A BLOCK BIDIANOZATION METHOD FOR
FIXED-PRECISION LOW-RANK MATRIX APPROXIMATION∗
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Abstract. We present randUBV, a randomized algorithm for matrix sketching based on the block Lanczos bidiagonalization process. Given a matrix \( A \), it produces a low-rank approximation of the form \( UBV^T \), where \( U \) and \( V \) are approximately orthogonal and \( B \) is block bidiagonal. It is closely related to the randQB algorithms of Yu, Gu, and Li (2018) in that the entries of \( B \) are incrementally generated and the Frobenius norm approximation error may be efficiently estimated. Our algorithm is therefore suitable for the fixed-precision problem, and so is designed to terminate as soon as a user input error tolerance is reached. Numerical experiments suggest that the block Lanczos method can be competitive with or superior to algorithms that use power iteration, even when \( A \) has significant clusters of singular values.

Key words. randomized algorithm, low-rank matrix approximation, fixed-precision problem, block Lanczos

AMS subject classifications. 15A18, 15A23, 65F15, 65F30, 68W20

1. Introduction. In this paper we consider the problem of finding a quality low-rank approximation \( \tilde{A}_r \) to a given matrix \( A \in \mathbb{R}^{m \times n} \), where we assume that \( m \geq n \). In particular we consider the fixed-precision problem, where the desired truncation rank \( r \) is not known in advance, but we rather want to find the smallest possible \( r \) such that \( \|A - \tilde{A}_r\|_F < \tau \) for some tolerance \( \tau \).

The optimal approximation can be found by computing and truncating the SVD of \( A \), but when \( A \) is large this method may be impractically expensive. It is therefore increasingly common to use randomized techniques to find an approximation to the dominant subspace of \( A \): that is, to find an orthogonal matrix \( Q \in \mathbb{R}^{m \times r} \) so that [11]

\[
A \approx QB,
\]

where \( B \) is an \( r \times n \) matrix satisfying

\[
B = Q^T A.
\]

Two variants on this basic approach are randomized subspace iteration and randomized block Lanczos. Algorithms 1.1 and 1.2 present prototype code for each of these methods for the fixed-rank problem, where \( r \) is specified in advance.

Algorithm 1.1 Randomized Subspace Iteration (randQB)

Input: \( A \in \mathbb{R}^{m \times n} \), rank \( l \geq r \), power parameter \( p \geq 0 \)
Output: \( Q, B \)
1: \( \Omega = \text{randn}(n, l) \)
2: \( Q = \text{orth}((AA^T)^p A\Omega) \)
3: \( B = Q^T A \)

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Algorithm 1.2 Randomized Block Lanczos

Input: $A \in \mathbb{R}^{m \times n}$, block size $b$, number of iterations $q$ with $(q+1)b \geq r$
Output: $Q$, $B$

1: $\Omega = \text{randn}(n, b)$
2: $Q = \text{orth}([A\Omega, (AA^T)A\Omega, \ldots, (AA^T)^qA\Omega])$
3: $B = Q^TA$

Extensions of these algorithms to the fixed-precision problem make use of the fact that the matrices $Q$ and $B$ can be computed incrementally rather than all at once. The process can then be terminated once the desired error threshold has been reached, assuming the error can be efficiently computed or estimated. Algorithms for the fixed-precision problem have been proposed in [11, 17], and more recently by Yu, Gu, and Li in [33]. The latter algorithm, randQB_EI, is currently the foundation for the MATLAB function svdsketch [18].

The algorithms cited above all rely on subspace iteration rather than the block Lanczos method, despite the fact that Krylov subspace methods are “the classical prescription for obtaining a partial SVD” [11], as with svds in MATLAB. The reasons for this are discussed partially in [11], but we list a few considerations here:

- Subspace iteration is easier to parallelize and more generally easier to optimize for a given architecture.
- In situations where accessing $A$ is expensive, it may be desirable to design sketching algorithms that minimize the number of passes over $A$, such as in [11, 28, 32]. A pass-efficient version of randomized subspace iteration is presented in [33] as randQB_FP, but these techniques cannot be readily extended to Krylov subspace methods.
- The error analysis is simpler for randomized subspace iteration. In particular, the block Lanczos method converges much more slowly when the spectrum of $A$ has a cluster larger than the block size $b$. In fact, if $A$ has a singular value with multiplicity greater than $b$, then in exact arithmetic Algorithm 1.2 will recover at most $b$ of the associated singular vectors.

1.1. Contributions. Our main contribution is to propose randUBV, an algorithm that uses the block Lanczos method to solve the fixed precision problem. It yields a factorization of the form $UBV^T$, where $U$ and $V$ are approximately orthogonal and $B$ is block bidiagonal. It has at worst the same cost per iteration as randQB_EI when the latter is run with power parameter $p = 0$, and empirically tends to have better convergence properties. The three most prominent implementation issues that we must consider are reorthogonalization, deflation, and dealing with clusters, and we handle them in the following manner.

- We propose to use one-sided reorthogonalization, wherein $V$ is reorthogonalized but $U$ is not. This technique leads to considerable cost savings when $A$ is rectangular with $m \gg n$ without degrading the accuracy of the error estimate.
- Deflation occurs when the blocks produced by the block Lanczos method are rank-deficient or nearly so, and results in a reduction in the block size. Our implementation handles deflation in a way that naturally takes advantage of BLAS 3 operations, and employs a technique which augments the Krylov space in order to keep the block size constant. We will refer to this technique as reinflation.
Our algorithm takes no special action whatsoever to account for singular value clusters. Clusters of singular values may slow the convergence of the block Lanczos method, but thanks to roundoff error they cannot entirely prevent it. This observation is not novel, but we provide empirical evidence to support it.

Numerical experiments on synthetic and real data suggest that \texttt{randUBV} is generally competitive with its \texttt{randQB} relatives, at least on modestly sized problems.

### 1.2. Outline

The paper is organized as follows. In section 2, we review the background of \texttt{QB} algorithms for the fixed-precision background as well as the block Lanczos method. In section 3 we discuss several implementation details, including how to handle deflation, reorthogonalization, and singular value clusters. We present our main algorithm in section 4 and establish the accuracy of the error indicator. Our numerical experiments are in section 5, and section 6 offers our concluding remarks and some avenues for future exploration.

