Classification of linear mappings between indefinite inner product spaces

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Abstract

Let $A:U \rightarrow V$ be a linear mapping between vector spaces $U$ and $V$ over a field or skew field $F$ with symmetric, or skew-symmetric, or Hermitian forms $B:U \times U \rightarrow F$ and $C:V \times V \rightarrow F$.

We classify the triples $(A,B,C)$ if $F$ is $\mathbb{R}$, or $\mathbb{C}$, or the skew field of quaternions $\mathbb{H}$. We also classify the triples $(A,B,C)$ up to classification of symmetric forms and Hermitian forms if the characteristic of $F$ is not 2.

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1. Introduction

We consider a triple

$$A:U \rightarrow V, \quad B:U \times U \rightarrow F, \quad C:V \times V \rightarrow F$$

consisting of a linear mapping $A$ and two forms $B$ and $C$ on finite-dimensional vector spaces $U$ and $V$ over a field or skew field $F$ of characteristic not 2. Each of the forms $B$ and $C$ is either symmetric or skew-symmetric if $F$ is a field, or both the forms are Hermitian with respect to a fixed nonidentity involution in $F$.

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A canonical form of the triple of matrices of (1) over a field $F$ of characteristic not 2 was obtained in the deposited manuscript [22] up to classification of Hermitian forms over finite extensions of $F$. The aim of this paper is to give a detailed exposition of this result and extend it to triples (1) over a skew field of characteristic not 2. We give canonical matrices of (1) over $\mathbb{R}$, $\mathbb{C}$, and the skew field of quaternions $\mathbb{H}$.

Other canonical matrices of (1) with nonsingular forms $B$ and $C$ over the fields $\mathbb{R}$ and $\mathbb{C}$ were given by Mehl, Mehrmann, and Xu [14, 15, 16], and by Bolshakov and Reichstein [2].

Following [22], we represent the triple (1) by the graph

$$
\begin{array}{ccc}
\bigcirc & \bigcirc & \bigcirc \\
U & B & V \\
A \downarrow & \downarrow C \\
\end{array}
$$

in which $\varepsilon = +$ if $B$ is symmetric or Hermitian and $\varepsilon = -$ if $B$ is skew-symmetric; $\delta = +$ if $C$ is symmetric or Hermitian and $\delta = -$ if $C$ is skew-symmetric.

Choosing bases in $U$ and $V$, we give (1) by the triple $(A, B, C)$ of matrices of $A$, $B$, and $C$. Changing bases, we can reduce it by transformations

$$(A, B, C) \mapsto (S^{-1}AR, R^sBR, S^sCS),$$

in which $R$ and $S$ are nonsingular and

$$M^s = M^T \quad \text{or} \quad M^s = \tilde{M}^T$$

with respect to a fixed involution $a \mapsto \tilde{a}$ in $F$. Thus, we consider the canonical form problem for matrix triples under transformations (3). We represent the matrix triple $(A, B, C)$ by the graph

$$
\begin{array}{ccc}
\bigcirc & \bigcirc & \bigcirc \\
m & B & n \\
A \downarrow & \downarrow & C \\
\end{array}
$$

$m := \dim U, n := \dim V.$

The direct sum of matrices is $A \oplus B := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and of matrix triples is

$$
\begin{array}{ccc}
\bigcirc & \bigcirc & \bigcirc \\
m_1 \oplus B_1 & m_2 \oplus B_2 & m_1 + m_2 \oplus B_1 \oplus B_2 \\
A_1 \downarrow & \downarrow A_2 \downarrow & \downarrow A_1 \oplus A_2 \\
\bigcirc & \bigcirc & \bigcirc \\
n_1 \oplus C_1 & n_2 \oplus C_2 & n_1 + n_2 \oplus C_1 \oplus C_2 \\
\end{array}
$$
The main result will be formulated in Section 2. In the following theorem, we formulate it in the most important case: for linear mappings between indefinite inner product spaces (an indefinite inner product space is a complex vector space with scalar product given by a nonsingular Hermitian form). Mehl, Mehrmann, and Xu [14] gave another classification of linear mappings between indefinite inner product spaces; their classification is presented in [16, Section 6.5]. This classification problem was also studied by Bolshakov and Reichstein [2, Section 6]. We refer the reader to Gohberg, Lancaster, and Rodman’s book [5] for a recent account of the indefinite linear algebra.

**Theorem 1.** For each triple (2) consisting of a linear mapping $A : U \rightarrow V$ and nonsingular Hermitian forms $B$ and $C$ on complex vector spaces $U$ and $V$, there exist bases of $U$ and $V$ in which the triple (4) of matrices of $A, B, C$ is a direct sum, determined uniquely up to permutation of summands, of triples of the form

$$
\begin{bmatrix}
0 & 1 \\
\lambda & 1 \\
\end{bmatrix} \quad \text{for } r \in \{1, 2, \ldots\}, \quad a \in \{1, -1\}, \quad \text{and } J_r(\mu) \text{ is an upper-triangular Jordan block.}
$$

Denote by $0_{pq}$ the $p \times q$ zero matrix with $p, q \in \{0, 1, 2, \ldots\}$. Note that

$$A_{mn} \oplus 0_{pq} = \begin{bmatrix} A_{mn} & 0_{mq} \\ 0_{pn} & 0_{pq} \end{bmatrix} = \begin{bmatrix} A_{mn} & 0_{pq} \\ 0_{pn} & 0_{pq} \end{bmatrix}, \quad A_{mn} \oplus 0_p = \begin{bmatrix} A_{mn} & 0_{m0} \\ 0_{pn} & 0_{pq} \end{bmatrix} = \begin{bmatrix} A_{mn} \\ 0_{pq} \end{bmatrix}
$$

for any $m \times n$ matrix $A_{mn}$. The triples (6) with $r = 1$ have the form

$$
\begin{bmatrix}
0 & 1 \\
\lambda & 1 \\
\end{bmatrix} \quad \begin{bmatrix}
0 & 1 \\
\lambda & 1 \\
\end{bmatrix} \quad \begin{bmatrix}
0 & 1 \\
\lambda & 1 \\
\end{bmatrix}
$$
We obtain our canonical form using the procedure developed by Roiter and Sergeichuk in [20, 21, 22, 23] and presented in [9, 27, 28]. If $F$ is a skew field (which can be commutative) of characteristic not 2 that is finite dimensional over its center, then this procedure reduces the problem of classifying any system of linear mappings and bilinear or sesquilinear forms over $F$ to the problems of classifying (i) some system of linear mappings over $F$ and (ii) Hermitian forms over finite extensions of the center of $F$. The solution of problem (ii) is given by the law of inertia if $F$ is $\mathbb{R}$, or $\mathbb{C}$, or the skew field of quaternions $\mathbb{H}$.

