A NOTE ON CR MAPPINGS OF POSITIVE CODIMENSION

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(Communicated by Franc Forstneric)

Abstract. We prove the following Artin type approximation theorem for smooth CR mappings: given $M \subset \mathbb{C}^N$ a connected real-analytic CR submanifold that is minimal at some point, $M' \subset \mathbb{C}^{N'}$ a real-analytic subset, and $H : M \to M'$ a $C^\infty$-smooth CR mapping, there exists a dense open subset $\mathcal{O} \subset M$ such that for any $q \in \mathcal{O}$ and any positive integer $k$ there exists a germ at $q$ of a real-analytic CR mapping $H^k : (M, q) \to M'$ whose $k$-jet at $q$ agrees with that of $H$ up to order $k$.

1. Introduction

Given germs of real-analytic submanifolds $M$ and $M'$ embedded in complex spaces, a fundamental question is to decide whether the formal equivalence of $M$ and $M'$ implies their biholomorphic equivalence. While this need not be in general the case in view of a well known example due to Moser-Webster [12] (see also [7, 8]), recent results due to Baouendi, Mir, Rothschild and Zaitsev [5, 3] provide a partial positive answer when the submanifolds are furthermore assumed to be CR. In [5, 3], the positive solution is obtained by approximating in the Krull topology a given formal holomorphic equivalence by a convergent one, following the spirit of Artin’s approximation theorem [1]. In this paper, we prove the following Artin type approximation theorem for arbitrary smooth CR mappings of any positive codimension.

Theorem 1.1. Let $M \subset \mathbb{C}^N$ be a connected real-analytic CR submanifold that is minimal at some point, $M' \subset \mathbb{C}^{N'}$ be a real-analytic subset, and $H : M \to M'$ be a $C^\infty$-smooth CR mapping. Then there exists a dense open subset $\mathcal{O} \subset M$ such that for any $q \in \mathcal{O}$ and any positive integer $k$ there exists a germ at $q$ of a real-analytic CR mapping $H^k : (M, q) \to M'$ whose $k$-jet at $q$ agrees with that of $H$ up to order $k$.

Here minimality is meant in the sense of Tumanov (see Section 2 for the precise definition). To the author’s knowledge, Theorem 1.1 is the first result of its kind for mappings of positive codimension between arbitrary real-analytic submanifolds. When the target is a real-algebraic set instead of a real-analytic set, then Theorem 1.1 follows from the work of Meylan, Mir and Zaitsev [10]. Observe that
Theorem 1.1 is also new even in the case $N = N'$ since there is no rank assumption on the mapping under consideration (compare with [3, 3, 13]). On the other hand, we do not know whether one may choose in Theorem 1.1 the dense open subset $O \subset M$ to be a Zariski open subset independent of the mapping $H$. Note that when $M'$ is real-algebraic, such a choice is possible and follows from the main result of [10]. For more details related to Artin type approximation in CR geometry, we refer the reader to the survey paper [11].

In this paper we shall give a rather elementary and self-contained proof of Theorem 1.1. For this, we will use several main steps of [6] for which we will provide simplified proofs of the results needed for this paper.

We will organize the paper as follows. Section 2 contains some basic definitions and technical lemmas used in Section 3. In Section 3, we give some elementary properties of a complex-analytic set invariantly attached to a graph of a smooth CR-mapping. The last section is devoted to the proof of Theorem 1.1.

2. Preliminaries

In this section we first recall some basic definitions and prove a lemma used in Section 3. For basic background on CR analysis, we refer the reader to [2]. Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold of codimension $d$. Let us recall that $M$ is said to be minimal at $p \in M$ if there is no germ of a real submanifold $S \subset M$ through $p$ such that the complex tangent space of $M$ at $q$ is tangent to $S$ at every $q \in S$ and $\dim \mathbb{R} S < \dim \mathbb{R} M$ (see [2]).

Following [6], for a $C^\infty$-smooth CR mapping $f: M \to \mathbb{C}^{N'}$ and for $p \in M$, we denote $T_p(f)$ as the germ of the smallest complex analytic set in $\mathbb{C}^{N+N'}$ containing the germ of the graph of $f$ at $(p, f(p))$. The integer $\dim T_p(f) - N$ will be called the degree of partial analyticity of $f$ at $p$ and denoted by $\deg_p f$. We may observe that, if $M$ is minimal at $p$, the degree of partial analyticity of $f$ at $p$ is non-negative (see Remark 3.2).

We will also use the notion of wedge. For $p \in M$, we consider an open neighborhood $U$ of $p$ in $\mathbb{C}^N$ and a local defining real-analytic function $\rho: U \to \mathbb{R}^d$ of $M$ near $p$. If $\Gamma$ is an open convex cone in $\mathbb{R}^d$ with vertex at the origin, an open set $W$ of the form \{ $z \in U, \rho(z, \bar{z}) \in \Gamma$ \} is called a wedge of edge $M$ in the direction $\Gamma$ centered at $p$.

