THE LAWRENCE–KRAMMER REPRESENTATION IS UNITARY

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Abstract. We show that the Lawrence–Krammer representation is unitary. We explicitly present the non-singular matrix representing the sesquilinear pairing invariant under the action. We show that reversing the orientation of a braid is equivalent to the transposition of its Lawrence–Krammer matrix followed by a certain conjugation. As corollaries it is shown that the characteristic polynomial of the Lawrence–Krammer matrix is invariant under substitution of its variables with their inverses up to multiplication by units, and is not a complete conjugacy invariant for braids.

1. Introduction

The Lawrence–Krammer representation $K: B_n \to GL(\mathbb{Z}[t^{\pm 1}, q^{\pm 1}], n(n-1)/2)$ was first introduced by Lawrence [Law90] and proved to be faithful by Bigelow [Big01] and Krammer [Kra00]. As a braid invariant $K$ is strong enough to distinguish all braids. The characteristic polynomial $f_\beta = \det(zI_{n(n-1)/2} - K(\beta))$, where $I_{n(n-1)/2}$ denotes the $n(n-1)/2$ dimensional identity matrix, of the Lawrence–Krammer matrix $K(\beta)$ of a braid $\beta \in B_n$ appears to be rather good as a conjugacy invariant of braids. The author observed that $f$ does not detect the orientation reversal of strings of braids, hence is not a complete conjugacy invariant, and that the polynomial $f_\beta$ has a symmetry $f_\beta(z, t, q) = f_\beta(z^{-1}, t^{-1}, q^{-1})$ just like the Alexander polynomial of links (see Corollary 1 and 2). Actually this paper arised in search of an explanation for these phenomena.

Let $D_n$ be an oriented disk with $n$ holes in the complex plane. The boundary $\partial D_n$ consists of $n$ puncture boundary components and the outer boundary component. Let $C$ denote the space $\{(x, y) \mid x, y \in D_n\}$ of all unordered pairs of points in $D_n$. Let $N(\text{diag } C)$ be an open regular neighborhood of the subset $\text{diag } C = \{(x, x) \mid x \in D_n\}$. Let $C = \overline{C} \setminus N(\text{diag } C)$. Then $C$ is a 4-manifold with boundary.

The fundamental group $\pi_1(C)$ is isomorphic to the subgroup $B_n, 2$ of $B_{n+2}$ consisting of $(n+2)$-braids whose first $n$ strands go straight down without winding and only the last two strands freely wind around. Let $\phi: B_{n, 2} \to \{t^a q^b \mid a, b \in \mathbb{Z}\}$ be the homomorphism to the abelian group of monomials defined by $\phi(\gamma) = ts^{-2l}q^l$, where $s$ is the exponent sum of $\gamma$ in Artin generators $\sigma_i$ and $l$ is the linking number of the last two strands with the first $n$ strands. Note that $\phi(\beta^{-1}\gamma\beta) = \phi(\gamma)$ for any $n$-braid $\beta \in B_n \subset B_{n, 2}$. Let $\tilde{C}$ be the covering space of $C$ whose fundamental group is the kernel of $\phi$.

Let $\Lambda$ denote the ring $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ of two variable Laurent polynomials. The homology group $H_2(\tilde{C})$ admits a $\Lambda$–module structure. It is a free $\Lambda$–module with rank $n(n-1)/2$ (see Big01, Section 4). The Lawrence–Krammer representation refers to the braid action on $H_2(\tilde{C})$ as $\Lambda$–module automorphisms.

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By the Blanchfield duality theorem \cite{Bla57, Kaw90}, Appendix E, we have a non-degenerate sesquilinear pairing \((\cdot, \cdot) : BH_2(C, \partial C) \times BH_2(C) \rightarrow \Lambda\) given by
\[
\langle X, y \rangle = \sum_{a, b \in \mathbb{Z}} t^a q^b (X \cdot t^a q^b y)
\]
where \((\cdot, \cdot)\) denotes the ordinary intersection number and \(BH\) denotes the torsion free part of a \(\Lambda\)-module \(H\). By identifying \(BH_2(C, \partial C)\) with \(\text{Hom}(H_2(C), \Lambda)\), we obtain a pairing \((\cdot, \cdot) : H_2(\tilde{C}) \times H_2(\tilde{C}) \rightarrow \Lambda\). Since the braid action on \(H_2(\tilde{C})\) is induced from self-homeomorphisms of \(\tilde{C}\), clearly it should preserve the pairing \((\cdot, \cdot)\), i.e., the Lawrence–Krammer representation is unitary. This could be a non-constructive and terse proof of the following theorem.

**Theorem 1.** There exists a non-singular \(n(n-1)/2 \times n(n-1)/2\) matrix \(J\) over \(\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]\) such that for each Lawrence–Krammer matrix \(M = \kappa(\beta)\) of an arbitrary \(n\)-braid \(\beta \in B_n\) and its conjugate transpose \(M^*\), the equality \(M J M^* = J\) holds.

The symmetry property of the characteristic polynomial \(f_\beta\) naturally follows.

In order to find the explicit matrix \(J\) representing the pairing, we need the concrete description of a base set of \(H_2(\tilde{C})\) and its dual base in \(BH_2(C, \partial C)\). In \cite{Kra00a, Big01} the forks were used to express relative 2-cycles in \(H_2(C, \partial C)\). Bigelow showed that given each relative 2-cycle \(X \in H_2(\tilde{C}, \partial \tilde{C})\) defined by a fork, \((1-q)^2(1+qt)X\) is an image of a 2-cycle from a closed surface immersed in \(\tilde{C}\). Using the set \(\{v_{i,j} \mid 1 \leq i < j \leq n\}\) of relative 2-cycles corresponding to standard forks as a base, he presented the explicit matrices for the Lawrence–Krammer representation. Let \(y_{i,j}\) be the 2-cycle in \(H_2(\tilde{C})\) which maps to \((1-q)^2(1+qt)v_{i,j}\). We can define a \(n(n-1)/2 \times n(n-1)/2\) matrix \(J_1\) whose entries are given by \(\langle v_{k,l}, y_{i,j} \rangle\) for \(1 \leq i < j \leq n\) and \(1 \leq k < l \leq n\). Then for the Lawrence–Krammer matrix \(M\) of an \(n\)-braid \(\beta\) the equality \(M J_1 M^* = J_1\) follows from \(\langle v_{k,l}, y_{i,j} \rangle = \langle \beta(v_{k,l}), \beta(y_{i,j}) \rangle\).

The author did not succeed in obtaining a concrete description of \(y_{i,j}\) as a surface or in calculating the the matrix \(J_1\). The proof of Theorem 1 in this paper involves cumbersome matrix calculations on biforks defined in Section 2. The matrix \(J\) of Theorem 1 is presented in Lemma 1 as a multiplication table of an algebra. Our proof has an advantage over the non-constructive one in that the same method also works for the proof of the following theorem, which relates the orientation reversal of a braid to the matrix transposition. As a corollary of this theorem we have \(f_\beta = f_{\beta^\text{rev}}\), where \(\text{rev} : B_n \rightarrow B_n\) is the anti-automorphism given by \(\sigma_i^\text{rev} = \sigma_i\).

**Theorem 2.** There exists a non-singular \(n(n-1)/2 \times n(n-1)/2\) matrix \(V\) over \(\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]\) such that for each \(n\)-braid \(\beta \in B_n\) the equality \(K(\beta^\text{rev})V = VK(\beta)^T\) holds.

