DUALIZING COMPLEX OF THE INCIDENCE ALGEBRA OF A FINITE REGULAR CELL COMPLEX

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Abstract. Let $\Sigma$ be a finite regular cell complex with $\emptyset \in \Sigma$, and regard it as a poset (i.e., partially ordered set) by inclusion. Let $R$ be the incidence algebra of the poset $\Sigma$ over a field $k$. Corresponding to the Verdier duality for constructible sheaves on $\Sigma$, we have a dualizing complex $\omega^\bullet \in D^b(\text{mod} R \otimes_k R)$ giving a duality functor from $D^b(\text{mod} R)$ to itself. This duality is somewhat analogous to the Serre duality for projective schemes ($\emptyset \in \Sigma$ plays a similar role to that of “irrelevant ideals”). If $H^i(\omega^\bullet) \neq 0$ for exactly one $i$, then the underlying topological space of $\Sigma$ is Cohen-Macaulay (in the sense of the Stanley-Reisner ring theory). The converse also holds if $\Sigma$ is a meet-semilattice as a poset (e.g., $\Sigma$ is a simplicial complex). $R$ is always a Koszul ring with $R! \sim R^\text{op}$. The relation between the Koszul duality for $R$ and the Verdier duality is discussed. This result is a variant of a theorem of Vybornov.

1. Introduction

Let $\Sigma$ be a finite regular cell complex, and $X := \bigcup_{\sigma \in \Sigma} \sigma$ its underlying topological space. The order given by $\sigma > \tau \overset{\text{def}}{\iff} \overline{\sigma} \supset \tau$ makes $\Sigma$ a finite partially ordered set (poset, for short). Here $\overline{\sigma}$ is the closure of $\sigma$ in $X$. Let $R$ be the incidence algebra of the poset $\Sigma$ over a field $k$. For a ring $A$, $\text{mod}_A$ denotes the category of finitely generated left $A$-modules. In this paper, we study the bounded derived category $D^b(\text{mod} R)$ using the theory of constructible sheaves (e.g., Verdier duality). For the sheaf theory, consult [6, 7, 14]. We basically use the same notation as [6].

Let $\text{Sh}_c(X)$ be the category of $k$-constructible sheaves on $X$ with respect to the cell decomposition $\Sigma$. We have an exact functor $(-)^{\dagger} : \text{mod}_R \to \text{Sh}_c(X)$. For $M \in \text{mod}_R$, we have a natural decomposition $M = \bigoplus_{\sigma \in \Sigma} M_{\sigma}$ as a $k$-vector space. If $p \in \sigma \subset X$, the stalk $(M^\dagger)_p$ of $M^\dagger$ at the point $p$ is isomorphic to $M_{\sigma}$.

Let $\Sigma' := \Sigma \setminus \emptyset$ be an induced subposet of $\Sigma$, and $T$ the incidence algebra of $\Sigma'$ over $k$. Then we have a category equivalence $\text{mod}_T \cong \text{Sh}_c(X)$, which is well-known to specialists (see for example [8, 11, 14]). But, in this paper, $\emptyset \in \Sigma$ plays a role. Although $\text{mod}_R \not\cong \text{Sh}_c(X)$, $\text{mod}_R$ has several interesting properties which $\text{mod}_T$ does not possess. In some sense, $\emptyset$ is analogous to the “irrelevant ideal” of a commutative noetherian homogeneous $k$-algebra.

We have a left exact functor $\Gamma_\emptyset : \text{mod}_R \to \text{vect}_k$ defined by $\Gamma_\emptyset(M) = \{ x \in M_\emptyset \mid Rx \subset M_\emptyset \}$. We denote its $i$th right derived functor by $H^i_\emptyset(-)$. For $M \in \text{mod}_R$,
Theorem 2.2 states that:

\[ H^i(X, M^\uparrow) \cong H^{i+1}_\emptyset (M) \quad \text{for all } i \geq 1, \]

\[ 0 \to H^0_\emptyset (M) \to M_\emptyset \to H^0(X, M^\uparrow) \to H^1_\emptyset (M) \to 0 \quad \text{(exact)}. \]

Here \( H^\bullet (X, M^\uparrow) \) stands for the sheaf cohomology (c.f. \([7, 6]\)).

Let \( A \) and \( B \) be \( k \)-algebras. Recently, several authors study a dualizing complex \( C^\bullet \in D^b(\text{mod}_{A \otimes k B}) \) giving duality functors between \( D^b(\text{mod}_A) \) and \( D^b(\text{mod}_B) \).

(Note that if \( M \in \text{mod}_A \) and \( N \in \text{mod}_{A \otimes k B} \) then \( \text{Hom}_A(M, N) \) has a left \( B \)-module structure.) In typical cases, it is assumed that \( B = A^{\text{op}} \). But, in this paper, from Verdier’s dualizing complex \( D^*_X \in D^b(\text{Sh}_c(X)) \) on \( X \), we construct a dualizing complex \( \omega^\bullet \in D^b(\text{mod}_{R \otimes R}) \) which gives the duality functor \( \mathbf{R} \text{Hom}_R(-, \omega^\bullet) \) from \( D^b(\text{mod}_R) \) to itself. Theorem 3.2 states that

\[ \mathbf{R} \text{Hom}_R(M^\bullet, \omega^\bullet)^\dagger \cong \mathbf{R} \text{Hom}((M^\bullet)^\dagger, D^*_X) \]

in \( D^b(\text{Sh}_c(X)) \) for all \( M^\bullet \in D^b(\text{mod}_R) \). The dualizing complex \( \omega^\bullet \) satisfies the Auslander condition in the sense of \([19]\).

Corollary 3.3 states that

\[ \text{Ext}^i_R(M^\bullet, \omega^\bullet)_{\emptyset} \cong H^{i-1}_\emptyset (M^\bullet)^\vee. \]

This corresponds to the (global) Verdier duality on \( X \). But, since \( H^i_\emptyset (-) \) can be seen as an analog of a local cohomology over a commutative noetherian homogeneous \( k \)-algebra, the above isomorphism can be seen as an imitation of the Serre duality. In Theorem 5.3 (1), \( \emptyset \in \Sigma \) is also essential. It states that, for a simplicial complex \( \Sigma \), \( H^i(\omega^\bullet) = 0 \) for all \( i \neq - \dim X \) if and only if \( X \) is Cohen-Macaulay in the sense of the Stanley-Reisner ring theory. If we use the convention that \( \emptyset \notin \Sigma \), then the Cohen-Macaulay property can not be characterized in this way.

Under the assumption that a subset \( \Psi \) of \( \Sigma \) gives the open subset \( U_\Psi := \bigcup_{\sigma \in \Psi} \sigma \) of \( X \), Theorem 5.3 describes the cohomology \( H^i(U_\Psi, M^\uparrow|_{U_\Psi}) \) using the duality functor \( \mathbf{R} \text{Hom}_R(-, \omega^\bullet) \). Note that the cohomology with compact support \( H^i_c(U_\Psi, M^\uparrow|_{U_\Psi}) \) is much easier to treat in our context as shown in Lemma 5.1.

We can regard \( R \) as a graded ring in a natural way. Then \( R \) is always Koszul, and the quadratic dual ring \( R^! \) is isomorphic to the opposite ring \( R^{\text{op}} \) (Proposition 6.1). Koszul duality (c.f. \([1]\)) gives an equivalence \( D^b(\text{mod}_R) \cong D^b(\text{mod}_{R^{\text{op}}}) \) of triangulated categories. The functors giving this equivalence coincide with the compositions of the duality functors \( \mathbf{R} \text{Hom}_R(-, \omega^\bullet) \) and \( \text{Hom}_k(-, k) \). This result is an “augmented” version of Vybornov \([14]\).

It is well known that the Möbius function of a finite poset is a very important tool in combinatorics. In Proposition 6.1, generalizing \([13]\) Proposition 3.8.9, we describes the Möbius function \( \mu(\sigma, \hat{1}) \) of the poset \( \Sigma := \Sigma \Pi \{1\} \) in terms of cohomology with compact support. As shown in \([2]\), some finite posets arising from purely combinatorial/algebraic topics (e.g., Bruhat order) are isomorphic to the posets of finite regular cell complexes. So the author expects that the results in the present paper will play a role in combinatorial study of these posets.
2. Preparation

A finite regular cell complex (c.f. §6.2 and §1) is a non-empty topological space $X$ together with a finite set $\Sigma$ of subsets of $X$ such that the following conditions are satisfied:

(i) $\emptyset \in \Sigma$ and $X = \bigcup_{\sigma \in \Sigma} \sigma$;
(ii) the subsets $\sigma \in \Sigma$ are pairwise disjoint;
(iii) for each $\sigma \in \Sigma$, $\sigma \neq \emptyset$, there exists a homeomorphism from an $i$-dimensional disc $B^i = \{x \in \mathbb{R}^i \mid ||x|| \leq 1\}$ onto the closure $\overline{\sigma}$ of $\sigma$ which maps the open disc $U^i = \{x \in \mathbb{R}^i \mid ||x|| < 1\}$ onto $\sigma$.

An element $\sigma \in \Sigma$ is called a cell. We regard $\Sigma$ as a poset by $\sigma \supset \tau \iff \sigma \supset \tau$. Combinatorics on posets of this type is discussed in [2]. If $\sigma \in \Sigma$ is homeomorphic to $U^i$, we write $\dim \sigma = i$ and call $\sigma$ an $i$-cell. Here $\dim \emptyset = -1$. Set $d := \dim X = \max\{\dim \sigma \mid \sigma \in \Sigma\}$.

A finite simplicial complex is a primary example of finite regular cell complexes. When $\Sigma$ is a finite simplicial complex, we sometimes identify $\Sigma$ with the corresponding abstract simplicial complex. That is, we identify a cell $\sigma \in \Sigma$ with the set $\{\tau \mid \tau \in \Sigma, \tau \subseteq \sigma \}$. In this case, $\Sigma$ is a subset of the power set $2^V$, where $V$ is the set of the vertices (i.e., 0-cells) of $\Sigma$. Under this identification, for $\sigma \in \Sigma$, set $st_\sigma := \{\tau \in \Sigma \mid \tau \cap \sigma \subseteq \tau \} \{\tau \in \Sigma \mid \tau \cap \sigma = \emptyset\}$ to be subcomplexes of $\Sigma$.

