Units and Augmentation Powers in Integral Group Rings

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Abstract
The augmentation powers in an integral group ring $\mathbb{Z}G$ induce a natural filtration of the unit group of $\mathbb{Z}G$ analogous to the filtration of the group $G$ given by its dimension series $\{D_n(G)\}_{n \geq 1}$. The purpose of the present article is to investigate this filtration, in particular, the triviality of its intersection.

Keywords: integral group rings, augmentation powers, unit group, lower central series, dimension series.

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1 Introduction

Given a group $G$, the powers $\Delta^n(G)$, $n \geq 1$, of the augmentation ideal $\Delta(G)$ of its integral group ring $\mathbb{Z}G$ induce a $\Delta$-adic filtration of $G$, namely, the one given by its dimension subgroups defined by setting

$$D_n(G) = G \cap (1 + \Delta^n(G)), \ n = 1, 2, 3, \ldots$$

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The dimension series of $G$ has been a subject of intensive research (see [Pas79, Gup87, MP09]). This filtration of a group $G$ (group of trivial units in $\mathbb{Z}G$), suggests its natural extension to the full unit group $\mathcal{V}(\mathbb{Z}G)$ of normalized units, i.e., the group of units of augmentation one in $\mathbb{Z}G$, by setting

$$\mathcal{V}_n(\mathbb{Z}G) = \mathcal{V}(\mathbb{Z}G) \cap (1 + \Delta^n(G)), \ n = 1, 2, 3 \ldots$$

It is easy to see that $\{\mathcal{V}_n(\mathbb{Z}G)\}_{n \geq 1}$ is a central series in $\mathcal{V}(\mathbb{Z}G)$ and consequently, for every $n \geq 1$, the $n^{th}$ term $\gamma_n(\mathcal{V}(\mathbb{Z}G))$ of the lower central series $\{\gamma_n(\mathcal{V}(\mathbb{Z}G))\}_{n=1}^{\infty}$ of $\mathcal{V}(\mathbb{Z}G)$ is contained in $\mathcal{V}_n(\mathbb{Z}G)$. Thus the triviality of the $\Delta$-adic residue of $\mathcal{V}(\mathbb{Z}G)$

$$\mathcal{V}_{\omega}(\mathbb{Z}G) := \cap_{n=1}^{\infty} \mathcal{V}_n(\mathbb{Z}G)$$

implies the residual nilpotence of $\mathcal{V}(\mathbb{Z}G)$, and so, in particular, motivates its further investigation.

The paper is structured as follows: We begin by collecting some results about the above stated filtration in Section 2. In Section 3, we take up the investigation of groups $G$ with $\mathcal{V}_{\omega}(\mathbb{Z}G) = \{1\}$. Section 4 deals with the groups having trivial $\mathcal{D}$-residue, i.e., with groups whose dimension series intersect in identity.

We now mention some of our main results.

If $G$ is a finite group, then the filtration $\{\mathcal{V}_n(\mathbb{Z}G)\}_{n \geq 1}$ terminates, i.e., $\mathcal{V}_n(\mathbb{Z}G) = \{1\}$, for some $n \geq 1$, if and only if

(i) $G$ is an abelian group of exponent 2, 3, 4 or 6; or
(ii) $G = K_8 \times E$, where $K_8$ denotes the quaternion group of order 8 and $E$ denotes an elementary abelian 2-group (Theorem 2). Further, $\mathcal{V}(\mathbb{Z}G)$ has trivial $\Delta$-adic residue if, and only if,

(i) $G$ is an abelian group of exponent 6; or
(ii) $G$ is a $p$-group (Theorem 3).

It is interesting to compare the constraints obtained on finite groups satisfying the specified conditions.

For an arbitrary group $G$, if $\mathcal{V}(\mathbb{Z}G)$ has trivial $\Delta$-adic residue, then $G$ cannot have an element of order $pq$ with primes $p < q$, except possibly when $(p, q) = (2, 3)$ (Theorem 4). Furthermore, if $G$ is a nilpotent group which is $\{2, 3\}$-torsion-free, then, for $\Delta$-adic residue of $\mathcal{V}(\mathbb{Z}G)$ to be trivial, the torsion subgroup $T$ of $G$ must satisfy one of the following conditions:

(i) $T = \{1\}$ i.e., $G$ is a torsion-free nilpotent group, or;
(ii) $T$ is a $p$-group with no element of infinite $p$-height (Theorem 5).
In addition, if $G$ is abelian, then it turns out that $\mathcal{V}(\mathbb{Z}G)$ possesses trivial $\Delta$-adic residue, if and only if $\mathcal{V}(\mathbb{Z}T)$ does (Theorem 10).

