VAN DER WAALS $\sigma$-MODEL
AND TOPOLOGICAL EXITATIONS
WITH LOGARITHMICAL ENERGY.

S.A. Bulgadaev

L.D. Landau Institute for Theoretical Physics
Kosygin Str. 2, Moscow, 117334, RUSSIA

A talk given at seminar
"Properties and Dynamics of Defects
in Liquid Crystals"

23 August 1999, MPIPKS, Dresden, Germany
I. INTRODUCTION

The stable topological excitations (TE) can exist in systems with degenerate minima, which form some manifold $\mathcal{M}$ with nontrivial topology. Such TE take place in many physical systems, in particular, in superfluid $^3$He and $^4$He, in different liquid crystals, in magnets. The two main problems are connected with the existence of TE:

1. Their classification and properties.
2. Their influence on the properties of the systems.

The most important property of the TE is connected with their influence on the behaviour of correlations in the systems. In general, this influence depends on interrelations between such fundamental properties of the systems as symmetry and topology. For example, the influence of the TE is especially strong in systems with initial scale invariance, since the TE introduce in the theory a new effective mass scale

$$m \sim a^{-1} \exp(-S_{TE}),$$

where $a$ is some UV cut-off parameter (a core radius or a lattice constant) and $S_{TE}$ is the dimensionless "energy" (or action) of the TE. In the cases, when one has many TE, they can form some textures, ordered or disordered, which also influence on behaviour of the systems. In these cases the influence depends also on the interaction between TE, which, in its turn, is determined again by interrelations between symmetries and geometry of the systems. If an interaction of TE is strong enough, it can induce the topological phase transition (TPT) in the system of TE, which can drastically change a character of correlations in system.

This effect, connected with TPT, is well studied in low-dimensional ($D \leq 2$) systems [2-4,5-9]. In order to understand its main reasons let us resume properties of such low-dimensional systems.

II. THE TE WITH LOGARITHMIC ENERGY IN LOW-DIMENSIONAL NS-MODELS

For investigation of the topological properties it is convenient to consider nonlinear $\sigma$-models (NSM) on the spaces $\mathcal{M}$, which are the long-wave approximation of the corresponding Ginzburg - Landau (GL) type theories. The topological properties of the space $\mathcal{M}$ are described by the homotopic groups $\pi_i(\mathcal{M}), i = 0, 1,\ldots$.

1. $D = 2$

The most popular two-dimensional NSM are the next:

1. NS-model on a circle $S^1$, which is defined by the action

$$S = \frac{1}{2\alpha} \int d^2x |\partial \psi|^2, \quad \psi = \exp(i\phi) \in S^1.$$
The relevant homotopical groups are
\[ \pi_i(S^1) = 0, \ i \neq 1, \ \pi_1(S^1) = \mathbb{Z}. \] (3)

Due to this, only one type of TE, the vortices, exist in this model. Under a vortex one can understand any TE (with a nontrivial vorticity or pure potential), corresponding to \( \pi_1(S^1) = \mathbb{Z} \). The energy of one vortex with topological charge \( e \in \mathbb{Z} \) is logarithmically divergent
\[ E = \frac{e^2}{2\alpha} 2\pi \ln \frac{R}{a}, \] (4)
where \( R \) is a space radius. The energy of \( N \)-"vortex" solution, \( E_N \), with the full topological charge \( e = \sum_{i}^N e_i = 0 \) is finite and equals
\[ E_N = \frac{2\pi}{2\alpha} \sum_{i \neq k}^N e_i e_k \ln \frac{|x_i - x_k|}{a} + C(a) \sum_{i}^N e_i^2, \] (5)
here \( C(a) \) is some nonuniversal constant, determining "self-energy" (or a core energy) of vortices and depending on type of a core regularization. From (5) it follows that vortices interact through logarithmic potential. Just this interaction of vortices induces the TPT in all 2D systems with continuous abelian symmetry, described by this NSM [5-7].

