Exact sum rules for inhomogeneous strings

Paolo Amore

Facultad de Ciencias, CUICBAS, Universidad de Colima,
Bernal Díaz del Castillo 340, Colima, Colima, Mexico

Abstract

We derive explicit expressions for the sum rules of the eigenvalues of inhomogeneous strings with arbitrary density and with different boundary conditions. We show that the sum rule of order $N$ may be obtained in terms of a diagrammatic expansion, with $(N-1)!/2$ independent diagrams. These sum rules are used to derive upper and lower bounds to the energy of the fundamental mode of an inhomogeneous string; we also show that it is possible to improve these approximations taking into account the asymptotic behaviour of the spectrum and applying the Shanks transformation to the sequence of approximations obtained to the different orders. We discuss three applications of these results.

Keywords: Helmholtz equation; inhomogeneous string; perturbation theory; collocation method

1. Introduction

In this paper we consider the problem of obtaining exact results for the spectral zeta functions of inhomogeneous strings at positive integer values:

$$Z(s) = \sum_{n=1}^{\infty} \frac{1}{E_n^s}, \quad s = 1, 2, \ldots$$

(1)

where $E_n$ are the eigenvalues of the string, obeying different boundary conditions at the extremities. Remarkably, the exact calculation of $Z(s)$ for integer values of $s$ does not require the explicit knowledge of the spectrum, as we have recently pointed out in Ref. [1]. However, the main focus of Ref. [1] was on the possibility of obtaining a perturbative expansion for $Z(s)$ for $s > d/2$ ($d$ is the number of dimensions) and use it to evaluate the Casimir energy of the system via its analytic continuation. Here we restrict our analysis to the one dimensional problem, aiming at obtaining an explicit expression for the sum rule of arbitrary integer order $n$. The extension of these results to higher dimensions is discussed in a companion paper [2].
As we have pointed out in [1] there is a good number of examples in the literature where sum rules have been obtained for different problems [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]: in particular Berry [5] and Crandall [10] have used the sum rules for the modes of certain two dimensional billiards and of one dimensional quantum problems respectively to obtain approximations to the fundamental mode of these systems. For the case of a quantum bouncer, Crandall has obtained the energy of the fundamental mode with a fractional error of $10^{-9}$, using the sum rule of order 30.

In this paper we show that it is possible to obtain an explicit expression for the sum rule of a string with arbitrary density, and provide a simple set of diagrammatic rules which allow one to write down the expression. The actual calculation can be conveniently carried out with the help of a computer and therefore sum rules of high order can be efficiently calculated. We show that very accurate approximations for the energy of the fundamental modes can be obtained using a sequence of sum rules. We discuss the application of these results in three examples.

The paper is organized as follows: in Section 2 we derive the general expression for the sum rule of order $n$ and state a set of diagrammatic rules for $Z(n)$; in Section 3 we discuss the application of the general results of Section 2 to three non–trivial problems; finally, in Section 4 we state the conclusions.

2. Exact sum rules

In this section we derive exact sum rules for inhomogeneous strings subject to different boundary conditions. We consider strings of length $a$ ($|x| \leq a/2$) and density $\Sigma(x)$ ($\Sigma(x) > 0$).

The Helmholtz equation for the inhomogeneous string is

$$-\frac{d^2}{dx^2} \psi_n(x) = E_n \Sigma(x) \psi_n(x),$$

(2)

where $E_n$ and $\psi_n(x)$ are the eigenvalues and eigenfunctions of the string. At the extremities of the string, $x = \pm a/2$, the eigenfunctions $\psi_n(x)$ fulfill Dirichlet-Dirichlet, Neumann-Neumann, Dirichlet-Neumann, Neumann-Dirichlet or periodic-periodic boundary conditions.

Defining $\phi_n(x) \equiv \sqrt{\Sigma(x)} \psi_n(x)$, as discussed in Ref. [1], eq. (2) becomes

$$\frac{1}{\sqrt{\Sigma(x)}} \left(-\frac{d^2}{dx^2}\right) \frac{1}{\sqrt{\Sigma(x)}} \phi_n(x) = E_n \phi_n(x).$$

(3)

Therefore $E_n$ and $\phi_n(x)$ are the eigenvalues and eigenfunctions of the hermitian operator $\hat{O} \equiv \frac{1}{\sqrt{\Sigma(x)}} \left(-\frac{d^2}{dx^2}\right) \frac{1}{\sqrt{\Sigma(x)}}$; the inverse operator is $\hat{O}^{-1} \equiv \sqrt{\Sigma(x)} \left(-\frac{d^2}{dx^2}\right)^{-1} \frac{1}{\sqrt{\Sigma(x)}}$. 

2
Alternatively we may define \( \xi_n(x) \equiv \left(-\frac{d^2}{dx^2}\right)^{1/2} \frac{1}{\sqrt{\Sigma(x)}} \phi_n(x) \), which fulfills the equation
\[
\left(-\frac{d^2}{dx^2}\right)^{1/2} \frac{1}{\Sigma(x)} \left(-\frac{d^2}{dx^2}\right)^{1/2} \xi_n(x) = E_n \xi_n(x) .
\] (4)

In this case \( E_n \) and \( \xi_n(x) \) are respectively the eigenvalues and eigenfunctions of the hermitian operator \( \hat{Q} \equiv \left(-\frac{d^2}{dx^2}\right)^{1/2} \frac{1}{\Sigma(x)} \left(-\frac{d^2}{dx^2}\right)^{1/2} \); its inverse operator is
\[
\hat{Q}^{-1} \equiv \left(-\frac{d^2}{dx^2}\right)^{-1/2} \Sigma(x) \left(-\frac{d^2}{dx^2}\right)^{-1/2} .
\]

Observe that
- the operators \( \hat{O} \) and \( \hat{Q} \) are isospectral and solving any of the eqs. (2), (3), or (4), is equivalent to solving the remaining two equations;
- the eigenvalues of \( \hat{O} \) and \( \hat{Q} \) grow quadratically in \( n \) for \( n \to \infty \), \( E_n \propto n^2 \);
- the spectrum of the inverse operators is bounded both from above and from below, \( 0 < \frac{1}{E_n} \leq \frac{1}{E_1} \), for \( n = 1, \ldots, \infty \);
- the trace of a hermitian operator is invariant under unitary transformations and therefore
\[
Z(s) \equiv Tr \left[ \hat{O}^{-1} \right]^s = \sum_n \frac{1}{E_n^s}
\] (5)
for \( s = 1, 2, \ldots \); because of this invariance the trace may be evaluated in any basis, in particular in the basis of an homogeneous string, \( |n\rangle \):
\[
Z(s) = \sum_n \langle n | \left[ \hat{O}^{-1} \right]^s |n\rangle
\] (6)
- because of the asymptotic behavior of \( E_n \) for \( n \to \infty \), the trace above is finite for \( s = 1, 2, \ldots \);

For practical purposes it is convenient to express Eq. (5) directly in terms of the Green’s functions of the negative 1D Laplacian, subject to different boundary conditions.

