Mechanics of a Particle in a Gauge Field

by

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Abstract

Classical motion of a charged particle in an electro-magnetic field is described by the Lorentz equation in terms of the field components. The field is defined by the infinitesimal gauge group elements associated with closed curves. The Lorentz equation may be derived from the action principle with an appropriate Lagrangian. The Lagrangian may also be used to associate group elements with curves in the space-time manifold. The action principle is shown here to be an equivalence relation between the infinitesimal elements so defined for a collection of closed curves and the identity element. This suggests a natural extension to require the equivalence of global elements with the identity and by considering all curves. The resulting equation, in addition to providing an extension of the Lorentz equation, also admits a straightforward generalization to non-Abelian gauge fields. The extended equation has an infinite number of trajectories as solutions. The properties of these paths are shown to impart wave-like properties to the particles in motion. In view of these results, the motion of a particle is formulated within the framework of the path integral formalism which yields a generalized Schrödinger type equation in a general gauge field. As a further implication of the properties of the trajectories assigned to a particle, this equation is shown to reduce to a set of equations, one of them being the Klein-Gordon equation.
1. Introduction

The notion of gauge transformation was introduced by Weyl [1,2] initially in an attempt to develop a unified theory of electro-magnetic and gravitational fields. In spite of a lack of success in achieving the original aim, the formulation has evolved into a central principle in the description of fields acting on microscopic particles (for an extensive bibliography, see ref.[3]). Weyl’s proposal provides, essentially, a mechanism to associate an element of a one dimensional group $G$ with an arbitrary curve $\rho(AB)$ joining the point $A$ to $B$ in the space-time manifold $M$. An infinitesimal group element associated with the displacement from $x$ to $(x + dx)$ is defined by $U_{(x+dx)x} = (1 + \alpha \phi_\mu dx^\mu)$, where $\phi_\mu$, $\mu = 0, 1, 2, 3$, are functions on $M$ and $\alpha$ is a constant. The elements of the type $U_{BA}(\rho)$ associated with $\rho(AB)$ are computed by repeated multiplications of the infinitesimal elements. A scalar function $\Phi_{BA}(\rho)$ may now be associated with $\rho(AB)$ by

$$\Phi_{BA}(\rho) = U_{BA}(\rho)\Phi_A \quad (1)$$

where $\Phi_A = \Phi_{AA}$ is given. The group elements associated with infinitesimal closed curves, $\rho_c$, define a gauge field with components $f_{\mu\nu}$ :

$$U_{ADCBA}(\rho) = \left(1 + \alpha f_{\mu\nu}d\sigma^{\mu\nu}\right) = \left(1 + \alpha v_A(d\sigma)\right) \quad (2)$$

where $d\sigma$ is the area enclosed by $\rho_c(ABCDA)$.

Although the concepts introduced above are valid with any $\alpha$, Weyl assumed it to be a real, non-zero constant. This assumption leads to a non-compact group. Also, $\Phi_{BA}(\rho)$ was interpreted as the length at $B$ of a ‘rigid’ measuring rod transported along $\rho(AB)$ provided that the length at $A$ is $\Phi_A$. The Jacobi identity satisfied by $f_{\mu\nu}$ led Weyl to conclude that the functions $\phi_\mu$ may be identified with the electro-magnetic potentials. In an attempt to relate Weyl’s formulation to the formalism of quantum mechanics, London [4] arbitrarily set $\alpha = i$ in natural units, leading to the description of electro-magnetism in terms of the group $U(1)$. Weyl later accepted the $U(1)$ description, and satisfactorily described the interaction of a charged particle with an electro-magnetic field by requiring the field-free quantum mechanical equations to be gauge covariant. This results in replacing $\partial_\mu$ by $(\partial_\mu - i\phi_\mu)$,
the gauge covariant derivative in trivial coupling with $U(1)$ as fibre. The procedure was later extended to include non-Abelian fields [5]. This coupling scheme assumes the availability of the field-free equations which are obtained by quite independent considerations. In addition to the generalization of the coupling scheme, Weyl's original construction described by (1) and (2), has also been extended to include an arbitrary differentiable manifold $\mathcal{M}$ and non-Abelian fields by letting $\phi_\mu = \phi^i_\mu X_l$ where $(\alpha X_l)$, $l = 1, 2, \ldots, n$, are the generators of a Lie group $\mathcal{G}$ [6]. In an irreducible matrix representation of an $n$-dimensional $\mathcal{G}$, $X_l$ are $n \times n$ matrices. The identity element will still be denoted by 1.

The motion of a particle coupled to an electro-magnetic field is described in terms of the gauge-field components $f_{\mu\nu}$, by the Lorentz equation (see e.g., ref.[7]):

$$m\dot{u}_\mu(s) = f_{\mu\nu}u^\nu$$

where $u_\mu = \dot{x}_\mu(s)$ and the dot denotes the derivative with respect to the indicated argument which in the case of (3) is the arc-length. The infinitesimal arc-length is given by $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, where $g^{00} = 1, g^{\mu\nu} = -1, \mu = 1, 2, 3$, and $g^{\mu\nu} = 0$ otherwise. Here $x_0 = x^0$ denotes the time. The analogue of (3) for non-Abelian fields [8] also describes the motion in terms of the gauge-field components, which in group theoretical terms, are associated with the infinitesimal elements corresponding to closed curves as defined by (2). A description in terms of the global elements should be expected to be more complete as they contain more information than their infinitesimal constituents.

Motion of a particle has been described in terms of the gauge group elements for the $n = 1$ case [9]. In this scheme, (3) is interpreted in terms of the group elements associated with a class of infinitesimal closed curves. While the field components are already expressed in this manner by (2), a representation of the complete equation requires an additional construction. With such a representation available, an extension is obtained by an appropriate replacement of the elements. In the process, the restriction imposed by the action principle on the curves becomes redundant which is, therefore, dropped. Although guided by the present interpretation of (3), the extension termed the gauge mechanical principle in itself forms a basic assumption of the formulation. In this development, the value of $\alpha$ is initially allowed to be an arbitrary complex constant. However, if the extended equation is to have any meaningful solutions, then $\alpha$ must be purely imaginary, providing a
derivation of its value. With $\alpha$ purely imaginary, the equation has an infinite number of trajectories as solutions termed the ‘physical paths’. The physical paths that a particle is allowed to follow are shown to possess wave-like coherences that impart similar properties to a moving particle. The wave-like behaviour of particles and the multiplicity of allowed paths form the basic assumptions of Feynman’s path integral formulation [10,11]. Therefore, the procedure of the path integral formalism is used to derive a five-dimensional generalized Schrödinger type equation. This equation was initially conjectured by Stückelberg [12,13] to provide a more satisfactory description of a relativistic particle with integral spin than the existing equations. In order to base it on a more solid foundation, the equation has been derived from a variational principle [14,15]. The present treatment provides a systematic derivation of this equation [9]. Thus the formulation provides a deductive formalism for these results which have been considered more satisfactory on intuitive grounds.

While the above extension is formulated without any restriction on the curves, not all conceivable trajectories are the solutions of the resulting equation. In addition to playing a crucial role in the derivation of some of the above results, the properties of the solutions lead to another result that was previously conjectured, as described below. In addition to accepting the conjecture of Stückelberg, Feynman [16] arbitrarily selected some periodic solutions of the generalized Schrödinger equation in order to deduce the Klein-Gordon equation. The properties of the particle paths implied by the present principle lead to a boundary condition on the five dimensional equation derived by the present methods which in all other respects is the same as Stückelberg’s generalized Schrödinger equation. As a result of the boundary condition, the solutions of this equation are confined to a collection of periodic functions. This enables one to decompose the five dimensional equation into a set of four dimensional equations, one of them being the Klein-Gordon equation. These results provide further justification for the use of the extended principle in studying the motion of particles.

In the present article the original formalism for the $n = 1$ case [9] is generalized to include the non-Abelian gauge fields. While the generalization is formally quite straightforward, a lack of commutability poses some technical difficulties, requiring some improvement in the previous techniques. In Sec.2, the classical variational principle is expressed in group theoretical terms. This provides a generalization of the corresponding result of Vatsya [9] to include
a larger class of Lagrangians and the associated Euler-Lagrange equations. A representation of (3) is obtained as a special case. In Sec.3, guided by the form of this representation, the gauge mechanical principle is formulated for a general gauge field. Relevant properties of the solutions of the extended equation are studied and used to describe the motion of a free particle, the double-slit experiment, and the Aharonov-Bohm effect. In Sec.4, an equation of motion for a particle coupled to an arbitrary gauge field is derived. In spite of a loss of commutability, the equation is remarkably similar to the generalized Schrödinger equation with the boundary condition for the \( n = 1 \) case.

