The high-density electron gas: How its momentum distribution $n(k)$ and its static structure factor $S(q)$ are mutually related through the off-shell self-energy $\Sigma(k, \omega)$

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It is shown in detail how the ground-state self-energy $\Sigma(k, \omega)$ of the spin-unpolarized uniform electron gas (with the density parameter $r_s$) in its high-density limit $r_s \to 0$ determines: the momentum distribution $n(k)$ through the Migdal formula, the kinetic energy $t$ from $n(k)$, the potential energy $v$ through the Galitskii-Migdal formula, the static structure factor $S(q)$ from $e = t + v$ by means of a Hellmann-Feynman functional derivative. The ring-diagram partial summation or random-phase approximation is extensively used and the results of Macke, Gell-Mann/Brueckner, Daniel/Vosko, Kulik, and Kimball are summarized in a coherent manner. There several identities were brought to the light.
I. INTRODUCTION

Although not present in the Periodic Table the homogeneous electron gas (HEG) is still an important and so far unsolved model system for electronic structure theory, cf. e.g. [1]. In its spin-unpolarized version, the HEG ground state is characterized by only one parameter \( r_s \), such that a sphere with the radius \( r_s \) contains on average one electron [2]. It determines the Fermi wave number as \( k_F = 1/(\alpha r_s) \) in atomic units (a.u.) with \( \alpha = [4/(9\pi)]^{1/3} \approx 0.521062 \) and it measures simultaneously both the interaction strength and the density such that high density corresponds to weak interaction and hence weak correlation [3]. For recent papers on this limit cf. [4, 5, 6, 7, 8, 9, 10, 11]. Usually the total ground-state energy per particle is written as (here and in the following are wave numbers measured in units of \( k_F \) and energies in units of \( k_F^2 \))

\[
e = e_0 + e_x + e_c, \quad e_0 = \frac{3}{5} \frac{1}{2}, \quad e_x = -\frac{3}{4} \frac{\alpha r_s}{\pi}, \quad e_c = (\alpha r_s)^2[a \ln r_s + (b + b_{2x}) + O(r_s)], \tag{1.1}
\]

where \( e_0 \) is the energy of the ideal Fermi gas, \( e_x \) is the exchange energy in lowest (1st) order (the corresponding direct term is zero, because the system is neutral), and \( e_c \) is referred to as correlation energy. The constants \( a \) and \( b \) arise from the ring-diagram summation as explained in the following. Naively one should expect that in the high-density limit the Coulomb repulsion \( e^2/r \) can be treated as perturbation. But in the early theory of the HEG, Heisenberg has shown [14], that ordinary perturbation theory with \( e_c = e_2 + e_3 + \cdots \) and \( e_n \sim (\alpha r_s)^n \), where the subscript \( n \) is the perturbation order, does not apply. Namely, in 2nd order, there is a direct term \( e_{2d} \) and an exchange term \( e_{2x} \), so that \( e_2 = e_{2d} + e_{2x} \). Unfortunately the direct term \( e_{2d} \) logarithmically diverges along the Fermi surface (i.e. for vanishing transition momenta \( q \)): \( e_{2d} \to \ln q \) for \( q \to 0 \) [14]. This failure of perturbation theory has been repaired by Macke [15] with an appropriate partial summation of higher-order terms \( e_{3r}, e_{4r}, \cdots \) (the subscript “r” means “ring diagram”) up to infinite order. The result is Eq. (1.1) with \( a = (1 - \ln 2)/\pi^2 \approx 0.031091 \) after Macke [15] and \( b \approx -0.0711 \) after Gell-Mann and Brueckner [16]. The latter means, that there is a pure 2nd-order remainder of the ring-diagram summation in addition to the non-analyticity \( r_s^2 \ln r_s \). A consistent description up to this order requires to take into account all other terms of the same order, i.e. \( e_{2x} \sim r_s^2 \). After Onsager, Mittag, and Stephen [12] it is \( e_{2x} = (\alpha r_s)^2 b_{2x} \) with \( b_{2x} = (1/6) \ln 2 - 3\zeta(3)/(2\pi)^2 \approx +0.02418 \). Thus part of the direct term \( b \) is compensated by the exchange term \( b_{2x} = -0.34 b \sim b + b_{2x} = 0.66 b \). The physically plausible partial
summation of higher-order perturbation terms used by Macke, respectively Gell-Mann and Brueckner is called ring-diagram summation with its characteristic particle-hole-pair excitations ($k \rightarrow k + q, |k| < 1, |k + q| > 1$, $q =$ momentum transfer), also known as the RPA = random phase approximation. It is the simplest approximation which simultaneously describes the closely related phenomena of screening effects and plasma oscillations as well as plasmon propagation with dispersion. (For their damping one has to go beyond RPA with local field corrections.) In the high-density limit, correlation “c” starts with RPA or ring-diagram terms “r”, symbolically written as $c = r + \cdots$. In the following the screening parameter $q_c^2 = 4 \alpha r_s / \pi$ (being the interaction strength $\alpha r_s$ times $4 / \pi$) is used.

In terms of this screening wave number $q_c$ the electron-gas plasma frequency (measured in units of $k_F^2$) is $\omega_{pl} = q_c / \sqrt{3} = \sqrt{4 \alpha r_s / (3 \pi)}$. The building elements of the RPA Feynman diagrams are shown in Figs. 1a,b. The diagrams for $e_r$ and $e_{2x}$ are in Figs. 1c,d, middle parts.

The non-analytical behavior of the total energy $e$ at the high-density limit carries over to its kinetic and potential components, $t$ respectively $v$, through the virial theorem [17] and to the chemical potential $\mu$ through the Seitz theorem [18]:

$$v = r_s \frac{d}{dr_s} e, \quad t = -r_s^2 \frac{d}{dr_s} \frac{1}{r_s} e, \quad \mu = \left( \frac{5}{3} - \frac{1}{3} r_s \frac{d}{dr_s} \right) e. \quad (1.2)$$

In the high-density limit, this means

$$t = t_0 + t_x + t_c, \quad t_0 = \frac{3}{5} \cdot \frac{1}{2}, \quad t_x = 0, \quad t_c = -(\alpha r_s)^2 \left[ a \ln r_s + (a + b + b_{2x}) + O(r_s) \right], \quad (1.3)$$

$$v = v_0 + v_x + v_c, \quad v_0 = 0, \quad v_x = -\frac{3}{4} \frac{\alpha r_s}{\pi}, \quad v_c = (\alpha r_s)^2 \left[ 2a \ln r_s + (a + 2b + 2b_{2x}) + O(r_s) \right], \quad (1.4)$$

$$\mu = \mu_0 + \mu_x + \mu_c, \quad \mu_0 = \frac{1}{2}, \quad \mu_x = -\frac{\alpha r_s}{\pi}, \quad \mu_c = (\alpha r_s)^2 \left[ a \ln r_s + \left( \frac{1}{3}a + b + b_{2x} \right) + O(r_s) \right]. \quad (1.5)$$

Fundamental relations between the simplest quantum-kinematicallytional quantities (momentum
distribution \(n(k)\) and static structure factor \(S(q)\) and the energy components \(t\) and \(v\) are

\[
t = \int_0^{\infty} d(k^3) \ n(k) \frac{k^2}{2}, \quad \int_0^{\infty} d(k^3) \ n(k) = 1, \quad 0 \leq n(k) \leq 1, \quad (1.6)
\]

\[
z_F = n(1^-) - n(1^+), \quad 0 \leq z_F = 1 - O(r_s), \quad n(k \to \infty) \to \frac{A(r_s)}{k^8} + \cdots,
\]

\[
v = -\frac{1}{3 \cdot 4} \int_0^{\infty} d(q^3) \ \left[1 - S(q)\right] \frac{q^2}{q^2} \ , \quad S(q \to 0) = \frac{q^2}{2\omega_{pl}} + \cdots , \quad [1 - S(q \to \infty)] \to \frac{B(r_s)}{q^4} + \cdots \quad (1.7)
\]

with the static 1-body quantity \(n(k)\), the momentum distribution, and with the static 2-body quantity \(S(q)\), the static structure factor (SSF). The Fourier transform of \(1 - S(q)\) is \(1 - g(r)\) with \(g(r) \geq 0\) being the pair density (PD), see Sec.V. The SSF \(S(q)\) behaves at transition momenta \(|q| = 2\) non-analytically, because there occurs a topological change from two overlapping to two non-overlapping Fermi spheres. This causes asymptotic Friedel oscillations of the PD \(g(r \to \infty)\), whereas cusp singularities \(S(q \to 0) \sim q^{\text{odd}}\) let emerge non-oscillatory asymptotic terms of \(g(r \to \infty)\). For the asymptotic coefficients the sum rules

\[
A(r_s) = \frac{1}{2} \omega_{pl}^4 \ g(0) \quad \text{and} \quad B(r_s) = 2 \omega_{pl}^2 \ g(0) \quad \text{with} \quad 1 - g(0) = \frac{1}{2} \int_0^{\infty} d(q^3) \ [1 - S(q)] \quad (1.8)
\]

hold with the on-top PD \(g(0) = 1/2 - O(r_s)\) \([19, 20]\) (for \(n(k)\)) and \([19, 21, 22]\) (for \(S(q)\)). \(g(0)\) is given by the normalisation of \(1 - S(q)\) and describes short-range correlations together with the peculiar behavior of \(g(r \ll 1/q_c)\), besides it determines the large-wave-number asymptotics of \(n(k)\) and \(S(q)\).

In view of (1.6) and (1.7), one may ask, which peculiarities of these lowest-order quantum-kinematical quantities cause the non-analyticities of \(t\) and \(v\). The above mentioned drastic changes, when switching on the Coulomb interaction, show up in the redistribution of the non-interacting momentum distribution \(n_0(k) = \Theta(1 - k)\) within thin layers inside and outside the Fermi surface \(|k| = 1\) and a remaining finite discontinuity \(z_F > 0\) (Migdal theorem [24, 25]). They show up also in the plasmon behavior of \(S(q)\) within a small spherical region around the origin of the reciprocal space, what causes an inflexion point \(q_{infl}, S_{infl} \sim \omega_{pl}\). All these reconstructions describe the long-range correlation (screening, collective mode called plasmon), characteristic for the Coulomb interaction. They
are treated in lowest order again by ring-diagram summations with the replacements 
\[ n_{2d}(k) \rightarrow n_t(k) \text{ and } S_{1d}(q) \rightarrow S_t(q). \]
(Note, that \( t_{2d} \) arises from \( n_{2d}(k) \), but \( v_{2d} \) from \( S_{1d}(q) \).) How these replacements lead to the \( r_s^2 \ln r_s \) terms of \( t \) and \( v \) is shown in [6]. However, because the redistributions take place essentially only in the mentioned sensitive regions \( |k| - 1 \ll q_c \) and \( |q| \ll q_c \), the \( r \)-terms approach the original \( d \)-terms far off these regions. Thus, to be consistent up to this order, the corresponding \( x \)-terms must be taken into account: 
\[ n_c(k) = n_t(k) + n_{2x}(k) + \cdots, \quad S_c(q) = S_t(q) + S_{1x}(q) + \cdots. \]
These \( x \)-terms compensate part of the \( r \)-terms, similarly as this is the case for \( e_c, t_c, v_c, \mu_c \) with the ratios of \( b_{2x} \) to \( b, a + b, a/2 + b, -a/3 + b \) being \(-0.34, -0.6, -0.43, -0.3\), respectively. The ring-diagram summation for \( n(k) \) has been developed in [27, 28], an analytical extrapolation is given in [30], for the spin-polarized case see [31]. The ring-diagram summation for \( S(q) \) has been done in [32, 33]. In [34] both \( n(k) \) and \( S(q) \) are considered on the same footing.