In \cite{Zin01} it was shown that the Lawrence–Krammer representation is isomorphic to a summand of the representation on the Birman–Murakami–Wenzl (BMW) algebra \cite{BWS, Wen90, Mur87}. This implies that the Artin generators \(\sigma_i\) satisfy certain cubic relations in the representation.

**Proposition 1.** The Lawrence–Krammer representation satisfies the cubic relations
\[
(K(\sigma_i) - I_{n(n-1)/2})(K(\sigma_i) + qI_{n(n-1)/2})(K(\sigma_i) + tq^2 I_{n(n-1)/2}) = 0
\]
for \(i = 1, \ldots, n-1\).

The eigenvalue \(-tq^2\) is associated to a 1–dimensional eigenspace. Therefore \(\sigma_i^{-1}(\sigma_i - 1)(\sigma_i + q) = \sigma_i - q\sigma_i^{-1} - (1-q)\) maps to a projector on to the 1–dimensional
subspace. This fact led the author to define the following skein relation of geometric biforks.

\[ X - qX = (1 - q)(-X) \]

Although all the proofs in this paper were solely done by matrix calculations independent of the geometric biforks, the definition of biforks and the main idea of the proofs came from the geometric biforks.

In [LTW98] it was shown that as a braid invariant the Burau representation \( B : B_n \to GL(n - 1, \mathbb{Z}[t^{\pm 1}]) \) is dominated by the finite type invariants. This is due to the fact that \( B(\sigma_i - \sigma_i^{-1})(1) = 0 \), i.e., \( t = 1 \) the Burau matrix is a finite type invariant of order 0. Likewise the Lawrence–Krammer matrix becomes a permutation matrix at \( t = -1 \) and \( q = 1 \). In Section 3 we show that the matrix \( \frac{\partial^k t^l}{\partial q^{k+l}} K(\beta)(-1, 1) \) is of finite type of order \( k + l \). Therefore the Lawrence–Krammer representation is dominated by the finite type invariants. With the faithfulness of the representation together, we obtain the following well-known fact.

**Proposition 2** ([Koh87] [BN95]). The finite type invariants separate braids.

The author is interested in the question whether the finite type conjugacy invariant \( f_3 \) of the Lawrence–Krammer matrix is rather strong as a conjugacy invariant, and is dominated by the finite type invariants. After observing that \( f_3 \) does not detect the orientation reversal of a braid, we raise the following question.

**Question 1.** Is there a finite type conjugacy invariant \( v \) of braids such that \( v(\beta) \neq v(\beta^{\text{rev}}) \) for some braid \( \beta \)?

## 2. Proofs of main results

Let \( \Lambda = \mathbb{Z}[t^{\pm 1}, q^{\pm 1}] \) be the ring of two variable Laurent polynomials in \( t \) and \( q \). The Lawrence–Krammer module \( V_n \) is a \( n(n - 1)/2 \)-dimensional free \( \Lambda \)-module generated by the set of standard forks \( \{ X_{i,j} \mid 1 \leq i < j \leq n \} \). The right action of \( B_n \) on \( V_n \) defined by \([1]\) gives the faithful representation \( K : B_n \to GL(n(n - 1)/2, \Lambda) \) [Kra00, Big01]. We follow the convention of [Big01] on the sign of variable \( t \).

\[
X_{i,j} \sigma_m = \begin{cases} 
X_{i,j} & \text{if } m < i - 1 \text{ or } i < m < j - 1 \text{ or } j < m \\
X_{i-1,j} & \text{if } m = i - 1 \\
X_{i,j-1} & \text{if } i < m = j - 1 \\
-tq^2X_{i,j} & \text{if } m = i = j - 1 \\
qX_{i+1,j} + (1 - q)X_{i,j} + tq(1 - q)X_{i,i+1} & \text{if } m = i < j - 1 \\
qX_{i,j+1} + (1 - q)X_{i,j} - q(1 - q)X_{j,j+1} & \text{if } m = j \end{cases} \tag{1}
\]

Let \( \{ (i, j) \mid 1 \leq i < j \leq n \} \to \{ \lambda \mid 1 \leq \lambda \leq n(n - 1)/2 \} \) be the bijection of index sets defined by \( b(i, j) = (2n - i)(i - 1)/2 + (j - i) \). This is the lexicographic order on the double indices. We identify \( X_{i,j} \) with the \( b(i, j) \)-th standard base element \( Z_\lambda \), \( \lambda = b(i, j) \), which is the row vector whose \( \lambda \)-th entry is 1 and uniquely non-zero. \( X_{i,j} \sigma_m \) denotes the \( b(i, j) \)-th row of the matrix \( K(\sigma_m) \).

**Proof of Proposition 4.** We need only to show the relation holds for \( i = 1 \) since \( \sigma_1 \)'s are all conjugate to each other.

Firstly it can be routinely verified for \( j > 2 \) that

\[
qX_{1,j} \sigma_1^{-1} = X_{1,j} - (1 - q)X_{2,j} - (1 - q^{-1})X_{1,2}.
\]
From this we calculate the images of $X_{1,j}$ and $X_{2,j}$ by $(\mathcal{K}(\sigma_1) - I_{n(n-1)/2})(\mathcal{K}(\sigma_i) + qI_{n(n-1)/2})$ as follows.

\[
X_{2,j}(\mathcal{K}(\sigma_1) - I_{n(n-1)/2})(\mathcal{K}(\sigma_i) + qI_{n(n-1)/2}) \\
= X_{2,j}(\mathcal{K}(\sigma_1) - q\mathcal{K}(\sigma_1^{-1}) - (1 - q)I_{n(n-1)/2})(\mathcal{K}(\sigma_i) \\
= (X_{2,j}\sigma_1 - qX_{2,j}\sigma_1^{-1} - (1 - q)X_{2,j})\mathcal{K}(\sigma_i) \\
= (X_{1,j} - (1 - q)X_{2,j} - (1 - q^{-1})X_{1,2}) - (1 - q)X_{2,j})\mathcal{K}(\sigma_1) \\
= (1 - q^{-1})(-tq^2)X_{1,2}
\]

\[
X_{1,j}(\mathcal{K}(\sigma_1) - I_{n(n-1)/2})(\mathcal{K}(\sigma_i) + qI_{n(n-1)/2}) \\
= X_{2,j}\sigma_1(\mathcal{K}(\sigma_1) - I_{n(n-1)/2})(\mathcal{K}(\sigma_i) + qI_{n(n-1)/2}) \\
= X_{2,j}(\mathcal{K}(\sigma_1) - I_{n(n-1)/2})(\mathcal{K}(\sigma_i) + qI_{n(n-1)/2})\mathcal{K}(\sigma_1) \\
= (1 - q^{-1})(-tq^2)X_{1,2}\sigma_1 \\
= (1 - q^{-1})t^2q^4X_{1,2}
\]

We also have

\[
X_{1,2}(\mathcal{K}(\sigma_1) - I_{n(n-1)/2})(\mathcal{K}(\sigma_1) + qI_{n(n-1)/2}) = (-tq^2 - 1)(-tq^2 + q)X_{1,2} \\
X_{i,j}(\mathcal{K}(\sigma_1) - I_{n(n-1)/2}) = 0 \text{ for } 2 < i < j \leq n
\]

The previous calculations exhibit that $(\mathcal{K}(\sigma_1) - I_{n(n-1)/2})(\mathcal{K}(\sigma_1) + qI_{n(n-1)/2})$ projects $V_n$ to the submodule generated by $X_{1,2}$. The equality

\[
(\mathcal{K}(\sigma_1) - I_{n(n-1)/2})(\mathcal{K}(\sigma_1) + qI_{n(n-1)/2})\mathcal{K}(\sigma_1) + t^2qI_{n(n-1)/2} = 0
\]

follows from $X_{1,2}(\mathcal{K}(\sigma_1) + tq^2I_{n(n-1)/2}) = 0$. □

From the proof of Proposition [5] we can see that $-(\mathcal{K}(\sigma_1) - I_{n(n-1)/2})(\mathcal{K}(\sigma_1) + qI_{n(n-1)/2})\mathcal{K}(\sigma_1^{-1})$ is an idempotent. We denote this matrix by $X^*_{1,2}X_{1,2}$.

