A Kenmotsu metric as a conformal \( \eta \)-Einstein soliton

By

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Abstract

The object of the present paper is to study some properties of Kenmotsu manifold whose metric is conformal \( \eta \)-Einstein soliton. We have studied some certain properties of Kenmotsu manifold admitting conformal \( \eta \)-Einstein soliton. We have also constructed a 3-dimensional Kenmotsu manifold satisfying conformal \( \eta \)-Einstein soliton.

Key words : Einstein soliton, \( \eta \)-Einstein soliton, conformal \( \eta \)-Einstein soliton, \( \eta \)-Einstein manifold, Kenmotsu manifold.

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1. Introduction

The notion of Einstein soliton was introduced by G. Catino and L. Mazzieri \(^3\) in 2016, which generates self-similar solutions to Einstein flow,

\[
\frac{\partial g}{\partial t} = -2(S - \frac{r}{2}g),
\]

(1.1)

where \( S \) is Ricci tensor, \( g \) is Riemannian metric and \( r \) is the scalar curvature. The equation of the \( \eta \)-Einstein soliton \(^2\) is given by,

\[
\mathcal{L}_\xi g + 2S + (2\lambda - r)g + 2\mu \eta \otimes \eta = 0,
\]

(1.2)

where \( \mathcal{L}_\xi \) is the Lie derivative along the vector field \( \xi \), \( S \) is the Ricci tensor, \( r \) is the scalar curvature of the Riemannian metric \( g \), and \( \lambda \) and \( \mu \) are real constants. For \( \mu = 0 \), the data \((g, \xi, \lambda)\) is called Einstein soliton.

In 2018, Mohd Danish Siddiqi \(^5\) introduced the notion of conformal \( \eta \)-Ricci soliton \(^7\) as:

\[
\mathcal{L}_\xi g + 2S + [2\lambda - (p + \frac{2}{n})]g + 2\mu \eta \otimes \eta = 0,
\]

(1.3)

where \( \mathcal{L}_\xi \) is the Lie derivative along the vector field \( \xi \), \( S \) is the Ricci tensor, \( \lambda \), \( \mu \) are constants, \( p \) is a scalar non-dynamical field(time dependent scalar field) and \( n \) is the dimension of manifold. For \( \mu = 0 \), conformal \( \eta \)-Ricci soliton becomes conformal Ricci soliton \(^6\).

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In [8], Roy, Dey and Bhattacharyya have defined conformal Einstein soliton, which can be written as:

$$\mathcal{L}_V g + 2S + [2\lambda - r + (p + \frac{2}{n})]g = 0, \quad (1.4)$$

where $\mathcal{L}_V$ is the Lie derivative along the vector field $V$, $S$ is the Ricci tensor, $r$ is the scalar curvature of the Riemannian metric $g$, $\lambda$ is real constant, $p$ is a scalar non-dynamical field (time dependent scalar field) and $n$ is the dimension of manifold.

So we introduce the notion of conformal $\eta$-Einstein soliton as:

**Definition 1.1:** A Riemannian manifold $(M, g)$ of dimension $n$ is said to admit conformal $\eta$-Einstein soliton if

$$\mathcal{L}_\xi g + 2S + [2\lambda - r + (p + \frac{2}{n})]g + 2\mu \eta \otimes \eta = 0, \quad (1.5)$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field $\xi$, $\lambda$, $\mu$ are real constants and $S$, $r$, $p$, $n$ are same as defined in (1.4).

