Kummer-type constructions of almost Ricci-flat 5-manifolds

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Abstract
A smooth closed manifold \( M \) is called almost Ricci-flat if
\[
\inf_{g} ||Ric_{g}||_{\infty} \cdot \text{diam}_{g}(M)^{2} = 0
\]
where \( Ric_{g} \) and \( \text{diam}_{g} \), respectively, denote the Ricci tensor and the diameter of \( g \) and \( g \) runs over all Riemannian metrics on \( M \). By using Kummer-type method, we construct a smooth closed almost Ricci-flat nonspin 5-manifold \( M \) which is simply connected. It is minimal volume vanishes; namely, it collapses with sectional curvature bounded.

Keywords
Almost Ricci-flat · Kummer-type construction · Collapsing

Mathematics Subject Classification
Primary 53C20 · Secondary 53C25

1 Introduction

M. Gromov [9] considered an almost flat manifold \( M \) which is defined as a smooth closed manifold satisfying
\[
\inf_{g} ||Rm_{g}||_{\infty} \cdot \text{diam}_{g}(M)^{2} = 0
\]
where \( Rm_{g} \) and \( \text{diam}_{g} \), respectively, denote the Riemann curvature tensor and the diameter of \( g \) and \( g \) runs over all smooth Riemannian metrics on \( M \). Such a manifold has to be a collapsing manifold and finitely covered by a compact nilmanifold. Moreover a volume-noncollapsed almost flat manifold is actually flat, more precisely given \( n \in \mathbb{Z}^{+} \) and \( \nu > 0 \) there is some \( \varepsilon = \varepsilon(n, \nu) > 0 \) so that any Riemannian \( n \)-manifold \( (M, g) \) with \( \text{Vol}_{g}(M) \geq \nu \), \( \text{diam}_{g}(M) \leq 1 \), and \( ||Rm_{g}|| \leq \varepsilon \) where \( \text{Vol}_{g} \) denotes the volume of \( g \) admits a flat metric.

In an analogous way, Kapovitch and Lott [15, 16] considered almost Ricci-flat manifolds with the aim of obtaining a Ricci-flat manifold. A smooth closed manifold \( M \) is called almost...
Ricci-flat if \( \mu(M) = 0 \) for

\[
\mu(M) := \inf_g \|\text{Ric}_g\|_{\infty} \cdot \text{diam}_g(M)^2
\]

where \( \text{Ric}_g \) denotes the Ricci tensor of \( g \) and \( g \) runs over all smooth Riemannian metrics on \( M \). They found some topological conditions to ensure that a volume-noncollapsed almost Ricci-flat manifold admits a Ricci-flat metric.

By the Cheeger–Gromoll splitting theorem [6], any smooth closed Ricci-flat manifold must be finitely covered by the product of a flat \( m \)-torus \( T^m \) and a compact simply connected Ricci-flat manifold. According to Berger’s classification [3, 23], an irreducible simply connected Ricci-flat \( n \)-manifold is either locally symmetric or has holonomy \( SU(n/2) \) or \( Sp(n/4) \) or \( G_2 \) for \( n = 7 \) or \( \text{Spin}(7) \) for \( n = 8 \), or \( SO(n) \). In case that it is locally symmetric, it must be flat, and there have been found many examples of compact simply connected Ricci-flat manifolds with special holonomy.

There are also examples of almost Ricci-flat manifolds which do not admit a Ricci-flat metric. Anderson [1] constructed such examples in dimension 4 by performing surgeries along circles of \( T^4 \) with \( (D^2 \times S^2, g_S) \) where \( g_S \) is the Riemannian Schwarzschild metric. While it is perhaps remote from the complete understanding of almost Ricci-flat manifolds, it is worthwhile to have different types of examples of almost Ricci-flat manifolds at hand.

Gluing techniques are major tools to construct geometric structures in differential geometry. In particular, Kummer-type construction has been useful for constructing simply connected Ricci-flat manifolds with special holonomy such as hyper-Kähler \( K3 \) surfaces [18, 27, 28], Joyce’s \( G_2 \)-manifolds [12], and Joyce’s \( \text{Spin}(7) \)-manifolds [13]. Brendle and Kapouleas [5] used this method to construct simply connected almost Ricci-flat 4-manifolds. In this article, we use Kummer-type construction to construct simply connected almost Ricci-flat 5-manifolds. More precisely we prove

**Theorem 1.1** There exists a smooth closed almost Ricci-flat nonspin 5-manifold \( X \) which is simply connected and has a nontrivial torsion subgroup of \( H_2(X, \mathbb{Z}) \). A sequence of metrics on \( X \) realizes \( \mu(X) = 0 \) has a positive lower bound of volumes, but the minimal volume \( \text{MinVol}(X) \) of \( X \) vanishes so that \( X \) collapses with sectional curvature bounded.

The minimal volume introduced by Gromov [10] is defined as

\[
\text{MinVol}(X) := \inf_g \{ \text{Vol}_g(X) | K_g | \leq 1 \}
\]

where \( K_g \) denotes the sectional curvature of \( g \) and \( g \) runs over all Riemannian metrics on \( X \). The vanishing of \( \text{MinVol}(X) \) implies that all the characteristic numbers of \( X \) are zero via Chern-Weil theory and Gromov’s simplicial volume

\[
\|\text{X}\| := \inf \{ \sum_i |r_i| | r_i \text{ are the coefficients of a real cycle representing } [X] \}
\]

of \( X \) is also zero by the inequality

\[
\|\text{X}\| \leq (n - 1)^n n! \text{MinVol}(X)
\]

where \( n = \dim(X) \).

Like other Kummer-type constructions, this example may shed light on the study of canonical geometry in dimension 5. However, for now we do not know whether it admits a Ricci-flat metric or not.
2 Construction of X

We start with a flat 5-dimensional torus $T^5 = \mathbb{R}^5 / \mathbb{Z}^5$ where
\[ \mathbb{Z}^5 = \{ (x_1, \cdots, x_5) | x_i \in \mathbb{Z} \} \].

For convenience, we always express a point of $T^5$ as $x = (x_1, \cdots, x_5) \in \mathbb{R}^5$ modulo $\mathbb{Z}^5$.

For the following 3 isometric involutions
\[ \alpha : x \mapsto (x_1, -x_2, -x_3, \frac{1}{2} - x_4, -x_5), \]
\[ \beta : x \mapsto (-x_1, \frac{1}{2} - x_2, -x_3, x_4, -x_5), \]
\[ \gamma : x \mapsto (-x_1, -x_2, \frac{1}{2} - x_3, -x_4, x_5) \]
of $T^5$, one can easily check that they all commute and hence generate
\[ \Gamma := (\mathbb{Z}_2)^3 = \langle \alpha \rangle \oplus \langle \beta \rangle \oplus \langle \gamma \rangle. \]

