A STRONG BOREL–CANTIELI LEMMA FOR RECURRENCE

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ABSTRACT. Consider a dynamical systems ([0,1], T, µ) which is exponentially mixing for L^1 against bounded variation. Given a non-summable sequence \langle m_k \rangle of non-negative numbers, one may define r_k(x) such that \mu(B(x, r_k(x))) = m_k. It is proved that for almost all x, the number of k \leq n such that T^k(x) \in B(x) is approximately equal to \sum_{i=1}^n m_i. This is a sort of strong Borel–Cantelli lemma for recurrence.

A consequence is that

\lim_{r \to 0} \frac{\log \tau_{B(x,r)}(x)}{- \log \mu(B(x,r))} = 1

for almost every x, where \tau is the return time.

1. Recurrence

A classical subject of study in the theory of dynamical systems is the concept of recurrence, which goes back to the well known Poincaré recurrence theorem, proved by Carathéodory [4].

Boshernitzan proved that if \langle X, T, \mu \rangle is a measure preserving system of a metric space X which is \sigma-finite with respect to the \alpha dimensional Hausdorff measure, then

\lim \inf_{k \to \infty} k^{\frac{1}{\alpha}} d(T^k(x), x) < \infty

for \mu almost every x. In other words, for almost every x, there is a c > 0 such that T^k(x) \in B(x, ck^{-\frac{1}{\alpha}}) for infinitely many k. Yet another way to say this is that for almost all x, there is a constant c > 0 such that

\sum_{k=1}^{\infty} 1_{B(x, c k^{-\frac{1}{\alpha}})}(T^k(x)) = \infty,

where 1_E denotes the indicator function of the set E.
Recent improvements of Boshernitzan’s result for particular classes of dynamical systems can be found in papers by Pawelec \cite{14}; Chang, Wu and Wu \cite{5}; Baker and Farmer \cite{1}; Hussain, Li, Simmons and Wang \cite{8}; and by Kirsebom, Kunde and Persson \cite{10}.

In this paper we shall refine the investigation and consider the typical growth speed of
\[
\sum_{k=1}^{n} 1_{B(x,r_k)}(T^k(x))
\]
for certain sequences $r_k$. That is, we shall count the number of close returns before time $n$ of a point $x$ to itself.

In the setting of hitting a shrinking target $B(y_k,r_k)$, one can sometimes prove that for almost all $x$ there are infinitely many $k$ such that $T^k(x) \in B(y_k,r_k)$ provided $\sum_{k=1}^{\infty} \mu(B(y_k,r_k)) = \infty$. Such results are called dynamical Borel–Cantelli lemmata. It is sometimes possible to prove the stronger result that
\[
\sum_{k=1}^{n} 1_{B(y_k,r_k)}(T^k(x)) \sim \sum_{k=1}^{n} \mu(B(y_k,r_k))
\]
for almost all $x$, see for instance Kim \cite{9}. Such results are called strong dynamical Borel–Cantelli lemmata. Note that here the centre $y_k$ of the target $B(y_k,r_k)$ may depend on $k$, but does not depend on $x$ as in the results for recurrence mentioned above.

The result of this paper is a sort of strong Borel–Cantelli lemma for recurrence. In a certain sense, we shall obtain that
\[
\sum_{k=1}^{n} 1_{B(x,r_k)}(T^k(x)) \sim \sum_{k=1}^{n} \mu(B(x,r_k))
\]
holds for almost all $x$, under some rather mild assumptions.

For a measure preserving dynamical system $([0,1], T, \mu)$, we say that correlations decay exponentially for $L^1$ against $BV$, if there are constants $c, \tau > 0$ such that
\[
\left| \int f \circ T^n g \, d\mu - \int f \, d\mu \int g \, d\mu \right| \leq c e^{-\tau n} \|f\|_1 \|g\|_{BV},
\]
where $\|f\|_1 = \int |f| \, d\mu$ and $\|g\|_{BV}$ is the bounded variation norm, defined by
\[
\|g\|_{BV} = \sup |g| + \text{var } g,
\]
where \text{var } $g$ is the total variation of $g$. It is well known that correlations decay exponentially for $L^1$ against $BV$, for instance if $T$ is a piecewise uniformly expanding map of $[0,1]$ and $\mu$ is a Gibbs measure \cite{12,15,13}. In that case, it is often the case that there are constants $c, s > 0$ such that $\mu(B(x,r)) \leq cr^s$ always hold.
The result of this paper is the following theorem, which can be thought of as a strong Borel–Cantelli lemma for recurrence.

**Theorem.** Suppose that $T: [0,1] \to [0,1]$ has an invariant measure $\mu$ such that correlations for $L^1$ against $BV$ decay exponentially, and that there are constants $c, s > 0$ such that
\[
\mu(B(x, r)) \leq cr^s
\]
holds for all points $x$ and all $r > 0$.

