I. INTRODUCTION

For quantum information science and technologies, it is crucial to build the fundamental quantum logic gates [1]. Together with the basic single bit logic gates, the non-trivial two bit gates constitutes the fundamental blocks for the quantum network of quantum computing [2]. Physically, the two-bit quantum logic gate are based on the entanglement of two-qubit through the controllable interaction between them [3].

In most scenarios to build the two-qubit logic gates, two qubits interact with a common object as data bus. When the normal frequency of the data bus is off resonance with respect to the energy level spacing of the qubits, the variables of the data bus can be removed and then induce an effective interaction between the two qubits, which can create an entanglement based on two-qubit logic gate. Most recently, this idea was used to construct the two-qubit logic gates of the superconductor Josephson junction where the data bus is implemented as a large junction. Since the recent experiments [4] have shown longer time quantum coherence, the Josephson junction qubits (JJ qubits) including charge qubits, flux qubit and the single junction qubit, the practical schemes to implement the two-gate operations of JJ qubit become very important. We notice that many of these schemes depend on the approximate approach to treat the large junction as a harmonic oscillation. We call this approximation the harmonic approximation (HA). Due to the linearity of its coupling to the two charge qubits, the variable of the harmonic oscillator (HO) is easy to remove in a dynamic way or adiabatically [6].

Now the validity of HA becomes the focus in search of the implementation of the JJ qubit based quantum computation. In this paper, we will tackle this problem by the numerical calculation and the analytical considerations.

II. THE MODEL FOR THE COUPLING OF A LARGE JUNCTION TO A CHARGE QUBIT

We consider a system of coupled charge-phase qubits [7, 8] as illustrated by the electronic circuit shown in Fig.1. We are interested in the limit case that the large junction (of capacitance $C$) stays at a low excitation or even a thermal state. The Cooper pair box is a small junction of capacitance $C' = C''$ that form a superconducting loop. $C_g$ is the capacitance of the gate, and $E_J$ is the Josephson coupling energy. The total Hamiltonian containing the reduced Coulomb energy and the three Josephson coupling energies [8] reads

$$H = E_c (n' - n_g)^2 - E'_J \cos \phi' + \cos \phi'' + E_c n^2 - E_J \cos \theta$$

where $E_c = 2e^2/C_\Sigma$, $n_g = C_g V_g/2e$, $E_c = 2e^2/C$ and $C_\Sigma = C_g + C' + C''$. Here $n'$ is the number of Cooper pair on the island, while $n$ is the number of Cooper pair on the Coulomb island connected with the large junction. $\phi'$, $\phi''$, and $\theta$ are superconducting-phase differences across the relevant junction. They are related through the fluxoid quantization condition around the loop $\theta + \phi' - \phi'' = 2\Theta$ and $\Theta_k = \pi \Phi_k / \phi_0$. Introducing $\phi = (\phi' + \phi'')/2$, we rewrite the Hamiltonian as

$$H = E_c (n' - n_g)^2 + E_c n^2 - E_J \cos \theta - 2E'_J \cos(\frac{\theta}{2} - \Theta) \cos \phi$$

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For charge-phase qubit, when \( n_g = 0.5 \) and \( E_J \) is not much larger than \( E_c \), the above Hamiltonian can be reduced as

\[
H = E_c(n^2 - T \cos \theta + T' \sin \frac{\theta}{2} \sigma_3)
\]

where \( T = E_J/E_c, T' = E_J'/E_c \) and \( \sigma_3 \) is Pauli matrix. The operators \( n \) and \( \theta \) satisfy the quantization condition \([\theta, n] = i\).

To consider the validity of the HA we only focus on the large junction together with its coupling to the charge qubit. In this sense we write down the corresponding Hamiltonian

\[
H = \hat{n}^2 - T \cos \theta + T' \sin \frac{\theta}{2} \sigma_3
\]

\[
\equiv \begin{bmatrix}
H^{(+)} & 0 \\
0 & H^{(-)}
\end{bmatrix}
\]

Since the relation \( H^{(+)}(\theta) = H^{(-)}(-\theta) \) holds, we know that these two Hamiltonians \( H^{(+)}, H^{(-)} \) have the same spectrum. Thus one can get all the eigenstates of (1) from the eigenstates of \( H^{(+)} \). In the following, we will concentrate on the eigenstates of the Hamiltonian in the qubits states spanned by discrete charge states \( |n\rangle >, n = 0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2, \cdots, \pm \infty \). The periodic potential can be expanded around the equilibrium points \( \theta_0 \), where \( \sin(\theta_0/2) = \pm \frac{T'}{4T} \) at which potential has minima. Then \( H^{(+)} \) can be written as

\[
H^{(+)} = \begin{cases}
\hat{n}^2 + \frac{16T^2 - T'^2}{32T} [\theta - (2\pi i) + 2 \arcsin(-T'/4T)]^2 - \frac{8T^2 + T'^2}{8T}, \\
\theta \in [(2i - 1)\pi, (2i + 1)\pi], i = \text{even}
\end{cases}
\]