We briefly examine the class $\mathcal{C}$ of groups $G$ with $\mathcal{V}_\omega(\mathbb{Z}G) = \{1\}$, and prove that a group $G$ belongs to $\mathcal{C}$ if all its quotients $G/\gamma_n(G)$ do so (Theorem 14), and that this class is closed under discrimination (Theorem 15). Finally, we examine the groups $G$ which have the property that the dimension series $\{D_n,\mathbb{Q}(G)\}_{n \geq 1}$ over the rationals has non-trivial intersection while $\{D_n(G)\}_{n \geq 1}$, the one over integers, has trivial intersection (Theorem 16).

For basic results on units and augmentation powers in group rings, we refer the reader to [Seh78] and [Pas79].

2 The filtration $\mathcal{V}_n(\mathbb{Z}G)$

As mentioned in the introduction, the filtration $\{\mathcal{V}_n(\mathbb{Z}G)\}_{n=1}^\infty$ of the group $\mathcal{V}(\mathbb{Z}G)$ of the normalized units is given by

$$\mathcal{V}_n(\mathbb{Z}G) = \mathcal{V}(\mathbb{Z}G) \cap (1 + \Delta^n(G)), \ n = 1, 2, 3 \ldots$$

and for every $n \geq 1$,

$$\gamma_n(\mathcal{V}(\mathbb{Z}G)) \subseteq \mathcal{V}_n(\mathbb{Z}G),$$

where $\gamma_n(\mathcal{V}(\mathbb{Z}G))$ denotes the $n$th term of the lower central series $\{\gamma_n(\mathcal{V}(\mathbb{Z}G))\}_{n=1}^\infty$.

In this section, we collect some results about this filtration.

(a) It is well-known that, for every group $G$, the map

$$g \mapsto g - 1 + \Delta^2(G), \ g \in G,$$

induces an isomorphism $G/\gamma_2(G) \cong \Delta(G)/\Delta^2(G)$. This map extends to $\mathcal{V}(\mathbb{Z}G)$ to yield

$$\mathcal{V}(\mathbb{Z}G) = G\mathcal{V}_2(\mathbb{Z}G), \ \mathcal{V}(\mathbb{Z}G)/\mathcal{V}_2(\mathbb{Z}G) \cong G/\gamma_2(G) \cong \Delta(G)/\Delta^2(G). \ \ (1)$$

(b) In case $G$ is an abelian group, then

$$\mathcal{V}(\mathbb{Z}G) = G \oplus \mathcal{V}_2(\mathbb{Z}G), \ \ (2)$$

and $\mathcal{V}_2(\mathbb{Z}G)$ is torsion-free (see [Seh93], Theorem 45.1).
One of the generic constructions of units in $V(ZG)$ is that of bicyclic units. For $g, h \in G$, define

$$u_{g,h} := 1 + (g - 1)h\hat{g},$$

where $\hat{g} = 1 + g + \cdots + g^{n-1}$, $n$ being the order of $g$. The unit $u_{g,h}$ is called a bicyclic unit in $ZG$; it is trivial if, and only if, $h$ normalizes $\langle g \rangle$, and is of infinite order otherwise. Since $u_{g,h} = 1 + (g - 1)(h - 1)\hat{g} \equiv 1 \pmod{\Delta^2(G)}$, it follows that all bicyclic units $u_{g,h} \in V_2(ZG)$.

Moreover, if $g, h$ are of relatively prime orders, then

$$u_{g,h} \in V_\omega(ZG),$$

for, in that case, $(g - 1)(h - 1) \in \Delta^\omega(G) := \cap_{n=1}^\infty(\Delta^n(G))$.

Another generic construction of units in $V(ZG)$ is that of Bass units. Given $g \in G$ of order $n$ and positive integers $k, m$ such that $k^m \equiv 1 \pmod{n}$,

$$u_{k,m}(g) := (1 + g + \cdots + g^{k-1})^m + \frac{1 - k^m}{n}(1 + g + \cdots + g^{n-1}),$$

is a unit which is trivial, i.e., an element of $G$ if, and only if, $k \equiv \pm 1 \pmod{n}$.

We observe the following elementary but useful fact.

**Lemma 1.** If $G$ is a finite group, and $V_n(ZG) = \{1\}$ for some $n \geq 1$, then all units of $V(ZG)$ must be torsion; in particular, all Bass and bicyclic units must be trivial.

**Proof.** Let $u \in V(ZG)$. As $u \in 1 + \Delta(G)$ and $G$ is finite, therefore, for any $n \geq 1$, there exists $m_n \in \mathbb{N}$ such that $u^{m_n} \in 1 + \Delta^n(G)$, i.e., $u^{m_n} \in V_n(ZG) = \{1\}$. Hence $u$ is a torsion element. 

A well known result in the theory of units in group rings states that, for a finite group $G$, all central torsion units in $ZG$ must be trivial (see e.g. [Sch93], Corollary 1.7). Further, a group $G$, such that all central units in $ZG$ are trivial, is termed as a cut-group; this class of groups is currently a topic of active research ([BMP17], [Mah18], [Bacak18], [BCJM18], [BMP19], [Tre19]). In this article, we quite often use the characterization of abelian cut-groups; namely, an abelian group $G$ is a cut-group, if, and only if, its exponent divides 4 or 6.

