ON THE RATIONAL BREDON COHOMOLOGY OF
EQUIVARIANT CONFIGURATION SPACES

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Abstract. Bredon cohomology is a cohomology theory that applies to
topological spaces equipped with the group actions. For any group $G$,
given a real linear representation $V$, the configuration space of $V$ has
a natural diagonal $G$-action. In the paper we study this group action
on the configuration space and give a decomposition of the homology
Bredon coefficient system of the configuration space and apply this to
compute Bredon cohomology of the configuration space for small non-
abelian group $G$.

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1. Introduction

The objects of study in equivariant algebraic topology are spaces equipped
with an action by a topological group $G$. The category of Bredon coeffi-
cient systems over a commutative ring $R$, denote by $C^R_G$, is the category of
contravariant functors from the canonical orbit category of $G$ to the cate-
gory of $R$-modules. It is an Abelian category with enough injectives. Using
this category $G^R$. Bredon defined a homology and a cohomology theory for $G$-spaces [8].

Through this paper we will assume that $G$ is a finite group. Given a $G$-space $X$, in order to compute Bredon cohomology, we need to access the homology of the fixed points sets $X^H$ for all subgroups $H$ of $G$ as well as an injective resolution for an arbitrary Bredon coefficient system.

For an arbitrary $G$-space $X$, it might be a too difficult task to give a closed formula for the fixed point sets of all subgroups $H < G$. However, for particular types of spaces, we can obtain such formulas, which makes computing Bredon cohomology possible. In [24], the author computed the fixed point sets for polyhedral products. In this paper, the objects to study is equivariant configuration spaces.

**Definition 1.1.** Let $G$ be a finite group. For an $R$-linear representation $V$, the equivariant configuration space of $V$ is the space $\text{Conf}(V,q)$ with the diagonal $G$-action.

We prove the following fixed point theorem in Section 7.

**Theorem 7.1.** For any subgroup $H < G$, the $H$-fixed point $\text{Conf}(V,q)^H$ is $\text{Conf}(V^K, q)$.

Define the Weyl group $WH$ for any subgroup $H$ of $G$ by

$$WH = N_G(H)/H.$$  

**Definition 1.2.** For any subgroup $H < G$, let $V_H$ be a left $\mathbb{Q}(WH)$-module. Define a Bredon coefficient system $I(V_H)$ by

$$I(V_H)(G/K) = \text{Hom}_{\mathbb{Q}(WH)}(\mathbb{Q}((G/K)^H), V_H).$$

For a $G$-map $f : G/K \rightarrow G/K'$, the map $\overline{f} : \mathbb{Q}((G/K)^H) \rightarrow \mathbb{Q}((G/K')^H)$ is induced by $f$. Then

$$I(V_H)(f) : I(V_H)(G/K') \rightarrow I(V_H)(G/K)$$

is defined by $I(V_H)(f)(g) = g \circ \overline{f}$ where $g \in I(V_H)(G/K')$.

In [15], Doman proved Bredon coefficient systems in the form $I(V_H)$ are injective. In addition, he provide an injective envelope for any Bredon coefficient system over $\mathbb{Q}$ using those injective coefficient systems.

**Theorem 1.1** ([15]). $f : M \rightarrow \oplus I(V_H)$ is a injective envelope of $M$, where direct sum is over all conjugacy classes of $G$. 
One of the tools for computing Bredon cohomology is the universal coefficient spectral sequences.

**Theorem 1.2** ([8], [20]). There is a universal coefficient spectral sequence that converges to Bredon cohomology

\[ E_2^{p,q} = \text{Ext}^{p,q}_{CG}(H_\ast(X), M) \Rightarrow H_{G}^{p+q}(X, M), \]

and a universal coefficient spectral sequence that converges to Bredon homology

\[ E_2^{p,q} = \text{Tor}^{p,q}_{CG}(H_\ast(X), N) \Rightarrow H_{G}^{p+q}(X, M). \]

For equivariant spaces, the homology Bredon coefficient system has a decomposition.

**Definition 1.3.** The Bredon coefficient system \( \underline{1}_H \) over \( \mathbb{Q} \) is given by

\[ \underline{1}_H(G/K) = \begin{cases} \mathbb{Q}, & \text{if } K \text{ conjugates to } H, \\ 0, & \text{otherwise.} \end{cases} \]

In section 7, we have the following result for the Bredon cohomology of equivariant configuration spaces.

**Theorem 7.2.** For a finite group \( G \), \( V = \mathbb{R}[G] \) is the regular \( \mathbb{R} \)-linear representation of \( G \). The following is a decomposition of the rational homology Bredon coefficient system of \( \text{Conf}(V, q) \),

i) For \( n > 0 \),

\[ H_n(\text{Conf}(V, q)) = \bigoplus_{H < G} \beta_{H,n} \underline{1}_H \]

where \( \beta_{H,n} \) is the \( n \)-th Betti number of \( \text{Conf}(V^H, q) \).

ii) For \( n = 0 \),

\[ H_0(\text{Conf}(V, q)) = \mathbb{Q} \oplus (q! - 1) \underline{1}_{[e]} \]

where \( \mathbb{Q} \) is the constant \( \mathbb{Q} \) coefficient system.

This decomposition allows us to compute Bredon cohomology of equivariant configuration spaces. In chapter 5, we compute the universal coefficient spectral sequence for the Bredon cohomology of \( \text{Conf}(\mathbb{R}[G], 3) \) and \( \text{Conf}(\mathbb{R}[G], 4) \) with the coefficient system \( \underline{1}_0 \).
2. Bredon coefficient systems

Definition 2.1. The canonical orbit category of group $G$, denoted by $\mathcal{O}_G$, is the category whose objects are $G$-spaces $G/H$ and morphisms are $G$-maps.

There is a $G$-map $f : G/H \to G/K$ if and only if $gHg^{-1} < K$. Notice that if $f(eH) = gK$ for some $G \in G$, then

$$eK = f(g^{-1}H) = f(g^{-1}hg \cdot g^{-1}H) = g^{-1}hg \cdot f(g^{-1}H) = g^{-1}hg \cdot eK$$

for any $h \in H$. Hence $g^{-1}Hg < K$.

