Relativistic recoil effects for energy levels in a muonic atom within a Grotch-type approach: An application to the one-loop electronic vacuum polarization

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We continue our account of relativistic recoil effects in muonic atoms and present explicitly analytic results at first order in electron-vacuum-polarization effects. The results are obtained within a Grotch-type approach based on an effective Dirac equation. Some expressions are cumbersome and we investigate their asymptotic behavior. Previously relativistic two-body effects due to the one-loop electron vacuum polarization were studied by several groups. Our results found here are consistent with the previous result derived within a Breit-type approach (including ours) and disagree with a recent attempt to apply a Grotch-type approach.

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I. INTRODUCTION

An analytic calculation of relativistic recoil effects in a hydrogen-like atom to order $(Z\alpha)^4m^2/M$ is possible through the equation
de 

$$E = m + m_R \left( f_C(Z\alpha) - 1 \right) - \frac{m_R^2}{2M} \left( f_C(Z\alpha) - 1 \right)^2,$$

where $f_C(Z\alpha)$ is the dimensionless energy of a Dirac-Coulomb equation, which is indeed well known (see, e.g., [1]). The corrections are of order $O((Z\alpha)^4 (m/M)^2 \bar{m})$ and $O((Z\alpha)^5 (m/M)m)$. The terms in $(Z\alpha)^5$ are due to effects of multiphoton exchange. It is even possible to provide a complete calculation of the $m/M$ recoil effects for pure Coulomb two-body systems by taking into account multiphoton exchange contributions exactly in $(Z\alpha)^3$ [2, 3].

As was shown in our previous paper [3], one can consider a more general problem and the result for the energy takes the form

$$E = m + m_R \left( f_{CN}(Z\alpha, \kappa) - 1 \right) - \frac{m_R^2}{2M} \left( f_{CN}(Z\alpha, \kappa) - 1 \right)^2 - \frac{m_R^2}{2M} \frac{\partial}{\partial \ln \kappa} \left( f_{CN}(Z\alpha, \kappa) - 1 \right)^2 - \langle \psi | \left( \frac{V^2}{2M} + \frac{1}{4M} [V, [p^2, W]] \right) | \psi \rangle,$$

where $\kappa = Z \alpha m_R / \mu$, the potential is $V = V_C + V_N$, where in certain sense $V_N \sim \varepsilon V_C$, $\varepsilon \ll 1$. Here, $W$ is a specific auxiliary potential, $\psi$ is the wave function of the Dirac problem with the reduced mass and $f_{CN}(Z\alpha, \kappa)$ is the dimensionless energy for the potential $V_C + V_N$. The momentum scale (i.e. the characteristic inverse radius) of $V_N$ is $\mu^{-1}$. For the case of the Uhlenberg potential the scale parameter is defined as $\mu = m_e$.

In this paper we study a correction to the energy in the first order of $V_N$, so we can write

$$f_{CN}(Z\alpha, \kappa) = f_C(Z\alpha) + f_N(Z\alpha, \kappa),$$

where $f_N(Z\alpha, \kappa)$ is the corresponding dimensionless correction.

Since we are interested only in terms of order $\varepsilon (Z\alpha)^4 m^2/M$, we can further simplify this expression

$$E = m + m_R \left( f_{CN}(Z\alpha, \kappa) - 1 \right) + \Delta E,$$

where

$$\Delta E = \frac{m_R^2}{2M} \left( f_{CN}(Z\alpha, \kappa) - 1 \right)^2 - \frac{m_R^2}{2M} \frac{\partial}{\partial \ln \kappa} \left( f_{CN}(Z\alpha, \kappa) - 1 \right)^2 - \langle \psi_{NR} | \left( \frac{V^2}{2M} + \frac{1}{4M} [V, [p^2, W]] \right) | \psi_{NR} \rangle,$$

where it is sufficient to apply the nonrelativistic approximation to the energy in the term with derivative

$$\left( f_{CN}(Z\alpha, \kappa) - 1 \right) \bigg|_{\text{nonrel}} = \frac{E^{(NR)}(\kappa)}{m_R},$$

as well as to the wave function in the last term.

It is remarkable that in certain respects the relativistic recoil correction beyond the Dirac equation with the reduced mass, $\Delta E$, is simpler than the solution of the

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Dirac equation. To order \(\varepsilon(Z\alpha)^4m^2/M\) it requires only nonrelativistic evaluation. In particular, the leading recoil correction, being expressed in pure nonrelativistic terms, does not depend on the total angular momentum, \(j\), but only on the angular momentum \(l\). That means that this correction may contribute to the Lamb splitting (a difference between states with the same \(j\), but different \(l\), such as the \(2p_{1/2} - 2s_{1/2}\) difference), but not to the fine-structure interval (a difference between states with the same \(l\), but different \(j\), such as the \(2p_{3/2} - 2p_{1/2}\) difference).

