INVARIANCE OF A LENGTH ASSOCIATED TO A REDUCTION

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Abstract. Let $(A, m)$ be a $d$-dimensional Cohen-Macaulay local ring with infinite residue field and let $J$ be a minimal reduction of $m$. We show that $\lambda(m^3/Jm^2)$ is independent of $J$.

The notion of minimal reduction of an ideal in a local Noetherian ring $(A, m)$ was introduced by Northcott and Rees [5]. It has significant applications in the theory of Hilbert function of $m$-primary ideals in $A$, particularly in the case when $A$ is Cohen-Macaulay. Minimal reductions of an ideal $I$ are highly non-unique. In fact if the residue field of $A$ is infinite and $I$ is $m$-primary then any $d$ general linear combinations of generators of $I$, (here $d = \dim A$), gives a minimal reduction of $I$.

However when $A$ is Cohen-Macaulay with infinite residue field there are some invariants of the ring and the ideal $I$ which are independent of minimal reductions of $I$. To state it let us fix some notation. Let $I$ be an $m$-primary ideal of $A$ and let $J$ be a minimal reduction of $I$. We let $\lambda(N)$ denote the length of an $A$-module $N$ and $\mu(N)$ number of its minimal generators. Let $G_I(A) = \oplus_{n \geq 0} I^n/I^{n+1}$ be the associated graded ring of $A$ with respect to $I$. We let $e_0^0(A)$ denote the multiplicity of $A$ with respect to $I$.

It is well known that

1. $\lambda(I/J) = e_0^0(A) - \lambda(A/I)$ due to Serre, cf. [1, Theorem 4.7.6].
2. $\lambda(I^2/JI) = e_0^0(A) + (d - 1)\lambda(A/I) - \lambda(I/I^2)$ [6, Lemma 1].

Furthermore if depth $G_I(A) \geq d - 1$ then $\lambda(I^{n+1}/JI^n)$ is independent of $J$ [4, Corollary 3.9]. Furthermore in this case the coefficients of the Hilbert polynomial of $A$ with respect to $I$ can also be expressed in terms of $\lambda(I^{n+1}/JI^n)$, with $n \geq 0$ [3, Corollary 2.1]. So it is of some interest to see how these length behaves when we do not have any assumptions on depth $G_I(A)$. In general $\lambda(I^3/JI^2)$ is not independent of minimal reduction (see Example 2).

Our result shows that $\lambda(m^3/Jm^2)$ is invariant of minimal reduction $J$ of $m$.

The fact essentially used in our proof is the following result: let $I$ be an $m$-primary ideal of $A$ and let $J$ be a reduction of $I$, and suppose that $J$ is minimally generated by $x_1, \ldots, x_n$. Then $J$ is a minimal reduction of $I$ if and only if the elements $x_1, \ldots, x_n$ are analytically independent in $I$ and $n = \dim A$. Recall that $x_1, \ldots, x_n$ are analytically independent in $I$ if whenever $f(X_1, \ldots, X_n)$ is a homogeneous polynomial of degree $m$ in...
such that \( f(x_1, \ldots, x_n) \in I^m m \) then all coefficients of \( f \) are in \( m \).

**Theorem 1.** Let \((A, m)\) be a Cohen-Macaulay local ring of dimension \( d \geq 1 \) with infinite residue field. If \( J \) is a minimal reduction of \( m \) then we have an equality

\[
\lambda(m^3/Jm^2) = e + (d - 1)\mu(m) - \mu(m^2) - \binom{d - 1}{2}.
\]

**Proof.** When \( d = 1 \) we have to show that \( \mu(m^2) + \lambda(m^3/Jm^2) = e \). This is well known cf. [11, Theorem 6.18]. So we assume that \( \dim A \geq 2 \). We assert that it suffices to construct an exact sequence

\[
(1) \quad 0 \to \left( \frac{A}{m} \right)_{m^d} \xrightarrow{\psi_d} \left( \frac{m}{m^2} \right)^d \xrightarrow{\phi_d} \frac{Jm}{Jm^2} \to 0
\]

