ORDER AND INTERVAL TOPOLOGIES ON COMPLETE BOOLEAN ALGEBRAS

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Abstract. Problem 76 of Birkhoff’s Lattice Theory [1] asks whether for complete Boolean algebras the order topology and the interval topology coincide. We answer this question in the negative.

1. Introduction

On p. 166 of Birkhoff’s Lattice Theory [1], the following problem is stated:

Does the order topology and the interval topology coincide for a complete Boolean algebra?

In the following we introduce the concepts necessary to understand and answer the question.

A partially ordered set (or poset for short) is a set $X$ with a binary relation $\leq$ that is reflexive, transitive, and anti-symmetric (i.e., $x, y \in X$ with $x \leq y$ and $y \leq x$ implies $x = y$). Often, a poset is denoted by $(X, \leq)$. A subset $D \subseteq X$ is called a down-set if it is “closed under going down”, that is $d \in D, x \in X, x \leq d$ jointly imply $x \in D$. A special case of a down-set is the set

$$\downarrow_P x = \{ y \in X : y \leq x \}$$

for $x \in X$. (Sometimes we just write $\downarrow x$ if the poset $P$ is clear from the context.) Down-sets of this form are called principal. If $S \subseteq X$ we say $S$ has a smallest element $s_0 \in S$ if $s_0 \leq s$ for all $s \in S$. Note that anti-symmetry of $\leq$ implies that a smallest element is unique (if it exists at all!). Similarly, we define a largest element. Moreover, we set

$$S^u = \{ x \in X : x \geq s \text{ for all } s \in S \}$$

to be the set of upper bounds of $S$. The set of lower bounds $S^l$ is defined analogously.

We say that a subset $S \subseteq X$ of a poset $(X, \leq)$ has an infimum or largest lower bound if

2010 Mathematics Subject Classification. 05A18, 06B23.

Key words and phrases. Lattice theory, Boolean algebra, order topology, interval topology, Birkhoff.
(1) $S^c \neq \emptyset$, and
(2) $S^c$ has a largest element.

Again, an infimum (if it exists) is unique by anti-symmetry of the ordering relation, and it is denoted by $\inf(S)$ or $\bigwedge_X S$. The dual notion (everything taken “upside down” in the poset) is called supremum and is denoted by $\sup(S)$ or $\bigvee_X S$. The infimum of the empty set is defined to be the largest element of $X$ if it has one, and the supremum is the smallest element of $X$.

A poset $(X, \leq)$ in which infima and suprema exist for all $S \subseteq X$ is called a complete lattice. A lattice has suprema and infima for finite non-empty subsets. If $(X, \leq)$ is a poset and $x, y \in X$ we use the following notation

$$x \lor y := \bigvee_X \{x, y\},$$
and $x \land y$ is defined analogously. To emphasize the binary operations $\lor, \land$, a lattice $(L, \leq)$ is sometimes written as $(L, \lor, \land)$. A lattice is distributive if for all $x, y, z \in L$ we have

$$x \land (y \lor z) = (x \lor y) \land (x \lor z).$$

**Definition 1.1.** Given a poset $(X, \leq)$, the interval topology $\tau_i(X)$ is given by the subbase

$$\mathcal{S} = \{X \setminus (\downarrow x) : x \in X\} \cup \{X \setminus (\uparrow x) : x \in X\}.$$

2. Convergence spaces

We need the notion of order convergence expressed with filters (Birkhoff uses nets, and filters offer an equivalent, but more concise approach to convergence [4]). The underlying concept is that of a convergence space.

