LOCAL SMOOTH SOLUTIONS OF THE NONLINEAR KLEIN-GORDON EQUATION

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ABSTRACT. Given any $\mu_1, \mu_2 \in \mathbb{C}$ and $\alpha > 0$, we prove the local existence of arbitrarily smooth solutions of the nonlinear Klein-Gordon equation $\partial_{tt} u - \Delta u + \mu_1 u = \mu_2 |u|^{\alpha} u$ on $\mathbb{R}^N, N \geq 1$, that do not vanish, i.e. $|u(t, x)| > 0$ for all $x \in \mathbb{R}^N$ and all sufficiently small $t$. We write the equation in the form of a first-order system associated with a pseudo-differential operator, then use a method adapted from [Commun. Contemp. Math. 19 (2017), no. 2, 1650038]. We also apply a similar (but simpler than in the case of the Klein-Gordon equation) argument to prove an analogous result for a class of nonlinear Dirac equations.

1. Introduction. We study the local existence of smooth solutions for the nonlinear Klein-Gordon equation on $\mathbb{R}^N, N \geq 1$,

$$\begin{cases}
\partial_{tt} w - \Delta w + \mu_1 w = \mu_2 |w|^{\alpha} w, \\
w(0, x) = w_0(x), \ w_t(0, x) = w_1(x),
\end{cases}$$

(1.1)

where $\alpha > 0$ and $\mu_1, \mu_2 \in \mathbb{C}$. Note that if $\mu_1 = 0$, then (1.1) is in fact the nonlinear wave equation.

Local well-posedness of the Cauchy problem (1.1) in the energy space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ is established in [15, 17] in the subcritical case $(N - 2)\alpha < 4$. Local existence in $H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$ with $0 \leq s < \frac{N}{2}$ is established in [46], provided $(N - 2s)\alpha \leq 4$, and also that the nonlinearity is sufficiently smooth. Typically, it is required that $\alpha > [s - 1]$. There are many more references on this topic but, as far as we are aware, they do not cover the case of all powers $\alpha > 0$ in all spatial dimensions $N \geq 1$. Under appropriate assumptions on $\mu_1, \mu_2$ and $\alpha$, it is known that the solutions of (1.1) are global and scatter as $|t| \to \infty$ (i.e., they behave like solutions to the linear equation) for small initial values (low energy scattering),

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or for all initial values (asymptotic completeness). For asymptotic completeness, see for instance [35, 4, 5, 16, 18, 38]. There is a considerable literature on the subject, of which we mention only a small fraction. For low energy scattering, see for instance [53, 54, 44, 45, 24, 50, 43, 36, 37, 23, 12]. In particular, one expects low energy scattering when \( \alpha > \frac{2}{N} \). In the case \( \alpha = \frac{2}{N} \) one expects low energy modified scattering, i.e. that small solutions of (1.1) behave as linear solutions modulated by a phase. This is known in space dimensions \( N = 1, 2 \). See [21] for the case \( \alpha > \frac{2}{N} \), and [11, 28, 19, 34] for the case \( \alpha = \frac{2}{N} \). (See also [40, 42] for the corresponding initial-boundary value problems.) In higher space dimensions \( N \geq 3 \), up to our knowledge, the best available result is the existence of scattering for power nonlinearities \( \alpha \geq \alpha_0 \) for some \( \alpha_0 > 2/N \). Therefore, there is a gap between the expectation and what is actually known.

In this paper, we construct for every \( N \geq 1 \) and \( \alpha > 0 \) a class of initial values for which there exists a local, highly-regular, non-vanishing solution of (1.1). The difficulty is that, since \( \alpha > 0 \) can be small with respect to the required regularity, the nonlinearity is not smooth enough. See [8, 9] for a discussion on this regularity issue.

In [9, 10] we proved similar results for the Schrödinger equation with nonlinearity \( |u|^a u \). These results were useful, via the pseudo-conformal transformation, to study the scattering problem for NLS with \( \alpha \geq 2/N \) close to the critical power \( \alpha = 2/N \). The highly-regular solutions were also used to prove local existence for the generalized derivative Schrödinger equation [26, 27] and to the generalized Korteweg-de Vries equation [25]. We expect that the results in the present paper will be useful to derive similar results for the nonlinear Klein-Gordon equation (1.1).

Before stating our results, we introduce some notation taken from [9]. We fix \( \alpha > 0 \), we consider three integers \( k, m, n \) such that

\[
k > \frac{N}{2}, \quad n > \max \left\{ \frac{N}{2} + 1, \frac{N}{2\alpha} \right\}, \quad 2m \geq k + n + 3
\]

and we let

\[
J = 2m + 2 + k.
\]

Let \( d \geq 1 \). We define the space \( \mathcal{X}_d \) by

\[
\mathcal{X}_d = \left\{ u \in H^J (\mathbb{R}^N, \mathbb{C}^d); \langle x \rangle^n D^\beta u \in L^\infty (\mathbb{R}^N, \mathbb{C}^d) \text{ for } 0 \leq |\beta| \leq 2m - 2; \right.
\]

\[
\left. \langle x \rangle^n D^\beta u \in L^2 (\mathbb{R}^N, \mathbb{C}^d) \text{ for } 2m - 1 \leq |\beta| \leq J \right\}.
\]

and we equip \( \mathcal{X}_d \) with the norm

\[
\| u \|_{\mathcal{X}_d} = \sum_{|\beta| = 0}^{2m-2} \| \langle x \rangle^n D^\beta u \|_{L^\infty (\mathbb{R}^N, \mathbb{C}^d)} + \sum_{|\beta| = 2m-1}^{J} \| \langle x \rangle^n D^\beta u \|_{L^2 (\mathbb{R}^N, \mathbb{C}^d)}
\]

where

\[
\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}.
\]

By standard considerations it follows that \( (\mathcal{X}_d, \| \cdot \|_{\mathcal{X}_d}) \) is a Banach space. Moreover, \( 2n > N + 2 \) by (1.2), so that \( \| \langle x \rangle u \|_{L^2} \leq C \| \langle x \rangle^n u \|_{L^\infty} \); and so

\[
\mathcal{X}_d \hookrightarrow H^J (\mathbb{R}^N, \mathbb{C}^d).
\]

It is straightforward to check that \( \mathcal{S}(\mathbb{R}^N, \mathbb{C}^d) \subset \mathcal{X}_d \), and that \( z\langle x \rangle^{-p} \in \mathcal{X}_d \) for all \( p \geq n \) and \( z \in \mathbb{C}^d \).

Our main result for equation (1.1) is the following.
Theorem 1.1. Let $\alpha > 0$ and $\mu_1, \mu_2 \in \mathbb{C}$. Assume (1.2)-(1.3) and let $X_1$ be defined by (1.4)-(1.5) with $d = 1$. Let $w_0, w_1 \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ satisfy $w_0 \in X_1$, $\langle i\nabla \rangle^{-1} w_1 \in X_1$, where $\langle i\nabla \rangle = (I - \Delta)^{1/2} = F^{-1} \langle \xi \rangle F$. If
\[
\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |w_0(x)| + |\langle i\nabla \rangle^{-1} w_1(x)| > 0, \tag{1.7}
\]
then there exist $T > 0$ and a unique solution $w \in C([-T, T], X_1)$ of (1.1). In addition, if
\[
\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |w_0(x)| > 0, \tag{1.8}
\]
then there exists $0 < T_1 < T$ such that the solution $w(t)$ does not vanish for all $|t| \leq T_1$, more precisely there exists $\eta > 0$ such that
\[
\inf_{-T_1 \leq t \leq T_1} \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |w(t, x)| \geq \eta. \tag{1.9}
\]

Remark 1.2. Here are some comments on Theorem 1.1.

(i) The parameters $k, m, n$ are arbitrary as long as they satisfy (1.2). In particular, $n$ can be any integer satisfying the second condition in (1.2).

(ii) There are no restrictions on the size of the initial value in Theorem 1.1. Besides the smoothness and decay imposed by the assumption $w_0, \langle i\nabla \rangle^{-1} w_1 \in X_1$, the only limitation is condition (1.7).

(iii) Theorem 1.1 applies to the initial data
\[
w_0(x) = z_0 \langle x \rangle^{-n} + \psi_0(x), \quad w_1(x) = z_1 \langle i\nabla \rangle \langle x \rangle^{-n} + \psi_1(x),
\]
where $z_j \in \mathbb{C}$, $j = 1, 2$, $n > \max\{\frac{N}{2} + 1, \frac{N}{2d}\}$, and $\psi_j \in \mathcal{S}(\mathbb{R}^N)$ satisfies $\|\langle \cdot \rangle^n \psi_j\|_{L^{\infty}} < 1$, for $j = 1, 2$.

(iv) We cannot guarantee that the solution $w(t)$ does not vanish if condition (1.8) is not fulfilled. We do not know if condition (1.8) is necessary.

