Classical and Quantum Theory with A New Symmetry

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May 14, 2009

Abstract

A formal symmetry between generalized coordinates and momenta is postulated to formulate classical and quantum theories of a particle coupled to an Abelian gauge field. It is shown that the symmetry (a) requires the field to have dynamic degrees of freedom and to be a connection in a non-flat space-time manifold, and (b) leads to a quantum theory free of the measurement problem. It is speculated that gravitomagnetism could be a possible source of the gauge field.

PACS: 02.40.-k, 03.50.-z, 03.65.Ca, 03.65Ta

Key Words: classical mechanics, quantum, mechanics, canonical invariance, differential geometry, gauge connection, measurement problem

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1 Introduction

Quantum theory is usually formulated by using either the canonical method or Feynman’s path-integral method. The latter has the advantage of preserving all relevant symmetries whereas the former is usually regarded as breaking Poincare invariance through an explicit choice of a ‘time’ coordinate. However, as shown by Witten and Zuckerman [1, 2], the canonical formalism can also be developed in a way that preserves all its relevant symmetries including Poincare invariance. This approach will be adopted to formulate a theory of classical and quantum systems with a new postulated symmetry between coordinates and momenta.

In Hamiltonian mechanics the generalized coordinates $q_i$ and $p_i$ are accorded equal status and their Lagrange and Poisson brackets are canonical invariants. This symmetry between the coordinates and momenta is lost when the particles are coupled to fields like the electromagnetic potential $A_\mu$ because the canonical four-momentum $p_\mu$ of a charged particle is then replaced by the ‘kinetic momentum’ $\pi_\mu = p_\mu - (e/c)A_\mu$ whereas its canonical coordinates $q_\mu$ remain unchanged [3]. The aim of this paper is to see under what conditions the symmetry between the coordinates and momenta can be restored in this case, and to explore its consequences in quantum mechanics.

The starting point is the classical phase space defined as the space of solutions of the classical equations. One can always, if one wishes, choose a coordinate system with a time coordinate and identify the classical solutions with the initial data in that coordinate system, but there is no necessity to make such a non-covariant choice. The notion of a ‘symplectic structure on phase space’ is a more intrinsic concept than the idea of choosing coordinates $q_i$ and $p_i$ [2].

2 Differential Geometric Preliminaries

Differential geometry has played a very useful role in the formulation of physical theories. Since certain basic concepts of differential geometry underlie the formulation of the present paper, it will be helpful to start by recapitulating them and developing them with a view to apply them to the problem at hand. Consider the configuration space of a classical system which is generally a manifold $\mathcal{M}$ with local charts $(U, x)$, $x(m, m \in U) = q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$. One can define the tangent vectors $X_i^q = \partial/\partial q^i$, $q \in \mathbb{R}^n$ and via the in-
verse mapping $x^{-1}$ the tangent vectors $X^m_i = \partial/\partial x^i, m \in U$. These tangent vectors span the tangent space at the point $m \in U$ and are fibres on $\mathcal{M}$. The fibres on all points on $\mathcal{M}$ together with $\mathcal{M}$ constitute the tangent bundle $T\mathcal{M}$. The dual to the tangent bundle is called the cotangent bundle $T^*\mathcal{M}$ with $\pi : T^*\mathcal{M} \to \mathcal{M}$ the projection. One can define a canonical one-form $\theta$ on $T^*\mathcal{M}$ by

$$\theta(\alpha)w = \alpha.T\pi(w)$$

where $\alpha \in T^*\mathcal{M}$ and $w \in T_{\pi}(T^*\mathcal{M})$. The canonical two-form is defined by $\omega = -d\theta, d\omega = 0$. This is a reflection of the fact that $T^*\mathcal{M}$ is a symplectic manifold. If $\mathcal{M}$ is finite dimensional, the formula for $\theta$ in a local chart $(U, x)$ may be written as $\theta = \sum_i p_i dq^i$ where the exterior derivatives $dq^i$ span the cotangent space and are dual to $X^i_q$: $\langle dq^i, X^q_j \rangle = \delta^i_j$. The $p_i$ are the momenta conjugate to the coordinates $q_i$. The two-form $\omega(q, p) = -d\theta = \sum_i dq^i \wedge dp_i$ and it is closed, i.e., $d\omega = 0$. It is well-known that one can always associate a Poisson manifold $(T^*\mathcal{M}, \{,\})$ with the sympletic manifold $T^*\mathcal{M}$. The fundamental Poisson brackets of $q_i$ and $p_j$ in a chart $(U, x)$ are

$$\{q_i, p_j\} = \delta_{ij}.$$  

$T^*\mathcal{M}$ can be regarded as a $2n$ dimensional manifold called ‘phase space’ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n) \in U$ rather than a bundle.

