A determination of the top-quark $\overline{\text{MS}}$ running mass via its perturbative relation to the on-shell mass with the help of principle of maximum conformality

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Abstract

In the present work, we study the properties of the top quark $\overline{\text{MS}}$ running mass computed from its on-shell mass by using both the four-loop $\overline{\text{MS}}$-on-shell relation and the principle of maximum conformity (PMC) scale-setting approach. The PMC adopts the renormalization group equation to set the correct magnitude of the strong running coupling of the perturbative series, its prediction avoids the conventional renormalization scale ambiguity, and thus a more precise pQCD prediction can be achieved. After applying the PMC to the four-loop $\overline{\text{MS}}$-on-shell relation and taking the top-quark on-shell mass $M_t = 172.9 \pm 0.4 \text{ GeV}$ as an input, we obtain the renormalization scale invariant $\overline{\text{MS}}$ running mass at the scale $m_t$, e.g. $m_t(m_t) = 162.6 \pm 0.4 \text{ GeV}$, in which the error is squared average of those from $\Delta \alpha_s(M_Z)$, $\Delta M_t$, and the approximate error from the uncalculated five-loop terms predicted by using the Padé approximation approach.

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I. INTRODUCTION

In Quantum Chromodynamics (QCD), the quark masses are elementary input parameters of the QCD Lagrangian. There are three light quarks (up, down and strange) and three heavy ones (charm, bottom and top). Comparing with other quarks, the top quark is special. It decays before hadronization, which can be almost considered as free quark. Therefore, the top quark on-shell (OS) mass, or equivalently the pole mass, can be determined experimentally. The direct measurements are based on analysis techniques which use top-pair events provided by Monte Carlo (MC) simulation for different assumed values of the top-quark mass. Applying those techniques to data yields a mass quantity corresponding to the top-quark mass scheme implemented in the MC, thus it is usually referred to as the “MC mass”. Since the top-quark MC mass is within $\sim 1 \text{ GeV}$ of its OS mass [1], one can treat the MC mass as the OS one [2–6]. Detailed discussions on the top quark OS mass can be found in Refs. [7–9]. As shown in Particle Data Group (PDG) [10], an average of various measurements at the Tevatron and the LHC gives the OS mass $M_t = 172.9 \pm 0.4 \text{ GeV}$.

Practically, one usually adopts the modified minimal subtraction scheme (the $\overline{\text{MS}}$ scheme) to do the pQCD calculation, and the $\overline{\text{MS}}$ running quark mass is introduced. As for the top quark $\overline{\text{MS}}$ running mass, it can be related to the OS mass perturbatively which has been computed up to four-loop level [11–19]. Using this relation and the measured OS mass, we are facing the chance of determining a precise value for the top quark $\overline{\text{MS}}$ running mass.

In using the relation, an important thing is to determine the exact value of the strong coupling constant ($\alpha_s$). The scale running behavior of $\alpha_s$ is controlled by the renormalization group equation (RGE) or the $\beta$-function [20–23], which is now known up to five-loop level [24]. Using the PDG reference point $\alpha_s(M_Z) = 0.1181 \pm 0.0011$ [10], we can fix its value at any scale. And thus the remaining task for achieving the precise value of the perturbative series of the $\overline{\text{MS}}$ running mass over the OS mass is to determine the correct momentum flow and hence the correct $\alpha_s$ value of the perturbative series.

Conventionally, people uses the guessed renormalization scale as the momentum flow of the process and varies it within an arbitrary range to estimate its uncertainty for the pQCD predictions. This naive treatment leads to the mismatching of the strong coupling constant with its coefficients, well breaking the renormalization group invariance [25–27] and leading to renormalization scale and scheme ambiguities. And the effectiveness of this treatment depends heavily on the perturbative convergence of the pQCD series. Sometimes, the scale is chosen so as to eliminate the large logarithmic terms or to minimize the contributions from high-order terms. And sometimes, the scale is so chosen directly to achieve the prediction in agreement with the data. Such kind of guessing work depresses the predicative power of the pQCD theory, and sometimes is misleading, since there may have new physics beyond the standard model.

