On a Factorization of Symmetric Matrices and Antilinear Symmetries

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Abstract

We present a simple proof of the factorization of (complex) symmetric matrices into a product of a square matrix and its transpose, and discuss its application in establishing a uniqueness property of certain antilinear operators.

1 Introduction

One of the interesting results of linear algebra is that every square matrix may be factored into the product of two symmetric matrices [2, 7]. The factorization of symmetric matrices into a product of a square matrix and its transpose is however less known. In fact, there seems to be no mention of this factorization in modern texts on linear algebra. The purpose of this note is to present a simple derivation of this particular factorization of symmetric matrices and to discuss its application in establishing a uniqueness property of certain antilinear operators.

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2 Notation and Definitions

In this note we shall express the complex-conjugate, the transpose, and the conjugate-
transpose (adjoint) of a matrix (an operator) \(m\) by \(\bar{m}\), \(m^T\), and \(m^*\), respectively, and identify
the elements of \(\mathbb{C}^\ell\) by columns of \(\ell\) complex numbers. Then the Euclidean inner-product of
two vectors \(\vec{w}_1\) and \(\vec{w}_2\) takes the form \(\vec{w}_2^* \vec{w}_1\).

**Definition 1:** Let \(S\) be a set, then the Kronecker delta function \(\delta : S^2 \to \{0,1\}\) is
defined by

\[
\forall a, b \in S, \quad \delta(a, b) = \delta_{ab} := \begin{cases} 
1 & \text{for } a = b \\
0 & \text{for } a \neq b.
\end{cases}
\]

**Definition 2:** A function \(\mathcal{X} : \mathcal{H} \to \mathcal{H}\) acting in a complex vector space \(\mathcal{H}\) is said to
be an antilinear operator if for all \(x, y \in \mathbb{C}\) and \(\phi, \psi \in \mathcal{H}\), \(\mathcal{X}(x\phi + y\psi) = \bar{x}\mathcal{X}\phi + \bar{y}\mathcal{X}\psi\),
where \(\mathcal{X}\phi\) means \(\mathcal{X}(\phi)\).

**Definition 3:** An antilinear operator \(\mathcal{X} : \mathcal{H} \to \mathcal{H}\) acting in a complex inner-product
space \(\mathcal{H}\) with inner-product \((\ ,\ )\) is said to be symmetric or Hermitian if for all \(\phi, \psi \in \mathcal{H}\),
\((\mathcal{X}\phi, \psi) = (\mathcal{X}\psi, \phi)\).

**Definition 4:** A linear operator \(H : \mathcal{H} \to \mathcal{H}\) acting in a complex inner-product space
\(\mathcal{H}\) is said to have a symmetry generated by a function \(X : \mathcal{H} \to \mathcal{H}\) or simply a \(X\)-
symmetry if \(H\) and \(X\) commute, i.e., \([H, X] := HX - XH = 0\), where \(0\) stands for the
zero operator. A symmetry generated by an antilinear operator is called an antilinear
symmetry.

**Definition 5:** Let \(H : \mathcal{H} \to \mathcal{H}\) be a linear operator acting in a complex inner-product
space \(\mathcal{H}\) and \(G : \mathcal{H} \to \mathcal{H}\) be an Hermitian invertible linear or antilinear operator. Then
\(H\) is said to be \(G\)-Hermitian \([\Box]\) or \(G\)-pseudo-Hermitian \([\Box]\) if \(H^* = GHG^{-1}\).
3 Motivation: Consequences of antilinear symmetries

Consider a diagonalizable linear operator $H : \mathcal{H} \rightarrow \mathcal{H}$ acting in a finite-dimensional complex inner-product space $\mathcal{H}$ with inner-product $(\ , \ )$. Let $n$ label the eigenvalues $E_n$ of $H$, $\mu_n$ be the multiplicity of $E_n$, and $\psi_{n,a}$ be the eigenvectors corresponding to the eigenvalue $E_n$ where $a \in \{1, 2, \ldots, \mu_n\}$ is the degeneracy label. Then it is well-known that the adjoint $H^*$ of $H$ is diagonalizable; the eigenvalues $\tilde{E}_n$ of $H^*$ are complex conjugate of those of $H$, i.e., $\tilde{E}_n = \overline{E}_n$; the multiplicity of $\tilde{E}_n$ is equal to $\mu_n$; and one can choose the eigenvectors $\phi_{n,a}$ of $H^*$ in such a way that for all spectral labels $m, n$ and degeneracy labels $a, b$, $(\phi_{n,a}, \psi_{n,a}) = \delta_{n,m} \delta_{a,b}$.

