Biharmonic homomorphisms of Riemannian Lie groups

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1. Introduction

Let \( \phi : (M, g) \rightarrow (N, h) \) be a smooth map between two Riemannian manifolds with \( m = \dim M \) and \( n = \dim N \). We denote by \( \nabla^M \) and \( \nabla^N \) the Levi-Civita connections associated respectively to \( g \) and \( h \) and by \( T^d N \) the vector bundle over \( M \) pull-back of \( TN \) by \( \phi \). It is an Euclidean vector bundle and the tangent map of \( \phi \) is a bundle homomorphism \( d\phi : TM \rightarrow T^d N \). Moreover, \( T^d N \) carries a connection \( \nabla^d \) pull-back of \( \nabla^N \) by \( \phi \) and there is a connection on the vector bundle \( \text{End}(TM, T^d N) \) given by

\[
(\nabla_X A)(Y) = \nabla_X^M A(Y) - A(\nabla^M_X Y), \quad X, Y \in \Gamma(TM), A \in \Gamma(\text{End}(TM, T^d N)).
\]

The map \( \phi \) is called harmonic if it is a critical point of the energy \( E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g \). The corresponding Euler-Lagrange equation for the energy is given by the vanishing of the tension field

\[
\tau(\phi) = tr_g \nabla d\phi = \sum_{i=1}^m (\nabla_{E_i} d\phi)(E_i),
\]

where \((E_i)_{i=1}^m\) is a local frame of orthonormal vector fields. Note that \( \tau(\phi) \in \Gamma(T^d N) \). The map \( \phi \) is called biharmonic if it is a critical point of the bienergy of \( \phi \) defined by \( E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g \). The corresponding Euler-Lagrange equation for the bienergy is given by the vanishing of the bitension field

\[
\tau_2(\phi) = -tr_g (\nabla^d)^2 \tau(\phi) - tr_g R^N(\tau(\phi), d\phi(\cdot)) d\phi(\cdot) = -\sum_{i=1}^m \left( (\nabla^d)^2_{E_i, E_i} \tau(\phi) + R^N(\tau(\phi), d\phi(E_i)) d\phi(E_i) \right),
\]

where \((E_i)_{i=1}^m\) is a local frame of orthonormal vector fields, \((\nabla^d)^2_{X,Y} = \nabla^d_X \nabla^d_Y - \nabla^d_{[X,Y]} \) and \( R^N \) is the curvature of \( \nabla^N \) given by

\[
R^N(X,Y) = \nabla^N_X \nabla^N_Y - \nabla^N_Y \nabla^N_X - \nabla^N_{[X,Y]}.
\]
The theory of biharmonic maps is old and rich and has gained a growing interest in the last decade see \cite{3, 17} and others. The theory of biharmonic maps into Lie groups, symmetric spaces or homogeneous spaces has been extensively studied related to the integrable systems by many mathematicians (see for examples \cite{6, 19, 20}). In particular, harmonic maps of Riemann surfaces into compact Lie groups equipped with a bi-invariant Riemannian metric are called principal chiral models and intensively studied as toy models of gauge theory in mathematical physics \cite{21}. Curiously, there is no detailed study of harmonic or biharmonic homomorphisms between Riemannian Lie groups. To our knowledge, the only works on this topic are \cite{13, 12} where the authors studied harmonic inner automorphisms of a compact semi-simple Lie group endowed with a left invariant Riemannian metric. In this paper, we investigate biharmonic homomorphisms between Riemannian Lie groups. At first sight, this class can be mainly used to build examples but, as we will see through this paper, its study gives rise to some interesting mathematical problems in the theory of Riemannian Lie groups. This class can be enlarged non trivially without extra work as follows. Let $\phi : G \rightarrow H$ be a biharmonic homomorphism between two Riemannian Lie groups, $\Gamma_1$ and $\Gamma_2$ are two discrete subgroups of $G$ and $H$, respectively, with $\phi(\Gamma_1) \subset \Gamma_2$. Then $\Gamma_1/G$ and $\Gamma_2/H$ carry two Riemannian metrics and $\pi \circ \phi : G \rightarrow \Gamma_2/H$ factor to a smooth map $\phi : \Gamma_1/G \rightarrow \Gamma_2/H$ which is biharmonic (harmonic if $\phi$ is harmonic).

The paper is organized as follows. In Section 2, we characterize at the level of Lie algebras harmonic and biharmonic group satisfies the property that any inner automorphism is harmonic i.

2. Biharmonic homomorphisms between Riemannian Lie groups: general properties and first examples

Let $(G, g)$ be a Riemannian Lie group. If $g = T_e G$ is its Lie algebra and $\langle , \rangle_g = g(e)$ then there exists a unique bilinear map $A : g \times g \rightarrow g$ called the Levi-Civita product associated to $(g, \langle , \rangle_g)$ given by the formula:

$$2\langle A_u v, w \rangle_g = \langle [u, v]^g, w \rangle_g + \langle [w, u]^g, v \rangle_g + \langle [w, v]^g, u \rangle_g. \quad (3)$$

A is entirely determined by the following properties:

1. for any $u, v \in g$, $A_u v - A_v u = [u, v]^g$,
2. for any $u, v, w \in g$, $\langle A_u v, w \rangle_g + \langle v, A_u w \rangle_g = 0.$

\footnote{A biharmonic homomorphism between Riemannian Lie groups is a homomorphism of Lie groups $\phi : G \rightarrow H$ which is also biharmonic where $G$ and $H$ are endowed with left invariant Riemannian metrics.}
If we denote by $u^\ell$ the left invariant vector field on $G$ associated to $u \in \mathfrak{g}$ then the Levi-Civita connection associated to $(G, \mathfrak{g})$ satisfies $\nabla_{u^\ell} v^\ell = (A_u v)^\ell$. The couple $(\mathfrak{g}, \langle \cdot, \cdot \rangle_\mathfrak{g})$ defines a vector say $U^\mathfrak{g} \in \mathfrak{g}$ by

$$\langle U^\mathfrak{g}, v \rangle_\mathfrak{g} = \text{tr}(\text{ad}_v), \quad \text{for any } v \in \mathfrak{g}. \quad (4)$$

One can deduce easily from (3) that, for any orthonormal basis $(e_i)_{i=1}^n$ of $\mathfrak{g}$,

$$U^\mathfrak{g} = \sum_{i=1}^n A_e e_i. \quad (5)$$

Note that $\mathfrak{g}$ is unimodular iff $U^\mathfrak{g} = 0$. We denote by $\text{Isin}(\mathfrak{g})$ the subgroup of $G$ consisting of $a \in G$ such that $i_a$ is an isometry. We have

$$\text{Isin}(\mathfrak{g}) = \left\{ a \in G, \text{Ad}_a^\mathfrak{g} \circ \text{Ad}_a = \text{id}_{\mathfrak{g}} \right\}.$$  

Thus $\text{Isin}(\mathfrak{g})$ is a closed subgroup containing the center $Z(G)$. We denote by $\text{Kill}(\mathfrak{g})$ its Lie algebra given by

$$\text{Kill}(\mathfrak{g}) = \{ u \in \mathfrak{g}, \text{ad}_u + \text{ad}_u^* = 0 \}.$$  

Remark that $\text{Kill}(\mathfrak{g})$ can be identified with the Lie algebra of left invariant Killing vector fields of $\mathfrak{g}$ and if $G$ is nilpotent then $\text{Isin}(\mathfrak{g}) = Z(G)$.

Let $\phi : (G, \mathfrak{g}) \rightarrow (H, \mathfrak{h})$ be a Lie group homomorphism between two Riemannian Lie groups. The differential $\xi : \mathfrak{g} \rightarrow \mathfrak{h}$ of $\phi$ at $e$ is a Lie algebra homomorphism. There is a left action of $G$ on $\Gamma(T^\mathfrak{h}H)$ given by

$$(a.X)(b) = T_{\phi(a)b}L_{\phi(a^{-1})}X(ab), \quad a, b \in G, X \in \Gamma(T^\mathfrak{h}H).$$

A section $X$ of $T^\mathfrak{h}H$ is called left invariant if, for any $a \in G$, $a.X = X$. For any left invariant section $X$ of $T^\mathfrak{h}H$, we have for any $a \in G$, $X(a) = (X(e))^\ell(\phi(a))$. Thus the space of left invariant sections is isomorphic to the Lie algebra $\mathfrak{h}$.

Since $\phi$ is a homomorphism of Lie groups and $g$ and $h$ are left invariant, one can see easily that $\tau(\phi)$ and $\tau_2(\phi)$ are left invariant and hence $\phi$ is harmonic (resp. biharmonic) iff $\tau(\phi)(e) = 0$ (resp. $\tau_2(\phi)(e) = 0$). Now, one can see easily that

$$\tau(\xi) := \tau(\phi)(e) = U^\mathfrak{g} - \xi(U^\mathfrak{g}) \quad \text{and} \quad \tau_2(\xi) := \tau_2(\phi)(e) = -\sum_{i=1}^n \left( B_{\xi(e_i)} B_{\xi(e_i)} \tau(\xi) + K^H(\tau(\xi), \xi(e_i))\xi(e_i) \right) + B_{\xi(U^\mathfrak{g})} \tau(\xi),$$

where $B$ is the Levi-Civita product associated to $(\mathfrak{h}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$.

$$U^\mathfrak{g} = \sum_{i=1}^n B_{\xi(e_i)} \xi(e_i), \quad (7)$$

$(e_i)_{i=1}^n$ is an orthonormal basis of $\mathfrak{g}$ and $K^H$ is the curvature of $B$ given by $K^H(u, v) = [B_u, B_v] - B_{[u, v]}$. So we get the following proposition.

**Proposition 2.1.** Let $\phi : G \rightarrow H$ be an homomorphism between two Riemannian Lie groups. Then $\phi$ is harmonic (resp. biharmonic) iff $\tau(\xi) = 0$ (resp. $\tau_2(\xi) = 0$), where $\xi : \mathfrak{g} \rightarrow \mathfrak{h}$ is the differential of $\phi$ at $e$.

**Remark 1.**

1. It is obvious from the definition of $\tau(\xi)$ that if $\xi$ preserves the Levi-Civita products then $\phi$ is harmonic.

2. Actually, one can define the harmonicity and biharmonicity of homomorphisms $\phi : G \rightarrow H$ where $G$ is a Riemannian Lie group and $H$ is an affine Lie group, i.e., a Lie group with a left invariant connection. In this case, in the expressions of $\tau(\xi)$ and $\tau_2(\xi)$ $B$ is the product associated to the left invariant connection on $H$. One can see [8] where harmonicity in this sense is investigated.

The following proposition follows easily from the definition of $\tau(\xi)$. It is a particular case of Proposition 2.5.

