Compactification and decompactification by weights on Bergman spaces

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Abstract. We characterize the symbols $\varphi$ for which there exists a weight $w$ such that the weighted composition operator $M_wC_{\varphi}$ is compact on the weighted Bergman space $\mathcal{B}_\alpha^2$. We also characterize the symbols for which there exists a weight $w$ such that $M_wC_{\varphi}$ is bounded but not compact. We also investigate when there exists $w$ such that $M_wC_{\varphi}$ is Hilbert-Schmidt on $\mathcal{B}_\alpha^2$.

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1 Introduction

It is known (see [4] for instance) that “weightening” a composition operator $C_{\varphi}$ on the Hardy space $H^2$ by some weight $w$, we can improve its compactness properties, and even its membership in Schatten classes $S_p$, or the decay of its approximation numbers ([9, Theorem 2.3], [12]), or at the opposite make a compact composition operator non compact ([12]).

In this paper, we consider weighted composition operators $M_wC_{\varphi}$ on the weighted Bergman spaces $\mathcal{B}_\alpha^2$, with $\alpha > -1$. Note that for such an operator to be bounded from $\mathcal{B}_\alpha^2$ into itself, it is necessary that $w \in \mathcal{B}_\alpha^2$ (since $w = (M_wC_{\varphi})(1)$).

We show in Section 3 that $C_{\varphi}$ can be weighted to become compact on $\mathcal{B}_\alpha^2$ if and only if the set where $\varphi$ has an angular derivative has null measure.

In Section 4, we show that the exists a weight $w$ such that $M_wC_{\varphi}$ is bounded but not compact on $\mathcal{B}_\alpha^2$ if and only if $\|\varphi\|_\infty = 1$.

In Section 5, we study when $M_wC_{\varphi}$ can be Hilbert-Schmidt on $\mathcal{B}_\alpha^2$ for some weight $w$. 
2 Notation and background

The weighted Bergman space $\mathcal{B}_\alpha^2$, with $\alpha > -1$, is the space of all analytic functions $f: D \to \mathbb{C}$ on the unit disk $D$ such that

$$\|f\|_{\mathcal{B}_\alpha^2}^2 = (\alpha + 1) \int_D |f(z)|^2 (1 - |z|^2)^\alpha \, dA(z) < \infty,$$

where $A$ is the normalized area measure on $D$. When $\alpha = 0$, we write simply $\mathcal{B}^2$ instead of $\mathcal{B}_0^2$ and call it the Bergman space.

Every analytic self-map $\varphi: D \to D$ defines a bounded composition operator $C_\varphi: f \mapsto f \circ \varphi$ from $\mathcal{B}_\alpha^2$ into itself ([16, Proposition 3.4]).

The pull-back measure $A_\varphi$ of $\varphi$ is defined as:

$$A_\varphi(B) = A[\varphi^{-1}(B)] \quad \text{for all Borel sets } B \subseteq D.$$

Let $\mu$ be a finite Borel measure on $D$. For $\beta > 1$, the measure $\mu$ is said a $\beta$-Carleson measure if:

(2.1) \[ \sup_{|\xi|=1} \mu[S(\xi,h)] = O(h^\beta), \]

where

$$S(\xi,h) = \{z \in D; \ |z - \xi| < h\}$$

is the Carleson box of size $h$ centered at $\xi \in \mathbb{T} = \partial D$. The measure $\mu$ is said a vanishing $\beta$-Carleson measure if:

(2.2) \[ \sup_{|\xi|=1} \mu[S(\xi,h)] = o(h^\beta) \quad \text{as } h \to 0. \]

Recall the following result (see [7] and [16 Theorem 4.3]).

**Theorem 2.1.** Let $\mu$ be a finite Borel measure on $D$. Then:

(a) $\mathcal{B}_\alpha^2 \subseteq L^2(\mu)$ if and only if $\mu$ is an $(\alpha + 2)$-Carleson measure.

Moreover, when this happens, the canonical inclusion $J_\mu: \mathcal{B}_\alpha^2 \to L^2(\mu)$ is bounded.

(b) The canonical inclusion $J_\mu: \mathcal{B}_\alpha^2 \to L^2(\mu)$ is compact if and only if $\mu$ is a vanishing $(\alpha + 2)$-Carleson measure.

**Corollary 2.2.** Let $\varphi: D \to D$ be an analytic self-map and $w \in \mathcal{B}_\alpha^2$. Set, for every Borel set $B$ in $D$:

(2.3) \[ \mu_{w,\varphi}(B) = \int_{\varphi^{-1}(B)} |w(z)|^2 (1 - |z|^2)^\alpha \, dA(z). \]

Then:
(a) The weighted composition operator $M_wC_\varphi: \mathcal{B}_\alpha^2 \to \mathcal{B}_\alpha^2$, defined as:

$$(2.4) \quad (M_wC_\varphi)f = w(f \circ \varphi),$$

is bounded if and only if $\mu_{w,\varphi}$ is an $(\alpha + 2)$-Carleson measure.

(b) The weighted composition operator $M_wC_\varphi: \mathcal{B}_\alpha^2 \to \mathcal{B}_\alpha^2$ is compact if and only if $\mu_{w,\varphi}$ is a vanishing $(\alpha + 2)$-Carleson measure.

**Proof.** Observe that, for all $f \in \mathcal{B}_\alpha^2$, we have

$$\| (M_wC_\varphi)f \|_{\mathcal{B}_\alpha^2}^2 = \int_D |f(\varphi(z))|^2 |w(z)|^2 (1 - |z|^2)\alpha \, dA(z) = \| f \|_{L^2(\mu_{w,\varphi})}^2.$$

3 Compactification

Recall the following definitions (see [20, Section 4.1]).

**Definition 3.1.** A holomorphic self-map $\varphi: \mathbb{D} \to \mathbb{D}$ has an angular limit (or a non-tangential limit) $l$ at $\xi \in \mathbb{T}$ if $\varphi(z)$ converges to $l$ whenever $z$ tends to $\xi$ inside any angular sector in $\mathbb{D}$ whose vertex is $\xi$. Then $l$ is called the angular limit of $\varphi$ at $\xi$ and is denoted:

$$l = \angle \lim_{z \to \xi} \varphi(z).$$

**Definition 3.2.** A holomorphic self-map $\varphi: \mathbb{D} \to \mathbb{D}$ has an angular derivative at $\xi \in \mathbb{T}$ if it has an angular limit $\zeta$ at $\xi$, with $|\zeta| = 1$ and:

$$\angle \lim_{z \to \xi} \frac{\varphi(z) - \zeta}{z - \xi}$$

exists and is finite. This limit is called the angular derivative of $\varphi$ at $\xi$ and is denoted by $\varphi'(\xi)$.

Let us also recall that the Julia-Carathéodory theorem (see [20, Section 4.2]), says that $\varphi$ has an angular derivative at $\xi \in \mathbb{T}$ if and only if:

$$\delta := \liminf_{z \to \xi} \left| \frac{1 - |\varphi(z)|}{1 - |z|} \right| < +\infty,$$

or, equivalently:

$$\limsup_{z \to \xi} \left| \frac{1 - |z|}{1 - |\varphi(z)|} \right| > 0,$$

and, when this happens, we have $\delta > 0$ and $\varphi'(\xi) = \xi \bar{\zeta} \delta$, so $|\varphi'(\xi)| = \delta$.

We define

$$\mathcal{AD}(\varphi) = \{ \xi \in \mathbb{T}; \varphi \text{ has an angular derivative at } \xi \}.$$
and we call it the angular derivative set of \( \varphi \).

B. MacCluer and J. Shapiro proved (\cite[Theorem 3.5]{16}) that, for \( \alpha > -1 \), the composition operator \( C_\varphi : B^2_\alpha \to B^2_\alpha \) is compact if and only if:

\[
(3.4) \quad AD(\varphi) = \emptyset.
\]

Asking for a compactification, we have the following result.

**Theorem 3.3.** Let \( \varphi : D \to D \) be an analytic self-map. Then the following assertions are equivalent, for the weighted Bergman space \( B^2_\alpha \), with \( \alpha > -1 \):

1) there exists a holomorphic function \( w \), with \( w \not\equiv 0 \), such that the weighted composition operator \( M_w C_\varphi : B^2_\alpha \to B^2_\alpha \) is compact;
2) there exists a weight \( w \in H^\infty \), with \( w \not\equiv 0 \), such that the weighted composition operator \( M_w C_\varphi : B^2_\alpha \to B^2_\alpha \) is compact;
3) the angular derivative set of \( \varphi \) has null measure:

\[
(3.5) \quad m[AD(\varphi)] = 0,
\]

where \( m \) is the normalized Lebesgue measure on \( \mathbb{T} = \partial D \);
4) \( \lim_{z \to \xi} \frac{1 - |z|}{1 - |\varphi(z)|} = 0 \) for almost all \( \xi \in \mathbb{T} \).

For example, if \( \varphi(z) = \frac{1 + z}{2} \), then \( C_\varphi \) is not compact on \( B^2_\alpha \), but it is compactifiable by a weight in \( H^\infty \).

When the equivalent conditions of Theorem 3.3 are satisfied, we say that composition operator \( C_\varphi \) is compactifiable.

The proof will be based on the following result of Moorhouse (\cite[Corollary 1]{17}; see also \cite[Proposition 1]{2}).

