Ranking fuzzy numbers based on weighted distance

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Abstract. We discuss how to rank fuzzy numbers using weighted distance in this paper. Basic properties of this ranking method are investigated. Further, we prove that the proposed order can be generated by a ranking index. At last, in the case of a pair of specific weights, we compute the ranking index in some common cases of practical applications.

1. Introduction
Fuzziness is a very important and common uncertainty. Fuzzy number plays a significant and important role in illustrating fuzziness. The ranking/ordering of fuzzy numbers is a key step in many fuzzy information processing problems in artificial intelligence, decision theory, optimization, etc [3-6].

Grzegorzewski [4] has proposed an important ranking method on fuzzy numbers. Suppose that \( U \) is a subset of fuzzy numbers. At first, find a lower horizon of \( U \). Then compute the distances between each fuzzy number in this set and this lower horizon. At last rank the fuzzy numbers in this set by the distance.

This work is very important and has inspired much of work in the topic of ranking fuzzy number [1, 6].

In [4], a type of distance is used that does not weight the \( \alpha \)-cuts. It's worth mentioning that weighted distances, which take non-weighted distance as a special case, apply a wider range of situations. Many fuzzy information processing problems are discussed by weighting the level set [2].

In this paper, we consider how to rank fuzzy numbers based on weighted distance. Further, we show that this kind of order can be generated by ranking index. This means that the ranking method proposed here has great advantages [1]. Finally, we have calculated the ranking index in some common cases of practical applications.

2. Fuzzy numbers and weighted distance \( d_{q,\phi,\psi} \)
This section reviews some elementary concepts about fuzzy number and introduces a type of weighted distances \( d_{q,\phi,\psi} \) between fuzzy numbers. The readers can find further details on fuzzy numbers in [3, 7].

We use \( F(i) \) to represent all fuzzy sets on \( i \), where \( i \) is the set of all real numbers. A fuzzy set can be seen as a function \( \nu: i \rightarrow [0,1] \), \( [\nu]_{\alpha}, \alpha \in [0,1] \), is the \( \alpha \)-cut of \( \nu \) which is defined as

\[
[\nu]_{\alpha} = \begin{cases} \{ x \in i : \nu(x) \geq \alpha \}, & \alpha \in (0,1], \\ \{ x \in i : \nu(x) > 0 \}, & \alpha = 0. \end{cases}
\]
For \( U \in \mathcal{F}(i) \), \( u \) is called a fuzzy number if \( u \neq \phi \), and \( [u]_\alpha = [u^-(\alpha), u^+(\alpha)] \) are bounded intervals of \( i \) for all \( \alpha \in [0,1] \). We use \( \mathcal{E} \) to denote all fuzzy numbers.

Grzegorzewski [4] introduced the following metric on fuzzy numbers.

Let \( q \in (0,1) \). \( d_q \) is a metric on \( \mathcal{E} \) defined as

\[
d_q(u,w) = (1-q) \int_0^1 |u^-(\alpha) - w^+(\alpha)| \, d\alpha + q \int_0^1 |u^+(\alpha) - w^-(\alpha)| \, d\alpha
\]

for any \( u, w \in \mathcal{E} \).

However, the metric \( d_q \) does not consider the differences of the importance of the \( \alpha \)-cuts and therefore it’s not fully adapted to many practical cases. So, in this paper, we introduce weighted distance \( d_{q,\phi,\psi} \) on fuzzy numbers, which allows weighting the \( \alpha \)-cuts of fuzzy numbers.

\( \phi : [0,1] \to \mathbb{R} \) is said to be a weight function if \( \int_0^1 \phi(\alpha) \, d\alpha = 1 \).

**Definition 2.1.** Let \( \phi, \psi \) be weight functions and \( q \in (0,1) \). We define weighted distance \( d_{q,\phi,\psi} \) on \( \mathcal{E} \) by

\[
d_{q,\phi,\psi}(u,w) = (1-q) \int_0^1 \phi(\alpha) |u^-(\alpha) - w^+(\alpha)| \, d\alpha + q \int_0^1 \psi(\alpha) |u^+(\alpha) - w^-(\alpha)| \, d\alpha
\]

for any \( u, w \in \mathcal{E} \).

