POISSON STATISTICS FOR MATRIX ENSEMBLES AT LARGE TEMPERATURE

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Abstract. In this article, we consider $\beta$-ensembles, i.e., collections of particles with random positions on the real line having joint distribution

$$\frac{1}{Z_N(\beta)}|\Delta(\lambda)|^\beta e^{-\frac{N\beta}{2} \sum_{i=1}^N \lambda_i^2} d\lambda,$$

in the regime where $\beta \to 0$ as $N \to \infty$. We briefly describe the global regime and then consider the local regime. In the case where $N\beta$ stays bounded, we prove that the local eigenvalue statistics, in the vicinity of any real number, are asymptotically to those of a Poisson point process. In the case where $N\beta \to \infty$, we prove a partial result in this direction.

1. Introduction

General $\beta$-ensembles are collections of particles with random positions on the real line with joint distribution

$$\frac{1}{Z_N(\beta)}|\Delta(\lambda)|^\beta e^{-\sum_{i=1}^N V(\lambda_i)} d\lambda,$$

where $d\lambda$ denotes the Lebesgue measure on $\mathbb{R}^N$, $\Delta(\lambda) := \prod_{1 \leq i < j \leq N} (\lambda_j - \lambda_i)$, $V$ is a potential with enough growth at infinity (like the Gaussian potential $V_G(x) = \frac{x^2}{2}$) and $Z_N(\beta)$ is a normalizing constant. Their study is initially motivated by some considerations from physics: the probability distribution can be viewed as the equilibrium measure of a one dimensional Coulomb gas, but they actually appear to be connected to a broad spectrum of mathematics and physics, such as random matrices, number theory, lattice gas theory, quantum mechanics and Selberg-type integrals. In the case where $\beta = 1, 2$ or 4, the probability measure of (1) is the joint distribution of the eigenvalues of a random $N \times N$ matrix $M$ with density proportional to $e^{-\text{Tr} V(M)}$ on the space of respectively real symmetric, complex Hermitian or quaternionic Hermitian matrices (see e.g. [6]). Besides, it was proved by Dumitriu and Edelman in [15] that when $V$ is the Gaussian potential, for any $\beta > 0$, the probability measure of (1) is the joint distribution of the eigenvalues of the
random $N \times N$ tridiagonal matrix

$$H = \begin{pmatrix}
g_1 & X_2 & X_3 & \cdots & \cdots & X_N \\
X_2 & g_2 & \cdots & \cdots & \cdots & \cdots \\
& X_3 & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots \\
& & & & & g_N
\end{pmatrix},$$  \hspace{1cm} (2)

where the $g_i$’s are some $N(0, 1)$ variables and for all $i$, $X_i = \sqrt{Y_i}$, with $Y_i$ distributed thanks to the $\Gamma((i - 1)/2)$ law, everything being independent. For general potential $V$, there is no random matrix representation.

In the classical cases ($\beta = 1, 2, 4$), the combinatorial structure and repulsive interaction has been well–understood for a long time via the theory of determinantal or Pfaffian processes (see e.g. [6] for references). The understanding of these asymptotic spectral statistics to the full class of parameters $\beta > 0$ has recently mobilized a lot of research. For general $\beta$, despite the lack of structure, some enormous progress has been accomplished recently. For fixed $\beta$, a few results are now known. First, it is known from [8] that the empirical eigenvalue distribution of the rescaled matrix $\frac{1}{\sqrt{N}}H$ converges weakly as $N \to \infty$ to a probability measure which is the semi-circle distribution in the case of Gaussian potential. The local eigenvalue statistics in the large $N$-limit are also quite well understood. In the Gaussian setting, at the edge of the spectrum, Ramírez, Rider and Virág have shown in [21] that the eigenvalues of $N^{1/6}(H - 2 \sqrt{N} I)$ converge in distribution to those of the so-called stochastic Airy operator. In the bulk of the spectrum, the limiting spectral statistics are asymptotically defined in terms of the Sine-$\beta$ process, which is again defined as the solution of a stochastic equation by Valkó and Virág in [26]. In particular the authors show that the Sine-$\beta$, which is translation invariant, has a geometric description in terms of the Brownian carousel, a deterministic function of the Brownian motion in the hyperbolic plane. Some advances on $\beta$-ensembles have also been made by Sosoe and Wang [22, 23] and Bao and Su in [7].

The question of universality for these statistics has now become an important matter of interest: some enormous progress has recently been accomplished by Bourgade, Erdös and Yau in [11, 12, 13, 10]. Therein the authors consider general $\beta$-ensembles (when the potential $V$ is $C^4$ and regular, or, in the first papers, convex and analytic). Assuming that the limiting spectral distribution (which depends on $V$) is supported on a single interval, they prove that the limiting eigenvalue statistics at the edge of the spectrum are given by the $\beta$-Tracy-Widom distribution. The universality in the bulk of the spectrum is also proved. Another point of view to tackle $\beta$-ensembles and in particular the quantitative aspect of the repulsion between eigenvalues has been developed in particular by Allez, Bouchaud and Guionnet in [3, 1]. They show in particular that when $\beta \leq 2$, $\beta$-ensembles can be seen as an $N$-dimensional process whose evolution is a mixing of that of $N$ independent real Brownian motions and of that of a $\beta$-Dyson Brownian motion.
The scope of this article is to understand the spectral behavior, at microscopic scale, of \( \beta \)-ensembles in the case where \( \beta \to 0 \) and \( N \to \infty \) (so that \( \beta \) depends on the dimension \( N \)). At macroscopic scale, such ensembles have been considered recently by [25] (see also the close model studied in [2]). Therein it is proved that when \( \beta N \to c \) for some constant \( c > 0 \), the scaled empirical eigenvalue distribution of \( \frac{1}{\sqrt{\beta}} H \) converges to the spectral measure of a deterministic Jacobi matrix, the density of which is explicit. When \( \beta N \to \infty \), the limiting empirical eigenvalue distribution of converges to the semi-circle distribution. Local eigenvalue statistics have not been considered yet.

We here also consider the regime where \( \beta \to 0 \) and \( N \to \infty \), but study the local eigenvalue statistics. In [18], Killip and Stoiciu have considered the same question for circular \( \beta \)-ensembles. More precisely they study CMV matrices (which are discrete one-dimensional Dirac-type operators) with random decaying coefficients. For rapidly decreasing coefficients, the eigenvalues have rigid spacing while in the case of slow decrease, the eigenvalues are distributed according to a Poisson process. More precisely, they prove that local eigenvalue statistics of \( \beta \)-circular ensembles when \( \beta \to 0 \) are in the large \( N \) limit those of a Poisson process.

For real-symmetric ensembles, the same question has recently been considered from a formal point of view. Indeed, in [4, 5], Allez and Dumaz considered the \( \beta \to 0 \) limit of the Sine-\( \beta \) process and of the \( \beta \)-Tracy-Widom distribution. The \( \beta \to 0 \) limit of the Sine-\( \beta \) is also considered by Leblé and Sefaty in [19]. The approach used by [19] is based on approximation theory while [4] use the diffusion representation of the Sine-\( \beta \) process to consider the limit \( \beta \to 0 \).

One would expect again to prove that when \( \beta \to 0 \) simultaneously to \( N \to \infty \), the eigenvalues in the vicinity of a point \( u \) in the bulk of the spectrum exhibit Poisson statistics. In this text, we prove that this is true when \( N \beta \) stays bounded as \( N \to \infty \). In the case where \( \beta \to 0 \) but \( N \beta \to \infty \), we have a partial result which formally implies the Poisson statistics in the bulk, but does not allow to get a complete proof.

In Figure 1, we compare this result with numerical simulations, giving a numerical evidence of the fact that the Poisson approximation works well (but gets less accurate as \( \beta \) grows).

**Notation.** For \( u = u(N) \) and \( v = v(N) \) some sequences,

\[
u \lesssim v \iff \frac{u}{v} \to 0 \quad \text{as} \quad N \to \infty;
\]

\[
u \asymp v \iff \frac{u}{v} \to 1 \quad \text{as} \quad N \to \infty.
\]

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2. Statement of results

2.1. Presentation of the model. For any $\alpha, \beta > 0$ and any $N \geq 1$, we define

$$Z_N(\alpha, \beta) := \int_{\lambda \in \mathbb{R}^N} \Delta(\lambda)^\beta e^{-\frac{\beta}{2} \sum_{i=1}^N \lambda_i^2} d\lambda,$$

with $\Delta(\lambda) := \prod_{1 \leq i < j \leq N} |\lambda_j - \lambda_i|$.

Let us now consider an exchangeable family $(\lambda_1, \ldots, \lambda_N)$ of random variables with joint law

$$P^{(N)}_{\alpha, \beta}(d\lambda_1, \ldots, d\lambda_N) := \frac{1}{Z_N(\alpha, \beta)} \Delta(\lambda)^\beta e^{-\frac{\beta}{2} \sum_{i=1}^N \lambda_i^2} d\lambda_1 \ldots d\lambda_N$$

with $Z_N(\alpha, \beta)$ the normalization constant defined at (3).
2.2. **Tridiagonal model and relation between $\alpha$ and $\beta$.** Let
\[
H = \frac{1}{\sqrt{\alpha}} \begin{pmatrix}
g_1 & X_2 & & \\
X_2 & g_2 & X_3 & \\
& X_3 & \ddots & \ddots \\
& & \ddots & X_N \\
g_N & & & X_N
\end{pmatrix},
\]
where the $g_i$’s are some $N(0,1)$ variables and for all $i$, $X_i = \sqrt{Y_i}$, with $Y_i$ distributed thanks to the $\Gamma((i - 1)\beta/2)$ law. We known, by [15] or Section 4.5 of [6], that $P_{\alpha,\beta}^{(N)}$ is the joint law of the eigenvalues of $H$.

