Abstract. Let $X$ and $Y$ be two graphs with vertex set $[n]$. Their friends-and-strangers graph $FS(X, Y)$ is a graph with vertices corresponding to elements of the group $S_n$, and two permutations $\sigma$ and $\sigma'$ are adjacent if they are separated by a transposition $\{a, b\}$ such that $a$ and $b$ are adjacent in $X$ and $\sigma(a)$ and $\sigma(b)$ are adjacent in $Y$. Specific friends-and-strangers graphs such as $FS(\text{Path}_n, Y)$ and $FS(\text{Cycle}_n, Y)$ have been researched, and their connected components have been enumerated using various equivalence relations such as double-flip equivalence. A spider graph is a collection of path graphs that are all connected to a single center point. In this paper, we delve deeper into the question of when $FS(X, Y)$ is connected when $X$ is a spider and $Y$ is the complement of a spider or a tadpole.

1. Introduction

Suppose that a set of $n$ people are sitting on $n$ chairs, playing a swapping game. Each pair of chairs can be adjacent or not, and a swap between two people sitting in adjacent chairs is only valid if the two people are friends (where friendship is symmetric). The set of all permutations of the $n$ people on the $n$ chairs produces a graph, where two permutations are adjacent if they can be obtained from each other by an allowable swap.

We acquaint ourselves with some basic graph theoretical terminology before formally introducing friends-and-strangers graphs. We write $V(G)$ and $E(G)$ for the vertex set and edge set, respectively, of a graph $G$. We define a subgraph of a graph to be a graph whose vertex and edge sets are subsets of the original graph. The complement of a graph $G$, denoted $\overline{G}$, is the graph with vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G}) = \{\{a, b\} : \{a, b\} \notin E(G)\}$. Additionally, a graph $G$ is connected if there exists a sequence of adjacent vertices from $a$ to $b$ for all vertices $a, b \in G$.

We can formalize the swapping game notion as follows. Let $X$ and $Y$ be graphs with $V(X) = V(Y) = [n] = \{1, \ldots, n\}$. The friends-and-strangers graph $FS(X, Y)$ has a vertex set isomorphic to the elements of the permutation group $S_n$, and two vertices (permutations) $\sigma, \phi$ are connected if there exists an edge $\{a, b\} \in E(X)$ such that the following conditions are met:

- $\{\sigma(a), \sigma(b)\} \in E(Y)$
- $\sigma(a) = \phi(b)$ and $\sigma(b) = \phi(a)$
- $\sigma(c) = \phi(c)$ for all $c \in V(X) \setminus \{a, b\}$.

To avoid confusion when discussing the vertices of $X$ and $Y$, we will refer to the vertices of $Y$ as labels.

We also define special graphs that will be frequented in the following sections. Note that all of the following graphs have vertex set $[n]$, where $n$ can be changed as necessary.

- The star graph $\text{Star}_n$ is the graph with edge set $E(\text{Star}_n) = \{\{1, b\} : b \in [n], 1 < b\}$.
- The path graph $\text{Path}_n$ is the graph with edge set $E(\text{Path}_n) = \{\{k, k + 1\} : k \in [n - 1]\}$.  

• The cycle graph $\text{Cycle}_n$ is the path graph with the additional edge $\{n, 1\}$.

• The tadpole graph $\text{Tad}_{c,n-c}$ is a cycle graph with $c \geq 3$ edges joined together with a path graph with $n - c$ edges. Its vertex set is $[n]$ and its edge set is $\{\{k, k + 1\} : k \in [n - 1]\} \cup \{\{1, c\}\}$. We also denote the sole vertex of degree 3 as the triple point. Additionally, letting $n = c$ results in $\text{Tad}_{c,n-c} \cong \text{Cycle}_n$.

Example. Let $X = \text{Star}_4$ and $Y = \text{Tad}_{3,1}$.

Friends-and-strangers graphs help explain how permutations are related. For example, the fact that the 15-puzzle is not solvable is due to the fact that $\text{FS}(\text{Star}_{16}, \text{Grid}_{4 \times 4})$ is not connected, where $\text{Grid}_{n \times n}$ is a square grid graph with $n^2$ vertices. This example was studied by Wilson in [8], where the connectivity of friends-and-strangers graphs of the form $\text{FS}(\text{Star}_n, Y)$ was studied. Extremal aspects of friends-and-strangers graphs have been the primary question for related research, Jeong’s recent paper shows that diameters of friends-and-strangers graphs are not polynomially bounded [5].

