Dirac Constraint Quantization of a Dilatonic Model of Gravitational Collapse

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Abstract

We present an anomaly-free Dirac constraint quantization of the string-inspired dilatonic gravity (the CGHS model) in an open 2-dimensional spacetime. We show that the quantum theory has the same degrees of freedom as the classical theory; namely, all the modes of the scalar field on an auxiliary flat background, supplemented by a single additional variable corresponding to the primordial component of the black hole mass. The functional Heisenberg equations of motion for these dynamical variables and their canonical conjugates are linear, and they have exactly the same form as the corresponding classical equations. A canonical transformation brings us back to the physical geometry and induces its quantization.

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1. Introduction

The formation of black holes by collapsing matter fields and their subsequent Hawking evaporation are well understood only within the semiclassical approximation. Unfortunately, an analysis of the final stages of black hole evaporation, its possible remnants, the information problem, and the fate of a final singularity, requires quantum gravity. In its absence, the best one can do is to turn to simplified models of gravitational collapse.

The least idealized of such models is obtained by a dimensional reduction of spacetime geometry coupled to a collapsing massless scalar field under the assumption of spherical symmetry. This was set up by Berger, Chitre, Moncrief, and Nutku (BCMN) [1] in 1972, and later corrected by Unruh [2]. However, even this simple model is not classically exactly solvable, and its quantization remains problematic.

By what formally appears to be a minor modification of the BCMN action, Callan, Giddings, Harvey, and Strominger (CGHS) [3] turned the BCMN model into one whose general classical solution is explicitly known. Their choice of the action was motivated by the effective action describing spherical modes of extremal dilatonic black holes in four or higher dimensions [4] and the spacetime action for noncritical strings [5]. Recalling its origin and laudibly avoiding the acronym, the CGHS model is often referred to as ‘string-inspired dilatonic gravity’. It is viewed as a genuine 2-dimensional theory.

Surprisingly, the quantization of even this classically exactly solvable model is far from trivial. It has been discussed from different points of view by numerous investigators [6]. The aim of this paper is to show that the canonical Dirac constraint quantization of the dilatonic model admits an exact solution of its quantum dynamics such that its physical degrees of freedom exactly correspond to those of the classical theory.

Canonical quantization of the string-inspired dilatonic gravity has been previously studied by Miković [7, 8, 9] and by Jackiw and his collaborators [10, 11, 12, 13].

Miković quantized the model after the ADM (Arnowitt, Deser, and Misner [17]) reduction to a constant extrinsic time foliation. He focused on the

\[^1\] Put into a general setting of Poisson \(\sigma\)-models, it got a lucid treatment by Strobl, Klösch, and Schaller [14]. Primordial (matter-free) dilatonic black holes are canonically quantized in [15]. Other relevant references are [16].
clarification of issues connected with the generation of Hawking radiation and unitarity of quantum black hole evolution. By fixing the foliation, however, he sidestepped the problems of possible non-integrability of the functional Schrödinger equation caused by anomalies.

In a series of carefully written papers, Jackiw and his coworkers discussed in detail different constraint quantization techniques (in particular the functional Schrödinger quantization and the BRST (Becchi-Rouet-Stora-Tyutin) quantization) and the discrepancies of the resulting quantum theories. In [12], they concluded that the functional Schrödinger equation approach seems unable to produce a large (or infinite) number of physical states, while the BRST approach leads to an infinite number of states corresponding to a single complete set of oscillators complemented by two homogeneous modes. More recently [13], they noticed the existence of a complete set of oscillator states in the functional Schrödinger equation approach, which reduces the mentioned discrepancy to an extra homogeneous mode in the BRST approach.

In the above papers, the explicit handling of the quantum model is made possible by a series of transformations from the original CGHS variables. The first of these is a rescaling of the physical 2-dimensional metric into an auxiliary flat metric by the dilaton [12]. The second is a canonical transformation that casts the constraints into those of a bosonic string in a 3-dimensional target space [11]. (This form is the starting point for the BRST quantization.) A long time ago, one of us showed [18] that this type of constraints can be further simplified by a third canonical transformation that brings the constraints into the form appropriate for the parametrized theory [19] of a single scalar field propagating on a 2-dimensional Minkowskian background. (This was the starting point for a functional Schrödinger equation treatment of the bosonic string [20].) The same approach was adopted in [12] to quantize the dilatonic model.

But by performing the canonical transformation to the would-be embedding variables in a series of steps, one can lose sight of their basic geometric significance. We present an alternative derivation of a transformation that ensures that our embedding variables $X^\pm(x)$ have the proper physical interpretation on the physical spacetime manifold equipped with the double null Minkowskian coordinates $X^\pm$ of its auxiliary flat metric (section 2c). This enables us to maintain the physical interpretation of the functional Heisenberg equations of motion (or the equivalent functional Schrödinger equation)
in the quantum theory.

Still, even the Dirac constraint quantization of a parametrized scalar field is not straightforward. Commutators of the constraint operators develop an anomaly, which leads to inconsistencies when one imposes the constraints as operator restrictions on physical states. One of us faced this problem earlier 21 when studying the Dirac constraint quantization of a massless scalar field propagating on a flat Minkowskian cylinder $\mathbb{R} \times S^1$. There it was shown how to remove the anomaly by an embedding-dependent factor ordering of the constraints. An analogous modification of the constraints was used in 12, 13 to quantize the dilatonic model. But unfortunately, the change of topology from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R} \times S^1$ is in conflict with the original geometric interpretation of dilatonic gravity as a theory of black hole formation in the physical curved spacetime (see the Appendix). 2 For this reason, we pay close attention to the open-space boundary conditions in dilatonic gravity, and point out that only they enable us to turn the transformation between the geometric and embedding variables into a truly bona fide, one-to-one, canonical transformation (section 3). This leads us finally to a consistent Dirac constraint quantization of the original dilatonic model (section 4). The removal of the anomaly is best understood by passing to the Heisenberg picture (sections 2d and 4a,b) in which the constraints are manifestly anomaly-free. 3 The quantum dynamics of the dilatonic model is entirely explicit in this picture, but it can also be recast into the Schrödinger picture, which is traditionally associated with the Dirac constraint quantization (sections 4c,d,e). We conclude that the quantum theory of the string-inspired dilatonic gravity has exactly the same degrees of freedom as the classical theory; namely, all the modes of the scalar field on the auxiliary open flat background, supplemented by a single additional degree of freedom corresponding to the primordial component of the black hole mass. This conclusion is in general agreement with the $\mathbb{R} \times S^1$ results obtained in 13. Moreover, the functional Heisenberg equations of motion for these dynamical variables and their canonical conjugates are linear, and they have exactly the same form as the corresponding

2 How to put the CGHS model on $\mathbb{R} \times S^1$ is discussed in 11. A thorough study of possible $\mathbb{R} \times S^1$ compactifications in different versions of dilatonic gravity and their consequences (like geodesic incompleteness or the presence of closed timelike curves) was undertaken in 14.

3 This conceptual strategy is clarified for a finite-dimensional parametrized system in 22.
classical equations.

One can return by the canonical transformation from these dynamical variables back to the original geometric variables. This enables us in the end to pose some relevant questions about the quantum physical geometry of dilatonic gravity (section 5).

Notation

Besides standard conventions, we will use the following notation throughout this paper: Arguments of functions will be enclosed in round brackets (e.g., \(y(X)\)), while arguments of functionals will be enclosed in square brackets (e.g., \(S[y, \gamma_{\alpha\beta}, f]\)). If a quantity is simultaneously a function of some variables, say \(x\), and a functional of other variables, say \(X\), we will use both round and square brackets as in \(n^\alpha(x; X)\), with the semi-colon separating the function and functional dependence. In the double null coordinates \(X^\alpha = (X^+, X^-)\), many quantities depend only on \(X^+\) or \(X^-\), but not on both variables. We will emphasize this by using only \(X^+\) or \(X^-\) as an argument of that function or functional. For example, while \(f(X)\) means that \(f\) is a function of both \(X^+\) and \(X^-\), \(f_+(X^+)\) and \(f_-(X^-)\) mean that the derivatives \(f_+\) and \(f_-\) depend only on \(X^+\) and \(X^-\), respectively. Moreover, \(f_{\pm}(X^\pm)\) will serve as a shorthand notation to denote the function dependence of both \(f_+\) and \(f_-\) simultaneously. Finally, \(A_{\pm}(x; X)\) means that \(A_+\) and \(A_-\) are functions of \(x\) and functionals of both \(X^+\) and \(X^-\). This is to be contrasted with the functional dependence of \(h_+\) and \(h_-\) as indicated by \(h_{\pm}(x; X^\pm, f, \pi_f)\).

2. Classical theory

2a. Spacetime action and equations of motion

We take, as our starting point, the spacetime action for dilatonic gravity written in the form

\[
S[y, \gamma_{\alpha\beta}, f] = \frac{1}{2} \int d^2X |\gamma|^{\frac{1}{2}} \left( y R[\gamma] + 4\kappa^2 - \gamma^{\alpha\beta} f_\alpha f_{\beta} \right).
\]

(1)

Here \(y\) is the dilaton field, \(\gamma_{\alpha\beta}\) is the spacetime metric (signature \((-++))\), and \(f\) is a conformally coupled scalar field. \(R[\gamma]\) denotes the scalar curvature
of $\gamma_{\alpha\beta}$, and $\kappa$ is a constant having the dimensions of inverse length. To interpret the theory, we will treat $\gamma_{\alpha\beta}$ as an auxiliary metric and

$$\check{\gamma}_{\alpha\beta} := y^{-1}\gamma_{\alpha\beta}$$

(2)
as the physical “black hole” metric. The action (1) is obtained from the original CGHS action (which is a functional of the physical metric $\check{\gamma}_{\alpha\beta}$) by the rescaling (2).