The fixed point sets of $\alpha$ are 16 copies of circle
\[ S_\alpha := S^1 \times \{ (p_2, \cdots, p_5) \} \]
where $p_2, p_3, p_5$ are either 0 or $\frac{1}{2}$ and $p_4$ is either $\frac{1}{4}$ or $\frac{3}{4}$, and similarly we have 16 copies of $S_\beta$ and 16 copies of $S_\gamma$. All together they are disjoint 48 circles which we shall denote by $Z :=$ the union of 48 singular circles

and these are all the singular locus of the $\Gamma$ action. Indeed
\[ \alpha \beta : x \mapsto (-x_1, \frac{1}{2} + x_2, x_3, \frac{1}{2} - x_4, x_5), \]
\[ \beta \gamma : x \mapsto (x_1, \frac{1}{2} + x_2, \frac{1}{2} + x_3, -x_4, -x_5), \]
\[ \alpha \gamma : x \mapsto (-x_1, x_2, \frac{1}{2} + x_3, \frac{1}{2} + x_4, -x_5), \]

and
\[ \alpha \beta \gamma : x \mapsto (x_1, \frac{1}{2} - x_2, \frac{1}{2} - x_3, \frac{1}{2} + x_4, x_5), \]

act without any fixed point. Moreover $\langle \alpha, \beta \rangle$ acts freely on the set of 16 $S_\gamma$ identifying them into 4 $S_\gamma$. Similarly $\langle \beta, \gamma \rangle$ identifies 16 $S_\alpha$ to 4 of them, and $\langle \alpha, \gamma \rangle$ identifies 16 $S_\beta$ to 4 of them. Thus the singular set of $T^5 / \Gamma$ is a disjoint union of 12 copies of $S^1$, and the singularity at each $S^1$ is modeled on $S^1 \times (\mathbb{C}^2/\{\pm 1\})$.

To resolve these singularities, we delete a neighborhood $S^1 \times (B_{\frac{1}{10}} / \{\pm 1\})$ of each singular $S^1$ where $B_{\frac{1}{10}}$ is a ball of radius $\frac{1}{10}$ around $0 \in \mathbb{C}^2$, and then graft $S^1 \times Y$ along each boundary component $S^1 \times \mathbb{R}P^3$, where $Y$ is a disk bundle of $T^*S^2$. The resulting manifold is our desired manifold X. Let’s set
\[ \tilde{T} := T^5 / \Gamma - U \]
where $U$ is the union of 12 neighborhoods $S^1 \times (B_{\frac{1}{10}} / \{\pm 1\})$. 

\[ \tilde{T} \]
There is another description of $X$ viewed as the $\Gamma$-quotient of a “blow-up” of $T^5$ along $Z$. For the quotient map

$$\pi : T^5 \to T^5 / \Gamma,$$

the boundary of $\pi^{-1}(\hat{T})$ consists of 48 $S^1 \times S^3$, so one can glue $48 \, S^1 \times \hat{Y}$ along the boundary, where $\hat{Y}$ is the associated disk bundle of the Hopf fibration over $S^2$. (Indeed $\hat{Y}$ is the algebro-geometric blow-up of 4-ball $B^4$ at the origin, where the origin is replaced by a sphere of self-intersection $-1$.) By the “blow-up” of $T^5$, we mean this resulting manifold which we denote by $\hat{T}^5$. Since the $\mathbb{Z}_2$-quotient of $\hat{Y}$ is $Y$, the $\Gamma$-action on $T^5$ can be (smoothly) extended to $\hat{T}^5$ with the quotient space equal to $X$. For the later purpose, we denote this quotient map by

$$\hat{\pi} : \hat{T}^5 \to X.$$

Now we will construct the metric on $X$. First note that $\hat{T}$ has a flat metric coming from the flat orbifold metric on $T^5 / \Gamma$, and we dilate this metric by multiplying $(20 d)^2$ for a constant $d \gg 1$. $T^* S^2$ has an ALE Ricci-flat Kähler metric known as the Eguchi-Hanson metric

$$g_{EH} = \frac{d r^2}{1 - r^{-4}} + r^2 (1 - r^{-4}) \sigma_3^2 + r^2 (\sigma_1^2 + \sigma_2^2)$$

for $r \in [1, \infty)$ and the left-invariant coframe $\{\sigma_1, \sigma_2, \sigma_3\}$ of $S^3$ satisfying

$$d \sigma_i = 2 \varepsilon_{ijk} \sigma_j \wedge \sigma_k.$$

It asymptotically approaches the Euclidean metric so that there exists a constant $C_I$ such that outside a neighborhood of the zero section,

$$| \frac{\partial |I|}{\partial x_I} ((g_{EH})_{\mu\nu} - \delta_{\mu\nu}) | \leq \frac{C_I}{|x|^{4+|I|}}$$

for all $\mu, \nu = 1, \ldots, 5$, where $I := (i_1, \ldots, i_n)$ denotes a multi-index with all $i_j \geq 0$. By the curvature formula

$$R_{ijk}^l = -\partial_j \Gamma_{ik}^l + \partial_i \Gamma_{j}^l - \Gamma_{ik}^m \Gamma_{jm}^l + \Gamma_{jk}^m \Gamma_{im}^l$$

where

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_l g_{ij} + \partial_j g_{il} - \partial_i g_{lj})$$

the curvature $R^l_{ijk}$ of $g_{EH}$ decays to zero as $O(r^{-6})$.

Take any smooth cutoff function $\rho_1(r) : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ satisfying

$$\rho_1(r) = \begin{cases} 1 & \text{for } r \leq 1 \\ 0 & \text{for } r \geq 2. \end{cases}$$

Then our desired cutoff function is $\rho_d(r) := \rho_1(\frac{r}{d})$ which then satisfies

$$|D^m \rho_d | \leq \frac{c_m}{d^m}$$
for any positive integer \( m \) where \( c_m \) is a constant depending on \( m \).

Consider a metric
\[
\tilde{g}_d := \rho_d \ g_{EH} + (1 - \rho_d) g_E
\]
on \( T^*S^2 \), which is equal to the Euclidean metric \( g_E \) for \( r \geq 2d \), and put this metric on \( Y \) using a diffeomorphism to \( T^*S^2 \). We now glue the metrics on \( \tilde{T} \) and \( S^1(20d) \times Y \) where \( S(20d) \) denotes the circle of length \( 20d \) to get a smooth metric \( g_d \) on \( X \). The important property of \( g_d \) is that there exists a constant \( \tilde{C}_I \) independent of \( d \) such that
\[
\left| \frac{\partial |I|}{\partial x_I} ((g_d)_{\mu\nu} - \delta_{\mu\nu}) \right| \leq \frac{\tilde{C}_I}{d^{4+|I|}}
\]
on the gluing regions for all \( \mu, \nu = 1, \cdots, 5 \) and hence \( |\text{Ric}_{g_d}| \) is \( O(\frac{1}{d^6}) \) by (2).

By shrinking back \( (X, g_d) \) by the factor \( (\frac{1}{20d})^2 \), we get a desired metric \( g \) on \( X \) satisfying
\[
||\text{Ric}_g|| \leq \frac{C_1}{d^4}, \quad \text{Vol}_g(X) \geq C_2, \quad \text{diam}_g(X) \in [1, 1 + C_3]
\]
for constants \( C_i > 0 \) independent of \( d \). Taking \( d \) arbitrarily large, one can achieve the almost \( \text{Ricci-flatness} \) on \( X \).

### 3 Topology of \( X \)

In this section, any path and loop are meant to be continuous.

