FINITE-DIMENSIONALITY OF Z-BOUNDARIES

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Abstract. An interesting feature of CAT(0) and hyperbolic group boundaries is that they must be finite-dimensional, even if the space the group acts on is infinite-dimensional. When Bestvina axiomatized the theory of group boundaries to larger classes of groups, he included a condition which forced these boundaries, known as Z-boundaries, to be finite-dimensional. In generalizing the definition of group boundaries to permit the study of groups with torsion, Dranishnikov removed the condition that, a priori, forced the finite-dimensionality of these boundaries. In this paper, we prove that a very general class of Z-boundaries, which include the type as defined by Dranishnikov, must be finite-dimensional. In doing so, we unify and improve the treatments of group boundaries put forth by Bestvina and Dranishnikov.

1. Introduction

It is easy to construct a proper CAT(0) space with infinite-dimensional boundary, but a result by Swenson [Swe99] shows that such a space cannot admit a cocompact action by isometries. Thus, every boundary of a CAT(0) group must be finite-dimensional. This observation mirrors an earlier theorem by Gromov and Swenson [Swe95] which asserts that boundaries of hyperbolic groups are finite-dimensional.

The rich study of CAT(0) and hyperbolic group boundaries led Bestvina to formalize the concept of group boundaries for wider classes of groups [Bes96]. Using the CAT(0) and hyperbolic cases as models, he included a hypothesis in his definition which forces these boundaries, known as Z-boundaries, to be finite-dimensional. Later, when Dranishnikov generalized Bestvina’s work to allow for groups with torsion [Dra06], he omitted the requirement in Bestvina’s original definition that forced the boundaries to be finite-dimensional. As a result, some of the results in [Dra06] are, a priori, not as strong as their analogs in [Bes96]. In particular, Dranishnikov related the cohomological dimension...
of a group to the cohomological dimension of its $\mathbb{Z}$-boundary, but not to the Lebesgue covering dimension of that boundary.

In this paper, we prove a generalization of Swenson’s theorem that applies to a more general class of groups. One of the consequences of this result is a more unified treatment of group boundaries put forth by Bestvina and Dranishnikov. We show that there is no advantage in restricting our attention to finite-dimensional spaces as in [Bes96]. In particular, we may weaken the requirement that the group act on an ER to an analogous action on an AR, without losing any applications. (Recall that an ER is just a finite-dimensional AR). In regards to [Dra06], all conclusions about the cohomological dimension of group boundaries can be extended to results about the Lebesgue covering dimension of these boundaries.

We close this introduction with an outline of this paper along with statements of some of our main results. In Section 2, we give a few preliminary definitions and more precise definitions of $\mathbb{Z}$-boundaries. In Section 3, we prove a generalization of Swenson’s theorem:

**Theorem A.** If a proper metric ANR $(X, d)$ admits a metric $\mathbb{Z}$-structure $(\hat{X}, \mathbb{Z})$, then $\mathbb{Z}$ is finite-dimensional.

In Section 4, we prove the following, using a generalization of Bestvina’s Boundary Swapping Theorem [Bes96]:

**Theorem B.** If a torsion-free group $G$ admits an AR $\mathbb{Z}$-structure, then $G$ admits a $\mathbb{Z}$-structure, as defined in [Bes96].

Due to its relevance to our work, we provide the details for an alternative proof of Bestvina’s Boundary Swapping Theorem as suggested by Ferry [Fer00] in Appendix A.

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2. Preliminaries

We begin with a few preliminary definitions, leading to Bestvina’s original definition of a $\mathbb{Z}$-structure. We then present various generalizations of this definition with explanations, justifications, and consequences of the changes.

We suppose that our spaces are locally compact, separable, and metrizable.

**Definition 2.1.** A closed subset, $A$, of an ANR, $X$, is a $\mathbb{Z}$-set if either of the following equivalent conditions hold:

- There exists a homotopy $H : X \times [0,1] \to X$ such that $H_0 = \text{id}_X$ and $H_t(X) \subseteq X - A$ for every $t > 0$.
- For every open subset $U$ of $X$, the inclusion $U - A \hookrightarrow U$ is a homotopy equivalence.

The standard example of a $\mathbb{Z}$-set is the boundary of an $n$-manifold, $M^n$. We can use a collared neighborhood of the boundary in $M^n$ to define a homotopy, which instantly pushes the boundary off itself.

**Definition 2.2.** A $\mathbb{Z}$-compactification of a space $Y$ is a compactification $\hat{Y}$ such that $\hat{Y} - Y$ is a $\mathbb{Z}$-set in $\hat{Y}$. We call $\hat{Y} - Y$ a $\mathbb{Z}$-boundary for $Y$.

Implicitly in this definition, we are assuming that $\hat{Y}$ is an ANR. Because open subsets of an ANR are ANRs, we require a space $Y$ to be an ANR before beginning to ask whether or not $Y$ is $\mathbb{Z}$-compactifiable. Once we do know $\hat{Y}$ is a $\mathbb{Z}$-compactification of $Y$, then $\hat{Y}$ and $Y$ will have the same homotopy type. In the realm of compactifications, $\mathbb{Z}$-compactifications are particularly nice as they are sensitive to the overall geometry of the original space. This preservation of geometry explains the choice to use $\mathbb{Z}$-compactifications in the theory of group boundaries as we now see with Bestvina’s original definition of a $\mathbb{Z}$-structure on a group $G$.

