How successful can the scalar-tensor theory be in understanding the accelerating universe?

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Abstract
The accelerating universe is closely related to today’s version of the cosmological constant problem; fine-tuning and coincidence problems. We show how successfully the scalar-tensor theory, a rather rigid theoretical idea, provides us with a simple and natural way to understand why today’s observed cosmological constant is small only because we are old cosmologically, without fine-tuning theoretical parameters extremely.

1 Introduction

Nearly a decade ago, the universe was discovered to be accelerating, driven by the cosmological constant represented by

$$\Omega_\Lambda = \frac{\Lambda_{\text{obs}}}{\rho_{\text{cr}}} \approx 0.7, \quad \text{with} \quad \rho_{\text{cr}} = 3H_0^2,$$

where $H_0$ is today’s value of the Hubble parameter. Note also that we are using the reduced Planckian units with $c = \hbar = M_P = (8\pi G)^{-1/2} = 1$. We then expect $H_0 \sim t_0^{-1}$, hence

$$\Lambda_{\text{obs}} \sim t_0^{-2},$$

where the present age of the universe is given by $t_0 = 1.37 \times 10^{10} \text{y} \approx 10^{60.2}$ in units of the Planck time. The result $\Lambda_{\text{obs}} \sim 10^{-120}$ poses a fine-tuning problem in today’s version of the cosmological constant problem: Is our theory good enough to the accuracy of 120 orders of magnitude? In addition, the relation (2), to be called $\Lambda$-$t_0$ Correlation, raises an even more serious question: Can we understand the numerical similarity between both sides beyond a mere coincidence? Is this supported by any fundamental theory? We will provide an affirmative reply in terms of the scalar-tensor theory invented first by Jordan in 1955 [2,3].

2 Scalar-tensor theory

The basic Lagrangian in what is called the Jordan (conformal) frame is

$$\mathcal{L} = \sqrt{-g} \left( \frac{1}{2} \xi \phi^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \Lambda + L_{\text{matter}} \right),$$

where $\phi$ is the scalar field, assumed to show itself as dark energy. We use two parameters $\epsilon$ and $\xi$ related to the better known symbol $\omega$ in the way of $\epsilon = \pm 1 = \text{Sgn}(\omega)$ and $\xi = 1/(4|\omega|) > 0$. The first term is a well-known nonminimal coupling term with the effective gravitational “constant,” $G_{\text{eff}}(x) = (8\pi \xi \phi^2)^{-1}$. We included $-\Lambda$. On the matter Lagrangian $L_{\text{matter}}$ we have a special comment.

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In 1961, Brans and Dicke came up with an additional assumption that $\phi$ never enters $L_{\text{matter}}$, because they could save the idea of Weak Equivalence Principle (WEP) only in this way \[4\]. Since then the word “Brans-Dicke theory” has been used widely for the entire theory, though we prefer to use another name, the “Brans-Dicke model,” in a more restricted context, partly because, as we are going to argue later, a departure from their assumption appears to be required eventually for the theory to be applied successfully to the accelerating universe \[3\].

For the later convenience let us give an illustrative example of $L_{\text{matter}}$ for a free massive Dirac field as a convenient representative of matter fields;

$$L_{\text{matter}} = -\bar{\psi} (\partial + m) \psi, \quad \text{where} \quad m = \text{const}. \quad (4)$$

Due to the assumed absence of $\phi$, the mass of this Dirac field is $m$, a pure constant. Generally speaking, the constancy of masses of matter particles is a unique feature of the BD model.

By applying the conformal transformation $g_{\mu\nu} \rightarrow g_{*\mu\nu} = \Omega^2 g_{\mu\nu}$ with a special choice $\Omega^2 = \xi \phi^2$, we move to what is called the Einstein frame, in which the same Lagrangian is re-expressed as

$$L = \sqrt{-g_{*}} \left( \frac{1}{2} R_{*} - \text{Sgn}(\zeta^2) \frac{1}{2} g_{*}^{\mu\nu} \partial_{\mu} \sigma \partial_{\nu} \sigma - V(\sigma) + L_{*\text{matter}} \right), \quad (5)$$

without nonminimal coupling term, as designed, hence a purely constant $G$ as in the standard Einstein-Hilbert term.