**Lemma 3.1** *For any base point \( p_0 \in T^5 \) and any loop \( \tau : [0, 1] \rightarrow T^5 \) based at \( p_0 \), \( \pi(\tau) \) is homotopic to the constant loop at \( \pi(p_0) \).*

**Proof** One can take generators \( [\rho_1], \cdots, [\rho_5] \) of \( \pi_1(T^5, p_0) \cong \mathbb{Z}^5 \) such that each \( \rho_i \) is parallel to \( x_i \)-axis, meaning that
\[
\rho_i(t) = p_0 + t e_i \mod \mathbb{Z}^5
\]
for \( t \in [0, 1] \) where \( \{e_1, \cdots, e_5\} \) is the standard basis of \( \mathbb{R}^5 \). Then there exists \( \rho_{i_1}, \cdots, \rho_{i_k} \) and \( e_1, \cdots, e_k \in \{\pm 1\} \) so that \( \tau = [\rho_{i_1}^{e_1} \cdots \rho_{i_k}^{e_k}] \) where \((\cdot)^{-1}\) denotes the inverse path.

It is enough to show that each \( \pi(\rho_1) \) is homotopic to the constant loop at \( \pi(p_0) \). First let’s see the case of \( \pi(\rho_1) \). Note that the loop \( \rho_1 \) is freely homotopic to the loop \( \tilde{\rho}_1 \) based at \( p_1 = (0, \frac{1}{4}, 0, 0, 0) \in T^5 \) defined by
\[
\tilde{\rho}_1(t) = p_1 + t e_1 \mod \mathbb{Z}^5
\]
for \( t \in [0, 1] \). (The translation by \( s(p_1 - p_0) \) for \( s \in [0, 1] \) gives the free homotopy of two loops.) The reason for this particular choice of \( p_1 \) will become obvious right next.

Since
\[
\beta(\tilde{\rho}_1(t)) = p_1 - te_1 \equiv p_1 + (1 - t)e_1 \mod \mathbb{Z}^5,
\]
\( \pi(\tilde{\rho}_1(t)) = \pi(p_1 + (1 - t)e_1) \). Then writing \( \pi(\tilde{\rho}_1) \) as
\[
\pi(\tilde{\rho}_1(t)) = \begin{cases} \pi(p_1 + te_1) & \text{for } t \in [0, \frac{1}{2}] \\ \pi(p_1 + (1 - t)e_1) & \text{for } t \in [\frac{1}{2}, 1], \end{cases}
\]
it is homotopic to the constant loop at $\pi(p_1)$ by the lemma below. Thus $\pi(\rho_1)$ is freely homotopic to the constant loop at $\pi(p_1)$, and this implies that $[\pi(\rho_1)] = 0 \in \pi_1(T^5/\Gamma, \pi(p_0))$.

Other cases can be verified in the same way. For instance, for the case of $\pi(\rho_2)$, instead of $p_1$ one can use $p_2 = (0, 0, 0, \frac{1}{3}, 0)$ fixed by $\alpha$ which inverts the direction of $\tilde{\rho}_2$. \hfill \Box

We name the following elementary fact as an lemma for convenience.

**Lemma 3.2** Let $M$ be any topological space and $\tau : [0, \frac{1}{2}] \to M$ be a path. Then the loop $\tilde{\tau} : [0, 1] \to M$ defined as

$$
\tilde{\tau}(t) = \begin{cases} 
\tau(t) & \text{for } t \in [0, \frac{1}{2}] \\
\tau(1-t) & \text{for } t \in [\frac{1}{2}, 1]
\end{cases}
$$

is homotopic to the constant loop at $\tau(0)$.

**Proposition 3.3** $X$ is simply connected.

**Proof** We shall first show that $\pi_1(T^5/\Gamma, \pi(p_0)) = 0$ where $p_0$ is chosen from $T^5 - Z$. Specifically we choose $p_0$ to be $(0, 0, 0, \frac{1}{4} + \epsilon, 0)$ for $\epsilon \in (0, \frac{1}{100})$ near one of $S_\alpha$ which is precisely

$$
S_\alpha := S^1 \times \{(0, 0, \frac{1}{4}), 0\}.
$$

Let $[\sigma]$ be any element in $\pi_1(T^5/\Gamma, \pi(p_0))$. Since $S^1 \times \{pt\}$ is a strong deformation retract of $S^1 \times (B_{\frac{1}{10}}/\{|\pm1|\})$ in an obvious way, one can use this deformation to choose a loop $\sigma : [0, 1] \to T^5/\Gamma$ representing $[\sigma]$ such that $\sigma$ dose not intersect with $\pi(Z)$. Since

$$
\pi : T^5 - Z \to T^5/\Gamma - \pi(Z)
$$

is a covering map, one can lift $\sigma$ to $\tilde{\sigma} : [0, 1] \to T^5 - Z$ such that $\tilde{\sigma}(0) = p_0$.

There are two possibilities for $\tilde{\sigma}$, whether it is a loop or a path with $\tilde{\sigma}(0) \neq \tilde{\sigma}(0)$. In the former case, by Lemma 3.1 $\sigma = \pi(\tilde{\sigma})$ must be homotopically trivial.

Now we deal with the latter case which needs an analysis of further subcases. There are $|\Gamma| = 8$ points in $\pi^{-1}(\pi(p_0))$. They are all the $\epsilon$ distance away from $\pi^{-1}(\pi(S_\alpha^1))$ which is the union of

$$
S_\alpha^1, S_\alpha^2 := \beta(S_\alpha^1), S_\alpha^3 := \gamma(S_\alpha^1), S_\alpha^4 := \beta\gamma(S_\alpha^1).
$$

(Recall that $\langle \beta, \gamma \rangle$ identifies 16 $S_\alpha$ to 4 of them.) We label the points of $\pi^{-1}(\pi(p_0))$ as $q_1, \ldots, q_8$ such that

$$
q_1 = p_0, q_2 = \alpha(p_0), q_3 = \beta(p_0), q_4 = \beta\alpha(p_0), q_5 = \gamma(p_0), q_6 = \gamma\alpha(p_0), q_7 = \beta\gamma(p_0), q_8 = \beta\gamma\alpha(p_0).
$$

Observe that $q_{2i-1}, q_{2i}$ are away from $S_\alpha^i$ by the $\epsilon$ distance. We have 7 subcases according to where $\tilde{\sigma}(1)$ lands.

In the 1st case when $\tilde{\sigma}(1) = q_2$, we take a path $\tilde{\sigma}_2 : [0, 1] \to T^5$ such that $\tilde{\sigma}_2(0) = p_0, \tilde{\sigma}_2(1) = q_2$ and $\text{Im(}\tilde{\sigma}_2)$ lies on a round 3-sphere

$$
\{ x = (0, x_2, \ldots, x_5) \in T^5 | ||x - (0, 0, 0, \frac{1}{4}, 0)|| = \epsilon \}.
$$
(Observe that $q_1$ and $q_2$ are antipodal to each other on this 3-sphere.) Then by Lemma 3.1 $\pi(\tilde{\sigma} \cdot \tilde{\sigma}_2^{-1}) = \sigma \cdot (\pi(\tilde{\sigma}_2))^{-1}$ is homotopic to constant so that $[\sigma] = [\pi(\tilde{\sigma}_2)] \in \pi_1(T^5/\Gamma, \pi(p_0))$. Note that $\pi(\tilde{\sigma}_2)$ is placed in a 4-orbifold $\{pt\} \times (B_{\frac{1}{m}}/\pm 1)$ which is contractible, so

$$[\pi(\tilde{\sigma}_2)] = 0.$$  \tag{4}

Therefore we proved that $[\sigma] = 0$ in this first case.