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\theta \in [(2i - 1)\pi, (2i + 1)\pi], i = \text{odd}.
\end{cases}
\]

In large \( T \) limit, the tunnelling between two neighbor wells is forbidden. The eigen functions of \( a_i^\dagger a_i \) are standard Harmonic wave functions with the argument \( [\theta - (2\pi i) \pm 2 \arcsin(-T'/4T)] \) when \( i \) is even or odd. Obviously, the spectrum of the Hamiltonian \( \tilde{H} \) is the same as the simple harmonic oscillator, i.e.

\[
E_m = (m + \frac{1}{2}) \sqrt{\frac{16T^2 - T'^2}{8T}} - T - \frac{T'^2}{8T}
\]

\((m = 0, 1, 2, \cdots)\).
but the eigenstates are multi-fold degenerate. Notice that the above discussion is based on the assumption that there is no tunnelling between two individual parabolic potential wells. The validity of this assumption need to be investigated analytically and numerically.

III. VARIATIONAL SOLUTION FOR THE MOTION EQUATION

In this section, we solve the Schrödinger equation governed by large junction Hamiltonian (2). In the Hilbert space spanned by the eigen state \( |p\rangle \) of \( \hat{n} \), where \( \hat{n}|p\rangle = p|p\rangle \) and \( p \in [-\infty, \infty] \), the Hamiltonian can be written as

\[
H = \int_{-\infty}^{\infty} |p><p|p^2dp
- \frac{T}{2} \int_{-\infty}^{\infty} [|p + 1><p| + |p><p + 1|]dp
- i\frac{T'}{2} \int_{-\infty}^{\infty} (|p + \frac{1}{2}><p| - |p><p + \frac{1}{2}|)dp.
\]  

(7)

Based on the transformation \( p \rightarrow n + k \), \( \int_{-\infty}^{\infty} dp \rightarrow \sum_n \int_{0}^{1/2} dk \), where \( n \) is integer and half-integer, this Hamiltonian is written as

\[
H = \frac{1}{2} \int_{0}^{1/2} H^k dk
= \frac{1}{2} \sum_n \int_{0}^{1/2} [(n + k)|n + k + (n + k)^2
- \frac{T}{2} (|n + k + 1><n + k|
+ |n + k><n + k + 1|)
- i\frac{T'}{2} (|n + k + \frac{1}{2}><n + k|
- |n + k><n + k + \frac{1}{2}|)] dk.
\]

(8)

The Hamiltonian is invariant under the transformation \( k \rightarrow N/2 + k \), where \( N \) is integer. It indicates that the whole space can be decomposed into the invariant subspaces denoted by \( k \). Therefore the eigenstates of (4) can be written as

\[
|\psi\rangle = \prod_{0 \leq k < 1/2} |\psi^k\rangle
\]

(9)

This means that one can obtain the eigenstates in any subspace from \( |\psi\rangle \) by the projection operator \( P_k = \sum_n |n + k"><n + k| \), whenever \( P_k |\psi\rangle \neq 0 \). What we concern is the eigenstates in \( k = 0 \) subspace. Since it is not so easy to tackle the eigen problem analytically in a subspace spanned by discrete basis, our strategy is that we first study the eigenstate of (3) by variational approach, then project the eigenstate onto the \( k = 0 \) subspace.

In order to investigate the eigenstates of the Hamiltonian in large \( T' \) limit, we introduce a set of trial harmonic wave function,

\[
|\varphi_n\rangle = N_n \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\alpha^2} p^2 + i\beta p\right) H_n(\alpha p) dp |p\rangle
\]

(10)

where \( \alpha, \beta \) are undetermined constants, \( N_n = \frac{\alpha}{\sqrt{\pi} 2^{m_n}} \) is renormalization factor. We assume \( \alpha \ll 1 \). A direct calculation yields the following matrix elements

\[
\langle \varphi_m | H |\varphi_n\rangle = A\delta_{m,n} + B \sqrt{m(m+1)}\delta_{m,n+2} + \sqrt{n(n+1)}\delta_{m,n-2}
+ iC (m\alpha N_m N_n \delta_{m,n+1} - n\alpha N_n N_m \delta_{m,n-1})
\]

