CONTINUUM LIMITS OF “INDUCED QCD”:
LESSONS OF THE GAUSSIAN MODEL AT d=1
AND BEYOND

I.I. Kogan, A. Morozov, G.W. Semenoff and N. Weiss

Department of Physics, University of British Columbia
Vancouver, B.C., Canada V6T1Z1

1This work is supported in part by the Natural Sciences and Engineering Research Council of Canada
2Permanent Address: ITEP, 117259, Moscow, Russia.
3Address after September 1, 1992: Physics Department, Jadwin Hall, Princeton University, Princeton, NJ 08544, USA
Abstract

We analyze the scalar field sector of the Kazakov–Migdal model of induced QCD. We present a detailed description of the simplest one dimensional ($d=1$) model which supports the hypothesis of wide applicability of the mean–field approximation for the scalar fields and the existence of critical behaviour in the model when the scalar action is Gaussian. Despite the occurrence of various non–trivial types of critical behaviour in the $d = 1$ model as $N \to \infty$, only the conventional large-$N$ limit is relevant for its continuum limit. We also give a mean–field analysis of the $N = 2$ model in any $d$ and show that a saddle point always exists in the region $m^2 > m^2_{crit}(= d)$. In $d = 1$ it exhibits critical behaviour as $m^2 \to m^2_{crit}$. However when $d>1$ there is no critical behaviour unless non–Gaussian terms are added to the scalar field action. We argue that similar behaviour should occur for any finite $N$ thus providing a simple explanation of a recent result of D. Gross. We show that critical behaviour at $d>1$ and $m^2 > m^2_{crit}$ can be obtained by adding a logarithmic term to the scalar potential. This is equivalent to a local modification of the integration measure in the original Kazakov–Migdal model. Experience from previous studies of the Generalized Kontsevich Model implies that, unlike the inclusion of higher powers in the potential, this minor modification should not substantially alter the behaviour of the Gaussian model.
1 Introduction

The Kazakov–Migdal model \[1\] for induced QCD (to be further referred to as the KMM) has recently attracted much attention \[2\]–[8]. It is interesting as a tractable example of a matrix model where the the integration over angular variables plays a non–trivial role and as such, it is the natural next step in the theoretical investigation of matrix models. It is also a lattice gauge theory and there is the possibility that, provided the appropriate scaling limit exists, its large wave-length behaviour is described by Yang-Mills theory. The KMM contains two kinds of fields, \(N \times N\) Hermitean matrices \(\Phi\) which live on the sites of a \(d\)-dimensional lattice and \(N \times N\) special unitary matrices \(U\) which live on links. At large distances, the latter are associated with the gauge fields of Yang-Mills theory, while the former play an essential role at short distances and presumably disappear from the spectrum in the continuum limit.

The most direct approach to finding a continuum limit of the KMM is to assume that \(N\) is large and to use the mean–field approximation for the \(\Phi\)-fields. It is natural to assume that the mean field is constant in space-time. The first results in the framework of this program were obtained in \[2\]. There an equation for the density of eigenvalues, \(\rho(\phi)\), of the master-field \(\Phi\) was deduced from a highly non–trivial saddle point equation with the help of an especially simple subset of the Schwinger–Dyson equations. Using a power-like ansatz, \(\rho(\phi) \sim \phi^\alpha\), a solution of this equation was found. However, in \[7\] an exact solution of the saddle point equations was found for the case when the action for the scalar fields is quadratic. In that case the eigenvalues had a semi-circle distribution \(\pi \rho(\phi) \sim \sqrt{2\mu - \mu^2 \phi^2}\). It was further argued that for \(d > 1\) there is no critical behaviour \((\mu \rightarrow \infty)\) if the critical point is approached from the strong coupling phase \((m^2 > m^2_{\text{crit}})\).

One must rather approach the critical point from the weak coupling phase \((m^2 < m^2_{\text{crit}})\). In order to make the effective potential for the scalar field stable in this situation one needs to introduce higher order (at least quartic) terms in the bare action. The resulting critical behaviour will generically be first order though second order critical points may be present for some special values of the parameters. The \(N = 2\) version of the KMM with quartic terms in the potential has been examined in the region \(m^2 < m^2_{\text{crit}}\) in \[5\] where such critical behaviour was indeed found. Moreover the results of computer simulations demonstrated a nice agreement between the \(N=2\)
KMM and the predictions of the mean field analysis which has no a priori reason to be reliable for $N = 2$.

In this paper we shall present a more extensive analysis of the easily solvable versions of the KMM. We believe that this is necessary before abandoning the simplest version of the model with a quadratic potential for the scalar field or accepting the suggestion (implied in [3] and [4]) that there may be second order critical behaviour in the instability region $m^2 < m_{\text{crit}}^2$. (We shall call the model with such a quadratic potential a “Gaussian” model despite the fact that the potential is only quadratic in the scalar ($\Phi$) fields. Its interaction with the gauge fields ($U$-variables) is highly non-linear.)

We begin by analyzing the exactly solvable 1–dimensional KMM in detail. There we observe some amusing types of critical behaviour which occur when $N \to \infty$ but which do not survive in the thermodynamic limit (when the size of the system increases to infinity). We shall also use the explicit results in $d=1$ to demonstrate the applicability of the mean field approximation both for these different kinds of critical behaviour as well as for the simplified $N = 2$ version of the KMM. The mean–field results for the continuum limit of the $N = 2$ model are then generalized to any $d > 1$. In this case the critical behaviour disappears in the region $m^2 \geq m_{\text{crit}}^2$ (and survives only for $m^2 < m_{\text{crit}}^2$). Thus it seems necessary to deal with the upside–down harmonic oscillator potential and possibly to introduce higher order terms in the scalar potential. We then argue that a similar conclusion applies to all $N > 2$. This provides us with a more transparent explanation of the result of ref.[7] concerning the absence of criticality in the Gaussian model when $d > 1$. The origin of this phenomenon is just a logarithmic increase of the effective potential at infinity due to the Van-der-Monde determinants. It is possible to compensate for this growth by introducing logarithmic terms into the bare potential. These can be interpreted as a change of the measure of the integration over matrices. Changes of the measure of this kind have been investigated in the somewhat simpler context of the generalized Kontsevich Model [4] in which case it is known to preserve all of the nice features of the model with a quadratic potential [10]. This may also be the case for the KMM though further investigation is required.

Our analysis of the various types of critical behaviour (even those irrelevant for the continuum limit but, rather, associated with the large $N$ limit in finite volume) indicates a wide applicability of the mean–field approximation for the $\Phi$-fields. The result for the $N = 2$ model also supports the
suggestion that the critical behaviour for $m^2 \to m_{\text{crit}}^2 + 0$ (whenever it occurs) is associated with a mean field $\Phi$ which is large, i.e. $\Phi \gg 1$. (This is not necessarily true if the critical behaviour is associated with the tuning of higher order terms in the bare potential for $\Phi$.) Both of these features provide some justification for the assumptions which were made in [6] in our analysis of the correlators of physical observables in the KMM.

We begin in Section 2 with a review of the KMM. Sections 3 and 4 are devoted to a detailed discussion of the 1-dimensional KMM. In Section 5 we consider the $N = 2$ model in any dimension and investigate its critical behaviour in the mean field approximation.

2 KMM: Some Generalities and the Mean–field Approximation

The KMM contains Hermitean matrices $\Phi(x)$ and special unitary matrices $U(x,y)$ which are defined on the sites (denoted by $x$) and on the links (denoted by $<x,y>$) of a $d$–dimensional lattice, respectively. The partition function is given by

$$Z_D \equiv \int \Phi[d\Phi] U[dU] \exp \left( -\sum_x \text{tr}V(\Phi(x)) + \sum_{<x,y>} \text{tr}\Phi(x)U(x,y)\Phi(y)U^\dagger(x,y) \right)$$

(1)

where $[dU]$ is the invariant Haar measure for unitary matrices. For most of this Paper we shall consider the case where the action for the scalar fields is quadratic i.e. $V(\Phi) = m^2 \Phi^2$. Higher order terms could be used to cut off the fluctuations of large $\Phi$-fields near the critical point and would affect the detailed properties of the continuum limit. We shall argue that in $d=1$ these additional terms are unnecessary. We shall also argue that in $d>1$ at least some additional logarithmic terms (which could also be viewed as modifying the measure rather than the action in (1)) are needed to produce the appropriate critical behaviour.

