INTEGRAL BASES OF CLUSTER ALGEBRAS AND REPRESEN TATIONS OF TAME QUIVERS

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Abstract. In [CK] and [SZ], the authors constructed the bases of cluster algebras of finite types and of type \( A_1,1 \), respectively. In this paper, we will deduce \( \mathbb{Z} \)-bases for cluster algebras of affine types.

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1. INTRODUCTION

Cluster algebras were introduced by S. Fomin and A. Zelevinsky [FZ] in order to develop a combinatorial approach to study problems of total positivity in algebraic groups and canonical bases in quantum groups. The link between acyclic cluster algebras and representation theory of quivers were first revealed in [MRZ]. In [BMRRRT], the authors introduced the cluster categories as the categorification of acyclic cluster algebras. Let \( Q \) be an acyclic quiver with vertex set \( Q_0 = \{1, 2, \ldots, n\} \). Let \( A = \mathbb{C}Q \) be the path algebra of \( Q \) and we denote by \( P_i \) the indecomposable projective \( \mathbb{C}Q \)-module with the simple top \( S_i \) corresponding to \( i \in Q_0 \) and \( I_i \) the indecomposable injective \( \mathbb{C}Q \)-module with the simple socle \( S_i \). Let \( \mathcal{D}^b(Q) \) be the bounded derived category of \( \text{mod}\mathbb{C}Q \) with the shift functor \([1]\) and the AR-translation \( \tau \). The cluster category associated to \( Q \) is the orbit category \( \mathcal{C}(Q) := \mathcal{D}^b(Q)/F \) with \( F = [1] \circ \tau^{-1} \). Let \( Q(x_1, \ldots, x_n) \) be a transcendental extension of \( \mathbb{Q} \) in the indeterminates \( x_1, \ldots, x_n \). The Caldero-Chapoton map of an acyclic quiver \( Q \) is the map

\[
X_Q^Q : \text{obj}(\mathcal{C}(Q)) \to Q(x_1, \ldots, x_n)
\]

defined in [CC] by the following rules:

Key words and phrases. \( \mathbb{Z} \)-basis, cluster algebra, tame quiver.

The research was partially supported by NSF of China (No. 10631010) and the Ph.D. Programs Foundation of Ministry of Education of China (No. 200800030058).
(1) if $M$ is an indecomposable $\mathbb{C}Q$-module, then

$$X^Q_M = \sum_{e} \chi(Gr_e(M)) \prod_{i \in Q_0} x_i^{\langle e, s_i \rangle - (s_i, \dim M - e)};$$

(2) if $M = P_i[1]$ is the shift of the projective module associated to $i \in Q_0$, then

$$X^Q_M = x_i;$$

(3) for any two objects $M, N$ of $\mathcal{C}_Q$, we have

$$X^Q_{M \oplus N} = X^Q_M \cdot X^Q_N.$$ 

Here, we denote by $\langle -, - \rangle$ the Euler form on $\mathbb{C}Q$-mod and $Gr_e(M)$ is the $e$-Grassmannian of $M$, i.e. the variety of submodules of $M$ with dimension vector $e$. $\chi(Gr_e(M))$ denotes its Euler-Poincaré characteristic. For any object $M \in \mathcal{C}(Q)$, $X^Q_M$ will be called the generalized cluster variable for $M$.

We note that the indecomposable $\mathbb{C}Q$-modules and $P_i[1]$ for $i \in Q_0$ exhaust the indecomposable objects of the cluster category $\mathcal{C}(Q)$:

$$\text{ind}\mathcal{C}(Q) = \text{ind}\mathbb{C}Q \cup \{ P_i[1] : i \in Q_0 \}.$$ 

Each object $M$ in $\mathcal{C}(Q)$ can be uniquely decomposed in the following way:

$$M = M_0 \oplus P_M[1]$$

where $M_0$ is a module and $P_M$ is a projective module. Let $P_M = \bigoplus_{i \in Q_0} m_i P_i$. We extend the definition of the dimension vector $\dim$ on modules in $\text{mod}\mathbb{C}Q$ to objects in $\mathcal{C}(Q)$ by setting

$$\dim M = \dim M_0 - (m_i)_{i \in Q_0}.$$ 

Let $R = (r_{ij})$ be a matrix of size $|Q_0| \times |Q_0|$ satisfying

$$r_{ij} = \dim \text{Ext}^1(S_i, S_j)$$

for any $i, j \in Q_0$. The Caldero-Chapoton map can be reformulated by the following rules (see [Xu] or [Hu1]):

(1) $X_{\tau P} = X_{P[1]} = x^{\dim P/\text{rad}P}$, $X_{\tau^{-1} I} = X_{I[-1]} = x^{\dim \text{soc} I}$ for any projective $\mathbb{C}Q$-module $P$ and any injective $\mathbb{C}Q$-module $I$;

(2) $X_M = \sum_{e} \chi(Gr_e(M)) x^{R + (\dim M - e) R'^t - \dim M}$

where $M$ is a $\mathbb{C}Q$-module, $R'^t$ is the transpose of the matrix $R$ and $x^v = x_1^{v_1} \cdots x_n^{v_n}$ for $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$.

Let $\mathcal{A}H(Q)$ be the subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by $\{ X_M, X_{\tau P} \mid M, P \in \text{mod}\mathbb{C}Q, P \text{ is projective module} \}$ and $\mathcal{E}H(Q)$ be the subalgebra of $\mathcal{A}H(Q)$ generated by $\{ X_M \mid M \in \text{ind}\mathcal{C}(Q), \text{Ext}^1_{\mathcal{C}(Q)}(M, M) = 0 \}$. 

In [CK2], the authors showed that the Caldero-Chapoton map induces a one to one correspondence between indecomposable objects in $\mathcal{C}(Q)$ without self-extension and the cluster variables of the cluster algebra $\mathcal{A}(Q)$. If $Q$ is a simply laced Dynkin quiver, $\mathcal{A}H(Q)$ coincides with $\mathcal{E}H(Q)$. In [CK], the authors showed that the set

$$\{ X_M \mid M \in \mathcal{C}(Q), \text{Ext}^1_{\mathcal{C}(Q)}(M, M) = 0 \}$$

is a $\mathbb{Z}$-basis of $\mathcal{E}H(Q)$, i.e. the cluster monomials is a $\mathbb{Z}$-basis of $\mathcal{E}H(Q)$. If $Q$ is a quiver of $\tilde{A}_{1,1}$, $\mathcal{A}H(Q)$ also coincides with $\mathcal{E}H(Q)$ (see [SZ], [CZ]). Furthermore, in [CZ], the authors gave a $\mathbb{Z}$-basis of $\mathcal{A}H(Q)$ called the semicanonical basis. If $Q$ is
a quiver of type $\tilde{D}_4$, $A\mathcal{H}(Q)$ is still equal to $E\mathcal{H}(Q)$ and a Z-basis is given in [DX1] (see also Section 6.3). Recently, in [Du2], Dupont introduced generic variables for any acyclic cluster algebra and conjectured that generic variables constitute a Z-basis for any acyclic cluster algebra. He proved that the conjecture is true for a cluster algebra of type $\tilde{A}_{p,q}$ and for any affine type, deduce the conjecture to a certain difference property. The difference property has been confirmed for type $\tilde{A}_{p,q}$ in [Du2]. At the same time, we independently constructed various Z-bases for cluster algebras of tame quivers with alternating orientations (i.e. any vertex is a sink or a source) in the first version of this paper (see [DXX]). The present paper is the strengthen version of [DXX]. It is self-contained. One need not refer to [DXX].

The main goal of the present paper is to construct various Z-bases for the cluster algebra of a tame quiver $Q$ with any acyclic orientation. Let us describe it in detail. Let $Q$ be a tame quiver. Then the underlying graph of $Q$ is of affine type $\tilde{A}_{p,q}, \tilde{D}_n (n \geq 4)$ or $\tilde{E}_m (m = 6, 7, 8)$ and $Q$ contains no oriented cycles. There are many references about the theory of representations of tame quivers, for example, see [DR] or [CB]. Define the set

$$\mathcal{D}(Q) = \{ d \in \mathbb{N}_0^{|E|} \mid \exists \text{ a regular module } T \oplus R \text{ such that } \dim(T \oplus R) = d \}.$$ 

$T$ indecomposable, $\text{Ext}^1_{C(Q)}(T, T) \neq 0, \text{Ext}^1_{C(Q)}(T, R) = \text{Ext}^1_{C(Q)}(R, R) = 0 \}$. We make an assignment, i.e., a map

$$\phi : \mathcal{D}(Q) \to \text{obj}(C(Q))$$

and set

$$X^\phi_d := X_{\phi(T \oplus R)}.$$ 

It is clear that the above assignment is not unique. For simplicity and without confusion, we omit $\phi$ in the notation $X^\phi_d$. The main theorem of this paper is as follows.

**Theorem 1.1.** Let $Q$ be a tame quiver and fix an assignment. Then the set

$$B(Q) := \{ X_L, X_{d\mathcal{F}} \mid L \in C(Q), d \in \mathcal{D}(Q), \text{Ext}^1_{C(Q)}(L, L) = 0 \}$$

is a Z-basis of $E\mathcal{H}(Q)$.

Given an assignment, we obtain a Z-basis. In fact, by Theorem 1.1, we will see that $\mathbb{Z}^{\mathcal{F}} = \mathcal{D}(Q) \cup \mathcal{E}(Q)$ where $\mathcal{E}(Q) = \{ d \in \mathbb{Z}^{\mathcal{F}} \mid \exists L \in C(Q) \text{ satisfies } \dim L = d \text{ and } \text{Ext}^1_{C(Q)}(L, L) = 0 \}$ (Corollary 5.10). Let $d = n\delta + d'$ be the canonical decomposition of $d$ (see [Kac]) and $E$ be an indecomposable regular simple module of dimension vector $\delta$. We set $T := E[n]$ to be the indecomposable regular module with quasi-socle $E$ and quasi-length $n$. Then we obtain another Z-basis

$$B'(Q) := \{ X_L, X_{E[n] \oplus R} \mid L \in C(Q), \text{Ext}^1_{C(Q)}(L, L) = 0, \text{Ext}^1_{C(Q)}(R, R) = 0 \}.$$ 

The paper is organized as follows. In Section 2, we recall various cluster multiplication formulas. In Section 3 and Section 4, we characterize the generalize cluster variables $X_M$ for $M \in C(Q)$ and a quiver $Q$ with an alternating orientation. In particular, we compare the generalized cluster variables of modules in homogeneous tubes and non-homogeneous tubes (see Proposition 4.4). Note that the proof of Proposition 4.4 does not depend on the difference property unless $Q$ is of type $\tilde{A}$. In Section 5, we prove Theorem 1.1 by using the BGP-reflection functor between cluster categories defined in [Zhu] and the construction of Z-bases for cluster algebras of type $\tilde{A}_{p,q}$ where $p \neq q$ in [Du2]. The BGP-reflection functor between cluster categories admits that one can drop the assumption of alternating orientation. We illustrate Theorem 1.1 by two examples of type $\tilde{A}_{1,1}$ and $\tilde{D}_4$ in Section 6. Section 7 shows the inductive formulas for the multiplication between generalized cluster
variables of modules in a tube. In order to compare our bases with generic variables in [Dn2], we prove the difference property for any tame quiver (Theorem S.2) in the appendix of the present paper (Section 8). A direct corollary is that the set of generic variables (denoted by $B_g(Q)$) defined in [Dn2] is a $\mathbb{Z}$-basis of a cluster algebra for affine type. There is a unipotent transition matrix from $B_g(Q)$ to $B'(Q)$ (Corollary S.3).

After the present paper appear, the latest advance is that Geiss-Leclerc-Schröer proved the above Dupont’s conjecture (in the more general context) via the construction of the dual semicanonical bases for preprojective algebras (arXiv:1004.2781).

2. THE CLUSTER MULTIPLICATION FORMULAS

In this section, we recall various cluster multiplication formulas. In Section 7, we will show inductive cluster multiplication formulas for a tube.

