + \theta X^\theta(t)G(X(t - \tau_{r_i}), r_i)] dt + \theta \sigma_{r_i}X^\theta(t) dB(t)
$$

$$
\Delta V(X(t), r_i) dt + \theta \sigma_{r_i}X^\theta dB(t).
$$

(2.5)

It follows from $X(t) > 0$ a.s. on $[0, \Lambda_\varepsilon), \beta_{r_i} > 0$ and Lemma (1.4) that $\Delta V(X(t), r_i) \leq -\beta_{r_i}X^\theta(t) + \theta \gamma_{r_i}X^{\theta-1}(t) \leq M_{r_i}$. Thus, we have $dV(X(t), r_i) \leq M_{r_i} dt + \theta \sigma_{r_i}X^\theta(t) dB(t)$. For $t \in [0, \tau_{r_0}] \subset [0, t_0]$, integrating both sides of above inequality on $[0, \Lambda_\varepsilon \wedge t]$ and taking expectation lead to

$$
\int_0^{\Lambda_\varepsilon \wedge t} dV(x(t), r_i) \leq \int_0^{\Lambda_\varepsilon \wedge t} M_{r_0} dt + \int_0^{\Lambda_\varepsilon \wedge t} \theta \sigma_{r_0}X^\theta(s) dB(s),
$$

which yields that $x(\cdot \wedge t) \leq M_{r_0} + \eta \phi(t)$ for some constant $\eta > \sup_{s \in [0, t_0]} \phi(s)$.
then \( \mathbb{E}[V(A_n \wedge t, r_0)] \leq V(\phi(0)) + M_{r_0}\mathbb{E}[(A_n \wedge t)] \leq V(\phi(0)) + M_{r_0} \tau_{r_0} \), which yields \( \Lambda_\infty \geq \tau_{r_0} \) a.s. Similarly, we can obtain that \( \Lambda_\infty \geq t_0 \). Repeating this procedure, we can show \( \Lambda_\infty \geq t_1, \ldots, \Lambda_\infty \geq t_{n-1} + j_n \tau_{r_n} \) a.s. for any integer \( j_n \), which implies \( \Lambda_\infty = \infty \) a.s.

**Theorem 2.2.** The solution \( X(t) \) of equation (1.2) with initial value \( \text{E} \) has the following properties:

\[
\limsup_{t \to \infty} \mathbb{E}[X^\theta(t)] \leq \mathbb{M}, \quad \limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}[X^\theta(s)] ds \leq W/\alpha, \quad \text{where } \theta \in [1,1+2\hat{\rho}]; \quad \text{and} \quad -\hat{d} \leq \liminf_{t \to \infty} \frac{1}{t} \log X(t) \leq \limsup_{t \to \infty} \frac{1}{t} \log X(t) \leq C/2 \quad \text{a.s.}
\]

**Proof.** It follows from Itô formula that \( \mathbb{E}[e^t V(X(t))] \leq V(\phi(0)) + \int_0^t M\epsilon^s ds = V(\phi(0)) + M(e^t - 1) \), which yields \( \limsup_{t \to \infty} \mathbb{E}[X^\theta(t)] \leq \mathbb{M} \) a.s. From equation (2.5) we have \( dV(t) + \alpha X^\theta(t) dt \leq W_r dt + \theta \sigma_i X^\theta(s) dB(s) \).

This leads to the desired assertion. It follows from Itô formula and equation (1.3) that

\[
\log(1 + X^2(t)) \leq \log(1 + \phi^2(0)) + \int_0^t C_r ds + \log \sqrt{k}.
\]

Thus, for all \( \omega \in \Omega_0 \) and \( t \in [k-1, k] \) we have

\[
\frac{1}{t} \log(1 + X^2(t)) \leq \frac{1}{k-1} \left[ \log(1 + \phi^2(0)) + \log \sqrt{k} \right] + \frac{1}{t} \int_0^t C_r ds.
\]

By letting \( t \to \infty \), we get

\[
\limsup_{t \to \infty} \frac{1}{t} \log(X^2(t)) \leq \limsup_{t \to \infty} \frac{1}{t} \log(1 + X^2(t)) \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t C_r ds \leq C \quad \text{a.s.}
\]

This leads to the desired assertion. It follows from Itô formula and equation (1.3) that

\[
\log(X(t)) = \log(\phi(0)) - \int_0^t \left( \delta_r + \frac{1}{2} \sigma^2_r \right) ds + \int_0^t p_r X(s - \tau_{r_s}) e^{-a_r X(s - \tau_{r_s})} \frac{X(s)}{X(0)} ds + \int_0^t \sigma_r dB(s),
\]

which together with the large number theorem for martingales yields \( \liminf_{t \to \infty} \frac{1}{t} \log X(t) \geq -\hat{d} \).