Comments on the proof of Theorem 1.1. We do not work directly with equation (1.1). Instead, we reduce it to a first order in time system and study the resulting “half-wave equation”. Namely, given $w_0, w_1$ let
\[
u_0 = 2^{-1} \left( w_0 a + i \langle |i\nabla|^{-1} w_1 \rangle b \right),
\]
where
\[
a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{1.10}
\]
We consider the first order system
\[
\begin{cases}
i \frac{\partial u}{\partial t} - \langle i\nabla \rangle u = \mathcal{M}(u) + \mathcal{N}(u), \\
u(0) = u_0,
\end{cases} \tag{1.11}
\]
where $\gamma$ is the Pauli matrix
\[
\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.12}
\]
and
\[
\mathcal{M}(u) = \mu_1 - \frac{1}{2} \langle i\nabla \rangle^{-1} \langle a \cdot u \rangle \rangle b, \quad \mathcal{N}(u) = -\frac{\mu_2}{2} \langle i\nabla \rangle^{-1} \langle |a \cdot u|^2 a \cdot u \rangle \rangle b. \tag{1.13}
\]
In particular, $\langle i\nabla \rangle$ with domain $H^1(\mathbb{R}^N, \mathbb{C}^2)$ is self-adjoint on $L^2(\mathbb{R}^N, \mathbb{C}^2)$, and $(e^{it\langle i\nabla \rangle})_{t \in \mathbb{R}}$ is a group of isometries on $L^2(\mathbb{R}^N, \mathbb{C}^2)$. Moreover, since $\langle i\nabla \rangle$ commutes with any power of $(I - \Delta)$, $(e^{it\gamma \langle i\nabla \rangle})_{t \in \mathbb{R}}$ is also a group of isometries on
Moreover, \( w \) is given in terms of \( u \) by
\[
w = a \cdot u.
\]
Therefore, we concentrate our attention on the problem (1.11). Similarly to the non-relativistic case of the Schrödinger equation [9], we observe that the possible defect of smoothness of the nonlinearity \( \mathcal{N}(u) \) is only at \( u = 0 \). This observation suggests to look for a solution to (1.11) that does not vanish. We follow the approach of [9] to construct such solutions. In order to explain our strategy, we consider as initial data the case of the concrete function \( \psi(x) = \langle x \rangle^{-n}a \), where \( n > \frac{N}{2} + 1 \). For this choice of \( n \) it follows that \( \psi \in H^1(\mathbb{R}^N, \mathbb{C}^2) \). Let \( v(t) = e^{it\gamma \langle i\nabla \rangle} \psi \) be the solution of the linear problem
\[
\begin{aligned}
iv_t &= \gamma \langle i\nabla \rangle v, \\
v(0, x) &= \psi(x).
\end{aligned}
\] (1.15)
We want to control \( \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v(t, x)| \). For this purpose we integrate (1.15) on \([0, t]\), and we obtain
\[
v(t, x) = \psi(x) - it \int_0^t \langle i\nabla \rangle v(s, x) \, ds.
\] (1.16)
Hence,
\[
\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v(t, x)| \geq \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |\psi(x)| - t \| \langle x \rangle^n \langle i\nabla \rangle v \|_{L^\infty((0, t) \times \mathbb{R}^N)}.
\] (1.17)
It follows that in order to control \( \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v(t, x)| \) from below, we need to estimate the last term on the right-hand side of (1.17). In [9] we use Taylor’s formula with integral remainder involving derivatives of the solution \( v \) of sufficiently large order, which we estimate in the Sobolev space \( H^s \) where \( s > \frac{N}{2} \) and \( k \) is sufficiently large. In the case of equation (1.16) we have
\[
\| \langle x \rangle^n \langle i\nabla \rangle v \|_{L^\infty} = \| \langle x \rangle^n \langle i\nabla \rangle^{-1} (1 - \Delta) v \|_{L^\infty}.
\] (1.18)
The Taylor’s formula applied to \( v(t) = e^{it\gamma \langle i\nabla \rangle} \psi \) yields terms of the form \( \langle i\nabla \rangle^j \psi \) and a remainder involving the operator \( (1 - \Delta)^\ell \), for some \( \ell \) large. To estimate the terms \( \langle i\nabla \rangle^j \psi \), we need to control the pseudo-differential operator \( \langle i\nabla \rangle^j \). This is done in Lemma 2.3 below by using the theory of Bessel potentials. In turn, expanding \( (1 - \Delta)^\ell \) to extract the part corresponding to \( v \) itself, for small times we can estimate the solution in terms of its higher-order derivatives. Hence, we can control \( v \) in terms of derivatives of \( \psi \) plus a high-order derivative of \( v \), \( \sup_{|\beta|=\ell} \| \langle x \rangle^n D^\beta v \|_{L^\infty} \), for some \( \ell \) large, which is estimated via the Sobolev’s embedding \( H^s \hookrightarrow L^\infty \) for \( s > \frac{N}{2} \). This first step is achieved in Lemma 2.4 below. In order to control
\[
\sup_{|\beta|=\ell} \| \langle x \rangle^n D^\beta v \|_H,
\]
we use energy estimates. Since the equation for \( v \) involves a first order pseudo-differential operator, we can estimate \( \langle x \rangle^n D^\beta v \) in terms of \( D^\beta v \), which then can be controlled by usual energy estimates. This second step is achieved in Lemma 2.6 below. As in the non-relativistic case of [9], the corresponding space \( X_0 \) involves weighted \( L^\infty \)-norms of the derivatives of the function up to a certain order, then weighted \( L^2 \)-norms of the derivatives of higher order. However, we stress that in the present case we are able to close the estimates using lower order derivatives, compared with the case of the Schrödinger equation. Combining Lemmas 2.4 and 2.6
we obtain the linear estimate that we use in this paper. This estimate is presented in Proposition 2.1 below. In particular, it follows from the linear estimate that if \( \langle x \rangle^n |\psi| \) is bounded from below, then the function \( \langle x \rangle^n |v(t, x)| \) remains positive for small times. The nonlinear estimate is provided by Proposition 3.1 below. We show that we can close the nonlinear estimates in the space \( X_2 \) via the control of the Bessel potential provided by Lemma 2.3. Using the linear and nonlinear estimates, we prove the existence of a solution \( u \in X_2 \) for (1.11) by a contraction mapping argument. This is the result of Proposition 4.1 below. Then, Theorem 1.1 follows from the transformation (1.14), which relates the problems (1.1) and (1.11).

It turns out that the method we use to study (1.1) similarly applies to the nonlinear Dirac equation

\[
\begin{aligned}
&i \Psi_t = H \Psi + \mu_3 |\Psi|^\alpha \Psi, \\
&\Psi(0, x) = \Psi_0(x),
\end{aligned}
\]  

(1.19)
on \( \mathbb{R}^N \). Here, \( \mu_3 \in \mathbb{C}, \Psi(t, x) \in \mathbb{C}^{2^\ell} \), where

\[
\ell = \left\lfloor \frac{N + 1}{2} \right\rfloor,
\]  

(1.20)
with \([x]\) the integer part of \( x \), and \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( \mathbb{C}^n \). The \( N \)-dimensional free Dirac operator \( H \) is defined by

\[
H = -i \sum_{k=1}^{N} \gamma_k \partial_k + \eta,
\]  

(1.21)
where the \( 2^\ell \times 2^\ell \) Hermitian matrices \( \gamma_k, \eta \) satisfy the anticommutation relations

\[
\begin{aligned}
&\gamma_j \gamma_k + \gamma_k \gamma_j = 2 \delta_{jk} I, \quad j, k \in \{1, 2, \ldots, N\}, \\
&\gamma_j \eta + \eta \gamma_j = 0, \\
&\eta^2 = I.
\end{aligned}
\]  

(1.22)
If

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};
\]

are the Pauli matrices, then a standard choice in dimension one is \( \gamma_1 = \sigma_1, \eta = \sigma_3 \). In the case of dimension two, the usual convention is \( \gamma_1 = \sigma_1, \gamma_2 = \sigma_2, \eta = \sigma_3 \). In dimension three, the standard convention is

\[
\gamma_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad 1 \leq j \leq 3, \quad \eta = \begin{pmatrix} I \\ 0 \end{pmatrix},
\]

where in the above formula \( I \) is the \( 2 \times 2 \) identity matrix. In higher dimensions \( N \geq 4 \), the full set of explicit matrices \( \gamma_j, \eta \) satisfying the anti-commutation relations (1.22) can be constructed by iteration (see e.g. the Appendix in [22]).

The Dirac equation with power nonlinearity like (1.19) was studied in particular in [13, 14, 57, 29, 30]. However, to the best of our knowledge, and as for equation (1.1), the available local well-posedness results do not cover the case of all powers \( \alpha > 0 \) in all spatial dimensions \( N \geq 1 \). Our main result for equation (1.19) is the following.

**Theorem 1.3.** Let \( \alpha > 0, \mu_3 \in \mathbb{C} \), and let \( \ell \) be given by (1.20). Assume (1.2)-(1.3) and let \( X_{2^\ell} \) be defined by (1.4)-(1.5) with \( d = 2^\ell \). Assume \( \Psi_0 \in X_{2^\ell} \) satisfies

\[
\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |\Psi_0(x)| > 0.
\]
It follows that there exist $T > 0$ and a unique solution $\Psi \in C([-T, T], X_2)$ of (1.19). In addition, there exists $\eta > 0$ such that

$$\inf_{-T \leq t \leq T} \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |\Psi(t, x)| \geq \eta.$$  

**Remark 1.4.** The Dirac equation is often studied with the nonlinearity $\langle \eta \Psi, \Psi \rangle \eta \Psi$, in this case it is known as the Thirring model [56] (see [51] for three space dimensions). Many results are available for the Thirring model, see for instance [32, 31, 33, 49, 6, 47, 2, 48, 3, 7] (see also [39, 41]). This type of nonlinearity is not accessible to the method we use in this paper. Indeed, we would need lower estimates of $|\langle \eta \Psi, \Psi \rangle|$, which do not follow immediately from the method we use to prove Proposition 2.9.

Comments on the proof of Theorem 1.3. The strategy of the proof is the same as in the case of the problem (1.11) above. Indeed, let $v(t) = e^{itH} \psi$ be the solution of the linear problem

$$\begin{cases}
iv_1 = Hv, \\
v(0, x) = \psi(x).
\end{cases}$$

Integrating, we have

$$v(t, x) = \psi(x) - \int_0^t (Hv)(s, x) ds.$$  

Hence,

$$\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v(t, x)| \geq \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |\psi(x)| - t\|\langle x \rangle^n Hv\|_{L^\infty((0, t) \times \mathbb{R}^N)}.$$  

As $H$ is a first-order differential operator, we are in a similar situation as for (1.11). In fact, the case of the Dirac equation results somehow easier to handle, since there is no pseudo-differential operator involved. See Proposition 2.9 below.

As mentioned earlier, a natural application of the above results would be to address the existence of scattering or modified scattering for the nonlinear Klein-Gordon (1.1) and Dirac (1.19) equations in any spatial dimensions $N \geq 1$, similar to the results obtained in [9, 10] for the case of the Schrödinger equation. This question is more challenging in these cases because the corresponding conformal-type transforms involve pseudo-differential operators in the case of the Klein-Gordon equations, and a system of equations in the case of the Dirac equation. Therefore, the action on the nonlinearity of these conformal-type transforms results to be difficult to control.

The rest of this paper is organized as follows. In Section 2 we establish estimates for the group $(e^{it\gamma(|\nabla|)})_{t \in \mathbb{R}}$ in the space $X_2$ (Proposition 2.1), and of the group $(e^{itH})_{t \in \mathbb{R}}$ in the space $X_{2\ell}$ (Proposition 2.9). In Section 3 we estimate the nonlinearity $\mathcal{N}(u)$ in $X_2$ and $\mathcal{N}_1(\Psi) = \mu_3|\Psi|^3\Psi$ in $X_{2\ell}$. Finally, in Section 4 we complete the proofs of Theorems 1.1 and 1.3.

**Notation.** We denote by $L^p(U, \mathbb{C}^d)$, for $1 \leq p \leq \infty$ and $U = \mathbb{R}^N$ or $U = (0, T) \times \mathbb{R}^N$, $0 < T \leq \infty$, the usual $\mathbb{C}^d$-valued Lebesgue spaces. We use the standard notation that $\|u\|_{L^p} = \infty$ if $u \in L^p_{\text{loc}}(U, \mathbb{C}^d)$ and $u \notin L^p(U, \mathbb{C}^d)$. $H^s(\mathbb{R}^N, \mathbb{C}^d)$, $s \in \mathbb{R}$, is the usual $\mathbb{C}^d$-valued Sobolev space. (See e.g. [1] for the definitions and properties of these spaces.) We will often write $L^p(U)$ and $H^s(\mathbb{R}^N)$ for $L^p(U, \mathbb{C}^d)$ and $H^s(\mathbb{R}^N, \mathbb{C}^d)$, respectively. We denote by $(e^{-it\gamma(|\nabla|)})_{t \in \mathbb{R}}$ the group associated to the equation (1.15). As is well known, $(e^{-it\gamma(|\nabla|)})_{t \in \mathbb{R}}$ is a group of isometries on $L^2(\mathbb{R}^N, \mathbb{C}^d)$, and on $H^s(\mathbb{R}^N, \mathbb{C}^d)$ for all $s \in \mathbb{R}$.
2. Weighted estimates for the linear equations. We first estimate the action of the group \(e^{-it\gamma(i\nabla)}\) on the space
\[
\mathcal{X} = \mathcal{X}_2.
\] (2.1)
We prove the following result.