The first column of $X^*_{1,2}X_{1,2}$ is $Z^*_1Z_2$ and only non-zero column of $X^*_{1,2}X_{1,2}$. We denote by $X^*_i$ the column vector given by the first column of $X^*_{1,2}X_{1,2}$ so that the expression $X^*_iX_{1,2}$ makes sense also as a matrix multiplication.

Let $M(n(n-1)/2, \Lambda)$ be the set of matrices representing endomorphisms of $V_n$. $M(n(n-1)/2, \Lambda)$ assumes a $B_n$-bimodule structure by multiplications from the right side and from the left. Let $M_n$ be the $B_n$-submodule of $M(n(n-1)/2, \Lambda)$ generated by the matrix $X^*_{1,2}X_{1,2}$, i.e.,

\[
M_n = \text{span}_\Lambda \{\mathcal{K}(\beta)X^*_{1,2}X_{1,2}\mathcal{K}(\gamma) \in M(n(n-1)/2, \Lambda) \mid \beta, \gamma \in B_n\}.
\]

For $1 \leq i < j \leq n$, let $A_{\pi(i,j)}$ denote the permutation braid $(\sigma_{i-1} \cdots \sigma_1)(\sigma_{j-1} \cdots \sigma_2)$ [ERM94]. $A_{\pi(i,j)}$ induces the permutation $\pi(i,j)$ on $\{1,2,\ldots,n\}$ which maps $i$ to $1$, $j$ to 2, and whose all inversions involve either $i$ or $j$. Let $X^*_{i,j}X_{k,l} = \mathcal{K}(A_{\pi(i,j)})X^*_{i,2}X_{1,2}\mathcal{K}(A_{\pi(k,l)}^{-1})$. We call $X^*_{i,j} = \mathcal{K}(A_{\pi(i,j)})X^*_{1,2}$ a standard dual fork. We call $X^*_{i,j}X_{k,l}$ a standard bifork. We denote $\mathcal{K}(\beta)X^*_{i,j}X_{k,l}\mathcal{K}(\gamma)$ by $\beta X^*_{i,j}X_{k,l}\gamma$ in emphasis of the right and left braid action on $M_n$.

The following lemma is the multiplication table for standard biforks.
Lemma 1. For $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, the $1 \times 1$ matrix $X_{k,l}X_{i,j}^*$ is given as follows.

\[
X_{k,l}X_{i,j}^* = \begin{cases}
0 & \text{if } (i-k)(i-l)(j-k)(j-l) > 0 \\
(-t^{-1} + q)(q^{-1} + qt) & \text{if } i = k < j = l \\
t^{-1}(1 - q^{-1}) & \text{if } i = k < l < j \text{ or } k < i < j = l \\
-(1 - q^{-1}) & \text{if } i < j = k < l \\
-tq(1 - q) & \text{if } i < k < j = l \text{ or } i = k < j < l \\
q(1 - q) & \text{if } k < l = i < j \\
-(1 - q)^2(t^{-1}q^{-1} + 1) & \text{if } k < i < l < j \\
(1 - q)^2(q^{-1} + t) & \text{if } i < k < j < l
\end{cases}
\]

Proof. This multiplication table can be verified routinely from the definition of $X_{i,j}$, $X_{i,l}^*$, and the Lawrence–Krammer representation. One may first calculate the table on $B_4$ (see (3)) in which every case of the double index correlations listed in the table occurs, and then use it for general $n$-braids.

We will not exhibit all the complicated calculations which could be too distracting. In Section 5 we introduce geometric bifork algebra and calculate the same multiplication table. For each picture-based calculation line in the proof of Theorem 3 the same calculation on biforks can be done in parallel by translating the isotopies to braid relations.

\[\Box\]

Lemma 2. The set of standard biforks $B = \{X_{i,j}^*X_{k,l} | 1 \leq i < j \leq n, 1 \leq k < l \leq n\}$ generates the algebra $\mathcal{M}_n$ as a $\Lambda$–module.

Proof. We need to show that for all $\beta, \gamma \in B_n$, $\beta X_{1,2}^*X_{1,2}\gamma$ is a linear combination of standard biforks. $X_{k,l}^*\gamma$ is a linear combination of standard forks from the definition of the Lawrence–Krammer representation. From the following formula we can see that $\beta X_{i,j}^*$ is also a linear combination of standard dual forks.

\[\sigma_m^{-1}X_{i,j}^* = \begin{cases}
X_{i,j}^* & \text{if } m < i - 1 \text{ or } i < m < j - 1 \text{ or } j < m \\
X_{i,j}^*_{i-1,j} & \text{if } m = i - 1 \\
X_{i,j}^*_{i,j-1} & \text{if } i < m = j - 1 \\
-t^{-1}q^{-2}X_{i,j}^* & \text{if } m = i = j - 1 \\
q^{-1}X_{i+1,j}^* + (1 - q^{-1})X_{i,j}^* & \text{if } m = i < j - 1 \\
q^{-1}X_{i,j}^*_{i,j+1} + (1 - q^{-1})X_{i,j}^*_{j,i} - q^{-1}(1 - q^{-1})X_{j,i+1}^* & \text{if } m = j
\end{cases}
\]

a. If $m < i - 1$, then

\[
\sigma_m^{-1}X_{i,j}^*X_{1,2} = \sigma_m^{-1}A_{\pi(i,j)}X_{1,2}^*X_{1,2} = A_{\pi(i,j)}\sigma_m^{-1}X_{1,2}^*X_{1,2} = A_{\pi(i,j)}X_{1,2}^*X_{1,2} = X_{i,j}^*X_{1,2}
\]

If $i < m < j - 1$, then $\sigma_m^{-1}A_{\pi(i,j)} = A_{\pi(i,j)}\sigma_m^{-1}$. If $j < m$, then $\sigma_m^{-1}A_{\pi(i,j)} = A_{\pi(i,j)}\sigma_m^{-1}$. For these two cases, the same calculation works as for the case $m < i - 1$. 