Let $\sigma, \sigma' \in \Sigma$. If $\dim \sigma = i + 1$, $\dim \sigma' = i - 1$ and $\sigma' \not< \sigma$, then there are exactly two cells $\sigma_1, \sigma_2 \in \Sigma$ between $\sigma'$ and $\sigma$. (Here $\dim \sigma_1 = \dim \sigma_2 = i$.) A remarkable property of a regular cell complex is the existence of an incidence function $\varepsilon$ (c.f. II. Definition 1.8]). The definition of an incidence function is the following.

(i) To each pair $(\sigma, \sigma')$ of cells, $\varepsilon$ assigns a number $\varepsilon(\sigma, \sigma') \in \{0, \pm 1\}$.
(ii) $\varepsilon(\sigma, \sigma') \neq 0$ if and only if $\dim \sigma' = \dim \sigma - 1$ and $\sigma' \not< \sigma$.
(iii) If $\dim \sigma = 0$, then $\varepsilon(\sigma, \emptyset) = 1$.
(iv) If $\dim \sigma = i + 1$, $\dim \sigma' = i - 1$ and $\sigma' \not< \sigma$, $\sigma_1 \not< \sigma, \sigma_1 \not= \sigma_2, \sigma_1 \not= \sigma_2$, then we have $\varepsilon(\sigma, \sigma_1) \varepsilon(\sigma_1, \sigma') + \varepsilon(\sigma, \sigma_2) \varepsilon(\sigma_2, \sigma') = 0$.

We can compute the (co)homology groups of $X$ using the cell decomposition $\Sigma$ and an incidence function $\varepsilon$.

Let $P$ be a finite poset. The incidence algebra $R$ of $P$ over a field $k$ is the $k$-vector space with a basis $\{e_{x,y} \mid x, y \in P \text{ with } x \geq y\}$. The $k$-bilinear multiplication defined by $e_{x,y} e_{z,w} = \delta_{y,z} e_{x,w}$ makes $R$ a finite dimensional associative $k$-algebra. Set $e_x := e_{x,x}$. Then $1 = \sum_{x \in P} e_x$ and $e_x e_y = \delta_{x,y} e_x$. We have $R \cong \bigoplus_{x \in P} Re_x$ as a left $R$-module, and each $Re_x$ is indecomposable.

Denote the category of finitely generated left $R$-modules by $\text{mod}_R$. If $N \in \text{mod}_R$, we have $N = \bigoplus_{x \in P} N_x$ as a $k$-vector space, where $N_x := e_x N$. Note that $e_{x,y} N_y \subseteq N_x$ and $e_{x,y} N_z = 0$ for $y \neq z$. If $f : N \to N'$ is a morphism in $\text{mod}_R$, then $f(N_x) \subseteq N'_x$.

For each $x \in P$, we can construct an indecomposable injective module $E_R(x) \in \text{mod}_R$. (When confusion does not occur, we simply denote it by $E(x)$.) Let $E(x)$ be the $k$-vector space with a basis $\{e(x)_y \mid y \leq x\}$. Then we can regard $E(x)$ as a
left $R$-module by

$$e_{z,w}e(x)_y = \begin{cases} e(x)_z & \text{if } y = w \text{ and } z \leq x, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $E(x)_y = ke(x)_y$ if $y \leq x$, and $E(x)_y = 0$ otherwise. An indecomposable injective in $\text{mod}_R$ is of the form $E(x)$ for some $x \in P$. Since $\dim_k R < \infty$, $\text{mod}_R$ has enough projectives and injectives. It is well-known that $R$ has finite global dimension.

Let $\Sigma$ be a finite regular cell complex, and $X$ its underlying topological space. We make $\Sigma$ a poset as above. In the rest of this paper, $R$ is the incidence algebra of $\Sigma$ over $k$. For $M \in \text{mod}_R$, we have $M = \bigoplus_{\sigma \in \Sigma} M_{\sigma}$ as a $k$-vector space, where $M_{\sigma} := e_{\sigma}M$.

Let $\text{Sh}(X)$ be the category of sheaves of finite dimensional $k$-vector spaces on $X$. We say $F \in \text{Sh}(X)$ is a constructible sheaf with respect to the cell decomposition $\Sigma$, if $F|_{\sigma}$ is a constant sheaf for all $\emptyset \neq \sigma \in \Sigma$. Here, $F|_{\sigma}$ denotes the inverse image $j^*F$ of $F$ by the embedding map $j : \sigma \to X$. Let $\text{Sh}_c(X)$ be the full subcategory of $\text{Sh}(X)$ consisting of constructible sheaves with respect to $\Sigma$. It is well-known that $D^b(\text{Sh}_c(X)) \cong D^b_{\text{sh}_c}(\text{Sh}(X))$. (See [7, Theorem 8.1.11]. There, it is assumed that $\Sigma$ is a simplicial complex. But this assumption is irrelevant. In fact, the key lemma [7, Corollary 8.1.5] also holds for regular cell complexes. See also [11, Lemma 5.2.1].) So we will freely identify these categories.

There is a functor $(-)^\dag : \text{mod}_R \to \text{Sh}_c(X)$ which is well-known to specialists (see for example [14, Theorem A]). But we give a precise construction here for the reader’s convenience. See [14, 17] for detail.

For $M \in \text{mod}_R$, set

$$\text{Sp}^e(M) := \bigcup_{\emptyset \neq \sigma \in \Sigma} \sigma \times M_{\sigma}.$$ 

Let $\pi : \text{Sp}^e(M) \to X$ be the projection map which sends $(p, m) \in \sigma \times M_{\sigma} \subset \text{Sp}^e(M)$ to $p \in \sigma \subset X$. For an open subset $U \subset X$ and a map $s : U \to \text{Sp}^e(M)$, we will consider the following conditions:

($\ast$) $\pi \circ s = \text{Id}_U$ and $s_{pq} = e_{\tau, \sigma} \cdot s_{p}$ for all $p, q \in \tau$ with $\tau \geq \sigma$. Here $s_p$ (resp. $s_q$) is the element of $M_{\sigma}$ (resp. $M_{\tau}$) with $s(p) = (p, s_p)$ (resp. $s(q) = (q, s_q)$).

($\ast\ast$) There is an open covering $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ such that the restriction of $s$ to $U_\lambda$ satisfies ($\ast$) for all $\lambda \in \Lambda$.

Now we define a sheaf $M^\dag \in \text{Sh}_c(X)$ from $M$ as follows. For an open set $U \subset X$, set

$$M^\dag(U) := \{ s : U \to \text{Sp}^e(M) \text{ is a map satisfying } (\ast\ast) \}$$

and the restriction map $M^\dag(U) \to M^\dag(V)$ is the natural one. It is easy to see that $M^\dag$ is a constructible sheaf. For $\sigma \in \Sigma$, let $U_\sigma := \bigcup_{\tau \geq \sigma} \tau$ be an open set of $X$. Then we have $M^\dag(U_\sigma) \cong M_{\sigma}$. Moreover, if $\sigma \leq \tau$, then we have $U_\sigma \supset U_\tau$ and the restriction map $M^\dag(U_\sigma) \to M^\dag(U_\tau)$ corresponds to the multiplication map $M_{\sigma} \ni x \mapsto e_{\tau, \sigma}x \in M_{\tau}$. For a point $p \in \sigma$, the stalk $(M^\dag)_p$ of $M^\dag$ at $p$ is isomorphic to $M_{\sigma}$. This construction gives the functor $(-)^\dag : \text{mod}_R \to \text{Sh}_c(X)$. Let
0 \to M' \to M \to M'' \to 0 \text{ be a complex in } \text{mod}_R. \text{ The complex } 0 \to (M')^\dagger \to M^\dagger \to (M'')^\dagger \to 0 \text{ is exact if and only if } 0 \to M'_\sigma \to M_\sigma \to M''_\sigma \to 0 \text{ is exact for all } \emptyset \neq \sigma \in \Sigma. \text{ Hence } (-)^\dagger \text{ is an exact functor. We also remark that } M_\emptyset \text{ is irrelevant to } M^\dagger.

For example, we have \( E(\sigma)^\dagger \cong j_! k_{\bar{\sigma}} \), where \( j \) is the embedding map from the closure \( \bar{\sigma} \) of \( \sigma \) to \( X \) and \( k_{\bar{\sigma}} \) is the constant sheaf on \( \bar{\sigma} \). Similarly, we have \( (R e_\sigma)^\dagger \cong h_! k_{U_\sigma} \), where \( h \) is the embedding map from the open subset \( U_\sigma = \bigcup_{\tau \supset \sigma} \tau \) to \( X \).

**Remark 2.1.** Let \( \Sigma' := \Sigma \setminus \emptyset \) be an induced subposet of \( \Sigma \), and \( T \) its incidence algebra over \( k \). Then we have a functor \( \text{mod}_T \to \text{Sh}_c(X) \) defined by a similar way to \((-)^\dagger\), and it gives an equivalence \( \text{mod}_T \cong \text{Sh}_c(X) \) (c.f. [14, Theorem A]). On the other hand, by virtue of \( \emptyset \in \Sigma \), our \((-)^\dagger : \text{mod}_R \to \text{Sh}_c(X) \) is neither full nor faithful. But we will see that \( \text{mod}_R \) has several interesting properties which \( \text{mod}_T \) does not possess.

For \( M \in \text{mod}_R \), set \( \Gamma_0(M) := \{ x \in M_0 \mid R x \subset M_\emptyset \} \). It is easy to see that \( \Gamma_0(M) \cong \text{Hom}_R(k, M) \). Here we regard \( k \) as a left \( R \)-module by \( e_{\sigma, \tau} k = 0 \) for all \( e_{\sigma, \tau} \neq 0 \). Clearly, \( \Gamma_0 \) gives a left exact functor from \( \text{mod}_R \) to itself (or \( \text{vect}_k \)). We denote the \( i \)th right derived functor of \( \Gamma_0(-) \) by \( H^i_0(-) \). In other words, \( H^i_0(-) = \text{Ext}^i_R(k, -) \).

**Theorem 2.2** (c.f. [14, Theorem 3.3]). For \( M \in \text{mod}_R \), we have an isomorphism \( H^i(X, M^\dagger) \cong H^{i+1}_0(M) \) for all \( i \geq 1 \), and an exact sequence

\[
0 \to H^0_0(M) \to M_0 \to H^0(X, M^\dagger) \to H^1_0(M) \to 0.
\]

Here \( H^\bullet(X, M^\dagger) \) stands for the cohomology with coefficients in the sheaf \( M^\dagger \).