2. The Variational Principle

In this section, some standard results of the variational calculus are expressed in group theoretical terms. This leads to a formulation of the classical action principle and the corresponding Euler-Lagrange equation that is well suited for its extension.

Let \( L'(\dot{x}, x, \tau) \) be a Lagrangian defined on curves in \( M \). For a path \( \rho(AB) = x(\tau) \) with \( x(\tau_1) = A, x(\tau_2) = B \), the action functional \( S'_{BA}(\rho) = S'(\tau_1, \tau_2) \) is given by

\[
S'(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} L'(\dot{x}, x, \tau) d\tau
\]

(4)

Without loss of generality, one may take a point A' as the reference point and set \( x(0) = A' \). By letting \( x(\tau) = B' \) a variable point, one obtains \( S'_{B'A'}(\rho) = S'(0, \tau) \) for the path \( \rho(A', B') \), which defines a one parameter group with elements

\[
U'_{B'A'}(\rho) = \exp(\alpha S'_{B'A'}(\rho)).
\]

(5)

The element infinitesimally close to the identity for a displacement \( dx \) will be denoted by \( (1 + \alpha v'_{A'}(dx)) \), which is obtained by retaining terms up to the first order in \( dx \) in the expansion of \( U'_{B'A'}(\rho) \). Corresponding Weyl’s scalar \( \Phi'_{B'A} \) may now be defined as \( \Phi_{B'A} \) above, i.e., \( \Phi'_{B'A'}(\rho) = U'_{B'A'}(\rho)\Phi'_{A'} \) where \( \Phi'_{A'} \) is given.

Some conceptual clarity is gained in describing the variational principle by considering the analogue of \( x(\tau) \) in \( M' \) obtained from \( M \) by including \( \tau \) among the co-ordinate variables (see e.g., ref[17], ch.1). Thus the curve \( x(\tau) \)
in $\mathcal{M}$ corresponds to the set of points $(x(\tau), \tau)$ in $\mathcal{M}'$, eliminating a need for an explicit reference to the parameterization. A metrical structure on $\mathcal{M}'$ is not needed for this purpose. For convenience, the curves in $\mathcal{M}'$ will also be denoted by $\rho$ and $\rho_\lambda$ with an indication whenever needed. It is pertinent to remark here that a curve closed in $\mathcal{M}'$ is described by a double valued function $x(\tau)$. Also, Weyl's construction may be expressed in terms of the curves in $\mathcal{M}'$ by replacing $dx^\mu$ by $\dot{x}^\mu d\tau$ without altering the results.

The Lagrangian $L'(\dot{x}, x, \tau)$ may be treated as a member of the family $L'(\dot{x} + \lambda \dot{y}, x + \lambda y, \tau)$ defined on curves $\rho_\lambda = (x(\tau) + \lambda y(\tau))$ where the values $\tau_1, \tau_2$ still correspond to the points $A, B$ respectively. This defines a family of actions $S'_{\lambda ABA}(\rho_\lambda) = S'(\tau_1, \tau_2, \lambda)$ obtained by substituting $L'(\dot{x} + \lambda \dot{y}, x + \lambda y, \tau)$ for $L'(\dot{x}, x, \tau)$ in (4). Let $\rho_\lambda(ABA)$ be the union of $\rho_\lambda(AB)$ and $\rho_\lambda(BA)$. The action $S'_{\lambda ABA}(\rho_\lambda)$ on the curve $\rho_\lambda$ is given by

$$S'_{\lambda ABA}(\rho_\lambda) = \begin{cases} S'(\tau_1, \tau_2, 0) + S'(\tau_1, \tau_2, \lambda) \\ S'(\tau_1, \tau_2, 0) - S'(\tau_1, \tau_2, \lambda) \end{cases}$$

(6)

It is clear that $\rho_\lambda$ is closed in $\mathcal{M}'$ and hence in $\mathcal{M}$. By definition, the corresponding group element given by

$$U'_{\lambda ABA}(\rho_\lambda) = \exp(\alpha S'_{\lambda ABA}(\rho_\lambda))$$

(7)

is defined in terms of the two parameters, $\tau$ and $\lambda$. The infinitesimal element $(1 + \alpha v'_BA(d\sigma))$ is obtained by retaining terms up to the first order in $\lambda$, equivalently in $d\sigma$, i.e., $v'_BA(d\sigma)$ is the first term in the expansion of $S'_{\lambda ABA}(\rho_\lambda)$.

The above construction for closed curves is closely related to the variational principle as follows. It is clear from (6), that the deformation of $x(\tau)$ by $\delta x(\tau) = \lambda y(\tau)$ results in the change $\delta S'_{\lambda ABA}(\rho_\lambda) = S'_{\lambda ABA}(\rho_\lambda)$ in the action. Hence, from (7), $\delta S'_{\lambda ABA}(\rho_\lambda) = 0$ is equivalent to $U'_{\lambda ABA}(\rho_\lambda) = 1$. The variational principle requires the equality $\delta S'_{\lambda ABA}(\rho_\lambda) = 0$ to hold only up to the first order in $\lambda$, as it varies in an arbitrary but small neighbourhood of zero, and $y$ is any given, reasonably smooth function. Since $v'_BA(d\sigma) = S'_{\lambda ABA}(\rho_\lambda)$ is $\delta S'_{\lambda ABA}(\rho_\lambda)$ up to the first order in $\lambda$, the variational principle is equivalent to $v'_BA(d\sigma) = 0$ for all curves closed in $\mathcal{M}'$ joining the points $(A, \tau_1)$ and $(B, \tau_2)$ with sufficiently small $d\sigma$. The curves $\rho_\lambda(AB)$ and $\rho_\lambda(AB)$ are usually referred to as the virtual paths and the action associated with the configuration is significant only to the extent that it yields the classical trajectory. This construction may be characterized by more elementary curves, as shown below.
For a given point \( x \) on \( \rho(AB) \), let \( \rho_\lambda(x, x + dx, x) \) be a curve closed in \( \mathcal{M}' \) obtained as \( \rho_\lambda(ABA) \). As \( dx \) tends to zero, it is sufficient to retain terms up to the first order in \( dx \) in defining the corresponding infinitesimal group element \( (1 + \alpha v'_z(d\sigma)) \). Since an arbitrary curve \( \rho(ABA) \) may be expressed as a union of the curves of the type \( \rho_z(x, x + dx, x) \), we have that \( \delta S'_{BA}(\rho_\lambda) = S'_{ABA}(\rho_\lambda) = \sum v'_z(d\sigma) \) where \( \sum \) denotes the sum over all of these curves. This is sufficient to conclude that an element \( U'_{ABA}(\rho_\lambda) \) may be expressed as a product of infinitesimal elements of the type \( (1 + \alpha v'_z(d\sigma)) \). This implies that if \( v'_z(d\sigma) = 0 \) for each \( x \) on \( \rho(AB) \), and each \( \rho_\lambda(x, x + dx, x) \), then \( v'_z(d\sigma) = 0 \) for each \( \rho_\lambda(ABA) \). The converse is obvious as \( \{ \rho_\lambda(ABA) \} \) includes the set \( \{ \rho_\lambda(x, x + dx, x) \} \). Thus the variational principle is equivalent to \( (1 + \alpha v'_z(d\sigma)) = 1 \) as long as the equation is required to hold for all curves of the type \( \rho_\lambda(x, x + dx, x) \).

We have shown above that setting \( \delta S'_{BA}(\rho_\lambda) = 0 \) up to the first order in \( \lambda \) with an arbitrary \( y(\tau) \) is equivalent to requiring \( (1 + \alpha v'_z(d\sigma)) = 1 \) for all arbitrary curves of the type \( \rho_\lambda(x, x + dx, x) \). This condition yields the Euler-Lagrange equation of the variational problem, if it has a solution. The Euler-Lagrange equation is satisfied if and only if the action is stationary. The classical action principle assigns the solution \( \rho_\delta(AB) \) of this equation to a particle. The action \( S'_{BA}(\rho_\delta) = S'(A, B, \tau_1, \tau_2) \) along \( \rho_\delta(AB) \), termed Hamilton’s principal function, depends only on the end points of the curve and the parameter \( \tau \). The principal function may be obtained as a solution of the associated Hamilton-Jacobi equation. While the elements \( U'_{BA}(\rho) \) and \( v'_z(\rho) \) may be computed directly from equations (4) to (7), they may also be constructed in terms of the action \( S'(A, B, \tau_1, \tau_2) \) as follows. In addition to providing a clear expression for the value of the Lie algebra element \( v'_z(dx) \), quite frequently this construction is more convenient, especially if only a few terms in the expansion of the group elements are needed.