**Proposition 2.1.** Assume (1.2)-(1.3) with \(\alpha = 1\), and let the space \(\mathcal{X}\) be defined by (1.4)-(1.5) and (2.1). It follows that \(e^{-it\gamma(i\nabla)}\psi \in C(\mathbb{R}, \mathcal{X})\) for all \(\psi \in \mathcal{X}\). Moreover, there exist \(C > 0\) and \(t_0 > 0\) such that
\[
\|e^{-it\gamma(i\nabla)}\psi\|_{\mathcal{X}} \leq C(1 + |t|)^{2m+n+1}\|\psi\|_{\mathcal{X}},
\] (2.2)
and
\[
\sup_{|\beta| \leq 2m} \|\langle x \rangle^\nu D^\beta (e^{-it\gamma(i\nabla)}\psi - \psi)\|_{L^\infty} \leq C|t|(1 + |t|)^{2m+n+1}\|\psi\|_{\mathcal{X}},
\] (2.3)
for all \(|t| \leq t_0\) and all \(\psi \in \mathcal{X}\).

Before proving Proposition 2.1, we first establish a weighted \(L^\infty\) estimate. Before doing this, we prepare two estimates. First, we recall an interpolation estimate (see Lemma A.2 of [9]):

**Lemma 2.2.** Given \(j \in \mathbb{N}\) and \(\nu \in \mathbb{R}\), there exists a constant \(C\) such that
\[
\sup_{|\beta| = j+1} \|\langle x \rangle^\nu D^\beta u\|_{L^\infty} \leq C(\sup_{|\beta| = j} \|\langle x \rangle^\nu D^\beta u\|_{L^\infty} + \sup_{|\beta| = j+2} \|\langle x \rangle^\nu D^\beta u\|_{L^\infty})
\] (2.4)
for all \(u \in C^{j+2}(\mathbb{R}^N)\).

Also, we need the following control of the Bessel potential \(\langle i\nabla \rangle^{-1}\).

**Lemma 2.3.** Let \(n \in \mathbb{N}\) and \(1 \leq p \leq \infty\). For \(f \in L^p\) the estimate
\[
\|\langle x \rangle^n \langle i\nabla \rangle^{-1} f\|_{L^p} \leq C\|\langle x \rangle^n f\|_{L^p}
\] (2.5)
is true for some \(C > 0\).

**Proof.** We use the theory of Bessel potentials applied to \(\langle i\nabla \rangle^{-1}\). We have (see relation (26), Chapter V of [52])
\[
\langle i\nabla \rangle^{-1} f = G * f = \int_{\mathbb{R}^N} G(x - y) f(y) dy,
\]
where
\[
G(x) = \frac{1}{2\pi} \int_0^\infty \frac{e^{-\frac{\sqrt{x^2 + y^2}}{\theta}} - \frac{\theta}{\pi} y}{\theta^{1 + \frac{N}{2}}} d\theta.
\] (2.6)
Since \(\langle x \rangle^n \leq C(1 + |x|^n)\), we estimate
\[
\|\langle x \rangle^n \langle i\nabla \rangle^{-1} f\|_{L^p} \leq C\|G * f\|_{L^p} + \|\langle x \rangle^n (G * f)\|_{L^p},
\] (2.7)
for \(x \in \mathbb{R}^N\). We take into account the estimate for the Bessel potentials (see relation (36), Chapter V of [52])
\[
\|G * f\|_{L^p} \leq \|f\|_{L^p}, \quad \text{for } 1 \leq p \leq \infty.
\] (2.8)
Using this estimate, we control the first term in the right-hand side of (2.7) by \(\|f\|_{L^p}\). Now we use that
\[
|x|^n \leq (|x| + |y|)^n = \sum_{j=0}^n C^n_j |x - y|^{n-j} |y|^j,
\]
where \(C^n_j\) are the binomial coefficients. Then
\[
||x||^n(G*f)(x)| \leq C \sum_{j=0}^{n} \int_{\mathbb{R}^N} |x-y|^{n-j} G(x-y) |y|^j |f(y)| dy.
\]

Since \(G\) given by (2.6) has the singularity \(|x|^{-n+1/2}\) at \(x = 0\) and decays exponentially as \(|x| \to \infty\) (see [52, Chapter V, formulas (29)-(30)]), we see that \(|\cdot|^m G \in L^1\), for all \(m \geq 0\). Hence, by Young inequality we show that
\[
||x||^n(G*f)(\cdot)||_{L^p} \leq C||\cdot||^n f||_{L^p}, \quad \text{for } 1 \leq p \leq \infty.
\]

Using (2.8) and (2.9) in (2.7) we obtain (2.5). \(\square\)

We now are in position to prove the weighted \(L^\infty\) estimate for the linear flow. We have the following result.

**Lemma 2.4.** Assume (1.2)-(1.3) with \(\alpha = 1\). There exist \(C > 0\) and \(t_0 > 0\) such that
\[
\sum_{|\beta| = 2m-2}^{2m-2} \sup_{t \in [0, t_0]} \|\langle x \rangle^m D^\beta e^{-is\gamma(\xi)} \psi\|_{L^\infty([0, t_0])} (R^N) 
\leq C(1 + t)^{2m} \sum_{|j| \leq 2m} ||\langle x \rangle^m D^\beta \psi||_{L^\infty} (2.10)
\]

for all \(0 \leq t \leq t_0\) and all \(\psi \in H^J(R^N)\).

**Proof.** Set \(v(t) = e^{-it\gamma(\xi)}\). Since \(\langle \xi \rangle^j \psi \in H^{J-j}(R^N)\) for \(0 \leq j \leq 2m\), we have \(v \in C^j([0, \infty), H^{J-j}(R^N))\) and \(\frac{dv}{dt} = (i\gamma)^j (i\xi)^j v(t)\) for all \(0 \leq j \leq 2m\). Given \(0 \leq \ell \leq m - 1\), we apply Taylor’s formula with integral remainder involving the derivative of order \(2(m - \ell)\) to the function \(v\), and we obtain
\[
v(t) = \sum_{j=0}^{2m-2\ell-1} \frac{(it\gamma)^j}{j!} \langle i\xi \rangle^j \psi
\]
\[
+ \frac{(it\gamma)^{2m-2\ell}}{(2m-2\ell-1)!} \int_0^t (t-s)^{2m-2\ell-1} (1-\Delta)^{m-\ell} v(s) ds
\]
\[
\text{for all } t \geq 0.\] Applying now \(D^\beta\) with \(|\beta| = 2\ell\), we deduce that
\[
D^\beta v(t) = \sum_{j=0}^{2m-2\ell-1} \frac{(it\gamma)^j}{j!} D^\beta \langle i\xi \rangle^j \psi
\]
\[
+ \frac{(it\gamma)^{2m-2\ell}}{(2m-2\ell-1)!} \int_0^t (t-s)^{2m-2\ell-1} D^\beta (1-\Delta)^{m-\ell} v(s) ds.
\]

Developing the binomial \((1-\Delta)^{m-\ell}\), we obtain
\[
D^\beta v(t) = \sum_{j=0}^{2m-2\ell-1} \frac{(it\gamma)^j}{j!} D^\beta \langle i\xi \rangle^j \psi
\]
\[
+ \frac{(it\gamma)^{2m-2\ell}}{(2m-2\ell-1)!} \sum_{j=0}^{m-\ell} (-1)^j C_j^{m-\ell} \int_0^t (t-s)^{2m-2\ell-1} D^\beta (1-\Delta)^{m-\ell} v(s) ds.
\]
Identity (2.11) holds in $C([0, \infty), H^k(\mathbb{R}^N))$, hence in $C([0, \infty) \times \mathbb{R}^N)$ by Sobolev’s embedding. We multiply (2.11) by $\langle x \rangle^n$ and take the supremum in $x$, then in $t$, to obtain

$$
\| \langle x \rangle^n D^\beta v \|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq C(1 + t)^{2m} \sum_{j=0}^{2m-2\ell-1} \| \langle x \rangle^n \langle t \nabla \rangle^j D^\beta \psi \|_{L^\infty} + Ct(1 + t)^{2m} \sum_{j=0}^{m-\ell} \| \langle x \rangle^n D^\beta \Delta^j v \|_{L^\infty((0,t) \times \mathbb{R}^N)},
$$

(2.12)

We note that if $j$ is even, then $\langle i \nabla \rangle^j = (1 - \Delta)^{j/2}$ and that if $j$ is odd, then $\langle i \nabla \rangle^j = (i \nabla)^{-1}(1 - \Delta)^{j/2}$. Therefore, for $0 \leq j \leq 2m - 2\ell - 1$, and since $|\beta| = 2\ell$, we have (using (2.5) if $j$ is odd)

$$
\| \langle x \rangle^n \langle i \nabla \rangle^j D^\beta \psi \|_{L^\infty} \leq \sum_{|\rho| \leq 2m} \| \langle x \rangle^n D^\rho \psi \|_{L^\infty}.
$$

It follows that

$$
\sup_{|\rho| = 2\ell} \| \langle x \rangle^n D^\rho v \|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq C(1 + t)^{2m} \sum_{|\rho| \leq 2m} \| \langle x \rangle^n D^\rho \psi \|_{L^\infty} + Ct(1 + t)^{2m} \sum_{j=\ell}^{m} \sup_{||\rho|=2j} \| \langle x \rangle^n D^\rho v \|_{L^\infty((0,t) \times \mathbb{R}^N)}.
$$

(2.13)

We now fix $t_0 > 0$ sufficiently small so that

$$
Ct_0(1 + t_0)^{2m} \leq \frac{1}{2},
$$

(2.14)

and we set

$$
A = C(1 + t_0)^{2m} \sum_{|\rho| \leq 2m} \| \langle x \rangle^n D^\rho \psi \|_{L^\infty}.
$$

(2.15)

Moreover, for $0 < t \leq t_0$, we set

$$
B(t) = Ct(1 + t)^{2m} \sup_{|\rho| = 2m} \| \langle x \rangle^n D^\rho v \|_{L^\infty((0,t) \times \mathbb{R}^N)}.
$$

(2.16)

With this notation, we see that for $0 < t \leq t_0$,

$$
\sup_{|\rho| = 2\ell} \| \langle x \rangle^n D^\rho v \|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq A + Ct(1 + t)^{2m} \sum_{j=\ell+1}^{m} \sup_{||\rho|=2j} \| \langle x \rangle^n D^\rho v \|_{L^\infty((0,t) \times \mathbb{R}^N)}
$$

$$
+ \frac{1}{2} \sup_{|\rho| = 2\ell} \| \langle x \rangle^n D^\rho v \|_{L^\infty((0,t) \times \mathbb{R}^N)},
$$

so that

$$
\sup_{|\rho| = 2\ell} \| \langle x \rangle^n D^\rho v \|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq 2A + 2B(t).
$$

(2.17)

We first apply (2.17) with $\ell = m - 1$, and we obtain using (2.16)

$$
\sup_{|\rho| = 2(m-1)} \| \langle x \rangle^n D^\rho v \|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq 2A + 2B(t).
$$

(2.18)
Next, we apply (2.17) with \( \ell = m - 2 \), together with (2.18) and (2.14), to obtain

\[
\sup_{|\rho|=2(m-2)} \| \langle x \rangle^n D^\rho v \|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq 4A + 4B(t).
\]

An obvious iteration shows that

\[
\sup_{|\rho|=2(m-j)} \| \langle x \rangle^n D^\rho v \|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq 2^j A + 2^j B(t)
\]

for all \( 1 \leq j \leq m \). Thus we see that

\[
\sup_{0 \leq \ell \leq m-1} \| \langle x \rangle^n D^\rho v \|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq 2^\ell A + 2^\ell B(t).
\]

The derivatives of even order in the left-hand side of (2.10) are estimated by (2.19). Finally, we use the interpolation estimate (2.4) to control the derivatives of odd order in the left-hand side of (2.10) by the derivatives of even order. This completes the proof of (2.10).