2.1. We recall the cluster multiplication theorem in [XX] and [Xu]. It is a generalization of the cluster multiplication theorem for finite type [CK] and for affine type [Hu1], [Hu2]. First, we introduce some notations in [XX] and [Xu]. Let $Q = (Q_0, Q_1, s, t)$ be a finite acyclic quiver, where $Q_0$ and $Q_1$ are the finite sets of vertices and arrows, respectively, and $s, t : Q_1 \to Q_0$ are maps such that any arrow $\alpha$ starts at $s(\alpha)$ and terminates at $t(\alpha)$. For any dimension vector $d = (d_\alpha)_{\alpha \in Q_0}$, we consider the affine space over $\mathbb{C}$

$$E_d = E_d(Q) = \bigoplus_{\alpha \in Q_1} \text{Hom}_{\mathbb{C}}(C^{d_{\alpha}(\alpha)}, C^{d_{\alpha}(\alpha)}).$$

Any element $x = (x_\alpha)_{\alpha \in Q_1}$ in $E_d$ defines a representation $M(x) = (C^d, x)$ where $C^d = \bigoplus_{\alpha \in Q_0} C^{d_\alpha}$. We set $(M(x))_1 = C^d$. Naturally we can define the action of the algebraic group $G_d(Q) = \prod_{\alpha \in Q_0} GL(C^{d_\alpha})$ on $E_d$ by $g.x = (g_{t(\alpha)}x_{s(\alpha)}g_{s(\alpha)}^{-1})_{\alpha \in Q_1}$. Let $e$ be the dimension vector $\dim M + \dim N$. For $L \in E_d$, we define

$$(L) := \{L' \in E_d \mid \chi(Gr_d(L')) = \chi(Gr_d(L)) \text{ for any } d\}.$$

There exists a finite subset $S(E_d)$ of $E_d$ such that ([Xu] Corollary 1.4)

$$E_d = \bigsqcup_{L \in S(E_d)} \langle L \rangle.$$

Let $O$ is a $G_{\mathbb{C}}$-invariant constructible subset of $E_d$. Define

$$\text{Ext}^1(M, N)_O = \{0 \to N \to L \to M \to 0 \in \text{Ext}^1(M, N) \setminus \{0\} \mid L \in O\}.$$

There is a $\mathbb{C}^*$-action on $\text{Ext}^1(M, N)$. The orbit space is denoted by $\mathbb{P} \text{Ext}^1(M, N)$.

Given $\mathbb{C}Q$-modules $U, V$ and an injective module $J$, define $\text{Hom}_{\mathbb{C}Q}(N, \tau M)(V) \oplus (U) \oplus J[-1]$. Then there exist two finite partitions

$$\text{Hom}(M, I) = \bigsqcup_{I' \in S(d_I(I'))} \text{Hom}(M, I)_{(V) \oplus I'[-1]},$$

$$\text{Hom}(P, M) = \bigsqcup_{P' \in S(d_P(P'))} \text{Hom}(P, M)_{P'[V] \oplus (U)},$$

where $d_I(I') = \dim I + \dim I' - \dim M$, $d_P(P') = \dim P + \dim P' - \dim M$,

$$\text{Hom}(M, I)_{(V) \oplus I'[-1]} = \{f \in \text{Hom}(M, I) \mid \ker(f) \in (V), \text{Coker}(f) = \tau U' \oplus J \text{ for some } U' \in (U)\}.$$
and

$$\text{Hom}(P,M)_{P \oplus (U)} = \{ g \in \text{Hom}(P,M) \mid \text{Ker}(g) = P', \text{Coker}(g) \in \langle U \rangle \}. $$

**Theorem 2.1.** Let $Q$ be an acyclic quiver. Then

1. for any $\mathbb{C}Q$-modules $M,N$ such that $M$ contains no projective summand, we have

$$\dim \text{Ext}^1_{\mathbb{C}Q}(M,N)X_MX_N = \sum_{I \in S(\mathbb{C}Q)} \chi(\text{PExt}^1_{\mathbb{C}Q}(M,N)(U))X_L$$

$$+ \sum_{I \in S(\mathbb{C}Q)} \sum_{V \in S(I)} \chi(\text{PHom}_{\mathbb{C}Q}(N,\tau M)_{(V) \oplus (U) \oplus t(-1)})X_V \oplus V \oplus t(-1)$$

where $I = \dim M + \dim N$ and

2. for any $\mathbb{C}Q$-module $M$ and projective $\mathbb{C}Q$-module $P$, we have

$$\dim \text{Hom}_{\mathbb{C}Q}(P,M)X_MX_P = \sum_{I \in S(\mathbb{C}Q)} \chi(\text{PExt}^1_{\mathbb{C}Q}(M,I)_{(V) \oplus P \oplus t(-1)})X_V \oplus P \oplus t(-1)$$

$$+ \sum_{I \in S(\mathbb{C}Q)} \chi(\text{PExt}^1_{\mathbb{C}Q}(P,M)_{P \oplus (U) \oplus t(-1)}X_P \oplus t(-1)$$

where $I = \text{DHom}_{\mathbb{C}Q}(P,\mathbb{C}Q)$, $I'$ is injective, and $P'$ is projective.

We have the following simplified version of Theorem 2.1.

**Theorem 2.2.** Let $kQ$ be an acyclic quiver and $\mathbb{C}(Q)$ be the cluster category of $kQ$. Then for $M,N \in \mathbb{C}(Q)$, there exists a finite subset $Y$ of the middle term of extensions in $\text{Ext}^1_{\mathbb{C}(Q)}(M,N)$ such that if $\text{Ext}^1_{\mathbb{C}(Q)}(M,N) \neq 0$, we have

$$\chi(\text{PExt}^1_{\mathbb{C}(Q)}(M,N))X_MX_N = \sum_{Y \in Y} (\chi(\text{PExt}^1_{\mathbb{C}(Q)}(M,N)_{(Y)}) + \chi(\text{PExt}^1_{\mathbb{C}(Q)}(N,M)_{(Y)}))X_Y.$$

2.2. Let us illustrate the cluster multiplication theorem by the following example. Let $Q$ be a tame quiver with minimal imaginary root $\delta$. Assume there is a sink $e \in Q_0$ such that $\delta_e = 1$. Let $P_e$ be simple projective $\mathbb{C}Q$-module at $e$. There exists unique preinjective module $I$ with dimension vector $\delta - \dim P_e$. Then we have $\dim \text{Ext}^1(I,P_e) = 2$. The set of indecomposable regular $\mathbb{C}Q$-modules consists of tubes indexed by the projective line $P^1$. Let $T_1, \ldots, T_l$ be all non-homogeneous tubes. Note that $l \leq 3$. Any $\mathbb{C}Q$-module $L$ with $\text{Ext}^1(I,P_e)_{(L)} \neq 0$ belongs to a tube. Conversely, for any tube $T$, up to isomorphism, there is unique $L \in \mathcal{T}$ such that $\text{Ext}^1(I,P_e)_{(L)} \neq 0$. Let $\mathcal{T}$ and $\mathcal{T}'$ be two homogeneous tubes and $E, E'$ be the regular simple modules in $\mathcal{T}$ and $\mathcal{T}'$, respectively. In Section 4.1, we will prove $\chi(Gr_e(E)) = \chi(Gr_e(E'))$ for any $\mathcal{T}$. Using this result, we have $X_E = X_{E'}$. By Theorem 2.1, we obtain

$$2X_{E_i}X_I = (2 - t)X_E + \sum_{i=1}^t X_{E_i} + \sum_{I' \in S(I)} \chi(\text{PExt}^1(P_e,\tau I)_{(U) \oplus t(-1)}X_U \oplus t(-1))$$

where $E$ is any regular simple module with $\dim E = \delta$ and $E_i$ is the unique regular module in $T_i$ such that $\text{Ext}^1(I,P_e)_{E_i} \neq 0$. Here $I'$ is an injective $\mathbb{C}Q$-module and $\chi(\text{Ext}^1_{\mathbb{C}Q}(P_e,\tau I)_{(U) \oplus t(-1)})X_U \oplus t(-1)$$

2.3. We give some variants of the cluster multiplication theorem. In fact, we will mainly meet the cases in the following theorems when we construct integral bases of the cluster algebras of affine types.

**Lemma 2.3.** Let $Q$ be an acyclic quiver and $\mathbb{C}(Q)$ be the cluster category associated to $Q$. Let $M$ and $N$ be indecomposable $\mathbb{C}Q$-modules. Then

$$\text{Ext}^1_{\mathbb{C}(Q)}(M,N) \cong \text{Ext}^1_{\mathbb{C}Q}(M,N) \oplus \text{Hom}_{\mathbb{C}Q}(M,\tau N).$$
Lemma 3.4. Let \( Q \) be an acyclic quiver and \( M \) any indecomposable non-projective \( \mathbb{C}Q \)-module, then
\[
X_M X_M = 1 + X_E
\]
where \( E \) is the middle term of the Auslander-Reiten sequence ending in \( M \).

Theorem 2.4. [CC] Let \( Q \) be an acyclic quiver and \( M,N \) be any two objects in \( \mathcal{C}(Q) \) such that \( \dim \mathcal{C} \operatorname{Ext}^1_{\mathcal{C}(Q)}(M,N) = 1 \), then
\[
X_M X_N = X_B + X_B'
\]
where \( B \) and \( B' \) are the unique objects such that there exists non-split triangles
\[
M \to B \to N \to M[1] \quad \text{and} \quad N \to B' \to M \to N[1].
\]

3. Numerators of Laurent expansions in generalized cluster variables

Let \( Q \) be an acyclic quiver and \( E_i[n] \) be the indecomposable regular module with quasi-socle \( E_i \) and quasi-length \( n \). For any \( \mathbb{C}Q \)-module \( M \), we denoted by \( d_M,i \) the \( i \)-th component of \( \dim M \).

Definition 3.1. For \( M,N \in \mathcal{C}(Q) \) with \( \dim M = (m_1,\ldots,m_n) \) and \( \dim N = (r_1,\ldots,r_n) \), we write \( \dim M \prec \dim N \) if \( m_i \leq r_i \) for \( 1 \leq i \leq n \). Moreover, if there exists some \( i \) such that \( m_i < r_i \), then we write \( \dim M \triangleleft \dim N \).

For any \( d \in \mathbb{Z}^Q \), define \( \bar{d}^+ = (d_i^+)_{i \in Q_0} \) such that \( d_i^+ = d_i \) if \( d_i > 0 \) and \( d_i^+ = 0 \) if \( d_i \leq 0 \) for any \( i \in Q_0 \). Dually, we set \( \bar{d}^- = \bar{d}^+ - \bar{d} \).

By Theorem 2.4, we have the following easy lemma.

Lemma 3.2. With the above notation in Theorem 2.1 and \( \operatorname{Ext}^1_{\mathcal{C}(Q)}(M,N) \neq 0 \), we have \( \dim(V \oplus U \oplus I[-1]) \prec \dim(M \oplus N) = \dim L \).

According to the definition of the Caldero-Chapoton map, we consider the Laurent expansions in generalized cluster variables \( X_M = \frac{P(x)}{\prod_{1 \leq i \leq n} x_i^{d_i-1}} \) for \( M \in \mathcal{C}(Q) \) such that the integral polynomial \( P(x) \) in the variables \( x_i \) is not divisible by any \( x_i \). We define the denominator vector of \( X_M \) as \( (m_1,\ldots,m_n) \) [FZ1]. The following theorem is called as the denominator theorem.

Theorem 3.3. [CK2] Let \( Q \) be an acyclic quiver. Then for any object \( M \) in \( \mathcal{C}(Q) \), the denominator vector of \( X_M \) is \( \dim M \).

The orientation of a quiver \( Q \) is called alternating if every vertex of \( Q \) is a sink or a source. We note that there exists an alternating orientation for a quiver of type \( \tilde{A}_0,\tilde{A}_n (n \geq 4) \) or \( \tilde{E}_m (m = 6,7,8) \). According to Theorem 3.3, we can prove the following propositions.

Lemma 3.4. Let \( Q \) be a tame quiver with the alternating orientation. If \( M \) is either \( P_i \) or \( I_i \) for \( 1 \leq i \leq n \), then \( X_M = \frac{P(x)}{\prod_{1 \leq i \leq n} x_i^{d_i-1}} \) where the constant term of \( P(x) \) is \( 1 \).

Proof. 1) If \( i \) is a sink point, we have the following short exact sequence:
\[
0 \to P_i \to I_i \to I' \to 0
\]
Then by Theorem 2.4, we have:
\[
X_{\tau P_i} X_{P_i} = x_i^{\dim \soc I'} + 1
\]
Thus the constant term of numerator in \( X_{P_i} \) as an irreducible fraction of integral polynomials in the variables \( x_i \) is \( 1 \) because of \( \tau X_{P_i} = x_i \).
If $i$ is a source point, we have the following short exact sequence:

$$0 \longrightarrow P' \longrightarrow P_i \longrightarrow I_i \longrightarrow 0$$

Similarly we have:

$$X_{τP_i}X_{P_i} = X_{P'} + 1$$

Thus we can finish it by induction on $P'$.

2) For $X_{I_i}$, it is totally similar. □

Note that $X_{τP_i} = x_i = \frac{1}{x_i}$ and $\dim(τP_i) = (0, \cdots, 0, -1, 0, \cdots, 0)$ with $i$-th component 1 and others 0. Hence we denote the denominator of $X_{τP_i}$ by $x_i^{-1}$, and assert the constant term of numerator in $X_{τP_i}$ is 1. With these notations, we have the following Proposition 3.5.

**Proposition 3.5.** Let $Q$ be a tame quiver with the alternating orientation. For any object $M \in C(Q)$, then $X_M = \frac{P(x)}{\dim(x)}$ where the constant term of $P(x)$ is 1.

**Proof.** It is enough to consider the case that $M$ is an indecomposable module.

1) If $M$ is an indecomposable preprojective module, then by the exchange relation in Theorem 2.3 we have

$$X_MX_{τM} = \prod_i X_{B_i} + 1.$$  

Thus by Lemma 3.4 we can prove that $X_M = \frac{P(x)}{\dim(x)}$ where the constant term of $P(x)$ is 1 by induction and the directness of the preprojective component of the Auslander-Reiten quiver of indecomposable $CQ$-modules. It is similar for indecomposable preinjective modules.