To validate applicability of this expression for the electron-vacuum-polarization (eVP) effects we should prove that the relativistic recoil effects can be reduced to the evaluation of the one-photon exchange (see Fig. 1) and present explicit expressions for related contributions to \(V\) and \(W\).

\[
\mu \quad e \quad \{ \quad \}
\]

\[
N
\]

FIG. 1: One-photon-exchange diagram for the eVP contributions. It is responsible for the the Uehling-potential corrections to orders \(\alpha(Z\alpha)^2m\) and \(\alpha(Z\alpha)^4m^2\).

Apparently, the correction to the potential is the Uehling potential, which can be presented, e.g., in the form [6]

\[
V_U(r) = -\frac{\alpha(Z\alpha)}{\pi} \int_0^1 dv \rho_e(v) \frac{e^{-\lambda r}}{r},
\]

where

\[
\lambda = \frac{2m_e}{\sqrt{1-v^2}},
\]

\[
\rho_e(v) = \frac{v^2(1-v^2/3)}{1-v^2}.
\]

The factor \(\alpha/\pi\) plays a role of the parameter \(\varepsilon\) in our general consideration [5] and \(\mu = m_e\). Meanwhile, a construction of \(W\), which is to be preceded by a choice of an appropriate gauge is not trivial (see the discussion in [7]).

In principle, the correction \(\Delta E\) can be treated relativistically without any nonrelativistic reduction of the energy and the wave function. However, the higher-order effects which are incorporated in this case are smaller than possible effects of two-photon corrections. In particular, for the eVP contributions the higher-order relativistic-recoil contributions to \(\Delta E\) are of order \(\alpha(Z\alpha)^6m^2/M\), while the two-photon-exchange diagrams contribute to order \(\alpha(Z\alpha)^5m^2/M\).

In the following sections we briefly reproduce this discussion and present appropriate results for a relativistic and nonrelativistic correction to the energy due to a Dirac equation with a potential which accounts for the eVP effects. Using them, we present an analytic expression for eVP relativistic recoil corrections in the general case as well as for most interesting particular cases, such as circular and low-lying states. For both kinds of the states we also derive their large-kappa asymptotics. In conclusion we discuss a comparison with an alternative technique for relativistic recoil corrections based on the Breit-type equation.

II. GROTCH-TYPE EXPRESSION FOR THE EVP CORRECTIONS IN FIRST ORDER IN \(\alpha\)

A choice of gauge for the photon propagator is crucial for the explicit presentation of the one-photon contribution and for the value of the two-photon contribution. We have already discussed that in part in [5] and in detail in [7].

Indeed, due to the gauge invariance of quantum electrodynamics, any final complete result for any physical calculation does not depend on the choice of the gauge. However, the technical origin of different contributions to such a final result may be different in different gauges. In particular, the physical result for a relativistic recoil correction to order \(\alpha(Z\alpha)^4m^2/M\) does not necessarily come only from the static part of the one-photon-exchange term.

Following [7], we use the Coulomb gauge for the free photon propagator, while the eVP correction to the propagator takes the form

\[
D_{00}^0 = -\frac{\alpha}{\pi} \int_0^1 dv \rho_e(v) \frac{1}{(k^2 + \lambda^2)} ,
\]

\[
D_{ij}^0 = 0 ,
\]

\[
D_{ij}^\nu = -\frac{\alpha}{\pi} \int_0^1 dv \rho_e(v) \frac{1}{(k^2 - \lambda^2)} \times \left( \delta_{ij} - \frac{k_ik_j}{(k^2 + \lambda^2)} \right) .
\]

We note that similarly to the Coulomb gauge the \(D_{00}\) component of the photon propagator does not depend on the energy transfer and \(D_{ij} = 0\). This choice is sufficient for vanishing \(\alpha(Z\alpha)^4m^2/M\) contributions from two-photon exchanges (see Fig. 2) and thus the problem of calculations of relativistic recoil effects at this order is reduced to consideration of the one-photon-exchange diagrams (see Fig. 1).

The Groetch-type calculations of the eVP contribution of order \(\alpha(Z\alpha)^3m^3/M^2\) were considered some time ago [8,10] (for earlier evaluations see [11,12]). However, the gauge was not appropriate and two-photon-exchange corrections should be added. Those corrections were missed in [8,10], which produces a discrepancy between the Breit-type calculation [13] and the Groetch-type ones. The situation was clarified in [7].

Once an appropriate gauge is chosen, we can restrict our consideration to the static part of the one-photon-
exchange contribution, including the vacuum polarization. Treating the nucleus nonrelativistically we arrive at the same equation as for the free one-photon-exchange

$$\left( \alpha \cdot p + \beta m + \frac{p^2}{2M} + V + \frac{1}{2M} \{ \alpha \cdot p, V \} \right) \psi(r) = E \psi(r). \quad (8)$$

where, however, the effective potentials \( W \) and \( V \) include eVP effects and, in particular,

$$V(r) = V_C(r) + V_U(r),$$

$$V_U(k) = -4\alpha(Z\alpha) \int_0^1 dv \frac{\rho_e(v)}{k^2 + \lambda^2}, \quad (9)$$

