Interval probability density functions constructed from a generalization of the Moore and Yang integral

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Abstract

Moore and Yang defined an integral notion, based on an extension of Riemann sums, for inclusion monotonic continuous interval functions, where the integrations limits are real numbers. This integral notion extend the usual integration of real functions based on Riemann sums. In this paper, we extend this approach by considering intervals as integration limits instead of real numbers and we abolish the inclusion monotonicity restriction of the interval functions and this notion is used to determine interval probability density functions.

Keywords: Interval Mathematics, Moore and Yang integrals, Riemann sums, interval probability density functions.
1 Introduction

The classical probability theory provides a mathematical model for the study of uncertainties of a random nature, also called uncertain knowledge that, even if produced identically, presents to each experiment, results that vary unpredictably. In it, random events must be precisely defined, which is often not the case in real situations. Therefore, in the classical theory of probabilities consider an absolute knowledge without consider the uncertainties generated from incomplete or imprecise knowledge present in many real situations and to deal with these uncertainties, were proposal several ways to extend the notion of probabilities \[2, 13, 20, 27, 46, 49\]. Within this context, many researchers have developed studies on inaccurate probabilities, in order to consider problems that are not contemplated by classical theory \[23, 29\]. Among them, the interval mathematics has served as a theoretical framework for this purpose as one can see on the works Yager \[38\], Sarveswaram et al. \[40\], Tanaka and Sugihara \[44\], Intan \[22\], Zhang, et al. \[50\], Campos and dos Santos \[8\], and Jamison and Lodwick \[24\].

In this work we are interested in studying the interval case of the probability density functions constructed from integrals. Also known as density of a continuous random variable, these functions describe the probability of a random variable assuming that the value of the random variable would equal that sample the given value in the sample space, which can be calculated from the integral of the density of that variable in a given range, i.e. given a continuous variable \(X\) and the values set \(X(S)\), the probability density function \(f\) assigns to each element \(x \in X(S)\) a number \(f(x)\) satisfying properties: (1) \(f(x) > 0\) and (2) \(\int f(x) = 1\). In this framework some researchers have been studied the interval version of probability density functions in order to provide a model where the uncertainty of the variable is measured by an interval. For instance, Berleant in \[4\] has developed an system to evaluate the reasoning automatically by using an intervals probability density functions. Ramírez \[37\] used a data sampling interval to analyze its influence in the estimation of the parameters of the Weibull wind speed probability density distribution. In both cases authors has been used interval to estimate the uncertainty considering the usual version of probability density function, i.e. they not consider an interval version of probability density function indeed. Here, we propose an interval probability density interpretation based on a new way to define interval integrals.

In \[34\] Moore and Yang defined the first approach to interval integrals. In \[33\] further properties of this interval integral notions and their relation with integration of ordinary real functions are established. The Moore-Yang integration of interval functions approach is based on a generalization of Riemann sums for real integrals. Moore in \[32\] defined integrals of continuous function from \(\mathbb{R}\) into \(I(\mathbb{R})\) which can be seen as restrictions of continuous and monotonic interval functions to degenerate intervals (this is guarantee
by the dependency on the degenerate intervals of interval integral [33]).

There exist some other few interval integration approaches in the literature. Among other, are the work of Caprani, Madsen and Rall in [9], they defined integrals for functions (not necessarily continuous) from an interval to the set of interval of extended real numbers (i.e. real numbers with $-\infty$ and $+\infty$) based on Darboux integrals. The Caprani, Madsen and Rall approach have Moore integrals (as defined in [32]) as an special case. Lately, Rall in [36], considering this interval integration approach, partially solved the problem of assignment of infinite intervals to some improper intervals known to be finite. Corliss in [11] extend the Caprani, Madsen and Rall notion of integral for considering interval valued limits. More recently, we can cite the following works on integration of interval-valued functions [5, 10, 24].

The Escardó approach in [18] adapted the Edalat work [17], who extended the Riemann integrals using Scott domain theory, for the domain of intervals, as seen in [14], resulting in a new version of the Moore-Yang integrals, which consider Scott continuous functions instead of continuous (w.r.to Moore metrics) and inclusion monotonic functions (notice that all Scott continuous interval functions are inclusion monotonic). The relationship between both continuity notion can be found in [1, 3, 39].

The Moore-Yang integrals have two restrictions:

1. The interval function must be inclusion monotonic and
2. The limits of integration are real numbers.

Moore, Strother and Yang in [33] (pp. A-5) asseverate that:

“Theorem 4 (theorem 5.1 in this paper) suggests a more general definition for $\int F$ may be feasible – namely the right side of the equality in theorem 4. This conclusion could lead to a deletion of the condition $A \subset B \Rightarrow F(A) \subset F(B)$.”

So, the first restriction, is not necessary and can be suppressed considering a more general definition of the integrals based on a characterization of the interval integral in term of an interval of real integrals. But, taken this definitions as primitive implies that the integral interval notion is not a generalization of a real integral approach, resulting in a notion with a poor mathematical foundation.

This paper define an integral for interval functions which extend the Moore-Yang approach eliminating both restrictions. We also give a characterization of this extension in terms of the extremes of the limits of integration which could simplify its computation.

Finally, we use this integral notion to determine probability density functions [19] in the context of interval probability [7, 30, 45].
2 Preliminary

Let $I(\mathbb{R})$ be the set of real closed intervals, or simply intervals. We will use upper letters at start of alphabet to indicate an interval. The left and right extremes of an interval $A$ will be denoted by $a$ and $b$ respectively, thus $A = [a, b]$. We define two projections for intervals: $\pi_1(A) = a$ and $\pi_2(A) = b$.

The partial order that we will use for intervals will be the Kulisch-Miranker one \cite{28}, i.e.

$$A \leq B \Leftrightarrow a \leq b \quad \text{and} \quad a \leq \pi_2(A) = b.$$

Particularly,

$$A \ll B \Leftrightarrow a < b \quad \text{and} \quad a < \pi_2(B).$$

The partial order used in the Moore and Yang integral approach was the inclusion of sets, i.e.

$$A \subseteq B \Leftrightarrow \pi_2(A) \leq \pi_2(B).$$

The arithmetical operations on intervals are defined as follow

$$A \oplus B = \{x \oplus y \mid x \in A \quad \text{and} \quad y \in B\},$$

where $\oplus$ is any one of the usual arithmetical operations. The unique restriction is that in the case of division, $B$ can not contain 0. Each operations can be characterized in terms of their extremes as follows:

$$A \oplus B = \left[\min\{a \oplus b, a \oplus b, a \oplus b, a \oplus b\}, \max\{a \oplus b, a \oplus b, a \oplus b, a \oplus b\}\right].$$

For an abuse of language, given a real number $r$, we will write $A \oplus r$ and $r \oplus A$ instead of $A \oplus [r, r]$ and $[r, r] \oplus A$, respectively.

In the case of addition, this expression can be abbreviated as

$$A + B = [a + b, a + b].$$

If either $a \geq 0$ or $b \geq 0$ then

$$AB = [ab, ab].$$

Functions whose domain and co-domain are subsets of $I(\mathbb{R})$ are called interval functions. Let $F : I(\mathbb{R}) \rightarrow I(\mathbb{R})$ be an interval function. Define the functions $\underline{F} : I(\mathbb{R}) \rightarrow \mathbb{R}$ and $\overline{F} : I(\mathbb{R}) \rightarrow \mathbb{R}$ by $\underline{F}(A) = \pi_1(F(A))$ and $\overline{F}(A) = \pi_2(F(A))$. Trivially, $F(A) = [\underline{F}(A), \overline{F}(A)]$. Sometimes $F$ will be denoted by $[\underline{F}, \overline{F}]$. An interval function $F$ is said to be an inclusion monotonic function if it is monotonic w.r.to the inclusion, i.e. $A \subseteq B \Rightarrow F(A) \subseteq F(B)$. 