**Definition 2.3.** [Bes96] A $\mathbb{Z}$-structure on a group $G$ is a pair of spaces $(\hat{X}, Z)$ satisfying the following four conditions:

1. $\hat{X}$ is a compact ER,
2. $\hat{X}$ is a $\mathbb{Z}$-compactification of $X = \hat{X} - Z$,.
(3) $G$ acts properly, cocompactly, and freely on $X$, and

(4) $\hat{X}$ satisfies a nullity condition with respect to the action of $G$ on $X$. That is, for every compactum $C$ of $X$ and any open cover $\mathcal{U}$ of $\hat{X}$, all but finitely many $G$ translates of $C$ lie in an element of $\mathcal{U}$.

There are a few things to notice about this definition. First, since an ER is a finite-dimensional AR and an AR is a contractible ANR, $\hat{X}$ is a compact, contractible, finite-dimensional ANR. The requirement that $\hat{X}$ be finite-dimensional is what forces $Z \subseteq \hat{X}$ to be finite-dimensional. In fact, a simple argument using $\varepsilon$-maps in the next section will show that $\dim Z \leq \dim X < \infty$.

Secondly, our spaces are metrizable, but we must take care to distinguish between the metric on $X$ and the metric on the compactification $\hat{X}$. These two metrics will look very different and are not necessarily related. Having a metric on $\hat{X}$ does allow us to restate the nullity condition as follows: for every $\varepsilon > 0$, all but finitely many $G$-translates of any compact subset $C$ of $X$ have diameter less than $\varepsilon$ (in the metric from the compactification). Thus, if we look at the translates using the metric on $X$, they may stay the same size (in particular if $G$ acts by isometries), but with the metric on $\hat{X}$, the nullity condition forces the translates of every compact set to become arbitrarily small when pushed towards the boundary.

Lastly, we say that $Z$ is a boundary (or $Z$-boundary) of $G$ if there is a $Z$-structure $(\hat{X}, Z)$ on $G$. This boundary is not unique; there can be multiple $Z$-structures for a given group $G$. However, any two boundaries of $G$ will have the same shape [Bes96].

**Examples:**

(1) Suppose $G$ acts properly, freely and cocompactly on a finite-dimensional CAT(0) space, $X$. Then $\hat{X} = X \cup \partial_\infty X$ is a $Z$-structure on $G$, where $\partial_\infty X$ denotes the visual boundary. We take care here to not say that all torsion-free CAT(0) groups admit $Z$-structures as it is still an open question whether or not a group acting geometrically on an infinite-dimensional CAT(0) space also acts geometrically on a finite-dimensional CAT(0) space.
(2) If \( G \) is a torsion-free \( \delta \)-hyperbolic group, \( G \) admits a \( \mathcal{Z} \)-structure. Let \( P_{\rho}(G) \) be an appropriately chosen Rips complex of \( G \) and \( \partial G \) the Gromov boundary. Then, with an appropriate topology, \( \hat{P}_{\rho}(G) = P_{\rho}(G) \cup \partial G \) is a \( \mathcal{Z} \)-structure for \( G \).

(3) In [Bes96], Bestvina outlines a method of placing a \( \mathcal{Z} \)-structure on the Baumslag-Solitar group \( BS(1, 2) \) by modifying the traditional universal cover of the presentation 2-complex. As \( BS(1, 2) \) is neither CAT(0) nor \( \delta \)-hyperbolic, this example illustrates how \( \mathcal{Z} \)-structures allow us to approach the study of group boundaries for different classes of groups.

(4) Osajda and Przytycki in [OP09] prove that all torsion-free systolic groups admit \( \mathcal{Z} \)-structures.

(5) Trel [Tir11] showed that if two groups \( G \) and \( H \) admit \( \mathcal{Z} \)-structures, so do \( G \times H \) and \( G \ast H \).

(6) Dahmani [Dah03] showed that if a group \( G \) is hyperbolic relative to a collection of subgroups, and each of these subgroups admit a \( \mathcal{Z} \)-structure, then \( G \) admits a \( \mathcal{Z} \)-structure.

(7) Martin [Mar14] provides conditions for building a \( \mathcal{Z} \)-structure for the fundamental group of a complex of groups over a finite simplicial complex.

As each of the above examples illustrates, groups must be torsion-free if they are to admit a \( \mathcal{Z} \)-structure. In [Dra06], Dranishnikov generalized Bestvina’s definition to allow for groups with torsion. In particular, he omitted the requirement that \( G \) act freely on \( X \) and replaced it with the requirement that the action of \( G \) is geometric (that is, proper, cocompact, and by isometries). He also loosened the restriction that \( \hat{X} \) be an ER to being an AR. This change permits \( \hat{X} \) to be infinite-dimensional. Using Dranishnikov’s definition then, we need not be as restrictive in our above example for CAT(0) spaces. We may now say that CAT(0) groups admit \( \mathcal{Z} \)-structures (in the sense of Dranishnikov). There is one immediate drawback in permitting infinite-dimensionality of \( \hat{X} \): \( \mathcal{Z} \) could potentially be infinite-dimensional. We will show in the next section that this is not the case, but the proof is not immediate as in the case of Bestvina’s original definition. Because it was unknown if \( \mathcal{Z} \)-boundaries had finite covering dimension, Dranishnikov used cohomological
dimension of the boundary to state and prove many of his results, unlike Bestvina’s results which used Lebesgue covering dimension. In proving Theorem A, one can then easily go back and restate Dranishnikov’s results from [Dra06] by replacing the cohomological dimension of the boundary with the Lebesgue covering dimension of the boundary. (We note that this replacement is valid as it is a standard fact that in a space with finite Lebesgue covering dimension, covering dimension and cohomological dimension coincide. See for example [Wal81, Theorem 3.2(b)]). Thus, Theorem A provides a connection between the results on \( \mathbb{Z} \)-boundaries as presented by Bestvina and Dranishnikov.