The equations of motion of dilatonic gravity are derived, as usual, by varying the spacetime action (1) with respect to all of its arguments. (See also [12] and [3].) Variation of $y$ implies

$$R[\gamma] = 0 .$$

(3)

Variation of $f$ implies

$$\Box_{\gamma} f := \left( |\gamma|^\frac{1}{2}\gamma^{\alpha\beta} f_{,\alpha} \right)_{,\beta} = 0 .$$

(4)

Variation of the (contravariant) spacetime metric $\gamma^{\alpha\beta}$ implies

$$y_{;\alpha\beta} - \gamma_{\alpha\beta} \Box_{\gamma} y = - \left( T_{\alpha\beta} + 2\kappa^2 \gamma_{\alpha\beta} \right) ,$$

(5)

where

$$T_{\alpha\beta} := f_{,\alpha} f_{,\beta} - \frac{1}{2} \gamma_{\alpha\beta} \gamma^{\mu\nu} f_{,\mu} f_{,\nu}$$

(6)

4In special relativity, $c = 1$ and the basic dimensions are mass, $M$, and length, $L$. Action has the dimension $[S] = ML$. We let the metric $\gamma_{\alpha\beta}$ and the dilaton $y$ be dimensionless; the spatial dimension $L$ is carried by the coordinates $X$. To match the dimension of $R[\gamma]$, the classical constant $\kappa$ must have the dimension $[\kappa] = L^{-1}$. The scalar field with dimension $[f] = M^\frac{1}{2} L^\frac{3}{2}$ yields the correct dimensionality for the matter action. Matter is coupled to gravity by Newton’s constant $G$, which in a 2d spacetime has the dimension $[G] = M^{-1} L^{-1}$. The coupling constant $G^{-1}$ in front of the dilatonic part of the action (1) restores its proper dimensionality. By setting $G = 1$, we agree to measure mass in units of $[M] = L^{-1}$. By the same decision, the action, which is measured in the units of $G^{-1}$, and the scalar field measured in the units of $G^{-\frac{1}{2}}$, become dimensionless. The only remaining basic dimension is $L$. None of these dimensional considerations has anything to do with quantum theory.

5If we put $\lambda = 4\kappa^2$, $\eta = y$, and change the signature of $\gamma_{\alpha\beta}$, we recover the dilatonic action given in [12]. If we put $\lambda = \kappa$, $\phi = -\frac{1}{2} \ln y$, and $\check{\gamma}_{\alpha\beta} = y^{-1} \gamma_{\alpha\beta}$, we recover (up to a boundary term) the dilatonic action given in [3].
is the energy-momentum tensor of the scalar field $f$. Equation (3) can be simplified by contracting it with $\gamma^{\alpha\beta}$. This yields

$$\Box_y = 4\kappa^2. \quad (7)$$

Substituting (7) back into (3), we find

$$y;_{\beta} = -\left(T_{\alpha\beta} - 2\kappa^2\gamma_{\alpha\beta}\right). \quad (8)$$

We solve the equations of motion (3), (4), (8) as follows: In 2-dimensions, $R[\gamma] = 0$ implies that spacetime is flat. Thus, we can introduce Minkowskian coordinates $(T, Z)$, or the equivalent double-null coordinates

$$X^{\pm} := Z \pm T, \quad (9)$$

for which the spacetime line elements takes the form

$$ds^2 = dX^+dX^{-}. \quad (10)$$

The solution of the wave equation (4) is then simply

$$f(X) = f_{+}(X^{+}) + f_{-}(X^{-}). \quad (11)$$

The energy-momentum tensor (3) takes the form

$$T_{\pm\pm}(X^{\pm}) = \left(f_{\pm}(X^{\pm})\right)^2, \quad T_{+-} = 0. \quad (12)$$

and (8) can be solved for $y$:

$$y(X) = \kappa^2X^{+}X^{-} + y_{+}(X^{+}) + y_{-}(X^{-}). \quad (13)$$

Here

$$y_{\pm}(X^{\pm}) = -\int^{X^{\pm}}_{\bar{X}^{\pm}} d\bar{X}^{\pm} \int^{X^{\pm}}_{\bar{X}^{\pm}} d\bar{X}^{\pm} T_{\pm\pm}(\bar{X}^{\pm}). \quad (14)$$

Note that the only non-trivial dynamics is contained in the scalar field $f$. The spacetime metric $\gamma_{\alpha\beta}$ is flat, while (13)-(14) show that the dilaton field $y$ is completely determined (up to constants of integration) by $f$. The physical metric (2) is dynamical via its dependence on $y$. Singularities in $\gamma_{\alpha\beta}$ usually occur where $y = 0$. 

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2b. Canonical form of the dilatonic action

The spacetime action (1) is cast into canonical form by the standard ADM decomposition [17]. Given an arbitrary foliation $X^\alpha = X^\alpha(t, x^a)$ of a spacetime by ($t =$ const) spacelike hypersurfaces, one has the general decomposition formula [23]:

$$|\gamma|^{\frac{1}{2}} R[\gamma] = N g^{\frac{1}{2}} \left( K_{ab} K^{ab} - K^2 + R[g] \right) - 2 g^{\frac{1}{2}} \Delta_g N + 2(g^{\frac{1}{2}} K N^a)_a - 2(g^{\frac{1}{2}} K)^\gamma. \quad (15)$$

In 1+1 dimensions, this reduces to

$$|\gamma|^{\frac{1}{2}} R[\gamma] = -2 g^{\frac{1}{2}} \Delta_g N + 2(g^{\frac{1}{2}} K N^1)' - 2(g^{\frac{1}{2}} K)^\gamma, \quad (16)$$

where the prime $'$ denotes partial derivative with respect to the (single) spatial coordinate $x$ ($x \in (-\infty, \infty)$), and the dot $\dot{}$ denotes partial derivative with respect to the time coordinate $t$. Here, $g$ is the determinant of the induced spatial metric, and $K$ is the trace of the extrinsic curvature of the $t =$ const hypersurfaces. $N$ and $N^1$ are the lapse function and shift vector, respectively.

Since the induced spatial metric is 1-dimensional, it has only one independent component

$$\sigma^2 := g_{11} = X^{\alpha'} X^{\beta'} \gamma_{\alpha\beta}. \quad (17)$$

Similarly, the extrinsic curvature tensor is completely specified by

$$K_{11} = -X^{\alpha'} X^{\beta'} n_{\alpha;\beta}, \quad (18)$$

where $n_\alpha$ is the unit (covariant) timelike normal to the spacelike hypersurfaces. Since in one dimension tensor fields of contravariant rank $r$ and covariant rank $s$ transform as scalar densities of weight $(s - r)$, it follows that $\sigma$ transforms as a scalar density of weight +1, while $K := g^{11} K_{11}$ and the lapse function $N$ transform as ordinary scalars. The shift vector $N^1$ transforms as a scalar density of weight $-1$.

If we substitute (17) into the spacetime action (1), integrate by parts, and discard the boundary terms, we find

$$S[y, \sigma, f, N, N^1]$$

$$= \int dt \int dx \ \left( \sigma K(\dot{y} - N^1 y') - N(\sigma \Delta_g y - 2 \kappa^2 \sigma) \right) + \text{matter}, \quad (19)$$
where $\sigma K$ may be thought of as shorthand notation for
\[ \sigma K = N^{-1} \left( -\dot{\sigma} + (N^1\sigma)' \right) . \]  
(20)

The matter contribution to the action is given by
\[ \text{matter} = \int dt \int dx \left( -\frac{1}{2} N\sigma^{-1} f'^2 + \frac{1}{2} N^{-1} \sigma (\dot{f} - N^1 f')^2 \right) . \]  
(21)

As usual, $N$ and $N^1$ play the role of Lagrange multipliers of the theory. The dynamical variables are $y$, $\sigma$, and $f$.

The momenta conjugate to $y$, $\sigma$, and $f$ are
\[ \pi_y = \sigma K = N^{-1} \left( -\dot{\sigma} + (N^1\sigma)' \right) , \]  
(22)
\[ p_\sigma = N^{-1} (-\dot{y} + N^1 y') , \]  
(23)
\[ \pi_f = N^{-1} \sigma (\dot{f} - N^1 f') . \]  
(24)

The notation we have chosen is such that the canonical variables denoted by Latin symbols $(y, f, p_\sigma)$ transform as ordinary scalars, while those denoted by Greek symbols $(\sigma, \pi_y, \pi_f)$ transform as scalar densities of weight +1. The above equations for the momenta can be inverted, yielding expressions for the velocities in terms of the momenta. It is then a straightforward exercise to cast the action (19)-(21) into Hamiltonian form
\[ S[y, \pi_y, \sigma, p_\sigma, f, \pi_f, N, N^1] \]
\[ = \int dt \int dx \left( \pi_y \dot{y} + p_\sigma \dot{\sigma} + \pi_f \dot{f} - NH - N^1 H_1 \right) , \]  
(25)

where
\[ H := -\pi_y p_\sigma + \sigma \Delta_g y - 2\kappa^2 \sigma + \frac{1}{2} \sigma^{-1} (\pi_f^2 + f'^2) , \]  
(26)
\[ H_1 := \pi_y y' - \sigma p_\sigma' + \pi_f f' \]  
(27)

are the super-Hamiltonian and supermomentum, which are constrained to vanish ($H \approx 0 \approx H_1$) as a consequence of variations of $N$ and $N^1$.

