FINITE GROUPS WITH SUBMULTIPLIATIVE SPECTRA

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Abstract. We study abstract finite groups with the property, called property (s), that all of their subrepresentations have submultiplicative spectra. Such groups are necessarily nilpotent and we focus on p-groups. p-groups with property (s) are regular. Hence, a 2-group has property (s) if and only if it is commutative. For an odd prime p, all p-abelian groups have property (s), in particular all groups of exponent p have it. We show that a 3-group or a metabelian p-group (p ≥ 5) has property (s) if and only if it is V-regular.

1. Introduction

In recent years a number of properties of matrix groups (and semigroups) were studied (see e.g. [20, 21]). We wish to propose a program to explore implications these results on matrix groups might have for the theory of abstract groups: Given a property (P) of matrix groups we say that an abstract group G has property (P) if all the finite-dimensional (irreducible) subrepresentations of G have property (P). We call a representation of a subgroup of G a subrepresentation of G. In this paper we commence our program by studying the so-called property (s).

Assume that F is an algebraically closed field of characteristic zero. A matrix group G ⊆ GLn(F) (or matrix semigroup G ⊆ Mn(F)) has submultiplicative spectrum or, in short, it has property (s) if for each pair A, B ∈ G every eigenvalue of the product AB is equal to a product of an eigenvalue of A and an eigenvalue of B. Such groups and semigroups were first studied by Lambrou, Longstaff, and Radjavi [14]. (See also [11, 12, 13, 19].) If G is an irreducible group with property (s) then it is nilpotent and essentially finite [21 Thms. 3.3.4 and 3.3.5], i.e., G ⊆ F*G0 for some finite nilpotent group G0. Here F* is the group of invertible elements in F. A group is nilpotent if and only if it is the direct product of its Sylow p-groups. So it is not a restriction to study only p-groups.

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A finite group $G$ has property (\(\hat{s}\)) if all its irreducible subrepresentations have property (\(s\)). Such groups are necessarily nilpotent and we focus on $p$-groups with the property. We show that a 2-group has property (\(\hat{s}\)) if and only if it is commutative. For an odd prime $p$ we show that all $p$-abelian groups have property (\(\hat{s}\)). A $p$-group $G$ is called $p$-abelian if $(xy)^p = x^py^p$ for all $x, y \in G$. In particular, all groups of exponent $p$ have property (\(\hat{s}\)). We characterize all the metabelian $p$-groups with property (\(\hat{s}\)). We show that 3-groups and metabelian $p$-groups ($p \geq 5$) have property (\(\hat{s}\)) if and only if they are V-regular. In our proofs we use several results on abstract $p$-groups, in particular those of Alperin [1, 2], Mann [15, 16, 17], and Weichsel [25, 26, 27].

Let us remark that it would be interesting to consider a weaker property than (\(\hat{s}\)). Namely, the property, called (\(\tilde{s}\)), that every irreducible representation of $G$ has property (\(s\)). We do not know whether the properties (\(\hat{s}\)) and (\(\tilde{s}\)) are equivalent for a finite $p$-group.

2. Preliminaries

We assume throughout that $G$ is a finite group and that $F$ is an algebraically closed field of characteristic 0. We denote by $|g|$ the order of an element $g \in G$. The exponent $e(G)$ of the group $G$ is the least common multiple of these orders. In particular, if $G$ is a $p$-group, then $e(G)$ is the maximum of the orders of its elements.

If $g : G \to GL_n(F)$ is a representation, then $\mathcal{G} = g(G)$ is a finite matrix group. For $A \in \mathcal{G}$ we denote by $\sigma(A)$ the spectrum of $A$. We say that $\mathcal{G}$ has property (\(s\)) if

\[
\sigma(AB) \subseteq \sigma(A)\sigma(B) = \{\lambda\mu; \lambda \in \sigma(A), \mu \in \sigma(B)\}
\]

for all $A, B \in \mathcal{G}$. An (abstract) group $G$ has property (\(s\)) if all its irreducible subrepresentations have property (\(s\)). Here we call a representation of a subgroup $H$ of $G$ a subrepresentation of $G$. Groups with property (\(s\)) are nilpotent [21 Thm. 3.3.5]. It is well known that an irreducible representation of a nilpotent group is equivalent to a monomial representation (see e.g. [22 Thm. 16, p. 66], [23 Lemma 6, p. 207] or [6 Cor. 6.3.11]). Since each representation of a finite group $G$ is completely reducible it follows that a group $G$ has property (\(s\)) if and only if all its subrepresentations have property (\(s\)). We use this fact later in the proofs, e.g., in the proof of Proposition 3.2.

Recall that all irreducible representations of a finite abelian group have degree 1. Hence we have:

**Lemma 2.1.** Every finite abelian group has property (\(\hat{s}\)).

The subgroups in the lower central series of $G$ are denoted by $G^{(i)}$, i.e. $G^{(0)} = G$, $G^{(1)} = G' = [G, G]$, and $G^{(i)} = [G^{(i-1)}, G]$ for $i \geq 2$. We write $c = c(G)$ for the class of $G$, i.e., $c$ is the least integer such that $G^{(c)} = 1$. 


Lemma 2.2. If \( G \) has property \((\hat{s})\) then all its subgroups and quotients have property \((\hat{s})\). Furthermore, all its sections have property \((\hat{s})\).

Proof. It is enough to show that property \((\hat{s})\) is inherited by quotients. Suppose that \( H \trianglelefteq G \) and that \( \varrho : G/H \to GL_n(F) \) is a representation. Then \( \hat{\varrho} : G \to GL_n(F) \) defined by \( \hat{\varrho}(g) = \varrho(gH) \) is a representation of \( G \) called the inflation representation (see [5, p.2]). By our assumption \((G, \hat{\varrho})\) has property \((s)\), and hence so does \((G/H, \varrho)\). \(\square\)

The following lemma is an easy consequence of a theorem of Burnside [21, Thm. 1.2.2].

Lemma 2.3. If \( G_j \subseteq GL_{n_j}(F), j = 1,2, \) are two irreducible matrix groups then \( G_1 \otimes G_2 \subseteq GL_{n_1 n_2}(F) \) is also irreducible.