and

$$W = W_C + W_U,$$

$$W_U(k) = -\frac{2V_U(k)}{k^2 + \lambda^2},$$

$$W_U(r) = \frac{\alpha(Z\alpha)}{\pi} \int_0^1 dv \rho_e(v) e^{-\lambda r}, \quad (10)$$

and, as given in [1, 5]

$$W_C(r) = -Z\alpha r,$$

$$W_C(k) = \frac{8\pi Z\alpha}{k^2}. \quad (11)$$

With an expression for \( W \) in hand, we can rewrite the addition to the Hamiltonian, which in the leading order of \( \alpha \), takes the form

$$\delta H = - \left( \frac{V_C^2}{2M} + \frac{1}{4M} [V_C, [p^2, W_C]] \right)_{v=0}$$

$$- \frac{V_U V_C}{M} \frac{1}{4M} [V_C, [p^2, W_U]]$$

$$- \frac{1}{4M} [V_U, [p^2, W_C]]. \quad (12)$$

The expression turns out to be equal to zero for the pure Coulomb case.

For the case of \( V_N = V_U \) and \( f_N = f_U \) Eq. 3 leads to the expression for the following correction of the first order in \( \alpha \)

$$E_U = \frac{m_R f_U(Z\alpha, \kappa)}{m} - \frac{m_R^2}{M} (f_C(Z\alpha) - 1) f_U(Z\alpha, \kappa)$$

$$- \frac{m_R^2}{M} (f_C(Z\alpha) - 1) \frac{\partial}{\partial \ln \kappa} f_U(Z\alpha, \kappa)$$

$$- \langle \psi | \left( \frac{V_U V_C}{M} \frac{1}{4M} [V_C, [p^2, W_U]] \right)$

$$+ \frac{1}{4M} [V_U, [p^2, W_C]] \rangle | \psi \rangle. \quad (13)$$

The expression includes a nonrelativistic term of order \( \alpha(Z\alpha)^2 m^2/M \) (exact in \( m/M \)), a pure relativistic one \( \alpha(Z\alpha)^4 m \) and a relativistic recoil correction to order \( \alpha(Z\alpha)^4 m^2/M \). It also contains a higher-order \( \alpha(Z\alpha)^6 m \) non-recoil term which may be numerically comparable to \( \alpha(Z\alpha)^4 m^2/M \) in a certain range of \( Z \).

This expression may be approached analytically or by numerical means. Below we express it in terms of certain base integrals which we evaluate in closed analytic form following [14, 16, 21].

The required expressions for the dimensionless \( f_C(Z\alpha) \) energy and Dirac-Coulomb wave functions \( \psi \) are summarized in the Appendix of our previous paper [5].

To obtain final results we have to find some derivatives and it is important to have an expression for \( f_U(Z\alpha, \kappa) \) (the dimensionless Uehling corrections to the energy levels of the Dirac-Coulomb equation with the reduced mass) in a form suitable for differentiation and it is also available. Not only analytic expressions for relativistic [14, 16] (see Sec. III) and nonrelativistic [16–19] (see Sec. III) corrections are known, but also their various asymptotics [14, 16, 18, 21].

For the analytical differentiation one can take into account

$$\frac{\partial}{\partial \ln \kappa} f(\kappa/n) = \frac{\partial}{\partial \ln \kappa_n} f(\kappa_n),$$

where \( \kappa_n = \kappa/n \) is a combination which naturally appears in various analytic expressions (see Sec. III). For numerical evaluations of the derivative a more useful relation is

$$\frac{\partial}{\partial \ln \kappa} f(\kappa) = -\frac{\partial}{\partial \ln m_e} f(\kappa).$$

III. THE UEHLING-POTENTIAL CORRECTION TO THE ENERGY LEVELS OF THE DIRAC-COULOMB EQUATION

The Uehling correction to the energy of the Dirac-Coulomb+Uehling problem was addressed in [14] for circular states and later was generalized for an arbitrary
The result reads as

\[
f_U(Z\alpha, \kappa) = -\frac{\alpha(Z\alpha)}{\pi m R} \times \int_0^1 dv \rho_v(\psi | \psi) e^{-\lambda v r} | \psi\rangle = \frac{\alpha(Z\alpha)^2}{\pi n^2} F_{nlj}(\kappa_n),
\]

where

\[
\kappa_n' = \frac{\eta m R_{\text{m}}}{m_e}, \quad \eta = \frac{Z\alpha}{\sqrt{(n_r + \zeta)^2 + (Z\alpha)^2}}, \quad \zeta = \sqrt{\nu^2 - (Z\alpha)^2}, \quad \nu = -(1)^{j+1/2}(j + 1/2), \quad n_r = n - |\nu|.
\]

The parameter \(\kappa_n'\) is different from \(\kappa_n = \kappa/n = Z\alpha m/nn_e\). However, in a nonrelativistic approximation

\[
kappa_n' = \frac{Z\alpha m R_{\text{m}}}{m_e \sqrt{(n_r + \zeta)^2 + (Z\alpha)^2}} \approx \frac{Z\alpha m R_{\text{m}}}{m_e \sqrt{(n_r + |\nu|)^2 + (Z\alpha)^2}} = \frac{Z\alpha m R}{m_e n} = \kappa_n.
\]

The function \(F_{nlj}(\kappa)\) can be expressed either in terms of a one-dimensional integral over elementary functions or in terms of a hypergeometric function \(3F_2\) [14, 16]. Indeed, the correction \(f_U\) can be computed numerically for any desired state. However, because of a required expansion and various further considerations, such as examination of the asymptotic behavior, we prefer here analytic or semi-analytic results.

