NEW ALGEBRAS OF FUNCTIONS ON TOPOLOGICAL GROUPS ARISING FROM G-SPACES

E. GLASNER AND M. MEGRELISHVILI

Abstract. For a topological group $G$ we introduce the algebra $SUC(G)$ of strongly uniformly continuous functions. It contains the algebra $WAP(G)$ of weakly almost periodic functions as well as the algebras $LE(G)$ and $Asp(G)$ of locally equicontinuous and Asplund functions respectively. For the Polish groups of order preserving homeomorphisms of the unit interval and of isometries of the Urysohn space of diameter 1, $SUC(G)$ is trivial. We study the Roelcke algebra (= $UC(G)$ = right and left uniformly continuous functions) and $SUC$ compactifications of the groups $S(\mathbb{N})$, of permutations of a countable set, and $H(C)$, the group of homeomorphisms of the Cantor set. For the first group we show that $WAP(G) = SUC(G) = UC(G)$ and also provide a concrete description of the corresponding metrizable (in fact Cantor) semitopological semigroup compactification. For the second group, in contrast, $SUC(G)$ is properly contained in $UC(G)$ and for this group $UC(G)$ does not yield a right topological semigroup compactification.

We introduce the notion of fixed point on a class $P$ of flows ($P$-fpp) and study in particular groups which are $SUC$-amenable and groups with the $SUC$-fpp ($SUC$-extreme amenability). We show that every Polish group $G$ with metrizable $M(G)$ is $SUC$-amenable and if, in addition, $M(G)$ is proximal, then $G$ is $SUC$-extremely amenable.

Contents

1. Introduction 2
2. Actions and $G$-compactifications 4
3. Cyclic $G$-systems and point-universality 7
4. Strong Uniform Continuity 10
5. $SUC$, homogeneity and the epimorphism problem 15
6. Representations of groups and $G$-spaces on Banach spaces 17
7. Dynamical complexity of functions 21
8. The group $H_+[0,1]$ 27
9. Matrix coefficient characterization of $SUC$ and $LE$ 29
10. Some conclusions about $H_+[0,1]$ and $Iso(\mathbb{U}_1)$ 37
11. Relative extreme amenability: $SUC$-fpp groups 39
12. The Roelcke compactification of the group $S(\mathbb{N})$ 41
13. The homeomorphisms group of the Cantor set 43

Date: January 2025.
Key words and phrases. Asplund function, fixed point property, $G$-compactification, locally equicontinuous, matrix coefficient, proximal flow, right topological semigroup compactification, strongly uniformly continuous, universal minimal flow.

Research partially supported by BSF (Binational USA-Israel) grant no. 2006119.
1. Introduction

In this paper we introduce the property of Strong Uniform Continuity (in short: SUC) of $G$-spaces and the associated notion of SUC functions. For every compact $G$-space $X$ the corresponding orbit maps $\tilde{x}: G \to X, g \mapsto gx$ are right uniformly continuous for every $x \in X$. If all the maps $\{\tilde{x}\}_{x \in X}$ are also left uniformly continuous then we say that $X$ is SUC. Every right uniformly continuous bounded real valued function $f: G \to \mathbb{R}$ comes from some compact $G$-space $X$. That is, there exist a compact $G$-space $X$, a continuous function $F: X \to \mathbb{R}$, and a point $x_0 \in X$ such that $f = F \circ \tilde{x}_0$. We say that $f$ is SUC if it comes from a compact $G$-space which is SUC. Denote by $\text{SUC}(G)$ the corresponding class of functions on $G$. The class $\text{SUC}(G)$ forms a uniformly closed $G$-invariant subalgebra of the algebra $\text{UC}(G) := \text{RUC}(G) \cap \text{LUC}(G)$ of (right and left) uniformly continuous functions. Of course we have $\text{SUC}(G) = \text{UC}(G) = \text{RUC}(G) = \text{LUC}(G)$ when $G$ is either discrete or abelian so that the notion of strong uniform continuity can be useful only when one deals with non-abelian non-discrete topological groups. Mostly we will be interested in Polish non-locally compact large groups, but some of the questions we study are of interest in the locally compact case as well.

In our recent work [18] we investigated, among other topics, the algebras of locally equicontinuous $\text{LE}(G)$, and Asplund functions $\text{Asp}(G)$ on a topological group $G$. The inclusions $\text{UC}(G) \supset \text{SUC}(G) \supset \text{LE}(G) \supset \text{Asp}(G) \supset \text{WAP}(G)$ hold for an arbitrary topological group $G$. In the present article we provide a characterization of the elements of $\text{SUC}(G)$ and $\text{LE}(G)$ in terms of matrix coefficients for appropriate Banach representations of $G$ by linear isometries.

Intuitively the dynamical complexity of a function $f \in \text{RUC}(G)$ can be estimated by the topological complexity of the cyclic $G$-flow $X_f$ (the pointwise closure of the left $G$-orbit $\{gf\}_{g \in G}$ of $f$) treating it as a subset of the Banach space $\text{RUC}(G)$. This leads to a natural dynamical hierarchy (see Theorem 7.12) where $\text{SUC}(G)$ plays a basic role. In some sense $\text{SUC}(G)$ is the largest “nice subalgebra” of $\text{UC}(G)$. It turns out that $f \in \text{SUC}(G)$ iff $X_f$ is a subset of $\text{UC}(G)$. Moreover the algebra $\text{SUC}(G)$ is point-universal in the sense of [18] and every other point-universal subalgebra of $\text{UC}(G)$ is contained in $\text{SUC}(G)$. Recall that a $G$-algebra $A \subset \text{RUC}(G)$ is point-universal if and only if the associated $G$-compactification $G \to G^A$ is a right topological semigroup compactification.

As an application we conclude that the algebra $\text{UC}(G)$ is point-universal if and only if it coincides with the algebra $\text{SUC}(G)$ and that the corresponding Roelcke compactification $G \to G^{\text{UC}}$ is in general not a right topological semigroup compactification.
compactification of $G$ (in contrast to the compactification $G \to G^{\text{SUC}}$ determined by the algebra $\text{SUC}(G)$).

For locally compact groups, $\text{SUC}(G)$ contains the subalgebra $C_0(G)$ consisting of the functions which vanish at infinity, and therefore determines the topology of $G$. The structure of $\text{SUC}(G)$ — in contrast to $\text{RUC}(G)$ which is always huge for non-precompact groups — is “computable” for several large groups like: $H_+[0,1]$, $\text{Iso} \,(U_1)$ (the isometry group of the Urysohn space of diameter one $U_1$), $U(H)$ (the unitary group on an infinite dimensional Hilbert space), $S_\infty = S(\mathbb{N})$ (the Polish infinite symmetric group) and any noncompact connected simple Lie group with finite center (e.g., $\text{SL}_n(\mathbb{R})$). For instance, $\text{SUC}(G) = \text{WAP}(G)$ for $U(H)$, $S_\infty$ and $\text{SL}_n(\mathbb{R})$. In the first case we use a result of Uspenskij [58] which identifies the Roelcke completion of $U(H)$ as the compact semigroup of contracting operators on the Hilbert space $H$. For $S_\infty$ see Section 12, and for $\text{SL}_n(\mathbb{R})$ this follows from an old result of Veech [61].

The group $H_+[0,1]$ of orientation preserving homeomorphisms of the closed unit interval, endowed with the compact open topology is a good test case in the class of “large” yet “computable” topological groups. See Section 8 for more details on this group. In particular recall the result from [39] which shows that $H_+[0,1]$ is WAP-trivial: Every weakly almost periodic function on $G := H_+[0,1]$ is a constant. Equivalently, $G$ is reflexively trivial, that is, every continuous representation $G \to \text{Iso} \,(V)$ where $V$ is a reflexive Banach space is trivial.

Here we show that $G$ is even “SUC-trivial” — that is, the algebra $\text{SUC}(G)$ (and hence, also the algebras $\text{LE}(G)$ and $\text{Asp}(G)$) consists only of constant functions — and that every continuous representation of $G$ into the group of linear isometries $\text{Iso} \,(V)$ of an Asplund Banach space $V$ is trivial. Since in general $\text{WAP}(G) \subset \text{Asp}(G)$ and since every reflexive Banach space is Asplund these results strengthen the main results of [39]. SUC-triviality implies that every adjoint continuous (see Section 6) representation is trivial for $H_+[0,1]$. The latter fact follows also from a recent unpublished result of Uspenskij (private communication).

From the WAP-triviality (equivalently, reflexive triviality) of $H_+[0,1]$ and results of Uspenskij about $\text{Iso} \,(U_1)$, Pestov deduces in [51, Corollary 1.4] the fact that the group $\text{Iso} \,(U_1)$ is also WAP-trivial. Using a similar idea and the matrix coefficient characterization of $\text{SUC}$ one can conclude that $\text{Iso}(U_1)$ is $\text{SUC}$-trivial. It is an open question whether the group $H([0,1]^{\omega})$ is $\text{SUC}$-trivial (or, WAP-trivial).

The above mentioned description of $\text{SUC}(G)$ and $\text{LE}(G)$ in terms of matrix coefficients (Section 9) is nontrivial. The proof is based on a dynamical modification of a well known interpolation technique of Davis, Figiel, Johnson and Pelczyński [10].
In the last two sections we study the Roelcke and SUC compactifications of the groups \( S_\infty \) and \( H(C) \). For the first group we show that \( \text{WAP}(S_\infty) = \text{SUC}(S_\infty) = \text{UC}(S_\infty) \) and also provide a concrete description of the corresponding metrizable (in fact Cantor) semitopological semigroup compactification. For the latter group \( G := H(C) \), in contrast, we have \( \text{SUC}(G) \subseteq \text{UC}(G) \) from which fact we deduce that the corresponding Roelcke compactification \( G \to G^{\text{UC}} \) is not a right topological semigroup compactification of \( G \).

Finally let us note that although in this work we consider, for convenience, algebras of real-valued functions, it seems that there should be no difficulty in extending our definitions and results to the complex case.

In section 11 we introduce a notion of (amenability) extreme amenability with respect to a class of flows. In particular we examine extreme SUC-amenability and extreme SUC-amenable groups. Namely those groups which have a fixed point property on compact SUC \( G \)-spaces. Several natural groups, like \( \text{SL}_2(\mathbb{R}), S_\infty, H(C) \) (the homeomorphisms group of the Cantor set), and \( H_+(\mathbb{T}) \), which fail to be extremely amenable, are however extremely SUC-amenable.

By Theorem 14.3 if \( G \) is a Polish topological group such that the universal minimal \( G \)-flow \( M(G) \) is metrizable and proximal, then \( M(G) \) is SUC-trivial (hence, \( G \) is extremely SUC-amenable). The same result can be derived by using Theorem 14.8. Furthermore, Theorem 14.3 leads to Proposition 14.10 which asserts that every Polish \( G \) with metrizable \( M(G) \) is SUC-amenable.

2. Actions and \( G \)-compactifications

Unless explicitly stated otherwise, all spaces in this paper are at least Tychonoff. A (left) action of a topological group \( G \) on a topological space \( X \) is defined by a function \( \pi: G \times X \to X, \pi(g, x) := gx \) such that always \( g_1(g_2x) = (g_1g_2)x \) and \( ex = x \) hold, where \( e = e_G \) is the neutral element of \( G \). Every \( x \in X \) defines an orbit map \( \hat{x}: G \to X, \ g \mapsto gx \). Also every \( g \in G \) induces a \( g \)-translation \( \pi^g: X \to X, \ x \mapsto gx \). If the action \( \pi \) is continuous then we say that \( X \) is a \( G \)-space (or a \( G \)-system or a \( G \)-flow). Sometimes we denote it as a pair \((G, X)\). If the orbit \( Gx_0 \) of \( x_0 \) is dense in \( X \) for some \( x_0 \in X \) then the \( G \)-space \( X \) is point transitive (or just transitive) and the point \( x_0 \) is a transitive point. If \( X \) in addition is compact then the pair \((X, x_0)\) is said to be a pointed system or a \( G \)-ambit. If every point \( x \) in a compact \( G \)-space \( X \) is transitive then \( X \) is said to be minimal.

Let \( G \) act on \( X_1 \) and on \( X_2 \). A continuous map \( f: X_1 \to X_2 \) is a \( G \)-map (or a homomorphism of dynamical systems) if \( f(gx) = gf(x) \) for every \((g, x) \in G \times X_1\).

A right action \( X \times G \to X \) can be defined analogously. If \( G^{op} \) is the opposite group of \( G \) with the same topology then the right \( G \)-space \((X, G)\) can be
treated as a left $G^{\text{op}}$-space $(G^{\text{op}}, X)$ (and vice versa). A map $h : G_1 \to G_2$ between two groups is a co-homomorphism (or, an anti-homomorphism) if $h(g_1 g_2) = h(g_2) h(g_1)$. This happens iff $h : G^{\text{op}}_1 \to G_2$ (the same assignment) is a homomorphism.

The Banach algebra (under the supremum norm) of all continuous real valued bounded functions on a topological space $X$ will be denoted by $C(X)$. Let $(G, X)$ be a left (not necessarily compact) $G$-space. Then it induces the right action $C(X) \times G \to C(X)$, with $(fg)(x) = f(gx)$, and the corresponding co-homomorphism $h : G \to \text{Iso}(C(X))$. While the $g$-translations $C(X) \to C(X)$ (being isometric) are continuous, the orbit maps $\tilde{f} : G \to C(X)$, $g \mapsto fg$ are not necessarily continuous. The function $f \in C(X)$ is right uniformly continuous if the orbit map $G \to C(X)$, $g \mapsto fg$ is norm continuous. The set $\text{RUC}(X)$ of all right uniformly continuous functions on $X$ is a uniformly closed $G$-invariant subalgebra of $C(X)$. Here and in the sequel “subalgebra” means a uniformly closed unital (containing the constants) subalgebra. A “$G$-subalgebra” is an algebra which is invariant under the natural right action of $G$.

Every topological group $G$ can be treated as a $G$-space under the left regular action of $G$ on itself. In this particular case $f \in \text{RUC}(G)$ iff $f$ is uniformly continuous with respect to the right uniform structure $\mathcal{R}$ on $G$ (furthermore this is also true for coset $G$-spaces $G/H$).

Thus, $f \in \text{RUC}(G)$ iff for every $\varepsilon > 0$ there exists a neighborhood $V$ of the identity element $e \in G$ such that $\sup_{g \in G} |f(vg) - f(g)| < \varepsilon$ for every $v \in V$.

Analogously one defines right translations $(gf)(x) := f(xg)$, and the algebra $\text{LUC}(G)$ of left uniformly continuous functions. These are the functions which are uniformly continuous with respect to the left uniform structure $\mathcal{L}$ on $G$.

A $G$-compactification of a $G$-space $X$ is a $G$-map $\nu : X \to Y$ into a compact $G$-space $Y$ with $\text{cl} \nu(X) = Y$. A compactification is proper when $\nu$ is a topological embedding. Given a compact $G$-space $X$ and a point $x_0 \in X$ the map $\nu : G \to X$ defined by $\nu(g) = gx_0$ is a compactification of the $G$-space $G$ (the left regular action) in the orbit closure $\text{cl} G x_0 \subset X$.

We say that a $G$-compactification $\nu : G \to S$ of $X := G$ is a right topological semigroup compactification of $G$ if $S$ is a right topological semigroup (that is, $S$ is a compact semigroup such that for every $p \in S$ the map $S \to S, s \mapsto sp$ is continuous) and $\nu$ is a homomorphism of semigroups.

There exists a canonical 1-1 correspondence between the $G$-compactifications of $X$ and $G$-subalgebras of $\text{RUC}(X)$ (see for example [62]). The compactification $\nu : X \to Y$ induces an isometric $G$-embedding of $G$-algebras $j_{\nu} : C(Y) = \text{RUC}(Y) \hookrightarrow \text{RUC}(X)$, $\phi \mapsto \phi \circ \nu$ and the algebra $A_{\nu}$ is defined as the image $j_{\nu}(C(Y))$. Conversely, if $A$ is a $G$-subalgebra of $\text{RUC}(X)$, then denote by $X^A$ or by $|A|$ the corresponding Gelfand space treating it as a weak star compact subset
of the dual space $A^\ast$. It has a structure of a $G$-space $(G, |A|)$ and the natural map $\nu_A: X \to X^A$, $x \mapsto ev_a$, where $ev_a(\varphi) := \varphi(x)$, is the evaluation at $x$ (a multiplicative functional), defines a $G$-compactification. If $\nu_1: X \to X^A_1$ and $\nu_2: X \to X^A_2$ are two $G$-compactifications then $A_{\nu_1} \subset A_{\nu_2}$ iff $\nu_1 = \alpha \circ \nu_2$ for some $G$-map $\alpha: X^{A_2} \to X^{A_1}$. The algebra $A$ determines the compactification $\nu_A$ uniquely, up to the equivalence of $G$-compactifications. The $G$-algebra $\text{RUC}(X)$ defines the corresponding Gelfand space $|\text{RUC}(X)|$, which we denote by $\beta_G X$, and the maximal $G$-compactification $i_{\beta}: X \to \beta_G X$. Note that this map may not be an embedding even for Polish $G$ and $X$ (see [35]); it follows that there is no proper $G$-compactification for such $X$. If $X$ is a compact $G$-space then $\beta_G X$ can be identified with $X$ and $C(X) = \text{RUC}(X)$.

Denote by $G^{\text{RUC}}$ the Gelfand space of the $G$-algebra $\text{RUC}(G)$. The canonical embedding $u: G \to G^{\text{RUC}}$ defines the greatest ambit $(G^{\text{RUC}}, u(\epsilon))$ of $G$.

It is easy to see that the intersection $\text{UC}(G) := \text{RUC}(G) \cap \text{LUC}(G)$ is a left and right $G$-invariant closed subalgebra of $\text{RUC}(G)$. We denote the corresponding compactification by $G^{\text{UC}}$. Denote by $\mathcal{L} \wedge \mathbb{R}$ the lower uniformity of $G$. It is the infimum (greatest lower bound) of the left and right uniformities on $G$; we call it the Roelcke uniformity. Clearly, for every bounded function $f: G \to \mathbb{R}$ we have $f \in \text{UC}(G)$ iff $f: (G, \mathcal{L} \wedge \mathbb{R}) \to \mathbb{R}$ is uniformly continuous. Recall the following important fact (in general the infimum $\mu_1 \wedge \mu_2$ of two compatible uniform structures on a topological space $X$ need not be compatible with the topology of $X$).

**Lemma 2.1.**

1. (Roelcke-Dierolf [53]) For every topological group $G$ the Roelcke uniform structure $\mathcal{L} \wedge \mathbb{R}$ generates the given topology of $G$.

2. For every topological group $G$ the algebra $\text{UC}(G)$ separates points from closed subsets in $G$.

**Proof.** (1) See Roelcke-Dierolf [53, Proposition 2.5].

(2) Follows from (1). \qed

By a uniform $G$-space $(X, \mu)$ we mean a $G$-space $(X, \tau)$ where $\tau$ is the (completely regular) topology defined by the uniform structure $\mu$ and the $g$-translations $(g \in G)$ are uniform isomorphisms.

Let $X := (X, \mu)$ be a uniform $G$-space. A point $x_0 \in X$ is a point of equicontinuity (notation: $x_0 \in Eq_X$) if for every entourage $\varepsilon \in \mu$, there is a neighborhood $O$ of $x_0$ such that $(gx_0, gx) \in \varepsilon$ for every $x \in O$ and $g \in G$. The $G$-space $X$ is equicontinuous if $Eq_X = X$. $(X, \mu)$ is uniformly equicontinuous if for every $\varepsilon \in \mu$ there is $\delta \in \mu$ such that $(gx, gy) \in \varepsilon$ for every $g \in G$ and $(x, y) \in \delta$. For compact $X$ (equipped with the unique compatible uniformity), equicontinuity and uniform equicontinuity coincide. Compact (uniformly) equicontinuous $G$-space $X$ is also
said to be Almost Periodic (in short: AP); see also Section 7. If Eq\_X is dense in X then (X, \(\mu\)) is said to be an almost equicontinuous (AE) G-space [2].

The following definition is standard (for more details see for example [18]).

**Definition 2.2.**

1. A function \(f \in C(X)\) on a G-space X comes from a compact G-system Y if there exist a G-compactification \(\nu : X \rightarrow Y\) (so, \(\nu\) is onto if X is compact) and a function \(F \in C(Y)\) such that \(f = F \circ \nu\) (i.e., if \(f \in A_\nu\)). Then necessarily, \(f \in \text{RUC}(X)\).

\[
\begin{array}{ccc}
X & \xrightarrow{\nu} & Y \\
\downarrow{f} & & \downarrow{F} \\
\downarrow{} & & \downarrow{} \\
\mathbb{R} & & \mathbb{R}
\end{array}
\]

2. A function \(f \in \text{RUC}(G)\) comes from a pointed system \((Y, y_0)\) if for some continuous function \(F \in C(Y)\) we have \(f(g) = F(gy_0), \ \forall g \in G\). Notation: \(f \in A(Y, y_0)\). Defining \(\nu : X = G \rightarrow Y\) by \(\nu(g) = g y_0\) observe that this is indeed a particular case of 2.2.1.

3. Let \(\Gamma\) be a class of compact G-spaces. For a G-space X denote by \(\Gamma(X)\) the class of all functions on X which come from a G-compactification \(\nu : X \rightarrow Y\) where the G-system Y belongs to \(\Gamma\).

