Maximizing Monotone Submodular Functions over the Integer Lattice

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Abstract The problem of maximizing non-negative monotone submodular functions under a certain constraint has been intensively studied in the last decade. In this paper, we address the problem for functions defined over the integer lattice.

Suppose that a non-negative monotone submodular function \( f : \mathbb{Z}_n^+ \rightarrow \mathbb{R}_+ \) is given via an evaluation oracle. Assume further that \( f \) satisfies the diminishing return property, which is not an immediate consequence of submodularity when the domain is the integer lattice. Given this, we design polynomial-time \( (1 - 1/e - \epsilon) \)-approximation algorithms for a cardinality constraint, a polymatroid constraint, and a knapsack constraint. For a cardinality constraint, we also provide a \( (1 - 1/e - \epsilon) \)-approximation algorithm with slightly worse time complexity that does not rely on the diminishing return property.

Keywords submodular functions · integer lattice · DR-submodular functions

1 Introduction

Submodular functions have been intensively studied in various areas of operations research and computer science, as submodularity naturally arises in many problems in these fields [12,14,18]. In the last decade, the maximization

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of submodular functions in particular has attracted interest. For example, one can find novel applications of submodular function maximization in the dissemination of influence through social networks [17], text summarization [19, 20], and optimal budget allocation for advertisements [1].

Most past works in the area have considered submodular functions defined over a set—submodular functions which take a subset of a ground set as the input and return a real value. However, in many practical scenarios, it is more natural to consider submodular functions over a multiset or, equivalently, submodular functions over the integer lattice \( \mathbb{Z}^E \) for some finite set \( E \). We say that a function \( f : \mathbb{Z}^E \to \mathbb{R} \) is (lattice) submodular if \( f(x) + f(y) \geq f(x \lor y) + f(x \land y) \) for all \( x, y \in \mathbb{Z}^E \), where \( x \lor y \) and \( x \land y \) denote the coordinate-wise maximum and minimum, respectively. Such a generalized form of submodularity arises in maximizing the spread of influence with partial incentives [9], optimal budget allocation, sensor placement, and text summarization [26].

When designing algorithms for maximizing submodular functions, the diminishing return property often plays a crucial role. A set function \( f : 2^E \to \mathbb{R} \) is said to satisfy the diminishing return property if \( f(X + e) - f(X) \geq f(Y + e) - f(Y) \) for all \( X \subseteq Y \subseteq E \) and \( e \notin Y \). For example, the simple greedy algorithm for cardinality constraints proposed by Nemhauser et al. [22] works because of this property. For set functions, it is well-known that submodularity is equivalent to the diminishing return property. For functions over the integer lattice, however, lattice submodularity only implies a weaker variant of the inequality. This causes difficulty in designing approximation algorithms; even for a single cardinality constraint, we need a more complicated approach such as partial enumeration [1, 26].

Fortunately, objective functions appearing in practical applications admit the diminishing return property in the following sense. We say that a function \( f : \mathbb{Z}^E \to \mathbb{R} \) is diminishing return submodular (DR-submodular) if \( f(x + \chi_e) - f(x) \geq f(y + \chi_e) - f(y) \) for arbitrary \( x \leq y \) and \( e \in E \), where \( \chi_e \) is the \( i \)-th unit vector. Any DR-submodular function is lattice submodular; i.e., DR-submodularity is stronger than lattice submodularity.\(^1\) The problem of maximizing DR-submodular functions over \( \mathbb{Z}^E \) naturally appears in the submodular welfare problem [16, 24] and the budget allocation problem with decreasing influence probabilities [26]. Nevertheless, only a few studies have considered this problem. In fact, it was not known whether we \((1 - 1/e)\)-approximation can be obtained in polynomial time under a single cardinality constraint.

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\(^1\) Note that \( f \) is DR-submodular if and only if it is lattice submodular and satisfies the coordinate-wise concave condition: \( f(x + \chi_e) - f(x) \geq f(x + 2\chi_e) - f(x + \chi_e) \) for any \( x \) and \( e \in E \) (see [27, Lemma 2.3]).
1.1 Main Results

In this paper, we develop polynomial-time approximation algorithms for maximizing monotone DR-submodular functions under cardinality constraints, polymatroid constraints, and knapsack constraints. Let \( f : \mathbb{Z}^E \rightarrow \mathbb{R} \) be a non-negative monotone DR-submodular function unless explicitly stated otherwise. Then given any small constant \( \epsilon > 0 \), our algorithms find \((1 - 1/e - \epsilon)\)-approximate solutions under these constraints. The details are described below.

Cardinality Constraint: The objective is to maximize \( f(x) \) subject to \( 0 \leq x \leq c \) and \( x(E) \leq r \), where \( c \in \mathbb{Z}^E_+ \), \( r \in \mathbb{Z}_+ \), and \( x(E) = \sum_{e \in E} x(e) \). We design a deterministic approximation algorithm with \( O(n^{\epsilon} \log \|c\|_\infty \log 1/\epsilon) \) running time, which is the first polynomial time algorithm for this problem.

Cardinality Constraint (lattice submodular case): For cardinality constraints, we also show a \((1 - 1/e - \epsilon)\)-approximation algorithm for a monotone lattice submodular function \( f \). This algorithm runs in \( O(n^{\epsilon} \log \|c\|_\infty \log \frac{r}{\epsilon} \log \tau) \) time, where \( \tau \) is the ratio of the maximum value of \( f \) to the minimum positive increase in the value of \( f \).

Polymatroid Constraint: The objective is to maximize \( f(x) \) subject to \( x \in P \cap \mathbb{Z}^E_+ \), where \( P \) is a polymatroid given via an independence oracle. Our algorithm runs in \( O(\frac{r}{\epsilon} \log \|c\|_\infty \log \frac{r}{\epsilon} \log \tau) \) time, where \( \tau \) is the ratio of the maximum value of \( f \) to the minimum positive increase in the value of \( f \).

Knapsack Constraint: The objective is to maximize \( f(x) \) subject to \( 0 \leq x \leq c \) and a single knapsack constraint \( w^\top x \leq 1 \), where \( w \in (0, 1]^E \). We devise an approximation algorithm with \( O\left(\frac{n^4}{\epsilon^3} \log^3 \tau \left[ \log^3 \|c\|_\infty + \frac{n}{\tau} \log \|c\|_\infty \log \frac{1}{\epsilon w_{\min}} \right]\right) \) running time, where \( \tau \) is the ratio of the maximum value of \( f \) to the minimum positive increase in the value of \( f \), and \( w_{\min} \) is the minimum entry of \( w \). This is the first polynomial time algorithm for this problem.

1.2 Technical Contribution

In order to devise polynomial-time algorithms instead of pseudo-polynomial time algorithms, we need to combine several techniques carefully. Our algorithms adapt the “decreasing-threshold greedy” framework recently introduced by Badanidiyuru and Vondrak [2], and work in the following way. We maintain a feasible solution \( x \in \mathbb{R}^E \) and a threshold \( \theta \in \mathbb{R} \) during the algorithm. Starting from \( x = 0 \), we greedily increase each component of \( x \) if the average gain in the increase is above the threshold \( \theta \), with consideration of constraints. Slightly decreasing the threshold \( \theta \), we repeat this greedy process until \( \theta \) becomes sufficiently small. We combine this framework with pseudo-polynomial time greedy algorithms to design a polynomial time algorithm for cardinality constraints. We also need to incorporate the partial enumeration technique [26, 29] to obtain a polynomial time algorithm for knapsack constraints. In order
to develop a polynomial time algorithm for polymatroid constraints, we follow the continuous greedy approach [7]; instead of the discrete problem, we consider the problem of maximizing a continuous extension of the original objective function. After the greedy phase, we round the current fractional solution to an integral solution if needed.

As described above, our algorithms share some ideas with the algorithms of [2, 7, 26, 29]. However, we attain several improvements and introduce new ideas, mainly due to the essential difference between set functions and functions over the integer lattice.

Binary Search in the Greedy Phase: In most previous algorithms, the greedy step works as follows: find the direction of the maximum marginal gain and move the current solution along this direction with a unit step size. However, it turns out that a naive adaptation of this greedy strategy only yields a pseudo-polynomial time algorithm. To circumvent this issue, we perform a binary search to determine the step size in the greedy phase. Combined with the decreasing threshold framework, this technique significantly reduces the time complexity.

New Continuous Extension: To execute the continuous greedy algorithm, we need a continuous extension of functions over the integer lattice. Note that the multilinear extension [6] cannot be directly used because the domain of the multilinear extension is only the hypercube $[0, 1]^E$. In this paper, we propose a new continuous extension of a function over the integer lattice for polymatroid constraints. This continuous extension has similar properties to the multilinear extension when $f$ is DR-submodular, and is carefully designed so that we can round fractional solutions without violating polymatroid constraints. To the best of our knowledge, this continuous extension in $\mathbb{R}_+^E$ has not been proposed in the literature so far.

Rounding without violating polymatroid constraints: Rounding fractional solutions in $\mathbb{R}_+^E$ without violating polymatroid constraints is non-trivial. We show that the rounding can be reduced to rounding in a matroid polytope; therefore we can use existing rounding methods for a matroid polytope.

Modification to the conference version An extended abstract of this paper appeared in [28]. Unfortunately, the algorithm for a knapsack constraint presented there is quite complicated and has a technical flaw: the correct time complexity is not as stated. In this paper, we provide another much simpler algorithm for a knapsack constraint. A main difference is that the algorithm in this paper use partial enumeration, whereas the algorithm in [28] used continuous greedy.