**Proof.** Let \( I^\bullet \) be an injective resolution of \( M \), and consider the exact sequence

\[
0 \to \Gamma_0(I^\bullet) \to I^\bullet \to I^\bullet/\Gamma_0(I^\bullet) \to 0
\]

of cochain complexes. Put \( J^\bullet := I^\bullet/\Gamma_0(I^\bullet) \). Each component of \( J^\bullet \) is a direct sum of copies of \( E(\sigma) \) for various \( \emptyset \neq \sigma \in \Sigma \). Since \( E(\sigma)^\dagger \) is the constant sheaf on \( \bar{\sigma} \) which is homeomorphic to a closed disc, we have \( H^i(X, E(\sigma)^\dagger) = H^i(\bar{\sigma} ; k) = 0 \) for all \( i \geq 1 \). Hence \( (J^\bullet)^\dagger \cong (I^\bullet)^\dagger \) gives a \( \Gamma(X, -) \)-acyclic resolution of \( M^\dagger \). It is easy to see that \( [J^\bullet]_0 \cong \Gamma(X, (J^\bullet)^\dagger) \). So the assertions follow from (2.2), since \( H^0(I^\bullet) \cong M \) and \( H^i(I^\bullet) = 0 \) for all \( i \geq 1 \).

**Remark 2.3.** (1) If \( M_\emptyset = 0 \), then we have \( H^i(X, M^\dagger) \cong H^{i+1}_0(M) \) for all \( i \).

(2) We regard a polynomial ring \( S := k[x_0, \ldots, x_n] \) as a graded ring with \( \text{deg}(x_i) = 1 \) for each \( i \). Let \( I \subset S \) be a graded ideal, and set \( A := S/I \). For a graded \( A \)-module \( M \), we have the algebraic quasi-coherent sheaf \( \tilde{M} \) on the projective scheme \( Y := \text{Proj} A \). It is well-known that \( H^i(Y, M) \cong [H^{i+1}_m(M)]_0 \) for all \( i \geq 1 \), and

\[
0 \to [H^0_m(M)]_0 \to M_0 \to H^0(Y, \tilde{M}) \to [H^1_m(M)]_0 \to 0 \quad \text{(exact)}.
\]
Here $H^i_m(M)$ stands for the local cohomology module with support in the irrelevant ideal $m := (x_0, \ldots, x_n)$, and $[H^i_m(M)]_0$ is its degree 0 component ($H^i_m(M)$ has a natural $\mathbb{Z}$-grading). See also Remark 4.4 (2) below.

(3) Assume that $\Sigma$ is a simplicial complex with $n$ vertices. The Stanley-Reisner ring $k[\Sigma]$ of $\Sigma$ is the quotient ring of the polynomial ring $k[x_1, \ldots, x_n]$ by the squarefree monomial ideal $I_\Sigma$ corresponding to $\Sigma$ (see [3, 12] for details). In [16], we defined squarefree $k[\Sigma]$-modules which are certain $\mathbb{N}^n$-graded $k[\Sigma]$-modules. For example, $k[\Sigma]$ itself is squarefree. The category $\text{Sq}(\Sigma)$ of squarefree $k[\Sigma]$-modules is equivalent to $\text{mod}_R$ of the present paper (see [18]). Let $\Phi : \text{mod}_R \to \text{Sq}(\Sigma)$ be the functor giving this equivalence. In [17], we defined a functor $(-)^+ : \text{Sq}(\Sigma) \to \text{Sh}_c(X)$. For example, $k[\Sigma]^+ \cong k[X]$. The functor $(-)^+$ is essentially same as the functor $(-)^{\dagger} : \text{mod}_R \to \text{Sh}_c(X)$ of the present paper. More precisely, $(-)^{\dagger} \cong (-)^+ \circ \Phi$. For $M \in \text{mod}_R$, we have $H^i_\emptyset(M) \cong H^i_m(\Phi(M))_0$. So the above theorem is a variation of [17] Theorem 3.3.

3. Dualizing complexes

Let $D^b(\text{mod}_R)$ be the bounded derived category of $\text{mod}_R$. For $M^\bullet \in D^b(\text{mod}_R)$ and $i \in \mathbb{Z}$, $M^\bullet[i]$ denotes the $i$th translation of $M^\bullet$, that is, $M^\bullet[i]$ is the complex with $M^\bullet[i]^j = M^{i+j}$. So, if $M \in \text{mod}_R$, $M[i]$ is the cochain complex $\cdots \to 0 \to M \to 0 \to \cdots$, where $M$ sits in the $(-i)$th position.

In this section, from Verdier’s dualizing complex $D^*_X \in D^b(\text{Sh}_c(X))$, we construct a cochain complex $\omega^\bullet$ of injective left $(R \otimes_k R)$-modules which gives a duality functor from $D^b(\text{mod}_R)$ to itself. Let $M$ be a left $(R \otimes_k R)$-module. When we regard $M$ as a left $R$-module via a ring homomorphism $R \ni x \mapsto x \otimes 1 \in R \otimes_k R$ (resp. $R \ni x \mapsto 1 \otimes x \in R \otimes_k R$), we denote it by $R M$ (resp. $M_{R^{op}}$).

For $i \leq 1$, the $i$th component $\omega^i$ of $\omega^\bullet$ has a $k$-basis

$$\{ e(\sigma)^\tau^\rho \mid \sigma, \tau, \rho \in \Sigma, \dim \sigma = -i, \sigma \geq \tau, \rho \},$$

and its module structure is defined by

$$(e_{\sigma', \tau'} \otimes 1) \cdot e(\sigma)^\tau^\rho = \begin{cases} e(\sigma)^{\tau'}_{\sigma'} & \text{if } \tau' = \rho \text{ and } \sigma' \leq \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(1 \otimes e_{\sigma', \tau'}) \cdot e(\sigma)^\tau^\rho = \begin{cases} e(\sigma)^{\sigma'}_{\rho'} & \text{if } \tau' = \tau \text{ and } \sigma' \leq \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $R(\omega^i) \cong (\omega^i)^{R^{op}} \cong \bigoplus_{\dim \sigma = -i} E(\sigma)^{\mu(\sigma)}$ as left $R$-modules, where $\mu(\sigma) := \#\{\tau \in \Sigma \mid \tau \leq \sigma\}$. Note that $R \otimes_k R$ is isomorphic to the incidence algebra of the poset $\Sigma \times \Sigma$. For each $\sigma \in \Sigma$ with $\dim \sigma = -i$, set $I(\sigma)$ to be the subspace $\langle e(\sigma)^\tau^\rho \mid \tau, \rho \leq \sigma \rangle$ of $\omega^i$. Then, as a left $R \otimes_k R$-module, $I(\sigma)$ is isomorphic to the injective module $E_{R \otimes_k R}(\sigma, \sigma)$, and $\omega^i \cong \bigoplus_{\dim \sigma = -i} I(\sigma)$. Thus $\omega^\bullet$ is of the form

$$0 \to \omega^{-d} \to \omega^{-d+1} \to \cdots \to \omega^1 \to 0,$$
Lemma 3.1. For $M \in \text{mod}_R$, we have $\text{Hom}_R(M, RI(\sigma)) \cong E(\sigma) \otimes_k (M_\sigma)^\vee$ as left $R$-modules. Here $(M_\sigma)^\vee$ is the dual vector space $\text{Hom}_k(M_\sigma, k)$ of $M_\sigma$.

Proof. First, we show that if $M_\sigma = 0$ then $\text{Hom}_R(M, RI(\sigma)) = 0$. Assume the contrary. If $0 \neq f \in \text{Hom}_R(M, RI(\sigma))$, there is some $x \in M_\tau$, $\tau < \sigma$, such that $f(x) \neq 0$. But we have $f(e_{\sigma, \tau}x) = e_{\sigma, \tau}f(x) \neq 0$. It contradicts the fact that $e_{\sigma, \tau}x \in M_\sigma = 0$.

For a general $M \in \text{mod}_R$, set $M_{\geq \sigma} = \bigoplus_{\tau \in \Sigma, \tau \geq \sigma} M_\tau$ to be a submodule of $M$. By the short exact sequence $0 \rightarrow M_{\geq \sigma} \rightarrow M \rightarrow M/M_{\geq \sigma} \rightarrow 0$, we have

$$0 \rightarrow \text{Hom}_R(M/M_{\geq \sigma}, RI(\sigma)) \rightarrow \text{Hom}_R(M, RI(\sigma)) \rightarrow \text{Hom}_R(M_{\geq \sigma}, RI(\sigma)) \rightarrow 0.$$ Since $(M/M_{\geq \sigma})_\sigma = 0$, we have $\text{Hom}_R(M_{\geq \sigma}, RI(\sigma)) = \text{Hom}_R(M_{\geq \sigma}, RI(\sigma))$. So we may assume that $M = M_{\geq \sigma}$. Let $\{f_1, \ldots, f_n\}$ be a $k$-basis of $(M_\sigma)^\vee$. Since $(RI(\sigma))_\tau = 0$ for $\tau > \sigma$, $\text{Hom}_R(M_{\geq \sigma}, RI(\sigma))$ has a $k$-basis $\{e(\sigma)^\tau \otimes f_i | \tau \leq \sigma, 1 \leq i \leq n\}$. By the module structure of $I(\sigma)_{R \otimes}$, we have the expected isomorphism.

Since each $R \omega^j$ is injective, $\textbf{D}(\sigma) := \text{Hom}_R^*(\sigma, R \omega^\sigma) \cong R \text{Hom}_R^*(-, R \omega^\sigma)$ gives a contravariant functor from $D^b(\text{mod}_R)$ to itself. In the sequel, we simply denote $\text{Hom}_R^*(-, R \omega^\sigma)$ by $\text{Hom}_R^*(-, \omega^\sigma)$, etc.