The following result is a lemma from [6] for which we provide a more elementary proof.
Lemma 2.2. Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold, minimal at $p \in M$, let $F: (M, p) \rightarrow \mathbb{C}^s$ and $u: (M, p) \rightarrow \mathbb{C}^t$ be two germs of $\mathcal{C}^\infty$-smooth CR mappings and let \( \psi: \left( \mathbb{C}^{2N+t+s}, (p, \bar{p}, u(p), F(p)) \right) \rightarrow \mathbb{C} \) be a germ at \( (p, \bar{p}, u(p), F(p)) \) of a holomorphic function. Assume that \( \psi \left( z, \bar{z}, u(z), F(z) \right) \equiv 0 \) for $z \in M$ near $p$ and that the function \( (z, w) \mapsto \psi \left( z, \bar{z}, u(z), w \right) \) is not identically zero near $(p, F(p))$ in $M \times \mathbb{C}^s$. Then there exists $q \in M$ as close as we want to $p$ such that $\deg_q F < s$.

Proof. This result will be proved by induction on the integer $s$. First, we consider the case where $s = 1$.

Let $A$ be the set of points $p_1$ near $p$ in $M$ such that the holomorphic function $\mathbb{C} \ni w \mapsto \psi \left( p_1, \bar{p}_1, u(p_1), w \right)$ is not identically zero near $F(p_1)$ in $\mathbb{C}$. We may find $p_1 \in A$ as close as we want to $p$. Indeed, since $\psi \left( z, \bar{z}, u(z), w \right)$ is not identically zero near $(p, F(p))$ in $M \times \mathbb{C}$, there exists $(p_1, p_1')$ as close as we want to $(p, F(p))$ such that $\psi \left( p_1, \bar{p}_1, u(p_1), p_1' \right) \neq 0$. So, the holomorphy of $\mathbb{C} \ni w \mapsto \psi \left( p_1, \bar{p}_1, u(p_1), w \right)$ implies it cannot be identically zero near $F(p_1)$ in $\mathbb{C}$. Moreover, for $z \in A$ fixed, since the holomorphic function $\mathbb{C} \ni w \mapsto \psi \left( z, \bar{z}, u(z), w \right)$ doesn’t vanish identically near $F(z)$ in $\mathbb{C}$, there exists a unique positive integer $k_z$ such that $\frac{\partial^{k_z} \psi}{\partial w^{k_z}} \left( z, \bar{z}, u(z), F(z) \right) \neq 0$ and, for any integer $k < k_z$, $\frac{\partial^k \psi}{\partial w^k} \left( z, \bar{z}, u(z), F(z) \right) = 0$.

Now we fix a sufficiently small open neighborhood $V$ of $p$ in $M$ and consider the integer

$$K = \min \{ k_z, z \in A \cap V \}.$$

We may pick $p_1 \in A \cap V$ such that $k_{p_1} = K$, and we have

\begin{equation}
\frac{\partial^{K-1} \psi}{\partial w^{K-1}} \left( p_1, \bar{p}_1, u(p_1), F(p_1) \right) = 0,
\end{equation}

\begin{equation}
\frac{\partial^K \psi}{\partial w^K} \left( p_1, \bar{p}_1, u(p_1), F(p_1) \right) \neq 0.
\end{equation}

Since $\psi$ is holomorphic near $\left( p, \bar{p}, u(p), F(p) \right)$ in $\mathbb{C}^{2N+t+1}$ by the implicit function theorem, there exists a germ at \( (p_1, \bar{p}_1, u(p_1)) \) of a holomorphic function

$$\Theta:\left( \mathbb{C}^{2N+t}, (p_1, \bar{p}_1, u(p_1)) \right) \rightarrow \mathbb{C}$$

such that the zeros of $\frac{\partial^{K-1} \psi}{\partial w^{K-1}}$ near $(p_1, \bar{p}_1, u(p_1), F(p_1))$ in $\mathbb{C}^{2N+t+1}$ are given by the equation $w = \Theta \left( z, \bar{z}, \nu \right)$. On the other hand, we may observe that the function $\frac{\partial^{K-1} \psi}{\partial w^{K-1}} \left( z, \bar{z}, u(z), F(z) \right)$ is identically zero near $p_1$ in $M$. Suppose, in order to reach a contradiction, that it is false. In this case, we may find $p_2$ as close as we want to $p_1$ in $M$ such that $\frac{\partial^{K-1} \psi}{\partial w^{K-1}} \left( p_2, \bar{p}_2, u(p_2), F(p_2) \right) \neq 0$, so there is a point $p_2 \in A \cap V$ with $k_{p_2} < K$. This is a contradiction in view of the definition of $K$. So $\frac{\partial^{K-1} \psi}{\partial w^{K-1}} \left( z, \bar{z}, u(z), F(z) \right) \equiv 0$ for $z \in M$ near $p_1$, and, from the remark on the zeros of $\frac{\partial^{K-1} \psi}{\partial w^{K-1}}$, we obtain that $F(z) = \Theta \left( z, \bar{z}, u(z) \right)$ near $p_1$ in $M$. But if $p_1$ is close enough to $p$, we may assume that $M$ is minimal at $p_1$ (since $M$ is real-analytic and minimal at $p$), and consequently we may apply Proposition 2.1 to obtain the
existence of a holomorphic extension $\tilde{F}$ of $F$ near $p_1$ in $\mathbb{C}^N$. Thus the graph of $F$ is contained, near $p_1$, in the graph of $\tilde{F}$, which is a complex analytic set of dimension $N$. Consequently, the dimension of $T_{p_1}(F)$ is less than or equal to $N$. So we proved that, for an arbitrary small neighborhood $V$ of $p$ in $M$, there exists $p_1 \in V$ such that $\deg_{p_1} F \leq 0$. This finishes the proof of the lemma for $s = 1$.