It is easy to see that \( d_{q,\phi,\psi} \) on \( \mathcal{E} \) is non-negative and symmetric and satisfies the triangle inequality.

When \( \phi = \psi = 1 \), \( d_{q,\phi,\psi} \) degenerates to \( d_q \).

**3. Ranking fuzzy numbers based on \( d_{q,\phi,\psi} \)**

Grzegorzewski [4] introduced a ranking method on fuzzy numbers. In this section, we modify this method by using the weighted distance \( d_{q,\phi,\psi} \). Properties of the new ranking methods are investigated. Further, we show that the new order can be generated by ranking index.

We say \( u \leq w \) if \( u^- (\alpha) \leq w^+ (\alpha) \) and \( u^+ (\alpha) \leq w^- (\alpha) \). Let \( U \) be a set of fuzzy numbers. We call \( w \) a lower (upper, resp.) fuzzy number of \( U \) if \( w \leq u \) (\( w \geq u \) resp.) for each \( u \in U \). In the case that \( w \) is a real number, we call \( w \) a lower (upper, resp.) real number of \( U \).

**Definition 3.1.** Let \( v \in \mathcal{E} \). Then the set

\[
v(+) := \{ u \in \mathcal{E} : v \leq u \}
\]

is called the upper set of \( v \).

**Definition 3.2.** Let \( v \in \mathcal{E} \). Then the set

\[
v(-) := \{ u \in \mathcal{E} : u \leq v \}
\]

is called the lower set of \( v \).

**Remark 3.3.** Grzegorzewski [4] introduced lower (upper) horizon of fuzzy number set \( U \) and the lower-dominated set \( W_l \) and the upper-dominated set \( W_u \) of a fuzzy number \( w \).

We can see that \( u \) is a lower (upper) horizon of \( U \) must also be a lower (upper) fuzzy number of \( U \). So \( W_l \subseteq w(+) \) and \( W_u \subseteq w(-) \).

Grzegorzewski [4] introduced the orders \( \succ_L \) and \( \succ_U \) using the metric \( d_q \). In this paper, we define new kinds of orders \( \succeq_+ \) and \( \succeq_- \) based on the weighted metric \( d_{q,\phi,\psi} \).

Let \( U \) be fuzzy number set and let \( w \) and \( z \) be lower fuzzy number and upper fuzzy number of \( U \),
respectively. Suppose that $u, v \in U$. We say that

$$u \geq_{q, \phi, \psi} v \iff d_{q, \phi, \psi}(u, w) \geq d_{q, \phi, \psi}(v, w)$$

(1)

$$u \geq_{q, \phi, \psi} v \iff d_{q, \phi, \psi}(u, z) \leq d_{q, \phi, \psi}(v, z)$$

(2)

**Theorem 3.4.** The orders $\geq_{q, \phi, \psi}$ and $\geq_{q, \phi, \psi}$ are reflexive, transitive and comparable.

**Proof.** The desired results follow immediately from (1) and (2).

**Theorem 3.5.**

(i) The order $\geq_{q, \phi, \psi}$ is independent of the choice of lower fuzzy number.

(ii) The order $\geq_{q, \phi, \psi}$ is independent of the choice of upper fuzzy number.

