Random Matrix Theory Analysis of Cross Correlations in Financial Markets

Akihiko Utsugi¹, Kazusumi Ino², Masaki Oshikawa¹

¹ Department of Physics, Tokyo Institute of Technology, Oh-okayama 2-12-1, Meguro-ku, Tokyo, 152-8551, Japan
² Department of Pure and Applied Sciences, University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo, 153-8902, Japan

We confirm universal behaviors such as eigenvalue distribution and spacings predicted by Random Matrix Theory (RMT) for the cross correlation matrix of the daily stock prices of Tokyo Stock Exchange from 1993 to 2001, which have been reported for New York Stock Exchange in previous studies. It is shown that the random part of the eigenvalue distribution of the cross correlation matrix is stable even when deterministic correlations are present. Some deviations in the small eigenvalue statistics outside the bounds of the universality class of RMT are not completely explained with the deterministic correlations as proposed in previous studies. We study the effect of randomness on deterministic correlations and find that randomness causes a repulsion between deterministic eigenvalues and the random eigenvalues. This is interpreted as a reminiscent of “level repulsion” in RMT and explains some deviations from the previous studies observed in the market data. We also study correlated groups of issues in these markets and propose a refined method to identify correlated groups based on RMT. Some characteristic differences between properties of Tokyo Stock Exchange and New York Stock Exchange are found.

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I. INTRODUCTION

The price changes of securities such as stocks involve various economic backgrounds as well as interaction between securities. They seem to be quite complicated. Conventionally financial economists model the price changes of securities by stochastic processes (random walks). It is a basic ingredient of modern portfolio theory. Although the use of stochastic processes is common in finance, the validity of such a formulation should be empirically tested e.g. by statistical properties of the markets, since the underlying ergodic property of a market may be hard to be established.

Recently the statistical characterizations of financial markets based on physics concepts and methods attract considerable attentions. Given that a stochastic model is valid, some statistical properties of the market should be derived as outsets of stochasticity. For example, the cross correlation matrix among $N$ securities can be regarded as a random matrix and it may be legitimate to expect that it shares universal properties of a corresponding ensemble of Random Matrix Theory (RMT) in an appropriate large $N$-limit (since $N$ is usually large). This has been confirmed by several studies on actual stock markets. The bulk of the eigenvalue distribution of the cross correlation matrix of a major index (S&P500) of New York Stock Exchange (NYSE) is found to follow the eigenvalue distribution of the Wishart matrix, which is a random correlation matrix constructed from mutually uncorrelated time series. The eigenvalue spacing statistics are found to follow those of the Gaussian Orthogonal Ensemble (GOE).

The aim of this paper is to yield further supports on the applicability of RMT to analysis of stock markets. In Sec. II we give a brief review on the relevant results of RMT. We describe our data sample in Sec. III. In Sec. IV we test predictions of RMT for the cross-correlation matrix for the daily prices of the issues in Tokyo Stock Exchange (TSE) from 1993 to 2001. The quantities we calculated are the distribution of the eigenvalues, the nearest and next-nearest neighbor spacings, rigidity and a certain moment of eigenvector components. We find a good agreement with the real data within the RMT bounds for the eigenvalues. Indeed there are clear deviations outside the bounds which indicate the presence of deterministic correlations among issues. In Sec. V we consider random walks with deterministic correlations and show that the bulk part of the eigenvalue distribution of the correlation matrix is stable. In Sec. VI we closely examine the distribution of the moment of eigenvector component. Eigenvectors corresponding to the eigenvalues outside the RMT bounds deviate from the RMT prediction. According to Ref., the deviating eigenvalues at the lower edge are a consequence of the strong correlations among a few issues. However, we find that the observed data is not explained quantitatively by this reasoning alone. Therefore we analyze the effect of randomness on deterministic correlations between issues and find an interplay between deterministic correlations and randomness. We argue that it gives a refined explanation on the deviations. In Sec. VII we identify groups...
of strongly correlated issues from the information of the non-random eigenvectors. The ways of grouping in TSE and NYSE show some differences.