2. NS-model on a sphere \( S^2 \) is defined by action
\[ S = \frac{1}{2\alpha} \int d^2x (\partial \mathbf{n})^2, \quad \mathbf{n} \in S^2. \] (6)

The relevant homotopical groups are
\[ \pi_i(S^2) = 0, \ i < 2, \ \pi_2(S^2) = \mathbb{Z}. \] (7)

There is also only one type of TE, the instantons, with topological charges \( q \in \mathbb{Z} \) equal to the degree of the corresponding mapping \( S^2 \to S^2 \). The energy of \( N \) instantons with topological charges \( q_i, \ i = 1,..N \)
\[ E_N = \frac{4\pi}{\alpha} \sum_{i=1}^N |q_i| \] (8)
Thus, the instantons do not interact and only weak, dipole-dipole like, interaction exist between instantons and anti-instantons [10]. The different types of interactions of TE in these models is determined by different character of TE, corresponding to groups \( \pi_1(S^1) \) and \( \pi_2(S^2), \) in \( D = 2 \).

In 2D the TE, described by \( \pi_1(S^1) \), correspond to the open space boundary and have logarithmic energy.
while the TE, described by $\pi_2(S^2)$, correspond to the boundary shrunk into one point and for this they have finite energy.

In principle, in both models the TE of vortex and instanton types are possible. In the first model the "neutral" configurations of vortices also correspond to the shrunk space boundary and could be classified by $\pi_2(S^1)$. But, since $\pi_2(S^1) = 0$, all neutral configurations belong to one type. Analogously, in the second model the vortices (or merons) are possible. Due to $\pi_1(S^3) = 0$, they are unstable. Their "neutral", dipole-like, configurations can have different topological structure, corresponding to $\pi_2(S^2) = \mathbb{Z}$, and are stable.

Analogous situation takes place in

2. 1D systems with long-range interaction of type

$$J(r) \sim 1/r^2.$$  \hspace{1cm} (9)

This form of $J(x)$ is determined by the scale (even conformal) invariance \[12\].

The corresponding 1D NS-models have the next Hamiltonian

$$\mathcal{H} = \frac{1}{2\alpha} \int dx dx' J(x-x')(s(x)-s(x'))^2.$$  \hspace{1cm} (10)

Here, due to $D = 1$, the important manifolds and homotopical groups are

$$\mathcal{M} = \{s_i\}, \quad i = 1, \ldots, p, \quad \pi_0(\{s_i\}) \neq 0, \quad \pi_i = 0, \quad i \geq 1;$$

$$\mathcal{M} = S^1, \quad \pi_1(S^1) = \mathbb{Z},$$

where $s$ is a classical spin and a number of discrete spin states $p$ can be finite or infinite \[2, 3, 4\]. To the discrete set $\{s_i\}$ and the open boundary correspond domain walls or kinks, which connect different minima, and to the compactified space $\mathbb{R}^1$, i.e. $S^1$, correspond 1D instantons, which are really the vortices, since they correspond to $\pi_1(S^1)$ \[12, 13\].

In 1D case the kinks described by $\pi_0(\{s\})$ have logarithmic energy and can induce TPT, while instantons have finite energy and do not interact between themselves.

**Resume:**

In low-dimensional systems ($D \leq 2$) the TE with logarithmic energy exist in systems described by

1) **conformal invariant NS-models,**

2) **defined on spaces with discrete abelian group** $\pi_{D-1} \neq 0$. 

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These TE can induce TPT, which describe such important physical phenomena as localization-delocalization [14], coherence-decoherence [15], melting [8] and others. All these transitions change a character of correlations in systems.

Then, the two main **Questions** appear:

1) Can such TE exist in higher-dimensional systems?

2) Can they influence on correlations so strong?

In this talk we propose one model which helps us to answer on the first question and can help answer on the second question.

**III. 3D VAN DER WAALS NS-MODEL ON \( S^2 \)**

1. Motivation.

An investigation of the possibility of existence of the TE with logarithmic energy in 3D scale invariant systems and their influence on correlations.

2. Model.

Let us consider 3D lattice with the order parameter (OP) \( n \) in each lattice site. The OP can be:

1) a unit vector \( n, \quad n^2 = 1, \quad n \in S^2 \), it can represent a magnet;
2) a unit rod or a director \( n \in S^2/Z_2 = RP^2 \), it can represent a liquid crystal or molecular crystal.