If the integral over $\Phi$ is taken first in eq.(1) it gives rise to an effective action for the gauge fields

$$S_{\text{eff}}(U) = -\frac{1}{2} \sum_{\Gamma} \frac{|\text{tr}U[\Gamma]|^2}{m^{2L(\Gamma)}L(\Gamma)}$$

(2)
where the sum is over all loops $\Gamma$ and $U[\Gamma]$ is the product of $U$-matrices associated with links in $\Gamma$. Besides the conventional symmetry under gauge transformations,

$$\Phi(x) \to V(x)\Phi(x)V^\dagger(x), \quad U(x, y) \to V(x)U(x, y)V^\dagger(y)$$  \hspace{1cm} (3)

with $V(x) \in SU(N)$ there is also an invariance under multiplication of link operators by an element of the center of the gauge group $[3]$,

$$U(x, y) \to \omega(x, y)U(x, y)$$  \hspace{1cm} (4)

where, for the U(N) group $\omega(x, y)$ is a unimodular complex number and for the SU(N) group it is an element of $Z_N$, i.e. $(\omega(x, y))^N = 1$. This symmetry has important implications for the properties of observables and their correlators $[6]$.

In the “naive continuum limit” this effective theory resembles Yang-Mills theory provided $D = 4$ $[1]$. However, this limit is too naive in the sense that, unlike the conventional Wilson formulation of lattice QCD which has a single, or finite number of terms like

$$\frac{1}{g^2} \text{tr}U(\Gamma)$$  \hspace{1cm} (5)

in the action, the effective action (2) has the special property that it is not sharply peaked in the vicinity of the trivial configuration $U = I$ $[1]$. This leads to significant differences between the procedure for taking the continuum limit in this gauge theory and in conventional lattice QCD.

Remarkably it is possible to do the gauge field integral first in (1). The result is an effective action for the $\Phi$-fields,

$$S_{\text{eff}}(\Phi) = m^2 \sum_x \text{tr}\Phi^2(x) - \sum_{<x,y>} \log I(\Phi(x), \Phi(y))$$  \hspace{1cm} (6)

where

$$I(\Phi, \Psi) \equiv \int [dU]e^{\text{tr}\Phi U\Psi U^\dagger} = V_N \frac{\det e^{\phi_i\psi_j}}{\Delta(\phi)\Delta(\psi)}$$  \hspace{1cm} (7)

denotes the integral over the $U$-matrix on a given link and we denote by
\[ V_N = \text{Vol}(U(N)) \text{ the volume of the unitary group (in the Haar measure } [dU]) : \]

\[ V_N = (2\pi)^{N(N+1)/2} \prod_{k=1}^N k! \]  

(8)

The integral depends only on the eigenvalues of the \( \Phi \)-fields, denoted here by \( \phi_i \) and

\[ \Delta(\phi) = \prod_{i<j}(\phi_i - \phi_j) \]  

(9)

is the Van-der-Monde determinant. This explicit formula for \( I(\Phi, \Psi) \) has been known for a long time [11]. In the large \( N \) limit the resulting effective theory for the \( \Phi \)-field can be analyzed in the mean field approximation.

\[ A \text{ simple way to derive this formula is provided by the theory of random matrices. The Gaussian integral over Hermitean matrices } H, \]

\[ J_N = \int dH e^{-\frac{1}{2}H^2} = \prod_{i=1}^N \int dH_i e^{-H_i^2} \prod_{i>j} \int dH_{ij} d\overline{H}_{ij} e^{-2|H_{ij}|^2} = \pi^{N^2/2} \]

can alternatively be expressed by diagonalizing \( H \), integrating over orbits of diagonal \( H \) and the orthogonal polynomial method to compute the remaining integral as

\[ J_N = \text{Vol}[U(N)/U(1)^N] \int \prod_{i=1}^N dh_i \Delta^2(h) e^{-\sum_{i=1}^N h_i^2} = \]

\[ \text{Vol}[U(N)/U(1)^N] N! \int \prod_{i=1}^N dh_i e^{-\sum h_i^2} \prod_{j=0}^{N-1} H_j^2(h_i) \]

The last product on the right-hand-side contains the norms of orthogonal Hermite polynomials normalized so that

\[ H_k(h) := \frac{1}{2^k} e^{\frac{h^2}{2}} (h - \frac{d}{dh}) e^{-\frac{h^2}{2}} = h^k + \ldots \]

Then \( ||H_k||^2 = \sqrt{\pi} k! / 2^k \). The \( N! \) factor in front of the integral comes from the permutations of the eigenvalues \( h_i \) while the volume factor arises from integration over the flag manifold \( U(N)/U(1)^N \) of unitary matrices \( U(N) \) modulo the Cartan (diagonal) elements of \( U(N) \) which commute with the matrix \( H \) once it is diagonalized. Comparison of these two expressions for \( J_N \) gives (8).
The main idea of ref. [2] was to analyze the classical equations of motion for the effective action \( S_{\text{eff}}(\Phi) \),

\[
m\Phi(x) = \frac{1}{2} \sum_{y \in \langle x,y \rangle} \left( \frac{\partial}{\partial \Phi(x)} \right) \log I(\Phi(x), \Phi(y))
\]

(10)

*together* with some simple Ward identities for \( I(\Phi, \Psi) \),

\[
\text{tr} \left( \frac{\partial}{\partial \Phi} \right)^k I(\Phi, \Psi) = \text{tr} \Psi^k I(\Phi, \Psi).
\]

(11)

It is important to note that this is *not* a complete set of Ward identities. They *do not* define \( I(\Phi, \Psi) \) unambiguously. For example

\[
I_0(\Phi, \Psi) = e^{\text{tr} \Phi \Psi}
\]

(12)

is also a solution of (11). A complete set of identities would contain analogues of these equations with operators like \( \text{tr} \left( \Phi^l \left( \frac{\partial}{\partial \Phi} \right)^k \right) \) (where \( l \neq 0 \)) on the left hand side. There may also be independent identities with mixed \( \Phi \) and \( \Psi \) derivatives. Unfortunately most of them (with the exception of (11)) have a complicated form.

Since the reduced set of identities (11) is incomplete, it is not clear whether *arbitrary* solutions to the equations (11)

\[
\mathcal{R}(\lambda) = P \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi i} \log \frac{\lambda - \frac{1}{2d} V'(\nu) - \frac{d-1}{d} \mathcal{R}(\nu) + i\pi \rho(\nu)}{\lambda - \frac{1}{2d} V'(\nu) - \frac{d-1}{d} \mathcal{R}(\nu) - i\pi \rho(\nu)} =
\]

\[
\int_{-\infty}^{+\infty} \frac{d\nu}{\pi} \arctan \frac{\pi \rho(\nu)}{\lambda - \frac{1}{2d} V'(\nu) - \frac{d-1}{d} \mathcal{R}(\nu)}
\]

(13)

with

\[
\mathcal{R}(\mu) \equiv P \int \frac{\rho(\nu)d\nu}{\mu - \nu},
\]

(14)

derived from them will have anything to do with the actual KMM.\footnote{In any case it is unclear how to apply the same trick to estimate quantities other than the eigenvalue distribution. To study correlation functions in the KMM one must use some more substantial information about \( I(\Phi, \Psi) \). The feature (11) is insufficient (see [3]).} While the results of Migdal and Gross might still be relevant to the KMM for some less
obvious reason this reason needs to be clearly identified. It was shown in [7] that (13) posses a solution for $\rho(\mu, d)$ in the form of a semi-circle distribution

$$\pi \rho(\mu, d) = \sqrt{2\mu(d) - \mu^2(d)} \varphi^2$$

which is usually characteristic of matrix models with purely quadratic potentials. This is not surprising since the information input in the derivation of (13) does not distinguish between the integral $I(\Phi, \Psi)$ in (7) and the triv-ial quadratic potential $I_0(\Phi, \Psi)$ in (12). One reason why (13) may still be relevant is that for large $\phi_i$ (see (59)), $I(\Phi, \Psi)$ is not very different from $I_0(\Phi, \Psi)$.

The explicit expression

$$\mu(d) = \frac{d}{2d - 1} \left[ m^2(d - 1) \pm \sqrt{d^2(m^4 - 1) + (d - 1)^2} \right]$$

was derived in [7] by substituting the semi-circle ansatz (13) into (13). This expression shows that while for $d = 1$ $\mu$ goes like $\sqrt{m^2 - m_{\text{crit}}^2}$ in the vicinity of the critical point this is no longer true for $d > 1$ in which case the branch of $\mu$ which is positive in the region $m^2 > m_{\text{crit}}^2 = d$ does not vanish when $m^2 = d$.

We shall see in Section 5 below that this result has a very simple explanation and we suggest a way to restore the critical behaviour for $d > 1$.

3 Partition Functions for Various Models in One Dimension

In one dimension the KMM can be solved using elementary techniques. It essentially reduces to the well known problem of a matrix-valued harmonic oscillator. This problem has recently been reviewed in the context of matrix models in the recent papers [12], [4] as well as in [3] and [7]. There are as many as four slightly different Gaussian matrix models in one dimension. They can be defined either on an open or on a closed lattice (chain), and they can either contain only hermitean matrices as in conventional matrix models or they can couple to unitary matrices to form a gauge invariant model.

