REALIZATIONS AND FACTORIZATIONS OF POSITIVE
DEFINITE KERNELS

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Abstract. Given a fixed sigma-finite measure space \((X, B, \nu)\), we shall study an associated family of positive definite kernels \(K\). Their factorizations will be studied with view to their role as covariance kernels of a variety of stochastic processes. In the interesting cases, the given measure \(\nu\) is infinite, but sigma-finite. We introduce such positive definite kernels \(K(\cdot, \cdot)\) with the two variables from the subset of the sigma-algebra \(B\), sets having finite \(\nu\) measure. Our setting and results are motivated by applications. The latter are covered in the second half of the paper. We first make precise the notions of realizations and factorizations for \(K\); and we give necessary and sufficient conditions for \(K\) to have realizations and factorizations in \(L^2(\nu)\). Tools in the proofs rely on probability theory and on spectral theory for unbounded operators in Hilbert space. Applications discussed here include the study of reversible Markov processes, and realizations of Gaussian fields, and their Itô-integrals.

1. Introduction

We study a family of problems from measurable dynamics and their connection to the theory of positive definite functions (kernels). In particular, the aim of the present paper is two-fold: (i) An extension of the traditional reproducing kernel Hilbert space (RKHS) theory, from the more traditional context of Aronszajn [Aro43, Aro50] to a measurable category. (ii) A characterization of when a positive definite kernel assumes realization/factorization in the general setting.

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1
Such an extension will adapt much better to a host of applications, including problems from probability theory, from stochastic processes [AJ12, JT16a], from mathematical physics [HKL+14, RAKK05], and for the latter, especially to the context of reversible processes (see, e.g., [TB13, CXY15, Sko13, Her12, DJ11, Rob11], and also [BP17, PSS16]).

For applications to random processes, a kernel in the sense of Aronszajn will typically represent a covariance kernel. Our applications include a new spectral theoretic analysis of (a) transient Markov processes, and of (b) generalized Gaussian fields and their Ito-integrals.

Now in the standard approach to RKHSs of Aronszajn, one starts with a positive definite (p.d.) function, $K$ on $X \times X$ (often called a p.d. kernel), where $X$ is a given set. The term “reproducing” refers to the fact that for every $f$ in $H$, the values $f(x)$ can be reproduced from the inner Hilbert-product in $H$. With a standard construction, starting with $X$ and $K$, one then arrives at a Hilbert space $H$ of functions on $X$, the so called reproducing kernel Hilbert space (RKHS). It depends on the pair $(X,K)$ of course; so is denoted $H(K)$ when the kernel is not given from the context. A priori, the set $X$ is not given any additional structure, but a key point is that both $K$ and the functions $f$ in the RKHS $H$ are defined everywhere on $X$. If for example, $X$ is a complex domain, in interesting applications, then the functions in $H(K)$ will be analytic, or in the case of the familiar RKHS of Bargmann, the functions in $H(K)$ will be entire analytic. If $X$ has a topology, and if $K$ is assumed continuous, then the functions in $H(K)$ will then also be continuous.

A novelty in the present paper is the starting point being a fixed measure space $(X,\mathcal{B},\nu)$ where $\nu$ is assumed sigma-finite. Set $\mathcal{B}_{\text{fin}} := \{A \in \mathcal{B} : \nu(A) < \infty\}$. We shall then consider p.d. functions $K$ on $\mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}}$. Our main result is a characterization of the case when the corresponding RKHS $H(K)$ may be realized in $L^2(\nu)$.

In addition to a number of applications, we also give necessary and sufficient conditions on $K$ which yield factorizations

$$K(A,B) = \int_X k_A(x) \overline{k_B(x)} d\nu(x), \quad \forall A, B \in \mathcal{B}_{\text{fin}},$$

where $\{k_A\}_{A \in \mathcal{B}_{\text{fin}}}$ is an $L^2(\nu)$ system.

But up to now, many of the applications have focused on Hilbert spaces of regular functions. If for example, a kernel represents a Green’s function for an elliptic partial differential operator (PDO), then the associated RKHS will consist of functions which have some degree of smoothness.

The Aronszajn approach has serious limitations: Often functions will be defined only almost everywhere with respect to some measure which is prescribed on the set $X$; for example, if $X$ represents time, in one or more dimensions, the prescribed measure $\nu$ is often Lebesgue measure. For fractal random fields, $\nu$ may be a fractal measure. For this reason, and others (to be outlined inside the paper), it is useful to instead let $X$ be a measure space, say $(X,\mathcal{B},\nu)$. If $X$ is a locally compact Hausdorff space, then $\mathcal{B}$ is the corresponding sigma-algebra of Borel sets, and $\nu$ is a fixed positive measure, and assumed to be a regular measure on $(X,\mathcal{B})$. The modification in the resulting new definition of p.d. kernels $K$ in this context is subtle. Here we just mention that, for a p.d. system $(X,\mathcal{B},\nu)$ and kernel $K$ in the measurable category, the associated RKHS $H(K)$ will now instead be a Hilbert
space of measurable functions on $X$, more precisely, measurable with respect to $\mathcal{B}$, and locally in $L^2(\nu)$. We shall say that $\mathcal{H}(K)$ is contained in $L^2_{\nu_{\text{loc}}}(\nu)$. The p.d. kernel $K$ itself will be a random family of signed measures on $(X, \mathcal{B})$.

