Random matrix theory and multivariate statistics

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Abstract
Some tools and ideas are interchanged between random matrix theory and multivariate statistics. In the context of the random matrix theory, classes of spherical and generalised Wishart random matrix ensemble, containing as particular cases the classical random matrix ensembles, are proposed. Some properties of these classes of ensemble are analysed. In addition, the random matrix ensemble approach is extended and a unified theory proposed for the study of distributions for real normed division algebras in the context of multivariate statistics.

1 Introduction
Random matrices first appeared in studies of mathematical statistics in the 1930s but did not attract much attention at the time. However, in time the foundations were laid for one of the main areas of present-day statistics, Multivariate Statistics or Multivariate Statistical Analysis. The advances made have been compiled in excellent books such as Roy (1957) and Anderson (1958), later in Srivastava and Khatri (1973) and Muirhead (1982) and recently in Gupta and Nagar (2000) and Eaton (2007), among many other important texts. In general, these writings are based on the multivariate normal (Gaussian) distribution. Nevertheless, over the last 20 years, a new current has been developed and consolidated to constitute what is now known as Generalised Multivariate Analysis, in which the multivariate normal distribution is replaced by a family of elliptical distributions, containing as particular cases the Normal, Student-\textit{t}, Pearson type VII, Logistic and Kotz distributions, among many others, see Fang \textit{et al.} (1990), Fang and Zhang (1990) and Gupta and Varga (1993). This family contains many distributions with heavier or lighter tails than the Gaussian distribution, thus offering a more flexible framework for modelling phenomena and experiments in more general situations.

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Random matrices are not only of interest to statisticians; they are studied in many disciplines of science, engineering and finance. This approach has been consolidated and is generally known as Random Matrix Theory, see Metha (1991), Forrester (2009), Ratnarajah et al. (2005a) and Edelman and Rao (2005), among many others. Fundamentally, this theory is concerned with the following question: consider a large matrix whose elements are random variables with given probability laws. Then, what can be said about the probabilities of a few of its eigenvalues or of a few of its eigenvectors? This question is relevant to our understanding of the statistical behaviour of slow neutron resonances in nuclear physics, a field in which the issue was first addressed in the 1950s and where it is intensively studied by physicists. The question was later found to be important in other areas of physics and mathematics, such as the characterisation of chaotic systems, the elastodynamic properties of structural materials, conductivity in disordered metals, the distribution of the values of the Riemann zeta function on the critical line, the enumeration of permutations with certain particularities, the counting of certain knots and links, quantum gravity, quantum chromodynamics, string theory, and others, see Metha (1991) and Forrester (2009). This approach has also motivated the study of random linear systems (operators), random matrix calculus (calculus of Jacobians of matrix factorisations) and numerical stochastic algorithms, among other mathematical fields, see Edelman and Rao (2005). A very interesting characteristic of the results obtained in random matrix theory is that they are valid for real, complex and quaternionic random matrices.

From the standpoint of the multivariate statistics, the main focus of statistical studies has been developed for real random matrices. However, many texts have proposed the extension of some of these tests to the case of complex random matrices, see for example Wooding (1956), Goodman (1963), James (1964), Khatri (1965), Hannan (1970, Section 6.6, p. 295), Methai (1997) and Micheas et al. (2006), among many others. Studies examining the case of quaternionic random matrices case are much less common, see Bhavsar (2000) and Li and Xue (2009).

From the point of view of the statistical and random matrix theory, the relevance of the octonions is not clear. An excellent review of the history, construction and many other properties of the octonions is Baez (2002), where it is stated that:

"Their relevance to geometry was quite obscure until 1925, when Élie Cartan described ‘triality’ – the symmetry between vector and spinors in 8-dimensional Euclidean space. Their potential relevance to physics was noticed in a 1934 paper by Jordan, von Neumann and Wigner on the foundations of quantum mechanics...Work along these lines continued quite slowly until the 1980s, when it was realized that the octonions explain some curious features of string theory... However, there is still no proof that the octonions are useful for understanding the real world. We can only hope that eventually this question will be settled one way or another."

Others problems, such as the existence of real or imaginary eigenvalues for $2 \times 2$ and $3 \times 3$ octonionic Hermitian matrices are treated in Dray and Manogue (1998) and Dray and Manogue (1999). For the sake of completeness, in the present study the case of the octonions is considered, but the veracity of the results in this case can only be conjectured. In particular, Forrester (2009, Section 1.4.5, pp. 22-24) it is proved that the bi-dimensional density function of the eigenvalue, for a Gaussian ensemble of a $2 \times 2$ octonionic matrix, is obtained from the general joint density function of the eigenvalues for the Gaussian ensemble, assuming $m = 2$ and $\beta = 8$, see Section 3.

In the present study, we propose an interchange between some ideas and tools of random matrix theory and multivariate statistics. Specifically, in Section 2 general expressions are given for the Jacobians of certain matrix transformation, together with the factorisation for
real normed division algebras, a question that is fundamental in the study of general random matrix variate distributions. Vector-spherical, spherical, vector-spherical-Laguerre and spherical-Laguerre random matrix ensembles are studied in Section 3 where it is observed that the classical random matrix ensembles are obtained as particular cases of the above general random matrix variate distributions. Finally, the density function of the matrix variate normal, Wishart, elliptical, generalised Wishart and beta type I and II distributions are studied for real normed division algebras.

2 Jacobians

In this section we discuss the formulas containing the Jacobians of familiar matrix transformations and factorisations using the parameter \( \beta \) (defined below). First, however, some notation must be established.

The parameter \( \beta \) has traditionally been used to count the real dimension of the underlying normed division algebra. In other branches of mathematics, the parameter \( \alpha = 2/\beta \) is used.

| \( \beta \) | \( \alpha \) | Normed division algebra |
|------|------|------------------|
| 1    | 2    | real (\( \mathbb{R} \)) |
| 2    | 1    | complex (\( \mathbb{C} \)) |
| 4    | 1/2  | quaternionic (\( \mathbb{H} \)) |
| 8    | 1/4  | octonion (\( \mathbb{O} \)) |

**Remark 2.1.** In Baez (2002) the following remarks are made:

“There are exactly four normed division algebras: the real numbers (\( \mathbb{R} \)), complex numbers (\( \mathbb{C} \)), quaternions (\( \mathbb{H} \)), and octonions (\( \mathbb{O} \)). The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete. The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings. But the octonions are the crazy old uncle nobody lets out of the attic: they are nonassociative.”

In addition, take into account that \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) and \( \mathbb{O} \) are the only normed division algebras; moreover, they are the only alternative division algebras, and all division algebras have a real dimension of 1, 2, 4 or 8, see Baez (2002, Theorems 1, 2 and 3).

Let \( \mathcal{L}_{m,n}^\beta \) be the linear space of all \( n \times m \) matrices of rank \( m \leq n \) over \( \mathfrak{F} \) with \( m \) distinct positive singular values, where \( \mathfrak{F} \) denotes a real finite-dimensional normed division algebra. Let \( \mathfrak{F}^{n\times m} \) be the set of all \( n \times m \) matrices over \( \mathfrak{F} \). The dimension of \( \mathfrak{F}^{n\times m} \) over \( \mathbb{R} \) is \( \beta mn \).

Let \( A \in \mathfrak{F}^{n\times m} \), then \( A^* = \overline{A}^T \) denotes the usual conjugate transpose. The set of matrices \( H_1 \in \mathfrak{F}^{n\times m} \) such that \( H_1^* H_1 = I_m \) is a manifold denoted \( \mathcal{V}_{m,n}^{\beta} \), termed Stiefel manifold (\( H_1 \) is also known as semi-orthogonal (\( \beta = 1 \)), semi-unitary (\( \beta = 2 \)), semi-symplectic (\( \beta = 4 \)) and semi-exceptional type (\( \beta = 8 \)) matrices, see Dray and Manogue (1999). The dimension of \( \mathcal{V}_{m,n}^{\beta} \) over \( \mathbb{R} \) is \( [\beta mn - (m+1)\beta/2 - m] \). In particular, \( \mathcal{V}_{m,m}^{\beta} \) with dimension over \( \mathbb{R} \), \( [m(m+1)\beta/2 - m] \), is the maximal compact subgroup \( \mathcal{U}^{\beta}(m) \) of \( \mathcal{L}_{m,m}^{\beta} \) and consist of all matrices \( H \in \mathfrak{F}^{m\times m} \) such that \( H^* H = I_m \). Therefore, \( \mathcal{U}^{\beta}(m) \) is the real orthogonal group \( O(m) \) (\( \beta = 1 \)), the unitary group \( U(m) \) (\( \beta = 2 \)), compact symplectic group \( Sp(m) \) (\( \beta = 4 \)).
or exceptional type matrices $O_o(m)$ ($\beta = 8$), for $\mathfrak{f} = \mathbb{R}$, $\mathfrak{c}$, $\mathfrak{j}$ or $\mathfrak{d}$, respectively. Denote by $\mathfrak{S}_m^\beta$ the real vector space of all $S \in \mathfrak{S}^{m \times m}$ such that $S = S^*$, let $\mathfrak{P}_m^\beta$ be the cone of positive definite matrices $S \in \mathfrak{S}^{m \times m}$, then $\mathfrak{P}_m^\beta$ is an open subset of $\mathfrak{S}_m^\beta$. Over $\mathfrak{c}$, $\mathfrak{S}_m^\beta$ consist of symmetric matrices; over $\mathfrak{c}$, Hermitian matrices; over $\mathfrak{j}$, quaternionic Hermitian matrices (also termed self-dual matrices) and over $\mathfrak{d}$, octonionic Hermitian matrices. Generically, the elements of $\mathfrak{S}_m^\beta$ are termed as Hermitian matrices, irrespective of the nature of $\mathfrak{f}$. The dimension of $\mathfrak{S}_m^\beta$ over $\mathbb{R}$ is $[m(m-1)\beta+2]/2$. Let $\mathfrak{D}_m^\beta$ be the diagonal subgroup of $\mathfrak{L}_m^\beta$, consisting of all $D \in \mathfrak{S}^{m \times m}$, $D = \text{diag}(d_1, \ldots, d_m)$. Let $\mathfrak{T}_L^\beta(m)$ be the subgroup of all upper triangular matrices $T \in \mathfrak{S}^{m \times m}$ such that $t_{ij} = 0$ for $1 < i < j \leq m$; and let $\mathfrak{T}_L^\beta(m)$ be the opposed lower triangular subgroup $\mathfrak{T}_L^\beta(m) = \left(\mathfrak{T}_L^\beta(m)\right)^T$. For any matrix $X \in \mathfrak{S}^{n \times m}$, $dX$ denotes the matrix of differentials $(dx_{ij})$. Finally, we define the measure or volume element $(dX)$ when $X \in \mathfrak{S}^{m \times n}$, $\mathfrak{S}_m^\beta$, $\mathfrak{D}_m^\beta$ or $\mathfrak{V}_m^\beta$, see [Dimitriu 2002].

