ON THE EQUALITY OF OPERATOR VALUED WEIGHTS

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This paper is dedicated to the memory of Uffe Haagerup.

Abstract. G. K. Pedersen and M. Takesaki have proved in 1973 that if \( \phi \) is a faithful, semi-finite, normal weight on a von Neumann algebra \( M \), and \( \psi \) is a \( \sigma \phi \)-invariant, semi-finite, normal weight on \( M \) equal to \( \phi \) on the positive part of a weak\(^*\)-dense \( \sigma \phi \)-invariant \( * \)-subalgebra of \( \mathfrak{M}_\phi \), then \( \psi = \phi \).

In 1978 L. Zsidó extended the above result by proving: if \( \phi \) is as above, \( a \geq 0 \) belongs to the centralizer \( M^\varphi \) of \( \varphi \), and \( \psi \) is a \( \sigma \phi \)-invariant, semi-finite, normal weight on \( M \), equal to \( \phi(a^{1/2} \cdot a^{1/2}) \) on the positive part of a weak\(^*\)-dense \( \sigma \phi \)-invariant \( * \)-subalgebra of \( \mathfrak{M}_\phi \), then \( \psi = \varphi_a \).

Here we will further extend this latter result, proving criteria for both the inequality \( \psi \leq \varphi_a \) and the equality \( \psi = \varphi_a \). Particular attention is accorded to criteria with no commutation assumption between \( \varphi \) and \( \psi \), in order to be used to prove inequality and equality criteria for operator valued weights.

Concerning operator valued weights, it is proved that if \( E_1, E_2 \) are semi-finite, normal operator valued weights from a von Neumann algebra \( M \) to a von Neumann subalgebra \( N \), and they are equal on \( \mathfrak{M}_{E_1} \), then \( E_2 \leq E_1 \). Moreover, it is shown that this happens if and only if for any (or, if \( E_1, E_2 \) have equal supports, for some) faithful, semi-finite, normal weight \( \theta \) on \( N \) the weights \( \theta \circ E_2, \theta \circ E_1 \) coincide on \( \mathfrak{M}_{\theta \circ E_1} \).

Introduction

For the equality of two normal positive forms on a \( W^* \)-algebra it is enough that they coincide on a weak\(^*\)-dense subset. For unbounded normal weights equality can follow from equality on a weak\(^*\)-dense subset only under additional conditions on the weights and/or on the subset.

A first criterion of this type, which served as a model for subsequent criteria, is \cite{13}, Proposition 5.9: Let \( \varphi, \psi \) be semi-finite, normal weights on a \( W^* \)-algebra \( M \), \( \varphi \) faithful and \( \psi \sigma \varphi \)-invariant. If \( \psi(x^*x) = \varphi(x^*x) \) for all \( x \) in a weak\(^*\)-dense, \( \sigma \varphi \)-invariant \( * \)-subalgebra of \( \mathfrak{M}_\varphi \), then \( \psi = \varphi \). This criterion was further extended in \cite{18}, Theorem 2.3 as follows: Let \( \varphi, \psi \) be as above, and \( a \) a positive element of the centralizer of \( \varphi \). If \( \psi(x^*x) = \varphi(a^{1/2}x^*xa^{1/2}) \) for \( x \) in a weak\(^*\)-dense, \( \sigma \varphi \)-invariant \( * \)-subalgebra of \( \mathfrak{M}_\varphi \), then \( \psi = \varphi(a^{1/2} \cdot a^{1/2}) \).

In both of the above results the \( \sigma \varphi \)-invariance of \( \psi \) is assumed. Therefore they are not useful to prove equality criteria for operator valued weights. Indeed, if \( M \) is a \( W^* \)-algebra and \( 1_M \in N \subset M \) a \( W^* \)-subalgebra, then, according to \cite{11}, Lemma 4.8, the equality of two faithful, semi-finite, normal operator valued weights \( E_1, E_2 \).
from $M$ to $N$ would be implied by the equality of $\theta \circ E_1 = \theta \circ E_2$ for some faithful, semi-finite, normal weight $\theta$ on $N$. However, there is no known way to derive the $\sigma^{\theta \circ E_1}$-invariance of $\theta \circ E_2$ from appropriate properties of $E_1$, $E_2$ and a suitable choice of $\theta$. Consequently, having in view equality criteria for normal operator valued weights, it is of interest to prove criteria for the equality of normal weights with no commutation assumption for them.

Similar considerations support the interest to prove inequality criteria for normal weights without making any commutation assumption.

After a preliminary first section concerning notations and used results, the second section is dedicated to the so-called regularizing nets, a useful tool in the modular theory of von Neumann algebras. It was used in the proof of [13], Lemma 5.2 and of [18], Lemma 2.1, on which [13], Proposition 5.9 resp. [18], Theorem 2.3 are based. It will be an essential ingredient also of the proof of the main result of Section 3.

We will call regularizing net for a faithful, semi-finite, normal weight $\varphi$ on a $W^*$-algebra $M$ (cf. [18], §1) any net $(u_i)_i$ in the Tomita algebra of $\varphi$ such that

(i) $\sup_{x \in K} \|e^{(\varphi(x) + \varphi(y))}_{\varphi}(u_i)\| < +\infty$ and $\sup_{x \in K} \|e^{(\varphi(x) - \varphi(y))}_{\varphi}(u_i)\| < +\infty$ for each compact $K \subset \mathbb{C}$;

(ii) $\pi(e^{\varphi(x)}_{\varphi})(u_i) \to 1_M$ in the $s^*$-topology for all $x \in \mathbb{C}$.

They can be derived from a bounded net $(x_i)_i$ in $\mathcal{N}_\varphi \cap \mathcal{R}_\varphi$ such that $x_i \to 1_M$ in the $s^*$-topology by setting

\begin{equation}
(2.1) \quad u_i = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \pi(s_{\varphi}^t(x_i)) dt.
\end{equation}

However the verification that the such obtained net $(u_i)_i$ satisfies (ii), done in the proof of [13], Lemma 5.2, works only if the net $(x_i)_i$ is increasing, while in the proof of [18], Lemma 2.1 no verification is given. If $(x_i)_i$ would be a sequence, we could use the dominated convergence theorem of Lebesgue, but the general case needs a complete treatment. We will prove that, even if we assume only $x_i \in \mathcal{N}_\varphi$, formula (2.1) yields always a regularizing net for $\varphi$ (Theorem 2.4).

Section 3 is dedicated to criteria for inequalities and equality between normal weights. The main result is an extension of [13], Lemma 2.1. If $\varphi, \psi$ are semi-finite, normal weights on a $W^*$-algebra $M$ with $\varphi$ assumed faithful, $a \geq 0$ is an element of the centralizer $M^\sigma = \varphi(a^{1/2} \cdot a^{1/2})$, $\psi(x^*x) = \varphi(a(x^*x))$ holds for every $x$ in a weak* dense, $\sigma\varphi$-invariant *-subalgebra of $\mathfrak{H}_{\varphi \psi}$, then $\psi \leq \varphi_a$ (Theorem 3.1). If we assume additionally that $\psi$ is $\sigma\varphi$-invariant, then the equality $\psi = \varphi_a$ follows (Theorem 3.3). But the above inequality criterion implies also the following equality criterion: If $\varphi, \psi$ are faithful, semi-finite, normal weights on a $W^*$-algebra $M$, $a, b \geq 0$ are elements of $M^\sigma$ resp. $M^\psi$, and $\psi_b(x^*x) = \varphi_a(x^*x)$ holds for all $x$ in a weak* dense *-subalgebra of $\mathfrak{H}_{\varphi_b} \cap \mathfrak{H}_{\psi_b}$, which is both $\sigma\varphi$- and $\sigma\psi$-invariant, then $\psi_b = \varphi_a$ (Theorem 3.8). We underline that, neither in the inequality criterion nor in the above result $\sigma\varphi$- or $\sigma\psi$-invariance of the other weight is required.

In the last Section 4 we discuss inequality and equality criteria for semi-finite, normal operator valued weights $E_1, E_2$ from a $W^*$-algebra $M$ to a $W^*$-subalgebra $1_M \in N \subset M$. Reduction to the case of scalar weights, considered in Section 3, is used.
The main result of the section is Theorem 4.5 which consists of two parts. In part (i) it is proved that if \( \mathfrak{M}_{E_1} \subset \mathfrak{M}_{E_2} \) and \( E_1, E_2 \) coincide on \( \mathfrak{M}_{E_1} \), then \( E_2 \leq E_1 \), while in part (ii) it is shown that \( \mathfrak{M}_{E_1} \subset \mathfrak{M}_{E_2} \) and \( E_1 = E_2 \) on \( \mathfrak{M}_{E_1} \), happens if and only if \( \mathfrak{M}_{\theta_0 E_1} \subset \mathfrak{M}_{\theta_0 E_2} \) and \( \theta \circ E_1 = \theta \circ E_2 \) on \( \mathfrak{M}_{\theta_0 E_1} \) for all faithful, semi-finite, normal weights \( \theta \) on \( N \). We notice that the difficulty in proving (ii) arises from the fact that there is no generic relation between \( \mathfrak{M}_{E_1} \) and \( \mathfrak{M}_{\theta_0 E_1} \).

If \( E_1 \) and \( E_2 \) have equal supports, for \( \mathfrak{M}_{E_1} \subset \mathfrak{M}_{E_2} \) and \( E_1 = E_2 \) on \( \mathfrak{M}_{E_1} \) another characterization, based on the Connes cocycle, is provided (Theorem 4.8). Most likely the support condition is here redundant.

1. Preliminaries

We will use the terminology of [15] and [14]. In particular,

- \((\cdot,\cdot)\) will denote the inner product of a Hilbert space and it will be assumed linear in the first variable and antilinear in the second variable;
- \(B(H)\) will denote the algebra of all bounded linear operators on the Hilbert space \( H \), with the identity operator denoted by \( 1_H \);
- for \( \xi \) in a Hilbert space \( H \), \( \omega_\xi \) will denote the positive vector functional \( B(H) \ni x \mapsto (x\xi|\xi) \);
- the unit of a \( W^* \)-algebra \( M \) will be denoted by \( 1_M \);
- the \( w \)-topology on a \( W^* \)-algebra \( M \) is the weak\(^*\) topology, the locally convex topology defined by the semi-norms
  \[ M \ni x \mapsto |\varphi(x)|, \varphi \text{ a normal positive form on } M; \]
- the \( s\)-topology on a \( W^* \)-algebra \( M \) is the locally convex topologies defined by the semi-norms
  \[ p_\varphi : M \ni x \mapsto \varphi(x^*x)^{1/2} + \varphi(xx^*)^{1/2}, \varphi \text{ a normal positive form on } M; \]
- \( s(A) \) denotes the support projection of a self-adjoint operator \( A \) affiliated to a \( W^* \)-algebra \( M \) (does not matter in which spatial representation);
- \( s(\varphi) \) denotes the support projection of a normal positive form or a normal weight \( \varphi \) on a \( W^* \)-algebra \( M \);
- \( M^+ \) denotes the extended positive part of a \( W^* \)-algebra \( M \).

As usual, for a weight \( \varphi \) on a \( W^* \)-algebra \( M \) we use the notations

\[ \mathfrak{N}_\varphi = \{ x \in M; \varphi(x^*x) < +\infty \}; \]
\[ \mathfrak{M}_\varphi = \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*, \]
\[ \mathfrak{M}_\varphi = \text{linear span of } \mathfrak{N}_\varphi \mathfrak{N}_\varphi. \]

Then

\[ \mathfrak{M}_\varphi \cap M^+ = \{ a \in M^+; \varphi(a) < +\infty \}. \]

We notice that, for \( \varphi \) a weight on a \( W^* \)-algebra \( M \) and \( e \in M \) a projection, \n
\[ \varphi(1_M - e) = 0 \quad \Longrightarrow \quad \varphi(x^*x) = \varphi(e x^* x e) \quad \text{for all } x \in M. \tag{1.1} \]

Indeed, since \( x - xe = x (1_M - e) \in \mathfrak{N}_\varphi \), we have \( x \in \mathfrak{N}_\varphi \iff x e \in \mathfrak{N}_\varphi \). Thus, if \( x \notin \mathfrak{N}_\varphi \), then \( \varphi(x^*x) = +\infty = \varphi(e x^* x e) \). On the other hand, if \( x \) and \( xe \) belong to \( \mathfrak{N}_\varphi \), then the Schwarz inequality entails

\[
\begin{align*}
|\varphi((1_M - e) x^* x e)|^2 &\leq \varphi((1_M - e) x^* x (1_M - e)) \varphi(e x^* x e) \\
&\leq ||x^*x|| \varphi(1_M - e) \varphi(e x^* x e) = 0
\end{align*}
\]
and, similarly, \( \varphi(e x^* x (1_M - e)) = 0 \). Consequently,

\[
\varphi(x^* x) = \varphi(e x^* x e) + \varphi((1_M - e) x^* x e) + \varphi(e x^* x (1_M - e)) + \varphi((1_M - e) x^* x (1_M - e)) = \varphi(e x^* x e).
\]

For \( \varphi \) a semi-finite, normal weight on a \( W^* \)-algebra \( M \), then \( \pi_\varphi : M \rightarrow B(H_\varphi) \) denotes its GNS representation, and \( x_\varphi \) the canonical image of \( x \in \mathfrak{M}_\varphi \) in \( H_\varphi \). If \( \varphi \) is also faithful, then \( S_\varphi \) stands for the closure of the antilinear operator

\[
H_\varphi \ni (\mathfrak{A}_\varphi)_\varphi \ni x_\varphi \mapsto (x^*)_\varphi \in H_\varphi,
\]

\( \Delta_\varphi = S_\varphi^* S_\varphi \) for the modular operator of \( \varphi \), and \( J_\varphi \) for its modular conjugation, so that \( S_\varphi = J_\varphi \Delta_\varphi^{1/2} \). Further, \( (\sigma_t^z)_{t \in \mathbb{R}} \) will denote the modular automorphism group of \( \varphi \):

\[
\pi_\varphi(\sigma_t^z(x)) = \Delta_\varphi^{it} \pi_\varphi(x) \Delta_\varphi^{-it}, \quad t \in \mathbb{R}, x \in M.
\]

An element \( x \in M \) will be called \( \sigma_\varphi \)-entire if \( \mathbb{R} \ni t \mapsto \sigma_t^z(x) \in M \) has an entire extension, denoted \( \mathbb{C} \ni z \mapsto \sigma_t^z(x) \in M \), and we consider the *-subalgebras of \( M \)

\[
M_\infty^z = \{ x \in M; x \text{ is } \sigma_\varphi \text{-entire} \},
\]

\[
M^z = \{ x \in M; \sigma_t^z(x) = x \text{ for all } t \in \mathbb{C} \} \quad (\text{centralizer of } \varphi),
\]

\[
\mathfrak{T}_\varphi = \{ x \in M_\infty^z; \sigma_t^z(x) \in \mathfrak{A}_\varphi \text{ for all } t \in \mathbb{R} \} \quad (\text{maximal Tomita algebra of } \varphi).
\]

For \( x \in \mathfrak{T}_\varphi \), the vector \( x_\varphi \) belongs to the domain of each \( \Delta_t^z, z \in \mathbb{C} \), and

\[
\sigma_t^z(x_\varphi) = \Delta_t^z x_\varphi, \quad z \in \mathbb{C}
\]

(see e.g. [15], Sections 10.20 and 10.21).