In this paper we expand upon the results shown in [3], specifically those relating to connectedness of $\text{FS}(\text{Spider}, Y)$.

In Section 4, we focus on finding graphs $X$ such that whenever $Y$ is a spider with at most $n$ legs, $\text{FS}(X, Y)$ is connected. We prove necessary and sufficient conditions guaranteeing the connectedness of $\text{FS}(X, Y)$. These conditions concern whether or not $X$ contains a specific graph.

In Section 5, we propose some more questions regarding the connectedness of friends-and-stranger graphs $\text{FS}(X, Y)$ where $X$ is a modified type of spider called the spycle, as well as if $Y$ is the complement of a spycle graph.

2. Background

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ be integers. The spider graph $\text{Spider}(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a graph on $n = 1 + \sum_{i=1}^{k} \lambda_i$ vertices. Each vertex of the form $1, 1 + \lambda_1, 1 + \lambda_1 + \lambda_2, \ldots$ is adjacent to the
vertex \( n \). For all vertices \( m \neq n \), the vertex \( m \) is adjacent to vertex \( m + 1 \) if and only if the vertex \( m + 1 \) is not adjacent to vertex \( n \). Each of the paths that have been adjoined to form the spider graph are referred to as legs, and the vertex of degree \( k \) is referred to as the center. Additionally, the vertices of degree 1 are referred to as feet. A picture of \( \text{Spider}(5,3,1) \) is below.

The idea of a fruit graph \( \text{Cycle}_n^k \) was also introduced in [3], where \( \text{Cycle}_n^k = \text{Tad}_{n-1,1} \). The following results were shown.

**Theorem 2.1** ([3]). Let \( \lambda_1 \geq \cdots \geq \lambda_k \) be positive integers such that \( \lambda_1 + \cdots + \lambda_k + 1 = n \geq 4 \). The friends-and-strangers graph \( \text{FS}(\text{Spider}(\lambda_1, \ldots, \lambda_k), \overline{\text{Cycle}_n}) \) is connected if and only if \( (\lambda_1, \ldots, \lambda_k) \) is not of the form \( (\lambda_1, 1, 1) \) and is not in the following list:

\[
(1,1,1,1), \ (2,2,1), \ (2,2,2), \ (3,2,1), \ (3,3,1), \ (4,2,1), \ (5,2,1).
\]

In particular, we use the following consequence of Theorem 2.1 to extend results related to connectedness.

**Corollary 2.2.** Let \( X \) be a connected graph on \( n \geq 6 \) vertices with at least one vertex of degree 4 or more. Then \( \text{FS}(X, \overline{\text{Cycle}_n}) \) is connected.

In [3] the following results regarding the connectedness of \( \text{FS}(\text{Spider}, \overline{\text{Cycle}_n^k}) \) were shown. In this paper, we expand upon this result by considering if the graph \( Y \) is the complement of a tadpole graph, which yields a general result that encompasses when \( Y \) is the complement of a 3-legged spider.

**Theorem 3.1** ([3]). Let \( \lambda_1 \geq \cdots \geq \lambda_k \) be positive integers such that \( k \geq 3 \) and \( \lambda_1 + \cdots + \lambda_k + 1 = n \). Then \( \text{FS}(\text{Spider}(\lambda_1, \ldots, \lambda_k), \overline{\text{Cycle}_n}) \) is disconnected if and only if \( (\lambda_1, \ldots, \lambda_k) \) is of one of the following forms:

\[
(\lambda_1,1,1,1), \ (\lambda_1,\lambda_2,1), \ (2,2,2).
\]

**Corollary 2.4.** Let \( a \) and \( b \) be positive integers with \( a \geq b \). Then \( \text{FS}(\text{Spider}(a,b,1,1), \overline{\text{Cycle}_n^k}) \) is connected if and only if \( b \geq 2 \).

### 3. Fruits to Tadpoles

Notice that the fruit graph \( \text{Cycle}_n^k \) from before is simply \( \text{Tad}_{1,n-1} \). In this section, we expand upon the results in [3], generalizing the fruit graphs that consist of a cycle together with an edge connecting a new vertex with an existing one into tadpole graphs.

**Theorem 3.1.** Let \( X = \text{Spider}(a,b,1,1) \) be a graph on \( n = a + b + 3 \) vertices with \( a \geq b \geq 2 \). Then \( \text{FS}(X, \overline{\text{Tad}_{c,n-c}}) \) is connected for any \( c \geq 3 \).