If $X \in \mathfrak{S}^{n \times m}$ then $(dX)$ (the Lebesgue measure in $\mathfrak{S}^{n \times m}$) denotes the exterior product of the $\beta mn$ functionally independent variables

$$(dX) = \bigwedge_{i=1}^{\beta} \bigwedge_{j=1}^{m} \bigwedge_{k=1}^{n} \beta dx_{ij}^{(k)}.$$ 

**Remark 2.2.** Note that for $x_{ij} \in \mathfrak{f}$

$$dx_{ij} = \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}.$$ 

In particular for $\mathfrak{f} = \mathbb{R}$, $\mathfrak{c}$, $\mathfrak{j}$ or $\mathfrak{d}$ we have

- $x_{ij} \in \mathbb{R}$ then

$$dx_{ij} = \bigwedge_{k=1}^{1} dx_{ij}^{(k)} = dx_{ij}.$$ 

- $x_{ij} = x_{ij}^{(1)} + ix_{ij}^{(2)} \in \mathfrak{c}$, then

$$dx_{ij} = dx_{ij}^{(1)} \wedge dx_{ij}^{(2)} = \bigwedge_{k=1}^{2} dx_{ij}^{(k)}.$$ 

- $x_{ij} = x_{ij}^{(1)} + ix_{ij}^{(2)} + jx_{ij}^{(3)} + kx_{ij}^{(4)} \in \mathfrak{j}$, then

$$dx_{ij} = dx_{ij}^{(1)} \wedge dx_{ij}^{(2)} \wedge dx_{ij}^{(3)} \wedge dx_{ij}^{(4)} = \bigwedge_{k=1}^{4} dx_{ij}^{(k)}.$$ 

- $x_{ij} = x_{ij}^{(1)} + e_1 x_{ij}^{(2)} + e_2 x_{ij}^{(3)} + e_3 x_{ij}^{(4)} + e_4 x_{ij}^{(5)} + e_5 x_{ij}^{(6)} + e_6 x_{ij}^{(7)} + e_7 x_{ij}^{(8)} \in \mathfrak{d}$, then

$$dx_{ij} = dx_{ij}^{(1)} \wedge dx_{ij}^{(2)} \wedge dx_{ij}^{(3)} \wedge dx_{ij}^{(4)} \wedge dx_{ij}^{(5)} \wedge dx_{ij}^{(6)} \wedge dx_{ij}^{(7)} \wedge dx_{ij}^{(8)} = \bigwedge_{k=1}^{8} dx_{ij}^{(k)}.$$ 

If $S \in \mathfrak{S}_m^\beta$ (or $S \in \mathfrak{T}_L^\beta(m)$) then $(dS)$ (the Lebesgue measure in $\mathfrak{S}_m^\beta$ or in $\mathfrak{T}_L^\beta(m)$) denotes the exterior product of the $m(m+1)\beta/2$ functionally independent variables (or denotes the
exterior product of the $m(m-1) \beta/2 + n$ functionally independent variables, if $s_{ii} \in \mathbb{R}$ for all $i = 1, \ldots, m$)

$$(dS) = \begin{cases} 
\bigwedge_{i \leq j} \bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}, \\
\bigwedge_{i=1}^{m} \bigwedge_{i<j}^{m} \bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}, \text{ if } s_{ii} \in \mathbb{R}.
\end{cases}$$

**Remark 2.3.** Since generally the context establishes the conditions on the elements of $S$, that is, if $s_{ij} \in \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. It is considered that

$$(dS) = \bigwedge_{i \leq j} \bigwedge_{k=1}^{\beta} ds_{ij}^{(k)} \equiv \bigwedge_{i=1}^{m} \bigwedge_{i<j}^{m} \bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}.$$ Observe, too, that for the Lebesgue measure $(dS)$ defined thus, it is required that $S \in P_{\beta}^m$, that is, $S$ must be a non singular Hermitian matrix (Hermitian definite positive matrix). In the real case, when $S$ is a positive semidefinite matrix its corresponding measure is studied in [Uhlig (1994), Díaz-García and Gutiérrez (1997), Díaz-García and González-Farías (2005a) and Díaz-García and González-Farías (2005b)] under different coordinate systems.

If $\Lambda \in D_{\beta}^m$ then $(d\Lambda)$ (the Lebesgue measure in $D_{\beta}^m$) denotes the exterior product of the $\beta m$ functionally independent variables

$$(d\Lambda) = \bigwedge_{i=1}^{n} \bigwedge_{k=1}^{\beta} d\lambda_i^{(k)}.$$ If $H_1 \in V_{m,n}^\beta$ then

$$(H_1^* dH_1) = \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{m} h_i^* d h_j.$$ where $H = (H_1 | H_2) = (h_1, \ldots, h_m | h_{m+1}, \ldots, h_n) \in \mathcal{U}^\beta(m)$. It can be proved that this differential form does not depend on the choice of the matrix $H_2$ and that it is invariant under the transformations

$$H_1 \rightarrow QHP_1, \quad Q \in \mathcal{U}^\beta(n) \text{ and } P \in \mathcal{U}^\beta(m).$$ When $m = 1$; $V_{1,n}^\beta$ defines the unit sphere in $\mathbb{R}^n$. This is, of course, an $(n-1)\beta$-dimensional surface in $\mathbb{R}^n$. When $m = n$ and denoting $H_1$ by $H$, $(H^* dH)$ is termed the Haar measure on $\mathcal{U}^\beta(m)$ and defines an invariant differential form of a unique measure $\nu$ on $\mathcal{U}^\beta(m)$ given by

$$\nu(\mathcal{M}) = \int_{\mathcal{M}} (H^* dH).$$ It is unique in the sense that any other invariant measure on $\mathcal{U}^\beta(m)$ is a finite multiple of $\nu$ and invariant because is invariant under left and right translations, that is

$$\nu(Q\mathcal{M}) = \nu(\mathcal{M}Q) = \nu(\mathcal{M}), \quad \forall Q \in \mathcal{U}^\beta(m).$$ The surface area or volume of the Stiefel manifold $V_{m,n}^\beta$ is

$$\text{Vol}(V_{m,n}^\beta) = \int_{H_1 \in V_{m,n}^\beta} (H_1^* dH_1) = \frac{2^m m! m \pi^{\frac{n\beta}{2}} \Gamma_{\frac{n\beta}{2}}}{\Gamma_{\frac{\beta}{2}}}, \quad (2)$$
where $\Gamma^\beta_m[a]$ denotes the multivariate Gamma function for the space $\mathbb{S}^\beta(m)$, and is defined by

$$\Gamma^\beta_m[a] = \int_{A \in \mathbb{P}^\beta_m} \text{etr}(-A)|A|^{a-(m-1)\beta/2-1}(dA)$$

$$= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a-(i-1)\beta/2],$$

where $\text{etr}(\cdot) = \exp(\text{tr}(\cdot))$, $|\cdot|$ denotes the determinant and $\text{Re}(a) \geq (m-1)\beta/2$, see Gross and Richards (1987).

Dimitriu (2002, p. 35) first assigned the responsibility of calculating some Jacobianos to researchers of random matrix theory, although in the end this fact is not especially important, as the Jacobian of the spectral factorisation (and of SVD among others) for the symmetric matrix was first computed by James (1954) and not by Wigner (1958).

There exist an extensive bibliography on the computation of Jacobians of matrix transformations and decompositions in both random matrix theory and statistics. We now summarised diverse Jacobians in terms of the parameter $\beta$, some based on the work of Dimitriu (2002), while other results are proposed as extensions of real or complex cases, see Deemer and Olkin (1951), Olkin (1953), Roy (1957), James (1954), Anderson (1958), James (1964), Khatri (1965), Muirhead (1982), Metha (1991), Olkin (2002), Ratnareja et al. (2005a) and Ratnareja et al. (2005b). An excellent reference for real and complex cases is Methal (1997), which includes many of the Jacobian that have been published. Some Jacobians in the quaternionic case are obtained in Li and Xue (2009). We also included a parameter count (or number of functionally independent variables, $#\text{fiv}$), that is, if $A$ is factorised as $A = BC$, then the parameter count is written as $#\text{fiv}$ in $A = [#\text{fiv in } B] + [#\text{fiv in } C]$, see Dimitriu (2002).