The one dimensional Green’s functions have the general form
\[
G(x, y) = G_-(x, y) \theta(y - x) + G_+(x, y) \theta(x - y) ,
\] (7)
where
\[
G_+(x, y) = G_-(x, y) , \quad G_+(y, x) = G_-(x, y) .
\]

In Appendix A we report the explicit expressions for the Green’s functions for different boundary conditions.
Using these Green’s functions we are able to write the spectral zeta function at $s = 1, 2, \ldots$ as:

$$Z(n) = \int_{-a/2}^{a/2} dx_1 \int_{-a/2}^{a/2} dx_2 \ldots \int_{-a/2}^{a/2} dx_{n-1} \int_{-a/2}^{a/2} dx_n \cdot G(x_1, x_2)G(x_2, x_3) \ldots G(x_{n-1}, x_n)G(x_n, x_1) \Sigma(x_1) \ldots \Sigma(x_n). \quad (8)$$

An equivalent expression for $Z(n)$ is easily obtained using the ”x-ordered” product of the Green’s functions:

$$Z(n) = \int_{-a/2}^{a/2} dx_1 \int_{-a/2}^{x_1} dx_2 \ldots \int_{-a/2}^{x_{n-1}} dx_n \cdot G(x_1, \ldots, x_n) \Sigma(x_1) \ldots \Sigma(x_n), \quad (9)$$

where we have defined the n-point Green’s function

$$G(x_1, \ldots, x_n) \equiv [G(x_1, x_2)G(x_2, x_3) \ldots G(x_{n-1}, x_n)G(x_n, x_1)]_p \quad (10)$$

and

$$[f(x_1, x_2, \ldots, x_n)]_p \equiv \sum_{\text{permutations}} f(x_{p_1}, \ldots, x_{p_n}).$$

For example:

$$[f(x_1, x_2)]_p \equiv f(x_1, x_2) + f(x_2, x_1)$$

$$[f(x_1, x_2, x_3)]_p \equiv f(x_1, x_2, x_3) + f(x_1, x_3, x_2) + f(x_2, x_1, x_3)$$

$$\quad + f(x_2, x_3, x_1) + f(x_3, x_1, x_2) + f(x_3, x_2, x_1).$$

We write explicitly $G$ up to order 5:

$$G(x_1) = G_+(x_1, x_1)$$

$$G(x_1, x_2) = 2 \left[G_+(x_1, x_2)\right]^2$$

$$G(x_1, x_2, x_3) = 6 G_+(x_1, x_2)G_+(x_1, x_3)G_+(x_2, x_3)$$

$$G(x_1, x_2, x_3, x_4) = 8 \left[G_+(x_1, x_2)G_+(x_1, x_4)G_+(x_2, x_3)G_+(x_3, x_4)\right] + G_+(x_1, x_3)G_+(x_1, x_4)G_+(x_2, x_3)G_+(x_2, x_4)$$

$$\quad + G_+(x_1, x_2)G_+(x_1, x_3)G_+(x_2, x_4)G_+(x_3, x_4)$$

$$G(x_1, x_2, x_3, x_4, x_5) = 10 \left[G_+(x_1, x_4)G_+(x_1, x_5)G_+(x_2, x_3)G_+(x_2, x_5)G_+(x_3, x_4)\right] + G_+(x_1, x_3)G_+(x_1, x_5)G_+(x_2, x_4)G_+(x_2, x_5)G_+(x_3, x_4)$$

$$\quad + G_+(x_1, x_2)G_+(x_1, x_5)G_+(x_2, x_4)G_+(x_3, x_5)G_+(x_3, x_4)$$

$$\quad + G_+(x_1, x_2)G_+(x_1, x_3)G_+(x_2, x_5)G_+(x_3, x_5)G_+(x_3, x_4)$$

$$\quad + G_+(x_1, x_2)G_+(x_1, x_3)G_+(x_2, x_4)G_+(x_4, x_5)G_+(x_3, x_4)$$

$$\quad + G_+(x_1, x_2)G_+(x_1, x_3)G_+(x_2, x_4)G_+(x_4, x_5)G_+(x_3, x_4)$$

$$\quad + G_+(x_1, x_4)G_+(x_1, x_5)G_+(x_2, x_3)G_+(x_2, x_4)G_+(x_3, x_5).$$
We can therefore calculate $Z(n)$ with the diagrammatic rules:

- Draw $n$ points $x_1, \ldots, x_n$ on a line;
- Connect each point to any two other points in all possible inequivalent ways excluding the disconnected diagrams and the diagrams corresponding to a cyclic permutation of the points;
- Associate a density $\Sigma(x_i)$ at each point $x_i$ ($i = 1, \ldots, n$);
- Associate a factor $G_+(x_i, x_j)$ to each line connecting $x_i$ to $x_j$ ($i < j$);
- Multiply the result by a factor $2^n$, corresponding to the $n$ cyclic permutations of each inequivalent configuration and to the 2 possible directions in which each diagram can be traveled;
- Integrate the expression obtained from the steps above over the internal points:

$$
\int_{-a/2}^{a/2} dx_1 \int_{-a/2}^{x_1} dx_2 \cdots \int_{-a/2}^{x_{n-2}} dx_{n-1} \int_{-a/2}^{x_{n-1}} dx_n
$$

It is easy to convince oneself that working to order $n$ there are $n!/2^n = (n - 1)!/2$ independent diagrams (for $n > 2$). For example in Fig. 1 we plot the diagrams for $G(x_1)$, $G(x_1, x_2)$ and $G(x_1, x_2, x_3)$; in Fig. 2 we plot the diagrams for $G(x_1, x_2, x_3, x_4)$: in this case there are $4!/8 = 3$ inequivalent diagrams.

We can now use any of the Eqs. (6), (8) or (9) to evaluate the sum rules for the string, although Eq. (9) is to be preferred for practical purposes, because of its simple diagrammatic representation.

---

1In Ref. [1] we have used Eq. (6) to evaluate explicitly the sum rules of certain strings.
It is particularly simple to evaluate $Z(1)$, which reads

$$Z^{(DD)}(1) = \int_{-a/2}^{+a/2} \left( \frac{a}{4} - \frac{x^2}{a} \right) \Sigma(x) \, dx$$  \hspace{1cm} (11)$$

$$Z^{(NN)}(1) = \int_{-a/2}^{+a/2} \left( \frac{a}{12} + \frac{x^2}{a} \right) \Sigma(x) \, dx$$  \hspace{1cm} (12)$$

$$Z^{(DN)}(1) = \int_{-a/2}^{+a/2} \left( \frac{a}{2} + x \right) \Sigma(x) \, dx$$  \hspace{1cm} (13)$$

$$Z^{(ND)}(1) = \int_{-a/2}^{+a/2} \left( \frac{a}{2} - x \right) \Sigma(x) \, dx$$  \hspace{1cm} (14)$$

$$Z^{(PP)}(1) = \int_{-a/2}^{+a/2} \frac{a}{12} \Sigma(x) \, dx .$$  \hspace{1cm} (15)$$

Therefore we obtain the interesting relations:

$$Z^{(DD+NN)}(1) = Z^{(DD)}(1) + Z^{(NN)}(1) = \frac{a^2}{3} \langle \Sigma \rangle$$  \hspace{1cm} (16)$$

$$Z^{(ON+ND)}(1) = Z^{(ON)}(1) + Z^{(ND)}(1) = a^2 \langle \Sigma \rangle$$  \hspace{1cm} (17)$$

$$Z^{(PP)}(1) = \frac{a^2}{12} \langle \Sigma \rangle ,$$  \hspace{1cm} (18)$$

where $\langle \Sigma \rangle \equiv \frac{1}{a} \int_{-a/2}^{+a/2} \Sigma(x) \, dx$ is the average density of the string. Apparently these simple relations have not been found previously.

The diagrammatic rules that we have enunciated allow one to obtain the spectral zeta functions of a string at large integer values using a computer; as we shall see soon, these sum rules are useful since they allow one to derive approximations to the energy of the ground state.