### 1.3. Notation

Matrices, vectors, integers, and scalars will be respectively denoted by \(A\), \(a\), \(a\), and \(\alpha\). We use \(\|A\|_F\) and \(\|A\|_2\) for the Frobenius norm and operator norm, and \(I_n\) for the \(n \times n\) identity matrix. We use MATLAB notation for matrix indices: i.e., \(A(i,j)\) and \(A(:,j)\) represent the \(i,j\) element and the \(j\)-th column of \(A\), respectively.

For the cost analysis of our algorithm we use the same notation as in \cite{17,33}: \(C_{mm}\) and \(C_{qr}\) will represent constants so that the cost of multiplying two dense matrices of sizes \(m \times n\) and \(n \times l\) is modeled as \(C_{mm}mnl\) and the cost of computing the QR factorization of an \(m \times n\) matrix with \(m \geq n\) is modeled as \(C_{qr}mn^2\).

### 2. Background

In this section we review the fixed-precision \texttt{QB} factorization algorithm \texttt{randQB\_EI} and the block Lanczos bidiagonalization process.

#### 2.1. A fixed-precision \texttt{QB} algorithm

In order to extend Algorithm 1.1 to the fixed-precision problem, Yu, Gu, and Li \cite{33} make use of two key ideas. First, for a given block size \(b\) the matrix \(\Omega\) can be generated \(b\) columns at a time rather than all at once, allowing the resulting factors \(Q\) and \(B\) to be generated incrementally rather than all at once. Second, since \(Q\) is orthogonal and \(B = Q^T A\), it follows \cite[Thm. 1]{33} that

\[
\|A - QB\|_F^2 = \|A - QQ^T A\|_F^2 = \|A\|_F^2 - \|QQ^T A\|_F^2 = \|A\|_F^2 - \|B\|_F^2.
\]

As long as \(Q\) is kept close to orthogonal, the Frobenius norm error can be efficiently estimated at each step simply by updating \(\|B\|_F\). It is therefore possible to compute the \texttt{QB} factorization and cheaply estimate its error without ever forming the matrix \(A - QB\) explicitly. The first idea is also used by the earlier algorithm \texttt{randQB\_b} by Martinsson and Voronin \cite{17}. The second allows \texttt{randQB\_EI} to estimate the error without forming \(A - QB\) explicitly, which lets it run significantly faster when \(A\) is sparse.

Algorithm 2.1 presents code for \texttt{randQB\_EI}, which in exact arithmetic will output the same \texttt{QB} factorization as \texttt{randQB} when run to the same rank. It is noted in \cite{11} that a stable implementation of Algorithm 1.1 should include a reorthogonalization step after each application of \(A\) or \(A^T\). The reorthogonalization step in Line 10 provides further stability.

Suppose that we run the algorithm with power parameter \(p = 0\) and stop after \(t\) iterations, and set \(l = tb\). Then assuming orthogonalization is done using unpivoted
Algorithm 2.1 Blocked randQB algorithm (randQB_EI)

Input: \( A \in \mathbb{R}^{m \times n} \), block size \( b \), power parameter \( p \geq 0 \), tolerance \( \tau \)
Output: \( Q, B \), such that \( \| A - QB \|_F < \tau \)

1. \( Q = [ ] \), \( B = [ ] \)
2. \( E = \| A \|_F^2 \)
3. for \( k = 1, 2, 3, \ldots \) do
4. \( \Omega_k = \text{randn}(n, b) \)
5. \( Q_k = \text{orth}(A\Omega_k - Q(B\Omega_k)) \)
6. for \( j = 1 : p \) do
7. \( Q_k = \text{orth}(A^TQ_k - B^T(Q^TQ_k)) \)
8. \( Q_k = \text{orth}(AQ_k - Q(BQ_k)) \)
9. end for
10. \( Q_k = \text{orth}(Q_k - Q(Q^TQ_k)) \)
11. \( B_k = Q_k^TA \)
12. \( Q = [Q, Q_k] \)
13. \( B = [B^T, B_k^T]^T \)
14. \( E = E - \| B_k \|_F^2 \)
15. if \( E < \tau^2 \) then stop
16. end for

QR, the runtime of randQB_EI can be approximated as

\[
T_{\text{randQB}_\text{EI}} \sim 2C_{mn}ml + \frac{1}{2}C_{mm}(3m + n)l^2 + \frac{2}{l}C_{qr}ml^2.
\]

By comparison, the cost of the fixed-rank algorithm randQB can be approximated as

\[
T_{\text{randQB}} \sim 2C_{mn}ml + C_{qr}ml^2.
\]

For general power parameter \( p \geq 0 \) and a fixed number of iterations \( t \), the runtime of randQB_EI will increase approximately proportionally to \( p + 1 \).

2.2. Block Lanczos bidiagonalization. Here we describe a block Lanczos method for reducing a matrix to block bidiagonal form. Since this method generalizes the single-vector algorithm by Golub and Kahan [6] commonly known as the Golub-Kahan-Lanczos process, we will abbreviate it as bGKL.

The bGKL process was introduced by Golub, Luk, and Overton [7] to find the largest singular values and associated singular vectors of a large and sparse matrix. Since then, it has been applied to both least squares problems [14, 27] and total least squares problems [2, 13] with multiple right-hand sides.

The bGKL process takes a matrix \( A \in \mathbb{R}^{m \times n} \) and orthogonal matrix \( V_1 \in \mathbb{R}^{n \times b} \), and after \( k \) steps produces orthogonal matrices \( U(k) = [U_1 \cdots U_k] \) and \( V(k+1) = [V_1 \cdots V_{k+1}] \) such that

\[
\text{Span} \left\{ U(k) \right\} = \text{Span} \left\{ AV_1, A(A^TA)V_1, \ldots, A(A^TA)^{k-1}V_1 \right\},
\]

\[
\text{Span} \left\{ V(k+1) \right\} = \text{Span} \left\{ V_1, (A^TA)V_1, \ldots, (A^TA)^kV_1 \right\}.
\]
Furthermore, it produces the $kb \times (k+1)b$ block bidiagonal matrix

\[
B_k = \begin{bmatrix}
R_1 & L_2 & \cdots & \cdots & L_k \\
& R_2 & \cdots & \cdots & R_k \\
& & \ddots & \ddots & L_{k+1} \\
& & & R_k & L_{k+1}
\end{bmatrix}
\]

so that at each step of the process the relations

\[
AV_{(k)} = U_{(k)}B_k(:,1:kb) \quad \text{and} \quad A^T U_{(k)} = V_{(k+1)}B_k^T
\]

are satisfied. Assuming no loss of rank, the blocks $R_k$ and $L_k$ are $b \times b$ upper and lower triangular, respectively.

