SECOND ORDER MODIFIED OBJECTIVE FUNCTION
METHOD FOR TWICE DIFFERENTIABLE VECTOR
OPTIMIZATION PROBLEMS OVER CONE CONSTRAINTS

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ABSTRACT. In the paper, a vector optimization problem with twice differentiable functions and cone constraints is considered. The second order modified objective function method is used for solving such a multiobjective programming problem. In this method, for the considered twice differentiable multi-criteria optimization problem, its associated second order vector optimization problem with the modified objective function is constructed at the given arbitrary feasible solution. Then, the equivalence between the sets of (weakly) efficient solutions in the original twice differentiable vector optimization problem with cone constraints and its associated modified vector optimization problem is established. Further, the relationship between an (weakly) efficient solution in the original vector optimization problem and a saddle-point of the second order Lagrange function defined for the modified vector optimization problem is also analyzed.

1. Introduction. The extremum problem, in which more than one objective functions are optimized simultaneously, is known in the optimization literature as a vector optimization problem or multiobjective programming problem. Recently, vector optimization problems have wide applications, for example, in product and process design, finance, aircraft design, climate change, the oil and gas industry, automobile design, evaluation of equity portfolios, etc. In a vector optimization problem, objectives are often in conflicting nature and consequently, there is no a single solution which optimizes all objectives simultaneously. The concept of efficiency has a vital role in solving such a type of optimization problems.

Convexity notion is a useful tool in proving the fundamental results in optimization theory for such vector optimization problems in which the involved functions are convex. However, there exist nonconvex operations research (O.R.) problems

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and, therefore, the convexity notion cannot be used for such practical optimization problems in proving, for example, sufficient optimality conditions for (weakly) efficiency of a feasible solution. Therefore, in recent years, some generalizations of the convexity notion were proposed in the literature, also in the case when the functions involved in an optimization problem are twice differentiable (see, for example, [5, 6]). Aghezzaf and Hachimi [1] developed second order necessary conditions for optimality for a vector optimization problem. Under the second order invexity assumptions, Gulati et al. [9] established duality theorems for a class of multiobjective programming problems with twice differentiable functions. Recently, Suneja et al. [13] introduced the concepts of second order ρ-invexity over cones and established some second-order multiobjective symmetric duality results over arbitrary cones.

Saddle-point criteria, which have an important role in order to find (weakly) efficient solutions in vector optimization problems, have been studied by several authors (see, for example, [3, 4, 8, 11, 12]). A new approach with the so-called η-modified objective function for solving a smooth vector optimization problem was first introduced by Antczak [2]. Thereafter, Chen et al. [7], based on the η-modified objective function method proposed by Antczak [2], established the equivalence between efficient solutions of the original vector optimization problem and η-saddle-points in its associated vector optimization problem with modified objective function under assumption that the involved functions are generalized invex functions over cone. Recently, Suneja et al. [14] constructed an η-modified vector optimization problem by modifying the objective function for the considered nondifferentiable vector optimization problem and discussed the relationship between efficient solutions and η-saddle-points of these two multiobjective programming problems over cone constraints.

Inspired and motivated by the above research works, we use a second order modified objective function method for the considered vector optimization problem involving second order cone convex functions. In this method, we select a point from the feasible set of the original vector optimization problem and then we construct its associated second order vector optimization problem with the modified objective function at such a fixed feasible point. Then we analyze the relationship between (weakly) efficient solutions in the original vector optimization problem and its associated vector optimization problem constructed in the used approach under second order cone convexity assumptions. Further, the relationship between an (weakly) efficient solution in the original vector optimization problem and a saddle-point of the modified Lagrange function defined for the associated vector optimization problem with the modified second order objective function is also investigated.

This paper is organized as follows: in Section 2, we recall some preliminary results which are useful in proving the main results. Also we present the formulation of the considered vector optimization problem with cone constraints in which the involved functions are twice differentiable and the second order necessary optimality conditions for such multi-criteria optimization problems. In Section 3, we use the second order modified objective function method for solving the considered twice differentiable vector optimization problem with cone constraints. Therefore, in this section, we construct a second order vector optimization problem with the modified objective function by modifying the objective function in the original twice differentiable multiobjective programming problem at the given arbitrary, but fixed feasible solution. Then, under the concept of second order cone convexity, we establish the
equivalence between the sets of (weakly) efficient solutions in aforesaid vector optimization problems. Further, the relationship between (weakly) efficient solutions in the original twice differentiable vector optimization problem and a second order saddle-points of the second order Lagrange function defined for the associated vector optimization problem with the modified objective function is analyzed in Section 4. Finally, we conclude our paper in Section 5.

2. Notations and Preliminaries. For any $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$;
(ii) $x < y$ if and only if $x_i < y_i$ for all $i = 1, 2, \ldots, n$;
(iii) $x \leq y$ if and only if $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$;
(iv) $x \leq y$ if and only if $x \leq y$ and $x \neq y$.

