ON COMPACT PSEUDOCONECAVE SETS

ZBIGNIEW SLODKOWSKI

ABSTRACT. Replying to three questions posed by N. Shcherbina, we show that a compact
cseudoconcave set can have the core smaller than itself, that the core of a compact set must
be pseudoconcave, and that it can be decomposed into compact pseudoconcave sets on which
all smooth plurisubharmonic functions are constant.

0. Introduction

This paper is inspired by the recent work of N. Shcherbina [Sh55], where he asks under
what condition on a compact subset $K$ of a complex manifold $M$ there is a smooth strictly
plurisubharmonic function defined in some neighborhood of $K$. In the spirit of [ST04] and
[HST17], he defines a subset $c(K)$ of $K$, which (if nonempty) constitutes an obstacle to
existence of such a function. This subset, called the core of $K$, consists of all points of $p$ of
$K$, such that no smooth plurisubharmonic function on $K$ can be strictly plurisubharmonic at
$p$. The main result of Shcherbina in [Sh21] is that a strictly plurisubharmonic function on $K$
exists (ie. $c(K)$ is nonempty), if and only if $K$ does not contain any compact pseudoconcave
set (cf. Def. 1.1). However, he does not prove this by studying the core, but by examining
another subset of $K$, the nucleus, $n(K)$, which turns out to be the maximal pseudoconcave
subset of $K$.

Shcherbina poses three questions concerning the core. Question 5.1 [Sh21]: "Is
$n(K) \subset c(K)$?" We give an example showing it is not true in general (Ex. 2.3). This circumstance
makes pertinent his Question 5.2: "Is the core of a compact set always pseudoconcave?" We
show this is true, and we also show, answering his Question 5.3, that the core decomposes
into pairwise disjoint pseudoconcave sets on which every smooth plurisubharmonic function
is constant (in analogy with similar results in [PS19], and [Sl19]). These facts constitute our
main result, Th. 2.1. It might be of some interest that with these properties of the core
established, an alternative proof of Shcherbina’s main theorem follows easily, cf. Cor. 2.3.
in Section 1 we collect some facts about compact pseudoconcave sets that we need.

0.1. Some conventions in terminology and notation. Following Shcherbina, we say
that a function is smooth plurisubharmonic on set $K$, if it has a smooth plurisubharmonic
extension to some neighborhood of $K$. By smooth we mean any smoothness class $C^k$, with
$k = 0, 1, \ldots, \infty$. All arguments in Sections 1 and 2 are are correct for all these classes. (In case
of $k = 0, 1$, strictly plurisubharmonic should be understood as strongly plurisubharmonic,
in the sense of Richberg [Ri68]). Although nucleus is clearly independent on the smoothness
class under consideration, this is unknown for the core (and seems to me unlikely). So the
notation $c^k(K)$ might be more accurate, but, like Shcherbina, we suppress the $k$.

A neighborhood will always be an open set of $M$.

$M$ will denote a complex manifold.

For a subset $A$ of $M$, $bA$ denotes $\overline{\text{A}}$.

Date: October 2022.
**Acknowledgement.** I am grateful to Nikolai Shcherbina for attracting my attention to these problems, and helpful correspondence concerning Question 5.1.

1. **Background on pseudoconcave sets and Liouville sets**

1.1. **Definition.** (a) Let $X$ be a locally closed subset of $M$ (i.e. $X \cap (X \setminus X) = \emptyset$). We say that $X$ is a local maximum set, if there do not exist: a point $p \in X$, a neighborhood $V$ of $p$ in $M$, and a $C^\infty$-smooth strictly plurisubharmonic function $v$ on $V$, such that $v|V \cap X$ has strict maximum at $p$.

(b) If a local maximum set is closed in $M$, we call it pseudoconcave.

Below we give just several facts about pseudoconcave sets, that we use in this article. For more background we refer first of all to [Sh21](since we continue that paper), and then to [Ro55], [Sl86], [ST04], [HST17, 20, 21], [PS19], and [Sl19].