In particular, as it was found in [16]

\[
F_{nlj}(\kappa) = -\sum_{i,k=0}^{n_r} B_{ik} K_{2,2\zeta+i+k}(\kappa),
\]

where

\[
B_{ik} = \left(\frac{m}{Z\alpha}\right)^2 \frac{(-1)^{i+k}(n_r)!}{i!(n_r - i)!k!(n_r - k)!} \times \frac{\Gamma(2\zeta + n_r + 1)\Gamma(2\zeta + i + k)}{\Gamma(2\zeta + i + 1)\Gamma(2\zeta + k + 1)} \frac{1}{\frac{Z\alpha}{\eta} - \nu} \times \left\{ \left(\frac{Z\alpha}{\eta} - \nu\right)^2 + (n_r - i)(n_r - k) \right\} - \frac{E_C(nlj)}{m} \left(\frac{Z\alpha}{\eta} - \nu\right) (2n_r - i - k).
\]

The base integrals, defined as [14, 16]

\[
K_{abc}(\kappa) = \int_0^1 dv \frac{v^{2\alpha}}{(1 - v^2)^{n/2}} \frac{\left(\frac{\kappa\sqrt{1 - u^2}}{1 + \kappa\sqrt{1 - u^2}}\right)^{c}}{1},
\]

\[
K_{bc}(\kappa) = K_{1bc}(\kappa) - \frac{1}{3} K_{2bc}(\kappa).
\]

It is easy to obtain for the first derivative of \(K\)

\[
\frac{\partial K_{bc}}{\partial \kappa} = \frac{c}{\kappa^2} K_{b+1,c+1}(\kappa).
\]

The integrals \(K\) can be also expressed in a closed form [14, 16]

\[
K_{abc}(\kappa) = \frac{\kappa^c}{2} B\left(a + \frac{1}{2}, 1 - b + \frac{c}{2}\right) \times 3F_2\left(\frac{c}{2}, \frac{c}{2} + 1, 1 - b + \frac{c}{2}; \frac{a + 3}{2} - b + \frac{c}{2}; \kappa^2\right) - \frac{c}{2} K_{b+1,c+1}(\kappa)
\]

\[
\times 3F_2\left(\frac{c}{2} + 1, \frac{c}{2} + 1, 3 - b + \frac{c}{2}; 3 - b + \frac{c}{2}; \kappa^2\right) - \frac{c}{2} K_{b+1,c+1}(\kappa).
\]

where \(B(\alpha, \beta)\) is the beta function and \(3F_2(\alpha, \beta, \gamma; \delta, \epsilon; z)\) stands for the generalized hypergeometric function (see, e.g., [22]).

The solution above is a solution of the Dirac equation for a particle with mass \(m\). However, as we see from Eq. (3), the two-body energy is the easiest to express in terms of a Dirac equation with the reduced mass, introducing corrections.

In muonic hydrogen for \(n = 2\) the argument of the hypergeometric function, \(\kappa_n^2\), is less than unity (\(\approx 0.7\)), and the hypergeometric series converges well. For \(n = 1\) in muonic hydrogen or \(n = 2\) in muonic helium, one has to use analytic continuation of the hypergeometric series or integral representation of the hypergeometric function.

To calculate the term with derivative and the term with \(\delta H\) we need efficient nonrelativistic expressions. The Uehling correction in the nonrelativistic limit is

\[
f_U^{(\text{NR})}(Z\alpha, \kappa) = \frac{\alpha(Z\alpha)^2}{\pi n^2} f_{nl}^{(\text{NR})}(\kappa_n),
\]

where

\[
F_{nlj}(\kappa_n) = -\sum_{i,k=0}^{n_r-1} B_{ik}^{(\text{NR})} K_{2,2\zeta+i+k+2}(\kappa_n),
\]

\[ B_{ik}^{(NR)} = \frac{(-1)^i k (n - l + 1)!}{2! (n - l - i + 1)! (n - l - k - 1)!} \times \frac{(n + l)! (2l + i + k + 1)!}{(2l + i + 1)!(2l + k + 1)!} \tag{24} \]
can be expressed in terms of elementary functions. Alternative expressions for the nonrelativistic correction can be found in [16, 19].

