Some effects of nonlinear vacuum electrodynamics in strong magnetic and gravitational fields of the pulsar

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Abstract. We consider the propagation of X-ray and gamma ray emissions in strong magnetic and gravitational fields of the pulsar in nonlinear vacuum electrodynamics. We show that the radiation will spread from the pulsar to the detecting device in the form of two normal modes polarized in mutually orthogonal planes, and having different velocities. We have calculated the delay between the two modes, as they propagate from the pulsar to the detecting device.

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1 Introduction

The theory [1] and the experiment [2] shows that electrodynamics in vacuum is nonlinear. Therefore, researches [3-10] on various manifestations of nonlinearity in electrodynamics are of undoubted interest.

In a number of recent studies [11-19] various effects of nonlinear vacuum electrodynamics has been considered along with the possibility of their measurement in the laboratory. However, due to the fact that magnetic fields that can be created in the laboratory $B \sim 10^5$ G, are significantly smaller than the quantum value $B_q = \frac{m^2 c^3}{(e \hbar)} = 4.41 \cdot 10^{13}$ G, their observation will only be possible in the future, after further development of measurement technology. Therefore, at present the main interest lies in the astrophysical effects of nonlinear vacuum electrodynamics occurring in the magnetic fields $B \sim B_q$ of pulsars and collapsars, which were studied in [20-22]. Calculations have shown that the beams of electromagnetic waves in strong magnetic fields of pulsars must bend [22] and, moreover, passing through the magnetic field of a pulsar must [23] undergo nonlinear-electrodynamic birefringence, i.e. it should be split into two normal modes with mutually orthogonal polarizations and propagate as two non-matching beams with different speeds. As shown in [24-26], in some special cases (propagation of electromagnetic wave in the planes of magnetic equator and magnetic meridian of pulsars), the main non-linear electrodynamics’ effect happens to be [26-27] the speed difference between the normal modes in magnetic field of a pulsar. As a result of this effect, the two electromagnetic signals radiated at the same time from the same source, but polarized in mutually perpendicular planes, arrive to a detector through the different beams at different times $t_2 \neq t_1$. For observation of this effect detectors of electromagnetic radiation of pulsars must be equipped with devices that would measure the polarization state of the radiation. In order to successfully conduct these measurements, detailed calculation of the laws of propagation of electromagnetic waves in magnetic and gravitational fields of a pulsar in the most generic case will be required.

However, in papers published earlier [24-25, 27], such a calculation was carried out only for a few simple cases. So let us make a calculation of this effect in the most generic case, when the beam of electromagnetic wave enters the magnetic field of a pulsar in a random direction. As in these papers, we consider hard radiation (X-ray and gamma spectrum) for which the distorting effect of the pulsar’s magnetosphere is negligible.
2 The equations of nonlinear vacuum electrodynamics and gravitation

Consider nonlinear post-Maxwell electrodynamics, which is a direct consequence of quantum electrodynamics [1]. Its Lagrangian [28] has the form:

\[
L = \sqrt{-g} \left\{ 2I_2 + \xi \left[ (\eta_1 - 2\eta_2)I_2^2 + 4\eta_2I_4 \right] \right\} - \frac{\sqrt{-g}}{c} j^\beta A_\beta,
\]

where \( j^\beta \) is the current density four-vector, \( g \) - determinant of the metric tensor, \( \xi = 1/B^2 \), \( I_2 = F_{\beta\sigma}F^{\beta\sigma} \) and \( I_4 = F_{\beta\sigma}F^{\sigma\nu}F_{\nu\mu}F^{\mu\beta} \) - invariants of the electromagnetic tensor \( F_{\beta\sigma} \) and according to quantum electrodynamics \( \eta_1 = e^2/(45\pi\hbar c) = 5.1 \cdot 10^{-5} \), \( \eta_2 = 7e^2/(180\pi\hbar c) = 9.0 \cdot 10^{-5} \).

The field equations derived from this Lagrangian have the form:

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\beta} \left\{ \sqrt{-g} Q^{\sigma\beta} \right\} = -\frac{4\pi}{c} j^\sigma,
\]

(2.1)

\[
Q^{\sigma\beta} = 8\pi \frac{\partial L}{\partial F^{\sigma\beta}} = \left\{ 1 + \xi (\eta_1 - 2\eta_2)I_2 \right\} F^{\sigma\beta} + 4\xi \eta_2 F^{\sigma\nu}F_{\nu\mu}F^{\mu\beta}.
\]

The second pair of equations of electrodynamics coincides with the corresponding equations of Maxwell’s theory:

\[
\frac{\partial F_{\mu\beta}}{\partial x^\nu} + \frac{\partial F_{\nu\mu}}{\partial x^\beta} + \frac{\partial F_{\nu\beta}}{\partial x^\mu} = 0.
\]