Over a field $F$ of characteristic not 2, Sergeichuk [22] obtained canonical forms, up to classification of Hermitian forms over finite extensions of $F$, for matrices of

(a) bilinear and sesquilinear forms,

(b) pairs of symmetric, or skew-symmetric, or Hermitian forms,

(c) self-adjoint or isometric operators in a space with scalar product given by a nonsingular form that is symmetric, or skew-symmetric, or Hermitian,

(d) systems represented by the graphs

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{graph1.png}}
\end{array}
\end{array}
\end{align*}
\]

in which each line between two vertices is an arrow $\rightarrow$ or $\leftarrow$. The vertices are vector spaces over $F$, the arrows are linear mappings, and the loops are symmetric, or skew-symmetric, or Hermitian forms.

These canonical matrices were also published in [23] for (a), (b), and (c), and in [24] for (d). The procedure and the graphs (7) are presented in the survey article [27, Theorem 3.2]; see also [6, 7, 8, 28].

\[1\] This is a deposited manuscript. There were very few mathematical journals in the USSR and publications abroad were forbidden. However, a manuscript could be deposited in an institute of information by recommendation of a scientific council, which was considered as a publication. The abstracts of all deposited manuscripts on mathematics were published in the RZhMat, which is the Russian analogue of MathRev. Everyone can order a copy of each deposited manuscript.
Using the canonical matrices of the system \( \mathbf{A} \xrightarrow{\varepsilon} \mathbf{B} \), Sergeichuk [25] classified pairs of subspaces in a space with scalar product given by a symmetric, or skew-symmetric, or Hermitian form. The canonical matrices of the system \( \mathbf{A} \xrightarrow{\varepsilon} \mathbf{B} \) are given in the next section.

The procedure developed in [20, 22] is based on Roiter’s quivers with involution [20], which were also studied by Derksen and Weyman [4] (they use the term “symmetric quivers”), Bocklandt [1], Shmelkin [29], and Zubkov [31].

2. Main result

Let \( \mathbb{F} \) be a skew field (which can be commutative) of characteristic not 2 with a fixed involution \( a \mapsto \tilde{a} \); that is, a bijection \( \mathbb{F} \to \mathbb{F} \) satisfying

\[
\tilde{a} + b = \tilde{\alpha} + \tilde{b}, \quad \tilde{ab} = \tilde{b}\tilde{a}, \quad \tilde{\tilde{a}} = a
\]

for all \( a, b \in \mathbb{F} \). This involution can be the identity if \( \mathbb{F} \) is a field. Elementary linear algebra over a skew field can be found in Bourbaki [3, Chapter II].

All vectors spaces that we consider are finite dimensional right vector spaces over \( \mathbb{F} \). Each linear mapping \( A : U \to V \) satisfies \( A(ua + vb) = (Aua + (Av)b \) and each sesquilinear form \( B : U \times U \to \mathbb{F} \) satisfies

\[
B(ua + vb, w) = \tilde{a}B(u, w) + \tilde{b}B(v, w), \quad B(u, va + wb) = B(u, v)a + B(u, w)b
\]

for all \( a, b \in \mathbb{F} \) and \( u, v, w \in U \). A form \( B : U \times U \to \mathbb{F} \) is Hermitian if \( B(u, v) = \tilde{B}(v, u) \) and skew-Hermitian if \( B(u, v) = -\tilde{B}(v, u) \) for all \( u, v \in U \).

We consider a triple

\[
\mathcal{T} : \quad U \xrightarrow{\varepsilon} \mathbf{B} \xleftarrow{\delta} V \quad \varepsilon, \delta \in \{+, -\}
\]

(8)

that consists of a linear mapping \( A : U \to V \) between vector spaces over \( \mathbb{F} \) and two sesquilinear forms \( B : U \times U \to \mathbb{F} \) and \( C : V \times V \to \mathbb{F} \) satisfying

\[
B(u, u') = \varepsilon\tilde{B}(u', u), \quad C(v, v') = \delta\tilde{C}(v', v)
\]

(9)

for all \( u, u', v, v' \). For simplicity, we suppose that the forms \( B \) and \( C \) are Hermitian (i.e., \( \varepsilon = \delta = +\)) if the involution \( a \mapsto \tilde{a} \) is not the identity. This
condition is not restrictive if $F$ is a field since then each skew-Hermitian form can be made Hermitian by multiplying by any $a - \tilde{a} \neq 0; a \in F$.