**Proposition 2.2.** Let $\phi : (G, \mathfrak{g}) \rightarrow (H, \mathfrak{h})$ be an homomorphism between two Riemannian Lie groups where $H$ is abelian. Then $\phi$ is biharmonic and it is harmonic when $\mathfrak{g}$ is unimodular. In particular, any character $\chi : G \rightarrow \mathbb{R}$ is biharmonic and it is harmonic when $\mathfrak{g}$ is unimodular.
It is a well known result that a Riemannian immersion is harmonic iff it is minimal (see [9]). We recover this fact in our context in an easy way. Let \((g,\langle \cdot,\cdot \rangle_g)\) be an Euclidean Lie algebra and \(g_0\) a subalgebra of \(g\). If \(A\) is the Levi-Civita product of \((g,\langle \cdot,\cdot \rangle_g)\), then for any \(u, v \in g_0\),
\[
    A u v = A^0 u v + h(u, v),
\]
where \(A^0\) is the Levi-Civita product of \((g_0,\langle \cdot,\cdot \rangle_{g_0})\) (\(\langle \cdot,\cdot \rangle_{g_0}\) is the restriction of \(\langle \cdot,\cdot \rangle_g\) to \(g_0\)) and \(h : g_0 \times g_0 \rightarrow g_0^+\) is bilinear symmetric. It is called the second fundamental form and its trace with respect to \(\langle \cdot,\cdot \rangle_g\) is the vector \(H^{g_0} \in g_0^+\) given by
\[
    H^{g_0} = \sum_{i=1}^n h(e_i, e_i),
\]
where \((e_i)\) is an orthonormal basis of \(g_0\). This vector is called the mean curvature vector of the inclusion of \(g_0\) in \((g,\langle \cdot,\cdot \rangle_g)\). So we get the following proposition.

**Proposition 2.3.** Let \(\phi : G \rightarrow H\) be an homomorphism between two Riemannian Lie groups which is also a Riemannian immersion. Then \(\phi\) is harmonic iff \(H^{(g)} = 0\).

The following proposition gives an useful expression of \(U^\xi\) and \(\tau^\xi(\xi)\).

**Proposition 2.4.** With the notations above, we have for any \(u \in \mathfrak{h}\),
\[
    \langle U^\xi, u \rangle_{\mathfrak{h}} = \text{tr}(\xi^* \circ \text{ad}_u \circ \xi),
\]
\[
    \langle \tau^\xi(\xi), u \rangle_{\mathfrak{h}} = \text{tr}(\xi^* \circ (\text{ad}_u + \text{ad}_u^*) \circ \xi) - \langle \{u, \tau(\xi)\}_g, \tau(\xi) \rangle_{\mathfrak{h}} - \langle \{\tau(\xi), U^\xi\}_g, u \rangle_{\mathfrak{h}}.
\]
In particular, \(U^\xi \in (\text{Kill}(\mathfrak{h}))^\perp\).

**Proof.** Let \((e_i)_{i=1}^n\) be an orthonormal basis of \(g\). For any \(u \in \mathfrak{h}\), we have
\[
    \langle U^\xi, u \rangle_{\mathfrak{h}} = \sum_{i=1}^n \langle B_{\xi(e_i)} e_i, e_i \rangle_{\mathfrak{h}} = \sum_{i=1}^n \langle \xi^* \circ \text{ad}_u \circ \xi, e_i \rangle_{g} = \text{tr}(\xi^* \circ \text{ad}_u \circ \xi),
\]
which gives the first relation. From this relation, one can deduce that if \(\text{ad}_u^* = -\text{ad}_u\) then \(\langle U^\xi, u \rangle_{\mathfrak{h}} = 0\) and hence \(U^\xi \in (\text{Kill}(\mathfrak{h}))^\perp\). Now put
\[
    Q = - \sum_{i=1}^n \left( \langle B_{\xi(e_i)} B_{\xi(e_i)} \tau(\xi), u \rangle_{\mathfrak{h}} + \langle K^1(\tau(\xi), \xi(e_i))\xi(e_i), u \rangle_{\mathfrak{h}} \right) + \langle B_{\xi(U^\xi)} \tau(\xi), u \rangle_{\mathfrak{h}}.
\]
We have
\[
    \sum_{i=1}^n \langle K^1(\tau(\xi), \xi(e_i))\xi(e_i), u \rangle_{\mathfrak{h}} = \sum_{i=1}^n \left( \langle B_{\xi(e_i)} B_{\xi(e_i)} \tau(\xi), u \rangle_{\mathfrak{h}} - \langle B_{\xi(e_i)} B_{\xi(e_i)} \xi(e_i), u \rangle_{\mathfrak{h}} - \langle B_{\xi(\tau(\xi)\xi(e_i))} \xi(e_i), u \rangle_{\mathfrak{h}} \right)
\]
\[
    = \langle B_{\tau(\xi) U^\xi} \tau(\xi), u \rangle_{\mathfrak{h}} - \sum_{i=1}^n \left( \langle B_{\xi(e_i)} B_{\xi(e_i)} \tau(\xi), u \rangle_{\mathfrak{h}} + \langle B_{\tau(\xi)\xi(e_i)} \xi(e_i), u \rangle_{\mathfrak{h}} \right).
So

\[ Q = -\langle B_{\tau(\xi)} U^b, u \rangle_h + \langle B_{\xi(u)} \tau(\xi), u \rangle_h - \sum_{i=1}^n \left( \langle B_{\xi(e_i)} \xi(e_i), \tau(\xi) \rangle^b, u \rangle_h - \langle B_{\tau(\xi),\xi(e_i)} \xi(e_i), u \rangle_h \right) \]

\[ = -\langle B_{\tau(\xi)} U^b, u \rangle_h + \langle B_{U \tau(\xi)} \tau(\xi), u \rangle_h - \sum_{i=1}^n \left( \langle [\xi(e_i)], [\xi(e_i), \tau(\xi)]^b \rangle, u \rangle_h - 2\langle B_{\tau(\xi),\xi(e_i)} \xi(e_i), u \rangle_h \right) \]

\[ = -\langle (u, \tau(\xi))^b, \tau(\xi) \rangle_h - \langle (\tau(\xi), U^b)^h, u \rangle_h - \sum_{i=1}^n \left( \langle [\xi(e_i)], [\xi(e_i), \tau(\xi)]^b \rangle, u \rangle_h - \langle [u, \xi(e_i)]^b, [\tau(\xi), \xi(e_i)]^b \rangle - \langle [u, [\tau(\xi), \xi(e_i)]^b, \xi(e_i)]^b \rangle \right) \]

\[ = \text{tr}(\xi^* \circ \text{ad}_u + \text{ad}_u^* \circ \text{ad}_{\tau(\xi)} \circ \xi) - \langle (u, \tau(\xi))^b, \tau(\xi) \rangle_h - \langle (\tau(\xi), U^b)^h, u \rangle_h. \]

So we get the second relation. \(\square\)

As an immediate consequence of this proposition we get the following result.

**Proposition 2.5.** Let \( \phi : G \rightarrow H \) be an homomorphism between two Riemannian Lie groups. Then:

(i) If the metric on \( G \) is bi-invariant and \( \phi \) is a submersion then \( \phi \) is harmonic.

(ii) If the metric on \( H \) is bi-invariant then \( \phi \) is biharmonic, it is harmonic when \( g \) is unimodular.

**Proof.** (i) Since \( \xi \) is an homomorphism of Lie algebras, for any \( u \in g, \xi \circ \text{ad}_u = \text{ad}_{\xi(u)} \circ \xi \) and hence

\[ (U^b, \xi(u))^b_h = \text{tr}(\xi^* \circ \text{ad}_{\xi(u)} \circ \xi) = \text{tr}(\xi^* \circ \xi \circ \text{ad}_u) = 0, \]

since \( \text{ad}_u \) is skew-symmetric and \( \xi^* \circ \xi \) is symmetric. Thus \( U^b = 0 \). Now we have also \( U^b = 0 \) and hence \( \tau(\xi) = 0 \).

(ii) If the metric on \( H \) is bi-invariant then \( \text{Kill}(h) = \mathfrak{h} \) and hence, according to Proposition 2.4, \( U^b = 0 \). By using the expression of \( \tau(\xi) \) given in Proposition 2.4 one can see easily that \( \tau(\xi) = 0 \). \(\square\)

**Example 1.** There is an interesting situation where we can apply Proposition 2.5. Let \( H \) be a compact connected semisimple Lie group and \( \pi : G \rightarrow H \) a covering homomorphism of \( H \) by a Lie group \( G \). Then \( G \) is compact and hence unimodular. Then, for any left invariant Riemannian \( g \) on \( G \) and any bi-invariant Riemannian metric \( h_0 \) on \( H \), \( \pi : (G, g) \rightarrow (H, h_0) \) is harmonic. Moreover, for any left invariant Riemannian metric \( h \) on \( H \) and any bi-invariant Riemannian metric \( g_0 \) on \( G \), \( \pi : (G, g_0) \rightarrow (H, h) \) is harmonic.

The following two propositions can be used to build examples of minimal Riemannian immersions.

**Proposition 2.6.** Let \( \phi : G \rightarrow H \) be an homomorphism between two Riemannian Lie groups. Suppose that \( \phi \) is a Riemannian immersion, both \( g \) and \( h \) are unimodular and \( \dim H = \dim G + 1 \). Then \( \phi \) is harmonic.

**Proof.** Choose an orthonormal basis \( (e_1, \ldots, e_n) \) of \( g \) and complete by \( f \) to get an orthonormal basis \( (\xi(e_1), \ldots, \xi(e_n), f) \) of \( h \). On the other hand, we have \( \tau(\xi) = H^b(\partial) = \alpha f \). We have, by using (3),

\[ \tau(\xi) = \sum_{i=1}^n \langle B_{\xi(e_i)} \xi(e_i) \rangle = U^b - B f = -B f. \]

So \( \langle \tau(\xi), \tau(\xi) \rangle_h = -\alpha \langle B f, f \rangle_h \) \(\square\). Thus \( \tau(\xi) = 0 \).
Proposition 2.7. Let $\phi : G \rightarrow H$ be an homomorphism between two Riemannian Lie groups. Suppose that $\phi$ is a Riemannian immersion, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, any derivation of $\mathfrak{g}$ is inner and $\xi(\mathfrak{g})$ is an ideal of $\mathfrak{h}$. Then $\phi$ is harmonic.