Let $K \subseteq \mathbb{R}^m$ be a closed convex cone with the nonempty interior and let $\text{int} K$ denote the interior of $K$. The positive dual cone $K^*$ of $K$ is defined as follows:

$$K^* = \{x^* \in \mathbb{R}^m : x^T x^* \geq 0, \forall x \in K\},$$

where the symbol “$T$” denotes the transpose of a vector.

**Lemma 2.1.** (Chen et al. [7]): Let $K \subseteq \mathbb{R}^m$ be a convex cone with nonempty interior and $K^*$ is the dual cone of $K$. Then the following statements hold:

(i) If $u \in \text{int} K$, then $x^T u > 0$, $\forall x \in K^* \setminus \{0\}$,
(ii) If $x \in \text{int} K^*$, then $x^T u > 0$, $\forall u \in K \setminus \{0\}$.

Now, we define the second order convex functions with respect to a cone which will be used in proving main results of the paper.

**Definition 2.2.** (Suneja et al. [15]): A twice differentiable function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is said to be second order $K$-convex at $\bar{x} \in \mathbb{R}^n$ on $\mathbb{R}^n$ if, for every $x \in \mathbb{R}^n$,

$$f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x}) - (x - \bar{x})^T \nabla^2 f(\bar{x}) d + \frac{1}{2} d^T \nabla^2 f(\bar{x}) d \in K, \forall d \in \mathbb{R}^n,$$

where $\nabla f(\bar{x})$ is the $(m \times n)$ Jacobian matrix. For any $d, h \in \mathbb{R}^n$, we have

$$d^T \nabla^2 f(x)h = \begin{pmatrix} d^T \nabla^2 f_1(x)h \\ d^T \nabla^2 f_2(x)h \\ \vdots \\ d^T \nabla^2 f_m(x)h \end{pmatrix}.$$

**Definition 2.3.** (Suneja et al. [15]): A twice differentiable function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is said to be strictly second order $K$-convex at $\bar{x} \in \mathbb{R}^n, (x \neq \bar{x})$ on $\mathbb{R}^n$ if, for every $x \in \mathbb{R}^n$,

$$f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x}) - (x - \bar{x})^T \nabla^2 f(\bar{x}) d + \frac{1}{2} d^T \nabla^2 f(\bar{x}) d \in \text{int} K, \forall d \in \mathbb{R}^n.$$

In the paper, consider the following vector optimization problem (VOP):

$$\text{(VOP)} \quad K - \text{minimize } f(x)$$
subject to $-g(x) \in Q,$
where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m, g : \mathbb{R}^n \rightarrow \mathbb{R}^l \) are twice differentiable functions on \( \mathbb{R}^n \) and \( K \) and \( Q \) are closed convex cones with nonempty interiors in \( \mathbb{R}^m \) and \( \mathbb{R}^l \), respectively. Let \( S \) denote the set of all feasible solutions of (VOP), i.e.,
\[
S = \{ x \in \mathbb{R}^n : -g(x) \in Q \}.
\]
The set of active constraints at the feasible point \( \bar{x} \) is denoted by \( J(\bar{x}) \) and is defined by
\[
J(\bar{x}) = \{ j \in J = \{1, 2, \ldots, l\} : g_j(\bar{x}) = 0 \}.
\]
For such optimization problems, minimization means obtaining of (weakly) efficient solutions in the following sense [14]:

**Definition 2.4.** Let \( \bar{x} \in S \). Then:
(a) \( \bar{x} \) is said to be an efficient solution in (VOP) if, for every \( x \in S \),
\[
 f(x) - f(\bar{x}) \notin -K \setminus \{0\}.
\]
(b) \( \bar{x} \in S \) is said to be a weakly efficient solution in (VOP) if, for every \( x \in S \),
\[
 f(x) - f(\bar{x}) \notin -\text{int } K.
\]

**Definition 2.5.** (Aghezzaf and Hachimi [1]): A direction \( d \in \mathbb{R}^n \) is said to be a critical direction for a feasible point \( \bar{x} \in S \) if it satisfies the following conditions:
(a) \( \nabla f(\bar{x}) d \leq 0 \),
(b) \( d^T \nabla f_i(\bar{x}) = 0 \) for at least one \( i \in I = \{1, 2, \ldots, m\} \),
(c) \( d^T \nabla g_j(\bar{x}) \leq 0, \ j \in J(\bar{x}) \).

The set of all critical directions at \( \bar{x} \) is denoted by \( A(\bar{x}) \).

Motivated by Aghezzaf and Hachimi [1], we present the following modified version of the second order necessary optimality conditions for efficiency of a feasible point \( \bar{x} \) in (VOP).