To eliminate the artificially introduced renormalization scale and scheme ambiguities, the principle of maximum conformity (PMC) scale-setting approach has been suggested [28–32]. The purpose of the PMC is to determine the effective $\alpha_s$ of a pQCD series by using the known $\beta$-terms of the pQCD series. The argument of the effective $\alpha_s$ is called as the PMC scale, which corresponds to the effective momentum flow of the process. It has been found that the magnitude of the determined effective $\alpha_s$ is independent to any choice of renormalization scale, thus the conventional renormalization scale ambiguity is
eliminated by applying the PMC. The PMC shifts all non-conformal $\beta$-terms into the strong coupling constant at all orders, and it reduces to the Gell-Mann and Low scale-setting approach [33] in the QED Abelian limit [34]. Furthermore, after adopting the PMC to fix the $\alpha_s$ running behavior, the remaining perturbative coefficients of the resultant series match the series of conformal theory, leading to a renormalization scheme independent prediction. Using the PMC single-scale approach [35], it has recently been demonstrated that the PMC prediction is scheme independent up to any fixed order [36].

The residual scale dependence due to the uncalculated higher-order term is highly suppressed by the combined $\alpha_s$-suppression and exponential suppression [37]. Due to the elimination of the divergent renormalon terms like $n!\beta^n_0 a^n_s$ [38–40], the convergence of the pQCD series is naturally improved, which leads to a more accurate prediction. Moreover, the renormalization scale-and-scheme independent series is also helpful for estimating the contribution of the unknown higher-orders, some examples can be found in Refs.[41–43].

II. CALCULATION TECHNOLOGY

The renormalized mass under the $\overline{\text{MS}}$ scheme or the OS scheme can be related to the bare mass ($m_0$) by

$$ m_0 = Z_m^R m, $$

where $R = \overline{\text{MS}}$ or OS. Under the $\overline{\text{MS}}$ scheme, one can derive the expression of $Z_m^R$ by requiring the renormalized propagator to be finite, which has been calculated up to five-loop level [44–47]. Under the OS scheme, the expression of $Z_m^{\text{OS}}$ can be obtained by requiring the quark two-point correlation function vanish at the position of OS mass, whose one-, two- and three-loop QCD corrections have been given in Refs.[11–15, 48], and the electro-weak effects have also been considered in Refs.[49–58]. Generally, the relation between the $\overline{\text{MS}}$ quark mass and OS quark mass can be written as

$$ z_m(\mu_r) = \frac{m(\mu_r)}{M} = \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}} = \sum_{n=0}^{\infty} z_m^{(n)}(\mu_r)a^n_s(\mu_r), $$(2)

where $a_s(\mu_r) = \alpha_s(\mu_r)/4\pi$, $m(\mu_r)$ is the $\overline{\text{MS}}$ running mass with $\mu_r$ being the renormalization scale, and the $M$ is OS quark mass. The perturbative coefficients $z_m^{(n)}$ have been known up to four-loop level [16, 17], and the $\overline{\text{MS}}$ running mass at the scale $M$ takes the following perturbative form:

$$ m(M) = M \left\{ 1 + z_m^{(1)}(M)a_s(M) + z_m^{(2)}(M)a_s^2(M) + z_m^{(3)}(M)a_s^3(M) + z_m^{(4)}(M)a_s^4(M) + \cdots \right\}, $$

where the coefficients $z_m^{(i)}(M)$ ($i = 1, \ldots, 4$) can be read from Ref.[17]. Using the displacement relation which relates the $\alpha_s$ value at the scale $\mu_1$ with its value at any other scale $\mu_2$,

$$ a_s(\mu_1) = a_s(\mu_2) + \frac{1}{\ln \mu_2^\beta_0} \sum_{n=1}^{\infty} \frac{1}{n!} \left( \ln \frac{\mu_2}{\mu_1} \right)^n a_n(\mu_2) (-\delta)^n, $$

where $\delta = \ln \mu_2^\beta_0/\mu_1^\beta_0$, one can obtain the relation at any renormalization scale $\mu_r$, i.e.