Definition 2.2. Let $R$ be a commutative ring and $\text{Mod}_R$ be the category of $R$-modules. A Bredon coefficient system over $R$ is a contravariant functor $\mathcal{O}_G \to \text{Mod}_R$. The category of Bredon coefficient systems over $R$ is denoted by $\mathcal{C}^R_G$. When $R = \mathbb{Z}$, we use $\mathcal{C}_G$ for simplicity.

Example 2.1. Given a based $G$-CW-complex, the $n$-th equivariant homotopy group $\pi_n(X)$ for $n \geq 2$, is a Bredon coefficient system given by

$$\pi_n(X)(G/H) = \pi_n(X^H)$$

Example 2.2. Let $X$ be a $G$-CW-complex, we can define the cellular chain complex of coefficient systems $C_\ast(X)$, where

$$C_n(X)(G/H) = H_n((X^n)^H, (X^{n-1})^H, R).$$

Define

$$H_n(X) = H_n(C_\ast(X)).$$

By a general categorical argument, the category $\mathcal{C}_G^R$ is an Abelian category with enough injectives. So we could talk about homological algebra concept such as homology and cohomology of chains and cochains in this category.

Definition 2.3. Let $M$ be a Bredon coefficient system and $X$ be a $G$-CW-complex. Define a cochain complex

$$C^n(X; M) = \text{Hom}_{\mathcal{C}_G^R}(C_n(X), M),$$

Its cohomology,

$$H^*_G(X, M) := H^*(C_\ast(X; M))$$

is called the Bredon cohomology of $X$ with coefficient $M$. 
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To define Bredon homology, we need a covariant functor $\mathcal{N} : \mathcal{O}_G \to \text{Mod}_R$ as a coefficient system. Define the cellular chain by

$$C_n(X; \mathcal{N}) = C_n(X) \otimes_{\mathcal{O}_G} \mathcal{N} = \int_{G/H} C_n(X)(G/H) \otimes_R \mathcal{N}(G/H)$$

In other word, the tensor product $\otimes_{\mathcal{O}_G}$ is given by the coend of the two functors.

**Definition 2.4.** Bredon homology of $X$ with coefficient $N$ is given by

$$H^G_n(X, \mathcal{N}) = H_n(C_*(X; \mathcal{N}))$$

**Theorem 2.1** ([8], [20]). There are universal coefficient spectral sequences

$$E_2^{p,q} = \text{Ext}_{\mathcal{O}_G}^{p,q}(H_*(X), M) \Rightarrow \mathcal{H}_n^G(X, M),$$

and

$$E_2^{p,q} = \text{Tor}_{\mathcal{O}_G}^{p,q}(H_*(X), N) \Rightarrow \mathcal{H}_n^G(X, M).$$

3. BREDON COEFFICIENT SYSTEMS IN THE LANGUAGE OF PATH ALGEBRAS

Using the language of path algebra, we could also give a more “algebraic” description of Bredon cohomology and homology. In this subsection, we will define path algebra and give an alternative definition of Bredon coefficient systems as well as Bredon homology and cohomology. The readers who are interested in this topic could check for more details of path algebras and the related representation theory in the summary book [2].

**Definition 3.1.** A quiver $Q = (V, E, s, t)$ is a directed graph, i.e. a graph that associates each edge a direction, where $V$ is the set of vertices and $E$ is the set of edges along with two maps $s, t : E \to V$ that for each edge $\alpha \in E$, the images $s(\alpha)$ and $t(\alpha)$ are the source and target of the edge $\alpha$ respectively.

**Example 3.1.** A small category $\mathcal{C}$ is naturally a quiver whose objects and morphisms are vertices and edges of the quiver respectively. We denote the quiver associated with $\mathcal{C}$ by $Q_\mathcal{C}$.

**Definition 3.2.** For any quiver $Q = (V, E, s, t)$, the set of paths of $Q$, denoted by $P_Q$, consists of the following elements:

i) For each vertex $v \in V$, there is a trivial constant path $e_v$, and set $s(e_v) = t(e_v) = v$;
ii) All the finite sequences $\alpha_n\alpha_{n-1}\cdots\alpha_1$ where $\alpha_i \in E$ for each $i$ and $t(\alpha_k) = s(\alpha_{k+1})$ for $k = 1, 2, \cdots, n - 1$. In other word, an actual path on the quiver.

Moreover, we can define a multiplication $\circ$ on $P_Q$. For any two paths $\alpha_n\alpha_{n-1}\cdots\alpha_1, \beta_m\beta_{m-1}\cdots\beta_1$ in $P_Q$,

$$\alpha_n\cdots\alpha_1 \circ \beta_m\cdots\beta_1 = \begin{cases} \alpha_n\cdots\alpha_1\beta_m\cdots\beta_1 & \text{if } t(\beta_m) = s(\alpha_1), \\ 0 & \text{otherwise}. \end{cases}$$

**Definition 3.3.** Let $R$ be a commutative ring, the path algebra $RQ$ is an associative algebra with basis $P_Q$ whose multiplication is linearly induced by the multiplication $\circ$ on $P_Q$ over $R$. In addition, if $V$ is an finite set, then the algebra $RQ$ has a multiplicative identity

$$1_{RQ} = \sum_{v \in V} e_v,$$

and these $e_v$’s are nonzero idempotents of $RQ$.

**Example 3.2.** Let $Q$ be the quiver given by Figure 1. Then over a field $k$, the path algebra $kQ$ is a $k$-algebra of dimension $n^2 + n$. And it is isomorphic to the algebra of $n \times n$ upper triangular matrices over $k$.

**Figure 1.** Dynkin diagram of type $A_n$

**Definition 3.4.** A relation of a quiver $Q$ is a subspace of $RQ$ spanned by linear combinations of paths having a common source and a common target. Let $S$ be a set of relations of $Q$. The path algebra with relation $S$, denoted by $RQ_S$ is $RQ/I_S$ where $I_S$ is the two-sided ideal generated by $S$.

For an algebra $A$ over $R$, Let $\text{Mod}_A$ be the category of right-$A$-modules and $\text{A-Mod}$ be the category of left-$A$-modules.