In the 2nd case when $\tilde{\sigma}(1)$ is $q_3$, we choose any point $q \in T^5$ which is fixed by $\beta$ and take any path $\tilde{\sigma} : [0, \frac{1}{2}] \to T^5$ such that $\tilde{\sigma}(0) = p_0, \tilde{\sigma}(\frac{1}{2}) = q$. Define a path $\tilde{\sigma}_3 : [0, 1] \to T^5$ as

$$\tilde{\sigma}_3(t) = \begin{cases} \tilde{\sigma}(t) & \text{for } t \in [0, \frac{1}{2}] \\ \beta(\tilde{\sigma}(1-t)) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}

Then again by Lemma 3.1 $\pi(\tilde{\sigma} \cdot \tilde{\sigma}_3^{-1}) = \sigma \cdot (\pi(\tilde{\sigma}_3))^{-1}$ is homotopic to a constant so that

$$[\sigma] = [\pi(\tilde{\sigma}_3)] = 0$$

where the second equality is due to Lemma 3.2, finishing in the 2nd case.

In the 3rd case when $\tilde{\sigma}(1)$ is $q_4$, this time we take $\tilde{\sigma} : [0, \frac{1}{2}] \to T^5$ such that $\tilde{\sigma}(0) = q_2, \tilde{\sigma}(\frac{1}{2}) = q$ where $q$ was as above, and define a path $\tilde{\sigma}_4 : [0, 1] \to T^5$ in the same way as $\tilde{\sigma}_3$ with this new $\tilde{\sigma}$. Then again by Lemma 3.1 $\pi(\tilde{\sigma} \cdot (\tilde{\sigma}_2 \cdot \tilde{\sigma}_4)^{-1}) = \sigma \cdot (\pi(\tilde{\sigma}_2) \cdot \pi(\tilde{\sigma}_4))^{-1}$ is homotopic to a constant so that

$$[\sigma] = [\pi(\tilde{\sigma}_2) \cdot \pi(\tilde{\sigma}_4)] = 0$$

where the second equality is due to (4) and Lemma 3.2, finishing the 3rd case.

In the 4th case when $\tilde{\sigma}(1)$ is $q_5$, the proof of $[\sigma] = 0$ is almost verbatim to the above 2nd case except that $\beta$ is replaced by $\gamma$ and one has to use a point $q'$ which enters into this case has no fixed points. Using $q' \in T^5$ chosen above and choosing any path $\tilde{\sigma} : [0, \frac{1}{2}] \to T^5$ such that $\tilde{\sigma}(0) = q_3, \tilde{\sigma}(\frac{1}{2}) = q'$, define $\tilde{\sigma}_7 : [0, 1] \to T^5$ as

$$\tilde{\sigma}_7(t) = \begin{cases} \tilde{\sigma}(t) & \text{for } t \in [0, \frac{1}{2}] \\ \gamma(\tilde{\sigma}(1-t)) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}

Then by Lemma 3.1 $\pi(\tilde{\sigma} \cdot (\tilde{\sigma}_3 \cdot \tilde{\sigma}_7)^{-1}) = \sigma \cdot (\pi(\tilde{\sigma}_3) \cdot \pi(\tilde{\sigma}_7))^{-1}$ is homotopic to constant so that

$$[\sigma] = [\pi(\tilde{\sigma}_3) \cdot \pi(\tilde{\sigma}_7)] = 0$$

where Lemma 3.2 is used twice, finishing the 6th case.

In the 5th case when $\tilde{\sigma}(1)$ is $q_6$, an additional treatment is needed because $\beta \gamma = \gamma \beta$ which enters into this case has no fixed points. Using $q' \in T^5$ chosen above and choosing any path $\tilde{\sigma} : [0, \frac{1}{2}] \to T^5$ such that $\tilde{\sigma}(0) = q_4, \tilde{\sigma}(\frac{1}{2}) = q'$, define $\tilde{\sigma}_8 : [0, 1] \to T^5$ in the same way as $\tilde{\sigma}_7$ with this new $\tilde{\sigma}$. Then by Lemma 3.1 $\pi(\tilde{\sigma} \cdot (\tilde{\sigma}_2 \cdot \tilde{\sigma}_4 \cdot \tilde{\sigma}_8)^{-1}) = \sigma \cdot (\pi(\tilde{\sigma}_2) \cdot \pi(\tilde{\sigma}_4) \cdot \pi(\tilde{\sigma}_8))^{-1}$ is homotopic to a constant so that

$$[\sigma] = [\pi(\tilde{\sigma}_2) \cdot \pi(\tilde{\sigma}_4) \cdot \pi(\tilde{\sigma}_8)]$$
where the second equality is due to (4) and Lemma 3.2, finishing the 7th case. This completes the proof that \( \pi_1(T^5/\Gamma, \pi(p_0)) \) is trivial.

Now recall that \( X \) is obtained by replacing \( 12 S^1 \times (B_{\frac{1}{m}}/\{\pm 1\}) \) with \( 12 S^1 \times Y \). Since the zero section \( S^2 \) is a strong deformation retract of \( Y \), \( \pi_1(Y) = 0 \), and hence \( S^1 \times (B_{\frac{1}{m}}/\{\pm 1\}) \) and \( S^1 \times Y \) have the same fundamental group \( \mathbb{Z} \). Moreover their boundaries are the same, i.e., \( S^1 \times \mathbb{R}P^3 \) and the induced inclusion maps on \( \pi_1 \) are also the same. Therefore we can conclude from the Seifert-Van Kampen theorem that these 12 replacement processes do not change fundamental groups all the way, and this proves that \( X \) is simply connected.

\[ \Box \]

**Proposition 3.4** \( H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{13} \).

**Proof** To prove \( b_2(X) = 13 \), first recall that \( H^2(T^5/\Gamma, \mathbb{R}) \) is isomorphic to the space \( H^2(T^5/\Gamma, \mathbb{R}) \) of the harmonic 2-forms on an orbifold \( T^5/\Gamma \) with a flat orbifold metric. (The deRham theorem and the Hodge theorem hold true in an orbifold too.) Since the \( \Gamma \) action preserves the flat metric of \( T^5 \), \( H^2(T^5/\Gamma, \mathbb{R}) \) is just the projection of the \( \Gamma \)-invariant elements of \( H^2(T^5, \mathbb{R}) \). By direct checking the only 12 \( \Gamma \)-invariant harmonic 2-forms of \( T^5 \) are constant multiples of \( dx_2 \wedge dx_3 \), so \( H_2(T^5/\Gamma, \mathbb{R}) \simeq H^2(T^5/\Gamma, \mathbb{R}) \simeq \mathbb{R} \).

In the followings, all (co)homology groups are over \( \mathbb{R} \), unless otherwise specified. First \( H_1(\tilde{T}) \simeq H^1(\tilde{T}) \) vanishes, since \( -j^* \) in the following Mayer-Vietoris sequence is an isomorphism:

\[
\begin{align*}
H^1(T^5/\Gamma) & \overset{(k^*, l^*)}{\longrightarrow} H^1(\tilde{T}) \oplus H^1(U) \overset{i^* - j^*}{\longrightarrow} H^1(\partial \tilde{T}) \\
\{0\} & \longrightarrow H^1(\tilde{T}) \oplus \mathbb{R}^{12} \longrightarrow \mathbb{R}^{12}
\end{align*}
\]

where \( i, j, k, l \) are obvious inclusion maps.

