SYMPLECTIC DEFORMATIONS, NON COMMUTATIVE SCALAR FIELDS AND FRACTIONAL QUANTUM HALL EFFECT

M. Daoud\textsuperscript{a} and A. Hamama\textsuperscript{b}

\textsuperscript{a} Max Planck Institute for Physics of Complex Systems, Nöthnitzer Str. 38, D-01187 Dresden, Germany

\textsuperscript{b} High Energy Laboratory, Faculty of Sciences, University Mohamed V, P.O. Box 1014, Rabat, Morocco

Abstract

We clearly show that the symplectic structures deformations lead, upon quantization, to quantum theories of non commutative fields. Two variants of deformations are considered. The quantization is performed and the modes expansions of the quantum fields are derived. The Hamiltonians are given and the degeneracies lifting induced by the deformation is also discussed. As illustration, we consider the noncommutative chiral boson fields in the context of fractional quantum Hall effect. A generalized fractional filling factor is derived and shown to reproduce the Jain Hall states. We also show that the coupling of left and right edge excitations of a quantum Hall sample, gives rise a noncommutative chiral boson theory. The coupling or the non-commutativity induces a shift of the chiral components velocities. A non linear dispersion relation is obtained corroborating some recent analytical and numerical analysis.

\textsuperscript{1}Faculté des Sciences, Département de Physique, Agadir, Morocco; email: m_ daoud@hotmail.com
1 Introduction

The noncommutative space time [1-2] has attracted a great deal of attention during the last decade due to its relevance to quantum aspects of gravity and its connection with the low energy description of string theory in the presence of a constant NS-NS $B$-field [3-4]. Field theories on the Moyal noncommutative space time have various features which are noticeably different from the models on the commutative space-time as for instance the UV/IR mixing [5], twisted symmetries [6-9], twisted statistics [10-15], etc. Beside the high energy physics, noncommutative field theory has found applications in condensed matter issues. The most studied example is the planar system of a collection of fermions evolving under a strong magnetic field. It is now well established that the ideas of noncommutative geometry are relevant in the context of quantum hall effect in the plane [16] as well as other spaces with different geometries and of higher dimensions [17-18].

Usually we refer to field theories in noncommutative space-time as "noncommutative field theories". More recently, a "quantum theory of noncommutative fields" was introduced [19-25] as a generalization of noncommutative quantum mechanics [26-27] and it is different from the quantum field theory constructed over noncommutative space-time. The theory of noncommutative fields is unrelated to space-time non-commutativity. The interest on these new kind of quantum field theories is mainly motivated by the violation of lorentz invariance and may provide a description of the experimentally observed matter/antimatter asymmetry.

In this work, we shall first be concerned with the quantum theory of noncommutative scalar fields where the fields and their conjugate momenta obey deformed equal time commutations rules. More precisely, unlike the usual case, the quantum fields cease to commute among themselves. In the first part of this paper we propose a noncommutative formulation of two dimensional scalar field theory. This formulation is done from a purely symplectic point of view. Indeed, we introduce a deformed symplectic two-form of the phase space associated with (1 + 1)-noncommutative field theory described by two canonical pairs $\varphi^i(x, t)$ and $\pi^i(x, t)$ ($i = 1, 2$). Two variants of symplectic deformations are considered. The first one

$$
\Omega = \sum_i \int dx \delta \varphi^i \wedge \delta \pi^i + \frac{1}{2} \sum_{ij} E_{ij} \int dx \delta \pi^i \wedge \delta \pi^j,
$$

gives rise upon quantization a theory of noncommutative fields with equal time commutation relations given

$$
[\varphi^i(x, t), \varphi^j(y, t)] = -i E_{ij} \delta(x-y) \quad [\pi^i(x, t), \pi^j(y, t)] = 0 \quad [\varphi^i(x, t), \pi^j(y, t)] = i \delta^{ij} \delta(x-y)
$$

where the non-commutativity is encoded in the constant parameter $E_{ij} = -E_{ji}$. The second kind of symplectic deformation

$$
\Omega = \sum_i \int dx \delta \varphi^i \wedge \delta \pi^i - \frac{1}{2} \sum_{ij} B_{ij} \int dx \delta \varphi^i \wedge \delta \varphi^j,
$$

leads to non-commutativity in the momentum space

\[
[\varphi^i(x, t), \varphi^j(y, t)] = 0 \quad [\pi^i(x, t), \pi^j(y, t)] = iB_{ij}\delta(x - y) \quad [\varphi^i(x, t), \pi^j(y, t)] = i\delta^{ij}\delta(x - y)
\]

where the deformation tensor \(B_{ij}\) is antisymmetric and can be viewed as the dual tensor of \(E_{ij}\). In quantizing the theory, we perform a transformation mapping the deformed symplectic two-form in a canonical one. It is remarkable that this map turn out to be identical to the so-called dressing transformation introduced in [25]. Consequently, the dynamics becomes described by a Hamiltonian involving terms encoding the deformation effect and are responsible of the degeneracy lifting of the energy levels of the scalar fields. It is important to stress that the formulation, presented in this first part of our work, start at classical level by deforming the symplectic structure.

In other hand, it is well known that planar fermions in a strong magnetic field are confined in lowest Landau levels and behave like a rigid droplet of liquid. This is the incompressible quantum fluid picture proposed by Laughlin [28] which constitutes the basis of the main advances in this field of research, especially its connection with noncommutative geometry. Indeed, it was shown that Laughlin states at filling factor \(1/k\) can be provided by an appropriate noncommutative finite Chern-Simons matrix model at level \(k\) and hence reproduces the basic features of quantum Hall states [29-30]. In this vain, as mentioned above the ideas of noncommutative geometry were useful to study the quantum Hall phenomenon in different geometries and for arbitrary dimensions [16-18] showing that the effective action for the edge excitations of a quantum hall droplet is generically given by a chiral boson action. For instance, in system with boundaries, like the disc geometry, the excitations reside on the edge of the droplet and the associated dynamics is described by a \((1+1)\) chiral boson theory (see for instance [31]).