**Algorithm 2.2** Block Lanczos bidiagonalization process (bGKL)

**Input:** $A \in \mathbb{R}^{m \times n}$, orthogonal $V_1 \in \mathbb{R}^{n \times b}$

1. $U_0 = 0; L_1 = 0$
2. for $k = 1, 2, \ldots$ do
3. \hspace{1em} $U_k R_k = \text{qr}(A V_k - U_{k-1} L_k)$
4. \hspace{1em} $V_{k+1} L_{k+1}^T = \text{qr}(A^T U_k - V_k R_k^T)$
5. end for

The basic outline of the process is given in Algorithm 2.2. We note that the original algorithm in [7] is organized so that $B_k$ is square at the end of each iteration, with trailing block $R_k$. Our current presentation more directly mimics the QB factorization, since $U_{(k)} B_k V_{(k+1)}^T = U_{(k)} U_{(k)}^T A$ by the second relation in (2.5). It follows that in exact arithmetic the identity

\[
\|A - U_{(k)} B_k V_{(k+1)}^T\|_F^2 = \|A\|_F^2 - \|B_k\|_F^2
\]

will hold, and so the bGKL process can be readily adapted to find a fixed-precision approximation to $A$.

Suppose that we stop the process after $t$ iterations and set $l = tb$. Treating the matrices $R_k$ and $L_k$ as dense, the runtime of the bGKL process is no greater than

\[
T_{\text{bGKL}} \sim 2C_{mn} mn l + C_{mn} n b^2 + C_{qr} n b^2
\]

It is not fair to compare this quantity to the cost of (2.2) because we have not yet accounted for the cost of reorthogonalization, which is necessary for stability. Nonetheless, it suggests that we may be able to obtain an algorithm based on bGKL that costs no more per iteration than randQB_EI with power parameter $p = 0$.

3. **Implementation details.** In this section we discuss how to handle several important issues in the implementation of our fixed-precision algorithm. The first is the matter of ensuring that $U_{(k)}$ and $V_{(k)}$ remain close to orthogonal. The second is deflation, which occurs when the blocks $R_k$ or $L_k$ are rank-deficient, and the third concerns the difficulty the Lanczos method encounters when $A$ has large singular value clusters.
3.1. Reorthogonalization. In exact arithmetic, the matrices $U(k)$ and $V(k)$ will be orthogonal. In practice, they will quickly lose orthogonality due to roundoff error, and so we must take additional steps to ensure that our algorithm remains practical.

For the single-vector case $b = 1$, Simon and Zha [26] observed that it may not be necessary to orthogonalize both $U(k)$ and $V(k)$ in order to obtain a good low-rank approximation. Instead, if $V(k)$ alone is kept close to orthogonal, then $U(k)B_kV(k+1)^T$ will remain a good approximation to $A$ regardless of the orthogonality of $U(k)$. Separately, experiments by Fong and Saunders [5] in the context of least-squares problems suggested that keeping $V(k)$ orthogonal to machine precision $\varepsilon_{\text{mach}}$ might be enough to keep $U(k)$ orthogonal to at least $O(\sqrt{\varepsilon_{\text{mach}}})$ at least until the least-squares solver terminated at a relative backward error of $\sqrt{\varepsilon_{\text{mach}}}$. For the sake of computational efficiency, we therefore choose to explicitly reorthogonalize $V(k)$ but not $U(k)$ (assuming that $m \geq n$).

Reorthogonalization can take up a significant portion of the runtime of our algorithm, particularly if $A$ is sparse. However, it is known for the Lanczos process that orthogonality is lost only in the direction of eigenvectors that have already converged [21]. Thus in a high-quality implementation, it should be possible to save time by orthogonalizing each $V_k$ against a smaller carefully chosen set of vectors rather than all of $V(k)$ (see [22, 9, 25] for a few such proposals). In our implementation, we orthogonize $V_k$ against all of $V(k-1)$ for simplicity. We note that even if $A$ is square, reorthogonalization will cost no more than the equivalent step in randQB_EI.

3.2. Deflation. In practice, the bGKL process may yield blocks $R_k$ or $L_k$ that are rank-deficient or nearly so. Here and with other block Krylov methods, it is typical to reduce the block size $b$ in response so that $R_k$ and $L_k$ retain full row rank and column rank, respectively. This process is known as deflation. For more background, we refer the reader to the survey paper by Gutknecht [10] and the references therein.

In the context of solving systems with multiple right-hand sides, Gutknecht stresses that deflation is highly desirable. Indeed, when solving a block system such as $AX = B$, it is precisely the dimension reduction resulting from deflation that gives block methods an advantage over methods that solve each right hand side separately. In this context, deflation might occur if $B$ is itself rank-deficient, or if $B$ has some notable rank structure in relation to the matrix $A$.

When running the bGKL process with a randomly chosen starting matrix $V_1$ (e.g., if $V_1 = \text{orth}(\Omega)$, where $\Omega$ is a standard Gaussian matrix), deflation will be less common. Nonetheless, a good implementation should be prepared for the possibility, and so we examine the details here. Various authors [2, 13, 12] have proposed computing the necessary QR factorizations by using Householder reflections without column pivoting, so that the factors $R_k$ and $L_k$ will have the form (e.g., for $b = 4$ and $\text{rank}(R_k) = 3$)

$$ R_k = \begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \\ 0 & 0 & \times \\ \end{bmatrix}.$$

(3.1)

Hnětynková et al. [13] call the resulting bidiagonal matrix $B_k$ a $b$-wedge shaped matrix.
Below we show an example of such a matrix for \( b = 3 \), taken from [13]:

\[
\begin{bmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
0 & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
0 & \times & \times & \times \\
\end{bmatrix}
\]

(3.2)

The zeros represent deflations, with those occurring in the matrices \( R_k \) called lower deflations and those in \( L_k \) called upper deflations. As Björck [2] notes, the recurrence in Lines 3–4 of Algorithm 2.2 still works in the presence of deflation. After \( b \) deflations, the bidiagonalization process will terminate.

