A numerical study of heat and mass transport by double-diffusive magnetoconvection in an electrically conducting fluid under sinusoidal/non-sinusoidal rotational modulation

Munyaradzi Rudziva | Precious Sibanda | Osman A. I. Noreldin | Sicelo P. Goqo

School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Pietermaritzburg, South Africa

Correspondence
Munyaradzi Rudziva, School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X01 Scottsville, Pietermaritzburg 3209, South Africa. Email: munyarudziva@gmail.com

Abstract
In this paper, we study the effects of rotational modulation on heat and mass transport due to double-diffusive magnetoconvection in an electrically conducting fluid. In the study, different modes of rotational modulation are considered. Using a perturbation method, a Ginzburg–Landau equation which is nonautonomous is obtained for the modulation amplitude. The effect of rotational modulation on the heat and mass transfer is studied. For the investigation of the effect of buoyancy ratio on the onset of convection, the system of Lorenz type equations is solved using the recent multistage spectral relaxation method. Furthermore, the influence of other fluid parameters on thermal convection and heat and mass transport is analyzed and presented graphically. We find among other results that the effect of increasing the Taylor number is to reduce the Nusselt and Sherwood numbers. A comparison between the sinusoidal and non-sinusoidal modes of rotational modulation on their influence on heat and mass transport in a
magneto-diffusive fluid layer is made. We show that the square wave rotational modulation is the most destabilizing type of modulation in the sense that it leads to the early onset of thermal instabilities while the triangular modulation is the most stabilizing type of modulation. The effect of the buoyancy ratio on the nature of the flow pattern has been determined using streamlines, isotherms, and isoconcentrations.

KEYWORDS
Ginzburg–Landau, magnetoconvection, multistage spectral relaxation, sinusoidal and non-sinusoidal modulation

1 | INTRODUCTION

Convective motions produced due to only temperature gradient in the fluid system heated from below are now well-understood. When more than one diffusive component with different gradients is present, a whole new set of phenomena can emerge, and empiricism based on simple thermal convection can be deceptive. For example, in many practical applications of interest, instabilities can develop even when the net density decreases upwards, and thus the system is hydrostatically stable in a single-component fluid. Diffusion, which has a stabilizing effect in a fluid containing solutes, can act in the case of a double or multiple components to discharge potential energy in the component that is heaviest at the top. The presence of two or more components with distinct molecular diffusivities in the fluid, as well as opposing contributions to the vertical density gradient, are two prerequisites for the existence of multicomponent-diffusive convection. Generally, the buoyant force in double-diffusive convection is affected by temperature and fluid concentration differences. According to Huppert and Turner and Stern, when the solute is added from below, the instability manifests as diffusive instabilities, whereas when the solute is added from above, the instability manifests as finger instabilities.

The study of rotating double-diffusive convection under the influence of an imposed vertical magnetic field is important in a variety of scientific and technology fields. It is important in chemical aspects of engineering and petroleum reservoir operations where the magnetic field is generated by the earth. Magnetoconvection occurs when an electrically conducting fluid interacts with a magnetic field. Chandrasekhar and Thompson were among the pioneers to investigate the magnetic effect in a fluid layer. Lortz discussed the stability criterion for steady finite-amplitude convection in the presence of an external magnetic field. Lortz’s work was motivated by mathematical criticism of the so-called “relative stability” criterion introduced by Malkus and Veronis. The instability criterion discussed by Malkus and Veronis was explained although the stability analysis lacked detail.

Magnetoconvection has been studied by researchers such as Alchaar et al., Bian et al., Borglin et al., Patil and Rudraiah, Sibanda and Noreldin, and many others. Rudraiah studied both linear and nonlinear magnetoconvection in a porous medium. He found among other results that the Chandrasekhar number has a significant influence on the stability of the system. Oreper and Szekely found that introducing a magnetic field could
inhibit mass transfer although total suppression is not an easy task to achieve. Srivastava et al.\textsuperscript{6} investigated the Soret influence on magnetoconvection in an anisotropic porous media considering the generalized Darcy model. Sharma\textsuperscript{28} studied the rotational and magnetic effects on the stability of a ferromagnetic fluid. Sharma concluded that magnetization acts as a destabilizing agent whereas rotation stabilizes the fluid system. Badhauria and Srivastava\textsuperscript{29} investigated thermal instability in an electrically conducting double-diffusive saturated-porous medium with time-dependent temperature. The influence of an external magnetic field on the stability of a viscoelastic layer fluid was investigated by Narayana et al.\textsuperscript{30} They concluded that the applied magnetic field stabilizes the fluid system.

The investigation of the influence of rotation in electrically conducting and nonconducting fluid layers has received significant attention due to its relevance in geophysical, oceanic flows, food processing, solidification, and centrifugal casting of metals. When a protein solution with a slow diffusion rate is layered on top of a sucrose solution with a higher diffusion rate, a doubly-diffusive instability occurs, as described by Brakke.\textsuperscript{31} Nason et al.\textsuperscript{32} showed that the instability described by Brakke can be reduced by rotation in the ultracentrifuge. The influence of uniform rotation on the stability of a binary fluid was studied numerically by Antoranz and Velarde.\textsuperscript{33} They concluded that thermal diffusion enhances diffusive and convective mass transport. Rudraiah et al.\textsuperscript{34} studied the influence of rotation on double-diffusive convection which they considered a non-Darcy equation. They concluded that rotation and porous parameters aid in reducing the instability region.

Mikhailenko et al. have performed some significant research into the effect of uniform rotation on thermal convection in closed systems, which can include internal heat generation. Recently, Mikhailenko et al.\textsuperscript{35} investigated the effects of uniform rotation on unrestricted convection in a closed porous layer with a local heat generation. They concluded that the mass, momentum, and heat transport are all suppressed as the Taylor number increases, whereas the heat transfer rate decreases as the porous layer thickness increases. Mukhailenko et al.\textsuperscript{36} went on further to investigate thermal convection within a rotating square chamber with a local isothermal element, convective nanofluid circulation, and thermal transmission. They considered a wide range of Rayleigh numbers, Taylor numbers, and nanoparticle concentrations. They discovered, among other results, that as the Taylor number increases, thermal transmission decreases, but moderately high Taylor numbers can significantly increase convective circulation. Concurrently, as the Rayleigh number increases, the Taylor number describing the transition between convective, and conduction modes increases. Furthermore, Mikhailenko and Sheremet\textsuperscript{37} investigated thermogravitational convective energy transfer in a uniformly rotating partially porous cavity with a local heat-generating source. The authors discovered that a weakly uniform rotation can enhance the heater’s cooling process, whereas an intensive rotation reduces energy transport and fluid motion. In consideration of these applications and findings, it is important to understand how rotation affects the stability of a fluid system.

There has been a surge of interest in externally modulated fluid systems over the past few decades. Venezian\textsuperscript{38} was among the first to study the influence of infinitesimal thermal modulation on the onset of instability in a fluid system. He observed that temperature modulation is capable of advancing or delaying the onset of convection in the system. Siddheshwar and Pranesh\textsuperscript{39,40} investigated the transverse magnetic field effect on thermal instability in an electrically conducting fluid with the inclusion of internal angular momentum under temperature and gravity modulation. They concluded that temperature modulation leads to the onset of subcritical motion while gravity modulation delays the onset of convection.
Bhadauria and Kiran\textsuperscript{41} presented a weakly nonlinear stability analysis of double-diffusive magnetoconvection in a viscoelastic fluid layer. Bhadauria and Sherani\textsuperscript{42} investigated how thermal modulation affects the onset of magnetoconvection in a saturated porous medium. They found that an externally applied magnetic field stabilizes the system and modulates the temperature and the gravity field significantly affects the rate of heat and mass transport. Maleshetty and Swamy\textsuperscript{43} presented the impact of temperature modulation and rotation on the onset of thermal convection in a porous layer concerning the concept of stability exchanges. They concluded that at low frequency in-phase and lower-wall modulation there is an advancement of convection whereas for out-of-phase modulation convection is delayed significantly. Siddheshwar et al.\textsuperscript{44} investigated magnetoconvection in a Newtonian fluid exposed to time-periodic boundary temperature and gravity modulation.

Many studies have been conducted in recent decades on turbulent convection in Rayleigh–Bernard configurations rotating at a constant rate about the vertical axes. There are many scenarios of practical significance in science and engineering in which rotation is a function of space and time. The effect of modulating the rotation plays important role in the heat and mass transport concepts. In their recent study, Rudziva et al.\textsuperscript{45} investigated only sinusoidal rotational modulation on Darcy porous medium saturated with a couple stress fluid. They did not consider the effects of non-sinusoidal rotational modulation on magnetoconvection. Therefore, we extend previous research by investigating the effects of time-varying rotational rates on heat and mass transport in an electrically conducting fluid.

As sinusoidal and non-sinusoidal wave modulations are possible in a variety of industrial processes, in this study we focus on the question of how the nature and type of rotation influence heat and mass transfer rates in the fluid system. We have been motivated by the recent work of Siddheshwar and Kanchana\textsuperscript{46} where they pioneered the investigations of the influence of different modes of gravitational modulation in Newtonian liquids and nanoliquids. Their investigation was the first such attempt to theoretically explain the effects of different wave-type modulations. They revealed that all modes of gravitational modulation affect the rate of heat and mass transfer. Other recent investigations on different modes of gravity modulation are found in Meghana et al.\textsuperscript{47} and Pranesh et al.\textsuperscript{48} Rudziva et al.\textsuperscript{49} made a comparison between sinusoidal and non-sinusoidal rotational modulations in triple diffusive convection. They concluded that non-sinusoidal modulations affect the thermal stability of the fluid system.

It can be noted from the mentioned literature that the interaction between the magnetic field, nonuniform rotation and solute concentration where the magnetic field acts as a third diffusing component has received little attention. Many previous double-diffusive magnetoconvection studies assumed uniform temperature and solute gradients and rotation. We may consider rotation as a function of both space and time. For instance, in crystal growth of metallic alloys, the time-dependent Coriolis force influences heat and mass transport, hence the alloy's structure and quality. Many previous studies focused solely on sinusoidal trigonometric sine/cosine rotational oscillations. In the study by Rudziva et al.,\textsuperscript{49} the comparison of different modes of rotational modulation did not include the influence of an imposed vertical magnetic field.

There are many geophysics processes in which fluid flow is affected by buoyancy due to thermal and concentration gradients, for example, evaporation from a lake and cooling of a rotating airstream. Mixed convection due to the combined buoyancy forces has received considerably limited attention. Lin et al.\textsuperscript{50} studied the influence of vaporization of a thin liquid layer on the tube wall in laminar convection flows with thermal and mass diffusion buoyancy effects. Orfi et al.\textsuperscript{51} investigated the buoyancy effects in lamina flow in a horizontal pipe. Their
study proved that the buoyancy ratio has an effect on heat and mass transfer. Therefore, it is important to study the effects of thermal and solutal buoyancy forces on the onset of convection in rotating double-diffusive magnetoconvection.