In the present paper we study conformal $\eta$-Einstein soliton on Kenmotsu manifold. The paper is organized as follows:

After introduction, section 2 is devoted for preliminaries on $(2n+1)$ dimensional Kenmotsu manifold. In section 3, we have studied conformal $\eta$-Einstein soliton on Kenmotsu manifold. Here we proved if a $(2n+1)$ dimensional Kenmotsu manifold admits conformal $\eta$-Einstein soliton then the manifold becomes $\eta$-Einstein. We have also characterized the nature of the manifold if the manifold is Ricci symmetric and the Ricci tensor is $\eta$-recurrent. Also we have discussed about the condition when the manifold has cyclic Ricci tensor. Then we have obtained the condition in a $(2n+1)$ dimensional Kenmotsu manifold admitting Conformal $\eta$-Einstein soliton when a vector field $V$ is pointwise co-linear with $\xi$ and a $(0,2)$ tensor field $h$ is parallel with respect to the Levi-Civita connection associated to $g$. We have also examined the nature of a Ricci-recurrent Kenmotsu manifold admitting conformal $\eta$-Einstein soliton.

In last section we have given an example of a 3-dimensional Kenmotsu manifold satisfying conformal $\eta$-Einstein soliton.

## 2. Preliminaries

Let $M$ be a $(2n+1)$ dimensional connected almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$ where $\phi$ is a $(1,1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is the compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \phi = 0, \phi \xi = 0, \quad (2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.2)$$
\[ g(X, \phi Y) = -g(\phi X, Y), \] (2.3)
\[ g(X, \xi) = \eta(X), \] (2.4)

for all vector fields \( X, Y \in \chi(M) \).

An almost contact metric manifold is said to be a Kenmotsu manifold \([4]\) if
\[ (\nabla_X \phi) Y = -g(X, \phi Y) \xi - \eta(Y) \phi X, \] (2.5)
\[ \nabla_X \xi = X - \eta(X) \xi, \] (2.6)
where \( \nabla \) denotes the Riemannian connection of \( g \).

In a Kenmotsu manifold the following relations hold \([1]\):
\[ \eta(R(X, Y)Z) = g(X, Z) \eta(Y) - g(Y, Z) \eta(X), \] (2.7)
\[ R(X, Y) \xi = \eta(X) Y - \eta(Y) X, \] (2.8)
\[ R(X, \xi) Y = g(X, Y) \xi - \eta(Y) X, \] (2.9)
where \( R \) is the Riemannian curvature tensor.

\[ S(X, \xi) = -2n \eta(X), \] (2.10)
\[ S(\phi X, \phi Y) = S(X, Y) + 2n \eta(X) \eta(Y), \] (2.11)
\[ (\nabla_X \eta) Y = g(X, Y) - \eta(X) \eta(Y), \] (2.12)
for all vector fields \( X, Y, Z \in \chi(M) \).

Now we know,
\[ (\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi), \] (2.13)
for all vector fields \( X, Y, \in \chi(M) \).
Then using (2.6) and (2.13), we get,
\[ (\mathcal{L}_\xi g)(X, Y) = 2[g(X, Y) - \eta(X) \eta(Y)]. \] (2.14)

3. Conformal \( \eta \)-Einstein soliton on Kenmotsu manifold

Let \( M \) be a \((2n+1)\) dimensional Kenmotsu manifold. Consider the conformal \( \eta \)-Einstein soliton (1.5) on \( M \) as:
\[ (\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + [2\lambda - r + (p + \frac{2}{2n + 1})]g(X, Y) + 2\mu \eta(X) \eta(Y) = 0, \] (3.1)
for all vector fields \( X, Y, \in \chi(M) \).
Then using (2.14), the above equation becomes,
\[ S(X, Y) = -[\lambda - \frac{r}{2} + \frac{(p + \frac{2}{2n + 1})}{2} + 1]g(X, Y) - (\mu - 1) \eta(X) \eta(Y). \] (3.2)
Taking \( Y = \xi \) in the above equation and using (2.10), we get,
\[
r = \left( p + \frac{2}{2n+1} \right) - 4n + 2\lambda + 2\mu,
\] (3.3)
since \( \eta(X) \neq 0 \), for all \( X \in \chi(M) \).
Also from (3.2), it follows that the manifold is \( \eta \)-Einstein.
This leads to the following:

**Theorem 3.1.** If the metric of a \((2n+1)\) dimensional Kenmotsu manifold is a conformal \( \eta \)-Einstein soliton then the manifold becomes \( \eta \)-Einstein and the scalar curvature is \( ( p + \frac{2}{2n+1} ) - 4n + 2\lambda + 2\mu \).