We now give a classification of finite groups for which this filtration terminates with identity after a finite number of steps.
Theorem 2. Let $G$ be a finite group. Then $V_n(ZG) = \{1\}$ for some $n \geq 1$ if, and only if, either

(i) $G$ is an abelian cut-group; or

(ii) $G = K_8 \times E$, where $K_8$ denotes quaternion group of order 8 and $E$ denotes elementary abelian 2-group.

Proof. Let $G$ be a finite group such that $V_n(ZG) = \{1\}$, for some $n \geq 1$. Since $\gamma_n(V(ZG)) \subseteq V_n(ZG)$, we have that $V(ZG)$ is nilpotent. By classification of finite groups $G$ with nilpotent unit group $V(ZG)$ ([Mil76], Theorem 1), $G$ must be either an abelian group or $G = K_8 \times E$. Now, if $G$ is a finite abelian group which is not a cut-group, then there exists a non-trivial Bass unit in $V(ZG)$, which in view of Lemma 1, is not possible.

Conversely, if $G$ is an abelian cut-group, then $V_2(ZG) = \{1\}$ (see (2)) and if $G = K_8 \times E$, then by Berman-Higman theorem, we have $V(ZG) = G$ implying $D_n(G) = V_n(ZG)$, $n \geq 1$. Moreover, in this case $G$ is a nilpotent group of class 2, i.e., $\gamma_3(G) = \{1\}$. Since $D_3(G) = \gamma_3(G)$ (see [Pas79], Theorem 5.10, p. 66), we obtain $V_3(ZG) = \{1\}$. 

3 Groups $G$ with $\Delta$-adic residue of $V(ZG)$ trivial

The triviality of $\Delta$-adic residue of $V(ZG)$ naturally restricts the structure of the group $G$, and consequently also of its subgroups, it being a subgroup closed property. In this section, we explore the structure of groups with this property.

Before proceeding further, observe that if, for a group $G$, $\Delta^\omega(G) = \{0\}$, then trivially $V_\omega(ZG) = \{1\}$. A characterization for a group $G$ to have the property $\Delta^\omega(G) = \{0\}$ is known; we recall it for the convenience of reader. To this end, we need few definitions, which we give next.

A group $G$ is said to be discriminated by a class $C$ of groups if, for every finite subset $g_1, g_2, \ldots, g_n$ of distinct elements of $G$, there exists a group $H \in C$ and a homomorphism $\varphi : G \to H$, such that $\varphi(g_i) \neq \varphi(g_j)$ for $i \neq j$. For a class of groups $C$, a group $G$ is said to be residually in $C$ if it satisfies:

For every $1 \neq x \in G$, there exists a normal subgroup $N_x$ of $G$ such that $x \notin N_x$ and $G/N_x \in C$.

Clearly, if $G \in C$, then $G$ is residually in $C$. Note that if a class $C$ is closed under subgroups and finite direct sums, then to say that $G$ is residually in $C$ is equivalent to saying that $G$ is discriminated by the class $C$. 

5
Theorem 3. ([Lic77], see also [Pas79], Theorem 2.30, p. 92) For a group $G$, $\Delta^\omega(G) = \{0\}$ if, and only if, either $G$ is residually ‘torsion-free nilpotent’, or $G$ is discriminated by the class of nilpotent $p_i$-groups, $i \in I$, of bounded exponents, where $\{p_i, i \in I\}$ is some set of primes.

We now characterise the finite groups $G$ for which $\mathcal{V}(\mathbb{Z}G)$ has trivial $\Delta$-adic residue. For this, we first prove the following:

**Lemma 4.** Let $G, H$ be finite groups of relatively prime orders. If $\mathbb{Z}G$ has a non-zero nilpotent element, then $\mathcal{V}(\mathbb{Z}[G \oplus H])$ does not have trivial $\Delta$-adic residue.

**Proof.** Let $G, H$ be finite groups of relatively prime order and let $\alpha(\neq 0)$ be a nilpotent element in $\mathbb{Z}G$, so that $\alpha^k = 0$, for some $k \in \mathbb{N}$ and hence $\alpha \in \Delta(G)$. Further, if $h(\neq 1) \in H$, then $h - 1 \in \Delta(H)$. By assumption on the orders of $G$ and $H$, we obtain $\alpha(h - 1) \in \Delta^\omega(G \oplus H)$. Note that as $\alpha$ is nilpotent and as $\alpha$ and $h - 1$ commute, $\alpha(h - 1)$ is a non-zero nilpotent element in $\mathbb{Z}[G \oplus H]$. Consequently, $1 + \alpha(h - 1)$ is a non-trivial unit in $\mathcal{V}_\omega(\mathbb{Z}[G \oplus H])$. □

**Theorem 5.** Let $G$ be a finite group. Then $\mathcal{V}_\omega(\mathbb{Z}G) = \{1\}$ if, and only if, either

(i) $G$ is an abelian group of exponent 6, or;

(ii) $G$ is a $p$-group.