**Proof.** (i) Suppose that $U$ is a subset of fuzzy numbers, and that $\xi$ is an arbitrarily chosen lower fuzzy number of $U$. Then

$$d_{q, \phi, \psi}(u, \xi) = (1-q)\int_0^1 \phi(\beta)[u^-(\beta) - \xi^- (\beta)] d\beta + q\int_0^1 \psi(\beta)[u^+(\beta) - \xi^+(\beta)] d\beta$$

$$= (1-q)\int_0^1 \phi(\beta)(u^-(\beta) - \xi^- (\beta)) d\beta + q\int_0^1 \psi(\beta)(u^+(\beta) - \xi^+(\beta)) d\beta$$

$$= \int_0^1 (1-q)\phi(\beta)u^-(\beta) + q\psi(\beta)u^+(\beta) d\beta - \int_0^1 (1-q)\phi(\beta)\xi^- (\beta) + q\psi(\beta)\xi^+(\beta) d\beta$$

So $u \geq_{q, \phi, \psi} v$ which is equivalent to

$$d_{q, \phi, \psi}(u, \xi) \geq d_{q, \phi, \psi}(v, \xi)$$

if and only if

$$\int_0^1 (1-q)\phi(\beta)u^-(\beta) + q\psi(\beta)u^+(\beta) d\beta \geq \int_0^1 (1-q)\phi(\beta)v^-(\beta) + q\psi(\beta)v^+(\beta) d\beta$$

Thus $u \geq_{q, \phi, \psi} v$ is independent of the choice of $\xi$.

(ii) Suppose that $U$ is a subset of fuzzy numbers, and that $\eta$ is an arbitrarily chosen upper fuzzy number of $U$. Then

$$d_{q, \phi, \psi}(u, \eta) = (1-q)\int_0^1 \phi(\beta)[u^- (\beta) - \eta^- (\beta)] d\beta + q\int_0^1 \psi(\beta)[u^+ (\beta) - \eta^+ (\beta)] d\beta$$

$$= (1-q)\int_0^1 \phi(\beta)(\eta^- (\beta) - u^- (\beta)) d\beta + q\int_0^1 \psi(\beta)(\eta^+ (\beta) - u^+ (\beta)) d\beta$$

$$= \int_0^1 (1-q)\phi(\beta)\eta^- (\beta) + q\psi(\beta)\eta^+ (\beta) d\beta - \int_0^1 (1-q)\phi(\beta)u^- (\beta) + q\psi(\beta)u^+ (\beta) d\beta.$$  

So $u \geq_{q, \phi, \psi} v$ which is equivalent to

$$d_{q, \phi, \psi}(u, \eta) \leq d_{q, \phi, \psi}(v, \eta)$$

if and only if

$$\int_0^1 (1-q)\phi(\beta)u^- (\beta) + q\psi(\beta)u^+ (\beta) d\beta \geq \int_0^1 (1-q)\phi(\beta)v^- (\beta) + q\psi(\beta)v^+ (\beta) d\beta.$$  

Thus $u \geq_{q, \phi, \psi} v$ does not depend on $\eta$.

An important type of orders on fuzzy numbers is generated by ranking indices. A ranking index is a function from fuzzy numbers to real values. This type of orders have significant advantage (see[1]). Below we show that the orders $\geq_{q, \phi, \psi}$ and $\geq_{q, \phi, \psi}$ are in fact induced from a ranking index.

**Definition 3.6.** Let $u \in E$. Define a parametric expected value $E_{q, \phi, \psi}(u)$ of $u$ as
\[ E_{q,\phi,\psi}(u) = \int_0^1 (1-q)\phi(a)u^-(a) + q\psi(a)u^+(a)da. \]

Then \( E_{q,\phi,\psi} \) can be used as a ranking index, i.e. we obtain an order \( \succeq_{q,\phi,\psi} \) induced by \( E_{q,\phi,\psi} \) which is defined as

\[ u \succeq_{q,\phi,\psi} v \iff E_{q,\phi,\psi}(u) \geq E_{q,\phi,\psi}(v). \]

The following Theorem 3.7 indicates that the order \( \succeq_{q,\phi,\psi} \) both are generated by the ranking index \( E_{q,\phi,\psi} \).

**Theorem 3.7.** Let \( u, v \in E \). Suppose that the orders \( \succeq_{q,\phi,\psi} \) are based on \( d_{q,\phi,\psi} \). Then the following are equivalent statements.

