ABSTRACT: We characterize the double Veronese embedding of $\mathbb{P}^n$ as the only variety that, under certain general conditions, can be isomorphically projected from the Grassmannian of lines in $\mathbb{P}^{2n+1}$ to the Grassmannian of lines in $\mathbb{P}^{n+1}$.

0. Introduction.

In [S], Severi proved that the only nondegenerate (i.e. not contained in a hyperplane) smooth (complex) surface in $\mathbb{P}^5$ that can be isomorphically projected to $\mathbb{P}^4$ is the Veronese surface. More recently, Zak extended Severi’s result and proved that, for $n \geq 2$ the only nondegenerate $n$-dimensional smooth subvariety of $\mathbb{P}^{n(n+3)}$ that can be isomorphically projected to $\mathbb{P}^{2n}$ is the $n$-uple Veronese embedding of $\mathbb{P}^n$ (see [Z2], or [Åd] for a similar statement).

In [A-S], there is a classification of all smooth surfaces in $G(1,3)$ (the Grassmann variety of lines in $\mathbb{P}^3$) that are non-trivial projection of a surface in $G(1,4)$ (by non-trivial we mean that the corresponding surface in $G(1,4)$ is nondegenerate in the sense that there is no hyperplane in $\mathbb{P}^4$ containing all the lines parametrized by the surface). In particular, this classification shows that the only nondegenerate smooth surface in $G(1,5)$ that can be isomorphically projected to $G(1,3)$ is a Veronese surface, more precisely the embedding of $\mathbb{P}^2$ in $G(1,5)$ by the vector bundle $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$.

More generally, consider the embedding of $\mathbb{P}^n$ in $G(1,2n + 1)$ given by the vector bundle $\mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$ (in this context we will refer to it as the $n$-dimensional Veronese variety in $G(1,2n + 1)$). We showed in [Ar] (see also Example 1.1) that this Veronese variety can be isomorphically projected into $G(1,n+1)$, and made the following:

**Conjecture 0.1:** For any $n \geq 1$, the only nondegenerate smooth complex $n$-dimensional subvariety $X$ of $G(1,2n + 1)$ that can be isomorphically projected into $G(1,n + 1)$ is the Veronese variety.
In this paper we prove this conjecture (see Theorem 3.1) under the extra assumption that \( X \) is what we will call *uncompressed*, i.e. that the union in \( \mathbb{P}^{2n+1} \) of all lines parametrized by \( X \) has the expected dimension \( n + 1 \). I have not been able to remove this condition, although I am sure that it is not necessary. In fact, the conjecture is known to be true for \( n \leq 2 \), and we can also prove it in case \( n = 3 \) (see Corollary 4.1). However the kind of proof required for the compressed case seems to be completely different from the techniques introduced in this paper.

The steps in the proof will follow the same as in [Z2] for the projective case. Surprisingly, the difficulty for each step in the Grassmannian case seems to be complementary to the difficulty in the projective case. For example, the most tricky part (probably the only one) in our case is to prove that the projectability of a variety implies that the appropriate secant variety has small dimension (Lemma 2.2). Our approach to this result consists of an infinitesimal study, which makes our result to depend strongly on the characteristic zero assumption. But on the other hand, the main point in Zak’s proof is the so-called Terracini’s lemma, used to prove some tangency result. However in our case this tangency condition (Lemma 2.5) follows immediately from the geometry of the Grassmannian of lines.

In some sense, Grassmannians of lines seem to provide a much more natural context to study these projection properties. For instance, any \( G(1, n+1) \) has dimension \( 2n \), so it is natural to expect that few smooth \( n \)-dimensional subvarieties of it are projected from bigger Grassmannians. Also our result works even for the case of curves (\( n = 1 \)), in which the theorem of Zak does not give a characterization of the corresponding Veronese variety (i.e. a conic).

The organization of the paper is as follows. In section 1 we give some preliminaries and recall some facts from [Ar]. In section 2 we prove some lemmas we will need to prove our theorem. In section 3 we state and prove our main theorem. Finally in section 4 we discuss the extra hypothesis we added to our theorem. We also discuss some general results of what could be a deeper study of projection properties in any Grassmannian of lines. In fact I hope that this paper will be just the starting point for such a study, even for general Grassmannians.

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1. Preliminaries.

We will work over an algebraically closed field $k$ of characteristic zero. We will denote by $G(r, m)$ the Grassmann variety of $r$-linear spaces in $\mathbb{P}^m$. A linear projection $G(1, m)\longrightarrow G(1, m')$ will mean the natural (rational) map induced by the corresponding linear projection $\mathbb{P}^m\longrightarrow\mathbb{P}^{m'}$.

**Notation:** We will denote the elements of any Grassmannian of lines by small letters, say $\ell$, and use the corresponding capital letter, say $L$, for the line in projective space that they define.