Let \( \rho_\lambda \) be the union of \( \rho_\delta(x, x + dx) \) and \( \rho_\lambda(x + dx, x) \) for a small value of \( \lambda \) and an arbitrary \( y(\tau) \). Also, let \( (1 + \alpha v'_z(d\sigma)) \) be the infinitesimal group element associated with the closed curve \( \rho_\lambda \). Since the action \( S'(x, x + dx, \tau, \tau + d\tau) = dS'(x, \tau) \) is stationary under the deformation \( \rho_\lambda(x, x + dx) \), equivalently \( v'_z(d\sigma) = 0 \), we have that \( S'_{x+dx,\tau}(\rho_\lambda) = S(x, x + dx, \tau, \tau + d\tau) \) up to the first order in \( \lambda \). Therefore the infinitesimal group element \( (1 + \alpha v'_z(dx)) \) for an arbitrary deformation of \( \rho_\delta(x, x + dx) \) is given by \( (1 + \alpha dxS'(x, \tau)) \) where the terms only up to the first order in \( dx \) have to be retained. Since the global group element \( U'_{BA}(\rho) \) associated with \( \rho(AB) \)
may be obtained as a product of elements of the type \((1 + \alpha v'(dx))\), it may be computed by multiplying the elements \((1 + \alpha dS'(x, \tau))\), yielding

\[
U'_{BA}(\rho) = \exp(\alpha \int_{\rho(AB)} dS'(x, \tau))
\]  

This construction evaluates \(S'_{BA}(\rho)\) by expressing it as a sum. Each term of the sum is the Hamilton’s principal function for points \(x\) and \((x + dx)\) on \(\rho(AB)\). The sum approaches \(S'_{BA}(\rho)\) as \(dx\) tends to zero.

As a prelude to applying the above results to the Lorentz equation, consider the classical motion of a free particle which is usually described by the homogeneous Lagrangian \(L' = L^0(\dot{x}, x, s) = m\sqrt{\dot{x}_{\mu}(s)\dot{x}^\mu(s)}\). There is some difficulty with a straightforward application of the Hamilton-Jacobi construction in case of a homogeneous Lagrangian (see e.g., ref.[17], ch.3). One procedure to circumvent the difficulty is to take \(L' = L^P = \frac{1}{2}m(\dot{x}_{\mu}(\tau)\dot{x}^\mu(\tau) + 1)\) with \(\tau\) an independent parameter and after the construction, set \(\tau = s\), equivalently \(\dot{x}_{\mu}\dot{x}^\mu = 1\) [18,19]. Let the elements \(U'_{BA}(\rho)\) and \(v'_x(d\sigma)\) obtained by substituting \(L^P\) for \(L'\) in (4), (5) and (7) be denoted by \(U^P_{BA}(\rho)\) and \(v^P_x(d\sigma)\) respectively. It is clear from the above that the classical motion of a free particle in a neighbourhood of \(x\) is described by \(v^P_x(d\sigma) = 0\) for all closed curves of the type \(\rho_c(x, x + dx, x)\).

The Lorentz equation, (3), may be obtained as the Euler-Lagrange equation of the variational principle with the Lagrangian \(L' = L^0 - \phi_{\mu}\dot{x}^\mu(s)\) [7]. However, for the same reason as with the free particle, the Lagrangian \(L' = L^P - \phi_{\mu}\dot{x}^\mu(\tau)\) is preferable [18,19]. With this choice of \(L'\) in equations (4) to (7), the elements are defined by \(U^P_{BA}(\rho) = U'_{BA}(\rho)U^P_{BA}(\rho)\) and \(v^P_x(d\sigma) = (v^P_x(d\sigma) - v_x(d\sigma))\), where \(U_{BA}(\rho)\) and \(v_x(d\sigma)\) are as defined by (1) and (2). Thus the Lorentz equation is implied by, and implies, the condition \(v^P_x(d\sigma) = 0\), i.e., \(v^P_x(d\sigma) = v_x(d\sigma)\) for all curves of the type \(\rho_c(x, x + dx, x)\). This shows that the classical motion of a charged particle in an electromagnetic field is characterized by the equality \((1 + \alpha v^P_x(d\sigma)) = (1 + \alpha v_x(d\sigma))\), between the infinitesimal group elements. This equality is required to hold for all curves of the type \(\rho_c(x, x + dx, x)\).

For later reference, the Weyl’s scalar associated with \(U^P_{BA}(\rho)\) will be denoted by \(\Phi^P_{BA}\). For \(n > 1\), \(U^P_{BA}\) will be assumed to be multiplied by the identity without an explicit indication and \(\Phi^P_{BA}(\rho)\) will be taken to be an \(n\)-vector. This makes \(U^P_{BA}(\rho)\) and \(\Phi^P_{BA}(\rho)\) compatible with \(U_{BA}(\rho)\) and \(\Phi_{BA}(\rho)\).
respectively. Hamilton’s principal function will not be needed for any Lagrangian other than $L^P$. In the following, $S(\cdots)$ and $dS(\cdots)$ will denote this function and its infinitesimal value, respectively.

3. Physical Paths

The characterizations of the variational principle and the Lorentz equation described in Sec.2 indicate that a description of motion within the framework of the classical action principle is incomplete in group theoretical terms. One of the inherent limitations is due to the equality $(1 + \alpha \nu_x(d\sigma)) = (1 + \alpha \nu_x(d\sigma))$ that equates the corresponding global elements only up to the first order in $d\sigma$. Additional information that may be available in the global elements is not utilized in the action principle. Also, the equivalence is required to hold for the curves of the type $\rho(x, x + dx, x)$, i.e., only the curves with closed images in $\mathcal{M}'$ which imposes an additional restriction on the solutions. In this section, we extend the classical treatment by requiring, essentially, the equivalence of $U_{BA}^P(\rho)$ and $U_{BA}(\rho)$ and by considering all curves in $\mathcal{M}$, which is a natural extension by completion of the above equality and thus of the action principle. An explicit introduction of the parameter is more convenient for some of the following treatment which will be used accordingly. With appropriate construction of the group elements, closed curves are included in the collection $\{\rho_{AB}\}$.

3.1 The gauge mechanical principle. A group element $U_{BA}(\rho)$ for a general gauge field is given by the path-ordered exponential $\text{Exp}(\cdot)$ [20]:

$$U_{BA}(\rho) = \text{exp} \left( \alpha \int_{\rho(AB)} \phi_\mu(x) dx^\mu \right)$$

$$= \sum_{l=0}^{\infty} \alpha^l \int_0^1 d\tau_1 \cdots \int_0^{\tau_{l-1}} d\tau_l \left( \hat{\gamma}_1^{\mu_1}(\tau_1) \cdots \hat{\gamma}_l^{\mu_l}(\tau_l) \right)$$

$$\times \phi_{\mu_1}(\gamma(\tau_1)) \cdots \phi_{\mu_l}(\gamma(\tau_l))$$

(9)

where $\gamma(\tau_j)$ is a parameterization of $\rho(AB)$ with $\gamma(0) = A$ and $\gamma(1) = B$. If $\{X_i\}$ forms a commuting set, then $\text{Exp}(\cdot)$ reduces to the ordinary exponential. The elements $U_{BA}^P(\rho)$ are given by (8):

$$U_{BA}^P(\rho) = \exp \left( \alpha \int_{\rho(AB)} dS(x, \tau) \right)$$

(10)
As explained above, Weyl’s vectors $\Phi^P_{BA}(\rho)$ and $\Phi_{BA}(\rho)$ are defined by $U^P_{BA}(\rho)$ and $U_{BA}(\rho)$ with their values $\Phi^P_A$ and $\Phi_A$ at $A$ being given, by appropriate substitutions in (1).

Weyl’s vectors $\Phi_A$ and $\Phi^P_A$ may be assumed to be related by $\Phi^P_A = \kappa(A)\Phi_A$ where $\kappa(A)$ is an invertible matrix depending on the physical properties of the system at $A$. A particle coupled to a gauge field will be assumed to follow the paths defined by

$$\Phi^P_{BA}(\rho) = \kappa(B)\Phi_{BA}(\rho)$$

(11)
i.e., a path $\rho(AB)$ is allowed (physical) if and only if there are (physical) points $A$ and $B$ and a trajectory $\rho$ joining them such that (11) is satisfied.

Eq.(11) is essentially the statement of the gauge mechanical principle which provides the present extension of the classical description of motion. Its alternative characterizations and necessary explanations are given below.