The main purpose of the second part of this paper is to give a description of the edge excitations of a quantum Hall droplet using the noncommutative chiral boson theory. An appropriate noncommutative action will be defined and shown to reproduce the basic features of Hall states. The quantization of the proposed model is performed and shown to imply a fractional filling involving the the non-commutativity parameter. This provides us with a model which also reproduces the Jain states [32] for appropriate values of the deformation parameter. It will be also pointed out that the action describing the coupling between excitations of left and right edges on a quantum Hall sample is equivalent to a noncommutative chiral action. The coupling strength between the left and right sectors play the role of the deformation parameter. We show that the coupling of excitations living on right and left edges modify the velocities of the chiral modes. Finally, we will discuss how the noncommutative chiral boson fields induces a nonlinear dispersion relation. This result is corroborated by the field description of quantum Hall edge reconstruction recently proposed in [33].
The paper is organized as follows. In section 2, a brief review of the symplectic structure and classical analysis of the massless scalar field theory is presented. The section 3 concerns the $E$ deformed phase space. We quantize the theory and we give the mode expansions of deformed scalar fields. Similarly, in section 4, the $B$ deformed phase space is defined. The quantization is performed and the quantum theory of $B$-deformed scalar field is given. We discuss the lifting degeneracy induced by the deformation. A general action describing the noncommutative $(1+1)$ chiral boson theory is derived in section 5. The non-commutativity induces a coupling between left and right chiral sectors. The obtained action presents many similitude with one introduced in [31] to classify different hierarchies in abelian fractional Hall effect. The connection between noncommutative chiral theory and fractional Hall effect is established in section 6. In fact, using an appropriate deformed metric, we obtain a generalized fractional filling factor in term of the deformation tensor strength. We also discuss the anyonic like statistics of the electron operators and the deviation from the fermionic statistics caused by the deformation. In section 7, we write the effective action describing the coupling between the left and right components of a scalar field. Using the tools developed in section 5, we show that the coupling induces a shift of the modes velocities. In section 8, we derive a nonlinear dispersion relation of noncommutative edge excitations modes of a quantum Hall droplet. Concluding remarks close this paper.

2 General considerations

As we will essentially concerned with deformed phase space of scalar fields, let us begin by recalling briefly some elements of symplectic structures of the phase space for $(1+1)$ scalar fields theory. More precisely, we introduce the symplectic two form to define the basic geometric structures. This will be useful in building the phase space of scalar fields with modified symplectic structures. For this end, We consider the real massless bosonic field

$$\varphi : \Sigma \longrightarrow \mathbb{R}^2$$

$$(x,t) \longrightarrow (\varphi^1(x,t), \varphi^2(x,t))$$

(1)

described by the action

$$S = \int_{\Sigma} dt dx L = \frac{1}{2l} \sum_{i=1,2} \int_{\Sigma} dt dx ((\partial_t \varphi^i)^2 - (\partial_x \varphi^i)^2)$$

(2)

where the space-time region $\Sigma$ will be taken to be of the form $[0,l] \times [t_i, t_f]$. The fields are confined in a line segment of length $l$. To simplify we fix $l = 2\pi$. With the periodic boundary conditions we can write the fields and their time derivatives as expansions of a set of modes functions

$$\varphi^i(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} q^i_n(t) \exp(-inx)$$

(3)
\[ \partial_t \varphi^i(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} p^i_n(t) \exp(-inx). \] (4)

The normal or Fourier modes \( q_n \) and \( p_n \) satisfy the conditions \( q_{-n} = q_n^* \) and \( p_{-n} = p_n^* \) required by the reality of the fields and their time derivatives. To equip the phase space of the system, whose variables (coordinates) are the scalar fields \( \varphi^i(x,t) \) and the canonical momentum \( \pi^i(x,t) = \partial_t \varphi^i(x,t) \), with a symplectic structure, one introduce the the canonical one form

\[ A(t) = \sum_{i=1,2} \int dx \partial_t \varphi^i \delta \varphi^i \] (5)

where \( \delta \) denotes the exterior derivative on the field space and the time derivatives of the fields must be treated as variables. The corresponding two-form is defined by

\[ \Omega_0 = \delta A(t) = \sum_{i=1,2} \int dx \delta \varphi^i \wedge \delta \pi^i. \] (6)

We may think \( \Omega_0 \) as field strength corresponding to one-form \( A \) viewed as \( U(1) \) gauge potential. It can also be written as

\[ \Omega_0 = \frac{1}{2} \sum_{I,J} \int dx dx' (\Omega_0)^{IJ}(x,x') \delta \xi^I (x) \wedge \delta \xi^J (x') \] (7)

where we denote the phase space coordinates by \( \xi^I (x) \equiv \xi^{i'i'} (x) \) (resp. \( \pi^i (x) \)) for \( i' = 1 \) (resp. for \( i' = 2 \)) and

\[ (\Omega_0)^{IJ}(x,x') \equiv (\Omega_0)_{IJ} \delta(x-x') = \delta_{ij} \epsilon_{i'j'} \delta(x-x'). \]

It follows that the Poisson brackets of two functionals \( \mathcal{F} \) and \( \mathcal{G} \) is given by

\[ \{ \mathcal{F}, \mathcal{G} \} = \sum_{I,J} \int dx (\Omega_0)^{IJ} \frac{\delta \mathcal{F}}{\delta \xi^I} \frac{\delta \mathcal{G}}{\delta \xi^J} = \sum_{i=1,2} \int dx \frac{\delta \mathcal{F}}{\delta \varphi^i} \frac{\delta \mathcal{G}}{\delta \pi^i} - \frac{\delta \mathcal{F}}{\delta \pi^i} \frac{\delta \mathcal{G}}{\delta \varphi^i} \] (8)

where \( \Omega_0^{IJ} \) are the elements of the inverse matrix of \( \Omega_0_{IJ} \). Using the equations (3-4) and (6), the symplectic two-form \( \Omega_0 \) becomes

\[ \Omega_0 = \sum_{i=1,2} \sum_{n \in \mathbb{Z}} \delta q_n^i \wedge \delta p_{-n}^i \] (9)

and the Poisson brackets take the simple form

\[ \{ \mathcal{F}, \mathcal{G} \} = \sum_{i=1,2} \sum_{n \in \mathbb{Z}} \frac{\delta \mathcal{F}}{\delta q_n^i} \frac{\delta \mathcal{G}}{\delta q_n^j} - \frac{\delta \mathcal{F}}{\delta p_{-n}^i} \frac{\delta \mathcal{G}}{\delta p_{-n}^j}. \] (10)

In particular, the Poisson brackets corresponding to the canonical coordinates of the phase space \( q_n^i \) and \( p_n^i \) are given by

\[ \{ q_n^i, q_m^j \} = 0 \quad \{ p_n^i, p_m^j \} = 0 \quad \{ q_n^i, p_m^j \} = \delta_{i,j} \delta_{m+n,0}. \] (11)
The Hamiltonian given by

\[ H_0 = \sum_{i=1,2} \int dx \pi^i \partial_t \varphi^i - \int dx \mathcal{L} = \frac{1}{2} \sum_m (p^i_m p^i_{-m} + n^2 q^i_n q^i_{-n}). \]  

is the generator of time translation. The time evolution of a function \( \mathcal{F} \) is given by the Hamilton’s equation of motion \( \dot{\mathcal{F}} = \{ \mathcal{F}, H_0 \} \) that gives

\[ \frac{dq^i_n}{dt} = \{ q^i_n, H_0 \} = p^i_n \quad \frac{dp^i_n}{dt} = \{ p^i_n, H_0 \} = -n^2 q^i_{-n}. \]  

To pass over the quantum theory in the Heisenberg picture, all canonical variables become Heisenberg operators satisfying commutation relations corresponding to Poisson brackets as

(Poisson Bracket) \( \rightarrow -i \) (commutator).

