SMALL-SCALE ANISOTROPY OF THE COSMIC BACKGROUND RADIATION AND SCATTERING BY CLOUDY PLASMA

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ABSTRACT

If the first stars formed soon after decoupling of baryons from the thermal cosmic background radiation (CBR), the radiation may have been last scattered in a cloudy plasma. We discuss the resulting small-scale anisotropy of the CBR in the limit where the plasma clouds are small compared to the mean distance between clouds along a line of sight. This complements the perturbative analysis valid for mildly nonlinear departures from homogeneity at last scattering. We conclude that reasonable choices for the cloud parameters imply CBR anisotropy consistent with the present experimental limits, in agreement with the perturbative approach. This means the remarkable isotropy of the CBR need not contradict the early small-scale structure formation predicted in some cosmogonies.

Subject headings: cosmic microwave background — galaxies: formation

1. INTRODUCTION

Cosmogonies in which the baryons were concentrated in clouds at the epoch of last scattering of the thermal cosmic background radiation (CBR), as in isocurvature models (Peebles 1994, 1997a), are in qualitative agreement with the appearance in quasar absorption-line spectra of a well-advanced state of structure formation at redshift \( z \approx 5 \), but the large amplitude of the density fluctuations may produce significant small-scale anisotropy in the CBR. The analysis of the CBR anisotropy in second-order perturbation theory was introduced by Ostriker & Vishniac (1986). The more detailed investigation by Vishniac (1987) has been confirmed by Hu, Scott, & Silk (1994), Dodelson & Jubas (1995), and Hu & White (1997). The analysis and numerical evaluation are extended in Persi (1995) and Persi et al. (1995). The application of perturbation theory may be uncertain if early structure formation produced highly nonlinear departures from homogeneity at the epoch of last scattering, however. To investigate this, we have developed a nonperturbative model for scattering in a cloudy distribution of plasma. The expression for the CBR anisotropy \( \delta T / T \) in the strongly cloudy limit bears a close resemblance to the Ostriker-Vishniac relation, and the numerical results for \( \delta T / T \) accordingly are similar.

Our analysis, which extends previous discussions by Hogan & Partridge (1989), Peebles (1990), Aghanim et al. (1996), Gruzinov & Hu (1998), and Knox, Scoccimarro, & Dodelson (1998), assumes that the CBR was last scattered by free electrons in clouds with density contrast well above unity, so the mean free distance \( t_f \) between intersections of gas clouds along a line of sight is much larger than the typical cloud size \( d_{cl} \). The simplification offered by this limit is that the details of the matter distribution and motion on scales from \( d_{cl} \) to \( t_f \) are not important, because a line of sight on average samples only one cloud over the distance \( t_f \). Thus we can model the clouds as an inhomogeneous random Poisson process. In the process the joint distribution of cloud motions and scattering optical depths as a function of position along the line of sight is determined by the plasma density and velocity fields averaged through a window of width \( t_f \). If \( t_f \) is larger than the scale of nonlinear density fluctuations, then the CBR anisotropy \( \delta T / T \) is the sum in quadrature of a perturbative contribution and a shot noise term.

In the next section we present a qualitative explanation of the similarity of the expressions for the perturbative Ostriker & Vishniac (1986) relation and the shot noise contribution to \( \delta T / T \) from strongly nonlinear clustering of plasma in clouds. In § 3 we generalize the analysis of scattering in clouds to an inhomogeneous random Poisson process that takes account of cosmic evolution of the cloud parameters and the large-scale correlations of cloud positions and motions. The latter have little effect on the small-scale CBR anisotropy, for as Sunyaev (1978) noted, the CBR averages the perturbations of the peculiar motions across the Hubble length \( t_f \) at the epoch of last scattering, suppressing the contribution to the anisotropy. The effect is further suppressed by the anticorrelation of the peculiar motions on either side of a density fluctuation, as confirmed by Kaiser (1984). In § 4 we apply the formalism developed in § 3 to a simplified model that helps clarify the physics behind the contributions to \( \delta T / T \) from shot noise and correlated motions of the gas clouds. This provides a convenient illustration of how a positive velocity correlation reduces the effective number of statistically independent steps in the random walk of \( \delta T / T \) and so increases the rms temperature perturbation. In § 5 we present numerical examples of the expected CBR anisotropy.

2. SHOT NOISE AND THE OSTRIKER-VISHNIAC ANISOTROPY

Ostriker & Vishniac (1986) showed that the CBR anisotropy at angular resolution \( \theta \) produced by the peculiar motion of the plasma at last scattering at proper expansion time \( t_s \) is

\[
\left( \frac{\delta T}{T} \right)_\theta \sim v_0^2 \delta_\theta^2 \frac{r_\theta}{t_s}.
\]

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Here $r_\theta$ is the proper length subtended at the epoch of last scattering by the observed angle $\theta$, and $\delta_0$ and $v_\theta$ are the rms values of the density contrast and the line-of-sight component of the peculiar velocity averaged through a window of width $r_\theta$. Ostriker & Vishniac based this relation on the lowest nontrivial order of perturbation theory. When the CBR is scattered in discrete and well-isolated clouds of plasma, the analogous expression is obtained as follows.

Suppose the CBR is scattered in clouds with typical optical depth

$$\tau = \alpha \Sigma_e,$$

(2)

where $\alpha$ is the Thomson cross section and $\Sigma_e$ is the characteristic free electron column density of a cloud. If $\tau \ll 1$, the CBR temperature perturbation observed at angular resolution smaller than a cloud size may be approximated as

$$\frac{\delta T}{T} \sim \int_{\theta} \sigma n_e v dt \sim \sum \tau v_n.$$

(3)

In the integral $n_e$ and $v$ are the free electron number density and streaming velocity as functions of position along a line of sight, and the integral is over a proper distance comparable to the expansion time $t_e$ at the epoch where the universe last is optically thick, $\sigma n_e t_e \approx 1$. (Here and below, the velocity of light is set to unity.) The sum is over $N \sim \tau^{-1} \approx t_e/t_\tau$ clouds, where $t_\tau$ is the mean distance between cloud intersections along a line of sight. If $t_\tau$ is larger than the nonlinear clustering length, then the sum in equation (3) adds as a random walk, and the mean square temperature perturbation along a line of sight is

$$\left( \frac{\delta T}{T} \right)^2 \sim v^2 \tau^2 N \sim v^2 \tau .$$

(4)