**Proposition 3.4** (Moorhouse). Let \( \alpha > -1 \). Let \( \varphi \) and \( w \) be analytic functions on \( D \). Then:

1) If the weighted composition operator \( M_w C_\varphi \) is compact on \( B^2_\alpha \), we have:

\[
(3.6) \quad \lim_{|z| \to 1} |w(z)|^2 \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} = 0.
\]

2) When \( w \) is bounded, \( M_w C_\varphi \) is compact on \( B^2_\alpha \) if and only if

\[
(3.7) \quad \lim_{|z| \to 1} |w(z)|^2 \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.
\]

For 1), we compute:

\[
\| (M_w C_\varphi)^\ast (k_z) \|^2_{(B^2_\alpha)^\ast} = |w(z)|^2 \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2},
\]
where $k_z$ is the normalized reproducing kernel of $\mathcal{B}_\alpha^2$, and, using that $k_z$ weakly converges to 0 as $|z| \to 1$, we obtain:

$$\lim_{|z| \to 1} |w(z)|^2 \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} = 0.$$ 

To obtain the necessary condition in 2), we use the following, easily checked, fact which shows that (3.7) is equivalent to (3.6), since $w$ is bounded in 2).

**Lemma 3.5.** Let $f, g: \mathbb{D} \to [0, \infty)$ be two bounded functions. Then the following assertions are equivalent:

a) $\lim_{|z| \to 1} f(z) g(z) = 0$;

b) $\lim_{|z| \to 1} \min[f(z), g(z)] = 0$;

c) $\lim_{|z| \to 1} [f(z)]^a [g(z)]^b = 0$, for all $a, b > 0$.

The sufficient condition in 2) is proved by [17, Lemma 1].

**Proof of Theorem**

The implication 2) $\Rightarrow$ 1) needs no comment.

3) $\Rightarrow$ 2) Assume that $m[\mathcal{A}\mathcal{D}(\varphi)] = 0$. A theorem of Privalov (see [21] Vol. I, bottom of page 276), asserts the existence a function $w \not\equiv 0$ in $H^\infty$ such that

$$(3.8) \quad \lim_{z \to \xi} w(z) = 0 \quad \text{for all } \xi \in \mathcal{A}\mathcal{D}(\varphi).$$

The Schwarz-Pick lemma (see [11] Corollary 2.40}) tells that

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \leq 2 \quad \frac{1 - |z|}{1 - |\varphi(z)|} \leq 2 \quad \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}.$$

Hence for $\xi \in \mathcal{A}\mathcal{D}(\varphi)$, we have

$$\lim_{z \to \xi} |w(z)|^2 \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

For $\xi \not\in \mathcal{A}\mathcal{D}(\varphi)$, thanks to the Julia-Carathéodory theorem and [36,2], we also have

$$\lim_{z \to \xi} |w(z)|^2 \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

Hence

$$(3.9) \quad \lim_{z \to \xi} |w(z)|^2 \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0 \quad \text{for all } \xi \in \mathbb{T}.$$

By a compactness argument, we obtain that

$$(3.10) \quad \lim_{|z| \to 1} |w(z)|^2 \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0;$$
in fact, if (3.10) failed, there would be a sequence \((z_n)\) such that \(|z_n| \to 1\) and for which

\[
\limsup_{n \to \infty} |w(z_n)|^2 \frac{1 - |z_n|^2}{1 - |\varphi(z_n)|^2} > 0;
\]

by compactness a subsequence converges to some \(\xi \in \partial \mathbb{D}\), and that would contradict (3.9).

Since \(w\) is bounded, it follows from Proposition 3.4 that \(M_w C_\varphi\) is compact on \(B^2\).

1) \(\Rightarrow\) 3) Assume that \(M_w C_\varphi\) is compact with \(w\) analytic and \(w \not\equiv 0\). By Proposition 3.4, we have:

\[
(3.11) \quad \lim_{|z| \to 1} |w(z)|^2 \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha + 2} = 0.
\]

In particular, for every \(\xi \in T\):

\[
(3.12) \quad \lim_{z \to \xi} |w(z)|^2 \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha + 2} = 0.
\]

Now, for every \(\xi \in AD(\varphi)\), we have, if \(\zeta\) is the angular limit of \(\varphi\) at \(\xi\):

\[
\lim_{z \to \xi} \frac{\varphi(z) - \zeta}{z - \xi} = |\varphi'(\xi)| < \infty.
\]

If \(z\) belongs to an angular sector \(S_\xi\) of vertex \(\xi\), there is a positive constant \(C\), depending only on this sector, such that \(|z - \xi| \leq C(1 - |z|)\); hence

\[
\liminf_{z \to \xi, z \in S_\xi} \frac{1 - |z|}{1 - |\varphi(z)|} \geq \liminf_{z \to \xi, z \in S_\xi} C \frac{|z - \xi|}{|\varphi(z) - \zeta|} = C \frac{|\varphi'(\xi)|}{|\varphi'(\xi)|} > 0.
\]

Then, it follows, with (3.12), that \(\lim_{z \to \xi, z \in S_\xi} w(z) = 0\). Since the angular sector \(S_\xi\) is arbitrary, we get that \(\angle \lim_{z \to \xi} w(z) = 0\).

By another theorem of Privalov (see [21, Chapter XIV, Theorem 1.1 and Theorem 1.9]), [6, Chapter VI, Theorem 2.3], or [5, Chapter II, Exercise 10], where it is called “local Fatou theorem”, it follows, since \(w \not\equiv 0\), that \(m[AD(\varphi)] = 0\).

3) \(\iff\) 4) follows from the Julia-Caratheodory theorem, as stated in (3.2).

**Remark 1.** The implication 2) \(\Rightarrow\) 3) can be proved using the classical F. and M. Riesz theorem (see [4, Theorem 2.2]) instead of Privalov’s theorem.

**Remark 2.** Condition (3.11) is necessary for the compactness of \(M_w C_\varphi\); however, it is not sufficient in general without this assumption that \(w \in H^\infty\). An example is given in [2] Section 5, Corollary 4 for which \(M_w C_\varphi\), with \(w = \varphi'\), is not even bounded on \(B^2\).
4 Decompactification

4.1 The main result

In the sequel, as usual, $\alpha > -1$.

**Definition 4.1.** We say that the composition operator $C_\varphi: \mathcal{B}_\alpha^2 \to \mathcal{B}_\alpha^2$ is decompactifiable if there exists a weight $w \in \mathcal{B}_\alpha^2$ such that the weighted composition operator $M_w C_\varphi: \mathcal{B}_\alpha^2 \to \mathcal{B}_\alpha^2$ is bounded but not compact.

Our main result is the following.

**Theorem 4.2.** Let $\varphi: \mathbb{D} \to \mathbb{D}$ be an analytic self-map. Then the composition operator $C_\varphi: \mathcal{B}_\alpha^2 \to \mathcal{B}_\alpha^2$ is decompactifiable if and only if $\|\varphi\|_\infty = 1$.

It is a consequence of this other theorem, whose proof is postponed.

**Theorem 4.3.** Let $\gamma > 1$ and $\nu$ be a vanishing $\gamma$-Carleson measure on $\mathbb{D}$. Assume that $\nu$ satisfies the following property:

\[(4.1) \quad \forall t > 0, \quad \exists \zeta \in \partial \mathbb{D} \text{ such that } \nu[S(\zeta, t)] > 0.\]

Then there exists a holomorphic function $u: \mathbb{D} \to \mathbb{C}$ such that

(i) $u \in L^2(\nu)$;

(ii) $\sup_{|\xi|=1, 0<h<1} \frac{1}{h^{\gamma}} \int_{S(\xi, h)} |u|^2 d\nu < \infty$;

(iii) there exist $\delta > 0$ and two sequences $(\zeta_n)$ in $\partial \mathbb{D}$ and $(t_n)$ in $(0, 1)$ with $t_n \to 0^+$ such that

\[(4.2) \quad \frac{1}{t_n^{\alpha}} \int_{S(\zeta_n, t_n)} |u|^2 d\nu \geq \delta, \quad \text{for all } n \geq 1.\]

**Proof of Theorem 4.2.** It is plain that if $\|\varphi\|_\infty < 1$, then $M_w C_\varphi$ is compact for every weight $w \in \mathcal{B}_\alpha^2$. In fact, if $\mu_{w, \varphi}$ is the measure defined in (4.3), then $\mu_{w, \varphi}[S(\xi, h)] = 0$ for $0 < h < 1 - \|\varphi\|_\infty$; hence Corollary 2.2 gives the result.

Conversely, assume that $\|\varphi\|_\infty = 1$.

Note that if $C_\varphi: \mathcal{B}_\alpha^2 \to \mathcal{B}_\alpha^2$ is not compact, it suffices to take $w = 1$; so we assume that $C_\varphi$ is compact. Then $\nu := (A_\alpha)_\varphi = \varphi(d\mu_{\alpha})$ is a vanishing $(\alpha + 2)$-Carleson measure.

Since $\|\varphi\|_\infty = 1$, condition (4.1) is satisfied. Set $\gamma = \alpha + 2$ and $u$ be the holomorphic function given by Theorem 4.3 and set $w = u \circ \varphi$. We have

\[
\int_{\mathbb{D}} |w|^2 dA_\alpha = \int_{\mathbb{D}} |u \circ \varphi|^2 dA_\alpha = \int_{\mathbb{D}} |u|^2 d\nu < \infty;
\]

so $w \in \mathcal{B}_\alpha^2$. 

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Now, for every ξ ∈ ∂D and h ∈ [0, 1), we have, with μ = φ(|w|^2 dA_α):
\[
\mu[S(ξ, h)] = \int_{\partial S(ξ, h)} |w|^2 dA_\alpha = \int_D (\mathbf{1}_{S(ξ, h)} \circ φ) |u \circ φ|^2 dA_\alpha = \int_D \mathbf{1}_{S(ξ, h)} |u|^2 dν.
\]
Hence the properties (ii) and (iii) of Theorem 4.3 show that μ is a non-vanishing (α + 2)-Carleson measure, and therefore that \( M_w C_\phi : \mathcal{B}_\alpha^2 \to \mathcal{B}_\alpha^2 \) is bounded but not compact.

4.2 Proof of Theorem 4.3

To prove Theorem 4.3, we need several auxiliary results.