3.2.1. Reinflation. In the context of matrix sketching, deflation is less desirable. Consider an extreme example where the columns of \( V_1 \) are right singular vectors of \( A \): the Lanczos process will terminate after a single iteration, returning an approximation of the form \( A \approx U_1 \Sigma V_1^T \). Termination at this point would likely be disastrous if we wish to solve the fixed-precision approximation problem.

As mentioned before, we do not expect deflation to be a common occurrence if \( V_1 \) is chosen randomly. However, if we do not make any further adjustments for deflation our algorithm would fail to converge on a test case as simple as \( A = I \). In order to make our method more robust, we use what we will call reinflation, wherein we replace any deflated vectors with new randomly drawn ones in order to keep the block size constant. Similar techniques have been used to prevent breakdown in the case of the nonsymmetric Lanczos process [30] and GMRES [23]. For example, if we obtain a QR factorization \( U_k R_k \) where \( R_k \) is the \( 4 \times 4 \) matrix shown in (3.1), we would replace the final column of \( U_k \) with a random Gaussian vector \( u' \) and orthogonalize it explicitly against \( U_{(k-1)} \) and the remaining vectors in \( U_k \). Reinflation for deflations in \( L_k \) would be done analogously.

We claim that the bidiagonalization process can then proceed as normal, and \( B \) will remain a banded matrix with bandwidth \( b \). With reinflation, the leading blocks of the wedge-shaped matrix from (3.2) would look as follows:

\[
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
0 & \times & \times & \times \\
0 & \otimes & \otimes & \otimes \\
\times' & \times' & \otimes & \otimes \\
\times' & \otimes & \otimes \\
\end{bmatrix}
\]

(3.3)

Elements marked \( \otimes \) are created as a result of the reinflation, and those marked \( \times' \) have been changed.

The following theorem shows that we can add any vector we like without disrupting the bidiagonalization process, so long as it is orthogonal to the previous ones.
Theorem 3.1. Let \( u' \) be any vector such that \( U^T_{(k)} u' = 0 \). Then \( V^T_{(k)} (A^T u') = 0 \).

Similarly, let \( v' \) be any vector such that \( V^T_{(k+1)} v' = 0 \). Then \( U^T_{(k)} (Av') = 0 \).

Proof. The first and second claim follow respectively from the relations \( AV_{(k)} = U_{(k)} B_k \) and \( A^T U_{(k)} = V_{(k)} B^T_k + V_{k+1} L^T_{k+1} \) in (2.5).

We observe that if reinflation is performed, the resulting spaces \( \text{Span} \{ U_{(k)} \} \) and \( \text{Span} \{ V_{(k)} \} \) will not be Krylov spaces, but will instead be the sum of multiple Krylov spaces with different dimensions. As of the time of writing we are not aware of any convergence results for this more general case.

3.2.2. Choosing a deflation tolerance. Since exact deflation is rare, we will in practice choose a tolerance \( \delta > 0 \) and employ deflation if we encounter an entry with absolute value less that \( \delta \). It would be reasonable to set \( \delta \) to be slightly larger than \( \epsilon_{\text{mach}} \|A\|_2 \), where \( \epsilon_{\text{mach}} \) is the machine precision.

3.2.3. BLAS 3 implementation. Ideally, we would like to use Householder reflections to compute a QR factorization such that in the event of deflation the R factor would have the form shown in (3.1). In theory, such a task is unproblematic and should cost no more than the standard unpivoted QR factorization. In practice, a naive implementation is unlikely to be competitive with more standard routines such as Matlab’s qr, even if it were written in a way that used blocking to take advantage of BLAS 3 operations.

In the absence of a high-quality implementation, we compute the QR factorization without pivoting using standard routines, then use Householder reflections to reduce the upper triangular factor to the desired form. Accounting for a single deflation in either case would cost \( O(mb) \) (or \( O(nb) \)), which for larger \( b \) will be small compared to the \( O(mb^2) \) cost of the QR factorization. Pseudocode for this procedure is provided in Algorithm (3.1), where code for the Householder function house_gen in line 10 can be found at [19]. We note that the Householder matrix \( H_u \) need not be formed explicitly.

3.3. Singular value clusters. It is known that if \( A \) has a singular value with multiplicity greater than the block size \( b \), then in exact arithmetic the block Lanczos process will recover at most \( b \) of those singular values. More generally, if \( A \) has a cluster of size greater than \( b \) then the approximate singular vectors recovered by the Lanczos process may converge slowly. This behavior stands in stark contrast to that of blocked subspace iteration methods such as randQB_EI, whose outputs do not in exact arithmetic depend on \( b \).

For the first situation—singular values with multiplicity greater than \( b \)—classical results tend to examine a restricted problem. Saad [24] noted that the Lanczos process would simply behave as though it were being performed on a restricted matrix \( A|_S \) whose eigenvalues had multiplicity at most \( b \). There was therefore “no loss of generality” in assuming that the eigenvalues of \( A \) had multiplicity bounded by \( b \) for the purpose of analyzing convergence rates.

Other more recent works have restricted their attention to the case where the cluster size was bounded by \( b \) [16], or where \( b \) was greater than or equal to the target rank \( k \) [20, 29, 4]. The analysis of Yuan, Gu, and Li [34] made an important advancement by allowing for cluster sizes (though not multiplicity) greater than \( b \), and showing that even within a large cluster the singular values will asymptotically converge quite quickly as the number of Lanczos iterations increases.

Emboldened by the latter result, we opt to take no extra action whatsoever to account for potential clusters. Even for cluster sizes greater than the block size, we
Algorithm 3.1 Deflated and Reinflated QR (reinfQR)

Input: Matrix $X \in \mathbb{R}^{m \times b}$, deflation tolerance $\delta$
Output: $Q$, $R$, number of deflations $d$

1: $QR = \text{qr}(X)$ \hspace{1cm} (Unpivoted QR)
2: $d = 0$ \hspace{1cm} (Deflation counter)
3: for $j = 1 : b$ do
4: $i = j - d$
5: $r = R(i : j, j)$
6: if $\|r\|_2 < \delta$ then
7: $R(i : j, j) = 0$
8: $d = d + 1$
9: else if $d \geq 1$ then
10: $u = \text{house\_gen}(r)$ \hspace{1cm} (Find the Householder vector)
11: $R(i : j, b) = H_u R(i : j, b)$ \hspace{1cm} (Apply Householder reflection $H_u$)
12: $Q(:, i : j) = Q(:, i : j) H_u$
13: end if
14: end for
15: if $d > 0$ then
16: $i = b - d$ \hspace{1cm} (Reinflation, if needed)
17: $\Omega = \text{randn}(m, d)$
18: $Q' = \text{orth}(\Omega - Q(:, 1 : i)(Q(:, 1 : i)^T \Omega))$
19: $Q(:, i + 1 : end) = Q'$
20: end if

can obtain a good approximation simply by increasing the number of Lanczos iterations. Our numerical experiments suggest that this decision is defensible: although we can construct synthetic examples for which randUBV is inferior to methods that use subspace iteration, our algorithm performs quite well on a real-world example with large clusters.