In a particular case of the ground state the result has a simple form \[17\]
\[ F_{10}^{(NR)}(\kappa) = -\frac{1}{3} \left\{ \frac{4 + \kappa^2 - 2 \kappa^4}{\kappa^3} \cdot A(\kappa) + \frac{4 + 3 \kappa^2}{\kappa^3} - \frac{\pi}{2} - \frac{12 + 11 \kappa^2}{3 \kappa^2} \right\}, \tag{25} \]
where
\[ A(\kappa) = \frac{\arccos(\kappa)}{\sqrt{1 - \kappa^2}} = \frac{\ln(\kappa + \sqrt{\kappa^2 - 1})}{\sqrt{\kappa^2 - 1}}. \]

The nonrelativistic kernels \( K_{bc} \) have only integer subscripts and that allows useful recurrence relations (see [14, 18]). Applying them we arrive at \[18\]
\[ F_{nl}^{(NR)}(\kappa_n) = \frac{(n + l)!}{(n - l - 1)!(2n - 1)!} \sum_{i=0}^{n-l-1} \frac{1}{(2l + i + 1)! i!} \]
\[ \times \left( \frac{(n - l - 1)!}{(n - l - i)!} \right)^2 \left( \frac{1}{\kappa_n} \right)^{2(n-l-i)} \]
\[ \times \left( \kappa_n^2 \frac{\partial}{\partial \kappa_n} \right)^{(2n-l-i-1)} \left( \kappa_n^{2(l+i+1)} \frac{\partial}{\partial \kappa_n} \right)^{2(l+i)} F_{10}^{(NR)}(\kappa_n). \tag{26} \]

**IV. THE ANALYTIC RESULT FOR THE RELATIVISTIC RECOIL UEHLING CORRECTION**

Using the expression for the relativistic Uehling energy Eq. \[14\] in terms of base integrals \( K_{bc} \) and their various properties \[14,16,18,21\], for the third term of Eq. \[14\] we arrive at
\[ \frac{\alpha(Z\alpha)^2 m_R^2}{\pi n^2} \frac{(f_C - 1)}{M} \sum_{i,k=0}^{n_r} B_{ik} \]
\[ \times \left[ \frac{2 \zeta + i + k}{\kappa_n^{2(2l+i+1)}} K_{2,2\zeta+i+k+1}(\kappa_n') \right]. \tag{27} \]

The last term of Eq. \[12\] corresponds to a matrix element of the additional Hamiltonian \[12\]. For the relativistic wave functions it can be rewritten in the form
\[ - \langle \psi | \left( V_U V_C + \frac{1}{4M} [V_C, [p^2, W_U]] + \frac{1}{4M} [V_U, [p^2, W_C]] \right) | \psi \rangle = \frac{\alpha(Z\alpha)^2}{2M\pi} \int_0^1 dv \rho_c(v) \langle \psi | \lambda \left[ e^{-\lambda r} \right] | \psi \rangle, \tag{28} \]
which has the same structure of integration over \( r \) as \( f_U \) (cf. Sec. \[11\]) and one can readily obtain for it an expression which differs from one for the Uehling correction \[17\] only by a factor and the second indices of \( K_{bc} \), arriving at
\[ \frac{\alpha(Z\alpha)^4 m_R^2}{\pi M Z\alpha n^2} \eta \sum_{i,k=0}^{n_r} B_{ik} K_{3,2\zeta+i+k}(\kappa_n'), \tag{29} \]
or, for the nonrelativistic case
\[ \frac{\alpha(Z\alpha)^4 m_R^2}{\pi M n^3} \sum_{i,k=0}^{n_r} F_{10}^{(NR)} K_{3,2\zeta+i+k+2}(\kappa_n'). \tag{30} \]

Combining the different parts of the expression \[13\], we obtain for the correction to the first order of \( \alpha \)
\[ E_U = m_R f_U(Z\alpha, \kappa) + \Delta E_U, \]
where its relativistic recoil part is
\[ \Delta E_U = \frac{\alpha(Z\alpha)^2 m_R^2}{\pi n^2} M \sum_{i,k=0}^{n_r} B_{ik} \]
\[ \times \left[ \frac{2 \zeta + i + k}{\kappa_n} K_{2,2\zeta+i+k+1}(\kappa_n') \right]. \tag{31} \]

Neglecting higher-order relativistic corrections, we find in order \( \alpha(Z\alpha)^4 m^2 / M \)
\[ \Delta E_{U^{(NR)}} = \frac{\alpha(Z\alpha)^4 m_R^2}{\pi n^3} M \sum_{i,k=0}^{n_r} B_{ik}^{(NR)} \]
\[ \times \left[ -\frac{1}{2n} K_{3,2l+i+k+2}(\kappa_n) - \frac{2l + i + k + 2}{2n \kappa_n} K_{3,2l+i+k+3}(\kappa_n) + \frac{1}{\kappa_n} K_{3,2l+i+k+2}(\kappa_n') \right]. \tag{32} \]

**V. RESULTS FOR PARTICULAR STATES**

There are several classes of states of interest. In this section we consider two of them, namely circular states \( (l = n - 1) \) and low lying states \( (n = 1, 2) \). The former are quite insensitive to the nuclear structure, allowing accurate \textit{ab initio} calculations, and may be of a “metrological” interest \[23, 24\], while the latter are the most sensitive to the nuclear structure and may be applied to measure the nuclear charge radius \[23, 26\].
Below we present results in closed form in terms of the
generalized hypergeometric function \( _3F_2 \) for arbitrary \( Z \) and \( \kappa_n \) and additionally the asymptotic behavior for large \( \kappa_n \) is investigated.