If the CBR is observed in a beam with angular size $\theta$ larger than the angular diameter $\theta_{cl}$ characteristic of clouds with proper width $d_{cl}$, the mean square anisotropy is

$$\left( \frac{\delta T}{T} \right)^2_{\theta} \sim \frac{\theta_{cl}^2}{\theta^2} \sim v^2 \left( \frac{d_{cl}}{r_\theta} \right)^2 ,$$

(5)

because we are ignoring correlations among cloud positions and motions. As in equation (1), $r_\theta$ is the proper length subtended by the beam size $\theta$ at last scattering. The number of free electrons in a cloud is $\sim \Sigma_e d_{cl}^2$, so the number density $n_{cl}$ of clouds satisfies

$$\Sigma_e d_{cl}^2 n_{cl} \sim \bar{n}_e \sim 1/(t_e \alpha) ,$$

(6)

where $\bar{n}_e$ is the large-scale mean number density of free electrons. Equations (5) and (6) give

$$\left( \frac{\delta T}{T} \right)^2_{\theta} \sim v^2 \delta_{cl}^2 , \quad \delta_{cl}^2 = \frac{1}{n_{cl} r_\theta^2} .$$

(7)

This is the same form as the Ostriker-Vishniac expression in equation (1), where $v$ is the rms line-of-sight peculiar streaming velocity of the material within a cloud and $\delta_{cl}$ is the shot noise contribution to the rms value of the fractional fluctuation in the number of clouds found within a randomly placed sphere of width $r_\theta$.

3. POISSON MODEL FOR THE SCATTERING CLOUDS

Here we generalize the analysis in § 2 to take account of the cosmic evolution of the cloud parameters and the correlations in cloud positions and motions. The starting assumption is that the distance between successive intersections of clouds along a line of sight is large. This suggests that we model the distribution of clouds as an inhomogeneous random Poisson process with probability distribution that evolves according to linear perturbation theory. The Poisson process is defined in equation (10) below, and the probability distribution in cloud position, optical depth, and line-of-sight peculiar velocity is defined in equation (11).

3.1. The Inhomogeneous Poisson Process

A suitable approximation to the transfer equation for the CBR temperature perturbation observed at epoch $t_\theta$ along a line of sight through a given electron distribution $n_e$ and streaming velocity field with a line-of-sight component $v$ is

$$\frac{\delta T}{T} = \int_0^{t_\theta} \sigma n_e v dt \exp \left( - \int_0^{t_\theta} \sigma n_e v dt \right) .$$

(8)

This expression ignores the gravitational perturbation to the CBR, which is small in the small-angle limit of interest here. It also ignores the inhomogeneity in the space distribution of the radiation at last scattering, for the observations tell us $\delta T/T \approx 10^{-5}$ at last scattering, while the velocity term in equation (8) is $v/c \approx 10^{-3}$. In the perturbative computation the analog of equation (8) is the velocity-density coupling, which Hu et al. (1994) and Dodelson & Jubas (1995) show is the dominant term in second-order perturbation theory.

We are assuming that the CBR is last scattered in well-separated clouds of plasma. If the optical depth of a cloud on the line of sight is $\tau$, the probability that the cloud has scattered a particular photon into the beam is $1 - e^{-\tau}$. If the peculiar velocity of
In the last line, the term is averaged over the distribution of optical depths for clouds found at position \( i \). We rewrite equation (8) where the integral runs over all the plasma intersected by the line of sight subsequent to this cloud. Thus in our cloud model of the visibility function in equation (10)

\[
Averages across the Poisson process may be computed independently for each different cell label \( iab \) where if there is a cloud with optical depth in the range \( \tau_a \) to \( \tau_a + d\tau_a \) and peculiar velocity in the range \( v_b \) to \( v_b + dv_b \) at position \( t_i \) to \( t_i + dt_i \) along the line of sight, and \( n_{iab} = 0 \) otherwise.

Equation (10) represents the strongly nonlinear cloud-like distribution of the scattering plasma by the inhomogeneous random Poisson process \( n_{iab} \). The inhomogeneity of the process includes the time evolution of the mean cloud properties and the large-scale structure in the distribution and motion of the clouds. Thus we compute statistical averages in two steps: first average over the Poisson process for given smooth fields that represent the large-scale structure, and then use perturbation theory to compute the average over the ensemble of smooth fields.

For a given realization of the smoothed fields the expectation value of the Poisson process \( n_{iab} \) in equation (10) is

\[
\langle n_{iab} \rangle = f_{iab} \tau_a \; dv_b \; dt_i / t_f(i) .
\]

The probability of finding a cloud at the position labeled by \( t_i \) in the interval \( dt_i \) is \( dt_i / t_f(i) \), where \( t_f(i) \) is the local mean free distance between cloud intersections in the given large-scale structure. If there is a cloud at \( t_i \), the probability distribution in its optical depth and velocity is \( f_{iab} \), with the normalization \( \int f_{iab} \; dt \; dv = 1 \). Thus the average over the Poisson process of the optical depth of a cloud found at point \( t_i \) on the line of sight is

\[
\langle \tau \rangle_i = \int f_{iab} \tau_a \; dv_b \; dt_i / t_f(i) = \sigma n(i) t_f(i) .
\]

In the last expression the mean plasma density \( n(i) \) is a function of position \( t_i \) along the line of sight, and averaged over the mean free path \( t_f(i) \). If the mean optical depth \( \langle \tau \rangle_i \) is independent of position this equation just says the smoothed electron density varies inversely as the mean free distance between clouds.

The mean value of the product of the electron density and streaming velocity \( v(i) \) at \( t_i \) is

\[
\langle n_e v \rangle = \frac{1}{\sigma dt_i} \sum_{ab} \langle n_{iab} \rangle \tau_a v_b
\]

\[
= \frac{1}{\sigma t_f(i)} \int d\tau_a dv_b \tau_a v_b f_{iab}
\]

\[
= \langle \tau v \rangle_i / \langle \tau \rangle_i .
\]

The last line, which uses equation (12), expresses the average of \( n_e v \) across the Poisson process as the product of the smoothed electron density field and the smoothed cloud velocity field weighted by the cloud optical depth.

\[3.2. \text{The Variance of the CBR Temperature}\]

In the computation of the mean of \( (\delta T/T)^2 \) we consider first the average over the Poisson process for a given realization of the smoothed fields that describe the large-scale structure of the plasma distribution.

