Abstract. We investigate conditions for logarithmic complete monotonicity of a quotient of two products of gamma functions, where the argument of each gamma function has different scaling factor. We give necessary and sufficient conditions in terms of nonnegativity of some elementary function and more practical sufficient conditions in terms of parameters. Further, we study the representing measure in Bernstein’s theorem for both equal and non-equal scaling factors. This leads to conditions on parameters under which Meijer’s G-function or Fox’s H-function represents an infinitely divisible probability distribution on positive half-line. Moreover, we present new integral equations for both G-function and H-function. The results of the paper generalize those due to Ismail (with Bustoz, Muldoon and Grinshpan) and Alzer who considered previously the case of unit scaling factors.

Keywords: gamma function, digamma function, completely monotone functions, logarithmic complete monotonicity, Meijer’s G-function, Fox’s H-function, Laplace transform, infinite divisibility

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1. Introduction. Recall that a nonnegative function $f$ defined on $(0, \infty)$ is called completely monotone (c.m.) if it has derivatives of all orders and $(-1)^n f^{(n)}(x) \geq 0$ for $n \in \mathbb{Z}_{>0}$ and $x > 0$ [20, Definition 1.3]. This inequality is known to be strict unless $f$ is a constant. By the celebrated Bernstein theorem a function is completely monotone if and only if it is the Laplace transform of a nonnegative measure [20, Theorem 1.4]. A positive function $f$ is said to be logarithmically completely monotone (l.c.m.) if $-(\log f)'$ is completely monotone [20, Definition 5.8]. The class of logarithmically completely monotone functions is a proper subset of the class of completely monotone functions. Their importance stems from the fact

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that they represent Laplace transforms of infinitely divisible probability distributions, see [20, Theorem 5.9] and [19, Section 51]. The study of complete monotonicity of the ratio

$$U(x) = \prod_{i=1}^{p} \frac{\Gamma(x + a_i)}{\Gamma(x + b_i)},$$

where $\Gamma$ stands for Euler’s gamma function and $p \in \mathbb{Z}_{>0}$, has been initiated by Bustoz and Ismail who demonstrated in their 1986 paper [4] that for $p = 2$, $a_1 = 0$ and $a_1 + a_2 = b_1 + b_2$ this function is logarithmically completely monotone on $(0, \infty)$. Eight years later Ismail and Muldoon showed in [7] that $U(x)$ is l.c.m. on $(0, \infty)$ for general $p$ if $\sum a_i = \sum b_i$ and $b_1 = b_2 = \cdots = b_p > 0$, $a_i \geq 0$, or $b_1 = b_2 = \cdots = b_{p-1} = 0$, $b_p > 0$, $a_i \geq 0$. Their result is, in fact, formulated for the ratio of $q$-gamma functions, but the proof works for $U(x)$ just as well. The subject was further pursued by Alzer who showed in 1997 [1] that $U(x)$ is logarithmically completely monotone on $(0, \infty)$ if

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_p, \quad 0 \leq b_1 \leq b_2 \leq \cdots \leq b_p,$$

and $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i$ for $k = 1, 2, \ldots, p.$

These inequalities are known as weak supermajorization [14, Definition A.2] and are abbreviated as $b \prec^W a$, where $a = (a_1, \ldots, a_p)$, $b = (b_1, \ldots, b_p)$. In their 2006 paper [5] Grinshpan and Ismail found new sufficient conditions for logarithmic complete monotonicity of $U(x)$ when $p = 2^n$ or $p = n!/2$, $n \in \mathbb{Z}_{>0}$. We will explain and slightly generalize their results in the next section. Finally, in 2009 Guo and Qi [6] used another approach to investigate logarithmic complete monotonicity of $U(x)$ for arbitrary real values of $a_i, b_i$. Their results, however, lead to complete monotonicity of $U(x)$ on some subinterval of $(0, \infty)$ of the form $(\gamma, \infty)$, where $\gamma > 0$.

The purpose of the present paper is to investigate complete monotonicity of the ratio

$$W(x) = \frac{\prod_{i=1}^{p} \Gamma(A_i x + a_i)}{\prod_{j=1}^{q} \Gamma(B_j x + b_j)},$$

where $A = (A_1, \ldots, A_p)$ and $B = (B_1, \ldots, B_q)$ are strictly positive scaling factors, while $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_q)$ are nonnegative. We find a necessary and sufficient condition for $W$ to be l.c.m. on $(0, \infty)$ in terms of nonnegativity of some function which, unfortunately, is not easy to verify. Further, we supply several simple sufficient conditions for such nonnegativity in terms of the vectors $A, B, a, b$ as well as some necessary conditions. When $W$ is completely monotone, we proceed by deriving its representing measure in the Bernstein theorem which leads us to studying some new properties of Fox’s $H$-function. We begin, however, by revisiting the ratio defined in (1) which we call the unweighted case. In the following section we refine some of the known results for $U(x)$ and discuss its representing measure in Bernstein’s theorem.

2. The unweighted case revisited. All proofs of complete monotonicity of $U(x)$ we are aware of are based on the following two observations:
(1) if \(-\log U(x)\)' is completely monotone then so is \(U(x)\);
(2) the logarithmic derivative of the gamma function enjoys the integral representation
[13, p.16]
\[
\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-tz}}{1 - e^{-t}} dt,
\]
where \(\gamma\) is Euler-Mascheroni constant.

The second fact lead Grinshpan and Ismail to formulate Lemma 2.1 in [5] stating that
\[
q(x) = \prod_{k=1}^p \Gamma^{\sigma_k}(x + \lambda_k) \text{ with } \sum_{k=1}^p \sigma_k = 0 \text{ and } \lambda_k \geq 0 \text{ is logarithmically completely monotone on } (0, \infty) \text{ if and only if }
\]
\[
v(t) = \sum_{k=1}^p \sigma_k t^{\lambda_k} \geq 0 \text{ for } t \in (0,1].
\]

In fact, looking at asymptotic expansion of \(-\log q(x)\)' as \(x \to \infty\) it is easy to sharpen this lemma as follows: \(q(x)\) is logarithmically completely monotone on \((0, \infty)\) iff \(v(t) \geq 0\) on \((0,1]\) and \(\sum_{k=1}^p \sigma_k = 0\).