Note that $\text{Tr } H$ is centered and $\alpha \mathbb{E} \text{Tr } H^2 = N + \beta \frac{N(N-1)}{2}$, so that for
\[
\alpha \sim 1 + N\beta/2,
\]
the empirical eigenvalue distribution of $H$ has asymptotic first moments 0 and 1.

2.3. **Global and local regime for bounded $N\beta$.** The following proposition gives the limit of the empirical distribution of the $\lambda_i$’s. The probability measure $\mu_\gamma$ in question here has been studied in [1, 25].

**Proposition 2.1** (Global regime for bounded $N\beta$). Suppose that $N\beta \to 2\gamma \geq 0$ as $N \to \infty$ and $\alpha \to \gamma + 1$.

a) Then under the law $P_{\alpha,\beta}^{(N)}$, the random probability measure
\[
\frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}
\]
converges in probability to an even probability measure $\mu_\gamma$ on $\mathbb{R}$, depending only on $\gamma$, with moments $m_k$ defined at (26), satisfying $m_2 = 1$.

b) The measure $\mu_\gamma$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with a density $f_{\mu_\gamma}$ that is bounded on any compact set, and satisfies, for all $x > 0$,
\[
\mu_\gamma(\mathbb{R} \setminus [-x,x]) \leq C_\gamma \frac{e^{-\gamma} x^2}{x}
\]
where $C_\gamma$ is a constant depending only on $\gamma$.

c) $\mu_\gamma$ depends continuously on $\gamma \geq 0$, is equal to $N(0,1)$ if and only if $\gamma = 0$, and tends to the semicircle law with support $[-2,2]$ as $\gamma \to \infty$.

d) For each $k$, $\text{Var}(N^{-1} \sum_{i=1}^{N} \lambda_i^k) = O(N^{-1})$. 


The following theorem gives the limit local behavior of the $\lambda_i$’s.

**Theorem 2.2** (Poisson limit for bounded $N\beta$). Suppose that $N\beta \to 2\gamma \geq 0$ as $N \to +\infty$ and that $\alpha \sim N\beta/2 + 1$. Fix $E \in \mathbb{R}$. As $N \to \infty$, the point process

$$
\sum_{i=1}^{N} \delta_{N(\lambda_i - E)}
$$

with $(\lambda_1, \ldots, \lambda_N) \sim P^{(N)}_{\alpha, \beta}$, converges in distribution to the law of a Poisson point process with intensity $\theta \, dx$ on $\mathbb{R}$, for

$$
\theta := \frac{(\gamma + 1)^{\gamma + \frac{1}{2}}}{\sqrt{2\pi \Gamma(\gamma + 1)}} \exp \left\{ -\frac{\gamma + 1}{2} E^2 + 2\gamma \int \log |E - x| d\mu_\gamma(x) \right\},
$$

with $\mu_\gamma$ is in Proposition 2.1.

**Remark 2.3.** Note that the formula of $\theta$ given at (7) should agree with the density of $\mu_\gamma$ at $E$ as given in [1, 25], but we were not able to prove it so far.

2.4. **Case where $\beta \gg N^{-1}$**. Using the fact that $N^2\beta \gg N$, one can easily adapt the proof of the following theorem from [6].

**Theorem 2.4** (LDP for $\beta \gg N^{-1}$). Suppose that as $N \to \infty$, $\alpha = \alpha(N)$ and $\beta = \beta(N)$ are such that $\alpha \sim N\beta/2$, $N\beta \gg 1$ and $\beta$ is bounded. Then for $(\lambda_1, \ldots, \lambda_N)$ distributed according to $P^{(N)}_{\alpha, \beta}$, the sequence of random probability measures $L_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ satisfies a LDP in the set of probability measure on $\mathbb{R}$ endowed with the weak topology with speed $N^2\beta$ and good rate function $I$ defined by

$$
I(\mu) := \int \int f(x, y) \mu^{\otimes 2}(dx, dy) - \frac{3}{8},
$$

with $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ is the function defined by

$$
f(x, y) = \frac{x^2 + y^2}{8} - \frac{1}{2} \log |x - y|.
$$

Moreover, the unique minimum of $I$ is achieved at the semicircle law

$$
\sigma := \frac{1}{2\pi} \sqrt{4 - x^2} 1_{|x| \leq 2} dx
$$

and we have

$$
\lim_{N \to \infty} \frac{1}{N^2\beta} \log Z_N(\alpha, \beta) = -\int \int f(x, y) \sigma^{\otimes 2}(dx, dy) = -\frac{3}{8}.
$$

In the case where $N\beta \gg 1$, as far as the local regime is concerned, we only have the following partial result, inspired from Johansson’s work in [17]. Below, we explain how formally, it allows to prove the convergence of local statistics to the ones of a Poisson point process and to identify its density.
Theorem 2.5. Let $\beta = \beta(N)$ and $\alpha = \alpha(N)$ positive such that
\[ \frac{1}{N} \ll \beta \ll \frac{1}{\log N}; \quad N\beta - 2\alpha \ll 1. \quad (11) \]
Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded function having 9 continuous bounded derivatives and $(\lambda_1, \ldots, \lambda_N)$ be distributed according to $P_{\alpha,\beta}^{(N)}$. Then as $N \to \infty$, we have
\[ \log \int e^{\beta \sum_{i=1}^{N} h(\lambda_i)} P_{\alpha,\beta}^{(N)}(d\lambda_1, \ldots, d\lambda_N) - N\beta \int h(t)d\sigma(t) \to \int h(t)d\nu(t) \quad (12) \]
for
\[ \nu := \frac{1}{2}(\delta_{-2} + \delta_2) - \frac{1}{\pi} \frac{1}{\sqrt{4 - x^2}} dx \quad (13) \]

Remark 2.6. The measure $\nu$ of (13) is a classic correction to the semi-circle law (see e.g. [17, Rem. 2.5] or, more recently, [16]).

Let us now explain how, on the formal level, Theorem 2.5 gives, for any $E \in (-2, 2)$, the convergence of the point process
\[ \sum_{i=1}^{N} \delta_{N(\lambda_i - E)} \]
to a Poisson point process with density $\theta dx$ on $\mathbb{R}$, for
\[ \theta := \frac{\sqrt{4 - x^2}}{2\pi}, \quad (14) \]
The first thing one has to notice is that
\[ \theta = \frac{1}{2\pi} \exp \int \log |E - t|d\nu(t) \quad (15) \]
for $\nu$ being as in (13) (the proof goes along the same lines as [8, Lem. 2.7]).

To prove it rigorously, we would need to prove that for
\[ \overline{R}^{(N)}_k(x_1, \ldots, x_k) := \int e^{\beta \sum_{i=1}^{k} \sum_{j=1}^{N-k} \log |E + \frac{x_i}{N} - \lambda_j|} P_{\alpha,\beta}^{(N-k)}(d\lambda_1, \ldots, d\lambda_{N-k}), \quad (16) \]
for any $x_1, \ldots, x_k \in \mathbb{R}$,
\[ \frac{Z_{N-k}(\alpha, \beta)}{Z_N(\alpha, \beta)} e^{-\frac{N}{2} \sum_{i=1}^{k} (E + \frac{x_i}{N})^2} \overline{R}^{(N)}_k(x_1, \ldots, x_k) \to \theta^k \quad (17) \]
and that we have an upper bound of the type of (64).

First, it can be proved (see Section 5.1) that as $N \to \infty$, for any fixed $k$,
\[ \frac{Z_{N-k}(\alpha, \beta)}{Z_N(\alpha, \beta)} \sim \left(\frac{e^{N\beta/2}}{2\pi}\right)^k. \quad (18) \]
Moreover, there is a universal positive constant $M$ independent of $N$ such that uniformly on $N, k$, 
\[
\mathbb{1}_{1 \leq k \leq N} \frac{Z_{N-k}(\alpha, \beta)}{Z_N(\alpha, \beta)} \leq M^k \left( \frac{e^{N\beta/2}}{2\pi} \right)^k.
\]  
(19)