We shall now list the four models and give the expression for the partition function for each. The relevant parameter in this Gaussian problem is $q_{\pm} = 6$Here we shall use our notation, for example our $m^2$ is two times smaller then in [7]
\[ m^2 \pm \sqrt{m^4 - 1} \] (rather than the “bare mass” \( m \)). In the closed chain models log \( q \) is additive in the length \( L \) of the chain. We thus define \( q_L \equiv q^L \). In the discussion below we use the branch \( q_- \).

### 3.1 Gaussian Multimatrix Model on an Open Chain

This model contains a chain of coupled matrices with free boundary conditions. The partition function was computed in [6]:

\[
\mathcal{Z}_A \equiv \int \prod_{x=1}^L d\Phi_x e^{-m^2 \text{tr}\Phi_x^2} \prod_{x=1}^{L-1} \text{e}^{\text{tr}\Phi_x \Phi_{x+1}} = \left( \frac{(1-q^2)q^L}{1-q^{2L+2}} \right)^{N^2/2} \tag{17}
\]

### 3.2 Gaussian KMM on an Open Chain

The partition function for KMM on an open chain is essentially the same as the partition function for the previous case.

\[
\mathcal{Z}_B \equiv \int \prod_{x=1}^L d\Phi_x e^{-m^2 \text{tr}\Phi_x^2} \prod_{x=1}^{L-1} [dU_{x,x+1}] \text{e}^{\text{tr}\Phi_x U_{x,x+1} \Phi_{x+1} U_{x,x+1}^\dagger} =
\]

\[
= [\text{Vol}(U(N))]^L \mathcal{Z}_A = V_N^L \left( \frac{(1-q^2)q^L}{1-q^{2L+2}} \right)^{N^2/2} \tag{18}
\]

The reason for this is that the gauge transformations (3) can be used to substitute unit matrices \( I \) for all of the \( U_{x,x+1} \) variables on the chain. The resulting integral is identical to that of the Gaussian multimatrix model with the exception of the factor \( V_N \) which is the volume of the group \( U(N) \) with the Haar measure \( [dU] \).

### 3.3 Gaussian Multimatrix Model on a Closed Chain

We now consider the multimatrix model on a closed chain with periodic boundary conditions \( \Phi_{L+1} = \Phi_1 \). The partition function for this model can be easily evaluated as

\[
\mathcal{Z}_C \equiv \int_{\Phi_{L+1} = \Phi_1} \prod_{x=1}^L d\Phi_x e^{-m^2 \text{tr}\Phi_x^2 + \text{tr}\Phi_x \Phi_{x+1}} = \left( \frac{q^L}{(1-q^L)^2} \right)^{N^2/2} \tag{19}
\]
3.4 Gaussian KMM on a Closed Chain

For the Gaussian KMM on a closed chain the partition function is given by \[4\]:

\[
Z_D \equiv \int \Phi_{L+1=\Phi_1} \prod_{x=1}^{L} d\Phi_x [dU_{x,x+1}] e^{-m^2 \text{tr} \Phi_x^2 + \text{tr} \Phi_x U_{x,x+1} \Phi_{x+1} U_{x,x+1}^\dagger} = \\
= V_N^{L} \frac{q_L^{N^2/2}}{\prod_{n=1}^{N} (1 - q_L^n)}
\]

(20)

In this case the gauge transformations \[3\] can be used to eliminate (i.e. put equal to \(I\)) all the \(U\)-matrices except for one. This last \(U\) can only be diagonalized so that \(U_{ab} = e^{i\theta_a} \delta_{ab}\). Using this idea one can derive another expression for \(Z_D \ [5], [4] \):

\[
Z_D = (2\pi)^N V_N^{L-1} \frac{q_L^{N^2/2}}{(1 - q_L)^N} \int \prod_{a=1}^{N} d\theta_a \prod_{i<j}^{N} \left| 1 - e^{i(\theta_a - \theta_b)} \right|^2
\]

(21)

The right hand side of (20) can be considered as the result of performing the integration in (21).

3.5 Comments and Applications

We now make several observations which are important for our further analysis of the \(d=1\) model as well as for its generalization to higher dimensions. First of all the factor

\[
q_L^{N^2/2} = q^{LN^2/2}
\]

(22)

is common to all four partition functions. The main difference between them is their behaviour at the critical point \(q_{\text{crit}} = 1\) (i.e. \(m_{\text{crit}}^2 = d = 1\)). Note that both the functions \(Z_A\) and \(Z_B\) for the open chain models are not singular at this point. On the other hand the partition functions for the closed chains do possess a singularity as \(q \to 1\). The nature of this singularity is, however, quite different in the two models. In the conventional multimatrix model

\[
Z_C \sim (1 - q)^{-N^2} \quad \text{as } q \to 1
\]

(23)
while in the KMM

\[ Z_D \sim (1 - q)^{-N} \quad \text{as} \ q \to 1 \quad (24) \]

The presence of this singularity reflects the occurrence of zero modes in the integral as \( m^2 \to d=1 \). In the model \( C \), any constant (\( x \)-independent) matrix \( \Phi(x) = \Phi \) is a zero mode in this limit. This results in the power \( N^2 \) in (23). Things are however different for the KMM \( D \). Integration over the \( U \)-matrices can, and indeed does, eliminate some of the zero modes which occur at \( U = I \). (This was the essence of our argument in \cite{6} against the “naive continuum limit” for the \( U \)-variables which would imply that the vicinity of \( U = I \) gives the main contribution to the integrals over \( U \).) The key observation is that in fact not all of the zero modes disappear. The zero modes in Cartan subalgebra survive. Indeed if the \( U \)'s can be diagonalized by a gauge transformation (3) (as we have seen is the case in the 1-dimensional model) then the diagonal elements of the matrices \( \Phi \) do not feel \( U \)'s. In fact

\[ \text{tr}\Phi U \Psi U^\dagger \rightarrow \sum_{a,b} \Phi_{ab} \Psi_{ba} e^{i(\theta_a - \theta_b)} \quad (25) \]

so that all coupling between the diagonal elements of \( \Phi \) and \( \Psi \) and the \( \theta \)'s disappears. This fact, combined with the absence of coupling between the diagonal and the off-diagonal elements of \( \Phi \) and \( \Psi \) in (23) ensures that the zero modes \( \Phi_{aa}(x) = \Phi_{aa} = \text{const} \) cannot be eliminated by the \( U \)-integration.

We can understand how the off-diagonal zero modes are eliminated by looking at eq.(21). Note that the integrand is not singular at \( q_L = 1 \). The singularity coming from the action is completely canceled by the zeros of the measure. (This is a very unusual situation which is specific to the KMM. It can be understood as being due to the very slow i.e. logarithmic increase of the action near its minimum. This can in turn be traced back to the fact that contours of arbitrarily large length contribute to the effective action for \( U \)-variables. See \cite{3} for details.) The net result is that the order of the singularity in (24) is just \( N = \text{rank} U(N) \). The generalization of this result to \( d > 1 \) is discussed in Section 4.

We now briefly discuss the implications of this singularity. According to \cite{6} the fact that the singularity is not of the power \( N^2 \) in the KMM (as it was in the model \( C \)) implies that there is no “naive continuum limit” for the \( U \)-variables (It is still possible that a “less naive” continuum limit may exist.)
Note however that the singularity still does survive. In fact it is of order $N$ which, although much less than $N^2$, is still much greater than 1. In the next section we shall argue that this fact implies that non-naive large $N$ limits may exist in the $\Phi$-sector even for finite volume. The resulting models are of no interest when taking the continuum limit since they do not survive in the limit of large volume. They are however interesting examples of critical behaviour and they should thus be kept in mind in future investigations. We shall also see that these limits are consistent with the mean-field approximation for $\Phi$ and they thus provide us with additional material to study its features.

4 Different Types of Critical Behaviour for the Gaussian KMM in One Dimension

Our goal in this section is to discuss the various kinds of critical behaviour which are possible in $d=1$ and to study the applicability of mean-field theory in each case. Since we have exact expressions for the various partition functions we can easily test the validity of the mean-field approximation. The main effect which results when mean-field theory is exact is the factorization property for correlators such as

$$\left\langle \left( \text{tr} \Phi^2(x_1) \cdots \text{tr} \Phi^2(x_k) \right) \right\rangle = \left( \left\langle \text{tr} \Phi^2 \right\rangle \right)^k.$$  \hspace{2cm} (26)

These particular correlators are especially suited for our purposes since

$$\left\langle \left( N \sum_x \text{tr} \Phi^2(x) \right)^k \right\rangle = \frac{1}{Z} \left( -N \frac{\partial}{\partial m^2} \right)^k Z.$$ \hspace{2cm} (27)

Thus if the factorization property (26) is exact we should find, in the large $N$ limit, that

$$\frac{1}{Z} \left( \frac{N}{L} \frac{\partial}{\partial m^2} \right)^k Z = \left( \left\langle N \text{tr} \Phi^2 \right\rangle \right)^k.$$ \hspace{2cm} (28)

Note that we have introduced a normalization factor $N$ in the definition of these operators. The conventional choice (corresponding to the “naive continuum (or large $N$) limit” which was examined in [4]) is $N_{\text{nc}} = 1/N^2$. However our experience in [3] indicates that one should be very cautious
when fixing the value of $\mathcal{N}$. The correct choice of $\mathcal{N}$ can (and does) depend on precisely how the limit is taken.