(2.2)

Metric tensor in equations (2.2) satisfy Einstein equations [29]:

\[
R_{\beta\sigma} - \frac{1}{2} g_{\beta\sigma} R = -\frac{8\pi G}{c^4} T_{\beta\sigma},
\]

(2.3)

where \( R_{\beta\sigma} = R^{\nu}_{\beta\nu} \) – Ricci tensor, \( T_{\beta\sigma} \) – energy-momentum tensor of the matter and all fields, including electromagnetic. The system of equations (2.1) - (2.3) in our problem will be sought by the method of successive approximations with a precision linear in the small dimensionless parameters: the gravitational potential and post-Maxwell amendments. The gravitational field of the pulsar will be assumed to be spherically symmetric, and in the harmonic Fock coordinates [29] metric will be expanded in the small parameter \( \alpha/r \) with the required accuracy:

\[
g_{00} = 1 - \frac{2\alpha}{r}, \quad g_{rr} = -1 - \frac{2\alpha}{r}, \quad g_{\theta\theta} = r^2 g_{rr}, \quad g_{\phi\phi} = g_{\theta\theta} \sin^2 \theta,
\]

(2.4)

where \( \alpha = \gamma M/c^2 \), \( \gamma \) – gravitational constant, and \( M \) – mass of the pulsar.

Suppose that at time \( t = 0 \) from the point \( r = r_0 \) of the pulsar magnetosphere hard radiation impulse was emitted. Then, in magnetic field of the pulsar that impulse, because of birefringence, will split [30] into two impulses with orthogonal polarizations and moving at different speeds.

For the convenience of further calculations, we introduce the spherical coordinate system as follows. Consider a beam of the first normal mode and draw a tangent to it at the point \( r = r_0 \). Axis of the spherical coordinate system will be directed in such a way, so that the tangent to the chosen beam and the center of the pulsar would be lying in the same plane, and \( \theta = \pi/2 \), and the azimuthal coordinate \( \phi \) of the source of hard radiation would be equal to \( \phi = 0 \).
Without loss of generality, we assume that in this coordinate system, the vector of the magnetic dipole moment of the pulsar $\mathbf{m}$ is directed to a point with spherical coordinates $\theta_0$ and $\phi_0$. Then the Cartesian components of the magnetic dipole moment $\mathbf{m}$ take the form:

$$m_x = |\mathbf{m}| \sin \theta_0 \cos \phi_0, \quad m_y = |\mathbf{m}| \sin \theta_0 \sin \phi_0, \quad m_z = |\mathbf{m}| \cos \theta_0.$$ 

As it is accepted [31] in the problems of celestial mechanics, instead of the radial coordinate $r$ we introduce the coordinate $u = 1/r$. Then, the non-zero components of the dipole electromagnetic field tensor of the pulsar, in the coordinate system $u, \theta, \phi$, with the required for our purposes accuracy will be:

$$F_{u\theta} = -F_{\theta u} = |\mathbf{m}| \sin \theta_0 \sin(\phi - \phi_0),$$

$$F_{u\phi} = -F_{\phi u} = |\mathbf{m}| \sin \theta \left[ \sin \theta_0 \cos \theta \cos(\phi - \phi_0) - \sin \theta \cos \theta_0 \right],$$

$$F_{\phi \theta} = -F_{\theta \phi} = 2|m|u \sin \theta \left[ \sin \theta_0 \sin \theta \cos(\phi - \phi_0) + \cos \theta \cos \theta_0 \right].$$

(2.5)

In electrodynamics when solving most of the problems the eikonal method [29-30, 32-33] is used, which allows studying the motion of electromagnetic impulses by their beams. Application of this method to nonlinear electrodynamics has shown [30,34] that the propagation of electromagnetic waves in external electromagnetic and gravitational fields in nonlinear electrodynamics with field equations (2.1) - (2.3) is equivalent to the propagation of the normal modes through the isotropic geodesics in effective space-time for which metric tensor $G_{\nu \mu}^{\text{eff}(1,2)}$ has the form:

$$G_{\nu \mu}^{\text{eff}(1,2)} = g_{\nu \mu} - 4\eta_{(1,2)} \xi F_{\nu \beta} g^{\beta \sigma} F_{\sigma \mu}. \quad (2.6)$$

Therefore the study of the laws of propagation of electromagnetic impulses in magnetic (2.5) and gravitational (2.4) fields of a pulsar is conveniently carried out not by using equations (2.1) - (2.2), but based on the analysis of isotropic geodesics in space-time with the metric tensor (6).