All further efforts of mathematicians who dealt with this problem were directed at finding sufficient conditions for nonnegativity of the Müntz polynomial \(v(t)\). When \(q(x) = U(x)\), where \(U\) is defined in (1),
\[
v(t) = \sum_{k=1}^p (t^{a_k} - t^{b_k}).
\]

As mentioned above Alzer noticed that \(v(t) \geq 0\) on \((0,1]\) under majorization conditions (2) which follows from 1949 result of Tomič [14, Proposition 4.B.2]. Let us remark that for \(p = 2\) conditions \(0 \leq \min(a_1,a_2) \leq \min(b_1,b_2)\) and \(a_1 + a_2 \leq b_1 + b_2\), equivalent to (2), are necessary and sufficient for nonnegativity of \(v(t)\). Necessity follows by considering the asymptotics of \(v(t)\) as \(t \to 0\) and \(t \to 1\). Formal proof of a more general result is given in Corollary 1 below.

Grinshpan and Ismail took another path in [5] and considered two types of factorization
\[
v(t) = \prod_{1 \leq i < j \leq n} (t^{\alpha_j} - t^{\alpha_i}) \text{ and } v(t) = \prod_{i=1}^n (1 - t^{\alpha_i}).
\]

It clear that for \(\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0\) for the first factorization and \(\alpha_i > 0\) for the second we get expressions nonnegative on \((0,1]\). The corresponding value of \(p\) is \(n!/2\) for the first factorization and \(2^{n-1}\) for the second. In both cases the authors of [5] found explicit combinatorial descriptions of the vectors \(a, b\) in (1) which lead to the above factorizations. We would like to remark here that taking \(v\) in the form
\[
v(t) = \prod_{i=1}^n (t^{\beta_i} - t^{\alpha_i})
\]
with \(\alpha_i \geq \beta_i \geq 0\) for \(i = 1, \ldots, n\) certainly results in nonnegative \(v\). This, of course, can be reduced to the second factorization of Grinshpan and Ismail by factoring out \(t^{\sum_i \beta_i}\). However, the description of the initial ratio \(U(x)\) turns out to be different and simpler.
**Theorem 1** Let \( I = \{1, 2, \ldots, n\} \) and suppose \( \mathcal{I}_{\text{odd}} \) (\( \mathcal{I}_{\text{even}} \)) comprises all subsets of \( I \) having odd (even) number of elements, \( \emptyset \in \mathcal{I}_{\text{even}} \). Suppose that \( \alpha_i \geq \beta_i \geq 0 \) for \( i = 1, \ldots, n \). Then

\[
U(x) = \frac{\prod_{J \in \mathcal{I}_{\text{even}}} \Gamma \left( x + \sum_{i \in J} \alpha_i + \sum_{i \in I \setminus J} \beta_i \right)}{\prod_{J \in \mathcal{I}_{\text{odd}}} \Gamma \left( x + \sum_{i \in J} \alpha_i + \sum_{i \in I \setminus J} \beta_i \right)}
\]

is logarithmically completely monotone on \((0, \infty)\).

**Example 1.** For \( \alpha_i \geq \beta_i \geq 0 \) for \( i = 1, 2, 3 \) the function

\[
U(x) = \frac{\Gamma(x + \beta_1 + \beta_2 + \beta_3)\Gamma(x + \beta_1 + \alpha_2 + \alpha_3)\Gamma(x + \alpha_1 + \beta_2 + \alpha_3)\Gamma(x + \alpha_1 + \alpha_2 + \beta_3)}{\Gamma(x + \alpha_1 + \beta_2 + \beta_3)\Gamma(x + \beta_1 + \alpha_2 + \beta_3)\Gamma(x + \beta_1 + \beta_2 + \alpha_3)\Gamma(x + \alpha_1 + \alpha_2 + \alpha_3)}
\]

is logarithmically completely monotone on \((0, \infty)\).

Further, changing variable in [18, formula 2.24.2.1] or in particular case of [11, Theorem 2.2] we get

\[
\prod_{i=1}^{p} \frac{\Gamma(x + a_i)}{\Gamma(x + b_i)} = \int_{(0, \infty)} e^{-tx} G_{p,p}^{p,0} \left( e^{-t} \middle| \begin{array}{c} b \\ a \end{array} \right) dt \tag{5}
\]

provided that \( \psi = \sum_{i=1}^{p} (b_i - a_i) > 0 \). Here \( G_{p,p}^{p,0} \) is Meijer’s G-function defined by the contour integral

\[
G_{p,p}^{p,0}(z \middle| \begin{array}{c} b \\ a \end{array} ) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(a_1 + s) \ldots \Gamma(a_p + s)}{\Gamma(b_1 + s) \ldots \Gamma(b_p + s)} z^{-s} ds.
\]

This function is a particular case of Fox’s H-function defined in (14) below on setting \( p = q \) and \( A_i = B_i = 1, i = 1, \ldots, p \) in that definition. A detailed description of the contour \( \mathcal{L} \) can be found below formula (14). Concise description is as follows: it begins and ends at infinity and leaves all the poles of the integrand on the left. Further, it follows from an expansion due to Nørlund [16, (2.28)] that for \( \psi = 0 \) the above formula must be modified as

\[
\prod_{i=1}^{p} \frac{\Gamma(x + a_i)}{\Gamma(x + b_i)} = \int_{[0, \infty)} e^{-tx} \left\{ \delta_0 + G_{p,p}^{p,0} \left( e^{-t} \middle| \begin{array}{c} b \\ a \end{array} \right) \right\} dt, \tag{6}
\]

where \( \delta_0 \) denotes the unit mass concentrated at zero. More information about Nørlund’s expansions can be found in [9]. Representation (5) was previously observed by us in [10].

Next we give an integral equation for Meijer’s G-function which we believe to be new.

**Theorem 2** Suppose \( a, b \geq 0 \) and \( \psi \geq 0 \). Then Meijer’s G-function satisfies the following integral equation:

\[
\log(1/x) G_{p,p}^{p,0} \left( x \middle| \begin{array}{c} b \\ a \end{array} \right) = \int_{x}^{1} G_{p,p}^{p,0} \left( t \middle| \begin{array}{c} b \\ a \end{array} \right) \sum_{k=1}^{p} \left( \frac{x^{a_k}}{t^{a_k}} - \frac{x^{b_k}}{t^{b_k}} \right) \frac{dt}{t - x}
\]

for \( 0 < x < 1 \).
Proof. Writing $U'(x) = U(x)(\log U(x))'$ and noting that each function here is a Laplace transform we can derive the above equation by the convolution theorem. Detailed proof of a more general result is given below in Theorem 7.