Estimate (2.10) shows that we can control the linear solution in terms of the initial data and a high-order derivative of this solution. Since we can estimate the \( L^2 \) norm from the equation via energy estimates, we now control the uniform norm of the term in the right-hand side of (2.10) which involves the linear flow by Sobolev’s embedding. The last is done by establishing an appropriate weighted \( L^2 \) estimate (Lemma 2.6 below). We first introduce some notation. Assume (1.2)-(1.3) with \( \alpha = 1 \). We define the space

\[
\mathcal{Y} = \{ u \in H^J(\mathbb{R}^N, \mathbb{C}^2); \langle x \rangle^n D^\beta u \in L^2(\mathbb{R}^N, \mathbb{C}^2) \text{ for } 2m - 1 \leq |\beta| \leq J \}. \tag{2.20}
\]

We equip \( \mathcal{Y} \) with the norm

\[
\| u \|_{\mathcal{Y}} = \sum_{|\beta|=0}^{2m-2} \| D^\beta u \|_{L^2} + \sum_{|\beta|=2m-1}^{J} \| \langle x \rangle^n D^\beta u \|_{L^2}. \tag{2.21}
\]

Note that, using (1.6),

\[
\mathcal{X} \hookrightarrow \mathcal{Y}. \tag{2.22}
\]

We observe that \( (\mathcal{Y}, \| \cdot \|_\mathcal{Y}) \) is a Banach space and that \( \mathcal{S}(\mathbb{R}^N, \mathbb{C}^2) \) is dense in \( \mathcal{Y} \). We will use the following commutation relation.

**Lemma 2.5.** Given any integer \( \ell \geq 1 \), there exist an integer \( \nu \geq 1 \) and functions

\[
(a_j)_{1 \leq j \leq \nu}, (b_j)_{1 \leq j \leq \nu}, (c_j)_{1 \leq j \leq \nu} \subset C^\infty(\mathbb{R}^N)
\]

satisfying

\[
|a_j(x)| \leq C \langle x \rangle^{\ell-1}, \quad |b_j(\xi)| \leq C, \quad |c_j(x)| \leq C \langle x \rangle^{\ell}, \tag{2.23}
\]

for \( 1 \leq j \leq \nu \), such that

\[
\langle \cdot \rangle^{2\ell} \langle i\nabla \rangle f = \langle i\nabla \rangle (\langle \cdot \rangle^{2\ell} f) + \sum_{j=1}^{\nu} a_j F^{-1} [b_j F(c_j f)], \tag{2.24}
\]

for all \( x \in \mathbb{R}^N \) and \( f \in \mathcal{S}(\mathbb{R}^N) \).
Proof. Since \(|x|^{2\ell} = \sum_{j=0}^{\ell} C_j^\ell |x|^{2j}\), we have
\[
\mathcal{F}[|x|^{2\ell} \langle i\nabla \rangle f] = \sum_{j=0}^{\ell} C_j^\ell \mathcal{F}[|x|^{2j} \langle i\nabla \rangle f] = \sum_{j=0}^{\ell} C_j^\ell (-\Delta)^j \mathcal{F}[\langle i\nabla \rangle f] = \sum_{j=0}^{\ell} C_j^\ell (-\Delta)^j [\langle \xi \rangle \hat{f}(\xi)].
\]
We observe that
\[
(-\Delta)^j (uv) = u(-\Delta)^j v + \sum_{|\beta_1|+|\beta_2|=2j, |\beta_1| \geq 1} \gamma_{\beta_1,\beta_2} [D^{\beta_1} u][D^{\beta_2} v]
\]
for some coefficients \(\gamma_{j_1,j_2}\). Therefore, we may write
\[
\mathcal{F}[|x|^{2\ell} \langle i\nabla \rangle f] = I_1(\xi) + I_2(\xi),
\]
where
\[
I_1(\xi) = \sum_{j=0}^{\ell} C_j^\ell \langle \xi \rangle [(-\Delta)^j \hat{f}]
\]
and
\[
I_2(\xi) = \sum_{|\beta_1|+|\beta_2| \leq 2\ell, |\beta_1| \geq 1} C_{\beta_1,\beta_2} [D^{\beta_1} (\xi)][D^{\beta_2} \hat{f}]
\]
for some coefficients \(C_{j_1,j_2}\). We have
\[
I_1(\xi) = \sum_{j=0}^{\ell} C_j^\ell \langle \xi \rangle \mathcal{F}[|x|^{2j} f] = \langle \xi \rangle \mathcal{F}[\langle x \rangle^{2\ell} f] = \mathcal{F}[\langle i\nabla \rangle (\langle x \rangle^{2\ell} f)].
\]
Taking the inverse Fourier transform in the expression for \(I_2(\xi)\) we obtain
\[
\mathcal{F}^{-1} I_2 = \sum_{|\beta_1|+|\beta_2| \leq 2\ell, |\beta_1| \geq 1} C_{\beta_1,\beta_2} w_{\beta_1,\beta_2}, \tag{2.27}
\]
where
\[
w_{\beta_1,\beta_2} = \mathcal{F}^{-1} [(D^{\beta_1} (\xi))(D^{\beta_2} \hat{f})] = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{ix\xi} (D^{\beta_1} (\xi))(D^{\beta_2} \hat{f}) d\xi. \tag{2.28}
\]
We recall that
\[
|D^\beta (\xi)| \leq C \tag{2.29}
\]
for all \(|\beta| \geq 1\). (See e.g. [10, formula (A.2)].) If \(|\beta_2| \leq \ell\), we write \(w_{\beta_1,\beta_2} = a \mathcal{F}^{-1} [b \mathcal{F}(c f)]\) with \(a(x) \equiv 1\), \(b(\xi) \equiv D^{\beta_1} (\xi)\) and \(c(x) = x^{\beta_2}\). Using (2.29), we see that this is a term allowed by (2.23). If \(|\beta_2| \geq \ell + 1\), we write \(\beta_2 = \beta_2' + \beta_2''\) with \(|\beta_2'| = \ell\) and \(|\beta_2''| \leq \ell - 1\) (recall that \(|\beta_2| \leq 2\ell - 1\) in the sum (2.27)). After integration by parts, we obtain
\[
w_{\beta_1,\beta_2} = (-1)^{|\beta_2'|} (2\pi)^{-N} \int_{\mathbb{R}^N} D^{\beta_2''} [e^{ix\xi} (D^{\beta_1} (\xi))](D^{\beta_2} \hat{f}) d\xi
\]
\[
\quad = (-1)^{|\beta_2'|} (2\pi)^{-N} \sum_{\beta_3 + \beta_4 = \beta_2''} \int_{\mathbb{R}^N} e^{ix\xi} (D^{\beta_1 + \beta_4} (\xi))(D^{\beta_2} \hat{f}) d\xi.
\]
Using again (2.29), we see that each of the terms in the above series has the appropriate form. The result now follows from (2.25), (2.26) and (2.27).
Proof. We first prove estimate (2.30) for \( e^{-it\gamma(i\nabla)}\psi \in C([0,\infty),Y) \). Moreover, there exists a constant \( C \) such that
\[
\|e^{-it\gamma(i\nabla)}\psi\|_Y \leq C(1 + t)^n \|\psi\|_Y
\] (2.30)
for all \( t \geq 0 \) and all \( \psi \in Y \).

We now prove the following:

Lemma 2.6. Assume (1.2)-(1.3) with \( \alpha = 1 \). It follows that, given any \( \psi \in Y \), \( e^{-it\gamma(i\nabla)}\psi \in C([0,\infty),Y) \). Moreover, there exists a constant \( C \) such that
\[
\|e^{-it\gamma(i\nabla)}\psi\|_Y \leq C(1 + t)^n \|\psi\|_Y
\] (2.30)
for all \( t \geq 0 \) and all \( \psi \in Y \).

Proof. We first prove estimate (2.30) for \( \psi \in S(\mathbb{R}^N) \). Let \( \psi \in S(\mathbb{R}^N) \) and set \( v(t) = e^{-it\gamma(i\nabla)}\psi \). It follows by standard Fourier analysis that \( v \in C^\infty([0,\infty),S(\mathbb{R}^N)) \).