**Definition 3.5.** If $G$ is a finite group, the canonical orbit category $\mathcal{O}_G$ is a small category with finitely many objects. Let $Q = Q_{\mathcal{O}_G}$ be the quiver associated with $\mathcal{O}_G$. And $S$ is the set of relations given by the equivalences of morphisms in the category $\mathcal{O}_G$. $RQ_S$ is the corresponding path algebra.
Lemma 3.1. There is an equivalence of abelian categories $\mathcal{C}_G^R \rightarrow \text{Mod}_{RQ_S}$. Then a Bredon coefficient system could be treated as a right-$RQ_S$-module and vice versa.

Proof. We define two functors $F : \mathcal{C}_G^R \rightarrow \text{Mod}_{RQ_S}$ and $G : \text{Mod}_{RQ_S} \rightarrow \mathcal{C}_G^R$ as follows:

i) For any Bredon coefficient system $\underline{M} \in \mathcal{C}_G^R$, let

$$F(\underline{M}) := \bigoplus_{H < G} \underline{M}(G/H).$$

The path algebra $RQ_S$ action on $F(\underline{M})$ is induced by structure maps of $\underline{M}$. Since the Bredon coefficient system is a contravariant functor $\mathcal{O}_G \rightarrow \text{Mod}_R$, the R-linear space $F(\underline{M})$ admits a right $RQ_S$-module structure.

ii) For any object $G/H$ in $\mathcal{O}_G$, we have the trivial path $e_{G/H}$. It is also an idempotent element of the path algebra $RQ_S$. For any right $RQ_S$-module $N$ in $\text{Mod}_{RQ_S}$, define a Bredon coefficient system $G(N)(G/H) = N.e_{G/H}$.

The two functors $F$ and $G$ give the natural equivalence

$$F : \mathcal{C}_G^R \leftrightarrow \text{Mod}_{RQ_S} : G.$$ 

□

Example 3.3. Let $G = C_2$, the cyclic group of order 2, the canonical orbit category is shown in Figure 2. The path algebra with relations corresponds to $\mathcal{O}_G$ has dimension 5, and a basis is given by $\{e_{C_2/C_2}, e_{C_2/0}, \alpha, \beta, \beta\alpha\}$. Notice that the loop $\beta$ is given by the Weyl group action. And since the Weyl group for trivial group is the whole group $C_2$, then set of relations in this case is $\{\beta^2\}$.

![Figure 2](image-url)
Similarly, a covariant functor $N : \mathcal{O}_G \to \text{Mod}_R$ is equivalent to a left-$RQ_S$-module and the coend of a contravariant functor with a covariant functor is the tensor product of a right $RQ_S$-module with a left $RQ_S$-module over $RQ_S$. We could restate the definition of Bredon homology using the path algebra terminology.

**Definition 3.6.** Let $X$ be a $G$-CW-complex and $N$ be a left $RQ_S$-module. The cellular chain complex of coefficient systems $C_\ast(X)$, where

$$C_n(X)(G/H) = H_n((X^n)^H, (X^{n-1})^H, R).$$

Apply Lemma 3.1, $F(C_n(X))$ is a right $RQ_S$-module. Define the chain of $R$-module $C_\ast(X; N)$ by

$$C_n(X; N) = F(C_n(X)) \otimes_{RQ_S} N$$

The Bredon homology of $X$ with coefficient system $N$ is given by

$$H_n^G(X, N) = H_n(C_\ast(X; N)).$$

4. **Reduced Bredon coefficient systems**

The computational complexity of Bredon cohomology is strongly related the complexity of the canonical orbit category $\mathcal{O}_G$. In this section, we reduce the Bredon coefficient system to a simpler form. This reduction is quite useful when we compute Bredon cohomology.

Since the conjugation of subgroups in $G$ induces isomorphisms in the canonical orbit category $\mathcal{O}_G$, it is enough that we just consider one subgroup of $G$ for each conjugacy class to obtain a category which is equivalent to $\mathcal{O}_G$ but with fewer objects. Let $\widehat{\mathcal{O}}_G$ be the full subcategory of $\mathcal{O}_G$ whose object set consists of one and only one object for each isomorphism class in $\mathcal{O}_G$. This definition is not canonical, but $\widehat{\mathcal{O}}_G$ is unique up to category equivalence.

**Definition 4.1.** A reduced Bredon coefficient system over a commutative ring $R$ is a contravariant functor $M : \widehat{\mathcal{O}}_G \to \text{Mod}_R$.

We denote the category of reduced Bredon coefficient systems by $\widehat{\mathcal{C}}_G^R$.

**Lemma 4.1.** The category $\widehat{\mathcal{C}}_G^R$ is equivalent to $\mathcal{C}_G^R$. 
Proof. The two functor categories $\widehat{\text{C}}_G^R$ and $\text{C}_G^R$ have the same target category $\text{Mod}_R$. Their source categories $\widehat{\text{O}}_G$ and $\text{O}_G$ are equivalent. Hence the functor categories are equivalent as well. The equivalence is induced by the equivalence between $\widehat{\text{O}}_G$ and $\text{O}_G$. □

Therefore in the future discussion, we will only talk about reduced Bredon coefficient systems and the reduced canonical orbit category. In order to simplify the notation, denote the reduced canonical orbit category again by $\text{O}_G$.