**Discussion of the literature.** The theory of RKHS and their applications is vast, and below we only make a selection. Readers will be able to find more cited applications. Given a positive definite kernel $K$ mentioned applications. Given a positive definite kernel, but are then subsequently specialized to the above mentioned applications. Given a positive definite kernel $K$ on $X \times X$ where $X$ is a fixed set, we first study families of factorizations, and the extension to the complex case is relatively simple.

**Definition 2.1.** If $K : X \times X \to \mathbb{C}$ is a given positive definite kernel, and $(\Omega, \mathcal{M}, \mu)$ is a measure space, a function $k : X \to L^2(\Omega, \mathcal{M}, \mu)$ is said to be a factorization iff (Defn.)

$$k(x, y) = \langle k_x, k_y \rangle_{L^2(\Omega, \mathcal{M}, \mu)}, \quad \forall x, y \in X.$$ 

Let $(X, \mathcal{B}, \nu)$ denote a measure space. Our starting assumptions are as follows:

(i) $\mathcal{B}$ is a fixed sigma-algebra of subsets of $X$, and

(ii) $\nu$ is a positive $\sigma$-finite measure defined on $\mathcal{B}$.

Given (i) & (ii), we set $\mathcal{B}_{\text{fin}} = \{A \in \mathcal{B} : \nu(A) < \infty\}$.

We shall consider positive definite kernels $K$ on $\mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}}$. While our results are valid in the case of function spaces consisting of complex-valued functions, we shall restrict attention here to the real-valued case. This is motivated by applications, and the extension to the complex case is relatively simple.

**Definition 2.2.** A function $K$ on $\mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}}$ (mapping into $\mathbb{R}$) is said to be **positive definite** iff (Defn.), for $\forall n \in \mathbb{N}$, $\forall \{\alpha_i\}_{1}^{n} \subset \mathbb{R}$, and all $\{A_i\}_{1}^{n}, A_i \in \mathcal{B}_{\text{fin}}$, we have

$$\sum_{i} \sum_{j} \alpha_i \alpha_j K(A_i, A_j) \geq 0. \quad (2.1)$$

When a positive definite kernel $K$ is given, we shall denote the corresponding reproducing kernel Hilbert space (RKHS) by $\mathcal{H}(K)$.

Recall the RKHS $\mathcal{H}(K)$ is a Hilbert space of functions $F$ defined on $\mathcal{B}_{\text{fin}}$ such that, for all $A \in \mathcal{B}_{\text{fin}}$, the mapping

$$\mathcal{H}(K) \ni F \mapsto F(A) \quad (2.2)$$

is norm-continuous. When this holds, there is a unique representation $\varphi_A \in \mathcal{H}(K)$ such that

$$F(A) = (F, \varphi_A)_{\mathcal{H}(K)}. \quad (2.3)$$

It is well-known that one may take $\varphi_A(\cdot) = K(\cdot, A)$, and that then

$$\langle K(\cdot, A), K(\cdot, B) \rangle_{\mathcal{H}(K)} = K(A, B), \quad \forall A, B \in \mathcal{B}_{\text{fin}}. \quad (2.4)$$
**Definition 2.3.** We shall say that a given positive definite kernel \( K \) on \( B_{\text{fin}} \times B_{\text{fin}} \) has a realization (as factorization) in \( L^2(\nu) \) iff (Defn.) \( \exists \{ k_A \}_{A \in B_{\text{fin}}} \), such that

\[
K(A, B) = \langle k_A, k_B \rangle_{L^2(\nu)} = \int_X k_A(x) k_B(x) \, d\nu(x).
\]  

(2.5)

**Theorem 2.4.** Let \( X, B, \nu, \) and \( K \) be as above, i.e., we assume that \( K \) is a positive definite kernel defined on \( B_{\text{fin}} \times B_{\text{fin}} \).

Then the following two conditions are equivalent:

(i) For all \( B \in B_{\text{fin}} \), \( K(\cdot, B) \) is a sigma-finite signed measure on \( B \), and

\[
K(\cdot, B) \ll \nu
\]

where \( \ll \) refers to “absolute continuity,” i.e., that the following implication holds:

\[
\nu(A) = 0 \implies K(A, B) = 0.
\]

(ii) The positive definite kernel \( K \) in (i) has an \( L^2(\nu) \)-realization, i.e., (2.5) is satisfied for a system \( \{ k_A \}_{A \in B_{\text{fin}}} \).

Before we turn to the proof, we shall need the following lemma. It clarifies the notation of condition (2.6) in (i).

**Lemma 2.5.** Let \( (X, B, \nu), K \) and \( B_{\text{fin}} \) be as stated in the theorem. Let \( A \in B_{\text{fin}} \) be given. If \( \nu(A) = 0 \implies K(A, A) = 0 \), then it follows automatically that also \( K(A, B) = 0 \) for all \( B \in B_{\text{fin}} \).