Another relevant ground state property is Dyson’s self-energy \( \Sigma(k, \omega) \), a dynamical 1-body quantity. Its on-shell value (as a function of \( r_s \)) is related to the chemical-potential shift through the Hugenholtz-van Hove (Luttinger-Ward) theorem \( \mu - \mu_0 = \Sigma(1, \mu) \) [35]. Besides this, with the off-shell self-energy \( \Sigma(k, \omega) \) also the 1-body Green’s function
\[
G(k, \omega) = \frac{1}{\omega - t(k) - \Sigma(k, \omega)} \quad \text{with} \quad \Sigma(k, \omega) \rightarrow \text{sign}(k - 1) i\delta \quad \text{for} \quad r_s \rightarrow 0 \quad (1.9)
\]
is known, from which follow the momentum distribution (Migdal formula [24])
\[
n(k) = \int_{C_+} \frac{d\omega}{2\pi i} e^{i\omega\delta} G(k, \omega) \sim \int_0^\infty d(k^3) n(k) = 1, \quad t = \int_0^\infty d(k^3) n(k) \frac{k^2}{2} \quad (1.10)
\]
and the potential energy (Galitskii-Migdal formula [37], for its use in total-energy calculations cf. [38] and refs. therein)
\[
v = \frac{1}{2} \int_0^\infty d(k^3) \int_{C_+} \frac{d\omega}{2\pi i} e^{i\omega\delta} G(k, \omega) \Sigma(k, \omega) \quad (1.11)
\]
with \( \delta \geq 0 \) and \( C_+ \) means the closing of the contour in the upper complex \( \omega \)-plane.

Note that \( \Sigma(k, \omega) \) and thus also \( G(k, \omega), n(k) \) and \( t \), as well as \( v \) are functionals of \( t(k) \) and \( v(q) \). Supposed the ground state energy \( e \) is available as such a functional, then the
(generalized) Hellmann-Feynman theorems \[17\]

\[
n(k) = \frac{4\pi}{3} \frac{\delta e}{\delta t(k)}, \quad S(q) - 1 = 16\pi \frac{\delta e}{\delta v(q)}
\]

(1.12)

hold, cf. App. A. These are equivalent writings of expressions given in \[27, 34\] for \(n(k)\) and in \[22, 32\] for \(S(q)\). The quantities \(e, n(k), t, S(q)\), and \(v\) result as functions of \(\rho_s\) from the replacements \(t(k) \rightarrow k^2/2\) and \(v(q) \rightarrow q^2/\pi^2\). These fundamental relations permit the following procedure (see Fig. 2): If \(\Sigma(k, \omega)\) is available as a functional of \(t(k)\) and \(v(q)\) from perturbation theory or otherwise, then \(n(k)\) can be calculated with the Migdal formula (1.10). Therefrom follows \(t\) with (1.6) and \(v\) with the Galitskii-Migdal formula (1.11). Finally from their sum \(t + v = e\) and the functional derivative (1.12) the SSF \(S(q)\) results (and \(n(k)\) may be checked once more for consistency). So the dynamical 1-body quantity \(\Sigma(k, \omega)\) provides the static 2-body quantity \(S(q)\).

Whereas in \[9\] the on-shell self-energy \(\Sigma(1, \mu)\) has been studied, here the off-shell self-energy \(\Sigma(k, \omega)\) is considered. The problem in \[9\] was, to find out the correct diagrammatic sum for \(\Sigma(1, \mu)\) on the rhs of the Luttinger-Ward theorem, which makes it an identity in the high-density limit \(\rho_s \rightarrow 0\). The answer: the GW approximation with \(W = \text{RPA}\) and \(G = \text{an appropriately renormalized particle-hole line}\) yield the correct \(\rho_s\)-behavior, which agrees with the lhs as it follows from \(\mu(\rho_s \rightarrow 0)\), cf. (1.5). Here it is shown in detail how the off-shell self-energy \(\Sigma(k, \omega)\) for the model case \(\rho_s \rightarrow 0\) yields \(n(k), t, v,\) and \(S(q)\) step by step according to the procedure of Fig. 2. So it is shown how \(n(k)\) and \(S(q)\), which have their common origin in the 2-body density matrix, are indirectly linked mutually through the self-energy \(\Sigma(k, \omega)\).

Perturbation theory for \(\Sigma(k, \omega)\) means \(\Sigma = \Sigma_x + \Sigma_c\) with \(\Sigma_c = \Sigma_2 + \Sigma_3 + \cdots\), where \(\Sigma_2 = \Sigma_{2d} + \Sigma_{2x}\). The divergence of \(\Sigma_{2d}\) is corrected by the ring-diagram summation (RPA): \(\Sigma_{2d} + \cdots \rightarrow \Sigma_r\), so \(\Sigma_c = \Sigma_r + \Sigma_{2x} + \cdots\). The building elements of the RPA Feynman diagrams (cf. Figs. 1a and 1b) are the Coulomb repulsion \(\pi^2 v(q) \rightarrow \pi^2 q_c^2/q^2\) with the coupling constant \(q_c^2 = 4\alpha \rho_s/\pi\) and the one-body Green’s function of free electrons with \(t(k) \rightarrow k^2/2\),

\[
G_0(k, \omega) = \frac{\Theta(k - 1)}{\omega - t(k) + i\delta} + \frac{\Theta(1 - k)}{\omega - t(k) - i\delta}, \quad \delta \geq 0.
\]

(1.13)
From $G_0(k, \omega)$ follows the particle-hole propagator $Q(q, \eta)$ in RPA according to
\[
Q(q, \eta) = -\int \frac{d^3 k}{4\pi} \int \frac{d\omega}{2\pi i} G_0(k, \omega) G_0(|k+q|, \omega + \eta)
\]
with the result
\[
Q(q, \eta) = \int_{|k|<1, |k+q|>1} \frac{d^3 k}{4\pi} \left[ \frac{1}{t(k+q) - t(k) - \eta - i\delta} + \frac{1}{t(k+q) - t(k) + \eta - i\delta} \right].
\]

The denominators contain the excitation energy to create a hole with $k$ inside the Fermi sphere and a particle with $k + q$ outside the Fermi sphere. Thus, $Q(q, \eta)$ is a functional of $t(k)$ with the functional derivative (A.5) and with $R(q, u) = Q(q, iqu)$ defining a real function, see (B.1). In the following the self-energy contributions $\Sigma_x$ (Sec. II), $\Sigma_r$ (Sec. III), $\Sigma_{2x}$ (Sec. VI) are explicitly given as they result from the diagram rules and it is derived, what follows from them: $n(k)$, $t$, $v$, $e$, $S(q)$. For (here not considered) issues as $S(q, \omega)$, $\varepsilon(k, \omega)$, GW-approximation etc. cf. e.g. [38, 39, 40, 41, 42] and refs. therein. Sec. V deals with the short-range correlation following from $S(q)$, cf. (1.8). Although the high-density spin-unpolarized HEG is only a marginal corner in the complex field of electron correlation, its ring-diagram summation gives deeper insight through rigorous theorems, how energies and low-order quantum-kinematics are functionally related, what should be of a more general interest.

\section{II. FIRST ORDER}

The 1st-order direct term vanishes because of the neutralizing positive background. So the expansion of $\Sigma(k, \omega)$ starts with the 1st-order exchange term
\[
\Sigma_x(k, [v(q)]) = -\int_{|k+q|<1} \frac{d^3 q}{(2\pi)^3} \pi^2 v(q) = -\frac{1}{8\pi} \int_{|k+q|<1} d^3 q \ v(q).
\]

Its peculiarity is, that it does not depend on $\omega$, what makes $n_x(k)$ vanishing identically, because of Eq. (1.10) with $G \approx G_0 \Sigma_x G_0$. This is in agreement with $t_x = 0$ as a consequence of the virial theorem (1.2). One may therefore conclude, that all the features of $n(k)$ start with the 2nd order. But this is not true. The peculiar redistribution of the non-interacting momentum distribution $n_0(k) = \Theta(1-k)$ due to the Coulomb repulsion makes the discontinuity jump $z_F = n(1^-) - n(1^+)$ to deviate from its non-interacting value of 1 in 1st order:
\[ z_F = 1 - 0.18 \, r_s + \cdots \]. But the corresponding kinetic energy starts with \( t_c \sim r_s^2 \ln r_s \). The potential energy \( v_x \) follows again as a functional of \( v(q) \) from the Galitskii-Migdal formula (1.11):

\[
v_x[v(q)] = \frac{3}{8\pi} \int d^3 k \, \Sigma_x(k, [v(q)]) = -\frac{3}{(8\pi)^2} \int_{|k| < 1, |k + q| < 1} d^3 k \int d^3 q \, v(q) . \tag{2.2}
\]

This is, because of \( t_x = 0 \), also the total energy in this order: \( e_x[v(q)] = v_x[v(q)] \). Thus the functional derivative (1.12) yields

\[
S_0(q) = 1 - \frac{3}{4\pi} \int_{|k| < 1, |k + q| < 1} d^3 k = \left[ \frac{3q}{2} - \frac{1}{2} \left( \frac{q}{2} \right)^3 \right] \Theta(2 - q) + \Theta(q - 2). \tag{2.3}
\]

The integral arises (for \( q \leq 2 \)) from two overlapping Fermi spheres. For \( q \geq 2 \) they do not overlap, thus a drastic change of the topology occurs, when passing \( q = 2 \). After Fourier transformation, see (5.1), this non-analyticity causes the asymptotic Friedel oscillations of the non-interacting PD \( g_0(r \to \infty) \sim 1 - \cos 2r, \sin 2r \). Another non-analyticity, namely the cusp singularities \( S_0(q \to 0) \sim q, q^3 \) make the non-oscillatory terms of \( g_0(r \to \infty) \). The above integral \( \int_{|k| < 1} d^3 k \) is just the volume of two calottes with the height \( h = 1 - q/2 \), hence \( \int_{|k| < 1} d^3 k = 2 \cdot \frac{\pi}{3} h^2(3 - h) \). If in Eqs. (2.1) and (2.2) the general interaction line \( v(q) \) is replaced by the Coulomb interaction \( q_c^2/q^2 \), then

\[
\Sigma_x \left( k, \left[ \frac{q_c^2}{q^2} \right] \right) = -\frac{q_c^2}{4} \left( 1 + \frac{1 - k^2}{2k} \ln \left| \frac{k + 1}{k - 1} \right| \right) \quad \text{and} \quad v_x = -\frac{3}{16} q_c^2 . \tag{2.4}
\]

This shows how the procedure of Fig. 2 works in lowest order.

### III. SECOND ORDER: THE DIRECT TERM \( \Sigma_{2d} \) AND ITS RPA CORRECTION

#### A. The self-energy \( \Sigma_r(k, \omega) \)

In 2nd order there is a direct term (d) and an exchange term (x), see Figs. 1c,d, left:

\[
\Sigma_2(k, \omega) = \Sigma_{2d}(k, \omega) + \Sigma_{2x}(k, \omega) .
\]

The direct term diverges along the Fermi surface, i.e. for vanishing transition momenta \( q_0 \to 0 \). This flaw is repaired by the ring-diagram summation (Fig. 1c, left) with the result

\[
\Sigma_r(k, \omega) = \frac{1}{8\pi} \int_{q > q_0} d^3 q \int \frac{d\eta}{2\pi i} \frac{v^2(q)Q(q, \eta)}{1 + v(q)Q(q, \eta)} G_0(|k + q|, \omega + \eta), \quad q_0 \geq 0 .
\]
Note, that this defines a functional of \( v(q) \) and note also
\[
\frac{v(q)}{1 + v(q)Q(q, \eta)} \rightarrow \frac{q^2}{1 + q^2 Q(q, \eta)} \quad \text{for} \quad v(q) \rightarrow \frac{q_c^2}{q^2},
\]
where the screening (or “Yukawa”) term \( q_c^2 Q(q, \eta) \) in the denominator makes the bare Coulomb repulsion renormalized and removes the above mentioned divergence of \( \Sigma_{2d}(k, \omega) \).

\( G_0(k, \omega) \) and \( Q(q, \eta) \) are functionals of \( t(k) \), see (1.13), (1.15). Use of Eq. (1.13) leads to
\[
\Sigma_r(k, \omega) = \frac{1}{8\pi} \int d^3q \int \frac{dq}{2\pi i} \frac{v^2(q)Q(q, \eta)}{1 + v(q)Q(q, \eta)} \times \left[ \frac{\Theta(|k + q| - 1)}{\omega + \eta - \frac{1}{2}k^2 - q \cdot (k + \frac{1}{2}q) + i\delta} + \frac{\Theta(1 - |k + q|)}{\omega + \eta - \frac{1}{2}k^2 - q \cdot (k + \frac{1}{2}q) - i\delta} \right].
\]
(3.1)

In the following it is shown, how \( n_r(k) \) and \( t_r \) result from this \( \Sigma_r(k, \omega) \).