If \( \varphi \) is a not necessarily faithful semi-finite, normal weight on a \( W^* \)-algebra \( M \), then the modular automorphisms are considered acting on the reduced algebra \( s(\varphi) M s(\varphi) \) and \( M_\infty^z, M^z, \mathfrak{T}_\varphi \) will be *-subalgebras of \( s(\varphi) M s(\varphi) \).

Let \( \Omega \) be a locally compact topological space, \( \mu \) a Radon measure on \( \Omega \), \( M \) a \( W^* \)-algebra, and \( F : \Omega \rightarrow M \) a \( w \)-continuous function such that

\[
\int_\Omega \| F(\omega) \| \, d\mu(\omega) < +\infty.
\]

Then \( M_* \ni \varphi \mapsto \int_\Omega \varphi(F(\omega)) \, d\mu(\omega) \in \mathbb{C} \) is a bounded linear functional, hence an element of \( (M_*)^* = M \), denoted by \( \int_\Omega F(\omega) \, d\mu(\omega) \). Therefore

\[
(1.2) \quad \varphi \left( \int_\Omega F(\omega) \, d\mu(\omega) \right) = \int_\Omega \varphi(F(\omega)) \, d\mu(\omega)
\]

for every \( \varphi \in M_* \). If \( F(\Omega) \subset M^+ \) then (1.2) holds also for any normal weight \( \varphi \) on \( M \) (see [13], Lemma 3.1). Indeed, according to [10], we have

\[
\varphi(a) = \sum_i \varphi_i(a), \quad a \in M^+
\]

for some family \( \{ \varphi_i \} \) of normal positive forms on \( M \) and, taking into account that the monotone convergence theorem for lower semicontinuous positive functions and regular Borel measures applies to arbitrary upward directed families (see e.g. [4],
Proposition 7.4.4), we deduce:
\[
\varphi\left(\int_{\Omega} F(\omega) d\mu(\omega)\right) = \sum_i \varphi_i \left(\int_{\Omega} F(\omega) d\mu(\omega)\right) = \sum_i \int_{\Omega} \varphi_i(F(\omega)) d\mu(\omega) = \int_{\Omega} \sum_i \varphi_i(F(\omega)) d\mu(\omega) = \int_{\Omega} \varphi(F(\omega)) d\mu(\omega).
\]

We notice also that
\[
(1.3) \quad p_{\varphi}\left(\int_{\Omega} F(\omega) d\mu(\omega)\right) \leq \int_{\Omega} p_{\varphi}(F(\omega)) d\mu(\omega),
\]
where \(\varphi\) is an arbitrary normal positive form on \(M\) and \(p_{\varphi}\) denotes the semi-norm \(M \ni x \mapsto \varphi(x^+x)^{1/2} + \varphi(x^+)\). Inequality (1.3) is consequence of
\[
(1.4) \quad p_{\varphi,j}\left(\int_{\Omega} F(\omega) d\mu(\omega)\right) \leq \int_{\Omega} p_{\varphi,j}(F(\omega)) d\mu(\omega), \quad j = 1, 2,
\]
where the semi-norms \(p_{\varphi,1}\) and \(p_{\varphi,2}\) on \(M\) are defined by
\[
p_{\varphi,1}(x) = \varphi(x^+x)^{1/2}, \quad p_{\varphi,2}(x) = \varphi(x^+)\quad x \in M.
\]
Now (1.4) easily follows by using (1.2) and the relations
\[
p_{\varphi,1}(x) = \sup_{y \in M, \varphi(yy^*) \leq 1} \Re \varphi(y^*x), \quad n_{\varphi,2}(x) = \sup_{y \in M, \varphi(yy^*) \leq 1} \Re \varphi(xy^*), \quad x \in M.
\]

Let \(\varphi\) be a faithful, semi-finite, normal weight on a \(W^*\)-algebra \(M\). Defining, for \(a \in (M^e)^+\), the semi-finite, normal weight \(\varphi_a\) on \(M\) by
\[
(1.5) \quad \varphi_a(x) = \varphi(a^{1/2}xa^{1/2}), \quad x \in M^+,
\]
if \(A\) is a positive, self-adjoint operator affiliated to \(M^e\), then the formula
\[
\varphi_a(x) = \sup_{\epsilon > 0} \varphi_A(1+\epsilon A)^{-1}(x) = \lim_{\epsilon < 0 \to 0} \varphi_A(1+\epsilon A)^{-1}(x), \quad x \in M^+
\]
defines a \(\sigma^e\)-invariant, semi-finite, normal weight \(\varphi_A\) on \(M\) and
\[
(1.6) \quad \varphi_A^e(x) = A^t \varphi_A^e(x) A^{-t}, \quad t \in \mathbb{R}, x \in s(A)M s(A)
\]
(see [13], Section 4). We notice that the notation is not contradictory because for bounded \(A\) the above definition yields \(\varphi_A(x) = \varphi(A^{1/2}x A^{1/2})\) for all \(x \in M^+\).

Actually formula (1.6) defines a normal weight \(\varphi_a\) for any normal weight \(\varphi\) on \(M\) and any \(a \in M^+\), but if \(\varphi\) is not semi-finite and \(a\) does not belong to \(M^e\), we cannot be sure that the weight \(\varphi_a\) is semi-finite.

If \(\varphi\) is a faithful, semi-finite, normal weight on a \(W^*\)-algebra \(M\), and \(A, B\) are commuting positive, self-adjoint operators affiliated to \(M^e\), such that \(s(B) \leq s(A)\), then \(B\) is affiliated also to \(M^e\) and \((\varphi_A)B = \varphi_B\) (see [13], Proposition 4.3).

Similarly as in the case of weights, if \(N\) is a \(W^*\)-subalgebra of a \(W^*\)-algebra \(M\), and \(E : M^+ \to N^+\) an operator valued weight, then we use the notations
\[
(1.7) \quad \Omega_E = \{x \in M; E(x^+x) \in N^+\},
\]
\[
(1.8) \quad \mathfrak{M}_E = \text{linear span of } \Omega_E \mathfrak{M}_E
\]
and notice that
\[
\mathfrak{M}_E \cap M^+ = \{a \in M^+; E(a) \in N^+\}.
\]
Statement (11) holds true also for operator valued weights: If \( E : M^+ \longrightarrow \mathcal{N}^+ \)

is an operator valued weight and \( e \in M \) is a projection, then

\[
E(1_M - e) = 0 \implies E(x^*x) = E(ex^*xe) \quad \text{for all } x \in M.
\]

Indeed, by the very definition of the extended positive part of a \( W^* \)-algebra (11, Definition 1.1) and of operator valued weights (11, Definition 2.1), the equality of the operator valued weights \( E \) and \( E(e \cdot e) \) means the equality

\[
\varphi(E(x^*x)) = \varphi(E(ex^*xe)) \quad \text{for all } x \in M \text{ and all } \varphi \in M^*_+,
\]

that is the equality, for every \( \varphi \in M^*_+ \), of the weights \( \varphi \circ E \) and \( (\varphi \circ E)(e \cdot e). \) But this is an immediate consequence of (11).

The support \( s(E) \) of a normal operator valued weight \( E : M^+ \longrightarrow \mathcal{N}^+ \) is \( 1_M - q \), where \( q \) is the greatest projection in \( M \) satisfying \( E(q) = 0 \), and it belongs to the relative commutant \( N' \cap M \) (see [11], Definition 2.8).

If \( E : M^+ \longrightarrow \mathcal{N}^+ \) is a faithful, semi-finite, normal operator valued weight, then

- \( E(\mathfrak{M}_E) \) is a w-density two-sided ideal in \( N \) (11, Proposition 2.5).
- For any faithful, semi-finite, normal weight \( \varphi \) on \( N \), the composition \( \varphi \circ E \) is a faithful, semi-finite, normal weight on \( M \) (11, Proposition 2.3) and

\[
E(\sigma^\varphi\circ E(a)) = \sigma^\varphi(E(a)), \quad a \in M^+, t \in \mathbb{R}
\]

(11, Proposition 4.9). In particular, \( \sigma^\varphi\circ E(\mathfrak{M}_E) = \mathfrak{M}_E \) for every \( t \in \mathbb{R} \).

Moreover, \( \varphi \circ E \) determines uniquely \( E \) in the following sense:

- If \( E_1, E_2 : M^+ \longrightarrow \mathcal{N}^+ \) are faithful, semi-finite, normal weights such that \( \varphi \circ E_1 = \varphi \circ E_2 \) for some faithful, semi-finite, normal weight \( \varphi \) on \( N \), then \( E_1 = E_2 \) (11, Lemma 4.8).

2. Regularizing nets

Let \( M \) be a \( W^* \)-algebra, and \( \varphi \) a faithful, semi-finite, normal weight on \( M \). We will call (slightly differently as in [18], §1), regularizing net for \( \varphi \) any net \((u_n)\)_n in \( \mathfrak{T}_\varphi \) such that

\[
\begin{align*}
(i) & \sup_{z \in K} \| \sigma^\varphi_z(u_n) \| < +\infty \text{ and } \sup_{z \in K} \| \sigma^\varphi_z(u_n) \varphi \| < +\infty \text{ for each compact } K \subset \mathbb{C}; \\
(ii) & \sigma^\varphi_z(u_n) \xrightarrow{\text{s}^*} 1_M \text{ in the } s^*\text{-topology for all } z \in \mathbb{C}.
\end{align*}
\]

Regularizing nets are useful in the modular theory of faithful, semi-finite, normal weights. Usually they are constructed starting with a bounded net \((x_n)\)_n in \( \mathfrak{M}_\varphi \) such that \( x_n \xrightarrow{\text{s}^*} 1_M \) in the \( s^*\)-topology and then getting it ”mollified”, for example, by the mollifier \( e^{-t^2} \), that is passing to the net \((u_n)\)_n with

\[
u_n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} \sigma^\varphi_z(x_n) \, dt.
\]

The verification of (i) is straightforward, more troublesome is to verify the inclusion \( u_n \in \mathfrak{T}_\varphi \) and the convergence (ii).

Concerning the verification of (ii), if the net \((x_n)\)_n would be increasing, we could proceed as in the proof of [133], Lemma 5.2 by using Dini’s theorem. But there are situations in which we cannot restrict us to the case of increasing \((x_n)\)_n. For
example, it is not clear whether every $s^*$-dense, $\sigma^*$-invariant (not necessarily hereditary) $*$-subalgebra of $M$ contains some increasing net $(x_i)_i$ with $x_i \xrightarrow{\sigma} 1_M$ in the $s^*$-topology, as used in the proof of [13, Lemma 5.2].

On the other hand, if the net $(x_i)_i$ would be a sequence, then we could use the dominated convergence theorem of Lebesgue. similarly as, for example, in the proof of [13, Theorem 2.16. But again, unless $M$ is countably decomposable (and so its unit ball $s^*$-metrizable), the unit ball of not every $s^*$-dense $*$-subalgebra of $M$ contains a sequence $s^*$-convergent to $1_M$.

Therefore, in order to cover also the case of non countable nets $(x_i)_i$, we have to verify (ii) directly, taking advantage of the particularities of the situation.

In this section we will prove that, starting with a bounded net $(x_i)_i$ even in $\mathcal{N}_\varphi$, formula (2.3) furnishes a regularizing net $(u_i)_i$.

Let us begin with recalling some facts concerning the modular theory of faithful, semi-finite, normal weights. The next lemma is [2], (2.27):

**Lemma 2.1.** Let $\varphi$ be a faithful, semi-finite, normal weight on a $W^*$-algebra $M$. If $x \in \mathcal{N}_\varphi$ and $f \in L^1(\mathbb{R})$, then

$$\int_{-\infty}^{+\infty} f(t)\sigma^\varphi_t(x)dt \in \mathcal{N}_\varphi$$

and

$$\int_{-\infty}^{+\infty} f(t)\sigma^\varphi_t(x)dt = \int_{-\infty}^{+\infty} f(t)\Delta^\varphi_{x}\varphi dt.$$

Let $\varphi$ be a faithful, semi-finite, normal weight on a $W^*$-algebra $M$, and $z \in \mathbb{C}$.

We define the linear operator $\sigma^\varphi_x : M \supset \mathcal{D}(\sigma^\varphi_x) \ni x \mapsto \sigma^\varphi_x(x) \in M$ as follows:

the pair $(x, \sigma^\varphi_x(x))$ belongs to its graph whenever the map $\mathbb{R} \ni t \mapsto \sigma^\varphi_t(x) \in M$ has a $w$-continuous extension on the closed strip

$$\{\zeta \in \mathbb{C}; 0 \leq |\text{Im}\zeta| \leq |\text{Im}z|, (\text{Im}\zeta)(\text{Im}z) \geq 0\},$$

analytic in the interior and taking the value $\sigma^\varphi_x(x)$ at $z$.

It is easily seen (see e.g. [17, Theorem 1.6]) that, for each $z \in \mathbb{C}$,

$$\mathcal{D}(\sigma^\varphi_x^*) = \mathcal{D}(\sigma^\varphi_{x^*}) \quad \text{and} \quad \sigma^\varphi_x(x^*) = \sigma^\varphi_{x^*}(x^*) \quad \text{for every } x \in \mathcal{D}(\sigma^\varphi_x).$$

We recall that $x \in M$ belongs to $\mathcal{D}(\sigma^\varphi_x)$ if and only if the operator $\Delta^\varphi_{x} \pi^\varphi_{x}(x) \Delta^{-\varphi}_{-xz}$ is defined and bounded on a core of $\Delta^{-\varphi}_{-xz}$, in which case

$$\mathcal{D}(\Delta^\varphi_{x} \pi^\varphi_{x}(x) \Delta^{-\varphi}_{-xz}) = \mathcal{D}(\Delta^{-\varphi}_{-xz}) \quad \text{and} \quad \overline{\Delta^\varphi_{x} \pi^\varphi_{x}(x) \Delta^{-\varphi}_{-xz}} = \pi^\varphi_{x}(\sigma^\varphi_{x}(x))$$

that is

$$\pi^\varphi_{x}(x) \Delta^{-\varphi}_{-xz} \subset \Delta^{-\varphi}_{-xz} \pi^\varphi_{x}(\sigma^\varphi_{x}(x))$$

(see [3], Theorem 6.2 or [2], Theorem 2.3).

