Abstract
We develop tools to analyze and compare the Brauer groups of spectra such as periodic complex and real $K$-theory and topological modular forms, as well as the derived moduli stack of elliptic curves. In particular, we prove that the Brauer group of $\text{TMF}$ is isomorphic to the Brauer group of the derived moduli stack of elliptic curves. Our main computational focus is on the subgroup of the Brauer group consisting of elements trivialized by some étale extension, which we call the local Brauer group. Essential information about this group can be accessed by a thorough understanding of the Picard sheaf and its cohomology. We deduce enough information about the Picard sheaf of $\text{TMF}$ and the (derived) moduli stack of elliptic curves to determine the structure of their local Brauer groups away from the prime 2. At 2, we show that they are both infinitely generated and agree up to a potential error term that is a finite 2-torsion group.
1. INTRODUCTION

The Brauer group \( \text{Br}(R) \) of an \( E_{\infty} \)-ring spectrum \( R \) was introduced by Baker–Richter–Szymik [8] following previous work of Baker–Lazarev [6] and Toën [70]. The group classifies Azumaya algebras over \( R \) up to Morita equivalence; equivalently, it classifies invertible \( R \)-linear stable \( \infty \)-categories. These can be seen as twisted versions of \( R \)-modules, and thus, \( \text{Br}(R) \) classifies all possible twists of \( \text{Mod}_R \). One can actually replace \( \text{Mod}_R \) here with any symmetric monoidal \( \infty \)-category, like quasi-coherent sheaves on a scheme or stack. In the most classical case of vector spaces over a field, Azumaya algebras are just central simple algebras (i.e., matrix algebras over a central division algebra) and the corresponding Brauer group was introduced by Brauer around 1930.

Classically, Brauer groups can often be computed as étale cohomology groups. They thus allow cohomological control of natural occurrences of Azumaya algebras (e.g., as endomorphism algebras of representations [63, Section 12.2]) or twisted sheaves (like in the theory of moduli of stable sheaves [16]). Another \( \infty \)-categorical example is given by the relevance of twists of parametrized spectra in Seiberg–Witten Floer homotopy theory [26, 38]. On the other hand, Brauer groups also allow algebraic or geometric interpretations of cohomology classes, as utilized, for example, in the classic Artin–Mumford example of a nonrational unirational variety [5] or the Merkurjev–Suslin theorem [31]. Brauer groups give also one of the approaches to class field theory [55, 61, 72] and form the basis of the Brauer–Manin obstruction for rational points [21]. Thus, the study of Brauer groups of ring spectra might be interesting for possible theories of étale cohomology on \( E_{\infty} \)-ring spectra, and can be seen as a contribution to the nascent subject of arithmetic of \( E_{\infty} \)-ring spectra. Moreover, the Brauer space provides a natural delooping of the Picard space, like the Picard space is a natural delooping of the space of units of an \( E_{\infty} \)-ring spectrum.
When $R$ is a connective $\mathbb{E}_\infty$-ring spectrum, $\text{Br}(R)$ depends only on $\pi_0 R$ and

$$\text{Br}(R) \cong H^1(\text{Spec } \pi_0 R, \mathbb{Z}) \times H^2(\text{Spec } \pi_0 R, G_m),$$  \hspace{1cm} (1.1)$$

where all cohomology is étale unless otherwise specified; see [3, 70]. For example, for a prime $p$, we have $\text{Br}(S[1/p]) \cong \mathbb{Z}/2$, so there is a “twisted form” of finite spectra after inverting $p$. These twisted forms are $\infty$-categories of modules for spherical quaternion algebras. In case that $R$ is a classical ring, $\text{Br}(R)$ might actually be larger than the classical Brauer group of $R$ because of the presence of derived, nonclassical Azumaya algebras.

The role of connectivity is to ensure (by an argument of Toën) that Brauer classes on connective $\mathbb{E}_\infty$-rings are étale-locally trivial. This fact enables the cohomological calculation of the Brauer group as in (1.1). We will show in Example 5.7 that this fails in general for nonconnective ring spectra. Thus, we will differentiate between $\text{Br}(R)$ and its subgroup $L\text{Br}(R)$ of Brauer classes that are étale-locally trivial, that is, become trivial after some faithful étale extension in the sense of [47, Definition 7.5.0.4]. Two of our main themes are that $L\text{Br}(R)$ is quite computable (up to the general difficulty of computing differentials), and that sometimes we may enlarge $L\text{Br}(R)$ by allowing more general extensions to kill Brauer classes. We can say something about the resulting subgroups of $\text{Br}(R)$, which may or may not coincide with $L\text{Br}(R)$.

Our main examples are real $K$-theory and topological modular forms. Let us begin with the former.

**Theorem 1.2** (Theorem 3.13). There is an isomorphism $L\text{Br}(K\text{O}) \cong \mathbb{Z}/2$.\footnote{Our actual definition of the Brauer group of a nonconnective spectral DM stack in Definition 4.10 is slightly different, but coincides in this case.}

The nontriviality of $\text{Br}(K\text{O})$ goes back to [30], where Gepner and Lawson compute the subgroup $\text{Br}(KU|K\text{O}) \subseteq \text{Br}(K\text{O})$ of classes split by the faithful $\mathbb{Z}/2$-Galois extension $KO \to KU$ to be $\mathbb{Z}/2$. It is not hard to check that $L\text{Br}(KU) = 0$, and thus, we find, in fact, that $L\text{Br}(KO) = \text{Br}(KU|KO)$ as subgroups of $\text{Br}(KO)$ although one a priori might expect $\text{Br}(KU|KO)$ to be bigger. In particular, we show that the nontrivial class $\alpha \in \text{Br}(KU|KO)$ is split by the faithful étale extension $KO \to KO[[\frac{1}{2}, \zeta_4]] \times KO[[\frac{1}{3}, \zeta_3]]$.

Regarding the spectrum $\text{TMF}$ of topological modular forms, we recall that Goerss, Hopkins, and Miller have defined a sheaf of $\mathbb{E}_\infty$-ring spectra $\mathcal{O}$ on the moduli stack $\mathcal{M}$ of elliptic curves [25]. The pair $(\mathcal{M}, \mathcal{O})$ defines a nonconnective spectral Deligne–Mumford stack in the sense of [48] and $\text{TMF}$ is the spectrum of global sections of $\mathcal{O}$. We may define $\text{Br}(\mathcal{M}, \mathcal{O})$ as the Brauer group of $\text{QCoh}(\mathcal{M}, \mathcal{O})$,\footnote{Our actual definition of the Brauer group of a nonconnective spectral DM stack in Definition 4.10 is slightly different, but coincides in this case.} which coincides with $\text{Mod}_{\text{TMF}}$ as $(\mathcal{M}, \mathcal{O})$ is 0-affine by [50]. On the other hand, we may define $L\text{Br}(\mathcal{M}, \mathcal{O})$ as the subgroup of Brauer classes that become trivial after pulling back to an étale cover of $\mathcal{M}$, and this group is potentially bigger than $L\text{Br}(\text{TMF})$. As by [49, Theorem 10.4], all faithful Galois extensions of localizations of $\text{TMF}$ arise from étale covers of $\mathcal{M}$, this local Brauer group $L\text{Br}(\mathcal{M}, \mathcal{O})$ is a natural analogue of $\text{Br}(KU|KO)$ above.

**Theorem 1.3** (Theorem 8.2, Theorem 8.3, Theorem 8.7). After inverting 2, the inclusion $L\text{Br}(\text{TMF}) \subseteq L\text{Br}(\mathcal{M}, \mathcal{O})$ becomes an equality and both groups are isomorphic to $\mathbb{Z}/3$.

After localizing at 2, the inclusion $L\text{Br}(\text{TMF}) \subseteq L\text{Br}(\mathcal{M}, \mathcal{O})$ has finite cokernel and both groups admit surjections to $(\mathbb{Z}/2)^\infty$ with kernel of order at most 8. In particular, $L\text{Br}(\text{TMF})$ is an infinitely generated torsion abelian group, and hence, $\text{Br}(\text{TMF})$ is infinitely generated as well.
For the (partial) determination of $LBr(KO)$ and $LBr(TM)$, our most important tool is an exact sequence for $LBr(R)$ (with mild assumptions on $\pi_0 R$) of the form

$$Br(\pi_0 R) \to LBr(R) \to H^1(Spec \pi_0 R, \pi_0 \text{Pic}_R),$$

which will be proven in a more precise form in Proposition 2.25. Here, $\pi_0 \text{Pic}_R$ is the Picard sheaf of $R$. It arises as the étale sheafification of the presheaf sending each étale extension $A$ of $\pi_0 R$ to $\text{Pic}(R_A)$, where $R \to R_A$ is the unique étale extension realizing $\pi_0 R \to A$. We determine the Picard sheaf of $KO$ in Proposition 3.8 and give a partial determination of the Picard sheaf of $TMF$ in Theorem 6.5. The remaining uncertainties lie in our inability to compute long differentials in the sheafy Picard spectral sequence or, essentially equivalently, in our inability to compute $\text{Pic}(TMF_{(2)} [\zeta_{2^n-1}])$ for $n \geq 2$. For possible subleties arising in such computations, we refer to Remark 3.12.

For the (partial) determination of $LBr(\mathcal{M}, \mathcal{O})$, a crucial point is to compare the (local) Brauer groups of a nonconnective spectral Deligne–Mumford stack $(\mathcal{X}, \mathcal{O})$ with the following variant: The cohomological Brauer group $Br'(\mathcal{X}, \mathcal{O})$ is defined using descent from the affine case and we also obtain a subgroup $LBr'(\mathcal{X}, \mathcal{O})$. The Brauer group of $(\mathcal{X}, \mathcal{O})$ is the subgroup $Br(\mathcal{X}, \mathcal{O}) \subseteq Br'(\mathcal{X}, \mathcal{O})$ of Brauer classes representable by Azumaya algebras. If $(\mathcal{X}, \mathcal{O}) \simeq Spec R$ is affine, then $Br(R) \cong Br(\text{Spec } R) \cong Br'(\text{Spec } R)$ and likewise for $LBr$, but $Br$ and $Br'$ might be different in general. In many cases of interest, we show however (extending work of Toën [70] and Hall–Rydh [35]) that the cohomological Brauer agrees with the usual Brauer group. This applies in particular to $(\mathcal{M}, \mathcal{O})$.

**Theorem 1.4** ($Br = Br'$, Theorem 4.17). If $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a nonconnective spectral DM stack satisfying some mild conditions stated in the body of the paper, then $Br(\mathcal{X}, \mathcal{O}_\mathcal{X}) \cong Br'(\mathcal{X}, \mathcal{O}_\mathcal{X})$ and $LBr(\mathcal{X}, \mathcal{O}_\mathcal{X}) \cong LBr'(\mathcal{X}, \mathcal{O}_\mathcal{X})$.

Since by definition, $Br'$ and $LBr'$ are approachable via descent, this result allows us to calculate $LBr(\mathcal{M}, \mathcal{O})$ via the Picard spectral sequence of [51], where it will be visible in the $(-1)$-column. Up to two differentials, we can analyze this column using [51] and the vanishing of the classical Brauer group of $\mathcal{M}$ from [4].

**Question 1.5.** Is the inclusion $LBr(\mathcal{M}, \mathcal{O}) \subseteq Br(TM)$ an equality?

A similar question can be asked for every 0-affine nonconnective spectral DM stack $(\mathcal{X}, \mathcal{O})$ where $\mathcal{O}$ is even-periodic and the underlying stack of $\mathcal{X}$ is regular noetherian. Here, we would replace $LBr$ by a Brauer–Wall type extension as in Remark 2.30 (which makes no difference in the case of $(\mathcal{M}, \mathcal{O})$). Inspired by the case of KO, we also want to pose the question.

**Question 1.6.** For $(\mathcal{X}, \mathcal{O})$ as above, is $LBr(\mathcal{O}(\mathcal{X})) \subseteq LBr(\mathcal{X}, \mathcal{O})$ always an equality?

**Conventions**

We will always use the étale topology. Thus, $H^s$ means étale cohomology if applied to a scheme or Deligne–Mumford stack, and $R^sf_*$ will like-wise refer to the $s$th higher direct image with respect to the étale topology. The notation $\Gamma$ will always refer to the global sections of some étale sheaf with values in an appropriate $\infty$-category; in particular, when applied to a sheaf of abelian groups.
$\mathcal{F}$ on a site, we will view $\mathcal{F}$ as a sheaf of spaces so that $\pi_{-i}I(\mathcal{F})$ is the $i$th cohomology group of $\mathcal{F}$. Moreover, if $\mathcal{F}$ is a sheaf of spaces or spectra, $\pi_i \mathcal{F}$ will always refer to the étale sheafification of the presheaf of homotopy groups.

Generally, we will work in an $\infty$-categorical context. In particular, a *commutative ring spectrum* will mean for us a commutative algebra in the $\infty$-category of spectra, that is, what is also called an $E_\infty$-ring (spectrum). If $R$ is a commutative ring spectrum, we will always equip $\text{Spec} R$ with the étale topology. In an $\infty$-category $\mathcal{C}$, we will write $\text{Map}_{\mathcal{C}}(x, y)$ for the mapping space between $x, y \in \mathcal{C}$; if $\mathcal{C}$ is a stable $\infty$-category or an $\infty$-category of quasi-coherent sheaves, then we will write $\text{Map}_{\mathcal{C}}(x, y)$ or simply $\text{Map}(x, y)$ for the mapping spectrum or the internal mapping spectrum.

The following infinity categories will be used in some of the theoretical results:

- $\text{Pr}^L_\infty$, the infinity category of presentable $\infty$-categories and left adjoint morphisms;
- $\hat{\text{Cat}}_\infty$, the infinity category of possibly large $\infty$-categories.

See [45] for details.

## 2 THE LOCAL BRAUER GROUP IN THE AFFINE CASE

After reminding the reader about the classical Brauer group of a commutative ring, we recall in this section the definition of the Brauer group and Brauer space of a commutative ring spectrum and introduce the notion of the local Brauer group. We will prove several basic properties (in particular that Brauer spaces define an étale hypersheaf) and provide basic tools for the computation of local Brauer groups.

### 2.1 The classical Brauer group

In this subsection, we will give a short introduction to the classical Brauer group. For more background, we refer, for example, to [21, 31] and the series of articles starting with [32].

Let $R$ be a commutative ring. An $R$-algebra $A$ is called *Azumaya* if one of the following equivalent conditions holds:

1. $A$ is finitely generated, faithful, and projective as an $R$-module and the map

   $$A \otimes_R A^{\text{op}} \to \text{End}_R(A), \quad a \otimes b \mapsto (x \mapsto axb)$$

   is an isomorphism.

2. étale locally, $A$ is isomorphic to the matrix algebra $\text{Mat}_n(R)$.

Two Azumaya algebras $A$ and $B$ are called *Morita equivalent* if their module categories are equivalent.

**Definition 2.1.** The *classical Brauer group* $\text{Br}^{\text{cl}}(R)$ of $R$ is the set of Azumaya algebras over $R$ up to Morita equivalence.
Remark 2.2. Instead of working with Morita equivalence classes of Azumaya algebras, one can also directly define the Brauer groups via their module categories. This is the approach we will take in Definition 2.11.

In the case that $R$ is regular noetherian, $\text{Br}^{\text{cl}}(R)$ coincides with what we later introduce as $\text{Br}(R)$; thus, we will drop the superscript in this case. Moreover, a result of Gabber identifies $\text{Br}(R)$ in the regular noetherian case with $H^2(\text{Spec } R; \mathbb{G}_m)$ [43, Corollary 3.1.4.2]. As $\text{Pic}(R) \cong H^1(\text{Spec } R; \mathbb{G}_m)$, this gives one perspective on why Brauer groups are a higher variant of Picard groups.

If $R = k$ is a field, every finite-dimensional division $k$-algebra with center $k$ is Azumaya. Conversely, every Azumaya $k$-algebra is Morita equivalent to a unique such. Thus, $\text{Br}(k)$ is in bijection with isomorphism classes of finite-dimensional division $k$-algebras with center $k$. For example, $\text{Br}(\mathbb{R}) \cong \{[\mathbb{R}], [\mathbb{H}]\} \cong \mathbb{Z}/2$ and $\text{Br}(\mathbb{C}) = 0$. In contrast, the Brauer group of a nonarchimedean local field $K$ (like $\mathbb{Q}_p$) is isomorphic to $\mathbb{Q}/\mathbb{Z}$.

It will be important for our later calculations to understand the Brauer groups of rings like $\mathbb{Z}$ or $\mathbb{Z}[ \frac{1}{2}, \zeta_4 ]$. More generally, we consider a number field $K$ and let $R$ be a localization of the ring of integers of $K$. In this case, by [34, Proposition 2.1], there is an exact sequence

$$0 \to \text{Br}(R) \to \text{Br}(K) \to \bigoplus_{\mathfrak{p} \in \text{Spec } R^{(1)}} \text{Br}(\text{Spec } K_{\mathfrak{p}}),$$

where $\text{Spec } R^{(1)}$ denotes the set of closed points of $\text{Spec } R$ and $K_{\mathfrak{p}}$ denotes the completion. This exact sequence is compatible with the exact sequence

$$0 \to \text{Br}(K) \to \bigoplus_{\mathfrak{p}} \text{Br}(\text{Spec } K_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z} \to 0$$

of class field theory (see [58, Theorem 8.1.17]). The sum ranges over the finite and the infinite places of $K$, and the map $\text{Br}(\text{Spec } K_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$ is the isomorphism described above when $\mathfrak{p}$ is a finite place, the natural inclusion $\mathbb{Z}/2 \to \mathbb{Q}/\mathbb{Z}$ when $K_{\mathfrak{p}} \cong \mathbb{R}$, and the natural map $0 \to \mathbb{Q}/\mathbb{Z}$ when $K_{\mathfrak{p}} \cong \mathbb{C}$.

**Example 2.4.** One can deduce the following vanishing results from the exact sequences above:

1. $\text{Br}(\mathbb{Z}) = 0$;
2. $\text{Br}\left( \mathbb{Z}\left[ \frac{1}{6} \right] \right) \cong \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}/2$;
3. $\text{Br}\left( \mathbb{Z}\left[ \frac{1}{p}, \zeta_p^n \right] \right) = 0$ for a prime $p$ and a natural number $n$.

We will only give an argument for the last vanishing and only if $p^n \geq 3$. Let $R = \mathbb{Z}\left[ \frac{1}{p}, \zeta_p^n \right]$. The field $\mathbb{Q}(\zeta_p^n)$ is totally imaginary and there is a unique prime ideal $\mathfrak{p} \subset \mathbb{Z}[\zeta_p^n]$ lying over $(p) \subset \mathbb{Z}$ (since $p$ is totally ramified). Thus, for every place $q$ of $\mathbb{Q}(\zeta_p^n)$, we have either $\text{Br}(K_q) = 0$, $q \in \text{Spec } R^{(1)}$, or $q = \mathfrak{p}$. Thus, $\text{Br}\left( \mathbb{Z}\left[ \frac{1}{p}, \zeta_p^n \right] \right)$ can be identified with the kernel of the map $\text{Br}(\mathbb{Q}(\zeta_p^n))_{\mathfrak{p}} \to \mathbb{Q}/\mathbb{Z}$, which is zero.

Brauer groups have several nice properties, three of which we will summarize in the next theorem.
Theorem 2.5. Let $R$ be a regular noetherian ring.

(1) If $\text{Spec } R \left[\frac{1}{f}\right] \subset \text{Spec } R$ is dense, then $\text{Br}(R) \to \text{Br} \left( R \left[\frac{1}{f}\right] \right)$ is injective. If $\frac{1}{p} \in R$ and $f$ is a nonzero divisor, we have more precisely a short exact sequence

$$0 \to \text{Br}(R)_{(p)} \to \text{Br} \left( R \left[\frac{1}{f}\right] \right)_{(p)} \to H^1 \left( R/f; \mathbb{Q}/\mathbb{Z} \right)_{(p)} \to 0.$$ 

If there is a ring homomorphism right inverse of $R \to R/f$, the sequence is split, sending $[\chi]$ to the Brauer class of the cyclic algebra $(\chi, f)$.

(2) If $\text{Spec } R \left[\frac{1}{p}\right] \subset \text{Spec } R$ is dense, then $\text{Br}(R)_{(p)} \to \text{Br}(R[x])_{(p)}$ is an isomorphism.

Proof. The first part in item (1) follows from [43, Proposition 3.1.3.3]. The rest is contained in [4, Propositions 2.14 and 2.16].

For item (2), a proof in the case $\frac{1}{p} \in R$ can be found, for example, in [4, Proposition 2.5]. To show that $\text{Br}(R)_{(p)} \cong \text{Br}(R[x])_{(p)}$ in general, consider the diagram

$$
\begin{array}{c}
\text{Br}(R[x])_{(p)} \\
\downarrow \\
\text{Br}(R)_{(p)} \\
\end{array}
\quad
\begin{array}{c}
\text{Br}(R[x])_{(p)} \\
\downarrow \cong \\
\text{Br}(R)_{(p)} \\
\end{array}
$$

induced by the morphism $R[x] \to X$, sending $x$ to $0$. The right vertical morphism is an isomorphism since $\frac{1}{p} \in R \left[\frac{1}{p}\right]$. The horizontal arrows are injections by the first point. Thus, $\text{Br}(R[x])_{(p)} \to \text{Br}(R)_{(p)}$ must be an injection as well. On the other hand, it is a split surjection, using the map $R \to R[x]$. This implies that it is an isomorphism. \qed

Corollary 2.6. Let $S \subset \mathbb{Q}$ be a subring. Then, the morphism

$$\Phi : \text{Br}(S) \oplus H^1(S; \mathbb{Q}/\mathbb{Z}) \to \text{Br}(S[j^{\pm 1}])$$

sending $[\chi] \in H^1(S; \mathbb{Q}/\mathbb{Z})$ to the cyclic algebra $[(\chi, j)]$ is an isomorphism. In particular, $\text{Br}(\mathbb{Z}[j^{\pm 1}]) = 0$.

Proof. Let $p$ be a prime and assume first that $S \subset \mathbb{Q}$ with $\frac{1}{p} \in S$. By Theorem 2.5, we obtain a split short exact sequence

$$0 \to \text{Br}(S[j])_{(p)} \to \text{Br}(S[j^{\pm 1}])_{(p)} \to H^1(S; \mathbb{Q}/\mathbb{Z})_{(p)} \to 0.$$ 

By $\mathbb{A}^1$-invariance, $\text{Br}(S[j]) \cong \text{Br}(S)$. This proves the claim $p$-locally if $\frac{1}{p} \in S$. Thus, we obtain it for $S = \mathbb{Q}$ without localization. As for a general $S \subset \mathbb{Q}$, the maps $\text{Br} \left( S \left[j^{\pm 1}\right] \right) \to \text{Br} \left( \mathbb{Q} \left[j^{\pm 1}\right] \right)$ and $H^1(S; \mathbb{Q}/\mathbb{Z}) \to H^1(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ are injections, $\Phi$ is an injection in general.

Next, we will show the statement in the case $S = \mathbb{Z}_{(p)}$. Let $(a, \chi) \in \text{Br}(\mathbb{Q}) \oplus H^1(\mathbb{Q}; \mathbb{Q}/\mathbb{Z})$. If $\Phi(a, \chi)$ lies in $\text{Br} \left( \mathbb{Z}_{(p)} \left[j^{\pm 1}\right] \right)$, the image $\Phi_\ast(a, \chi)$ of the class $j_\ast^\ast \Phi(a, \chi) \in \text{Br}(\mathbb{Z}_{(p)})$ (for an
arbitrary \( u \in \mathbb{Z}_p \oplus \mathbb{Z}_p \) inducing \( j_u : \text{Spec } \mathbb{Z}(p) \rightarrow \text{Spec } \mathbb{Z}(p)[j^{\pm 1}] \) must have image zero in \( \text{Br}(\mathbb{Q}_p) \) since \( \text{Br}(\mathbb{Z}_p) = 0 \).

\[
\begin{CD}
\text{Br}(\mathbb{Z}(p)[j^{\pm 1}]) @>{j_u}>> \text{Br}(\mathbb{Z}(p)) \rightarrow \text{Br}(\mathbb{Z}_p) \rightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\text{Br}(\mathbb{Q}[j^{\pm 1}]) @>>> \text{Br}(\mathbb{Q}) @>>> \text{Br}(\mathbb{Q}_p)
\end{CD}
\]

Assume now that \((a, \chi) \notin \text{Br}(\mathbb{Z}(p)) \oplus H^1(\mathbb{Z}(p); \mathbb{Q}/\mathbb{Z})\). If \( a \notin \text{Br}(\mathbb{Z}(p)) \), then its image \( b \in \text{Br}(\mathbb{Q}_p) \) is nontrivial and thus \( \Phi_1(a, \chi) = b \neq 0 \). Now suppose \( a \in \text{Br}(\mathbb{Z}(p)) \), but \( \chi \notin H^1(\mathbb{Z}(p); \mathbb{Q}/\mathbb{Z}) \). Then, the corresponding extension \( K \) of \( \mathbb{Q}_p \) must be ramified. Hence, the image \( N(K) = N_K|_{\mathbb{Q}_p} (K^\times) \subseteq \mathbb{Q}_p^\times \cong \mathbb{Z}_p^\times \times \mathbb{Z} \) cannot contain \( \mathbb{Z}_p^\times \). As \( N(K) \) is of finite index, the surjections \( \mathbb{Z}_p^\times \rightarrow (\mathbb{Z}/p^n)^\times \) imply that there exists a \( u \in \mathbb{Z}_p^\times \) such that \( u \notin N(K) \). By [31, Corollary 4.7.4], the class \( \Phi_u(a, \chi) = [(\chi, u)] \) is thus nonzero in \( \text{Br}(\mathbb{Q}_p) \). This shows that \( \Phi \) is indeed surjective if \( S = \mathbb{Z}(p) \).

