AN EXAMPLE OF MELKERSSON SUBCATEGORY WHICH IS NOT CLOSED UNDER INJECTIVE HULLS

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Abstract. The Melkersson subcategory is a special Serre subcategory which satisfies useful conditions $C_I$ defined in [1]. It was proved that a Serre subcategory which is closed under injective hulls is a Melkersson subcategory. However, it has been an open question whether the contrary implication holds. In this paper, we shall show that this question has a negative answer in general.

1. Introduction

Throughout this paper, all rings are commutative noetherian ring, all modules are unitary and $R$ denotes a ring. We assume that all full subcategories $S$ of the modules category $R$-Mod and the finitely generated $R$-modules category $R$-mod are closed under isomorphisms, that is if $M$ is in $S$ and $R$-module $N$ is isomorphic to $M$ then $N$ is in $S$.

In [1], M. Aghapournahr and L. Melkersson gave a useful condition $C_I$ on the Serre subcategory $S$ of $R$-Mod where $I$ is an ideal of $R$. It is said that $S$ satisfies the condition $C_I$ if the following condition holds: if $M = I\Gamma_I(M)$ and $(0 :_M I)$ is in $S$, then $M$ is in $S$. They showed that local cohomology modules and Serre subcategories which satisfy such a condition have affinity for each other. After of this, the Serre subcategory which satisfies the condition $C_I$ for all ideals $I$ of $R$ was named Melkersson subcategory by M. Aghapournahr, A. J. Taherizadeh and A. Vahidi in [2]. For example, all Serre subcategories which are closed under injective hulls are Melkersson subcategory. So it is natural to ask the following question which was given in [1]:

Question. Is Melkersson subcategory closed under injective hulls?

In this paper, we shall show that this question has a negative answer in general. To be more precise, we denote by $S_{f.g}$ the Serre subcategory of all finitely generated $R$-modules and by $M_{f.s}$ the Serre subcategory of all $R$-modules with finite support. We shall see that a class

$$(S_{f.g}, M_{f.s}) = \left\{ X \in R\text{-Mod} \mid \text{there are } S \in S_{f.g} \text{ and } M \in M_{f.s}, \text{ such that } 0 \to S \to X \to M \to 0 \text{ is exact.} \right\}$$

is Melkersson subcategory which is not closed under injective hulls on the ring of formal power series $R = k[[x, y]]$ in the indeterminate $x$ and $y$ with the coefficients in a field $k$.

The organization of this paper is as follows.

In section 2, we shall recall definitions of Melkersson subcategory and classes $(S_1, S_2)$ of extension modules of a Serre subcategory $S_1$ by another Serre subcategory $S_2$. In section 3, we shall give a proof of main result. In Section 4, we shall see several remarks on Melkersson subcategory.

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2. Preliminaries

In this section, we shall recall several definitions which are necessary to prove the main result of this paper.

A class $S$ of $R$-modules is called a Serre subcategory of $R$-Mod if it is closed under submodules, quotients and extensions. We also say that a Serre subcategory $S$ of $R$-Mod is a Serre subcategory of $R$-mod if $S$ consists of finitely generated $R$-modules.

In [1], M. Aghapournahr and L. Melkersson gave the following condition on Serre subcategories of $R$-Mod.

**Definition 2.1.** Let $S$ be a Serre subcategory of $R$-Mod and $I$ be an ideal of $R$. We say that $S$ satisfies the condition $C_I$ if the following condition satisfied:

$(C_I)$ If $M = I_1(M)$ and $(0 :_M I)$ is in $S$, then $M$ is in $S$.

The following special Serre subcategory was named Melkersson subcategory by M. Aghapournahr, A. J. Taherizadeh and A. Vahidi in [2].

**Definition 2.2.** Let $S$ be a Serre subcategory of $R$-Mod.

1. $S$ is called a Melkersson subcategory with respect to an ideal $I$ of $R$ if $S$ satisfies the condition $C_I$.

2. $S$ is called a Melkersson subcategory if $S$ satisfies the condition $C_I$ for all ideals $I$ of $R$.

It has already shown that any Serre subcategory which is closed under injective hulls is the Melkersson subcategory with respect to all ideals $I$ of $R$, so that it is a Melkersson subcategory. (See [1, Lemma 2.2].)

Next, we consider classes of extension modules of Serre subcategory by another one.

