One-parameter generalised Fisher information

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Abstract

We introduce the generalised Fisher information or the one-parameter extended class of the Fisher information. This new form of the Fisher information is obtained from the intriguing connection between the standard Fisher information and the variational principle together with the non-uniqueness property of the Lagrangian. Furthermore, one could treat this one-parameter Fisher information as a generating function for obtaining what is called Fisher information hierarchy. The generalised Cramér-Rao inequality is also derived. The interesting point is about the fact that the whole Fisher information hierarchy, except for the standard Fisher information, does not follow the additive rule. This could suggest that there is an indirect connection between the Tsallis entropy and the one-parameter Fisher information. Furthermore, the whole Fisher information hierarchy is also obtained from the two-parameter Kullback-Leibler divergence.

1 Introduction

There is no doubt that we are now in the “information era”. The information is physical [1] and plays an essential role in modern physics ranging from thermodynamics, statistical mechanics, quantum mechanics to relativity. The birth of information theory can be traced back to the seminal paper of Shannon [2] on communication. The key quantity in this context is the entropy or more precisely “Shannon entropy” as the mean value of information or uncertainty inherent in the possible outcomes. The interesting point is that the Shannon entropy is in the same form as the Gibbs-Boltzmann entropy in the context of statistical mechanics if one ignores the Boltzmann’s constant, which measures the configuration of the microscopic states. However, the notion of entropy was first introduced in the context of thermodynamics the second law of thermodynamics—which is a bit more abstract relating to the heat flow in or out and the temperature of the system. However, if we trace back long before the breakthrough work of Shannon, Fisher purposed another information quantity, later known as Fisher information [3], as a measurement uncertainty on estimating unknown parameters in the system. This means that the Fisher information allows us to probe into the internal structure of the system. At this point, Shannon entropy and Fisher information provides a complete description of the system in the sense that the Fisher information can give an insight of what the system is made of and the Shannon entropy gives the system behaviour in the big picture. Moreover, the Shannon differential entropy and Fisher information are connected which was first observed by Kullback [4]. With the Kullback insight, the Fisher information matrix can be obtained from the second derivative of the Kullback-Leibler divergence (or the relative entropy).

The generalised version of the Shannon entropy was first introduced by Renyi [5]. The Renyi entropy comes with a parameter and with a suitable limit, the Shannon entropy can be recovered. Then one could think that the Renyi entropy is a one-parameter extended class of the Shannon entropy. On the statistical mechanical side, the generalised version of the Gibbs-Boltzmann entropy was proposed by Tsallis [6]. Again, this Tsallis entropy comes with a parameter and the Gibbs-Boltzmann entropy can be recovered...
with a suitable limit. One main feature of the Tsallis entropy is the non-additive property directly related to the non-extensivity of the system. Consequently, this leads to a new kind of research area known as the Tsallis statistics with a wide range of applications in statistics, physics, chemistry, economics, and biosciences [7]. On the other hand, several extensions of the Fisher information have been proposed with different aspects [8]-[13] to serve different uses in statistics. Then in this contribution, we propose another one-parameter extended class of the Fisher information. The key motivation and derivation come from the intriguing connection between Fisher information and variational principle observed by Frieden [14]–[16] together with one-parameter extended class of the Lagrangian [18].

The remaining body of this work is the following. In section 2 a brief review of the standard Fisher information together with notations will be given. In section 3 one parameter extended class of the Fisher information is derived by employing the connection with the variational principle. The Fisher information hierarchy will be so obtained. In section 4 the extended Cramér-Rao inequality and non-additive property are given. In section 5 the connection between two-parameter Kullback-Leibler divergence and Fisher information hierarchy will be established. The last section will be about the conclusion and discussion.