II. BRIEF REVIEW ON RANDOM MATRIX THEORY

A. Wishart Matrix

Let \( S_i(t) \) be a price at time \( t \) of a stock labeled by \( i (i = 1, 2, \cdots, N, t = 1, 2, \cdots, T) \). The change of price at time \( t \) can be measured by

\[
G_i(t) \equiv \ln S_i(t + 1) - \ln S_i(t).
\]

Here we take logarithm of the prices because the fluctuation of stock prices is typically given by the geometric Brown motion. Since

\[
G_i(t) \approx \frac{S_i(t + 1) - S_i(t)}{S_i(t)}
\]

\( G_i(t) \) is approximately the return of the issue \( i \) from \( t \) to \( t + 1 \). We also define the normalized return \( g_i(t) \) as follows.

\[
g_i(t) \equiv \frac{G_i(t) - \langle G_i \rangle_T}{\sigma_i}. \tag{3}
\]

\( \langle \cdots \rangle_T \) indicates the time series average of \( T \) steps and the dispersion \( \sigma_i \) is given by

\[
\sigma_i \equiv \sqrt{\langle G_i^2 \rangle_T - \langle G_i \rangle_T^2}. \tag{4}
\]

Then the correlation matrix \( C \) is expressed in terms of \( g_i(t) \)

\[
C_{ij} \equiv \langle g_i g_j \rangle_T. \tag{5}
\]

\( C \) is a real symmetric matrix with positive eigenvalues.

We will model the price of stocks as a stochastic process (random walk). For \( N \) random walks \( x_i(t), \quad (i = 1, 2, \cdots, N) \), a matrix \( M \) which is defined by \( M_{ti} = x_i(t) \) is a \( T \times N \) matrix. The cross correlation matrix \( W \) is defined as follows

\[
W_{ij} \equiv \langle x_i x_j \rangle_T = \frac{1}{T} M^t M,
\]

where \( M^t \) is the transposition of \( M \). A purely random case with a uniform dispersion \( \sigma \) is given by

\[
\langle x_i(t) \rangle = 0, \tag{7}
\]

\[
\langle x_i(t) x_j(\tau) \rangle = \sigma^2 \delta_{ij} \delta_{t\tau}. \tag{8}
\]

Here \( \langle \cdots \rangle \) indicates the average over the random variable phase space. In this case, \( W \) is called the Wishart matrix \( \mathbb{W} \). We can include “true” correlations among issues by replacing \( \delta_{ij} \) in \( \mathbb{W} \) by a non-diagonal matrix \( \tilde{C} \). We will call \( \tilde{C} \) as deterministic correlation while we call \( C \) or \( W \) as cross correlation.

B. Eigenvalue Statistics of Random Matrices

Let us summarize the relevant results of RMT to which we will refer in this paper.

In the limit \( N \to \infty, T \to \infty \) with \( Q \equiv T/N \) fixed, the eigenvalue distribution \( \rho(\lambda) \) for the Wishart matrix becomes \( \rho \)

\[
\rho(\lambda) = \frac{Q}{2\pi\sigma^2} \sqrt{(\lambda_{\text{max}} - \lambda)(\lambda - \lambda_{\text{min}})} \frac{1}{\lambda}
\]

\[
\lambda_{\text{max}}^\text{max} = \sigma^2 (1 + \frac{1}{Q}) \pm 2 \sqrt{\frac{T}{Q}} \tag{10}
\]

\[
\lambda_{\text{min}}^\text{min} = \sigma^2 (1 + \frac{1}{Q}) \pm 2 \sqrt{\frac{T}{Q}}
\]
is exact at $N \to \infty$, $T \to \infty$ with $Q \equiv T/N = \text{const}$. It is approximately valid at finite $N$ and $T$ when $N$ and $T$ are not small. According to (9), the eigenvalues of the Wishart matrix distribute only in the range $(\lambda_{\text{min}}, \lambda_{\text{max}})$.

Next we consider the Gaussian ensembles of random matrices. In the Gaussian ensembles, the probability of a matrix $H$ to be in the infinitesimal volume element $dH$ ($dH$ is given by the product of infinitesimal of independent elements) is given by $P(H)dH$ where $P(H)$

$$P(H) = A \exp(-a \sum_i |\lambda_i|^2).$$

Here $a$ is a parameter which characterizes the ensemble, $\lambda_i$ is the eigenvalue of $H$ and $A$ is the normalization constant. For general ensembles, one replaces the term $\sum_i |\lambda_i|^2$ by $\sum_i V(\lambda_i)$ with a function $V(\lambda)$. For example, one can add the quartic or higher order terms, but it is known that, in the large $N$-limit ($N$ is the size of $H$), the model flows to the Gaussian model. The Gaussian models are classified by the symmetry of the matrix as i) Gaussian Orthogonal Ensemble (GOE), the ensemble invariant under the orthogonal group, ii) Gaussian Symplectic Ensemble (GSE), the ensemble invariant under the symplectic group, and iii) Gaussian Unitary Ensemble (GUE), the ensemble invariant under the unitary group. Since the correlation matrix $C$ is real symmetric, we expect that its statistical properties of the eigenvalue are given by

$$\text{const} \cdot \text{exp}\left(\frac{-1}{\beta} \sum_i V(\lambda_i)\right).$$