Since the relevant homotopy groups are [14]

\[
\pi_2(S^2) = \pi_2(RP^2) = \mathbb{Z},
\]

\[
\pi_3(S^2) = \pi_3(RP^2) = \mathbb{Z},
\]

it will be more convenient to consider the OP \( n \in S^2 \). All results will takes place with non-essential modifications for \( n \in RP^2 \) too. Due to its vectorness, the OP can interact by different type of interactions:

1) exchange type \( \sim (n_r \cdot n_{r'})V(r - r') \),
2) dipole-like \( \sim (n^i_r D_{ik}(r - r') n^k_{r'})V(r - r') \quad V(r) \sim 1/r^3 \),
3) van der Waals type \( V_{vdW}(r) \sim 1/r^6 \).

All they can be represented in the next form

\[
E \sim (n^i_r D_{ik} n^k_{r'})V(r - r'), \quad V(r) \sim 1/r^6.
\]
where $\sigma$ defines an asymptotic behaviour of the potential $V(r)$.

As a first approximate attempt to the problem one can compose from all these types of interaction the simplified one, which must conserve the two main properties:

1) a scale invariance of the corresponding Hamiltonian $H$,
2) a vectorness of the OP.

In result one gets the lattice vector van der Waals model with $H$

$$H = -\frac{J}{2} \sum_{r \neq r'} (\mathbf{n}_r \cdot \mathbf{n}_{r'}) V_{vdW}(r - r'). \quad (14)$$

The analogous approximation, for example, was used by Nelson [8] in the theory of 2D melting.

In long-wave continuous approximation our model passes into the vector van der Waals NS-model with a partition function

$$Z_{vdW} = \int D\mathbf{n} e^{-S_{vdW}[\mathbf{n}]};$$

$$S_{vdW}[\mathbf{n}] = -\frac{1}{2\alpha} \int d^3 x d^3 x' (\mathbf{n}(x) \mathbf{n}(x')) V_{vdW}(x - x'), \quad (15)$$

$$V_{vdW}(x) = \int \frac{d^3 k}{(2\pi)^3} e^{i(kx)} |k|^3 f(ka) \sim 1/|x|^6 \quad (16)$$

where

$$\mathbf{n}^2 = 1, \quad \alpha \sim 1/J \beta, \quad (17)$$

$f(ka)$ is a regularizing function with next asymptotics

$$f(ka)_{ka < 1} \simeq 1 + O(ka), \quad f(ka)_{ka > 1} \rightarrow 0. \quad (18)$$

From now on we omit an index vdW for brevity. A scale invariance of the model at large distances follows immediately from large-distance asymptotics of $V(x)$ and dimensionlessness of the OP $\mathbf{n}$. Moreover, $S$ is conformal invariant at large distances, i.e. it is invariant under conformal transformations:

$$x_i \rightarrow x_i' = x_i/x^2, \quad x \rightarrow x' = 1/x, \quad x_i/x = x_i'/x',$$

$$d^3 x \rightarrow d^3 x/|x|^6, \quad \frac{1}{|x_1 - x_2|^6} \rightarrow \frac{|x_1|^6 |x_2|^6}{|x_1 - x_2|^6} \quad (19)$$

and, consequently,

$$S \sim \int d^3 x_1 d^3 x_2 \frac{(\mathbf{n}_1 \cdot \mathbf{n}_2)}{|x_1 - x_2|^6} \rightarrow S$$

For this reason this model can be named also the 3D conformal NS-model [17].
The corresponding Euler-Lagrange equation has a form
\[ \int V(x - x') n(x') d^3x' - n(x) \int (n(x)n(x')) V(x - x') d^3x' = 0. \] (20)

The Green function \( G(x) \) of the conformal kernel \( V(x) \) we define by next equation
\[ \int V(x - x'') G(x'' - x') d^3x'' = \delta(x - x') \] (21)

It has the following form
\[ G(x) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i(kx)}}{k^3 f(ka)} \Bigg|_{r \gg a} \simeq -\frac{2\pi}{(2\pi)^3(2)^{1/2} \Gamma(3/2)} \ln(r/R) \] (22)

and logarithmic asymptotic behaviour.