Theorem 2.5 can be rewritten as follows: for each fixed $h$ as in the theorem, 
\[
\mathbb{E}_{P_{\alpha, \beta}^{(N)}} e^{\beta \sum_{i=1}^N h(\lambda_i)} = \exp \left\{ N\beta \int h(t) d\sigma(t) + \int h(t) d\nu(t) + \varepsilon_N(h) \right\},
\]  
(20)

with $\varepsilon_N(h) \ll 1$. By (20), cutting on the right thanks to Lemma 5.1 and making as if the function $h_N : \lambda \mapsto \sum_{i=1}^k \log |E + \frac{x_i}{N} - \lambda|$ were $C^9$ (and close enough to the function $h : \lambda \mapsto k \log |E - \lambda|$), we should have 
\[
\tilde{R}_k^{(N)}(x_1, \ldots, x_k) = \int e^{N\beta \sum_{i=1}^{N-k} h_N(\lambda_i)} P_{\alpha, \beta}^{(N-k)}(d\lambda_1, \ldots, d\lambda_{N-k}) 
\approx \exp \left\{ N\beta \int h_N(t) d\sigma(t) + \int h_N(t) d\nu(t) + \varepsilon_N(h) \right\}
\] 
(21)

for $\nu$ as in (13). But by [8] p. 529, we know that for any $E \in (-2, 2)$,
\[
\int k^{-1} h(t) d\sigma(t) = \int \log |E - t| d\sigma(t) = \frac{E^2}{4} - \frac{1}{2},
\]

so that we should have
\[
\tilde{R}_k^{(N)}(x_1, \ldots, x_k) \approx \exp \left\{ N\beta \left( \frac{kE^2}{4} - \frac{k}{2} \right) + \int h(t) d\nu(t) + \varepsilon_N(h) \right\}.
\]  
(21)

Besides, by (18), we have 
\[
\frac{Z_{N-k}(\alpha, \beta)}{Z_N(\alpha, \beta)} \sim \left( \frac{e^{N\beta/2}}{2\pi} \right)^k.
\]  
(22)

Putting together (21), (22) and the fact that $\alpha \sim \frac{N\beta}{2}$, we should have 
\[
\frac{Z_{N-k}(\alpha, \beta)}{Z_N(\alpha, \beta)} e^{-\frac{d}{2} \sum_i (E + \frac{x_i}{N})^2} \tilde{R}_k^{(N)}(x_1, \ldots, x_k) \approx \left( \frac{e^{N\beta/2}}{2\pi} \right)^k e^{-\frac{Nk}{4} E^2} e^{N\beta \left( \frac{kE^2}{4} - \frac{k}{2} \right) + \int h(t) d\nu(t)}
\] 
\[
= \theta^k
\]  
(23)

with $\theta$ as in (15).
3. Proof of Proposition 2.1

Let $H$ be as in (5). We shall prove that for any $k$, $N^{-1} \mathbb{E} \text{Tr} H^k$ tends to $m_k$, that the $m_k$’s satisfy Carleman’s criterion and that d) holds. By Skorohod’s representation theorem (see e.g. [14, Th. 2.3.2]) and a diagonal extraction, it will imply a). Part c) will be clear from the proof. Note first that if $Y$ is a $\Gamma(t)$-distributed variable, then for all $k \geq 0$,

$$\mathbb{E} Y^k = \frac{\Gamma(t+k)}{\Gamma(t)}.$$  \hspace{1cm} (24)

Let us fix $k \geq 0$, $u \in [0,1]$, let $i = i(N)$ be such that $i/N \rightarrow u$ and compute $\mathbb{E}(H^k)_{ii}$. We have

$$(H^k)_{ii} = \sum_{\varepsilon} H_{\varepsilon(0),\varepsilon(1)} \cdots H_{\varepsilon(k-1),\varepsilon(k)},$$

where the sum is taken over paths $\varepsilon : \{0,\ldots,k\} \rightarrow \{1,\ldots,n\}$ such that

- $\varepsilon(0) = \varepsilon(k) = i,$
- for all $\ell = 1,\ldots,k$, $\varepsilon(\ell) - \varepsilon(\ell-1) = -1, 0$ or $1$, in which case we say that $\ell$ belongs respectively to $D(\varepsilon)$, $F(\varepsilon)$ or $U(\varepsilon)$.

Note first that for such a path $\varepsilon$,

$$\# D(\varepsilon) = \# U(\varepsilon), \quad \# D(\varepsilon) + \# F(\varepsilon) + \# U(\varepsilon) = k.$$  \hspace{1cm} (25)

For any $j \in \{1,\ldots,N\}$, we introduce

- $F_j(\varepsilon) := \{\ell \in \{1,\ldots,k\}; \varepsilon(\ell-1) = \varepsilon(\ell) = j\}$
- $U_j(\varepsilon) := \{\ell \in U(\varepsilon); \varepsilon(\ell) = j\}$

Then one can easily see, using (24), that

$$\alpha^{k/2} \mathbb{E} H_{\varepsilon(0),\varepsilon(1)} \cdots H_{\varepsilon(k-1),\varepsilon(k)} = \prod_j (\mathbb{E}^H_{\# F_j(\varepsilon)} \mathbb{E} Y_{\# U_j(\varepsilon)})$$

$$= \prod_j \left( \mathbb{1}_{\# F_j(\varepsilon) \text{ is even}} \times \Gamma \left( \frac{(j-1)\beta}{2} + \# U_j(\varepsilon) \right) \Gamma \left( \frac{(j-1)\beta}{2} \right) \right)$$

$$\rightarrow_{N \rightarrow \infty} \prod_j \left( \mathbb{1}_{\# F_j(\varepsilon) \text{ is even}} \times \gamma u(\gamma u + 1) \cdots (\gamma u + \# U_j(\varepsilon) - 1) \Gamma \left( \frac{(j-1)\beta}{2} + \# U_j(\varepsilon) \right) \Gamma \left( \frac{(j-1)\beta}{2} \right) \right)$$
By the dominated convergence theorem, it follows that $N^{-1} \mathbb{E} \text{Tr } H^k$ converges to

$$m_k := \sum_{\varepsilon} \frac{(\gamma + 1)^{-k/2}}{\gamma} \int_{u=0}^1 \prod_j \{ \# F_j(\varepsilon)!! \times \gamma u (\gamma u + 1) \cdots (\gamma u + \# U_j(\varepsilon) - 1) \} \, du,$$  \hspace{1cm} (26)

where the sum runs over paths $\varepsilon : \{0, \ldots, k\} \to \mathbb{Z}$ whose steps are in $\{-1, 0, 1\}$, such that $\varepsilon(0) = \varepsilon(k) = 0$ and for all $k$, $\# F_j(\varepsilon)$ is even.

Note that $\sum_j \# F_j(\varepsilon) = \# F(\varepsilon)$, whose parity is the one of $k$ by (25), so that when $k$ is odd, $\mathbb{E}(H^k)_{ii} = 0$. Using (25) again, we see that when $k$ is even, for any $\varepsilon$, for any $u$,

$$\prod_j (\# F_j(\varepsilon)!! \times \gamma u (\gamma u + 1) \cdots (\gamma u + \# U_j(\varepsilon) - 1)) \leq \max\{1, \gamma\}^k,$$

so that the $m_k$'s satisfy Carleman’s criterion. It follows that the $m_k$'s are the moments of a unique measure $\mu_\gamma$ which depends continuously on $\gamma$. Besides, d) follows from the fact that $\text{Cov}((H^k)_{ii}, (H^k)_{jj}) = 0$ as soon as $|j - i| > 2k$.

If $\gamma = 0$, then the only way for the term associated to $\varepsilon$ in (26) to be non zero is that $k$ is even and $\varepsilon$ is the constant path equal to $i$. This proves that $\mu_0 = N(0, 1)$. The reciprocal is obvious, as the fact that $\mu_\gamma$ tends to the semicircle law when $\gamma \to \infty$ (using the formula of the moments of the semicircle law in terms of Dyck paths, as in [6] or [20]).

To prove the first part of b), we use Lemma 4.2 below. For any $a \in \mathbb{R}$ and $\varepsilon \in (0, 1]$, we have

$$\mu((a - \varepsilon, a + \varepsilon)) \leq \liminf_{N \to \infty} P^{(N)}_{a,\beta} (|\lambda_1 - a| < \varepsilon) \leq C \varepsilon e^{\frac{2a^2}{N}}.$$

The second part of b) is a direct consequence of Lemma 4.3 below.

4. Proof of Theorem 2.2

4.1. Correlation functions. To prove the theorem, according to Proposition 6.1, we introduce the correlation functions of the point process $\sum_{i=1}^N \delta_{N(\lambda_i - E)}$, given by the formulas

$$R_k^{(N)}(x_1, \ldots, x_k) := \frac{Z_{N-k}(\alpha, \beta)}{Z_N(\alpha, \beta)} \frac{N^{-k} N! \Delta(x_1, \ldots, x_k)^\beta}{(N-k)!} e^{-\frac{1}{2} \sum_i (E + \frac{z_i}{N} - \lambda)^2} R_k^{(N)}(x_1, \ldots, x_k)$$

with

$$\tilde{R}_k^{(N)}(x_1, \ldots, x_k) := \int e^{\beta \sum_{i=1}^N \sum_{j=1}^{N-1} \log |E + \frac{z_i}{N} - \lambda_j|} P^{(N-k)}_{\alpha,\beta}(d\lambda_1, \ldots, d\lambda_{N-k})$$

First of all, we know that

$$\mathbb{1}_{k \leq N} \frac{N^{-k} N!}{(N-k)!} \leq 1$$

(29)
and that as $N \to \infty$, for each fixed $k$,
\begin{equation}
\frac{N^{-k}N!}{(N-k)!} \xrightarrow{N \to \infty} 1.
\end{equation}