**Proof.** The assertion follows from the assumed positive definite property for \( K \). If \( \mathcal{H}(K) \) denotes the RKHS, then the Schwarz inequality holds for the \( \mathcal{H}(K) \)-inner product. Hence

\[
|K(A, B)|^2 \leq \|K(\cdot, A)\|_{\mathcal{H}(K)}^2 \|K(\cdot, B)\|_{\mathcal{H}(K)}^2 = \text{by (2.3) and (2.4)} \quad K(A, A) K(B, B);
\]

and the conclusion of the lemma follows. \( \square \)

**Proof of Theorem 2.4.** (i)\(\implies\)(ii). With (i) assumed, we shall denote the indexed family of Radon-Nikodym derivatives \( g(\cdot, B) \); so an indexed family of locally integrable functions on \( (X, B) \) such that

\[
\frac{dK(\cdot, B)}{d\nu} = g(\cdot, B);
\]  

(2.7)

or equivalently,

\[
\int_A g(x, B) \, d\nu(x) = K(A, B), \quad \forall A, B \in B_{\text{fin}}.
\]  

(2.8)

Let \( B_{\text{fin}} \) denote the finite linear combinations

\[
\sum_{i=1}^n \alpha_i \chi_{A_i}, \quad \alpha_i \in \mathbb{R}, \ A_i \in B_{\text{fin}}.
\]  

(2.9)
Recall that \( \mathcal{D}_{\text{fin}} \) is dense in \( L^2(\nu) \) relative to the \( L^2(\nu) \)-norm. For \( \varphi = \sum_i \alpha_i \chi_{A_i}, \psi = \sum_j \beta_j \chi_{B_j} \in \mathcal{D}_{\text{fin}}, \alpha_i, \beta_j \in \mathbb{R}, A_i, B_j \in \mathcal{B}_{\text{fin}}; \) then

\[
\varphi, \psi \rightarrow \sum_i \sum_j \alpha_i \beta_j \int_{A_i} g(x, B_j) \, d\nu(x)
\]  

(2.10)
defines a positive definite bilinear mapping in \( L^2(\nu) \), with \( \mathcal{D}_{\text{fin}} \) as its dense domain. One easily checks that it is closable. Hence an application of the Kato-Friedrichs theorem (see, e.g., [DS88, Kat95, JT17c]) yields a positive selfadjoint operator \( T \) in \( L^2(\nu) \), \( \mathcal{D}_{\text{fin}} \subset \text{dom}(T) \) such that the expression in (2.10) takes the following form

\[
\text{RHS}(2.10) = \int_X \varphi(x) \langle T \psi \rangle(x) \, d\nu(x)
\]  

(2.11)

This yields a solution to (ii) with \( k_A(x) = \langle T^\frac{1}{2} \chi_A \rangle(x), A \in \mathcal{B}_{\text{fin}}, x \in X. \)

We now turn to (ii) \( \Rightarrow \) (i). Assuming \( \{k_A\}_{A \in \mathcal{B}_{\text{fin}}} \) is a solution to (ii), see also (2.5), we may define an operator \( S \) with dense domain \( \mathcal{D}_{\text{fin}} \) in \( L^2(\nu) \), by setting

\[
S \left( \sum_i \alpha_i \chi_{A_i} \right) = \sum_i \alpha_i k_{A_i}.
\]  

(2.12)

From the condition in (ii) we conclude that \( S \) is closable, and that \( S^* S \) is defined on \( \mathcal{D}_{\text{fin}} \). Hence for \( \varphi, \psi \in \mathcal{D}_{\text{fin}} \); see (2.9)-(2.10), we get

\[
\sum_i \sum_j \alpha_i \beta_j K(A_i, B_j) = \int_X \varphi(x) \langle S^* S \psi \rangle(x) \, d\nu(x).
\]

The conclusion in (i) is now immediate:

For the associated Radon-Nikodym derivative (see (2.7)) we get

\[
\frac{dK(x, B)}{d\nu(x)} = g(x, B) = \langle S^* S \chi_B \rangle(x).
\]

\[\square\]

**Corollary 2.6.** Let \((X, \mathcal{B}, \nu)\) be a fixed sigma-finite measure-space, as specified in Theorem 2.4 above. We set \( \mathcal{B}_{\text{fin}} := \{A \in \mathcal{B} : \nu(A) < \infty\} \), and we consider a positive kernel \( K \) defined on \( \mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}} \). Suppose \( K \) has a factorization \( \{k_A\}_{A \in \mathcal{B}_{\text{fin}}} \) in \( L^2(\nu) \), i.e.,

\[
K(A, B) = \langle k_A, k_B \rangle_{L^2(\nu)}
\]  

(2.13)

where the RHS in (2.13) refers to the \( L^2(\nu) \)-inner product.

Define a function \( b \) by

\[
b(K(\cdot, A)) = k_A
\]

from \( \mathcal{H}(K) (= \text{the RKHS}) \) into \( L^2(\nu) \), and then extend by linearity:

\[
b \left( \sum_i \alpha_i K(\cdot, A_i) \right) = \sum_i \alpha_i k_{A_i};
\]  

(2.14)

then \( b \) is an isometry \( \mathcal{H}(K) \rightarrow L^2(\nu). \)
(i) When $b$ is extended by closure, we get $b(\mathcal{H}(K)) \subseteq L^2(\nu)$ as a closed subspace.