Lemma 2.4. If \( G_j \subseteq GL_{n_j}(F), j = 1,2, \) are two matrix groups with property \((s)\) then also \( G_1 \otimes G_2 \subseteq GL_{n_1 n_2}(F) \) has property \((s)\).

Proof. Observe that \( \sigma(A \otimes B) = \sigma(A)\sigma(B) \). \(\square\)

Corollary 2.5. If \( G_1 \) and \( G_2 \) are finite groups with property \((\hat{s})\) then also the direct product \( G_1 \times G_2 \) has property \((\hat{s})\).

Since each finite nilpotent group is a direct product of its Sylow \( p \)-groups we can limit our attention to \( p \)-groups.

Proposition 2.6. A finite group \( G \) has property \((\hat{s})\) if and only if for each pair of elements \( x, y \in G \) the subgroup \( \langle x, y \rangle \) has property \((\hat{s})\).

Proof. If \( G \) has property \((\hat{s})\) then by definition every subgroup, in particular every two-generated subgroup, has property \((\hat{s})\).

Conversely, assume that every two generated subgroup of \( G \) has property \((\hat{s})\). Let \( \varrho : K \to GL_n(F) \) be an irreducible representation of a subgroup \( K \subseteq G \). Choose \( x, y \in K \) and let \( H = \langle x, y \rangle \). The restriction \( \varrho : H \to GL_n(F) \) is a representation of \( H \). By assumption it has property \((s)\) and thus \( \sigma(\varrho(x)\varrho(y)) \subseteq \sigma(\varrho(x))\sigma(\varrho(y)) \). \(\square\)

3. The Power Structure of \( p \)-Groups with Property \((\hat{s})\)

Suppose that \( G \) is a finite \( p \)-group of exponent \( p^e \). Then for \( k = 1,2, \ldots, e \)
\[
\triangle_k(G) = \{ g \in G; g^{p^k} = 1 \}
\]
is the set of all the elements of order dividing \( p^k \), and 
\[
\nabla_k(G) = \{ g \in G; g = h^{p^k} \text{ for some } h \in G \}
\]
is the set of all \( p^k \)-th powers. We denote by \( \Omega_k(G) \) the subgroup generated by \( \triangle_k(G) \) and by \( \Upsilon_k(G) \) the subgroup generated by \( \nabla_k(G) \).
A $p$-group $G$ has property $(P1)$ if for all the sections $H$ of $G$ and all $k$ we have

$$\nabla_k(H) = \Omega_k(H).$$

A $p$-group $G$ has weak property $(P2)$ – denoted by $(wP2)$ – if

$$\Delta_k(G) = \Omega_k(G)$$

for $k = 1, 2, \ldots, e$. A $p$-group $G$ has property $(P2)$ if all sections of $G$ have property $(wP2)$.

Properties $(P1)$ and $(P2)$ were introduced by Mann [17]. We refer to [17, 28] for further details.

**Proposition 3.1.** If a matrix group $G \subseteq \text{GL}_n(F)$ has property $(s)$ then it has property $(wP2)$.

**Proof.** The submultiplicativity condition $\sigma(AB) \subseteq \sigma(A)\sigma(B)$ implies that the order $|AB|$ divides $\max\{|A|, |B|\}$. Hence, if $A, B \in \Delta_k(G)$ then also $AB, A^{-1} \in \Delta_k(G)$. □

**Proposition 3.2.** If a $p$-group $G$ has property $(\hat{s})$ then it has property $(P2)$.

**Proof.** Suppose that $K$ is a section of $G$. By Lemma [22] it follows that $K$ has property $(\hat{s})$. Take a faithful representation $\rho : K \to \text{GL}_n(F)$, e.g. the regular representation. It has property $(s)$ and by Proposition 3.1 it has property $(wP2)$. Hence, $G$ has property $(P2)$. □

Next we prove the main result of this section and one of our main results. We begin by recalling some definitions.

A $p$-group is called regular if for every pair $x, y \in G$ there is an element $z$ in the commutator group $\langle x, y \rangle^\prime$ such that

$$(xy)^p = x^py^pz^p.$$ 

Note that [10, Satz III.10.8(g)] shows that the above definition of a regular $p$-group is equivalent to the more common one [10, p. 321].

A regular group $G$ is called $V$-regular if any finite direct product of copies of $G$ is regular. Not every regular $p$-group is $V$-regular – see Wielandt’s example [10, Satz III.10.3(c)]. A $p$-group $G$ is $V$-regular if and only if all the finite groups in the variety of $G$ are regular [18, 26]. For the definitions of a variety of groups and a variety of a given group $G$ we refer to Hanna Neumann’s book [18]. Further properties of regular $p$-groups can be found in [10, 24].

**Theorem 3.3.** If a $p$-group has property $(\hat{s})$ then it is regular. Moreover, it is $V$-regular.

**Proof.** Assume that $G$ is a $p$-group with property $(\hat{s})$ and that the exponent of $G$ is equal to $p^e$. If $e = 1$ then it is regular by [10, Satz III.10.2(d)]. Suppose that $e \geq 2$. Now $G$ and the cyclic group $C_{p^e}$ of order $p^e$ both have property $(\hat{s})$. The direct product $G \times C_{p^e}$ has property $(\hat{s})$ by Corollary [25].
property \((P2)\) by Proposition \[3.2\] and property \((P1)\) by \[17\] Cor. 4. Finally, Theorem 25 of \[17\] implies that \(G\) is regular. The group \(G\) is \(V\)-regular since by Corollary \[2.5\] the direct product of any finite number of copies of \(G\) has property \((\hat{s})\) and thus it is regular. □

Properties of regular groups \[10\] Satz III.10.3(a),(b)] are used to prove the following corollaries.