We know,
\[
(\nabla_X S)(Y,Z) = X(S(Y,Z)) - S(\nabla_X Y,Z) - S(Y,\nabla_X Z),
\] (3.4)
for all vector fields \( X,Y,Z \) on \( M \) and \( \nabla \) is the Levi-Civita connection associated with \( g \).
Now replacing the expression of \( S \) from (3.2), we obtain,
\[
(\nabla_X S)(Y,Z) = -(\mu - 1)[\eta(Z)(\nabla_X \eta)Y + \eta(Y)(\nabla_X \eta)Z].
\] (3.5)
for all vector fields \( X,Y,Z \) on \( M \).

Let the manifold \( M \) be Ricci symmetric i.e \( \nabla S = 0 \).
Then from (3.5), we get,
\[
-(\mu - 1)[\eta(Z)(\nabla_X \eta)Y + \eta(Y)(\nabla_X \eta)Z] = 0,
\] (3.6)
for all vector fields \( X,Y,Z \in \chi(M) \).
Taking \( Z = \xi \) in the above equation and using (2.12), (2.1), we obtain,
\[
\mu = 1.
\] (3.7)
Then from (3.3), we get,
\[
r = \left( p + \frac{2}{2n+1} \right) - 4n + 2\lambda + 2.
\] (3.8)
So we can state the following theorem:

**Theorem 3.2.** If the metric of a \((2n+1)\) dimensional Ricci symmetric Kenmotsu manifold is a conformal \( \eta \)-Einstein soliton then \( \mu = 1 \) and the scalar curvature is \( ( p + \frac{2}{2n+1} ) - 4n + 2\lambda + 2 \).

Now if the Ricci tensor \( S \) is \( \eta \)-recurrent, then we have,
\[
\nabla S = \eta \otimes S,
\] (3.9)
which implies that,
\[
(\nabla_X S)(Y,Z) = \eta(X)S(Y,Z),
\] (3.10)
for all vector fields \( X,Y,Z \) on \( M \).
Using (3.5), the above equation reduces to,
\[
-(\mu - 1)[\eta(Z)(\nabla_X \eta)Y + \eta(Y)(\nabla_X \eta)Z] = \eta(X)S(Y,Z).
\] (3.11)
Taking $Y = \xi, Z = \xi$ in the above equation and using (2.12),(3.2), we get,

$$[\lambda + \mu - \frac{r}{2} + \frac{p + \frac{2}{2n+1}}{2}]\eta(X) = 0,$$

which implies that,

$$r = 2\lambda + 2\mu + (p + \frac{2}{2n+1}).$$

Then we can state the following:

**Theorem 3.3** If the metric of a $(2n+1)$ dimensional Kenmotsu manifold is a conformal $\eta$-Einstein soliton and the Ricci tensor $S$ is $\eta$- Recurrent, then the scalar curvature is $2\lambda + 2\mu + (p + \frac{2}{2n+1})$.

Similarly from (3.5), we get,

$$(\nabla_Y S)(Z, X) = - (\mu - 1)[\eta(Y)(\nabla_Y \eta)Z + \eta(Z)(\nabla_X \eta)Y],$$

and

$$(\nabla_Z S)(X, Y) = - (\mu - 1)[\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y].$$

for all vector fields $X, Y, Z$ on $M$.