Since we will be working with many of these deviations from Bestvina’s original definition of a \( \mathbb{Z} \)-structure, we introduce some notation in hopes of highlighting what conditions have been changed or removed. We will always use the notation \( \mathbb{Z} \)-structure to denote Bestvina’s original definition. If we remove the requirement that the action be free, we say \( G \) admits a \( \mathbb{Z}^{n.f.} \)-structure. If \( \hat{X} \) is an AR, rather than an ER, we say \( G \) admits a \( \mathbb{Z}_{AR} \)-structure. Thus, a \( \mathbb{Z}_{AR}^{n.f.} \)-structure on a group is a \( \mathbb{Z} \)-structure in which \( \hat{X} \) is an AR and the group need not be torsion-free.

As mentioned above, we show in the next section that the dimension of \( \mathbb{Z} \)-boundaries (in the sense of Dranishnikov) is finite. This fact was already known in the case of CAT(0) and hyperbolic group boundaries (see [Swe99], [Swe95]). Because these two special cases served as models for the definition of \( \mathbb{Z} \)-boundaries, proving finite-dimensionality of Dranishnikov’s \( \mathbb{Z} \)-boundaries seemed promising. In fact, our main result was motivated by attempting to generalize the following theorem of Swenson on boundaries of CAT(0) spaces:

**Theorem 2.4.** [Swe99] If \( X \) is a CAT(0) space which admits a cocompact action by isometries, then \( \partial X \) is finite-dimensional.

In the statement of Swenson’s theorem, the action of the group on \( X \) is not required to be proper. Thus, to obtain a full generalization of Swenson’s theorem, we must also omit the properness condition that is contained in both Bestvina and Dranishnikov’s definition of a \( \mathbb{Z} \)-structure. Simply removing properness from either definition will not suffice because the statement of the nullity condition is dependent on having a proper
group action. In particular, it would be possible for infinitely many translates of any compact set \( C \) to intersect \( C \). Therefore, we present one final generalized definition of a \( \mathbb{Z} \)-structure which will be used in Theorem A in the next section.

**Definition 2.5.** Let \((X, d)\) be a metric space. A metric \( \mathbb{Z} \)-structure on \( X \), denoted \( \mathbb{M} \mathbb{Z} \)-structure, is a pair of spaces \((\hat{X}, Z)\) satisfying the following conditions:

1. \( \hat{X} \) is a compact \( AR \),
2. \( \hat{X} \) is a \( \mathbb{Z} \)-compactification of \( X = \hat{X} - Z \),
3. \( X \) admits a cocompact action by isometries by some group \( G \), and
4. \( \hat{X} \) satisfies a nullity condition with respect to the action of \( G \): for every \( \epsilon > 0 \) and for each bounded subset \( U \) of \( X \) (bounded in the \( d \) metric), there exists a compact subset \( C \) of \( X \) such that any \( G \)-translate of \( U \) that does not intersect \( C \) has diameter less than \( \epsilon \) (in the metric on the compactification).

3. **Finite-Dimensionality Results**

Recall that with Bestvina’s original definition of a \( \mathbb{Z} \)-structure, the boundary must be finite-dimensional. In fact, if \((\hat{X}, Z)\) is a \( \mathbb{Z} \)-structure on a group \( G \), the dimension of \( X = \hat{X} - Z \) serves as an upper bound for the dimension of \( Z \). This argument is rather simple and we present it now for completeness and to highlight why such a simple argument cannot be used in the case of \( \mathbb{M} \mathbb{Z} \)-structures. The rest of the section is dedicated to obtaining finite-dimensionality in the more general case.

We first recall the definition of Lebesgue covering dimension.

**Definition 3.1.** The Lebesgue covering dimension, or covering dimension, of a topological space \( X \) is the minimal integer \( n \) such that for every open cover \( \mathcal{U} \) of \( X \), there exists a refinement \( \mathcal{V} \) of \( \mathcal{U} \) where the order of \( \mathcal{V} \) is at most \( n + 1 \). In this case, we write \( \dim X = n \). If no such integer exists, \( X \) is said to be infinite-dimensional.

When we say that the order of an open covering \( \mathcal{U} \) of a space \( X \) is at most \( n \), we mean that each \( x \in X \) is in at most \( n + 1 \) elements of \( \mathcal{U} \). If \((X, d)\) is a metric space, we define the mesh of a cover \( \mathcal{U} \) as \( \text{mesh}(\mathcal{U}) = \sup\{\text{diam}(U) | U \in \mathcal{U}\} \). One can easily check that an equivalent definition of the covering dimension of a compact metric space can be
formulated as follows: a metric space $X$ satisfies $\dim X \leq n$ if and only if for every $\epsilon > 0$, there exists an open cover $\mathcal{U}$ with $\text{mesh}(\mathcal{U}) < \epsilon$ and $\text{order}(\mathcal{U}) \leq n$.

There is another method to determine if $\dim X < \infty$ for $X$ a compact metric space using $\epsilon$-maps. This method provides a quick proof that $\mathbb{Z}$-boundaries are finite-dimensional.

**Definition 3.2.** Let $\epsilon > 0$, $(X, d)$ a metric space, and $Y$ a topological space. A continuous map $f : X \to Y$ is an $\epsilon$-mapping if $\text{diam}(f^{-1}(\{y\})) < \epsilon$ for every $y \in Y$.

**Theorem 3.3.** (See [Eng78, Theorem 1.10.12]) If $X$ is a compact metric space and for every $\epsilon > 0$ there exists an $\epsilon$-mapping $f : X \to Y$ where $Y$ is a compact metric space with $\dim Y \leq n$, then $\dim X \leq n$.

**Proposition 3.4.** If a group $G$ admits a $\mathbb{Z}$-structure $(\hat{X}, \mathbb{Z})$, then $\dim \mathbb{Z} \leq \dim \hat{X}$.