Now we prove a criterion for an element of $\mathcal{N}_\varphi$ to belong to $\mathcal{N}_\varphi$, hence to $\mathfrak{A}_\varphi$:

**Lemma 2.2.** Let $\varphi$ be a faithful, semi-finite, normal weight on a $W^*$-algebra $M$, and $x \in \mathcal{N}_\varphi$. Then

$$x \in \mathcal{D}(\sigma^\varphi_{-x}) \quad \text{and} \quad \sigma^\varphi_{-x}(x) \in \mathcal{N}_\varphi \implies x \in \mathcal{N}_\varphi \quad \text{and} \quad \sigma^\varphi_{-x}(x) = \Delta^\varphi_{x} x_\varphi,$$

that is

$$x \in \mathfrak{A}_\varphi \quad \text{and} \quad S_\varphi x_\varphi = J_\varphi \sigma^\varphi_{-x}(x)_\varphi.$$

**Proof.** Let $y \in \mathcal{N}_\varphi$ be arbitrary. Then

$$\pi^\varphi_*(x^*)J^\varphi y_\varphi = \pi^\varphi_*(x^*)J^\varphi_\varphi(S^\varphi(y^*)_\varphi) = \pi^\varphi_*(x^*) \Delta^{\varphi^2_\varphi}_x (y^*)_\varphi.$$
Application of (2.2) with $z = \frac{i}{2}$ yields $x^* \in D(\sigma_{\frac{3}{2}})$ and $\sigma_{\frac{3}{2}}(x^*) = \sigma_{\frac{3}{2}}(x)^*$, so, applying (2.3) to $x^*$ and $z = \frac{i}{2}$, we deduce

\[(2.5) \quad \pi(\sigma_{\frac{3}{2}}(x^*)^\phi) \Delta_{\phi}^{1/2} \subset \Delta_{\phi}^{1/2} \pi(\sigma_{\frac{3}{2}}(x^*)) = \Delta_{\phi}^{1/2} \pi(\sigma_{\frac{3}{2}}(x)^*).
\]

By (2.4) and (2.5) we conclude:

\[
\pi(\sigma_{\frac{3}{2}}(x^*)) J_{\phi} y_{\phi} = \Delta_{\phi}^{1/2} \pi(\sigma_{\frac{3}{2}}(x^*)) (y^*)_\phi = \Delta_{\phi}^{1/2} \pi(\sigma_{\frac{3}{2}}(x^*)) (y^*_\phi)
\]

\[
= J_{\phi} S_{\phi} \left( (y \sigma_{\frac{3}{2}}(x^*))^* \right) = J_{\phi} (y \sigma_{\frac{3}{2}}(x)^*)^* \phi 
\]

\[
= J_{\phi} \pi(\sigma_{\frac{3}{2}}(x)^*) \phi.
\]

By the above

\[\| \pi(\sigma_{\frac{3}{2}}(x^*)) J_{\phi} y_{\phi} \| \leq \| \sigma_{\frac{3}{2}}(x^*) \phi \| \cdot \| y \|, \quad y \in M_{\phi},\]

so we can apply [2], Lemma 2.6 (1) to deduce that $x^* \in \mathfrak{N}_\phi \iff x \in \mathfrak{N}_\phi^\phi$.

Taking into account that $x \in \mathfrak{A}_\phi$ and $y \in M_{\phi} \subset \mathfrak{A}_\phi$, and using [2], (2.5), as well as the above (2.3) with $z = \frac{i}{2}$, we deduce:

\[
\pi(\sigma_{\frac{3}{2}}(x^*)) J_{\phi} y_{\phi} = J_{\phi} \pi(\sigma_{\frac{3}{2}}(x)^*) J_{\phi} y_{\phi} = J_{\phi} \pi(\sigma_{\frac{3}{2}}(x)^*) J_{\phi} S_{\phi}(y^*) \phi 
\]

\[
= J_{\phi} \pi(\sigma_{\frac{3}{2}}(x)^*) \Delta_{\phi}^{1/2} (y^*) \phi = J_{\phi} \Delta_{\phi}^{1/2} \pi(\sigma_{\frac{3}{2}}(x)^*) (y^*) \phi 
\]

\[
= S_{\phi}(y x^*) \phi = (y x^*) \phi = \pi(\sigma_{\frac{3}{2}}(x)^*) \phi = \pi(\sigma_{\frac{3}{2}}(x)^*) \phi
\]

\[
= \pi(\sigma_{\frac{3}{2}}(x)^*) \phi = \Delta_{\phi}^{1/2} \phi.
\]

Since $\pi(\mathfrak{M}_\phi)$ is $\omega$-dense in $M$, it follows the equality $\sigma_{\frac{3}{2}}(x^*)^\phi = \Delta_{\phi}^{1/2} \phi$. 

The above two lemmas can be used to produce elements of the Tomita algebra $\mathfrak{T}_\phi$ by "regularizing" elements of $\mathfrak{N}_\phi$ (not only elements of $\mathfrak{A}_\phi$, as customary: see in [15] the comments after the proof of Theorem 10.20 on page 347):

**Lemma 2.3.** Let $\phi$ be a faithful, semi-finite, normal weight on a $W^*$-algebra $M$. For each $x \in M$,

\[u = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} \sigma_{\frac{1}{2}}(x) dt
\]

belongs to $M_{\infty}^x$ and

\[(2.6) \quad \sigma_{\frac{1}{2}}(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(t-z)^2} \sigma_{\frac{1}{2}}(x) dt, \quad z \in \mathbb{C}.
\]

**Assuming that** $x \in \mathfrak{N}_\phi$, we have $u \in \mathfrak{T}_\phi$.

**Proof.** Since

\[
\Re \ni s \mapsto \sigma_{\frac{1}{2}}(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} \sigma_{\frac{1}{2}}(x) dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} \sigma_{\frac{1}{2}}(x) dt
\]

allows the entire extension.
Proof.

(i) is an immediate consequence of Lemma 2.3.

Then

\[ \sigma(z) = \sigma(z)_{\|u\|}, \quad z \in \mathbb{C}. \]

Using (2.8) it is easy to see that

\[ \sigma^z_t(\sigma^z_t(u)) = \sigma^z_{t+1}(u), \quad z \in \mathbb{C}, t \in \mathbb{R}, \]

so

\[ (2.7) \quad \sigma^z_t(u) \in M^z_t \quad \text{and} \quad \sigma^z_t(\sigma^z_t(u)) = \sigma^z_{t+\zeta}(u), \quad z, \zeta \in \mathbb{C}, \]

For each \( z \in \mathbb{C} \), applying Lemma 2.1 with \( f(t) = \frac{1}{\sqrt{\pi}} e^{-(t-z)^2} \), we deduce that \( \sigma^z_t(u) \in \mathcal{N}_\varphi \). Since \( z \in \mathbb{C} \) is here arbitrary, also \( \sigma^z_{t-i/2}(u) \in \mathcal{N}_\varphi \) holds true. But by (2.7) we have \( \sigma^z_{t-i/2}(\sigma^z_t(u)) = \sigma^z_{t-i/2}(u) \), so \( \sigma^z_{t-i/2}(\sigma^z_t(u)) \in \mathcal{N}_\varphi \). Applying now Lemma 2.2 we conclude that \( \sigma^z_t(u) \) belongs also to \( \mathcal{N}_\varphi^c \), hence \( \sigma^z_t(u) \in \mathcal{N}_\varphi^c \).

□

Next we prove a dominated convergence theorem for integrals of the form (2.6) and nets of arbitrary cardinality:

Lemma 2.4. Let \( \varphi \) be a faithful, semi-finite, normal weight on a \( W^* \)-algebra \( M \), and \( (x_i) \), a net in the closed unit ball of \( M \) such that \( x_i \rightharpoonup 1_M \) in the \( s^* \)-topology. Let the net \( (u_i) \), be defined by the formula

\[ u_i = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} \sigma^z_t(x_i) \, dt. \]

Then

(i) \( u_i \in M^z_t \) for all \( i \);

(ii) \( \|\sigma^z_t(u_i)\| \leq e^{(1+m)^2} \) for all \( i \) and \( z \in \mathbb{C} \);

(iii) \( \sigma^z_t(u_i) \rightharpoonup 1_M \) in the \( s^* \)-topology for all \( z \in \mathbb{C} \).

Proof. (i) is an immediate consequence of Lemma 2.3.

For (ii), let \( i \) and \( z \in \mathbb{C} \) be arbitrary. By Lemma 2.3 we have

\[ (2.8) \quad \sigma^z_t(u_i) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(t-z)^2} \sigma^z_t(x_i) \, dt. \]

Since \( \|\sigma^z_t(x_i)\| = \|x_i\| \leq 1 \) for all \( t \in \mathbb{R} \), it follows

\[ \|\sigma^z_t(u_i)\| \leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(t-z)^2} \, dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(t-\text{Re}(z))^2 + (1+m)^2} \, dt = e^{(1+m)^2}. \]

The more involved issue is (iii). For fixed \( z \in \mathbb{C} \), we have to show that

\[ \sigma^z_t(u_i) - 1_M \rightharpoonup 1_M \quad \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(t-z)^2} \sigma^z_t(x_i - 1_M) \, dt \rightharpoonup 0 \]

in the \( s^* \)-topology. Since the \( s^* \)-topology is defined by the semi-norms
\( p_\psi : M \ni x \rightarrow \psi(x^*x)^{1/2} + \psi(x^*)^{1/2} \), \( \psi \) a normal positive form on \( M \),
this means that
\[
p_\psi \left( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{e^2}{2}} \sigma_t^\psi (x_t - 1_M) \, dt \right) \xrightarrow{t \to 0} 0
\]
for every normal positive form \( \psi \) on \( M \).

For let \( \psi \) be any normal positive form on \( M \). Since, according to (1.3),
\[
p_\psi \left( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{e^2}{2}} \sigma_t^\psi (x_t - 1_M) \, dt \right) \leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \psi(e^{-\frac{e^2}{2}} \sigma_t^\psi (x_t - 1_M)) \, dt
\]
the proof is done if we prove the convergence
\[
\int_{-\infty}^{+\infty} \left| e^{-\frac{e^2}{2}} \sigma_t^\psi (x_t - 1_M) \right| \, dt \xrightarrow{t \to 0} 0,
\]
which of course is consequence of
\[
(2.9) \quad \int_{-\infty}^{+\infty} e^{-\frac{e^2}{2}} \psi((x_t - 1_M)^*(x_t - 1_M) + (x_t - 1_M)(x_t - 1_M)^*) \, dt \xrightarrow{t \to 0} 0,
\]
because \( |e^{-\frac{e^2}{2}}| = e^{-\frac{e^2}{2} + (\text{Im} e)^2} \) and
\[
p_\psi(\sigma_t^\psi (x_t - 1_M)) = (\psi \circ \sigma_t^\psi)((x_t - 1_M)^*(x_t - 1_M) + (x_t - 1_M)(x_t - 1_M)^*)^{1/2}
\]
\[
\leq \sqrt{2}(\psi \circ \sigma_t^\psi)\left((x_t - 1_M)^*(x_t - 1_M) + (x_t - 1_M)(x_t - 1_M)^*\right)^{1/2}
\]
We go to complete the proof by verifying (2.9).

Since \( x_t \xrightarrow{t \to 0} 1_M \) in the \( s^* \)-topology and \( \|x_t\| \leq 1 \) for all \( t \), we have that
\[
\left((x_t - 1_M)^*(x_t - 1_M) + (x_t - 1_M)(x_t - 1_M)^*\right)
\]
is a bounded net, convergent to 0 in the \( s^* \)-topology. According to a theorem due to C. A. Akemann (see \cite{1}, Theorem II.7 or \cite{16}, Corollary 8.17), on bounded subsets of \( M \) the \( s^* \)-topology coincides with the Mackey topology \( \tau_w \) associated to the \( w \)-topology, that is with the topology of the uniform convergence on the weakly compact absolutely convex subsets of the predual \( M_s \). Since, by the classical Krein-Šmulian theorem (see e.g. \cite{9}, Theorem V.6.4), the closed absolutely convex hull of every weakly compact set in a Banach space is still weakly compact, \( \tau_w \) is actually the topology of the uniform convergence on the weakly compact subsets of \( M_s \). Therefore
\[
(2.10) \quad \sup_{\theta \in K} \left| \theta((x_t - 1_M)^*(x_t - 1_M) + (x_t - 1_M)(x_t - 1_M)^*)\right| \xrightarrow{t \to 0} 0
\]
for every weakly compact \( K \subset M_s \).
Now let $\varepsilon > 0$ be arbitrary. Choose some $t_0 > 0$ such that
\begin{equation}
(2.11) \quad \int_{|t| > t_0} e^{-(t - \operatorname{Re} z)^2} dt \leq \frac{\varepsilon}{4 \sqrt{2} \|\psi\|}.
\end{equation}
Since $K_{t_0} = \{\psi \circ \sigma_t^r: |t| \leq t_0\}$ is a weakly compact subset of $M_*$, (2.10) holds true with $K = K_{t_0}$. Thus there exists some $t_0$ such that
\begin{equation}
(2.12) \quad \sup_{|t| \leq t_0} \left| (\psi \circ \sigma_t^r) \left( (x_i - 1_M)^* (x_i - 1_M) + (x_i - 1_M) (x_i - 1_M)^* \right) \right| \leq \frac{\varepsilon}{2 \sqrt{\pi}}
\end{equation}
for all $t \geq t_0$. (2.11) implies
\begin{align*}
\int_{|t| > t_0} e^{-(t - \operatorname{Re} z)^2} \left( \psi \circ \sigma_t^r \right) \left( (x_i - 1_M)^* (x_i - 1_M) + (x_i - 1_M) (x_i - 1_M)^* \right)^{1/2} dt & \\
\leq \int_{|t| > t_0} e^{-(t - \operatorname{Re} z)^2} \left( 8 \|\psi\| \right)^{1/2} dt \leq \frac{\varepsilon}{4 \sqrt{2} \|\psi\|} \left( 8 \|\psi\| \right)^{1/2} = \frac{\varepsilon}{2},
\end{align*}
while using (2.12) we deduce for every $t \geq t_0$:
\begin{align*}
\int_{|t| \leq t_0} e^{-(t - \operatorname{Re} z)^2} \left( \psi \circ \sigma_t^r \right) \left( (x_i - 1_M)^* (x_i - 1_M) + (x_i - 1_M) (x_i - 1_M)^* \right)^{1/2} dt & \\
\leq \int_{|t| \leq t_0} e^{-(t - \operatorname{Re} z)^2} \frac{\varepsilon}{2 \sqrt{\pi}} dt \leq \frac{\varepsilon}{2 \sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(t - \operatorname{Re} z)^2} dt = \frac{\varepsilon}{2 \sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} dt = \frac{\varepsilon}{2}.
\end{align*}
Consequently, for every $t \geq t_0$,
\begin{align*}
\int_{-\infty}^{+\infty} e^{-(t - \operatorname{Re} z)^2} \left( \psi \circ \sigma_t^r \right) \left( (x_i - 1_M)^* (x_i - 1_M) + (x_i - 1_M) (x_i - 1_M)^* \right)^{1/2} dt & \\
= \int_{|t| > t_0} + \int_{|t| \leq t_0} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{align*}
\[\square\]

Remark 2.5. The proof of statement (iii) in Lemma 2.4 was rather cumbersome, we could not avoid to use the deep result of Akemann concerning $s^*$-topology.