b. If $m = i - 1$, then

\[
\sigma_m^{-1}X_{i,j}^*X_{1,2} = \sigma_{i-1}^{-1}A_{\pi(i,j)}X_{1,2}^*X_{1,2} = A_{\pi(i-1,j)}X_{1,2}^*X_{1,2} = X_{i-1,j}^*X_{1,2}
\]
c. If \( i < m = j - 1 \), then
\[
\sigma_m^{-1}X^*_{i,j}X_{1,2} = \sigma_j^{-1}A_{\pi(i,j)}X^*_{1,2}X_{1,2} = A_{\pi(i,j-1)}X^*_{1,2}X_{1,2} = X^*_{i,j-1}X_{1,2}.
\]
d. If \( m = i = j - 1 \), then
\[
\sigma_m^{-1}X^*_{i,j}X_{1,2} = \sigma_i^{-1}A_{\pi(i,j)}X^*_{1,2}X_{1,2} = A_{\pi(i,j)}\sigma_1^{-1}X^*_{1,2}X_{1,2}
= A_{\pi(i,j)}X^*_{1,2}X_{2}\sigma_1^{-1} = t^{-1}q^{-2}A_{\pi(i,j)}X^*_{1,2}X_{1,2}
= t^{-1}q^{-2}X^*_{i,j}X_{1,2}.
\]
e. If \( m = i < j - 1 \), then
\[
\sigma_m^{-1}X^*_{i,j}X_{1,2} = \mathcal{K}(\sigma_i^{-1})X^*_{i,j}X_{1,2}
= (q^{-1}X^*_{i+1,i}X_{i,i+1} + q^{-1}\mathcal{K}(\sigma_i) + (1 - q^{-1})I_n(n-1)/2) X^*_{i,j}X_{1,2}
= q^{-1}X^*_{i+1,i}X_{i,i+1}X^*_{i,j}X_{1,2} + q^{-1}\sigma_iA_{\pi(i,j)}X^*_{1,2}X_{1,2}
+ (1 - q^{-1})X^*_{i,j}X_{1,2}
= q^{-1}X^*_{i+1,i}(t^{-1}(1 - q^{-1}))X_{1,2} + q^{-1}\sigma_iA_{\pi(i,j)}X^*_{1,2}X_{1,2}
+ (1 - q^{-1})X^*_{i,j}X_{1,2}
= t^{-1}q^{-1}(1 - q^{-1})X^*_{i+1,i}X_{1,2} + q^{-1}X^*_{i+1,i}X_{1,2}
+ (1 - q^{-1})X^*_{i,j}X_{1,2}.
\]
f. If \( m = j \), then
\[
\sigma_m^{-1}X^*_{i,j}X_{1,2} = \mathcal{K}(\sigma_j^{-1})X^*_{i,j}X_{1,2}
= (q^{-1}X^*_{j,j+1}X_{j,j+1} + q^{-1}\mathcal{K}(\sigma_j) + (1 - q^{-1})I_n(n-1)/2) X^*_{i,j}X_{1,2}
= q^{-1}X^*_{j,j+1}X_{j,j+1}X^*_{i,j}X_{1,2} + q^{-1}\sigma_jA_{\pi(i,j)}X^*_{1,2}X_{1,2}
+ (1 - q^{-1})X^*_{i,j}X_{1,2}
= q^{-1}X^*_{j,j+1}(-1 - q^{-1})X_{1,2} + q^{-1}\sigma_jA_{\pi(i,j+1)}X^*_{1,2}X_{1,2}
+ (1 - q^{-1})X^*_{i,j}X_{1,2}
= -q^{-1}(1 - q^{-1})X^*_{j,j+1}X_{1,2} + q^{-1}X^*_{j,j+1}X_{1,2}
+ (1 - q^{-1})X^*_{i,j}X_{1,2}.
\]

The formula (3) of \( \sigma_m^{-1}X^*_{i,j} \) for the left action of \( B_n \) on \( \mathcal{M}_n \), presented in the proof of Lemma 3, strikingly exhibits the duality between forks and dual forks, and between the right action and the left action of braids. If we replace in (3) \( \sigma_m^{-1} \) with \( \sigma_m \), the left multiplication with the right one, \( t \) with \( t^{-1} \), \( q \) with \( q^{-1} \), and \( X^*_{i,j} \) with \( X_{i,j} \), then we obtain exactly the same formula (3) used in the definition of the Lawrence–Krammer representation. This observation interprets that if \( Z_\lambda \sigma_m = \sum_\mu M_{\lambda,\mu}Z_\mu \) then
\[
\sigma_m^{-1}Z^*_\lambda = \sum_\mu \overline{M}_{\lambda,\mu}Z_\mu
\]
where \( \overline{M}(t,q) \) denotes the matrix \( M(t^{-1},q^{-1}) \).

**Lemma 3.** The set of standard biforks \( B \) is linearly independent.

**Proof.** Let \( J = (J_{\lambda,\mu}) \) be the \( n(n-1)/2 \times n(n-1)/2 \) matrix defined by \( J_{\lambda,\mu} = Z_\lambda Z^*_\mu \), the multiplication table given in Lemma 3. At \( q = 1 \), \( J \) is a diagonal matrix with non-zero diagonal entries, which implies det \( J \neq 0 \) even for generic \( q \).
Suppose that $\sum a_{\lambda\mu} Z_{\lambda}^* Z_{\mu} = 0$ for some $n(n-1)/2 \times n(n-1)/2$ matrix $A = (a_{\lambda\mu})$. Then for each $1 \leq \nu, o \leq n(n-1)/2$

$$0 = Z_{\nu} \left( \sum a_{\lambda\mu} Z_{\lambda}^* Z_{\mu} \right) Z_{\nu}^* = \sum a_{\lambda\mu} Z_{\lambda}^* Z_{\mu} Z_{\nu}^* = \sum a_{\lambda\mu} Z_{\lambda}^* Z_{\nu} Z_{\mu}^* = \sum a_{\lambda\mu} J_{\nu\lambda} J_{\mu\nu},$$

which implies $J AJ = 0$. Since $J$ is non-singular, $a_{\lambda\mu} = 0$ for each $1 \leq \lambda, \mu \leq n(n-1)/2$.

Lemma 3 shows in particular that the set of standard dual forks \{\(Z_{\lambda}^* | 1 \leq \lambda \leq n(n-1)/2\)\} is mutually linearly independent. Given a matrix $M(t, q) \in M_n$, $M^*$ denotes the conjugate transpose $M(t^{-1}, q^{-1})^T$. Here we assumed $t$ and $q$ are evaluated at generic unit complex numbers so that $t^{-1}$ and $q^{-1}$ are complex conjugates of $t$ and $q$.

**Proof of Theorem 1.** Let $J = \sum_{1 \leq i < j \leq n} X_{i,j} X_{i,j} = \sum_{1 \leq \lambda \leq n(n-1)/2} Z_{\lambda}^* Z_{\lambda}$.

$$J_{\lambda\mu} = Z_{\lambda} J Z_{\mu}^T = Z_{\lambda} \left( \sum_{\nu} Z_{\nu} Z_{\nu}^T \right) Z_{\mu}^T = \sum_{\nu} Z_{\lambda} Z_{\nu}^* Z_{\nu} Z_{\mu}^T = \sum_{\nu} Z_{\lambda} Z_{\nu}^* \delta_{\nu\mu} = Z_{\lambda} Z_{\mu}^*$$

where $\delta_{\nu\mu} = 1$ and $\delta_{\nu\mu} = 0$ if $\nu \neq \mu$. As shown above, $J$ is the same matrix given as the multiplication table in the proof of Lemma 3 where we showed that $J$ is non-singular.

It suffices to show the identity $JM^* = M^{-1}J$ for the Lawrence–Krammer matrices $M = K(\sigma_m)$ of the Artin generators $\sigma_m$ of $B_n$.

$$M^{-1}J = \sum_\lambda \sigma_m^{-1} Z_{\lambda}^* Z_{\lambda} = \sum_\lambda \left( \sum_{\mu} Z_{\lambda}^* M_{\lambda\mu} \right) Z_{\lambda} \quad \text{by (3)}$$

$$= \sum_{\mu} Z_{\mu}^* \left( \sum_{\lambda} Z_{\lambda} M_{\lambda\mu} \right) = \sum_{\mu} Z_{\mu}^* \left( Z_{\mu} M^T \right) = \left( \sum_{\mu} Z_{\mu}^* Z_{\mu} \right) M^*$$

$$= JM^*$$

**Corollary 1.** Let $f_\beta(z, t, q) = \det(z I_{n(n-1)/2} - K(\beta)(t, q))$ be the characteristic polynomial of $K(\beta)$. Then $f_\beta(z^{-1}, t^{-1}, q^{-1})$ equals $f_\beta(z, t, q)$ up to multiplication by units in $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$.