**Proof.** Let $\epsilon > 0$ and $H : \hat{X} \times [0, 1] \to \hat{X}$ be a homotopy associated to the $\mathbb{Z}$-compactification. Since $H$ is uniformly continuous and $H_0 = \text{id}_{\hat{X}}$, there is some $t_\epsilon \in (0, 1]$ such that $H_{t_\epsilon}^{H_{t_\epsilon}} : \mathbb{Z} \to H_{t_\epsilon}(\mathbb{Z})$ is an $\epsilon$-map. $\mathbb{Z}$ is compact, being a closed subset of $\hat{X}$, and thus, by continuity, $H_{t_\epsilon}(\mathbb{Z})$ is also compact. Since $t_\epsilon > 0$, $H_{t_\epsilon}(\mathbb{Z}) \subseteq X$ and $\dim X < \infty$, then $\dim H_{t_\epsilon}(\mathbb{Z}) \leq \dim X < \infty$. Applying Theorem 3.3, $\dim \mathbb{Z} \leq \dim X$.

**Remark 1.** The above statement is true in the case that $\hat{X}$ is a $\mathbb{Z}$-compactification and $\mathbb{Z}$ is a $\mathbb{Z}$-set, as we only made use of the existence of a homotopy. Furthermore, the inequality in Proposition 3.4 is strict. Bestvina proves this in [Bes96] using a cohomological approach and Guilbault and Tirel prove this in [GT13] using tools from standard dimension theory.

The goal of the remainder of this section is to prove:

**Theorem A.** Let $(X, d)$ be a metric space which admits a $M\mathbb{Z}$-structure $(\hat{X}, \mathbb{Z})$. Then $\dim \mathbb{Z} < \infty$.

Notice that we cannot use Theorem 3.3 to prove the main result as it requires the range space of the $\epsilon$-mapping to be finite-dimensional. Our proof relies on the existence of a particular uniformly bounded open cover, $\mathcal{U}$, with finite order. Once such a cover exists,
because of the nullity condition, elements of the cover near infinity become arbitrarily small. Thus, we can think of small neighborhoods of infinity of the boundary as being like finite-dimensional sets. Using these covers of neighborhoods of infinity, we define covers of the boundary with arbitrarily small mesh and order bounded above by the order of \( U \).

With the sketch of the proof in mind, we are now ready to provide the details, beginning with the existence of such a cover. We will use the notation that \( B(x, r) \) is the open ball of radius \( r \) centered at \( x \) and \( \overline{B(x, r)} \) is the closed ball of radius \( r \) centered at \( x \).

**Lemma 3.5.** Suppose \( G \) acts cocompactly by isometries on a proper metric space \( X \). Then there exists a uniformly bounded open cover \( \mathcal{U} \) of \( X \) with finite order.

**Proof.** As the action of \( G \) on \( X \) is cocompact, there exists a compact subset \( C \) of \( X \) such that \( GC = X \). Choose \( r > 0 \) large enough so that \( C \subset B(x_0, r) \) for some \( x_0 \in X \). Let \( G_x = \{gx_0 | g \in G \} \) be the orbit of \( x_0 \) and let \( A \subset G_x \) be a maximal \( r \)-separated subset of \( G_x \). That is for all \( x, y \in A \) with \( x \neq y \), \( d(x, y) \geq r \) and \( A \) is maximal with respect to this property. Let \( \mathcal{U} = \{B(x, 2r) | x \in A \} \). Clearly, \( \mathcal{U} \) consists of uniformly bounded open sets, which are just translates of \( B(x_0, 2r) \). To show that \( \mathcal{U} \) is a cover, let \( y \in X \). There is some isometry \( g \in G \) so that \( gy \in C \). As \( C \subset B(x_0, r) \), then \( d(gy, x_0) < r \). Since \( g \) is an isometry, \( d(y, g^{-1}x_0) < r \). The element \( g^{-1}x_0 \in G_x \), so by maximality of \( A \), there is some \( x \in A \) such that \( d(x, g^{-1}x_0) < r \). Applying the triangle inequality: \( d(x, y) \leq d(x, g^{-1}x_0) + d(g^{-1}x_0, y) < r + r = 2r \). Hence, \( y \in B(x, 2r) \) for some \( x \in A \). To see that \( \mathcal{U} \) has finite order, first observe that the number of \( r \)-separated points in \( \overline{B(x_0, 4r)} \) must be finite (by compactness). If we let \( n \) be this maximal number of \( r \)-separated points, then \( \text{order}\mathcal{U} \leq n \). Otherwise, there are points \( x_1, x_2, ...x_{n+1} \in A \) with \( \cap_{i=1}^{n+1}B(x_i, 2r) \neq \emptyset \). Thus, \( r \leq d(x_i, x_j) < 4r \) for \( i \neq j \) and \( i, j \in \{1, 2, ..., n+1\} \). Choosing an isometry \( g \in G \) with \( gx_1 = x_0 \), the points \( gx_1, gx_2, ..., gx_{n+1} \) are \( r \)-separated and contained in \( B(x_0, 4r) \), a contradiction. Hence, \( \text{order}\mathcal{U} \leq n \). \( \square \)

We will call a cover as described in the proof of Lemma 3.5 an \( r \)-separated covering of order \( n \).
Proof of Theorem A. Let $H : \hat{X} \times [0, 1] \to \hat{X}$ be a $Z$-set homotopy with $H_0 = id\hat{X}$ and $H_t(\hat{X}) \cap Z = \emptyset$ for every $t > 0$. Let $\epsilon > 0$ and fix a metric $\hat{d}$ on $\hat{X}$.

Using Lemma 3.5, choose an $r$-separated covering $U$. Let $k < \infty$ be the order of $U$ and choose $U \in U$. Recall that all remaining elements in the cover are certain $G$-translates of $U$. Thus, by the nullity condition, we may choose a compact set $K \subseteq X$ such that $\text{diam} \, V < \epsilon/2$ for every $V \in U$ with $V \cap K = \emptyset$.