**Example 1.1:** For the sake of completeness, let us recall here the example provided by Proposition 3.4 in [Ar]. Consider the natural embedding of $\mathbb{P}^n$ in $G(1, 2n + 1)$ defined by $O_{\mathbb{P}^n}(1) \oplus O_{\mathbb{P}^n}(1)$. In coordinates, it can be described by associating to each $(t_0 : \ldots : t_n) \in \mathbb{P}^n$ the line spanned by the rows of the matrix

\[
\begin{pmatrix}
t_0 & \ldots & t_n & 0 & \ldots & 0 \\
0 & \ldots & 0 & t_0 & \ldots & t_n
\end{pmatrix}
\]

We consider now the linear projection $\mathbb{P}^{2n+1}\longrightarrow\mathbb{P}^{n+1}$ defined by

\[
(x_0 : \ldots : x_{2n+1}) \mapsto (x_0 : x_1 + x_{n+1} : \ldots : x_n + x_{2n} : x_{2n+1})
\]

This projection induces a projection from $G(1, 2n + 1)$ to $G(1, n + 1)$ and the image of $\mathbb{P}^n$ corresponds to the lines spanned by the rows of the matrix

\[
\begin{pmatrix}
t_0 & t_1 & \ldots & t_n & 0 \\
0 & t_0 & \ldots & t_{n-1} & t_n
\end{pmatrix}
\]

This is still an embedding of $\mathbb{P}^n$ in $G(1, n + 1)$, since the maximal minors of the above matrix (which give the image of $\mathbb{P}^n$ after the Plücker embedding of $G(1, n+1)$ in $\mathbb{P}^{(n+2)\frac{n}{2} - 1}$) define the double Veronese embedding of $\mathbb{P}^n$ in $\mathbb{P}^{(n+2)\frac{n}{2} - 1}$. By this reason we will call this subvariety of $G(1, 2n + 1)$ (or of $G(1, n + 1)$) the $n$-dimensional Veronese variety.

For the rest of the paper (except for the last section) our setting will be the one needed for proving our main theorem. Most of the results could be formulated in a more general setting, but we will leave this kind of comments for the last section. So we will consider $X$ to be a smooth irreducible $n$-dimensional subvariety of $G(1, 2n + 1)$. For short we will sometimes call a line of $X$ to a line in $\mathbb{P}^{2n+1}$ parametrized by a point of $X$. We will say that $X$ is nondegenerate if the union of all the lines of $X$ is not contained in a hyperplane. We will also say that $X$ is compressed if the union in $\mathbb{P}^{2n+1}$ of all of its lines has dimension at most $n$. Otherwise, if this union has dimension is $n + 1$, we will say that $X$ is uncompressed.
Remark 1.2: The notion of compressedness is related with the following fact. Consider the incidence variety \( I_X := \{(\ell, p) \in X \times \mathbb{P}^{2n+1} \mid p \in L\} \) and the projection \( p_X : I_X \to \mathbb{P}^{2n+1} \). Then \( X \) is uncompressed if and only if this projection is generically finite over its image. In other words, for a general hyperplane \( H \) of \( \mathbb{P}^{2n+1} \) the rational map \( p_H : X \dashrightarrow H \) that assigns to each line of \( X \) (not contained in \( H \)) its intersection with \( H \) is generically finite over its image.

2. Some previous results.

From now on, \( X \) will be a smooth irreducible \( n \)-dimensional uncompressed nondegenerate subvariety of \( G(1, 2n + 1) \) that can be isomorphically projected to \( G(1, n + 1) \). This means in particular that there is an \((n - 1)\)-dimensional center of projection \( \Lambda \) verifying the following two conditions:

\((*) \) Any two skew lines of \( X \) (probably infinitely close) span a three-dimensional linear space that meets \( \Lambda \) at most in one point.

\((**) \) If two lines of \( X \) (probably infinitely close) span only a two-dimensional linear space, then this span does not meet \( \Lambda \).

Condition \((*)\) is easier to handle in the sense that can be described in terms of an irreducible variety, namely an open subset of \( X \times X \). Our first task is to see that this set is non-empty. This is the statement of the following (easy and well-known) lemma.

Lemma 2.1. Two general lines of \( X \) are skew.

Proof: If any two lines of \( X \) meet, then take two of them, say \( L_1, L_2 \). They meet in a point \( P \in \mathbb{P}^{2n+1} \) and span a plane \( \Pi \). Then, from the irreducibility of \( X \), either all lines of \( X \) are contained in \( \Pi \) or pass through \( P \). The first possibility is impossible, either by dimensional reasons (if \( n \geq 3 \)) or by the hypothesis that \( X \) is nondegenerate. In the second possibility, \( X \) will consist of the generators of a cone with vertex \( P \) over a projective \( n \)-dimensional subvariety \( Y \). The nondegeneracy hypothesis implies that \( Y \) spans a hyperplane in \( \mathbb{P}^{2n+1} \), and the fact that \( X \) can be projected implies that \( Y \) can be isomorphically projected into \( \mathbb{P}^n \). But this implies that \( Y \) is a linear space, contradicting again the nondegeneracy hypothesis. \( \square \)

Definition: We will call secant variety to \( X \) to the variety \( SX \subset G(3, 2n + 1) \) consisting of the closure of the set of linear spaces spanned by pairs of skew lines of \( X \). In other words, \( SX \) is the closure of the image of the rational map \( p : X \times X \dashrightarrow G(3, 2n + 1) \) that associates to each pair of skew lines its linear span. We will call the secant defect of \( X \) to
the dimension $\delta$ of $Y_{\Pi}$ for a general $\Pi \in SX$, where $Y_{\Pi}$ is the set of lines of $X$ contained in $\Pi$. It is clear, by looking at the map $p$, that $\dim(SX) = 2n - 2\delta$.

