Two-dimensional $N = 1, 2$ Supersymmetric Chiral and Dual Models

C.P. Constantinidis$^{(a1)}$, F.P. Devecchi$^{(b)}$ and F. Toppan$^{(a2)}$

$^{(a1,a2)}$ UFES, CCE Depto de Física, Goiabeiras cep 29060-900, Vitória (ES) Brasil
$^{(b)}$ UFPR, Depto de Física, c.p. 19091, cep 81531-990, Curitiba (PR) Brasil

Abstract

Two-dimensional $N = 1, 2$ supersymmetric chiral models and their dual extensions are introduced and canonically quantized. Working within a superspace formalism, the non-manifest invariance under 2D-superPoincaré transformations is proven. The $N = 1, 2$ superVirasoro algebras are recovered as current algebras. The non-anomalous quantum invariances under 1D-superdiffeomorphisms (for chiral models) and $N = 1, 2$ superconformal transformations (for dual models) are shown to be a consequence of an $N = 1, 2$ super-Coulomb gas representation.

E-Mails:
$^{(a1)}$ clisthen@cce.ufes.br
$^{(b)}$ devecchi@fisica.ufpr.br
$^{(a2)}$ toppan@cce.ufes.br

DF/UFES-P001/98
1 Introduction.

The Schwarz-Sen electromagnetic dual model \cite{1} is a four-dimensional member of a more general family of theories which are manifestly invariant under duality transformations \cite{2}. Its two-dimensional version is known as Tseytlin model \cite{3}, introduced at first in the string theory context. It is a conformal theory and as such can be decomposed into two independent (for left and right movers) Floreanini-Jackiw (FJ) \cite{4} chiral boson models. FJ models have peculiar features which have been extensively investigated in the last ten years \cite{5}. In particular, despite the fact they are not-manifestly Lorentz-invariant, they turn out to be 2-dimensional Poincaré invariant. The quantum hamiltonian structure of the FJ model was analyzed in \cite{5}, while in \cite{6} this analysis was extended to the Tseytlin model, proving in particular the closure of 2D Poincaré algebra both in the classical and in the quantum case. In \cite{6} it was furthermore proposed a hamiltonian supersymmetric theory which coincides with a 2-dimensional reduction of the supersymmetric extension of the original Schwarz-Sen model \cite{1}. A complete analysis of its symmetries, as well as a manifest supersymmetric formulation, was however not carried out in that paper. The purpose of our present work is to fully investigate the properties of supersymmetric extensions of both chiral (FJ) and dual (Tseytlin) models. Our aim is to provide the algebraic setting underlying dimensional reductions of supersymmetric 4-dimensional dual models.

In this paper we construct for both FJ and Tseytlin models their $N=1$ and $N=2$ supersymmetric extensions by using a superfield formalism$^1$. We show that their symmetries generate $N=1,2$ SuperVirasoro algebras and are in connection with the $N=1,2$ Coulomb gas formulation (see \cite{6} and references therein). The closure of 2$D$ ($N=1,2$) superPoincaré algebra is proven in both classical and quantum cases.

It is worth mentioning that the construction of supersymmetric extensions must be carefully performed, which means their investigation is quite interesting. As an example we just mention that the equations of motion for a system involving a chiral boson and a chiral fermion can be derived by using two different hamiltonian pictures. Only one of them leads to a supersymmetric theory, while the other does not. Further topics of this kind will be discussed in the text.

The scheme of the paper is as follows:

In the next section we review the formulation of the bosonic FJ and Tseytlin models. Despite the fact that most of the material presented is nowadays standard, some results presented are new.

In section 3 we introduce the $N=1$ superfield formalism and derive the corresponding super-FJ and superTseytlin models. Since it is not possible to derive the equations of motion directly from a manifestly supersymmetric 2-dimensional action, we supersymmetrize the space coordinate only, leaving the time an ordinary bosonic variable. Our formulation differs from a previously constructed version \cite{8} in which one light-cone variable was supersymmetrized, and is more suitable for analyzing the Tseytlin model which deals with both chiralities. The invariances under 1-dimensional superdiffeomorphisms and 2-dimensional superPoincaré transformations are proven. The Dirac bracket analysis is performed and the Noether supercurrents are derived. They realize an $N=1$ superVirasoro algebra (for both chiralities in the Tseytlin model) with central charge $c = \frac{3}{2}$ in the quantum case.

$^1$ In reality the “$N=2$” Tseytlin model as a quantum mechanical system is globally $N=4$ supersymmetric. We discuss this point in detail in the following.
The section 4 deals with the $N = 2$ extensions of the above models. They are constructed by making use of $N = 2$ chiral and antichiral superfields and mimicking the procedure employed in the previous case. The total field content (in the $N = 2$ FJ model) consists of two ordinary chiral bosons and two ordinary chiral fermions. Invariances under 2-dimensional $N = 2$ superPoincaré transformations and 1-dimensional $N = 2$ superdiffeomorphisms are proven. The Dirac’s brackets for the conserved Noether currents generate an $N = 2$ superVirasoro algebra with central charge $c = 3$ in the quantum case.