Since the linear flow is isometric on \( H^{2m-2}(\mathbb{R}^N) \), we need only estimate the weighted terms \( \|\langle x \rangle^n D^\beta v\|_{L^2} \), with \( 2m-1 \leq |\beta| \leq J \). We fix \( 2m-1 \leq |\beta| \leq J \) and we prove that
\[
\|\langle x \rangle^n D^\beta v\|_{L^2} \leq C(1 + t)^n \sum_{|\nu|=2m-1} \|\langle x \rangle^n D^\nu v\|_{L^2}.
\] (2.31)

Let \( \ell \in \{1, \cdots, n\} \). Applying \( D^\beta \) to equation (1.15), multiplying by \( \langle x \rangle^{2\ell} D^\beta \), integrating on \( \mathbb{R}_N \), and taking the imaginary part, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\langle x \rangle^{\ell} D^\beta v\|_{L^2}^2 = \Im \int_{\mathbb{R}_N} \langle x \rangle^{2\ell} \gamma(i\nabla) D^\beta v \cdot D^\beta \tau.
\] (2.32)

We use now the commutation relation (2.24) in (2.32). We have
\[
\frac{1}{2} \frac{d}{dt} \|\langle x \rangle^{\ell} D^\beta v\|_{L^2}^2 = \Im \int_{\mathbb{R}_N} \langle x \rangle^{2\ell} \gamma(i\nabla) D^\beta v \cdot D^\beta \tau
\]
\[
+ \sum_{j=1}^\nu \Im \int_{\mathbb{R}_N} (\gamma a_j b_j (\nabla) [c_j D^\beta v]) \cdot D^\beta \tau.
\] (2.33)

Since \( \gamma(i\nabla) \) is self-adjoint, by using again (2.24) we have
\[
\Im \int_{\mathbb{R}_N} \langle x \rangle^{2\ell} D^\beta v \cdot D^\beta \tau
\]
\[
= \frac{1}{2i} \left( \int_{\mathbb{R}_N} \langle x \rangle^{2\ell} [\gamma(i\nabla)] D^\beta v \cdot [\gamma(i\nabla)] D^\beta \tau - \int_{\mathbb{R}_N} \langle x \rangle^{2\ell} D^\beta v \cdot [\gamma(i\nabla)] D^\beta \tau \rangle \right)
\]
\[
= \frac{1}{2i} \left( \int_{\mathbb{R}_N} \langle x \rangle^{2\ell} [\gamma(i\nabla)] D^\beta v \cdot D^\beta \tau - \int_{\mathbb{R}_N} \langle x \rangle^{2\ell} [\gamma(i\nabla)] D^\beta v \cdot D^\beta \tau \right)
\] (2.34)
\[
= -\frac{1}{2i} \sum_{j=1}^\nu \Im \int_{\mathbb{R}_N} (\gamma a_j b_j (\nabla) [c_j D^\beta v]) \cdot D^\beta \tau.
\]

It follows from (2.33), (2.34), (2.23), and \( \ell \leq n \), that
\[
\frac{1}{2} \frac{d}{dt} \|\langle x \rangle^{\ell} D^\beta v\|_{L^2} \leq C \sum_{j=1}^\nu \int_{\mathbb{R}_N} \langle x \rangle^{\ell-1} |b_j(\nabla) [c_j D^\beta v] \cdot D^\beta \tau|
\]
\[
\leq C \sum_{j=1}^\nu \|b_j(\nabla) [c_j D^\beta v]\|_{L^2} \|\langle \cdot \rangle^{\ell-1} D^\beta v\|_{L^2}
\] (2.35)
\[
\leq C \|\langle \cdot \rangle^{\ell} D^\beta v\|_{L^2} \|\langle \cdot \rangle^{\ell-1} D^\beta v\|_{L^2}.
\]
Hence
\[
\| \langle x \rangle^t D^\beta v(t) \|_{L^2} \leq \| \langle x \rangle^t D^\beta \psi \|_{L^2} + Ct \sup_{0 < \tau < t} \| \langle x \rangle^{t-1} D^\beta v(\tau) \|_{L^2} \\
\leq \| \langle x \rangle^n D^\beta \psi \|_{L^2} + Ct \sup_{0 < \tau < t} \| \langle x \rangle^{t-1} D^\beta v(\tau) \|_{L^2}.
\]
We apply successively the above estimate with \( t = 1, \cdots, n \). Since \( \| D^\beta v(\tau) \|_{L^2} = \| D^\beta \psi \|_{L^2} \), we conclude that
\[
\| \langle x \rangle^n D^\beta v(t) \|_{L^2} \leq C(1 + t)^n \| \langle x \rangle^n D^\beta \psi \|_{L^2}.
\]
This last estimate proves (2.31).

Let now \( \psi \in \mathcal{Y} \) and \((\psi_n)_{n \geq 1} \subset \mathcal{S}(\mathbb{R}^N)\) such that \( \psi_n \to \psi \) in \( \mathcal{Y} \) as \( n \to \infty \). Applying (2.31) with \( \psi \) replaced by \( \psi_m - \psi_n \), we deduce that for every \( T > 0 \), 
\[ e^{-it\gamma(\nu)} \psi_n \] is a Cauchy sequence in \( L^\infty([-T,T), \mathcal{Y}) \). It follows that \( e^{-it\gamma(\nu)} \psi \) belongs to \( C([-T,T], \mathcal{Y}) \) and satisfies (2.31) for all \( t \in [-T,T] \). Since \( T > 0 \) is arbitrary, this completes the proof.

**Lemma 2.7.** Assume (1.2)-(1.3) with \( \alpha = 1 \). It follows that there exists a constant \( C \) such that
\[
\sup_{2m-1 \leq |\beta| \leq 2m} \| \langle \cdot \rangle^n D^\beta u \|_{L^\infty} \leq C \| u \|_{\mathcal{Y}} \tag{2.36}
\]
for all \( u \in \mathcal{Y} \).

**Proof.** By density of \( \mathcal{S}(\mathbb{R}^N) \) in \( \mathcal{Y} \), the result follows if we prove estimate (2.36) for \( u \in \mathcal{S}(\mathbb{R}^N) \). Let \( u \in \mathcal{S}(\mathbb{R}^N) \) and \( 2m-1 \leq |\beta| \leq 2m \). Since \( H^k(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N) \) by (1.2), we have
\[
\| \langle \cdot \rangle^n D^\beta u \|_{L^\infty} \leq C \| \langle \cdot \rangle^n D^\beta u \|_{H^k} \leq C \sum_{|\rho| \leq k} \| D^\rho \langle \cdot \rangle^n D^\beta u \|_{L^2}.
\]
Moreover (see e.g. [9, Lemma A.1]),
\[
|D^\rho \langle \cdot \rangle^n D^\beta u| \leq C \sum_{j=0}^{|\rho|} \langle x \rangle^{n-|\rho|+j} \sum_{|\rho'|=j} |D^{\rho'+\beta} u| \leq C \langle x \rangle^n \sum_{|\rho'|=|\beta|} |D^{\rho'} u|.
\]
It follows that
\[
\| \langle \cdot \rangle^n D^\beta u \|_{L^\infty} \leq C \sum_{|\rho|=2m-1} \| \langle x \rangle^n D^{\rho} u \|_{L^2}.
\]
Hence (2.36) holds.

**Proof of Proposition 2.1.** Since \( e^{it\gamma(\nu)} \psi = e^{-it\gamma(\nu)} \overline{\psi} \) and the map \( \psi \mapsto \overline{\psi} \) is isometric \( \mathcal{X} \to \mathcal{X} \), we can restrict ourselves to the case \( t \geq 0 \).

We let \( \psi \in \mathcal{X} \). As before, we set \( v(t) = e^{-it\gamma(\nu)} \psi \). We begin by proving that if \( t_0 > 0 \) is given by Lemma 2.4, then \( v(t) \in \mathcal{X} \) for all \( 0 \leq t \leq t_0 \) and (2.2) holds. By the definition (1.5) of the norm in the space \( \mathcal{X} \), the estimate (2.36), and the embedding (2.22), we have
\[
\sum_{|\beta|=0}^{2m} \| \langle x \rangle^n D^\beta \psi \|_{L^\infty} \leq C \| \psi \|_{\mathcal{X}}.
\]
Then, using (2.10) we have
\[
\sum_{j=0}^{2m} \sup_{|\beta|=j} \|\langle x \rangle^n D^\beta v\|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq C(1 + t)^{2m} \|\psi\|_X
\]
\[+ Ct(1 + t)^{2m} \sup_{|\beta|=2m} \|\langle x \rangle^n D^\beta v\|_{L^\infty((0,t) \times \mathbb{R}^N)},
\]
for all $0 \leq t \leq t_0$. We estimate the last term in the above inequality by (2.36) and (2.30). For $|\beta| = 2m$ we have
\[
\|\langle x \rangle^n D^\beta v\|_{L^\infty((0,t) \times \mathbb{R}^N)} = \sup_{0<s<t} \|\langle x \rangle^n D^\beta v(s)\|_{L^\infty}
\leq C \sup_{0<s<t} \|v(s)\|_Y
\leq C(1 + t)^n \|\psi\|_Y.
\]
Using (2.22), we deduce that
\[
\sum_{j=0}^{2m} \sup_{|\beta|=j} \|\langle x \rangle^n D^\beta v\|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq C(1 + t)^{2m+n+1} \|\psi\|_X.
\]
(2.37)
Using again (2.30), we conclude that $v(t) \in X$ for all $0 \leq t \leq t_0$ and that estimate (2.2) holds.

Let us prove now (2.3). We consider a multi-index $\beta$ with $0 \leq |\beta| \leq 2m$. Using (1.16) and (2.5) we estimate
\[
\|\langle x \rangle^n D^\beta (v(t) - \psi)\|_{L^\infty} \leq \sum_{j=0}^{2m} \int_0^t \sup_{|\beta|=j} \|\langle x \rangle^n (i\nabla)^{-1} D^\beta v(s)\|_{L^\infty} ds
\leq t \sum_{j=0}^{2m} \sup_{|\beta|=j} \|\langle x \rangle^n D^\beta v\|_{L^\infty((0,t) \times \mathbb{R}^N)}.
\]
Using (2.37) we conclude that (2.3) holds.

By Lemma 2.6, $v \in C([0, \infty), Y)$. Moreover, by (2.3) $v$ is continuous at $t = 0$ in weighted $L^\infty$ norms. Thus, $v$ is continuous at $t = 0$ in $X$ norm. By the semigroup property we conclude that $v \in C([0, \infty), X)$. This completes the proof. 

**Remark 2.8.** In particular, Proposition 2.1 and (1.17) show that for $\psi \in X$ satisfying $\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |\psi(x)| > 0$, the estimate from below $\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |e^{-it\gamma(i\nabla)} \psi(x)| > 0$ holds, for all $|t|$ sufficiently small. We do not know if this small time requirement is necessary.

We now prove estimates for the linear Dirac group $(e^{-itH})_{t \in \mathbb{R}}$ on the space
\[
\tilde{X} = X_{2\ell}
\]
(3.8)
similar to the ones established in Proposition 2.1 for the group $(e^{-it\gamma(i\nabla)})_{t \in \mathbb{R}}$. The free Dirac operator $H$ defined by (1.21) with domain $D(H) = H^1(\mathbb{R}^N, \mathbb{C}^{2\ell})$ is a self-adjoint operator on $L^2(\mathbb{R}^N, \mathbb{C}^{2\ell})$, see [55]. Then, $(e^{itH})_{t \in \mathbb{R}}$ is a group of isometries on $L^2(\mathbb{R}^N, \mathbb{C}^{2\ell})$, and on $H^s(\mathbb{R}^N, \mathbb{C}^{2\ell})$ for all $s \in \mathbb{R}$. We have the following.

**Proposition 2.9.** Assume (1.2)- (1.3) with $\alpha = 1$, and let the space $\tilde{X}$ be defined by (1.4)- (1.5), (1.20) and (3.8). It follows that $e^{-itH} \Psi \in C(\mathbb{R}, \tilde{X})$ for all $\Psi \in \tilde{X}$. Moreover, there exist $C > 0$ and $t_0 > 0$ such that
\[
\|e^{-itH} \Psi\|_{\tilde{X}} \leq C(1 + |t|)^{2m+n+1} \|\Psi\|_{\tilde{X}}.
\]
and
\[ \sup_{|\beta| \leq 2m} \| \langle x \rangle^m D^\beta (e^{-itH} \Psi - \psi) \|_{L^\infty} \leq C |t| (1 + |t|)^{2m+n+1} \| \Psi \|_{\tilde{X}}, \]
for all \( |t| \leq t_0 \) and all \( \Psi \in \tilde{X} \).