According to the presented literature and to the best of our knowledge, there have been no studies on the use of a multistage spectral relation method to analyze the influence of sinusoidal and non-sinusoidal rotational modulations on thermal convection in a double-diffusive electrically conducting fluid. We believe this study is noteworthy and its applicability to various engineering and scientific systems, as discussed above has motivated us. Therefore, the objective of this study is to find out how different modes of rotational modulation affect the rate of heat and mass transport in an electrically conducting Newtonian liquid with free–free boundaries. We give a comparison between sinusoidal and non-sinusoidal modulations on a system with a vertically imposed magnetic field. We solve the Lorenz equations using the multistage spectral relaxation method (MSRM)\(^{52-54}\) which was found to be efficient, consistent, and user friendly.

2 | MATHEMATICAL FORMULATION

An electrically conducting fluid layer situated between two free surfaces is considered. We use cartesian coordinates with the fluid confined between infinitely extended planes at \(z = 0\) and \(z = d\). A uniform vertical magnetic field is maintained across the fluid system. The fluid layer is considered to be rotating about the \(z\)-axis with a nonuniform rotational speed \( \Omega(t) \hat{k} \) as shown in Figure 1.

The governing equations considered in this study are\(^{20,44,55}\)

\[
\begin{align*}
\nabla \times \vec{u} &= 0, \\
\nabla \times \vec{H} &= 0, \\
\frac{\partial \vec{u}}{\partial t} + \vec{u} \times \vec{u} - 2\vec{\Omega} \times \vec{u} - \frac{\mu_m H}{\rho_0} \nabla \vec{H} &= \frac{1}{\rho_0} \nabla P + \frac{\rho g}{\rho_0} + \frac{\mu}{\rho_0} \nabla^2 \vec{u},
\end{align*}
\]

\[\text{FIGURE 1} \quad \text{Geometry of the problem.}\]
\[
\frac{\partial T}{\partial t} + \vec{u} \times \nabla T = \kappa_f \nabla^2 T, \quad (4)
\]
\[
\frac{\partial S}{\partial t} + (\vec{u} \times \nabla) S = \kappa_S \nabla^2 S, \quad (5)
\]
\[
\frac{\partial \vec{H}}{\partial t} + \vec{u} \times \nabla \vec{H} - \nabla \times \vec{H} = \mu \nabla^2 \vec{H}, \quad (6)
\]
\[
\rho = \rho_0 [1 - \alpha(T - T_0) + \beta(S - S_0)], \quad (7)
\]

where \(\vec{u} = (u, v, w)\) is the velocity and \(\vec{H} = (H_x, H_y, H_z)\) is the magnetic field intensity. Other variables and parameters are listed in the nomenclature with their usual meanings.

The fluid layer is kept at a uniform temperature gradient \(\Delta T/d\), with the thermal boundary conditions

\[
T = T_0 + \Delta T \quad \text{at} \quad z = 0,
\]
\[
T = T_0 \quad \text{at} \quad z = d. \quad (8)
\]

For concentration, we consider the solute concentration boundary conditions

\[
S = S_0 + \Delta S, \quad \text{at} \quad z = 0,
\]
\[
S = S_0, \quad \text{at} \quad z = d. \quad (9)
\]

The time-varying rotational modulation term is given as

\[
\vec{\Omega} = \Omega_0 [1 + \epsilon^2 \delta_1 f] \hat{k},
\]

where \(\delta_1\) denotes the modulation amplitude, \(\epsilon\) is a small perturbation parameter, and \(f\) to be defined at a later stage.

The basic state of the fluid system is quiescent and is given by

\[
\vec{u} = (0, 0, 0), \quad \rho = \rho_b(z), \quad P = P_b(z), \quad S = S_b(z),
\]
\[
T = T_b(z), \quad \vec{H} = \vec{H}_b. \quad (10)
\]

Using (10) in Equations (1)–(7), gives

\[
\frac{d^2 T_b}{dz^2} = 0, \quad \frac{d^2 S_b}{dz^2} = 0, \quad \frac{dP_b}{dz} = -\rho_b g, \quad (11)
\]
\[
\rho_b = \rho_0 [1 - \alpha(T_b - T_0) + \beta(S_b - S_0)]. \quad (12)
\]
Solving Equations (11) and (12), the solutions for the basic temperature and solute concentration are

\[ T_b = T_0 + \Delta T \left(1 - \frac{z}{d}\right), \]
\[ S_b = S_0 + \Delta S \left(1 - \frac{z}{d}\right). \]

To investigate the behavior of infinitesimal disturbances, we perturb the basic state as

\[ \vec{u} = \vec{u}', (u', v', w'), T = T_b + T', S = S_b + S', P = P_b + p', \]
\[ \rho = \rho_b + \rho', \hat{H} = \hat{H}_b + \hat{h}', \]

where \( \hat{h}' \) is the magnetic field perturbation generated by the motions and induced currents in the field. The perturbed quantities \( \vec{u}', \phi', \rho', \) and \( \rho' \) are infinitesimally small. The fluid’s magnetic permeability and thermal conductivity are assumed to be constant with operating conditions such as temperature, concentration, and pressure. Substituting (15) into (1)–(7) and using basic state solutions we obtain

\[ \nabla \times \vec{u}' = 0, \]
\[ \nabla \times \vec{H}' = 0, \]
\[ \frac{\partial \vec{u}'}{\partial t} + \vec{u}' \times \nabla \vec{u}' - 2\Omega \times \vec{u}' - \frac{\mu_m \vec{h}'}{\rho_0} \times \nabla \vec{h}' = -\frac{1}{\rho_0} \nabla P' + \frac{\rho'_g}{\rho_0} + \nu \nabla^2 \vec{u}', \]
\[ + \frac{\mu_m}{\rho_0} \vec{H}_b \times \frac{\partial \vec{h}'}{\partial z}, \]

\[ \frac{\partial T'}{\partial t} + \vec{u}' \times \nabla T' + w' \frac{\partial T_b}{\partial z} = \kappa_T \nabla^2 T', \]
\[ \frac{\partial S'}{\partial t} + (\vec{u}' \times \nabla) S' + w' \frac{\partial S_b}{\partial z} = \kappa_S \nabla^2 S', \]
\[ \frac{\partial \vec{h}'}{\partial t} + \vec{u} \times \nabla \vec{h}' - \vec{h}' \times \nabla \vec{u}' - \vec{H}_b \frac{\partial w'}{\partial z} = \mu \nabla^2 \vec{h}', \]
\[ \rho' = -\rho_0 (\alpha T' - \beta S'). \]
Two-dimensional disturbances are considered where stream function $\psi$ and magnetic potential $\phi$ are defined as

$$(u', w') = \left( \frac{\partial \psi}{\partial z}, -\frac{\partial \psi}{\partial x} \right), (h'_x, h'_z) = \left( \frac{\partial \phi}{\partial z}, -\frac{\partial \phi}{\partial x} \right).$$  \hspace{1cm} (23)$$

We can eliminate the density term from Equations (18)–(22) and the pressure term can be eliminated by taking the curl of the resulting equation. The equations are nondimensionalized using

$$(x', y', z') = d(x^*, y^*, z^*), t = \frac{d^2}{\kappa_T} t^*, T_b = \Delta T \times T_b^*, T' = \Delta T \times T^*,$$

$$S_b = \Delta S \times S_b^*, S' = \Delta S \times S^*, \vec{u}' = \frac{\kappa_T}{d} \vec{u}^*, \psi' = \kappa_T \psi^*, \phi' = dH_b \phi^*,$$

$$\xi = \frac{\kappa_T}{d} \xi^*, \zeta = H_b \zeta^*.$$  

Dropping the asterisk for convenience, we get the nondimensionalized system of equations

$$- \nabla^4 \psi - \sqrt{Ta} (1 + \epsilon \hat{f}) \frac{\partial \xi}{\partial z} + Ra_f \frac{\partial T}{\partial x} - Ra_s \frac{\partial S}{\partial x} - QPm \frac{\partial \nabla^2 \phi}{\partial z} = -\frac{1}{Pr} \frac{\partial \nabla^2 \psi}{\partial t} + \frac{1}{Pr} \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, z)} - QPm \frac{\partial (\phi, \nabla^2 \phi)}{\partial (x, z)},$$  \hspace{1cm} (24)$$

$$- \nabla^2 \xi + \sqrt{Ta} (1 + \epsilon \hat{f}) \frac{\partial \psi}{\partial z} - QPm \frac{\partial \xi}{\partial z} = -\frac{1}{Pr} \frac{\partial \xi}{\partial t} + \frac{1}{Pr} \frac{\partial (\psi, \xi)}{\partial (x, z)} - QPm \frac{\partial (\phi, \xi)}{\partial (x, z)},$$  \hspace{1cm} (25)$$

$$\frac{\partial \psi}{\partial x} - \nabla^2 T = -\frac{\partial T}{\partial t} + \frac{\partial (\psi, T)}{\partial (x, z)},$$  \hspace{1cm} (26)$$

$$\frac{\partial \psi}{\partial x} - \frac{1}{Le} \nabla^2 S = -\frac{\partial S}{\partial t} + \frac{\partial (\psi, S)}{\partial (x, z)},$$  \hspace{1cm} (27)$$

$$-\frac{\partial \psi}{\partial z} - Pm \nabla^2 \phi = -\frac{\partial \phi}{\partial t} + \frac{\partial (\psi, \phi)}{\partial (x, z)},$$  \hspace{1cm} (28)$$

$$-\frac{\partial \xi}{\partial z} - Pm \nabla^2 \zeta = -\frac{\partial \xi}{\partial t} + \frac{\partial (\psi, \xi)}{\partial (x, z)},$$  \hspace{1cm} (29)$$

where the $z$-component vorticity and current density are $\xi$ and $\zeta$, respectively. The nondimensional parameters are $Ta = \left( \frac{2 \Omega d^2}{\nu} \right)^2$ the Taylor number, $Q = \frac{\mu_m H_b^2 d^2}{\rho_0 \nu \gamma m}$ the Chandrasekhar number, $Pm = \frac{\nu_0}{\kappa_T}$ the magnetic Prandtl number, $Pr = \frac{\nu}{\kappa_T}$ the Prandtl number, $Le = \frac{\kappa_T}{\kappa_S}$ the Lewis number, $Ra_f = \frac{\alpha \Delta T \nu_g}{\kappa_T \nu}$ the thermal Rayleigh number, and $Ra_s = \frac{\alpha S \nu_g}{\kappa_S \nu}$ the solutal Rayleigh number.

