Random numbers from vacuum fluctuations

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We implement a quantum random number generator based on a balanced homodyne measurement of vacuum fluctuations of the electromagnetic field. The digitized signal is directly processed with a fast randomness extraction scheme based on a linear feedback shift register. The random bit stream is continuously read in a computer at a rate of about 480 Mbit/s and passes an extended test suite for random numbers.

I. INTRODUCTION

Various cryptographic schemes, classical or quantum, require high quality and trusted random numbers for key generation and other aspects of the protocols. In order to keep up with data rates in modern communication schemes, these random numbers need to be generated at a high rate. Equally, large amounts of random numbers are at the core of Monte Carlo simulation methods. Algorithmically generated pseudo-random numbers are available at very high rates, but are deterministic by definition and are unsuitable for cryptographic purposes, as they may contain backdoors in the particular algorithm used to generate them. For applications that require unpredictable random numbers, physical random number generators (PRNG) have been used in the past and more recently. These involve measuring noisy physical processes and conversion of the outcome into random numbers. Since it is either practically (e.g. for thermal noise sources) or fundamentally (for certain quantum processes) impossible to predict the outcome of such measurements, these physically generated random numbers are considered “truly” random.

Quantum random number generators (QRNG) belong to a class of physical random number sources where the source of randomness is the fundamentally unpredictable outcome of a quantum measurement. Early PRNG of this class were based on observing the decay statistics of radioactive nuclei. More recently, similar PRNG based on Poisson statistics in optical photon detection were implemented. Different schemes use the randomness of a single photon scattered by a beam splitter into either of two output ports. Since the reflection/transmission of the photon is intrinsically random due to the quantum nature of the process, the unpredictability of the generated numbers is ensured. Other implementations of QRNGs measure the amplified spontaneous emission, the vacuum fluctuations of the electromagnetic field, or the intensity and phase noise of different light sources.

In this paper we report on a quantum random number generator based on measuring vacuum fluctuations as the raw source of randomness. Such measurements have a very high bandwidth compared to schemes based on photon counting, and have a much simpler optical setup compared to phase noise measurements. Coupled with an efficient randomness extractor, we obtain an unbiased, uncorrelated stream of random bits at high speed.

II. IMPLEMENTATION

Figure schematically shows the setup of our QRNG. A continuous wave laser (wavelength 780 nm) is used as the local oscillator (LO) for the vacuum fluctuations of the electromagnetic field entering the beam splitter at the empty port. The output of the beam splitter is directed onto two pin photodiodes, and the photocurrent difference is processed further. This setup is known as a balanced homodyne detector and maps the the electrical field in the second mode entering the beam splitter to the photocurrent difference. Here, the second input port is empty, so the homodyne measurement is probing the vacuum state of the electromagnetic field. This field fluctuates, and it is seen as the source of randomness. As the vacuum field is independent of external

FIG. 1. Schematic of the quantum random number generator. A polarizing beam splitter (PBS) distributes the light of a 780 nm laser diode equally onto two fast photodiodes, generating photocurrents $i_1$ and $i_2$. The fluctuations in the photocurrent difference $i_1 - i_2$ are amplified, digitized, and sent to a randomness extractor to generate unbiased “true” random numbers.
i is measured from the photocurrent difference width $B$.

FIG. 2. Noise levels measured after amplification into a bandwidth $B = 1$ kHz. Between 20 and 120 MHz, the total noise is measured from the photocurrent difference $i_1 - i_2$ with a balanced optical power impinging on both photodiodes and approaches the theoretical shot noise level of -52 dBm (dashed trace) given by (1). The current $i_1$ of a single photodiode reveals colored classical amplitude noise. The electronic noise is measured without any optical input.

Physical quantities, it can not be tampered with. Since the optical power impinging on the two photodiodes is balanced, any power fluctuation in the local oscillator will be simultaneously detected by the two diodes, and therefore cancel in the photocurrent difference $i_1 - i_2$. In an alternative view, the laser beam can be seen as generating photocurrents $i_1, i_2$ with a shot noise power proportional to the average optical power. The shot noise currents from the two diodes will add up because they are uncorrelated, while amplitude fluctuations in the laser intensity (referred to as classical noise) represented by the average current of the photodiodes does not affect the photocurrent difference.

The power between the two output ports is balanced by rotating the laser diode in front of a polarizing beam splitter (PBS). The output light leaving the PBS is detected by a pair of rearward biased silicon pin photodiodes (Hamamatsu S5972) connected in series to perform the current subtraction. The balancing of the photocurrents is monitored by observing the voltage drop across a resistor $R_D$ providing a DC path for the current difference from the common node to ground. The fluctuations above 20 MHz are amplified by a transimpedance amplifier with a calculated effective transimpedance of $R_{\text{eff}} \approx 540$ kΩ.