2 Fisher information

We shall start in this section to give a brief review of the basic Fisher information and the connection with Kullback-Leibler divergence. Given a set of an observable random variable \( X = (x_1, x_2, ..., x_N) \) that belongs in the sample space \( \Omega (X \in \Omega) \) and associated probability density functions \( p(x_i \mid \theta) \), which is identical and independent, we define the likelihood as

\[
L(\theta \mid X) = \prod_{i=1}^{N} p(x_i \mid \theta),
\]

where \( \theta \) is an unknown parameter in the probability model. In principle, the likelihood measures how good the statistical model is comparing to the sample of data \( X \) for given the values of the unknown parameter \( \theta \). Then what we need is to find maximum value of the likelihood \( \frac{\partial L(\theta \mid X)}{\partial \theta} = 0 \). However, in practise, the likelihood might not be well behave in calculation. Since, logarithm is monotonic increasing function, we shall work with the log-likelihood \( \frac{\partial \log L(\theta \mid X)}{\partial \theta} = 0 \) (also called the score function), which gives the same maximum points. What we are interested is the nature of the maximum peak as an indicator for how good it is for the estimated value. If the peak is thin, the estimated value is fairly determined. On the other hand, if the peak is broad, the estimated value is uncertainly determined. With this reason, we therefore consider the second derivative of the log-likelihood averaged over all possible random variables as the measure called the Fisher information

\[
I(\theta) \equiv \left\langle \frac{\partial^2}{\partial \theta^2} \log L(\theta \mid X) \right\rangle. \tag{2.1}
\]

Employing the fact that

\[
\frac{\partial}{\partial \theta} \log L(\theta \mid X) = \frac{1}{L(\theta \mid X)} \frac{\partial L(\theta \mid X)}{\partial \theta} \quad \text{and} \quad \left\langle \frac{\partial}{\partial \theta} \log L(\theta \mid X) \right\rangle = 0, \tag{2.2}
\]

Fisher information can be re-expressed as

\[
I(\theta) = \left\langle \frac{\partial^2}{\partial \theta^2} \log L(\theta \mid X) \right\rangle = -\left\langle \left( \frac{\partial}{\partial \theta} \log L(\theta \mid X) \right)^2 \right\rangle = -\text{Var} \left[ \frac{\partial}{\partial \theta} \log L(\theta \mid X) \right].
\]

Here, we can conclude that Fisher information tells us how much we know about the internal structure from data with a given space of outcomes (sample space) \( \Omega \).

Fisher information also provides a information lower bound on the variance of an unbiased estimator for a parameter. This relation is known as the Cramér-Rao inequality. To obtain such relation, one can start to consider the unbiased estimator

\[
B(\hat{\theta}) \equiv \left\langle \hat{\theta} - \theta \right\rangle = \int_{\Omega} ... \int_{\Omega} (\hat{\theta} - \theta) L(\theta \mid X) dX = 0, \tag{2.3}
\]

where \( \hat{\theta} = h(x_1, x_2, ..., x_N) \) is a point estimator. For now on, we might neglect subscription \( \Omega \) on integrating for our convenient. Next, we consider the derivative of (2.3) with respect to the parameter \( \theta \) resulting in
Using the fact that \( \int ... \int L(X \mid \theta) dX = 1 \), we obtain
\[
\int ... \int \left[ \frac{\partial}{\partial \theta} \hat{L}(\theta \mid X) \cdot L^{1/2}(\theta \mid X) \right] dX = 1 .
\] (2.5)

Applying Cauchy–Schwarz inequality, we obtain
\[
\frac{1}{\int ... \int (\hat{L}(\theta - \theta)^2 L(\theta \mid X) dX)} \leq \int ... \int \left( \frac{\partial}{\partial \theta} \log L(X \mid \theta) \right)^2 L(\theta \mid X) dX \leq \frac{1}{\text{Var}(\hat{\Theta})} \leq \mathbf{I}(\theta) .
\] (2.6)

What we have in (2.6) is that the variance of any such estimator is at least as much as the inverse of the Fisher information.