Then adding (3.5),(3.14), (3.15) and using (2.12), (2.2), we obtain,

$$\begin{align*}
(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) & = -2(\mu - 1)[\eta(X)g(\phi Y, \phi Z) \allowbreak \, + \eta(Y)g(\phi Z, \phi X) \allowbreak \, + \eta(Z)g(\phi X, \phi Y)].
\end{align*}$$

Now if the manifold $M$ has cyclic Ricci tensor i.e $(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$, then from (3.16), we have,

$$(\mu - 1)[\eta(X)g(\phi Y, \phi Z) + \eta(Y)g(\phi Z, \phi X) + \eta(Z)g(\phi X, \phi Y)] = 0. $$

Taking $X = \xi$ in the above equation and using (2.1), we get,

$$\mu = 1.$$ 

Again if we take $\mu = 1$ in (3.16), we obtain $(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$, i.e the manifold $M$ has cyclic Ricci tensor.

Hence we can state the following:

**Theorem 3.4** Let the metric of a $(2n+1)$ dimensional Kenmotsu manifold $M$ is a conformal $\eta$-Einstein soliton. Then $M$ has cyclic Ricci tensor iff $\mu = 1$.

Now if $\mu = 1$, then from (3.3) we obtain,

$$r = (p + \frac{2}{2n+1}) - 4n + 2\lambda + 2.$$

Then we have,

**Corollary 3.5.** If a $(2n+1)$ dimensional Kenmotsu manifold $M$ has a cyclic Ricci tensor and the metric is a conformal $\eta$-Einstein soliton then the scalar curvature is $(p + \frac{2}{2n+1}) - 4n + 2\lambda + 2$. 
Let a conformal $\eta$-Einstein soliton is defined on a $(2n+1)$ dimensional Kenmotsu manifold $M$ as,

$$\mathcal{L}_V g + 2S + [2\lambda - r + (p + \frac{2}{2n+1})]g + 2\mu \eta \otimes \eta = 0,$$  \hfill (3.20)

where $\mathcal{L}_V$ is the Lie derivative along the vector field $V$, $S$ is the Ricci tensor, $r$ is the scalar curvature of the Riemannian metric $g$, $\lambda$, $\mu$ are real constants, $p$ is a scalar non-dynamical field (time dependent scalar field).

Let $V$ be pointwise co-linear with $\xi$, i.e $V = b\xi$, where $b$ is a function on $M$.

Then (3.20) becomes,

$$(\mathcal{L}_b \xi)g(X, Y) + 2S(X, Y) + [2\lambda - r + (p + \frac{2}{2n+1})]g(X, Y) + 2\mu \eta(X)\eta(Y) = 0,$$  \hfill (3.21)

for all vector fields $X, Y$ on $M$.

Applying the property of Lie derivative and Levi-Civita connection we have,

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) + 2S(X, Y)$$

$$+ [2\lambda - r + (p + \frac{2}{2n+1})]g(X, Y) + 2\mu \eta(X)\eta(Y) = 0.$$  \hfill (3.22)

Now using (2.6), we get,

$$2bg(X, Y) - 2b\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X, Y)$$

$$+ [2\lambda - r + (p + \frac{2}{2n+1})]g(X, Y) + 2\mu \eta(X)\eta(Y) = 0.$$  \hfill (3.23)

Taking $Y = \xi$ in the above equation and using (2.1),(2.4),(2.10), we obtain,

$$(Xb) + (\xi b)\eta(X) - 4n\eta(X) + [2\lambda - r + (p + \frac{2}{2n+1})]\eta(X) + 2\mu \eta(X) = 0.$$  \hfill (3.24)

Then by putting $X = \xi$, the above equation reduces to,

$$\xi b = 2n + \frac{r}{2} - \lambda - \mu - \frac{p + \frac{2}{2n+1}}{2}.$$  \hfill (3.25)

Using (3.25), (3.24) becomes,

$$(Xb) + [\lambda + \mu + \frac{(p + \frac{2}{2n+1})}{2}] - 2n - \frac{r}{2} = 0.$$  \hfill (3.26)

Applying exterior differentiation in (3.26), we have,

$$[\lambda + \mu + \frac{(p + \frac{2}{2n+1})}{2}] - 2n - \frac{r}{2} d\eta = 0.$$  \hfill (3.27)

Now we know,

$$d\eta(X, Y) = \frac{1}{2}[(\nabla_X \eta)Y - (\nabla_Y \eta)X],$$  \hfill (3.28)

for all vector fields $X, Y$ on $M$.