Let us substitute expressions (2.4) and (2.5) into (2.6) and write down the components of the metric tensor of the effective space-time $G_{\nu \mu}^{\text{eff}(1,2)} \equiv G_{\nu \mu}^{(1,2)}$ explicitly:

$$G_{\nu \mu}^{(1,2)} = 1 - 2\alpha u,$$

$$G_{u u}^{(1,2)} = -\frac{1 + 2\alpha u}{u^2} - 4m^2 \xi \eta_{1,2} u^4 \left\{ \sin^2 \theta_0 \sin^2(\phi - \phi_0) + \left[ \sin \theta_0 \cos \theta \cos(\phi - \phi_0) - \sin \theta \cos \theta_0 \right]^2 \right\},$$

$$G_{u \theta}^{(1,2)} = 8m^2 \xi \eta_{1,2} u^3 \left[ \sin \theta_0 \sin \theta \cos(\phi - \phi_0) + \cos \theta \cos \theta_0 \right] \left[ \sin \theta_0 \cos \theta \cos(\phi - \phi_0) - \sin \theta \cos \theta_0 \right],$$

$$G_{u \phi}^{(1,2)} = -8m^2 \xi \eta_{1,2} u^3 \left[ \sin \theta_0 \sin \theta \cos(\phi - \phi_0) + \cos \theta \cos \theta_0 \right] \sin \theta \sin \theta_0 \sin(\phi - \phi_0),$$

$$G_{\theta \theta}^{(1,2)} = \frac{1 + 2\alpha u}{u^2} - 4m^2 \xi \eta_{1,2} u^4 \left\{ \sin^2 \theta_0 \sin^2(\phi - \phi_0) + 4\left[ \sin \theta_0 \sin \theta \cos(\phi - \phi_0) + \cos \theta \cos \theta_0 \right]^2 \right\},$$

$$G_{\theta \phi}^{(1,2)} = -4m^2 \xi \eta_{1,2} u^4 \left[ \sin \theta_0 \cos \theta \cos(\phi - \phi_0) - \sin \theta \cos \theta_0 \right] \sin \theta \sin \theta_0 \sin(\phi - \phi_0),$$

$$G_{\phi \phi}^{(1,2)} = -\left\{ \frac{1 + 2\alpha u}{u^2} + 4m^2 \xi \eta_{1,2} u^4 \left\{ \sin \theta_0 \cos \theta \cos(\phi - \phi_0) - \sin \theta \cos \theta_0 \right]^2 - 4\left[ \sin \theta_0 \sin \theta \cos(\phi - \phi_0) + \cos \theta \cos \theta_0 \right]^2 \right\} \sin^2 \theta. \quad (2.7)$$
Equations for isotropic geodesics in the effective space-time with the metric tensor (2.7) will be written in the form where differentiation is performed not with respect to the affine parameter $\sigma$, but with respect to the azimuthal angle $\phi$:

$$\frac{d^2 x^0}{d\phi^2} + \left\{ \Gamma^0_{\beta \mu} - \frac{dx^0}{d\phi} \Gamma^3_{\beta \mu} \right\} \frac{dx^\beta}{d\phi} \frac{dx^\mu}{d\phi} = 0,$$

$$\frac{d^2 u}{d\phi^2} + \left\{ \Gamma^1_{\beta \mu} - \frac{du}{d\phi} \Gamma^3_{\beta \mu} \right\} \frac{dx^\beta}{d\phi} \frac{dx^\mu}{d\phi} = 0,$$

$$\frac{d^2 \theta}{d\phi^2} + \left\{ \Gamma^2_{\beta \mu} - \frac{d\theta}{d\phi} \Gamma^3_{\beta \mu} \right\} \frac{dx^\beta}{d\phi} \frac{dx^\mu}{d\phi} = 0,$$

where $\Gamma^\nu_{\beta \mu}$ - Christoffel symbols defined in effective space-time with the metric tensor (2.7).

The system of equations (2.8) has a first integral:

$$G_{(1,2)}^{\beta \mu} \frac{dx^\beta}{d\phi} \frac{dx^\mu}{d\phi} = 0. \tag{2.9}$$

Equations (2.8) and (2.9) are non-linear, but in our case there are small parameters $\alpha u$ and $m^2 \xi \eta_{1,2} \mu^6$. Therefore, the solution of these equations will be sought by the method of successive approximations in these small parameters. Since the angles of nonlinear-electrodynamic bending of the beams of the first and second normal modes are different even in the meridian and equatorial planes [24-26], the boundary conditions for them should be different to ensure that the two impulses arrive in the same detector located on the near-Earth spacecraft [35]

Let us require the chosen beam of the first normal mode to start at $u = u_0 = 1/r_0$, $\theta = \pi/2$, $\phi = 0$ at $t = 0$, and the coordinate $u$ at the pericenter to be equal to some given value $u_p = 1/r_p > u_0$, where $r_0$ - distance from the center of the pulsar to the source of high-energy radiation, $r_p$ - distance from the pulsar to the pericenter of the beam. We also assume that $r_p > R_n$, where $R_n$ - radius of the neutron star.