Proof. Choose an orthonormal basis $(e_i)_{i=1}^n$ of $\mathfrak{g}$. By using Proposition 2.4 we get for any $u \in \xi(\mathfrak{g})$,

$$\langle H^{\xi(g)}, u \rangle_{\mathfrak{h}} = \sum_{i=1}^n \langle \text{ad}_u \xi(e_i), \xi(e_i) \rangle_{\mathfrak{h}} = \text{tr}(\text{ad}_u),$$

where $\text{ad}_u$ is the restriction of $\text{ad}_u$ to $\xi(\mathfrak{g})$. Now $\xi(\mathfrak{g})$ being an ideal, $\text{ad}_u$ is a derivation of $\xi(\mathfrak{g})$ an hence from the hypothesis it is inner and $\text{tr}(\text{ad}_u) = 0$. Finally, $H^{\xi(\mathfrak{g})} = 0$ which proves the proposition.

Remark 2. 1. The class of Lie algebras $\mathfrak{g}$ satisfying $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and any derivation of $\mathfrak{g}$ is inner contains the class of semi-simple Lie algebras and, actually, it is more large than this subclass (see [5]).

2. In Proposition 2.4 if $H$ is nilpotent then both $\mathfrak{g}$ and $\mathfrak{h}$ are unimodular. This can be used to construct many examples of minimal Riemannian immersions into Riemannian nilmanifolds. For instance let $\mathfrak{h}$ be the 5-dimensional nilpotent Lie algebra whose Lie brackets are given by

$$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5.$$

$\mathfrak{g} = \text{span}\{e_1, e_2, e_3, e_5\}$ is a subalgebra of $\mathfrak{h}$. If $G$ is the connected and simply connected Lie group associated to $\mathfrak{g}$ and $H$ the connected subgroup associated to $\mathfrak{h}$ then, for any left invariant Riemannian metric on $G$, the inclusion $H \hookrightarrow G$ is a minimal Riemannian immersion. Moreover, according to Malcev’s Theorem (see [10]) $G$ has uniform lattices, i.e., there exists a discrete subgroup $\Gamma$ of $G$ such that $\Gamma/G$ is compact. Thus we get a minimal immersion into a compact nimanifold $H \hookrightarrow \Gamma/G$.

Recall that a Kählerian Lie group is a Lie group $G$ endowed with a left invariant Kähler structure. This is equivalent to the existence on $\mathfrak{g}$ of a complex structure $J : \mathfrak{g} \rightarrow \mathfrak{g}$ and an Euclidean product $\langle , \rangle$ such that, for any $u, v \in \mathfrak{g}$,

$$\langle Ju, Jv \rangle = \langle u, v \rangle \quad \text{and} \quad A_u Jv = JA_u v,$$

where $A$ is the Levi-Civita associated to $(\mathfrak{g}, \langle , \rangle)$. An homomorphism $\phi : (G, g, J) \rightarrow (H, h, K)$ between two Kählerian Lie groups is holomorphic iff, for any $u \in \mathfrak{g}$, $\xi(Ju) = K\xi(u)$. The following result is a particular case of a general well-known result (see [11]).

Proposition 2.8. Let $\phi : (G, g, J) \rightarrow (H, h, K)$ be an homomorphism between two Kählerian Lie groups. If $\phi$ is holomorphic then $\phi$ is harmonic.

Proof. There exists an orthonormal basis of $\mathfrak{g}$ having the form $(e_i, Je_i)_{i=1}^n$. We have

$$\tau(\xi) = \sum_{i=1}^n (B_{\xi(e_i)}(e_i) - \xi(Ae_i)) + \sum_{j=1}^n (B_{\xi(\xi(e_j))}(Je_j) - \xi(\xi(Ae_j)))$$

$$= \sum_{i=1}^n (B_{\xi(e_i)}(e_i) - \xi(Ae_i)) + \sum_{i=1}^n (B_{\xi(\xi(e_i))}K\xi(e_i) - \xi(\xi(Ae_i)))$$

$$= \sum_{i=1}^n (B_{\xi(e_i)}(e_i) - \xi(Ae_i)) + \sum_{i=1}^n (KB_{\xi(\xi(e_i))}\xi(e_i) - K\xi(Ae_i))$$

$$= \sum_{i=1}^n (B_{\xi(e_i)}(e_i) - \xi(Ae_i)) + \sum_{i=1}^n (K[K\xi(e_i), \xi(e_i)] + KB_{\xi(e_i)}K\xi(e_i) - K\xi(Je_i, e_i)) - K\xi(Ae_i, Je_i))$$

$$= \sum_{i=1}^n (B_{\xi(e_i)}(e_i) - \xi(Ae_i)) + \sum_{i=1}^n (K[K\xi(e_i), \xi(e_i)] - B_{\xi(e_i)}\xi(e_i) - K[\xi(Je_i), \xi(e_i)] + \xi(Ae_i, e_i))$$

$$= 0.$$
Proposition 2.9. Let \( g_1, g_2 \) be two left invariant Riemannian metrics on \( E(1) \). Then the following holds:

(i) A Riemannian immersion \( i : \mathbb{R} \rightarrow (E(1), g_1) \) is minimal iff \( d_0 i(\mathbb{R}) \) is orthogonal to \([g_2, g_2]\).

(ii) Any homomorphism \( \chi : (E(1), g_1) \rightarrow \mathbb{R} \) is biharmonic never harmonic unless it is constant.

(iii) If \( \phi : (E(1), g_1) \rightarrow (E(1), g_2) \) is a non constant homomorphism which is harmonic then there exists a constant \( \lambda > 0 \) such that \( \phi^* g_2 = \lambda g_1 \).

(iv) If \( \phi : (E(1), g_1) \rightarrow (E(1), g_2) \) is an homomorphism which is biharmonic non harmonic then there exists a Riemannian immersion \( i : \mathbb{R} \rightarrow (E(1), g_2) \) and a biharmonic homomorphism \( \chi : (E(1), g_1) \rightarrow \mathbb{R} \) such that \( \phi = i \circ \chi \).

Proof. Note first that \( g_2 \) is not unimodular. Put \( \langle \cdot, \cdot \rangle_i = g_i(iId_{\mathbb{R}^2}) \) for \( i = 1, 2 \).

(i) Denote by \( A \) the Levi-Civita product associated to \( (g_2, \langle \cdot, \cdot \rangle) \). To show the assertion it suffices to show that for any \( u \in g_2 \setminus \{0\}, A_0 u = 0 \) iff \( u \in [g_2, g_2]^\perp \). Indeed, if \( v \) is such that \( \{u, v\} \) is a basis of \( g_2 \), we have

\[
\langle A_0 u, u \rangle_1 = 0 \quad \text{and} \quad \langle A_0 u, v \rangle_1 = \langle u, [u, v] \rangle_{g_1}.
\]

Since \( \{u, v\} \) is a generator of \([g_2, g_2]\) we can conclude.

(ii) We have shown in Proposition 2.4 that \( \chi \) is biharmonic. If \( \xi : g_2 \rightarrow \mathbb{R} \) denote the differential of \( \chi \), we can see from the first relation in Proposition 2.4 that \( U^\xi = 0 \). So \( \tau(\xi) = -\xi(U^\xi) \). Now \( U^\theta \in [g_2, g_2]^\perp \) and \([g_2, g_2] \subset \ker \xi \) so \( \tau(\xi) = 0 \) iff \( \xi = 0 \).

(iii) Suppose that \( \phi \) is harmonic and denote by \( \xi : g_2 \rightarrow g_2 \) the differential of \( \phi \). Then there exists an \( \langle \cdot, \cdot \rangle_1 \)-orthonormal basis \( (e, f) \) such that \( [e, f] = af \). Denote by \( A \) and \( B \) the Levi-Civita product associated respectively to \( g_1 \) and \( g_2 \). One can see easily that \( U^\xi = -af \). The harmonicity of \( \phi \) is equivalent to

\[
B_{\xi(e)}\xi(e) + B_{\xi(f)}\xi(f) = \xi(U^\xi) = -a\xi(f).
\]

Since \( e \) is a generator of \([g_2, g_2]\) and \( \xi \) is a Lie algebra homomorphism then \( \xi(e) = ae \). Put \( \xi(f) = pe + qf \). Since \( \xi \) is a Lie algebra homomorphism then

\[
\xi([e, f]) = [\xi(e), \xi(f)] = aae = [ae, pe + qf] = aae.
\]

If \( a = 0 \) then from (h) we get \(-a\xi(f) = B_{\xi(f)}\xi(f) \) and hence

\[
-a(\xi(f), \xi(f))_{g_2} = -(B_{\xi(f)}\xi(f), \xi(f))_{g_2} = 0,
\]

and hence \( \xi(f) = 0 \). If \( a \neq 0 \) then \( q = 1 \) and from (h) we get

\[
-a\xi(f) = a^2 B_e e + B_{\xi(f)}\xi(f).
\]

So

\[
-a(\xi(f), \xi(f))_{g_2} = a^2(\xi(f), e)_{g_2} = -aa^2(e, e)_{g_2},
\]

and

\[
-a(\xi(f), e)_{g_2} = \langle a, \xi(f), e \rangle_{g_2} = a(e, \xi(f))_{g_2}.
\]

So

\[
\langle \xi(f), \xi(f) \rangle_{g_2} = \langle \xi(e), \xi(e) \rangle_{g_2} = a^2(e, e)_{g_2} \quad \text{and} \quad \langle \xi(e), \xi(f) \rangle_{g_2} = 0,
\]

which completes the proof of the assertion.
(iv) By multiplying \( \langle , \rangle_2 \) by a positive constant if necessary, we can suppose that there exists a generator \( e \) of \([g_2, g_2]\) such that \( \langle e, e \rangle_1 = \langle e, e \rangle_2 = 1 \). Choose \( f \) orthogonal to \( e \) with respect to \( \langle , \rangle_1 \), \( \langle f, f \rangle_1 = 1 \) and \( f' \) orthogonal to \( e \) with respect to \( \langle , \rangle_2 \) with \( \langle f', f' \rangle_2 = 1 \). We have \( [e, f] = ae \) and \( [e, f'] = be \) and

\[
Aee = -af, \ Aef = ae, \ Afe = 0, \ A_{f}f = 0
\]

and

\[
B_{e}e = -bf', \ B_{e}f' = be, \ B_{f}e = 0, \ B_{f}f' = 0
\]

if \( R \) is the curvature of \((g_2, \langle , \rangle_2)\) then

\[
R(e, f')e = B_{e}B_{f'}e - B_{f}B_{e}e - bB_{e}e = b^2f',
\]

\[
R(e, f')f' = B_{e}B_{f'}f' - B_{f}B_{e}f' - bB_{e}f' = -b^2e.
\]

Now \( \phi \) is biharmonic iff

\[
B_{[e\langle , \rangle e]}B_{\xi(e)}\tau(\xi) + B_{[e\langle f \rangle f]}B_{\xi(f)}\tau(\xi) + R(\tau(\xi), \xi(e))\xi(e) + R(\tau(\xi), \xi(f))\xi(f) = B_{\xi(U_{\xi})}\tau(\xi)
\]

where

\[
\tau(\xi) = B_{e\langle , \rangle e}\xi(e) + B_{e\langle f \rangle f}\xi(f) + a\xi(f).
\]