Using (2.12), the above equation becomes,

$$d\eta = 0.$$  \hfill (3.29)
Hence the 1-form $\eta$ is closed. So from (3.27), either $r = 2\lambda + 2\mu + (p + \frac{2}{2n+1}) - 4n$ or $r \neq 2\lambda + 2\mu + (p + \frac{2}{2n+1}) - 4n$. If $r = 2\lambda + 2\mu + (p + \frac{2}{2n+1}) - 4n$, (3.26) reduces to,

$$ (Xb) = 0. $$

(3.30)

This implies that $b$ is constant.

So we can state the following theorem:

**Theorem 3.6.** Let $M$ be a $(2n+1)$ dimensional Kenmotsu manifold admitting a conformal $\eta$-Einstein soliton $(g, V)$, $V$ being a vector field on $M$. If $V$ is point-wise co-linear with $\xi$, a vector field on $M$, then $V$ is a constant multiple of $\xi$, provided the scalar curvature is $2\lambda + 2\mu + (p + \frac{2}{2n+1}) - 4n$.

Let $h$ be a symmetric tensor field of $(0,2)$ type which we suppose to be parallel with respect to the Levi-Civita connection $\nabla$ i.e $\nabla h = 0$.

Applying the Ricci commutation identity, we have,

$$ \nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; W, Z) = 0. $$

(3.31)

for all vector fields $X, Y, Z, W$ on $M$.

From (3.31), we obtain the relation,

$$ h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0. $$

(3.32)

Replacing $Z = W = \xi$ in the above equation and using (2.8), we get,

$$ \eta(X)h(Y, \xi) - \eta(Y)h(X, \xi) = 0. $$

(3.33)

Replacing $X = \xi$ and using (2.1), the above equation reduces to,

$$ h(Y, \xi) = \eta(Y)h(\xi, \xi), $$

(3.34)

for all vector fields $Y$ on $M$.

Differentiating the above equation covariantly with respect to $X$, we get,

$$ \nabla_X (h(Y, \xi)) = \nabla_X (\eta(Y)h(\xi, \xi)). $$

(3.35)

Now expanding the above equation by using (3.34), (2.6),(2.12) and the property that $\nabla h = 0$, we obtain,

$$ h(X, Y) = h(\xi, \xi)g(X, Y), $$

(3.36)

for all vector fields $X, Y$ on $M$.

Let us take,

$$ h = \mathcal{L}_\xi g + 2S + 2\mu \eta \otimes \eta. $$

(3.37)

Then from (2.14),(3.2), we get,

$$ h(\xi, \xi) = -2\lambda - (p + \frac{2}{2n+1}) + r. $$

(3.38)

Then using (3.37), (3.36) becomes,

$$ (\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + [2\lambda - r + (p + \frac{2}{2n+1})]g(X, Y) + 2\mu \eta(X)\eta(Y) = 0, $$

(3.39)
which is the Conformal $\eta$-Einstein soliton. This leads to,

**Theorem 3.7.** In a $(2n+1)$ dimensional Kenmotsu manifold assume that a symmetric $(0,2)$ tensor field $\mathbf{h} = \mathbf{L}_\xi g + 2S + 2\mu \eta \otimes \eta$ is parallel with respect to the Levi-Civita connection associated to $g$. Then $(g, \xi)$ yields a conformal $\eta$-Einstein soliton.