Similarly, for infinite dimensional systems like fields one considers the manifold $\mathcal{B}$ of potentials $B_\mu$. The corresponding phase space is then the cotangent bundle $T^*\mathcal{B}$ with the canonical symplectic structure. Since the Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4\chi} F_B^{\mu\nu} F_B^{\mu\nu},$$

the canonical momentum is $\pi_B^\mu = (1/c \chi) F_B^{\mu\nu} d^3x = (1/c \chi) F_B^{\mu\lambda} \eta^{\lambda\lambda} d^3x$ with $\eta^{\lambda\lambda} = -1$. The canonical symplectic structure $\omega$ on $T^*\mathcal{B}$ is

$$\omega((B_1, \pi_{B1}), (B_2, \pi_{B2})) = \int_{\mathbb{R}^3} (\pi_{B2} \cdot B_1 - \pi_{B1} \cdot B_2) d^3x,$$

and the associated fundamental Poisson bracket is

$$\{F, H\}_{(B, \pi_B)} = \int_{\mathbb{R}^3} \left( \frac{\delta F}{\delta B} \frac{\delta H}{\delta \pi_B} - \frac{\delta F}{\delta \pi_B} \frac{\delta H}{\delta B} \right) d^3x$$

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Figure 1: The commutative diagram illustrating the pull-back map $d\varphi^*_m$

where $\delta F/\delta B$ is the vector field defined by

$$D_B F(B, \pi_B) . B' = \int \frac{\delta F}{\delta B} . B' d^3x \tag{6}$$

with the vector field $\delta F/\delta \pi_B$ defined similarly. For further details, see [4].

If a manifold $M$ is 'curved', the tangent spaces $T_m M$ and $T_{m'} M$ at two infinitesimally separated neighbouring points $m, m' \in M$ are disjoint. The connection is a mapping of these tangent spaces. Let $\varphi : m \to m'$ be a map. Then $d\varphi_m : T_m M \to T_{\varphi(m)} M$, i.e., the tangent vectors to $M$ at $m$ are mapped to the tangent vectors to $M$ at $m'$ by the differential or covariant derivative $d\varphi_m$ which consists of the ordinary partial derivative plus the connection. This is the connection map $d\varphi_m$ (Fig. 1). Let $\tau_m : T^*_m M \to T_m M$ and $\tau_{m'} : T^*_{m'} M \to T_{m'} M$. Then $d\varphi^*_m : T^*_m M \to T^*_{m'} M$ is the pullback map. In terms of local charts on the manifold let the map

$$p \to \pi = p - \frac{2m}{c} B \tag{7}$$

correspond to the pullback map $d\varphi^*_m$, and let the coordinates of the $T^*_m M$ bundle be $(Q, \pi)$. Since $(q, p)$ and $(B, \pi_B)$ are canonical pairs, we have

$$Q = q - \frac{c}{2m} \pi_B. \tag{8}$$

One can regard $(q, Q) \in V \times V$ where $V$ is a vector space and $(p \otimes \pi) \in V \times V$. Then one has a nondegenerate symplectic two-form on $V \times V \times V \times V$ given by