Let $X \neq \emptyset$ be a set. By a set filter $\mathcal{F}$ on $P$ we mean a collection of subsets of $P$ such that:

- $\emptyset \notin \mathcal{F};$
- $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F};$
- $U \in \mathcal{F}, U' \subseteq P$ and $U' \supseteq U$ implies $U' \in \mathcal{F}.$

If $\mathcal{B}$ is a collection of subsets of $X$ such that

- $\emptyset \notin \mathcal{B},$
- for $A, B \in \mathcal{B}$ there is $C \in \mathcal{B}$ with $C \subseteq A \cap B,$

then we call $\mathcal{B}$ a filter base. The filter generated by $\mathcal{B}$ is the collection of subsets of $X$ that contain some member of $\mathcal{B}.$
If \( F \subseteq G \) are filters on \( X \) we say that \( G \) is a super-filter of \( F \). An ultrafilter is a maximal filter with respect to set inclusion (i.e. it has no proper super-filters). Given \( x_0 \in X \) we set

\[
P_{x_0} = \{ A \subseteq X : x_0 \in A \},
\]

which is easily seen to be an ultrafilter.

By \( \Phi(X) \) we denote the set of filters on \( X \).

**Definition 2.1.** A convergence space is a pair \((X, \rightarrow)\) where \( X \) is a non-empty set and \( \rightarrow \subseteq \Phi(X) \times X \) is a relation satisfying the following properties:

1. If \( F \subseteq G \) are elements of \( \Phi(X) \) such that \( F \rightarrow x \), then \( G \rightarrow x \), and
2. for all \( x \in X \) we have \( P_x \rightarrow x \).

(Note that we write \( F \rightarrow x \) instead of \((F, x) \in \rightarrow\).)

If \( F \rightarrow x \) for some \( F \in \Phi(X) \) and \( x \in X \) we say that \( F \) converges to \( x \).

To every convergence relation \( \rightarrow \) as described above we can associate a topology on the base set \( X \) by setting

\[
\tau_{\rightarrow} = \{ U \subseteq X : \text{if } F \in \Phi(X) \text{ and } u \in U \text{ with } F \rightarrow u \text{ we have } U \in F \}.
\]

It is a routine exercise to verify that \( \tau_{\rightarrow} \) is a topology.

We will use the following fact later on:

**Fact 2.2.** If a filter \( F \) contains all open neighborhoods of \( x \) in the topological space \((X, \tau_{\rightarrow})\) then \( F \rightarrow x \).

Interestingly, many topological properties such as compactness, Hausdorffness, and more can be put in terms of convergence spaces. We will need the following later on:

**Proposition 2.3.** If \((X, \rightarrow)\) is a convergence space, then the following are equivalent:

1. every \( F \in \Phi(X) \) converges to at most one point \( x \in X \);
2. \((X, \tau_{\rightarrow})\) is Hausdorff.

**Proof.** \((1) \Rightarrow (2).\) Assume that \((X, \tau_{\rightarrow})\) is not Hausdorff. Then there are \( x \neq y \) such that every neighborhood \( U \) of \( x \) intersects every neighborhood \( V \) of \( y \). We let \( F \) be the collection of all \( U \cap V \) where \( U \) is a neighborhood of \( x \) and \( V \) is a neighborhood of \( y \). A routine verification shows that \( F \) is indeed a filter, and Fact [2.2] implies that both \( F \rightarrow x \) and \( F \rightarrow y \), so \( F \) does not converge to a unique point.

\((2) \Rightarrow (1).\) To be done.  \(\square\)
3. ORDER CONVERGENCE

Let \((P, \leq)\) be a poset. If \(F\) is a set filter on \(P\), then we set \(F^u = \bigcup\{F^u : F \in F\}\) and define \(F^\ell\) similarly. For \(x \in P\) and \(F\) a set filter on \(P\) we write \(F \to_o x\) iff \(\bigwedge F^u = x = \bigvee F^\ell\) and say \(F\) order-converges to \(x\).

It is easy to verify that the order convergence relation is indeed a convergence relation as described in definition 2.1. The topology \(\tau \to_o\) associated to \(\to_o\) on the poset \((P, \leq)\) is denoted by \(\tau_o(P)\), and we call it the order (convergence) topology.