We can describe $\textbf{D}(M^\bullet)$ explicitly. Since $\omega^j \cong \bigoplus_{\dim \sigma = -i} I(\sigma)$, we have

$$\text{Hom}_R(M, \omega^j) \cong \bigoplus_{\dim \sigma = -i} \text{Hom}_R(M, I(\sigma)) \cong \bigoplus_{\dim \sigma = -i} E(\sigma) \otimes_k (M_\sigma)^\vee$$

for $M \in \text{mod}_R$ by Lemma 3.1. So we can easily check that $\textbf{D}(M)$ is of the form

$$\textbf{D}(M) : 0 \rightarrow \textbf{D}^{-d}(M) \rightarrow \textbf{D}^{-d+1}(M) \rightarrow \cdots \rightarrow \textbf{D}^1(M) \rightarrow 0,$$

$$\textbf{D}^i(M) = \bigoplus_{\dim \sigma = -i} E(\sigma) \otimes_k (M_\sigma)^\vee.$$

Here the differential sends $e(\sigma)^\tau \otimes f \in E(\sigma) \otimes_k (M_\sigma)^\vee$ to

$$\sum_{\tau \in \Sigma, \tau \geq \rho} \varepsilon(\sigma, \tau) \cdot e(\tau)^\rho \otimes f(e_{\sigma, \tau} - ) \in \bigoplus_{\dim \tau = \dim \sigma - 1} E(\tau) \otimes_k (M_\tau)^\vee.$$

For a bounded cochain complex $M^\bullet$ of objects in $\text{mod}_R$, we have

$$\textbf{D}^i(M^\bullet) = \bigoplus_{i-j=t} \textbf{D}^i(M^j) = \bigoplus_{-\dim \sigma-j=t} E(\sigma) \otimes_k (M_\sigma)^\vee.$$
Lemma 3.3. For each $D$ in Definition 1.1. Hence, treated in [14, §2.4], and also follows from the author’s previous paper [17] (and [18]). The general case can be reduced to the simplicial complex case using the barycentric subdivision.

Shepard’s description of $\mathcal{R}Hom((M^\bullet)^\dagger, D_X^\bullet)$ is the same thing as the above mentioned description of $D(M^\bullet)$ under the functor $(-)^\dagger$. □

Lemma 3.3. For each $\sigma \in \Sigma$, the natural map $E(\sigma) \to D \circ D(E(\sigma))$ is an isomorphism in $D^b(\text{mod}_R)$.

Proof. We may assume that $\sigma \neq \emptyset$. Let $\Sigma|_\sigma := \{ \tau \in \Sigma \mid \tau \leq \sigma \}$ be a subcomplex of $\Sigma$. It is easy to see that $D(E(\sigma)|_\sigma)$ is isomorphic to the chain complex $C_*(\Sigma|_\sigma, k)$ of $\Sigma|_\sigma$. Thus $H^i(D(E(\sigma)|_\sigma)) = \tilde{H}^i(\sigma; k)$ for all $i$, where $\tilde{H}^i(\sigma; k)$ stands for the reduced homology group of the closure $\sigma$ of $\sigma$. Hence $H^i(D(E(\sigma)|_\sigma)) = 0$ for all $i$.

By Theorem 3.2 and the Poincaré-Verdier duality, we have

$$D(E(\sigma))^\dagger \cong \mathcal{R}Hom(j_!k_\sigma^\bullet, D_X^\bullet) \cong j_!k_\sigma^\bullet \text{dim} \sigma.$$ 

Here $j : \sigma \to X$ is the embedding map.

Let $M$ be a simple $R$-module with $M = M_\sigma \cong k$. Combining the above observations, we have $D(E(\sigma)) \cong M[\text{dim} \sigma]$. So $D \circ D(E(\sigma)) \cong D(M[\text{dim} \sigma]) \cong E(\sigma)$, and the natural map $E(\sigma) \to D \circ D(E(\sigma))$ is an isomorphism. □

Theorem 3.4. (1) $\omega^\bullet \in D^b(\text{mod}_{R_{G_{\text{fin}}}})$ is a dualizing complex in the sense of [19] Definition 1.1. Hence $D(-)$ is a duality functor from $D^b(\text{mod}_R)$ to itself.

(2) The dualizing complex $\omega^\bullet$ satisfies the Auslander condition in the sense of [19] Definition 2.1. That is, if we set

$$j_\omega(M) := \inf \{ i \mid \text{Ext}^i_R(M, \omega^\bullet) \neq 0 \} \in \mathbb{Z} \cup \{ \infty \},$$

then, for all $i \in \mathbb{Z}$ and all $M \in \text{mod}_R$, any submodule $N$ of $\text{Ext}^i_R(M, \omega^\bullet)$ satisfies $j_\omega(N) \geq i$.

Proof. (1) The conditions (i) and (ii) of [19] Definition 1.1 obviously hold in our case. So it remains to prove the condition (iii). To see this, it suffices to show that the natural morphism $R \to D \circ D(R)$ is an isomorphism. But it follows from “Lemma on Way-out Functors” ([5] Proposition 7.1) and Lemma 3.3.
(2) We may assume that $M \neq 0$. By the description of $D(M)$, we have
\[ j_\omega(M) = -\max\{ \dim \sigma \mid \sigma \in \Sigma, M_\sigma \neq 0 \} \]
and $\Ext^i_R(M, \omega^*)_\sigma = 0$ for $\sigma \in \Sigma$ with $\dim \sigma > -i$. Hence, any submodule $N \subset \Ext^i_R(M, \omega^*)$ satisfies $j_\omega(N) \geq i$.

**Corollary 3.5.** We have $\Ext^i_R(M^*, \omega^*)_0 \cong H^{i+1}_0(M^*)^\vee$ for all $i \in \mathbb{Z}$ and all $M^* \in D^b(\text{mod}_R)$.

**Proof.** Since $D \circ D(M^*)$ is an injective resolution of $M^*$, we have $R\Gamma_0(M^*) = \Gamma_0(D \circ D(M^*))$. By the structure of $D(-)$, we have $\Gamma_0(D \circ D(M^*)) = (D(M^*)_0)^\vee[-1]$. So we are done. □

4. Categorical Remarks

For $M, N \in \text{mod}_R$ and $\sigma \in \Sigma$, set $\overline{\Hom}_R(M, N)_\sigma := \Hom_R(M_{\geq \sigma}, N)$. We make $\Hom_R(M, N)_\Sigma := \bigoplus_{\sigma \in \Sigma} \overline{\Hom}_R(M, N)_\sigma$ a left $R$-module as follows: For $f \in \Hom_R(M, N)_\sigma$ and a cell $\tau$ with $\tau \geq \sigma$, set $e_{\tau, \sigma}f$ to be the restriction of $f$ into the submodule $M_{\geq \tau}$ of $M_{\geq \sigma}$.

**Lemma 4.1.** For $M \in \text{mod}_R$, we have $\overline{\Hom}_R(M, E(\sigma)) \cong E(\sigma) \otimes_k (M_\sigma)^\vee$.

**Proof.** Similar to Lemma 3.1. □

If a complex $M^*$ in $\text{mod}_R$ is exact, then so is $\overline{\Hom}_R(M^*, E(\sigma))$ by Lemma 4.1. By the usual argument on double complexes, if $M^*$ is bounded and exact, and $I^*$ is bounded and each $I^*$ is injective, then $\overline{\Hom}_R(M^*, I^*)$ is exact.

Note that $\Sigma$ is a *meet-semilattice* (see [13, §3.3]) as a poset if and only if, for any two cells $\sigma, \tau \in \Sigma$ with $\sigma \cap \tau \neq \emptyset$, there is a cell $\rho \in \Sigma$ with $\sigma \cap \tau = \rho$. If $\Sigma$ is a simplicial complex, or more generally, a polyhedral complex, then it is a meet-semilattice. If $\Sigma$ is a meet-semilattice, for two cells $\sigma, \tau \in \Sigma$, either there is no upper bound of $\sigma$ and $\tau$ (i.e., no cell $\rho \in \Sigma$ satisfies $\rho \geq \sigma$ and $\rho \geq \tau$), or there is the least element $\sigma \lor \tau$ in $\{ \rho \in \Sigma \mid \rho \geq \sigma, \rho \geq \tau \}$ (c.f. [13 Proposition 3.3.1]).

Assume that $\Sigma$ is a meet-semilattice. Consider $\overline{\Hom}_R(Re_\sigma, N)_\tau$ for $N \in \text{mod}_R$ and $\tau \in \Sigma$. If $\sigma \lor \tau$ exists, then we have $\overline{\Hom}_R(Re_\sigma, N)_\tau = N_{\sigma \lor \tau}$. Otherwise, there is no upper bound of $\sigma$ and $\tau$, and $\overline{\Hom}_R(Re_\sigma, N)_\tau = 0$. Hence the complex $\overline{\Hom}_R(Re_\sigma, N^*)$ is exact for an exact complex $N^*$. Hence if $N^*$ is bounded and exact, and $P^*$ is bounded and each $P^*$ is projective, then $\overline{\Hom}_R(P^*, N^*)$ is exact.

By the above remarks, we have the following lemma (see [7, I.1.10] for the derived functor of a bifunctor).

**Lemma 4.2.** For $M^*, N^* \in D^b(\text{mod}_R)$, we have the following.

1. If $I^*$ is an injective resolution of $N^*$, then
   \[ R\overline{\Hom}_R(M^*, N^*) \cong \overline{\Hom}_R(M^*, I^*) \]

2. If $\Sigma$ is a meet-semilattice as a poset (e.g., $\Sigma$ is a simplicial complex), then
   \[ R\overline{\Hom}_R(M^*, N^*) \cong \overline{\Hom}_R(P^*, N^*) \]
for a projective resolution $P^*$ of $M^*$. 

Example 4.3. The additional assumption in Lemma 4.2 (2) is really necessary. That is, $\text{RHom}_R(M^*, N^*) \neq \text{Hom}_R(P^*, N^*)$ in general.

For example, let $X$ be a closed 2 dimensional disc, and $\Sigma$ a regular cell decomposition of $X$ consisting of one 2-cell (say, $\sigma$), two 1-cells (say, $\tau_1, \tau_2$), and two 0-cells (say, $\rho_1, \rho_2$). Since $\rho_1 \neq \rho_2$ does not exist, $\Sigma$ is not a meet-semilattice.

Let $N$ be a left $R$-module with $N = N_\sigma = k$. Then the injective resolution of $N$ is of the form

$$I^\bullet : 0 \to E(\sigma) \to E(\tau_1) \oplus E(\tau_2) \to E(\rho_1) \oplus E(\rho_2) \to E(\emptyset) \to 0.$$  

We have $\text{Hom}_R(Re_{\rho_1}, E(\sigma))_{\rho_2} = \text{Hom}_R(Re_{\rho_1}, E(\tau_1))_{\rho_2} = \text{Hom}_R(Re_{\rho_1}, E(\tau_2))_{\rho_2} = k$ and $\text{Hom}_R(Re_{\rho_1}, E(\rho_1))_{\rho_2} = \text{Hom}_R(Re_{\rho_1}, E(\rho_2))_{\rho_2} = 0$. Thus $\text{Ext}_R^1(Re_{\rho_1}, N)_{\rho_2} = H^1(\text{Hom}(Re_{\rho_1}, I^\bullet))_{\rho_2} \neq 0$, while $Re_{\rho_1}$ is a projective module.