As early as 1776, Waring in his "Meditationes Analyticae" considered the roots $\alpha$, $\beta$, $\gamma$, ... of an equation (with $\alpha > \beta > \gamma > ...$) and discussed the convergence of $(\alpha^{2n} + \beta^{2n} + \gamma^{2n} + ...)^{1/2n}$ towards the largest root $\alpha^2$. This

\footnote{This is originally reported by Whittaker and Robinson in Ref. [19] and by Watson in Ref. [20]. This result appears at pag. 375 in the edition of 1785, freely available at google books [21].}
result may be expressed as

\[ E_1 = \lim_{s \to \infty} Z(s)^{-1/s}. \]

(19)

Little later, in 1781, Euler used the inequalities

\[ Z(s)^{-1/s} \leq E_1 \leq \frac{Z(s)}{Z(s + 1)}, \]

(20)
to approximate the lowest Bessel zero (see also [19, 20, 5]).

Berry [5] and Crandall [10] have used these formulas to obtain precise approximations to the energy of the ground state of certain systems: in particular, Berry in Ref. [5] has introduced a "semiclassical zeta approximation", which amounts to evaluate the energy of the ground state with the formula

\[ E_1 = \left[ Z(2) - Z_+^+(2) \right]^{-1/2}, \]

(21)

where, following Berry’s notation,

\[ Z(2) = \sum_{j=1}^{N} \frac{1}{E_j} + \sum_{j=N+1}^{\infty} \frac{1}{E_j} = Z_0^-(2) + Z_0^+(2). \]

(22)

Here \( Z_0^+(2) \) is estimated using a semiclassical approximation for the higher part of spectrum. In this way Berry has obtained the lowest energy of certain two dimensional billiards within a 1% error.

The approach followed by Berry is particularly convenient in our case, both because we have at our disposal explicit formulas for the spectral zeta functions at integer positive values, and because we know the first few terms of the asymptotic behavior of the spectrum of the string.

We may generalize Eq. (21) to

\[ E_1 = \left[ Z(q) - Z_1^+(q) \right]^{-1/q}, \]

(23)

where \( q = 1, 2, \ldots \); notice that since \( Z(q) > Z_1^+(q) \), one recovers Eq. (19) in the limit \( q \to \infty \), neglecting \( Z_1^+(q) \).

The procedure followed by Berry may also be applied to the calculation of the eigenvalues of the excited states: in this case the energy of the \( n^{th} \) state is obtained as

\[ E_n = \left[ Z(q - n + 1) - Z_n^+(q - n + 1) - \sum_{j=1}^{n-1} \frac{1}{E_j^{q-n+1}} \right]^{-1/(q-n+1)}. \]

(24)

---

3 The asymptotic behavior of the spectrum of an inhomogeneous string with Dirichlet boundary conditions is \( E_n \approx A_1 n^2 + A_2 + A_3/n^2 + \ldots \). The coefficients \( A_1 \) and \( A_2 \) may be obtained using the WKB method; in Ref. [22] we have discussed a method which allows one to obtain \( A_3 \) and higher order coefficients.
For example, the second eigenvalue is obtained using Eq. (23) and it reads:

\[
E_2 = \left[ Z(q-1) - Z_2^+(q-1) - \left( Z(q) - Z_1^+(q) \right)^{(q-1)/q} \right]^{-1/(q-1)}. \tag{25}
\]

Notice that as long as the \(Z_n^+(q-n+1)\) are evaluated exactly, these equations are also exact; for sufficiently large \(q\) (with \(q \gg n\), \(Z_n^+(q-n+1)\) may be estimated accurately using the asymptotic behavior of the spectrum.

In Appendix B we show that, for \(n \to \infty\), for any of the boundary conditions considered

\[
E_n \to \alpha \epsilon_n + \beta \\
\equiv \frac{a^2}{\sigma(a/2)^2} \epsilon_n + \frac{\int_{-a/2}^{a/2} \frac{4\Sigma(x)\Sigma''(x)-5\Sigma'(x)^2}{16\Sigma(x)^2} \, dx}{\int_{-a/2}^{a/2} \sqrt{\Sigma(x)} \, dx}, \tag{26}
\]

where \(\epsilon_n\) are the eigenvalues of a homogeneous string of length \(a\) and \(\sigma(x) \equiv \int_{-a/2}^{a/2} \sqrt{\Sigma(y)} \, dy\). Notice that, for the case of Dirichlet-Dirichlet boundary conditions, this formula provides the coefficients \(A_1\) and \(A_2\) of ref. \[22\].

Thus we are interested in calculating the semiclassical approximation to \(Z_n^+(s)\), which we call \(\tilde{Z}_n^+(s)\):

\[
\tilde{Z}_n^+(s) = \sum_{j=n+1}^{\infty} (\alpha \epsilon_j + \beta)^{-s} \tag{27}
\]

For instance, for \(s = 1\):

\[
\tilde{Z}_0^{(DD)+}(1) = \frac{a \coth \left( \frac{a \sqrt{\alpha}}{2 \sqrt{\beta}} \right)}{2 \sqrt{\alpha \beta}} - \frac{1}{2\beta}, \tag{28}
\]

\[
\tilde{Z}_0^{(NN)+}(1) = \frac{a \tanh \left( \frac{a \sqrt{\alpha}}{2 \sqrt{\beta}} \right)}{4 \sqrt{\alpha \beta}} + \frac{a \coth \left( \frac{a \sqrt{\alpha}}{2 \sqrt{\beta}} \right)}{4 \sqrt{\alpha \beta}} - \frac{1}{2\beta}, \tag{29}
\]

\[
\tilde{Z}_0^{(DN)+}(1) = \frac{a \tanh \left( \frac{a \sqrt{\alpha}}{2 \sqrt{\beta}} \right)}{2 \sqrt{\alpha \beta}}, \tag{30}
\]

\[
\tilde{Z}_0^{(PP)+}(1) = \frac{a \coth \left( \frac{a \sqrt{\alpha}}{2 \sqrt{\beta}} \right)}{4 \sqrt{\alpha \beta}} - \frac{1}{2\beta}. \tag{31}
\]

Explicit expressions for \(s = 2, 3, \ldots\) can be also obtained, although we do not report them here.

When we use the formulas above setting \(\beta = 0\) we obtain the interesting relations:

\[
\tilde{Z}_0^{(DD+NN)+}(1) = \frac{a^2}{3\alpha} = \frac{\sigma(a/2)^2}{3}, \tag{33}
\]

\[
\tilde{Z}_0^{(DD+NN)+}(1) = \frac{a^2}{3\alpha} = \frac{\sigma(a/2)^2}{3}, \tag{33}
\]

\[
\tilde{Z}_0^{(DD+NN)+}(1) = \frac{a^2}{3\alpha} = \frac{\sigma(a/2)^2}{3}, \tag{33}
\]

\[
\tilde{Z}_0^{(DD+NN)+}(1) = \frac{a^2}{3\alpha} = \frac{\sigma(a/2)^2}{3}, \tag{33}
\]
\[ \tilde{Z}_0^{(DN+ND)^+}(1) = \frac{a^2}{\alpha} = \frac{\sigma(a/2)^2}{12} \] (34)

\[ \tilde{Z}_0^{(PP)^+}(1) = \frac{a^2}{12\alpha} = \frac{\sigma(a/2)^2}{12} \] (35)

which have the same form of eqs. (16), (17), (18) apart from \( \sigma(a/2)^2 \leftrightarrow a^2(S) \).

We will now discuss a different method to obtain accurate approximations to the energy of the fundamental mode of an inhomogeneous string of arbitrary density. As we have seen, it is possible to obtain explicit expressions for the sum rules \( Z(n) \) corresponding to different boundary conditions: the only limitation to the calculation of \( Z(n) \) is the factorial growth of the number of terms in \( G \) contributing to \( Z(n) \) when \( n \) is large and the difficulty in performing analytically the integrations over the coordinates \( x_1, \ldots, x_n \).