**Proof.** Let $G$ be a finite group with $\mathcal{V}_\omega(\mathbb{Z}G) = \{1\}$. If $G$ is not a $p$-group, then let $z \in G$ be an element of order $pq$, with $p, q$ primes, $p < q$. The cyclic subgroup $H := \langle z \rangle$ of $G$ satisfies $\mathcal{V}_\omega(\mathbb{Z}H) = \{1\}$. On expressing $z$ as $z = xy$ with elements $x, y$ of orders $p$ and $q$ respectively, we have an exact sequence

$$1 \to \mathcal{V}_\omega(\mathbb{Z}H) \to \mathcal{V}(\mathbb{Z}H) \to \mathcal{V}(\mathbb{Z}\langle x \rangle) \oplus \mathcal{V}(\mathbb{Z}\langle y \rangle) \to 1,$$

induced by the natural projections $H \to \langle x \rangle, \ H \to \langle y \rangle$. Recall that for a finite abelian group $A$, the torsion-free rank $\rho(\mathcal{V}(\mathbb{Z}A))$ of the unit group $\mathcal{V}(\mathbb{Z}A)$ is given by the following formula (see [Seh78], Theorem 3.1, p. 54):

$$\rho(\mathcal{V}(\mathbb{Z}A)) = \frac{1}{2} \{ |A| + n_2 - 2c_A + 1 \},$$

where $n_2$ is the number of elements of order 2 in $A$ and $c_A$ is the number of cyclic subgroups in $A$. Thus,

$$\rho(\mathcal{V}(\mathbb{Z}(x))) = \begin{cases} 0, & \text{if } p = 2, \\ \frac{p-3}{2}, & \text{otherwise} \end{cases}, \quad \rho(\mathcal{V}(\mathbb{Z}(y))) = \frac{q-3}{2}.$$
and

\[
\rho(\mathcal{V}(\mathbb{Z}H)) = \begin{cases} 
q - 3, & \text{if } p = 2, \\
\frac{pq - 7}{2}, & \text{otherwise}.
\end{cases}
\]

Therefore,

\[
\rho(\mathcal{V}(\mathbb{Z}H)) > \rho(\mathcal{V}(\mathbb{Z}\langle x \rangle)) + \rho(\mathcal{V}(\mathbb{Z}\langle y \rangle)),
\]

except possibly when \( p = 2 \) and \( q = 3 \), implying that \( G \) must be a \((2,3)\)-group.

Moreover, as \( \mathcal{V}(\mathbb{Z}G) \) has trivial \( \Delta \)-adic residue, it follows that it is residually nilpotent. Hence, by [MWS2], \( G \) is a nilpotent group with its commutator subgroup a \( p \)-group. Thus, if the exponent of \( G \) is not 6, then either \( G \) (i) has an element \( x \) of order 4 and an element \( y \) of order 3; or (ii) has an element \( x \) of order 2 and an element \( y \) of order 9. In both the cases, rank considerations, as above yield \( \mathcal{V}_\omega(\mathbb{Z}G) \neq \{1\} \). Consequently, \( G = E \oplus S \), where \( E \) is an elementary abelian 2-group and \( S \) is a group of exponent 3. As we assume \( \mathcal{V}_\omega(\mathbb{Z}G) = \{1\} \), it follows from Lemma [4] that neither \( \mathbb{Z}E \) nor \( \mathbb{Z}S \) can have non-zero nilpotent element. By the classification of finite groups whose integral group rings do not have non-zero nilpotent elements [Sch75], \( S \) must be abelian, i.e., \( G \) is an abelian group of exponent 6.

Conversely, if \( G \) is an abelian group of exponent 6, then \( \mathcal{V}(\mathbb{Z}G) = G \) and therefore \( \mathcal{V}_\omega(\mathbb{Z}G) = \{1\} \). Also, if \( G \) is a \( p \)-group, then by Theorem [3] \( \Delta^\omega(G) = \{0\} \), and hence \( \mathcal{V}_\omega(\mathbb{Z}G) = \{1\} \).

As already observed in [3], the existence of non-trivial bicyclic units \( u_{g,h} \in \mathbb{Z}G \), with the elements \( g, h \in G \) of relatively prime orders, implies that the \( \Delta \)-adic residue of \( \mathcal{V}(\mathbb{Z}G) \) is non-trivial. Thus, if \( G \) is a group with \( \mathcal{V}_\omega(\mathbb{Z}G) = \{1\} \), then either \( G \) does not have elements of relatively prime orders, or every bicyclic unit \( u_{g,h} \), with the elements \( g, h \) of relatively prime orders, is trivial.