Clearly, both sets of eigenvectors $\psi_{n,a}$ of $H$ and eigenvectors $\phi_{a,n}$ of $H^*$ form bases of $\mathcal{H}$; $\{\psi_{n,a}, \phi_{n,a}\}$ is a complete biorthonormal system. Recently, we have shown that if the eigenvalues of $H$ are real, then $H$ has an antilinear symmetry. More generally, we proved the following theorem.

**Theorem 1:** The presence of an antilinear symmetry of $H$ is a necessary and sufficient condition for the eigenvalues of $H$ to either be real or come in complex conjugate pairs.

The proof of Theorem 1 uses the following lemma.

**Lemma 1:** Every diagonalizable linear operator $H : \mathcal{H} \rightarrow \mathcal{H}$ acting in a finite-dimensional complex inner-product space $\mathcal{H}$ is $T$-Hermitian,

$$H^* = THT^{-1}, \tag{1}$$

for some Hermitian, invertible, antilinear operator $T : \mathcal{H} \rightarrow \mathcal{H}$.

It turns out that any such $T$ may be expressed in terms of the eigenvectors $\phi_{n,a}$ of $H^*$ according to

$$\forall \zeta \in \mathcal{H}, \quad T\zeta = \sum_n \sum_{a,b=1}^{\mu_n} c_{ba}^{(n)}(\phi_{n,a}, \zeta)\phi_{n,b}, \tag{2}$$

where $c_{ab}^{(n)}$ are the entries of symmetric invertible $\mu_n \times \mu_n$ matrices $c^{(n)}$. 

3
Note that Theorem 1 and Lemma 1 have infinite-dimensional generalizations for linear operators $H$ admitting a complete biorthonormal system of eigenvectors $\mathcal{E}$.

Next, consider a general basis transformation,

$$
\phi_{n,a} \rightarrow \phi'_{n,a} := \sum_{b=1}^{\mu_n} v_{ba}^{(n)} \phi_{n,b}
$$

where $v_{ab}^{(n)}$ are the entries of an invertible $\mu_n \times \mu_n$ matrix $v^{(n)}$. In terms of the transformed basis vectors $T$ has the form: $\forall \zeta \in \mathcal{H}, T\zeta = \sum_n \sum_{a,b=1}^{\mu_n} c'_{ba}(\phi'_{n,a}, \zeta) \phi'_{n,b}$, where $c'(n)$ are related to $c(n)$ according to $c(n) = v(n) c'(n) v(n)^T$. This equation indicates that the issue of the uniqueness of $T$ for a given $H$ is related to whether one can find for each $c(n)$ an invertible matrix $v(n)$ such that $c(n) = v(n) v(n)^T$. In the remainder of this note we shall give a proof of the fact that this is indeed possible, and one can transform to a basis in which $T$ has the (canonical) form (2) with $c_{ab}^{(n)} = \delta_{ab}$ for all $n$.

4 Factorization of Symmetric Matrices

**Theorem 2:** A square matrix $c$ is symmetric if and only if it can be written as $c = vv^T$ for some square matrix $v$.

**Proof:** If $c = vv^T$ then clearly $c$ is symmetric. To prove the converse we use induction on the dimension $n$ of the matrix $c$. For $n = 1$, $c = vv^T = v^2$ is trivially satisfied by letting $v := \sqrt{c}$. By induction hypothesis we assume that for all $k \in \{2, \cdots, n\}$, every $k \times k$ symmetric matrix $c$ can be written in the form $c = vv^T$ for some $k \times k$ matrix $v$. Now let $C$ be an $(n+1) \times (n+1)$ symmetric matrix. Then $C$ has at least one eigenvector $\vec{e}$ i.e., there are $\vec{e} \in \mathbb{C}^{n+1} - \{0\}$ and $\lambda \in \mathbb{C}$ such that