**Corollary 3.4.** A 2-group has property \((\hat{s})\) if and only if it is abelian.

*Proof.* If a 2-group is regular then it is abelian \[10\] Satz III.10.3(a)]. The converse follows since, by Lemma \[2.1\] all abelian groups have property \((\hat{s})\). □

Let us point out that if \(k = 1\) then finite matrix 2-groups in \(GL_2k(F)\) with property \((s)\) are always commutative \[14\]; however, they need not be commutative if \(k \geq 2\) \[11\] 19. Moreover, finite irreducible matrix 2-groups in \(GL_2k(F)\) with property \((s)\) are constructed in \[19\] for \(k = 3\) and in \[11\] for \(k \geq 4\).

**Corollary 3.5.** If a 3-group has property \((\hat{s})\) then it is metabelian.

*Proof.* This follows from a result of Alperin \[1\] Thm. 1]. □

**Proposition 3.6.** If \(G\) has property \((\hat{s})\) then any finite group in the variety of \(G\) has property \((\hat{s})\).

*Proof.* By \[18\] Cor. 32.32] a finite group \(H\) in the variety of \(G\) is a section of a finite direct product of copies of \(G\). Corollary \[2.5\] implies that the direct product has property \((\hat{s})\) and Lemma \[2.2\] implies that \(H\) has property \((\hat{s})\) as well. □

In §6 we show that the converse of Theorem \[3.3\] is true for metabelian \(p\)-groups. We do not know the answer to the general question: Does a finite \(V\)-regular \(p\)-group have property \((\hat{s})\)? We know that the direct product of finitely many groups with property \((\hat{s})\) again has property \((\hat{s})\). If the direct product of two \(V\)-regular groups is not \(V\)-regular then the answer to the above question is negative. The question if the direct product of two \(V\)-regular groups is \(V\)-regular was studied by Groves \[8\].

4. Matrix Groups in \(GL_p(F)\) with Property \((s)\)

In this section we consider an irreducible matrix \(p\)-group \(G\) in \(GL_p(F)\). We assume hereafter that \(p\) is an odd prime. The main result of the section is the following: if \(G\) has property \((s)\) then the class \(c(G)\) is at most \(p - 1\).

Assume first that \(G \subseteq GL_p(F)\) is an irreducible \(p\)-group of exponent \(e(G) = p\). Then we may assume without loss that \(G\) is monomial. Each element of \(G\) is either diagonal or of the form \(DP^k\), where \(D\) is diagonal,
\[ k \in \{1, 2, \ldots, p - 1\} \text{ and} \]
\[
P = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix}.
\]

Since \( e(G) = p \) and \((DP^k)p = (\det D)I\) it follows that \( \det D = 1 \). Note that an element of the form \( DP, D \) diagonal of determinant 1, is diagonally similar to \( P \). Therefore we may assume without loss that \( P \in G \). We denote by \( D \) the subgroup of all the diagonal elements in \( G \). A simple matrix computation shows that \( G' \subseteq D \). Let \( \omega \) be a primitive \( p \)-th root of 1 and \( \Gamma_1 \) the set of all the \( p \)-th roots of 1. Further we denote by \( \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \) the finite field with \( p \) elements. We define a map \( \chi : D \to \mathbb{Z}_p^p \) by
\[
\chi = (k_1, k_2, \ldots, k_p).
\]

**Lemma 4.1.** \( \chi \) is a homomorphism of abelian groups and its image \( \text{im} \chi \) is invariant under the cyclic permutation \( \pi : \mathbb{Z}_p^p \to \mathbb{Z}_p^p \) given by
\[
\pi(k_1, k_2, \ldots, k_p) = (k_2, k_3, \ldots, k_p, k_1).
\]

**Proof.** It is an easy observation that \( \chi \) is a homomorphism and that its image is a vector subspace. Since
\[
P^{-1} \begin{bmatrix}
\omega^{k_1} & 0 & 0 & \cdots & 0 \\
0 & \omega^{k_2} & 0 & \cdots & 0 \\
0 & 0 & \omega^{k_3} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega^{k_p}
\end{bmatrix}
= P \begin{bmatrix}
\omega^{k_2} & 0 & 0 & \cdots & 0 \\
0 & \omega^{k_3} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega^{k_p}
\end{bmatrix}
\]

it follows that \( \pi(\text{im} \chi) \subseteq \text{im} \chi. \)

**Lemma 4.2.** There are exactly \( p + 1 \) subspaces in \( \mathbb{Z}_p^p \) invariant under \( \pi \), one in each dimension \( j = 0, 1, \ldots, p \). They are \( \text{im} (I - \pi)^{p-j} \) for \( j = 0, 1, \ldots, p - 1 \), and \( \mathbb{Z}_p^p \). Also, \( (I - \pi)^p = 0 \).
Proof. The linear maps $\pi$ and $I - \pi$ have the same invariant subspaces. Since $\pi^p = I$ it follows that $(I - \pi)^p = 0$. The matrix

$$I - \pi = \begin{bmatrix} 1 & 0 & 0 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

has rank equal to $p - 1$. Hence

$$\mathbb{Z}_p^l \supset \text{im}(I - \pi) \supset \text{im}(I - \pi)^2 \supset \cdots \supset \text{im}(I - \pi)^{p-1} \supset 0$$

is the chain of all the distinct invariant subspaces of $I - \pi$. $\square$

Lemma 4.3. For $j \geq 1$ we have $\chi(G^{(j)}) \subseteq \text{im}(I - \pi)^j$.

Proof. Since $G$ is monomial and $P \in G$ it follows that each element of $G$ can be written in the form

$$(4.1) \quad P^l D_1 = D_2 P^l$$

for some $l \in \{0, 1, \ldots, p - 1\}$ and $D_1, D_2 \in D$.