### A. Circular states

For states with maximal orbital and angular momenta, i.e. \( l = n - 1 \) and \( j = l + 1/2 \) there is no difference between \( \kappa_n \) and its relativistic analogue \( \kappa'_n \), and \( n_s = 0 \).

The expression (31) in this case can be transformed to

\[
\Delta E_U = \frac{\alpha(Z\alpha)^2}{\pi n\zeta} \frac{m_R^2}{M} \times \left[ \frac{1}{\kappa_n} \left( K_{2,2\zeta}(\kappa_n) + \frac{2\zeta}{\kappa_n} K_{3,1+2\zeta}(\kappa_n) \right) \right.
\]

or, neglecting higher-order terms in \( Z\alpha \),

\[
\Delta E_{U_{NR}}^{(NR)} = \frac{\alpha(Z\alpha)^4}{\pi n^3} \frac{m_R^2}{M} \times \left[ \frac{1}{\kappa_n} \right] \left( -\frac{\kappa_n}{2} K_{2,2\zeta}(\kappa_n) + n K_{3,2n}(\kappa_n) \right.
\]

\[
-\left. n K_{3,1+2n}(\kappa_n) \right] \right), \quad (33)
\]

The asymptotics of the last expression for large \( \kappa_n \) is

\[
\Delta E_{U_{NR}}^{(NR)} = \frac{\alpha(Z\alpha)^4}{\pi n^3} \frac{m_R^2}{M} \times \left[ \frac{1}{3n} \left( \ln(2\kappa_n) - \psi(2n) + \psi(1) + \frac{1}{6} \right) \right.
\]

\[
+ \frac{2}{3(2n - 1)} - \frac{\pi}{4\kappa_n} \left. + \frac{2n - 1}{4\kappa_n^2} - \frac{\pi}{4\kappa_n} \left( \frac{n - 2}{18\kappa_n^2} \right) + O \left( \frac{1}{\kappa_n^4} \right) \right] \right), \quad (35)
\]

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \).

### B. The low-lying states

The above results can be applied to the case of the \( 1s_{1/2} \) and \( 2p_{3/2} \) states. In particular, for the nonrelativistic case

\[
\Delta E_{U_{NR}}^{(NR)}(1s) = \frac{\alpha(Z\alpha)^4}{\pi} m_R^2 \times \left[ \frac{1}{36\kappa^3(\kappa^2 - 1)} \right]
\]

\[
-6(2\kappa^6 - 3\kappa^4 - 12\kappa^2 + 10)A(\kappa) \right.
\]

\[
+ 2\kappa(5\kappa^4 + 16\kappa^2 - 30)
\]

\[
- 3\pi(3\kappa^4 + 7\kappa^2 - 10) \right], \quad (36)
\]

or, for large \( \kappa \),

\[
\Delta E_{U_{NR}}^{(NR)}(1s) = \frac{\alpha(Z\alpha)^4}{\pi} m_R^2 \times \left[ \frac{1}{36\kappa^3(\kappa^2 - 1)} \right]
\]

\[
-6(2\kappa^6 - 3\kappa^4 - 12\kappa^2 + 10)A(\kappa) \right.
\]

\[
+ 2\kappa(5\kappa^4 + 16\kappa^2 - 30)
\]

\[
- 3\pi(3\kappa^4 + 7\kappa^2 - 10) \right] \right) \right), \quad (36)
\]

or, for large \( \kappa \),

\[
\Delta E_{U_{NR}}^{(NR)}(1s) = \frac{\alpha(Z\alpha)^4}{\pi} m_R^2 \times \left[ \frac{1}{36\kappa^3(\kappa^2 - 1)} \right]
\]

\[
-6(2\kappa^6 - 3\kappa^4 - 12\kappa^2 + 10)A(\kappa) \right.
\]

\[
+ 2\kappa(5\kappa^4 + 16\kappa^2 - 30)
\]

\[
- 3\pi(3\kappa^4 + 7\kappa^2 - 10) \right] \right) \right), \quad (36)
\]

Other particular cases of interest are \( 2s_{1/2} \) and \( 2p_{1/2} \) states. For the nonrelativistic case relations for these states can be written in a unified form\(^1\)

\[
\Delta E_{U_{NR}}^{(NR)}(2l) = \frac{\alpha(Z\alpha)^4}{\pi} m_R^2 \times \left[ \frac{1}{36\kappa^3(\kappa^2 - 1)} \right]
\]

\[
-6(2\kappa^6 - 3\kappa^4 - 12\kappa^2 + 10)A(\kappa) \right.
\]

\[
+ 2\kappa(5\kappa^4 + 16\kappa^2 - 30)
\]

\[
- 3\pi(3\kappa^4 + 7\kappa^2 - 10) \right] \right) \right), \quad (36)
\]