In what follows, it is more convenient to work with a rescaled super-Hamiltonian and lapse function:
\[ \bar{H} := \sigma H , \quad \bar{N} := \sigma^{-1} N . \]  
(28)
Since
\[ \sigma^2 \Delta_y := \sigma^2 g^{-\frac{1}{2}} (g^\frac{1}{2} g^{11} y')' = y'' - \sigma^{-1} \sigma' y' , \]
we have
\[ \tilde{H} = -\pi_y \sigma p_\sigma + y'' - \sigma^{-1} \sigma' y' - 2 \kappa^2 \sigma^2 + \frac{1}{2}(\pi_f^2 + f'^2) . \]
Both \( \tilde{H} \) and \( H_1 \) transform as scalar densities of weight +2.

2c. Canonical transformation to embedding variables

As shown in section 2a, the equations of motion imply that the spacetime metric \( \gamma_{\alpha \beta} \) is flat:
\[ ds^2 = dX^+ dX^- . \]
This means that given an arbitrary foliation \( X^\pm = X^\pm(t, x) \) of spacetime by \( (t = \text{const}) \) spacelike hypersurfaces, the induced spatial metric and the trace of the extrinsic curvature are
\[ g_{11} = X^+ X^- , \]
\[ g^\frac{1}{2} K = -\frac{1}{2} \left[ \ln \left( \frac{X^+}{X^-} \right) \right]' . \]
Since \( \sigma^2 = g_{11} \) and \( \pi_y = \sigma K \), equations (32) and (33) imply
\[ \sigma = \sqrt{X^+ X^-} , \]
\[ \pi_y = -\frac{1}{2} \left[ \ln \left( \frac{X^+}{X^-} \right) \right]' . \]

The connection (34)-(35) between the null coordinates \( X^\pm \) and the geometry of embeddings \( \sigma, \pi_y \) holds only modulo the field equations. This does not prevent us, however, from introducing the embedding variables \( X^\pm(x) \) as new canonical coordinates on phase space such that (34)-(35) are satisfied even prior to varying the action. Following this strategy, we want to complete (34)-(35) into a canonical transformation
\[ (y(x), \pi_y(x), \sigma(x), p_\sigma(x)) \leftrightarrow (X^\pm(x), \Pi^\pm(x)) . \]
Since the matter variables remain unchanged, they are not mentioned in what follows. In this section, we will ignore important issues related to falloff conditions on the field variables, and corresponding boundary terms and constants of integration. These issues will be addressed in section 3, where we give a detailed discussion of asymptotics.

We proceed in a series of steps:

(i) We first replace \((y, \pi_y)\) by \((z, \pi_z)\) via the canonical transformation

\[
y(x) = -\int_x^x d\bar{x} \, z(\bar{x}) \pi_z(\bar{x}) \, , \tag{37}
\]

\[
\pi_y = -(\ln z)' \, . \tag{38}
\]

This brings the super-Hamiltonian \(\bar{H}\) into a more symmetric form

\[
\bar{H} = z^{-1} z' \sigma \rho + \sigma^{-1} \sigma' z \pi_z - (z \pi_z)' - 2\kappa^2 \sigma^2 + \frac{1}{2} (\pi_j^2 + f'^2) \, . \tag{39}
\]

(ii) We then express \((\sigma, z)\) as combinations of the density variables \(\xi^\pm\):

\[
\sigma \ = \ \sqrt{\xi^+ \xi^-} \, , \tag{40}
\]

\[
z \ = \ \sqrt{\frac{\xi^+}{\xi^-}} \, . \tag{41}
\]

Equations (40), (41) can be completed into a point canonical transformation \((\sigma, p_\sigma, z, \pi_z) \leftrightarrow (\xi^\pm, p_\pm)\), with

\[
\sigma p_\sigma = \xi^+ p_+ + \xi^- p_- \, , \tag{42}
\]

\[
z \pi_z = \xi^+ p_+ - \xi^- p_- \, . \tag{43}
\]

Because

\[
\frac{\sigma'}{\sigma} = \frac{1}{2} \left( \frac{\xi^+'}{\xi^+} + \frac{\xi^-'}{\xi^-} \right) \, , \tag{44}
\]

\[
\frac{z'}{z} = \frac{1}{2} \left( \frac{\xi^+'}{\xi^+} - \frac{\xi^-'}{\xi^-} \right) \, . \tag{45}
\]

\(\bar{H}\) assumes the form

\[
\bar{H} = -\xi^+ p_+ ' + \xi^- p_- ' - 2\kappa^2 \xi^+ \xi^- + \frac{1}{2} (\pi_j^2 + f'^2) \, . \tag{46}
\]
(iii) We interchange the roles of the coordinates $\xi^\pm$ and momenta $p^\pm$ by the canonical transformation $(\xi^\pm, p^\pm) \leftrightarrow (X^\pm, \Pi^\pm)$:

\begin{align*}
\xi^\pm &= X^\pm', \\
p^\pm(x) &= -\int^x d\bar{x} \, \Pi^\pm(\bar{x}).
\end{align*}

(47) (48)

This gives

$$\tilde{H} = \Pi_+ X'^+ - \Pi_- X'^- - 2\kappa^2 X'^+ X'^- + \frac{1}{2}(\pi f^2 + f'^2).$$

(49)

(iv) Finally, we absorb the term $2\kappa^2 X'^+ X'^-$ by a redefinition of the momenta $\Pi^\pm \rightarrow \Pi^\pm$:

$$\Pi^\pm = \Pi^\pm \pm \kappa^2 X^{\pm'}.$$  

(50)

This yields

$$\tilde{H} = \Pi_+ X'^+ - \Pi_- X'^- + \frac{1}{2}(\pi f^2 + f'^2).$$

(51)

Since $X^\pm$ and $f$ transform as ordinary spatial scalars, the supermomentum $H_1$ necessarily takes the form

$$H_1 = \Pi_+ X'^+ + \Pi_- X'^- + \pi f f'.$$

(52)

The super-Hamiltonian and supermomentum constraints can then be combined into the Virasoro pair:

$$H^\pm := \frac{1}{2}(\tilde{H} \pm H_1) = \pm \Pi^\pm X^{\pm'} + \frac{1}{4}(\pi f \pm f')^2 \approx 0.$$  

(53)

Our derivation of the canonical transformation (37)-(38), (40)-(43), (47)-(48), and (50) makes it clear that the embedding variables $X^\pm(x)$ on the space of solutions really describe embeddings in the double null Minkowskian coordinates $X^\pm$ of the flat background metric $\gamma_{\alpha\beta}$.

2d. Commuting constraints and Heisenberg variables

By scaling (53), we obtain an equivalent set of commuting constraints:

$$\Pi^\pm(x) := \Pi^\pm(x) + h^\pm(x; X^\pm, f, \pi_f) \approx 0,$$

(54)
where
\[ h_\pm(x; X^\pm, f, \pi_f) := \pm \frac{1}{4} \left( X^{\pm'}(x) \right)^{-1} \left( \pi_f(x) \pm f'(x) \right)^2 , \]  
(55)
and
\[ \{ \Pi_\pm(x), \Pi_\mp(x) \} = 0 = \{ \Pi_+(x), \Pi_-(\bar{x}) \} . \]  
(56)
Because
\[ X^{\pm'}(x)f_\pm(X^\pm(x)) = \frac{1}{2} \left( f'(x) \pm \pi_f(x) \right) , \]  
(57)
the energy flux (55) is simply related to the null components (12) of the energy-momentum tensor:
\[ h_\pm(x; X^\pm, f, \pi_f) = \pm X^{\pm'}(x)T_{\pm\mp}(X^\pm(x)) . \]  
(58)
Because the dynamical variables \( \Pi_\pm(x) \) commute, (56), we can choose them as new momenta. The embedding variables remain their canonically conjugate coordinates:
\[ X^\pm(x) = X^\pm(x) . \]  
(59)
To complete the canonical variables \((X^\pm(x), \Pi_\pm(x))\) into a canonical chart on phase space, we need to replace the conjugate field variables \((f(x), \pi_f(x))\) by new conjugate variables \((f(x), \pi_f(x))\) which commute with \(X^\pm(x)\) and \(\Pi_\pm(x)\). The commutation with \(\Pi_\pm(x)\), i.e., with the constraints, means that \(f(x)\) and \(\pi_f(x)\) are constants of the motion. They can be identified with initial data on a fixed embedding \(X^{\pm}(x) = X^{\pm}(0(x))\). In flat spacetime, it is natural to choose \(X^{\pm}(0(x))\) as the \(T = 0\) hypersurface parametrized by the Cartesian coordinate \(Z\):
\[ X^{\pm}(0(x)) = x . \]  
(60)
The canonical transformation
\[ (X^\pm(x), \Pi_\pm(x), f(x), \pi_f(x)) \leftrightarrow (X^\pm(x), \Pi_\pm(x), f(x), \pi_f(x)) \]  
(61)
is closely connected with the passage from the Schrödinger to the Heisenberg picture. We shall call the boldface canonical variables the fundamental Heisenberg variables, and the lightface canonical variables the fundamental Schrödinger variables. The canonical transformation between the respective embedding coordinates and momenta is given by (59) and (54). When we write the commutation relation \(\{ f(x), \Pi_\pm(\bar{x}) \} = 0\) between the Schrödinger
variables in the Heisenberg canonical chart, we get the Heisenberg equation of motion

\[
\frac{\delta f(x; X, f, \pi_f)}{\delta X^\pm(x)} = \{f(x; X, f, \pi_f), h_\pm(x; X, f, \pi_f)\}
\]

for the field \(f(x; X, f, \pi_f)\). A similar equation holds for the field momentum \(\pi_f(x; X, f, \pi_f)\). The solution of the Heisenberg equations of motion under the initial condition that \(f(x), \pi_f(x)\) match \(f(x), \pi_f(x)\) at the initial embedding \(X^\pm(x) = X^\pm(0)\) gives the canonical transformation from the Heisenberg to the Schrödinger field data.