Let $(m_k)$ be a sequence such that
\[
m_k \geq \frac{(\log k)^{4+\epsilon}}{k}
\]
for some $\epsilon > 0$, and such that
\[
\lim_{\rho \to 1^-} \limsup_{k \to \infty} \frac{m_k}{m_{\lfloor \rho k \rfloor}} = 1.
\]

Let $B_k(x)$ be the ball with centre $x$ and $\mu(B_k(x)) = m_k$. Then $\mu$ almost every $x$ satisfies
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} 1_{B_k(x)}(T_k(x))}{\sum_{k=1}^{n} \mu(B_k(x))} = 1.
\]

The last section of this paper contains two examples of systems for which results of this type do not hold.

2. **Overview of the proof**

Let $E_k = \{ x : T_k(x) \in B_k(x) \}$. Then $T_k(x) \in B_k(x)$ if and only if $x \in E_k$. Hence, the goal is to estimate the sum
\[
\sum_{k=1}^{n} 1_{E_k}(x).
\]

It turns out that it is difficult to get good enough estimates to handle this sum directly. The idea is therefore to instead consider the sum
\[
S_n(x) = \sum_{k=1}^{n} \frac{1_{E_k}(x)}{\mu(E_k)},
\]
which has the useful property that
\[
\int S_n \, d\mu = n.
\]

Using estimates on correlations, one obtains through Lemma 1 from Section 4 the asymptotic estimate
\[
S_n(x) = \sum_{k=1}^{n} \frac{1_{B_k(x)}(T_k(x))}{\mu(E_k)} \sim \sum_{k=1}^{n} \frac{1_{B_k(x)}(T_k(x))}{\mu(B_k(x))} \sim n
\]
for almost all $x$. Under some assumptions on the sequence $m_k = \mu(B_k(x))$, Lemma 2 of the same section, implies that a consequence of (1) is that

$$\frac{\sum_{k=1}^{n} 1_{B_k(x)}(T^k(x))}{\sum_{k=1}^{n} \mu(B_k(x))} \to 1$$

as $n \to \infty$.

The above mentioned lemmata are stated and proved in Section 4. The proof of the Theorem is in Section 5.

3. An application to return times

We give an application of the Theorem in this paper. As mentioned in the introduction, a strong Borel–Cantelli lemma is a statement of the type that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} 1_{B(y, r_k)}(T^k(x)) = 1$$

holds for almost every $x$ when $\sum_{k=1}^{\infty} \mu(B(y, r_k)) = \infty$. Galatolo and Kim [6] used strong Borel–Cantelli lemmata, to conclude that

$$\lim_{n \to \infty} \frac{\log \tau_{B(y, r_n)}(x)}{\log \mu(B(y, r_n))} = 1$$

holds for almost every $x$, where $\tau_B$ is the hitting time to $B$, defined by

$$\tau_B(x) = \min \{ k > 0 : T^k(x) \in B \}.$$  

When $x \in B$, the time $\tau_B(x)$ is usually called the return time of $x$ to $B$.

Adapting the proof of Galatolo and Kim, one obtains the following result on return times as a corollary to the Theorem.

**Corollary.** Under the assumptions of the Theorem, we have

$$\lim_{r \to 0} \frac{\log \tau_{B(x, r)}(x)}{\log \mu(B(x, r))} = 1$$

for $\mu$ almost every $x$.

The proof of this corollary is in Section 6.

If we let

$$d_\mu(x) = \lim_{r \to 0} \inf \frac{\log \mu(B(x, r))}{\log r}$$

and

$$\overline{d}_\mu(x) = \lim_{r \to 0} \sup \frac{\log \mu(B(x, r))}{\log r}$$

be the lower and upper pointwise dimensions of the measure $\mu$ at the point $x$, then it follows from the Corollary that

$$d_\mu(x) = \lim_{r \to 0} \inf \frac{\log \tau_{B(x, r)}(x)}{-\log r} \leq \lim_{r \to 0} \sup \frac{\log \tau_{B(x, r)}(x)}{-\log r} = \overline{d}_\mu(x)$$
holds for almost every \(x\). In many cases it is known that \(d_\mu(x) = \overline{d}_\mu(x)\) holds for almost all \(x\) so that in fact we have

\[
(2) \quad d_\mu(x) = \lim \inf_{r \to 0} \frac{\log \tau_{B(x,r)}(x)}{-\log r} = \lim \sup_{r \to 0} \frac{\log \tau_{B(x,r)}(x)}{-\log r} = \overline{d}_\mu(x)
\]

for almost all \(x\). The equalities in (2) have been proved by Barreira and Saussol [2] for \(C^{1+\alpha}\)-diffeomorphisms when \(\mu\) is an equilibrium measure.

4. LEMMATA

The following lemma is a variation on a lemma by Sprindžuk [16, Lemma 10, page 45]. The assumptions are a little bit different, and the proof is similar, but easier.