Besides, for any $M > 0$, for any $k \geq 1$ and any $x_1, \ldots, x_k \in [-M, M]$, we have
\begin{equation}
1^k \leq N \Delta(x_1, \ldots, x_k)^{\beta} \leq \left( \frac{(2M)^{\beta}}{N^{N/2}} \right)^k
\end{equation}
and as soon as $\beta \ll (\log N)^{-1}$, for any fixed $k$ and any fixed $x_1, \ldots, x_k$,
\begin{equation}
\frac{\Delta(x_1, \ldots, x_k)^{\beta}}{N^{\beta k(k-1)/2}} \xrightarrow{N \to \infty} 1.
\end{equation}

4.2. Partition functions. We know, by [6, Cor. 2.5.9], that
\begin{equation}
Z_N(\alpha, \beta) = \alpha^{-\left( \frac{(N-1)^{\beta}+N}{2} \right)}(2\pi)^{N/2} \prod_{j=1}^{N} \frac{\Gamma(1+j\beta/2)}{\Gamma(1+\beta/2)}
\end{equation}
Hence
\begin{equation}
1^k \leq N \frac{Z_{N-k}(\alpha, \beta)}{Z_N(\alpha, \beta)} = \alpha^{\frac{k}{2}} \alpha^\beta (\frac{N(N-1)-(N-k)(N-k-1)}{(2\pi)^{-1/2}} \prod_{j=N-k+1}^{N} \frac{\Gamma(1+j\beta/2)}{\Gamma(1+\beta/2)}
\end{equation}
By hypothesis, $N\beta$ is bounded and so is $\alpha$. Let $C \geq 1$ be such that $N\beta + \alpha \leq C$. Then we have, uniformly in $k$,
\begin{equation}
1^k \leq N \frac{Z_{N-k}(\alpha, \beta)}{Z_N(\alpha, \beta)} \leq \left( \frac{C^{1+C}}{\sqrt{2\pi}} \max_{[1,C]} \frac{\Gamma}{\min_{[1,C]} \Gamma} \right)^k
\end{equation}
Besides, as $N\beta \to 2\gamma \geq 0$, for each fixed $k$,
\begin{equation}
\frac{Z_{N-k}(\alpha, \beta)}{Z_N(\alpha, \beta)} \xrightarrow{N \to \infty} \left( \frac{(\gamma + 1)^{\gamma + 1/2}}{\sqrt{2\pi} \Gamma(\gamma + 1)} \right)^k.
\end{equation}

4.3. Uniform upper-bound on the correlation functions.

Lemma 4.1. Let $K$ be a compact subset of $\mathbb{R}$. There is a constant $C$ depending only on $K$ and on the upper bounds on the sequences $N\beta, \alpha$ such that for all $k, N$ and all $x_1, \ldots, x_k \in K$, we have
\begin{equation}
1^k \leq N \bar{P}^{(N)}_k(x_1, \ldots, x_k) \leq C^k e^{\frac{1}{2} \sum_{i=1}^{k} x_i^2}.
\end{equation}
Proof. Note that by (36) and (37), for each \( i \in \{1, \ldots, k \} \) and \( j \in \{1, \ldots, N - k \} \), we have
\[
|E + \frac{x_i}{N} - \lambda_j|^\beta \leq (|E + \frac{x_i}{N}| + |\lambda_j|)^\beta \leq 2^\beta \exp\{\frac{\beta (E + \frac{x_i^2}{N})^2 + \lambda_j^2}{8}\}.
\]
Hence for \( C \) a constant (that might change from line to line) as in the statement of the lemma,
\[
\tilde{R}_k^{(N)}(x_1, \ldots, x_k) \leq 2^{kN \beta} \exp\{\frac{N \beta}{8} \sum_{i=1}^k (E + \frac{x_i}{N})^2\} \int e^{\frac{\beta}{N} \sum_{j=1}^{N-k} \lambda_j^2} P_{\alpha, \beta}^{(N-k)}(d\lambda_1, \ldots, d\lambda_{N-k})
\leq \frac{Z_{N-k}(\alpha - k\beta/4, \beta)}{Z_{N-k}(\alpha, \beta)} C^k e^{kC^2}
\leq C^k e^{kC^2}
\]
where we used (33) and the fact that for any \( x \in [0, 1/2] \), \((1 - x)^{-1} \leq 4^x\).

Hence by (29), (31), (34) and the previous lemma, we have proved that b) of Proposition 6.1 is satisfied. It remains to prove a) for \( \theta \) given by (7).

4.4. Preliminary estimates.

Lemma 4.2 (Bulk eigenvalues). There is a constant \( C \) depending only on the upper bounds on the sequences \( \alpha \) and \( N \beta \) such that for any \( a \in \mathbb{R} \) and \( \varepsilon \in (0, 1] \),
\[
P_{\alpha, \beta}^{(N)}(|\lambda_1 - a| \leq \varepsilon) \leq C \varepsilon e^{\frac{N \beta}{4(1 - \varepsilon)}} a^2,
\]
Proof. We have
\[
P_{\alpha, \beta}^{(N)}(|\lambda_1 - a| \leq \varepsilon) = \frac{1}{Z_N(\alpha, \beta)} \int_{\lambda_1 \in [a \pm \varepsilon]} d\lambda_1 e^{-\frac{\beta}{2} \lambda_1^2} \int_{\mu \in \mathbb{R}^{N-1}} \Delta(\mu)^\beta \prod_{j} |\lambda_1 - \mu_j|^\beta e^{-\frac{\beta}{2} \sum_i \mu_j^2} d\mu
\]
Note that if \(|\lambda_1 - a| \leq \varepsilon\), the for any \( j \), \(|\lambda_1 - \mu_j| \leq \varepsilon + |\mu_j - a|\). Moreover, for all \( x \in \mathbb{R} \), \( t > 0 \), we have \(|x| \leq t^{-1/2} e^{\frac{x^2}{2t^2}}\), hence
\[
|x| \leq 2 e^{\frac{x^2}{4t}}.
\]
Using also the fact that
\[
(x + y)^2 \leq 2x^2 + 2y^2,
\]
we get that if \(|\lambda_1 - a| \leq \varepsilon\),
\[
|\lambda_1 - \mu_j|^\beta \leq (\varepsilon + |\mu_j - a|\)^\beta \leq 2^\beta e^{\beta \frac{\varepsilon^2}{8} (\mu_j - a)^2}.
\]
Hence
\[
P_{\alpha, \beta}^{(N)}(|\lambda_1 - a| \leq \varepsilon) \leq \frac{2^{N \beta} e^{\frac{N \beta \varepsilon^2}{8}}}{Z_N(\alpha, \beta)} \int_{\lambda_1 \in [a \pm \varepsilon]} d\lambda_1 e^{-\frac{\beta}{2} \lambda_1^2} \int_{\mu \in \mathbb{R}^{N-1}} \Delta(\mu)^\beta e^{-\frac{\beta}{2} \sum_i (\mu_j^2 - \frac{\beta (\mu_j - a)^2}{4N})} d\mu
\]
We have
\[ \mu_j^2 - \beta \frac{(\mu_j - a)^2}{4\alpha} = \frac{4\alpha - \beta}{4\alpha} \left( \mu_j + \frac{\beta a}{4\alpha - \beta} \right)^2 - \frac{\beta}{4\alpha - \beta} a^2 \]
Hence
\[
P_{\alpha,\beta}^{(N)}(|\lambda_1 - a| \leq \varepsilon) \leq \frac{2^{N\beta} e^{\frac{N\beta^2}{8} e^{\frac{N \alpha \beta}{4\alpha - \beta} a^2}}}{Z_N(\alpha, \beta)} \int_{|\lambda_1| \leq a + \varepsilon} d\lambda_1 e^{-\frac{a^2}{2}\lambda_1^2} \prod_{\mu \in \mathbb{R}^{N-1}} \Delta(\mu)^{\beta} e^{-\frac{\beta}{2} \sum_j \lambda_j^2} \int_{\mu \in \mathbb{R}^{N-1}} d\mu.
\]
we used (33) to upper bound partition functions quotient.

**Lemma 4.3** (Largest eigenvalues). There is a constant $C$ depending only on the upper bounds on the sequences $\alpha$ and $N\beta$ such that for all $x > 0$,
\[
P_{\alpha,\beta}^{(N)}(|\lambda_1| \geq x) \leq C e^{-\frac{(4\alpha - \beta)^2 x^2}{8}}.
\]  
(38)

**Proof.** We have
\[
P_{\alpha,\beta}^{(N)}(|\lambda_1| \geq x) = \frac{1}{Z_N(\alpha, \beta)} \int_{|\lambda_1| \geq x} d\lambda_1 e^{-\frac{a^2}{2}\lambda_1^2} \int_{\mu \in \mathbb{R}^{N-1}} \Delta(\mu)^{\beta} \prod_j |\lambda_1 - \mu_j|^\beta e^{-\frac{\beta}{2} \sum_j \mu_j^2} d\mu.
\]
Note that by (36) and (37),
\[ |\lambda_1 - \mu_j|^\beta \leq (|\lambda_1| + |\mu_j|)^\beta \leq 2^\beta e^{\frac{\lambda_1^2 + \mu_j^2}{2}}. \]
Hence
\[
P_{\alpha,\beta}^{(N)}(|\lambda_1| \geq x) \leq \frac{2^{N\beta}}{Z_N(\alpha, \beta)} \int_{|\lambda_1| \geq x} d\lambda_1 e^{-\frac{1}{2} (\alpha - \beta)^2 \lambda_1^2} \int_{\mu \in \mathbb{R}^{N-1}} \Delta(\mu)^{\beta} e^{-\frac{1}{2} (\alpha - \beta) \sum_j \mu_j^2} d\mu
\]
\[ = \frac{2^{N\beta}}{\sqrt{\alpha - \beta} / 4} Z_{N-1}(\alpha - \beta / 4, \beta) \int_{|\lambda_1| \geq x \sqrt{\alpha - \beta / 4}} e^{-\frac{\lambda_1^2}{2}} d\lambda_1. \]
Then we conclude using (33) and the fact that for all $y > 0$,
\[
\int_y^{+\infty} e^{-\frac{t^2}{2}} dt \leq \frac{e^{-\frac{x^2}{2}}}{y}.
\]  
(39)