We claim that \( H_2(\tilde{T}) \simeq H_2(T^5/\Gamma) \). In the Mayer-Vietoris sequence of \( T^5/\Gamma \):

\[
\begin{align*}
H_2(\partial \tilde{T}) & \overset{(i^*, -j^*)}{\longrightarrow} H_2(\tilde{T}) \oplus H_2(U) \overset{k^* + i^*}{\longrightarrow} H_2(T^5/\Gamma) \overset{\partial_\ast}{\longrightarrow} H_1(\partial \tilde{T}) \overset{(i^*, -j^*)}{\longrightarrow} H_1(\tilde{T}) \oplus H_1(U) \\
\{0\} & \longrightarrow H_2(\tilde{T}) \oplus \{0\} \overset{k^* + i^*}{\longrightarrow} H_2(T^5/\Gamma) \overset{\partial_\ast}{\longrightarrow} \mathbb{R}^{12} \overset{(i^*, -j^*)}{\longrightarrow} \{0\} \oplus \mathbb{R}^{12}
\end{align*}
\]

one can see that

\[ -j_\ast : H_1(\partial \tilde{T}) \to H_1(U) \]

is an isomorphism, so \( \partial_\ast = 0 \) implying that

\[ k_\ast : H_2(\tilde{T}) \to H_2(T^5/\Gamma) \]

is an isomorphism.

In the Mayer-Vietoris sequence of \( X \):

\[
\begin{align*}
H_2(\partial \tilde{T}) & \overset{(i^*, -j^*)}{\longrightarrow} H_2(\tilde{T}) \oplus H_2(N) \overset{k^* + i^*}{\longrightarrow} H_2(X) \overset{\partial_\ast}{\longrightarrow} H_1(\partial \tilde{T}) \overset{(i^*, -j^*)}{\longrightarrow} H_1(\tilde{T}) \oplus H_1(N) \\
\{0\} & \longrightarrow \mathbb{R} \oplus \mathbb{R}^{12} \overset{k^* + i^*}{\longrightarrow} H_2(X) \overset{\partial_\ast}{\longrightarrow} \mathbb{R}^{12} \overset{(i^*, -j^*)}{\longrightarrow} \{0\} \oplus \mathbb{R}^{12}
\end{align*}
\]

where \( N \) denotes the union of 12 \( S^1 \times Y \),

\[ -j_\ast : H_1(\partial \tilde{T}) \to H_1(N) \]
is an isomorphism, so \( \partial_* = 0 \) implying that

\[
k_* + l_* : H_2(\tilde{T}) \oplus H_2(N) \to H_2(X)
\]

is an isomorphism. Thus we get \( b_2(X) = 13 \), and hence \( H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{13} \) is obtained by the universal coefficient theorem and \( H_1(X, \mathbb{Z}) = \{0\} \).

\[\Box\]

**Proposition 3.5** \( X \) is nonspin, i.e., \( w_2(X) \neq 0 \).

**Proof** Assume to the contrary that \( X \) is spin. Then so is its subset \( \tilde{T} \). By using the global trivialization of the tangent bundle

\[
TT^5 \cong T^5 \times \mathbb{R}^5 \quad \text{where} \quad \mathbb{R}^5 = \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_5} \rangle,
\]

the orthonormal frame bundle \( PSO \) of \( T^5 \) can be trivialized as \( T^5 \times SO(5) \), so its double cover \( PSpin \) is trivialized as \( T^5 \times Spin(5) \). The orientation-preserving isometric \( \Gamma \) action on \( T^5 \) obviously lifts to \( PSO \).

Let’s consider this induced \( \Gamma \) action on \( PSO \) over \( \pi^{-1}(\tilde{T}) \). Since \( \tilde{T} \) is spin, this action must also lift to its double cover \( PSpin \) over \( \pi^{-1}(\tilde{T}) \). We shall show that whatever we take a choice of the lifts for the 3 generators \( \alpha, \beta, \gamma \) of \( \Gamma = (\mathbb{Z}_2)^3 \) to bundle maps of \( PSpin \), the lifted maps do not satisfy commutativity, which is contradictory.

With respect to the above trivializations, \( \alpha_*(x, v) \) of \( (x, v) \in T^5 \times \mathbb{R}^5 \cong TT^5 \) where \( \alpha_* \) denotes the derivative map of \( \alpha \) is given by \( (\alpha(x), M_\alpha v) \) where \( M_\alpha \) is the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}
\]

and the induced bundle map \( \alpha_*(x, A) \) of \( (x, A) \in T^5 \times SO(5) \cong PSO \) is given by \( (\alpha(x), M_\alpha A) \).

There are two elements in \( p^{-1}(M_\alpha) \subset Spin(5) \) where \( p : Spin(5) \to SO(5) \) is the double covering map. Recall that \( Spin(n) \) can be expressed as the multiplicative subgroup

\[
\{ v_1, \ldots, v_{2m} | v_i \in \mathbb{R}^n, ||v_i|| = 1, m \geq 0 \}
\]

of the Clifford algebra \( Cl(n) \). (cf. [19]) In this expression,

\[
p^{-1}(M_\alpha) = \{ \pm e_2 \cdot e_3 \cdot e_4 \cdot e_5 \}
\]

where \( \{ e_1, \ldots, e_5 \} \) is the standard orthonormal basis of \( \mathbb{R}^n \).

As a trial, let’s take any one of them to be the lift \( \hat{M}_\alpha \) of \( M_\alpha \), and define the principal bundle map \( \alpha_* : PSpin \to PSpin \) as

\[
\alpha_*(x, B) = (\alpha(x), \hat{M}_\alpha B)
\]

for \( (x, B) \in \pi^{-1}(\tilde{T}) \times Spin(5) \). Likewise let’s choose \( \hat{M}_\beta \) among

\[
p^{-1}(M_\beta) = \{ \pm e_1 \cdot e_2 \cdot e_3 \cdot e_5 \}
\]
where $M_\beta$ is
\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
and define the bundle map $\beta_\bullet$. Whatever the choice may be, it turns out that

\[
\hat{M}_\alpha \cdot \hat{M}_\beta = - \hat{M}_\beta \cdot \hat{M}_\alpha,
\]

since

\[
(e_2 \cdot e_3 \cdot e_4 \cdot e_5) \cdot (e_1 \cdot e_2 \cdot e_3 \cdot e_5) = -(e_1 \cdot e_2 \cdot e_3 \cdot e_5) \cdot (e_2 \cdot e_3 \cdot e_4 \cdot e_5).
\]

Thus

\[
\alpha_\bullet \circ \beta_\bullet = - \beta_\bullet \circ \alpha_\bullet,
\]
yielding the desired contradiction to that the $\Gamma$ action is lifted to $PSpin$ over $\pi^{-1}(\tilde{T})$. \qed

Smooth closed simply-connected 5-manifolds $M$ are classified up to diffeomorphism by Smale [25] and D. Barden [2]. In particular if $H_2(M, \mathbb{Z})$ is torsion-free, then such $M$ is completely classified by $k := b_2(M)$ and $w_2(M)$, namely

\[
M \simeq \begin{cases} 
S^5 \# k(S^2 \times S^3) & \text{if } w_2(M) = 0 \\
(S^2 \times S^3) \# (k-1)(S^2 \times S^3) & \text{if } w_2(M) \neq 0
\end{cases}
\]

where $S^2 \times S^3$ denotes the\(^1\) nontrivial $S^3$-bundle over $S^2$. However, $H_2$ of our $X$ has nontrivial torsion.