In view of (10), $U^P_{BA}(\rho)$ is a constant multiple of the identity matrix and hence commutes with all $n \times n$ matrices. Consequently (11) is equivalent to

$$V_{BA}(\rho)\Phi^P_A = \Phi^P_A$$

(12)

where $V_{BA}(\rho) = \kappa(B)V'_{BA}(\rho)\kappa^{-1}(A)$ with $V'_{BA}(\rho) = [U^P_{BA}(\rho)]^{-1}U_{BA}(\rho) = U^P_{AB}(\rho)U_{BA}(\rho)$. Eqs. (11) and (12) may also be expressed as

$$V'_{BA}(\rho)\Phi_A = \kappa^{-1}(B)\kappa(A)\Phi_A$$

(13)

As indicated by (12), the present assumption stated in (11), is essentially a requirement of an equivalence between $V_{BA}(\rho)$ and the identity element. Similarly (13) is a statement of equivalence between $V'_{BA}(\rho)$ and the identity. It follows from the definitions that both of these are equivalent to an equivalence between $U^P_{BA}(\rho)$ and $U_{BA}(\rho)$. This equivalence does not reduce to a strict equality even for the Abelian case. For the Abelian groups, as is the case with $n = 1$, and for closed curves $\rho_c(ABA)$, this does reduce to a strict equality. Classical action principle was characterized in Sec.2 by a strict equality for general curves. However, only an equivalence can be inferred for general curves from an equality for the closed ones.

Eqs. (12) and (13) are eigenvalue equations, implying that their solutions, and hence a physical path is independent of the magnitudes of $\Phi^P_A$ and $\Phi_A$. Therefore, $\Phi^P_A$ and $\Phi_A$ may be assumed to be normalized in the complex $n$-vector space $l^2(\mathbb{C}^n)$. This restricts $\kappa$ to the class of norm-preserving, i.e.,
unitary matrices. It will be seen below that a real $n$-vector space is too restrictive to describe the physical paths. Thus, further restrictions on $\Phi_A$, $\Phi_A^P$ and $\kappa$ are not possible. The matrix $\kappa(A)$ embodies the properties of the physical system under consideration at $A$, which should remain unchanged if $\phi_\mu = 0$. Therefore it will be assumed that if $\phi_\mu = 0$, for each $\mu$ and all paths in a set large enough to include physical paths joining $A$ and $B$, then $\kappa(A) = \kappa(B)$.

The constant $\alpha$ in the above is as yet undetermined. Its value was determined in ref.[9] by considering the motion of a charged particle in an electromagnetic field. The same result is obtained by considering the motion of a free particle, as follows.

For a free particle, i.e., in the limit of $\phi_\mu = 0$, the physical paths are defined by
\[
\exp (-\alpha S_{BA}(\rho)) \Phi_A^P = \Phi_A^P
\]
(14)

Since $S_{BA}(\rho)$ is in general real, it follows that $\alpha$ must be purely imaginary. The trivial solution $\alpha = 0$ is excluded for it implies that neither the gauge field nor the geometry of $\mathcal{M}$ has any effect on the motion.

Present formulation does not yield the magnitude of $\alpha$ which must be determined experimentally. The elements of the type $U_{BA}(\rho)$ with an imaginary value of $\alpha$ are called the phase-factors and the resulting group is compact. By comparing with the usual definition of the phase-factors that conforms with the experimental observations, $\alpha = 2\pi i/h$ where $h$ is Planck’s constant. Thus, $\alpha$ may be set equal to $i$ by selecting units with $h = 2\pi$. This identification will be seen to be consistent with the rest of the treatment to follow. Natural units are used elsewhere also.

Solutions of (12) are identified by their equivalence classes as follows. Let $\{B_j\}$ be a set of points on $\rho(AB)$ such that $V_{B_jA}(\rho) \Phi_A^P = \Phi_A^P$. If one member of $\{B_j\}$ is a physical point with respect to $\{\rho(AB), \Phi_A^P\}$, then this is also the case for each $j$. Thus the equivalence class $\{B_j\}$ so defined characterizes the solutions $\{\rho(AB_j)\}$. A natural order is defined on $\{B_j\}$ by setting $B_j$ to be the $j$th closest member to $A$. Let $\{B_j^k\}$, $k = 1, 2, ..., n$ be such ordered equivalence classes with respect to $\{\rho(AB^k), \Phi_A^P\}$. The set $\zeta_j = \{B_j^k\}$ defines a physical ‘surface’ for each $j$.

For a free particle, the physical paths are the solutions of (14) which
reduces to
\[ \exp \left( -im \int_{\rho(AB)} u_\mu dx^\mu \right) \Phi^P_A = \Phi^P_A \]  
(15)
The equivalent points \( \{B_j\} \) on these curves satisfy
\[ m \int_{\rho(B_j,B_{j+1})} u_\mu dx^\mu = 2\pi \]
Along the paths characterized by a constant velocity \( \bar{u} \), \( B_j \) and \( B_{j+1} \) are thus separated by the de Broglie wavelength \( 2\pi/m\bar{u} \) and the length of a physical path is its integral multiple.

Consider a source-detector system with source at \( A \) and detector at \( B \). A curve \( \rho(AB) \) will be called monotonic if the parameter value increases or decreases monotonically along the curve. By convention, \( \tau \) will be assumed to increase from \( A \) to \( B \). A particle starting at \( A \) and confined to \( \rho(AB) \) is observable at \( B \) if and only if \( \rho(AB) \) is physical. If \( \theta \) is the intensity associated with \( \rho(AB) \) at \( A \) then the intensity transmitted to \( B \) by this path must be equal to \( \theta \).

A union of physical paths is obviously physical. Also a union of non-physical monotonic curves can be physical. For example, let \( \rho(AB) \) be a monotonic physical path with the associated physical points \( \{B_j\} \) and let \( C \) be a point in the interior of \( \rho(B_j,B_{j+1}) \). Then the union of \( \rho(B_jC) \) and \( \rho(CB_{j+1}) \) is \( \rho(B_jB_{j+1}) \) and the union of \( \rho(B_{j-1}C) \) and \( \rho(CB_j) \) is \( \rho(B_{j-1}B_j) \), both of which are physical. However, these trivial constructions are redundant as they are indistinguishable from the paths of the type \( \rho(B_kB_{k+1}) \). A significant, non-trivial class of such paths is described below.

Consider a configuration of two curves \( \rho(AB) \) and \( \rho'(AB) \) with \( \rho'(ABA) \) being the union of \( \rho'(AB) \) and \( \rho(BA) \), i.e., the type of the virtual paths encountered in the action principle. In classical mechanics, these curves serve only to define the classical trajectory. According to the present prescription, if (11) is satisfied then this is a physical configuration. In that case, \( \rho(AB) \) and \( \rho'(AB) \) offer equally likely alternatives for the transmission of a particle from \( A \) to \( B \), even if \( \rho(AB) \) and \( \rho'(AB) \) may not be physical. The case of the alternatives of the type \( \rho(AB) \) and \( \rho'(CB) \) is treated similarly. To be precise, let the parameter value at \( B \) be \( \tau_B \). According to the above convention, \( \tau \) increases from \( C \) to \( B \) along \( \rho'(CB) \) and decreases from \( B \) to \( A \) along \( \rho(BA) \). The group elements associated with such configurations may be computed
by integrating along $\rho'(CB)$ and then along $\rho(BA)$. If $V_{ABA}(\rho'')\Phi_A^P = \Phi_A^P$, where $\rho''$ is the union of $\rho'(CB)$ and $\rho(BA)$, then $\rho(AB)$ and $\rho'(CB)$ offer likely alternatives. Such configurations of trajectories are referred to as the interfering alternatives. The intensity of particles transmitted to $B$ by the equally likely alternatives must be equal to the sum of the intensities at $A$ and $C$ associated with the respective trajectories. A separate treatment of such configurations is not necessary for a general theory as they are represented by well defined curves in $M'$, but it provides clearer understanding of some physical phenomena described in the sequel.

As a prelude to a precise treatment of motion in Sec.4, an approximate description of a few phenomena is given in Secs.3.2, 3.3 and 3.4, which also clarifies the properties of a multiplicity of physical trajectories.

3.2 Motion of a free particle. Consider a physical system described by a Lagrangian $L'(\dot{x}, x)$ with $\rho_s$ being the resulting classical path. For convenience, it is assumed that $L'$ does not depend on $\tau$ explicitly. However, $\tau$-dependence may be included without a significant change in the following analysis. For a free particle, $L' = L^p$. For an undisturbed particle, the equivalent points on $\rho_s$ are given by

$$S'(B_j, B_{j+1}, \tau_j, \tau_{j+1}) = 2\pi$$

where $S'(\ )$ denotes Hamilton's principal function.