The equations of motion (13) can be obtained directly from Euler-Lagrange equations. The symplectic procedure discussed in this section has the advantage, as it will be clear below, to be more adequate to deal with modified or deformed symplectic structures.

3 \( \mathcal{E} \)-Deformed scalar fields

3.1 \( \mathcal{E} \)-Deformed phase space

In this section, we consider two abelian fields \( \varphi^1(x, t) \) and \( \varphi^2(x, t) \) described by the Hamiltonian (12) and we suggest to replace the usual symplectic two-form \( \Omega_0 \) by

\[ \Omega = \Omega_0 + \frac{1}{2} \mathcal{E}_{ij} \int dx \delta \xi^i(x) \wedge \delta \xi^j(x). \]  

The new closed two-form \( \Omega \) rewrites in a compact form as

\[ \Omega = \frac{1}{2} \int dx \Omega_{IJ} \delta \xi^I(x) \wedge \delta \xi^J(x). \]  

The constant antisymmetric tensor \( \mathcal{E}_{ij} = \theta \epsilon_{ij} \) encodes the modification or the deformation of the symplectic structure providing the non trivial metric

\[ \Omega_{IJ} = \delta_{ij} \epsilon_{\mathcal{E}ij} + \mathcal{E}_{ij} \delta \xi^i \delta \xi^j \]  

which is non degenerate (\( \det \Omega \neq 0 \)). In order to establish the connection between the classical and quantum theory, we introduce the Poisson brackets defined by

\[ \{ \mathcal{F}, \mathcal{G} \} = \sum_{I,J} \int dx \Omega^{IJ} \frac{\delta \mathcal{F}}{\delta \xi^I} \frac{\delta \mathcal{G}}{\delta \xi^J} \]  

where \( \Omega^{IJ} \) is the inverse matrix of \( \Omega_{IJ} \) (16). Using the equations (3-4), the symplectic form \( \Omega \) and the associated Poisson brackets rewrite

\[ \Omega = \delta q^i_n \wedge \delta p^i_{-n} + \frac{1}{2} \mathcal{E}_{ij} \delta p^i_n \wedge \delta p^j_{-n} \]  

\[ \{ F, G \} = \sum_{in} \frac{\delta F}{\delta q_n^i} \frac{\delta G}{\delta p_{-n}^i} - \frac{\delta F}{\delta p_{-n}^i} \frac{\delta G}{\delta q_n^i} - \sum_{ijn} \epsilon_{ij} \frac{\delta F}{\delta q_n^i} \frac{\delta G}{\delta q_j^{2n}} \]  

respectively, in terms of the normals modes and their conjugate momenta. The Hamiltonian vector fields \( X_F \), associated to the functional \( F \), is given by

\[ X_F = \sum_{in} X_n^i \frac{\delta}{\delta q_n^i} + Y_n^i \frac{\delta}{\delta p_{-n}^i} \]

where

\[ X_n^i = \frac{\delta F}{\delta p_{-n}^i} - \epsilon_{ij} \frac{\delta F}{\delta q_j^{2n}} \quad Y_n^i = -\frac{\delta F}{\delta q_n^{2i}} \]

such that the interior contraction of \( \Omega \) with \( X_F \) verifies \( i(X_F)\Omega = \delta F \). In particular, from the equation (19), one gets the Poisson brackets of the phase space variables

\[ \{ q_n^i, q_n^j \} = -\epsilon_{ij} \delta_{m+n,0} \quad \{ p_n^i, p_n^j \} = 0 \quad \{ q_n^i, p_{-m}^j \} = \delta_{m,n} \delta_{i,j} \]  

reflecting a deviation from the canonical brackets. At this stage, it is remarkable that the symplectic form and the Poisson brackets can be converted in the canonical ones by mean of the following transformation

\[ Q_n^i = q_n^i - \frac{1}{2} \epsilon^{ik} p_n^k \quad P_n^i = p_n^i. \]

with summation over \( k \). Indeed, one can check that

\[ \{ Q_n^i, Q_n^j \} = 0 \quad \{ P_n^i, P_n^j \} = 0 \quad \{ Q_n^i, P_{-m}^j \} = \delta_{m+n,0} \delta_{ij}, \]

and that the symplectic two-form (18) becomes

\[ \Omega = \sum_{in} dQ_n^i \wedge dP_{-n}^i. \]  

The Hamiltonian (12) is now given by

\[ H = \frac{1}{2} \left[ \sum_{in} \frac{1}{4} (4 + \theta^2 n^2) P_n^i P_{-n}^i + n^2 Q_n^i Q_{-n}^i + n^2 \theta \sum_j \epsilon^{ij} P_{-n}^i Q_{-n}^j \right] \]  

in terms of the new dynamical modes of the theory. Evidently the \( \theta \)-dependent term in \( H \) arises from the deformation of the symplectic structure. This suggests that the fields \( \varphi^1 \) and \( \varphi^2 \) interacts with a certain internal potential \( V_{int} \) to have \( H = H_0 + V_{int} \) where \( H_0 \) is the free hamiltonian given by (12) modulo the substitution \( (q, p) \to (Q, P) \). Then, the interacting potential \( V_{int} \) induces a modification of the symplectic structure and a deformation of the phase space geometry. Further, it is clear that the classical scalar field theory described by the modified symplectic two-form (14) and the free Hamiltonian (12) is equivalent to the description given by the canonical symplectic form \( \Omega \) (23) and the Hamiltonian \( H \) (24) expressed in terms of the new phase space variables \( (Q_n^i, P_n^i) \) and involving interacting harmonic modes.
3.2 Quantization

Upon replacing the variables by operators with commutation rules given by $i$ times the Poisson brackets ($\{,\} \rightarrow -i[ , ]$), we get

$$[Q^i_n, Q^j_m] = 0 \quad [P^i_n, P^j_m] = 0 \quad [Q^i_n, P^j_{-m}] = i\delta_{m+n,0}\delta_{i,j}$$ (25)

It follows that the operators $q^i_n$ and $p^j_n$, corresponding to the dynamical variables of the classical theory, satisfy the following commutation relations

$$[q^i_n, q^j_m] = -i\mathcal{E}^{ij}\delta_{m+n,0} \quad [p^i_n, p^j_m] = 0 \quad [q^i_n, p^j_{-m}] = i\delta_{i,j}\delta_{n,m},$$ (26)

and the equal time commutation relations are given by

$$[\varphi^i(x,t), \varphi^j(y,t)] = -i\mathcal{E}^{ij}\delta(x-y)$$ (27)

$$[\pi^i(x,t), \pi^j(y,t)] = 0$$ (28)

$$[\varphi^i(x,t), \pi^j(y,t)] = i\delta_{i,j}\delta(x-y).$$ (29)