**Lemma 2.1.** Let $X$ and $Y \in \mathcal{L}^\beta_{m,n}$, and let $Y = XAB + C$, where $A \in \mathcal{L}^\beta_{n,n}$, $B \in \mathcal{L}^\beta_{m,m}$ and $C \in \mathcal{L}^\beta_{m,n}$ are constant matrices. Then

$$(dY) = |A^*A|^{\beta m/2}|B^*B|^{\beta n/2}(dX).$$

**Lemma 2.2.** Let $X$ and $Y \in \mathcal{P}^\beta_m$, and let $Y = XAX^* + C$, where $A$ and $C \in \mathcal{L}^\beta_{m,m}$ are constant matrices. Then

$$(dY) = |A^*A|^{\beta (m-1)/2+1}(dX).$$

**Lemma 2.3.** Let $J \in \mathcal{T}^\beta_U(m)$ and write $J = BG$ where $B = \text{diag}(b_1, \ldots, b_m) \in \mathcal{D}^\beta_m$, and $G \in \mathcal{T}^\beta_U(m)$, with $g_{ii} = 1$ for all $i = 1, \ldots, m$. Then

- parameter count: $\beta m(m+1)/2 = [\beta m] + [\beta m(m-1)/2]$ and

$$(dJ) = \prod_{i=1}^m |b_i|^{\beta (m-i)}(dB)(dG),$$

where

$$(dG) = \bigwedge_{i<j} \bigwedge_k^{\beta} dg_{ik}.$$
where \((d\Delta) = \bigwedge_{i,j=1}^{m} d\delta_{ij}^{(k)}\).

**Lemma 2.5** (LU decomposition, Crout’s version). Let \(X \in \mathcal{L}^\beta_{m,m}\), such that \(X = \Delta Y\), where \(\Delta \in \mathcal{T}^\beta_{L}(m)\) and \(Y \in \mathcal{T}^\beta_{U}(m)\) such that \(\psi_{ii} = 1\) for all \(i = 1, \ldots, m\). Then

- parameter count: \(\beta m^2 = [\beta m(m+1)/2] + [\beta m(m-1)/2]\)

\[
(dX) = \prod_{i=1}^{m} |\delta_{ii}|^{\beta(m-1)}(dY)(d\Delta),
\]

where \((dY) = \bigwedge_{i<j}^{m} d\psi_{ij}^\beta\).

As a consequence of Lemmas 2.3 and 2.4 there follows

**Lemma 2.6** (LDM decomposition). Let \(X \in \mathcal{L}^\beta_{m,m}\), such that \(X = \Psi \Pi \Xi\), where \(\Psi \in \mathcal{T}^\beta_{L}(m)\) with \(\psi_{ii} = 1\) for all \(i = 1, \ldots, m\), \(\Pi = \text{diag}(\pi_1, \ldots, \pi_m) \in \mathcal{D}^\beta_{m}\), and \(\Xi \in \mathcal{T}^\beta_{U}(m)\) with \(\xi_{ii} = 1\) for all \(i = 1, \ldots, m\). Then

- parameter count: \(\beta m^2 = [\beta m(m-1)/2] + [\beta m] + [\beta m(m-1)/2]\)

\[
(dX) = \prod_{i=1}^{m} |\pi_{ii}|^{2\beta(m-1)}(d\Psi)(d\Pi)(d\Xi),
\]

**Lemma 2.7** (QR decomposition). Let \(X \in \mathcal{L}^\beta_{m,m}\), there then exists an \(H_1 \in \mathcal{V}^\beta_{m,n}\) and a \(T \in \mathcal{U}^\beta_{L}(m)\) with real \(t_{ii} > 0\), \(i = 1, 2, \ldots, q\) such that \(X = H_1 T\). Then

- parameter count: \(\beta mn = [\beta mn - \beta m(m-1)/2 - m] + [\beta m(m-1)/2 + m]\)

\[
(dX) = \prod_{i=1}^{m} t_{ii}^{\beta(n-1)+1-1}(H_1^*dH_1)(dT).
\]

Now, from Lemmas 2.5 and 2.6 it follows that

**Lemma 2.8** (Modified QR decomposition (QDR)). Let \(X \in \mathcal{L}^\beta_{m,n}\), then there exist \(H_1 \in \mathcal{V}^\beta_{m,n}\), a diagonal matrix \(N = \text{diag}(n_1, \ldots, n_m) \in \mathcal{D}^1_{m}\), with \(n_1 > \cdots > n_m > 0\) and \(\Omega \in \mathcal{T}^\beta_{U}(m)\) with \(\omega_{ii} = 1\), \(i = 1, 2, \ldots, m\) such that \(X = H_1 N\Omega\). Then

- parameter count: \(\beta mn = [\beta mn - \beta m(m-1)/2 - m] + [m] + [\beta m(m-1)/2]\)

\[
(dX) = 2^{-m} \prod_{i=1}^{m} n_{i}^{\beta(n+i-1)-1}(H_1^*dH_1)(dN)(d\Omega).
\]

**Lemma 2.9** (Polar decomposition). Let \(X \in \mathcal{L}^\beta_{m,n}\) and write \(X = P_1 R\), with \(P_1 \in \mathcal{V}^\beta_{m,n}\) and \(R \in \mathcal{Q}^\beta_{m}\). Then

- parameter count: \(\beta mn = [\beta mn - \beta m(m-1)/2 - m] + [\beta m(m-1)/2 + m]\)

\[
(dX) = |R|^{\beta(n+1)-1} \prod_{i<j}^{m}(d_i + d_j)^{\beta}(dR)(P_1^*dP_1),
\]

where \(R = QDQ^*\) is the spectral decomposition of \(R\), \(Q \in \mathcal{U}^\beta(m)\) and \(D = \text{diag}(d_1, \ldots, d_m) \in \mathcal{D}^1_{m}\) with \(d_1 > \cdots > d_m > 0\).

**Lemma 2.10** (Singular value decomposition, SVD). Let \(X \in \mathcal{L}^\beta_{m,n}\), such that \(X = V_1 D^*\) with \(V_1 \in \mathcal{V}^\beta_{m,n}\), \(W \in \mathcal{U}^\beta(m)\) and \(\bar{D} = \text{diag}(d_1, \ldots, d_m) \in \mathcal{D}^1_{m}\), \(d_1 > \cdots > d_m > 0\). Then
\begin{itemize}
  \item parameter count: $\beta mn = [\beta mn - \beta m(m-1)/2 - m - (\beta - 1)m] + [m] + [\beta m(m-1)/2 + m]$ and
  \begin{equation}
  (dX) = 2^{-m} \pi^2 \prod_{i=1}^{m} d_i^{\beta(n-m+1)-1} \prod_{i<j}^{m} (d_i^2 - d_j^2)^{\beta} (dD)(V_i^* dV_i)(W^* dW), \tag{12}
  \end{equation}
  where
  \[\tau = \begin{cases}
    0, & \beta = 1; \\
    -m, & \beta = 2; \\
    -2m, & \beta = 4; \\
    -4m, & \beta = 8.
  \end{cases}\]

  \textbf{Lemma 2.11} (Cholesky’s decomposition). Let $S \in \mathcal{P}_m$ and write $S = T^* T$, where $T \in \mathcal{T}_{\beta}^2(m)$ with $t_{ii} > 0$, $i = 1, 2, \ldots, m$. Then
  \begin{itemize}
    \item parameter count: $\beta m(m-1)/2 + m = \beta m(m-1)/2 + m$ and
    \begin{equation}
    (dS) = 2^m \prod_{i=1}^{m} i^{\beta(m-i)+1} (dT). \tag{13}
    \end{equation}
  \end{itemize}

  From Lemmas 2.11 and 2.4 it follows that

  \textbf{Lemma 2.12} ( $L^* D L$ decomposition). Let $S \in \mathcal{P}_m^\beta$ and write $S = \Omega^* O \Omega$, where $\Omega \in \mathcal{T}_{\beta}^2(m)$ with $\omega_{ii} = 1$, $i = 1, 2, \ldots, m$ and a diagonal matrix $O = \text{diag}(o_1, \ldots, o_m) \in \mathfrak{D}_m^1$, with $o_1 > \cdots > o_m > 0$. Then
  \begin{itemize}
    \item parameter count: $\beta m(m-1)/2 + m = [\beta m(m-1)/2] + [m]$ and
    \begin{equation}
    (dS) = \prod_{i=1}^{m} o_i^{\beta(m-i)} (d\Omega)(dO). \tag{14}
    \end{equation}
  \end{itemize}

  \textbf{Lemma 2.13} (Hermitian positive definite square root). Let $S$ and $R \in \mathcal{P}_m^\beta$, such that $S = R^2$. Then
  \begin{itemize}
    \item parameter count: $\beta m(m-1)/2 + m = \beta m(m-1)/2 + m$ and
    \begin{equation}
    (dS) = 2^m |R| \prod_{i<j}^{m} (d_i + d_j)^{\beta} (dR). \tag{15}
    \end{equation}
  \end{itemize}

  where $R = QDQ^*$ is the spectral decomposition of $R$, $Q \in \mathcal{U}_{\beta}^m$ and $D = \text{diag}(d_1, \ldots, d_m) \in \mathfrak{D}_m^1$ with $d_1 > \cdots > d_m > 0$.

  \textbf{Lemma 2.14} (Spectral decomposition). Let $S \in \mathcal{P}_m^\beta$. Then the spectral decomposition can be written as $S = W \Lambda W^*$, where $W \in \mathcal{U}_{\beta}^m$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m) \in \mathfrak{D}_m^1$, with $\lambda_1 > \cdots > \lambda_m > 0$. Then
  \begin{itemize}
    \item parameter count: $\beta m(m-1)/2 + m = [\beta m(m+1)/2 - m - (\beta - 1)m] + [m]$ and
    \begin{equation}
    (dS) = 2^{-m} \pi^2 \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} (d\Lambda)(W^* dW), \tag{16}
    \end{equation}
  \end{itemize}

  where $\tau$ is given in Lemma 2.10.