**Lemma 1.** Let \((f_k)\) be a sequence of integrable functions and \((\phi_k)\) a sequence of numbers such that \(\phi_k \geq 1\) for all \(k\).

If for all \(m < n\), we have

\[
(3) \quad \int \left( \sum_{m < k \leq n} \left( f_k - \int f_k \, d\mu \right) \right)^2 \, d\mu \leq \sum_{m < k \leq n} \phi_k,
\]

then for every \(\varepsilon > 0\), we have

\[
\sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} \int f_k \, d\mu + O\left( (\log \Phi(n+1))^{\frac{3+\varepsilon}{2}} \Phi(n+1)^{\frac{1}{2}} \right),
\]

for \(\mu\) almost every \(x\), where

\[
\Phi(n) = \sum_{k=1}^{n} \phi_k.
\]

Before proving Lemma 1, we state the following lemma.

**Lemma 2.** Let \((a_k)\) be a decreasing sequence of non-negative numbers such that

\[
\sum_{k=1}^{\infty} a_k = \infty
\]

and

\[
(4) \quad \lim_{\rho \to 1^+} \limsup_{k \to \infty} \frac{a_k}{\rho^{1-k}} = 1.
\]

Suppose that \((x_k)\) is a sequence of non-negative numbers such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{x_k}{a_k} = 1.
\]

Then

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} x_k}{\sum_{k=1}^{n} a_k} = 1.
\]
We shall now prove the two lemmata, and start with Lemma 1.

Proof of Lemma 1. If \( I = (m, n] \) and \( m \) and \( n \) are integers, then we write

\[
\Phi(I) = \sum_{k \in I} \phi_k, \quad \text{and} \quad F(I, x) = \sum_{k \in I} f_k(x). 
\]

Let \( \Phi(n) = \Phi([0, n]) \) and \( F(n, x) = F([0, n], x) \).

When \( u \) is a natural number, we let \( n_u \) be the largest \( n \) for which \( \Phi(n) < u \). Hence \( \Phi(n_u) < u \leq \Phi(n_u + 1) \). If \( u < v \), the interval \( I = \langle u, v \rangle \) is non-empty, but it can happen that \( \langle n_u, n_v \rangle \) is empty.

We define \( \sigma([u, v]) = \langle n_u, n_v \rangle \). It follows that if \( I = I_1 \cup I_2 \), then \( \sigma(I) = \sigma(I_1) \cup \sigma(I_2) \). Moreover, if the union \( I_1 \cup I_2 \) is disjoint, then so is \( \sigma(I_1) \cup \sigma(I_2) \).

For integers \( 0 \leq s \leq r \), we let \( J_{r,s} \) be the set of intervals of the form \( \langle i2^s, (i + 1)2^s \rangle \), \( 0 \leq i < 2^{r-s} \). Then

\[
\bigcup_{I \in J_{r,s}} \sigma(I) = \sigma([0, 2^r]) = [0, n_{2^r}],
\]

is a disjoint union. Therefore,

\[
\sum_{I \in J_{r,s}} \Phi(\sigma(I)) = \Phi\left( \bigcup_{I \in J_{r,s}} \sigma(I) \right) = \Phi(n_{2^r}) < 2^r. 
\]

With \( J_r = \bigcup_{s=0}^{r} J_{r,s} \) we then have

\[
\sum_{I \in J_r} \Phi(\sigma(I)) = \sum_{s=0}^{r} \sum_{I \in J_{r,s}} \Phi(\sigma(I)) < (r + 1)2^r. 
\]

Put

\[
g(r, x) = \sum_{I \in J_r} \left( F(\sigma(I), x) - \int F(\sigma(I), x) \, d\mu(x) \right)^2. 
\]

Then by (3), we have that

\[
\int g(r, x) \, d\mu(x) \leq \sum_{I \in J_r} \Phi(\sigma(I)) < (r + 1)2^r. 
\]

It follows that

\[
\mu \{ x : g(r, x) \geq (r + 1)2^{1+2\varepsilon}2^r \} < r^{-1-2\varepsilon}
\]

where \( \varepsilon > 0 \). By the Borel–Cantelli lemma we then get that for almost all \( x \),

\[
g(r, x) < (r + 1)2^{1+2\varepsilon}2^r
\]

holds for all large \( r \).
An interval \([0, v]\) is a union of at most \(r = \lfloor \log_2 v \rfloor + 1\) intervals from \(J_r\). We let \(J_r(v)\) denote the set of those intervals. Then \((0, n_v]\)

is a disjoint union of the intervals \(\sigma(I), I \in J_r(v)\). We have

\[
F(n_v, x) - \int F(n_v, x) \, d\mu(x) = \sum_{I \in J_r(v)} \left( F(\sigma(I), x) - \int F(\sigma(I), x) \, d\mu(x) \right).
\]