$$\omega((q, Q)_1, (p \otimes \pi)_1, (q, Q)_2, (p \otimes \pi)_2) = (p_1 \otimes \pi_1)(q_1, Q_1) - (p_2 \otimes \pi_2)(q_2, Q_2) = \pi_1(q_1) . p_1(Q_1) - \pi_2(q_2) . p_2(Q_2). \tag{9}$$
Consider the phase spaces \((Q, p) \in V_Q \times V_Q^*\) and \((q, \pi) \in V_q \times V_q^*\). The total space is \(V_T = V_Q \times V_Q^* \times V_q \times V_q^*\). Now consider the projections \(P_1 V_T = V_Q \times V_Q^* \times V_q^* \equiv V'\) and \(P_2 V' = V_Q \times V_q^*\). Then \(P_2 P_1 V_T = V_Q \times V_q^*\), and \((Q, \pi) \in V_Q \times V_q^*\). This shows that the space \(Q-\pi\) is a projection of the higher dimensional phase space \(V_T\) that allows Hamiltonian flows. Similarly, consider the projections \(P_3 V_T = V_Q^* \times V_q \times V_q^* \equiv V''\) and \(P_4 V'' = V_Q^* \times V_q\). Then \(P_4 P_3 V_T = V_Q^* \times V_q\), and \((p, q) \in V_Q^* \times V_q\), which shows that the space \(p-q\) is another projection of \(V_T\) whose dual is the space \(Q-\pi\).

### 3 Classical Theory

With this differential geometric background let us consider the phase space of a relativistic particle coupled to a gauge field \(B\). Let \(Q^i, I = 1, 2, \ldots, 2N\) with \(Q^i = \pi^i\) for \(i \leq N = 4\) and \(Q^i = q^{i-N}\) for \(i > N\). Then the closed two-form is \(\omega(q, \pi) = \sum_i dq^i \wedge d\pi_i\) and the \(2N \times 2N\) antisymmetric matrix \(\omega_{IJ}\) whose non-zero matrix elements are \(\omega_{i, i+N} = -\omega_{i+N, i} = 1\) is invertible. One can define the Poisson bracket of any two functions \(F(Q^I)\) and \(G(Q^J)\) by

\[
[F, G] = \omega^{IJ} \frac{\partial F}{\partial Q^I} \frac{\partial G}{\partial Q^J} \quad (10)
\]

where \(\omega^{IJ}\) is the inverted matrix \([2]\). Corresponding definitions can be given for the field \(B\) following the previous section.

Let the ‘kinetic’ momentum of the particle be

\[
\pi^\mu = p^\mu - \frac{2m}{c} B_\mu \quad (11)
\]

where \(2m\) is the “charge” of the particle. Following the arguments of the previous section (Eqns. (7) and (8)), let us define the new variable \(Q\), the ‘kinetic coordinate’, by

\[
Q^\mu = q^\mu - \frac{c}{2m} \pi^\mu_B . \quad (12)
\]

Using the canonical Poisson brackets

\[
\{q^\mu, p^\nu\} = g^{\mu\nu} , \quad (13)
\]

\[
\{q^\mu, \pi^\nu\} = g^{\mu\nu} , \quad (14)
\]

\[
\{B^\mu, \pi^\nu_B\} = g^{\mu\nu} , \quad (15)
\]
one obtains
\[ \{ Q^\mu, Q^\nu \} = 0, \quad (16) \]
\[ \{ Q^\mu, \pi^\nu \} = \{ q^\mu, p^\nu \} + \{ \pi_B^\mu, B^\nu \} = g^{\mu\nu} - g^{\mu\nu} = 0 \quad (17) \]
and
\[ \{ Q^\mu, p^\nu \} = \{ q^\mu, p^\nu \} - \left( \frac{c}{2m} \pi_B^\mu, p^\nu \right) = g^{\mu\nu}. \quad (18) \]

It follows from these results that in addition to \((q, \pi)\) one can also choose \((Q, p)\) as a canonical pair.

As explained in the previous section, Eqn. (11) is the momentum space representation (or pullback map) of the covariant derivative \(d\phi_m\) which takes the coordinate form
\[ D^\mu = \partial^\mu - \frac{i}{2m\chi} B^\mu. \quad (19) \]
with \(B\) as the connection. \(\pi_B\) connects the two coordinates \(q\) and \(Q\) (Eqn. 12) just as \(B\) connects the two momenta \(p\) and \(\pi\) (Eqn. 11). Just as \(\pi\) is the kinematic momentum of the particle conjugate to its canonical position \(q\), the Poisson bracket (18) implies that \(Q\) is the kinematic position conjugate to the canonical momentum \(p\). \(Q\) and \(\pi\) carry global information about the configuration manifold of the particle through the connection \(B\) and its conjugate \(\pi_B\).