**Proof.** The proof is similar to the proof of Proposition 2.1. We only point out the differences. To prove an estimate similar to Lemma 2.4, for a given \( 0 \leq k \leq m - 1 \) we apply Taylor’s formula with integral remainder involving the derivative of order \( 2(m - k) \) to the function \( v = e^{-itH} \Psi \). Using the commutation relations (1.22) we see that \( H^2 = -\Delta + I \). It follows that \( v(t) = e^{-itH} \Psi \) satisfies
\[
v(t) = \sum_{j=0}^{2m-2k-1} \frac{(it\gamma)^j}{j!} H^j \Psi + \frac{(it\gamma)^{2m-2k}}{(2m-2k-1)!} \int_0^t (t-s)^{2m-2k-1} (1 - \Delta)^{m-k} v(s) \, ds.
\]
Then, similarly to (2.12), we deduce that
\[
\| \langle x \rangle^m D^\beta v \|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq C (1 + t)^{2m} \sum_{j=0}^{2m-2k-1} \| \langle x \rangle^n H^j D^\beta \Psi \|_{L^\infty}
+ C t (1 + t)^{2m} \sum_{j=0}^{m-k} \| \langle x \rangle^n D^\beta \Delta^j v \|_{L^\infty((0,t) \times \mathbb{R}^N)},
\]
where the \( L^\infty \) norms are for \( C^{2^\ell} \)-valued functions. Since the matrices \( \gamma_j \) and \( \eta \) are unitary, we have
\[
\| \langle x \rangle^n H^j D^\beta \Psi \|_{L^\infty} \leq C \sum_{|\rho| \leq 2m} \| \langle x \rangle^n D^\rho \Psi \|_{L^\infty}.
\]
Then, continuing as in the proof of Lemma 2.4 we obtain that there exist \( C > 0 \) and \( t_0 > 0 \) such that
\[
\sum_{j=0}^{2m-2} \sup_{|\beta| = j} \| \langle x \rangle^m D^\beta e^{-isH} \Psi \|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq C (1 + t)^{2m} \sum_{|j| \leq 2m} \| \langle x \rangle^n D^\beta \Psi \|_{L^\infty}
+ C t (1 + t)^{2m} \sup_{|\beta| = 2m} \| \langle x \rangle^n D^\beta e^{-isH} \Psi \|_{L^\infty((0,t) \times \mathbb{R}^N)},
\]
for all \( 0 \leq t \leq t_0 \) and all \( \Psi \in H^j(\mathbb{R}^N) \).

Next, we consider the space \( \tilde{\mathcal{V}} \) defined similarly to \( \mathcal{V} \) (i.e. by (2.20)-(2.21)), but for \( C^{2^\ell} \)-valued functions instead of \( C^2 \)-valued functions. We claim that there exists a constant \( C \) such that
\[
\| e^{-itH} \Psi \|_{\tilde{\mathcal{V}}} \leq C (1 + t)^n \| \Psi \|_{\tilde{\mathcal{V}}},
\]
for all \( t \geq 0 \) and all \( \Psi \in \tilde{\mathcal{V}} \). To see this, we consider \( \Psi \in S(\mathbb{R}^N)^{2^\ell} \) and we set \( v(t) = e^{-itH} \Psi \), so that \( v \in C([0,\infty), S(\mathbb{R}^N)^{2^\ell}) \). Let \( k \in \{1, \cdots, n\} \). Since \( H \) is self-adjoint, similarly to (2.32) we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \langle x \rangle^k D^\beta v \|_{L^2}^2 = 3 \int_{\mathbb{R}^N} \langle x \rangle^{2k} H D^\beta v \cdot D^\beta v.
\]
We now use the commutator relation

$$[\langle x \rangle^{2k}, \mathcal{H}] = 2k i \langle x \rangle^{2k-2} \sum_{j=1}^{N} \gamma_{j} x_{j},$$

to prove similarly to (2.35) that

$$\frac{1}{2} \frac{d}{dt} \|\langle \cdot \rangle^{k} D^{\beta} v\|_{L^{2}}^{2} \leq C \|\langle \cdot \rangle^{k} D^{\beta} v\|_{L^{2}} \|\langle \cdot \rangle^{k-1} D^{\beta} v\|_{L^{2}}.$$  

Following the proof of Lemma 2.6, we deduce (2.40).

Finally, we can use (2.39) and (2.40) to complete the proof of Proposition 2.9 like the proof of Proposition 2.1.

3. The nonlinear estimates. With the linear estimates at our disposal, we now establish estimates of the nonlinear terms.

We first estimate $\mathcal{N}(u)$ in the space $\mathcal{X}$.

**Proposition 3.1.** Let $\alpha > 0$ and assume \((1.2)-(1.3)\). Let $\mathcal{X}$ be defined by (1.4)-(1.5) and (2.1), and let $\mathcal{N}(u)$ be defined by (1.13). For every $\eta > 0$ and $u \in \mathcal{X}$ satisfying

$$\eta \inf_{x \in \mathbb{R}^{N}} (\langle x \rangle^{\alpha} |u(x)|) \geq 1$$

it follows that $\mathcal{N}(u) \in \mathcal{X}$. Moreover, there exists a constant $C$ such that

$$\|\mathcal{N}(u)\|_{\mathcal{X}} \leq C(1 + \eta \|u\|_{\mathcal{X}})^{2J} \|u\|_{\mathcal{X}}^{\alpha+1}$$

for all $\eta > 0$ and $u \in \mathcal{X}$ satisfying (3.1). Furthermore,

$$\|\mathcal{N}(u_{1}) - \mathcal{N}(u_{2})\|_{\mathcal{X}} \leq C (1 + \eta(\|u_{1}\|_{\mathcal{X}} + \|u_{2}\|_{\mathcal{X}}))^{2J+1} (\|u_{1}\|_{\mathcal{X}} + \|u_{2}\|_{\mathcal{X}})^{\alpha} \|u_{1} - u_{2}\|_{\mathcal{X}}$$

for all $\eta > 0$ and $u, u_{1}, u_{2} \in \mathcal{X}$ satisfying (3.1).

**Proof.** Without loss of generality, we assume $\mu_{2} = 1$. First of all by (2.5) we have

$$\|\mathcal{N}(u)\|_{\mathcal{X}} \leq C \|u\|_{\mathcal{X}}^{\alpha}$$

and

$$\|\mathcal{N}(u_{1}) - \mathcal{N}(u_{2})\|_{\mathcal{X}} \leq C \|u_{1} - u_{2}\|_{\mathcal{X}}^{\alpha}.$$  

Therefore, it is suffices to show that

$$\|u\|_{\mathcal{X}}^{\alpha} \leq C(1 + \eta \|u\|_{\mathcal{X}})^{2J} \|u\|_{\mathcal{X}}^{\alpha+1}$$

and

$$\|u_{1} - u_{2}\|_{\mathcal{X}}^{\alpha} \leq C ((1 + \eta \|u_{1}\|_{\mathcal{X}} + \|u_{2}\|_{\mathcal{X}}))^{2J+1} (\|u_{1}\|_{\mathcal{X}} + \|u_{2}\|_{\mathcal{X}})^{\alpha} \|u_{1} - u_{2}\|_{\mathcal{X}}.$$  

First, we calculate $D^{\beta}(|u|^{\alpha} u)$ with $1 \leq |\beta| \leq J$. We have

$$D^{\beta}(|u|^{\alpha} u) = \sum_{\gamma + \rho = \beta} c_{\gamma, \rho} D^{\gamma}(|u|^{\rho}) D^{\rho} u,$$

where $c_{\gamma, \rho}$ are given by Leibnitz’s rule. We write $|u|^{\alpha} = (u \pi)^{\frac{\alpha}{2}}$. Thus, the development of $D^{\beta}(|u|^{\alpha} u)$ contains on the one hand the term

$$A = |u|^{\alpha} D^{\beta} u,$$  

(3.8)
and on the other hand, terms of the form

$$B = |u|^\alpha - 2p D^\rho u \prod_{j=1}^{p} D^{\gamma_j} u D^{\gamma_j} u$$

(3.9)

where

$$\gamma + \rho = \beta, \quad 1 \leq p \leq |\gamma|, \quad |\gamma_1 + \gamma_2| \geq 1, \quad \sum_{j=0}^{p} (\gamma_1 + \gamma_2) = \gamma.$$

First, we prove (3.6). There are two possibilities. If $|\beta| \leq 2m - 2$, we need to estimate the terms $(x)^n A$ and $(x)^n B$ in $L^\infty$. On the other hand, if $2m - 1 \leq |\beta| \leq J$, we need to control the terms $(x)^n A$ and $(x)^n B$ in $L^2$. Observe that the terms corresponding to $A$ contribute by $\|u\|_{L^\infty}^2 \|u\|_X$. Thus, we see that these terms are controlled by $(1 + \eta\|u\|_X)^{2j} \|u\|_{X_j}^{\alpha + 1}$. Let us focus on the terms corresponding to $B$. Using the lower bound (3.1) we have

$$|u|^{\alpha - 2p} \leq \eta^{2p} (x)^{2p\eta} |u|^\alpha \leq \eta^{2p} (x)^{(2p - \alpha)\eta} \|u\|_{X}^\alpha,$$

hence

$$|B| \leq \eta^{2p} (x)^{(2p - \alpha)\eta} \|u\|_{X}^\alpha \|D^\rho u\| \prod_{j=1}^{p} \|D^{\gamma_j} u\| \|D^{\gamma_j} u\|.$$  

(3.10)

We now consider separately the cases $|\beta| \leq 2m - 2$ and $2m - 1 \leq |\beta| \leq J$.