Introducing the operators

$$a^i_n = \sqrt{\frac{\Delta_n}{2}}(Q^i_n + iP^i_{n}) \quad a^{i+}_n = \sqrt{\frac{\Delta_n}{2}}(Q^i_n - iP^i_{n})$$ (30)

with $\Delta_n = 2\vert n\vert(4 + \theta^2n^2)^{-\frac{1}{2}}$, the Hamiltonian $H$ can be also written as

$$H = H_{n=0} + \frac{1}{2}\sum_{n \neq 0} \left[\frac{\vert n\vert}{2}\sqrt{4 + \theta^2n^2}(a^i_n a^{i+}_n + a^{i+}_n a^i_{-n}) + i\sum_j \theta n^2\mathcal{E}^{ij}a^i_n a^j_{-n}\right]$$ (31)

where

$$H_{n=0} = \frac{1}{2}[(P^1_0)^2 + (P^2_0)^2]$$ (32)

stands for the zero mode energy contribution. The diagonalization of the Hamiltonian $H - H_{n=0}$ can be performed by means of the new Weyl operators

$$A^1_n = \frac{1}{\sqrt{2}}(a^1_n - ia^2_n) \quad A^{1+}_n = \frac{1}{\sqrt{2}}(a^{1+}_n + ia^{2+}_n) \quad A^2_n = \frac{1}{\sqrt{2}}(a^1_n + ia^2_n) \quad A^{2+}_n = \frac{1}{\sqrt{2}}(a^{1+}_n - ia^{2+}_n).$$ (33)

Substituting (33) in (31), one obtain

$$H - H_{n=0} = \sum_{n \neq 0} (\omega_n A^{1+}_n A^1_n + \omega^+_n A^{2+}_n A^2_n)$$ (34)

where

$$\omega^\pm_n = \frac{\vert n\vert}{2}\sqrt{4 + \theta^2n^2} \pm \theta n^2$$ (35)
It is easy seen that the deformation induces a lifting of the degeneracies of the spectrum. Note that the modes frequencies satisfy the relations $\omega_n^\pm = \omega_{-n}^\pm$ and $\omega_n^+(-\theta) = \omega_n^-(\theta)$.

The dynamics of this system is described by the Heisenberg equations given by

$$\frac{dA_n^1}{dt} = -i[A_n^1, H] = -i\omega_n A_n^1 \quad \frac{dA_n^2}{dt} = -i[A_n^2, H] = -i\omega_n A_n^2,$$

and the corresponding solutions are

$$A_n^1(t) = \hat{A}_n^1 \exp(-i\omega_n t) \quad A_n^2(t) = \hat{A}_n^2 \exp(-i\omega_n^+ t)$$

where the operators $\hat{A}_n^1$ and $\hat{A}_n^2$ are time-independent. Consequently, using the equations (21), (30) and (33), we obtain the normal modes as

$$q_n^1(t) = \frac{1}{2} \left[ \Lambda_n^+ \left( \hat{A}_n^1 \exp(-i\omega_n t) + \hat{A}_n^1 \exp(+i\omega_n t) \right) + \Lambda_n^- \left( \hat{A}_n^2 \exp(-i\omega_n^+ t) + \hat{A}_n^2 \exp(+i\omega_n^+ t) \right) \right]$$

and

$$q_n^2(t) = \frac{i}{2} \left[ \Lambda_n^+ \left( \hat{A}_n^1 \exp(-i\omega_n t) - \hat{A}_n^1 \exp(+i\omega_n t) \right) + \Lambda_n^- \left( \hat{A}_n^2 \exp(+i\omega_n^+ t) - \hat{A}_n^2 \exp(-i\omega_n^+ t) \right) \right]$$

where the $\theta$-dependent constants $\Lambda_n^\pm$ are defined by

$$\Lambda_n^\pm = \frac{1}{\sqrt{\Delta_n}} \pm \frac{\theta}{2\sqrt{\Delta_n}}.$$

It is clear from equation (34) that the non-commutativity induces a lifting of the energy levels degeneracies. This feature is very similar to the Landau problem in quantum mechanics. The Hamiltonian (34) is given by a sum of two independents one dimensional harmonic oscillators. For $\theta = 0$, we have $\omega_n^+ = \omega_n^-$ and the Hamiltonian becomes a superposition of multi-modes two dimensional harmonic oscillators.

### 4 $\mathcal{B}$-Deformed scalar fields

#### 4.1 $\mathcal{B}$-Deformed phase space

Hereafter we use the symbols $\Omega, \theta, P, Q, a$ and $A$ that do not should be confused with ones introduced in the previous section. To begin we consider the closed non-degenerate symplectic two-form

$$\Omega = \Omega_0 - \theta \int dx \delta \varphi^1(x) \wedge \delta \varphi^2(x)$$

where the added term involves only the fields $\varphi^1$ and $\varphi^2$. It can also be writing as

$$\Omega = \Omega_0 - \frac{1}{2} \mathcal{B}_{ij} \int dx \delta \xi^{i1}(x) \wedge \delta \xi^{j1}(x)$$

where the constant antisymmetric tensor $\mathcal{B}_{ij}$ is defined by

$$\mathcal{B}_{ij} = \theta \epsilon_{ij}$$
in term of the non-commutativity parameter $\theta$. As discussed in the section 2, it is more appropriate to rewrite $\Omega$ in the following compact form

$$\Omega = \frac{1}{2} \sum_{IJ} \int dxdx' \Omega_{IJ}(x, x') \delta \xi^I(x) \wedge \delta \xi^J(x')$$  \hspace{1cm} (43)$$

where

$$\Omega_{IJ}(x, x') = \Omega_{IJ} \delta(x - x') = (\delta_{ij} \epsilon_{i'j'} - \delta_{i'j} B_{ij}) \delta(x - x').$$  \hspace{1cm} (44)$$

Using the modes expansions given by equations (3) and (4), the modified symplectic two-form can be expressed as

$$\Omega = \frac{1}{2} \sum \delta q_i^n \wedge \delta p_{-n}^i - \frac{1}{2} \sum_{ijn} B_{ij} \delta q_i^j \wedge \delta q_{-n}^j.$$  \hspace{1cm} (45)$$

Using the definition (17), the Poisson brackets reads

$$\{F, G\} = \sum_{in} \frac{\delta F}{\delta q_i^n} \frac{\delta G}{\delta p_{-n}^i} - \frac{1}{2} \sum_{ijn} \frac{B_{ij}}{\delta q_i^n} \frac{\delta G}{\delta p_{-n}^j},$$  \hspace{1cm} (46)$$

and in particular the fundamental Poisson brackets are given by

$$\{q_i^n, q_j^m\} = 0 \hspace{1cm} \{q_i^n, p_j^m\} = \delta_{ij} \delta_{m+n,0} \hspace{1cm} \{p_i^n, p_j^m\} = B_{ij} \delta_{m+n,0}.$$  \hspace{1cm} (47)$$

Contrary to the previous section, the Poisson brackets between the variables $q_i^n$ give zero and the canonical momentum variables acquire non-vanishing Poisson brackets and upon quantization they become non-commuting operators. This reflects the non-commutativity in the momentum space induced by the deformation of the canonical symplectic two-form $\Omega_0$ (see equation (41)).