The proof of this theorem relies heavily on induction and can be split into multiple base cases as well as inductive steps.
• **Base Case:** All graphs of the form $\text{FS}(\text{Spider}(2,2,1,1), \text{Tad}_{c,7-c})$ with $3 \leq c \leq 7$ are connected. This can be verified using a computer program. Past results in [3] also prove connectedness for $c = 6, 7$. We will also need that $\text{FS}(\text{Spider}(a, b, 1, 1), \text{Tad}_{c,n-c})$ is connected for $c = 0, n$ for all $n \geq 7$. However, this just reduces to the cases of $\text{FS}(\text{Spider}(a, b, 1, 1), \text{Cycle}_n)$ and $\text{FS}(\text{Spider}(a, b, 1, 1), \text{Path}_n)$, both of which are connected by Theorem 2.1.

• **Inductive Step:** A lemma similar to [3, Corollary 4.3], shown below.

**Lemma 3.2.** Let $\text{FS}(\text{Spider}(a, b, 1, 1), \text{Tad}_{c-1,n-c})$ be connected with $a, b \geq 2$ and $n > c$, and let $\text{FS}(\text{Spider}(a, b, 1, 1), \text{Tad}_{c,n-c-1})$ also be connected. Let $X$ be a graph obtained by adding an additional vertex to $\text{Spider}(a, b, 1, 1)$ and a single edge that connects the new vertex in $X$ to one of the existing ones. Then $\text{FS}(X, \text{Tad}_{c,n-c})$ is also connected.

**Proof.** Consider $\text{FS}(X, \text{Tad}_{c,n-c})$. If we were to remove a label $l$ in $\text{Tad}_{c,n-c}$ (namely the one occupying the new vertex of $X$), the graph $\text{Tad}_{c,n-c}$ would be reduced to a graph on $n - 1$ vertices isomorphic one of the following:

- $\text{Spider}(n - c, i - 1, c - i - 1)$ for $1 \leq i \leq \left\lfloor \frac{c - 1}{2} \right\rfloor$, if the removed vertex is $i \geq 2$ away from the triple point and on the cycle
- $\text{Path}_{n-1}$ if the removed vertex is 1 away from the triple point and on the cycle
- $\text{Path}_{n-1} \cup \text{Tad}_{c,i-1}$ if the removed vertex is a distance $i \geq 1$ from the triple point and on the path
- $\text{Path}_{c-1} \cup \text{Path}_{n-c}$, if the removed vertex is the triple point.

Considering each one at a time, we see that all of their complements actually contain either $\text{Tad}_{c,n-c-1}$ or $\text{Tad}_{c-1,n-c}$.

- Notice that $\text{Tad}_{c-1,n-c}$ contains all spiders of the form $\text{Spider}(n - c, i - 1, c - i - 1)$ with $1 \leq i \leq \left\lfloor \frac{c - 1}{2} \right\rfloor$ as subgraphs (the former can be attained by adding an edge between the two endpoints of the latter two legs). Thus, given that $\text{FS}(\text{Spider}(a, b, 1, 1), \text{Tad}_{c-1,n-c})$ is connected, we can also conclude that $\text{FS}(\text{Spider}(a, b, 1, 1), \text{Spider}(n - c, i - 1, c - i - 1))$ is connected for the values of $i$ concerned.
- Additionally, we know that $\text{Path}_{n-1}$ is a subgraph of $\text{Cycle}_{n-1}$, and it is known from Theorem 2.1 that $\text{FS}(\text{Spider}(a, b, 1, 1), \text{Cycle}_{n-1})$ is connected.
- Furthermore, $\text{Path}_{n-1} \cup \text{Tad}_{c,i-1}$ is a subgraph of $\text{Tad}_{c,n-c-1}$ for all values of $i$ mentioned. However, we know that $\text{FS}(\text{Spider}(a, b, 1, 1), \text{Tad}_{c,n-c-1})$ is connected.
- Finally, $\text{Path}_{c-1} \cup \text{Path}_{n-c}$ is also a subgraph of $\text{Cycle}_{n-1}$, so we have a similar result as in the second $\text{Path}_{n-1}$ case.

Because all four of the resultant friends-and-strangers graphs are connected, we can perform the necessary swaps to bring one of the friends (i.e., labels not among those that are adjacent in $\text{Tad}_{c,n-c}$ to the label of the new vertex in $X$) next to the new vertex in $X$. Then, we may swap the label $l$ currently occupying the new vertex in $X$ with the friend labeled $l'$ in $X$. Now consider the remaining friends-and-strangers graph $\text{FS}(X \setminus \{n\}, \text{Tad}_{c,n-c} \setminus \{l'\})$, which is connected regardless of the label $l'$ that is removed. Then, we can repeat again by performing the necessary swaps to bring friend of $l'$, say $l''$, to the vertex adjacent to the $n$ in $X$.