We prove the lemma by induction on $j$. Assume $j = 1$. It is an easy consequence of the form (4.1) that elements of $G^{(1)}$ are products of elements of the form $DP^l D^{-1} P^{-l}$ for some $l \in \{1, \ldots, p - 1\}$ and $D \in D$. If

$$D = \begin{bmatrix} \omega^{k_1} & 0 & 0 & \cdots & 0 \\ 0 & \omega^{k_2} & 0 & \cdots & 0 \\ 0 & 0 & \omega^{k_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{k_p} \end{bmatrix}$$

then

$$DP^l D^{-1} P^{-l} = \begin{bmatrix} \omega^{k_1 - k_{l+1}} & 0 & 0 & \cdots & 0 \\ 0 & \omega^{k_2 - k_{l+2}} & 0 & \cdots & 0 \\ 0 & 0 & \omega^{k_{l+3}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{k_p - k_l} \end{bmatrix},$$

where the index $s$ of $k_s$ is computed modulo $p$. It follows that $\chi([D, P^l]) \subseteq \text{im}(I - \pi^l)$. Since $I - \pi^l = (I - \pi)(I + \pi + \pi^2 + \cdots + \pi^{l-1})$ we see that $\text{im}(I - \pi^l) \subseteq \text{im}(I - \pi)$ for $l = 1, 2, \ldots, p - 1$. Therefore, $\chi(G^{(1)}) \subseteq \text{im}(I - \pi)$.

Assume now that $D \in G^{(j-1)}$. The induction hypothesis is that $\chi(D) \in \text{im}(I - \pi)^{j-1}$. An easy matrix computation shows that each element of $G^{(j)}$ is a product of elements of the form $DP^l D^{-1} P^{-l}$ for some $l \in \{1, \ldots, p - 1\}$ and $D \in G^{(j-1)}$. Then we prove, in a way similar to the case $j = 1$, that $\chi(DP^l D^{-1} P^{-l}) \in \text{im}(I - \pi)^j$ and thus $\chi(G^{(j)}) \subseteq \text{im}(I - \pi)^j$. $\square$
Corollary 4.4. If $G \subseteq GL_p(F)$ is an irreducible $p$-group of exponent $p$ then its class is at most $p - 1$.

Proof. Lemma 4.2 and Lemma 4.3 with $l = p$ imply that $\chi(G^{(l)}) \subseteq \text{im}(I - \pi)^p = 0$. Therefore, $G^{(p)} = 1$. □

Proposition 4.5. If $G \subseteq SL_p(F)$ is an irreducible $p$-group with property ($wP2$) then the exponent of $G$ is equal to $p$.

Proof. We denote by $D$ the subgroup of all the diagonal elements of $G$. Assume that $A \in G \setminus D$. Then $A^p = I$ since $\det A = 1$. We may assume that

$$
P = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix}
$$

is in $G$. Suppose now that $D \in D$. Since $\det D = 1$ it follows that $(DP)^p = 1$. Hence $P, DP \in \Omega_1(G)$. Since $G$ has property ($wP2$) it follows that $D \in \Omega_1(G)$ and hence $D^p = I$. □

Theorem 4.6. If $G \subseteq GL_p(F)$ is an irreducible $p$-group with property ($s$) then its class is at most $p - 1$.

Proof. Assume that the exponent of $G$ is equal to $p^s$. Let $\theta$ be a primitive $p^{s+1}$-th root of 1. We enlarge $G$ to $\mathcal{H}$ by multiplying all the elements by $\theta^j$, $j = 1, 2, \ldots, p^{s+1}$. Note that this only enlarges the center, all other quotients of two consecutive elements of the upper central series of $G$ and $\mathcal{H}$ are equal. Hence the classes of both groups are equal. Next we consider the subgroup $\mathcal{K} = \{A \in \mathcal{H}; \det A = 1\}$. Since the exponent of $G$ is equal to $p^s$ it follows that for each $A \in G$ there is an integer $k(A)$ such that $\theta^{k(A)} A \in \mathcal{K}$. The elements of $G'$ are products of commutators $[A, B]$. Note that each commutator $[A, B]$ has determinant equal to 1. Since $[A, B] = [\theta^{k(A)} A, \theta^{k(B)} B]$ it follows that $G^{(j)} = \mathcal{K}^{(j)}$ for $j = 1, 2, \ldots, p$, and hence the classes $c(G)$ and $c(\mathcal{K})$ are equal. By Proposition 3.1 property ($wP2$) follows from property ($s$). Next, Proposition 4.3 implies that the exponent of $\mathcal{K}$ is equal to $p$ and Corollary 3.4 implies that $c(\mathcal{K}) \leq p - 1$. □

5. $p$-Abelian Groups Have Property ($s$)

Theorem 5.1. If the exponent of $G$ is equal to $p$ then $G$ has property ($s$).

Proof. It suffices to show that each finite irreducible matrix group $G \subseteq GL_{p^k}(F)$, $k \geq 0$, of exponent $p$ has property ($s$). Assume that $G$ is monomial and denote by $D$ the subgroup of all the diagonal matrices. Since the exponent of $G$ is equal to $p$ it follows that $\sigma(D) \subseteq \Gamma_1$ for all $D \in D$. Each element of $G$ is of the form $DP$ for some $D \in D$ and a permutation matrix $P$ of order dividing $p$. 
We choose two elements \( A_1 = D_1P_1 \) and \( A_2 = D_2P_2 \) in \( G \). Here \( D_1, D_2 \in \mathcal{D} \) and \( P_1, P_2 \) are permutation matrices. Observe that our assumptions imply that if \( P_i \neq I \) then \( \sigma(A_i) = \Gamma_1 \).

To show submultiplicativity of spectra we treat three cases:

- If \( P_1 = P_2 = I \) then the submultiplicativity is obvious.
- If \( P_1P_2 \neq I \) then one of \( P_1, P_2 \) is not equal to \( I \). We assume \( P_1 \neq I \), the case \( P_1 = I \) and \( P_2 \neq I \) is done in a similar way. Then
  \[
  \sigma(A_1A_2) = \Gamma_1 = \Gamma_1\sigma(A_2) = \sigma(A_1)\sigma(A_2).
  \]
- If \( P_1P_2 = I \), but neither of \( P_1, P_2 \) is equal to \( I \), then
  \[
  \sigma(A_1A_2) \subseteq \Gamma_1 = \Gamma_1\Gamma_1 = \sigma(A_1)\sigma(A_2).
  \]

\[\square\]

A finite \( p \)-group \( G \) is called \( p \)-abelian if \((xy)^p = x^py^p \) for all \( x, y \in G \). Now we recall a characterization of such groups [2, 27]: A finite group is \( p \)-abelian if and only if it is a section of a direct product of an abelian \( p \)-group and a group of exponent \( p \). By Lemma [2,1] abelian groups have property \((\hat{s})\). So we have the following consequence of Corollary [2,5] and Theorem [5.1]

**Corollary 5.2.** A finite \( p \)-abelian group has property \((\hat{s})\).

The following result is of interest on its own, and it will be used later as the first step of a proof by induction.