**B. How \( n_r(k) \) results from \( \Sigma_r(k, \omega) \) and \( t_r \) from \( n_r(k) \)**

According to the Migdal formula (1.10), \( n(k) \) follows from \( G(k, \omega) \). In RPA it is
\[
G(k, \omega) = G_0(k, \omega) + G_0(k, \omega)\Sigma_r(k, \omega)G_0(k, \omega) + \cdots
\]
(3.2)
The first term yields the momentum distribution of the ideal Fermi gas
\[
n_0(k) = \int_C \frac{d\omega}{2\pi i} \left[ \frac{\Theta(k - 1)}{\omega - \frac{1}{2}k^2 + i\delta} + \frac{\Theta(1 - k)}{\omega - \frac{1}{2}k^2 - i\delta} \right] = \Theta(1 - k)
\]
(3.3)
being for small \( r_s \) additively corrected by the RPA expression \( n_r(k) \), which follows from the second term of Eq. (3.2) according to
\[
n_r(k) = \int \frac{d\omega}{2\pi i} G_0(k, \omega)\Sigma_r(k, \omega)G_0(k, \omega).
\]
(3.4)
Using \( \Sigma_r(k, \omega) \) of Eq. (3.1) it is
\[
n_r(k) = \frac{1}{8\pi} \int d^3q \int \frac{dq}{2\pi i} \frac{v^2(q)Q(q, \eta)}{1 + v(q)Q(q, \eta)} f(k, q, \eta).
\]
(3.5)
The case “2d” appears, when the “Yukawa” term \( v(q)Q(q, \eta) \) in the denominator is deleted (‘descreening’). The contour integrations are comprised [using \( \Theta(k - 1)\Theta(1 - k) = 0 \)] in
\[
f(k, q, \eta) = \int \frac{d\omega}{2\pi i} \left[ \frac{\Theta(k - 1)}{(\omega - \frac{1}{2}k^2 + i\delta_1)(\omega - \frac{1}{2}k^2 + i\delta_2)} + \frac{\Theta(1 - k)}{(\omega - \frac{1}{2}k^2 - i\delta)(\omega - \frac{1}{2}k^2 - i\delta)} \right] \times \left[ \frac{\Theta(|k + q| - 1)}{\omega + \eta - \frac{1}{2}k^2 - q \cdot (k + \frac{1}{2}q) + i\delta} + \frac{\Theta(1 - |k + q|)}{\omega + \eta - \frac{1}{2}k^2 - q \cdot (k + \frac{1}{2}q) - i\delta} \right].
\]
(3.6)
Only the combinations $\Theta(k - 1)\Theta(1 - |k + q|)$ with a pole in the upper $\omega$-plane and $\Theta(1 - k)\Theta(|k + q| - 1)$ with a pole in the lower plane contribute. Next the contour integration along the real $\omega$-axis is closed by half-circles in the upper, respectively lower plane. Note that $\int_{C_{\pm}} d\omega / (\omega - \omega_{\pm}) = \pm 2\pi i$, where $\text{Im} \, \omega_{\pm} \gtrless 0$. The result is

$$f(k, q, \eta) = \frac{\Theta(k, q)}{[\eta - q(k + \frac{1}{2}q)]^2} \quad (3.7)$$

with $\Theta(k, q) = \pm 1$ for $|k| \gtrless 1$, $|k + q| \lessgtr 1$ and 0 otherwise. This provides with the particle-number conservation, because of

$$\left( \int_{|k| > 1, |k + q| < 1} - \int_{|k| < 1, |k + q| > 1} \right) \frac{d^3k}{[\eta - q(k + \frac{1}{2}q)]^2} = 0 \quad \text{or} \quad \int d^3k \, n_t(k) = 0. \quad (3.8)$$

Here, with the replacement $k \rightarrow -k - q$, the denominator of the second integral transforms to $[\eta + q(k + \frac{1}{2}q)]^2$. The second integral is identical to the first one. This is because of the property $Q(q, -\eta) = Q(q, \eta)$ in the term in front of $f(k, q, \eta)$ in Eq. (3.5).

Next it is shown, how $t_r$ results from $n_t(k)$, starting with Eq. (3.5). With the identity (A.5) it can be written as

$$n_t(k) = -\frac{1}{4\pi} \int d^3q \int \frac{dn}{2\pi} \sum_{n=1}^{\infty} (-1)^{n+1} v^{n+1} q^n \frac{\delta Q(q, \eta)}{\delta t(k)} Q^n(q, \eta) \frac{\delta Q^n(q, \eta)}{\delta t(k)} \quad (3.9)$$

Next $t_r = 3/(4\pi) \int d^3k \, n_t(k) \, t(k)$ is combined with the identity (A.6). It results in

$$t_r = \frac{3}{16} \int d^3q \int \frac{dn}{2\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1} v^{n+1} q^n Q^{n+1}(q, \eta) \quad (3.10)$$

This is the contribution of the ring-diagram summation to the kinetic energy. For the total energy contribution $e_r$ the potential energy $v_r$ is needed.
C. How \( v_r \) results from \( \Sigma_r(k, \omega) \) and \( S_r(q) \) from \( e_r \)

Equations (1.9) and (3.1) inserted into the Galitskii-Migdal formula (1.11) and the \( \omega \)–integration performed yields

\[
v_r = \frac{1}{16\pi} \int d^3q \int \frac{d\eta}{2\pi i} \ \frac{v^2(q)Q(q, \eta)}{1 + v(q)Q(q, \eta)} \times \\
\times \int \frac{3}{4\pi} d^3k \left[ \Theta(|k + q| - 1)\Theta(1 - k) \Theta(1 - k) \right] .
\]

(3.11)

With the definition (1.15) of \( Q(q, \eta) \) it results

\[
v_r = -\frac{3}{16\pi} \int d^3q \int \frac{d\eta}{2\pi i} \ \frac{v^2(q)Q^2(q, \eta)}{1 + v(q)Q(q, \eta)} .
\]

(3.12)

The power-series expansion (its 1st term is \( v_{2d} \))

\[
v_r = -\frac{3}{16\pi} \int d^3q \int \frac{d\eta}{2\pi i} \sum_{n=1}^{\infty} (-1)^{n+1}v^{n+1}(q)Q^{n+1}(q, \eta) .
\]

makes it better comparable with Eq. (3.10) for \( t_r \). Their sum yields [with \( -\frac{n}{n+1} + 1 = \frac{1}{n+1} \)] the well-known RPA expression for the total energy (after Macke)

\[
e_r = \frac{3}{16\pi} \int d^3q \int \frac{d\eta}{2\pi i} [\ln(1 + v(q)Q(q, \eta)) - v(q)Q(q, \eta)] .
\]

(3.13)

(The 1st term of the power-series expansion gives \( e_{2d} \).) Note that \( r_s d e_r / d r_s \) agrees with \( v_r \) of Eq. (3.12): virial theorem (1.2). By means of functional derivatives [see Appendix A and Eqs. (1.12)] follow the SSF \( S_r(q) \) and the momentum distribution \( n_r(k) \).

Indeed, \( S_r(q) \) results from \( e_r[t(k), v(q)] \) as

\[
S_r(q) = 16\pi \frac{\delta e_r}{\delta v(q)} = -3 \int \frac{d\eta}{2\pi i} \ \frac{v(q)Q^2(q, \eta)}{1 + v(q)Q(q, \eta)},
\]

(3.14)

cf. Fig. 1c, right. If this is multiplied by \( v(q)/(16\pi) \) and integrated \( \int d^3q \) [according to (1.7)], then \( v_r \) turns out, as it should.

The analog procedure for \( n_r(k) \) is

\[
n_r(k) = 4\pi \frac{\delta e_r}{3 \delta t(k)} = -\frac{1}{4} \int d^3q \frac{d\eta}{2\pi i} \ \frac{v^2(q)Q(q, \eta)}{1 + v(q)Q(q, \eta)} \frac{\delta Q(q, \eta)}{\delta t(k)} ,
\]

(3.15)
in agreement with Eq. (3.5), as it should.

In the following it is shown, how Eqs. (3.12-3.15) really yield the high-density results of Macke [15], Gellmann/Brueckner [16], Daniel/Vosko [27], Kulik [28], and Kimball [32].

D. How the complex $Q(q, \eta)$ is replaced by the real $R(q, u)$ and how $v_r$, $S_r(q)$, and $n_r(k)$ behave for $r_s \to 0$

The replacement $\eta \to iqu$ turns the contour integration in the RPA expressions (3.12-3.15) into one along the real axis. Besides it is a useful trick to introduce the velocity $u$ instead of the frequency $\eta$. With the replacement $v(q) \to q_c^2/q^2$ they take the form (note $q \, d^3q = 2\pi q^2 \, d(q^2)$):

$$e_r = \frac{3}{8\pi} \int_0^\infty du \int_0^\infty d(q^2) \, q^2 \left[ \ln \left( 1 + \frac{q_c^2}{q^2} R(q,u) \right) - \frac{q_c^2}{q^2} R(q,u) \right], \quad (3.16)$$

$$v_r = -\frac{3q_c^4}{8\pi} \int_0^\infty du \int_0^\infty d(q^2) \, \frac{R^2(q,u)}{q^2 + q_c^2 R(q,u)}, \quad (3.17)$$
\[ S_r(q) = -\frac{3q_c^2}{\pi} q \int_0^\infty du \frac{R^2(q, u)}{q^2 + q_c^2 R(q, u)}, \]  

(3)

\[ n_r(k) = \frac{q_c^4}{8\pi} \int_0^\infty du \int_0^\infty d(q^2) \frac{R(q, u)}{q^2 + q_c^2 R(q, u)} \int_{-1}^1 d\zeta \frac{\Theta(k, q, \zeta)}{q^2 [iu - (k\zeta + \frac{1}{2}q)]^2} \]  

(4)

with \( \Theta(k, q, e_k e_q) = \Theta(k, q), \) see (3.7). In the following, these four RPA quantities are discussed in detail in the high-density limit \( r_s \to 0. \)

(1) Let us first consider the total energy \( e_r \) of Eq. (3.16). The power expansion leads in lowest order to

\[ e_{2d} = -\frac{3q_c^4}{16\pi} \int_0^\infty du \int_{q_0}^\infty d(q^2) \frac{R^2(q, u)}{q^2} = -\frac{3q_c^4}{(8\pi)^2} \int_{q_0}^\infty dq \frac{q^2}{q^2} I(q). \]  

(3.20)

For the momentum transfer function \( I(q) \), to be referred to as Macke function, see (C.11). Its property \( I(q \to 0) \sim q \) makes \( e_{2d} \) to diverge for \( q_0 \gtrless 0 \). Vice versa this flaw of the 2nd-order perturbation theory is rectified by the RPA summation (3.16). How this \( e_r \) behaves for small \( r_s \) with the result (1.1) has been shown by Macke [13] and Gell-Mann/Brueckner [16].