To illustrate the various possibilities for critical behaviour let us proceed directly to the most interesting case – the KMM on a closed chain (model $D$ from the previous section). The partition function $\mathcal{Z}$ is given in eq. (21). We begin by evaluating the correlators (28). Since $\mathcal{Z}$ depends on $q_L=q^L$ and we wish to differentiate it with respect to $m^2$ we must use the fact that

$$-rac{\partial \log q_L}{L \partial m^2} = -\frac{\partial \log q}{\partial m^2} = \frac{1}{\sqrt{m^4-1}} = \frac{2q}{1-q^2} \quad \text{(29)}$$

Then

$$\ll \mathcal{N} \text{tr}\Phi^2 \gg = \mathcal{N} \frac{2q}{1-q^2} \left( \frac{N^2}{2} + \sum_{n=1}^{N} \frac{nq^n_L}{1-q^n_L} \right) \quad \text{(30)}$$

and

$$\ll \left( \frac{\mathcal{N}}{L} \sum_x \text{tr}\Phi^2(x) \right)^2 \gg = \mathcal{N}^2 \left( \frac{2q}{1-q^2} \right)^2 \left\{ \left( \frac{N^2}{2} + \sum_{n=1}^{N} \frac{nq^n_L}{1-q^n_L} \right)^2 + \sum_{n=1}^{N} \frac{n^2q^n_L}{(1-q^n_L)^2} + \frac{1+q^2}{L} \left( \frac{N^2}{2} + \sum_{n=1}^{N} \frac{nq^n_L}{1-q^n_L} \right) \right\} \quad \text{(31)}$$

We are now ready to discuss the different limits. We are concerned with the limit $N \to \infty$, $q \to 1$ and possibly $L \to \infty$. Let us define $\epsilon_L=1-q_L=1-(1-\epsilon)^L$ and $\epsilon \equiv \epsilon_1 = 1-q$. We shall always be concerned with the limit $\epsilon \to 0$ so that $\epsilon$ is assumed to be small. We shall also assume that $\epsilon L \ll 1$ so that $\epsilon L \ll 1$.

**4.1 Limit (a): $N\epsilon_L \gg 1$**

Let us first consider the case when $N\epsilon_L \gg 1$. This is the naive large $N$ limit (considered in [7]). In this limit we pick up only the terms with the highest powers of $N$. If we choose the normalization $\mathcal{N} = 1/N^2$ these are the terms which are finite in the limit $N \to \infty$. From the above formulae we find in this limit

$$\ll \left( \frac{\mathcal{N}}{L} \sum_x \text{tr}\Phi^2(x) \right)^k \gg = \left( \frac{\mathcal{N} N^2}{2\sqrt{m^4-1}} \right)^k \left(1 + \mathcal{O}(1/N)\right) =$$
\[
\left( \langle \mathcal{N} \text{tr}\Phi^2 \rangle \right)^k (1 + \mathcal{O}(1/N)),
\]

(32)

We thus see that the factorization property is valid in this limit. Furthermore the particular value of \( \langle \mathcal{N} \text{tr}\Phi^2 \rangle = 1/2\sqrt{m^4 - 1} \) as well as the other averages (see below) are consistent with the prediction of the semicircular distribution \( \pi\rho(\phi) = \sqrt{2\epsilon - \epsilon^2\phi^2} \).

### 4.2 The limit (b): \( N \to \infty, N\epsilon_L \ll 1 \)

The limit \( N \to \infty \) with \( N\epsilon_L \ll 1 \) can be quite different than the previous one since terms in (31) which have less powers of \( N \) can be more singular as \( \epsilon_L \to 0 \). In fact since

\[
\sum_{n=1}^{N} \frac{nq_L^n}{1 - q_L^n} = \sum_{n=1}^{N} \frac{n}{n\epsilon_L} (1 + \mathcal{O}(n\epsilon_L)) = \frac{N}{\epsilon_L} (1 + \mathcal{O}(N\epsilon_L)),
\]

(33)

and

\[
\sum_{n=1}^{N} \frac{n^2q_L^n}{(1 - q_L^n)^2} = \frac{N}{\epsilon_L^2} (1 + \mathcal{O}(N\epsilon_L))
\]

(34)

we have:

\[
\langle \mathcal{N} \text{tr}\Phi^2 \rangle \equiv \frac{N}{\epsilon_L} \left( \frac{N^2}{2} + \frac{N}{\epsilon_L} (1 + \mathcal{O}(N\epsilon_L)) \right) = \frac{NN}{\epsilon_L^2} (1 + \mathcal{O}(N\epsilon_L))
\]

(35)

and

\[
\ll \left( \frac{N}{L} \sum_x \text{tr}\Phi^2(x) \right)^2 \gg = \frac{N^2}{\epsilon_L^2} \left\{ \left( \frac{N^2}{2} + \frac{N}{\epsilon_L} (1 + \mathcal{O}(N\epsilon_L)) \right)^2 + \frac{N}{\epsilon_L^2} (1 + \mathcal{O}(N\epsilon_L)) + \frac{1}{L\epsilon_L} \left( \frac{N^2}{2} + \frac{N}{\epsilon_L} (1 + \mathcal{O}(N\epsilon_L)) \right) \right\} =
\]

(36)
We see that in this limit the results are somewhat different. Firstly in order for a reasonable limit to exist $\mathcal{N}$ should be equal to $1/N$ (rather than $1/N^2$). Secondly the averages behave as $\epsilon^{-2k}_L$ (rather than $\epsilon^{-k}_L$). But despite these differences the factorization property still remains valid for $k \ll N$. Note however that the relevant master field is clearly different from that of ref.[7] and there is no reason to believe that it is described by a semicircular distribution. Our analysis so far which is based on the calculation of the specific correlators $\langle (\sum \text{tr}\Phi^2) \rangle$ is not sufficient to completely rule out this possibility.

To demonstrate the failure of the semicircular distribution, one should evaluate the more complicated averages $\langle \mathcal{N}_k \text{tr} \Phi^{2k} \rangle$ which cannot be directly extracted from the above partition functions. If the distribution of eigenvalues obeys the semicircle law these averages should be equal to

$$\langle \mathcal{N}_k \text{tr} \Phi^{2k} \rangle_{\text{semicircular}} = \frac{(2k)!}{k!(k+1)!} \frac{1}{(2\epsilon)^k}$$

(37)

Any deviation of these averages from this formula indicates that the distribution is not semicircular. We compute these averages below.

Before we proceed to the calculation note that the normalization factor $\mathcal{N}_k$ is equal to

$$\mathcal{N}_k \equiv \frac{1}{\mathcal{N}} (\mathcal{N}\mathcal{N})^k.$$  

(38)

This will be true for all the types of limits which we consider. The reason for this is that every trace should be accompanied by a factor of $1/N$ if it is to be substituted by an integral with an eigenvalue-distribution function $\frac{1}{N} \text{tr}... \rightarrow \int d\phi \rho(\phi)...$ normalized so that $\int d\phi \rho(\phi) = 1$. The factor $(\mathcal{N}\mathcal{N})^k$ can be absorbed into the normalization of the $\Phi$-field in which case the role of the effective Plank constant in the integral would be played by $(\mathcal{N}\mathcal{N})/\mathcal{N} = \mathcal{N}$ (the factor of $N$ in denominator is associated with traces in the action). Applicability of the saddle point approximation requires $\mathcal{N}$ to be small.