It may be noted that the triviality of the \( \Delta \)-adic residue of \( \mathcal{V}(\mathbb{Z}G) \), for an arbitrary (not necessarily finite) group \( G \), has an impact on its torsion elements. Following the arguments of Theorem [3], we record the following result which brings out constraints on torsion elements of groups with the property under consideration.

**Theorem 6.** If \( G \) is a group with \( \Delta \)-adic residue of \( \mathcal{V}(\mathbb{Z}G) \) trivial, then \( G \) cannot have an element of order \( pq \) with primes \( p < q \), except possibly when \((p, q) = (2, 3)\); in particular, if the group \( G \) is either 2-torsion-free or 3-torsion-free, then every torsion element of \( G \) has prime-power order.
Recall that an element $g$ of a group $G$ is said to have infinite $p$-height in $G$ if, for every choice of natural numbers $i$ and $j$, there exist elements $x \in G$ and $y \in \gamma_j(G)$ such that $x^p = gy$. The set of elements of infinite $p$-height in $G$ forms a normal subgroup; we denote it by $G(p)$.

**Theorem 7.** Let $G$ be a nilpotent group with $V_\omega(ZG) = \{1\}$, and let $T$ be its torsion subgroup. Then one of the following statements holds:

(i) $T = \{1\}$;

(ii) $T$ is a $(2,3)$ group of exponent 6;

(iii) $T$ is a $p$-group, $T(p) \neq T$, and $T(p)$ is an abelian $p$-group of exponent at most 4.

In particular, if $G$ is a nilpotent group with its torsion subgroup $\{2,3\}$-torsion-free, then, $V(ZG)$ has trivial $\Delta$-adic residue only if either $G$ is a torsion-free group or its torsion subgroup is a $p$-group which has no element of infinite $p$-height.

**Proof.** Let $G$ be a nilpotent group with $V_\omega(ZG) = \{1\}$, and let $T$ be its torsion subgroup. If $G$ is not torsion-free, then, in view of Theorem 6, either $T$ is a $p$-group, or must be a $(2,3)$ group. Moreover, in the latter case, arguing as in Theorem 5, it follows that $T$ must be a $(2,3)$ group of exponent 6.

Suppose $T (\neq \{1\})$ is a $p$-group, $T(p) \neq \{1\}$ and let $x, y \in T(p)$. Then ([Pas79], Theorem 2.3, p. 97)

$$ (x - 1)(y - 1) \in \Delta^\omega(T). $$

Consequently,

$$ V_2(Z[T(p)]) \subseteq V_\omega(ZT). $$

Since $V_\omega(ZT) = \{1\}$, we have

$$ V_2(Z[T(p)]) = \{1\}. $$

Therefore, by (1),

$$ V(Z[T(p)]) = T(p). $$

Moreover, by ([Mal49], Theorem 1), $T(p)$ is a central subgroup of $T$. Thus $T(p)$ is a $p$-group which is an abelian cut-group and thus, its exponent is 2, 3 or 4 and $T \neq T(p)$.

**Remark 8.** Towards converse of the above result, it may be mentioned that if $G$ is a nilpotent group which is either (i) torsion-free, or (ii) finitely generated having no $p'$-element for some prime $p$, or (iii) $G(p) = \{1\}$, then $\Delta^\omega(G) = \{0\}$, and hence $V_\omega(ZG) = \{1\}$.  

8
Remark 9. The triviality of the $\Delta$-adic residue $V_\omega(ZG)$ does not, in general, imply the triviality of $\Delta^\omega(G)$. For instance, observe that 
\[ V(Z[C_2 \oplus Z]) = C_2 \oplus Z \quad (\text{[Sch78], p. 57}, \] 
and therefore, 
\[ V(Z[C_2 \oplus Q]) = C_2 \oplus Q \cong V(ZC_2) \oplus V(ZQ), \] 
where $Z$ is an infinite cyclic group and $Q$ is the additive group of rationals. Consequently, 
\[ V_\omega(Z[C_2 \oplus Q]) = \{1\}. \] 
On the other hand, it is easy to see that $\Delta^\omega(C_2 \oplus Q) \neq \{0\}$. 

We next proceed to analyse the case of abelian groups.

Theorem 10. Let $G$ be an abelian group and let $T$ be its torsion subgroup. Then, 
\[ V_\omega(ZG) = \{1\} \text{ if, and only if, } V_\omega(ZT) = \{1\}. \]

Proof. Let $G$ be an abelian group and let $T$ be its torsion subgroup. Clearly, if 
\[ V_\omega(ZG) = \{1\}, \] 
then 
\[ V_\omega(ZT) = \{1\}. \]

For the converse, let $u \in V_\omega(ZG)$. Then, as $V(ZG) = V(ZT)G$ (\text{[Sch78], p. 56}), 
we have that $u = vg$, with $v \in V(ZT)$, $g \in G$. Since $V_\omega(Z[G/T]) = \{1\}$, $G/T$ 
being torsion-free, projecting $u$ onto $G/T$ yields $g \in T$, and hence $u \in V(ZT)$. The 
result now follows from the fact that, 
\[ V(ZT) \cap V_\omega(ZG) = V_\omega(ZT). \]