The action $S'_{BA}'(\rho')$ along a trajectory $\rho'(A'B)$ in a small neighbourhood of $\rho(AB)$ may be expressed as

$$S'_{BA}'(\rho') = S_{BA}(\rho) = \int_{\rho(AB)} \delta x^\mu \left[ \frac{\partial L'}{\partial x^\mu} - \frac{d}{d\tau} \frac{\partial L'}{\partial \dot{x}^\mu} \right] d\tau$$

by standard methods [21]. Here $\delta'x^\mu$, $\delta'\tau$ correspond to the variation of the end points $A$, $B$ to $A'$, $B'$, and $H$ is the Hamiltonian. The term $O(\delta^2)$ is the integral along $\rho(AB)$ of $J$, containing functions of second or higher order in $(\delta x)$ and $(\delta \dot{x})$.

If $\rho = \rho_s$, then the first term on the right side of (16) is equal to zero. Hence $S'_{BA}'(\rho') = S_{BA}(\rho_s)$ for some values of $\delta'x = O(\delta^2)$. Therefore the trajectories in a $\delta x$ neighbourhood of a physical classical path $\rho_s(B_jB_{j+k})$ are
also physical and their end points are confined to \((\delta^2)\) neighbourhoods of \(B_j\) and \(B_{j+k}\). Thus the intensity transmitted by paths in a \(\delta x\) neighbourhood of a classical trajectory is concentrated in \((\delta^2)\) neighbourhoods of the equivalent points on \(\rho_s\). Let \(\rho\) be a path transmitting intensity outside \((\delta^2)\) neighbourhood of \(\{B_j\}\), i.e. \(\rho\) is outside \((\delta x)\) neighbourhood of \(\rho_s\). Since \(\rho\) is not a solution of the Euler-Lagrange equation, the first term in (16) dominates which is \(O(\delta x)\). Repeating the above argument, we have that the intensity transmitted by trajectories in a \(\delta x\) neighbourhood of \(\rho\) is spread over a \(\delta x\) neighbourhood of points outside \((\delta^2)\) neighbourhood of \(\{B_j\}\). Further, the magnitude of the first term in (16) increases as \(\rho\) is removed farther from the classical trajectory. Therefore the contribution to the intensity decreases accordingly. Some intensity is also transmitted by the interfering alternatives whose monotonic segments are non-physical. In a homogeneous space, such paths are roughly evenly distributed about the classical trajectory implying a uniform distribution of the associated intensity. The properties of these paths will be described in more detail in Sec.3.3 where their impact is greater.

Assuming that the particles originate in a small region about a point \(A\), intensity should be expected to be higher near the points equivalent to \(A\) and to decrease away from them, creating a wave-like pattern over a uniform background. On a classical scale, the segments between \(B_j\) and \(B_{j+1}\) are negligibly small. Also for macroscopic trajectories, the contribution of the first term in (16) is enormous as one moves away from a purely classical trajectory, owing to the large interval of integration. Therefore, the contribution to the variation of the intensity over a wavelength, between \(B_j\) and \(B_{j+1}\), must come from extremely small neighbourhoods of the the long trajectories, and from larger neighbourhoods of the shorter ones, which are still small on a classical scale. Thus on a macroscopic scale, the particles from \(A\) to \(B\) travel along narrow beams centered about the classical trajectories. A classical path has an uncertainty associated with it to this extent. In view of the above, the intensity should be concentrated in a small neighbourhood of \(B\) decreasing away from the central point.

3.3 The double-slit experiment. The interfering alternatives play a prominent role in the double slit experiment. The this setup, identical particles are allowed to pass through two slits at \(A\) and \(A'\), and collected on a distant screen at a point \(B\). As explained in Sec.3.2, the particle paths may be assumed concentrated about the classical trajectories from \(A\) to \(B\)
and from $A'$ to $B$. If one of the beams is blocked, then the intensity observed in a neighbourhood of $B$ should behave almost as deduced in Sec.3.2 for a free particle. However, if the intensity is transmitted by both of the beams, then a multitude of the interfering alternatives is allowed. Existence of such paths and their influence on the intensity distribution is studied next.

In view of the physical equivalence of $A$ and $A'$ and that of the particles, one has that $\kappa(A) = \kappa(A')$, $\Phi_A = \Phi_{A'}$, and hence $\Phi_A = \Phi_A'$. However, because of an interaction with the detecting instrument at $B$, $\kappa(B)$ may not be equal to $\kappa(A)$. For monotonic trajectories, this dislocates only one physical point about $B$, having no significant impact on the above conclusions. For the interfering alternatives, $\kappa(B)$ cancels out and the paths are the solutions of

$$\exp\left[i \left( \int_{\rho(AB)} dS(x, \tau) - \int_{\rho'(A'B)} dS(x, \tau) \right) \right] \Phi_A = \Phi_A$$

(17)

For the classical trajectories, $\rho = \rho_s$ and $\rho' = \rho'_s$, (17) is solved by

$$\left( S'_{BA}(\rho_s) - S'_{BA}(\rho'_s) \right) = 2\pi j$$

where $j$ is an arbitrary integer and the action in this case is Hamilton’s principal function or the arc-length in $\mathcal{M}$. Classical paths are characterized by a constant velocity $\bar{u}$. This reduces the solution to $\Delta r = 2\pi j/m\bar{u}$, where $\Delta r$ is the difference between the path-lengths of $\rho_s(AB)$ and $\rho'_s(A'B)$. Therefore $\rho_s(AB)$ and $\rho'_s(A'B)$ are interfering alternatives whenever $\Delta r = 2\pi j/m\bar{u}$.

Let $B(\varepsilon)$ be the point on the screen such that

$$\left( \left( S'_{BA}(\rho_s) - S'_{B(\varepsilon)A'}(\rho'_s) \right) = 2\pi(j + \varepsilon) \right)$$

(18)

for a fixed $j$ and each $0 \leq \varepsilon \leq 1/2$. In the following we study the variation of intensity as $\varepsilon$ varies in the prescribed interval which is sufficient to describe it on the entire screen.

It follows from the analysis of Sec.3.2, that $S'_{CA}(\rho) = S'_{B(\varepsilon)A}(\rho_s)$, $S'_{C'A'}(\rho') = S'_{B(\varepsilon)A'}(\rho'_s)$, for $\rho$, $\rho'$ in a $\delta x$ neighbourhoods of $\rho_s$, $\rho'_s$ respectively, where $C$ and $C'$ vary over a $(\delta^2)$ neighbourhood of $B(\varepsilon)$ on the screen for a fixed $\varepsilon$. Therefore, by varying the paths over a $(\delta x)$ width of the beam and over a $(\delta^2)$ neighbourhood of $B(\varepsilon)$ it is possible to satisfy

$$\left( S'_{DA}(\rho) - S'_{D'\rho'} \right) = 2\pi(j + \varepsilon)$$
for most of the paths. In fact cancellations favour this equality which can be easily seen, in particular for the cases when $\rho_s$, $\rho'_s$ are extremals as is presently the case. This conclusion is valid for other points in the vicinity of $A$ and $A'$ also. For $\varepsilon = 0$, this implies that there is a large concentration of interfering alternatives reaching about $B(0)$ and hence the intensity in a $(\delta^2)$ neighbourhood of $B(0)$ is almost equal to the intensity in $\delta x$ neighbourhoods of $\rho_s(AB(0))$ and $\rho'_s(A'B(0))$. For $\varepsilon \neq 0$, the configuration of the paths $\rho_s(AB(\varepsilon))$ and $\rho'_s(A'B(\varepsilon))$ is obviously non-physical. From the above argument, a large number of paths in $\delta x$ neighbourhoods of $\rho_s(AB(\varepsilon))$ and $\rho'_s(A'B(\varepsilon))$ are excluded from combining to form the interfering alternatives and hence unable to transmit the intensity in a $(\delta^2)$ neighbourhood of $B(\varepsilon)$. However, still there are many paths capable of transmitting intensity about $B(\varepsilon)$ for $\varepsilon \neq 0$, which are described below.