2 The bosonic FJ and Tseytlin models.

Let us start introducing the bosonic FJ and Tseytlin models. They are defined in terms of the following lagrangian densities:

$$L_{FJ} = \partial_0 \phi \partial_1 \phi - (\partial_1 \phi)^2$$

for the FJ model and

$$L_{Ts} = \partial_0 \phi \partial_1 \tilde{\phi} + \partial_0 \tilde{\phi} \partial_1 \phi - (\partial_1 \phi)^2 - (\partial_1 \tilde{\phi})^2$$

for the Tseytlin model.

The above lagrangians coincide with the ones given in the literature up to an overall normalization factor.

We work in the 2-dimensional Minkowski spacetime; the time coordinate $t$ will also be denoted as $x^0 (\partial_0 = \partial_t)$ and the space coordinate $x$ as $x^1 (\partial_1 = \partial_{x^1})$. We will also make use of the light-cone coordinates $z_\pm$ defined as

$$z_\pm = x \pm t, \quad \partial_\pm = \frac{1}{2}(\partial_1 \pm \partial_0).$$

The equations of motion are given by

$$(\partial_0 - \partial_1)\partial_1 \phi = 0$$

in the FJ case and

$$\partial_1^2 \tilde{\phi} - \partial_0 \partial_1 \phi = 0, \quad \partial_1^2 \phi - \partial_0 \partial_1 \tilde{\phi} = 0$$

in the Tseytlin case.

The lagrangian density is invariant under duality transformations, i.e. exchanging $\phi \leftrightarrow \tilde{\phi}$. The Tseytlin model can be decomposed into two (chiral and antichiral) FJ models as it is evident from the positions

$$\phi_\pm = \frac{1}{\sqrt{2}}(\phi \pm \tilde{\phi}).$$

The lagrangian $L_{Ts}$ can therefore be rewritten as

$$L_{Ts} = \partial_1 \phi_+ (\partial_1 - \partial_0) \phi_+ + \partial_1 \phi_- (\partial_1 + \partial_0) \phi_-.$$
Both the FJ and the Tseytlin model are invariant under Poincaré 2-dimensional transformations, with $\phi$, $\tilde{\phi}$ transforming as scalar fields. A basic difference between the FJ theory and its dual version consists in the fact that in the chiral case the Hamiltonian $H \equiv P^0$ coincides with the space-translation generator $P^1$ while they are different for the Tseytlin Lagrangian.

Another set of invariances is provided by a full class of transformations dependent on a parameter $\lambda$. Such kind of transformations are very well studied in the context of Coulomb gas picture [9]. However, they have not been considered for the models we are dealing with. Indeed the action correspondent to the FJ Lagrangian is invariant under the infinitesimal transformations:

$$\delta_\lambda \phi(x,t) = \epsilon(z_+) \partial_1 \phi + \lambda \partial_t \epsilon(z_+)$$

for any value of $\lambda$. The same transformation applied to the $\phi_+$ field leaves invariant the Tseytlin action while a similar transformation applied on $\phi_-$ depends on an infinitesimal parameter $\epsilon(z_-)$.

By expanding $\epsilon(z_+)$ in Laurent series ($\epsilon(z_+) = - \sum_{n \in \mathbb{Z}} \epsilon_n z_+^{n+1}$) we can introduce for any $\lambda$ the operators $l_n(\lambda)$ given by

$$l_n(\lambda) = -(z_+^{n+1} \partial_1 + \lambda(n+1)z_+^n),$$

therefore

$$\delta_\lambda \phi = \sum_{n \in \mathbb{Z}} \epsilon_n l_n(\lambda) \phi.$$ (9)

For any fixed value of $\lambda$ the algebra generated by the $l_n(\lambda)$ operators is the 1-dimensional diffeomorphisms algebra (centerless Virasoro algebra):

$$[l_n(\lambda), l_m(\lambda)] = (n-m)l_{n+m}(\lambda),$$ (10)

therefore the set of invariances of the FJ model includes the 1D-diffeomorphisms, while in the Tseytlin case we have the direct sum of two copies of 1D-diffeomorphisms, one for each chirality.

The analysis of the Hamiltonian dynamics, the structure of primary constraints and the construction of Dirac’s brackets has been performed in [5] in the FJ case and in [6] in the Tseytlin case.

These results lead to the Hamiltonian

$$H_{F,J} = \int dx \theta_{F,J}^{00}(x),$$ (11)

with the current $\theta_{F,J}^{00}$ given by (in the quantum case)

$$\theta_{F,J}^{00} = - : (\partial_1 \phi)^2 :.$$ (12)

The equal-time Dirac’s brackets [10] are computed and give

$$[\phi(x), \phi(y)]_D = \frac{i}{2} \delta_y^{-1} \delta(x-y),$$ (13)

In the following formulae the double dots denote the standard normal ordering; moreover for our purposes, in order to avoid complications arising from boundary conditions, we assume the space coordinate $x$ being compactified on a circle $S^1$ with periodic boundary conditions, or living on $\mathbb{R}$ and the fields being fast-decreasing at the infinities.
where the standard delta-function appears in the r.h.s.