As we noticed at the beginning of this section, if $(x_i)_{i}$ would be an increasing net of positive elements, then we could make use of Dini’s theorem, as it was done in the proof of [13], Lemma 5.2. On the other hand, if the net $(x_i)_{i}$ would be a sequence, then we could use the Lebesgue dominated convergence theorem.

Lemmas 2.4 and 2.8 yield immediately:

**Theorem 2.6.** Let $\varphi$ be a faithful, semi-finite, normal weight on a $W^*$-algebra $M$, and $(x_i)_{i}$ a net in the closed unit ball of $M$ such that $x_i \xrightarrow{s^*} 1_M$ in the $s^*$-topology. Let the net $(u_i)_{i}$ be defined by the formula
\begin{equation*}
u_i = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} \sigma_t^r (x_i) dt.
\end{equation*}
Then

(i) \( u_\iota \in M_\infty^\infty \) for all \( \iota \);
(ii) \( \| \sigma_\zeta(u_\iota) \| \leq e^{(\Im z)^2} \) for all \( \iota \) and \( z \in \mathbb{C} \);
(iii) \( \sigma_\zeta(u_\iota) \rightharpoonup 1_M \) in the \( * \)-topology for all \( z \in \mathbb{C} \).

Moreover, if \( x_\iota \in N_\varphi \) for all \( \iota \), then \( u_\iota \) belongs to \( T_\iota \) for every \( \iota \) and therefore \( (u_\iota)_\iota \) is a regularizing net for \( \varphi \).

\[ \square \]

3. Inequalities and equality between weights

The main tool by proving criteria for inequalities and equalities between weights is the following generalization of \([18],\) Lemma 2.1.

We recall that a \( * \)-subalgebra \( \mathcal{M} \) of a \( W^* \)-algebra \( M \) is called facial subalgebra or hereditary subalgebra whenever \( \mathcal{M} \cap M^+ \) is a face, that is a convex cone satisfying

\[ M^+ \ni b \leq a \in \mathcal{M} \cap M^+ \implies b \in \mathcal{M} \cap M^+ , \]

and \( \mathcal{M} \) is the linear span of it (see e.g. \([15],\) Section 3.21).

**Theorem 3.1.** Let \( M \) be a \( W^* \)-algebra, \( \varphi \) a faithful, semi-finite, normal weight on \( M \), \( a \in (M^\varphi)^+ \), and \( \psi \) a normal weight on \( M \). Assume that there exists a \( w \)-dense, \( \sigma^\varphi \)-invariant \( * \)-subalgebra \( \mathcal{M}_a \) of \( M_\varphi^\varphi \) such that

\[ \psi(x^* x) = \varphi_a(x^* x), \quad x \in \mathcal{M}. \]

Then

\[ (3.1) \quad \psi \leq \varphi_a. \]

Additionally, there exists a \( \sigma^\varphi \)-invariant, hereditary \( * \)-subalgebra \( \mathcal{M}_0 \) of \( M_\varphi^\varphi \) such that

\[ \mathcal{M} \cap M^+ \subset \mathcal{M}_0 \cap M^+, \]

\[ \psi(b) = \varphi_a(b), \quad b \in \mathcal{M}_0 \cap M^+. \]

The difference between the above Theorem \( 3.1 \) and \([18],\) Lemma 2.1 consists in the fact that in \([18],\) Lemma 2.1 is additionally assumed that

(i) \( \psi \) is semi-finite and \( \sigma^\varphi \)-invariant and
(ii) \( \mathcal{M} \) is contained already in \( M_\varphi^\varphi \) (which of course, according to \([13],\) Theorem 3.6, is a subset of \( M_\varphi^\varphi \)).

However the proof of \([18],\) Lemma 2.1 does not use assumption (i) and, on the other hand, we can adapt it to work with the assumption \( \mathcal{M} \subset M_\varphi^\varphi \) instead of \( \mathcal{M} \subset \mathcal{M}_\varphi^\varphi \).

**Proof.** Let \( x \in \mathcal{M} \subset M_\varphi^\varphi \) be arbitrary. Since \( \psi(x^* x) = \varphi_a(x^* x) < +\infty \), we have \( x \in \mathcal{M}_0 \cap M_\varphi^\varphi \) and therefore \( \psi(x^* \cdot x) \) and \( \varphi_a(x^* \cdot x) \) are normal positive forms on \( M \). Taking into account that

\[ \psi(x^* y^* yx) = \varphi_a(x^* y^* yx), \quad y \in \mathcal{M} \]

and \( \mathcal{M} \) is \( w \)-dense in \( M \), we deduce that

\[ (3.2) \quad \psi(x^* \cdot x) = \varphi_a(x^* \cdot x). \]
Now, by the Kaplansky density theorem there exists a net \((a_i)\), in \(\mathcal{M}\) such that 
\(0 \leq a_i \leq 1_M\) for all \(i\) and \(a_i \xrightarrow{s^*} 1_M\). Set, for each \(i\),
\begin{equation}
(3.3) \quad u_i = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} \sigma^*_i(a_i) dt \in M^+.
\end{equation}

Clearly, \(0 \leq u_i \leq 1_M\) for all \(i\). According to Lemma 2.4, \(u_i \in M^{\mathcal{E}}_\infty\) for all \(i\) and
\begin{equation}
(3.4) \quad \sigma^*_z(u_i) \xrightarrow{\mathcal{E}} 1_M \text{ in the } s^*\text{-topology for all } z \in \mathbb{C}.
\end{equation}

Since \(a^{1/2} \in M^\mathcal{E}\), also
\begin{equation}
(3.5) \quad u_i a^{1/2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} \sigma^*_i(a_i) a^{1/2} dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} \sigma^*_i(a_i a^{1/2}) dt
\end{equation}
belongs to \(M^{\mathcal{E}}_\infty\) for each \(i\). Furthermore, \(a_i \in \mathcal{M} \subset \mathcal{M}_{\varphi}\) yields
\(\varphi((a_i a^{1/2})^*(a_i a^{1/2})) = \varphi(a_i^2) < +\infty\),
hence \(a_i a^{1/2} \in \mathcal{M}_{\varphi}\). Taking into account (3.5) and applying Lemma 2.3, we deduce that \(u_i a^{1/2} \in \mathcal{M}_{\varphi}\) for all \(i\).

Let \(y \in \mathcal{M}\) and \(\epsilon\) be arbitrary. Since \(a_i \in \mathcal{M}\) and \(\mathcal{M}\) is \(\sigma^\varphi\)-invariant, application of (3.2) yields for every \(t, s \in \mathbb{R}\) and \(k = 0, 1, 2, 3:\)
\begin{align*}
\psi \left( (\sigma^\varphi_i(a_i) + i^k \sigma^\varphi_s(a_i))^* y (\sigma^\varphi_i(a_i) + i^k \sigma^\varphi_s(a_i)) \right) \\
= \varphi_a \left( (\sigma^\varphi_i(a_i) + i^k \sigma^\varphi_s(a_i))^* y (\sigma^\varphi_i(a_i) + i^k \sigma^\varphi_s(a_i)) \right).
\end{align*}

Applying (1.2) with 
\(F(t, s) = \frac{1}{\pi} e^{-t^2 - s^2} (\sigma^\varphi_i(a_i) + i^k \sigma^\varphi_s(a_i))^* y (\sigma^\varphi_i(a_i) + i^k \sigma^\varphi_s(a_i))\),
it follows for \(k = 0, 1, 2, 3:\)
\begin{align*}
\psi \left( \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-t^2 - s^2} (\sigma^\varphi_i(a_i) + i^k \sigma^\varphi_s(a_i))^* y (\sigma^\varphi_i(a_i) + i^k \sigma^\varphi_s(a_i)) dt ds \right) \\
= \varphi_a \left( \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-t^2 - s^2} (\sigma^\varphi_i(a_i) + i^k \sigma^\varphi_s(a_i))^* y (\sigma^\varphi_i(a_i) + i^k \sigma^\varphi_s(a_i)) dt ds \right).
\end{align*}

Since, by (3.3),
\[
\psi(u_i y^* y u_i) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-t^2 - s^2} \sigma^\varphi_i(a_i) y y^* \sigma^\varphi_i(a_i) dt ds
\]
\[
= \frac{1}{4} \sum_{k=0}^{3} i^k \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-t^2 - s^2} (\sigma^\varphi_i(a_i) + i^k \sigma^\varphi_s(a_i))^* y y^* (\sigma^\varphi_i(a_i) + i^k \sigma^\varphi_s(a_i)) dt ds,
\]
we conclude that
\begin{equation}
(3.6) \quad \psi(u_i y^* y u_i) = \varphi_a(u_i y^* y u_i).
\end{equation}
Next let $y \in \mathcal{N}_c$ be arbitrary. Using (3.6) and applying [6], Lemma 7 (b) or [18], Proposition 1.1, we deduce for every $i$

$$\psi(u_i y^* y u_i) = \varphi_\alpha(u_i y^* y u_i) = \varphi(a^{1/2} u_i y^* y u_i a^{1/2}) = \|(y u_i a^{1/2})_\varphi\|^2$$

$$= \|J_\varphi \pi_\varphi(\sigma^-_1(a^{1/2} u_i)) J_\varphi y_\varphi\|^2 = \|J_\varphi \pi_\varphi(a^{1/2}) \pi_\varphi(\sigma^-_1(u_i)) J_\varphi y_\varphi\|^2$$

Since $u_i y^* y u_i \rightarrow y^* y$ and $\sigma^-_1(u_i) \rightarrow 1_M$ in the $s^*$-topology, and $\psi$ is lower semicontinuous in the $s^*$-topology, we get

$$\psi(y^* y) \leq \lim_{i \to \infty} \psi(u_i y^* y u_i) = \lim_{i \to \infty} \|J_\varphi \pi_\varphi(a^{1/2}) \pi_\varphi(\sigma^-_1(u_i)) J_\varphi y_\varphi\|^2$$

$$= \|J_\varphi \pi_\varphi(a^{1/2}) J_\varphi y_\varphi\|^2.$$ 

Applying now [18], Corollary 1.2, we conclude:

$$(3.7) \quad \psi(y^* y) \leq \|(ya^{1/2})_\varphi\|^2 = \varphi(a^{1/2} y^* y a^{1/2}) = \varphi_\alpha(y^* y).$$

In order to have (3.1) proved, we should show that (3.7) actually holds for every $y \in \mathcal{N}_{\varphi_\alpha}$. This follows by repeating word for word the corresponding part of the proof of [18], Lemma 2.1. We report it for sake of completeness.

For every $y \in \mathcal{N}_c$, since $s(a) \in M^\varphi$ and $\mathcal{N}_c M^\varphi \subset \mathcal{N}_c$, (3.7) yields

$$\psi((1_M - s(a)) y^* y(1_M - s(a))) \leq \varphi(a^{1/2} (1_M - s(a)) y^* y(1_M - s(a)) a^{1/2}) = 0.$$ 

$\mathcal{N}_c$ being $\psi$-dense in $M$, it follows $\psi((1_M - s(a))) = 0$, what means $s(\psi) \leq s(a)$.

For $n \geq 1$ we consider the projection $e_n = \chi_{[1/n, +\infty)}(a) \in M^\varphi$, where $\chi_{[1/n, +\infty)}$ stands for the characteristic function of $[1/n, +\infty)$. Then $e_n \not\in s(a)$. We consider also the inverse $b_n$ of $a^{1/2} e_n$ in the reduced algebra $e_n M^\varphi e_n$: $b_n = f_n(a) \in M^\varphi$ with $f_n(t) = \frac{1}{\sqrt{t}} \chi_{[1/n, +\infty)}(t)$.

Now let $y \in \mathcal{N}_{\varphi_\alpha}$ be arbitrary. Then $ya^{1/2} \in \mathcal{N}_c$, so

$$y e_n = (ya^{1/2}) b_n \in \mathcal{N}_c M^\varphi \subset \mathcal{N}_c, \quad n \geq 1.$$ 

Applying (3.7) and [18], Corollary 1.2, we obtain for every $n \geq 1$

$$\psi(e_n y^* y e_n) \leq \varphi(a^{1/2} e_n y^* y e_n a^{1/2}) = \|(y e_n a^{1/2})_\varphi\|^2 = \|(ya^{1/2} e_n)_\varphi\|^2$$

$$= \|J_\varphi \pi_\varphi(e_n) J_\varphi(ya^{1/2})_\varphi\|^2.$$ 

Since $s(\psi) \leq s(a), e_n \not\in s(a)$ and $\psi$ is lower semicontinuous in the $s^*$-topology, it follows

$$\psi(y^* y) = \psi((ya^{1/2} s(a))_\varphi) = \lim_{n \to \infty} \psi(e_n y^* y e_n)$$

$$\leq \lim_{n \to \infty} \|J_\varphi \pi_\varphi(e_n) J_\varphi(ya^{1/2})_\varphi\|^2$$

$$= \|J_\varphi \pi_\varphi((ya^{1/2})_\varphi\|^2.$$ 

Applying [18], Corollary 1.2 again, we conclude:

$$\psi(y^* y) \leq \|(ya^{1/2} s(a))_\varphi\|^2 = \|(ya^{1/2})_\varphi\|^2 = \varphi(a^{1/2} y^* y a^{1/2}) = \varphi_\alpha(y^* y).$$

To complete the proof of the theorem, we have to find a $\sigma^\varphi$-invariant, hereditary *-subalgebra $\mathcal{M}_0$ of $\mathcal{M}_{\varphi_\alpha}$ such that

$$\mathcal{M} \cap M^+ \subset \mathcal{M}_0 \cap M^+,$$

$$\psi(b) = \varphi_\alpha(b), \quad b \in \mathcal{M}_0 \cap M^+.$$ 

For we first notice:
(i) \( \{ b \in M \cap M^+; \psi(b) = \varphi_a(b) \} \subset M \) is a face.
(ii) \( \psi(b) = \varphi_a(b) \) for all \( b \in M \cap M^+ \).

Since \( \{ b \in M \cap M^+; \psi(b) = \varphi_a(b) \} \) is clearly a convex cone, for (i) we have only to verify the implication
\[
M^+ \ni b \leq c \in M \cap M^+, \quad \psi(c) = \varphi_a(c) \implies \psi(b) = \varphi_a(b).
\]

It follows immediately by using
\[
\psi(b) \leq \varphi_a(b), \quad \psi(c - b) \leq \varphi_a(c - b),
\]
\[
\psi(b) + \psi(c - b) = \psi(c) = \varphi_a(c) = \varphi_a(b) + \varphi_a(c - b) < +\infty.
\]

For (ii) let \( b \in M \cap M^+ \) be arbitrary. Without loss of generality we can assume that \( \|b\| \leq 1 \). Denoting \( b_n := 1_M - (1_M - b)^* \in M \cap M^+, n \geq 1 \), we obtain an increasing sequence \( (b_n)_{n \geq 1} \) which is \( s^* \)-convergent to the support \( s(b) \) of \( b \) (see e.g. \cite{15}, Section 2.22). Since all \( b_n \) belong to the commutative \( C^* \)-subalgebra of \( M \) generated by \( b \), the sequence \( (b_n b_n)_{n \geq 1} \) is still increasing and it is \( s^* \)-convergent to \( b \). Now we deduce successively:

- \( \psi(b_n b_n) = \varphi_a(b_n b_n) \) for all \( n \geq 1 \) by the assumption on \( M \);
- \( \psi(b_n b_n b_n) = \varphi_a(b_n b_n b_n) \) for all \( n \geq 1 \) by applying (3.8) with \( b = b_n b_n \) and \( c = b_n b_n \);
- \( \psi(b) = \lim_{n \to \infty} \psi(b_n b_n) = \lim_{n \to \infty} \varphi_a(b_n b_n) = \varphi_a(b) \) by the normality of \( \psi \) and \( \varphi_a \).