**Theorem 2.6.** Let \( f \) and \( g \) be twice differentiable functions on \( \mathbb{R}^n \). Assume that \( \bar{x} \in S \) is an (weakly) efficient solution in (VOP) at which the second order Abadie constraint qualification (ACQ) [1] is satisfied. Then, for every \( d \in A(\bar{x}) \), there exist \( \lambda \in K^* \setminus \{0\} \) and \( \mu \in Q^* \) such that
\[
\begin{align*}
\lambda^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}) &= 0, \\
\lambda^T d^T \nabla^2 f(\bar{x}) d + \mu^T d^T \nabla^2 g(\bar{x}) d &\geq 0, \\
\mu^T g(\bar{x}) &= 0, \\
\lambda^T \nabla f(\bar{x}) d &= 0, \\
\mu^T_j d^T \nabla g_j(\bar{x}) &= 0, \ \forall \ j \in J(\bar{x}).
\end{align*}
\]

We need the following lemma in order to prove the main results in the subsequent sections.

**Lemma 2.7.** (Li and Li [10]): Let \( K \) be a closed convex cone of topological vector space \( X \) with nonempty interior. Then, for any \( x, y \in X \), the following statements are true:
(i) \( y - x \in K, \ y \in -K \Rightarrow x \in -K \),
(ii) \( y - x \in K, \ y \in -\text{int } K \Rightarrow x \in -\text{int } K \),
(iii) \( y - x \in K, \ x \not\in - \text{int} K \Rightarrow y \not\in - \text{int} K \),
(iv) \( y - x \in K, \ y \in - K \setminus \{0\} \Rightarrow x \in - K \setminus \{0\} \),
(v) \( y - x \in \text{int} K, \ y \in - K \setminus \{0\} \Rightarrow x \in - \text{int} K \).

3. Vector optimization problem with the second order modified objective function and optimality conditions. Let \( \bar{x} \) be an arbitrary given feasible solution to the considered vector optimization problem (VOP). Now, for the considered multiobjective optimization problem (VOP), we construct a vector optimization problem (VOP)\(^2\)(\( \bar{x} \)) with the second order modified objective function as follows:

\[
\text{(VOP)}^2(\bar{x}) K \ 	ext{minimize} \ f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x})
\]

subject to \(- g(x) \in Q\),

where the functions \( f, g \) and cones \( K \) and \( Q \) are defined in the similar way as for problem (VOP).

**Remark 1.** Note that the set of all feasible solutions of \( \text{(VOP)}^2(\bar{x}) \) is the same as in (VOP).

Now, we prove the equivalence between an (weakly) efficient solution in the original multiobjective programming problem (VOP) and an (weakly) efficient solution in its associated vector optimization problem (VOP)\(^2\)(\( \bar{x} \)) with the second order modified objective function.

**Theorem 3.1.** Let \( \bar{x} \) be a weakly efficient solution in (VOP) at which the second order \((ACQ)\) is satisfied. Assume that the constraint function \( g \) is second order \(Q\)-convex at \( \bar{x} \) on \( S \). Then \( \bar{x} \) is also a weakly efficient solution in (VOP)\(^2\)(\( \bar{x} \)).

**Proof.** Since \( \bar{x} \) is a weakly efficient solution in (VOP) at which the second order (ACQ) is satisfied, therefore, the second order necessary optimality conditions (1)-(5) hold together with multipliers \( \bar{\lambda} \in K^* \setminus \{0\} \) and \( \bar{\mu} \in Q^* \). Suppose, contrary to the result, that \( \bar{x} \) is not a weakly efficient solution in (VOP)\(^2\)(\( \bar{x} \)). Then, by Definition 2.4 (b), there exists another solution \( \hat{x} \in S \) such that

\[
\left( \nabla f(\bar{x})(\hat{x} - \bar{x}) + \frac{1}{2}(\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})(\hat{x} - \bar{x}) \right)
- \left( \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x}) \right) \in - \text{int} K.
\]

Thus,

\[
\left( \nabla f(\bar{x})(\hat{x} - \bar{x}) + \frac{1}{2}(\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})(\hat{x} - \bar{x}) \right) \in - \text{int} K.
\]

Since \( \bar{\lambda} \in K^* \setminus \{0\} \), therefore, using Lemma 2.1 (i), the above relation can be written as

\[
\bar{\lambda}^T \nabla f(\bar{x})(\hat{x} - \bar{x}) + \frac{1}{2} \bar{\lambda}^T (\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})(\hat{x} - \bar{x}) < 0.
\] (6)

On the other hand, since \( \bar{\mu} \in Q^* \), therefore, by the feasibility of \( \hat{x} \) and (3), we have