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A distance between two intervals is defined by:

\[ d_M(A, B) = \max \{|b - a|, |\overline{b} - \overline{a}|\}. \]

In [31] was proved that \(d_M\) is a metric. So, an interval function \(F\) is continuous, if it is continuous w.r.to the metric \(d_M\).

## 3 Moore Approach

In this section we will overview the main definitions of the Moore and Yang interval integral approach.

**Definition 3.1** Let \(A\) be a real interval. A partition of \(A\) is a sequence \(T = \{a = x_0, x_1, \ldots, x_n = a\}\) such that for each \(i = 0, \ldots, n - 1\), \(x_i < x_{i+1}\). The set of all partition of \(A\) will be denoted by \(\mathfrak{T}(A)\).

**Definition 3.2** Let \(A\) be a real interval. A partition \(T_1\) of \(A\) is finer than the partition \(T_2\) of \(A\), denoted by \(T_1 \preceq T_2\), if \(T_2 \subseteq T_1\). Clearly, \(\preceq\) is a partial order on \(\mathfrak{T}(A)\).

**Proposition 3.1** Let \(A\) be a real interval. \(\langle \mathfrak{T}(A), \preceq \rangle\) is a lattice with greatest element.

**Proof:** Let \(T_1\) and \(T_2\) be partitions of \(A\). Then trivially, \(T_1 \cap T_2\) and \(T_1 \cup T_2\) are the supremum and infimum, respectively, of \(T_1\) and \(T_2\). The greatest element of \(\mathfrak{T}(A)\) is the partition \(\{a, a\}\).

**Definition 3.3** Let \(A\) be an interval and \(F\) be an inclusion monotonic continuous interval function. The Riemann sum of \(F\) w.r.to a partition \(T\) of \(A\) is defined by:

\[ \sum(F, T) = \sum_{k=0}^{n-1} F([x_k, x_{k+1}]) d(x_k, x_{k+1}), \]

where \(d\) is the usual metric on the real numbers, i.e. \(d(x, y) = |x - y|\).

The Moore and Yang integral of \(F\) at the interval \(A\) is defined by:

\[ \int_A F(X) dX = \bigcap_{T \in \mathfrak{T}(A)} \sum(F, T). \]

**Theorem 3.1** (Characterization theorem) Let \(A\) be an interval and \(F\) be an inclusion monotonic continuous interval function. Then

\[ \int_A F(X) dX = \left[ \int_{a}^{\overline{a}} f_l(x) dx, \int_{\overline{a}}^{\overline{a}} f_r(x) dx \right], \]

where \(f_l(x) = \pi_1 F[x, x]\) and \(f_r(x) = \pi_2 F[x, x]\).

**Proof:** See [33].
4 Our Extension

Definition 4.1 Let $A$ and $B$ be two real intervals such that $A \ll B$. Define,

$$I_{[A,B]} = \{ X \in \mathbb{I} \mid A \leq X \leq B \}$$

and $A\overline{B} = \{ X \mid x = x \text{ and } x = \overline{x} + \frac{\overline{b} - \overline{a}}{a - b}(x - a) \text{ for some } x \in [a, b] \}$.

Since $A \ll B$, then $a < b$ and $\overline{a} < \overline{b}$. In what follows, without lost of generality, we will suppose that $a < b$.

Given a metric space $(X, d)$ and a subset $A$ of $X$ we define the diameter of $A$, denoted by $\text{diam}(A)$, by

$$\text{diam}(A) = \sup \{ d(a, b) \mid a, b \in A \}.$$  

Definition 4.2 A subset $A$ of a metric space $(X, d)$ is **bounded** if $\text{diam}(A)$ is finite.

Definition 4.3 A function of a non-empty set into a metric space is called a **bounded function** if its image is a bounded set.

Clearly a subset $I$ of $\mathbb{I} \mid \mathbb{R}$ is bounded if, and only if, it is contained in $I_{[A,B]}$ for some $A, B \in \mathbb{I} \mid \mathbb{R}$. Thus, an interval function $F : I \longrightarrow \mathbb{I} \mid \mathbb{R}$, where $I \subseteq \mathbb{I} \mid \mathbb{R}$, is bounded if, and only if, its image is contained in $I_{[A,B]}$ for some $A, B \in \mathbb{I} \mid \mathbb{R}$. That is, if $A \leq F(X) \leq B \forall X \in I$. Therefore, the topological and order based notions of boundedness for an interval function $F : I \longrightarrow \mathbb{I} \mid \mathbb{R}$ coincides.

Let $\phi : \mathbb{I} \mid \mathbb{R} \longrightarrow \mathbb{R}^2$ be the function defined by, $\phi([a, b]) = (a, b)$. Clearly the function $\phi$ is continuous and injective. Thus, we can identify $\mathbb{I} \mid \mathbb{R}$ with a subspace of $\mathbb{R}^2$, where $\mathbb{R}^2$ is considered here with the metric,

$$d((a, \overline{a}), (b, \overline{b})) = \max \{ | \overline{b} - \overline{a} |, | \overline{b} - a | \}.$$  

This is not restrictive since usual metrics in $\mathbb{R}^n$ are equivalents, from the topological point-of-view.

Notice that $I_{[A,B]}$ is closed and bounded in $\mathbb{R}^2$, hence, by the Heine-Borel theorem [43] p. 119, it is compact.

Lemma 4.1 (Completeness Lemma) Let $X$ be a bounded subset of $\mathbb{I} \mid \mathbb{R}$. Define $\underline{X} = \{ c \mid C \in X \}$ and $\overline{X} = \{ x \mid C \in X \}$. Then, $\bigcup X = \bigcup \underline{X} \bigcup \overline{X}$ and $\bigcap X = \bigcap \underline{X} \bigcap \overline{X}$ where $\bigcap \underline{X} = \inf X$ and $\bigcup \overline{X} = \sup X$.

Proof: If $X$ is a bounded subset of $\mathbb{I} \mid \mathbb{R}$ then $X \subseteq I_{[A,B]}$ for some $A, B \in \mathbb{I} \mid \mathbb{R}$. Clearly $a$ is a lower bound of $\underline{X}$ and $\overline{X}$ and $\overline{b}$ is an upper bound of $\underline{X}$ and $\overline{X}$. Since each upper bounded subset of the real numbers has
a supremum and each lower bounded subset of the real numbers has an infimum, then \( X \) and \( \overline{X} \) has supremum and infimum. It is easy to see that
\[
\bigcup X = [\bigcup X, \bigcup \overline{X}] \quad \text{and} \quad \bigcap X = [\bigcap X, \bigcap \overline{X}].
\]

**Corollary 4.1** Let \( I \subseteq \mathbb{I}(\mathbb{R}) \) and \( F : I \rightarrow \mathbb{I}(\mathbb{R}) \) be a bounded interval function. Then, the set \( F(I) = \{ F(X) \in \mathbb{I}(\mathbb{R}) \mid X \in I \} \) has supremum and infimum. In fact, if \( F(X) = [F(X), \overline{F(X)}] \) then,
\[
\bigcup F(I) = [\bigcup F(I), \bigcup \overline{F(I)}]
\]
and
\[
\bigcap F(I) = [\bigcap F(I), \bigcap \overline{F(I)}].
\]

**Proof:** Since \( F \) is a bounded interval function we have that \( F(I) \subseteq I \) for some \( A, B \in \mathbb{I}(\mathbb{R}) \). The statements follows from the Completeness lemma, by taking \( X = F(I) \).