Now, we assume that the lemma holds for $s-1$, and for any $t \in \mathbb{N}$, any germs of CR mappings $F: (M, p) \to \mathbb{C}^{s-1}$, $u: (M, p) \to \mathbb{C}^t$ and any germ at $(p, \tilde{p}, u(p), F(p))$ of a holomorphic function $\psi: \left(\mathbb{C}^{2N+ts-s-1}, (p, \tilde{p}, u(p), F(p))\right) \to \mathbb{C}$. Our aim is to prove the same result for $(M, p)$.

First, we consider the case where $\psi \left(\bar{z}, \nu, w, F'(z), w_0\right) \equiv 0$ for $(z, w_0) \in M \times \mathbb{C}$ near $(p, F_s(p))$. Taking the Taylor series of $\psi$ in $w_0$ at $F_s(p)$, we obtain that, for any $k \in \mathbb{N}$, $\psi_k \left(z, \bar{z}, u(z), F'(z)\right) \equiv 0$ for $z \in M$ near $p$ and that there exists $k_0$ such that $\psi_{k_0} \left(z, \bar{z}, u(z), w\right)$ doesn’t vanish identically near $(p, F'(p))$ in $M \times \mathbb{C}^{s-1}$. So, by the induction hypothesis, there exists $q \in M$ as close as we want to $p$ such that $\deg_q F' < s-1$, which implies $\deg_q F < s$. This completes the proof for this case.

To finish the proof, we have to consider the case where $\psi \left(z, \bar{z}, u(z), F'(z), w_0\right)$ doesn’t vanish identically near $(p, F_s(p))$ in $M \times \mathbb{C}$. By the same method as in the case $s = 1$, we show that, for point $p_1$ as close as we want to $p$ where $M$ is minimal, there exists a germ at $\left(p_1, \bar{p}_1, u(p_1), F'(p_1)\right)$ of a holomorphic function $\Theta: \left(\mathbb{C}^{2N+ts-s-1}, (p_1, \bar{p}_1, u(p_1), F'(p_1))\right) \to \mathbb{C}$ such that

$$F_s(z) = \Theta \left(z, \bar{z}, u(z), F'(z)\right)$$

near $p_1$ in $M$. Let $\Theta(z, \bar{z}, u(z)) = \sum_{\alpha \in \mathbb{N}^{s-1}} \Theta_{\alpha}(z, \bar{z}, u(z)) (w' - F'(p_1))^\alpha$ be the Taylor series of $\Theta$ in $w'$ at $F'(p_1)$.

If every $\Theta_{\alpha}(z, \bar{z}, u(z))$ is CR near $p_1$ in $M$, then by Proposition 4.1 (recall that $M$ is minimal at $p_1$) the function $M \times \mathbb{C}^{s-1} \ni (z, w') \mapsto \Theta \left(z, \bar{z}, u(z), w'\right) \in \mathbb{C}$ can be holomorphically extended near $(p_1, F'(p_1))$ in $\mathbb{C}^{N+s-1}$. We denote the extension by $\tilde{\Theta}$. The graph of $F$ is contained in the graph of $\tilde{\Theta}$, which is a complex submanifold of $\mathbb{C}^{N+s}$ of dimension $N + s - 1$. This is equivalent to saying that the degree of partial analyticity of $F$ at $p_1$ is smaller than $s - 1$.