\[
\bar{\mu}^T g(\hat{x}) - \bar{\mu}^T g(\bar{x}) \leq 0.
\] (7)
By assumption, \( g \) is second order \( Q \)-convex at \( \bar{x} \in S \). Hence, by Definition 2.2, we have
\[
g(\hat{x}) - g(\bar{x}) - \nabla g(\bar{x})(\hat{x} - \bar{x}) - (\hat{x} - \bar{x})^T \nabla^2 g(\bar{x})d + \frac{1}{2}d^T \nabla^2 g(\bar{x})d \in Q, \quad \forall d \in \mathbb{R}^n.
\]
In particular, this is also true for \( d = \hat{x} - \bar{x} \in \mathbb{R}^n \). Thus,
\[
g(\hat{x}) - g(\bar{x}) - \nabla g(\bar{x})(\hat{x} - \bar{x}) - \frac{1}{2}(\hat{x} - \bar{x})^T \nabla^2 g(\bar{x})(\hat{x} - \bar{x}) \in Q.
\]
Since \( \bar{\mu} \in Q^* \), the above relation gives
\[
\bar{\mu}^T g(\hat{x}) - \bar{\mu}^T g(\bar{x}) - \bar{\mu}^T \nabla g(\bar{x})(\hat{x} - \bar{x}) - \frac{1}{2} \bar{\mu}^T (\hat{x} - \bar{x})^T \nabla^2 g(\bar{x})(\hat{x} - \bar{x}) \geq 0.
\]
By using (7), the above inequality yields
\[
\bar{\mu}^T \nabla g(\bar{x})(\hat{x} - \bar{x}) + \frac{1}{2} \bar{\mu}^T (\hat{x} - \bar{x})^T \nabla^2 g(\bar{x})(\hat{x} - \bar{x}) \leq 0. \tag{8}
\]
Combining (6) and (8), we get
\[
(\bar{\lambda}^T \nabla f(\hat{x}) + \bar{\mu}^T \nabla g(\bar{x}))(\hat{x} - \bar{x}) + \frac{1}{2} \left( \bar{\lambda}^T (\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})(\hat{x} - \bar{x}) + \bar{\mu}^T (\hat{x} - \bar{x})^T \nabla^2 g(\bar{x})(\hat{x} - \bar{x}) \right) < 0.
\]
The above inequality together with condition (1) imply that the following inequality
\[
\bar{\lambda}^T (\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})(\hat{x} - \bar{x}) + \bar{\mu}^T (\hat{x} - \bar{x})^T \nabla^2 g(\bar{x})(\hat{x} - \bar{x}) < 0
\]
holds, which contradicts (2) for \( d = \hat{x} - \bar{x} \). This completes the proof. \( \square \)

**Theorem 3.2.** Let \( \bar{x} \) be an efficient solution of the original multiobjective optimization problem (VOP) at which the necessary optimality conditions (1)-(5) are fulfilled with multipliers \( \lambda \) and \( \mu \) and, moreover, the second order Abadie Constraint Qualifications (ACQ) \( 1 \) is satisfied. Assume that the constraint function \( g \) is second order \( Q \)-convex at \( \bar{x} \) on \( S \). If the Lagrange multiplier \( \bar{\lambda} \) satisfies \( \bar{\lambda} \in \text{int}K^* \), then \( \bar{x} \) is also an efficient solution in the associated vector optimization problem (VOP)**\( 2(\bar{x}) \) with the second order modified objective function.

**Proof.** By assumption, \( \bar{x} \) is an efficient solution in problem (VOP) at which the second order (ACQ) is satisfied. Hence, the second order necessary optimality conditions (1)-(5) are fulfilled at \( \hat{x} \) with Lagrange multipliers \( \bar{\lambda} \in K^* \setminus \{0\} \) and \( \bar{\mu} \in Q^* \). Suppose, contrary to the result, that \( \hat{x} \) is not an efficient solution in problem (VOP)**\( 2(\bar{x}) \). Then, by Definition 2.4 (a), there exists a point \( \hat{x} \in S \) such that
\[
\left( \nabla f(\bar{x})(\hat{x} - \bar{x}) + \frac{1}{2}(\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})(\hat{x} - \bar{x}) \right) - \left( \nabla f(\bar{x})(\bar{x} - \bar{x}) + \frac{1}{2}(\bar{x} - \bar{x})^T \nabla^2 f(\bar{x})(\bar{x} - \bar{x}) \right) \in -K \setminus \{0\}.
\]
Thus, the above relation can be re-written as follows:
\[
\nabla f(\bar{x})(\hat{x} - \bar{x}) + \frac{1}{2}(\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})(\hat{x} - \bar{x}) \in -K \setminus \{0\}.
\]
Since the Lagrange multiplier \( \bar{\lambda} \) is assumed to satisfy \( \bar{\lambda} \in \text{int} K^* \), therefore, using Lemma 2.1 (ii), the above relation can be written as
\[
\bar{\lambda}^T \nabla f(\bar{x})(\hat{x} - \bar{x}) + \frac{1}{2} \bar{\lambda}^T (\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})(\hat{x} - \bar{x}) < 0.
\]
The last part of this proof is the same as the proof of Theorem 3.1 and, therefore, it has been omitted in the paper.

Now, we prove the converse results to those ones established in Theorems 3.1 and 3.2.

**Theorem 3.3.** Let the objective function $f$ be second order $K$-convex at $\bar{x}$ on $S$. If $\bar{x}$ is a weakly efficient solution in the vector optimization problem (VOP) $^2(\bar{x})$ with the second order modified objective function, then $\bar{x}$ is also a weakly efficient solution in the original twice differentiable vector optimization problem (VOP).