Lemma 2.3 and the fact that every filter on a poset order-converges to at most one point by definition jointly imply the following:

**Corollary 3.1.** If \((P, \leq)\) is a poset, then \((P, \tau_o(P))\) is Hausdorff.

4. INTERVAL TOPOLOGY ON COMPLETE BOOLEAN ALGEBRAS

**Proposition 4.1.** The interval topology of a complete atomless Boolean algebra (such as \(\mathcal{P}(\omega)/\text{fin}\)) is not Hausdorff.

**Proof.** The set of principal ideals and principal filters form a subbasis of closed sets for the topology on \(B\), so a typical basic closed set has the form \(I \cup F\) where \(I\) is a finitely generated order ideal and \(F\) is a finitely generated order filter. Our goal is to show that there do not exist proper basic closed subsets \(C_0 = I_0 \cup F_0\) containing 0 and not 1 and \(C_1 = I_1 \cup F_1\) containing 1 and not 0, whose union \(C_0 \cup C_1\) equals \(B\). To obtain a contradiction, assume that \(B = C_0 \cup C_1\) for proper basic closed sets satisfying \(0 \in C_0 - C_1\) and \(1 \in C_1 - C_0\).

Since \(1 \notin C_1\) and \(0 \notin C_0\) we get that \(C_0 = I_0\) is a proper f.g. order ideal and \(C_1 = F_1\) is a proper f.g. order filter.

Suppose that \(C_0\) is the order ideal generated by some finite set \(X\) and that \(C_1\) is the order filter generated by some finite set \(Y\). Let \(C\) be the subalgebra of \(B\) generated by \(X \cup Y\). It is finite. If the set of atoms of \(C\) is \(V = \{v_1, \ldots, v_p\}\), then for \(u_i := \bigvee_{j \neq i} v_j\) we get that \(U = \{u_1, \ldots, u_p\}\) is the set of coatoms of \(C\). Note that \(V\) consists of the cells in a finite partition of unity, and \(U\) consists of the complementary elements.

The f.g. order ideal \(D_0\) generated by \(U\) contains \(C_0\) and still does not contain 1, while the f.g. order filter \(D_1\) generated by \(V\) contains \(C_1\) but still does not contain 0. This reduces the original situation to one in which \(D_1\) is a proper order filter generated by the cells in a finite partition of unity, while
$D_0$ is the proper f.g. order ideal generated by the complements of single cells in the same partition of unity.

The set $V = \{v_1, \ldots, v_p\}$ consists of nonempty, pairwise disjoint, elements of $B$. If $B$ is atomless, then we can choose $w_i$ such that $0 < w_i < v_i$ for all $i$. Let $W = \bigvee w_i$. For each $i$ we have $v_i \not\leq W$, since $v_i \cap W = v_i \cap w_i = w_i < v_i$. Moreover, $W \not\leq u_j$ for any $j$, since $0 < w_j \leq W$ and $w_j \not\leq u_j$. Thus $W \not\in D_0 \cup D_1$. Hence also $W \not\in C_0 \cup C_1$.

In fact, the argument can be relativized to intervals to prove that if $x < y$, then $x$ cannot be separated from $y$ unless there is an atom below $y$ disjoint from $x$. This happens for every $x < y$ only when $B$ is an atomic Boolean algebra. To restate this: if the topology is Hausdorff, then the Boolean algebra must be atomic. 

As the order topology on any poset is Hausdorff, proposition 4.1 provides a negative answer to Birkhoff’s question.

**Corollary 4.2.** On $\mathcal{P}(\omega)/\text{fin}$ the order topology and the interval topology do not agree.

Note that on $\mathcal{P}(\omega)/\text{fin}$ is the order topology is strictly finer than the interval topology, since on any poset, the order topology contains the interval topology, see [3].

5. **Acknowledgement**

I want to thank Keith A. Kearnes (University of Colorado, Boulder) for his help on Proposition 4.1 [2].

**References**

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