This is because 
\[ V(ZT) \cap V_n(ZG) = V_n(ZT), \text{ for all } n \geq 1. \]

Note that for the last assertion, it is enough to prove for finitely generated abelian 
groups, and there it follows from the fact that the group splits over its torsion 
subgroup. \hfill \box

Theorems 7 and 10 yield the following:

Corollary 11. For an abelian group $G$ which is $\{2, 3\}$-torsion-free, $V(ZG)$ has 
trivial $\Delta$-adic residue if, and only if, it is either torsion-free or its torsion subgroup 
is a $p$-group which has no element of infinite $p$-height.

Remark 12. If $G$ is a nilpotent group and $T$ is the torsion subgroup of $G$ such 
that idempotents in $QT$ are central in $QG$, then 
\[ V(ZG) = V(ZT)G. \]

The above requirement on idempotents holds true, for instance, if $QG$ has no non- 
zero nilpotent elements (\text{[Sch78], p. 194}).
Proposition 13. Let $G$ be a nilpotent group and $T$ its torsion subgroup. If $G/T$ is finitely generated, then

$$V_\omega(ZG) \cap ZT = V_\omega(ZT).$$

Proof. Let $G$ be a nilpotent group of class $c$, $Z$ an infinite cyclic group, and

$$1 \to H \to G \to Z \to 1$$

a split exact sequence, so that

$$G = HZ, \ H \triangleleft G, \ H \cap Z = \{1\}.$$

Regard the integral group ring $ZH$ as a left $ZG$-module with $Z$ acting on $H$ by conjugation and $H$ acting on $ZH$ by left multiplication. Then, we have by Swan’s Lemma (see [Pas79], Theorem 2.3, p. 79),

$$\Delta^{m^c}(G)ZH \subseteq \Delta^{m}(H), \text{ for all } m \geq 1.$$

Now if $u = n_1g_1 + n_2g_2 + \ldots n_rg_r \in ZG$ is an element of augmentation 1, and $g_i = h_iz_i$, with $h_i \in H$, $z_i \in Z$, then we have

$$u^{-1} = \sum n_i(h_i - 1) \in \Delta(H).$$

Consequently, if $u \in V_\omega(ZG)$, then

$$u^{-1} \in \Delta^\omega(H).$$

It thus follows that we have

$$V_\omega(ZG) \cap ZH = V_\omega(ZH).$$

Next, let $T$ be torsion subgroup of $G$. If $G/T$ is finitely generated, then we have a series

$$T \triangleleft H_1 \triangleleft H_2 \triangleleft \ldots \triangleleft H_n = G$$

with $H_{i+1}/H_i$ infinite cyclic, $1 \leq i \leq n - 1$. Induction yields the desired result, i.e.,

$$V_\omega(ZG) \cap ZT = V_\omega(ZT).$$

We next shift our analysis to the quotients of a residually nilpotent group by elements of lower central series.
Theorem 14. Let $G$ be a residually nilpotent group, and let $\{\gamma_n(G)\}_{n \geq 1}$ be its lower central series. If $V_\omega(Z[G/\gamma_n(G)]) = \{1\}$ for all $n \geq 1$, then $V_\omega(ZG) = \{1\}$.

Proof. Let $\pi_n : G \to G/\gamma_n(G)$, $n \geq 1$, be the natural projection. Extend $\pi_n$ to the group ring $ZG$ by linearity and let the extended map still be denoted by $\pi_n$. Let $u = \sum n_i g_i$ be an element of $V_\omega(ZG)$ with $n_i \in \mathbb{Z}$ and all $g_i \in G$ distinct. By hypothesis, $\pi_n(u) = \overline{1}$ for all $n \geq 1$. Choose $m \geq 1$ such that $g_i^{-1} g_j \notin \gamma_m(G)$ for $i \neq j$. Since $\pi_m(u) = \overline{1}$, it follows that $u = g \in \gamma_m(G)$. From $\overline{1} = \pi_n(u) = \overline{g}$ for all $n \geq 1$ it follows that $g \in \gamma_n(G)$ for all $n \geq 1$. Since $G$ is residually nilpotent, $g = 1$. Hence $V_\omega(ZG) = \{1\}$. □

We next prove that the class of groups $G$ with $V_\omega(ZG) = \{1\}$ is closed with respect to discrimination.

Let $C$ denote the class of groups $G$ such that $V(ZG)$ has trivial $\Delta$-adic residue, and let $\text{Disc } C$ denote the class of groups discriminated by the class $C$.

Theorem 15. $C = \text{Disc } C$.

Proof. Clearly, by definition, $C \subseteq \text{Disc } C$.