For \( S \subseteq \mathbb{Q} \) general, we have

\[
\text{Br}(S) \oplus H^1(\mathbb{Q}; \mathbb{Q}/\mathbb{Z}) = \left( \text{Br}(S[1/p]) \oplus H^1(S[1/p]; \mathbb{Q}/\mathbb{Z}) \right) \cap (\text{Br}(\mathbb{Z}(p)) \oplus H^1(\mathbb{Z}(p); \mathbb{Q}/\mathbb{Z}))
\]
as subgroups of \( \text{Br}(\mathbb{Q}) \oplus H^1(\mathbb{Q}; \mathbb{Q}/\mathbb{Z}) \). As we have for every \( p \) with \( \frac{1}{p} \neq S \) a containment

\[
\text{Br}(S[j^{\pm 1}](p)) \subseteq \text{Br}(S[j^{\pm 1}](p)) \cap \text{Br}(\mathbb{Z}(p)[j^{\pm 1}](p)),
\]
we see that \( \Phi(p) \) is actually surjective for every \( S \subseteq \mathbb{Q} \) and every prime \( p \), which proves the claim.

In some of our examples below, we will also use the following classical results, which will help us compute Brauer groups of various ring spectra. The first is Grothendieck’s rigidity result for the Brauer group [32, Corollaire 6.2]

**Theorem 2.7.** Suppose that \( R \) is Hensel local with residue field \( k \); then \( \text{Br}(R) \cong \text{Br}(k) \). If \( R \) is also regular, then \( \text{Br}(R) \cong H^2(\text{Spec } R, \mathbb{G}_m) \) so that \( H^2(\text{Spec } R, \mathbb{G}_m) \cong H^2(\text{Spec } k, \mathbb{G}_m) \).

The next is a corollary of the affine analog of proper base change as proved in Gabber–Huber [29, 40], see also [11, Corollary 1.18(a)].

**Theorem 2.8.** If \( R \) is a Hensel local ring with residue field \( k \), then \( H^* (\text{Spec } R, \mathcal{A}) \cong H^* (\text{Spec } k, i^* \mathcal{A}) \) for all torsion étale sheaves \( \mathcal{A} \) on \( \text{Spec } R \), where \( i : \text{Spec } k \hookrightarrow \text{Spec } R \).

The next result can be found in [23, Corollaire II.3.5, Proposition II.3.6] or [54, Corollary II.3.6] and we will use it several times in the setting of closed immersions.

**Theorem 2.9.** Let \( f : X \rightarrow Y \) be a finite morphism of schemes. If \( \mathcal{F} \) is an étale sheaf on \( X \), then \( R^q f_* = 0 \) for \( q > 0 \) and hence \( H^i(X; \mathcal{F}) \cong H^i(Y; f_* \mathcal{F}) \) for all \( i \geq 0 \).
2.2 Brauer groups of ring spectra

In this subsection, we will recall the Brauer group and Brauer space of a commutative ring spectrum, which were first introduced by [8] and [68]. For our purposes, an approach will be convenient that sees Brauer groups as categorified Picard groups. Let us thus first recall the definition of the Picard group and Picard space.

If $\mathcal{C}$ is a symmetric monoidal $\infty$-category, its underlying $\infty$-groupoid $\iota \mathcal{C}$ naturally admits the structure of an $E_\infty$-space and the counit map $\iota \mathcal{C} \to \mathcal{C}$ is symmetric monoidal. We define the Picard space $\text{Pic}(\mathcal{C})$ to be the maximal grouplike $E_\infty$-groupoid in $\iota \mathcal{C}$. In other words, $\text{Pic}(\mathcal{C})$ is the space of $\otimes$-invertible objects of $\mathcal{C}$ and equivalences. The Picard group of $\mathcal{C}$ is $\text{Pic}(\mathcal{C}) = \pi_0 \text{Pic}(\mathcal{C})$. We refer to [51] for more background on Picard groups and spaces.

**Example 2.10.** If $R$ is a commutative ring spectrum, its Picard space $\text{Pic}(R)$ is $\text{Pic}({\text{Mod}}_R)$ and its Picard group is $\text{Pic}(R) = \pi_0 \text{Pic}(R)$.

Next, we introduce the Brauer group $\text{Br}(R)$ of a commutative ring spectrum $R$ as a categorification of the Picard group. In the case that $R$ is a regular noetherian ring, this will agree with the classical Brauer group (see Remark 2.26).

**Definition 2.11.** Let $R$ be a commutative ring spectrum and let $\text{Cat}_R$ denote the presentably symmetric monoidal $\infty$-category of compactly generated $R$-linear stable $\infty$-categories and compact object-preserving left adjoint functors.†

(a) We let $\text{Br}(R) = \text{Pic}(\text{Cat}_R)$ denote the Brauer space of $R$. The Brauer group of a commutative ring spectrum $R$ is $\text{Br}(R) = \pi_0 \text{Br}(R)$.

(b) If $A$ is an $R$-algebra, we say that $A$ is an Azumaya algebra over $R$ if $\text{Mod}_A$ defines a point of $\text{Br}(R)$.

(c) An Azumaya $R$-algebra $A$ is trivial if $\text{Mod}_A \simeq \text{Mod}_R$, that is, if $A$ is $R$-linearly (derived) Morita equivalent to $R$.

(d) An Azumaya $R$ algebra $A$ is étale-locally trivial if there is an étale cover $R \to S$ such that $S \otimes_R A$ is trivial.

This definition of an Azumaya algebra is due to Toën [70]. It agrees with the original definition of an Azumaya algebra in this setting due to [8]. See [3] for more details; see particularly Theorem 3.15 of [3] for the statement that every Brauer class is represented by an Azumaya algebra.

**Lemma 2.12.** If $R$ is a commutative ring spectrum, then there is a natural equivalence $\text{Pic}(R) \simeq \Omega \text{Br}(R)$, where $\Omega \text{Br}(R)$ is computed via loops based at the trivial Brauer class.

**Proof.** By construction, $\Omega \text{Br}(R)$ is the space of autoequivalences of the unit object of $\text{Cat}_R$. The unit object is $\text{Mod}_R$ and the autoequivalences must be $R$-linear, so they correspond to tensoring with invertible $R$-modules. \qed

† Background on this can, for example, be found in [3, Section 3.1], where the notation $\text{Cat}_{R,\omega}$ is used.
We will prove in the next section that $R \mapsto \text{Br}(R)$ is a Postnikov complete étale sheaf. To do so, we first establish that $R \mapsto \text{Cat}_R$ is an étale sheaf (with values in $\text{Pr}^L$). The result was discovered in the context of the present project, but appeared first in [2, Thm. 2.16].

**Proposition 2.13.** The presheaf $R \mapsto \text{Cat}_R$ is an étale sheaf with values in $\text{Pr}^L$.

**Proof.** The fact that each $\text{Cat}_R$ is presentable follows from [12, Corollary 4.25]. That for any map $R \to S$ of commutative ring spectra, the induced functor $\text{Cat}_R \to \text{Cat}_S$ is a left adjoint follows because $\text{Cat}_S \simeq \text{Mod}_{\text{Mod}_R}(\text{Cat}_R)$. Thus, since the forgetful functor $\text{Pr}^L \to \hat{\text{Cat}}_{\infty}$ preserves limits by [45, Prop. 5.5.3.13], it suffices to see that $R \mapsto \text{Cat}_R$ is an étale sheaf with values in $\hat{\text{Cat}}_{\infty}$. This is part of [2, Thm. 2.16].

**Corollary 2.14.** The construction $R \mapsto \text{Br}(R)$ is an étale sheaf on $\text{CAlg}^{op}$.

**Proof.** The construction $C \mapsto \text{Pic}(C)$ preserves limits as a functor from symmetric monoidal $\infty$-categories to spaces [51, Proposition 2.2.3]. As limits of symmetric monoidal $\infty$-categories are computed on the level of underlying $\infty$-categories, the result follows from Proposition 2.13.

### 2.3 The local Brauer group

While in classical algebras, Azumaya algebras are always étale-locally Morita equivalent to the ground ring, this is no longer true in the spectral setting. In this subsection (and actually the whole article), we will concentrate on those that are étale-locally trivial.

**Definition 2.15.** Let $\pi_0\text{Br}$ denote the étale sheaf of connected components of $\text{Br}$. We let $\text{LBr}$ be the fiber of the natural map $\text{Br} \to \pi_0\text{Br}$ in étale sheaves. The space $\text{LBr}(R)$ is the *local Brauer space* of $R$ and $\text{LBr}(R) = \pi_0(\text{LBr}(R))$ is the *local Brauer group* of $R$.

**Remark 2.16.** Thanks to Lemma 2.12, we could equivalently have defined $\text{LBr}$ as $\text{BPic}$, the étale classifying space of $\text{Pic}$, computed in étale sheaves. However, note that the functor $R \mapsto \text{BPic}(R)$, sending $R$ to the classifying space of its Picard space, is not a sheaf, and $\text{BPic}$ is its sheafification.

The name “local Brauer group” is short-hand for “locally-trivial Brauer group,” which is justified by the following lemma.

**Lemma 2.17.** Let $R$ be a commutative ring spectrum.

(a) The natural map $\text{LBr}(R) \to \text{Br}(R)$ is an injection, and hence, $\text{LBr}(R) \to \text{Br}(R)$ is the inclusion of a subspace of connected components.

(b) An element $\alpha \in \text{Br}(R)$ is contained in $\text{LBr}(R)$ if and only if there is a faithful étale map $R \to S$ such that $\alpha$ maps to zero in $\text{Br}(S)$.

**Proof.** For (a), use the fiber sequence

$$\text{LBr}(R) \to \text{Br}(R) \to \Gamma(\text{Spec } R, \pi_0\text{Br})$$
of spaces. Recall here that $\Gamma$ denotes the space (as opposed to the set) of global sections of an étale sheaf and $\pi_0 Br$ denotes the étale-sheafified homotopy group. Since $\pi_i \Gamma(\text{Spec } R, \pi_0 Br) = 0$ for $i > 0$, the first claim follows from the long exact sequence in homotopy. Thus, we can identify $\text{LBr}(R)$ as a subgroup of $\text{Br}(R)$.

If $\alpha \in \text{LBr}(R)$, then $\alpha$ is étale-locally trivial since $\pi_0 \text{LBr} = 0$. Conversely, if $\alpha \in \text{Br}(R)$ is such that there exists a faithful étale map $R \to S$ such that $\alpha_S = 0 \in \text{Br}(S)$, then the image of $\alpha$ in $\pi_0 R \Gamma(\text{Spec } R, \pi_0 Br)$ is zero. Thus, $\alpha \in \text{LBr}(R)$. This proves (b).

If $R$ is a commutative ring spectrum, we will always equip Spec $R$ with the étale topology. The small étale sites of Spec $R$ and Spec $\pi_0 R$ agree, so we can compute cohomology of sheaves of abelian groups on Spec $R$ via étale cohomology on Spec $\pi_0 R$: given a sheaf of abelian groups $\mathcal{A} \in \text{Shv}_{Sp}(\text{Spec } R)^\circ$, we have

$$\Gamma(\text{Spec } R, \mathcal{A}) \simeq \Gamma(\text{Spec } \pi_0 R, \mathcal{A}),$$

and thus,

$$\pi_{-i} \Gamma(\text{Spec } R, \mathcal{A}) \cong \pi_{-i} \Gamma(\text{Spec } \pi_0 R, \mathcal{A}) \cong H^i(\text{Spec } \pi_0 R; \mathcal{A}).$$

This will be used constantly below.

Specifically, to compute $\text{LBr}(R)$, we can restrict $\text{LBr}$ and $\text{Br}$ to étale sheaves $\text{LBr}_\Theta$ and $\text{Br}_\Theta$ on the small étale site of $R$. Note that these are already sheaves and no additional sheafification is necessary. While we may and will view the $\text{LBr}_\Theta$ and $\text{Br}_\Theta$ as sheaves on Spec $\pi_0 R$, they certainly depend crucially on $R$, not only on $\pi_0 R$. We use the notation $\text{Pic}_\Theta$ also for the restriction of $\text{Pic}$ to Spec $R$.

**Lemma 2.18.** Let $\text{LBr}_\Theta$ be the local Brauer space sheaf constructed above on Spec $R$ for a commutative ring spectrum $R$. The homotopy sheaves of $\text{LBr}_\Theta$ are given by

$$\pi_t \text{LBr}_\Theta \cong \begin{cases} 0 & \text{if } t = 0, \\ \pi_0 \text{Pic}_\Theta & \text{if } t = 1, \\ \mathbb{G}_m & \text{if } t = 2, \text{ and} \\ \pi_{t-2} \Theta & \text{if } t \geq 3, \end{cases}$$

where $\mathbb{G}_m$ is the étale sheaf $\mathbb{G}_m(S) \cong (\pi_0 S)^\times$ for an étale commutative $R$-algebra $S$. In particular, $\pi_t \text{LBr}_\Theta$ is quasi-coherent for $t \geq 3$.

**Proof.** This follows from Lemma 2.12, using that étale sheafification commutes with restriction along the morphism $\text{CAlg}_{\Theta R}^{\text{ét}} \to \text{CAlg}$ from commutative étale $R$-algebras to all commutative ring spectra.

**Remark 2.19.** Analysis of $\pi_t \text{LBr}_\Theta \cong \pi_0 \text{Pic}_\Theta$ is often the most difficult part of local Brauer group computations.

**Definition 2.20.** Let $\mathcal{C}$ be a prestable $\infty$-category in the sense of [48, Appendix C] having all limits, which is automatically the nonnegative part of a $t$-structure on a stable $\infty$-category. We
say that $X \in \mathcal{C}$ is $\omega$-connective if $\text{Map}(X, Y) \simeq 0$ for every truncated object $Y$. An object $Y$ of $\mathcal{C}$ is hypercomplete if $\text{Map}(X, Y) \simeq 0$ for every $\omega$-connective object $X$. Finally, $Y$ is Postnikov complete if the natural map $Y \to \lim_n \tau_{\leq n} Y$ is an equivalence; this occurs if and only if $\lim_n \tau_{\geq n+1} Y \simeq 0$ as fiber sequences are closed under limits.

Postnikov complete objects are hypercomplete, but the converse is not always true. The significance of Postnikov completeness is that it allows us to compute global sections by using descent spectral sequences. As our prestable $\mathcal{C}$, we will use sheaves with values in grouplike $E_\infty$-spaces (i.e., connective spectra). Note that the forgetful functor from grouplike $E_\infty$-spaces to spaces preserves and detects limits.

**Proposition 2.21.** The assignments $R \mapsto \text{LBr}(R)$ and $R \mapsto \text{Br}(R)$ define Postnikov complete étale sheaves of grouplike $E_\infty$-spaces on $\text{CAlg}^{\text{op}}$. Similarly, $\text{LBr}_\emptyset$ and $\text{Br}_\emptyset$ are Postnikov complete étale sheaves on $\text{Spec} R$ for any commutative ring spectrum $R$.

**Proof.** The proofs of all four cases are the same, so we give it only for $\text{Br}$ on $\text{CAlg}^{\text{op}}$. As $\text{Br}$ is Postnikov complete if and only if $\lim_n \tau_{\geq n+1} \text{Br} \simeq *$, it is enough to prove Postnikov completeness for any sufficiently connective cover of $\text{Br}$. We prove that $\tau_{\geq 3} \text{Br}$ is Postnikov complete in two steps. First, $R \mapsto \text{Mod}_R$ satisfies hyperdescent (see [48, Corollary D.6.3.3]), so $\text{Pic}$ preserving limits as a functor from symmetric monoidal $\infty$-categories implies $\text{Pic}$ being hypercomplete. This implies that $\text{LBr} \simeq \text{BPic}$ is hypercomplete: On 1-connective étale sheaves, $\Omega$ is fully faithful. If $X$ is an $\omega$-connective étale sheaf, then

$$\text{Map}(X, \text{LBr}) \simeq \text{Map}(\Omega X, \text{Pic}) \simeq 0,$$

since $\Omega X$ is still $\omega$-connective.

Second, the fact that $\text{LBr}$ is hypercomplete implies that its 3-connective cover $\tau_{\geq 3} \text{LBr}$ is hypercomplete. However, the homotopy sheaves $\pi_* \tau_{\geq 3} \text{LBr}$ are all quasi-coherent by Lemma 2.18. Therefore, there are enough objects (affines, e.g.) of cohomological dimension $\leq 0$ with $\{\pi_* \tau_{\geq 3} \text{LBr}\}$-coefficients in the sense of [19, Def. 2.8]. By [19, Prop. 2.10], it follows that $\tau_{\geq 3} \text{LBr}$ is, in fact, Postnikov complete, which is what we wanted to show. 

**Remark 2.22.** A form of Proposition 2.21 was proved in [3, Section 7] in the special case of connective commutative ring spectra using a different argument, although the proof in the connective case and of [46, Proposition 6.5], which is used in the proof of Proposition 2.13, are closely related under the hood. The main point of [3] is that when $R$ is connective, every Azumaya $R$-algebra is étale locally trivial. This is not true in general, as Example 5.7 below shows.

As the sheaves $\text{Pic}$, $\text{LBr}$, and $\text{Br}$ take values in grouplike $E_\infty$-spaces, we can deloop them to presheaves of spectra. Sheafifying these results in sheaves $\text{pic}$, $\text{lbr}$, and $\text{br}$ and the restrictions $\text{pic}_\emptyset$, $\text{lbr}_\emptyset$, and $\text{br}_\emptyset$ when working on the étale site of $\text{Spec} R$. Note that by construction, $\text{lbr}_\emptyset \simeq \text{pic}[1]$. Note, moreover, that $\pi_n \text{lbr} \simeq \pi_n \text{LBr}$, but the global sections can acquire additional negative homotopy groups.
Corollary 2.23. There is a convergent spectral sequence

\[ E_{2}^{s,t} \cong H^{s}(\text{Spec } \pi_{0}R, \pi_{t}\text{LB}_{\mathcal{O}}) \implies \pi_{t-s}\text{lb}_{\mathcal{O}}(\text{Spec } R) \cong H^{t-s}(\text{Spec } R, \pi_{0}\text{LB}_{\mathcal{O}}) \]  

with differentials \( d_{r} : E_{r}^{s,t} \to E_{r}^{s+r,t+r-1} \), which degenerates at the \( E_{3} \)-page.

**Proof.** This spectral sequence is the descent spectral sequence for the sheaf \( \text{lb}_{\mathcal{O}} \) of spectra, associated to the tower of global sections of the truncations of \( \text{lb}_{\mathcal{O}} \). Convergence follows from the fact that \( \pi_{t}\text{LB}_{\mathcal{O}} \) is quasi-coherent for \( t \geq 3 \) by Lemma 2.18 so that \( E_{2}^{s,t} = 0 \) for \( t \geq 3 \) and \( s \geq 1 \), and hence, the spectral sequence degenerates at the \( E_{3} \)-page.

The next proposition is our main tool to attack the local Brauer group of a commutative ring spectrum. Recall to that purpose that a commutative ring spectrum \( R \) is weakly \( 2k \)-periodic if \( \pi_{2k}R \) is an invertible \( \pi_{0}R \)-module and \( \pi_{2k}R \otimes_{\pi_{0}R} \pi_{n}R \to \pi_{2k+n}R \) is an isomorphism for all \( n \in \mathbb{Z} \).

**Proposition 2.25.** Let \( R \) be a commutative ring spectrum.

(i) There is a natural exact sequence

\[ 0 \to H^{1}(\text{Spec } \pi_{0}R, \mathbb{G}_{m}) \to \text{Pic}(R) \to H^{0}(\text{Spec } \pi_{0}R, \pi_{1}\text{LB}_{\mathcal{O}}) \to H^{2}(\text{Spec } \pi_{0}R, \mathbb{G}_{m}) \to \text{LB}_{\mathcal{O}}(R) \to H^{1}(\text{Spec } \pi_{0}R, \pi_{1}\text{LB}_{\mathcal{O}}) \to H^{3}(\text{Spec } \pi_{0}R, \mathbb{G}_{m}). \]

(ii) If \( R \) is connective, then there are natural identifications \( \text{LB}_{\mathcal{O}}(R) = \text{Br}(R) \cong H^{1}(\text{Spec } \pi_{0}R, \mathbb{Z}) \times H^{2}(\text{Spec } \pi_{0}R, \mathbb{G}_{m}) \).

(iii) Fix \( k \geq 1 \). If \( R \) is weakly \( 2k \)-periodic, with \( \pi_{1}R = 0 \) for \( i \) not divisible by \( 2k \), and such that \( \pi_{0}R \) is regular noetherian, then there is a natural exact sequence

\[ 0 \to H^{2}(\text{Spec } \pi_{0}R, \mathbb{G}_{m}) \to \text{LB}_{\mathcal{O}}(R) \to H^{1}(\text{Spec } \pi_{0}R, \mathbb{Z}/2k) \to H^{3}(\text{Spec } \pi_{0}R, \mathbb{G}_{m}). \]

**Proof.** Part (i) is the exact sequence of low-degree terms of the spectral sequence (2.24) using that \( \pi_{0}\text{LB}_{\mathcal{O}} = 0 \) and that the spectral sequence degenerates at the \( E_{3} \)-page.

Part (ii) is the content of [3, Theorem 5.11 and Corollary 7.13]. Note that the exact sequence from part (i) splits in this case as \( \mathbb{Z} \) is the free grouplike \( E_{1} \)-space, which leads us to split the map \( \text{LB}_{\mathcal{O}} \to \mathbb{B}_{1}\text{LB}_{\mathcal{O}} \cong \mathbb{B}\mathbb{Z} \) in the connective case. Here, we use that \( \text{Pic}(R) \cong \text{Pic}(\pi_{0}R) \cong \text{Pic}(\pi_{0}R) \times \mathbb{Z} \) by a result of [7, Theorem 21] and [51, Theorem 2.4.4] and that the functor \( R \mapsto \text{Pic}(\pi_{0}R) \times \mathbb{Z} \) sheafifies in the \( \text{étale} \) topology to the constant sheaf \( \mathbb{Z} \) since every Picard group element is \( \text{étale} \) locally trivial.

For Part (iii), we first claim that \( \pi_{1}\text{LB}_{\mathcal{O}} \cong \pi_{0}\text{Pic}_{\mathcal{O}} \cong \mathbb{Z}/2k \) when \( R \) satisfies the conditions of (iii). Indeed, \( \text{étale} \)-locally we can assume that \( R \) is \( 2k \)-periodic, and thus, we obtain \( \text{Pic}(R) \cong \text{Pic}(\pi_{0}R) \times \mathbb{Z}/2k \) by [7, Theorem 37] when \( k = 1 \) and [51, Corollary 2.4.7] in general. As above, this sheafifies to \( \mathbb{Z}/2k \). Hence, \( \text{Pic}(R) \to H^{0}(\text{Spec } \pi_{0}R, \mathbb{Z}/2k) \) is surjective. \[\square\]
Remark 2.26. If \( R \) is a classical commutative ring, then \( \text{LBr}(R) = \text{Br}(R) \) differs from the classical notion of the Brauer group, because we allow derived Azumaya algebras. In this case,

\[
\text{Br}(R) \cong H^1(\text{Spec } R, \mathbb{Z}) \times H^2(\text{Spec } R, \mathbb{G}_m)
\]

by part (ii) of Proposition 2.25 or by [70, Theorem 1.1]. The Brauer group of ordinary Azumaya algebras is given by \( H^2(\text{Spec } R, \mathbb{G}_m)_{\text{tors}} \), by Gabber [28]. As before, we write \( \text{Br}^{\text{cl}}(R) \) for the classical Brauer group of ordinary Azumaya algebras when \( R \) is a commutative ring. If \( R \) is regular noetherian, then \( \text{Br}^{\text{cl}}(R) = \text{Br}(R) \) since in this case, the \( H^1(\text{Spec } R, \mathbb{Z}) \) term vanishes because \( R \) is normal (see [24, 2.1]) and since \( H^2(\text{Spec } R, \mathbb{G}_m) \) is all torsion (see [33, Cor. 1.8]).

Example 2.27. When \( \pi_0 R = \mathbb{Z} \) and \( R \) is either connective or satisfies the conditions of (iii) in Proposition 2.25, then the proposition implies that \( \text{LBr}(R) = 0 \). Indeed, \( H^1(\text{Spec } \mathbb{Z}, \mathbb{Z}) = 0 = H^1(\text{Spec } \mathbb{Z}, \mathbb{Z}/2k) \) and Grothendieck proved that \( H^2(\text{Spec } \mathbb{Z}, \mathbb{G}_m) = 0 \) in [34]. This covers the sphere spectrum \( S \), the complex cobordism ring \( \mu \), the periodic complex \( K \)-theory spectrum, as well as connective complex or real \( K \)-theory and connective topological modular forms.