**Lemma 4.4** (Tail of the empirical spectral law). There are some constants $C, c$ depending only on the upper bounds on the sequences $N\beta, \alpha$ such that for all $N$ and all $x > 0$,
\[
P_{\alpha,\beta}^{(N)}\left( \frac{\lambda_1^2 + \cdots + \lambda_N^2}{N} \geq x \right) \leq C e^{-cx}. \]
Proof. We use again the tridiagonal matrix model of (5) for $P_{\alpha,\beta}^{(N)}$ of the $\lambda_i$’s. We know that $\lambda_1^2 + \cdots + \lambda_N^2$ has the same law as $\text{Tr} \; H^2$, with $H$ the matrix introduced as (5). Note that by the well known convolution relations between Gamma-distributed variables, $\frac{1}{2} \text{Tr} \; H^2$ has a $\Gamma(\varphi)$-distribution for $\varphi := N(1 + (N - 1)\beta) / 2$. Hence

$$P_{\alpha,\beta}^{(N)}(\frac{\lambda_1^2 + \cdots + \lambda_N^2}{N} \geq x) \leq \mathbb{P}(G \geq \frac{\alpha N x}{2}).$$

Then, one concludes using the concentration inequalities for Gamma variables (see [9] p. 28-29) which say that for all $u \geq 0$,

$$P(G \geq \varphi(1 + u)) \leq e^{-\varphi(1 + u - \sqrt{1 + 2u})}.$$

\[ \Box \]

Lemma 4.5. For $C, c$ as in the previous lemma, for any $u, M, \theta$ such that $cM^2 > 2\theta \geq 0$,

$$1 \leq \int e^{\theta \sum_i \log(|\lambda_i - u| M \vee 1)} P_{\alpha,\beta}^{(N)}(d\lambda_1, \ldots, d\lambda_N) \leq 1 + \frac{2C\theta e^{cu^2}}{cM^2 - 2\theta}.$$

Proof. The integral above rewrites

$$1 + \theta \int_0^{+\infty} e^{\theta x} P_{\alpha,\beta}^{(N)}(\frac{1}{N} \sum_i \log(|\lambda_i - u| M \vee 1) \geq x) dx. \quad (40)$$

Now, note that as $\log(|\lambda - u| \vee 1) \leq (\lambda - u)^2$,

$$P_{\alpha,\beta}^{(N)}(\frac{1}{N} \sum_i \log(|\lambda_i - u| M \vee 1) \geq x) \leq P_{\alpha,\beta}^{(N)}(\frac{(\lambda_1 - u)^2 + \cdots + (\lambda_N - u)^2}{N} \geq M^2 x) \leq P_{\alpha,\beta}^{(N)}(\frac{2\lambda_1^2 + 2u^2 + \cdots + 2\lambda_N^2 + 2u^2}{N} \geq M^2 x) \leq P_{\alpha,\beta}^{(N)}(\frac{\lambda_1^2 + \cdots + \lambda_N^2}{N} \geq \frac{M^2 x - u^2}{2}).$$

Then one concludes using (40) and the previous lemma. \[ \Box \]

4.5. Convergence of the correlation functions. Let us now prove a) of Proposition 6.1 for $\theta$ given by (7). Note first that by b) of Proposition 2.1, we know that

$$\int \log |E - x| d\mu_\gamma(x) < \infty. \quad (41)$$

Besides, by (27), (30), (32) and (35), it suffices to prove that for each $k$ and each $x_1, \ldots, x_k \in \mathbb{R}$, the quantity $\tilde{R}_k^{(N)}(x_1, \ldots, x_k)$ defined at (28) satisfies, as $N \to \infty$,

$$\tilde{R}_k^{(N)}(x_1, \ldots, x_k) \xrightarrow{N \to \infty} \exp \left\{ 2\gamma k \int \log |E - x| d\mu_\gamma(x) \right\}. \quad (42)$$
4.5.1. Upper-bound. Let us prove that for any fixed $k$ and $x_1, \ldots, x_k$,
\[
\limsup_{N \to \infty} \tilde{R}^{(N)}_k(x_1, \ldots, x_k) \leq \exp \left\{ 2\gamma k \int \log |E - x| d\mu_\gamma(x) \right\}.
\]
(43)

For $\varepsilon > 0$, set
\[
\tilde{R}^{(N,\varepsilon)}_k(x_1, \ldots, x_k) := \int e^{\beta \sum_{i=1}^k \sum_{j=1}^{N-k} \log(|E + \frac{x_i}{N} - \lambda_j| \vee \varepsilon)} P^{(N-k)}_{\alpha,\beta}(d\lambda_1, \ldots, d\lambda_{N-k}).
\]
As $\tilde{R}^{(N)}_k \leq \tilde{R}^{(N,\varepsilon)}_k$ and, by (41),
\[
\inf_{\varepsilon > 0} \int \log(|x - E| \vee \varepsilon) d\mu_\gamma(x) = \int \log |x - E| d\mu_\gamma(x),
\]
it suffices to prove that for any $\varepsilon > 0$ small enough, we have
\[
\limsup_{N \to \infty} \tilde{R}^{(N,\varepsilon)}_k(x_1, \ldots, x_k) \leq \exp \left\{ 2\gamma k \int \log |E - x| \vee \varepsilon) d\mu_\gamma(x) \right\}.
\]
(44)

Note now that for any $M > 0$ large enough, as
\[
\log(|E + x_i/N - x| \vee \varepsilon) = \log \left\{ (|E + x_i/N - x| \vee \varepsilon) \wedge M \right\} + \log \left( \frac{|E + x_i/N - x|}{M} \vee 1 \right)
\]
the function $\tilde{R}^{(N,\varepsilon)}_k(x_1, \ldots, x_k)$ rewrites
\[
\tilde{R}^{(N,\varepsilon)}_k(x_1, \ldots, x_k) = \int e^{\beta \sum_{i=1}^k \sum_{j=1}^{N-k} \log((|E + \frac{x_i}{N} - \lambda_j| \vee \varepsilon) \wedge M)} e^{\beta \sum_{i=1}^k \sum_{j=1}^{N-k} \log\left( \frac{|E + \frac{x_i}{N} - \lambda_j|}{M} \vee 1 \right)} P^{(N-k)}_{\alpha,\beta}(d\lambda_1, \ldots, d\lambda_{N-k})
\]
\[
= \int e^{\beta \sum_{i=1}^k \sum_{j=1}^{N-k} \log((|E + \frac{x_i}{N} - \lambda_j| \vee \varepsilon) \wedge M)} P^{(N-k)}_{\alpha,\beta}(d\lambda_1, \ldots, d\lambda_{N-k}) + \int e^{\beta \sum_{i=1}^k \sum_{j=1}^{N-k} \log((|E + \frac{x_i}{N} - \lambda_j| \vee \varepsilon) \wedge M)} \left( e^{\beta \sum_{i=1}^k \sum_{j=1}^{N-k} \log\left( \frac{|E + \frac{x_i}{N} - \lambda_j|}{M} \vee 1 \right)} - 1 \right) P^{(N-k)}_{\alpha,\beta}(d\lambda_1, \ldots, d\lambda_{N-k})
\]

By Proposition 2.1, we know that under the law $P^{(N-k)}_{\alpha,\beta}$, the $L^\infty$-bounded sequence of random variables $\beta \sum_{i=1}^k \sum_{j=1}^{N-k} \log((|E + \frac{x_i}{N} - \lambda_j| \vee \varepsilon) \wedge M)$ converges in probability, as $N \to \infty$, to
\[
2\gamma k \int \log \left\{ (|E - x| \vee \varepsilon) \wedge M \right\} d\mu_\gamma(x)
\]
(one gets rid of the $\frac{x_i}{N}$’s by noticing, for example, that the convolution of probability measures is continuous with respect to the weak topology and that $\delta_{\lambda_i/N}$ converges to $\delta_0$).

Note that by choosing $M$ large enough, one can make $\int \log \left\{ (|E - x| \vee \varepsilon) \wedge M \right\} d\mu_\gamma(x)$ as close as we want from $\int \log |E - x| d\mu_\gamma(x)$. Moreover, one can easily adapt the
proof of Lemma 4.1 to see that \( \int X^2 P_{\alpha,\beta}^{(N-k)}(d\lambda_1, \ldots, d\lambda_{N-k}) \) is bounded by a constant independent of \( M \), hence by Cauchy-Schwartz, to prove (44), it suffices to prove that \( \int Y^2 P_{\alpha,\beta}^{(N-k)}(d\lambda_1, \ldots, d\lambda_{N-k}) \) can be made as small as we want if \( M \) is large enough. Note that for

\[
Y_i := e^{\beta \sum_{j=1}^{N-k} \log \left( \frac{|E + x_j - \lambda_i|}{|v_1|} \right)},
\]

we have \( Y = Y_1 \cdots Y_k - 1 \). Thus by the Hölder inequality, it is enough to prove that for \( M \) large enough, each \( Y_i \) can have its \( k \)-th and \( 2k \)-th moment as close as we want from 1, which is a direct consequence of Lemma 4.5.