An important and well-known property of covariant derivatives is that they do not commute,
\[ [D^\mu, D^\nu] = -\frac{i}{2m\chi} F_B^{\mu\nu}, \quad (20) \]
the commutator being the curvature \(F_B^{\mu\nu}\) of the connection. This non-commutativity of the covariant derivatives is a classical (i.e., non-quantum theoretic) result following from the geometrical fact that parallel transporting a vector around a closed loop on a curved manifold results in a different vector. This failure to return to the initial vector is known as holonomy. Thus, the covariant derivatives carry global information about the manifold.

Thus, the postulated symmetry between the ‘kinematic’ momenta and coordinates of the particle requires the connection \(B\) to be dynamical (Eqn. 15) and the space-time manifold to be non-flat (Eqn. 20). However, the local charts on which the equations are written are all flat.
The total Hamiltonian of the interacting system is therefore

\[ H = H_p(p - (2m/c)A_g = \pi) + H_{\text{GEM}}(A_g, \pi_g - (2m/c)q = -(2m/c)Q). \] (21)

The equations of motion for the particle in terms of the variables \((Q,p)\) are then

\[ \dot{Q}_\mu = \{Q_\mu, H\}_{(Q,p)} = \frac{\partial H}{\partial p^\mu}, \quad \dot{p}_\mu = \{p_\mu, H\}_{(Q,p)} = -\frac{\partial H}{\partial Q^\mu}, \] (22)

and in terms of the variables \((q,\pi)\), they are

\[ \dot{q}_\mu = \{q_\mu, H\}_{(q,\pi)} = \frac{\partial H}{\partial \pi^\mu}, \quad \dot{\pi}_\mu = \{\pi_\mu, H\}_{(q,\pi)} = -\frac{\partial H}{\partial q^\mu}. \] (23)

These equations show that \((q,p)\) act as the fundamental canonical variables ensuring Hamiltonian flows underlying the evolution of the kinematic variables \((Q,\pi)\). The \(Q-\pi\) space of the particle is a projection of a higher dimensional phase space \(V_T\)—it is dual to the canonical phase space \(p-q\) (see previous section).

4 Quantum Theory

It is now straightforward to construct the quantum theory of the system by adopting the standard canonical procedure of replacing the classical Poisson brackets \((13)\) through \((18)\) by commutators. One gets

\[ [q^\mu, \pi^\nu] = [q^\mu, \hat{p}^\nu] = i\hbar g^{\mu\nu}, \] (24)

\[ [B^\mu, \pi_B^\nu] = i\hbar g^{\mu\nu}, \] (25)

\[ [\hat{Q}^\mu, \pi^\nu] = [q^\mu, \hat{p}^\nu] + [\hat{p}_B^\mu, B^\nu] = 0, \] (26)

\[ [\hat{Q}^\mu, \hat{p}_B^\nu] = i\hbar g^{\mu\nu}. \] (27)

It follows from these commutators that \(\hat{p}^\mu = -i\hbar \partial\mu\), \(\hat{p}_B^\mu = -i\hbar \delta / \delta B_\mu\), \(q^\mu\), \(\hat{Q}^\mu\) and \(\pi^\mu\) are all hermitian. Hence, \(\pi^\mu = -i\hbar \partial\mu - (2m/c)B^\mu\) and therefore

\[ [\pi^\mu, \pi^\nu] = \frac{2i\hbar}{c} F_B^{\mu\nu} \] (28)

but

\[ [\hat{Q}^\mu, \hat{Q}^\nu] = 0. \] (29)
A comparison of (20) and (28) shows that the latter is a consequence of the curvature and holonomy of the connection $B$. Thus, this commutator carries global information about the configuration manifold. For example, it vanishes in flat space-time regions where $F^\mu{}_{\nu} = 0$ but $B^\mu \neq 0$ and $\hbar \neq 0$.