**Proposition 3.6** $H_2(X, \mathbb{Z})$ has a nontrivial torsion subgroup.

**Proof** To make the argument simple, we prove by contradiction, so assume to the contrary that $H_2(X, \mathbb{Z})$ is torsion-free. Recall from the proof of Proposition 3.4 that $H_2(X, \mathbb{R}) \simeq \mathbb{R}^{13}$ is generated by $H_2(\tilde{T}, \mathbb{R}) \simeq \mathbb{R}$ and $H_2(N, \mathbb{R}) \simeq \mathbb{R}^{12}$. Let's denote the generator (unique up to $\pm 1$) of the torsion-free part of $H^2(\tilde{T}, \mathbb{Z})$ by $\omega$. Thus $\omega$ is the pull-back from $H^2(X, \mathbb{Z})$ under the inclusion map $k : \tilde{T} \to X$.

Since $H_1(X, \mathbb{Z}) = \{0\}$ and $H_2(X, \mathbb{Z}) \simeq \mathbb{Z}^{13}$,

\[
H^2(X, \mathbb{Z}) \simeq (\mathbb{Z}_2)^{13} \simeq H^2(X, \mathbb{Z}) \otimes \mathbb{Z}_2
\]

by the universal coefficient theorem. Hence $w_2(X) \neq 0 \in H^2(X, \mathbb{Z}_2)$ is the mod 2 reduction of a torsion-free integral cohomology class. In the proof of Proposition 3.5, we showed that $\tilde{T}$ is nonspin, so $w_2(\tilde{T}) \neq 0$. Thus

\[
w_2(\tilde{T}) = k^*w_2(X) \equiv \omega \mod 2,
\]

and hence

\[
\pi^*w_2(\tilde{T}) \neq 0,
\]

\(^1\) It is unique up to bundle isomorphism because $\text{Diff}(S^3)$ deformation retracts onto $O(4)$ and $\pi_1(SO(4)) \simeq \mathbb{Z}_2$. 

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because $\pi^*\omega$ as a real cohomology is
\[ [dx_2 \wedge dx_3] \neq 0 \in H^2(\pi^{-1}(\tilde{T}), \mathbb{R}) \cong H^2(T^5, \mathbb{R}) \]
where $\cong$ is justified by the following Mayer-Vietoris sequence of real cohomology groups:
\[
H^1(\pi^{-1}(U)) \oplus H^1(\pi^{-1}(\tilde{T})) \to H^1(\partial(\pi^{-1}(U))) \delta^* \to H^2(T^5)
\]
\[
\to H^2(\pi^{-1}(U)) \oplus H^2(\pi^{-1}(\tilde{T})) \to H^2(\partial(\pi^{-1}(U)))
\]
\[
\mathbb{Z}^{48} \oplus H^1(\pi^{-1}(\tilde{T})) \to \mathbb{Z}^{48} \delta^* \to H^2(T^5) \to \{0\} \oplus H^2(\pi^{-1}(\tilde{T})) \to \{0\}
\]
where $\pi^{-1}(U)$ is the union of 48 copies of $S^1 \times B^4$ and all homomorphisms except the connecting homomorphism $\delta^*$ are induced by obvious inclusion maps. (From $\delta^* = 0$, the desired isomorphism follows.)

On the other hand
\[ \pi^*w_2(\tilde{T}) = w_2(\pi^{-1}(\tilde{T})) = 0, \]
since $\pi^{-1}(\tilde{T}) \subset T^5$ is spin. This is the desired contradiction. \qed

As a consequence of this, $X$ cannot admit an effective $T^3$ action by Oh’s theorem [20]. However, it turns out that $X$ still admits an $\mathcal{F}$-structure to be explained in the next section.

4 $\mathcal{F}$-structure on $X$

An $\mathcal{F}$-structure introduced by Cheeger and Gromov [7, 8] generalizes an effective $T^k$-action on a manifold.

**Definition 4.1** An $\mathcal{F}$-structure on a smooth manifold $M$ is given by data $(U_i, \hat{U}_i, T^{k_i})$ for $i \in I$ with the following conditions:

1. $\{U_i | i \in I\}$ is a locally finite open cover of $M$.
2. Each $\pi_i : \hat{U}_i \to U_i$ is a finite Galois covering with covering group $\Gamma_i$.
3. Each torus $T^{k_i}$ of dimension $k_i$ acts smoothly and effectively on $\hat{U}_i$ in a $\Gamma_i$-covariant way, i.e., there exists a homomorphism $\Psi_i : \Gamma_i \to \text{Aut}(T^{k_i})$ so that $\gamma(tx) = \Psi_i(\gamma)(t)x$ for any $\gamma \in \Gamma_i$, $t \in T^{k_i}$, and $x \in \hat{U}_i$.
4. If $U_i \cap U_j \neq \emptyset$, then there is a common covering of $\pi_i^{-1}(U_i \cap U_j)$ and $\pi_j^{-1}(U_i \cap U_j)$ such that it is invariant under the lifted actions of $T^{k_i}$ and $T^{k_j}$, and they commute.

As a special case, an $\mathcal{F}$-structure is called polarized if the torus actions defined on the finite coverings are locally free. The minimum of the dimensions of the orbits is called the rank of the $\mathcal{F}$-structure.

**Proposition 4.2** $X$ admits a polarized $\mathcal{F}$-structure.
Proof Paternain and Petean [22] showed that well-known manifolds obtained by Kummer-type constructions have $\mathcal{F}$-structures. We shall follow their method.

To define an open covering of $X$ for an $\mathcal{F}$-structure, consider the following open sets in $T^5 = \mathbb{R}^5 / \mathbb{Z}^5$:

$$W^\alpha(\varepsilon) := \{(x_1, \cdots, x_5) \in T^5 | ||(x_2, x_3) - (a_2, a_3)|| < \varepsilon, a_i = 0, \frac{1}{2}, i = 2, 3\}$$

$$W^\beta(\varepsilon) := \{(x_1, \cdots, x_5) \in T^5 | ||(x_2, x_3) - (a_2, a_3)|| < \varepsilon, a_2 = \frac{1}{4}, \frac{3}{4}, a_3 = 0, \frac{1}{2}\}$$

$$W^\gamma(\varepsilon) := \{(x_1, \cdots, x_5) \in T^5 | ||(x_2, x_3) - (a_2, a_3)|| < \varepsilon, a_2 = 0, \frac{1}{2}, a_3 = \frac{1}{4}, \frac{3}{4}\}.$$

For $\varepsilon \in (0, \frac{1}{100})$ these sets are disjoint, left invariant under the $\Gamma$ action, and each of them has 4 connected components which are copies of $T^3 \times D^2$. Observe that $\pi(W^\alpha(\varepsilon))$, $\pi(W^\beta(\varepsilon))$, and $\pi(W^\gamma(\varepsilon))$ are all connected open subsets of $T^5 / \Gamma$. After the surgery resolving the singularities, they got modified into open subsets of $X$, which are denoted by $U^\alpha(\varepsilon)$, $U^\beta(\varepsilon)$, $U^\gamma(\varepsilon)$, respectively.