**The case $|\beta| \leq 2m - 2$.** We need to estimate $\|\langle x \rangle^n B\|_{L^\infty}$. Since $|\beta| \leq 2m - 2$, all the derivatives in the right-hand side of (3.10) are of order less than or equal to $2m - 2$. Hence, all the derivatives in (3.10) are estimated by $\langle x \rangle^{-n} \|u\|_X$; and so

$$|B| \leq (\eta\|u\|_X)^{2p} \|u\|_{X}^{\alpha + 1} \leq C(1 + \eta\|u\|_X)^{2J} \|u\|_{X}^{\alpha + 1}.$$  

(3.11)

so that

$$\|\langle x \rangle^n B\|_{L^\infty} \leq (\eta\|u\|_X)^{2p} \|u\|_{X}^{\alpha + 1} \leq C(1 + \eta\|u\|_X)^{2J} \|u\|_{X}^{\alpha + 1}.$$  

The case $2m - 1 \leq |\beta| \leq J$. We need to estimate $\|\langle x \rangle^n B\|_{L^2}$. Suppose that one of the derivatives in the right-hand side of (3.10) is of order $\geq 2m - 1$, for instance $|\gamma_1| \geq 2m - 1$. The sum of the orders of all derivatives in (3.10) is equal to $|\beta|$. On the other hand, by (1.2), $k + n \geq 2m - 3$ and $n \geq 2$, which implies that $|\beta| \leq J = 2m + 2 + k \leq 2m + k + n \leq 4m - 3$. Thus, we conclude that all other derivatives in (3.10) must have order $\leq 2m - 2$. Hence, they are controlled by $\langle x \rangle^{-n} \|u\|_X$. Therefore, from (3.10) we get

$$|B| \leq (\eta\|u\|_X)^{2p} \|u\|_{X}^{\alpha + 1} \|D^{\gamma_1} u\|.$$  

(3.12)

Since $2m - 1 \leq |\gamma_1| \leq J$, we estimate $\|\langle x \rangle^n D^{\gamma_1} u\|_{L^2} \leq \|u\|_X$. Hence, from (3.12) we deduce

$$\|\langle x \rangle^n B\|_{L^2} \leq (\eta\|u\|_X)^{2p} \|u\|_{X}^{\alpha + 1} \leq C(1 + \eta\|u\|_X)^{2J} \|u\|_{X}^{\alpha + 1}.$$  

Next, suppose that all the derivatives in the right-hand side of (3.10) are of order $\leq 2m - 2$. In this case, we obtain (3.11) again. Multiplying (3.11) by $\langle x \rangle^\alpha$ we get

$$\langle x \rangle^\alpha |B| \leq (\eta\|u\|_X)^{2p} \|u\|_{X}^{\alpha + 1}.$$  

(3.13)

We need to estimate the $L^2$ norm of the last inequality. By the definition of $\alpha$ in (1.2) we have $\alpha n > \frac{2m}{2}$. Then, $\langle x \rangle^{-\alpha n} \in L^2(\mathbb{R}^N)$. Thus, it follows from (3.13) that

$$\|\langle x \rangle^n B\|_{L^2} \leq C(1 + \eta\|u\|_X)^{2J} \|u\|_{X}^{\alpha + 1}.$$
Taking into account all the estimates, we obtain (3.6); and using (3.4), we deduce (3.2).

Let us now prove (3.3). We develop both $D^\beta(|u_1|^\alpha u_1)$ and $D^\beta(|u_2|^\alpha u_2)$ and use the expressions (3.8) and (3.9) to expand the difference $D^\beta(|u_1|^\alpha u_1) - D^\beta(|u_2|^\alpha u_2)$. On the one hand, due to (3.8), we get the term $|u_1|^\alpha D^\beta u_1 - |u_2|^\alpha D^\beta u_2$. We write this term as

$$\|u_1|^\alpha D^\beta u_1 - |u_2|^\alpha D^\beta u_2 = |u_1|^\alpha (D^\beta u_1 - D^\beta u_2) + (|u_1|^\alpha - |u_2|^\alpha)D^\beta u_2. \quad (3.14)$$

Similarly to the proof of (3.6), we separate the cases $|\beta| \leq 2m - 2$ and $2m - 1 \leq |\beta| \leq J$ and estimate the $L^\infty$ and $L^2$ norms of (3.14), respectively. We see that the first term in the right-hand side of (3.14) can be controlled by $\|u_1\|_{L^\infty}^\alpha \|u_1 - u_2\|_X$ and hence by the right-hand side of (3.7). In turn, the second term in the right-hand side of (3.14) is estimated by $\|u_1|^\alpha - |u_2|^\alpha\|_{L^\infty} \|u_2\|_X$. By (3.1)

$$|\|u_1|^\alpha - |u_2|^\alpha\| \leq C(|u_1|^{-1} + |u_2|^{-1})(|u_1| + |u_2|)^\alpha |u_1 - u_2|$$

$$\leq C\eta(x)\eta(|u_1| + |u_2|)^\alpha |u_1 - u_2|$$

$$\leq C\eta\|u_1\|_X + \|u_2\|_X)^\alpha \|u_1 - u_2\|_X,$$

and we obtain again a term which is controlled by the right-hand side of (3.7).

Let us now estimate the terms that correspond to the difference of terms of the form (3.9) for $u_1$ and $u_2$. Each of these terms can be written as

$$(|u_1|^{\alpha-2p} - |u_2|^{\alpha-2p})D^\beta u_1 \prod_{j=1}^p D^{\gamma_j, j} u_2 D^{\gamma_2, j} \bar{w}_2 \quad (3.15)$$

plus a sum of terms of the form

$$|u_1|^{\alpha-2p} D^\beta u_1 \prod_{j=1}^p D^{\gamma_j, j} w_{1,j} D^{\gamma_2, j} \bar{w}_{2,j} \quad (3.16)$$

where $w$, $w_{1,j}$, $w_{2,j}$ are all equal to either $u_1$ or $u_2$, except one of them which is equal to $u_1 - u_2$. The terms of the form (3.16) are controlled by the right-hand side of (3.7), by using (3.1). To estimate the term (3.15) we use that

$$|\|u_1|^{\alpha-2p} - |u_2|^{\alpha-2p}| \leq C(|u_1|^{-2p-1} + |u_2|^{-2p-1})(|u_1| + |u_2|)^\alpha |u_1 - u_2|$$

$$\leq C\eta(2p+1)(x)^{(2p+1)\alpha}(|u_1| + |u_2|)^\alpha |u_1 - u_2|$$

$$\leq C\eta(2p+1)(x)^{(2p+1)\alpha} \|u_1\|_X + \|u_2\|_X)^\alpha \|u_1 - u_2\|_X.$$

Then, proceeding as in the proof of (3.6), we control (3.16) by the right-hand side of (3.7). Thus, we see that (3.7) hold. Using (3.5) we get (3.3). This completes the proof of Proposition 3.1.

Now, we estimate

$$\mathcal{N}_1(\Psi) = \mu_3 |\Psi|^{\alpha} \Psi \quad (3.17)$$

in the space $\vec{X}$.

**Proposition 3.2.** Let $\alpha > 0$ and assume (1.2)-(1.3). Let $\vec{X}$ be defined by (1.4)-(1.5), (1.20) and (2.38), and let $\mathcal{N}_1(\Psi)$ be defined by (3.17). For every $\eta > 0$ and $\Psi \in \vec{X}$ satisfying

$$\eta \inf_{x \in \mathbb{R}^N} (x)^{\alpha} |\Psi(x)| \geq 1 \quad (3.18)$$
it follows that $N_1(\Psi) \in \tilde{X}$. Moreover, there exists a constant $C > 0$ such that
$$\|N_1(\Psi)\|_{\tilde{X}} \leq C(1 + \eta\|\Psi\|_{\tilde{X}})^{2J}\|\Psi\|_{\tilde{X}}^{\alpha+1},$$
and
$$\|N_1(\Psi_1) - N_1(\Psi_2)\|_{\tilde{X}} \leq C(1 + \eta(\|\Psi_1\|_{\tilde{X}} + \|\Psi_2\|_{\tilde{X}}))^{2J+1}(\|\Psi_1\|_{\tilde{X}} + \|\Psi_2\|_{\tilde{X}})^{\alpha}\|\Psi_1 - \Psi_2\|_{\tilde{X}},$$
for all $\eta > 0$ and $\Psi, \Psi_1, \Psi_2 \in \tilde{X}$ satisfying (3.18).

Proof. The proof of Proposition 3.1 uses only formulas (3.4) and (3.5). Since $N_1$ clearly satisfy these, the result follows. \[ \square \]

4. Proofs of Theorems 1.1 and 1.3. We are now in position to prove our main results. Theorem 1.1 will be consequence of the following existence result for the Cauchy problem
\[
\begin{aligned}
    \left\{ \begin{array}{l}
    i\partial_t u - \gamma (i\nabla) u = \mathcal{A}(u), \\
    u(0) = u_0,
    \end{array} \right.
\end{aligned}
\] (4.1)
where
$$\mathcal{A}(u) = \mathcal{M}(u) + \mathcal{N}(u),$$
which we study in the equivalent form (Duhamel’s formula)
$$u(t) = e^{-it\gamma (i\nabla)}u_0 - i \int_0^t e^{-i(t-s)\gamma (i\nabla)}\mathcal{A}(u) \, ds. \tag{4.2}$$

Proposition 4.1. Let $\alpha > 0$ and $\mu_2 \in \mathbb{C}$. Assume (1.2)-(1.3) and let the space $\mathcal{X} = \mathcal{X}_2$ be defined by (1.4)-(1.5) and (2.1). If $u_0 \in \mathcal{X}$ satisfies
$$\inf_{x \in \mathbb{R}^N} (x)^n|u_0(x)| > 0, \tag{4.3}$$
then there exist $T > 0$ and a unique solution $u \in C([-T, T], \mathcal{X})$ of (4.1). Moreover,
$$\inf_{-T \leq t \leq T} \inf_{x \in \mathbb{R}^N} (x)^n|u(t, x)| > 0. \tag{4.4}$$

Proof. We first prove uniqueness. Suppose $T > 0$ and $u_1, u_2 \in C([-T, T], \mathcal{X})$ are two solutions of (4.2). Using (1.13), (2.5) (with $n = 0$), and $\mathcal{X} \hookrightarrow H^J(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ we see that
$$\|\mathcal{N}(u_1) - \mathcal{N}(u_2)\|_{L^2} \leq C\|a \cdot u_1|a|u_1 - |a \cdot u_2|a \cdot u_2\|_{L^2} \leq C\|u_1 - u_2\|_{L^2},$$
and
$$\|\mathcal{M}(u)\|_{L^2} \leq C\|u\|_{L^2}.$$ 
Since $(e^{-it\gamma (i\nabla)})_{t \in \mathbb{R}}$ is a group of isometries on $L^2(\mathbb{R}^N)$, we deduce that
$$\|u_1(t) - u_2(t)\|_{L^2} \leq C\int_0^t \|u_1(s) - u_2(s)\|_{L^2} ds,$$
and uniqueness follows by Gronwall’s inequality.