$$ m(\mu_r) = \mu_0 \left\{ 1 + c_{1,0} a_s(\mu_r) + (c_{2,0} + c_{2,1} n_f) a_s^2(\mu_r) + (c_{3,0} + c_{3,1} n_f + c_{3,2} n_f^2) a_s^3(\mu_r) + (c_{4,0} + c_{4,1} n_f + c_{4,2} n_f^2 + c_{4,3} n_f^3) a_s^4(\mu_r) + \cdots \right\} \cdot, $$

where $m_0$ is the top quark $\overline{\text{MS}}$ mass and $M_t$ is the top quark OS mass. To apply the PMC to fix the $\alpha_s$ value with the help of RGE, we first transform the $n_f$ series as the $\{\beta_i\}$-series by using the degeneracy relations which is the general properties of a non-Abelian gauge theory [59],

$$ m_t(M_t) = M_t \left\{ 1 + r_{1,0} a_s(\mu_r) + (r_{2,0} + \beta_r r_{2,1}) a_s^2(\mu_r) + (r_{3,0} + \beta_r r_{3,1} + 2 \beta_\rho r_{3,2}) a_s^3(\mu_r) + (r_{4,0} + \beta_r r_{4,1} + 2 \beta_\rho r_{4,2} + \beta_\rho^2 r_{4,3}) a_s^4(\mu_r) + \cdots \right\}. $$

The coefficients $r_{i,j}$ can be obtained from the known coefficients $c_{i,j}$ ($i > j \geq 0$) by applying basic PMC formulas listed in Ref.[31, 32]. The conformal coefficients $r_{1,0}$ are independent of $\mu_r$, and the non-conformal coefficients $r_{i,j}$
(j ≠ 0) are functions of μ_r, i.e.

\[ r_{i,j} = \sum_{k=0}^{j} C^k_j \hat{r}_{i-k,j-k} \ln^k(\mu_r^2/M_t^2), \]  

(8)

where the reduced coefficients \( \hat{r}_{i,j} = r_{i,j}|_{\mu_r=M_t} \), the combination coefficients \( C^k_j = j!/[k!(j-k)!] \), and \( i, j, k \) are the polynomial coefficients. For convenience, we put the reduced coefficients \( \hat{r}_{i,j} \) in the Appendix.

Applying the standard PMC single-scale approach \cite{35}, the effective coupling \( \alpha_s(Q) \) can be obtained by using all the non-conformal terms and the perturbative series \( 7 \) changes to the following conformal series,

\[ m_r(M_t)|_{\text{PMC}} = M_t \left\{ 1 + \hat{r}_{1,0}a_s(Q) + \hat{r}_{2,0}a_s^2(Q) \right\} \]  

where \( Q \) is the PMC scale, which corresponds to the effective momentum flow of the process and is determined by requiring all the non-conformal terms vanish. The PMC scale \( Q^* \), or \( \ln Q^2/M_t^2 \), can be expanded as a perturbative series, and up to next-to-next-to-leading log (NNLL) accuracy, we have

\[ \ln \frac{Q^2}{M_t^2} = T_0 + T_1 a_s(M_t) + T_2 a_s^2(M_t) + \mathcal{O}(a_s^3), \]  

(10)

where the coefficients are

\[ T_0 = -\frac{\hat{r}_{2,1}}{r_{1,0}}, \]  

(11)

\[ T_1 = \frac{\beta_0 (r_{2,1}^2 - r_{1,0}^2 r_{2,3,2})}{r_{1,0}^2} + \frac{2(r_{2,0}r_{2,1} - r_{1,0}^2 r_{3,1})}{r_{1,0}^2} \]  

and

\[ T_2 = 3\beta_1 (r_{2,1}^3 - r_{1,0}^2 r_{3,2}) - \frac{4(r_{1,0}^2 r_{2,1}^2 - r_{2,0}^4 r_{2,1}) + 3(r_{1,0}^2 r_{2,1} r_{3,0} - r_{1,0}^3 r_{2,1}^2)}{r_{1,0}^2} \]  

(12)

++ \beta_0\left( 4r_{2,1} r_{3,1} r_{1,0} - 3r_{2,0}^2 r_{2,1}^2 + 2r_{2,0} r_{2,3,2} r_{1,0} - 3r_{2,0} r_{2,3,1} \right) + \frac{\beta_0^2 (2r_{1,0} r_{3,2} r_{2,1} - r_{3,2} r_{1,0}^2)}{r_{2,1}^2} + \frac{\beta_0^2 (2r_{1,0} r_{3,2} r_{2,1} - r_{3,2} r_{1,0}^2)}{r_{2,1}^2} \]  

(13)

Using the present known four-loop relations, we can fix the PMC scale up to NNLL accuracy. It can be found that \( Q_s \) is independent to the choice of the renormalization scale \( \mu_r \) at any fixed-order, and the conventional renormalization scale ambiguity is eliminated. This indicates that one can finish the a fixed-order perturbative calculation by choosing any renormalization scale as a starting point, and the PMC scale \( Q_s \), and hence the PMC prediction shall be independent to such choice.