(i) \( u \succeq_{q,\phi,\psi} v \).

(ii) \( u \preceq_{q,\phi,\psi} v \).

(iii) \( E_{q,\phi,\psi}(u) \geq E_{q,\phi,\psi}(v) \).

**Proof.** From the proof of Theorem 3.5, \( u \succeq_{q,\phi,\psi} v \) and \( u \preceq_{q,\phi,\psi} v \) are equivalent to \( E_{q,\phi,\psi}(u) \geq E_{q,\phi,\psi}(v) \). So we obtain the desired results.

4. Calculations of the Ranking index

In practical applications, the most often used fuzzy numbers are triangular fuzzy numbers and trapezoidal fuzzy numbers. In this section, we compute the ranking index \( E_{q,\phi,\psi}(u) \) when \( \phi(\alpha) = \psi(\alpha) = 2\alpha \) and \( u \) is a triangular fuzzy number or a trapezoidal fuzzy number.

Let \( L \leq M \leq R \). \((L, M, R)\) represents the triangular fuzzy number

\[ u(x) = \begin{cases} \frac{x-L}{M-L}, & L \leq x \leq M \\ \frac{R-x}{R-M}, & M \leq x \leq R. \end{cases} \]

Let \( L \leq M_1 \leq M_2 \leq R \). \((L, M_1, M_2, R)\) denotes the trapezoidal fuzzy number

\[ v(x) = \begin{cases} \frac{x-L}{M_1-L}, & L \leq x \leq M_1, \\ 1, & M_1 \leq x \leq M_2, \\ \frac{R-x}{R-M_2}, & M_2 \leq x \leq R. \end{cases} \]

In the sequel, we will take \( \phi(\alpha) = \psi(\alpha) = 2\alpha \).

**Example 4.1.** Let \( u = (L, M, R) \) be a triangular fuzzy number. Then

\[
\int_0^1 \phi(\alpha)u^-(\alpha)d\alpha = \int_0^1 2\alpha u^- (\alpha)d\alpha \\
= \int_0^1 2\alpha(L + \alpha(M - L))d\alpha \\
= \frac{2}{3}(M - L) + L 
\]

and
\[
\int_{0}^{1} \psi(\alpha)u^+(\alpha) d\alpha = \int_{0}^{1} 2\alpha u^+(\alpha) d\alpha \\
= \int_{0}^{1} 2\alpha(R + \alpha(M - R)) d\alpha \\
= \frac{2}{3}(M - R) + R.
\]
So
\[
E_{q,\psi,\varphi}(u) = (1 - q)(\frac{2}{3}(M - L) + L) + q(R + \frac{2}{3}(M - R)).
\]

**Example 4.2.** Let \(v = (L, M_1, M_2, R)\) be a trapezoidal fuzzy number. Then
\[
\int_{0}^{1} \phi(\alpha)v^-(\alpha) d\alpha = \int_{0}^{1} 2\alpha v^-(\alpha) d\alpha \\
= \int_{0}^{1} 2\alpha(L + \alpha(M_1 - L)) d\alpha \\
= \frac{2}{3}(M_1 - L) + L,
\]
and
\[
\int_{0}^{1} \psi(\alpha)v^+(\alpha) d\alpha = \int_{0}^{1} 2\alpha v^+(\alpha) d\alpha \\
= \int_{0}^{1} 2\alpha(R + \alpha(M_2 - R)) d\alpha \\
= \frac{2}{3}(M_2 - R) + R.
\]
So
\[
E_{q,\psi,\varphi}(v) = (1 - q)(\frac{2}{3}(M_1 - L) + L) + q(R + \frac{2}{3}(M_2 - R)).
\]
It's easy to see that if \(M_1 = M_2\) then trapezoidal fuzzy number \(v\) degenerates to triangular fuzzy number \(u\). In this case, \(E_{q,\psi,\varphi}(v)\) also degenerates to \(E_{q,\phi,\psi}(u)\).

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