Example 2.28. Let \( k \) be a perfect field of positive characteristic \( p \) and let \( G \) be a one-dimensional formal group law of height \( n \) on \( k \). Let \( E_n(G, k) \) be the Lubin–Tate spectrum associated to \( G \) so that \( \pi_n E_n(G, k) \cong \mathbb{W}(k)[u_1, ..., u_{n-1}][u^{\pm 1}] \), where \( \mathbb{W}(k) \) denotes the ring of \( p \)-typical Witt vectors of \( k \), each \( u_i \) has degree 0, and \( u \) has degree 2. We want to show that the local Brauer group \( \text{LBr}(E_n(G, k)) \) is typically nonzero. Note that this is related to but different than the results of Hopkins and Lurie in [39], who look at the Brauer group of \( K(n) \)-local \( E_n(G, k) \)-modules, which is different from that of \( E_n(G, k) \)-modules. Moreover, they study the Brauer group and not just the local Brauer group.

Since \( E_n(G, k) \) is \( 2 \)-periodic, part (iii) of Proposition 2.25 applies. To compute the groups that build \( \text{LBr}(E_n(G, k)) \), note first that \( H^2(\text{Spec } \pi_0 E_n(G, k), \mathbb{G}_m) \cong H^2(\text{Spec } k, \mathbb{G}_m) \) by Theorem 2.7 and \( H^2(\text{Spec } k, \mathbb{G}_m) \cong \text{Br}(k) \) is typically nonzero. Moreover, \( H^1(\text{Spec } \pi_0 E_n(G, k), \mathbb{Z}/2) \cong H^1(\text{Spec } k, \mathbb{Z}/2) \) by Theorem 2.8.

If \( k = \mathbb{F}_{p^r} \) is finite, the contribution from \( H^2(\text{Spec } \mathbb{F}_{p^r}, \mathbb{G}_m) \cong \text{Br}(\mathbb{F}_{p^r}) \) vanishes. Indeed, by two theorems of Wedderburn, every Azumaya algebra over a field is Morita equivalent to a finite-dimensional central division algebra, and finite division algebras are commutative; thus, the central ones are isomorphic to the base field. The group \( H^1(\text{Spec } \mathbb{F}_{p^r}, \mathbb{Z}/2) \) equals \( \mathbb{Z}/2 \), as there is a unique \( \mathbb{Z}/2 \)-Galois extension of \( \mathbb{F}_{p^r} \). Thus, we obtain an exact sequence

\[
0 \to \text{LBr}(E_n(G, \mathbb{F}_{p^r})) \to \mathbb{Z}/2 \to H^3(\text{Spec } \pi_0 E_n(G, \mathbb{F}_{p^r}), \mathbb{G}_m),
\]

which implies that \( \text{LBr}(E_n(G, \mathbb{F}_{p^r})) \) is either zero or \( \mathbb{Z}/2 \).

We can be more specific if we assume that \( n = 1 \), since in that case, we have \( \pi_0 E_1(G, \mathbb{F}_{p^r}) \cong \mathbb{W}(\mathbb{F}_{p^r}) \) and \( H^3(\text{Spec } \mathbb{W}(\mathbb{F}_{p^r}), \mathbb{G}_m) = 0 \) by [52, 1.7a]. Thus, \( \text{LBr}(E_1(G, \mathbb{F}_{p^r})) \) must be \( \mathbb{Z}/2 \). Note that \( E_1(G, \mathbb{F}_{p^r}) \) is closely related to periodic complex \( K \)-theory.

For a general height \( n \), we claim that \( H^3(\text{Spec } \pi_0 E_n(G, \mathbb{F}_{p^r}), \mathbb{G}_m) \) is \( p \)-local. Indeed, for \( l \) prime to \( p \), we have a short exact sequence

\[
0 \to \mu_l \to G_m \xrightarrow{l} G_m \to 0
\]
of étale sheaves. By Theorem 2.8, we have \( H^i(\text{Spec } \pi_0 E_n(G, F_{p^r}); \mu_l) \cong H^i(\text{Spec } F_{p^r}; \mu_l) \). The latter group is zero for \( i \geq 2 \) since the absolute Galois group \( \hat{\mathbb{Z}} \) of \( F_{p^r} \) has cohomological dimension 1. Thus, multiplication by \( l \) is an isomorphism on \( H^3(\text{Spec } \pi_0 E_n(G, \mathbb{G}_m)) \), proving the claim. As a consequence, the map

\[
\mathbb{Z}/2 = H^1(\text{Spec } \pi_0 E_n(G, F_{p^r}), \mathbb{Z}/2) \rightarrow H^3(\text{Spec } \pi_0 E_n(G, F_{p^r}), \mathbb{G}_m)
\]
must be zero in the case of odd primes \( p \), and again, we obtain that \( \text{LBr}(E_n(G, F_{p^r})) \cong \mathbb{Z}/2 \) at all heights if \( p \) is odd.

On the other hand, if we work over a separably closed (rather than finite) field \( k_{sep} \), then \( H^2(\text{Spec } k_{sep}, \mathbb{G}_m) \) is again zero, but so is \( H^1(\text{Spec } k_{sep}, \mathbb{Z}/2) \). This results in the fact that \( \text{LBr}(E_n(G, k_{sep})) \) is zero, and in particular, it implies that any nontrivial Brauer class in \( \text{LBr}(E_n(G, k)) \) is necessarily split by \( E_n(G, k) \rightarrow E_n(G, k_{sep}) \).

**Remark 2.29.** We do not, in fact, know an example where the differential

\[
H^1(\text{Spec } \pi_0 R, \pi_1 \text{LBr}_\emptyset) \rightarrow H^3(\text{Spec } \pi_0 R, \mathbb{G}_m)
\]
from Proposition 2.25(iii) is nonzero. Similarly, it would be informative to know if there is a commutative ring spectrum \( R \) such that

\[
\text{Pic}(R) \rightarrow H^0(\text{Spec } \pi_0 R, \pi_1 \text{LBr}_\emptyset) = H^0(\text{Spec } \pi_0 R, \pi_0 \text{Pic}_\emptyset)
\]
is not surjective.

**Remark 2.30.** We are primarily interested in integral results as we want to understand contributions to the Brauer group for commutative ring spectra such as the various forms of topological modular forms. Nevertheless, when \( R \) is even and weakly 2-periodic and if additionally 2 is a unit in \( R \), then there is an identifiable non-étale-locally trivial contribution to the Brauer group in general. If \( R \) is, in fact, 2-periodic and \( u \in \pi_2 R \) a unit, let \( A_u \) be the Azumaya algebra constructed in [30, Example 7.2]: it is an \( R \)-algebra with \( \pi_* A_u = \pi_* R[x] \) where \( |x| = 1 \). We let \( \text{BrW}_\emptyset \) be the sheafification of the components of \( \text{Br}_\emptyset \) containing 0 and the \( [A_u] \) for units \( u \in \pi_2 S \) for étale extensions \( S \) of \( R \). Using that \( [A_u] + [A_v] \) lies in \( \text{LBr}(S) \) for units \( u, v \in \pi_2 S \), there is a natural fiber sequence

\[
\text{LBr}_\emptyset \rightarrow \text{BrW}_\emptyset \rightarrow \mathbb{Z}/2
\]
of sheaves on \( \text{CAlg}^{et}_R \) by [30, Cor. 7.6] (using that all local Brauer classes over \( R \) are algebraic in the sense of [30]). More generally, if 2 is not a unit on \( R \) (but \( R \) is still even and weakly periodic), then we can construct an extension

\[
\text{LBr}_\emptyset \rightarrow \text{BrW}_\emptyset \rightarrow j_! \mathbb{Z}/2,
\]
where \( j : \text{Spec } \pi_0 R \left[ \frac{1}{2} \right] \rightarrow \text{Spec } \pi_0 R \). An easy check using [30, Proposition 7.6] verifies that the algebraic Brauer group of \( R \), as defined in [30], is a subgroup of \( \text{LBrW}(R) = \pi_0(\text{BrW}_\emptyset(R)) \).
3 | THE PICARD SHEAF AND LOCAL BRAUER GROUP OF KO

The aim of this section is to show that the local Brauer group of KO is $\mathbb{Z}/2$. By the previous section, the key is to understand the étale Picard sheaf $\pi_0 \text{Pic}_{\mathcal{O}_{KO}}$ on Spec KO. To achieve that, we essentially rerun the calculations of $\text{Pic}(KO)$ from Gepner–Lawson and Mathew–Stojanoska, but this time in sheaves of spaces on Spec KO. As an aside we will also compute $\text{Pic}(KO_R)$ for any étale extension $R$ of $\mathbb{Z}$, where $KO_R$ denotes the étale extension of KO lifting $R$. (We will use similar notation for other ring spectra as well.)

Recall that $KO \to KU$ is a $C_2$-Galois extension, and consequently, $\text{pic}(KO) \simeq \tau_{\geq 0} (\text{pic}(KU)^{hC_2})$ by Galois descent. Similarly, if $R$ is an étale $\mathbb{Z}$-algebra, then

$$\text{pic}(KO_R) \simeq \tau_{\geq 0} (\text{pic}(KU_R)^{hC_2}).$$

Thus, there is an equivalence

$$\text{pic}_{\mathcal{O}_{KO}} \simeq \tau_{\geq 0} \left( \text{pic}_{\mathcal{O}_{KU}}^{hC_2} \right)$$

of sheaves of connective spectra on Spec KO, which results in a homotopy fixed point descent spectral sequence with signature

$$E_2^{s,t} = H^s(C_2, \pi_t \text{pic}_{\mathcal{O}_{KU}}) \Rightarrow \pi_{t-s} \text{pic}_{\mathcal{O}_{KU}}^{hC_2}. \quad (3.1)$$

The notation $E_2^{s,t} \equiv H^s(C_2, \pi_t \text{pic}_{\mathcal{O}_{KU}})$ means that the $C_2$-cohomology is taken in étale sheaves, and the differentials are

$$d_r^{s,t} : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

Note that in the figures below, we will depict this spectral sequence with the Adams indexing convention, that is, in the $(t-s, s)$-plane.

The action of $C_2$ on the homotopy sheaves of $\text{pic}_{\mathcal{O}_{KU}}$ is as follows:

$$\pi_i \text{pic}_{\mathcal{O}_{KU}} = \begin{cases} \mathbb{Z}/2, \text{ with trivial action, when } i = 0, \\ \mathcal{O}^\times, \text{ with trivial action, when } i = 1, \\ \mathcal{O}, \text{ with trivial action, when } i > 1, i \equiv 1 \mod 4, \\ \mathcal{O}, \text{ with sign action, when } i > 1, i \equiv 3 \mod 4. \end{cases}$$

This allows us to compute $C_2$-cohomology and hence the $E_2$-page of (3.1).

Example 3.2. The action of $C_2$ on $\pi_0 \mathcal{O}^\times$ is trivial, so the cohomology sheaves are

$$H^s(C_2, \mathcal{O}^\times) \cong \begin{cases} \mathcal{O}^\times & \text{if } s = 0, \\ \mu_2 & \text{if } s > 0 \text{ is odd, and} \\ \omega_2 & \text{if } s > 0 \text{ is even}, \end{cases}$$
where $\mu_2$ and $\omega_2$ fit into the exact sequence

$$0 \to \mu_2 \to \mathcal{O}^\times \xrightarrow{x \mapsto x^2} \mathcal{O}^\times \to \omega_2 \to 0.$$ 

Note that on $\text{Spec } \mathbb{Z}$, the sheaf $\mu_2$ is isomorphic to the constant sheaf $\mathbb{Z}/2$; indeed, every étale extension of $\mathbb{Z}$ is a product of integral domains with $2 \neq 0$.

The following identification will not be necessary for our computation of $\text{LBr}(\text{KO})$, but we add it for completeness.

**Lemma 3.3.** The sheaf $\omega_2$ (from Example 3.2) is isomorphic to $\mathcal{O}/2$ on $\text{Spec } \mathbb{Z}$.

**Proof.** Since $\omega_2$ is supported only at 2 with stalk given by $A^\times/(A^\times)^2$ where $A = \mathbb{Z}^{\text{sh}}_{(2)}$ is the strict Henselization, it is enough to compute the value of this group with its structure as a module over the absolute Galois group $\hat{\mathbb{Z}}$ of $\mathbb{F}_2$ (cf. [54, Corollary II.3.11]). Let $\mathcal{W} = \mathcal{W}(\mathbb{F}_2)$ be the ring of Witt vectors. There is an injection $A \hookrightarrow \mathcal{W}$ and $\mathcal{W}$ is the 2-adic completion of $A$. We will see that the induced map $A^\times/(A^\times)^2 \to \mathcal{W}^\times/(\mathcal{W}^\times)^2$ will turn out to be an isomorphism.

To prove that this map is injective, it suffices to show that if $u \in A^\times$ is a square in $\mathcal{W}^\times$, then it is already a square in $A^\times$. To see this, let $R = A[x]/(x^2 - u)$. This is a finite $A$-algebra with 2-adic completion $R^\wedge_2 \cong \mathcal{W}[x]/(x^2 - u)$. By the Hensel property for $A$ and $\mathcal{W}$, the ring $R$ is a product of either 1 or 2 local rings (see, e.g., [66, Tag 04GG]) and $R^\wedge_2$ is a product of the same number by looking at fraction fields. If $u$ is a square in $\mathcal{W}$, then $R^\wedge_2$ is a product of 2 local rings, but then the same is true of $R$.

Next, we explicitly describe $\mathcal{W}^\times/(\mathcal{W}^\times)^2$, which will help us prove surjectivity of the above quotient map. Let $U_n = \{u \in \mathcal{W}^\times : v_2(u - 1) \geq n\}$, where $v_2$ denotes the 2-adic valuation. One has $\mathcal{W}^\times/U_1 \cong \mathcal{W}^\times$ and $U_n/U_{n+1} \cong \mathcal{W}^\times$ for $n > 2$. The snake lemma for the diagram

$$
\begin{array}{cccccc}
0 & \to & U_1 & \to & \mathcal{W}^\times & \to & \mathcal{W}^\times \cong \mathcal{W}^\times \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & U_1 & \to & \mathcal{W}^\times & \to & \mathcal{W}^\times \\
\end{array}
$$

where the vertical maps are all given by squaring, gives an isomorphism $U_1/U_2^2 \cong \mathcal{W}^\times/(\mathcal{W}^\times)^2$, as squaring is an isomorphism on $\mathcal{F}_2^\times$. This also shows that the kernel of squaring on $U_1$ is isomorphic to $\mathbb{Z}/2$, identified as $\{\pm 1\} \subset \mathcal{W}^\times$.

Classical results imply that there is a logarithmic isomorphism $U_2 \cong \mathcal{W}$ (see, e.g., the argument in [62, Sec. II.3]). Thus, there is an exact sequence

$$0 \to \mathcal{W} \cong U_2 \to U_1 \to \mathcal{F}_2 \to 0.$$ 

Using the snake lemma for the squaring map for this sequence as above gives an exact sequence

$$0 \to \mathbb{Z}/2 \to \mathcal{F}_2^\delta \to \mathcal{F}_2 \to U_1/U_1^2 \to \mathcal{F}_2 \to 0,$$

where we have used that $\mathcal{F}_2$ is 2 torsion.
### TABLE 1
An assortment of étale sheaves.

| Symbol | □ | □^x | • | O |
|--------|---|-----|---|---|
| Sheaf  | O | O^x | O/2 | Z/2 |

The boundary map $\partial$ is computed by lifting $x \in \overline{F}_2$ to $U_1$ as $1 + 2\tilde{x}$ for some $\tilde{x}$ lifting $x$ to $\mathcal{W}$ and then squaring, to find $1 + 4\tilde{x} + 4\tilde{x}^2 = 1 + 4(\tilde{x} + \tilde{x}^2)$, which is in $U_2$ with residue modulo squares given by $x + x^2$. We see that $\partial$ is surjective and that $\mathcal{W}^x/(\mathcal{W}^x)^2 \cong U_1/U_1^2 \cong \overline{F}_2$. Explicitly, this isomorphism sends $1 + 2\tilde{x}$ to $\tilde{x} \mod 2$.

Returning to the question of surjectivity of $A^x/(A^x)^2 \to \mathcal{W}^x/(\mathcal{W}^x)^2$, since the residue fields of $A$ and $\mathcal{W}$ agree, we can lift any element of $\overline{F}_2$ in the map above to an element of the form $1 + 2\tilde{x}$ with $\tilde{x} \in A \subset \mathcal{W}$; moreover, $1 + 2\tilde{x}$ will be in $A^x$ by the Hensel property. It follows that $A^x/(A^x)^2 \to \mathcal{W}^x/(\mathcal{W}^x)^2$ is surjective as well and that both groups are Galois-equivariantly isomorphic to $\mathbb{F}_2$.

### Remark 3.4
The above result can also be read off from a much more sophisticated result due to Clausen, Mathew, and Morrow. They show in [20, Thm. A] that if $R$ is $p$-torsion free, henselian along $p$, and $R/p$ is perfect, then $K(R)/p \cong TC(R)/p$. Let $A = \mathbb{Z}_{sh}^{(2)}$ and $\mathcal{W} = \mathcal{W}(\mathbb{F}_2)$, the 2-completion of $A$. Applying the Clausen–Mathew–Morrow result to $A$ and $\mathcal{W}$, one obtains

$$A^x/(A^x)^2 \cong K_1(A)/2 \cong TC_1(A)/2 \cong TC_1(\mathcal{W})/2 \cong K_1(\mathcal{W})/2 \cong \mathcal{W}^x/(\mathcal{W}^x)^2,$$

using that for any local ring $R$, we have an isomorphism $K_1(R) \cong R^x$ and that $TC(R)/p \cong TC(R^p)/p$ for any $R$ and any prime $p$, for example, by [20, Lem. 5.3].

To depict the spectral sequence (3.1), we will use symbols to denote the various sheaves and Table 1 can be used as a legend.

Figures 1 and 2 show the spectral sequence (3.1). Several lemmas explain the nature of the differentials and the calculation of the $E_4$-page.

#### Lemma 3.5
The $E_4$-page is zero in column 0 above row 3.

**Proof.** Note that our spectral sequence consists on the $E_2$-page of quasi-coherent sheaves above the antidiagonal $x + y = t = 1$. We will identify quasi-coherent sheaves on $\text{Spec} \pi_0KO$ with their abelian groups of global sections.

Since our spectral sequence can be seen as the sheafification of a presheaf of Picard homotopy fixed-point spectral sequences, we can freely use the tools from [51]. In particular, [51, Comparison Tool 5.2.4] implies that any $d_3$-differential originating from above the $x + y = t = 3$ antidiagonal can be directly read off its counterpart in the homotopy fixed-point spectral sequence for $KU^{hC_2} \cong KO$. As in [51, Example 7.1.1], the claim follows. □

#### Lemma 3.6 ($d_3^{1,3}$)
The differential $d_3^{1,3} : • \to •$ is given by $x \mapsto x + x^2$. In particular, it is a surjective map of sheaves and the kernel is $i_*O$, where $i : \text{Spec} F_2 \to \text{Spec} \mathbb{Z}$ is the closed inclusion.

**Proof.** The first claim follows from [51, Theorem 6.1.1], see also Example 7.1.1 in [51] for the worked example in the case of the abelian group version of the spectral sequence (3.1). Using that $• = O/2$
FIGURE 1 The $\mathcal{ℰ}_2$-page of the spectral sequence (3.1). All differentials on all pages above the antidiagonal line $x + y = 4$ agree with their linear counterparts by [51]. Not all information is shown in degrees $\leq -2$. Dashed black arrows potentially differ from their linear partners, but they do not figure into the calculation of $\pi_0 \text{Pic}_\mathcal{O}_{\text{KO}}$. The dashed and dotted red arrow is nonlinear and figures into the calculation of $\pi_0 \text{Pic}_\mathcal{O}_{\text{KO}}$.

FIGURE 2 A part of the $\mathcal{ℰ}_4$-page of the spectral sequence (3.1).

is of characteristic 2, we obtain an exact sequence

$$i_\ast \circ \rightarrow \ast \rightarrow \ast.$$  

Since all terms vanish away from 2, it suffices to check that this sequence is short exact at the stalk at 2. Here, the sequence becomes

$$\mathbb{F}_2 \rightarrow \mathbb{F}_2 \xrightarrow{x \mapsto x + x^2} \mathbb{F}_2,$$

which is indeed short exact. □
Remark 3.7. By [30, Proposition 7.15], the differentials $d^{1,0}_2, d^{2,0}_2$, and $d^{2,1}_3$ are nonzero on global sections (where our spectral sequence is isomorphic, at least before differentials, to the usual Picard spectral sequence for KO). The first two differentials have $\mathbb{Z}/2$ as source and are thus determined by global sections: $d^{1,0}_2$ is an isomorphism and $d^{2,0}_2$ is the unique injection $\mathbb{Z}/2 \to \mathcal{O}/2$. The differential $d^{2,1}_3 : \mathcal{O}/2 \to \mathcal{O}/2$ is not determined by global sections, however, and thus remains unresolved. None of these differentials are needed for our computation of the Picard sheaf and hence of $\text{LBr}(\text{KO})$, though their result on global sections is used in the Gepner–Lawson computation of $\text{Br}(\text{KU}|\text{KO})$, which we will come back to in Remark 3.15.

These computations determine the associated graded of $\pi_0\text{pic}_{\text{KO}}$, but we can also resolve the extension problems as follows.

**Proposition 3.8.** There is a filtration on $\pi_0\text{pic}_{\text{KO}}$ with associated graded pieces $\mathbb{Z}/2$, $\mathbb{Z}/2$, and $i_*\mathbb{Z}/2$, where $i$ is the closed inclusion $\text{Spec} \, \mathbb{F}_2 \to \text{Spec} \, \mathbb{Z}$. There is a surjective map from the constant sheaf $\mathbb{Z}/8$ to $\pi_0\text{pic}_{\text{KO}}$, resulting in a nontrivial extension

$$0 \to i_*\mathbb{Z}/2 \to \pi_0\text{pic}_{\text{KO}} \to \mathbb{Z}/4 \to 0. \quad (3.9)$$

**Proof.** The first statement was proved in the lemmas above, namely, we get a filtration on the $E_\infty$-page of the spectral sequence (3.1) with

$$
\begin{array}{ccc}
\mathbb{F}_2^\infty & \xrightarrow{\sim} & \mathbb{F}_1^\infty \\
i_*\mathbb{Z}/2 & \downarrow & \mathbb{Z}/2 \\
 & \downarrow & \mathbb{Z}/2.
\end{array}
$$

This filtration gives an inclusion $i_*\mathbb{Z}/2 \cong \mathbb{F}_2^\infty \to \pi_0\text{pic}_{\text{KO}}$, and we need to identify the quotient $Q$ with $\mathbb{Z}/4$. This quotient sits in an extension

$$0 \to \mathbb{Z}/2 \to Q \to \mathbb{Z}/2 \to 0. \quad (3.10)$$

The filtration implies that the group of global sections $H^0(\text{Spec} \, \mathbb{Z}, \pi_0\text{pic}_{\text{KO}})$ is a finite group of cardinality at most 8. On the other hand, Proposition 2.25 implies that the homomorphism $\text{Pic}(\text{KO}) \to H^0(\text{Spec} \, \mathbb{Z}, \pi_0\text{pic}_{\text{KO}})$ is an injection since $H^1(\text{Spec} \, \mathbb{Z}, G_m) = \text{Pic}(\mathbb{Z}) = 0$. Composing with the isomorphism

$$\mathbb{Z}/8 \to \text{Pic}(\text{KO}), \quad [1] \to \Sigma \text{KO},$$

we obtain a map of sheaves $\mathbb{Z}/8 \to \pi_0\text{pic}_{\text{KO}}$, which must be an isomorphism on global sections.

The above also gives a map $\mathbb{Z}/8 \to Q$ that is the surjection $\mathbb{Z}/8 \to \mathbb{Z}/4$ on global sections, implying that the extension (3.10) is nontrivial. But the only nontrivial extension of $\mathbb{Z}/2$ by $\mathbb{Z}/2$ on $\text{Spec} \, \mathbb{Z}$, which has $\mathbb{Z}/4$ as global sections, is the constant sheaf $\mathbb{Z}/4$.† This identifies the

† Indeed, $\text{Ext}_{\text{Spec} \, \mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2) \cong H^1(\text{Spec} \, \mathbb{Z}; \mathbb{Z}/2) = 0$, and thus, the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

implies that $\mathbb{Z}/2$ has no nontrivial extensions by $\mathbb{Z}/2$. Therefore, the only nontrivial extension of $\mathbb{Z}/2$ by $\mathbb{Z}/2$ is the constant sheaf $\mathbb{Z}/4$, which is the desired extension.
quotient in (3.9), and to see that this extension is also not split, we again compare with the global sections.

□

**Corollary 3.11.** Let $R$ be an étale extension of $\mathbb{Z}$. Then there is a short exact sequence

$$0 \to \text{Pic}(R) \to \text{Pic}(KO_R) \to (\pi_0 \text{pic}_{KO})(R) \to 0.$$  

If $\text{Spec} R$ is connected, the last term sits in an extension of the form

$$0 \to (\mathbb{Z}/2)^d \to (\pi_0 \text{pic}_{KO})(R) \to \mathbb{Z}/4 \to 0,$$

where $d$ is the number of factors when decomposing $R/2$ as a product of fields.

