Some properties of surfaces of finite III-type

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Abstract: - In this paper, we firstly investigate some relations regarding the first and the second Laplace operators corresponding to the third fundamental form III of a surface in the Euclidean space \( E^3 \). Besides, we introduce the finite Chen type surfaces of revolution with nonvanishing Gauss curvature with respect to the third fundamental form. We present a special case of this family of surfaces of revolution in \( E^3 \), namely, surfaces of revolution with \( R \) is constant, where \( R \) denotes the sum of the radii of the principal curvature of a surface.

Key-Words: - Surfaces in the Euclidean 3-space, Surfaces of finite Chen-type, Laplace operator, Surfaces of revolution.

1 Introduction

One of the most interesting and profound aspects of differential geometry is the idea of surfaces of finite type which was born by B-Y. Chen in the early 1970s and since then, it has become a source of interest for many researchers in this field. The reader can refer to [17] for more details. In the framework of this kind of study, the first-named author with S. Stamatakis has given in [28] a new generalization to this area of study by giving a similar definition of surfaces of finite type.

Let \( x \) be an isometric immersion of a surface \( S \) in the 3-dimensional Euclidean space \( E^3 \). We represent by \( \Delta \) the Laplacian operator of \( S \) acting on the space of smooth functions \( C^\infty(S) \). Then \( S \) is said to be of finite \( J \)-type, \( J = I, II, III \), if the position vector \( x \) of \( S \) can be decomposed as a finite sum of eigenvectors of \( \Delta \) of \( S \), that is

\[
x = x_0 + x_1 + x_2 + ... + x_k, \quad (1)
\]

where

\[
\Delta x_i = \lambda_i x_i, \quad i = 1, ..., k,
\]

\( x_0 \) is a fixed vector and \( \lambda_1, \lambda_2, ..., \lambda_k \) are eigenvalues of the operator \( \Delta \).

Surfaces of finite type in \( E^3 \) regarding the second fundamental form were investigated for some important classes of surfaces. More precisely, the class of ruled surfaces was studied in [7], while in [3], H. AL-Zouabi studied tubular surfaces in \( E^3 \). Other classes such as translation surfaces, Quadric surfaces, surfaces of revolution, helicoidal surfaces, cyclides of Dupin, and spiral surfaces, the classification of its finite II-type surfaces still unknown. According to the third fundamental form, ruled surfaces in [4], tubes in [5], and quadric surfaces [6] are the only classes were investigated in \( E^3 \).

This type of study can be also extended to any smooth map, not necessary for the position vector of the surface, for example, the Gauss map of a surface. Regarding this see [8, 9].

Another generalization to the above, one can study surfaces in \( E^3 \) whose position vector \( x \) satisfies the following condition

\[
\Delta x = Ax, \quad J = I, II, III, \quad (2)
\]

where \( A \in \mathbb{R}^{3 \times 3} \).

Related to this, in [29] it was proved that the spheres and the catenoids are the only surfaces of revolution satisfying the above equation. Similarly, in [1] it was shown that helicoids and spheres are the only quadric surfaces in \( E^3 \) that satisfy (2). Next, in [2] condition (2) was studied for the class of translation surfaces. In fact, authored ascertained that Scherk's surface is the only translation surface in the Euclidean 3-space that satisfies (2), finally, in [24] the authors studied the class of translation surfaces in \( \text{Sol}_3 \) satisfying (2). Surfaces satisfying condition (2) are said to be of coordinate finite \( J \)-type.

Another interesting study is to find surfaces in \( E^3 \) whose Gauss map \( N \) satisfies the relation (2) that is
\[ \Delta'N = AN, \quad J = I, \, II, \, III, \]

For this problem, readers can be referred to [10, 11, 13, 18, 19, 20, 21].

Interesting research also one can follow the idea in [23,26] by defining the first and second Laplace operator using the definition of the fractional vector operators.

In order to achieve our goal, we briefly introduce a formula for \( \Delta^I x \) and \( \Delta^II N \) by using tensors calculations. Further, in the last section, we contribute to the solution of our main result.