**Theorem 2.3.** Assume (A) holds, then the solution \( X(t) \) of equation (1.2) with (1.3) has the properties:

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}[X^\theta(s)] ds \leq \frac{1}{\alpha} \sum_{i \in \mathcal{S}} \pi_i W_i, \quad \text{where } \theta \in [1,1+2\hat{\rho}]; \quad \text{and} \quad -\sum_{i \in \mathcal{S}} \pi_i d_i \leq \liminf_{t \to \infty} \frac{1}{t} \log X(t) \leq \limsup_{t \to \infty} \frac{1}{t} \log X(t) \leq \frac{1}{2} \sum_{i \in \mathcal{S}} \pi_i C_i \quad \text{a.s.}
\]
Theorem 2.6. \( T > \) then the solution \( \lambda > \)  

Proof. The Markov \( r_t \) has an invariant distribution \( \pi = (\pi_i, i \in S) \), thus we have \( \limsup_{t \to \infty} \frac{1}{t} \int_0^t W_r ds = \sum_{i \in S} \pi_i W_i \), \( \limsup_{t \to \infty} \frac{1}{t} \int_0^t C_r ds = \sum_{i \in S} \pi_i C_i \) and \( \limsup_{t \to \infty} \frac{1}{t} \int_0^t (\delta r + \frac{1}{2} \sigma_r^2) ds = \sum_{i \in S} \pi_i d_i \). \( \square \)

Remark 2.1. These estimations presented in Theorem 2.3 are dependent on the stationary distribution \( \pi \) of the Markov chain \( (r_t)_{t \geq 0} \). Especially, these estimations are also suitable for the case of \( \delta \leq \frac{1}{2} \sigma^2 \), which fills in the gap of the corresponding results in Wang et al. [3].

Theorem 2.4. Assume (A) holds, then the solution \( X(t) \) of model (1.2) with initial value (1.3) satisfies 

\[
0 \leq \liminf_{t \to \infty} X(t) \leq \sum_{i \in S} \pi_i \gamma_i / \sum_{i \in S} \pi_i d_i := N^* \text{ a.s.} \quad (2.9)
\]

Remark 2.2. Inequality (2.9) means that if the daily egg production rate goes to 0, then the limit inferior of the population of the sheep-blowfly will go to 0. In fact, one can see from (2.9) that \( p_i \to 0, \forall i \in S \) lead to \( N^* \to 0 \), which yields \( \inf X(t) \to 0 \) as \( t \to \infty \) a.s.

Proof. If \( \liminf_{t \to \infty} X(t) > N^* \text{ a.s.} \), then there exists a positive constant \( \varepsilon \) such that \( \liminf_{t \to \infty} X(t) = N^* + 2\varepsilon \text{ a.s.} \), and for this \( \varepsilon \) there exists a constant \( T > 0 \) so that \( X(t) \geq N^* + \varepsilon \) for all \( t > T \) a.s. It follows from the Itô formula and equation (1.2) with (1.3) that

\[
\log X(t) \leq \log \phi(0) + \int_0^t \left( -d r_s + \frac{\gamma r_s}{X(s)} \right) ds + \int_0^t \sigma_r dB(s) \text{ a.s.}
\]

Thus

\[
\limsup_{t \to \infty} \frac{1}{t} \log X(t) \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t \left( -d r_s + \frac{\gamma r_s}{N^* + \varepsilon} \right) ds = \sum_{i \in S} \pi_i \left( -d_i + \frac{\gamma_i}{N^* + \varepsilon} \right) < 0 \text{ a.s.}
\]

which yields \( \limsup_{t \to \infty} X(t) = 0 \), a contradiction. \( \square \)

Theorem 2.5. Assume (A) holds, then the solution \( X(t) \) of equation (1.2) with initial value (1.3) has the properties: if \( \forall i \in S, p_i = 0 \), then

\[
\limsup_{t \to \infty} \frac{1}{t} \log X(t) \leq - \sum_{i \in S} \pi_i d_i < 0 \text{ and } \lim_{t \to \infty} X(t) = 0 \text{ a.s.}
\]

if \( \tau_i = 0, \forall i \in S \), then

\[
\limsup_{t \to \infty} \frac{1}{t} \log X(t) \leq - \sum_{i \in S} \pi_i (d_i - p_i) := - \lambda
\]

Further if \( \lambda > 0 \), then \( \lim_{t \to \infty} X(t) = 0 \); and if \( \lambda = 0 \), then \( \limsup_{t \to \infty} X(t) \leq 1 \) a.s.