$$
C\vec{e} = \lambda \vec{e}.
$$

Now let $\mathcal{V} := \{\vec{w} \in \mathbb{C}^{n+1} | \vec{w}^* \vec{e} = 0\}$ be the orthogonal complement of $\vec{e}$. Clearly $\mathcal{V}$ is an $n$-dimensional vector subspace of $\mathbb{C}^{n+1}$. Next, consider the following two possibilities.
(i) \( \vec{e} \notin \mathcal{V} \). In this case, choose a basis \( \{\vec{e}_1, \vec{e}_n, \cdots, \vec{e}_n\} \) of \( \mathcal{V} \) and let \( \vec{e}_{n+1} := \vec{e} \). Then \( \{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n, \vec{e}_{n+1}\} \) is a basis of \( \mathbb{C}^{n+1} \) and the matrix \( A := (\vec{e}_1, \vec{e}_n, \cdots, \vec{e}_n, \vec{e}_{n+1}) \) is invertible. Note that for all \( \ell \in \{1, 2, \cdots, n\} \), \( \vec{e}_\ell \in \mathcal{V} \), and \( \vec{e}_\ell^T \vec{e}_{n+1} = 0 \). This in turn implies that the matrix \( A^TCA \) which is symmetric has the block form

\[
A^TCA = \begin{pmatrix}
\vec{c} & \vec{0} \\
\vec{0}^T & \lambda^2
\end{pmatrix},
\]

where \( \vec{c} \) is a symmetric \( n \times n \) matrix. By induction hypothesis there is an \( n \times n \) matrix \( \tilde{\vec{v}} \) such that \( \vec{c} = \tilde{\vec{v}}^T \tilde{\vec{v}} \). Now let \( B \) be the \( (n + 1) \times (n + 1) \) matrix

\[
B := \begin{pmatrix}
\tilde{\vec{v}}^T & \vec{0} \\
\vec{0}^T & \lambda
\end{pmatrix},
\]

and \( V := (BA^{-1})^T \). Then in view of (3) and (4), \( B^T B = A^TCA \) and

\[
VV^T = (BA^{-1})^T BA^{-1} = A^{-1T} B^T BA^{-1} = C.
\]

This completes the proof for case (i).

(ii) \( \vec{e} \in \mathcal{V} \), i.e., \( \vec{e}^T \vec{e} = 0 \). In this case, let \( \mathcal{V}' := \{\vec{w} \in \mathcal{V} | \vec{w}^T \vec{e} = 0\} \) be the orthogonal complement of \( \vec{e} \) in \( \mathcal{V} \), \( \{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_{n-1}\} \) be a basis of \( \mathcal{V}' \), \( \vec{e}_n = \vec{e} \), and \( \vec{e}_{n+1} = \vec{e} \). Then \( \{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n, \vec{e}_{n+1}\} \) is a basis of \( \mathbb{C}^{n+1} \) and the matrix \( A' := (\vec{e}_1, \vec{e}_n, \cdots, \vec{e}_n, \vec{e}_{n+1}) \) is invertible. Note that for all \( \ell \in \{1, 2, \cdots, n - 1\} \), \( \vec{e}_\ell \in \mathcal{V} \), and \( \vec{e}_\ell^T \vec{e}_{n+1} = 0 \). Furthermore, \( \vec{e}_{n+1}^T \vec{e}_{n+1} = \vec{e}^T \vec{e} = 0 \) and \( \alpha := \vec{e}_n^T \vec{e}_{n+1} = \vec{e}^T \vec{e} \in \mathbb{R}^+ \).

In view of these relations and (4),

\[
C' := A'^TCA' = \begin{pmatrix}
\vec{c}_{1,1} & \vec{c}_{1,2} & \cdots & \vec{c}_{1,n-1} & \vec{c}_{1,n} & 0 \\
\vec{c}_{2,1} & \vec{c}_{2,2} & \cdots & \vec{c}_{2,n-1} & \vec{c}_{2,n} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & 0 \\
\vec{c}_{n-1,1} & \vec{c}_{n-1,2} & \cdots & \vec{c}_{n-1,n-1} & \vec{c}_{n-1,n} & 0 \\
\vec{c}_{n,1} & \vec{c}_{n,2} & \cdots & \vec{c}_{n,n-1} & \vec{c}_{n,n} & \lambda \alpha \\
0 & 0 & \cdots & 0 & \lambda \alpha & 0
\end{pmatrix},
\]

where \( \vec{c}_{i,j} := \vec{e}_i^T \vec{c} \vec{e}_j \) are the entries of a symmetric \( n \times n \) matrix \( \vec{c} \). Now if \( \lambda = 0 \), \( C' \) is block-diagonal and the argument given in case (i) leads to a proof of the
theorem. This leaves the case $\lambda \neq 0$. In this case, let $A := A'D$ where $D$ is the $(n+1) \times (n+1)$ matrix