Put \( \xi(e) = ae \) and \( \xi(f) = pe + qf' \). We have

\[
\xi([e, f]) = [\xi(e), \xi(f)] = aae = [ae, pe + qf'] = abqe.
\]

We have

\[
\tau(\xi) = B_{e\langle , \rangle e}\xi(e) + B_{e\langle f \rangle f}\xi(f) + a\xi(f)
\]

\[
= -a^2bf' + p(-bf' + qbe) + (pe + qf')
\]

\[
= p(a + qb)e + (aq - bp^2 - a^2b)f'.
\]

So

\[
\tau(\xi) = p(a + qb)e + (aq - bp^2 - a^2b)f' = Qe + Pf.
\]

Moreover,

\[
B_{e\langle , \rangle e}B_{\xi(e)}\tau(\xi) = a^2B_{e}(-Qbf' + Pbe) = a^2(-Qb^2e - Pb^2f')
\]

\[
= -a^2b^2\tau(\xi),
\]

\[
B_{e\langle f \rangle f}B_{\xi(f)}\tau(\xi) = -p^2b^2\tau(\xi),
\]

\[
R^2(\tau(\xi), \xi(e))\xi(e) = a^2PR^2(f', e)e = -a^2Pb^2f',
\]

\[
R^2(\tau(\xi), \xi(f))\xi(f) = (Qq - Pp)R^2(e, f')\xi(f) = (Qq - Pp)(pb^2f' - qb^2e),
\]

\[
-B_{\xi(U_{\xi})}\tau(\xi) = abB_{e\langle f \rangle f}\tau(\xi) = ap(-Qbf' + Pbe).
\]

We have

\[
\langle R(\tau(\xi), \xi(e))\xi(e), \tau(\xi) \rangle_2 = -a^2P^2b^2,
\]

\[
\langle R(\tau(\xi), \xi(f))\xi(f), \tau(\xi) \rangle_2 = b^2(Qq - Pp)(Pp - Qq).
\]

So if \( \tau(\xi) \neq 0 \) then \( a = p = 0 \) and hence \( \xi(e) = 0 \) and \( \xi(f) = qf' \). If we define \( i_0 : \mathbb{R} \rightarrow g_2 \) and \( \xi_0 : g_2 \rightarrow \mathbb{R} \) by \( i_0(1) = qf' \), \( \xi_0(e) = 0 \) and \( \xi_0(f) = 1 \) then \( \xi = i_0 \circ \xi_0 \) and we can integrate \( i_0 \) and \( \xi_0 \) to get the desired homomorphisms.

\]
3. Harmonic automorphisms of a Riemannian Lie group

We denote by \( \mathcal{M}(G) \) the set of all left invariant Riemannian metrics on a Lie group \( G \) and fix \( g \in \mathcal{M}(G) \). Recall that \( \text{Isin}(g) \) denotes the subgroup of \( G \) consisting of \( a \in G \) such that \( i_a = L_a \circ R_{(0)} \) is an isometry, \( \text{Kill}(g) = \{ u \in \mathfrak{g}, \text{ad}_u + \text{ad}_u^* = 0 \} \) its Lie algebra. We denote by \( H(g) \) the set consisting of \( a \in G \) such that \( i_a \) is harmonic. We have obviously \( Z(G) \subset \text{Isin}(g) \subset H(g) \), \( \text{Isin}(g)H(g) \subset H(g) \) and \( H(g)\text{Isin}(g) \subset H(g) \).

Remark that since \( G \) is unimodular then \( \text{Isin}(g) \) is open in \( H(g) \), i.e., there exists an open set \( U \subset G \) such that \( U \cap H(g) = \text{Isin}(g) \).

**Theorem 3.1.** If \( G \) is unimodular then \( \text{Isin}(g) \) is open in \( H(g) \), i.e., there exists an open set \( U \subset G \) such that \( U \cap H(g) = \text{Isin}(g) \).

**Proof.** Since \( G \) is unimodular, according to the first relation in Proposition [2.4] \( a \in H(g) \) iff

\[
\forall u \in \mathfrak{g}, \; \text{tr}(\text{Ad}_u^* \circ \text{ad}_u \circ \text{Ad}_u) = 0.
\]

Define \( \alpha : G \rightarrow g^* \) by

\[
\alpha(a)(u) = \text{tr}(\text{Ad}_u^* \circ \text{ad}_u \circ \text{Ad}_u).
\]

Thus \( H(g) = \alpha^{-1}(0) \). The differential of \( \alpha \) at \( a \in H(g) \) is given by

\[
d_a \alpha(T_R(a))(v) = \text{tr}(\text{Ad}_v^* \circ \text{ad}_v \circ \text{Ad}_v) + \text{tr}(\text{Ad}_u^* \circ \text{ad}_v \circ \text{Ad}_u).
\]

If \( a \in \text{Isin}(g) \) then \( \text{Ad}_a^* = \text{Ad}_{a^{-1}} \) and hence

\[
d_a \alpha(T_R(a))(v) = \text{tr}(\text{ad}_v^* \circ \text{ad}_v \circ \text{Ad}_v).
\]

So \( T_R(a)(u) \in \ker d_a \alpha \) iff, for any \( v \in \mathfrak{g}, \text{tr}(\text{ad}_v^* + \text{ad}_v \circ \text{Ad}_v) = 0 \). In particular, we get \( \text{tr}(\text{ad}_v^* + \text{ad}_v \circ \text{Ad}_v) = 0 \). By using the properties of the trace we get also \( \text{tr}(\text{ad}_v^* + \text{ad}_v \circ \text{Ad}_v) = 0 \) and hence \( \text{ad}_v^* + \text{ad}_v = 0 \). Thus \( \ker d_a \alpha = T_R(a)(\text{Kill}(g)) \).

Now, it is easy to see that \( \alpha \) factor to give a smooth map \( \bar{\alpha} : G/\text{Isin}(g) \rightarrow g^* \) and from what above, we get that \( \bar{\alpha} \) is an immersion at \( \pi(e) \) and hence there exists an open set \( \bar{U} \subset G/\text{Isin}(g) \) containing \( \pi(e) \) such that the restriction of \( \bar{\alpha} \) to \( \bar{U} \) is injective. Thus \( U = \pi^{-1}(\bar{U}) \) satisfies the conclusion of the theorem.

Since both \( \text{Isin}(g) \) and \( H(g) \) are closed in \( G \), we get the following corollary.

**Corollary 3.1.** If \( G \) is unimodular then for any \( a \in \text{Isin}(g) \) the connected component of \( \text{Isin}(g) \) containing \( a \) coincides with the connected component of \( H(g) \) containing \( a \).

In [15], Park showed that if for a left invariant Riemannian metric \( g \) on \( SU(2) \) any inner automorphism is harmonic then \( g \) is actually bi-invariant. The following theorem generalizes this result to any connected Lie group.

**Theorem 3.2.** Let \( (G, g) \) be a connected Riemannian Lie group such that \( H(g) = G \). Then \( g \) is bi-invariant.

**Proof.** The hypothesis of the theorem is equivalent to:

\[
\forall a \in G, \; \forall u \in \mathfrak{g}, \; \langle U^{\text{Ad}_a^*}u, u \rangle = \langle U^0, \text{Ad}_a^*u \rangle = 0.
\]

According to the first relation in Proposition [2.4] this is equivalent to

\[
\forall a \in G, \; \forall u \in \mathfrak{g}, \; \text{tr}(\text{Ad}_u^* \circ \text{ad}_u \circ \text{Ad}_u) = \text{tr}(\text{ad}_{\text{Ad}_a^*u}).
\]

By taking \( a = \exp(nv) \) and differentiating this relation, we get

\[
\forall u, v \in \mathfrak{g}, \; \text{tr}(\text{ad}_v^* \circ \text{ad}_u \circ \text{Ad}_u) = \text{tr}(\text{ad}_{\text{ad}_u^*u}).
\]

Remark that since \( U^0 \in [{\mathfrak{g}, \mathfrak{g}}]^L \) then \( \text{ad}_{U^0}^* U^0 = 0 \) and hence

\[
\text{tr}(\text{ad}_{U^0} + \text{ad}_{U^0}^*) = 0,
\]

which is equivalent to \( \text{ad}_{U^0} + \text{ad}_{U^0}^* = 0 \) and hence \( \langle U^0, U^0 \rangle = \text{tr}(\text{ad}_{U^0}) = 0 \). Thus \( g \) is unimodular and we can apply Theorem [3.1] to conclude.
Theorem 3.3. If $G$ is abelian or 2-step nilpotent then $H(g) = \text{Isin}(g) = Z(G)$.

Proof. The theorem is obvious when $G$ is abelian. Suppose now that $G$ is 2-step nilpotent. Then $\exp : g \rightarrow G$ is a diffeomorphism. An element $\exp(u) \in H(g)$ is equivalent to the determination of the group $\text{Aut}(g)$

$$\forall v \in g, \quad \text{tr}(\text{Ad}_{\exp(u)} \circ \text{ad}_v \circ \text{Ad}_u) = 0.$$ 

Now, $\text{Ad}_{\exp(u)} = \exp(\text{ad}_u) = \text{Id}_g + \text{ad}_u$ and for any $v, w \in g$, $\text{ad}_v \circ \text{ad}_u = 0$. So $\exp(u) \in H(g)$ iff

$$\forall v \in g, \quad \text{tr}(\text{ad}_v \circ \text{ad}_u) = 0.$$ 

By taking $v = u$, we get $\text{ad}_u = 0$ which achieves the proof. \hfill \□

Example 2. In [13], Park determined harmonic inner automorphisms of $(\text{SU}(2), g)$ for every left invariant Riemannian metric $g$. In this example, we consider $G = \text{SL}(2, \mathbb{R})$ and $g = \text{sl}(2, \mathbb{R})$ and we give the equations determining harmonic inner automorphisms for a particular class of left invariant Riemannian metrics. Put

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ 

We have

$$[e, f] = h, \quad [h, e] = 2e, \quad \text{and} \quad [h, f] = -2f.$$ 

We consider the Euclidean product on $g$ for which $(h, e, f)$ is orthogonal and $\langle h, h \rangle = \alpha_1$, $\langle e, e \rangle = \alpha_2$ and $\langle f, f \rangle = \alpha_3$ and we denote by $g$ the associated left invariant metric on $G$. Put $\alpha_{ij} = \alpha_i(\alpha_j)^{-1}$. A direct computation shows that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H(g) \iff \begin{cases} 8(a^2b^2c_2 - c^2d^2c_3) + 2(a^4 - d^4 + b^4c_2c_3 - c^4c_3) = 0, \\ 2(ad + bc)(2abc_2 + cd) + ac(c_2c_1 + 2d_2) + bd(d_2d_1 + 2b^2c_2) = 0, \\ 2(ad + bc)(ab + 2d_1d_2) + ac(a^2c_1 + 2c^2c_2) + bd(b^2d_1 + 2d^2) = 0. \end{cases}$$

We consider now the following problem. Given a Riemannian Lie group $(G, g)$ one can aim to determine all the couples $(\phi, h)$ where $\phi$ is an automorphism of $G$ and $h \in \mathcal{A}(G)$ such that $\phi : (G, g) \rightarrow (G, h)$ is harmonic. By remarking that $\phi : (G, g) \rightarrow (G, h)$ is harmonic iff $\text{Id}_G : (G, g) \rightarrow (G, \phi h)$ is harmonic, the solution of the problem is equivalent to the determination of the group $\text{Aut}(G)$ and the set $CH(g)$ of the left invariant Riemannian metric $h$ on $G$ such that $\text{Id}_G : (G, g) \rightarrow (G, h)$ is harmonic.