3. Taking care of asymptotic conditions

In the transformation to the embedding variables \((X^\pm(x), \Pi^\pm(x))\), we did not pay any attention to boundary terms at the two spatial infinities. In this section we shall complete the analysis by evaluating these contributions. To do this, we need to know the asymptotic behavior of the phase space variables (section 3a). These are easiest to motivate for the embedding variables \((X^\pm(x), \Pi^\pm(x))\). We then choose the falloff conditions on the multipliers \((\tilde{N}(x), N^1(x))\) to ensure differentiability of the Hamiltonian. The falloff conditions for the original geometric variables \((y(x), \pi_y(x), \sigma(x), p_\sigma(x))\) are deduced from the asymptotic behavior of the spacetime solution. In section 3b, we show that the transformation between the geometric and embedding variables that respects their falloff conditions requires parametrization of the infinities by proper times \(\tau_L, \tau_R\), and complementation of the embedding variables by a new canonical pair \(m_R, p\):

\[
(y(x), \pi_y(x), \sigma(x), p_\sigma(x); \tau_L, \tau_R) \leftrightarrow (X^\pm(x), \Pi^\pm(x); m_R, p) .
\]

The physical meaning of all of these variables will naturally emerge from the examination of the action.

In the rest of the paper, we distinguish the terms at the left and right infinities by the subscripts \(L\) and \(R\). Their suppression means that the discussion is valid at both infinities.
3a. Asymptotic conditions

Before discussing the embedding variables, let us briefly discuss the asymptotic behavior of the matter variables. With a view towards quantum theory, we impose such falloff conditions that the Klein-Gordon symplectic norm associated with points on the matter phase space is well-defined. To achieve this, it suffices to require that $f(x), \pi_f(x) \in \mathcal{S}$ lie in the Schwartz space $\mathcal{S}$ of smooth functions with rapid decay at infinity. One can check that with these conditions on the matter variables and our subsequent choice of falloff conditions on the multipliers $\bar{N}(x), N^1(x)$ the matter part of the action is well-defined and differentiable on phase space.

Falloff conditions on the embedding variables $X^\pm(x)$ are chosen by examining the asymptotic behavior of the classical solution (13)-(14) for $y$. To leading order in $X^\pm(x)$ at the spatial infinities

$$y(x) = \left(\kappa X^+(x) + \frac{A^+}{\kappa}\right)\left(\kappa X^-(x) + \frac{A^-}{\kappa}\right) + \frac{m}{\kappa}, \quad (64)$$

where $A^\pm$ and $m$ are constants. Equation (64) can be verified by substituting the falloff conditions of the scalar field variables $f(x), \pi_f(x)$ into the explicit solution for $y(x)$. The form of (64) (although not the values of $A^\pm$) is left invariant by translations

$$X^\pm(x) \to X^\pm(x) + \xi^\pm \quad (65)$$

of the Minkowskian coordinates $X^\pm(x)$, and by their boosts

$$X^\pm(x) \to e^{\pm\tau}X^\pm(x). \quad (66)$$

The physical interpretation of these transformations is as follows: The physical metric, $\gamma_{\alpha\beta}$, is asymptotically flat. A clock moving near spatial infinity along the orbits of the asymptotic stationary Killing field of the physical metric measures the parameter $\tau$. The transformations (64) correspond to asymptotic time translations along the orbits of this Killing field. We will refer to $\tau$ as the Killing time measured at infinity. We fix the Minkowskian translational freedom (65) by imposing $A^+_R = 0 = A^-_L$.

A deeper analysis of the role of $A^\pm$ in the canonical theory reveals that they are related to the generators of translations (65) of $X^\pm(x)$ (see (96)-(97)). Hence, in the canonical description, setting the values of $A^+_R$ and $A^-_L$ to
zero must be accompanied by freezing the Minkowskian translational freedom in $X^+(x)$ at right infinity and $X^-(x)$ at left infinity. Thus we require that

\begin{align}
X^-(x) &= e^{-\tau_L}x + O(x^{-2}) , \\
X^+(x) &= e^{\tau_L}x + \xi_L^+ + O(x^{-2})
\end{align}

(67) near left infinity, and

\begin{align}
X^-(x) &= e^{-\tau}x + \xi_R^- + O(x^{-2}) , \\
X^+(x) &= e^{\tau}x + O(x^{-2})
\end{align}

(68) near right infinity. Here, $\tau_L$ and $\tau$ are the Killing times of the physical geometry at the left and right infinities. The parameters $\xi_L^+$ and $\xi_R^-$ correspond to the residual translational freedom (65) in $X^+(x)$ at left infinity and $X^-(x)$ at right infinity.

For $\Pi_{\pm}(x)$ to generate cotangent maps from the space of $X^\pm(x)$, we require that

\[ \Pi_{\pm}(x) = O(x^{-3}) \]

(71) at both infinities.

For the smeared constraint functionals

\[ \bar{H}[^N] := \int_{-\infty}^{\infty} dx \, \bar{N}(x)\bar{H}(x) , \quad H_1[^N] := \int_{-\infty}^{\infty} dx \, N^1(x)H_1(x) \]

(72) to be functionally differentiable and to preserve (67)-(71), we put

\begin{align}
\bar{N}(x) &= \alpha_L x + \nu_L + O(x^{-2}) , \\
N^1(x) &= \nu_L + O(x^{-2})
\end{align}

(73) at the left infinity, and

\begin{align}
\bar{N}(x) &= \alpha_R x - \nu_R + O(x^{-2}) , \\
N^1(x) &= \nu_R + O(x^{-2})
\end{align}

(75) at the right infinity.

6The symbol $\tau$ is an exception to the $L$ and $R$ subscript convention; i.e., even though $\tau$ is the Killing time at $+\infty$, it carries no $R$ subscript. The symbol $\tau_R$ is reserved for the parametrization time at $+\infty$, which will be introduced later.

We can make the parametrization time and the Killing time coincide at one of the infinities, but not at both. We shall later choose to make them coincide at the left infinity. Hence $\tau_L$ has a dual interpretation of Killing time and parametrization time at $-\infty$, while $\tau$ and $\tau_R$ are physically distinct quantities associated with $+\infty$. 

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at the right infinity. Note that the infinitesimal changes (generated by the constraint functionals) of \((\tau_L, \tau, \xi^+_L, \xi^+_R)\) are \((\alpha_L, \alpha_R, \nu_L, \nu_R)\).

Next, we motivate and state our choice of the asymptotic behavior of the geometric variables \((y(x), \pi_y(x), \sigma(x), p_\sigma(x))\). By substituting the falloff conditions \((67)-(70)\) in the asymptotic form of the spacetime solution \((64)\) for \(y(x)\), we obtain

\[
y(x) = \kappa^2 x^2 + B_Lx + \frac{m_L}{\kappa} + O(x^{-1})
\]

at the left infinity, and

\[
y(x) = \kappa^2 x^2 + B_Rx + \frac{m_R}{\kappa} + O(x^{-1})
\]

at the right infinity. Here

\[
B_L = (A_L^+ + \kappa^2 \xi^+_L)e^{-\tau_L},
\]

\[
B_R = (A_R^- + \kappa^2 \xi^+_R)e^{\tau},
\]

and \(m_L\) and \(m_R\) are the values of the parameter \(m\) in \((64)\) at the left and right infinities.

By substituting the falloff conditions \((67)-(70)\) in \((35)\) and \((34)\) for \(\pi_y(x)\) and \(\sigma(x)\), we obtain

\[
\pi_y(x) = O(x^{-4}),
\]

\[
\sigma(x) = 1 + O(x^{-3})
\]

at both infinities.

Finally, by substituting the falloffs \((77)\), \((78)\), and \((73)-(76)\) in \((23)\), we obtain

\[
p_\sigma(x) = B_L + O(x^{-2})
\]

at the left infinity, and

\[
p_\sigma(x) = -B_R + O(x^{-2})
\]

at the right infinity.

Equations \((77)-(84)\) constitute our choice of boundary conditions on the \((y(x), \pi_y(x), \sigma(x), p_\sigma(x))\) variables. Although they have been deduced from the asymptotic behavior of a solution to the field equations, it can be checked that their imposition on the entire phase space leads to a consistent Hamiltonian formulation. In particular, all the relevant Hamiltonian flows preserve these falloff conditions.
3b. Canonical action in the geometric and embedding variables

It can be checked that with the asymptotic conditions (77)-(84) on the geometric variables, and the falloff conditions (73)-(76) on the lapse and shift multipliers, the action (25) must be complemented by a surface term to be functionally differentiable:

\[
S[y, \pi_y, \sigma, p_\sigma, f, \pi_f, N, N^1] = \int dt \int_{-\infty}^{\infty} dx \left( \pi_y \dot{y} + p_\sigma \dot{\sigma} + \pi_f \dot{f} - \bar{N} \bar{H} - N^1 H_1 \right) + \int dt \left( -\alpha_R \frac{m_R}{\kappa} + \alpha_L \frac{m_L}{\kappa} \right). \tag{85}
\]

We see that the mass parameters generate the asymptotic Killing time translations (66) of the physical metric at the left and right infinities. From the viewpoint of canonical Hamiltonian theory, this property of \( m_L \) and \( m_R \) justifies their identification with the left and right mass of the system.\(^7\) The equations of motion which follow from this action preserve our choice of asymptotic behavior of the \((y(x), \pi_y(x), \sigma(x), p_\sigma(x))\) variables.