Although pseudoconcavity is the principal concept, the more relaxed notion of a local maximum set ads occasionally some flexibility to arguments that would be cumbersome otherwise. See Example 2.3, where the following two properties are used. (They follow immediately from the definition.)

1.2. **Proposition.** (a) The union of a family of local maximum subsets of $M$ is pseudoconcave, provided it is closed.

(b) If $Z \subset X \subset M$, $Z$ relatively closed in $X$, and $X$ a local maximum subset of $M$, then $X \setminus Z$ is a local maximum subset of $M$ as well.

1.3. **Definition** [HST20]. We say that a closed subset $X$ of $M$ is a Liouville set, if every smooth plurisubharmonic function $u$ on $X$, such that $u|X$ is bounded from the above, must be constant on $X$.

1.4. **Proposition.** (a) Every Liouville set is pseudoconcave.

(b) If $K \subset M$ is compact, and $L$ is a compact Liouville subset of $M$, then $L \subset c(K)$. In particular, $c(L) = L$.

*Comments on proof.* For (a) see [HST20, Lemma 4.1].

(b) Suppose, to the contrary, that there is a smooth plurisubharmonic function $u$ on a neighborhood $U$ of $X$, which is strictly plurisubharmonic at some neighborhood $V$ of $p \in X$. We can assume $\nabla_1 \subset U$ and $X \setminus V \neq \emptyset$. Choose a nonnegative $C^\infty$ function $\rho$ on $U$, such that $\rho(p) > 0$ and $\text{supp}(\rho) \subset V$. By the standard argument, for some positive $\epsilon$, the function $v := u + \epsilon \rho$ is plurisubharmonic on $U$. Since $X$ is a compact Liouville set, both $u|X$, and $v|X$ must be constant, which is not possible (as they differ by $\epsilon \rho$).

We will say that $X$ is a *minimal* pseudoconcave set, if it does not contain any proper pseudoconcave subset $Z$. Below we will consider only compact minimal pseudoconcave sets. (A pseudoconcave subset of a compact set is automatically compact.)

1.5. **Proposition.** Every compact pseudoconcave subset of manifold $M$ contains a minimal compact pseudoconcave subset.
ON COMPACT PSEUDOCONCAVE SETS

This is very similar to [Sl19, Cor.1.11], and has practically the same proof, but we need this formally stronger statement. (It will be only a consequence of Th.2.1 that the two facts are actually equivalent.)

Proof. Let \( K \subset M \) be a given compact pseudoconcave set. We will define sets \( L_t \) by transfinite induction. Let \( L_0 = K \). If sets \( L_s \) up to ordinal \( t \) have already been defined, we let \( L_{t+1} \) to be any proper compact pseudoconcave subset of \( L_t \), if such exists; if not the induction stops and \( L_t \) is the answer. If \( t \) is a limit ordinal, we let

\[
L_t := \bigcap \{ L_s : s < t \}.
\]

Clearly \( L_t \) is compact and nonempty; we have to check it is a local maximum set. Suppose not, and let \( p, V, v : V \leftarrow R \) be as in Def.1.1(i). Choose \( B := B(p, r), r > 0 \), a ball in a coordinate neighborhood of \( p \), and such that \( \overline{B} \subset V \). Let \( F := \{ x \in \overline{B} : v(x) \geq v(p) \} \). Since \( F \) is compact, and

\[
\{ p \} = F \cap L_t = F \cap \bigcap_{s < t} L_s,
\]

there is \( s_0 < t \), such that \( F \cap L_{s_0} \subset B(p, r/2) \). Thus

\[
\max v|L_{s_0} \cap B(p, r/2) \geq v(p) > \max v|L_{s_0} \cap bB(p, r).
\]

But this violates another, equivalent, form of local maximum property, namely (ii) in [Sl86, Prop.2.3], so \( L_{s_0} \) would not be pseudoconcave. This contradiction shows that \( L_t \) is pseudoconcave. For cardinality reasons, the induction has to stop at some ordinal \( t \), yielding a compact minimal pseudoconcave subset of \( K \).