Example 4.1. Let $G = \Sigma_3$, the (reduced) canonical orbit category is shown in Figure 3. Notice that $\sigma, \tau$ are the generators of the Weyl group $W\{e\} = \Sigma_3$ and $\iota$ is the generator of $W\langle(123)\rangle = \mathbb{Z}_2$. The set of relations is \{\beta\alpha - \delta\gamma, \iota^2, \sigma^3, \tau^2, \tau\sigma\tau - \sigma^2, \sigma\tau\sigma - \tau\}.

\[
\begin{array}{c}
\Sigma_3/\Sigma_3 \\
\downarrow \beta \\
\Sigma_3/\langle(123)\rangle \\
\downarrow \gamma \\
\Sigma_3/\langle(123)\rangle \\
\downarrow \alpha \\
\Sigma_3/\{e\} \\
\downarrow \gamma \\
\Sigma_3/\{e\} \\
\downarrow \sigma \\
\Sigma_3/\{e\}
\end{array}
\]

Figure 3. Reduced canonical category of $\Sigma_3$

5. Rational Bredon coefficient systems

In order to apply the universal coefficient spectral sequence, we need to find an injective resolution for a given coefficient system. However, it could be a difficult task for a general underlying ring $R$. It is not known to the author whether there is a general method to construct an injective resolution for any coefficient system over the integers. However, over $\mathbb{Q}$, in [15], Doman constructed the injective envelope for any Bredon coefficient system over the rationals.
Definition 5.1. For any subgroup $H < G$, let $V_H$ be a left $\mathbb{Q}(WH)$-module. Define a Bredon coefficient system $I(V_H)$ by

$$I(V_H)(G/K) = \text{Hom}_{\mathbb{Q}(WH)}(\mathbb{Q}((G/K)_H), V_H).$$

For a $G$-map $f : G/K \to G/K'$, the map $\overline{f} : \mathbb{Q}((G/K)_H) \to \mathbb{Q}((G/K')_H)$ is induced by $f$. Then

$$I(V_H)(f) : I(V_H)(G/K') \to I(V_H)(G/K)$$

is defined by $I(V_H)(f)(g) = g \circ \overline{f}$ where $g \in I(V_H)(G/K')$.

Lemma 5.1 ([15]). The coefficient system $I(V_H)$ is an injective object in the category $\mathcal{C}^\mathbb{Q}_G$.

Moreover, the injective coefficient system $I(V_H)$ could be used to construct an injective envelope for any Bredon coefficient system $\underline{M}$ over $\mathbb{Q}$. Let $V_{\{e\}} = \underline{M}(G/\{e\})$. In general, let

$$V_H = \bigcap_{K < H} \text{Ker} \underline{M}(f_{K,H})$$

where $f_{K,H} : G/K \to G/H$ is the projection in the canonical orbit category.

Theorem 5.2 ([15]). For any coefficient system $\underline{M}$ over $\mathbb{Q}$, the map $f : \underline{M} \to \bigoplus I(V_H)$ is an injective envelope of $\underline{M}$, where the direct sum is over all conjugacy classes of $G$.

Corollary 5.3. The global (injective) dimension of $\mathcal{C}^\mathbb{Q}_G$ is less than $L - 1$ where $L$ is the largest length of subgroup chains in $G$.

Let $\underline{\mathbb{Q}}$ be the constant $\mathbb{Q}$ coefficient system.

Lemma 5.4. The constant coefficient system $\underline{\mathbb{Q}}$ is injective.

Proof. Take $H = \{e\}$, it follows that $WH = G$ and $(G/K)_H = G/K$. Then for trivial $WH$-module $V_H = \mathbb{Q}$, we have

$$\text{Hom}_{\mathbb{Q}(WH)}(\mathbb{Q}(G/K), V_H) = \mathbb{Q}.$$

Hence $I(V_H) = \mathbb{Q}$. \quad $\square$

Corollary 5.5. The universal coefficient spectral sequence

$$E_2^{p,q} = \text{Ext}^p_{\mathcal{C}^\mathbb{Q}_G}(H_q(X), \underline{\mathbb{Q}}) \Rightarrow H^p_G(X, \underline{\mathbb{Q}}).$$
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collapses at the $E_2$-page. Therefore

$$H^*_G(X,\mathbb{Q}) = \text{Hom}_{\mathbb{Q}}(H_*(X),\mathbb{Q}).$$

6. ORDERED CONFIGURATION SPACES OF EUCLIDEAN SPACES

In this section we will introduce the classical computations on the homology and cohomology of configuration spaces. For more details, the reader can refer to [11], [12].

Definition 6.1. Given a space $M$, the ordered configuration space of $q$-tuples of distinct points in $M$ is

$$\text{Conf}(M,q) = \{(x_1, \cdots, x_q) \in M^q | x_i \neq x_j, \text{for } i \neq j\}.$$ 

Example 6.1. When $M = \mathbb{R}^2$, $\text{Conf}(\mathbb{R}^2, q)$ is an Eilenberg-Mac Lane space of type $K(B_q, 1)$ where $B_q$ is the Artin’s Braid groups on $q$ strands.

We will mainly focus on the case $M = \mathbb{R}^n$ in order to compute the Bredon cohomology of equivariant configuration spaces. The following theorem gives the structure of integral cohomology ring of $\text{Conf}(\mathbb{R}^n, q)$

Theorem 6.1 ([12]). The integral cohomology ring of configuration spaces $\text{Conf}(\mathbb{R}^n, q)$ is given by the following data:

i) For $n = 1$, $\text{Conf}(\mathbb{R}, q)$ is homotopy equivalent to the symmetric group of degree $q$ as a discrete space, or in other word, a finite discrete space with $q!$ points.

ii) For $n \geq 2$, the integral cohomology ring $H^*(\text{Conf}(\mathbb{R}^n, q))$ is generated by elements

$$A_{i,j}, 1 \leq j < i \leq q,$$

where $A_{i,j}$ is of degree $n - 1$ and the relations are given by

1) $A_{i,j}^2 = 0.$

2) $A_{i,j}A_{i,k} = A_{k,j}(A_{i,k} - A_{i,j})$ for $j \leq k$; and

3) associativity and graded commutativity.

Corollary 6.2. i) Let $n \geq 2$, for $k = 1, \cdots, q-1$, $H^{k(n-1)}(\text{Conf}(\mathbb{R}^n, q))$, as an abelian group, has a basis

$$\{A_{i_1,j_1} \cdots A_{i_k,j_k} | i_1 < i_2 < \cdots < i_k, \text{and } j_l < i_l \text{ for } l = 1, 2, \cdots, k\}.$$
ii) There is an abelian group isomorphism

\[ H^k(\text{Conf}(\mathbb{R}^n, q)) \cong H_k(\text{Conf}(\mathbb{R}^n, q)) \]

for each \( k \).