**Proof.** By Theorem 1, $K(\beta)^* = J^{-1}K(\beta)^{-1}J$ as matrices over the quotient field $\mathbb{Q}(t, q)$.

$$f_\beta(z^{-1}, t^{-1}, q^{-1}) = \det(z^{-1} I_{n(n-1)/2} - K(\beta)^*)$$

$$= \det(z^{-1} I_{n(n-1)/2} - J^{-1}K(\beta)^{-1}J)$$

$$= \det(z^{-1} I_{n(n-1)/2} - K(\beta)^{-1})$$

$$= (-z)^{-n(n-1)/2} \det(K(\beta)^{-1}) \det(z I_{n(n-1)/2} - K(\beta))$$

$$= (-z)^{-n(n-1)/2} \det(K(\beta)^{-1}) f_\beta(z, t, q)$$
Let $Y_{i,j} = X_{n+1-j,n+1-i} \Delta_n$ for $1 \leq i < j \leq n$, where $\Delta_n$ denote the square root of the full twist that generates the center of $B_n$. For each $\sigma_m \in B_n$, we have $\Delta_n \sigma_m \Delta_n^{-1} = \sigma_{n-m}$.

**Lemma 4.** With the base set $\{Y_{i,j} \mid 1 \leq i < j \leq n\}$, the Lawrence–Krammer representation is given by following formula.

$$Y_{i,j} \sigma_m^{-1} = \begin{cases} Y_{i,j} & \text{if } m < i - 1 \text{ or } i < m < j - 1 \text{ or } j < m \\ Y_{i-1,j} & \text{if } m = i - 1 \\ Y_{i,j-1} & \text{if } i < m = j - 1 \\ -t^{-1} q^{-2} Y_{i,j} & \text{if } m = i = j - 1 \\ q^{-1} Y_{i+1,j} + (1 - q^{-1}) Y_{i,j} + t^{-1} q^{-1} (1 - q^{-1}) Y_{i,i+1} & \text{if } m = i < j - 1 \\ q^{-1} Y_{i,j+1} + (1 - q^{-1}) Y_{i,j} - q^{-1} (1 - q^{-1}) Y_{j,j+1} & \text{if } m = j \end{cases}$$

(4)

**Proof.** From the definition (3) of the Lawrence–Krammer representation the formula for $X_{i,j} \sigma_m^{-1}$ can be easily derived as follows.

$$X_{i,j} \sigma_m^{-1} = \begin{cases} X_{i,j} & \text{if } m < i - 1 \text{ or } i < m < j - 1 \text{ or } j < m \\ X_{i+1,j} & \text{if } m = i < j - 1 \\ X_{i,j+1} & \text{if } j = m \\ -t^{-1} q^{-2} X_{i,j} & \text{if } m = i = j - 1 \\ q^{-1} X_{i-1,j} + (1 - q^{-1}) X_{i,j} - q^{-1} (1 - q^{-1}) X_{i-1,i} & \text{if } m = i - 1 \\ q^{-1} X_{i,j+1} + (1 - q^{-1}) X_{i,j} + t^{-1} q^{-1} (1 - q^{-1}) X_{j,j+1} & \text{if } i < m = j - 1 \end{cases}$$

(5) (6) (7) (8) (9) (10)

The formula of this lemma is verified by the following routine calculations.

a. If either $m < i - 1$, $i < m < j - 1$ or $j < m$, then $n + 1 - i < n - m$, $n + 1 - j < n - m < (n + 1 - i) - 1$, or $n - m < (n + 1 - j) - 1$ respectively. Therefore

$$Y_{i,j} \sigma_m^{-1} = X_{n+1-j,n+1-i} \Delta_n \sigma_m^{-1} = X_{n+1-j,n+1-i} \sigma_{n-m}^{-1} \Delta_n$$

$$= X_{n+1-j,n+1-i} \Delta_n = Y_{i,j} \quad \text{by (3)}.$$

b. If $m = i - 1$, then $n - m = n + 1 - i$ so that by (3)

$$Y_{i,j} \sigma_m^{-1} = X_{n+1-j,n+1-i} \sigma_{n-m}^{-1} \Delta_n$$

$$= X_{n+1-j,(n+1-i)+1} \Delta_n = Y_{i-1,j}.$$

c. If $i < m = j - 1$, then $n - m = n + 1 - j < (n + 1 - i) - 1$ so that by (3)

$$Y_{i,j} \sigma_m^{-1} = X_{n+1-j,n+1-i} \sigma_{n-m}^{-1} \Delta_n$$

$$= X_{(n+1-j)+1,n+1-i} \Delta_n = Y_{i,j-1}.$$

d. If $m = i = j - 1$, then $n - m = n + 1 - j = (n + 1 - i) - 1$ so that by (3)

$$Y_{i,j} \sigma_m^{-1} = X_{n+1-j,n+1-i} \sigma_{n-m}^{-1} \Delta_n$$

$$= -t^{-1} q^{-2} X_{n+1-j,n+1-i} \Delta_n = -t^{-1} q^{-2} Y_{i,j}.$$
If $m = i < j - 1$, then $n + 1 - j < n - m = (n + 1 - i) - 1$ so that by (3)

\[ Y_{i,j}^{-1} = X_{n+1-j, n+1-i} \sigma_n^{-1} \Delta_n \]

\[ = (q^{-1}X_{n+1-j, (n+1-i)+1} + (1-q^{-1})X_{n+1-j, n+1-i}) 
   + t^{-1}q^{-1}(1-q^{-1})X_{n+1-i, (n+1-i)+1} \Delta_n \]

\[ = q^{-1}Y_{i-1,j} + (1-q^{-1})Y_{i,j} + t^{-1}q^{-1}(1-q^{-1})Y_{i-1,j}. \]

If $m = j$, then $n - m = (n + 1 - j) - 1$ so that by (3)

\[ Y_{i,j}^{-1} = X_{n+1-j, n+1-i} \sigma_n^{-1} \Delta_n \]

\[ = (q^{-1}X_{(n+1-j)-1, n+1-i} + (1-q^{-1})X_{n+1-j, n+1-i}) 
   - q^{-1}(1-q^{-1})X_{(n+1-j)-1, n+1-j} \Delta_n \]

\[ = q^{-1}Y_{i,j+1} + (1-q^{-1})Y_{i,j} - q^{-1}(1-q^{-1})Y_{j,j+1}. \]

For a word $W$, $W^{\text{rev}}$ denotes the reverse word of $W$. For a braid $\beta = W(\sigma_i) \in B_n$, written as a word in Artin generators, we define $\beta^{\text{rev}} = W^{\text{rev}}(\sigma_i)$. In other words $\beta \mapsto \beta^{\text{rev}}$ is the anti-isomorphism given by $\sigma_i^{\text{rev}} = \sigma_j$. Geometrically this equals reversing the orientations of the strings of a braid.

Observe in the formula (3) of Lemma 4 that if one replaces $\sigma_n^{-1}$, $Y_{i,j}$, $t^{-1}$ and $q^{-1}$ with $\sigma_m$, $X_{i,j}$, $t$ and $q$, then one obtains exactly the same formula (3).

**Theorem 3.** There exists an invertible $n(n-1)/2 \times n(n-1)/2$ matrix $R$ over $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ such that for each $n$-braid $\beta \in B_n$, the equality

\[ K(\beta^{-1})(t, q) = R^{-1}K(\beta^{\text{rev}})(t^{-1}, q^{-1})R \]

holds.