Theorem 2.6. Assume \( \forall i \in S, \tau_i = \tau \) and \( a_i = a \) are independent of the Markov chain \( (r_t)_{t \geq 0} \), and \( \mu > 0 \), then the solution \( X(t) \) of equation (1.3) with initial value (1.5) has the properties:

\[
\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[X^2(t)] \leq - \kappa < 0 \text{, } \limsup_{t \to \infty} \frac{1}{t} \log X(t) \leq - \kappa/2 < 0 \text{ and } \lim_{t \to \infty} X(t) = 0 \text{ a.s.,}
\]

where \( \kappa \in (0, \mu) \) is the unique root of the equation \( \kappa \vartheta e^{\kappa \tau} + \kappa = \mu \), and \( \vartheta > \varrho \) is a constant.

Proof. Let \( e^{\kappa t} V(x, t) = \int_0^t \vartheta \int_{t-\tau}^t g(x(s)) ds \) where \( g(x) = x^2 e^{-2ax} \), by applying Itô formula we have

\[
d[e^{\kappa t} V(x(t), t)] \leq e^{\kappa t} \left[ \kappa \vartheta \int_{t-\tau}^t g(X(s)) ds + (p_r + \sigma^2_r - 2\delta_r + \kappa) X^2(t) \right] dt + 2e^{\kappa t} \varrho^2 X^2(t) dB(t)
\]

Integrating both sides of this inequality from 0 to \( T > 0 \) and taking expectation we obtain

\[
e^{\kappa T} \mathbb{E}[V(X(T), T)] \leq \mathbb{E}[V(\phi(0), 0)] + \mathbb{E} \int_0^T e^{\kappa t} \left[ \kappa \vartheta \int_{t-\tau}^t g(X(s)) ds + (p_r + \sigma^2_r - 2\delta_r + \kappa) X^2(t) \right] dt.
\]

On the other hand, we have

\[
\int_0^T e^{\kappa t} \int_{t-\tau}^t g(X(s)) ds dt = \int_{-\tau}^T \int_{t\neq 0} e^{\kappa t} dt \int_{t\neq 0} g(X(s)) ds \leq \tau e^{\kappa \tau} \int_{-\tau}^T e^{\kappa t} g(X(s)) ds
\]

\[
\leq \tau e^{\kappa \tau} \int_{-\tau}^T e^{\kappa t} X^2(s) ds + \tau^2 e^{\kappa \tau - 2} \varrho^2,
\]

\[
(2.10)
\]
which together with \[2.10\] gives
\[e^{\kappa T}E(V(X(T), T) \leq V(\phi(0), 0) + \kappa \sigma^2 e^{\kappa t - 2}/a^2 + E \int_0^T e^{\kappa t} (\kappa \sigma^2 e^{\kappa t} + \kappa - \mu) X^2(t)dt \leq V(\phi(0), 0) + \kappa \sigma^2 e^{\kappa t - 2}/a^2.\]

Hence,
\[e^{\kappa T}E[X^2(T)] \leq V(\phi(0), 0) + \kappa \sigma^2 e^{\kappa t - 2}/a^2,
which implies \(\limsup_{t \to \infty} \frac{1}{t} \log(E[X^2(t)]) \leq -\kappa.\)

The rest of the proof uses the same arguments as those in the proof of Mao [7] (p.373-374). For completeness, we include the details here. For arbitrary \(\epsilon \in (0, \kappa/2)\), there must exist a constant \(K > 0\) so that
\[E[X^2(t)] \leq Ke^{-(\kappa - \epsilon)}, \quad \forall t \geq \tau.\]

For \(n = 1, 2, ...,\) by the Doob martingale inequality and Holder inequality, we get
\[E\left[\sup_{n \leq t \leq n+1} X^2(t)\right] \leq 3E[X^2(n)] + C \int_n^{n+1} (E[X^2(t)] + E[X^2(t - \tau)]) dt,
where \(C\) is positive constant independent of \(n\), and allowed to be different in different lines. Therefore, we obtain
\[E[\sup_{n \leq t \leq n+1} X^2(t)] \leq Ce^{-(\kappa - \epsilon)n},\]
which together with Chebyshev’s theorem yields
\[P\{\omega: \sup_{n \leq t \leq n+1} X^2(t) > e^{-(\kappa - \epsilon)n}\} \leq Ce^{-\epsilon n}.\]