In the Tseytlin case we have respectively

\[ H_{T_s} = -\int dx (\partial_1 \phi^+)^2 + : (\partial_1 \phi^-)^2 : ), \]  

(14)

and the following Dirac’s brackets

\[ [\phi_+(x), \phi_\pm(y)]_D = \pm \frac{i}{2} \partial_y^{-1} \delta(x - y), \quad [\phi_+(x), \phi_-(y)]_D = 0. \]  

(15)

The 2-dimensional Poincaré algebra defined by the translation generators \( P^0, P^1 \) and Lorentz boost \( M \), with structure constants

\[ [M, P^0] = iP^1, \quad [M, P^1] = iP^0 \]  

(16)

(and vanishing otherwise), is reproduced by the conserved charges computed with the standard Nöther methods. In the FJ case we have

\[ P^1 = P^0 \equiv H_{FJ}, \quad M = \int dx M(x), \]  

(17)

with \( M(x) \) given by

\[ M(x) = (x + t)\theta_{FJ}^{00}(x, t). \]  

(18)

We can compute the commutation relations of the \( \theta_{FJ}^{00}(x) \) currents by using OPE techniques. The result is the following

\[ [\theta_{FJ}^{00}(x), \theta_{FJ}^{00}(y)] = -\frac{1}{12} \partial_y^3 \delta(x - y) + 2i \theta_{FJ}^{00}(y) \partial_y \delta(x - y) + i \partial_y \theta_{FJ}^{00}(y) \cdot \delta(x - y). \]  

(19)

The above algebra corresponds to the Virasoro algebra and the first term in the r.h.s. gives the central extension. In the classical case such term is not present and the algebra coincides with the algebra of 1-dimensional diffeomorphisms. By reexpressing \((19)\) in standard quantum OPE form we realize that the value of the central charge \( c \) corresponds to \( c = 1 \).

By setting \( P^0 = P^1 = P \) the 2D-Poincaré algebra can be recovered from the single commutation relation

\[ [M, P] = iP. \]  

(20)

We finish this part devoted to the FJ model by discussing its invariance properties under 1D-diffeomorphisms. For any given \( \lambda \) the \((7)\) transformations are generated by the Nöther currents \( \theta_\lambda(x, t) \) given by

\[ \theta_\lambda(x) = - (\partial_1 \phi)^2 + \lambda (\partial_1)^2 \phi. \]  

(21)

The conserved charges are given by \( L_n(\lambda) \),

\[ L_n(\lambda) = \int dx(x + t)^{(n+1)} \theta_\lambda(x, t). \]  

(22)
The $L_n(\lambda)$ charges satisfy a closed algebra structure generated by Dirac’s brackets. It coincides with the Virasoro algebra with central charge $c = 1 - 6i\lambda^2$:

$$[L_n(\lambda), L_m(\lambda)] = i(n-m)L_{n+m}(\lambda) - \frac{c}{12} n(n^2 - 1)\delta_{n+m,0}.$$  

The extra term $\lambda(\partial_1)^2\phi$ corresponds to the well-known Feigin-Fuchs term in the Coulomb gas picture. Its purpose there consists in providing a bosonization for conformal theories with any specific value of the central charge $c$. In the present context it tells us the following feature of the model under consideration. It keeps an invariance under 1D-diffeomorphisms even in the quantum case because it is possible to find a particular value of $\lambda$ ($\lambda = \sqrt{-\frac{5i}{6}}$) in such a way that the central charge is vanishing, which leads to a non-anomalous quantum theory.

For what concerns the Tseytlin model its Noether currents $\theta_{Ts}^{00}$, $\theta_{Ts}^{01}$ associated to the space-time translations can be decomposed as

$$\theta_\pm = \theta_{Ts}^{00} \pm \theta_{Ts}^{01} = \theta_\pm = -:\!(\partial_1\phi_\pm)^2:\!,$$

while the current $\mathcal{M}$ associated to the Lorentz boost is

$$\mathcal{M} = x\theta_{Ts}^{00} + t\theta_{Ts}^{01}.$$  

The 2-D Poincaré algebra is realized by the commutators of the conserved charges

$$P^0 = \int dx\theta_{Ts}^{00}, \quad P^1 = \int dx\theta_{Ts}^{01}, \quad M = \int dx\mathcal{M}.$$  

The currents $\theta_\pm$ make explicit the fact that the model under consideration is conformally invariant since their commutation relations satisfy the following algebraic relations

$$[\theta_\pm(x), \theta_\pm(y)] = -1/12\partial_\mp^3\delta(x-y) + 2i\theta_\pm(y)\partial_y\delta(x-y) + i\partial_y\theta_\pm(y)\cdot\delta(x-y),$$

$$[\theta_+(x), \theta_-(y)] = 0,$$

which correspond to two separated copies (one for each chirality) of the Virasoro algebra, both with central charge $c = 1$ in the quantum case. We wish to mention that this analysis corrects a statement in claiming the absence of the central extension (the proof there furnished of the Poincaré invariance remains valid because not affected by the presence of the central term).