Another condition follows from the formulation of the problem: if at chosen orientation of the coordinate axes the beam starts from the point $\phi = 0$, $u = u_0$, $\theta = \pi/2$ touching the plane $\theta = \pi/2$. It follows that for the beam of the first normal mode at $\phi = 0$ the following must be satisfied: $d\theta/d\phi = 0$.

3 Solution of the equations for beams of the first normal mode

In the zeroth approximation in small parameters the beam under mentioned boundary conditions, will be a straight line in the plane $\theta = \pi/2$, passing through the point $u = u_0$, $\phi = 0$ and take a value $u = u_p$ at pericenter. It follows that in this approximation the expression $d\theta/d\phi = 0$ for all point of the selected beam and the system of equations (2.8)-(2.9) takes the next form:

$$\frac{d^2 x^0}{d\phi^2} = -\frac{2}{u} \left( \frac{du}{d\phi} \right) \left( \frac{dx^0}{d\phi} \right), \quad \frac{d^2 u}{d\phi^2} = -u, \quad u^4 \left( \frac{dx^0}{d\phi} \right)^2 - u^2 - \left( \frac{du}{d\phi} \right)^2 = 0.$$

Solving this system of equations with the boundary conditions, we arrive at the relations:

$$u(\phi) = u_p \sin(\phi + \psi), \quad x^0(\phi) = ct = \frac{\cos \psi}{u_p \sin \psi} - \frac{\cos(\phi + \psi)}{u(\phi)} + \frac{\cos(\phi + \psi)}{u(\phi)}, \quad \theta(\phi) = \frac{\pi}{2} \tag{3.1}$$
where $\psi$ is defined from: $\sin \psi = u_0/u_p$. Since within the magnetosphere $r_0 < 100R_n$, then the angle $\psi$ meets the conditions: $0 < \psi < \pi$.

Using an expression (3.1), we search the solution of the system of equations (2.8)-(2.9) for the first normal wave in the ordinary form for equations of that type:

$$
u(\phi) = u_p \sin(\phi + \psi) + \alpha u_p^2 \Phi_1(\phi) + m^2 \xi_1 u_p^5 \Phi_2(\phi),$$

$$\theta(\phi) = \frac{\pi}{2} + \alpha u_p \Phi_3(\phi) + m^2 \xi_1 u_p^6 \Phi_4(\phi),$$

$$x^0(\phi) = \frac{\cos \psi}{u_p \sin \psi} - \frac{\cos(\phi + \psi)}{u(\phi)} + \alpha \Phi_5(\phi) + m^2 \xi_1 u_p^5 \Phi_6(\phi), \quad (3.2)$$

where $\Phi_a(\phi)$, $a = 1 - 6$ are unknown functions of the azimuthal angle $\phi$, having zero order of smallness.

Substituting (3.2) into the left-hand sides of equations (2.8), expanding them in the small parameters to first order inclusive, we obtain the following equation for $\nu$:

$$m^2 \xi_1 u_p^7 \left\{ \Phi_2' + \Phi_2 - 6 \sin^4(\phi + \psi) \{ 2 \sin(\phi + \psi) + \left[ 14 \sin(\phi + \psi) - 
- \sin 2(\phi - \phi_0) \cos(\phi + \psi) [ 7 \sin^2(\phi + \psi) - 3 ] + 2 \sin^2(\phi - \phi_0) \sin(\phi + \psi) \times
\times [ 8 \sin^2(\phi + \psi) - 9 ] - 12 \sin^3(\phi + \psi) \} \sin^2 \theta_0 \} \right\} + \alpha u_p^2 \left\{ \Phi_6'' + \Phi_6 - 2 \right\} = 0. \quad (3.3)$$

The equation for determining the angle $\theta$ takes the form:

$$m^2 \xi_1 u_p^6 \left\{ \Phi_4' + \Phi_4 + 6 \sin(\phi + \psi) [ 3 \cos(\phi - \phi_0) \cos^2(\phi + \psi) - 
- \sin(\phi - \phi_0) \sin 2(\phi + \psi) \sin 2 \theta_0 \right\} + \alpha u_p \left\{ \Phi_3'' + \Phi_3 \right\} = 0. \quad (3.4)$$

We now write the first integral (2.9) in this approximation:

$$m^2 \xi_1 u_p^5 \left\{ - \Phi_6' - 2 \sin^4(\phi + \psi) \{ 2 \sin(\phi + \psi) \sin 2(\phi - \phi_0) + 
+ 4 \sin^2(\phi + \psi) + [ 1 - 5 \sin^2(\phi + \psi) ] \sin^2(\phi - \phi_0) \} \sin^2 \theta_0 - 
- 2 \sin^4(\phi + \psi) \cos^2 \theta_0 \right\} - 2 \alpha \{ \Phi_6' \sin(\phi + \psi) + 2 \} = 0. \quad (3.5)$$

Equation for determining the $x^0$, we will not write, as it is a consequence of (3.3)-(3.5).