**Observation:** These definitions are different from the “natural” generalization of the notion of projective secant variety and defect. However, their behavior will play a similar role as their corresponding projective concepts, as we will see throughout the paper.

**Lemma 2.2.** The variety $SX$ has dimension at most $2n - 2$, or equivalently, $X$ has positive secant defect.

**Proof:** We need to study the dominant rational map $p : X \times X \rightarrow SX$ and show that its differential $dp(\ell_1, \ell_2)$ at a general point $(\ell_1, \ell_2)$ has rank at most $2n - 2$. So assume for contradiction that $dp$ is injective for a general $(\ell_1, \ell_2) \in X \times X$. From Lemma 2.1 we know that the corresponding lines $L_1, L_2$ are skew. We choose projective coordinates $x_0, \ldots, x_{2n+1}$ so that these two lines are $L_1 : x_2 = \ldots = x_{2n+1} = 0$ and $L_2 : x_0 = x_1 = x_4 = \ldots = x_{2n+1} = 0$. An affine chart for $G(1, 2n + 1)$ around $\ell_1$ consists of the lines in $\mathbf{P}^{2n+1}$ spanned by the points whose coordinates are the rows of the matrix

$$
\begin{pmatrix}
1 & 0 & a_{02} & \ldots & a_{0,2n+1} \\
0 & 1 & a_{12} & \ldots & a_{1,2n+1}
\end{pmatrix}
$$

(the coordinates of the chart are the $a_{ij}$’s). Similarly, an affine chart for $G(1, 2n + 1)$ around $\ell_2$ consists of the lines in $\mathbf{P}^{2n+1}$ spanned by the points whose coordinates are the rows of the matrix

$$
\begin{pmatrix}
b_{00} & b_{01} & 1 & 0 & b_{04} & \ldots & b_{0,2n+1} \\
b_{10} & b_{11} & 0 & 1 & b_{14} & \ldots & b_{1,2n+1}
\end{pmatrix}
$$

Take also as an affine chart for $G(3, 2n + 1)$ around $<L_1, L_2>$ to be the set of all three-spaces spanned by the rows of the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & p_{04} & \ldots & p_{0,2n+2} \\
0 & 1 & 0 & 0 & p_{14} & \ldots & p_{1,2n+2} \\
0 & 0 & 1 & 0 & p_{24} & \ldots & p_{2,2n+2} \\
0 & 0 & 0 & 1 & p_{34} & \ldots & p_{3,2n+2}
\end{pmatrix}
$$

Then it is not difficult to check that the differential $dp$ at $(\ell_1, \ell_2)$ – the origin in our system of coordinates – is given in these coordinates by the equation

$$
(a_{02}, \ldots, a_{0,2n+1}, a_{12}, \ldots, a_{1,2n+1}; b_{00}, \ldots, b_{0,2n+1}, b_{10}, \ldots, b_{1,2n+1}) \mapsto (a_{04}, \ldots, a_{0,2n+1}, a_{14}, \ldots, a_{1,2n+1}, b_{04}, \ldots, b_{0,2n+1}, b_{14}, \ldots, b_{1,2n+1})
$$

The fact that $dp$ is injective is equivalent to the fact that the maps $T_{\ell_1}X \rightarrow k^{4(n-1)}$ and $T_{\ell_2}X \rightarrow k^{4(n-1)}$ defined respectively by

$$
(a_{02}, \ldots, a_{0,2n+1}, a_{12}, \ldots, a_{1,2n+1}) \mapsto (a_{04}, \ldots, a_{0,2n+1}, a_{14}, \ldots, a_{1,2n+1})
$$
\[ (b_{00}, \ldots, b_{0,2n+1}, b_{10}, \ldots, b_{1,2n+1}) \mapsto (b_{04}, \ldots, b_{0,2n+1}, b_{14}, \ldots, b_{1,2n+1}) \]

are injective.

On the other hand, for general points of \( X \), it is very easy to see that we can assume that the maps \( T_{\ell_1}X \rightarrow k^{2n} \) and \( T_{\ell_2}X \rightarrow k^{2n} \) defined respectively by

\[
(a_{02}, \ldots, a_{0,2n+1}, a_{12}, \ldots, a_{1,2n+1}) \mapsto (a_{02}, \ldots, a_{0,2n+1})
\]

\[
(b_{00}, \ldots, b_{0,2n+1}, b_{10}, \ldots, b_{1,2n+1}) \mapsto (b_{00}, b_{01}, b_{04}, \ldots, b_{0,2n+1})
\]

are injective. Indeed, since \( X \) is uncomressed, from Remark 1.2 we know that the rational map \( p_H : X \rightarrow H \) is generically finite over its image for a general hyperplane \( H \subset \mathbb{P}^{2n+1} \).

If we take \( \ell_1 \) not to be a ramification point of \( p_H \) and choose coordinates so that \( H \) has equation \( x_1 = 0 \) then we get the wanted hypotheses for \( \ell_1 \). We proceed similarly for \( \ell_2 \).