Proposition 3.1. Let $(G, g)$ be a Riemannian Lie group. Then $h \in CH(g)$ iff: for any $u \in g$,

$$\text{tr}(J \circ \text{ad}_u) = \text{tr}(\text{ad}_u), \quad (11)$$

where $J$ is given by $h(u, v) = g(Ju, v)$. In particular, $CH(g)$ is a convex cone which contains $g$.

Proof. Denote $\langle ., . \rangle_1 = g(e), \langle ., . \rangle_2 = h(e), A$ the Levi-Civita product of $(g, \langle ., . \rangle_1)$ and $B$ the Levi-Civita product of $(g, \langle ., . \rangle_2)$. We have, for any $u \in g$, and for any orthonormal basis $(e_i)$ of $\langle ., . \rangle_1$

$$\langle \tau(\text{Id}_g), u \rangle_2 = \sum_{i=1}^n \langle B_i, e_i, u \rangle_2 - \sum_{i=1}^n \langle A_i, e_i, u \rangle_2$$

$$= \sum_{i=1}^n \langle (u, e_i), e_i \rangle_2 - \sum_{i=1}^n \langle A_i, e_i, Ju \rangle_1$$

$$= \text{tr}(J \circ \text{ad}_u) - \text{tr}(\text{ad}_u).$$

\hfill \□
Definition 3.1. We call \( CH(g) \) the harmonic cone of \( g \) and \( \dim CH(g) \) the harmonic dimension of \( g \), where \( \dim CH(g) \) is the dimension of the subspace generated by \( CH(g) \).

This is an invariant of \( g \) in the following sense. If \( \phi : (G, g_1) \to (G, g_2) \) is an automorphism such that \( g_2 = \alpha g_1 \) with \( \alpha \) is positive constant then \( CH(g_2) = (\phi^{-1})^*CH(g_1) \) and hence \( \dim CH(g_2) = \dim CH(g_1) \). The following proposition is a consequence of Theorem 3.2.

Proposition 3.2. Let \( (G, g) \) be a Riemannian Lie group. Then \( CH(g) = M^2(G) \) iff \( g \) is bi-invariant.

Proof. If \( g \) is bi-invariant then, according to Proposition 2.9 for any left invariant metric \( h, Id_G : (G, g) \to (G, h) \) is harmonic. Suppose now that \( CH(g) \) contains all the left invariant Riemannian metrics on \( G \). Then, for any \( a \in G \), \( Ad_a^* \in CH(g) \) and hence \( Ad_a : (G, g) \to (G, g) \) is harmonic. By applying Theorem 3.2 we get that \( g \) is bi-invariant. \( \square \)

Example 3. 1. Let \( E(1) \) be the 2-dimensional Lie group of rigid motions of the real line. Then, according to Proposition 2.9, for any left invariant Riemannian metric \( g \) on \( E(1) \), \( CH(g) = \{ \alpha g, \alpha > 0 \} \). Thus the harmonic dimension of any left invariant Riemannian metric on \( E(1) \) is equal to 1.

2. Let \( H_3 \) be the 3-dimensional Heisenberg group and \( g \) a left invariant Riemannian metric on \( g \). Denote by \( h_3 \) its Lie algebra and \( \langle \cdot, \cdot \rangle_1 \) be the Euclidean inner product. There exists an \( \langle \cdot, \cdot \rangle_1 \)-orthonormal basis \( (z, f, g) \) such that \( [f, g] = \alpha z \). A direct computation solving (11) shows that \( h \in CH(g) \) iff \( h(u, v)(e) = \langle Ju, v \rangle_1 \) where the matrix of \( J \) in the basis \( (z, f, g) \) has the form

\[
\begin{pmatrix}
a & 0 & 0 \\
0 & d & e \\
0 & e & h
\end{pmatrix}, \quad a > 0, \quad d + h > 0 \quad \text{and} \quad dh - e^2 > 0.
\]

Thus the harmonic dimension of any left invariant Riemannian metric on \( H_3 \) is equal to 4.

3. Let \( G = SO(3, \mathbb{R}) \) and \( g = so(3) \) its Lie algebra. Fix \( g \) a left invariant Riemannian metric on \( G \) and denote by \( \langle \cdot, \cdot \rangle \) the bi-invariant Euclidean product given

\[
\langle A, B \rangle_0 = -\text{tr}(AB).
\]

Define \( J_0 \) by \( \langle u, v \rangle = \langle Ju, v \rangle_0 \). There exists an \( \langle \cdot, \cdot \rangle_0 \)-orthonormal basis \( (X_1, X_2, X_3) \) of \( g \) such that \( J_0 X_i = \alpha_i X_i \) with \( i = 1, 2, 3 \) and \( \alpha_i > 0 \). Since \( \langle \cdot, \cdot \rangle_0 \) is bi-invariant it is easy to see that there exists a constant \( c \) such that

\[
[X_1, X_2] = cX_3, \quad [X_2, X_3] = cX_1, \quad [X_3, X_1] = cX_2. \quad \text{and} \quad \langle X_i, X_j \rangle_1 = \delta_{ij}\alpha_i.
\]

Denote by \( M \) the matrix of \( \langle \cdot, \cdot \rangle_1 \) in this basis. By identifying an endomorphism with its matrix in \( (X_1, X_2, X_3) \), we have

\[
ad_{X_1} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -c \\
0 & c & 0
\end{pmatrix}, \quad ad_{X_2} = \begin{pmatrix}
0 & 0 & c \\
0 & 0 & 0 \\
-c & 0 & 0
\end{pmatrix}, \quad ad_{X_3} = \begin{pmatrix}
0 & -c & 0 \\
c & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

and

\[
J = \begin{pmatrix}
a & b & c \\
b' & d & e \\
c' & e' & f
\end{pmatrix}.
\]

Then (11) is equivalent to \( b = b' \), \( c = c' \) and \( e = e' \). The condition that \( J \) is symmetric with respect to \( \langle \cdot, \cdot \rangle_1 \) is equivalent to \( MJ = JM \) which is equivalent to

\[
(a_1 - a_2)b = (a_3 - a_1)c = (a_3 - a_2)e = 0.
\]

If the \( \alpha_i \) are distinct then \( \dim CH(g) = 3 \). If \( \alpha_i = \alpha_j \neq \alpha_k \) then \( \dim CH(g) = 5 \). If \( \alpha_1 = \alpha_2 = \alpha_3 \) then \( g \) is bi-invariant and \( \dim CH(g) = 6 \).

Remark 3. On all the examples above, we have the following formula

\[
\dim CH(g) = \frac{n(n-1)}{2} + \dim \text{Kill}(g).
\]

where \( n \) is the dimension of the Lie group \( G \). We conjecture that this formula holds in general.
4. Biharmonic submersions between Riemannian Lie groups

Let $\phi : (G, g) \longrightarrow (H, h)$ be a submersion between two Riemannian Lie groups. Then $G_0 = \ker \phi$ is a normal subgroup of $G$, $G/G_0$ is a Lie group and $\bar{\phi} : G/G_0 \longrightarrow H$ is an isomorphism. Let $\pi : g \longrightarrow g/G_0$ the natural projection. If $\xi : g \longrightarrow h$ is the differential of $\phi$ at $e$, the restriction of $\pi$ to $\ker \xi^\perp$ is an isomorphism onto $g/G_0$ and we denote by $\tau : g/G_0 \longrightarrow \ker \xi^\perp$ its inverse. Thus $\tau^\perp(\cdot , \cdot)_g$ is an Euclidean product on $g/G_0$ which defines a left invariant Riemannian metric $g_0$ on $G/G_0$. We denote by $\xi$ the differential of $\bar{\phi}$ at $e$.

**Proposition 4.1.** With the notation above, we have

$$\tau(\xi) = \tau(\xi) - \xi(H_{\ker \xi}),$$

where $\tau : (g/G_0, \tau^\perp(\cdot , \cdot)_g) \longrightarrow (h, (\cdot , \cdot)_h)$.

**Proof.** We have $g = \ker \xi \oplus \ker \xi^\perp$. Choose an orthonormal basis $(f_i)_{i=1}^p$ of $\ker \xi$ and an orthonormal basis $(v_i)_i$ of $\ker \xi^\perp$. If $A$ and $B$ denote the Levi-Civita products of $g$ and $h$ respectively, we have

$$\tau(\xi) = \sum_{i=1}^p B_{\xi(f_i)}(v_i) - \sum_{i=1}^p B_{\xi(v_i)}(f_i).$$

If we put, for any $u, v \in \ker \xi$, $A_u v = A^0 u + h(u, v)$ where $A^0$ is the Levi-Civita product of $\ker \xi$, we get

$$\sum_{i=1}^p \xi(A_{f_i} f_i) = \xi(H_{\ker \xi}).$$

Denote by $\pi : g \longrightarrow g/G_0$ the natural project. Then $(\pi(v_i))_{i=1}^q$ is an orthonormal basis of $g/G_0$ and hence

$$U^\xi = \sum_{i=1}^q B_{\pi(v_i)}(\xi) = \sum_{i=1}^q B_{\xi(\pi(v_i))} = U^{\bar{\xi}}.$$