Fundamental system of solutions of homogeneous equations for the system (3.3) - (3.4) can be conveniently represented in the form: $y_1 = \sin(\phi + \psi)$, $y_2 = \cos(\phi + \psi)$. Wronskian [36] of this system: $W = y_1'y_2 - y_2'y_1 = 1$. Therefore, the general solution of these equations has the form:

$$\Phi_1(\phi) = 2 + S_1 \sin \phi + C_1 \cos \phi, \quad \Phi_2(\phi) = \frac{1}{64} \left\{ f_2(\phi) + S_2 \sin \phi + C_2 \cos \phi \right\},$$

$$\Phi_3(\phi) = S_3 \sin \phi + C_3 \cos \phi, \quad \Phi_4(\phi) = \frac{\sin 2 \theta_0}{64} \left\{ f_4(\phi) + S_4 \sin \phi + C_4 \cos \phi \right\}, \quad (3.6)$$

where $S_1, S_2, S_3, S_4, C_1, C_2, C_3, C_4$ are integration constants and for the convenience of further calculations we use the notation:

$$f_2(\phi) = \sin^2 \theta_0 \left\{ \cos 2(\phi_0 + \psi) \left[ 195 \phi \cos(\phi + \psi) + 65 \sin^3(\phi + \psi) + 26 \sin^5(\phi + \psi) + 152 \sin^7(\phi + \psi) - 
- 5 \right] \right\}.$$
\[-144\sin^9(\phi + \psi) + 2\sin 2(\phi_0 + \psi)\left[72\sin^8(\phi + \psi) - 40\sin^6(\phi + \psi) - 26\sin^4(\phi + \psi) - 39\sin^2(\phi + \psi)\right] + \cos(\phi + \psi) + 39\phi(\phi + \psi) + 32\sin^7(\phi + \psi) - 24\sin^5(\phi + \psi) - 60\sin^3(\phi + \psi) - 180\phi\cos(\phi + \psi)\right] - 16\left[2\sin^5(\phi + \psi) + 5\sin^3(\phi + \psi) + 10\phi\cos(\phi + \psi)\right] - 40\sin^7(\phi + \psi) - 10\sin^5(\phi + \psi) + 3\phi(\phi + \psi) - 3\sin^2(\phi + \psi) + 2\sin^4(\phi + \psi) + 40\sin^6(\phi + \psi)\cos(\phi + \psi)\right]\cos(\phi + \psi).

Solving the equations for functions \(\Phi_5(\phi)\) and \(\Phi_6(\phi)\), belong to (3.5), we have:

\[
\Phi_5(\phi) = A_5 - 2\ln\left|\frac{1 - \cos(\phi + \psi)}{\sin(\phi + \psi)}\right|, \quad \Phi_6(\phi) = \frac{1}{64}\left\{A_6 + f_6(\phi)\right\}, \tag{3.7}
\]

where \(A_5, A_6\) are integration constants and as shorthand used:

\[
f_6(\phi) = \left\{16\sin 2(\phi_0 + \psi)\sin^6(\phi + \psi)\left[4 - 9\sin^2(\phi + \psi)\right] - \cos 2(\phi_0 + \psi)\right\}\left\{144\sin^7(\phi + \psi) + 8\sin^5(\phi + \psi) + 26\sin^3(\phi + \psi) + 39\sin(\phi + \psi)\right\}\cos(\phi + \psi) - 39\phi\right\} + 4\left[8\sin^5(\phi + \psi) + 6\sin^3(\phi + \psi) + 9\sin(\phi + \psi)\right]\cos(\phi + \psi) - 36\phi\right\}\sin^2\theta_0 + 8\left[3 + 2\sin^2(\phi + \psi)\right]\sin 2(\phi + \psi) - 48\phi.
\]

By virtue of boundary conditions the functions \(\Phi_1(\phi)\) and \(\Phi_2(\phi)\) for beams of the first normal mode should become zero at \(\phi = 0\) and \(\phi = \pi/2 - \psi\), and constants of the expressions (3.6) will take the following values:

\[
C_1 = -2, \quad S_1 = -\frac{2\cos \psi}{(1 + \sin \psi)}, \quad C_2 = -f_2(0), \quad S_2 = f_2(0)\tan \psi - \frac{1}{\cos \psi}\left\{99\cos 2(\phi_0 + \psi) + 39(\pi - 2\psi)\sin 2(\phi_0 + \psi) - 52\right\}\sin^2\theta_0 - 112\right\}. \tag{3.8}
\]