2 Fundamentals

We consider a smooth surface \( S \) in \( E^3 \) given by a patch \( x = x(u^1, u^2) \) on a region \( U = (a, b) \times S \) of \( \mathbb{R}^3 \) in which does not contain parabolic points. We denote by

\[ I = g_{ij} du^i du^j, \quad II = b_{ij} du^i du^j, \quad III = e_{ij} du^i du^j \]

the three fundamental forms of \( S \). For any two differentiable functions \( f(u^1, u^2) \) and \( g(u^1, u^2) \) on \( S \), the first differential parameter of Laplace regarding the fundamental form \( J \) is defined by [12]

\[ \nabla^I(f, g) = d^I f_i g_j, \]  
where \( f_i = \frac{\partial f}{\partial u^i} \) and \( (d^I) \) denotes the inverse tensor of \((g_{ij}), (b_{ij}) \) and \((e_{ij}) \) for \( J = I, \, II \) and \( III \) respectively.

We first prove the following relations:

\[ \nabla^I(f, x) + \nabla^II(f, N) = 0, \quad (4) \]
\[ \nabla^II(f, x) + \nabla^III(f, N) = 0. \quad (5) \]

For the proof of (5) we use (3) and the Weingarten equations

\[ N_j = -e_{jk} b_{km} \, x_{m}, \quad (6) \]
to obtain

\[ \nabla^II(f, N) = b_{ij} f_j N_i = b_{ij} f_j b_{km} \, x_{m}, \]
\[ = -g^{im} f_j x_{m} = -\nabla^I(f, x), \]

being (4). We have similarly

\[ \nabla^III(f, N) = e_{ij} f_j N_i = -e_{ij} f_j e_{jk} b_{km} \, x_{m}, \]
\[ = -b^{im} f_j x_{m} = -\nabla^II(f, x), \]

which is (5).

The second Laplace operator according to the fundamental form \( J = I, \, II, \, III \) of \( S \) is defined by [10]

\[ \Delta'f = -d^I \nabla^I f_i, \]

where \( f \) is a sufficiently differentiable function, \( \nabla^I \) is the covariant derivative in the \( u^i \) direction with respect to the fundamental form \( J \) [12]. For \( J = III \) we have

\[ \Delta^III f = -e^{ij} \nabla^III f_{ij}, \quad (7) \]

We now compute \( \Delta^III x \) and \( \Delta^II N \). From (7) and the equations [19, p.128]

\[ \nabla^III x_j = -b^{km} \nabla^I_{m} b_{ij} x_{k} + b_{ij} N \]

we get

\[ \Delta^III x = e^{ij} b^{km} \nabla^I_{m} b_{ij} x_{k} - e^{ij} b_{ij} N. \quad (8) \]

Denote by

\[ A^{ij}_k = \frac{1}{2} e^{km} (-e_{ijm} + e_{imj} + e_{jim}), \]

the Christoffel symbols of the second kind regarding the third fundamental form. We put

\[ T^{ij}_k = \Gamma^{ij}_k - \Pi^{ij}_k, \]
\[ \tilde{T}^{ij}_k = A^{ij}_k - \Pi^{ij}_k. \]

It is known that [19, p.22]

\[ T^{ij}_k = -\frac{1}{2} b^{km} \nabla^I_{m} b_{ij}, \quad (9) \]
\[ \tilde{T}^{ij}_k = \frac{1}{2} b^{km} \nabla^III_{m} b_{ij} \quad (10) \]

and

\[ T^{ij}_k + \tilde{T}^{ij}_k = 0. \quad (11) \]

Besides, using Ricci’s Lemma

\[ \nabla^III e^{ij} = 0 \]

and the formula

\[ R = \frac{2H}{K} = e^{ik} b_{ik}, \quad (12) \]

where \( K \) is the Gauss curvature and \( H \) is the mean curvature of \( S \) respectively we have

\[ R_{im} = \nabla^III_{m} (e^{ik} b_{ik}) = e^{ik} \nabla^III_{m} b_{ik}. \quad (13) \]

From (9), (10), (11) and (13) we find
\[ e^{ij} b_{km} \nabla^I_m b_{ij} = -2e^{ij} T^k_{ij} = 2e^{ij} \tilde{T}^k_{ij} \]
\[ = -e^{ij} b_{km} \nabla^m_x b_{ij} = -b_{km} R_{ln} \]

so and
\[ e^{ij} b_{km} \nabla^I_m b_{ij} x_{ik} - b_{km} R_{ln} x_{ik} = -\nabla^M(R, x). \quad (14) \]