  Finally, by combining Lemmas 2.10, 2.4, 2.7 and 2.8 with Lemmas 2.11, 2.13, 2.11 and 2.12 respectively, the following result is obtained.
Lemma 2.15. Let $X \in \mathcal{L}^\beta_{m,n}$, and write $X = V_1D W^*$ (SVD), $X = V_1R$ (Polar decomposition), $X = V_1Q$ (QR decomposition) or $X = V_1N\Omega$ (modified QR decomposition) and let $S = X'X \in \mathcal{P}^\beta_m$. Then

\[
(dX) = 2^{-m/\beta} |S|^{|\beta(n-m+1)/2-1|} (dS) (V_1' dV_1). \tag{17}
\]

Note that the Lebesgue measure $(dS)$ in (17), can be factorised in terms of Cholesky, $L^*DL$, Hermitian positive definite square root or spectral decompositions, thus obtaining this way alternative explicit expressions of $(dS)$, given by [13], [14], [15] and [16], respectively. For the corresponding coordinates and bases, see Eaton (2007, Chapter 6).

3 Elliptical ensemble

In this section we employ the nomenclature recently proposed in random matrix theory for some of the most commonly-studied random matrices. These terms are *Hermite*, *Laguerre*, *Jacobi* and *Fourier* instead of Gaussian, Wishart, MANOVA (or beta type I) and circular, see Dimitriu and Edelman (2002), Edelman and Rad (2005), Edelman and Sutton (2008). These alternative names are proposed in view of the fact that in random matrix theory we are interested in matrices with joint eigenvalue density proportional to $\prod_{i<j} (\lambda_i - \lambda_j)$, where $|\Delta(\Lambda)| = \prod_{i<j} (\lambda_i - \lambda_j)$ is the absolute value of the Vandermonde determinant and $w(\lambda)$ is a weight function of a system of orthogonal polynomials. In particular, when the joint eigenvalue density is calculated under Gaussian, Wishart and MANOVA random matrices, these weight functions correspond to the Hermite, Laguerre and Jacobi orthogonal polynomials, respectively.

Under generalised multivariate analysis, four classes of matrix variate elliptical distributions have been defined and studied, see Fang and Zhang (1990). A random matrix $X \in \mathcal{L}^\beta_{m,n}$ is said to have a matrix variate left-elliptical distribution, the largest of the four classes of matrix variate elliptical distributions, if its density function is given by

\[
c(m,n) \left| \Sigma^{1/2} \Theta \right|^{m/2} h \left( \Sigma^{-1/2} (X - \mu)^T \Theta^{-1} (X - \mu) \Sigma^{-1/2} \right),
\]

where $h$ is a real function, $c(m,n)$ denotes the normalisation constant which might be a function of another parameter implicit in the function $h$, $\Sigma \in \mathcal{P}^\beta_m$, $\Theta \in \mathcal{P}^\beta_n$ and $\mu \in \mathbb{R}^{n \times m}$. This fact is denoted as

\[
X \sim \mathcal{ELS}_{m \times n} (\mu, \Sigma, \Theta, h).
\]

One might wish to consider situations where $X$ has a density function of the following form, see Fang and Zhang (1990) and Fang and Li (1999),

\[
c(m,n) \left| \Sigma^{1/2} \Theta \right|^{m/2} h \left( \Sigma^{-1} (X - \mu)^T \Theta^{-1} (X - \mu) \right). \tag{18}
\]

This condition is equivalent to considering the function $h$ as a symmetric function, i.e. $g(g(AB)) = g(BA)$ for any symmetric matrices $A$ and $B$. This condition is equivalent to that in which $h(\Lambda)$ depends on $A$ only through its eigenvalues, in which case the function $h(\Lambda)$ can be expressed as $h(\lambda(\Lambda))$, where $\lambda(\Lambda) = \text{diag}(\lambda_1, \ldots, \lambda_m)$ and $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of $\Lambda$. Two subclasses of matrix variate elliptical distributions are of particular interest: matrix variate vector-spherical and spherical elliptical distributions. For these distributions, $\lambda(\Lambda) \equiv \text{tr}(\Lambda)$ and $\lambda(\Lambda)$ represent any function of eigenvalues of $\Lambda$, and are denoted as $X \sim \mathcal{EVS}_{m \times n} (\mu, \Sigma, \Theta, h)$ and $X \sim \mathcal{ESS}_{m \times n} (\mu, \Sigma, \Theta, h)$, respectively. Note that matrix variate vector-spherical elliptical distributions are a subclass of matrix variate

\[
9
\]
spherical elliptical distributions. Many well-known distributions are examples of these sub-
classes; one such is the matrix variate \textit{Hermite distribution}. Other variants include matrix
variates of vector-spherical or spherical distributions, e.g. Pearson type II, Pearson type VII, Kotz
type, Bessel and Logistic, among many others, see \cite{Gupta1993}. Further cases
are those of matrix variate spherical elliptical distributions, e.g. Pearson Type II, Pearson
type VII and Kotz type, among many others, see \cite{Fang1990, Fang1999}.

When in matrix variate vector-spherical and spherical elliptical distributions it is as-
sumed that \( \Sigma = \mathbf{I}_m, \Theta = \mathbf{I}_n \) and \( \mu = \mathbf{0}_{n \times m} \) we obtain the matrix variate vector-spherical
(denoted as \( \mathcal{V} \mathcal{S}_{m \times n}(0, \mathbf{I}_m, \mathbf{I}_n, h) \equiv \mathcal{V} \mathcal{S}_{m \times n}(h) \)) and spherical distributions (denoted as
\( \mathcal{S} \mathcal{S}_{m \times n}(0, \mathbf{I}_m, \mathbf{I}_n, h) \equiv \mathcal{S} \mathcal{S}_{m \times n}(h) \)), which are known to be \textit{invariant distrubutions under orthogonal transforma-
tions}, because \( X \) and \( \mathbf{Q} \mathbf{X} \mathbf{P} \) have the same distribution when \( \mathbf{P} \in \mathbb{U}_1(m) \) and \( \mathbf{Q} \in \mathbb{U}_1(n) \). In particular, note that if \( x \) is a random variable with vector-spherical or
spherical distribution it is denoted as \( x \sim \mathcal{V} \mathcal{S}(0,1,h) \equiv \mathcal{V} \mathcal{S}(h) \) or \( x \sim \mathcal{S} \mathcal{S}(0,1,h) \equiv \mathcal{S} \mathcal{S}(h) \), respectively.

We now define \textit{spherical} and \textit{generalised Laguerre ensembles} based on vector-spherical
and spherical distributions. As can be seen, these ensembles contain as particular cases the
classical Hermite, Laguerre, Jacobi and Fourier ensembles, as well as many others that have
been studied in the literature of random matrix theory, see \cite{Forrester2009}.

\section{Vector-spherical and spherical random matrices}

\( \mathcal{V} \mathcal{S}^\beta(n,m,h) \) and \( \mathcal{S} \mathcal{S}^\beta(n,m,h) \) are \( n \times m \) matrices of non-correlated and identically dis-
tributed (n-c.i.d.) with entries \( \mathcal{V} \mathcal{S}^\beta(0,1,h) \) and \( \mathcal{S} \mathcal{S}^\beta(0,1,h) \), respectively.

1. If \( \mathbf{A} \in \mathcal{L}^\beta_{n,m} \) is an \( n \times m \) vector-spherical random matrix \( \mathcal{V} \mathcal{S}^\beta(n,m,h) \) then its joint
element density is given by

\[
e^\beta(m,n)h(\text{tr}(\mathbf{A}^* \mathbf{A}))
\]

where

\[
e^\beta(m,n) = \frac{\Gamma(\beta mn/2)}{2^{\beta mn/2}} \left\{ \int_{\mathbb{R}_1^m} u^{\beta mn-1} h(u^2) du \right\}^{-1},
\]

2. If \( \mathbf{A} \in \mathcal{L}^\beta_{m,n} \) is an \( n \times m \) spherical random matrix \( \mathcal{S} \mathcal{S}^\beta(n,m,h) \) then its joint element
density is given by

\[
d^\beta(m,n)h(\lambda(\mathbf{A}^* \mathbf{A}))
\]

where \( d^\beta(m,n) \) is given by

\[
\frac{\Gamma^\beta(m/2) \Gamma^\beta(n/2)}{\pi^{(m+n+1)/2}} \left\{ \int_{l_1 \geq \cdots \geq l_m > 0} |L|^{(\beta(n-m+1))/2-1} \Delta(L)^{\beta/2} h(L)(dL) \right\}^{-1},
\]

with \( L = \text{diag}(l_1, \ldots, l_m) \) and \( l_i \) is the \( i \)-th eigenvalue of \( (\mathbf{A}^* \mathbf{A}) \) and \( \tau \) is given in
Lemma \ref{lem:segments} for the real case see \cite{Fang1990}, \cite{Fang1999} and
\cite{Gupta1993}.

Note that \( \mathbf{A} \) and \( \mathbf{Q}\mathbf{A}\mathbf{P} \) have the same distribution when \( \mathbf{P} \in \mathbb{U}_1(m) \) and \( \mathbf{Q} \in \mathbb{U}_1(n) \).
That is, the distributions of \( \mathcal{V} \mathcal{S}^\beta(n,m,h) \) and \( \mathcal{S} \mathcal{S}^\beta(n,m,h) \) are \textit{invariant} under orthogonal
(\( \beta = 1 \)), unitary (\( \beta = 2 \)), symplectic (\( \beta = 4 \)) and exceptional type (\( \beta = 8 \))
transformations.