**Proof.** Let $\bar{x}$ be a weakly efficient solution in problem (VOP) $^2(\bar{x})$. Suppose, contrary to the result, that $\bar{x}$ is not a weakly efficient solution in problem (VOP). Then, by Definition 2.4 (b), there exists a point $\hat{x} \in S$ such that
\[
f(\hat{x}) - f(\bar{x}) \in -\text{int } K.\tag{9}
\]
Since the objective function $f$ is second order $K$-convex at $\bar{x}$ on $S$, therefore, by Definition 2.2, we have
\[
f(\hat{x}) - f(\bar{x}) - \nabla f(\bar{x})(\hat{x} - \bar{x}) - (\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})d + \frac{1}{2} d^T \nabla^2 f(\bar{x})d \in K, \ \forall d \in R^n.
\]
Hence, the above relation is also true for $d = \hat{x} - \bar{x} \in R^n$. Thus, the above relation yields
\[
f(\hat{x}) - f(\bar{x}) - \nabla f(\bar{x})(\hat{x} - \bar{x}) - \frac{1}{2}(\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})(\hat{x} - \bar{x}) \in K.\tag{10}
\]
Combining (9), (10) and using Lemma 2.7 (ii), we get that the following relation holds, which contradicts the assumption that $\bar{x}$ is a weakly efficient solution in (VOP). Thus, the proof of this theorem is completed.

**Theorem 3.4.** Let the objective function $f$ be second order $K$-convex at $\bar{x}$ on $S$. If $\bar{x}$ is an efficient solution in the vector optimization problem (VOP) $^2(\bar{x})$ with the second order modified objective function, then $\bar{x}$ is also an efficient solution in the original twice differentiable vector optimization problem (VOP).

**Proof.** Let $\bar{x}$ be an efficient solution in problem (VOP) $^2(\bar{x})$, but it is not efficient in problem (VOP). Then, by Definition 2.4 (a), there exists a point $\hat{x} \in S$ such that
\[
f(\hat{x}) - f(\bar{x}) \in -K\backslash \{0\}.\tag{11}
\]
Since the objective function $f$ is second order $K$-convex at $\bar{x}$ on $S$, therefore, by Definition 2.2, we have
\[
f(\hat{x}) - f(\bar{x}) - \nabla f(\bar{x})(\hat{x} - \bar{x}) - (\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})d + \frac{1}{2} d^T \nabla^2 f(\bar{x})d \in K, \ \forall d \in R^n.
\]
In particular, this is also true for $d = \hat{x} - \bar{x} \in R^n$. Hence, the above relation gives
\[
f(\hat{x}) - f(\bar{x}) - \nabla f(\bar{x})(\hat{x} - \bar{x}) - \frac{1}{2}(\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})(\hat{x} - \bar{x}) \in K.\tag{12}
\]
On combining (11), (12) and using Lemma 2.7 (iv), we get
\[
\nabla f(\bar{x})(\hat{x} - \bar{x}) + \frac{1}{2}(\hat{x} - \bar{x})^T \nabla^2 f(\bar{x})(\hat{x} - \bar{x}) \in -K\backslash \{0\},
\]
which contradicts the assumption that $\bar{x}$ is an efficient solution in (VOP). This completes the proof of the theorem.
Now, we give an example to illustrate the result established in Theorem 3.4.

**Example 3.1** Let $K = \{(x, y) \in R^2 : x \leq 0, x \leq -y\}$ and $Q = \{(x, y) \in R^2 : -x \leq y, y \geq 0\}$ be two closed convex cones in $R^2$. Let us consider the following vector optimization problem:

(VOP1) $K - \text{minimize } f(x) = (-x^2 - x^4 \arctan x)^2, 1 + x^2 + x^3 \cos x$

subject to $-g(x) = -(x^2 + 2x + 1, -x - 1) \in Q$,

where $f : R \to R^2$, $g : R \to R^2$. Note that the set of all feasible solutions of (VOP1) is given by $S = \{x \in R : -1 \leq x \leq 0\}$. One can easily verify at $\bar{x} = 0$ that

$$f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x}) - (x - \bar{x})^T \nabla^2 f(\bar{x})d + \frac{1}{2}d^T \nabla^2 f(\bar{x})d$$

$$= (-x - d)^2 - x^4 \arctan x)^2, (x - d)^2 + x^3 \cos x) \in K, \forall d \in R.$$
Proof. Since \( \bar{g}(VOP) \) for the second order Lagrange function in the modified vector optimization problem (VOP) at which the second order (ACQ) holds for any \( g \in S \). Then there exist \( \bar{\lambda} \in K^* \backslash \{0\} \) and \( \bar{\mu} \in Q^* \) such that \( (x, \bar{\lambda}, \bar{\mu}) \) is a second order saddle-point for the second order Lagrange function in the modified vector optimization problem (VOP) with the second order modified objective function.