To simplify our evaluation of $\langle \mathcal{N}_k \text{tr} \Phi^{2k} \rangle$ we shall begin with the case $L=1$. It is straightforward to generalize to the case for general $L$. Up to terms
which are negligible since they are of higher orders in $1/L$ and $1/N$, it can
be obtained by substituting $q_L$ for $q$. The integral of interest
\[ \int d\theta_a \Delta^2(e^{i\theta_a}) \int d\phi_{ab} \exp(-\sum_{a,b} |\phi_{ab}|^2(m^2 - \cos \theta_{ab})) \text{ tr}\Phi^{2k} \quad (39) \]
can be greatly simplified in the vicinity of the critical point (This integral
should, of course, be divided by a similar integral with $k=0$ in order to give
\( \ll N_k \text{tr}\Phi^{2k} \)). Indeed, for $k=0$ this integral is just equal to (24), and the
integrand is non-singular at $q=1$. At the limiting point $q=1$ the integral
is simply equal to unity. It follows that the measure of integration over $d\theta$
in (39) becomes trivial at the critical point. Thus in order to compute our
correlators we need only evaluate the contributions of propagators
\[ \ll \phi_{ab}\phi_{cd} \gg = (2(m^2 - \cos \theta_{ab}))^{-1}\delta_{ad}\delta_{bc} \quad (40) \]
Since we are interested in the most singular terms at small $\epsilon = 1 - q \sim
\sqrt{2(m^2 - 1)}$, we can take all the differences $\theta_{ab} = \theta_a - \theta_b$ in the propagators
to be small, so that
\[ \ll \phi_{ab}\phi_{cd} \gg \to (\theta_{ab}^2 + 2(m^2 - 1))^{-1}\delta_{ad}\delta_{bc} \quad (41) \]
The propagator for the diagonal components of $\Phi$ is independent of $\theta$
and equals $1/\epsilon^2$. The contribution of any off-diagonal component
\[ \sim \int d\theta_{ab} (\theta_{ab}^2 + \epsilon^2)^{-1}/ \int d\theta_{ab} = (\pi/2\epsilon)(\pi)^{-1} = 1/2\epsilon \quad (42) \]
It remains to compute the number of diagonal and off-diagonal components
arising from $\text{tr}\Phi^{2k}$. Their contributions are easily distinguished since those
of the diagonal components give rise to higher powers of $1/\epsilon$ but to lower
powers of $N$. There are thus $N$ diagonal components and $N^2(1 + O(1/N))$
off-diagonal ones. Let us first consider the limit (a) where we retain only the
terms of order $N^2$. Now, since $k \ll N$ we can assume that integrations over
different off-diagonal elements are independent. Using a combinatorial argument (see below, this gives the contribution of the off-diagonal components
to the correlator as:
\[ N_k s_k^0 N^{k+1}/(2\epsilon)^k \quad (43) \]
with \( s_k^0 = (2k)!/k!(k + 1)! \). With the proper choice of normalization factor,

\[
\mathcal{N} = 1/N^2
\]

thus

\[
\mathcal{N}_k = 1/N^{k+1}
\]

this coincides with (44), and confirms the result of [4] for the limit (a) in one dimension.

In the limit (b) we instead take into account the contribution of only the diagonal elements of \( \Phi \). In contrast to the off-diagonal case, the contribution to \( \text{tr} \Phi^{2k} \) is given by the \( N \) times the average of one component to the power \( 2k \). This gives a factor of \( N \) (in accordance with the proper normalization \( \mathcal{N} = 1/N \) in this case) times

\[
\int d\phi(\phi)^{2k} e^{-\epsilon^2 \phi^2/2} \sim (2k - 1)!! e^{-2k} = s_k^k e^{-2k}
\]

In this limit we clearly get a distribution function of the form

\[
\rho(\phi) = \frac{\epsilon}{\sqrt{2\pi}} e^{-\epsilon^2 \phi^2/2}
\]

rather than the semicircular distribution. Note that this \( \rho(\phi) \) is non-vanishing on the entire axis.

**4.3 Limit (c): \( N \epsilon L \sim 1 \)**

We shall now consider the limit where \( N \) is large, \( \epsilon \) is small but \( N \epsilon \) is of order 1. This is a kind of a “double scaling” limit somewhat different from the type usually encountered in matrix models of gravity where \( N \) and \( L \) are correlated and \( N \) can be interpreted as a new continuum variable giving rise to a new space–time dimension \( d \to d + 1 \). The formulae for the correlators in this limit are

\[
\ll \mathcal{N}_k \text{tr} \Phi^{2k} \gg = (NN)^k \sum_{j=0}^{k} s_k^{(j)} \frac{(N/2)^{k-j}}{e^{k+j}} (1 + \mathcal{O}(1/N)).
\]

In the limits (a) and (b) the first \((j = 0)\) and the last \((j = k)\) terms in this sum are dominating respectfully. The coefficients \( s_k^{(j)} \) can be defined
from the following combinatorial problem. Take \( \text{tr} \phi^{2k} = \sum_{\{a_i\}} \phi_{a_1a_2}...\phi_{a_{2k}a_1} \) and consider all the pair contractions of \( \Phi \)'s (Wick rule), each contraction being \( \langle \phi_{ab}\phi_{cd} \rangle \equiv \delta_{ad}\delta_{bc}(\alpha + \beta\delta_{ab}) \), where \( \alpha = 1/2\epsilon \) and \( \beta = 1/\epsilon^2 \). The \( s^{(j)}_k \) is by definition the leading term (with the highest possible power of \( N \)) in the coefficient in front of \( \alpha^{k-j}\beta^j \). Let us draw a picture (Fig.1a) where dots correspond to the \( \Phi \)-matrices. Every solid-line contraction (Fig.1b) of dots, gives rise to a factor of \( \alpha \) and makes an identification \( a = d, b = c \); while every wavy-line contraction (Fig.1c) contributes a factor of \( \beta \) and identifies \( a = b = c = d \). One can easily check that any intersection of two solid lines “eats up” a factor of \( N^2 \), while any intersection of a solid line and a wavy line eliminates one \( N \). Thus the leading-\( N \) contribution arises in the “planar” limit, when the only lines of contraction which are allowed to intersect are wavy lines. Thus, estimation of \( s^{(j)}_k \) is a clear combinatorial problem of enumeration of pair contractions of the \( 2k \) points on a circle by \( k-j \) solid and \( j \) wavy lines, such that only wavy lines are allowed to intersect. One can easily derive a recurrent relation:

\[
s^{(i)}_{k+1} = \frac{k+1}{k+1-i} \sum_{j=0}^{i} \sum_{l=j}^{K-i+j} s^{(j)}_l s^{(i-j)}_{k-l}; \quad i \leq k. \tag{49}
\]

It is enough to note, that the first solid line separates the circle into two new circles of the lengths \( l \) and \( k-l \). The factor \( \frac{k+1}{k+1-i} \) arises because there are \( 2(k+1) \) points to which the first solid line can be attached, while at the end there are as many as \( 2(k+1-i) \) points at the ends of all solid lines. The relations (49) should be supplemented by “initial conditions”:

\[
s^{(0)}_0 = 1; \quad s^{(k)}_k = (2k-1)!! \text{, the first one is obvious, the second is just the number of all possible (intersections allowed) contractions of 2k dots by k wavy lines. This information is enough to find any } s^{(i)}_k \text{. In particular,}
\]

\[
s^{(0)}_k = 2^k \frac{(2k-1)!!}{(k+1)!} = \frac{(2k)!}{k!(k+1)!} \tag{50}
\]

and

\[
s^{(i)}_k = s^{(0)}_k \frac{k!}{(k-i)!!(k+2)\ldots(k+i)} \frac{P_i[k]}{(k-i)!(k+i)!!} = \frac{(2k)!}{(k-i)!(k+i)!!} P_i[k] \tag{51}
\]

where \( P_i[x] \) are certain polynomials of degree \( i-1 \):

\[
P_0 = \frac{1}{x+1}, \quad P_1 = 1, \quad P_2 = x + 1,
\]

17
\[ P_3 = x^2 + 11x + 48, \quad P_4 = x^3 + 21x^2 + 218x + 1248, \ldots \] (52)

4.4 Limit (d): The Modular form

Finally there seems to be one more non-trivial (and, perhaps, attractive) possibility. It comes to mind when one looks at the formula (20) for \( Z_D \). Clearly there should exist a large \( N \) limit, when

\[ Z_D \to \frac{q_L^s}{\eta(qL)} \] (53)

Naively \( s = N^2/2 + 1/24 \), but one can think about reinterpretation of the \( U(N) \)-dimension \( N^2 \) as, say,

\[ N^2 = N + 2 \frac{N(N-1)}{2} = \sum_{n=1}^{N} 1 + 2 \sum_{n=1}^{N-1} n \to \infty - \frac{1}{2} + 2 \left( -\frac{1}{12} \right), \] (54)

i.e. as being regularized with the \( \zeta \)-function technique, thus giving \( s \) in (53) some finite value \( (s = -1/8) \). More interesting, the elliptic function \( \eta(q) \equiv q^{1/24} / \prod_{n=1}^{\infty} (1 - q^n) \) in the denominator in (53) has an essential singularity as \( q \to 1 \):

\[ \frac{1}{\eta(qL)} \sim \sqrt{(1-qL)} \exp \frac{\pi^2/6}{1-qL}. \] (55)

This gives rise to expressions for the correlators like

\[ \ll (\text{tr} \Phi^2)^k \gg = \left( \frac{\partial \log q}{\partial m^2} \frac{\pi^2/6}{(1-qL)^2} \right)^k (1 + O(1-qL)), \] (56)

which again obey the factorization property in the leading approximation. The role of \( N \) of the previous cases is now played by \( N_{\text{eff}} \sim \frac{1}{1-qL} = \frac{1}{\alpha} \), which is large near the critical point; in this sense this limit can be just some particular case of (c).