**Example 6.2.** The rank of \( H^{n-1}(\text{Conf}(\mathbb{R}^n, q)) \) is \( \frac{q(q-1)}{2} \) and the rank of \( H^{(q-1)(n-1)}(\text{Conf}(\mathbb{R}^n, q)) \) is \( (q - 1)! \).

We can compute the rank of a general degree using Poincaré series.

**Proposition 6.3** ([6]). The Poincaré series of \( H^*(\text{Conf}(\mathbb{R}^n, q)) \) is given by

\[ \prod_{m=1}^{q-1} (1 + mt^{n-1}) \]

Hence rank of \( H^m(n-1)(\text{Conf}(\mathbb{R}^n, q)) \) is

\[ \sum_{i_1 < i_2 < \cdots < i_m} i_1 \cdot i_2 \cdots \cdot i_m \]

**Proposition 6.4.** For \( m < n \), let \( \iota : \mathbb{R}^m \to \mathbb{R}^n \) be a linear embedding. It induces a natural embedding of configuration spaces

\[ \tau : \text{Conf}(\mathbb{R}^m, q) \to \text{Conf}(\mathbb{R}^n, q) \]

Moreover the induced map on homology

\[ \tau_* : H_*(\text{Conf}(\mathbb{R}^m, q)) \to H_*(\text{Conf}(\mathbb{R}^n, q)) \]

is trivial.

**Proof.** It is directly from the following commutative diagram.

\[ \begin{array}{ccc}
  H_*(\text{Conf}(\mathbb{R}^n, q)) & \xrightarrow{A_{i,j}} & H_*(S^{n-1}) \\
  \uparrow & & \uparrow 0 \\
  H_*(\text{Conf}(\mathbb{R}^m, q)) & \xrightarrow{A_{i,j}} & H_*(S^{m-1})
\end{array} \]

The two horizontal maps are surjections given by cohomology class \( A_{i,j} \) for any \( 1 \leq j < i \leq q \).
7. Fixed point sets of equivariant configuration spaces

Definition 7.1. Let $G$ be a finite group, for an $\mathbb{R}$-linear representation $V$, the equivariant configuration space of $V$ is the space $\text{Conf}(V, q)$ with the diagonal $G$-action.

Lemma 7.1. For any subgroup $H < G$, the $H$-fixed point $\text{Conf}(V, q)^H$ is $\text{Conf}(V^H, q)$.

Proof. For any $h \in H$ and any point $(x_1, \cdots, x_q) \in \text{Conf}(V, q)^H$,
\[(x_1, \cdots, x_q) = h.(x_1, \cdots, x_q) = (h.x_1, \cdots, h.x_q).
\]Hence $h.x_i = x_i$ for each $i$. Then $x_i \in V^H$. \qed

Since $V$ is a linear representation, the fixed point set $V^H$ is a linear subspace of $V$. Moreover, if given $K$ and $H$ are two subgroups of $G$ such that $K < H < G$. Then the the embedding of fixed point sets $\text{Conf}(V, q)^H \hookrightarrow \text{Conf}(V, q)^K$ is induced by the embedding $V^H \hookrightarrow V^K$. By Proposition 6.3, the embedding is trivial on homology as long as the embedding $V^H \hookrightarrow V^K$ is proper.

If $V = \mathbb{R}[G]$ is the regular representation of $G$. The above argument leads to an important decomposition of $H_*(\text{Conf}(V, q))$, the homology Bredon coefficient system of $\text{Conf}(V, q)$. To state the decomposition theorem, we need to introduce the following class of “1-dimensional” coefficient system.

Definition 7.2. The Bredon coefficient system $\mathbf{1}_H$ over $\mathbb{Q}$ is given by
\[
\mathbf{1}_H(G/K) = \begin{cases} 
\mathbb{Q}, & \text{if } K \text{ conjugates to } H, \\
0, & \text{otherwise}.
\end{cases}
\]

Theorem 7.2. For a finite group $G$, $V = \mathbb{R}[G]$ is the regular $\mathbb{R}$-linear representation of $G$. We have the following decomposition of the rational homology Bredon coefficient system of $\text{Conf}(V, q)$,

i) For $n > 0$,
\[
H_n(\text{Conf}(V, q)) = \bigoplus_{H < G} \beta_{H,n} \mathbf{1}_H
\]
where $\beta_{H,n}$ is the $n$-th Betti number of $\text{Conf}(V^H, q)$.

ii) For $n = 0$,
\[
H_0(\text{Conf}(V, q)) = \mathbb{Q} \oplus (q! - 1) \mathbf{1}_{(e)}
\]
where \( \mathbb{Q} \) is the constant \( \mathbb{Q} \) coefficient system.

**Proof.** For any subgroup \( H \), \( V^H = \mathbb{R}^{[G:H]} \). In addition, if \( K \) is a proper subgroup of \( H \), \( V^K \) is a proper subspace of \( V^K \). Hence when \( n > 0 \), by Proposition 6.3, the connection maps of \( H_n(\text{Conf}(V, q)) \) are all trivial except those isomorphisms induced by conjugation. Therefore \( H_n(\text{Conf}(V, q)) \) could be written as the direct sum of \( 1_H \)'s. The case when \( n = 0 \) is a direct computation.

**Corollary 7.3.** If \( V \) is finitely generated free representation, i.e. \( V = (\mathbb{R}[G])^s \) for some \( s > 1 \). Then

\[
H_n(\text{Conf}(V, q)) = \begin{cases} 
\bigoplus_{H < G} \beta_{n,H}1_H, & \text{for } n > 0 \\
\mathbb{Q}, & \text{for } n = 0
\end{cases}
\]

**Example 7.1.** Let \( G = D_8 \), and \( V = \mathbb{R}[G] \), the regular representation of \( G \). For any subgroup \( H \), \( V^H = \mathbb{R}^{[G:H]} \). In this example we compute the homology Bredon coefficient system of \( \text{Conf}(V, 3) \). To simplify the notation, we denote denote subgroup of \( G \) and canonical orbit category the same as in Section ??.