\textbf{Remark 3.1.} Note that the unique case in which the elements \( a_{ij} \) of the random matrix
\( \mathbf{A} \) are i.i.d. is when \( \mathcal{V} \mathcal{S}^\beta(0,1,h) = \mathcal{S} \mathcal{S}^\beta(0,1,h) = \mathcal{N}^\beta(0,1) \), this is, when \( a_{ij} \) has a standard
normal (Gaussian or of Hermite) distribution, see \cite{Fang1999, Gupta1993} Theorem 2.7.1, p.
72) and \cite{Gupta1993} Theorems 6.2.1 and 6.2.2 p.193).
Some particular vector-spherical random matrices and their corresponding density functions are summarised in the following result, where \( u \equiv \text{tr} A^*A \).

**Corollary 3.1.** Let \( A \in \mathcal{L}_{m,n}^\beta \) with density function (19) then

1. **(random matrix of Hermite)** If \( h(u) = \exp(-\beta u/2) \), it is said that \( A \) is a random matrix of Hermite with density function
   \[
   \frac{1}{(2\pi^{\beta-1})^{3mn/2}} \text{etr}\{-\beta A^*A/2\}.
   \]

2. **(Type I random matrix)** If \( h(u) = (1 + u/\nu)^{-\beta(mn+\nu/2)} \), it is said that \( A \) is a Type I random matrix with density function
   \[
   \frac{\Gamma[\beta(mn+\nu)/2]}{\pi^{\beta mn/2}\Gamma[\beta\nu/2]} (1 + \text{tr} A^*A)^{-\beta(mn+\nu)/2},
   \]
   where \( \nu > 0 \). This distribution is also termed the Pearson type VII matrix variate distribution, see Gupta and Varga [1993] and Diaz-Garcia and Gutierrez [2009].

3. **(Gegenbauer type I random matrix)** If \( h(u) = (1-u)^{-\beta q} \), it is said that \( A \) is a Gegenbauer type I random matrix with density function
   \[
   \frac{\Gamma[\beta mn/2 + \beta q + 1]}{\pi^{\beta mn/2}\Gamma[\beta q + 1]} (1 - \text{tr} A^*A)^{-\beta q},
   \]
   where \( q > -1 \) and \( \text{tr} A^*A \leq 1 \). This distribution is known in statistical bibliography as the inverted Type II or Pearson type II matrix variate distribution, see Gupta and Varga [1993], Press [1982] and Diaz-Garcia and Gutierrez [2009].

For the spherical random matrices we have.

**Corollary 3.2.** Let \( A \in \mathcal{L}_{m,n}^\beta \) with density function (19) then

1. **(Type II random matrix)** If \( h(\lambda(A^*A)) = |I_m + A^*A|^{-\beta(n+\nu)/2} \), it is said that \( A \) is a Type II random matrix with density function
   \[
   \frac{\Gamma_m[\beta(n+\nu)/2]}{(\pi^{\beta mn/2}\Gamma_m[\beta\nu/2]} |I_m + A^*A|^{-\beta(n+\nu)/2},
   \]
   where \( \nu > m \). This distribution is also termed the Type II or Pearson type VII matrix variate distribution, see Dickey [1967], Press [1982], Fang and Li [1999] and Diaz-Garcia and Gutierrez [2009].

2. **(Gegenbauer type II random matrix)** If
   \[
   h(\lambda(A^*A)) = |I - A^*A|^{\beta(\nu-m+1)/2-1},
   \]
   it is said that \( A \) is a Gegenbauer type II random matrix with density function
   \[
   \frac{\Gamma_m[\beta(n+\nu)/2]}{\pi^{\beta mn/2}\Gamma_m[\beta\nu/2]} |I - A^*A|^{\beta(\nu-m+1)/2-1},
   \]
   where \( \nu > (m-1)/2 \) and \( A^*A \in \mathcal{P}_m^\beta \). This distribution is known in statistical bibliography as the inverted Type II or Pearson type II matrix variate distribution, see Press [1982], Fang and Li [1999] and Diaz-Garcia and Gutierrez [2009].
3.2 Construction of the vector-spherical and spherical random matrix ensembles

The vector-spherical and generalised vector-spherical-Laguerre ensembles are constructed from \( V S^\beta(n, m, h) \) as follows.

**Vector-spherical orthogonal ensemble (VSOE):** symmetric \( m \times m \) matrix obtained as \((A + A^T)/2\), where \( A \) is \( V S^1(m, m, h) \). The diagonal entries are n-c.i.d. with distribution \( V S^1(0, 1, h) \), and the off-diagonal entries are n-c.i.d. (subject to the symmetry) with distribution \( V S^1(0, 1/2, h) \).

**Vector-spherical unitary ensemble (VSUE):** Hermitian \( m \times m \) matrix obtained as \((A + A^*)/2\), where \( A \) is \( V S^2(m, m, h) \) and \(*\) denotes the Hermitian transpose of a complex matrix. The diagonal entries are n-c.i.d. with distribution \( V S^1(0, 1, h) \), and the off-diagonal entries are n-c.i.d. (subject to the symmetry) with distribution \( V S^2(0, 1/2, h) \).

**Vector-spherical symplectic ensemble (VSSE):** quaternionic Hermitian (self-dual) \( m \times m \) matrix obtained as \((A + A^*)/2\), where \( A \) is \( V S^4(m, m, h) \) and \(*\) denotes the quaternionic Hermitian (or dual) transpose of a quaternion matrix. The diagonal entries are n-c.i.d. with distribution \( V S^1(0, 1, h) \), and the off-diagonal entries are n-c.i.d. (subject to the symmetry) with distribution \( V S^4(0, 1/2, h) \).

**Vector-spherical exceptional type ensemble (VSETE):** octonionic Hermitian \( m \times m \) matrix obtained as \((A + A^*)/2\), where \( A \) is \( V S^8(m, m, h) \) and \(*\) denotes the octonionic Hermitian transpose of a octonionic matrix. The diagonal entries are n-c.i.d. with distribution \( V S^1(0, 1, h) \), and the off-diagonal entries are n-c.i.d. (subject to the symmetry) with distribution \( V S^8(0, 1/2, h) \).

Analogously, by replacing \( VS \) and \( LS \) by \( SS \) and \( SS \), respectively, in the four previous definitions we obtain the **Spherical orthogonal ensemble (SOE), Spherical unitary ensemble (SU), Spherical symplectic ensemble (SSSE), and Spherical exceptional type ensemble (SSETE).**

Similarly, generalised vector-spherical-Laguerre and spherical-Laguerre ensembles can be defined as follows.

**Generalised vector-spherical-Laguerre \((V S L^\beta(n, m, h), n \geq m)\):** Symmetric/Hermitian/ quaternionic Hermitian/ octonionic Hermitian \( m \times m \) matrix which can be obtained as \( A^* A \), where \( A \) is \( V S^\beta(n, m, h) \).

Again, by replacing \( V S^\beta \) by \( SS^\beta \) in the previous definition we obtain the **Generalised spherical-Laguerre \((S L^\beta(n, m, h), n \geq m)\).**

As shown later, the Jacobi ensemble is a particular case of \( V S L^\beta(n, m, h) \) or \( S L^\beta(n, m, h) \) and the Fourier ensemble is a particular case of \( S S^\beta(m, m, h) \).

3.3 Computing the joint element densities

Let \( A \) be an \( m \times m \) matrix from the vector-spherical ensemble, that is

\[
a_{ij} \sim \begin{cases} 
V S^\beta(0, 1, h), & i = j, \\
V S^\beta(0, 1/2, h), & i > j.
\end{cases}
\]  

Then, it is straightforward to see that its joint element density is

\[
2^{m(m-1)\beta/4} c^\beta(m, m) h \left( \text{tr} A^2 \right),
\]

where

\[
c^\beta(m, m) = \frac{\Gamma((m + m(m - 1)\beta/2)/2)}{2\pi^{m+m(m-1)\beta/2}/2} \left\{ \int_{\mathbb{R}_+^2} u^{m+m(m-1)\beta/2-1} h(u) du \right\}^{-1}.
\]
Remark 3.2. In the general case, from (18) we have $|\Sigma|^{\beta n/2}|\Theta|^{\beta m/2} = |\Theta \otimes \Sigma|^{\beta/2}$. Under (19), $\Theta \otimes \Sigma$ is a diagonal matrix with $m$-times ones and $m(m-1)/2\times 1/2$, considering that $A$ is symmetric, from which $|\Theta \otimes \Sigma|^{\beta/2} = (1/2)^{m(m-1)/4}$.

Similarly, let $A$ be an $m \times m$ matrix from the spherical ensemble, that is

$$a_{ij} \sim \begin{cases} SS^\beta(0,1,h), & i = j, \\ SS^\beta(0,1/2,h), & i > j. \end{cases}$$

And then it follows that its joint element density is

$$2^{m(m-1)/2}d^\beta(m,m)h(\lambda(A^2)), \quad (25)$$

where

$$d^\beta(m,m) = \frac{\Gamma_m^\beta[\beta m/2]}{2^{m(m-1)/4}\pi^m\beta^{m/2}} \left\{ \int \cdots \int |\Delta(L)|^{\beta/2}h(L^2)(dL) \right\}^{-1}, \quad (26)$$

with $L = \text{diag}(l_1, \ldots, l_m)$, where $l_i$ is the $i$th eigenvalue of $A$ and $\tau$ is as given in Lemma 2.1.

Consider the generalised vector-spherical-Laguerre ensemble $VL(n,m,h) = S = A^*A$, where $A = VS(n,m,h)$. Its joint element density can be computed in diverse ways. For example, following the approach of Herz (1965), let $A = V_1R$ be the polar decomposition of $A$, then $S = A^*A = R^2$, from Lemma 17 and (19), the joint density of $S$ and $V_1$ is

$$c^\beta(m,n)h(\text{tr}(S))^{2-\nu}S^{\beta(n-\nu+1)/2-1}(dS)(V_1^*dV_1).$$

On integrating with respect to $V_1$ using (2) the marginal density of $S$ is found to be

$$c^\beta(m,n)\pi^{\beta mn/2} \frac{\Gamma_m^\beta[\beta n/2]}{\Gamma_m^\beta[\beta n/2]} S^{\beta(n-\nu+1)/2-1}h(\text{tr}(S)). \quad (27)$$

Note that the same result can be obtained from Lemma 17 and taking into account the SV, QR or MQR decomposition instated of the polar decomposition in the previous procedure.