Also (changing coordinates if necessary) we can assume that the composition of both maps with the same linear projection \( k^{2n} \rightarrow k^n \) is an isomorphism. Summing up, we can assume from our hypothesis that the two linear maps \( d_1 : T_{\ell_1}X \rightarrow k^n \) and \( d_2 : T_{\ell_2}X \rightarrow k^n \) defined by

\[
d_1(a_{02}, \ldots, a_{0,2n+1}, a_{12}, \ldots, a_{1,2n+1}) = (a_{0,n+2}, \ldots, a_{0,2n+1})
\]

\[
d_2(b_{00}, \ldots, b_{0,2n+1}, b_{10}, \ldots, b_{1,2n+1}) = (b_{0,n+2}, \ldots, b_{0,2n+1})
\]

are bijective.

Now we will relate these maps with the differential of another map. More precisely, let us consider \( IX \) to be the closure of the set

\[ \{(\ell_1, \ell_2, \Lambda) \in X \times X \times G(n-1, 2n+1) \mid \dim < L_1, L_2 >= 3, \dim(\Lambda \cap < L_1, L_2 >) \geq 1 \} \]

and let \( q : IX \rightarrow G(n-1, 2n+1) \) be the natural projection. Both varieties have the same dimension \( n^2 + 2n \) (for the dimension of \( IX \) consider the projection onto \( X \times X \), whose general fibers are Schubert varieties of dimension \( n^2 \)). The hypothesis that \( X \) can be projected to \( G(1, n+1) \) means, by condition (*) that \( q \) is not surjective, equivalently that its differential map at any point \( (\ell_1, \ell_2, \Lambda) \in IX \) is not injective. Let us study this differential map at the point \( (\ell_1, \ell_2, \Lambda) \), where \( \Lambda \) is the linear subspace of equations \( \Lambda : x_0 = x_3, \ x_1 = x_{n+2} = \ldots = x_{2n+1} = 0 \). We clearly have that \( (l_1, l_2, \Lambda) \) belongs to \( IX \). We take the affine chart for \( G(n-1, 2n+1) \) around \( \Lambda \) consisting of the \( (n-1) \)-linear subspaces of \( \mathbb{P}^{2n+1} \) spanned by the points whose coordinates are the rows of the \( n \times (2n+2) \)-matrix

\[
\begin{pmatrix}
    x_{00} & x_{01} & 1 & 0 & \ldots & 0 & x_{0,n+2} & \ldots & x_{0,2n+1} \\
    1 + x_{10} & x_{11} & 0 & 1 & \ldots & 0 & x_{1,n+2} & \ldots & x_{1,2n+1} \\
        \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots \\
    x_{n-1,0} & x_{n-1,1} & 0 & 0 & \ldots & 1 & x_{n-1,n+2} & \ldots & x_{n-1,2n+1}
\end{pmatrix}
\]
Locally at the point \((\ell_1, \ell_2, \Lambda) \in X \times X \times G(n-1, 2n+1)\) –which is the origin in our coordinates– the equations for \(IX\) are given by the maximal minors of the \((n+3) \times (2n+2)\)-matrix

\[
\begin{pmatrix}
1 & 0 & a_{02} & a_{03} & a_{04} & \ldots & a_{0,n+1} & a_{0,n+2} & \ldots & a_{0,2n+1} \\
0 & 1 & a_{12} & a_{13} & a_{14} & \ldots & a_{1,n+1} & a_{1,n+2} & \ldots & a_{1,2n+1} \\
b_{i0} & b_{i1} & 1 & 0 & b_{i4} & \ldots & b_{i,n+1} & b_{i,n+2} & \ldots & b_{i,2n+1} \\
x_{00} & x_{01} & 1 & 0 & 0 & \ldots & 0 & x_{0,n+2} & \ldots & x_{0,2n+1} \\
1 + x_{10} & x_{11} & 0 & 1 & 0 & \ldots & 0 & x_{1,n+2} & \ldots & x_{1,2n+1} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
x_{n-1,0} & x_{n-1,1} & 0 & 0 & 0 & \ldots & 1 & x_{n-1,n+2} & \ldots & x_{n-1,2n+1}
\end{pmatrix}
\]

for \(i = 0, 1\). In particular, the tangent space of \(IX\) at the origin (identifying the coordinates in the tangent space with the affine coordinates) is easily seen to be given by the equations

\[x_{0j} = b_{0j} \quad \text{for} \quad j = n + 2, \ldots, 2n + 1\]

\[x_{1j} = a_{0j} + b_{1j} \quad \text{for} \quad j = n + 2, \ldots, 2n + 1\]

(To see this, just take the initial terms of the maximal minors defined by the first \(n + 2\) columns of the above matrix and any of the other columns). But now the injectivity of the above maps \(d_1\) and \(d_2\) easily implies the injectivity of \(dq\), so that we get a contradiction. \(\square\)

We can improve the above result by proving that the inequality for the dimension is in fact an equality. We prefered to separate this result in two parts for a later discussion of both facts in relation with the use of the uncompressedness hypothesis. The precise statement is the following.

**Lemma 2.3.** For a general \(\Pi \in SX\), \(Y_\Pi\) is the curve in \(G(1, \Pi)\) consisting of the lines of one of the rulings of a smooth quadric in \(\Pi\).

**Proof:** We claim first that it cannot happen that any line of \(Y_\Pi\) meets another line of \(Y_\Pi\) (maybe infinitely close). If this happens, \(Y_\Pi\) would be the union of planes containing two (maybe infinitely close) lines of \(X\). From condition (***) we know that the union of such planes in \(\mathbb{P}^{2n+1}\) has dimension at most \(n + 1\). As a consequence, the union \(Z\) of all the \(Y_\Pi\)'s has dimension at most \(n + 1\). Since \(X\) is uncompressed, this implies that \(Z\) is also the union of all lines of \(X\).