2) If $M$ is an indecomposable regular module, it is enough to prove that the proposition holds for any regular simple module according to the exchange relation. If $M$ is in a homogeneous tube, then $M \cong τM$. It is enough to consider the case of $\dim M = δ = (δ_i)_{i \in Q_0}$ by Theorem 2.4. Note that there exists a vertex $e \in Q_0$ such that $δ_e = 1$. Thus we have

$$\dim_C Ext^1_{CQ}(M, P_e) = \dim_C Hom_{CQ}(P_e, M) = 1.$$  

Then we obtain the following two non-split exact sequences:

$$0 \longrightarrow P_e \longrightarrow L \longrightarrow M \longrightarrow 0$$  

and

$$0 \longrightarrow L' \longrightarrow P_e \longrightarrow M \longrightarrow L'' \longrightarrow 0$$

where $L$ and $L'$ are preprojective modules and $L''$ is a preinjective module. Using Theorem 2.1 or Theorem 2.5 we have

$$X_MX_{P_e} = X_L + X_{L'}X_{τ-1L''}$$

where $\dim(L' \oplus τ^{-1}L'') < \dim(P_e \oplus M)$. We have already proved that the constant terms of the numerators of $X_{P_e}, X_L, X_{L'}$ and $X_{τ-1L''}$ as irreducible fractions of integral polynomials in the variables $x_i$ is 1 by the discussion in 1), then the constant term of the numerator in $X_M$ as an irreducible fraction must be 1. Let $T$ be a non-homogeneous tube of rank $r$ with regular simple module $E_i$ for $1 \leq i \leq r$. By Theorem 2.4 we only need to prove the constant term of the numerator in $X_{E_i}$ is 1 for $1 \leq i \leq r$. We assume $d_{E_{i,e}} ≠ 0$ and $τE_2 = E_1, \cdots, τE_1 = E_r$. Therefore $\dim_C Ext^1(E_1, P_e) = 1$, then we have the following non-split exact sequences combining the relation $τE_1 = E_r$

$$0 \longrightarrow P_e \longrightarrow L \longrightarrow E_1 \longrightarrow 0$$  

and

$$0 \longrightarrow L' \longrightarrow P_e \longrightarrow E_r \longrightarrow L'' \longrightarrow 0$$


where \( L \) and \( L' \) are preprojective modules and \( L'' \) is a preinjective module. Then we have

\[
X_{E_1}X_{P_e} = X_L + X_{L'}X_{\tau^{-1}L''}
\]

where \( \dim(L' \oplus \tau^{-1}L'') < \dim(P_e \oplus E_1) \). Hence, the constant term of the numerator in \( X_{E_1} \) must be 1. Note that \( d_{E_1,2,e} = d_{E_1,e} + d_{E_1,e} = 1 \), by similar discussions, we can obtain the constant term of the numerator in \( X_{E_1[2]} \) must be 1. Thus by \( X_{E_1}X_{E_2} = X_{E_1[2]} + 1 \), we obtain that the constant term of the numerator in \( X_{E_2} \) must be 1. Using the same method, we can prove the constant term of the numerator in \( X_{E_i} \) must be 1 for \( 3 \leq i \leq r \). \( \square \)

4. Generalized cluster variables on tubes

Let \( Q \) be a tame quiver with the minimal imaginary root \( \delta = (\delta_i)_{i \in Q_0} \). Then tubes of indecomposable regular \( \mathbb{C}Q \)-modules are indexed by the projective line \( \mathbb{P}^1 \). Let \( \lambda \) be an index of a homogeneous tube and \( E(\lambda) \) be the regular simple \( \mathbb{C}Q \)-module with dimension vector \( \delta \) in this homogeneous tube. Assume there are \( t(\leq 3) \) non-homogeneous tubes for \( Q \). We denote these tubes by \( T_1, \ldots, T_t \). Let \( r_i \) be the rank of \( T_i \) and the regular simple modules in \( T_i \) be \( E_1^{(i)}, \ldots, E_{r_i}^{(i)} \) such that \( \tau E_1^{(i)} = E_2^{(i)}, \ldots, \tau E_{r_i}^{(i)} = E_1^{(i)} \) for \( i = 1, \ldots, t \). If we restrict the discussion to one tube, we will omit the index \( i \) for convenience. Given a regular simple \( E \) in a non-homogeneous tube, \( E[i] \) is the indecomposable regular module with quasi-socle \( E \) and quasi-length \( i \) for any \( i \in \mathbb{N} \). Let \( X_M \) be the generalized cluster variable associated to \( M \) by the reformulation of the Caldero-Chapoton map. Set \( X_{n\delta_{\lambda,j}} = X_{E_{j}[nr_i]} \) for \( n \in \mathbb{N} \).

Proposition 4.1. [Dy2] Lemma 3.14] Let \( \lambda \) and \( \mu \) be in \( \mathbb{C} \) such that \( E(\lambda) \) and \( E(\mu) \) are two regular simple modules of dimension vector \( \delta \). Then \( \chi(Gr_{E}(E(\lambda))) = \chi(Gr_{E}(E(\mu))) \).

Proposition 4.2. Let \( M \) be the regular simple \( \mathbb{C}Q \)-module with dimension vector \( \delta \) in a homogeneous tube. For any \( m, n \in \mathbb{N} \) and \( m \geq n \), we have

\[
X_M[n]X_M[n] = X_M[m+n] + X_M[m+n-2] + \cdots + X_M[m-n+2] + X_M[m-n].
\]

Proof. If \( n = 1 \), we know \( \dim_{\mathbb{C}}\text{Ext}^1(M[m], M) = \dim_{\mathbb{C}}\text{Hom}(M, M[m]) = 1 \). The involving non-split short exact sequences are

\[
0 \rightarrow M \rightarrow M[m+1] \rightarrow M[m] \rightarrow 0
\]

and

\[
0 \rightarrow M \rightarrow M[m] \rightarrow M[m-1] \rightarrow 0.
\]

Thus by Theorem 2.1 or Theorem 2.3 and the fact \( \tau M[k] = M[k] \) for any \( k \in \mathbb{N} \), we obtain the equation

\[
X_M[n]X_M = X_M[m+1] + X_M[m-1].
\]

Suppose that it holds for \( n \leq k \). When \( n = k + 1 \), we have

\[
X_M[n]X_M[k+1] = X_M[n]X_M[k]X_M[k]X_M[k-1] = X_M[n]X_M[k]X_M[k]X_M[k]X_M[k-1]
\]

\[
= \sum_{i=0}^{k} X_M[m+k-2i]X_M^k - \sum_{i=0}^{k-1} X_M[m+k-1-2i]
\]

\[
= \sum_{i=0}^{k} (X_M[m+k-1-2i] + X_M[m+k-1-2i]) - \sum_{i=0}^{k-1} X_M[m+k-1-2i]
\]

\[
= \sum_{i=0}^{k+1} X_M[m+k+1-2i].
\]
By Proposition 4.1 and Proposition 4.2, we can define $X_{n\delta} := X_{M[n]}$ for $n \in \mathbb{N}$.

Now we consider non-homogeneous tubes.

**Proposition 4.3.** Let $Q$ be a tame quiver with the alternating orientation. Then $X_{n\delta,i,k} = X_{n\delta,i,k} + \sum_{\dim L < n\delta} a_L X_L$ in non-homogeneous tubes where $a_L \in Q$.

**Proof.** Denote $\delta = (v_1, v_2, \ldots, v_n)$ and $\dim E_j^{(i)} = (v_{j1}, v_{j2}, \ldots, v_{jn})$, then $\delta = \sum_{1 \leq j \leq r} \dim E_j^{(i)}$. Thus by Theorem 2.1, Lemma 3.2 and the fact that for any dimension vector there is at most one exceptional module up to isomorphism, we have

$$X_{S_1}^v X_{S_2}^v \cdots X_{S_n}^v = (X_{S_1}^{v_1} X_{S_2}^{v_2} \cdots X_{S_n}^{v_n}) (X_{S_1}^{v_{r_1}} X_{S_2}^{v_{r_2}} \cdots X_{S_n}^{v_{r_n}})$$

$$= (a_1 X_{E_1^{(i)}} + \sum_{\dim L < \dim E_1^{(i)}} a_L X_L) \cdots (a_r X_{E_r^{(i)}} + \sum_{\dim L < \dim E_r^{(i)}} a_L X_L)$$

$$= a_1 \cdots a_r X_{\delta,i,k} + \sum_{\dim M < \delta} a_M X_M.$$

Note that by Proposition 3.3, the left hand side of the above equation have a term $\prod_{i=1}^{n} x_i$, which cannot appear in $X_M$ for $\dim M < \delta$, we thus have $a_i's \neq 0$. Similarly we have

$$X_{S_1}^{v_1} X_{S_2}^{v_2} \cdots X_{S_n}^{v_n} = (b_1 X_{E_1^{(i)}} + \sum_{\dim T < \dim E_1^{(i)}} b_T X_T) \cdots (b_r X_{E_r^{(i)}} + \sum_{\dim T < \dim E_r^{(i)}} b_T X_T)$$

$$= b_1 \cdots b_r X_{\delta,i,l} + \sum_{\dim N < \delta} b_N X_N,$$

where $b_i's \neq 0$. Thus we have

$$a_1 \cdots a_n X_{\delta,i,k} = b_1 \cdots b_r X_{\delta,i,l} + \sum_{\dim N < \delta} b_N X_N - \sum_{\dim M < \delta} a_M X_M.$$

Therefore by Proposition 3.3 and Theorem 3.3 we have

$$X_{\delta,i,k} = X_{\delta,i,l} + \sum_{\dim N < \delta} b_N X_N.$$

Now, suppose the proposition holds for $m \leq n$, then on the one hand

$$X_{n\delta,i,k} X_{\delta,i,k} = X_{(n+1)\delta,i,k} + \sum_{\dim L < (n+1)\delta} b_L X_L.$$ 

On the other hand

$$X_{n\delta,i,k} X_{\delta,i,k} = (X_{n\delta,i,j} + \sum_{\dim L < n\delta} a_L X_L) X_{\delta,i,k}$$

$$= (X_{n\delta,i,j} + \sum_{\dim L < n\delta} a_L X_L)(X_{\delta,i,j} + \sum_{\dim N < \delta} b_N X_N)$$

$$= X_{(n+1)\delta,j} + \sum_{\dim L < (n+1)\delta} b_L X_L.$$
Therefore, we have
\[ X_{(n+1)\delta_{i,k}} = X_{(n+1)\delta_{j,l}} + \sum_{\text{dim} L'' > (n+1)\delta} b_{L''}X_{L''} - \sum_{\text{dim} L' < (n+1)\delta} b_{L'}X_{L'}. \]

Thus the proof is finished. \(\square\)

**Proposition 4.4.** Let \(Q\) be a tame quiver with the alternating orientation. Then \(X_{n\delta} = X_{n\delta_{i,1}} + \sum_{\text{dim} L < n\delta} a_LX_L\), where \(a_L \in \mathbb{Q}\).

**Proof.** Let \(Q\) be of type \(\tilde{D}_n(n \geq 4)\) or \(\tilde{E}_m(m = 6, 7, 8)\) and \(\delta = (v_1, v_2, \ldots, v_n)\), as in the proof of Proposition 4.3, we have
\[
X_{v_1}X_{v_2}\cdots X_{v_n} = (a_1X_{P_1}) + \sum_{\text{dim} L' < \text{dim} P_1} a_{L'}X_{L'}(a_{r_1}X_{P_1}) + \sum_{\text{dim} L'' < \text{dim} P_1} a_{L''}X_{L''}
\]
\[= a_1\cdots a_rX_{\delta_{1,1}} + \sum_{\text{dim} M < \delta} a_MX_M.\]

where \(a_i'\neq 0\). On the other hand, by the discussion in Section 2.2 and Proposition 4.3, we have
\[
X_{v_1}X_{v_2}\cdots X_{v_n} = (b_1X_{P_1} + \sum_{\text{dim} L' < \text{dim} P_1} b_{L'}X_{L'})(b_2X_I + \sum_{\text{dim} L'' < \text{dim} I} b_{L''}X_{L''})
\]
\[= \frac{1}{2}b_1b_2(\sum_{k=1}^3 X_{\delta_{1,1}}) + \sum_{\text{dim} N < \delta} b_NX_N
\]
\[= -\frac{1}{2}b_1b_2X_{\delta} + \frac{3}{2}b_1b_2X_{\delta_{1,1}} + \sum_{\text{dim} N' < \delta} b_{N'}X_{N'}.
\]

where \(b_i'\neq 0\). Thus
\[a_1\cdots a_nX_{\delta_{1,1}} + \sum_{\text{dim} M < \delta} a_MX_M = -\frac{1}{2}b_1b_2X_{\delta} + \frac{3}{2}b_1b_2X_{\delta_{1,1}} + \sum_{\text{dim} N' < \delta} b_{N'}X_{N'}.
\]

This deduces the following identity
\[
\frac{1}{2}b_1b_2X_{\delta} = (\frac{3}{2}b_1b_2 - a_1\cdots a_n)X_{\delta_{1,1}} + \sum_{\text{dim} N' < \delta} b_{N'}X_{N'} - \sum_{\text{dim} M < \delta} a_MX_M.
\]

Note that \(b_i'\neq 0\). Therefore by Proposition 3.5 and Theorem 3.3, we have
\[X_{\delta} = X_{\delta_{1,1}} + \sum_{\text{dim} M < \delta} a_MX_M.
\]

Then we can finish the proof by induction as in the proof of Proposition 4.3.