It follows from (16) that for trajectories $\rho(AB(\varepsilon))$, $\rho'(A'B(\varepsilon))$ in $\delta x$ neighbourhoods of $\rho_s(AB(\varepsilon))$, $\rho'_s(A'B(\varepsilon))$ respectively,

$$(S'_{B(\varepsilon)A}(\rho) - S'_{B(\varepsilon)A}(\rho_s)) = O(\delta^2)$$

and

$$(S'_{B(\varepsilon)A'}(\rho') - S'_{B(\varepsilon)A'}(\rho'_s)) = O(\delta^2)$$

We have used the fact that the first term on the right side of (16) is zero as the curves are varied about the classical trajectories and the second term is zero as the end points are kept fixed. For these curves, we have

$$(S'_{B(\varepsilon)A}(\rho) - S'_{B(\varepsilon)A'}(\rho')) = 2\pi j + O(\delta^2)$$

(19)

Since there are distortions for which $O(\delta^2)$ term is non-zero and its magnitude is large in natural units, it is possible to adjust the curves $\rho$, $\rho'$ such that

$$(S'_{B(\varepsilon)A}(\rho) - S'_{B(\varepsilon)A'}(\rho')) = 2\pi k$$

(20)

with $k = j$ or $(j+1)$, most likely $j$. This implies that $\rho(AB(\varepsilon))$ and $\rho'(A'B(\varepsilon))$ form a pair of interfering alternatives. Since $\rho(AB(\varepsilon))$, $\rho'(A'B(\varepsilon))$ are non-classical trajectories, it follows as in Sec.3.2 that while there is a multitude of paths satisfying (20), in $\delta x$ neighbourhoods of the central paths, their end points are spread over a $\delta x(\varepsilon)$ neighbourhood of $B(\varepsilon)$. This implies that the amount of intensity that is concentrated in a $(\delta^2)$ neighbourhood of $B(0)$ is
spread over a $\delta x(\varepsilon)$ neighbourhood of $B(\varepsilon)$. Consequently, a rapid decrease in the intensity is expected as $\varepsilon$ increases away from zero.

As $\varepsilon$ increases further, it is seen from (19) that the neighbourhood $\delta x$ must be increased to satisfy (20), i.e. $\rho(AB(\varepsilon))$, $\rho'(A'B(\varepsilon))$ must be moved farther away from the solutions of the Euler-Lagrange equations. Thus the magnitude of the first term on the right side of (16) integrated along $\rho(AB(\varepsilon))$, $\rho'(A'B(\varepsilon))$ increases as $\varepsilon$ increases for each fixed variation $\delta x$. As above, $O(\delta x(\varepsilon))$ increases with $\varepsilon$, implying a decrease in the intensity.

The above arguments also imply a symmetric intensity distribution as $\varepsilon$ is varied over the interval zero to $-1/2$, and a repeat of the pattern as $j$ is varied over the integers. Thus an interference pattern should be observed on the screen over a background of almost uniform but relatively low intensity as the major contributions have been estimated here.

Similar arguments may be used to estimate the variations in the intensity about peaks as $j$ varies, resulting in a decrease in the intensity as $|j|$ increases. This result is based on the fact that the term $O(\delta^2)$ for each $j$, may be expressed as a sum of two terms, one being $j$-independent and the other, directly proportional to $|j|$.

Availability of two interfering beams originating at $A, A'$ and the equivalence of the physical conditions at these points have played a crucial role in the above analysis. As explained before, if one of the beams is blocked, the interference pattern is destroyed. Also, such a distribution should not be expected to result if the equivalence of $A$ and $A'$ is violated.

Above considerations indicate a wave-like behaviour of microscopic particles observed macroscopically as a collection while behaving as particles individually. This is in agreement with the observed behaviour [11, pp. 2-5]. These results obtained here from (11), are known to inspire the formalism of quantum mechanics.

3.4 The Aharonov-Bohm effect. Additional insight into the behaviour of the particles as implied by the present extension may be gained by considering their response to a non-zero gauge field, as follows. The phase-factor obtained from (9) by replacing $\phi_\mu$ by $\hat{\phi}_\mu$ will be denoted by $\hat{U}_{BA}(\rho)$. Let $\{\rho\}$, $\{\hat{\rho}\}$ be the collections of the solutions of (11), equivalently, the solutions of (12) with $\phi_\mu$, $\hat{\phi}_\mu$ respectively. Assume that $U_{BA}(\rho) \neq \hat{U}_{BA}(\rho)$ for a solution $\rho(AB)$. If $U_{BA}(\rho)$ is replaced by $\hat{U}_{BA}(\rho)$ in (11), then $\rho(AB)$ is no longer a solution. The same conclusion holds for a path $\hat{\rho}(A'B')$. Thus,
if the inequality holds for some of the solutions of (11) with \( \phi_\mu \), or with \( \hat{\phi}_\mu \), then the collections \( \{ \rho \} \), and \( \{ \hat{\rho} \} \) of the physical paths are not identical. Therefore a change of potentials from \( \phi_\mu \) to \( \hat{\phi}_\mu \) should in general produce an observable effect. However, if \( U_{BA}(\rho') = \hat{U}_{BA}(\rho') \) for each \( \rho'(AB) \) in a collection \( \{ \rho' \} \) large enough to include the union of \( \{ \rho \} \) and \( \{ \hat{\rho} \} \), then (12) remains the same equation under the change from \( \phi_\mu \) to \( \hat{\phi}_\mu \). Consequently, a change of potential from \( \phi_\mu \) to \( \hat{\phi}_\mu \) would not change the solutions \( \{ \rho \} \). Since the set of physical paths remains the same under this change, the response of the particles must remain unchanged also. Therefore, such a change of potentials will not alter the outcome of an experimental observation.

As an application, consider the Aharonov-Bohm effect [22]. In the corresponding experimental setup, the electrons travel in beams centered about paths \( \rho(ACB) \) and \( \rho'(ADB) \), enclosing a non-zero magnetic field but shielded from it. Chambers used reflectors at \( C \) and \( D \) to obtain a configuration of piece-wise classical narrow beams centered about \( \rho(AC) \), \( \rho(CB) \), \( \rho'(AD) \) and \( \rho'(DB) \) [23]. The magnetic field was generated by placing a long coil carrying an electric current between the reflectors and perpendicular to the plane of the beams with one end in the plane. The electron beams were further shielded from the magnetic field. As the current in the coil is varied, the magnetic field varies accordingly. The classical Lagrangian for this system is the same as for the Lorentz equation.

As in the case of the double slit experiment, most of the electrons are transmitted by the interfering alternatives with parameter value increasing from \( A \) to \( B \) along \( \rho(ACB) \) and decreasing from \( B \) to \( A \) along \( \rho'(BDA) \), taking value \( \tau_B \) at \( B \). The estimates obtained in the treatment of the double slit experiment are valid for the present case as they were not restricted to a free particle. Some consideration should be given to the reflectors at \( C \) and \( D \). Because of the continuity of the physical paths at points about \( A \), \( B \), \( C \), and \( D \), \( \kappa(\ ) \) cancels out. From Sec.3.2, we have that most of the intensity transmitted along \( \rho(AC) \) reaches a small neighbourhood of \( C \) which remains almost within a macroscopically narrow beam. By the same argument, most of this intensity reaches a small neighbourhood of \( B \). The same comment is valid for \( \rho'(ADB) \). The intensity along both of the beams is assumed equal. Consequently, the arguments of Sec.3.3 can be used to conclude the existence of a similar interference pattern on the screen.

It follows from (12) that the interfering alternatives for an electro-magnetic
potential $\phi_\mu$ are the solutions of (21):

$$\exp \left[ -i \oint (dS(x, \tau) - \phi_\mu dx^\mu) \right] \Phi_A = \Phi_A$$

(21)

where the integration is along the closed curves $\rho_c(ACBDA)$. Here the phase-factor $U_{BA}(\rho)$ is given by

$$U_{BA}(\rho) = \exp(i \int_{\rho(AB)} \phi_\mu dx^\mu)$$

It is clear that $\phi_\mu$-dependent part in (21) is $U_{ABA}(\rho_e)$ which is given by $U_{ABA}(\rho) = \exp(i F(\phi))$ where $F(\phi)$ is the magnetic flux enclosed by $\rho$. As $\rho_e$ is distorted, $F(\phi)$ remains unchanged as long as the distorted closed path encloses the flux, which covers all of the paths of significance here as all of them surround the coil.

As $F(\phi)$ varies to $F(\hat{\phi})$, $U_{ABA}(\rho_e) \neq \hat{U}_{ABA}(\rho_e)$ for any $\rho_e$ unless

$$(F(\phi) - F(\hat{\phi})) = \oint (\phi_\mu - \hat{\phi}_\mu) dx^\mu = 2\pi j$$

(22)

with an arbitrary integer $j$. When (22) is satisfied, $U_{ABA}(\rho_e) = \hat{U}_{ABA}(\rho_e)$ for each curve $\rho_e$ and hence the experimental observation with $\hat{\phi}_\mu$ must be the same as with $\phi_\mu$. Thus the interference pattern on the screen should repeat itself periodically as the potential is varied continuously. The period is defined by (22).