To analyze the stationary convection, we write the nonlinear equations (24)–(29) in matrix form, assuming small variations in time and rescaling using $\tau = \epsilon^2 t$ to get
The stress-free plane surfaces are assumed to be ideal heat and salt conductors with boundary conditions

\[ \psi = T = S = \zeta = \frac{\partial \psi}{\partial z} = \frac{\partial \xi}{\partial z} = \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = 0, 1. \]  

(31)

3 | LINEAR STABILITY ANALYSIS

For our study to be self-contained, we provide a linear stability analysis for the modulated system. Stationary and oscillatory convection thresholds are predicted using linear theory where nonlinear terms in Equations (24)–(29) are neglected. The linear equations are

\[
\begin{bmatrix}
-\nabla^4 & -\sqrt{Ta} \frac{\partial}{\partial z} & Ra_f \frac{\partial}{\partial x} & -Ra_s \frac{\partial}{\partial x} & -QPM \frac{\partial \nabla^2}{\partial z} & 0 \\
\sqrt{Ta} \frac{\partial}{\partial z} & -\nabla^2 & 0 & 0 & 0 & QPM \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & 0 & -\nabla^2 & 0 & 0 & 0 \\
\frac{\partial}{\partial x} & 0 & 0 & -\frac{1}{Le} \nabla^2 & 0 & 0 \\
-\frac{\partial}{\partial z} & 0 & 0 & 0 & -Pm\nabla^2 & 0 \\
0 & -\frac{\partial}{\partial z} & 0 & 0 & 0 & -Pm\nabla^2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{\epsilon^2}{Pr} \frac{\partial \nabla^2 \psi}{\partial \tau} + \frac{\epsilon^2}{Pr} \sqrt{Ta} f \frac{\partial \xi}{\partial \tau} + \frac{1}{Pr} \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, z)} - QPM \frac{\partial (\phi, \nabla^2 \phi)}{\partial (x, z)} \\
-\frac{\epsilon^2}{Pr} \frac{\partial \xi}{\partial \tau} - \frac{\epsilon^2}{Pr} \sqrt{Ta} f \frac{\partial \psi}{\partial \tau} + \frac{1}{Pr} \frac{\partial (\psi, \xi)}{\partial (x, z)} - QPM \frac{\partial (\phi, \xi)}{\partial (x, z)} \\
-\frac{\epsilon^2}{Pr} \frac{\partial T}{\partial \tau} + \frac{\partial (\psi, T)}{\partial (x, z)} \\
-\frac{\epsilon^2}{Pr} \frac{\partial S}{\partial \tau} + \frac{\partial (\psi, S)}{\partial (x, z)} \\
-\frac{\epsilon^2}{Pr} \frac{\partial \phi}{\partial \tau} + \frac{\partial (\psi, \phi)}{\partial (x, z)} \\
-\frac{\epsilon^2}{Pr} \frac{\partial \zeta}{\partial \tau} + \frac{\partial (\psi, \zeta)}{\partial (x, z)}
\end{bmatrix}
\]

\[
\begin{align*}
\left( \frac{1}{Pr} \frac{\partial}{\partial \tau} - \nabla^2 \right) \nabla^2 \psi - \sqrt{Ta} (1 + ef) \frac{\partial \xi}{\partial z} & = -Ra_f \frac{\partial T}{\partial x} + Ra_s \frac{\partial S}{\partial x} + QPM \frac{\partial \nabla^2 \phi}{\partial z}, \\
\left( \frac{1}{Pr} \frac{\partial}{\partial \tau} - \nabla^2 \right) \xi & = QPM \frac{\partial \xi}{\partial z} - \sqrt{Ta} (1 + ef) \frac{\partial \psi}{\partial z},
\end{align*}
\]

(30)
Eliminating $\xi, T, S, \phi,$ and $\zeta$ from Equations (32)–(37), to obtain a single differential equation for $\psi$ as

\[
\left(\frac{\partial}{\partial t} - \nabla^2\right) T = -\frac{\partial \psi}{\partial x}, \quad (34)
\]

\[
\left(\frac{\partial}{\partial t} - \frac{1}{Le} \nabla^2\right) S = -\frac{\partial \psi}{\partial x}, \quad (35)
\]

\[
\left(\frac{\partial}{\partial t} - Pm \nabla^2\right) \phi = \frac{\partial \psi}{\partial z}, \quad (36)
\]

\[
\left(\frac{\partial}{\partial t} - Pm \nabla^2\right) \zeta = \frac{\partial \xi}{\partial z}, \quad (37)
\]

The boundary conditions (31) together with Equations (33)–(37) gives the following conditions in terms of $\psi$ as

\[
\psi = \frac{\partial^2 \psi}{\partial z^2} = 0 \text{ at } z = 0, 1. \quad (39)
\]

3.1 Method of solution

Now we seek the eigenfunctions $\psi$ and eigenvalues $Ra_T$ of Equation (38). The Rayleigh number is obtained using the perturbation technique where the stream function and thermal Rayleigh number are expanded as below

\[
\psi = \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \cdots \quad (40)
\]
Substituting Equations (40) and (41) into (38) and equating the coefficients of different powers of $\epsilon$, we obtain the following system of equations up to the order of $\epsilon^2$

$$L\psi_0 = 0,$$

$$L\psi_1 = -2TaL_2L_3L_4^2f\frac{\partial^2\psi_0}{\partial z^2} + L_3L_4L_5Ra_{T_1}\frac{\partial^2\psi_0}{\partial x^2},$$

$$L\psi_2 = -TaL_2L_3L_4^2\left(2f\frac{\partial^2\psi_1}{\partial z^2} + f^2\frac{\partial^2\psi_0}{\partial z^2}\right) + L_5L_3L_4\left(Ra_{T_2}\frac{\partial^2\psi_0}{\partial x^2} + Ra_{T_1}\frac{\partial^2\psi_1}{\partial x^2}\right),$$

where

$$L = L_1L_2L_3L_4L_5\nabla^2 + TaL_2L_3L_4^2\frac{\partial^2}{\partial z^2} - L_3L_4L_5Ra_{T_0}\frac{\partial^2}{\partial x^2} + L_2L_4L_5Ra_S\frac{\partial^2}{\partial x^2} - QPMl_2L_3L_5\frac{\partial^2\nabla^2}{\partial z^2},$$

with

$$L_1 = \left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right), L_2 = \left(\frac{\partial}{\partial t} - \nabla^2\right), L_3 = \left(\frac{\partial}{\partial t} - \frac{1}{Le}\nabla^2\right), L_4 = \left(\frac{\partial}{\partial t} - Pm\nabla^2\right), L_5 = \left(L_1L_4 - QPM\frac{\partial^2}{\partial z^2}\right),$$

and $\psi_0, \psi_1, \psi_2$ must satisfy the boundary conditions of Equation (39). It is observed that Equation (42) is the one used in the study for the double-diffusive magnetoconvection in an electrically conducting fluid with uniform rotation.

We now assume the marginally stable solutions for Equation (42) as $\psi_0 = \sin\alpha x \sin\pi z$, where $\alpha$ is the wavenumber. The Rayleigh number for stationary convection is given as

$$Ra_{T_0} = \frac{\delta^2(\delta^4 + Q\pi^2) + Ra_S Le\alpha^2}{\alpha^2} + \frac{Ta\pi^2\delta^4}{\alpha^2(\delta^4 + Q\pi^2)},$$

where $\delta^2 = \pi^2 + \alpha^2$. This is the same result obtained by Bhadauria et al.\textsuperscript{55,56} when $Ta = 0$. Furthermore, for $Ta = Ra_S = 0$, we obtain the classical result as in Chandrasekhar\textsuperscript{8} and Siddheshwar et al.\textsuperscript{44}

$Ra_{T_0}$ given by Equation (47) attains its critical value $Ra_{T_0}^{c}$ at critical wavenumber $\alpha^2 = \alpha_c^2$, where $\alpha$ satisfies the following equation:

$$\alpha_c^4\left(2\alpha_c^2 + 3\pi^2\right) - \pi^4(\pi^2 + Q) + \pi^2Ta\left(\alpha_c^4 - \pi^4\right) = 0.$$  

\(48\)
Rotation and magnetic field are observed to have an effect on the critical wavenumber, while additional diffusing component has no significant effect on the critical wavenumber. We note that when rotation and magnetic field are not considered, we have

\[ 2\alpha_c^2 + 3\alpha_c^4 \pi^2 - \pi^6 = 0, \tag{49} \]

which gives \( \alpha_c = \frac{\pi}{\sqrt{2}} \) and \( Ra_{c0}^e = 657.5 \) which is the classical result given by Chandrasekhar\(^8\) and Drazin.\(^57\)

Table 1 validates the results of this study to those of Chandrasekhar’s earlier works for classical Rayleigh–Bernard convection in the presence of rotation and magnetic field. The table shows that adding rotation and a magnetic field increases the critical Rayleigh number hence delaying the onset of convection in the system. It is clear from the table that the results predicted in the present study are in good agreement with the earlier works of Chandrasekhar.\(^8\) We now seek to obtain the correction Rayleigh number. We use \( \psi_0 = \sin \alpha x \sin \pi z \) as the starting point of our other solutions.