Because \( n_{iab} \) in equation (11) is either zero or unity, the mean of any positive power of \( n_{iab} \) is the same as the mean of \( n_{iab} \). By expanding the exponential as a power series, one sees that

\[
\langle e^{-n_{iab} \tau_a} \rangle = 1 - \langle n_{iab} \rangle (1 - e^{-\tau_a}) .
\]

Averages across the Poisson process may be computed independently for each different cell label \( iab \), so the expectation value of the visibility function in equation (10) is

\[
\langle \exp (-n_{iab} \tau_a) \rangle = \prod_i \langle \exp (-n_{iab} \tau_a) \rangle
\]

\[
= \prod_i \left[ 1 - \langle n_{iab} \rangle (1 - e^{-\tau_a}) \right]
\]

\[
= \exp \left[ - \int dt_i \langle e^{-\tau_a} \rangle / t_f(i) \right] .
\]

In the last line, the term \( e^{-\tau_a} \) is averaged over the distribution \( f_{iab} \) of optical depths for clouds found at position \( i \) along the line of sight, as in equation (12). If the cloud optical depths are large, so that \( \langle e^{-\tau} \rangle \ll 1 \), equation (15) is just the probability that the line of sight intersects no clouds in the Poisson process. If \( \langle \tau \rangle \ll 1 \), equation (15) is the usual expression in perturbation
theory,

\[ \left\langle \exp \left( -\sum n_{ab} \tau_i \right) \right\rangle = \exp \left( -\int d\tau \langle \tau \rangle \right) = \exp \left( -\int n_\tau \, dt \right). \]  

Here \( n_\tau \) is the local smoothed free electron number density defined in equation (12).

The average of the transfer equation (10) across the Poisson distribution for a given realization of the large-scale structure is

\[ \left\langle \frac{\delta T}{T} \right\rangle = \int_0^T \frac{dt}{t} \left\langle \left(1 - e^{-\tau } \right) \nu_\tau \right\rangle \exp \left[ -\int_t^T \frac{dt'}{t'} \left(1 - \langle e^{-\tau } \rangle \right) \right]. \]  

Here again the averages are over the distribution \( f_{\text{obs}} \) of optical depths and line-of-sight velocities of clouds found at a given position \( t \) along the line of sight. If the cloud optical depths are small, we can use the expression for the smoothed electron number density \( n_\tau \) in equation (12) to rewrite equation (17) as

\[ \left\langle \frac{\delta T}{T} \right\rangle = \int_0^T \sigma n_\tau \, dt \nu(t) \exp \left( -\int_t^T \sigma n_\tau \, dt' \right), \quad \nu(t) = \frac{\langle \tau \rangle }{\langle \tau \rangle_t}. \]  

This is the form one would write down in perturbation theory, where the smoothed field \( \nu(t) \) as a function of position \( t \) along the line of sight is the mean peculiar velocity weighted by the cloud optical depth, as in equation (13).

In the expression for the mean square value of the temperature perturbation along a line of sight, we have to consider separately the squared terms and the cross terms from the sum over the space position index \( i \) in equation (10). Thus we write the average over the Poisson process as

\[ \left\langle \left( \frac{\delta T}{T} \right)^2 \right\rangle = \left( \frac{\delta T}{T} \right)_s^2 + \left( \frac{\delta T}{T} \right)_c^2. \]  

As we now discuss, the squared terms generalize the shot noise in equation (4) and the cross terms approximate the variance of \( \delta T/T \) in perturbation theory for the smoothed fields.

The sum of the squared terms in equation (10) is

\[ \left( \frac{\delta T}{T} \right)_s^2 = \sum_{iabc} \left\langle n_{iab} n_{cde} (1 - e^{-\tau })(1 - e^{-\tau } \nu_\tau ) v_d \prod_{k > i, e f} e^{-2n_{kfe} \tau_k} \right\rangle. \]  

In the last product we have taken account of the fact that each exponential factor \( e^{-2n_{kfe} \tau_k} \) appears twice, once from each factor of \( \delta T/T \). As in equation (15) we can compute separately the average over the Poisson process for each different cell \( iab \). Also, we have

\[ \left\langle n_{iab} n_{cde} \right\rangle = \left\langle n_{iab} \right\rangle \delta_{ac} \delta_{bd}, \]  

because the terms where \( a \ne c \) or \( b \ne d \) are of second order in \( dt \). Thus equation (20) is

\[ \left( \frac{\delta T}{T} \right)_s^2 = \int_0^T \frac{dt}{t} \left\langle \nu^2 (1 - e^{-\tau })^2 \right\rangle \left[ -\int_t^T \frac{dt'}{t'} \left(1 - \langle e^{-2\tau} \rangle \right) \right]. \]  

We complete the discussion of this shot noise term in § 3.3, after dealing with the cross terms in equation (19).

The sum of the cross terms in equation (10) is

\[ \left( \frac{\delta T}{T} \right)_c^2 = 2 \sum_{i \ne j} \sum_{abcd} \left(1 - e^{-\tau } \nu_\tau \right) \nu_\tau \prod_{k > i, e f} \left\langle n_{jcd} \prod_{ef} e^{-n_{jfe} \tau_e} \prod_{i < k < j} \prod_{gh} e^{-n_{kfe} \tau_k} \prod_{l > j, mn} e^{-2n_{mn} \tau_m} \right\rangle. \]  

The first expectation value in this equation refers to the earlier time \( t_i \) along the line of sight. The second expectation value contains the exponential factors at the later time \( t_j \) along the line of sight. In the next group of expectation values the exponential factors are evaluated at times between and \( t_j \). In the last group, at times greater than \( t_j \), each exponential factor appears twice, from the two factors of \( \delta T/T \). We have

\[ \left\langle n_{jcd} \prod_{ef} e^{-n_{jfe} \tau_e} \right\rangle = \left\langle n_{jcd} \right\rangle e^{-\tau}. \]  

The first step follows because the exponential factors with \( ef \) not equal to \( cd \) introduce terms of second order in \( dt_j \). The last step follows by the argument used in equation (14). With equations (11) and (15) we get

\[ \left( \frac{\delta T}{T} \right)_c^2 = 2 \int_{t < j} \frac{dt_i dt_j}{t_i t_j} I J, \]

\[ I = \langle \langle (1 - e^{-\tau}) \nu \rangle \rangle \langle (1 - e^{-\tau}) \nu^{-1} \rangle, \]

\[ J = \exp \left[ -\int_{t_i}^{t_j} \frac{dt_k (1 - \langle e^{-\tau} \rangle)}{t_k} \right] \exp \left[ -\int_{t_j}^T \frac{dt (1 - \langle e^{-2\tau} \rangle)}{t} \right]. \]  

(25)
In this equation the outermost brackets denote an average over an ensemble of large-scale velocity fields, while the inner brackets denote an average over the Poisson process for each cell $iab$ for a given realization of the ensemble of velocity fields.