(ii) The co-isometry $L^2(\nu) \xrightarrow{b^*} \mathcal{H}(K)$ is given as follows: For all $\varphi \in L^2(\nu)$, and $A \in B_{\text{fin}}$, set
\[
(b^*\varphi)(A) = \langle \varphi, k_A \rangle_{L^2(\nu)} = \int_X \varphi(x) k_A(x) \, d\nu(x),
\]
then $b^*$ is the adjoint operator to $b$ from (i).

(iii) For every orthonormal basis (ONB) in $L^2(\nu)$, $\{\varphi_n\}_{n \in \mathbb{N}}$, we have the corresponding factorization of $K(\cdot, \cdot)$ on $B_{\text{fin}} \times B_{\text{fin}}$:
\[
K(A, B) = \sum_{n \in \mathbb{N}} \langle \varphi_n, k_A \rangle_{L^2(\nu)} \langle k_B, \varphi_n \rangle_{L^2(\nu)}.
\]

Proof. The key step is the isometry-property via formula (2.14), i.e., the fact that
\[
\left\| \sum_i \alpha_i K(\cdot, A_i) \right\|^2_{\mathcal{H}(K)} = \left\| \sum_i \alpha_i k_{A_i} \right\|^2_{L^2(\nu)}.
\]
To prove this, fix $\{\alpha_i\}^n_{i=1}$, $\{A_i\}^n_{i=1}$ as specified; then
\[
\text{LHS} (2.17) = \sum_i \sum_j \alpha_i \alpha_j \langle K(\cdot, A_i), K(\cdot, A_j) \rangle_{\mathcal{H}(K)}
= \sum_i \sum_j \alpha_i \alpha_j K(A_i, A_j) \quad \text{(the reproducing property)}
= \int_X \left| \sum_i \alpha_i k_{A_i}(x) \right|^2 \, d\nu(x) = \text{RHS} (2.17),
\]
the desired conclusion.

Hence $b$ extends by closure to an isometry, also denoted by $b$, $\mathcal{H}(K) \rightarrow L^2(\nu)$, and we conclude that $b(\mathcal{H}(K))$ is closed in $L^2(\nu)$.

Proof of (ii). To begin with we first establish that $b^*$ from (2.15) yields a well defined operator from $L^2(\nu)$ into $\mathcal{H}(K)$. Note that this makes reference to the respective inner products on the two Hilbert spaces. In particular for $\mathcal{H}(K)$, a finite linear combination (see Definition 2.2),
\[
\left\| \sum_i \alpha_i K(\cdot, A_i) \right\|^2_{\mathcal{H}(K)} = \sum_i \sum_j \alpha_i \alpha_j K(A_i, A_j),
\]
with $\alpha_i \in \mathbb{R}$, and $A_i \in B_{\text{fin}}$.

To show that (2.15) defines an operator as stated into $\mathcal{H}(K)$, we must show that $b^* \varphi \in \mathcal{H}(K)$, and
\[
\left| \sum_i \alpha_i (b^* \varphi)(A_i) \right|^2 \leq \|\varphi\|^2_{L^2(\nu)} \sum_i \sum_j \alpha_i \alpha_j K(A_i, A_j).
\]
We have
\[
\text{LHS} (2.19) \quad \text{(by (2.15))} \quad \left| \langle \varphi, \sum_i \alpha_i k_{A_i} \rangle_{L^2(\nu)} \right|^2
\leq \|\varphi\|^2_{L^2(\nu)} \left\| \sum_i \alpha_i k_{A_i} \right\|^2_{L^2(\nu)}.
\]
Hence $b^* \varphi \in \mathcal{H}(K)$ by Aronszajn [Aro50], and (2.17).
Indeed, by (i), we also have
\[ \left\| \sum_i \alpha_i k_A \right\|_{L^2(\nu)}^2 = \sum_i \sum_j \alpha_i \alpha_j \langle k_A, k_{A_j} \rangle_{L^2(\nu)} \]
and the result follows; see also (2.18).

To show that \( b^* \) as an operator is the adjoint, we must check that
\[ \langle b^* \varphi, F \rangle_{\mathcal{H}(K)} = \langle \varphi, bF \rangle_{L^2(\nu)}, \quad \forall \varphi \in L^2(\nu), \forall F \in \mathcal{H}(K). \]
But by [Aro50], it is enough to do this for the special case when \( F = K(\cdot, A), A \in \mathcal{B}_{fin} \); and in this case, it follows from (i).

**Proof of (iii).** Let \( \{\varphi_n\} \) be an ONB in \( L^2(\nu) \). By the assumptions, and Theorem 2.4, we get
\[ K(A, B) = \langle k_A, k_B \rangle_{L^2(\nu)} \]
\[ = (\text{Parseval}) \sum_{n \in \mathbb{N}} \langle k_A, \varphi_n \rangle_{L^2(\nu)} \langle \varphi_n, k_B \rangle_{L^2(\nu)} \]
which is the desired conclusion. \( \square \)

**Remark 2.7.** It follows from the argument of the proof that \( b(\mathcal{H}(K)) = L^2(\nu) \) iff \( \{k_A\}_{A \in \mathcal{B}_{fin}} = 0 \), where \( \perp \) refers to the \( L^2(\nu) \)-inner product. See Example 4.2 for case when \( b(\mathcal{H}(K)) \) is one-dimensional.

### 3. Minimal Factorization

Below we show that any solution \( \{k_A\}_{A \in \mathcal{B}_{fin}} \) to the factorization problem from Theorem 2.4 must necessarily be minimal in a sense we defined below; see also [JT17a].