For GOE, $\beta = 1$ for GOE, $\beta = 2$ for GUE and $\beta = 4$ for GSE. Thus the eigenvalue distribution for a Gaussian ensemble is determined by $\beta$. By this way, we get the eigenvalue distribution for a general potential $V$ as follows.

$$P(\lambda_1, \lambda_2, \cdots, \lambda_N) = A' \exp(-\beta(\sum_{k=1}^{N} \frac{V(\lambda_k)}{\beta} - \sum_{i<j} \ln |\lambda_i - \lambda_j|)).$$

where $A'$ is the normalization constant. From (14), one sees that the statistical properties at the short spacing between eigenvalues are dominated by $-\ln |\lambda_i - \lambda_j|$ and the total potential is negligible. Thus $\beta$ determines the eigenvalue spacing at short distance. For each $\beta$, the level spacing has been closely studied. As the correlation matrix is real symmetric, we expect that its statistical properties of the eigenvalue spacing are given by $\beta = 1$. One can characterize the statistical properties of eigenvalue spacing by the nearest neighbor spacing $P_{nn}$, the next-nearest neighbor spacing $P_{nnn}$, and the "rigidity" $\Delta(L)$. $P_{nn}$ and $P_{nnn}$ are for short-range correlations while $\Delta(L)$ is for long-range correlations. $\Delta(L)$ is defined as

$$\Delta(L) \equiv \frac{1}{L} \min_{A,B} \left\{ \int_{\lambda - \frac{L}{2}}^{\lambda + \frac{L}{2}} (F(\lambda') - AX - B)^2 d\lambda' \right\},$$

where $F(\lambda)$ is given by

$$F(\lambda) = \sum_k \Theta(\lambda - \lambda_k)$$

with the Heaviside function $\Theta$. $F(\lambda)$ counts the number of eigenvalues below $\lambda$. The meaning of $\Delta(L)$ is that one fits $F(\lambda)$ by a line in an interval with a width $L$ around each eigenvalue, and take the average of the deviations of the fit. $\Delta(L)$ is small when the eigenvalue spacing has a uniform distribution.

For GOE, $P_{nn}, P_{nnn}$ and $\Delta(L)$ are given by

$$P_{nn}(s) = \frac{\pi s}{2} \exp(-\frac{\pi}{4} s^2)$$

(17)
\[
P_{\text{nnn}}(s) = \frac{2^{18}}{3^9 \pi^3} s^4 \exp \left( -\frac{64}{9 \pi} s^2 \right)
\]

\[
\Delta(L) = \frac{1}{15} L^{-4} \int_0^L du (L-u)^3(2L^2 - 9Lu - 3u^2)
\times \left( \frac{1}{2} \delta(u) - Y(u) \right)
\]

\[
Y(u) = \left( \frac{\sin(\pi u)}{\pi u} \right)^2 + \frac{d}{du} \left( \frac{\sin(\pi u)}{\pi u} \right) \int_u^\infty \frac{\sin(\pi t)}{\pi t} dt.
\]

According to RMT, the distribution of components of an eigenvector of GOE is the normal distribution
with mean 0 and dispersion \( N \). A useful quantity in characterizing the distribution of components is the
Inverse Participation Ratio (IPR) \( [11, 12] \). For each eigenvector \( u_k \), IPR is defined by the following formula.

\[
I_k \equiv \sum_{i=1}^N u_{ki}^2,
\]

where \( u_{ki} \) is the \( i \)-th component of \( u_k \). For example, let us consider the case \( u_{ki} = 1/\sqrt{L} \) for \( 1 \leq i \leq L \)
and 0 for the other \( i \)'s. This gives \( I_k = 1/L \). Thus IPR can be interpreted as the inverse of the number of
components which differ from zero significantly. In RMT, the expectation value of IPR is

\[
\langle I_k \rangle = \frac{N}{\sqrt{2\pi N}} \exp \left( -\frac{u_{ki}^2}{2N} \right) du_{ki} = \frac{3}{N}.
\]

III. MARKET DATA

The data we analyzed are daily stock prices of i) Tokyo Stock Exchange (TSE) from 1993 January to 2001
June and ii) S&P 500 index of New York Stock Exchange (NYSE) from 1991 January to 2001 July. As for
S&P, the daily price data for a different period has been analyzed by Laloux et al. \[4\]. Also the 30 minute
price data for NYSE has been studied by Plerou et al. \[5, 6\]. In the TSE data, the number of data points
(the days that the market is open) is 1848. We analyze, among all the issues in TSE, the 493 issues which
are traded in all of the 1848 days. We select the data of these issues and analyze it. For this data, \( N = 493 \)
and \( T = 1848 \). We fit the distributions by