Suppose we have the exact sequence as claimed. Then we prove the result as follows: The exact sequence \((1)\) gives that

\[
(2) \quad \lambda \left( \frac{Jm}{Jm^2} \right) = d\mu(m) - \binom{d}{2}
\]

We also have

\[
\lambda(Jm/Jm^2) = \lambda(A/Jm^2) - \lambda(A/Jm) = \lambda(A/m^3) + \lambda(m^3/Jm^2) - \lambda(A/m^2) - \lambda(m^2/Jm) = \lambda(m^2/m^3) + \lambda(m^3/Jm^2) - \lambda(m^2/Jm)
\]

\[
= \mu(m^2) + \lambda(m^3/Jm^2) - (e - (1 + \mu(m) - d))
\]

So by \((2)\) we get

\[
\mu(m^2) + \lambda(m^3/Jm^2) = e - (1 + \mu(m) - d) + d\mu(m) - \binom{d}{2} = (d - 1)\mu(m) + e + d - 1 - \binom{d}{2} = (d - 1)\mu(m) + e - \binom{d - 1}{2}
\]

So it remains to construct the exact sequence \((1)\).

The author thanks the referee for indicating a simpler proof than the original.

Denote the Koszul complex on \( x_1, \ldots, x_d \) by \( K_A(x) \). Consider part of \( K_A(x) \)

\[
A_{m^d} \xrightarrow{\psi_d} A^d \xrightarrow{\phi_d} A.
\]

Then Image \( \Psi_d \subseteq m^{\oplus d} \) and \( \Phi_d(m^{\oplus d}) \subseteq Jm \), which gives a right exact sequence

\[
A_{m^d} \to m^{\oplus d} \to Jm \to 0.
\]
Now tensoring this by $A/\mathfrak{m}$ we get the sequence (1) which is right exact.

Hence we only need to prove that $\psi_d$ is injective. Let $\{e_i \mid 1 \leq i \leq d\}$ and $\{e_i \wedge e_j \mid i < j\}$ be the canonical bases of $(A/\mathfrak{m})^d$ and $(A/\mathfrak{m})^\binom{d}{2}$ respectively. Let $\sum_{i<j} m_{ij} e_i \wedge e_j \in \ker \psi_d$ then $\psi_d(\sum m_{ij} e_i \wedge e_j) = \sum \lambda_i (m_{i+1, i+1} x_{i+1} + \cdots + m_{d, i} x_d)$.

By analyticity we get $m_{ij} \in \mathfrak{m}$ for all $j$ and $i$. Therefore we have constructed an exact sequence (1) as claimed and as noted before this completes the proof. □

Next we give an example to show that if $I \neq \mathfrak{m}$ then the assertion of Theorem 1. does not hold. This example was constructed by Huckaba [2, 3.1] to show that reduction number of $I$ with respect to to minimal reduction $J$ of $I$, depends on $J$.

**Example 2.** Let $A = k[[x, y]]$ and $I = (x^7, x^6 y, x^2 y^5, y^7)A$. The ideals $J_1 = (x^7, y^7)A$ and $J_2 = (x^7, x^6 y + y^7)$ are minimal reductions of $I$. Note that all the ideals involved are homogeneous. So to compute lengths $\lambda(I^3/J_1 I^2)$ and $\lambda(I^3/J_2 I^2)$ we may work in the polynomial ring $k[x, y]$ and use some computer algebra package (we used ‘Singular’) to get $\lambda(I^3/J_1 I^2) = 3$, and $\lambda(I^3/J_2 I^2) = 2$.

Our theorem also prompts the following:

**Question 3.** Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ with infinite residue field. If $J$ is a minimal reduction of $\mathfrak{m}$ then is $\lambda(\mathfrak{m}^4/J \mathfrak{m}^3)$ independent of $J$?

Even though I personally think that this question does not have a positive answer, I have not been able to get a counter-example.

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