Proposition 4.4. If $M^* \in D^b(Sh_c(X))$, then $D(M^*) \cong \text{RHom}_R(M^*, D(Re_\emptyset))$.

Proof. Since $D(Re_\emptyset)$ is of the form $0 \to D^{-d} \to D^{-d+1} \to \cdots \to D^1 \to 0$ with $D^i = \bigoplus_{\dim \sigma = i} E(\sigma)$, the assertion follows from Lemmas 4.1 and 4.2. \qed

Since $(Re_\emptyset)^{\dagger} \cong \overline{k_X}$, we have $D_X^\bullet \cong D(\overline{k_X}) \cong D(Re_\emptyset) \dagger$ by Proposition 4.4.

If $F, G \in Sh_c(X)$, then it is easy to see that $\text{Hom}(F, G) \in Sh_c(X)$. For $M, N \in \text{mod}_R$ and $\emptyset \neq \sigma \in \Sigma$, we have $\text{Hom}(M^\dagger, N^\dagger)(U_\sigma) = \text{Hom}(M^\dagger |_{U_\sigma}, N^\dagger |_{U_\sigma}) = \text{Hom}_R(M_{>\sigma}, N_{>\sigma}) = \text{Hom}_R(M_{>\sigma}, N) = \text{Hom}_R(M, N)$. Hence

$$\text{Hom}_R(M, N) \dagger \cong \text{Hom}(M^\dagger, N^\dagger).$$

For $F^\bullet, G^\bullet \in D^b(Sh_c(X))$, then it is known that $\text{RHom}(F^\bullet, G^\bullet) \in D^b(Sh_c(X))$ (see [11 Proposition 8.4.10]). Thus we can use an injective resolution of $G^\bullet$ in $D^b(Sh_c(X))$ to compute $\text{RHom}(F^\bullet, G^\bullet)$. If $I^\bullet$ is an injective resolution of $N^\bullet \in D^b(\text{mod}_R)$, then $(I^\bullet)^{\dagger}$ is an injective resolution of $(N^\bullet)^{\dagger}$ in $D^b(Sh_c(X))$. Hence we have the following.

Proposition 4.5 ([11 Theorem 5.2.5]). If $M^\bullet, N^\bullet \in D^b(\text{mod}_R)$, then

$$\text{RHom}_R(M^\bullet, N^\bullet)^{\dagger} \cong \text{RHom}((M^\bullet)^{\dagger}, (N^\bullet)^{\dagger}).$$

By Lemma 4.1 (2), if $\Sigma$ is a meeting-semilattice, then $\text{RHom}(F^\bullet, G^\bullet)$ for $F^\bullet, G^\bullet \in D^b(Sh_c(X))$ can be computed using a projective resolution of $F^\bullet$ in $D^b(Sh_c(X))$.

Remark 4.6. (1) Let $J$ be the left ideal of $R$ generated by $\{ e_{\sigma, \emptyset} \mid \emptyset \neq \emptyset \}$. Note that $J^\dagger \cong \overline{k_X}$. Then we have that $\text{Hom}_R(J, M)^{\dagger} \cong M^\dagger$ and $\text{Hom}_R(J, M)_{\emptyset} \cong \Gamma(X, M^\dagger)$. Moreover, we have $\text{Ext}_R^i(J, M) = \text{Ext}_R^i(J, M)_{\emptyset} \cong H^i(X, M^\dagger)$ for all $i \geq 1$ by an argument similar to Theorem 2.2.

(2) Let $\text{mod}_0$ be the full subcategory of $\text{mod}_R$ consisting of modules $M$ with $M_\sigma = 0$ for all $\emptyset \neq \emptyset$. Then $\text{mod}_0$ is a dense subcategory of $\text{mod}_R$. That is, for a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\text{mod}_R$, $M$ is in $\text{mod}_0$ if and only if $M'$ and $M''$ are in $\text{mod}_0$. So we have the quotient category $\text{mod}_R / \text{mod}_0$ by [11 Theorem 4.3.3]. Let $\pi : \text{mod}_R \to \text{mod}_R / \text{mod}_0$ be the canonical functor. It is easy to see that $\pi(M) \cong \pi(M')$ if and only if $M_\emptyset \cong M'_\emptyset$. Moreover, we have $\text{Sh}_c(X) \cong \text{mod}_R / \text{mod}_0$. 

Let the notation be as in (1) of this remark. Then \( \text{Hom}_R(J, -) \) gives a functor \( \eta : \text{mod}_R / \text{mod}_\emptyset \to \text{mod}_R \) with \( \pi \circ \eta = \text{Id} \). Moreover, \( \eta \) is a section functor (c.f. [10] §4.4) and \( \text{mod}_\emptyset \) is a localizing subcategory of \( \text{mod}_R \).

Let \( A = \bigoplus_{i \geq 0} A_i \) be a commutative noetherian homogeneous \( k \)-algebra as in Remark 2.3 (2) and \( \text{Gr}_A \) the category of graded \( A \)-modules. We say \( M \in \text{Gr}_A \) is a torsion module, if for all \( x \in M \) there is some \( i \in \mathbb{N} \) with \( A_{i+1}x = 0 \). Let \( \text{Tor}_A \) be the full subcategory of \( \text{Gr}_A \) consisting of torsion modules. Clearly, \( \text{Tor}_A \) is dense in \( \text{Gr}_A \).

It is well-known that the category \( \text{Qco}(Y) \) of quasi-coherent sheaves on the projective scheme \( Y : = \text{Proj} A \) is equivalent to the quotient category \( \text{Gr}_A / \text{Tor}_A \), and we have the section functor \( \text{Qco}(Y) \to \text{Gr}_A \) given by \( F \mapsto \bigoplus_{j \in \mathbb{Z}} H^0(Y, F(i)). \) So \( \text{Tor}_A \) is a localizing subcategory of \( \text{Gr}_A \). In this sense, our \( \text{Sh}_c(X) \cong \text{mod}_R / \text{mod}_\emptyset \) is a small imitation of \( \text{Qco}(Y) \cong \text{Gr}_A / \text{Tor}_A \).

5. Cohomologies of sheaves on open subsets

Let \( \Psi \subset \Sigma \) be an order filter of the poset \( \Sigma \). That is, \( \sigma \in \Psi, \tau \in \Sigma, \) and \( \tau \geq \sigma \) imply \( \tau \in \Psi \). Then \( U_\Psi := \bigcup_{\sigma \in \Psi} \sigma \) is an open subset of \( X \). If \( M \in \text{mod}_R, M_\Psi := \bigoplus_{\sigma \in \Psi} M_\sigma \) is a submodule of \( M \). It is easy to see that \( (M_\Psi) \dagger \cong \bigoplus_{\sigma \in \Psi} M_\sigma \). where \( j : U_\Psi \to X \) is the embedding map. If \( \Psi = \{ \tau \mid \tau \geq \sigma \} \) for some \( \sigma \in \Sigma \), then \( U_\Psi \) and \( M_\Psi \) are denoted by \( U_\sigma \) and \( M_{\geq \sigma} \) respectively.

Lemma 5.1. Let \( \Psi \subset \Sigma \) be an order filter with \( \Psi \neq \emptyset \). Then we have the following isomorphisms for all \( i \in \mathbb{Z} \) and \( M \in \text{mod}_R \).

1. \( H^{i+1}_\emptyset (M_\Psi) \cong H^i(U_\Psi, M^\dagger | U_\Psi) \) for all \( i \).
2. \( \text{Ext}^i_R(M, \omega^*)_\sigma \cong H^{-i-1}(M_{\geq \sigma})^\vee \cong H^{-i}(U_\sigma, M^\dagger | U_\sigma)^\vee \) for all \( \emptyset \neq \sigma \in \Sigma \).

Proof. (1) We have \( H^{i+1}_\emptyset (M_\Psi) \cong H^i(U_\Psi, M^\dagger | U_\Psi) \cong H^i(X, M^\dagger (M_\Psi)^\dagger) \cong H^i_c(U_\Psi, M^\dagger | U_\Psi) \).

Here, by Remark 2.3 (1), the first isomorphism holds even if \( i = 0 \).

(2) By the description of \( D(M) \), we have \( D(M)_\sigma \cong D(M_{\geq \sigma}) \). Hence we have \( \text{Ext}^i_R(M, \omega^*)_\sigma \cong \text{Ext}^i_R(M_{\geq \sigma}, \omega^*)_\emptyset \cong H^{-i-1}_\emptyset (M_{\geq \sigma})^\vee \cong H^{-i}_c(U_\sigma, M^\dagger | U_\sigma)^\vee \).

Here the second isomorphism follows from Corollary 3.3.

Proposition 5.2. For any \( \sigma \in \Sigma \), \( D(Re_\sigma)^\dagger \cong Rj_* D^*_U \) where \( j : U_\sigma \to X \) is the embedding map. In particular, \( D(Re_\emptyset)^\dagger \cong D^*_X \).

Proof. Set \( U := U_\sigma \). Since \( (Re_\sigma)^\dagger \cong j_* k_U \), we have

\[
D(Re_\sigma)^\dagger \equiv R\text{Hom}(j k_U^\dagger, D^*_X) \quad \text{(by Theorem 3.2)}
\equiv Rj_* R\text{Hom}(k_U^\dagger, j^* D^*_X) \quad \text{(by [3] VII. Theorem 5.2)}
\equiv Rj_* R\text{Hom}(k_U^\dagger, D_U^\dagger) \equiv Rj_* D_U^\dagger.
\]

Motivated by Lemma 5.1, we will give a formula on the ordinal (not compact support) cohomology \( H^i(U_\Psi, M^\dagger | U_\Psi) \).
Theorem 5.3. Let $\Psi \subset \Sigma$ be an order filter with $\Psi \not\supset \emptyset$. We have
\[ H^i(U_\Psi, M^j|_{U_\Psi}) \cong [\Ext^i_R(D(M)_\Psi, \omega^\bullet)]_0 \]
for all $i \in \mathbb{N}$ and $M \in \mod_R$.