IV. UNIVERSAL RANDOM PROPERTIES OF CROSS CORRELATIONS IN STOCK
MARKETS

In Refs. \[4, 5\], the cross correlation matrices of NYSE data are analyzed and found that they exhibit
remarkable agreement with the predictions of universality properties of RMT for the small eigenvalues'
distribution, their nearest and next-nearest neighbor spacings, rigidity and IPR. In this section, we perform
a similar analysis on the TSE data and confirm these properties. We also use the S&P data for comparison.

We diagonalize the correlation matrices of TSE and S&P, to obtain the eigenvalues and the eigenvectors \( u_k \)
\((k = 1, \cdots , N)\). \( k \) is smaller for a large eigenvalue. For TSE, \( \sigma^2 = 1 \) and \( Q = N/T = 3.75 \) give \( \lambda_{\text{min}} = 0.23 \)
and \( \lambda_{\text{max}} = 2.30 \), also for S&P, \( Q = 8.75 \) gives \( \lambda_{\text{min}} = 0.43 \) and \( \lambda_{\text{max}} = 1.79 \). We fit the distributions by
optimizing \( \sigma^2 \) smaller than 1, as discussed in Ref. \[8\]. Fig\[11\] shows the eigenvalue distribution for TSE. We
see that the small eigenvalue distribution of the correlation matrix of TSE is well reproduced by RMT. There
are large eigenvalues beyond the bound \( [\lambda_{\text{min}}, \lambda_{\text{max}}] \) predicted by the Wishart matrix. The largest eigenvalue
we obtain is 121.6 (52.2) for TSE (S&P) and is interpreted as the factor for market trend as readily verified
by examining the corresponding eigenvector. The multitude of this factor to the price changes of individual
stocks is given by \( \lambda_1/N \), which is 0.247 (0.176) for TSE (S&P). Thus TSE is more correlated with the trend
factor than S&P.
FIG. 1: The figure shows the eigenvalue distribution for the correlation matrix of TSE. The line in each figure is for the real data and the dotted line is for the Wishart matrix. We use $N'$ multiplied by $N'/N$ for fitting where $N'$ is the number of eigenvalues within $[\lambda_{\text{min}}, \lambda_{\text{max}}]$. $\sigma^2$ is fitted to the optimized value by the least square method. $\sigma^2 = 0.47(0.53)$ for TSE (S&P). For TSE (S&P), a Kolmogorov-Smirnov test in the fitted region cannot reject the hypothesis that the RMT prediction is the correct description at the 30% (60%) confidence level.

FIG. 2: The figures are the nearest and the next-nearest neighbor spacing distribution for TSE compared to the prediction of RMT indicated by the dotted line. A Kolmogorov-Smirnov test cannot reject the hypothesis that the GOE prediction is the correct description at the 30% (80%) confidence level for the nearest neighbor spacing for TSE (S&P), at the 80% (60%) confidence level for the next-nearest neighbor spacing for TSE (S&P).
Next we compare spacings of the nearest neighbor and the next-nearest neighbor eigenvalues, and the rigidity with the predictions of RMT. To examine the statistics of the eigenvalue spacing, we first do the “unfolding” transformation on the data. The “unfolding” transformation is described in [6]. After doing the “unfolding” transformation on the eigenvalues below $\lambda_{\text{max}}$, we compare their nearest-neighbor and next-nearest neighbor spacing distributions to the ones for GOE. The theoretical predictions for the nearest neighbor spacing and the next-nearest spacing are given in (17) and (18) respectively. We show in Fig. 2 the spacings of small eigenvalues for TSE. It shows a good agreement with the prediction of RMT.

For the rigidity $\Delta(L)$, the theoretical prediction is given in eq. (19). The rigidity of the eigenvalues of the cross correlation matrix for TSE below $\lambda_{\text{max}}$ is compared to RMT in Fig. 3. Fig. 3 shows that the rigidity agrees well with the prediction of RMT.

In Fig. 4, we plot the calculated IPR for the eigenvectors of the cross correlation matrix of TSE. One sees that IPR agrees with the prediction of RMT around 1. There are also eigenvectors whose IPR are larger than the RMT prediction. These eigenvalues are from deterministic correlations. As in Fig. 1 such deviations can be seen at the large eigenvalues. However one also sees that there is a deviation in small eigenvalues. This deviation is concentrated at the lower edge. A simple model was constructed by Plerou et al. [6]. We will study this deviation closely in Sec. VI.