To ensure that the fluctuations at the output of the amplifier are dominated by quantum fluctuations of the vacuum field, the spectral power density at the output of the amplifier is measured (see Fig. 2). With an optical power of 3.1 mW received by each photodiode corresponding to an average photocurrent $I = 1.7$ mA, a noise power of $P = -53.5$ dBm (at 75 MHz) in a bandwidth of $B = 1$ kHz was measured. This is about 1.5 dB lower than the theoretically expected shot noise value (dashed trace) of

$$P = \frac{4eIBR_{\text{eff}}^2}{Z} \approx -52 \text{ dBm,}$$

where $e$ is the electron charge and $Z = 50\Omega$ the load impedance. The difference is compatible with uncertainties in determining the transimpedance of the amplifier. The measured total noise after the amplifier has a relatively flat power density in the range of 20 to 120 MHz, while the high pass filters in the circuit suppress low frequency fluctuations. The high end of the pass band is defined by the cutoff frequency of the amplifier. To illustrate the effectiveness of removing classical noise in the photocurrents, the spectral power density of a photocurrent generated from a single diode is also shown. Strong spectral peaks at various radio frequencies appear that enter the system probably via the laser diode current. For completeness, the spectral power density of the electronic noise of the amplifier is recorded without any light input, and found to be at least 10 dB below the total noise level, i.e., the total noise is dominated by quantum fluctuations.

The amplified total noise signal is digitized into signed 16 bit wide words $x_i$ at a sampling rate of 60 MHz with an analog to digital converter (ADC). The sampling rate is set to be lower than the cut-off frequency of the noise signal in order to avoid temporal correlation between samples. As shown in Fig. 3, the normalized autocorrelation

$$A(d) = \langle x_i x_{i+d} \rangle_\text{n}/\langle x_i^2 \rangle_\text{n} \quad (2)$$

evaluated over $n = 10^6$ measured samples falls into the expected 2σ confidence interval which indicates no significant correlation between samples.

III. ENTROPY ESTIMATION

The total noise we measured before the ADC consists of both quantum noise and the electronic noise of the de-
tector. To determine how much randomness we can safely extract from the system in the sense that it originates from a quantum process, it is necessary to quantitatively estimate the entropy contributed by the quantum noise.

To estimate the entropy of the quantum noise $H(X_q)$, we assume that the measured total noise signal $X_t = X_q + X_e$ is the sum of independent random variables $X_q$ for the quantum noise, and $X_e$ for the electronic noise \cite{20,30}. Furthermore, all three variables $X_q$, $X_e$ and $X_t$ are assumed to have discrete values between $-2^{15}$ and $2^{15} - 1$. Since the origin of electronic noise is uncertain, we take the worst case scenario that the adversary gains full knowledge of the electronic noise, i.e., is able to predict the exact outcome of variable $X_e$ at any moment. In this case, the accessible amount of randomness in the acquired total noise signal is quantified by the conditional entropy $H(X_t|X_e)$, i.e. the amount of entropy left in the total signal, given full knowledge of the electronic noise $X_e$. As the variables are assumed to be additive and independent, the conditional entropy is calculated as $H(X_t|X_e) = H(X_q + X_e|X_e) = H(X_q|X_e) = H(X_q)$.

The variance of the total noise, $\sigma_t^2$, is given by the sum of the variances $\sigma_q^2$ for the quantum noise, and $\sigma_e^2$ of the electronic noise. In an ensemble of $10^6$ samples, we find $\sigma_t = 4504.41$ and $\sigma_e = 1481.8$, which is measured by switching off the laser (see Fig. 4). Note that for the total noise, the observed distribution is slightly skewed compared to a Gaussian distribution [solid line in Fig. 4(a)]. We believe this is due to a distortion in the digitizer. Assuming the quantum noise $X_q$ has a Gaussian distribution \cite{28}, we would assign $\sigma_q^2 = \sigma_t^2 - \sigma_e^2 \approx 4253.7^2$. To estimate the entropy for a Gaussian distribution, we use the Shannon entropy

$$H(X_q) = \sum_{x = -2^{15}}^{2^{15} - 1} -p_q(x) \log_2 p_q(x),$$

where $p_q(x)$ is the probability distribution of the quantum noise $X_q$ with variance $\sigma_q^2$. Since $\sigma_q \gg 1$, $H(X_q)$ can be well approximated by

$$\int_{-\infty}^{+\infty} -f(x) \log_2 f(x) \, dx = \log_2(\sqrt{2\pi e \sigma_q}),$$

where $f(x)$ is a Gaussian probability density function with variance $\sigma_q^2$ and $e$ the base of the natural logarithm\cite{31}. This yields 14.1 bits of entropy per 16-bit sample.