**Corollary 5.3.** Suppose \( G \subseteq SL_p(F) \) is an irreducible \( p \)-group. Then the following are equivalent:

1. \( G \) has property \((s)\).
2. \( G \) has property \((wP2)\).
3. \( e(G) = p \).

**Proof.** The implication (1) \( \Rightarrow \) (2) follows by Proposition [3.1]. the implication (2) \( \Rightarrow \) (3) by Proposition [1.5] and (3) \( \Rightarrow \) (1) by Theorem [5.1] \[\square\]

### 6. Metabelian Groups with Property \((\hat{s})\)

In this section we assume that \( G \) is a metabelian \( p \)-group, i.e. we assume that \( G' \) is abelian. Our result extends a result of Weichsel [26] that characterizes metabelian V-regular \( p \)-groups. We show that a finite metabelian group has property \((\hat{s})\) if and only if it is V-regular.

Let us introduce some notation. We write

\[
P_k = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]
for the cyclic matrix of order $p^k$ in $GL_{p^k}(F)$. If $D \in GL_{p^k}(F)$ is a diagonal matrix of order $p^l$ for some $l$ then an element of the form $DP_k$ is called a big cycle.

**Lemma 6.1.** Suppose that a monomial $p$-group $G \subseteq SL_{p^k}(F)$ is generated by a big cycle and a diagonal matrix. If $G$ is irreducible with property (wP2) then the exponent of $G$ is equal to $p^k$.

**Proof.** Assume that the generators of $G$ are a big cycle $DP_k$ and a diagonal matrix $B$. Here $D$ is a diagonal matrix, too. Since $\det(DP_k) = \det D = 1$ it follows that $DP_k$ is similar to $P_k$ using a diagonal similarity. Without loss we may assume that $D = I$, i.e., that $P_k \in G$. We denote by $\mathcal{D}$ the subgroup of all the diagonal matrices in $G$. Then each element of $G$ is of the form $EP_k^j$ for some $E \in \mathcal{D}$ and some integer $j$.

We prove the lemma by induction on $k$. The case $k = 1$ was proved in Proposition 4.5. Assume now that $k \geq 2$. Suppose that the subgroup $H \subseteq G$ consists of all the elements of the form $EP_k^j$, where $E \in \mathcal{D}$ and $j$ is a multiple of $p$. We may assume that, up to a permutational similarity, the elements of $H$ are all of the form

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_p \end{bmatrix},$$

where $A_j \in GL_{p^k-1}(F)$, $j = 1, 2, \ldots, p$. We denote by $H_1$ the subgroup in $GL_{p^k-1}(F)$ generated by all the blocks $A_1$ of all elements $A \in H$. Observe that it is irreducible. Let

$$\tilde{H}_1 = \{ \theta B; \det(\theta B) = 1, \theta \in F, B \in H_1 \}.$$ 

Then the group $\tilde{H}_1$ is an irreducible $p$-group in $GL_{p^k-1}(F)$ such that $\det B = 1$ for all $B \in \tilde{H}_1$. By the inductive hypothesis the exponent $e(\tilde{H}_1)$ is equal to $p^{k-1}$. Choose an element $C \in G \setminus H$. Without loss we may assume that

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & U \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix},$$

where $U \in GL_{p^k-1}(F)$. Observe that $\det U = 1$ since $\det C = 1$ and that $C^p \in H$. Hence $U \in \tilde{H}_1$. Then $|U|$ divides $p^{k-1}$ and $|C|$ divides $p^k$. Next assume that $A \in H$ is of form \((6.1)\). Since $\det A = 1$ it follows that $\prod_{j=1}^p \det A_j = 1$. In the same way as we did for $C$ we prove that $|AC|$ divides $p^k$. Since $G$ has property (wP2) it follows that $|A|$ also divides $p^k$. This shows that $e(G)$ divides $p^k$. Since $|P_k| = p^k$ it follows that $e(G) = p^k$. \hfill $\Box$
We denote by $\Gamma_k$ the set of all $p^k$-th roots of 1. If $\eta \in \Gamma_k$ is a scalar and $i$ a positive integer then

$$D_k(i, \eta) = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \eta^{(i)} & 0 & \cdots & 0 \\
0 & 0 & \eta^{(i)} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \eta^{(p^k-1)}
\end{bmatrix}$$

is a diagonal matrix in $GL_{p^k}(F)$. Here we assume that $(\frac{i}{i}) = 0$ if $j < i$.

**Lemma 6.2.** Assume that a 2-generated irreducible monomial $p$-group $G \subseteq SL_{p^k}(F)$ has class $c \leq p - 1$. Suppose further that one of the generators is a big cycle and the other is a diagonal matrix. Then $G$ has property (P2), its exponent is equal to $p^k$, and each element of the $(c-i)$-th subgroup $G^{(c-i)}$ in the lower central series of $G$, $i = 1, 2, \ldots, c - 1$, is a product of elements of the form

$$\alpha_0 I, \text{ and } \alpha_j D_k(j, \eta_j), \ j = 1, 2, \ldots, i - 1,$$

for some $\alpha_0 \in F$, $\alpha_j \in \Gamma_k$, $\eta_j \in \Gamma_k$, $j = 1, 2, \ldots, i - 1$.