A ‘boundary action’ similar to that in (85) appears in the study of the Schwarzschild black holes [24]. By following the method introduced in [24, 25], we parametrize the asymptotic time translations (66) at the spatial infinities by introducing into the action two additional parameters \( \tau_R \) and \( \tau_L \):

\[
S[y, \pi_y, \sigma, p_\sigma, f, \pi_f, N, N^1; \tau_L, \tau_R] = \int dt \int_{-\infty}^{\infty} dx \left( \pi_y \dot{y} + p_\sigma \dot{\sigma} + \pi_f \dot{f} - \bar{N} \bar{H} - N^1 H_1 \right) + \int dt \left( -\dot{\tau}_R \frac{m_R}{\kappa} + \dot{\tau}_L \frac{m_L}{\kappa} \right). \tag{86}
\]

Note that the reading \( \tau_R \) of the right parametrization clock does not necessarily coincide with the proper time \( \tau \) identified from the geometry. The equations of motion, however, insure that \( \tau_R \) and \( \tau \) are running at the same rate, so that the difference is due only to an initial setting.

\(^7\)Our identification of the generators of the transformations (66) with the mass of the system differs from that made by Miković [8]. In [8], certain combinations of the generators of Minkowskian translations (65) of \( X^\pm(x) \) are identified with the mass.
The dynamical variables in the action (86) are the original field variables \(y(x), \pi_y(x), \sigma(x), p_\sigma(x)\) and the parameters \((\tau_L, \tau_R)\). To define a one-to-one, invertible transformation to the embedding variables, we need to complement \((X^\pm(x), \Pi^\pm(x))\) by a pair of parameters. One of these is simply the mass parameter \(m_R\) at the right infinity. The second parameter is
\[
p := \frac{\tau_R - \tau}{\kappa}.
\] (87)
Its physical meaning will follow from the transformation equations. We can cast the parametrized action (86) into manifestly canonical form by transforming \((y(x), \pi_y(x), \sigma(x), p_\sigma(x); \tau_L, \tau_R)\) into \((X^\pm(x), \Pi^\pm(x); m_R, p)\):
\[
y(x) = \kappa^2 X^+(x) X^-(x) \\
- \int_x^\infty d\tilde{x} \; X^{-\prime}(\tilde{x}) \int_{-\infty}^x d\bar{x} \; \Pi_-(\bar{x}) + \int_x^\infty d\bar{x} \; X^+\prime(\bar{x}) \int_{-\infty}^x d\tilde{x} \; \Pi_+(\tilde{x}) \\
+ \int_{-\infty}^\infty dx \; X^+(x) \Pi_+(x) + \frac{m_R}{\kappa},
\] (88)
\[
\pi_y(x) = -\frac{1}{2} \left[ \ln \left( \frac{X^{+\prime}(x)}{X^{-\prime}(x)} \right) \right]^{\prime},
\] (89)
\[
\sigma(x) = \sqrt{X^+\prime(x)X^-\prime(x)},
\] (90)
\[
p_\sigma(x) = \frac{1}{\sqrt{X^+\prime(x)X^-\prime(x)}} \left( \kappa^2 \left( X^+(x)X^-\prime(x) - X^+\prime(x)X^-(x) \right) \right. \\
- \left. X^-\prime(x) \int_x^\infty d\tilde{x} \; \Pi_-(\tilde{x}) - X^+\prime(x) \int_{-\infty}^x d\bar{x} \; \Pi_+(\bar{x}) \right),
\] (91)
\[
\tau_L = -\lim_{x \to -\infty} \ln \left( \frac{X^-(x)}{x} \right),
\] (92)
\[
\tau_R = \kappa p + \lim_{x \to \infty} \ln \left( \frac{X^+(x)}{x} \right).
\] (93)
(The above transformation equations are obtained by following steps (i)-(iv) in section 2c, paying proper attention to boundary terms.) It can be checked (using (87)-(88) and (91)-(93)) that up to finite total time derivatives
\[
\int_{-\infty}^\infty dx \; \left( \pi_y \dot{y} + p_\sigma \dot{\sigma} \right) - \dot{\tau_R} \frac{m_R}{\kappa} + \dot{\tau_L} \frac{m_L}{\kappa} = \int_{-\infty}^\infty dx \; \left( \Pi_+ \dot{X}^+ + \Pi_- \dot{X}^- \right) + p \dot{m}_R.
\] (94)
It is best to confine $x$ to a finite interval $x_L < x < x_R$ and to take the limits $x_L \to -\infty$ and $x_R \to \infty$ only at the end.

To summarize, from equations (86) and (94) we conclude that up to an unimportant finite total time derivative

\begin{align*}
S[y, \pi_y, \sigma, p_\sigma, f, \pi_f, \bar{N}, N^1; \tau_L, \tau_R) &= S[X^\pm, \Pi_\pm, f, \pi_f, \bar{N}, N^1; p, m_R) \\
&= \int dt \int_{-\infty}^{\infty} dx \left( \Pi^*_+ X^+ + \Pi^*_- X^- + \pi_f \dot{f} - \bar{N} \bar{H} - N^1 H_1 \right) \\
&\quad + \int dt \ p \dot{m}_R ,
\end{align*}

(95)

with $\bar{H}$, $H_1$ taking the form of the constraints for a parametrized massless scalar field on a 2-dimensional Minkowski spacetime as in (51) and (52).

It is easy to see that the right mass $m_R$ and its conjugate momentum $p$ are constants of the motion. From (97), we see that $p$ can be interpreted as the difference between the proper time $\tau_R$ as measured by the right parametrization clock and the proper time $\tau$ reconstructed from the geometry.

Finally, we can express the parameters $A^+_L$, $A^-_R$ and the left mass $m_L$ in terms of the new set of variables:

\begin{align*}
A^+_L &= \int_{-\infty}^{\infty} dx \ \Pi_-(x) , \quad (96) \\
A^-_R &= \int_{\infty}^{-\infty} dx \ \Pi_+(x) , \quad (97)
\end{align*}

and

\begin{equation}
\frac{m_L}{\kappa} = \frac{m_R}{\kappa} + \int_{-\infty}^{\infty} dx \ X^+(x)\Pi_+(x) - \int_{\infty}^{-\infty} dx \ X^-(x)\Pi_-(x) . \quad (98)
\end{equation}

Equations (96)-(97) show that $A^+_L$ and $A^-_R$ generate Minkowskian translations (65) of $X^- (x)$ and $X^+ (x)$, respectively, through the Poisson brackets.

\[ \text{For more about this interpretation see \[24, 25]. In fact, the parameters } \tau, \tau_L, \tau_R \text{ correspond to those referred to as Killing time and left and right parametrization time in \[27].} \]
4. Constraint quantization of the dilatonic model

The constraints (53) of dilatonic gravity have the same form as those of a parametrized massless scalar field propagating on a flat 2d background. This reflects the conformal invariance of the scalar wave equation in two dimensions: While the scalar field curves the spacetime in which it propagates, the curvature does not affect the propagation. Because every 2-geometry is conformally flat, one can consider the propagation as taking place on an auxiliary flat background, rather than in the physical curved spacetime. The way in which we have reconstructed the embedding variables $X^\pm(x)$ from the geometric variables $g_{11}$ and $K_{11}$ guarantees that they coincide with the (double-null) Cartesian coordinates on this auxiliary background.

In the Dirac constraint quantization, the canonical variables $X^\pm(x)$, $\Pi^\pm(x)$, $f(x)$, $\pi_f(x)$, $m_R$, $p$ should be replaced by corresponding operators, and the constraints imposed as restrictions on physical states $\Psi[X^\pm, f; m]$.

The problem which needs to be overcome is that commutators of the energy-momentum tensor operators acquire Schwinger terms which, because the constraints contain projections of the energy-momentum tensor, enter into the commutators of the constraints as anomalies. The imposition of constraints on the states then leads to inconsistencies. In a previous work on the Dirac constraint quantization of a massless scalar field propagating on a flat Minkowskian cylinder $\mathbb{R} \times S^1$, we have shown how to remove the anomaly by a covariant (but embedding-dependent) factor ordering of the constraints. The closing of space $S^1$ has many formal advantages (the discrete spectra and the removal of the infrared problem), but it is not appropriate in the original geometric framework of dilatonic gravity (see the Appendix).

We now explain how the same procedure, adopted to the open space...
$\mathbb{R} \times \mathbb{R}$ boundary conditions, leads to a consistent Dirac constraint quantization of the dilatonic model.