As in the classical theory, $\hat{Q}$ and $\hat{\pi}$ are the ‘kinematic’ coordinate and momentum operators respectively of the interacting particle which carry global information about the manifold, whereas $\hat{p}$ and $\hat{q}$ are the local canonical momentum and coordinate operators respectively that enable underlying Hamiltonian (or Schrödinger) evolutions to occur. Significantly, $\hat{Q}$ and $\hat{\pi}$ have simultaneous eigenvalues because of (26). This implies that quantum theory admits ‘trajectories’ of the particle in the $Q-\pi$ space. One can define the density operator

$$\hat{\rho} = \sum_j p_j |\psi_j\rangle \langle \psi_j|,$$

with $|\psi_j\rangle$ forming a complete set of states that are simultaneous eigenstates of $\hat{Q}$ and $\hat{\pi}$. It satisfies the evolution equation

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [H, \hat{\rho}].$$

Hence, one can define the density function of the trajectories as

$$G(Q, \pi, t) = \langle Q, \pi | \hat{\rho}(t) | Q, \pi \rangle$$

with

$$\hat{\rho}(t) = U(t) \hat{\rho}(0) U^\dagger(t), \quad U(t) = e^{-iHt/\hbar}.$$

It is important to point out here that despite the existence of ‘trajectories’ in the theory, there are important differences from the Bohm theory [5]. The Bohm theory imposes an additional condition, namely the guidance condition on standard quantum mechanics to define trajectories in configuration space (the “hidden variables”). The trajectories in the $Q-\pi$ space are, on the other hand, consequences of the postulated new symmetry. Furthermore, the ‘guidance condition’ in the Bohm theory results in trajectories in configuration space whose initial distribution must be chosen to be identical with the quantum mechanical distribution (the “quantum equilibrium hypothesis”). The continuity equation then ensures that this identity is preserved in time. The trajectories in $Q-\pi$ space, on the other hand, are consequences of the commutation relations (26) which are preserved in time, and the distribution of the trajectories is automatically determined by the theory for all times.
The Measurement Problem

The postulated new symmetry as well as gauge covariance imply that \( \hat{Q} \) and \( \hat{\pi} \) rather than \( \hat{q} \) and \( \hat{p} \) must be regarded as the ‘observables’ for the particle. It follows from this that the trajectories of the particles in \( Q - \pi \) space can account for the occurrence of individual stochastic events in space-time. There is no need therefore for any additional hypothesis like an external observer or ‘collapse/reduction’ of the state vector. The theory is thus free of the so called ‘measurement problem’ in a manner analogous to Bohm’s theory \[6\].

One might suspect that the trajectories imply conflict with the Heisenberg uncertainty relations. This is not the case for the following reason. The standard commutation relations from which the uncertainty relations are believed to follow still hold for the canonical position \( q \) and momentum \( p \) of the particle \([24]\) and also for \( q \) and \( \pi \) \([24]\) and \( Q \) and \( p \) \([27]\). Although there is no such restriction on \( Q \) and \( \pi \) \([26]\) and a particle can have simultaneous sharp values of these variables, actual measurements of \( Q \) and \( \pi \) over an ensemble of trajectories will nevertheless exhibit the statistical ‘scatter relation’ \( \Delta Q \Delta \pi \geq \hbar / 2 \) because the trajectory distribution in the ensemble is determined by the density function \([32]\) at all times \([8]\).

5 Concluding Remarks

We have seen that the postulated new symmetry between the coordinates and momenta of particles coupled to an Abelian gauge field requires the field to be a dynamical connection in a non-flat space-time manifold. An attractive feature of this new symmetry is the natural occurrence of trajectories in \( Q - \pi \) space in the quantum theory, which frees the theory of the measurement problem characteristic of quantum mechanics in flat space-time. Since non-flat space-time is characteristic of gravity, it is tempting to speculate that a possible physical interpretation of the gauge potential \( B \) is gravito-electromagnetism (GEM) in which the canonical momentum of a particle is given by the relation \([11]\) with \( 2m \) as the “gravitational charge” and \( \chi = G \), the Newtonian gravitational constant. As shown by several authors, GEM is a consequence of splitting the general relativistic space-time manifold into space and time locally \([9, 10, 11, 12, 13]\). If this interpretation is indeed possible, it would be indicative of a deep connection between quantum theory and gravity, at least in its post-Newtonian linearized form. However, it
is premature to draw any conclusion about full quantum gravity from these considerations.

This paper is a sequel to “A Relativistic Particle and Gravitoelectromagnetism” and is an alternative approach to it in which the role of gravitomagnetism is subsidiary to the postulated new symmetry.

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