or, for large \( \kappa \),

\[
\Delta E_{U_{NR}}^{(NR)}(2l) = \frac{\alpha(Z\alpha)^4}{\pi} m_R^2 \times \left[ \frac{1}{36\kappa^3(\kappa^2 - 1)} \right]
\]

\[
-6(2\kappa^6 - 3\kappa^4 - 12\kappa^2 + 10)A(\kappa) \right.
\]

\[
+ 2\kappa(5\kappa^4 + 16\kappa^2 - 30)
\]

\[
- 3\pi(3\kappa^4 + 7\kappa^2 - 10) \right] \right) \right), \quad (36)
\]

or, for large \( \kappa \),

\[
\Delta E_{U_{NR}}^{(NR)}(2l) = \frac{\alpha(Z\alpha)^4}{\pi} m_R^2 \times \left[ \frac{1}{36\kappa^3(\kappa^2 - 1)} \right]
\]

\[
-6(2\kappa^6 - 3\kappa^4 - 12\kappa^2 + 10)A(\kappa) \right.
\]

\[
+ 2\kappa(5\kappa^4 + 16\kappa^2 - 30)
\]

\[
- 3\pi(3\kappa^4 + 7\kappa^2 - 10) \right] \right) \right), \quad (36)
\]

The numerical results for the low-lying states in light muonic atoms, obtained from (36) and (39) are summarized in Table I.

| Atom   | 1s     | 2s     | 2p\(_{1/2}\) | 2p\(_{1/2}\) |
|--------|--------|--------|-------------|-------------|
| \(\mu^0H\) | 0.182  | 0.0381 | 0.000901    | 0.000901    |
| \(\mu^0D\)  | 0.180  | 0.0388 | 0.000968    | 0.000968    |
| \(\mu^0He\) | 0.122  | 0.0459 | 0.00184     | 0.00184     |
| \(\mu^0He\) | 0.121  | 0.0459 | 0.00184     | 0.00184     |

TABLE I: Relativistic recoil eVP corrections for the low-lying levels in muonic hydrogen. The units are \((\alpha/\pi)(Z\alpha)^4m_R^2/M^3\).

\(^1\) To come to this form from Eq. (32) one can use the relation

\[
K_{\kappa} = \frac{1}{\kappa} K_{\kappa+1,\kappa+1} + K_{\kappa-1,\kappa+1} \right], \quad (38)
\]
VI. COMPARISON WITH THE BREIT-TYPE CALCULATIONS

Evaluations of relativistic recoil effects within the Groth-type approach developed in this paper and the standard Breit-type technique (see, e.g., [11, 13, 27]) are complementary. Both produce for the eVP effects not only the leading term of order $\alpha(Z\alpha)^4m^2/M$, but also certain higher-order corrections. While the Groth-type calculation provides us with partial account of $\alpha(Z\alpha)^6m^2/M$ contributions, the Breit-type calculation leads to an exact in $(m/M)$ result for the $\alpha(Z\alpha)^4m$ contribution.

The additional $\alpha(Z\alpha)^6m^2/M$ terms in the Groth-type approach are not so important as a simplification of pure recoil contributions. The technique allows us to easily separate the leading $\alpha(Z\alpha)^4m^2/M$ term from the higher-order effects and calculate it much more easily than by means of the Breit-type evaluation.

Meanwhile, such an evaluation is completely separated from the non-recoil relativistic term. On the contrary, the Breit-type evaluation produces both recoil and non-recoil relativistic contributions within the same calculations, which allows additional crosstalks.

In this section we describe a rearrangement of the Breit-type evaluation which would allow a direct comparison between the Groth-type and Breit-type results.

In paper [27] we have calculated the relativistic recoil correction in question for low-lying states in light muonic atoms by both mentioned methods. In both cases we can expand the correction by powers of $m/M$

$$E_U = \frac{\alpha}{\pi} (Z\alpha)^4m R \left[ C_0 + C_1 \frac{mR}{M} \right] + C_2 \left( \frac{mR}{M} \right)^2 + \ldots , \tag{41}$$

where $C_0$ corresponds to the known non-recoil Uehling correction to the energy, $C_1$ is found by the Groth method, and the Breit method provides both $C_1$ and $C_2$. The coefficients $C_1$ calculated by the two methods agree. That is a unique expansion and the $C_0$ and $C_1$ coefficients obtained by both methods should be the same. The Groth-type approach produces $C_0$ and $C_1$, but not $C_2$.

The conventional Breit-type calculation does not produce the result in such a form which makes the direct comparison difficult.

The Breit approach is based on an unperturbed Hamiltonian

$$H^{(0)} = \frac{p^2}{2mR} + V(r)$$

and thus the wave function does not include the muon and nuclear mass separately, but only in a combination in the form of the reduced mass $\phi^{(0)} = \phi(r; mR)$.