**Proof.** Let $W_{\delta(i,j)} = Y_{i,j}$ for $1 \leq i < j \leq n$ and define the matrix $R$ by $R_{\lambda\mu} = (W_{\lambda})_{\mu}$, the $\mu$-th entry of $W_{\lambda}$ for $1 \leq \lambda, \mu \leq n(n-1)/2$.

It suffices to show the equality for $M = K(\sigma_m)$. The previous observation on the similarity between Lemma 3 and the definition of the Lawrence–Krammer representation interprets that $W_{\lambda}M^{-1} = \sum_{\mu} M_{\lambda\mu}W_{\nu}$. Taking the $\mu$-th entry of each side, we have

\[ (W_{\lambda}M^{-1})_{\mu} = \sum_{\nu} M_{\lambda\nu}(W_{\nu})_{\mu} \]

\[ (RM^{-1})_{\lambda\mu} = \sum_{\nu} M_{\lambda\nu}R_{\nu\mu} \]

\[ RM^{-1} = \overline{MR}. \]

From the definition of $Y_{i,j}$, $RK(\Delta_n^{-1}) = P$ for some permutation matrix $P$. Hence $R$ is invertible.

**Proof of Theorem 3.** Let $V = \overline{RJ}$. Then we have

\[ K(\beta^{\text{rev}})V = K(\beta^{\text{rev}})RJ = \overline{K(\beta^{\text{rev}})}RJ \]

\[ = RK(\beta^{-1})J \quad \text{by Theorem 3} \]

\[ = RJK(\beta)^T \quad \text{by Theorem 3} \]

\[ = RJK(\beta)^T = VK(\beta)^T. \]

Since transposition and conjugation do not alter the characteristic polynomial of a matrix, we obtain the following corollary from Theorem 3.
Corollary 2. $\mathcal{K}(\beta)$ and $\mathcal{K}(\beta^{rev})$ have the same characteristic polynomial.

3. Explicit Matrices

In this section we exhibit how to compute the matrices $J$ and $V$ of the main theorems for low braid index.

Let $J_4 = \sum_{1 \leq i < j \leq 4} X_{i,j}^* X_{i,j} \in \mathcal{M}_4$. $B_4$ is generated by the two elements $\sigma_1$ and $\delta_4 = \sigma_3 \sigma_2 \sigma_1$. Note that $\sigma_2 = \delta_4^{-1} \sigma_1 \delta_4$ and $\sigma_3 = \delta_4^{-2} \sigma_1 \delta_4^2$. For the proof of Theorem 4 for 4-braids, it is enough to verify the equalities $\mathcal{K}(\sigma_1) J_4 \mathcal{K}(\sigma_1)^* = \mathcal{K}(\delta_4) J_4 \mathcal{K}(\delta_4)^* = J$. The three matrices $J_4$, $\mathcal{K}(\sigma_1)$ and $\mathcal{K}(\delta_4)$ can be explicitly written as follows:

$$\mathcal{K}(\sigma_1) = \begin{bmatrix} -tq^2 & 0 & 0 & 0 & 0 & 0 \\ tq(1-q) & 1-q & 0 & q & 0 \\ tq(1-q) & 0 & 1-q & 0 & q \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{K}(\delta_4) = \begin{bmatrix} 0 & 0 & 0 & q^2 & 0 \\ 0 & 0 & 0 & 0 & q^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -tq^2 & 0 & 0 & 0 \\ 0 & 0 & -tq^2 & 0 & 0 \end{bmatrix}$$

$$J_4 = \begin{bmatrix} X_{1,2}^* & X_{1,3}^* & X_{1,4}^* & X_{2,3}^* & X_{2,4}^* & X_{3,4}^* \end{bmatrix}$$  \hspace{1cm} (11)$$

where the $6 \times 1$ matrices $X_{i,j}^*$ are given by

$$X_{1,2}^* = \begin{bmatrix} (-t^{-1} + q) (q^{-1} + qt) \\ -tq(1-q) \\ -tq(1-q) \\ -(1-q^{-1}) \\ -(1-q^{-1}) \\ 0 \end{bmatrix}$$

and

$$X_{1,3}^* = \mathcal{K}(\sigma_2) X_{1,2}^*$$
$$X_{1,4}^* = \mathcal{K}(\sigma_3 \sigma_2) X_{1,2}^*$$
$$X_{2,3}^* = \mathcal{K}(\sigma_1 \sigma_2) X_{1,2}^*$$
$$X_{2,4}^* = \mathcal{K}(\sigma_1 \sigma_3 \sigma_2) X_{1,2}^*$$
$$X_{3,4}^* = \mathcal{K}(\sigma_2 \sigma_1 \sigma_3 \sigma_2) X_{1,2}^*.$$

The Lawrence–Krammer matrices of Artin generators $\sigma_1, \sigma_2$ of $B_3$ can written as follows:

$$\mathcal{K}(\sigma_1) = \begin{bmatrix} -tq^2 & 0 & 0 \\ tq(1-q) & 1-q & q \\ 0 & 1 \end{bmatrix}$$

$$\mathcal{K}(\sigma_2) = \begin{bmatrix} 1-q & q & -q(1-q) \\ 1 & 0 & 0 \\ 0 & 0 & -tq^2 \end{bmatrix}$$

Let $Y_{1,2}^* = X_{1,2}^*$ be the $3 \times 1$ matrix taken from the first column of $X_{1,2}^* X_{1,2} \in \mathcal{M}_3$.

Let $V_3 = \begin{bmatrix} Y_{1,2}^* & Y_{1,3}^* & Y_{2,3}^* \end{bmatrix}$ where $Y_{1,3}^* = \mathcal{K}(\sigma_2^{-1}) Y_{1,2}^*$ and $Y_{2,3}^* = \mathcal{K}(\sigma_1^{-1} \sigma_2^{-1}) Y_{1,2}^*$.

The three columns of $V_3$ are explicitly written as follows:
Let $\beta = \sigma_2\sigma_1^{-1}\sigma_2^{-1}$ in $B_3$. Then $\beta^{rev} = \sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-2}$ is related to $\beta$ by a flype move, which changes the conjugacy class while preserving the link type of a closed braid. One may check the equality $K(\beta^{rev})V_3 = V_3K(\beta)^T$ and that $K(\beta^{rev})$ shares with $K(\beta)$ the same characteristic polynomial.

4. The Burau Representation

In this section we review the Squier’s result that the Burau representation is unitary. The reduced Burau representation $\mathcal{B}: B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$ is defined by these two $(n-1) \times (n-1)$ matrices:

$$\mathcal{B}(\sigma_1) = I_{n-1} + te_{1,2} - (1 + t)e_{1,1}$$

$$\mathcal{B}(\delta_n) = -t^{n-1}e_{n-1,1} + \sum_{1 \leq i \leq n-2} (-t^i e_{i,1} + e_{i,i+1})$$

where $e_{i,j}$ denotes the elementary matrix whose only non-zero entry is the $(i,j)$ entry with value 1, and $\delta_n = \sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 \in B_n$.

**Theorem 4** (Squier). There exists a nonsingular $(n-1) \times (n-1)$ matrix $J_0$ over $\mathbb{Z}[t^{\pm 1}]$ such that for each $\beta$ in $B_n$ it follows that $\mathcal{B}(\beta)^* J_0 \mathcal{B}(\beta) = J_0$.

The reduced Burau representation can be interpreted as the action on the homology group $H_1(\tilde{D}_n; \mathbb{Z}[t^{\pm 1}])$ of the infinite cyclic cover $\tilde{D}_n$ induced by the braid homeomorphism. It is natural to expect the reduced Burau representation is unitary because homeomorphisms should preserve intersection forms. We clarify this point in the following proof. The original proof in [Squ84] was done by giving the matrix $J_0$ and directly evaluating the equality.