Furthermore, there is also one more important feature of the Fisher information known as the additive property. From the right hand side of above inequality (2.6), the Fisher information is given by
\[
\mathbf{I}(\theta) = \sum_{i,j=1}^{N} \int \frac{\partial p(x_i \mid \theta)}{\partial \theta} \frac{\partial p(x_j \mid \theta)}{\partial \theta} dX_{i}dX_{j} + \sum_{j=1}^{N} \int \frac{1}{p(x_j \mid \theta)} \left( \frac{\partial p(x_j \mid \theta)}{\partial \theta} \right)^2 dx_j .
\] (2.7)

Here we are dealing with identical and independent random variables. Then the Fisher information (2.7) can be simplified as
\[
\mathbf{I}(\theta) = \sum_{j=1}^{N} \int \left( \frac{\partial}{\partial \theta} \log p(x_j \mid \theta) \right)^2 p(x_j \mid \theta) dx_j = \sum_{j=1}^{N} I_j(\theta) .
\] (2.8)

With many, independent data, random variables, the Fisher information can be splitted as the summation of all Fisher information of each random variable. Therefore, Fisher information possesses the additive property.

In general, the estimated parameters could come in a set i.e., \( \theta = (\theta_1, \theta_2, ..., \theta_n) \). Then the Fisher information becomes \( \mathbf{I}(\theta) = [I_{ij}(\theta)] \), where
\[
I_{ij}(\theta) = - \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log L(\theta \mid X) \right) ,
\] (2.9)

which is known as the Fisher information matrix. The interesting point is that the Fisher matrix can also be obtained by considering the relative entropy or Kullback-Leibler divergence between two distribution \( p(X) \) and \( q(X) \) on the probability manifold
\[
KL(p\|q) = \int ... \int p(X) \log \left( \frac{p(X)}{q(X)} \right) dX .
\] (2.10)

Then the Kullback-Leibler divergence between two probability distributions \( L(\theta \mid X) \) and \( L(\theta' \mid X) \), parameterised by \( \theta \), is given by
\[
D(\theta, \theta') \equiv KL(L(\theta \mid X)||L(\theta' \mid X)) = \int ... \int L(\theta \mid X) \log \left( \frac{L(\theta \mid X)}{L(\theta' \mid X)} \right) dX .
\] (2.11)

For \( \theta \) being fixed, the Kullback-Leibler divergence can be expanded around \( \theta \) as
\[
D(\theta, \theta') = \frac{1}{2} (\theta' - \theta)^T \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} D(\theta, \theta') \right) \bigg|_{\theta=\theta'} (\theta' - \theta) + O((\theta' - \theta)^2) ,
\] (2.12)
where the second order derivative is
\[
\left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} D(\theta, \theta') \right) \bigg|_{\theta = \theta'} = - \int \cdots \int \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} L(\theta' | X) \right) \bigg|_{\theta' = \theta} L(\theta | X) dX = [I_{ij}(\theta)] .
\]
(2.13)

With this connection, one may intuitively interpret the Fisher information as the metric between two points on the probability manifold. However, the Kullback-Leibler divergence is not symmetric and does not follow the triangle inequality \[19\]. Then the Fisher information cannot be treated as a true metric.

### 3 One-parameter generalised Fisher information

In this section, we will derive a new type of the Fisher information called one parameter generalised Fisher information. One of the key features which will play an important role to obtain such new Fisher information is the variational principle. The intriguing connection between the variational principle and Fisher information was proposed by Frieden \[16\]. Another feature is the non-uniqueness property of the Lagrangian which leads to the non-standard form of the Lagrangian called the Multiplicative form \[18\].