### 4.5.2. Lower bound

To obtain the analogous lower bound

\[
\liminf_{N \to \infty} \log \bar{R}_k^{(N)}(x_1, \ldots, x_k) \geq \exp \left\{ 2 \gamma k \int \log |E - x| \, d\mu_\gamma(x) \right\}, \tag{45}
\]

we observe that first by Jensen’s inequality and then by exchangeability,

\[
\log \bar{R}_k^{(N)}(x_1, \ldots, x_k) \geq \beta \sum_{i=1}^{k} \sum_{j=1}^{N-k} \log \left| E + \frac{x_i}{N} - \lambda_j \right| P_{\alpha,\beta}^{(N-k)}(d\lambda_1, \ldots, d\lambda_{N-k})
\]

\[
= (N-k) \beta \int \sum_{i=1}^{k} \log \left| E + \frac{x_i}{N} - \lambda_1 \right| P_{\alpha,\beta}^{(N-k)}(d\lambda_1, \ldots, d\lambda_{N-k})
\]

Hence as \( (N-k)\beta \to 2\gamma \) and the triplet \((N, \beta(N), \alpha(N))\) satisfies the same hypotheses as \((N-k, \beta(N), \alpha(N))\), it suffices to prove that for any fixed \( x \), we have

\[
\liminf_{N \to \infty} \int \log \left| E + \frac{x}{N} - \lambda_1 \right| P_{\alpha,\beta}^{(N)}(d\lambda_1, \ldots, d\lambda_N) \geq \int \log |E - t| \, d\mu_\gamma(t).
\]

As, by exchangeability, \( \mu_\gamma \) is also the weak limit of the distribution of \( \lambda_1 \) under \( P_{\alpha,\beta}^{(N)} \), we know that for any \( \varepsilon > 0 \),

\[
\liminf_{N \to \infty} \int \log \left| E + \frac{x}{N} - \lambda_1 \right| \vee \varepsilon \right) P_{\alpha,\beta}^{(N)}(d\lambda_1, \ldots, d\lambda_N) \geq \int \log |E - t| \, d\mu_\gamma(t)
\]

(and one can get rid of \( \frac{\varepsilon}{N} \) for the same reason as in Section 4.5.1 above). Hence it suffices that for \( \varepsilon \) small enough,

\[
\limsup_{N \to \infty} \left| \int \left( \log \left| E + \frac{x}{N} - \lambda_1 \right| \vee \varepsilon \right) - \log \left| E + \frac{x}{N} - \lambda_1 \right| \right) P_{\alpha,\beta}^{(N)}(d\lambda_1, \ldots, d\lambda_N)
\]

can be made as small as desired. But for any random variable \( X > 0 \),

\[
\mathbb{E}[\log(X \vee \varepsilon) - \log(X)] = \mathbb{E}[\log \varepsilon - \log(X)] 1_{X \leq \varepsilon} = \int_0^\varepsilon \frac{\mathbb{P}(X \leq t)}{t} \, dt.
\]

Here, by Lemma 4.2, there is a constant \( C \) such that

\[
P_{\alpha,\beta}^{(N)}(|E + \frac{x}{N} - \lambda_1| \leq t) \leq Ct,
\]
which allows to get the desired bound.

5. Case where \( N^{-1} \ll \beta \ll 1/\log(N) \)

5.1. Partition functions: proofs of (18) and (19). It follows from (33) that for all \( N \), for all \( 1 \leq k \leq N \),

\[
\frac{Z_{N-k}(\alpha, \beta)}{Z_N(\alpha, \beta)} = \alpha^k \alpha^\frac{q(2k-N^2+k)}{(2\pi)^{k/2}} \prod_{\ell=0}^{k-1} \frac{1}{\Gamma(1+(N-\ell)\beta/2)}
\]

But by the Stirling formula, for \( z \to +\infty \),

\[
\Gamma(1+z) \sim \sqrt{2\pi z} \exp \left\{ z \log(z) - z \right\}.
\]

Note that our hypothesis on \( \beta \) implies that for any fixed \( \ell = 0, \ldots, k-1 \), we have

\[
(N-\ell)\beta/2 \log((N-\ell)\beta/2) - (N-\ell)\beta/2 = N/2 \log(N/2) - N/2 + o(1),
\]

so that, as (11) implies that \( N\beta \log \frac{\alpha}{N\beta} \ll 1 \) and \( \beta \log \alpha \ll 1 \),

\[
\frac{Z_{N-k}(\alpha, \beta)}{Z_N(\alpha, \beta)} \sim \left( \frac{e^{N/2}}{2\pi} \right)^k \times \left( \exp \left\{ \frac{N\beta}{2} \left[ \log \alpha - \log(N/2) \right] \right\} \right)^k
\]

By (11), we get (18). The upper bound (19) comes in the same way, noticing that the error in (46) is uniformly bounded on \( z \geq 0 \).

5.2. Tail estimate.

**Lemma 5.1.** Let \( \beta = \beta(N) \) and \( \alpha = \alpha(N) \) be satisfying (11). Then there is a constant \( C \) depending only on the sequences \( \alpha \) and \( \beta \) such that for all \( N \), for all \( x > 0 \),

\[
P_{\alpha, \beta}^{(N)}(|\lambda_1| \geq x) \leq C e^{(-\frac{x^2}{2}+C)^{N^{1/2}}}.
\]

**Proof.** We have

\[
P_{\alpha, \beta}^{(N)}(|\lambda_1| \geq x) = \frac{1}{Z_{N}(\alpha, \beta)} \int_{|\lambda_1| \geq x} d\lambda_1 e^{-\frac{1}{2} \lambda_1^2} \int_{\mu \in \mathbb{R}^{N-1}} \Delta(\mu)^\beta \prod_j |\lambda_1 - \mu_j|^\beta e^{-\frac{1}{2} \sum_i \mu_i^2} d\mu
\]

Note that by (36) and (37),

\[
|\lambda_1 - \mu_j|^\beta \leq (|\lambda_1| + |\mu_j|)^\beta \leq 2^\beta e^{\frac{\lambda_1^2 + \mu_j^2}{2}}.
\]
Hence
\[
P^{(\alpha,\beta)}_{\lambda_1}(\lambda_1 \geq x) \leq \frac{2^{N\beta}}{Z_N(\alpha,\beta)} \int_{|\lambda_1| \geq x} d\lambda_1 e^{-\frac{\alpha-\beta/4}{2} \lambda_1^2} \int_{\mu \in \mathbb{R}^{N-1}} \Delta(\mu)^\beta e^{-\frac{\beta}{2} \sum \mu_i^2} d\mu
\]
\[
eq \frac{2^{N\beta}}{Z_N(\alpha,\beta)\sqrt{\alpha - \beta/4}} \int_{|\lambda_1| \geq x} e^{-\frac{\lambda_1^2}{2}} d\lambda_1 \times Z_{N-1}(\alpha - \beta/4,\beta)
\]
\[
\leq \frac{2^{N\beta+1}e^{-x^2(\alpha-\beta/4)}}{x(\alpha - \beta/4)} Z_{N-1}(\alpha - \beta/4,\beta)
\]
\[
= \frac{2^{N\beta+1}e^{-x^2(\alpha-\beta/4)}}{x(\alpha - \beta/4)} (1 - \beta/(4\alpha))^{-\left(\frac{(N-1)(\alpha-2\beta)}{4} + \frac{N-1}{2}\right)} Z_{N-1}(\alpha,\beta) Z_N(\alpha,\beta)
\]
Let us now use (19) and for example the fact that \((1 - y)^{-1} \leq e^{2y}\) when \(y \in [0,1/2]\). We get
\[
P^{(\alpha,\beta)}_{\lambda_1}(\lambda_1 \geq x) \leq \frac{M}{2\pi x\alpha} \exp\left\{-\frac{x^2}{2} (\alpha - \beta/4) + N^2\beta^2/(8\alpha) + N\beta/(2\alpha) + N\beta/2\right\},
\]
which allows to conclude, as we already noticed that (11) implies that \(2\alpha \sim N\beta \gg 1\). □

5.3. Proof of Theorem 2.5. We first define the probability measure on \(\mathbb{R}^N\)
\[
P^{(N,h)}_{\alpha,\beta}(dx) := \frac{1}{Z_N^h(\alpha,\beta)} \Delta(x)^\beta \exp\left\{-\sum_{i=1}^N x_i^2 + \beta \sum_{i=1}^N h(x_i)\right\} dx_1 \cdots dx_N,
\]
where
\[
Z_N^h(\alpha,\beta) = \int \Delta(x)^\beta e^{-\frac{\alpha}{2} \sum x_i^2 + \beta \sum h(x_i)} dx_1 \cdots dx_N
\]
is the normalisation constant. Let, for \(i = 1, 2,\)
\[
u_{N,i}^{(h)}(x_1, \ldots, x_i) := \int_{x_{i+1}, \ldots, x_N} \rho_{\alpha,\beta}^{(N,h)}(x_1, \ldots, x_N) dx_{i+1} \cdots dx_N
\]
be the \(i\)-th correlation function of \(\rho_{\alpha,\beta}^{(N,h)}\).