The action (15) can be represented in a form, admitting a small deviation expansion of \( n \)
\[ S[n] = \frac{1}{4\alpha} \int d^3x d^3x' (n(x) - n(x'))^2 V(x - x'). \] (23)

In this case in its Fourier form can appear a usual, \( \sim k^2 \), term which breaks a scale invariance. Then one can consider a more general model, including a local gradient term in an explicit form
\[ S_{loc} = \frac{1}{2\alpha'} \int d^3x (\partial n)^2. \] (24)

This term is not scale invariant in 3D and for this reason the whole action
\[ S_{tot} = S + S_{loc} \] (25)

is also not scale invariant. Its form in Fourier space will be
\[ S_{tot} = S_{loc} + S = \int \frac{d^3k}{(2\pi)^3} |n(k)|^2 \left( \left( \frac{1}{2\alpha'} + C \right) k^2 + \frac{1}{2\alpha} k^3 + \ldots \right), \] (26)

A similar action was obtained earlier in the theory of liquid crystals [18]. If in the system there is a point, where a "rigidity" \( \left( \frac{1}{2\alpha'} + C \right) = 0 \), then in this point one obtains a scale invariant action (15) with a first term \( \sim k^3 \). From a point of view of the GL theory this is similar to the tricritical point, where a term \( \sim \psi^4 \) is absent in the expansion of effective potential of the theory on nonlinearities [19].
IV. THE TE WITH LOGARITHMIC ENERGY
IN 3D VAN DER WAALS NS-MODEL ON $S^2$

A nontriviality of $\pi_2(S^2) = \mathbb{Z}$ means that in the model there are point-like TE, corresponding to the open boundary. The simplest such excitation is the "hedgehog". It is a solution of equation (20) and has the next asymptotic form

$$ n(x)|_{|x| > a} \simeq \frac{x^i}{|x|}. $$

(27)

The action $S$ of this solution is

$$ S = 4\frac{(4\pi)^2}{2\alpha} \frac{4\pi}{(2\pi)^3} \int \frac{dk}{k} f(ka) $$

$$ \approx 4\frac{(4\pi)^2}{2\alpha} \frac{4\pi}{(2\pi)^3} \ln(R/a). $$

(28)

Its topological charge $q \in \mathbb{Z}$ is the degree of the corresponding mapping. The energy of two "hedgehogs" with a full topological charge $q = 0$ is finite and the interaction of two "hedgehogs" with charges $q_1$ and $q_2$ on large distances has a form of the Green function $G(x)$ of the kernel $V(x)$

$$ H_{12}(r) = e_1 e_2 G(r) \simeq -e_1 e_2 \frac{2\pi}{(2\pi)^3 (2)^{3/2} \Gamma(3/2)} \ln(r/R). $$

(29)

Note that in the usual local 3D NS-model (24) such TE have the energy linear in $R$

$$ E \sim \frac{R}{\alpha}. $$

(30)

A nontriviality of another homotopical group $\pi_3(S^2) = \mathbb{Z}$ means that the "neutral" configurations, having a full topological charge equal to 0 and corresponding to the shrinked boundary, can also have different topological structures. They are characterized by topological invariant, coinciding with the Hopf invariant $H \in \mathbb{Z}$ of the corresponding mapping $S^3 \to S^2$. This invariant is connected with linking number and can be expressed through the integrals over $\mathbb{R}^3$ or over compactified space $\mathbb{R}^3 \simeq S^3$. Some its properties are described in Appendix. Just this invariant is an analog of the topological charge of 2D instantons. Note, that this additional topological invariant classifies "neutral" configurations in all 3D NS-models defined on sphere $S^2$, in particular, in the usual local model (24). Thus, the "hedgehog" excitations in 3D van der Waals NSM have properties reminiscent of the mixed properties of the two-dimensional vortices and instantons:

1) their topology is described by $\pi_2(S^2)$,
2) their "neutral" configurations are classified by integer topological Hopf invariant $H$. 

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V. DISCUSSION, APPLICATIONS

1. A possibility of TPT

The TE with logarithmic energy can induce TPT in system of such TE. Simple arguments by Kosterlitz and Thouless show this for any dimensional case. Let us admit that in D-dimensional system there is such TE with energy

\[ E = A \ln R/a. \]

Consider the corresponding free energy \( F \) per one such TE

\[ F = E - TS, \]

where \( T \) is a temperature, \( S \) is an entropy. One can estimate the entropy of the TE as a logarithm of the number of possible places of the TE in space

\[ S = k_B \ln(R/a)^D = k_B D \ln(R/a). \]

Then one has for free energy of one TE

\[ F = E - TS = (A - k_B D T) \ln(R/a) \]

Free energy \( F \) becomes equal to 0 and changes it sign at

\[ T_{KT} = A/k_B D. \]

It means that at \( T > T_{KT} \) it becomes energetically favorable to birth such TE and they can generate spontaneously, while at \( T < T_{KT} \) one needs positive free energy to birth these TE. More detailed study of this transition in low-dimensional systems has shown that these arguments correspond to the first order approximation in the renorm-group (RG) approach. In higher orders of RG the contributions, taking into account more detailed information about geometry of space \( \mathcal{M} \), type of the corresponding topological charges and their interaction, appear in RG equations. In principle, than more complicated topological structures appear, then different complications, destroying this TPT, become more probable.