To achieve the proof, we must show that

$$\xi(U^\chi) = \left(\sum_{i=1}^q \xi(\pi(v_i))\right).$$

This is a consequence of more general formula. If $\overline{A}$ is the Levi-Civita product on $g/G_0$, then for any $u, v \in \ker \xi^\perp$, $\pi(A_u v) = \overline{A}_{\pi(u)} \pi(v)$. To establish this relation note first that, for any $u, v \in \ker \xi^\perp$, we have $\xi([u, v]) = r([\pi(u), \pi(v)])^{\perp(\cdot, \cdot)}_{\tau} + \omega(u, v)$ where $\omega(u, v) \in \ker \xi$. Now, for any $u, v, w \in \ker \xi^\perp$, we have

$$2(\overline{A}_{\pi(u)} \pi(v), \pi(w))_{g/G_0} = ([\pi(u), \pi(v)]^{\perp(\cdot, \cdot)}_{\tau} \pi(w))_{g/G_0} + ([\pi(w), \pi(u)]^{\perp(\cdot, \cdot)}_{\tau} \pi(v))_{g/G_0} + ([\pi(w), \pi(v)]^{\perp(\cdot, \cdot)}_{\tau} \pi(u))_{g/G_0}$$

$$= (r([\pi(u), \pi(v)]^{\perp(\cdot, \cdot)}_{\tau} \pi(w)), v)_{g} + (r([\pi(w), \pi(u)]^{\perp(\cdot, \cdot)}_{\tau} \pi(v)), v)_{g} + (r([\pi(w), \pi(v)]^{\perp(\cdot, \cdot)}_{\tau} \pi(u)), v)_{g}$$

$$= ([u, v])^{\perp(\cdot, \cdot)}_{\tau} + ([w, u])^{\perp(\cdot, \cdot)}_{\tau} + ([w, v])^{\perp(\cdot, \cdot)}_{\tau}$$

$$= 2([A_{u} v, w])_{g}$$

$$= 2(\pi(A_{u} v), \pi(w))_{g/G_0}.$$

The following proposition is an immediate consequence of Proposition 4.1

**Proposition 4.2.** Let $\phi : (G, g) \longrightarrow (H, h)$ be a submersion between two Riemannian Lie groups. Then:

(i) If $\ker \xi$ is minimal then $\phi$ is harmonic (resp. biharmonic) iff $\bar{\phi}$ is harmonic (resp. biharmonic).

(ii) If $\bar{\phi}$ is harmonic then $\phi$ is harmonic iff $\ker \xi$ is minimal.
Let $\phi : (G, g) \rightarrow (H, h)$ be a submersion between two connected Riemannian Lie groups. The connectedness implies that $\phi$ is onto and $\overline{\phi} : G/G_0 \rightarrow H$ is an isomorphism. So $\phi$ is harmonic (resp. biharmonic) iff $\Pi : (G, g) \rightarrow (G/G_0, \overline{\phi}h)$ is harmonic (resp. biharmonic). So the study of harmonic or biharmonic submersion between two connected Riemannian Lie groups is equivalent to the study of the projections $\Pi : (G, g) \rightarrow (G/G_0, h)$ where $(G, g)$ is a connected Lie group, $G_0$ is a normal subgroup and $h$ is left invariant Riemannian metric on $G/G_0$. To build harmonic or biharmonic such projections, let first understand how $G$ can be constructed from $G/G_0$ and $G_0$.

Fix $\Pi : (G, g) \rightarrow (H, h)$ where $H = G/G_0$ denote by $\pi : g \rightarrow g/g_0$ the natural projection and $r : h \rightarrow (ker \xi)^\perp$ the inverse of the restriction of $\pi$ to $(ker \xi)^\perp$. In this context the formula in Proposition 4.1 has the following simpler form:

$$\tau(\pi) = \tau(Id_h) - \pi(H^{ker \xi}),$$

where $Id_h : (h, (\ , )_h) \rightarrow (h, (\ , )_h)$ where $(\ , )_h = r^*(\ , )_g$.

For any $u \in g$ we denote by $ad_u$ the restriction of $ad_g$ to $ker \xi$. We define $\rho : h \rightarrow Der(ker \xi)$ and $\omega \in \wedge^2 h^* \otimes ker \xi$ by

$$\rho(h) = \overline{ad}_{\pi(h)} \quad \text{and} \quad \omega(h_1, h_2) = [\tau(h_1), \tau(h_2)] - \pi(\tau(h_1), \tau(h_2)),$$

where $Der(ker \xi)$ is the space of derivations of $ker \xi$. A direct computation using Jacobi identity of $\{ \ , \ \}^g$ and $\{ \ , \ \}^h$ shows that

$$\rho([h_1, h_2]^h) = \rho(h_1), \rho(h_2) - \overline{ad}_{\rho(h_1), h_2} \quad \text{and} \quad d_x \omega = 0,$$

where

$$d_x \omega(h_1, h_2, h_3) = \oint (\rho(h_1)(\omega(h_2, h_3)) - \omega([h_1, h_2]^h, h_3)).$$

The symbol $\oint$ stands for circular permutations. Let give a characterization of $\tau(\xi)$ using the formalism above.

**Proposition 4.3.** For any $h \in h$, we have

$$\langle \pi(H^{ker \xi}), h \rangle = tr(\rho(h)).$$

**Proof.** We have $H^{ker \xi} = \sum_{i=1}^p \langle A^0_f, f_i \rangle = A^0_f$, where $\{f_i\}_{i=1}^p$ is an orthonormal basis of $ker \xi$, $A$ is the Levi-Civita product of $g$ and $A^0$ is the Levi-Civita product of $ker \xi$. So

$$\langle \pi(H^{ker \xi}), h \rangle = \langle H^{ker \xi}, \tau(h) \rangle_g = \sum_{i=1}^p \langle A^0_f, \tau(h) \rangle_g \sum_{i=1}^p \langle \tau(h), f_i \rangle_g = tr(\rho(h)).$$

The following proposition is an interesting consequence of (15).

**Proposition 4.4.** Let $G$ be a connected Riemannian Lie group and $g_0$ a semisimple normal subgroup of $G$. Then $G_0 \subset G$ is minimal and $\Pi : G \rightarrow G/G_0$ is harmonic when $G/G_0$ is endowed with the quotient metric $g_0$. Moreover, for any left invariant Riemannian metric $h$ on $G/G_0$, $\Pi : (G, g) \rightarrow (G/G_0, h)$ is harmonic (resp. biharmonic) iff $Id_{G/G_0} : (G/G_0, g_0) \rightarrow (G/G_0, h)$ is harmonic (resp. biharmonic).

**Proof.** This is a consequence of (12), (15) and the fact that $g_0$ being semisimple, for any $u \in g/g_0$, the derivation $\rho(u)$ is inner and hence $tr(\rho(u)) = 0$.

Let study now the converse of our study above. Let $(n, h)$ be two Lie algebras such that $n$ carries an Euclidean product and $h$ two Euclidean products $(\ , )_1$ and $(\ , )_2, \rho : h \rightarrow Der(n)$ and $\omega \in \wedge^2 h^* \otimes n$ satisfying (14). Define on $g = n \oplus h$ the bracket $[\ , ]^g$

$$[u, v]^g = \begin{cases} [u, v]^n & \text{if } u, v \in n, \\ [u, v]^h + \omega(u, v) & \text{if } u, v \in h, \\ \rho(u)(v) & \text{if } u \in h, v \in n. \end{cases}$$

(16)
Then \((\mathfrak{g}, [ , ]_\mathfrak{g}, \langle , \rangle_\mathfrak{g}) = (\langle , \rangle_\mathfrak{g})_0 \otimes (\langle , \rangle_1)\) is an Euclidean Lie algebra and the projection \(\pi : \mathfrak{g} \rightarrow \mathfrak{h}\) is an homomorphism of Lie algebras. Let \(G\) be the connected and simply connected Lie group associated to \(\mathfrak{g}\) and \(H\) any connected Riemannian Lie group associated to \(\mathfrak{h}\). Then there exists a unique homomorphism of Lie groups \(\phi : G \rightarrow H\) such that \(d_{\phi} \mathbf{1} = \pi\). If we endow \(G\) and \(H\) by the left invariant Riemannian metrics associated respectively to \(\langle , \rangle_\mathfrak{g}\) and \(\langle , \rangle_2, \phi\) become a submersion. Moreover, \(\tau(\phi)(e)\) is given by

\[
\tau(\phi)(e) = \tau(\mathbf{1}_\mathfrak{h}) = H^0,
\]

where \(\mathbf{1}_\mathfrak{h} : (\mathfrak{h}, \langle , \rangle_1) \rightarrow (\mathfrak{h}, \langle , \rangle_2)\) and \(H^0\) is given by \((H^0, u)_1 = \text{tr}(\rho(u))\) for any \(u \in \mathfrak{h}\). So we have shown the following proposition.

**Proposition 4.5.** There is a correspondence between the set of submersions with a connected and simply-connected domain and the set of \((\mathfrak{n}, \mathfrak{h}, \rho, \omega)\) where \(\mathfrak{n}\) is an Euclidean Lie algebra, \(\mathfrak{h}\) is a Lie algebra having two Euclidean products, \(\rho : \mathfrak{h} \rightarrow \text{Der}(\mathfrak{n})\) and \(\omega \in \wedge^2 \mathfrak{h}^* \otimes \mathfrak{n}\) satisfying (14).

**5. Biharmonic Riemannian submersions between Riemannian Lie groups**

The following proposition follows easily from the last section’s study.

**Proposition 5.1.** Let \(\phi : G \rightarrow H\) be an homomorphism between two Riemannian Lie groups which is a Riemannian submersion. Then \(\phi\) is harmonic in each of the following cases:

(i) Both \(\mathfrak{g}\) and \(\mathfrak{h}\) are unimodular.
(ii) \(\ker \xi\) is unimodular and the Lie algebra \(\mathfrak{h}\) of \(H\) satisfies \([\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}\).
(iii) \(\ker \xi\) satisfies \([\ker \xi, \ker \xi] = \ker \xi\) and \(\text{Der}(\ker \xi) = \text{ad}_{\ker \xi}\).

**Proof.** (i) This a consequence of the definition of \(\tau(\xi) = U^\xi - \xi(U^\mathfrak{g})\) and the fact that \(U^\xi = U^\mathfrak{h}\) when \(\phi\) is a Riemannian submersion.

(ii) According to (14) and (15), we have for any \(h_1, h_2 \in \mathfrak{h}\),

\[
(\tau(\xi), [h_1, h_2]_\mathfrak{h})_\mathfrak{h} = -\text{tr}(\rho([h_1, h_2])) = \text{tr}(\text{ad}_{[h_1, h_2]}(\tau(\xi))) = 0,
\]

and hence \(\phi\) is harmonic.

(iii) From the hypothesis, \(\ker \xi\) is unimodular and any derivation of \(\ker \xi\) is interior and hence, for any \(h \in \mathfrak{h}\),

\[
\text{tr}(\rho(h)) = 0
\]

and \(\xi\) gives the result.

\(\square\)

The following proposition gives an useful characterization of biharmonic Riemannian submersions between Riemannian Lie groups.