To achieve the physical implications of the modification of the symplectic two-form $\Omega_0$, we follow a similar procedure that one used in the previous section. In this sense, we start by converting $\Omega$ in a canonical two-form to perform the quantization of the model. Indeed, the modified symplectic two-form $\Omega$ can be expressed as

$$\Omega = \sum_{in} \delta Q_i^n \wedge \delta P_{-n}^i$$  \hspace{1cm} (48)$$

in terms of the new dynamical variables $Q_i^n$ and $P_i^n$ defined as follows

$$Q_i^n = q_i^n \hspace{1cm} P_i^n = p_i^n - \frac{1}{2} B_{ij} q_j^n,$$  \hspace{1cm} (49)$$

with a summation over repeated indices. The Poisson bracket (46) becomes canonical and in particular, we have the canonical relations

$$\{Q_i^n, Q_j^m\} = 0 \hspace{1cm} \{P_i^n, P_j^m\} = 0 \hspace{1cm} \{Q_i^n, P_j^m\} = \delta_{m+n,0} \delta_{ij}.$$  \hspace{1cm} (50)$$

Using (49), the Hamiltonian (12) reads

$$H = \frac{1}{2} \sum_{in} \left[ P_i^n P_{-n}^i + \frac{1}{4} (4n^2 + \theta^2) Q_i^n Q_{-n}^i + \theta \sum_j \epsilon^{ij} P_n^i P_{-n}^j \right].$$  \hspace{1cm} (51)$$

At this level, we have derived the necessary tools needed to quantize the model under consideration.
4.2 Quantization

The quantization analysis now, though identical in spirit to the previous one, differs in detail. The correspondence principle leads to the commutation rules

\[ [Q_n^i, Q_m^j] = 0 \quad [P_n^i, P_m^j] = 0 \quad [Q_n^i, P_m^j] = i\delta_{ij}\delta_{m+n,0}. \] (52)

Consequently, we have the equal time commutation relations

\[ [\phi_i(x, t), \phi_j(y, t)] = 0 \quad [\pi_i(x, t), \pi_j(y, t)] = iB_{ij}\delta(x - y) \quad [\phi_i(x, t), \pi_i(y, t)] = i\delta_{ij}\delta(x - y). \] (53)

Passing in the Schwinger representation, the quantized hamiltonian \( H \) becomes

\[ H = \sum_n \frac{1}{2}\omega_n (a_n^i a_n^{i+} + a_n^{i+} a_n^i ) + i\theta \sum_{nij} \epsilon_{ij} a_n^{i+} a_n^j \] (54)

in terms of the creation and annihilation operators defined by

\[ a_n^i = \sqrt{\frac{\omega_n}{2}} (Q_n^i + i\frac{P_n^i}{\omega_n}) \quad a_n^{i+} = \sqrt{\frac{\omega_n}{2}} (Q_n^i - i\frac{P_n^i}{\omega_n}) \] (55)

where \( 2\omega_n = \sqrt{4n^2 + \theta^2} \). To remove the angular momentum contribution from \( H \) (the last term in (54)), we use the Weyl operators defined by (33) and we get

\[ H = \sum_n \omega_n^\pm A_n^1 A_n^1 + \omega_n A_n^2 A_n^2 \] (56)

where \( \omega_n^\pm = \omega_n \pm \theta/2 \). Solving the Heisenberg equations of motion satisfied by the operators \( A_n^i \) (\( i = 1, 2 \)), one can see the normal modes of the quantum fields \( \phi^1(x, t) \) and \( \phi^2(x, t) \) are given by

\[ q_n^1(t) = \frac{1}{2\sqrt{\omega_n}} \left( \hat{A}_n^1 \exp(-i\omega_n^\pm t) + \hat{A}_n^{1+}\exp(+i\omega_n^-t) + \hat{A}_n^2 \exp(-i\omega_n^+t) + \hat{A}_n^{2+}\exp(+i\omega_n^+t) \right) \] (57)

and

\[ q_n^2(t) = \frac{i}{2\sqrt{\omega_n}} \left( \hat{A}_n^1 \exp(-i\omega_n^-t) - \hat{A}_n^{1+}\exp(+i\omega_n^-t) + \hat{A}_n^2 \exp(+i\omega_n^+t) - \hat{A}_n^{2+}\exp(-i\omega_n^+t) \right) \] (58)

Here again the operators \( \hat{A}_n^i \) are time independents. It is easily seen from (56) that due to the phase space non-commutativity, the energy levels are not degenerate. Thus, it seems that the deformation induces a coupling between the fields which removes the degeneracies of the energy levels.

5 Noncommutative chiral field space

5.1 Commutative chiral field space

This section is devoted to chiral boson theory in connection with quantum Hall effect. In this respect, let us give a brief review of some basic results in order to explain our purpose. Consider
a Quantum Hall state on a disc (of unit radius) with filling factor $\nu$. Following the Laughlin picture, we known that the edge excitations might have many branches $I = 1, 2, \cdots, N$ residing on the edge of the quantum droplet. It is also now well established that the edge dynamics, associated with disc geometry, can be described by a chiral boson field compactified on a circle. Thus, let us consider a massless scalar field $\phi$ in (1+1) Minkowski space-time where the metric has the diagonal elements $(1, -1)$. The field

$$\phi = \phi_+ + \phi_-$$

is a superposition of the left ($\phi_- \equiv \phi(x-t)$) and right ($\phi_+ \equiv \phi(x+t)$) moving components which are known as chiral bosons. We define the partial derivatives $\partial_0$ and $\partial_1$ to refer to differentiations with $t$ and $x$. The action describing the dynamics of left or right chiral boson is given by

$$S_s = \int dx dt [s\partial_1 \phi_s \partial_0 \phi_s - (\partial_1 \phi_s)^2]$$ (59)

where $s = +$ or $-$ respectively and the velocity is normalized to unity. The corresponding Hamiltonians are

$$H_s = \int dx (\partial_1 \phi_s)^2.$$ 

Since the fields live on a compact space, they must satisfy the boundary condition

$$\phi_s(2\pi, t) - \phi_s(0, t) = 2s\pi.$$ 

The general solutions of the equations of motion arising from (59) and compatible with the boundary conditions are

$$\phi_s(x, t) = \alpha_0^s + \tilde{\alpha}_0^s(t + sx) + i \sum_{n \neq 0} \frac{\alpha_n^s}{n} e^{-in(t+sx)}.$$ (60)