In 3D van der Waals NS-model there are some additional complications, which can remove the TPT or complexify its study:

1) additional topological invariant, the Hopf invariant \( H \), which distinguishes different "neutral" configurations;

2) one needs to conserve a condition, equalizing a rigidity to 0, in order a logarithmic interaction can not be screened during renormalization;

3) a conformal symmetry, which is, on our opinion, a hidden reason of existence of TPT in low-dimensional systems, is finite in spaces with \( D > 2 \), while in spaces with \( D \leq 2 \) it is infinite-dimensional;
4) other, not so evident, complications.

2. Generalizations.

1. D-dimensional generalization.

The first condition from the resume defines form of action \( S \) in any dimension and the second one defines partially a topology of \( \mathcal{M} \)

\[
S = \frac{1}{2\alpha} \int d^Dx d^Dx' \psi_a(x) \bar{\psi}^{(D)}_{ab} (x - x') \psi_b(x'),
\]

where \( \psi \in \mathcal{M} \), \( a, b = 1, 2, \ldots, n \), \( n \) is a dimension of \( \mathcal{M} \) and a form of kernel \( \bar{\psi} \) depends on dimension of space \( D \). For decoupled internal and physical spaces \( \bar{\psi} \) can be decomposed

\[
\bar{\psi}^{(D)}_{ab} (x) = g_{ab} \Box_D (x),
\]

where \( g_{ab} \) is the Euclidean metric of the space \( \mathbb{R}^{N(n)} \), in which a manifold \( \mathcal{M} \) can be embedded. In the momentum space, for small \( k \)

\[
\Box_D (k) \simeq |k|^D (1 + a_1 (k a) + ...),
\]

where \( a \) is a UV cut-off parameter. Action \( S \) can be named \( D \)-dimensional conformal nonlinear \( \sigma \)-model. The kernel \( \Box_D \) generalizes an usual local and conformal kernel of two-dimensional \( \sigma \)-model

\[
\Box (k) \equiv \Box_2 (k) = k^2.
\]

For local models an expression for \( \bar{\psi} \) can be defined in terms of manifold \( \mathcal{M} \) only

\[
\bar{\psi} = g_{ab} (\phi) \Box (\delta(x))
\]

In odd dimensions \( \Box_D \) is nonlocal

\[
\Box_D \sim 1/x^{2D}.
\]

The simplest spaces with properties, satisfying the second condition of the resume, are the spheres \( S^{D-1} \). An existence of topological excitations with logarithmic energy follows from the invariance of the kernel \( \Box_D \) and the simplest topologically nontrivial excitations \( n_i = x_i/r \) under scale and conformal transformations

\[
x_i \rightarrow x'_i = x_i/r^2, \quad r \rightarrow r' = 1/r, \quad x_i/r = x'_i/r',
\]

\[
d^Dx \rightarrow d^Dx/|x|^{2D}, \quad \frac{1}{|x_1 - x_2|^{2D}} \rightarrow \frac{|x_1|^{2D} |x_2|^{2D}}{|x_1 - x_2|^{2D}},
\]

and, consequently,

\[
S \sim \int d^Dx_1 d^Dx_2 \frac{(x_1 x_2)}{r_1 r_2} \frac{1}{|x_1 - x_2|^{2D}}
\]
is invariant and dimensionless. By direct calculation one can show that

\[ S[n] \sim C_D \ln \frac{R}{a}, \]

and the different excitations interact through potential \( G_D(r) \), inverse to \( \Box_D(r) \), which has a logarithmic behaviour at large distances

\[ G_D(r) \sim \ln \frac{r}{R}. \]

Just these conformal kernels and their logarithmic Green functions have appeared under consideration of logarithmic gases and equivalent field theories in \[20\].