The equation for \( \psi_1 \) takes the form

\[ L\psi_1 = 2\pi^2 TaL_2 L_3 L_4^2 f\psi_0 - \alpha^2 L_3 L_4 L_5 Ra_{c1} \psi_0, \tag{50} \]

It is observed that Equation (50) is inhomogeneous and includes a resonance term. If the solution of this equation is to exist, the right-hand side must be orthogonal to the null space of the operator \( L \), and this implies \( Ra_{c1} = 0 \) due to the solvability condition. It follows that all the odd coefficients \( Ra_{c1}, Ra_{c3}, Ra_{c5}, \ldots \) in Equation (41) must vanish. For further simplification, we have chosen the trigonometric cosine waveform to explain the linear stability analysis. Therefore, Equation (50) becomes

\[ L\psi_1 = 2\pi^2 TaL_2 L_3 L_4^2 Re\{e^{-i\omega t}\} \psi_0, \tag{51} \]

where \( Re \) stands for the real part. We solve Equation (51) for \( \psi_1 \) by inverting the operator \( L \) term by term. For convenience, we define

| Ta | \( Q \) | Chandrasekhar\(^8\) | Present problem |
|----|------|-----------------|----------------|
| 0  | 0    | 2.233 657.5     | 2.233 657.5    |
| 10 | 2.590| 923.070 2.590   | 923.070 2.590  |
| 100| 3.702| 2653.7 3.702    | 2653.7 3.702   |
| 10 | 2.270| 677.1 2.271     | 677.1 2.271    |
| 10 | 3.76 | 2886.0 3.68     | 2886.2 3.68    |
| 100| 5.66 | 15,080.0 5.66   | 15,080.0 5.66  |

TABLE 1 Validation of results of the present study with that of Chandrasekhar\(^8\) for the effect of rotation on magnetoconvection.
\[ L(\omega, n) = \Delta_1 + i\Delta_2, \] (52)

where

\[ \Delta_1 = \frac{\delta^2 \omega^6}{Pr^2} + \frac{\omega^4}{Pr} \{ \alpha^2(Ra_{T0} - Ra_s) - 2\pi^2 \delta^2 QPm - \lambda_3 \delta^6 \} \]

\[ - \omega^2 \{ \delta^{10} \lambda_4 - \delta^6 \lambda_6 \pi^2 QPm + \alpha^2 QPm(\delta^4 + Q\pi^2)(Ra_{T0} - Ra_s) \] \[ + \alpha^2 \delta^2 \left[ \frac{1}{Le} + Pm \left( 1 + \frac{2 + Le}{PrLe} \right) \right] Ra_{T0} - \alpha^2 \delta^2 \left[ 1 + Pm + \frac{Pm(2 + Pr)}{Pr} \right] Ra_s \] \[ - \delta^2 \left[ \frac{1}{Le} + \left( 1 + \frac{2 + Le}{PrLe} \right) \right] + (\delta^4 + Q\pi^2) \left[ QPm \pi^2 \delta^2 \right] \} \]

\[ + \frac{Pm}{Le} \left\{ \delta^{10} (\delta^4 - QPm \pi^2) + Pm^2 \delta^4 \alpha^2 (\delta^4 + Q\pi^2) Ra_{T0} \right\} - Pm \delta^4 \alpha^2 (\delta^4 + Q\pi^2) Ra_s \]

\[ - \frac{\delta^6 QPm \pi^2}{Le} (\delta^4 + Q\pi^2), \] (53)

\[ \Delta_2 = \delta^4 \lambda_1 \omega^5 + \omega^3 \left\{ \delta^4 (\delta^4 \lambda_3 - QPm \pi^2 \lambda_5) + \delta^2 \alpha^2 \left[ 1 + \frac{2PmLe + 1}{PrLe} \right] Ra_{T0} \] \[ - \delta^2 \alpha^2 \left[ 1 + \frac{2Pm + 1}{Pr} \right] Ra_s - QPm \pi^2 \delta^4 \left[ 1 + \frac{Le + 1}{PrLe} \right] \} \]

\[ - \omega \left\{ \delta^8 (\delta^4 \lambda_5 - QPm \pi^2 \lambda_3) + \delta^2 \left[ \frac{Pm(LePm + 1)(\delta^4 + Q\pi^4)}{Le} + \frac{(Pr + Pm)Pm \delta^4}{PrLe} \right] \right\} \]

\[ \times \alpha^2 Ra_{T0} - \delta^2 \left[ Pm(1 + Pm)(\delta^4 + Q\pi^4) + \frac{(Pr + Pm)Pm \delta^4}{Pr} \right] \frac{\alpha^2 Ra_s}{PrLe} \]

\[ - \delta^4 QPm \pi^2 \left\{ \left( 1 + Le \right)(\delta^4 + Q\pi^4) + \frac{\delta^4 (1 + Pr)}{PrLe} \right\}. \] (54)

where

\[ \lambda_1 = \frac{1}{Pr} \left\{ 1 + \frac{1}{PrLe} (1 + (1 + Pr + 2Pm)Le) \right\}, \] (55)

\[ \lambda_2 = \frac{1}{Pr} \left\{ \left( 2 \frac{1}{Pr} \right) Pm - 1 + \left( 3 + 2Pm + Pr + \frac{2Pm - 1}{Pr} \right) \frac{1}{Le} \right\}, \] (56)

\[ \lambda_3 = \frac{1}{Pr} \left\{ Pr + Pm \left[ 2 \frac{1}{Pr} \right] \left( 1 + \frac{1}{Le} \right) \right\} \] \[ - \frac{1}{Le} \left\{ Pr + 2Pm + \frac{2Pm}{Pr} \right\}, \] (57)

\[ \lambda_4 = Pm \left\{ 1 - \left( \frac{1}{Pr} + 2 \right) \frac{1}{Le} + 1 \right\} + \frac{1}{Pr} \left\{ 1 - \left( \frac{1}{Pr} + 1 \right) \frac{1}{Le} \right\} \] \[ - \left( \frac{2Pm}{Pr} + 1 \right) \frac{1}{Le}, \] (58)
\[ \lambda_5 = P_m \left( 1 - \frac{1}{Le} \right) \]  

(59)

\[ \lambda_6 = 1 + P_m + \frac{1 + P_m(1 + Le) + Pr}{Pr Le}, \]  

(60)

\[ \lambda_7 = 1 + \frac{1 + Le(1 + P_m)}{Pr Le}, \]  

(61)

\[ \lambda_8 = P_m + \frac{1}{Le} \left( 1 + P_m + \frac{P_m}{Pr} \right), \]  

(62)

It is easily observed that

\[ Le^{-i\omega t} \sin n\pi \zeta = L(\omega, n)e^{-i\omega t} \sin n\pi \zeta \]  

(63)

so that (51) becomes

\[ \psi_1 = 2\pi^2 Ta L_2 L_3 \Lambda_4^2 \Re \left\{ \frac{e^{-i\omega t}}{L(\omega)} \right\} \psi_0. \]  

(64)

We need

\[ L_2 L_3 \Lambda_4^2 f \psi_0 = \Re \{\Lambda\} \psi_0, \]  

(65)

where

\[ \Lambda = \Lambda_1 + i \Lambda_2, \]

and

\[ \Lambda_1 = \omega^4 - \left[ P_m(P_m + 2) + \frac{1}{Le}(2P_m + 1) \right] \omega^2 \delta_4^4 + \frac{P_m^2}{Le} \delta_8^8, \]

\[ \Lambda_2 = \left( 1 + 2P_m + \frac{1}{Le} \right) \pi^2 \omega^3 - \frac{P_m}{Le}(P_m(1 - Le) - 2) \pi^6 \omega. \]

Substituting Equation (65) in Equation (64) we get

\[ \psi_1 = 2\pi^2 Ta \Re \left\{ \frac{\Lambda e^{-i\omega t}}{L(\omega)} \right\} \psi_0. \]  

(66)

Now, the equation for \( \psi_2 \) becomes

\[ L\psi_2 = Ta\pi^2 L_2 L_3 \Lambda_4^2 \Re \left\{ \frac{\pi^2 Ta\Lambda}{L(\omega)} + 1 \right\} \psi_0 - \alpha^2 Ra_T L_5 L_3 \Lambda_4 \psi_0. \]  

(67)
We use Equation (67) to determine $Ra_{T2}$ the first nonzero correction to $Ra_{T}$. The solubility condition requires that the steady part of the right-hand side should be orthogonal to $\sin \pi z$. Therefore, to simplify Equation (67) for $\psi_2$, we need

$$L_5 L_3 L_4^2 \psi_0 = \frac{\delta^4 Pm^2 (\delta^4 + Q \pi^2)}{Le} \psi_0,$$

(68)

$$L_2 L_3 L_4^2 f^2 \psi_0 = Re \{ \Gamma \} f^2 \psi_0,$$

(69)

where

$$\Gamma = \Gamma_1 + i \Gamma_2,$$

and

$$\Gamma_1 = 16 \omega^4 - 4 \left[ Pm (Pm + 2) + \frac{1}{Le} (2Pm + 1) \right] \omega^2 \delta^4 + \frac{Pm^2}{Le} \delta^8,$$

$$\Gamma_2 = 8 \left( 1 + 2Pm + \frac{1}{Le} \right) \pi^2 \omega^3 - 2 \frac{Pm}{Le} (Pm (1 - Le) - 2) \pi^6 \omega.$$

Substituting Equations (68) and (70) in Equation (67) we get

$$L \psi_2 = \frac{\pi^2 T a \Gamma_1 \left( 4 \pi^2 T a (\Lambda_1 \Delta_1 + \Lambda_2 \Delta_2) + \left( \Delta_1^2 + \Delta_2^2 \right) \right)}{\left( \Delta_1^2 + \Delta_2^2 \right)} f^2 \psi_0 - Ra_{T2} \frac{\alpha^2 \delta^4 Pm^2 (\delta^4 + Q \pi^2)}{Le} \psi_0.$$

(70)

We do not need to solve this equation, but rather use it to calculate $Ra_{T2}$, the first nonzero correction to $Ra_{T0}$. The solvability condition requires that the time-independent part of the right-hand side of Equation (68) be orthogonal to $\sin \pi z$, therefore,

$$Ra_{T2} = \frac{2 \pi^2 T a \Gamma_1 Le \left( 4 \pi^2 T a (\Lambda_1 \Delta_1 + \Lambda_2 \Delta_2) + \left( \Delta_1^2 + \Delta_2^2 \right) \right)}{\alpha^2 \delta^4 Pm^2 (\delta^4 + Q \pi^2) \left( \Delta_1^2 + \Delta_2^2 \right)} \int_0^1 \bar{f}^2 \sin^2 \pi z dz,$$

(71)

where the bar denotes a time average. Finally, Equation (71) yields

$$Ra_{T2} = \frac{2 \pi^2 T a \Gamma_1 Le \left( 4 \pi^2 T a (\Lambda_1 \Delta_1 + \Lambda_2 \Delta_2) + \left( \Delta_1^2 + \Delta_2^2 \right) \right)}{2 \alpha^2 \delta^4 Pm^2 (\delta^4 + Q \pi^2) \left( \Delta_1^2 + \Delta_2^2 \right)}.$$

(72)

4  |  WEAKLY NONLINEAR STABILITY ANALYSIS

Heat and mass transport in the system are determined using a weakly nonlinear analysis.
4.1 Nonautonomous Ginzburg–Landau equation

The nonlinear stability analysis uses a conventional asymptotic perturbation method. The thermal Rayleigh number, the current density, the solute Rayleigh number, the vorticity, and the stream function are expanded in a power series in terms of \( \epsilon \ll 1 \)^{11,29,56}

\[
Ra_T = Ra_{T_0} + \epsilon^2 Ra_{T_2} + \epsilon^4 Ra_{T_4} + ..., \tag{73}
\]

\[
\psi = \epsilon \psi_1 + \epsilon^2 \psi_2 + \epsilon^3 \psi_3 + ..., \tag{74}
\]

\[
\xi = \epsilon \xi_1 + \epsilon^2 \xi_2 + \epsilon^3 \xi_3 + ..., \tag{75}
\]

\[
T = \epsilon T_1 + \epsilon^2 T_2 + \epsilon^3 T_3 + ..., \tag{76}
\]

\[
S = \epsilon S_1 + \epsilon^2 S_2 + \epsilon^3 S_3 + ..., \tag{77}
\]

\[
\phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \epsilon^3 \phi_3 + ..., \tag{78}
\]

\[
\zeta = \epsilon \zeta_1 + \epsilon^2 \zeta_2 + \epsilon^3 \zeta_3 + ..., \tag{79}
\]

where \( Ra_{T_0} \) is the critical Rayleigh number. We now solve the system for various orders of \( \epsilon \).