**Definition 3.1.** Let \((X, \mathcal{B}, \nu)\), and \(K\) be as in the statement of Theorem 2.4. Let \((\Omega, \mathcal{M}, \mu)\) be a sigma-finite measure space. Suppose we have two factorizations
\[ \mathcal{B}_{fin} \ni A \mapsto k_A \in L^2(X, \nu), \quad \text{and} \]
\[ \mathcal{B}_{fin} \ni A \mapsto m_A \in L^2(\Omega, \mu), \]
such that
\[ K(A, B) = \int_X k_A(x) k_B(x) \, d\nu(x) \]
\[ = \int_{\Omega} m_A(\omega) m_B(\omega) \, d\mu(\omega), \quad \forall A, B \in \mathcal{B}_{fin}. \]
We say that \( k \ll m \) if (Defn.) there is a measurable mapping \( F : (\Omega, \mathcal{M}) \rightarrow (X, \mathcal{B}) \) such that
\[ \mu \circ F^{-1} = \nu. \quad (3.3) \]

**Theorem 3.2.** Solution \( \{k_A\}_{A \in \mathcal{B}_{fin}} \) to the factorization problem (2.5), if they exist, are minimal in the sense that whenever some other solution \( \{k'_A\}_{A \in \mathcal{B}_{fin}} \) in a sigma-finite measure space \((X', \mathcal{B}', \nu')\) satisfies \( k' \ll k \), then the two measure spaces \((X, \mathcal{B}, \nu)\) and \((X', \mathcal{B}', \nu')\) are isomorphic.

**Proof.** We refer the reader to our [JT17a, JT17c]. \( \square \)
4. Two applications

The positive definite kernels, and their factorizations, considered in Sections 2-3, include as special cases covariance kernels of general white noise processes, of transient Markov chains, and of generalized Gaussian fields. Below we discuss applications of our factorization results to these three cases. Our treatment of generalized Gaussian fields includes an extension of the more traditional setting for Itô calculus. This is in the last section of the paper.

4.1. The Generalized Wiener-process. Here we consider the following kernel $K$ on $\mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}}$: Set

\[ K(A, B) = \nu(A \cap B), \quad A, B \in \mathcal{B}_{\text{fin}}, \]  

where the measure space $(X, \mathcal{B}, \nu)$ is specified as above.

\begin{proposition}
\begin{enumerate}
\item $K = K^{(\nu)}$ in (4.1) is positive definite.
\item $K^{(\nu)}$ is the covariance kernel for the stationary Wiener process $W = W^{(\nu)}$ indexed by $\mathcal{B}_{\text{fin}}$, i.e., Gaussian, mean zero, and

\[ \mathbb{E}(W_A W_B) = K^{(\nu)}(A, B) = \nu(A \cap B). \]  

\item If $f \in L^2(\nu)$, and $W_f = \int_X f(x) \, dW_x$ denotes the corresponding Itô-integral, then

\[ \mathbb{E}\left( |W_f|^2 \right) = \int_X |f|^2 \, d\nu; \]

in particular, if $f = \sum_i \alpha_i \chi_{A_i}$, then

\[ \sum_i \sum_j \alpha_i \alpha_j K^{(\nu)}(A_i, A_j) = \int_X \left| \sum_i \alpha_i \chi_{A_i} \right|^2 \, d\nu. \]

\item The RKHS $\mathcal{H}(K^{(\nu)})$ of the positive definite kernel in (4.1) consists of functions $F$ on $\mathcal{B}_{\text{fin}}$ represented by $f \in L^2(\nu)$ via

\[ F(A) = F_f(A) = \int_A f \, d\nu, \quad A \in \mathcal{B}_{\text{fin}}; \]

and

\[ \|F_f\|_{\mathcal{H}(K)}^2 = \|f\|_{L^2(\nu)}^2 = \int_X |f|^2 \, d\nu. \]  

\item The isometry $b : \mathcal{H}(K) \rightarrow L^2(\nu)$ from Corollary 2.6, is specified by

\[ b\left(K^{(\nu)}(\cdot, A)\right) = b(\nu(\cdot \cap A)) = \chi_A, \quad \forall A \in \mathcal{B}_{\text{fin}}. \]

\end{enumerate}
\end{proposition}

\begin{proof}
The details can be found at various places in the literature; see e.g., [AJ12, JT16b, JT17a, JT17b].
\end{proof}

Example 4.2. Below we illustrate the role of kernels $K$, as in Definition 2.2, in accounting for correlation. The case of $K^{(\nu)}$ in (4.1) yields a RKHS describing generalized white noise processes; see also (4.2). By contrast, if $(X, \mathcal{B}, \nu)$ is given, and we set

\[ K_2(A, B) = \nu(A) \nu(B), \quad \forall A, B \in \mathcal{B}_{\text{fin}} \]  

\]
then the corresponding RKHS $\mathcal{H}(K_2)$ is one-dimensional; and hence trivial. (We sketch the argument below in the special case when $\nu$ is finite. The reader can easily extend it.)