Let us now define \( \theta \) as actual value of a measurement quantity, \( X = (x_1, x_2, \ldots, x_N) \) as \( N \) outcomes of the quantity and \( Y = (y_1, y_2, \ldots, y_N) \) as random errors associated with each measurement. Then we have

\[
x_i = \theta + y_i .
\]
(3.14)

Next, let \( p(x_i | \theta) \) be a probability distribution, whose support is a set \( \Omega \), over the \( x \)'s with respect to \( \theta \). We recall the Fisher information \[1\] (2.8)

\[
I(\theta) = \sum_{i=1}^{N} \int \left( \frac{\partial}{\partial \theta} \ln p(x_i | \theta) \right)^2 p(x_i | \theta) dx_i
\]
(3.15)

In the case \( N = 1 \), we do have \( p(x | \theta) = p(x - \theta) = p(y) \). The Fisher information is simply reduced to

\[
I[p(y)] = \int p(y) \left( \frac{d}{dy} \ln p(y) \right)^2 dy = \int \left( \frac{p'(y)}{p(y)} \right)^2 dy \quad \text{when} \quad p'(y) = \frac{dp(y)}{dy} .
\]
(3.16)

Next, we do a transformation such that \( q(y) = \sqrt{p(y)} \) resulting in

\[
I[q(y)] = 4 \int q'^2(y) dy, \quad \text{when} \quad q'(y) = \frac{dq(y)}{dy} .
\]
(3.17)

We find that the Fisher information is now a functional with the input function \( q(y) \). Here comes to an interesting point. If we define \( I[q] \equiv S[q] \) as an action functional and \( \mathcal{L}(q', q; y) \equiv 4q'^2(y) \) as the Lagrangian, the variational principle would give the Euler-Lagrange equation

\[
\frac{\partial \mathcal{L}(q', q; y)}{\partial q(y)} - \frac{d}{dy} \left( \frac{\partial \mathcal{L}(q', q; y)}{\partial q'(y)} \right) = 0 ,
\]
(3.18)

resulting in \(-8q''(y) = 0 \). This second order differential equation describes how a position \( q \) change with time \( y \) for a free particle. This would mean that the Fisher information \[3.17\] could be remarkably treated as the action functional for the free particle and of course, in the absence of the interaction, the equation of motion in physics is a direct result of extremising the Fisher information: \( \delta I[q] = 0 \).

Next, what we would like to focus is the property of the Lagrangian known as the non-uniqueness property. Commonly, two Lagrangians differing by the total derivative with respect to time of some function \( F(q, y) \) would give the identical equation of motion on extremising the action. However, one could ask an inverse question as follows. Imposing the equation of motion, could we solve all possible Lagrangians directly from

\[\text{Here we prefer the natural logarithm function.}\]
By considering the limit \( \lambda \rightarrow 0 \), one finds that \( \lim_{\lambda \rightarrow 0} \mathcal{L}_\lambda = \mathcal{L}(q', q; y) = 4q'^2(y) \). Then we shall seek a transformation to express the higher order Fisher information in terms of infinite series as

\[
I_n[\theta] = \sum_{j=1}^{\infty} \frac{\lambda^{n-1}}{n!} I'_n[\theta] = \sum_{j=1}^{\infty} \frac{\lambda^{n-1}}{n!} I_j[\theta].
\]

At this point, we may treat \( I_n[\theta] \) as the generating function for the entire hierarchy of the Fisher information by expanding with respect to the parameter \( \lambda \).
4 Generalised Cramér-Rao inequality and non-additive property

Here in this section, we will construct the Cramér-Rao inequality associated with the Fisher information hierarchy given in the previous section. To achieve the goal, we start with

\[ \langle \hat{\Theta} - \theta \rangle = \int (\hat{\Theta} - \theta) p^\theta(x \mid \theta) dx = 0 , \] (4.26)

which is known as the \( q \)-expectation value \[20\] \[22\]. Taking the 1st derivative, we obtain

\[
\frac{\partial}{\partial \theta} \langle \hat{\Theta} - \theta \rangle = \int \frac{\partial}{\partial \theta} (\hat{\Theta} - \theta) p^\theta(x \mid \theta) dx + \int (\hat{\Theta} - \theta) \frac{\partial}{\partial \theta} p^\theta(x \mid \theta) dx \\
= - \int p^\theta(x \mid \theta) dx + q \int (\hat{\Theta} - \theta) p^{\theta - 1}(x \mid \theta) \frac{\partial p(x \mid \theta)}{\partial \theta} dx \\
= - \int p^\theta(x \mid \theta) dx + q \int (\hat{\Theta} - \theta) p^{\theta - 1}(x \mid \theta) p(x \mid \theta) \frac{\partial \ln p(x \mid \theta)}{\partial \theta} dx \\
= - Q_q + q J = 0 ,
\] (4.27)