and by the Cauchy–Bunyakovsky–Schwarz inequality, we have that

\[
\left( F(n_v, x) - \int F(n_v, x) \, d\mu(x) \right)^2 \\
\leq \left( \sum_{I \in J_r(v)} 1^2 \right) \sum_{I \in J_r(v)} \left( F(\sigma(I), x) - \int F(\sigma(I), x) \, d\mu(x) \right)^2 \\
\leq r \sum_{I \in J_r(v)} \left( F(\sigma(I), x) - \int F(\sigma(I), x) \, d\mu(x) \right)^2 \\
\leq r g(r, x),
\]

since the sum has at most \(r\) terms. Hence, for almost every \(x\), we have

\[
\sum_{k=1}^{n_v} f_k(x) = F(n_v, x) \leq \int F(n_v, x) \, d\mu(x) + \sqrt{rg(r, x)}
\]

\[
= \int F(n_v, x) \, d\mu(x) + O\left( r^{\frac{1}{2} + \epsilon} 2^\frac{1}{2} \right) \\
= \sum_{k=1}^{n_v} \int f_k(x) \, d\mu(x) + O\left( (\log v)^{\frac{1}{2} + \epsilon} v^{\frac{1}{2}} \right) \\
= \sum_{k=1}^{n_v} \int f_k(x) \, d\mu(x) + O\left( (\log \Phi(n_v + 1))^{\frac{1}{2} + \epsilon} \Phi(n_v + 1)^{\frac{1}{2}} \right).
\]

Consider an arbitrary positive integer \(n\). Since \(\phi_k \geq 1\), there is a \(v\) such that \(n = n_v\). This finishes the proof.

\(\square\)

**Proof of Lemma** 2  Put

\[
\sigma_n = \frac{1}{n} \sum_{k=1}^{n} \frac{x_k}{a_k}
\]

and let \(\epsilon > 0\). Take \(n_1\) such that \(|\sigma_n - 1| < \epsilon\) for all \(n \geq n_1\).

Let \(\rho > 1\) and put \(n_k = \lceil \rho^{k-1} n_1 \rceil\). For every \(k\), the number

\[
\Delta_k := n_{k+1} \sigma_{n_{k+1}} - n_k \sigma_{n_k} = \sum_{l=n_k+1}^{n_{k+1}} \frac{x_l}{a_l}
\]
satisfies
\[ \Delta_k \leq n_{k+1}(1 + \varepsilon) - n_k(1 - \varepsilon) \]
\[ = (n_{k+1} - n_k)(1 + \varepsilon) + 2\varepsilon n_k \]
and
\[ \Delta_k \geq (n_{k+1} - n_k)(1 - \varepsilon) - 2\varepsilon n_k. \]

We have
\[ n_k \leq \rho^{k-1} n_1 = \frac{1}{\rho - 1}(\rho^k n_1 - \rho^{k-1} n_1) \]
\[ \leq \frac{1}{\rho - 1}(n_{k+1} - n_k + 1) \leq \frac{2}{\rho - 1}(n_{k+1} - n_k). \]

It now follows that
\[ \frac{1}{a_{n_k}} \sum_{l=n_k+1}^{n_{k+1}} x_l \leq \sum_{l=n_k+1}^{n_{k+1}} \frac{x_l}{a_l} = \Delta_k \leq (n_{k+1} - n_k)\left(1 + \varepsilon + \frac{4\varepsilon}{\rho - 1}\right) \]
and
\[ \frac{1}{a_{n_k}} \sum_{l=n_k+1}^{n_{k+1}} x_l \geq (n_{k+1} - n_k)\left(1 - \varepsilon - \frac{4\varepsilon}{\rho - 1}\right). \]

Letting \( c_\rho \) be so that
\[ 1 \leq \frac{a_n}{a_{[n]}} \leq c_\rho \]
for all \( n \geq n_1 \), we have
\[ \sum_{l=n_k+1}^{n_{k+1}} x_l \leq \sum_{l=n_k+1}^{n_{k+1}} a_l c_\rho \left(1 + \varepsilon + \frac{4\varepsilon}{\rho - 1}\right) \]
and
\[ \sum_{l=n_k+1}^{n_{k+1}} x_l \geq \sum_{l=n_k+1}^{n_{k+1}} a_l c_\rho \left(1 - \varepsilon - \frac{4\varepsilon}{\rho - 1}\right). \]

Hence
\[ c_\rho \left(1 - \varepsilon - \frac{4\varepsilon}{\rho - 1}\right) \leq \frac{\sum_{l=n_{k+1}+1}^{n_k} x_l}{\sum_{l=n_{k+1}+1}^{n_k} a_l} \leq c_\rho \left(1 + \varepsilon + \frac{4\varepsilon}{\rho - 1}\right) \]
holds for any \( m > 1 \).