1.3 Related Work

Studies on maximizing monotone submodular functions were pioneered by Neumhauser, Wolsey, and Fisher [22]. They showed that a greedy algorithm
achieves a \((1 - 1/e)\)-approximation for maximizing a monotone and submodular set function under a cardinality constraint, and a \(1/2\)-approximation under a matroid constraint. Their algorithm provided a prototype for subsequent work. For knapsack constraints, Sviridenko [29] devised the first \((1 - 1/e)\)-approximation algorithm with \(O(n^3)\) running time. Whereas these algorithms are combinatorial and deterministic, the best known algorithms for matroid constraints are based on a continuous and randomized method. The first \((1 - 1/e)\)-approximation algorithm for a matroid constraint was provided by [6], and employed the continuous greedy approach: first solve a continuous relaxation problem and obtain a fractional approximate solution; then round it to an integral feasible solution. In their framework, the multilinear extension of a submodular set function was used as the objective function in the relaxation problem. They also provided the \textit{pipage rounding} to obtain an integral feasible solution. Chekuri, Vondrák, and Zenklusen [7] designed a simple rounding method—\textit{swap rounding}—based on the exchange property of matroid base families. Badanidiyuru and Vondrák [2] recently devised \((1 - 1/e - \epsilon)\)-approximation algorithms for any fixed constraint \(\epsilon > 0\), with significantly lower time complexity for various constraints. For the inapproximability side, Nemhauser et al. [22] proved that no algorithm making polynomially many queries to a value oracle of \(f\) can achieve an approximation ratio better than \(1 - 1/e\) under any of the constraints mentioned so far. Furthermore, Feige [10] showed that, even if \(f\) is given explicitly, \((1 - 1/e)\)-approximation is the best possible unless \(P = NP\).

Generalized forms of submodularity have been studied in various contexts. Fujishige [12] discussed submodular functions over a distributive lattice and its related polyhedra. In the theory of discrete convex analysis by Murota [21], a subclass of submodular functions over the integer lattice was considered. The maximization problem has also been studied for variants of submodular functions. Shioura [24] investigated the maximization of discrete convex functions. Soma et al. [26] provided a \((1 - 1/e)\)-approximation algorithm for maximizing a monotone lattice submodular function under a knapsack constraint. However, its running time is pseudo-polynomial. Although this paper focuses on monotone submodular functions, there is a large body of work on maximization of \textit{non-monotone} submodular functions [11,4,5,3]. Gottschalk and Peis [13] provided a \(1/3\)-approximation algorithm for maximizing a lattice submodular function over a (bounded) integer lattice. Recently, \textit{bisubmodular} functions and \(k\)-\textit{submodular} functions, other generalizations of submodular functions, have been studied as well, and approximation algorithms for maximizing these functions can be found in [15,25,30].

1.4 Organization of This Paper

The rest of this paper is organized as follows. In Section 2, we provide our notations and basic facts on submodular functions and polymatroids. Section 3 describes our algorithm for cardinality constraints. In Section 4, we provide
the continuous extension for polymatroid constraints and our approximate algorithm. We present our algorithm for knapsack constraints in Section 5.

2 Preliminaries

Notation We denote the sets of non-negative integers and non-negative reals by \( \mathbb{Z}_+ \) and \( \mathbb{R}_+ \), respectively. We denote the set of positive integers by \( \mathbb{N} \). For a positive integer \( k \in \mathbb{N} \), \( [k] \) denotes the set \( \{1, \ldots, k\} \).

Throughout this paper, \( E \) denotes a ground set of size \( n \). We denote the \( i \)-th entry of a vector \( x \in \mathbb{R}^n \) by \( x(i) \). The \( i \)-th standard unit vector is denoted by \( \chi_i \). The zero vector is denoted by \( 0 \) and the all-one vector by \( 1 \). We denote the characteristic vector of \( X \subseteq E \) by \( \chi_X \). For \( f : \mathbb{R}^E \to \mathbb{R} \) and \( x, y \in \mathbb{R}^E \), we define \( f(x | y) := f(x + y) - f(y) \).

\( \supp(x) \) denotes the set \( \{e \in E \mid x(e) > 0\} \). For \( x \in \mathbb{R}^E \), \( \{x\} \) denotes the multiset where the element \( e \) appears \( x(e) \) times. For arbitrary two multisets \( \{x\} \) and \( \{y\}, \) we define \( \{x\} \setminus \{y\} := \{(x - y) \vee 0\} \). For a multiset \( \{x\}, \) we define \( |\{x\}| := x(E) \).

Lemma 1 (Relative+Additive Chernoff’s bound, [2]) Let \( X_1, \ldots, X_m \) be independent random variables such that for each \( i, X_i \in [0, 1] \). Let \( X = \frac{1}{m} \sum X_i \) and \( \mu = \mathbb{E}[X] \). Then

\[
\Pr[X > (1 + \alpha)\mu + \beta] \leq e^{-\frac{m\alpha^2}{\beta^2}},
\]

\[
\Pr[X < (1 - \alpha)\mu - \beta] \leq e^{-\frac{m\alpha^2}{\beta^2}}.
\]

2.1 Submodularity and the Diminishing Return Property

We say that a function \( f : \mathbb{Z}_+^E \to \mathbb{R} \) is lattice submodular if it satisfies \( f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \) for all \( x, y \in \mathbb{Z}_+^E \), where \( x \vee y \) and \( x \wedge y \) denote the coordinate-wise maximum and minimum, respectively, i.e., \( (x \vee y)(e) = \max\{x(e), y(e)\} \) and \( (x \wedge y)(e) = \min\{x(e), y(e)\} \) for each \( e \in E \). A function \( f : \mathbb{Z}_+^E \to \mathbb{R} \) is monotone if \( f(x) \leq f(y) \) for all \( x \) and \( y \) with \( x \leq y \). We say that \( f : \mathbb{Z}_+^E \to \mathbb{R} \) is diminishing return submodular (DR-submodular) if \( f(x + \chi_i) - f(x) \geq f(y + \chi_i) - f(y) \) for every \( x \leq y \) and \( i \in E \), where \( \chi_i \) denotes the \( i \)-th unit vector. We note that the lattice submodularity of \( f \) does not imply DR-submodularity when the domain is the integer lattice. Throughout this paper, we assume that \( f(0) = 0 \) without loss of generality.

If a function \( f : \mathbb{Z}_+^E \to \mathbb{R} \) satisfies \( f(x \vee k\chi_i) - f(x) \geq f(y \vee k\chi_i) - f(y) \) for any \( i \in E, k \in \mathbb{Z}_+, x \) and \( y \) with \( x \leq y \), then we say that \( f \) satisfies the
weak diminishing return property. Any monotone lattice submodular function satisfies the weak diminishing return property [26].

2.2 Polymatroid

Let \( \rho : 2^E \to \mathbb{Z}_+ \) be a monotone submodular set function with \( \rho(\emptyset) = 0 \). The (integral) polymatroid associated with \( \rho \) is the polytope \( P = \{ x \in \mathbb{R}_+^E : x(X) \leq \rho(X) \ \forall X \subseteq E \} \), and \( \rho \) is called the rank function of \( P \). The base polytope of polymatroid \( P \) is defined as \( B := \{ x \in P : x(E) = \rho(E) \} \). The set of integral points in \( B \) satisfies the following simultaneous exchange property:

For any \( x, y \in B \cap \mathbb{Z}_+^E \) and \( s \in \text{supp}^+(x - y) \), there exists \( t \in \text{supp}^+(y - x) \) such that \( x - \chi_s + \chi_t \in B \cap \mathbb{Z}_+^E \) and \( y + \chi_s - \chi_t \in B \cap \mathbb{Z}_+^E \).

The following lemma can be derived by the simultaneous exchange property.

Lemma 2 Let \( x, y \in B \cap \mathbb{Z}_+^E \) and \( I(x) := \{(e, i) : e \in \text{supp}^+(x), 1 \leq i \leq x(e)\} \). Then there exists a map \( \phi : I(x) \to \text{supp}^+(y) \) such that \( x - \chi_e \in B \cap \mathbb{Z}_+^E \) for each \( e \in \text{supp}^+(x) \) and \( 1 \leq i \leq x(e) \), and \( y = \sum_{(e, i) \in I(x)} \chi_{\phi(e, i)} \).

Proof Induction on \( |\{x\} \setminus \{y\}| \). If \( |\{x\} \setminus \{y\}| = 0 \), then \( \phi(e, i) := e \) satisfies the condition. Let us assume that \( |\{x\} \setminus \{y\}| > 0 \). Let us fix \( s \in \text{supp}^+(x - y) \) arbitrarily. By the simultaneous exchange property, we can find \( t \in \text{supp}^+(y - x) \) such that \( x - \chi_s + \chi_t \in B \cap \mathbb{Z}_+^E \) and \( y' := y + \chi_s - \chi_t \in B \cap \mathbb{Z}_+^E \). By the induction hypothesis, we can obtain \( \phi' : I(x) \to \text{supp}^+(y') \) satisfying the conditions. The desired \( \phi \) can be obtained by modifying \( \phi' \) as \( \phi'(s, y(s) + 1) := t \).

3 Cardinality Constraint

In this section, we consider cardinality constraints. We provide approximation algorithms for maximizing monotone DR-submodular and lattice submodular functions in Sections 3.1 and 3.2, respectively.

3.1 Maximization of Monotone DR-Submodular Function

We start with the case of a DR-submodular function. Let \( f : \mathbb{Z}_+^E \to \mathbb{R}_+ \) be a monotone DR-submodular function. Let \( c \in \mathbb{Z}_+^E \) and \( r \in \mathbb{Z}_+ \). We want to maximize \( f(x) \) under constraints \( 0 \leq x \leq c \) and \( x(E) \leq r \). The pseudocode description of our algorithm, based on the decreasing threshold greedy framework, is shown in Algorithm 1.
Algorithm 1 Cardinality Constraint/DR-Submodular

Input: \( f : \mathbb{R}_+^E \to \mathbb{R}_+ \), \( c \in \mathbb{R}_+^E \), \( r \in \mathbb{Z}_+ \), and \( \epsilon > 0 \).
Output: \( y \in \mathbb{Z}_+^E \).
\begin{itemize}
  \item 1: \( y \leftarrow 0 \) and \( d \leftarrow \max_{e \in E} f(x_e) \).
  \item 2: for \( (\theta = d; \theta \geq \frac{d}{2}; \theta \leftarrow \theta(1 - \epsilon)) \) do
  \item 3: for all \( e \in E \) do
  \item 4: Find maximum \( k \leq \min \{ \epsilon(e) - y(e), r - y(E) \} \) with \( f(kx_e \mid y) \geq k\theta \) with binary search.
  \item 5: if such \( k \) exists then
  \item 6: \( y \leftarrow y + kx_e \).
  \item 7: return \( y \).
\end{itemize}

Lemma 3 Let \( x^* \) be an optimal solution. When we are adding \( kx_e \) to the current solution \( y \) in Line 6, the average gain satisfies the following:
\[ \frac{f(kx_e \mid y)}{k} \geq \frac{(1 - \epsilon)}{r} \sum_{s \in \{x^* \} \setminus \{y\}} f(x_s \mid y). \]

Proof Due to DR-submodularity, the marginal values can only decrease as we add elements. When we are adding a vector \( kx_e \) and the current threshold value is \( \theta \), the following inequalities hold:

Claim \( f(kx_e \mid y) \geq k\theta \), and \( f(x_s \mid y) \leq \frac{\theta}{1 - \epsilon} \) for any \( s \in \{x^* \} \setminus \{y\} \).