Upon quantization the field $\phi_s$ and its corresponding canonical momenta $\pi_s$ (its space derivative) satisfy the commutation rules

$$[\phi_s(x), \phi_{s'}(x')] = -is\delta_{ss'}\varepsilon(x - x')$$ (61)

$$[\phi_s(x), \pi_{s'}(x')] = i\delta_{ss'}\delta(x - x'),$$ (62)

$$[\pi_s(x), \pi_{s'}(x')] = is\delta_{ss'}\delta'(x - x')$$ (63)

where $\varepsilon(x - y)$ represents the Heaviside function and $\delta'(x - y)$ denote the spatial derivative of the delta function. The coefficients $\alpha_0^s$, $\tilde{\alpha}_0^s$ and $\alpha_n^s$ of the expansion (60) become operators satisfying the commutations relations

$$[\alpha_0^s, \tilde{\alpha}_0^{s'}] = i\delta_{ss'}$$  

$$[\alpha_n^s, \alpha_m^{s'}] = n\delta_{ss'}\delta_{n+m} \quad \text{others} = 0$$ (64)

acting on a bosonic Fock space, whose vacuum $|0\rangle$ is defined by

$$\alpha_n^s|0\rangle = 0.$$
5.2 Noncommutative space of fields

To obtain a quantum theory of noncommutative chiral fields describing the edge excitations having many branches \((I = 1, 2, \cdots, N)\), we start by considering \(N\) left moving chiral fields \(\Phi_I\) travelling, on the edge of an incompressible Hall droplet, with velocities normalized to unity. They satisfy the commutations rules

\[
[\Phi_I(x), \Phi_J(x')] = -i\delta_{IJ}\varepsilon(x - x')
\]

and are described by the action

\[
S = \sum_I \int dt dx (\partial_0 \Phi_I)(\partial_1 \Phi_I) - (\partial_1 \Phi_I)(\partial_1 \Phi_I).
\]

We suggest to replace the commutation relation (65) by

\[
[\Phi_I(x), \Phi_J(x')] = -i\Omega_{IJ}(\theta)\varepsilon(x - x').
\]

This corresponds to a deformation of the symplectic structure of the fields space. From a physical point of view, this deformation can be viewed as a sort of coupling between the \(N\) quantum Hall branches. The parameter \(\theta\) encodes the non-commutativity of the fields space and the matrix \(\Omega\) reduces to unit matrix when the deformation vanishes \((\Omega_{IJ} \rightarrow \delta_{IJ})\). The matrix \(\Omega\) in (67) is real, symmetric and it is assumed to be invertible. We assume also that the dynamics is governed by the Hamiltonian

\[
H = \sum_I \int dx : (\partial_1 \Phi_I)^2 :
\]

where the symbol :: denotes the normal ordering. The Heisenberg equation

\[
-i\frac{d\Phi_I}{dt} = [H, \Phi_I]
\]

gives

\[
\partial_0 \Phi_I = \sum_J \Omega_{IJ}\partial_1 \Phi_J
\]

which rewrites also as

\[
\partial_0^2 \Phi_I = \sum_J (\Omega^2)_{IJ}\partial_1^2 \Phi_J.
\]

The last equation is easily solved by mean of a unitary matrix \(U\) which diagonalizes the matrix \(\Omega^2\). Indeed, setting

\[
\Psi_I = \sum_J U_{IJ}\Phi_J,
\]

one has

\[
\partial_0^2 \Psi_I = \lambda_I \partial_1^2 \Psi_I
\]
where \( \lambda_I \) are the eigenvalues of \( \Omega^2 \). Consequently, it clear that the velocities of the fields \( \Phi_I \) are given by

\[
v_I = \pm \sqrt{\lambda_I}
\]

(73)

where the positive \( v_I \) correspond to a left moving branch and negative value to a right moving one. It is easily seen from the equations (72) and (73) that the velocities of the chiral fields are modified. This is the effect of the deformation of symplectic structure. This result agrees with one recently obtained in [19]. Obviously, the eigenvalues \( \lambda_I \) and the matrix elements \( U_{IJ} \) are \( \theta \)-dependents and reduce respectively to 1 and \( \delta_{IJ} \) for \( \theta = 0 \).

Since the matrix \( \Omega \) is invertible, the action can be simply read off from the symplectic structure evident in the deformed commutation relation (67):

\[
S = \sum_{IJ} \int dt dx (\partial_0 \Phi_I)(\Omega^{-1})_{IJ}(\partial_1 \Phi_J) - (\partial_1 \Phi_I)\delta_{IJ}(\partial_1 \Phi_J).
\]

(74)

It is easy to check that the above action implies the equations of motion (70-71). At this stage some remarks are in order. First, notice that the action (74) is formally similar to one derived in [31]

\[
\sum_{IJ} \int dt dx (\partial_0 \Phi_I)K_{IJ}(\partial_1 \Phi_J) - (\partial_1 \Phi_I)V_{IJ}(\partial_1 \Phi_J)
\]

(75)

to characterize the topological orders and to classify the different hierarchies in abelian fractional Hall effect. In equation (75), the matrix \( K \) is symmetric and \( V \) must be a positive definite matrix. According to this formal similarity, the action given by (74) provides us with an useful bridge between the quantum theory of noncommutative chiral fields and fractional quantum Hall effect.

Of course, since the matrix \( \Omega \) is arbitrary, the action (74) offers the possibility to cover different fractional Hall hierarchies. As first simple illustration, we consider the matrix \( \Omega \) having the following form

\[
\Omega_{IJ}(\theta) = a_{IJ}(\theta)\delta_{IJ} + (I - J)(b_{IJ}(\theta) - b_{JI}(\theta))
\]

(76)

where \( a_{IJ} \) and \( b_{IJ} \) are real parameters. In particular, for \( N = 2 \) and

\[
a_{IJ} = -\exp(i\pi/2)(I + J) \quad b_{IJ} = \frac{\theta}{2} \epsilon_{IJ},
\]

we obtain

\[
\Omega_{IJ} = (-)^{I+1}\delta_{IJ} + (I - J)\theta \epsilon_{IJ}.
\]

(77)

This is exactly the metric considered in [19]. We will next study the effects of the non-commutativity of chiral fields space in the context of fractional quantum Hall effect when the matrix encoding the deformation is given by (77). This particular choice generates, as it is shown in the next section, a deviation from the Laughlin hierarchy and the obtained model can be used to describe others hierarchies like Jain one.
6 Generalized fractional quantum Hall filling