If we assume that this complications may be overcome up to some \( N \), then one has a sequence of sum rules, \( Z(1), Z(2), \ldots, Z(N) \), which can be used to obtain a sequence of approximations to \( E_1 \), as explained before.

It is easy to convince oneself that the terms in this sequence converge exponentially to \( E_1 \): for example, we notice that for \( s \gg 1 \), we have

\[ Z(s)^{-1/s} \approx E_1 - \frac{E_1}{s} \left( \frac{E_1}{E_2} \right)^s + \ldots \] (36)

where we have only kept the leading correction. A similar behavior can also be inferred for the sequence of ratios \( Z(s)/Z(s+1) \):

\[ \frac{Z(s)}{Z(s+1)} \approx E_1 + E_1 \left( \frac{E_1}{E_2} \right)^s \frac{E_2 - E_1}{E_2} + \ldots \] (37)

Notice that the lower bound is more accurate than the upper bound.

Sequences with transient behavior of this kind can be efficiently extrapolated using the Shanks transformation \[23\]: in this way, starting with the sequence of values of \( Z(n)^{-1/n} \) one obtains a new sequence

\[ \frac{Z(n-1)^{-1/n} Z(n+1)^{-1/n} - Z(n)^{-1/n}}{Z(n-1)^{-1/n} Z(n+1)^{-1/n} - 2Z(n)^{-1/n}} \]

which converges more rapidly to \( E_1 \). The new sequence has \( N - 2 \) terms.

Notice that the Shanks transformation can be applied repeatedly, as long as the sequence at one’s disposal has at least three terms. Therefore, one can in principle obtain a large gain in precision by eliminating several transient behaviors from the original sequence. The advantage of this procedure is that we are dealing with exact sum rules and therefore we do not have to worry about round-off errors which would necessarily be present in a numerical calculation: moreover, the \( Z(n) \) are calculated explicitly as functions of the physical parameters in the problem and therefore the Shanks transformation will also provide an analytical expression in terms of the physical parameters.

We will see several applications of this method in the following section.
3. Applications

3.1. Isospectral strings

Isospectral strings are strings with different densities, but with the same spectrum. A well known example was discovered long time ago by Borg \[24\]; this string has a density

\[
\Sigma(x) = \frac{(1 + \alpha)^2}{(1 + \alpha(x + 1/2))^4}, \quad |x| \leq 1/2 ,
\]

with \(\alpha > -1\). Borg proved that for \(\alpha > -1\), all the strings have the same Dirichlet spectrum of a string of constant density, corresponding to \(\alpha = 0\).

Using the sum rules obtained before it is easy to see that these strings are only isospectral to the uniform string for Dirichlet boundary conditions. As a matter of fact already for \(s = 1\), the sum rule for Neumann-Neumann, Neumann-Dirichlet, Dirichlet-Neumann and periodic-periodic all depend on \(\alpha\):

\[
Z^{(NN)}(1) = \frac{\alpha(2\alpha + 3) + 3}{18(\alpha + 1)} \quad (39)
\]

\[
Z^{(DN)}(1) = \frac{\alpha + 3}{6\alpha + 6} \quad (40)
\]

\[
Z^{(ND)}(1) = \frac{1}{6}(2\alpha + 3) \quad (41)
\]

\[
Z^{(PP)}(1) = \frac{\alpha(\alpha + 3) + 3}{36(\alpha + 1)} \quad (42)
\]

This result can be better understood noticing that the average density of the string depends on \(\alpha\): \(\langle \Sigma \rangle = \frac{\alpha^2 + 3\alpha + 3}{2\alpha + 1}\).

Gottlieb has proved in Ref.\[25\] that, given a string of length \(a\) and with density \(\Sigma(x)\), the strings with density \(\tilde{\Sigma}(x) = \xi'(x)^2 \Sigma(\xi(x))\), are isospectral to the first string for Dirichlet bc. Notice that \(\xi(x)\) maps the interval \((-a/2, a/2)\) onto itself. The case discussed by Borg is a special case of eq.(43) and corresponds to \(\Sigma(x) = 1\).

It is now easy to check the isospectrality of the strings with Dirichlet bc: it is essential to notice that for the transformation of eq.(44)

\[
G^{(DD)}_+(x, y) = G^{(DD)}_+(\xi(x), \xi(y)) \sqrt{\xi'(x)\xi'(y)} .
\]

\[
G^{(DD)}_+(x, y) = G^{(DD)}_+(\xi(x), \xi(y)) \sqrt{\xi'(x)\xi'(y)} .
\]
Using this property we may express the spectral zeta function for the string with density eq. (43) at arbitrary integer values $n$ as

$$Z^{(DD)}_{\tilde{\Sigma}}(n) = \int_{-a/2}^{a/2} dx_1 \int_{-a/2}^{x_1} dx_2 \cdots \int_{-a/2}^{x_{n-1}} dx_{n-1} \int_{-a/2}^{x_{n-2}} dx_n \cdots$$

$$\cdot G^{(DD)}(x_1, \ldots, x_n) \xi'(x_1) \xi'(x_2) \cdots \xi'(x_{n-1}) \xi'(x_n) \Sigma(\xi(x_1)) \ldots \Sigma(\xi(x_n))$$

$$= \int_{-a/2}^{a/2} dx_1 \int_{-a/2}^{\xi_1} d\xi_2 \cdots \int_{-a/2}^{\xi_{n-2}} d\xi_{n-1} \int_{-a/2}^{\xi_{n-1}} d\xi_n$$

$$\cdot G^{(DD)}(\xi_1, \ldots, \xi_n) \Sigma(\xi_1) \ldots \Sigma(\xi_n) = Z^{(DD)}_{\Sigma}(n), \quad (46)$$

where we have used the notation $Z^{(DD)}_{\tilde{\Sigma}}(n)$ and $Z^{(DD)}_{\Sigma}(n)$ for the spectral zeta functions of the strings with Dirichlet bc and with density $\tilde{\Sigma}$ and $\Sigma$, respectively.

This result holds for arbitrary integer $n$ and arbitrary real $\alpha > -1$ and it is consistent with the isospectrality of the two strings.

It is straightforward to see that the Green’s functions corresponding to different boundary conditions do not obey the transformation (45) and therefore they are not isospectral.

Notice that starting from order 2 the sum rules for Neumann bc contain a non polynomial dependence on $\alpha$. For example $Z^{(NN)}(2)$ is

$$Z^{(NN)}(2) = \frac{1}{810\alpha^4(\alpha + 1)^2} \left[ 10\alpha^8 + 12\alpha^7 + 93\alpha^6 + 1422\alpha^5 + 6021\alpha^4 + 12420\alpha^3 + 14220\alpha^2 + 2160 \right]$$

$$- \frac{2(\alpha + 1)(\alpha + 2)(\alpha(\alpha + 2) + 2)}{3\alpha^5} \log(\alpha + 1)$$

We do not report here higher order sum rules because of their lengthy expressions.

In Fig 3 we plot the bounds for the energy of the fundamental mode of the Borg string with Neumann boundary conditions as function of the parameter $\alpha$. The shaded area is the allowed region. The bounds are obtained using the sum rules of order 3 and 4.

### 3.2. An exactly solvable string

We consider a string of density

$$\Sigma(x) = \frac{9}{12x + 10}, \quad |x| \leq 1/2,$$  

which belongs to a family of inhomogeneous strings first studied by Horgan and Chan [26]. The frequencies of these strings can be calculated with arbitrary precision, since they are solutions to a transcendental equation.