Lemma 4.4. For every ω ∈ ∂D and r ∈ (0, 1), there exists a bounded analytic function \( F \in H^\infty \) such that, for all \( z \in D \):

a) \( \Re F(z) > 0 \);

b) \( 1/2 \leq |F(z)| \leq 2 \);

c) \( |F(z)| < 1 \) when \( |z - ω| > r \);

d) \( |F(z)| > 1 \) when \( |z - ω| < r \).

Proof. By composing with a rotation, we can, and do, assume that ω = 1.

Let \( C = 1 - r \) and let A and B be the points of the intersection of the unit circle \( T = \partial D \) with the circle of center 1 and radius r, with \( \Im A > 0 \) and \( \Im B < 0 \). Consider the Möbius transformation \( T \) sending A to 0, C to 1, and B to \( \infty \). The images by \( T \) of \( \partial D \) and \( \partial D(1, r) \) are straight lines passing through 0. In fact the image of \( \partial D(1, r) \) is the extended real line \( \mathbb{R}_\infty = \mathbb{R} \cup \{ \infty \} \). Moreover \( T[D(1, r)] \) is the open upper half-plane.

Define \( g(z) = \sqrt{T(z)} \), where \( \sqrt{\cdot} \) is the principal branch of the square root. Then, for \( z \in D \):

\[
\begin{cases}
\arg [g(z)] \in (0, \pi/2) & \text{if } z \in D(1, r), \\
\arg [g(z)] \in (-\pi/2, 0) & \text{if } z \in D \setminus \overline{D(1, r)}.
\end{cases}
\]

Let now \( U \) be the Möbius transformation sending 0 to \( i/2 \), \( \infty \) to \( -i/2 \), and 1 to 0. We have

- \( |U[g(z)]| < 1/2 \) for all \( z \in D \);
- \( \Re U[g(z)] > 0 \) for all \( z \in D \cap D(1, r) = S(1, r) \);
- \( \Re U[g(z)] < 0 \) for all \( z \in D \setminus \overline{D(1, r)} \).

Finally, the function \( F \) defined as \( F(z) = \exp U[g(z)] \) suits.
Lemma 4.5. Let $\gamma \geq 1$, $\nu$ be a $\gamma$-Carleson measure on $\mathbb{D}$ and $F \in H^\infty$ such that $1/2 \leq |F(z)| \leq 2$ for all $z \in \mathbb{D}$. For given $\beta \in (0,1]$, we define the function $\Phi : \mathbb{R}^+_* \to \mathbb{R}^+_*$ as:

$$
\Phi(\delta) = \sup_{|\xi|=1, 0 < h \leq \beta} \frac{1}{h^\gamma} \int_{S(\xi,h)} |F|^{2\delta} \, d\nu, \quad \text{for all } \delta > 0.
$$

Then $\Phi$ is continuous.

**Proof.** First, we have $\Phi(\delta) < +\infty$ for all $\delta > 0$ because $\nu$ is a $\gamma$-Carleson measure; indeed, for all $\xi \in \partial \mathbb{D}$ and all $h \in (0,1]$:

$$
\frac{1}{h^\gamma} \int_{S(\xi,h)} |F|^{2\delta} \, d\nu \leq 4 \delta \frac{\nu[S(\xi,h)]}{h^\gamma} \leq C 4^\delta < +\infty.
$$

Now, observe that, since $1/2 \leq |F(z)| \leq 2$, we have, for all $h \in (0,1]$, all $\xi \in \partial \mathbb{D}$, and all $t \in \mathbb{R}$:

$$
\frac{1}{4^{|t|}} \frac{1}{h^\gamma} \int_{S(\xi,h)} |F|^{2\delta} \, d\nu \leq \frac{1}{h^\gamma} \int_{S(\xi,h)} |F|^{2\delta+2t} \, d\nu \leq 4^{|t|} \frac{1}{h^\gamma} \int_{S(\xi,h)} |F|^{2\delta} \, d\nu.
$$

Taking the supremum, we get

$$
4^{-|t|} \Phi(\delta) \leq \Phi(\delta + t) \leq 4^{|t|} \Phi(\delta),
$$

and that proves the continuity of $\Phi$, since $\Phi(\delta) < +\infty$. \hfill \Box

**Proposition 4.6.** Let $\nu$ be a finite $\gamma$-Carleson measure on $\mathbb{D}$ with property (1.1). Then, for every $\beta \in (0,1]$ and every $\varepsilon \in (0,1)$, there exists a function $v \in H^\infty$ satisfying:

(a) $|v(z)| < \varepsilon$ for all $z \in \mathbb{D}$ such that $|z| < 1 - \beta$;

(b) $\frac{1}{h^\gamma} \int_{S(\xi,h)} |v|^2 \, d\nu \leq 1$ for all $h \in (0,1]$ and all $\xi \in \partial \mathbb{D}$;

(c) $\frac{1}{h^\gamma} \int_{S(\xi,h)} |v|^2 \, d\nu \leq \varepsilon^2$ for all $h \in (\beta,1]$ and all $\xi \in \partial \mathbb{D}$;

(d) there exists $t \in (0,\beta]$ and $\zeta \in \partial \mathbb{D}$ such that

$$
\frac{1}{h^\gamma} \int_{S(\xi,t)} |v|^2 \, d\nu \geq \left(\frac{3}{4}\right)^2.
$$

**Proof.** Since $\nu$ is a $\gamma$-Carleson measure, there exists a positive constant $C$ (and we can and do assume that $C \geq 1$) such that:

$$
\nu[S(\xi,h)] \leq C h^\gamma, \quad \forall h \in (0,1], \ \forall \xi \in \partial \mathbb{D}.
$$

Take $r = \beta (\varepsilon^2/2)^{1/\gamma}$.
By (4.1), there exists \( \omega \in \partial \mathbb{D} \) such that
\[ \nu[S(\omega, r)] > 0. \]

Let \( F \) be the function given by Lemma 4.4.
We define:
\[ \Phi(\delta) = \sup_{0 < h \leq \beta, |\xi| = 1} \int_{S(\xi, h)} |F|^{2\delta} \, d\nu. \]

Thanks to (4.3), we have, for all \( \xi \in \partial \mathbb{D} \) and all \( h \in (0, \beta) \):
\[ \int_{S(\xi, h)} |F|^2 \, d\nu \leq 4 \nu[S(\xi, h)] \frac{1}{h^{\delta}} \leq 4C, \]
and we get \( \Phi(1) \leq 4C \).

On the other hand, for all \( \delta > 0 \):
\[ \Phi(\delta) \geq \int_{S(\omega, r)} |F|^{2\delta} \, d\nu. \]
Since \( |F(z)| > 1 \) for \( z \in S(\omega, r) \) and \( \nu[S(\omega, r)] > 0 \), we get
\[ \lim_{\delta \to +\infty} \int_{S(\omega, r)} |F|^{2\delta} \, d\nu = +\infty, \]
and consequently \( \lim_{\delta \to +\infty} \Phi(\delta) = +\infty \). Since \( \Phi(1) \leq 4C < (2C/\varepsilon)^2 \) and, thanks to Lemma 4.5 \( \Phi \) is continuous, there exists \( \delta_0 > 1 \) such that \( \Phi(\delta_0) = (2C/\varepsilon)^2 \).

Define
\[ v = \left( \varepsilon/2C \right) F^{\delta_0}. \]

Observe that \( r < \beta \); so \( |z| < 1 - \beta \) implies \( |z| < 1 - r \); hence \( z \notin S(\omega, r) \) and \( |F(z)| < 1 \). That means that \( |v(z)| < \varepsilon/(2C) < \varepsilon \), and we have proved (a).

By definition of \( \delta_0 \), (b) is satisfied for all \( h \in (0, \beta) \). It will be satisfied as well for \( h \in (\beta, 1) \) once we have proved (c).

Let us prove (c). Take \( \beta < h \leq 1 \). Since \( |F(z)| \leq 1 \) for \( z \in S(\xi, h) \setminus S(\omega, r) \), we have:
\[
\int_{S(\xi, h)} |v|^2 \, d\nu \leq \int_{S(\xi, h) \setminus S(\omega, r)} \left( \frac{\varepsilon}{2C} \right)^2 \, d\nu + \int_{S(\omega, r)} |F|^{2\delta_0} \, d\nu \\
\leq \left( \frac{\varepsilon}{2C} \right)^2 \nu[S(\xi, h)] + \left( \frac{\varepsilon}{2C} \right)^2 r^\gamma \Phi(\delta_0) \\
\leq \left( \frac{\varepsilon}{2C} \right)^2 C h^\gamma + \frac{r^\gamma}{2} \left( \frac{\varepsilon}{2C} \right)^2 C h^\gamma + \beta^\gamma \frac{\varepsilon^2}{2} \\
\leq \left( \frac{\varepsilon}{2C} \right)^2 C h^\gamma + h^\gamma \frac{\varepsilon^2}{2} \leq \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{2} h^\gamma \leq \varepsilon^2 h^\gamma. 
\]

Finally, by definition of \( \delta_0 \), there exist \( t \in (0, \beta) \) and \( \zeta \in \partial \mathbb{D} \) such that
\[ \frac{1}{t^{\gamma}} \int_{S(\zeta, t)} |v|^2 \, d\nu = \left( \frac{\varepsilon}{2C} \right)^2 \frac{1}{t^{\gamma}} \int_{S(\zeta, t)} |F|^{2\delta_0} \, d\nu \geq \left( \frac{3}{4} \right)^2, \]
and (d) if proved. \( \square \)
Proof of Theorem 4.3. Consider a sequence \((\varepsilon_n)_{n \geq 1}\) of positive numbers such that \(\sum_{n=1}^{\infty} \varepsilon_n < 1/4\).