Lemma 5.2. Let \(\psi : \mathbb{R} \to \mathbb{C}\) be a \(C^1\) function on \(\mathbb{R}\) such that the real and imaginary parts of \(\psi'\) are bounded below. Then we have
\[
\beta \frac{N-1}{2} \int \frac{\psi(t) - \psi(s)}{t-s} u_{N}^{2,h}(s,t) ds dt + \int_{\mathbb{R}} (\beta h'(t) - \alpha t) \psi(t) u_{N}^{1,h}(t) dt
\]
\[
+ \int_{\mathbb{R}} \psi(t) u_{N}^{1,h}(t) dt = 0. \quad (50)
\]
Proof. As (50) is linear in $\psi$, one can suppose $\psi$ to be real-valued. Then for $\theta \geq 0$ small enough, the function $y + \theta \psi(y)$ is an homeomorphism on $\mathbb{R}$, hence one can make the change of variable $x_i = y_i + \theta \psi(y_i)$ in (49). We get

$$Z_N^h = \int_{y \in \mathbb{R}^N} \prod_{1 \leq i < j \leq N} |y_j - y_i + \theta(\psi(y_j) - \psi(y_i))|^\beta$$

(51)

$$e^{-\frac{N}{2} \sum_{i=1}^N (y_i^2 + 2\theta y_i \psi(y_i) + \theta^2 \psi(y_i)^2) + \beta \sum_{i=1}^N h(y_i + \theta \psi(y_i))} \prod_{i=1}^N (1 + \theta \psi'(y_i)) dy_1 \cdots dy_N$$

Let us compute the derivative, with respect to $\theta$, at $\theta = 0$, of the RHT of (51). We have

$$\partial_{\theta, \theta=0} |y_j - y_i + \theta(\psi(y_j) - \psi(y_i))|^\beta = \beta \frac{\psi(y_j) - \psi(y_i)}{y_j - y_i} |y_j - y_i|^\beta,$$

we also have

$$\partial_{\theta, \theta=0} e^{-\frac{N}{2} \sum_{i=1}^N (y_i^2 + 2\theta y_i \psi(y_i) + \theta^2 \psi(y_i)^2)} = -\alpha \sum_{i=1}^N y_i \psi(y_i),$$

$$\partial_{\theta, \theta=0} e^{\beta \sum_{i=1}^N h(y_i + \theta \psi(y_i))} = \beta \sum_{i=1}^N h'(y_i) \psi(y_i),$$

and

$$\partial_{\theta, \theta=0} \prod_{i=1}^N (1 + \theta \psi'(y_i)) = \sum_{i=1}^N \psi'(y_i).$$

Hence

$$\frac{\partial_{\theta, \theta=0} \text{(RHT of (51))}}{Z_N^h} = \beta \sum_{1 \leq i < j \leq N} \int_{y \in \mathbb{R}^N} \frac{\psi(y_j) - \psi(y_i)}{y_j - y_i} \rho_N^h(y_1, \ldots, y_N) dy_1 \cdots dy_N$$

$$-\alpha \sum_{i=1}^N \int_{y \in \mathbb{R}^N} y_i \psi(y_i) \rho_N^h(y_1, \ldots, y_N) dy_1 \cdots dy_N$$

$$+\beta \sum_{i=1}^N \int_{y \in \mathbb{R}^N} h'(y_i) \psi(y_i) \rho_N^h(y_1, \ldots, y_N) dy_1 \cdots dy_N$$

$$+ \sum_{i=1}^N \int_{y \in \mathbb{R}^N} \psi'(y_i) \rho_N^h(y_1, \ldots, y_N) dy_1 \cdots dy_N$$

We get exactly (50).
Now, we define, for $z$ such that $\Re z > 0$,
\[ H_N(z) := \int \frac{h'(t)}{z-t} u_N^{1h}(t)dt \quad ; \quad U_N(z) := \int \frac{1}{z-t} u_N^{1h}(t)dt \]
and
\[ K_N(z) := N \int \int \left( \frac{1}{(z-t)(z-s)} - \frac{1}{2(z-t)^2} - \frac{1}{2(z-s)^2} \right) k_N(s,t)dsdt \]  
(52)

We also introduce
\[ U(z) := \frac{1}{2}(z - \sqrt{z^2 - 4}) \quad (\Re z > 0), \]  
(53)

where when $\Re(z) \geq 0$ (resp. $\leq 0$), $\sqrt{z^2 - 4}$ is computed with the determination of the square root on $\mathbb{C}\setminus(-\infty,0)$ (resp. on $\mathbb{C}\setminus(0,\infty)$) with positive values on $[0,\infty)$ (resp. such that $\sqrt{-1} = i$). It is well known that $U$ is the Stieltjes transform of the semicircle law $\sigma$.

**Lemma 5.3.** On the upper half-plane, we have
\[ N\beta(U_N - U)(2U - \frac{2\alpha}{N\beta}z + U_N - U) = \Delta_N := \beta N^{-1}K_N - 2\beta H_N + (2 - \beta)U_N' + (2\alpha - N\beta)U^2. \]  
(54)

**Proof.** We shall apply the previous lemma with $\psi(t) = \frac{1}{z-t}$. Note that $\frac{\psi(t) - \psi(s)}{t-s} = \frac{1}{(z-t)(z-s)}$, so that we have:

\[ N(N-1) \int \int \frac{\psi(t) - \psi(s)}{t-s} u_N^{2h}(s,t)dsdt \]

\[ = \int \int \frac{1}{(z-t)(z-s)} N(N-1) u_N^{2h}(s,t)dsdt \]

\[ = \int \int \frac{1}{(z-t)(z-s)} (-Nk_N(s,t) + N^2 u_N^{1h}(s) u_N^{1h}(t))dsdt \]

\[ = N^2 U_N(z)^2 - N \int \int \frac{1}{(z-t)(z-s)} k_N(s,t)dsdt \]

\[ = N^2 U_N(z)^2 - N \int \int \left( \frac{1}{(z-t)(z-s)} - \frac{1}{2(z-t)^2} - \frac{1}{2(z-s)^2} \right) k_N(s,t)dsdt \]

\[ - N \int \int \frac{1}{(z-t)^2} k_N(s,t)dsdt \]

\[ = N^2 U_N(z)^2 - K_N(z) + nU_N'(z) \]
(where we use the fact that for any function $f(t)$, $\int \int f(t)k_N(s,t)dsdt = \int f(t)u_N^{1,h}(t)dt$).

We also have

$$\int -t\psi(t)u_N^{1,h}(t)dt = 1 - zU_N(z) ;$$

$$\int \psi'(t)u_N^{1,h}(t)dt = -U'_N(z),$$

so $2N^{-2}\beta^{-1} \times (50)$ rewrites

$$U_N(z)^2 - N^{-2}K_N(z) + N^{-1}U'_N(z) + 2N^{-1}H_N(z) + \frac{2\alpha}{N\beta}(1 - zU_N(z)) - \frac{2}{N\beta}U'_N(z) = 0$$

i.e.

$$U_N(z)^2 - \frac{2\alpha}{N\beta}zU_N(z) + \frac{2\alpha}{N\beta} = N^{-2}K_N(z) - 2N^{-1}H_N(z) + \left( \frac{2}{N\beta} - \frac{1}{N} \right)U'_N(z). \quad (55)$$

One gets $(54)$, using the well known equation $U(z)^2 - zU(z) + 1 = 0$ (see [6, Eq. (2.4.6)]). □

A key step in the proof of the theorem will be to prove that as $N \to \infty$,

$$\beta K_N(z) \ll N. \quad (56)$$

We shall now prove $(56)$. Let $(y_1, \ldots, y_N)$ be a random vector with distribution $P_{\alpha,\beta}^{(N,h)}$ and for $g \in C_b(\mathbb{R}, \mathbb{C})$, define the random variable

$$\hat{\mu}_N^h(g) := \sum_{i=1}^N g(y_i),$$

with variance $\Sigma_N(g) := \mathbb{E}[|\hat{\mu}_N^h(g) - \mathbb{E}[\hat{\mu}_N^h(g)]|^2]$. As

$$K_N(z) = \mathbb{E}[|\hat{\mu}_N^h(g) - \mathbb{E}[\hat{\mu}_N^h(g)]|^2] \quad (57)$$

for $g(t) = (z - t)^{-1}$, we have

$$|K_N(z)| \leq \Sigma_N(\frac{1}{z - t}). \quad (58)$$

Note that as

$$\mathbb{E}[\hat{\mu}_N^h(g)] = N \int u_N^{1,h}(t)g(t)dt$$

$$\mathbb{E}[|\hat{\mu}_N^h(g)|^2] = \sum_{i,j=1}^N \mathbb{E}[g(y_i)g(y_j)]$$

$$= N(N - 1) \int \int g(s)g(t)u_N^{1,h}(s,t)dsdt + N \int u_N^{1,h}(t)|g(t)|^2dt,$$
we deduce
\[
\Sigma_N(g) = N(N-1) \int \int g(s)\overline{g(t)}(u_N^{2h}(s,t) - u_N^{1h}(s)u_N^{1h}(t))dsdt \\
+ N \left( \int u_N^{1h}(t)|g(t)|^2dt - |\int u_N^{1h}(t)g(t)dt|^2 \right)
\]