In this section, we assume the left and right components of a quantum Hall droplets are coupled. The matrix encoding this coupling is given by (77) where $I = 1$ (resp. $I = 2$) is replaced by $s = +$ (resp. $s = -$). Thus, the left and right chiral fields satisfy the commutation relation

$$[\Phi_s(x), \Phi_{s'}(y)] = is(\epsilon_{ss'}\theta - \delta_{ss'})\varepsilon(x - y)$$  \hspace{1cm} (78)

and from the equation (74) the action reads

$$S = \frac{1}{\theta} \sum_s \int dx dt \left[ \sum_{s'} \partial_1 \Phi_s [s\delta_{ss'} - s\theta\epsilon_{ss'}] \partial_0 \Phi_{s'} - \bar{\theta}(\partial_1 \Phi_s)^2 \right]$$  \hspace{1cm} (79)

where $\bar{\theta} = \sqrt{1 + \theta^2}$. It is important to note that this result can be achieved using the following transformation

$$\Phi_s(x) = \sqrt{\bar{\theta} + 1} \frac{1}{2} \left( \phi_s(x) - \sqrt{\bar{\theta} - 1} \epsilon_{ss'} \phi_{s'}(x) \right)$$  \hspace{1cm} (80)

where the $\phi_s$’s stands for un-deformed chiral fields defined in the previous section. Remark that the hamiltonian (68) remains unchanged under (80)

$$H = \sum_s H_s = \sum_s \int dx : (\partial_1 \Phi_s)^2 = \sum_s \int dx : (\partial_1 \phi_s)^2 :$$  \hspace{1cm} (81)

Using the relation (80) and equations (70) and (78), we expand the chiral fields

$$\Phi_s(x, t) = \beta_0^s + \bar{\beta}_0^{s'}(\bar{\theta}t + sx) + i \sum_{n \neq 0} \beta_n^s e^{-in(\bar{\theta}t + sx)}$$  \hspace{1cm} (82)

where the operators $\beta$ satisfy the relations

$$[\beta_0^s, \bar{\beta}_0^{s'}] = i(\delta_{ss'} - \theta\epsilon_{ss'}) \hspace{1cm} [\beta_n^s, \bar{\beta}_m^{s'}] = n\delta_{ss'}\delta_{n+m} + \theta n\delta_{n-m}\epsilon_{ss'} \hspace{1cm} others = 0$$  \hspace{1cm} (83)

of two non-commuting copies of $U(1)$ Kac-Moody algebra. The non-commutativity between right and left sectors disappears for $\theta = 0$ and the relations (83) give (64) as it is expected. We note that a mapping can be established between the deformed oscillations operators (83) and un-deformed ones (64). This is

$$\beta_0^s = \sqrt{\bar{\theta} + 1} \frac{1}{2} \left( \alpha_0^s - \sqrt{\bar{\theta} - 1} \epsilon_{ss'} \alpha_0^{s'} \right)$$  \hspace{1cm} (84)

$$\bar{\beta}_0^s = \sqrt{\bar{\theta} + 1} \frac{1}{2} \left( \bar{\alpha}_0^s - \sqrt{\bar{\theta} - 1} \epsilon_{ss'} \bar{\alpha}_0^{s'} \right)$$  \hspace{1cm} (85)

$$\beta_n^s = \sqrt{\bar{\theta} + 1} \frac{1}{2} \left( \alpha_n^s - \sqrt{\bar{\theta} - 1} \epsilon_{ss'} \alpha_n^{s'} \right)$$  \hspace{1cm} (86)

15
To compute the Hall filling factor associated to the edge excitations described by the action (79), we define the operators

\[ O_s(x) = \eta : e^{i \frac{1}{2} \gamma \Phi_s(x)} : \]  

(87)

where \( \eta \) is constant. Using

\[ : e^A :: e^B := e^{<AB>} : e^A e^B :, \]

it is easy to verify that the operators (87) satisfy the exchange relation

\[ O_s(x)O_s'(x') = e^{i \gamma^2 \Omega_{ss'} \epsilon(x-x')} O_s'(x')O_s(x). \]  

(88)

Since \( O_s(x) \) is identified as an electron operator, it must satisfy

\[ O_s(x)O_s(x') = -O_s(x')O_s(x). \]

(89)

Thus

\[ \gamma^2 = (2m + 1) \quad m \in \mathbb{N} \]

is an odd integer. However, for \( s \neq s' \) we have

\[ O_s(x)O_{s'}(x') = e^{-i(2m+1)\pi \theta s_{ss'} \epsilon(x-x')} O_{s'}(x')O_s(x). \]

(90)

The exchange relation (90) is a consequence of the non-commutativity between left and right sectors. The non local operators \( O_s(x) \) describe hard core objects \((O_s(x))^2 = 0\) which obey the standard (fermionic) anticommutation relations. In view of the exchange relation (90), we shall refer to the objects \( O_s(x) \) as generalized fermionic operators. Note also that the relation (90) is similar to the exchange relation for anyonic particles.

Now we come the Green correlation functions which play an important role since they give the filling factor the quantum Hall system described by the action (79). They can be easily evaluated. Indeed, using the mapping (84-86) expressing the deformed modes in term of the un-deformed ones and the transformation (80), the correlation functions can be expressed in terms of the usual Green functions

\[ \langle \phi_s(x)\phi_{s'}(0) \rangle \sim -8\delta_{ss'} \ln |x + st|. \]

(91)

Thus after some algebra and by rescaling \( \gamma^2 \to \frac{2}{\pi} \gamma^2 \) we get

\[ \langle O_s^\dagger(x)O_s(0) \rangle \sim \frac{1}{(x + \theta t)(2m+1)\theta}. \]

(92)

The first thing we see that the electron propagator on the edge of a fractional Hall droplet acquires a non trivial exponent \((2m + 1)\theta \) not equal to one. This implies that the electrons on the edge are strongly correlated. Using the terminology of quantum Hall physics, we will call this type of an electron state the generalized Luttinger liquid. For \( \theta = 0 \), the exponent



text continues...
the total action $S = S_0 + S_c$ can be expressed in the following compact form:

$$S = \int dt dx \sum_{ss'} (\partial_1 \phi_s) \omega_{ss'} (\partial_0 \phi_{s'}) - \sum_s (\partial_1 \phi_s)^2$$  \hspace{1cm} (96)

where

$$\omega_{ss'} = sk_o \delta_{ss'} + sk\epsilon_{ss'}.$$  \hspace{1cm} (97)

It is clear from the last two equations that the action (96) is similar to one given by (79). The coupling constant play the role of the non-commutativity parameter. The model described by the action (96) is exactly solvable. In fact, by introducing the fields

$$\phi_s = \sum_{s'} U_{ss'} \hat{\phi}_{s'}$$  \hspace{1cm} (98)

by mean of the matrix $U$ which diagonalizes the matrix $\omega$, it is simply verified that the action (96) becomes

$$S = \int dt dx \sum_s [\lambda_s (\partial_1 \hat{\phi}_s) (\partial_0 \hat{\phi}_s)) - (\partial_1 \hat{\phi}_s)^2]$$  \hspace{1cm} (99)

where

$$\lambda_\pm = \frac{1}{2}(k_\pm - k_-) \pm \frac{1}{2} \sqrt{4k^2 + (k_+ + k_-)^2}$$  \hspace{1cm} (100)

are the the eigenvalues of the matrix $\omega$. Notice that under the transformation $U$, we get a system of two uncoupled chiral fields travelling with different velocities $\lambda_+ \neq \lambda_-$. Further, since