(2) In the following their method is applied to the potential energy \( v_r \) of Eq. (3.17). Deleting the term \( q_c^2 R(q, u) \) in the denominator cancels the RPA partial summation and the diverging expression

\[ v_{2d} = -\frac{3}{8\pi} q_c^4 \int_0^\infty du \int_{q_0}^\infty d(q^2) \frac{R^2(q, u)}{q^2} = -\frac{2 \cdot 3q_c^4}{(8\pi)^2} \int_{q_0}^\infty dq \frac{q^2}{q^2} I(q). \]  

(3.21)

results. It remains to show, how to extract from Eq. (3.17) the constants \( c_1 \) and \( c_2 \) of \( v_r = (\alpha r_s)^2 [c_1 \ln r_s + c_2 + O(r_s)] \) to be compared with \( v_c = v_r + O(r_s^3) \) of Eq. (1.4). Whereas \( c_1 \) follows from \( v_{2d} \), \( c_2 \) results from the peculiar behavior of \( v_r \) for small transition momenta \( q \). Therefore one can approximate \( R(q, u) \approx R_0(u) \ldots \) and restrict \( q \) to \( q < q_1 \), where \( q_1 \)
is a small (non-vanishing) momentum:

\[ v_r^0 = \frac{3}{8\pi} q_1^3 \int_0^\infty du \int_0^{q_1^2} d(q^2) \frac{R_0^2(u)}{q^2 + q_1^2 R_0(u)} \]

\[ = \frac{3}{8\pi} q_1^4 \int_0^\infty du R_0^2(u) \left\{ \ln[q_1^2 + q_c^2 R_0(u)] - \ln[q_c^2 R_0(u)] \right\} \]

\[ = \frac{3}{8\pi} q_1^4 \int_0^\infty du R_0^2(u) \left\{ \ln[q_1^2 + O(r_s)] - \ln[q_c^2 R_0(u)] \right\} . \]  \hspace{1cm} (3.22)

With the constants \( a, b'_t \) defined in Appendix B it is

\[ v_r^0 = (\alpha r_s)^2 \left[ 2a \ln r_s + 2(a \ln \frac{4\alpha}{\pi} + b'_t - a \ln q_1^2) \right] + O(r_s^3) , \]  \hspace{1cm} (3.23)

thus \( c_1 = 2a \). To find also \( c_2 \) the difference \( \Delta v_{2d} = v_{2d} - v_{2d}^0 \) between the correct 2nd-order term of Eq. (3.22) and the first term in the expansion of \( v_r^0 \), namely

\[ v_{2d}^0 = - \frac{3}{2\pi^4} \int_0^{q_1} \frac{dq}{q^2} 2 \cdot 4\pi q \frac{\pi^3}{3} a \]  \hspace{1cm} (3.24)

has to be considered (exploiting the trick of Gell-Mann/Brueckner \textit{mutatis mutandi}):

\[ \Delta v_{2d} = -(\alpha r_s)^2 \left\{ \int_{q_0}^{\infty} \frac{dq}{q^2} I(q) - \int_{q_0}^{q_1} \frac{dq}{q^2} 2 \cdot 4\pi q \frac{\pi^3}{3} a \right\} + O(r_s^3) \]

\[ = (\alpha r_s)^2 \left\{ \frac{3}{2\pi^4} \int_{q_0}^{\infty} \frac{dq}{q^2} I(q) - \frac{8\pi^4 a}{3} \Theta(1 - q) + 4a \int_{1}^{q_1} \frac{dq}{q} \right\} + O(r_s^3) . \]  \hspace{1cm} (3.25)

The first integral does no longer diverge for \( q_0 \to 0 \), therefore one can set \( q_0 = 0 \). Besides

\[ \Delta v_{2d} = (\alpha r_s)^2 2(b_{2d} + a \ln q_1^2) + O(r_s^3) \]  \hspace{1cm} (3.26)

shows [for \( b_{2d} \) see (B.8)], that for \( r_s \to 0 \) the sum \( v_r^0 + \Delta v_{2d} \) does not depend on the arbitrary cut-off \( q_1 \):

\[ v_r = (\alpha r_s)^2 2 \left[ a \ln r_s + \left( a \ln \frac{4\alpha}{\pi} + b'_t + b_{2d} \right) \right] + O(r_s^3) . \]  \hspace{1cm} (3.27)

This has to be compared with Eq. (1.4). Indeed the constant direct term beyond \( \ln r_s \) yields

\[ a + 2b = 2(a \ln \frac{4\alpha}{\pi} + b'_t + b_{2d}) , \]  \hspace{1cm} (3.28)
which defines $b$. Its value $b \approx -0.0711$ agrees with what is given in [16], as it should. - The appearance of a 2nd-order term $\sim r_s^2$ means that - to be consistent - all other terms of the same order contribute to the small-$r_s$ behavior of $e$. There is only one such term, namely $v_{2s}$, as treated in Sec. IV.

(3) Next the static structure factor of Eq. (3.18) is considered. In the lowest order ($r \rightarrow 1d$) it is - again with the Macke function $I(q)$ of Eq. (C.11) and with $\omega_{pl} = q_c/\sqrt{3}$

$$S_{1d}(q) = -2 \frac{\omega_{pl}^2}{(4\pi/3)^2} \frac{I(q)}{q^2} \oslash$$

$$S_{1d}(q \rightarrow 0) = -3(1 - \ln 2 ) \frac{\omega_{pl}^2}{q} + O(1/q^3), \quad S_{1d}(q \rightarrow \infty) = -2 \frac{\omega_{pl}^2}{q^4} + O(1/q^6).$$

At $q = 2$, the non-interacting value $S_{1d}(2) = 1$ and its discontinuity jump $\Delta S_{1d}'(2) = 3/4$, which arises from the Fermi edge, are reduced by [with $I(2), \Delta I''(2)$ from [9], Eq. (C.2)]

$$S_{1d}(2) = -(13 - 16 \ln 2 ) \frac{3}{20} \omega_{pl}^2, \quad \Delta S_{1d}'(2) = - \left( \frac{3}{4} \right)^2 \omega_{pl}^2. \quad (3.30)$$

Note, that $S_{1x}(q)$ compensates part (ca. half) of $S_{1d}(q)$, cf. Eq. (4.7), and note that $S(q) = S_0(q) + S_{1d}(q) + S_{1x}(q) + \cdots$ decreases with increasing $r_s$ for a given value of $q$. Whereas for $q \gg q_c$ the perturbative treatment $S_r(q) = S_{1d}(q) + O(r_s^2)$ holds (screening effects are not so important for large momentum transfers $q$), for $q \ll q_c$ there is a big difference between the ‘bare’ $S_{1d}(q)$, which behaves unphysically because of $I(q \rightarrow 0) \sim q$, cf. Eq. (C.11), and its renormalized counterpart $S_r(q)$, where the ring-diagram summation ameliorates the above mentioned flaw of $S_{1d}(q)$, cf. [6], Fig. 3. For small $q$, the approximation $R(q, u) = R_0(u) + \cdots$ is sufficient. So $S_r(q) \approx -q_c L(q/q_c)$ with

$$L(y) = \frac{3}{\pi} y \int_0^\infty du \frac{R_0^2(u)}{y^2 + R_0(u)} \oslash \quad L(y \rightarrow 0) = \frac{3}{4} y - \frac{\sqrt{3}}{2} y^2 + \frac{9\sqrt{3}}{20} y^4 + \cdots, \quad (3.31)$$

to be referred to as Kimball function, for its properties cf. [6]. As a consequence, the expression

$$S_r(q \ll q_c) = - \frac{3}{4} q + \frac{q^2}{2 \omega_{pl}} - \frac{3}{20} \frac{q^4}{\omega_{pl}^3} + \cdots, \quad (3.32)$$

(i) eliminates the divergence of $S_{1d}(q \rightarrow 0)$ and (ii) simultaneously replaces in the sum $S_0(q) + S_r(q)$ the linear term of $S_0(q)$ with a quadratic one, which is in agreement with the plasmon sum rule [43, 44]. Higher-order terms arising from the difference $R(q, u) - R_0(u) =$
\[ q^2 R_1(u) + \cdots \] and from local field corrections beyond RPA have to kill also the cubic term \(-q^3/16\) of \(S_0(q)\), to substitute the coefficient of the term \(\sim q^4/\omega_{\text{pl}}^4\) of \(S_r(q)\) correspondingly, and to add a term \(\sim q^5\), such that \[ S(q \ll q_c) = \frac{q^2}{2\omega_{\text{pl}}} + s_4 \frac{q^4}{\omega_{\text{pl}}^3} + s_5 \frac{q^5}{\omega_{\text{pl}}^4} + \cdots, \quad s_4 < 0, \quad \frac{1}{12} < |s_4| < \frac{3}{20}. \] (3.33)

A direct consequence of these replacements is the appearance of an inflexion point, which moves for \(r_s \to 0\) towards the origin with \(q_{\text{infl}, S} \sim \omega_{\text{pl}} = \sqrt{4ar_s/(3\pi)}\). \(S_r(q)\) of (3.18) realizes the smooth transition from the \(k_F(\sim 1/r_s)\)-scaling “far off” the origin [reasonably approximated by \(I(q)\) of (3.29)] to the \(k_F q_c(\sim 1/\sqrt{r_s})\)-scaling near the origin [reasonably approximated by the Kimball-function \(L(q/q_c)\) of (3.31)]. This transition causes the non-analyticity \(v_r \sim r_s^2 \ln r_s\). \[ (4) \]

Finally the **momentum distribution** (3.19) is considered. The expression \(q^2 f(k, q, i\omega)\), see (3.14), is developed in the following way

\[
\frac{1}{|u - (k\zeta + \frac{1}{2} q)|^2} = \frac{1}{|u + i(k\zeta + \frac{1}{2} q)|^2} = \frac{\partial}{\partial u} \left( \frac{1}{u + i(k\zeta + \frac{1}{2} q)} \right) \to \frac{\partial}{\partial u} \left( \frac{u}{u^2 + (k\zeta + \frac{1}{2} q)^2} \right). \quad (3.34)
\]

This allows to write Eq. (3.19) as

\[
n_r(k) = \frac{\omega_{\text{pl}}^4}{(4\pi/3)^2} F_r(k) \quad \text{with} \quad F_r(k) = \int_0^\infty du \int \frac{d^3q}{q^2} \frac{R(q, u)}{q^2 + q_c^2 R(q, u)} F(k, q, u) \quad (3.35)
\]

and

\[
F(k, q, u) = \frac{\partial}{\partial u} \left( u \int_{-1}^{+1} d\zeta \Theta(k, q, \zeta) \frac{\Theta(k, q, \zeta)}{u^2 + (k\zeta + \frac{1}{2} q)^2} \right). \quad (3.36)
\]

Next the \(\zeta\)-integration has to be performed. The boundary conditions \(k \gtrsim 1\) and \(|k + q| \lesssim 1\) mean \(k^2 + q^2 + 2kq\zeta \lesssim 1\) or \(\zeta \lesssim \zeta_0\) with \(\zeta_0 = (1 - k^2 - q^2)/(2kq)\). So we have

\[
F(k > 1, q, u) = -\frac{1}{k} \left( \frac{k\zeta + \frac{1}{2} q}{(k\zeta + \frac{1}{2} q)^2 + u^2} \right)_{-1}^{+1}, \quad F(k < 1, q, u) = \frac{1}{k} \left( \frac{k\zeta + \frac{1}{2} q}{(k\zeta + \frac{1}{2} q)^2 + u^2} \right)_{-1}^{+1}. \quad (3.37)
\]

For \(k > 1\) the function \(F(k, q, u)\) is non-zero only within a certain stripe of the \(k - q\)-plane (see Fig. 3), namely \(k-1 < q < k+1\) with \(\zeta = -1 \cdots \zeta_0\) (area I). For \(k < 1\) the corresponding areas are the triangle \(1 - k < q < 1 + k\) with \(\zeta = \zeta_0 \cdots + 1\) (area II) and the stripe \(q > 1 + k\). \[ (3.37) \]
with $\zeta = -1 \cdots + 1$ (area III). With $k\zeta_0 + \frac{1}{2}q = \frac{1-k^2}{2q}$ (in area I and II) it follows

\[
F_1(k > 1, q, u) = + \frac{1}{k} \left[ \frac{k^2 - \frac{1}{2}q}{(k^2 - \frac{1}{2}q)^2 + u^2} - \frac{k - \frac{1}{2}q}{(k - \frac{1}{2}q)^2 + u^2} \right] \quad \text{for } (k, q) \text{ in I ,}
\]

\[
F_{II}(k < 1, q, u) = - \frac{1}{k} \left[ \frac{\frac{1}{2}q - k}{(\frac{1}{2}q)^2 + u^2} - \frac{\frac{1}{2}q + k}{(\frac{1}{2}q + k)^2 + u^2} \right] \quad \text{for } (k, q) \text{ in II ,}
\]

\[
F_{III}(k < 1, q, u) = - \frac{1}{k} \left[ \frac{\frac{1}{2}q - k}{(\frac{1}{2}q - k)^2 + u^2} - \frac{\frac{1}{2}q + k}{(\frac{1}{2}q + k)^2 + u^2} \right] \quad \text{for } (k, q) \text{ in III . (3.38)}
\]