As in equation (18), when the cloud optical depths $\tau_i$ are much less than unity we can rewrite equation (25) as

$$
\left(\frac{\delta T}{T}\right)_i^2 = \int_0^{t_0} dt_1 \sigma n_e dt_1 \exp \left( -\int_{t_1}^{t_0} \sigma n_e dt_t \right) \exp \left( -\int_{t_1}^{t_0} \sigma n_e dt_t \right) .
$$

(26)

This is the expression one would write down in perturbation theory based on the smoothed velocity and density fields defined in equations (12), (13), and (18). If $t_f$ is large compared to the scale of nonlinear clustering, this smoothed velocity field is well approximated by linear perturbation theory.

3.3. The Shot Noise Contribution to $\delta T/T$

We consider here some simplified forms for the shot noise term in equation (22). We assume the universe is ionized back to high redshift where the scattering optical depth to the present is large. If the distributions of $\tau_i$ and $v_i$ are independent of time, the integral is

$$
\left(\frac{\delta T}{T}\right)_s^2 = \frac{\langle v^2(1 - e^{-\tau})^2 \rangle}{1 - \langle e^{-2\tau} \rangle} .
$$

(27)

We see that if $\tau \gg 1$ the temperature perturbation is just the rms velocity of the last cloud along the line of sight. If the dispersion in $\tau$ is small, equation (27) is

$$
\left(\frac{\delta T}{T}\right)_s^2 = \frac{\langle v^2 \rangle}{1 + e^{-\tau}} = \frac{\langle v^2 \rangle}{2 - q} ,
$$

where the probability that a photon is scattered by a cloud is

$$
q = 1 - e^{-\tau} .
$$

(28)

(29)

Another derivation of equation (28) is given in Peebles (1990).

In the numerical examples in §5 the cloud optical depths $\tau$ are small. Here, following equations (18) and (26), we can write equation (22) as

$$
\left(\frac{\delta T}{T}\right)_s^2 = \int_0^{t_0} dt \frac{\langle \tau^2 v^2 \rangle}{\langle \tau \rangle} \sigma n_e \exp \left( -2 \int_t^{t_0} \sigma n_e dt' \right) .
$$

(30)

This is the average across the Poisson process for a given realization of the smoothed fields that represent the large-scale structure. In linear perturbation theory the average across the smoothed fields just replaces the density and velocity dispersion with the global mean values as functions of world time. One could numerically evaluate the resulting time integrals, but if $\langle \tau^2 v^2 \rangle/\langle \tau \rangle$ is not a rapidly varying function of time the integral is well approximated by the value of this expression at the epoch of last scattering. In this approximation the shot noise term is

$$
\left(\frac{\delta T}{T}\right)_s^2 = \frac{\langle \tau^2 v^2 \rangle_s}{2 \langle \tau \rangle_s} = \frac{\tau_s v_s^2}{2} .
$$

(31)

The factors in the last expression are suitably weighted values of the mean cloud optical depth and mean square peculiar velocity evaluated at the epoch of last scattering of the CBR, under the assumption that the baryons are concentrated in clouds back to this epoch.

To describe the effect of scattering at lower redshifts, we use the mean scattering optical depth,

$$
\bar{\tau}(z) = \int_0^z \sigma n_e \left( \frac{dt}{dz} \right) dz ,
$$

(32)

where $\bar{n}_e$ is the cosmic mean density of free electrons in optically thin clouds. In the “saturated” case, the integral in equation (32) increases indefinitely with $z$, and the redshift of last scattering, $z_s$, is defined by $\bar{\tau}(z_s) = 1$. Equation (31) assumes this saturated ionization case. Another possibility is that $\bar{\tau}(z)$ does not grow indefinitely with $z$ but rather reaches a maximum value $\tau_s < 1$ at $z = z_s$, and then remains constant back to the hydrogen recombination epoch. Here the approximation to the shot noise term in equation (30) changes from equation (31) to

$$
\left(\frac{\delta T}{T}\right)_s^2 = \frac{\tau_s v_s^2}{2} \left( \frac{v_s}{v_0} \right)^2 (1 - e^{-2\bar{\tau}}) .
$$

(33)

The plasma streaming velocity $v_s$ at last scattering, at redshift $z_s$, has been scaled to the present value $v_0$ by the factor $v_s/v_0$ computed in linear perturbation theory. It will be recalled that the typical optical depth per cloud is $\tau_s$.

To compare our results with experimental data and theoretical predictions for primary anisotropies, it is convenient to expand the CBR temperature two-point angular correlation function in Legendre polynomials:

$$
C(\Theta) = \langle \frac{\delta T}{T} (1) \frac{\delta T}{T} (2) \rangle = \sum_{\ell} \frac{2\ell + 1}{4\pi} C_\ell P_\ell(\cos \Theta) ,
$$