**Definition 4.4** A set \( P \) is a partition of \( I_{[A,B]} \) if there exists partitions \( T_1 \) and \( T_2 \) of \([a,b]\) and \([\overline{a}, \overline{b}]\), respectively, such that \( P = (T_1 \times T_2) \cap \mathbb{I}(\mathbb{R}) \).

**Remark 4.1** Sometimes we say that the partition \( P \) of \( I_{[A,B]} \) comes from the partitions \( T_1 \) and \( T_2 \) of \([a,b]\) and \([\overline{a}, \overline{b}]\), respectively, or simply, that \( P \) of \( I_{[A,B]} \) comes from \( T_1 \times T_2 \).

**Definition 4.5** We say that a partition \( P' \) of \( I_{[A,B]} \), is finer than the partition \( P \) of \( I_{[A,B]} \), denoted by \( P' \preceq P \), if \( P \subseteq P' \).

**Lemma 4.3** Let \( \mathcal{P}[A,B] = \{ P \mid P \text{ is a partition of } I_{[A,B]} \} \). Then \( \langle \mathcal{P}[A,B], \preceq \rangle \) is a lattice with greatest element.

**Proof:** Let \( P \) and \( P' \) be partitions of \( I_{[A,B]} \) coming from \( T_1 \times T_2 \) and \( T'_1 \times T'_2 \), respectively. Then trivially, from definition of \( \preceq \) and proposition 3.1 we have that \( P \cup P' \) is the partition coming from \( T_1 \times T_2 \cap T'_1 \times T'_2 \). Analogously, we have that \( P \cap P' \) is the partition coming from \( T_1 \times T_2 \cup T'_1 \times T'_2 \). The greatest element of \( \mathcal{P}[A,B] \) is the partition \( P_\uparrow = \{ A, B \} \) which came from \( [a, b] \times [\overline{a}, \overline{b}] \).
Definition 4.6 Let \( F : I_{[A,B]} \to \mathbb{I}(\mathbb{R}) \) be a bounded interval function. Given a partition \( \mathcal{P} \) of \( I_{[A,B]} \), we define the following Riemann sums of \( F \) w.r.to \( \mathcal{P} \) namely:

- **Lower Riemann sum**

  \[
  \sigma(F, \mathcal{P}) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} F(I_{[P_{i,j}, P_{i+1,j}]}) \, d_M(P_{i,j}, P_{i+1,j+1})
  \]

- **Upper Riemann sum**

  \[
  \sum(F, \mathcal{P}) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \bigcup F(I_{[P_{i,j}, P_{i+1,j}]}) \, d_M(P_{i,j}, P_{i+1,j+1})
  \]

where \( P_{i,j} = [x_i, y_j] \) and, as usual, \( F(I_{[P_{i,j}, P_{i+1,j}]}) = \{ F(X) \mid X \in I_{[P_{i,j}, P_{i+1,j}]}) \} \).

Lemma 4.4 Let \( F : I_{[A,B]} \to \mathbb{I}(\mathbb{R}) \) be a bounded function and let \( \mathcal{P} \) be a partition of \( I_{[A,B]} \). Then

\[
\sigma(F, \mathcal{P}) \leq \sum(F, \mathcal{P}).
\]

**Proof:** Clearly, for each \( i = 0, \ldots, n-1 \) and \( j = 0, \ldots, m-1 \),

\[
\bigcup F(I_{[P_{i,j}, P_{i+1,j}]}) \leq \bigcup F(I_{[P_{i,j}, P_{i+1,j}]}) \quad \text{and} \quad d_M(P_{i,j}, P_{i+1,j+1}) > 0.
\]

Therefore,

\[
\sigma(F, \mathcal{P}) \leq \sum(F, \mathcal{P}).
\]

Proposition 4.1 Let \( F : I_{[A,B]} \to \mathbb{I}(\mathbb{R}) \) be a bounded function and let \( \mathcal{P}' \) and \( \mathcal{P} \) be partitions of \( I_{[A,B]} \). If \( \mathcal{P}' \preceq \mathcal{P} \) then

\[
\sigma(F, \mathcal{P}) \leq \sigma(F, \mathcal{P}') \leq \sum(F, \mathcal{P}') \leq \sum(F, \mathcal{P}).
\]

**Proof:** Let \( \mathcal{P} \) be a partition comes from \( T_1 \times T_2 \) where \( T_1 = \{ a = x_0, \ldots, x_n = b \} \). With no lost of generality we may assume that the partition \( \mathcal{P}' \) comes from \( T'_1 \times T_2 \) where \( T'_1 = \{ a = x_0, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n = b \} \).

Let \( Q_j = [z, y_j] \) then

\[
d_M(P_{i-1,j}, P_{i,j+1}) \leq d_M(P_{i-1,j}, Q_j) + d_M(Q_{j+1}, P_{i,j+1}).
\]
On the other hand, we have that

\[
\prod F(\mathcal{I}_{[p_{i,j}]}(P_{i-1,j}, P_{i,j+1})) \leq \prod F(\mathcal{I}_{[p_{i,j}]}(Q_{j})) \quad \text{and} \quad \prod F(\mathcal{I}_{[p_{i,j}]}(Q_{j+1}, P_{i,j+1})) \leq \prod F(\mathcal{I}_{[q_{j}]}(P_{i,j+1})).
\]

Therefore, we have that

\[
\sigma(F, \mathcal{P}') = \sum_{r=1}^{i-1} \sum_{s=0}^{m-1} F(\mathcal{I}_{[p_{r-1,s}, p_{r,s+1}]}(P_{r-1,s}, P_{r,s+1})) d_M(P_{r-1,s}, P_{r,s+1}) + \\
\sum_{s=0}^{m-1} F(\mathcal{I}_{[p_{i-1,s}]}(Q_{s+1}, Q_{s+1}) d_M(P_{i-1,s}, Q_{s+1})) + \\
\sum_{s=0}^{n-1} F(\mathcal{I}_{[q_{s}]}(p_{s+1}, Q_{s}) d_M(P_{s+1}, Q_{s})) + \\
\sum_{r=1}^{i-1} \sum_{s=0}^{n-1} F(\mathcal{I}_{[q_{s}]}(p_{r,s+1}, p_{r,s+1}) d_M(P_{r,s+1}, p_{r,s+1})).
\]

The second inequality was proved in Lemma 4.4 and the third inequality follows by the same token of the first.