Let $G \in \text{Disc } C$, and let $u \in V_\omega(ZG)$, $u = \alpha_1 g_1 + \ldots + \alpha_n g_n$, $g_i \neq g_j$ for $i \neq j$. Let $\varphi : G \to H$ be a homomorphism with $H \in C$ such that $\varphi(g_i), \ldots, \varphi(g_n)$ are distinct elements of $H$. Extend $\varphi$ to $ZG$ by linearity. Then $\varphi(u) \in V_\omega(ZH) = \{1\}$, and it thus follows that $u = g \in \ker \varphi$. In case $g \neq 1$, we have a homomorphism $\psi : G \to K$ with $K \in C$ such that $\psi(g) \neq 1 = \psi(1)$. Since $g - 1 \in \Delta^\omega(G)$, extension of $\psi$ to $ZG$ shows that $\psi(g - 1) = 0$, a contradiction. Hence, it follows that $g = 1$ and so $G \in C$. □

4 Groups with trivial $D$-residue

We next study groups $G$ whose $D$-residue, namely,

$$D_\omega(G) := \cap_{n=1}^\infty D_n(G)$$

is trivial. Since

$$\gamma_n(G) \subseteq D_n(G) \subseteq V_n(ZG),$$

the triviality of the $\Delta$-adic residue of a group always implies that of its $D$-residue.

Over the field $\mathbb{Q}$ of rationals, the following statements for an arbitrary group $G$ are equivalent ([Pas79], Theorem 2.26, p. 90):
(i) \( G \) is residually torsion-free nilpotent;
(ii) \( \Delta_Q^\omega(G) = \{0\} \);
(iii) \( D_{\omega,Q}(G) = \{1\} \).

Here \( \Delta_Q^\omega(G) \) denotes the augmentation ideal of the group algebra \( \mathbb{Q}G \),

\[
D_{\omega,Q}(G) = \cap_{n=1}^\infty D_{n,Q}(G), \quad D_{n,Q}(G) = G \cap (1 + \Delta_Q^n(G)).
\]

**Theorem 16.** Let \( G \) be a group such that

(i) \( D_{\omega,Q}(G) \neq \{1\} \); and
(ii) \( D_{\omega}(G) = \{1\} \).

Then, for any finitely many distinct elements \( g_1, g_2, \ldots, g_n \in G \), and the subgroup \( W \) generated by the left-normed commutators \([w_1, w_2, \ldots w_r], r \geq 1\), with

\[
w_i \in \{g_j^{-1}g_k, \ 1 \leq j, k \leq n, \ j \neq k\}, \ 1 \leq i \leq r,
\]

and such that for every \( 1 \leq j, k \leq n, \ j \neq k \), either \( g_j^{-1}g_k \) or \( g_k^{-1}g_j \) equals some \( w_i \), one of the following holds:

(a) The subgroup \( W \) is contained in \( C_G(D_{\omega,Q}(G)) \), the centralizer of \( D_{\omega,Q}(G) \) in \( G \);
(b) \( g_1, g_2, \ldots, g_n \) are discriminated by the class

\[
\mathcal{K} := \cup_{p \text{ prime}} \mathcal{K}_p,
\]

where \( \mathcal{K}_p \) denotes the class of nilpotent \( p \)-groups of bounded exponent.

In order to prove the above result, we need the following:

**Lemma 17.** Let \( G \) be a group which is not residually “torsion-free nilpotent” and let \( 1 \neq g \in D_{\omega,Q}(G) \). If \( 1 \neq g_1 \in G \) is such that \( 1 \) and \( g_1 \) are not discriminated by the class \( \mathcal{K} \), then \([g_1, g] \in D_{\omega}(G)\).

**Proof.** Let \( G, g_1 \) and \( g \) be as in the statement of the Lemma. Since

\[
[g_1, g] - 1 = g_1^{-1}g_2^{-1}[(g_1 - 1)(g - 1) - (g - 1)(g_1 - 1)],
\]

it suffices to prove that, for all \( n \geq 1 \), \((g_1 - 1)(g - 1)\) and \((g - 1)(g_1 - 1)\) belong to \( \Delta^n(G) \).
Let $n \geq 1$ be fixed. Note that $1 \neq g \in D_{n, \mathbb{Q}}(G)$ implies that
\[ m_n(g - 1) \in \Delta^n(G), \]
for some $m_n \in \mathbb{N}$. Let $m_n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime factorization of $m_n$. Since 1 and $g_1$ are not discriminated by the class $\mathcal{K}$, for any group $H \in \mathcal{K}$, and for any homomorphism $\phi : G \to H$, $\phi(g_1) = 1_H$. In particular, for all natural numbers $l$, $k$ and primes $p$, since $G/\gamma_l(G)G^{p, k} \in \mathcal{K}_p$, we have that
\[ g_1 \in \cap_{l,k} \gamma_l(G)G^{p, k}. \]
Consequently, $g_1 - 1 \in \Delta^n(G) + p^\ell \Delta(G)$ for all primes $p$ and natural numbers $\ell \geq 1$. It then easily follows that both $(g - 1)(g_1 - 1)$ and $(g_1 - 1)(g - 1)$ lie in $\Delta^n(G)$.