Repeating this process for $l''$, $l'''$, and so forth will allow us to rotate through all possible permutations of $[n + 1]$. Notice that because $n \geq 4$, we can always perform the swaps in a
way that avoids returning the same label back to the new vertex in $X$. Thus $FS(X,Y)$ is connected. \hfill \square

**Proof of Theorem 3.7.** We induct on both $c$ and $n-c$, the lengths of the cycle and path in $\overline{Y}$, respectively. Our base cases tell us that all possible graphs $FS(Spider(2,2,1,1),\overline{Tad,c,7-c})$ for $3 \leq c \leq 7$ are connected. With these base cases (as well as the fact that $FS(X,Cycle_n)$ is connected for all 4-legged spiders $X$ with $n \geq 6$ vertices by Theorem 2.1), we can repeatedly use Lemma 3.2 to obtain the desired result. \hfill \square

### 4. Generalizations to Spiders

Notice that $FS(Spider(\lambda_1,\lambda_2,\ldots,\lambda_k),\overline{Spider(\lambda_1,\lambda_2,\ldots,\lambda_k)})$ is not connected, as the identity permutation id: $V(Spider(\lambda_1,\lambda_2,\ldots,\lambda_k)) \to V(Spider(\lambda_1,\lambda_2,\ldots,\lambda_k))$ is not adjacent to any other vertices in the friends-and-strangers graph. Thus, if $X$ is an $n$-vertex spider such that $FS(X,Y)$ is connected for every $n$-vertex graph $Y$ that is the complement of a spider with at most $k$ legs, then $X$ must have at least $k+1$. In Theorem 4.3, we show that for each $k \geq 3$, there exists such a spider $X$ with exactly $k+1$ legs. Additionally, the following claim shows that we cannot take $X$ to be the $(k+1)$-legged spider $Spider(2,1,\ldots,1)$.

**Claim 4.1.** Let $X = Spider(\lambda_1,\lambda_2,\ldots,\lambda_{k+1})$ and $Y$ be a graph such that $\overline{Y}$ has maximum degree $\lambda_2 + \lambda_3 + \cdots + \lambda_{k+1}$. Then $FS(X,Y)$ is not connected.

The following theorem trivializes the proof of the claim. A **cut vertex** is a vertex that if removed, leaves behind a disconnected graph.

**Theorem 4.2.** [4] Let $X$ and $Y$ be graphs on $n$ vertices. Suppose $x_1\cdots x_d$ ($d \geq 1$) is a path in $X$, where $x_1$ and $x_d$ are cut vertices and each of $x_2,\ldots,x_{d-1}$ has degree exactly 2. If the minimum degree of $Y$ is at most $d$, then $FS(X,Y)$ is disconnected.

**Proof of Claim 4.7.** We take the path $x_1\ldots x_d$ to be the longest spider leg excluding the foot, together with the center vertex so that $d = (\lambda_1 - 1) + 1 = \lambda_1$. Let the center vertex be $x_1$ and the vertex adjacent to the foot of the longest leg be $x_d$. It is not hard to see that the conditions in Theorem 4.2 are satisfied, and that

\[
\text{(minimum degree of } Y \text{)} \leq \left(\text{(maximum possible degree of } Y \text{)} - 1\right) - \text{(maximum degree of } \overline{Y} \text{)} = (\lambda_1 + \lambda_2 + \cdots + \lambda_{k+1} + 1 - 1) - (\lambda_2 + \lambda_3 + \cdots + \lambda_{k+1}) = \lambda_1 \leq \lambda_1,
\]

so $FS(X,Y)$ is indeed disconnected. \hfill \square

**Theorem 4.3.** Let $X = Spider(2,2,1\ldots,1)$ be a spider on $k+1$ legs with $n = k + 4$ vertices. The following three graphs are connected:

- $FS(X,Spider(2,2,2,1\ldots,1))$
- $FS(X,Spider(3,2,1\ldots,1))$
- $FS(X,Spider(4,1\ldots,1))$

Here, there are $k$ legs in each of the complement spider graphs.