Note that the scalar field $\phi$ in \[3\] and the same $\sigma$ in \[5\] are related to each other by

$$\phi = \xi^{-1/2} e^{\zeta \sigma}, \quad \text{with} \quad \zeta^2 = 6 + \epsilon \xi^{-1} = 6 + 4 \omega. \quad (6)$$

Also the constant term $\Lambda$ in \[3\] has been converted to the potential $V(\sigma) = \Lambda e^{-4\epsilon \sigma}$ in \[5\]. Otherwise, we put the symbol $*$ nearly everywhere. Also again, however, for the later convenience to discuss simple cosmology, we add

$$a_* = \Omega a, \quad \text{and} \quad dt_* = \Omega dt, \quad (7)$$

for the scale factor $a$ and the cosmic time $t$, respectively. According to the first of \[7\] the way of cosmological expansion differs from frame to frame.

We also transform the matter fields like \[5\]

$$L_{*\text{matter}} = -\bar{\psi}_* (\partial + m_*) \psi_*, \quad \psi_* = \Omega^{-3/2} \psi, \quad (8)$$

together with the transformation rule for the mass;

$$m_* = \Omega^{-1} m. \quad (9)$$

We compare \[3\] with string theory. For the closed strings, we find an effective Lagrangian

$$L_{\text{string}} = \sqrt{-\bar{g} e^{-2\Phi}} \left( \frac{1}{2} \bar{R} + 2 g_{\bar{\mu}\bar{\nu}} \partial_{\bar{\mu}} \Phi \partial_{\bar{\nu}} \Phi - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right), \quad (10)$$

as Eq. (3.4.58) of \[6\], in higher-dimensional spacetime. Note the presence of a scalar field $\Phi$, called dilaton. By introducing $\phi = 2 e^{-\Phi}$, we can re-express the first two terms with the result precisely the
same as the first two terms in \( \text{(3)} \), provided \( \epsilon = -1 \) and \( \xi = 1/4 \), or \( \omega = -1 \). This suggests that our Jordan frame corresponds to the world in which unification is realized. This also justifies the way of including \( \Lambda \) of the Planckian size in \( \text{(3)} \). We may thus call the Jordan frame a “string” frame or “theoretical” frame. Also it may even appears as if Jordan’s theory had to wait for decades before it was rediscovered later by string theory.

Then how about the “physical” or “observational” frame? According to Dicke in this connection, the conformal transformation is a local change of units. Let us emphasize this view on the units.

Suppose we use an atomic clock, measuring time in reference to the frequency of certain atomic transition, in which we have the fundamental unit provided by the electron mass \( m_e \). Then we find no way to detect any change, if any, of \( m_e \) itself, as long as we continue to use the atomic clock. This might be re-expressed by a more general term, “own-unit-insensitivity principle”. Using atomic clocks then implies that we are in the physical frame in which \( m_e \) is kept constant. According to what we pointed out on the BD model, the constancy of \( m_e \) indicates the Jordan frame, implying the physical frame is precisely the Jordan frame. The same simple argument can be extended to a wider class of astronomical observations based on measuring redshift of atomic spectra, again with the fundamental unit provided by \( m_e \) as in using atomic clocks. In this sense we repeat the above statement on the identification of the Jordan frame with the physical frame, as far as we accept the BD model.