To see this, we identify $\{k_A\}_{A \in \mathcal{B}_{fin}}$ as an $L^2(\nu)$-solution to (2.13). Let $I$ be the constant function “one” on $(X, \mathcal{B})$. Assume $\nu(X) = 1$, and set

$$k_A(x) = \nu(A), \quad (4.7)$$

and so, by Corollary 2.6, $\mathcal{H}(K)$ is isometrically identified with a one-dimensional subspace of $L^2(\nu)$, i.e., the subspace spanned by the constant function $1$ in $L^2(\nu)$.

4.2. Transient Markov Processes. Below we identify a class of Markov processes which have naturally associated reproducing kernel Hilbert spaces in the sense of Section 2. Hence, for each of these Markov processes there is then a positive definite kernel $K$ as per Definitions 2.2 and 2.3. Our starting point is again a fixed sigma-finite measure space $(X, \mathcal{B}, \nu)$. The Markov transition probabilities $P$ will be defined relative to $(X, \mathcal{B}, \nu)$. The two conditions we place on the Markov process is that it be (i) reversible relative to $\nu$, and that it be (ii) transient; see definitions below. The reversible property allows us to bring in spectral theory in $L^2(\nu)$, and the transient property yields a well defined Greens function which in turn is the key tool in our consideration of the associated positive definite kernel $K$; and we show that our results in Section 2 will then apply.

Let $(X, \mathcal{B}, \nu)$ be as specified above, and let $P$ denote a Markov transition measure, i.e.,

$$X \ni x \mapsto P(x, \cdot) \in M_1(X, \mathcal{B})$$

assumed measurable, where $M_1(X, \mathcal{B}) = \text{probability measures on } (X, \mathcal{B})$.

We shall assume that the indexed Markov process is reversible (see, e.g., [JT15a, JT15b]), i.e., that for all $A, B \in \mathcal{B}_{fin}$, we have

$$\int_A P(x, B) d\nu(x) = \int_B P(y, A) d\nu(y).$$

Set, for $x \in X, A \in \mathcal{B}_{fin}, n \in \mathbb{N}$,

$$P_{n+1}(x, A) = \int_X P(x, dy) P_n(y, A),$$

and

$$G(x, A) = \sum_{n=0}^{\infty} P_n(x, A).$$

We say that the Markov process is transient iff the sum in (4.11) is pointwise a.e. (w.r.t $\nu$) convergent, and

$$\int_A G(x, B) d\nu(x)$$

is finite for all $A, B \in \mathcal{B}_{fin}$. (Also see [SBM07, Kor08].)

**Proposition 4.3.** Let $G$ be as in (4.11). Setting now

$$K(A, B) = \int_A G(x, B) d\nu(x),$$

then $K$ is positive definite on $\mathcal{B}_{fin} \times \mathcal{B}_{fin}$. 
Moreover, the condition in Corollary 2.6 holds. The \( \{ k_A \}_{A \in A_{\text{fin}}} \) system satisfying \( K (A, B) = \int_X k_A (x) k_B (x) \, d \nu (x) \) is

\[
k_A = (I - P_{(\nu)})^{-\frac{1}{2}} (\chi_A)
\]

where \( P_{(\nu)} \) denotes the selfadjoint operator in \( L^2 (\nu) \) given by

\[
(P_{(\nu)} \phi) (x) = \int_X \phi (y) P (x, dy)
\]

Proof. With the stated assumptions, the conclusion follows from a direct application of the results in Section 2, i.e., Theorem 2.4, and Corollary 2.6.

The justification for the formula (4.14) is as follows: First \( P_{(\nu)} \) is a bounded operator in \( L^2 (\nu) \) as follows. It is selfadjoint on account of (4.10). To show that it is contractive, consider \( \phi \in L^2 (\nu) \), and estimate:

\[
\left| \left\langle \phi, P_{(\nu)} \phi \right\rangle_{L^2(\nu)} \right| = \left| \iint \phi (x) P (x, dy) \phi (y) d \nu (x) \right|
\]

\[
\leq \left( \text{Jensen} \right) \left( \iint P (x, dy) |\phi (y)|^2 \right)^{\frac{1}{2}} d \nu (x)
\]

\[
\leq \left( \text{Schwarz} \right) \left( \iint |\phi|^2 d \nu \right)^{\frac{1}{2}} \left( \iint P (x, dy) |\phi (y)|^2 d \nu (x) \right)^{\frac{1}{2}}
\]

and so for the corresponding operator norms, we have: \( \| P_{(\nu)} \|_{L^2 \to L^2} \leq 1 \), which is the asserted contractivity.

One further shows that the transience assumption implies that \( \lambda = 1 \) is not in the point-spectrum of \( P_{(\nu)} \) as an operator in \( L^2 (\nu) \). Hence

\[
(I - P_{(\nu)}) L^2 (\nu) \subset L^2 (\nu).
\]

(dense subspace)

It follows that \( (I - P_{(\nu)})^{-1} \) is unbounded and selfadjoint in \( L^2 (\nu) \), and so the Spectral Theorem applies. Hence \( (I - P_{(\nu)})^{-\frac{1}{2}} \) is well defined.