As it turns out, even singular values with multiplicity greater than $b$ are not fatal to the Lanczos process. Parlett [22] notes that due to roundoff error, even repeated eigenvectors will eventually be found. Simon and Zha [26] additionally note that the eigenvectors will not converge in consecutive order, as the Lanczos process will likely find several smaller eigenvalues of $A$ before it finds copies of the larger ones. What we should expect in practice is that a singular value of multiplicity greater than $b$ (or a cluster of comparable size) will delay convergence, but not prevent it entirely.

3.3.1. Adaptive block size. An alternate method for dealing with clusters was offered in [31] and explored further in [1, 35]: instead of keeping the block size constant, we may periodically increase $b$ in order to better approximate clusters. The rough idea would be to monitor the singular values of $B_k$, and to increase $b$ so that it remains larger than the largest cluster in $B_k$.

The cost of finding the singular values of $B_k$ will be tolerable as long as we exploit its banded structure. For the sake of keeping the implementation of our algorithm simple, we leave this extension for future exploration.

4. Fixed-precision algorithm. Algorithm 4.1 presents code for randUBV.

The cost is equal to the cost of bGKL plus the cost of reorthogonalizing $V_k$ in Line 16. Thus if we stop the process after $t$ iterations, set $l = tb$, and treat $R_k$ and $L_k$ as
Algorithm 4.1 Blocked Bidiagonalization algorithm (randUBV)

Input: $A \in \mathbb{R}^{m \times n}$, block size $b$, relative error $\tau$, deflation tolerance $\delta$

Output: $U, B, V$, such that $\|A - UBV^T\|_F < \tau$

1: $E = \|A\|_F^2$
2: $\Omega = \text{randn}(n, b)$
3: $V_1 = \text{qr}(\Omega)$
4: $U_1 = 0$
5: $L_1 = 0$
6: $V = V_1$
7: $U = U_1$
8: for $k = 1, 2, 3, \ldots$ do
9:   $U_kR_k = \text{reinfQR}(AV_k - U_{k-1}L_k, \delta)$
10:   if deflation then
11:     Orthogonalize the reinflated vectors against $U$
12:   end if
13:   $E = E - \|R_k\|_F^2$
14:   $V_{k+1} = A^TU_k - V_kR_k^T$
15:   $V_{k+1} = V_{k+1} - (V_{k+1}V_{k+1}^T)(k-1)C_{mm}mb^2$
16:   $L_{k+1} = \text{reinfQR}(V_{k+1}, \delta)$
17:   if deflation then
18:     Orthogonalize the reinflated vectors against $V$
19:   end if
20:   $V = [V, V_k]$
21:   $E = E - \|L_{k+1}\|_F^2$
22: if $E < \tau^2\|A\|_F^2$ then stop
23: end for

dense, the total runtime can be modeled as

$$T_{\text{randUBV}} \sim 2C_{mm}mn l + C_{mm}nl^2 + \frac{1}{t}C_{mm}(m + n)l^2 + \frac{1}{t}C_{qr}(m + n)l^2.$$

Comparing this quantity to (2.2), we see that randUBV requires fewer floating points operations than randQB_EI when run for the same number of iterations, even when the latter is run with power parameter $p = 0$. Furthermore, the cost of one-sided reorthogonalization is $C_{mm}nl^2$ rather than the $C_{mm}ml^2$ used by randQB_EI. We can therefore expect that if $A$ is sparse and $m \gg n$, randUBV may run significantly faster.

Since our focus is on the fixed-precision algorithm, however, different algorithms (and for randQB_EI, different power parameters $p$) will converge after different numbers of iterations. We must therefore consider not just the cost per iteration, but how quickly the approximations converge. We discuss this matter further along with the numerical experiments in section 5.

4.1. Approximation accuracy. It is noted in [33] that due to roundoff error, the computed value of $E = \|A\|_F^2 - \|B\|_F^2$ may be inaccurate when $E$ is very small. In order to estimate the error $E$ to within a relative tolerance of $\gamma$ (say, $\gamma = 1\%$), the authors suggest that the absolute accuracy tolerance $\tau$ for the QB factorization
should be set large enough to satisfy

\[ (4.2) \quad \tau > \sqrt{E} \geq \sqrt{\frac{\epsilon_{\text{mach}}}{\gamma}} \|A\|_F, \]

where \( \epsilon_{\text{mach}} \) is the machine precision. In short, the proposed method of error estimation cannot reliably estimate a relative error below \( 2\sqrt{\epsilon_{\text{mach}}} \).

We provide a similar analysis in order to account for deflations and loss of orthogonality of \( U_{(k)} \). In particular, we show that the accuracy of the error estimate depends only on the local orthogonality of \( U_{(k)} \), beginning with the following lemma.

**Lemma 4.1.** Assume that \( U_1, \ldots, U_k \) are all orthogonal, set \( U_0 = 0 \), and define

\[ (4.3) \quad \epsilon_k = \max_{i < k} \|U_{i-1}^T U_k\|_2. \]

Then with \( U_{(k)} \) and \( B_k \) defined as in the rest of this paper, we have that

\[ \|U_{(k)} B_k\|_F^2 = (1 + \theta) \|B_k\|_F^2, \quad |\theta| \leq \epsilon_k. \]

**Proof.** Since \( B_k \) is block bidiagonal, the squared Frobenius norm of the \( i \)-th block column of \( U_{(k)} B_k \) is given by

\[ (4.4) \quad \|U_{i-1} L_i + U_i R_i\|_F^2 = \|L_i\|_F^2 + \|R_i\|_F^2 + 2 \text{tr} R_i^T U_i^T U_{i-1} L_i. \]

The exception is the final column, where \( \|U_k L_{k+1}\|_F^2 = \|L_{k+1}\|_F^2 \) exactly. Then

\[
2 \text{tr} R_i^T U_i^T U_{i-1} L_i \leq 2 \|R_i^T U_i^T U_{i-1} L_i\|_F \leq 2 \|U_i^T U_{i-1}\|_2 \|R_i\|_F \|L_i\|_F \\
\leq 2 \epsilon_k \|R_i\|_F \|L_i\|_F \\
\leq \epsilon_k (\|R_i\|_F^2 + \|L_i\|_F^2),
\]

so the desired relative error bound holds for each block column. The main claim then follows by summing over the block columns.