(34)
where $\Theta$ is the angle between directions 1 and 2, while
\begin{equation}
C_\ell = \langle |a_\ell^m|^2 \rangle ,
\end{equation}
and $a_\ell^m$ is a coefficient in the spherical harmonic expansion of the CBR temperature distribution. The shot noise contribution to the CBR temperature autocorrelation function is $C(\Theta) = (\Delta T/T)^2_\ell$ at angular separation $\Theta$ substantially smaller than the typical angular size $\theta_{cl}$ of a cloud at redshift $z_c$, and $C(\Theta) = 0$ at $\Theta > \theta_{cl}$. The angle $\theta_{cl}$ subtended by a cloud of diameter $d_{cl}$ at redshift $z_c$ is
\begin{equation}
\theta_{cl} = H_0 d_{cl}(1 + z_c)/y ,
\end{equation}
where $y$ is the usual dimensionless angular size distance (Peebles 1993, eq. [13.29]),
\begin{equation}
y(z_c, \Omega, \Lambda, H_0) = H_0 a_0 r ,
\end{equation}
with $r$ equal to the comoving angular size distance to a cloud at $z_c$. The autocorrelation function of the temperature perturbation produced by a typical cloud is bell shaped, and a reasonable approximation is a Gaussian,
\begin{equation}
C(\Theta) = (\Delta T/T)^2_\ell \exp \left( -\frac{\Theta^2/\theta_{cl}^2}{4} \right) .
\end{equation}
If the shot noise dominates, and $\ell \gg 1$, the resulting power spectrum of CBR fluctuations is given by the usual Hankel transform,
\begin{equation}
C_\ell = \int_0^\infty C(\Theta)J_0(\ell \Theta)2\pi \Theta d\Theta = \pi \theta_{cl}^2 \left( \frac{\Delta T}{T} \right)^2_s \exp \left( -\frac{\ell^2 \theta_{cl}^2}{4} \right) ,
\end{equation}
where $J_0$ is the zeroth-order Bessel function. In these models the power per octave, rises with a growing multipole number like $\ell^2$, reaching its peak value $\approx (\Delta T/T)^2_\ell$ near $\ell \approx \theta_{cl}^{-1}$, and then drops. This maximum signal is diluted in experiments with antenna beamwidth greater than $\theta_{cl}$. The mean square CBR temperature anisotropy averaged through a Gaussian beam response of dispersion $\theta/2$, corresponding to full width at half-maximum of 1.2$\theta$, is
\begin{equation}
\left( \frac{\Delta T}{T} \right)^2_s = \frac{\theta_{cl}^2}{\theta^2 + \theta_{cl}^2} \left( \frac{\Delta T}{T} \right)^2_s .
\end{equation}
The image of the microwave sky produced with a radio telescope beam $\theta \gg \theta_{cl}$ will be resolution-limited, with the rms $\Delta T/T$ reduced by the factor $\theta_{cl}/\theta$. Equations (33) and (40) are the analogs of the Ostriker-Vishniac expression in equation (1), as we discussed in connection with equation (5).

3.4. The Contribution from Correlated Motions

Here we consider the term induced by the correlations in the peculiar velocity field. We ignore the correlations of cloud positions in our analysis for the following reason.

If the power spectrum of density fluctuations at zero redshift is given by $P(k)$, where $k$ is the wavenumber, then the linearized continuity equation says that the present power spectrum of the peculiar velocity field is $P_s(k) = H_0^2 a_0^2 f^2(\Omega, \Lambda)P(k)/k^2$, where $f \approx \Omega^{0.6}$. (e.g., Peebles 1993, § 13). Because of the factor $k^{-2}$, the coherence length of the density field is always significantly shorter than that of the velocity. Moreover, density correlations tend to decrease more rapidly with increasing redshift than velocity correlations. Thus, in an Einstein–de Sitter universe, $P_s(k, z) = P_s(k)(1 + z)^{-2}$ while $P_s(k, z) = P_s(k)(1 + z)^{-1}$, so at high redshift
\begin{equation}
(v_s/v_0)^2 \equiv P_s(k, z)/P_s(0) \gg P_s(k, z)/P(k) .
\end{equation}
A similar inequality holds for a wider class of models, with $\Omega < 1$ and/or $\Lambda \neq 0$ (Peebles 1993, § 13). For all models under consideration here, the left-hand side of the above inequality is larger than the right-hand side by a factor $\sim z_c \gg 1$. In other words, the spatial correlations between the clouds can be safely ignored.

As in § 3.3, to simplify our calculations further, we now assume that the dispersion in the cloud optical depths is small and that the probability of Thomson scattering per cloud is given by equation (29), where $q$ and $r$ do not depend on redshift (while in more realistic models $\tau$ may be a function of the cloud and the redshift). Instead of the cloud coordinates along the past light cone, $(t, t_i, t_j)$, it is more convenient to use the mean and relative position, given by
\begin{equation}
t = t(i, j) = (t_i + t_j)/2 , \quad u = u(i, j) = t_i - t_j ,
\end{equation}
as in the usual derivation of the Limber equation (see, e.g., Peebles 1993, § 7). After this coordinate transformation, equations (22) and (25) become
\begin{equation}
\left( \frac{\Delta T}{T} \right)^2_s = q^2 \int_0^\infty \frac{dt}{t f(t)} \exp \left[ -2q - q^2 \right] \Pi(0, t) ,
\end{equation}
and
\begin{equation}
\left( \frac{\Delta T}{T} \right)^2_s = 2q^2(1 - q) \int_0^\infty \frac{dt}{t f(t)} \exp \left[ -2q - q^2 \right] \int_0^\infty \frac{du}{t f(t)} \exp \left[ -uq/t f(t) \right] \Pi(u, t) ,
\end{equation}
Here $\Pi$ is the radial component of the velocity correlation tensor,
\begin{equation}
\Pi(u, t) = (v_s/v_0)^2 \Pi(r) ,
\end{equation}
The variable $i$ wavenumber $r$ dispersion) is expression in square brackets. Hence the source term of the shot noise contribution (i.e., the one-dimensional velocity dispersion) is

$$
\Pi(r) = \frac{(a_0 H_0 f)^2}{2\pi^2} \int_0^\infty P(k) \left[ j_0(kr) - \frac{2 j_1(kr)}{kr} \right] dk ,
$$

(46)

where the $j_n$ are spherical Bessel functions (Groth, Juszkiewicz, & Ostriker 1989; Peebles 1993). At $r = 0$, the value of the expression in square brackets is $\frac{1}{2}$. In our analysis of the relative importance of the contribution from correlated motion, this expression should be compared with the inner integral in equation (44), which can be written as

$$
\int_0^\infty \frac{du}{u} e^{-\omega t/\Pi(u, t)} = \frac{(v_z/v_0)^2}{2\pi^2} \int_0^\infty P(k) W(kt_j/qa) dk ,
$$

(47)

where the window function $W$ is

$$
W(k) = \kappa^{-1} \int_0^\infty dx e^{-\kappa x} \left[ j_0(x) - \frac{2 j_1(x)}{x} \right] .
$$

(49)

The variable $\kappa(k, t) = kt_j/qa$ is the ratio of the photon mean free path, $t_j(t)/q$ to the proper length $a(t)/kr$ corresponding to the wavenumber $k$. The integral in equation (49) can be expressed in terms of elementary functions (see the Appendix),

$$
W(k) = \kappa^{-2} - \kappa^{-3} \arctan \kappa \approx (3 + \kappa^2)^{-1} .
$$

(50)