This together with (3.35) is the momentum distribution in the ring-diagram summation [27, 28], see also [4, 30]. For $k = 0$ only the last line for III contributes with $F_{III}(0, q, u) = -8(q^2 - u^2)/(q^2 + u^2)^2$ giving $F_r(0) = -4.112335 + 1.35595 r_s + \cdots$. If in the denominator of Eq. (3.35) the “Yukawa”-term $q^2 R(q, u)$ is deleted, then

\[
n_{2d}(k) = \frac{\omega_{11}^4}{(4\pi/3)^2} F_{2d}(k) \quad \text{and} \quad F_{2d}(k) = 4\pi \int_0^\infty \frac{dq}{q^2} \int_0^\infty du \, R(q, u) \, F(k, q, u)
\]

arises with $F_{2d}(k > 1) = F_1(k)$ and $F_{2d}(k < 1) = F_{II}(k) + F_{III}(k)$. [Note that the definition of $F_{2d}(k)$ with $F_{2d}(k \geq 1) \geq 0$ differs from what is used in [4, 30], where $F(k) > 0$ for all $k \geq 1$]. The function (with $u \sim \tan \varphi$ the $u$-integration is replaced by an angular integration [4])

\[
F_{2d}(k > 1) = + \frac{4\pi}{k} \int_{k-1}^{k+1} \frac{dq}{q^2} \int_0^{\pi/2} d\varphi \left[ R(q, \frac{k^2 - 1}{2q} \tan \varphi) - R(q, (k - \frac{1}{2}q) \tan \varphi) \right],
\]

\[
F_{2d}(k < 1) = - \frac{4\pi}{k} \int_{1-k}^{1+k} \frac{dq}{q^2} \int_0^{\pi/2} d\varphi \left[ R(q, \frac{1-k^2}{2q} \tan \varphi) - R(q, (k + \frac{1}{2}q) \tan \varphi) \right]
\]

\[
- \frac{4\pi}{k} \int_{1+k}^\infty \frac{dq}{q^2} \int_0^{\pi/2} d\varphi \left[ R(q, (\frac{1}{2}q - k) \tan \varphi) - R(q, (\frac{1}{2}q + k) \tan \varphi) \right] \quad (3.40)
\]

possesses the properties (see also [4, 29] and note that $F_{2d}(k)$ is of the same order, cf. Eq. (4.3))

\[
F_{2d}(k \to 0) = - \left( 4.112335 + 8.984 k^2 + \cdots \right), \quad F_{2d}(k \to \infty) = \frac{1}{2} \frac{(4\pi/3)^2}{k^8} + \cdots,
\]

\[
F_{2d}(k \to 1^\pm) = \pm \frac{\pi^2}{3} \left( 1 - \ln 2 \right) \frac{1}{(k - 1)^2} + \cdots. \quad (3.41)
\]

Consequently, $n_{2d}(k)$ approximates $n_r(k)$ for $k \ll 1$ and $k \gg 1$ (far off the Fermi surface, where the screening effect is not so important), but it diverges near the Fermi surface
square-inversely as $\pm 1/(k-1)^2$ for $k \geq 1$. Vice versa, this divergence is removed through the ring-diagram partial summation with the replacement $q^3 \to q[q^2 + q^2 R(q, u)]$ in Eq. (3.39). For $k$ near the Fermi edge, it holds [6]

\[
F_t(k \to 1^\pm) \to \pm \frac{2\pi}{q_c^2 k^2} G \left( \frac{|k-1|}{q_c} \right) \quad \text{for} \quad 1 \leq k \leq 1 \pm \sqrt{q_c} \quad \text{(3.42)}
\]

with $G(x)$ being the Kulik function (B.15). Thus the discontinuity jump for $r_s \to 0$ is described by (the higher-order terms are different for outside/inside the Fermi surface)

\[
n_r(k \to 1^\pm) = n_r(1^\pm) + \frac{\omega_{pl}^4}{(4\pi/3)^2 q_c^2} G(0) + \cdots = \pm \frac{\omega_{pl}^2}{(4\pi/3)^2} \frac{2\pi}{3} G(0) + \cdots = \pm 0.088519 r_s + \cdots \quad \text{(3.43)}
\]

Comparison of (3.41) with (3.42) shows that the divergence at $k \to 1^\pm$ is eliminated and replaced by the non-analytical behavior of $G(x \to 0)$, see (B.16). Near the Fermi surface the distribution is "symmetrical" and shows a logarithmical "snuggling" with infinite slopes:

\[
n_r(k \to 1^\pm) = n_r(1^\pm) \pm \frac{\omega_{pl}^2}{8 \left( \frac{\sqrt{3\pi}}{4} + 3 \right)} |k-1| \ln |k-1| + O(k-1).
\]

$F_t(k)$ of (3.35) realizes the smooth transition from the $k_F(\sim 1/r_s)$-scaling "far off" the Fermi surface [reasonably approximated by $F_{2d}(k)$ of (3.39)] to the $k_F q_c(\sim 1/\sqrt{r_s})$-scaling near the Fermi surface [reasonably approximated by the Kulik-function $G(|k-1|/q_c)$, see (3.42)]. This transition causes the non-analyticity of $t_r \sim r_s^2 \ln r_s$, [6]. From (3.43) follows $z_F = 1 - 0.177038 r_s + \cdots$, what is in agreement with the Luttinger formula [25], which relates the quasi-particle weight $z_F$ directly to the self-energy $\Sigma(k, \omega)$:

\[
z_F = \frac{1}{1 - \Sigma'(1, \mu)} , \quad \Sigma'(1, \mu) = \text{Re} \left. \frac{\partial \Sigma(1, \omega)}{\partial \omega} \right|_{\omega=\mu} . \quad \text{(3.44)}
\]

Indeed, with the ring-diagram approximation (3.1) it becomes

\[
\Sigma_t'(1, \frac{1}{2}) = \frac{1}{8\pi} \int d^3q \int \frac{d\eta}{2\pi i} \frac{v^2(q) Q(q, \eta)}{1 + v(q) Q(q, \eta)} \frac{\partial}{\partial \eta} \frac{1}{q \cdot (e + \frac{1}{2}q) \pm i\delta} , \quad |e + q| \geq 1 . \quad \text{(3.45)}
\]

For $r_s \to 0$ it behaves as [47]

\[
\Sigma_t'(1, \frac{1}{2}) = \frac{\alpha r_s}{\pi^2} \int_0^\infty du \frac{R_0'(u)}{\sqrt{R_0(u)}} \arctan \frac{1}{u} + \cdots \approx -0.177038 r_s + \cdots , \quad \text{(3.46)}
\]

in agreement with (3.43) and (B.17). (In [48], $z_F = 1 - 0.12 r_s + \cdots$ is claimed, instead of the RPA figure 0.18.) The linear behavior of $z_F(r_s)$ is an example for how higher-order partial
summation may create lower-order terms. Calculations beyond RPA with (3.44) have been done in [49]. Calculations of $z_F$ for $r_s \leq 55$ have been done in [39]. The strength of the correlation tail, i.e. the relative number of particles [with $k > 1$ and using (B.19)] is \[ \approx 0.05383 \, r_s^{3/2} + \ldots \] \tag{3.47}

The $u$-integral is 1.06252. How does $n_{2x}(k)$ change the above results for $z_F$ and $N$?

**IV. SECOND ORDER: THE EXCHANGE TERM $\Sigma_{2x}$ AND ITS CONSEQUENCES**

As already mentioned above and stressed by Geldart [49], for a consistent small-$r_s$ description up to terms $\sim r_s^2 \ln r_s$ and $\sim r_s^2$ the exchange terms $v_{2x}$, $e_{2x}$, $n_{2x}(k)$, $S_{1x}(q)$ are needed. Here it is shown, how they arise from $\Sigma_{2x}(k, \omega)$, cf. Fig. 1d, left.

The self-energy in the second order of exchange is

\[
\Sigma_{2x}(k, \omega) = \frac{q_c^4}{(8\pi)^2} \int \frac{d^3q_1 d^3q_2}{q_1^2 q_2^2} \int \frac{d\eta_1 d\eta_2}{(2\pi i)^2} x \times G_0(|k + q_1 + q_2|, \omega + \eta_1, \eta_2) G_0(|k + q_1, \omega + \eta_1|).
\]

Use of (1.9) yields

\[
\Sigma_{2x}(k, \omega) = -\frac{q_c^4}{(8\pi)^2} \int \frac{d^3q_1 d^3q_2}{q_1^2 q_2^2} \left[ \frac{\Theta(|k + q_1 + q_2| - 1)\Theta(1 - |k + q_1|)\Theta(1 - |k + q_2|)}{\omega - \frac{1}{2}k^2 + q_1 \cdot q_2 - i\delta} + \frac{\Theta(1 - |k + q_1 + q_2|)\Theta(|k + q_1| - 1)\Theta(|k + q_2| - 1)}{\omega - \frac{1}{2}k^2 + q_1 \cdot q_2 + i\delta} \right],
\]

see also [10], Eq. (A.5). This together with (1.9) used in the Galitskii-Migdal formula (1.11) gives (cf. Fig. 1d, middle) after the $\omega$-integration has been performed

\[
v_{2x} = -\frac{3q_c^4}{(8\pi)^3} \text{Re} \left[ \int_A \frac{d^3k d^3q_1 d^3q_2}{q_1^2 q_2^2} \frac{1}{q_1 \cdot q_2 + i\delta} + \int_B \frac{d^3k d^3q_1 d^3q_2}{q_1^2 q_2^2} \frac{1}{q_1 \cdot (-q_2) + i\delta} \right]. \tag{4.3}
\]

It is easy to show with the help of the substitutions $q_1 \rightarrow q_1', q_2 \rightarrow -q_2', k \rightarrow -(k' + q_1')$ that the second term equals the first one. The virial theorem $v_{2x} = 2e_{2x}$ gives the 2nd-order
exchange energy $e_{2x}$ to be compared with the 2nd-order direct energy $e_{2d}$ [see Eq. (3.20)]:

$$e_{2x} = - \frac{3q_{t}^4}{(8\pi)^3} \int_{A} d^3k d^3q_{1} d^3q_{2} \frac{P}{q_{1}^2 q_{2}^2} \quad , \quad e_{2d} = +2 \frac{3q_{t}^4}{(8\pi)^3} \int_{A} d^3k d^3q_{1} d^3q_{2} \frac{P}{q_{1} \cdot q_{2}} .$$

$P$ means the Cauchy principle value. Note the replacement $1/q_{1}^2 \to 1/q_{2}^2$ and the addition of a factor $-1/2$, when going from the direct term $e_{2d}$ to the corresponding exchange term $e_{2x}$. Having $e_{2x}$ available, the functions $n_{2x}(k)$ and $S_{1x}(q)$ follow by means of the functional derivatives (1.12).