2. Multicomponent generalization for \( D > 2 \).

A simple generalization of the sphere \( S^{D-1} \), analogous to the torus \( T^n \) in 2D case, is a bouquet of \( n \) spheres

\[ B_n^{D-1} = S_1^{D-1} \lor \ldots \lor S_n^{D-1} \]

and all spaces \( \mathcal{M} \) with this first nontrivial topological cell complex. But, the vector topological charges, corresponding to different spheres will not interact as in a case of torus \[21\]. For obtaining an interaction of topological charges in 3D case one needs to consider NS-models on deformed \( B_n^{D-1} \). In particular, 3D van der Waals NS-models on the maximal flag spaces \( F_G = G/T_G \) of the simple compact groups \( G \), with \( \pi_2(F_G) = \mathbb{L}_v \), (note that a sphere \( S^2 \) is a particular case of \( F_G \): \( S^2 = SU(2)/U(1) \)) will also have topological excitations with interacting vector topological charges \( Q \in \mathbb{L}_v \) and logarithmic energy. Since \( \pi_3(F_G) = \pi_3(G) = \mathbb{Z} \), in this case the ”neutral” configurations will also have different topological structures described by group \( \pi_3(F_G) \). Thus, in these models the TE will have again the mixed properties of the two-dimensional vortices and instantons: their vector topological charges, connected with \( \pi_2(F_G) \), will interact logarithmically as vortices and their ”neutral” configurations will have additional topological structure.

Another interesting possible generalization is the 3D conformal NS-models on compact groups \( G \). Since \( \pi_2(G) = 0, \pi_3(G) = \mathbb{Z} \), then the instanton-like TE can only exist in these models.

3. Applications.

The van der Waals NSM, by its construction, is a simplified model of real systems, consisting from the rod-like molecules, interacting through the van der Waals potential. But, since topological characteristics depend on rough qualitative properties, not on some inessential details, one can hope that
the proposed van der Waals model can describe the qualitative properties of some real systems. An application of ideas, developed under investigation of this model, to the more realistic, taking into account an anisotropy of the liquid crystals, model from [18] may be especially interesting.

CONCLUSIONS

1. The TE with logarithmic energies have the most strong influence on correlations in low-dimensional systems with conformal invariance. They correspond to the open boundary and to group \( \pi_{D-1}(\mathcal{M}) \).

2. 3D Van der Waals \( \sigma \)-model is conformal invariant.

3. 3D Van der Waals \( \sigma \)-model on \( S^2 \) has the pointlike TE, the hedgehogs, with logarithmic energy and topological charges \( q \), corresponding to homotopical group \( \pi_2(S^2) = \mathbb{Z} \).

4. These TE have the mixed properties of the 2D vortices and instantons. Their "neutral" configurations have different topological structures characterized by the Hopf invariant.

5. The possibility of the TPT in 3D systems induced by these TE is discussed.

6. A generalization of the van der Waals NS-model on other dimensions and multicomponent systems is proposed.

PROBLEMS, PERSPECTIVES

1. General solutions, their classifications.

2. Renormalization and TPT.

3. Generalizations and modifications.

4. Real systems and experiment.

APPENDIX. THE HOPF INVARIANT \( H \) AND RELATED FACTS

The Hopf invariant \( H \) classifies the mappings [16]

\[ S^3 \to S^2 = S^3/S^1. \]

The last equality means that these mappings are a projection, which projects some circles \( S^1 \subset S^3 \) into points \( \in S^2 \).
An example of such projection.

Let us take $S^3$ as a sphere in $\mathbb{R}^4 \cong \mathbb{C}^2$

\[
\frac{z_1}{(|z_1|^2 + |z_2|^2)^{1/2}}, \quad \frac{z_2}{(|z_1|^2 + |z_2|^2)^{1/2}}, \quad z_{1,2} \in \mathbb{C}^2,
\]

and a sphere $S^2$ as a complex projective space $\mathbb{C}P^1$, where any pairs of the next type

\[
z_2 = Cz_1
\]
gives the same point (here $C$ is arbitrary complex number).

The Hopf mapping $f : S^3 \rightarrow S^2$ can be defined as

\[
f : (z_1, z_2) \rightarrow z_1/z_2
\]
for any $z_2 \neq 0$. It gives $\mathbb{C}P^1$, under this all

\[
\lambda z_1, \lambda z_2,
\]
with $|\lambda| = 1$ gives the same point.