$$\lambda_+ \lambda_- = \det(\omega) < 0$$

the fields $\hat{\phi}_1$ and $\hat{\phi}_2$ are travelling in opposites directions. Note also that the transformation $U$ leaves the Hamiltonian

$$H = \sum_s (\partial_1 \phi_s)^2 = \sum_s (\partial_1 \hat{\phi}_s)^2$$

unchanged. The new fields $\hat{\phi}_s$ satisfy the deformed commutations relations

$$[\hat{\phi}_s, \hat{\phi}_{s'}] = -i \Delta_{ss'} \varepsilon(x - y)$$  \hspace{1cm} (101)

where

$$\Delta_{ss'} = \sum_{s''} s'' (U^{-1})_{ss'} (U^{-1})_{s's''},$$

and the matrix $U$ is given by

$$U = \begin{pmatrix} \frac{k_- + \lambda_+}{\sqrt{k^2 + (k_- + \lambda_+)^2}} & \frac{k}{\sqrt{k^2 + (k_- + \lambda_-)^2}} \\ \frac{k}{\sqrt{k^2 + (k_+ + \lambda_-)^2}} & \frac{k_- + \lambda_-}{\sqrt{k^2 + (k_+ + \lambda_-)^2}} \end{pmatrix},$$  \hspace{1cm} (102)

in terms of the shifted velocities $\lambda_+$ and $\lambda_-$. Subsequently, the system of the coupled chiral fields $\phi_+$ and $\phi_-$ can be described by the action (96) where the fields satisfy the canonical commutation relations (61) or alternatively by the action (99) with chiral fields satisfying deformed
commutation rules \(101\). The two descriptions are equivalents. It is important to notice that the coupling between the left and right chiral fields induces a modification of their velocities
\[ k_+ \to \lambda_+ \quad k_- \to \lambda_- \]
Finally, it is remarkable that for \(k = 0\), we have \(U_{ss'} \to \delta_{ss'}\) and \(\Delta_{ss'} \to s \delta_{ss'}\). In this limit the commutations relations \(101\) reduce to un-deformed ones \(61\). This agrees with the generalized deformation procedure discussed in the subsection 5.2.

### 8 Nonlinear chiral boson dispersion

Recently there has been considerable interest in the generalization of the chiral Luttinger liquid (CLL). This is essentially motivated by experimental observations which argue a noticeably difference with CLL prediction \[34-35\]. From a theoretical point of view, the generalization is based on the interplay between the electron-electron interaction and the confining potential at the edge. This gives rise new additional low energy modes. In this sense, in a recent works, it was shown analytically \[33\] and numerically \[36\] that the chiral boson dispersion becomes nonlinear when the non local nature of the electron-electron interaction is incorporated in the CLL theory.

In this short section, we shall show that the theory of noncommutative fields gives rise the nonlinear dispersion relation corroborating the results mentioned above. For this, let us consider the fields
\[ \hat{\Phi}_s = \phi_s + \theta \sum_{s'} \epsilon_{ss'} \pi_{s'} \tag{103} \]
expressed in terms of the un-deformed ones. This transformation is formally similar to one given by equation \(21\) or \(49\). The new fields satisfy the deformed commutation rules
\[ [\hat{\Phi}_s(x), \hat{\Phi}_{s'}(x')] = -2i\theta\epsilon_{ss'}\delta(x - x') - i\delta_{ss'}s(\varepsilon(x - x') + \theta^2 \delta'(x - x')) \tag{104} \]
which reduce the canonical ones in the limiting case \(\theta = 0\). We assume that the Hamiltonian takes the form
\[ H = \frac{1}{4\pi} \sum_s \int dx : (\partial_1 \hat{\Phi}_s)^2 : \tag{105} \]
in term of the new fields. Substituting \(103\) in \(105\), \(H\) can be cast in the following form
\[ H = \frac{1}{4\pi} \sum_s \int dx : [(\partial_1 \phi_s)^2 + \theta^2 (\partial_1^2 \phi_s)^2 + \theta \sum_{s'} \epsilon_{ss'} (\partial_1 \phi_s)(\partial_1^2 \phi_{s'})] : \tag{106} \]
in term of the un-deformed fields. Clearly edge dynamics described by the Hamiltonian \(106\) with fields satisfying the canonical commutation relations is equivalent to the description provided by \(H\) Eq.(105) in terms of deformed fields obeying the commutation rules \(104\). The
effect of non-commutativity is now apparent in the expression of the Hamiltonian. Using the
mapping (103) and the fields expansion (60), we obtain

\[ H = \frac{1}{2} \sum_n |(\bar{\alpha}_n^0)^2 + 2 \sum_{n>0} (1 + \theta^2 n^2) \alpha^+_n \alpha_n^0| \]  

(107)
describing chiral bosons with nonlinear dispersion relation

\[ E_n \sim n + \theta^2 n^3 \]  

(108)
which coincides with the result obtained in [33] in which the author incorporates the electron-electron interaction in the chiral Luttinger liquid theory. This constitutes another illustration of our interest for the quantum theory of noncommutative fields.

9 Concluding remarks

Two facets of quantum theory of noncommutative scalar fields are considered.
The first one, developed in the first part of this paper, concerns the theory of noncommutative scalar fields \( \varphi^1(x, t) \) and \( \varphi^2(x, t) \). We have shown that the non-commutativity at the quantum level is deeply related to the deformation of the symplectic structures of the classical phase space. The quantization of the theory is performed thanks to a mapping which is called dressing transformation. We stress that our approach is based on the symplectic deformation and establish clearly that the noncommutative theory of fields originates from the deformation of the corresponding phase space.

The second facet related to noncommutative fields deals with the non-commutativity between left and right sectors of a chiral boson fields. We develop a general approach to introduce a theory of noncommutative chiral fields. As a particular case, we recover the model discussed in [19] which is shown to be useful in the context of fractional quantum Hall effect. Indeed, the non-commutativity generates a deviation from the Laughlin fractional filling factor and gives, for some selected values of deformation parameter, the factors associated with Jain sequence.

Note that the action (74) is formally similar to (75) which constitutes now the basis of our understanding of the Hall hierarchies classification [31]. We derived the action describing the coupling between the right and left excitations living on the edges of a quantum Hall sample. It is remarkable that the coupled system, described by the action (96), is equivalent to one describing noncommutative chiral field theory. The coupling induces a shift of the velocities of chiral components. Finally, we show that the non-commutativity in the fields space generate a nonlinear dispersion relation. This is in agreement with recent analytical and numerical analysis [33,36]. To close this paper, we mention that due to the similarity between the action (74) and (75), the quantum theory of noncommutative fields constitutes another suitable approach to classify the Hall hierarchies. The results obtained throughout this paper illustrate the interest of the quantum theory of noncommutative fields, especially in the context of fractional quantum
Hall effect. This open the way for further investigations. Note also that the present study can be extended to the situation where the fields are self interacting which constitutes one of the main features of quantum field theory. We hope to report on these issues in a forthcoming work.

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