We say that two triples $\mathcal{T}$ and $\mathcal{T}'$ of the form (8) with the same $\varepsilon$ and $\delta$ are isomorphic and write $\mathcal{T} \simeq \mathcal{T}'$ if there exist linear bijections $\varphi : U \to U'$ and $\psi : V \to V'$ that transform $\mathcal{T}$ to $\mathcal{T}'$:

$$
\begin{array}{c}
U \xrightarrow{\varphi} U' \\
A \\
V \xrightarrow{\psi} V'
\end{array}
\begin{array}{c}
B \\
\varepsilon
\end{array} \xrightarrow{A'=\psi A \varphi^{-1}} \begin{array}{c}
B' \\
\varepsilon'
\end{array}
\quad \begin{array}{l}
\mathcal{B}(u_1, u_2) = \mathcal{B}'(\varphi u_1, \varphi u_2) \text{ for all } u_1, u_2 \in U, \\
\mathcal{C}(v_1, v_2) = \mathcal{C}'(\psi v_1, \psi v_2) \text{ for all } v_1, v_2 \in V.
\end{array}
$$

The direct sum of triples is defined as in (5):

$$
\begin{array}{c}
U_1 \oplus U_2 \\
\varepsilon \\
V_1 \oplus V_2
\end{array} \xrightarrow{A_1 \oplus A_2} \begin{array}{c}
B_1 \oplus B_2 \\
\varepsilon
\end{array} \quad \begin{array}{l}
\mathcal{A}_1 \oplus \mathcal{A}_2 \colon= \mathcal{A}_1 \oplus \mathcal{A}_2, \\
\mathcal{C}_1 \oplus \mathcal{C}_2.
\end{array}
$$

We say that a triple (8) is regular if $A$ is bijective. Each regular triple is isomorphic to a triple of the form

$$
\begin{array}{c}
U \xrightarrow{A} U' \\
A \\
V \xrightarrow{\psi} V'
\end{array}
\begin{array}{c}
B \\
\varepsilon
\end{array} \xrightarrow{A'=\psi A \varphi^{-1}} \begin{array}{c}
B' \\
\varepsilon'
\end{array}
\quad \begin{array}{l}
\mathcal{B}(u_1, u_2) = \mathcal{B}'(\varphi u_1, \varphi u_2) \text{ for all } u_1, u_2 \in U, \\
\mathcal{C}(v_1, v_2) = \mathcal{C}'(\psi v_1, \psi v_2) \text{ for all } v_1, v_2 \in V.
\end{array}
$$

which we call strictly regular. We say that a triple is strictly singular if it is not isomorphic to a direct sum with a regular direct summand.

For $r \in \{0, 1, 2, \ldots \}$, define the $r \times r$ matrices

$$
Z_r = Z_{r, \varepsilon} := \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}, \quad \text{and } Z_{r, -\varepsilon} := \begin{bmatrix} 0 & -Z_{r/2} \\ Z_{r/2} & 0 \end{bmatrix} \text{ if } r \text{ is even. (10)}
$$

For $r \in \{1, 2, \ldots \}$, define the $(r - 1) \times r$ matrices

$$
F_r := \begin{bmatrix} 1 & 0 & 0 \\ & \ddots & \ddots \\ 0 & 1 & 0 \end{bmatrix}, \quad G_r := \begin{bmatrix} 0 & 1 & 0 \\ & \ddots & \ddots \\ 0 & 0 & 1 \end{bmatrix}. \quad (11)
$$
In particular, $F_1 = G_1 = 0_{01}$.

Let

$$
\begin{pmatrix}
\varepsilon A^\circ \\
\varepsilon [0,0]
\end{pmatrix}
$$

in which $\varepsilon \in \{+, -\}$ and $A^\circ = \bar{A}^\top$ is the adjoint matrix.

We can now formulate our main result, which was given over a field in [22, p. 44].

**Theorem 2.** Let $\mathbb{F}$ be a skew field (which can be commutative) of characteristic not 2 with a fixed involution $a \mapsto \bar{a}$. Let us fix $\varepsilon, \delta \in \{+, -\}$ if the involution is the identity, and put $\varepsilon = \delta = +$ if the involution is not the identity.

(a) Let $(\mathcal{F})$ be a triple, consisting of a linear mapping $A : U \to V$ between right vector spaces over $\mathbb{F}$ and two sesquilinear forms $B : U \times U \to \mathbb{F}$ and $C : V \times V \to \mathbb{F}$ satisfying $(\mathcal{I})$. Then the triple $(\mathcal{F})$ is isomorphic to a direct sum of a strictly regular triple and a strictly singular triple; these summands are uniquely determined, up to isomorphism.

(b) Two strictly regular triples are isomorphic if and only if their forms are simultaneously equivalent:

$$
\begin{array}{c}
\varepsilon_B = \varepsilon_{B'} \\
\delta_C = \delta_{C'}
\end{array}
\iff
\begin{array}{c}
B(U, \varepsilon) \cong B'(U', \varepsilon') \\
C(U, \delta) \cong C'(U', \delta')
\end{array}
$$

(c) Each strictly singular triple $(\mathcal{F})$ possesses bases of $U$ and $V$, in which the triple $(\mathcal{I})$ of its matrices is a direct sum of triples of the types

$$
F_r^1(a) : F_r^1 \bigoplus_{r-1} a Z_{r-1,\varepsilon} \\
F_r^7(a) : F_r^7 \bigoplus_{r} a Z_{r,\delta}
$$

(13)

$$
F_r^{10} \bigoplus_{2r-2} I^\circ_r \\
F_r^{11} \bigoplus_{2r-1} I^\circ_r
$$

(14)

$$
F_r^{12} \bigoplus_{2r-2} G_r^\circ \\
F_r^{13} \bigoplus_{2r-1} G_r^\circ
$$

(15)

$$
F_r^{14} \bigoplus_{2r-1} (G_r^\circ)^\circ \\
F_r^{15} \bigoplus_{2r} (G_r^\circ)^\circ
$$

(16)
in which \( r \in \{1, 2, \ldots \} \) and \( 0 \neq a = \tilde{a} \in \mathbb{F} \).