**Proof.** Consider the pairing $\langle \cdot, \cdot \rangle: H_1(\tilde{D}_n, \partial \tilde{D}_n) \times H_1(\tilde{D}_n) \to \mathbb{Z}[t^{\pm 1}]$ defined by

$$\langle x, y \rangle = \sum_k t^k(t^k x \cdot y)$$

where $(\cdot \cdot)$ denotes the usual algebraic intersection number. The pairing $\langle \cdot, \cdot \rangle$ is sesquilinear. The equalities $\langle tx, y \rangle = t^{-1}\langle x, y \rangle$ and $\langle x, ty \rangle = t\langle x, y \rangle$ follow from the definition. If an automorphism $h_* \in H_1(\tilde{D}_n, \partial \tilde{D}_n)$ is induced by a homeomorphism $h: \tilde{D}_n \to \tilde{D}_n$, then $\langle h_*(x), h_*(y) \rangle = \langle x, y \rangle$.

We embed $\tilde{D}_n$ in the complex plane $\mathbb{C}$ so that the $i$-th hole is placed around the point $i \in \mathbb{C}$. Let $x_i \in \pi_1(\tilde{D}_n)$ denote the standard generator represented by the closed contour winding only around the $i$-th hole once. Let $y_i \in H_1(\tilde{D}_n)$ be the cycle which is the lift of $x_i x_{i+1} \in \pi_1(D_n)$. Then $\{y_i \mid 1 \leq i \leq n-1\}$ is a base for the free $\mathbb{Z}[t^{\pm 1}]$-module $H_1(\tilde{D}_n)$. Let $w_i \in H_1(\tilde{D}_n, \partial \tilde{D}_n)$ denote the lift of the relative cycle connecting the $i$-th puncture boundary to the $(i+1)$-st one by a straight segment, for $1 \leq i \leq n-1$. Note that $y_i$ maps to $(t^{-1})w_i$ by the inclusion $H_1(\tilde{D}_n) \to H_1(\tilde{D}_n, \partial \tilde{D}_n)$.
Let $M = B(\beta)$ so that $\beta(w_i) = \sum_k M_{ki}w_k$ and $\beta(y_j) = \sum_l M_{lj}y_l$. Let $(J_0)_{ij} = \langle w_i, y_j \rangle$. Then the equality $J_0 = M^t J_0 M$ follows as below:

$$
\langle w_i, y_j \rangle = \langle \beta(w_i), \beta(y_j) \rangle = \sum_{k,l} (M_{ki}w_k, M_{lj}y_l) = \sum_{k,l} M_{ki} \langle w_k, y_l \rangle M_{lj}
$$

That $\det J_0 \neq 0$ can be shown easily by evaluating $J_0$ at $t = 0$. \hfill \Box

Given a relative cycle $v \in H_1(\widetilde{D_n}, \partial \widetilde{D_n})$, the action of $\sigma_i \in B_n$ on $w$ is determined by the intersection number $\langle w_i, v \rangle$ as follows:

$$
\sigma_i(v) = v + \langle w_i, v \rangle y_i
$$

as pointed in \cite{Big99}. From this we obtain

$$
\sigma_i(w_j) = w_j + \langle w_i, w_j \rangle y_i = w_j + \langle w_i, w_j \rangle (t - 1)w_i = w_j + \langle w_i, (t - 1)w_j \rangle w_i = w_j + \langle w_i, y_j \rangle w_i.
$$

This equation implies that the $i$-th row of the matrix $B(\sigma_i) - I_{n-1}$ is the unique non-zero row with $\langle w_i, y_j \rangle$ as the $j$-th entry. Therefore we calculate the intersection pairing $J_0$ by

$$
J_0 = \sum_{1 \leq i \leq n-1} (B(\sigma_i) - I_{n-1}).
$$

The analogous formula for the Lawrence–Krammer representation obtained in the previous section is:

$$
J = \sum_{1 \leq i < j \leq n} X_{i,j}^* X_{i,j} = \sum_{1 \leq i < j \leq n} (I_{(n+1)/2} - K(b_{i,j}))(I_{(n+1)/2} + qK(b_{i,j}^{-1}))
$$

where $b_{i,j}$ denotes the band generator $A_{\pi(i,j)} \sigma_1 A_{\pi(i,j)}^{-1}$.

5. Geometric biforks

In this section, we define geometric biforks and a skein algebra $\cal{T}_n$ generated by the geometric biforks.

Let $P = \{p_i \in D^2 \mid 1 \leq i \leq n\}$ be a set of $n$ distinct points in a disk $D^2$. Then $P \times \{1, 0\} \subset D^2 \times [0, 1]$ is the set of $2n$ distinct points on the top and bottom of the solid cylinder $D^2 \times [0, 1]$. A geometric $n$-braid is a disjoint union of $n$ strings in $D^2 \times [0, 1]$ having no local maxima or minima with their end points fixed in $P \times \{1, 0\}$. A geometric bifork is a disjoint union of $n$ strings in $D^2 \times [0, 1]$ having exactly one local maximum and one local minimum with one additionally attached string, which we call a handle, connecting the maximum point to the minimum point without touching the other $n - 2$ strings nor making a local extremum. We distinguish the handle from the other strings by drawing it with a wavy line as in Figure $[\text{Big}]$. The two strings to which the handle is attached are called times. Two geometric biforks related by an isotopy which does not create a new local extremum are considered to be the same.

We construct an arbitrary geometric bifork as follows. Connect $p_1 \times \{1\}$ to $p_2 \times \{1\}$ by a string with exactly one local minimum in $D^2 \times [0, 1]$ and connect $p_1 \times \{0\}$ to $p_2 \times \{0\}$ by a string with exactly one local maximum. Then connect the local minimum to local maximum by a straight wavy line. For $3 \leq i \leq n$,
connect \( p_i \times \{1\} \) to \( p_i \times \{0\} \) by straight strings. Now we obtained a simple geometric bifork \( x_{1,2}^*x_{1,2} \) which looks like a generator of the Birman–Murakami–Wenzl algebra except that it has a handle attached. We may attach arbitrary braids \( \beta \) and \( \gamma \) to the top and the bottom of the geometric bifork to obtain a general geometric bifork \( \beta x_{1,2}^*x_{1,2}\gamma \) where \( \beta, \gamma \in B_n \). One can easily see that every geometric bifork can be written as \( \beta x_{1,2}^*x_{1,2}\gamma \).

We define a \( \Lambda \)–algebra \( T_n \) generated by geometric biforks with the following relations.

\[
\begin{align*}
&x_{i,j} - qx_{i,j} = (1-q)x_{i,j} \quad (12) \\
&x_{k,l} = -tq^2x_{k,l} \quad (13) \\
&xx = \sigma x_{i,j} = x_{i,j}x = 0 \quad (14) \\
&x_i^2 = q^2x_i^2, \quad \bigcup x_{i,j} = q(1-q)x_{i,j}, \quad \bigcap x_{i,j} = q(1-q)x_{i,j} \quad (15)
\end{align*}
\]

The multiplication in \( T_n \) is given from concatenation as in the braid groups.

The relations \((12)(13)\) come from the definition that \( X_{1,2}^*X_{1,2} = -\mathcal{K}(\sigma_1) + q\mathcal{K}(\sigma_1^{-1}) + (1-q)L_{n(n-1)/2} \) and \( X_{1,2}\sigma_1 = -tq^2X_{1,2} \). The relations \((14)\) mean that \( X_{1,2}\sigma_m = X_{1,2} \) for \( m > 2 \). The first relation in \((15)\) reflects that \( X_{1,2}^*X_{1,2}\sigma_1\sigma_2 = q^2X_{1,2}X_{1,2} \) and the second one is from \( X_{1,2}X_{2,3} = q(1-q) \).