Now we set
\[
\mathfrak{F}_0 := \{ b \in M \cap M^+; 0 \leq b \leq c \text{ for some } c \in M \cap M^+ \},
\]
\[
\mathfrak{M}_0 := \{ x \in M; x^* x \in \mathfrak{F}_0 \},
\]
\[
M_0 := \text{linear span of } \mathfrak{M}_0 \mathfrak{M}_0.
\]

Then \( \mathfrak{F}_0 \) is a face, \( \mathfrak{M}_0 \) is a \( * \)-subalgebra of \( M \), \( \mathfrak{M}_0 \cap M^+ = \mathfrak{F}_0 \), and \( \mathfrak{M}_0 \) is the linear span of \( \mathfrak{F}_0 \) (see e.g. \cite{15}, Proposition 3.21). Thus \( \mathfrak{M}_0 \) is a hereditary \( * \)-subalgebra of \( \mathfrak{M}_a \) and \( \mathfrak{M} \cap M^+ \subset \mathfrak{F}_0 = \mathfrak{M}_0 \cap M^+ \). Since \( \mathfrak{M} \cap M^+ \) is \( \sigma^* \)-invariant, also \( \mathfrak{F}_0 \), and therefore \( \mathfrak{M}_0 \) is \( \sigma^* \)-invariant. Finally, the above (ii) and (i) imply that we have \( \psi(b) = \varphi_a(b) \) for all \( b \in \mathfrak{F}_0 \).

\[\square\]

**Remark 3.2.** If \( a \) is assumed only affiliated to \( M^e \) and not necessarily bounded, the statement of Theorem [3.1] is not more true. Counterexamples can be obtained using [13], Proposition 7.8 or [6], Exemple 8.

In both papers two faithful, semi-finite, normal weights \( \psi_0, \psi \) are constructed on \( B(\ell^2) \) such that
\[
\psi_0 \leq \psi \text{ and } \psi_0 \neq \psi, \text{ but } \psi_0(x) = \psi(x) \text{ for } x \in M \cap M^+,
\]
where \( M \) is a \( \sigma \)-dense \( * \)-subalgebra of \( M_\psi \) (in [6], Exemple 8, the construction delivers \( M = M_\psi \)).

Now let \( \varphi \) be a faithful, semi-finite, normal trace on \( B(\ell^2) \). By [13], Theorem 5.12 there exists a positive, self-adjoint operator \( A \) on \( \ell^2 \), necessarily affiliated to \( B(\ell^2)^e = B(\ell^2) \), such that \( \psi_0 = \varphi_A \). Then

- \( \varphi \) is a faithful, semi-finite, normal trace on \( M = B(\ell^2) \),
- \( A \) is a positive, self-adjoint operator affiliated to \( M^e = B(\ell^2) \),
- \( \psi \) is a \( \sigma^* \)-invariant, faithful, semi-finite, normal weight on \( M \).
Using now the inequalities
\begin{equation}
\phi(x^*x) = \phi_A(x^*x) \quad \text{for } x \in \mathcal{M}, \text{ where } \mathcal{M} \text{ is a } w\text{-dense } \ast\text{-subalgebra of } \mathcal{M}_\psi \subset \mathcal{M}_\psi = \mathcal{M}_{\psi_A},
\end{equation}
but \( \psi \not\leq \phi_A \), because otherwise it would follow \( \psi \leq \phi_A = \psi_0 \), hence \( \phi = \psi_0 \), in contradiction to \( \psi \neq \psi_0 \).

Remark 3.3. If in Theorem 3.1 it is additionally assumed that \( 1_M - s(\psi) \) belongs to the \( w \)-closure of \( \{ y \in \mathcal{M}_\phi : y s(\psi) = 0 \} \) (that happens, for example, if \( s(\psi) \in M_\phi^* \), because \( \mathcal{M}_\phi M_\phi^* \subset \mathcal{M}_\phi \)), then it follows also the equality \( s(\psi) = s(a) \).

Since \( s(\psi) \leq s(a) \) trivially, we have to verify that for any \( y \in \mathcal{M}_\phi \) with \( y s(\psi) = 0 \), that is with \( \psi(y^*y) = 0 \), we have \( y s(a) = 0 \). We can argue as in the proof of [13], Lemma 5.2:

By (3.4), by the lower semicontinuity of \( \phi_a \) in the \( s^* \)-topology, and by (3.3), we have
\begin{equation}
\phi_a(y^*y) \leq \lim_{i} \phi_a(u_i y^* y u_i) = \lim_{i} \psi(u_i y^* y u_i).
\end{equation}
Using now the inequalities
\begin{align*}
u_i y^* y u_i &\leq (2 \cdot 1_M - u_i) y^* y (2 \cdot 1_M - u_i) + u_i y^* y u_i \\
&= 2 \left( (1_M - u_i) y^* y (1_M - u_i) + y^* y \right),
\end{align*}
and \( \psi \leq \phi_a \), as well as [6], Lemma 7 (b) or [18], Proposition 1.1, we obtain
\begin{align*}
\phi_a(y^*y) &\leq 2 \lim_{i} \psi \left( (1_M - u_i) y^* y (1_M - u_i) \right) \\
&\leq 2 \lim_{i} \phi_a \left( (1_M - u_i) y^* y (1_M - u_i) \right) = 2 \lim_{i} \| (y (1_M - u_i) a^{1/2}) \phi \|^2 \\
&= 2 \lim_{i} \| J_\phi \pi_\phi (\sigma_{i/2}^\varphi (a^{1/2} (1_M - u_i))) J_\phi y \phi \|^2 \\
&= 2 \lim_{i} \| J_\phi \pi_\phi (a^{1/2}) \pi_\phi (1_M - \sigma_{i/2}^\varphi (u_i)) J_\phi y \phi \|^2.
\end{align*}
Since, by (3.4), \( \sigma_{i/2} (u_i) \xrightarrow{i} 1_M \) in the \( s^* \)-topology, we conclude that \( \phi_a(y^*y) = 0 \),
what is equivalent to \( ya^{1/2} = 0 \iff y s(a) = 0 \).

The next theorem is a slight extension of [18], Theorem 2.3:

Theorem 3.4. Let \( M \) be a \( W^* \)-algebra, \( \phi \) a faithful, semi-finite, normal weight on \( M \), \( a \in (M^*)^+ \), and \( \psi \) a \( \sigma^* \)-invariant, normal weight on \( M \). If there exists a \( w \)-dense, \( \sigma^* \)-invariant \( \ast \)-subalgebra \( \mathcal{M} \) of \( \mathcal{M}_{\phi_a} \) such that
\begin{equation}
\psi(x^*x) = \phi_a(x^*x), \quad x \in \mathcal{M},
\end{equation}
then
\begin{equation}
\psi = \phi_a.
\end{equation}

Proof. By Theorem 5.1 we have \( \psi \leq \phi_a \). In particular, \( \psi \) is semi-finite.

On the other hand, by [13], Theorem 5.12 there exists a positive, self-adjoint operator \( A \), affiliated to \( M^* \), such that \( \psi = \phi_A \). Since \( \phi_A = \psi \leq \phi_a \), Lemma [13], 2.2 yields \( A \leq a \). In particular, \( A \) is bounded.

Since \( \mathcal{M}_{\phi_A} \) is the linear span of \( \{ b \in M^+ : \phi_A(b) < +\infty \} \), \( \mathcal{M}_{\phi_a} \) is the linear span of \( \{ b \in M^+ : \phi_a(b) < +\infty \} \), and \( \phi_a \leq \phi_a \), we have \( \mathcal{M} \subset \mathcal{M}_{\phi_a} \subset \mathcal{M}_{\phi_A} \). Therefore we can apply Theorem 3.1 again and deduce that \( \phi_a \leq \phi_A = \psi \).

□
An equivalent, but slightly more symmetric form of Theorem 3.4 is

**Theorem 3.5.** Let $M$ be a $W^*$-algebra, $\varphi$ a faithful, semi-finite, normal weight on $M$, $a, b \in (M^\varphi)^+$, and $\psi$ a $\sigma^\varphi$-invariant, normal weight on $M$. If there exists a $w$-dense, $\sigma^\varphi$-invariant $*$-subalgebra $\mathcal{M}$ of $\mathcal{M}_{\varphi_a}$ such that

$$\psi_b(x^*x) = \varphi_a(x^*x), \quad x \in \mathcal{M},$$

then

$$\psi_b = \varphi_a.$$

**Proof.** Since $\psi$ is $\sigma^\varphi$-invariant and $b \in (M^\varphi)^+$, the normal weight $\psi_b$ is still $\sigma^\varphi$-invariant: we have for every $t \in \mathbb{R}$ and every $x \in M^+$

$$\psi_b(\sigma_t^\varphi(x)) = \psi(b^{1/2} \sigma_t^\varphi(x)b^{1/2}) = \psi(\sigma_t^\varphi(b^{1/2}xb^{1/2})) = \psi(b^{1/2}xb^{1/2}) = \psi_b(x).$$

Thus we can apply Theorem 3.4 with $\psi$ replaced by $\psi_b$.

□

An immediate consequence of Theorems 3.4 and 3.5 is [13], Proposition 5.9:

**Corollary 3.6.** Let $M$ be a $W^*$-algebra, $\varphi$ a faithful, semi-finite, normal weight on $M$, and $\psi$ a $\sigma^\varphi$-invariant, normal weight on $M$. If there exists a $w$-dense, $\sigma^\varphi$-invariant $*$-subalgebra $\mathcal{M}$ of $\mathcal{M}_{\varphi_a}$ such that

$$\psi(x^*x) = \varphi(x^*x), \quad x \in \mathcal{M},$$

then

$$\psi = \varphi.$$

□

The next result is a balanced counterpart of Theorem 3.5:

**Theorem 3.7.** Let $M$ be a $W^*$-algebra, $\varphi, \psi$ faithful, semi-finite, normal weights on $M$, and $a \in (M^\varphi)^+$, $b \in (M^\psi)^+$. Assume that there are a $w$-dense, $\sigma^\varphi$-invariant $*$-subalgebra $\mathcal{M}_1$ of $\mathcal{M}_{\varphi_a}$ and a $w$-dense, $\sigma^\psi$-invariant $*$-subalgebra $\mathcal{M}_2$ of $\mathcal{M}_{\psi_b}$ such that

$$\psi_b(x^*x) = \varphi_a(x^*x), \quad x \in \mathcal{M}_1 \cup \mathcal{M}_2.$$

Then

$$\psi_b = \varphi_a.$$

**Proof.** We have just to apply twice Theorem 3.4.

□

Immediate consequences of Theorem 3.7 are:

**Theorem 3.8.** Let $M$ be a $W^*$-algebra, $\varphi, \psi$ faithful, semi-finite, normal weights on $M$, and $a \in (M^\varphi)^+$, $b \in (M^\psi)^+$. Assume that there exists a $w$-dense, both $\sigma^\varphi$- and $\sigma^\psi$-invariant $*$-subalgebra $\mathcal{M}$ of $\mathcal{M}_{\varphi_a} \cap \mathcal{M}_{\psi_b}$ such that

$$\psi_b(x^*x) = \varphi_a(x^*x), \quad x \in \mathcal{M},$$

Then

$$\psi_b = \varphi_a.$$

□
Corollary 3.9. Let $M$ be a $W^*$-algebra and $\varphi, \psi$ faithful, semi-finite, normal weights on $M$. If there exists a $w$-dense, both $\sigma^\varphi$- and $\sigma^\psi$-invariant $^*$-subalgebra $\mathcal{M}$ of $\mathcal{M}_\varphi \cap \mathcal{M}_\psi$ such that
\[ \psi(x^*x) = \varphi(x^*x), \quad x \in \mathcal{M}, \]
then
\[ \psi = \varphi. \]

There exist also criteria of different kind for equality and inequalities between faithful, semi-finite, normal weights, due to A. Connes. They are in terms of the Connes cocycle (see [5], Section 1.2) or [15], Theorem 10.28 and C.10.4): if $\varphi$ and $\psi$ are faithful, semi-finite, normal weights on a $W^*$-algebra, the Connes cosycle of $\psi$ with respect to $\varphi$ will be denoted by $(D\psi : D\varphi)_t, t \in \mathbb{R}$.

The next theorem frames up results from [5] and [6]. It will be used in the next section.

Theorem 3.10. Let $M$ be a $W^*$-algebra, and $\varphi, \psi$ faithful, semi-finite, normal weights on $M$.

(i) $\psi = \varphi$ if and only if $(D\psi : D\varphi)_t = 1_M$ for all $t \in \mathbb{R}$.
(ii) $\psi \leq \varphi$ if and only if $\mathbb{R} \ni t \mapsto (D\psi : D\varphi)_t \in M$ allows a $w$-continuous extension \( \{ \zeta \in \mathbb{C} : -\frac{1}{2} \leq \text{Im}\zeta \leq 1 \} \ni \zeta \mapsto (D\psi : D\varphi)_\zeta \in M \), which is analytic in the interior and satisfies $\|(D\psi : D\varphi)_{-\frac{1}{2}}\| \leq 1$.
(iii) $\psi(a) = \varphi(a)$ for all $a \in \mathcal{F}_\varphi$ if and only if $\mathbb{R} \ni t \mapsto (D\psi : D\varphi)_t \in M$ has a $w$-continuous extension \( \{ \zeta \in \mathbb{C} : -\frac{1}{2} \leq \text{Im}\zeta \leq 1 \} \ni \zeta \mapsto (D\psi : D\varphi)_\zeta \in M \), which is analytic in the interior and such that $(D\psi : D\varphi)_{-\frac{1}{2}}$ is isometric.

Proof. (i) is consequence of [5], Lemme 1.2.3 (b) (see also [14], Corollary 3.6), (ii) is [6], Théorème 3 (see also [14], Corollary 3.13), and (iii) is [6], Théorème 4 (see also [14], Corollary 3.14).