The last expression is exact at $\kappa = 0$ and in the limit $\kappa \to \infty$. For any intermediate values of $\kappa$, the accuracy of the approximate expression is better than 20%. The window function $W[\kappa(k, t)]$ acts as a low pass filter, damping the contribution to $\delta T/T$, induced by correlated motions on characteristic scales that are shorter than the photon mean free path. Whether or not $\delta T/T$, can be neglected in comparison to $\delta T/T$, thus depends on the ratio of the velocity coherence length to the photon mean free path at $z$. To be more specific, let us model the power spectrum as

$$
P(k) = \frac{Ak}{1 + k^2 r_c^2} ,
$$

(51)

where $A$, $r_c$, and $\beta$ are constant parameters. The APM data suggest $\beta = 1.2$ and $r_c = 33 h^{-1}$ Mpc (Baugh & Gaztañaga 1996). We trade precision for simplicity and for the purpose of our order-of-magnitude estimates set

$$
\beta = 1 .
$$

(52)

To make this spectrum applicable for our linear theory expression for $\Pi(t)$, we need a high wavenumber cutoff to account for the stabilization of clustering by virialization, occurring on comoving scales smaller than $k_{nl}^{-1} \ll r_c$, where the density contrast exceeds unity. We therefore set $P(k) = 0$ for $k > k_{nl}$. Now the integrals in equations (47) and (48) become trivial, and the ratio of the correlated source term to the shot noise source term is

$$
\frac{\int_0^{k_{nl}} PW dk}{\int_0^{k_{nl}} P dk} = \frac{\ln \left[ (k_{nl} \lambda \lambda^2 + 3)(k_{nl} \lambda r_c^2 + 1) \right]}{q (\lambda/r_c^2) - 3 \ln (1 + k_{nl} r_c^2)} ,
$$

(53)

where $\lambda = t_j/qa$, and $r_c/\lambda$ is the ratio of the velocity coherence length in proper coordinates, $r_c a(z)$, to the mean free path of the CBR photons, $t_j(z)/q$. The correlated term becomes negligibly small when the condition

$$
\frac{1}{q} \left[ \frac{qr_c}{t_j(z)} \right]^2 \ll 1
$$

(54)

is satisfied. Note that to make the correlated term dominant, it is not enough to require that $r_c a(z) \gg t_j(z)/q$; the clouds have to be optically thin as well ($q \ll 1$). The coherence of the velocity field can amplify the net CBR temperature fluctuation by causing the effects of individual clouds to add with the same sign and not as a random walk. This, however, can happen only if (1) the velocity coherence length is larger than the photon mean free path and (2) the rescattering cutoff in the sum appears sufficiently far away from the observer that amplification by coherent motions is not destroyed by rescattering. The second condition means good visibility, or $q \ll 1$. Indeed, when $r_c/\lambda \gg 1$, the ratio of the correlated term to the shot noise term is

$$
\frac{\int_0^{k_{nl}} PW dk}{\int_0^{k_{nl}} P dk} \approx \frac{\ln (\sqrt{3} r_c/\lambda)}{3q \ln (k_{nl} r_c)} .
$$

(55)

Hence, unless the optical depth per cloud is small, the correlated term will be subdominant, even in the large-$r_c$ limit.

4. A SIMPLIFIED MODEL

We present here a simplified model that helps clarify the physics behind the contributions to $\delta T/T$ from shot noise and correlated motions of the gas clouds. The model assumes the universe is static, each cloud has the same scattering probability $q = 1 - e^{-t}$ (eq. [29]), and the mean free path between clouds is large enough that we can neglect the correlation in cloud
positions. The model takes account of the broader correlation of peculiar velocities. We characterize the strength and range of the velocity correlations by an “effective” correlation function,

$$\Pi(r) = \sigma_v^2 e^{-r/r_v},$$  \hspace{1cm} (56)

where $r_v$ is the velocity coherence length and $\sigma_v^2 = \Pi(0)$ is the one-dimensional velocity dispersion. The assumption that the correlation of cloud positions may be neglected means the probability distribution of relative separations $r$ between pairs of clouds, intersected along a line of sight, and numbered $i$ and $i+j$, is the Poisson expression

$$dp(r, j) = \frac{(r/r_f)^{j-1}}{(j-1)!} e^{-r/r_f} \frac{dr}{r_f},$$  \hspace{1cm} (57)

where

$$r_f = \int_0^\infty r dp(r, 1) = \frac{4}{n\pi d^2_{cl}}$$  \hspace{1cm} (58)

is the mean separation between clouds along the line of sight, $n$ is the spatial density of clouds, $d_{cl}$ is the cloud diameter, and the cloud numbers increase with distance from the observer. The perturbation to the CBR along a given line of sight is (Peebles 1990)

$$\frac{\delta T}{T} = q v_i + q(1-q)v_2 + q(1-q)^2v_3 + \cdots.$$  \hspace{1cm} (59)

The line-of-sight component of the $j$th velocity is $v_j$, counting back from the present. The prefactor $q(1-q)^{j-1}$ is the probability $q$ that a photon is scattered into our line of sight by the cloud $j$, multiplied by the probability that it is not scattered out of our line of sight by the remaining $j-1$ clouds in the foreground. If the universe is ionized back to large redshift, equation (59) in effect is a convergent infinite series, meaning the influence of consecutive terms decreases with increasing cloud number $j$. The attenuation by rescattering out of the line of sight means the CBR perturbation is determined by the last $\sim N$ cloud intersections, where

$$N \approx q^{-1}.$$  \hspace{1cm} (60)

The attenuation by rescattering is most severe when the clouds are opaque ($q = 1$), and all $j > 1$ terms vanish. The nearest cloud obscures the more distant ones, the anisotropy is determined only by the velocity of the nearest cloud, $\delta T/T = v_i$, and

$$\left\langle \left( \frac{\delta T}{T} \right)^2 \right\rangle = \sigma_v^2.$$  \hspace{1cm} (61)

Let us now return to the more interesting case when $q < 1$. Squaring the expression in equation (59), and averaging, we get

$$\left\langle \left( \frac{\delta T}{T} \right)^2 \right\rangle = \sigma_v^2 \sum_{i=1}^\infty \sum_{j=1}^\infty (1-q)^{i+j-2} \langle \Pi \rangle_{ij},$$  \hspace{1cm} (62)

where $\langle \Pi \rangle_{ij} = \sigma_v^2$ and for $j > i$ the mean of the velocity autocorrelation function (eq. [56]) for the Poisson distribution in the distance between cloud intersections $i$ and $j$ (eq. [57]) is

$$\langle \Pi \rangle_{ij} = \sigma_v^2 \int_0^{\infty} \frac{du}{r_f} \left( \frac{u}{r_f} \right)^{j-i-1} \frac{1}{(j-i-1)!} e^{-u(1/r_f+1/r_v)} = \frac{\sigma_v^2}{(1 + r_f/r_v)^{j-i}}.$$  \hspace{1cm} (63)

With this expression the sums in equation (62) are readily evaluated to get

$$\left\langle \left( \frac{\delta T}{T} \right)^2 \right\rangle = \sigma_v^2 q \left[ 1 + \frac{2(1-q)(r_f/r_v)}{1+q(r_f/r_v)} \right].$$  \hspace{1cm} (64)

The first term in the square brackets comes from the sum of the squared terms in equation (62) and the second from the sum of the cross terms.