Proof The first inequality is trivial. The second inequality is also trivial by DR-submodularity if \( \theta = d \). Thus we assume that \( \theta < d \), i.e., there was at least one threshold update. Let \( s \in \{x^* \} \setminus \{y\} \), \( k' \) be the increment in the previous threshold (i.e., \( \frac{\theta}{1 - \epsilon} \)), and \( y' \) be the variable \( y \) at the time. Suppose that \( f(x_s \mid y) > \frac{\theta}{1 - \epsilon} \). Then \( f((k' + 1)x_s \mid y') \geq f(x_s \mid y) + f(k'x_s \mid y') > \frac{\theta}{1 - \epsilon} + \frac{k'\theta}{1 - \epsilon} = \frac{(k' + 1)\theta}{1 - \epsilon} \), which contradicts the fact that \( k' \) is the largest value with \( f(k'x_s \mid y') \geq \frac{k'\theta}{1 - \epsilon} \).

The above inequalities imply that \( \frac{f(kx_e \mid y)}{k} \geq \frac{(1 - \epsilon)}{r} \sum_{s \in \{x^* \} \setminus \{y\}} f(x_s \mid y) \) for each \( s \in \{x^* \} \setminus \{y\} \). Taking the average over these inequalities we obtain
\[ \frac{f(kx_e \mid y)}{k} \geq \frac{(1 - \epsilon)}{\|x^*\| \setminus \{y\}} \sum_{s \in \{x^* \} \setminus \{y\}} f(x_s \mid y) \geq \frac{1 - \epsilon}{r} \sum_{s \in \{x^* \} \setminus \{y\}} f(x_s \mid y) \]

\( \square \)

Theorem 1 Algorithm 1 achieves an approximation ratio of \( 1 - \frac{1}{\epsilon} - \epsilon \) in \( O(\frac{\epsilon}{\epsilon} \log \|c\|_\infty \log \frac{1}{\epsilon}) \) time.

Proof Let \( y \) be the output of Algorithm 1. Without loss of generality, we can assume that \( y(E) = r \). To see this, consider a modified version of the algorithm in which the threshold is updated until \( y(E) = r \). Let \( y' \) be the output of this modified algorithm. Since the marginal gain of increasing any coordinate of \( y \)
by one is at most $e^{\frac{1}{\epsilon}}$, \( f(y') - f(y) \leq cd \leq \epsilon \text{OPT} \). Therefore, it suffices to show that \( y' \) is a \((1 - 1/e - \epsilon)\)-approximate solution.

Let \( y_i \) be the vector \( y \) following the \( i \)-th update. We define \( y_0 = 0 \). Let \( k_i \chi_{x_i} \) be the vector added during the \( i \)-th update. That is, \( y_i = \sum_{j=1}^{i} k_j \chi_{x_j} \).

By Lemma 3, for any \( i \in \mathbb{N} \),

\[
\frac{f(k_i \chi_{x_i} | y_{i-1})}{k_i} \geq \frac{1 - \epsilon}{r} \sum_{s \in \{x^* \} \setminus \{y_{i-1}\}} f(\chi_s | y_{i-1}).
\]

By DR-submodularity, \( \sum_{s \in \{x^* \} \setminus \{y_{i-1}\}} f(\chi_s | y_{i-1}) \geq f(x^* \cup y_{i-1}) - f(y_{i-1}) \) holds. Therefore by monotonicity,

\[
f(y_i) - f(y_{i-1}) = f(k_i \chi_{x_i} | y_{i-1}) \\
\geq \frac{(1 - \epsilon)k_i}{r} (f(x^* \cup y_{i-1}) - f(y_{i-1})) \\
\geq \frac{(1 - \epsilon)k_i}{r} (\text{OPT} - f(y_{i-1})).
\]

Hence, we can show by induction that \( f(y) \geq \left( 1 - \prod_i \left( 1 - \frac{(1 - \epsilon)k_i}{r} \right) \right) \text{OPT} \).

Since \( \prod_i \left( 1 - \frac{(1 - \epsilon)k_i}{r} \right) \leq \prod_i \exp\left(-\frac{(1 - \epsilon)k_i}{r} \right) = \exp\left( -\frac{(1 - \epsilon)\sum_i k_i}{r} \right) = e^{-(1 - \epsilon)} \leq \frac{1}{e} + \epsilon \), we obtain \((1 - \frac{1}{e} - \epsilon)\)-approximation. \( \square \)

### 3.2 Maximization of Monotone Lattice Submodular Function

We now consider the case of a lattice submodular function. Let \( f : \mathbb{Z}_+^E \rightarrow \mathbb{R}_+ \) be a monotone lattice submodular function, \( c \in \mathbb{Z}_+^E \), and \( r \in \mathbb{Z}_+ \). We want to maximize \( f(x) \) under the constraints \( 0 \leq x \leq c \) and \( x(E) \leq r \).

The main issue is that we cannot find \( k \) such that \( f(k \chi_e | x) \geq k \theta \) by naive binary search. However, we can find \( k \) such that \( f(k \chi_e | x) \geq (1 - \epsilon)k \theta \) in polynomial time (if exists). The key idea is guessing the value of \( f(k \chi_e) \) by iteratively decreasing the threshold and checking whether the desired \( k \) exists with binary search. See Algorithm 2 for details.

We have the following properties:

**Lemma 4** Algorithm 2 satisfies the following:
Algorithm 3 Cardinality Constraint/Lattice Submodular

Input: $f: \mathbb{Z}_+^E \rightarrow \mathbb{R}_+, c \in \mathbb{Z}_+^E, r \in \mathbb{Z}_+, \epsilon > 0$.
1: $y \leftarrow 0$ and $d_{\text{max}} \leftarrow \max_{e \in E} f(e)X_e$.
2: for $(\theta = d_{\text{max}}; \theta \geq \frac{1}{2}d_{\text{max}}; \theta \leftarrow \theta(1 - \epsilon))$ do
3:  for all $e \in E$ do
4:   invoke $\text{BinarySearchLattice}(f(\cdot | y), e, \theta, \min\{c(e) - y(e), r - y(E)\}, \epsilon)$
5: if $\text{BinarySearchLattice}$ did not fail and returned $k \in \mathbb{N}$ then
6:   $y \leftarrow y + kX_e$
7: return $y$.

\begin{enumerate}
\item Suppose that there exists $0 \leq k^* \leq k_{\text{max}}$ such that $f(k^*X_e) \geq k^*\theta$. Then, Algorithm 2 returns $k$ with $k_{\text{min}} \leq k \leq k_{\text{max}}$ such that $f(kX_e) \geq (1 - \epsilon)k\theta$.
\item Suppose that Algorithm 2 outputs $0 \leq k \leq k_{\text{max}}$. Then, $f(k'X_e) < \max\{\frac{f(kX_e)}{1 - \epsilon}, k\theta\}$ for any $k < k' \leq k_{\text{max}}$.
\item If Algorithm 2 does not fail, then the output $k$ satisfies $f(kX_e) \geq (1 - \epsilon)k\theta$.
\item Let $k_{\text{min}} = \min\{k \mid f(kX_e) > 0\}$. Then, Algorithm 2 runs in $O(\frac{1}{\epsilon} \log \frac{f(k_{\text{max}}X_e)}{f(k_{\text{min}}X_e)})$ time. If no such $k_{\text{min}}$ exists, then Algorithm 2 runs in $O(\log k_{\text{max}})$ time.
\end{enumerate}

Proof (1) Let $H \coloneqq \{1 - \epsilon)^s f(k_{\text{max}}X_e) : s \in \mathbb{Z}_+, (1 - \epsilon)^s f(k_{\text{max}}X_e) \geq (1 - \epsilon) f(k_{\text{min}}X_e)\}$. Let $h^*$ be the (unique) element in $H$ such that $h^* \leq f(k^*X_e) < \frac{h}{1 - \epsilon}$. Let $k$ be the minimum integer such that $f(kX_e) \geq h^*$. Then, $k \leq k^*$ and $f(kX_e) < \frac{f(kX_e)}{1 - \epsilon}$. Thus $k\theta \leq h^* \theta \leq f(kX_e) \leq \frac{f(kX_e)}{1 - \epsilon}$, which means that $(1 - \epsilon)k\theta \leq f(kX_e)$.

(2) Let $h$ and $h'$ be the unique elements in $H$ such that $h \leq f(kX_e) < \frac{h}{1 - \epsilon}$ and $h' \leq f(k'X_e) < \frac{h'}{1 - \epsilon}$, respectively. Note that $h \leq h'$. If $h = h'$ then $f(k'X_e) \leq \frac{f(kX_e)}{1 - \epsilon}$. If $h < h'$, let $k_1$ be the minimum $k_1$ such that $f(k_1X_e) \geq h'$. Then $\frac{f(k_1X_e)}{1 - \epsilon} \leq \frac{f(kX_e)}{1 - \epsilon} < h$, where the last inequality follows from the fact that $k_1$ is examined by the algorithm before $k$.