4. Inequalities and equality between operator valued weights

[11], Proposition 2.3, [11], Lemma 4.8 and [12]. Remark on page 360 indicate a way to reduce investigation of equality and inequalities between faithful, semi-finite, normal operator valued weights to verification of equality and inequalities between usual, scalar valued faithful, semi-finite, normal weight. Actually, most part of the above results holds, with essentially the same proof, also for not necessarily faithful weights and operator valued weights. Let us single out those statements which will be used in the sequel:

Proposition 4.1. Let $M$ be a $W^*$-algebra, $1_M \in N \subset M$ a $W^*$-subalgebra, and $E, E_1, E_2 : M^+ \to \overline{N}^+$ semi-finite, normal operator valued weights.

(i) If $\varphi$ is a semi-finite, normal weight on $N$, then $\varphi \circ E$ is a normal weight on $M$ such that $\mathcal{M}_E \cap \mathcal{M}_{\varphi \circ E}$ is a $w$-dense $^*$-subalgebra of $M$. In particular, $\varphi \circ E$ is a semi-finite, normal weight on $M$. Assuming additionally that $E$ and $\varphi$ are faithful, also $\varphi \circ E$ is faithful.
(ii) If $\varphi \circ E_2 \leq \varphi \circ E_1$ for every faithful, semi-finite, normal weight $\varphi$ on $N$, then $E_2 \leq E_1$. 
Proof. For (i) we have essentially to repeat the proof of \[11\], Proposition 2.3:

Since \( \varphi \) is semi-finite, there exists a bounded net \((x_i)_i\) in \(\mathfrak{M}_\varphi\) such that \(x_i \xrightarrow{\ast} 1_M\) in the \(s^*\)-topology. For every \(a \in \mathfrak{M}_E \cap M^+\) and every \(i\), we have

\[ E(x_i^*ax_i) = x_i^*E(a)x_i \in N, \]

so \(x_i^*ax_i \in \mathfrak{M}_E \cap M^+\). On the other hand,

\[ (\varphi \circ E)(x_i^*ax_i) = \varphi(x_i^*E(a)x_i) \leq \|E(a)\| \varphi(x_i^*x_i) < +\infty, \]

yields \(x_i^*ax_i \in \mathfrak{M}_{\varphi \circ E} \cap M^+\), hence \(x_i^*ax_i \in \mathfrak{M}_E \cap \mathfrak{M}_{\varphi \circ E}\). Since \(x_i^*ax_i \xrightarrow{\ast} a\) in the \(s^*\)-topology, it follows that \(a\) belongs to the \(s^*\)-closure of \(\mathfrak{M}_E \cap \mathfrak{M}_{\varphi \circ E}\).

On the other hand, the faithfulness of \(E\) and \(\varphi\) implies the faithfulness of \(\varphi \circ E\) trivially.

For the proof of (ii) we use the argumentation of the proof of \[11\], Lemma 4.8:

Let \(\psi\) be any normal positive form on \(N\). Choosing some semi-finite, normal weight \(\theta\) on \(N\) with support \(s(\theta) = 1_M - s(\psi)\), \(\varphi := \psi + \theta\) will be a faithful, semi-finite, normal weight on \(N\) such that \(\psi(b) = \varphi(s(\psi)b s(\psi))\) for all \(b \in N^+\), and consequently, for all \(b \in \overline{N}^+\). By the assumption on \(E_1\) and \(E_2\), we have \(\varphi \circ E_2 \leq \varphi \circ E_1\) and it follows for each \(a \in M^+\):

\[
\psi(E_2(a)) = \varphi(s(\psi)E_2(a)s(\psi)) = (\varphi \circ E_2)(s(\psi)a s(\psi)) \leq (\varphi \circ E_1)(s(\psi)a s(\psi)) = \varphi(s(\psi)E_1(a)s(\psi)) = \psi(E_1(a)).
\]

In conclusion, for every \(a \in M^+\) holds

\[
\psi(E_2(a)) \leq \psi(E_1(a)) \quad \text{for all normal positive forms } \psi \text{ on } N,
\]

what means the inequality \(E_2(a) \leq E_1(a)\) in \(\overline{N}^+\). In other words, \(E_2 \leq E_1\).

For two weights \(\varphi, \psi\) on a \(W^*\)-algebra \(M\) holds trivially: If \(\mathfrak{M}_\varphi \subset \mathfrak{M}_\psi\) and the restrictions of \(\varphi\) and \(\psi\) to \(\mathfrak{M}_\varphi\) coincide, then \(\psi \leq \varphi\). Indeed, at any element of \(M^+\) which does not belong to \(\mathfrak{M}_\varphi\), \(\varphi\) takes the value \(+\infty\) which majorizes any extended real value.

Here we will prove that the analogous statement holds true also for semi-finite, normal operator valued weights on \(W^*\)-algebras. The proof will use modular theory, it is not such trivial as in the case of scalar weights.

Let us first examine the situation \(E_2 \mid (\mathfrak{M}_{E_1} \cap M^+) = E_1 \mid (\mathfrak{M}_{E_1} \cap M^+)\) for semi-finite, normal operator valued weights \(E_1, E_2 : M \rightarrow \overline{N}^+, M\) a \(W^*\)-algebra and \(1_M \in N \subset M\) a \(W^*\)-subalgebra, in the case of faithful \(E_1\):

**Lemma 4.2.** Let \(M\) be a \(W^*\)-algebra, and \(1_M \in N \subset M\) a \(W^*\)-subalgebra. If \(E_1, E_2 : M^+ \rightarrow \overline{N}^+\) are semi-finite, normal operator valued weights, \(E_1\) faithful, such that

\[
\mathfrak{M}_{E_1} \subset \mathfrak{M}_{E_2} \quad \text{and } E_2(a) = E_1(a) \quad \text{for all } a \in \mathfrak{M}_{E_1} \cap M^+,
\]

then \(E_2 \leq E_1\). Moreover, for each faithful, semi-finite, normal weight \(\varphi\) in \(N\),

\[
\mathfrak{M}_{\varphi \circ E_1} \subset \mathfrak{M}_{\varphi \circ E_2} \quad \text{and } (\varphi \circ E_2)(a) = (\varphi \circ E_1)(a) \quad \text{for all } a \in \mathfrak{M}_{\varphi \circ E_1} \cap M^+.
\]

**Proof.** Let \(\varphi\) be an arbitrary faithful, semi-finite, normal weight on \(N\).

By Proposition 4.1 (i) and by the assumptions of the lemma, \(\varphi \circ E_1\) and \(\varphi \circ E_2\) are semi-finite, normal weights on \(M\), \(\varphi \circ E_1\) is faithful, and \(\mathfrak{M} := \mathfrak{M}_{E_1} \cap \mathfrak{M}_{\varphi \circ E_1}\)
is a $w$-dense $\ast$-subalgebra of $\mathfrak{M}_{\varphi \circ E_1}$ satisfying

$$(\varphi \circ E_2)(x^* x) = (\varphi \circ E_1)(x^* x), \quad x \in \mathfrak{M}.$$ 

We can apply Theorem 3.1 to deduce the inequality $\varphi \circ E_2 \leq \varphi \circ E_1$ once we verify the $\sigma^{\varphi \circ E_1}$-invariance of $\mathfrak{M}$. But this is a consequence of the $\sigma^{\varphi \circ E_1}$-invariance of both $\mathfrak{M}_{E_1}$ (see [11], Proposition 4.9) and $\mathfrak{M}_{\varphi \circ E_1}$.

Now, having $\varphi \circ E_2 \leq \varphi \circ E_1$ for any faithful, semi-finite, normal weight $\varphi$ on $N$, the inequality $E_1 \leq E_2$ follows by applying Proposition 4.1 (ii).

For the second statement of the lemma, let the faithful, semi-finite, normal weight $\varphi$ on $N$ and $a \in M^+$ verifying $\varphi(E_1(a)) < +\infty$ be arbitrary. We have to show that then

$$(4.1) \quad \varphi(E_2(a)) = \varphi(E_1(a)).$$

Let us consider $M$ in a spatial representation $M \subset B(H)$.

Since each normal positive form on a von Neumann algebra is a countable sum of positive vector functionals (see e.g. [8], Chap. I, Théorème 1 or [15], E.7.8) and each normal weight on a von Neumann algebra is a sum of normal positive forms (see [13], Theorem 7.2), $\varphi$ is a sum of positive vector functionals on $N$.

Since $E_2(a) \leq E_1(a)$ belong to $N^+$ and $\varphi(E_2(a)) \leq \varphi(E_1(a)) < +\infty$ with $\varphi$ faithful, by [11], Lemma 1.4 they correspond to positive self-adjoint operators $A_1$ and $A_2$ on $H$, affiliated with $N$ and such that, for every $\xi \in H$,

$$\omega_\xi(E_j(a)) = \begin{cases} \|A_j^{1/2}\xi\|^2 & \text{for } \xi \in \mathcal{D}(A_j^{1/2}) \\ \infty & \text{otherwise} \end{cases}, \quad j = 1, 2,$$

where $\mathcal{D}(A_j^{1/2})$ stands for the domain of $A_j^{1/2}$. Having $\omega_\xi(E_2(a)) \leq \omega_\xi(E_1(a))$ for all $\xi \in H$, it follows

$$\mathcal{D}(A_1^{1/2}) \subset \mathcal{D}(A_2^{1/2}) \quad \text{and} \quad \|A_2^{1/2}\xi\| \leq \|A_1^{1/2}\xi\| \quad \text{for all } \xi \in \mathcal{D}(A_1^{1/2}).$$

Consequently the formulas

$$U(A_1^{1/2}\xi) := A_2^{1/2}\xi \quad \text{for } \xi \in \mathcal{D}(A_1^{1/2}), \quad U\eta := 0 \text{ for } \eta \in (1_H - s(A_1))H$$

define a linear contraction $U : H \rightarrow H$ for which $U A_1^{1/2} \subset A_2^{1/2}$. Let us consider, for any $n \geq 1$, the spectral projection $f_n := \chi_{[1/n, n]}(A_1) \in N$ of $A_1$. We notice that $f_n \not\succeq s(A_1)$.

The boundedness of the operator $A_1 f_n$ yields $E_1(f_n a f_n) = f_n E_1(a) f_n \in N$, hence $f_n a f_n \in \mathfrak{M}_{E_1} \cap M^+ \subset \mathfrak{M}_{E_2} \cap M^+$ and $E_2(f_n a f_n) = E_2(f_n a f_n)$. For every $\xi \in H$ we deduce

$$\|A_1^{1/2} f_n \xi\|^2 = \omega_{f_n \xi}(E_1(a)) = \omega_{\xi}(f_n E_1(a) f_n) = \omega_{\xi}(E_1(f_n a f_n))$$

and

$$= \omega_{\xi}(E_2(f_n a f_n)) = \omega_{\xi}(f_n E_2(a) f_n) = \omega_{f_n \xi}(E_2(a))$$

$$= \|A_2^{1/2} f_n \xi\|^2 = \|U A_1^{1/2} f_n \xi\|^2.$$ 

Since $A_1^{1/2} f_n H = f_n H$, it follows that $U$ acts isometrically on $f_n H$.

Taking into account that $\bigcup_{n \geq 1} f_n H$ is dense in $s(A_1)H$, we conclude that $U$ acts isometrically on $s(A_1)H$, hence $U$ is a partial isometry with initial projection $s(A_1)$.

Therefore, for every $\xi \in \mathcal{D}(A_1^{1/2})$,

$$(4.2) \quad \omega_\xi(E_1(a)) = \|A_1^{1/2}\xi\|^2 = \|U A_1^{1/2}\xi\|^2 = \|A_2^{1/2}\xi\|^2 = \omega_{f_n \xi}(E_2(a)) \quad \text{for } \xi \in \mathcal{D}(A_1^{1/2}).$$
Let \((\xi_i)_i\) be a family of vectors in \(H\) such that \(\varphi\) is equal to the sum \(\sum_i \omega_{\xi_i}\).

Since \(\sum_i \omega_{\xi_i}(E_1(a)) = \varphi(E_1(a)) < +\infty\), each \(\xi_i\) should belong to \(D(A_1^{1/2})\). Indeed, 
\(\xi_i \notin D(A_1^{1/2})\) for some \(i\) would imply \(+\infty = \omega_{\xi_i}(E_1(a)) \leq \varphi(E_1(a))\). Thus (1.2) follows:
\[
\varphi(E_1(a)) = \sum_i \omega_{\xi_i}(E_1(a)) = \sum_i \omega_{\xi_i}(E_2(a)) = \varphi(E_2(a)).
\]

\(\square\)

The next lemma is an inverse to Lemma 1.2.

**Lemma 4.3.** Let \(M\) be a \(W^*\)-algebra, and \(1_M \in N \subset M\) a \(W^*\)-subalgebra. If \(E_1, E_2 : M^+ \to N^+\) are semi-finite, normal operator valued weights, \(E_1\) faithful, such that, for each faithful, semi-finite, normal weight \(\varphi\) in \(N\),
\[
\mathcal{M}_{\varphi \circ E_1} \subset \mathcal{M}_{\varphi \circ E_2} \quad \text{and} \quad (\varphi \circ E_2)(a) = (\varphi \circ E_1)(a) \quad \text{for all} \quad a \in \mathcal{M}_{\varphi \circ E_1} \cap M^+,
\]
then
\[
\mathcal{M}_{E_1} \subset \mathcal{M}_{E_2} \quad \text{and} \quad E_2(a) = E_1(a) \quad \text{for all} \quad a \in \mathcal{M}_{E_1} \cap M^+.
\]

**Proof.** First of all, by Proposition 1.1 (ii) we have \(E_2 \leq E_1\), so \(\mathcal{M}_{E_1} \subset \mathcal{M}_{E_2}\).

Now let \(a \in \mathcal{M}_{E_1} \cap M^+\) be arbitrary. Then \(E_2(a) \leq E_1(a) \in N\), in particular also \(E_2(a) \in N\). All we have now to show is the equality \(E_2(a) = E_1(a)\).

Since \(E_1(\mathcal{M}_{E_1})\) is a \(w\)-dense two-sided ideal in \(N\) (see [11], Proposition 2.5), there exists a family \((f_i)_{i \in I}\) of non-zero, countably decomposable projections in \(N\), belonging to \(E_1(\mathcal{M}_{E_1})\), such that \(\sum_i f_i = 1_M\).

For let us consider, by the Zorn lemma, a maximal family \((f_i)_{i \in I}\) of mutually orthogonal, non-zero, countably decomposable projections in \(N\), each one belonging to \(E_1(\mathcal{M}_{E_1})\).