As will be proved below, from the assumption \( \sum_{k=1}^\infty a_k = \infty \) it now follows that
\[ \text{(5)} \quad \limsup_{n \to \infty} \frac{\sum_{l=1}^n x_l}{\sum_{l=1}^n a_l} \leq \rho c_\rho \left(1 + \varepsilon + \frac{4\varepsilon}{\rho - 1}\right) \]
and
\[ \text{(6)} \quad \liminf_{n \to \infty} \frac{\sum_{l=1}^n x_l}{\sum_{l=1}^n a_l} \geq \frac{c_\rho}{\rho} \left(1 - \varepsilon - \frac{4\varepsilon}{\rho - 1}\right). \]
This is proved as follows. For \( n \) between \( n_m \) and \( n_{m+1} \) we have
\[
\frac{\sum_{l=1}^{n} x_l}{\sum_{l=1}^{n} a_l} \leq \frac{\sum_{l=1}^{n_{m+1}} x_l}{\sum_{l=1}^{n_{m+1}} a_l} \frac{\sum_{l=1}^{n_{m+1}} a_l}{\sum_{l=1}^{n_{m}}} \frac{\sum_{l=1}^{n_{m}} a_l}{\sum_{l=1}^{n} a_l}
\]
and
\[
\frac{\sum_{l=1}^{n_{m+1}} a_l}{\sum_{l=1}^{n_{m}} a_l} = 1 + \frac{\sum_{l=1}^{n_{m+1}} a_l}{\sum_{l=1}^{n_{m}} a_l} \leq 1 + \left( \frac{n_{m+1} - n_m}{n_m} a_{n_m} \right) = 1 + \frac{n_{m+1} - n_m}{n_m} \rightarrow \rho
\]
as \( m \rightarrow \infty \). This proves (5), and the proof of (6) is obtained in a similar way.

We first note that we can let \( c_\rho \) be fixed while increasing \( n_1 \). Letting \( \varepsilon \rightarrow 0 \), we may therefore keep \( \rho \) and \( c_\rho \) fixed. This proves that
\[
\limsup_{n \rightarrow \infty} \frac{\sum_{l=1}^{n} x_l}{\sum_{l=1}^{n} a_l} \leq \rho c_\rho
\]
and
\[
\liminf_{n \rightarrow \infty} \frac{\sum_{l=1}^{n} x_l}{\sum_{l=1}^{n} a_l} \geq \frac{c_\rho}{\rho}.
\]
Because of (4), we can make \( \rho \) and \( c_\rho \) as close to 1 as we desire, and hence
\[
\lim_{n \rightarrow \infty} \frac{\sum_{l=1}^{n} x_l}{\sum_{l=1}^{n} a_l} = 1
\]
which finishes the proof. \( \square \)

5. Proof of the Theorem

We start the proof of the Theorem by establishing the following estimate.

Proposition 1. Suppose that \( \mu \) is a probability measure and that \( (E_n) \) is a sequence of sets that satisfy the estimate
\[
\mu(E_k \cap E_{k+1}) \leq \mu(E_k) \mu(E_{k+1}) + c e^{-\eta k} + c e^{-\eta l},
\]
for some \( c, \eta > 0 \). Then there is a constant \( c_2 \) such that for any \( m < n \)
\[
\int \left( \sum_{m < k \leq n} (\mu(E_k)^{-1} 1_{E_k} - 1) \right)^2 d\mu \leq c_2 \sum_{m < k \leq n} \frac{1}{k^2 \mu(E_k)} \sum_{m < k \leq n} \log k \mu(E_k).
\]
Proof. We have of course that \( \mu(E_k \cap E_{k+1}) \leq \mu(E_{k+1}) \).

Put
\[
f_k = \frac{1}{\mu(E_k)} 1_{E_k}
\]
and

\[ S = \int \left( \sum_{m < k \leq n} (\mu(E_k)^{-1}1_{E_k} - 1) \right)^2 \, d\mu. \]

Then

\[ \int f_k \, d\mu = 1 \quad \text{and} \quad S = \sum_{m < j, k \leq n} \left( \int f_k f_j \, d\mu - 1 \right). \]

Let \( c_0 \) be a positive constant that will be chosen later. We write \( S \) as a sum of three parts, \( S = A + B + C \), where

\[ A = 2 \sum_{m < k \leq n} \sum_{j = m + c_0 \log k}^{k - c_0 \log k} \left( \int f_k f_j \, d\mu - 1 \right), \]

\[ B = 2 \sum_{m < k \leq n} \sum_{j = k - c_0 \log k + 1}^{k} \left( \int f_k f_j \, d\mu - 1 \right), \]

\[ C = 2 \sum_{m < k \leq n} \sum_{j = m + 1}^{m + c_0 \log k} \left( \int f_k f_j \, d\mu - 1 \right). \]

We shall now estimate \( A, B \) and \( C \) separately.

