A Conjecture on Induced Subgraphs of Cayley Graphs

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Abstract

In this paper, we propose the following conjecture which generalizes a theorem proved by Huang [Hua19] in his recent breakthrough proof of the sensitivity conjecture. We conjecture that for any Cayley graph $X = \Gamma(G, S)$ on a group $G$ and any generating set $S$, if $U \subseteq G$ has size $|U| > |G|/2$, then the induced subgraph of $X$ on $U$ has maximum degree at least $\sqrt{|S|}/2$.

Using a recent idea of Alon and Zheng [AZ20], who proved this conjecture for the special case when $G = \mathbb{Z}_n^2$, we prove that this conjecture is true whenever $G$ is abelian. We also observe that for this conjecture to hold for a graph $X$, some symmetry is required: it is insufficient for $X$ to just be regular and bipartite.

1 Introduction

Huang [Hua19] gave a remarkably simple and elegant proof of the Sensitivity Conjecture of Nisan and Szegdy [NS94]. His proof showed that any subset $U$ of the $n$-dimensional Boolean cube of size $|U| > 2^{n-1}$ induces a subgraph with maximum degree at least $\sqrt{n}$, which improves a lower bound of Chung et al. [CFG88] exponentially and is known to imply the Sensitivity Conjecture [GL92].

The proof has attracted considerable attention and there are already a number of extensions and generalizations to both the technique and the result itself. Among those, Alon and Zheng [AZ20] proved that Huang’s result implies the same phenomenon for not just the Boolean cube, but any Cayley graph of $\mathbb{Z}_2^n$. In this paper, we prove that the same phenomenon holds for any Cayley graph of any abelian group and conjecture that it holds for any Cayley graph.

1.1 Basic definitions and statement of the conjecture

We need the following basic definitions. All the groups we consider in this paper are finite.

Definition 1.1 (Cayley Graphs). Given a group $G$ and a set of non-identity elements $S$ of $G$, the Cayley graph $X = \Gamma(G, S)$ is the graph with vertices $V(X) = G$ and edges $E(X) = \{(g, sg) : g \in G, s \in S\}$. Here we take $X = \Gamma(G, S)$ to be undirected and we assume that $S$ is symmetric, i.e. if $s \in S$ then $s^{-1} \in S$.

Remark. Without loss of generality, we can assume that $S$ generates $G$ as otherwise, letting $G'$ be the subgroup of $G$ which is generated by $S$, $\Gamma(G, S)$ consists of $|G|/|G'|$ disjoint copies of $\Gamma(G', S)$.

Definition 1.2 (Boolean Hypercube). We define $Q_n$ to be the $n$ dimensional boolean hypercube, i.e. $Q_n = \Gamma(\mathbb{Z}_2^n, \{e_i : i \in [n]\})$.

Definition 1.3 (Induced Subgraphs). Given a graph $X$ and a subset of vertices $U \subseteq V(X)$, we define $X(U)$ to be the induced subgraph of $X$ on $U$, i.e. $V(X(U)) = U$ and $E(X(U)) = \{(u_1, u_2) \in E(X) : u_1, u_2 \in U\}$.

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With these definitions, we now state our conjecture.

**Conjecture 1.** For any Cayley graph \( X = (G, S) \) and any \( U \subseteq G \) such that \( |U| > |G|/2 \), the induced subgraph \( X(U) \) of \( X \) on \( U \) has maximum degree at least \( \sqrt{|S|/2} \).

## 2 Proof of the conjecture for abelian groups

In this section, we prove that Conjecture 1 is true for abelian groups. More precisely, we prove the following theorem.

**Theorem 2.1.** For any Cayley graph \( X = \Gamma(G, S) \) such that \( G \) is abelian and any \( U \subseteq G \) of size \(|U| > |G|/2\), the induced subgraph \( X(U) \) of \( X \) on \( U \) has maximum degree at least \( \sqrt{|S|+t}/2 \) where \( t \) is the number of elements in \( S \) of order 2.

We prove this theorem in two steps:

1. We show that Huang’s theorem [Hua19] implies that the same property holds for products of cycles.
2. We generalize the argument used by Alon and Zheng [AZ20] to prove Conjecture 1 for \( G = \mathbb{Z}_2^n \) to prove the result for all abelian \( G \).

### 2.1 From the Boolean hypercube to products of cycles

We recall Huang’s theorem and show how it implies the same property for products of cycles.

**Theorem 2.2 ([Hua19]).** For every integer \( n \geq 1 \), if \( H \) is an induced subgraph of \( Q_n \) with at least \( (2^n - 1 + 1) \) vertices, then the maximum degree of \( H \) is at least \( \sqrt{n} \).

**Corollary 2.3.** Let \( G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_d} \), \( S = \{\pm e_1, \ldots, \pm e_d\} \), and \( X = \Gamma(G, S) \). For any \( U \subseteq G \) of size \(|U| > |G|/2\), there is an element \( u \in U \) and \( k \geq \sqrt{d} \) distinct indices \( i_1, \ldots, i_k \in [d] \) such that for all \( j \in [k] \), either \( u + e_{i_j} \in U \) or \( u - e_{i_j} \in U \).