Note also, that \( \eta(q) \) in the denominator of (53) is equal to the square root of Laplace operator on two-dimensional surface (with two real dimensions, \( q \) plays the role of the modular parameter, describing the shape (complex structure, to be exact) of the torus).
4.5 Alternative Large–N Limits Beyond $d = 1$

We saw in the previous Sections that the critical behaviour in a Gaussian model arose from the presence of zero modes of the quadratic form in the action. We also saw that in the Gaussian $d=1$ model exactly the $N$ (= rank $U(N)$) of the $N^2$ (= dim $U(N)$) zero modes of the $\Phi$-field survive after the $U$–integration. Here, we shall argue that the case of $d > 1$ is much more subtle.

Let us first consider an analogue of the model $C$ from Section 3 in $d$ dimensions. The partition function is given by

$$Z_C^{(d)} = \int \prod_x d\Phi_x e^{-m^2 \text{tr} \Phi_x^2} \prod_{<x,y>} e^{\text{tr} \Phi_x \Phi_y},$$

(57)

The result of the integral is of course the determinant of the lattice Laplace operator raised to the power $-N^2/2$. Zero modes on a lattice without boundaries (such as the $d$–dimensional torus) are constant ($x$-independent) matrices $\Phi$. There are thus $N^2$ of them. On a rectangular lattice they arise when the mass vanishes i.e. when the parameter $m^2 = m_{\text{crit}}^2 = d$.

If we proceed to the KMM (1) the theory becomes non–linear and the number of zero modes is no longer $N^2$. In fact when $d>1$ it is no longer trivial to show that there are $N$ zero modes associated with the diagonal matrices $\Phi$ (i.e. those belonging to the Cartan subalgebra of $U(N)$). This is already clear because diagonal matrices are not $a \text{ priori}$ distinguished in eq.(1). In the one–dimensional model $D$ they were distinguished but only after the $U$–matrices were diagonalized. Then the diagonal components of $\Phi$ decoupled from $U$–fields (since $|U_{ii}|^2 = 1$ for diagonal $U_{ij}$), and the $N$ corresponding zero modes survived the integration over $U$. Unfortunately there is no analogue of this procedure in the higher–dimensional case. Gauge transformations (3) are not enough to make all the $U$’s at all the links diagonal. (The most one can do is to eliminate all the $U$’s on any maximal tree of the lattice and diagonalize one more $U$.) It is however possible to diagonalize $\Phi$ at all the sites. (The difference is that for $d>1$ there are more links than sites). As a result the $U$–integral depends only on the eigenvalues of $\Phi$. The resulting formula [11] is given by

$$I(\Phi, \Psi) = V_N \frac{\det (\Phi) e^{\phi \psi}}{\Delta(\phi) \Delta(\psi)} = V_N \frac{\sum_{P} (-1)^P e^{\sum_{i=1}^{N} \phi_i \psi_{P(i)}}}{\Delta(\phi) \Delta(\psi)},$$

(58)
where \( \Delta(\phi) = \prod_{i<j}^N (\phi_i - \phi_j) \) and the sum is over all the \( N! \) permutations \( P \) of the \( N \) variables \( \{\psi_1, \ldots, \psi_N\} \).

Despite the above argument we still claim that there are \( N \) zero modes even when \( d > 1 \). To understand the idea of the derivation we begin with a simple model which imitates some of the features of the \( d \)-dimensional lattice. We consider the KMM on a lattice consisting of a single site with \( d \) links attached to it. This “lattice” is sometimes called a “bouquet” and is shown in Fig. 2. After the \( U \)-integrations the partition function for the KMM on this lattice is

\[
Z_D[bouquet] = \int \prod_{i=1}^N d\phi_i e^{-m^2 \phi_i^2} \left( \sum_P (-)^P e^{\sum_{i=1}^N \phi_i \phi_P(i)} \right)^d. \tag{59}
\]

This integrand is non-singular as \( \phi_i - \phi_j \to 0 \) since the order of the zeros in the numerator is more than enough to cancel the zeros in the denominator. Zero modes of the \( \Phi \)-field contribute to the integral at large \( \phi_i \).

To find the dominant singularity let us simplify the problem even more by considering first the case \( N=2 \). In this case the eigenvalues \( \phi \) of \( \Phi \) can be written as \( \phi_{1,2} = \frac{1}{\sqrt{2}} (\phi_{tr} \pm \varphi) \). The variable \( \phi_{tr} \) which is associated with the \( U(1) \) factor of \( U(N) \) decouples (for any \( N \)) and gives rise (for any \( N \)) to a factor

\[
\int d\phi_{tr} e^{-(m^2 - d) \phi_{tr}^2} \sim \frac{1}{\sqrt{N}} \frac{1}{(m^2 - d)} \tag{60}
\]

in the partition function. (This is why we usually have one trivial zero mode if we consider \( U(N) \) instead of the \( SU(N) \).) For \( N=2 \) the contribution of \( \varphi \) to the partition function is

\[
Z_D[bouquet] \sim \int d\varphi e^{-m^2 \varphi^2} \frac{(e^{2\varphi^2} - e^{-2\varphi^2})^d}{\varphi^{2(d-1)}} \sim \sum_{k=0}^d (-1)^{d-k} \frac{d!}{k!(d-k)!} (m^2 + d - 2k)^{d-3/2} \tag{61}
\]

The first singularity which arises as \( m^2 \) decreases from infinity is at \( m^2 = m_{\text{crit}}^2 = d \). Note that, unlike the situation in \( d = 1 \) where the partition function itself is singular at the critical point, here it is non–analytic so that high enough orders of its derivatives are singular. In \( d=4 \) the only the third and
higher order derivatives by \( m^2 \) become singular, the standard behaviour of a third order phase transition. This non-analyticity comes from the large \( \phi \) behaviour of the integrand which contains the factor \( e^{(d-m^2)\phi^2} \). It is associated with the limit where large \( \phi \) is damped only by a logarithmic potential in the action, rather than a Gaussian. This is a result of the \( \varphi^{2(d-1)} \) term in the denominator which comes from the Van–der–Monde determinants. We clearly see how the \( U \)-integration smooths the singularity at \( m^2 = d \): If this were a Gaussian Hermitean matrix model the Van–der–Monde determinant would appear in the numerator rather than the denominator and the integration would really be singular at \( m^2 = d \).

It is clear that the partition function can be made more singular at \( m^2 = d \) by inserting a power of \( \varphi \) in the measure for the \( \Phi \) integration. This option, which in the general case amounts to inserting a power of \( \det \Phi \) in the integration measure, is discussed in more detail in Section 5.

It is straightforward to generalize this to arbitrary \( N \). The main singularity comes from the contribution of the identity permutation \( P = I \) in the sums (59) on every link. The corresponding contribution is \( \sim (m^2 - d)^{N(N-1)(d-1)/2-N/2} \). (We have include the \( U(1) \) factor of \( U(N) \)). As a byproduct this calculation reveals an interesting feature of the model. Despite the non-linearity of the Gaussian KMM the critical value of \( m^2 \) is just the same as in the quasiclassical approximation. This observation is consistent with our hopes that this model may be exactly soluble.

Note that one should not be confused by the fact that the term \( P = I \) is only one of \( N! \) terms in the sum (59) and that there is also a product over all of the links of the lattice. In fact the situation is similar to evaluating \( (1 + (N!\epsilon^N)^{-1})^{\# \text{of links}} \) and it may seem that the second term is negligibly small. We saw however in the previous section that this does not happen. (The factor of \( N! \) is hidden in the \( \prod_{n=1}^{N}(1 - q^n) \sim N!(1 - q)^N \) as \( q \sim 1 \). Indeed note that \( N!\epsilon^N \sim (N\epsilon)^N \) and in the limit (b), when \( N\epsilon \ll 1 \) this term obviously dominates.

Unfortunately a similar analysis of the most intriguing limit (d) is difficult to do, even approximately, in higher dimensions. It almost certainly requires an exact solution of the \( d \)-dimensional KMM (which may not be an absolutely hopeless problem) and it will be left for future investigations.