Defining

$$V := T^5 - cl(W^\alpha(\varepsilon) \cup W^\beta(\varepsilon) \cup W^\gamma(\varepsilon))$$

where $cl(\cdot)$ denotes the closure, $\Gamma$ acts freely on $V$ such that $V / \Gamma$ is an open subset of $T^5 / \Gamma$. Thus $V / \Gamma$ considered as a subset of $X$ and $U^\alpha(\varepsilon)$, $U^\beta(\varepsilon)$, $U^\gamma(\varepsilon)$ give an open covering of $X$, over which we define an $\mathcal{F}$-structure.

Let’s denote the circle actions $x_i \mapsto x_i \pm \theta$ for $\theta \in \mathbb{R} / \mathbb{Z}$ on $T^5$ by $A_{i \pm}$. An important fact is that the $T^3$ action given by $A_{1+} \times A_{4+} \times A_{5+}$ leaves invariant $W^\alpha(\varepsilon)$, $W^\beta(\varepsilon)$, $W^\gamma(\varepsilon)$, $V$, respectively.

To define an $S^1$ action on $U^\alpha(\varepsilon)$, note that $(\beta, \gamma)$ swaps the 4 connected components of $W^\alpha(\varepsilon)$ while $\alpha$ preserves each component, and denote 4 components of $W^\alpha(\varepsilon)$ by

$$W^\alpha_1(\varepsilon) := \{(x_1, \cdots, x_5) \in T^5 | ||(x_2, x_3) - (0, 0)|| < \varepsilon\},$$

$$W^\alpha_2(\varepsilon) := \beta(W^\alpha_1(\varepsilon)), W^\alpha_3(\varepsilon) := \gamma(W^\alpha_1(\varepsilon)), W^\alpha_4(\varepsilon) := \gamma(\beta(\gamma(W^\alpha_1(\varepsilon)))).$$

We define an $S^1$-action on $W^\alpha_1(\varepsilon)$, $\cdots$, $W^\alpha_4(\varepsilon)$ by using $A_{1+}$, $A_{-1}$, $A_{1-}$, $A_{1+}$, respectively. Then this $S^1$-action on $W^\alpha(\varepsilon)$ is $\Gamma$-equivariant, namely commutes with the $\Gamma$-action. Therefore this action descends to $\pi(W^\alpha(\varepsilon))$, and hence $U^\alpha(\varepsilon)$, since the surgeries do not affect the first coordinate $x_1$ on which $A_{1 \pm}$ act.

In a similar way one can define a circle action on $U^\beta(\varepsilon)$ and $U^\gamma(\varepsilon)$ by using $A_{4 \pm}$ and $A_{5 \pm}$, respectively.

On $V / \Gamma$ we cannot define a circle action directly, but we put the $T^3 = A_{1+} \times A_{4+} \times A_{5+}$ action on its Galois cover $V$. Indeed it can be made $\Gamma$-covariant by using the homomorphism

$$\Psi_V : \Gamma \to \text{Aut}(T^3)$$

given by

$$\Psi_V(\alpha)(t_1, t_2, t_3) = (t_1, -t_2, -t_3)$$

$$\Psi_V(\beta)(t_1, t_2, t_3) = (-t_1, t_2, -t_3)$$

$$\Psi_V(\gamma)(t_1, t_2, t_3) = (-t_1, -t_2, t_3).$$

It remains to check the last condition for $\mathcal{F}$-structure. The overlaps consist of 3 regions

$$(V / \Gamma) \cap U^\alpha(\varepsilon), (V / \Gamma) \cap U^\beta(\varepsilon), (V / \Gamma) \cap U^\gamma(\varepsilon).$$
For the common Galois cover $V \cap W^\alpha(e)$ of the 1st one, the lifted actions of $T^3 = A_{1+} \times A_{4+} \times A_{5+}$ and $S^1 = A_{1\pm}$ obviously commute. Likewise for the 2nd and the 3rd ones.

Finally we have a well-defined $\mathcal{F}$-structure on $X$, and it is polarized, since all actions are locally free.

By the Cheeger–Gromov theorem [7], any manifold admitting a polarized $F$-structure collapses with sectional curvature bounded, implying that its minimal volume vanishes. Thus we have

**Corollary 4.3** $\text{MinVol}(X) = 0$.

**Remark 4.4** The above $\mathcal{F}$-structure on $X$ certainly has positive rank. Cheeger and Gromov [7] also proved that if an $\mathcal{F}$-structure on a manifold has positive rank then the Euler characteristic of the manifold must be 0. Indeed our $X$ has zero Euler characteristic.

As a consequence of $\text{MinVol}(X) = 0$, one can also deduce the vanishing of the minimal entropy and various types of other minimal volumes of $X$ (cf. [21, 26])

### 5 More examples

One can use this Kummer-type method to construct other examples with such properties. For example, let the group $\Gamma \equiv \langle \alpha \rangle \oplus \langle \beta \rangle \oplus \langle \gamma \rangle = (\mathbb{Z}_2)^3$ act on $T^5$ as follows:

\[
\begin{align*}
\alpha : x & \mapsto (x_1, -x_2, -x_3, -x_4, \frac{1}{2} + x_5), \\
\beta : x & \mapsto (x_1, -x_2, -x_3, -x_4, -x_5), \\
\gamma : x & \mapsto (-x_1, -x_2, \frac{1}{2} - x_3, -x_4, x_5), \\
\alpha\beta : x & \mapsto (x_1, x_2, x_3, x_4, \frac{1}{2} + x_5), \\
\beta\gamma : x & \mapsto (-x_1, x_2, \frac{1}{2} + x_3, x_4, -x_5), \\
\alpha\gamma : x & \mapsto (-x_1, x_2, \frac{1}{2} + x_3, x_4, \frac{1}{2} - x_5),
\end{align*}
\]

and

\[
\begin{align*}
\alpha\beta\gamma : x & \mapsto (-x_1, -x_2, \frac{1}{2} - x_3, -x_4, \frac{1}{2} + x_5).
\end{align*}
\]

Again the fixed point sets are 16 copies of $S_\alpha$, $S_\beta$, and $S_\gamma$, respectively, which are all disjoint. As before, $\langle \beta, \gamma \rangle$ acts freely on the set of 16 $S_\alpha$ identifying them into 4 of them, and similarly $\langle \alpha, \gamma \rangle$ identifies 16 $S_\beta$ to 4 of them. But both of $\alpha$ and $\beta$ act freely on the set of 16 $S_\gamma$ identifying them into 8 $S_\gamma$, while $\alpha\beta$ preserves each of $S_\gamma$ acting as a translation

\[
(p_1, \ldots, p_4, x_5) \mapsto (p_1, \ldots, p_4, \frac{1}{2} + x_5)
\]

of a circle $S_\gamma$. Thus the singular set of $T^5 / \Gamma$ is a disjoint union of 8 copies of $S^1$ whose singularity is modeled on $S^1 \times \mathbb{C}^2/\{\pm 1\}$ and 8 copies of $S^1$ whose singularity is modeled on $(\mathbb{C}^2/\{\pm 1\} \times S^1)/\mathbb{Z}_2$ which is also diffeomorphic to $S^1 \times \mathbb{C}^2/\{\pm 1\}$.