In Ref. [22] we have calculated the first 10000 Dirichlet eigenvalues of the string (47), each with a precision of 200 digits. Using these numerical results,
we have extracted the leading asymptotic behavior of the Dirichlet spectrum of this string

\[ E_n \approx \pi^2 n^2 + \frac{3}{8} - \frac{165}{512 \pi^2 n^2} + \frac{73179}{81920 \pi^4 n^4} - \frac{81997443}{14680064 \pi^6 n^6} + \ldots \]  

(48)

The first three coefficients of this expansion were also obtained analytically using a WKB-perturbation expansion, developed in Ref. [22]. We can now use these numerical and analytical results to test our sum rules; using Eq. (9) with the help of Mathematica we obtain the exact sum rules for Dirichlet boundary conditions:

\[
\begin{align*}
Z^{(DD)}(1) &= \frac{5}{8} - \frac{2}{3} \log(2) \\
Z^{(DD)}(2) &= -\frac{13}{64} + \frac{4}{9} \log^2(2) \\
Z^{(DD)}(3) &= -\frac{105}{1024} - \frac{8}{27} \log^3(2) + \frac{7}{24} \log(2) \\
Z^{(DD)}(4) &= \frac{131}{46080} + \frac{16}{81} \log^4(2) - \frac{7}{27} \log^2(2) + \frac{95}{864} \log(2) \\
Z^{(DD)}(5) &= \frac{9521}{589824} - \frac{32}{243} \log^5(2) + \frac{35}{162} \log^3(2) \\
&\quad - \frac{475}{5184} \log^2(2) - \frac{917}{27648} \log(2) \\
Z^{(DD)}(6) &= \frac{11466667}{2752512000} + \frac{64}{729} \log^6(2) - \frac{14}{81} \log^4(2) + \frac{95}{1296} \log^3(2)
\end{align*}
\]
The factor $\log(2)$ in these expressions is related to the average density of the string, $\langle \Sigma \rangle = \int_{-1/2}^{1/2} \Sigma(x) \, dx = \frac{3}{2} \log(2)$. Notice that the largest sum rule obtained here corresponds to a diagrammatic expansion involving 20160 inequivalent diagrams.

In Table 1 we report the error $Z^{(DD)}(q) - Z^{(num)}_{num}(q)$, where $Z^{(DD)}_{num}(q)$ is approximated with $\tilde{Z}^{(DD)}_{1}(q)$. The underlined digits are exact.

In Table 2 we report the estimates of $E^{(DD)}_1$ using Eqs. (19) and (23), where $Z^{(DD)}_1(q)$ is approximated with $\tilde{Z}^{(DD)}_1(q)$. The underlined digits are exact.

In Table 3 we report the estimates of $E^{(DD)}_1$ using repeated Shanks transformations of the sequence in the second column of Table 2 corresponding to Eq. (19).

In Table 4 we report the estimates of $E^{(DD)}_1$ using repeated Shanks transformations of the sequence in the third column of Table 2 corresponding to Eq. (23). The energy of the fundamental mode is obtained with 17 digits of precision.

3.3. A string with rapidly oscillating density

We consider a string with density

$$\Sigma(x) = 2 + \sin \left( \frac{2\pi(x + 1/2)}{\epsilon} \right), \quad (50)$$
Table 1: $Z^{(DD)}(q) - Z_{num}^{(DD)}(q)$.

| $q$ | $Z^{(DD)}(q) - Z_{num}^{(DD)}(q)$ |
|-----|-----------------------------------|
| 1   | $6.8 \cdot 10^{-50}$             |
| 2   | $1.2 \cdot 10^{-58}$             |
| 3   | $1.5 \cdot 10^{-67}$             |
| 4   | $1.8 \cdot 10^{-76}$             |
| 5   | $2.1 \cdot 10^{-85}$             |
| 6   | $2.3 \cdot 10^{-94}$             |
| 7   | $2.5 \cdot 10^{-103}$            |
| 8   | $2.6 \cdot 10^{-112}$            |
| 9   | $2.8 \cdot 10^{-121}$            |

Table 2: Estimates of $E_1^{(DD)}$ using Eq. (19) and Eq. (23).

| $q$ | $(Z^{(DD)}(q))^{-1/q}$ | $(Z^{(DD)}(q) - Z_1^{(DD)}(q))^{-1/q}$ |
|-----|-------------------------|----------------------------------------|
| 1   | 6.13866459              | 10.22002206                            |
| 2   | 9.80124983              | 10.21851148                            |
| 3   | 10.15503866             | 10.21820809                            |
| 4   | 10.20660399             | 10.21813692                            |
| 5   | 10.21580556             | 10.21819311                            |
| 6   | 10.21762510             | 10.21814861                            |
| 7   | 10.21806050             | 10.21811373                            |
| 8   | 10.21808942             | 10.21813444                            |
| 9   | 10.21810790             | 10.21811337                            |
Table 3: Estimates of $E_1^{(DD)}$ using repeated Shanks transformations of the sequence in the second column of Table 2 corresponding to Eq. (19).

| $S_1$        | $S_2$        | $S_3$        | $S_4$        |
|-------------|-------------|-------------|-------------|
| 10.19286707426 | 10.21809078335 | 10.21811465291 | 10.21811334408 |
| 10.21540206670 | 10.21810761972 | 10.21811333956 | -           |
| 10.21780418009 | 10.21811258058 | 10.21811334410 | -           |
| 10.21807358764 | 10.21811323885 | -            | -           |
| 10.21810765046 | 10.21811332959 | -            | -           |
| 10.21811245123 | -            | -            | -           |
| 10.21811319374 | -            | -            | -           |

Table 4: Estimates of $E_1^{(DD)}$ using repeated Shanks transformations of the sequence in the third column of Table 2 corresponding to Eq. (23).

| $S_1$        | $S_2$        | $S_3$        | $S_4$        |
|-------------|-------------|-------------|-------------|
| 10.2181318565099322 | 10.21811333603626791 | 10.218113346663210 | 10.2181133446659408 |
| 10.2181151161288641 | 10.21811333457026718 | 10.218113346659633 | -           |
| 10.2181135270306264 | 10.2181133447356183 | 10.218113346659411 | -           |
| 10.2181133642743735 | 10.2181133446706434 | -            | -           |
| 10.2181133468298224 | 10.2181133446662585 | -            | -           |
| 10.2181133449084580 | -            | -            | -           |
| 10.218113344933714 | -            | -            | -           |
with $\epsilon \to 0^+$ and $|x| \leq 1/2$. This example was studied by Castro and Zuazua in Ref. [27], obtaining the asymptotic behavior of the Dirichlet spectrum for $\epsilon \to 0^+$ using the WKB method. More recently, these results have been reproduced in Ref. [22], using an alternative approach, developed by the author: in particular, the energy of the fundamental mode, which had been calculated by Castro and Zuazua to order $\epsilon^4$ has been obtained in Ref. [22] to order $\epsilon^5$ and reads

$$E_1^{(DD)} \approx \frac{\pi^2}{2} - \frac{\pi^2}{64}\epsilon^2 + \frac{1}{4}\pi \sin^2 \left( \frac{\pi}{\epsilon} \right) \epsilon^3 - \frac{15\pi^2}{1024}\epsilon^4$$

$$+ \frac{\pi \left( 5 \sin \left( \frac{4\pi}{\epsilon} \right) - 116 \cos \left( \frac{4\pi}{\epsilon} \right) + 116 \right)}{1024}\epsilon^5 + O \left[ \epsilon^6 \right]. \quad (51)$$

In Fig. 4 we plot the upper and lower bounds obtained with the sum rules of order 4 and 5 for the energy of the fundamental mode of the string (50) with Dirichlet boundary conditions as function of $\epsilon$. In Fig. 5 we compare the exact asymptotic behavior of $E_1^{(DD)}$ for $\epsilon \ll 1$ of Eq. (51) with the approximation obtained using the Shanks transformation

$$S \equiv \frac{Z^{(DD)}(3)^{-1/3}Z^{(DD)}(5)^{-1/5} - Z^{(DD)}(4)^{-1/2}}{Z^{(DD)}(3)^{-1/3} + Z^{(DD)}(5)^{-1/5} - 2Z^{(DD)}(4)^{-1/4}}.$$ 