If there is a multi-index $\alpha \in \mathbb{N}^{s-1}$ such that the mapping $\Theta_{\alpha}(z, \bar{z}, u(z))$ is not CR, then there exists a vector field $\tilde{L} = \sum_{j=1}^N a_j(z, \bar{z}) \frac{\partial}{\partial w_j}$ near $p$ in $\mathbb{C}^N$, where $a_1, \ldots, a_N$ are real-analytic functions near $p$, such that $\tilde{L}|_M$ is a CR vector field and

$$\tilde{L} \left(\Theta \left(z, \bar{z}, u(z), w'\right)\right) \neq 0$$
near \((p_1, F'(p_1))\) in \(M \times \mathbb{C}^{s-1}\). Using the chain rule, we may observe that there exists a holomorphic function \(\Psi_1\) near \((p_1, p_1, \bar{L}(u(z)) \mid_{z=p_1}, F'(p_1))\) in \(\mathbb{C}^{2N+t+s-1}\) such that
\[
(2.5) \quad \bar{L} \left( \Theta \left( z, \bar{z}, u(z), w' \right) \right) = \Psi_1 \left( z, \bar{z}, \bar{L}(u(z)), w' \right)
\]

near \((p_1, F'(p_1))\) in \(M \times \mathbb{C}^{s-1}\). Since \(M\) is minimal at \(p\), Tumanov’s extension theorem (see [2]) implies that there exists a holomorphic extension \(\bar{u}\) of \(u\) in a wedge \(\mathcal{W}\) of edge \(M\) centered at \(p\). We may assume that the mapping \(\bar{u}\) is \(C^{\infty}\)-smooth on \(\mathcal{W}\) up to the edge \(M\). Moreover, for any \(z \in \mathcal{W} \cup (M \cap V)\), where \(V\) is a sufficiently small neighborhood of \(p\) in \(\mathbb{C}^N\),
\[
\bar{L} \left( \bar{u}(z) \right) = \sum_{j=1}^{N} a_j(z, \bar{z}) \frac{\partial \bar{u}}{\partial z_j}(z).
\]

Now, for any \(j \in \{1, \ldots, N\}\), \(\frac{\partial \bar{u}}{\partial z_j}\) is holomorphic in \(\mathcal{W}\) and \(C^{\infty}\) up to the edge \(M\). Thus, the mapping \(U\) whose components are the restrictions to \(M\) near \(p\) of the derivatives of \(\bar{u}\) is a CR mapping near \(p\) in \(M\). Consequently, from the identity \((2.5)\), and since \(\bar{L}(u(z)) = \bar{L}(\bar{u}(z))\), for \(z \in M\) close enough to \(p\), we may find a germ at \((p_1, p_1, U(p_1), F'(p_1))\) of a holomorphic function
\[
\Psi: \left( \mathbb{C}^{2N+t+N+t+s-1}, (p_1, p_1, U(p_1), F'(p_1)) \right) \rightarrow \mathbb{C}
\]
such that
\[
\bar{L} \left( \Theta \left( z, \bar{z}, \bar{u}(z), w' \right) \right) = \Psi \left( z, \bar{z}, \bar{U}(z), F'(z) \right)
\]

near \((p_1, F'(p_1))\) in \(M \times \mathbb{C}^{s-1}\). Moreover, we know that \(\bar{L} \left( \Theta(z, \bar{z}, \bar{u}(z), F'(z)) \right) \equiv \bar{L}(\bar{F}_s(z)) \equiv 0\) near \(p_1\) in \(M\), since \(F\) is CR; i.e. \(\Psi \left( z, \bar{z}, \bar{U}(z), F'(z) \right) \equiv 0\) near \(p_1\) in \(M\). So, since we saw that \(\Psi \left( z, \bar{z}, U(z), w' \right) \neq 0\) near \((p_1, F'(p_1))\) in \(M \times \mathbb{C}^{s-1}\), the induction assumption implies that the degree of partial analyticity of \(F'\) is strictly smaller than \(s-1\) for points in \(M\) as close as we want to \(p_1\) and therefore as close to \(p\). This finishes the proof of Lemma 2.2. \(\square\)

As in [6], one gets from Lemma 2.2 the following result.

**Lemma 2.3.** Let \(M \subset \mathbb{C}^N\) be a real-analytic generic submanifold, minimal at \(p \in M\), let \(F: (M, p) \rightarrow \mathbb{C}^s\) be a germ of a CR mapping and let
\[
\psi: \left( \mathbb{C}^{2N+2s}, (p, \bar{p}, \bar{F}(p), F(p)) \right) \rightarrow \mathbb{C}
\]
be a germ at \((p, \bar{p}, \bar{F}(p), F(p))\) of a holomorphic function. Assume that
\[
\psi \left( z, \bar{z}, \bar{F}(z), F(z) \right) \equiv 0
\]
for \(z \in M\) near \(p\) and that the function \((z, v, w) \mapsto \psi(z, \bar{z}, v, w)\) is not identically zero near \((p, \bar{F}(p), F(p))\) in \(M \times \mathbb{C}^{2s}\). Then, there exists \(q \in M\) as close as we want to \(p\) such that \(\deg_q F < s\).
Proof. First, we assume that \( \psi(z, \bar{z}, \overline{F(z)}, w) \) does not vanish identically near \((p, F(p))\) in \( M \times \mathbb{C}^s \). Since \( \psi(z, \bar{z}, \overline{F(z)}, F(z)) \equiv 0 \) for \( z \in M \) near \( p \), we may apply Lemma 2.2 and deduce that the degree of partial analyticity of \( F \) at \( q \) is strictly smaller than \( s \) for \( q \) arbitrarily close to \( p \).