As mentioned, we also performed the same analysis on the S&P data for comparison. Results for rigidity and IPR are shown in Figs. 3 and 4. We found that the conclusions of Plerou et al. [6] for 30 minutes data of NYSE on eigenvalue spacings also hold for our daily S&P data.
V. STABILITY OF EIGENVALUE DISTRIBUTION OF THE WISHART MATRIX IN THE PRESENCE OF DETERMINISTIC CORRELATIONS

In the previous section, we found that the small eigenvalue distributions of the cross correlation matrices of TSE and S&P are reproduced well by the ones of the Wishart matrix, as previously found in Ref. [4]. The Wishart matrix is generated by the random walks without any deterministic correlations while the real stock data has a distribution of large eigenvalues, showing a deviation from the Wishart matrix. This indicates the existence of deterministic correlations.

Thus, in this section, we examine the stability of the random eigenvalue distribution of the cross correlation matrix $W$ of random walks when one includes deterministic correlations.

Let us consider a set of random walks whose deterministic correlation matrix has a finite number of large eigenvalues and other eigenvalues are small. We assume that $T \times N$ matrix $\{M_{nt} = x_i(t)\}$ has a deterministic
The present case was studied in Ref. [9]. Using the replica method, a Dyson-type equation for \( N \) correlation matrix equals eigenvectors for \( \lambda \) at \( N, T \). Let us split the trace as \( \text{Tr} = \text{Tr}_S + \text{Tr}_L \) and small eigenvalues \( \lambda \).

We also assume no temporal correlations thus set \( D = 1/T \) and taking the trace of (27)

\[
\text{Tr} G(\lambda) = \frac{N}{\lambda - \sigma^2 \left( 1 + \frac{1}{\text{Tr}_L CG} \right)^{-1}}.
\]

Solving this second-order algebraic equation for \( \text{Tr} G(\lambda) \) and putting the solution to (26) yields (9) and (10).

Now we assume that \( C \) has \( L \) large eigenvalues \( \lambda^C_k \) (\( k = 1, 2, \cdots, L \)) and the other \( N - L \) eigenvalues \( \lambda^C_k \) (\( k = L + 1, \cdots, N \)). We set \( \lambda^{CG}_k \) (\( k = L + 1, \cdots, N \)) to be a same value \( \lambda^{CG}_k \). Since the trace of the cross correlation matrix equals \( N \) by definition, we have

\[
\lambda^C_s = \frac{N - \sum_{k=1}^L \lambda^C_k}{N - L}.
\]

We also assume no temporal correlations thus set \( D = 1/T \).

From (27), the eigenvalues \( \lambda^G_k (\lambda) \) of \( G(\lambda) \) are given by

\[
\lambda^G_k (\lambda) = \frac{1}{\lambda - \lambda^C_k \left( 1 + \frac{1}{\text{Tr}_L CG + \text{Tr}_L CG} \right)^{-1}}.
\]

Let us split the trace as \( \text{Tr} = \text{Tr}_L + \text{Tr}_S \). Here \( \text{Tr}_L \) and \( \text{Tr}_S \) are the trace over the eigenspace spanned by the eigenvectors for \( \lambda^G_k \) (\( k = 1, \cdots, L \)), \( \lambda^G_k \) (\( k = L + 1, \cdots, N \)) respectively. Summation over \( k = L + 1, \cdots, N \) gives

\[
\text{Tr}_S \lambda^G_k (\lambda) = \frac{N - L}{\lambda - \lambda^C_s \left( 1 + \frac{1}{\text{Tr}_L CG + \text{Tr}_L CG} \right)^{-1}}.
\]

For \( N \) large, \( \rho(\lambda) \) should have finite supports around \( \lambda^C_k \) in the real axis of \( \lambda \). We denote supports for large and small eigenvalues \( D_S \) and \( D_L \). We assume that the case

\[
\lambda^{CG}_s \ll \lambda^{CG}_k \text{ (} k = 1, \cdots, L \text{)},
\]

when \( D_S \) and \( D_L \) don’t have an overlap. In that case, \( \lambda^G_k \) (\( k = 1, \cdots, L \)) is analytic in \( D_S \) while \( \lambda^G_k \) (\( k = L + 1, \cdots, N \)) has a branch cut. Thus in \( D_S \), \( \rho(\lambda) \) is determined by the imaginary part of \( \text{Tr}_S G \). For \( \text{Tr}_S G \), the contribution from \( \lambda^G_k \) (\( k = 1, \cdots, L \)) comes from the right hand of (31). Since \( \lambda^G_k \) (\( k = 1, \cdots, L \)) is analytic in the neighborhood of \( D_S \), \( \text{Tr}_L G \) is bounded by a constant. Since \( \lambda^G_k \) is an algebraic function of
$N$ and the scaling behavior consistent with (30) is $O(1)$, the constant can be taken to be independent of $N$. Thus if for $k = 1, \cdots, L$