Let \( \bar{x} \) be an (weakly) efficient solution in (VOP) at which the second order (ACQ) is satisfied. Further, the necessary optimality conditions (1)-(5) are fulfilled with Lagrange multipliers \( \bar{\lambda} \in K^* \backslash \{0\} \) and \( \bar{\mu} \in Q^* \). Further, the constraint function \( g \) is second order \( Q \)-convex at \( \bar{x} \) on \( S \). Hence, by Definition 2.2, it follows that
\[
g(x) - g(\bar{x}) - \nabla g(\bar{x})(x - \bar{x}) - (x - \bar{x})^T \nabla^2 g(\bar{x})d + \frac{1}{2} d^T \nabla^2 g(\bar{x})d \in Q, \ \forall \ d \in \mathbb{R}^n, \ x \in S.
\]
In particular, the above relation is also true for \( d = x - \bar{x} \in \mathbb{R}^n \). Thus,
\[
g(x) - g(\bar{x}) - \nabla g(\bar{x})(x - \bar{x}) - \frac{1}{2} (x - \bar{x})^T \nabla^2 g(\bar{x})(x - \bar{x}) \in Q.
\]
Since \( \bar{\mu} \in Q^* \), the above relation gives
\[
\bar{\mu}^T g(x) - \bar{\mu}^T g(\bar{x}) \geq \bar{\mu}^T \nabla g(\bar{x})(x - \bar{x}) + \frac{1}{2} \bar{\mu}^T (x - \bar{x})^T \nabla^2 g(\bar{x})(x - \bar{x}).
\]
Using \( \bar{\mu} \in Q^* \) together with the feasibility of \( x \) and the necessary optimality condition (3), we obtain
\[
\bar{\mu}^T \nabla g(\bar{x})(x - \bar{x}) + \frac{1}{2} \bar{\mu}^T (x - \bar{x})^T \nabla^2 g(\bar{x})(x - \bar{x}) \leq 0. \tag{13}
\]
Hence, by the necessary optimality conditions (1) and (2), (13) implies
\[
\bar{\lambda}^T \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2} \bar{\lambda}^T (x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x}) \geq 0. \tag{14}
\]
Thus, (14) yields
\[
\bar{\lambda}^T f(\bar{x}) + \bar{\mu}^T g(\bar{x}) + \bar{\lambda}^T \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2} \bar{\lambda}^T (x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x})
\]
\[
\geq \bar{\lambda}^T f(\bar{x}) + \bar{\mu}^T g(\bar{x}) + \bar{\lambda}^T \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2} \bar{\lambda}^T (x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x}).
\]
Thus, by the definition of the second order Lagrange function (Definition 4.1), the following inequality
\[
L^2(\bar{x}, \bar{\lambda}, \bar{\mu}) \leq L^2(x, \bar{\lambda}, \bar{\mu}), \ \forall \ x \in S \tag{15}
\]
holds. Further, by the necessary optimality condition (3) and feasibility of \( \bar{x} \), the following inequality
\[
\mu^T g(\bar{x}) \leq \bar{\mu}^T g(\bar{x}) \tag{16}
\]
holds for any \( \mu \in Q^* \). Thus, (16) gives
\[
\bar{\lambda}^T f(\bar{x}) + \bar{\mu}^T g(\bar{x}) + \bar{\lambda}^T \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2} \bar{\lambda}^T (x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x})
\]
\[
\leq \bar{\lambda}^T f(\bar{x}) + \bar{\mu}^T g(\bar{x}) + \bar{\lambda}^T \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2} \bar{\lambda}^T (x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x}).
\]
Hence, by the definition of the second order Lagrange function (Definition 4.1), we get
\[
L^2(x, \bar{\lambda}, \mu) \leq L^2(\bar{x}, \bar{\lambda}, \bar{\mu}), \ \forall \ \mu \in Q^*. \tag{17}
\]
Thus, by (15) and (17), we conclude that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a saddle-point of the Lagrange function defined in problem $(VOP)^2(\bar{x})$. This completes the proof. \hfill \Box

**Theorem 4.4.** Let $\bar{x} \in S$ and the objective function $f$ be second order strictly $K$-convex at $\bar{x}$ on $S$. If $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in S \times K^* \setminus \{0\} \times Q^*$ is a second order saddle-point of the second order Lagrange function for the vector optimization problem $(VOP)^2(\bar{x})$ with the second order modified objective function, then $\bar{x}$ is an efficient solution in $(VOP)$.

**Proof.** Assume that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a saddle-point of the Lagrange function defined in the vector optimization problem $(VOP)^2(\bar{x})$ with the second order modified objective function. Hence, by Definition 4.2 (i), we have

$$L^2(\bar{x}, \bar{\lambda}, \mu) \leq L^2(\bar{x}, \bar{\lambda}, \bar{\mu}), \ \forall \mu \in Q^*.$$  

By the definition of the Lagrange function in problem $(VOP)^2(\bar{x})$, it follows that

$$\bar{\lambda}^T f(\bar{x}) + \mu^T g(\bar{x}) + \bar{\lambda}^T \nabla f(\bar{x})(\bar{x} - \bar{x}) + \frac{1}{2} \bar{\lambda}^T (\bar{x} - \bar{x})^T \nabla^2 f(\bar{x})(\bar{x} - \bar{x})$$

$$\leq \bar{\lambda}^T f(\bar{x}) + \bar{\mu}^T g(\bar{x}) + \bar{\lambda}^T \nabla f(\bar{x})(\bar{x} - \bar{x}) + \frac{1}{2} \bar{\lambda}^T (\bar{x} - \bar{x})^T \nabla^2 f(\bar{x})(\bar{x} - \bar{x}).$$