Next, we observe that with one-sided reorthogonalization of \( V_{(k)} \) and in the absence of deflation, the first relation in \((2.5)\) will remain accurate to machine precision regardless of the orthogonality of \( U \) (as noted in \[26\], the second relation will not). In the presence of deflation, the first relation must be amended slightly. We rewrite it as

\[ (4.5) \quad AV_{(k)} = U_{(k)} B'_k + D_k, \]

where \( B'_k \) is shorthand for \( B_k(:, 1 : kb) \) and \( D_k \) is a matrix accounting for all lower deflations. The deflation matrix \( D_k \) will have \( d \) nonzero columns, where \( d \) is the total number of lower deflations. Each column will have 2-norm bounded by the deflation tolerance \( \delta \). Presumably, \( \delta \) will be somewhat larger than \( \epsilon_{\text{mach}} \|A\|_2 \).

We now show that the error estimate \( E = \|A\|_F^2 - \|B_k\|_F^2 \) will remain accurate. Assuming orthogonality of \( V_{(k+1)} \) and ignoring other roundoff errors of size \( O(\epsilon_{\text{mach}} \|A\|_F) \), we find that

\[ (4.6) \quad \|A - U_{(k)} B_k V_{(k+1)}^T\|_F^2 = \|A\|_F^2 + \|U_{(k)} B_k\|_F^2 - 2 \text{tr} AV_{(k+1)} B'_k U_{(k)}^T. \]
Examining the final term and using (4.5), we then have
\[
\text{tr } AV_{(k+1)}B_k^T U_k^T = \text{tr } (U_{(k+1)}B_{k+1} + [D_k, 0]) B_k^T U_k^T
= \|U_{(k)}B_k\|^2_F + \text{tr } U_k^T U_{k+1} R_{k+1} L_{k+1} + \text{tr } D_k B_k^T U_k^T.
\]
Assuming \(\|B_k\|_F \leq \|A\|_F\), the second term in the final line is bounded in magnitude by \(\varepsilon_{k+1}\|A\|^2_F\) and the third term by \(\|D_k\|_F \|U_{(k)}B_k\|_F\). Returning to (4.6) we conclude that
\[
\|A - U_{(k)}B_k V_{(k+1)}^T\|^2_F = \|A\|^2_F + \|U_{(k)}B_k\|^2_F - 2 \text{tr } AV_{(k+1)}B_k^T U_k^T
\leq \|A\|^2_F - \|U_{(k)}B_k\|^2_F + 2\varepsilon_{k+1}\|A\|^2_F + 2\|D_k\|_F \|U_{(k)}B_k\|_F
\leq E + 3\varepsilon_{k+1}\|A\|^2_F + 2\delta \sqrt{d}(1 + \varepsilon_k)\|A\|_F.
\]
Thus as long as local orthogonality is maintained for \(U_{(k)}\) and as long as the number of deflations is not too large, we can expect \(E\) to remain an accurate estimate of the Frobenius norm approximation error. We summarize this result in the form of the following theorem.

**Theorem 4.2.** Let \(E = \|A\|^2_F - \|B_k\|^2_F\), let \(\varepsilon_k\) be as defined in (4.3), let \(\delta\) be the deflation tolerance, and let \(d\) be the number of lower deflations performed. Assume that \(U_1, \ldots, U_k\) and \(V_{k+1}\) are each orthogonal. If \(E > 0\), then
\[
\|A - U_{(k)}B_k V_{(k+1)}^T\|^2_F \leq E + 3\varepsilon_{k+1}\|A\|^2_F + 2\delta \sqrt{d}(1 + \varepsilon_k)\|A\|_F + O(\varepsilon_{mach}\|A\|^2_F).
\]

**4.2. Postprocessing of \(B\).** Recall that our original goal for the fixed-precision problem was not just to find a factorization that satisfies the bound \(\|A - UBV\|^2_F < \tau\), but to find the factorization with the smallest rank that does so. In order to accomplish this, we may compute the SVD of \(B\) as \(B = U\Sigma V^T\), truncate it to the smallest rank \(r\) such that \(\|A - \hat{U}\Sigma_r\hat{V}_r^T\|^2_F < \tau\), then approximate the left and right singular vectors of \(A\) by \(\hat{U}\hat{U}_r\) and \(VV_r\). It should be noted that since \(B\) is a banded matrix, its SVD can be computed more efficiently than if \(B\) were dense. Algorithms for computing the SVD typically first reduce the matrix to bidiagonal form [6], and \(B\) can be efficiently reduced to this form using band reduction techniques as in [15].

This postprocessing step takes on additional importance when dealing with the block Lanczos method rather than subspace iteration. Where subspace iteration will yield a matrix \(B\) whose singular values are all decent approximations of the top singular values of \(A\), the factor \(B\) produced by the Lanczos method will contain approximations to the smallest singular values of \(A\) as well [8]. It is therefore possible that the matrix \(B\) produced by \texttt{randUBV}\ can be truncated significantly without diminishing the quality of the approximation.

In fact, if one has the goal of obtaining a factorization whose rank is as small as possible, we recommend setting the stopping tolerance \(\tau\) slightly smaller than the desired approximation tolerance (or similarly, running the algorithm for a few more iterations after the approximation tolerance has already been satisfied). Doing so will may significantly reduce the rank \(r\) of the truncated SVD, which will in turn pay dividends by reducing the cost of computing \(\hat{U}\hat{U}_r\) and \(VV_r\).

**5. Numerical experiments.** Here we report the results of numerical experiments on synthetic and real test cases. We will generally compare the cost and accuracy of \texttt{randUBV}\ and \texttt{randQBEI}\ with power iterations \(p = 0, 1, 2\) for the latter. We also observe the extent to which orthogonality is lost in \(U\), the frequency with
which deflation occurs, and the behavior of our algorithm on matrices with clustered singular values. All experiments were carried out in MATLAB 2020b on an 8-core Apple M1 CPU with 8GB RAM.