III. NUMERICAL RESULTS

To do the numerical calculation, we adopt \cite{10}:

\[ \alpha_s(M_Z^2) = 0.1181 \pm 0.0011 \]  

and \( M_t = 172.9 \pm 0.4 \text{ GeV} \).

A. Properties of the top quark \( \overline{\text{MS}} \) running mass

FIG. 1. The top quark \( \overline{\text{MS}} \) running mass, \( m_r(M_t) \), up to four-loop QCD corrections under the conventional (Conv.) and PMC scale-setting approaches. The renormalization scale \( \mu_r \in [\frac{1}{2} M_t, 2 M_t] \).

By setting all input parameters to be their central values into Eqs. (5, 9), we present the top quark \( \overline{\text{MS}} \) running mass at the scale \( M_t \) under conventional and PMC scale-setting approaches in FIG. 1. It shows that the conventional renormalization scale dependence becomes small when we have known more loop terms. Numerically, we obtain \( m_r(M_t)|_{\text{Conv.}} = [162.170, 162.462] \text{ GeV for } \mu_r \in [\frac{1}{2} M_t, 2 M_t] \text{ GeV, and } m_r(M_t)|_{\text{Conv.}} = [162.052, 162.522] \text{ GeV for } \mu_r \in [\frac{1}{2} M_t, 3 M_t] \); e.g. the net scale errors are only \( \sim 0.2\% \text{ and } \sim 0.3\% \), respectively. We should point out that such small net scale dependence for the four-loop prediction is due to the well convergent behavior of the perturbative series, e.g. the relative magnitudes of LO: NLO: N^2LO: N^3LO: N^4LO=1: 4.6\%: 1\%: 0.3\%: 0.1\% for the case of \( \mu_r = M_t \), and also due to the cancellation of the scale dependence among different orders. The scale errors for each loop terms are unchanged and large, for example, the \( m_r(M_t) \) has the following perturbative series up to four-loop level,

\[ m_r(M_t)|_{\text{Conv.}} = 172.9 - 7.903 \pm 0.834 - 1.854 \pm 0.391 \]  

(14)

where the central values are for \( \mu_r = M_t \), the upper and lower errors are for \( \mu_r = M_t/2 \) and \( \mu_r = 2 M_t \), respectively. It shows that the absolute scale errors are 18\%, 36\%, 63\% and 70\% for the NLO-terms, N^2LO-terms, N^3LO-terms and N^4LO-terms, respectively.
On the other hand, FIG. 1 shows that after applying the PMC, the relative magnitudes of LO: NLO: N^2LO: N^3LO: N^4LO of the pQCD series changes to 1: 7.2\%: 0.5\%: 0.3\%: < 0.1\%. And there is no renormalization scale dependence for \( m_t(M_t) \) at any fixed order,

\[
\begin{align*}
m_t(M_t)|_{\text{PMC}} &= 172.9 - 12.497 + 0.919 + 0.551 - 0.095 \text{ (GeV)} \\
&= 161.778 \text{ (GeV)},
\end{align*}
\]

which is unchanged for any choice of renormalization scale. The PMC scale, or equivalently the effective momentum flow of the process, is \( Q_\star = 12.30 \text{ GeV} \), which is fixed up to NNLL accuracy,

\[
\ln \frac{Q^2}{M_t^2} = -4.686 - 51.890a_s(M_t) - 2126.558a_s^2(M_t)
\]

\[
= -4.686 - 0.445 - 0.156.
\]

The relative magnitudes of each loop terms are 1 : 9\% : 3\%, which shows a good convergence. As a conservative estimation, if using the last known term as the magnitude of its unknown NNLL term, the change of momentum flow is small, \( \Delta Q_\star \simeq (-0.92) \text{ GeV} \).