3 The $N = 1$ supersymmetric FJ and dual models.

The theory of a chiral boson $\phi$ and a chiral fermion $\psi$ consists in the system of equations of motion

$$\partial_-\partial_1\phi = 0, \quad \partial_-\psi = 0.$$  

The above system of equations can be recovered from a single superfield equation where both spacetime coordinates $x, t$ (or more commonly the lightcone coordinates $z_\pm$) have been supersymmetrized. However one can easily realize that such an equation cannot be derived from a 2D manifestly supersymmetric action principle. It is nevertheless possible to make use of a superaction principle where only one coordinate has been supersymmetrized, while the other
has been kept ordinary. Such procedure has been employed in [8] to define the super-Siegel model [12] in light-cone coordinates (one made supersymmetric). Here we adopt the point of view of leaving the time variable $t$ unchanged while supersymmetrizing the space coordinate $x$ (now denoted as $X \equiv x, \theta$, with $\theta$ a Grassmann variable). This approach is quite natural when dealing with hamiltonian systems which single out the time coordinate and more suitable for application to supersymmetric dual models; indeed one can obtain them from a single dynamics instead of being obliged to introduce two separate dynamics, one for each chirality.

In our conventions $X \equiv x, \theta$ and the $N = 1$ supersymmetric derivative is

$$D = \frac{\partial}{\partial \theta} + i \theta \partial_1, \quad D^2 = i \partial_1.$$  \hfill (29)

We introduce the superfield $\Phi(X,t)$:

$$\Phi(X,t) = \phi(x) + \theta \psi(x).$$  \hfill (30)

The $N = 1$ supersymmetric action $S$ is given by

$$S = -i \int dX dt (\partial_0 \Phi - \partial_1 \Phi) D \Phi,$$  \hfill (31)

which implies for $\Phi$ the equation of motion, equivalent to (28),

$$\partial_- D \Phi = 0.$$  \hfill (32)

In components the supersymmetric lagrangian $L_{SFJ}$ is

$$L_{SFJ} = \partial_0 \phi \partial_1 \phi - (\partial_1 \phi)^2 + i \psi (\partial_0 \psi - \partial_1 \psi)$$  \hfill (33)

and the global supersymmetry transformation is given by

$$\delta \phi = \epsilon \psi, \quad \delta \psi = i \epsilon \partial_1 \phi.$$  \hfill (34)

The hamiltonian analysis of the above action can be straightforwardly done. At first we compute the supermomentum $\Pi$

$$\Pi + i D \Phi = \Omega \approx 0,$$  \hfill (35)

which represents one primary superconstraint ($\Omega$) of second class.

The total hamiltonian is

$$H_T = H_c + \mu \Omega,$$  \hfill (36)

where $\mu$ is an arbitrary multiplier and

$$H_c = -i \int dX \partial_1 \Phi D \Phi.$$  \hfill (37)

The following generalized Poisson algebra is satisfied by the superfields

$$\{ \Phi(X), \Pi(Y) \} = \delta(X,Y) \equiv \delta(x-y)(\theta_x - \theta_y)$$  \hfill (38)
(all the other Poisson brackets being zero).

The constraints \( \{\Omega(X), \Omega(Y)\} \) are found to be second class

\[
\Delta(X, Y) = \{\Omega(X), \Omega(Y)\} = 2iD_X \delta(X, Y)
\]

and they do not generate secondary constraints.

In order to find the correct equations of motion we construct the reduced phase space structure following Dirac [10]. The inverse of (39) is

\[
\Delta^{-1}(X, Y) = -\frac{1}{2}D_X \partial^{-1}_x \delta(X, Y).
\]

The Dirac brackets can now be computed for any couple of superfie lds \( A(X), B(X) \):

\[
\{A(X), B(Y)\}_D = \{A(X), B(Y)\} - \int dZdW \{A(X), \Omega(Z)\} \Delta^{-1}(Z, W) \{\Omega(W), B(Y)\}.
\]

In particular we obtain, as fundamental algebraic relation

\[
\{\Phi(X), \Phi(Y)\}_D = \frac{1}{2}D_X \partial^{-1}_x \delta(X, Y).
\]

Finally we get the classical hamiltonian equations of motion

\[
\partial_0 \Phi(X) = \{\Phi(X), H\}_D = \partial_1 \Phi(X).
\]

At the quantum level the equal-time anti-commutator in superfield notation is

\[
\{D_X \Phi(X), D_Y \Phi(Y)\} = \frac{1}{2}D_Y \delta(X, Y),
\]

which in components reads

\[
[\partial_x \phi(x), \partial_y \phi(y)] = -\frac{i}{2} \partial_y \delta(x - y), \quad \{\psi(x), \psi(y)\} = \frac{1}{2} \delta(x - y).
\]

The supercurrent

\[
\vartheta^{00}(X) = i : \partial_1 \Phi D \Phi : = q(x) + \theta l(x) = i : (\partial_1 \phi) \cdot \psi : + \theta(- : (\partial_1 \phi)^2 : + i : \partial_1 \psi \cdot \psi :)
\]

gives the \( c = \frac{3}{2} N = 1 \) superVirasoro algebra.