The summands of types \( F^\sigma_r(a) \) with \( \sigma \in \{1, \tilde{}\} \) are determined up to replacement of the whole group of summands

\[
F^\sigma_r(a_1) \oplus \cdots \oplus F^\sigma_r(a_k)
\]

with the same \( r \) and \( \sigma \) by any direct sum

\[
F^\sigma_r(b_1) \oplus \cdots \oplus F^\sigma_r(b_k), \quad b_1 = \tilde{b}_1, \ldots, b_k = \tilde{b}_k \in \mathbb{F} \setminus \{0\}
\]

with the same \( r \) and \( \sigma \) such that the Hermitian forms

\[
\tilde{x}_1 a_1 x_1 + \cdots + \tilde{x}_k a_k x_k, \quad \tilde{x}_1 b_1 x_1 + \cdots + \tilde{x}_k b_k x_k
\]

are equivalent over \( \mathbb{F} \). The other summands are uniquely determined up to permutation.

(d) Let \( \mathbb{F} \) be \( \mathbb{R} \), or \( \mathbb{C} \), or the skew field of quaternions \( \mathbb{H} \). Then each strictly singular triple \( (8) \) possesses bases of \( U \) and \( V \), in which the triple \( (11) \) of its matrices is a direct sum, uniquely determined up to permutation of summands, of triples of types \( (13) \)–\( (16) \), in which

- \( a = 1 \) if \( \mathbb{F} \) is \( \mathbb{C} \) with the identity involution, or \( \mathbb{H} \) with involution that differs from the quaternion conjugation

\[
\alpha + \beta i + \gamma j + \delta k \mapsto \alpha - \beta i - \gamma j - \delta k \quad (\alpha, \beta, \gamma, \delta \in \mathbb{R}); \quad (17)
\]

- \( a \in \{-1, 1\} \) if \( \mathbb{F} \) is \( \mathbb{R} \), or \( \mathbb{C} \) with complex conjugation, or \( \mathbb{H} \) with the quaternion conjugation.

The triples \( (13) \)–\( (16) \) are the triples \( (A''_2)'1), (A''_2)2), (A''_2)a), and (A''_2)3) from [22, p. 44] and the triples that are “dual” to them (they are obtained by replacing the vector spaces by the dual vector spaces \( (18) \) and the linear mappings by the adjoint mappings \( (19) \); the triples with forms on the dual spaces are also classified in [22]). Each involution in \( \mathbb{H} \) is either \( (17) \), or \( \alpha + \beta i + \gamma j + \delta k \mapsto \alpha + \beta i + \gamma j - \delta k \) in a suitable set \( \{i, j, k\} \) of orthogonal imaginary units; see [19, Theorem 2.4.4(c)].

Thus, Theorem 2 reduces the problem of classifying triples \( (8) \) over \( \mathbb{F} \) up to isomorphism

(i) to the problem of classifying pairs of forms \( \bigotimes \mathbb{F} \) over \( \mathbb{F} \), and
(ii) (if $F$ is not $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$) to the problem of classifying Hermitian forms over $F$.

We do not consider the problem (i); its solution is given in [23, Theorem 4] over a field $F$ of characteristic not 2 up to classification of Hermitian forms over finite extensions of $F$, which gives its full solution if $F$ is $\mathbb{R}$ or $\mathbb{C}$ due to the law of inertia. The pairs $\pm \pm$ over $\mathbb{R}$ and $\mathbb{C}$ are also classified in [7, 12, 13, 30] and other papers. The pairs $\pm \pm$ over $\mathbb{H}$ are classified in [11, 17, 18, 19].

3. The reducing procedure

The procedure that reduces the problem of classifying systems of linear mappings and forms to the problem of classifying systems of linear mappings is described in [23, Section 1] and [28, Section 3.5]. In this section, we present it for the problem of classifying triples (8).

For each right vector space $V$ over $F$,

\[ V^\ast \text{ is the right dual space of semilinear functionals,} \]
\[ \text{that is, mappings } \varphi : V \to F \text{ satisfying} \]
\[ \varphi(ua + vb) = \bar{a}\varphi(u) + \bar{b}\varphi(v) \text{ for all } a, b \in F \text{ and } u, v \in V. \]

For each linear mapping $A : U \to V$,

\[ A^\ast : V^\ast \to U^\ast \text{ is the adjoint mapping defined by} \]
\[ (A^\ast \varphi)(u) := \varphi(Au) \text{ for all } u \in U \text{ and } \varphi \in V^\ast. \] (19)

Each sesquilinear form $B : U \times V \to F$ defines

- the linear mapping (we denote it by the same letter)
  \[ B : V \to U^\ast, \quad v \mapsto B(\cdot, v), \] (20)

- the adjoint linear mapping
  \[ B^\ast : U \to V^\ast, \quad u \mapsto \overline{B(u, \cdot)}. \]

If the form $B$ is Hermitian, then the mapping (20) is self-adjoint (i.e., $B = B^\ast$).
Thus, the triple \( \mathcal{T} \) in (8) defines in a one-to-one manner the *quadruple of linear mappings* \( \mathcal{T} : \)

\[
\begin{array}{c}
U \\
\downarrow_{\beta \circ \beta^*} \\
\downarrow_{\varepsilon B^*} \\
\downarrow_{\varepsilon} \\
A \\
\downarrow_{C = \delta C^*} \\
\downarrow_{\gamma} \\
\downarrow_{\gamma^*} \\
V \\
\end{array}
\]

(21)

(in terms of [20, 23, 9], the quadruple (21) is a self-dual representation of the quiver \( \mathcal{G} : \)

\[
\begin{array}{c}
u \\
\downarrow_{\alpha} \\
\downarrow_{\alpha^*} \\
u^* \\
a \\
\end{array}
\]

\[
\begin{array}{c}
\beta \circ \beta^* \\
\downarrow_{\varepsilon} \\
\gamma \circ \gamma^* \\
\end{array}
\]

(22)

with involutions in the set of vertices and in the set of arrows).

Thus, we can consider quadruples of the form (21) instead of the triples (8). We will classify the quadruples (21) using the classification of *arbitrary* quadruples of linear mappings \( \mathcal{P} : \)

\[
\begin{array}{c}
U_1 \\
\downarrow_{B} \\
\downarrow_{U_2} \\
\downarrow_{B^*} \\
A_1 \\
\downarrow_{C} \\
\downarrow_{A_2} \\
\downarrow_{C^*} \\
V_1 \\
\end{array}
\]

(23)

(that is, representations of the quiver (22), which we consider as a quiver without involutions).