To estimate \( A \), we use the assumption (7), which implies that

\[ \mu(E_k \cap E_j) \leq \mu(E_k) + ce^{-\eta_k} + ce^{-\eta(k-j)} \]

for \( j < k \). Hence

\[ \int f_k f_j \, d\mu \leq 1 + \frac{1}{\mu(E_k)\mu(E_j)} (ce^{-\eta_k} + ce^{-\eta(k-j)}). \]

We may therefore estimate \( A \) by

\[ A \leq 2 \sum_{m < k \leq n} \frac{1}{\mu(E_k)} \sum_{j = m + c_0 \log k}^{k - c_0 \log k} \frac{(ce^{-\eta_k} + ce^{-\eta(k-j)})}{\mu(E_j)} \]

\[ \leq 2 \sum_{m < k \leq n} \frac{ce^{-\eta_k} + ce^{-\eta_0 \log k}}{\mu(E_k)} \sum_{m < j \leq k} \frac{1}{\mu(E_j)}. \]

Choose \( c_0 \) so large that \( e^{-\eta_0 \log k} \leq k^{-3} \). Then

\[ A \leq c_1 \left( \sum_{m < k \leq n} \frac{1}{\mu(E_k)} \right) \left( \sum_{m < k \leq n} \frac{1}{k^3 \mu(E_k)} \right), \]

for some constant \( c_1 \).

The terms \( B \) and \( C \) are both estimated in the following way. In the sums defining \( B \) and \( C \), there are not more than \( c_0 \log k \) terms in the sum over \( j \). We use the trivial estimate

\[ \int f_k f_j \, d\mu \leq \frac{1}{\mu(E_k)} \int f_j \, d\mu = \frac{1}{\mu(E_k)}. \]
and obtain that
\[ B, C \leq 2 \sum_{m<k\leq n} \frac{c_0 \log k}{\mu(E_k)}. \]

Combining all estimates, we get that
\[ S \leq c_2 \sum_{m<k\leq n} \frac{1}{k^2 \mu(E_k)} \sum_{m<k\leq n} \frac{\log k}{\mu(E_k)} \]
for some constant \( c_2 \).

We are now ready to prove the Theorem.

Let
\[ E_k = \{ x : d(T^k(x), x) < r_k(x) \} = \{ x : T^k(x) \in B_k(x) \} \]
with \( r_k(x) \) defined by the relation
\[ \mu(B(x, r_k(x))) = m_k. \]

Then certain estimates hold. More precisely, under the assumptions of the Theorem, we have that
\[ |\mu(E_k) - m_k| \leq ce^{-\eta k}, \]
\[ \mu(E_k \cap E_{k+j}) \leq \mu(E_k)\mu(E_{k+j}) + ce^{-\eta k} + ce^{-\eta j} \]
for some constants \( c, \eta > 0 \) [Lemma 4.1 and 4.2].

We put \( f_k = \mu(E_k)^{-1} 1_{E_k} \). By Proposition 1 we have
\[ \int \left( \sum_{m<k\leq n} (\mu(E_k)^{-1} 1_{E_k} - 1)^2 \right) d\mu \leq c_2 \sum_{m<k\leq n} \frac{1}{k^2 \mu(E_k)} \sum_{m<k\leq n} \frac{\log k}{\mu(E_k)}. \]

Since \( \mu(E_k)^{-1} \leq k(\log k)^{-4-\varepsilon} \), we have
\[ \int \left( \sum_{m<k\leq n} (\mu(E_k)^{-1} 1_{E_k} - 1)^2 \right) d\mu \leq c_4 \sum_{m<k\leq n} k(\log k)^{-3-\varepsilon}. \]

From Lemma 1 follows that for almost all \( x \)
\[ \sum_{k=1}^{n} \frac{1}{\mu(E_k)} 1_{B_k(x)}(T^k(x)) = \sum_{k=1}^{n} \frac{1}{\mu(E_k)} 1_{E_k}(x) \]
\[ = n + O\left( \Phi(n + 1)^{\frac{1}{2}}(\log \Phi(n + 1))^{\frac{3}{2} + \varepsilon} \right), \]
where
\[ \Phi(n) = \sum_{k=1}^{n} c_5 k(\log k)^{-3-\varepsilon} \leq c_5 n^2(\log n)^{-3-\varepsilon}. \]

Hence
\[ \sum_{k=1}^{n} \frac{1}{\mu(E_k)} 1_{B_k(x)}(T^k(x)) = n + O\left( n(\log n)^{-\frac{3}{2}} \right). \]
In particular, since \( \mu(E_k)/\mu(B_k(x)) \to 1 \), we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\mu(B_k(x))} \mathbf{1}_{B_k(x)}(T^k(x)) = 1
\]
for almost all \( x \). Finally, Lemma \( \ref{lem:correlation} \) implies that
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} \mathbf{1}_{B_k(x)}(T^k(x))}{\sum_{k=1}^{n} \mu(B_k(x))} = 1
\]
for almost every \( x \).

6. Proof of the Corollary

In this section, we assume that \( \mu \) is a non-atomic probability measure, not necessarily invariant under \( T \). Under this assumption we prove two propositions, from which the Corollary of this paper immediately follows.