**Proof.** Since $c \leq p - 1$ it follows that $G$ is a regular group [10, p. 322], and hence it has properties (P2) and (P2) [17]. By Lemma 6.1 its exponent is equal to $p^k$. The irreducibility of $G$ implies that its center consists of scalar matrices, which have order dividing $p^k$. Assume that $B$ is the diagonal generator and the other generator is $C = DP_k$, where $D$ is a diagonal matrix. Since $\det A = 1$ for all $A \in G$ it follows that $\det C = \det D = 1$. So $C$ is similar to $P_k$ using a diagonal similarity. Without loss we may assume that $C = P_k \in G$. We denote by $D$ the subgroup of all diagonal matrices in $G$. Since $G$ is monomial and $P_k \in G$ it follows that each element of $G$ can be written in the form

$$P_k^l D_1 = D_2 P_k^l$$

for some $l \in \{0, 1, \ldots, p^k - 1\}$ and $D_1, D_2 \in D$. It is an easy consequence of the form (6.3) that elements of $G^{(1)}$ are products of elements of the form $DP_k^l D^{-1} P_k^{-l}$ for some $l \in \{0, 1, \ldots, p^k - 1\}$ and $D \in D$. In particular, it follows that $G^{(1)} \subset D$. Observe that $G^{(c-i)}$ is a nontrivial subgroup of the center $Z(G)$ of $G$.

Let $\omega = e^{2\pi i}$ be a primitive $p^k$-th root of 1 and thus $\Gamma_k = \{\omega^j, j = 0, 1, \ldots, p^k - 1\}$. Further we denote by $Z_{p^k} = \mathbb{Z}/p^k \mathbb{Z}$ the finite quotient ring of $\mathbb{Z}$ by the principal ideal generated by $p^k$. We define a map $\chi : D \to Z_{p^k}$
by
\[
\chi \begin{bmatrix}
\omega l_1 & 0 & 0 & \cdots & 0 \\
0 & \omega l_2 & 0 & \cdots & 0 \\
0 & 0 & \omega l_3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega l_{p^k}
\end{bmatrix} = (l_1, l_2, \ldots, l_{p^k}).
\]

A proof similar to the proof of Lemma 4.1 shows that \( \chi \) is a homomorphism of abelian groups. We define the cyclic permutation \( \pi : \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^k} \) by

\[
\pi(l_1, l_2, \ldots, l_{p^k}) = (l_2, l_3, \ldots, l_{p^k}, l_1).
\]

If a matrix \( C_2 \in G^{(c-2)} \) is such that \([P_k, C_2] \neq I\) then

\[
[P_k, C_2] \in G^{(c-1)} \subset Z(G)
\]

and so

\[
(6.4) \quad [P_k, C_2] = \omega^t I
\]

for some \( t \) such that \( 1 \leq t \leq p^k - 1 \). If we write

\[
C_2 = \begin{bmatrix}
\omega l_1 & 0 & 0 & \cdots & 0 \\
0 & \omega l_2 & 0 & \cdots & 0 \\
0 & 0 & \omega l_3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega l_{p^k}
\end{bmatrix}
\]

then (6.4) implies that

\[
(6.5) \quad l_{j+1} - l_j = t, \quad j = 1, 2, 3, \ldots, p^k - 1
\]

and

\[
(6.6) \quad l_1 - l_{p^k} = t.
\]

This can be viewed as a simple linear difference equation (6.5) for an infinite sequence \( \{l_j\}_{j=1}^{\infty} \). Its solution is of the form

\[
(6.7) \quad l_j = t_1j + t_0 = t_1 \binom{j}{1} + t_0 \binom{j}{0}, \quad j = 1, 2, 3, \ldots, p^k - 1,
\]

for some \( t_1, t_0 \in \mathbb{Z} \). Observe that condition (6.6) is satisfied modulo \( p^k \).

Using (6.7) we obtain that \( C_2 = \alpha D_k(1, \eta) \) for some scalars \( \alpha, \eta \in \Gamma_k \). Note that the expression for \( l_j \) in (6.7) is linear in \( j \).

We prove the structure result for elements in \( G^{(c-i)} \) by induction on \( i \). Our inductive assumption is that for each \( C_i \in G^{(c-i)} \) the elements \( l_j \) of the sequence \( \chi(C_i) = (l_j)_{j=1}^{p^k} \) are given as a linear combination of binomial expressions \( \binom{j}{u} \), \( u = 0, 1, \ldots, i - 1 \), with integer coefficients. The case \( i = 2 \) was proved above.
Now we take an element \( C_{i+1} \in G^{(c-i-1)} \) and denote by \((l_j)_{j=1}^{p^k}\) the image \( \chi(C_{i+1}) \). Then \([C_{i+1}, P_k]\) is in \( G^{(c-i)} \) and we have
\[
\chi([C_{i+1}, P_k]) = (I - \pi)\chi(C_{i+1}). \tag{6.8}
\]
The inductive assumption implies that the elements of \( (6.8) \) are given as a linear combination of binomial expressions \((\binom{j}{u})\), \( u = 0, 1, \ldots, i - 1 \), with integer coefficients. As before, we can view the components of \( (6.8) \) as a simple linear difference equation. Its solution, i.e. the elements of \( \chi(C_{i+1}) \) are then given by
\[
l_j = \sum_{u=0}^{i} s_u \binom{j}{u}, \quad j = 1, 2, 3, \ldots, p^k. \tag{6.9}
\]
Since any polynomial which has integer values if the argument is an integer can be written as a \( \mathbb{Z} \)-linear combination in the binomial basis \( (\binom{j}{u}) \) (confer [III, p. 2]) it follows that the coefficients \( s_u \) in \( (6.9) \) are integers. Then
\[
l_{p^m+j} - l_j = \sum_{u=0}^{i} s_u \left( \binom{p^m+j}{u} - \binom{j}{u} \right) \tag{6.10}
\]
for \( m = 1, 2, \ldots \). Since \( 0 \leq u \leq i < p \), it is clear that
\[
\binom{p^m+j}{u} - \binom{j}{u}
\]
is divisible by \( p^m \) and hence \( p^m \) divides \( l_{p^m+j} - l_j \). In particular,
\[
l_{p^k} - l_1 = (l_{p^k} - l_{p^k+1}) + (l_{p^k+1} - l_1) \equiv (l_{p^k} - l_{p^k+1}) \pmod{p^k}. \]
This implies that \( l_{p^k+1} - l_1 \) is divisible by \( p^k \). In particular, this implies that also the equation given by the \( p^k \)-th component of \( (6.8) \) is satisfied modulo \( p^k \). Finally, since \( l_j \) can be written in the basis given by the binomial expressions \( (\binom{j}{u}) \) it follows that \( C_{i+1} \) is a product of elements of the forms
\[
o_0 I, \quad \alpha_j D_k(j, \eta_j), \quad j = 1, 2, \ldots, i,
\]
where \( o_0, \alpha_j, \eta_j \in \Gamma_k, \quad j = 1, 2, \ldots, i. \)