Thus, the above inequality gives

$$\mu^T g(\bar{x}) \leq \bar{\mu}^T g(\bar{x}), \ \forall \mu \in Q^*. \quad (18)$$

If we set $\mu = 0$ in the above inequality, then we obtain

$$\bar{\mu}^T g(\bar{x}) \geq 0. \quad (19)$$

Since $\bar{\mu} \in Q^*$, therefore, from the feasibility of $\bar{x}$, we have

$$\bar{\mu}^T g(\bar{x}) \leq 0. \quad (20)$$

On combining (19) and (20), we get

$$\bar{\mu}^T g(\bar{x}) = 0.$$  

Now, suppose, contrary to the result, that $\bar{x}$ is not an efficient solution in the original twice differentiable vector optimization problem $(VOP)$. Then, by Definition 2.4 (a), there exists $\bar{x} \in S$ such that

$$f(\bar{x}) - f(\bar{x}) \in -K \setminus \{0\}. \quad (21)$$

Since the objective function $f$ is second order strictly $K$-convex at $\bar{x}$ on $S$, therefore, by Definition 2.3, it follows that the following relation

$$f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x}) - (x - \bar{x})^T \nabla^2 f(\bar{x})d + \frac{1}{2} d^T \nabla^2 f(\bar{x})d \in \text{int } K, \ \forall d \in \mathbb{R}^n \quad (22)$$

holds for all $x \in S$. In particular, it is also fulfilled for $x = \bar{x} \in S$ and $d = \bar{x} - \bar{x} \in \mathbb{R}^n$. Thus, (22) yields

$$f(\bar{x}) - f(\bar{x}) - \nabla f(\bar{x})(\bar{x} - \bar{x}) - \frac{1}{2} (\bar{x} - \bar{x})^T \nabla^2 f(\bar{x})(\bar{x} - \bar{x}) \in \text{int } K. \quad (23)$$

Combining (21), (23) and using Lemma 2.7 (v), we get

$$\nabla f(\bar{x})(\bar{x} - \bar{x}) + \frac{1}{2} (\bar{x} - \bar{x})^T \nabla^2 f(\bar{x})(\bar{x} - \bar{x}) \in -\text{int } K. \quad (24)$$

Since $\bar{\lambda} \in K^* \setminus \{0\}$, by Lemma 2.1 (i), (24) gives

$$\bar{\lambda}^T \nabla f(\bar{x})(\bar{x} - \bar{x}) + \frac{1}{2} \bar{\lambda}^T (\bar{x} - \bar{x})^T \nabla^2 f(\bar{x})(\bar{x} - \bar{x}) < 0.$$
Thus,
\[
\lambda^T f(\bar{x}) + \mu^T g(\bar{x}) + \lambda^T \nabla f(\bar{x})(\bar{x} - \bar{x}) + \frac{1}{2} \lambda^T (\bar{x} - \bar{x})^T \nabla^2 f(\bar{x})(\bar{x} - \bar{x}) < \lambda^T f(\bar{x}) + \mu^T g(\bar{x}) + \lambda^T \nabla f(\bar{x})(\bar{x} - \bar{x}) + \frac{1}{2} \lambda^T (\bar{x} - \bar{x})^T \nabla^2 f(\bar{x})(\bar{x} - \bar{x}).
\]

Now, using the definition of the second order Lagrange function (Definition 4.1), we get
\[
L^2(\bar{x}, \lambda, \mu) < L^2(\bar{x}, \bar{\lambda}, \bar{\mu}).
\]
This contradicts inequality (ii) in the definition of a second order saddle-point of the second order Lagrange function defined for (VOP). Hence, \( \bar{x} \) is an efficient solution in (VOP). This completes the proof.

\[\square\]

**Theorem 4.5.** Let \( \bar{x} \in S \) and the objective function \( f \) be second order \( K \)-convex at \( \bar{x} \) on \( S \). If \( (\bar{x}, \bar{\lambda}, \bar{\mu}) \in S \times K^* \setminus \{0\} \times Q^* \) is a second order saddle-point of the second order Lagrange function defined for the vector optimization problem \((VOP)^2(\bar{x})\) with the second order modified objective function, then \( \bar{x} \) is a weakly efficient solution in (VOP).

**Proof.** Proof of this theorem is similar to that of Theorem 4.4 and hence it is omitted in the paper. \( \square \)

Now, we give an example of a twice differentiable vector optimization problem with cone constraints in order to illustrate the result established in Theorem 4.4.