Let us prove now by induction on \(k\) that the (closure of the) union of the spans of \(k + 1\) lines of \(X\) is again \(Z\) for any \(k\) (this would give a contradiction since there exists a value of \(k\) for which the span of \(k\) general lines of \(X\) must be \(\mathbb{P}^{2n+1}\)). We just proved
this for \( k = 1 \). So take now \( p \) to be a point of \( \mathbb{P}^{2n+1} \) that is in the span of \( k + 1 \) general lines \( L_1, \ldots, L_{k+1} \) of \( X \) and assume \( k > 1 \). Then \( p \) is in the span of \( L_{k+1} \) and a point \( p' \in < L_1, \ldots, L_k > \). By induction hypothesis, then \( p' \) is in \( Z \), i.e. there exists a line \( \ell \) of \( X \) passing through it. But this means that \( p \) is in the span of \( L \) and \( L_{k+1} \), which means in turn that \( p \) is in \( Z \), as wanted. This proves the claim.

We know from Lemma 2.2 that \( Y_\Pi \) has positive dimension. It cannot be a surface, since this would imply that any line of \( Y_\Pi \) would meet infinitely many others. So assume that \( Y_\Pi \) is a curve of degree \( d \). The proof that its degree is \( d = 2 \) will follow a very standard argument (see for example the proof of Lemma 5.3 in [A-S]). Take a general line \( \ell \) of \( Y_\Pi \) and consider the Schubert variety \( Z_\ell \) in \( G(1, \Pi) \) of all lines meeting \( L \). This is a quadratic cone with vertex \( \ell \) (after the Pl"ucker embedding) and its intersection number with \( Y_\Pi \) is \( d \). If \( Y_\Pi \) and \( Z_\ell \) are transversal at \( \ell \), the intersection multiplicity at that point is two, and hence there are other \( d - 2 \) lines (counted with multiplicity) of \( X \) in \( Y_\Pi \) meeting \( L \). If the intersection is not transversal, the tangent line at \( \ell \) of \( Y_\Pi \) is a generator of the cone. Hence there is a plane containing \( \ell \) such that the intersection of \( X \) with the Schubert variety of the lines contained in that plane contains a subscheme of length at least two.

Therefore the only possibility (after the claim) is that \( d = 2 \) and one easily checks also that \( Y_\Pi \) must consist of one the rulings of a smooth quadric.

This result proves that \( X \) contains too many conics. For dimension \( n = 2 \) this is the way of showing that \( X \) is the Veronese surface. For general dimension, we need to find a lot of “special” divisors in \( X \). For this we will need to generalize the above results to the span of more than two lines of \( X \). We need first to generalize a few definitions.

**Definition:** For any \( k = 1, \ldots, n \) let \( r_k \) be the dimension of the span of \( k + 1 \) general lines of \( X \). We define the \( k \)-secant variety to \( X \) to be the subvariety \( S^k X \subset G(r_k, 2n+1) \) defined as the closure of the \( r_k \)-linear subspaces in \( \mathbb{P}^{2n+1} \) spanned by \( k + 1 \) general lines of \( X \). If \( \Pi \) is a general \( r_k \)-space in \( S^k X \) we define the set \( Y_\Pi \) of \( X \) as the set of lines contained in \( \Pi \). Of course for \( k = 1 \) we have \( S^1 X = SX \).

The wanted divisors will be the subsets \( Y_\Pi \) for \( \Pi \in S^{n-1} X \). For this, we will need first to show that these are indeed divisors. This is the purpose of the next result, which more generally gives the dimension of any secant variety.

**Proposition 2.4.** For any \( k = 1, \ldots, n \), the span \( \Pi \) of \( k + 1 \) general lines of \( X \) has dimension \( 2k + 1 \), \( \dim(Y_\Pi) = k \) and \( \dim(S^k X) = (k+1)(n-k) \).

**Proof:** First we observe that it is enough to show that, if \( k < n \) and \( \ell_1, \ldots, \ell_{k+1} \) are \( k + 1 \) general lines of \( X \) and \( \Pi' = < L_1, \ldots, L_k > \), \( \Pi = < L_1, \ldots, L_{k+1} > \), then:

\[
\dim(Y_\Pi) \geq \dim(Y_{\Pi'}) + 1 \tag{2.1}
\]
Indeed this inequality implies that \( \dim(Y_\Pi) \geq k \) for a general \( \Pi \in S^kX \); in particular, \( \dim(Y_\Pi) \geq n \) for \( \Pi \in S^nX \). Since \( X \) has dimension \( n \) this shows that \( Y_\Pi = X \) for \( \Pi \in S^nX \) and we have that all inequalities in (2.1) are equalities. This proves \( \dim(Y_\Pi) = k \). Also, since \( Y_\Pi = X \) for \( \Pi \in S^nX \) and \( X \) is nondegenerate, it must be \( \dim(\Pi) = 2n + 1 \), from which we conclude that for \( \Pi \in S^kX \) it is \( \dim(Y_\Pi) = 2k + 1 \). Finally, we immediately see that the dimension of the fiber of \( X \times \dim(\Pi) = 2n + 1 \) (the rational map assigning to \( k + 1 \) general lines of \( X \) its linear span) has dimension \( k(k+1) \). This proves that \( \dim(S^kX) = (k+1)(n-k) \).