**Corollary 4.2** Let \( F: \mathcal{I}_{[A,B]} \rightarrow \mathcal{I}(\mathbb{R}) \) be a bounded function and let \( \mathcal{P} \) and \( \mathcal{Q} \) be partitions of \( \mathcal{I}_{[A,B]} \). Then,

\[
\sigma(F, \mathcal{P}) \leq \Sigma(F, \mathcal{Q}).
\]

**Proof:** Since the partition \( \mathcal{P} \cup \mathcal{Q} \) refines \( \mathcal{P} \) and \( \mathcal{Q} \) we have that

\[
\sigma(F, \mathcal{P}) \leq \sigma(F, \mathcal{P} \cup \mathcal{Q}) \leq \Sigma(F, \mathcal{P} \cup \mathcal{Q}) \leq \Sigma(F, \mathcal{Q}).
\]

**Definition 4.7** Let \( F: \mathcal{I}_{[A,B]} \rightarrow \mathcal{I}(\mathbb{R}) \) be a bounded function. We define the **lower integral** of \( F \) w.r.t. \( A \) and \( B \), denoted by \( \int_{A}^{B} F(X) dX \), by

\[
\int_{A}^{B} F(X) dX = \bigcup_{\mathcal{P} \in \mathcal{P}[A,B]} \sigma(F, \mathcal{P}).
\]
and the upper integral of $F$ w.r.to $A$ and $B$, denoted by $\int_A^B F(X)dX$, by
\[
\int_A^B F(X)dX = \prod_{P \in \mathcal{P}[A,B]} \sum (F, P).
\]

**Proposition 4.2** Let $F : \mathcal{I}_{[A, B]} \to \mathbb{I}[:, \mathbb{R}]$ be a bounded function such that $C \leq F(X) \leq D \forall X \in \mathcal{I}_{[A, B]}$. Then, for any partition $\mathcal{P}$ of $\mathcal{I}_{[A, B]}$ we have that
\[
Cd_M(A, B) \leq \sigma(F, \mathcal{P}) \leq \int_A^B F(X)dX \leq \int_A^B F(X)dX \leq \sum (F, P) \leq Dd_M(A, B).
\]

**Proof:** Let $\mathcal{P}_\top = \{A, B\}$ be the trivial partition of $\mathcal{I}_{[A, B]}$. Then, by Lemma 4.1 we have that $\sigma(F, \mathcal{P}_\top) \leq \sigma(F, \mathcal{P})$. But, by definition, we have that $\sigma(F, \mathcal{P}_\top) = \prod F(\mathcal{I}_{[A, B]}) dM(A, B) \geq Cd_M(A, B)$, which proves the first inequality. The last inequality follows by the same token.

The inequality $\int_A^B F(X)dX \leq \int_A^B F(X)dX$ follows by Corollary 4.2. The remaining inequalities follows by definition of the lower and upper integrals.

**Proposition 4.3** Let $\mathcal{P}'$ and $\mathcal{P}''$ be subset of $\mathcal{P}[A, B]$ satisfying the following properties:

(•) $\forall P \in \mathcal{P}[A, B] \exists P' \in \mathcal{P}'$ and $P'' \in \mathcal{P}''$ such that $\sigma(F, P') \leq \sigma(F, P)$ and $\sum (F, P'') \leq \sum (F, P)$.

Then,
\[
\int_A^B F(X)dX = \bigcup_{P' \in \mathcal{P}'} \sigma(F, P')
\]
and
\[
\int_A^B F(X)dX = \prod_{P' \in \mathcal{P}''} \sum (F, P'').
\]

**Proof:** This follows by general properties of supremum and infimum.

**Corollary 4.3** Let $A \ll B$, $C \in \overline{AB}$ and let $\mathcal{P}'[A, B]$ be the subset of $\mathcal{P}[A, B]$ consisting of partition containing $C$. Then,
\[
\int_A^B F(X)dX = \bigcup_{P \in \mathcal{P}'[A, B]} \sigma(F, P)
\]
and

\[
\int_A^B F(X) dX = \prod_{P \in \Psi'[A,B]} \sum (F, P).
\]

**Proof:** From a partition \( P \in \Psi[A,B] \) build the partition \( P' = P \cup \{C\} \), containing \( C \). Since \( P' \) is finer than \( P \) we have that \( \sigma(F, P) \leq \sigma(F, P') \) and \( \sum(F, P') \leq \sum(F, P) \). Thus, \( \Psi'[A,B] \) satisfies the condition (●) of the above proposition. \(\blacksquare\)

**Corollary 4.4** Let \( \mathcal{P}_n = \{A = X_0, X_1, \ldots, X_n = B\} \) be the partition of \( \mathcal{I}_{[A,B]} \) coming from the partitions \( T = \{a = t_0, t_1, \ldots, t_n = b\} \) and \( S = \{\bar{a} = s_0, s_1, \ldots, s_n = \bar{b}\} \) of \([a, b]\) and \([\bar{a}, \bar{b}]\), where \( t_k = a + \frac{k}{n}(b - a) \) and \( s_k = \bar{a} + \frac{k}{n}(\bar{b} - \bar{a}) \), respectively. Then,

\[
\int_A^B F(X) dX = \bigcup_{n \in \mathbb{N}} \sigma(F, \mathcal{P}_n)
\]

and

\[
\int_A^B F(X) dX = \prod_{n \in \mathbb{N}} \sum (F, \mathcal{P}_n).
\]

**Proof:** Let \( \mathcal{P} \in \Psi[A,B] \) be a partition of \( \mathcal{I}_{[A,B]} \). Clearly, \( \exists n \in \mathbb{N} \) such that \( \mathcal{P}_n \preceq \mathcal{P} \). Therefore, \( \sigma(F, \mathcal{P}) \leq \sigma(F, \mathcal{P}_n) \) and \( \sum(F, \mathcal{P}_n) \leq \sum(F, \mathcal{P}) \). Thus, \( \Psi = \{\mathcal{P}_n \mid n \in \mathbb{N}\} \) satisfies the condition (●) \(\blacksquare\)

**Definition 4.8** A bounded function \( F : \mathcal{I}_{[A,B]} \to \mathbb{R} \) is said to be an integrable function if

\[
\int_A^B F(X) dX = \int_A^B F(X) dX.
\]

This common value is called the **interval integral** of \( F \) w.r.t \( A \) and \( B \) and it is denoted by \( \int_A^B F(X) dX \).

**Definition 4.9** Let \( F : \mathcal{I}_{[A,B]} \to \mathbb{R} \) be a bounded function. Define the left and right spectrum of \( F \), denoted by \( F_l \) and \( F_r \) respectively, by

\[
F_l(x) = \pi_1 F \left[ x, \bar{a} + \frac{\bar{b} - \bar{a}}{b - a} (x - \bar{a}) \right]
\]

and

\[
F_r(x) = \pi_2 F \left[ x, \bar{a} + \frac{\bar{b} - \bar{a}}{b - a} (x - \bar{a}) \right],
\]
where $\pi_1$ and $\pi_2$ are the left and right projections from $\mathbb{I}(\mathbb{R})$ to $\mathbb{R}$ and $x \in [a, b]$.

**Theorem 4.1** Let $F : \mathcal{I}_{[A,B]} \rightarrow \mathbb{I}(\mathbb{R})$ be a continuous function. Then,

$$\int_A^B F(X) dX = \left[ \int_a^b F_1(x) dx, \int_a^b F_r(x) dx \right] \frac{dM(A, B)}{b - a}$$

and

$$\int_A^B F(X) dX = \left[ \int_a^b F_l(x) dx, \int_a^b F_r(x) dx \right] \frac{dM(A, B)}{b - a}.$$

**Proof:** We will only prove the first equality since the second one follows analogously.

Let $\mathcal{P}_n = \{ A = X_0, X_1, \ldots, X_n = B \}$ be the partition of $\mathcal{I}_{[A,B]}$ coming from the partitions $\mathcal{T} = \{a = t_0, t_1, \ldots, t_n = b\}$ and $\mathcal{S} = \{\pi = s_0, s_1, \ldots, s_n = \bar{b}\}$ of $[a, b]$ and $[\bar{a}, \bar{b}]$, where $t_k = a + \frac{k}{n}(b - a)$ and $s_k = \bar{a} + \frac{k}{n}(\bar{b} - \bar{a})$, respectively.