4a. Fundamental Heisenberg operators

We want to turn the fundamental Heisenberg variables into operators acting on a suitable function space. Let

$$\Psi = \{ \Psi_n(k_1, \ldots, k_n) \} , \quad n = 0, 1, 2, \ldots$$

be a sequence of complex-valued functions symmetric in their arguments $k_1, \ldots, k_n$, with $\Psi_0 \in \mathbb{C}$ being a complex number. Define the norm of $\Psi$ by

$$||\Psi||^2_{\text{Fock}} := \Psi_0^* \Psi_0 + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{dk_1}{|k_1|} \cdots \int_{-\infty}^{\infty} \frac{dk_n}{|k_n|} \Psi_n^*(k_1, \ldots, k_n) \Psi_n(k_1, \ldots, k_n) .$$

The finite-norm sequences $\Psi$ are elements of the familiar Fock space $\mathcal{F}_{\text{Fock}}$. On $\mathcal{F}_{\text{Fock}}$, we introduce the standard annihilation $\hat{a}(k)$ and creation $\hat{a}^*(k)$ operators

$$\left( \hat{a}(k) \Psi \right)_n(k_1, \ldots, k_n) = \sqrt{n+1} \Psi_{n+1}(k, k_1, \ldots, k_n) ,$$
$$\left( \hat{a}^*(k) \Psi \right)_n(k_1, \ldots, k_n) =$$
$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} |k| \delta(k - k_i) \Psi_{n-1}(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n) ,$$

which satisfy the commutation relations

$$[ \hat{a}(k), \hat{a}^*(\bar{k}) ] = |k| \delta(k - \bar{k}) .$$

We want to represent the fundamental Heisenberg operators $\hat{f}(x)$ and $\hat{\pi}_f(x)$ on the Fock space $\mathcal{F}_{\text{Fock}}$. To see how this is done, it is best to introduce first the scalar field operator

$$\hat{f}(X) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{dk}{|k|} e^{i k_\alpha x^\alpha} \hat{a}(k) + \text{c.c.} ,$$

where

$$\gamma_{\alpha\beta} k^\alpha k^\beta = 0 .$$
In the double null coordinates $X^\pm$, the wave vector $k_\alpha$ has components $k_\pm = \frac{1}{2}(k \mp |k|)$. A spacelike embedding $X^\alpha(x)$ carries the canonical data

$$\hat{f}(x) = \hat{f}(X(x)) \quad \hat{\pi}_f(x) = g^{\frac{1}{2}}(x; X)n^\alpha(x; X)\hat{f}_\alpha(X(x)),$$

where $n^\alpha(x; X)$ is the unit (future-pointing) contravariant normal to the embedding, and $g(x; X)$ is the determinant of the induced spatial metric. By virtue of (103), these data satisfy the canonical commutation relations. In particular, the initial embedding (60) carries the fundamental Heisenberg data

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{dk}{|k|} e^{ikx} \hat{a}(k) + c.c. \quad (107)$$

$$\hat{\pi}_f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} dk e^{ikx} \hat{a}(k) + c.c. \quad (108)$$

This is the desired representation of $\hat{f}(x)$ and $\hat{\pi}_f(x)$ on $F_{\text{Fock}}$.

To have a space which would be able to carry a representation of the remaining fundamental Heisenberg operators, we extend $F_{\text{Fock}}$ into a larger space $F$ by allowing $\Psi_n(k_1, \ldots, k_n, m; X^\alpha)$ to be also functions of $m$ and functionals of the embeddings $X^\alpha(x)$. On $F$, we represent $\hat{m}_R$ and $\hat{X}^\alpha(x)$ by multiplication operators, and $\hat{p}$ and $\hat{\Pi}_\alpha(x)$ by differentiation operators:

$$\hat{m}_R = m \times, \quad \hat{p} = -i \frac{\partial}{\partial m};$$

$$\hat{X}^\alpha(x) = X^\alpha(x) \times, \quad \hat{\Pi}_\alpha(x) = -i \frac{\delta}{\delta X^\alpha(x)}.$$  

We have thus represented all the fundamental Heisenberg operators on $F$.

4b. Constraint quantization in the Heisenberg picture

In the Dirac constraint quantization, constraints are imposed as operator restrictions on physical states:

$$\hat{\Pi}_\alpha(x) \Psi = 0.$$  

\footnote{One may want the spectrum of the mass operator $\hat{m}_R$ to be non-negative. If so, one needs to replace the operator representation (109) by one appropriate for the affine group, and modify appropriately the inner product (113). The details of this approach are clearly discussed by Isham \cite{Isham}.}
In the Heisenberg picture, $\hat{\Pi}_\alpha(x)$ coincide with the embedding momenta and, as such, they are represented by variational derivatives (110). The constraint equation has a simple solution; it implies that physical states do not depend on embeddings:
\[
\Psi = \{ \Psi_n(k_1, \cdots, k_n, m) \}.
\]
This is, of course, the trademark of the Heisenberg picture: States do not depend on time.

The space $\mathcal{F}_0$ of physical states can now be equipped with the norm
\[
||\Psi||^2 = \int_{-\infty}^{\infty} dm \ ||\Psi(m)||^2_{\text{Fock}}
\]
which determines the statistical predictions of the theory. Observables are to be constructed from the Schrödinger field variables $\hat{f}(x)$, $\hat{\pi}_f(x)$ on an embedding $X^\alpha(x)$. In the Heisenberg picture, these variables are expressed as functionals of the fundamental Heisenberg operators $\hat{X}^\alpha(x)$, $\hat{f}(x)$, $\hat{\pi}_f(x)$:
\[
\hat{f}(x) = f(x; \hat{X}, \hat{f}, \hat{\pi}_f), \quad (114)
\]
\[
\hat{\pi}_f(x) = \pi_f(x; \hat{X}, \hat{f}, \hat{\pi}_f). \quad (115)
\]
They depend explicitly on time, i.e., on the embeddings $X^\alpha(x)$. The time dependence is determined by the Heisenberg equations of motion. These are the operator versions of the classical equations (62), (58):
\[
i\delta \hat{f}(x; X) \over\delta X^\pm(x) = [ \hat{f}(x; X), h_\pm(\bar{x}; X^\pm, \hat{f}, \hat{\pi}_f) ] = \pm[ \hat{f}(x; X), X^{\pm'}(\bar{x}) \hat{T}_{\pm\pm}(X^\pm(\bar{x})) ] ,
\]
and similarly for $\hat{\pi}_f(x; X)$. They are to be solved under the initial condition that the field operators $\hat{f}(x; X)$ and $\hat{\pi}_f(x; X)$ match the fundamental Heisenberg operators $\hat{f}(x)$ and $\hat{\pi}_f(x)$ on the initial embedding. The energy-momentum operator $\hat{T}_{\pm\pm}(X^\pm(\bar{x}))$ in (117) is assumed to be normal ordered in the annihilation and creation operators (101) and (102). As one can expect of a linear field theory, the solution of the Heisenberg equations of motion can be constructed from the scalar field operator (104) by differentiations, projections, and restrictions to the embedding according to equations (106).

We see that the Dirac constraint quantization of the dilatonic model in the Heisenberg picture leads to the standard Fock space and scalar field
operator of linear field theory on a flat background. The only extra feature is the presence of a single additional degree of freedom \( m \) in the state (112). This corresponds to the primordial component of the black hole mass which remains undetermined by the matter degrees of freedom. Indeed, the matter degrees of freedom fix only the difference \( (m_L - m_R) \) between the masses at the left and right infinities in accordance with (98). Because the true Hamiltonian in the canonical action (95) is equal to zero, both \( m_R \) and \( p \) are constants of the motion. The same is true about the corresponding operators in quantum theory:

\[
\frac{\delta \hat{m}_R}{\delta X^\pm(x)} = 0 = \frac{\delta \hat{p}}{\delta X^\pm(x)}.
\]

(118)

We conclude that the quantum theory in the Heisenberg picture has the same degrees of freedom as the classical theory, and that the Heisenberg equations of motion (which are linear) have exactly the same form as the classical equations.

**4c. Anomaly**

The Dirac constraint quantization of the dilatonic model is simplest when carried out in the Heisenberg picture. Historically, however, the Dirac procedure has always been associated with the Schrödinger picture. In this framework, its naive application to the dilatonic model leads to inconsistencies because the constraints develop an anomaly. To settle the question whether the Dirac procedure can be consistently implemented in the Schrödinger picture, we shall trace the origins of the anomaly and show how to remove it from the Schrödinger form of the constraints.

To pass from the Heisenberg to the Schrödinger picture, we need to transform the fundamental Heisenberg operators into fundamental Schrödinger operators. We have already seen that (106) and (104) solve this problem for the field operators. Because the embedding variables of the two pictures are the same, (59), the only remaining task is to find the Schrödinger embedding momenta \( \Pi^\pm(x) \). The classical equation (54) leads us to a natural candidate for \( \Pi^\pm(x) \), namely

\[
\hat{\Pi}^\pm(x) = \hat{\Pi}^\pm(x) - \hbar^\pm(x; X^\pm, \hat{f}, \hat{\pi}_f) \\
= \Pi^\pm(x) \mp X^\pm(x) \hat{T}^\pm(\Pi^\pm(x)).
\]

(119)

(120)
where $\hat{T}_{\pm \pm}(X^\pm(x))$ is the normal ordered energy-momentum tensor operator. This choice does not work, however, because the operators $\hat{\Pi}_{\pm}(x)$ do not commute.