**Lemma 6.3.** The matrices \( D_k(i, \eta) \), for \( i = 1, 2, \ldots, p-2 \) and \( \eta \in \Gamma_k \), have the following properties:

1. \( \det D_k(i, \eta) = 1 \),
2. \( D_k(i, \eta) \) is permutationally similar to
\[
\tilde{D}_k(i, \eta) = \begin{bmatrix}
\alpha_1 E_1 & 0 & \cdots & 0 \\
0 & \alpha_2 E_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_p E_p
\end{bmatrix},
\]

where the matrices \( E_1, E_2, \ldots, E_p \) are diagonal with the determinant equal to 1 and each is a product of elements of the form \( D_{k-1}(i, \theta) \) for some scalars \( \theta \in \Gamma_{k-1} \). The similarity between \( D_k(i, \eta) \) and \( \tilde{D}_k(i, \eta) \).
is induced by the reordering of the standard basis \((e_1, e_2, \ldots, e_p)\) to 
\((e_1, e_{p^k-1+1}, \ldots, e_{(p-1)p^k-1+1}, e_2, e_{p^k-1+2}, \ldots, e_{(p-1)p^k-1+2}, \ldots, \
e_p, e_{p^k-1+p}, \ldots, e_{p^k})\).

**Proof.** Property (1) follows from the identity
\[
\sum_{j=1}^{n} \binom{j}{i} = \binom{n+1}{i+1},
\]
which holds for all positive integers \(i\) and \(n\) and can be verified by a counting argument.

Property (2) follows from the fact that the elements of the sequence \(\chi(D_k(i, \eta))\) are given by an expression of the form (6.9), which satisfies relation (6.10). Taking \(m = 1\) we see that \(E_j\) are products of elements of the form \(D_{k-1}(i, \theta)\) for some scalars \(\theta \in \Gamma_{k-1}.\)

**Proposition 6.4.** Suppose that \(G \subseteq SL_{p^k}(F)\) is an irreducible monomial \(p\)-group that is generated by a big cycle and a diagonal matrix. If \(G\) has class at most \(p-1\) then it has property \((s)\).

**Proof.** Assume that \(DP_k\) is the big cycle generator, where \(D\) is a diagonal matrix. Since \(1 = \det DP_k = \det D\) it follows that \(DP_k\) is similar, in fact by a diagonal similarity, to \(P_k\). Thus, we may further assume that \(P_k \in G\). We denote by \(D\) the subgroup of all the diagonal matrices in \(G\). Each element of \(G\) is of the form \(DP_k^j\) for some integer \(j\) and matrix \(D \in D\).

Observe that Lemma 6.2 implies that \(G\) has exponent equal to \(p^k\) and it has property \((wP2)\). We prove the proposition by induction on \(k\). For \(k = 1\) the claim follows by Corollary 5.3. Assume that our claim is true for the subgroups of \(SL_{p^l}(F)\) with \(l < k\). Choose two elements \(A_1 = DP_k^{j_1}\) and \(A_2 = DP_k^{j_2}\) in \(G\). We consider several cases:

- If \(j_1 = j_2 = 0\) then the submultiplicativity is obvious.
- If \(j_1 + j_2\) is not divisible by \(p\) then one of \(j_1, j_2\) is not divisible by \(p\).
  
  We assume that \(j_1\) is not divisible by \(p\). The case \(j_1\) is divisible by \(p\) and \(j_2\) is not divisible by \(p\) is done in a similar way.) Then
  \[
  \sigma(A_1 A_2) = \Gamma_k = \Gamma_k \sigma(A_2) = \sigma(A_1) \sigma(A_2).
  \]

- If \(j_1 + j_2\) is 0 modulo \(p^k\), but neither of \(j_1, j_2\) is 0 or divisible by \(p\), then
  \[
  \sigma(A_1 A_2) \subseteq \Gamma_k = \Gamma_k \Gamma_k = \sigma(A_1) \sigma(A_2).
  \]

It remains to consider the case when both \(j_1\) and \(j_2\) are divisible by \(p\). By Lemma 6.2 it follows that each element of the subgroup \(G^{(c-i)} i = 1, 2, \ldots, c - 1\), is equal to a product of elements of the following possible forms:

\[
\beta_0 I, \beta_j D_k(j, \eta_j), j = 1, 2, \ldots, i-1,
\]
where \( \beta_j \in \Gamma_k \) and \( \eta_j \in \Gamma_k \). By Lemma 6.3 each element of the form \( D_k(j, \eta_j) \) for \( j \leq p - 2 \), is permutationally similar to a matrix of the form

\[
\begin{bmatrix}
\alpha_1 E_1 & 0 & \cdots & 0 \\
0 & \alpha_2 E_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_p E_p
\end{bmatrix},
\]

where the matrices \( E_1, E_2, \ldots, E_p \) are diagonal with determinant equal to 1 and each is a product of matrices \( D_k(i, \theta) \) for \( \theta \in \Gamma_{k-1} \). The same permutational similarity brings \( P_k^p \) to

\[
\begin{bmatrix}
P_{k-1} & 0 & \cdots & 0 \\
0 & P_{k-1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & P_{k-1}
\end{bmatrix}.
\]

Now \( P_{k-1} \) and the diagonal blocks generate a metabelian group in \( GL_{p^{k-1}}(F) \) generated by a big cycle and diagonal matrices that are scalar multiples of matrices with the determinant equal to 1. Recall that property \( (s) \) depends only on 2-generated subgroups by Proposition 2.6 and does not depend on multiplication of elements of the group by scalars. Hence, the remaining case then follows by induction. \( \square \)

**Lemma 6.5.** Suppose that \( \mathcal{G} \subseteq GL_{p^k}(F) \) is an irreducible monomial \( p \)-group of class at most \( p - 1 \) that is generated by a big cycle and a diagonal matrix. Then it has property \( (s) \).