Therefore,

$$\int_A^B F(X) dX = \bigcup_{n \in \mathbb{N}} \sigma(F, \mathcal{P}_n)$$

$$= \bigcup_{n \in \mathbb{N}} \bigcap_{k=0}^{n-1} F(\mathcal{I}_{[X_k, X_{k+1}]}) dM(X_{k+1}, X_k)$$

$$= dM(A, B) \left[ \bigcup_{n \in \mathbb{N}} \left( \frac{1}{n} \right) \sum_{k=0}^{n-1} F(\mathcal{I}_{[X_k, X_{k+1}]}) \right]$$

$$= dM(A, B) \left[ \bigcup_{n \in \mathbb{N}} \left( \frac{1}{n} \right) \sum_{k=0}^{n-1} F(\mathcal{I}_{[X_k, X_{k+1}]}) \right] \bigcup_{n \in \mathbb{N}} \left( \frac{1}{n} \right) \sum_{k=0}^{n-1} F(\mathcal{I}_{[X_k, X_{k+1}]})\right].$$

On the other side, we have that

$$\int_a^b F_l(x) dx = \bigcup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} F_l(t_k) d(t_k, t_{k+1})$$

$$= \bigcup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} F_l(t_k) \left( \frac{1}{n} \right) (b - a)$$

$$= (b - a) \bigcup_{n \in \mathbb{N}} \left( \frac{1}{n} \right) \sum_{k=0}^{n-1} F_l(X_k).$$
Analogously we have that
\[ \int_a^b F_r(x) \, dx = (b - a) \sum_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=0}^{n-1} F(X_k). \]

Therefore, it is enough to prove that
\[ \bigcup_{n \in \mathbb{N}} \left( \frac{1}{n} \sum_{k=0}^{n-1} F(I_{[X_k, X_{k+1}]}) \right) = \bigcup_{n \in \mathbb{N}} \left( \frac{1}{n} \sum_{k=0}^{n-1} F(X_k) \right) \]
and
\[ \bigcup_{n \in \mathbb{N}} \left( \frac{1}{n} \sum_{k=0}^{n-1} F(I_{[X_k, X_{k+1}]}) \right) = \bigcup_{n \in \mathbb{N}} \left( \frac{1}{n} \sum_{k=0}^{n-1} F(X_k) \right). \]

By similarity, we will only prove the former.

Clearly we have that
\[ F(I_{[X_k, X_{k+1}]}) \leq F(X_k). \]

Thus,
\[ \bigcup_{n \in \mathbb{N}} \left( \frac{1}{n} \sum_{k=0}^{n-1} F(I_{[X_k, X_{k+1}]}) \right) \leq \bigcup_{n \in \mathbb{N}} \left( \frac{1}{n} \sum_{k=0}^{n-1} F(X_k) \right). \]

Conversely, since \( I_{[X_k, X_{k+1}]} \) is compact and \( F \) is a continuous function, we have that \( \exists Y_k \in I_{[X_k, X_{k+1}]} \) such that \( \prod_{n \in \mathbb{N}} F(I_{[X_k, X_{k+1}]}) = F(Y_k). \)

Since \( F \) is a continuous function and \( I_{[A,B]} \) is compact we have that \( F \) is uniformly continuous.

Let \( \epsilon > 0 \). Since \( F \) is uniformly continuous, we have that \( \exists \delta > 0 \) such that if \( d_M(X,Y) \leq \delta \) then \( d(F(X), F(Y)) \leq \epsilon. \)

Let \( n_0 \in \mathbb{N} \) be such that if \( n \geq n_0 \) then \( \frac{d_M(A,B)}{n} \leq \delta. \)

Since \( Y_k \in I_{[X_k, X_{k+1}]} \) we have that
\[ d_M(X_k,Y_k) \leq d_M(X_k, X_{k+1}) = \frac{d_M(A,B)}{n} \leq \delta. \]

Thus, since \( F \) is uniformly continuous, we have that \( d(F(X_k), F(Y_k)) \leq \epsilon. \) Therefore,
\[
\left( \frac{1}{n} \right) \sum_{k=0}^{n-1} F(X_k) - \left( \frac{1}{n} \right) \sum_{k=0}^{n-1} F(I[X_k, X_{k+1}]) = \left( \frac{1}{n} \right) \sum_{k=0}^{n-1} F(X_k) - F(Y_k)
\leq \left( \frac{1}{n} \right) \sum_{k=0}^{n-1} \epsilon = \epsilon,
\]

which proves that

\[
\bigcup_{n \in \mathbb{N}} \left( \frac{1}{n} \right) \sum_{k=0}^{n-1} F(I[X_k, X_{k+1}]) = \bigcup_{n \in \mathbb{N}} \left( \frac{1}{n} \right) \sum_{k=0}^{n-1} F(X_k).
\]

**Corollary 4.5** [characterization theorem] If \( F : I_{[A,B]} \rightarrow \mathbb{R} \) is a continuous interval function then, \( F \) is an integrable function and

\[
\int_A^B F(X) dX = \left[ \int_a^b F_l(x) dx, \int_a^b F_r(x) dx \right] dM(A, B).
\]

**Proof:** Since \( F \) is continuous, we have that \( F_l \) and \( F_r \) are continuous as well. Therefore,

\[
\int_a^b F_l(x) dx = \int_a^b F_l(x) dx
\]

and

\[
\int_a^b F_r(x) dx = \int_a^b F_r(x) dx.
\]

Therefore, by the above proposition, we have that

\[
\int_A^B F(X) dX = \int_A^B F(X) dX.
\]

Thus, \( F \) is an integrable function and

\[
\int_A^B F(X) dX = \left[ \int_a^b F_l(x) dx, \int_a^b F_r(x) dx \right] dM(A, B).
\]

**Corollary 4.6** If \( A = [a, a] \) and \( B = [b, b] \) with \( a < b \) then

\[
\int_A^B F(X) dX = \int_{[a,b]} F(X) dX.
\]
Proof: Notice that when $A$ and $B$ are degenerated intervals, $F_l = f_l$, $F_r = f_r$ and $\frac{d_M(A, B)}{b - a} = 1$. So, this corollary is straightforward from Corollary 4.3 and Theorem 3.1.

Therefore our approach generalize the Moore and Yang approach.

Proposition 4.4 For fixed $A \in \mathbb{I}(\mathbb{R})$ and an interval continuous function $F$, The function $G : \mathcal{I}_A \rightarrow \mathbb{I}(\mathbb{R})$ defined by

$$G(Y) = \int_A^Y F(X)dX,$$

where $\mathcal{I}_A = \{Y \in \mathbb{I}(\mathbb{R}) \mid A \leq Y\}$, is continuous.

Proof: It is enough to show that $\pi_1 \circ G$ and $\pi_2 \circ G$ are continuous. Since $F$ is continuous we have that $F_l$ and $F_r$ are continuous as well. Therefore, the real functions $f_l(y) = \int_y^a F_l(x)dx$ and $f_r(y) = \int_y^a F_r(x)dx$ are continuous. On the other hand, the function $H : \mathcal{I}_A \rightarrow \mathbb{R}$ given by $H(Y) = \frac{d_M(A, Y)}{y - a} = \max \left\{1, \frac{y - a}{y - a} \right\}$ is clearly continuous. Thus, the functions $\pi_1 \circ G = H \cdot (f_l \circ \pi_1)$ and $\pi_2 \circ G = H \cdot (f_r \circ \pi_1)$ are continuous as well.