At \( O(\epsilon) \), we have the system

\[
\begin{bmatrix}
-\nabla^4 & -\sqrt{Ta} \frac{\partial}{\partial z} & Ra_{T_0} \frac{\partial}{\partial x} & -Ra_s \frac{\partial}{\partial x} & -QPM \frac{\partial \nabla^2}{\partial z} & 0 \\
\sqrt{Ta} \frac{\partial}{\partial z} & -\nabla^2 & 0 & 0 & 0 & QPM \frac{\partial}{\partial z} \\
0 & 0 & -\nabla^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{Le} \nabla^2 & 0 & 0 \\
0 & 0 & 0 & 0 & -Pm \nabla^2 & 0 \\
0 & -\frac{\partial}{\partial z} & 0 & 0 & 0 & -Pm \nabla^2 \\
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\xi_1 \\
T_1 \\
S_1 \\
\phi_1 \\
\zeta_1 \\
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}, \tag{80}
\]

The solutions subject to the boundary conditions (31) are

\[
\psi_1 = A(\tau) \sin \alpha_c x \sin \pi z, \tag{81}
\]

\[
\xi_1 = -\frac{\pi \delta^2 \sqrt{Ta}}{Q \pi^2 + \delta^4} A(\tau) \sin \alpha_c x \cos \pi z, \tag{82}
\]

\[
T_1 = -\frac{\alpha_c}{\delta^2} A(\tau) \cos \alpha_c x \sin \pi z, \tag{83}
\]
\[ S_1 = -\frac{Le\alpha_c}{\delta^2} A(\tau) \cos \alpha_c x \sin \pi z, \quad (84) \]
\[ \phi_1 = \frac{\pi}{Pm\delta^2} A(\tau) \sin \alpha_c x \cos \pi z, \quad (85) \]
\[ \xi_1 = \frac{\pi^2 \sqrt{T\alpha}}{Pm(Q\pi^2 + \delta^4)} A(\tau) \sin \alpha_c x \sin \pi z, \quad (86) \]

where \( \delta^2 = \alpha_c^2 + \pi^2 \) and \( \alpha_c \) is the critical wavenumber. The critical Rayleigh number for the onset of magnetoconvection is found to be

\[ Ra_{\xi_0}^c = \frac{\delta^2(\delta^4 + Q\pi^2) + Ra_\xi \alpha_c^2}{\alpha_c^2} + \frac{T\alpha \pi^2 \delta^4}{\alpha_c^2(Q\pi^2 + \delta^4)}. \quad (87) \]

Confirming the result obtained in Subsection 3.1.

At \( O(\varepsilon^2) \), we have the system of equations

\[
\begin{bmatrix}
-\nabla^4 & -\sqrt{T\alpha} \frac{\partial}{\partial z} & Ra_{\xi_0}^c \frac{\partial}{\partial x} & -Ra_\xi \frac{\partial}{\partial x} & -Q\text{Pm} \frac{\partial^2}{\partial z^2} & 0 \\
-\sqrt{T\alpha} \frac{\partial}{\partial z} & -\nabla^2 & 0 & 0 & 0 & Q\text{Pm} \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & 0 & -\nabla^2 & 0 & 0 & 0 \\
\frac{\partial}{\partial x} & 0 & 0 & -\frac{1}{Le} \nabla^2 & 0 & 0 \\
-\frac{\partial}{\partial z} & 0 & 0 & 0 & -P\text{m}\nabla^2 & 0 \\
0 & -\frac{\partial}{\partial z} & 0 & 0 & 0 & -P\text{m}\nabla^2 
\end{bmatrix}
\begin{bmatrix}
\psi_2 \\
\xi_2 \\
T_2 \\
S_2 \\
\phi_2 \\
S_2
\end{bmatrix} = \begin{bmatrix}
R_{21} \\
R_{22} \\
R_{23} \\
R_{24} \\
R_{25} \\
R_{26}
\end{bmatrix}, \quad (88)
\]

where

\[ R_{21} = R_{26} = 0, \quad (89) \]
\[ R_{22} = \frac{\sqrt{T\alpha} \pi^2 \alpha_c}{2\delta^2(Q\pi^2 + \delta^4)} \left( \frac{\delta^4}{Pr} - \frac{Q\pi^2}{Pm} \right) A^2(\tau) \sin 2\alpha_c x, \quad (90) \]
\[ R_{23} = -\frac{\alpha_c^2 \pi}{2\delta^2} A^2(\tau) \sin 2\pi z, \quad (91) \]
\[ R_{24} = -\frac{Le\alpha_c^2 \pi}{2\delta^2} A^2(\tau) \sin 2\pi z, \quad (92) \]
\[ R_{25} = -\frac{\alpha_c \pi}{2Pm\delta^2} A^2(\tau) \sin 2\alpha_c x. \quad (93) \]
The second-order system (88) gives the solution

\[ \psi_2 = \xi_2 = 0, \]  
\[ \xi_2 = \frac{\sqrt{T_a \pi^2}}{8\alpha c\delta^2 (Q\pi^2 + \delta^4)} \left( \frac{\delta^4}{Pr} - \frac{Q\pi^2}{Pm} \right) A^2(\tau) \sin 2\alpha_c x, \]  
\[ T_2 = -\frac{\alpha_c^2}{8\pi\delta^2} A^2(\tau) \sin 2\pi z, \]  
\[ S_2 = -\frac{Le\alpha_c^2}{8\pi\delta^2} A^2(\tau) \sin 2\pi z, \]  
\[ \phi_2 = -\frac{\pi^2}{8Pm^2\alpha_c\delta^2} A^2(\tau) \sin 2\alpha_c x. \]  

At \(O(\varepsilon^3)\), the following system is obtained

\[
\begin{bmatrix}
-\nabla^4 & -\sqrt{T_a} \frac{\partial}{\partial z} & Ra_{\gamma_0} \frac{\partial}{\partial x} & -Ra_s \frac{\partial}{\partial x} & -Q_{\mbox{pm}} \frac{\partial \nabla^2}{\partial z} & 0 \\
\sqrt{T_a} \frac{\partial}{\partial z} & -\nabla^2 & 0 & 0 & 0 & Q_{\mbox{pm}} \frac{\partial \nabla^2}{\partial z} \\
\frac{\partial}{\partial x} & 0 & -\nabla^2 & 0 & 0 & 0 \\
\frac{\partial}{\partial x} & 0 & 0 & -\frac{1}{Le} \nabla^2 & 0 & 0 \\
-\frac{\partial}{\partial z} & 0 & 0 & 0 & -P_{\mbox{m}} \nabla^2 & 0 \\
0 & -\frac{\partial}{\partial z} & 0 & 0 & 0 & -P_{\mbox{m}} \nabla^2 \\
\end{bmatrix}
\begin{bmatrix}
\psi_3 \\
\xi_3 \\
T_3 \\
S_3 \\
\phi_3 \\
\xi_3 \\
\end{bmatrix}
= \begin{bmatrix}
R_{31} \\
R_{32} \\
R_{33} \\
R_{34} \\
R_{35} \\
R_{36} \\
\end{bmatrix},
\]  

where

\[
R_{31} = \frac{Q\pi^4}{Pm^2\delta^2} \left( \frac{1}{4} - \frac{\alpha_c^2}{\delta^2} \right) A^3(\tau) \sin \alpha_c x \sin \pi z \cos 2\alpha_c x + \frac{\delta^2}{Pr} \frac{dA(\tau)}{dt} \sin \alpha_c x \sin \pi z \\
+ \frac{T_{\alpha\pi^2\delta^2}}{Q\pi^2 + \delta^4} fA(\tau) \sin \alpha_c x \sin \pi z - Ra_{\gamma_2} \frac{\alpha_c^2}{\delta^2} A(\tau) \sin \alpha_c x \sin \pi z,
\]

\[
R_{32} = -\frac{\sqrt{T_a \pi^3}}{4Q^2(Q\pi^2 + \delta^4)} \left( \frac{\delta^4}{Pr} - Q\pi^2 \frac{Q_{\mbox{pm}}}{{P}_{\mbox{pm}}^2} \right) A^3(\tau) \sin \alpha_c x \cos \pi z \cos 2\alpha_c x \\
+ \frac{\sqrt{T_a \pi^3}}{Q(Pr\pi^2 + \delta^4)} \frac{dA(\tau)}{dt} \sin \alpha_c x \cos \pi z - \sqrt{T_a f\pi} A(\tau) \sin \alpha_c x \cos \pi z,
\]

\[
R_{33} = \frac{\alpha_c}{\delta^2} \frac{dA(\tau)}{dt} \cos \alpha_c x \sin \pi z - \frac{\alpha_c^2}{4Q^3} A^3(\tau) \cos \alpha_c x \sin \pi z \cos 2\pi z,
\]

\[
R_{34} = \frac{Le_\alpha}{\delta^2} \frac{dA(\tau)}{dt} \cos \alpha_c x \sin \pi z - \frac{Le_\alpha^3}{4Q^3} A^3(\tau) \cos \alpha_c x \sin \pi z \cos 2\pi z,
\]

\[
R_{35} = \frac{\pi^3}{4Pm^2\delta^2} A^3(\tau) \sin \alpha_c x \cos 2\alpha_c x \cos \pi z - \frac{\pi}{Pm\delta^2} \frac{dA(\tau)}{dt} \sin \alpha_c x \cos \pi z,
\]

\[
R_{36} = -\frac{\sqrt{T_a \pi^3}}{Pm(Q\pi^2 + \delta^4)} \frac{dA(\tau)}{dt} \sin \alpha_c x \sin \pi z.
\]
For the existence of a solution, the Fredholm solvability condition\textsuperscript{58–60} is used to obtain the nonautonomous Ginzburg–Landau equation

\begin{equation}
    a_1 A'(\tau) - a_2 A(\tau) + a_3 A^3(\tau) = 0,
\end{equation}

where

\begin{equation}
    a_1 = \frac{\delta^2}{Pr} + \frac{\alpha_a^2}{\delta^4} \left( Ra_{\tau 0}^c - Ra_S L e^2 \right) - \frac{Q \pi^2}{P m^2} + \frac{T a \pi^2}{(Q \pi^2 + \delta^4)^2} \left( \frac{\delta^4}{Pr} - \frac{\pi^2 \delta^2}{P m^2} \right),
\end{equation}

\begin{equation}
    a_2 = \frac{Ra_{T 2} \alpha_a^2}{\delta^2} - \frac{2 f T a \pi^2 \delta^2}{Q \pi^2 + \delta^4},
\end{equation}

\begin{equation}
    a_3 = \frac{\alpha_a^4}{8 \delta^4} \left( Ra_{\tau 0}^c - Ra_S L e^2 \right) + \frac{Q \pi^4}{2 P m^2 \delta^2} \left( \frac{\alpha_a^2}{\delta^2} - \frac{1}{2} \right).
\end{equation}

Equation (100) is a Bernoulli type equation and due to its nonautonomous character, obtaining an analytical solution is difficult. As a result, it is solved using Mathematica’s built-in function NDSolve, with the initial condition \( A(0) = a_0 \), where \( a_0 \) is a specified initial amplitude of convection. For mathematical simplicity, we use the approximation \( Ra_{T 2} = Ra_{\tau 0}^c \) in computations.