The Nöther conserved charges which generates a constrained (\( P^0 = P^1 \)) version of the 2D superPoincaré algebra are

\[
P = P^0 = P^1 = \int dx l(x), \quad M = \int dx(x + t) l(x), \quad Q = \int dx q(x).
\]

The non-vanishing (anti)-commutators are

\[
[M, P] = iP, \quad [M, Q] = i\frac{1}{2}Q, \quad \{Q, Q\} = \frac{1}{2} P.
\]
Just like its bosonic counterpart, the supersymmetric action (31) is classically invariant under a class of \( \lambda \)-parametrized 1D-superdiffeomorphisms transformations:

\[
\delta \Phi(X,t) = \epsilon(z_+, \theta) \partial_1 \Phi - \frac{i}{2} \lambda \partial_1 \epsilon(z_+, \theta),
\]

(49)

where the infinitesimal variation \( \epsilon \) is function of \( z_+, \theta \) only (\( \partial_- \epsilon = 0 \)). By performing the same analysis as in the bosonic case we find the fermionic supercurrent \( \vartheta_\lambda(X) \) which generates the transformations above (49):

\[
\vartheta_\lambda(X) = i : \partial_1 \Phi \partial \Phi : -i \lambda \partial_1 \partial \Phi .
\]

(50)

The (anti)-commutation relations satisfied by \( \vartheta_\lambda \) produce the \( N = 1 \) superVirasoro algebra with central extension \( c = \frac{3}{2} - 6i \lambda^2 \):

\[
\{ \vartheta_\lambda(X), \vartheta_\lambda(Y) \} = -\frac{1}{8} (i + 4 \lambda^2) D_y (\partial_y)^2 \delta(X,Y) - \frac{3}{2} i \vartheta_\lambda(Y) \partial_y \delta(X,Y) - \frac{1}{2} D \vartheta_\lambda(Y) \cdot D \delta(X,Y) ,
\]

(51)

The conserved charges are computed as in the bosonic case and the non-anomalous 1D-superdiffeomorphisms invariance is recovered for the value \( \lambda = \sqrt{\frac{3}{4}} \).

We point out that, for the real-valued component fields \( \phi, \psi \), the equations of motion (28) can also be obtained from the hamiltonian

\[
H = - \int dx \left( : (\partial_1 \phi)^2 : + i : \partial_1 \psi : \right)
\]

(52)

with (anti)-commutation relations

\[
[\partial_x \phi(x), \partial_y \phi(y)] = -\frac{i}{2} \partial_x \delta(x - y) , \quad \{ \phi(x), \psi(y) \} = -\frac{1}{2} \delta(x - y) .
\]

(53)

The above formulas are recovered from the previous ones after setting \( \psi \mapsto i \psi \).

The hamiltonian \( H \) of eq. (52) however, unlike \( P^0 \) in eq. (47), is not supersymmetric because the fermionic hermitian operator \( Q = i \sqrt{2} : \partial_1 \phi : \) in this case leads to \( \{Q,Q\} = -H \). Notice the presence of the “wrong” minus sign. When dealing with extended supersymmetries or supersymmetric dual models one has to be very careful in picking up the “correct” supersymmetric hamiltonian.

We devote the last part of this section to discuss the superextension of the Tseytlin model. As in the bosonic case the supersymmetric dual model can be decomposed into two independent, respectively chiral and antichiral, \( N = 1 \) FJ models. The supersymmetric action which coincides with a dimensional reduction of the 4-dimensional super-Schwarz-Sen model is up to a normalizing factor [8]

\[
S = \int d^2 x \left[ \partial_0 \phi \partial_1 \phi + \partial_0 \tilde{\phi} \partial_1 \phi - (\partial_1 \phi)^2 - (\partial_1 \tilde{\phi})^2 \right.
\]

\[
+i \psi \partial_0 \psi + i \tilde{\psi} \partial_0 \tilde{\psi} - i \psi \partial_1 \psi - i \tilde{\psi} \partial_1 \tilde{\psi} \] ,
\]

(54)
and can be rewritten in superfield notations as

$$ S = -i \int dX dt (\partial_0 \Phi_+ - \partial_1 \Phi_+) D \Phi_+ - (\partial_0 \Phi_- + \partial_1 \Phi_-) D \Phi_- , $$  \hspace{1cm} (55)

where

$$ \Phi_+ = \phi_+ + \theta \psi_+ , \quad \Phi_- = \phi_- + i \theta \psi_- $$  \hspace{1cm} (56)

are chiral (antichiral) superfields and

$$ \phi_\pm = \frac{1}{\sqrt{2}} (\phi \pm \tilde{\phi}) , \quad \psi_\pm = \frac{1}{\sqrt{2}} (\psi \pm \tilde{\psi}) . $$  \hspace{1cm} (57)

Notice that the presence of an extra “i” in the decomposition of $\Phi_-$ is in order to make the hamiltonian for the antichiral sector supersymmetric, as explained above.

The duality invariance corresponds to the exchange $\Phi_\pm \leftrightarrow \pm \Phi_\mp$.

The Nöther analysis is recovered from the results of the $N = 1$ FJ model. The anticommutation relations are

$$ [D \Phi_\pm (X), D \Phi_\pm (Y)] = \mp \frac{1}{2} D_\gamma \delta (X, Y) , \quad [D \Phi_+ (X), D \Phi_- (Y)] = 0 . $$  \hspace{1cm} (58)

Two independent $c = \frac{3}{2}$ superVirasoro algebras result from the supercurrents $\vartheta_\pm$:

$$ \vartheta_\pm = \gamma_\pm : \partial_1 \Phi_\pm \cdot D \Phi_\pm : = q_\pm - i \gamma_\pm \theta l_\pm , $$  \hspace{1cm} (59)

where $\gamma_+ = i$ and $\gamma_- = -1$.