Choose $\delta_1 \in (0, 1]$ small enough such that $H_\delta(Z)$ is covered by open sets $V \in U$ with $\text{diam} \, V < \epsilon/2$ for all $\delta \leq \delta_1$. This may be accomplished because of the nullity condition.

As $U$ is a bounded open cover of $X$, there are open sets in $U$ that do not intersect $K$ and cover a neighborhood of infinity $X - K'$ where $K'$ is a compact set containing $K$. Thus, we can choose $\delta_1$ so that for every $\delta \leq \delta_1$, $H_\delta(Z) \subseteq X - K'$.

Moreover, $H : \hat{X} \times [0, 1] \to \hat{X}$ is uniformly continuous, so we may choose a $\delta_2 \in (0, 1]$ so that for every $\delta \leq \delta_2$ and for each $z \in Z$, $\hat{d}(z, H_\delta(z)) < \epsilon/4$.

Take $t_\epsilon = \min\{\delta_1, \delta_2\}$. Note then that:

1. $H_{t_\epsilon}(Z)$ is covered by open sets $V \in U$ having diameters less than $\epsilon/2$ because $t_\epsilon \leq \delta_1$ and
2. $\hat{d}(z, H_{t_\epsilon}(z)) < \epsilon/4$ for every $z \in Z$, since $t_\epsilon \leq \delta_2$.

Consider $\mathcal{V}_\epsilon = \{V \in U | V \cap H_{t_\epsilon}(Z) \neq \emptyset \text{ and } V \cap K = \emptyset\}$. That is, $\mathcal{V}_\epsilon$ is an open cover of $H_{t_\epsilon}(Z)$ with mesh bounded by $\epsilon/2$ and order bounded by $k$.

Define $\mathcal{W}_\epsilon = \{H_{t_\epsilon}|^{-1}_Z(V) | V \in \mathcal{V}_\epsilon\}$.

Clearly, $\mathcal{W}_\epsilon$ forms a cover of $Z$ since $\mathcal{V}_\epsilon$ forms a cover of $H_{t_\epsilon}(Z)$. Each $W \in \mathcal{W}_\epsilon$ is also open as it is the pre-image of an open set under a continuous map.

We now show that $\text{diam} W < \epsilon$ for every $W \in \mathcal{W}_\epsilon$. Let $z_1, z_2 \in W$. Then

$$\hat{d}(z_1, z_2) \leq \hat{d}(z_1, H_{t_\epsilon}(z_1)) + \hat{d}(H_{t_\epsilon}(z_1), H_{t_\epsilon}(z_2)) + \hat{d}(H_{t_\epsilon}(z_2), z_2) < \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon$$

The last inequality is due to (1) and (2) from above. Every pair of points in $V_g$ is within a distance of $\epsilon$ from one another, so $\text{diam} W < \epsilon$ for all $W \in \mathcal{W}_\epsilon$. 
Lastly, the order of the cover $\mathcal{V}_\epsilon$ of $H_{t,\epsilon}(Z)$ is at most $k$. Since $\mathcal{W}_\epsilon$ is the set of pre-images of $\mathcal{V}_\epsilon$ under the continuous map $H_{t,\epsilon}|_Z$, then $\mathcal{W}_\epsilon$ also has order at most $k$.

\[\square\]

**Remark 2.** Theorem 2.4 now follows directly from Theorem A.

**Remark 3.** If desired, we could give an upper bound on the dimension of $Z$ by letting $m$ be the minimum over all orders of $r$-separated coverings of $X$. Then $\dim Z \leq m - 1$.

In practice, we do not need the complete generality of Theorem A. In particular, we usually require our group actions to be proper, cocompact, and by isometries (i.e. geometric). Adding properness eliminates the need to use Lemma 3.5 to obtain a single uniformly bounded cover with finite order. We conclude this section with a few words about this particular case and what it means for the dimension of $Z$-boundaries in Dranishnikov’s definition of a $Z$-structure.

Suppose $G$ admits a $Z_{n.f.}^{AR}$-structure. Since $G$ acts cocompactly on $X$, there is an open subset $U$ of $X$ such that $\overline{U}$ is compact and the set of all $G$-translates of $U$ cover $X$, that is $\bigcup_{g \in G} gU = X$. Furthermore, since the action of $G$ on $X$ is proper, the set $\{g \in G | gU \cap U \neq \emptyset\}$ is finite. Let $k \in \mathbb{Z}^+$ be this finite number of $G$-translates of $U$ that intersect $U$. Since the cover of $X$ was formed in a “nice” way by the group action, the cover looks the same everywhere. That is, any translate of $U$ can also intersect only $k$ other translates. Thus, the order of a cover constructed in this way is at most $k$. We will call this type of cover a **covering by translations of order at most $k$**.

**Theorem 3.6.** Let $G$ be a group which admits a $Z_{n.f.}^{AR}$-structure $(\hat{X}, Z)$. Then the dimension of $Z$ is bounded above by $k - 1$, where $k$ is the minimum over orders of covering by translations of $X$.

**Proof.** Repeat the proof of Theorem A choosing the cover $\mathcal{U}$ to be a covering by translations of order at most $k$. \[\square\]

**Remark 4.** We could in fact lower the bound in Theorem 3.6 to $k - 2$ by combining the strategy used here with a technique found in [GT13].