**Proof.** We first show the second part. The long exact sequence in cohomology associated to the extension in Proposition 3.8 takes the form

$$0 \to (\mathbb{Z}/2)^d \to (\pi_0 \text{pic}_{KO})(R) \to \mathbb{Z}/4 \to H^1(R; i_* \mathbb{Z}/2) \to \cdots$$

The composite $\text{Pic}(KO_R) \xrightarrow{q} (\pi_0 \text{pic}_{KO})(R) \xrightarrow{r} \mathbb{Z}/4$ sends $\Sigma KO_R$ to $[1]$ and is thus a surjection. This implies the second claim.

For the first part, we can assume that $\text{Spec} R$ is connected and thus $R$ a regular integral domain. From Proposition 2.25, we have a natural exact sequence

$$0 \to \text{Pic}(R) \to \text{Pic}(KO_R) \xrightarrow{q} (\pi_0 \text{pic}_{KO})(R) \xrightarrow{\partial_R} \text{Br}(R).$$

Since $rq : \text{Pic}(KO_R) \to \mathbb{Z}/4$ is surjective and $\partial_R q = 0$, it suffices to show that the restriction $\partial'_R : (\mathbb{Z}/2)^d \to \text{Br}(R)$ is zero. The map $R \to R \left[ \frac{1}{2} \right]$ induces a commutative square

$$\begin{array}{ccc}
(\mathbb{Z}/2)^d & \xrightarrow{} & \text{Br}(R) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{} & \text{Br}(R \left[ \frac{1}{2} \right]),
\end{array}$$

in which the horizontal arrows are the restricted boundaries $\partial'$ for $R$ and $R \left[ \frac{1}{2} \right]$, respectively. The right-hand vertical map is an injection by Theorem 2.5 since $\text{Spec} R \left[ \frac{1}{2} \right] \subset \text{Spec} R$ is dense. Thus, $\partial'_R = 0$.

□

**Remark 3.12.** As a consequence of the preceding corollary, we see that it is not true that for every étale extension $\mathbb{Z} \subset R$ with $\text{Spec} R$ connected, we have $\text{Pic}(KO_R) \cong \text{Pic}(R) \times \mathbb{Z}/8$ or $\text{Pic}(R) \times \mathbb{Z}/4$.

For example, take the field $K = \mathbb{Q}(\sqrt{17})$, whose ring of integers is $\mathbb{Z}[\omega]$, where $\omega = \frac{1+\sqrt{17}}{2}$, and set $R = \mathbb{Z}[\omega] \left[ \frac{1}{17} \right]$. Here, we have $2 = -(1 + \omega)(2 - \omega)$ and thus $R/2 \cong \mathbb{F}_2 \times \mathbb{F}_2$. We obtain

implies that $\text{Ext}_{\text{Spec } \mathbb{Z}/2}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \text{coker}(\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/2) \cong \mathbb{Z}/2$. 

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\[ \text{Ext}_{\text{Spec } \mathbb{Z}/2}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \text{coker}(\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/2) \cong \mathbb{Z}/2. \]
Pic(KOₜ) ≅ ℤ/8 × ℤ/2. In the Picard spectral sequence for KOₜ, the “exotic” elements arise as the kernel of the $d_3$-differential

$$d_3 : R/2 \cong H^3(C_2; \pi_2 KU) \to H^6(C_2; \pi_4 KU) \cong R/2, \quad x \mapsto x + x^2$$

is bigger than ℤ/2, namely, $(ℤ/2)^2$ in our example.

How can we understand these additional classes? Let us sketch a conjectural general picture of the filtration on Pic(A) from the Picard spectral sequence for a faithful $G$-Galois extension $A \to B$. Let $M \in \text{Pic}(A)$. The 0-line detects the image $M \otimes_A B \in \text{Pic}(B)$. If $M \otimes_B B \cong B$ (and such an equivalence is chosen), the 1-line $H^1(G; \pi_0 B)$ describes how the $G$-action on $\pi_*(M \otimes_A B)$ is twisted in comparison to that on $\pi_*B$. Thus, the $E_2$-term of the homotopy fixed-point spectral sequence for $(M \otimes_A B)^{hC_2} \cong M$ is isomorphic to that for $B^{hC_2} \cong A$ if $M$ has filtration at least 2, which we will assume now. We fix such an isomorphism. We conjecture that if $M$ has filtration $i \geq 2$, its reduction to $H^i(G, \pi_0 \text{Pic}(B)) \cong H^i(G; \pi_{i-1} B)$ equals $d_i(1)$ in the homotopy fixed-point spectral sequence for $(M \otimes_A B)^{hG} \cong M$.

Back to our example, this means that the three nontrivial classes in Pic(KOₜ) of filtration 3 correspond conjecturally to invertible KOₜ-modules $M$ such that $d_3^M(1)$ is 1, $\omega$, and $1 + \omega$ respectively.

The identification of $\text{Pic}_{\text{KO}}$ allows us to compute the local Brauer group of KO. Recall in this context that Gepner and Lawson proved in [30, Proposition 7.17] that the subgroup $\text{Br}(KU|KO) \subseteq \text{Br}(KO)$ of classes killed by the extension KO → KU is isomorphic to ℤ/2. We will show that $L\text{Br}(KO)$ is also ℤ/2, and, in fact, it will be equal to $\text{Br}(KU|KO)$.

**Theorem 3.13.** There is an isomorphism $L\text{Br}(KO) \cong ℤ/2$. The unique nontrivial class is killed by the étale cover

$$KO \to KO\left[\frac{1}{2}, \zeta_4\right] \times KO\left[\frac{1}{3}, \zeta_3\right].$$

Here, we use that the cyclotomic fields $ℚ(\zeta_4)$ and $ℚ(\zeta_3)$ are ramified only at the primes (2) and (3), respectively, to produce $KO\left[\frac{1}{2}, \zeta_4\right]$ and $KO\left[\frac{1}{3}, \zeta_3\right]$ as commutative ring spectra.

**Proof.** To use the exact sequence in Proposition 2.25, we first need to compute $H^1(\text{Spec } Z, \pi_0 \text{Pic}_{\text{KO}})$, which we will do using the long exact sequence

$$H^0(\text{Spec } Z, \pi_0 \text{Pic}_{\text{KO}}) \to H^0(\text{Spec } Z, ℤ/4) \to H^1(\text{Spec } Z, i_* ℤ/2) \to H^1(\text{Spec } Z, \text{Pic}_{\text{KO}})$$

$$\to H^1(\text{Spec } Z, ℤ/4)$$

associated to the short exact sequence (3.9) in Proposition 3.8.

Since there is a unique $ℤ/2$-Galois extension of Spec $𝔽_2$, Theorem 2.9 implies

$$H^1(\text{Spec } Z, i_* ℤ/2) \cong H^1(\text{Spec } ℱ_2, ℤ/2) \cong ℤ/2.$$
Moreover, \( H^1(\text{Spec } \mathbb{Z}, \mathbb{Z}/4) = 0 \) as there are no unramified \( \mathbb{Z}/4 \)-Galois extensions of \( \mathbb{Q} \). Since by Corollary 3.11,

\[
H^0(\text{Spec } \mathbb{Z}, \text{Pic}_{\mathbb{Q}^{\text{et}}}) \to H^0(\text{Spec } \mathbb{Z}, \mathbb{Z}/4) \cong \mathbb{Z}/4
\]

is surjective, the long exact cohomology sequence (Equation (3.14)) implies that \( H^1(\text{Spec } \mathbb{Z}, \pi_0 \text{Pic}_{\mathbb{Q}^{\text{et}}}) \) is isomorphic to \( \mathbb{Z}/2 \).

To conclude \( \text{LBr}(\mathbb{K}) \cong \mathbb{Z}/2 \) using Proposition 2.25, it remains to show the vanishing of the differential

\[
H^1(\text{Spec } \mathbb{Z}, \pi_0 \text{Pic}_{\mathbb{Q}^{\text{et}}}) \to H^3(\text{Spec } \mathbb{Z}, \mathbb{G}_m).
\]

We show this by comparison to \( \mathbb{KU} \): The map \( \mathbb{K} \to \mathbb{KU} \) induces a map of presheaves \( \text{Br}_{\mathbb{K}} \to \text{Br}_{\mathbb{KU}} \) on the étale site of \( \text{Spec } \mathbb{Z} \), which we identify with either of the étale sites of \( \mathbb{K} \) and \( \mathbb{KU} \) using the isomorphism \( \pi_0 \mathbb{K} \cong \mathbb{Z} \cong \pi_0 \mathbb{KU} \). Thus, we get an induced map of descent spectral sequences and in particular a commutative diagram

\[
\begin{array}{ccc}
H^1(\text{Spec } \mathbb{Z}, \pi_0 \text{Pic}_{\mathbb{Q}^{\text{et}}}) & \to & H^3(\text{Spec } \mathbb{Z}, \mathbb{G}_m) \\
\downarrow & & \downarrow \\
H^1(\text{Spec } \mathbb{Z}, \pi_0 \text{Pic}_{\mathbb{KU}^{\text{et}}}) & \to & H^3(\text{Spec } \mathbb{Z}, \mathbb{G}_m),
\end{array}
\]

where the right vertical map is an equality. Since \( H^1(\text{Spec } \mathbb{Z}, \pi_0 \text{Pic}_{\mathbb{Q}^{\text{et}}}) \cong H^1(\text{Spec } \mathbb{Z}, \mathbb{Z}/2) = 0 \), we see that the top differential must vanish. Therefore, \( \text{LBr}(\mathbb{K}) \cong H^1(\text{Spec } \mathbb{Z}, \pi_1 \text{BPic}_{\mathbb{Q}^{\text{et}}}) \cong \mathbb{Z}/2 \).

For the second part of our claim, note first that \( \text{Br}(\mathbb{Z}[\frac{1}{2}, \zeta_4]) \) and \( \text{Br}(\mathbb{Z}[\frac{1}{3}, \zeta_3]) \) vanish by Example 2.4. The Brauer groups of \( \mathbb{Z}[\frac{1}{2}, \zeta_4] \) and \( \mathbb{Z}[\frac{1}{3}, \zeta_3] \) agree with the second étale cohomology with \( \mathbb{G}_m \)-coefficients because the rings are regular and noetherian. Using Proposition 2.25 again, we thus see that the nontrivial class in \( \text{LBr}(\mathbb{K}) \) must be killed by the extension \( \mathbb{K} \to \mathbb{KU} \) if the image of the nontrivial element of \( H^1(\text{Spec } \mathbb{Z}, i_* \mathbb{Z}/2) \) vanishes in \( H^1(\text{Spec } \mathbb{Z}[\frac{1}{2}, \zeta_4], i_* \mathbb{Z}/2) \) and \( H^1(\text{Spec } \mathbb{Z}[\frac{1}{3}, \zeta_3], i_* \mathbb{Z}/2) \). This is clear in the first case as \( i_* \mathbb{Z}/2 \) restricted to \( \text{Spec } \mathbb{Z}[\frac{1}{2}] \) vanishes. In the second case, we use that the extension \( \mathbb{F}_2 \subset \mathbb{F}_4 \cong \mathbb{F}_2[\zeta_3] \) kills the nontrivial element of \( H^1(\text{Spec } \mathbb{F}_2, \mathbb{Z}/2) \).

\[\square\]

Remark 3.15. Note that since \( \text{LBr}(\mathbb{K}) = 0 \), functoriality of the local Brauer group implies that the nonzero class \( \alpha \in \text{LBr}(\mathbb{K}) \cong \mathbb{Z}/2 \) is killed by the \( \mathbb{Z}/2 \)-Galois extension \( \mathbb{K} \to \mathbb{KU} \), that is, lies in the relative Brauer group \( \text{Br}(\mathbb{KU}/\mathbb{K}) \). By the main result of [30], \( \text{LBr}(\mathbb{K}) \) thus agrees with \( \text{Br}(\mathbb{KU}/\mathbb{K}) \) though a priori we only get an inclusion. This gives a new proof of the fact that the Galois-cohomological Brauer class found in [30, Proposition 7.15] is representable by an Azumaya algebra, which Gepner and Lawson prove instead with an unstable descent spectral sequence. See also Example 4.15 for another perspective.

We urge the reader to consider the analog of the descent spectral sequence computation of \( \text{Br}(\mathbb{KU}/\mathbb{K}) \) as in [30, Figure 7.2] in the case of the relative Brauer group of \( \mathbb{K} \left[ \frac{1}{2}, \zeta_4 \right] \) with
respect to $KU \left[ \frac{1}{3}, \zeta_3 \right]$. As the class in filtration six contributing to $Br(KU|KO)$ has to die in $Br \left( KO \left[ \frac{1}{3}, \zeta_3 \right] \right)$, there must be a new $d_3$ killing it. This $d_3$ is given by the formula in [51, Theorem 6.1.1], the point being that the image of $x \mapsto x + x^2$ on $\mathbb{Z} \left[ \frac{1}{3}, \zeta_3 \right]/2 \cong \mathbb{F}_4$ is $\mathbb{Z}/2 = \mathbb{F}_2 \subseteq \mathbb{F}_4$.

4 | BRAUER GROUPS OF NONCONNECTIVE SPECTRAL DM STACKS

In this section, we turn to Brauer groups of nonconnective spectral Deligne–Mumford (DM) stacks. A significant difference will be that the Brauer group is in general no longer $\pi_0$ of the global sections of the Brauer sheaf, yielding a distinction between the Brauer group and the cohomological Brauer group, which we will explain below.

To fix notation, we recall the following definition from Lurie [48].

**Definition 4.1.** A nonconnective spectral DM stack is a spectrally ringed $\infty$-topos $(\mathcal{X}, \mathcal{O})$ such that there exists a covering $\coprod_{i \in I} U_i \to \ast$ of the final object where for each $i$, there is an equivalence $(\mathcal{X}/U_i, \mathcal{O}|_{\mathcal{X}/U_i}) \cong \text{Spec} R_i$ for some commutative ring spectrum $R_i$.\(^{\dagger}\) If $\mathcal{O}$ is connective, we say that $(\mathcal{X}, \mathcal{O})$ is a connective spectral DM stack; if $\mathcal{O}$ is discrete, we say that $(\mathcal{X}, \mathcal{O})$ is a classical DM stack.

**Remark 4.2.**

(a) In [48], Lurie calls connective spectral DM stacks simply spectral DM stacks.

(b) Given a nonconnective spectral DM stack $(\mathcal{X}, \mathcal{O})$, there is a diagram $(\mathcal{X}, \mathcal{O}) \to (\mathcal{X}, \tau_{\geq 0} \mathcal{O}) \leftarrow (\mathcal{X}, \pi_0 \mathcal{O})$ of nonconnective spectral DM stacks.

**Construction 4.3.** For a nonconnective spectral DM stack, étale sheaves on $\mathcal{X}$ are equivalent to étale sheaves on the site $\text{Aff}_{\text{ét}}/((\mathcal{X}, \mathcal{O}))$ of étale maps $\text{Spec} R \to (\mathcal{X}, \mathcal{O})$ for some commutative ring spectrum $R$. Restricting the sheaves $\text{Pic}, Br,$ and $\text{LBr}$ on $\text{CAAlg}_{S}^{\text{op}} \cong \text{Aff}$ from Section 2, we obtain sheaves $\text{Pic}_\mathcal{O}, Br_\mathcal{O},$ and $\text{LBr}_\mathcal{O}$ on $\text{Aff}_{\text{ét}}/((\mathcal{X}, \mathcal{O}))$ or, equivalently, on $\mathcal{X}$.

**Remark 4.4.** There is a natural map $B\text{Pic}_\mathcal{O} \to Br_\mathcal{O}$ that induces an equivalence $B\text{Pic}_\mathcal{O} \cong \text{LBr}_\mathcal{O}$, since again this can be checked locally. The computation of the homotopy sheaves of $\text{LBr}_\mathcal{O}$ given in Lemma 2.18 goes through verbatim here.

**Example 4.5.** In general, $\pi_0 Br_{\tau_{\geq 0} \mathcal{O}} = \pi_0 Br_{\pi_0 \mathcal{O}} = 0$ since Brauer classes on connective commutative ring spectra are étale-locally trivial by [3, Theorem 5.11]. We also have $\pi_1 Br_{\tau_{\geq 0} \mathcal{O}} \cong \pi_1 Br_{\pi_0 \mathcal{O}} \cong \mathbb{Z}$ by the computation of Picard groups of connective commutative ring spectra (see Proposition 2.25). On the other hand, $\pi_0 Br_{\mathcal{O}}$ and $\pi_1 Br_{\mathcal{O}}$ are highly dependent on the nature of $\mathcal{O}$ itself.

**Definition 4.6.** We let $Br'(\mathcal{X}, \mathcal{O}) = \pi_0 \Gamma(\mathcal{X}, Br_{\mathcal{O}}) = \pi_0 (Br_{\mathcal{O}}(\mathcal{X}))$. This is the cohomological Brauer group of $\mathcal{X}$. Similarly, the cohomological local Brauer group of $(\mathcal{X}, \mathcal{O})$ is $\text{LBr}'(\mathcal{X}, \mathcal{O}) = \pi_0 B\text{Pic}_\mathcal{O}(\mathcal{X})$.

\(^{\dagger}\) Lurie writes $\text{Spét} R$ for what we write as $\text{Spec} R$. 
We call the space of global sections $\text{Br}_\mathcal{O}(\mathcal{X})$ the Brauer space and similarly for the local Brauer space $\text{BPic}_\mathcal{O}(\mathcal{X}) \simeq \text{LBr}_\mathcal{O}(\mathcal{X})$.

Remark 4.7. The subgroup $\text{LBr}'(\mathcal{X}, \mathcal{O}) \subseteq \text{Br}'(\mathcal{X}, \mathcal{O})$ consists of those cohomological Brauer classes that are étale locally trivial on $\mathcal{X}$. Since $(\mathcal{X}, \mathcal{O})$ is a nonconnective spectral DM stack this means that for $\alpha \in \text{LBr}'(\mathcal{X}, \mathcal{O})$, there is a surjective family of étale maps $\{p_i : \text{Spec } R_i \to (\mathcal{X}, \mathcal{O})\}_{i \in I}$ such that $p_i^* \alpha = 0$ for all $i$.

Construction 4.8. In order to compute $\text{Br}'(\mathcal{X}, \mathcal{O})$ and $\text{LBr}'(\mathcal{X}, \mathcal{O})$, it is convenient to deloop $\text{Br}_\mathcal{O}$ and $\text{LBr}_\mathcal{O}$ and view them as presheaves of spectra; étale sheafification yields sheaves $\mathbf{br}_\mathcal{O}$ and $\mathbf{lbr}_\mathcal{O}$. As such we have $\pi_t \mathbf{br}_\mathcal{O} \cong \pi_t \text{Br}_\mathcal{O}(\mathcal{X})$ and similarly for $\mathbf{lbr}_\mathcal{O}$; in particular, the homotopy sheaves vanish for $t < 0$. We have $\Omega^\infty \mathbf{br}_\mathcal{O}(\mathcal{X}) \cong \text{Br}_\mathcal{O}(\mathcal{X})$. Analogously to Proposition 2.21, we argue that $\mathbf{br}_\mathcal{O}$ and $\mathbf{lbr}_\mathcal{O}$ are Postnikov complete. Thus, we obtain a descent spectral sequence $E_2^{s,t} = H^s(\mathcal{X}, \pi_t \mathbf{br}_\mathcal{O}) \Rightarrow \pi_{t-s} \mathbf{br}_\mathcal{O}(\mathcal{X}) \cong t-s \geq 0 \pi_{t-s} \text{Br}_\mathcal{O}(\mathcal{X})$ and similarly, for $\mathbf{lbr}_\mathcal{O}(\mathcal{X})$.

For the following definition, recall that a quasi-coherent sheaf is perfect if it is dualizable or, equivalently, if it becomes a compact object whenever restricted to an affine.

Definition 4.9. A quasi-coherent sheaf $\mathcal{A}$ of $\mathcal{O}$-algebras on a nonconnective spectral DM stack $(\mathcal{X}, \mathcal{O})$ is an Azumaya algebra if the following equivalent conditions hold:

(i) $\mathcal{A}$ is perfect, locally generates $\text{QCoh}(\mathcal{X}, \mathcal{O})$, and the natural map $\mathcal{A}^{\text{op}} \otimes_\mathcal{O} \mathcal{A} \to \text{End}_\mathcal{O}(\mathcal{A})$ is an equivalence;

(ii) there is an étale cover $\{\text{Spec } R_i \overset{p_i}{\to} (\mathcal{X}, \mathcal{O})\}_{i \in I}$ such that $p_i^* \mathcal{A}$ is an Azumaya $R_i$-algebra for all $i$.

Any Azumaya algebra $\mathcal{A}$ on $(\mathcal{X}, \mathcal{O})$ defines a point of $\text{Br}_\mathcal{O}$ and hence an element $[\mathcal{A}]$ of $\text{Br}'(\mathcal{X}, \mathcal{O})$, called the class of $\mathcal{A}$. If $\mathcal{A}$ is an Azumaya algebra, then so is the opposite algebra $\mathcal{A}^{\text{op}}$ and we have $[\mathcal{A}^{\text{op}}] = -[\mathcal{A}]$; if $\mathcal{B}$ is a second Azumaya algebra, then $\mathcal{A} \otimes_\mathcal{O} \mathcal{B}$ is Azumaya and $[\mathcal{A} \otimes_\mathcal{O} \mathcal{B}] = [\mathcal{A}] + [\mathcal{B}]$. These assertions may be verified locally using Definition 4.9(ii) and Definition 2.11(b).

Definition 4.10. Let $\text{Br}(\mathcal{X}, \mathcal{O}) \subseteq \text{Br}'(\mathcal{X}, \mathcal{O})$ be the subgroup consisting of the classes of Azumaya algebras. Let $\text{LBr}(\mathcal{X}, \mathcal{O}) = \text{LBr}'(\mathcal{X}, \mathcal{O}) \cap \text{Br}(\mathcal{X}, \mathcal{O})$ inside $\text{Br}'(\mathcal{X}, \mathcal{O})$. We call these the Brauer and local Brauer groups of $(\mathcal{X}, \mathcal{O})$.

Example 4.11. For any commutative ring spectrum $R$, Proposition 2.21 implies $\text{Br}'(\text{Spec } R) = \text{Br}(\text{Spec } R)$.

Definition 4.12. Let $(\mathcal{X}, \mathcal{O})$ be a nonconnective spectral DM stack and let $\alpha \in \text{Br}'(\mathcal{X}, \mathcal{O})$. Denote by $\text{Cat}_\mathcal{O}$ the sheaf of $\infty$-categories sending each étale $\text{Spec } R \to (\mathcal{X}, \mathcal{O})$ to $\text{Cat}_R$; cf. Proposition 2.13. Using the inclusion $\text{Br}_\mathcal{O} \to \text{Cat}_\mathcal{O}$, the section $\alpha \in \text{Br}'(\mathcal{X}, \mathcal{O})$ defines a section of $\text{Cat}_\mathcal{O}$ and hence a stack of stable presentable $\infty$-categories, $\text{QCoh}_{\mathcal{O}, \alpha}$. This is the stack of $\alpha$-twisted quasi-coherent...
sheaves on \((\mathcal{X}, \mathcal{O})\). The stable \(\infty\)-category of global sections will be denoted by \(\text{QCoh}(\mathcal{X}, \alpha)\). An object \(\mathcal{F} \in \text{QCoh}(\mathcal{X}, \alpha)\) is perfect if for every étale map \(p: \text{Spec} \, R \to (\mathcal{X}, \mathcal{O})\), the complex \(p^* \mathcal{F}\) is a compact object of \(\text{QCoh}(\text{Spec} \, R, p^* \alpha)\). Note that the latter stable \(\infty\)-category is equivalent to \(\text{Mod}_A\) where \(A\) is any Azumaya \(R\)-algebra with Brauer class \(p^* \alpha\). We say that \(\mathcal{F}\) is a perfect local generator if it is perfect and \(p^* \mathcal{F}\) generates \(\text{QCoh}(\text{Spec} \, R, p^* \alpha)\) for any \(\text{Spec} \, R \to (\mathcal{X}, \mathcal{O})\).

**Lemma 4.13.** Let \((\mathcal{X}, \mathcal{O})\) be a nonconnective spectral DM stack. An \(\alpha \in \text{Br}'(\mathcal{X}, \mathcal{O})\) lies in \(\text{Br}(\mathcal{X}, \mathcal{O}) \subseteq \text{Br}'(\mathcal{X}, \mathcal{O})\) if and only if there exists a perfect local generator of \(\text{QCoh}(\mathcal{X}, \alpha)\).

**Proof.** If \(\mathcal{A}\) is an Azumaya algebra representing \(\alpha\), define \(\text{QCoh}(\mathcal{X}, \mathcal{A})\) as the limit of \(\text{Mod}_\mathcal{A}(\text{ShvSp}(\mathcal{X}))\) over all étale maps \(\text{Spec} \, R \to (\mathcal{X}, \mathcal{O})\); this can be identified with a full subcategory of \(\text{Mod}_\mathcal{A}(\text{ShvSp}(\mathcal{X}))\). We have \(\text{QCoh}(\mathcal{X}, \mathcal{A}) \cong \text{QCoh}(\mathcal{X}, \alpha)\) and under this equivalence, \(\mathcal{A}\) corresponds to a perfect local generator. Conversely, given a perfect local generator \(\mathcal{F}\) of \(\text{QCoh}(\mathcal{X}, \alpha)\), the sheaf of endomorphisms \(\text{End}_\mathcal{O}(\mathcal{F})\) is an Azumaya algebra with class \(\alpha\). \(\square\)

Here is one example where every cohomological Brauer class is representable by an Azumaya algebra.