Let us introduce the currents

$$ \vartheta^{00} = l_+ + l_- , \quad \vartheta^{01} = l_+ - l_- , \quad q^{01} = q_+ + q_- , \quad q^{02} = q_+ - q_- . $$  \hspace{1cm} (60)

The superPoincaré algebra is realized by the bosonic conserved charges

$$ P^0 = \int dx \vartheta^{00} , \quad P^1 = \int dx \vartheta^{01} , \quad M = \int dx (x \vartheta^{00} + t \vartheta^{01}) , $$  \hspace{1cm} (61)

together with the supercharges

$$ Q^1 = \int dx q^{01} , \quad Q^2 = \int dx q^{02} . $$  \hspace{1cm} (62)

As a quantum mechanical system, the “$N = 1$” Tseytlin model is globally $N = 2$ supersymmetric since $Q_1, Q_2$ satisfy

$$ \{ Q^1, Q^1 \} = \{ Q^2, Q^2 \} = H , \quad \{ Q^1, Q^2 \} = 0 , $$  \hspace{1cm} (63)

where $H$ is the hamiltonian.

Explicitly $Q^1, Q^2$ generate two supersymmetry transformations

$$ \delta_1 \phi_\pm = \frac{\epsilon_1}{2} \psi_\pm , \quad \delta_1 \psi_\pm = \pm \frac{i}{2} \epsilon_1 \partial_1 \phi_\pm ; \quad \delta_2 \phi_\pm = \pm \frac{\epsilon_2}{2} \psi_\pm , \quad \delta_2 \psi_\pm = \frac{i}{2} \epsilon_2 \partial_1 \phi_\pm . $$  \hspace{1cm} (64)
The model is superconformally invariant and non-anomalous even in the quantum case as a trivial consequence of the 1D superdiffeomorphisms invariance of the $N = 1$ FJ theory.

4 The $N = 2$ extensions.

The $N = 2$ extensions of the FJ and the (globally $N = 4$ invariant) Tseytlin model can be constructed by mimicking the previous constructions in a manifest $N = 2$ superfield formulation. Since the analysis proceeds as before we limit ourselves to write the results.

When dealing with $N = 2$ superfields we have at first to establish if real or constrained (anti)-chiral superfields are employed. It turns out that chiral-antichiral superfields make the job.

The theories will be defined by leaving the time $t$ ordinary while the space coordinate will be $N = 2$ supersymmetrized with the introduction of $\theta, \overline{\theta}$ Grassmann variables. Our $N = 2$ conventions (see also [11]) are as follows. The fermionic derivatives $D, \overline{D}$ are

$$D = \frac{\partial}{\partial \theta} - \frac{i}{2} \overline{\theta} \partial_1, \quad \overline{D} = \frac{\partial}{\partial \overline{\theta}} - \frac{i}{2} \theta \partial_1.$$  \hfill (65)

They satisfy the equations

$$D^2 = \overline{D}^2 = i \partial_1 , \quad \{D, \overline{D}\} = 0 . \hfill (66)$$

$N = 2$ chiral ($\Phi$) and antichiral ($\overline{\Phi}$) superfields satisfy the constraints

$$D \Phi = 0 , \quad \overline{D} \Phi = 0 . \hfill (67)$$

In components we have

$$\Phi = \phi + \overline{\theta} \psi + \frac{i}{2} \overline{\theta} \theta \partial_1 \phi , \quad \overline{\Phi} = \overline{\phi} + \theta \overline{\psi} - \frac{i}{2} \theta \overline{\theta} \partial_1 \overline{\phi} . \hfill (68)$$

The $N = 2$-invariant action for the FJ model is given by the sum of two pieces involving separately $N = 2$ chiral and antichiral superfields:

$$S = i \int dtdX_L \left( \partial_0 \Phi - \partial_1 \Phi \right) D \overline{\Phi} + i \int dtdX_R \left( \partial_0 \overline{\Phi} - \partial_1 \overline{\Phi} \right) \overline{D} \Phi , \hfill (69)$$

where $dX_L \equiv dx d\theta$, $dX_R \equiv dx d\overline{\theta}$ denote integration over the chiral (antichiral) variables.

In components the lagrangian $\mathcal{L}$ is

$$\mathcal{L} = \partial_0 \phi \partial_1 \overline{\phi} + \partial_0 \overline{\theta} \partial_1 \phi - 2 \partial_1 \phi \partial_1 \overline{\phi} + i (\partial_0 \psi - \partial_1 \psi) \overline{\psi} + i (\partial_0 \overline{\psi} - \partial_1 \overline{\psi}) \psi . \hfill (70)$$

The reality condition sets $\phi^\dagger = \overline{\phi}$, $\psi^\dagger = \overline{\psi}$.

At the level of the equations of motion we obtain two copies of the supersymmetric FJ equations. Indeed

$$\partial_1 \partial_- \phi = \partial_1 \partial_- \overline{\phi} = 0 , \quad \partial_- \psi = \partial_- \overline{\psi} = 0 . \hfill (71)$$

The (anti)-commutation relations which define the hamiltonian dynamics are given by

$$[\partial_1 \overline{\phi}(x), \partial_1 \phi(y)] = \frac{i}{2} \partial_y \delta(x - y) , \quad \{\overline{\psi}(x), \psi(y)\} = \frac{1}{2} \delta(x - y) . \hfill (72)$$
and vanishing otherwise.