When \(Q\) is of type \(A_{p,p}\), we can prove it by Theorem 5.2. \(\square\)

**Proposition 4.5.** Let \(Q\) be a tame quiver with the alternating orientation. If \(\text{dim}(T_1 \oplus R_1) = \text{dim}(T_2 \oplus R_2)\) where \(R_i\) are 0 or any regular modules, \(T_i\) are 0 or any indecomposable regular modules with self-extension in non-homogeneous tubes and there are no extension between \(R_i\) and \(T_i\), then
\[
X_{T_1 \oplus R_1} = X_{T_2 \oplus R_2} + \sum_{\text{dim} R < \text{dim}(T_2 \oplus R_2)} a_RX_R
\]
where $a_R \in \mathbb{Q}$.

**Proof.** Suppose $\dim(T_1 \oplus R_1) = (d_1, d_2, \cdots, d_n)$, as in the proof of Proposition 3.3, we have

\[
X^{d_1}_1X^{d_2}_2\cdots X^{d_n}_n = (a_1X_{E_1} + \sum_{\dim L < \dim E_1} a_LX_L) \cdots (a_sX_{E_s} + \sum_{\dim L < \dim E_s} a_LX_L) \\
\times (a_{R_1}X_{R_1} + \sum_{\dim L < \dim R_1} a_LX_L) \\
= aX_{T_1 \oplus R_1} + \sum_{\dim L < \dim (T_1 \oplus R_1)} a_LX_L,
\]

where $a \neq 0$ as the discussion in the proof of Proposition 3.3. Similarly, we have

\[
X^{d_1}_1X^{d_2}_2\cdots X^{d_n}_n = (b_1X_{E_1} + \sum_{\dim M < \dim E_1} b_MX_M) \cdots (b_sX_{E_s} + \sum_{\dim M < \dim E_s} b_MX_M) \\
\times (b_{R_2}X_{R_2} + \sum_{\dim M < \dim R_2} b_MX_M) \\
= bX_{T_2 \oplus R_2} + \sum_{\dim L < \dim (T_2 \oplus R_2)} b_MX_M,
\]

where $b \neq 0$. Thus

\[
aX_{T_1 \oplus R_1} + \sum_{\dim L < \dim (T_1 \oplus R_1)} a_LX_L = bX_{T_2 \oplus R_2} + \sum_{\dim L < \dim (T_2 \oplus R_2)} b_MX_M.
\]

Therefore by Proposition 3.3 and Theorem 5.3, we have

\[
X_{T_1 \oplus R_1} = X_{T_2 \oplus R_2} + \sum_{\dim R < \dim (T_2 \oplus R_2)} aRX_R.
\]

\[\square\]

### 5. A $\mathbb{Z}$-Basis for the Cluster Algebra of Affine Type

In this section, we will construct a $\mathbb{Z}$-basis for the cluster algebra of a tame quiver.

**Definition 5.1.** Let $Q$ be an acyclic quiver and $B = R - R^r$. The quiver $Q$ is called graded if there exists a linear form $\epsilon$ on $\mathbb{Z}^n$ such that $\epsilon(B\alpha_i) < 0$ for any $1 \leq i \leq n$ where $\alpha_i$ denotes the $i$-th vector of the canonical basis of $\mathbb{Z}^n$.

**Theorem 5.2.** Let $Q$ be a graded quiver and $\{M_1, \cdots, M_r\}$ a family objects in $C(Q)$ such that $\dim M_i \neq \dim M_j$ for $i \neq j$, then $X_{M_1}, \cdots, X_{M_r}$ are linearly independent over $\mathbb{Q}$.

Now let $Q$ be a tame quiver with the alternating orientation. Note that the quiver $Q$ we consider is graded. Define the set $S(Q)$ to be

\[
\{X_L, X_{T_1 \oplus R} | \dim (T_1 \oplus R_1) \neq \dim (T_2 \oplus R_2), \text{Ext}^1_C(T, R) = 0, \text{Ext}^1_C(L, L) = 0\}
\]

where $L$ is any non-regular exceptional object, $R$ is 0 or any regular exceptional module and $T$ is 0 or any indecomposable regular module with self-extension.

To prove Theorem 5.6, Firstly we need to prove the dimension vectors of these objects associated to the corresponding elements in $S(Q)$ are different by Theorem 5.2.
Proposition 5.3. Let $T \oplus R$ satisfy $X_{T \oplus R} \in \mathcal{S}(Q)$. If $R \neq 0$, then $\dim M \neq m\delta$ for $m \in \mathbb{N}$.

Proof. Let $R_0$ be an indecomposable regular exceptional module as a non-zero direct summand of $R$. We set $M = T \oplus R = M' \oplus R_0$ and $m\delta = \dim M$. Since $\Ext^1_{\mathcal{C}(Q)}(T, R) = 0$ and $\Ext^1_{\mathcal{C}(Q)}(R, R) = 0$, we have

$$(\dim M, \dim R_0) = (\dim M', \dim R_0) + (\dim R_0, \dim R_0) > 0$$

where $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ for $\alpha, \beta \in \mathbb{Z}^n$. It is a contradiction to $(m\delta, \dim R_0) = 0$. \hfill $\square$

Proposition 5.4. Let $M$ be a regular module associated to some element in $\mathcal{S}(Q)$ and $L$ be a non-regular exceptional object in $\mathcal{C}(Q)$. Then $\dim M \neq \dim L$.

Proof. If $L$ contains some $\tau P_i$, as its direct summand, we know that

$$\dim(\tau P_i) = (0, \ldots, 0, -1, 0, \ldots, 0)$$

where the $i$-th component is $-1$. Suppose $L = \tau P_i \oplus \tau P_i \cdots \tau P_i \oplus N$ where $N$ is an exceptional module. Because $L$ is an exceptional object, $X_{\tau P_i} X_N = X_{\tau P_i \oplus N}$ i.e. $\dim_{\mathcal{C}} \Hom(P_i, N) = 0$. Thus we have $\dim_{\mathcal{C}} (i) = 0$ and $\dim_{\mathcal{C}} (\tau P_i \oplus \tau P_i \cdots \tau P_i \oplus N)(i) < -1$. However, $\dim M \geq 0$. Therefore, $\dim M \neq \dim L$.

If $L$ is a module. Suppose $\dim M = \dim L$. Because $L$ is an exceptional module, we know that $M$ belongs to the orbit of $L$ and then $M$ is a degeneration of $L$. Hence, there exists some $\mathbb{C}Q$-module $U$ such that

$$0 \rightarrow M \rightarrow L \oplus U \rightarrow U \rightarrow 0$$

is an exact sequence. Choose minimal $U$ so that we cannot separate the following exact sequence

$$0 \rightarrow 0 \rightarrow U_1 \rightarrow U_1 \rightarrow 0$$

from the above short exact sequence. Thus $M$ has a non-zero map to every direct summand of $L$. Therefore $L$ has no preprojective modules as direct summand because $M$ is a regular module.

Dually there exists a $\mathbb{C}Q$-module $V$ such that

$$0 \rightarrow V \rightarrow V \oplus L \rightarrow M \rightarrow 0$$

is an exact sequence. We can choose minimal $V$ so that one cannot separate the following exact sequence

$$0 \rightarrow V_1 \rightarrow V_1 \rightarrow 0 \rightarrow 0$$

from the above short exact sequence. Thus every direct summand of $L$ has a non-zero map to $M$. Therefore $L$ has no preinjective modules as direct summand because $M$ is a regular module.

Therefore $L$ is a regular exceptional module, it is a contradiction. \hfill $\square$

Secondly, we need to prove that $\mathcal{S}(Q)$ is a $\mathbb{Z}$-basis of $\mathcal{A}H(Q)$.

Proposition 5.5. $X_M X_N$ must be a $\mathbb{Z}$-combination of elements in the set $\mathcal{S}(Q)$ for any $M, N \in \mathcal{C}(Q)$.

Proof. Let $M, N$ be in $\mathcal{C}(Q)$. By Lemma 3.2, we know that $X_M X_N$ must be a $\mathbb{Q}$-linear combination of elements in the set

$$\{X_L, \Ext^1_{\mathcal{C}(Q)}(T, R) = 0, \Ext^1_{\mathcal{C}(Q)}(L, L) = 0\}$$

where $L$ is any non-regular exceptional object, $R$ is 0 or any regular exceptional module and $T$ is 0 or any indecomposable regular module with self-extension.
From those propositions in Section 4, we can easily find that $X_MX_N$ is a $\mathbb{Q}$-linear combination of elements in the set $S(Q)$. Thus we have
$$X_MX_N = b_Y X_Y + \sum_{\dim Y' > \dim (M \oplus N)} b_{Y'} X_{Y'},$$
where $\dim Y = \dim (M \oplus N)$, $X_Y, X_{Y'} \in S(Q)$, and $b_Y, b_{Y'} \in \mathbb{Q}$. Therefore by Proposition 5.3 and Theorem 5.3, we have $b_Y = 1$. Note that there exists a partial order on these dimension vectors by Definition 5.1. Thus in this remained $Y'$, we choose these maximal elements denoted by $Y'_1, \cdots, Y'_s$. Then by $b_Y = 1$ and the coefficients of Laurent expansions in generalized cluster variables are integers, we obtain that $b_{Y'_1}, \cdots, b_{Y'_s}$ are integers. Using the same method, we have $b_{Y'} \in \mathbb{Z}$. 

**Theorem 5.6.** The set $S(Q)$ is a $\mathbb{Z}$-basis of $\mathcal{AH}(Q)$.

**Proof.** The proof follows from Proposition 5.3, Proposition 5.4, and Proposition 5.5. 

**Corollary 5.7.** $S(Q)$ is a $\mathbb{Z}$-basis of the cluster algebra $\mathcal{EH}(Q)$.

**Proof.** Let $E_1, \cdots, E_s$ be regular simple modules in a tube $T$ of rank $r > 1$. It is obvious that $X_{E_i} \in \mathcal{EH}(Q)$. Then by [Dm1], Proposition 6.2, one can show that $X_{E_i[n]} \in \mathcal{EH}(Q)$ for any $n \in \mathbb{N}$. Thus by Proposition 5.3, we can prove that $X_{n \delta} \in \mathcal{EH}(Q)$. By definition, $X_L \in \mathcal{EH}(Q)$ for $L$ satisfying $\text{Ext}_{\mathcal{C}(Q)}^1(L, L) = 0$. Thus $S(Q) \subset \mathcal{EH}(Q)$. It follows that $S(Q)$ is a $\mathbb{Z}$-basis of the cluster algebra $\mathcal{EH}(Q)$ by Theorem 5.6. 

According to Theorem 5.6 and Corollary 5.7, we have

**Corollary 5.8.** Let $Q$ be an alternating tame quiver. Then $\mathcal{EH}(Q) = \mathcal{AH}(Q)$.

**Proposition 5.9.** Let $Q$ be an alternating tame quiver with $Q_0 = \{1, 2, \cdots, n\}$. For any $d = (d_1)_{1 \leq i \leq n} \in \mathbb{Z}^n$, we have
$$\prod_{i=1}^n X_{S_i}^{d_i} X_{P_i[1]}^{d_i} = X_M(d_1, \cdots, d_n) + \sum_{\dim M = (d_1, \cdots, d_n) \in \mathbb{Z}^n} b_L X_L,$$
where $X_M(d_1, \cdots, d_n)$ and $X_L \in S(Q)$, $\dim M = (d_1, \cdots, d_n) \in \mathbb{Z}^n$ and $b_L \in \mathbb{Z}$.

**Proof.** It follows from Proposition 5.5, Theorem 5.3, and Theorem 5.6. Note that $\{X_M(d_1, \cdots, d_n) : (d_1, \cdots, d_n) \in \mathbb{Z}^n\}$ is a $\mathbb{Z}$-basis of $\mathcal{AH}(Q)$, then we have the following Corollary 5.10 by Proposition 5.9.

**Corollary 5.10.** The set $\{\prod_{i=1}^n X_{S_i}^{d_i} X_{P_i[1]}^{d_i} \mid (d_1, \cdots, d_n) \in \mathbb{Z}^n\}$ is a $\mathbb{Z}$-basis of $\mathcal{AH}(Q)$.

Suppose $Q$ is an acyclic quiver. Define the reflected quiver $\sigma_i(Q)$ by reversing all the arrows ending at $i$. The mutations can be viewed as generalizations of reflections i.e. if $i$ is a sink or a source in $Q_0$, then $\mu_i(Q) = \sigma_i(Q)$ where $\mu_i$ denotes the mutation in the direction $i$. Thus there is a natural isomorphism of cluster algebras
$$\Phi : \mathcal{EH}(Q) \rightarrow \mathcal{EH}(Q')$$
where $Q'$ is a quiver mutation equivalent to $Q$, and $\Phi$ is called the canonical cluster algebras isomorphism.