(3) and (4) are obvious. \hfill \Box

Our algorithm for maximizing monotone lattice submodular functions under a cardinality constraint is based on the decreasing threshold greedy framework and uses Algorithm 2 to find elements whose marginal gain is as large as the current threshold. The pseudocode description is shown in Algorithm 3.

Let $\theta_i$ be $\theta$ in the $i$-th iteration of the outer loop. Let $k_{i,e}$ be $k$ when handling $e \in E$ in the $i$-th iteration of the outer loop. Let $y_{i,e}$ be the vector $y$ right before adding the vector $k_{i,e}X_e$. For notational simplicity, we define $y_{0,e} = 0$ and $k_{0,e} = 0$ for all $e \in E$. Let $\Delta_{i,e} = (x^* - y_{i,e}) \vee 0$. The following lemma shows that $f(\Delta_{i,e}(a), y_{i,e}) \leq \Delta_{i,e}(a)\theta_i$ for any $i \in \mathbb{N}$ and $e, a \in E$, which is a crucial property to guarantee the approximation ratio of Algorithm 3, except that there are several small error terms.

Lemma 5 For any $i \in \mathbb{N}$ and $e, a \in E$,
\[
f(\Delta_{i,e}(a)X_a \mid y_{i,e}) \leq \max\left\{\frac{\epsilon}{1 - \epsilon} f(k_{i-1,a}X_a \mid y_{i-1,a}), (e k_{i-1,a} + \Delta_{i,e}(a)) \theta_i \right\}.
\]
Proof We define $\Delta = \Delta_{i,e}$, $k' = k_{i-1,a}$, $y = y_{i,a}$, $y' = y_{i-1,a}$, and $\theta' = \frac{\theta}{r}$ for notational simplicity. We assume $\Delta(a) > 0$ as otherwise the statement is trivial. When $i \geq 2$, from (2) in Lemma 4,

$$f((k' + \Delta(a))x_a \mid y') \leq \max\left\{ \frac{f(k'x_a \mid y')}{1 - \epsilon}, (k' + \Delta(a))\theta' \right\}.$$ 

We can verify that this inequality also holds when $i = 1$ because

$$f((k' + \Delta(a))x_a \mid y') = f(\Delta(a)x_a) \leq d_{\max} \leq (k' + \Delta(a))\theta'.$$

Since $f((k' + \Delta(a))x_a \mid y') \geq f(\Delta(a)x_a \mid y) + f(k'x_a \mid y')$ holds from lattice submodularity, it follows that

$$f(\Delta(a)x_a \mid y) \leq \max\left\{ \frac{\epsilon}{1 - \epsilon}f(k'x_a \mid y'), (k' + \Delta(a))\theta' - f(k'x_a \mid y') \right\}.$$ 

Since $f(k'x_a \mid y') \geq (1 - \epsilon)k'\theta'$ from (1) of Lemma 4, we obtain the claim. □

Lemma 6

$$f(k_{i,e}x_e \mid y_{i,e}) \geq \frac{(1 - \epsilon)^2k_{i,e}}{(1 + \epsilon)r} \left( \sum_{a \in E} f(\Delta(a)x_a \mid y_{i,e}) - \frac{\epsilon}{1 - \epsilon}f(y_{i,e}) \right).$$

Proof Note that for any $i \in \mathbb{N}$ and $e \in E$, $f(k_{i,e}x_e \mid y_{i,e}) \geq (1 - \epsilon)k_{i,e}\theta_i$ from (1) of Lemma 4. Hence, by summing up the inequalities of Lemma 5 over all $a \in E$, we obtain

$$\sum_{a \in E} f(\Delta(a)x_a \mid y_{i,e})$$

$$\leq \sum_{a \in E} (\epsilon k_{i-1,a} + \Delta(a)) \frac{\theta_i}{1 - \epsilon} + \sum_{a \in E} \frac{\epsilon}{1 - \epsilon}f(k_{i-1,a}x_a \mid y_{i-1,a})$$

$$\leq (1 + \epsilon)r \frac{\theta_i}{1 - \epsilon} + \frac{\epsilon}{1 - \epsilon}f(y_{i,e})$$

$$\leq \frac{(1 + \epsilon)r}{(1 - \epsilon)^2k_{i,e}}f(k_{i,e}x_e \mid y_{i,e}) + \frac{\epsilon}{1 - \epsilon}f(y_{i,e}).$$

We obtain the claim by rearranging the inequality. □

Theorem 2 Algorithm 3 achieves an approximation ratio of $1 - \frac{1}{r} - O(\epsilon)$ in $O\left( \frac{1}{\epsilon^2} \log ||c||_{\infty} \log \frac{1}{\epsilon} \right)$ time, where $\tau = \frac{\max_{a \in E} f(\hat{x}(e)x_e)}{\min_{a \in E} \frac{f(\hat{x}(e)x_e)}{d_{\max}(x_e)}}$.

Proof Let $y$ be the final output of Algorithm 3. We can assume that $y(E) = r$. To see this, consider the modified algorithm in which $\theta$ is updated until $y(E) = r$. The output $y'$ of this modified algorithm satisfies $y'(E) = y$.

By a similar argument as above, we can show that $\sum_{a \in E} f((y' - y)(a)x_a \mid y) \leq (1 + \epsilon)\frac{\theta_{\min}}{r} + \frac{1}{r^2}f(y)$, where $\theta_{\min} = \epsilon d_{\max}$. This yields $f(y') \leq (1 + O(\epsilon))f(y) + O(\epsilon)\text{OPT}$. Hence it suffices to show that $y'$ gives $(1 - 1/e - \epsilon)$-approximation.

Proof of Theorem 2 When $\theta_i \geq 1$ and $y_i = 0$, we have $y'(E) = r$. Thus, we can assume that $\theta_i < 1$ and $y_i > 0$. Then, $y''(E) = u_i \cdot y_i$, where $u_i = \frac{\theta_i}{\theta}$. We have $y''(E) = y''(E)$. Thus, we can assume that $y''(E) = y''(E)$. Therefore, $y''(E) = y''(E)$.

By a similar argument as above, we can show that $\sum_{a \in E} f((y'' - y')(a)x_a \mid y) \leq (1 + \epsilon)\frac{\theta_{\min}}{r} + \frac{1}{r^2}f(y)$, where $\theta_{\min} = \epsilon d_{\max}$. This yields $f(y') \leq (1 + O(\epsilon))f(y) + O(\epsilon)\text{OPT}$. Hence it suffices to show that $y'$ gives $(1 - 1/e - \epsilon)$-approximation.
Let $\alpha = \frac{(1-c)^2}{1-\epsilon}$ and $\beta = \frac{\epsilon}{1-\epsilon}$. By Lemma 6 and lattice submodularity of $f$,

$$f(k_{i,e}x_e | y_{i,e}) \geq \frac{\alpha k_{i,e}}{r} \left( \sum_{a \in E} f(\Delta_{i,e}(a)x_a | y_{i,e}) - \beta f(y_{i,e}) \right)$$

$$\geq \frac{\alpha k_{i,e}}{r} \left( (f(x^* \lor y_{i,e}) - f(y_{i,e})) - \beta f(y_{i,e}) \right)$$

$$\geq \frac{\alpha k_{i,e}}{r} (\text{OPT} - (1 + \beta) f(y_{i,e}))$$

$$= \frac{\alpha(1 + \beta) k_{i,e}}{r} (\text{OPT} - f(y_{i,e}))$$

$$= \frac{(1 - O(\epsilon)) k_{i,e}}{r} (\text{OPT} - f(y_{i,e})),$$

where $\text{OPT} = \text{OPT} / (1 + \beta) = (1 - c)\text{OPT}$. By the same argument in the case of DR-submodular functions, we can show that $f(y) \geq (1 - 1/e - O(\epsilon))\text{OPT} = (1 - 1/e - O(\epsilon))\text{OPT}$. The analysis of the time complexity is straightforward.

\[\square\]

## 4 Polymatroid Constraint

Let $P$ be a polymatroid with a ground set $E$ and the rank function $\rho : 2^E \to \mathbb{Z}_+$. The objective is to maximize $f(x)$ subject to $x \in P \cap \mathbb{Z}^E$, where $f$ is a DR-submodular function. In what follows, we denote $\rho(E)$ by $r$. We assume that $P$ is contained in the interval $[0, c] = \{ x \in \mathbb{R}^E_+ : 0 \leq x \leq c \}$.

We start by describing a continuous extension for polymatroid constraints in Section 4.1 and then state and analyze our algorithm in Section 4.2.

### 4.1 Continuous extension for polymatroid constraints

For $x \in \mathbb{R}^E$, let $\lfloor x \rfloor$ denote the vector obtained by rounding down each entry of $x$. For $a \in \mathbb{R}$, let $\lfloor a \rfloor$ denote the fractional part of $a$, that is, $(a) := a - \lfloor a \rfloor$.

For $x \in \mathbb{R}^E$, we define $C(x) := \{ y \in \mathbb{R}^E : |x| \leq y \leq \lfloor x \rfloor + 1 \}$ as the hypercube to which $x$ belongs.

For $x \in \mathbb{R}^E$, we define $D(x)$ as the distribution from which we sample $\bar{x}$ such that $\bar{x}(i) = \lfloor x(i) \rfloor$ with probability $1 - \langle x(i) \rangle$ and $\bar{x}(i) = \lfloor x(i) \rfloor$ with probability $\langle x(i) \rangle$, for each $i \in E$. We define the continuous extension $F : \mathbb{R}^E_+ \to \mathbb{R}_+$ of $f : \mathbb{Z}^E_+ \to \mathbb{R}_+$ as follows. For each $x \in \mathbb{R}^E_+$, we define

$$F(x) := \mathbb{E}_{x \sim D(x)} [f(\bar{x})] = \sum_{S \subseteq E} f(|x| + \chi_S) \prod_{i \in S} \langle x(i) \rangle \prod_{i \notin S} (1 - \langle x(i) \rangle). \quad (1)$$

We call this type of continuous extension the continuous extension for polymatroid constraints. Note that $F$ is obtained by gluing the multilinear extension of $f$ restricted to each hypercube. If $f : \{0, 1\}^E \to \mathbb{R}_+$ is a monotone submodular function, it is known that its multilinear extension is monotone and
concave along non-negative directions. We can show similar properties for the continuous extension of a function $f : \mathbb{Z}^E_+ \rightarrow \mathbb{R}$ if $f$ is monotone and DR-submodular.