As in the generalised vector-spherical-Laguerre ensemble case, it is obtained that the joint element density of the generalised spherical-Laguerre ensemble is

$$\frac{d^\beta(m,n)\pi^{\beta mn/2}}{\Gamma_m^\beta[\beta n/2]} |S|^{\beta(n-\nu+1)/2-1}h(\lambda(S)). \quad (28)$$

As was done for Corollary 3.1, some particular vector-spherical random matrix ensembles and their corresponding density functions are summarised in the following result.

Corollary 3.3. Let $A \in \mathcal{P}_m^\beta$ with density function (23) then

1. (Classical Hermite ensemble) If $h(u) = \exp(-\beta u/2)$, it is said that $A$ is a Hermite ensemble with density function

$$\frac{1}{2^{m/2}(\pi^{-1})^{m/2}m(m-1)/4} \text{etr}\{-\beta A^2/2\}.$$

2. (T type I ensemble) If $h(u) = (1 + u/\nu)^{-m^2-\beta(m(m-1)/2+\nu)/2}$, it is said that $A$ is a $T$ type I ensemble with density function

$$\frac{2^{m(m-1)/4}\Gamma(m(m-1)/2+\nu)/2}{\pi^{m^2/2}m(m-1)/4\Gamma(\beta/2)}(1 + \text{tr} A^2)^{-m^2-\beta(m(m-1)/2+\nu)/2},$$

where $\nu > 0$. This ensemble could be termed the Pearson type VII matrix variate ensemble.
3. **(Gegenbauer type I ensemble)** If \( h(u) = (1 - u)^{-\beta q} \), it is said that \( A \) is a Gegenbauer type I ensemble with density function

\[
\frac{2^{m(m-1)\beta/4} \Gamma[m/2 + m(m-1)\beta/4 + \beta q + 1]}{\pi^{m/2 + m(m-1)\beta/4} \Gamma[\beta q + 1]} (1 - \text{tr} A^2)^{-\beta q},
\]

where \( q > -1 \) and \( \text{tr} A^2 \leq 1 \). In this case, too, the ensemble might be termed the inverted T or Pearson type II matrix variate ensemble.

For the spherical random matrix ensembles we have the following.

**Corollary 3.4.** Let \( A \in \Psi_m^\beta \) with density function (23) then

1. **(T type II ensemble)** If \( h(\lambda(A^2)) = |I_m + A^2|^{-\beta(n+\nu)/2} \), it is said that \( A \) is a T type II ensemble with density function

\[
2^{m(m-1)\beta/4} c^\beta(m, m) |I_m + A^2|^{-\beta(n+\nu)/2},
\]

where \( \nu > m \). This ensemble could possibly be termed the Pearson type VII matrix variate ensemble.

2. **(Gegenbauer type II ensemble)** If \( h(\lambda(A^2)) = |I_m - A^2|^{\beta(\nu-(m+1)/2-1)} \), it is said that \( A \) is a Gegenbauer type II ensemble with density function

\[
2^{m(m-1)\beta/4} c^\beta(m, m) |I_m - A^2|^{\beta(\nu-(m+1)/2-1)},
\]

where \( \nu \geq (m-1)\beta \) and \( A^* A \in \Psi_m^\beta \). This ensemble could be termed an inverted T or Pearson type II matrix variate ensemble.

Analogously, some particular cases for vector-spherical and spherical-Laguerre random matrix ensembles are obtained in the following two results.

**Corollary 3.5.** Let \( S \in \Psi_m^\beta \) with density function (27) then

1. **(Classical Laguerre ensemble)** If \( h(u) = \exp(-\beta u/2) \), it is said that \( S \) is a Laguerre ensemble with density function

\[
\frac{1}{(2\beta-1)^{\beta mn/2} \Gamma[m\beta mn/2]} |S|^{\beta(n-m+1)/2-1} \text{etr}\{-\beta S/2\}
\]

where \( n \geq (m-1)\beta \).

2. **(T-Laguerre type I ensemble)** If \( h(u) = (1 + u/\nu)^{-\beta(n+\nu)/2} \), it is said that \( S \) is a T-Laguerre type I ensemble with density function

\[
\frac{\Gamma[\beta mn + \nu/2]}{\Gamma[\beta\nu/2] \Gamma[m\beta n/2]} |S|^{\beta(n-m+1)/2-1} (1 + \text{tr} S)^{-\beta(n+\nu)/2},
\]

where \( \nu > 0 \) and \( n \geq (m-1)\beta \).

3. **(Gegenbauer-Laguerre type I ensemble)** If \( h(u) = (1 - u)^{-\beta q} \), it is said that \( S \) is a Gegenbauer-Laguerre type I ensemble with density function

\[
\frac{\Gamma[\beta mn + 2 + \beta q + 1]}{\Gamma[\beta q + 1] \Gamma[m\beta n/2]} |S|^{\beta(n-m+1)/2-1} (1 - \text{tr} S)^{\beta q},
\]

where \( q > -1 \), \( n > (m-1)\beta/2 \) and \( \text{tr} S \leq 1 \).
Finally, note that

Corollary 3.6. Let \( S \in \mathcal{W}_m^\beta \) with density function (28) then

1. (T-Laguerre type II ensemble) If \( h(\lambda(S)) = |I_m + S|^{-\beta(n+\nu)/2} \), it is said that \( S \) is a T-Laguerre type II ensemble with density function

\[
\frac{1}{\mathcal{B}_m^\beta[\beta m/2, \beta n/2]} |S|^{\beta(m-n+1)/2-1} |I_m + S|^{-\beta(n+\nu)/2},
\]

where \( \nu \geq (m-1)\beta \) and \( n \geq (m-1)\beta \). This distribution is also known as the Studentised Wishart distribution, see [Olkin and Rubin (1964)].

2. (Gegenbauer-Laguerre type II ensemble) If \( h(\lambda(S)) = |I_m - S|^{\beta(n-m+1)/2-1} \), it is said that \( S \) is a Gegenbauer-Laguerre type II ensemble with density function

\[
\frac{1}{\mathcal{B}_m^\beta[\beta n/2, \beta \nu/2]} |S|^{\beta(n-m+1)/2-1} |I_m - S|^{\beta(n-m+1)/2-1},
\]

where \( \nu \geq (m-1)\beta \) and \( n \geq (m-1)\beta \).

Finally, note that \( \mathcal{B}_m^\beta[a, b] \), defined as

\[
\mathcal{B}_m^\beta[a, b] = \frac{\Gamma_m^\beta[a] \Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b]},
\]

is the multivariate beta function, where \( \text{Re}(a) > (m-1)\beta/2 \) and \( \text{Re}(b) > (m-1)\beta/2 \), see [Herz (1955)].

Remark 3.3. 1. Note that, from [11], \( H_1 \in \mathcal{V}_{m,n}^\beta \) is a spherical random matrix. Moreover, (see Fang and Zhang (1990, Lemma 3.1.3(iii), p. 94)), the differential form of its density function is

\[
\frac{1}{\text{Vol}(\mathcal{V}_{m,n}^\beta)} (H_1^dH_1) = \frac{\Gamma_m^\beta[\beta n/2]}{2^{m\beta m/2}(\Gamma[\beta/2])^m} (H_1^dH_1).
\]

In the context of random matrix theory, \( H_1 \) is a Fourier random matrix. Now, let \( \mathcal{U}_S^\beta(m) \) be the group of orthogonal Hermitian matrices \( H \). Then \( H \) is a Fourier ensemble with a differential form of its density function

\[
\frac{1}{\text{Vol}(\mathcal{U}_S^\beta(m))} (H^dH) = \frac{\Gamma_m^\beta[\beta m/2] \Gamma[\beta m/2 + 1]}{2^{m\beta m^2/2+1} (\Gamma[\beta/2 + 1])^m} (H^dH).
\]

The value of \( \text{Vol}(\mathcal{U}_S^\beta(m)) \) is found in next subsection.

2. Some \( g(A) \) functions, where \( A \) is a spherical random matrix, have invariant distributions under the corresponding class of spherical distribution, under certain conditions; in other words, \( Y = g(A) \) has the same distribution for each particular spherical distribution. An analogous situation is true for the class of vector-spherical distribution, see [Fang and Zhang (1994, Section 5.1, pp. 154-156) and Gupta and Varga (1993, Section 5.1, pp. 182-189)]. In particular note the following: let \( A \in \mathcal{C}_{m,n}^\beta \), be an \( n \times m \) spherical random matrix \( SS^\beta(n, m, h) \), such that \( A = (A_1^T | A_2^T)^T \) with \( A_i \in \mathcal{C}_{m,n_i}^\beta \), \( n_i \geq m \ i = 1, 2 \) and \( n = n_1 + n_2 \). Then
(a) the $T$ type II random matrix defined as

$$T = A_1(A_2^*A_1 + A_2^*A_2)^{-1/2},$$

(b) the Gegenbauer type II random matrix defined as

$$R = A_1(A_1^*A_1 + A_2^*A_2)^{-1/2},$$

(c) the classical Jacobi random matrix defined as

$$B = (A_1^*A_1 + A_2^*A_2)^{-1/2}(A_1^*A_1)(A_1^*A_1 + A_2^*A_2)^{-1/2},$$

(d) and the $F$ random matrix defined as

$$F = (A_2^*A_2)^{-1/2}(A_1^*A_1)(A_1^*A_1 + A_2^*A_2)^{-1/2},$$

are invariant under the class of spherical distributions, see Fang and Zhang (1990, Section 3.5, pp. 110-116). Analogous results are concluded for the class of vectorspherical distributions, see Gupta and Varga (1993, Theorem 5.3.1, p.182). Moreover, the classical Jacobi and $F$ random matrices have the same joint element distribution if $A$ is a vector-spherical or spherical random matrix, see Fang and Zhang (1990, Theorems 3.5.1 and 3.5.5) and Gupta and Varga (1993, Theorem 5.3.1).