Let $i$ be a sink in $Q_0$, $Q' = \sigma_i(Q)$ and $R_i^+ : \mathcal{C}(Q) \rightarrow \mathcal{C}(Q')$ be the extended BGP-reflection functor defined in [Zhu]. Denote by $X_Q^{\mathcal{C}}$ (resp. by $X_Q^{\mathcal{C}}$) the Caldero-Chapoton map associated to $Q$ (resp. to $\sigma(Q)$).

Then the following hold.
Lemma 5.11. \textsc{Zhu} Let $Q$ be an acyclic quiver and $i$ be a sink in $Q$. Then $R_i^+$ induces a triangle equivalence

$$R_i^+: C(Q) \to C(\sigma_i Q)$$

We note that Lemma 5.11 plays an essential importance to obtain the following lemmas.

Lemma 5.12. \textsc{Du2} Let $Q$ be a tame quiver and $i$ be a sink in $Q$. Denote by $\Phi_i : A(Q) \to A(\sigma_i Q)$ the canonical cluster algebra isomorphism and by $R_i^+: C(Q) \to C(\sigma_i Q)$ the extended BGP-reflection functor. Then

$$\Phi_i(X^Q_{\text{C}}M) = X^Q_{R_i^+ M}$$

where $M$ is any rigid object in $C(Q)$ or any regular module in non-homogeneous tubes.

Lemma 5.13. Let $Q$ be a tame quiver and $E$ be any regular simple module of dimension vector $\delta$. Then we have $\Phi_e(X^{e\sigma_i Q}_E) = X^{Q'}_{R_i^+ (E)}$ where $e$ is a sink or source in $\sigma_i Q$.

Proof. We only consider the case $e$ is a sink in $\sigma_i Q$. If $Q$ is of type $A_{p,q}$, then the lemma follows \textsc{Du2} Proposition 4.6]. Now we assume that $Q$ is of type $D_m(n \geq 4)$ or $E_m(m = 6, 7, 8)$. Thus there are three non-homogeneous tubes for $\sigma_i Q$ denoted by $T_1, T_2, T_3$. Let $E^{i(0)}_1[n_1]$ be the unique indecomposable regular module in $T_i$ such that $\dim E^{i(0)}_1[n_1] = \delta$ and $\text{reg.top}(E^{i(0)}_1[n_1]) \neq 0$ for $i = 1, 2, 3$. Let $P_e$ and $I$ be $\sigma_i Q$-modules such that $\dim I = \delta - e$. Using Theorem 2.1 (1), we know that the product $2X^{Q}_{P_e}X^{Q'}_{R_i^+}$ is equal to

$$-X^{\sigma_i Q}_E + 3X^{\sigma_i Q}_{E^{i(0)}_1[n_1]} + \sum_{i', U \in S(\text{dim} I')} \chi(\text{Hom}(P_e, \tau I_{(U)\oplus I'}[-1]))X^{Q'}_{R_i^+ (U\oplus I')[-1]}.$$

Since $U \oplus I'[-1]$ is a preinjective module, applying the isomorphism $\Phi_e$ to two side, by Lemma 5.12, we know that the product $2X^{Q'}_{R_i^+ (I)}$ is equal to

$$-\Phi_e(X^{\sigma_i Q}_E) + 3X^{Q'}_{R_i^+ (E^{i(0)}_1[n_1])} + \sum_{i', U \in S(\text{dim} I')} \chi(\text{Hom}(P_e, \tau I_{(U)\oplus I'}[-1]))X^{Q'}_{R_i^+ (U\oplus I')[-1]}.$$

We note that $P_e^Q$ is the projective $\mathbb{C}Q$-module at $e$ and $\dim R_i^+(I) = \delta + e$. Let $I'^Q$ be the simple injective module at $e$. Then any nonzero morphism from $R_i^+(I)$ to $I'^Q$ is epic and any nonzero morphism from $P_e^Q$ to $R_i^+ I$ is monic. Applying Theorem 2.1 (2) to the product $X^{Q'}_{P_e^Q[I]}X^{Q'}_{R_i^+ (I)}$, we obtain that the product $2X^{Q'}_{P_e^Q[I]}$ is equal to

$$-X^{Q'}_{R_i^+(E)} + 3X^{Q'}_{R_i^+(E^{i(0)}_1[n_1])} + \sum_{i', U \in S(\text{dim} I')} \chi(\text{Hom}(P_e, \tau I_{(U)\oplus I'}[-1]))X^{Q'}_{R_i^+ (U\oplus I')[-1]}.$$

Hence, we have $\Phi_e(X^{\sigma_i Q}_E) = X^{Q'}_{R_i^+(E)}$.

Theorem 5.14. Let $Q$ be an alternating tame quiver and $Q' = \sigma_i \cdots \sigma_i(Q)$. Then a $\mathbb{Z}$-basis for the cluster algebra of $Q'$ is the following set (denoted by $S(Q')$):

$$\{ X_{L'}, X_{L' \oplus P} | \dim(T' \oplus R') \neq \dim(T' \oplus R'), \Ext^1_{\mathbb{C}(Q')}(T', R') = 0, \Ext^1_{\mathbb{C}(Q')}(L', L') = 0 \}$$

where $L'$ is any non-regular exceptional object, $R'$ is 0 or any regular exceptional module, $T'$ is 0 or any indecomposable regular module with self-extension.
Proof. If $Q'$ is a quiver of type $\tilde{A}_{1,1}$, it is obvious that

$$\{X_M, X_{n\delta} | M \in C(Q), \text{Ext}^1(M, M) = 0\}$$

is a $\mathbb{Z}$-basis for cluster algebra of $\tilde{A}_{1,1}$, which is called the semicanonical basis in $\mathbb{Z}$. If $Q'$ is not a quiver of type $\tilde{A}_{1,1}$ and $Q$ is an alternating tame quiver, then in Theorem 5.6, we have already obtained a $\mathbb{Z}$-basis for cluster algebra of $Q$, denoted by $\mathcal{S}(Q)$:

$$\{X_L, X_{T \oplus R} | \dim(T_1 \oplus R_1) \neq \dim(T_2 \oplus R_2), \text{Ext}^1_{C(Q)}(T, R) = 0, \text{Ext}^1_{C(Q)}(L, L) = 0\}$$

where $L$ is any non-regular exceptional object, $R$ is 0 or any regular exceptional module and $T$ is 0 or any indecomposable regular module with self-extension. Thus $\Phi(S(Q))$ is a $\mathbb{Z}$-basis for the cluster algebra of $Q'$ because $\Phi : \mathcal{E}(Q) \rightarrow \mathcal{E}(Q')$ is the canonical cluster algebras isomorphism. Then by Lemma 5.12 and Lemma 5.13 we know that $\Phi(S(Q))$ is exactly the basis $\mathcal{S}(Q')$. □

Proof of Theorem 1.1. If $Q$ is reflection equivalent to a quiver with alternating orientation, then the proof follows from Theorem 5.13. If any orientation of the underlying graph of $Q$ does not satisfy that any vertex is a sink or a source, then $Q$ is a quiver of type $\tilde{A}_{p,q}$ with $p \neq q$. The theorem follows [Du2, Theorem 4.8]. □

For any dimension vector $d$, let $d = k\delta + d_0$ be the canonical decomposition of $d$ (see [Kac]) and $E$ be an indecomposable regular simple module of dimension vector $\delta$. There exists a regular rigid module $R$ of dimension vector $d_0$. Then Theorem 1.1 implies that the set

$$B'(Q) := \{X_L, X_{E[i] \oplus R} | L \in C(Q), \text{Ext}^1_{C(Q)}(L, L) = 0, \text{Ext}^1_{C(Q)}(R, R) = 0\}$$

is a $\mathbb{Z}$-basis for $\mathcal{E}(Q)$.

As in Corollary 5.7 let $E_1, \cdots, E_r$ be regular simple modules in a tube $T$ of rank $r > 1$. It is obvious that $X_{E[i]} \in \mathcal{E}(Q)$ for $1 \leq i \leq r$ and $n \in \mathbb{N}$. Thus, we have

Corollary 5.15. Let $Q$ be a tame quiver. Then $\mathcal{E}(Q) = \mathcal{A}(Q)$.

Recall the set $E(Q)$ was defined in the introduction. For any $d \in \mathbb{Z}^Q$, we have

$$\prod_{i=1}^n X_{S_i}^{d_i} X_{P_{j[1]}}^{d_j} = b_M X_M + \sum_{\dim d < (d_1, \cdots, d_n)} b_L X_L$$

where $X_M$ and $X_L \in \mathcal{B}(Q)$, and $b_M, b_L \in \mathbb{Z}$. According to Theorem 2.1, we know that $b_M \neq 0$ and $\dim M = (d_1, \cdots, d_n) \in \mathbb{Z}^n$. Thus we have the following corollary.

Corollary 5.16. Let $Q$ be a tame quiver. Then $\mathbb{Z}^Q = \mathcal{D}(Q) \cup E(Q)$.

6. Examples

In this section, we will give some examples for special cases to explain the $\mathbb{Z}$-bases explicitly.

6.1. $\mathbb{Z}$-basis for finite type. For finite type, we know that there are no regular modules in mod-$\mathcal{C}Q$. Thus the $\mathbb{Z}$-bases are exactly

$$\{X_M | M \in C(Q), \text{Ext}^1(M, M) = 0\}$$

in [CK].
6.2. **Z-basis for the Kronecker quiver.** Consider the Kronecker quiver $K$. Let $M$ be a regular simple $\mathbb{C}K$-module and $M[n]$ be the regular module with regular socle $M$ and regular length $n \in \mathbb{N}$. Since $X_M[n] = X_{M'}[n]$ for any regular simple $M'$. We set $X_{n\delta} := X_{M[n]}$. In this case, there is a $\mathbb{Z}$-basis of $\mathcal{C}(K)$ is

$$\mathcal{B}'(K) = \{X_M, X_{n\delta} \mid M \in \mathcal{C}(Q), \operatorname{Ext}^1(M, M) = 0\}$$

which is called the semicanonical basis in [CZ]. If we modify:

$$z_1 := X_\delta, \quad z_n := X_{n\delta} - X_{(n-2)\delta}.$$  

Then $\{X_M, z_n \mid M \in \mathcal{C}(Q), \operatorname{Ext}^1(M, M) = 0\}$ is the canonical basis for cluster algebra of Kronecker quiver in [SZ].

6.3. **Z-basis for $\tilde{D}_4$.** Let $Q$ be the tame quiver of type $\tilde{D}_4$ as follows

```
2
/\ /
3 1 5
4
```

We denote the minimal imaginary root by $\delta = (2, 1, 1, 1)$. The regular simple modules of dimension vector $\delta$ are

$$\mathbb{C} \xrightarrow{\alpha_1} \mathbb{C} \xrightarrow{\alpha_2} \mathbb{C} \xrightarrow{\alpha_3} \mathbb{C} \xrightarrow{\alpha_4} \mathbb{C}$$

with linear maps

$$\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

where $\lambda/\mu \in \mathbb{P}^1$, $\lambda/\mu \neq 0, 1, \infty$. Let $M$ be any regular simple $\mathbb{C}Q$-module of dimension vector $\delta$ and $X_M$ be the generalized cluster variable associated to $M$ by the reformulation of the Caldero-Chapoton map. Then we have

**Proposition 6.1.** [DX1] $X_M = \frac{1}{x_1^2 x_2 x_3 x_4 x_5} + \frac{4}{x_1 x_2 x_3 x_4 x_5} + x_1^2 x_2 x_3 x_4 x_5 + x_2 x_3 x_4 x_5 + \frac{2}{x_1^4} + \frac{4}{x_1^2}.$

We define $X_{n\delta} := X_{M[n]}$ for $n \in \mathbb{N}$. Now, we consider three non-homogeneous tubes labelled by the subset $\{0, 1, \infty\}$ of $\mathbb{P}^1$. The regular simple modules in non-homogeneous tubes are denoted by $E_1, E_2, E_3, E_4, E_5, E_6$, where

$$\operatorname{dim} E_1 = (1, 1, 0, 0, 0), \quad \operatorname{dim} E_2 = (1, 0, 0, 1, 1), \quad \operatorname{dim} E_3 = (1, 1, 0, 1, 0),$$

$$\operatorname{dim} E_4 = (1, 0, 1, 0, 1), \quad \operatorname{dim} E_5 = (1, 0, 1, 1, 0), \quad \operatorname{dim} E_6 = (1, 1, 0, 0, 1).$$

We note that $\{E_1, E_2\}, \{E_3, E_4\}$ and $\{E_5, E_6\}$ are pairs of the regular simple modules at the mouth of non-homogeneous tubes labelled by $1, \infty$ and $0$, respectively. We set $X_{n\delta} := X_{E_i[n]}$ for $1 \leq i \leq 6$.

**Proposition 6.2.** [DX1] For any $n \in \mathbb{N}$, we have $X_{n\delta_1} = X_{n\delta_2} = X_{n\delta_3} = X_{n\delta_4} = X_{n\delta_5} = X_{n\delta_6}$ and $X_{n\delta_1} = X_{n\delta} + X_{(n-1)\delta}$ where $X_0 = 1$. 