Using Proposition 4.6, we are going to construct by induction four sequences \((v_n)_n\) in \(H^\infty\), \((\beta_n)_n\), with \(\beta_1 = 1\), and \((t_n)_n\) in \((0, 1]\), and \((\zeta_n)_n\) in \(\partial \mathbb{D}\) such that, for all \(n \geq 1:\)

(S1) \(\beta_n \geq t_n > \beta_{n+1} \geq t_{n+1};\)
(S2) \(|v_n(z)| < \varepsilon_n\) for \(|z| < 1 - \beta_n;\)
(S3) for all \(\xi \in \partial \mathbb{D}\) and all \(h \in (0, \beta_{n+1}] \cup [\beta_n, 1]:\)

\[
\frac{1}{h^\gamma} \int_{S(\xi, h)} |v_n|^2 \, d\nu \leq \varepsilon_n^2;
\]

(S4) \(\frac{1}{h^\gamma} \int_{S(\xi, h)} |v_n|^2 \, d\nu \leq 1\) for all \(h \in (\beta_{n+1}, \beta_n]\) and all \(\xi \in \partial \mathbb{D};\)
(S5) \(\frac{1}{t_n^\gamma} \int_{S(\zeta_n, t_n)} |v_n|^2 \, d\nu \geq \left(\frac{3}{4}\right)^2,\)

and

(S6) \(\lim_{n \to \infty} \beta_n = \lim_{n \to \infty} t_n = 0.\)

Take \(\beta_1 = 1\). With \(\beta = \beta_1\) and \(\varepsilon = \varepsilon_1\), let \(v_1 = v\) be the function given by Proposition 4.6 and \(\xi_1 = \xi\) and \(t_1 = t \leq \beta_1\) the numbers given by part (d) of that proposition. By Proposition 4.6 (b) and (d) respectively, conditions (S4) and (S5) are satisfied for \(n = 1\). Condition (S2) is void for \(n = 1\). For condition (S3), note that since \(\nu\) is a vanishing \(\gamma\)-Carleson measure, there exists \(\beta_2 > 0\) such that

\[
\frac{\nu(S(\xi, h))}{h^\gamma} \leq \varepsilon_1^2 (1 + \|v_1\|_\infty^2)^{-1}
\]

for all \(h \in (0, \beta_2]\) and all \(\xi \in \partial \mathbb{D}\). This implies, for these \(h\)'s and \(\xi\)'s:

\[
\frac{1}{h^\gamma} \int_{S(\xi, h)} |v_1|^2 \, d\nu \leq \frac{\|v_1\|_\infty^2 \nu(S(\xi, h))}{h^\gamma} \leq \varepsilon_1^2.
\]

It follows, with (c) of Proposition 4.6, that (S3) is satisfied for \(n = 1\).

We can of course ask that \(\beta_2 \leq 1/2.\)

Now, assume that \(v_1, \ldots, v_{n+1}, \beta_1, \ldots, \beta_{n+1}, t_1, \ldots, t_{n+1}\) and \(\zeta_1, \ldots, \zeta_{n+1}\) satisfying (S1), (S2), (S3), (S4) and (S5) have been constructed.

As above, since \(\nu\) is a vanishing \(\gamma\)-Carleson measure, there exists a positive number \(\beta_{n+2} \leq \min(\beta_{n+1}, 1/(n + 2))\) such that

\[
\frac{\nu(S(\xi, h))}{h^\gamma} \leq \varepsilon_{n+1}^2 (1 + \|v_{n+1}\|_\infty^2)^{-1}
\]

for all \(h \in (0, \beta_{n+2}]\) and all \(\xi \in \partial \mathbb{D}\). Using Proposition 4.6 with \(\beta = \beta_{n+2}\) and \(\varepsilon = \varepsilon_{n+1}\), we get \(v_{n+2} = v \in H^\infty, \zeta_{n+2} = \xi \in \partial \mathbb{D}\) and \(t_{n+2} = t \in (0, \beta_{n+2}]\) and the induction step follows.
We now set:

\[ u(z) = \sum_{n=1}^{\infty} v_n(z), \quad z \in \mathbb{D}. \]

Thanks to (S 2), this series converges uniformly on compact subsets of \( \mathbb{D} \), so \( u \) is analytic in \( \mathbb{D} \).

Take \( \xi \in \partial \mathbb{D} \) and \( h \in (0, 1] \). There exists a unique \( n \geq 1 \) such that \( h \in (\beta_n + 1, \beta_n] \). By the triangle inequality:

\[
\left( \frac{1}{h^\gamma} \int_{S(\xi, h)} |u|^2 \, d\nu \right)^{1/2} \leq \sum_{k=1}^{\infty} \left( \frac{1}{h^\gamma} \int_{S(\xi, h)} |v_k|^2 \, d\nu \right)^{1/2} + \left( \frac{1}{h^\gamma} \int_{S(\xi, h)} |v_n|^2 \, d\nu \right)^{1/2}.
\]

Now, since:

\[
(4.4) \quad h \in (0, \beta_k + 1] \cup (\beta_k, 1] \quad \text{for every } k \neq n,
\]

we get, by (S 3) and (S 4):

\[
\left( \frac{1}{h^\gamma} \int_{S(\xi, h)} |u|^2 \, d\nu \right)^{1/2} \leq \left( \sum_{k \neq n} \varepsilon_k \right) + 1 \leq \frac{1}{4} + 1 = \frac{5}{4}.
\]

Consequently, \( u \) satisfies (ii) of Theorem 4.3 Then condition (ii) implies (i) because \( \nu \) is a finite measure, \( u \) is bounded on \((1/2) \mathbb{D}\), and \( \mathbb{D} \setminus (1/2) \mathbb{D} \) can be covered by a finite number of boxes \( S(\xi, 1) \), with \( \xi \in \partial \mathbb{D} \).

To obtain (iii), we use (4.4) again, with \( h = t_n \), to get:

\[
\left( \frac{1}{t_n^\gamma} \int_{S(\xi, t_n)} |u|^2 \, d\nu \right)^{1/2} \geq \left( \frac{1}{t_n^\gamma} \int_{S(\xi, t_n)} |v_n|^2 \, d\nu \right)^{1/2} - \sum_{k \neq n} \left( \frac{1}{t_n^\gamma} \int_{S(\xi, t_n)} |v_k|^2 \, d\nu \right)^{1/2}
\]

\[
\geq \frac{3}{4} - \sum_{k \neq n} \varepsilon_k \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2},
\]

and we have (iii). \( \square \)

5 Hilbert-Schmidt regularization

We remarked in Section 3 that if \( \varphi(z) = \frac{1 + z}{2} \), then \( C_\varphi \) is compactifiable on \( \mathcal{B}_2 \) by a weight in \( H^\infty \). Actually, since \( |\varphi(e^{it})| = \cos(t/2) \), we have \( \int_{-\pi}^{\pi} \log \frac{1}{|\varphi(e^{it})|} \, dm(t) < \infty \), and \cite{12} Theorem 4.1 tells that the composition
operator $C_\varphi$ can be weighted to have a Hilbert-Schmidt operator on $H^2$; a fortiori, this weighted composition operator is Hilbert-Schmidt on $\mathfrak{B}^2_\alpha$ (see [13, Theorem 3.12]). We can be more specific on an example, but unfortunately this example shows no difference between the Hardy and Bergman spaces.

**Proposition 5.1.** Let $\varphi : \mathbb{D} \to \mathbb{D}$ be defined by $\varphi(z) = \frac{1+z}{2}$, and let $w(z) = (1 - z)^\beta$ with $\beta > -1/2$, so that $w \in H^2$. Then the weighted composition operators $M_wC_\varphi : H^2 \to H^2$ and $M_wC_\varphi : \mathfrak{B}^2 \to \mathfrak{B}^2$ are Hilbert-Schmidt if and only if $\beta > 1/2$.

**Proof.** The first item was proved in [9, Proposition 2.4]. For the second item, we have to determine those $\beta$ such that

$$I := \int_D \frac{|w(z)|^2}{(1 - |\varphi(z)|^2)^2} \, dA(z),$$

where $\Delta = \mathbb{D} \cap D(1,1)$. Passing in polar coordinates centered at 1, we write, for $z \in \Delta$: $z = 1 - re^{i\theta}$ with $|\theta| < \pi/2$ and $r < 2 \cos \theta$. Then, $|\frac{1+z}{2}|^2 = 1 + \frac{r^2}{4} - r \cos \theta$ and

$$I = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \frac{r^{2\beta+1}}{r^2 \cos \theta - r^2 \cos \theta \cos \theta} \, dr \, d\theta = \frac{2}{\pi} \int_0^{\pi/2} \int_0^{2 \cos \theta} \frac{r^{2\beta-1}}{(\cos \theta - r^2 \cos \theta)^2} \, dr \, d\theta.$$

Making the change of variable $r = 2t \cos \theta$, $0 \leq t \leq 1$ in the inner integral and observing that $1 \geq 1 - t/2 \geq 1/2$, we see that

$$I \approx \int_0^{\pi/2} \int_0^1 \frac{t^{2\beta-1} \cos \theta}{\cos^2 \theta} \, dt \, d\theta = \left( \int_0^1 t^{2\beta-1} \, dt \right) \left( \int_0^{\pi/2} (\sin \theta)^{2\beta-2} \, d\theta \right).$$

So, clearly, $I < \infty$ if and only if $\beta > 1/2$. \hfill $\square$

Fortunately, examples showing the difference between Hardy and Bergman spaces exist.

**Theorem 5.2.** There exists a Blaschke product $B$ which can be Hilbert-Schmidt regularized, and more, on $\mathfrak{B}^2$, but not on $H^2$.

**Proof.** Any Blaschke product $B$ is an inner function, i.e. $|B^*| = 1$ $m$-almost everywhere on the unit circle, implying, by [12, Theorem 3.1], that $M_wC_B$ is compact on $H^2$ for no weight $w \in H^2$, with $w \neq 0$.

On the other hand, as a consequence of [10 Theorem 3.1], we proved ([11 Theorem 4.4]; see also [14 Theorem 13]) that there exist Blaschke products $B$ (which we called slow Blaschke products) such that $C_B$ is compact on the Bergman-Orlicz space $\mathfrak{B}^{\Psi^2}$, and hence belong to every Schatten class $S_p$ of $\mathfrak{B}^2$. \hfill $\square$
Moreover, we can give the following quantitative precision to Theorem 5.2.