By combining (8), (12), and (14) we obtain [22]
\[ \Delta^{III} x = -\nabla^{III}(R, x) - RN. \]

Finally, using (5) we arrive at
\[ \Delta^{III} x = \nabla^{III}(R, N) - RN. \quad (15) \]

For the normal vector \( N \) we have
\[ (1.23) \quad \nabla^{III}_k N_{ih} = -e_{ik} N \]

we have
\[ \Delta^{III} N = -e^{ik} \nabla^{III} N_{ih} = e^{ik} e_{ik} N, \]

so that we conclude
\[ \Delta^{III} N = 2 N. \]

From the last equation, it can be seen that the Gauss map of every surface \( S \) in \( E^3 \) is of finite \( III \)-type 1, the corresponding eigenvalue is 2. Now we prove some relations according to the third fundamental form of \( f \).

For any differentiable function \( f(u^1, u^2) \) it can be easily shown that
\[ \Delta^{III}(f) = (\Delta^{III} f) x + f \Delta^{III} x - 2 \nabla^{III}(f, x) \]
\[ = (\Delta^{III} f) x + f \nabla^{III}(R, N) - fR N - 2 \nabla^{III}(f, x) \]

Similarly
\[ \Delta^{III}(f N) = (\Delta^{III} f) N + f \Delta^{III} N - 2 \nabla^{III}(f, N) \]
\[ = (\Delta^{III} f) N + 2f N + 2 \nabla^{III}(f, x) \]

Denote by \( W = -\langle x, N \rangle \) the support function of \( S \), where \( \langle \cdot, \cdot \rangle \) is the Euclidean inner product. Applying relation (7) for the function \( W \), it can be easily verified that
\[ \Delta^{III} W = e^{ik} \nabla^{III} x_{ik} = e^{ik} \nabla^{III} x_{ih} = -2 \nabla^{III}(x, N) \]
\[ = -2 \nabla^{III}(x, N) + \nabla^{III}(x, N) \]
\[ = -2e^{ik} x_{ih}, N > = -\Delta^{III} x, N > \]
\[ - \langle x, \Delta^{III} N \rangle + 2e^{ik} x_{ih}, b_{ik} g^{ij} x_{j} > \]
\[ = -\langle N, \nabla^{III}(R, N) - RN \rangle \]
\[ = -\langle x, 2N \rangle - 2e^{ik} x_{ih}, b_{ik} g^{ij} x_{j} > \]
\[ = 2W - R. \quad (16) \]

We consider now the surface \( S \) of finite \( III \)-type 1. Then we have \( \Delta^{III} x = kx \), where \( k \) is a constant eigenvalue.

From (15) we get \( \nabla^{III}(R, N) = RN \) \( \langle x, N \rangle \). Taking the inner product of both sides of this equation with \( N \) we find \( R = kW \), From the formula (16) we find that
\[ \Delta^{III} W = (2 - k)W, \Delta^{III} R = (2 - k)R. \]

Thus, we have proved the following:

**Theorem 1.** Let \( S \) be a surface in \( E^3 \) of finite \( III \)-type 1 with corresponding eigenvalue \( 1 \). Then the support function \( W \) and the sum of the principal radii of curvature \( R \) are of eigenfunctions of the Laplacian \( \Delta^{III} \) with corresponding eigenvalue 0.

Let now \( S \) be a minimal surface. Then we have
\[ R = \frac{2H}{K} = 0. \]

Thus from the equation (16) we get \( \Delta^{III} W = 2W \). So we have

**Corollary 1.** Let \( S \) be a minimal surface. Then the support function \( W \) is of an eigenfunction of \( \Delta^{III} \) with corresponding eigenvalue 0.

Let \( S' \) be a parallel surface of \( S \) (in directed) distance \( \mu = \text{const.} \neq 0 \), so that \( 1 - 2\mu H + \mu^2 K \neq 0 \). Then \( S' \) possesses the position vector \( x' = x + \mu N \).