**Lemma 7** For a monotone DR-submodular function $f$, the continuous extension $F$ for the polymatroid constraint is a non-decreasing concave function along any line of direction $d \geq 0$.

To prove this, we need the following useful fact from convex analysis.

**Lemma 8** (Rockafellar [23], Theorem 24.2) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a non-increasing function and $a \in \mathbb{R}$. Then

$$f(x) := \int_a^x g(t)dt$$

is a concave function.

We will use the following notation from [23]: For a function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ and $a \in \mathbb{R}$, let us define

$$\phi'_+(a) := \lim_{\epsilon \downarrow 0} \frac{\phi(a + \epsilon) - \phi(a)}{\epsilon} \quad \text{and} \quad \phi'_-(a) := \lim_{\epsilon \uparrow 0} \frac{\phi(a + \epsilon) - \phi(a)}{\epsilon},$$

if the limits exist. For a multivariate function $F : \mathbb{R}^E \rightarrow \mathbb{R}$, $\nabla_+ F$ and $\nabla_- F$ are defined in a similar manner. For $i \in E$, the $i$-th entry of $\nabla_+ F$ equals $\phi'_+$ where $\phi = \frac{\partial F}{\partial x_i}$ and $\nabla_- F$ is defined accordingly.

**Proof (of Lemma 7)** Let $p \in \mathbb{R}^E_+$ and $\phi(\xi) := f(p + \xi d)$ for $\xi \geq 0$. To show that $\phi$ is a non-decreasing concave function, we will find a non-increasing non-negative function $g$ such that $\phi(\xi) = \phi(0) + \int_0^\xi g(t)dt$. Note that if $\phi$ is differentiable for all $\xi \geq 0$, then $g := \phi'$ satisfies the condition. However, $\phi$ may be non-differentiable at $\xi$ if $p + \xi d$ contains an integral entry.

Let us denote $x := p + \xi d$ and suppose that $x_i \in \mathbb{Z}$ for some $i$. Then one can check that $\phi'_-(\xi_0) = (\nabla F_-(x))_i d$ and $\phi'_+(\xi_0) = (\nabla F_+(x))_i d$. For each $j$,

$$\nabla F_-(j)|_{\xi_0} = \sum_{i:j \not\in S} f(x_i \mid [x] - x_i + x_S) \prod_{j \in S} \langle x(j) \rangle \prod_{j \geq S, j \not\in i} (1 - \langle x(j) \rangle),$$

$$\nabla F_+(j)|_{\xi_0} = \sum_{i:j \not\in S} f(x_i \mid [x] + x_S) \prod_{j \in S} \langle x(j) \rangle \prod_{j \geq S, j \not\in i} (1 - \langle x(j) \rangle).$$

On the other hand, from DR-submodularity,

$$f(x_i \mid [x] - x_i + x_S) \geq f(x_i \mid [x] + x_S).$$

Since $d \geq 0$, we have $\phi'_-(\xi_0) \geq \phi'_+(\xi_0)$.

Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ be an arbitrary function such that $\phi'_-(\xi) \geq g(\xi) \geq \phi'_+(\xi)$ for $\xi \geq 0$. Note that $g(\xi) = \phi'(\xi)$ if $\phi$ is differentiable at $\xi$. Then $g$ is non-decreasing and non-negative. Since $g(\xi) = \phi'(\xi)$ almost everywhere, $\phi(\xi) = \phi(0) + \int_0^\xi g(t)dt$. The lemma now directly follows from the non-negativity of $g$ and Lemma 8. \qed
Algorithm 4 Polymatroid Constraint/DR-Submodular

Input: \( f: \mathbb{Z}^E_+ \rightarrow \mathbb{R}^+ \), \( P \subseteq \mathbb{R}^E_+ \), and \( \epsilon > 0 \).
Output: A vector \( \bar{x} \in P \cap \mathbb{Z}^E_+ \).

1: \( x^0 \leftarrow 0 \).
2: for (\( t \leftarrow 1; t \leq \lfloor \frac{1}{\epsilon} \rfloor \); \( t \leftarrow t + 1 \)) do
3: \( y_t \leftarrow \text{DirectionPolymatroid}(f, x^{t-1}, \epsilon, P) \).
4: \( x^t \leftarrow x^{t-1} + \epsilon y_t \).
5: \( \bar{x} \leftarrow \text{RoundingPolymatroid}(x^1/\epsilon, P) \).
6: return \( \bar{x} \).

Algorithm 5 DirectionPolymatroid(\( f, x, \epsilon, P \))

Input: \( f: \mathbb{Z}^E_+ \rightarrow \mathbb{R}^+ \), \( x \in \mathbb{R}^E_+ \), \( \epsilon \in [0, 1] \), \( P \subseteq \mathbb{R}^E_+ \).
Output: A vector \( y \in P \cap \mathbb{Z}^E_+ \).

1: \( y \leftarrow 0 \) and \( d \leftarrow \max_{e \in E} f(\chi_e) \).
2: \( N \leftarrow \text{the solution to } N = n \left\lceil \log_{1/1-\epsilon} \frac{1}{\epsilon} \right\rceil \). Note that \( N = O\left( \frac{n}{\epsilon} \log \frac{n}{\epsilon} \right) \).
3: for (\( \theta = d; \theta \geq \frac{d}{N}; \theta \leftarrow \theta(1 - \epsilon) \)) do
4: for all \( e \in E \) do
5: \( k_{\text{max}} \leftarrow \max \{ k : x + y + k \chi_e \in P \} \).
6: \( k \leftarrow \text{BinarySearchPolymatroid}(f, x, e, \theta, \alpha_5, \beta_5, \delta_5, k_{\text{max}}) \).
7: if \( k \geq 1 \) then
8: \( y \leftarrow y + k \chi_e \).
9: return \( y \).

For a continuous extension \( F \) for polymatroid constraints and \( x, y \in \mathbb{R}^E \), we define \( F(x \mid y) = F(x + y) - F(y) \). The following is immediate.

Proposition 1 If \( x \in \mathbb{Z}^E_+ \), then \( F(x \mid y) = \mathbb{E}_{z \sim D(y)}[f(x \mid z)] \).

4.2 Continuous greedy algorithm for polymatroid constraints

In this section, we describe and analyze our algorithm for the polymatroid constraint, whose pseudocode description is presented in Algorithm 4.

At a high level, our algorithm computes a sequence \( x^0, x^1, \ldots, x^{1/\epsilon} \) in \( P \) (recall that we have assumed that \( 1/\epsilon \) is an integer). For each time step \( t \in [1/\epsilon] \), given \( x^{t-1} \), we determine a direction \( y^t \) by calling a subroutine DirectionPolymatroid, and we update as \( x^t := x^{t-1} + \epsilon y^t \). Intuitively speaking, \( y^t \) is the direction that maximizes the marginal gain while keeping \( x^t \in P \). Finally, we execute a rounding algorithm RoundingPolymatroid to the fractional solution \( x^{1/\epsilon} \), and obtain an integral solution \( \bar{x} \). The detailed description on DirectionPolymatroid and RoundingPolymatroid are given in Sections 4.2.1 and 4.2.2, respectively.

4.2.1 Computing an approximation to the gradient

As mentioned above, given a vector \( x \in P \), we want to compute a direction \( y \) that maximizes the marginal gain \( F(\epsilon y \mid x) \) and satisfies \( x + \epsilon y \in P \). In order to efficiently approximate such \( y \), again we use the decreasing-threshold greedy
Algorithm 6 BinarySearchPolymatroid($f, x, e, \theta, \alpha, \beta, \delta, k_{\text{max}}$)

Input: $f : \mathbb{Z}_{E}^{+} \rightarrow \mathbb{R}_{+}$, $x \in \mathbb{R}_{E}^{+}$, $e \in E$, $\theta \in \mathbb{R}_{+}$, $\alpha \in (0, 1/2)$, $\beta, \delta \in (0, 1)$, $k_{\text{max}} \in \mathbb{Z}_{+}$
Output: $k \in \mathbb{Z}_{+}$.

1: $\ell \leftarrow 1$, $u \leftarrow k_{\text{max}}$.
2: while $\ell < u$ do
3: \hspace{1em} $m = \lfloor \frac{\ell + u}{2} \rfloor$.
4: \hspace{1em} $\tilde{F}(m \chi_e | x) \leftarrow$ the estimate of $F(m \chi_e | x)$ obtained by averaging $O\left(\frac{\log(k_{\text{max}}/\delta)}{\alpha \beta}\right)$ random samples.
5: \hspace{1em} if $\tilde{F}(m \chi_e | x) \geq m \theta$ then
6: \hspace{2em} $\ell \leftarrow m + 1$.
7: \hspace{1em} else
8: \hspace{2em} $u \leftarrow m$.
9: $k \leftarrow \ell - 1$.
10: return $k$.

framework. The pseudocode of DirectionPolymatroid is given in Algorithm 5. The parameters $\alpha_5$, $\beta_5$, and $\delta_5$ will be determined later.