In this study, the rotational modulation is represented by \( f = f(\omega, \tau) \), where \( \omega \) is the frequency of modulation. This study takes into consideration both sinusoidal and non-sinusoidal rotational modulations. Non-sinusoidal modulations are periodic oscillations that are neither cosine nor sine waveforms, they are obtained by transposing two or more sinusoidal waveforms of a specific frequency, amplitude, and phase. They have applications in the electronics and food manufacturing industry. For non-sinusoidal modulations, we consider triangle, square, and sawtooth waveforms.

- **Case (a) Sinusoidal rotational modulation**
  This is given by the trigonometric cosine waveform: \( f(\omega, \tau) = \cos(\omega \tau) \)

- **Case (b) Non-sinusoidal rotational modulation**
  1. Square wave-form:

\begin{equation}
    f(\omega, \tau) = \sum_{n=1,3,5}^{\infty} \frac{4}{n \pi} \sin(n \omega \tau).
\end{equation}

  2. Sawtooth wave-form:

\begin{equation}
    f(\omega, \tau) = \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} (1 - \cos(n \pi)) \cos(n \omega \tau).
\end{equation}

  3. Triangular wave-form:

\begin{equation}
    f(\omega, \tau) = \frac{8}{\pi^2} \sum_{n=1,3,5,...}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin(n \omega \tau).
\end{equation}
4.2 Nonautonomous Lorenz equations analysis

The Lorenz equations type are used to investigate the effect of the buoyancy effect on the onset of diffusive convection in the fluid system. Following Rudraiah et al.,\textsuperscript{26,34,61} we perform the analysis using a truncated double Fourier series for velocity, vorticity, temperature, solutal and magnetic fields. We assume the following truncated Fourier series solutions for \( \psi, \xi, T, S, \phi, \) and \( \zeta \)

\[
\psi = A(t) \sin \alpha_c x \sin \pi z, \tag{104}
\]

\[
\xi = B_1(t) \sin \alpha_c x \cos \pi z + B_2(t) \sin 2\alpha_c x, \tag{105}
\]

\[
T = C_1(t) \cos \alpha_c x \sin \pi z + C_2(t) \sin 2\pi z, \tag{106}
\]

\[
S = D_1(t) \cos \alpha_c x \sin \pi z + D_2(t) \sin 2\pi z, \tag{107}
\]

\[
\phi = E_1(t) \sin \alpha_c x \cos \pi z + E_2(t) \sin 2\alpha_c x, \tag{108}
\]

\[
\zeta = F(t) \sin \alpha_c x \sin \pi z, \tag{109}
\]

where amplitudes \( A(t), B_1(t), C_1(t), D_1(t), E_1(t)(i = 1, 2) \) and \( F \) are generic time functions which are determined by the system's dynamics. The generalized Lorenz model is obtained by applying the truncated Fourier series to nondimensionalized equations (24)–(29) and defines new variables as

\[
X_0 = \frac{\alpha_c \tau}{\delta^2} A(t), \quad X_2 = \frac{\pi^2 \alpha_c \sqrt{Ta}}{\delta^2} B_1(t), \quad X_3 = \frac{\pi^2 \alpha_c \sqrt{Ta}}{\delta^2} B(t)_2,
\]

\[
X_4 = -R\pi C_1(t), \quad X_5 = -R\pi C_2(t), \quad X_6 = -R\pi D_1(t + ),
\]

\[
X_7 = -R\pi D_2(t), \quad X_8 = \frac{\pi^2 \alpha_c}{\delta^2} E_1(t), \quad X_9 = \frac{\alpha_c \tau}{\delta^2} E_2(t),
\]

\[
X_{10} = \frac{\pi^2 \alpha_c \sqrt{Ta}}{\delta^2} F(t), \quad R = \frac{\alpha_c \tau}{\delta^2}, \quad t = \frac{\tau}{\delta^2}.
\]

This leads to the following coupled nonlinear Lorenz type equations

\[
\frac{dX_1}{d\tau} = -Pr (X_1 - (1 + \epsilon^2 \delta_1 f) X_2 - X_4 + NX_6 + QPrmX_8) + QPrm \left( \delta^2 - 4\alpha_c^2 \right) X_8 X_9, \tag{110}
\]

\[
\frac{dX_2}{d\tau} = -Pr \left( X_2 + Ta^* (1 + \epsilon^2 \delta_1 f) X_1 + \frac{X_1 X_3}{Pr} \right) + QPrm \left( \frac{\pi^2}{\delta^4} X_{10} - \frac{\pi^2}{\delta^2} X_8 X_{10} \right), \tag{111}
\]

\[
\frac{dX_3}{d\tau} = -Pr \left( 4\alpha_c^2 \frac{X_3}{\delta^2} + \frac{X_1 X_5}{2Pr} + \frac{QPrm}{2} X_8 X_{10} \right), \tag{112}
\]

\[
\frac{dX_4}{d\tau} = RX_1 + X_3 X_5 - X_4. \tag{113}
\]
\[
\frac{dX_5}{d\tau} = \frac{X_5X_4}{2} - \frac{4\pi^2}{\delta^2} X_5,
\]

(114)

\[
\frac{dX_6}{d\tau} = RX_1 + X_1X_7 - \frac{X_6}{Le},
\]

(115)

\[
\frac{dX_7}{d\tau} = \frac{X_7X_6}{2} - \frac{4\pi^2}{Le\delta^2} X_7,
\]

(116)

\[
\frac{dX_8}{d\tau} = \frac{\pi^2}{\delta^4} X_1 - PmX_8 - \frac{\pi^2}{\delta^2} X_1X_9,
\]

(117)

\[
\frac{dX_9}{d\tau} = -\frac{\delta^2}{2\pi^2} X_1X_8 - 4Pm\frac{\alpha_c^2}{\delta^2} X_9,
\]

(118)

\[
\frac{dX_{10}}{d\tau} = -PmX_{10} - X_2,
\]

(119)

where \(N = \frac{Ra}{Ra_r}\) and \(Ta^* = \frac{\pi^2 Ta}{\delta^8}\).

As this system of nonlinear autonomous equations cannot be solved analytically, a numerical method will be used. Qualitative predictions, on the other hand, can be made, as detailed below. Equations (110)–(119) are uniformly bounded in time and exhibit many of the same features as the whole system. Equations (110)–(119) must also be dissipative, just like the original equations (16)–(22). As a result, the volume in the phase space is expected to contract. Therefore, to verify volume contraction, we illustrate that the velocity field is characterized by a uniform negative divergence. Thus the uniform rate of contraction is given by

\[
\sum_{n=1}^{10} \frac{\partial X_n}{\partial X_n} = -2\left(2 + \frac{2\alpha_c^2}{\delta^2}\right)(Pr + Pm) + 1 + \frac{1}{Le} + \frac{4\pi^2}{\delta^2}\left(1 + \frac{1}{Le}\right),
\]

(120)

where \(\dot{X}_i = \frac{dX_i}{d\tau}\).

Equation (120) is always negative if all physically important parameters in the square bracket are nonnegative, suggesting that the system is bounded and dissipative. Therefore, the trajectories are either attracted to a fixed point, a limit attractor, or a strange attractor. After some notable time \(\tau > 0\), the endpoints of the corresponding trajectories populate the volume

\[
V(\tau) = V(0)e^{-2\left(2 + \frac{2\alpha_c^2}{\delta^2}\right)(Pr + Pm) + 1 + \frac{1}{Le} + \frac{4\pi^2}{\delta^2}\left(1 + \frac{1}{Le}\right)\tau}.
\]

(121)

Equation (121) shows that the volume decreases in time exponentially. It is observed that \(Pm\) and \(Pr\) enhance the dissipation rate, which aligns with the results shown by Xu et al.\(^{62}\)
4.3 | Numerical method

The MSRM was used to solve the coupled nonlinear Lorenz equations (110)–(119). The solution sets were generated using an initial condition of 0.5. Generally, the method is ideal for chaotic systems of first-order IVPs. Many studies53,63–65 provide in-depth explanations and validation of the MSRM, as well as a detailed verification of the method. The time interval \([0, T]\) is subdivided into small subintervals \(\Omega = [t_{i-1}, t_i]\), where \(i = 1, 2, 3, \ldots p\). The new interval is again transformed to the numerical interval \([-1, 1]\) by employing the linear transformation

\[
t = \frac{(t_i - t_{i-1})\tau}{2} + \frac{t_i + t_{i-1}}{2}.
\]  

(122)

To discretize \(\Omega\), we use the Chebyshev–Gauss–Lobatto collocation points

\[
\tau_j^i = \cos\left(\frac{\pi j}{M}\right), j = 0, 1, 2, \ldots M.
\]  

(123)

A system with \(m\) nonlinear first-order differential equations typically takes the form

\[
\dot{X}_r^i + \alpha_{r,k}X_k^i + (1 - \delta_{rk})\sum_{k=1}^{m} \alpha_{r,k}X_k^i + \Gamma_r\left[X_1^i, \ldots, X_{r-1}^i, X_{r+1}^i, \ldots, X_m^i\right] = g_r, r = 1, 2, \ldots, m
\]  

(124)

subject to initial conditions

\[
X_r^i(t_{i-1}) = X_r^{i-1}(t_{i-1}),
\]  

(125)

where \(\delta_{rk}\) is the Kronecker delta and the dot denotes time derivative, \(X_r\) represents unknown variables, \(\alpha_{r,k}\) and \(g_r\) are parameters which are known and \(\Gamma_r\) is the nonlinear term.