**Corollary 3.7.** If $G$ admits a $Z$-structure $(\hat{X}, Z)$ in the sense of [Dra06], then $\dim Z < \infty$. 
4. Consequences of Finite-Dimensionality of $\mathbb{Z}$-Boundaries

The goal of this section is to demonstrate how knowing the covering dimension of the various formulations of $\mathbb{Z}$-boundaries can serve to unify the theories of group boundaries presented by Bestvina and Dranishnikov. First, any result about the cohomological dimension of the boundary may now be replaced with a statement concerning the Lebesgue covering dimension (or vice-versa). This fact allows us to see the results from [Bes96] and [Dra06] within the context of a consistent dimension theory. Secondly, we show that there is no advantage in restricting ourselves to working with an ER rather than an AR by proving the following:

**Theorem B.** Suppose a group $G$ admits a $\mathbb{Z}_{AR}$-structure. Then $G$ admits a $\mathbb{Z}$-structure.

The proof of Theorem B relies on a more general version of Bestvina’s boundary swapping theorem. Given that $G$ admits a $\mathbb{Z}$-structure, the original version of boundary swapping [Bes96, Lemma 1.4] provides a method to take the $\mathbb{Z}$-boundary from the $\mathbb{Z}$-structure and place it on another finite-dimensional space admitting an action by $G$ to obtain a new $\mathbb{Z}$-structure on $G$. Since Bestvina only worked in the ER setting, if we are to prove Theorem B, we need a more general version of boundary swapping that allows one of the spaces to be infinite-dimensional. We can easily obtain this more general version by knowing finite-dimensionality of $\mathbb{Z}_{AR}$-boundaries. We should note here that in Appendix A, we present an alternative approach to boundary swapping suggested by Ferry [Fer00] that does not rely on finite-dimensionality of the boundary. This second version could also be used to prove Theorem B.

**Theorem 4.1** (Boundary Swapping: Version 1). Let $G$ be a group acting properly, cocompactly, and freely on an ER $X_1$ and an AR $X_2$. Assume that $X_1$ and $X_2$ are $G$-homotopy equivalent and $\hat{X}_2 = X_2 \cup Z$ is a $\mathbb{Z}_{AR}$-structure on $G$. Then $(\hat{X}_1, Z)$ is a $\mathbb{Z}$-structure on $G$.

*Proof.* Since $G$ admits a $\mathbb{Z}_{AR}$-structure, by Theorem 3.6, $\dim Z < \infty$. A key component in Lemma 1.4 of [Bes96] is proving that $\hat{X}_1$ is an ANR. This is accomplished by showing that $\hat{X}_1$ is locally-contractible. The equivalence of local-contractibility and being an ANR
requires finite-dimensionality (see [Hu65, Page 168, Theorem 7.1]). Since we know $Z$ is finite-dimensional, even if $(\hat{X}_2, Z)$ in an infinite-dimensional $\mathcal{Z}_{AR}$-structure, $\hat{X}_1$ is finite-dimensional, so Bestvina’s original proof is still valid in this more general setting.

The last ingredient in the proof of Theorem B is the following deep result by West from ANR theory which will provide the ER onto which we may add the desired boundary:

**Theorem 4.2.** [Wes77] Every compact ANR is homotopy equivalent to a finite complex.

**Proof of Theorem B.** Let $(\hat{X}, Z)$ be a $\mathcal{Z}_{AR}$-structure for $G$. The map $p : X \rightarrow X/G$ is a covering projection, so $X/G$ is a compact ANR. Theorem 4.2 says that $X/G$ is homotopy equivalent to a finite complex $Y$. Lifting the homotopies to the universal cover $\tilde{Y}$, an ER, we obtain a $G$-equivariant homotopy equivalence between $X$ and $\tilde{Y}$. Applying Theorem 4.1, $\tilde{Y} \cup Z$ is a $\mathcal{Z}$-structure for $G$. □

The proof of Theorem B provides a different upper bound for the dimension of the boundary. Recall that a group $G$ has finite geometric dimension if there exists a finite-dimensional $K(G, 1)$ space. In that case, $gdG = \min\{\dim K | K$ is a $K(G, 1)\}$. Thus, we obtain the following:

**Corollary 4.3.** Suppose a group $G$ admits a $\mathcal{Z}_{AR}$-structure $(\hat{X}, Z)$, then $\dim Z \leq gdG$

**Proof.** Suppose that $(\hat{X}, Z)$ is a $\mathcal{Z}_{AR}$-structure on $G$. The finite complex $Y$ in the proof of Theorem B is a finite $K(G, 1)$ space for $G$. Since $\tilde{Y} \cup Z$ is a $\mathcal{Z}$-structure for $G$, Proposition 3.4 ensures $\dim Z \leq \dim Y$, and therefore the dimension of $Z$ is bounded above by the geometric dimension of $G$. □

**Remark 5.** Again, we can get the inequality in Corollary 4.3 to be strict using the result that $\dim Z < \dim Y$ [Bes96, GT13].

While Theorem B shows that there is no reason to limit our attention to ER’s, and is therefore one step closer towards bridging Bestvina and Dranishnikov’s definitions, it does not completely bridge the gap. One would hope to generalize Theorem B as follows: If a group $G$, not necessarily torsion-free, admits a $\mathcal{Z}$-structure in the sense of [Dra06], then
$G$ admits a $\mathbb{Z}$-structure in the sense of [Bes96] (modulo the freeness requirement found therein.)

However, notice that the proof of Theorem B relies heavily on the use of covering space theory. In particular, once we permit groups with torsion, we cannot obtain the required equivariant homotopies using lifting theorems. One idea to fix this complication is to use the theory of $EG$ complexes. When $EG$ complexes exist, they are well-defined up to $G$-equivariant homotopy equivalence. Thus, we leave the reader with an important open question that, if answered in the affirmative, would help with the unification of the theory of $\mathbb{Z}$-boundaries found in [Bes96] and [Dra06].