Let $\phi_\mu(\varepsilon)$ be a one parameter family of potentials with $0 \leq \varepsilon \leq 1$, such that $(F(\phi(\varepsilon)) - F(\phi(0))) = 2\pi$, i.e., $\varepsilon$ covers one period. The interference patterns corresponding to $\phi_\mu(0)$ and $\phi_\mu(1)$ are indistinguishable. Let the solutions of (21) with $\phi_\mu$ replaced by $\phi_\mu(\varepsilon)$ be $\{\rho(\varepsilon)\}$. Owing to the continuity of $F(\phi(\varepsilon))$ with respect to $\varepsilon$, $\{\rho(\varepsilon)\}$ should vary continuously, implying a continuous variation of the corresponding interference pattern. As $\varepsilon$ approaches one, the distribution of the intensity must return to the same as for $\varepsilon = 0$. Thus, each interference fringe should be expected to shift as $\varepsilon$ varies from zero to one, from its position to the original location of the next.

Above conclusion agrees with the experimental observation [23,24]. It is pertinent to remark that the indistinguishability of $\phi_\mu$ and $\hat{\phi}_\mu$ that satisfy (22) is a direct consequence of (21) which is obtained from (11) and the fact
that the physical paths in this case are closed in $\mathcal{M}'$. For this part of the conclusion, no estimates are needed. Dirac’s quantization of the magnetic monopole is also derivable from (22) [25].

The Aharonov-Bohm effect is an implication of the quantum mechanical equations [22] which were developed from different premises than the present formalism. Ingredients of the quantum mechanical deduction of this effect are the representation of the momenta $p_\mu$ by $-i\partial_\mu$ and the corresponding extension of the classical coupling scheme $(p_\mu - \phi_\mu)$. The former was inspired by the observed wave-like behaviour of particles and the later, in addition to being intuitive, sets $\alpha = i$ in the London-Weyl description of electromagnetism. Here the major aspects of the Aharonov-Bohm effect are deduced directly from (11) without an appeal to any other theory.

4. Equation of Motion

As explained in Sec. 2, the action principle includes a collection of curves into consideration, but assigns a unique trajectory to a particle in motion between two points. The present extension (11), on the other hand, assigns many paths, but not all curves are allowed. Since it is impossible to assign a unique trajectory to a particle, as an alternative, one may describe its motion in terms of the intensity of the particles transmitted to a region in $\mathcal{M}$ or $\mathcal{M}'$ by the physical trajectories. Equivalently, a probability density may be assigned to a given region and compute it from the contributions from all of the physical paths passing through the region. This was done in Sec. 3 for a beam of free particles and for the double-slit experiment, but only approximately. Approximations were made in obtaining the estimates and by retaining only the major contributions. In a complete theory, all physical paths must be included and the contributions must be computed exactly.

Wave-like behaviour of particles and a possibility of describing their motion in terms of the probability densities associated with a collection of trajectories led Feynman to develop his path integral formulation of non-relativistic quantum mechanics [10,11]. The formalism was extended in an analogous manner by introducing a proper time-like evolution parameter [16]. The wave-like behaviour of the particles was used to conclude that the intensity is the absolute square of the amplitude obtained by the law of superposition. The amplitude associated with a path $\rho(AB)$ was taken to be proportional to $\exp(iS'_{BA}(\rho))$ which was based on a deduction by Dirac [26] of the behaviour
of a quantum mechanical particle. This treatment also implies a multiplicity of paths for a particle. Present formulation associates a phase-factor equal to \( \exp(iS'_{BA}(\rho)) \) with \( \rho(AB) \) whenever a classical description is possible in terms of a Lagrangian. In the process, an alternative method to compute the phase-factors up to the required order is developed that has some advantages over the standard procedure [11]. The phases associated with a multiplicity of paths are shown to interfere in a manner that imparts wave-like properties to the particles in motion. A precise determination of a multitude of physical trajectories follows from (11). Thus all of the necessary assumptions required for the formulation of Feynman’s postulates have been deduced from (11). Having yielded its basic assumptions, the gauge mechanical principle finds a natural expression within the framework of the path integral formalism. However, only the physical paths should be included in the computation of the total contribution.

Above deduction is based on a treatment of some physical systems that can be described classically within the framework of the action principle. This includes the response of a charged particle to an electro-magnetic field. However, the extension stated in (11) is not subject to this restriction which is, therefore, dropped in the following. Also, the original postulates formulated for \( n = 1 \) [9] are extendable in a standard manner to include the non-Abelian gauge fields, as stated below.

**Postulate 1.** The probability of finding a particle in a region of space-time is the square of the \( l_2(C^n) \)-norm of the sum of contributions from each physical path or its segment in the region.

**Postulate 2.** The contribution at a point \( C \) of a physical path \( \rho(AB) \) is equal to \( K\kappa^{-1}(C)V_{CA}(\rho)\Phi^P_A = KV_{CA}(\rho)\Phi_A \) where \( K \) is a path-independent constant.

Since the assumptions underlying the above postulates are deduced from (11), the formalism is self-consistent and based essentially on one assumption. Postulate 2. provides a mechanism for an accurate computation of the total contribution from all trajectories by the techniques developed originally for the path-integral formulation. An equation of motion is developed below by this procedure and by isolating the contribution of the physical paths. Postulate 1. provides a means to obtain experimentally observable quantities
from the solutions of the equation of motion.

Consider a point $C$ on a physical path $\rho(AB)$. Let $\rho(AB)$ be the shortest segment of $\rho(A'B')$ containing $C$ such that $A$ and $B$ are equivalent to $A'$ and $B'$ respectively. Consider the pair of points $A$ and $A'$. The pair $B, B'$ is treated similarly. In view of the equivalence, $V_{AA'}(\rho)\Phi_A = \Phi_{A'} = \Phi_A$. Thus the contribution from $\rho(A'C)$ is the same as that from $\rho(AC)$. Therefore it is sufficient to consider the minimal curves $\rho(AB)$ instead of any larger physical paths containing $\rho(AB)$.

The next step is to parametrize the minimal physical paths in a way that enables one to isolate their contribution. Since a single parameter is needed for all of the curves, standard parametrization by arc-length is inadequate. A suitable parameter was found in ref.[9] as follows. Let $u'_\mu = \sum u_\mu$ where $\sum$ denotes the sum over all paths of the type $\rho(AB)$ with $A$ being a variable point. For any such collection of curves, there is a Lorentz frame $\mathcal{L}$ in which $u'_\mu = 0$ for $\mu = 1, 2, 3$. A particle may thus be treated as being located at the origin of $\mathcal{L}$. Incidentally, the origin of $\mathcal{L}$ coincides with the centre of mass of a fluid of uniform density and total mass $m$ with an infinitesimal element flowing along each of $\rho(AB)$. Let $z(\tau)$ be a parameterization of each path $\rho(AB)$ with $z(0) = A$, where $\tau$ is the proper time of $\mathcal{L}$. In $\mathcal{L}$, each of the curves $\rho(AB)$ coincides with the straight line along $\tau$. Therefore, $V_{CA}(\rho) = \exp(im\tau)$ and hence $B = z(2\pi/m)$. From Postulate 2, the contribution $\psi'(x, \tau)$ at $C = x$ is given by

$$\psi'(x, \tau) = \sum K'V'[x, z(\tau)]\Phi[z(0)]$$  \hspace{1cm} (23)

where the sum is over all paths passing through $x$ at $\tau$; $\Phi[z(0)] = \Phi_A$ and for each $z(\tau), V'[x, z(\tau)] = V'_{CA}[z(\tau)]$. The sum is the limit of a finite one with constant $K'$ depending on the number of terms. Because of the continuity of the paths, the number of curves for $\tau = 0$ is the same as for $\tau = 2\pi/m$. Also, for each physical path $z(\tau), V[x, z(\tau)] = V[x, z(2\pi/m)] = 1$, i.e., $V'[x, z(0)] = V'[x, z(2\pi/m)] = \kappa^{-1}(C)\kappa(A)$. It follows that

$$\psi'(x, 0) = \psi'(x, 2\pi/m)$$  \hspace{1cm} (24)

The boundary condition given by [24] provides a means to retain the contribution in (23) from the physical paths. Thus the proper time $\tau$ of $\mathcal{L}$ acquires a physical significance, which is treated below as an independent parameter as in [18] and [19]. The following derivation is essentially the same
as in the standard path integral formulation, except for some technicalities arising out of the non-Abelian nature of the fields under consideration.