$$L \lambda_k^C \ll N \lambda_s^C, \quad (33)$$

then

$$\text{Tr}_S \tilde{C} G = \lambda_s^C \text{Tr}_S G \gg \text{Tr}_L \tilde{C} G$$

for $N$ large because $\text{Tr}_S G$ gets large as $N \to \infty$. Then (31) is approximated by

$$\text{Tr}_S G(\lambda) = \frac{N - L}{\lambda - \lambda_s^C} \frac{1}{1 - \lambda_s^C \text{Tr}_S G(\lambda)} \quad (34)$$

(34) is equal to (28) when $\sigma^2 = \lambda_s^C$ and $N$ is replaced by $N - L$. By putting the solution of (34) to (26), we get

$$\rho(\lambda) \simeq \frac{N - L}{N} \frac{Q}{2 \pi \lambda_s^C} \sqrt{\frac{(\lambda_{\max} - \lambda)(\lambda - \lambda_{\min})}{\lambda}} \quad (35)$$

This formula is valid under (32) and (33). Note that there is a trade-off between $N, L, \lambda_s^C, \lambda_k^C (k = 1, \cdots, L)$ under (32) and (33). Thus $N - L$ eigenvalue distribution of this model can be approximated by the one for the Wishart matrix.

To conclude, the distribution of the small eigenvalues remains the same in the $N \to \infty$, as long as the numbers of the large eigenvalues of the deterministic correlation $\tilde{C}$ is finite and they appear only outside of $D_S$.

To confirm the validity of the approximation, we performed a Monte-Carlo simulation with 6 large eigenvalues. We choose the large eigenvalues to be 121.6, 14.5, 11.4, 7.9, 4.7, 4.0 which are the observed large eigenvalues of TSE. The result is shown in Fig.5. We see that the large eigenvalues correspond to the large eigenvalues of the real correlation matrix while the small eigenvalue distribution is well reproduced by the one for the Wishart matrix. We also examined other values of large eigenvalues and obtained similar results. Moreover, the probability of observed eigenvalues has a finite width by the effect of randomness. The width of an observed eigenvalue is wider for a larger eigenvalue.
VI. LEVEL REPULSION OF DETERMINISTIC CORRELATIONS BY RANDOMNESS

According to Plerou et al. [5], the deviation at small eigenvalues arises from strong correlations among a small number of issues. This is illustrated well by the following model. We consider a model that $N$ issues have an equal correlation $c$:

$$\tilde{C} = \begin{pmatrix}
1 & c & \cdots & c \\
c & 1 & \ddots & \\
\vdots & \ddots & \ddots & c \\
c & \cdots & c & 1
\end{pmatrix}$$

(36)

$\tilde{C}$ has an eigenvalue $1 + (N - 1)c$ with no degeneracy and an eigenvalue $1 - c$ with degeneracy $N - 1$. The eigenvalue $1 - c$ becomes small if $c$ is close to 1 i.e. strong correlation. Its eigenvectors have non-zero components at the correlated issues, resulting in a large IPR.

However this reasoning of large IPR eigenvectors at small eigenvalues is not sufficient to explain two facts. Firstly, eigenvectors with large IPR appear only below the bulk of the eigenvalue distribution of the Wishart matrix, concentrating at the lower edge. Since the correlation $c$ should be distributed in a wide range, eigenvectors with large IPR should also be distributed in a wide range. Thus the absence of
small eigenvalues with large IPR within the bulk is puzzling. Secondly, each eigenvector with large IPR is observed at a smaller value than expected from the model above. As the largest non-diagonal element of the correlation matrix of TSE (S&P) is 0.74 (0.83), eq. (36) tells that the eigenvector with large IPR should be observed at 0.26 (0.18). Actually the smallest eigenvalue with large IPR is observed at 0.11 (0.14) which is smaller than the lower bound of the eigenvalue distribution of the Wishart matrix.