3. Observe that the Gegenbauer-Laguerre type II ensemble is indeed the classical Jacobi random matrix ensemble. Also note that, if $F$ is an $F$ random matrix, then $B = I_m - (I_m + F)^{-1}$, and if $B$ is a classical Jacobi random matrix ensemble then $F = (I_m - B)^{-1} - I_m$. Therefore $F$ can be termed a modified Jacobi random matrix ensemble. Similarly, the $T$-Laguerre type II random matrix ensemble is indeed the modified Jacobi random matrix.

4. There remain two final remarks:

(a) As Edelman and Sutton (2008) observed, the classical Jacobi random matrix can be obtained from the Fourier random matrix. Moreover, this procedure can be applied to any function $g(A)$, invariant under the class of spherical distributions. Form the statistician’s point of view, this approach has been used by Khatri (1970) and Cadet (1996), in the real case. The latter studied the $T$, inverted $T$, $F$ and beta distributions.

(b) Finally, note that the matrix factorisation associated with classical and modified Jacobi ensembles could be the singular value decompositions, but applied to Gegenbauer and $T$ random matrices, respectively, see Edelman and Rao (2005).

3.4 Joint eigenvalue densities

**Theorem 3.1.** Let $A$ be an $m \times m$ matrix from the vector-spherical ensemble. Then its joint eigenvalue density function is

$$f^\beta_\Lambda(A) = \frac{e^{\beta(m,n)}2^{m(m-1)/2}\pi^{m/2}}{\Gamma_m[\beta m/2]} \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta \frac{1}{h} \left( \sum_{i=1}^m \lambda_i^2 \right),$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$. 

16
Proof. From (23) and Lemma 2.14, the joint density of $W \in \mathcal{U}_\beta(m)$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$, with $A = \Lambda W W^*$ is

$$2^{m(m-1)/4} c^\beta(m,m) h(\text{tr} \Lambda^2) 2^{-m} \pi^\beta \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta (d\Lambda)(W^*dW).$$

The marginal density desired is found by integrating over $W \in \mathcal{U}_\beta(m)$ using (2).

Similarly,

Theorem 3.2. Let $A$ be an $m \times m$ matrix from the spherical ensemble. Then its joint eigenvalue density function is

$$f_\beta^A(\Lambda) = \frac{c^\beta(m,n) 2^{m(m-1)/2} \pi^\beta m^2/2 + \tau}{\Gamma_m[\beta m/2]^{\beta m/2}} \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta \times h(\text{diag}(\lambda_1^2, \ldots, \lambda_m^2))$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$.

Proof. The proof is analogous to that given for (30); note however that $h(\lambda(A^2)) = h(\lambda((\Lambda W W^*)^2)) = h(\lambda(\Lambda^2)) = h(\text{diag}(\lambda_1^2, \ldots, \lambda_m^2))$.

Theorem 3.3. Let $S$ be an $m \times m$ matrix from the vector-spherical-Laguerre ensemble. Then its joint eigenvalue density function is

$$f_\beta^S(\Lambda) = \frac{c^\beta(m,n) \pi^\beta m^2/2 + \tau}{\Gamma_m[\beta n/2]^{\beta n/2}} \prod_{i=1}^m \lambda_i^\beta (n-m+1)/2 - 1 \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta \times h\left(\sum_{i=1}^m \lambda_i\right),$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$.

Proof. Let $S = \Lambda W W^*$, from (23) and Lemma 2.14, then the joint density of $W \in \mathcal{U}_\beta(m)$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$ is

$$\frac{c^\beta(m,n) \pi^\beta m^2/2}{\Gamma_m[\beta n/2]^{\beta m/2}} |S|^{\beta(n-m+1)/2 - 1} h(\text{tr}(S)) 2^{-m} \pi^\beta \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta (d\Lambda)(W^*dW).$$

The marginal density desired is found by integrating over $W \in \mathcal{U}_\beta(m)$ using (2).

Similarly,

Theorem 3.4. Let $S$ be an $m \times m$ matrix from the spherical-Laguerre ensemble. Then its joint eigenvalue density function is

$$f_\beta^S(\Lambda) = \frac{c^\beta(m,n) \pi^\beta m^2/2 + \tau}{\Gamma_m[\beta n/2]^{\beta m/2}} \prod_{i=1}^m \lambda_i^\beta (n-m+1)/2 - 1 \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta \times h(\text{diag}(\lambda_1, \ldots, \lambda_m)),$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$. 
Given the differential form of the joint element density function of the Fourier ensemble, the corresponding joint eigenvalue density function cannot be obtained by applying the Theorem 3.2. This density function has been obtained by Metha (1991, Lemma 10.4.4, p. 198) and Forrester (2009, Proposition 2.3, p. 61). Based on the wedge product, this is now obtained and $\text{Vol} \left( \mathcal{P}_S(\beta) \right)$ is calculated indirectly, see (29).

Let $H \in \mathcal{P}_S(\beta)$, then there exist $U \in \mathcal{P}_S(\beta)$ such that $H = U E U^*$, where $E = \text{diag}(\exp(i \theta_1), \ldots, \exp(i \theta_m))$, $\exp(i \theta_i)$ are $m$ complex numbers on the unit circle, see Metha (1991) and Forrester (2009). Then the joint density function of $\theta_1, \ldots, \theta_m$ is

$$c^\beta(m) \prod_{i \leq j} |\exp(i \theta_i) - \exp(i \theta_j)|^\beta, \quad \theta_j \in (-\pi, \pi).$$

From (29)

$$dF_H(H) = \frac{1}{\text{Vol} \left( \mathcal{P}_S(\beta) \right)} (\text{HdH}).$$

Let $H = U E U^*$, then by Forrester (2009, Exercise 2 (iii), p. 63), recalling that $H = H^*$

$$(\text{HdH}) = \prod_{i \leq j} |\exp(i \theta_i) - \exp(i \theta_j)|^\beta \left( \bigwedge_{l=1}^m \theta_l \right) (U^* dU).$$

Note that it is necessary to divide the measure by $2^{-m} \pi^\tau$ to normalise the arbitrary phases of the $m$ elements in the first row of $U$, where $\tau$ is given in Lemma 2.10. Then the joint density of $\text{diag}(\theta_1, \ldots, \theta_m)$ and $U$ is

$$\frac{2^{-m} \pi^\tau}{\text{Vol} \left( \mathcal{P}_S(\beta) \right)} \prod_{i \leq j} |\exp(i \theta_i) - \exp(i \theta_j)|^\beta \left( \bigwedge_{l=1}^m \theta_l \right) (U^* dU).$$

By integrating with respect to $U$ we obtain the marginal of $\theta_1, \ldots, \theta_m$

$$\frac{2^{-m} \pi^\tau \text{Vol} \left( \mathcal{P}_S(\beta) \right)}{\text{Vol} \left( \mathcal{P}_S(\beta) \right)} \prod_{i \leq j} |\exp(i \theta_i) - \exp(i \theta_j)|^\beta \left( \bigwedge_{l=1}^m \theta_l \right).$$

(34)

From the Morris integral Forrester (2009, Equation (3.4), p. 122)

$$M_m[0,0,\beta/2] = (2\pi)^{-m} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{i \leq j} |\exp(i \theta_i) - \exp(i \theta_j)|^\beta \bigwedge_{l=1}^m \theta_l$$

$$= \frac{(2\pi)^{-m} \Gamma[\beta m/2 + 1]}{(\Gamma[\beta/2 + 1])^m},$$

then

$$c^\beta(m) = \frac{\Gamma[\beta m/2 + 1]}{(\Gamma[\beta/2 + 1])^m}.$$
particular, that its joint normalised eigenvalue density function is invariant under all classes of vector-functions and from Theorems 3.1, 3.2, 3.3 and 3.4. The corresponding normalisation constants are obtained from their joint element density in particular ensembles studied in the subsection 3.3, using the names proposed in Remark 3.3.

This problem has been studied in the context of shape theory and the corresponding density function was obtained by Díaz-Garcia et al. (2003).

Table 2 summarises the kernels of the joint eigenvalue density function of all the particular ensembles studied in the subsection 3.3, using the names proposed in Remark 3.3. The corresponding normalisation constants are obtained from their joint element density functions and from Theorems 3.1, 3.2, 3.3 and 3.4.