So let us prove inequality (2.1). Consider \( \ell_1, \ldots, \ell_k \) to be general lines of \( X \). Since \( X \) is nondegenerate and the corresponding lines \( L_1, \ldots, L_k \) span at most a linear space \( \Pi' \) of dimension \( 2k - 1 < 2n + 1 \) it holds that \( Y_{\Pi'} \) is not the whole \( X \). So we can take a general \( \ell_{k+1} \) in \( X \) that is not in \( Y_{\Pi'} \). Consider the set \( J \) to be the closure in \( Y_{\Pi} \times Y_{\Pi'} \) of

\[
\{(\ell, \ell') \in Y_{\Pi} \times Y_{\Pi'} \mid L \subset \langle L', L_{k+1} \rangle, \dim \langle L', L_{k+1} \rangle = 3\}
\]

The fiber of the natural projection \( J \to Y_{\Pi'} \) over a general \( \ell' \in Y_{\Pi'} \) is the set \( Y_{\langle L', L_{k+1} \rangle} \), which has dimension one after Lemma 2.3. Hence \( \dim(J) = \dim(Y_{\Pi'}) + 1 \). So it is enough to show that the projection map from \( J \) to \( Y_{\Pi} \) is generically finite over its image. This is so because, given a general \( \ell \in Y_{\Pi} \), a general element \( \ell' \in Y_{\Pi'} \) verifies that \( L \subset \langle L', L_{k+1} \rangle \) if \( L' \subset \langle L, L_{k+1} \rangle \). If the intersection of \( \langle L, L_{k+1} \rangle \) with \( \Pi' \) is just one line, this is precisely \( L' \). If the intersection is a plane, \( L' \) is a line in this plane that is also in \( X \). But from Lemma 2.3, since \( L \) and \( L'_{k+1} \) are general, the set of lines of \( X \) that are in \( \langle L, L_{k+1} \rangle \) is one of the rulings of a quadric. So there is only one of these lines contained in a plane.

The following easy lemma can be considered as a (partial) generalization to our context of Terracini’s lemma.

**Lemma 2.5.** Let \( \Pi \) be a general element of \( S^{n-1}X \). Consider the divisor \( H_{\Pi} \) of \( G(1, 2n+1) \) given by the Schubert cycle consisting of all lines meeting \( \Pi \). Then the intersection cycle of \( X \) with \( H_{\Pi} \) contains \( Y_{\Pi} \) with multiplicity at least two.

**Proof:** This follows immediately from the observation that the singular locus of \( H_{\Pi} \) is the Schubert cycle of all lines contained in \( \Pi \), so that \( Y_{\Pi} \) is contained in that singular locus.

3. The main theorem.

We can now state and prove the main theorem of this paper.
Theorem 3.1. Let $X$ be a smooth irreducible subvariety of $G(1, 2n+1)$ ($n \geq 1$). Assume that $X$ is uncompressed and nondegenerate and that it can be isomorphically projected to $G(1, n+1)$. Then $X$ is a Veronese variety as in Example 1.1.

Proof: We keep the same notation as in the previous sections. We will assume $n \geq 2$ since the proof for $n = 1$ goes differently and it is much easier (see [Ar]). The idea is to prove that the linear system corresponding to the hyperplanes of $\mathbb{P}^{2n+1}$ is the set of divisors $Y_\Pi$ for $\Pi \in S^{n-1}X$ (it is a nice exercise to contrast each step of the proof with the actual behavior of the Veronese embedding).

Take a general $\Pi \in S^{n-1}X$. From Lemma 2.5 we have that

$$H_\Pi|_X = rY_\Pi + E_\Pi$$

(3.1)

where $r \geq 2$ and the support of $E_\Pi$ does not contain $Y_\Pi$.

Take now a general $\Pi' \in SX$. We know from Lemma 2.3 that $Y_{\Pi'}$ consists of one of the rulings of a smooth quadric in $\Pi'$. Hence intersecting with $Y_{\Pi'}$ in (3.1) we obtain that $2 = rY_\Pi \cdot Y_{\Pi'} + E_\Pi \cdot Y_{\Pi'}$. From this we see that $r = 2$, $Y_\Pi \cdot Y_{\Pi'} = 1$ and $E_\Pi \cdot Y_{\Pi'} = 0$. This last equality easily implies that $E_\Pi = 0$. Indeed, if there exists $\ell \in E_\Pi$, it is not difficult to find a $\Pi'$ such that $Y_{\Pi'}$ is irreducible and is not contained in $E_\Pi$; therefore the intersection of $Y_{\Pi'}$ and $E_\Pi$ would be proper, hence empty since the intersection number is zero, which is a contradiction.