As we saw above the fact that logarithmic complete monotonicity of $U$ implies its complete monotonicity translates into the next implication:

$$v(t) = \sum_{k=1}^{p} (t^a_k - t^b_k) \geq 0 \text{ on } [0, 1) \Rightarrow G_{p,p}^{0,0} \left( x \left| \begin{array}{c} b \cr a \end{array} \right. \right) \geq 0 \text{ on } (0, 1)$$

under additional conditions $a_i \geq 0$, $i = 1, \ldots, p$, and $\sum_{i=1}^{p} (b_i - a_i) \geq 0$. The following much stronger assertion is supported by numerical evidence:

**Conjecture 1** If $a_i \geq 0$, $i = 1, \ldots, p$ and $\sum_{i=1}^{p} (b_i - a_i) \geq 0$ then

$$\# \left\{ \text{zeros of } G_{p,p}^{0,0} \left( x \left| \begin{array}{c} b \cr a \end{array} \right. \right) \text{ on } (0, 1) \right\} \leq \# \left\{ \text{zeros of } v(t) \text{ on } (0, 1) \right\}.$$ 

In fact, we were also able to demonstrate that if $G_{p,p}^{0,0}(x)$ has a zero on $(0, 1)$ then $v(t)$ also has at least one zero on this interval.

3. Generalized gamma ratio. For the positive vectors $A = (A_1, \ldots, A_p)$, $B = (B_1, \ldots, B_q)$ and nonnegative vectors $a = (a_1, \ldots, a_p)$, $b = (b_1, \ldots, b_q)$, consider the positive function

$$W(x) = \frac{\prod_{i=1}^{p} \Gamma(A_i x + a_i)}{\prod_{j=1}^{q} \Gamma(B_j x + b_j)}.$$ 

defined on $(0, \infty)$. The next theorem translates the study of logarithmic complete monotonicity of $W$ into the study of nonnegativity of the function $P$ defined by (8).

**Theorem 3** The function $W$ is logarithmically completely monotone if and only if

$$\sum_{j=1}^{q} B_j = \sum_{i=1}^{p} A_i, \quad \rho = \prod_{i=1}^{p} A_i^{A_i} \prod_{j=1}^{q} B_j^{-B_j} \leq 1 \quad (7)$$

and

$$P(u) = \sum_{i=1}^{p} e^{-a_i u / A_i} - \sum_{i=1}^{q} e^{-b_i u / B_i} \geq 0 \text{ for all } u > 0. \quad (8)$$

In the affirmative case

$$f(x) = -(\log W)' = \int_{0}^{\infty} e^{-xu} P(u) du + \log(1/\rho). \quad (9)$$
Proof. We prove necessity first. Compute

\[ f(x) = -(\log W)' = \sum_{j=1}^{q} B_j \psi(B_j x + b_j) - \sum_{i=1}^{p} A_i \psi(A_i x + a_i), \]

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \) denotes the logarithmic derivative of the gamma function. Using [2, Corollary 1.4.5]

\[ \psi(z) = \log(z) - \frac{1}{2z} + O(z^{-2}) \text{ as } z \to \infty \]

we get, after simple calculation,

\[
\begin{align*}
    f(x) &= \sum_{j=1}^{q} \left\{ B_j \log(B_j x + b_j) - \frac{B_j}{2(B_j x + b_j)} \right\} - \sum_{i=1}^{p} \left\{ A_i \log(A_i x + a_i) - \frac{A_i}{2(A_i x + a_i)} \right\} \\
    &\quad + O(x^{-2}) = \log(x) \left\{ \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i \right\} + \left\{ \sum_{j=1}^{q} B_j \log(B_j) - \sum_{i=1}^{p} A_i \log(A_i) \right\} \\
    &\quad + \frac{1}{x} \left\{ \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2} \right\} + O(x^{-2}) \text{ as } x \to \infty,
\end{align*}
\]

where we have applied the relations

\[ \log(x + \alpha) = \log(x) + \frac{\alpha}{x} + O(x^{-2}) \text{ and } \frac{1}{\alpha x + \beta} = \frac{1}{\alpha x} + O(x^{-2}) \text{ as } x \to \infty. \]

Since each completely monotonic function is decreasing and has a nonnegative limit as \( x \to \infty \), we conclude that for \( f \) to be completely monotonic conditions (7) are necessary. Next we apply the the integral representation (see [2, Theorem 1.6.1], [13, p.16] or [1, Proof of Theorem 10])

\[ \psi'(x) = \int_{0}^{\infty} \frac{te^{-xt}}{1-e^{-t}} dt, \quad x > 0, \]

to derive the representation

\[ f'(x) = -\int_{0}^{\infty} e^{-xu} uP(u)du, \quad (10) \]
where $P(u)$ is given by (8). Indeed, exchanging the sum and the integral and making change of variable $u = A_i t$ or $u = B_j t$ in the appropriate integrals we get

$$f'(x) = \sum_{j=1}^{q} B_j^2 \psi'(B_j x + b_j) - \sum_{i=1}^{p} A_i^2 \psi'(A_i x + a_i)$$

$$= \sum_{j=1}^{q} B_j^2 \int_{0}^{\infty} \frac{te^{-(B_j x + b_j)t}dt}{1 - e^{-t}} - \sum_{i=1}^{p} A_i^2 \int_{0}^{\infty} \frac{te^{-(A_i x + a_i)t}dt}{1 - e^{-t}}$$

$$= \sum_{j=1}^{q} \int_{0}^{\infty} u e^{xu - ub_j/B_j} du - \sum_{i=1}^{p} \int_{0}^{\infty} u e^{xu - ua_i/A_i} du$$

$$= - \int_{0}^{\infty} e^{-xu} \left\{ \sum_{i=1}^{p} \frac{e^{-a_i u/A_i}}{1 - e^{-u/A_i}} - \sum_{j=1}^{q} \frac{e^{-b_j u/B_j}}{1 - e^{-u/B_j}} \right\} du.$$

Integrating we get

$$f(\infty) - f(x) = \int_{x}^{\infty} f'(t)dt = - \int_{0}^{\infty} u P(u)du \int_{x}^{\infty} e^{-tu} dt = - \int_{0}^{\infty} e^{-xu} P(u)du.$$

By asymptotic expansion of $f$ given above we obtain

$$f(\infty) = \sum_{j=1}^{q} B_j \log(B_j) - \sum_{i=1}^{p} A_i \log(A_i) = - \log \rho,$$

where $\rho$ is defined in (7). Substituting $f(\infty)$ into the formula above yields representation (9). According to [20, Proposition 1.2] this representation is unique. Hence, if $f$ is completely monotonic then $P(u) \geq 0$ by Bernstein’s theorem [20, Theorem 1.4].