The vector

\[
\dim \mathcal{P} := (\dim U_1, \dim U_2, \dim V_2, \dim V_1)
\]

(24)

is called the *dimension* of (23).

A *homomorphism* \( \phi = (\varphi_1, \varphi_2) : \mathcal{P} \to \mathcal{P}' \)

(25)

of quadruples \( \mathcal{P} \) and \( \mathcal{P}' \) is a sequence \( \varphi_1, \varphi_2, \psi_1, \psi_2 \) of linear mappings

\[
\begin{array}{c}
\begin{array}{c}
\varphi_1 \\
\downarrow_{\varphi_2} \\
\downarrow_{\varphi_2} \\
\downarrow_{\varphi_2} \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\varphi_1 \\
\downarrow_{\varphi_2} \\
\downarrow_{\varphi_2} \\
\downarrow_{\varphi_2} \\
\end{array}
\end{array}
\]

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such that
\[ \psi_1 A_1 = A'_1 \varphi_1, \quad \varphi_2 A_2 = A'_2 \psi_2, \quad \varphi_2 B = B' \varphi_1, \quad \psi_2 C = C' \psi_1. \]

A homomorphism (25) is called an isomorphism (we write \( \phi : \mathcal{P} \cong \mathcal{P}' \) or \( \mathcal{P} \simeq \mathcal{P}' \)) if \( \varphi_1, \varphi_2, \psi_1, \psi_2 \) are bijections.

Let \( \varepsilon \) and \( \delta \) be as in (8). For each quadruple \( \mathcal{P} \) of the form (23), we define the dual quadruple

\[ \mathcal{P}^\circ : \begin{array}{ccc}
A_2^\circ & \xrightarrow{\varepsilon^{B^\circ} \delta^{C^\circ}} & A_1^\circ \\
V_2^\circ & \downarrow & V_1^\circ
\end{array} \]  \quad (26)

The quadruple (21) is self-dual.

For each homomorphism (25), we define the dual homomorphism \( \phi^\circ := \left( \varphi_2^\circ, \psi_1^\circ \right) : \mathcal{P}^\circ \to \mathcal{P}^\circ \) between the dual quadruples:

Define the direct sum of quadruples:

\[ \mathcal{P} \oplus \mathcal{P}' : \begin{array}{ccc}
U_1 \oplus U_1' & \xrightarrow{B \oplus B'} & U_2 \oplus U_2' \\
V_1 \oplus V_1' & \downarrow & V_2 \oplus V_2'
\end{array} \]

A quadruple is indecomposable if it is not isomorphic to a direct sum of quadruples of smaller dimensions. Let \( \text{ind}(\mathcal{G}) \) be any set of indecomposable quadruples such that each indecomposable quadruple is isomorphic to exactly one quadruple from \( \text{ind}(\mathcal{G}) \) (we give \( \text{ind}(\mathcal{G}) \) in Lemma 2). The procedure of constructing indecomposable canonical triples (8) consists of three steps; see details in [23, §1] and [28, Section 3].
Step 1. We replace each quadruple in \(\text{ind}(G)\) that is isomorphic to a self-dual quadruple by a self-dual quadruple. Let \(\text{ind}_0(G) \subset \text{ind}(G)\) be the set of obtained self-dual quadruples \(\mathcal{P} = \mathcal{P}^\circ\). Denote by \(\text{ind}_1(G)\) a set consisting of

- all \(Q \in \text{ind}(G) \setminus \text{ind}_0(G)\) such that \(Q \simeq Q^\circ\), and
- one quadruple from each pair \(\{Q, R\} \subset \text{ind}(G)\) such that \(Q \neq R\) and \(Q^\circ \simeq R^\circ\).

We have obtained a new set \(\text{ind}(G)\) partitioned into 3 subsets:

\[
\text{ind}(G) = \begin{cases} 
\mathcal{P} = \mathcal{P}^\circ, & \mathcal{P} \in \text{ind}_0(G), \\
Q, & Q \in \text{ind}_1(G). 
\end{cases}
\] (27)

Step 2. Let \(\mathcal{P} \in \text{ind}_0(G)\). Since \(\mathcal{P}\) is an indecomposable quadruple, the algebra \(\text{End}(\mathcal{P})\) of its endomorphisms is local, its radical \(R\) consists of all non-invertible endomorphisms (see [23, Lemma 1]), and so \(\mathbb{T}(\mathcal{P}) := \text{End}(\mathcal{P})/R\) is a skew field. The mapping

\[
\phi + R \mapsto (\phi + R)^\circ := \phi^\circ + R, \quad \phi \in \text{End}(\mathcal{P}),
\]

is an involution in \(\mathbb{T}(\mathcal{P})\). For each self-dual automorphism \(\phi = \phi^\circ := \begin{pmatrix} \varphi & \varphi^* \\ \psi & \psi^* \end{pmatrix}: \mathcal{P} \simeq \mathcal{P}\), we denote by \(\mathcal{P}^\phi\) the self-dual quadruple such that

\[
\mathcal{P} \begin{pmatrix} \varphi^1 \\ \psi^1 \end{pmatrix} \rightarrow \mathcal{P}^\phi \begin{pmatrix} 1 & \varphi^* \\ 1 & \psi^* \end{pmatrix} \rightarrow \mathcal{P}.
\] (28)