The top quark \( m_t \) at LO is about 21\% of the exact \( m_t \) LO, the PAA predicted N^4LO-term is about 70\% – 88\% of the exact N^4LO term, and the PAA predicted N^6LO is about 42\% – 43\% of the exact N^4LO term. On the other hand, the PAA predictions with the help of the renormalization scheme and scale invariant PMC conformal series is much more reliable. TABLE I shows that by using the conventional pQCD series under the orders of \( \mu_r \in [M_t/2, 2M_t] \), the PAA predicted N^4LO-term is about 91\% of the exact N^4LO term, and the PAA predicted N^6LO is about 21\% of the exact N^4LO term (showing better convergence). Thus the approximate top quark \( \overline{\text{MS}} \) mass up to N^5LO level becomes

\[
m_t(M_t)|_{\text{Conv.}} = 162.288 - 0.181 + 0.049 \text{ (GeV)} [1/2] - \text{type(17)}
\]

\[
= 162.298 + 0.183 \text{ (GeV)} [2/1] - \text{type(18)}
\]

\[
m_t(M_t)|_{\text{PMC}} = 161.758 \text{ (GeV)}.
\]

As a final remark, the top quark \( \overline{\text{MS}} \) running mass at two scales \( \mu_1 \) and \( \mu_2 \) can be related via the following equation [45],

\[
m_t(\mu_1) = m_t(\mu_2) \frac{c_t(4a_s(\mu_1))}{c_t(4a_s(\mu_2))}
\]

where the function \( c_t(x) = x^\tau (1 + 1.3980 x + 1.7935 x^2 - 0.6834 x^3 - 3.5365 x^4) \). Using Eqs.(14, 15, 20), we obtain the top quark \( \overline{\text{MS}} \) running mass at the scale \( m_t \):

\[
m_t(m_t)|_{\text{Conv.}} = 163.182 + 0.081 - 0.190 \text{ (GeV)};
\]

\[
m_t(m_t)|_{\text{PMC}} = 162.629 \text{ (GeV)}.
\]
B. Theoretical uncertainties

After eliminating the renormalization scale uncertainty via using the PMC approach, there are still several error sources, such as the $\alpha_s$ fixed-point error $\Delta\alpha_s(M_Z)$, the error of top quark OS mass $\Delta M_t$, the unknown contributions from six-loop and higher order terms, and etc. The uncertainty of the four-loop coefficient $z_n^{(4)}(M)$ has been discussed in Ref.[17], whose magnitude is negligibly small. For convenience, when discussing one uncertainty, the other input parameters shall be set as their central values.

As for the $\alpha_s$ fixed-point error, by using $\Delta\alpha_s(M_Z) = 0.0011$ together with the four-loop $\alpha_s$-running behavior, we obtain $\Lambda_{\overline{QCD}, n_f=5} = 209.5^{+13.2}_{-12.6}$ MeV and $\Lambda_{\overline{QCD}, n_f=6} = 88.3^{+6.5}_{-5.6}$ MeV. Then we obtain the top quark $\overline{MS}$ running mass at the scale $m_t$

$$m_t(m_t)|_{\text{Conv.}} = 163.182^{+0.103}_{-0.103} \ (\text{GeV}),$$

$$m_t(m_t)|_{\text{PMC}} = 162.629^{+0.018}_{-0.019} \ (\text{GeV}).$$

Eqs. (23, 24) show that the PMC prediction is more sensitive to the value of $\Delta\alpha_s(M_Z)$. This is reasonable since the purpose of PMC is to achieve an accurate $\alpha_s$ value of the process, and inversely, a slight change of its running behavior derived from RGE may lead to sizable alterations. Numerically, the determined effective momentum $Q_*$ $\simeq 12$ GeV is much smaller than the guessed momentum $O(M_t)$, and the strong coupling constant is more sensitive to the variation of $\Lambda_{\overline{QCD}}$.

As for the error from the choice of the top quark OS mass $\Delta M_t = \pm 0.4$ GeV, we obtain

$$m_t(m_t)|_{\text{Conv.}} = 163.182^{+0.380}_{-0.380} \ (\text{GeV}),$$

FIG. 2 shows that the top quark $\overline{MS}$ running mass $m_t(m_t)$ depends almost linearly on its OS mass, whose error is at the same order of $O(\Delta M_t)$.