**Theorem 5.3.** For any sequence $(\varepsilon_n)$ of positive numbers with limit zero, there is a Blaschke product $B$ such that

$$a_n(C_B \colon \mathfrak{B}^2 \to \mathfrak{B}^2) \lesssim e^{-n\varepsilon_n}.$$  

**Proof.** We can assume that $\varepsilon_n$ decreases and that $n\varepsilon_n \uparrow \infty$ with $n\varepsilon_n \geq \sqrt{n}$.

For a given symbol $\varphi$, we set

$$\chi(h) = A(\{z : |\varphi(z)| \geq 1 - h\}).$$

We use [15, Theorem 5.1] which implies that

$$a_n(C_{\varphi}) \lesssim \inf_{0 < h < 1} \left[ \sqrt{n} e^{-nh} + \sqrt{\chi(h)/h^2} \right].$$

Let $\delta : (0, 1) \to (0, 1)$ be a non-increasing and piecewise linear map, decreasing to 0 so slowly at the origin that

$$\delta(1 - |z|) \leq 4 \varepsilon_n \implies 1 - |z| \leq \varepsilon_n^2 \exp(-2n\varepsilon_n).$$

By [10, Theorem 3.1] again, there exists a Blaschke product $B$ such that $|B(z)| \leq \exp(-\delta(1 - |z|))$. Take $\varphi = B$ in (5.1) and observe that, for $h = 2\varepsilon_n \leq 1/2$, we have

$$|B(z)| \geq 1 - h \implies \exp(-\delta(1 - |z|)) \geq 1 - h \geq \exp(-2h).$$

Hence

$$\delta(1 - |z|) \leq 4 \varepsilon_n \quad \text{and} \quad 1 - |z| \leq \varepsilon_n^2 \exp(-2n\varepsilon_n)$$

and

$$\chi(h) \leq 2 \varepsilon_n^2 \exp(-2n\varepsilon_n).$$

Inserting this in (5.1), we get the result. \qed

In order to find a necessary and sufficient condition for a symbol can be weighted in a Hilbert-Schmidt operator, we make some observations.

As recalled, a weighted composition operator which is Hilbert-Schmidt on $H^2$ is also Hilbert-Schmidt on $\mathfrak{B}^2$. We know that $C_{\varphi}$ is Hilbert-Schmidt on $H^2$ if and only if

$$\int_{\mathbb{T}} \frac{1}{1 - |\varphi|^2} dm < \infty.$$  

Equivalently:

$$(5.2) \quad \sum_{n=0}^{\infty} \|\varphi^n\|_{H^2}^2 < \infty.$$
On the other hand, $C_\phi$ can be weighted to become a Hilbert-Schmidt operator on $H^2$ if and only if
\[ \int_\mathbb{T} \log \frac{1}{1-|\phi|} \, dm < \infty, \]
\text{(12, Theorem 4.1)}, which is equivalent to
\[ (5.3) \quad \sum_{n=0}^\infty \frac{1}{n+1} \|C_\phi(e_n)\|^2_{H^2} < \infty. \]

Now, writing $e_n(z) = z^n$, and since \((n+1)(\alpha+1)/2 e_n)\) is an orthonormal basis of $\mathcal{B}_\alpha^2$, $C_\phi$ is Hilbert-Schmidt on $\mathcal{B}_\alpha^2$ if and only if
\[ \sum_{n=0}^\infty (n+1)^{\alpha+1} \|\phi^n\|^2_{\mathcal{B}_\alpha^2} = \sum_{n=0}^\infty \|C_\phi((n+1)^{\alpha+1/2} e_n)\|^2_{\mathcal{B}_\alpha^2} < \infty. \]

By comparison with (5.3), we might think that $C_\phi$ can be weighted to become a Hilbert-Schmidt operator on $\mathcal{B}_\alpha^2$ if and only if
\[ (5.4) \quad \sum_{n=0}^\infty (n+1)^{\alpha} \|\phi^n\|^2_{\mathcal{B}_\alpha^2} < \infty. \]

Since
\[ (5.5) \quad \sum_{n=0}^\infty (n+1)^{\alpha} \|\phi^n\|^2_{\mathcal{B}_\alpha^2} = \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\alpha+1}} \, dA(z), \]
this guesswork takes the following form: is it true that there exists a weight $w$ such that $M_wC_\phi : \mathcal{B}_\alpha^2 \to \mathcal{B}_\alpha^2$ is Hilbert-Schmidt if and only if
\[ (5.6) \quad \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\alpha+1}} \, dA(z) < \infty? \]

We do not know if (5.6) implies the existence of a weight $w \neq 0$ for which $M_wC_\phi$ is Hilbert-Schmidt, but, in any case, it implies that $C_\phi$ is compactifiable.

**Proposition 5.4.** If, for $\alpha > -1$, we have
\[ \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\alpha+1}} \, dA(z) < \infty, \]
then
\[ \lim_{z \to \xi} \frac{1-|z|}{1-|\phi(z)|} = 0 \quad \text{for almost all } \xi \in \mathbb{T}. \]

Recall that, by Theorem 4.3, this last condition means that $C_\phi$ is compactifiable on $\mathcal{B}_\alpha^2$.

**Proof.** We set:
\[ g(z) = \left( \frac{1-|z|^2}{1-|\phi(z)|^2} \right)^{\alpha+1}. \]
Let \( r_n = 1 - 2^{-n} \) and:

\[
\Gamma_n = \{ z \in \mathbb{D} : r_n \leq |z| < r_{n+1} \}.
\]

Now, \( 1/(1 - |\varphi|^2)^{\alpha+1} \) is subharmonic (and even logarithmically-subharmonic), because we can write \( 1/(1 - |\varphi|^2)^{\alpha+1} = \sum_{k=0}^{\infty} c_k(\alpha)|\varphi|^{2k} \) with \( c_k(\alpha) \geq 0 \). Hence, we have, since \( A(\Gamma_n) \approx 1 - r_n^2 \):

\[
\int_{\Gamma_n} g(r_n e^{i\theta}) d\theta = \int_{\Gamma_n} \frac{(1 - r_n^2)^{\alpha+1}}{(1 - |r_n e^{i\theta}|^2)^{\alpha+1}} d\theta \\
\quad \leq \frac{1}{1 - r_n^2} \int_{\Gamma_n} \frac{(1 - r_n^2)^{\alpha+1}}{(1 - |r_n e^{i\theta}|^2)^{\alpha+1}} dA(z) \\
\quad \leq \int_{\Gamma_n} \frac{(1 - |z|^2)^{\alpha}}{(1 - |\varphi(z)|^2)^{\alpha+1}} dA(z)
\]

(we used that \( 1 - r_n^2 \leq 2(1 - r_n) = 2 \cdot 2^{-n} = 4(1 - r_{n+1}) \leq 4(1 - |z|) \leq 4(1 - |z|^2) \) for \( z \in \Gamma_n \), so \( (1 - r_n^2)^{\alpha} \leq (1 - |z|^2)^{\alpha} \) when \( \alpha \geq 0 \), and, when \(-1 < \alpha < 0\), we used that \( 1 - |z|^2 \leq 1 - r_n^2 \) for \( z \in \Gamma_n \)). The sets \( \Gamma_n \) being disjoint, we get that:

\[
\int_{\mathbb{T}} \left( \sum_{n=0}^{\infty} g(r_n e^{i\theta}) \right) d\theta = \sum_{n=0}^{\infty} \int_{\mathbb{T}} g(r_n e^{i\theta}) d\theta \\
\quad \leq \sum_{n=0}^{\infty} \int_{\Gamma_n} \frac{(1 - |z|^2)^{\alpha}}{(1 - |\varphi(z)|^2)^{\alpha+1}} dA(z) \\
\quad = \int_{\mathbb{D}} \frac{(1 - |z|^2)^{\alpha}}{(1 - |\varphi(z)|^2)^{\alpha+1}} dA(z) < \infty,
\]

meaning that the function \( \sum_{n=0}^{\infty} g(r_n \cdot) \) is integrable on \( \mathbb{T} \). It follows that \( g(r_n \cdot) \xrightarrow{n \to \infty} 0 \) almost everywhere. Since the existence of a radial limit implies that of an angular limit, we obtain that \( \angle \lim_{z \to \xi} \frac{1 - |z|}{1 - |\varphi(z)|} = 0 \) for almost all \( \xi \in \mathbb{T} \). By the Julia-Caratheodory theorem, it follows that \( \lim_{z \to \xi} \frac{1 - |z|}{1 - |\varphi(z)|} = 0 \) for almost all \( \xi \in \mathbb{T} \).

An a priori different condition than \([5.6]\) appears in the following theorem.

**Theorem 5.5.** Let \( d\lambda_\alpha(r) = 2(\alpha+1)(1-r^2)^\alpha r \, dr \), be the marginal probability measure on \([0,1]\) of \( dA_\alpha \), and

\[
(5.7) \quad G(\theta) = \int_{0}^{1} \frac{d\lambda_\alpha(r)}{(1 - |r e^{i\theta}|^2)^{\alpha+2}}.
\]

Then:

1. If \( \log G \in L^1(0,2\pi) \), then there exists \( w \in H^\infty \), \( w \neq 0 \), such that \( M_w C_\varphi \) is Hilbert-Schmidt on \( \mathcal{B}_2^2 \).

2. Conversely, if there exists such a weight \( w \), then \( \log G \in L^{1,\infty}(0,2\pi) \).
Note that $G \geq 1$, so $\log G \geq 0$.

Recall that $L^{1,\infty}(\mu)$ is the space of (classes of) measurable functions $f$ such that $\sup_{a>0} a \, m(\{|f|>a\}) < \infty$, and that $L^1(\mu) \subseteq L^{1,\infty}(\mu)$, by Markov’s inequality.