Let \(P\) be a class of compact G-spaces which is preserved by G-isomorphisms, products and closed G-subspaces. It is well known (see for example, [18, Proposition 2.9]) that for every G-space X there exists a universal (maximal) G-compactification \(X \rightarrow X^P\) such that \(X^P\) lies in P. More precisely, for every (not necessarily compact) G-space X denote by \(\mathcal{P} \subset C(X)\) the collection of functions coming from G-spaces having property \(P\). Then \(\mathcal{P}\) is a uniformly closed, G-invariant subalgebra of \(\text{RUC}(X)\) and the maximal G-compactification of X with property \(P\) is the corresponding Gelfand space \(X^P := |\mathcal{P}|\). If X is compact then \((G, X^P)\) is the maximum factor of \((G, X)\) with property \(P\). In particular let \(P\) be one of the following natural classes of compact G-spaces: a) Almost Periodic (=equicontinuous); b) Weakly Almost Periodic; c) Hereditarily Not Sensitive; d) Locally Equicontinuous; e) all compact G-spaces. Then, in this way the following maximal (in the corresponding class) G-compactifications are: a) \(G^{\text{AP}}\); b) \(G^{\text{WAP}}\); c) \(G^{\text{Asp}}\); d) \(G^{\text{LE}}\); e) \(G^{\text{RUC}}\).

For undefined concepts and more details see Section 7 and also [18].

3. **Cyclic G-systems and point-universality**

Here we give some background material about cyclic compact G-systems \(X_f\) defined for \(f \in \text{RUC}(X)\). These G-spaces play a significant role in many aspects
of topological dynamics and are well known at least for the particular case of $X := G$. We mostly use the presentation and results of [18] (see also [63]).

As a first motivation note a simple fact about Definition 2.2. For every $G$-space $X$ a function $f: X \to \mathbb{R}$ lies in $\text{RUC}(X)$ iff it comes from a compact $G$-flow $Y$.

We can choose $Y$ via the maximal $G$-compactification $\nu: X \to \beta X$. This is the largest possibility in this setting. Among all possible $G$-compactifications $\nu: X \to Y$ of $X$ there exists also the smallest one. Take simply the smallest $G$-subalgebra $A_f$ of $\text{RUC}(X)$ generated by the orbit $fG$ of $f$ in $\text{RUC}(X)$. Denote by $X_f$ the Gelfand space $|A_f| = X_f$ of the algebra $A_f$. This is the smallest $G$-subalgebra $A_f$ of $\text{RUC}(X)$ generated by the orbit $fG$ of $f$ in $\text{RUC}(X)$. Denote by $X_f$ the Gelfand space $|A_f| = X_f$ of the algebra $A_f$. Then the corresponding $G$-compactification $X \to Y := X_f$ is the desired one. We call $A_f$ and $X_f$ the cyclic $G$-algebra and cyclic $G$-system of $f$, respectively.

Next we provide an alternative construction and some basic properties of $X_f$.

Let $X$ be a (not necessarily compact) $G$-space. Given $f \in \text{RUC}(X)$ let $I = [-\|f\|,\|f\|] \subset \mathbb{R}$ and $\Omega = I^G$, the product space equipped with the compact product topology. We let $G$ act on $\Omega$ by $g\omega(h) = \omega(hg)$, $g, h \in G$. Define the continuous map

$$f_\sharp: X \to \Omega, \quad f_\sharp(x)(g) = f(gx)$$

and the closure $X_f := \text{cl}(f_\sharp(X))$ in $\Omega$. Note that $X_f = f_\sharp(X)$ whenever $X$ is compact.

Denoting the unique continuous extension of $f$ to $\beta X$ by $\tilde{f}$ (it exists because $f \in \text{RUC}(X)$) we now define a map

$$\psi: \beta X \to X_f \quad \text{by} \quad \psi(y)(g) = \tilde{f}(gy), \quad y \in \beta X, \ g \in G.$$ 

Let $pr_e: \Omega \to \mathbb{R}$ denote the projection of $\Omega = I^G$ onto the $e$-coordinate and let $F_e := pr_e \mid X_f: X_f \to \mathbb{R}$ be its restriction to $X_f$. Thus, $F_e(\omega) := \omega(e)$ for every $\omega \in X_f$.

As before denote by $A_f$ the smallest (closed and unital, of course) $G$-invariant subalgebra of $\text{RUC}(X)$ which contains $f$. There is then a naturally defined $G$-action on the Gelfand space $X^{A_f} = |A_f|$ and a $G$-compactification (morphism of dynamical systems if $X$ is compact) $\pi_f: X \to |A_f|$. Next consider the map $\pi: \beta X \to |A_f|$, the canonical extension of $\pi_f$ induced by the inclusion $A_f \subset \text{RUC}(X)$.

The action of $G$ on $\Omega$ is not in general continuous. However, the restricted action on $X_f$ is continuous for every $f \in \text{RUC}(X)$. This follows from the second assertion of the next fact.

**Proposition 3.1.** (See for example [18])

1. Each $\omega \in X_f$ is an element of $\text{RUC}(G)$. That is, $X_f \subset \text{RUC}(G)$. 

(2) The map $\psi: \beta_G X \to X_f$ is a continuous homomorphism of $G$-systems. The dynamical system $(G, |A_f|)$ is isomorphic to $(G, X_f)$ and the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i_g} & \beta_G X \\
\downarrow{\pi} & & \downarrow{f} \\
|A_f| & \xleftarrow{\psi} & X_f \\
\end{array}
$$

commutes.

(3) $f = F \circ f_\beta$. Thus every $f \in \text{RUC}(X)$ comes from the system $X_f$. Moreover, if $f$ comes from a system $Y$ and a $G$-compactification $\nu: X \to Y$ then there exists a homomorphism $\alpha: Y \to X_f$ such that $f_\nu = \alpha \circ \nu$. In particular, $f \in A_f \subset A_\nu$.

If $X := G$ with the usual left action then $X_f$ is the pointwise closure of the $G$-orbit $Gf := \{gf\}_{g \in G}$ of $f$ in $\text{RUC}(G)$. Hence $(X_f, f)$ is a transitive pointed $G$-system.

As expected by the construction the cyclic $G$-systems $X_f$ provide “building blocks” for compact $G$-spaces. That is, every compact $G$-space can be embedded into the $G$-product of $G$-spaces $X_f$.

Let us say that a topological group $G$ is uniformly Lindel"of if for every nonempty open subset $O \subset G$ countably many translates $g_nO$ cover $G$ (there are several alternative names for this notion: $\omega$-bounded, $\omega$-bounded, $\omega$-narrow, $\omega$-precompact). It is well known that $G$ is uniformly Lindel"of iff $G$ is a topological subgroup in a product of second countable groups. When $G$ is uniformly Lindel"of (e.g. when $G$ is second countable) the compactum $X_f$ is metrizable.

The question “when is $X_f$ a subset of $\text{UC}(G)$?” provides another motivation for introducing the notion of strongly uniformly continuous (SUC) functions (see Definition 4.2 and Theorem 4.12).

The enveloping (or Ellis) semigroup $E = E(X)$ of a compact $G$-space $X$ is defined as the closure in $X^X$ (with its compact pointwise convergence topology) of the set $\{\pi^g: X \to X\}_{g \in G}$ of translations considered as a subset of $X^X$. With the operation of composition of maps this is a right topological semigroup. Moreover, the map

$$j = j_X: G \to E(X), \ g \mapsto \pi^g$$

is a right topological semigroup compactification of $G$. The compact space $E(X)$ becomes a $G$-space with respect to the natural action

$$G \times E(X) \to E(X), \ (gp)(x) = gp(x).$$

Moreover the pointed $G$-system $(E(X), j(e))$ is point-universal in the following sense.
Definition 3.2. ([18]) A pointed $G$-system $(X,x_0)$ is point-universal if it has the property that for every $x \in X$ there is a homomorphism $\pi_x: (X,x_0) \to (\text{cl}(Gx),x)$. The $G$-subalgebra $\mathcal{A} \subset \text{RUC}(G)$ is said to be point-universal if the corresponding $G$-ambit $(G^\mathcal{A}, u_\mathcal{A}(e))$ is point-universal.

We will use the following characterization of point-universality from [18].

Lemma 3.3. Let $(X,x_0)$ be a transitive compact $G$-system. The following conditions are equivalent:

1. The system $(X,x_0)$ is point-universal.
2. The orbit map $G \to X, g \mapsto gx_0$ is a right topological semigroup compactification of $G$.
3. $(X,x_0)$ is $G$-isomorphic to its enveloping semigroup $(E(X), j(e))$.
4. $\mathcal{A}(X,x_0) = \bigcup_{x \in X} \mathcal{A}(\text{cl}(Gx),x)$.
5. $X_f \subset \mathcal{A}(X,x_0)$ for every $f \in \mathcal{A}(X,x_0)$ (where $\mathcal{A}(X,x_0)$ is the corresponding subalgebra of $\text{RUC}(G)$ coming from the $G$-compactification $\nu: G \to X, \nu(g) = gx_0$).

In particular, for every right topological semigroup compactification $\nu: G \to S$ the pointed $G$-space $(S, \nu(e))$ is point-universal. For other properties of point-universality see [18] and Remark 9.4.

4. Strong Uniform Continuity

Let $G$ be a topological group. As before denote by $\mathcal{L}$ and $\mathcal{R}$ the left and right uniformities on $G$. We start with a simple observation.

Lemma 4.1. For every compact $G$-space $X$ the corresponding orbit maps $\bar{x}: (G, \mathcal{R}) \to (X, \mu_X), \ g \mapsto gx$ are uniformly continuous for every $x \in X$.

Proof. Let $V$ be an open neighborhood of the diagonal $\Delta \subset X \times X$. In order to obtain a contradiction suppose that for every neighborhood $U$ of $e \in G$ there is $u_U \in U$ and $x_U \in X$ such that $(x_U, u_Ux_U) \notin V$. For a convergent subnet we have $\lim(x_U, u_Ux_U) = (x, x') \notin V$ contradicting the joint continuity of the $G$-action. $\square$

In general, for non-commutative groups, one cannot replace $\mathcal{R}$ by the left uniformity $\mathcal{L}$ (see Remark 4.4). This leads us to the following definition.

Definition 4.2. Let $G$ be a topological group.

1. We say that a uniform $G$-space $(X, \mu)$ is strongly uniformly continuous at $x_0 \in X$ (notation: $x_0 \in \text{SUC}_X$) if the orbit map $\bar{x}_0: G \to X, \ g \mapsto gx_0$
is $(\mathcal{L}, \mu)$-uniformly continuous. Precisely, this means that for every $\varepsilon \in \mu$ there exists a neighborhood $U$ of $e \in G$ such that

$$(gux_0, gx_0) \in \varepsilon$$

for every $g \in G$ and every $u \in U$. If $\text{SUC}_X = X$ we say that $X$ is strongly uniformly continuous.

(2) If $X$ is a compact $G$-space then there exists a unique compatible uniformity $\mu_X$ on $X$. So, $\text{SUC}_X$ is well defined. By Lemma 4.1 it follows that a compact $G$-space $X$ is SUC at $x_0$ iff $\bar{X}_0 : G \to X$ is $(\mathcal{L} \wedge \mathcal{R}, \mu_X)$-uniformly continuous. We let $\text{SUC}$ denote the class of all compact $G$-systems such that $(X, \mu_X)$ is SUC.

(3) A function $f \in C(X)$ is strongly uniformly continuous (notation: $f \in \text{SUC}(X)$) if it comes from a SUC compact dynamical system.

(4) Let $x_1$ and $x_2$ be points of a $G$-space $X$. Write $x_1 \sim \text{SUC} x_2$ if these points cannot be separated by a SUC function on $X$. Equivalently, this means that these points have the same images under the universal SUC compactification $G$-map $X \to X^{\text{SUC}}$ (see Lemma 4.6.1 below).

**Lemma 4.3.** $\text{SUC}(G) \subset \text{UC}(G)$.

**Proof.** Let $f : G \to \mathbb{R}$ belong to $\text{SUC}(G)$. Then it comes from a function $F : X \to \mathbb{R}$, where $\nu : G \to X$ is a $G$-compactification of $G$ such that $f(g) = F(\nu(g))$, and $F \in \text{SUC}(X)$. Clearly, $F$ is uniformly continuous because $X$ is compact. Then $f \in \text{RUC}(G)$ by Lemma 4.1. In order to see that $f \in \text{LUC}(G)$, choose $x_0 := \nu(e) \in X$ in the definition of SUC. \qed

**Remark 4.4.** Recall that if $\mathcal{L} = \mathcal{R}$ then $G$ is said to be SIN group. If $G$ is a SIN group then $X \in \text{SUC}$ for every compact $G$-space $X$. It follows that for a SIN group $G$ we have $\text{SUC}(X) = \text{RUC}(X)$ for every, not necessarily compact, $G$-space $X$ and also $\text{SUC}(G) = \text{UC}(G) = \text{RUC}(G)$. For example this holds for abelian, discrete, and compact groups.

A special case of a SUC uniform $G$-space is obtained when the uniform structure $\mu$ is defined by a $G$-invariant metric. If $\mu$ is a metrizable uniformity then $(X, \mu)$ is uniformly equicontinuous iff $\mu$ can be generated by a $G$-invariant metric on $X$. A slightly sharper property is the local version: $\text{SUC}_X \supset \text{Eq}_X$ (see Lemma 7.8).

We say that a compactification of $G$ is Roelcke if the corresponding algebra $A$ is a $G$-subalgebra of $\text{UC}(G)$, or equivalently, if there exists a natural $G$-morphism $G^{\text{UC}} \to G^A$.

**Lemma 4.5.** We collect here the following properties of $\text{SUC}$.

(1) $f(\text{SUC}_X) \subset \text{SUC}_Y$ for every uniformly continuous $G$-map $f : (X, \mu) \to (Y, \eta)$. 

(2) The class SUC is closed under products, subsystems and quotients.

(3) Let \( \alpha: G \to Y \) be a Roelcke compactification. Then \( \alpha(G) \subset \text{SUC}_Y \).

(4) Let \( X \) be a not necessarily compact \( G \)-space and \( f \in \text{SUC}(X) \). Then for every \( x_0 \in X \) and every \( \varepsilon > 0 \) there exists a neighborhood \( U \) of \( e \) such that
\[
|f(gux_0) - f(gx_0)| < \varepsilon \quad \forall \ (g, u) \in G \times U.
\]

Proof. (1) and (2) are straightforward.

(3): Follows directly from (1) because the left action of \( G \) on itself is uniformity equicontinuous with respect to the left uniformity and \( \alpha: (G, \mathcal{L}) \to (Y, \mu_Y) \) is uniformly continuous for every Roelcke compactification.

(4): There exist: a compact SUC \( G \)-space \( Y \), a continuous function \( F: Y \to \mathbb{R} \) and a \( G \)-compactification \( \nu: X \to Y \) such that \( f = F \circ \nu \). Now our assertion follows from the fact that \( \nu(x_0) \in \text{SUC}_Y \) for every \( x_0 \in X \) (taking into account that \( F \) is uniformly continuous).

By Lemmas 4.5.2, 3.3 and the standard subdirect product construction (see [18, Proposition 2.9.2]) we can derive the following facts.

Lemma 4.6.

1. The collection \( \text{SUC}(X) \) is a \( G \)-subalgebra of \( \text{RUC}(X) \) for every \( G \)-space \( X \) and the corresponding Gelfand space \( |\text{SUC}(X)| = X^{\text{SUC}} \) is the maximal \( \text{SUC} \)-compactification of \( X \).

2. A compact \( G \)-system \( X \) is SUC if and only if \( C(X) = \text{SUC}(X) \).

3. For every \( f \in \text{RUC}(X) \) we have \( f \in \text{SUC}(X) \) if and only if \( X_f \) is SUC.

4. \( \text{SUC}(G) \) is a point-universal closed \( G \)-subalgebra of \( \text{RUC}(G) \).

5. The canonical compactification \( j: G \to G^{\text{SUC}} \) is always a right topological semigroup compactification of \( G \).

We need also the following link between SUC functions and cyclic \( G \)-spaces.

Lemma 4.7. Let \( X \) be a \( G \)-space, \( f \in \text{RUC}(X) \) and \( h \in X_f \). Then the following are equivalent:

1. \( h \in \text{LUC}(G) \).

2. \( h \in \text{UC}(G) \).

3. \( h \), as a point in the \( G \)-flow \( \text{Y} := X_f \), is in \( \text{SUC}_\text{Y} \).

Proof. We know by Proposition 3.1.1 that \( X_f \subset \text{RUC}(G) \). Thus (1) \( \iff \) (2).

For (1) \( \iff \) (3) observe that
\[
|h(gu) - h(g)| < \varepsilon \quad \forall \ g \in G \quad \iff \quad |h(t_k gu) - h(t_k g)| < \varepsilon \quad \forall \ g \in G, \ k = 1, \ldots, n
\]
for arbitrary finite subset \( \{t_1, \ldots, t_n\} \) of \( G \). □
The following result shows that SUC(\(X\)) can be described by internal terms for every compact \(G\)-space \(X\).

**Proposition 4.8.** Let \(X\) be a compact \(G\)-space. The following are equivalent:

1. \(f \in \text{SUC}(X)\).
2. For every \(x_0 \in X\) and every \(\varepsilon > 0\) there exists a neighborhood \(U\) of \(e\) such that 
   \[|f(gux_0) - f(gx_0)| < \varepsilon \quad \forall (g, u) \in G \times U.\]

**Proof.** (1) \(\implies\) (2): Apply Lemma 4.5.4.

(1) \(\iff\) (2): By Lemma 4.6.3 we have to show that the cyclic \(G\)-space \(X_f\) is SUC. Fix an arbitrary element \(\omega \in X_f\). According to Lemma 4.7 it is equivalent to verify that \(\omega \in \text{LUC}(G)\). The \(G\)-compactification map \(f: X \to X_f\) is onto because \(X\) is compact. Choose \(x_0 \in X\) such that \(f(x_0) = \omega\). Then \(\omega(g) = f(gx_0)\) for every \(g \in G\). By assertion (2) for \(x_0 \in X\) and \(\varepsilon\) we can pick a neighborhood \(U\) of \(e\) such that 
   \[|f(gux_0) - f(gx_0)| \leq \varepsilon \quad \forall (g, u) \in G \times U.\]
holds. Now we can finish the proof by observing that 
   \[|f(gux_0) - f(gx_0)| = |\omega(gu) - \omega(g)| \leq \varepsilon \quad \forall (g, u) \in G \times U.\]

\[\square\]

The following result emphasizes the differences between RUC(\(G\)) and SUC(\(G\)).

**Theorem 4.9.** Let \(\alpha = \alpha_A: G \to S\) be a \(G\)-compactification of \(G\) such that the corresponding left \(G\)-invariant subalgebra \(A\) of RUC(\(G\)) is also right \(G\)-invariant. Consider the following conditions:

1. \(A \subset \text{UC}(G)\) (that is, \(\alpha: G \to S\) is a Roelcke compactification).
2. The induced right action \(S \times G \to S, (s, g) \mapsto s\alpha(g)\) is jointly continuous.
3. \(A \subset \text{SUC}(G)\).

Then

(a) always, 1 \(\iff\) 2 and 3 \(\implies\) 1.

(b) if, in addition, \(S\) is a right topological semigroup and \(\alpha: G \to S\) is a right topological semigroup compactification of \(G\) then 1 \(\iff\) 2 \(\iff\) 3.

**Proof.** (a) 1 \(\iff\) 2: By our assumption \(A\) is \(G\)-invariant with respect to left and right translations (that is, the functions \((fg)(x) := f(gx)\) and \((gf)(x) := f(xg)\) lie in \(A\) for every \(f \in A\) and \(g \in G\)). Then the corresponding (weak star compact) Gelfand space \(S := X^A \subset A^*\), admits the natural dual left and right actions (see also Definition 6.2 and Remark 6.7) \(\pi_l: G \times S \to S\) and \(\pi_r: S \times G \to S\) such that \((g_1 s)g_2 = g_1 (sg_2)\) for every \((g_1, s, g_2) \in G \times S \times G\). It is easy to see that this right action \(S \times G \to S\) is jointly continuous if and only if \(A \subset \text{LUC}(G)\). On the
other hand, since \( \alpha: G \to S \) is a \( G \)-compactification of left \( G \)-spaces we already have \( A \subset \text{RUC}(G) \).

3 \( \Rightarrow \) 1: By Lemma 4.3 we have \( \text{SUC}(G) \subset \text{UC}(G) \).

(b) We have to verify that 1 \( \Rightarrow \) 3 provided that \( \alpha: G \to S \) is a right topological semigroup compactification of \( G \). The latter condition is equivalent to the fact that the system \((G, S)\) is point universal (Lemma 3.3) and thus for every \( x_0 \in S \) there exists a homomorphism of \( G \)-ambits \( \phi: (S, \alpha(e)) \to (\text{cl}(Gx_0), x_0) \). By Lemma 4.5 we conclude that the point \( x_0 = \phi(\alpha(e)) \in S \) is a point of \( \text{SUC} \) in the \( G \)-system \( \text{cl}(Gx_0) \) (and, hence, in \( S \)). Since \( x_0 \) is an arbitrary point in \( S \) we get that \( \text{SUC}_S = S \) and hence \( S \) is an \( \text{SUC} \) system. Since every function \( f \in A \) on \( G \) comes from the compactification \( \alpha: G \to S \) we conclude that \( A \subset \text{SUC}(G) \).

\[ \square \]

**Corollary 4.10.** The \( G \)-compactification \( j: G \to G^{\text{SUC}} \) is a right topological semigroup compactification of \( G \) such that the right action \( G^{\text{SUC}} \times G \to G^{\text{SUC}} \) is also jointly continuous.

Proof. Apply Proposition 4.6.5 and Theorem 4.9. \[ \square \]

**Corollary 4.11.** The following conditions are equivalent:

1. \( i: G \to G^{\text{UC}} \) is a right topological semigroup compactification.
2. \((G^{\text{UC}}, i(e))\) is a point universal \( G \)-system.
3. \( G^{\text{UC}} \) is \( \text{SUC} \).
4. \( \text{SUC}(G) = \text{UC}(G) \).