To see this, one evaluates first the commutators of the components $\hat{T}_{\pm \pm}(X^\pm)$ at two spacetime events, $X^\pm$ and $\bar{X}^\pm$. A somewhat involved but straightforward calculation based on equations (12), (104), and (103) reveals that the commutator differs from the classical Poisson bracket by a Schwinger term proportional to the triply differentiated $\delta$-function:

$$
\frac{1}{i} [ \hat{T}_{\pm \pm}(X^\pm), \hat{T}_{\pm \pm}(\bar{X}^\pm) ] = \pm \hat{T}_{\pm \pm}(X^\pm)\delta_\pm(X^\pm - \bar{X}^\pm) - (X^\pm \leftrightarrow \bar{X}^\pm) \mp \frac{1}{12 \pi} \delta_\pm\pm\pm(X^\pm - \bar{X}^\pm),
$$

(121)

while

$$
\left[ \hat{T}_{++}(X^+) , \hat{T}_{--}(\bar{X}^-) \right] = 0.
$$

(122)

The Schwinger term leads then to the anomaly

$$
F_{\pm \pm}(x, \bar{x}; X^\pm) = \pm \frac{1}{12 \pi} \partial_x \left( \left( X^{\pm}(x) \right)^{-1} \partial_x \left( \left( X^{\pm}(x) \right)^{-1} \partial_x \delta(x, \bar{x}) \right) \right),
$$

(123)

in the commutator of (120):

$$
\frac{1}{i} [ \hat{\Pi}_{\pm}(x) , \hat{\Pi}_{\pm}(\bar{x}) ] = -F_{\pm \pm}(x, \bar{x}; X^\pm).
$$

(124)

The details of these as well as of the following calculations can be found in [21].

To summarize, the operators $\hat{\Pi}_{\pm}(x)$ commute with the Schrödinger fields $\hat{f}(x)$ and $\hat{\pi}_f(x)$—this is the content of the Heisenberg equations of motion (117)—and they also have the correct commutators with the embeddings:

$$
\frac{1}{i} [ \hat{X}^\pm(x) , \hat{\Pi}_{\pm}(\bar{x}) ] = \delta(x, \bar{x}).
$$

(125)

\[12\] The details of the calculation on the cylindrical background $\mathbb{R} \times S^1$ are given in [28] or [24]. The calculation in the open case $\mathbb{R} \times \mathbb{R}$ is similar and leads to exactly the same Schwinger term. The Casimir term present on $\mathbb{R} \times S^1$ is absent in the open case. The sign of the anomaly and its potential given in [21] should be corrected from $+$ to $-$. After this is done, to convert the signs of [21] into those in the present paper, one needs to keep track of the switch from the $T^\pm := T \pm Z$ variables used in [24] to the $X^\pm$ variables of equation (9).
However, they do not commute among themselves. This prevents us from identifying them with the Schrödinger embedding momenta.

4d. Removing the anomaly

Let us show how to amend $\tilde{\Pi}_\pm(x)$ into commuting operators which retain the correct commutators with the rest of the fundamental Schrödinger variables. The clue is provided by the Jacobi identity of the commutator (124). Because the anomaly depends only on the embedding variables, we conclude that

$$\frac{\delta F_{\alpha\beta}(x, \bar{x}; X)}{\delta X^\gamma(\bar{x})} + \text{cyclic permutations } (\alpha x, \beta \bar{x}, \gamma \bar{x}) = 0. \quad (126)$$

This means that the anomaly is a closed 2-form on the space of embeddings. More than that, the anomaly is exact: There exists a potential $A_\alpha(x; X)$ (we shall call it the anomaly potential) whose exterior derivative generates $F_{\alpha\beta}(x, \bar{x}; X)$:

$$F_{\alpha\beta}(x, \bar{x}; X) = \frac{\delta A_\beta(\bar{x}, X)}{\delta X^\alpha(x)} - \frac{\delta A_\alpha(x, X)}{\delta X^\beta(\bar{x})}. \quad (127)$$

Of course, $A_\alpha(x; X)$ is determined by (127) only up to a functional gradient. It is easy to check that

$$A_\pm(x; X^\pm) = \pm \frac{1}{24 \ 2\pi} \left( \left( X^{\pm}(x) \right)^{-1} \right)^{\prime \prime} \quad (128)$$

is one anomaly potential; indeed, the exterior derivative (127) of (128) gives the anomaly (123). Unfortunately, (128) does not transform as a scalar density under spatial diffeomorphisms. To improve this, we can gauge (128) by a functional gradient into a new potential

$$A_\pm(x; X) = \frac{1}{24 \ 2\pi} \left( \left( X^{\pm}(x) \right)^{-1} \left( X^{-\prime}(x) \frac{X^{-}(x)}{X^{-\prime}(x)} - \frac{X^{+}(x)}{X^{+\prime}(x)} \right) \right)^{\prime} = \frac{1}{12 \ 2\pi} \left( \left( X^{\pm}(x) \right)^{-1} g^2(x; X) K(x; X) \right)^{\prime}, \quad (129)$$

which is a scalar density and generates the same anomaly. As indicated, the potential (129) is simply related to the mean extrinsic curvature $K$ of the embedding.
Our old operators \( \hat{\Pi}_\pm(x) \) have correct commutation relations with the Schrödinger fields \( \hat{f}(x), \hat{\pi}_f(x) \) and with \( \hat{X}^\pm(x) \), but they do not commute among themselves. By subtracting from them the anomaly potential, we change them into commuting operators

\[
\hat{\Pi}_\pm(x) := \hat{\Pi}_\pm(x) - A_\pm(x; X) ,
\]
which retain the correct commutation relations with \( \hat{f}(x), \hat{\pi}_f(x) \), and \( \hat{X}^\pm(x) \). These we can identify with the Schrödinger embedding momenta.

4e. Imposing constraints in the Schrödinger picture

At this point, we can express the constraints in terms of the fundamental Schrödinger variables:

\[
\hat{\Pi}_\pm(x) = \hat{\Pi}_\pm(x) + h_\pm(x; X^\pm, \hat{f}, \hat{\pi}_f) + A_\pm(x; X) \approx 0 .
\]

To find an explicit form of \( h_\pm(x; X^\pm, \hat{f}, \hat{\pi}_f) \), we split the spacetime field operators \( \hat{f}_\pm(X^\pm) \) into their positive-frequency (the Heisenberg annihilator \( \hat{a}(k) \)) and negative-frequency (the Heisenberg creator \( \hat{a}^\ast(k) \)) parts:

\[
\hat{f}_\pm(X^\pm) = (+)\hat{f}_\pm(X^\pm) + (-)\hat{f}_\pm(X^\pm).
\]

This is achieved by the positive- and negative-frequency parts \((\pm)\delta\) of the \(\delta\)-function

\[
(\pm)\delta(X) := \frac{1}{2\pi} \int_0^\infty dk \ e^{\mp ikX} ,
\]
which act as kernels of integral operators

\[
(\pm)\hat{f}_\pm(X^\pm) = \int_{-\infty}^{\infty} d\bar{X}^\pm (\pm)\delta(X^\pm - \bar{X}^\pm) \hat{f}_\pm(\bar{X}^\pm) .
\]

This decomposition allows us to perform the Heisenberg normal ordering of the energy-momentum tensor by the ordering kernel

\[
\mathcal{N}(X^\pm; \bar{X}^\pm, \tilde{X}^\pm) = (\pm)\delta(X^\pm - X^\pm)(\pm)\delta(X^\pm - \bar{X}^\pm) + (-)\delta(X^\pm - X^\pm)(-\pm)\delta(X^\pm - \bar{X}^\pm)
\]

\[
+ (\pm)\delta(X^\pm - \bar{X}^\pm)(\pm)\delta(X^\pm - \tilde{X}^\pm) + (-)\delta(X^\pm - \bar{X}^\pm)(-\pm)\delta(X^\pm - \tilde{X}^\pm)
\]

\[
\mathcal{N}(X^\pm; \bar{X}^\pm, \tilde{X}^\pm) =
\]

\[
(130)
\]

\[
(131)
\]

\[
(132)
\]

\[
(133)
\]

\[
(134)
\]

\[
(135)
\]
in an integral operator

\[ \hat{T}_{\pm \pm}(X^\pm) = \int_{-\infty}^{\infty} d\bar{X}^\pm \int_{-\infty}^{\infty} \bar{d}\tilde{X}^\pm \mathcal{N}(X^\pm; \bar{X}^\pm, \tilde{X}^\pm) \hat{f}_{\pm \pm}(X^\pm) \hat{\bar{f}}_{\pm \pm}(\tilde{X}^\pm) \]  

(136)

The connection (57) between the field operators \( \hat{f}_{\pm \pm}(x) \) and the fundamental Schrödinger operators then finishes our task:

\[ h_{\pm}(x; X^\pm, \hat{f}, \hat{\pi}_f) = \pm \frac{1}{4} X^{\pm t}(x) \int_{-\infty}^{\infty} d\bar{x} \int_{-\infty}^{\infty} \bar{d}x \times \]

\[ \times \mathcal{N}(X^\pm(x); X^\pm(\bar{x}), X^\pm(\tilde{x}))(\hat{f}'(\bar{x}) \pm \hat{\pi}_f(\bar{x}))(\hat{\bar{f}}'(\tilde{x}) \pm \hat{\pi}_f(\tilde{x})) \]  

(137)

The fundamental Schrödinger operators satisfy the appropriate commutation relations:

\[ \frac{1}{i}[\hat{X}^\pm(x), \hat{\Pi}_\pm(\bar{x})] = \delta(x, \bar{x}) \]  

(138)

\[ \frac{1}{i}[\hat{f}(x), \hat{\pi}_f(\bar{x})] = \delta(x, \bar{x}) \]  

(139)

with all other commutators vanishing. In the Schrödinger picture, \( \hat{X}^\pm(x) \) and \( \hat{\Pi}_\pm(x) \) are represented as multiplication and differentiation operators

\[ \hat{X}^\pm(x) = X^\pm(x) \times, \quad \hat{\Pi}_\pm(x) = -i \frac{\delta}{\delta X^\pm(x)} \]  

(140)

and the quantum constraint (111) yields the functional Schrödinger equation

\[ i \frac{\delta \Psi[X]}{\delta X^\pm(x)} = \left( h_{\pm}(x; X^\pm, \hat{f}, \hat{\pi}_f) + A_{\pm}(x; X) \right) \Psi[X] = 0. \]  

(141)

This determines the dependence of the Schrödinger state \( \Psi[X] \) on the embeddings. If one also decides to work in the Schrödinger representation, the field operators get represented by multiplication and differentiation operators, and (141) becomes an equation for the state functional \( \Psi[X, f] \). The anomaly potential ensures the functional integrability of (141).