**Proposition 2.** Suppose that correlations are summable for \( L^1 \) against \( BV \). For almost every \( x \) we have that
\[
\lim \inf_{r \to 0} \frac{\log \tau_{B(x,r)}(x)}{-\log \mu(B(x,r))} \geq 1.
\]

**Proof.** Suppose that \( x \) is such that
\[
\lim \inf_{r \to 0} \frac{\log \tau_{B(x,r)}(x)}{-\log \mu(B(x,r))} < \theta < 1.
\]
We may then choose sequences \( r_n \to 0 \) and \( M_n \to \infty \) such that
\[
\frac{\log \tau_{B(x,r_n)}(x)}{-\log \mu(B(x,r_n))} < \theta \quad \text{and} \quad 2^{-M_n-1} \leq \mu(B(x,r_n)) \leq 2^{-M_n},
\]
for all \( n \).

Define the function \( \rho_k \) such that
\[
\mu(B(y,\rho_k(y))) = 2k^{-\theta}
\]
for all \( y \). Put \( k_n = 2^{[M_n+1]\theta} \). Then, for the point \( x \), we have
\[
B(x,r_n) \subset B(x,\rho_{k_n})
\]
since \( \mu(B(x,\rho_{k_n})) = 2^{-M_n} \geq \mu(B(x,r_n)) \). We then have
\[
\tau_{B(x,\rho_{k_n}(x))} \leq \tau_{B(x,r_n)}(x) \leq \mu(B(x,r_n))^{-\theta} \leq 2^{[M_n+1]\theta} = k_n.
\]
It follows that there are infinitely many \( k \) such that \( T^k(x) \in B(x,\rho_k(x)) \). But the set of \( x \) with this property is a set of zero measure since \( k^{-\frac{1}{\theta}} \) is a summable sequence [10, Theorem C]. \( \square \)

It now remains to prove the following proposition, for which no mixing assumption is needed.
Proposition 3. Suppose that $x$ is a point such that

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} 1_{B(x, r_k)}(T^k x)}{\sum_{k=1}^{n} \mu(B(x, r_k))} = 1,$$

where $r_k$ is such that $\mu(B(x, r_k)) = \frac{(\log k)^5}{k}$. Then

$$\limsup_{r \to 0} \frac{\log \tau_{B(x,r)}(x)}{-\log \mu(B(x,r))} \leq 1.$$

**Proof.** To get a contradiction, suppose that $x$ is such that

$$\limsup_{r \to 0} \frac{\log \tau_{B(x,r)}(x)}{-\log \mu(B(x,r))} > \theta > 1.$$

Put $m_n = \frac{(\log n)^5}{n}$. There is a sequence $\rho_n$ such that

$$m_{n+1} \leq \mu(B(x, \rho_n)) \leq m_n$$

and a sequence $n_j$ such that

$$\tau_{B(x, \rho_{n_j})}(x) \geq \mu(B(x, \rho_{n_j}))^{-\theta}.$$

It then follows that

$$\tau_{B(x, \rho_{n_j})}(x) \geq \frac{n_j^\theta}{(\log n_j)^{5\theta}}.$$

This means that if we put

$$S_n(x) = \sum_{k=1}^{n} 1_{B(x, \rho_k)}(T^k x),$$

then $S_{n_j}(x) = S_{N_j}(x)$, where $N_j = \frac{n_j^\theta}{(\log n_j)^{5\theta}}$.

However, we have

$$\sum_{k=1}^{n_j} \mu(B(x, \rho_k)) \leq \sum_{k=1}^{n_j} m_k = \sum_{k=1}^{n_j} m_k + \sum_{k=n_j+1}^{n} m_k \leq 1 + K,$$

for some constant $K > 0$, and

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} \mu(B(x, r_k))}{\sum_{k=1}^{n} \mu(B(x, r_k))} = 1.$$

Now, because $S_{n_j}(x) = S_{N_j}(x)$ we have

$$\frac{S_{n_j}(x)}{\sum_{k=1}^{n_j} \mu(B(x, r_k))} = \frac{S_n(x)}{\sum_{k=1}^{n} \mu(B(x, r_k))} \frac{\sum_{k=1}^{n_j} \mu(B(x, \rho_k)) \sum_{k=1}^{n_j} \mu(B(x, r_k))}{\sum_{k=1}^{N_j} \mu(B(x, \rho_k)) \sum_{k=1}^{n_j} \mu(B(x, r_k))}.$$

and it follows from (8) and (9) that we cannot have

$$\lim_{n \to \infty} \frac{S_n(x)}{\sum_{k=1}^{n} \mu(B(x, r_k))} = 1,$$
which is a contradiction. □

7. Slowly mixing systems

For the types of systems considered in the Theorem, it is known [10, Theorem C] that if \( \sum m_k < \infty \), then for almost every \( x \) we have that \( T^k(x) \in B(x, r_k(x)) \) for at most finitely many \( k \). For some systems, it is furthermore known that if \( \sum m_k = \infty \), then for almost all \( x \) we have that \( T^k(x) \in B(x, r_k(x)) \) for infinitely many \( k \) [8, [10, Theorem D].