Proof. Apply assertion (b) of Theorem 4.9 to \( A = \text{UC}(G) \) taking into account Lemmas 3.3 and 4.3. \[ \square \]

Particularly interesting examples of groups \( G \) with \( \text{SUC}(G) = \text{UC}(G) \) are the Polish groups \( U(H) \) of all unitary operators (Example 7.13), and the group \( S_\infty(\mathbb{N}) \) (Theorem 12.2). In both cases we actually have \( \text{SUC}(G) = \text{UC}(G) = \text{WAP}(G) \). Note that these groups are not SIN (compare Remark 4.4).

For the next result see also Veech [61, Section 5].

**Theorem 4.12.** Let \( f \in \text{RUC}(X) \). The following conditions are equivalent:

1. \( X_f \subset \text{UC}(G) \).
2. \( (G, X_f) \) is \( \text{SUC} \).
3. \( f \in \text{SUC}(X) \).

Proof. 1 \( \Rightarrow \) 2: Let \( h \in X_f \). By our assumption we have \( h \in \text{UC}(G) \subset \text{LUC}(G) \). Then by Lemma 4.7, \( h \), as a point in the \( G \)-flow \( Y := X_f \), is in \( \text{SUC}_Y \). So \( \text{SUC}_Y = Y \). This means that \( (G, X_f) \) is \( \text{SUC} \).

2 \( \Rightarrow \) 3: Let \( (G, X_f) \) be \( \text{SUC} \). By Proposition 3.1.3, the function \( f: X \to \mathbb{R} \) comes from the \( G \)-compactification \( f_x: X \to X_f \). By Definition 4.2.3 this means that \( f \in \text{SUC}(X) \).
3 ⇒ 1: Let \( f \in \text{SUC}(X) \). Then Lemma 4.6.3 says that \( X_f \) is SUC. By Lemma 4.7 we have \( X_f \subset \text{UC}(G) \). □

5. SUC, HOMOGENEITY AND THE EPI-MORPHISM PROBLEM

We say that a \( G \)-space \( X \) is a \textit{coset} \( G \)-\textit{space} if it is \( G \)-isomorphic to the usual coset \( G \)-space \( G/H \) where \( H \) is a closed subgroup of \( G \) and \( G/H \) is equipped with the quotient topology. We say that a \( G \)-space \( X \) is \textit{homogeneous} if for every \( x, y \in X \) there exists \( g \in G \) such that \( gx = y \). A homogeneous \( G \)-space \( X \) is a coset \( G \)-space if and only if the orbit map \( \tilde{x} : G \to X \) is open for some (equivalently, every) \( x \in X \). Furthermore, \( \tilde{x} : G \to X \) is open iff it is a quotient map. Recall that by a well known result of Effros every homogeneous \( G \)-space with Polish \( G \) and \( X \) is necessarily a coset \( G \)-space.

**Proposition 5.1.** Let \( X = G/H \) be a compact coset \( G \)-space.

(1) If \( X \) is SUC then \( X \) is equicontinuous (that is, almost periodic).

(2) \( \text{SUC}(X) = \text{AP}(X) \).

**Proof.** (1): Indeed let \( x_0H \in G/H \) and let \( \varepsilon \) be an element of the uniform structure on the compact space \( X \). By Definition 4.2 we can choose a neighborhood \( U \) of \( e \) such that

\[
(gux_0H, gx_0H) \in \varepsilon \quad \forall (g, u) \in G \times U.
\]

By the definition of coset space topology the set \( O := Ux_0H \) is a neighborhood of the point \( x_0H \) in \( G/H \). We obtain that \( (gxH, gx_0H) \in \varepsilon \) whenever \( xH \in O \). This proves that \( x_0H \) is a point of equicontinuity of \( X = G/H \). Hence \( X \) is AP.

(2): Every equicontinuous compact \( G \)-space is clearly SUC. This implies that always, \( \text{SUC}(X) \supset \text{AP}(X) \). Conversely, let \( f \in \text{SUC}(X) \). This means that \( f = \alpha \circ F \) for a \( G \)-compactification \( \alpha : X \to Y \) where \( Y \) is SUC and \( F \in C(Y) = \text{SUC}(Y) \). We can suppose that \( \alpha \) is onto because \( X \) is compact. Then \( \alpha \) is a quotient map. On the other hand \( X \) is a coset space \( G/H \). It follows that the natural onto map \( G \to Y \) is also a quotient map. Therefore, \( Y \) is also a coset space of \( G \). Now we can apply (1). It follows that \( Y \) is almost periodic. Hence \( f \) comes from an AP \( G \)-factor \( Y \) of \( X \). Thus, \( f \in \text{AP}(X) \). □

Next we discuss a somewhat unexpected connection between SUC, free topological \( G \)-groups and an epimorphism problem. Uspenskij has shown in [56] that in the category of Hausdorff topological groups epimorphisms need not have a dense range. This answers a longstanding problem by K. Hofmann. Pestov established [48, 50] that the question completely depends on the free topological \( G \)-groups \( F_G(X) \) of a \( G \)-space \( X \) in the sense of Megrelishvili [36]. More precisely, the inclusion \( i : H \hookrightarrow G \) of topological groups is epimorphism iff the free
topological $G$-group $F_G(X)$ of the coset $G$-space $X := G/H$ is trivial. Triviality means, ‘as trivial as possible’, isomorphic to the cyclic discrete group.

For a $G$-space $X$ and points $x_1, x_2 \in X$ we write $x_1 \overset{\text{Aut}}{\sim} x_2$ if these two points have the same image under the canonical $G$-map $X \to F_G(X)$. If $d$ is a bounded compatible $G$-invariant metric on a $G$-space $X$ then $(X, d)$ is isometrically $G$-linearizable using Arens-Eells embedding. Therefore, in this case $x_1 \overset{\text{Aut}}{\sim} x_2$ iff $x_1 = x_2$.

**Theorem 5.2.** Let $H$ be a closed subgroup of $G$.

1. If $x_1 \overset{\text{Aut}}{\sim} x_2$ for $x_1, x_2$ in the $G$-space $X := G/H$ then $x_1 \overset{\text{SUC}}{\sim} x_2$.

2. If the inclusion $H \hookrightarrow G$ is an epimorphism then the coset $G$-space $G/H$ is SUC-trivial.

**Proof.** (1) Assuming the contrary let $f : G/H \to \mathbb{R}$ be SUC function which separates the points $x_1 := a_1 H$ and $x_2 := a_2 H$. Then the bounded $G$-invariant pseudometric $\rho_f$ on $G/H$ defined by $\rho_f(xH, yH) := \sup_{g \in G} |f(gxH) - f(gyH)|$ also separates these points. We show that $\rho_f$ is continuous. Indeed let $\varepsilon > 0$ and $x_0 H \in G/H$. By virtue of Lemma 4.5.4 we can choose a neighborhood $U$ of $e$ such that $|f(gux_0H) - f(gx_0H)| < \varepsilon \quad \forall (g, u) \in G \times U$.

By the definition of coset space topology the set $O := Ux_0H$ is a neighborhood of the point $x_0 H$ in $G/H$. We obtain that $\rho_f(xH, x_0H) < \varepsilon$ whenever $xH \in O$. This proves the continuity of $\rho_f$.

Consider the associated metric space $(Y, d)$ and the canonical distance preserving onto $G$-map $X \to Y, x \mapsto [x]$. The metric $d$ on $Y$ (defined by $d([x], [y]) := \rho_f(x, y)$) is $G$-invariant. Since $d([x_1], [x_2]) > 0$ we conclude that $x_1$ and $x_2$ have different images in $F_G(X)$ (see the discussion above). This contradicts the assumption $x_1 \overset{\text{Aut}}{\sim} x_2$.

(2) Assume that $G/H$ is not SUC-trivial. By Assertion (1) we get that the free topological $G$-group $F_G(G/H)$ of $G/H$ is not trivial. Therefore by the above mentioned result of Pestov [48] we can conclude that the inclusion $H \hookrightarrow G$ is not an epimorphism.

**Remark 5.3.**

1. The converse to Theorem 5.2.2 is not true (take $G := H_+[0, 1], \ H := \{e\}$ and apply Theorem 8.3).

2. As a corollary of Theorem 5.2.2 one can get several examples of SUC-trivial (compact) $G$-spaces. For example, by [36] the free topological $G$-group $F_G(X)$ of $X := G/H$ with $G := H(\mathbb{T}), \ H := St(z)$ (where $z \in \mathbb{T}$ is an arbitrary point of the circle $\mathbb{T}$) is trivial. In fact it is easy to see that the same is true for the smaller group $G := H_+(\mathbb{T})$ (and the subgroup $H := St(z)$) (cf., Proposition 5.1).
(3) It is a well known result by Nummela [47] that if \( G \) is a SIN group then the inclusion of a closed proper subgroup \( H \hookrightarrow G \) is not an epimorphism. This result easily follows from Theorem 5.2.2. Indeed if \( G \) is SIN then by Remark 4.4 for the coset \( G \)-space \( G/H \) we have \( \text{SUC}(G/H) = \text{RUC}(G/H) \). Hence if \( G/H \) is SUC-trivial then necessarily \( H = G \) because \( \text{RUC}(G/H) \) is non-trivial for every closed proper subgroup \( H \) of \( G \).

6. REPRESENTATIONS OF GROUPS AND \( G \)-SPACES ON BANACH SPACES

For a real normed space \( V \) denote by \( B_V \) its closed unit ball \( \{ v \in V : ||v|| \leq 1 \} \). Denote by \( \text{Iso}(V) \) the topological group of all linear surjective isometries \( V \rightarrow V \) endowed with the strong operator topology. This is just the topology of pointwise convergence inherited from \( V^V \). Let \( V^* \) be the dual Banach space of \( V \) and

\[
\langle \cdot, \cdot \rangle : V \times V^* \rightarrow \mathbb{R}, \quad (v, \psi) \mapsto \langle v, \psi \rangle = \psi(v)
\]

is the canonical (always continuous) bilinear mapping.

A representation (co-representation) of a topological group \( G \) on a normed space \( V \) is a homomorphism (resp. co-homomorphism) \( h : G \rightarrow \text{Iso}(V) \). Sometimes it is more convenient to describe a representation (co-representation) by the corresponding left (resp. right) linear isometric actions \( \pi_h : G \times V \rightarrow V \), \( (g, v) \mapsto gv = h(v)(g) \) (resp., \( V \times G \rightarrow V \), \( (v, g) \mapsto vg = h(v)(g) \)). The (co)representation \( h \) is continuous if and only if the action \( \pi_h \) is continuous.

**Remark 6.1.** Many results formulated for co-representations remain true also for representations (and vice versa) taking into account the following simple fact: for every representation (co-representation) \( h \) there exists an associated co-representation (representation) \( h^{\text{op}} : G \rightarrow \text{Iso}(V) \), \( g \mapsto h(g^{-1}) \).

**Definition 6.2.** Let \( \pi : G \times V \rightarrow V \) be a continuous left action of \( G \) on \( V \) by linear operators. The adjoint (or, dual) right action \( \pi^* : V^* \times G \rightarrow V^* \) is defined by \( \psi(gv) := \psi(g) \). The corresponding adjoint (dual) left action is \( \pi^* : G \times V^* \rightarrow V^* \), where \( g\psi(v) := \psi(g^{-1}v) \). Similarly, if \( \pi : V \times G \rightarrow V \) is a continuous linear right action of \( G \) on \( V \) (e.g., induced by some co-representation), then the corresponding adjoint (dual) action \( \pi^* : G \times V^* \rightarrow V^* \) is defined by \( g\psi(v) := \psi(vg) \).

The main question considered in [37] was whether the dual action \( \pi^* \) of \( G \) on \( V^* \) is jointly continuous with respect to the norm topology on \( V^* \). When this is the case we say that the action \( \pi \) (and, also the corresponding representation \( h : G \rightarrow \text{Iso}(V) \), when \( \pi \) is an action by linear isometries) is adjoint continuous. This name was suggested by V. Uspenskij.

**Remark 6.3.** In general, not every continuous representation is adjoint continuous (see for example [37]). A standard example is the representation of the
circle group $G := \mathbb{T}$ on $V := C(\mathbb{T})$ by translations. Here the Banach space $V$ is separable but with “bad” geometry. The absence of adjoint continuity may happen even for relatively “good” (for instance, for separable Radon-Nikodým) Banach spaces like $V := l_1$. Indeed if we consider the symmetric group $G := S_\infty$, naturally embedded into $\text{Iso}(V)$ (endowed with the strong operator topology) as the group of “permutation of coordinates” operators, then the dual action of $G$ on $l_1^* = l_\infty$ is not continuous (see [38]).

It turns out that the situation in that respect is the best possible for the important class $\mathcal{A}_{sp}$ of Asplund Banach spaces. The investigation of this class and the closely related Radon-Nikodým property is among the main themes in Banach space theory. Recall that a Banach space $V$ is an Asplund space if the dual of every separable linear subspace is separable, iff every bounded subset $A$ of the dual $V^*$ is (weak*,norm)-fragmented, iff $V^*$ has the Radon-Nikodým property. Reflexive spaces and spaces of the type $c_0(\Gamma)$ are Asplund. For more details cf. [9, 14]. For the readers convenience we recall also the definition of fragmentability.

**Definition 6.4.** (Jane and Rogers [34]) Let $(X, \tau)$ be a topological space and $\rho$ be a metric on the set $X$. Then $X$ is said to be $(\tau, \rho)$-fragmented if for every nonempty $A \subset X$ and every $\varepsilon > 0$ there exists a $\tau$-open subspace $O$ of $X$ such that $O \cap A$ is nonempty and $\varepsilon$-small in $(X, \rho)$.

Namioka’s Joint continuity theorem implies that every weakly compact set in a Banach space is norm fragmented. This explains why every reflexive space is Asplund.

**Theorem 6.5.** [37, Corollary 6.9] Let $V$ be an Asplund Banach space. If a (not necessarily isometric) linear action $\pi : G \times V \to V$ is continuous then the dual right action $\pi^* : V^* \times G \to V^*$ is also continuous.

Certainly, this result remains true for dual left actions $\pi^* : G \times V^* \to V^*$, where $g\psi(v) := \psi(g^{-1}v)$, as well as for dual actions defined by a right action $\pi : V \times G \to V$. The obvious reason is the continuity of the map $G \to G, g \mapsto g^{-1}$.

The following definition provides a flow version of the group representation definitions discussed above. It differs from the usual notion of $G$-linearization in that here we represent the phase space of the flow as a subset of the dual space $V^*$ (with respect to the dual action and weak star topology) rather than as a subset of $V$.

**Definition 6.6.** [41] Let $X$ be a $G$-space. A continuous (proper) representation of $(G, X)$ on a Banach space $V$ is a pair

$$(h, \alpha) : G \times X \ni \text{Iso}(V) \times B^*$$
where \( h: G \to \text{Iso}(V) \) is a strongly continuous co-homomorphism and \( \alpha: X \to B^* \) is a weak star continuous G-map (resp. embedding) with respect to the dual action \( G \times V^* \to V^* \), \( (g\varphi)(v) := \varphi(h(g)(v)) \). Here \( B^* \) is the weak star compact unit ball of the dual space \( V^* \).

Alternatively, one can define a representation in such a way that \( h \) is a homomorphism and the dual action \( G \times V^* \to V^* \) is defined by \( (g\varphi)(v) := \varphi(h(g^{-1})(v)) \).

**Remark 6.7.** Let \( X \) be a \( G \)-space and let \( A \) be a Banach (closed, unital) subalgebra of \( C(X) \). Associated with \( A \) we have the canonical \( A \)-compactification \( \nu_A: X \to X^A \) of \( X \), where \( X^A = |A| \) is the Gelfand space of \( A \). Here \( X^A \) is canonically embedded into the weak star compact unit ball \( B^* \) of the dual space \( A^* \). If \( A \) is \( G \)-invariant (that is, the function \( (fg)(x) := f(gx) \) lies in \( A \) for every \( f \in A \) and \( g \in G \) then \( X^A \) admits the natural adjoint action \( G \times X^A \to X^A \) with the property that all translations \( \tilde{g}: X^A \to X^A \) are continuous and such that \( \alpha_A: X \to X^A \subset B^* \) is \( G \)-equivariant. We obtain in this way a representation (where \( h \) is not necessarily continuous)

\[
(h, \alpha_A): (G, X) \Rightarrow (\text{Iso}(A), B^*)
\]

on the Banach space \( A \), where \( h(g)(f) := fg \) (and \( \alpha_A(x)(f) := f(x) \)). We call it the canonical (or, regular) \( A \)-representation of \( (G, X) \). It is continuous iff \( A \subset \text{RUC}(X) \) (see for example [41, Fact 2.2] and [42, Fact 7.2]). The regular \( \text{RUC}(X) \)-representation leads to the maximal \( G \)-compactification \( X \to \beta_G X \) of \( X \). It is proper if and only if \( X \) is \( G \)-compactifiable.

The following observation due to Teleman is well known (see also [50]).

**Fact 6.8.** (Teleman [55]) Every topological group can be embedded into \( \text{Iso}(V) \) for some Banach space \( V \).

**Proof.** It is well known that \( \text{RUC}(G) \) determines the topology of \( G \). Hence the regular \( V := \text{RUC}(G) \)-representation \((h, \alpha): (G, G) \Rightarrow (\text{Iso}(V), B^*)\) is proper. That is, the map \( \alpha \) is an embedding. In fact it is easy to see that the co-homomorphism \( h \) is an embedding of topological spaces. The representation \( h^{op}: G \to \text{Iso}(V), g \mapsto h(g^{-1}) \) is then a topological group embedding. \( \square \)

**Definition 6.9.** Let \( \mathcal{K} \) be a “well behaved” subclass of the class \( \mathcal{B}an \) of all Banach spaces. Typical and important particular cases for such \( \mathcal{K} \) are: \( \mathcal{H}ilb \), \( \mathcal{R}ef \) or \( \mathcal{A}sp \), the classes of Hilbert, reflexive or Asplund Banach spaces respectively.

1. A topological group \( G \) is \( \mathcal{K} \)-representable if there exists a (co)representation \( h: G \to \text{Iso}(V) \) for some \( V \in \mathcal{K} \) such that \( h \) is topologically faithful (that is, an embedding). Notation: \( G \in \mathcal{K}_r \).

2. In the opposite direction, we say that \( G \) is \( \mathcal{K} \)-trivial if every continuous \( \mathcal{K} \)-representation (or, equivalently, co-representation) \( h: G \to \text{Iso}(V) \) is trivial.
(3) We say that a topological group $G$ is SUC-trivial if $\text{SUC}(G) = \{\text{constants}\}$. Analogously can be defined WAP-trivial groups. $G$ is WAP-trivial iff $G$ is reflexively trivial ($\text{Ref}$-trivial in the sense of Definition 6.9.2). Similarly, $\text{Asp}(G) = \{\text{constants}\}$ iff $G$ is Asp-trivial. These equivalences follow for instance from Theorem 9.3 below.

**Remark 6.10.**

(1) By Teleman’s theorem (Fact 6.8) every topological group is “Banach representable”. Hence, $\{\text{Topological Groups}\} = \text{Ban}_r$.

(2) $\{\text{Topological Groups}\} = \text{Ban}_r \supset \text{Asp}_r \supset \text{Ref}_r \supset \text{Hilb}_r$.

(3) By Herer and Christensen [31] (see also Banasczyk [5]) abelian (even monothetic) groups can be Hilb-trivial. Note also that $c_0 \notin \text{Hilb}_r$, [40].

(4) [40] The additive group $L_4[0,1]$ is reflexively but not Hilbert representable.

(5) [39] $H_+[0,1] \notin \text{Ref}_r$. It was shown in [39] that every weakly almost periodic function on the topological group $G := H_+[0,1]$ is constant and that $G$ is $\text{Ref}$-trivial. By Pestov’s observation (see [51, Corollary 1.4] and Lemma 10.2) the same is true for the group $\text{Iso}(\mathbb{U}_1)$.

(6) Theorem 10.3.3 shows that $H_+[0,1]$ is even Asp-trivial. In fact we show that every “adjoint continuous” representation of that group is trivial (Theorem 10.3.2). This result was obtained also by Uspenskij (unpublished). Furthermore, we prove a stronger result by showing that $H_+[0,1]$ (and also $\text{Iso}(\mathbb{U}_1)$) are SUC-trivial.

**Problem 6.11.** (See also [42] and [39])

(1) Distinguish $\text{Asp}_r$ and $\text{Ref}_r$ by finding $G \in \text{Asp}_r$ such that $G \notin \text{Ref}_r$.

(2) Find an abelian $G \notin \text{Ref}_r$.

Now we turn to the “well behaved actions”. Recall the dynamical versions of Eberlein and Radon-Nikodým compact spaces.

**Definition 6.12.** [18, 41] Let $X$ be a $G$-space.

(1) $(G, X)$ is a **Radon-Nikodým system** (RN for short) if there exists a proper representation of $(G, X)$ on an Asplund Banach space $V$. If we can choose $V$ to be reflexive, then $(G, X)$ is called an **Eberlein system**. The classes of Radon-Nikodým and Eberlein compact systems will be denoted by RN and Eb respectively.

(2) $(G, X)$ is called an **RN-approximable system** ($\text{RN}_{\text{app}}$) if it can be represented as a subdirect product of RN systems.

Note that compact spaces which are not Eberlein are necessarily non-metrizable, while even for $G := \mathbb{Z}$, there are many natural metric compact $G$-systems which are not RN.
Definition 6.13.

(1) A representation \((h, \alpha)\) of a \(G\)-space \(X\) on \(V\) is adjoint continuous if the dual action \(G \times V^* \to V^*\) is also continuous (or, equivalently, if the group corepresentation \(h: G \to \text{Iso}(V)\) is adjoint continuous).

(2) Denote by \(\mathcal{A}dj\) the class of compact \(G\)-systems which admit a proper adjoint continuous representation on some Banach space \(V\). Theorem 6.5 implies that \(\text{RN} \subset \mathcal{A}dj\).