Finally, another interesting property of the vector-spherical random matrix ensemble is that its joint normalised eigenvalue density function is invariant under all classes of vector-spherical distribution, where the normalised eigenvalue can be defined as \( \delta_i = \lambda_i / r \). In particular, \( r \) can be defined as

\[
r = \left( \sum_{i=1}^{m} \lambda_i^2 \right)^{1/2}.
\]

| Ensemble                           | Kernel                                                                 |
|------------------------------------|------------------------------------------------------------------------|
| Hermite \( \lambda_i \in (-\infty, \infty) \) | \( \exp \left\{ \frac{\beta}{2} \sum_{i=1}^{m} \lambda_i^2 \right\} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} \) |
| T type I \( \lambda_i \in (-\infty, \infty) \) | \( \prod_{i=1}^{m} \left( 1 + \sum_{i=1}^{m} \lambda_i^2 \right)^{-\beta/2} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} \) |
| Gegenbauer type I \( \lambda_i \in (-1, 1) \) | \( \prod_{i=1}^{m} \left( 1 - \sum_{i=1}^{m} \lambda_i^2 \right)^{\beta} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} \) |
| T type II \( \lambda_i \in (-\infty, \infty) \) | \( \prod_{i=1}^{m} \left( 1 - \lambda_i^2 \right)^{-\beta(n+\nu)/2} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} \) |
| Gegenbauer type II \( \lambda_i \in (-1, 1) \) | \( \prod_{i=1}^{m} \lambda_i^{\beta(n-m+1)/2-1} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} \) |
| Laguerre \( \lambda_i \in (0, \infty) \) | \( \prod_{i=1}^{m} \lambda_i^{\beta(n-m+1)/2-1} \exp \left\{ \frac{\beta}{2} \sum_{i=1}^{m} \lambda_i \right\} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} \) |
| Modified Jacobi type I \( \lambda_i \in (0, \infty) \) | \( \prod_{i=1}^{m} \lambda_i^{\beta(n-m+1)/2-1} \left( 1 + \sum_{i=1}^{m} \lambda_i \right)^{-\beta(n+\nu)/2} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} \) |
| Jacobi type I \( \lambda_i \in (0, 1) \) | \( \prod_{i=1}^{m} \lambda_i^{\beta(n-m+1)/2-1} \left( 1 - \sum_{i=1}^{m} \lambda_i \right)^{\beta(n-m+1)/2-1} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} \) |
| Modified Jacobi \( \lambda_i \in (0, \infty) \) | \( \prod_{i=1}^{m} \lambda_i^{\beta(n-m+1)/2-1} \left( 1 + \lambda_i \right)^{-\beta(n+\nu)/2} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} \) |
| Jacobi \( \lambda_i \in (0, 1) \) | \( \prod_{i=1}^{m} \lambda_i^{\beta(n-m+1)/2-1} (1 - \lambda_i)^{\beta(n-m+1)/2-1} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} \) |
| Fourier \( \lambda_i \in (-\pi, \pi) \) | \( \prod_{i<j} \left| \exp(i\theta_i) - \exp(i\theta_j) \right|^\beta \) |

Table 2: Kernels of the joint eigenvalue density function of all the particular ensembles.
4 Multivariate statistics

Many areas of multivariate statistics (and of statistics in general), such as multiple time series or econometrics, can be enriched with the use of the tools and ideas of random matrix theory, see Hannan (1970). However, in this section these tools and ideas are used to develop a unified theory of multivariate distributions for normed division algebras.

**Theorem 4.1.** Let \( Z \in \mathcal{L}_{m,n}^\beta \) be a matrix variate left elliptical distribution with density function

\[
c^\beta(m,n) h(Z^*Z).
\]

Therefore if \( X = AZB + \mu \), with \( A \in \mathcal{L}_{n,n}^\beta \), \( B \in \mathcal{L}_{m,m}^\beta \) and \( \mu \in \mathcal{L}_{m,n}^\beta \), constant matrices such that \( A^*A = \Theta \in P_{\beta n} \) and \( B^*B = \Sigma = (\Sigma^{1/2})^2 \in P_{\beta m} \), then

\[
\frac{c^\beta(m,n)}{|\Sigma|^{\beta n/2}|\Theta|^{\beta m/2}} h\left(\Sigma^{-1/2}(X - \mu)^*\Theta^{-1}(X - \mu)\Sigma^{-1/2}\right).
\]

where \( c^\beta(m,n) \) is a normalisation constant.

**Proof.** The proof is analogous to that given in the real case, but considering the Jacobian of the transformation \( Y = AXB + \mu \) defined by Lemma 2.1.

**Corollary 4.1.** Under the hypothesis of Theorem 4.1,

1. for the matrix variate spherical elliptical distribution, its density function is

\[
\frac{c^\beta(m,n)}{|\Sigma|^{\beta n/2}|\Theta|^{\beta m/2}} h\left(\lambda\left(\Sigma^{-1}(X - \mu)^*\Theta^{-1}(X - \mu)\right)\right),
\]

this fact being denoted as \( X \sim \text{SSE}_{n\times m}^\beta(\mu, \Sigma, \Theta, h) \).

2. for the matrix variate vector-spherical elliptical distribution, its density function is

\[
\frac{c^\beta(m,n)}{|\Sigma|^{\beta n/2}|\Theta|^{\beta m/2}} h\left(\text{tr}\left(\Sigma^{-1}(X - \mu)^*\Theta^{-1}(X - \mu)\right)\right),
\]

this fact being denoted as \( X \sim \text{VSE}_{n\times m}^\beta(\mu, \Sigma, \Theta, h) \).

3. and for the matrix variate normal distribution, its density function is (taking \( h(u) = \exp(-\beta u/2) \))

\[
\frac{1}{(2\pi\beta^{-1})^{\beta mn/2}|\Sigma|^{\beta n/2}|\Theta|^{\beta m/2}} \text{etr}\left\{-\frac{\beta}{2}\Sigma^{-1}(X - \mu)^*\Theta^{-1}(X - \mu)\right\},
\]

denoting this fact as \( X \sim \text{N}_{n\times m}^\beta(\mu, \Sigma, \Theta) \).

Many other particular vector-spherical or spherical elliptical distributions can be obtained by simply specifying the function \( h(\cdot) \) in a similar way to Corollaries 3.3 and 3.4.

Elliptical and, in particular, normal symmetric random matrices have received less attention in multivariate statistics, but analogous results can be obtained in a similar way using Lemma 2.2.

**Theorem 4.2.** Let us define \( S = X^*\Theta^{-1}X \in \mathcal{P}_m^\beta \), then
Proof. The desired results are obtained from Theorem 4.2 directly, after defining $S$ for the hypothesis of Theorem 4.2 it follows that, Corollary 4.2. Let $X \sim \text{SSE}_m(0, \Sigma, \Theta, h)$, the density function of $S$ is 

$$d^\beta(m,n)\pi^{\beta mn/2}\frac{|S|^{\beta(n-m+1)/2-1}h(\lambda(S^{-1}S))}{\Gamma_m[\beta n/2]\Sigma[\beta n/2]},$$

This distribution is known as the spherical-generalised-Wishart distribution and it is denoted as $S \sim \text{SGW}_m^\beta(n, \Sigma, h)$.

2. if $X \sim \text{VSE}_m(0, \Sigma, \Theta, h)$, the density function of $S$ is 

$$d^\beta(m,n)\pi^{\beta mn/2}\frac{|S|^{\beta(n-m+1)/2-1}h(\lambda(S^{-1}S))}{\Gamma_m[\beta n/2]\Sigma[\beta n/2]},$$

This distribution is known as the vector-spherical-generalised-Wishart distribution and it is denoted as $S \sim \text{VSGW}_m^\beta(n, \Sigma, h)$.

Proof. Let $S = X^*\Theta^{-1}X$, and note that 

$$Y = \Theta^{-1/2}X \sim \text{SSE}_{n,m}(0, \Sigma, I_n, h)$$

with $(\Theta^{-1/2})^2 = \Theta$. Therefore $S = X^*\Theta^{-1}X = Y^*Y$. The desired results are follow from [28] and [27], respectively. □

Corollary 4.2. Under the hypothesis of Theorem 4.2 it follows that,

1. the Wishart random matrix has a density function

$$\frac{1}{(2\pi)^{\beta m n/2}}\frac{|S|^{\beta(n-m+1)/2-1}e^{\text{tr}\{-\beta S^{-1}S/2\}}}{\Gamma_m[\beta n/2]\Sigma[\beta n/2]},$$

where $n \geq (m-1)\beta$ and this is denoted as $S \sim \mathcal{W}_m^\beta(n, \Sigma)$;

2. the matrix variate beta type I distribution has a density function

$$\frac{1}{B_m[\beta n/2, \beta n/2]}\frac{|S|^{\beta(n-m+1)/2-1}|I_m - S|^{\beta(n-m+1)/2-1}},$$

where $\nu \geq (m-1)\beta$ and $n \geq (m-1)\beta$ and this is denoted as $S \sim \mathcal{B}_m^\beta(n, \nu)$;

3. and the matrix variate beta type II distribution has a density function

$$\frac{1}{B_m[\beta n/2, \beta n/2]}\frac{|S|^{\beta(n-m+1)/2-1}|I_m + S|^{-\beta(n+\nu)/2}},$$

where $\nu \geq (m-1)\beta$ and $n \geq (m-1)\beta$ and this is denoted as $S \sim \mathcal{BII}_m^\beta(n, \nu)$.

Proof. The desired results are obtained from Theorem 4.2 directly, after defining $h(\text{tr}(\Sigma^{-1}S)) = e^{\text{tr}\{-\beta S^{-1}S/2\}}$, $h(\lambda(S^{-1}S)) = |I_m - S|^{\beta(n-m+1)/2-1} (\Sigma = I_m)$ and $h(\lambda(S^{-1}S)) = |I_m + S|^{-\beta(n+\nu)/2} (\Sigma = I_m)$, respectively. □
Conclusions

Although most results about Jacobians are known in the context of random matrix theory, in general for statisticians they are less familiar, as are the corresponding technical tools in the context of normed division algebras. Thus the importance of addressing this area of study.

In the field of random matrix theory, various classes of ensembles are proposed, including many of those studied previously, see [Forrester (2009, Section 4.1.4, p. 177)]. What is most important is that these new classes of ensembles contain many other ensembles of potential interest, which may enable us to study phenomena and experiments under more general conditions.

Analogously to current random matrix theory, see [Edelman and Rao (2005)], in the present study we propose a unified theory of matrix variate distribution for normed division algebra, that is, for real, complex, quaternion, and octonion cases.

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