Proof. For the simplicity, set $U := U_\Psi$. Let $\mathcal{F}^* \in D^b(\text{Sh}(U))$. Taking a complex in the isomorphic class of $\mathcal{F}^*$, we may assume that each component $\mathcal{F}^i$ is a direct sum of sheaves of the form $h_\cdot^\bullet_U$, where $V$ is an open subset of $U$ with the embedding map $h : V \to U$ (see [6 II. Proposition 2.4]). Since each component $\mathcal{D}^i_U$ of $\mathcal{D}^*_U$ is an injective sheaf, $h^*\mathcal{D}^i_U$ is also injective by [6 II. Corollary 6.10], and we have
\[ \text{Hom}(h_\cdot^\bullet_U, \mathcal{D}^i_U) \cong \check{R}h_*\text{Hom}(h_\cdot^\bullet_U, h^*\mathcal{D}^i_U) \cong \check{R}h_*(h^*\mathcal{D}^i_U) \cong h_*h^*\mathcal{D}^i_U \]
by [6 VII. Theorem 5.2]. Since the sheaf $h_*h^*\mathcal{D}^i_U$ is flabby, $\text{Hom}^\bullet(\mathcal{F}^*, \mathcal{D}^*_U)$ is a complex of flabby sheaves. Hence we have
\[ \Ext^i_{\text{Sh}(U)}(\mathcal{F}^*, \mathcal{D}^*_U) \cong H^i(\Gamma(U, \check{R}\text{Hom}^\bullet(\mathcal{F}^*, \mathcal{D}^*_U))) \]
\[ \cong \check{R}\Gamma(U, \check{R}\text{Hom}(\mathcal{F}^*, \mathcal{D}^*_U))) \]
Since $\check{R}\text{Hom}(\check{R}\text{Hom}(M^i|_U, \mathcal{D}^*_U), \mathcal{D}^*_U) \cong M^i|_U$ in $D^b(\text{Sh}(U))$, we have
\[ H^i(U, M^i|_U) \cong \check{R}\Gamma(U, \check{R}\text{Hom}(\mathcal{F}^*, \mathcal{D}^*_U))) \]
\[ \cong \Ext^i_{\text{Sh}(U)}(\mathcal{F}^*, \mathcal{D}^*_U)) \]
\[ \cong \check{R}^{-i}\Gamma_c(U, \check{R}\text{Hom}(M^i|_U, \mathcal{D}^*_U))/U (\text{by } [6 V. \text{Theorem } 2.1]) \]
\[ \cong \check{R}^{-i}\Gamma_c(U, \check{R}\text{Hom}(M^i, \mathcal{D}^*_U)|_U) \]
\[ \cong \check{R}^{-i}\Gamma_c(U, \check{R}\text{Hom}(M^i|_U, \mathcal{D}^*_U)) \]
\[ \cong (\Ext^i_{\check{R}}(M(\mathcal{F}_\Psi), \omega^\bullet)\check{\cdot}_0) (\text{by Corollary } 3.5). \]
\[ \square \]

Example 5.4. Assume that $X$ is a $d$-dimensional manifold (in this paper, the word “manifold” always means a manifold with or without boundary, as in [6]) and $\Psi \subset \Sigma$ is an order filter with $\Psi \not\supset \emptyset$. We denote the orientation sheaf of $X$ over $k$ (c.f. [6 V.3]) by $\mathcal{O}_X$. Thus we have $\mathcal{O}_X[d] \cong \mathcal{D}^*_X$ in $D^b(\text{Sh}(X))$. Let $U := U_\Psi$ be an open subset with the embedding map $j : U \to X$. We have $(D(Re_\emptyset)\Psi)^i \cong j^iD(Re_\emptyset)^i \cong j^i\mathcal{D}^*_X \cong j^i\mathcal{D}^*_U \cong (j^i\mathcal{O}_U)[d]$. Thus $[\Ext^i_{\check{R}}(D(Re_\emptyset)\Psi, \omega^\bullet)]_0 \cong H_{c}^{d-i}(D(Re_\emptyset)_\Psi)^i \cong H^{d-i}(U, \mathcal{O}_U)^i$ by the Poincaré duality. So the equality in Theorem 5.3 actually holds.

For a finite poset $P$, the order complex $\Delta(P)$ is the set of chains of $P$. Recall that a subset $C$ of $P$ is a chain if any two elements of $C$ are comparable. Obviously, $\Delta(P)$ is an (abstract) simplicial complex. The geometric realization of the order complex $\Delta(\Sigma') := \Sigma \setminus \emptyset$ is homeomorphic to the underlying space $X$ of $\Sigma$.

We say a finite regular cell complex $\Sigma$ is Cohen-Macaulay (resp. Buchsbaum) if $\Delta(\Sigma')$ is Cohen-Macaulay (resp. Buchsbaum) over $k$ in the sense of [12 II.§§3-4] (resp. [12 II.§8]). (If $\Sigma$ itself is a simplicial complex, we can use $\Sigma$ directly instead of $\Delta(\Sigma')$.) These are topological properties of the underlying space $X$. In fact,
of the underlying space $X$. For example, if $X$ is homeomorphic to a $d$-dimensional sphere, then $\Sigma$ is Cohen-Macaulay.

Remark 5.6. By [17, Proposition 4.10], Proposition 5.5 (1) states that if $\Sigma$ is a Cohen-Macaulay simplicial complex, the relative simplicial complex $(\Sigma, \text{del}_\Sigma(\sigma))$ is Cohen-Macaulay in the sense of [12, III.5.7] for all $\sigma \in \Sigma$. Here $\text{del}_\Sigma(\sigma) := \{ \tau \in \Sigma \mid \tau \not\supseteq \sigma \}$ is a subcomplex of $\Sigma$.

Example 5.7. (1) We say a finite regular cell complex $\Sigma$ of dimension $d$ is Gorenstein* over $k$ (see [12, p.67]), if the order complex $\Delta := \Delta(\Sigma')$ of $\Sigma' := \Sigma \setminus \emptyset$ is Cohen-Macaulay over $k$ (i.e., $\check{H}_i(\text{lk}_\Delta; k) = 0$ for all $i \in \Sigma$ and all $i \neq d - \dim \sigma - 1$) and $\check{H}_{d-\dim \sigma - 1}(\text{lk}_\Delta; k) = k$ for all $\sigma \in \Delta$. (If $\Sigma$ itself is a simplicial complex, we can use $\Sigma$ directly instead of $\Delta$.) This is a topological property of the underlying space $X$. For example, if $X$ is homeomorphic to a $d$-dimensional sphere, then $\Sigma$ is Gorenstein*.

It is easy to see that $\text{D}(R \Theta_0) \cong (R \Theta_0)[d]$ in $D^b(\text{mod}_R)$ if and only if $\Sigma$ is Gorenstein* if $\Sigma$ is a Gorenstein* simplicial complex, then $\omega^* \cong \Omega[d]$ for some $\Omega \in \text{mod}_{R \Theta_0 R}$ by Proposition 5.5. Moreover, we can describe $\Omega$ explicitly. In fact, $\Omega$ has a $k$-basis $\{ e_\sigma | \sigma \in \Sigma, \sigma \cup \tau \in \Sigma \}$ and its module structure is defined by

$$(e_{\sigma'}, \tau \otimes 1) \cdot e_\rho = \begin{cases} e_{\tau'} & \text{if } \tau' = \rho \text{ and } \sigma' \cup \tau \in \Sigma, \\ 0 & \text{otherwise}, \end{cases}$$
and
\[
(1 \otimes e_{\sigma'}, \tau') \cdot e_{\tau}^\sigma = \begin{cases} e_{\sigma'}^\tau & \text{if } \tau' = \tau \text{ and } \sigma' \cup \rho \in \Sigma, \\ 0 & \text{otherwise.} \end{cases}
\]

To check this, note that the "\(\tau\)-\(\rho\) component" \((\omega^\bullet)^\rho_\tau\) of \(\omega^\bullet = \langle e(\sigma)^\rho_\tau | \sigma \geq \tau, \rho \rangle\) is isomorphic to \(\check{C}_{-n-\bullet}(\text{lkk}(\tau \cup \rho))\) as a complex of \(k\)-vector spaces, where \(\check{C}_{\bullet}(\text{lkk}(\tau \cup \rho))\) is the augmented chain complex of \(\text{lkk}(\tau \cup \rho)\) and \(n = \text{dim}(\tau \cup \rho) + 1\). So the description follows from the Gorenstein\(^*\) property of \(\Sigma\). It is easy to see that \(\Omega := \text{mod}_R \text{lk}_R \text{lk}_{\hat{\Sigma}}[d]\), where \(j : \text{lk}_\sigma \rightarrow \text{lk}_\sigma\) is the embedding map of the closure \(\text{lk}_\sigma\) of \(U_\sigma\).

(2) Let \(\Sigma\) be a finite simplicial complex of dimension \(d\), and \(V\) the set of its vertices. Assume that \(\Sigma\) is Gorenstein in the sense of [12, II.\S5]. Then there is a subset \(W \subset V\) and a Gorenstein\(^*\) simplicial complex \(\Delta \subset 2^V \setminus W\) such that \(\Sigma = 2^W \ast \Delta\), where "\(\ast\)" stands for simplicial join. (The Gorenstein property depends on the particular simplicial decomposition of \(X\).) Since a Gorenstein simplicial complex is Cohen-Macaulay, there is \(\Omega \in \text{mod}_R \text{lk}_R \text{lk}_{\hat{\Sigma}}\) such that \(\omega^\bullet \cong \Omega[d]\). By the argument similar to (1), \(\Omega\) has a \(k\)-basis \(\{ e^\sigma_\tau | \sigma \cup \tau \in \Sigma, \sigma \cup \tau \supset W \}\) and its left \(R \otimes_k R\)-module structure is given by the similar way to (1).

Assume that \(\Sigma\) is the \(d\)-simplex \(2^V\). Then \(\Sigma\) is Gorenstein and \(\Omega\) has a \(k\)-basis \(\{ e^\sigma_\tau | \sigma \cup \tau = V \}\). Moreover, we have a ring isomorphism given by \(\varphi : R \ni e_\sigma \tau \mapsto e^\sigma_\tau \in R^\text{op}\), where \(R^\text{op}\) is the opposite ring of \(R\), and \(\sigma^c := V \setminus \sigma\). Thus \(R\) has a left \((R \otimes_k R)\)-module structure given by \((x \otimes y) \cdot r = x \cdot r \cdot \varphi(y)\). Then a map given by \(R \ni e_\sigma \tau \mapsto e^\sigma_\tau \in \Omega\) is an isomorphism of \((R \otimes R)\)-modules. So \(R\) is an Auslander regular ring in this case. See [13, Remark 3.3].