Fig. 5.1: Convergence tests. Left: the Frobenius norm approximation errors. Right: local orthogonality of $U_{(k)}$ is maintained better than global orthogonality.

5.1. Synthetic data. For our first set of test cases we created matrices of size $2000 \times 2000$ with the form $A = U\Sigma V^T$, where $U$ and $V$ were random orthogonal
matrices and $\Sigma$ was set in the following manner:

- (Matrix 1) Slow decay, in which $\sigma_j = 1/j^2$.
- (Matrix 2) Fast decay, in which $\sigma_j = \exp(-j/20)$.
- (Matrix 3) Step function decay, in which $\sigma_j = 10^{-0.6(j/30)-1}$. Here, each singular value of $A$ (except for the smallest) has multiplicity 30.

In all three cases, we ran the sketching algorithms to a maximum rank $k = 200$ using block size $b = 10$. The deflation tolerance was set at $\delta = 10^{-12} \sqrt{\|A\|_1 \|A\|_\infty}$, but we did not encounter deflation for any of the three cases.

Results are shown in Figure 5.1. In the first two test cases, the approximation error for randUBV was smaller than that of randQB_EI (with power parameter $p = 0$) for every iteration after the first. It lagged somewhat behind randQB_EI with $p = 1$ or $p = 2$, both of which were quite close to optimal. In the third case, where the singular values of $A$ were chosen to have multiplicity larger than the block size, randUBV lagged significantly behind even randQB_EI with $p = 0$. We note that algorithm randUBV did nonetheless converge, which would not have been possible in exact arithmetic.

As was expected, without explicit reorthogonalization the matrices $U(k)$ lost orthogonality fairly quickly. The loss of local orthogonality, as measured by $\varepsilon_k$ from (4.3), was notably smaller than the global loss $\|U^T(k)U(k) - I\|_2$. Based on the relations in (2.5), $U(k)$ should remain locally close to orthogonal even as global orthogonality is lost.

Finally, we offer a snapshot of the singular values of $B_{200}$ after the algorithms have terminated. Results for Matrices 1 and 3 are shown in Figure 5.2. We note that the leading singular values returned by randUBV are more accurate than those returned by randQB_EI with $p = 0$ and comparable to the cases $p = 1$ or $p = 2$. The smallest singular values for randUBV are much smaller than their randQB counterparts, which suggests that the rank of $B_k$ can be truncated without losing much approximation accuracy when running randUBV.

![Fig. 5.2: Singular values of $B_k$ after termination. Left: slow decay. Right: step function decay.](image)

5.1.1. Speed Comparison. For the second set of test cases, we compared the runtime of the sketching algorithms on two larger matrices: the first was a dense $4000 \times 4000$ matrix with random Gaussian entries, and the second a random $16000 \times 16000$ sparse matrix with approximately 1% of the entries nonzero. Using block size $b = 20$ for both cases, we ran the sketching algorithms to ranks $100, 200, \ldots, 1000$ for the dense matrix and $40, 80, \ldots, 400$ for the sparse matrix.
For each approximation rank, the average runtime over 10 trials is shown in Figure 5.3. For both test matrices, randUBV takes almost exactly the same amount of time as randQB_{EI} with \( p = 0 \). Using \( p = 1 \) takes approximately twice as long, and \( p = 2 \) takes approximately three times as long. Presumably, for both of these test matrices the cost of computing matrix products with \( A \) and \( A^T \) dominates the cost of the algorithm.

5.2. Real data. The final experiment was performed on two matrices drawn from real data. The first is a dense \( 3168 \times 4752 \) matrix, representing the grayscale image of a spruce pine. The second, \( \text{lp\_cre\_b} \), comes from a linear programming problem from the SuiteSparse collection \([3]\), and is a \( 9648 \times 77137 \) sparse matrix with 260,785 nonzero elements and at most 9 nonzero elements per column. This second matrix has several sizeable clusters of singular values: for example, \( \sigma_{268} \approx 71.10 \) and \( \sigma_{383} \approx 70.77 \). The median relative gap \((\sigma_k - \sigma_{k+1})/\sigma_{k+1}\) among the first 800 singular values is about \( 8.6 \times 10^{-5} \), and the smallest relative gap is about \( 2.3 \times 10^{-8} \). Prior to running the sketching algorithms, both matrices were transposed in order to have more rows than columns.

The procedure went as follows: in the first step, each sketching algorithm was run until the Frobenius norm approximation error dropped below a set tolerance \( \tau \). In the second step, the SVD of \( B \) was then computed and truncated as \( B_r = U_r \Sigma_r V_r^T \) to the smallest rank such that \( \|A - B_r\|_F \leq \tau \|A\|_F \), and the singular vectors of \( A \)
computed as \( \mathbf{U} \mathbf{U}^T \) and \( \mathbf{V} \mathbf{V}^T \) (or as \( \mathbf{Q} \mathbf{U} \) for rand\( \mathbf{Q} \mathbf{B} \_\mathbf{E} \_\mathbf{I} \)). We computed the SVD of \( \mathbf{B} \) in all cases by using \texttt{svd}, and did not take advantage of its banded structure in the case of rand\( \mathbf{U} \mathbf{B} \mathbf{V} \).

The time required for each of these two stages was recorded using \texttt{tic} and \texttt{toc}. For algorithm rand\( \mathbf{U} \mathbf{B} \mathbf{V} \), we also tried setting a slightly stricter stopping tolerance \( \tau_0 \) than the truncation tolerance \( \tau \), with the goal of obtaining better compression in the second step by running the Lanczos process longer in the first step.

| Method | \( \tau_0 \) | \( \tau \) | \( t_{\text{fac}} \) | \( t_{\text{svd}} \) | \( t_{\text{total}} \) | \( k \) | \( r \) |
|--------|---------------|---------------|----------------|----------------|----------------|----|----|
| SVD    | –             | 0.1           | –             | 9.42           | 9.42           | –  | 388|
| UBV    | 0.09          | 0.1           | 0.70          | 0.11           | 0.81           | 520 | 439|
| UBV    | 0.1           | 0.1           | 0.81          | 0.16           | 0.97           | 600 | 392|
| QB(\( P=0 \)) | 0.1 | 0.1 | 1.15 | 0.36 | 1.51 | 700 | 663|
| QB(\( P=1 \)) | 0.1 | 0.1 | 1.25 | 0.22 | 1.47 | 440 | 420|
| QB(\( P=2 \)) | 0.1 | 0.1 | 1.75 | 0.17 | 1.91 | 420 | 398|

Fig. 5.5: Results for image data with approximation tolerance \( \tau = 0.1 \).