\( \square \)

1.6. Proposition. Let \( Z \) be a pseudoconcave subset of \( M \) and let \( u \) be a plurisubharmonic function on \( Z \). Suppose \( u \) is bounded from the above on \( Z \) and attains maximum value at some point of \( Z \). Then the (nonempty) set \( Y := \{ x \in Z : u(x) = \sup u|Z \} \) is pseudoconcave.

. Note \( u \) is does not have to be smooth, just uppersemicontinuous. The other assumptions about it hold automatically when \( Z \) is compact.

Proof. Since \( u \) is uppersemicontinuous, \( Y \) is closed. Assume WLOG that \( \max u|Z = 0 \), and suppose that \( Y \) is not a local maximum set. By Def.1.1 there are: a point \( p \in Y \), a neighborhood \( V \) of \( p \), a smooth plurisubharmonic function \( v \) on \( V \), and a coordinate ball \( B(p, r), r > 0 \), such that : \( \nabla v \cap Z \) is compact, \( \overline{B(p, r)} \subset V \), \( v(p) = 0 \), and \( v|Y \cap bB(p, r) \leq -1 \). So there is a neighborhood \( W \) of \( Y \cap bB(p, r) \), such that \( v|W \leq -1/2 \). Let \( F := Z \cap bB(p, r) \setminus W \). Since \( F \) is disjoint from \( Y \), \( u \) is negative on \( F \). Let \( c_1 := \max u|F \), then \( c_1 < 0 \) (as \( F \) is compact and \( u \) usc). Similarly, \( v \) is bounded on \( F \) by some finite \( c \).

Choose now \( n \in N \) large enough so that \( c + nc_1 < -1/2 \), and define a plurisubharmonic function on \( V \) by \( \phi := v + nu \). Now \( \phi \leq v \leq -1/2 \) on \( W \cap bB(p, r) \), but \( \phi \leq c + nc_1 < -1/2 \) on \( F \). Thus \( \phi|Z \cap bB(p, r) < -1/2 \), while \( \phi(p) = 0 \). This contradicts (one of the forms) of local maximum property of \( Z \), cf.[Sl86, Prop.2.3(ii)].

\( \square \)

1.7. Corollary. Every minimal compact pseudoconcave set \( Z \subset M \) has the Liouville property (with respect to all plurisubharmonic functions on \( Z \).
Proof. Let \( u \) be a plurisubharmonic function defined in a neighborhood of \( Z \). As \( Z \) is compact, \( u|Z \) has maximum value \( c \). If \( u \) were not constant on \( Z \), then the set \( Y := \{ x \in Z : u(x) = c \} \) would be different from \( Z \). Since, by last proposition, \( Y \) is pseudoconcave, we would obtain contradiction with the minimality of \( Z \).

\[ \square \]

We Recall the properties of the regularized maximum function.

1.8. Proposition. [HM88, 4.13-14]. For every \( \delta > 0 \) there is a smooth convex function \( \phi_{\delta} \) on \( \mathbb{R}^2 \), with both partial nonnegative partial derivatives, and such that
\[
\phi_{\delta}(t, s) = \max(t, s), \text{ when } |t - s| \geq \delta.
\]

2. Pseudoconcave decomposition of the core

2.1. Theorem. Let \( K \subset M \) be compact, and \( p \in K \). Let \( Y \) be the set of all points \( x \in K \) such that \( u(x) = u(p) \) for all smooth plurisubharmonic functions \( u \) on \( K \).

(a) If \( Y = \{ p \} \), then then \( p \) does not belong to the core \( c(K) \).

(b) If \( Y \) is not a singleton, then \( Y \) is a (compact) Liouville set. In particular, \( Y \) is pseudoconcave, and \( c(Y) = Y \).