In the above subsection, we have predicted the magnitude of the uncalculated $N^5$LO-terms. If treating the absolute value of the PAA predicted $N^5$LO magnitude as a conservative estimation of the error of present $N^4$LO prediction, we shall have an extra error from the unknown perturbative terms, e.g.

$$\Delta m_t(m_t)|_{\text{Conv.}} = \pm 0.080(\text{GeV}), \ [1/2] - \text{type (27)}$$

$$\Delta m_t(m_t)|_{\text{Conv.}} = \pm 0.072(\text{GeV}), \ [2/1] - \text{type (28)}$$

$$\Delta m_t(m_t)|_{\text{PMC}} = \pm 0.018(\text{GeV}), \ [0/3] - \text{type (29)}$$

IV. SUMMARY

In the present paper, we have presented a more accurate prediction of the top quark $\overline{MS}$ running mass from the experimentally measured top quark OS mass by applying the PMC to eliminate the conventional renormalization scale ambiguity. As a combination, we obtain

$$m_t(m_t)|_{\text{Conv.}} = 163.182^{+0.410}_{-0.445}(\text{GeV}), \ [1/2] - \text{type (30)}$$

$$m_t(m_t)|_{\text{Conv.}} = 163.182^{+0.404}_{-0.445}(\text{GeV}), \ [2/1] - \text{type (31)}$$

$$m_t(m_t)|_{\text{PMC}} = 162.629^{+0.397}_{-0.400}(\text{GeV}), \ [0/3] - \text{type (32)}$$

where the errors are squared averages of those from $\Delta\alpha_s(M_Z)$, $\Delta M_t$, and the uncalculated $N^5$LO terms predicted by using the PAA. Among the errors, the one caused by $\Delta M_t$ is dominant, and we need more accurate data to suppress this uncertainty. The conventional predictions have also the renormalization scale uncertainty by varying $\mu_r \in [M_t/2, 2M_t]$, even though its magnitude is small due to the cancellation of scale errors among different orders. Up to the present known $N^4$LO-level, the predictions under the PMC and conventional scale-setting approaches are consistent with each order. However, it has been found that after applying the PMC, a scale-invariant and more convergent pQCD series, and a more reliable prediction of contribution from unknown higher-order terms can be achieved. Thus we think the PMC is an important approach for achieving precise pQCD predictions, since its prediction is independent to the choice of renormalization scale. It should extremely important for lower fixed-order pQCD predictions, when there is not enough terms to suppress the large scale uncertainty of each loop terms.

\footnote{As an addendum, if taking the small difference of the OS mass and the Monte-Carlo mass into consideration \cite{66}, $M_t = M_t^{MC} - |0.29 \text{ GeV}, 0.85 \text{ GeV}|$, then the above central values of $m_t(m_t)$ shall be altered by about $[0.11, -1.19] \text{ GeV}$ for both conventional and PMC scale-setting approaches.}
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**APPENDIX: THE PMC REDUCED PERTURBATIVE COEFFICIENTS \( \hat{r}_{i,j} \)**

In this appendix, we give the required PMC reduced coefficients \( \hat{r}_{i,j} \) for the perturbative series of the top quark \( \overline{\text{MS}} \) running mass over its OS mass up to four-loop level, i.e.