**Proposition 5.2.** Let \(\phi : G \rightarrow H\) be an homomorphism between two Riemannian Lie groups which is a Riemannian submersion. Then \(\phi\) is biharmonic iff one of the following equivalent conditions holds:

(i) For an orthonormal basis \((e_i)_{i=1}^{\mathfrak{g}}\) of \(\mathfrak{h}\),

\[
\sum_{i=1}^{\mathfrak{g}} B_{e_i} \tau(\xi) + \text{ric}^\mathfrak{h}(\tau(\xi)) - B_{\rho(U^\xi)} \tau(\xi) = 0,
\]

where \(\text{ric}^\mathfrak{h}\) is the Ricci operator.

(ii) For any \(u \in \mathfrak{h}\),

\[
\text{tr}((\text{ad}_u + \text{ad}_u^\mathfrak{h}) \circ \text{ad}_{\tau(\xi)}) - \langle [u, \tau(\xi)], \tau(\xi) \rangle_\mathfrak{h} - \langle [\tau(\xi), U^\mathfrak{h}], u \rangle_\mathfrak{h} = 0.
\]

**Proof.** It is a consequence of the fact that \(\phi\) is a Riemannian submersion, [5] and Proposition 2.4

We can now state this interesting result.
Theorem 5.1. Let \( \phi : G \rightarrow H \) be an homomorphism between two Riemannian Lie groups which is a Riemannian submersion. Then:

(i) When \( h \) is unimodular then \( \phi \) is biharmonic iff \( \tau(\xi) \) is a Killing vector field.

(ii) When \( \ker \xi \) is unimodular or \( \omega = 0 \) then \( \phi \) is biharmonic iff \( \tau(\xi) \) is a parallel vector field (\( \omega \) is given by (13)).

Proof. (i) Suppose that \( \phi \) is biharmonic. By taking \( u = \tau(\xi) \) in (19), we get, since \( U^b = 0 \),

\[
\text{tr}((\text{ad}_{\tau(\xi)} + \text{ad}_{\tau(\xi)}') \circ \text{ad}_{\tau(\xi)}) = 0.
\]

This equivalent to \( \text{ad}_{\tau(\xi)} + \text{ad}_{\tau(\xi)}' = 0 \) and hence \( \tau(\xi) \) is a Killing vector field. The converse follows easily from (19).

(ii) Suppose that \( \phi \) is biharmonic. We get from (18)

\[
- \sum_{i=1}^{g} \langle B_i \tau(\xi), B_i \tau(\xi) \rangle_h + \langle \text{ric}^h(\tau(\xi)), \tau(\xi) \rangle_h = 0.
\]

By using the fact that \( \ker \xi = 0 \) or \( \omega = 0 \), (14) and (15), one can see easily that \( \tau(\xi) \in [h, h]^+ \). It follows (see [13] Lemma 2.3) that

\[
\langle \text{ric}^h(\tau(\xi)), \tau(\xi) \rangle_h = -\text{tr}((\text{ad}_{\tau(\xi)} + \text{ad}_{\tau(\xi)}') \tau(\xi)) \leq 0.
\]

So \( B\tau(\xi) = 0 \) which is equivalent to the fact that \( \tau(\xi) \) is parallel. The converse follows from the fact that if \( B\tau(\xi) = 0 \) then \( \text{ad}_{\tau(\xi)} + \text{ad}_{\tau(\xi)}' = 0 \).

\[\square\]

Let \( H \) be a Riemannian Lie group. \( TH \) has natural Lie group structure for which the Sasaki metric is left invariant and the projection \( \pi : TH \rightarrow H \) is a Riemannian submersion. We have the following result.

Proposition 5.3. The following assertions are equivalent:

(i) The projection \( \pi : TH \rightarrow H \) is harmonic.

(ii) The projection \( \pi : TH \rightarrow H \) is biharmonic.

(iii) \( h \) is unimodular.

Proof. In this case \( \ker \xi = h \), \( \rho \) is the adjoint representation of \( h \), \( \omega = 0 \) and from (15) we deduce that \( \tau(\xi) = -U^b \) and the equivalence of (i) and (iii) follows. Since \( \omega = 0 \), according to Theorem 5.1 \( \pi \) is biharmonic iff \( \text{ad}_{U^h} + \text{ad}_{U^h}^\prime = 0 \) this implies that \( \text{tr}((\text{ad}_{U^h}) \circ (U^b, U^b)_h = 0 \) and the equivalence of (i) and (ii) follows.

\[\square\]

Theorem 5.2. Let \( \phi : G \rightarrow H \) be a Riemannian submersion between two Riemannian Lie groups. Suppose that the metric on \( H \) is flat and \( \ker \xi \) is unimodular or the metric on \( H \) is flat and \( \omega = 0 \). Then \( \phi \) is biharmonic.

Proof. One can deduce easily from (15) and (14) that if \( \ker \xi \) is unimodular or \( \omega = 0 \) then \( \tau(\xi) \in [h, h]^+ \). Now if the metric on \( H \) is flat, it was shown in [1] that for any \( u \in [h, h]^+ \) \( u^\prime \) is parallel and Theorem 5.1 permits to conclude.

\[\square\]

We end this section by an important remark involving Riemannian submersion between Riemannian Lie groups. Let \( \phi : G \rightarrow H \) and \( \psi : H \rightarrow K \) two homomorphisms between Riemannian Lie groups. Suppose that \( \phi \) is a Riemannian submersion and denote by \( \xi \) and \( \rho \) the differential at the neutral element of \( \phi \) and \( \psi \), respectively. We have

\[
\tau(\rho \circ \xi) = \tau(\rho) + \rho(\tau(\xi)).
\]

This formula implies that if \( \phi \) is harmonic then \( \psi \) is biharmonic (resp. harmonic) iff \( \psi \circ \phi \) is biharmonic (resp. harmonic).
6. When harmonicity and biharmonicity are equivalent

The following result is similar to Jiang’s Theorem where compacity is replaced by unimodularity.

**Theorem 6.1.** Let \( \phi : G \rightarrow H \) be a homomorphism between two Riemannian Lie groups such that \( R^H \leq 0 \) and \( g \) is unimodular. Then \( \phi \) is harmonic iff it is biharmonic.

**Proof.** Suppose that \( \phi \) is biharmonic. Then according to [6] we get

\[
\sum_{i=1}^{n} \left( B_{\xi(e_i)}(\tau(\xi)), B_{\xi(e_i)}(\tau(\xi)) \right)_h + \left( R^H(\tau(\xi), \xi(e_i))\xi(e_i), \tau(\xi) \right)_h = 0.
\]

Since the curvature is negative we deduce that \( B_{\xi(e_i)}(\tau(\xi)) = 0 \) for any \( i = 1, \ldots, n \). Now since \( g \) is unimodular \( U^g = 0 \) and hence \( \tau(\xi) = U^\xi \) so

\[
\langle \tau(\xi), \tau(\xi) \rangle_h = \sum_{i=1}^{n} \langle B_{\xi(e_i)}\xi(e_i), \tau(\xi) \rangle_h = -\sum_{i=1}^{n} \langle \xi(e_i), B_{\xi(e_i)}\tau(\xi) \rangle_h = 0
\]

and hence \( \phi \) is harmonic. \( \square \)

**Corollary 6.1.** Let \( \phi : G \rightarrow H \) be a homomorphism between two Riemannian Lie groups such that \( R^H \leq 0 \) and \( \text{Ric}^H \geq 0 \). Then \( \phi \) is harmonic iff it is biharmonic.

**Proof.** This is a consequence of Theorem 6.1 and the fact that a Lie group which admits a left invariant Riemannian metric with non-negative Ricci curvature must be unimodular (see [13] Lemma 6.4).

**Remark 4.** Actually, this corollary follows from a general theorem (see [2] Theorem 3.1). The Lie groups which admit left invariant Riemannian metrics with \( R \leq 0 \) have been classified by Azencott and Wilson [2] and are all solvable.

Since \( \tau(\phi) = \text{cst} \) for any homomorphism of Riemannian Lie groups, we get the following results proved in a more general siting by Oniciuc [14] in Propositions 2.2, 2.4, 2.5, 4.3.

**Theorem 6.2.** Let \( \phi : G \rightarrow H \) be a homomorphism between two Riemannian Lie groups. In each of the following cases, \( \phi \) is biharmonic iff it is harmonic:

1. \( R^H \leq 0 \) and \( \phi \) is a Riemannian immersion.
2. \( \text{Ric}^H \leq 0 \), \( \phi \) is a Riemannian immersion and \( \dim H = \dim G + 1 \).
3. \( R^H < 0 \) and \( \text{rank} \xi > 1 \).
4. \( \text{Ric}^H < 0 \) and \( \phi \) is a Riemannian submersion.

The following results are specific to our context.

**Theorem 6.3.** Let \( \phi : G \rightarrow H \) be a homomorphism between two Riemannian Lie groups. In the following cases the harmonicity of \( \phi \) and its biharmonicity are equivalent:

1. \( H \) is 2-step nilpotent and \( g \) is unimodular.
2. \( \phi \) is a Riemannian submersion and \( g \) is unimodular.
3. \( \phi \) is a Riemannian submersion, \( \ker \xi^\perp \) is a subalgebra of \( g \) and \( g \) is unimodular.
4. \( \phi \) is a Riemannian submersion, \( \ker \xi \) is unimodular, \( \dim H = 2 \) and \( H \) is non abelian.

**Proof.** 1. Since \( g \) is unimodular then \( \tau(\xi) = U^\xi \). Now since \( h \) is 2-step nilpotent then \( [h, h] \subset Z(h) \) and \( \text{ad}_u \circ \text{ad}_v = 0 \) for any \( u, v \in h \). From Proposition 2.4 we deduce that \( U^\xi \in Z(h)^{\perp} \subset [h, h]^\perp \) and hence

\[
\text{tr}(\xi^\perp \circ \text{ad}_{U^\xi} \circ \text{ad}_{U^\xi} \circ \xi) = 0.
\]

This equivalent to \( \text{ad}_{U^\xi} \circ \xi = 0 \). So

\[
\langle U^\xi, U^\xi \rangle_h = \sum_{i=1}^{n} \langle B_{\xi(e_i)}\xi(e_i), U^\xi \rangle_h = \sum_{i=1}^{n} \langle \xi(e_i), [U^\xi, \xi(e_i)]h \rangle_h = 0.
\]

and hence \( \phi \) is harmonic.
2. Suppose that \( \phi \) is biharmonic. Since \( g \) is unimodular \( \tau(\xi) = U^\xi \), \( \ker \xi \) is unimodular and according to Theorem 5.1 \( U^\xi \) is parallel an hence the Killing. But we have seen in Proposition 2.4 that \( U^\xi \) is orthogonal to the space \( \text{Kill}(h) \) and hence \( U^\xi = 0 \).