**Question:** If $G$ admits a $\mathbb{Z}_{AR}^{n,f}$-structure $(\hat{X}, Z)$, does there exists a cocompact $EG$ complex? Furthermore, must $X$ be $G$-equivariantly homotopic to that complex?

**Appendix A. An Alternative Approach to Boundary Swapping**

In [Fer00, Remark 1.7(ii)], Ferry suggests an alternative approach to boundary swapping. In that remark, he restricts attention to $\mathbb{Z}$-compactifications of universal covers of finite $K(G, 1)$ complexes. However, we will show that the suggested proof applies more generally. Furthermore, since Hanner’s Criterion [Han51], rather than local-contractibility, is used to detect the ANR property, the hypothesis that the boundary be finite-dimensionality is not required. As a corollary, one may obtain an alternative proof of Theorem B and also a new approach to proving finite-dimensionality of $\mathbb{Z}_{AR}$-boundaries.

Because of its relevance to this paper, we use this appendix to fill in the details to Ferry’s approach to boundary swapping and discuss its connections to finite-dimensionality of boundaries.

We begin with the statement of the second version of boundary swapping and then introduce a few results that are needed for the proof.

**Theorem A.1** (Boundary Swapping: Version 2). Let $G$ be a group acting properly and cocompactly on ARs $X$ and $Y$. Suppose that $f : X \to Y$ is a $G$-equivariant homotopy equivalence. If $\hat{Y} = Y \cup Z$ is a $\mathbb{Z}_{AR}$-structure on $G$, then we may topologize $\hat{X} = X \cup Z$ so that $(\hat{X}, Z)$ is also a $\mathbb{Z}_{AR}$-structure on $G$. 
We first describe the topology on $\hat{X} = X \cup Z$:

**Definition A.2.** Let $f : X \to Y$ be a proper map between ANRs. If $\hat{Y} = Y \cup Z$ is a $Z$-compactification of $Y$, define $\overline{f} : \hat{X} = X \cup Z \to \hat{Y}$ to be the identity on $Z$ and $f$ on $X$. The topology on $\hat{X}$ is generated by the open subsets of $X$ and sets of the form $\overline{f}^{-1}(U)$ where $U \subset \hat{Y}$ is open.

The foundation of the proof of the second version of boundary swapping is the following theorem from Ferry which describes when we know a closed subset of a space is a $Z$-set. We point out here that Ferry’s definition of a $Z$-set is not restricted to ANR’s, but allows for any metric space. We use the same terminology for both cases, but will take care to distinguish between the two in the proof of Theorem A.1 as we ultimately need to work in the ANR setting.

**Theorem A.3.** [Fer00] Let $(\hat{X}, Z)$ and $(\hat{Y}, Z)$ be compact metric spaces that are homotopy equivalent rel$Z$ by maps and homotopies which are the identity on $Z$ and which take the complement of $Z$ to the complement of $Z$. Then $Z$ is a $Z$-set in $\hat{X}$ if and only if $Z$ is a $Z$-set in $\hat{Y}$.

Now we are ready to give the proof of Theorem A.1.

**Proof of Theorem A.1.** By assumption we have $G$-equivariant maps and homotopies

\[
\begin{align*}
  f &: X \to Y \\
  h &: Y \to X \\
  H &: Y \times [0,1] \to Y, \quad H_0 = f \circ h, \quad H_1 = id_Y \\
  F &: X \times [0,1] \to X, \quad F_0 = h \circ f, \quad F_1 = id_X
\end{align*}
\]

We claim that $\overline{f} = f \cup id_Z$ is a homotopy equivalence rel$Z$.

Define maps:

\[
\begin{align*}
  \overline{f} &: \hat{Y} \to \hat{X}, \text{ by } \overline{f} = f \cup id_Z \\
  \overline{H} &: \hat{Y} \times [0,1] \to \hat{Y}, \text{ by } \overline{H}(y,t) = H(y,t) \text{ for } y \notin Z \text{ and identity else.} \\
  \overline{F} &: \hat{X} \times [0,1] \to \hat{X}, \text{ by } \overline{F}(x,t) = F(x,t) \text{ for } x \notin Z \text{ and identity else.}
\end{align*}
\]
Since the maps are the identity on $Z$, send complements of $Z$ to complements of $Z$, and $\overline{H}_0 = \overline{f} \circ \overline{h}$, $\overline{H}_1 = \text{id}_{\hat{Y}}$, $F_0 = \overline{H} \circ \overline{f}$, and $F_1 = \text{id}_{\hat{X}}$, all we must check is the continuity of the maps at points of $Z$. Then, applying Theorem A.3, we will have a homotopy $H : \hat{X} \times [0,1] \to \hat{X}$ such that $H_0 = \text{id}_{\hat{X}}$ and $H_t(\hat{X}) \cap Z = \emptyset$ for all $t > 0$. As mentioned above, Theorem A.3 only requires $\hat{X}$ to be a metric space, but the existence of such a homotopy together with an application of Hammner’s Criterion for ANRs [Han51] proves that $\hat{X}$ is indeed an ANR and thus $\hat{X}$ is a $Z$-compactification of $X$.

First, note that any open set containing $z \in Z \cap \hat{X}$ must be of the form $\overline{f}^{-1}(U)$ where $U$ is an open set in $\hat{Y}$ containing $z$. So, rather than picking arbitrary open sets in $\hat{X}$ for continuity arguments on $Z$, we can pick arbitrary open sets in $\hat{Y}$.