In manifest $N = 2$ superfield notation they are written as

$$\{D\Phi(X), \overline{D\Phi}(Y)\} = \frac{1}{2} D_X \overline{D_Y} \delta(X, Y), \quad (73)$$

here $\delta(X, Y) = \delta(x - y)(\theta_x - \theta_y)(\overline{\theta}_x - \overline{\theta}_y)$ is the $N = 2$ supersymmetric delta-function.

The hamiltonian $H$ is given by

$$H = \int dt dX : D\Phi \cdot \overline{D\Phi} : \quad . \quad (74)$$

The supercurrent $J(X) =: D\Phi \cdot \overline{D\Phi} := j + \theta q + \overline{\theta}\overline{q} + \theta \overline{\theta}l$, where

$$j(x) := \psi \overline{\psi} : \quad q(x) = i : \partial_x \phi \cdot \overline{\psi} : \quad \overline{q}(x) = -i : \partial_x \overline{\phi} \cdot \psi : \quad$$

$$l(x) := i : \partial_x \phi \cdot \partial_x \overline{\phi} : + \frac{i}{2} : \partial_x \psi \cdot \overline{\psi} : + \frac{i}{2} : \partial_x \overline{\psi} \cdot \psi : \quad (75)$$

satisfy the $N = 2$ superVirasoro algebra with central charge $c = 3$ in the quantum case (the standard OPE conventions are recovered by rescaling $l \mapsto \tilde{l} = -2il$, $(q, \overline{q}) \mapsto (\tilde{q}, \overline{\tilde{q}}) = \sqrt{-8i}(q, \overline{q})$, $j \mapsto \tilde{j} = 2j$):

$$[l(x), l(y)] = -\frac{3}{48} \partial_y^3 \delta(x - y) + il(y) \partial_y \delta(x - y) + \frac{i}{2} \partial_y l(y) \cdot \delta(x - y),$$

$$[l(x), q(y)] = \frac{3i}{4} q(y) \partial_y \delta(x - y) + \frac{i}{2} \partial_y q(y) \cdot \delta(x - y),$$

$$[l(x), \overline{q}(y)] = \frac{3i}{4} \overline{q}(y) \partial_y \delta(x - y) + \frac{i}{2} \partial_y \overline{q}(y) \cdot \delta(x - y),$$

$$[l(x), j(y)] = \frac{i}{2} j(y) \partial_y \delta(x - y) + \frac{i}{2} \partial_y j(y) \cdot \delta(x - y),$$

and vanishing otherwise.

The above currents are the building blocks to construct the $N = 2$ superPoincaré generators just like in the previous cases.

In particular the $N = 2$ global hermitian charges are

$$Q_1 = \text{def} \int dx \ (q(x) + \overline{q}(x)) , \quad Q_2 = \text{def} \ i \int dx \ (q(x) - \overline{q}(x)) ,$$

which satisfy $\{Q_1, Q_2\} = 0$, $\{Q_1, Q_1\} = \{Q_2, Q_2\} = H$.

The (non-anomalous) invariance under 1-dimensional $N = 2$ diffeomorphisms is implied in the quantum case by the existence of modified currents $u_{x, \tau} (u$ denotes either $j, q, \overline{q}$ or $l)$, which satisfy an $N = 2$ superVirasoro where all central charges are vanishing. The modified currents
cannot be accommodated into an \( N = 2 \) superfield formalism and we are obliged to write them in components. We get
\[
\begin{align*}
    j_{\lambda \overline{\lambda}}(x) &= : \psi \overline{\psi} : - i \lambda \partial_1 \phi - i \overline{\lambda} \partial_1 \overline{\phi} , \\
    q_{\lambda \overline{\lambda}}(x) &= i : \partial_1 \phi \cdot \overline{\psi} : - i \overline{\lambda} \partial_1 \overline{\phi} , \\
    l_{\lambda \overline{\lambda}}(x) &= : \partial_1 \phi \cdot \partial_1 \overline{\phi} : + \frac{i}{2} : \partial_1 \psi \cdot \overline{\psi} : + \frac{\lambda}{2} \partial_1^2 \phi - \frac{\overline{\lambda}}{2} \partial_1^2 \overline{\phi} .
\end{align*}
\]
(77)

If \( \lambda, \overline{\lambda} \) are chosen in such a way that \( \lambda \cdot \overline{\lambda} = - \frac{i}{4} \) then all central charges are vanishing; as an example in particular
\[
[j_{\lambda \overline{\lambda}}(x), j_{\lambda \overline{\lambda}}(y)] = - i (\lambda \overline{\lambda} + \frac{i}{4}) \partial_y \delta(x - y) . \tag{78}
\]

We notice that the modified current \( l_{\lambda \overline{\lambda}} \) is no longer hermitian if the above constraint is taken into account (however on abstract level the closed algebraic structure it satisfies is compatible with a hermitian condition). This feature is not specific of \( N = 2 \) but is already present in the bosonic and \( N = 1 \) cases.