(3) Denote by \(\text{adj}(G)\) the collection of functions on \(G\) which come from a compact \(G\)-space \(X\) such that \((G, X)\) is in the class \(\mathcal{A}dj\). In fact this means that \(f\) can be represented as a generalized matrix coefficient (see Section 9) of some adjoint continuous representation of \(G\).

Proposition 6.14. \(\text{Asp}(G) \subset \text{adj}(G)\) for every topological group \(G\).

Proof. By [41, Theorem 7.11] (or Proposition 7.5) \(f \in \text{Asp}(G)\) iff \(f\) comes from a \(G\)-compactification \(G \to X\) of \(G\) with \(X \in \text{RN}\). Now observe (as in Definition 6.13.2) that \(\text{RN} \subset \mathcal{A}dj\) by Theorem 6.5. \(\square\)

7. Dynamical complexity of functions

In this section we introduce a hierarchy of dynamical complexity of functions on a topological group \(G\) which reflects the complexity of the \(G\)-systems from which they come. Our main tool is the cyclic \(G\)-system \(X_f\) corresponding to a function \(f: X \to \mathbb{R}\). Recall that when \(X := G\), the space \(X_f\) is the pointwise closure of the orbit \(Gf\) in \(\text{RUC}(G)\). The topological nature of \(X_f\) in the Banach space \(\text{RUC}(G)\) relates to the dynamical complexity of \(f\) and leads to a natural hierarchy of complexity (see Theorem 7.12 below). In particular we will examine the role that \(\text{SUC}\) functions play in this hierarchy.

Periodic orbits and the profinite compactification

The most elementary dynamical system is a finite (periodic) orbit. It corresponds to a clopen subgroup \(H < G\) of finite index. These subgroups form a directed set and the corresponding compact inverse limit \(G\)-system

\[ X^{PF} = \lim_{\leftarrow} G/H \]

is the profinite compactification of \(G\).

Almost Periodic functions and the Bohr compactification

The weaker requirement that \(X_f\) be norm compact in \(\text{RUC}(G)\) leads to the well known definition of almost periodicity. A function \(f \in C(X)\) on a \(G\)-space \(X\) is almost periodic if the orbit \(fG := \{fg\}_{g \in G}\) forms a precompact subset of the Banach space \(C(X)\). The collection \(\text{AP}(X)\) of AP functions is a \(G\)-subalgebra
in RUC\((X)\). The universal almost periodic compactification of \(X\) is the Gelfand space \(X^{\text{AP}}\) of the algebra \(\text{AP}(X)\). When \(X\) is compact this is the classical *maximal equicontinuous factor* of the system \(X\). A compact \(G\)-space \(X\) is equicontinuous iff \(X\) is almost periodic (AP), that is, iff \(C(X) = \text{AP}(X)\). For a \(G\)-space \(X\) the collection \(\text{AP}(X)\) is the set of all functions which come from equicontinuous (AP) \(G\)-compactifications.

For every topological group \(G\), treated as a \(G\)-space, the corresponding universal AP compactification is the well known *Bohr compactification* \(b: G \to bG\), where \(bG\) is a compact topological group.

**Theorem 7.1.** *Let \(X\) be a \(G\)-space. For \(f \in \text{RUC}(X)\) the following conditions are equivalent:*

1. \(f \in \text{AP}(X)\).
2. \((G, X_f)\) is equicontinuous.
3. \(X_f\) is norm compact in \(\text{RUC}(G)\).

**Proof.** (1) \(\iff\) (2): \(f \in \text{AP}(X)\) iff the cyclic algebra \(A_f\) (which, by Proposition 3.1, generates the compactification \(X \to X_f\)) is a subalgebra of \(\text{AP}(X)\).

(2) \(\iff\) (3): It is easy to see that the \(G\)-space \(X_f\) is equicontinuous iff the norm and pointwise topologies coincide on \(X_f \subset \text{RUC}(G)\). \(\square\)

**Weakly Almost Periodic functions**

A function \(f \in C(X)\) on a \(G\)-space \(X\) is called *weakly almost periodic* (WAP for short; notation: \(f \in \text{WAP}(X)\)) if the orbit \(fG := \{fg\}_{g \in G}\) forms a weakly precompact subset of \(C(X)\). A compact \(G\)-space \(X\) is said to be *weakly almost periodic* \([11]\) if \(C(X) = \text{WAP}(X)\). For a \(G\)-space \(X\) the collection \(\text{WAP}(X)\) is the set of all functions which come from WAP \(G\)-compactifications. The universal WAP \(G\)-compactification \(X \to X^{\text{WAP}}\) is well defined. The algebra \(\text{WAP}(G)\) is a point-universal \(G\)-algebra containing \(\text{AP}(G)\). The compactification \(G \to G^{\text{WAP}}\) (for \(X := G\)) is the universal semitopological semigroup compactification of \(G\).

A compact \(G\)-space \(X\) is WAP iff it admits sufficiently many representations on reflexive Banach spaces \([41]\). Furthermore if \(X\) is a metric compact \(G\)-space then \(X\) is WAP iff \(X\) admits a proper representation on a reflexive Banach space. That is, iff \(X\) is an Eberlein \(G\)-space.

**Theorem 7.2.** *Let \(X\) be a \(G\)-space. For \(f \in \text{RUC}(X)\) the following conditions are equivalent:*

1. \(f \in \text{WAP}(X)\).
2. \((G, X_f)\) is WAP.
3. \(X_f\) is weak compact in \(\text{RUC}(G)\).
4. \((G, X_f)\) is Eberlein (i.e., reflexively representable).
Proof. (1) $\Longleftrightarrow$ (2): $f \in \text{WAP}(X)$ iff the algebra $A_f$ is a subalgebra of $\text{WAP}(X)$.

(2) $\implies$ (3): Let $F_e : X_f \to \mathbb{R}$, $F_e(\omega) = \omega(e)$ be as in the definition of $X_f$. Consider the weak closure $Y := \text{cl}_w(F_eG)$ of the orbit $F_eG$. Then $Y$ is weakly compact in $C(X_f)$ because $F_e \in C(X_f) = \text{WAP}(X_f)$ is weakly almost periodic. If $\omega_1$ and $\omega_2$ are distinct elements of $X_f$ then $(F_eg)(\omega_1) = \omega_1(g) \neq \omega_2(g) = (F_eg)(\omega_2)$ for some $g \in G$. This means that the separately continuous evaluation map $Y \times X_f \to \mathbb{R}$ separates points of $X_f$. Now $X_f$ can be treated as a pointwise compact bounded subset in $C(Y)$. Hence by Grothendieck’s well known theorem [30] we get that $X_f$ is weakly compact in $C(Y)$. Since $G \to Y$, $g \mapsto gF_e$ is a $G$-compactification of $G$, we have a natural embedding of Banach algebras $j : C(Y) \hookrightarrow \text{RUC}(G)$. It follows that $X_f = j(X_f)$ is also weakly compact as a subset of $\text{RUC}(G)$.

(3) $\implies$ (4): The isometric action $G \times \text{RUC}(G) \to \text{RUC}(G)$, $(g,f) \mapsto gf$ induces a representation $h : G \to \text{Iso}(\text{RUC}(G))$. If the $G$-subset $X_f$ is weakly compact in $\text{RUC}(G)$ then one can apply Theorem 4.11 (namely, the equivalence between (i) and (iii)) of [41] which guarantees that the $G$-space $X_f$ is Eberlein.

(4) $\implies$ (1): $f \in \text{WAP}(X)$ because it comes from $(G,X_f)$ (Proposition 3.1) which is WAP (being reflexively representable).

Asplund functions, “sensitivity to initial conditions” and Banach representations

The following definition of “sensitivity to initial conditions” is essential in several definitions of chaos in dynamical systems, mostly for $G := \mathbb{Z}$ or $\mathbb{R}$ actions on metric spaces (see for instance papers of Guckenheimer, Auslander and Yorke, Devaney, Glasner and Weiss).

Definition 7.3. [18] Let $(X, \mu)$ be a uniform $G$-space.

(1) We say that $X$ is sensitive to initial conditions (or just sensitive) if there exists an $\varepsilon \in \mu$ such that for every nonempty open subset $O$ of $X$ the set $gO$ is not $\varepsilon$-small for some $g \in G$. Otherwise, $X$ is non-sensitive (for short: NS).

(2) $X$ is Hereditarily Non Sensitive (HNS) if every closed $G$-subspace of $X$ is NS.

Denote by HNS the class of all compact HNS systems. The following result says that a compact $G$-system $X$ is HNS iff $(G,X)$ admits sufficiently many representations on Asplund spaces.

Theorem 7.4. [18]

(1) $\text{HNS} = \text{RN}_{\text{app}}$.

(2) If $X$ is a compact metric $G$-space then $X$ is HNS iff $X$ is RN (that is, Asplund representable).
A function \( f : X \to \mathbb{R} \) on a \( G \)-space \( X \) is Asplund (notation: \( f \in \text{Asp}(X) \)) [41] if it satisfies one of the following equivalent conditions.

**Proposition 7.5.** Let \( f : X \to \mathbb{R} \) be a function on a \( G \)-space \( X \). The following conditions are equivalent:

1. \( f \) comes from a \( G \)-compactification \( \nu : X \to Y \) where \((G,Y)\) is HNS.
2. \( f \) comes from a \( G \)-compactification \( \nu : X \to Y \) where \((G,Y)\) is RN.
3. \( f \) comes from a \( G \)-compactification \( \nu : X \to Y \) and a function \( F : Y \to \mathbb{R} \) where the pseudometric space \((Y,\rho_{H,F})\) with
   \[
   \rho_{H,F}(x,x') = \sup_{h \in H} |F(hx) - F(hx')|
   \]
   is separable for every countable (equivalently, second countable) subgroup \( H \subset G \).

The collection \( \text{Asp}(X) \) is always a \( G \)-subalgebra of \( \text{RUC}(X) \). It defines the maximal HNS-compactification \( X \to X^{\text{Asp}} = |\text{Asp}(X)| \) of \( X \). For every topological group \( G \) the algebra \( \text{Asp}(G) \) (as usual, \( X := G \) is a left \( G \)-space) is point-universal.

**Theorem 7.6.** Let \( X \) be a \( G \)-space. For every \( f \in \text{RUC}(X) \) the following conditions are equivalent:

1. \( f \in \text{Asp}(X) \).
2. \((G,X_f)\) is RN.
3. \( X_f \) is norm fragmented in \( \text{RUC}(G) \).

**Proof.** If \( X \) is compact then the proof follows directly from [18, Theorem 9.12]. Now observe that one can reduce the case of an arbitrary \( G \)-space \( X \) to the case of a compact \( G \)-space \( X_f \) by considering the cyclic \( G \)-system \((X_f)_{F_e} \) (defined for \( X := X_f \) and \( f := F_e \)) which can be naturally identified with \( X_f \). \( \square \)

Explicitly the fragmentability of \( X_f \) means that for every \( \varepsilon > 0 \) every nonempty (closed) subset \( A \) of \( X_f \subset \mathbb{R}^G \) contains a relatively open (in the pointwise topology) nonempty subset \( O \cap A \) which is \( \varepsilon \)-small in the Banach space \( \text{RUC}(G) \).

As we already mentioned every weakly compact set is norm fragmented so that \( \text{WAP}(X) \subset \text{Asp}(X) \) for every \( G \)-space \( X \). In particular, \( \text{WAP}(G) \subset \text{Asp}(G) \).

**Locally equicontinuous functions**

During the last decade various conditions weakening the classical notion of equicontinuity were introduced and studied (see e.g. [24], [1], [2], [3]). The following definition first appears in a paper of Glasner and Weiss [24].

**Definition 7.7.** [24] Let \((X,\mu)\) be a uniform \( G \)-space. A point \( x_0 \in X \) is a point of local equicontinuity (notation: \( x_0 \in \text{LE}_X \)) if \( x_0 \) is a point of equicontinuity in
the uniform $G$-subspace $\text{cl}(Gx_0)$. We have $x_0 \in \text{LE}_X$ iff $x_0 \in \text{LE}_Y$ iff $x_0 \in \text{Eq}_Y$ where $Y$ is the orbit $Gx_0$ of $x_0$ (see Lemma 7.8.1). If $\text{LE}_X = X$, then $X$ is locally equicontinuous (LE).

**Lemma 7.8.**

1. Let $Y$ be a dense $G$-subspace of $(X, \mu)$ and $y_0 \in Y$. Then $y_0 \in \text{Eq}_X$ if and only if $y_0 \in \text{Eq}_Y$.
2. $\text{SUC}_X \supset \text{LE}_X \supset \text{Eq}_X$.
3. $\text{SUC}(X) \supset \text{LE}(X)$.

**Proof.** (1) Let $\epsilon \in \mu$. There exists $\delta \in \mu$ such that $\delta$ is a closed subset of $X \times X$ and $\delta \subset \epsilon$. If $y_0 \in \text{Eq}_Y$ there exists an open set $U$ in $X$ such that $y_0 \in U$ and $(gy, gy_0) \in \delta$ for all $g \in U \cap Y$ and $g \in G$. Since $Y$ is dense in $X$ and $U$ is open we have $U \subset \text{cl}(U \cap Y)$. Since every $g$-translation $X \to X$, $x \mapsto gx$ is continuous and $\delta$ is closed we get $(gx, gy_0) \in \delta \subset \epsilon$ for every $g \in G$ and $x \in U$.

(2): Let $x_0 \in \text{LE}_X$. For every $\epsilon \in \mu$ there exists a neighborhood $O(x_0)$ such that $(gx, gx_0) \in \epsilon$ for every $x \in O(x_0)$ and $g \in G$. Choose a neighborhood $U(e)$ such that $Ux_0 \subset O$. Then $(gx_0, gx_0)$ is $\epsilon$-small, too. This proves the non-trivial part $\text{SUC}_X \supset \text{LE}_X$.

(3): Directly follows from (2). □

The collection $\text{LE}(X)$ forms a $G$-subalgebra of $\text{RUC}(X)$. Always, $\text{Asp}(X) \subset \text{LE}(X)$. The algebra $\text{LE}(X)$ defines the maximal LE-compactification $X \to X^{\text{LE}}$ of $X$. For every topological group $G$ the algebra $\text{LE}(G)$ is point-universal [18].

**Theorem 7.9.** [18] Let $X$ be a $G$-space. For $f \in \text{RUC}(X)$ the following conditions are equivalent:

1. $f \in \text{LE}(X)$.
2. $(G, X_f)$ is LE.
3. $X_f$ is orbitwise light in $\text{RUC}(G)$ (that is, for every function $\psi \in X_f$ the pointwise and norm topologies coincide on the orbit $G\psi$).

**Proof.** (1) $\iff$ (2) Directly follows from [18, Theorem 5.15.1]. On the other hand by [18, Lemma 5.18] we have (2) $\iff$ (3). □

The dynamical hierarchy

**Theorem 7.10.**

1. Let $X$ be a (not necessarily compact) $G$-space. We have the following inclusions of $G$-subalgebras:

\[
\text{RUC}(X) \supset \text{SUC}(X) \supset \text{LE}(X) \supset \text{Asp}(X) \supset \text{WAP}(X) \supset \text{AP}(X)
\]

and the corresponding chain of $G$-factor maps

\[
\beta_G X \to X^{\text{SUC}} \to X^{\text{LE}} \to X^{\text{Asp}} \to X^{\text{WAP}} \to X^{\text{AP}}
\]
(2) For every topological group \( G \) we have the following inclusions of \( G \)-subalgebras:

\[
\text{RUC}(G) \supset \text{UC}(G) \supset \text{SUC}(G) \supset \text{LE}(G) \supset \text{Asp}(G) \supset \text{WAP}(G) \supset \text{AP}(G)
\]

and the corresponding chain of \( G \)-factor maps

\[
G^{\text{RUC}} \to G^{\text{UC}} \to G^{\text{SUC}} \to G^{\text{LE}} \to G^{\text{Asp}} \to G^{\text{WAP}} \to G^{\text{AP}}.
\]

**Proof.** For the assertions concerning \( \text{SUC}(X) \) and \( \text{SUC}(G) \) see Lemmas 4.3 and 7.8. For the other assertions see [18]. \( \square \)

**Remark 7.11.** The compactifications \( G^{\text{AP}} \) and \( G^{\text{WAP}} \) of \( G \) are respectively a topological group and a semitopological semigroup. The compactifications \( G^{\text{RUC}} \) and \( G^{\text{Asp}} \) are right topological semigroup compactifications of \( G \) (see [18]). The same is true for the compactification \( j: G \to G^{\text{SUC}} \) (Lemma 4.6.5). Below (Theorem 10.3.5) we show that the *Roelcke compactification* \( i: G \hookrightarrow G^{\text{UC}} \) (which is always proper by Lemma 2.1) is not in general a right topological semigroup compactification. That is, \( \text{UC}(G) \) is not in general point-universal.

We sum up our results in the following dynamical hierarchy theorem where we list dynamical properties of \( f \in \text{RUC}(X) \) and the corresponding topological properties of \( X_f \subset \text{RUC}(G) \) (cf. [18, Remark 9.13]).

**Theorem 7.12.** For every \( G \)-space \( X \) and a function \( f \in \text{RUC}(X) \) we have

\[
X_f \text{ is norm compact} \iff f \text{ is AP}
\]

\[
X_f \text{ is weakly compact} \iff f \text{ is WAP}
\]

\[
X_f \text{ is norm fragmented} \iff f \text{ is Asplund}
\]

\[
X_f \text{ is orbitwise light} \iff f \text{ is LE}
\]

\[
X_f \subset \text{UC}(G) \iff f \text{ is SUC}
\]

**Example 7.13.** Let \( G \) be the unitary group \( U(H) = \text{Iso}(H) \) where \( H \) is an infinite dimensional Hilbert space. Then \( \text{UC}(G) = \text{SUC}(G) = \text{LE}(G) = \text{Asp}(G) = \text{WAP}(G) \). Indeed the completion of \( (G, \mathcal{L} \setminus \mathcal{R}) \) can be identified with the compact semitopological semigroup \( \Theta(H) \) of all nonexpansive linear operators (Uspenskij [58]). It follows that \( G^{\text{UC}} \) can be identified with \( \Theta(H) \). The latter is a reflexively representable \( G \)-space (see for example [41, Fact 5.2]). Therefore \( \text{UC}(G) \subset \text{WAP}(G) \). The reverse inclusion is well known (see for instance Theorem 7.10). Hence, \( \text{UC}(G) = \text{WAP}(G) \).

Let \( \mathcal{C}_u = \{ f \in \text{UC}(G) : X_f \subset \text{UC}(G) \} \). The collection of functions \( \mathcal{C}_u \) was studied by Veech in [61]. He notes there that \( \text{WAP}(G) \subset \mathcal{C}_u \) and proves the following theorem.
Theorem 7.14. (Veech [61, Proposition 5.4]) Let $G$ be a semisimple analytic Lie group with finite center and without compact factors. If $f \in \mathcal{C}_u$ then every limit point of $Gf$ in $X_f$; i.e. any function of the form $h(g) = \lim_{g_n \to \infty} f(gg_n)$, is a constant function.

(A sequence $g_n \in G$ “tends to $\infty$” if each of its “projections” onto the simple components of $G$ tends to $\infty$ in the usual sense.) He then deduces the fact that for $G$ which is a direct product of simple groups the algebra $\text{WAP}(G)$ coincides with the algebra $W^*$ of continuous functions on $G$ which “extend continuously” to the product of the one-point compactification of the simple components of $G$ ([61, Theorem 1.2]). By our Proposition 4.12, $\mathcal{C}_u = \text{SUC}(G)$. Taking this equality into account, Veech’s theorem implies now the following result.

Corollary 7.15. For every simple noncompact connected Lie group $G$ with finite center (e.g., $\text{SL}_n(\mathbb{R})$) we have $\text{SUC}(G) = \text{WAP}(G) = W^*$. In particular the corresponding universal SUC (and hence WAP) compactification is equivalent to the one point compactification of $G$.

8. The group $H_+[0,1]$

Consider the Polish topological group $G := H_+[0,1]$ of all orientation preserving homeomorphisms of the closed unit interval, endowed with the compact open topology. Here is a list of some selected known results about this group:

1. $G$ is topologically simple.
2. $G$ is not Weil-complete; that is, the right uniform structure $\mathcal{R}$ of $G$ is not complete. The completion of the uniform space $(G, \mathcal{R})$ can be identified with the semigroup of all continuous, nondecreasing and surjective maps $[0,1] \to [0,1]$ endowed with the uniform structure of uniform convergence (Roelcke-Dierolf [53, p. 191]).
3. $G$ is Roelcke precompact (that is the Roelcke uniformity $\mathcal{L} \wedge \mathcal{R}$ on $G$ is precompact) [53].
4. The completion of $(G, \mathcal{L} \wedge \mathcal{R})$ can be identified with the curves that connect the points $(0,0)$ and $(1,1)$ and “never go down” (Uspenskij [60], see Lemma 8.4 below).
5. Every weakly almost periodic function on $G$ is constant and every continuous representation $G \to \text{Iso}(V)$, where $V$ is a reflexive Banach space, is trivial (Megrelishvili [39]).
6. $G$ is extremely amenable; that is every compact Hausdorff $G$-space has a fixed point property (Pestov [49]).

We are going to show that $H_+[0,1]$ is SUC-trivial and hence also $\text{Aspl}$-trivial. Since every reflexive Banach space is Asplund these results strengthen the main results of [39] (results mentioned in item (5) above).
Definition 8.1. Let \((X, \mu)\) be a compact \(G\)-space. We say that two points \(a, b \in X\) are SUC-proximal if there exist nets \(s_i\) and \(g_i\) in \(G\) and a point \(x_0 \in X\) such that \(s_i\) converges to the neutral element \(e\) of \(G\), the net \(g_ix_0\) converges to \(a\) and the net \(g_is_ix_0\) converges to \(b\).