Now we treat the case where \( \psi(z, \bar{z}, \overline{F(z)}, w) \equiv 0 \) for \((z, w) \in M \times \mathbb{C}^s \) near \((p, F(p))\). For this we consider the Taylor series of \( \psi \) in \( w \) at \( F(p) \):

\[
\psi(z, \zeta, v, w) = \sum_{\alpha \in \mathbb{N}^s} \psi_\alpha(z, \zeta, v) (w - F(p))^\alpha.
\]

The assumption implies that, for any \( \alpha \in \mathbb{N}^s \), \( \psi_\alpha \left( z, \bar{z}, \overline{F(z)} \right) \equiv 0 \) near \( p \) in \( M \). However, by assumption, there is a multi-index \( \alpha_0 \) such that \( \psi_{\alpha_0}(z, \bar{z}, v) \neq 0 \) near \((p, F(p))\) in \( M \times \mathbb{C}^s \), and Lemma 2.2 gives the desired result. \( \square \)

3. Properties of \( T_p(H) \)

In this section, we fix a real-analytic generic submanifold \( M \subset \mathbb{C}^N \) and a \( C^\infty \)-smooth CR mapping \( H : M \to M' \) on \( M \) with values in a real-analytic set \( M' \subset \mathbb{C}^{N'} \). We shall give some properties of the degree of partial analyticity of \( H \) and of the complex analytic set \( T_p(H) \) for \( p \in M \). All the results of this section can be found in [6], but since the proofs we shall give are rather elementary compared to [6], we include them in this paper for completeness.

The following lemma is a direct consequence of the boundary uniqueness theorem.

Lemma 3.1. Let \( M, M' \) and \( H \) be as above. If \( H \) admits a holomorphic extension \( \overline{H} \) on a wedge of edge \( M \) centered at \( p \), then the graph of \( \overline{H} \) near \((p, H(p))\) is contained in \( T_p(H) \).

Remark 3.2. If \( M \) is minimal at \( p \in M \), Tumanov’s extension theorem implies that, in the above setting, there is a unique extension \( \overline{H} \) of \( H \) holomorphic on a wedge of edge \( M \) centered at \( p \). Thus \( T_p(H) \) contains the graph of \( \overline{H} \) near \((p, H(p))\). This implies that the dimension of \( T_p(H) \) at \( p \) is greater than \( N \), i.e. that the degree of partial analyticity of \( H \) at \( p \) is non-negative.

The two following lemmas describe the regular points of the complex analytic set \( T_p(H) \).

Lemma 3.3. In the above setting, assume that the degree of partial analyticity of \( H \) is constant near some point \( p \in M \). Then there exists an open neighborhood \( U_p^1 \) of \( p \) in \( M \) such that the set \( \Sigma_p^1 \subset M \cap U_p^1 \) of points \( q \in U_p^1 \) for which \( T_p(H) \) is not regular at \((q, H(q))\) is a closed set with empty interior.

Proof. We may find an open neighborhood \( U_p^1 \) of \( p \) in \( M \) on which \( T_p(H) \) is a complex analytic set and on which the degree of partial analyticity of \( H \) is constant equal to \( s \). Let \( \Sigma_p^1 \subset M \cap U_p^1 \) be the set of points \( q \in U_p^1 \) for which \( T_p(H) \) is not regular at \((q, H(q))\). In view of the classical definition of regular points of a complex analytic set, \( \Sigma_p^1 \) is a closed subset of \( M \cap U_p^1 \). To prove that its interior is empty, assume by contradiction that we may find an open subset \( V \) of \( M \) contained in \( \Sigma_p^1 \). Thus, for any \( q \in V \), the graph of \( H \) near \((q, H(q))\) is contained in the set of
the singular points of $\mathcal{T}_p(H)$, which is a complex analytic set of dimension strictly smaller than $N + s$. So, the dimension of $\mathcal{T}_q(H)$ is also strictly smaller than $N + s$ for any $q \in V$. This is impossible, since the degree of partial analyticity of $H$ is constant equal to $s$ on $V$. □

Lemma 3.4. In the above setting, assume that $M$ is minimal at $p \in M$, that the degree of partial analyticity of $H$ is constant equal to $s$ near $p$ in $M$, and write $t = N' - s$. Then there are an open neighborhood $U_p \subset U_2^p$ of $p$ in $M$ (with $U_p^1$ given by Lemma 3.3) and a closed set with empty interior $\Sigma_p^2 \subset M \cap U_2^p$ such that, for any $q \in U_2^p \setminus \Sigma_p^2$, there are holomorphic coordinates $(u',v') \in \mathbb{C}^s \times \mathbb{C}^t$ near $H(q)$ for which $H = (F,G) \in \mathbb{C}^s \times \mathbb{C}^t$, and a germ at $(q,F(q))$ of a holomorphic mapping $T_q$: $(\mathbb{C}^{N+s},(q,F(q))) \to \mathbb{C}^t$ such that $\mathcal{T}_p(H)$ is given near $(q,H(q))$ by the equation $v' = T_q(z,u')$. In particular $\mathcal{T}_p(H)$ is regular at $(q,H(q))$.