Recall that, in the framework, we want to determine how many copies of an element can be added when the average marginal gain is restricted to be as large as the current threshold. Given $x \in \mathbb{R}_{E}^{+}$, $e \in E$, and $\theta \in \mathbb{R}_{+}$, BinarySearchPolymatroid (Algorithm 6) performs binary search to find the largest $k$ such that $F(k \chi_e | x) \geq \theta$. Since we can only estimate the value of $F(\cdot)$ in polynomial time, the output $k$ of Algorithm 6 only satisfies the following weak conditions:

**Lemma 9** Algorithm 6 satisfies the following properties:

1. Suppose that Algorithm 6 outputs $k \in \mathbb{Z}_{+}$. Then, with probability at least $1 - \delta$,
   \[ F(k \chi_e | x) \geq (1 - \alpha)k\theta - \beta f(k \chi_e) \]
   and
   \[ F((k + 1) \chi_e | x) < \frac{(k + 1)\theta}{1 - \alpha} + 2\beta f((k + 1) \chi_e). \]
2. Suppose that Algorithm 6 outputs $k \in \mathbb{Z}_{+}$. Then, with probability at least $1 - \delta$, for any $k < k' \leq k_{\text{max}}$,
   \[ F(k' \chi_e | x) < \frac{k'\theta}{1 - \alpha} + 2\beta f(k_{\text{max}} \chi_e). \]
3. Algorithm 6 runs in $O(\log k_{\text{max}} \cdot \frac{\log(k_{\text{max}}/\delta)}{\alpha \beta})$ time.

**Proof** (1) For $k \in \mathbb{Z}$, let $s_m = F(m \chi_e | x) = E_{z \sim D(x)}[f(m \chi_e | z)]$ and $\tilde{s}_m$ be its estimation. Note that $0 \leq f(m \chi_e | z) \leq f(m \chi_e)$ for any $z$ in the support of $D(x)$. By Lemma 1 and the union bound, with probability at least $1 - \delta$,
   \[ |\tilde{s}_m - s_m| \leq \alpha s_m + \beta f(m \chi_e). \]
for every estimation in the process. In what follows, we suppose this happens.
Suppose \( s_m < \frac{1}{1 + \alpha}(m\theta - \beta f(m\chi_e)) \). Then, \( \tilde{s}_m < m\theta \) and we reach Line 8.

On the other hand, Suppose \( s_m \geq \frac{1}{1 - \alpha}(m\theta + \beta f(m\chi_e)) \). Then, \( \tilde{s}_m \geq m\theta \) and we reach Line 6. Hence after the while loop,

\[
\begin{align*}
s_{\ell - 1} & \geq \frac{1}{1 + \alpha}((\ell - 1)\theta - \beta f((\ell - 1)\chi_e)) \geq (1 - \alpha)(\ell - 1)\theta - \beta f((\ell - 1)\chi_e) \\
s_{\ell} & < \frac{1}{1 - \alpha}(\ell\theta + \beta f(\ell\chi_e)) = \frac{\ell\theta}{1 - \alpha} + \frac{\beta f(\ell\chi_e)}{1 - \alpha} \leq \frac{\theta}{1 - \alpha} + 2\beta f(\ell\chi_e),
\end{align*}
\]

where the last inequality uses the fact that \( \alpha < 1/2 \). Hence, we obtain (1), (2) holds due to (1) and the concavity of \( F \), and (3) is obvious. \( \Box \)

**Lemma 10** Let \( x^* \) be an optimal solution. \( \text{DirectionPolymatroid}(f, x, \epsilon, P) \) produces a vector \( y \in \mathbb{R}_+^E \) such that \( x' := x + y \) satisfies

\[
F(x') - F(x) \geq \epsilon \left((1 - 5\epsilon)f(x^*) - F(x')\right)
\]

with probability at least \( 1 - \epsilon/3 \).

**Proof** We choose \( \alpha_5 = \epsilon \), \( \beta_5 = \epsilon / 2N(o+1) \), and \( \delta_5 = \epsilon / 2N \). With probability at least \( 1 - \epsilon/3 \), all the binary searches succeed. In what follows, we assume that this has happened. Assume for now that \( \text{DirectionPolymatroid} \) returns \( y \) with \( y(E) = r \). If not, we add dummy elements of value 0 so that \( y(E) = r \).

Let \( y_i \) be the vector \( y \) after the \( i \)-th update in the execution of \( \text{DirectionPolymatroid} \). We define \( y_0 = 0 \). Let \( b_i \) and \( k_i \) be the element of \( E \) and the step size chosen in the \( i \)-th update, respectively; i.e., \( y_i = y_{i-1} + k_i\chi_{b_i} \). Applying Lemma 2 to \( y \) and \( x^* \), we obtain a mapping \( \phi : I(y) \rightarrow \text{supp}^+(x^*) \). Let \( e^{i,j}_* := \phi(b_i, y_{i-1}(b_i) + j) \) for \( j = 1, \ldots, k_i \). Then, by construction, \( y_{i-1} - \chi_{b_i} + \chi_{e^{i,j}_*} \in P \) for \( j = 1, \ldots, k_i \).

For each \( i \in \mathbb{N} \), let \( \theta_i \) be the threshold used in the \( i \)-th update and let \( x_i = x + y_i \). By (1) of Lemma 9,

\[
F(k_i\chi_{b_i} | x_{i-1}) \geq (1 - \epsilon)k_i\theta_i - \beta_5 f(k_i\chi_{b_i}) - \frac{\epsilon d}{N},
\]

where the last term \( \frac{\epsilon d}{N} \) comes from the case that \( b_i \) is a dummy element.

For each \( i \in \mathbb{N} \) and \( e \in E \), we define \( e_{i,e}^* \in \mathbb{Z}_+ \) as the number of the occurrence of \( e \) in the sequence \( e_{i,1}^*, \ldots, e_{i,k_i}^* \). When the vector \( k_i\chi_{b_i} \) is added, any \( e \in E \) has been a candidate element when the threshold was \( \frac{\theta}{1 - \epsilon} \). (This is a valid argument only when \( i > 1 \). We consider the case of \( i = 1 \) presently.)

Thus by (2) of Lemma 9, for each \( e \in E \),

\[
F(k_{i,e}^*\chi_e | x_{i-1}) \leq \frac{k_{i,e}^*\theta_i}{(1 - \epsilon)^2} + 2\beta_5 f(k_{\text{max}}\chi_e).
\]

This inequality also holds when \( i = 1 \) because \( F(k_{i,e}^*\chi_e | x_{i-1}) = F(k_{i,e}^*\chi_e) \leq k_{i,e}^*d \leq k_{i,e}^*\theta_1 \).
Rephrasing the inequality yields
\[ \theta_i \geq \frac{(1 - \epsilon)^2}{k^{*\epsilon}_i} \left( F(k^{*\epsilon}_i x | x_{i-1}) - 2\beta_5 f(k_{\max} x) \right). \]

By averaging over \( e \in E \), we have
\[ \theta_i \geq \frac{(1 - \epsilon)^2}{k_i} \sum_e \left( F(k^{*\epsilon}_i x | x_{i-1}) - 2\beta_5 f(k_{\max} x) \right). \]  

Combining (2), (3), and the fact that \( f(x^*) \geq d \), we obtain
\[ F(k_i x_b | x_{i-1}) \]
\[ \geq (1 - \epsilon)^3 \sum_e \left( F(k^{*\epsilon}_i x | x_{i-1}) - 2\beta_5 f(k_{\max} x) \right) - \frac{\epsilon d}{N} \]
\[ = (1 - \epsilon)^3 \sum_e F(k^{*\epsilon}_i x | x_{i-1}) - 2\beta_5 \left( f(k_i x_b) + \sum_e f(k_{\max} x) \right) - \frac{\epsilon d}{N} \]
\[ \geq (1 - \epsilon)^3 \sum_e F(k^{*\epsilon}_i x | x_{i-1}) - 2\beta_5 \left( f(x^*) + \sum_e f(x^*) \right) - \frac{\epsilon d}{N} \]
\[ \geq (1 - \epsilon)^3 \sum_e F(k^{*\epsilon}_i x | x_{i-1}) - \frac{2\epsilon}{N} f(x^*). \]

Since \( F \) is concave along non-negative directions, for any \( \epsilon > 0 \),
\[ F(\epsilon k_i x_b | x_{i-1}) \geq \epsilon (1 - \epsilon)^3 \sum_e F(k^{*\epsilon}_i x | x_{i-1}) - \frac{2\epsilon^2}{N} f(x^*). \] (4)

Using the above inequality and the fact that \( N \) is an upper bound on the total number of updates, we bound the improvement at each time step as follows:
\[ F(x') - F(x) = \sum_i (F(x + \epsilon y_i) - F(x + \epsilon y_{i-1})) = \sum_i F(\epsilon k_i x_b | x_{i-1}) \]
\[ \geq \sum_i \left( \epsilon (1 - \epsilon)^3 \sum_e F(k^{*\epsilon}_i x | x_{i-1}) - \frac{\epsilon^2}{N} f(x^*) \right) \] (By (4))
\[ = \epsilon (1 - \epsilon)^3 \sum_{i \in E} \sum_{e \in E} \mathbb{E}_{z \sim D(x_{i-1})} [f(z + k^{*\epsilon}_i x) - f(z)] - \sum_i \frac{\epsilon^2}{N} f(x^*) \]
\[ \geq \epsilon (1 - \epsilon)^3 \sum_{z \sim D(x^*)} [f(z \vee x^*) - f(z)] - 2\epsilon f(x^*) \] (from DR-submodularity)
\[ \geq \epsilon \left( (1 - \epsilon)^3 (f(x^*) - F(x')) - 2\epsilon f(x^*) \right). \] (from monotonicity)
\[ \geq \epsilon \left( (1 - 5\epsilon) f(x^*) - F(x') \right). \]  

\[ \square \]

\textbf{Lemma 11} DirectionPolymatroid runs in time \( \tilde{O}(n^2 N \log \frac{n}{\epsilon} \log r \log \frac{rN}{\epsilon}) \).
Algorithm 6 takes $O\left(\frac{n^2}{\epsilon^3} \log r \log \frac{n}{\epsilon} \right)$ time. The outer loop iterates $O\left(\frac{1}{\epsilon^3} \log \frac{n}{\epsilon} \right)$ times, while the inner loop contains the execution of Algorithm 6 $n$ times. □

**Lemma 12** At the end of Algorithm 4, $F(x) \geq (1 - 1/e - O(\epsilon))OPT$ with probability at least $2/3$. Moreover the time complexity is $O\left(\frac{n^2}{\epsilon^3} \log \frac{n}{\epsilon} \log \frac{r^2}{\epsilon} \right)$.