**Proof.** To prove this, we cover \( X \) with copies of the Boolean hypercube \( Q_d \).

**Definition 2.4.** Let \( U_r = \{r + \sum_{i \in T} e_i : T \subseteq [d]\} \).

Observe that \( \mathbb{E}_{r}[U_r \cap U] > 2^{n-1} \) where the expectation is over uniform random \( r \in G \). Thus, there must be some \( g \in G \) that satisfies \( |U_g \cap U| > 2^{n-1} \). Since \( X(U_g) \) is isomorphic to the Boolean cube \( Q_d \) of dimension \( d \), by Huang’s theorem, the induced subgraph \( X(U_g \cap U) \) of \( X \) on \( U_g \cap U \) has maximum degree at least \( \sqrt{d} \). \( \Box \)

![Figure 1: An illustration of some of the sets \( \{U_r : r \in G\} \) when \( d = 2 \) and \( m_1 = m_2 = 3 \). The sets \( U_{02}, U_{12}, U_{20}, U_{21}, U_{22} \) wrap around and are not shown.](image-url)
2.2 From products of cycles to abelian Cayley graphs

We now apply an argument of Alon and Zheng [AZ20] to prove Theorem 2.1.

Proof of Theorem 2.1. We can assume without loss of generality that \( G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k} \). Denote \( S = \{s_1, \ldots, s_k, s_{k+1}, \ldots, s_{d}\} \) and let \( m = \text{lcm}(m_1, \ldots, m_k) \). We consider the Cayley graph \( X' = \Gamma(\mathbb{Z}_m, T) \) where \( T = \{e_1, \ldots, e_d\} \). Let \( A : \mathbb{Z}_m^d \to G \) be a linear map defined by \( A(e_i) = s_i \). Note that \( A \) is well-defined because \( \text{ord}(s_i)| m \) for all \( i \in [d] \).

Since \( S \) is a generating set of \( G \), the linear map \( A \) is onto. Thus, for all \( g \in G \), \( A^{-1}(g) \) has size \( m^d/|G| \). It follows that \( A^{-1}(U) \) has size \( |A^{-1}(U)| = (m^d/|G|)|U| > (m^d/|G|)(|G|/2) = m^d/2 \). By Corollary 2.3, there is a vertex \( h \in A^{-1}(U) \) and \( k \geq \sqrt{d} \) distinct indices \( i_1, \ldots, i_k \in [d] \) such that for all \( j \in [k] \), either \( h + e_{i_j} \) or \( h - e_{i_j} \) is in \( A^{-1}(U) \). Take \( h_j \in \{h + e_{i_j}, h - e_{i_j}\} \) so that \( h_j \in U \) (if both elements are in \( U \) then this choice is arbitrary) and observe that for all \( j' \neq j \in [k] \),

\[
A(h_{j'}) - A(h_{j}) = A(h) \pm A(e_{i_{j'}}) - A(h) \mp A(e_{i_j}) = \pm s_{i_{j'}} \mp s_{i_j} \neq 0.
\]

Thus, all \( A(h_1), \ldots, A(h_k) \) are distinct, contained in \( U \), and adjacent to \( A(h) \in U \) in \( X(U) \) where \( X = \Gamma(G, S) \). Finally, \( |S| = t + 2(d - t) \) and hence \( d = (|S| + t)/2 \), as desired. \( \Box \)

3 A counterexample for regular, bipartite graphs

In this section, we observe that for Conjecture 1 to hold, it is not sufficient for \( X \) to be regular and bipartite. In particular, we construct a regular bipartite graph \( X = (L, R, E) \) such that there exists a subset of size \( |L| + 1 \) that induces a subgraph with maximum degree 1.

![Figure 2: An illustration of the graph \( X \) for \( n = 2 \).](image)

Let \( A, B, C, D \) be disjoint sets of size \( |A| = |C| = n + 1 \) and \( |B| = |D| = n \). Let \( L = A \cup B \) and \( R = C \cup D \). Let \( E \) be the union of a perfect matching between \( A \) and \( C \), the set of all edges between \( A \) and \( D \), and the set of all edges between \( B \) and \( C \). It is straightforward to check that \( X \) is \((n + 1)\)-regular, but the set \( A \cup B \) has size \( 2(n + 1) = |L| + 1 \) and induces a subgraph with maximum degree 1. A concrete drawing of such graph for \( n = 2 \) is shown in Figure 2.

4 Open problems

The obvious open problem is to prove or disprove Conjecture 1. There are a few more problems that we find interesting:
1. Our counterexample shows that we cannot replace being Cayley by just regular in Conjecture 1. What about vertex-transitive? Can we find a counterexample where $X$ is vertex-transitive or find evidence that being vertex-transitive is sufficient?

2. Huang actually proved a stronger claim that the Boolean cube $Q_n$ admits an orthogonal signing, which is a signed adjacency matrix of $Q_n$ whose eigenvalues are either $-\sqrt{n}$ or $\sqrt{n}$. Alon and Zheng [AZ20] considered the more general unitary signing and showed that any Cayley graph of $\mathbb{Z}_2^n$ with respect to $S$ of size at most $n+1$ admits such a signing. They also observed that some Cayley graphs of $\mathbb{Z}_2^n$ of degree $2n+1$ do not admit such signings, which implies that the tight maximum degree bound of induced subgraph cannot always be proved by finding a good signing. Despite that, finding a signing such that all eigenvalues have large modulus still seems to be an interesting problem. For Cartesian products of $d$ many even-length cycles, it is not difficult to find a signing with all eigenvalues at least $\Omega_k(\sqrt{d})$ in absolute values, where $k$ is the maximum length of the cycles in the product. Can we find a signing for any finite bipartite abelian Cayley graph $\Gamma(G, S)$ with all eigenvalues at least $|S|^{\Omega(1)}$ in absolute value?

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