By the choice of orientation the axes of the spherical coordinate system, the boundary conditions for the functions \(\Phi_3\) and \(\Phi_4\) have next form:

\[
\Phi_3(0) = \Phi_4(0) = 0, \quad \frac{d\Phi_3}{d\phi} \bigg|_{\phi=0} = \frac{d\Phi_4}{d\phi} \bigg|_{\phi=0} = 0. \tag{3.9}
\]

Hence \(S_3 = C_3 = 0\) and therefore \(\Phi_3 = 0\). The constants \(C_4\) and \(S_4\) according to (3.9) should have the form:

\[
C_4 = -f_4(0), \quad S_4 = 5\sin(\phi_0 + \psi)\cos \psi\left[56\sin^6 \psi - 10\sin^4 \psi - 15\sin^2 \psi - 15\right] + \cos(\phi_0 + \psi)\sin \psi\left[3 - 280\sin^6 \psi + 230\sin^4 \psi - \sin^2 \psi\right]. \tag{3.10}
\]

The constants \(A_5\) and \(A_6\) are found from the condition \(\Phi_5(\phi) = \Phi_6(\phi) = 0\) at \(\phi = 0\). Then we obtain form (3.7):

\[
A_5 = 2\ln\left|\frac{1 - \cos \psi}{\sin \psi}\right|, \quad A_6 = -f_6(0) \tag{3.11}.
\]
The result of the expressions (3.2) will take the form:

\[
\begin{align*}
  u(\phi) &= u_p \sin(\phi + \psi) - 2\alpha u_p^2 \left[ \cos \phi + \frac{\sin \phi \cos \psi}{1 + \sin \psi} \right] + \\
  &+ \frac{m^2 \xi \eta u_p^2}{64} \left\{ f_2(\phi) + \left[ f_2(0) \right] \psi - \frac{1}{\cos \psi} \left\{ 99 \cos(2\phi + \psi) + \\
  &+ 39(\pi - 2\psi) \cos 2(\phi + \psi) - 52 \sin^2 \theta_0 - 112 \right\} \sin \phi - f_2(0) \cos \phi \right\}, \\
  \theta(\phi) &= \frac{\pi}{2} + \frac{m^2 \xi \eta u_p^6 \sin 2\theta_0}{64} \left\{ f_4(\phi) - f_4(0) \cos \phi + S_4 \sin \phi \right\}, \\
  x^0(\phi) &= \frac{\cos \psi}{u_p \sin \psi} - \frac{\cos(\phi + \psi)}{u(\phi)} + 2\alpha \left\{ \ln \left| \frac{1 - \cos \psi}{\sin \psi} \right| - \\
  &- \ln \left| \frac{1 - \cos(\phi + \psi)}{\sin(\phi + \psi)} \right| \right\} + \frac{m^2 \xi \eta u_p^6}{64} \left\{ f_6(\phi) - f_6(0) \right\}. \\
\end{align*}
\tag{3.12}
\]

The beam of the first normal mode after exiting the vicinity of the pulsar has to be detected by measuring device located in Earth orbit. Since the nearest pulsars locate \([37]\) at considerable distance \((r \sim 10 \text{ kps} \gg R_n)\) from the Earth, it is possible to assume that in the chosen coordinate system our measuring device has the coordinate \(u_1 = 1/r_1 \ll u_p\). This condition allows everyone to simply define the required angular coordinates \(\phi_1\) and \(\theta_1\) of the device with an aim to register the beam of the first normal mode. We assume \(\phi_1 = \pi - \psi + \beta_1\), where \(\beta_1 \ll 2\pi\).

Substituting this value of \(\phi_1\) in the equation \(u(\phi_1) = u_1\), and deriving it up to the first order with respect to \(\beta_1\), we will have:

\[
\begin{align*}
  \beta_1 &= -\frac{u_1}{u_p} + 2\alpha u_p \left[ 1 + \frac{\cos \psi}{1 + \sin \psi} \right] + \frac{m^2 \xi \eta u_p^6}{64} N_2, \\
  N_2 &= S_2 \sin \psi - C_2 \cos \psi + f_2(\phi = \pi - \psi) = \\
  &= \frac{\sin^2 \theta_0}{\cos \psi} \left\{ \sin 2(\phi_0 + \psi) \left\{ 144 \sin^8 \psi - 80 \sin^6 \psi - 52 \sin^4 \psi - 78 \sin^2 \psi \right\} \cos \psi + \\
  &+ 39(2\psi - \pi) \sin \psi \right\} + \cos 2(\phi_0 + \psi) \left\{ 152 \sin^7 \psi - 144 \sin^9 \psi + 26 \sin^5 \psi + 65 \sin^3 \psi - 99 \sin \psi + \\
  &+ 195(\psi - \pi) \cos \psi \right\} + 4(8 \sin^7 \psi - 6 \sin^5 \psi - 15 \sin^3 \psi + 13 \sin \psi + 45(\pi - \psi) \cos \psi) + \\
  &+ \frac{16}{\cos \psi} \left\{ 15(\pi - \psi) \cos \psi + 7 \sin \psi - 5 \sin^3 \psi - 2 \sin^5 \psi \right\}. \tag{3.13}
\end{align*}
\]

From the second expression (21) one can obtain \(\theta_1\):

\[
\theta_1 = \theta(\phi_1) = \frac{m^2 \xi \eta u_p^6}{64} N_4 \sin 2\theta_0,
\]

\[
N_4 = S_4 \sin \psi + f_4(\phi = \pi - \psi) + f_4(\phi = 0) = 5 \sin(\phi_0 + \psi) \left\{ 48 \sin^7 \psi - 8 \sin^5 \psi - \\
- 10 \sin^3 \psi - 15 \sin \psi \right\} \cos \psi + 15(\psi - \pi) \right\} + 48 \cos(\phi_0 + \psi) \left\{ 4 \sin^6 \psi - 5 \sin^8 \psi \right\}. \tag{3.14}
\]

This implies that the gravitational field bends the beams only in one plane.
4 Solution of the equations for beams of the second normal mode

For the beam on which the pulse propagates carried by the second normal mode, the expressions (3.2) take the form:

\[
\begin{align*}
    u(\psi) &= u_p \sin(\psi + \phi) + \alpha u_p^2 \Phi_1(\psi) + m^2 \xi_2 u_p \Phi_2(\psi), \\
    \theta(\psi) &= \frac{\pi}{2} + \alpha u_p \Phi_3(\psi) + m^2 \xi_2 u_p^6 \Phi_4(\psi), \\
    x^0(\psi) &= \frac{\cos \psi}{u_p \sin \psi} - \frac{\cos(\phi + \psi)}{u(\psi)} + \alpha \Phi_5(\psi) + m^2 \xi_2 u_p^5 \Phi_6(\psi),
\end{align*}
\]

with the same functions \(\Phi_a(\psi)\), which were used for the first normal mode (3.6)-(3.7).

Integration constants for beam of the second normal mode are defined from boundary conditions: at \(\phi = 0\) and \(t = 0\) the beam should begin at the point \(u = u_0\), \(\theta = \pi/2\) and asymptotically go to spatial infinity \((r \to \infty, u \to 0)\). Therefore, the integration constants \(C_1, C_2, C_3, C_4, A_5\) and \(A_6\) will be defined by equations (3.8), (3.10)-(3.11).

For the aim of finding values of the integration constants \(S_1, S_2, S_3\) and \(S_4\), we should define the angle \(\phi = \phi_2\), at which \(u = u_1\). Substituting \(\phi = \phi_2 = \pi - \psi + \beta_2\) in the first equation of (4.1), and equating it to \(u_1\), we obtain:

\[
\beta_2 = \frac{u_4}{u_p} + \alpha u_p \left[2 + 2 \cos \psi + S_1 \sin \psi\right] + \frac{m^2 \xi_2 u_p^6}{64} N_{22},
\]

\[
N_{22} = S_2 \sin \psi - C_2 \cos \psi + f_2(\phi = \pi - \psi) = S_2 \sin \psi + \\
+ \sin^2 \theta_0 \left[2 \sin 2(\phi_0 + \psi) \left[112 \sin^8 \psi - 72 \sin^{10} \psi - 14 \sin^6 \psi + 13 \sin^4 \psi - 39 \sin^2 \psi\right]\right] + \\
+ \cos 2(\phi_0 + \psi) \left[(152 \sin^7 \psi - 144 \sin^9 \psi + 26 \sin^5 \psi + 65 \sin^3 \psi) \cos \psi + 195(\psi - \pi)\right] + 2 \left[16 \sin^7 \psi - 12 \sin^5 \psi - 30 \sin^3 \psi\right] \cos \psi - 180(\psi - \pi) - 16 \sin^3 \psi[5 + 2 \sin^2 \psi] \cos \psi - 240(\psi - \pi). \quad (4.2)
\]