**Figure 4.** Repeated figure of \( \mathcal{O}_{D_8} \)

Then homology Bredon coefficient system is given by the Figure 5.
By Corollary 6.2 when \( k > 1 \),

\[
H_n(\text{Conf}(\mathbb{R}^k, 3)) = \begin{cases} 
\mathbb{Q} & \text{for } n = 0 \\
\mathbb{Q}^3, & \text{for } n = k - 1 \\
\mathbb{Q}^2, & \text{for } n = 2(k - 1) 
\end{cases}
\]

Table 1 is the summary of the decomposition of the rational homology Bredon coefficient system of equivariant configuration space \( \text{Conf}(\mathbb{R}[G], 3) \).

| \( n \) | Decomposition |
|----------|---------------|
| 0        | \( \mathbb{Q} \oplus 5 \mathbb{L}_7 \) |
| 1        | \( 3\mathbb{L}_4 \oplus 3\mathbb{L}_6 \) |
| 2        | \( 2\mathbb{L}_4 \oplus 2\mathbb{L}_5 \oplus 2\mathbb{L}_6 \) |
| 3        | \( 3\mathbb{L}_1 \oplus 3\mathbb{L}_2 \oplus 3\mathbb{L}_3 \) |
| 6        | \( 2\mathbb{L}_1 \oplus 2\mathbb{L}_2 \oplus 2\mathbb{L}_3 \) |
| 7        | \( 3\mathbb{L}_0 \) |
| 14       | \( 2\mathbb{L}_0 \) |

**Table 1.** Decomposition of \( H_n(\text{Conf}(\mathbb{R}[G], 3)) \)
8. Rational Bredon cohomology of equivariant configuration spaces

In this section we will apply universal coefficient spectral sequence to compute the rational Bredon cohomology of $\text{Conf}(V, q)$. If $V$ is a free $G$-representation, we have given a direct sum decomposition of the homology coefficient system. Firstly, we have the following lemma.

**Lemma 8.1.** Given two Bredon coefficient system $M$ and $N$. If $M = \bigoplus_k M_k$, then

$$\text{Hom}_{\mathcal{C}_G}(M, N) = \bigoplus_k \text{Hom}_{\mathcal{C}_G}(M_k, N)$$

Thus, the computation $E_2$-page of the universal coefficient spectral sequence, we need to compute the $\mathbb{Q}$ dimension of

$$\text{Hom}_{\mathcal{C}_G}(1_H, N)$$

**Proposition 8.2.** Let $G$ be a finite group, $H$ is a subgroup of $G$, for any rational coefficient system $N \in \mathcal{C}_G^\mathbb{Q}$,

$$\text{Hom}_{\mathcal{C}_G}(1_H, N) \cong N(G/H) \bigcap_{K<H} \text{Ker}(i^H_K)$$

where $i^H_K : N(G/H) \to N(G/K)$ is the map induced by the natural projection $i : G/K \to G/H$.

**Proof.** Consider the following commutative diagram,

$$
\begin{array}{ccc}
\mathbb{Q} & \xrightarrow{h} & W \\
\downarrow & & \downarrow f \\
0 & \xrightarrow{} & V
\end{array}
$$

We have $f \circ h = 0$. Namely, $h$ maps $\mathbb{Q}$ into the Kernel of $f$. \qed

8.1. **Computation for 3-configuration spaces.** Similar to the polyhedral products case, in this subsection we consider the 1-dimensional coefficient system $1_0$ given by Figure 6.
By Theorem 5.2, $I_0$ admits an injective resolution

$$0 \to I_0 \to \mathbb{Q} \to I_1 \to I_2 \to 0$$

The two new injectives $I_1$ and $I_2$ is given by Figure 7 and Figure 8.

```
( 1 0 0 )
( 0 1 0 )
```

```
( 0,1,0 )
```

```
( 0 1 0 )
( 0 0 1 )
```

Figure 6. $I_0$

Figure 7. $I_1$
By Proposition 8.2 we have the following result.

1. $\text{Hom}_{\mathcal{C}}(Q, Q) = Q$;
2. $\text{Hom}_{\mathcal{C}}(I_0, Q) = Q$;
3. $\text{Hom}_{\mathcal{C}}(I_k, Q) = 0$ for $k = 1, 2, \ldots, 7$;
4. The map between $Q$ and $I_1$ is given in Figure 9.
(a) We first assume \( f_1 = s, f_2 = t, f_3 = r \).
(b) Next we set \( f_4 = (a_1, a_2)^T \), then by the commutative diagram

\[
\begin{array}{c}
\mathbb{Q} \\
\downarrow \\
\mathbb{Q} \\
\downarrow \\
\mathbb{Q}
\end{array}
\begin{array}{c}
(1, a_2)^T \\
\downarrow \\
(1, 0) \\
\downarrow \\
(0, 1)
\end{array}
\begin{array}{c}
\mathbb{Q} \\
\downarrow \\
\mathbb{Q} \\
\downarrow \\
\mathbb{Q}
\end{array}
\begin{array}{c}
f_1 = s \\
\downarrow \\
f_2 = t
\end{array}
\]

we have \( a_1 = s \). And by another commutative diagram

\[
\begin{array}{c}
\mathbb{Q} \\
\downarrow \\
\mathbb{Q} \\
\downarrow \\
\mathbb{Q}
\end{array}
\begin{array}{c}
(1, a_2)^T \\
\downarrow \\
(0, 1) \\
\downarrow \\
(0, 1)
\end{array}
\begin{array}{c}
\mathbb{Q} \\
\downarrow \\
\mathbb{Q} \\
\downarrow \\
\mathbb{Q}
\end{array}
\begin{array}{c}
f_2 = t \\
\downarrow \\
f_3 = r
\end{array}
\]

we have \( a_2 = t \). Hence \( f_4 = (s, t)^T \).
(c) Next consider \( f_5 \), by the commutative diagram

\[
\begin{array}{c}
\mathbb{Q} \\
\downarrow \\
\mathbb{Q} \\
\downarrow \\
\mathbb{Q}
\end{array}
\begin{array}{c}
f_5 \\
\downarrow \\
f_2 = t \\
\downarrow \\
f_3 = r
\end{array}
\]

we have \( f_5 = t \).
(d) For \( f_6 \), assume that

\[ f_6 = (b_1, b_2)^T. \]