**Definition 2.3.** Let $S_1$ and $S_2$ be Serre subcategories of $R$-Mod. We denote by $(S_1, S_2)$ the class of all $R$-modules $M$ with some $R$-modules $S_1 \in S_1$ and $S_2 \in S_2$ such that a sequence $0 \to S_1 \to M \to S_2 \to 0$ is exact, that is

$$(S_1, S_2) = \left\{ M \in R\text{-Mod} \mid \text{there are } S_1 \in S_1 \text{ and } S_2 \in S_2 \text{ such that } 0 \to S_1 \to M \to S_2 \to 0 \text{ is exact} \right\}.$$

We shall refer to $(S_1, S_2)$ as a class of extension modules of $S_1$ by $S_2$.

For example, a class $(S_{f.g.}, S_{Artin})$ is the set of all Minimax $R$-modules where $S_{f.g.}$ denotes the Serre subcategory consists of all finitely generated $R$-modules and $S_{Artin}$ denotes the Serre subcategory consists of all Artinian $R$-modules. We note that a class $(S_1, S_2)$ is not necessarily Serre subcategory. (For more detail, see [7].)

3. Main result

In this section, we shall give an example of Melkersson subcategory which is not closed under injective hulls. We denote by $M_{f.s.}$ the class of $R$-modules with finite support. A class $M_{f.s.}$ is Serre subcategory of $R$-Mod which is closed under injective hulls, so that $M_{f.s.}$ is a Melkersson subcategory. (See [1, Example 2.4].) Furthermore, a class $(S_{f.g.}, M_{f.s.})$ is a Serre subcategory of $R$-Mod by [7, Corollary 4.3 or 4.5].

The main result in this paper is as follows.
Theorem 3.1. Let \((R, \mathfrak{m})\) be a local ring with a maximal ideal \(\mathfrak{m}\). Then the following assertions hold.

1. If \(R\) has infinite many prime ideals, then \((\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})\) is not closed under injective hulls.

2. If \(R\) is a 2-dimensional local domain, then \((\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})\) is a Melkersson subcategory.

In particular, if \(R\) is a 2-dimensional local domain with infinite many prime ideals, then \((\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})\) is a Melkersson subcategory which is not closed under injective hulls.

Proof. (1) We assume that \(R\) has infinite many prime ideals. (We note that the dimension of \(R\) must be at least two.) Since the set \(\text{Min}(R)\) of all minimal prime ideals of \(R\) is finite set, there exists a prime ideal \(p \in \text{Min}(R)\) such that \(V(p) = \{q \in \text{Spec}(R) \mid p \subseteq q\}\) is infinite set. We fix this prime ideal \(p\).

We assume that \((\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})\) is closed under injective hulls and shall derived a contradiction. Since \(R/\mathfrak{p}\) is in \((\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})\), the injective hull \(E_R(R/\mathfrak{p})\) of \(R/\mathfrak{p}\) is also in \((\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})\) by assumption. Therefore, there exists a short exact sequence

\[
0 \to F \to E_R(R/\mathfrak{p}) \to M \to 0
\]

with \(F \in \mathcal{S}_{f.g.}\) and \(M \in \mathcal{M}_{f.s.}\). Since \(V(p)\) is finite set and \(\text{Supp}(M)\) is finite set, we can choose a prime ideal \(n \in V(p) \setminus (\text{Supp}(M) \cup \{\mathfrak{p}\})\). Here, we set \(T = R_n\) and \(q = pR_n = pT\).

We note that \(T\) is local ring with at least dimension one and \(q\) is a minimal prime ideal of \(T\).

Now here, we claim that \(E_{T/q}(T/q)\) is a finitely generated as \(T/q\)-module and shall show this. By applying the exact functor \((-) \otimes_R T\) to the above short exact sequence, we see that it holds

\[
F_n \cong E_R(R/\mathfrak{p}) \otimes_R T \cong E_T(T/q).
\]

(Also see [4, Lemma 3.2.5].) Furthermore, it holds

\[
E_{T/q}(T/q) \cong (0 :_{E_T(T/q)} q) \cong (0 :_{F_n} q)
\]

by the above isomorphisms. (Also see [3, 10.1.15 Lemma].) Since \(F\) is a finitely generated \(R\)-module, \(F_n\) is so as \(T\)-module. Thus \(E_{T/q}(T/q)\) is a finitely generated \(T\)-module. Consequently, we see that \(E_{T/q}(T/q)\) is a finitely generated as \(T/q\)-module.