**Proposition 4.14.** Let \((\mathcal{X}, \mathcal{O})\) be a nonconnective spectral DM stack. If \((\mathcal{X}, \mathcal{O})\) admits a finite étale cover \(\pi: \text{Spec} \, R \to (\mathcal{X}, \mathcal{O})\), then \(\text{Br}(\mathcal{X}, \mathcal{O}) = \text{Br}'(\mathcal{X}, \mathcal{O})\).

**Proof.** There is a compact generator \(\mathcal{F}\) of \(\text{QCoh}(\text{Spec} \, R, \pi^* \alpha)\) by Example 4.11 and Lemma 4.13. The pushforward \(\pi_* \mathcal{F}\) is a perfect local generator of \(\text{QCoh}(\mathcal{X}, \alpha)\), as one can check étale locally. \(\square\)

**Example 4.15.** If a finite group \(G\) acts on a commutative ring spectrum \(R\), we obtain a finite étale map \(\text{Spec} \, R \to [\text{Spec} \, R/G]\) to the stack quotient. In particular, \(\text{Br}([\text{Spec} \, R/G]) \cong \text{Br}'([\text{Spec} \, R/G])\) by the preceding proposition. This is especially interesting if \(R_{\text{ring}}^G \to R\) is a faithful \(G\)-Galois extension, when Galois descent implies that \(\text{Mod}_{R_{\text{ring}}^G} \cong \text{QCoh}([\text{Spec} \, R/G])\). Examples include \(\text{KO} \to \text{KU}, \text{TMF} \left[\frac{1}{2}\right] \to \text{TMF}(2),\) and \(\text{TMF} \left[\frac{1}{3}\right] \to \text{TMF}(3)\) (see [60] and [50]).

Proposition 4.14 will not be enough to show the agreement of \(\text{Br}'\) and \(\text{Br}\) for the derived moduli stack of elliptic curves because the moduli stack of elliptic curves does not have a finite étale cover by an affine scheme [71]. This issue will be solved by Theorem 4.17 below. Before we state it, we introduce the following definition needed for its proof.

**Definition 4.16.** Let \((\mathcal{X}, \mathcal{O})\) be a nonconnective spectral DM stack. Let \(\alpha \in \text{Br}'(\mathcal{X}, \mathcal{O})\) be a Brauer class and let \(\mathcal{F} \in \text{QCoh}(\mathcal{X}, \alpha)\) be a perfect local generator. We say that \(\mathcal{F}\) is a global generator if \(\mathcal{F}\) is compact and if \(\text{QCoh}(\mathcal{X}, \alpha)\) is compactly generated by \(\mathcal{F}\).

**Theorem 4.17.** Let \((\mathcal{X}, \mathcal{O})\) be a nonconnective spectral DM stack and fix \(\alpha \in \text{Br}'(\mathcal{X}, \mathcal{O})\). If \(\mathcal{X}\) admits a Zariski open cover \(\{\mathcal{U}_i\}_{i=1}^n\) such that

(a) for each \(1 \leq i, j \leq n\), the kernel of \(\text{QCoh}(\mathcal{U}_i, \mathcal{O}) \to \text{QCoh}(\mathcal{U}_i \cap \mathcal{U}_j, \mathcal{O})\) is generated by a single compact object \(\mathcal{X}_{i,j}\), and

(b) there is a global generator \(\mathcal{F}_i\) of \(\text{QCoh}(\mathcal{U}_i, \alpha)\) for each \(i = 1, \ldots, n\),

then \(\alpha \in \text{Br}(\mathcal{X}, \mathcal{O})\) and there is a global generator of \(\text{QCoh}(\mathcal{X}, \alpha)\).
The proof follows the work of [70] and [3], which uses older arguments of Bökstedt–Neeman [14] and Bondal–van den Bergh [15] who showed that for a quasi-compact and quasi-separated scheme $X$, the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves admits a single compact generator, which is global in the sense above. Other important examples of $Br = Br'$ in the non-derived and derived context have been established in [18, 22, 28, 35].

Proof. Note first that each $\mathcal{U}_i \subseteq X$ is relatively scalloped in the sense of [48, 2.5.4.1].†

We glue local perfect generators as in [3, Theorem 6.11] or [70, Proposition 5.9], taking care in each step to produce a global generator. Let $Y_k$ be the union $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_k$ in $X$. It is enough to prove that there is a global generator of $\text{QCoh}(Y_k, \alpha)$ for each $k = 1, \ldots, n$, and hence, $\alpha|_{Y_k}$ is in $\text{Br}(Y_k, \mathcal{O})$ for each $k$. The base case follows from assumption (b). Suppose that the conclusion holds for some $1 \leq k < n$. Set $W = Y_k \cap \mathcal{U}_{k+1}$ and consider the pullback square

\[
\begin{array}{ccc}
\text{QCoh}(Y_{k+1}, \alpha) & \to & \text{QCoh}(\mathcal{U}_{k+1}, \alpha) \\
\downarrow & & \downarrow \\
\text{QCoh}(Y_k, \alpha) & \to & \text{QCoh}(W, \alpha)
\end{array}
\] (4.18)

of stable presentable $\infty$-categories. As each inclusion $W \subseteq \mathcal{U}_{k+1}$, $W \subseteq Y_k$, $Y_k \subseteq Y_{k+1}$, and $\mathcal{U}_{k+1} \subseteq Y_{k+1}$ is quasi-affine and hence relatively scalloped, it follows from [48, 2.5.4.3], or rather its proof, that the functors in (4.18) preserve compact objects.

We want to show that the kernel of $\text{QCoh}(\mathcal{U}_{k+1}, \alpha) \to \text{QCoh}(W, \alpha)$ is generated by the compact object

\[ G := \mathcal{F}_{k+1} \otimes \mathcal{H}_{k+1, 1} \otimes \cdots \otimes \mathcal{H}_{k+1, k}. \]

Compactness follows by construction. For generation, suppose that $\mathcal{M}$ is an object of $\text{QCoh}(\mathcal{U}_{k+1}, \alpha)$ that restricts to zero on $\text{QCoh}(W, \alpha)$ and suppose additionally that the mapping spectrum $\text{Map}(G, \mathcal{M})$ is zero. We want to show that $\mathcal{M} \simeq 0$. But,

\[ 0 \simeq \text{Map}(G, \mathcal{M}) \simeq \text{Map}(\mathcal{H}_{k+1, 1} \otimes \cdots \otimes \mathcal{H}_{k+1, k}, \text{Map}(\mathcal{F}_{k+1}, \mathcal{M})) \]

by adjunction, where $\text{Map}(\mathcal{F}_{k+1}, \mathcal{M})$ denotes the internal mapping spectrum, a quasi-coherent sheaf on $\mathcal{U}_{k+1}$. Since the $\mathcal{H}_{k+1, j}$ are compact generators of the kernels of $\text{QCoh}(\mathcal{U}_{k+1}, \mathcal{O}) \to \text{QCoh}(\mathcal{U}_{k+1} \cap \mathcal{U}_j, \mathcal{O})$, their tensor product is a compact generator of the kernel of $\text{QCoh}(\mathcal{U}_{k+1}) \to \text{QCoh}(W)$. Denoting the inclusion $W \to \mathcal{U}_{k+1}$ by $i$, it follows that $\text{Map}(\mathcal{F}_{k+1}, \mathcal{M}) \to i^* \text{Map}(\mathcal{F}_{k+1}, \mathcal{M})$ is an equivalence since its fiber lies in the kernel of $\text{QCoh}(\mathcal{U}_{k+1}) \to \text{QCoh}(W)$. But, this implies that $\text{Map}(\mathcal{F}_{k+1}, \mathcal{M}) \simeq \text{Map}(\mathcal{F}_{k+1}, \mathcal{M}|_W)$. The latter is zero as $\mathcal{M}|_W \simeq 0$, so $\text{Map}(\mathcal{F}_{k+1}, \mathcal{M}) \simeq 0$ and hence the mapping spectrum $\text{Map}(\mathcal{F}_{k+1}, \mathcal{M})$ is zero, which, in turn, implies that $\mathcal{M} \simeq 0$ since $\mathcal{F}_{k+1}$ is a compact generator of $\text{QCoh}(\mathcal{U}_{k+1}, \alpha)$.

† Quasi-affine morphisms are relatively scalloped; these will be enough for our applications.

† One just has to repeat the proof in the twisted setting and use that a left adjoint preserves compact objects if its right adjoint preserves filtered colimits.
Using that the square (4.18) is a pullback, the vertical fibers are equivalent stable $\infty$-categories. Thus, $\mathcal{G}$ corresponds to a compact object of $\text{QCoh}(\mathcal{Y}_{k+1}, \alpha)$ that vanishes on $\mathcal{Y}_k$. On the other hand, by induction, there is a global generator $\mathcal{H}$ of $\text{QCoh}(\mathcal{Y}_k, \alpha)$. Our goal will be to lift $\mathcal{H}$ to $\mathcal{Y}_{k+1}$.

The fact that $\text{QCoh}(\mathcal{U}_{k+1}, \alpha) \to \text{QCoh}(\mathcal{W}, \alpha)$ is a localization and preserves compact objects implies that $\text{QCoh}(\mathcal{W}, \alpha)$ is generated by the image of $\mathcal{F}_{k+1}$. Since the kernel is compactly generated by a compact object of $\text{QCoh}(\mathcal{U}_{k+1}, \alpha)$, we are in the setting of Thomason’s extension proposition [69, 5.2.2] (see [56, Corollary 0.9] for the generality needed here), which says that if $\mathcal{B} \to \mathcal{C} \to \mathcal{D}$ is a Verdier sequence of idempotent complete stable $\infty$-categories, then an object $\mathcal{M} \in \mathcal{D}$ lifts to $\mathcal{C}$ if and only if its class $[\mathcal{M}] \in K_0(\mathcal{C})$ lifts to $K_0(\mathcal{D})$. Thus, possibly by replacing $\mathcal{H}$ by $\mathcal{H} \oplus \Sigma \mathcal{H}$ (which always has vanishing class in $K_0$), we see that the restriction of $\mathcal{H}$ to $\text{QCoh}(\mathcal{W}, \alpha)$ lifts to a compact object $\mathcal{H}_{k+1}$ of $\text{QCoh}(\mathcal{Y}_{k+1}, \alpha)$. Gluing $\mathcal{H}$ and $\mathcal{H}_{k+1}$ via the pullback (4.18), we obtain a compact object $\mathcal{E}$ of $\text{QCoh}(\mathcal{X}, \alpha)$. Let $\mathcal{D} = \mathcal{E} \oplus \mathcal{G}$. We claim that $\mathcal{D}$ is a global generator of $\text{QCoh}(\mathcal{Y}_{k+1}, \alpha)$. Verification is standard and left to the reader. □

**Corollary 4.19.** If a nonconnective spectral DM stack $(\mathcal{X}, \mathcal{O})$ satisfies the assumptions of Theorem 4.17 for every $\alpha \in S \subset \text{Br}'(\mathcal{X}, \mathcal{O})$, we have

1. $\text{Br}(\mathcal{X}, \mathcal{O}) = \text{Br}'(\mathcal{X}, \mathcal{O})$ if $S = \text{Br}'(\mathcal{X}, \mathcal{O})$,
2. $\text{LBr}(\mathcal{X}, \mathcal{O}) = \text{LBr}'(\mathcal{X}, \mathcal{O})$ if $S = \text{LBr}'(\mathcal{X}, \mathcal{O})$, and
3. $\text{BrW}(\mathcal{X}, \mathcal{O}) = \text{BrW}'(\mathcal{X}, \mathcal{O})$ if $\mathcal{O}$ is weakly 2-periodic and $S = \text{LBrW}'(\mathcal{X}, \mathcal{O})$.

This corollary will be applied in Proposition 8.1 to the derived moduli stack of elliptic curves.

### 5 THE 0-AFFINE CASE

Let $(\mathcal{X}, \mathcal{O})$ be a nonconnective spectral DM stack.

**Definition 5.1.** We say that $(\mathcal{X}, \mathcal{O})$ is 0-affine if the global sections functor

$$\Gamma : \text{QCoh}(\mathcal{X}, \mathcal{O}) \to \text{Mod}_\Gamma(\mathcal{X}, \mathcal{O})$$

is an equivalence; equivalently, $(\mathcal{X}, \mathcal{O})$ is 0-affine if $\mathcal{O}$ is a compact generator of $\text{QCoh}(\mathcal{X}, \mathcal{O})$.

In classical algebraic geometry, there are few 0-affine DM stacks. If $X$ is a scheme, $X$ is 0-affine if and only if it is quasi-affine, which is to say quasi-compact and can be embedded as an open subscheme of Spec $A$ for some $A$. In this case, one can take $A = \text{H}^0(X, \mathcal{O})$. More generally, quasi-affine connective spectral DM stacks are 0-affine.

Remarkably, in the theory nonconnective spectral DM stacks, there is an additional wealth of nonclassical examples, as supplied by the following theorem of [50].

**Theorem 5.2 [50].** Let $(\mathcal{X}, \mathcal{O})$ be a nonconnective spectral DM stack such that $\mathcal{O}$ is weakly 2-periodic. Suppose that $(\mathcal{X}, \pi_0 \mathcal{O})$ is separated and noetherian and that the associated map $(\mathcal{X}, \pi_0 \mathcal{O}) \to \mathcal{M}_{FG}$ to the moduli of formal groups is quasi-affine and flat. Then $(\mathcal{X}, \mathcal{O})$ is 0-affine.

Our main example will be $(\mathcal{M}, \mathcal{O})$, where $\mathcal{M}$ is the moduli stack of elliptic curve and $\mathcal{O}$ is the weakly 2-periodic sheaf of $\mathcal{E}_\infty$-ring spectra defined by Goerss, Hopkins, and Miller [25]. Later, the
nonconnective spectral DM stack $(\mathcal{M}, \mathcal{O})$ was reinterpreted and reconstructed by Lurie to classify oriented spectral elliptic curves [44], and we will refer to it as the derived moduli stack of elliptic curves.

**Corollary 5.3** [50]. The derived moduli stack $(\mathcal{M}, \mathcal{O})$ of elliptic curves is 0-affine, that is, $\Gamma : \text{QCoh}(\mathcal{M}, \mathcal{O}) \xrightarrow{\sim} \text{Mod}_{\text{TMF}}$ is an equivalence.

The main point of this section is to show that for a 0-affine spectral DM stack, the canonical map $p : (\mathcal{X}, \mathcal{O}) \to \text{Spec} \Gamma(\mathcal{X}, \mathcal{O})$ induces an isomorphism $p^* : \text{Br}(\text{Spec} \Gamma(\mathcal{X}, \mathcal{O})) \cong \text{Br}(\mathcal{X}, \mathcal{O})$.

**Theorem 5.4.** If $(\mathcal{X}, \mathcal{O})$ is a 0-affine nonconnective spectral DM stack with $R = \Gamma(\mathcal{X}, \mathcal{O})$ and $p : (\mathcal{X}, \mathcal{O}) \to \text{Spec} R$, then $p^* : \text{Br}(R) \to \text{Br}(\mathcal{X}, \mathcal{O})$ is an isomorphism.

**Proof.** By hypothesis, the functors $p^* : \text{Mod}_R \to \text{QCoh}(\mathcal{X}, \mathcal{O})$ and $p_* : \text{QCoh}(\mathcal{X}, \mathcal{O}) \to \text{Mod}_R$ are symmetric monoidal adjoint equivalences. The functor $p^*$ preserves Azumaya algebras. It is enough to show that $p_*$ preserves Azumaya algebras. The condition that for an $\mathcal{O}$-algebra $\mathcal{A}$, we have $\mathcal{A}^{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{A} \simeq \text{End}(\mathcal{A})$ is preserved by $p_*$ since it is symmetric monoidal and hence also preserves internal mapping objects. We must see that if $\mathcal{A}$ is a perfect local generator of $\text{QCoh}(\mathcal{X}, \mathcal{O})$, then $p_* \mathcal{A}$ is a compact generator of $\text{Mod}_R$. However, since $\mathcal{O}$ is compact by the definition of 0-affineness, it follows that every perfect object is compact. In particular, $\mathcal{A}$ is compact in $\text{QCoh}(\mathcal{X}, \mathcal{O})$, and hence, $p_* \mathcal{A}$ is compact in $\text{Mod}_R$. Now, we need to see that $p_* \mathcal{A}$ generates $\text{Mod}_R$. But, if $M \in \text{Mod}_R$ is such that $\text{Map}_R(p_* \mathcal{A}, M) \simeq 0$, then $\text{Map}_R(\mathcal{A}, p^* M) \simeq 0$ and hence $p^* M \simeq 0$ (since $\mathcal{A}$ is a perfect local generator). But $M \simeq p_*(p^* M)$ so finally $M \simeq 0$. Thus, $p_* \mathcal{A}$ is a compact generator. □

**Corollary 5.5.** If $(\mathcal{X}, \mathcal{O})$ is a 0-affine nonconnective spectral DM stack with $R = \Gamma(\mathcal{X}, \mathcal{O})$, then the isomorphism $\text{Br}(R) \cong \text{Br}(\mathcal{X}, \mathcal{O})$ restricts to an injection $\text{LBr}(R) \subseteq \text{LBr}(\mathcal{X}, \mathcal{O})$.

**Remark 5.6.** The proof of Theorem 5.4 uses Azumaya algebras and does not say anything about cohomological Brauer classes.

Suppose that $(\mathcal{X}, \mathcal{O})$ is a quasi-affine nonconnective spectral scheme. Thus, $(\mathcal{X}, \mathcal{O})$ is a quasi-compact open inside Spec $S$ for some commutative ring spectrum $S$. Let $R = \Gamma(\mathcal{X}, \mathcal{O})$. Then, $(\mathcal{X}, \mathcal{O})$ is 0-affine by [48, Proposition 2.4.1.4], so we see that $\text{Br}(\mathcal{X}, \mathcal{O}) \cong \text{Br}(R)$. Moreover, in this case, we have $\text{Br}(\mathcal{X}, \mathcal{O}) \cong \text{Br}((\mathcal{X}, \mathcal{O}))$ by Theorem 4.17, which applies because $(\mathcal{X}, \mathcal{O})$ has a finite cover by affine schemes. In the next example, we use this to completely compute the Brauer group of a particular nonconnective $E_{\infty}$-ring.

**Example 5.7.** Let $(\mathcal{X}, \mathcal{O}_0)$ be the classical quasi-affine scheme given by the complement of 0 inside the affine space $\mathbb{A}^4_k = \text{Spec} k[x_1, x_2, x_3, x_4]$ where $k$ is some algebraically closed field. Let $\mathcal{O} = \mathcal{O}_0[S^1] = \mathcal{O}_0 \otimes S^S S^1$ be the sheaf of $E_{\infty}$-rings on $\mathcal{X}$ given by $S^1$-chains on $\mathcal{O}_0$. Thus, $\pi_0 \mathcal{O} \cong \mathcal{O}_0$, $\pi_1 \mathcal{O} \cong \mathcal{O}_0$, and all other homotopy sheaves vanish. Since $(\mathcal{X}, \mathcal{O}_0)$ is normal, $H^1(\mathcal{X}, \mathcal{O}) = 0$. By purity for the Brauer group, $H^2(\mathcal{X}, G_m) \cong H^2(\mathbb{A}^4_k, G_m) = 0$ [17]. Since $\mathcal{O}$ is connective, [3] gives that $\text{Br}(\mathcal{X}, \mathcal{O}) \cong \text{LBr}(\mathcal{X}, \mathcal{O})$ (cf. Proposition 2.25). Thus, the only contribution to $\text{Br}(\mathcal{X}, \mathcal{O})$ in the descent spectral sequence in Construction 4.8 comes from $H^3(\mathcal{X}, \pi_1 \mathcal{O}) \cong H^3(\mathcal{X}, \mathcal{O}_0) \cong$
All differentials out must vanish and the only thing that can hit this is $H^1(\mathcal{X}, \mathcal{G}_m) = \text{Pic}(\mathbb{A}^4_k)$, which vanishes as $\text{Pic}(\mathbb{A}^4_k) = 0$ and every line bundle extends (cf., e.g., the argument after (5.6) in [4]). Thus, with $R = \Gamma(\mathcal{O}_\mathcal{X})$, we obtain

$$\text{Br}(R) \cong \text{Br}(\mathcal{X}, \mathcal{O}) \cong \text{Br}^t(\mathcal{X}, \mathcal{O}) \cong k[x_1^{-1}, x_2^{-1}, x_3^{-1}, x_4^{-1}] \cdot (x_1 \cdots x_4)^{-1}.$$ 

Note that the descent sequence computing $\pi_* R$ degenerates so that

$$\pi_n R \cong \begin{cases} 
  k[x_1, x_2, x_3, x_4] & \text{if } n = 0, 1, \\
  k[x_1^{-1}, x_2^{-1}, x_3^{-1}, x_4^{-1}] \cdot (x_1 \cdots x_4)^{-1} & \text{if } n = -3, -2, \text{ and } \\
  0 & \text{otherwise.}
\end{cases}$$

Our computation gives examples of Brauer classes on a commutative ring spectrum $R$ which are not killed by an étale cover. To see this, pick $\alpha \in \text{Br}(R)$ and suppose that $\alpha$ is killed by an étale cover $R \to S$. Then, $S \otimes_R \mathcal{O}$ defines a new quasi-affine connective spectral DM stack $(\mathcal{Y}, \mathcal{O})$. (The underlying $\infty$-topos is naturally equivalent to $\mathcal{X} \times_{\mathbb{A}^4_k} \text{Spec} \pi_0 S$.) By quasi-affineness, it follows that $\alpha$ restricts to 0 on $(\mathcal{Y}, \mathcal{O})$. However, the induced map on the Brauer group is

$$\text{Br}(\mathcal{X}, \mathcal{O}) \cong H^3(\mathcal{X}, \mathcal{O}_0) \to H^3(\mathcal{Y}, \mathcal{O}_0) \cong \text{Br}(\mathcal{Y}, \mathcal{O}).$$

This map is equivalent to

$$k[y_1, y_2, y_3, y_4] \to k[y_1, y_2, y_3, y_4] \otimes_{k[x_1, x_2, x_3, x_4]} \pi_0 S,$$

which is injective as $R \to S$ is faithfully flat. Thus, $\alpha = 0$ and so no nonzero class in the Brauer group can be killed by an étale cover.

### 6 THE PICARD SHEAF OF TMF

To compute the local Brauer group of TMF, it is necessary to first compute the Picard sheaf of TMF, which we will attack in this section. Our key tool is a sheafy version of the Picard spectral sequence used in [51], which we will introduce next.

Let $(\mathcal{M}, \mathcal{O})$ be the derived moduli stack of elliptic curves, where $\mathcal{O}$ denotes the Goerss–Hopkins–Miller–Lurie sheaf of $E_{\infty}$-ring spectra. By Proposition 6.6, the descent spectral sequence identifies $\pi_0 \text{TMF}$ with $H^0(\mathcal{M}, \pi_0 \mathcal{O})$ and one computes the latter to be $\mathbb{Z}[j]$. Thus, the underlying classical morphism of $(\mathcal{M}, \mathcal{O}) \to \text{Spec} \text{TMF}$ is the map $j : \mathcal{M} \to \mathbb{A}^1 = \text{Spec} \mathbb{Z}[j]$; we will denote $(\mathcal{M}, \mathcal{O}) \to \text{Spec} \text{TMF}$ by $j$ as well.

For every étale map $f : \text{Spec} R \to \mathbb{A}^1$, we obtain an induced sheaf of $E_{\infty}$-ring spectra $\mathcal{O}_R$ on the base change $\mathcal{M}_R = \mathcal{M} \times_{\mathbb{A}^1} \text{Spec} R$. Let $\text{Pic}_{\mathcal{O}_R}$ denote the Picard sheaf corresponding to $\mathcal{O}_R$ on $\mathcal{M}_R$ (with subscript left out if $\text{Spec} R = \mathbb{A}^1$). We obtain a Picard spectral sequence $H^t(\mathcal{M}_R; \pi_t \text{Pic}_{\mathcal{O}_R}) \Rightarrow$
Sheafification thus yields a spectral sequence

\[ E_2^{s,t} = R^s j_* \pi_t \mathcal{P}_{\mathcal{O}} \Rightarrow \pi_{t-s} \mathcal{P}_{\mathcal{O}} \cong \pi_{t-s} j_* \mathcal{P}_{\mathcal{O}} \] (6.1)

in the abelian category of étale sheaves of abelian groups on \( \text{Spec} \mathbb{Z}[j] = \mathbb{A}^1 \). We note that \( \mathcal{P}_{\mathcal{M}}(\mathcal{O}_R) \cong \mathcal{P}_{\text{TMF}_R} \) where \( \text{TMF}_R \) is the étale extension of \( \text{TMF} \) realizing \( f \). Indeed: the natural map \( \text{TMF}_R \to \mathcal{O}_R(\mathcal{M}) \cong (\mathcal{O} \otimes_{\text{TMF}} \text{TMF}_R)(\mathcal{M}) \) is an equivalence since taking global sections and \( \mathcal{O} \otimes \text{TMF} \) are inverse equivalences between \( \text{QCoh}(\mathcal{M}, \mathcal{O}) \) and \( \text{Mod}_{\text{TMF}} \) by Corollary 5.3; thus the equivalence of Picard spaces. Moreover, \( (\mathcal{M}, \mathcal{O}) \) is 0-affine by Theorem 5.2 and thus \( \text{QCoh}(\mathcal{M}, \mathcal{O}) \cong \text{Mod}_{\text{TMF}_R} \). It follows that

\[ j_* \mathcal{P}_{\mathcal{O}_{\mathcal{M}}} \cong \mathcal{P}_{\text{TMF}_\mathcal{O}} \] (6.2)

as sheaves of grouplike \( \mathbb{E}_\infty \)-spaces on \( \text{Spec} \mathbb{Z}[j] \).