Proof. Suppose that we are given a neighborhood \( V \) of \( Y \), and a smooth plurisubharmonic function \( v : V \rightarrow [1/2, 1] \). During most of the proof we will construct, for the pair \( (V, v) \), a smooth plurisubharmonic function \( \chi \) on \( K \), with special properties (cf. Conclusion below).

Then a pair \( (V, v) \) will be chosen differently in cases (a) and (b), and using corresponding \( \chi \), we will obtain (a) and (b) immediately.

By the definition of \( Y \), there is a family of smooth plurisubharmonic functions \( \phi_t : U_t \rightarrow \mathbb{R} \), for \( t \in T \), such that
\[
Y = \bigcap_{t \in T} \{ x \in K : \phi_t(x) = 0 \}.
\]

Since the sets in this intersection are compact, we can choose finitely many of them so that (simplifying the notation slightly) we get
\[
Y \subset Z := \bigcap_{i=1}^{n} \{ x \in K : \phi_i(x) = 0 \} \subset V,
\]
with \( \phi_i : U_i \rightarrow R, K \subset U_i, i = 1, ..., n \). Choose a relatively compact open set \( U \), and such that
\[
K \subset U \subset \overline{U} \subset U_0 := \bigcap_{i=1}^{n} U_i.
\]

Shrinking \( U_i \)’s if needed, we we can assume WLOG that all \( \phi_i \)’s are uniformly bounded on \( U_0 \).

In this part of the proof we will follow the beginning of the proof of Lemma 3.5 in [Sl19] (preserving the notation). Define first two smooth plurisubharmonic functions \( \phi \) and \( \mu \) on \( U_0 \). Let
\[
\phi(x) := \phi_1(x) + ... + \phi_n(x), \text{ for } x \in U_0.
\]

To define \( \mu \), let first \( v_{\epsilon}(t_1, ..., t_n) := t_1 + ... + t_n + \epsilon(t_1^2 + ... + t_n^2) \) which defines a smooth convex function on \( \mathbb{R}^n \), for \( \epsilon > 0 \). Since \( m_0 := \inf \{ \phi_i(x) : x \in U_0, i = 1, ..., n \} \) is finite (by the choice of \( U_0 \)), we can choose an \( \epsilon > 0 \), such that \( 1 + \epsilon m_0 > 0 \). Then \( \frac{\partial \phi}{\partial t_i}(t) > 0 \)
when \( \min(t_1, \ldots, t_n) > m_0 \), in particular on the joint range of the vector valued function \((\phi_1, \ldots, \phi_n)\). So, by Lemma 1.13(iii) in [Sl19], the function \( \mu(x) := v_n(\phi_1(x), \ldots, \phi_n(x)) \) is smooth plurisubharmonic on \( U_0 \). Similarly as in [Sl19], we have \( \phi(x) \leq \mu(x) \), for \( x \in U_0 \), and

\[
\{ x \in U_0 : \phi(x) = \mu(x) \} = \bigcap_{i=1}^n \{ x \in U_0 : \phi_i(x) = 0 \}.
\]

Observe that

\[
Z = K \cap \bigcap_{c>0} \{ x \in \overline{U} : |\phi_i(x)| \leq c, i = 1, \ldots, n \}.
\]

Since the sets \( \{ x \in \overline{U} : |\phi_i(x)| \leq c, i = 1, \ldots, n \} \) are compact and decreasing with \( c \), there is \( c > 0 \), such that \( K \cap \overline{H} \subset V \), where \( H := \bigcap_{i=1}^n \{ x \in \overline{U} : |\phi_i(x)| < c \} \). Finally, there is an open set \( W \) such that

\[
K \subset W \subset \overline{W} \subset U, \text{ and } \overline{W} \cap \overline{H} \subset V.
\]

Observe that

\[
U \setminus H \subset \bigcup_{i+1}^n \{ x \in U : |\phi_i(x)| = 1 \} \subset \{ x \in U : \mu(x) > \phi(x) \}.
\]