Proposition 4.5 Let $F_2$ and $F_1$ be integrable functions from $\mathcal{I}_{[A,B]}$ to $\mathbb{I}(\mathbb{R})$. If

$$F_2 \left[ x, \overline{\alpha} + \frac{b - \alpha}{b - a}(x - \alpha) \right] = F_1 \left[ x, \overline{\alpha} + \frac{b - \alpha}{b - a}(x - \alpha) \right]$$

for all $x \in [a,b]$ then

$$\int_A^B F_2(X)dX = \int_A^B F_1(X)dX.$$

Proof: Straightforward from Corollary 4.3 and definition of $F_l$ and $F_r$ functions.

4.1 Improper Interval Integrals

The definite integral is defined to integrate that continuous functions on a limited and closed interval. However, for those functions that has only one point of discontinuity (actually, for a enumerate number of point of discontinuity) it is also possible to integrate it by means using the concept of improper integral. Such cases arise for the following types of intervals: $[a, +\infty)$, $(-\infty, b]$ or $(-\infty, +\infty)$. 

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Definition 4.10 Let \( F : I_{[A,B]} \rightarrow I(\mathbb{R}) \) be an interval function. An interval \( L \in I(\mathbb{R}) \) is called to be the limit of \( F \) when \( X \in I_{[A,B]} \) tends to the interval \( C \) and it is denoted by \( L = \lim_{X \to C} F(X) \), if for every given \( \varepsilon > 0 \) there exists an \( \delta > 0 \) such that \( d_M(F(X), L) < \varepsilon \) whereas \( 0 < d_M(X, C) < \delta \) and \( X \in I_{[A,B]} \).

Some researchers have been worked on the definition of limits of an interval function and its properties. Here we will not deal with this subject in depth, but for the most interested reader on this topic we recommend [47].

Considering the intervals \([\infty, x] = [x, +\infty] \) and \([-\infty, x] \) for some \( x \in \mathbb{R} \). Particularly, when \( x = +\infty \) interval \([\infty, +\infty] = [+, +\infty] \) will be denoted just by \([+, +\infty] \). Similarly \([-\infty, -\infty] \) will be denoted just by \([-\infty, -\infty] \). Using that notation, we are able to define limit for infinite. So it follows that

\[
I_{A,[a,\infty]} = \{ X \in I(\mathbb{R}) \mid A \leq X \leq [a, +\infty] \text{ and } a \leq b \}
\]

and

\[
I_{[\infty, a, B]} = \{ X \in I(\mathbb{R}) \mid [-\infty, a] \leq X \leq B \text{ and } a \leq b \}
\]

Definition 4.11 Let \( F : I_{[A,B]} \rightarrow I(\mathbb{R}) \) be an interval function and \( L \in I(\mathbb{R}) \). Thus

1. If \( B = [\infty]^b \) for some \( b \in \mathbb{R} \) then \( L = \lim_{X \to [\infty]^b} F(X) \) if given \( \varepsilon > 0 \) there exists an \( 0 < K \in I_{[A,B]} \) such that for every \( X > K \) it follows that \( d_M(F(X), L) < \varepsilon \);

2. If \( A = [\infty]^a \) for some \( a \in \mathbb{R} \) then \( L = \lim_{X \to [\infty]^a} F(X) \) if given \( \varepsilon > 0 \) there exists an \( 0 < K \in I_{[A,B]} \) such that for every \( X < K \) it follows that \( d_M(F(X), L) < \varepsilon \);

It is clear that in case \( B = [+\infty] \) and \( A = [-\infty] \) the limits on infinite \( \lim_{X \to [+\infty]} F(X) \) and \( \lim_{X \to [-\infty]} F(X) \) are particular cases of the Definition 4.11.

So now we have the main conditions to define the notion of improper interval integral as follows.

Definition 4.12 Let \( F : I_{[A,B]} \rightarrow I(\mathbb{R}) \) be an interval function. Thus

1. If \( B = [\infty]^b \) for some \( b \in \mathbb{R} \) then

\[
\int_{A}^{[\infty]^b} F(X) dX = \lim_{C \to [\infty]^b} \int_{A}^{C} F(X) dX
\]

2. If \( A = [\infty]^a \) for some \( a \in \mathbb{R} \) then

\[
\int_{[\infty]^a}^{B} F(X) dX = \lim_{C \to [\infty]^a} \int_{C}^{B} F(X) dX
\]
3. For $A = [-\infty]$ and $B = [+\infty]$ then
\[
\int_{[-\infty]}^{[+\infty]} F(X) dX = \lim_{C \to [\infty]} \int_{C}^{[+\infty]} F(X) dX.
\]

It is worth to note that
\[
\int_{[-\infty]}^{[+\infty]} F(X) dX = \lim_{D \to [\infty]} \int_{[-\infty]}^{D} F(X) dX.
\]

5 Interval Probability Density Functions

Interval probability, is an extension of classical probability by consider interval-values to represent the probability of some event. This area is not new, in fact there are several approach for interval probability, among them we have [7, 12, 14, 38, 45, 46]. The interval probability of an event is natural in some situations:

1. When is working with interval data as for example in [8, 26].

2. When is used a Von Mises frequentist approach to probability which is obtained via observations by projection of a future stability, once “the relative frequency of the observed attributes would tend to a fixed limit if the observations were continued indefinitely by sampling from a collective” [42]. Nevertheless, since we can not observed indefinitely, this projection include an error which could be captured by an interval.

3. When the probability of an event is not representable finitely, and therefore is necessary use an approximation of this probability [7].

4. When we has an uncertainty in the probability value, for example, when only partial information about error distributions is available and standard statistical approaches cannot be applied [27].

An important notion in standard probability theory is the concept of probability density function of a continuous random variable, which is a function that describes the relative likelihood for the continuous random variable to occur at a given point in the observation space.

5.1 Interval Probability Spaces

Let $\mathcal{F}$ be a $\sigma$-algebra in the standard sense over a set $\Omega$ (the sample space). A (positive) interval measure $\mu : \mathcal{F} \to \mathbb{I}(\mathbb{R})^+$, where $\mathbb{I}(\mathbb{R})^+ = \{X \in \mathbb{I}(\mathbb{R}) \mid [0,0] \subseteq X\}$, is a function that assigns a (positive) real interval to each element of $\mathcal{F}$ such that the following properties are satisfied:
(i) The empty set has measure zero: \( \mu(\emptyset) = [0, 0] \)

(ii) Countable additivity: if \( \{A_i \mid i \in I\} \subseteq \mathcal{F} \) is a set of pairwise disjoint
     for some countable index set \( I \), then

\[
\mu \left( \bigcup_{i \in I} A_i \right) = \sum_{i \in I} \mu(A_i) \tag{2}
\]

An interval measurable space is defined as a tuple \( \langle \Omega, \mathcal{F}, \mu \rangle \), where \( \Omega \) is
a set, \( \mathcal{F} \) is a \( \sigma \)-algebra over \( \Omega \) and \( \mu \) is an interval measure on \( \langle \Omega, \mathcal{F} \rangle \). The
elements of \( \mathcal{F} \) are called interval measurable sets.

An interval probability space \( \langle \Omega, \mathcal{F}, \mathcal{P} \rangle \) is a tuple consisting of a sample
space \( \Omega \), a \( \sigma \)-algebra \( \mathcal{F} \) of subsets of \( \Omega \), and a positive, interval measure \( \mathcal{P} \)
on \( \langle \Omega, \mathcal{F} \rangle \) satisfying \( \mathcal{P}(\Omega) = [1, 1] \). In this case, \( \Omega \) is known as the outcome
space or sample space, \( \mathcal{F} \) is called the set of events and \( \mathcal{P} \) is called an interval
probability measure or by simplicity an interval probability.