Meantime, the standard Breit equation (see, e.g., [27])

$$V_{Br} = - \left( \frac{1}{m^3} + \frac{1}{M^3} \right) \frac{p^4}{8} + \frac{Z\alpha}{8} \left( \frac{1}{m^2} + \frac{1}{M^2} \right) 4\pi\delta^3(r) + \frac{Z\alpha}{4mR} \frac{L \cdot \sigma}{r^3} + \frac{Z\alpha}{2mR} \frac{4\pi\delta^3(r)}{r} + \frac{Z\alpha}{2mR} \frac{1}{r^3} L^2 - \frac{p^2}{r} - \frac{1}{r} p^2 \right] \tag{42}$$

explicitly depends on the muon mass $m$ and the nuclear mass $M$. The eVP addition to the Breit Hamiltonian is of the form [27]

$$V_{Br}^{\text{VP}} = \left( \frac{1}{8m^2} + \frac{1}{8M^2} \right) \nabla^2 V_U + \left( \frac{1}{4m^2} + \frac{1}{2mR} \right) \frac{V_U'}{r} L \cdot \sigma + \frac{1}{2mR} \left[ \frac{V_U'}{r} L^2 + \frac{p^2}{2} (V_U - rV_U') \right] + \left( V_U - rV_U' \right) \frac{p^2}{2} \right] \tag{43}$$

As a result, the matrix element of $V_{Br}$ and $V_{Br}^{\text{VP}}$ over $\phi^{(0)}(r; mR)$ depends on $m, M, mR$ and is not suited for presentation in the form of [11].

To adjust the Breit Hamiltonian to this form one has to rewrite it as a function of $mR$ and $M$, but not $m$. The related corrections to the Hamiltonian take the form

$$V_{Br} = - \left( \frac{1}{mR} - \frac{3}{mR^3} \right) \frac{p^4}{8} + \frac{Z\alpha}{8} \left( \frac{1}{mR^2} - \frac{2}{mR^3} \right) \frac{4\pi\delta^3(r)}{r} + \frac{Z\alpha}{4mR} \frac{L \cdot \sigma}{r^3} + \frac{Z\alpha}{2mR^3} \frac{4\pi\delta^3(r)}{r} + \frac{Z\alpha}{2mR} \frac{1}{r^3} L^2 - \frac{p^2}{r} - \frac{1}{r} p^2 \right] , \tag{44}$$

and

$$V_{Br}^{\text{VP}}(r) = \left( \frac{1}{8mR^2} - \frac{1}{4mR^3} \right) \nabla^2 V_U + \frac{V_U'}{4mR} L \cdot \sigma + \frac{1}{2mR} \left[ \frac{V_U'}{r} L^2 + \frac{p^2}{2} (V_U - rV_U') \right] + \left( V_U - rV_U' \right) \frac{p^2}{2} \right] , \tag{45}$$
and here all terms which contribute to order $(Z\alpha)^4 m^3/M^2$ and $\alpha(Z\alpha)^4 m^3/M^2$ are neglected.

The perturbations in $\alpha$ directly lead to eVP results in $\alpha$. We realized such a rearrangement in [7] and obtained results by the Breit-type approach, which completely agree with our Grotch-type approach within an uncertainty of numerical integration (cf. Table I).

Such a rearrangement is applicable not only in the first order in $\alpha$. In particular, we applied it in [28] to second order in $\alpha$ and obtained in that work relativistic recoil corrections (of the first order in $m/M$) consistent with our calculation of the relativistic recoil by the Grotch-type method [29].

VII. CONCLUSIONS

In this paper a method for a calculation of relativistic recoil effects developed previously [3] was applied perturbatively for the one-loop electronic vacuum-polarization corrections. With the results of the Dirac problem with Coulomb+Uehling potential already known, the evaluation of the additional recoil correction has dealt only with nonrelativistic wave functions. It was performed in closed analytic form in terms of the same base integrals as required for the calculation of the Uehling correction itself (cf. [14, 16, 19]).

We also found asymptotics for high $Zm/(m_n e)$ for the circular and low-lying states. The low-lying states, 1s, 2s, 2p, are of particular interest in light muonic atoms, because they provide us with perhaps the best opportunity to determine the nuclear charge radius.

Some time ago, certain results were obtained within the Breit-type [13, 27] and Grotch-type approaches [10]. They have been discussed in part by us in [7].

Both calculations contain not only the leading eVP relativistic recoil term of order $\alpha(Z\alpha)^4 m^2/M$, but also certain higher-order corrections. Here we explain in details how to compare those calculations. A modification of the effective Breit Hamiltonian is described, which allows us to avoid any higher-order effects in the Breit type approach. As a result [7], we agree with [13] and disagree with [9, 10] and [27]. A discrepancy with the former is due to an inappropriate gauge used in that work, while the discrepancy with the latter is most probably due to a numerical error there.

Here, we applied the technique based on the presentation of the results in terms of base integrals. However, this is not necessary. If it is desired, one can solve the related Dirac equation numerically. As we mentioned for the relativistic recoil correction by itself even the Dirac equation is not necessary. A nonrelativistic Schrödinger equation with an appropriate potential and subsequent nonrelativistic perturbation theory is sufficient.

The approach can be extended further and it has been extended. In a subsequent paper [29] it is successfully applied to the second-order eVP relativistic recoil corrections to order $\alpha^2(Z\alpha)^4 m^2/M$.

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