(3) Assume that \(\Sigma\) is a simplicial complex and \(X\) is a \(d\)-dimensional manifold which is orientable (i.e., \(or_X \cong k_X\)) and connected. Then \(H^i(\omega^\bullet) = 0\) for all \(i \neq -d\). It is easy to see that \(\Omega := H^{-d}(\omega^\bullet) \in \text{mod}_R \text{lk}_{\hat{\Sigma}}\) has a \(k\)-basis \(\{ e^\sigma_\tau | \sigma \cup \tau \in \Sigma \}\) and the module structure is give by the same way as (1).

6. THE MÖBIUS FUNCTION OF THE POSET \(\hat{\Sigma}\)

The Möbius function of a finite poset \(P\) is a function \(\mu : \{ (x, y) | x \leq y \text{ in } P \} \rightarrow \mathbb{Z}\) defined by the following way

\[
\mu(x, x) = 1 \quad \text{for all } x \in P \quad \text{and} \quad \mu(x, y) = - \sum_{x \leq z < y} \mu(x, z) \quad \text{for all } x < y \text{ in } P.
\]

See [13, Chapter 3] for a general theory of this function.

For a finite regular cell complex \(\Sigma\), let \(\hat{\Sigma}\) be the poset obtained from \(\Sigma\) adjoining the greatest element \(\hat{1}\) (even if \(\Sigma\) already possess the greatest element, we add the new one). Then the Möbius function \(\mu\) of \(\hat{\Sigma}\) has a topological meaning. For example, we have \(\mu(\emptyset, \hat{1}) = \chi(X)\), where \(\chi(X)\) is the reduced Euler characteristic \(\sum (-1)^i \dim_k H^i(X; k)\) of \(X\). When the underlying space \(X\) is a manifold, the Möbius function of \(\hat{\Sigma}\) is completely determined in [13, Proposition 3.8.9]. Here we study the general case.
For $\sigma \in \Sigma$ with $\dim \sigma > 0$, \( \{ \sigma' \in \Sigma \mid \sigma' < \sigma \} \) is a regular cell decomposition of $\bar{\sigma} - \sigma$ which is homeomorphic to a sphere of dimension $\dim \sigma - 1$. Hence we have $\mu(\tau, \sigma) = (-1)^{l(\tau, \sigma)}$ for $\tau \in \Sigma$ with $\tau \leq \sigma$ by [13, Proposition 3.8.9], where $l(\tau, \sigma) := \dim \sigma - \dim \tau$. So it remains to determine $\mu(\sigma, \hat{1})$ for $\sigma \neq \emptyset$.

**Proposition 6.1.** For a cell $\emptyset \neq \sigma \in \Sigma$ with $j := \dim \sigma$, we have

$$
\mu(\sigma, \hat{1}) = \sum_{i \geq j} (-1)^{i-j+1} \dim_K H^i_c(U_{\sigma}; k).
$$

Here $H^i_c(U_{\sigma}; k)$ is the cohomology with compact support of the open set $U_{\sigma} = \bigcup_{\rho \geq \sigma} \rho$ of $X$.

**Proof.** The assertion follows from the next computation.

\[
\mu(\sigma, \hat{1}) = - \sum_{\rho \in \Sigma, \rho \geq \sigma} \mu(\sigma, \rho) \\
= \sum_{i \geq j} (-1)^{i-j+1} \# \{ \rho \in \Sigma \mid \rho \geq \sigma, \dim \rho = i \} \\
= \sum_{i \geq j} (-1)^{i-j+1} \dim_K \mathcal{H}^{-i}(D_X^\bullet)(U_{\sigma}) \\
= \sum_{i \geq j} (-1)^{i-j+1} \dim_K H^i_c(U_{\sigma}; k).
\]

The second equality follows the fact that $\mu(\sigma, \rho) = (-1)^{l(\sigma, \rho)}$. The third equality follows from $D_X^\bullet \cong D(RE_\emptyset)^\dagger$ and the description of $D(RE_\emptyset)$. Recall also that $M^\dagger(U_{\sigma}) \cong M_\sigma$. And the last equality follows from the Verdier duality. \( \square \)

Assume that $X$ is a manifold of dimension $d$. If $\sigma \neq \emptyset$ is contained in the boundary of $X$, then $U_\sigma$ is homeomorphic to $(\mathbb{R}^{d-1} \times \mathbb{R}_{\geq 0})$ and $H^i_c(U_{\sigma}; k) = 0$ for all $i$. Thus $\mu(\sigma, \hat{1}) = 0$ in this case. If $\sigma$ is not contained in the boundary of $X$, then $U_\sigma$ is homeomorphic to $\mathbb{R}^d$ and $H^i_c(U_{\sigma}; k) = 0$ for all $i \neq d$ and $H^d_c(U_{\sigma}; k) = k$. Hence we have $\mu(\sigma, \hat{1}) = (-1)^{d - \dim \sigma + 1}$. So Proposition 6.1 recovers [13, Proposition 3.8.9].

### 7. Relation to Koszul duality

Let $A = \bigoplus_{i \geq 0} A_i$ be an $\mathbb{N}$-graded associative $k$-algebra such that $\dim_k A_i < \infty$ for all $i$ and $A_0 \cong k^n$ for some $n \in \mathbb{N}$ as an algebra. Then $\mathfrak{r} := \bigoplus_{i \geq 0} A_i$ is the graded Jacobson radical. We say $A$ is **Koszul**, if a left $A$-module $A/\mathfrak{r}$ admits a graded projective resolution

\[
\cdots \rightarrow P^{-2} \rightarrow P^{-1} \rightarrow P^0 \rightarrow A/\mathfrak{r} \rightarrow 0
\]

such that $P^{-i}$ is generated by its degree $i$ component as an $A$-module (i.e., $P^{-i} = AP^{-i}$). If $A$ is Koszul, it is a quadratic ring, and its *quadratic dual ring* $A^!$ (see [11, Definition 2.8.1]) is Koszul again, and isomorphic to the opposite ring of the Yoneda algebra $\text{Ext}_A^\bullet(\mathfrak{r}, A/\mathfrak{r})$.

Note that the incidence algebra $R$ of $\Sigma$ is a graded ring with $\deg(\epsilon_{\sigma, \sigma'}) = \dim \sigma - \dim \sigma'$. So we can discuss the Koszul property of $R$. 

Proposition 7.1 (c.f. [18 Lemma 4.5]). The incidence algebra $R$ of a finite regular cell complex $\Sigma$ is always Koszul. And the quadratic dual ring $R^1$ is isomorphic to $R^{\text{op}}$.

When $\Sigma$ is a simplicial complex, the above result was proved by Polishchuk [8] in much wider context (but, $\emptyset \not\in \Sigma$ in his convention). More precisely, he put the new partial order on the set $\Sigma \setminus \emptyset$ associated with a perversity function $p$, and construct two rings from this new poset. Then he proved that these two rings are Koszul and quadratic dual rings of each other. Our $R$ and $R^{\text{op}}$ correspond to the case when $p$ is a bottom (or top) perversity. In the middle perversity case, $\Sigma$ has to be a simplicial complex to make their rings Koszul.

Proof. By [9, 15], $R$ is Koszul if and only if the order complex $\Delta(I)$ is Cohen-Macaulay over $k$ for any open interval $I$ of $\Sigma$. Set $\Sigma' := \Sigma \setminus \emptyset$. Note that $\Delta(I) = \text{lk}_{\Delta(\Sigma')} F$ for some $F \in \Delta(\Sigma')$ containing a maximal cell $\sigma \in \Sigma$. Set $\Delta := \text{st}_{\Delta(\Sigma')} \sigma$. Then $\Delta(I) = \text{lk}_\Delta F$. Since the underlying space of $\Delta$ is the closed disc $\bar{\sigma}$, $\Delta$ is Cohen-Macaulay. Hence $\text{lk}_\Delta F$ is also. So $R$ is Koszul.

Let $I := T_{R_0} R_1 = R_0 \oplus R_1 \oplus (R_1 \otimes_{R_0} R_1) \oplus \cdots = \bigoplus_{i \geq 0} R_1^i$ be the tensor ring of $R_1 = \{ e_{\sigma, \tau} \mid \sigma, \tau \in \Sigma, \sigma > \tau, \dim \sigma = \dim \tau + 1 \}$ over $R_0$. Then $R \cong T/I$, where

$I = (e_{\sigma, \rho_1} \otimes e_{\rho_2, \tau} - e_{\sigma, \rho_2} \otimes e_{\rho_1, \tau} \mid \sigma, \tau, \rho_1, \rho_2 \in \Sigma, \sigma > \rho_i > \tau, \dim \sigma = \dim \tau + 2 )$ is a two sided ideal. Let $R_1^* := \text{Hom}_{R_0}(R_1, R_0)$ be the dual of the left $R_0$-module $R_1$. Then $R_1^*$ has a right $R_0$-module structure such that $(fa)(v) = (f(v))a$, and a left $R_0$-module structure such that $(af)(v) = f(va)$, where $a \in R_0$, $f \in R_1^*$, $v \in R_1$. As a left (or right) $R_0$-module, $R_1^*$ is generated by $\{ e_{\tau, \sigma}^* \mid \sigma > \tau, \dim \sigma = \dim \tau + 1 \}$, where $e_{\tau, \sigma}^*(e_{\tau', \sigma'}) = \delta_{\tau, \tau'} \cdot e_{\sigma, \sigma'}$.