So we have arrived to the equality $H_\Pi|_X = 2Y_\Pi$. This easily implies that all the divisors $Y_\Pi$ are linearly equivalent. Let us study the complete linear system $|Y_\Pi|$. It clearly has no base points, since for any point $\ell \in X$ we know from Proposition 2.4 that we can find $\ell_1, \ldots, \ell_n$ such that $L, L_1, \ldots, L_n$ span $\mathbb{P}^{2n+1}$. This means that $l \notin Y_\Pi$, where $\Pi = \langle L_1, \ldots, L_n \rangle$. Hence $|Y_\Pi|$ defines a regular map

$$\varphi : X \to \mathbb{P}^N$$

where $N = \dim |Y_\Pi|$. Let us denote by $X'$ the image of $\varphi$. From what we have just seen, $|2Y_\Pi|$ is the hyperplane section of $X$ (after the Plücker embedding), so that $|Y_\Pi|$ is ample, and therefore the map $\varphi$ is finite over $X'$. Recall that we have also got from (3.1) the equality $Y_\Pi \cdot Y_{\Pi'} = 1$ for general $\Pi \in S^{n-1}X$, $\Pi' \in SX$. Hence the image of a general $Y_{\Pi'}$ is a line in $\mathbb{P}^N$. This proves that two general points of $X'$ can be joined by a line, and hence $X' = \mathbb{P}^n$ and $N = n$.

On the other hand, let $E$ be the rank-two vector bundle on $X$ giving the embedding of $X$ in $G(1, 2n+1)$. Since $X$ is nondegenerate we have that

$$m + 1 := h^0(X, E) \geq 2n + 2$$
(Also we could conclude a priori that equality holds, since the proof of Proposition 2.4 works in fact for $X$ in any $G(1, m)$ with $m \geq n + 3$, and we could show then that all lines are contained in the linear span of $n + 1$ general lines, hence $m = 2n + 1$, from the nondegeneracy hypothesis). We also have that $\bigwedge^2 E = \mathcal{O}_X(2Y_{\Pi})$. Now let us show that the equality $H_{\Pi}|_X = 2Y_{\Pi}$ implies the splitting

$$E \cong \mathcal{O}_X(Y_{\Pi}) \oplus \mathcal{O}_X(Y_{\Pi}) \quad (3.2)$$

Indeed take a general $\Pi \in S^{n-1}X$. It is a linear space of codimension two in $\mathbb{P}^{2n+1}$, hence there are two independent sections of $E$ vanishing on $Y_{\Pi}$ (therefore we can consider them as sections of $E(-Y_{\Pi})$). But the equality $H_{\Pi}|_X = 2Y_{\Pi}$ means that the dependency locus of these two sections is precisely $Y_{\Pi}$. In other words, there are two independent sections of $E(-Y_{\Pi})$ whose dependency locus is empty. This proves $E(-Y_{\Pi}) \cong \mathcal{O}_X \oplus \mathcal{O}_X$, which is (3.2).

Now (3.2) implies that there is a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & \mathbb{P}^n \\
\downarrow & \searrow & \downarrow \\
G(1, m) & \longrightarrow & G(1, n + 1)
\end{array}
$$

Here the vertical maps are the respective embeddings of $X$ an $\mathbb{P}^n$ given by the bundles $E \cong \mathcal{O}_X(Y_{\Pi}) \oplus \mathcal{O}_X(Y_{\Pi})$ and $\mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$; the horizontal dashed arrow is the projection induced by a linear projection $\mathbb{P}^m \longrightarrow \mathbb{P}^{2n+1}$; and the composed diagonal morphism is the given inclusion of $X$ in $G(1, 2n + 1)$. Therefore $\varphi$ is also an embedding, hence an isomorphism, so that $X$ is the Veronese variety in $G(1, 2n + 1)$.

4. Remarks and questions on projections of Grassmannians.

One of my main goals when I started to think of this problem was to prove Conjecture 0.1 for $n = 3$. In this dimension, it follows easily from Theorem 3.1, as we show next.

**Corollary 4.1.** Let $\bar{X}$ be a smooth irreducible threefold of $G(1, 4)$ that is a projection of a nondegenerate threefold $X$ in $G(1, 7)$. Then $\bar{X}$ is the Veronese threefold.

**Proof:** If $\bar{X}$ is not the Veronese threefold, we know from Theorem 3.1 that $X$ must be compressed. In other words, through a general point of $\mathbb{P}^4$ there passes no line of $\bar{X}$. But smooth threefolds in $G(1, 4)$ verifying this property are classified in [ABT] and it is easy to check that none of them comes from $G(1, 7)$.

In general, the natural way of approaching compressed subvarieties of Grassmannians is the philosophy that a projective variety containing too many lines either has a bounded
degree or contains many linear varieties of bigger dimension. This is in fact the method used in [ABT] in dimension three. However, for higher dimension, I do not know of any sufficiently strong result for $n$-dimensional varieties containing an $n$-dimensional family of lines.

There are only two places in which we used the uncompressedness hypothesis: in Lemmas 2.2 and 2.3. It does not seem so important its use in Lemma 2.3. More precisely, it seems possible to prove first Proposition 2.4 without using Lemma 2.3 (we used this lemma just for a small detail at the end of the proof of the proposition); then one could deduce the lemma from the general position statement in the proposition (in fact I had originally an incorrect proof in this way, and I decided to fix the gap in the way shown in the paper as soon as I realized that I would need uncompressedness anyway). The crucial point where uncompressedness is used seems to be Lemma 2.2, at least the proof strongly needs this condition. I do not know of any example of a compressed $n$-variety projectable from $G(1,m)$ to $G(1,n+1)$ with $m \geq n+3$. If such an example exist, it would be interesting to see whether the secant defect is positive or not. The only example I know is the following for $m = n+2$.