Proof of Theorem 16. We first consider the case when the given elements of $G$ are 1 and $g_1 \neq 1$, so that $W = \langle g_1 \rangle$.

If $g_1 \notin C_G(D_{\omega, \mathbb{Q}}(G))$, then there exists $1 \neq g \in D_{\omega, \mathbb{Q}}(G)$ such that $[g_1, g] \neq 1$. But then, if 1 and $g_1$ are not discriminated by the class $\mathcal{K}$, by Lemma 17, $1 \neq [g_1, g] \in D_{\omega}(G)$, which contradicts the assumption (ii). Hence, 1 and $g_1$ must be discriminated by the class $\mathcal{K}$.

Next, let $g_1, g_2, \ldots, g_n$, $n \geq 2$, be distinct elements of $G$. In case the subgroup $W$ is not contained in $C_G(D_{\omega, \mathbb{Q}}(G))$, then there exists one of the generating left-normed commutators of $W$, say $u = [w_1, \ldots, w_r]$, which does not belong to $C_G(D_{\omega, \mathbb{Q}}(G))$. By the case considered above the elements 1 and $u$ are discriminated by the class $\mathcal{K}$. Thus there exists a homomorphism $\varphi : G \to H$ with $H \in \mathcal{K}$ such that $\varphi(u) \neq 1$. Since
\[ \varphi(u) = [\varphi(w_1), \ldots, \varphi(w_r)], \]
it follows that $\varphi(w_i) \neq 1$ for $i = 1, \ldots, r$. Consequently, $\varphi(g_j^{-1}g_k) \neq 1$ for $j \neq k$, since either $g_j^{-1}g_k$ or $g_k^{-1}g_j$ equals one of the $w'_i$s. Hence the elements $g_1, \ldots, g_n$ are discriminated by the class $\mathcal{K}$. □

Since, for every torsion group $G$, we have $D_{\omega, \mathbb{Q}}(G) = G$, the above theorem yields the following:

Corollary 18. Let $G$ be a torsion group with trivial $\mathcal{D}$-residue, then, every non-central element is discriminated from the identity element by the class $\mathcal{K}$.

For any group $G$, an analogous analysis done on $\mathcal{V}(\mathbb{Z}G)$ yields the following results of which we omit the proofs.
Theorem 19. Let $G$ be a group such that

(i) $\mathcal{V}_{\omega, Q}(ZG) \neq \{1\}$, where $\mathcal{V}_{\omega, Q}(ZG) := \mathcal{V}(ZG) \cap (1 + \Delta^Q(G))$; and

(ii) $\mathcal{V}_{\omega}(ZG) = \{1\}$

Then, for finitely many distinct elements $v_1, v_2, \ldots, v_n \in \mathcal{V}(ZG)$, and the subgroup $W$ generated by the left-normed commutators $[w_1, \ldots, w_r]$, $r \geq 1$, with

$$w_i \in \{v_j^{-1}v_k \mid 1 \leq j, k \leq n, j \neq k\}, \quad 1 \leq i \leq r,$$

and such that for every $1 \leq j, k \leq n, j \neq k$, either $v_j^{-1}v_k$ or $v_k^{-1}v_j$ equals some $w_i$, one of the following holds:

(i) the subgroup $W$ is contained in $C_{\mathcal{V}}(\mathcal{V}_{\omega, Q}(ZG))$, the centralizer of $\mathcal{V}_{\omega, Q}(ZG)$ in $\mathcal{V} := \mathcal{V}(ZG)$; or

(ii) there exists a prime $p$ and integers $k, \ell$ such that none of the elements $v_1, v_2, \ldots, v_n$ belong to $\mathcal{V}_{p, k, \ell}(ZG) := \{v \in \mathcal{V}(ZG) : v - 1 \in \Delta^k(G) + p^\ell ZG\}$.

Corollary 20. If the group $G$ is such that $\mathcal{V}(ZG) = \mathcal{V}_{\omega, Q}(ZG)$ and $\mathcal{V}_{\omega}(ZG) = \{1\}$, then every non central unit $u \in \mathcal{V}(ZG)$ does not belong to some $\mathcal{V}_{p, k, \ell}(ZG)$.

In case $G$ is a finite group, then clearly $\mathcal{V}_{\omega, Q}(ZG) \neq \{1\}$. Thus we immediately have the following:

Corollary 21. If $G$ is a finite cut-group with $\Delta$-adic residue of $\mathcal{V}(ZG)$ trivial, then $G$ is nilpotent and every non-central unit is missed by some $\mathcal{V}_{p, k, \ell}(ZG)$.

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