Let $z \in Z$ and $U$ any open neighborhood of $z$ in $\hat{Y}$. Choose $\epsilon > 0$ such that $\overline{B}(z, \epsilon) \subseteq U$. By cocompactness, choose compact sets $K_Y \subset Y$ and $K_X \subset X$ with $GK_Y = Y$ and $GK_X = X$. Let $L = \overline{f} \circ \overline{F}(K_X \times [0,1]) \cup \overline{H}(K_Y \times [0,1])$, a compact subset of $Y$. Since $\mathcal{U} = \{U, \hat{Y} - \overline{B}(z, \epsilon)\}$ forms an open cover of $\hat{Y}$, by the nullity condition, there exists a finite subset $\Gamma \subset G$ such that $\forall g \in G - \Gamma$, $gL \subseteq U$ or $gL \cap \overline{B(z, \epsilon)} = \emptyset$. Set

$$V = B(z, \epsilon) - \left( \bigcup_{g \in \Gamma} gL \right)$$

**Claim 1:** $\overline{h}(V) \subseteq \overline{f}^{-1}(U)$, so $\overline{h}$ is continuous at $z \in Z$.

If $y \in V \cap Y$, choose $g \in G$ such that $y \in gK_Y$. Then

$$\overline{f} \circ \overline{h}(y) \in \overline{f} \circ \overline{h}(gK_Y)$$

$$= g\overline{f} \circ \overline{h}(K_Y)$$

$$\subseteq gL$$

$$\subseteq U$$

**Claim 2:** $\overline{H}(V \times [0,1]) \subseteq U$, so $\overline{H}$ is continuous at $z \times [0,1]$ for all $z \in Z$.

If $y \in V \cap Y$, choose $g \in G$ such that $y \in gK_Y$. Then

$$\overline{H}(y \times [0,1]) \subseteq \overline{h}(gK_Y \times [0,1])$$

$$= g\overline{H}(K_Y \times [0,1])$$
FINITE-DIMENSIONALITY OF Z-BOUNDARIES 17

\[ \subseteq gL \]
\[ \subseteq U \]

Claim 3: \( \overline{F}(f^{-1}(V) \times [0, 1]) \subseteq \overline{f}^{-1}(U) \), so \( \overline{F} \) is continuous at \( z \times [0, 1] \) for \( z \in Z \).

If \( x \in \overline{f}^{-1}(V) \cap X \), choose \( g \in G \) such that \( x \in gK \). Then

\[
\overline{f} \circ \overline{F}(x \times [0, 1]) \subseteq \overline{f} \circ \overline{F}(gK \times [0, 1])
\]

\[
= g \overline{f} \circ \overline{F}(K \times [0, 1])
\]

\[ \subseteq gL \]
\[ \subseteq U \]

All that remains to be shown is the nullity condition. Let \( K \) be a compact subset of \( X \) and \( \mathcal{U} \) any open cover of \( \hat{X} \). Let

\[ \mathcal{V} = \{ U \in \mathcal{U} | U = \overline{f}^{-1}(W), W \text{ open in } \hat{Y}, U \cap Z \neq \emptyset \} \]

\[ \mathcal{W} = \{ W | \overline{f}^{-1}(W) \in \mathcal{V} \} \]

\( \mathcal{V} \) and \( \mathcal{W} \) are open covers of \( Z \) in \( \hat{X} \) and \( \hat{Y} \), respectively.

\[ A = \hat{X} - \bigcup_{V \in \mathcal{V}} V \]

is a compact subset of \( X \), so by properness of the action, there exists a finite subset \( \Gamma_1 \subseteq G \) such that \( gK \cap A = \emptyset \) for all \( g \in G - \Gamma_1 \).

Now we must fill out the open cover in \( \hat{Y} \). Let

\[ B = \hat{Y} - \bigcup_{W \in \mathcal{W}} W \]

\[ \mathcal{B} = \{ B(y, r_y) | y \in B, r_y = \frac{1}{2}d(y, Z) \} \]

Set \( \mathcal{W}' = \mathcal{W} \cup \mathcal{B} \), an open cover of \( \hat{Y} \). Since \( B \) is compact and \( \mathcal{B} \) is an open cover of \( B \), there exists a finite subcover \( \mathcal{B}' \subseteq \mathcal{B} \). Set

\[ C = \bigcup_{\mathcal{B}} \overline{B(y, r_y)} \]

By properness, there exists a finite subset \( \Gamma_2 \subseteq G \) such that for all \( g \in G - \Gamma_2 \), \( g \overline{f}(K) \cap C = \emptyset \). Furthermore, by the nullity condition, there exists a finite subset \( \Gamma_3 \subseteq G \)
such that for all \( g \in G - \Gamma_3 \), \( g\overline{f}(K) \subseteq W \) for \( W \in \mathcal{W}' \).

Thus, let \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \), a finite subset of \( G \). Then \( \forall g \in G - \Gamma \), \( g\overline{f}(K) \subseteq W \) for \( W \in \mathcal{W} \), so by equivariance of the action, \( gK \subseteq \overline{f}^{-1}(W) \in \mathcal{V} \subseteq \mathcal{U} \).

\[ \square \]

Using the second version of Boundary Swapping, we recover finite-dimensionality of the boundary, but in a much less generalized form than is found in the statement of Theorem A. In particular, the second version of Boundary Swapping proves that a \( \mathcal{Z}_{AR} \)-boundary must be finite-dimensional. The proof is essentially the same as the proof of Corollary 4.3. Suppose that \((\hat{X}, Z)\) is a \( \mathcal{Z}_{AR} \)-structure on \( G \). Then \( X/G \) is homotopy equivalent to a finite \( K(G,1) \) space for \( G \). Lifting the homotopies gives a proper \( G \)-equivariant homotopy equivalence between \( X \) and \( \tilde{K} \), the universal cover of \( K \). By Theorem A.1, \((\tilde{K} \cup Z, Z)\) is a \( \mathcal{Z} \)-structure on \( G \). Proposition 3.4 ensures \( \text{dim}Z \leq \text{dim}K < \infty \).
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