Denoting by \( K' \) and \( H' \) the Gauss and mean curvature of \( S' \) respectively, we mention the following relations
\[ K' = \frac{K}{1 - 2\mu H + \mu^2 K}, \]
\[ H' = \frac{H - \mu K}{1 - 2\mu H + \mu^2 K}. \]

Hence we get
\[ R' = \frac{2H'}{K'} = R - 2\mu. \quad (17) \]
On the other hand, the surfaces $S$, $S'$ have common unit normal vector and spherical image. Thus $III = III'$ and $\Delta^\text{III} = \Delta^\text{III'}$. We prove now the following theorem for later use.

**Theorem 2.** Let $S$ be a minimal surface in $E^3$. Then $S'$ is a parallel surface of $S$ if and only if the sum of the principal radii of curvature $R'$ of $S'$ is constant.

**Proof.** Suppose that $S$ is a minimal surface in $E^3$, which is defined on a simply connected domain $D$ in the $(u^1, u^2)$-plane. Let

$$S': x' = x + \mu N, \mu \neq 0$$

be parallel surface of $S$. From (17) and taking into account $H = 0$, we find $R' = -2\mu = \text{const.}$. Hence the first part of the theorem is proved.

Conversely, let $R' = \text{const.} \neq 0$. Then from Theorem (4.4) (see [29]), $S'$ is of null $III$-type 2. Therefore from (1) there exist nonconstant eigenvectors $x_1(u^1, u^2)$ and $x_2(u^1, u^2)$ defined on the same domain $D$ such that

$$x' = x_1 + x_2,$$  

(18)

where $\Delta^\text{III} x_1 = \lambda_1 x_1$, $\Delta^\text{III} x_2 = \lambda_2 x_2$, and here we have $\lambda_1 = 0$ because $S'$ is of null $III$-type 2.

Once we have $\Delta^\text{III} x' = \Delta^\text{III} x_1 + \Delta^\text{III} x_2$, it then follows that

$$\Delta^\text{III} x' = \lambda_2 x_2.$$  

(19)

Besides, since $R' = \text{const.} \neq 0$, we find

$$\Delta^\text{III} x' = -R' N.$$  

(20)

Thus from (19) and (20), one finds

$$\lambda_2 x_2 = -R' N$$

or $x_2 = c N$, where $c = -\frac{R'}{\lambda_2}$, and then (18) becomes

$$x' = x_1 + c N.$$  

(21)

The differential of the above equation is

$$dx' = dx_1 + cdN.$$  

(22)

Taking the inner product of both sides of (22) with $N$ yields

$$<dx_1, N> = 0.$$  

(23)

Now we want to show that $x_1(u^1, u^2)$ is a regular parametric representation of a surface in $E^3$. It is enough to prove that

$$x_1[1] \times x_1[2] \neq 0, \quad \forall (u^1, u^2) \in D,$$

where $\times$ is the Euclidean cross product. We have

$$x_1 = x^* - \mu N,$$  

(24)

Using the Weingarten equations

$$N_\beta = -b_{ij}g^\beta x^*_i x^*_j,$$

and the equation (24), it follows that

$$x_1[1] \times x_1[2] = (x^*_1 - \mu N_1) \times (x^*_2 - \mu N_2)$$

$$= (x^*_1 \times x^*_2) - \mu (x^*_1 \times N_2) + \mu (x^*_2 \times N_1) + \mu^2 (N_1 \times N_2)$$

$$= (1 - 2\mu H + \mu^2 K)(x^*_1 \times x^*_2) \neq 0,$$

$$\forall (u^1, u^2) \in D.$$  

(25)

Hence, on account of (23) and (25), we conclude that $x_1(u^1, u^2)$ is a regular parametric representation of a surface in $E^3$ with $N$ its Gauss map.

Since $\Delta^\text{III} x_1 = 0$. Consequently, from Theorem (3.1) (see [29]), $x_1(u^1, u^2)$ is a minimal surface. Thus from (21), we obtain that $S'$ is a parallel surface of a minimal. Now we mention and prove our main theorem.

**Theorem 3.** The only surfaces of revolution in $E^3$ of which the sum of the radii of the principal curvature $R$ is constant are

- parts of spheres which are of finite $III$-type 1,
- catenoids which are of finite null $III$-type 1, and
- the parallel surfaces to the catenoids, which are of finite null $III$-type 2.