Warning 6.3. In contrast to the descent spectral sequence for \( \pi_* \text{TMF} \), the Picard spectral sequence will in general not be \( \mathbb{Z}[j] \)-linear even in the range where its \( E_2 \)-term agrees with a shift of the descent spectral sequence (i.e., for \( t \geq 2 \)). We do, however, have \( \mathbb{Z}[j] \)-linearity in the range specified by Proposition 6.7 below. This should be seen in light of (a sheafy analog of) [51, Corollary 5.2.3].

Remark 6.4. Alternatively, the sheafy Picard spectral sequence can be constructed as the relative descent spectral sequence for \( j_* \mathcal{P}_{\mathcal{O}} \), that is, the spectral sequence associated to applying (sheafy) \( \pi_* \) to the tower \( (j_* \tau_{\leq n} \mathcal{P}_{\mathcal{O}})_{n \in \mathbb{Z}} \). Indeed, the presheaf of Picard spectral sequences considered above is obtained by applying presheaf homotopy groups \( \pi_*^{\text{pre}} \) to the tower \( (j_* \tau_{\leq n} \mathcal{P}_{\mathcal{O}})_{n \in \mathbb{Z}} \), and thus, its sheafification agrees with the relative descent spectral sequence.

We will not compute the whole spectral sequence (6.1), but obtain the following result about the 0-stem, which will be crucial to our results about the local Brauer group.

Theorem 6.5. The spectral sequence (6.1) induces a complete decreasing filtration \( F^* \pi_0 j_* \mathcal{P}_{\mathcal{O}_{\mathcal{M}}} \) on \( \pi_0 j_* \mathcal{P}_{\mathcal{O}_{\mathcal{M}}} \) with

\[
\begin{align*}
(0) & \quad \text{gr}^0 \pi_0 j_* \mathcal{P}_{\mathcal{O}_{\mathcal{M}}} \cong \mathbb{Z}/2, \\
(1) & \quad \text{gr}^1 \pi_0 j_* \mathcal{P}_{\mathcal{O}_{\mathcal{M}}} \cong R^1 j_* \mathbb{G}_m, \text{ which sits in an exact sequence} \\
0 & \to (i_0)_* \mathbb{Z}/3 \oplus (i_{1728})_* \mathbb{Z}/2 \to R^1 j_* \mathbb{G}_m \to \mathbb{Z}/2 \to 0
\end{align*}
\]

as established in Proposition 6.9,
TABLE 2 An assortment of sheaves on Spec \( \mathbb{Z}[j] \).

| Symbol | Sheaf Structure Sheaf | Units in \( \mathcal{O} \) | \( \mathcal{O}/(2, j) \) | \( \mathcal{O}/2 \) | \( i_{1728,*} \mathcal{O}/2 \) | \( k_* \mathcal{O}/(3, j) \) | \( i_{0,*} \mathcal{O}/3 \) |
|--------|-------------------------|-----------------------------|-----------------------------|-----------------------------|--------------------------------|--------------------------------|--------------------------------|

(3) \( \text{gr}^3 \pi_0 j_* \text{pic}_{\mathcal{O}, \mathcal{M}} \cong k_* v_* \mathbb{Z}/2 \), where \( k \) and \( v \) denote the inclusions \( \text{Spec} \mathbb{F}_2[j] \hookrightarrow \text{Spec} \mathbb{Z}[j] \) and \( \text{Spec} \mathbb{F}_2[j^\pm 1] \hookrightarrow \text{Spec} \mathbb{F}_2[j] \), respectively.

(5) \( \text{gr}^3 \pi_0 j_* \text{pic}_{\mathcal{O}, \mathcal{M}} \) is a sum of \( b_* \mathbb{Z}/3 \) and a subsheaf of an abelian sheaf \( \mathcal{A} \), where \( \mathcal{A} \) sits in a nontrivial extension

\[
0 \to \mathcal{O}/(2, j) \to \mathcal{A} \to a_* \mathbb{Z}/2 \to 0,
\]

\( a \) is the closed inclusion of \( \text{Spec} \mathbb{F}_2 \) into \( \text{Spec} \mathbb{Z}[j] \) at \( j = 2 = 0 \) and \( b \) is the closed inclusion of \( \text{Spec} \mathbb{F}_3 \) into \( \text{Spec} \mathbb{Z}[j] \) at \( j = 3 = 0 \).

(7) \( \text{gr}^7 \pi_0 j_* \text{pic}_{\mathcal{O}, \mathcal{M}} \) is a subsheaf of \( \mathcal{O}/(2, j) \);

all other graded pieces vanish.

In fact, in the last two items, we describe the graded pieces as subsheaves of what we see on the \( E_6 \)-page, but there are (at most) two more potential differentials originating from these spots.

The rest of this section will be devoted to the proof of the theorem. We will use Table 2 for notation for sheaves on \( \text{Spec} \mathbb{Z}[j] \) appearing in the spectral sequence (6.1). Figure 3 shows the \( E_2 \)-page of the spectral sequence (6.1). The general pattern follows from the work of Mathew–Stojanoska [51] and the computations of the homotopy groups of TMF, as in Bauer [9].

To prove Theorem 6.5, we show in the next subsection that there are no contributions in filtration degrees above 7. Then, we analyze each remaining filtration in turn.

6.1 High filtrations

In this section, we use the comparison tool of [51] to narrow down the possible filtration degrees computing to \( \pi_0 j_* \text{pic}_{\mathcal{O}, \mathcal{M}} \).

We use the following facts about the large-scale structure of the additive spectral sequence

\[
E_2^{s,t} = H^s(\mathcal{M}, \pi_t \mathcal{O}, \mathcal{M}) \Rightarrow \pi_{t-s} \text{TMF},
\]

which can be read off from the charts in [9] for \( \text{tmf} \) or [42] for \( \text{Tmf} \) by inverting the discriminant modular form \( \Delta \) (or rather \( \Delta^{24} \) since only this is a permanent cycle).

Proposition 6.6.

(1) The \( E_\infty \)-page of the additive spectral sequence

(a) vanishes in columns \(-1 \) and \(-2 \) and

(b) vanishes above row 0 in column 0.

(2) The longest differential in the additive spectral sequence is \( d_{23} \).

Here, column \( n \) always refers to \( t - s = n \), that is, to the column if drawn in Adams grading. We recall the following key tool from [51].
The $E_3$-page of the sheafy spectral sequence (6.1) for Pic of the moduli stack. Above the $x + y = 1$ diagonal, only 2-primary torsion information is shown. Here, we use Adams grading, that is, $x = t - s$ and $y = s$. 
**Proposition 6.7** (Comparison Tool). For $2 \leq r \leq t - 1$, $d_r^{s,t}$ in the Pic spectral sequence “is” $d_r^{s,t-1}$ in the additive spectral sequence.

**Proof.** By [51, Comparison Tool 5.2.4], this is true for each term in the presheaf of Picard spectral sequence and is thus also true after sheafification. □

Using these results, we can indeed show that the Picard spectral sequence eventually vanishes in high enough degrees.

**Proposition 6.8.** Everything above row 7 in column 0 vanishes in the $E_\infty$-page of the Picard spectral sequence; likewise above row 30 in column $-1$. After inverting 2, the latter vanishing holds already above row 14.

**Proof.** The claim about column 0 follows from the Comparison Tool (Proposition 6.7) and the further claim that in the additive spectral sequence

$$E_2^{s,t} = H^s(M, \pi_t \mathcal{O}_M) \Rightarrow \pi_{t-s} TMF$$

every spot in the $(-1)$-column above row 7 is killed by or supports a $d_r$-differential, which is an isomorphism and satisfies $r \leq t$ for $t = x + y$ being the antidiagonal through the origin. Indeed, the corresponding differential also has to occur in the Picard spectral sequence and the isomorphism of groups becomes an isomorphism of quasi-coherent sheaves.

By inspection the further claim is true up to row 23 (on the $E_5$-page, there is only one class in column $-1$ between rows 7 and 24, namely, in row 19 and this is killed by a $d_9$). As noted in Proposition 6.6, the longest possible differential is a $d_{23}$ and the $E_\infty$-term vanishes; thus, everything above row 23 is killed by or supports a differential, which is at most a $d_{23}$. Moreover, by inspection, nothing in the additive spectral sequence in column 0 below row 23 supports a differential killing a class above row 23 in column $-1$.

The proof for column $-1$ of the Picard spectral sequence is analogous. □

The Proposition 6.8 implies that to prove Theorem 6.5, it is enough to analyze $gr^n \pi_0 j_* \text{pic}_{\mathcal{O}_M}$ for $0 \leq n \leq 7$.

### 6.2 Row 0

Since the geometric fibers of $j : \mathcal{M} \to \text{Spec } \mathbb{Z}[j]$ are connected and $\pi_0 \text{pic}_{\mathcal{O}_M} \cong \mathbb{Z}/2$, we have $R^0 j_* \mathbb{Z}/2 \cong \mathbb{Z}/2$. This term does not support any differentials since TMF[1] is a global section of the Picard sheaf that restricts to a generator of $\mathbb{Z}/2$ everywhere; this proves part (0) of Theorem 6.5.

### 6.3 Row 1 and the algebraic Picard sheaf

The next term to understand is $R^1 j_* \mathbb{G}_m$, which appears on the $E_2$-page at $(s, t) = (1, 1)$. This calculation is done on the classical moduli stack. The sheaf $R^1 j_* \mathbb{G}_m$ is the sheafification of the presheaf that sends every étale $U \to \text{Spec } \mathbb{Z}[j]$ to $\text{Pic}(\mathcal{M} \times_{\text{Spec } \mathbb{Z}[j]} U)$. Thus, our next lemma can be seen as
a sheafy analog of the classical computation that \( \text{Pic}(\mathcal{M}) \cong \mathbb{Z}/12 \) (see [27]), where a generator is given by the Hodge bundle \( \lambda \) that arises as the pushforward of the sheaf of differentials of the universal elliptic curve. We will indeed use the stronger result from [27] that the same isomorphism holds over any reduced and normal base ring \( R \) with vanishing Picard group. Moreover, we will use that \( \text{Pic}(\mathbb{A}^1_R) \cong \text{Pic}(R) \) for any regular noetherian \( R \), where \( \mathbb{A}^1_R = \mathbb{A}^1 \times \text{Spec} R \); this follows, for example, from the \( \mathbb{A}^1 \)-invariance of the divisor class group as in [36, Proposition 6.6, Corollary 6.16].

**Proposition 6.9.** Denote by \( i_t : \text{Spec} \mathbb{Z} \to \text{Spec} \mathbb{Z}[j] \) the inclusion corresponding to the value \( t \) of the function \( j \) on \( \text{Spec} \mathbb{Z}[j] \) and by \( u_t \) the inclusion of its complement.

The morphism \( \text{Pic}(\mathcal{M}) \cong \mathbb{Z}/12 \to R^1q_*\mathbb{G}_m \) is surjective with kernel \( (u_0)_*\mathbb{Z}/3 \oplus (u_{1728})_*\mathbb{Z}/2 \). Thus, \( R^1j_*\mathbb{G}_m \) sits in the extension

\[
0 \to (i_0)_*\mathbb{Z}/3 \oplus (i_{1728})_*\mathbb{Z}/2 \to R^1j_*\mathbb{G}_m \to \mathbb{Z}/2 \to 0
\]

that is pushed forward from the nontrivial extension of \( \mathbb{Z}/3 \oplus \mathbb{Z}/2 \) and \( \mathbb{Z}/2 \) of constant sheaves along the unit map \( \mathbb{Z}/3 \oplus \mathbb{Z}/2 \to (i_0)_*\mathbb{Z}/3 \oplus (i_{1728})_*\mathbb{Z}/2 \).

**Proof.** We will explain first why it suffices to show the surjectivity of \( \mathbb{Z}/12 \to R^1q_*\mathbb{G}_m \) and identify its kernel. Note that there is an exact sequence

\[
0 \to (u_t)_*u_t^*\mathcal{F} \to \mathcal{F} \to (i_t)_*i_t^*\mathcal{F} \to 0
\]

for any \( t \) and any étale sheaf \( \mathcal{F} \) (see, e.g., [54, Remark II.3.13]). Thus, we obtain the claimed extension from the proposition by quotienting the first two terms of the exact sequence

\[
0 \to \mathbb{Z}/3 \oplus \mathbb{Z}/2 \to \mathbb{Z}/12 \to \mathbb{Z}/2 \to 0
\]

by \( (u_0)_*\mathbb{Z}/3 \oplus (u_{1728})_*\mathbb{Z}/2 \) and using the snake lemma if indeed

\[
0 \to (u_0)_*\mathbb{Z}/3 \oplus (u_{1728})_*\mathbb{Z}/2 \to \mathbb{Z}/12 \to R^1j_*\mathbb{G}_m \to 0
\]

is exact. This exactness can be checked on the level of stalks, which is the content of the rest of the argument.

Let \( \overline{x} : \text{Spec} k \to \mathbb{A}^1 \) be a geometric point (corresponding to some \( j \in k \)) and \( x : \text{Spec} k \to \mathcal{M} \) its unique lift. We will show that \( \mathbb{Z}/12 \to (R^1j_*\mathbb{G}_m)_\overline{x} \) is surjective with the prescribed kernel. One can deduce from [1, Lemma 2.2.3] that the base change of \( \mathcal{M} \) to the étale stalk of \( \mathbb{A}^1 \) at \( \overline{x} \) is equivalent to the quotient stack \( \text{[Spec} S/\text{Aut}(x)] \), where \( S \) is strictly Henselian with residue field \( k \) and \( \text{Aut}(x) \) is acting trivially on \( k \) (cf. [53, Proposition 8]). We can compute the stalk \( (R^1j_*\mathbb{G}_m)_\overline{x} \) as \( \text{Pic}([\text{Spec} S/\text{Aut}(x)]) \cong H^1(\text{Aut}(x); \mathbb{G}_m(S)) \). For the values of \( \text{Aut}(x) \), we refer to [65, Section III.10, Proposition A.1.2]. We will proceed with a case distinction based on \( j \) and the characteristic of \( k \).

**Case 1:** \( j \neq 0, 1728 \):

If \( j \) is neither 0 nor 1728 in \( k \), we have \( \text{Aut}(x) \cong C_2 \), generated by \( [−1] \). As the \([−1]\)-automorphism is defined on all elliptic curves, \( [\text{Spec} S/\text{Aut}(x)] \) \( \cong BC_{2,S} \), that is, the \( C_2 \)-action on \( S \) is trivial (cf. [64, Lemma 3.2]). Hence, \( (R^1j_*\mathbb{G}_m)_\overline{x} \cong H^1(C_2; \mathbb{G}_m(S)) \cong \mu_2(S) = \mathbb{Z}/2 \).

**Case 2:** \( \text{char}(k) \neq 2, 3 \) and \( j = 0 \) or 1728:
In general, if $k$ is of characteristic $p \geq 0$, the group $G_m(S) \left( \frac{1}{p} \right)$ is divisible (where $\left( \frac{1}{p} \right)$ is understood not to have any effect). Using the structure theory of divisible abelian groups and [66, Tag 06RR], one can show that $G_m(S) \left( \frac{1}{p} \right)$ decomposes into a $\mathbb{Q}$-vector space and a torsion group, which maps isomorphically to $G_m(k) \left( \frac{1}{p} \right) \cong \mathbb{Q}/\mathbb{Z} \left( \frac{1}{p} \right)$ (cf. the proof of [53, Lemma 9]). We obtain

$$H^1(\text{Aut}(x); G_m(S)) \left( \frac{1}{p} \right) \cong H^1(\text{Aut}(x); \mathbb{Q}/\mathbb{Z} \left( \frac{1}{p} \right)) \cong \text{Hom}(\text{Aut}(x); \mathbb{Q}/\mathbb{Z} \left( \frac{1}{p} \right)).$$

If $k$ is of characteristic not 2 or 3, we have $\text{Aut}(x) \cong \mathbb{Z}/4$ if $j = 1728$ and $\text{Aut}(x) \cong \mathbb{Z}/6$ if $j = 0$, which implies that the corresponding stalks of $R^1j_*G_m$ are the Pontryagin duals of $\mathbb{Z}/4$ and $\mathbb{Z}/6$, that is, isomorphic to $\mathbb{Z}/4$ and $\mathbb{Z}/6$ as well. (Note that in these cases, $H^1(\text{Aut}(x); G_m(S))$ is 12-torsion, so inverting $p$ changes nothing.)

Concretely, the map $\mathbb{Z}/12 \cong \text{Pic}(\mathcal{M}) \to \text{Hom}(\text{Aut}(x); \mathbb{Q}/\mathbb{Z} \left( \frac{1}{p} \right))$ sends a line bundle $\mathcal{L}$ to the action of $\text{Aut}(x)$ on $\mathcal{L}_x$ by the roots of unity $\mathbb{Q}/\mathbb{Z} \left( \frac{1}{p} \right) \cong \mu_\infty \subset G_m(k)$. By the proof of [65, Theorem III.10.1], in our case, a generator of $\text{Aut}(x)$ acts by a fourth respectively a sixth root of unity on the invariant differential and thus on $\lambda_x$ (for $\lambda$ the standard generator of $\text{Pic}(\mathcal{M})$ as above). Thus, summarizing, we see that the map $\overline{\varphi} : \mathbb{Z}/12 \to (R^1j_*G_m)_{\overline{\mathbb{F}}}$ is surjective with the prescribed kernel unless $\text{char}(k) = 2, 3$ and $\overline{x}$ corresponds to $j = 0 \equiv 1728$. In particular, we see that $\varphi : \mathbb{Z}/12 \to R^1q_*G_m$ factors through $\overline{\mathbb{F}} = (\mathbb{Z}/12)/(\mu_0)\mathbb{Z}/3 \oplus (\mu_{1728})\mathbb{Z}/2$.

**Case 3:** $\text{char}(k) = 2$ or $3$ and $j = 0 \equiv 1728$:

From now on, let $\overline{x} : k \to \mathbb{A}^1$ be a geometric point with $\text{char}(k) = p$ for $p = 2, 3$ corresponding to $j = 0$. For a base ring $R$, denote by $\mathcal{M}_R$ the base change $\mathcal{M} \times \text{Spec} R$. We will show that $\varphi_{\overline{x}} : \mathbb{Z}/12 \to (R^1j_*G_m)_{\overline{x}}$ is an isomorphism by comparison with the known computation of the Picard group of $\text{Pic}(\mathcal{M}_R)$ for certain $R$. To that purpose, we will use the Leray spectral sequence

$$E_2^{s,t} = H^s(\mathbb{A}^1_R; R^tj_*^R G_m) \Rightarrow H^{s+t}(\mathcal{M}_R; G_m)$$

for the map $j^R : \mathcal{M}_R \to \mathbb{A}^1_R$. Let us display the part relevant for the computation of $\text{Pic}$.

$$
\begin{array}{ccc}
G_m(\mathbb{A}^1_R) & \to & \text{Pic}(\mathbb{A}^1_R) \\
G_m(\mathbb{A}^1_R) & \to & \text{Pic}(\mathbb{A}^1_R) \\
H^1(\mathbb{A}^1_R; R^1j_*^RG_m) & \to & H^2(\mathcal{M}_R; G_m) \\
H^0(\mathbb{A}^1_R; R^1j_*^RG_m) & \to & H^1(\mathbb{A}^1_R, R^1j_*^RG_m).
\end{array}
$$

Denoting by $R$ the strict Henselization of the image of $\overline{x}$ in $\text{Spec} \overline{\mathbb{F}}$, the spectral sequence implies that the map $\Phi_R : \mathbb{Z}/12 \cong \text{Pic}(\mathcal{M}_R) \to H^0(\mathbb{A}^1_R; R^1j_*^RG_m)$ is an isomorphism. Here, we use that as $\mathbb{Z}$ is regular noetherian, $R$ is so as well by [66, Tags 06LJ, 06LN], and thus, $\text{Pic}(\mathbb{A}^1_R) \cong \text{Pic}(R) = 0$ and

$$H^2(\mathbb{A}^1_R, G_m) \cong H^2(R, G_m) \cong H^2(\text{Spec} \mathbb{F}_p, G_m) = 0$$

by Theorem 2.5 and Theorem 2.7. The same argument shows that the map $\Phi_{\overline{x}_p} : \mathbb{Z}/12 \cong \text{Pic}(\mathcal{M}_{\overline{x}_p}) \to H^0(\mathbb{A}^1_{\overline{x}_p}; R^1j_*^F G_m)$ is an isomorphism.
If \( \varphi \) were not injective, there would be some \([m] \in \mathbb{Z}/12\) (namely, [4] or [6]) such that \( \varphi([m]) \) is zero in every étale stalk in characteristic \( p \), and hence, \( \Phi_{\overline{y}^p} \) would not be injective either. We see that \( \varphi_{\overline{y}} \) is thus indeed injective and thus \( \mathcal{F} \) agrees with the image of \( \varphi \).

As above we use the notation \( \mathcal{F} = (\mathbb{Z}/12)/(u_0)\mathbb{Z}/3 \oplus (u_{1728})\mathbb{Z}/2 \). Denote the cokernel of \( \mathbb{Z}/12 \to \mathcal{F} \to R^1j_*\mathbb{G}_m \) by \( \mathcal{C} \). Its base change \( \mathcal{C}_R \) agrees with the cokernel of \( \mathcal{F}_R \to R^1j^*_R\mathbb{G}_m \) and it suffices to show its vanishing. By the arguments of the first paragraph, we have a short exact sequence

\[
0 \to (i_0)_*\mathbb{Z}/3 \oplus (i_{1728})_*\mathbb{Z}/2 \to \mathcal{F} \to \mathbb{Z}/2 \to 0.
\]

Moreover, Theorem 2.9 implies \( H^1(\mathbb{A}^1_R; (i_0)_*\mathbb{Z}/3 \oplus (i_{1728})_*\mathbb{Z}/2) \cong H^1(R; \mathbb{Z}/3 \oplus \mathbb{Z}/2) = 0 \) since the étale cohomology of anything on \( R \) vanishes. Thus, \( \mathbb{Z}/12 \to H^0(\mathbb{A}^1_R; \mathcal{F}_R) \) is an isomorphism.

Moreover, \( H^1(\mathbb{A}^1_R; \mathbb{Z}/2) \cong H^1(\mathbb{A}^1_R; \mu_2) \) sits in a short exact sequence with \( H^1(\mathbb{A}^1_R; \mathbb{G}_m)/2 = 0 \) and \( H^2(\mathbb{A}^1_R; \mathbb{G}_m)[2] = 0 \) and thus has to vanish as well. We conclude that \( H^1(\mathbb{A}^1_R, \mathcal{F}_R) = 0 \).

Summarizing, we have an exact sequence

\[
0 \to H^0(\mathbb{A}^1_R; \mathcal{F}_R) \to H^0(\mathbb{A}^1_R; R^1j^*_R\mathbb{G}_m) \to H^0(\mathbb{A}^1_R, \mathcal{C}_R) \to 0 = H^1(\mathbb{A}^1_R, \mathcal{F}_R).
\]

We have seen above that the natural morphisms from \( \mathbb{Z}/12 \) to the first two nontrivial groups are isomorphisms, and thus, the map between them is an isomorphism. Thus, \( H^0(\mathbb{A}^1_R, \mathcal{C}_R) \) vanishes. As \( \mathcal{C}_R \) is supported at \( \overline{x} \), we see that its stalk at \( \overline{x} \) vanishes, and thus, that \( \varphi_{\overline{y}} \) is also surjective. \( \square \)

We claim that there are no differentials out of \( E^{1,1}_{1,2} = R^1j_*\mathbb{G}_m \). Indeed, by the preceding proposition

\[
\text{Pic}(\mathcal{M}, \pi_0\mathcal{O}_{\mathcal{M}}) \cong \mathbb{Z}/12 \to R^1j_*\mathbb{G}_m
\]

is a surjective map of sheaves on \( \mathbb{A}^1 = \text{Spec} \mathbb{Z}[j] \). Thus, as long as the classes of \( \mathbb{Z}/12 \) lift to invertible sheaves on the derived moduli stack, surjectivity of the map means there cannot be differentials. But, this \( \mathbb{Z}/12 \) is generated by TMF[2], which gives part (1) of Theorem 6.5.

### 6.4 3-Torsion in row 5

For higher filtrations, it is necessary to compute differentials. Differentials \( d^r_s \) in the sheafified Pic spectral sequence where \( r \leq t - 1 \) (i.e., where the “length” of the differential is smaller than the coordinate \( t = x + y \) of the antidiagonal through the origin) can be directly read off the descent spectral sequence computing \( \pi_*\text{TMF} \) by Proposition 6.7. We will use this fact without further comment.