From the commutative diagram

\[
\begin{array}{c}
\mathbb{Q} \\
\downarrow \\
\mathbb{Q} \\
\downarrow \\
\mathbb{Q}
\end{array}
\begin{array}{c}
f_6 \\
\downarrow \\
f_2 = t \\
\downarrow \\
(1, 0)
\end{array}
\]

we conclude that \( b_1 = t \). Similarly \( b_2 = r \). Hence \( f_6 = (t, r)^T \).
(e) Finally for \( f_7 \), assume that \( f_7 = (c_1, c_2, c_3)^T \). From the commutative diagram

\[
\begin{array}{c}
\mathbb{Q} \xrightarrow{f_7} \mathbb{Q}^3 \\
1 \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\mathbb{Q} \xrightarrow{f_4} \mathbb{Q}^2
\end{array}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

we have \( c_1 = s \) and \( c_2 = t \). From the commutative diagram

\[
\begin{array}{c}
\mathbb{Q} \xrightarrow{f_7} \mathbb{Q}^3 \\
1 \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\mathbb{Q} \xrightarrow{f_6} \mathbb{Q}^2
\end{array}
\]

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

we have \( c_2 = t, c_3 = r \). Hence \( f_7 = (s, t, r)^T \).

(f) In summary, there are three free variables in total and

\[
\text{Hom}_{\mathbb{C}_G}(\mathbb{Q}, I_1) = \mathbb{Q}^3.
\]

(5) \( \text{Hom}_{\mathbb{C}_G}(\text{10}, I_1) = 0 \);
(6) \( \text{Hom}_{\mathbb{C}_G}(\text{1k}, I_1) = \mathbb{Q} \) for \( k = 1, 2, 3 \);
(7) \( \text{Hom}_{\mathbb{C}_G}(\text{1k}, I_1) = 0 \) for \( k = 4, 5, 6, 7 \);
(8) The map between \( \overline{\mathbb{Q}} \) and \( \overline{I_2} \) is given in Figure 10.

(a) Firstly we assume that \( f_4 = r \) and \( f_6 = s \).

(b) From the following two commutative diagrams

\[
\begin{array}{c}
\mathbb{Q} \xrightarrow{f_7} \mathbb{Q}^2 \\
1 \downarrow \quad \quad \downarrow (1, 0) \downarrow \\
\mathbb{Q} \xrightarrow{r} \mathbb{Q}
\end{array}
\]

\[
\begin{array}{c}
\mathbb{Q} \xrightarrow{f_7} \mathbb{Q}^2 \\
1 \downarrow \quad \quad \downarrow (0, 1) \downarrow \\
\mathbb{Q} \xrightarrow{s} \mathbb{Q}
\end{array}
\]

We have \( f_7 = (s, t)^T \).

(c) In summary,

\[
\text{Hom}_{\mathbb{C}_G}(\mathbb{Q}, I_2) = \mathbb{Q}^2.
\]
(9) $\text{Hom}_{cG}(1_k, I_2) = 0$ for $k = 0, 1, 2, 3, 5$;
(10) $\text{Hom}_{cG}(1_k, I_2) = \mathbb{Q}$ for $k = 4, 6$;
(11) $\text{Hom}_{cG}(1_k, I_2) = 0$ for $k = 7$;

Combine with Table 4 we have the $E_2$-page of the universal coefficient spectral sequence

$$E_2^{p,q} = \text{Ext}_{cG}^q(H_p(\text{Conf}(V, 3)), I_0)$$

There is one possible $d_2$-differential and the spectral sequence collapses at $E_3$-page. Hence we have the Bredon cohomology for $\text{Conf}(\mathbb{R}[G], 3)$ with coefficient $I_0$ for most of the degree except in degree 3 and 4. Further
computation on the $d_2$-differential is needed to decide the rank on those degrees.

**Proposition 8.3.** The Bredon cohomology of $\text{Conf}(\mathbb{R}[G], 3)$ with coefficient system $\mathbb{1}_0$ is given by the following table.

| $n$ | $H^n_G(\text{Conf}(\mathbb{R}[G], 3), \mathbb{1}_0)$ |
|-----|---------------------------------|
| 0   | $\mathbb{Q}$                   |
| 1   | $\mathbb{Q}^3$                 |
| 2   | $\mathbb{Q}^2$                 |
| 3   | $\mathbb{Q}^{6-k}$             |
| 4   | $\mathbb{Q}^{13-k}$            |
| 7   | $\mathbb{Q}^9$                 |
| 14  | $\mathbb{Q}^2$                 |
| Otherwise | 0 |

**Table 2.** Bredon cohomology $H^n_G(\text{Conf}(\mathbb{R}[G], 3), \mathbb{1}_0)$

for some integer $k \leq 6$.

**8.2. Computation for 4-configuration spaces.** In the section we will compute the $E_2$-page of the universal coefficient spectral sequence for the equivariant configuration space $\text{Conf}(\mathbb{R}[G], 4)$ where $G = D_8$. For a general $q$-configuration spaces, the computation has no substantial difference. But there might be a few more nottrivial $d_2$ differential and $d_3$-differential.

By Corollary 6.2, Proposition 6.3, for $k \geq 1$, the rational homology of 4-configuration spaces is

$$H_n(\text{Conf}(\mathbb{R}^k, 4)) = \begin{cases} 
\mathbb{Q} & \text{for } n = 0 \\
\mathbb{Q}^6 & \text{for } n = k - 1 \\
\mathbb{Q}^{11} & \text{for } n = 2(k - 1) \\
\mathbb{Q}^6 & \text{for } n = 3(k - 1) 
\end{cases}$$

Then by Theorem 7.2, the decomposition of the rational Bredon coefficient system of $\text{Conf}(\mathbb{R}[G], 4)$ is given by the following table.
Using the result on the dimension of related homomorphism set from previous subsection, we have the $E_2$-page of the universal coefficient spectral sequence.

$$E_2^{p,q} = \Ext^q_{\mathcal{C}_0}(H_p(\text{Conf}(\mathbb{R}[G], 4)), \mathbb{F}_0)$$
Figure 12. $\text{Ext}_{\text{B}_{G}}^{q}(H_p((\text{Conf(}R[G], 4)), I_0)$