A local domain \(T/q\) is \(\text{dim }T/q \geq 1\) and has a finitely generated injective \(T/q\)-module \(E_{T/q}(T/q)\). So it follows from the Bass formula that it holds

\[
0 < \text{depth}_{T/q} T/q = \text{inj dim}_{T/q} E_{T/q}(T/q) = 0.
\]

This is a contradiction.

(2) We note that any minimal element in \(\text{Supp}(M)\) is in \(\text{Ass}(M)\) for any (not necessarily finitely generated) \(R\)-module \(M\). (e.g. see [5, Theorem 2.4.12].)

We assume that \(R\) is a 2-dimensional local domain and have to show that a Serre subcategory \((\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})\) satisfies the condition \(C_f\) for all ideals \(I\) of \(R\). We fix an ideal \(I\) of \(R\). We suppose that \(X\) is an \(R\)-module such that \(X = I_T(X)\) and \(0 :_X I\) is in \((\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})\), and shall show that \(X\) is in \((\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})\). There exists a short exact sequence

\[
0 \to F \to (0 :_X I) \to M \to 0
\]

with \(F \in \mathcal{S}_{f.g.}\) and \(M \in \mathcal{M}_{f.s.}\).

In the case of \(\text{dim } (0 :_X I) \leq 1\). Then it holds \(\text{Supp}(X) = \text{Ass}(X) \cup \{\mathfrak{m}\}\). Indeed, since it holds \(\text{Ass}(X) = \text{Ass}(0 :_X I)\), it is easy to see that the zero ideal \((0)\) of \(R\) does not belong to \(\text{Supp}(X)\). Therefore, if there exists a prime ideal \(p \in \text{Supp}(X) \setminus \{\mathfrak{m}\}\), \(p\) is minimal in \(\text{Supp}(X)\).
Thus $p$ is in $\text{Ass}(X)$, so we see that the above equality holds. On the other hand, it holds

$$\text{Ass}(X) = \text{Ass}((0 :_X I))$$

$$\subseteq \text{Ass}(F) \cup \text{Ass}(M)$$

$$\subseteq \text{Ass}(F) \cup \text{Supp}(M).$$

Since $F$ is a finitely generated $R$-module and $M$ is in $\mathcal{M}_{f.s.}$, $\text{Ass}(X)$ is finite set. Consequently, $\text{Supp}(X)$ is also finite set, so we see that $X$ is in $\mathcal{M}_{f.s.} \subseteq (S_{f.g.}, \mathcal{M}_{f.s.}).$

In the case of $\dim (0 :_X I) = 2$. Since $R$ is a 2-dimensional domain, the zero ideal $(0)$ of $R$ must be in $\text{Supp}((0 :_X I))$ and this is a minimal in $\text{Supp}((0 :_X I))$. It follows that $(0) \in \text{Ass}((0 :_X I)) = V(I) \cap \text{Ass}(X) \subseteq V(I).$

Therefore, it holds $I = (0)$. Consequently, $X = (0 :_X I)$ is in $(S_{f.g.}, \mathcal{M}_{f.s.}).$

The proof is completed. \hfill \Box

\textbf{Remark 3.2.} If $(R, m)$ is a local ring with at most one dimension, then $\text{Spec}(R)$ is finite set. Thus, any support of $R$-module is finite set, so we see $(S_{f.g.}, \mathcal{M}_{f.s.}) = R$-$\text{Mod}$. Therefore, in this case, $(S_{f.g.}, \mathcal{M}_{f.s.})$ is a Melkersson subcategory and is closed under injective hulls.

\textbf{Example 3.3.} Let $R$ be the ring of formal power series $k[[x, y]]$ in the indeterminate $x$ and $y$ with the coefficients in a field $k$. Then $R$ is a 2-dimensional local domain and has infinite many prime ideals $(x + y^n)$ for each non-negative integer $n$. Thus, in this case, $(S_{f.g.}, \mathcal{M}_{f.s.})$ is a Melkersson subcategory which is not closed under injective hulls by Theorem \ref{thm:melkersson}.

\section{Several remarks on Melkersson subcategories}

In this section, we assume that any full subcategory contains a non-zero $R$-module.