4.6 The Large Volume Limit

To end this section we comment on the large $L$ limit and we explain why only case (a) of the large $N$ limits considered above has anything to do with continuum limit of the KMM. The crucial point is that in this limit one may not assume that $q^L$ is close to unity ($\epsilon_L \ll 1$) whenever $q$ is near 1 ($\epsilon \ll 1$). One can say that the infrared critical point $q = 1$ is exponentially unstable under renormalization group evolution with increasing $L$. (Recall [3] that it is $q_L = M_L^2 \pm \sqrt{M_L^4 - 1}$ rather than the mass $M_L$ that is renormalized in the simple manner $q_L = q^L$.) It thus follows that our considerations in this section are not relevant for the large $L$ limit of the theory. This is made even clearer by considering the free energy

$$-\log Z_D = -\frac{N^2}{2} \log q_L + \sum_{n=1}^{N} \log(1 - q_L^n)$$ (62)

which in the normal thermodynamical limit should be proportional to the size $L$ of the system as $L \to \infty$. This is certainly true for the first term on the r.h.s. of the above equation since $\log q_L = L \log q$ for any $q$). But $\log(1 - q_L^n) \to 0$ as $L \to \infty$ unless $q$ is fine tuned with exponential accuracy so that $1 - q_L^n$, which normally $\sim (1 - e^{-L_N\epsilon})$ can be replaced by $nL\epsilon \ll 1$. This becomes even more spectacular in the limit (d) where the main contribution to the free energy is

$$\log \eta(q_L) \to \frac{\pi^2/6}{1 - q_L} \sim \frac{\pi^2/6}{L\epsilon}$$

which is inversely proportional to the size $L$ of the system. Thus for the study of the large $L$ limit only the first term in (62) is relevant. This is of course the one which is important for the limit (a), which was examined in [7] and which leads to the semicircular distribution.

One could think that this argument is important to explain the irrelevance of limits (b)-(d) only in one dimension. Indeed, there is a great difference between the KMM for $d=1$ and for $d>1$. The main feature of the KMM in dimensions greater than one is the presence of many closed contours on the lattice while there is only one such contour in model $D$ in one dimension. If there were no closed contours, all the $U$-variables could be gauged away by a gauge transformation [3]. This very reason makes the large $L$ limit of the
KMM in one dimension (where only one matrix $U$ survives) very different. For the study of KMM in higher dimensions it is more reasonable to look at the 1–dimensional KMM with small values of $L \sim 1$. However, one can easily see that this argument does not help the critical phenomena arising due to the zero-modes: all of them become irrelevant in infinite volume in any dimensions$^7$.

5 KMM Beyond One Dimension: From the KMM to a Modified KMM

Another simple example of the KMM which can be analyzed rather easily is the model with $N = 2$. We already used this example in the previous section, as well as in our considerations of the correlators in \[6\]. We shall now analyze the mean-field approximation in the $\Phi$-sector of this model and see that it is, first, in agreement with the exact solution at $d = 1$, second, it does not change much when $d$ is increased to be greater than one, and, third, implies the existence of the nice mean-field in $d > 1$ KMM. Of course these results do not need to mean too much for the large-$N$ limit of the KMM at $d > 1$, and the conclusion of ref.\[4\] still can be unavoidable in that situation. However, we do not see any obvious reason for this in our simplified analysis.

The basic formula for the study of the $\Phi$-sector of KMM for $N = 2$ is the integral (61). While we introduced this formula for a specific bouquet lattice, it has a more universal meaning: it describes the effective theory of a constant master field on a conventional rectangular $d$-dimensional lattice. The value of the master field is therefore defined from the equation of motion for effective potential

$$V_{\text{eff}}(\varphi) = m^2 \varphi^2 + (d - 1) \log \varphi^2 - d \log \sinh \varphi^2,$$

\[63\]

$^7$The only possibility for these types of critical points to survive can be some sophisticated limits (similar to conventional double scaling limit of the gravity-models), when $L$ and $N$ are adjusted to increase in a correlated fashion, so that $Ne^{-L\epsilon} = \text{const}$, or, in other words, when $N$ grows exponentially with the decrease of the lattice spacing $a$: $\log N \sim 1/a$. This is extremely unnatural from the point of view of QCD, where one can rather expect that the correlation (if any) is $N \sim 1/\log a$. 

23
which look like

$$m^2 + \frac{d - 1}{\varphi_0^2} = d \coth \varphi_0^2$$

(64)

while

$$\frac{\partial^2 V(\varphi_0)}{\partial \varphi^2} = 4 \varphi_0^2 \left( \frac{d}{\sinh^2 \varphi_0^2} - \frac{d - 1}{\varphi_0^4} \right).$$

(65)

If $m^2 < m_{\text{crit}}^2 = d$, this potential has a form of Fig.3a, it usually has a minimum, but needs some higher order terms to be stabilized at infinity (puncture line on the picture); while in the case of $m^2 > m_{\text{crit}}^2 = d$ it looks like Fig.3b, and has a clear minimum at $\varphi = \varphi_0$ and does not require anything in order to be stable.

In the case of $d = 1$ the logarithmic term in (64) disappears and one can get:

$$\varphi_0^2 = \frac{1}{2} \log \frac{m^2 + 1}{m^2 - 1}$$

(66)

and

$$Z_{\text{mean field}} \equiv e^{-V_{\text{eff}}(\varphi_0)} \det^{-1/2} \frac{\partial^2 V(\varphi_0)}{\partial \phi_x \partial \phi_y} \rightarrow$$

$$\frac{\exp \left( \frac{m^2}{2} \log \frac{m^2 - 1}{m^2 + 1} \right)}{(m^4 - 1) \log^{1/2} \frac{m^2 + 1}{m^2 - 1}} \sim 1/\sqrt{(m^2 - 1) \log^{1/2} \frac{m^2 + 1}{m^2 - 1}}$$

(67)

in agreement (up to the logarithmic factor) with the $m^2 \rightarrow 1$ limit of exact formula (20) for the partition function. This is remarkable, since there is no a priori reason to rely upon applicability of the mean-field approximation for $N = 2$. This opens the possibility to believe that analogous results can have something to do with reality at $d > 1$ as well.

The explanation of the picture Fig.3b is in fact very simple and seems essentially independent of $N$. From formula (59) it is clear that for infinitely large and different values of the eigenvalues $\phi_i$ the potential is grows quadratically as long as $m^2 > m_{\text{crit}}^2 = d$. When two eigenvalues come close to each other, the potential grows logarithmically and they repel. This is because of the obvious singularity, arising from $\Delta(\phi)$ in the denominator in
(59) (and could be interpreted as attraction of eigenvalues for \( d > 1 \)) is in fact overcome by the zero (of the order \( 2d \)) in the sum in brackets in (59) (or the numerator in (58). It looks very plausible that this behaviour, which is seen clearly in the case \( N = 2 \), shown in the Fig.3b (repulsion at coinciding \( \phi_i \)'s and quadratic well at infinity) should persist for all \( N \).

However, the properties of this minimum are drastically different at \( d = 1 \) and at \( d > 1 \). Indeed, at \( d = 1 \) the value of the master field \( \phi_0 \) tends to infinity as one approaches the critical point \( m_{\text{crit}}^2 = d \) from above, while \( \partial^2 V(\phi_0)/\partial \phi^2 \) tends to zero, thus giving rise to singularities and the critical behaviour of the partition function, but this is no longer true if \( d > 1 \). The logarithmic term \( +(d-1) \log \phi^2 \) in the effective potential (58) forces it to grow at large values of \( \phi^2 \) even when \( m^2 = d \). Thus the position of the minimum remains finite, the curvature \( \partial^2 V(\phi_0)/\partial \phi^2 \) at the minimum does not vanish even for infinitely small \( m^2 - d \) and no critical behaviour occurs. This makes the result of ref.[7] very natural. The earlier claim of ref.[2] that higher order terms in the \( \Phi \)-potential can be adjusted to produce critical behavior means that the coefficient of the \( \text{tr}\Phi^4 \) term in the potential can be adjusted to make the curvature small.

This, however, does not seem to be the simplest possibility to overcome the problem, found in [4]. Our interpretation of the loss of criticality at \( d > 1 \) attributes it to a logarithmic term in the effective potential (which arises from the Van-der-Monde determinants \( \Delta(\phi) \)). This effect can easily be compensated by adding a logarithmic term to the bare potential. This is equivalent to introducing \( (\text{det}\Phi)^\alpha \) in the measure of integration of original KMM (1):

\[
\tilde{Z}_D \equiv \int d\Phi(x)[dU(x,y)](\text{det}\Phi(x))^\alpha \exp \left( -\sum_x \text{tr} V(\Phi(x)) + \sum_{<x,y>} \text{tr}\Phi(x)U(x,y)\Phi(y)U^\dagger(x,y) \right). \tag{68}
\]

Clearly \( \alpha \) should be non-negative, \( \alpha \geq 0 \), in order to leave the integral over Hermitian matrices \( \Phi \) well defined. This is exactly what necessary in order to restore the critical behaviour (for \( m^2 \to d + 0 \) and \( d > 1 \)): we just need \( \alpha \) to be positive (and big enough). In fact for \( N = 2 \) the shape of the effective potential in Fig.3b remains the same for \( \alpha \neq 0 \), only the increase at the origin is a bit faster: \( -\log \phi^2 \to -(\alpha + 1) \log \phi^2 \), while the asymptotics at
infinity $(d-1) \log \phi^2 + (m^2 - d) \phi^2 \to (d-1-\alpha) \log \phi^2 + (m^2 - d) \phi^2$, is clearly dominated by the $\phi^2$ term if $\alpha \geq d - 1$. For $N > 2$ the picture becomes more complicated, since $\det \Phi$ and $\Delta(\Phi)$ are very different functions of the eigenvalues of $\Phi$. (Also note, that the trace part of $\Phi$ does not decouple as $\alpha \neq 0$.) However, for big enough $\alpha$ there usually will be critical behaviour at some minima with large and non-coinciding $\phi_i$’s.