### 3 Proof of Theorem 3

Let $C$ be a smooth curve lies on the $xz$-plane parametrized by

$$x(u) = (f(u), 0, g(u)), u \in J, (J \subset R),$$

where $f, g$ are smooth functions and $f$ is a positive function. When $C$ is revolved about the $z$-axis, the
resulting point set $S$ is called the surface of revolution generated by the curve $C$. In this case, the $z$-axis is called the axis of revolution of $S$, and $C$ is called the profile curve of $S$. On the other hand, a subgroup of the rotation group which fixes the vector $(0, 0, 1)$ is generated by

$$\begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

Then the position vector of $S$ is given by see ([14, 24])

$$x(u, v) = (f(u) \cos v, f(u) \sin v, g(u)),$$

$$u \in J, v \in [0, 2\pi).$$  

Without loss of generality, we may assume that $C$ has the arc-length parametrization, i.e., it satisfies

$$(f')^2 + (g')^2 = 1,$$  

where $':= \frac{d}{du}$. Furthermore if $f' g' = 0$, then $f = \text{const.}$ or $g = \text{const.}$ and $S$ would be a circular cylinder or part of a plane, respectively. A case that has been excluded since $S$ would consist only of parabolic points.

The partial derivatives of (26) are

$$x_u = (f' (u) \cos v, f' (u) \sin v, g' (u)),$$

and

$$x_v = (-f(u) \sin v, f(u) \cos v, 0).$$

The components $e_\theta$ of the first fundamental form in (local) coordinates are the following

$$g_{11} = \langle x_u, x_u \rangle = 1, \quad g_{12} = \langle x_u, x_v \rangle = 0,$$

$$g_{22} = \langle x_v, x_v \rangle = f'^2.$$  

Denoting by $\kappa$ the curvature of the curve $C$ and $r_1, r_2$ the principal radii of curvature of $S$, we have

$$r_1 = \frac{1}{\kappa}, \quad r_2 = \frac{f}{g'}.$$  

The Gauss curvature and the mean curvature of $S$ are respectively

$$K = \frac{1}{r_1 r_2} = \frac{g''}{f} = -\frac{f''}{f},$$

and

$$2H = \frac{1}{r_1} + \frac{1}{r_2} = \kappa + \frac{g'}{f}.$$  

The Gauss map $\mathbf{N}$ of $S$ is computed as follows

$$\mathbf{N}(u, v) = (-g' \cos v, -g' \sin v, -f').$$  

Now, by using the natural frame $\{\mathbf{N}_u, \mathbf{N}_v\}$ of $S$ defined by

$$\mathbf{N}_u = (-g'' \cos v, -g'' \sin v, f''),$$

and

$$\mathbf{N}_v = (g' \sin v, -g' \cos v, 0)$$

the components $e_\theta$ of the third fundamental form in (local) coordinates are the following

$$e_{11} = \langle \mathbf{N}_u, \mathbf{N}_u \rangle = (g'')^2 + (f'')^2,$$

$$e_{12} = \langle \mathbf{N}_u, \mathbf{N}_v \rangle = 0, \quad e_{22} = \langle \mathbf{N}_v, \mathbf{N}_v \rangle = (g'')^2.$$  

The Beltrami operator $\Delta^{\III}$ in terms of local coordinates $(u, v)$ of $S$ can be expressed as follows

$$\Delta^{\III} \mathbf{g} = -\frac{1}{\kappa^2} \frac{\partial}{\partial v} \left( \frac{\partial}{\partial u} \right) + \frac{g' \kappa' - g'' \kappa}{g' \kappa^3} \frac{\partial}{\partial u} - \frac{1}{g'^2} \frac{\partial^2}{\partial v^2}. \quad (29)$$

On account of (27) we put

$$f' = \cos \varphi, \quad g' = \sin \varphi,$$

where $\varphi = \varphi (u)$. Then $\kappa = \varphi'$ and the parametric representation (28) of the unit vector $\mathbf{N}$ of $S$ becomes

$$\mathbf{N}(u, v) = \{-\sin \varphi \cos v, -\sin \varphi \sin v, \cos \varphi \}. \quad (30)$$

Also relation (29) takes the following form

$$\Delta^{\III} \mathbf{g} = -\frac{1}{\varphi''} \frac{\partial}{\partial \varphi} \left( \frac{\partial}{\partial u} \right) + \left( \frac{\varphi'' - \cos \varphi}{\varphi'^3} \right) \frac{\partial}{\partial u} - \frac{1}{\sin^2 \varphi} \frac{\partial^2}{\partial v^2}. \quad (31)$$