Combining Eq. (1.12) with (4.4) yields $n_{2x}(k) = \frac{\omega_{pl}^4}{(4\pi/3)^{2}} F_{2x}(k)$ with

$$F_{2x}(k \geq 1) = \mp \frac{1}{4} \int d^3q_{1} d^3q_{2} \frac{1}{(q_{1} \cdot q_{2})^2} \quad \text{for} \quad |k + q_{1} + q_{2}| \geq 1, \quad |k + q_{1,2}| \leq 1 .$$

For comparison the same procedure with $e_{2d}$ yields $n_{2d}(k) = \frac{\omega_{pl}^4}{(4\pi/3)^{2}} F_{2d}(k)$ with

$$F_{2d}(k \geq 1) = \pm \frac{1}{2} \int d^3q_{1} d^3q_{2} \frac{1}{(q_{1} \cdot q_{2})^2} \quad \text{for} \quad |k + q_{1} + q_{2}| \geq 1, \quad |k + q_{1,2}| \leq 1 .$$

It follows from (C.17), (C.18), that (4.6) is equivalent with what arises from (3.5) for $r \to 2d$ (‘descreening’). Whereas the direct term drives the electrons outside the Fermi surface and decrease the quasi-particle weight $z_{F}$, the exchange process draws them back inside the Fermi surface and increases $z_{F}$ (49). Again the boundary conditions enforce $q_{1,2} \to \infty$ for $k \to \infty$, so the integral becomes simply $(4\pi/3)^{2}/k^{8}$. Hence $n_{2d}(k \to \infty) \to +2\omega_{pl}^{4}/(4k^{8})$ and $n_{2x}(k \to \infty) \to -\omega_{pl}^{4}/(4k^{8}) \sim [n_{r}(k) + n_{2x}(k)]_{k \to \infty} \to +\omega_{pl}^{4}/(4k^{8})$. Comparison with (4.8) shows $g(0) \approx g_{0}(0) = 1/2$. $F_{2d,2x}(0)$ are given in (C.22), (C.23), integral properties of $F_{2x}(k)$ are in (C.24), (C.25). What concerns $N_{2x}$ one should expect $N_{2x} \approx -1/2 N_{r}$, because of $n_{2x}(k \to \infty) = -1/2 n_{r}(k \to \infty)$ . Thus $N \approx N_{r}/2 + \cdots \approx 0.02691 r_{s}^{3/2} + \cdots$.

Combining Eqs. (1.12) and (4.4), the 1st-order exchange term of $S(q)$ is (Fig. 1d, right)

$$S_{1x}(q) = + \left( \frac{\omega_{pl}}{4\pi/3} \right)^{2} \frac{I_{x}(q)}{q^{2}} , \quad I_{x}(q) = - \int d^3k d^3q_{2} \frac{P}{q_{1} \cdot q_{2}} \bigg|_{q_{1} = q} .$$

For comparison with $S_{1d}(q)$ the Macke function $I(k)$ in Eq. (3.29) is rewritten with (C.11):

$$S_{1d}(q) = -2 \left( \frac{\omega_{pl}}{4\pi/3} \right)^{2} \frac{I(q)}{q^{2}} , \quad I(q) = - \int d^3k d^3q_{2} \frac{P}{q_{1} \cdot q_{2}} \bigg|_{q_{1} = q} .$$
They have the asymptotics $S_{1d}(q \to \infty) \to -2\omega_{pl}^2/q^4$ and $S_{1x}(q \to \infty) \to +\omega_{pl}^2/q^4 \sim [S_t(q) + S_{1x}(q)]_{q \to \infty} = -\omega_{pl}^2/q^4$. Thus the x-term again compensates half of the direct term. Comparison with Eq. (1.8) shows $g(0) \approx g_0(0) = 1/2$, as it should up to this order. How does $S_{1x}(q)$ influence the non-analyticity of $S(q)$ at $q = 2$ and the behavior of $S(q)$ at the origin $q = 0$?

V. PAIR DENSITY AND SHORT-RANGE CORRELATION

The SSF $S(q)$ and the PD $g(r)$ are mutually related through the Fourier transforms

$$1 - g(r) = \frac{1}{2} \int_0^\infty d(q^3) \frac{\sin qr}{qr} [1 - S(q)] \quad 1 - S(q) = \alpha^3 \int_0^\infty d(r^3) \frac{\sin qr}{qr} [1 - g(r)] . \quad (5.1)$$

$S(0) = 0$ expresses the perfect screening sum rule: the normalisation of $1 - g(r)$ is $9\pi/4$. About it, the plasmon sum rule says $S(q \to 0) = q^2/(2\omega_{pl}) + \cdots$. - For $r_s = 0$ (ideal Fermi gas) it is [see also Eq. (2.3)]

$$S_0(q \leq 2) = \frac{3}{2} \frac{q^2}{2} - \frac{1}{2} \left(\frac{q}{2}\right)^3, \quad S_0(q \geq 2) = 1 \sim g_0(r) = 1 - \frac{9}{2} \frac{\left(\sin r - r \cos r\right)^2}{r^3} \leq 1 . \quad (5.2)$$

This causes the potential energy in lowest order as $v_x = -(3/16) q_c^2 = -(3/4)^2 \omega_{pl}^2$. The non-analyticity of $S_0(q)$ at $q = 2$, namely the 2nd-order-derivative jump $\Delta S_0''(2) = 3/4$, causes - Fourier transformed - the non-interacting Friedel oscillations of $g_0(r)$. They are tiny: the 1st minimum at $r \approx 5.76$ is $1 - 0.0037$.

Short-range correlation means the behavior of the PD $g(r)$ for $r \ll 1/q_c$, to which belong also (i) the coalescing cusp and curvature theorems [21, 26] and (ii) the influence of the on-top PD $g(0)$ on the large-wave-number asymptotics of $n(k)$ and $S(q)$, Eq. (1.8).

The relations (5.1) between the SSF $S(q)$ and the PD $g(r)$ make, that the on-top value $g(0)$ follows from the normalization (1.7) of $1 - S(q)$. The expansion $S(q) = S_0(q) + S_t(q) + S_{1x}(q) + \cdots$ creates corresponding on-top terms $g_i(0)$ with $i = 0, r, 1x, \cdots$. This series starts with $g_0(0) = 1/2$ according to (5.2). The contribution of $S_t(q)$ consists of two parts, a term $S_{1d}(0)$, linear in $r_s$, and a term $S_{2r}(q)$, logarithmically non-analytic. The first order direct
term is

\[ g_{1d}(0) = \frac{1}{2} \int_{0}^{\infty} d(q^3) \ S_{1d}(q) = - \left( \frac{\omega_{pl}}{4\pi/3} \right)^2 \int_{0}^{\infty} d(q^3) \ I(q) = -2(\pi^2 + 6 \ln 2 - 3) \frac{\alpha r_s}{5\pi} \approx -0.73167 \ r_s, \]

where (C.11) is used. The factor 2 is killed by \( g_{1x}(0) = -\frac{1}{2} g_{1d}(0) \). So, \( g_{1}(0) = g_{1d}(0) + g_{1x}(0) = \frac{1}{2} g_{1d}(0) \approx -0.3658 \ r_s \) [32, 50]. Exactly this high-density behavior of \( g(0) \) results also from the ladder theory as a method to treat short-range correlation [51, 52]. The second term \( \sim r_s^2 \ln r_s \) follows from

\[ S_{2r}(q) = S_{r}(q) - S_{1d}(q) = \frac{3q^4}{\pi} \int_{0}^{\infty} d\left( \frac{1}{q} \right) \frac{R^3(q, u)}{q^2 + q^2 R(q, u)} \sim \]

\[ g_{2r}(0) = \frac{1}{2} \int_{0}^{\infty} d(q^3) \ S_{2r}(q) = \frac{9q^4}{2\pi} \int_{0}^{\infty} du \int_{0}^{\infty} dq \frac{R^3(q, u)}{q^2 + q^2 R(q, u)} = -2 \left( 3 - \frac{\pi^2}{4} \right) \left( \frac{3\alpha r_s}{2\pi} \right)^2 \ln r_s + \cdots, \]

where \( \int_{0}^{\infty} du \ R^3_0(u) = \frac{\pi}{8} (3 - \frac{\pi^2}{4}) \) is used. Again part \( \approx \text{half?} \) of \( g_{2r}(0) \) is compensated by a corresponding exchange term. Thus with \( x \approx 1/2 \) it is [32]

\[ g(0) = \frac{1}{2} - (\pi^2 + 6 \ln 2 - 3) \frac{\alpha}{5\pi} \ r_s - x \ 2 \left( 3 - \frac{\pi^2}{4} \right) \left( \frac{3\alpha r_s}{2\pi} \right)^2 \ r_s^2 \ln r_s + \cdots \]

\[ = \frac{1}{2} - 0.3658 \ r_s - x \ 0.032966 r_s^2 \ln r_s + \cdots. \]

The decrease of \( g(0) \) with increasing \( r_s \) describes the increase of the area between 1 and \( S(q) \), the amount of which is \( 1 - g(0) = 1/2 + 0.3658 \ r_s + \cdots \).

\[ \text{VI. SUMMARY} \]

Following the procedure of Fig. 2, it is shown for the ground state of the high-density electron gas (as an example), how the static 2-body quantity \( S(q) \), the static structure factor (SSF), follows from the dynamic 1-body quantity \( \Sigma(k, \omega) \), the Dyson self-energy, using rigorous theorems as the Migdal formula, the Galitskii-Migdal formula, the generalized Hellmann-Feynman theorem, the virial theorem. Along this way all the static RPA results are thoroughly revisited and summarized on a unified footing: the energy \( e \) [Eq. (3.16)] and its components \( t \) and \( v \), the momentum distribution \( n(k) \) [Eq. (3.19)], its behavior for \( k \to 0 \), \( k \to \infty \), and at \( k \to 1^\pm \) with the discontinuity \( z_F \), the SSF \( S(q) \) [Eq. (3.18)], its behavior
at \(q \rightarrow 0\), and the on-top pair density \(g(0)\). Several identities were found, e.g. the relation between the SSF \(S(q)\) and the Macke function \(I(q)\). \(S(q)\) and \(n(k)\), stemming from the 2-body density matrix, are simultaneously linked mutually through the self-energy \(\Sigma(k, \omega)\), see Fig. 2. An exercise would be to perform the “inverse” procedure \(S(q) \rightarrow v \rightarrow e \rightarrow n(k)\).

So far not solved problems: To have exactly all terms up to \(\sim r_s^2\) and \(r_s^2 \ln r_s\) available, the functions \(I_x(q)\) and \(F_{2x}(k)\) of Eqs. (C.19) and (4.5), respectively, have to be calculated. \(F_{2x}(k)\) has to be renormalized. Besides the exchange term, which compensates part of the direct term \(g_{2x}(0)\), Eq. (5.4), has to be specified and calculated.

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The last sentence of the Abstract has to be deleted.

Here it is shown, which peculiarities of the momentum distribution \( n(k) \) and of the SSF \( S(q) \) caused by RPA lead to the logarithmic non-analyticity of the high-density correlation energy \( e_c = (\alpha r_s)^2[a \ln r_s + \text{const} + O(r_s)] \), \( a = \) Macke number. - In Eq. (27) the correct prefactor is \( 2\alpha r_s/(5\pi) \).

In this work only terms are derived and discussed containing “\( \ln r_s \)”. The term “\( \text{const} \)” is determined in Ref. [9].

- In Appendix C the Macke function \( I(q) \) is explicitly given. The notation \( I(q) \) is related to the notation used in [32] and in [6] as \( \Delta S(q/2)_\text{D} = -3/(8\pi^2)(q_c/q)^2I(q) \) and \( K(q) = 3/(8\pi^2 q^2)I(q) \), respectively.

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APPENDIX A: FUNCTIONAL DERIVATIVES

In terms of diagrams, the relations (1.12) have a simple interpretation according to elementary rules of differentiation: \( \delta e / \delta v(q) \) means that in a (vacuum) diagram for \( e \) one after the other interaction line has to be cut out successively: \( \delta v(q') / \delta v(q) = \delta(q - q') \). So to each \( e \)--diagram corresponds a set of \( S(q) \)--diagrams. Because \( t(k) \) is in the energy dominator of \( G_0(k, \omega) \) (1.9), \( \delta e / \delta t(k) \) means, that one by one particle-hole line has to be cut up in two successively: \( \delta G_0(k', \omega) / \delta t(k) = -G_0(k, \omega) \delta(k - k')G_0(k', \omega) \). Thus each \( G_0 \) has to be replaced by \( -G_0G_0 \). The minus sign is because a closed loop is broken and the 2 \( G_0 \)'s are the outer ends of an \( n(k) \)--diagram, corresponding to \( G_0\Sigma G_0 \), see (3.4), (C.16).