The unit vector $n \in S^2$ under this projection can be defined as

\[
n_1 + in_2 = \frac{2z_1z_2^*}{|z_1|^2 + |z_2|^2}, \quad n_3 = \frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2}
\]

The Hopf invariant $H$ is a linking number \{\gamma_1, \gamma_2\} of two projected circles $\gamma_1, \gamma_2$ in $S^3$, corresponding to different points of $S^2$ in general positions, into which they are projected,

\[
H = \{\gamma_1, \gamma_2\}
\]
It can be represented in different integral forms. In $\mathbb{R}^3$

\[
\{\gamma_1, \gamma_2\} = \frac{1}{4\pi} \int_\gamma_1 \int_\gamma_2 \frac{<r_{12} \cdot [dr_1 \cdot dr_2]>}{|r_1 - r_2|^3}.
\]

For simple case of one winding of one circle around another $H = 1$. If a mapping projects each circle $q_i (i = 1, 2)$ times then $H = q_1q_2$.

In $S^3$ it can be also represented as

\[
H = \int_{S^3} \theta \wedge d\theta,
\]
where 1-form $\theta$ is defined as

\[
d\theta = f^{-1}(d\Omega).
\]
Here $d\Omega$ is a 2-form or an element of the area of $S^2$, $f^{-1}$ is a mapping inverse to the projection mapping.

I would like to thank the organizers of the seminar for the opportunity to give this talk. The conversations with M.Kleman and M.Monastyrsky were very useful for preparation of this talk.
References

[1] Landau L.D., Lifshits E.M., Quantum mechanics, Nauka, Moscow, 1974.
[2] Anderson P.W., Yuval G., Hamann D.R., Phys.Rev. B1 (1970) 4464.
[3] Cardy J.L., J.Phys. 14A (1981) 1407.
[4] Bulgadaev S.A., Phys.Lett., 86A (1981) 213; ibid., 102A (1984) 260; Theoret.Math.Phys., 51 (1982) 424; [hep-th/9808115]
[5] Berezinsky V.L., JETP 59 (1970) 907; 61 (1971) 1545.
[6] Kosterlitz J.M., Thouless J.P., J.Phys. C6 (1973) 118; Kosterlitz J.M., J.Phys. C7 (1974) 1046.
[7] Popov V.N., Feynman integrals in quantum field theory and statistical mechanics. Atomizdat, Moscow, 1976.
[8] Nelson D.R., Phys.Rev. B18 (1978) 2318; Nelson D.R., Halperin B.I., Phys.Rev. B19 (1979) 2457.
[9] Bulgadaev S.A., Phys.Lett. 86A (1981) 213; Theoret.Math.Phys. 49 (1981) 77; Nucl.Phys. B224 (1983) 349; JETP Letters 63 (1996) 780; [hep-th/9906091]
[10] Polyakov A.M., Gauge Fields and Strings, Harwood Academic Publishers, 1987.
[11] Belavin A.A., Polyakov A.M., Pisma v JETP, 22 (1975) 245.
[12] Bulgadaev S.A., Phys.Lett. A125 (1987) 299.
[13] Korshunov S., Pisma v ZETP 45 (1987) 342.
[14] Leggett A.J. et al., Rev.Mod.Phys. 59 (1987) 1; Schmid A., Phys.Rev.Lett. 51 (1983) 1506.
[15] Schon G., Zaikin A.D., Phys.Rep. 198 (1990) 237.
[16] Dubrovin B.A., Novikov S.P., Fomenko A.T., Modern geometry, part I,II. Nauka, Moscow, 1979; part III. Nauka, Moscow, 1984.
[17] Bulgadaev S.A., 3D conformal $\sigma$ -model and topological excitations. Landau Institute preprint 29/05/1997.
[18] Dzyaloshinskii I.E., Dmitriev S.G., Kats E.I., JETP 68 (1975) 2335.
[19] Patashinskii A.S., Pokrovskii V.L., Fluctuation theory of phase transitions, Nauka, Moscow, 1982.
[20] Bulgadaev S.A., Phys.Lett., 87B (1979) 47.
[21] Bulgadaev S.A., On topological interpretation of quantum numbers, [hep-th/9901036], JETP 116 N10 (1999).