We note that this numerical estimation of entropy only serves as an upper bound of extractable randomness, i.e. the maximum possible amount of entropy one can extract from the source of randomness under the assumption of a Gaussian distribution of the independent random variables $X_q$ and $X_e$. An alternative estimation of the entropy in $X_q$ assumes that electronic noise is not only known to a third party, but also could be tampered with\cite{17,32}.

![FIG. 4. Probability distribution of the measured total output noise with variance $\sigma_t^2$ (a), electronic noise with variance $\sigma_e^2$ (b), and the estimated quantum noise with variance $\sigma_q^2$ (c). The filled areas in (a), (b) show the actual measurements over $10^6$ samples, the solid lines approximate the Gaussian distributions.](image)

### IV. RANDOMNESS EXTRACTION

In many applications, random numbers are required to be not only unpredictable, but also uniformly distributed. As such, the raw data at the amplifier output cannot be directly used since they are non-uniformly distributed. Randomness extraction is the essential process required to convert our biased raw data into a uniformly distributed binary stream at the final output\cite{33}.

Various implementations of randomness extractors have been reported, such as Trevisian’s extractor and Toeplitz-hashing extractor\cite{30}, random-matrix multiplication\cite{20}, or the family of secure hashing algorithms (SHA)\cite{16}.

In this work, we use a randomness extractor based on a Linear Feedback Shift Register (LFSR). The LFSRs are well known for quickly generating long pseudo-random streams with little computational resources and are in widespread use in communication applications for spectrum whitening\cite{34,35}.

We use a maximum length LFSR with 63 memory cells and a two-element feedback path. Its state at any time step $t$ could be represented by 63 binary variables $s_j^t$, with a recursion relation

$$s_{j+1}^t = s_{j-1}^t \quad \text{for } j = 1 \ldots 62,$$

$$s_0^{t+1} = s_{62}^t \oplus s_{61}^t,$$

where $\oplus$ denotes an exclusive-or operation. The 16 bit ADC word is serially injected into the feedback path\cite{6} as $s_0$ with an exclusive or operation,

$$s_{0}^{t+1} = s_{62}^t \oplus s_{61}^t \oplus d^t,$$

where $d^t$ represents an input bit from the ADC word at time $t$. A reduced number of bits are extracted from $s_0$ obeying the entropy bound. To implement this efficiently
in parallel for each sampled value of the vacuum field, we add a second set of memory cells, \(m_j, j = 0 \ldots 62\), with the recursion relations

\[
\begin{align*}
    m_j^{t+1} &= s_j^t & \text{for } j = 0 \ldots 62, \\
    s_j^{t+1} &= m_j^t \oplus m_{j+1}^t \oplus d_j^t & \text{for } j = 0 \ldots 61, \\
    s_{62}^{t+1} &= m_{62}^t \oplus s_0^t
\end{align*}
\]

where \(d_j^t\) represents the \(j\)-th bit of the ADC word sampled at \(t\) for \(j < 16\), and \(d_j^t = 0\) for \(j \geq 16\). Recursion relations (8-10) are equivalent to the operation described in (7), but with all input bits \(d_j^t\) of one sampled word injected at once instead of serially. The output bit stream is a snapshot of eight cells \(m_j\) with \(j = 0, 2, 4 \ldots 14\), extracted at the ADC sampling rate (60 MHz). The extraction ratio of 50% is lower than 14.1/16 \(\approx 88\%\) from the entropy bound estimated in (4). The recursion equations (8-10) and the reduced rate extraction is implemented in a complex programmable logical device (CPLD, Model LC4256 from Lattice semiconductor).

A merit of this extractor is its low circuit complexity. Unlike many secure hashing algorithms, it can be easily implemented either in high speed or low power technology. Therefore, the extraction process does not limit the random number generation rate. This scheme can receive a parallel injection of up to 63 raw bits per clock cycle while still following the extractor equations [5] and [7]. With the CPLD operating at its maximum clock frequency (400 MHz), this algorithm would be able to process up to 25 \(\times 10^8\) raw input bits per second.

\section{VI. CONCLUSION}

In summary, we demonstrated a random number generation scheme by measuring the vacuum fluctuations of the electromagnetic field. By estimating the amount of usable entropy from quantum noise and using an efficient randomness extractor based on linear feedback shift registers, we are able to generate uniformly distributed random numbers at a high rate from a fundamentally unpredictable quantum measurement.

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