For each \(0 \neq a = a^\circ \in \mathbb{T}(\mathcal{P})\), we fix a self-dual automorphism \(\phi_a = \phi_a^\circ \in a\) (we can take \(\phi_a := (\phi + \phi^\circ)/2\) for any \(\phi \in a\) and denote by \(\mathcal{P}^a\) the triple \([8]\) that corresponds to the self-dual quadruple \(\mathcal{P}^{\phi_a}\). For each Hermitian form

\[
f(x) = x_1^a a_1 x_1 + \cdots + x_k^a a_k x_k, \quad a_1 = a_1^\circ, \ldots, a_k = a_k^\circ \in \mathbb{T}(\mathcal{P}) \setminus \{0\},
\]

over the skew field \(\mathbb{T}(\mathcal{P})\), we put

\[
\mathcal{P} f(x) := \mathcal{P}^{a_1} \oplus \cdots \oplus \mathcal{P}^{a_k}.
\]
Step 3. For each quadruple $Q \in \text{ind}_1(G)$, we take the direct sum

$$Q \oplus Q^\circ : \begin{bmatrix} A_1 & 0 \\ 0 & A_2^* \end{bmatrix} \oplus \begin{bmatrix} A_2 & 0 \\ 0 & A_1^* \end{bmatrix}$$

and make it self-dual by interchanging the summands in $U_2 \oplus U_1^*$ and in $V_2 \oplus V_1^*$.

The corresponding triple is

$$Q^+ : \begin{bmatrix} A_1 & 0 \\ 0 & A_2^* \end{bmatrix} \oplus \begin{bmatrix} 0 & \delta C^* \\ C & 0 \end{bmatrix}$$

(29)

The following lemma is a special case of [23, Theorem 1] (see also [28, Theorem 3.1]) about arbitrary systems of linear mappings and forms.

Lemma 1. Over a skew field $F$ of characteristic not 2, each triple (30) is isomorphic to a direct sum

$$\mathcal{P}_1^{f_1(x)} \oplus \cdots \oplus \mathcal{P}_m^{f_m(x)} \oplus Q_1^+ \oplus \cdots \oplus Q_n^+,$$

in which $\mathcal{P}_1, \ldots, \mathcal{P}_m \in \text{ind}_0(G)$, $\mathcal{P}_i \neq \mathcal{P}_i'$ if $i \neq i'$, each $f_i(x)$ is a Hermitian form over the skew field $\mathbb{T}(\mathcal{P}_i)$, and $Q_1, \ldots, Q_n \in \text{ind}_1(G)$. This sum is uniquely determined, up to permutation of summands and replacement of each $\mathcal{P}_i^{f_i(x)}$ by $\mathcal{P}_i^{g_i(x)}$, where $f_i(x)$ and $g_i(x)$ are equivalent Hermitian forms over $\mathbb{T}(\mathcal{P}_i)$.

If a skew field $F$ is finite-dimensional over its center $Z$ and $\mathcal{P}$ is a self-dual indecomposable quadruple, then $\mathbb{T}(\mathcal{P})$ is finite-dimensional over $Z$ under the natural imbedding of $Z$ into the center of $\mathbb{T}(\mathcal{P})$ and the involution in $\mathbb{T}(\mathcal{P})$ extends the involution $a \mapsto \bar{a}$ in $Z$. 

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If we have chosen a basis \( e_1, \ldots, e_n \) in a vector space \( V \), then we always choose in the dual space \( V^* \) the dual basis consisting of the semilinear functionals \( e_1^*, \ldots, e_n^* : V \to \mathbb{F} \) such that \( e_i^*(e_j) = 0 \) if \( i \neq j \) and \( e_i^*(e_i) = 1 \) for all \( i \).

If \( A \) is the matrix of a linear mapping \( A : U \to V \) in some bases of \( U \) and \( V \), then \( A^* := \overline{A^T} \) is the matrix of the adjoint mapping \( A^* : V^* \to U^* \) (see (19)) in the dual bases. If \( B : U \times V \to \mathbb{F} \) is a sesquilinear form, then its matrix in any bases of \( U \) and \( V \) coincides with the matrix of \( B : V \to U^* ; v \mapsto B(?, v) \), in the same basis of \( V \) and the dual basis of \( U^* \).

Thus,

\[
Q^+ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2^* \end{bmatrix} \begin{bmatrix} 0 & eB^* \\ B & 0 \end{bmatrix} \begin{bmatrix} m_1 + m_2 & 0 \\ n_1 + n_2 & C^* \end{bmatrix} \begin{bmatrix} 0 \\ \delta \end{bmatrix}
\]

(31)

(see (12)), in which \( m_1, m_2, n_1, n_2 \) are the dimensions of \( U_1, U_2, V_1, V_2 \).

4. Classification of quadruples of linear mappings

Choosing bases in the spaces of a quadruple (23), we can give it by the quadruple of its matrices

\[
P : \begin{array}{c}
\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \\
B \\
\begin{bmatrix} C \\ C \end{bmatrix}
\end{array}
\]

(33)

This quadruple is isomorphic to the quadruple (23). For abbreviation, we usually omit "\( P \)" in (33) (as in (4)). Two matrix quadruples are isomorphic (which means that

\[
(\Phi_1, \Phi_2) : \begin{array}{c}
\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \\
B \\
\begin{bmatrix} C \\ C \end{bmatrix}
\end{array} \leftrightarrow \begin{array}{c}
\begin{bmatrix} \Phi_1 A_1 \Phi_1^{-1} \\ \Phi_2 A_2 \Phi_2^{-1} \end{bmatrix} \\
\Phi_1 B \Phi_1^{-1} \\
\Phi_2 C \Phi_2^{-1}
\end{array}
\]

for some nonsingular matrices \( \Phi_1, \Phi_2, \Psi_1, \Psi_2 \) if and only if they give the same quadruple (23) in different bases.
By (26) and (31), the dual quadruple to (33) is the quadruple

\[ P^\circ : \begin{array}{c}
m_2 \\
\downarrow \\
A^\circ \\
\downarrow \\
C^\circ \\
\downarrow \\
A^\circ \end{array} \begin{array}{c}
m_1 \\
\downarrow \\
B^\circ \\
\downarrow \\
A^\circ \\
\downarrow \\
A^\circ \end{array} \begin{array}{c}
n_2 \\
\downarrow \\
n_1 \\
\downarrow \\
n_1 \\
\downarrow \\
n_1 \end{array} \]

in which \( A^\circ := \tilde{A}^T \) is the adjoint matrix.