### 5.2.1. Image data
For the image data, we ran all algorithms to a relative error of \( \tau = 0.1 \) with block size \( b = 20 \), and for UBV considered the stricter stopping tolerance \( \tau_0 = 0.09 \). In both cases, we found that the \( \mathbf{U} \) factor from the UBV factorization satisfied \( \| \mathbf{U}^T \mathbf{U} - \mathbf{I} \|_2 \leq 10^{-12} \), which is quite satisfactory.

Results are shown in Figure 5.5, with all time reported in seconds. There, \( t_{\text{fac}} \) is the time required for the QB or UBV factorization, \( t_{\text{svd}} \) is the time required to compute the SVD of \( \mathbf{B} \) and the new singular vectors of \( \mathbf{A} \), and \( t_{\text{total}} = t_{\text{fac}} + t_{\text{svd}} \). Finally, \( k \) is the rank at which the algorithm was terminated, and \( r \) the rank to which \( \mathbf{B} \) was truncated. The first line represents the time required to directly compute the SVD of \( \mathbf{A} \) and the optimal truncation rank.

We observe that rand\( \mathbf{U} \mathbf{B} \mathbf{V} \) ran significantly faster than rand\( \mathbf{Q} \mathbf{B} \_\mathbf{E} \_\mathbf{I} \) regardless of the value of the power parameter \( p \). Even though it required more iterations to converge than rand\( \mathbf{Q} \mathbf{B} \_\mathbf{E} \_\mathbf{I} \) with \( p = 1 \) or \( p = 2 \), it required fewer matrix products with \( \mathbf{A} \) or \( \mathbf{A}^T \) per iteration. Furthermore, running rand\( \mathbf{U} \mathbf{B} \mathbf{V} \) to a stopping tolerance that was slightly smaller than the truncation tolerance took somewhat longer but resulted in nearly optimal compression, even superior to subspace iteration with \( p = 2 \).

### 5.2.2. SuiteSparse data
For the matrix \texttt{lp\_cre\_b} from the SuiteSparse collection, we ran two trials. In the first, we ran all algorithms to the rather modest relative error of \( \tau = 0.5 \), and for UBV considered the stricter stopping tolerance \( \tau_0 = 0.45 \). In the second, we ran the algorithms to the stricter relative error of \( \tau = 0.15 \), and for UBV additionally considered \( \tau_0 = 0.14 \). We used block size \( b = 50 \) for both trials. In the first trial, the \( \mathbf{U} \) factor from the UBV factorization satisfied \( \| \mathbf{U}^T \mathbf{U}^T - \mathbf{I} \|_2 \leq 10^{-13} \), and in the second trial it satisfied \( \| \mathbf{U}^T \mathbf{U} - \mathbf{I} \|_2 \leq 10^{-12} \).
Results are shown in Figures 5.6 and 5.7, with all time reported in seconds. Due to the size of the matrix $A$, we did not attempt to compute its SVD directly but instead found the optimal truncation rank using the precomputed singular values available online [3].

Once again, randUBV ran faster than its subspace-iteration-based counterpart, and using a slightly smaller stopping tolerance $\tau_0$ improved the compression ratio without increasing the runtime by too much. The iteration $k$ at which randUBV terminated was significantly smaller than it was for randQB_EI with $p = 0$, but significantly larger than for randQB_EI with $p = 1$ or $p = 2$ (perhaps in part due to the singular value clusters).

It should be noted that the matrix $A$ in question is quite sparse with only about 0.03% of its entries nonzero, and fairly skinny with $m \approx 8n$. It is therefore worth exploring whether randQB_EI might save time on reorthogonalization costs if performed on $A^T$ instead. We re-ran the experiment for $\tau = 0.15$, and found that while the factorization time $t_{\text{fac}}$ did not change much, the second step $t_{\text{svd}}$ took around twice as long due to the matrix $B$ being $k \times m$ rather than $k \times n$.

6. Conclusions. We have proposed a randomized algorithm randUBV that takes a matrix $A$ and uses the block Lanzos method to find an approximation of the form $UBV^T$, where $U$ and $V$ are approximately orthogonal and $B$ is a banded block bidiagonal matrix. For square matrices it costs approximately the same per iteration as randUBV-type methods run with power parameter $p = 0$ while having better convergence properties. On rectangular matrices, it exploits one-sided reorthogonalization to run faster without much degrading the accuracy of the error estimator. We have also proposed a method to handle deflation using BLAS 3 operations without decreasing the block size. Numerical experiments suggest that randUBV is generally competitive with existing randUBV-type methods, at least as long as the problem is not so large
that it becomes important to minimize the number of passes over \( \mathbf{A} \).

A few avenues for future exploration are suggested. First and most importantly, roundoff error allows block Lanczos methods to handle repeated singular values, which they would be unable to do in exact arithmetic. The general reason for this has long been known—“rounding errors introduce components in all directions”, as Parlett [22] succinctly puts it—but we are not currently aware of any rigorous convergence bounds that account for finite precision. Second, reinflation or any more general method for adaptively changing the block size \( b \) will make the span of \( \mathbf{V} \) a sum of Krylov spaces of different dimensions. We are not aware of any convergence results that cover this more general setting. It is also worth examining how \texttt{randUBV} may be effectively parallelized, or whether it even remains competitive with \texttt{randQB} methods in a parallel setting. Smaller block sizes generally allow Lanczos methods to converge faster, but larger block sizes are more suitable for parallel computing.

Finally, we recall that the matrix \( \mathbf{U} \) computed by \texttt{randUBV} remained closer to orthogonal on the real test cases than it did on the synthetic ones. We conjecture that this is due to the fact that the real test cases were run to more modest stopping thresholds \( \tau \), since it is more generally known that loss of orthogonality happens only in the direction of the singular vectors that correspond to singular values that have already converged. It would be highly desirable to obtain a rigorous result establishing that one-sided reorthogonalization is safe as long as only a rough approximation is required, but we leave this goal for a future study.

MATLAB code is available at https://github.com/erhallma/randUBV, including our main algorithm \texttt{randUBV} as well as code used to reproduce the figures used in this paper.

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