To prove sufficiency assume that conditions (7) are satisfied and $P(u) \geq 0$. It has been shown above that the first condition in (7) implies representation (9). The second condition in (7) and $P(u) \geq 0$ imply that the measure representing $f$ via Laplace transform is nonnegative. Complete monotonicity of $f$ now follows by Bernstein’s theorem. □

**Remark 1.** By substituting $t = e^{-u}$ condition (8) can be also written in the form

$$Q(t) = \sum_{i=1}^{p} \frac{t^{a_i/A_i}}{1 - t^{1/A_i}} - \sum_{j=1}^{q} \frac{t^{b_j/B_j}}{1 - t^{1/B_j}} \geq 0 \text{ for } t \in (0, 1).$$

(11)

**Remark 2.** Expression of the form $S_A = \sum_{i=1}^{p} A_i \log(A_i)$ for positive numbers $A_i$ is known as Shannon’s entropy in information theory, so that the second condition in (7) can be restated as ”entropy of the vector $A$ does not exceed that of the vector $B”$.

We now collect the necessary conditions for logarithmic complete monotonicity of $W$ in the next corollary.

**Corollary 1** The following conditions are necessary for $W$ to be logarithmically completely monotone:

...
a) \( \sum_{j=1}^{q} B_j = \sum_{i=1}^{p} A_i \)

b) \( \rho \leq 1 \)

c) \( \mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{1}{2}(p-q) \geq 0 \)

d) \( \min_{1 \leq i \leq p} \left( a_i/A_i \right) \leq \min_{1 \leq j \leq q} \left( b_j/B_j \right) \)

**Proof.** Necessity of a) and b) has been demonstrated in Theorem 3. Next, straightforward computation shows that

\[
P(u) = \frac{1}{u} \left\{ \sum_{i=1}^{p} A_i - \sum_{j=1}^{q} B_j \right\} + \left\{ \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{1}{2}(p-q) \right\} + O(u) \text{ as } u \to 0. \quad (12)
\]

Since the first sum is zero by a) we conclude that c) is necessary for nonnegativity of \( P(u) \). Further, using representation (11), we easily compute

\[
Q(t) = t^\alpha (1 + o(1)) - t^\beta (1 + o(1)) \text{ as } t \to 0,
\]

where \( \alpha = \min \left( a_i/A_i \right) \) and \( \beta = \min \left( b_j/B_j \right) \). Hence, we conclude that condition d) is also necessary. \( \square \)

**Remark 3.** Another method to show the necessity of condition \( \mu \geq 0 \) (condition c) above) is as follows. Calculating further derivatives of \( f \) and using

\[
(-1)^{n+1} \psi^{(n)}(x) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + O(x^{-n-2}) \text{ as } x \to \infty,
\]

we get

\[
(-1)^n f^{(n)}(x) = (-1)^{n+1} \sum_{i=1}^{p} \left( A_i^{n+1} \psi^{(n)}(A_ix + a_i) \right) - (-1)^{n+1} \sum_{j=1}^{q} B_j^{n+1} \psi^{(n)}(B_jx + b_j))
\]

\[
= \sum_{i=1}^{p} \left\{ \frac{(n-1)!A_i^{n+1}}{\left( A_ix + a_i \right)^n} + \frac{n!A_i^{n+1}}{2\left( A_ix + a_i \right)^{n+1}} \right\} - \sum_{j=1}^{q} \left\{ \frac{(n-1)!B_j^{n+1}}{\left( B_jx + b_j \right)^n} + \frac{n!B_j^{n+1}}{2\left( B_jx + b_j \right)^{n+1}} \right\} + O(x^{-n-2})
\]

\[
= \sum_{i=1}^{p} \left\{ \frac{(n-1)!A_i}{x^n} \left( 1 - \frac{na_i}{A_ix} \right) + \frac{n!}{2x^{n+1}} \right\} - \sum_{j=1}^{q} \left\{ \frac{(n-1)!B_j}{x^n} \left( 1 - \frac{nb_j}{B_jx} \right) + \frac{n!}{2x^{n+1}} \right\} + O(x^{-n-2})
\]

\[
= \frac{(n-1)!}{x^n} \left\{ \sum_{i=1}^{p} A_i - \sum_{j=1}^{q} B_j \right\} + \frac{n!}{x^{n+1}} \left\{ \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{1}{2}(p-q) \right\} + O(x^{-n-2})
\]

\[
= \frac{n!}{x^{n+1}} \left\{ \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{1}{2}(p-q) \right\} + O(x^{-n-2}),
\]
where the first sum vanishes due to the necessary condition a) from Corollary 1. This shows that this must require that \( \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i \geq \frac{1}{2}(p - q) \geq 0 \) in order for \((-1)^n f^{(n)}(x)\) to be nonnegative for large \(x\).

Recall that a nonnegative function \( f \) is called Bernstein function if \( f' \) is completely monotone [20, Definition 3.1 and below].

**Corollary 2** Suppose \( A, B > 0 \) and \( a, b \geq 0 \). Then the function

\[
g(x) = \sum_{i=1}^{p} A_i \psi(A_i x + a_i) - \sum_{j=1}^{q} B_j \psi(B_j x + b_j)
\]

is a Bernstein function if and only if condition (8) holds.

**Proof.** Derivation of representation (10) in the proof of Theorem 3 is valid for \( A, B > 0 \) and \( a, b \geq 0 \). Once this representation holds the statement of the corollary follows from Bernstein’s theorem. \( \square \)

Given the vectors \( A, B, a, b \) condition (8) is not easy to verify. The next theorem provides some practical sufficient conditions.

**Theorem 4** Inequality (8) is true if any of the following conditions holds:

(a) \( \sum_{j=1}^{q} B_j = \sum_{i=1}^{p} A_i \) and \( \max_{1 \leq i \leq p} (a_i/A_i) \leq \min_{1 \leq j \leq q} (b_j - 1)/B_j \);

(b) \( p = q \), \( \sum_{i=1}^{p} B_i = \sum_{i=1}^{p} A_i \) with \( A_i \geq B_i \) for \( i = 1, \ldots, p - 1 \), and

\[
\max_{1 \leq k \leq p-1} b_k/B_k \leq (b_p - 1)/B_p, \quad a_i/A_i \leq (b_i - 1)/B_i \quad \text{for} \quad i = 1, \ldots, p;
\]