If the velocity field is uncorrelated, $r_v = 0$, the contribution from the cross terms vanishes and equation (64) reduces to equation (28). The cross terms also vanish if the nearest cloud along the line of sight is opaque, $q = 1$, and the anisotropy reduces to the “single cloud” limit (eq. [61]), independent of the velocity correlation length. The “single cloud” limit also applies when the velocity correlation length is large, $q r_v \gg r_f$, because the photon mean free path samples a single coherently moving set of clouds. More generally, we see that a positive velocity correlation increases the rms temperature perturbation.

We now note finally that one can take account of the Hubble expansion by replacing equation (57) with an inhomogeneous Poisson process,

$$dp(t) = \frac{dt}{t_f(t)} \left[ \int_t^\infty dt'/t_f(t') \right]^{i-1} \frac{1}{(i-1)!} e^{-\int_t^\infty \frac{dt'}{t_f(t')}} \left( - \int_t^\infty \frac{dt'}{t_f(t')} \right)^i,$$  \hspace{1cm} (65)

from which it is an interesting exercise to rederive equations (43) and (44).

5. NUMERICAL EXAMPLES

Since we know very little about the history of structure formation at redshifts $z \gtrsim 5$, a good strategy is to consider examples of what might have happened and how it would have affected observables such as the CBR temperature fluctuations. To keep
our discussion of possibilities simple and definite, we choose one set of cosmological parameters,

\[ \Omega = 0.2, \quad \Omega_b = 0.05, \quad h = 0.7 = \frac{H_0}{100 \text{ km s}^{-1} \text{ Mpc}^{-1}}. \]  

(66)

The low value of \( \Omega \) is in line with the observational evidence (Freedman et al. 1994; Peebles 1997b; Perlmutter et al. 1998; and references therein), and cosmogonies with the early structure formation we have assumed in this analysis seem to be most promising in a low-density cosmological model (Peebles 1997a). To emphasize the scattering effect, we have adopted a baryon density parameter near the high end of the range now under discussion (Copi, Schramm, & Turner 1995). Following Navarro, Frenk, & White (1996), we assume that at any epoch matter is concentrated in clouds with density contrast

\[ \delta \sim 200. \]  

(67)

Then a cloud mass \( M \) fixes a characteristic cloud diameter,

\[ d_{cl} = [6M/(\bar{\rho}_s \delta)]^{1/3}, \]  

(68)

where \( \bar{\rho}_s \) is the mean mass density at the scattering epoch \( z_s \). A measure of the physical length scale of nonlinear mass fluctuations is

\[ d_{nl} = \frac{15 \text{ Mpc}}{h(1 + z_s)} \left( \frac{D_x}{D_0} \right)^{2/3 + m}. \]  

(69)

Here the growing solution for the density contrast in linear perturbation theory is \( D(t) \propto \delta \rho / \rho \). A sphere with diameter \( d_{nl} \) contains cloud mass \( \sim M \). We have normalized to the rms fluctuation of galaxy counts, \( \delta N/N \sim 1 \) in a sphere of diameter 15 \( h^{-1} \) Mpc. The assumption that galaxies trace mass agrees with our low value for the density parameter \( \Omega \). Equation (69) is scaled by the power-law power spectrum

\[ P \propto k^m, \]  

(70)

to the epoch \( z_s \) using the linear perturbation theory growth factor for the density contrast, \( \delta \rho / \rho \propto D(t) \). Finally, we take the present rms peculiar velocity to be

\[ v_0 = 600 \text{ km s}^{-1}, \]  

(71)

and we scale this velocity with time by the linear perturbation theory relation \( v \propto a dD/dt \).

We consider first the effect of scattering by clouds at epoch \( 1 + z_s = 10 \). In a cosmologically flat model the density and velocity growth factors are

\[ 1 + z_s = 10, \quad D_x/D_0 = 0.14, \quad v_s/v_0 = 0.49. \]  

(72)

For a mass characteristic of the central part of a large galaxy (eqs. [69] and [70]), the model parameters are

\[ M_{cl} = 1 \times 10^{11} M_\odot, \quad m = -1.4, \quad r_f = 4 \text{ Mpc}, \quad d_{cl} = 33 \text{ kpc}, \quad y = 2.5, \quad \theta_{cl} = 6.4. \]  

(73)

Here \( r_f \) is the mean free path for intersection of clouds, \( d_{cl} \) and \( \theta_{cl} \) are the cloud diameter and the angle it subtends, and \( y \) is the usual angle size distance parameter. (For comparison, note that for \( \lambda = 0 \) and \( z_s \gg \Omega^{-1} \), the angular size distance is \( y = 2/\Omega \).) The shot noise contribution to the CBR anisotropy (eqs. [33] and [40]) is

\[ \delta T/T(\theta = 1) = 2 \times 10^{-6}. \]  

(74)

We are assuming all baryons are in optically thin plasma, so we can neglect the optical depth for scattering back to this redshift is \( \bar{\tau} = 0.1; \) the probability of scattering per cloud is \( \tau_s = 0.004. \) The contribution to the Compton-Thompson parameter in the CBR spectrum by the motions of the galaxies is \( y_c \sim \bar{\tau} v_s^2 = 1 \times 10^{-7} \), well within the COBE bound (Fixsen et al. 1996; note that \( v_s \), here is expressed in units of the velocity of light; in our units \( c = 1 \)).

With the parameters in the above example the mean baryon density within a cloud is \( n \sim 0.06 \) protons \( \text{cm}^{-3} \), and the net mass density is equivalent to about 0.2 protons \( \text{cm}^{-3} \). These numbers may not be unreasonable for a protogalaxy. The power-law index in equation (73) is comparable to that of the most recent version of the isocurvature CDM model, \( m = -1.8 \) (Peebles 1997a). If the characteristic cloud mass is reduced to \( M_{cl} = 1 \times 10^{10} M_\odot \) with all other parameters the same, we get \( m = -1.8 \) and \( \delta T/T = 6 \times 10^{-7} \) in a window of 1 degree. In an open universe (with zero cosmological constant) with \( M_{cl} = 1 \times 10^{11} M_\odot \) and all other parameters the same, \( \delta T/T \) is 1.5 times the value in equation (74).