5.2 Interval Density Functions

Let \( \Omega \) be a sample space. An interval random variable \( \mathcal{X} \) is a function
\( \mathcal{X} : \Omega \to I(\mathbb{R}) \), i.e. assign a real interval to each sample point in \( \Omega \). When
\( \mathcal{X} \) is a continuous interval function which also has a piecewise continuous
derivative \( dF_\mathcal{X}(A)/dA \) we called \( \mathcal{X} \) of continuous interval random variable.

Let \( \mathcal{X} \) be an interval random variable and \( A \) and \( B \) fixed intervals such
that \( A \leq B \). Then define the events:

\[
(\mathcal{X} = A) = \{\zeta \mid \mathcal{X}(\zeta) = A\} \tag{3}
\]
\[
(\mathcal{X} \leq A) = \{\zeta \mid \mathcal{X}(\zeta) \leq A\} \tag{4}
\]
\[
(\mathcal{X} \gg A) = \{\zeta \mid \mathcal{X}(\zeta) \gg A\} \tag{5}
\]
\[
(A \ll \mathcal{X} \leq B) = \{\zeta \mid A \ll \mathcal{X}(\zeta) = B\} \tag{6}
\]

Given an interval probability space \( \langle \Omega, \mathcal{F}, \mathcal{P} \rangle \), these events have associ-
ated the following interval probabilities

\[
\mathcal{P}(\mathcal{X} = A) = \mathcal{P}\{\zeta \mid \mathcal{X}(\zeta) = A\} \tag{8}
\]
\[
\mathcal{P}(\mathcal{X} \leq A) = \mathcal{P}\{\zeta \mid \mathcal{X}(\zeta) \leq A\} \tag{9}
\]
\[
\mathcal{P}(\mathcal{X} \gg A) = \mathcal{P}\{\zeta \mid \mathcal{X}(\zeta) \gg A\} \tag{10}
\]
\[
\mathcal{P}(A \ll \mathcal{X} \leq B) = \mathcal{P}\{\zeta \mid A \ll \mathcal{X}(\zeta) = B\} \tag{11}
\]

The interval distribution function of \( \mathcal{X} \) is the function \( F_\mathcal{X} : I(\mathbb{R}) \to I(\mathbb{R}) \)
defined by
\[ F_X(X) = P(\mathcal{X} \leq X). \] (12)

The following properties are straightforward from the fact that \( F_X \) is based on an interval probability space.

(F1) \([0, 0] \leq F_X(X) \leq [1, 1]\)

(F2) If \( X \leq Y \) then \( F_X(X) \leq F_Y(Y) \)

(F3) \( \lim_{X \to +\infty} F_X(X) = [1, 1] \)

(F4) \( \lim_{X \to -\infty} F_X(X) = [0, 0] \)

(F5) \( \lim_{X \to A^+} F_X(X) = F_X(A) \), where \( A^+ = \lim_{|0, 0| \ll \epsilon \to |0, 0|} A + \epsilon \)

From Eq. (12) is possible to obtain the probability of other events:

(P1) \( P(A \ll \mathcal{X} \leq B) = F_X(B) - F_X(A) \)

(P2) \( P(\mathcal{X} \gg A) = [1, 1] - F_X(A) \)

(P3) \( P(\mathcal{X} \ll B) = F_X(B^-) \) where \( B^- = \lim_{|0, 0| \ll \epsilon \to |0, 0|} B - \epsilon \).

Let \( \mathcal{X} \) be a continuous random variable. The function \( f_X : \mathbb{I}(\mathbb{R}) \to \mathbb{I}(\mathbb{R}) \)
defined by

\[ f_X(X) = \frac{dF_X(X)}{dX} \] (13)

is called the interval probability density function of \( \mathcal{X} \).

**Proposition 5.1** The interval probability density function of \( \mathcal{X} \) satisfy the following properties:

1. \( f_X(X) \geq [0, 0] \)

2. \( \int_{[\infty]} f_X(X)dX = [1, 1] \)

3. \( F_X \) is piecewise continuous

4. \( P(A \ll \mathcal{X} \leq B) = \int_A^B f_X(X)dX \)

**Proof:**

1. By considering Properties (F1) and (F2) and Eq. (13), it is clear that \( f_X(X) \geq [0, 0] \).
2. Taking into account Definition 4.12 and Properties (F3) and (F4) it follows that

\[
\int_{[-\infty]}^{[+\infty]} \int_{C}^{D} f_{X}(X) dX = \lim_{C \to [-\infty]} \lim_{D \to [+\infty]} \int_{C}^{D} f_{X}(X) dX
\]

\[
= \lim_{C \to [-\infty]} \lim_{D \to [+\infty]} \left| F_{X}(D) - F_{X}(C) \right|_{C}
\]

\[
= \left| \lim_{D \to [+\infty]} F_{X}(D) - \lim_{C \to [-\infty]} F_{X}(C) \right|
\]

\[
= [1, 1] - [0, 0] = [1, 1]
\]

3. Straightforward Definition of continuous interval random variable.

4. The distribution function \( F_{X} \) of a continuous interval random variable \( X \) can be obtained by

\[
F_{X}(X) = P(X \leq X) = \int_{[-\infty]}^{X} f_{X}(\xi) d\xi \quad (14)
\]

Thus, once when \( X \) is a continuous interval random variable \( P(X = X) = [0, 0] \), then

\[
P(A \ll X \leq B) = P(A \leq X \leq B) = P(A \ll X \ll B) = P(A \ll X \ll B) = \int_{A}^{B} f_{X}(X) dX \quad (15)
\]

Also, from Eq. (15) it possible to verify that

\[
P(A \ll X \leq B) = \int_{A}^{B} f_{X}(X) dX = F_{X}(B) - F_{X}(A) \quad (16)
\]

The mean or expected value of a continuous interval random variable \( X \), denoted by \( \mu_{X} \), is defined by

\[
\mu_{X} = \int_{[-\infty]}^{[+\infty]} X f_{X}(X) dX \quad (17)
\]

The \( n \)th moment of a continuous interval random variable \( X \), denoted by \( E(X^{n}) \), is defined by

\[
E(X^{n}) = \int_{[-\infty]}^{[+\infty]} X^{n} f_{X}(X) dX \quad (18)
\]

Thus, the mean of \( X \) is the first moment of \( X \).
The variance of a continuous interval random variable \( X \), denoted by \( \text{Var}(X) \), is defined by

\[
\text{Var}(X) = \int_{-\infty}^{+\infty} (X - \mu_X)^2 f_X(X) dX
\]

(19)

The standard deviation of a continuous interval random variable \( X \), denoted by \( \sigma_X \), is defined by

\[
\sigma_X(X) = \sqrt{\int_{-\infty}^{+\infty} (X - \mu_X)^2 f_X(X) dX}
\]

(20)

### 5.3 Uniform, Exponential and Gaussian Interval Probability Distribution Functions

Let \( X \) be a continuous interval random variable and \( A \) and \( B \) intervals such that \( A \ll B \). \( X \) is uniform over \((A, B)\) if

\[
f_X(X) = \begin{cases} 
[1, 1] & \text{if } A \ll X \ll B \\
[0, 0] & \text{otherwise}
\end{cases}
\]

(21)