Now, as \( U \setminus H \) is relatively compact in \( U \), we conclude that \( \Delta := \inf \{ (\mu - \phi)(x) : x \in W \setminus H \} > 0 \). So, choosing a \( \delta \in (0, \Delta/5) \), we obtain

\[
\mu \geq \phi + \Delta \geq (\phi + (\Delta/2)v) + \delta, \text{ on } W \cap V \setminus H.
\]

By this inequality and Prop.1.8, the function \( \chi \) defined as \( M_\delta(\phi + (\Delta/2)v, \mu) \) on \( W \cap V \), equals \( \mu \) on \( W \cap V \setminus \overline{H} \). Function \( \chi \) is smooth plurisubharmonic on \( W \cap V \). Define now function \( \chi^* \) as \( \chi \) on \( W \cap V \) and as \( \mu \) on \( W \setminus \overline{H} \). Since the latter two functions are equal on the intersection of their domains \( (V \cap W) \cap (W \setminus \overline{H}) = W \cap V \setminus \overline{H} \), function \( \chi^* \) is well defined smooth plurisubharmonic function on the union of these domains:

\[
(V \cap W) \cup (W \setminus \overline{H}) = W, \text{ since } W \cap H \subset V.
\]

Consider now \( \chi^* \) near \( Y \). Since \( \phi|Y = \mu|Y = 0 \), it follows that \( \phi + (\Delta/2)v > \mu + \delta \), on \( Y \), and so on some neighborhood of \( Y \), thus \( \chi^* = \phi + (\Delta/2)v \) near \( Y \).

**Conclusion.** For a smooth plurisubharmonic function \( 1/2 < v < 1 \) on a neighborhood of \( Y \), there is a smooth plurisubharmonic function \( \chi^* \) on \( K \) that is equal to \( \phi + (\Delta/2)v \) near \( Y \), and to \( (\Delta/2)v \) on \( Y \).

**Proof of (a).** If \( Y = \{ p \} \), we choose as \( V \) a small coordinate ball centered at \( P \), and as \( v \) a smooth strictly plurisubharmonic function on \( V \), bounded by \( 1/2 \) and \( 1 \). By the Conclusion, we obtain a smooth plurisubharmonic function \( \chi^* \) on \( K \), that is strictly plurisubharmonic near \( p \). Thus \( p \) is not in \( c(K) \).

**Proof of (b).** Suppose \( Y \) is not a Liouville set. Then there is a smooth plurisubharmonic function \( v \) on some open \( V \) containing \( Y \), and \( p, q \in Y \), such that \( v(p) \neq v(q) \) (WLOG \( 1/2 < v < 1 \)). By the Conclusion, there is a smooth strictly plurisubharmonic function \( \chi^* \) on \( K \), that is equal to \( (\Delta/2)v \) on \( Y \). But then \( \chi^*(p) \neq \chi^*(q) \), which contradicts the definition of \( Y \).

\( \Box \)
Observe that the last theorem gives positive answer to Shcherbina’s Question 5.3. Also, since \( c(K) \) is the union of sets \( Y \) having local maximum property by (b), \( c(K) \) is pseudoconcave by Proposition 1.2(a). Finally, Theorem 2.1 allows for an alternative proof of Shcherbina’s Main Theorem.

2.2. Corollary. Let \( K \) be a compact subset of a complex manifold. Then there is a smooth strictly plurisubharmonic function on \( K \), if and only if \( K \) does not contain any compact pseudoconcave subset.

Proof. If \( K \) contains a compact pseudoconcave set \( X \), then, by Proposition 1.5, \( X \) contains a minimal compact pseudoconcave set \( Z \) which by Proposition 1.7 is a Liouville set, and so, by Proposition 1.4(b), must be contained in the core \( c(K) \). Core being nonempty, a smooth strictly plurisubharmonic fuction on \( K \) cannot exist.