where

\[
Q_q = \int p^\theta(x \mid \theta) dx ,
\] (4.28)

\[
J = \int (\hat{\Theta} - \theta) p^{\theta - 1}(x \mid \theta) p(x \mid \theta) \frac{\partial \ln p(x \mid \theta)}{\partial \theta} dx .
\] (4.29)

Conventionally, the term \( Q_q \) is well known as information generating function \[22\] with Tsallis index \( q \) \[6\] (we are now interested in case \( q \geq 1 \)). It is also called incomplete normalization and \( p^\theta \) is called effective probability \[21\]. Next, we rewrite the \( J \) in the form

\[
J = \int [(\hat{\Theta} - \theta)] \left[ \frac{\partial \ln p(x \mid \theta)}{\partial \theta} p^{\theta - 1}(x \mid \theta) \right] p(x \mid \theta) dx ,
\] (4.30)

and applying the Hölder’s inequality \[23\] to \(4.31\), we obtain

\[
J \leq \left[ \int (\hat{\Theta} - \theta)^3 p(x \mid \theta) dx \right]^{1/3} \left[ \int \left( \frac{\partial \ln p(x \mid \theta)}{\partial \theta} \right)^\alpha (p^{\theta - 1}(x \mid \theta))^{\alpha} p(x \mid \theta) dx \right]^{1/\alpha} \\
= \left[ \int (\hat{\Theta} - \theta)^3 p(x \mid \theta) dx \right]^{1/3} \left[ \int \left( \frac{\partial \ln p(x \mid \theta)}{\partial \theta} \right)^\alpha p^{\alpha(q-1)+1}(x \mid \theta) dx \right]^{1/\alpha} ,
\] (4.31)

where Hölder conjugates \( \alpha \) and \( \beta \) are related with the condition \( 1/\alpha + 1/\beta = 1 \) for \( \alpha, \beta = [1, \infty] \). Finally, employing \(4.27\), the inequality \(4.31\) becomes

\[
\frac{Q_q}{q} = \frac{\int p^\theta(x \mid \theta) dx}{q} \leq \left[ \int (\hat{\Theta} - \theta)^3 p(x \mid \theta) dx \right]^{1/3} \left[ \int \left( \frac{\partial \ln p(x \mid \theta)}{\partial \theta} \right)^\alpha p^{\alpha(q-1)+1}(x \mid \theta) dx \right]^{1/\alpha} ,
\] (4.32)

which is our generalised Cramer-Rao inequality. It is not difficult to see that if one takes \( q = 1 \), \( \beta = 2 \) and \( \alpha = 2 \), the standard Cramer-Rao inequality can be recovered. For \( \alpha = 4 \), \( \beta = 4/3 \) and \( q = 5/4 \), we obtain

\[
\frac{4 \frac{Q_{5/4}}{q}}{5^4 \left[ (\hat{\Theta} - \theta)^{4/3} \right]^3} \leq I_2 ,
\] (4.33)

which is the Cramér-Rao inequality for the 2nd extended Fisher information. Basically, the inequality \(4.32\) provides the Cramér-Rao bound for the whole Fisher information hierarchy as shown in table 1.
The joint probability of the two subsystems is given by

\[ p(1, 2|\theta) = p(x_1|\theta)p(x_2|\theta) \]