In a local ring $R$, it is clear that any Serre subcategory of $R$-$\text{Mod}$ contains all finite length modules. On the other hand, we can see the following assertion holds.

\textbf{Proposition 4.1.} Let $(R, m)$ be a local ring and $\mathcal{M}$ be a Melkersson subcategory with respect to $m$. Then any Artinian module is in $\mathcal{M}$. In particular, Melkersson subcategory contains all Artinian modules.

\textbf{Proof.} Let $\mathcal{M}$ be a Melkersson subcategory with respect to $m$. Since all finite length $R$-modules belong to any Serre subcategory, we can see that the injective hull $E_R(R/m)$ of $R/m$ belongs to $\mathcal{M}$. Indeed, since it holds

$$\begin{cases} E_R(R/m) = \Gamma_m(E_R(R/m)) \text{ and} \\ (0 :_{E_R(R/m)} m) \cong \text{Hom}_R(R/m, E_R(R/m)) = R/m \text{ is in } \mathcal{M}, \end{cases}$$

it follows from the condition $C_m$ that $E_R(R/m)$ is in $\mathcal{M}$.

Let $M$ be an Artinian module. Then $M$ is embedded in $\bigoplus^n E_R(R/m)$ for some integer $n$. Therefore, since Melkersson subcategory is closed under finite direct sums and submodules, we see that $M$ is in $\mathcal{M}$. \hfill \Box

To see whether Serre subcategory is Melkersson subcategory, we have only to check that it satisfies the condition $C_I$ for all radical ideals $I$ of $R$.

\textbf{Proposition 4.2.} Let $\mathcal{M}$ be a Serre subcategory. Then following conditions are equivalent:

(1) $\mathcal{M}$ is a Melkersson subcategory;

(2) $\mathcal{M}$ is a Melkersson subcategory with respect to $\sqrt{I}$ for all ideals $I$ of $R$. 

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Proof. We assume that $M$ is a Melkersson subcategory with respect to $\sqrt{I}$ for all ideals $I$ of $R$. Let $I$ be an ideal of $R$ and shall show that $M$ satisfies condition $C_I$. We suppose that $M$ is an $R$-module such that $M = \Gamma_I(M)$ and $(0 :_M I)$ is in $M$. Then it holds $\Gamma_{\sqrt{I}}(M) = \Gamma_I(M) = M$. Furthermore, since $M$ is closed under submodules and $(0 :_M \sqrt{I}) \subseteq (0 :_M I)$, we see $(0 :_M \sqrt{I})$ is in $M$. If follows from the condition $C_{\sqrt{I}}$ that $M$ is in $M$. □

Serre subcategory is defined not only in the category $R$-Mod but also in the category $R$-mod. Therefore, it stands to reason that we consider the Melkersson subcategory of $R$-mod which is defined by considering the condition $C_I$ for only finitely generated $R$-modules as follows: the Serre subcategory $M$ of $R$-mod is Melkersson subcategory of $R$-mod if it satisfies the condition

$$(C_I) \quad \text{If } M = \Gamma_I(M) \in R\text{-mod and } (0 :_M I) \text{ is in } M, \text{ then } M \text{ is in } M$$

for all ideal $I$ of $R$. However, by the following proposition, we can see that it is not necessary to treat Serre subcategory which satisfies such a condition specially.

Proposition 4.3. Any Serre subcategory $S$ of $R$-mod is a Melkersson subcategory of $R$-mod in the above sense.

Proof. By [3, Theorem 4.1], there exists a specialization closed subset $W$ of Spec($R$) corresponding to the Serre subcategory $S$. In particular, we can denote

$$S = \{ M \in R\text{-mod} \mid \text{Supp}(M) \subseteq W \}$$

and $W = \bigcup_{M \in S} \text{Supp}(M)$.

Let $I$ be an ideal of $R$. We suppose that $M$ is a finitely generated $R$-module such that $M = \Gamma_I(M)$ and $(0 :_M I)$ is in $S$. Since $(0 :_M I)$ is in $S$, it holds $\text{Ass}(M) = \text{Ass}((0 :_M I)) \subseteq \text{Supp}((0 :_M I)) \subseteq W$, and so we have $\text{Supp}(M) \subseteq W$. Consequently, $M$ is in $S$. □

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