(c) \( p = q \), \( 0 \leq a_1/A_1 \leq a_2/A_2 \leq \cdots \leq a_p/A_p \), \( 0 \leq b_1/B_1 \leq b_2/B_2 \leq \cdots \leq b_p/B_p \), \( 0 < 1/A_1 \leq 1/A_2 \leq \cdots \leq 1/A_p \), \( 0 < 1/B_1 \leq 1/B_2 \leq \cdots \leq 1/B_p \) and

\[
\sum_{i=1}^{k} a_i/A_i \leq \sum_{i=1}^{k} b_i/B_i, \quad \sum_{i=1}^{k} 1/A_i \leq \sum_{i=1}^{k} 1/B_i \quad \text{for each integer} \quad k = 1, \ldots, p.
\]

**Proof.** (a) Denote \( a'_i = a_i/A_i, \ A'_i = 1/A_i, \ b'_i = b_i/B_i, \ B'_i = 1/B_i, \ x = e^u \). Then by the mean value theorem we have

\[
\sum_{i=1}^{p} \frac{e^{-a_i u/A_i}}{1 - e^{-u/A_i}} = \sum_{i=1}^{p} \frac{e^{-a'_i u}}{1 - e^{-u/A'_i}} = \sum_{i=1}^{p} \frac{1}{x^{a'_i} - x^{a'_i - A'_i}} = \sum_{i=1}^{p} \frac{1}{A'_i x^{c_i} \log x} = \frac{1}{\log x} \sum_{i=1}^{p} A_i x^{-c_i},
\]

where \( c_i \in (a'_i - A'_i, a'_i) \). Hence, for \( P(u) \) we get

\[
P(u) = \frac{1}{u} \left\{ \sum_{i=1}^{p} A_i e^{-c_i u} - \sum_{j=1}^{q} B_j e^{-d_j u} \right\},
\]

where \( d_j \) is some point of the interval \( (b'_j - B'_j, B'_j) \). Denote \( c = \max c_i, \ d = \min d_j \). Suppose \( \max(a_i/A_i) \leq \min(b_j - 1)/B_j \), which is equivalent to \( \max(a'_i) \leq \min(b'_j - B'_j) \). Then \( c < d \) and nonnegativity of \( P(u) \) follows from the relations

\[
\sum_{i=1}^{p} A_i x^{-c_i} \geq \sum_{i=1}^{p} A_i x^{-c} \geq \sum_{i=1}^{p} A_i x^{-d} \geq \sum_{j=1}^{q} B_j x^{-d_j} \geq \sum_{j=1}^{q} B_j x^{-d_j}.
\]
(b) Denote $S = \sum_{j=1}^{p} B_j = \sum_{i=1}^{p} A_i$. Nonnegativity of $P(u)$ follows from representation (13) and the following chain
\[
\sum_{i=1}^{p} A_i e^{-c_i u} \geq \sum_{i=1}^{p} A_i e^{-d_i u} = \sum_{i=1}^{p-1} A_i e^{-d_i u} + \left( S - \sum_{i=1}^{p-1} A_i \right) e^{-d_p u}
\]
\[
= \sum_{i=1}^{p-1} A_i (e^{-d_i u} - e^{-d_p u}) + S e^{-d_p u} \geq \sum_{i=1}^{p-1} B_i (e^{-d_i u} - e^{-d_p u}) + S e^{-d_p u} = \sum_{i=1}^{p} B_i e^{-d_i u}.
\]

The first inequality here is due to conditions $a_i/A_i \leq (b_i - 1)/B_i$, $i = 1, \ldots, p$ which imply that $c_i < d_i$. The second inequality follows from conditions $A_i \geq B_i$, $i = 1, \ldots, p$, combined with inequality $\max_{1\leq k\leq p-1} b_k/B_k \leq (b_p - 1)/B_p$ which ensures that each term $e^{-d_i u} - e^{-d_p u}$ is nonnegative.

(c) Expanding each expression like $(1 - e^{-u/A_i})^{-1}$ in geometric series and exchanging the order of summations we get
\[
P(u) = \sum_{l=0}^{\infty} \sum_{i=1}^{p} \left( e^{-a_i u/A_i - lu/A_i} - e^{-b_i u/B_i - lu/B_i} \right).
\]

Conditions c) guarantee that for any $l \geq 0$, $u \geq 0$ we have
\[
(b_1 u/B_1 + lu/B_1, \ldots, b_p u/B_p + lu/B_p) \prec_W (a_1 u/A_1 + lu/A_1, \ldots, a_p u/A_p + lu/A_p)
\]
(see(2) for definition of weak supermajorization $\prec_W$). Hence, according to [14, Proposition 4.B.2] or [1, Lemma 2] we conclude that
\[
\sum_{i=1}^{p} \left( e^{-a_i u/A_i - lu/A_i} - e^{-b_i u/B_i - lu/B_i} \right) \geq 0
\]
since $x \rightarrow e^{-x}$ is convex and decreasing.

\textbf{Remark 4.} Conditions (c) from Theorem 4 are only compatible with condition $\sum_{j=1}^{p} A_j = \sum_{j=1}^{p} B_j$ if $A = B$ (up to permutation). Indeed, majorization $B' \prec_W A'$ (recall that $A'_j = A_j^{-1}$, $B'_j = B_j^{-1}$) forms a part of conditions (c). According to [14, Proposition 4.B.2] this majorization implies that
\[
\sum_{j=1}^{p} B_j = \sum_{j=1}^{p} \frac{1}{B'_j} \leq \sum_{j=1}^{p} \frac{1}{A'_j} = \sum_{j=1}^{p} A_j,
\]
because the function $x \rightarrow 1/x$ is decreasing and convex. Further, [14, 3.A.6.a] says that for continuous strictly decreasing functions the above inequality is strict unless $A'$ is a permutation of $B'$. This brings us to the conclusion that for $A \neq B$ (modulo permutations) conditions (c) can be used to check whether $g$ from Corollary 2 is a Bernstein function, but they cannot be used to check whether $W$ is completely monotone. On the other hand, if $A = B$, conditions (c) reduce to checking the majorization $a' \prec_W b'$ ($a'_j = a_j A_j^{-1}$, $b'_j = b_j A_j^{-1}$). This
majorization is weaker than (a) or (b) in this particular situation. One reason why the case \( A = B \) might be important is applications in probability as explained in Remark 5 below.

**Example 2.** The function

\[
x \to \frac{\Gamma(2x + 0.4)\Gamma(3x + 2.4)\Gamma(x + 0.9)}{\Gamma(x + 2)\Gamma(5x + 6)}
\]

is logarithmically completely monotone since \( A_1 + A_2 + A_3 = B_1 + B_2 + B_3, \rho = 0.03456 \) and \( \max_{1 \leq i \leq 3}(a_i/A_i) = 0.9, \min_{1 \leq i \leq 2}((b_i - 1)/B_i) = 1 \). Hence, necessary conditions (7) are satisfied and \( P(u) \geq 0 \) according to Theorem 4(a). Note, that neither conditions (b) nor (c) from Theorem 4 can be applied here as \( p \neq q \).