Notice that, in this case the interval distribution function of \( X \), by considering Eq. (14), is

\[
F_X(X) = \begin{cases} 
\int_A^X f_X(\xi) d\xi & \text{if } A \ll X \ll B \\
[0, 0] & \text{if } \underline{X} \leq A \text{ or } \overline{X} \leq \overline{a}
\end{cases}
\]

where

\[
\int_A^X f_X(\xi) d\xi = \left[ \int_A^\overline{a} f_X(\xi) \frac{dM(A,X)}{\overline{a} - a} \right. \\
\left. + \int_{\overline{a}}^\overline{b} \pi_1(f_X([\overline{a}, \overline{b} - a](x - \overline{a}))) dx, \\
+ \int_{\overline{b}}^\overline{a} \pi_2(f_X([\overline{a}, \overline{b} - a](x - \overline{a}))) dx \right] \\
\left. \times \frac{dM(A,X)}{\overline{b} - \overline{a}} \right] \\
+ \int_{\overline{a}}^\overline{b} \frac{1}{B - A} dx, \int_{\overline{a}}^\overline{b} \pi_2([1, 1]) dx \right] \\
\left. \times \frac{dM(A,X)}{B - \overline{a}} \right]
\]

\[
= \int_{\overline{a}}^\overline{b} \frac{1}{B - a} dx, \int_{\overline{a}}^\overline{b} \pi_2([1, 1]) dx \right] \\
\left. \times \frac{dM(A,X)}{B - \overline{a}} \right]
\]
The uniform distribution is naturally used when has no previous knowledge of the begin and end for where a variable could take values but the probability that it take a value in the range is the same. For example, random numbers in an interval which are distributed according to the standard uniform distribution. But if the limits not are well defined, then in this case is more adequate use standard uniform interval distribution.

Let $X$ be a continuous interval random variable and $\lambda \in I(\mathbb{R})$ such that $[0, 0] \ll \lambda$. $X$ is exponential with parameter $\lambda$ if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } [0, 0] \ll X \\ [0, 0] & \text{otherwise} \end{cases} \quad (22)$$

where $e^X = [e^{X_l}, e^{X_r}]$. Thus, $f_X(x) = [\lambda e^{-\lambda x}, \lambda e^{-\lambda x}]$ when $X \gg [0, 0]$.

Notice that, in this case the interval distribution function of $X$, by consider the Eq. (14), is

$$F_X(X) = \int_{-\infty}^{X} f_X(\xi) d\xi$$

$$= \begin{cases} \int_{[0, 0]}^{X} f_X(\xi) d\xi & \text{if } [0, 0] \ll X \\ [0, 0] & \text{otherwise} \end{cases}$$

where

$$\int_{[0, 0]}^{X} f_X(\xi) d\xi = \left[ \int_{0}^{Z} f_{X_l}(x) dx, \int_{0}^{Z} f_{X_r}(x) dx \right] \frac{dM([0, 0], X)}{Z}$$

$$= \left[ \int_{0}^{Z} \pi_1(f_X([x, \frac{x}{X}])) dx, \int_{0}^{Z} \pi_2(f_X([x, \frac{x}{X}])) dx \right] \frac{\pi}{Z}$$

$$= \left[ \int_{0}^{Z} \frac{\lambda}{\lambda X} (e^{-\lambda x} - 1), - \frac{\lambda}{\lambda X} (e^{-\lambda x} - 1) \right] \frac{\pi}{Z}$$

$$= \left[ \frac{\lambda}{X} (1 - e^{-\lambda x}), \frac{\lambda X}{\lambda^2} (1 - e^{-\lambda x}) \right]$$

Let $X$ be a continuous interval random variable. $X$ is normal or Gaussian if

$$f_X(x) = \frac{1}{\sqrt{2\pi Var(X)}} e^{-\frac{(x - \mu)^2}{2Var(X)}} \quad (23)$$

Notice that, in this case

$$f_X(x) = \left[ \frac{1}{\sqrt{2\pi Var(X)}} e^{-\frac{(x - \mu)^2}{2Var(X)}} : \frac{1}{\sqrt{2\pi Var(X)}} e^{-\frac{(x - \mu)^2}{2Var(X)}} \right] \quad (24)$$
Notice that, in this case the interval distribution function of $X$, by consider the Eq. \[(14)\], is

$$F_X(X) = \int_{-\infty}^{X} f_X(\xi) d\xi = \int_{-\infty}^{X} f_{Xl}(x) dx, \int_{X}^{\infty} f_{Xr}(x) dx \right] \frac{dM(A,B)}{b-a}$$

$$= \left[ \int_{-\infty}^{\pi_1} f_X([x, \pi + \frac{b-a}{b-a}(x-a)]) dx, \int_{-\infty}^{\pi_2} f_X([x, \pi + \frac{b-a}{b-a}(x-a)]) dx \right] \frac{dM(A,B)}{b-a}$$

6 Final Remarks

As can be seen in an evaluation of the existing literature regarding probability density functions, the interval mathematics was still little explored in this scope, although it is an important way of estimating the imprecision of the parameters. The few published works on this topics (as one can see in [4, 37]) consider only the interval interpretation for measuring the uncertainty of the variables taking into account a classical view of probability density functions instead of an interval one. For this reason, the concept of interval probability density functions based on a new way of defining interval integrals presented in this paper creates a wide range of possibilities for probabilistic problems from the interval point of view. The results show that the theory is consistent and allows us to generate an interesting way to control the inaccuracies and uncertainties of the variables throughout the mathematical process of the model.

Under point of view of the interval integral theory presented here it is important to highlight that there exists functions which are continuous (according to the Moore topology) but are not inclusion monotonic, for example the interval function $F(X) = m(X) + \frac{1}{2}(X - m(X))$, where $m(X)$ is the middle point of $X$. However the (real) integrals based on Riemann sums are defined for all continuous functions, and therefore would be desirable that an interval extension of this notion consider all continuous interval functions (with respect to a suitable notion of continuity for interval functions) and not only those which are inclusion monotonic. So, the deletion of inclusion monotonic restriction, in Moore and Yang integral definition, made in our extension is fundamental in order to provide a robust extension and to consider the integral of this kind of interval functions.

\[1\]This function was used by Moore in [32] as an example of an interval valued function which is not inclusion monotonic, lately in [39] was proved that this function is continuous with respect to the Moore topology.
Another restriction in the Moore and Yang approach, which also occurs in Caprani, Madsen and Rall integrals in [9], is by considering only real numbers as integral limits. The Corliss extension in [11] of Caprani, Madsen and Rall integral in [9] for interval integral limits, consist in a couple of Caprani, Madsen and Rall integrals with real integration limits. Therefore, it could be used in order to extend any other notion of interval integral where the limits are real numbers by simply substituting the Caprani, Madsen and Rall integral for another notion. Nevertheless, these extension and any other of the same line, still computationally more easy to calculate, is not intrinsic and lack of mathematical foundations. Our extension follows the original spirit of Moore and Yang, i.e. it is a generalization of the usual Riemann sum integrals based on the extension of all elements used in such kind of integral and therefore is mathematically a well founded extension.

The Corollary 4.5 can be seen as a meta-algorithm to compute our extension of the Moore-Yang integrals, in the sense that we can use any usual method to compute the Riemann integrals for each integral in the extremes of the interval in equation (1). Obviously, in this case, as is usual in interval computing, we need to use directed rounding in each path of the computing.

Choquet integrals and generalizations of the Choquet integrals [6, 15, 16] are an important family of (pre)aggregation functions used to merge discrete real inputs. Nevertheless, in many real problems, the inputs arise in the continuum and therefore the standard Choquet integrals and their generalizations are not adequate. In [25] was proposed a way of merging Riemann integrable inputs from discrete Choquet integral. As future work we will extend such work by considering interval-valued extensions of Choquet integrals [5, 21, 34].

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