In particular for $\epsilon \to 0^+$ we have

$$S \approx 4.9347 - 0.1543\epsilon^2$$

$$+ \epsilon^3 \left( 0.3929 - 0.3929 \cos \left( \frac{2\pi}{\epsilon} \right) \right) - 0.1463\epsilon^4$$

$$+ \epsilon^5 \left( 0.0155 \sin \left( \frac{4\pi}{\epsilon} \right) - 0.3605 \cos \left( \frac{2\pi}{\epsilon} \right) + 0.3605 \right) \quad (52)$$

which should be compared with the exact asymptotic formula

$$E_1^{(DD)} \approx 4.9348 - 0.1542\epsilon^2$$

$$+ \epsilon^3 \left( 0.3927 - 0.3927 \cos \left( \frac{2\pi}{\epsilon} \right) \right) - 0.1446\epsilon^4$$

$$+ \epsilon^5 \left( 0.0153 \sin \left( \frac{4\pi}{\epsilon} \right) - 0.3559 \cos \left( \frac{2\pi}{\epsilon} \right) + 0.3559 \right) + \ldots \quad (53)$$

Notice that $S$ has the correct asymptotic behavior for $\epsilon \to 0^+$, with coefficients which approximate remarkably well the exact coefficients. Notice also that the lower bound $(Z^{(DD)}(5))^{-1/5}$ is more precise than the upper bound $Z^{(DD)}(4)/Z^{(DD)}(5)$. This behaviour is consistent with Eqs. (50) and (51).

In the case of Neumann boundary condition the exact asymptotic behavior for $\epsilon \to 0^+$ of the fundamental mode of the string (50) is not known. It is however straightforward to obtain rigorous upper and lower bounds for this energy, using the exact sum rules. In Fig. 6 we show the bounds obtained using $Z^{(NN)}(3)$ and $Z^{(NN)}(4)$, and the Shanks transformation obtained using using
Figure 4: Bounds for the energy of the fundamental mode of the string (50) with Dirichlet boundary conditions as function of $\epsilon$.

Figure 5: Comparison between Eq.(51) (solid line) and the Shanks transformation 
\[
\frac{Z^{(DD)}(3) - 1/3 - Z^{(DD)}(5) - 1/5 - Z^{(DD)}(4) - 1/2}{Z^{(DD)}(3) - 1/3 + Z^{(DD)}(5) - 1/5 - 2Z^{(DD)}(4) - 1/4}
\] (dashed line) as function of $\epsilon$. The lower and upper curves are the lower and upper bounds respectively.
Figure 6: Bounds for the energy of the fundamental mode of the string with Neumann boundary conditions as function of $\epsilon$.

$Z^{(NN)}(2), Z^{(NN)}(3)$ and $Z^{(NN)}(4)$. The shaded area corresponds to the allowed region.

If we expand the expression obtained with the Shanks transformation for $\epsilon \to 0^+$ we obtain

$$E^{(NN)}_1 \approx 4.9336 + \epsilon \left(0.7852 \cos \left(\frac{2\pi}{\epsilon}\right) - 0.7852\right)$$

$$+ \epsilon^2 \left(-0.3122 \cos \left(\frac{2\pi}{\epsilon}\right) + 0.01562 \cos \left(\frac{4\pi}{\epsilon}\right) - 0.1084\right) + \ldots (54)$$

Therefore we see that for Neumann boundary conditions the fundamental mode of the string is more sensible to the rapid oscillations of the density: as a matter of fact, $E^{(NN)}_1$ contains a term of order $\epsilon$. This term includes an oscillatory contribution $\cos \left(\frac{2\pi}{\epsilon}\right)$; in the case of Dirichlet boundary conditions the dependence on $\epsilon$ starts at order $\epsilon^2$, while the oscillatory contributions only start at order $\epsilon^3$. In other words, it is easier to observe the periodicity of the density looking at the Neumann rather than Dirichlet spectrum of the string.

4. Conclusions

We have obtained explicit expressions for the sum rules involving the eigenvalues of strings with arbitrary density for different boundary conditions and we have provided simple diagrammatic rules which allow one to obtain the expression corresponding to a given order. Despite the factorial growth of the number of diagrams to a given order, we have derived general expressions up to order 9, corresponding to 20160 diagrams, using Mathematica.
These sum rules can be used to obtain precise bounds on the lowest eigenvalue of the string. A more accurate determination of this eigenvalue can then be obtained taking into account the known asymptotic behaviour of the spectrum and by performing repeated Shanks transformations of the sequence of approximations. Since we deal with exact results, no numerical instability due to round-off errors is ever present. In this way we have been able to obtain the Dirichlet eigenvalues of a particular string, first discussed by Horgan and Chan, with 17 digits of precision.

For the case of the Borg string, we have proved that the sum rules reduce to the analogous sum rules for a homogeneous string only for Dirichlet boundary conditions. Therefore the Borg string is isospectral to the homogeneous string only in this case.

For the case of a string with rapidly oscillating density, we have used the sum rules to obtain bounds on the lowest eigenvalue: in the case of Dirichlet boundary conditions we have verified that the sum rule approximate very well the exact asymptotic behaviour of this eigenvalue for arbitrarily rapid oscillations of the density, providing the exact functional dependence on the physical parameter $\varepsilon$. We have then applied the same method to study the lowest eigenvalue for Neumann bc, for which the exact asymptotic result is not available, showing that it is more sensible to the oscillations of the density.

The extension of these results to higher dimensions is treated in a companion paper [2].

Acknowledgements

This research was supported by the Sistema Nacional de Investigadores (México).

Appendix A. Green’s functions

In this appendix we derive the explicit expressions for the Green’s functions of the negative laplacian in one dimension and with different boundary conditions.

We need to solve the equation

$$\frac{d^2}{dx^2} G(x, y) = \delta(x - y)$$

(A.1)

with $|x| \leq a/2$ and $|y| \leq a/2$.

Appendix A.1. Dirichlet boundary conditions

The eigenfunctions and eigenvalues of the negative 1D laplacian for Dirichlet boundary conditions are

$$\psi_n^{(DD)}(x) = \sqrt{\frac{2}{a}} \sin n\pi \left( \frac{x + a/2}{a} \right),$$

(A.2)

$$e_n^{(DD)} = \frac{n^2 \pi^2}{a^2}.$$  

(A.3)
The Green’s function is obtained as
\[
G^{(DD)}(x, y) = \sum_{n=1}^{\infty} \frac{\psi_n^{(DD)}(x)\psi_n^{(DD)}(y)}{\epsilon_n^{(DD)}}
\] (A.4)
and Eq.(A.1) follows from the completeness of the basis \( \{\psi_n^{(DD)}(x)\} \).

It is easy to see that
\[
G^{(DD)}(x, y) = \frac{(a - 2x)(a + 2y)}{4a} \theta(x - y) + \frac{(a + 2x)(a - 2y)}{4a} \theta(y - x)
= \frac{(a - 2 \max[x, y])(a + 2 \min[x, y])}{4a}
\] (A.5)

To verify this result we just need to check that, for \( x \neq y \),
\[
-\frac{d}{dx}G^{(DD)}(x, y) \bigg|_{x \rightarrow y^+} + \frac{d}{dx}G^{(DD)}(x, y) \bigg|_{x \rightarrow y^-} = 1
\] (A.6)
which follows from integrating Eq.(A.1) on an arbitrary interval containing \( y \).