Finally transience implies that (4.12) is finite. Moreover, for \( A \in \mathcal{B}_{\text{fin}} \), we have

\[
\int_A G (x, A) \, d \nu (x) \quad \text{(4.12)}
\]

\[
= \int_X \chi_A (x) (1 - P_{(\nu)})^{-1} (\chi_A) (x) \, d \nu (x)
\]

\[
= \| (1 - P_{(\nu)})^{-\frac{1}{2}} (\chi_A) \|^2_{L^2(\nu)}
\]

which justifies the assertion in (4.14). \( \square \)

5. Gaussian Fields

In this section we establish a close connection between generalized Gaussian fields, and associated Ito-integrals, on the one hand, and the class of positive definite kernels considered here, on the other. This section is motivated in part by a number of earlier works, for example [AD93, AJ12, BP17, EMESO17, Jr68, JT16a, JT17c, LMP09, MSF+16, PSS16, SBM07, SZ09, KS93, AL08, Tak11].
The setting below will be as in Section 2. Fix a sigma-finite measure space \((X, \mathcal{B}, \nu)\), and a positive definite kernel \(K\) on \(\mathcal{B}_{fin} \times \mathcal{B}_{fin}\). We further assume that \(K\) has an \(L^2(\nu)\) factorization; see (2.5) and (2.13). In the proof of Theorem 2.4, we saw that there is an unbounded selfadjoint operator \(S\), with dense domain \(D_{fin} = span\{\chi_A : A \in \mathcal{B}_{fin}\}\) such that we may take \(\{k_A\}_{A \in \mathcal{B}_{fin}}\) to have the form \(k_A = S(\chi_A)\), where \(\chi_A\) is the indicator function. Hence, we may take as core domain for the operator \(S\),

\[ \mathbb{P}_f := span\{\chi_A : A \in \mathcal{B}_{fin}\}. \]  

(5.1)

It is a core for the full domain of \(S\). Here, by “span”, we mean all finite linear combinations. (In interesting examples, see e.g. Section 4.2, the operator \(S\) will indeed be unbounded.

5.1. The path-space \(\Omega\). Let \(\hat{\mathbb{R}}\) denote the one-point compactification of \(\mathbb{R}\), and set \(\Omega := \hat{\mathbb{R}}^{\mathcal{B}_{fin}}\), (5.2) \(\mathcal{C} := \) the cylinder sigma-algebra of subsets of \(\Omega\); and \(\mathbb{P}\) be the Gaussian probability measure on \(\Omega\), defined on \(\mathcal{C}\), and indexed by \(K\).

In details, if \(F = \{A_i\}_{1}^{n}\) is a finite system, \(A_i \in \mathcal{B}_{fin}\), set

\[ \mathcal{C}_F := \prod_{i=1}^{n} A_i \times \prod_{x \in \mathbb{R}} \hat{\mathbb{R}}, \]  

(5.3)

(a cylinder subset); and let \(\mathcal{C}\) be the sigma-algebra of all subsets of \(\Omega\) which is generated by the cylinder sets.

To construct \(\mathbb{P}\) as a probability measure, and defined on \(\mathcal{C}\), we first specify its finite-dimensional joint distributions,

\[ \mathbb{P} (\cdot \mid \mathcal{C}_F) := \text{the Gaussian on } \mathbb{R}^n \text{ which has 0 mean, and} \]  

\[ \text{covariance matrix } (K(A_i, A_j))_{i,j=1}^{n}. \]  

(5.4)

By Kolmogorov’s consistency property, we then get a unique probability measure \(\mathbb{P}\) on \((\Omega, \mathcal{B})\) which satisfies (5.4).

For \(A \in \mathcal{B}_{fin}\), and \(\omega \in \Omega\) (see (5.2)), set

\[ W_A(\omega) := \omega(A). \]  

(5.5)

Then, by construction, the expectation \(E\), defined from \(\mathbb{P}\), satisfies \(E(W_A) = 0\),

\[ E(W_A W_B) = K(A, B). \]  

(5.6)

Specifically, each \(W_A\) is Gaussian with distribution \(N(0, K(A, A))\); and the joint distribution of \((W_A, W_{A_2}, \cdots, W_{A_n})\) is the Gaussian from (5.4).

Now, let \(\mathcal{A} := \{A_i\}\) be a countable partition of \(X\), \(A_i \in \mathcal{B}_{fin}\) \(A_i \cap A_j = \emptyset, i \neq j,\)

and let \(\mathcal{C}_\mathcal{A} := \) the sigma-subalgebra of \(\mathcal{C}\) which is generated by \(\{W_A\}, A_i \in \mathcal{A}\).

Introducing conditional expectations, we get an inductive system of Ito-integrals, indexed by the set of partitions \(\mathcal{A}\) (as above) where we use the usual ordering of partitions \(\mathcal{A} \leq \mathcal{A}'\) given by refinement: If \(\varphi = \sum \alpha_i \chi_{A_i}\),

\[ E \left( \left\| \int \varphi dW_x \right\|^2 \mid \mathcal{C}_\mathcal{A} \right) = \sum \alpha_i \alpha_j K(A_i, A_j). \]  

(5.8)
where the "|" stands for conditional expectation.

Passing to the limit, over the set of all partitions, we get a necessary and sufficient condition for when the Itô-integral \( \int_X \varphi(x) \, dW_x \) is well defined, and is in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \), i.e., when \( \mathbb{E}\left(\left|\int \varphi \, dW_x\right|^2\right) < \infty \).