Lemma 8.2. If the points \(a\) and \(b\) are SUC-proximal in a \(G\)-space \(X\) then \(a \sim b\).

Proof. A straightforward consequence of our definitions using Lemma 4.5.4. □

Theorem 8.3. Let \(G = H_+[0,1]\) be the topological group of orientation-preserving homeomorphisms of \([0,1]\) endowed with the compact open topology. Then \(G\) is SUC-trivial.

Proof. Denote by \(j: G \rightarrow G^{SUC}\) and \(i: G \rightarrow G^{UC}\) the \(G\)-compactifications (\(i\) necessarily is proper by Lemma 2.1) induced by the Banach \(G\)-algebras \(SUC(G) \subset UC(G)\). There exists a canonical onto \(G\)-map \(\pi: G^{UC} \rightarrow G^{SUC}\) such that the following diagram of \(G\)-maps is commutative:

\[
\begin{array}{ccc}
G & \xrightarrow{i} & G^{UC} \\
\downarrow{j} & & \downarrow{\pi} \\
G^{SUC} & & 
\end{array}
\]

We have to show that \(G^{SUC}\) is trivial for \(G = H_+[0,1]\). One of the main tools for the proof is the following identification.

Lemma 8.4. [60, Uspenskij] The dynamical system \(G^{UC}\) is isomorphic to the \(G\)-space \((G, \Omega)\). Here \(\Omega\) denotes the compact space of all curves in \([0,1] \times [0,1]\) which connect the points \((0,0)\) and \((1,1)\) and “never go down”, equipped with the Hausdorff metric. These are the relations \(\omega \subset [0,1] \times [0,1]\) where for each \(t \in [0,1]\), \(\omega(t)\) is either a point or a vertical closed segment. The natural action of \(G = H_+[0,1]\) on \(\Omega\) is \((g\omega)(t) = g(\omega(t))\) (by composition of relations on \([0,1]\)).

We first note that every “zig-zag curve” (i.e. a curve \(z\) which consists of a finite number of horizontal and vertical pieces) is an element of \(\Omega\). In particular the curves \(\gamma_c\) with exactly one vertical segment defined as \(\gamma_c(t) = 0\) for every \(t \in [0,c)\), \(\gamma_c(c) = \{c\} \times [0,1]\) and \(\gamma(t) = 1\) for every \(t \in (c,1]\), are elements of \(\Omega = G^{UC}\). Note that the curve \(\gamma_1\) is a fixed point for the left \(G\) action. We let \(\theta = \pi(\gamma_1)\) be its image in \(G^{SUC}\). Of course \(\theta\) is a fixed point in \(G^{SUC}\). We will show that \(\theta = j(e)\) and since the \(G\)-orbit of \(j(e)\) is dense in \(G^{SUC}\) this will show that \(G^{SUC}\) is a singleton.

The idea is to show that zig-zag curves are SUC-proximal in \(G^{UC}\). Then Lemma 8.2 will ensure that their images in \(G^{SUC}\) coincide. Choosing a sequence \(z_n\) of zig-zag curves which converges in the Hausdorff metric to \(i(e)\) in \(G^{UC}\) we will have \(\pi(z_n) = \pi(\gamma_1) = \theta\) for each \(n\). This will imply that indeed \(j(e) = \pi(i(e)) = \pi(\lim_{n \to \infty} z_n) = \lim_{n \to \infty} \pi(z_n) = \theta\).
First we show that \( \pi(\gamma_1) = \pi(\gamma_c) \) for any \( 0 < c < 1 \). As indicated above, since \( G^{\text{SUC}} \) is the Gelfand space of the algebra \( \text{SUC}(G) \), by Lemma 8.2, it suffices to show that the pair \( \gamma_1, \gamma_c \) is \( \text{SUC} \)-proximal in \( G^{\text{UC}} \). Since \( X^{\text{SUC}} = X \) for \( X := G^{\text{SUC}} \) we conclude that \( \pi(\gamma_1) = \pi(\gamma_c) \).

Let \( p \in G^{\text{UC}} \) be the curve defined by \( p(t) = t \) in the interval \([0, c]\) and by \( p(t) = c \) for every \( t \in [c, 1) \). Pick a sequence \( s_n \) of elements in \( G \) such that \( s_n \) converges to \( e \) and \( s_n c < c \). It is easy to choose a sequence \( g_n \) in \( G \) such that \( g_n s_n c \) converges to \( 0 \) and \( g_n c \) converges to \( 1 \). Then the sequences \( s_n \) and \( g_n \) are the desired sequences; that is, \( g_n p \to \gamma_c \), \( g_n s_n p \to \gamma_1 \) (see the picture below).

Denote \( \theta = \pi(\gamma_1) = \pi(\gamma_c) \). Using similar arguments (see the picture below, where \( a \sim b \) means \( \pi(a) = \pi(b) \)) construct a sequence \( z_n \in G^{\text{UC}} \) of zig-zag curves which converges to \( i(e) \) and such that \( \pi(z_n) = \theta \) for every \( n \).

In view of the discussion above this construction completes the proof of the theorem. \( \square \)

9. Matrix coefficient characterization of \( \text{SUC} \) and \( \text{LE} \)

**Definition 9.1.** Let \( h: G \to \text{Iso}(V) \) be a co-representation of \( G \) on a normed space \( V \) and let

\[
V \times G \to V, \quad (v, g) \mapsto vg := h(g)(v)
\]

be the corresponding right action. For a pair of vectors \( v \in V \) and \( \psi \in V^* \) the associated matrix coefficient is defined by

\[
m_{v, \psi}: G \to \mathbb{R}, \quad g \mapsto \psi(vg) = \langle vg, \psi \rangle = \langle v, g\psi \rangle.
\]

If \( h: G \to \text{Iso}(V) \) is a representation then the matrix coefficient \( m_{v, \psi} \) is defined similarly by

\[
m_{v, \psi}: G \to \mathbb{R}, \quad g \mapsto \psi(gv) = \langle gv, \psi \rangle = \langle v, \psi g \rangle.
\]
For example, if $V = H$ is a Hilbert space then $f = m_{u, \psi}$ is the Fourier-Stieltjes transform. In particular, for $u = \psi$ we get the positive definite functions.

We say that a vector $v \in V$ is $G$-continuous if the corresponding orbit map \( \tilde{v}: G \to V, \tilde{v}(g) = vg \), defined through $h: G \to \text{Iso}(V)$, is norm continuous. The continuous $G$-vector $\psi \in V^*$ are defined similarly with respect to the dual action.

**Lemma 9.2.** [41] Let $h: G \to \text{Iso}(V)$ be a co-representation of $G$ on $V$. If $\psi$ (resp.: $v \in V$) is norm $G$-continuous, then $m_{v, \psi}$ is left (resp.: right) uniformly continuous on $G$. Hence, if $v$ and $\psi$ are both $G$-continuous then $m_{v, \psi} \in \text{UC}(G)$.

In the next theorem we list characterizations of several subalgebras of $\text{RUC}(G)$ in terms of matrix coefficients. These characterizations also provide an alternative way to establish the inclusions in Theorem 7.10.2 for $X := G$.

**Theorem 9.3.** Let $G$ be a topological group and $f \in C(G)$.

1. $f \in \text{RUC}(G)$ iff $f = m_{v, \psi}$ for some continuous co-representation $h: G \to \text{Iso}(V)$, where $V \in \text{Ban}$.
2. $f \in \text{UC}(G)$ iff $f = m_{v, \psi}$ for some co-representation where $v$ and $\psi$ are both $G$-continuous iff $f = m_{v, \psi}$ for some continuous co-representation where $\psi$ is $G$-continuous.
3. $f \in \text{SUC}(G)$ iff $f = m_{v, \psi}$ for some continuous co-representation $h: G \to \text{Iso}(V)$, where $\varphi$ is norm $G$-continuous in $V^*$ for every $\varphi$ from the weak star closure $\text{cl}_{w^*}(\text{G}\psi)$.
4. $f \in \text{adj}(G)$ iff $f = m_{v, \psi}$ for some adjoint continuous co-representation.
5. $f \in \text{LE}(G)$ iff $f = m_{v, \psi}$ for some continuous co-representation $h: G \to \text{Iso}(V)$, where weak star and norm topologies coincide on each orbit $G\varphi$ where $\varphi$ belongs to the weak star closure $Y := \text{cl}_{w^*}(\text{G}\psi)$.
6. $f \in \text{Asp}(G)$ iff $f$ is a matrix coefficient of some continuous Asplund co-representation of $G$.
7. $f \in \text{WAP}(G)$ iff $f$ is a matrix coefficient of some continuous reflexive co-representation (or, representation) of $G$.

**Proof.** Claim (1) follows by taking in the regular $\text{RUC}(G)$-corepresentation $V := \text{RUC}(G), v := f$ and $\psi := \alpha(e)$. Claim (4) is a reformulation of Definition 6.13.3.

The remaining assertions are essentially nontrivial. Their proofs are based on an equivariant generalization of the Davis-Figiel-Johnson-Pelczyński interpolation technic [10]. For detailed proofs of (6) and (7) (for co-representations) see [41, Theorem 7.17] and [41, Theorem 5.1]. As to the “representations case” in (7) observe that a matrix coefficient of a (continuous) co-representation on a reflexive space $V$ can be treated as matrix coefficient of a (continuous) representation on the dual space $V^*$. The continuity of the dual action follows by Theorem 6.5.
For (2) apply Lemma 9.2 and take $V := UC(G)$ (or, see [43]). Below we provide the proof of the new assertions (3) and (5). See Theorems 9.8 and 9.10 respectively.

**Remark 9.4.** Let $A \subset RUC(G)$ be a point-universal $G$-subalgebra. Then $A$ is left $m$-introverted in the sense of [45], [8, Definition 1.4.11]. Indeed, we have only to check that every matrix coefficient $m_{v,\psi}$ of the regular $A$-representation of the action $(G,G)$ on the Banach space $A$ belongs again to $A$ whenever $v \in A$ and $\psi \in |A| \subset A^*$. Let $X := |A|$. Then the $G$-system $(X,eva(e))$ is point universal and $A = A(X,eva(e))$. The matrix coefficient $m_{v,\psi}$ comes from the subsystem $(cl (G\psi), \psi)$. In other words $m_{v,\psi} \in A(cl (G\psi), \psi)$. By Lemma 3.3 we have $A(cl (G\psi), \psi) \subset A(X,eva(e))$. Thus $m_{v,\psi} \in A(X,eva(e)) = A$.

**Definition 9.5.**

(1) Let $G$ be a topological group and $G \times X \to X$ and $Y \times G \to Y$ be respectively left and right actions. A map $\langle , \rangle : Y \times X \to \mathbb{R}$ is $G$-compatible if

$$\langle yg, x \rangle = \langle y, gx \rangle \quad \forall \ (y, g, x) \in Y \times G \times X.$$ 

(2) We say that a subset $M \subset Y$ is SUC-small at $x_0 \in X$ if for every $\varepsilon > 0$ there exists a neighborhood $U$ of $e$ such that

$$|\langle v, ux_0 \rangle - \langle v, x_0 \rangle| \leq \varepsilon \quad \forall (v, u) \in M \times U.$$ 

If $M$ is SUC-small at every $x \in X$ then we say that $M$ is SUC-small for $X$.

(3) Let $h : G \to Iso(V)$ be a continuous co-representation on a normed space $V$ and $h^* : G \to Iso(V^*)$ be the dual representation. Then we say that $M \subset V$ is SUC-small at $x_0 \in X \subset V^*$ if this happens in the sense of (2) regarding the canonical bilinear $G$-compatible map $\langle , \rangle : V \times V^* \to \mathbb{R}$.

For example, a vector $\psi \in V^*$ in the dual space $V^*$ is $G$-continuous iff the unit ball $B_V$ of $V$ is SUC-small at $\psi$ (see Lemma 9.7.3).

We give some useful properties of SUC-smallness.

**Lemma 9.6.**

(1) Let $Y_1 \times X_1 \to \mathbb{R}$ and $Y_2 \times X_2 \to \mathbb{R}$ be two $G$-compatible maps. Suppose that $\gamma_1 : X_1 \to X_2$ and $\gamma_2 : Y_2 \to Y_1$ are $G$-maps such that

$$\langle y, \gamma_1(x) \rangle = \langle \gamma_2(y), x \rangle \quad \forall (y, x) \in Y_2 \times X_1.$$ 

Then for every nonempty subset $M \subset Y_2$ the subset $\gamma_1(M) \subset Y_1$ is SUC-small at $x \in X_1$ if and only if $M$ is SUC-small at $\gamma_1(x) \in X_2$.

(2) Let $X$ be a (not necessarily compact) $G$-space. If $f \in SUC(X)$ then:
(a) $fG$ is SUC-small for $X$ with respect to the $G$-compatible evaluation map

$$fG \times X \to \mathbb{R}, \quad (fg, x) \mapsto f(gx).$$

(b) The subset $F_G$ of $C(X_f)$ is SUC-small for $X_f$ considered as a subset of $V^*$ where $V := C(X_f)$ (with respect to the canonical map $V \times V^* \to \mathbb{R}$ and the natural co-representation $G \to \text{Iso}(V)$).

**Proof.**

(1): Observe that for every triple $(m, u, x_0) \in M \times G \times X$ we have

$$\langle \gamma_1(m), ux_0 \rangle - \langle \gamma_1(m), x_0 \rangle = \langle m, \gamma_2(ux_0) \rangle - \langle m, \gamma_2(x_0) \rangle = \langle m, u\gamma_2(x_0) \rangle - \langle m, \gamma_2(x_0) \rangle.$$

(2): (a) Directly follows by Lemma 4.5.4.

(b): Let $f \in \text{SUC}(X)$. Then it comes by Proposition 3.1.3 from a compact $G$-system $X_f$ and the $G$-compactification $f_2: X \to X_f$. As we know there exists $F_e \in C(X_f)$ such that $f = F_e \circ f_2$. Theorem 4.12 implies that $X_f$ is SUC and $F_e \in \text{SUC}(X_f)$. By claim (a) it follows that $F_eG$ is SUC-small for $X_f$. The $G$-compatible map $F_eG \times X_f \to \mathbb{R}$ can be treated as a restriction of the canonical form $V \times V^* \to \mathbb{R}$, where $V := C(X_f)$ (considered $X_f$ as a subset of $C(X_f)^*$).

**Lemma 9.7.** Let $h: G \to \text{Iso}(V)$ be a continuous co-representation.

(1) For every subset $X$ of $V^*$ the family of SUC-small sets for $X$ in $V$ is closed under taking subsets, norm closures, finite linear combinations, finite unions and convex hulls.

(2) If $M_n \subset V$ is SUC-small at $x_0 \in V^*$ for every $n \in \mathbb{N}$ then so is the set

$$\bigcap_{n \in \mathbb{N}} (M_n + \delta_n B_V)$$

for every positive decreasing sequence $\delta_n \to 0$.

(3) For every $\psi \in V^*$ the following are equivalent:

(i) The orbit map $\psi: G \to V^*$ is norm continuous.

(ii) $B$ is SUC-small at $\psi$, where

$$B := \{ \hat{v}: V^* \to \mathbb{R}, \quad x \mapsto \langle v, x \rangle \}_{v \in B_V}.$$

**Proof.** Assertion (1) is straightforward.

(2): We have to show that the set $\bigcap_{n \in \mathbb{N}} (M_n + \delta_n B_V)$ is SUC-small at $x_0$. Let $\varepsilon > 0$ be fixed. Since $Gx_0$ is a bounded subset of $V^*$ one can choose $n_0 \in \mathbb{N}$ such that $|v(gx_0)| < \frac{\varepsilon}{4}$ for every $g \in G$ and every $v \in \delta_n B_V$. Since $M_{n_0}$ is SUC-small at $x_0$ we can choose a neighborhood $U(e)$ such that $|m(ux_0) - m(x_0)| < \frac{\varepsilon}{4}$ for every $u \in U$ and every $m \in M_{n_0}$. Now every element $w \in \bigcap_{n \in \mathbb{N}} (M_n + \delta_n B_V)$ has a form $w = m + v$ for some $m \in M_{n_0}$ and $v \in \delta_n B_V$. Then for every $u \in U$ we have

$$|w(ux_0) - w(x_0)| \leq |m(ux_0) - m(x_0)| + |v(ux_0)| + |v(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

(3): Use that $\|w\psi - \psi\| = \sup_{v \in B_V} |\langle v, uw \rangle - \langle v, \psi \rangle|$ and $B_V$ is $G$-invariant. □

**Theorem 9.8.** The following conditions are equivalent:
(1) $f \in \text{SUC}(G)$.

(2) $f = m_{v, \psi}$ for some continuous Banach co-representation $h: G \to \text{Iso}(V)$, $v \in V$ and $\psi \in V^*$, with the property that $\varphi$ is norm $G$-continuous for every $\varphi$ in the weak star closure $\text{cl}_{w^*}(G\psi)$.

Moreover, one can assume in (2) that $\text{cl}_{w^*}(G\psi)$ separates points of $V$.

Proof. $(2) \implies (1)$: Let $h: G \to \text{Iso}(V)$ be a continuous co-homomorphism such that $f$ is a matrix coefficient of $h$. That is, we can choose $v \in V$ and $\psi \in V^*$ such that $f(g) = \langle vg, \psi \rangle = \langle v, g\psi \rangle$ for every $g \in G$. One can assume that $||\psi|| = 1$. The strong continuity of $h$ ensures that the dual restricted (left) action of $G$ on the weak star compact unit ball $(V_1^*; w^*)$ is jointly continuous. Consider the orbit closure $X := \text{cl}_{w^*}(G\psi)$ in the compact $G$-space $(V_1^*, w^*)$. Define the continuous function $\hat{v}: X \to \mathbb{R}$ induced by the vector $v$. Precisely, $\hat{v}(x) = \langle v, x \rangle$ and in particular, $f(g) = \hat{v}(g\psi)$. So $f$ comes from $X$ and the compactification $\nu: G \to X, g \mapsto g\psi$. It suffices to show that the $G$-system $X$ is SUC. Let $x_0$ be an arbitrary point in $X$ and let $w$ be an arbitrary vector in $V$. By the definition of the weak star topology and the corresponding uniformity on the compact space $X$ it suffices to show that for every $\varepsilon > 0$ there exists a neighborhood $U(\varepsilon)$ such that $|\hat{w}(gux_0) - \hat{w}(gx_0)| \leq \varepsilon$ for every $g \in G$ and every $u \in U$. By simple computations we get

$$|\hat{w}(gux_0) - \hat{w}(gx_0)| = |\langle w, gux_0 \rangle - \langle w, gx_0 \rangle| = |\langle wg, ux_0 \rangle - \langle wg, x_0 \rangle|$$

$$\qquad \qquad = |\langle wg, ux_0 - x_0 \rangle| \leq ||wg|| \cdot ||ux_0 - x_0||.$$  

Take into account that $||wg|| = ||w||$. Since $x_0 \in \text{cl}(G\psi)$, by our assumption the orbit map $\hat{x}_0: G \to V^*$ is norm continuous with respect to the dual action of $G$ on $V^*$. Therefore, given $\varepsilon > 0$, there exists a neighborhood $U$ of $\varepsilon$ in $G$ such that $||ux_0 - x_0|| < ||w||^{-1}\varepsilon$ for every $u \in U$. Thus, $|\hat{w}(gux_0) - \hat{w}(gx_0)| \leq \varepsilon$. This shows that $X$ is SUC. Hence, $f \in \text{SUC}(G)$.

$(1) \implies (2)$: Let $f \in \text{SUC}(X)$. Then it comes by Proposition 3.1.3 from a compact transitive $G$-system $X_f$ and the $G$-compactification $f_z: G \to X_f$. There exists $F := F_e \in C(X_f)$ such that $f = F \circ f_z$. By Lemma 9.6.2 we conclude that $FG$ is SUC-small for $X_f \subset C(X_f)^*$. Let $M := \text{co}(-FG \cup FG)$ be the convex hull of the symmetric set $-FG \cup FG$. Then $M$ is a convex symmetric bounded $G$-invariant subset in $C(X_f)$. By Lemma 9.7.1 we know that $M$ is also SUC-small for $X_f$.

For brevity of notation let $E$ denote the Banach space $C(X_f)$. Since $X_f$ is a compact $G$-space the natural right action of $G$ on $E = C(X_f)$ (by linear isometries) is continuous.

Consider the sequence $K_n := 2^nM + 2^{-n}B_E$, where $B_E$ is the unit ball of $E$. Since $M$ is convex and symmetric, we can apply the construction of [10] (we mostly use the presentation and the development given by Fabian in the book
Let $\| \|_n$ be the Minkowski's functional of the set $K_n$. That is,

$$\|v\|_n = \inf \{ \lambda > 0 \mid v \in \lambda K_n \}$$

Then $\| \|_n$ is a norm on $E$ equivalent to the given norm of $E$ for every $n \in \mathbb{N}$. For $v \in E$, let

$$N(v) := \left( \sum_{n=1}^{\infty} \|v\|_n^2 \right)^{1/2} \text{ and } V := \{ v \in E \mid N(v) < \infty \}$$

Denote by $j: V \hookrightarrow E$ the inclusion map. Then $(V, N)$ is a Banach space, $j: V \to E$ is a continuous linear injection and $M \subset j(B_V) = B_V$

Indeed, if $v \in M$ then $2^n v \in K_n$. Therefore, $\|v\|_n \leq 2^{-n}$ and $N(v)^2 \leq \sum_{n \in \mathbb{N}} 2^{-2n} < 1$.

By our construction $M$ and $B_E$ are $G$-invariant. This implies that the natural right action $V \times G \to V$, $(v, g) \mapsto vg$ is isometric, that is $N(vg) = N(v)$. Moreover, by the definition of the norm $N$ on $V$ (use the fact that the norm $\| \|_n$ on $E$ is equivalent to the given norm of $E$ for each $n \in \mathbb{N}$) we can show that this action is norm continuous. Therefore, the co-representation $h: G \to \text{Iso}(V)$, $h(g)(v) := vg$ on the Banach space $(V, N)$ is well defined and continuous.