The RPA energy \( e_r \) of (3.13) is of the kind

\[
e_r[t(k), v(q)] = \int d^3q \int d\eta \ f(Q(q, \eta), v(q))
\]

with \( Q(q, \eta) \) being a functional of \( t(k) \). From the derivative

\[
\frac{\delta e_r}{\delta v(q)} = \int d^3q' \int d\eta \ \frac{\partial f(Q(q', \eta), v)}{\partial v} |_{v \rightarrow v(q')} \delta(q' - q) = \int d\eta \ \frac{\partial f(Q(q, \eta), v)}{\partial v} |_{v \rightarrow v(q)}
\]

(A.2)

follows \( S_r(q) \) of Eq. (3.14), as it should. From the derivative

\[
\frac{\delta e_r}{\delta t(k)} = \int d^3q \int d\eta \ \frac{\partial f(Q, v(q))}{\partial Q} \bigg|_{Q \rightarrow Q(q, \eta)} \frac{\delta Q(q, \eta, [t(k)])}{\delta t(k)} \cdot \frac{\delta Q(q, \eta, [t(k)])}{\delta t(k)}.
\]

(A.3)

follows \( n_r(k) \) as soon as the task \( \delta Q / \delta t \) is solved. This is done with the help of

\[
\frac{\delta}{\delta t(k)} \int \frac{d^3k'}{t(k' + q) - t(k') \pm \eta - i\delta} = -\int d^3k' \frac{\delta(k' + q - k) - \delta(k' - k)}{[t(k' + q) - t(k') \pm \eta - i\delta]^2}.
\]

(A.4)

This result combined with the definition (1.15) of \( Q(q, \eta) \) yields

\[
\frac{\delta Q(q, \eta)}{\delta t(k)} = -\frac{1}{2\pi} f(k, q, \eta).
\]

(A.5)

This identity used in (1.12) gives the expression (3.15), which has been derived from the RPA-self-energy \( \Sigma_r(k, \omega) \).

To derive the identity

\[
\int d^3k \ t(k) \frac{\delta}{\delta t(k)} \int d\eta \ Q^{n+1}(q, \eta) = -n \int d\eta \ Q^{n+1}(q, \eta).
\]

(A.6)
one must have in mind, that $Q(q, \eta)$ has the structure $\hat{Q}f(\eta)$ with $f(\eta) = 1/(\tau - \eta - i\delta) + 1/(\tau + \eta - i\delta)$ with the excitation energy $\tau = t(k + q) - t(k)$. After the contour integration (only) $n$ factors $f_i(\tau_{n+1})$ remain:

$$\int d\eta \, Q^{n+1}(q, \eta) = \hat{Q}_1 \cdots \hat{Q}_{n+1} \int d\eta \, f_1(\eta) \cdots f_{n+1}(\eta) = \hat{Q}_1 \cdots \hat{Q}_{n+1} f_1(\tau_{n+1}) \cdots f_n(\tau_{n+1}) .$$

(A.7)

Now the operation $\int d^3k \, t(k) \delta/\delta t(k)$ counts these remaining $n$ factors. Finally the energy denominators $1/(\tau_i + \tau_{n+1})$ cause the minus sign.

**APPENDIX B: THE FUNCTION $R(q, u)$ AND CHARACTERISTIC RPA CONSTANTS**

From the particle-hole propagator (1.15) follows the real function $R(q, u) = Q(q, iqu)$ with

$$R(q, u) = \frac{1}{2} \left[ 1 + \frac{1 + u^2 - \frac{q^2}{4}}{2q} \ln \left( \frac{\frac{q}{2} + 1)^2 + u^2}{(\frac{q}{2} - 1)^2 + u^2} - u \left( \arctan \frac{1 + \frac{q}{2}}{u} + \arctan \frac{1 - \frac{q}{2}}{u} \right) \right] \right].$$

(B.1)

It has the $q$-expansion $R(q, u) = R_0(u) + q^2 R_1(u) + \cdots$ with

$$R_0(u) = 1 - u \arctan \frac{1}{u} , \quad R_1(u) = -\frac{1}{12(1 + u^2)^2} , \cdots .$$

(B.2)

Its derivative with respect to $u$ is

$$R'(q, u) = -\frac{1}{2} \left( \arctan \frac{1 + \frac{q}{2}}{u} + \arctan \frac{1 - \frac{q}{2}}{u} \right) + \frac{u}{2q} \ln \left( \frac{\left( \frac{q}{2} + 1 \right)^2 + u^2}{\left( \frac{q}{2} - 1 \right)^2 + u^2} \right)$$

with the $q$-expansion $R'(q, u) = R'_0(u) + q^2 R'_1(u) + \cdots$ with

$$R'_0(u) = \frac{u}{1 + u^2} - \arctan \frac{1}{u} , \quad R'_1(u) = \frac{u}{3(1 + u^2)^3} , \cdots , \quad R'(q \to \infty, u) = -\frac{32u}{3q^4} + \cdots .$$

(B.4)
For the integral $\int_0^\infty du R^2(q,u)$ see (C.12). Important RPA constants are:

$$a = \frac{3}{\pi^3} \int_0^\infty du \, R^2_0(u) = \frac{1 - \ln 2}{\pi^2} \approx 0.031091, \quad (B.5)$$

$$b'_r = \frac{3}{\pi^3} \int_0^\infty du \, R^2_0(u) \ln R_0(u) \approx -0.0171202, \quad (B.6)$$

$$b_r = b'_r + a \left( \ln \frac{4\alpha}{\pi} - \frac{1}{2} \right) \approx -0.045423, \quad (B.7)$$

$$b_{2d} = -\frac{3}{4\pi^4} \int_0^1 dq \, \frac{q \left[ I(q) - 8\pi^4/3 \alpha \Theta(1-q) \right]}{q} = \frac{1}{4} + \frac{1}{\pi^2} \left[ -\frac{11}{6} - \frac{8}{3} \ln 2 + 2(\ln 2)^2 \right] \approx -0.025677, \quad (B.8)$$

$$b_{2x} = \frac{1}{6} \ln 2 - \frac{3 \zeta(3)}{4 \pi^2} \approx +0.02418. \quad (B.9)$$

Here is a list of integrals:

$$\int_0^\infty du \, R^2_0(u) = \frac{\pi^3}{3} a \approx 0.321336, \quad (B.10)$$

$$\int_0^\infty du \, \frac{R_0(u)}{1 + u^2} = \frac{\pi^3}{2} a \approx 0.482003, \quad (B.11)$$

$$\int_0^\infty du \, \frac{R'_0(u)}{R_0(u)} \arctan \frac{1}{u} \approx -3.353337, \quad \int_0^\infty du \, \frac{R''_0(u)}{R_0(u)} \arctan \frac{1}{u} \approx 4.581817. \quad (B.12)$$

$$\frac{2}{\pi^3} \int_0^\infty du \, R_0(u) \ln R_0(u) \left[ \frac{1}{1 + u^2} - \frac{3}{2} R_0(u) \right] = -\frac{1}{6} a, \quad (B.13)$$

$$\frac{4}{\pi} \int_0^\infty du \, \left[ \frac{uR'_0(u)}{4(1 + u^2)^2} - \left( \arctan \frac{1}{u} \right) R'_1(u) \right] = -\frac{1}{6}. \quad (B.14)$$

The Kulik function $G(x)$ approximates $F_r(k)$ for $k \approx 1$, see (3.32). It follows from $R_0(u)$ according to

$$G(x) = \int_0^\infty du \, \frac{R'_0(u)}{R_0(u)} \cdot \frac{u \cdot R_0(u) - R_0(y)}{u + y} \bigg|_{y=x/\sqrt{R_0(u)}}. \quad (B.15)$$

It has the small-$x$ behavior

$$G(x \ll 1) = G(0) + \left[ \pi \left( \frac{\pi}{4} + \sqrt{3} \right) x + O(x^2) \right] \ln x + O(x) \quad (B.16)$$
with
\[ G(0) = -\int_0^\infty du \frac{R_0'(u)}{R_0(u)} \arctan \frac{1}{u} \approx 3.353337. \] (B.17)

The coefficient of \( x \ln x \) is 7.908799 (the Kulik number). \( G(x) \) has the large \( x \)-behavior
\[ G(x \gg 1) = \frac{\pi}{6} (1 - \ln 2) \frac{1}{x^2} \approx 0.160668 \frac{1}{x^2}. \] (B.18)

The Kulik function is shown in [30], Fig. 1.

The relative number of particles, \( N_r \) of (3.47), follows from \( F_1(k) \) of (3.38) as
\[ N_r = -\frac{3q_c^4}{4\pi} \int_0^\infty du \int q dq k dk \frac{R'(q, u)}{[q^2 + q_c^2 R(q, u)]^2} \left[ \arctan \frac{u}{(k^2 - 1)/(2q)} - \arctan \frac{u}{k - q/2} \right]. \] (B.19)

In lowest order the decisive region is \( k \gg 1 \) and \( 0 < q \lesssim \delta \):
\[ N_r = -\frac{3}{4\pi} q_c^4 \int_0^\delta du \int q dq k dk \frac{R'(q, u)}{[q^2 + q_c^2 R(q, u)]^2} \left[ \arctan \frac{u}{(k^2 - 1)/(2q)} - \arctan \frac{u}{k - q/2} \right]. \] (B.20)

The \( k \)-integral yields (expanded in powers of \( q \))
\[ \frac{1}{2} \left( \arctan \frac{1}{u} - \arctan u - \frac{\pi}{2} \right) + \frac{1}{2} u \ln \left( 1 + \frac{1}{u^2} \right) q + \cdots. \]

The 1st (\( q \)-independent) term does not contribute, because of \( \int_0^\infty du \ R_0'(u)(\cdots) = 0 \). The \( q \)-integral yields in the limit \( r_s \to 0 \) the expression \( \pi/(4q_c \sqrt{R_0(u)}) \), giving (3.47).

**APPENDIX C: IDENTITIES, MACKE FUNCTION, MOMENTUM DISTRIBUTION AT \( k = 0 \)**

For the regions of the wave-number integrations the following short hands are introduced
\[ A = (k < 1, \ |k + q_1 + q_2| < 1, \ |k + q_{1,2}| > 1), \quad \bar{A} = (k_{1,2} < 1, \ |k_{1,2} + q| > 1), \] (C.1)
\[ B = (k > 1, \ |k + q_1 + q_2| > 1, \ |k + q_{1,2}| < 1), \quad \bar{B} = (k_{1,2} > 1, \ |k_{1,2} + q| < 1). \] (C.2)

For the Heisenberg energy \( e_{2d} \), the Onsager energy \( e_{2x} \), for the SSF’s \( S_{1d,1x}(q) \) and for the momentum distributions \( n_{2d,2x}(k) \) the following equivalent writings exist:
\[ e_{2d} : \int_A d^3k d^3q_1 d^3q_2 \frac{P}{q_1 \cdot q_2} = -\int_\bar{A} d^3q d^3k_1 d^3k_2 \frac{P}{q \cdot (k_1 + k_2 + q)}, \] (C.3)
\[ e_{2x} : \int_A d^3k d^3q_1 d^3q_2 \frac{P}{q_1 \cdot q_2} = - \int_A d^3q d^3k_1 d^3k_2 \frac{P}{q \cdot (k_1 + k_2 + q)} , \quad (C.4) \]

\[ S_{1d}(q) : \int_A d^3k d^3q_2 \frac{P}{q_1 \cdot q_2} \bigg|_{q_1 \rightarrow q} = - \int_A d^3k_1 d^3k_2 \frac{P}{q \cdot (k_1 + k_2 + q)} , \quad (C.5) \]

\[ S_{1x}(q) : \int_A d^3k d^3q_2 \frac{P}{q_1 \cdot q_2} \bigg|_{q_1 \rightarrow q} = - \int_A d^3k_1 d^3k_2 \frac{P}{(k_1 + k_2 + q)^2 \cdot (k_1 + k_2 + q)} , \quad (C.6) \]

\[ n_{2d}(k < 1) : \int_A d^3q d^3q_2 \frac{1}{(q_1 \cdot q_2)^2} = \int_A d^3q d^3k_2 \frac{P}{q^2 q_2^2 (q \cdot (k_1 + k_2 + q))^2} \bigg|_{k_1 \rightarrow k} \quad , \quad (C.7) \]

\[ n_{2d}(k > 1) : \int_B d^3q d^3q_2 \frac{1}{(q_1 \cdot q_2)^2} = \int_B d^3q d^3k_2 \frac{P}{q^2 q_2^2 (q \cdot (k_1 + k_2 + q))^2} \bigg|_{k_1 \rightarrow k} \quad , \quad (C.8) \]

\[ n_{2x}(k < 1) : \int_A d^3q d^3q_2 \frac{1}{(q_1 \cdot q_2)^2} = \int_A d^3q d^3k_2 \frac{P}{q^2 (k_1 + k_2 + q)^2 (q \cdot (k_1 + k_2 + q))^2} \bigg|_{k_1 \rightarrow k} \quad , \quad (C.9) \]

\[ n_{2x}(k > 1) : \int_B d^3q d^3q_2 \frac{1}{(q_1 \cdot q_2)^2} = \int_B d^3q d^3k_2 \frac{P}{q^2 (k_1 + k_2 + q)^2 (q \cdot (k_1 + k_2 + q))^2} \bigg|_{k_1 \rightarrow k} \quad . \quad (C.10) \]