It would not be unreasonable to suspect that for some systems we have the following stronger dichotomy: The conclusion of the Theorem holds whenever \( (m_k) \) is a sequence of non-negative numbers that satisfies \( \sum m_k = \infty \), and if \( \sum m_k < \infty \), then for almost every \( x \) we have that \( T^k(x) \in B(x, r_k(x)) \) for at most finitely many \( k \).

Because of technical reasons in the proof, we have not been able to prove this dichotomy. However, it is possible to provide examples of slowly mixing systems for which this dichotomy does not hold. The purpose of this section is to give two such examples.

Example (Rotations). Rotations are not mixing and one may therefore regard them as particularly slowly mixing. Let \( \alpha \) be a real number and let \( T^\alpha: S^1 \to S^1 \) be the rotation by the angle \( \alpha \) on the one dimensional circle \( S^1 \). The Lebesgue measure is invariant, but not mixing.

It is clear that for any \( x \in S^1 \), we have \( T^\alpha(x) \in B(x, r_k) \) if and only if \( T^\alpha(0) \in B(0, r_k) \).

If \( \alpha \) is a badly approximable number, then there is a \( c > 0 \) such that
\[
d(x, T^\alpha(x)) = d(0, T^\alpha(0)) > m_k : = \frac{c}{k}
\]
for all \( k > 0 \). In this case we have \( \sum m_k = \infty \), but for every \( x \), there is no \( k > 0 \) such that \( T^\alpha(x) \in B(x, r_k) \).

If \( \alpha \) is of a higher Diophantine class, say for instance such that \( d(0, T^\alpha(0)) < \frac{1}{k^2} \) for infinitely many \( k \) (as is also the case if \( \alpha \) is rational), then for these \( k \) we have
\[
d(x, T^\alpha(x)) = d(0, T^\alpha(0)) < m_k : = \frac{1}{k^2},
\]
so that \( \sum m_k < \infty \) but for every \( x \) we have \( T^\alpha(x) \in B(x, r_k) \) for infinitely many \( k \).

In fact [11], for almost all \( \alpha \), we have for all \( x \) that \( T^\alpha(x) \in B(x, r_k) \) for infinitely many \( k \), if and only if \( \sum \phi(k)r_k = \infty \), where \( \phi \) is Euler’s totient function.

Example (Certain skew products). Here we consider a simple special case of the type of systems considered by Galatolo, Rousseau...
and Saussol [7]. As in the previous example, $T_\alpha$ is the rotation by the angle $\alpha$. We consider the skew product transformation $T: [0,1) \times S^1 \to [0,1) \times S^1$ defined by

$$T(x,y) = (2x \mod 1, y + \alpha \mathbf{1}_{[0,\frac{1}{2})}(x)).$$

The two dimensional Lebesgue measure, which we denote by $\mu$, is invariant under $T$, and $\mu$ is mixing, with a rate which depends on $\alpha$.

Let $\gamma(\alpha)$ be the infimum of those $\beta$ such that there is a $c > 0$ with $d(0, T^k_\alpha(0)) > \frac{c}{k^\beta}$ for all $k > 0$. Then, as a special case of a theorem by Galatolo, Rousseau and Saussol [7, Theorem 23], we have for almost all $p = (x,y)$ that the number

$$R(p) = \liminf_{r \to 0} \frac{\log \tau_{B(p,r)}(p)}{-\log r}$$

satisfies $R(p) \leq 1 + \frac{2}{\gamma(\alpha)}$. Moreover, $T$ is mixing for Lipschitz continuous functions with a rate $O(n^{-\gamma(\alpha)})$ [7, Proposition 9].

We may choose $\alpha$ such that $\gamma(\alpha) > 2$ and then $R(p) \leq 1 + \frac{2}{\gamma(\alpha)} < 2$ holds for almost all $p$. Since $\mu$ is the two dimensional Lebesgue measure, we have

$$\liminf_{r \to 0} \frac{\log \tau_{B(p,r)}(p)}{-\log \mu(B(p,r))} = \frac{1}{2} R(p) \leq \frac{1}{2} + \frac{1}{\gamma} < 1$$

for almost every $p$. Take

$$\theta \in \left(\frac{1}{2} + \frac{1}{\gamma}, 1\right).$$

Following the beginning of the proof of Proposition 2 we obtain that for almost all $p$ we have $T^k(p) \in B(p, \rho_k)$ for infinitely many $k$, and that $m_k := \mu(B(p, \rho_k)) = 2k^{-\theta}$.

Hence, $\sum m_k < \infty$ but nevertheless we have for almost all $p$ that $T^k(p) \in B(p, \rho_k)$ for infinitely many $k$.

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