**Example 4.2:** Consider the embedding of $\mathbb{P}^1$ in $G(r,2r+2)$ given by $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, or equivalently the smooth rational normal scroll of dimension $r+1$ in $\mathbb{P}^{2r+2}$. This gives a one-dimensional family of pairwise disjoint $r$-spaces in $\mathbb{P}^{2r+2}$. Dually, we find a one-dimensional family of $(r+1)$-spaces in $\mathbb{P}^{2r+2}$ such that any two of them meet only at one point. Hence the set of lines contained in these $(r+1)$-spaces forms a smooth $(2r+1)$-subvariety $X$ in $G(1,2r+2)$, which is compressed if $r > 0$. This variety is in fact projected from $G(1,2r+3)$. Indeed the map $\mathbb{P}^1 \to G(r+1,2r+2)$ corresponding to the family of $(r+1)$-spaces is defined by the epimorphism appearing in the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^1}^{\oplus 2r+3} \to \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r+2} \to 0$$

where $\psi$ is the dual of the map defining the given embedding of $\mathbb{P}^1$ in $G(r,2r+2)$. Since $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r+2}) = 2r + 4$, it follows that this family of $(r+1)$-spaces, and hence $X$, comes from $\mathbb{P}^{2r+3}$.

It could be very risky to conjecture that any compressed $n$-variety of $G(1, n+1)$ that comes projected from a bigger $(1,m)$ is one of those in the above example 4.2. However, having a classification of such varieties would certainly prove (or maybe disprove?) Conjecture 0.1.

I would like to discuss now a little bit what should be the main items in a general theory of secant varieties in Grassmannians of lines. Of course in order for the theory
to work properly we need to make some general assumptions. This includes not only the uncompressedness hypothesis (which we were unable to avoid in our main theorem) but also it is sometimes useful to assume some general position hypothesis. In the particular case we studied in this paper, because of the hypothesis that our variety was nondegenerate in $G(1, 2n + 1)$ we were able to prove this general position statement (Proposition 2.4). Let us give the precise definition of this hypothesis.

**Definition:** We will say that a subvariety $X$ of $G(1, N)$ is in *general position* if for $k = 1, \ldots, \lceil \frac{N-1}{2} \rceil$ the span of $k + 1$ general lines of $X$ is a linear space of dimension $2k + 1$. Hence the $k$-th secant variety $S^k X$ is a subvariety of $G(2k + 1, N)$.

It is immediate to check that the same proof of Lemma 2.2 works to prove that if $N \geq n + 3$, $X$ is not compressed, is not a cone and can be isomorphically projected to $G(1, n + 1)$ then the secant variety $SX = S^1 X$ has dimension at most $2n - 2$. For general secant varieties, one can make the following definition:

**Definition:** Let $X$ be a subvariety of $G(1, N)$ in general position. We will call the $k$-th secant defect of $X$ to be the dimension $\delta_k$ of a general $Y_{<L_0, \ldots, L_{i+j}>}$, where $Y_{\Pi}$ denotes the set of lines of $X$ contained in a given linear space $\Pi$.

Looking at the image and fibers of the rational map $X \times \ldots \times X \to G(2k + 1, N)$ that associates to each $k + 1$ general lines its linear span we easily obtain that

$$\dim(S^k X) = (k + 1)(n - \delta_k)$$

Of course, if $X$ is uncompressed, then $X$ is projectable if and only if $\delta_1 > 0$.

There is another relation among the defects, which is the translation to Grassmannians of the so-called Zak’s superadditivity theorem (see [Z1], or [H-R], or [F]). It is just the generalization of inequality (2.1) in the proof of Proposition 2.4 and its proof is surprisingly easy, contrary to what happens in the projective case. It is the following.

**Proposition 4.3.** If $X$ is in general position, then, for $i + j \leq \lceil \frac{N-1}{2} \rceil$, one has

$$\delta_{i+j} \geq \delta_i + \delta_j$$

**Proof:** It is exactly the same as for (2.1), but easier since we are already assuming general position (statement that we proved simultaneously in Proposition 2.4). We take $\ell_1, \ldots, \ell_{i+j}$ to be general lines of $X$ and consider the linear spaces $\Pi' := < L_0, \ldots, L_i >$ and $\Pi := < L_0, \ldots, L_{i+j} >$. Define $J$ to be the closure in $Y_{\Pi} \times Y_{\Pi'}$ of the set

$$\{ (\ell, \ell') \in Y_{\Pi} \times Y_{\Pi'} \mid L \subset< L', L_{i+1}, \ldots, L_{i+j} >, \dim < L', L_{i+1}, \ldots, L_{i+j} > = 2j + 1 \}$$

For a general $\ell' \in Y_{\Pi'}$, the fiber of the natural projection $J \to Y_{\Pi'}$ is $Y_{<L', L_{i+1}, \ldots, L_{i+j}>}$. Hence $J$ has dimension $\delta_i + \delta_j$. Finally the other projection $J \to Y_{\Pi}$ is generically finite.
over its image. Indeed for a general $\ell$ in this image, from the general position hypothesis we have that the spaces $P_i' = \langle L_0, \ldots, L_i \rangle$ and $\langle L_{i+1}, \ldots, L_{i+j}, L \rangle$ meet only along a line $L'$. This gives a unique point $(\ell, \ell')$ in $J$ mapping to $\ell$.

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