\[
\begin{align*}
\hat{r}_{1,0} &= -4C_F, \\
\hat{r}_{2,0} &= C_AC_F(6\zeta_3 + 5\pi^2 - \frac{55}{4} - 4\pi^2 \ln 2) + C_F^2(\frac{7}{8} - 12\zeta_3 - 5\pi^2 + 8\pi^2 \ln 2) + (12 - 4\pi^2)C_FT_F, \\
\hat{r}_{2,1} &= -\frac{71}{8} - \pi^2)C_F, \\
\hat{r}_{3,0} &= C_AC_F^2(51\pi^2\zeta_3 + 219\zeta_3 - 130\zeta_5 - \frac{181\pi^2}{6} - \frac{53\pi^4}{30} - \frac{19027}{216} + \frac{16}{3}\pi^2 \ln 2) + C_AC_F^2\left[384\text{Li}_4\left(\frac{1}{2}\right) - 76\pi^2\zeta_3 \\
&- 112\zeta_3 + 180\zeta_5 + \frac{518\pi^2}{3} + \frac{5731}{12} - \frac{\pi^4}{15} + 16\ln^4 2 - 16\pi^2 \ln 2 - \frac{728}{3}\pi^2 \ln 2\right] + C_FT_F\left[8\pi^2\zeta_3 \\
&+ 88\zeta_3 - 40\zeta_5 - \frac{28\pi^4}{9} - \frac{4372\pi^2}{27} + 144 + \frac{32}{3}\pi^2 \ln 2 + \frac{1696}{9}\pi^2 \ln 2\right] + C_F^2\left(\frac{56\pi^4}{9} - 288\zeta_3 - \frac{2608\pi^2}{27} \\
&- 24 + \frac{64}{3}\pi^2 \ln 2 + \frac{1216}{9}\pi^2 \ln 2\right) + C_F^3\left[40\zeta_5 - 76\text{Li}_4\left(\frac{1}{2}\right) - 4\pi^2\zeta_3 - 32\zeta_3 - \frac{4\pi^4}{3} - \frac{613\pi^2}{3} - \frac{2969}{12} \\
&- 32\ln^4 2 + 32\pi^2 \ln 2 + 2 + 464\pi^2 \ln 2\right] - \frac{608}{45}\pi^2 C_FT_F^2, \\
\hat{r}_{3,1} &= C_AC_F\left[32\text{Li}_4\left(\frac{1}{2}\right) + \frac{73\zeta_3}{2} + \frac{26\pi^2}{3} - \frac{19\pi^4}{90} - \frac{20335}{432} + \frac{4}{3}\ln^4 2 + \frac{8}{3}\pi^2 \ln 2 - \frac{44}{3}\pi^2 \ln 2\right] + C_F^2\left(\frac{119\pi^4}{90} \\
&- 64\text{Li}_4\left(\frac{1}{2}\right) - 55\zeta_3 - \frac{95\pi^2}{6} - \frac{1927}{48} - \frac{16}{3}\pi^2 \ln 2 + \frac{88}{3}\pi^2 \ln 2\right] + C_FT_F\left(22 - 24\zeta_3 - \frac{26\pi^2}{3}\right), \\
\hat{r}_{3,2} &= C_F\left(- 14\zeta_3 - \frac{13\pi^2}{3} - \frac{2353}{216}\right), \\
\hat{r}_{4,0} &= -947.046C_A^3C_F^2 - 1269.84C_A^2C_F^2 + 2T_F(3216.18C_A^2C_F + 568.364C_AC_F^3 - 335.759C_F^3) + 3671.8C_AC_F^3 \\
&+ T_F^2(587.571C_F^2 - 2497.16C_AC_F) - 219.883C_Ad_F^{abcd}d_F^{abcd} \\
&+ 219.883C_Ad_F^{abcd}d_F^{abcd} \\
&+ 219.883C_Ad_F^{abcd}d_F^{abcd} \\
&+ 219.883C_Ad_F^{abcd}d_F^{abcd} \\
&+ 219.883C_Ad_F^{abcd}d_F^{abcd} \\
&+ 19.9893d_F^{abcd}d_F^{abcd}, \\
\hat{r}_{4,1} &= 932.846C_A^2C_F + T_F(81.3012C_F^2 - 834.037C_AC_F) - 473.22C_AC_F^2 - 21.945C_F^3 - 780.54C_FT_F^2 \\
&+ 569.768C_F^2 + 21.945C_F^3 - 780.54C_FT_F^2, \\
\hat{r}_{4,2} &= -112.694C_AC_F + 85.1974C_F^2 - 590.868C_FT_F, \\
\hat{r}_{4,3} &= -439.436C_F,
\end{align*}
\]

where

\[
\begin{align*}
N_C &= 3, T_F = \frac{1}{2}, \\
C_A &= N_C, C_F = \frac{N_C^2 - 1}{2N_C},
\end{align*}
\]

\[d_F^{abcd}d_F^{abcd} = \frac{(N_C^2 - 1)(N_C^2 - 6N_C^2 + 18)}{96N_C^2}, \]

\[d_F^{abcd}d_F^{abcd} = \frac{N_C(N_C^2 - 1)(N_C^2 + 6)}{48}.\]

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