3. The same argument as above.

4. According to Theorem 5.1, to prove this assertions it suffices to prove that for any left invariant metric on the 2-dimensional non abelian Lie group there is no non trivial parallel left invariant vector field. Suppose that \( \dim \mathfrak{h} = 2 \) non abelian. Then there exists an orthonormal basis \( (e, f) \) such that \( [e, f] = ae \). A direct computation gives

\[
B_e e = -af, \quad B_e f = ae, \quad B_f e = 0, \quad B_f f = 0.
\]

\( \alpha = a_1 e^* + a_2 f^* \) is parallel iff \(-aa_1 f + aa_2 e = 0 \). So \( \alpha = 0 \).

7. Some general methods for building examples

In this section, following our study in Sections 3 and 4, we give some general methods to builds large classes of harmonic and biharmonic homomorphisms.

7.1. How to build harmonic submersions between Riemannian Lie groups

1. Choose two Lie algebras \( \mathfrak{h} \) and \( \mathfrak{n} \) with two Euclidean products \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \) on \( \mathfrak{h} \) and an Euclidean product \( \langle \cdot, \cdot \rangle_\mathfrak{n} \) on \( \mathfrak{n} \).
2. Compute \( \tau(\mathfrak{Id}_n) \) where \( \mathfrak{Id}_n : (\mathfrak{h}, \langle \cdot, \cdot \rangle_1) \rightarrow (\mathfrak{h}, \langle \cdot, \cdot \rangle_2) \).
3. Construct \( \rho : \mathfrak{h} \rightarrow \text{Der}(\mathfrak{n}) \) and \( \omega \in \wedge^2 \mathfrak{h}^* \otimes \mathfrak{n} \) satisfying (14) and for any \( h \in \mathfrak{h} \), \( \text{tr}(\rho(h)) = \langle h, \tau(\mathfrak{Id}_n) \rangle_1 \).
4. The projection \( (\mathfrak{n} \oplus \mathfrak{h}, [\cdot, \cdot], \langle \cdot, \cdot \rangle_1) \rightarrow (\mathfrak{h}, [\cdot, \cdot], \langle \cdot, \cdot \rangle_2) \) is harmonic. The bracket \( [\cdot, \cdot] \) is given by (16).

7.2. How to build biharmonic submersions between Riemannian Lie groups

1. Choose two Lie algebras \( \mathfrak{h} \) and \( \mathfrak{n} \) with two Euclidean products \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \) on \( \mathfrak{h} \) and an Euclidean product \( \langle \cdot, \cdot \rangle_\mathfrak{n} \) on \( \mathfrak{n} \) such that \( \mathfrak{Id}_\mathfrak{n} : (\mathfrak{h}, \langle \cdot, \cdot \rangle_1) \rightarrow (\mathfrak{h}, \langle \cdot, \cdot \rangle_2) \) is biharmonic.
2. Construct \( \rho : \mathfrak{h} \rightarrow \text{Der}(\mathfrak{n}) \) and \( \omega \in \wedge^2 \mathfrak{h}^* \otimes \mathfrak{n} \) satisfying (14) and for any \( u \in \mathfrak{h} \), \( \text{tr}(\rho(u)) = 0 \).
3. The projection \( (\mathfrak{n} \oplus \mathfrak{h}, [\cdot, \cdot], \langle \cdot, \cdot \rangle_1) \rightarrow (\mathfrak{h}, [\cdot, \cdot], \langle \cdot, \cdot \rangle_2) \) is biharmonic. The bracket \( [\cdot, \cdot] \) is given by (16).

7.3. How to build biharmonic Riemannian submersions between Riemannian Lie groups: first method

1. Choose two Lie algebras \( \mathfrak{h} \) and \( \mathfrak{n} \) with two Euclidean products \( \langle \cdot, \cdot \rangle_1 \) on \( \mathfrak{h} \) and \( \langle \cdot, \cdot \rangle_\mathfrak{n} \) on \( \mathfrak{n} \).
2. Construct \( \rho : \mathfrak{h} \rightarrow \text{Der}(\mathfrak{n}) \) a representation such that \( \text{tr} \circ \rho \) is parallel, i.e., for any \( u, v \in \mathfrak{h} \), \( \text{tr}(\rho(A_n u v)) = 0 \), where \( A \) is the Levi-Civita product on \( \mathfrak{h} \).
3. The projection \( (\mathfrak{n} \oplus \mathfrak{h}, [\cdot, \cdot], \langle \cdot, \cdot \rangle_1) \rightarrow (\mathfrak{h}, [\cdot, \cdot], \langle \cdot, \cdot \rangle_1) \) is biharmonic. The bracket \( [\cdot, \cdot] \) is given by (16) with \( \omega = 0 \).

7.4. How to build biharmonic Riemannian submersions between Riemannian Lie groups: second method

1. Choose two Lie algebras \( \mathfrak{h} \) and \( \mathfrak{n} \) with two Euclidean products \( \langle \cdot, \cdot \rangle_1 \) on \( \mathfrak{h} \) and \( \langle \cdot, \cdot \rangle_\mathfrak{n} \) on \( \mathfrak{n} \). Take \( \mathfrak{n} \) unimodular.
2. Construct \( \rho : \mathfrak{h} \rightarrow \text{Der}(\mathfrak{n}) \) and \( \omega \in \wedge^2 \mathfrak{h}^* \otimes \mathfrak{n} \) satisfying (14) such that \( \text{tr} \circ \rho \) is parallel, i.e., for any \( u, v \in \mathfrak{h} \), \( \text{tr}(\rho(A_n u v)) = 0 \), where \( A \) is the Levi-Civita product on \( \mathfrak{h} \).
3. The projection \( (\mathfrak{n} \oplus \mathfrak{h}, [\cdot, \cdot], \langle \cdot, \cdot \rangle_1) \rightarrow (\mathfrak{h}, [\cdot, \cdot], \langle \cdot, \cdot \rangle_1) \) is biharmonic. The bracket \( [\cdot, \cdot] \) is given by (16).
7.5. How to build biharmonic Riemannian submersions between Riemannian Lie groups: third method

1. Choose two Lie algebras \( \mathfrak{h} \) and \( n \) with two Euclidean products \( \langle \cdot, \cdot \rangle_1 \) on \( \mathfrak{h} \) and \( \langle \cdot, \cdot \rangle_n \) on \( n \). Take \( \mathfrak{h} \) unimodular.

2. Construct \( \rho : \mathfrak{h} \rightarrow \text{Der}(n) \) and \( \omega \in \wedge^2 \mathfrak{h}^* \otimes n \) satisfying (14) such that \( \text{tr} \circ \rho \) is a Killing 1-form, i.e., for any \( u, v \in \mathfrak{h} \), \( \text{tr}(\rho(\text{ad}_u^\varepsilon v + \text{ad}_v^\varepsilon u)) = 0 \).

3. The projection \( (n \oplus \mathfrak{h}, [\cdot, \cdot], \langle \cdot, \cdot \rangle_1 \oplus \langle \cdot, \cdot \rangle_n) \rightarrow (\mathfrak{h}, [\cdot, \cdot]_n, \langle \cdot, \cdot \rangle_1) \) is biharmonic. The bracket \( [\cdot, \cdot]_n \) is given by (16).

On all the methods above, the crucial point is to solve (14), the following lemma gives an easy way of finding many solutions of these equations.

**Lemma 7.1.** Let \( (n, \mathfrak{h}) \) a couple of Lie algebras such that \( \text{Der}(n) = \text{ad}(n) \). Then \( \rho : \mathfrak{h} \rightarrow \text{Der}(n) \) and \( \omega \in \wedge^2 \mathfrak{h}^* \otimes n \) satisfy (14) iff there exists \( F : \mathfrak{h} \rightarrow n \) a linear map and \( \omega_0 \in \wedge^2 \mathfrak{h}^* \otimes Z(n) \) such that

\[
\rho(u) = \text{ad}_{F(u)}, \quad \omega(u, v) = F([u, v]) - [F(u), F(v)] + \omega_0(u, v) \quad \text{and} \quad d\omega_0 = 0.
\]

**Proof.** Since \( \text{Der}(n) = \text{ad}(n) \) then \( \rho(u) = \text{ad}_{F(u)} \) and from (14) we deduce that \( \omega \) must have the following form

\[
\omega(u, v) = F([u, v]) - [F(u), F(v)] + \omega_0(u, v),
\]

where \( \omega_0 \) takes its values in the center \( Z(n) \). We have

\[
\rho(u) \omega(v, w) = [F(u), \omega(v, w)]^n = [F(u), F([v, w]^b)] - [F(u), F(v), F(w)]^b,
\]

\[
\omega(u, [v, w]^b) = F([F(u), F([v, w]^b)] - [F(u), F(v), F(w)]^b + \omega_0(u, [v, w]^b).
\]

This shows that \( d\mu \omega = 0 \) iff \( d\omega_0 = 0 \). \( \square \)

We end this paper by giving an example where we use Lemma 7.1 to illustrate the first method.

**Example 4.** We take \( \mathfrak{h} \) the non abelian 2-dimensional Lie algebra endowed with two Euclidean products \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \). There exists an \( \langle \cdot, \cdot \rangle_1 \)-orthonormal basis \( (e_1, e_2) \) such that \( [e_1, e_2] = \alpha e_1 \). We have

\[
\tau(\text{Id}_h) = B_1 e_1 + B_2 e_2 + e_2.
\]

The condition \( \text{tr}(\rho(h)) = \langle h, \tau(\text{Id}_h) \rangle_1 \) is equivalent to

\[
\tau(\text{Id}_h) = \text{tr}(\rho(e_1)) e_1 + \text{tr}(\rho(e_2)) e_2.
\]

This is equivalent to the fact that \( (\text{tr}(\rho(e_1)), \text{tr}(\rho(e_2))) \) is solution of the system

\[
(S) \quad \begin{cases} 
\{ (e_1, e_1)_2 x + (e_1, e_2)_2 y = (\alpha + 1)(e_1, e_2)_2, \\
(e_1, e_2)_2 x + (e_2, e_2)_2 y = -\alpha (e_1, e_1)_2 + (e_2, e_2)_2.
\end{cases}
\]

Let \( n \) be a non unimodular Euclidean Lie algebra and \( F : n \rightarrow \mathfrak{h} \) an endomorphism such that \( F(U^n) = x_0 e_1 + y_0 e_2 \) where \( (x_0, y_0) \) is the unique solution of \( (S) \). Put \( \rho(h) = \text{ad}_{F(u)} \). For any \( \omega \in \wedge^2 \mathfrak{h}^* \otimes Z(n) \), \( (\rho, \omega) \) satisfy (14) where

\[
\omega(u, v) = F^*([u, v]) - [F^*(u), F^*(v)] + \omega_0(u, v),
\]

and \( F^* : h \rightarrow n \) is the adjoint of \( F \) with respect to \( \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2 \).

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