These two facts motivate us to study the interplay between deterministic correlations and randomness. We consider a model of random walks with a deterministic correlation matrix \( \tilde{C} \), and examine IPR of eigenvectors of the cross correlation matrix \( C \). As a simple model, we assume \( \tilde{C} \) to have a following form:

\[
\tilde{C} = \begin{pmatrix}
\tilde{C}_1 & 0 & \cdots & 0 \\
0 & \tilde{C}_2 & \vdots & \\
\vdots & \ddots & \ddots & \\
0 & \cdots & \cdots & \tilde{C}_L \\
\end{pmatrix}.
\]

Here \( \tilde{C}_l \) (\( l = 1, \cdots, L \)) and \( 1 \) are

\[
\tilde{C}_l = \begin{pmatrix}
1 & c_l & \cdots & c_l \\
c_l & 1 & \vdots & \\
\vdots & \ddots & \ddots & \\
c_l & \cdots & c_l & 1 \\
\end{pmatrix}
\quad \quad 1 = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \vdots & \\
\vdots & \ddots & \ddots & \\
0 & \cdots & 0 & 1 \\
\end{pmatrix}.
\]

The form of \( \tilde{C} \) assumes \( L \) groups of issues with strong correlations. We consider \( N \) random walks \( x_i(t) \) with

\[
\langle x_i(t)x_j(\tau) \rangle = \tilde{C}_{ij} \delta_{\tau t}
\]

and examine their \( T \)-step cross correlation matrix

\[
C_{ij} = \frac{1}{T} M^T M = \frac{1}{T} \sum_{t=1}^{T} x_i(t)x_j(t).
\]

We set \( N = 493 \) and \( T = 1847 \) following our TSE data. We set the number \( L \) of strongly correlated groups to be 4 and the number of issues \( M \) participating each group to be 6. We choose the correlations to be \( c_1 = 0.8, c_2 = 0.6, c_3 = 0.4, c_4 = 0.2 \). We performed a Monte Carlo simulation of this model. We present IPR of the eigenvalues in Fig. 4. Fig. 4 shows that eigenvalues with large IPR distribute outside the bounds of eigenvalue distribution from randomness as in the real stock data. In this model, there should be 20 (counting degeneracies) small eigenvalues with large IPR in the simple model above, but the observed ones with large IPR only amount to 10. This implies that, when small eigenvalues arising from a strong correlation appear within the bounds of the Wishart matrix, IPRs of their eigenvectors get smaller and cannot be distinguished from the random eigenvalues. This is one effect of randomness on deterministic correlations. We also note that even for the eigenvectors which have larger IPR than the RMT value, their IPRs are smaller than expected.

Moreover \( \tilde{C} \) has small eigenvalues 0.2 and 0.4 while the corresponding eigenvalues of \( C \) distribute in the vicinity of 0.14 and 0.22 respectively. On the other hand, the eigenvalues of \( C \) corresponding to the large eigenvalues of \( \tilde{C} \) are shifted to larger values than the original values. Namely the eigenvalues of \( C \) from the deterministic correlation are repelled from the distribution of the random eigenvalues. We performed Monte Carlo simulations by changing the parameters for \( \tilde{C} \) and got similar results. This may be interpreted as a manifestation of the universal effect of randomness, called “level repulsion”. According to RMT, the eigenvalues of random matrices are repelled from each other by the logarithmic potential \(-\ln |\lambda_i - \lambda_j|\) in eq. (39). Even when some deterministic terms are present, this logarithmic potential causes a repulsion between eigenvalues. This universal effect has been observed for various systems such as levels of complicated nuclei. In the present case, deterministic correlations between random walks are repelled from the bulk distribution of the random eigenvalues. The eigenvalues in the RMT bounds form a repulsive potential and it repels the eigenvalues outside them.
FIG. 6: The upper graph is the IPR of the eigenvectors of the real correlation matrix $\tilde{C}$ given by (37-40). The lower graph is the IPR for the eigenvector of $C$. In the simulation, we set $N = 493, T = 1847, M = 6, L = 4, c_1 = 0.8, c_2 = 0.6, c_3 = 0.4$ and $c_4 = 0.2$.

We can deduce this “level repulsion” by solving the Dyson-type equations (23-26) numerically. We assume for simplicity that the eigenvalues of $\tilde{C}$ are 1 except one eigenvalue smaller or larger than 1. We solve (23-26) numerically for $N = 293$ and $T = 1847$ and obtain the relation between the smaller (or larger) eigenvalue and the corresponding eigenvalue of $C$. The result is shown in Fig. 7. Fig. 7 shows that smaller (larger) eigenvalues of $\tilde{C}$ are repelled by the bulk distribution around 1 and are observed as smaller (larger) eigenvalues of $C$.