Theorem 6.3. [DX1] A $\mathbb{Z}$-basis for cluster algebra of $\bar{D}_4$ is the following set denoted by $B(Q)$:

$$\{X_L, X_{m\delta}, X_{E_i[2k+1]} \oplus R \mid \dim(E_i[2k+1] \oplus R_1) \neq \dim(E_i[2l+1] \oplus R_2),$$

$$\text{Ext}_{C(Q)}^1(L, L) = 0, \text{Ext}_{C(Q)}^1(E_i[2k+1], R) = 0, m, k, l \geq 0, 1 \leq i, j \leq 6$$

where $L$ is any non-regular exceptional object, $R$ is 0 or any regular exceptional module.

7. Inductive multiplication formulas for a tube

In this section, let $Q$ be any tame quiver. Now we fix a tube with rank $r$ and these regular simple modules are $E_1, \ldots, E_r$ with $\tau E_2 = E_1, \ldots, \tau E_1 = E_r$. Let $X_{E_i}$ be the corresponding generalized cluster variable for $1 \leq i \leq r$. With these notations, we have the following inductive cluster multiplication formulas.

Theorem 7.1. Let $i, j, k, l, m$ and $r$ be in $\mathbb{Z}$ such that $1 \leq k \leq mr + l, 0 \leq l \leq r - 1, 1 \leq i, j \leq r, m \geq 0$.

(1) When $j \leq i$, then

1) for $k + i \geq r + j$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[(m+1)r+i+j-i]}X_{E_i[k+i-r-j]} + X_{E_i[(r+j)-i]}X_{E_{k+i-1}(m+1)+j-k+i}$.

2) for $k + i < r + j$ and $i \leq l + j \leq k + i - 1$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[(mr+k+i-1)]}X_{E_j[(r+i-j)-i]} + X_{E_j[(m+1)r+i+j-i]}X_{E_{k+i-1}(k+i-l-j-1)}$.

3) for other conditions, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[k] \oplus E_i[mr+l]}$.

(2) When $j > i$, then

1) for $k + j \geq r - i$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[j-i]}X_{E_{k+i+j-k+i-1}+1}$ + $X_{E_i[(mr+k+i-1)]}X_{E_j[(r+i-j)-i]}$.

2) for $k < j - i$ and $i \leq l + j \leq k + i - 1$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[(m+1)r+k+i-j]}X_{E_j[(r+j)-i]} + X_{E_j[(m+1)r+i+j-i]}X_{E_{k+i-1}(k+i-l-j-1)}$.

3) for other conditions, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[k] \oplus E_i[mr+l]}$.

Proof. We only prove (1) and (2) is totally similar to (1).

1) When $k = 1$, by $k + i \geq r + j$ and $1 \leq j \leq i \leq r$ we have $X_{E_i[2]}X_{E_j[mr+l]} + X_{E_2[2]}X_{E_i[mr+l]}$.

When $k = 2$, by $k + i \geq r + j$ and $1 \leq j \leq i \leq r$ we have $i = r$ or $i = r - 1$.

For $i = r$ we have $j = 1$ or $j = 2$.

The case for $i = r$ and $j = 1$ we have

$$X_{E_2[2]}X_{E_1[mr+l]} = (X_{E_2[2]}X_{E_1[mr+l]} - X_{E_2[2]}X_{E_1[mr+l]})$$

The case for $i = r$ and $j = 2$ we have

$$X_{E_2[2]}X_{E_1[mr+l]} = (X_{E_2[2]}X_{E_1[mr+l]} - X_{E_2[2]}X_{E_1[mr+l]})$$
For $i = r - 1 \implies j = 1$:

$$X_{E_{r-1}[2]E_1[mr+t]} = (X_{E_{r-1}}E_r - 1)X_{E_1[mr+t]}$$

$$= X_{E_{r-1}}(X_{E_1[mr+t+1]} + X_{E_2[mr+t-1]}) - X_{E_1[mr+t]}$$

$$= (X_{E_{r-1}[mr+t+2]} + X_{E_1[mr+t]} + X_{E_{r-1}}X_{E_2[mr+t-1]} - X_{E_1[mr+t]}$$

$$= X_{E_{r-1}[mr+t+2]} + X_{E_{r-1}}X_{E_2[mr+t-1]}.$$

Now, suppose it holds for $k \leq n$, then by induction we have

$$X_{E_{[n+1]}X_{E_1[mr+t]}}$$

$$= (X_{E_1[n]}X_{E_{n+1}} - X_{E_1[1-n]})X_{E_1[mr+t]}$$

$$= X_{E_{n+1}}(X_{E_1[n]}X_{E_1[mr+t]}) - X_{E_1[n]}X_{E_1[mr+t]}$$

$$= X_{E_{n+1}}X_{E_1[(n+1)r+i+j-1]X_{E_1[n+i-r-1]}} + X_{E_1[1-r-i-1]}$$

$$= X_{E_1[(n+1)r+i+j-1]X_{E_1[n+i-r-1]}} + X_{E_1[1-r-i-1]}$$

$$+ X_{E_1[r+j-1]}X_{E_1[(n+1)r+i+j-1]}$$

$$- X_{E_1[(n+1)r+i+j-1]X_{E_1[n+i-r-1]}} + X_{E_1[1-r-i-1]}X_{E_1[(n+1)r+i+j-1]}$$

$$= X_{E_1[(n+1)r+i+j-1]X_{E_1[n+i-r-1]}} + X_{E_1[1-r-i-1]}X_{E_1[(n+1)r+i+j-1]}$$

$$+ X_{E_1[r+i-1]}X_{E_1[n+i-r-1]} + X_{E_1[r+j-1]}X_{E_1[(n+1)r+i+j-1]}$$

$$+ X_{E_1[r+i-1]}X_{E_1[n+i-r-1]} + X_{E_1[r+j-1]}X_{E_1[(n+1)r+i+j-1]}$$

$$+ X_{E_1[r+i-1]}X_{E_1[n+i-r-1]} + X_{E_1[r+j-1]}X_{E_1[(n+1)r+i+j-1]}$$

$$2) When k = 1, by i \leq l + j \leq k + i - 1 \implies i \leq l + j \leq i \implies i = l + j.$$

Then by Theorem 2.1 or Theorem 2.5, we have

$$X_{E_1}X_{E_1[mr+t]} = X_{E_{l+j+1}}X_{E_1[mr+t]} = X_{E_1[1-r+t+1]} + X_{E_1[mr+t-1]}$$

When $k = 2, by i \leq l + j \leq k + i - 1 \implies i \leq l + j \leq i + 1 \implies i = l + j or i + 1 = l + j$.

For $i = l + j$, we have

$$X_{E_1[2]E_1[1]} = X_{E_1[l+j]}X_{E_1[mr+t]}$$

$$= (X_{E_1[l+j]}X_{E_1[1]} - 1)X_{E_1[1]}$$

$$= (X_{E_1[l+j+1]} + X_{E_1[mr+l+1]}X_{E_1[l+j]}X_{E_1[1]} - X_{E_1[1]}$$

$$= X_{E_1[1]}X_{E_1[1]} + X_{E_1[1]}X_{E_1[1]} + X_{E_1[l+j+1]}X_{E_1[1]}X_{E_1[1]} - X_{E_1[1]}$$

$$= X_{E_1[l+j+1]} + X_{E_1[l+j+1]}X_{E_1[1]}.$$
Suppose it holds for $k \leq n$, then by induction we have

$$X_{E_1[n]}X_{E_2[n]} = \left( X_{E_1[n]}X_{E_2[n]} - X_{E_1[n]}X_{E_2[n]} \right)$$

$$= \left( X_{E_1[n]}X_{E_2[n]} - X_{E_1[n]}X_{E_2[n]} \right)$$

$$= \left( X_{E_1[n]}X_{E_2[n]} - X_{E_1[n]}X_{E_2[n]} \right)$$

$$= \left( X_{E_1[n]}X_{E_2[n]} - X_{E_1[n]}X_{E_2[n]} \right)$$

$$= \left( X_{E_1[n]}X_{E_2[n]} - X_{E_1[n]}X_{E_2[n]} \right)$$

Then by Theorem 7.1, then we have

$$= \left( X_{E_1[n]}X_{E_2[n]} - X_{E_1[n]}X_{E_2[n]} \right)$$

$$= \left( X_{E_1[n]}X_{E_2[n]} - X_{E_1[n]}X_{E_2[n]} \right)$$

$$= \left( X_{E_1[n]}X_{E_2[n]} - X_{E_1[n]}X_{E_2[n]} \right)$$

$$= \left( X_{E_1[n]}X_{E_2[n]} - X_{E_1[n]}X_{E_2[n]} \right)$$

3) It is trivial by the definition of the Caldero-Chapoton map.

We explain Theorem 7.1 by the following proposition involving a tube of rank 3.

**Proposition 7.2.**

1) For $n \geq 3m + 1$, then

$$X_{E_2[3m+1]}X_{E_1[n]} = X_{E_2[3m+1]}X_{E_1[n]} + X_{E_2[3m+1]}X_{E_1[n-3m-3]} + X_{E_2[3m+1]}X_{E_1[n-3m-6]} + X_{E_2[3m+1]}X_{E_1[n-3m-9]} + \cdots$$

$$= X_{E_2[3m+1]}X_{E_1[n]} + X_{E_2[3m+1]}X_{E_1[n-3m-3]} + X_{E_2[3m+1]}X_{E_1[n-3m-6]} + X_{E_2[3m+1]}X_{E_1[n-3m-9]} + \cdots$$

2) For $n \geq 3m + 2$, then

$$X_{E_2[3m+2]}X_{E_1[n]} = X_{E_2[3m+2]}X_{E_1[n]} + X_{E_2[3m+2]}X_{E_1[n-3m-7]} + X_{E_2[3m+2]}X_{E_1[n-3m-10]} + X_{E_2[3m+2]}X_{E_1[n-3m-13]} + \cdots$$

3) For $n \geq 3m + 3$, then

$$X_{E_2[3m+3]}X_{E_1[n]} = X_{E_2[3m+3]}X_{E_1[n]} + X_{E_2[3m+3]}X_{E_1[n-3m-9]} + X_{E_2[3m+3]}X_{E_1[n-3m-12]} + X_{E_2[3m+3]}X_{E_1[n-3m-15]} + \cdots$$

**Proof.**

1) By Theorem 7.1 then we have

$$X_{E_2[3m+1]}X_{E_1[n]} = X_{E_2[3m+1]}X_{E_1[n]} + X_{E_2[3m+1]}X_{E_1[n-3m-3]} + X_{E_2[3m+1]}X_{E_1[n-3m-6]} + X_{E_2[3m+1]}X_{E_1[n-3m-9]} + \cdots$$

2) By Theorem 7.1 then we have

$$X_{E_2[3m+2]}X_{E_1[n]} = X_{E_2[3m+2]}X_{E_1[n]} + X_{E_2[3m+2]}X_{E_1[n-3m-4]} + X_{E_2[3m+2]}X_{E_1[n-3m-7]} + X_{E_2[3m+2]}X_{E_1[n-3m-10]} + X_{E_2[3m+2]}X_{E_1[n-3m-13]} + \cdots$$

3) By Theorem 7.1 then we have

$$X_{E_2[3m+3]}X_{E_1[n]} = X_{E_2[3m+3]}X_{E_1[n]} + X_{E_2[3m+3]}X_{E_1[n-3m-5]} + X_{E_2[3m+3]}X_{E_1[n-3m-8]} + X_{E_2[3m+3]}X_{E_1[n-3m-11]} + X_{E_2[3m+3]}X_{E_1[n-3m-14]} + \cdots$$
Corollary 7.3. When $n = 3k + 1$, we can rewrite Proposition 7.2 as following:

(1) For $n \geq 3m + 1$, then
\[ X_{E_2[3m+1]} X_{E_1[n]} = X_{E_1[n+3m+1]} + X_{E_1[n+3m-1]} + \cdots + X_{E_2[n+3m-5]} + X_{E_1[n-3m+3] + X_{E_1[n-3m-1]} + X_{E_1[n-3m-3]}}. \]

(2) For $n \geq 3m + 2$, then
\[ X_{E_2[3m+2]} X_{E_1[n]} = X_{E_2[n+3m+2]} + X_{E_2[n+3m]} + X_{E_2[n+3m-2]} + X_{E_2[n+3m-4]} + \cdots + X_{E_2[n-3m+2]} + X_{E_2[n-3m]} + X_{E_2[n-3m-2]}. \]

(3) For $n \geq 3m + 3$, then
\[ X_{E_2[3m+3]} X_{E_1[n]} = X_{E_1[n+3m+3]} + X_{E_1[n+3m+1]} + X_{E_2[n+3m-1]} + X_{E_2[n+3m-3]} + X_{E_1[n-3m+3]} + X_{E_1[n-3m+1]} + X_{E_1[n-3m-1]} + X_{E_1[n-3m-3]}. \]

Proof. When $n = 3k + 1$, we have the following equations:
\[ X_{E_2} X_{E_1[n]} = X_{E_1[n+1]} + X_{E_1[n-1]} \]
\[ X_{E_2} X_{E_2[n-1]} = X_{E_2[n]} + X_{E_2[n-2]}. \]

Then the proof immediately follows from Proposition 7.2.