Let $j^*: E^* \to V^*$ be the adjoint map of $j: V \to E$. Now our aim is to check the $G$-continuity of every vector $\varphi \in j^*(X_f) = cl_{w^*}(G\psi)$, where $\psi := j^*(z)$ and $z$ denotes the point $f_2(e) \in X_f$. By Lemma 9.7.3 we have to show that $B_V$ is SUC-small for $j^*(X_f)$.

Claim : $j(B_V) \subset \bigcap_{n \in \mathbb{N}} K_n = \bigcap_{n \in \mathbb{N}} (2^n M + 2^{-n} B_E)$.

Proof. The norms $\| \|_n$ on $E$ are equivalent to each other. It follows that if $v \in B_V$ then $\|v\|_n < 1$ for all $n \in \mathbb{N}$. That is, $v \in \lambda_n K_n$ for some $0 < \lambda_n < 1$ and $n \in \mathbb{N}$. By the construction $K_n$ is a convex subset containing the origin. This implies that $\lambda_n K_n \subset K_n$. Hence $j(v) = v \in K_n$ for every $n \in \mathbb{N}$. \hfill $\square$

Recall now that $FG$ is SUC-small for $X_f \subset C(X_f)^*$. By Lemma 9.7.1 we know that also $M := \text{co}(-FG \cup FG)$ is SUC-small for $X_f \subset C(X_f)^*$. Moreover by Lemma 9.7.2 we obtain that $A := \bigcap_{n \in \mathbb{N}} (2^n M + 2^{-n} B_E) \subset C(X_f)$ is SUC-small for $X_f \subset C(X_f)^*$. The linear continuous operator $j: V \to C(X_f)$ is a $G$-map. Then by Lemma 9.6.1 it follows that $j^{-1}(A) \subset V$ is SUC-small for $j^*(X_f) \subset V^*$. The same is true for $B_V$ because by the above claim we have $j(B_V) \subset A$ (and hence, $B_V \subset j^{-1}(A)$). That is $B_V$ is SUC-small for $j^*(X_f)$. Now Lemma 9.7.3 shows that the orbit map $\tilde{\varphi}: G \to V^*$ is $G$-continuous for every $\varphi \in j^*(X_f) = cl_{w^*}(G\psi)$. By our construction $F \in j(V)$ (because $F \in M \subset j(B_V)$). Since $j$ is injective the element $v := j^{-1}(F)$ is uniquely determined in $V$. We show that $f = m_{v, \psi}$.
for the co-representation \( h \). Using the equality \( F \circ \alpha_f = f \) and the fact that \( \alpha_f \)
is a \( G \)-map we get

\[
\langle Fg, z \rangle = F(g \alpha_f(e)) = (F \circ \alpha_f)(g) = f(g).
\]

On the other hand,

\[
m_{v,\psi}(g) = \langle vg, \psi \rangle = \langle j^{-1}(F)g, j^*(z) \rangle = \langle j(j^{-1}(F))g, z \rangle = \langle Fg, z \rangle.
\]

Hence, \( f = m_{v,\psi} \), as required. Therefore we have proved that (1) \( \iff \) (2).

Finally we show that one can assume in (2) that \( cl_{w^*}(G\psi) = j^*(X_f) \) separates points of \( V \). If \( v_1, v_2 \) are different elements in \( V \) then \( j(v_1) \neq j(v_2) \). Since \( X_f \) separates \( C(X_f) \) then \( \langle j(v_1), \phi \rangle \neq \langle j(v_2), \phi \rangle \) for some \( \phi \in X_f \). Now observe that \( \langle j(v), \phi \rangle = \langle v, j^*(\phi) \rangle \) for every \( v \in V \).

\[\square\]

**Corollary 9.9.** \( Adj(G) \subset SUC(G) \).

Next we show how one can characterize \( LE(G) \) in terms of matrix coefficients.

**Theorem 9.10.** The following conditions are equivalent:

1. \( f \in LE(G) \).
2. \( f = m_{v,\psi} \) for some continuous co-representation \( h: G \to Iso(V) \), \( v \in V \) and \( \psi \in V^* \), for a Banach space \( V \) with the property that the weak * and the norm topologies coincide on the orbit \( G\varphi \) of every \( \varphi \) in the weak * closure \( Y := cl_{w^*}(G\psi) \).

Moreover, one can assume in (2) that \( Y \) separates points of \( V \).

**Proof.** (2) \( \Rightarrow \) (1): By definition \( f \) comes from \( Y := cl_{w^*}(G\psi) \). Hence it suffices to show that \( Y \) is LE. Equivalently, we need to show that \( Y \) is orbitwise light (see [18, Lemma 5.8.2]). Let \( \mu_Y \) be the uniform structure on the compact space \( Y \). Denote by \( (\mu_Y)_G \) the corresponding uniform structure of uniform convergence inherited from \( Y^G \) (see [18]). We have to show that \( \text{top}(\mu_Y)|_{G\varphi} = \text{top}(\mu_Y)|_{G\varphi} \) for every \( \varphi \in Y \). Observe that the topology \( (\mu_Y)_G \) on the orbit \( G\varphi \) is weaker than the norm topology. Since the latter is the same as the weak star topology (that is, \( top(\mu_Y)|_{G\varphi} \)) we get that indeed \( top(\mu_Y)|_{G\varphi} = top(\mu_Y)|_{G\varphi} \).

(1) \( \Rightarrow \) (2): The proof uses again the interpolation technique of [10], as in Theorem 9.8. The proof is similar so we omit the details. However we provide necessary definition and two lemmas (Definition 9.11 and Lemmas 9.12 and 9.13). They play the role of Lemmas 9.6 and 9.7.

For every set \( M \) denote by \( \mathbb{R}^M \) the set of all real valued functions \( M \to \mathbb{R} \). The topologies of pointwise and uniform convergence on \( \mathbb{R}^M \) will be denoted by \( \tau_p \) and \( \tau_u \) respectively.

**Definition 9.11.** Let \( \langle , \rangle : Y \times X \to \mathbb{R} \) be a \( G \)-compatible map (as in Definition 9.5) and \( M \) be a nonempty subset of \( Y \). Denote by \( j: X \to \mathbb{R}^M \), \( j(x)(m) := \langle m, x \rangle \) the associated map.
We say that a subset $A$ of $X$ is $M$-light if the pointwise and uniform topologies coincide on $j(A) \subset \mathbb{R}^M$.

(2) $M$ is LE-small at $x_0 \in X$ if the orbit $Gx_0$ is $M$-light.

(3) $M$ is LE-small for $X$ if the orbit $Gx$ is $M$-light at every $x \in X$ (compare Theorem 7.9).

We are going to examine this definition in a particular case of the canonical bilinear map $V \times V^* \to \mathbb{R}$ which is $G$-compatible for every co-representation $h: G \to \text{Iso}(V)$.

We collect here some useful properties of LE-smallness.

**Lemma 9.12.**

(1) Let $Y_1 \times X_1 \to \mathbb{R}$ and $Y_2 \times X_2 \to \mathbb{R}$ be two $G$-compatible maps. Suppose that $\gamma_1: X_1 \to X_2$ and $\gamma_2: Y_2 \to Y_1$ are $G$-maps such that

$$\langle y, \gamma_1(x) \rangle = \langle \gamma_2(y), x \rangle \quad \forall \ (y, x) \in Y_2 \times X_1.$$  

Then for every nonempty subset $M \subset Y_2$ the subset $\gamma_1(M) \subset Y_1$ is LE-small at $x \in X_1$ if and only if $M$ is LE-small at $\gamma_1(x) \in X_2$.

(2) Let $X$ be a (not necessarily compact) $G$-space. If $f \in \text{LE}(X)$ then the subset $F_eG$ of $C(X_f)$ is LE-small for $X_f$ considered as a subset of $V^*$ where $V := C(X_f)$ (with respect to the canonical map $V \times V^* \to \mathbb{R}$ and the natural co-representation $G \to \text{Iso}(V)$).

**Proof.** (1): Similar to Lemma 9.6.1.

(2): Let $f \in \text{LE}(X)$. Then it comes by Proposition 3.1.3 from a compact $G$-system $X_f$ and the $G$-compactification $f_\sharp: X \to X_f$. As we know there exists $F_e \in C(X_f)$ such that $f = F_e \circ f_\sharp$. Theorem 7.9 implies that $X_f$ is LE and $F_e \in \text{LE}(X_f)$. By the same theorem, $X_f$ is orbitwise light in $\text{RUC}(G)$. This means that pointwise and norm topologies in $\text{RUC}(G)$ agree on every $G$-orbit in $X_f$. On the other hand it is straightforward to see that for the $G$-compatible map

$$F_eG \times X_f \to \mathbb{R}$$

(Definition 9.11 with $M := F_eG$) the corresponding pointwise topology $\tau_p$ on $X_f$ coincides with the pointwise topology inherited from $\text{RUC}(G)$ and the uniform topology $\tau_u$ on $X_f$ coincides with the norm topology of $\text{RUC}(G)$. \qed

**Lemma 9.13.** Let $h: G \to \text{Iso}(V)$ be a continuous co-representation.

(a) For every $X \subset V^*$ the family of LE-small sets for $X$ in $V$ is closed under taking: subsets, norm closures, finite linear combinations, finite unions and convex hulls.

(b) If $M_n \subset V$ is LE-small at $x_0 \in V^*$ for every $n \in \mathbb{N}$ then so is

$$\bigcap_{n \in \mathbb{N}} (M_n + \delta_n B_V)$$
for every positive decreasing sequence $\delta_n \to 0$.

(c) The following are equivalent:

(i) The pointwise and norm topologies agree on the $G$-orbit $Gx$ for every $x \in X \subset V^*$.

(ii) $B$ is LE-small for $X$, where

$$B := \{ \tilde{v} : V^* \to \mathbb{R}, x \mapsto \tilde{v}(x) := \langle v, x \rangle \} \subseteq B_V.$$ 

\[ \square \]

**Question 9.14.** Do we have $\text{adj}(G) = \text{SUC}(G)$ for every topological group $G$?

The question seems to be open even for abelian non-discrete $G$ (say $G = \mathbb{R}$).

The equivalent question for abelian $G$ is whether $\text{adj}(G) = \text{UC}(G)$? Also, what is the relation between the algebras $\text{LE}(G)$ and $\text{adj}(G)$?

### 10. Some conclusions about $H_+[0,1]$ and $\text{Iso}(U_1)$

From the reflexive triviality of $H_+[0,1]$ and results of Uspenskij about $\text{Iso}(U_1)$ Pestov deduces in [51, Corollary 1.4] the fact that the group $\text{Iso}(U_1)$ is also reflexively trivial. Using a similar idea and the matrix coefficient characterization of SUC and LE one can conclude that $\text{Iso}(U_1)$ is SUC-trivial and LE-trivial.

Recall the following results of Uspenskij.

**Theorem 10.1.** (Uspenskij [57, 59]) The group $\text{Iso}(U_1)$ is topologically simple and contains a copy of every second countable topological group (e.g., $H_+[0,1]$).

Lemma 10.2 below is a generalized version of Pestov’s observation. Of course it is important here that the corresponding property admits a reformulation in terms of Banach space representations, which is the case, for instance, for SUC and LE.

**Lemma 10.2.** Let $G_1$ be a topological subgroup of a group $G_2$. Suppose that $G_2$ is $G_1$-simple, in the sense that, every non-trivial normal subgroup $N$ in $G_2$ containing $G_1$ is necessarily dense in $G_2$. Then if $G_1$ is either: 1) SUC-trivial, 2) LE-trivial, 3) adjoint continuous trivial or 4) $K$-trivial (where $K$ is a class of Banach spaces) then the same is true for $G_2$.

**Proof.** We consider only the case of SUC. Other cases are similar (and even easier for (3) and (4)).

We use Theorem 9.8. Let $h : G_2 \to \text{Iso}(V)$ be a continuous co-representation where $Y := cl_{w^*}(G\psi)$ separates points of $V$ and $\varphi$ is norm $G_2$-continuous in $V^*$ for every $\varphi \in Y$. It is enough to show that any such co-representation of $G_2$ is trivial. By Theorem 9.8 this will show that $G_2$ is SUC-trivial. First observe that the restriction $h|_{G_1}$ of $h$ to $G_1$ is trivial. In fact, otherwise $vg \neq v$ for some $(v, g) \in V \times G_1$ and by our assumption there exists $\varphi \in Y$ such that $\varphi(v) \neq \varphi(vg)$. Then
the restriction $m_{v,\varphi}|_{G_1}$ of the corresponding matrix coefficient $m_{v,\varphi}: G_2 \to \mathbb{R}$ to $G_1$ is not constant. However, by Theorem 9.8, $m_{v,\varphi}|_{G_1} \in \text{SUC}(G_1)$, contradicting our assumption that $G_1$ is SUC-trivial. Therefore, $h|_{G_1}: G_1 \to \text{Iso}(V)$ is trivial. Hence $G_1$ is a subgroup of the normal closed subgroup $N := \ker(h)$ of $G_2$. Since $G_2$ is $G_1$-simple it follows that $N = G_2$. Hence $h$ is trivial. □

Note that if $G_2$ is topologically simple, that is $\{e\}$-simple, then it is $G_1$-simple for every subgroup.

The following theorem sums up some of our results concerning the topological groups $H_+[0,1]$ and $\text{Iso}(U_1)$.

**Theorem 10.3.** Let $G$ be one of the groups $H_+[0,1]$ or $\text{Iso}(U_1)$.

1. The compactifications $G^{\text{SUC}}, G^{\text{LE}}, G^{\text{Asp}}, G^{\text{WAP}}$ are trivial.
2. Every adjoint continuous (co)representation of the group $G$ is trivial.
3. Every continuous Asplund (co)representation of the group $G$ is trivial.
4. Every (co)representation $h: G \to \text{Iso}(V)$ on a separable Asplund space $V$ is trivial.
5. The algebra $\text{UC}(G)$ and the ambit $(G^{\text{UC}}, i(e))$ are not point universal. In particular, the map $i: G \to G^{\text{UC}}$ is not a right topological compactification of $G$.

**Proof.**

1: $H_+[0,1]$ is SUC-trivial by Theorem 8.3. By results of Uspenskij (see Theorem 10.1) the group $\text{Iso}(U_1)$ is topologically simple and also a universal second countable group. In particular it contains a copy $G_1$ of $H_+[0,1]$ as a topological subgroup. It follows that $G_2 := \text{Iso}(U_1)$ is $G_1$-simple. Applying Lemma 10.2 we conclude that $G_2 := \text{Iso}(U_1)$ is also SUC-trivial. The rest follows by the inclusions of Theorem 7.10.

2: Every adjoint continuous (co)representation of $H_+[0,1]$ must be trivial. Otherwise, by Theorem 9.8 (or Corollary 9.9), it contains a nonconstant SUC function. Now Lemma 10.2 implies that $\text{Iso}(U_1)$ is also adjoint continuous trivial.

3: By Theorem 6.5 every continuous Asplund (co)representation of $G$ is adjoint continuous. Now apply (2).

4: By a recent result of Rosendal and Solecki [54, Corollary 3] every homomorphism of $G = H_+[0,1]$ into a separable group is necessarily continuous. Combining this result and our assertion (3) we obtain a proof in the case of $G = H_+[0,1]$. The case of $G = \text{Iso}(U_1)$ now follows by using again the $H_+[0,1]$-simplicity of $\text{Iso}(U_1)$.

5: Take a non-constant uniformly continuous function on $G$ (such a function necessarily exists by Lemma 2.1). Since $\text{SUC}(G) = \{\text{constants}\}$ we get $\text{SUC}(G) \neq \text{UC}(G)$. Now Corollary 4.11 finishes the proof. □

By Theorems 10.1 and 10.3 we get
Corollary 10.4. Every second countable group $G_1$ is a subgroup of a Polish SUC-trivial group $G_2$.

However the following questions are open (see also [42]).

Question 10.5.

(1) Find a nontrivial Polish group which is SUC-trivial (Ref-trivial, Asp-trivial) but does not contain a subgroup topologically isomorphic to $H_+\{0,1\}$.

(2) Is the group $H(I^-)$ SUC-trivial (Ref-trivial, Asp-trivial)?

And, a closely related question (see Lemma 10.2):

(3) Is the group $G_2 := H(I^-)$, $G_1$-simple for a subgroup $G_1 < G_2$ where $G_1$ is a copy of either $H_+\{0,1\}$ or of $\text{Iso}(U_1)$?

Theorem 10.6. Let $G$ be an Asplund trivial (e.g. $H_+\{0,1\}$ or $\text{Iso}(U_1)$) group. Then every metrizable right topological semigroup compactification of $G$ is trivial.

Proof. By Theorem 10.3.1, $G^{\text{Asp}}$ is trivial, so that every RN transitive $G$-space is trivial. If $G \rightarrow S$ is a right topological semigroup compactification of $G$, then the natural induced $G$-space $(G,S)$ is isomorphic to its own enveloping semigroup. By a recent work [20], a metric dynamical system $(G,X)$ is RN iff its enveloping semigroup is metrizable. Now if $S$ is metrizable then it follows that the transitive system $(G,S)$ is RN and therefore trivial. □

Recall, in contrast, that for every topological group $G$ the algebra $\text{RUC}(G)$ separates points and closed subsets on $G$ and therefore the maximal right topological semigroup compactification $G \hookrightarrow G^{\text{RUC}}$ is faithful.

11. RELATIVE EXTREME AMENABILITY: SUC-FPP GROUPS

Recall that a topological group $G$ has the fixed point on compacta property (fpp) (or is extremely amenable) if every compact $G$-space $X$ has a fixed point. It is well known that locally compact extremely amenable groups are necessarily trivial (see for example [28]). Gromov and Milman [29] proved that the unitary group $U(H)$ is extremely amenable. Pestov has shown that the groups $H_+\{0,1\}$ and $\text{Iso}(U_1)$ are extremely amenable (see [49, 52] for more information).

Consider the following relativization.

Definition 11.1. Let $P$ be a class of compact $G$-spaces.

(1) A $G$-space $X$ is $P$-fpp (or is extremely $P$-amenable) if every $G$-compactification $X \rightarrow Y$ such that $Y$ is a member of $P$ has a fixed point.

(2) A topological group $G$ is $P$-fpp (or is extremely $P$-amenable) if the $G$-space $X := G$ is $P$-fpp or equivalently, if every $G$-space $Y$ in $P$ has a fixed point.

Taking $P$ as the collection of all compact flows we get extreme amenability. With the class $P$ of compact affine flows we recover amenability. When $P$ is taken
to be the collection of equicontinuous (that is, almost periodic) flows we obtain the old notion of minimal almost periodicity (MAP). Minimal almost periodicity was first studied by von Neumann and Wigner [46] who showed that $\text{PSL}(2, \mathbb{Q})$ has this property. See also Mitchell [45] and Berglund, Junghenn and Milnes [8].

**Lemma 11.2.** Let $P$ be a class of compact $G$-spaces which is preserved by isomorphisms, products subsystems and quotients. Let $\mathcal{P}$ and $X^P$ be as in Section 2. The following conditions are equivalent:

1. $G$ is $P$-fpp.
2. The compact $G$-space $G^P$ is $P$-fpp.
3. Any minimal compact $G$-space in $P$ is trivial.
4. For every $f \in \mathcal{P}$ the $G$-system $X_f$ has a fixed point.
5. The algebra $\mathcal{P}$ is extremely left amenable (that is it admits a multiplicative left invariant mean).

**Proof.** Clearly each of the conditions (1) and (3) implies all the others. Use the fact that the $G$-space $G^P$ is point universal to deduce that each of (2) and (5) implies (3). Finally (4) implies (2) because $G^P$ has a presentation as a subsystem of the product of all the $X_f$, $f \in \mathcal{P}$. Note that $f \in \mathcal{P}$ iff $X_f$ has property $P$ (see [18, Proposition 2.9.3]).

**Remark 11.3.**

1. The smaller the class $P$ is one expects the property of being $P$-fpp to be less restrictive; however even when one takes $P$ to be the class of equicontinuous $\mathbb{Z}$-spaces (that is, cascades) it is still an open question whether $P$-fpp, that is minimal almost periodicity, is equivalent to extreme amenability (see [16]).

2. A minimal compact $G$-space $X$ is LE iff $X$ is AP. It follows by the inclusions $\text{LE} \supset \text{RN}_{\text{app}} \supset \text{WAP} \supset \text{AP}$ (cf. also Theorem 7.10) that $G$ is minimally almost periodic iff $G$ is $P$-fpp for each of the following classes: WAP, RN$_{\text{app}}$ or LE.

Here we point out two examples of topological groups $G$ which are SUC-extremely amenable (equivalently, SUC-fpp) but not extremely amenable. In the next two sections we will show that $S_\infty$ as well as the group $H(C)$ of homeomorphisms of the Cantor set $C$ are also SUC-fpp (both groups are not extremely amenable). See Corollaries 12.3 and 14.4 below.

**Example 11.4.** For every $n \geq 2$ the simple Lie group $\text{SL}_n(\mathbb{R})$, being locally compact, is not extremely amenable. However it is SUC-extremely amenable. This follows easily from Corollary 7.15.
12. The Roelcke compactification of the group $S(\mathbb{N})$

Let $G = S(\mathbb{N})$ be the Polish topological group of all permutations of the set $\mathbb{N}$ of natural numbers (equipped with the topology of pointwise convergence). Consider the one point compactification $X^* = \mathbb{N} \cup \{\infty\}$ and the associated natural $G$ action $(G, X^*)$. For any subset $A \subset \mathbb{N}$ and an injection $\alpha : A \to \mathbb{N}$ let $p_\alpha$ be the map in $(X^*)^{X^*}$ defined by

$$p_\alpha(x) = \begin{cases} 
\alpha(x) & x \in A \\
\infty & \text{otherwise}
\end{cases}$$

We have the following simple claim.