Proof. Since $M$ is minimal at $p$ and $H$ is CR on $M$, Tumanov’s extension theorem implies that there exists a holomorphic extension $\tilde{H}$ of $H$ in a wedge $W$ of edge $M$ centered at $p$ which is $C^\infty$-smooth up to the edge. By Lemma 3.1 the graph of $\tilde{H}$ is contained in $\mathcal{T}_p(H)$ near $(p,H(p))$. It means that we may choose an open neighborhood $\Delta$ of $p$ in $\mathbb{C}^N$ such that $(z,\tilde{H}(z)) \in \mathcal{T}_p(H)$ for any $z \in \Delta \cap W$.

On the other hand, if $U_p^1$ is the open neighborhood of $p$ given by Lemma 3.3, we define $U_p^2 = U_p^1 \cap \Delta$ and $\Sigma_p^2 = \Sigma_p^1 \cap U_p^2$. Now, for a fixed point $q \in U_p^2 \setminus \Sigma_p^2$, $\mathcal{T}_p(H)$ is regular at $(q,H(q))$. Moreover, since the degree of partial analyticity of $H$ is constant equal to $s$ on $U_p^1$, the dimension of $\mathcal{T}_p(H)$ at $(q,H(q))$ is $N + s$. So, there exist an open neighborhood $U \subset U_p^2$ of $q$ in $\mathbb{C}^N$, an open neighborhood $V$ of $H(q)$ in $\mathbb{C}^{N'}$ and a holomorphic mapping $f: U \times V \to \mathbb{C}^t$ of rank $t$ at $(q,H(q))$ such that

$$\mathcal{T}_p(H) = \{(z,z') \in U \times V, \ f(z,z') = 0\}.$$  

Thus, for $z \in U' = U \cap W \cap H^{-1}(V)$, we have

$$f(z,\tilde{H}(z)) = 0.$$

So $\frac{\partial f}{\partial z}(z,\tilde{H}(z)) + \frac{\partial f}{\partial z'}(z,\tilde{H}(z)) = 0$ in $U'$. Since the mapping $\tilde{H}$ is $C^\infty$-smooth up to the edge $M$, we have the following identity:

$$\frac{\partial f}{\partial z}(z,\tilde{H}(z)) + \frac{\partial f}{\partial z'}(z,\tilde{H}(z)) \cdot \frac{\partial \tilde{H}}{\partial z}(z) = 0$$

on $M \cap U'$. But $f$ is of rank $t$ at $(q,H(q))$ and the identity (3.3) shows that the columns of $\frac{\partial f}{\partial z}(q,H(q))$ are a linear combination of the columns of $\frac{\partial f}{\partial z'}(q,H(q))$. Consequently the rank of $\frac{\partial f}{\partial z}(q,H(q))$ is $t$. So, by the implicit function theorem there exist holomorphic coordinates $(u',v') \in \mathbb{C}^s \times \mathbb{C}^t$ for which $H = (F,G) \in \mathbb{C}^s \times \mathbb{C}^t$ and a holomorphic mapping $T_q$ near $(q,F(q))$ in $\mathbb{C}^{N+s}$ such that the zeros of $f$ are given by points of the form $(z,u',T_q(z,u')) \in \mathbb{C}^N \times \mathbb{C}^s \times \mathbb{C}^t$. □

For the next result, we will denote $\pi: \mathbb{C}^N \times \mathbb{C}^{N'} \to \mathbb{C}^N$ and $\pi': \mathbb{C}^N \times \mathbb{C}^{N'} \to \mathbb{C}^{N'}$ as the canonical projections.

Lemma 3.5. In the above setting, assume that $M$ is minimal at $p \in M$ and that the degree of partial analyticity of $H$ is constant equal to $s$ near $p$ in $M$. Then there are an open neighborhood $U_p^{3} \subset U_p^{2}$ of $p$ in $M$ (with $U_p^{2}$ given by Lemma 3.3) and
a closed set with empty interior $\Sigma^3_p \subset M \cap U^3_p$ such that, for any $q \in U^3_p \setminus \Sigma^3_p$, there is a neighborhood $\Omega_q$ of $(q, H(q))$ in $\mathbb{C}^{N+N'}$ satisfying

$$\pi' (T_p (H) |_{M \times \mathbb{C}^{N'} \cap \Omega_q}) \subset M'.$$

Proof. Let $\rho': \mathbb{C}^{N'} \to \mathbb{R}^{d'}$ be a local defining real-analytic function of $M'$ near $H(p)$. Since $H(M) \subset M'$, we have the mapping identity

$$(3.4) \quad \rho' \left( \frac{H(z), H(z)}{z} \right) \equiv 0$$

for $z \in M$ near $p$.