At the collocation points, the time derivative is given by

\[
\frac{dX_{r,s+1}^i(\tau^i)}{d\tau} = \sum_{k=0}^{N} D_{rk}X_{r,s+1}^i(\tau^i) = DX_{r,s+1}^i, j = 0, 1, 2, \ldots, m
\]  

(126)

where at the collocation points \(\tau_j^i\)

\[
D = \frac{2D}{(\tau_i - \tau_{i-1})} \quad \text{and} \quad X_{r,s+1}^i = \begin{bmatrix} X_{r,s+1}^i(t_0^i), X_{r,s+1}^i(\tau_1^i), \ldots, X_{r,s+1}^i(\tau_M^i) \end{bmatrix}^T.
\]
The system of nonlinear Equation (115) becomes

\[
\begin{align*}
X_{1,s+1}^i &+ \alpha_{1,1}X_{1,s+1}^i + \alpha_{1,2}X_{2,s}^i + \alpha_{1,4}X_{4,s}^i + \alpha_{1,6}X_{6,s}^i + \alpha_{1,8}X_{8,s}^i + \Gamma_1[X_{8,s}^i, X_{9,s}^i] = 0, \quad (127) \\
\dot{X}_{2,s+1}^i &+ \alpha_{2,1}X_{1,s+1}^i + \alpha_{2,2}X_{2,s+1}^i + \alpha_{2,10}X_{10,s}^i + \Gamma_2[X_{1,s+1}^i, X_{3,s}^i, X_{9,s}^i, X_{10,s}^i] = 0. \quad (128) \\
\dot{X}_{3,s+1}^i &+ \alpha_{3,3}X_{3,s+1}^i + \Gamma_3[X_{1,s+1}^i, X_{2,s+1}^i, X_{6,s}^i, X_{10,s}^i] = 0, \quad (129) \\
\dot{X}_{4,s+1}^i &+ \alpha_{4,1}X_{1,s+1}^i + \alpha_{4,4}X_{4,s+1}^i + \Gamma_4[X_{1,s+1}^i, X_{5,s}^i] = 0, \quad (130) \\
\dot{X}_{5,s+1}^i &+ \alpha_{5,5}X_{5,s+1}^i + \Gamma_5[X_{1,s+1}^i, X_{4,s+1}^i] = 0, \quad (131) \\
\dot{X}_{6,s+1}^i &+ \alpha_{6,1}X_{1,s+1}^i + \alpha_{6,6}X_{6,s}^i + \Gamma_6[X_{1,s+1}^i, X_{7,s}^i] = 0, \quad (132) \\
\dot{X}_{7,s+1}^i &+ \alpha_{7,7}X_{7,s+1}^i + \Gamma_7[X_{1,s+1}^i, X_{6,s+1}^i] = 0, \quad (133) \\
\dot{X}_{8,s+1}^i &+ \alpha_{8,1}X_{1,s+1}^i + \alpha_{8,8}X_{8,s+1}^i + \Gamma_8[X_{1,s+1}^i, X_{9,s}^i] = 0. \quad (134) \\
\dot{X}_{9,s+1}^i &+ \alpha_{9,9}X_{9,s+1}^i + \Gamma_9[X_{1,s+1}^i, X_{8,s+1}^i] = 0, \quad (135) \\
\dot{X}_{10,s+1}^i &+ \alpha_{10,2}X_{2,s+1}^i + \alpha_{10,10}X_{10,s+1}^i = 0. \quad (136)
\end{align*}
\]

Using the Chebyshev spectral collocation method,\(^63\) Equations (127)–(136) gives

\[
A_r X_{r,s+1}^i = B_r^i,
\]

and

\[
X_{r,s+1}^i(t_M^{s_i-1}) = X_r^{i-1}(t_M^{s_i-1}), \quad r = 1, 2, 3... 
\]

with

\[
A_r = D + \alpha_{r,r} I, \quad (137)
\]

and
Our study focuses strongly on the quantification of heat and mass transport. The onset of the convection is recognized with the increase in the Rayleigh number through its impact on heat and mass transport. Heat and mass transport occur only by conduction in a basic state. The objective is to investigate the effects of rotational modulation on heat and mass transfer in an electrically conducting double-diffusive system. Heat and mass transfer are quantified in terms of Nusselt number $Nu$ and Sherwood number $Sh$, respectively. A Nusselt number which is approximately equal to 1 represents heat transfer by pure conduction and this is probably in the basic state. Larger Nusselt and Sherwood numbers relate to more active convection with typically flow instability. Considering the stationary mode of convection, the Nusselt and Sherwood numbers are given by
\[ Nu(\tau) = 1 + \left( \frac{\alpha_c}{2\pi} \int_0^{2\pi} \frac{\partial^2 T}{\partial z^2} \, dx \right) \bigg|_{z=0} \] and \[ Sh(\tau) = 1 + \left( \frac{\alpha_c}{2\pi} \int_0^{2\pi} \frac{\partial^2 S}{\partial z^2} \, dx \right) \bigg|_{z=0}. \] (139)

It can be observed that \( f(\omega, \tau) \), is effective at third-order meaning that it can only affect \( Nu(\tau) \) and \( Sh(\tau) \) through the amplitude \( A(\tau) \) defined in Equation (100). Substituting \( T, S, T_b, \) and \( S_b \) into Equation (139), we obtain

\[ Nu(\tau) = 1 + \frac{\alpha_c^2}{4\delta^2} A^2(\tau) \quad \text{and} \quad Sh(\tau) = 1 + \frac{Le^2\alpha_c^2}{4\delta^2} A^2(\tau). \] (140)

To have a meaningful investigation of the effect of time-dependent rotation on heat and mass transfer, we define the time-averaged Nusselt number (mean Nusselt number) \( \overline{Nu(\tau)} \) and Sherwood number (mean Sherwood number), \( \overline{Sh(\tau)} \) as

\[ N\tilde{u}(\tau) = \frac{\omega}{2\pi} \int_0^{2\pi} N\mu(\tau) \, d\tau \quad \text{and} \quad S\tilde{h}(\tau) = \frac{\omega}{2\pi} \int_0^{2\pi} S\mu(\tau) \, d\tau, \]

where the chosen interval to calculate the mean Nusselt and Sherwood numbers is \([0, \frac{2\pi}{\omega}]\).

6 | RESULTS AND DISCUSSION

The study of double-diffusive convection in a rotating electrically conducting fluid layer involves the external regulation of heat and mass transfer characteristics. The objective of this study is to focus on three mechanisms, namely, an applied vertical magnetic field, the presence of a solute concentration, and rotational modulation, for promoting or inhibiting convective heat and mass transport. To analyze heat and mass transfer we use the nonlinear stability theory. The effect of various modes of rotational modulation is assumed to be of order \( O(\epsilon^2) \). For this reason we only consider small values of amplitude modulations \( 0.2 \leq \delta_1 \leq 0.8 \). For the parameters \( Q, Pm, Pr, \omega, Le, Ra_s, \) and \( Ta \), we have used the values in Siddheshwar et al.,44 Bhaduria et al.55,56 and Gupta et al.59 It is worth noting that the chosen governing parameter ranges allow for a better understanding of the impact of the appropriate mechanisms on convective heat and mass transfer. It is important to note that for heat and mass transport profiles, the values of average Nusselt \( \overline{Nu} \) and Sherwood \( \overline{Sh} \) numbers begin with a unit, indicating that heat and mass transfer across a resting fluid layer occurs primarily through conduction.

First and foremost, it is essential to comprehend the individual effects of the four types of rotation modulation on the onset of convection. Figure 2 shows the variation of the modulating frequency \( \omega \) on correction Rayleigh number \( Ra_{T_2} \) for trigonometric cosine, square, triangular, and sawtooth wave types of rotation modulation. It should be noted that small values of \( \delta_1 \) and moderate values of \( \omega \) are considered in this study because \( Ra_{T_2} \to 0 \) for higher values of \( \omega \). From Figure 2, we observe that \( Ra_{T_2} \) is positive for all values of \( \omega \) indicating that all modes of rotational modulation delay the onset of convection and hence stabilize the system. It can be concluded from the figure that
Unmodulated, Square, Trig cosine, Sawtooth, Sawtooth, Triangular. As a result, triangular wave-type modulation is the most stabilizing, while square wave-type modulation is the most destabilizing.

Figure 3 displays the influence of modulation amplitude on convection amplitude solutions. It is observed that increasing the amplitude of modulation increases the magnitude of amplitude of convection solutions. The solutions have an unlimited number of peaks with no patterns at larger modulation amplitudes, indicating that increasing rotation amplitude enhances convection in the fluid system. Figure 4 backs the conclusion made from Figure 3. The figure shows that increasing the amplitude of rotational modulation $\delta_1$ increases the average Nusselt and Sherwood numbers, resulting in increased heat and mass transport and a faster onset of thermal convection.

Figure 5 shows the effect of modulation frequency on convection amplitude solutions. The figure shows that increasing the modulation amplitude reduces convection amplitude. The effect of modulation frequency $\omega$ on average Nusselt and Sherwood numbers is depicted in Figure 6. It shows that for low values of modulation frequency, there is higher heat and mass transmission. Therefore,
Figures 5 and 6 suggest that the effect of $\omega$ is to delay the onset of thermal convection thus stabilizing the system. These are the same results obtained by Venezian.\(^{38}\)

Rotational influence on average Nusselt and Sherwood numbers is depicted in Figure 7. A decrease in heat and mass transfer is observed when the Taylor number increases. This conclusion is compatible with results given by Gupta et al.\(^{59}\)

The ratio of Lorentz force to viscous force ratio is given by the Chandrasekhar number $Q$. The Lorentz force is the result of electromagnetic fields combining electric and magnetic forces at a point charge. Figure 8 displays the effect of Chandrasekhar number on average Nusselt and Sherwood numbers keeping all other parameter values fixed. It is observed that when Chandrasekhar number increases, the average Nusselt and Sherwood numbers decrease, implying a reduction in heat and mass transfer hence delaying the advent of thermal convection and stabilizing the system which is in line with results obtained by Bhadauria and Kiran,\(^{55,56}\) Rudraiah et al.,\(^{20,66}\) and Siddheshwar et al.\(^{44}\)

The influence of Lewis number $Le$ on average Nusselt and Sherwood numbers is shown in Figure 9. The Lewis number tells the relationship between heat diffusivity and mass diffusivity. It arises in fluid flows in which heat and mass are transferred simultaneously. Figure 9 shows...
FIGURE 6  Changes in $\overline{\text{Nu}}$ and $\overline{\text{Sh}}$ with time $\tau$ for modulation frequency values $\omega = 4, 10, \text{and } 20$ with fixed parameters $Pr = 1.2$, $Le = 2.2$, $Pm = 1.2$, $Ta = 100$, $\delta_1 = 0.5$, $Ra_S = 20$, and $Q = 25$.