Next, we will investigate the additive property of the higher order Fisher information. For simplicity, we shall start with the second order Fisher information. Suppose a system composed of two independent identically subsystems that are defined its random variable \( X = (x_1, x_2) \), where superscript denote for subsystems. The joint probability of the two subsystems is given by \( p_{12} \equiv p(x_1, x_2|\theta) = p(x_1|\theta)p(x_2|\theta) = p_1p_2 \). What we have for the second order Fisher information is

\[
I_2[p_{12}] = \frac{4}{2^n} \int \left( \frac{\partial}{\partial \theta} \ln(p_{12}) \right)^2 p_1^2 p_2^2 dx_1 dx_2 \\
= \frac{4}{2^n} \left[ \int \left( \frac{\partial}{\partial \theta} \ln p_1 \right)^2 p_1^2 dx_1 \int p_2^2 dx_2 + 4 \int \left( \frac{\partial}{\partial \theta} \ln p_1 \right)^2 p_1^2 dx_1 \int \left( \frac{\partial}{\partial \theta} \ln p_2 \right)^2 p_2^2 dx_2 \\
+ 6 \int \left( \frac{\partial}{\partial \theta} \ln p_1 \right)^2 p_1^2 dx_1 \int \left( \frac{\partial}{\partial \theta} \ln p_2 \right)^2 p_2^2 dx_2 + 4 \int \left( \frac{\partial}{\partial \theta} \ln p_1 \right)^2 p_1^2 dx_1 \int \left( \frac{\partial}{\partial \theta} \ln p_2 \right)^2 p_2^2 dx_2 \\
+ \int p_1^2 dx_1 \int \left( \frac{\partial}{\partial \theta} \ln p_2 \right)^4 p_2^2 dx_2 \right] \\
= \frac{4}{2^n} Q_2(p_2) \int \left( \frac{\partial}{\partial \theta} \ln p_1 \right)^2 p_1^2 dx_1 + 6 \int \left( \frac{\partial}{\partial \theta} \ln p_1 \right)^2 p_1^2 dx_1 \int \left( \frac{\partial}{\partial \theta} \ln p_2 \right)^2 p_2^2 dx_2 \\
+ Q_2(p_1) \int \left( \frac{\partial}{\partial \theta} \ln p_2 \right)^4 p_2^2 dx_2 \\
= \frac{1}{4} \left[ Q_2(p_2)I_2(p_1) + Q_2(p_1)I_2(p_2) + 6I(p_1)I(p_2) \right]. \tag{4.34}
\]

Here see that the second order Fisher information does not follow the additive rule. With the result in (4.34), it is not difficult now to see that the \( n \)th order Fisher information could give

\[
I_n[p_{12}] = \frac{4}{2^n} \binom{2n}{0} Q_n(p(x_2|\theta)) \left( \frac{\partial}{\partial \theta} \ln p(x_1|\theta) \right)^{2n} p^{n}(x_1|\theta) dx_1 \\
+ \sum_{k=2}^{2n-2} \binom{2n}{k} \left( \frac{\partial}{\partial \theta} \ln p(x_1|\theta) \right)^{2n-k} p^{n}(x_1|\theta) dx_1 \left( \frac{\partial}{\partial \theta} \ln p(x_2|\theta) \right)^{k} p^{n}(x_2|\theta) dx_2 \\
+ \binom{2n}{2n} Q_n(p(x_1|\theta)) \left( \frac{\partial}{\partial \theta} \ln p(x_2|\theta) \right)^{2n} p^{n}(x_2|\theta) dx_2 , \tag{4.35}
\]

where the first and last terms refer to the Fisher information for each subsystem and the middle one is the crossing-term. Therefore, our Fisher information hierarchy does not follow the additive property, except for \( n = 1 \) the standard Fisher information.

Table 1: The \( n \)th Carmer-Rao inequalities and their associated three parameters.

| \( n \)th Carmer-Rao inequality | Parameters |
|--------------------------------|------------|
| 1st order                     | 1          |
| 2nd order                     | 5/4        |
| 3rd order                     | 4/3        |
| 4th order                     | 11/8       |
5 The Kullback–Leibler divergence revisited

Here in this section, we shall investigate on the connection between our Fisher information hierarchy and the Kullback–Leibler divergence. We shall begin with the Kullback–Leibler divergence

\[ D(p(y) \parallel q(y)) = \int p(y) \ln \left( \frac{p(y)}{q(y)} \right) dy , \]