**Claim 12.1.** The enveloping semigroup $E = E(G, X^*)$ of the $G$-system $(G, X^*)$ consists of the maps $\{p_\alpha : A \to \mathbb{Z}\}$ as above. Every element of $E$ is a continuous function so that by the Grothendieck-Ellis-Nerurkar theorem [12], the system $(G, X^*)$ is WAP.

**Proof.** Let $\pi_\nu$ be a net of elements of $S(\mathbb{N})$ with $p = \lim_\nu \pi_\nu$ in $E$. Let $A = \{n \in \mathbb{N} : p(n) \neq \infty\}$ and $\alpha(n) = p(n)$ for $n \in A$. Clearly $\alpha : A \to \mathbb{N}$ is an injection and $p = p_\alpha$.

Conversely given $A \subset \mathbb{N}$ and an injection $\alpha : A \to \mathbb{N}$ we construct a sequence $\pi_n$ of elements of $S(\mathbb{N})$ as follows. Let $A_n = A \cap [1, n]$ and $M_n = \max\{\alpha(i) : i \in A_n\}$. Next define an injection $\beta_n : [1, n] \to \mathbb{N}$ by

$$\beta_n(j) = \begin{cases} 
\alpha(j) & j \in A \\
\alpha(j) + M_n + n & \text{otherwise}
\end{cases}$$

Extending the injection $\beta_n$ to a permutation $\pi_n$ of $\mathbb{N}$, in an arbitrary way, we now observe that $p_\alpha = \lim_{n \to \infty} \pi_n$ in $E$. The last assertion is easily verified. □

**Theorem 12.2.**

(1) The two algebras $\text{UC}(G)$ and $\text{WAP}(G)$ coincide for $G = S(\mathbb{N})$.

(2) The universal WAP compactification $G^{\text{WAP}}$ of $G$ (and hence also $G^{\text{UC}}$), is isomorphic to $E = E(G, X^*)$. Thus the universal WAP (and Roelcke) compactification of $G$ is homeomorphic to the Cantor set.

**Proof.** Given $f \in \text{UC}(G)$ and an $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that — with $H = H(1, \ldots, k) = \{g \in G : g(j) = j, \forall 1 \leq j \leq k\}$ —

$$\sup_{u,v \in H} |f(ugv) - f(g)| < \varepsilon.$$

Set $\hat{f}(g) = \sup_{u,v \in H} f(ugv)$, then $\|\hat{f} - f\| \leq \varepsilon$. Clearly $\hat{f}$, being $H$-biinvariant, is both right and left uniformly continuous; i.e. $\hat{f} \in \text{UC}(G)$. Let

$$\mathbb{N}_k = \{(n_1, n_2, \ldots, n_k) : n_j \in \mathbb{N} \text{ are distinct}\} = \{\text{injections} : \{1, 2, \ldots, k\} \to \mathbb{N}\}$$
and let $G$ act on $\mathbb{N}_k^k$ by $g(n_1, n_2, \ldots, n_k) = (g^{-1}n_1, g^{-1}n_2, \ldots, g^{-1}n_k)$. The stability group of the point $(1, \ldots, k) \in \mathbb{N}_k^k$ is just $H$ and we can identify the discrete $G$-space $G/H$ with $\mathbb{N}_k^k$. Under this identification, to a function $f \in \text{UC}(G)$ which is right $H$-invariant (that is $f(gh) = f(g)$, $\forall g \in G, h \in H$), corresponds a bounded function $\omega_f \in \Omega_k = \mathbb{R}^{\mathbb{N}_k^k}$, namely
\[
\omega_f(n_1, n_2, \ldots, n_k) = f(g) \quad \text{iff} \quad g(j) = n_j, \quad \forall \ 1 \leq j \leq k.
\]
If we now assume that $f \in \text{UC}(G)$ is both right and left $H$-invariant (so that $f = \hat{f}$) then, as we will see below, $f$ and accordingly its corresponding $\omega_f$, admits only finitely many values, corresponding to the finitely many double $H$ cosets $\{HgH : g \in G\}$.

We set $Y_f = Y = \text{cl}\{g\omega_f : g \in G\} \subset \Omega_k = \mathbb{R}^{\mathbb{N}_k^k}$, where the closure is with respect to the pointwise convergence topology. $(G, Y_f)$ is a compact $G$-system which is isomorphic, via the identification $G/H \cong \mathbb{N}_k^k$, to $X_f \subset \mathbb{R}^G$. We will refer to elements of $\Omega_k = \mathbb{R}^{\mathbb{N}_k^k}$ as configurations. Consider first the case $k = 2$.

In the following figure we have a representation of the configuration $f = \omega_f = \omega_{1,2}$ and three other typical elements of $Y_f$. The configuration $\omega_{2,7} = \sigma\omega_{1,2}$ — where $\sigma$ is the permutation $\left(\frac{1}{2} \frac{7}{2} \right)$ — , admits seven values (the maximal number it can possibly have): “blank” at points $(m, n)$ with $m, n \notin \{2, 7\}$, the values $\bullet$ and $\ast$ at $(2, 7)$ and $(7, 2)$ respectively, and four more constant values on the two horizontal and two vertical lines. (The circled diagonal points $(2, 2)$ and $(7, 7)$ are by definition not in $\mathbb{N}_2^2$.) If we let $\pi_n$ be the permutation
\[
\pi_n(j) = \begin{cases} 
  j & j \notin \{1, n\} \\
  n & \text{for } j = 1 \\
  1 & \text{for } j = n 
\end{cases}
\]
and denote by $p = \lim_{n \to \infty} \pi_n$, the corresponding element of $E(G, X)$ then, e.g.
\[
\omega_{1,\infty} = p\omega_{1,2} = \lim_{n \to \infty} \pi_n\omega_{1,2} = \lim_{n \to \infty} \omega_{1,n}.
\]

The functions $\omega_{1,2}$, $\omega_{1,\infty}$, $\omega_{2,7}$ and $\omega_{2,\infty}$ in $\Omega_2$

Now it is not hard to see that the $G$ action on $Y$ naturally extends to an action of $E = E(G, X^*)$ on $Y$ where each $p \in E$ acts continuously. (Show that
the map \((a, b) \mapsto \omega_{a,b}\) is an isomorphism of \(G\)-systems from \(X^* \times X^* \setminus \Delta\) onto \(Y\), where \(\Delta = \{(n, n) : n \in \mathbb{N}\}\). It then follows that \(E = E(G, X^*)\) coincides with \(E(G, Y)\).

By the Grothendieck-Ellis theorem (see e.g. [17, Theorem 1.45]) these observations show that the \(G\)-space \(Y_f\) is WAP and therefore the function \(f\), which comes from \((G, Y_f)\), is a WAP function.

These considerations are easily seen to hold for any positive integer \(k\). For example, an easy calculation shows that for \(H_k = H(1, 2, \ldots, k)\) the number of double cosets \(\{H_k g H_k : g \in G\}\) is

\[
\sum_{j=0}^{k} \binom{k}{j} \frac{k!}{(k-j)!}
\]

Since the subgroups \(H_k = H(1, \ldots, k)\) form a basis for the topology at \(e\), as we have already seen, the union of the \(H_k\)-biinvariant functions for \(k = 2, 3, \ldots\) is dense in \(UC(G)\) and we conclude that indeed \(UC(G) = WAP(G)\).

Since for each \(H\)-biinvariant function \(f\) the enveloping semigroup of the dynamical system \((G, Y_f)\) is isomorphic to \(E(G, X^*)\) and since the Gelfand compactification of \(UC(G)\) is isomorphic to a subsystem of the direct product

\[
\prod \{Y_f : f \text{ is } H\text{-biinvariant for some } H(1, \ldots, k)\}
\]

we deduce that \(E(G, X^*)\) serves also as the enveloping semigroup of the universal dynamical system \(|WAP(G)| = |UC(G)|\). Finally, since \(|WAP(G)|\) is point universal we conclude (by Lemma 3.3) that \((G, |WAP(G)|)\) and \((G, E(G, X^*))\) are \(G\)-isomorphic.

**Corollary 12.3.** The Polish group \(G = S(\mathbb{N})\) is SUC-extremely amenable but not extremely amenable.

**Proof.** It was shown by Pestov [49] that \(G\) is not extremely amenable and the nontrivial universal minimal \(G\)-system was described in [25]. On the other hand the \(G\)-system \(G^{UC}\) described in Theorem 12.2 admits a unique minimal set which is a fixed point. Thus the SUC \(G\)-system \(G^{SUC}\), being \(G\)-isomorphic to \(G^{UC}\) (see Theorems 12.2 and 7.10.2), has a fixed point. \(\square\)

**13. The homeomorphisms group of the Cantor set**

In this section let \(C\) denote the classical Cantor set — i.e. the ternary subset of the interval \([0, 1]\). Thus \(C\) has the following representation:

\[
C = \bigcap_{n=0}^{\infty} I^n,
\]

where \(I^n = \bigcup_{j=1}^{2^n} I^n_j\), is the disjoint union of the \(2^n\) closed intervals obtained by removing from \(I = [0, 1]\) the appropriate \(2^n - 1\) open ‘middle third’ intervals. In
the sequel we will write \( I_j^m \) for the clopen subset \( I_j^m \cap C \) of \( C \). For each integer \( m \geq 1 \), \( \mathbf{I}^m = \{ I_j^m : 1 \leq j \leq 2^m \} \) denotes the basic partition of \( C \) into \( 2^m \) clopen “intervals”.

We let \( G = H(C) \) be the Polish group of homeomorphisms of \( C \) equipped with the topology of uniform convergence. For \( n \in \mathbb{N} \) we let

\[
H_n = \{ g \in G : gI_j^n = I_j^n, \; \forall 1 \leq j \leq 2^n \}.
\]

Each \( H_n \) is a clopen subgroup of \( G \) and we note that the system of clopen subgroups \( \{ H_n : n = 2, 3, \ldots \} \) forms a basis for the topology of \( G \) at the identity \( e \in G \).

For any fixed integer \( k \geq 1 \) consider the collection

\[
\mathcal{A}^k = \{ a = \{ A_1, A_2, \ldots, A_k \} : \text{a partition of } C \text{ into } k \text{ nonempty clopen sets} \}.
\]

In particular note that for \( k = 2^n \), \( \mathbf{I}^n \) is an element of \( \mathcal{A}^k \).

The discrete homogeneous space \( G/H_n \) can be identified with \( \mathcal{A}^k = \mathcal{A}^{2^n} \): an element \( gH_n \in G/H_n \) is uniquely determined by the partition

\[
a = \{ gI_j^n : 1 \leq j \leq 2^n = k \},
\]

and conversely to every partition \( a \in \mathcal{A}^k \) corresponds a coset \( gH_n \in G/H_n \). In fact, if \( a = \{ A_1, A_2, \ldots, A_k \} \) we can choose \( g \) to be any homeomorphism of \( C \) with \( A_j = gI_j^n \).

Thus for \( k = 2^n \) we have a parametrization of \( \mathcal{A}^k \) by the discrete homogeneous space \( G/H_n \).

Let

\[
\Omega^k = \mathbb{R}^{A^k} \cong \mathbb{R}^{G/H_n}.
\]

Via the quotient map \( G \to G/H_n, \; g \mapsto gH_n \), the Banach space \( \ell^\infty(\mathcal{A}^k) \) canonically embeds into the Banach space \( \text{RUC}(G) \) where the image consists of all the right \( H_n \)-invariant functions in \( \text{RUC}(G) \). Thus if \( f \in \text{RUC}(G) \) satisfies \( f(gh) = f(g) \) for all \( g \in G \) and \( h \in H_n \) then \( \omega_f(gI_1^n, \ldots, gI_k^n) = \omega_f(A_1, \ldots, A_k) = f(g) \), where \( A_j = gI_j^n \), is the corresponding configuration in \( \ell^\infty(\mathcal{A}^k) \).

We equip \( \Omega^k = \mathbb{R}^{A^k} \) with its product topology. The group \( G \) acts on the space \( \Omega^k \) as follows. For \( \omega \in \Omega^k \) and \( g \in G \) let

\[
g\omega(a) = \omega(g^{-1}A_1, g^{-1}A_2, \ldots, g^{-1}A_k),
\]

for any \( a = \{ A_1, A_2, \ldots, A_k \} \in \mathcal{A}^k \). Equivalently \( g\omega(g'H_n) = \omega(g^{-1}g'H_n) \), for every \( g'H_n \in G/H_n \). For each right \( H_n \)-invariant functions \( f \) in \( \text{RUC}(G) \) we denote the compact orbit closure of \( f = \omega_f \) in \( \Omega^k \) by \( Y_f \).

First let us consider the case \( n = 1 \), where \( k = 2 \),

\[
\mathcal{A}^2 = \{ a = \{ A, A^c \} : \text{a partition of } C \text{ into } 2 \text{ nonempty clopen sets} \},
\]

and

\[
H = H_1 = \{ g \in G : gI_j^1 = I_j^1, \; j = 1, 2 \}.
\]
Claim 13.1. There are exactly seven double cosets $HgH$, $g \in G$.

Proof. For a partition $(A, A^c) \in \Delta^2$ exactly one of the following five possibilities holds: (1) $A = I_1^1$, (2) $A = I_2^2$, (3) $A \subsetneq I_1^1$, (4) $A \supsetneq I_1^1$, (5) $A \supsetneq I_2^2$, (6) $A \supsetneq I_1^1$, (7) $A \cap I_1^1 \neq \emptyset \neq A \cap I_2^2$, and $A^c \cap I_1^1 \neq \emptyset \neq A^c \cap I_2^2$.

Clearly for any two partitions $(A, A^c), (B, B^c)$ we have $(B, B^c) = (hA, hA^c)$ for some $h \in H$ iff they belong to the same class. Our claim follows in view of the correspondence $G/H \cong \Delta^2$.

Define an element $\omega_f \in \Omega^2$ and the corresponding function $f \in \text{UC}(G)$ as follows:

$$\omega(A, A^c) = j \quad \text{if } (A, A^c) \text{ is of type } (j), \quad j = 1, \ldots, 7,$$

and $f(g) = \omega(gI_1^1, gI_2^2)$. Clearly $f$ is $H_1$-biinvariant and in particular an element of $\text{UC}(G)$. Let $X_f$ denote the (pointwise) orbit closure of $f$ in $\text{RUC}(G)$. Via the natural lift of $\Omega^2$ to $\text{RUC}(G)$ we can identify $X_f$ with $Y_f = \text{cl} \{g \omega_f : g \in G\} \subset \Omega^2$.

Next consider a sequence of homeomorphisms $h_n \in G$ satisfying the conditions

(i) $h_n(I_1^1) = (I_2^{2n})^c$,
(ii) $h_n((I_1^1)^c) = I_2^{2n}$ and
(iii) $h_n$ is order preserving.

It is then easy to check that the limit $\lim_{n \to \infty} h_n \omega_f = \omega_0$ exists in $\Omega^2$ where $\omega_0$ is defined by

$$\omega_0(A, A^c) = \begin{cases} 5 & \text{if } 0 \in A \\ 4 & \text{if } 0 \not\in A. \end{cases}$$

Now for any $g \in G$ we have

$$(g \cdot \omega_0)(A, A^c) = \omega_0(g^{-1}A, g^{-1}A^c) = \begin{cases} 5 & \text{if } g(0) \in A \\ 4 & \text{if } g(0) \not\in A. \end{cases}$$

For $x \in C$ set

$$(\omega_x)(A, A^c) = \begin{cases} 5 & \text{if } x \in A \\ 4 & \text{if } x \not\in A. \end{cases}$$

Then for $g \in G$ we have $g \omega_0 = \omega_g$. Moreover denoting $Y_0 = \text{cl} \{g \omega_0 : g \in G\} \subset \Omega^2$ we have $Y_0 = G \omega_0$ and the map $\phi : (G, C) \to (G, Y_0)$ defined by $\phi(x) = \omega_x$ is an isomorphism of $G$-spaces. We get the following lemma.

Lemma 13.2. Let $Y_0 = \text{cl} \{g \omega_0 : g \in G\}$ be the orbit closure of $\omega_0$ in $\Omega^2$, then the $G$-space $(G, Y_0)$ is isomorphic to $(G, C)$, the natural action of $G = H(C)$ on the Cantor set $C$.

Remark 13.3. An argument analogous to that of Lemma 13.2 will show that for every $n$ the number of $H_n$ double cosets is finite. As in the case of $S(\mathbb{N})$ in the previous section this shows the well known fact that $G = H(C)$ is Roelcke precompact (see [60]).
In contrast to Theorem 12.2 we obtain the following result.

**Theorem 13.4.** For $G = H(C)$ we have $\text{UC}(G) \not\subseteq \text{SUC}(G)$.

**Proof.** Consider the function

$$f_0(g) = \omega_0(g^{-1}I_1^1, g^{-1}I_2^1) = g\omega_0(I_1^1, I_2^1) = \omega g_0(I_1^1, I_2^1)$$

and let $h_n \in G$ be defined as above. Let $u_n$ be a sequence of elements of $G$ which converges to $e \in G$ and for which $h_n u_n 0 = 2/3$. Then, as $h_n 0 = 0$ for every $n$, we have

$$f_0(h_n) = \omega_0(h_n^{-1}I_1^1, h_n^{-1}I_2^1) = 5,$$

but as $h_n u_n 0 = 2/3$,

$$f_0(h_n u_n) = \omega_0(u_n^{-1}h_n^{-1}I_1^1, u_n^{-1}h_n^{-1}I_2^1) = \omega h_n u_n 0 (I_1^1, I_2^1) = 4.$$

Thus $f_0$ is not left uniformly continuous. Since $f_0 \in X_f \cong Y_0$, we conclude, by Theorem 4.12, that $f$ is not a SUC function. □

**Remark 13.5.** A similar argument will show that any two points $a, b \in C$ are SUC-proximal for the $G$-space $(G, C)$. Thus this $G$-space is SUC-trivial by Lemma 8.2. Letting $F : Y_0 \to \{4, 5\} \subset \mathbb{R}$ be the evaluation function $F(\omega) = \omega(I_1^1, I_2^1)$, we observe that

$$f_0(g) = \omega_0(g^{-1}I_1^1, g^{-1}I_2^1) = g\omega_0(I_1^1, I_2^1) = F(g\omega_0) = F(g\phi 0) = (F \circ \phi)(g 0).$$

Thus the function $f_0$ comes from the $G$ space $C$, via the continuous function $F \circ \phi: \mathbb{R} \to \mathbb{R}$ and the point $0 \in C$. This is another way of showing that $f_0$ and hence also $f$ are not SUC.

**Remark 13.6.** By Theorem 13.4 and Corollary 4.11 we obtain, in particular, that the algebra $\text{UC}(G)$ is not point universal and the corresponding Roelcke compactification $G \to G^{\text{UC}}$ is not a right topological semigroup compactification of $G$. The same is true for $G := H_+[0, 1]$ because $\text{UC}(G) \neq \text{SUC}(G)$. This follows from Theorem 8.3 and Lemma 2.1.

### 14. Topological 2-transitivity vs SUC

**Definition 14.1.** Let $\pi : G \times X \to X$ be a continuous action.

1. A point $x \in X$ is *transitive* if the $G$-orbit $Gx = \{gx : g \in G\}$ is dense in $X$. We denote by $\text{Trans}(X)$ the set of transitive points in $X$. The action is called *point transitive* when $\text{Trans}(X)$ is nonempty.

2. The action is called *topologically transitive* when the set $\{g \in G : gO_1 \cap O_2\}$ is nonempty for every pair of nonempty open subsets $O_1, O_2$ in $X$.

3. We say that the action on $X$ is *weakly mixing* when the diagonal action on $X^2 = X \times X$ is topologically transitive.
(4) A point \((x_1, x_2) \in X^2\) is 2-transitive if the \(G\)-orbit \(\{(gx_1, gx_2) : g \in G\}\) of \((x_1, x_2)\) is dense in \(X^2\). We denote by \(\text{Trans}_2(X)\) the set of 2-transitive points in \(X^2\).

(5) The action \(\pi\) is algebraically 2-transitive if the induced diagonal action \(G \times X^2 \setminus \Delta \to X^2 \setminus \Delta\) is algebraically transitive (one orbit).

When \(X\) is compact and metrizable and the action is topologically transitive, then the subset \(\text{Trans}(X)\) of all transitive points is an invariant dense \(G\)-subset of \(X\) (indeed, \(\text{Trans}(X) = \bigcap_{n \in \mathbb{N}}( \cup \{g^{-1}O_n : g \in G\})\) for every countable base \(\{O_n : n \in \mathbb{N}\}\) of \(X\)). Thus, in this case, \(X\) is topologically transitive if and only if \(X\) is point transitive. Therefore, if \(X\) is a compact metrizable \(G\)-flow, then \(X\) is weakly mixing if and only if \(\text{Trans}_2(X)\) is nonempty. Note that if \(\text{Trans}_2(X)\) is nonempty then it is a \(G\)-invariant dense subset of \(X\).

**Proposition 14.2.** Let \(\pi : G \times X \to X\) be a continuous action. Assume that there exists \(c_0 \in X\) such that:

1. its \(G\)-orbit \(Gc_0\) is dense in \(X\);
2. the orbit map \(G \to Gc_0\) is open;
3. \((Gc_0 \times Gc_0) \cap \text{Trans}_2(X)\) is dense in \(X \times X\).

Then the \(G\)-space \(X\) is SUC-trivial (hence, \(G\) has SUC-fpp).