We consider $U^2_p$ and $\Sigma^2_p$ respectively as the open neighborhood of $p$ in $M$ and the closed set with empty interior given by Lemma [3.4]. Let $U^3_p \subset U^2_p$ be a sufficiently small connected neighborhood of $p$ in $M$. Since $M$ is real-analytic and minimal at $p$, the set $\Sigma_1$ of points in $U^3_p$ where $M$ is not minimal is a closed set with empty interior. Thus, $\Sigma^3_p = (\Sigma^2_p \cap U^3_p) \cup \Sigma_1$ is a closed set with empty interior. Moreover, for any $q \in U^3_p \setminus \Sigma^3_p$, $M$ is minimal at $q$, and we may find holomorphic coordinates $(u', v') \in \mathbb{C}^s \times \mathbb{C}^t$ near $H(q)$ for which $H = (F, G) \in \mathbb{C}^s \times \mathbb{C}^t$, and a germ at $(q, F(q))$ of a holomorphic mapping $T_q: (\mathbb{C}^{N+s}, (q, F(q))) \to \mathbb{C}^t$ such that $T_p(H)$ is given near $(q, H(q))$ by the equation $v' = T_q(z, u')$. If $U^3_p$ is chosen small enough we may assume that the graph of $H$ is contained near $q \in U^3_p \setminus \Sigma^3_p$ in $T_p(H)$; i.e. we have the mapping identity

$$(3.5) \quad G(z) = T_q(z, F(z))$$

for $z \in M$ near $q$. We may also assume that the mapping defined by

$$(3.6) \quad \Psi(z, \zeta, u') = \rho' \left( u', T_q(z, u'), v', T_q(z, \Omega^{-1}) \right)$$

is holomorphic near $(q, \bar{q}, F(q), F(q))$ in $\mathbb{C}^{2N+2s}$ for any $q \in U^3_p \setminus \Sigma^3_p$.

From (3.5) and (3.6), we obtain

$$(3.7) \quad \rho' \left( F(z), T_q(z, F(z)), \bar{F}(z), T_q(z, \bar{F}(z)) \right) \equiv 0$$

for $z \in M$ near $q$. Thus, by (3.7), we have

$$(3.8) \quad \Psi \left( z, \bar{z}, \bar{F}(z), F(z) \right) \equiv 0$$

for $z \in M$ near $q$. Since the degree of analyticity of $H$ is constant equal to $s$ on $U^3_p$, from (3.5), we deduce that the degree of partial analyticity of $F$ is constant equal to $s$ on $U^3_p$. So, the identity (3.8) and Lemma [5.1] (recall that $M$ is minimal at $q$) imply that

$$(3.9) \quad \Psi(z, \bar{z}, u', v') \equiv 0$$

for $(z, v', u') \in M \times \mathbb{C}^{2s}$ near $(q, \bar{F}(q), F(q))$. This identity is equivalent to

$$(3.10) \quad \rho' \left( u', T_q(z, u'), \bar{u}', T_q(z, \bar{u}') \right) \equiv 0$$

for $(z, u', v') \in M \times \mathbb{C}^{2s}$ near $(q, \bar{F}(q), F(q))$. So taking $u' = \bar{u}'$ in (3.10), we obtain that $(u', T_q(z, u')) \in M'$ as soon as $z$ is close enough to $q$ in $M$ and $u'$ is close enough to $F(q)$ in $\mathbb{C}^s$. This finishes the proof of the lemma.

We conclude this section with a lemma which makes use of the upper semi-continuity of the partial analyticity degree; we leave the details to the reader.
Lemma 3.6. In the above setting, there exists a closed set with empty interior $\Sigma_2 \subset M$ such that the degree of partial analyticity of $H$ is constant on each connected component of $M \setminus \Sigma_2$.

4. Proof of Theorem 1.1

In this section, we keep the assumptions and the notation of Section 3. i.e. we consider a real-analytic generic submanifold $M$ in $\mathbb{C}^N$ and a $C^\infty$-smooth CR mapping $H: M \to M'$ where $M' \subset \mathbb{C}^{N'}$ is a real-analytic set. Theorem 1.1 will be a consequence of the following result.

Proposition 4.1. In the above setting, assume that $M$ is minimal at $p \in M$ and that the degree of partial analyticity of $H$ is constant equal to $s$ near $p$ in $M$. Let $U_p^3$ and $\Sigma_p^3 \subset M \cap U_p^3$ be, respectively, the open neighborhood of $p$ in $M$ and the closed set with empty interior given by Lemma 3.5. Then for any $q \in U_p^3 \setminus \Sigma_p^3$ and for any positive integer $k$ there exists a germ at $q$ of a real-analytic CR mapping $H^k: (M, q) \to M'$ whose $k$-jet at $q$ agrees with that of $H$ up to order $k$.