Next, we use the linear estimates of Proposition 2.1 and the nonlinear estimates of Proposition 3.1 to prove the local existence result by a contraction mapping argument. We let
$$\eta > 0, \quad K > 0, \quad 0 < T \leq t_0,$$
where $t_0$ is given by Proposition 2.1. We define the set $\mathcal{E}$ by
$$\mathcal{E} = \{ u \in C([-T, T], \mathcal{X}); \|u\|_{L^\infty((-T, T), \mathcal{X})} \leq K \}
\quad \text{and} \quad \eta \inf_{x \in \mathbb{R}^N} (x)^n |u(t, x)| \geq 1 \text{ for } -T < t < T \},$$
so that $\mathcal{E}$ equipped with the distance $d(u,v) = \|u - v\|_{L^\infty((-T,T),X)}$ is a complete metric space. For given $u \in \mathcal{E}$ and $u_0 \in X$, we set

$$\Phi_u(t) = -i \int_0^t e^{-i(t-s)\gamma(i\nabla)} \mathcal{A}(u(s)) \, ds$$

and

$$\Psi_{u_0,u}(t) = e^{-it\gamma(i\nabla)} u_0 + \Phi_u(t)$$

for $-T < t < T$. By the definition of $\mathcal{E}$ and Proposition 3.1 we see that if $u \in \mathcal{E}$, then $\mathcal{N}(u) \in C([-T,T],\mathcal{E})$ and

$$\|\mathcal{N}(u)\|_{L^\infty((-T,T),X)} \leq C(1 + \eta K)^{2J K^{\alpha+1}}.$$  \hspace{1cm} (4.5)

moreover, by (2.5),

$$\mathcal{M} \in \mathcal{L}(\mathcal{X}),$$

so that

$$\|\mathcal{M}(u)\|_{L^\infty((-T,T),X)} \leq CK.$$  \hspace{1cm} (4.6)

In addition, it follows from Proposition 2.1 and the semigroup property that $\Phi_u \in C([-T,T],\mathcal{X})$. From (2.2), (4.5) and (4.7) we estimate

$$\|\Phi_u\|_{L^\infty((-T,T),X)} \leq CTK_{t_0} [K + (1 + \eta K)^{2J K^{\alpha+1}}]$$

and

$$\|\Psi_{u_0,u}\|_{L^\infty((-T,T),X)} \leq C K_{t_0} \left(\|u_0\|_X + TK + T (1 + \eta K)^{2J K^{\alpha+1}}\right),$$

where

$$K_{t_0} = (1 + t_0)^{2m+n+1}.$$  \hspace{1cm} (4.9)

Arguing similarly and using (3.3), we estimate

$$\|\Phi_v - \Phi_w\|_{L^\infty((-T,T),X)} \leq CTK_{t_0} [1 + (1 + \eta K)^{2J K^{\alpha+1}}] d(v, w),$$

for all $v, w \in \mathcal{E}$. Next, using (2.3), (4.8) and the inequality $\langle \phi \rangle |u(\cdot)| \leq \|u\|_X$, we see that

$$\langle x \rangle^n |\Psi_{u_0,u}(t, x) \rangle \geq \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u_0(x)\rangle - CTK_{t_0} \|u_0\|_X - \|\Phi_u\|_X$$

$$\geq \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u_0(x)\rangle - CTK_{t_0} (\|u_0\|_X + K + (1 + \eta K)^{2J K^{\alpha+1}});$$

Having all the necessary estimates, we now argue as follows. Let $u_0 \in X$ be such that $\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u_0(x)\rangle > 0$. We let

$$\eta = \frac{2(\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u_0(x)\rangle)^{-1}}{2}$$

$$C = \frac{2 \bar{C} K_{t_0} \|u_0\|_X.}$$  \hspace{1cm} (4.13)

where $\bar{C}$ is the supremum of the constants $C$ in (4.8)–(4.11). In particular we see that $u(t) = u_0$ belongs to $\mathcal{E}$, so that $\mathcal{E} \neq \emptyset$. We let $T \in (0, t_0]$ be sufficiently small so that

$$\bar{C} T K_{t_0} [1 + (1 + \eta K)^{2J K^{\alpha+1}}] \leq \frac{1}{2}$$

$$\bar{C} T K_{t_0} (\|u_0\|_X + K + (1 + \eta K)^{2J K^{\alpha+1}}) \leq \frac{1}{\eta}.$$  \hspace{1cm} (4.14)

Then, applying (4.9), (4.13) and (4.14) we obtain

$$\|\Psi_{u_0,u}\|_{L^\infty((-T,T),X)} \leq K.$$
Moreover, inequalities \((4.11), (4.12)\) and \((4.15)\) imply that
\[
\eta \inf_{x \in \mathbb{R}^N} \langle x \rangle^n \langle \Psi_{u_0,u}(t,x) \rangle \geq 1
\]  
(4.16)
for \(-T \leq t \leq T\). It follows that \(\Psi_{u_0,u} \in \mathcal{E}\) for all \(u \in \mathcal{E}\). Using \((4.10)\) and \((4.14)\) we deduce that the map \(u \mapsto \Psi_{u_0,u}\) is a strict contraction \(\mathcal{E} \to \mathcal{E}\). Therefore, it has a fixed point, which is a solution of \((4.2)\), and estimate \((4.4)\) follows from \((4.16)\).

This completes the proof. \(\square\)

**Proof of Theorem 1.1.** Consider the problem \((1.1)\). Suppose that the initial data are such that \(w_0 \in \mathcal{X}_1, \langle i\nabla \rangle^{-1}w_1 \in \mathcal{X}_1\) and \((1.7)\) holds. It follows that \(u_0\) defined by
\[
u_0 = \frac{1}{2}(w_0a + i[\langle i\nabla \rangle^{-1}w_1]b)
\]  
(4.17)
belongs to \(\mathcal{X}\). Moreover, using \((1.10)\),
\[
|u_0|^2 = 2(|w_0|^2 + |\langle i\nabla \rangle^{-1}w_1|^2)
\]
so that \(u_0\) satisfies \((4.3)\). It follows from Proposition 4.1 that there exist \(T > 0\) and a solution
\[
u \in C([-T,T], \mathcal{X})
\]  
(4.18)
of \((4.1)\). Since \(\mathcal{X} \to H^J(\mathbb{R}^N)\) by \((1.6)\) we have \(\gamma \langle i\nabla \rangle u \in C([-T,T], H^{J-1}(\mathbb{R}^N))\). Moreover, \(A(u) \in C([-T,T], \mathcal{X})\) by Proposition 3.1 and \((4.6)\). Equation \((4.1)\), yields \(u \in C^1([-T,T], H^{J-2}(\mathbb{R}^N))\), so that \(\gamma \langle i\nabla \rangle u \in C^1([-T,T], H^{J-2}(\mathbb{R}^N))\). In addition, one verifies easily that \(\mathcal{N}(u) \in C([-T,T], H^{J-1}(\mathbb{R}^N))\) and
\[
\partial_t \mathcal{N}(u) = -\mu_2 \left[ \langle i\nabla \rangle^{-1} \left( \left. \frac{\alpha + 1}{2} |a \cdot u|^2 a \cdot \partial_t u + \frac{\alpha}{2} |a \cdot u|^2 (a \cdot a) \partial_t \nabla \right) \right] b.
\]
Furthermore, \(\mathcal{M} \in L(H^{J-1}(\mathbb{R}^N))\) by \((2.5)\) (with \(n = 0\) and \(p = 2\)), so that \(\mathcal{M}(u) \in C^1([-T,T], H^{J-1}(\mathbb{R}^N))\). Using again equation \((4.1)\), we conclude that
\[
u \in C^2([-T,T], H^{J-2}(\mathbb{R}^N)).
\]

We now define
\[
\nu \in C([-T,T], \mathcal{X}_1) \cap C^2([-T,T], H^{J-2}(\mathbb{R}^N))
\]  
(4.19)
by
\[
\nu = a \cdot \nu.
\]  
(4.20)
Since \(J - 2 > \frac{N}{2}\), we have in particular \(\nu \in C^2([-T,T] \times \mathbb{R}^N)\). Next, note that
\[
\gamma^2 = I, \quad \gamma a = b, \quad a \cdot b = 0, \quad \gamma b = a, \quad a \cdot a = 2.
\]  
(4.21)
Using equation \((4.1)\) and \((4.21)\) we see that
\[
w_{tt} - \Delta w + w = a \cdot (u_{tt} - \Delta u + u)
\]
\[
= -a \cdot [(i\partial_t + \gamma \langle i\nabla \rangle)(i\partial_t u - \gamma \langle i\nabla \rangle u)]
\]
\[
= -a \cdot [(i\partial_t + \gamma \langle i\nabla \rangle)A(u)] = -a \cdot [\gamma \langle i\nabla \rangle A(u)]
\]
\[
= -\mu_1 \frac{1}{2} a \cdot [\gamma (w b)] + \mu_2 \frac{1}{2} a \cdot [\gamma(|w|^\alpha w b)]
\]
\[
= (1 - \mu_1) w + \mu_2 |w|^\alpha w.
\]
Moreover, by \((4.17)\) and \((4.21)\),
\[
w(0) = a \cdot u_0 = \frac{1}{2} a \cdot (w_0 a + i[\langle i\nabla \rangle^{-1}w_1]b) = w_0.
\]
Similarly, using (4.1), (4.17) and (4.21),
\[ w_t(0) = a \cdot \partial_t u(0) = -i a \cdot (\gamma \langle i \nabla \rangle_u_0 + A(u_0)) \]
\[ = -i a \cdot (\gamma \langle i \nabla \rangle_u_0) = -\frac{i}{2} a \cdot (\gamma [\langle i \nabla \rangle w_0 a + i w_1 b]) = w_1. \]
Thus we see that \( v \) solves (1.1). This proves the existence part.

Since \( X_1 \hookrightarrow L^\infty(\mathbb{R}^N) \), uniqueness easily follows from standard energy estimates. Finally, suppose that \( w_0 \) satisfies (1.8). Taking the scalar product of equation (4.1) with \( a \), integrating in time and using (4.21) and (4.20), we obtain
\[ w(t) = w_0 - i \int_0^t \langle i \nabla \rangle b \cdot u. \]

Then,
\[ \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |w(t)| \geq \inf_{x \in \mathbb{R}^N} \langle x \rangle^n w_0 - t \| \langle x \rangle^n \langle i \nabla \rangle u \|_{L^\infty((0,t) \times \mathbb{R}^N)} \]
\[ = \inf_{x \in \mathbb{R}^N} \langle x \rangle^n w_0 - t \| \langle x \rangle^n \langle i \nabla \rangle^{-1} (1 - \Delta) u \|_{L^\infty((0,t) \times \mathbb{R}^N)}. \]

By Lemma 2.3
\[ \| \langle x \rangle^n \langle i \nabla \rangle^{-1} (1 - \Delta) u \|_{L^\infty((0,t) \times \mathbb{R}^N)} \leq C \| \langle x \rangle^n (1 - \Delta) u \|_{L^\infty((0,t) \times \mathbb{R}^N)}. \]
Using (4.18) we see that there is \( 0 < T_1 \leq T \) such that (1.9) holds.

Finally, we complete the proof of Theorem 1.3.

**Proof of Theorem 1.3.** We use Duhamel’s formula to reformulate equation (1.19) in the equivalent form
\[ \Psi(t) = e^{-it H} \Psi_0 - i \int_0^t e^{-i(t-s)H} \mathcal{N}_1(\Psi(s)) \, ds, \]
where \( \mathcal{N}_1(\Psi) \) is given by (3.17). Theorem 1.3 now follows from a standard contraction mapping argument (exactly as in the proof of Proposition 4.1) based on the linear estimates of Proposition 2.9 and the nonlinear estimates of Proposition 3.2. □

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