By the same method as before, one can construct $(X, g)$. Since the circle length of $\pi(S_\gamma) = S^1 \times \{\text{orbifold point}\}$ is the half of the circle lengths of $\pi(S_\alpha)$ and $\pi(S_\beta)$, the circle length of the
glued $S^1 \times Y$ should be correspondingly the half. The same properties listed in Theorem 1.1 hold in this example too, and they can be proved in the same way. The only difference is that $b_2(X) = b_3(X) = 17$ in this case and the $\Gamma$-invariant part of $H^3_{\text{DR}}(T^5) = \{c d x_3 \wedge d x_4 | c \in \mathbb{R} \}$.

To prove the existence of a polarized $F$-structure on $X$, one can use the following disjoint open sets in $T^5 = \mathbb{R}^5 / \mathbb{Z}$:

$$W^{\alpha \beta}(\varepsilon) := \{(x_1, \ldots, x_5) \in T^5 | ||(x_3, x_4) - (a_3, a_4)|| < \varepsilon, a_i = \frac{1}{2}, i = 3, 4 \}$$

$$W^{\gamma}(\varepsilon) := \{(x_1, \ldots, x_5) \in T^5 | ||(x_3, x_4) - (a_3, a_4)|| < \varepsilon, a_3 = \frac{1}{4}, a_4 = 0, \frac{1}{2} \}$$

$$V := T^5 - cl(W^{\alpha \beta}(\frac{\varepsilon}{2}) \cup W^{\gamma}(\frac{\varepsilon}{2}))$$

all of which are invariant under the $T^2 = A_{1^+} \times A_{5^+}$ action, and likewise define an $S^1$ action on $U^{\alpha \beta}(\varepsilon)$ and $U^{\gamma}(\varepsilon)$ by using $A_{1^\pm}$ and $A_{5^\pm}$, respectively.

6 Discussions and Questions

Every smooth closed simply connected 5-manifold $M$ admits a metric of positive scalar curvature by the well-known Gromov-Lawson surgery theorem [11]. What about a metric of positive Ricci curvature? If $H_2(M, \mathbb{Z})$ is torsion-free, then $M$ admits a metric of positive Ricci curvature [24], and even a toric Sasaki-Einstein metric (of positive scalar curvature) under the further assumption that $M$ is spin and $b_2(M)$ is odd [29]. It is left as an interesting question whether our constructed manifolds $X$ admit metrics of positive Ricci curvature or not. Since our $X$ are nonspin, they cannot admit a Sasaki-Einstein structure.

There are other ways of constructing simply connected almost Ricci-flat 5-manifolds. One may also use cylindrical construction as in [14].

It is a difficult task to find out whether these almost Ricci-flat 5-manifolds actually admit a Ricci-flat metric or not. Should one of them does, it would serve as a much-sought-after example of a compact simply connected Ricci-flat manifold with generic holonomy.(cf. [4])

References

1. Anderson, M.: Hausdorff perturbations of Ricci-flat manifolds and the splitting theorem. Duke Math. J. 68, 67–82 (1992)
2. Barden, D.: Simply connected five-manifolds. Ann. of Math. 82, 365–385 (1965)
3. Berger, M.: Sur les groupes d’holonomie homogènes de variétés à connexion affine et des variétés riemanniennes. Bull. Soc. Math. France 83, 279–330 (1955)
4. Berger, M.: A Panoramic View of Riemannian Geometry. Springer-Verlag, Berlin (2003)
5. Brendle, S., Kapouleas, N.: Gluing Eguchi–Hanson metrics and a question of Page. Comm. Pure Appl. Math. 70, 1366–1401 (2017)
6. Cheeger, J., Gromoll, D.: The splitting theorem for manifolds of nonnegative Ricci curvature. J. Diff. Geom. 6, 119–128 (1971)
7. Cheeger, J., Gromov, M.: Collapsing Riemannian manifolds while keeping their curvature bounded I. J. Diff. Geom. 23, 309–346 (1986)
8. Cheeger, J., Gromov, M.: Collapsing Riemannian manifolds while keeping their curvature bounded II. J. Diff. Geom. 32, 269–298 (1990)
9. Gromov, M.: Almost flat manifolds. J. Diff. Geom. 13, 231–241 (1978)
10. Gromov, M.: Volume and bounded cohomology. Math. IHES 56, 1–99 (1982)
11. Gromov, M., Lawosn, B.: The classification of simply connected manifolds of positive scalar curvature. Ann. Math. 111, 423–434 (1980)
12. Joyce, D.: Compact 7-manifolds with holonomy $G_2$ I, II. J. Diff. Geom. 43 no. 2, 291–328, 329–375 (1996)
13. Joyce, D.: Compact 8-manifolds with holonomy Spin(7). Invent. Math. 123(3), 507–552 (1996)
14. Kovalev, A.: Twisted connected sums and special Riemannian holonomy. J. Reine Angew. Math. 565, 125–160 (2003)
15. Kapovitch, V., Lott, J.: On noncollapsed almost Ricci-flat 4-manifolds. Amer. J. Math. 141, 737–755 (2019)
16. Lott, J.: The collapsing geometry of almost Ricci-flat 4-manifolds. Comment. Math. Helv. 95, 79–98 (2020)
17. LeBrun, C.: Edges, orbifolds, and Seiberg–Witten theory. J. Math. Soc. Jpn. 67, 979–1021 (2015)
18. LeBrun, C., Singer, M.: A Kummer-type construction of self-dual 4-manifolds. Math. Ann. 300, 165–180 (1994)
19. Morgan, J.: The Seiberg–Witten Equations and Applications to the Topology of Smooth Four-Manifold. Princeton University Press, New Jersey (1996)
20. Oh, H.: Toral actions on 5-manifolds. Trans. AMS. 278, 233–252 (1983)
21. Paternain, G., Petean, J.: Minimal entropy and collapsing with curvature bounded from below. Invent. Math. 151, 415–450 (2003)
22. Paternain, G., Petean, J.: Collapsing manifolds obtained by Kummer-type constructions. Trans. AMS. 361, 4077–4090 (2009)
23. Petersen, P.: Riemannian Geometry. Springer-Verlag, New York (1998)
24. Sha, J., Yang, D.: Examples of manifolds of positive Ricci curvature. J. Diff. Geom. 29, 95–104 (1989)
25. Smale, S.: On the structure of 5-manifolds. Ann. Math. 75, 38–46 (1962)
26. Sung, C.: Surgery, curvature, and minimal volume. Ann. Global Anal. Geom. 26, 209–229 (2004)
27. Topiwala, P.: A new proof of the existence of Kähler–Einstein metrics on K3, I. Invent. Math. 89, 425–448 (1987)
28. Topiwala, P.: A new proof of the existence of Kähler–Einstein metrics on K3, II. Invent. Math. 89, 449–454 (1987)
29. van Coevering, C.: Sasaki-Einstein 5-manifolds associated to toric 3-Sasaki manifolds. New York J. Math. 18, 555–608 (2012)

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