These equivalences are consequences of the invariance, when changing the integration variables according to

\[ k \leftrightarrow k_1 , \quad q_1 \leftrightarrow q , \quad q_2 \leftrightarrow -(k_1 + k_2 + q) . \]

Note that Eq. \( (C.5) \) defines a function \( I(q) \), which has been calculated explicitly by Macke:

\[ I(q) = - \int_A d^3k d^3q_2 \frac{P}{q_1 \cdot q_2} \bigg|_{q_1 \rightarrow q} \quad \text{or} \quad I(q) = + \int_A d^3k_1 d^3k_2 \frac{P}{q \cdot (k_1 + k_2 + q)} \quad \sim \]

\[ I(q \rightarrow 0) = \frac{8\pi^4}{3} aq + \cdots , \quad I(q \rightarrow \infty) = \frac{(4\pi/3)^2}{q^2} + \cdots , \quad \int_0^\infty dq I(q) = \frac{8\pi^2}{45} (\pi^2 + 6\ln 2 - 3) . \quad (C.11) \]

The linear behavior for \( q \rightarrow 0 \) causes the Heisenberg divergence of \( e_{2d} \), the large-\( q \) asymptotics makes \( S(q \rightarrow \infty) \sim 1/q^4 \). At \( q = 2 \), \( I(q) \) has an inflexion point and \( I''(q) \) has the
jump discontinuity $\Delta I''(2) = 2\pi^2$, besides $I(q) = \cdots + c_\pm \frac{\pi^2}{2}(q - 2)^3 \ln|q - 2| + \cdots$ with $c_+ = +1$ for $q > 2$ and $c_- = -1/2$ for $q < 2$. The non-analyticity at $q = 2$ influences the Friedel oscillations of $g(r \to \infty)$. For further properties see [9], Appendix C, [52]. The identity

$$8\pi q \int_0^\infty du \ R^2(q, u) = I(q) \quad (C.12)$$

is proven as it follows. On the one hand it is (with $\eta = iqu$ and $R(q, u) = Q(q, iqu)$)

$$8\pi^2 \int \frac{d\eta}{2\pi}Q^2(q, \eta) = 8\pi q \int_0^\infty du \ R^2(q, u) \quad (C.13)$$

On the other hand the lhs of (C.13) - with the definition (1.15) of $Q(q, \eta)$ - can be written as [with the particle-hole excitation energy $\tau_i = t(k_i + q) - t(k_i)$]

$$8\pi^2 \int \frac{d^3k_1d^3k_2}{(4\pi)^2} \int \frac{d\eta}{2\pi} \left[ \frac{1}{\tau_1 - \eta - i\delta} \cdot \frac{1}{\tau_2 + \eta - i\delta} + \frac{1}{\tau_1 + \eta - i\delta} \cdot \frac{1}{\tau_2 - \eta - i\delta} \right] \quad (C.14)$$

The frequency integration is performed as a contour integration with the result

$$8\pi q \int_0^\infty du \ R^2(q, u) = \int_A d^3k_1d^3k_2 \frac{P}{[t(k_1 + q) - t(k_1)] + [t(k_2 + q) - t(k_2)]} \quad (C.15)$$

With $t(k) = k^2/2$ the Macke function $I(q)$ of (C.11) turns out, qed.

With the help of the functional derivative

$$\frac{\delta}{\delta t(k)} \frac{1}{q \cdot (k_1 + k_2 + q)} = -\frac{\delta(k_1 + q - k) - \delta(k_1 - k) + \delta(k_2 + q - k) - \delta(k_2 - k)}{[q \cdot (k_1 + k_2 + q)]^2} \quad (C.16)$$

it is easy to show

$$\frac{\delta I(q)}{\delta t(k_1)} = \mp 2 \int \frac{d^3k_2}{[q \cdot (k_1 + k_2 + q)]^2} \quad \text{for} \quad k_{1,2} \geq 1, \ |k_{1,2} + q| \leq 1 \quad (C.17)$$

With the help of (C.12), (C.13), and (A.5) it is again easy to show also

$$\frac{\delta I(q)}{\delta t(k_1)} = -8qR(q, u)f(k_1, q, iqu) \quad (C.18)$$

(C.17) and (C.18) together with (C.7), (C.8) show the equivalence of (3.39) $[n_{2d}(k)$ derived from the self-energy $\Sigma_{2d}(k, \omega)]$ with (4.6) $[n_{2d}(k)$ derived from $e_{2d}]$. 

\[ S_{1x}(q) : \text{The exchange counterpart of } I(q), \text{ namely} \]
\[ I_x(q) = -q^2 \int_A \frac{d^3kd^3q_2}{q_2^2} \frac{P}{q_1 \cdot q_2} q_1 - q \]
\[ \text{or } I_x(q) = + \int_A \frac{d^3k_1d^3k_2}{(k_1 + k_2 + q)^2} q \cdot (k_1 + k_2 + q) \]  
(C.19)

has the asymptotics \( I_x(q \to \infty) \to (4\pi/3)^2/q^2 \), the same as \( I(q) \). The properties
\[ \int_0^\infty dq \ I_x(q) = \int_0^\infty dq \ I(q) \] and
\[ \int_0^\infty dq \ \frac{dq}{q^2} I_x(q) = \frac{8\pi^4}{3} \left[ \frac{1}{6} \ln 2 - 3 \frac{\zeta(3)}{(2\pi)^2} \right] \] (C.20)

follow from
\[ \int_A \frac{d^3kd^3q_1d^3q_2}{q_2^2} q_1 \cdot q_2 = \int_A \frac{d^3kd^3q_1d^3q_2}{q_1^2} q_1 \cdot q_2 \quad \text{and} \quad q_e^2 \int_0^\infty dq \ S_{1x}(q) = v_{2x} = 2b_{2x}(\alpha r_s)^2, \] (C.21)

respectively. One may guess, that \( I_x(q \to 0) \) starts with \( \sim q^6 \). This would contribute to \( s_4 \) of Eq. (3.33).

\[ n_{2d,2x}(k) : \text{Here } F_{2d,2x}(0) \text{ are calculated starting with (4.5), (4.6) for } k = 0. \text{ The condition} \]
\[ |q_1 + q_2| < 1 \text{ means with } \zeta = \cos \cdot (q_1 \cdot q_2) \text{ the restrictions } \zeta = -1 \cdots - (1 - q_1^2 - q_2^2)/(2q_1q_2) \]
\[ \text{and } q_1 - 1 < q_2 < q_1 + 1: \]
\[ F_{2d}(0) = -4\pi^2 \left( \int_1^2 \frac{dq_1}{q_1^4} \int_1^{q_1+1} dq_2 + \int_1^2 \frac{dq_1}{q_1^4} \int_{q_1-1}^{q_1+1} dq_2 \right) \frac{1 - (q_1 - q_2)^2}{q_1^2 + q_2^2 - 1} = -\frac{5\pi^2}{12}. \] (C.22)

For the corresponding exchange term it is similarly
\[ F_{2x}(0) = +2\pi^2 \left( \int_1^2 \frac{dq_1}{q_1^2} \int_1^{q_1+1} dq_2 + \int_1^2 \frac{dq_1}{q_1^2} \int_{q_1-1}^{q_1+1} dq_2 \right) \frac{1 - (q_1 - q_2)^2}{q_1^2 + q_2^2 - 1} = +2\pi^2(1 - \ln 2)^2, \] (C.23)

thus \( F_2(0) \approx -4.1123 + 1.8586 \approx -2.2537 \) and \( n_2(0) \approx -0.0063 \ r_s^2. \) - \( F_{2d,2x}(k) \) have the asymptotics \( \to (4\pi/3)^2/4k^8 \). The particle-number conservation
\[ \left( \int_B - \int_A \right) dk \ k^2 \int \frac{d^3q_1d^3q_2}{q_1^2q_2^2} \frac{1}{(q_1 \cdot q_2)^2} = 0 \] (C.24)

easily follows from the replacements in the second integral \( k \to -k - q_1 \) and \( q_2 \to -q_2. \)

The property
\[ \left( \int_B - \int_A \right) dk \ k^4 \int \frac{d^3q_1d^3q_2}{q_1^2q_2^2} \frac{1}{(q_1 \cdot q_2)^2} = \frac{8\pi^4}{9} \left[ \frac{1}{6} - 3 \frac{\zeta(3)}{(2\pi)^2} \right] \] (C.25)
follows from
\[
\frac{\omega_{pl}^4}{(4\pi/3)^2} \int_0^\infty d(k^3) F_{2x}(k) \frac{k^2}{2} = t_{2x} = -b_{2x}(\alpha r_s)^2.
\]  
(C.26)

How behaves \(F_{2x}(k)\) near the Fermi edge? It diverges, but weaker than \(F_{2d}(k)\), namely logarithmically, \(F_{2x}(k \to 1^\pm) \sim \pm \ln |k - 1|\). It is not strong enough to make the corresponding (Onsager) energy \(t_{2x}\) divergent:
\[
\left( \int_1^{1+\Delta} - \int_1^{1-\Delta} \right) dk \, k^4 \ln |k - 1| \sim \Delta^2 \ln \Delta \to 0,
\]  
(C.27)
saying that the lhs of Eq. (C.26) exists. The divergence of \(F_{2x}(k)\) is eliminated by the renormalization of at least one interaction line.

**Figures**

FIG. 1a: Feynman diagrams for the one-body Green’s function of the ideal Fermi gas \(G_0(k, \omega)\), the bare Coulomb repulsion \(v_0(q) = \pi^2 q_c^2 / q^2\), and the RPA polarization propagator \(Q(q, \eta)\) as defined in Eqs. (1.13)-(1.15).

FIG. 1b: Feynman diagrams for \(v_r = v_0 + v_0 Q v_r\), the screened Coulomb repulsion in RPA.

FIG.1c: Feynman diagrams for the ring-diagram-summed self-energy \(\Sigma_r\) (left) with \(n_r = G_0 \Sigma_r G_0\), for the energies \(e_r, v_r\) (middle), and for \(S_r\) (right) according to Eqs. (3.1), (3.4), (3.13), (3.14). “Descreening” \(v_r \to v_0\) makes the divergent quantities \(\Sigma_{2d}, n_{2d}, e_{2d}, v_{2d}, S_{1d}\).
FIG. 1d: Feynman diagrams for $\Sigma_{2x}$ (left) with $n_{2x} = G_0 \Sigma_{2x} G_0$, for the energies $e_{2x}$, $v_{2x}$ (middle), and for $S_{1x}$ (right) according to Eqs. (4.1)-(4.5), (4.7).

FIG. 2: Terms following from the off-shell self-energy $\Sigma(k, \omega)$ as a functional of $t(k) = k^2/2$ and $v(q) = q_e^2/q^2$: The Migdal formula (1.10) yields the momentum distribution $n(k)$, from which follows the kinetic energy $t$. The Galitskii-Migdal formula (1.11) yields the potential energy $v$. The Hellmann-Feynman functional derivative (1.12) of the total energy $e = t + v$ yields the SSF $S(q)$, its Fourier transform gives the PD $g(r)$. The cases $\Sigma_x$, $\Sigma_r$, $\Sigma_{2x}$ are studied in Secs. II, III, IV, respectively.

FIG. 3: The momentum distribution (3.38) and its $k$-$q$-plane.