Similar to the 3-configuration space case, there is one possible $d_2$-differential and the spectral sequence collapses at $E_3$-page. Hence we have the Bredon cohomology for $\text{Conf(}R[G], 4)$ with coefficient $I_0$ for most of the degree except in degree 3 and 4. Further computation on the $d_2$-differential is needed to decide the rank on those degrees.
Proposition 8.4. The Bredon cohomology of $\text{Conf}(\mathbb{R}[G], 4)$ with coefficient system $\mathbb{I}_0$ is given by the following table.

| $n$ | $H^n_G(\text{Conf}(\mathbb{R}[G], 4), \mathbb{I}_0)$ |
|-----|----------------------------------|
| $n = 0$ | $\mathbb{Q}$ |
| $n = 1$ | $\mathbb{Q}^3$ |
| $n = 2$ | $\mathbb{Q}^2$ |
| $n = 3$ | $\mathbb{Q}^{12-k}$ |
| $n = 4$ | $\mathbb{Q}^{40-k}$ |
| $n = 5$ | $\mathbb{Q}^{12}$ |
| $n = 7$ | $\mathbb{Q}^{39}$ |
| $n = 10$ | $\mathbb{Q}^{18}$ |
| $n = 14$ | $\mathbb{Q}^{11}$ |
| $n = 21$ | $\mathbb{Q}^6$ |
| Otherwise | $0$ |

Table 4. Bredon cohomology $H^*_G(\text{Conf}(\mathbb{R}[G], 4), \mathbb{I}_0)$

For some integer $k \leq 12$.

Notice that since the constant coefficient system $\mathbb{Q}$ is injective, the bottom rows in the universal coefficient spectral sequences in both Figure 11 and Figure 12 give the Bredon cohomology of $\text{Conf}(\mathbb{R}[G], 3)$ and $\text{Conf}(\mathbb{R}[G], 4)$ with constant $\mathbb{Q}$ coefficient system. Compared with the classic cohomology of the configuration spaces,

Corollary 8.5. The Bredon cohomology of $\text{Conf}(\mathbb{R}[G], 3)$ and $\text{Conf}(\mathbb{R}[G], 4)$ with constant $\mathbb{Q}$ coefficient system is isomorphic to the classic cohomology of the underlying configuration spaces. Namely,

$$H^*_G(\text{Conf}(\mathbb{R}[V], k), \mathbb{Q}) \cong H^*(\text{Conf}(\mathbb{R}^8), k)$$

for $k = 3, 4$.

References

[1] A. Al-Raisi. Equivariance, Module Structure, Branched Covers, Strickland Maps and Cohomology related to the Polyhedral Product Functor. PhD thesis, University of Rochester, 2014.

[2] I. Assem, A. Skowronski, and D. Simson. Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory. London Mathematical Society Student Texts. Cambridge University Press, 2006.
[3] A. Bahri, M. Bendersky, F. Cohen, and S. Gitler. The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces. *Advance in Math.*, 225, 2010.

[4] A. Bahri, M. Bendersky, F. Cohen, and S. Gitler. A spectral sequence for polyhedra products. *arXiv:1511.08292*, 2015.

[5] I. V. Baskakov. Cohomology of $K$-powers of spaces and the combinatorics of simplicial divisions. *Russian Mathematical Surveys*, 57(5):989–990, oct 2002.

[6] C.-F. Bășigheimer, F. Cohen, and L. Taylor. On the homology of configuration spaces. *Topology*, 28(1):111 – 123, 1989.

[7] S. Bouc, R. Stancu, and P. Webb. On the projective dimensions of mackey functors. *Algebras and Representation Theory*, 20, 2017.

[8] G. Bredon. *Equivariant Cohomology Theories*, volume 34 of Lecture Notes in Mathematics. Springer, 1967.

[9] V. Buchstaber and T. Panov. *Toric Topology*, volume 204 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2015.

[10] V. M. Bukhshtaber and T. E. Panov. Torus actions, combinatorial topology, and homological algebra. *Russian Mathematical Surveys*, 55(5):825–921, oct 2000.

[11] F. Cohen. On configuration spaces, their homology, and lie algebras. *Journal of Pure and Applied Algebra*, 100(1):19 – 42, 1995.

[12] F. Cohen, T. Lada, and P. May. *The Homology of Iterated Loop Spaces*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1976.

[13] H. Coxeter. Regular skew polyhedra in three and four dimension, and topological analogues. *Proceedings of the London Mathematical Society*, s2-43(1), 1938.

[14] M. W. Davis and T. Januszkiewicz. Convex polytopes, coxeter orbifolds and torus actions. *Duke Math. J.*, 62(2):417–451, 03 1991.

[15] R. Doman. On injective rational coefficient system. *Monatshefte für Mathematik*, 106, 1988.

[16] T. Ganea. A generalization of the homology and homotopy suspension. *Commentarii Mathematici Helvetici*, 39(1):295–322, Dec 1964.

[17] M. Goresky and R. MacPherson. *Stratified Morse Theory*, pages 3–22. Springer Berlin Heidelberg, Berlin, Heidelberg, 1988.

[18] M. Hochster. Cohen-macaulay rings, combinatorics, and simplicial complexes, in "ring theory ii". *Lect. Notes in Pure Appl. Math.*, (26):171–223, 1977.

[19] J. May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999.

[20] J. P. May. *Equivariant Homotopy and Cohomology Theory: Dedicated to the Memory of Robert J. Piacenza*, volume 91 of CBMS Regional Conference Series in Mathematics. American Mathematical Society, 1996.

[21] T. E. Panov. *Cohomology of face rings, and torus actions*, pages 165–201. London Mathematical Society Lecture Note Series. Cambridge University Press, 2007.

[22] G. J. Porter. The homotopy groups of wedges of suspensions. *American Journal of Mathematics*, 88(3):655–663, 1966.
[23] J. Thevenaz and P. Webb. The structure of mackey functors. *Transactions of the American Mathematical Society*, 347(6):1865–1961, 1995.

[24] Q. Zhu. Bredon cohomology of polyhedral products. *arXiv:1811.07076*, 2018.

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