(5.36)

where \( p \) and \( q \) are two different points on the probability manifold. If \( q = p + dp \), where \( dp \) is infinitesimal, we could have

\[ D(p(y) \parallel p(y) + p'(y)) = \int p(y) \ln \left( \frac{p(y)}{p(y) + p'(y)} \right) dy , \]

(5.37)

where \( p'(y) = dp/dy \). We then shall expand (5.37) with respect to \( p' \). Keeping only the first dominate term, we obtain

\[ D(p(y) \parallel p(y) + p'(y)) \approx \int \frac{1}{2} \left( \frac{p'(y)}{p(y)} \right)^2 dp \]

(5.38)

\[ \int \frac{1}{2} dp \]

We could see that the righthand side of (5.38) is nothing but the standard Fisher information.

Now we introduce two-parameter Kullback-Leibler divergence

\[ D_{q,q'} (p(y) \parallel p(y) + p'(y)) = \int p(y) \ln \left( \frac{p(y)}{p(y) + p'(y)} \right) dy . \]

(5.39)

Here we do again the expansion with respect to \( p' \) and we obtain the two-parameter generalisation of the Fisher information from the first dominant term [8]

\[ I_{a,b}[p] = \int p^a(y) \left( \frac{dp}{dy} \right)^b dy , \]

(5.40)

where \( a = q - q' - 1 \) and \( b = q' + 1 \) with the requirements \( q > 0 \) and \( q' > 0 \). We find that, with a suitable choice of parameters, our whole hierarchy of Fisher information can be identified as shown in the table [2]

**Table 2: Comparison our one-parameter Fisher information with two-parameter Fisher information.**

| \( n^{th} \) Fisher information | Parameters |
|---------------------------------|------------|
| \( 1^{st} \) order: \( I_1 \)   | 1          | 2          |
| \( 2^{nd} \) order: \( I_2 \)   | 2          | 4          |
| \( 3^{rd} \) order: \( I_3 \)   | 3          | 6          |
| \( 4^{th} \) order: \( I_4 \)   | 4          | 8          |

We note here that the quantities in (5.40), of course directly connected with our Fisher information hierarchy as we already mentioned, can be possibly viewed as the generalised Fisher matrices. However, there exist also other generalised Fisher matrices for different purposes and motivations [13][24].

6 Conclusion

We succeed to construct the one-parameter generalised Fisher information. The main method used to derive the one-parameter generalised Fisher information is the variational principle. We consider here the Fisher information as the action functional of free particle Lagrangian. With the new insight of the one-parameter
generalised Lagrangian [18], one can naturally obtain the one-parameter generalised Fisher information. Furthermore, we can treat our one-parameter generalised Fisher information as the generator to obtain what we call the Fisher information hierarchy. The generalised Cramér-Rao inequality is also obtained with the help of the Hölder’s inequality. We find that our Fisher information hierarchy does not follow the addition property, expect for the standard Fisher information. The interesting point is that this non-additive property pops up when the Tsallis index is not equal to 1 as shown in table [1]. Of course this point is quite interesting since this non-additive property has been widely discussed in the Tsallis statistic [6]. Let us point out possibly indirect connection with the Tsallis entropy by recalling our one-parameter generalised Fisher information

$$I_\lambda[p] = \frac{4}{\lambda} \int \left[ e^{\lambda \frac{4}{\lambda} p'(y)^2} - 1 \right] dy,$$

and Tsallis entropy

$$S_q[p] = \frac{1}{1 - q} \left[ \int p^q(y) - 1 \right] dy.$$

It might be seem a bit strange but these two quantities more or less similar in the sense that they both contain a parameter and under the suitable limit the standard quantities can be recovered. However, more direction connection with the entropy might be the relative entropy or the Kullback–Leibler divergence, more specifically two-parameter Kullback–Leibler divergence and our whole hierarchy Fisher information can be identified with the two-parameter Fisher information with the appropriate choice of parameters.

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