### 5.1 Proof of 1) of Theorem 5.5

For convenience, we set

$$U(z) = \frac{1}{(1 - |\varphi(z)|^2)^{\alpha+2}}.$$  

(5.8)

We will use two lemmas. For that, we denote $\rho$ the pseudo-hyperbolic metric on $D$. Recall that

$$\rho(u,v) = \frac{|u - v|}{1 - \bar{u}v}, \quad u,v \in D.

#### Lemma 5.6. There is a positive constant $C = C(\alpha)$ such that, for $u,v \in D$:

$$\rho(u,v) \leq 1 \Rightarrow \frac{1}{C} \leq \frac{U(u)}{U(v)} \leq C.$$  

(5.9)

**Proof.** Since

$$\left(\frac{1}{2} \times \frac{1 - |\varphi(v)|}{1 - |\varphi(u)|}\right)^{\alpha+2} \leq \frac{U(u)}{U(v)} \leq \left(2 \times \frac{1 - |\varphi(v)|}{1 - |\varphi(u)|}\right)^{\alpha+2},$$

it suffices to show that there is a positive constant such that

$$\frac{1}{C} \leq \frac{1 - |\varphi(v)|}{1 - |\varphi(u)|} \leq C$$

when $\rho(u,v) \leq 1/2$. Moreover, by the Schwarz-Pick inequality, we have:

$$\rho(|\varphi(u)|,|\varphi(v)|) \leq \rho(\varphi(u),\varphi(v)) \leq \rho(u,v),$$

it suffices to majorize $q := \frac{1-a}{1-b}$ when $\rho(a,b) \leq 1/2$ and $0 \leq a, b < 1$ (the minoration will come by exchanging $a$ and $b$).

If $a \geq b$, then $q \leq 1$

If $a < b$, we remark that $\rho(a,b) \leq 1/2$ writes $T_a(b) := \frac{a-b}{1-ab} \geq -1/2$. Since $T_a$ is decreasing on $[-1,1]$, we get $b \leq T_a(-1/2)$, i.e. $b \leq \frac{b\pm\sqrt{b}}{2+b}$ and $1-b \geq \frac{b-a}{2+b}$. Therefore $q \leq 2 + a \leq 3$.

Let, for $n \geq 0$:

$$r_n = \exp(-2^{-n})$$

and

$$\Gamma_n = \{ z \in D ; \ r_n \leq |z| < r_{n+1} \}.$$
Lemma 5.7. For $r_n \leq u, v \leq r_{n+1}$, we have $\rho(u e^{i\theta}, v e^{i\theta}) \leq 1/2$, for every $\theta \in \mathbb{R}$.

Proof. It is a simple computation:

$$\rho(u e^{i\theta}, v e^{i\theta}) = |u - v| |1 - w| \leq r_{n+1} - r_n = \frac{r_{n+1} - r_n^2}{1 - r_{n+1}^2} = \frac{r_{n+1} - 1}{1 + r_{n+1}} \leq \frac{1}{2}.$$

Now, we can finish the proof of 1) of Theorem 5.5.

We have to show that there exists a non-null function $w_0 \in H^\infty$ such that

$$\int \mathbb{D} |w_0|^2 U dA_\alpha < \infty,$$

where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)\alpha dA(z)$.

For every $w \in H^\infty$, we have:

$$\int \mathbb{D} |w|^2 U dA_\alpha = \int_{D(0,e^{-1})} |w|^2 U dA_\alpha + \sum_{n=0}^{\infty} \int_{\Gamma_n} |w|^2 U dA_\alpha$$

For every $n \geq 0$:

$$\int_{\Gamma_n} |w|^2 U dA_\alpha = 2 (\alpha + 1) \int_{r_{n+1}}^{r_n} \left( \frac{1}{2\pi} \int_0^{2\pi} |w(re^{i\theta})|^2 U(re^{i\theta}) d\theta \right) (1 - r^2)^\alpha r dr$$

As said in the proof of Proposition 5.3, $U$ is logarithmically-subharmonic; hence the function $|w|^2 U$ is also logarithmically-subharmonic; in particular, it is subharmonic; so we have (see [3] Theorem 1.6, page 9), for $r_n \leq r \leq r_{n+1}$:

$$\frac{1}{2\pi} \int_0^{2\pi} |w(re^{i\theta})|^2 U(re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |w(r_{n+1} e^{i\theta})|^2 U(r_{n+1} e^{i\theta}) d\theta.$$

By Lemma 5.6 and Lemma 5.7, we have $U(r_{n+1} e^{i\theta}) \leq C U(r_n e^{i\theta})$. But $r_n = r_{n+1}^2$, so $U(r_{n+1} e^{i\theta}) \leq C U(r_{n+1} e^{i\theta})$, and hence

$$\frac{1}{2\pi} \int_0^{2\pi} |w(re^{i\theta})|^2 U(re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |w(r_{n+1} e^{i\theta})|^2 U(r_{n+1} e^{i\theta}) d\theta.$$

By the subharmonicity of $|w|^2 U$ again, we obtain:

$$\frac{1}{2\pi} \int_0^{2\pi} |w(re^{i\theta})|^2 U(re^{i\theta}) d\theta \leq C \frac{1}{2\pi} \int_0^{2\pi} |w(e^{i\theta})|^2 U(r e^{i\theta}) d\theta.$$

Using Lemma 5.6 and Lemma 5.7 again, we have, for every $r_n \leq r < r_{n+1}$:

$$\frac{1}{2\pi} \int_0^{2\pi} |w(e^{i\theta})|^2 U(r_{n+1} e^{i\theta}) d\theta \leq C \frac{1}{2\pi} \int_0^{2\pi} |w(e^{i\theta})|^2 U(r e^{i\theta}) d\theta.$$
Therefore:
\[
\int_{r_{\alpha}} |w|^2 U dA_{\alpha} \leq C^2 \int_{r_{\alpha}} \left( \frac{1}{2\pi} \int_0^{2\pi} |w(e^{i\theta})|^2 U(r e^{i\theta}) \, d\theta \right) d\lambda_{\alpha}(r).
\]

Using the Fubini theorem, we finally obtain:
\[
\int_{D} |w|^2 U dA_{\alpha} \leq \int_{D(0,e^{-1})} |w|^2 U dA_{\alpha}
+ C^2 \frac{1}{2\pi} \int_0^{2\pi} \left( \int_{c^{-1}}^{1} U(r e^{i\theta}) \, d\lambda_{\alpha}(r) \right) |w(e^{i\theta})|^2 \, d\theta.
\]

Since
\[
G(\theta) = \int_0^1 U(r e^{i\theta}) \, d\lambda_{\alpha}(r),
\]
we have:
\[
(5.11) \quad \int_{D} |w|^2 U dA_{\alpha} \leq \int_{D(0,e^{-1})} |w|^2 U dA_{\alpha} + C^2 \frac{1}{2\pi} \int_0^{2\pi} G(\theta) |w(e^{i\theta})|^2 \, d\theta.
\]

We now use Szegö’s theorem (see [3, Theorem 3.1, Chapter IV, page 139], or [18, Section 8.3]):
\[
\inf_{w \in H^\infty, w(0)=1} \frac{1}{2\pi} \int_0^{2\pi} |w(e^{i\theta})|^2 G(\theta) \, d\theta = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log G(\theta) \, d\theta \right).
\]

Remarking that the hypothesis of the theorem writes:
\[
\int_0^{2\pi} \log G(\theta) \, d\theta < \infty,
\]
that shows that there exists \(w_0 \in H^\infty\) with \(w_0(0) = 1\) such that
\[
\frac{1}{2\pi} \int_0^{2\pi} |w_0(e^{i\theta})|^2 G(\theta) \, d\theta < \infty.
\]

With (5.11), that shows that \(w_0\) satisfies (5.10), and that ends the proof of 1) of Theorem 5.5.

Note that the proof shows that we can actually get a polynomial for \(w_0\).

### 5.2 Proof of 2) of Theorem 5.5

We may, and do, assume that \(||w||_\infty = 1\).

By hypothesis, we have
\[
\int_{D} |w(z)|^2 \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{\alpha+2}} \, dA(z) < \infty.
\]
Setting, with $U$ defined in (5.8):

(5.12) \[ \psi(\theta) = \int_{1/2}^{1} |w(re^{i\theta})|^2 U(r e^{i\theta}) \, d\lambda_\alpha(r), \]

we hence have $\psi \in L^1(0, 2\pi)$.

Let

\[ \tilde{G}(\theta) = \int_{1/2}^{1} U(r e^{i\theta}) \, d\lambda_\alpha(r) \]

and

\[ J(\theta) = \inf \{|w(r e^{i\theta})|^2; \ 1/2 \leq r < 1\}. \]

We have $\psi(\theta) \geq \tilde{G}(\theta) J(\theta)$, so

\[ \log \tilde{G} \leq \log \psi + \log(1/J) \leq \log^+ \psi + \log(1/J) \leq \log(1/J). \]

Since $U \geq 1$, we have $\tilde{G}(\theta) \geq C_\alpha$, with $C_\alpha = (3/4)^{\alpha+1} > 0$; hence $\log \tilde{G}(\theta) \geq \log C_\alpha > -\infty$. Therefore, to get $\log \tilde{G} \in L^{1,\infty}(0, 2\pi)$ and finish the proof of 2) of Theorem 5.5 it suffices to prove that $\log \tilde{G} \in L^{1,\infty}(0, 2\pi)$, and for that, to prove that $\log(1/J) \in L^{1,\infty}(0, 2\pi)$. This is the object of the following theorem.

**Theorem 5.8.** Let $v \in H^\infty$ such that $\|v\|_\infty = 1$ and set

(5.13) \[ I_v(\theta) = \inf \{|v(r e^{i\theta})|; \ 1/2 \leq r < 1\}. \]

Then $\log(1/I_v) \in L^{1,\infty}(0, 2\pi)$.