$$D = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & x_1 \\
0 & 1 & 0 & \cdots & 0 & 0 & x_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & x_{n-1} \\
0 & 0 & 0 & \cdots & 0 & 1 & x_n \\
y_1 & y_2 & y_3 & \cdots & y_{n-1} & y_n & 0
\end{pmatrix},$$

with $x_1, x_2, \cdots x_{n-1}$ being arbitrary complex numbers,

$$x_n := -(\lambda \alpha)^{-1},$$

and for all $i \in \{1, 2, \cdots, n\}$

$$y_i := \sum_{j=1}^{n} \tilde{c}_{i,j} x_j.$$

Then a simple computation shows that

$$A^TCA = D^T C'D = \begin{pmatrix}
\tilde{c}_{ij} & 0 \\
0 & \lambda' \\
\check{0} & \lambda' \check{2}
\end{pmatrix},$$

where $\tilde{c}_{i,j}$ are the entries of a symmetric $n \times n$ matrix $\tilde{c}$ and $\lambda' \in \mathbb{C}$. Therefore, we can use the argument given in case (i) to show the existence of an $(n+1) \times (n+1)$ matrix $B$ satisfying

$$A^TCA = B^TB.$$

The proof of the theorem will be complete if we show that the matrix $A = A'D$ is invertible. Because $A'$ is invertible, it suffices to show the existence of $x_1, x_2, \cdots x_{n-1}$ for which $\det D \neq 0$. We can use the properties of the determinant and Equations (8) and (10) to compute

$$\det D = -\sum_{i=1}^{n} x_i y_i$$

$$= -\sum_{i=1}^{n-1} \tilde{c}_{i,i} x_i^2 - 2 \sum_{i<j}^{n-1} \tilde{c}_{i,j} x_i x_j + 2(\lambda \alpha)^{-1} \sum_{i=1}^{n-1} \tilde{c}_{i,i} x_i - (\lambda \alpha)^{-2} \tilde{c}_{n,n}.$$
Suppose that for all values of \(x_1, x_2, \cdots, x_{n-1}\), \(\det D = 0\). This implies that \(\tilde{c}' = 0\) in which case \(C'\) will have the form

\[
C' = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & \lambda \alpha \\
0 & 0 & \ldots & 0 & \lambda \alpha & 0
\end{pmatrix}.
\]

By induction hypothesis we have a \(2 \times 2\) matrix \(m\) satisfying

\[
\begin{pmatrix}
0 & \lambda \alpha \\
\lambda \alpha & 0
\end{pmatrix} = m^T m.
\]

Therefore, setting

\[
B := \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & m_{1,2} & m_{1,2} \\
0 & 0 & \ldots & 0 & m_{2,1} & m_{2,2}
\end{pmatrix}
\]

and \(V := (BA^{-1})^T\), we have \(C' = B^T B\) and \(C = V V^T\). If \(\tilde{c}' \neq 0\) then there are values of \(x_1, x_2, \cdots, x_{n-1}\) for which \(D\) is a nonsingular matrix and \(A = A'D\) is invertible. Hence we can set \(V := (BA^{-1})^T\) and use (II) to show that \(C = V V^T\). □

5 Concluding remarks

1. The factorization \(c = vv^T\) established in Theorem 2 is invariant under the transformation \(v \rightarrow v' = vo\) where \(o\) is an arbitrary (complex) orthogonal matrix. In particular, one may choose \(o\) so that the factorizing matrix \(v'\) has a simple form.

2. In view of the discussion of Section 3, one has the following consequence of Theorem 2.
Corollary 1: Up to basis transformations (3), there is a unique antilinear operator $\mathcal{T}$ satisfying Equation (1), namely

$$\forall \zeta \in \mathcal{H}, \quad \mathcal{T}\zeta = \sum_n \sum_{a=1}^{\mu_n} (\phi_{n,a}, \zeta) \phi_{n,a},$$

(13)

3. For a self-adjoint linear operator $H$, one can set $\phi_{n,a} = \psi_{n,a}$ and use the completeness of the eigenvectors $\psi_{n,a}$ and (13) to deduce $\mathcal{T}^2 = I$, where $I$ is the identity operator. Furthermore, noting that in this case (1) is equivalent to $\mathcal{T}$-symmetry of $H$, one can prove the following.

Corollary 2: Every self-adjoint linear operator $H$ has an antilinear symmetry generated by a Hermitian, invertible, antilinear operator $\mathcal{T}$ satisfying $\mathcal{T}^2 = I$.

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