It follows from the Borel-Cantelli lemma that for almost all \(\omega \in \Omega\) there is a random integer \(n_0(\omega)\) such that for all \(n \geq n_0\),
\[\sup_{n \leq t \leq n+1} X^2(t) \leq e^{-(\kappa - \epsilon)n},\]
which yields \(\limsup_{t \to \infty} \frac{1}{t} \log(X(t)) \leq -\kappa/2 + \epsilon\) a.s. The arbitrariness of \(\epsilon\) implies the desired assertion.

### 3 Examples

In this section an example is given to check the results obtained in section 2 by numerical simulation.

Let \(S = \{1, 2, 3\}, Q = [-10, 4, 6, 2, -3, 1, 3, 5, -8]\), \(\delta = [2, 1, 4]\), \(p = [4, 2, 8]\), \(\tau = [1, 1, 1]\), \(\alpha = [0.4, 0.2, 0.3]\), \(\sigma = [1.5, 2, 3]\), and \(\phi(t) = 1, t \in [-1, 0]\), then the Markov chain \(r_t\) is irreducible and has a unique stationary distribution \(\pi = (0.1845, 0.6019, 0.2136)\) and one can see that \(\delta_1 > \sigma_2/2, \delta_2 = \sigma_2^2/2,\) and \(\delta_3 < \sigma_3^2/2\). It follows from Theorem 2.1 that the solution of equation 1.2 is global existence and positive almost surely on \(R_+\), Figure 2(left) shows this. Meanwhile, we get \(\limsup_{t \to \infty} E[X(t)] \leq 1.1221\) and \(\limsup_{t \to \infty} E[X^2(t)] \leq 0.3713\).

Furthermore, from Theorem 2.3 we have \(C \approx [2.1022, 3.6788, 10.3229]\), \(d = [3.125, 3, 8.5]\), \(N^* = 1.1883\) and the estimation of the sample Lyapunov exponent: \(-4.1978 \leq \liminf_{t \to \infty} \frac{1}{t} \log X(t) \leq \limsup_{t \to \infty} \frac{1}{t} \log X(t) \leq 2.4035\). One can see from Figure 2(right) that the value of \(\frac{1}{t} \log X(t)\) oscillates up and down at zero as \(t \to \infty\), which yields that the sample path of \(X(t)\) oscillates up and down at \(1\) as \(t \to \infty\). Thus, \(X(t) \to 0\) as \(t \to \infty\), that is, the population will neither go to extinction finally nor explode in finite time.

In order to verify the effectiveness of Theorem 2.6, we reset \(p = [0.2, 0.2, 0.4], \alpha = [0.4, 0.4, 0.4], \sigma = [1.5, 1, 2.5]\) then \(\hat{\mu} > 0\) holds. One can see from Figure 1(right) that \(X(t) \to 0\) and \(\frac{1}{t} \log X(t) < 0\) as \(t \to \infty\).
value of $K$ under what conditions there exists a $M$ such that model (1.2) can be extended to a general case:

They dependent on the Markov chain $G$ is uniformly bounded, and the methods applied in Theorems 2.1–2.5 are also suitable for it.

For model (1.2) with initial value (1.3), there are still some interesting problems left:

1) Find conditions for the almost surely strong (weak) persistence of the population, that is, under what conditions $\lim_{t \to \infty} X(t) > 0$ a.s. ($\lim_{t \to \infty} X(t) > 0$ a.s.); or more accurately, estimate the maximum value of $K \in \mathbb{R}^+$ such that $\lim_{t \to \infty} X(t) > K$ a.s.

2) In the case of $p > 0$, $\tau > 0$, the delay $\tau$ and parameter $a$ vary along the Markov chain $r_t$, that is, they depend on the Markov chain $r_t$. Find conditions for the upper boundedness of the population, that is, under what conditions there exists a $M \in \mathbb{R}^+$ such that $\limsup_{t \to \infty} X(t) \leq M$ a.s., and give the estimation of this upper bound $M$. Find conditions for the negativity of the Lyapunov exponent of $X(t)$, that is $\limsup_{t \to \infty} \frac{1}{t} \log X(t) < 0$ a.s.

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