**Example 3.** The function

\[
x \to \frac{\Gamma(2x + 0.8)\Gamma(3x + 8)\Gamma(1.4x + 2.3)}{\Gamma(x + 1.5)\Gamma(2.4x + 7.8)\Gamma(3x + 11)}
\]

is logarithmically completely monotone since \( A_1 + A_2 + A_3 = B_1 + B_2 + B_3, \rho = 0.783668, \) so that necessary conditions (7) are satisfied and \( A_1 > B_1, A_2 > B_2, \max_{1 \leq i \leq 2}b_i/B_i = 3.25 < (b_3 - 1)/B_3 = 3.33, a_1/A_1 = 0.4 < (b_1 - 1)/B_1 = 0.5, a_2/A_2 = 2.66 < (b_2 - 1)/B_2 = 2.83, a_3/A_3 = 1.643 < (b_3 - 1)/B_3 = 3.33, \) so that \( P(u) \geq 0 \) by Theorem 4(b). Note that conditions (a) and (c) of Theorem 4 are violated here.

**Example 4.** The function

\[
x \to 4\psi(4x + 0.7) + 2\psi(2x + 1.8) - 3\psi(3x + 0.6) - \psi(x + 1.2)
\]

is a Bernstein function, since \( A_1 = 4 > A_2 = 2, B_1 = 3 > B_2 = 1, A_1^{-1} = 0.25 < B_1^{-1} = 0.33, A_1^{-1} + A_2^{-1} = 0.75 < B_1^{-1} + B_2^{-1} = 1.33, a_1A_1^{-1} = 0.175 < b_1B_1^{-1} = 0.2, a_1A_1^{-1} + a_2A_2^{-1} = 1.075 < b_1B_1^{-1} + b_2B_2^{-1} = 1.4, \) so that \( P(u) \geq 0 \) by Theorem 4(c) and the claim follows from Corollary 2. Note, that the function \( W \) with parameters from this example is not logarithmically completely monotone since necessary conditions (7) are violated.

**Example 4.** The function

\[
x \to \frac{\Gamma(3x + 0.8)\Gamma(2.2x + 1.8)\Gamma(1.4x + 2.3)}{\Gamma(3x + 1.2)\Gamma(2.2x + 1.7)\Gamma(1.4x + 2.5)}
\]

is logarithmically completely monotone by Theorem 4(c). Indeed, \( A_i = B_i, i = 1, 2, 3, \) and majorization conditions are easily verified.

**4. The representing measure.** We will need a particular case of Fox’s \( H \)-function defined by

\[
H_{q,p}^{p,0}(\mathcal{L} \mid (\mathbf{B}, \mathbf{a}) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{k=1}^{p} \Gamma(A_k s + a_k)}{\prod_{j=1}^{q} \Gamma(B_j s + b_j)} z^{-s} ds, \quad (14)
\]

where \( A_k, B_j > 0 \) and \( a_k, b_j \) are real. The contour \( \mathcal{L} \) can be either the left loop \( \mathcal{L}_- \) starting at \( -\infty + i\alpha \) and ending at \( -\infty + i\beta \) for some \( \alpha < 0 < \beta \) such that all poles of the integrand lie inside the loop; or the right loop \( \mathcal{L}_+ \) starting at \( \infty + i\alpha \) and ending at \( \infty + i\beta \) and leaving
all poles on the left; or the vertical line \( L_{ic}, \Re z = c \), traversed upwards and leaving all poles of the integrand on the left. Denote the rightmost pole of the integrand by \( \gamma \):

\[
\gamma = - \min_{1 \leq k \leq p} (a_k/A_k).
\]

Recall the definition of \( \rho \) from (7) and the definition of \( \mu \) from Corollary 1(c):

\[
\rho = \prod_{k=1}^{p} A_k^{a_k} \prod_{j=1}^{q} B_j^{-B_j}, \quad \mu = \sum_{j=1}^{q} b_j - \sum_{k=1}^{p} a_k + \frac{p-q}{2}.
\]

Existence conditions of \( H \)-function under each choice of the contour \( L \) have been thoroughly considered in the book [11]. Under restrictions (7) and \( x > 0 \) Theorem 1.1 from [11] states that \( H_{q,p}^{\rho,0}(x) \) exists if we choose \( L = L_+ \) or \( L = L_{ic} \) under additional restriction \( \mu > 1 \). Only the second choice of the contour ensures the existence of the Mellin transform of \( H_{q,p}^{\rho,0}(x) \) as demonstrated in [11, Theorem 2.2]. In our next theorem we relax the condition \( \mu > 1 \) to \( \mu > 0 \) and demonstrate that the first condition in (7) leads to the finite support of \( H_{q,p}^{\rho,0}(x) \).

**Theorem 5** Suppose \( \mu > 0 \) and \( \sum_{k=1}^{p} A_k = \sum_{j=1}^{q} B_j \). Then the integral over the contour \( L_{ic} \) with \( c > \gamma \) in the definition of \( H_{q,p}^{\rho,0}(x) \) converges for \( x > 0 \) except possibly for \( x = \rho \) and

\[
H_{q,p}^{\rho,0} \left( x \left| \begin{align*}
(B, b) \\
(A, a)
\end{align*} \right. \right) = 0
\]

for \( x > \rho \). Moreover, under these restrictions, the Mellin transform exits for \( \Re s > \gamma \) and

\[
\int_{0}^{\rho} H_{q,p}^{\rho,0} \left( x \left| \begin{align*}
(B, b) \\
(A, a)
\end{align*} \right. \right) x^{s-1} dx = \frac{\prod_{k=1}^{p} \Gamma(A_k s + a_k)}{\prod_{j=1}^{q} \Gamma(B_j s + b_j)}.
\]

**Proof.** By applying Stirling's asymptotic formula and in view of \( \sum_{k=1}^{p} A_k = \sum_{j=1}^{q} B_j \) it is not difficult to derive the formula [17, (2.2.4)]

\[
\frac{\prod_{k=1}^{p} \Gamma(A_k s + a_k)}{\prod_{j=1}^{q} \Gamma(B_j s + b_j)} = A \rho^{s-\mu} + g(s),
\]

with \( g(s) = O(s^{-\mu-1}) \) as \( |s| \to \infty \) uniformly in \( |\arg s| \leq \pi - \epsilon \) for any \( 0 < \epsilon < \pi \) and

\[
A = (2\pi)^{(p-q)/2} \prod_{k=1}^{p} A_k^{a_k-1/2} \prod_{j=1}^{q} B_j^{1/2-b_j}.
\]