In the same way by using Theorem 7.1, we have the following proposition.

Proposition 7.4. (1) For $n \geq 3m + 1$, then
\[ X_{E_1[3m+1]} X_{E_1[n]} = X_{E_1[n+3m]} + X_{E_1[n+3m-3]} + X_{E_1[n+3m-6]} + X_{E_1[n+3m-9]} + \cdots + X_{E_1[n+3m-5]} + X_{E_1[n+3m-3]} + X_{E_1[n+3m-9]} + X_{E_1[n+3m-1]} + X_{E_1[n+3m-3]} + X_{E_1[n+3m-1]} + X_{E_1[n+3m-9]}. \]

(2) For $n \geq 3m + 2$, then
\[ X_{E_1[3m+2]} X_{E_1[n]} = X_{E_1[n+3m]} + X_{E_1[n+3m-6]} + \cdots + X_{E_1[n-3m-1]} \]
\[ X_{E_2[n+3m]} + X_{E_1[n+3m-1]} + X_{E_1[n+3m-3]} + X_{E_1[n+3m-6]} + \cdots + X_{E_1[n-3m-1]} + X_{E_2[n+3m]} + X_{E_1[n+3m-1]} + X_{E_1[n+3m-3]} + X_{E_1[n+3m-1]} + X_{E_1[n+3m-6]}. \]

Remark 7.5. (1) By the same method in Corollary 7.3, we can also rewrite Proposition 7.4 if we consider these different $n$. Here we omit it.

(2) In fact, we can also check Proposition 7.2 Corollary 7.3 and Proposition 7.4 by induction.

8. Appendix on the difference property

In this appendix, we show that the difference property holds for any tame quiver without oriented cycles.
8.1. Let $Q$ be a tame quiver without oriented cycles and $\delta$ be its minimal imaginary root. For any $CQ$-module $M$, define
\[ \partial(M) := (\delta, \dim M). \]
Let $(\dim M)_i$ be the $i$-th component of $\dim M$ for $i \in Q_0$. It is called the defect of $M$. A pair $(X, Y)$ of $CQ$-modules is called an orthogonal exceptional pair if $X$ and $Y$ are exceptional modules such that
\[ \text{Hom}_{CQ}(X, Y) = \text{Hom}_{CQ}(Y, X) = \text{Ext}^1_{CQ}(Y, X) = 0. \]
We denote by $S(A, B)$ the full subcategory of $\text{mod} CQ$ generated by $X$ and $Y$. The following result is well-known.

**Lemma 8.1.** Let $Q$ be a tame quiver. We fix a preprojective module $P$ and a preinjective module $I$ such that $\partial(P) = -1$ and $\dim P + \dim I = \delta$. Then the subcategory $S(P, I)$ is a hereditary abelian subcategory and there is an equivalence
\[ F : \text{mod} K \to S(P, I) \]
where $K$ is the Kronecker algebra $\mathbb{C} \begin{array}{cc} 1 & 2 \end{array}$. Every tube of $\text{mod} CQ$ contains a unique module $M$ as the middle term of the extension of $P$ by $I$.

Let $e \in Q_0$ be an extending vertex, i.e., $\delta_e = 1$. Let $P_e$ (resp. $I_e$) be the projective module (resp. injective module) corresponding to $e$. Let $Q'$ be the subquiver of $Q$ deleting the vertex $e$ and involving edges. Then $Q'$ is a Dynkin quiver. Let $I$ (resp. $P$) be the unique indecomposable preinjective (resp. preprojective) $CQ$-module of dimension vector $\delta - \dim P_e$ (resp. $\delta - \dim I_e$). We obtain a subcategory $S(P_e, I)$ (resp. $S(P, I_e)$). Consider the universal extensions
\[ 0 \to P^2_e \to L \to I \to 0 \quad \text{and} \quad 0 \to P \to L' \to I^2_e \to 0. \]
Here, $L$ (resp. $L'$) is unique preprojective (resp. preinjective) $CQ$-module with dimension vector $\delta + \dim P_e$ (resp. $\delta + \dim I_e$). Then we obtain a pair $(I, T^{-1}$ as the mutation of the pair $(P, I)$ where $T$ is the shift functor in $D^b(CQ)$. Indecomposable regular $CQ$-modules consist of tubes indexed by the projective line $\mathbb{P}^1$. As in [CB] Section 9), each tube contains a unique module in the set
\[ \Omega = \{ \text{iso.classes of indecomposable } X \mid \dim X = \delta, (\dim_{\text{reg.top}}(X))_e \neq 0 \}. \]
There are the following bijections:
\[ \text{PHom}(P_e, L) \quad \Omega \quad \text{PExt}^1(I, P_e) \]
induced by the map sending $0 \neq \theta \in \text{Hom}(P_e, L)$ to $\text{Coker} \theta$ and the map sending $\varepsilon \in \text{Ext}^1(I, P_e)$ to the isomorphism class of the middle term. We note that the bijection between $\text{PHom}(P_e, L)$ and $\text{PExt}^1(I, P_e)$ is induced by the universal extension. Note that $0 \neq \theta \in \text{Hom}(P_e, L)$ is mono. Dually, we also have an orthogonal exceptional pair $(P, I_e)$. Each tube contains a unique module in the set
\[ \Omega' = \{ \text{iso.classes of indecomposable } X \mid \dim X = \delta, (\dim_{\text{reg.socle}}(X))_e \neq 0 \}. \]
There are the following bijections:
\[ \text{PHom}(L', I_e) \quad \Omega' \quad \text{PExt}^1(I_e, P) \]
induced by the map sending $0 \neq \theta' \in \text{Hom}(L', I_e)$ to $\text{ker} \theta'$ and the map sending $\varepsilon' \in \text{Ext}^1(I_e, P)$ to the isomorphism class of the middle term. Note that $0 \neq \theta' \in \text{Hom}(L', I_e)$ is epic.

Now we assume $e$ is a sink in the quiver $Q$. Let $T$ be any tube of rank $n > 1$ and $E_1, \cdots, E_n$ be the regular simple modules in $T$. Without loss of generality, we assume that $\tau E_i = E_{i-1}$ for $i = 2, \cdots, n$, $\tau E_1 = E_n$ and $(E_n)_e \neq 0$. Let $E_i[r]$ be
the indecomposable regular module in $\mathcal{T}$ with quasi-socle $E_i$ and quasi-length $r$ for $i = 1, \cdots, n$. Then $E_1[n] \in \Omega$. It is easy to know that
\[ \dim_{\mathbb{C}} \text{Ext}_{\mathcal{C}}^1(E_1[n], P_e) = \dim_{\mathbb{C}} \text{Hom}_{\mathcal{C}}(P_e, E_1[n]) = \dim_{\mathbb{C}} \text{Hom}_{\mathcal{C}}(P_e, E_1[n]) = 1. \]
The corresponding short exact sequences are
\[ 0 \to P_e \to L \to E_1[n] \to 0, \]
\[ 0 \to P_e \to E_n[n] \to E_1[n-1] \oplus E_n/P_e \to 0 \]
and
\[ 0 \to P_e \to E_1[n] \to I \to 0. \]
The last exact sequence is induced by the following commutative diagram
\[
\begin{array}{ccccccccc}
0 & \rightarrow & P_e & \rightarrow & E_n[n] & \rightarrow & E_1[n] & \rightarrow & E_n/P_e & \rightarrow & 0 \\
0 & \rightarrow & P_e & \rightarrow & E_n[n] & \rightarrow & X & \rightarrow & 0.
\end{array}
\]
Indeed, since $E_n/P_e$ is a indecomposable preinjective module of defect 1 and using the snake lemma on the diagram, we have $X \cong E_1[n-1] \oplus E_n/P_e$. Hence, by Theorem 2.5 we have
\[ X_{P_e} X_{E_1[n]} = X_L + X_{E_2[n-1]} X_{\tau^{-1}(E_n/P_e)}. \]
Consider the commutative diagram of exact sequences
\[
\begin{array}{ccccccccc}
P_e & \rightarrow & P_e & \rightarrow & E_1[n-1] & \rightarrow & E_1[n] & \rightarrow & E_n & \rightarrow & 0 \\
0 & \rightarrow & E_1[n-1] & \rightarrow & I & \rightarrow & E_n/P_e & \rightarrow & 0.
\end{array}
\]
The short exact sequence at the bottom induces a triangle in $\mathcal{C}(Q)$
\[ E_2[n-1] \to \tau^{-1} I \to \tau^{-1}(E_n/P_e) \to E_1[n-1]. \]
The short exact sequence
\[ 0 \to E_2[n-2] \to E_2[n-1] \to E_n \to 0 \]
induces the short exact sequence
\[ 0 \to E_2[n-2] \oplus P_e \to E_2[n-1] \to E_n/P_e \to 0. \]
Hence, we have the triangle in $\mathcal{C}(Q)$
\[ \tau^{-1}(E_n/P_e) \to E_2[n-2] \oplus P_e \to E_2[n-1] \to E_n/P_e. \]
On the other hand, apply $\text{Hom}(-, E_1[n-1])$ to the short exact sequence
\[ 0 \to P_e \to E_n \to E_n/P_e \to 0 \]
to obtain $\dim_{\mathbb{C}} \text{Ext}_{\mathcal{C}}^1(E_n/P_e, E_1[n-1]) = \dim_{\mathbb{C}} \text{Ext}_{\mathcal{C}}^1(E_n, E_1[n-1]) = 1$. Since $\text{Hom}_{\mathcal{C}}(E_n/P_e, E_2[n-1]) = 0$, we deduce
\[ \text{Ext}_{\mathcal{C}}^1(E_n/P_e, E_2[n-1]) = \text{Ext}_{\mathcal{C}}^1(E_n/P_e, E_1[n-1]) = 1. \]
Hence, using Theorem 2.5 again, we have
\[ X_{E_2[n-1]} X_{\tau^{-1}(E_n/P_e)} = X_{\tau^{-1} I} + X_{E_2[n-2]} X_{P_e}. \]
Let $E$ be a regular simple module of dimension vector $\delta$. Then by the similar discussion as above and Theorem 2.5, we have

$$X_P X_E = X_L + X_{\tau^{-1} I}.$$  

Therefore, we obtain the identity

$$X_{E_1[n]} = X_E + X_{E_2(n-2)}.$$  

By Lemma 6.4 in [Du2], we have

$$X_{E_i[n]} = X_E + X_{\tau^{-1}(E_i(n-2))},$$

for $i = 1, \ldots, n$. This identity is called the difference property.

If $e$ is a source, then $I_e$ is a simple injective module and $(P, I_e)$ is an orthogonal exceptional pair. Let $T$ be as above. Then $\text{reg.socle}(E_n[n])_e = (E_n)_e \neq 0$ and then $E_n[n] \in \Omega'$. It is easy to know that

$$\dim \text{Ext}^1_C(Q(I_e, E_n[n])) = \dim \text{Hom}_C(E_1[n], I_e) = \dim \text{Hom}_C(Q(E_n[n], I_e) = 1.$$  

The corresponding short exact sequences are

$$0 \to E_n[n] \to L' \to I_e \to 0, \quad 0 \to P \to E_n[n] \to I_e \to 0$$

and

$$0 \to Y \to E_1[n] \to I_e \to 0.$$  

Let

$$0 \to P' \to E_1[n-1] \to 0.$$  

be a short exact sequence. Here, $P'$ is a preprojective module. Then we have

$$Y \cong P' \oplus E_1[n-1]$$

by the following commutative diagram

$$\begin{array}{ccc}
0 & \to & Y \\
\downarrow & & \downarrow \\
0 & \to & E_1[n] \\
& \downarrow & \\
& \downarrow & \\
0 & \to & I_e \\
& \downarrow & \\
& \downarrow & \\
0 & \to & E_n \\
& \downarrow & \\
& \downarrow & \\
0 & \to & I_e \\
& \downarrow & \\
& \downarrow & \\
0 & \to & 0.
\end{array}$$

The short exact sequence

$$0 \to Y \to E_1[n] \to I_e \to 0$$

induces a triangle in $C(Q)$

$$I_e \to \tau (P' \oplus E_n[n-1]) \to E_n[n] \to \tau I_e.$$  

Hence, by Theorem 2.5, we have

$$X_{I_e} X_{E_n[n]} = X_{L'} + X_{E_1[n-1]} X_{\tau P'}.$$  

The short exact sequence

$$0 \to P' \to P \to E_n[n-1] \to 0$$

induces the triangle in $C(Q)$

$$\tau P' \to \tau P \to E_n[n-1] \to \tau^2 P'.$$  

The short exact sequence

$$0 \to P' \to E_n[n-1] \to E_1[n-2] \oplus I_e \to 0$$

induces the triangle in $C(Q)$

$$E_n[n-1] \to E_1[n-2] \oplus I_e \to \tau P' \to E_1[n-1].$$  

Hence, we have

$$X_{E_n[n-1]} X_{\tau P'} = X_{\tau P} + X_{E_1[n-2]} X_{I_e}.$$
Let $E$ be a regular simple module of dimension vector $\delta$. Then by the similar discussion as above and Theorem 8.2, we have
\[ X_L X_E = X_{L'} + X_{T P}. \]
Therefore, we obtain the identity
\[ X_{E_i[n]} = X_E + X_{E_{i[n-2]}}. \]
By Lemma 6.4 in [Du2], we have
\[ X_{E_i[n]} = X_E + X_{T \tau^i(E_{i[n-2]})}. \]
for $i = 1, \cdots, n$. We have proved the following theorem

**Theorem 8.2.** Let $Q$ be a tame quiver without oriented cycles and $T$ be a tube of rank $n \geq 1$ with regular simple modules $E_1, \cdots, E_n$ such that $E_1 = \tau E_2, \cdots, E_n = \tau E_1$. Assume $\delta$ is the minimal imaginary root for $Q$. Then we have
\[ X_{E_i[n]} = X_E + X_{T \tau^i(E_{i[n-2]})} \]
for $i = 1, \cdots, n$ and any regular simple module $E$ of dimension vector $\delta$.