For the rest of the analysis, we will work separately with 2 and 3 inverted, to analyze the 3 and 2-torsion, respectively. Figure 4 depicts the potential 3-torsion in filtrations two and above; combined with Proposition 6.8, we conclude that the only possible contribution to 3-torsion in \( \pi_0j_*\mathcal{P}_{\mathcal{O},\mathcal{H}} \) is the kernel of the \( d_9 \) differential on \( E^{5,5}_g \). We will implicitly invert 2 throughout this subsection.
FIGURE 4  The $E_3$-page of the sheafy spectral sequence (6.1) for Pic of the moduli stack. Above the $x + y = 1$ diagonal, only 3-primary torsion information is shown.
Lemma 6.10. The differential

\[ d_9 : R^5 j_* \pi_5 \mathcal{O}_\mathcal{M} \cong R^5 j_* \pi_4 \mathcal{O}_\mathcal{M} \cong \mathcal{O}/(3, j) \to R^{14} j_* \pi_{13} \mathcal{O}_\mathcal{M} \cong R^{14} j_* \pi_{12} \mathcal{O}_\mathcal{M} \cong \mathcal{O}/(3, j). \]

is surjective and the kernel is \( b_* \mathbb{Z}/3 \), where \( b \) is the closed inclusion of \( \text{Spec} \mathbb{F}_3 \) into \( \text{Spec} \mathbb{Z}[j] \) at \( j = 3 = 0 \).

Proof. The differential is the first possible outside of the exponentiable range from Proposition 6.7. In the descent spectral sequence for \( \pi_* \text{TMF} \), the corresponding differential is an isomorphism (see [51, (8-4)]). In contrast, [37, Theorem 7.1] implies that we can write the differential \( d_9 : \mathcal{O}/(3, j) \to \mathcal{O}/(3, j) \) in the Picard spectral sequence as

\[ x \mapsto x + \xi \beta \mathcal{P}^2(x), \]

where \( \xi \) is a unit in \( \mathbb{F}_3 \) and \( \beta \mathcal{P}^2 \) is certain power operation on \( \mathbb{E}_\infty \)-rings in which \( 2 \) is invertible.† Moreover, \( x \mapsto \xi \beta \mathcal{P}^2(x) \) is Frobenius-semilinear in the sense that \( z x \mapsto \xi \beta \mathcal{P}^2(zx) = z^3 \xi \beta \mathcal{P}^2(x) \).

We know that our \( d_9 \) must be zero on global sections by [51, Sec. 8.1] (as else Pic(TMF)\(_{(3)} \) could have at most three elements). Thus, \( 1 + \xi \beta \mathcal{P}^2(1) = d_9(1) \) is zero on global sections and thus also everywhere, and hence \( \xi \beta \mathcal{P}^2(1) = -1 \). By Frobenius-semilinearity, we see that \( d_9(z) = z + \xi \beta \mathcal{P}^2(z \cdot 1) = z - z^3 \). It follows from Artin–Schreier theory that this differential is a surjective map of étale sheaves and that the kernel is \( b_* \mathbb{Z}/3 \) (cf. [54, Example 2.18c]).

As Figure 4 proves, the lemma shows that Part (5) of Theorem 6.5 holds with \( 2 \) inverted and all the other graded pieces vanish with \( 2 \) inverted. Thus, it remains to analyze the \( 2 \)-torsion and we will implicitly work \( 2 \)-locally everywhere. We will compute two further differentials affecting the zeroth column of the Picard spectral sequence and then give an outlook on what remains to be done to compute all remaining differentials.

6.5 | Row 3

There is a \( d_3 \)-differential,

\[ d_3 : R^3 j_* \pi_5 \mathcal{O}_\mathcal{M} \cong R^3 j_* \pi_2 \mathcal{O}_\mathcal{M} \cong \mathcal{O}/2 \to R^6 j_* \pi_5 \mathcal{O}_\mathcal{M} \cong R^6 j_* \pi_4 \mathcal{O}_\mathcal{M} \cong \mathcal{O}/2. \]

This differential is of the form

\[ x \mapsto x + jx^2 = x(1 + jx), \]

as shown in [51, Sec. 8.2]. Recall from [54, Corollary II.3.11] that \( k_* \) for \( k : \text{Spec} \mathbb{F}_2[j] \to \text{Spec} \mathbb{Z}[j] \) induces an exact equivalence between étale sheaves on \( \mathbb{F}_2[j] \) and étale sheaves on \( \mathbb{Z}[j] \) supported at the prime \( 2 \); hence, we can work directly on the étale site of \( \mathbb{F}_2[j] \). Under this equivalence, we have \( k_* \mathcal{O} \cong \mathcal{O}/2 \). We claim that the \( d_3 : \mathcal{O} \to \mathcal{O} \) is surjective (viewed as étale sheaves on \( \text{Spec} \mathbb{F}_2[j] \)). Indeed, given any étale morphism \( \mathbb{F}_2[j] \to R \) and element \( c \in R \), the extension

† The power operation \( \beta \mathcal{P}^2 \) arises from the derived functors of symmetric powers. For details, we refer the reader to [37, Proposition 7.5] and the original [59].
\[ R \to R[x]/(jx^2 + x - c) \text{ is étale and surjective on geometric points: base-changing along any morphism } R \to K \text{ to a field, } j \text{ becomes either invertible or zero and in either case } K[x]/(jx^2 + x - c) \text{ is nonzero. Thus, Spec } R[x]/(jx^2 + x - c) \to \text{Spec } R \text{ is an étale cover and } c \text{ has per construction a preimage under } d_3 \text{ on } R[x]/(jx^2 + x - c). \]

Any nonzero element in the stalk of the kernel must be of the form \( x = \frac{1}{j} \) (since all stalks of \( \mathcal{O} \) on \( \mathbb{F}_2[j] \) are integral domains); thus, the kernel corresponds to the étale sheaf \( v! \mathbb{Z}/2 \) on \( \mathbb{F}_2[j] \) (with \( v: \text{Spec } \mathbb{F}_2[j^{\pm 1}] \to \text{Spec } \mathbb{F}_2[j] \) being the inclusion) and is \( k^* v! \mathbb{Z}/2 \) as an étale sheaf on \( \mathbb{Z}[j] \). There will be no further differentials from this spot because all possible further targets are supported at \((2, j)\).

### 6.6  |  2-Torsion in row 5

The next differential (displayed in Figure 5) is the \( d_5 \)-differential,

\[ d_5 : R^5 j_* \pi_5 \text{pic}_{\mathcal{O}_{\mathcal{M}}} \cong R^5 j_* \pi_4 \mathcal{O}_{\mathcal{M}} \cong \mathcal{O} / (4, j) \to R^{10} j_* \pi_9 \text{pic}_{\mathcal{O}_{\mathcal{M}}} \cong R^{10} j_* \pi_8 \mathcal{O}_{\mathcal{M}} \cong \mathcal{O} / (2, j), \]

which factors through a map \( \mathcal{O} / (2, j) \to \mathcal{O} / (2, j) \). This differential is just outside of the exponentiable range and is given by [51, Thm. 6.1.1]. The map \( \mathcal{O} / (2, j) \to \mathcal{O} / (2, j) \) is given by \( x \mapsto x + x^2 \). This is a surjective map of étale sheaves by Artin–Schreier theory: given \( y \in \mathcal{O} / 2 \), the extension defined by \( y = x + x^2 \) is étale. The kernel is \( a_* \mathbb{Z}/2 \), where \( a: \text{Spec } \mathbb{F}_2 \to \text{Spec } \mathbb{Z}[j] \) is the inclusion at \( 2 = j = 0 \).

### 6.7  |  Long differentials

As already established in Proposition 6.8, in column 0, everything above row 7 must be zero on the \( E_\infty \)-page. As Figure 6 and the preceding discussion show, the only remaining possible differentials are a \( d_{13} \) and \( d_{25} \) originating in row 5 and a \( d_{11} \) and a \( d_{23} \) originating in row 7. We can show the vanishing of one of these differentials.

**Lemma 6.11.** The differential

\[ d_{11} : E^{7,7}_{4} \cong E^{7,7}_{11} \cong \mathcal{O} / (2, j) \to E^{18,17}_{11} \cong \mathcal{O} / (2, j). \]

is zero.

**Proof.** We use the sequence of maps of derived stacks

\[ \mathcal{M}(3)^{ss} \to \mathcal{M}(3) \to \mathcal{M} \left[ \frac{1}{3} \right], \]

where \( \mathcal{M}(3) \) is the moduli stack of elliptic curves with full level-3-structure (see, e.g., [67]) and \( \mathcal{M}(3)^{ss} \) is the completion of \( \mathcal{M}(3) \) at the ideal \((2, j)\). We write \( j \) for the map from any of these stacks to Spec \( \mathbb{Z}[j] \). Pushing the Picard sheaves down to Spec \( \mathbb{Z}[j] \), we obtain

\[ j_* \text{pic}_{\mathcal{O}_{\mathcal{M}}} \to j_* \text{pic}_{\mathcal{O}_{\mathcal{M}(1/3)}} \to j_* \text{pic}_{\mathcal{O}_{\mathcal{M}(3)^{ss}}}. \]
FIGURE 5  The $E_5$-page of the sheafy spectral sequence (6.1) for Pic of the moduli stack. Above the $x + y = 1$ diagonal, only 2-primary torsion information is shown.
FIGURE 6  The $E_2$-page of the sheafy spectral sequence (6.1) for Pic of the moduli stack. Above the $x + y = 1$ diagonal, only 2-primary torsion information is shown.
Since $\mathcal{M}(3) \to \mathcal{M}[\frac{1}{3}]$ is a Galois cover with group $\text{GL}_2(\mathbb{Z}/3)$, the first of these maps induces an equivalence

$$\tau_{\geq 0} j_+ \text{pic}_{\theta[^{1/3}]} \to \tau_{\geq 0} \left( j_+ \text{pic}^{h\text{GL}_2(\mathbb{Z}/3)}_{\theta[^{1/3}]} \right);$$

(6.12)

this kind of Galois descent follows from the sheaf property of $\text{pic}$. The 2-local Picard spectral sequence for $\text{TMF}$ is for $t > 1$ the same as the $\text{GL}_2(\mathbb{Z}/3)$-based relative descent spectral sequence for (6.12) (cf. Remark 6.4).

Completing $\mathcal{M}(3)$ at the ideal $(2, j) \subset \mathbb{Z}[j]$ results in the formal deformations space of a supersingular elliptic curve $C$ over $\mathbb{F}_4$, which can be coordinatized as $\text{Spec} W(\mathbb{F}_4)[[u]]$. Thus, completing $\mathcal{M}$ itself at $(2, j)$ becomes identified with the stack quotient of $\text{Spec} W(\mathbb{F}_4)[[u]]$ by $\text{GL}_2(\mathbb{Z}/3)$. As source and target of the $d_{11}$-differential we care about are supported at $(2, j)$, completion at $(2, j)$ does not lose information; more precisely, the composition of $(2, j)$-completion and pushing forward along $\text{Spf} \mathbb{Z}_2[[j]]$ to $\text{Spec} \mathbb{Z}[j]$ is an isomorphism at the relevant spots in the sheafy Picard spectral sequence. Note that the étale topos of $\text{Spf} \mathbb{Z}_2[[j]]$ agrees with that of $\mathbb{F}_2$. Thus, the $(2, j)$-completed sheafy Picard spectral sequence is the sheafification of the collection of Picard spectral sequences of the higher real $K$-theories, assigning $E_{2s,t}(\hat{C}, \mathbb{F}_4 \otimes \mathbb{F}_2 k)^{h\text{GL}_2(\mathbb{Z}/3)}$ to each finite extension $\mathbb{F}_2 \subset k$.

Denote the corresponding spectral sequence for $E_{2s,t}(\hat{C}, \mathbb{F}_4 \otimes \mathbb{F}_2 k)^{h\text{H}}$ with $H \subset \text{GL}_2(\mathbb{F}_3)$ by $E^{s,t}_{*, k, H}$, where we consider $k = \mathbb{F}_{2^r}$ or $\overline{\mathbb{F}}_2$. We will deduce from [13] that the restriction map $\text{res}_{\mathcal{C}_4}^{G_{48}} : E^{s,t}_{2\mathbb{F}_2, \text{GL}_2(\mathbb{Z}/3)} \to E^{s,t}_{2\mathbb{F}_2, \mathcal{C}_4}$ is zero for $(s,t) = (7,7)$ and an isomorphism for $(18,17)$ on the $E_{4}$-page (which agrees in this range in descent and Picard spectral sequence). Hence, the same is true for all $k$ in place of $\mathbb{F}_2$ (since the restriction maps are module maps). Moreover, we have established above that in the Picard spectral sequence, $E_4 = E_{11}$ in these spots. As $d_{11} \text{res}_{\mathcal{C}_4}^{G_{48}} = \text{res}_{\mathcal{C}_4}^{G_{48}} d_{11}$, this implies the vanishing of our $d_{11}$.

To read off our claim about the restriction map from [13], we will use their notation (noting that they write $G_{48}$ for $\text{GL}_2(\mathbb{Z}/3)$). Section 2.3 of [13] implies that the generator of $E^{7,7}_{2\mathbb{F}_2, \text{GL}_2(\mathbb{Z}/3)}$ is $\Delta^{-1} \eta^3 \overline{\kappa}$. Since $\Delta$ is a $d_3$-cycle, it will suffice for our vanishing claim about the restriction to show that $\text{res}_{\mathcal{C}_4}^{G_{48}}(\eta^3 \overline{\kappa})$ is hit by a $d_3$. This restriction equals $\eta^3 \delta \xi^2$ by the Table above Section 2.3 of [13]. We have the differential $d_3(\xi) = \delta^{-1} \eta^2 \xi^2$ by [10, Proposition 2.3.1], so $d_3(\delta^2 \eta^2 \xi^2) = \delta \eta^3 \xi^2 = \text{res}_{\mathcal{C}_4}^{G_{48}}(\eta^3 \overline{\kappa})$. Now we turn to the generator of $E^{18,17}_{4\mathbb{F}_2, \text{GL}_2(\mathbb{Z}/3)}$, which is $\Delta^{-4} \kappa \overline{\kappa}$. Using [10, Lemma 2.2.4, Corollary 2.3.2], $\text{res}_{\mathcal{C}_4}^{G_{48}}(\Delta)$ acts like $\delta^3$ on a torsion class like $\text{res}_{\mathcal{C}_4}^{G_{48}}(\kappa \overline{\kappa}) = \delta^5 \nu^2 \xi^8$. Thus, $\Delta^{-4} \kappa \overline{\kappa}$ restricts to $\delta^{17} \nu^2 \xi^8$. This can be used to show that the restriction is an isomorphism on $E^{18,17}_{4\mathbb{F}_2, \text{GL}_2(\mathbb{Z}/3)}$. Moreover, the class in $E^{18,17}_{4\mathbb{F}_2, \text{GL}_2(\mathbb{Z}/3)}$ is only hit by a $d_{15}$ and its restriction by a $d_{13}$ (namely, from $(\delta^2 \nu^2)\delta^9(\delta^2 \nu \xi)$; cf. [10, Proposition 2.3.9]). In particular, in these spots, the $E_{4}$-page equals the $E_{11}$-page. 

We included this lemma not primarily for its intrinsic importance, but rather to demonstrate that the remaining computational mysteries of the sheafy Picard spectral sequence are purely $K(2)$-local phenomena and might thus potentially be resolved purely in the setting of Lubin–Tate spectra.

1 As in [13], we will use [10] for information about differentials on the $\mathcal{C}_4$-level, using that $\text{TMF}_1(5)$ becomes a Lubin–Tate theory after $K(2)$-localization.
APPLICATIONS TO PICARD GROUPS

We can use Theorem 6.5 to compute Picard groups of various spectra related to TMF.

**Example 7.1.** Using Theorem 6.5, we want to compute $\text{Pic}(\text{TMF}[c_4^{-1}])$. Noting $\pi_0\text{TMF}[c_4^{-1}] \cong \mathbb{Z}[j^{\pm 1}]$, the relevant part of the exact sequence from Proposition 2.25 is

$$0 \to H^1(\text{Spec } \mathbb{Z}[j^{\pm 1}]; G_m) \to \text{Pic}(\text{TMF}[c_4^{-1}]) \to H^0(\text{Spec } \mathbb{Z}[j^{\pm 1}], \pi_0\text{Pic}_{\text{TMF}})$$

$$\to H^2(\text{Spec } \mathbb{Z}[j^{\pm 1}]; G_m) \to \cdots.$$  

The groups

$$H^1(\text{Spec } \mathbb{Z}[j^{\pm 1}]; G_m) = \text{Pic}(\text{Spec } \mathbb{Z}[j^{\pm 1}]) \subset \text{Pic}(\text{Spec } \mathbb{Z}[j]) \cong \text{Pic}(\mathbb{Z})$$

and

$$H^2(\text{Spec } \mathbb{Z}[j^{\pm 1}]; G_m) = \text{Br}(\text{Spec } \mathbb{Z}[j^{\pm 1}])$$

vanish since $\mathbb{Z}$ is a PID and Pic is $\mathbb{A}^1$-invariant by [36, Proposition II.6.6], and by Corollary 2.6. Thus,

$$\text{Pic}(\text{TMF}[c_4^{-1}]) \to H^0(\text{Spec } \mathbb{Z}[j^{\pm 1}]; \pi_0\text{Pic}_{\text{TMF}})$$

is an isomorphism. By Theorem 6.5, the restriction of $\pi_0\text{Pic}_{\text{TMF}}$ to Spec $\mathbb{Z}[j^{\pm 1}]$ has a filtration with associated graded $\mathbb{Z}/2, \mathbb{Z}/2, (i_{1728})_*\mathbb{Z}/2$ and $k_*\mathbb{Z}/2$, where $k : \text{Spec } \mathbb{F}_2[j^{\pm 1}] \to \text{Spec } \mathbb{Z}[j^{\pm 1}]$ is the inclusion. We obtain directly that Pic (TMF $[c_4^{-1}]$) is 2-power torsion.

Let $\mathcal{O}$ be the quotient of $(\pi_0\text{Pic}_{\text{TMF}})_{(2)}$ by everything of filtration at least 2. We obtain a short exact sequence

$$0 \to k_*\mathbb{Z}/2 \to u^*\pi_0\text{Pic}_{\text{TMF}} \to u^*\mathcal{O} \to 0,$$  

(7.2)

where $u : \text{Spec } \mathbb{Z}[j^{\pm 1}] \to \text{Spec } \mathbb{Z}[j]$ is the inclusion. Applying the long exact sequence in cohomology to this short exact sequence and to the analogous one for TMF, we obtain a diagram

$$0 \longrightarrow \mathbb{Z}/8 \longrightarrow \text{Pic}(\text{TMF})_{(2)} \longrightarrow H^0(\mathbb{Z}[j]; \mathcal{O}) \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \text{Pic}(\text{TMF}[c_4^{-1}]) \longrightarrow H^0(\mathbb{Z}[j^{\pm 1}]; u^*\mathcal{O}).$$

Investigating the associated graded pieces of the Picard sheaves, we see that the first vertical map is zero. The rightmost vertical map is an injection by the four-lemma because it is an isomorphism on global sections of graded pieces. Looking at the global sections of the graded pieces, we also obtain that source and target of this map have at most 8 elements. Since Pic(TM(2)) $\cong \mathbb{Z}/64$, we see that $H^0(\mathbb{Z}[j]; \mathcal{O})$ must be $\mathbb{Z}/8$ and thus the same is true for $H^0(\mathbb{Z}[j^{\pm 1}]; u^*\mathcal{O})$. As TMF[1] is sent
to a generator of $H^0(\mathbb{Z}[j]; \mathcal{O})$, we see that $\text{TMF}[c_{-1}^{-1}][1]$ is sent to a generator of $H^0(\mathbb{Z}[j^{\pm1}]; u^* \mathcal{O})$. Moreover, $\text{TMF}[c_{-1}^{-1}][1]$ generates a group of order 8 inside $\text{Pic}(\text{TMF}[c_{-1}^{-1}])$. Thus, the lower exact sequence is split short exact and $\text{Pic}(\text{TMF}[c_{-1}^{-1}]) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8$. The extra $\mathbb{Z}/2$ provides an example of an exotic Picard group element.

The following proposition is a higher (but less explicit) analog of Corollary 3.11.

**Proposition 7.3.** Denote by $\mathcal{O}$ the quotient of $\pi_0 \mathcal{P}_{\text{pic}}(\text{TMF})$ corresponding to (0) and (1) in Theorem 6.5 and by $\mathcal{J}$ the kernel of the quotient map. Let further $R$ be an étale extension of $\mathbb{Z}$. Then, there is a short exact sequence

$$0 \to \text{Pic}(R) \to \text{Pic}(\text{TMF}_R) \to H^0(\mathbb{A}^1_R; \pi_0 \mathcal{P}_{\text{pic}}(\text{TMF})) \to 0.$$ 

If $\text{Spec } R$ is connected, the last term fits into a short exact sequence

$$0 \to H^0(\mathbb{A}^1_R; \mathcal{J}) \to H^0(\mathbb{A}^1_R; \pi_0 \mathcal{P}_{\text{pic}}(\text{TMF})) \to \mathbb{Z}/24 \to 0.$$ 

**Proof.** We begin by proving the second claim. Applying the long exact sequence of cohomology to the extension

$$0 \to \mathcal{J} \to \pi_0 \mathcal{P}_{\text{pic}}(\text{TMF}_R) \to \mathcal{O} \to 0,$$

we obtain an exact sequence

$$0 \to H^0(\mathbb{A}^1_R; \mathcal{J}) \to H^0(\mathbb{A}^1_R; \pi_0 \mathcal{P}_{\text{pic}}(\text{TMF})) \to H^0(\mathbb{A}^1_R; \mathcal{O}). \quad (7.4)$$

The composition

$$\mathbb{Z} \to \text{Pic}(\text{TMF}_R) \to H^0(\mathbb{A}^1_R; \pi_0 \mathcal{P}_{\text{pic}}(\text{TMF})) \to H^0(\mathbb{A}^1_R; \mathcal{O})$$

$$n \mapsto \text{TMF}_R[n]$$

is a surjection by comparison to TMF (similar to the preceding example) since $H^0(\mathbb{A}^1_R; \mathcal{O}) \rightarrow H^0(\mathbb{A}^1_R; \mathcal{O})$ is an isomorphism (using a comparison on associated graded pieces and the five lemma). This implies that $H^0(\mathbb{A}^1_R; \mathcal{O}) \cong \mathbb{Z}/24$ and that Equation (7.4) is short exact.

For the first claim, we recall from Proposition 2.25 the exact sequence

$$0 \to \text{Pic}(\mathbb{A}^1_R) \to \text{Pic}(\text{TMF}_R) \to H^0(\mathbb{A}^1_R; \pi_0 \mathcal{P}_{\text{pic}}(\text{TMF})) \xrightarrow{\partial_R} \text{Br}(\mathbb{A}^1_R).$$

Note that $\text{Pic}(\mathbb{A}^1_R) \cong \text{Pic}(R)$ (e.g., by [36, Proposition II.6.6]). The arguments proceeds now exactly as in Corollary 3.11, using that $\text{Br}(\mathbb{A}^1_R)$ injects into $\text{Br}(\mathbb{A}^1_R)$ by Theorem 2.5 and

$$\mathcal{J}\left(\mathbb{A}^1_{R\left[\pi^{1/5}\right]}\right) = 0. \quad \square$$
8 | THE LOCAL BRAUER GROUPS OF TMF AND \((\mathcal{M}, \mathcal{O})\)

The aim of this section is to show that local Brauer groups of TMF and the derived moduli stack \((\mathcal{M}, \mathcal{O})\) are infinitely generated and to compute them up to finite ambiguity. First, we observe the coincidences of various Brauer groups pertinent to this example.

**Proposition 8.1.** If \((\mathcal{M}, \mathcal{O})\) is the derived moduli stack of elliptic curves, then

(i) \(\text{Br}(\mathcal{M}, \mathcal{O}) \cong \text{Br}^\prime(\mathcal{M}, \mathcal{O})\),

(ii) \(\text{Br}(\mathcal{M}, \mathcal{O}) \cong \text{Br}(\text{TMF})\), and

(iii) \(\text{LBr}(\mathcal{M}, \mathcal{O}) \cong \text{LBr}^\prime(\mathcal{M}, \mathcal{O})\).

**Proof.** Parts (i) and (iii) follow from Corollary 4.19. Indeed, we can use the cover with opens \(\mathcal{M}\left[\frac{1}{2}\right] = \mathcal{M} \times \text{Spec } \mathbb{Z}\left[\frac{1}{2}\right]\) and \(\mathcal{M}\left[\frac{1}{3}\right] = \mathcal{M} \times \text{Spec } \mathbb{Z}\left[\frac{1}{3}\right]\). Condition (a) of Theorem 4.17 follows because the kernels of \(\text{QCoh}(\mathcal{M}\left[\frac{1}{2}\right]) \to \text{QCoh}(\mathcal{M}\left[\frac{1}{6}\right])\) and \(\text{QCoh}(\mathcal{M}\left[\frac{1}{3}\right]) \to \text{QCoh}(\mathcal{M}\left[\frac{1}{6}\right])\) are generated by the compact objects \(\mathcal{O}/3\) and \(\mathcal{O}/2\), respectively. Moreover, both \(\mathcal{M}\left[\frac{1}{2}\right]\) and \(\mathcal{M}\left[\frac{1}{3}\right]\) admit a finite étale cover from an affine scheme, for example, the moduli stacks \(\mathcal{M}(4)\) and \(\mathcal{M}(3)\) of elliptic curves with full level 4 and full level 3 structures, respectively. Thus, by (the proof of) Proposition 4.14, \(\alpha\)-twisted sheaves on \(\mathcal{M}\left[\frac{1}{2}\right]\) and \(\mathcal{M}\left[\frac{1}{3}\right]\) admit a local perfect generator for every Brauer class \(\alpha\). Condition (b) follows because \(\mathcal{M}\left[\frac{1}{p}\right]\) is 0-affine by Theorem 5.2, and hence, a local perfect generator is a global generator by the proof of Theorem 5.4.