For the sum of the principal radii of curvature $R = r_1 + r_2 = \frac{2H}{\kappa}$, one finds

$$R = \frac{f}{\sin \varphi} + \frac{1}{\varphi'}. \quad (32)$$

Taking the derivative of (32) we find

$$R' = \frac{\varphi''}{\varphi'^3} - \frac{fg' \cos \varphi}{\sin^2 \varphi} + \frac{\cos \varphi}{\sin \varphi}. \quad (33)$$
Let \((x_1, x_2, x_3)\) be the coordinate functions of (26). By virtue of (31), we obtain

\[
\Delta_{III}^{x_1} = \Delta_{III}(f \cos v) = 
\left( \frac{\varphi'' \cos \varphi - \frac{1}{\varphi' \sin \varphi} \frac{2 \sin \varphi}{\varphi'} + \frac{f}{\sin \varphi} \right) \cos v
\]

\[
\Delta_{III}^{x_2} = \Delta_{III}(f \sin v) = 
\left( \frac{\varphi'' \cos \varphi - \frac{1}{\varphi' \sin \varphi} \frac{2 \sin \varphi}{\varphi'} + \frac{f}{\sin \varphi} \right) \sin v
\]

\[
\Delta_{III}^{x_3} = \Delta_{III}(g) = -\frac{2 \cos \varphi}{\varphi'} + \frac{\varphi'' \sin \varphi}{\varphi'^2}
\]

From (32) and (33), equations (34), (35) and (36) become respectively

\[
\Delta_{II}^{x_1} = \left( R \sin \varphi - \frac{R' \cos \varphi}{\varphi'} \right) \cos v
\]

\[
\Delta_{II}^{x_2} = \left( R \sin \varphi - \frac{R' \cos \varphi}{\varphi'} \right) \sin v
\]

\[
\Delta_{II}^{x_3} = -\frac{R' \sin \varphi}{\varphi'} - R \cos \varphi
\]

We obtain the following two cases:

**Case I.** \(R = 0\). Thus \(H \equiv 0\). Consequently \(S\), being a minimal surface of revolution, is a catenoid.

**Case II.** \(R \neq 0\). From (37), (38), and (39) we obtain

\[
\Delta_{II}^{x_1} = R \sin \varphi \cos v
\]

\[
\Delta_{II}^{x_2} = R \sin \varphi \sin v
\]

\[
\Delta_{II}^{x_3} = -R \cos \varphi
\]

Let \((N_1, N_2, N_3)\) be the coordinate functions of \(N\). From (29), relations (40) can be written

\[
\Delta_{III}^{x_1} = -RN_1, \quad \Delta_{III}^{x_2} = -RN_2, \quad \Delta_{III}^{x_3} = -RN_3,
\]

and hence

\[
\Delta_{III} x = -RN.
\]

In view of (7) and (41) we have

\[
(\Delta_{III}^0)x = -(2^{n-1})RN.
\]

Now, if \(S\) is of finite type \(k\), then there exist real numbers \(c_1, c_2, \ldots, c_k\) such that

\[
(\Delta_{III})^0 x + c_1(\Delta_{III})^{-1} x + \ldots + c_k x = 0.
\]

From (41) and (42), relation (43) becomes

\[
-2^{k-1}RN - 2^{k-2}c_1RN - \ldots - c_{k-1}RN + c_k = 0,
\]

or

\[
c N + c_k x = 0,
\]

where \(c = -R(2^{k-1} + 2^{k-2}c_1 + \ldots + c_{k-1}) = \text{const.}\).

Now, if \(c_k \neq 0\), then from (44) we have \(x = -\frac{1}{c_k} N\), and hence we get \(|x| = \frac{1}{|c|}\) and so \(S\) is a sphere. On account of Theorem (3.3) (see [23]), \(S\) is of finite \(III\)-type 1. If \(c_1 = 0\), then \(S\) is of null type \(k\). Since \(R = \text{const.}\), thus according to Theorem (4.4) (see [29]) and Theorem (2), \(S\) is of null \(III\)-type 2 which is a parallel surface of a minimal.

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