One can use now \(\phi = \phi_2 = \pi - \psi + \beta_2\) in the second equation of (4.1). It is simple to show that

\[
\theta_2 = \theta(\phi_2) = \frac{\pi}{2} + \alpha u_p S_3 \sin \psi + \frac{m^2 \xi_2 u_p^6}{64} N_{44} \sin 2\theta_0,
\]

\[
N_{44} = S_4 \sin \psi + f_4(\phi = \pi - \psi) + f_4(\phi = 0) = S_4 \sin \psi - 5 \sin(\phi_0 + \psi) \left[8 \sin^7 \psi - 2 \sin^5 \psi - 5 \sin^3 \psi\right] \cos \psi - 15(\psi - \pi) + \cos(\phi_0 + \psi) \left[40 \sin^8 \psi - 38 \sin^6 \psi + 3 \sin^4 \psi - 3 \sin^2 \psi\right]. \quad (4.3)
\]

Since at spatial infinity both beams have to get to the measuring device, the conditions which must be satisfied are the next: \(\beta_1 = \beta_2, \theta_1 = \theta_2\). By substituting in these expressions the equations (3.13)-(3.14) and (4.2)-(4.3), we obtain:

\[
S_1 = -\frac{2 \cos \psi}{(1 + \sin \psi)}, \quad S_2 = 0,
\]

\[
S_2 = \frac{u_p \eta_1}{u_0 \eta_2} N_2 - \frac{u_p \sin^2 \theta_0}{u_0} \left[2 \sin 2(\phi_0 + \psi) \left[112 \sin^8 \psi - 72 \sin^{10} \psi - 14 \sin^6 \psi + \right.ight. \\
+ \left.13 \sin^4 \psi - 39 \sin^2 \psi\right] + \cos 2(\phi_0 + \psi) \left[152 \sin^7 \psi - 144 \sin^9 \psi + 26 \sin^5 \psi + \right.ight.
\]
\[+65 \sin^3 \psi \cos \psi + 195(\psi - \pi) + 2\left[16 \sin^7 \psi - 12 \sin^5 \psi - 30 \sin^3 \psi\right] \cos \psi -
-180(\psi - \pi)\right]\]
\[+ \frac{16 \sin \psi}{u_0}\left[\left[5 + 2 \sin^2 \psi\right] \cos \psi \sin^3 \psi - 240(\psi - \pi)\right],\]
\[S_4 = \frac{u_p \eta_1}{u_0 \eta_2} N_4 + \frac{u_p}{u_0}\left\{5 \sin(\phi_0 + \psi)\left[8 \sin^7 \psi - 2 \sin^5 \psi - 5 \sin^3 \psi\right] \cos \psi -
-15(\psi - \pi)\right\} - \cos(\phi_0 + \psi)\left[40 \sin^8 \psi - 38 \sin^6 \psi + \sin^4 \psi - 3 \sin^2 \psi\right]\].

Thus, all integration constants for beam of the second mode are defined.

5 Calculation of the delay time

The last paragraph we define a time interval \(T_{adv} = t_1 - t_2\), which one normal mode ahead of another mode in the propagation of an electromagnetic pulse from the source to the measurement device. Using expressions (11) and (24), we obtain:

\[T_{adv} = \frac{m^2 \xi(n_1 - n_2) u_p^5}{64} \left\{\sin^2 \theta_0 \left[16 \sin 2(\phi_0 + \psi)\left[9 \sin^8 \psi - 4 \sin^6 \psi\right] + \cos 2(\phi_0 + \psi)\right](144 \sin^7 \psi +
+8 \sin^5 \psi + 26 \sin^3 \psi + 39 \sin \psi) \cos \psi + 39(\pi - \psi)\right] - 4\left(8 \sin^5 \psi +
+6 \sin^3 \psi + 9 \sin \psi\right) \cos \psi + 36(\psi - \pi)\right\} - 16\left[2 \sin^3 \psi + 3 \sin \psi\right] \cos \psi + 48(\psi - \pi)\right\}.\] (5.1)

We estimate the numerical value of \(T_{adv}\) in the case where the magnetic field source is a neutron star with field on a surface \(B \sim 10^{16}\) G (magnetar [38]). Due to the condition \(\xi m^2/r^6 << 1\), which must be satisfied for all points of considered beams, our calculation will be applicable only to the beams, for which the pericenter \(r_p\) exceeds ten radii of the neutron star. In this spatial region \(B(r) < 10^{13}\) G and \(\xi m^2/r^6 \leq 0.05\). Taking into account that the radius of a typical magnetar is 10 km, from the expression (5.1) we obtain in order of magnitude the value: \(T_{adv} \sim 10^{-8}\) sec.

6 Conclusion

The calculations show that the effect of delay has good prospects for its experimental measurement using spacecrafts. Therefore, it is necessary to conduct such experiment in the future exoatmospheric investigations of hard radiation coming from the neutron stars, in the interests of fundamental physics.

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