**Proof** Define $\Omega := (1 - 5\epsilon)f(x^*) = (1 - 5\epsilon)OPT$. Let $x^t$ be the variable $x$ after the $t$-th update. Substituting this into the result of Lemma 10, for any $t \in [1/\epsilon]$, we obtain

$$F(x^t) - F(x^{t-1}) \geq \epsilon(\Omega - F(x^t)).$$

Rephrasing the equation, we have

$$\Omega - F(x^t) \leq \frac{\Omega - F(x^{t-1})}{1 + \epsilon}.$$

Now applying induction to this equation, we obtain $\Omega - F(x^t) \leq \Omega/(1 + \epsilon)^t$. Substituting $t = 1/\epsilon$ and rewriting the equation we get the desired approximation ratio:

$$F(x) \geq \left(1 - \frac{1}{(1+\epsilon)^{1/\epsilon}}\right)\Omega \geq (1 - \frac{1}{e})(1 - 5\epsilon)OPT = \left(1 - \frac{1}{e} - O(\epsilon)\right)OPT.$$

By Lemma 11, the total time complexity is $O\left(\frac{n^2}{\epsilon^3} \log \frac{n}{\epsilon} \log \frac{r^2}{\epsilon} \right) = O\left(\frac{n^2}{\epsilon^3} \log \frac{n}{\epsilon} \log \frac{r^2}{\epsilon} \right)$. □

### 4.2.2 Rounding

We need a rounding procedure that takes a real vector $x^{1/\epsilon}$ as the input and returns an integral vector $\bar{x}$ such that $E[f(\bar{x})] \geq F(x^{1/\epsilon})$. There are several rounding algorithms in the $\{0, 1\}^E$ case [6, 7]. However, generalizing these rounding algorithms over integer lattice is a non-trivial task. Here, we show that rounding in the integer lattice can be reduced to rounding in the $\{0, 1\}^E$ case.

Suppose that we have a fractional solution $x$. The following lemma implies that we can round $x$ by considering the corresponding matroid polytope.

**Lemma 13** For $x \in P$, $P \cap C(x)$ is a translation of a matroid polytope.

**Proof** Let us consider a polytope $P' := \{(z) : z \in P \cap C(x)\}$. We can check that $P'$ can be obtained by translating $P \cap C(x)$ by $-\lfloor x \rfloor$ and restricting to $[0, 1]^E$. Therefore, $P' = \{z \in [0, 1]^E : z(X) \leq \rho'(X) \quad \forall X \subseteq E\}$, where $\rho'(X) := \min_{Y \subseteq X} \{(\rho - \lfloor x \rfloor)(Y) + 1(E \setminus Y)\}$. Then we can show that $\rho'$ is the rank function of a matroid by checking the axiom. □
The independence oracle of the corresponding matroid is simply the independence oracle of \( P \) restricted to \( C(x) \). Thus, the pipage rounding algorithm for \( P' = \{ (z) : z \in P \cap C(x^{1/\epsilon}) \} \) yields an integral solution \( \bar{x} \) with \( \mathbb{E}[f(\bar{x})] \geq F(x^{1/\epsilon}) \) in strongly polynomial time.

Slightly faster rounding can be achieved by swap rounding. Swap rounding requires that the given fractional solution \( x \) is represented by a convex combination of extreme points of the matroid polytope. In our setting, we can represent \( x^{1/\epsilon} \) as a convex combination of extreme points of \( P \cap C(x^{1/\epsilon}) \) using the algorithm of Cunningham \[8\]. Then, we run the swap rounding algorithm for the convex combination and \( P \cap C(x^{1/\epsilon}) \). The running time of this rounding algorithm is dominated by the complexity of finding a convex combination for \( x^{1/\epsilon} \), which is \( O(n^8) \) time. Adopting this algorithm as \texttt{RoundingPolymatroid} in Algorithm 4, we get the following:

**Theorem 3** Algorithm 4 finds an \((1 - 1/e - \epsilon)\)-approximate solution (in expectation) with probability at least \( 2/3 \) in \( O(n^3 \log^5 n \log^2 r + n^8) \) time.

### 5 Knapsack Constraint

In this section, we give an efficient approximation algorithm for maximizing a DR-submodular function over the integer lattice under a knapsack constraint. The problem we study is formalized as follows: given a monotone DR-submodular function \( f : \mathbb{Z}_+^E \to \mathbb{R}_+ \), \( c \in \mathbb{Z}_+^E \), and \( w \in [0, 1]^E \), we want to maximize \( f(x) \) subject to \( 0 \leq x \leq c \) and \( w^\top x \leq 1 \).

Our algorithm is twofold. The basic idea is similar to algorithms for cardinality constraint: we increase the current solution in a greedy manner using the decreasing threshold greedy framework. A difference is that the algorithm takes its initial solution as an input, whereas the algorithm for cardinality constraints always uses the zero vector as the initial solution. This greedy procedure is presented in Section 5.1. Obviously, the quality of the output of this greedy procedure depends on the choice of the initial solution. Here, we use partial enumeration, i.e., we try polynomially many initial solutions and return the best one among the outputs of the greedy procedure. This partial enumeration algorithm is described in Section 5.2. The entire algorithm is presented in Section 5.3. Throughout this section, \( x^* \) denotes an optimal solution.

#### 5.1 Greedy Procedure with Decreasing Threshold

Let us fix an initial solution \( x_0 \) and analyze the behavior of Algorithm 7 on \( x_0 \). Let us call an execution of Line 6 a trial. Let \( e_i \) and \( k_i \) be the value of \( e \) and \( k \) in the \( i \)-th trial. We denote by \( x_i \) the tentative solution \( x \) following the \( i \)-th trial. Assume that Algorithm 7 first has not updated the tentative solution \( x \) in the \( L \)-th trial. Equivalently, let \( L \) be the minimum number such that \( x_{L-1} = x_L \) and \( x_{i-1} < x_i \) for \( i = 1, \ldots, L - 1 \). We consider only such a
Lemma 14 Without loss of generality, we may assume that \( x_{L-1}(\epsilon L) + kL \leq x^*(\epsilon L) \).

Proof Suppose that \( x_{L-1}(\epsilon L) + kL > x^*(\epsilon L) \). Let us consider a modified instance in which \( c(\epsilon L) \) is reduced to \( x_{L-1}(\epsilon L) + kL - 1 \). The optimal value is unchanged by this modification because \( x^* \) is still feasible and optimal. Furthermore, Algorithm 7 returns the same solution. Thus, it suffices to analyze the algorithm in the modified instance. Repeating this argument completes the proof of this lemma. \( \square \)

Lemma 15 For \( i = 1, \ldots, L \), the average gain satisfies the following.

\[
\frac{f(k_i \chi_{x_{i-1}} | x_{i-1})}{k w(e_i)} \geq (1 - \epsilon) \frac{f(\chi_{x_{i-1}} | x_{i-1})}{w(s)} \quad (s \in \text{supp}^+(x^* - x))
\]

Proof The proof is similar to Lemma 3. For the sake of simplicity, let us fix \( i \) and denote \( x := x_{i-1}, e := e_i \), and \( k := k_i \). We first have \( f(k_i \chi_{x_{i-1}} | x) \geq k w(e) \theta \). Then, we show that \( f(\chi_{x_{i-1}} | x) \leq \frac{w(s)}{1 - \epsilon} \) for any \( s \in \text{supp}^+(x^* - x) \). This is trivial by DR-submodularity if \( \theta = d \). Thus we assume that \( \theta < d \), i.e., there is at least one threshold update. Let \( s \in \{x^*\} \setminus \{x\} \), \( k' \) be the increment in the \( s \)-th entry in the previous threshold (i.e., \( \frac{w(s)}{1 - \epsilon} \)), and \( x' \) be the variable \( x \) at the time. Suppose that \( f(\chi_{x} | x) > \frac{w(s)}{1 - \epsilon} \). Then \( f((k' + 1) \chi_{x} | x') \geq f(\chi_{x} | x) + f(k' \chi_{x} | x') > \frac{w(s)}{1 - \epsilon} + \frac{k' w(s) \theta}{1 - \epsilon} = \frac{(k' + 1) w(s) \theta}{1 - \epsilon} \), which contradicts the fact that \( k' \) was the largest value with \( f(k' \chi_{x} | x') \geq \frac{k' w(s) \theta}{1 - \epsilon} \). Eliminating \( \theta \) from these inequalities completes the proof. \( \square \)
Lemma 16 Let $x$ be the output of Algorithm 7 with an initial solution $x_0$. Then
\[ f(x) \geq \left( 1 - \frac{1}{e} - O(\epsilon) \right) \text{OPT} + \frac{f(x_0)}{e} - f(k_Lx_{e_L} \mid x_L). \quad (5) \]

Proof By monotonicity and DR-submodularity, we have
\[
\text{OPT} \leq f(x_i \lor x^*) \\
\leq f(x_i) + \sum_{s \in \{x^*\} \setminus \{x_i\}} f(x_s \mid x_{i-1}) \\
\leq f(x_i) + \sum_{s \in \{x^*\} \setminus \{x_i\}} \frac{w(s) f(k_i x_{e_i} \mid x_{i-1})}{1 - \epsilon k_i w(e_i)} \\
\leq f(x_i) + \frac{1}{1 - \epsilon} \frac{f(k_i x_{e_i} \mid x_{i-1})}{k_i w(e_i)}, \quad \text{(since } \sum_{s \in \{x^*\} \setminus \{x_i\}} w(s) \leq 1) \\
\]
for $i = 1, \ldots, L$. Rearranging the terms, we have
\[ f(k_i x_{e_i} \mid x_{i-1}) \geq (1 - \epsilon)w(e_i)k_i (\text{OPT} - f(x_i)) \quad (i = 1, \ldots, L) \]