5.2. \( L^2(\nu) \)-factorizations. We now introduce a particular \( L^2(\nu) \) factorization,

\[
K(A, B) = \langle k_A, k_B \rangle_{L^2(\nu)}, \quad A, B \in \mathcal{B}_{fin}
\]  

(5.9)
as in Section 2. We introduce the selfadjoint operator \( S \) from (2.12), so

\[
k_A = S(\chi_A).
\]  

(5.10)

For applications, it is not a serious restriction to further assume that \( S \) has a bounded inverse (in \( L^2(\nu) \)); see e.g., Section 4.2 above.

Theorem 5.1. Let \((X, \mathcal{B}, \nu), (\Omega, \mathcal{F}, \mathbb{P})\), and \( \{W_A\}_{A \in \mathcal{B}_{fin}} \) be as above. We shall consider the corresponding Itô-integral \( \int_X \varphi(x) \, dW_x \) for deterministic functions \( \varphi \) and \( \psi \) on \((X, \mathcal{B})\). We have:

(i) \( \int \varphi \, dW \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \) iff \( S(\varphi) \in L^2(\nu) \), i.e., iff \( \varphi \) is in the domain of \( S \).

(ii) When (i) holds, then

\[
\mathbb{E}\left(\left|\int \varphi \, dW\right|^2\right) = \|S(\varphi)\|_{L^2(\nu)}^2,
\]  

(5.11)

and

\[
\mathbb{E}\left(\left(\int \varphi \, dW\right) \left(\int \psi \, dW\right)\right) = \langle S\varphi, S\psi \rangle_{L^2(\nu)}.
\]  

(5.12)

Proof. In view of the discussion before the statement of the theorem, it is enough to prove (5.12). We shall do this by first establishing the formula when a partition is fixed. The general case will then follow from the inductive limit argument outlined above.

In detail: Since the Itô-integral is defined as an inductive limit on partitions as specified in (5.8), to prove (5.12), it is enough to fix a partition \( \mathcal{A} = \{A_i\} \), and functions \( \varphi = \sum_i \alpha_i \chi_{A_i} \), \( \psi = \sum_j \beta_j \chi_{A_j} \); then

\[
\mathbb{E}\left(\left(\int \varphi \, dW\right) \left(\int \psi \, dW\right)\right) = \sum_i \sum_j \alpha_i \beta_j \mathbb{E}(W_{A_i} W_{A_j})
\]

(by (5.6))

\[
= \sum_i \sum_j \alpha_i \beta_j K(A_i, A_j)
\]

(by (5.10))

\[
= \sum_i \sum_j \alpha_i \beta_j \langle S(\chi_{A_i}), S(\chi_{A_j}) \rangle_{L^2(\nu)}
\]

\[
= \langle S\left(\sum_i \alpha_i \chi_{A_i}\right), S\left(\sum_j \beta_j \chi_{A_j}\right) \rangle_{L^2(\nu)}
\]

= \langle S\varphi, S\psi \rangle_{L^2(\nu)}

which is the desired conclusion (5.12) when the partition \( \mathcal{A} \) is fixed.

The general result follows from use of the ordering of partitions, and the corresponding Kolmogorov inductive limit over all partitions; see (5.7). For more details, we refer to the cited literature on standard considerations for generalized Itô limits. \( \square \)
Let the setting be as in the statement of Theorem 5.1. We consider the limit over partitions \( \mathcal{A} \) as in (5.7):
\[
E \left( \left| \int \varphi dW \right|^2 \middle| \mathcal{C}_\mathcal{A} \right),
\]
where the limit is taken over \( \mathcal{A} \).

Here \( \mathcal{C}_\mathcal{A} := \) (the sigma-algebra generated by \( \{W_{A_i}\} \) for \( A_i \in \mathcal{A} \)); and we take monotone limit over refinements, written \( \mathcal{A} \leq \mathcal{A}' \). Note that
\[
E \left( \left| \int \varphi dW \right|^2 \middle| \mathcal{C}_\mathcal{A} \right) \leq E \left( \left| \int \varphi dW \right|^2 \middle| \mathcal{C}_\mathcal{A}' \right) \tag{5.14}
\]
holds if \( \mathcal{A} \leq \mathcal{A}' \).

**Corollary 5.2** (Monotonicity). We have:
\[
\sup_{\mathcal{A}} E \left( \left| \int \varphi dW \right|^2 \middle| \mathcal{C}_\mathcal{A} \right) = E \left( \left| \int \varphi dW \right|^2 \right) \tag{5.15}
\]
where the supreme in (5.15) over all partitions is finite iff \( \varphi \in \text{domain}(S) \).

**Proof.** From the construction, we get
\[
E \left( \cdot \middle| \mathcal{C}_\mathcal{A} \right) \leq E \left( \cdot \middle| \mathcal{C}_\mathcal{A}' \right)
\]
in the ordering of projections in \( L^2(\Omega, \mathcal{C}, \mathbb{P}) \). Equivalently, when \( \mathcal{A} \leq \mathcal{A}' \), we have:
\[
E \left( (F \middle| \mathcal{C}_\mathcal{A}') \middle| \mathcal{C}_\mathcal{A} \right) = E \left( F \middle| \mathcal{C}_\mathcal{A} \right),
\]
and the assertions in (5.14)-(5.15) follow immediately from this. \( \square \)

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