Assume that \(f_0 := 1_M - \sum_{i \in I} f_i \neq 0\). Since \(E_1(\mathcal{M}_{E_1})\) is \(w\)-dense, there is some \(y \in E_1(\mathcal{M}_{E_1})\) with \(y f_0 \neq 0\). Further, since \(E_1(\mathcal{M}_{E_1})\) is a two-sided ideal in \(N\), \(f_0 y^* y f_0 \neq 0\) belongs to \(E_1(\mathcal{M}_{E_1})\) and, choosing some \(0 < \lambda < \|f_0 y^* y f_0\|\), we have \(f_1 := \chi_{\{\lambda, +\infty\}}(f_0 y^* y f_0) \neq 0\). Choose some non-zero, countably decomposable projection \(f_2 \leq f_1\) in \(N\). Then \(f_2 \leq f_1 \leq \frac{1}{\lambda} f_0 y^* y f_0 \in E_1(\mathcal{M}_{E_1})\). Consequently \(0 \neq f_2 \leq f_0\) and, since \(E_1(\mathcal{M}_{E_1})\) is a two-sided ideal in \(N\), \(f_2 \in E_1(\mathcal{M}_{E_1})\). But this contradicts the maximality of the family \((f_i)_{i \in I}\). Consequently \(f_0\) should vanish, that is \(\sum_{i \in I} f_i = 1_M\).

Set \(f_F := \sum_{i \in F} f_i\) for any finite subset \(F \subset I\). \(f_F\) is then a countably decomposable projection belonging to \(E_1(\mathcal{M}_{E_1})\). Since \(f_F \not> 1_M\), for the equality \(E_2(a) = E_1(a)\) it is enough to show that \(f_F E_2(a) f_F = f_F E_1(a) f_F\) for every finite \(F \subset I\).

For let \(F\) be any finite subset of \(I\). \(f_F E_2(a) f_F = f_F E_1(a) f_F\) will follow if we show that \(\psi(f_F E_2(a) f_F) = \psi(f_F E_1(a) f_F)\) for every normal positive form \(\psi\) on \(N\) with \(s(\psi) = f_F\).
Proposition 4.4. Let $M$ be a $W^*$-algebra, $1_M \in N \subset M$ a $W^*$-subalgebra, and $E_0 : M^+ \to \mathcal{N}^+$ a semi-finite, normal operator valued weight. Let $p_0$ denote the central support of $s(E_0) \in N^\circ \cap M$ in $N$, and $\pi_0$ the $^*$-isomorphism $Np_0 \ni y \mapsto ys(E_0) \in Ns(E_0)$. Then there exists an one-to-one correspondence between the semi-finite, normal operator valued weights $E : M^+ \to \mathcal{N}^+$ of support $s(E) \leq s(E_0)$, and the semi-finite, normal operator valued weights $\widetilde{E} : s(E_0)M^+s(E_0) \to \overline{Ns(E_0)}^+$, such that

$$\widetilde{E}(a) = \pi_0^{-1}(\pi_0^{-1}(E(s(E_0)a s(E_0))))^{-1}, \quad a \in M^+. \tag{4.3}$$

Moreover, by this correspondence

$$\mathfrak{M}_{\widetilde{E}} = \mathfrak{M}_E \cap (s(E_0)Ms(E_0)), \quad \mathfrak{M}_E \cap M^+ = \{a \in M^+; s(E_0)a s(E_0) \in \mathfrak{M}_{\widetilde{E}}\}, \quad \omega(E) = s(E), \text{ in particular, } \widetilde{E}_0 \text{ is faithful.} \tag{4.4}$$

Proof. Let $E : M^+ \to \mathcal{N}^+$ be a semi-finite, normal operator valued weight with $s(E) \leq s(E_0)$.

Since $1_M - p_0 \leq 1_M - s(E_0) \leq 1_M - s(E)$, we have

$$E(1_M - p_0) \leq E(1_M - s(E_0)) \leq E(1_M - s(E)) = 0 \tag{4.5}$$

and (1.6) yields

$$E(a) = E(p_0ap_0) = p_0E(a)p_0 \in p_0Np_0^+ = \overline{Np_0}^+, \quad a \in M^+. \tag{4.6}$$

and

$$E(a) = E(s(E_0)a s(E_0)), \quad a \in M^+. \tag{4.7}$$

According to (4.3) the composition $\pi_0 \circ E$ is well defined, so an operator valued weight $\widetilde{E} : s(E_0)M^+s(E_0) \to \overline{Ns(E_0)}^+$ can be defined by the formula

$$\widetilde{E}(a) := \pi_0(E(a)), \quad a \in s(E_0)M^+s(E_0).$$

It is normal, semi-finite and satisfying $s(\widetilde{E}) = s(E)$.
where the last equality is trivial because $a$, and for each $s$

applications. Indeed, for each $s$

is a linear combination of the elements $s$

Then $E$

Finally we show that the associations $E \mapsto \tilde{E}$ and $\tilde{E} \mapsto E$ are mutually inverse applications. Indeed, for each $a \in M^+$,

and, for each $a \in s(E_0)M^+s(E_0)$,

where the last equality is trivial because $a \in s(E_0)M^+s(E_0)$,

The normality is obvious.

The semi-finiteness is consequence of the semi-finiteness of $E$ because, by (4.4), $x \in \mathfrak{M}_E \iff x(s(E_0)) \in \mathfrak{M}_E$, and therefore $\mathfrak{M}_E s(E_0) \subset \mathfrak{M}_E$.

By the definition of $\tilde{E}$, $\tilde{E}(s(E_0) - s(E)) = \pi_0\left(E(s(E_0) - s(E))\right) = 0$, so

$s(E_0) - s(E) \leq s(E_0) - s(\tilde{E}) \iff s(\tilde{E}) \leq s(E)$.

On the other hand, $0 = \tilde{E}(s(E_0) - s(\tilde{E})) = \pi_0\left(E(s(E_0) - s(\tilde{E}))\right)$ yields

$s(E) - s(\tilde{E})s(E) = (s(E_0) - s(\tilde{E}))s(E) = 0 \iff s(E) \leq s(\tilde{E})$.

For $\mathfrak{M}_{\tilde{E}} = \mathfrak{M}_E \cap (s(E_0)M^+s(E_0))$ it is enough to show that

(4.5) $\mathfrak{M}_{\tilde{E}} \cap (s(E_0)M^+s(E_0)) = \mathfrak{M}_E \cap (s(E_0)M^+s(E_0))$

because $\mathfrak{M}_E \cap (s(E_0)M^+s(E_0))$ is the linear span of $\mathfrak{M}_E \cap (s(E_0)M^+s(E_0))$. Indeed, each $x \in \mathfrak{M}_E \cap (s(E_0)M^+s(E_0))$ is the one hand linear combination of certain elements $a_j \in \mathfrak{M}_E \cap M^+, 1 \leq j \leq n$, and on the other hand $x = s(E_0)x(s(E_0))$, so $x$ is a linear combination of the elements $s(E_0)a_j(s(E_0) \in \mathfrak{M}_E \cap (s(E_0)M^+s(E_0))$.

Now (4.5) is consequence of the definition of $\tilde{E}$. Indeed, for $a \in s(E_0)M^+s(E_0)$ we have:

$a \in \mathfrak{M}_{\tilde{E}} \iff \tilde{E}(a) \in Ns(E_0) \iff E(a) \in Np_0 \iff E(a) \in N$.

Let next $\tilde{E} : s(E_0)M^+s(E_0) \rightarrow Ns(E_0)^+$ be a semi-finite, normal operator valued weight and define the operator valued weight $E : M^+ \rightarrow N^+$ by

$E(a) = \pi_0^{-1}\left(\tilde{E}(s(E_0)a s(E_0))\right)$, $a \in M^+$.

Then $E$ is normal, semi-finite and of support $s(E) \leq s(E_0)$:

The normality is obvious.

Since $\{x(y + 1_M - s(E_0)) : x \in M, y \in \mathfrak{M}_E\} \subset \mathfrak{M}_E$ and $s(E_0)$ belongs to the $w$-closure of $\mathfrak{M}_E$, it follows that every $x \in M$ belongs to the $w$-closure of $\mathfrak{M}_E$, that is $E$ is semi-finite.

Since $E$,

$E(1_M - s(E_0)) = \pi_0^{-1}\left(\tilde{E}(s(E_0)(1_M - s(E_0))s(E_0))\right) = 0$, we have $1_M - s(E_0) \leq 1_M - s(E) \iff s(E) \leq s(E_0)$.

The equality $\mathfrak{M}_E \cap M^+ = \{a \in M^+ : s(E_0)a(s(E_0)) \in \mathfrak{M}_E\}$ follows by using

$\mathfrak{M}_E \cap M^+ = \{a \in M^+ : s(E_0)a(s(E_0)) \in \mathfrak{M}_E\}$, consequence of (4.4), and

the equivalence $s(E_0)a(s(E_0)) \in \mathfrak{M}_E \iff s(E_0)a(s(E_0)) \in \mathfrak{M}_E$ for $a \in M^+$, consequence of the above proved equality $\mathfrak{M}_{\tilde{E}} = \mathfrak{M}_E \cap (s(E_0)M^+s(E_0))$.

Finally we show that the associations $E \mapsto \tilde{E}$ and $\tilde{E} \mapsto E$ are mutually inverse applications. Indeed, for each $a \in M^+$,

$\pi_0^{-1}\left(\pi_0\left(E(s(E_0)a s(E_0))\right)\right) = E(s(E_0)a s(E_0)) \iff E(a)$

and, for each $a \in s(E_0)M^+s(E_0)$,

$\pi_0\left(\pi_0^{-1}\left(\tilde{E}(s(E_0)a s(E_0))\right)\right) = \tilde{E}(s(E_0)a s(E_0)) = \tilde{E}(a)$,
Now we are ready to describe the situation \( E_2 \mid (\mathfrak{M}_{E_1} \cap M^+) = E_1 \mid (\mathfrak{M}_{E_1} \cap M^+) \) for general semi-finite, normal operator valued weights \( E_1, E_2 : M \to N^+ \), \( M \) a \( W^* \)-algebra and \( 1_M \in N \subset M \) a \( W^* \)-subalgebra. The proof consists in reduction, based on Proposition 4.4, to the case of faithful \( E_1 \). We prove also a characterization of this situation in terms of scalar valued weights.

**Theorem 4.5.** Let \( M \) be a \( W^* \)-algebra, \( 1_M \in N \subset M \) a \( W^* \)-subalgebra, and \( E_1, E_2 : M^+ \to N^+ \) semi-finite, normal operator valued weights.

(i) If \( \mathfrak{M}_{E_1} \subset \mathfrak{M}_{E_2} \) and \( E_2(a) = E_1(a) \) for all \( a \in \mathfrak{M}_{E_1} \cap M^+ \), then \( E_2 \leq E_1 \).

(ii) The condition

\[
\mathfrak{M}_{E_1} \subset \mathfrak{M}_{E_2} \quad \text{and} \quad E_2(a) = E_1(a) \quad \text{for all} \quad a \in \mathfrak{M}_{E_1} \cap M^+
\]

is equivalent to the condition

for every faithful, semi-finite, normal weight \( \varphi \) on \( N 
.

\[
\mathfrak{M}_{\varphi \circ E_1} \subset \mathfrak{M}_{\varphi \circ E_2} \quad \text{and} \quad (\varphi \circ E_2)(a) = (\varphi \circ E_1)(a) \quad \text{for all} \quad a \in \mathfrak{M}_{\varphi \circ E_1} \cap M^+.
\]

**Proof.** The assumptions in both (4.6) and (4.7) imply the inequality \( s(E_2) \leq s(E_1) \).

Indeed, assuming (4.6),

\[
1_M - s(E_1) \in \mathfrak{M}_{E_1} \implies E_2(1_M - s(E_1)) = E_1(1_M - s(E_1)) = 0
\]

\[
\implies 1_M - s(E_1) \leq 1_M - s(E_2) \iff s(E_2) \leq s(E_1).
\]

On the other hand, assuming (4.7), we have for any faithful, semi-finite, normal weight \( \varphi \) on \( N 
.

\[
1_M - s(E_1) \in \mathfrak{M}_{\varphi \circ E_1} \implies (\varphi \circ E_2)(1_M - s(E_1)) = (\varphi \circ E_1)(1_M - s(E_1)) = 0
\]

\[
\implies E_2(1_M - s(E_1)) = 0
\]

\[
\implies 1_M - s(E_1) \leq 1_M - s(E_2) \iff s(E_2) \leq s(E_1).
\]

Consequently, we can start the proof of both (i) and (ii) with the assumption \( s(E_2) \leq s(E_1) \). Let \( p_1 \) denote the central support of \( s(E_1) \in N \cap M \) in \( N 
. Using the notations from Proposition 4.4, we can consider the semi-finite, normal operator valued weights \( \overline{E_j} : s(E_1)M^+s(E_1) \to Ns(E_1)^+, j = 1, 2 \), defined by the formula

\[
\overline{E_j}(a) = \pi_1(E_j(a)), \quad a \in s(E_1)M^+s(E_1),
\]

where \( \pi_1 \) denotes the *-isomorphism \( Np_1 \ni y \mapsto ys(E_1) \in Ns(E_1) \). According to Proposition 4.4, the operator valued weight \( \overline{E_1} \) is also faithful,

\[
\mathfrak{M}_{\overline{E_1}} = \mathfrak{M}_{E_1} \cap (s(E_1)M^+s(E_1)), \quad j = 1, 2,
\]

\[
\mathfrak{M}_{E_j} \cap M^+ = \{a \in M^+ : s(E_1) a s(E_1) \in \mathfrak{M}_{E_j}\}, \quad j = 1, 2,
\]

and the inversion formula

\[
E_j(a) = \pi_1^{-1}(\overline{E_j}(s(E_1) a s(E_1))), \quad a \in M^+, \quad j = 1, 2
\]

holds true.

Now we pass to the proof of (i).

Assuming (4.6) and using (4.9) and (4.8), we deduce :

\[
\mathfrak{M}_{E_1} = \mathfrak{M}_{E_1} \cap (s(E_1)M^+s(E_1)) \subset \mathfrak{M}_{E_2} \cap (s(E_1)M^+s(E_1)) = \mathfrak{M}_{\overline{E_2}},
\]

and, for each \( a \in \mathfrak{M}_{E_1} \cap (s(E_1)M^+s(E_1)) \).
\[ \hat{E}_2(a) = \pi_1(E_2(a)) = \pi_1(E_1(a)) = \hat{E}_1(a). \]

Since \( \hat{E}_1 \) is faithful, Lemma 4.2 entails that \( \hat{E}_2 \leq \hat{E}_1 \) and using (4.11), we conclude that \( E_2 \leq E_1 \).

Concerning (ii), it will be implied by Lemmas 4.2 and 4.3 once we prove that (4.10) is equivalent to

\[(4.6\text{-bis}) \quad M_{\hat{E}_1} \subset M_{\hat{E}_2} \quad \text{and} \quad \hat{E}_2(a) = \hat{E}_1(a) \quad \text{for all} \quad a \in M_{\hat{E}_1} \cap (s(E_1)M^+s(E_1)) \]

and (4.7) is equivalent to

\[(4.7\text{-bis}) \quad \text{for every faithful, semi-finite, normal weight} \ \psi \ \text{on} \ \mathcal{N}s(E_1), \]

\[(4.7) \quad \text{is equivalent to} \quad (\psi \circ \hat{E}_2)(a) = (\psi \circ \hat{E}_1)(a) \quad \text{holds true} \]

for all \( a \in M_{\psi \circ \hat{E}_1} \cap (s(E_1)M^+s(E_1)) \).