Then by induction, we can prove
\[
\text{OPT} - \sum_{i=1}^L f(k_i x_i \mid x_{i-1}) \\
\leq (\text{OPT} - f(x_0)) \prod_{j=1}^L (1 - (1 - \epsilon)w(e_j)k_j) \\
\leq (\text{OPT} - f(x_0)) \exp \left( -(1 - \epsilon) \sum_{j=1}^L w(e_j)k_j \right) \\
\leq (\text{OPT} - f(x_0)) \exp (-1 + \epsilon) \quad \text{(since } \sum_{j=1}^L w(e_j)k_j > 1) \\
\leq (\text{OPT} - f(x_0)) \left( \frac{1}{e} + O(\epsilon) \right). \\
\]

By monotonicity,
\[
f(x) \geq f(x_L) \\
= f(x_0) + \sum_{i=1}^L f(k_i x_{e_i} \mid x_{i-1}) - f(k_L x_{e_L} \mid x_{L-1}) \\
\geq \left( 1 - \frac{1}{e} - O(\epsilon) \right) \text{OPT} + \frac{f(x_0)}{e} - f(k_L x_{e_L} \mid x_{L-1}), \\
\]
which completes the proof. □
Algorithm 8 PartialEnumeration\( (f, c, w, \epsilon) \)
\textbf{Input:} \( f : \mathbb{Z}^E_+ \rightarrow \mathbb{R}_+ \), \( c \in \mathbb{Z}^E_+ \), \( w \in [0,1]^E \), and \( \epsilon > 0 \).
\textbf{Output:} A set \( X \) consisting of \( x_0 \in \mathbb{Z}^E_+ \) with \( |\text{supp}(x)| \leq 3 \) and \( w^T x_0 \leq 1 \).
1: \( X \leftarrow \emptyset \).
2: for each ordered tuple \( X \) consisting of at most three elements in \( E \) do
3: \( Y \leftarrow \{0\} \).
4: for \( i = 1, \ldots, |X| \) do
5: Let \( e \) be the \( i \)-th element of \( X \).
6: \( Y \leftarrow \text{IncreaseSupport}(f, w, c, Y, \epsilon) \).
7: \( X' \leftarrow X \cup \{x \in Y : w^T x \leq 1\} \).
8: return \( X \).

Algorithm 9 IncreaseSupport\( (f, c, w, e, Y, \epsilon) \)
\textbf{Input:} \( f : \mathbb{Z}^E_+ \rightarrow \mathbb{R}_+ \), \( c \in \mathbb{Z}^E_+ \), \( w \in [0,1]^E \), \( e \in E \), \( Y \subseteq \mathbb{Z}^E_+ \), and \( \epsilon > 0 \).
\textbf{Output:} A set \( X \).
1: \( X \leftarrow \emptyset \).
2: for \( y \in Y \) do
3: Find \( k_{\min} \) with \( 0 \leq k_{\min} \leq c(e) \) such that \( f(k_{\min} c_e \mid y) > 0 \) by binary search.
4: if no such \( k_{\min} \) exists then continue.
5: for \( (b = f(c(e)) \mid y), h \geq (1 - \epsilon)f(k_{\min} c_e \mid y) \) do \( h = (1 - \epsilon)h \) do
6: Find the smallest \( k \) with \( k_{\min} \leq k \leq c(e) \) such that \( f(kc_e) \geq h \) by binary search.
7: Add \( y + k c_e \) to \( X \).
8: return \( X \).

5.2 Partial Enumeration

We now prove that the greedy procedure returns a \((1 - 1/\epsilon)\)-approximate solution for some \( x_0 \) and that such \( x_0 \) can be found in polynomial time. Here, we exploit partial enumeration technique. The pseudocode description of our algorithm is shown in Algorithm 8.

Lemma 17 There exists \( x_0 \) in the output of PartialEnumeration such that \( f(k_L X_{L-1} \mid X_{L-1}) \leq \frac{f(x_0)}{(1-\epsilon)^3} \), where \( x_{L-1}, k_L \), and \( e_L \) are defined as above for Algorithm 7 with the initial solution \( x_0 \).

Proof In what follows, we assume that \( n \geq 3 \) for simplicity.\(^2\) For a function \( g : \mathbb{Z}^E_+ \rightarrow \mathbb{R}_+ \), and an element \( e \in E \), we define

\[
H(g, e) := \{(1 - \epsilon)^s g(c(e)c) : s \in \mathbb{Z}_+, (1 - \epsilon)^s g(c(e)c) \geq g(k_{\min} c_e)\},
\]

where \( k_{\min} \) is the maximum \( k \in \mathbb{Z}_+ \) such that \( g(kc_e) > 0 \). Let us define \( h(g, e) \) to be the unique element in \( H(g, e) \) with \( h(g, e) \leq g(x^*(e)c) < \frac{h(g, e)}{1-\epsilon} \) and \( l(g, e) \) to be the minimum integer such that \( g(l(g, e)c_e) \geq h(g, e) \).

Then we define \( e_1^*, e_2^*, e_3^* \in E \) as follows:

\[
f_1 := f, \quad e_1^* := \arg\max_{e \in E} f_i(x^*(e)c), \quad l_i := l(f_i, e_1^*) \quad (i = 1, 2, 3),
\]

\[
f_1 := f \cdot \vee_{j=1}^3 l_j x_{e_j}^*, \quad e_i^* := \arg\max_{e \in E} f_i(x^*(e)c), \quad l_i := l(f_i, e_i^*) \quad (i = 2, 3).
\]

\(^2\) If \( n < 3 \), by a similar argument, one can show that there exists \( x_0 \) in the output of PartialEnumeration that attains \((1 - \epsilon)\)-approximation.
Let us define $x_0 := \sum_{i=1}^3 l_i x_{c_i}$. Note that $x_0$ is an element of the output of PartialEnumeration. By the definition, we have

$$f(l_i x_{c_i} \mid \forall_{j=1}^{i-1} l_j x_{c_j}) \leq f(x_i^*) x_{c_i} \mid \forall_{j=1}^{i-1} l_j x_{c_j}) \leq \frac{1}{1 - \epsilon} f(l_i x_{c_i} \mid \forall_{j=1}^{i-1} l_j x_{c_j}).$$

for $i = 1, 2, 3$. We now show this $x_0$ satisfies the required condition. We have

$$f(k_l x_{e_L} \mid x_{L-1}) \leq f(x^*(e_L)x_{e_L} \mid x_{L-1}) - f(x_{L-1}) \leq f(x^*(e_L)x_{e_L} \mid 0) - f(0) \leq f(x^*(e_1)x_{c_1}) \leq \frac{1}{1 - \epsilon} f(l_1 x_{c_1}).$$

Similarly, applying weak diminishing return with $l_1 x_{c_1}$ instead of $0$, we obtain

$$f(k_l x_{e_L} \mid x_{L-1}) \leq f(x^*(e_2)x_{e_L} \mid l_1 x_{c_1}) \leq \frac{1}{1 - \epsilon} f(l_2 x_{c_2} \mid l_1 x_{c_1}).$$

In the same way, we have

$$f(k_l x_{e_L} \mid x_{L-1}) \leq \frac{1}{1 - \epsilon} f(l_3 x_{c_3} \mid l_1 x_{c_1} \lor l_2 x_{c_2}).$$

Adding these inequalities, we obtain $3 f(k_l x_{e_L} \mid x_{L-1}) \geq f(x_0)/(1 - \epsilon)$.  

5.3 Final Algorithm

Our final algorithm for maximizing a monotone DR-submodular function subject to a knapsack constraint is shown in Algorithm 10.

**Algorithm 10** Knapsack Constraint/DR-Submodular

**Input:** $f : \mathbb{Z}_E^\mathbb{L} \rightarrow \mathbb{R}_+$, $c \in \mathbb{Z}_E^\mathbb{L}$, $w \in \{0, 1\}^E$, and $\epsilon > 0$.

**Output:** $x \in \mathbb{Z}_E^\mathbb{L}$.

1: $X \leftarrow$ PartialEnumeration$(f, c, w, \epsilon)$, $\mathcal{G} \leftarrow \emptyset$.
2: for each $x_0 \in X$ do
3:  $y \leftarrow$ GreedyKnapsack$(f, c, w, x_0, \epsilon)$
4:  Add $y$ to $\mathcal{G}$.
5: $x \leftarrow \arg\max_{y \in \mathcal{G}} f(y)$.
6: return $x$

**Theorem 4** Algorithm 10 finds a $(1 - 1/e - O(\epsilon))$-approximate solution in 

$$O\left(\frac{n^3 \log^3 \tau}{\epsilon^3} + \frac{n \log ||c||_\infty \log \frac{1}{\epsilon w_{\min}}}{\epsilon^3} \right)$$

time, where $\tau = \max_{c \leq \mathbb{E}} f(c(x) | x)$, $w_{\min} = \min_{c \in \mathbb{E}} w(c)$, and $0 < \epsilon < 1 - \epsilon/3$. 
Proof Let \( x_0 \) be the element in the output of \textit{PartialEnumeration} described in Lemma 17 and \( x_{L-1} \) be the corresponding variable for \textit{GreedyKnapsack} with the initial solution \( x_0 \). By Lemmas 16 and 17,

\[
    f(x_{L-1}) \geq \left( 1 - \frac{1}{e} - O(\epsilon) \right) \text{OPT} + \left( \frac{1}{e} - \frac{1}{3(1-\epsilon)} \right) f(y_0) \\
    \geq \left( 1 - \frac{1}{e} - O(\epsilon) \right) \text{OPT}.
\]

For the running time, \textit{PartialEnumeration} finds \( O(\frac{n^3}{\epsilon^3} \log^3 \tau) \) initial solutions in \( O(\frac{n^3}{\epsilon^3} \log \|c\|_\infty \log^3 \tau) \) time. For each initial solution, \textit{GreedyKnapsack} takes \( O(\frac{n}{\epsilon} \log \|c\|_\infty \log \frac{1}{\epsilon \min w}) \) time. Thus the total running time of Algorithm 10 is as claimed. \( \Box \)

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