### 3 Cosmology

Let us go on to discuss cosmology in the presence of \( \Lambda \) assumed positive and of the Planckian size, first in the Jordan frame. As usual we make simplifying assumptions of the metric in the radiation-dominated universe. We also assume \( \phi \) to be spatially uniform depending only on the cosmic time \( t \). We then write down equations and seek the asymptotic solutions, given in Chapter 4.4.1 of [3]:

\[
\begin{align*}
a &= \text{const}, \quad \text{or} \quad H = \frac{\dot{a}}{a} = 0, \\
\phi &= \sqrt{\frac{4\Lambda}{6\xi + \epsilon}} t, \\
\rho &= -3\Lambda \frac{2\xi + \epsilon}{6\xi + \epsilon} = \text{const}.
\end{align*}
\]

(11)

(12)

(13)

Skipping all the details, we focus upon (11). Rather unexpectedly this implies a static universe. We emphasize that this solution is not only asymptotic but also an attractor solution which any solution starting with whatever initial values tends to, as was reconfirmed by our recent reanalysis. Due to this unrealistic aspect, the Jordan frame can be hardly accepted as a physical frame, contrary to what we discussed before.

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1. The reduced mass should be preferred in principle. The required details are straightforward, but will be avoided for the time being for simplicity.
In this connection we point out that our solution fails to show a smooth behavior in the limit $\Lambda \to 0$, suggesting that in the presence of $\Lambda$, the solution can be different from what had been derived in its absence, resulting in a drastic change.

Now in the Einstein frame the asymptotic solutions are obtained in Chapter 4.4.2 of [3]:

$$a_* = t_*^{1/2},$$  \hspace{1cm} (14)

$$\sigma = \dot{\sigma} + \frac{1}{2} \zeta^{-1} \ln t_*, \quad \text{with} \quad \Lambda e^{-4\zeta \sigma} = \frac{1}{16} \zeta^{-2},$$  \hspace{1cm} (15)

$$\rho_\sigma = \frac{1}{2} \dot{\sigma} + V(\sigma) = \frac{3}{16} \zeta^{-2} t_*^{-2},$$  \hspace{1cm} (16)

$$\rho_* = \Omega^{-4} \rho = \frac{3}{4} \left(1 - \frac{1}{4} \zeta^{-2}\right) t_*^{-2},$$  \hspace{1cm} (17)

where $t_*$ is the cosmic time in the Einstein frame, obtained to be $t_* \sim t^2$ in accordance with the second of (7). We also have $\dot{\sigma} = d\sigma/dt_*$. According to (14) the universe now expands, as expected from the discussion on the first of (7), precisely in the same way as in the ordinary radiation-dominated universe\(^2\) hence tempting us to accept the Einstein frame as the physical frame. However, examining (7) and (9) together with the second of (4), we find\(^3\)

$$m_* \sim t_*^{-1/2},$$  \hspace{1cm} (18)

which fails to be constant obviously in contradiction with the own-unit-insensitivity principle, a condition necessary to be a physical frame. We again face an unrealistic universe, no way to accommodate a physical frame.

4 \hspace{1cm} **Leaving the Brans-Dicke model**

We then wonder if we find a way out by somehow demanding a static $m_*$, still keeping the expansion\(^4\) as it is, so that we can accept the Einstein frame as a physical frame. This forces us finally to decide to *leave* the BD model. In fact we decided to revise our previous choice of $L_{\text{matter}}$ as in (4), by replacing the mass term by the Yukawa-type coupling, as shown by

$$L_{\text{matter}} = -\overline{\psi} (\partial^2 + f \phi) \psi,$$  \hspace{1cm} (19)

where $f$ is a dimensionless coupling constant.

In this way we derive a constant $m_*$, hence achieving the goal to identify the Einstein frame as the physical frame. The simple feature of the dimensionlessness of $f$ in (19) is shown to be shared by any other terms in the basic Lagrangian\(^5\) except for the $\Lambda$ term. In a sense we are having a global scale-invariance, hence the name the “scale-invariant” model in place of the BD model\(^6\).

\(^2\)This turns out to be rather accidental, because the same behavior follows even for the dust-dominance\(^6\).