Lemma 5.4. There is $L > 0$ and $c > 0$ such that for any fixed function $g$,
\[
\Sigma_N(g) \leq N(N-1) \int \int_{[-L,L]^2} g(s)\overline{g(t)}(u_N^{2h}(s,t) - u_N^{1h}(s)u_N^{1h}(t))dsdt \\
+ N \left( \int_{-L}^L u_N^{1h}(t)|g(t)|^2dt - |\int_{-L}^L u_N^{1h}(t)g(t)dt|^2 \right) + 4N^2\|g\|_\infty e^{-cN^\beta}
\]

Proof. Using the fact that $|\sum_{i=1}^Nh(x_i)| \leq N\|h\|_\infty$, we see that the probability measure $P^{(N,h)}_{\alpha,\beta}$ defined at (48) and its normalization constant can be controlled thanks to the probability measure $P^{(N)}_{\alpha,\beta}$ and its normalization constant: for any Borel set $A \subset \mathbb{R}^N$, we have
\[
P^{(N,h)}_{\alpha,\beta}(A) \leq e^{2N^\beta\|h\|_\infty}P^{(N)}_{\alpha,\beta}(A).
\]
It follows that up to a change of the constant $C$, Lemma 5.1 is also true for $P^{(N,h)}_{\alpha,\beta}$, which allows to conclude. \qed

This lemma allows to reduce the problem to a compact set, and after rescaling, one can turn the compact set in question to $[-1/2,1/2]$ : we deduce, as in [17], that for $w := z/L$ and
\[
\rho_N^2(t,s) := u_N^{2h}(Lt, Ls), \quad \rho_N^1(t) := u_N^{1h}(Lt).
\]
we have
\[
\Sigma_N(\frac{1}{z-t}) \leq N(N-1) \int_{-1/2}^{1/2} \frac{1}{w-t}G_N(x \mapsto \frac{1}{w-x})'(t)dt + 4N^2e^{-cN^\beta}(3z)^{-2}, \quad (59)
\]
where $G_N$ is the operator on $L^2([-1/2,1/2], dx)$ defined by
\[
G_N(f)(t) = \int f(s)(\rho_N^2(t,s) - \rho_N^1(t)\rho_N^1(s))ds + \frac{\rho_N(t)}{N-1} \left( \frac{1}{L}f(t) - \int_{-1/2}^{1/2} \rho_N(s)f(s)ds \right).
\]

Thus to prove the estimate of interest (56), we have to upper bound:
\[
N(N-1) \int_{-1/2}^{1/2} \frac{1}{w-t}G_N(x \mapsto \frac{1}{w-x})'(t)dt. \quad (60)
\]
Following [17], we introduce the integral operator $P_w$ on $L^2([-1/2,1/2], dx)$ with kernel $P_w(t,s) = \frac{1}{(w-t)(w-s)}$. Then $(s,t) \mapsto \frac{1}{w-t}G_N(x \mapsto \frac{1}{w-x})(s)$ is an integral operator satisfying
the hypothesis of Theorem 2.12 of [24]. This trace class operator is nothing but $P_w G_N$, thus by this theorem, we have

$$\int_{-1/2}^{1/2} \frac{1}{w-t} G_N \left( \frac{1}{w-s} \right) dt = \text{Tr}(P_w G_N).$$

(61)

We will not here recall all the arguments used in [17] Lemma 3.12 and Proposition 3.9 to estimate this trace. The proof transfers to our setting using minor modifications (essentially replacing $N$ by $N\beta$). Note that the important feature of $h$ is that is Lipschitz on compact sets in our case. Thus we simply state the final estimate we will use in this article, namely the following lemma:

**Lemma 5.5.** We have $\text{Tr}(P_w G_N) \leq CN^{-1} \log(N)$ for some constant $C$.

It follows, by (58), (59) and (61), that

$$|\beta K_N(z)| \leq \beta \Sigma_N \left( \frac{1}{z-t} \right) \leq CN\beta \log(N) + 4N^2 \beta e^{-cN\beta}(\Im z)^{-2}.$$

This is of course $\ll N$, so the estimate of interest (56) is proved.

As $\beta \log(N) \ll 1$ and $N\beta - 2\alpha \ll 1$, by (54), we deduce that, uniformly on compact subsets of $\mathbb{C}^+$, one has that

$$N\beta (U_N(z) - U(z)) \to \frac{2U'(z)}{2U(z) - z} = \frac{z}{z^2 - 4} - \frac{1}{\sqrt{z^2 - 4}},$$

(62)

One recognizes easily that the RHT of (62) is the Stieltjes transform of the null mass signed measure $\nu$ of (13) (to do that, use the fact that $U(z)$, given by (53), is the Stieltjes transform of the semi-circle law and then use an integration by parts).

The rest of the proof of the theorem is an easy adaptation of the proof of Theorem 2.4 in [17] (p. 169-172). The main idea is to define

$$F(\lambda) := \log \mathbb{E} e^{\beta \lambda \Sigma_j h(x_j)} - N\beta \lambda \int h(t) d\sigma(t),$$

to notice that

$$\partial_\lambda F(\lambda) = N \int_R h(t) u_N^{1h}(t) dt,$$

to prove (12) for derivatives and to deduce (12) by dominated convergence. We use (62), namely the convergence

$$N\beta \int \frac{1}{z-t} \left( u_N^{1h}(t) - 1_{|t| \leq 2} \frac{\sqrt{4-t^2}}{2\pi} \right) dt \to \int \frac{1}{z-t} d\nu(t),$$

in Fourier transform manipulations, precisely via the formula

$$\int_0^\infty \delta_N(\xi) e^{ik\xi} d\xi = i \int \frac{1}{z-t} \delta_N(t) dt \quad (\Im z > 0)$$
with \( \delta_N(t) := N \beta(u^1_N(t) - 1_{|t| \leq 2} \frac{\sqrt{1-t^2}}{2 \pi}) \).

6. Appendix: Poisson limit for point processes

Let \( \mathcal{X} \) be a locally compact Polish space and \( \mu \) be a Radon measure on \( \mathcal{X} \). We consider an exchangeable random vector \( (\lambda_1, \ldots, \lambda_N) \) taking values on \( \mathcal{X} \) implicitly depending on \( N \), with density \( \rho^{(N)} \) with respect to \( \mu^{\otimes N} \). We define, for \( 1 \leq k \leq N \), the \( k \)-th correlation function on \( \mathcal{X}^k \) by the formula

\[
R^{(N)}_k(x_1, \ldots, x_k) := \frac{N!}{(N-k)!} \int_{(x_{k+1}, \ldots, x_N) \in \mathcal{X}^{N-k}} \rho^{(N)}(x_1, \ldots, x_N) d\mu^{\otimes N-k}(x_{k+1}, \ldots, x_N).
\]

Proposition 6.1. Suppose that there is \( \theta \geq 0 \) independent of \( N \) such that the correlation functions \( R^{(N)}_k \) satisfy:

a) For each \( k \geq 1 \), on \( \mathcal{X}^k \), we have the pointwise convergence

\[
R^{(N)}_k(x_1, \ldots, x_k) \xrightarrow{N \to \infty} \theta^k,
\]

(63)

b) For each compact \( K \subset \mathcal{X} \), there is \( \Theta_K \) such that for all \( k, N \), on \( K^k \), we have

\[
1_{k \leq N} R^{(N)}_k(x_1, \ldots, x_k) \leq \Theta^k_K
\]

(64)

Then the point process \( \sum_{i=1}^N \delta_{\lambda_i} \) converges in distribution to a Poisson point process with intensity \( \theta d\mu \) as \( N \to \infty \).

Proof. Note that the Poisson point process \( M \) with intensity \( \theta d\mu \) is characterized, among random random Radon measures on \( \mathcal{X} \), by the fact that for any compactly supported continuous function \( f \) on \( \mathcal{X} \), we have

\[
\mathbb{E} e^{\langle M, f \rangle} = \exp \left( \theta \int (e^{f(x)} - 1) d\mu(x) \right).
\]
So let us fix $f$ a compactly supported continuous function on $X$. Then, with the convention $R^{(N)}_0 = 1$,

$$
\mathbb{E} e^{\sum_{i=1}^N f(\lambda_i)} = \mathbb{E} \prod_{i=1}^N \left( 1 + (e^{f(\lambda_i)} - 1) \right)
= \sum_{P \subset \{1, \ldots, n\}} \mathbb{E} \prod_{i \in P} (e^{f(\lambda_i)} - 1)
= \sum_{k=0}^N \binom{N}{k} \mathbb{E} \prod_{i=1}^k (e^{f(\lambda_i)} - 1)
= \sum_{k=0}^N \frac{1}{k!} \int \prod_{i=1}^k (e^{f(x_i)} - 1) R^{(N)}_k(x_1, \ldots, x_k) d\mu^\otimes k(x_1, \ldots, x_k)
$$

This proves the proposition. □

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