If \( K \) does not contain any compact pseudoconcave set, then \( c(K) \) must be empty (it would be otherwise pseudoconcave), which means that a smooth strictly plurisubharmonic function on \( K \) does exist. \( \square \)

The next example illustrates Theorem 2.1, and gives answer to Shcherbina’s Question 5.1.

2.3. Example. Let \( M := \overline{C} \times C \) and \( K := \overline{C} \times S^1 \cup \{0\} \times \overline{D}(0, 2) \). Consider pluriharmonic functions \( u(z, w) = \Re w \), and \( v(z, w) = \Im w \), and let \( Y_{a, b} := \{(z, w) \in K : u(z, w) = a, v(z, w) = b\} \). If \( a^2 + b^2 = 1 \), then \( Y_{a, b} = \overline{C} \times \{a + ib\} \) and is a compact Liouville set. Otherwise \( Y_{a, b} = \{(0, a + ib)\} \) is a single point (when \( a^2 + b^2 < 1 \), or \( 1 < a^2 + b^2 \leq 4 \)). By Theorem 2.1 the core is the union of Liouville sets contained in \( K \). Hence \( c(K) = \overline{C} \times S^1 \).

Let now \( K_1 := c(K) \cup \{0\} \times \overline{D}(0, 1) \). Since this set is closed, and is the union of local maximum sets (the second not closed), it is pseudoconcave by Proposition 1.2(a), and so is contained in \( n(K) \). Thus in this case the nucleus and the core are not equal (which answers [Sh21,Question 5.1]). Observe yet that \( n(K) = K_1 \). Suppose \( n(K) \) contains \( K_1 \) properly. By Proposition 1.2(b), \( n(K) \setminus K_1 \) would be a relatively closed local maximum subset of the annulus \( \{0\} \times \{a + ib : 1 < a^2 + b^2 \leq 4\} \). But this is not possible.

References

[HST17] Harz, T., Shcherbina, N., Tomassini, G.: On defining functions and cores for unbounded domains I. Math. Z. 286 (2017), 987-1002.

[HST20] Harz, T., Shcherbina, N., Tomassini, G.: On defining functions and cores for unbounded domains II. J. Geom. Anal. 30 (2020), 2293-2325.

[HST21] Harz, T., Shcherbina, N., Tomassini, G.: On defining functions and cores for unbounded domains III. Mat. Sb. 212, 6 (2021), 126-156.

[HL88] Henkin,G.-M., Leiterer, J., G.: Andreotti-Grauert theory by integral formulas. Progress in Mathematics 74 (1988).

[MST18] Mongodi, S., Slodkowski, Z., Tomassini, G.: Weakly complete complex surfaces. Indiana Univ. Math. J. 67 (2018), 899-935.

[PS19] Poletsky, E. A., Shcherbina, N.: Plurisubharmonically separable complex manifolds. Proc. Amer. Math. Soc. 147 (2019), 2413 - 2424.

[Ri68] Richberg, R.: Stetige streng pseudokonveze Funktionen. Math. Ann. 175 (1968), 257-286.
[Ro55] Rothstein, W.: *Zur Theorie der analytischen Mannigfaltigkeiten in Raume von n komplexen Veränderlichen*. Math. Ann. **129** (1955), 96-138.

[Sh21] Shcherbina, N.V.: *On compacts possessing strictly plurisubharmonic functions*. Izvestiya : Mathematics **85** (2021), 605-618.

[ST04] Slodkowski, Z., Tomassini, G.: *Minimal kernels of weakly complete spaces*. J. Funct. Anal. **210** (2004), 125–147.

[Sl19] Slodkowski, Z.: *Pseudoconcave decompositions in complex manifolds*. Contemp. Math. **735** (2019), 239-259.

Zbigniew Slodkowski Department of Mathematics, University of Illinois at Chicago
851 South Morgan Street, Chicago, IL 60607, USA

*e-mail*: zbigniew@uic.edu