(11)
where

\[ A = \frac{2m+1}{2\alpha^2} + \left(\frac{m\alpha^2 T}{2} - \cos \beta \right) \exp\left(-\frac{1}{4}\alpha^2\right) \]
\[ + \left(\frac{m\alpha^2}{8} - 1\right) T' \sin \frac{\beta}{2} \exp\left(-\frac{1}{16}\alpha^2\right) \]  
\[ B = \frac{1}{2\alpha^2} \left[ 4T \cos \beta \exp\left(-\frac{1}{4}\alpha^2\right) \right] \]
\[ + T' \sin \frac{\beta}{2} \exp\left(-\frac{1}{16}\alpha^2\right) \]
\[ C = T \sin \beta \exp\left(-\frac{1}{4}\alpha^2\right) + \frac{T'}{2} \cos \frac{\beta}{2} \exp\left(-\frac{1}{16}\alpha^2\right) \]  

(12)

(13)

(14)

The trial wave functions are approximate wave functions of the Hamiltonian if all the off-diagonal elements vanish. Taking \( B = C = 0 \), we have

\[ \sin \frac{\beta}{2} = \frac{T'}{4T}, \quad \alpha^4 = \frac{32T}{16T'^2 - T'^2} \]  

(15)

The spectrum is

\[ E_m = (m + \frac{1}{2}) \sqrt{\frac{16T'^2 - T'^2}{8T}} - T - \frac{T'^2}{8T} \]  

(16)

What we concern is the eigenstate in the subspace \( k = 0 \), which can be obtained by the project operator \( P_0 = \sum_n |n><n| \)

\[ \sum_n |n><n| \varphi_m > = N_m \sum_n \exp\left(-\frac{1}{2}n^2\alpha^2 + i\beta n\right) H_m(\alpha n) |n\rangle \]  

(17)

where \( \beta = 2 \arcsin\left(\frac{T'}{4T}\right) \) and \( 2\pi - 2 \arcsin\left(\frac{T'}{4T}\right) \) which correspond to the minima of the potential \( T \cos \theta + T' \sin \theta \). Then in \( |n\rangle \) representation, the lower approximate eigen wave functions are

\[ \psi^L_m(n) = N_m \exp\left(-\frac{1}{2}n^2\alpha^2\right) H_m(\alpha'n) \exp\left[i\theta_0\right] \]
\[ \psi^R_m(n) = N_m \exp\left(-\frac{1}{2}n^2\alpha^2\right) H_m(\alpha'n) \exp\left[i\left(2\pi - \theta_0\right)\right] \]
\[ \theta_0 = 2 \arcsin\left(\frac{T'}{4T}\right) \]
\[ m = 0, 1, 2, \cdots, n = 0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2, \cdots, \pm \infty. \]  

(18)

Notice that each energy level in (16) corresponds to a narrow energy band for the exact spectrum of the Hamiltonian (2). Employing the Hellmann-Feynman theorem, one can evaluate the band width

\[ \Delta(m) = \int_0^{1/2} \left(\frac{\partial H}{\partial k}\right)_m dk = \frac{1}{4} \]  

(19)

which is much less than the gap between two neighbor bands. The eigenstates of \( H^k \) share only one level from each band and the approximate wave function \( |\varphi_n > \) is just the linear combination of all the eigenstates corresponding to \( E_n \).

**IV. NUMERICAL RESULTS**

In order to verify the validity of HA, numerical calculation is employed to compute the energy levels and wave functions. Since the whole Hilbert space is constitutive of invariant subspaces and in the practice system the charge
FIG. 2: The probability amplitudes of three lowest levels for $T = 100$, (a) $T' = 0$ and (b) $T' = 60$ obtained by numerical and variational methods. Solid (open) square, circle and triangle denote numerical (variational) results of eigenstates of $n = 0, 1, 2$. 
FIG. 3: The energy levels for $T = 100$, (a) $T' = 0$ and (b) $T' = 60$ obtained by numerical (solid square) and variational (open square) methods. It shows that the value of $T'$ doesn't effect the results obtained by two methods so much which agree quite well for lower levels.
number $n$ is restricted to be integer and half-integer, exact diagonalization is performed to get the lower energy spectrum and corresponding eigen functions of the Hamiltonian in $k = 0$ subspace. In Fig.2 and 3, the energy levels for systems with $T = 100, T' = 0$ and $T = 100, T' = 60$ obtained by numerical and variational methods are plotted. It shows that the HA is valid for wide range of $T'$. The sudden deviations occur around zero energy since the tunnelling effect between neighbor wells is non-neglectable. Under the threshold point numerical and variational results agree well. In Fig.4 and 5, the probability amplitudes of three lowest levels for systems with $T = 100, T' = 0$ and $T = 100, T' = 60$ obtained by numerical and variational methods are plotted. It shows that numerical and variational results agree well.

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