**Proof.** For sake of clarity, we assign the following names to the various vertices in $X = Spider(2,2,1\ldots,1)$:

- the center vertex is the vertex of degree $k+1$,
• any foot that is adjacent to the center is an \( A \)-type vertex,

• any vertex of degree 2 is a \( B \)-type vertex, and

• any foot that is not adjacent to the center is a \( C \)-type vertex.

An example graph is displayed in Figure 4 with appropriate color-coded vertex types.

Figure 2. The graph Spider(2, 2, 1, 1, 1, 1). The center vertex is brown, the \( A \)-type vertices are teal, the \( B \)-type vertices are red, and the \( C \)-type vertices are blue.

Additionally, let the three complement spider graphs generally be referred to as \( Y \). We focus on the center vertex of the spider \( Y \), which corresponds to the label \( n \). Theoretically we can think of “hiding” this vertex in the graph \( X \) so that many of the edges in \( Y \) are effectively canceled, thus making swaps between the other \( n - 1 \) labels much easier. To achieve this, we prove the following three claims. Combining the latter two claims (and noting that all series of swaps are reversible) shows that the label \( n \) can occupy any vertex in \( X \), and combining this fact with the first claim proves the desired result by showing all \( n! \) permutations are reachable.

**Claim 4.4.** If the label \( n \) is located at an \( A \) or \( C \)-type vertex, the remaining \( n - 1 \) labels can be freely swapped.

**Claim 4.5.** It is always possible to move the label \( n \) if it is located at the center or a \( B \)-type vertex to either an \( A \) or \( C \)-type vertex through a series of swaps.

**Claim 4.6.** If the label \( n \) currently occupies an \( A \) or \( C \)-type vertex, a series of swaps can move it to any other \( A \) or \( C \)-type vertex.

*Proof of Claim 4.4.* For either type of vertex, there remain exactly 3 edges in \( Y \) after the label \( n \) is removed, and in all three graphs the edges are subgraphs of \( \text{Path}_6 \), which is itself a subgraph of \( \text{Cycle}_{n-1} \). Thus, showing all \( (n-1)! \) permutations can be reached is equivalent to showing \( \text{FS}(\text{Spider}(2,2,\triangle),\overline{\text{Cycle}}_{n-1}) \) is connected for the \( A \)-type vertex and \( \text{FS}(\text{Spider}(2,1,\triangle),\overline{\text{Cycle}}_{n-1}) \) is connected for the \( C \)-type vertex, where the \( \triangle \) represents \( k-2 \) 1’s in both graphs. However, by Proposition 2.2 both graphs are connected. \( \square \)

*Proof of Claim 4.5.* We begin with the case of the \( B \)-type vertex. If the label \( n \) is not adjacent in \( Y \) to the label \( l \) currently occupying the adjacent \( C \)-type vertex, a swap between these two labels would complete the proof. Thus, we assume otherwise: that the labels \( n \) and \( l \) are adjacent in \( Y \). Consider the graph \( X' = \text{Spider}(2,1,\triangle) \) obtained from \( X \) by deleting the entire leg containing the label \( n \). Additionally, consider the graph \( Y' = Y \setminus \{l,n\} \). Note that the union of edges in \( Y' \) is a subgraph of \( \text{Cycle}_{n-2} \), so to show that the remaining \( n-2 \) labels in \( Y' \) can be free swapped around we show \( \text{FS}(X',Y') \) is connected. Once again by Proposition 2.2 this is true.
Thus, we can swap the labels so that the center vertex and an $A$-type vertex are both occupied by labels not adjacent to $n$ in $\overline{Y}$. From here, swap the label $n$ first with the center vertex label, then with the $A$-type vertex label so that the label $n$ now occupies an $A$-type vertex.

For the case of when the label $n$ is in the center, note that it is adjacent to exactly $n$ different labels in $\overline{Y}$, so at least one of the $n - 3$ $A$ or $B$-type vertices adjacent to the center vertex will have a label that can swap with it. Once this occurs, the label $n$ occupies either an $A$ or $B$-type vertex, and for the latter we just repeat the procedure for $B$-type vertices to complete the proof.

Proof of Claim 4.6. We show that the label $n$ can be moved from any $A$-type vertex to any other $A$-type vertex. We also show that it can be moved from any $C$-type vertex to any $A$-type vertex. Reversing the steps of the second method shows that moving from a $A$ to any $C$-type vertex as well as going from one $C$-type vertex to the other, is possible.