**Proof.** By Proposition 6.4 it follows that the group

\[
\tilde{\mathcal{G}} = \{ \theta A; \ A \in \mathcal{G}, \theta \in F, \ det(\theta A) = 1 \}
\]

has property \( (s) \). The lemma now follows since property \( (s) \) does not depend on multiplication of each element of the group by scalars. \( \square \)

Assume that \( c \) and \( e \) are positive integers. Suppose further that \( A \) is the direct product of \( c \) copies of the cyclic group of order \( p^e \), and that \( \{a_1, a_2, \ldots, a_c\} \) is a set of generators of \( A \). We denote by \( B_p(c, e) \) the split extension of \( A \) by an automorphism of order \( p^e \) defined by relations: \( b^{-1}a_ib = a_ia_{i+1}, \ i = 1, 2, \ldots, c - 1, \) and \( b^{-1}a_c = a_c \). It is easy to see that \( B_p(c, e) \) is a metabelian group of exponent \( p^e \) and class \( c \) and that it is generated by \( a = a_1 \) and \( b \). The groups \( B_p(c, e) \) are called basic groups.

Weichsel [25, p. 62] (see also Brisley [3]) showed that each finite metabelian \( p \)-group of class at most \( p - 1 \) is in the variety generated by a finite number of the basic groups \( B_p(c, e) \), \( c \leq p - 1 \).

**Proposition 6.6.** Suppose that \( G = B_p(c, e) \) is a basic metabelian \( p \)-group with \( c \leq p - 1 \) and that \( \mathcal{G} \subset GL_{p^k}(F) \) is an irreducible representation of \( G \). Then \( \mathcal{G} \) is a 2-generated monomial \( p \)-group such that one of the generators is a big cycle and the other is a diagonal matrix.
Proof. Assume that \( \psi : G \to GL_{p^k}(F) \) is an irreducible representation and that \( k \geq 1 \). Denote by \( \mathcal{G} \) the image of \( \psi \). Since \( G \) is generated by two elements it follows that \( \mathcal{G} \) is also generated by two elements. Observe that \( A \) is an abelian normal subgroup of \( G \) of index \( p^e \). By [22, Prop. 24, p. 61] and arguments in the proof of [22, Thm. 16, pp. 66-67] it follows that \( \psi \) is induced from a representation of \( A \). Since \( b^iA, i = 0, 1, \ldots, p^e - 1 \), is a complete set of cosets of \( A \) and since \( \rho \) is irreducible, \( G \) is monomial and one of the generators is diagonal, belonging to \( \rho(A) \). The group \( G = B_{p^e}(c,e) \) is a semi-direct product of \( A \) by a cyclic group \( C_{p^e} \) of order \( p^e \). By [22, Prop. 25, p. 62] all the representations of \( G \) are of the type \( \theta_i, \rho = \chi_i \otimes \rho \), where \( \chi_i \) is a representation of \( A \), and thus of degree 1, and \( \rho \) a representation of \( C_{p^e} \).

Since \( k \geq 1 \) and \( \mathcal{G} \) is irreducible monomial, the image \( \psi(b) = \chi_i(1) \otimes \rho(b) \) of the generator \( b \) of \( C_{p^e} \) is a big cycle. \[ \square \]

Next we prove the main result of the section. First, we introduce some notation. For two elements \( x, y \in G \) we define commutators \([x, ky]\) inductively as follows: \([x, 1y] = [x, y]\) and \([x, ky] = [[x, (k - 1)y], y]\) for \( k = 2, 3, \ldots \).

**Theorem 6.7.** Suppose that \( G \) is a metabelian \( p \)-group. Then the following are equivalent:

1. \( G \) has property \((\hat{s})\),
2. \( G \) is \( V \)-regular,
3. every two generated subgroup of \( G \) has class at most \( p - 1 \),
4. the variety of \( G \) is generated by a finite group of exponent \( p \) and a finite group of class at most \( p - 1 \),
5. \( G \) is a \((p - 1)\)-Engel group, i.e. \([x, (p - 1)y] = 1\) for and \( x, y \in G \),
6. the variety of \( G \) does not contain the wreath product of two cyclic groups of order \( p \).

Proof. The equivalence of (2), (3) and (4) was proved by Weichsel [26, Thm. 1.4]. The implication (1)\(\Rightarrow\)(2) follows from Theorem 3.3.

To prove the implication (3)\(\Rightarrow\)(1) we may without loss assume that \( G \) is a basic metabelian group \( B_{p^e}(c,e) \) of class \( c \leq p - 1 \). Suppose next that \( \mathcal{G} \subseteq GL_{p^k}(F) \) is an irreducible representation of \( G \). By Proposition 6.6, \( \mathcal{G} \) is monomial, generated by 2 elements one of which is a big cycle and the other a diagonal matrix. By Lemma 6.5 it follows that \( \mathcal{G} \) has property \((s)\).

We use [7, Thm. 3.7] to show that (2) and (3) imply (5) and (6). Finally [7, Lem. 3.2, Thms. 3.6 and 3.7] imply that either (5) or (6) imply (2). \[ \square \]

We remark that, in general, the class of a metabelian \( p \)-group with property \((\hat{s})\) can be larger than \( p - 1 \). See, for instance, the example given by Gupta and Newman in [9, (3.2)]. Corollary 3.5 implies that a 3-group \( G \) has property \((\hat{s})\) if and only if any of properties (2)--(6) of Theorem 6.7 holds for \( G \). In particular we have:

**Corollary 6.8.** A 3-group has property \((\hat{s})\) if and only if it is \( V \)-regular.
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