Part (ii) follows from Theorem 5.4 since \((\mathcal{M}, \mathcal{O})\) is 0-affine by Corollary 5.3. \(\square\)

**Theorem 8.2.** The local Brauer group \(\text{LBr}(\mathcal{M}, \mathcal{O})\) is a torsion group. There is no \(p\)-torsion for \(p > 3\). The 3-torsion is \(\mathbb{Z}/3\). Moreover, there is a surjection \(\text{LBr}(\mathcal{M}, \mathcal{O})_{(2)} \to (\mathbb{Z}/2)^{\infty}\) with kernel of order 8.

**Proof.** By the previous proposition, we can apply the spectral sequence from Construction 4.8 for the computation of \(\text{LBr}(\mathcal{M}, \mathcal{O}) = \pi_0 \text{LBr}_0(\mathcal{M})\). Up to a onefold shift, this agrees with the Picard spectral sequence for TMF from [51]. Note that \(H^1(\mathcal{M}; \mathbb{Z}/2) = 0\) (since \(\mathcal{M}\) has no finite covers) and \(H^2(\mathcal{M}, \mathbb{G}_m) = \text{Br}(\mathcal{M}) = 0\) by [4]. The nonsheafy version of Proposition 6.8 holds by the same arguments, and thus, only terms of filtration at most 30 can survive in the Picard spectral sequence in column \((-1)\). By the results from [51], we know all differentials from the 0-column to the \((-1)\)-column of the Picard spectral sequence: up to row 30, there are only \(d_3\) and they are 2-local (cf. especially Figures 6–10 in [51]). One thus observes that the \(p\)-torsion is as stated for \(p > 3\). In the \(E_{\infty}\)-term, we have 2-locally the kernel of an unknown \(d_5\)-differential from \(\text{coker}(d_1: \mathbb{F}_2[j] \to \mathbb{F}_2[j] \oplus \mathbb{Z}/2)\) in row 6 to a \(\mathbb{Z}/2\) in row 15 (which must be abstractly isomorphic to \((\mathbb{Z}/2)^{\infty}\), as in the proof of Equation (8.5) below), and further copies of \(\mathbb{Z}/2\) in rows 10, 18, and 30, which cannot support differentials. Here, we use [51, Comparison Tool 5.2.4], both to show that possible targets of differentials vanish and to show the vanishing of a possible \(d_3\) on the class in row 10. This implies the result. \(\square\)

Next, we give a similar (but less precise) computation for \(\text{LBr}(\text{TMF})\). Later, we will compare the two calculations.
**Theorem 8.3.** The local Brauer group \( LBr(TMF) \) is a torsion group. There is no \( p \)-torsion for \( p > 3 \). The 3-torsion is \( \mathbb{Z}/3 \). Moreover, there is a split surjection \( LBr(TMF) \to (\mathbb{Z}/2)^{\infty} \) with finite kernel.

**Proof.** By Proposition 2.25 and using \( Br(\pi_0 \text{TMF}) \cong Br(\mathbb{Z}[j]) \cong Br(\mathbb{Z}) = 0 \) by Theorem 2.5 and Example 2.4, \( LBr(TMF) \) is isomorphic to the kernel of the differential

\[
d^{\text{TMF}} : H^1(\text{Spec} \mathbb{Z}[j]; \pi_0 j^* \text{pic}_{\mathcal{O}_{\text{TMF}}}) \to H^3(\text{Spec} \mathbb{Z}[j]; \mathbb{G}_m),
\]

where we use (6.2) to identify \( \pi_0 \text{Pic}_{\text{TMF}} \) with \( \pi_0 j^* \text{pic}_{\mathcal{O}_{\text{TMF}}} \). We will first partially compute the source of the differential. Using Proposition 6.9, the facts that \( H^1(\text{Spec} \mathbb{Z}[j]; \mathbb{Z}/m) = H^1(\text{Spec} \mathbb{Z}; \mathbb{Z}/m) = 0 \) for any \( m \), and Theorem 2.9, we deduce first that \( H^1(\text{Spec} \mathbb{Z}[j]; R^1 j_* \mathbb{G}_m) \) vanishes. From Theorem 6.5, it thus follows that \( H^1(\text{Spec} \mathbb{Z}[j]; \pi_0 j^* \text{pic}_{\mathcal{O}_{\text{TMF}}}) \cong H^1(\text{Spec} [j]; F^3 \pi_0 j^* \text{pic}_{\mathcal{O}_{\text{TMF}}}) \), where \( F^3 \) refers to the third filtration. The sheaf \( F^3 \pi_0 j^* \text{pic}_{\mathcal{O}_{\text{TMF}}} \) sits in an extension

\[
0 \to F^5 \pi_0 j^* \text{pic}_{\mathcal{O}_{\text{TMF}}} \to F^3 \pi_0 j^* \text{pic}_{\mathcal{O}_{\text{TMF}}} \to k_* \mathbb{V}/2 \to 0.
\]

The extension must be split because \( (F^5 \pi_0 j^* \text{pic}_{\mathcal{O}_{\text{TMF}}})(2) \) is supported at \((2, j)\), while \( k_* \mathbb{V}/2 \) is only nonzero on étale maps \( U \to \mathbb{A}^1 \) whose image does not contain \((2, j)\).

To compute \( H^1(\mathbb{A}^1; k_* \mathbb{V}/2) \), recall that we obtained \( k_* \mathbb{V}/2 \) as the kernel of a surjective differential \( d_3 : \mathcal{O}/2 \to \mathcal{O}/2 \). Since \( \mathcal{O}/2 \) is quasi-coherent, its first cohomology vanishes and we can thus identify \( H^1(\mathbb{A}^1; k_* \mathbb{V}/2) \) with the cokernel of the map \( d_3 : H^0(\mathbb{A}^1; \mathcal{O}/2) \cong \mathbb{F}_2[j] \to \mathbb{F}_2[j] \cong H^0(\mathbb{A}^1; \mathcal{O}/2) \), which sends \( f \) to \( f + jf^2 \) (cf. Section 6.5). One checks that \( j^2, j^4, j^6, \ldots \) is a linearly independent subset in the cokernel and thus

\[
H^1(\mathbb{A}^1; k_* \mathbb{V}/2) \cong \mathbb{F}_2^{\infty}.
\]

Next, we turn to \( F^5 \pi_0 j^* \text{pic}_{\mathcal{O}_{\text{TMF}}} \). By Theorem 6.5, this vanishes if localized at primes bigger than 3, while 3-locally it is isomorphic to \( b_* \mathbb{Z}/3 \) for \( b : \text{Spec} \mathbb{F}_3 \to \mathbb{A}^1 \) the inclusion at \( j = 3 = 0 \). Thus,

\[
H^1(\mathbb{A}^1; F^5 \pi_0 j^* \text{pic}_{\mathcal{O}_{\text{TMF}}}) \cong H^1(\mathbb{A}^1; b_* \mathbb{Z}/3) \cong H^1(\mathbb{F}_3; \mathbb{Z}/3) \cong \mathbb{Z}/3.
\]

For the 2-local situation, recall from [54, Corollary II.3.11] that we can view sheaves supported at \((2, j)\) equivalently as étale sheaves on \( \text{Spec} \mathbb{F}_2 \), whose category is equivalent to (discrete) abelian groups with a continuous action by the absolute Galois group \( \text{Gal}(\mathbb{F}_2) \cong \hat{\mathbb{Z}} \); we refer to such as *discrete* \( \hat{\mathbb{Z}} \)-modules. Let \( \mathcal{F} \) be the class of discrete \( \hat{\mathbb{Z}} \)-modules where \( H^i(\hat{\mathbb{Z}}, -) \) is finite for all \( i \). From the fact that for discrete \( \hat{\mathbb{Z}} \)-modules, \( H^i(\hat{\mathbb{Z}}, -) \) vanishes for \( i > 1 \), one deduces that \( \mathcal{F} \) is closed under kernels, cokernels, and extensions. By Proposition 6.6, we know that only finitely many discrete \( \hat{\mathbb{Z}} \)-modules can contribute to \( \left( F^5 \pi_0 j^* \text{pic}_{\mathcal{O}_{\text{TMF}}} \right)(2) \) and they all lie in \( \mathcal{F} \); moreover, there are only finitely many possible targets and they also lie in \( \mathcal{F} \) (cf. Section 6.7). Thus, \( H^1(\mathbb{A}^1; F^5 \pi_0 j^* \text{pic}_{\mathcal{O}_{\text{TMF}}})(2) \) is finite.
It remains to study the differential
\[ d^{\text{TMF}} : H^1(\text{Spec } Z[j]; \pi_0 \mathcal{B}\text{Pic}_{\mathcal{O}_{\text{TMF}}}) \to H^3(\text{Spec } Z[j]; G_m). \]

Let \( U \) be the complement of the image of the closed immersion \( \text{Spec } Z/6 \to \text{Spec } Z[j] \) corresponding to \( j = 0 \). We obtain a commutative diagram

\[
\begin{array}{ccc}
H^1(\mathbb{A}^1; F^5 \pi_0 \mathcal{B}\text{Pic}_{\mathcal{O}_{\text{TMF}}}) & \to & H^1(\mathbb{A}^1; \pi_0 \mathcal{B}\text{Pic}_{\mathcal{O}_{\text{TMF}}}) \\
\downarrow & & \downarrow \\
H^1(\mathbb{A}^1; F^5 \pi_0 \mathcal{B}\text{Pic}_{\mathbb{A}^1_1}) & \to & H^1(U; \pi_0 \mathcal{B}\text{Pic}_{\mathbb{A}^1_1}) \\
\downarrow & & \downarrow \\
H^3(\mathbb{A}^1; \mathcal{B}\text{Pic}_{\mathbb{A}^1_1}) & \to & H^3(U; \mathcal{B}\text{Pic}_{\mathbb{A}^1_1})
\end{array}
\]

The rightmost lower horizontal arrow is the differential in the descent spectral sequence for \( \mathcal{B}\text{Pic} \) on \( (U, \mathcal{O}_{\text{TMF}}|_U) \). The rightmost vertical map is an injection by purity [34, Théorème 6.1b]. Moreover, the leftmost vertical map is zero since \( F^5 \pi_0 \mathcal{B}\text{Pic}_{\mathcal{O}_{\text{TMF}}} \) is supported at \( (2, j) \) and \( (3, j) \).

Thus, \( d^{\text{TMF}} \) vanishes when restricted to \( H^1(\mathbb{A}^1; F^5 \pi_0 \mathcal{B}\text{Pic}_{\mathcal{O}_{\text{TMF}}}) \) and the differential factors over \( H^1(\mathbb{A}^1; k^* v! \mathbb{Z}/2) \).

We can cover \( U \) by \( V = \text{Spec } Z[j^\pm, (j - 1728)^{-1}] \) and \( W = \text{Spec } Z[\frac{1}{6}, j] \). We obtain an exact sequence

\[ \cdots \to H^2(V \cap W; G_m) \to H^3(U; G_m) \to H^3(\mathbb{A}^1; \mathbb{Z}/2) \to \cdots \]

We claim that the image of \( d^{\text{TMF}} \) in \( H^3(V; G_m) \oplus H^3(W; G_m) \) is zero. Assuming this claim for the moment, we know that the image of \( d^{\text{TMF}} \) lies in the image of \( H^2(V \cap W; G_m) \to H^3(U; G_m) \). By Theorem 2.5, we have 2-locally an isomorphism

\[ H^2(V \cap W; G_m) \cong \text{Br} \left( Z \left[ \frac{1}{6}, j^\pm, (j - 1728)^{-1} \right] \right) \cong \text{Br} \left( Z \left[ \frac{1}{6} \right] \right) \oplus H^1 \left( Z \left[ \frac{1}{6} \right]; \mathbb{Q}/\mathbb{Z} \right) ^{\oplus 2}. \]

We use the following two computations:

- \( \text{Br} \left( Z \left[ \frac{1}{6} \right] \right) \cong \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}/2 \) by Example 2.4;
- \( H^1 \left( Z \left[ \frac{1}{6} \right]; \mathbb{Q}/\mathbb{Z} \right) \cong \text{Hom}(\text{Gal}(K/\mathbb{Q}), \mathbb{Q}/\mathbb{Z}) \cong \text{Hom} \left( Z^\times \mathbb{Z}/2 \times Z^\times \mathbb{Z}/3, \mathbb{Q}/\mathbb{Z} \right) \cong (Z/2)^3 \oplus \mathbb{Z}/2 \oplus \mathbb{Q}/\mathbb{Z}. \)

Here, \( K \) is the maximal abelian extension of \( \mathbb{Q} \), which is ramified only at 2 and 3. We use the Kronecker–Weber theorem (see [57, Thm. V.110]) to identify \( K \) with the field obtained by adjoining all 2- and 3-power roots of unity to \( \mathbb{Q} \), which has Galois group \( Z^\times \mathbb{Z}/2 \times Z^\times \mathbb{Z}/3 \).

Thus, the image of \( H^2(V \cap W; G_m) \to H^3(U; G_m) \) is 2-locally of a direct sum of a finite and a divisible abelian group. Since the image of \( d^{\text{TMF}} \) must be an \( F_2 \)-vector space (as the image of an \( F_2 \)-vector space), the image of \( d^{\text{TMF}} \) must be finite. We deduce that the kernel of \( d^{\text{TMF}} \) consists of the finite group \( H^1(\mathbb{A}^1; F^5 \pi_0 \mathcal{B}\text{Pic}_{\mathcal{O}_{\text{TMF}}}) \) plus an infinite-dimensional subspace of \( H^1(\mathbb{A}^1; k^* v! \mathbb{Z}/2) \cong F_2^\infty \), as claimed.

It remains to show that the restrictions of \( d^{\text{TMF}} \) to \( V \) and \( W \) are zero. The case of \( W \) is clear as \( k^* v! \mathbb{Z}/2 \) is supported outside of \( W \). For the case of \( V \), recall from [64, Lemma 3.2] that the base change \( \mathcal{M} \times_{\mathbb{A}^1} V \) is equivalent to \( V \times BC_2 \), that is, the stack quotient of \( V \) by the trivial \( C_2 \)-action;
this yields in particular an étale map $V \to \mathcal{M} \times_{\mathbb{A}^1} V \to \mathcal{M}$. We obtain a diagram

\[
\begin{array}{cccc}
H^1(\mathbb{A}^1; F^3\pi_0 \text{BPic}_{\mathcal{O}_V}) & \cong & H^1(\mathbb{A}^1; \pi_0 \text{BPic}_{\mathcal{O}_V}) & \xrightarrow{d_{\text{TMF}}} H^1(\mathbb{A}^1; \mathbb{G}_m) \\
H^1(V; F^3\pi_0 \text{BPic}_{\mathcal{O}_V}) & \to & H^1(V; \pi_0 \text{BPic}_{\mathcal{O}_V}) & \xrightarrow{d_{(V, \mathcal{O}_{\text{TMF}})}} H^3(V; \mathbb{G}_m) \\
& & & \text{id} \\
& & & H^3(V; \mathbb{G}_m).
\end{array}
\]

Here, $d_{(V)}$ refers to the boundary map in the long exact sequence from Proposition 2.25 for the ring spectrum $\mathcal{O}(V \to \mathcal{M})$, while $d_{(V, \mathcal{O}_{\text{TMF}})}$ uses the restriction of the spectral scheme structure of $\text{Spec TMF}$ to $V$; note that both affine spectral schemes here have underlying scheme $V$. In particular, the rightmost vertical map is an isomorphism. Note further that $F^3\pi_0 \text{BPic}_{\mathcal{O}_V} = 0$ since all terms in the sheafy Picard spectral sequence of filtration 3 and higher are of the form $H^{2i+1}(V; \pi_{2i})$ for $i \geq 1$, which all vanish since $V$ is an affine scheme. Thus, we see that $d_{\text{TMF}}$ is indeed zero after restricting to $V$. □

Our next goal is to compare $L\text{Br}(\mathcal{M}, \mathcal{O})$ with $L\text{Br}(\text{TMF})$. Clearly, we have maps

$$L\text{Br}(\text{TMF}) \to L\text{Br}(\mathcal{M}, \mathcal{O}) \to \text{Br}(\mathcal{M}, \mathcal{O}).$$

Since $\text{Br}(\text{TMF}) \to \text{Br}(\mathcal{M}, \mathcal{O})$ is an isomorphism, $L\text{Br}(\text{TMF}) \to L\text{Br}(\mathcal{M}, \mathcal{O})$ is an injection by Corollary 5.5. We want to describe how to obtain a computational handle on this injection. In conjunction with Theorem 8.2, this will also provide an alternative proof of Theorem 8.3, except for the splitting.

Consider the sheaf $j_* \text{lbr}_{\mathcal{O}}$ on $\mathbb{A}^1$. It assigns to every étale open $U$, the spectrum $\text{lbr}(U \times_{\mathbb{A}^1} \mathcal{M}, \mathcal{O})$. The relative descent spectral sequence (cf. Remark 6.4) for $j_* \text{lbr}_{\mathcal{O}}$ takes the form

$$E_2^{s,t} = R^sj_*\pi_t \text{lbr}_{\mathcal{O}} \Rightarrow \pi_{t-s}j_* \text{lbr}_{\mathcal{O}} \cong \pi_{t-s}L\text{Br}(\mathcal{O}),$$

and provides thus a method to compute $\pi_{t-s}j_* \text{lbr}_{\mathcal{O}}$. But since $\text{lbr}$ is just a suspension of $\text{pic}$, this spectral sequence is up to a shift actually the same as the sheafy Picard spectral sequence considered in Section 6. In particular, one observes that $\pi_{t-s}j_* \text{lbr}_{\mathcal{O}} \cong \pi_{t-1} \text{Pic}_{\mathcal{O}_{\text{TMF}}}$ for $t \geq 1$, but there are additionally interesting sheaves $\pi_t$ for $t \leq 0$, which are computed by the $(t-1)$-column of the sheafy Picard spectral sequence. We obtain a descent spectral sequence

$$E_2^{s,t} = H^s(\mathbb{A}^1; \pi_t j_* \text{lbr}_{\mathcal{O}}) \Rightarrow \pi_{t-s} \Gamma(j_* \text{lbr}_{\mathcal{O}}) \cong \pi_{t-s} \text{lbr}(\mathcal{M}, \mathcal{O}). \quad (8.6)$$

In particular, Proposition 8.1 gives that $\pi_0 \text{lbr}(\mathcal{M}, \mathcal{O}) = L\text{Br}'(\mathcal{M}, \mathcal{O}) = L\text{Br}(\mathcal{M}, \mathcal{O})$. Thus, we are indeed computing the local Brauer group of $(\mathcal{M}, \mathcal{O})$.

As observed above, the map $\text{lbr}_{\mathcal{O}_{\text{TMF}}} \to j_* \text{lbr}_{\mathcal{O}}$ identifies the source with $\tau_{\geq 1}j_* \text{lbr}_{\mathcal{O}}$. The associated map of descent spectral sequences is on $E_2$-pages an isomorphism on the antidiagonals $t = s + (t - s) \geq 1$ and converges to $\pi_{t-s}L\text{Br}(\text{TMF}) \to \pi_{t-s}L\text{Br}(\mathcal{M}, \mathcal{O})$ for $t - s \geq 0$. Figure 7 gives
Figure 7  Schematic comparison of descent spectral sequences computing $\text{LBr}(\text{TMF})$ (solid hexagons, in black) and $\text{LBr}(\mathcal{M}, \mathcal{O})$ (all the hexagons, including those beyond the picture).

A schematic picture of part of this map, with the image of the descent spectral sequence of $\text{lb}_{\text{TMF}}$ filled with black.

Theorem 8.7. The injection $\text{LBr}(\text{TMF}) \rightarrow \text{LBr}(\mathcal{M}, \mathcal{O})$ has finite cokernel and is an isomorphism after inverting 2.

Remark 8.8. Our proof does not rule out that the map $\text{LBr}(\text{TMF}) \rightarrow \text{LBr}(\mathcal{M}, \mathcal{O})$ is, in fact, an isomorphism. This would, for example, be the case if $\pi_0 j_*\text{lb}_\mathcal{O} = 0$. This homotopy sheaf is calculated by the $(-1)$-column in the sheafy Picard spectral sequence of Section 6. It would be interesting to compute the missing differentials to prove or disprove this vanishing.

Proof. In the spectral sequence (8.6), the only possible nonzero entries in the zeroth column are $H^s(\mathbb{A}^1, \pi_s j_*\text{lb}_\mathcal{O})$ for $0 \leq s \leq 2$. By the above discussion, every element in $\text{LBr}(\mathcal{M}, \mathcal{O})$ not coming from $\text{LBr}(\text{TMF})$ must be detected in $H^0(\mathbb{A}^1, \pi_0 j_*\text{lb}_\mathcal{O})$.

The charts in Figures 4 and 6 show the possible contributions in the $(-1)$-column of the sheafy Picard spectral sequence to $\pi_0 j_*\text{lb}_\mathcal{O}$. We first note that the two question marks corresponding to $R^1 j_*\mathbb{Z}/2$ and $R^2 j_*\mathbb{G}_m$ cannot contribute to $H^0$. Indeed, by the Leray spectral sequence, $0 = H^1(\mathcal{M}; \mathbb{Z}/2)$ surjects onto $H^0(\mathbb{A}^1; R^1 j_*\mathbb{Z}/2)$, which is thus zero as well. Likewise, from the Leray spectral sequence

$$E_2^{mn} = H^m(\mathbb{A}^1; R^n j_*\mathbb{G}_m) \Rightarrow H^{m+n}(\mathcal{M}; \mathbb{G}_m),$$

we see that $\text{Br}(\mathcal{M}) \cong H^2(\mathcal{M}; \mathbb{G}_m)$ surjects onto the cokernel of the differential $H^1(\mathbb{A}^1, j_*\mathbb{G}_m) \rightarrow H^0(\mathbb{A}^1; R^2 j_*\mathbb{G}_m)$. But $j_*\mathbb{G}_m \cong \mathbb{G}_m$ (since every function $\mathcal{M} \rightarrow \mathbb{A}^1$ factors through $j$, even after étale base change) and thus $H^1(\mathbb{A}^1, j_*\mathbb{G}_m) \cong \text{Pic}(\mathbb{A}^1) = 0$. Moreover, $\text{Br}(\mathcal{M}) = 0$ by one of the main results from [4]. Thus, $H^0(\mathbb{A}^1; R^2 j_*\mathbb{G}_m)$ vanishes.
Regarding the contributions in higher rows: even the $E_2$-term vanishes $p$-locally for $p > 3$. At $p = 3$, the only potential contribution is in row 14 (see Figure 4 and Proposition 6.6). As demonstrated in Lemma 6.10, this contribution is hit by a surjective $d_9$. Thus, $F^3 \pi_0 j_* \mathbb{LBr}_{\mathcal{O}}$ vanishes after inverting 2. We deduce $H^0 \left( \mathbb{A}^1, \pi_0 j_* \mathbb{LBr}_{\mathcal{O}} \right) \left[ \frac{1}{2} \right] = 0$ and hence that $\text{LBr}(\text{TMF}) \to \text{LBr}(\mathcal{M}, \mathcal{O})$ is an isomorphism after inverting 2.

Regarding the 2-local picture, Figure 6 shows that the only possible contributions are in rows 6, 18, and 30 and each of them is an $\mathcal{O}/(2, j)$ on the $E_7$-page. The same argument as provided in the proof of Theorem 8.3 for the finiteness of $H^1 \left( \mathbb{A}^1, F^5 \pi_0 j_* \mathbb{Pic}_{\mathcal{O}^{\text{et}}} \right)$ also shows the finiteness of $H^0 \left( \mathbb{A}^1, \pi_0 j_* \mathbb{LBr}_{\mathcal{O}} \right)$. This, in turn, implies the finiteness of the cokernel of $\text{LBr}(\text{TMF}) \to \text{LBr}(\mathcal{M}, \mathcal{O})$.

\section*{Acknowledgments}
We would like to thank Elden Elmanto, David Gepner, Tyler Lawson, Akhil Mathew, and Bertrand Toën for helpful conversations about the subject matter of this paper over the years. We further thank Sven van Nigtevecht and the anonymous referee for their remarks improving the exposition.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1440140, while the first and third authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2019 semester. The authors would also like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the program “Homotopy harnessing higher structures” when work on this paper was undertaken. This work was supported by EPSRC grant number EP/R014604/1. We moreover thank the Hausdorff Research Institute for Mathematics for its hospitality, as the completion of this project took place during the program “Spectral Methods in Algebra, Geometry, and Topology.”

The first author was supported by NSF Grants DMS-2120005 and DMS-2102010, and by a Simons Fellowship. The second author was supported by the NWO grant VI.Vidi.193.111. The third author was supported by NSF Grants DMS-1812122 and DMS-2304797, and by a Simons Fellowship.

\section*{Journal Information}
The Journal of Topology is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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