Let $T^* = T_{R_0} R_1^*$ be the tensor ring of $R_1^*$. Note that $e_{\tau, \sigma}^* \otimes e_{\tau', \sigma'}^* \in R_1^* \otimes_{R_0} R_1^*$ is non-zero if and only if $\sigma = \tau'$. We have that $(R_1^* \otimes_{R_0} R_1^*)^* = \text{Hom}_{R_0}(R_1^* \otimes_{R_0} R_1^*, R_0)$ via $(f \otimes g)(v \otimes w) = g(w(f(v)))$, where $f, g \in R_1^*$ and $v, w \in R_1$. In particular, $(e_{\tau, \sigma}^* \otimes e_{\tau', \sigma'}^*)(e_{\rho, \sigma} \otimes e_{\rho, \tau}) = e_{\sigma}$. Recall that if $\sigma, \tau \in \Sigma, \sigma > \tau$ and $\dim \sigma = \dim \tau + 2$, then there are exactly two cells $\rho_1, \rho_2 \in \Sigma$ between $\sigma$ and $\tau$. So easy computation shows that the quadratic dual ideal

$I_+ = \{ f \in R_1^* \otimes R_1^* \mid f(v) = 0 \text{ for all } v \in I_2 \subset R_1 \otimes R_1 = T_2 \} \subset T^*$

of $I$ is equal to

$(e_{\tau, \rho_1}^* \otimes e_{\rho_2, \tau}^* + e_{\tau, \rho_2}^* \otimes e_{\rho_1, \tau}^* \mid \sigma, \tau, \rho_1, \rho_2 \in \Sigma, \sigma > \rho_i > \tau, \dim \sigma = \dim \tau + 2 ).$

The $k$-algebra homomorphism $R \to R^1 = T^*/I^1$ defined by the identity map on $R_0 = T_0 = (T^*)_0 = (R^1)_0$ and $R_1 \ni e_{\sigma, \tau} \mapsto \varepsilon(\sigma, \tau) \cdot e_{\tau, \sigma}^* \in R_1^*$ is a graded isomorphism. Here $\varepsilon$ is a incidence function of $\Sigma$. \qed

Since $R^1 \cong R^{\text{op}}$, $\text{Hom}_k(\varepsilon, k)$ gives duality functors $D_k : \text{mod}_R \to \text{mod}_{R^1}$ and $D_k^\text{op} : \text{mod}_{R^1} \to \text{mod}_R$. These functors are exact, and they can be extended to the duality functors between $D^b(\text{mod}_R)$ and $D^b(\text{mod}_{R^1})$.

Note that $R^1$ is a graded ring with $\text{deg } e_{\tau, \sigma}^* = \dim \sigma - \dim \tau$. Let $gr_R$ (resp. $gr_{R^1}$) be the category of finitely generated graded left $R$-modules (resp. $R^1$-modules).
Note that we can regard the functor $\mathbf{D}$ (resp. $\mathbf{D}_k$ and $\mathbf{D}_k^{op}$) as the functor from $D^b(\text{gr}_R)$ to itself (resp. $D^b(\text{gr}_R) \to D^b(\text{gr}_R)$ and $D^b(\text{gr}_R) \to D^b(\text{gr}_R)$).

For each $i \in \mathbb{Z}$, let $\text{gr}_R(i)$ be the full subcategory of $\text{gr}_R$ consisting of $M \in \text{gr}_R$ with $\deg M_\sigma = \dim \sigma - i$. For any $M \in \text{gr}_R$, there are modules $M^{(i)} \in \text{gr}_R(i)$ such that $M \cong \bigoplus_{i \in \mathbb{Z}} M^{(i)}$. The forgetful functor gives an equivalence $\text{gr}_R(i) \cong \text{mod}_R$ for all $i \in \mathbb{Z}$, and $D^b(\text{gr}_R(i))$ is a full subcategory of $D^b(\text{gr}_R)$. Similarly, let $\text{gr}_{R'}(i)$ be the full subcategory of $\text{gr}_{R'}$ consisting of $M \in \text{gr}_R$ with $\deg M_\sigma = - \dim \sigma - i$. The above mentioned facts on $\text{gr}_R$ are (dim $R$, the left $R$-morphism given by $\text{Hom}_{\text{gr}_R}(R, N)$).

Let $\text{DF} : \text{mod}_R \to \text{mod}_{R'}$ and $\text{DG} : \text{mod}_{R'} \to \text{mod}_R$ be the functors defined in [1] Theorem 2.12.1. Since $R$ and $R'$ are artinian, $\text{DF}$ and $\text{DG}$ give an equivalence $D^b(\text{gr}_R) \cong D^b(\text{gr}_{R'})$ by the Koszul duality ([1] Theorem 2.12.6).

When $\Sigma$ is a simplicial complex the next result was given by Vybornov [14] (under the convention that $\emptyset \not\in \Sigma$). Independently, the author also proved a similar result ([15] Theorem 4.7]).

**Theorem 7.2** (c.f. Vybornov, [14] Corollary 4.3.5]). Under the above notation, if $M^\bullet \in D^b(\text{gr}_R(0))$, then we have $\text{DF}(M^\bullet) \in D^b(\text{gr}_{R'}(0))$. Similarly, if $N^\bullet \in D^b(\text{gr}_{R'}(0))$, then $\text{DG}(N^\bullet) \in D^b(\text{gr}_R(0))$. Under the equivalence $\text{gr}_R(0) \cong \text{mod}_R$ and $\text{gr}_{R'}(0) \cong \text{mod}_{R'}$, we have $\text{DF} \cong \text{D}_k \circ \text{D}$ and $\text{DG} \cong \text{D} \circ \text{D}_k^{op}$.

**Proof.** Recall that $(R')_0 = R_0$. Let $N \in \text{mod}_{R'}$. For the functor $\text{DG}$, we need the left $R$-module structure on $\text{Hom}_{R_0}(R, N_\sigma)$ given by $(x f)(y) := f(y x)$. The $R$-morphism given by $\text{Hom}_{R_0}(R, N_\sigma) \ni f \mapsto \sum_{\tau \leq \sigma} e(\sigma)_\tau \otimes_k f(e_{\sigma, \tau}) \in E(\sigma) \otimes_k N_\sigma$ gives an isomorphism $\text{Hom}_{R_0}(R, N_\sigma) \cong E(\sigma) \otimes_k N_\sigma$. Under this isomorphism, for cells $\tau < \sigma$, the morphism $\text{Hom}_{R_0}(R, N_\sigma) \to \text{Hom}_{R_0}(R, N_\tau)$ given by $f \mapsto [x \mapsto e^*_{\sigma, \tau} f(e_{\sigma, \tau})]$ corresponds to the morphism $E(\sigma) \otimes_k N_\sigma \to E(\tau) \otimes_k N_\tau$ given by $e(\sigma)_\rho \otimes y \mapsto e(\tau)_\rho \otimes e^*_{\tau, \sigma} y$. (Here $e(\tau)_\rho = 0$ if $\tau \not\leq \rho$.)

Let $N \in \text{gr}_{R'}$. By the explicit description of $\text{D}$ given in §3, we have

$$(\text{D} \circ \text{D}_k^{op})^i(N) = \bigoplus_{\sigma \in \Sigma} E(\sigma) \otimes_k N_\sigma$$

and the differential map defined by

$$E(\sigma) \otimes_k N_\sigma \ni e(\sigma)_\rho \otimes y \mapsto \sum_{\tau \leq \sigma} \varepsilon(\sigma, \tau) (e(\tau)_\rho \otimes e^*_{\tau, \sigma} y) \in (\text{D} \circ \text{D}_k^{op})^{i+1}(N).$$

So, if we forget the grading of modules, we have $\text{DG}(N) \cong (\text{D} \circ \text{D}_k^{op})(N)$. Similarly, we can check an isomorphism $\text{DG}(N^\bullet) \cong (\text{D} \circ \text{D}_k^{op})(N^\bullet)$ for a complex $N^\bullet \in D^b(\text{gr}_{R'})$.

Assume that $N \in \text{gr}_{R'}(0)$. Then the degree of $e(\sigma)_\tau \otimes y \in E(\sigma) \otimes_k N_\sigma \subset \text{DG}(N)$ is $(\dim \tau - \dim \sigma) + \dim \sigma = \dim \tau$ (see the proof of [1] Theorem 2.12.1] for the grading of $\text{DG}(N)$). Thus we have $\text{DG}(N) \in \text{gr}_R(0)$.

We can prove the statement on $\text{DF}$ by a similar (easier) way.

The results corresponding to Proposition 7.1 and Theorem 7.2 also hold for the incidence algebra of the poset $\Sigma \setminus \emptyset$. In other words, Vybornov [14] Corollary 4.3.5]
and the “top perversity case” of Polishchuk \cite{8} can be generalized directly into regular cell complexes.

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**References**

[1] A. Beilinson, V. Ginzburg and W. Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (1996), 473–527.
[2] A. Björner, Posets, regular CW complexes and Bruhat order, European J. Combin. 5 (1984), 7–16.
[3] W. Bruns and J. Herzog, Cohen-Macaulay rings, revised edition, Cambridge University Press, 1998.
[4] G. E. Cooke and R. L. Finney, Homology of Cell Complexes (Based on lectures by Norman E. Steenrod), Princeton Univ. Press, 1967.
[5] R. Hartshorne, Residues and duality, Lectures Notes in Math., vol. 20, Springer-Verlag, 1966.
[6] B. Iversen, Cohomology of sheaves. Springer-Verlag, 1986.
[7] M. Kashiwara and P. Schapira, Sheaves on manifolds, Springer-Verlag, 1990.
[8] A. Polishchuk, Perverse sheaves on a triangulated space. Math. Res. Lett. 4 (1997), 191–199.
[9] P. Polo, On Cohen-Macaulay posets, Koszul algebras and certain modules associated to Schubert varieties, Bull. London Math. Soc. 27 (1995), 425–434.
[10] N. Popescu, Abelian categories with applications to rings and modules, Academic Press, 1973.
[11] A. Shepard, A cellular description of the derived category of a stratified space, Ph.D. Thesis, Brown University (1984).
[12] R. Stanley, Combinatorics and commutative algebra, 2nd ed. Birkhäuser, 1996.
[13] R. Stanley, Enumerative combinatorics. Vol. 1. Cambridge University Press, 1997.
[14] M. Vybornov, Sheaves on triangulated spaces and Koszul duality, preprint \texttt{math.AT/9910150}.
[15] D. Woodcock, Cohen-Macaulay complexes and Koszul rings, J. London Math. Soc. 57 (1998), 398–410.
[16] K. Yanagawa, Alexander duality for Stanley-Reisner rings and squarefree \(N^n\)-graded modules, J. Algebra 225 (2000), 630–645.
[17] K. Yanagawa, Stanley-Reisner rings, sheaves, and Poincaré-Verdier duality, Mathematical Research Letters 10 (2003) 635–650.
[18] K. Yanagawa, Derived category of squarefree modules and local cohomology with monomial ideal support, J. Math. Soc. Japan 56 (2004) 289–308.
[19] A. Yekutieli and J. Zhang, Rings with Auslander dualizing complexes, J. Algebra 213 (1999) 1–51.

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