\(^3\)Combining these with the first of (11) leads to the condition $am = a_* m_* = \text{const}$, which implies that the universe, hence the inter-galactic distances, grow with the same rate as the microscopic meter-stick provided by $m^{-1}$ or $m_*^{-1}$, in totally inconsistency with today’s concept of the expanding universe.
This is, however, not the end of the story. We allowed $\phi$ to enter $L_{\text{matter}}$ in violation of the assumption due to Brans and Dicke. For this reason we face the WEP violating terms, which fortunately turn out to be unobservable in the classical limit. But quantum effects arising from the interactions among matter fields will regenerate the WEP violating terms, though occurring somewhat suppressed according to the estimates by means of quantum anomalies, a well-established technique in the relativistic quantum field theory [3]. We point out that this conclusion hinges upon the static universe solution of the cosmological equation in the presence of $\Lambda$, as well as the simple and straightforward analysis on how to define the physical conformal frames.

Finally we come back to the Correlation, mentioned at the beginning. As we point out, an important clue is already found in (16) in the solution in the Einstein frame, which has been shown to be a physical frame, where $\rho_\sigma$ is the dark energy density, hence is interpreted as $\Lambda_{\text{eff}}$, thus giving

$$\Lambda_{\text{eff}} \sim t_*^{-2},$$

(20)

also to be called Scenario of a decaying cosmological constant. This Scenario does include the Correlation [2] re-interpreted in the Einstein frame, now extended to much wider time span. Obviously, we reproduce our previous estimate $\Lambda_{\text{eff}} \sim 10^{-120}$ for $t_{*0} \sim 10^{60}$, though we no longer appeal to an extreme and unnatural fine-tuning process; the left-hand side is small nearly automatically, only because we are old enough as indicated on the right-hand side, somewhat a la Dirac [12]. Allowing us to deal with the number of this small so naturally is certainly a major success of the scalar-tensor theory, an advantage not shared by many other phenomenological approaches.

Unfortunately this is still short of a complete success because the smooth behavior $\rho_\sigma \sim t_*^{-2}$ in (16) is not the way we expect an extra acceleration of the universe, which we are now watching. But what we need can be only a relatively small deviation off the dominant behavior, though in a rather phenomenological way at this moment, probably with different laws for different kinds of behaviors.

Without entering into any further details, we show an example of the solutions plotted in Fig. 1, intended to be more realistic [3]. In the bottom panel, $\rho_\sigma$, the ordinary matter density and $\rho_s$, certain modification of $\rho_\sigma$ to give $\Lambda_{\text{eff}}$, are plotted against log $t_*$. Today is $\sim 60$. We find two densities falling off like $t_*^{-2}$ as a \textit{common overall} behavior. This is the way we inherit the Scenario, but also with sporadic interlacing behaviors, and ensuing extra accelerations of the scale factor including the period around today, as exhibited in the top panel. They represent \textit{non-smooth} behaviors. Summarizing we find that in spite of the non-smooth aspects off the dominant overall behaviors, the presence of the underlying trend based on the simple scalar-tensor theory is unmistakable.

\footnote{The same result had been obtained in [3]. See also footnote 12 of [10], and the second paragraph of section 1 in [3].}

\footnote{Even wider than in [11].}
Figure 1: Taken from Fig. 5.8 of [3]. In the bottom panel, we plot $\rho_\ast$, the ordinary matter density in the Einstein frame, and $\rho_s$, modified version of $\rho_\sigma$ including the contribution from the assumed added scalar field $\chi$, both in the logarithmic scale against $\log t_\ast$, which is $\sim 60$ today. Two densities fall off with a common overall behavior $t_\ast^{-2}$, but with sporadic interlacing behaviors. Correspondingly we find sudden but finite increases of the scale factor $a_\ast$, called a mini-inflations, representing extra accelerations, as shown in the top panel, together also with $\tilde{\alpha}$ for the effective exponent of the scale factor shown in the middle panel.

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