We thus see that the Dirac constraint quantization in the Schrödinger picture, though somewhat subtle to formulate, is consistent and equivalent to the more straightforward Heisenberg picture quantization.
5. How to quantize physical geometry?

We found a canonical transformation which reduced the constraints of the dilatonic model to those of a parametrized field theory on a flat background. Any reference to the original physical geometry disappeared from the description of the system. However, the interesting questions in dilatonic gravity are precisely those which are concerned with the physical spacetime. We must show how to pose such questions in the framework based on the flat background canonical variables.

We know that the physical metric $\bar{\gamma}_{\alpha\beta}$ is related to the auxiliary flat metric $\gamma_{\alpha\beta}$ by the dilaton factor $y^{-1}$, (2). In the null coordinates $X^{\pm}$, the physical interval $d\bar{s}$ is given by the formula

$$d\bar{s}^2 = \frac{dX^{+}dX^{-}}{y(X)}.$$

(142)

In the canonical theory, the dilaton is connected to the new canonical variables $(X^{\pm}(x), \Pi_{\pm}(x); m_R, p)$ by the canonical transformation (88). On the constraint surface (54), (58), the dilaton can be expressed as a functional of the energy-momentum tensor:

$$y(X) = \kappa^2 X^{+}X^{-} \left( -\int_{-\infty}^{X^{-}} dX^{-} \int_{-\infty}^{X^{-}} d\tilde{X}^{-} T_{--}(\tilde{X}^{-}) - \int_{-\infty}^{X^{+}} dX^{+} \int_{-\infty}^{X^{+}} d\tilde{X}^{+} T_{++}(\tilde{X}^{+}) \right)$$

$$- \int_{-\infty}^{\infty} dX^{+} X^{+}T_{++}(X^{+}) + \frac{m_R}{\kappa}.$$

(143)

This brings us back to the spacetime solution (13)-(14).

By quantizing the dilatonic model in the Heisenberg picture, we turn $y(X)$ into an operator on the Fock space $\mathcal{F}_0$ of physical states (112) with the norm (113). This turns the network of physical intervals (142) into operators. The operator version of equation (142) is thus the starting point of discussions about quantum geometry.

To make sense of the operator version of equation (142) is not entirely straightforward. First, in the old action, the dilaton field $y(x)$ is one of the canonical coordinates, and hence it commutes at any two points on a spacelike surface. When expressed as a functional (143) of the scalar field data, this is no longer necessarily true. This poses questions about simultaneous
measurability of different pieces of the quantum geometry. Second, one must
decide what is the correct factor ordering of the field operators in the energy-
momentum tensor in (143). Normal ordering is a natural candidate, but our
previous discussion (section 4c) raises the possibility that other options may
be more appropriate. Third, after the factor ordering decision is made, the
quantum geometry should be defined by spectral analysis. Finally, the classi-
cal dilaton field is required to be positive, to ensure the right signature (−, +)
of the physical metric. In quantum theory it is quite difficult to maintain the
positivity condition. Indeed, refraining from doing so opens the possibility
of evolving the physical geometry through what classically would appear to
be a singularity. These and other problems must be settled before one can
meaningfully speak about quantum physical geometry. We intend to address
them in a future paper.

Notwithstanding that the detailed resolution of issues connected with
the Hawking effect will depend on how we settle the problems mentioned
above, we stress that the quantum theory in section 4 is a standard unitary
quantum field theory on a Fock space; i.e., that we do not encounter any loss
of unitarity under evolution.

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Appendix

To see the problems of putting the CGHS model on \( \mathbb{R} \times S^1 \) in its simplest
setting, inspect the vacuum solution

\[
y(X) = \kappa^2 X^+ X^- + \frac{m}{\kappa}
\]  

(144)
for the dilaton field. This describes a primordial black hole of mass $m$ with the physical metric $\bar{\gamma}_{\alpha\beta} = y^{-1}(X)\gamma_{\alpha\beta}$ (see [3]).

Try to put this solution on a Minkowskian cylinder $\mathbb{R} \times S^1$ by identifying the points with Cartesian coordinates $X_{(1)}^\alpha = (T, Z - \xi)$ and $X_{(2)}^\alpha = (T, Z + \xi)$; i.e., with the double null coordinates differing by a translation:

$$X_{(2)}^\pm = X_{(1)}^\pm + 2\xi, \quad \xi \in (0, \infty). \quad (145)$$

The dilaton $y(X)$, the induced metric $g_{TT}(T) = -y^{-1}$, and the scalar curvature $R[\gamma] = 4mky^{-1}$ are continuous across the seam. However, the (mean) extrinsic curvature of the seam, as embedded in the physical geometry on its two sides, suffers a jump:

$$K(\xi) - K(-\xi) = 2\kappa \cdot \kappa \xi \cdot y^{-\frac{1}{2}}. \quad (146)$$

This indicates the presence of an unphysical sheet of matter. We would encounter the same situation if we tried to identify the left and the right openings of the Einstein-Rosen bridge at the same finite value of the spatial Kruskal coordinate (and hence the same value of the area coordinate $r$, which is analogous to the dilaton) in the Schwarzschild black hole. Our argument can easily be generalized to the dilatonic black hole (13)-(14) in the presence of a scalar field. This straightforward attempt of putting the dilatonic black hole on a Minkowskian cylinder thus fails.

The correct way of putting the vacuum black hole solution (144) on $\mathbb{R} \times S^1$ is to cut a wedge from the dynamical region $X^+ > 0, X^- < 0$ and wrap it into a cone.\footnote{The same construction can also be done in the past dynamical region $X^+ < 0, X^- > 0.$} Take a straight timelike line

$$X^\alpha(t) = t^\alpha t \quad (147)$$

in the Minkowskian plane parametrized by the Minkowskian proper time $t > 0$, with $t^\alpha$ being a constant, future-pointing, unit tangent vector with respect to the flat Minkowski metric. The vector field $k^\alpha$, having components

$$k^\pm = (\kappa X^+, -\kappa X^-), \quad (148)$$

is perpendicular to the radius vector $X^\pm = (X^+, X^-)$, and it is a Killing vector field of the physical metric. Because $k^\alpha$ is perpendicular to $t^\alpha$, the
extrinsic curvature of the line \( I \), as embedded in the physical spacetime, vanishes. Its induced metric is 
\[
g_{tt} = -\left( -\kappa^2 t^2 + \kappa^{-1} m \right)^{-1}.
\]

Now take any two lines, with tangent vectors \( t_{(1)}^\alpha \) and \( t_{(2)}^\alpha \) oriented in the clockwise direction with respect to one another, and identify their points labeled by the same \( t \). Instead of the Minkowskian cylinder \( I \), we get a Minkowskian cone
\[
X^+ \ (2) = X^+ \ (1) e^\eta, \quad X^- \ (2) = X^- \ (1) e^{-\eta},
\]
where the monodromy parameter \( \eta \) is the hyperbolic angle between the two lines in the Minkowskian plane. Because the physical metric is conformally related to the flat Minkowski metric, it is also the hyperbolic angle between the lines in the physical space. Both the induced metric and the extrinsic curvature match at the seam. The transformation equations
\[
X^\pm = \pm \kappa^{-1} \sqrt{\frac{m}{\kappa}} \exp \left( \frac{\eta}{2\pi} (\chi \pm \phi) \pm \eta_0 \right),
\]
where \( \eta_0 \) is the hyperbolic parameter of the midline between \( t_{(1)}^\alpha \) and \( t_{(2)}^\alpha \), brings us to dimensionless coordinates \( \chi \in (-\infty, 0), \phi \in (-\pi, \pi) \) and the physical line element
\[
ds^2 = \left( \frac{\eta}{2\pi \kappa} \right)^2 \left( e^{-\frac{\eta_0}{\kappa}} - 1 \right)^{-1} (-d\chi^2 + d\phi^2).
\]
The vector field \( \frac{\partial}{\partial \phi} \) is a Killing field of the physical metric.

The resulting \( \mathbb{R} \times S^1 \) spacetime covers only a segment of the interior of the black hole, and it is geodesically incomplete. The counterpart of this construction for the Schwarzschild black hole is the Kantowski-Sachs universe. Again, one can generalize this procedure to the dilatonic black hole \( (13)-(14) \) produced by a scalar source \( \delta \). For possible \( \mathbb{R} \times S^1 \) identifications in different dilatonic theories, see \( \delta \).

Thus, we see that by putting the CGHS model on \( \mathbb{R} \times S^1 \), we lose the physical picture of the black hole formation by the gravitational collapse of the matter field.
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