Indeed, one can think about the entire set of eigenvalues as of a system of $N$ point particles on a line (a variant of the Dyson gas). These particles interact through a logarithmic repulsion at short distances and a logarithmic attraction at long distances $(d-1) \log(\phi_i - \phi_j)^2$: a rather peculiar picture of a Coulomb gas with a logarithmic “hard core” which is also Coulomb-like. The particles also interact with the origin through a combination of a central harmonic potential and a logarithmic repulsion, $(m^2 - d) \phi_i^2 - \frac{\alpha}{2} \log \phi_i^2$. Furthermore, since the matrices are traceless, the center of mass of the particles is constrained to be at the origin. (Actually, we could consider matrices with a non-zero trace but, as we have discussed previously, the trace is an irrelevant degree of freedom.)

As long as $m^2 - d > 0$ the longest range part of the interaction is harmonic and the particles are confined in the vicinity of the origin with size of the order $1/\sqrt{m^2 - d}$. When $m^2 - d = 0$, the particles still have a Coulombic long-ranged attraction and if $\alpha$ is not too large, they still form a stable cloud. When $\alpha$ is large enough, the repulsive central potential destabilizes the cloud and repels the particles to infinity. This is the appropriate critical behavior. This system is a rather simple object both for gedanken experiments and computer simulations; moreover, as we tried to demonstrate, the most interesting qualitative results can be extracted from the study of the simplest cases, including $N = 2$.

The simple analogue of the model (68),

$$
\int d\Phi (\det \Phi)^\alpha e^{-\text{tr} V(\Phi)} e^{\text{tr} \Lambda \Phi},
$$

(69)

with a single $\Phi$-matrix and a slightly different interaction with the matrix-valued background field $\Lambda$ (analogue of $U$), which is known as the Generalized Kontsevich Model [9], have been analyzed in [10]. At least for this model the effect of non-vanishing $\alpha$ is known to create no problems for the solvability of the model, and is, moreover, natural from the view of its integrability structure. Introduction of the parameter $\alpha$ even proved useful for description
of continuum limits of the model [10]. All this can easily appear true for the modified KMM (68), where the dependence on the potential $V(\Phi)$ should be studied at least with the purely theoretical motivations (to reveal the intrinsic integrable structure, for example). This, however, is the deal for the future.

6 Conclusion

This paper was devoted to analysis of the $\Phi$-sector of the “Gaussian” KMM. The really interesting properties of the KMM (see [6]) are essentially independent of the structure of the $\Phi$-sector. It is only necessary that it guarantees the existence of some master-field $\phi$ and that the magnitude of the master field is large. We saw that these requirements were fulfilled in all of the situations we discussed above, even if the critical point has nothing to do with the continuum limit.

It is of course important to understand the variety of possible universality classes associated with the critical behavior. This is why it is also interesting to investigate the $\Phi$-sector of the model. A priori in the KMM it is not forbidden to look for a non–trivial universality class represented by a rather simple “Gaussian” model.

We showed that this model indeed exhibits a rich pattern of critical behavior even at $d = 1$, but, as we demonstrated, only the most naive large-$N$ limit has anything to do with the continuum limit: other types of the critical points (defined as the places in parameter space where the partition function is non-analytic) do not survive as the volume of the system become infinite (they are infrared unstable critical points). Thus, unconventional large $N$ limits cannot save the $d > 1$ Gaussian KMM from the conclusion of [7].

Therefore, it is necessary to adopt another line of reasoning. In Section 5 we proposed a simple explanation of the result of [7]: in the Gaussian model and when $d > 1$ the effective potential for the eigenvalues of $\Phi$ remains convex for all $N$. This was due to a logarithmic attraction of the eigenvalues at large distances. We can estimate the magnitude of the eigenvalues at their equilibrium positions and it is not so large, probably, of order 1-10, probably not large enough for the mean field approximation to be effective for the $U$-sector of the theory.

This reasoning also suggests a way to overcome this difficulty. We intro-
duce a repulsive logarithmic central potential which increases the magnitudes of the eigenvalues at their equilibrium positions and, if it is strong enough and when the curvature of the Gaussian vanishes, it gives a critical behavior where some or all of the eigenvalues take on infinite (or at least arbitrarily large) magnitudes. This logarithmic potential can be regarded as a modification of the integration measure in the partition function of the Gaussian KMM. This is a rather mild modification of the model and may not affect its chances to be exactly solvable.

Acknowledgments
A.M. acknowledges the hospitality of the theory group of the Physics Department at UBC.
References

[1] V.A. Kazakov and A.A. Migdal, “Induced QCD at Large N”, preprint PUPT - 1322, LPTENS -92/15, May, 1992.

[2] A.A. Migdal, “Exact Solution of Induced Lattice Gauge Theory at large N”, preprint PUPT - 1323, Revised, June, 1992; “1/N expansion and particle spectrum in induced QCD”, Princeton preprint PUPT-1332, July, 1992.

[3] I.I. Kogan, G.W. Semenoff and N.Weiss, “Induced QCD and Hidden Local $Z_N$ Symmetry”, preprint UBCTP 92-22, June, 1992.

[4] M.Caselle, A. D’Adda and S.Panzeri, “Exact solution of $D = 1$ Kazakov–Migdal induced gauge theory”, preprint DFTT 38/92, July, 1992.

[5] A.Gocksch and Yu.Shen, “The Phase Diagram of the $N = 2$ Kazakov–Migdal Model”, preprint BNL, July, 1992.

[6] I.I.Kogan et al. “Area Law and Continuum Limit in “Induced QCD”, preprint UBCTP 92-26, ITEP M6/92, July 1992

[7] D.Gross, “Some Remarks about Induced QCD”, preprint PUPT-1335, August, 1992.

[8] S.Khokhlachev and Yu.Makeenko, “The Problem of the Large-N Phase Transition in the Kazakov–Migdal Model of Induced QCD”, preprint ITEP-YM-5-92, July, 1992; Yu. Makeenko, “Large-N Reduction, Master Field and Loop Equations in Kazakov-Migdal Model”, preprint ITEP-YM-6-92.

[9] M.Kontsevich, Funk.Anal.and Appl. 25 (1991) 50; Yu.Makeenko and G.Semenoff, Mod.Phys.Lett. A6 (1991) 3455; D.J.Gross and M.J.Newman, Phys.Lett. 266B (1991) 291; E.Witten, “On the Kontsevich Model and Other Models of Two-Dimensional Gravity”, preprint IASSNS-HEP-91/24, June, 1991; S.Kharchev et al., Phys.Lett. 275B (1992) 311; Nucl.Phys. B380 (1992) 181.

29
[10] L.Chekhov and Yu.Makeenko, Phys. Lett. 278B (1992) 271;
    Mod.Phys.Lett. 7 (1992) 1223;
    S.Kharchev et al. "Generalized Kontsevich Model versus Toda Hierarchies and Discrete Matrix Models", preprint FIAN/TD/3-92, ITEP-M3/92, February 1992.
    L.Chekhov, preprint MIAN, May, 1992.

[11] Harish-Chandra, Amer. J. Math. 79 (1957) 87;
    C.Itzykson and J.B. Zuber, J. Math. Phys. 21 (1980), 411;
    M.L. Mehta, Comm. Math. Phys. 79 (1981) 327;
    A.A. Alekseev, L.D. Faddeev and S.L. Shatashvili, J. Geom. Phys. Volume dedicated to I.M. Gelfand, 3 (1989)

[12] D. Boulatov and V. Kazakov, “One-Dimensional string theory with vortices as the upside-down matrix oscillator”, preprint LPTENS 91/24, KUNS 1094 HE(TH) 91/14 (August 1991).
Figure Captions:

Fig. 1a. Pictorial representation of the trace of a product of $\Phi$-matrices. Dots correspond to the matrices and labels $a_1, \ldots$ correspond to pairs of indices in the matrix product.

Fig. 1b. Using Wick’s theorem to compute the correlation functions we must consider contractions of the matrices, the first kind of which we denote by a solid line and is depicted here. This contraction of two matrices identifies the the indices as shown.

Fig. 1c. The other contraction, denoted by a wavy line, identifies the indices as depicted here.

Fig. 2 The bouquet.

Fig. 3a The effective potential for SU(2) when $m^2 < m^2_{\text{crit}}$ when $d > 1$.

Fig. 3b The effective potential for SU(2) when $m^2 \geq m^2_{\text{crit}}$ when $d > 1$. 