In [Du2], the author defined the generic variables for any acyclic quiver $Q$. In particular, if $Q$ is a tame quiver, the author showed the the set of generic variables is the following set
\[ B_\delta(Q) := \{ X_L, X^m L | L \in C(Q), \text{Ext}^1(L, L) = 0, m \geq 1, R \text{ is a regular rigid module} \}. \]
The author proved that if the difference property holds for $Q$, then $B_\delta(Q)$ is a $\mathbb{Z}$-basis of the cluster algebra $\mathcal{E}\mathcal{H}(Q)$. If $Q$ is of type $\tilde{A}_{p,q}$, the author showed the difference property holds. Hence, by Theorem 8.2 we have the following corollary.

**Corollary 8.3.** The set $B_\delta(Q)$ is a $\mathbb{Z}$-basis of the cluster algebra $\mathcal{E}\mathcal{H}(Q)$. There is a unipotent transition matrix between $B_\delta(Q)$ and $B(Q)$.

**Proof.** With the notation in Theorem 8.2 we have
\[ X^m_L X_R = X^m_L X_R = (X_{E_i[n]} - X_{E_i[n-2]})^m X_R = (X_{E_i[n]} + \sum_{\text{dim } L = \text{m}\delta} n_L X_L) X_R \]
where $L$ is a regular module and $n_L \in \mathbb{Z}$. Using Lemma 3.2, Theorem 7.1 and Theorem 1.1 we obtain
\[ (X_{E_i[n]} + \sum_{\text{dim } L = \text{m}\delta} n_L X_L) X_R = X_{T \oplus R'} + \sum_{\text{dim } Y = \text{dim } (T \oplus R')} n_Y X_Y. \]
where both $T \oplus R'$ and $Y$ are regular modules such that $X_{T \oplus R'}, X_Y \in B(Q)$ and $n_Y \in \mathbb{Z}$. This completes the proof of the corollary.

**Corollary 8.4.** There is a unipotent transition matrix between $B_\delta(Q)$ and $B'(Q)$.

We will illustrate Theorem 8.2 by two examples in the following.

8.2. Let $Q$ be a quiver of type $\tilde{A}_{p,q}$ as follows.
\[
\begin{array}{c}
1 \\
\downarrow \quad \downarrow \\
p+q & \cdots & p+1
\end{array}
\]
\[
\begin{array}{c}
2 \\
p + 1 \\
p + 2
\end{array}
\]
The difference property has been proved to hold for $\tilde{A}_{p,q}$ by the different method in [Du2]. Here we give an alternative proof. There are two non-homogeneous tubes (denoted by $T_0, T_\infty$) in the set of indecomposable regular modules. The minimal
imaginary root of $Q$ is $(1, 1, \cdots, 1)$. Let $\lambda \in \mathbb{C}^*$ and $E(\lambda)$ be the regular simple module as follows

![Diagram](image_url)

Its proper submodules are of the forms as follows:

![Diagram](image_url)

The regular simple modules in $T_0$ are

$E_{t}^{(0)} = S_{p+t+1}$ for $1 \leq t \leq q - 1$, $E_{q}^{(0)}:

![Diagram](image_url)

The regular module $E_{1}^{(0)}[q]$ has the form as follows

![Diagram](image_url)

Its proper submodules are of the following two forms:

![Diagram](image_url)

and

![Diagram](image_url)

The proper submodules with the second form are the submodules of $E_{2}^{(0)}[q - 2]$ with regular socle $S_{p+2}$. Hence, we have

$$
\chi(Gr_{\varepsilon}(E_{1}^{(0)}[q])) = \chi(Gr_{\varepsilon}(E(\lambda))) + \chi(Gr_{\varepsilon}(E_{2}^{(0)}[q - 2]))
$$

for any dimension vector $\varepsilon$ and $\varepsilon' = \varepsilon - \dim S_{p+2}$. Indeed, by [Hu1] Lemma 1, we have

$$
\dim E_{1}^{(0)}R + \dim E_{q}^{(0)}R^{\varepsilon'} = \dim E_{1}^{(0)} + \dim E_{q}^{(0)}.
$$

By definition, we deduce the difference property for $\tilde{A}_{p,q}$. 
8.3. Let $Q$ be a tame quiver of type $\tilde{D}_m$ for $m \geq 4$ as follows

\[
\begin{array}{c}
m \\
\vdots \\
m - 1 \\
m + 1
\end{array} \xleftarrow{1} \begin{array}{c} m \rightarrow \cdots \rightarrow 4 \rightarrow 3 \\
\end{array} \xleftarrow{1} 2
\]

There are three non-homogeneous tubes (denoted by $T_0, T_1, T_\infty$). The minimal imaginary root of $Q$ is $(1, 1, 2, \cdots, 2, 1, 1)$. Let $\lambda \in \mathbb{C}$ and $E(\lambda)$ be the regular module as follows:

\[
\begin{array}{c}
\mathbb{C} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\mathbb{C}^2 \xleftarrow{1} \cdots \xleftarrow{1} \mathbb{C}^2 \\
\mathbb{C} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{array} \xleftarrow{1} \begin{array}{c} (1, 1) \\
(\lambda, 1)
\end{array} \xleftarrow{1} \mathbb{C}
\]

We note that $E(0) \in T_0$ and $E(1) \in T_1$. Replacing $(\lambda, 1)$ by $(1, 0)$, we obtain a regular module $E(\infty) \in T_\infty$. The regular simple modules in $T_1$ are

\[
E_t^{(1)} = S_{t+2} \text{ for } 1 \leq t \leq m - 3, \quad E_{m-2}^{(1)} =: 
\]

\[
\begin{array}{c}
\mathbb{C} \\
\mathbb{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xleftarrow{1} \cdots \xleftarrow{1} \mathbb{C} \xleftarrow{1} \mathbb{C}
\end{array}
\]

We note that $E(1) = E_1^{(1)}[m - 2]$.

Let $\mathbf{e} = (e_1, \cdots, e_{m+1})$ be a dimension vector satisfying one of the following condition:

1. $e_2 \neq 0$;
2. $e_2 = 0, e_3 = 0$;
3. $e_2 = 0, e_3 = 2$.

Let $M$ be a submodule of $E(\lambda)$ for $\lambda \in \mathbb{C}^*$ with $\dim M = \mathbf{e}$. Then it is clear that there are unique submodule $M(0)$ of $E(0)$ and submodule $M(1)$ of $E(1)$. Hence, we have

\[
\chi(\text{Gr}_\mathbf{e}(E(\lambda))) = \chi(\text{Gr}_\mathbf{e}(E(0))) = \chi(\text{Gr}_\mathbf{e}(E(1))) \text{ and } \chi(\text{Gr}_\mathbf{e}(E_1^{(1)}[m - 3])) = 0
\]

for $\lambda \in \mathbb{C}^*$. Now we consider the proper submodules $M$ of $E(1)$ with dimension vector $\mathbf{e}$ satisfying $e_2 = 0, e_3 = 1$. If $M = X \oplus Y$, we assume that $X$ satisfies $(\dim X)_3 = 0$, then $X$ is also the submodule of $E(\lambda)$ for any $\lambda \in \mathbb{C}^*$. Hence, we can assume that $M$ is indecomposable with $(\dim M)_3 = 1$ and $(\dim M)_2 = 0$. Then $M$ has the form as follows

\[
\begin{array}{c}
\cdots \xleftarrow{1} \mathbb{C} \xleftarrow{1} \cdots \\
0 \xleftarrow{1} \mathbb{C} \xleftarrow{1} 0
\end{array}
\]
Let \( N \) be an indecomposable submodule of \( E(\lambda) \) such that \((\dim N)_3 = 1\) and \((\dim N)_2 = 0\). Then \( N \) has the form as follows

\[
\begin{array}{c}
0 \\
\downarrow \\
C \\
\rightarrow \\
C \\
\rightarrow \\
\rightarrow \cdots \\
C \\
\rightarrow \\
0 \\
\end{array}
\]

It corresponds to the submodule of \( E(1) \)

\[
\begin{array}{c}
0 \\
\downarrow \\
C \\
\rightarrow \\
C \\
\rightarrow \\
\rightarrow \cdots \\
0 \\
\end{array}
\]

Note that \( \chi(\text{Gr}_E(E(1)[m-3])) = \chi(\text{Gr}_E(E(1)[m-4])) \). Hence, we have

\[
\chi(\text{Gr}_E(E(1))) = \chi(\text{Gr}_E(E(\lambda))) + \chi(\text{Gr}_E(\text{dim} S_3(E(1)[m-4])).
\]

REFERENCES

[BMRRT] A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, Tilting theory and cluster combinatorics. Advances in Math. 204 (2006), 572-618.

[CB] W. Crawley-Boevey, Lectures on representations of quivers, 1992.

[CC] P. Caldero and F. Chapoton, Cluster algebras as Hall algebras of quiver representations, Comm. Math. Helvetici, 81 (2006), 596-616.

[CK] P. Caldero and B. Keller, From triangulated categories to cluster algebras, Invent. math. 172 (2008), no. 1, 169-211.

[CK2] P. Caldero and B. Keller, From triangulated categories to cluster algebras II, Annales Scientifiques de l’Ecole Normale Superieure, 39 (4) (2006), 83-100.

[CZ] P. Caldero and A. Zelevinsky, Laurent expansions in cluster algebras via quiver representations, Moscow Math. J. 6 (2006), no. 2, 411-429.

[Du1] G. Dupont, Cluster multiplication in stable tubes via generalized Chebyshev polynomials, arXiv:math.RT/0801.3964.

[Du2] G. Dupont, Generic variables in Acyclic cluster algebras, arXiv:math.RT/0811.2909v1.

[DR] V. Dlab and C. M. Ringel, Indecomposable representations of graphs and algebras, Memories of the AMS, 173 (1976), 1-57.

[DX1] M. Ding and F. Xu, A \( Z \)-basis for the cluster algebra of type \( \tilde{D}_4 \), to appear in Algebra Colloquium.

[DX2] M. Ding and F. Xu, Notes on the cluster multiplication formulas for \( 2 \)-Calabi-Yau categories, arXiv:1004.1465.

[DXX] M. Ding, J. Xiao and F. Xu, A \( Z \)-basis for the cluster algebra associated to an affine quiver, arXiv:0811.3876.

[FZ] S. Fomin and A. Zelevinsky, Cluster algebras. I. Foundations. J. Amer. Math. Soc. 15 (2002), no. 2, 497-529.

[FZ1] S. Fomin and A. Zelevinsky, Cluster algebras. II. Finite type classification. Invent. math. 154 (2008), 63-121.

[GLS] C. Geiss, B. Leclerc, and J. Schröer, Cluster algebra structures and semi-canonical bases for unipotent groups, ArXiv:math/0709.0092, 2008.

[Hu1] A. Hubery, Acyclic cluster algebras via Ringel-Hall algebras, preprint.

[Hu2] A. Hubery, Hall polynomial for affine quivers, arXiv:math.RT/0706.1784v2.

[Kac] V.G. Kac, Infinite root systems, representations of graphs and invariant theory II, Journal of algebra, 78 (1982) 163–180.

[MRZ] R. Marsh, M. Reineke, and A. Zelevinsky, generalized associahedra via quiver representations. Trans.AMS, 355(1) (2003), 4171-4186.

[Pa] Y. Palu, Cluster characters II: a multiplication formula, arXiv:0903.3281v1, 2009.

[SZ] P. Sherman and A. Zelevinsky, Positivity and canonical bases in rank 2 cluster algebras of finite and affine types, Moscow Math. J. 4 (2004), no.4, 947-974.

[XX] J. Xiao and F. Xu, Green’s formula with \( C^* \)-action and Caldero-Keller’s formula, arXiv:math.RT/0707.1175. To appear in Prog. Math.

[Xu] F. Xu, On the cluster multiplication theorem for acyclic cluster algebras, 362(2) (2010), 753-776.
[Zhu] B. Zhu, *Equivalence between cluster categories*, Journal of algebra, 304:832-850, 2006.

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