This asymptotic behavior implies that \( t \to g(c+it) \) is absolutely integrable continuous function on the real line for any \( c > \gamma \) so that the integral \( v(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} g(s) ds \) exists and we are in the position to apply the Mellin inversion theorem yielding

\[
\int_{0}^{\infty} x^{s-1} v(x) dx = g(s).
\]

On the other hand, an application of [3, (6),§ 12] after a simple rearrangement gives

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} A \rho^{s-\mu} x^{-s} ds = \frac{A}{2\pi} \int_{-\infty}^{+\infty} e^{\log(\rho/x)(c+it)} dt = \begin{cases} 
\frac{A}{\Gamma(\mu)} (\log(\rho/x))^{-\mu-1}, & 0 < x < \rho, \\
A, & x > \rho.
\end{cases}
\]

12
Denote the function on the right by \( h(x) \). It then follows from (16) that

\[
H_{q,p}^{0,0} \left( x \mid (B, b) \right| (A, a) \right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \prod_{k=1}^{p} \frac{\Gamma(A_k s + a_k)}{\Gamma(B_j s + b_j)} x^{-s} ds = h(x) + v(x)
\]

and the above integral exists for all \( x > 0, x \neq \rho \). Direct computation gives

\[
\int_{0}^{\infty} x^{-1} H_{q,p}^{0,0} \left( x \mid (B, b) \right| (A, a) \right) dx = \frac{A}{\Gamma(\mu)} \int_{0}^{\rho} \left( \log \frac{\rho}{x} \right)^{\mu-1} x^{-1} dx + g(s) = A \rho^{\mu} s^{-\mu} + g(s),
\]

which is equivalent to

\[
\int_{0}^{\infty} x^{-1} H_{q,p}^{0,0} \left( x \mid (B, b) \right| (A, a) \right) dx = \prod_{k=1}^{p} \frac{\Gamma(A_k s + a_k)}{\Gamma(B_j s + b_j)}.
\]

Next we show that (15) holds so that the integration in the above formula is in fact over the interval \((0, \rho)\). According to the definition (14)

\[
H_{q,p}^{0,0} \left( z \mid (B, b) \right| (A, a) \right) = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{c-iR}^{c+iR} \prod_{k=1}^{p} \frac{\Gamma(A_k s + a_k)}{\Gamma(B_j s + b_j)} z^{-s} ds.
\]

By Cauchy’s theorem the last integral equals to the integral along the right semicircle of radius \( R \) centered at \( c \). Hence, we need to prove that

\[
I_R = \frac{R}{2\pi} \int_{-\pi/2}^{\pi/2} \prod_{k=1}^{p} \frac{\Gamma(A_k c + A_k Re^{i\varphi} + a_k)}{\Gamma(B_j c + B_j Re^{i\varphi} + b_j)} e^{-(c+R \cos \varphi + iR \sin \varphi) \log x + i \varphi} d\varphi
\]

goes to zero as \( R \to \infty \) for \( x > \rho \). Setting \( s = \Re^{i\varphi} \) in (16) we get

\[
\prod_{k=1}^{p} \frac{\Gamma(A_k c + A_k Re^{i\varphi} + a_k)}{\Gamma(B_j c + B_j Re^{i\varphi} + b_j)} = A^R \Re^{i\varphi} (\Re^{i\varphi})^{-\mu} (1 + O(R^{-1})) \quad \text{as} \quad R \to \infty.
\]

Hence, for sufficiently large \( R \)

\[
|I_R| \leq \frac{M e^{-c \log x}}{2\pi} \int_{-\pi/2}^{\pi/2} R^{-\mu + 1} e^{-R \log \frac{\rho}{\Re}} \Re^{i\varphi} d\varphi.
\]

Employing the elementary inequality \( \cos \varphi \geq 1 - \frac{2}{\pi} \varphi, 0 < \varphi < \pi/2 \), we obtain (recall that \( x > \rho \)):

\[
|I_R| \leq 2M_2 R^{-\mu + 1} \int_{0}^{\pi/2} e^{-R(1 - \frac{2}{\pi} \varphi) \log \frac{\rho}{\Re}} d\varphi = \pi M_2 R^{-\mu} \frac{(1 - e^{-R \log \frac{\rho}{\Re}})}{\log \frac{\rho}{\Re}}.
\]

Hence, \( \lim_{R \to \infty} I_R = 0 \) which completes the proof of the theorem. \( \square \)
Theorem 6 Suppose that $\mu > 0$ and

$$W(x) = \frac{\prod_{i=1}^{p} \Gamma(A_i x + a_i)}{\prod_{j=1}^{q} \Gamma(B_j x + b_j)}$$

is logarithmically completely monotone. Then

$$W(x) = \int_{\log(1/\rho)}^{\infty} e^{-tx} H_{q,p}^{p,0} \left( e^{-t} \left| \frac{(B, b)}{(A, a)} \right) \right) dt,$$

and the $H$-function in the integrand is nonnegative. In particular, the conclusion is true if conditions of Theorem 4 and inequality (7) are satisfied.

Proof. According to Theorem 3 logarithmic complete monotonicity of $W(x)$ implies $\sum_{k=1}^{p} A_k = \sum_{j=1}^{q} B_j$, so that we are in the position to apply Theorem 5 yielding the formula

$$W(x) = \prod_{i=1}^{p} \Gamma(A_i x + a_i) \prod_{j=1}^{q} \Gamma(B_j x + b_j) = \int_{0}^{\rho} H_{q,p}^{p,0} \left( u \left| \frac{(B, b)}{(A, a)} \right) \right) u x^{-1} du.$$

The claimed Laplace transform representation for $W(x)$ follows by substitution $u = e^{-t}$. Nonnegativity of $H_{q,p}^{p,0}(u)$ on $(0, \rho)$ follows from Bernstein theorem in view of uniqueness of the measure with given Laplace transform, see [20, Proposition 1.2] or [22, Theorem 6.3].

Theorem 6 requires $\mu$ to be strictly positive, while the necessary conditions from Corollary 1 allow for $\mu = 0$. In analogy with (6) in that case we believe in the validity of the following conjecture.