Implication (4.6) \( \implies \) (4.6-bis) was already shown and used in the proof of (i), it is immediate consequence of (4.9) and (4.8). Also the proof of the inverse implication (4.6-bis) \( \implies \) (4.6) is easy, it follows immediately by using (4.10) and (4.11).

Let us next prove (4.7) \( \iff \) (4.7-bis).

For let \( \psi \) be any faithful, semi-finite, normal weight on \( \mathcal{N}s(E_1) \). Choosing some faithful, semi-finite, normal weight \( \theta \) on \( \mathcal{N}(1_M - p_1) \), let us consider the faithful, semi-finite, normal weight \( \varphi := \psi \circ \pi_1 + \theta \) on \( N \). By (4.7) we have

\[(4.12) \quad M_{\varphi \circ E_1} \subset M_{\varphi \circ E_2} \quad \text{and} \quad (\varphi \circ E_2)(a) = (\varphi \circ E_1)(a) \quad \text{for all} \quad a \in M_{\varphi \circ E_1} \cap M^+. \]

Now let \( a \in M_{\varphi \circ E_1} \cap (s(E_1)M^+s(E_1)) \) be arbitrary. Since

\[ a \in s(E_1)M^+s(E_1) \subset p_1 M^+ p_1 \implies E_j(a) \in p_1 \mathcal{N}^+ p_1 \implies \theta(E_j(a)) = 0, \quad j = 1, 2, \]

we have

\[(4.8) \quad (\varphi \circ E_1)(a) = \psi(\pi_1(E_1(a))) + \theta(E_1(a)) = (\psi \circ \hat{E}_1)(a) < +\infty. \]

Thus \( a \in M_{\varphi \circ E_1} \subset M_{\varphi \circ E_2} \) and, using (4.12), we conclude :

\[(4.8) \quad (\psi \circ \hat{E}_1)(a) = (\varphi \circ E_1)(a) = (\varphi \circ E_2)(a) = \psi(\pi_1(E_2(a))) + \theta(E_2(a)) \]

Finally we prove also the inverse implication (4.7-bis) \( \implies \) (4.7).

For let \( \varphi \) be any faithful, semi-finite, normal weight on \( N \). We define the faithful, semi-finite, normal weight \( \psi \) on \( \mathcal{N}s(E_1) \) by the formula

\[ \psi(b) := \varphi(\pi_1^{-1}(b)), \quad b \in (\mathcal{N}s(E_1))^+. \]

By (4.7-bis) we have

\[(4.13) \quad M_{\psi \circ E_1} \subset M_{\psi \circ E_2} \quad \text{and} \quad (\psi \circ \hat{E}_2)(a) = (\psi \circ \hat{E}_1)(a) \quad \text{holds true} \]

for all \( a \in M_{\varphi \circ E_1} \cap (s(E_1)M^+s(E_1)) \).

Let \( a \in M_{\varphi \circ E_1} \cap M^+ \) be arbitrary. Then

\[ +\infty > (\varphi \circ E_1)(a) \quad \text{by (4.11)} \quad \psi(\pi_1^{-1}(E_j(s(E_1) a s(E_1)))) = (\psi \circ \hat{E}_1)(s(E_1) a s(E_1)). \]
Thus $s(E_1) a s(E_1) \in M_{\psi \circ \bar{E}_1} \subset M_{\psi \circ \bar{E}_2}$ and, using (4.13), we conclude:

$$(\varphi \circ E_1)(a) = (\psi \circ \bar{E}_1)(s(E_1) a s(E_1)) = (\psi \circ \bar{E}_2)(s(E_1) a s(E_1))$$

(4.11) \[ \psi(\pi_1(E_2(a))) = (\varphi \circ E_2)(a). \]

\[ \square \]

The next equality criterion is an immediate consequence of Theorem 3.10:

**Corollary 4.6.** Let $M$ be a $W^*$-algebra, and $1_M \in N \subset M$ a $W^*$-subalgebra. If $E_1, E_2 : M^+ \rightarrow N^+$ are semi-finite, normal operator valued weights such that

$$M_{E_1} = M_{E_2} \text{ and } E_2(a) = E_1(a) \text{ for all } a \in M_{E_1} \cap M^+ = M_{E_2} \cap M^+, \text{ then } E_2 = E_1.$$ 

\[ \square \]

For $E_1, E_2$ of equal supports, the situation $E_2 | (M_{E_1} \cap M^+) = E_1 | (M_{E_1} \cap M^+)$ is equivalent also to a weakened version of (4.7). Let us first consider the case of faithful operator valued weights and formulate the following variant of [12], Remark on page 360, using [6].

**Lemma 4.7.** Let $M$ be a $W^*$-algebra, $1_M \in N \subset M$ a $W^*$-subalgebra, and $E_1, E_2$ two faithful, semi-finite, normal operator valued weights $M^+ \rightarrow N^+$. If for some faithful, semi-finite, normal weight $\varphi$ on $N$ we have

$$M_{\varphi \circ E_1} \subset M_{\varphi \circ E_2} \text{ and } (\varphi \circ E_2)(a) = (\varphi \circ E_1)(a) \text{ for all } a \in M_{\varphi \circ E_1} \cap M^+, \text{ then } (4.14) \text{ holds true for every faithful, semi-finite, normal weight } \varphi \text{ on } N.$$ 

**Proof.** We have just to follow the reasoning in [12], Remark on page 360, using [6], Théorème 3 instead of [6], Théorème 3 (that is the above Theorem 3.10 (iii) instead of Theorem 3.10 (ii)). Let us sketch the details.

Assuming that (4.14) holds true for some faithful, semi-finite, normal weight $\varphi_1$ on $N$, let $\varphi_2$ be any other faithful, semi-finite, normal weight on $N$. According to Theorem 3.10 (iii), $\mathbb{R} \ni t \mapsto (D(\varphi_1 \circ E_2) : D(\varphi_1 \circ E_1))_t \in M$ has a $w$-continuous extension $\{ \zeta \in \mathbb{C}; -\frac{\pi}{2} \leq \text{Im} \zeta \leq 1 \} \ni \zeta \mapsto (D(\varphi_1 \circ E_2) : D(\varphi_1 \circ E_1))_\zeta \in M$, which is analytic in the interior and such that $D(\varphi_1 \circ E_2) : D(\varphi_1 \circ E_1)_{\frac{\pi}{2}}$ is isometric. On the other hand, by [12], Proposition 6.1 (2), we have

$$\{ D(\varphi_2 \circ E_2) : D(\varphi_2 \circ E_1) \}_t = (D(\varphi_2 \circ E_2) : D(\varphi_2 \circ E_1))_t \text{ for all } t \in \mathbb{R}.$$ 

Thus $\mathbb{R} \ni t \mapsto (D(\varphi_2 \circ E_2) : D(\varphi_2 \circ E_1))_t \in M$ allows the extension

$$\{ \zeta \in \mathbb{C}; -\frac{\pi}{2} \leq \text{Im} \zeta \leq 1 \} \ni \zeta \mapsto (D(\varphi_2 \circ E_2) : D(\varphi_2 \circ E_1))_\zeta = (D(\varphi_2 \circ E_2) : D(\varphi_2 \circ E_1))_\zeta,$$

which is $w$-continuous, analytic in the interior, and such that

$$(D(\varphi_2 \circ E_2) : D(\varphi_2 \circ E_1))_{\frac{\pi}{2}} = (D(\varphi_2 \circ E_2) : D(\varphi_2 \circ E_1))_{\frac{\pi}{2}} \text{ is isometric.}$$ 

Using now again Theorem 3.10 (iii), we infer that (4.14) holds true for $\varphi_2$.

\[ \square \]

Now we complete, in the case $E_1, E_2$ have equal supports, the characterization of the situation $E_2 | (M_{E_1} \cap M^+) = E_1 | (M_{E_1} \cap M^+)$ given in Theorem 4.5:
Theorem 4.8. Let $M$ be a $W^*$-algebra, $1_M \in N \subset M$ a $W^*$-subalgebra, and $E_1, E_2: M^+ \to \mathbb{N}^+$ semi-finite, normal operator valued weights of equal supports. Then the following conditions are equivalent:

(i) $\mathcal{M}_{E_1} \subset \mathcal{M}_{E_2}$ and $E_2(a) = E_1(a), a \in \mathcal{M}_{E_1} \cap M^+$ (implying, by Theorem 4.4 (i), $E_2 \leq E_1$).

(ii) $\mathcal{M}_{\varphi \circ E_1} \subset \mathcal{M}_{\varphi \circ E_2}$ and $(\varphi \circ E_2)(a) = (\varphi \circ E_1)(a), a \in \mathcal{M}_{\varphi \circ E_1} \cap M^+$ for every faithful, semi-finite, normal weight $\varphi$ on $N$.

(iii) $\mathcal{M}_{\varphi \circ E_1} \subset \mathcal{M}_{\varphi \circ E_2}$ and $(\varphi \circ E_2)(a) = (\varphi \circ E_1)(a), a \in \mathcal{M}_{\varphi \circ E_1} \cap M^+$ for some faithful, semi-finite, normal weight $\varphi$ on $N$.

Proof. Equivalence (i)$\iff$(ii) was proved in Theorem 4.4 (even without assuming the equality of the supports of $E_1$ and $E_2$), while implication (ii)$\implies$(iii) is trivial. Thus the proof will be done once we prove (iii)$\implies$(ii), what will be performed by reduction to Lemma 4.7.

Let $p$ denote the central support of $s := s(E_1) = s(E_2) \in N' \cap M$ in $N$. Using the notations from Proposition 4.4 we can consider the semi-finite, normal operator valued weights $E_j: sM^+s \to (Ns)^+, j = 1, 2$, defined by the formula

$$\widetilde{E}_j(a) = \pi(E_j(a)), \quad a \in sM^+s,$$

where $\pi$ denotes the $^*$-isomorphism $Np \ni y \mapsto ys \in Ns$. According to Proposition 4.4 the operator valued weights $\widetilde{E}_1$ and $\widetilde{E}_2$ are also faithful.

Now let us assume that

$$\mathcal{M}_{\varphi_1 \circ E_1} \subset \mathcal{M}_{\varphi_2 \circ E_2} \quad \text{and} \quad (\varphi_1 \circ E_2)(a) = (\varphi_1 \circ E_1)(a), a \in \mathcal{M}_{\varphi_1 \circ E_1} \cap M^+$$

holds true for some faithful, semi-finite, normal weight $\varphi_1$ on $N$. We have to show that, for any faithful, semi-finite, normal weight $\varphi_2$ on $N$,

$$\mathcal{M}_{\varphi_2 \circ E_1} \subset \mathcal{M}_{\varphi_2 \circ E_2} \quad \text{and} \quad (\varphi_2 \circ E_2)(a) = (\varphi_2 \circ E_1)(a), a \in \mathcal{M}_{\varphi_2 \circ E_1} \cap M^+$$

still holds true.

We define the faithful, semi-finite, normal weights $\psi_k, k = 1, 2$, on $Ns$ by the formula

$$\psi_k(b) := \varphi_k(\pi^{-1}(b)), \quad b \in (Ns)^+.$$ 

Then, for each $j = 1, 2$ and $k = 1, 2$, we have

$$\psi_k \circ \widetilde{E}_j(a) = \psi_k(\pi(E_j(a))) = \varphi_k(E_j(a)), \quad a \in sM^+s,$$

in particular

$$\mathcal{M}_{\psi_k \circ \widetilde{E}_j} \cap (sM^+s) = \mathcal{M}_{\varphi_k \circ E_j} \cap (sM^+s).$$

Now, by (4.19) and (4.16) we have

$$\mathcal{M}_{\psi_1 \circ \widetilde{E}_1} \cap (sM^+s) = \mathcal{M}_{\varphi_1 \circ E_1} \cap (sM^+s) \subset \mathcal{M}_{\varphi_1 \circ E_2} \cap (sM^+s) = \mathcal{M}_{\psi_1 \circ \widetilde{E}_2} \cap (sM^+s),$$

and by (4.18) and (4.16) we obtain for every $a \in \mathcal{M}_{\psi_1 \circ \widetilde{E}_1} \cap (sM^+s)$:

$$(\psi_1 \circ \widetilde{E}_2)(a) = \varphi_1(E_2(a)) = \varphi_1(E_1(a)) = (\psi_1 \circ \widetilde{E}_1)(a).$$

Thus we can apply Lemma 4.7 deducing

$$\mathcal{M}_{\varphi_2 \circ E_1} \subset \mathcal{M}_{\psi_2 \circ \widetilde{E}_2} \quad \text{and} \quad (\psi_2 \circ \widetilde{E}_2)(a) = (\psi_2 \circ \widetilde{E}_1)(a), a \in \mathcal{M}_{\psi_2 \circ \widetilde{E}_1} \cap (sM^+s).$$

Let $a \in \mathcal{M}_{\varphi_2 \circ E_1} \cap M^+$ be arbitrary. By (4.18) and (4.16) we have
\[ (\psi_k \circ \widetilde{E}_j)(sas) = (\varphi_2 \circ E_1)(sas) = (\varphi_2 \circ E_1)(a) < +\infty, \]

so \( sas \in \mathcal{M}_{\varphi_2 \circ E_1} \cap (sM^+ s) \). Using (1.1), (4.18) and (4.20), we obtain

\[ (\varphi_2 \circ E_2)(a) = (\varphi_2 \circ \widetilde{E}_2)(sas) = (\varphi_2 \circ \widetilde{E}_1)(sas) = (\varphi_2 \circ E_1)(a) < +\infty. \]

In other words \( a \in \mathcal{M}_{\varphi_2 \circ E_2} \cap M^+ \) and \( (\varphi_2 \circ E_2)(a) = (\varphi_2 \circ E_1)(a) \). This proves (4.17).

\[ \square \]

Remark 4.9. It is possible that the statement of Theorem 4.8 holds without the assumption \( s(E_1) = s(E_2) \). This would follow if Lemma 4.7 would hold without assuming the faithfulness of \( E_2 \).

Actually the cocycle \((D\psi : D\varphi)_t, t \in \mathbb{R}\), of a not necessarily faithful, semi-finite, normal weight \( \psi \) on a \( W^* \)-algebra \( M \) with respect to a faithful, semi-finite, normal weight \( \varphi \) on \( M \) was already considered by A. Connes and M. Takesaki in [7], I.1, pages 478-479 (see also [13], Theorem 3.1). If

- Theorem 3.10 would keep validity even in this setting and
- the proof of [12], Proposition 6.1 could be adapted to the situation of semi-finite, normal operator valued weights \( E_1, E_2 \) from \( M \) to a \( W^* \)-subalgebra \( 1_M \in N \subset M \), where only \( E_1 \) assumed to be faithful, and thus prove that the cocycle \((D(\psi \circ E_2) : D(\varphi \circ E_1))_t, t \in \mathbb{R}\), does not depend on the faithful, semi-finite, normal weight \( \theta \) on \( N \),

then the proof of Lemma 4.7 would work without assuming the faithfulness of \( E_2 \).

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