Notice that
\[
G^{(DD)}(x, x) = \frac{a}{4} - \frac{x^2}{a}
\] (A.7)

Appendix A.2. Neumann boundary conditions

The eigenfunctions and eigenvalues of the negative laplacian for Neumann boundary conditions are
\[
\psi_{n,u}^{(NN)}(x) = \begin{cases} 
\sqrt{\frac{2}{a}} \cos \frac{2n\pi x}{a}, & n > 0, \ u = 1 \\
\sqrt{\frac{2}{a}} \sin \frac{(2n-1)\pi x}{a}, & n \geq 0, \ u = 2
\end{cases}
\]
\[
\epsilon_{n,u}^{(NN)} = \begin{cases} 
\frac{4n^2\pi^2}{a^2}, & u = 1 \\
\frac{4n^2\pi^2}{a^2}, & u = 2
\end{cases}
\] (A.8)

The Green’s function is obtained as
\[
G^{(NN)}(x, y) = \sum_{n=0}^{\infty} \frac{\psi_n^{(NN)}(x)\psi_n^{(NN)}(y)}{\epsilon_n^{(NN)}} + \sum_{n=1}^{\infty} \frac{\psi_n^{(NN)}(x)\psi_n^{(NN)}(y)}{\epsilon_n^{(NN)}}
\] (A.9)
and Eq.(A.1) follows from the completeness of the basis \( \{\psi_n^{(NN)}(x)\} \). Notice that this expression is formally divergent, because of the zero mode which is
present in the Neumann spectrum. However the eigenfunction corresponding to 
\( n = 0 \) and \( u = 1 \) is a constant.

The derivation of an explicit expression for \( G^{(NN)}(x, y) \) requires a careful
discussion because of the presence of the zero mode. Let us write

\[
G^{(NN)}(x, y) = G_0^{(NN)}(x, y) + \tilde{G}^{(NN)}(x, y)
\]

where

\[
G_0^{(NN)}(x, y) \equiv \frac{\psi_{\ell_{n_1}}^{(NN)}(x)\psi_{\ell_{n_1}}^{(NN)}(y)}{\epsilon_{n_1}^{(NN)}} \quad (A.11)
\]

\[
\tilde{G}^{(NN)}(x, y) \equiv \sum_{n=1}^{\infty} \frac{\psi_{\ell_{n_1}}^{(NN)}(x)\psi_{\ell_{n_1}}^{(NN)}(y)}{\epsilon_{n_1}^{(NN)}} + \sum_{n=1}^{\infty} \frac{\psi_{\ell_{n_2}}^{(NN)}(x)\psi_{\ell_{n_2}}^{(NN)}(y)}{\epsilon_{n_2}^{(NN)}} \quad (A.12)
\]

Now

\[
-\frac{d^2}{dx^2} \tilde{G}^{(NN)}(x, y) = \sum_{n=1}^{\infty} \left[ \psi_{\ell_{n_1}}^{(NN)}(x)\psi_{\ell_{n_1}}^{(NN)}(y) + \psi_{\ell_{n_2}}^{(NN)}(x)\psi_{\ell_{n_2}}^{(NN)}(y) \right]
\]

\[
= \delta(x - y) - \psi_{\ell_{n_1}}^{(NN)}(x)\psi_{\ell_{n_1}}^{(NN)}(y)
\]

\[
= \delta(x - y) - \frac{1}{a}. \quad (A.13)
\]

Therefore, for \( x \neq y \) one must have

\[
-\frac{d^2}{dx^2} \tilde{G}^{(NN)}(x, y) = \frac{1}{a}. \quad (A.14)
\]

To ensure that the rhs of Eq. (A.1) is obtained we also need to impose that

\[
-\frac{d}{dx} G^{(NN)}(x, y) \bigg|_{x \to y^+} + \frac{d}{dx} G^{(NN)}(x, y) \bigg|_{x \to y^-} = 1 \quad (A.15)
\]

However since \( G_0^{(NN)}(x, y) \) is a constant, the equation above may be cast
directly as

\[
-\frac{d}{dx} \tilde{G}^{(NN)}(x, y) \bigg|_{x \to y^+} + \frac{d}{dx} \tilde{G}^{(NN)}(x, y) \bigg|_{x \to y^-} = 1 \quad (A.16)
\]

It is easy to see that

\[
\tilde{G}^{(NN)}(x, y) = \left( \frac{a^2 + 6a(y - x) + 6(x^2 + y^2)}{12a} \right) \theta(x - y)
\]

\[
+ \left( \frac{a^2 + 6a(x - y) + 6(x^2 + y^2)}{12a} \right) \theta(y - x)
\]

\[
= \left( \frac{a^2 - 6a|x - y| + 6(x^2 + y^2)}{12a} \right) \quad (A.17)
\]

and

\[
\tilde{G}^{(NN)}(x, x) = \frac{a}{12} + \frac{x^2}{a} \quad (A.18)
\]
Appendix A.3. Mixed boundary conditions: Dirichlet-Neumann

The eigenfunctions and eigenvalues of the negative laplacian for Dirichlet-Neumann boundary conditions are

$$\psi_{n}^{(DN)}(x) = \psi_{2n-1}^{(DD)} \left( \frac{-x + a/2}{2} \right), \quad n \geq 1$$

$$\epsilon_{n}^{(DN)} = \frac{(2n-1)^2 \pi^2}{4a^2}$$  \hspace{1cm} (A.19)

The Green’s function is obtained as

$$G^{(DN)}(x, y) = \sum_{n=1}^{\infty} \frac{\psi_{n}^{(DN)}(x)\psi_{n}^{(DN)}(y)}{\epsilon_{n}^{(DN)}}$$

The derivation of the Green’s function for this case is completely analogous to what done before for Dirichlet boundary conditions and therefore we only report the results:

$$G^{(DN)}(x, y) = (x + a/2)\theta(y - x) + (y + a/2)\theta(x - y)$$

$$= (\min[x, y] + a/2) \quad (A.20)$$

and

$$G^{(DN)}(x, x) = x + \frac{a}{2} \quad (A.21)$$

Appendix A.4. Mixed boundary conditions: Neumann-Dirichlet

The eigenfunctions and eigenvalues of the negative laplacian for Neumann-Dirichlet boundary conditions are

$$\psi_{n}^{(ND)}(x) = \psi_{2n-1}^{(DD)} \left( \frac{x + a/2}{2} \right), \quad n \geq 1$$

$$\epsilon_{n}^{(ND)} = \frac{(2n-1)^2 \pi^2}{4a^2}$$  \hspace{1cm} (A.22)

The Green’s function is obtained as

$$G^{(ND)}(x, y) = \sum_{n=1}^{\infty} \frac{\psi_{n}^{(ND)}(x)\psi_{n}^{(ND)}(y)}{\epsilon_{n}^{(ND)}}$$

Once again we only report the result:

$$G^{(ND)}(x, y) = (-x + a/2)\theta(x - y) + (-y + a/2)\theta(y - x)$$

$$= (-\max[x, y] + a/2) \quad (A.23)$$

and

$$G^{(ND)}(x, x) = -x + \frac{a}{2} \quad (A.24)$$
Appendix A.5. Periodic boundary conditions

The eigenfunctions and eigenvalues of the negative laplacian for periodic boundary conditions are

\[ \psi_n^{(PP)}(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos \frac{2n\pi x}{a}, & n > 0, \ u = 1 \\ \sqrt{\frac{2}{a}} \sin \frac{2n\pi x}{a}, & n \geq 0, \ u = 2 \end{cases} \]

\[ \epsilon_n^{(PP)} = \frac{4n^2\pi^2}{a^2} \]