Let \([0, 2\pi/m]\) be divided into \(N\) equal intervals \([\tau_j, \tau_{j+1}], j = 0, 1, ..., N - 1;\) with \(\tau_0 = 0\), and \(\tau_N = 2\pi/m\). Consider all of the paths with \(z(\tau_k) = (x)_k\). By the same argument as for \(n = 1\), the vector \(\psi'(x)_k, \tau_k\), for each \(k\), is given by

\[
\psi'(x)_k, \tau_k = \int U^P[(x)_0, (x)_1] \cdots U^P[(x)_{k-1}, (x)_k] \\
\times U[(x)_k, (x)_{k-1}] \cdots U[(x)_1, (x)_0] \Phi[z(0)] \\
\times \frac{d(x)_0}{Q} \cdots \frac{d(x)_{k-1}}{Q}
\]  

(25)

where \(U^P[(x)_{j+1}, (x)_j] = \{U^P[(x)_j, (x)_{j+1}]\}^{-1} = U_{B', A'}[z(\tau)]\), \(U[(x)_{j+1}, (x)_j] = U_{B', A'}[z(\tau)]\) with \(A' = (x)_j\), \(B' = (x)_{j+1}\), and \(Q\) is a normalization constant. Set \((x)_k = y, \tau_k = \tau, (x)_{k+1} = x\) and \(\tau_{k+1} = \tau_k + \epsilon\). Since \(U^P[(x)_j, (x)_{j+1}]\) for each \(j\), is a constant multiple of the identity matrix, it follows from (25) that

\[
\psi'(x, \tau + \epsilon) = \frac{1}{Q} \int U^P(y, x) U(x, y) \psi'(y, \tau) dy
\]  

(26)

A curve \(z(\tau)\) in \(\mathcal{M}\) may be arbitrarily closely approximated by \(z_N(\tau)\) for large enough \(N\), where \(z_N(\tau_j) = z(\tau_j), j = 0, 1, ..., N;\) and in each of the intervals \([\tau_j, \tau_{j+1}], z_N(\tau)\) is the geodesic line. The element \(U^P(y, x) = U_{y, x}[z(\tau)]\) may be approximated by

\[
U^P_{y, x}[z_N(\tau)] = \exp \left[ i S(x, y) \right]
\]

where \(S(x, y)\) is Hamilton’s principal function for a ‘free’ particle of mass \(m\) from \(x\) to a variable point \(y\) (see also Sec.2). Here the Lagrangian is \(L^P\) with \(\tau\) being the proper time of \(\mathcal{L}\). The action is given by \([18]\)

\[
S(x, y) = -\frac{m}{2\epsilon} g_{\mu\nu} \xi^\mu \xi^\nu - \frac{m}{2} \epsilon
\]

where \(\xi^\mu = (x^\mu - y^\mu)\). Also, \(U(x, y)\) is approximated by \(U_{x, y}[z_N(\tau)]\) up to the desired order which, from (9), is given by

\[
U_{x, y}[z_N(\tau)] = 1 + i \phi_\mu(x) \xi^\mu - \frac{1}{2} \left[i \phi_{\mu, \nu} + \phi_{\mu} \phi_\nu\right] \xi^\mu \xi^\nu \\
+ \text{higher order terms.}
\]
Let $\psi(x, \tau) = \exp(i m \tau/2) \psi'(x, \tau)$, then it follows from (24) that

$$\psi(x, 0) = -\psi(x, 2\pi/m)$$

(27)

With the above substitutions, from (26), we have

$$\psi(x, \tau + \epsilon) = \frac{1}{Q} \int \exp \left[ -\frac{i m}{2\epsilon} g_{\mu\nu} \xi^\mu \xi^\nu \right] U_{x,y}[z_N(\tau)] \psi(x - \xi, \tau) d\xi$$

(28)

Eq.(28) holds exactly in the limit of infinite $N$, equivalently $\epsilon = 0$. As such it holds up to the first order in $\epsilon$, which is sufficient for the present.

Expanding $\psi(x, \tau + \epsilon)$ and $\psi(x - \xi, \tau)$ in a Taylor series about the point $(x, \tau)$ and comparing the coefficients of $\epsilon^j$, $j = 0, 1$, yields $Q = -i(2\pi\epsilon/m)^2$ and

$$i \frac{\partial \psi}{\partial \tau} = -\frac{1}{2m} \Pi_\mu \Pi^\mu \psi$$

(29)

where $\Pi_\mu = (i \partial/\partial x^\mu \cdot 1 + \phi_\mu)$. In view of the boundary condition (27), $\psi$ may be expressed as

$$\psi(x, \tau) = \sum_{k = -\infty}^{\infty} \psi_k(x) \omega_k(\tau)$$

where for each $k$, $\omega_k(\tau) = \sqrt{m/2\pi} \exp[i(k + 1/2)m\tau]$ and the $n$-vector $\psi_k$ satisfies

$$\Pi_\mu \Pi^\mu \psi_k = (2k + 1)m^2 \psi_k$$

(30)

$k = 0, \pm 1, \pm 2, \ldots$. Eqs.(29) and (30) are remarkably similar to the corresponding equations for the case $n = 1$ [9]. For $n = 1$ and $k = 0$, (30) reduces to the Klein-Gordon equation in an electro-magnetic field.

Equation of motion (29) is an expected prescription to describe the motion in a general gauge field on intuitive grounds that could be inferred from the generalized Schrödinger equation conjectured by Stückelberg [12,13]. The boundary condition (27) is a direct result of the definition of the physical paths provided by (11). As shown above, this boundary condition is crucial in relating (29) to the Klein-Gordon equation. If all trajectories are allowed to contribute, the resulting equation is still (29) but without the boundary condition (27). Feynman [16] used this equation for $n = 1$ to deduce the Klein-Gordon equation by restricting the solution to the form $\psi_0(x) \omega_0(\tau)$. Present treatment relates (29) with the Klein-Gordon equation (30) quite naturally. Further to the arguments of Sec.3, this result provides additional support for the assumption (11).
5. Concluding Remarks

The variational principle determines a particle trajectory by requiring the action to be stationary under all small deformations. In group theoretical terms, this results in a requirement of equivalence between the elements associated with a subset of the closed curves up to the first order only. In this article, the classical action principle is extended to require the equivalence of the global elements associated with all of the curves. The resulting equation selects an infinite subset, termed the physical paths, to assign to a particle in motion.

Properties of the physical paths impart wave-like properties to a particle in motion. The wave-like behaviour of particles and the multiplicity of allowed paths form the basis of the path integral formulation. An imaginary value of $\alpha$ yielded by the present extension, implies the compactness of the gauge groups which is inherent in quantum mechanical equations in gauge fields. Consequent description of the influence of the field enclosed by a closed curve on the particles, as is the case with the Aharonov-Bohm effect, is described by (11) to a large extent without an appeal to any other theory. Thus the present formulation develops a coherent theory unifying various treatments underlying the existing quantum mechanics.

The above results lead naturally to Feynman’s path integral formalism with physical paths being the contributing members. The criterion imposed by (11) on the physical paths plays a crucial role in the deduction of the above results, some of which have been used to justify the use of the path integral formalism. Thus the present formulation is self-consistent.

In the present paper we have used a proper time-like parameter to convert the problem of isolating the contribution from the physical paths into a boundary condition on (29). This type of parameter was introduced in a rather ad hoc manner by several authors [12,18]. Here this parameter gains a clearer physical significance. A need for a five-dimensional relativistic wave equation has been felt for a long time, for the existing equations suffer from some conceptual difficulties. In response to this need, Stückelberg originally conjectured the generalized Schrödinger equation for a particle in an Abelian gauge field [12,13]. There is a renewed interest in this equation to interpret it in a more satisfactory framework than a conjecture, as well as to study its implications (see e.g., [27]). Present formalism provides a systematic derivation of the generalized Schrödinger equation [9]. In the present article this
derivation is generalized to include a general gauge field.

In addition to accepting the conjecture of Stückelberg [12,13], Feynman
selected a particular set of periodic solutions to deduce the Klein-Gordon
equation from the generalized Schrödinger equation. As pointed out above,
the physical paths are characterized by a boundary condition on (29). This
boundary condition confines the solution to a set described by a class of
periodic functions. As a consequence, the equation decomposes into count-
ably many four dimensional equations, one of them being the Klein-Gordon
equation. Thus the resulting boundary condition provides an additional jus-
tification for the present treatment.

The techniques and results developed in the process of this work pro-
vide also the ground work for a similar treatment of the spinors, and for an
extension of other theories based on the variational principle.
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