The relative size of the contribution to \( \delta T/T \) by correlated motions is determined by equation (53). With the numbers in equation (73) the scattering probability in a cloud is \( q = 0.004 \), so if the parameter \( g(r_c/z_s r_f)^2 \) were of order unity, the comoving velocity coherence length would have to be \( r_c \sim 700 \) Mpc, much larger than that suggested by the APM power spectrum (eq. [51]; Baugh & Gaztañaga 1996) or even the most radical interpretation of bulk flow observations. That is, \( \delta T/T \) in our model is dominated by the shot noise.

At a larger redshift in a cosmologically flat model,

\[ 1 + z_s = 30, \quad D_x/D_0 = 0.047, \quad v_s/v_0 = 0.28, \]  

(75)

...
the probability of scattering is $\tau_s = 0.007$ per cloud, the mean optical depth is $\bar{\tau} = 0.6$ if all baryons are in optically thin plasma, and a cloud model is

$$M_{cl} = 1 \times 10^9 \, M_\odot, \quad m = -1.4, \quad r_f = 300 \, \text{kpc}, \quad d_{cl} = 2.4 \, \text{kpc}, \quad y = 3.1, \quad \theta_{cl} = 1.1',$$

and

$$\frac{\delta T}{T}(\theta = 1') = 5 \times 10^{-7}. \quad (76)$$

At still larger redshift in a cosmologically flat model,

$$1 + z_s = 50, \quad D_s/D_0 = 0.028, \quad v_s/v_0 = 0.22,$$

we get $\tau_s = 0.004$, $\bar{\tau} = 1$, and a cloud model is

$$M_{cl} = 1 \times 10^7 \, M_\odot, \quad m = -1.7, \quad r_f = 40 \, \text{kpc}, \quad d_{cl} = 0.3 \, \text{kpc}, \quad y = 3.3, \quad \theta_{cl} = 0.2'. \quad (79)$$

Here the contribution to the CBR anisotropy is still smaller,

$$\frac{\delta T}{T}(\theta = 1') = 7 \times 10^{-8}. \quad (78)$$

and the contribution to the Compton-Thompson parameter is $y_c \sim 1 \times 10^{-7}$.

The temperature fluctuations in these examples are well below the measured bounds (Readhead et al. 1989; Fomalont et al. 1993; Subrahmanian et al. 1993; Church et al. 1997; Partridge et al. 1997; Andreani 1994). In Figure 1 we plot power spectra for CBR anisotropies for the above three models, calculated from equation (39). One sees that the measurements that resolve the clouds should detect secondary temperature fluctuations as large as primary anisotropies at their maximum.

6. DISCUSSION

The simplifying assumption for this analysis is that structure formation is so well advanced at the epoch of last scattering of the CBR by free electrons that the mean distance $t_f$ between intersections of clouds along a line of sight is large compared to the scale of nonlinear mass fluctuations. This allows us to model the clouds as an inhomogeneous random Poisson process determined by the mass density and peculiar velocity fields smoothed through a window of width $t_f$, and it leads to the shot noise contribution to the small-scale CBR anisotropy in equation (33). This approach is motivated by the isocurvature CDM model for structure formation (Peebles 1997a), in which structure formation could commence at decoupling at redshift $z \sim 1000$. A second important motivation has been to complement the usual perturbative analysis of the effect of the nonlinear growth of small-scale structure. The similarity of results from the perturbative (Persi et al. 1995) and nonperturbative approaches leads us to believe we have reliable methods for estimating the effect of early nonlinear structure formation on the CBR anisotropy.

The observations of young galaxies and the intergalactic medium at $z \sim 3$ indicate a situation intermediate between the perturbative and nonperturbative cases. The damped Ly$\alpha$ systems contain a significant baryon fraction, and the mean distance between intersections of these clouds is large (at $z = 3$ it is longer than the Hubble length). There also is a significant baryon fraction in the Ly-$\alpha$ forest, and these clouds have a relatively short mean free distance. It is not unreasonable to
speculate that the situation at much larger redshifts similarly calls for a combination of the two approaches to the analysis of the angular distribution of the CBR.

The CBR anisotropy produced by the Sunyaev-Zeldovich (1970) effect of the hot electrons in clusters of galaxies, which certainly is dominated by the shot noise term, offers an important constraint on the epoch of collection of the intracluster plasma. The evidence from the analysis of Persi et al. (1995) is that this constraint does not yet rule out the early structure formation picture. And our conclusion from the numerical examples in §5 is that within presently known observational constraints, structure formation could have commenced when the universe was optically thick to scattering of the CBR.

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**APPENDIX**

**THE THOMSON SCATTERING WINDOW FUNCTION**

The integral in equation (49) can be expressed in terms of two hypergeometric functions (Ryzhik & Gradshteyn 1994, hereafter RG, eq. [6.621.1]),

\[
W(\kappa) = (1 - \kappa^2)^{-1/2} \left[ \frac{\Gamma(1/2 + \kappa^2)}{\Gamma(1 + \kappa^2)} \right] \frac{1}{\kappa^2} F \left( \begin{array}{cc} 1/2 & 3/2 \\ 1 + \kappa^2 \end{array} ; \frac{\kappa^2}{1 + \kappa^2} \right).
\]  

(A1)

One of the hypergeometric functions turns out to be an elementary function in disguise (RG, eq. [9.121.26]):

\[
F \left( \begin{array}{cc} 1/2 & 3/2 \\ 1 + \kappa^2 \end{array} ; \frac{\kappa^2}{1 + \kappa^2} \right) = \frac{\arcsin z}{z},
\]  

(A2)

while the other can also be expressed in terms of elementary functions by differentiating equation (A2), and using the standard recurrence relation (Abramowitz & Stegun 1972, eq. [15.2.7])

\[
\frac{\partial}{\partial z} \left[ (1 - z)^{a} F(a, b; c; z) \right] = -\frac{(c - b)}{c} (1 - z)^{a - 1} F(a + 1, b; c + 1; z)
\]  

(A3)

for \((a, b, c) = (1/2, 1/2, 1/2)\). The final result of all this is equation (50).

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