**Example 4.1** Let \( K = \{(x, y) \in \mathbb{R}^2 : 2x - y \leq 0, y \geq 0\} \) and \( Q = \{(x, y) \in \mathbb{R}^2 : x \leq 0, y \leq 0\} \) be two closed convex cones in \( \mathbb{R}^2 \). Let us consider the following vector optimization problem with cone constraints:

\[
\text{(VOP2) } K \text{ minimize } f(x) = \left( x^2 \log(x + 1) - (\tan x)^3 + x \sin x, x^2 \arctan x + (\arctan x)^2 + (\arctan x)^3 + (\arcsin x)^2 + (\arcsin x)^3 \right)
\]

subject to \(- g(x) = -(x^3, x - x^2) \in Q,\)

where \( f : R \mapsto \mathbb{R}^2 \), \( g : R \mapsto \mathbb{R}^2 \). Note that the set of all feasible solutions of (VOP2) is given by \( S = \{x \in R : 0 \leq x \leq 1\}. \) Now, we show by Definition 2.3, that the objective function \( f \) is second order strictly \( K \)-convex at \( \bar{x} = 0 \) on \( S \). Indeed, we have

\[
f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x}) - (x - \bar{x})^T \nabla^2 f(\bar{x})d + \frac{1}{2} d^T \nabla^2 f(\bar{x})d
\]

\[
= (x \sin x - 2xd + x^2 \log(x + 1) - (\tan x)^3 + d^2, 2d^2 - 4xd + x^2 \arctan x + (\arctan x)^2 + (\arctan x)^3 + (\arcsin x)^2 + (\arcsin x)^3) \in \text{int} K,
\]

for all \( d \in R \). Hence, by Definition 2.3, \( f \) is second order strictly \( K \)-convex at \( \bar{x} = 0 \) on \( S \).

Now, the corresponding vector optimization problem \((VOP)^2(\bar{x})\) with the second order modified objective function is defined as follows:

\[
\text{(VOP2)} f(\bar{x}) \text{ K minimize } (x^2, 2x^2)
\]

subject to \(- g(x) = -(x^3, x - x^2) \in Q.\)
The second order Lagrangian $L^2 : S \times K^*\{0\} \times Q^* \rightarrow R$ defined in the modified vector optimization problem (VOP2)$^2(\bar{x})$ with the second order modified objective function is given by

$$L^2(x, \lambda, \mu) = (\lambda_1 + 2\lambda_2)x^2.$$ 

Let $\bar{\lambda} = (-1, 1) \in K^*\{0\}, \bar{\mu} = (0, -1) \in Q^*$. Now, by Definition 4.2, we show that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a second order saddle-point of the second order Lagrangian $L^2$ defined in (VOP2)$^2(\bar{x})$. We have

$$L^2(\bar{x}, \bar{\lambda}, \mu) - L^2(\bar{x}, \bar{\lambda}, \bar{\mu}) = (\bar{\lambda}_1 + 2\bar{\lambda}_2)\bar{x}^2 - (\bar{\lambda}_1 + 2\bar{\lambda}_2)x^2 = 0, \ \forall \mu \in Q^*,$$

and

$$L^2(\bar{x}, \bar{\lambda}, \mu) - L^2(x, \bar{\lambda}, \bar{\mu}) = (\bar{\lambda}_1 + 2\bar{\lambda}_2)x^2 - (\bar{\lambda}_1 + 2\bar{\lambda}_2)x^2 = -x^2 \leq 0, \ \forall x \in S.$$ 

Hence, we conclude $(\bar{x}, \bar{\lambda}, \bar{\mu}) = (0, (-1, 1), (0, -1))$ is a second order saddle-point of the second order Lagrangian $L^2$ defined in (VOP2)$^2(\bar{x})$. Since all hypotheses of Theorem 4.4 are fulfilled at $\bar{x} = 0$, by Theorem 4.4, $\bar{x} = 0$ is an efficient solution in the considered vector optimization problem (VOP2).

5. Conclusion. In this paper, the second order modified objective function method has been used for solving the considered vector optimization problem with twice differentiable functions and cone constraints. Then, the equivalence between an (weakly) efficient solution in the original vector optimization problem with cone constraints and an (weakly) efficient solution in its associated vector optimization problem with the second order modified objective function has been established under second order cone convexity hypotheses. Further, the relationship between an (weakly) efficient solution in the original vector optimization problem with cone constraints and a saddle-point of the second order Lagrange function defined for the vector optimization problem with the second order modified objective functions has been investigated also under assumption that the involved functions are second order cone convex. Also the relationship between an (weakly) efficient solution in the original vector optimization problem and a saddle-point of the modified Lagrange function defined for the associated vector optimization problem with the second order modified objective function has been analyzed. It has been illustrated that the original vector optimization problem is, in general, reduced to a simpler one in the used second order modified objective function method. Therefore, it is simpler to solve in such cases. The property of this method is important from the practical point of view. Hence, the second order modified objective function approach for vector optimization problems with twice differentiable functions and cone constraints turned out to be useful to determine (weakly) efficient solutions of a vector optimization problem with a complex objective function by the help of (weakly) efficient solutions and/or saddle points in its associated vector optimization problems with the second order modified objective functions. Furthermore, we shall extend this idea to an $\eta$-approximation approach which will orient the future work of the authors.

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