**Proof.** It is enough to show that every \(f \in \text{SUC}(X)\) is constant. Assume to the contrary that \(f : X \to \mathbb{R}\) is a nonconstant SUC function. Then there exist: \(\varepsilon > 0\) and \(x_1, x_2 \in X\) such that

\[|f(x_1) - f(x_2)| > \varepsilon.\]

By the density assertion (3) and the continuity of \(f\), there exists \((w_1, w_2) \in (Gc_0 \times Gc_0) \cap \text{Trans}_2(X)\) which is sufficiently close to \((x_1, x_2)\) in \(X^2\) such that

\[|f(w_1) - f(w_2)| > \varepsilon.\]

By Lemma 4.5.4 there exists an open neighborhood \(U(e)\) in \(G\) such that \(U^{-1} = U\) and

\[|f(guc_0) - f(gc_0)| < \frac{\varepsilon}{2} \quad \forall (g, u) \in G \times U.\]

The triangle inequality ensures that

\[|f(gu_1c_0) - f(gu_2c_0)| < \varepsilon \quad \forall (g, u_1, u_2) \in G \times U \times U.\]

By condition (2) the image \(Uc_0\) is open in \(Gc_0\). Let \(O\) be an open subset of \(X\) such that \(O \cap Gc_0 = Uc_0\).

Since \((w_1, w_2) \in \text{Trans}_2(X)\), we can choose \(g_0 \in G\) such that

\[(g_0w_1, g_0w_2) \in O \times O.\]
Then, since \((w_1, w_2) \in Gc_0 \times Gc_0\), we have
\[(g_0 w_1, g_0 w_2) \in (O \cap Gc_0) \times (O \cap Gc_0) = Uc_0 \times Uc_0.\]
Therefore, there exist \(u_1, u_2 \in U\) such that \(w_1 = g_0^{-1}u_1c_0\) and \(w_2 = g_0^{-1}u_2c_0\) for some \(u_1, u_2 \in U\). Now, Equation 14.1 implies that
\[|f(w_1) - f(w_2)| < \varepsilon,\]
which contradicts the choice of \((w_1, w_2)\).

**Theorem 14.3.** Let \(G\) be a Polish topological group such that \(M(G)\) is metrizable and proximal. Then \(M(G)\) is SUC-trivial (hence, \(G\) has the SUC-fpp).

**Proof.** Every minimal proximal flow is weakly mixing by [15, Chapter 2, Cor. 2.2]. Hence, \(M(G)\) is weakly mixing. As we already mentioned, when \(M(G)\) is compact metric, it is weakly mixing if and only if there exists a point of transitivity for the diagonal action of \(G\) on \(M(G) \times M(G)\). Then, \(\text{Trans}_2(M(G))\) is a dense \(G_\delta\)-subset of \(M(G) \times M(G)\).

On the other hand, since the universal minimal \(G\)-flow \(M(G)\) is metrizable, by a criterion due to Ben Yaacov, Melleray and Tsankov [7] (which for nonarchimedean groups \(G\) previously was proved by Zucker [64]), there exists a generic point \(c_0 \in M(G)\). That is, we have a dense \(G\)-orbit \(Gc_0 \subseteq M(G)\) which is a \(G_\delta\)-subset of \(M(G)\). By Effros’ theorem, the orbit map \(G \to Gc_0, g \mapsto gc_0\) is open. Then, the set \(Gc_0 \times Gc_0\) and also \((Gc_0 \times Gc_0) \cap \text{Trans}_2(M(G))\) are dense \(G_\delta\)-subsets of \(M(G) \times M(G)\). Now Proposition 14.2 finishes the proof. \(\square\)

**Corollary 14.4.** Let \(G = H(C)\) be the Polish homeomorphism group of the Cantor set \(C\). Then \(M(G)\) is SUC-trivial (hence, \(G\) has the SUC-fpp).

**Proof.** By [26], \(M(H(C))\) is metrizable and proximal. \(\square\)

**Remark 14.5.** Corollary 14.4 was formulated in [19, Theorem 13.8]. However, its proof was not correct. We thank Lionel Van Thé for pointing out this error.

**Corollary 14.6.** Let \(G := \text{Aut}(\mathbb{Q}, \circ)\) be the Polish group of all circular order preserving permutations of \(\mathbb{Q}_0\) with the pointwise topology, where \(\mathbb{Q}_0\) is the rational discrete circle. Then \(M(G)\) is SUC-trivial.

**Proof.** By [22, Thm 5.2], \(M(G)\) is a circularly ordered metrizable compactum which we get from the circle after splitting its rational points. Also, \(M(G)\) is a proximal \(G\)-flow by [22, Thm 4.9]. \(\square\)

If the action \(G \times X \to X\) is algebraically 2-transitive and \(X\) is perfect (i.e., has no isolated points) then the action is topologically 2-transitive and hence Proposition 14.2 applies. For many concrete homogeneous metric compact perfect spaces \(X\) the natural action of the topological group \(G = H(X)\) on \(X\) is algebraically
2-transitive. By Proposition 14.2 the flow \((G, X)\) admits only constant SUC functions and the corresponding SUC \(G\)-compartmentation \(X^{\text{SUC}}\) is trivial. This is the case, to mention some concrete examples, for \(X\) the Cantor set, the Hilbert cube and the circle \(\mathbb{T}\).

In the latter case even the subgroup \(G := H_+ (\mathbb{T}) < H (\mathbb{T})\) of all orientation-preserving homeomorphisms of the circle acts algebraically 2-transitive on \(\mathbb{T}\). Pestov has shown [49, 52] that the universal minimal dynamical \(G\)-system \(M(G)\) for \(G := H_+ (\mathbb{T})\) coincides with the natural action of \(G\) on \(\mathbb{T}\).

Combining these results with Proposition 14.2 we obtain

**Corollary 14.7.** The Polish group \(G = H_+ (\mathbb{T})\) of orientation preserving homeomorphisms of the circle has the SUC-fpp (is SUC-extremely amenable) but it is not extremely amenable.

This can be proved also using Corollary 14.6 because there exists a (dense) injective continuous homomorphism \(\text{Aut} (\mathbb{Q}, \circ) \to H_+ (\mathbb{T})\).

An alternative proof follows easily from Proposition 5.1. Moreover, the following result is stronger than Proposition 5.1 and leads to an additional explanation of Theorem 14.3.

**Theorem 14.8.** Let \(X\) be a compact minimal \(G\)-space which contains a topologically transitive point \(x_0 \in X\) such that the orbit map \(G \to Gx_0\) is open.

1. If \(X\) is SUC then \(X\) is AE.
2. If \(X\) is SUC and minimal then \(X\) is equicontinuous.

**Proof.** (1): Since the orbit map \(G \to Gx_0\) is open, the orbit \(Gx_0\) can be identified with the coset \(G\)-space \(G/H\), with the stabilizer subgroup \(H = \text{st}(x_0)\). As in the proof of Proposition 5.1 we can verify that \(x_0\) is a point of equicontinuity of the action of \(G\) on \((G/H, \mu)\), where \(\mu\) is the \(\mu\) precompact uniformity on \(G/H\) inherited from the compact space \(X\). Then \(x_0\) is a point of equicontinuity also in \(cl(Gx_0) = X\) (Lemma 7.8.1). So, \(Eq(X)\) is nonempty. In particular, \(X\) is nonsensitive. By Lemma [18, Lemma 9.2.3], \(\text{Trans}(X) \subseteq Eq(X)\). Clearly, \(\text{Trans}(X)\) is dense in \(X\) (containing \(Gx_0\)). Therefore, \(Eq(X)\) is dense in \(X\). This means that \(X\) is AE.

(2) Follows from (1) because \(\text{Trans}(X) \subseteq Eq(X)\). By the minimality, \(X = \text{Trans}(X)\). Hence, \(X = Eq(X)\). \(\square\)

Note again that for Polish groups \(G\) with metrizable \(M(G)\) there exists a generic point \(x_0 \in M(G)\). For every \(G\)-factor \(q: M(G) \to Y\) the point \(q(x_0)\) is generic in \(Y\) (see [4, Prop. 14.1]). These results, together with Theorem 14.8 imply that every SUC \(G\)-factor of a proximal metric \(M(G)\) is trivial (being proximal and equicontinuous). This gives one more verification of Theorem 14.3.
Remark 14.9. The SUC-fpp for topological groups can be otherwise described as extremal SUC-amenability (extreme amenability in the domain of SUC flows). We can similarly define SUC-amenability as the property of having a fixed point in every affine SUC $G$-flow $Q$. Note that, the existence of a fixed point in $Q$ is equivalent to the existence of an equicontinuous compact $G$-subspace of $Q$ (see [21, Prop. 2.1]). For SIN topological groups SUC-amenability and amenability are equivalent (see Remark 4.4).

Proposition 14.10. Let $G$ be a Polish group with metrizable $M(G)$. Then $G$ is SUC-amenable.

Proof. Let $Q$ be a compact affine SUC $G$-system. It is enough to show that there exists an equicontinuous compact (minimal) $G$-subspace $X$ of $Q$. Choose any minimal $G$-subsystem $X$ of $Q$. Then $X$ is SUC. Since $M(G)$ is metrizable and $X$ is a $G$-factor of $M(G)$, using again [4, Prop. 14.1], there exists a point $x_0 \in X$ such that the orbit map $G \to Gx_0$ is open. Theorem 14.8 guarantees that $X$ is equicontinuous.

Proposition 14.11. Every Roelcke precompact Polish group $G$ is SUC-amenable.

Proof. By a result of Ibarlucia [32, Theorem 2.9] for every Roelcke precompact Polish group $G$ holds SUC($G$) = WAP($G$). Choose any minimal $G$-subsystem $X$ of $Q$ and $x_0 \in X$. Then $X$ can be treated as a $G$-compactification $\nu: G \to X, g \mapsto gx_0$. Since $X$ is SUC, the corresponding algebra $A_\nu$ of this compactification is a subalgebra of SUC($G$). By our assumption, SUC($G$) = WAP($G$). Hence, $X$ is WAP. Being minimal and WAP it is necessarily equicontinuous [41, Cor. 6.11].

Remark 14.12. Propositions 14.10 and 14.11 provide two sufficient conditions of SUC-amenability involving two important subclasses of Polish groups. These two classes are incomparable. Indeed, by Pestov’s result [52], the isometry group $G :=$ Iso($U$) of the Urysohn space $U$ is extremely amenable (hence, $M(G)$ is metrizable) but $G$ is not Roelcke precompact. On the other hand (answering a question from [44]), there exist Roelcke precompact Polish groups $G$ such that $M(G)$ is not metrizable (see, [13] and [33]).

Remark 14.13. There exist nonamenable Polish groups (with metrizable $M(G)$) which is SUC-amenable but not SUC-extremely amenable. Indeed, take for example, the product $G := H_+\mathbb{T} \times K$, where $K$ is a compact metrizable non-trivial group. Note that $M(G_1 \times G_2) = M(G_1) \times M(G_2)$ for Polish groups $G_1, G_2$ with metrizable $M(G_1), M(G_2)$ (see [6, Example 3.5]). Hence, $M(G) =$
\[ M(H_+(\mathbb{T})) \times K = T \times K. \] Now, observe that \( G \) is SUC-amenable (Proposition 14.10), nonamenable (use the fact that \( H_+(\mathbb{T}) \) is non-amenable), not SUC-extremely amenable (\( M(G) \) has a nontrivial SUC \( G \)-factor \( K \) without fixed points).

References

1. E. Akin, J. Auslander and K. Berg, *When is a transitive map chaotic*, Convergence in Ergodic Theory and Probability, Walter de Gruyter & Co. 1996, pp. 25-40.
2. E. Akin, J. Auslander and K. Berg, *Almost equicontinuity and the enveloping semigroup*, Topological dynamics and applications, Contemporary Mathematics 215, a volume in honor of R. Ellis, 1998, pp. 75-81.
3. E. Akin and E. Glasner, *Residual properties and almost equicontinuity*, J. d'Anal. Math. 84 (2001), 243-286.
4. O. Angel, A.S. Kechris and R. Lyons, *Random orderings and unique ergodicity of automorphism groups*, J. Eur. Math. Soc. 16, 2059–2095.
5. W. Banaszczyk, *Additive subgroups of topological vector spaces*, Lecture Notes in Math. 1466, Springer-Verlag, 1991.
6. G. Basso and A. Zucker. *Topological dynamics beyond Polish groups*. Comment. Math. Helv. 96 (2021), 589–630.
7. I. Ben Yaacov, J. Melleray, T. Tsankov, *Metrizable universal minimal flows of Polish groups have a comeagre orbit*, Geometric and Functional Analysis, 27 (2017), 67–77.
8. J.F. Berglund, H.D. Junghenn and P. Milnes, *Compact right topological semigroups and generalizations of almost periodicity*, Lecture Notes in Math. 663, Springer-Verlag, 1978.
9. R.D. Bourgin, *Geometric Aspects of Convex Sets with the Radon-Nikodým Property*, Lecture Notes in Math. 993, Springer-Verlag, 1983.
10. W.J. Davis, T. Figiel, W.B. Johnson and A. Pelczyński, *Factoring weakly compact operators*, J. of Funct. Anal. 17 (1974), 311-327.
11. R. Ellis, *Equicontinuity and almost periodic functions*, Proc. Amer. Math. Soc. 10, (1959), 637-643.
12. R. Ellis and M. Nerurkar, *Weakly almost periodic flows*, Trans. Amer. Math. Soc. 313 (1989), 103-119.
13. D.M. Evans, J. Hubička and J. Nešetřil, *Automorphism groups and Ramsey properties of sparse graphs*, Proc. London Math. Soc. (3) 119 (2019), 515–546.
14. M. Fabian, *Gateaux differentiability of convex functions and topology. Weak Asplund spaces*, Canadian Math. Soc. Series of Monographs and Advanced Texts, A Wiley-Interscience Publication, New York, 1997.
15. E. Glasner, *Proximal flows*, Lecture Notes in Mathematics, 517, Springer-Verlag, 1976.
16. E. Glasner, *On minimal actions of Polish groups*, Topology Appl. 85 (1998), 119-125.
17. E. Glasner, *Ergodic theory via joinings*, AMS, Surveys and Monographs, 101, 2003.
18. E. Glasner and M. Megrelishvili, *Linear representations of hereditarily non-sensitive dynamical systems*, Colloq. Math. 104 (2006), no. 2, 223–283.
19. E. Glasner and M. Megrelishvili, *New algebras of functions on topological groups arising from G-spaces*, Fundam. Math. 20 (2008), 1–51.
20. E. Glasner, M. Megrelishvili and V.V. Uspenskij, *On metrizable enveloping semigroups*, Israel Journal of Math. 164 (2008), 317–332.
21. E. Glasner and M. Megrelishvili, *On fixed point theorems and nonsensitivity*, Israel Journal of Math. 190 (2012), 289–305.
22. E. Glasner, M. Megrelishvili, *Circular orders, ultra-homogeneous order structures and their automorphism groups*, AMS Contemporary Mathematics book series volume 772 “Topology, Geometry, and Dynamics: Rokhlin-100” (ed.: A.M. Vershik, V.M. Buchstaber, A.V. Malyutin) 2021, pp. 133–154. ArXiv:1803.06583.
23. E. Glasner and B. Weiss, *Sensitive dependence on initial conditions*, Nonlinearity 6 (1993), 1067-1075.
24. E. Glasner and B. Weiss, *Locally equicontinuous dynamical systems*, Colloquium Mathematicum, part 2, 84/85 (2000), 345-361.
25. E. Glasner and B. Weiss, *Minimal actions of the group $S(\mathbb{Z})$ of permutations of the integers*, Geom. and Funct. Anal. 12 (2002), 964–988.
26. E. Glasner and B. Weiss, *The universal minimal system for the group of homeomorphisms of the Cantor set*, Fund. Math. 176 (2003), no. 3, 277–289.
27. E. Glasner and B. Weiss, *On the lattice of factors of the universal minimal flow of the group of homeomorphisms of the Cantor set*, arxiv preprint, January, 2025.
28. E. Granirer and A.T. Lau, *Invariant means on locally compact groups*, Ill. J. Math. 15 (1971), 249-257.
29. M. Gromov and V.D. Milman, *A topological application of the isoperimetric inequality*, Amer. J. Math. 105 (1983), 843-854.
30. A. Grothendieck, *Critères de compacité dans les espaces fonctionelles généraux*, Amer. J. Math. 74 (1952), 168-186.
31. W. Herer and J.P.R. Christensen, *On the existence of pathological submeasures and the construction of exotic topological groups*, Math. Ann. 213 (1975), 203-210.
32. T. Ibarlucia, *The dynamical hierarchy of Roelcke precompact Polish groups*, Israel J. Math., 215 (2016), 965–1009.
33. A. Kwiatkowska, *Universal minimal flows of generalized Ważewski dendrites*, The Journal of Symbolic Logic, (83) 2018, 1618–1631.
34. J.E. Jayne and C.A. Rogers, *Borel selectors for upper semicontinuous set-valued maps*, Acta Math. 155, (1985), 41-79.
35. M. Megrelishvili, *A Tychonoff $G$-space not admitting a compact $G$-extension or a $G$-linearization*, Russ. Math. Surv. 43:2 (1988), 177-178.
36. M. Megrelishvili, *Free topological $G$-groups*, New Zealand J. of Math. 25 (1996), 59-72.
37. M. Megrelishvili, *Fragmentability and continuity of semigroup actions*, Semigroup Forum 57 (1998), 101-126.
38. M. Megrelishvili, *Operator topologies and reflexive representability*, In: “Nuclear groups and Lie groups” Research and Exposition in Math. series, vol. 24, Heldermann Verlag Berlin, 2001, 197-225.
39. M. Megrelishvili, *Every semitopological semigroup compactification of the group $H_+\times [0,1]$ is trivial*, Semigroup Forum 63:3 (2001), 357-370.
40. M. Megrelishvili, *Reflexively but not unitarily representable topological groups*, Topology Proceedings 25 (2002), 615-625.
41. M. Megrelishvili, *Fragmentability and representations of flows*, Topology Proceedings 27:2 (2003), 497-544. See also: www.math.biu.ac.il/~megereli.
42. M. Megrelishvili, *Topological transformation groups: selected topics*, in: Open Problems In Topology II (Elliott Pearl, ed.), Elsevier Science, 2007, 423-438.
43. M. Megrelishvili, *Every topological group is a group retract of a minimal group*, Topology Appl. 155 (2008), 2105–2127.
44. J. Melleray, L. Nguyen Van Thé, and T. Tsankov, *Polish groups with metrizable universal minimal flows*, International Mathematics Research Notices, vol. 2016 (2016), 1285–1307.
45. T. Mitchell, *Function algebras, means and fixed points*, Trans. AMS 130 (1968), 117-126.
46. J. von Neumann and E.P. Wigner, *Minimally almost periodic groups, Ann. of Math. 41 (1940), 746-750.
47. E.C. Nummela, *On epimorphisms of topological groups*, Gen. Top. and its Appl. 9 (1978), 155-167.
48. V. Pestov, *Epimorphisms of Hausdorff groups by way of topological dynamics*, New Zealand J. of Math. 26 (1997), 257-262.
49. V. Pestov, *On free actions, minimal flows and a problem by Ellis*, Trans. Amer. Math. Soc. 350 (1998), 4149-4165.
50. V. Pestov, *Topological groups: where to from here?* Topology Proceedings 24 (1999), 421-502. http://arXiv.org/abs/math.GN/9910144.
51. V. Pestov, *The isometry group of the Urysohn space as a $\mathbf{L}$evy group*, Topology Appl. 154 (2007), 2173–2184.
52. V. Pestov, *Dynamics of infinite-dimensional groups. The Ramsey-Devoetzky-Milman phenomenon*, University Lecture Series, v. 40, AMS, Providence, 2006.
53. W. Roelcke and S. Dierolf, *Uniform structures on topological groups and their quotients*, Mc Graw-hill, New York, 1981.
54. C. Rosendal and S. Solecki, *Automatic Continuity of Homomorphisms and Fixed Point on Metric Compacta*, Israel J. Math. 162 (2007), 349-371.
55. S. Teleman, *Sur la représentation linéaire des groupes topologiques*, Ann. Sci. Ecole Norm. Sup. 74 (1957), 319-339.
56. V.V. Uspenskij, *The epimorphism problem for Hausdorff topological groups*, Topology Appl. 57 (1994), 287-294.
57. V.V. Uspenskij, *On subgroups of minimal topological groups*, V. V. Uspenskij, *On subgroups of minimal topological groups*, Topology Appl. 155, (2008), 1580–1606.
58. V.V. Uspenskij, *The Roelcke compactification of unitary groups*, in: Abelian groups, module theory, and topology, Proceedings in honor of Adalberto Orsatti's 60th birthday (D. Dikranjan, L. Salce, eds.), Lecture notes in pure and applied mathematics, Marcel Dekker, New York e.a. 201 (1998), 411-419.
59. V.V. Uspenskij, *Compactifications of topological groups*, Proceedings of the Ninth Prague Topological Symposium (Prague, August 19–25, 2001). Edited by Petr Simon. Published April 2002 by Topology Atlas (electronic publication). Pp. 331–346, arXiv:math.GN/0204144.
60. V.V. Uspenskij, *The Roelcke compactification of groups of homeomorphisms*, Topology Appl. 111 (2001), 195-205.
61. W.A. Veech, *Weakly almost periodic functions on semisimple Lie groups*, Monat. Math. 88, (1979), 55-68.
62. J. de Vries, *Equivariant embeddings of G-spaces*, in: J. Novak (ed.), *General Topology and its Relations to Modern Analysis and Algebra IV*, Part B, Prague, 1977, 485-493.
63. J. de Vries, *Elements of Topological Dynamics*, Kluwer Academic Publishers, 1993.
64. A. Zucker, *Topological dynamics of automorphism groups, ultrafilter combinatorics, and the generic point problem*. Trans. Amer. Math. Soc. 368 (2016), 6715–6740.

Department of Mathematics, Tel-Aviv University, Ramat Aviv, Israel
*Email address*: glasner@math.tau.ac.il
*URL*: http://www.math.tau.ac.il/~glasner

Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel
*Email address*: megereli@math.biu.ac.il
*URL*: http://www.math.biu.ac.il/~megereli