FIGURE 7  Changes in $\overline{\text{Nu}}$ and $\overline{\text{Sh}}$ with time $\tau$ for Taylor numbers $Ta = 50, 100, \text{and } 150$ with fixed parameters $Pr = 1.2$, $Le = 2.2$, $Pm = 1.2$, $\delta_1 = 0.5$, $\omega = 2$, $Ra_S = 20$, and $Q = 25$.

FIGURE 8  $\overline{\text{Nu}}$ and $\overline{\text{Sh}}$ variations with time $\tau$ for Chandrasekhar numbers $Q = 15, 25, \text{and } 35$ with fixed parameters $Pr = 1.2$, $Le = 2.2$, $Pm = 1.2$, $\delta_1 = 0.5$, $\omega = 2$, $Ra_S = 20$, and $Ta = 100$. 
that increasing the Lewis number increases both $\overline{Nu}$ and $\overline{Sh}$. This improves the rate of heat and mass transfer in the fluid system.

Figure 10 shows that the magnetic Prandtl number $Pm$, which is the ratio of viscous to magnetic diffusion rates, has a significant impact on heat and mass transfer. Increasing the magnetic Prandtl number causes an increase in either the viscous or magnetic diffusion rates, resulting in an increase in heat and mass transfer in both cases. Therefore, magnetic Prandtl number advances the onset of thermal instabilities hence destabilizes the fluid system, Bhadhauria and Kiran $^{56}$ and Siddheshwar et al. $^{44}$ previously obtained similar results.

Figure 11 illustrates that the Prandtl number $Pr$ has the impact of increasing heat and mass transmission for lower time values. Prandtl number is the kinematic viscosity to thermal diffusivity ratio. As a result of this correlation, it is clear that as $Pr$ increases, either kinematic viscosity or thermal diffusivity increases, implying that heat transfer increases in both cases. It is observed that as time progresses, the significance of increasing the Prandtl number becomes negligible. Although the critical stationary Rayleigh number does not depend on the Prandtl
number, the time derivative has an impact on heat and mass transmission when $Pr$ is small. The Prandtl number was considered to be small (around 1) to preserve the time derivative in the momentum equation. Aniss et al.\textsuperscript{67} considered the Prandtl number above 5 for the magnetoconvection linear theory case and the influence of the time-derivative of stream functions is not observed.

Figure 12 displays the effect of the solutal Rayleigh number $Ra_S$ on average Nusselt and Sherwood numbers. It is shown that increasing the solutal Rayleigh number increases heat and mass transfer, causing the system to become unstable. Even though the gradient of the solute concentration prevents convection, significant finite-amplitude flows, which occur for large values of $Ra_T$, tend to mix and redistribute the solute, resulting in more neutrally stratified interior layers of the fluid.\textsuperscript{41,56} As a result, the solute gradient's impact is considerably reduced, and fluid will convect more, increasing heat and mass transfer.

Table 2 summarizes the effect of selected parameters on heat and mass transfer for different modes of modulation. From the table it is observed that heat and mass transfer rates are higher in square wave modulation and lower in triangular waveforms. Therefore, square waveform
modulation is the most destabilizing, while triangle waveform modulation is the most stabilizing. This result is consistent with the conclusion found in the case of linear stability analysis. Furthermore, Table 2 clearly shows that the modulation frequency, Taylor number, and the Chandrasekhar number are all stabilizing agencies while the amplitude of modulation has a destabilizing effect.

Figures 13–15 show the effects of $N$ on streamlines, isotherms, and isoconcentrations. Two distinct eddies can be seen where negative and positive values represent clockwise and anticlockwise flows, respectively. Different buoyancy ratios have been found to have significant effects on flow structure and thus the rate of heat transfer and mass transfer. It is noted that the contour lines are sparse for $N = -5$.

| Trig cosine | Square wave | Sawtooth | Triangular |
|-------------|-------------|----------|------------|
| $\delta_1$ | $\omega$ | $S_T$ | $S_H$ | $\delta_1$ | $\omega$ | $S_T$ | $S_H$ | $\delta_1$ | $\omega$ | $S_T$ | $S_H$ |
| 0.2 | 4 | $1.12269$ | $1.17668$ | $1.29781$ | $1.12256$ | $1.17547$ | $1.101130$ | $1.17001$ |
| 0.5 | 10 | $1.12417$ | $1.17881$ | $1.99637$ | $2.09956$ | $2.11305$ | $1.17691$ | $1.11231$ | $1.17546$ |
| 0.8 | 20 | $1.12585$ | $1.18123$ | $2.10021$ | $2.49544$ | $1.12440$ | $1.1810$ | $1.11270$ | $1.18034$ |

Table 2: Values of $\bar{N}_T$ and $\bar{S}_H$ for different modes of modulation.

Note: Parameter values are fixed as $Pr = 1.2$, $Ra_T = 20$, $Le = 2.2$, $Pm = 1.2$, $\delta_1 = 0.5$, $\omega = 2$, $Q = 25$, and $Ta = 100$, unless being varied.

Figure 13: Effect of buoyancy ratio on streamlines (A) $N = -5$, (B) $N = 0$, and (C) $N = 5$ for fixed parameters $Pr = 1.2$, $Ra_T = 20$, $Le = 2.2$, $Pm = 1.2$, $\delta_1 = 0.5$, $\omega = 2$, $Q = 25$, and $Ta = 100$. 
When the buoyancy ratio progresses from negative to positive values, more contour cells begin to form and become more dense. The flow is more intense near the center of the enclosure. At $N = 5$, the flow rate has increased as a result of the combined influence of thermal and solutal buoyancy forces, resulting in the creation of more contours that are densely packed together. In this instance, the thermal and solutal buoyancy effects reinforce one another, exhibiting a stronger stratification in the isotherms and isoconcentrations hence aiding heat and mass transfer. The isotherm and isoconcentration contours behave similarly as they are described by equations of similar form.

7 | CONCLUSION

The effect of various modes of rotational modulation on the rate of heat and mass transfer in double-diffusive magnetoconvection has been analyzed using the nonautonomous Ginzburg–Landau and nonlinear coupled Lorenz equations. A comparison between sinusoidal and non-sinusoidal rotational type modulations is made. The Lorenz equations were solved using a MSRM. This study produced the following conclusions.
1. By increasing the amplitude of rotation $\delta_1$, heat, and mass transfer profiles can be magnified, whereas increasing the frequency of modulation $\omega$ has the opposite effect. Therefore, rotational amplitude acts as a destabilizing factor, whereas modulation frequency acts as a stabilizing factor.

2. The effects of the Chandrasekhar number $Q$, and the Taylor number $Ta$ are to stabilize the convective system as the heat and mass profiles decrease with an increase in the parameters.

3. The effect of increasing the Prandtl number $Pr$, Lewis number $Le$, solute Rayleigh number $Ra_s$, and the magnetic Prandtl number $Pm$ is to increase the heat and mass transfer profiles, thus destabilizing the system.

4. In comparison to other types of rotational modulation, square wave rotational modulation is the most heat and mass transfer enhancer meaning it is the most destabilizing type of modulation. The relationship can be expressed as

$$[Nu/\dot{Sh}]_{\text{Triangular}} < [Nu/\dot{Sh}]_{\text{Sawtooth}} < [Nu/\dot{Sh}]_{\text{Trig cosine}} < [Nu/\dot{Sh}]_{\text{Square}}.$$  

5. Increased buoyancy ratio promotes heat and mass transfer in the fluid system.

In summary, we can show that the following relations hold:

1. $[Nu/\dot{Sh}]_{\delta=0.2} < [Nu/\dot{Sh}]_{\delta=0.50} < [Nu/\dot{Sh}]_{\delta=0.8}$
2. $[Nu/\dot{Sh}]_{\omega=4} < [Nu/\dot{Sh}]_{\omega=10} < [Nu/\dot{Sh}]_{\omega=20}$
3. $[Nu/\dot{Sh}]_{Q=35} < [Nu/\dot{Sh}]_{Q=25} < [Nu/\dot{Sh}]_{Q=15}$
4. $[Nu/\dot{Sh}]_{Ta=60} < [Nu/\dot{Sh}]_{Ta=40} < [Nu/\dot{Sh}]_{Ta=20}$
5. $[Nu/\dot{Sh}]_{Ra_s=20} < [Nu/\dot{Sh}]_{Ra_s=200} < [Nu/\dot{Sh}]_{Ra_s=300}$
6. $[Nu/\dot{Sh}]_{Le=1.2} < [Nu/\dot{Sh}]_{Le=2.2} < [Nu/\dot{Sh}]_{Le=3.2}$
7. $[Nu/\dot{Sh}]_{Pr=1.2} < [Nu/\dot{Sh}]_{Pr=1.4} < [Nu/\dot{Sh}]_{Pr=1.6}$
8. $[Nu/\dot{Sh}]_{Pm=1.2} < [Nu/\dot{Sh}]_{Pm=2.2} < [Nu/\dot{Sh}]_{Pm=3.2}$

**NOMENCLATURE**

Latin symbols

| Symbol | Description |
|--------|-------------|
| $d$    | distance between the plates |
| $\vec{g}$ | gravitational acceleration |
| $\vec{k}$ | unit vector in the $z$-direction |
| $N$    | buoyancy ratio |
| $P$    | pressure |
| $Q$    | Chandrasekhar number |
| $\vec{u}$ | velocity |
| $R$    | revised Rayleigh number |
| $Ra_T$ | thermal Rayleigh number |
| $Ra_S$ | salinity Rayleigh number |
| $S$    | solute concentration |
| $T$    | temperature |
| $Ta$   | Taylor number |
| $Ta^*$ | revised Taylor number |
GREEK SYMBOLS

\( \alpha \)  \hspace{2em} \text{thermal expansion coefficient}
\( \beta \)  \hspace{2em} \text{concentration expansion coefficient}
\( \kappa_T \)  \hspace{2em} \text{thermal diffusivity}
\( \kappa_S \)  \hspace{2em} \text{solute diffusivity}
\( \Omega \)  \hspace{2em} \text{angular velocity}
\( \mu \)  \hspace{2em} \text{fluid dynamic viscosity}
\( \mu_m \)  \hspace{2em} \text{magnetic permeability}
\( \nu \)  \hspace{2em} \text{kinematic viscosity} \left( \frac{\mu}{\rho_0} \right)
\( \nu_m \)  \hspace{2em} \text{magnetic viscosity} \left( \frac{\mu_m}{\rho_0} \right)

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DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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