**Proof.** We can write $v(z) = B(z) v_0(z)$, where $B$ is the Blaschke product whose zeros are those of $v$, and $v_0$ does not vanish. Since

\[ I_v \geq I_B \times I_{v_0}, \]

it suffices to prove that $\log(1/I_B) \in L^{1,\infty}(0, 2\pi)$ and $\log(1/I_{v_0}) \in L^{1,\infty}(0, 2\pi)$.

**Case of a non vanishing function.**

We can write $v_0 = \exp(-h)$, where $h : \mathbb{D} \to \{\Re z > 0\}$. We have $h = u + i\tilde{u}$, where $u = \Re h$ and $\tilde{u}$ is the conjugate function of $u$. Since $u > 0$, $u = P[\mu]$ is the Poisson integral of a positive measure $\mu$, and we have

\[ u(r e^{i\theta}) \leq C M_\mu(\theta) \quad \forall r \in [0, 1), \]

where $M_\mu$ is the Hardy-Littlewood maximal function of $\mu$. Then:

\[ |v_0(r e^{i\theta})| \geq \exp \left(- C M_\mu(\theta)\right); \]

so $I_{v_0}(\theta) \geq \exp \left(- C M_\mu(\theta)\right)$, and $\log \left(1/I_{v_0}(\theta)\right) \leq C M_\mu(\theta)$. Since $M_\mu \in L^{1,\infty}(0, 2\pi)$, by Kolmogorov’s theorem, we obtain that $\log(1/I_{v_0}) \in L^{1,\infty}(0, 2\pi)$. 

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Case of a Blaschke product.

This case will follow from the next result. We note $\arg z$ the principal argument of $z$: $-\pi < \arg z \leq \pi$.

**Proposition 5.9.** Let $B_0$ be a Blaschke product whose zeros $a_n$ have modulus greater or equal to some positive constant $c$, say $c = 3/4$. Then there exist $f \in L^1(-\pi, \pi)$ and $u = P[w]$, with $w \in L^1(-\pi, \pi)$, such that

\begin{equation}
\log \left( \frac{1}{|B_0(z)|} \right) \leq f(\arg z) + u(z), \quad \text{for all } z \in \mathbb{D}.
\end{equation}

For $a \in \mathbb{D}$, we denote $\Delta(a, 1/2)$ the pseudo-hyperbolic disk of center $a$ and radius $1/2$.

We begin by two lemmas.

**Lemma 5.10.** For $a \in \mathbb{D}$, we set

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z},$$

as well as $I_a = I_{\varphi_a}$ and $G_a = \log(1/I_a)$. Then, for every $a \in \mathbb{D}$, we have $G_a \in L^1(-\pi, \pi)$.

First, we have $G_a \geq 0$. Then:

$$|\varphi_a(z)| = \frac{|a - z|}{|1 - \overline{a}z|} \geq \frac{|z - a|}{2};$$

so, it suffices to give a lower estimate of $|z - a|$.

We separate two cases.

- First case: $|a| \leq 1/4$. Then we have $|re^{i\theta} - a| \geq 1/4$ when $1/2 \leq r < 1$; hence $G_a(\theta) \leq \log 8$ for all $\theta$ and $G_a \in L^1(-\pi, \pi)$.

- Second case: $|a| > 1/4$. We can assume that $1/4 < a < 1$. If $z = re^{i\theta}$, then, for $|\theta| \leq \pi/2$:

$$|z - a| \geq \text{dist} (a, R_\theta) = a |\sin \theta|,$$

where $R_\theta$ is the ray passing through 0 and $e^{i\theta}$, so

$$G_a(\theta) \leq \left| \log \left( \frac{a}{2 |\sin \theta|} \right) \right|$$

and $G_a \in L^1(-\pi, \pi)$. \qed

**Lemma 5.11.** There is a positive constant $C$ such that, for $3/4 \leq a < 1$ and $h = 1 - a$, we have:

\begin{equation}
|\theta| \leq Ch \quad \text{when } z = re^{i\theta} \in \Delta(a, 1/2).
\end{equation}
Proof. The pseudo-hyperbolic disk $\Delta(a, 1/2)$ is equal to the Euclidean disk $D(\tilde{a}, R)$, with

$$\tilde{a} = \frac{3}{4 - |a|^2}a \quad \text{and} \quad R = 2 \frac{1 - |a|^2}{4 - |a|^2}$$

(see [5, page 3]). For $0 < a < 1$ and $h = 1 - a$, we have $R \leq 4h/3$ and $3a/4 \leq \tilde{a} \leq a$. Hence $\Delta(a, 1/2)$ is contained in the angular sector of vertex 0 and half-angle $\theta_a$ such that $\sin \theta_a = R/\tilde{a} \leq (4h/3)/(3a/4)$ (see Figure 1). For $3/4 < a < 1$, that gives $\sin \theta_a \leq (64/27)h$. It follows that there is $C > 0$ ($C = 64\pi/54$ works) such that $|\theta| \leq Ch$ when $z = re^{i\theta} \in \Delta(a, 1/2)$.

**Proof of Proposition 5.4** We restrict ourselves, for the time, to $3/4 \leq a < 1$.

Note that, since $3/4 \leq a < 1$ and $a = 1 - h$, we have $0 < h \leq 1/4$.

- Let $z = re^{i\theta} \in \Delta(a, 1/2)$.

We write

$$|\varphi_a(z)| = \frac{|z - a|}{a} \geq \frac{|z - a|}{a} \left[ |z - a| + \left( \frac{1}{a} - a \right) \right] = \frac{1}{a + \frac{1 - a^2}{|z - a|}}$$

so, if $z \in \Delta(a, 1/2)$, and $z = re^{i\theta}$, we have $|\theta| < \pi/2$; hence, when $\theta \neq 0$:

$$|\varphi_a(z)| \geq \frac{1}{a + \frac{1 - a^2}{|a| \sin |\theta|}}$$

It follows from Lemma 5.11 that, for another constant $C$:

$$\frac{1}{I_a(\theta)} \leq a + \frac{1 - a^2}{a |\sin |\theta|} \leq C \frac{h}{|\theta|},$$

so

$$G_a(\theta) \leq \log \left( C \frac{h}{|\theta|} \right)$$

and

$$\int_0^{Ch} G_a(\theta) d\theta \leq \left[ \theta \log \left( C \frac{h}{\theta} \right) + \theta \right]_0^{Ch} = Ch.$$

Setting, for $|\theta| \leq \pi$ (recall that $a = 1 - h$):

$$f_a(\theta) = \log \left( C \frac{h}{|\theta|} \right) \mathbb{I}_{[-Ch, Ch]}(\theta),$$

we hence have:

$$\log \left( \frac{1}{|\varphi_a(z)|} \right) \leq f_a(\theta) \quad \text{for} \quad z = re^{i\theta} \in \Delta(a, 1/2)$$

(since then $|\theta| \leq Ch$). Moreover, we have

$$\|f_a\|_1 \leq 2Ch.$$
Now, let $z \in \mathbb{D} \setminus \Delta(a, 1/2)$.

Let $D_a$ be the (Euclidean) disk of diameter $[c, 1/c]$, where $c$ is the point of the segment $\partial \Delta(a, 1/2) \cap [0, 1)$ such that $0 < c < a$, and

\begin{equation}
A_a = \partial \mathbb{D} \cap D_a.
\end{equation}

We write simply $A_a = A$ thereafter.

We set

\begin{equation}
w_a = 2 \log 2 \mathbb{I}_A \quad \text{and} \quad u_a = \mathbb{P}[w_a],
\end{equation}

the Poisson integral of $w_a$.

We have, for some positive constant $C$:

\begin{equation}
\|w_a\|_1 = 2 \log 2 \mathbb{m}(A) \leq Ch.
\end{equation}

In fact, the diameter of $D_a$ is $\frac{1}{c} - c$ and $\frac{1}{2} = |\varphi_a(c)| = \frac{a-c}{1-ae}$, so

\begin{equation*}
c = \frac{2a-1}{2-a} = \frac{1-2h}{1+h} = 1 - 3h + o(h),
\end{equation*}

and the diameter of $D_a$ is equal to $6h + o(h)$.

![circles](image)

**Figure 1: circles**

**Lemma 5.12.** Let $0 < \theta_0 < \pi/2$ and $A$ be the arc of $\partial \mathbb{D}$ with end points $e^{-i\theta_0}$ and $e^{i\theta_0}$ and midpoint 1. Then, if $D_A$ is the disk orthogonal to $\partial \mathbb{D}$ passing through $e^{-i\theta_0}$ and $e^{i\theta_0}$, we have

\[ \mathbb{P}[\mathbb{I}_A] \geq 1/2 \quad \text{on} \quad \mathbb{D} \cap D_A. \]

**Proof.** Let $T : \overline{\mathbb{D}} \to \{z \in \mathbb{C} ; \Re z \geq 0\} \cup \{\infty\}$ be the conformal map mapping $A$ onto $\mathbb{R}_- \cup \{\infty\}$ and $\partial \mathbb{D} \setminus A$ onto $\mathbb{R}_+^*$. The unique bounded solution to the
We have $U(\zeta) \geq 1/2$ if and only if $\zeta$ is in the closed left-hand side upper quadrant $Q$. But $T^{-1}(i\mathbb{R})$ is the arc orthogonal to $\partial \mathbb{D}$ and passing through $e^{-i\theta_0}$ and $e^{i\theta_0}$; hence $T^{-1}(i\mathbb{R}) = A$ and, since $D_A$ is orthogonal to $\mathbb{D}$, we have $T((\mathbb{D} \cap D_A)) = Q \cup \{\infty\}$. Therefore $P[\mathbb{I}_A](z) \geq 1/2$ when $z \in \mathbb{D} \cap D_A$.