For the $A$ to $A$-type vertex swap, we use the fact that we can rearrange the remaining $n - 1$ labels in any way we want at the start. Thus, we arrange them so that the center vertex as well as the new $A$-type vertex contain two of the three labels that are not adjacent to the label $n$ in $\overline{Y}$. Swapping the label $n$ with the center vertex label and then the new $A$-type vertex label completes the procedure.

For the $C$ to $A$-type vertex swap, we once again use the rearranging method, this time to place all three labels not adjacent to the label $n$ in $\overline{Y}$ at (1) the $B$-type vertex adjacent to the current $C$-type vertex, (2) the center vertex, and (3) the desired $A$-type vertex. Swapping the label $n$ with the labels occupying these three vertices in order completes the procedure.

With these three claims, the proof is complete.

\textbf{Theorem 4.7.} Let $X$ be any connected graph on $n$ vertices that contains the $(k + 1)$-legged spider $\text{Spider}(2, 2, 1, \ldots, 1)$ as a subgraph. Let $Y$ be a graph so that $\overline{Y}$ is a spider with at most $k$ legs. Then $\text{FS}(X, Y)$ is always connected.

\textit{Proof.} We proceed by induction on the number of vertices, the base case for different values of $k$ given by Theorem 4.3. Assume that for any graphs $X'$ and $Y'$ with vertex set $[n - 1]$ that satisfy the constraints in the theorem statement, $\text{FS}(X', Y')$ is connected. We can add a single edge connecting any existing foot in $X'$ to a new vertex, simultaneously adding a new edge connecting an existing foot in $\overline{Y}'$ to a new label. Denote the new graphs formed as $X$ and $Y$, respectively.

We show that regardless of the label $l$ that occupies the newly added vertex in $X$, the graph $\text{FS}(X', \overline{Y} \setminus \{l\})$ is still connected. The graph $X'$ obviously satisfies Theorem 4.7 and $\overline{Y} \setminus \{l\}$ is always a subgraph of a spider with \textit{at most} $n$ legs. Thus, $\text{FS}(X', \overline{Y} \setminus \{l\})$ is always connected.

Since this implies that

- it is always possible to perform swaps so that all permutations of the remaining labels on $X'$ can be reached, and
- repeated swaps between different labels occupying the new vertex in $X$ and the adjacent foot in $X'$ will guarantee that every label could be swapped into the new vertex,
this is sufficient to prove that FS(X, Y) is connected. Since our choices of X and Y were arbitrary, this implies any graphs X and Y with n vertices that satisfy the conditions of Theorem 4.7 yield a connected FS(X, Y), completing the inductive step.

Example. The friends-and-strangers graph FS(Spider(10, 5, 4, 3, 1, 1), Y) is connected for any graph Y such that Y is a spider with at most 5 legs.

Considering the overlap between sections 3 and 4, we obtain a corollary of Theorems 3.1 and 4.7. Notice that Spider(a, b, c) is a subgraph of Tad_{a+b+1,c}, which allows us to extend the implications of Theorem 3.1 to 3-legged spiders as follows.

Corollary 4.8. Let X be a graph on n vertices that contains Spider(2, 2, 1, 1) as a subgraph. Let Y = Spider(a, b, c) so that a + b + c + 1 = n. Then FS(X, Y) is connected.

5. Concluding Remarks and Future Directions

A more generalized study of the tadpole graphs Tad_{c,n−c} in Section 3 brings forth a new type of graph. Let λ_1 ≥ λ_2 ≥ ⋯ ≥ λ_k and γ_1 ≥ γ_2 ≥ ⋯ ≥ γ_h ≥ 3 all be integers. We let the spycle graph SC(λ_1, λ_2, ⋯, λ_k; γ_1, γ_2, ⋯, γ_h) be the graph on n = 1 + ∑_{i=1}^{k} λ_i + ∑_{i=1}^{h} γ_i vertices. The vertex n serves as the center vertex of a spider with k legs of length λ_1, ⋯, λ_k, and the graph also contains h cycles with length γ_1, ⋯, γ_h, all intersecting at only n. Notice the spycle graph is also a generalization of the tadpole graph Tad_{c,n−c}, which can be expressed as SC(n − c; c). A picture of SC(2, 4, 4; 3, 5) is below.

Question 5.1. What conditions are required for Y to satisfy in order for FS(SC, Y) to be connected? If two legs of a spider X are connected at their feet by an edge to form a spycle X', what graphs Y yield a connected friends-and-strangers graph FS(X', Y) but a disconnected FS(X, Y)?

Similar to the method used to prove Theorem 4.3, an inductive approach is promising when investigating the connectedness of FS(SC, SC).

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