A quadruple (23) is a special case of a cycle of linear mappings

\[ V_1 \xrightarrow{A_1} V_2 \xrightarrow{A_2} \ldots \xrightarrow{A_{t-2}} V_{t-1} \xrightarrow{A_{t-1}} V_t \]  

in which each line is \( \rightarrow \) or \( \leftarrow \), \( V_1, \ldots, V_t \) are vector spaces, and \( A_1, \ldots, A_t \) are linear mappings. A canonical form of matrices of a cycle over a field is well known; see, for example, [26, Theorem 3.2]. The regularizing algorithm from [26] constructs for each cycle (34) its decomposition into a direct sum, in which the first summand is of the form

\[ \begin{array}{c}
V \\
\downarrow \\
A \\
\downarrow \\
V \end{array} \]

with a bijective \( A \) and each other summand is an indecomposable canonical cycle that contains a nonbijective linear mapping. The proof of this algorithm is given over a field but it holds over the skew field \( F \) too. Thus, a canonical form of the cycle (35) over \( F \) is obtained from the canonical form of a linear operator over a skew field, which is given in [10, Chapter 3, Section 12]. This ensures the following lemma.

**Lemma 2.** Let \( F \) be a skew field, which can be commutative. For each quadruple (23) over \( F \), there exist bases of \( U_1, U_2, V_1, V_2 \), in which the quadruple (33) of its matrices is a direct sum, uniquely determined up to permutations of summands, of indecomposable quadruples of the following types:

\[ \begin{array}{c}
F_r \\
\downarrow \\
I_r \\
\downarrow \\
F_r \end{array} \begin{array}{c}
G_r \\
\downarrow \\
I_r \\
\downarrow \\
G_r \end{array} \begin{array}{c}
F_r \\
\downarrow \\
I_r \\
\downarrow \\
F_r \end{array} \]

\[ \begin{array}{c}
G_r \\
\downarrow \\
I_r \\
\downarrow \\
G_r \end{array} \begin{array}{c}
F_r \\
\downarrow \\
I_r \\
\downarrow \\
F_r \end{array} \]

\[ \begin{array}{c}
F_r \\
\downarrow \\
I_r \\
\downarrow \\
F_r \end{array} \begin{array}{c}
G_r \\
\downarrow \\
I_r \\
\downarrow \\
G_r \end{array} \begin{array}{c}
F_r \\
\downarrow \\
I_r \\
\downarrow \\
F_r \end{array} \]

\[ \begin{array}{c}
G_r \\
\downarrow \\
I_r \\
\downarrow \\
G_r \end{array} \begin{array}{c}
F_r \\
\downarrow \\
I_r \\
\downarrow \\
F_r \end{array} \]

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and the quadruples dual to (37), in which \( r \in \{1, 2, \ldots \} \), \( F_r \) and \( G_r \) are defined in (11), and \( \Phi_r \) is an \( r \times r \) indecomposable canonical matrix under similarity over \( \mathbb{F} \).

Note that the dimensions (24) of indecomposable quadruples from Lemma 2 are

\[
(r, r, r, r), \quad (r, r, r, r - 1), \quad (r, r, r - 1, r - 1), \quad (r, r - 1, r - 1, r - 1)
\]

and their cyclic permutations. In each of these dimensions except for \( (r, r, r, r) \), there is exactly one, up to isomorphism, indecomposable quadruple.

5. Proof of Theorem 2

(a) Let \( T \) be a triple (8). By Lemma 1, \( T \) is isomorphic to some direct sum (30). Write this sum in the form \( T_1 \oplus T_2 \), in which \( T_1 \) is the direct sum of all regular summands and \( T_2 \) is the direct sum of the remaining summands. Let \( T_3 \) be a strictly regular triple that is isomorphic to \( T_1 \). Then the isomorphism \( T \simeq T_3 \oplus T_2 \) satisfies the statement (a) of Theorem 2.

(b) This statement of Theorem 2 is obvious.

(c) Let \( T \) be a strictly singular triple (8). Using \( \text{ind}(G) \) given in Lemma 2 and Steps 1–3 from Section 3, we can construct a direct sum (30), which is isomorphic to \( T \) by Lemma 1. Since \( T \) is strictly singular, all summands of this direct sum are strictly singular. Thus, they cannot be obtained from the first quadruple in (38) with nonsingular \( \Phi \) and from the last two quadruples in (38). Denote by \( \text{ind}'(G) \) the set of all remaining quadruples (36)–(38) and the quadruples dual to (37).

Let us apply Steps 1–3 from Section 3 to \( \text{ind}'(G) \).

**Step 1:** In this step, we construct a partition of \( \text{ind}'(G) \) into 3 subsets as in (27):

\[
\text{ind}'(G) = \begin{cases} \mathcal{P} \cap \mathcal{P}^\circ, & \mathcal{P} \in \text{ind}'(G), \\ \mathcal{Q} \cap \mathcal{Q}^\circ \text{ if } \mathcal{Q}^\circ \neq \mathcal{Q}, & \mathcal{Q} \in \text{ind}'(G). \end{cases}
\]
Let us prove that