By this lemma, we have

$$u_a \geq \log 2 \text{ on } \mathbb{D} \cap D_a.$$  

In particular, since $\Delta(a, 1/2) \subseteq \mathbb{D} \cap D_a$, we have $u_a \geq \log 2$ on $\partial \Delta(a, 1/2)$. Of course $u_a$ is equal to $\mathbb{I}_A$ on $\partial \mathbb{D}$, so is positive on $\partial \mathbb{D}$.

On the other hand, the function $\log(1/|\varphi_a|)$ is harmonic in $\mathbb{D} \setminus \Delta(a, 1/2)$ and is equal to 0 on $\partial \mathbb{D}$ and to $\log 2$ on $\partial \Delta(a, 1/2)$. Therefore, since $\partial \mathbb{D} \cup \partial \Delta(a, 1/2)$ is the boundary of $\mathbb{D} \setminus \Delta(a, 1/2)$, we obtain that

$$u_{a}(z) \geq \log \left(1/|\varphi_{a}(z)|\right) \quad \text{for } z \in \mathbb{D} \setminus \Delta(a, 1/2).$$

- It follows from (5.17) and (5.22) that

$$\log \left(1/|\varphi_{a}(z)|\right) \leq f_{a}(\arg z) + u_{a}(z) \quad \text{for all } z \in \mathbb{D}.$$

- We are now able to finish the proof.

We write

$$B_0 = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \varphi_{a_n}$$

with $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$ and $|a_n| \geq 3/4$. We have, by (5.22):

$$\log \left(1/|B_0(z)|\right) \leq \sum_{n=1}^{\infty} f_{|a_n|}(|\arg(\bar{a}_n z)|) + \sum_{n=1}^{\infty} u_{|a_n|}(z e^{-i\arg a_n}),$$

that is

$$\log \left(1/|B_0(z)|\right) \leq f(\arg z) + u(z),$$

with

$$f(\theta) = \sum_{n=1}^{\infty} f_{|a_n|}(|\arg(\bar{a}_n e^{i\vartheta})|)$$

and $u = P[w]$, where

$$w = 2 \log 2 \sum_{n=1}^{\infty} \mathbb{I}_{0 \leq |\arg a_n| \leq A|a_n|}.$$

We have $f, w \in L^1(-\pi, \pi)$, by invariance of the Lebesgue measure and since, we have $\|f_{|a_n|}\|_1 \leq C (1 - |a_n|)$ and $\|w_{|a_n|}\|_1 \leq C (1 - |a_n|)$, and $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$. That finishes the proof of Proposition 5.9. 

\[\square\]
Now, it is easy to end the proof of Theorem 5.8 and hence that of Theorem 5.5.

**End of the proof of Theorem 5.8.** We only have to write

\[ B = \left( \prod_{|a_n| < 3/4} \frac{|a_n|}{a_n} \right) \times B_0 \]

where \( B_0 \) is the Blaschke product made with the zeros of \( B \) of modulus \( \geq 3/4 \) (as usual \( |a_n|/a_n = 1 \) if \( a_n = 0 \)).

Then, with the notation of Lemma 5.10, if \( G = \sum_{|a_n| < 3/4} G_{a_n} \), we have, by Proposition 5.9, if \( |z| \geq 1/2 \):

\[
\log \left( \frac{1}{|B(z)|} \right) \leq G(\arg z) + f(\arg z) + u(z),
\]

with \( f \in L^1(-\pi, \pi) \) and \( u = P[w] \) with \( w \in L^1(-\pi, \pi) \).

Since the maximal radial function of \( u = P[w] \) is smaller than its Hardy-Littlewood maximal function \( M_u \) (actually equivalent: see [19], Theorem 11.20 and Exercise 19), by a well-known theorem of Hardy and Littlewood, we get:

\[
(5.25) \quad \sup_{1/2 \leq r < 1} \log \frac{1}{|B(re^{i\theta})|} \leq G(\theta) + f(\theta) + M_u(\theta).
\]

Now, \( G \in L^1(-\pi, \pi) \), by Lemma 5.10, and \( M_u \in L^{1,\infty}(-\pi, \pi) \), by the Kolmogorov theorem; therefore \( \log(1/|B|) \in L^{1,\infty}(-\pi, \pi) \), and that finishes the proof of Theorem 5.8. \( \square \)

**Remark.** The proofs of Theorem 5.5 and Theorem 5.8 show that if the weighted Bergman space \( B^2_U \) of analytic functions \( f \) such that \( \int_D |f|^2 U dA < \infty \), contains a function \( v \in H^\infty \), with \( v(0) = 1 \), then \( \log(1/I_v) \in L^{1,\infty}(0, 2\pi) \).

The result of Theorem 5.8 is essentially sharp, as said by the following result.

**Theorem 5.13.** There exists \( v \in H^\infty \), \( v \neq 0 \), such that \( \log(1/I_v) \notin L^1(0, 2\pi) \).

**Proof.** We start with

\[
\sigma(\theta) = \begin{cases} 
1/|\theta(\log \theta)^2|, & 0 < \theta \leq 1/2, \\
0, & \text{elsewhere.}
\end{cases}
\]

We have \( \sigma \in L^1(0, 2\pi) \), and we consider \( u = P[\sigma] \). Then \( u \) is positive and, since the Poisson kernel is positive and decreasing on \([0, \pi]\), we have, for \( 0 < \theta \leq 1/2 \):

\[
u(\rho e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_\rho(\theta - t) \sigma(t) dt \geq \frac{1}{2\pi} \int_0^\theta P_\rho(\theta - t) \sigma(t) dt \]

\[
\geq \frac{1}{2\pi} \int_0^\theta P_\rho(\theta) \sigma(t) dt = \frac{1}{2\pi} P_\rho(\theta) \frac{1}{\log(1/\theta)}.
\]
Taking $\rho = 1 - \theta$, we have, as $\theta$ goes to 0:

$$\frac{1 - \rho^2}{1 - 2\rho \cos \theta + \rho^2} = \frac{1 - \rho^2}{(1 - \rho)^2 + 2\rho(1 - \cos \theta)} \sim \frac{\theta(2 - \theta)}{\theta^2 + 2(1 - \theta) \theta^2/2} \sim \frac{1}{\theta}.$$ 

Hence:

$$u((1 - \theta) e^{i\theta}) \geq \frac{C}{\theta} \frac{1}{\log(1/\theta)}.$$

Therefore

$$\sup_{1/2 \leq \rho < 1} u(\rho e^{i\theta}) \geq \frac{C}{\theta \log(1/\theta)}.$$

Let now $g = u + \bar{u}$ and $v = \exp(-g)$. We have $|v| = e^{-u}$ and hence

$$I_v(\theta) \leq \exp \left( - \frac{C}{\theta \log(1/\theta)} \right)$$

and

$$\log(1/I_v) \geq \frac{C}{\theta \log(1/\theta)}.$$

Therefore $\log(1/I_v) \notin L^1(0, 2\pi)$.

5.3 An example

The fact that, in Theorem 5.5, the function $U$ has the particular form given in (5.8), in particular is logarithmically-subharmonic, is important. In fact, we have the following result.

Theorem 5.14. There exist a continuous function $U : \mathbb{D} \to \mathbb{C}$, with $U \geq 1$ and an analytic function $w : \mathbb{D} \to \mathbb{C}$, $w \not\equiv 0$, such that:

$$\int_{\mathbb{D}} |w|^2 U \, dA < \infty,$$

but

$$\int_0^1 U(r e^{i\theta}) \, dr = \infty, \quad \text{for almost all} \quad \theta.$$

To prove this, we use the following weak form of a result of Kahane and Katznelson [8].

Theorem 5.15 (Kahane-Katznelson). Given any positive increasing function $\omega : (0, 1) \to (0, \infty)$ such that $\omega(r) \to \infty$ and any pair of measurable functions $g, h : [0, 2\pi] \to \mathbb{R} = [-\infty, +\infty]$, there exists an analytic function $F : \mathbb{D} \to \mathbb{C}$ such that

1) $\max_{|z|=r} |F(z)| = o(\omega(r))$ as $r$ goes to 1;
2) $\lim_{r \to 1} \Re F(r e^{i\theta}) = g(\theta)$ and $\lim_{r \to 1} \Im F(r e^{i\theta}) = h(\theta)$, for almost all $\theta \in [0, 2\pi]$. 

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Proof of Theorem 5.14. The Kahane-Katznelson theorem shows that there exists a function \( w = \exp(-F) \) belonging to \( \mathcal{B}^2 \), and even in \( \bigcap_{\beta > 1} \mathcal{B}^2_{\beta} \), if we want, taking, for instance, \( \omega_0 \) for almost every \( \theta \). We have by c), and since \( w \) is continuous on \( \mathbb{T} \), because the sum is locally finite, since \( U_n = 0 \) out of \( G_n \). We have, by c), and since \( w \in \mathcal{B}^2 \):

\[
\int_{\mathbb{T}} |w|^2 U dA = \int_{\mathbb{T}} |w|^2 dA + \sum_{n=1}^{\infty} \int_{\mathbb{T}} |w|^2 U_n dA < \infty.
\]

Moreover, since \( \sum_{n=1}^{\infty} m(\mathbb{T} \setminus A_n) < \infty \), for almost all \( \theta \), there exists \( N(\theta) \geq 1 \) such that \( e^{i\theta} \in A_n \) for all \( n \geq N(\theta) \). Hence, for these \( \theta \):

\[
\int_{0}^{1} U(r e^{i\theta}) dr \geq \sum_{n=N(\theta)}^{\infty} \int_{\rho_n}^{\rho_n^u} U_n(r e^{i\theta}) dr \geq \sum_{n=N(\theta)}^{\infty} \int_{\rho_n}^{\rho_n^u} \frac{2}{1-\rho_n} dr
\]

\[
= \sum_{n=N(\theta)}^{\infty} (\rho_n^u - \rho_n) \frac{2}{1-\rho_n} \geq \sum_{n=N(\theta)}^{\infty} \frac{2}{3} = \infty.
\]

That finishes the proof of Theorem 5.14.
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