**Definition 3.8** A Kenmotsu manifold is said to be Ricci-recurrent manifold if there exists a non-zero 1-form $A$ such that

$$ (\nabla_W S)(Y, Z) = A(W)S(Y, Z), $$

for any vector fields $W, Y, Z$ on $M$. Replacing $Z$ by $\xi$ in the above equation and using (2.10), we get,

$$ (\nabla_W S)(Y, \xi) = -2nA(W)\eta(Y), $$

which implies that,

$$ WS(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi) = -2nA(W)\eta(Y). $$

Using (2.10) and (2.6), the above equation becomes,

$$ 2n(\nabla_W \eta)Y + 2n\eta(W)\eta(Y) + S(Y, W) = 2nA(W)\eta(Y). $$

Again using (2.12), the above equation reduces to,

$$ 2ng(W, Y) + S(Y, W) = 2nA(W)\eta(Y). $$

Taking $W = \xi$ in the above equation and using (3.2), we obtain,

$$ r = 2\lambda + 2\mu + (p + \frac{2}{2n+1}) + 4n(A(\xi) - 1). $$

So we can state,

**Theorem 3.9.** If the metric of a $(2n+1)$ dimensional Ricci-recurrent Kenmotsu manifold is a conformal $\eta$-Einstein soliton with the 1-form $A$, then the scalar curvature becomes $2\lambda + 2\mu + (p + \frac{2}{2n+1}) + 4n(A(\xi) - 1)$.

### 4. Example of a 3-dimensional Kenmotsu manifold admitting conformal $\eta$-Einstein soliton:

We consider the three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq (0,0,0)\}$, where $(x, y, z)$ are standard coordinates in $\mathbb{R}^3$. The vector fields

$$ e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = -z \frac{\partial}{\partial z} $$

are linearly independent at each point of $M$. Let $g$ be the Riemannian metric defined by

$$ g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0, $$

$$ g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1. $$
Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on $M$ and $\phi$ be the $(1, 1)$-tensor field defined by,

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$ 

Then using the linearity of $\phi$ and $g$, we have,

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$.

Let $\nabla$ be the Levi-Civita connection with respect to the Riemannian metric $g$. Then we have,

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$ 

The connection $\nabla$ of the metric $g$ is given by,

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([X, Y], Z) + g(Z, [X, Y]),$$

which is known as Koszul's formula.

Using Koszul's formula, we can easily calculate,

$$\nabla e_1 e_1 = -e_3, \quad \nabla e_1 e_2 = 0, \quad \nabla e_1 e_3 = e_1,$$

$$\nabla e_2 e_1 = 0, \quad \nabla e_2 e_2 = -e_3, \quad \nabla e_2 e_3 = e_2,$$

$$\nabla e_3 e_1 = 0, \quad \nabla e_3 e_2 = 0, \quad \nabla e_3 e_3 = 0.$$ 

From the above it follows that the manifold satisfies $\nabla_X \xi = X - \eta(X)\xi$, for $\xi = e_3$. Hence the manifold is a Kenmotsu Manifold.

Also, the Riemannian curvature tensor $R$ is given by,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$ 

Hence,

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_3 = -e_1, \quad R(e_2, e_1)e_1 = -e_2,$$

$$R(e_2, e_3)e_3 = -e_2, \quad R(e_3, e_1)e_1 = -e_3, \quad R(e_3, e_2)e_2 = -e_3,$$

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_1 = 0, \quad R(e_3, e_1)e_2 = 0.$$ 

Then, the Ricci tensor $S$ is given by,

$$S(e_1, e_1) = -2, \quad S(e_2, e_2) = -2, \quad S(e_3, e_3) = -2.$$ 

From (3.2), we have,

$$S(e_3, e_3) = -[\lambda + \mu - \frac{r}{2} + \frac{(p + \frac{2}{3})}{2}],$$

(4.1)

which implies that,

$$r = 2\lambda + 2\mu - 4 + (p + \frac{2}{3}).$$

(4.2)

Hence $\lambda$ and $\mu$ satisfies equation (3.3) and so $g$ defines a conformal $\eta$-Einstein soliton on the 3-dimensional Kenmotsu manifold $M$. 

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