- \( \text{ind}'_0(G) \) can be taken consisting of the quadruples

\[
\begin{align*}
\begin{array}{ccc}
F_r & \downarrow & Z_{r-1, \delta} \\
F_r & \downarrow & Z_{r-1, \delta} \\
r-1 & \rightarrow & r-1
\end{array}
\end{align*}
\begin{align*}
(r \text{ is even if } \varepsilon = - , \\
\quad \quad \quad \quad r \text{ is odd if } \delta = -)
\end{align*}
\begin{align*}
\begin{array}{ccc}
F_r & \downarrow & Z_{r-1, \delta} \\
F_r & \downarrow & Z_{r-1, \delta} \\
r-1 & \rightarrow & r-1
\end{array}
\end{align*}
\begin{align*}
(r \text{ is odd if } \varepsilon = - , \\
\quad \quad \quad \quad r \text{ is even if } \delta = -)
\tag{40}
\end{align*}
\]

in which \( Z_{r,+} \) and \( Z_{r,-} \) are defined in (10), and

- \( \text{ind}'_1(G) \) can be taken consisting of

\[(1^\circ)\] the first quadruple in (36), in which \( r \) is odd and \( \varepsilon = - \), or \( r \) is even and \( \delta = - \),

\[(2^\circ)\] the second quadruple in (36), in which \( r \) is even and \( \varepsilon = - \), or \( r \) is odd and \( \delta = - \),

\[(3^\circ)\] the quadruples (37),

\[(4^\circ)\] the first quadruple in (38) with \( \Phi_r = J_r(0)^\top \).

The horizontal arrows in the quadruples (36) are assigned by nonsingular matrices. Since every nonsingular skew-symmetric matrix is of even size, the quadruples \((1^\circ)\) and \((2^\circ)\) are not isomorphic to self-dual quadruples. The remaining quadruples (36) are isomorphic to the self-dual quadruples (40). For example, if \( r \) is odd and \( \varepsilon = + \), then the first quadruple in (36) is isomorphic to the first self-dual quadruple in (40) since \( Z_rG_r^\top Z_{r-1} = F_r^\top \).

The quadruples \((3^\circ)\) are not isomorphic to self-dual quadruples since if some quadruple (33) is isomorphic to a self-dual quadruple, then \( m_1 = m_2 \) and \( n_1 = n_2 \).

The quadruple \((4^\circ)\) is dual to the second quadruple in (38), and so it is not isomorphic to a self-dual quadruple.

**Steps 2:** Let us construct a set of triples \( \{ P^a | 0 \neq a = a^\circ \in T(P) \} \) for each \( P \in \text{ind}'_0(G) \).
Let \( P \) be the first quadruple in (40) with odd \( r \) and \( \varepsilon = + \). Let \( \left( \frac{R_1}{R_2} \frac{S_1}{S_2} \right) : P \to P \) be an endomorphism of \( P \). Then

\[
F_rR_1 = R_2F_r, \quad S_1F_r^\top = F_r^\top S_2, \quad S_1Z_r = Z_rR_1, \quad S_2Z_r^{-1,\delta} = Z_r^{-1,\delta}R_2.
\]

A straightforward computation shows that

\[
\left( R_1, R_2, S_1, S_2 \right) = \left( \left[ \begin{array}{ccc} a & 0 & 0 \\ s & \ddots & \ddots \\ \vdots & \ddots & \ddots \\ 0 & \ddots & a \\ \end{array} \right] \right), \quad a \in \mathbb{F}.
\]

Hence, we can identify \( T(P) \) and \( \mathbb{F} \). The mapping \( a \mapsto \phi_a := \left( \frac{aI_r}{aI_r \delta} \right) \) is an embedding of \( \mathbb{F} \) into \( \text{End}(P) \). If \( 0 \neq a = \tilde{a} \in \mathbb{F} \), then

\[
P_{\phi_a} : \quad \begin{array}{ccc}
\xrightarrow{F_r} & aZ_r & \xrightarrow{F_r^\top} \\
\xrightarrow{r} & r & \xrightarrow{r} \\
\xleftarrow{r^{-1}} & aZ_{r^{-1,\delta}} & \xleftarrow{r^{-1}} \\
\end{array}
\]

(see (28)) is a self-dual quadruple and the corresponding triple \( P^a \) is the first triple in (13) with odd \( r \) and \( \varepsilon = + \).

In the same manner, we can identify \( T(P) \) and \( \mathbb{F} \) for each quadruple \( P \) from (40). The mapping \( a \mapsto \phi_a := \left( \frac{aI_r}{aI_r \delta} \right) \) for the first quadruple in (40) and \( \phi_a := \left( \frac{aI_{r-1}}{aI_{r-1} \delta} \right) \) for the second quadruple in (40), is an embedding of \( \mathbb{F} \) into \( \text{End}(P) \). If \( 0 \neq a = \tilde{a} \in \mathbb{F} \), then the quadruple \( P_{\phi_a} \) is obtained from \( P \) by multiplying by \( a \) the matrices that correspond to the horizontal arrows. The triple \( P^a \) is the first triple in (13) or the first triple in (14).

**Step 3:** Let us construct the triple \( Q^+ \) for each \( Q \in \text{ind} \left( \rho(G) \right) \).

By (29) and the correspondence (32), if \( Q \) is \((1^o)\) or \((2^o)\), then \( Q^+ \) is the second triple in (13) or (14), respectively. The quadruples \((3^o)\) give the triples (15) and the first two triples in (16). The quadruple \((4^o)\) gives the last triple in (16).

This proves the statement (c) due to Lemma 1.

(d) This statement follows from (c) since by the law of inertia

- each symmetric form over \( \mathbb{C} \) and each Hermitian form over \( \mathbb{H} \) with involution that differs from (17) are reduced to exactly one form \( \bar{x}_1x_1 + \cdots + \bar{x}_kx_k \), and
• each symmetric form over $\mathbb{R}$, each Hermitian form over $\mathbb{C}$, and each Hermitian form over $\mathbb{H}$ with involution (17) are reduced to exactly one form $\tilde{x}_1 x_1 + \cdots + \tilde{x}_l x_l - \tilde{x}_{l+1} x_{l+1} - \cdots - \tilde{x}_k x_k$,

in which the involution $a \mapsto \tilde{a}$ is the identity if the form is symmetric; see [23, p. 484] for Hermitian forms over $\mathbb{H}$. This proves Theorem 2.

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