Let \( x_{i,j}^*x_{k,l} = A_{\pi(i,j)}x_{i,2}^*x_{1,2}A_{\pi(k,l)}^{-1} \). We call \( x_{i,j}^*x_{k,l} \) a standard geometric bifork. Figure 1 shows a typical one. Given an arbitrary geometric bifork \( g^*f \), we can express \( g^*f \) as a linear combination of other geometric biforks, which have less under-crossings of tines than \( g^*f \), by using the relations \((12)(13)\). Iterating this procedure, we express \( g^*f \) as a linear combination of standard geometric biforks. Therefore the set of standard geometric biforks generates \( T_n \) as a \( \Lambda \)–module. One may check that the formulas \((1)\) and \((2)\) with \( x_{i,j} \) and \( x_{k,l}^* \) in place of \( X_{i,j} \) and \( X_{k,l}^* \), also hold by applying the relations \((12)(13)\).

The previous observation on the relations \((12)(13)\) implies that there exists a surjective \( \Lambda \)–homomorphism \( \rho: T_n \to M_n, \rho(x_{i,j}^*x_{k,l}) = X_{i,j}^*X_{k,l} \).

**Theorem 5.** The geometric bifork algebra \( T_n \) is isomorphic to the matrix algebra \( M_n \) of biforks.

**Proof.** By Lemma 3, the algebra \( M_n \) is a free \( \Lambda \)–module with rank \( (n(n-1)/2)^2 \).

The fact that \( T_n \) is generated by \( (n(n-1)/2)^2 \) many elements implies that the surjective homomorphism \( \rho: T_n \to M_n \) is also injective.

In order to see \( \rho \) is an algebra homomorphism, we need \( (x_{u,v}^*x_{w,n})(x_{i,j}^*x_{o,p}) = (X_{k,l}X_{i,j}^*)x_{u,v}x_{o,p} \). In the following we verify the equality \( x_{k,l}^*x_{i,j} = X_{k,l}^*X_{i,j} \).
a. For \((i - k)(i - l)(j - k)(j - l) > 0\)
\[
\begin{align*}
\mathcal{U} \mathcal{U} = \mathcal{U} \mathcal{U} = \mathcal{U} \mathcal{U} = \mathcal{U} \mathcal{U} = 0
\end{align*}
\]

b. For \(i = k < j = l\)
\[
\begin{align*}
\mathcal{U} & = -\mathcal{U} + q \mathcal{U} + (1 - q) \mathcal{U} \\
& = t q^2 \mathcal{U} - t^{-1} q^{-1} \mathcal{U} + (1 - q) \mathcal{U} \\
& = (-t^{-1} + q)(q^{-1} + gt) \mathcal{U}
\end{align*}
\]

c. For \(i = k < l < j\) or \(k < i < j = l\)
\[
\begin{align*}
\mathcal{U} & = -t^{-1} q^{-2} \mathcal{U} = -t^{-1} q^{-1}(1 - q) \mathcal{U} \\
\mathcal{U} & = -t^{-1} q^{-2} \mathcal{U} = -t^{-1} q^{-1}(1 - q) \mathcal{U}
\end{align*}
\]

d. For \(i < j = k < l\)
\[
\begin{align*}
\mathcal{U} & = q(1 - q) \mathcal{U} = q^{-1}(1 - q) \mathcal{U}
\end{align*}
\]

e. For \(i < k < j = l\) or \(i = k < j < l\)
\[
\begin{align*}
\mathcal{U} & = -t q^2 \mathcal{U} = -t q^3(1 - q) \mathcal{U} = -t q(1 - q) \mathcal{U} \\
\mathcal{U} & = -t q^2 \mathcal{U} = -t q^3(1 - q) \mathcal{U} = -t q(1 - q) \mathcal{U}
\end{align*}
\]
f. For \(k < l = i < j\)
\[
\begin{align*}
\mathcal{U} & = q(1 - q) \mathcal{U}
\end{align*}
\]

g. For \(k < i < l < j\)
\[
\begin{align*}
\mathcal{U} & = q^{-1} \mathcal{U} + (1 - q^{-1}) \mathcal{U} + q^{-1} \mathcal{U} \\
& = 0 + (1 - q^{-1}) q(1 - q) \mathcal{U} - t^{-1} q^{-3} \mathcal{U} \\
& = -(1 - q)^2 \mathcal{U} - t^{-1} q^{-3} q^2(1 - q)^2 \mathcal{U} \\
& = (-1 - q^2)(1 - q t^{-1} q^{-1}) \mathcal{U} \\
& = -(1 - q)^2(t^{-1} q^{-1} + 1) \mathcal{U}
\end{align*}
\]
h. For $i < k < j < l$

\[
\begin{align*}
\{W\} & = q\{W\} + (1 - q)\{W\} - \{W\} \\
& = 0 + (1 - q)q(1 - q)\{W\} + tq^2 \{W\} \\
& = q^{-1}(1 - q)^2 \{W\} + tq^2 q^2 (1 - q)^2 \{W\} \\
& = (q^{-1}(1 - q)^2 + tq^4 (1 - q)^2 q^{-4}) \{W\} \\
& = (1 - q)^2(q^{-1} + t) \{W\}
\end{align*}
\]

\[
\square
\]

6. Dominance by finite type invariants

Let $I$ be the ideal of the integral group ring $\mathbb{Z}[B_n]$ generated by $\{\sigma_i - \sigma_i^{-1} \mid 1 \leq i \leq n-1\}$. Let $A$ be an abelian group. If a $\mathbb{Z}$–module homomorphism $v : \mathbb{Z}[B_n] \to A$ vanishes on $I^{(k+1)}$, we call $v$ a finite type invariant of order $k$. If $v(\gamma^{-1} \beta \gamma) = v(\beta)$ for each $\beta, \gamma \in B_n$, we call $v$ a conjugacy invariant.

Theorem 6. For each $k, l \geq 0$, the $n(n-1)/2 \times n(n-1)/2$ integral matrix invariant $\mathbb{Z}[B_n] \to M(n(n-1)/2, \mathbb{Z})$, 

\[
\beta \mapsto \frac{\partial^{k+l}}{\partial t^k \partial q^l} K(\beta)(-1, 1)
\]

is a finite type invariant of order $k + l$.

Proof. At $t = -1$ and $q = 1$, it is easy to check that $K(\sigma_m)(-1, 1) = K(\sigma_m^{-1})(-1, 1)$, so that at $t = -1$ and $q = 1$, $K$ is a finite type invariant of order 0. In the series expansion of $K(\sigma_m - \sigma_m^{-1}) = \sum_{k, l \geq 0} a_{kl} (t + 1)^k (q - 1)^l$ at $t = -1$ and $q = 1$, the lowest degree is at least 1. In other words, $K(W)$ has the lowest degree at least 1 for each $W \in I^{(1)}$.

If $W \in I^{(k+l+1)}$, then the lowest degree in the series expansion of $K(W)$ is at least $k + l + 1$ since $W$ is a linear combination of products of $(k + l + 1)$ elements of $I^{(1)}$. The theorem follows from 

\[
\frac{\partial^{k+l}}{\partial t^k \partial q^l} (t + 1)^i (q - 1)^j |_{t=-1, q=1} = 0
\]

for $i + j > k + l$.

\[
\square
\]

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