The Green’s function is obtained as

\[ G^{(PP)}(x,y) = \sum_{n=0}^{\infty} \frac{\psi_n^{(PP)}(x)\psi_n^{(PP)}(y)}{\epsilon_n^{(PP)}} + \sum_{n=1}^{\infty} \frac{\psi_n^{(PP)}(x)\psi_n^{(PP)}(y)}{\epsilon_n^{(PP)}} \]

In this case the same considerations done for the case of Neumann bc apply and we have

\[ \bar{G}^{(PP)}(x,y) = \frac{a^2 + 6a(x-y) + 6(x-y)^2}{12a} \theta(y-x) + \frac{a^2 + 6a(y-x) + 6(x-y)^2}{12a} \theta(x-y) \]

\[ = \frac{a^2 - 6a|x-y| + 6(x-y)^2}{12a} \]

and

\[ G^{(PP)}(x,x) = \frac{a}{12} \]

Appendix B. Asymptotic laws

We derive here the leading asymptotic behavior of the spectrum of an inhomogeneous string subject to any of the boundary conditions discussed in this paper. We follow the discussion of Ref. [22] and consider the operator

\[ \hat{O} = \frac{1}{\sqrt{\Sigma}} \left( -\frac{d^2}{dx^2} \right) \frac{1}{\sqrt{\Sigma}}, \]

whose spectrum coincides with the spectrum of a string of density \( \Sigma(x) \).

Let \( \epsilon_n \) and \( \psi_n(x) \) be the eigenvalues and eigenfunctions of the negative 1d laplacian:

\[ -\frac{d^2\psi_n}{dx^2} = \epsilon_n \psi_n(x). \]

Define:

\[ \Psi_n(x) = \sqrt{\frac{a}{\sigma(a/2)}} \Sigma(x)^{1/4} \psi_n \left( a \frac{\sigma(x)}{\sigma(a/2)} - \frac{a}{2} \right) \]

\[ \text{(B.1)} \]
where $\sigma(x) \equiv \int_{-a/2}^{x} \sqrt{\Sigma(y)} dy$. Notice that $\Psi_n(x)$ and $\psi_n(x)$ obey the same boundary conditions only for the Dirichlet case.

Then

$$\hat{O}\Psi_n(x) = \left[ \tilde{\epsilon}_n + \frac{4\Sigma(x)\Sigma''(x) - 5\Sigma'(x)^2}{16\Sigma(x)^3} \right] \Psi_n(x)$$

(B.2)

where $\tilde{\epsilon}_n = \frac{a^2}{\sigma(a/2)^2} \epsilon_n$ are the eigenvalues of a homogeneous string of length $\sigma(a/2)$.

Taking into account that $\tilde{\epsilon}_n \propto n^2$ for $n \to \infty$, we see that the $\Psi_n(x)$ tend to become eigenfunctions of $\hat{O}$ in this limit; we can therefore introduce the operator

$$\hat{P} \equiv \hat{O} - \frac{4\Sigma(x)\Sigma''(x) - 5\Sigma'(x)^2}{16\Sigma(x)^3},$$

(B.3)

such that

$$\hat{P}\Psi_n(x) = \tilde{\epsilon}_n \Psi_n(x),$$

(B.4)

and write

$$\hat{O} = \hat{P} + \frac{4\Sigma(x)\Sigma''(x) - 5\Sigma'(x)^2}{16\Sigma(x)^3},$$

(B.5)

We can now apply perturbation theory and calculate the corrections to the eigenvalues of the string, treating the second term in $\hat{O}$ as a perturbation:

$$E_n^{(0)} = \tilde{\epsilon}_n$$

(B.6)

$$E_n^{(1)} = \langle n | V | n \rangle = \int_{-a/2}^{a/2} \Psi_n^2(x) \frac{4\Sigma(x)\Sigma''(x) - 5\Sigma'(x)^2}{16\Sigma(x)^3} dx,$$

(B.7)

$$\ldots = \ldots$$

In the limit $n \to \infty$

$$E_n^{(1)} \to \frac{1}{\sigma(a/2)} \int_{-a/2}^{a/2} \frac{4\Sigma(x)\Sigma''(x) - 5\Sigma'(x)^2}{16\Sigma(x)^{5/2}} dx + O \left[ \frac{1}{n^2} \right].$$

(B.8)

We can thus read off the asymptotic coefficients $A_1$ and $A_2$ of the string; the calculation of the higher order coefficients can be performed in a similar way. For the reader interested in this calculation we refer to Ref.[23], where we have derived the analytical expression for $A_3$ (see Eq.(53)).

Therefore

$$E_n \to \tilde{\epsilon}_n + \frac{1}{\sigma(a/2)} \int_{-a/2}^{a/2} \frac{4\Sigma(x)\Sigma''(x) - 5\Sigma'(x)^2}{16\Sigma(x)^{5/2}} dx + O \left[ \frac{1}{n^2} \right] + \ldots$$

(B.9)
References

[1] P. Amore, J. Math. Phys. 53, 123519 (2012)

[2] P. Amore, "Exact sum rules for inhomogeneous drums", submitted to Annals of Physics (2013)

[3] A. Voros, Nucl. Phys. B 165, 209 (1980)

[4] F. Steiner, Phys. Lett. B 159, 397-402 (1985)

[5] M.V. Berry, J. Phys. A 19, 2281-2296 (1986)

[6] C. Itzykson, P. Moussa and J.M. Luck, J. Phys. A 19, L111-L115 (1986)

[7] F. Steiner, Fortschr. Phys. 35, 87-114 (1987)

[8] F. Steiner, Phys. Lett. B 188, 447-454 (1987)

[9] E. Elizalde, S. Leseduarte and A. Romeo, J. Phys. A 26, 2409-2419 (1993)

[10] R.E. Crandall, J. Phys. A 29, 6795-6816 (1996)

[11] A.A. Kvitsinsky, J. Phys. A 29, 6379-6393 (1996)

[12] A. Voros, J. Phys. A 32, 1301-1311 (1999)

[13] G.A. Mezincescu, J. Phys. A 33, 4911 (2000)

[14] C.M. Bender and Q. Wang, J. Phys. A 34, 3325 (2001)

[15] B. Dittmar, Math. Nachr. 237, 45-61 (2002)

[16] B. Dittmar, Journal d’Analyse Mathematique 95, 323 (2005)

[17] W. Arendt, R. Nittka, W. Peter and F. Steiner, "Weyl’s law: spectral properties of the Laplacian in Mathematics and Physics", in "Mathematical Analysis of Evolution, Information and Complexity", edited by W. Arendt et al., Wiley-VCH (2009), 1-71

[18] B. Dittmar and M. Hantke, Annales UMCS, Mathematica, 65(2), 29-44 (2011)

[19] E.T. Whittaker and G. Robinson, "The calculus of observations: a treatise on numerical mathematics", London, Glasgow, Blackie & Son (1944)

[20] G.N. Watson, "A treatise on the theory of Bessel functions", Cambridge University Press (1944)

[21] E. Waring, "Mediationes analyticae", Nicholson (1785)

[22] P. Amore, Annals of Physics 326, 23152355 (2011)
[23] C.M. Bender and S.A. Orszag, Advanced mathematical methods for scientists and engineers, McGraw-Hill (1978)

[24] G. Borg, Acta Math. 78, 1-96. (doi:10.1007/BF02421600) (1946)

[25] H.P.W. Gottlieb, Inverse Problems 18, 971-978 (2002)

[26] C.O. Horgan and A.M. Chan, Journal of Sound and Vibration 225, 503-513 (1999)

[27] C. Castro and E. Zuazua, SIAM J. Appl. Math. 60, 1205-1233 (2000)