The modified currents \( u_{\lambda \overline{\lambda}} \) are the generators of the \( N = 2 \) 1-dimensional diffeomorphisms invariances, the infinitesimal transformation being given by the commutators with
\[
\int dx \epsilon_u (z_+) u_{\lambda \overline{\lambda}}(x) . \tag{79}
\]

The dualized version of the \( N = 2 \) FJ model is now easily constructed by introducing a second set of superfields (antichiral in spacetime). The action is given by
\[
S = 2i \int dt dX_L \left( \partial_+ \Phi_+ \cdot D \overline{\Phi}_+ - \partial_+ \Phi_- \cdot D \overline{\Phi}_- \right) + 2i \int dt dX_R \left( \partial_- \overline{\Phi}_+ \cdot \overline{D} \Phi_- - \partial_- \overline{\Phi}_- \cdot \overline{D} \Phi_- \right) . \tag{80}
\]

The correct expansion for \( \Phi_\pm, \overline{\Phi}_\pm \) in hermitian component fields which leads to the supersymmetric hamiltonian (see the remark in section 3) is given by:
\[
\Phi_\pm = \phi_\pm - i \gamma_\pm \theta \psi_\pm + \frac{i}{2} \theta \theta \partial_1 \phi_\pm , \quad \overline{\Phi}_\pm = \overline{\phi}_\pm - i \gamma_\pm \theta \overline{\psi}_\pm - \frac{i}{2} \theta \theta \partial_1 \overline{\phi}_\pm \tag{81}
\]
(here again \( \gamma_+ = i, \gamma_- = -1 \)).

The lagrangian in components reads
\[
\mathcal{L} = 2(\partial_+ \phi_+ \cdot \partial_1 \overline{\phi}_+ + \partial_- \overline{\phi}_+ \cdot \partial_1 \phi_+ - \partial_+ \phi_- \cdot \partial_1 \overline{\phi}_- - \partial_+ \overline{\phi}_- \cdot \partial_1 \phi_- + i \partial_- \psi_+ \cdot \overline{\psi}_+ + i \partial_- \overline{\psi}_+ \cdot \psi_+ + i \partial_- \psi_- \cdot \overline{\psi}_- + i \partial_+ \overline{\psi}_- \cdot \psi_- ) . \tag{82}
\]

The non-vanishing (anti)-commutators are
\[
[\partial_x \overline{\phi}_\pm(x), \partial_y \phi_\pm(y)] = \pm \frac{i}{2} \partial_y \delta(x - y) , \quad \{ \overline{\psi}_\pm(x), \psi_\pm(y) \} = \frac{1}{2} \delta(x - y) . \tag{83}
\]

The conserved currents of the antichiral sector generate a second \( N = 2 \) superVirasoro algebra with (quantum) central charge \( c = 3 \).
Following the same reasoning as in the previous section we can construct four global supercharges $Q^i (i = 1, \ldots, 4)$ which lead to a global $N = 4$ supersymmetry ($\{Q^i, Q^j\} = H$ for any $i$ and $\{Q^i, Q^j\} = 0$ for $i \neq j$). The Coulomb gas realization for the $N = 2$ FJ model implies the full $N = 2$ superconformal invariance for the quantum dual model.

5 Conclusions.

In this paper $N = 1, 2$ extensions of chiral and dual models have been constructed and their symmetry properties analyzed. In particular their relativistic character was proven by computing global charges which close the 2D-superePoincaré algebra. The invariances under 1D-supercdiffeomorphisms and respectively superconformal transformations were furthermore investigated. It was shown that, due to $N = 1, 2$ Coulomb gas results, modified currents exist which lead to non-anomalous quantum theories.

The present work was mainly motivated to establish an algebraic framework for dimensional reductions of higher-dimensional supersymmetric dual models. The algebraic structures found however have an interest in their own and further investigations look promising. Currently under study, e.g. the $N = 4$ supersymmetric extensions seem related to non-abelian structures leading to a new $N = 4$ realization of the Coulomb gas.

Another topic which deserves to be studied, as suggested in [1], is the coupling of the above theories with 2D supergravity, with special attention to the presence of anomalies.

References

[1] J. H. Schwarz and A. Sen, Nucl. Phys. B 411 (1994) 35; J. H. Schwarz, Nucl. Phys. Proc. Suppl. 55 B (1997) 1.
[2] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, Phys.Lett. B 400 (1997) 80.
[3] A. Tseytlin, Nucl. Phys. B 350 (1991) 395.
[4] R. Floreanini and R. Jackiw, Phys. Rev. Lett. 59 (1987) 1873.
[5] F.P. Devecchi and H.O. Girotti, Phys. Rev. D 49 (1994) 4302, and references therein.
[6] C.P. Constantinidis and F.P. Devecchi, Mod. Phys. Lett. A 13, N. 8, (1998) 631.
[7] D. Nemeschansky and N.P. Warner, Nucl. Phys. B 442 (1996) 623; E. Ivanov, S. Krivonos and F. Toppan, Phys. Lett. B 405 (1997), 85.
[8] P. Pasti, D. Sorokin and M. Tonin, “Space-time supersymmetries in duality symmetric models”, hep-th9509052.
[9] V. Dotsenko and V.A. Fateev, Nucl. Phys. B 240 [FS 12] (1984) 312.
[10] P.A.M. Dirac, “Lectures on Quantum Mechanics” (Belfer Graduate School, Yeshiva University, New York, 1964).
[11] E. Ivanov and F. Toppan, Mod. Phys. Lett. A 9 (1994) 51.
[12] W. Siegel, Nucl. Phys. B 238 (1984) 307.