Conjecture 2 For $\mu = 0$ the representing measure is given by

$$W(x) = \int_{\log(1/\rho)\infty} e^{-tx} \left\{ \delta_{\log(1/\rho)} + H_{q,p}^{p,0} \left( e^{-t} \left| \frac{(B, b)}{(A, a)} \right) \right) \right\} dt,$$

where $\delta_{\log(1/\rho)}$ denotes the unit mass concentrated at the point $\log(1/\rho)$.

Unlike the case of $G$-function (6), no Nørlund’s expansion is available for $H$-function. A study of its behavior in the neighborhood of the singular point $\log(1/\rho)$ will be become a subject of our future work.

Corollary 3 Suppose $\mu > 0$, $a, b > 0$ and conditions (7), (8) are satisfied (the latter is in particular true under Theorem 4). Then the function

$$t \to \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)} H_{q,p}^{p,0} \left( e^{-t} \left| \frac{(B, b)}{(A, a)} \right) \right)$$

represents an infinitely divisible probability density on $(0, \infty)$. 14
Proof. The result follows from Theorem 6 combined with [20, Definition 5.6] and [20, Theorem 5.9]. See also [12, Theorem 7].

Remark 5. The case \( A = B \) is also connected to probability as follows. Suppose \( \zeta_k, k = 1, \ldots, p \) are independent beta-distributed random variables, so that \( \zeta_k \) has the density

\[
\frac{\Gamma(\alpha_k + \beta_k)}{\Gamma(\alpha_k)\Gamma(\beta_k)}t^{\alpha_k}(1-t)^{\beta_k}, \quad t \in (0,1).
\]

Then the random variable \( u = \prod_{k=1}^{p} \zeta_k^{A_k} \) with \( A_k > 0 \), has the following moments

\[
E(u^{x-1}) = \prod_{k=1}^{p} \frac{\Gamma(\alpha_k + \beta_k)}{\Gamma(\alpha_k)} \prod_{k=1}^{p} \frac{\Gamma(A_kx + \alpha_k - A_k)}{\Gamma(A_kx + \alpha_k + \beta_k - A_k)}, \quad x > 0.
\]

The probability density of \( u \) is expressed via Fox’s \( H \)-function:

\[
\prod_{k=1}^{p} \frac{\Gamma(\alpha_k + \beta_k)}{\Gamma(\alpha_k)} H_{p,0}^{p,p} \left( \begin{array}{c} (A_j, \alpha_j + \beta_j - A_j), j = 1, \ldots, p \\ (A_j, \alpha_j - A_j), j = 1, \ldots, p \end{array} \right)
\]

See [15, Section 4.2.1].

Theorem 7 Suppose \( A, B > 0, a, b \geq 0, \mu > 0, \rho \leq 1 \) and \( \sum_{i=1}^{p} A_i = \sum_{j=1}^{q} B_j \). Then for all \( x \in (0, \rho) \) the following identity holds:

\[
H_{q,0}^{p,p} \left( \begin{array}{c} (B, b) \\ (A, a) \end{array} \right) = \frac{1}{\log(\rho/x)} \int_{x/\rho}^{1} H_{q,0}^{p,p} \left( \begin{array}{c} (B, b) \\ (A, a) \end{array} \right) \frac{Q(u)}{u} du,
\]

where \( Q(u) \) is defined in (11).

Proof. Since \( -\log(\rho) = S_B - S_A \), where \( S_A \) (\( S_B \)) stands for the entropy of \( A \) (\( B \)), we adopt the notation \( \Delta S = S_B - S_A = -\log(\rho) \). Further, denote

\[
I(t) = \begin{cases} 1, & t \geq \Delta S \\ 0, & t < \Delta S \end{cases}
\]

and define the (signed) measure

\[
d\nu(u) = \Delta S\delta_0 + P(u)du,
\]

where \( \delta_0 \) is a unit mass at zero. It follows from the proof of Theorem 3 and the asymptotic formula (12) that representation (9) is true under conditions of the theorem. In terms of the measure \( d\nu \) (9) takes the form:

\[
W'(s) = W(\log W)' = -W \int_{[0,\infty)} e^{-su} d\nu(u).
\]

On the other hand, by Theorem 5 under the hypotheses of this theorem we have the representation

\[
W(s) = \int_0^\infty e^{-ts} H(e^{-t})I(t)dt, \quad \text{where} \quad H(x) = H_{q,p}^{p,0} \left( \begin{array}{c} (B, b) \\ (A, a) \end{array} \right).
\]
Hence,
\[ W'(s) = - \int_0^\infty t e^{-ts} H(e^{-t}) I(t) dt. \]

Putting these representations together yields
\[ \int_0^\infty t e^{-ts} H(e^{-t}) I(t) dt = \int_0^\infty e^{-ts} H(e^{-t}) I(t) dt \quad \int_{[0, \infty)} e^{-su} d\nu(u). \]

The convolution theorem for the Laplace transform [22, Theorem 11.4] then leads to the formula
\[ t H(e^{-t}) I(t) = \int_{[0, t]} H(e^{-t+\tau}) P(\tau) d\tau \]
for \( t \in (\Delta S, \infty). \)

Recalling the definition of \( d\nu \) we can rewrite the above relation as
\[ t H(e^{-t}) = \Delta S H(e^{-t}) + \int_0^{t-\Delta S} H(e^{-t+\tau}) P(\tau) d\tau \]
or
\[ H(e^{-t}) = \frac{1}{t - \Delta S} \int_0^{t-\Delta S} H(e^{-t+\tau}) P(\tau) d\tau. \]

Substituting \( x = e^{-t} \) and \( u = e^{-\tau} \) yields identity (18).

As we saw above the fact that logarithmic complete monotonicity of \( W \) implies its complete monotonicity translates into the next implication:
\[ Q(t) \geq 0 \text{ on } [0, 1) \Rightarrow H_{p,0}^{p,0} \left( \begin{array}{c} B, b \\ A, a \end{array} \right) \geq 0 \text{ on } (0, \rho). \]

The following much stronger assertion is supported by numerical evidence:

**Conjecture 3** Suppose \( A, B > 0, a, b \geq 0, \mu \geq 0, \rho \leq 1 \) and \( \sum_{i=1}^p A_i = \sum_{j=1}^q B_j. \) Then
\[ \# \left\{ \text{zeros of } H_{q,p}^{p,0} \left( \begin{array}{c} B, b \\ A, a \end{array} \right) \text{ on } (0, \rho) \right\} \leq \# \left\{ \text{zeros of } Q(t) \text{ on } (0, 1) \right\}. \]

Just like with Conjecture 1 we can demonstrate that this conjecture is true for the case when the left hand side is equal to one.

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