Proof. For any $q \in U_p^3 \setminus \Sigma_p^3$, we may choose holomorphic coordinates $z' = (u', v') \in \mathbb{C}^s \times \mathbb{C}^t$ and a germ at $(q, F(q))$ of a holomorphic mapping $T_q: (\mathbb{C}^{N+s}, (q, F(q))) \to \mathbb{C}^t$ such that $T_q(H)$ is given by the equation $v' = T_q(z, u')$ near $(q, H(q))$. We may also choose a neighborhood $\Omega_q$ of $(q, H(q))$ in $\mathbb{C}^{N+s}$ such that $\pi'(T_q(H)|_{M \times \mathbb{C}^{N'} \cap \Omega_q}) \subset M'$.

We fix a point $q \in U_p^3 \setminus \Sigma_p^3$ and a positive integer $k$. Since the graph of $H$ is contained in $T_q(H)$ near $(q, H(q))$, we have the following mapping identity:

\begin{equation}
G(z) = T_q(z, F(z))
\end{equation}

near $q$ in $M$. Let $F^k$ be the $k$-th order Taylor polynomial of $F$ at $q$ (that is holomorphic since $H$ is CR; see [2]). We define the holomorphic mapping $G^k$ near $q$ in $\mathbb{C}^N$ by setting

\begin{equation}
G^k(z) = T_q(z, F^k(z)).
\end{equation}

Thus $H^k = (F^k, G^k)$ is a holomorphic mapping near $q$ in $\mathbb{C}^N$ with values in $\mathbb{C}^{N'}$. Moreover, by definition of $H^k$ and from (1.1), the $k$-th derivatives of $H^k$ at $q$ coincide with that of $H$. So to complete the proof of Proposition 4.1, we have to show that $H^k$ sends $M$ into $M'$ near $q$. From the definition of $H^k$ and the local defining equation of $T_q(H)$ near $(q, H(q))$ in $\mathbb{C}^{N+s}$, we obtain that there exists an open neighborhood $\tilde{\Omega}_q$ of $(q, H(q))$ in $\mathbb{C}^{N+s}$ such that

\[ G_{H^k} \cap \left( M \times \mathbb{C}^{N'} \right) \cap \tilde{\Omega}_q \subset T_q(H)|_{M \times \mathbb{C}^{N'} \cap \Omega_q}, \]

where $G_{H^k}$ is the graph of the mapping $H^k$. Since $\pi'(T_q(H)|_{M \times \mathbb{C}^{N'} \cap \Omega_q}) \subset M'$ by Lemma 3.5, the conclusion of the proposition follows.\[\square\]

Now, we are able to prove our main result, Theorem 1.1.

Proof of Theorem 1.1. We first note that Theorem 1.1 holds in the case where $N = 1$ or $N' = 1$. We may therefore assume that $N, N' \geq 2$.

We first treat the case where $M$ is generic. Since $M$ is real-analytic and connected, there exists a real-analytic subvariety $\Sigma_1$ of $M$ such that $M$ is minimal at each point $p \in M \setminus \Sigma_1$. Moreover, from Lemma 3.5, there exists a closed set with empty interior $\Sigma_2 \subset M$ such that the degree of partial analyticity of $H$ is constant
on each connected components of $M \setminus \Sigma_2$. Thus, $\Sigma = \Sigma_1 \cup \Sigma_2$ is a closed subset of $M$ with empty interior. Fix a point $p \in M \setminus \Sigma$; by Proposition 4.1 we may find an open subset $U_p^3$ of $p$ in $M$ and a closed set with empty interior $\Sigma_p^3 \subset M \cap U_p^3$ such that, for every $q \in U_p^3 \setminus \Sigma_p^3$, the conclusion of Theorem 1.1 holds at $q$. Thus, the set $O = \bigcup_{p \in M \setminus \Sigma} \left( U_p^3 \setminus \Sigma_p^3 \right)$ does the job, and this finishes the proof of Theorem 1.1 for the generic case.

If $M$ is not generic, for any $p \in M \setminus \Sigma_1$ (where $\Sigma_1$ again denotes the set of nonminimal points of $M$) we may assume, thanks to a local holomorphic change near $p$, that $M = \tilde{M}_p \times \{0\} \subset \mathbb{C}^{N-r_1} \times \mathbb{C}^{r_1}$, where $r_1$ is a non-negative integer and $\tilde{M}_p$ is a connected real-analytic generic submanifold which is minimal (see [2]). From the generic case treated above, there exists a dense open subset $\tilde{O}_p \subset \tilde{M}_p$ such that, for any non-negative integer $k$ and any $\tilde{q} \in \tilde{O}_p$, there exists a germ at $\tilde{q}$ of a real-analytic CR mapping $H_{\tilde{q}}^k : (\tilde{M}_p, \tilde{q}) \rightarrow M'$ whose $k$-jet at $\tilde{q}$ agrees with that of $\tilde{M}_p \ni z_1 \mapsto H(z_1, 0)$. Since $\bigcup_{p \in M \setminus \Sigma_1} \left( \tilde{O}_p \times \{0\} \right)$ is a dense open subset of $M$, the proof of Theorem 1.1 is complete. □

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