Thus we found two interplays between deterministic correlations and randomness. Namely, when groups of issues have strong correlations, it results in large and small eigenvalues in the cross correlation matrix. Some of these eigenvalues are soaked up within the RMT bounds and their IPR becomes as small as the RMT value. They cannot be distinguished from random eigenvalues. On the other hand, eigenvalues from deterministic correlations outside the RMT bounds feel the repulsive potential generated by the bulk distribution of randomness. At the lower edge, they are shifted to smaller values. We believe that these give the explanation for two deviations we raised in this section.
FIG. 7: The effect of level repulsion on the eigenvalues of $C$. The horizontal axis is the small (large) eigenvalue of $\tilde{C}$ and the vertical axis is the corresponding eigenvalue of $C$. The upper (lower) graph is for the case where eigenvalues of $\tilde{C}$ are smaller (larger) than 1. The crosses are the result of a Monte-Carlo simulation based on eqs. 39 and 40. The straight line corresponds to the absence of the effect of randomness, when the eigenvalues of $C$ are identical to those of $\tilde{C}$. The eigenvalues of $C$ are repelled from the bulk vicinity of 1.

VII. GROUPS OF ISSUES FORMED BY STRONG CORRELATION

We have seen that the existence of a group of issues with strong correlation results in eigenvalues of the cross correlation matrix with large IPR. Conversely, by examining the eigenvectors with large IPR, we may identify groups formed by strong correlations.

For NYSE, Plerou et al. [5] examined the eigenvectors of large eigenvalues and distinguished strongly correlated issues by a criteria to have a large component in these eigenvectors. They found that the groups are formed according to the industrial sectors. However we found a difficulty to apply their method to TSE. Because eigenvectors for large eigenvalues have significant components not only from correlations but also from randomness, even if an issue has a large component in an eigenvector of large eigenvalue, it is difficult to tell whether it is from the effect of deterministic correlation or just from randomness. Especially for TSE, the effect of deterministic correlations is apparently not strong enough to make the separation straightforward. As we examined the eigenvectors of the large eigenvalues, we found it impossible to separate the group of strongly correlated issues. For example, Fig. 8 shows the component distribution of the eigenvector for the
The component distribution of the eigenvector for the six-th largest eigenvalue 4.0 of TSE. The components distribute continuously and it is hard to distinguish the components from correlations.

Therefore, here we propose a supplementary method to identify strongly correlated components. As we saw in Sec. VI when a group of issues is formed by strong correlations, they not only have a large component in the eigenvectors of the corresponding large eigenvalue, but also have a large component in the eigenvectors of the corresponding small eigenvalue. On the other hand, issues which do not have strong correlations with others should have the normal distribution in eigenvectors. Namely, the deviation from the normal distribution indicates the issue is correlated with others. To quantify how an issue has a distribution different from the normal distribution, we define a quantity $Z_i$ as follows.

$$Z_i = \sum_{k: I_{ki} \geq \delta_{th}} u_{ki}^2,$$

where $\delta_{th}$ is a threshold for IPR. $Z_i$ is the sum of the square of $i$-th component of the eigenvectors which have IPR $\geq \delta_{th}$. We set $\delta_{th} = 0.008(0.02)$ for TSE (S&P), which sort out 41(28) eigenvectors. If $i$-th issue has no true correlation with others, the components $u_{ki}$ of the eigenvectors follow the normal distribution, and hence the probability of having a large $Z_i$ should be small. Thus the $i$-th issue may be regarded as significantly correlated if $Z_i$ is larger than a certain threshold $\alpha_{th}$. We choose $\alpha_{th}$ so that the probability of $Z_i \geq \alpha_{th}$ is 1 % if the eigenvector components for the $i$-th issue follows the normal distribution. For our data, $\alpha_{th} = 0.131 (0.162)$ for TSE (S&P). If $i$-th issue has a large component in an eigenvector, we consider it to be in the corresponding group of strong correlations when $Z_i \geq \alpha_{th}$.

We applied this method to large eigenvalues observed in our market data. The results are shown in TABLE 1-2.

**TABLE 1 TSE issues with $Z_i \geq \alpha_{th}$**

| Eigenvector | TSE code | Company Name               | Sector          |
|-------------|----------|----------------------------|-----------------|
| $u_2$       | 6701     | NEC Electric Products      | Electric Products |
| $u_2$       | 6702     | Fujitsu Electric Products  | Electric Products |
| $u_2$       | 8035     | Tokyo Electron             | Electric Products |
| $u_3$       | 1888     | Wakachiku Construction     | Construction    |
| $u_3$       | 8834     | Douwa Real Estate          | Real Estate     |
| $u_4$       | 9501     | Tokyo Electric Power       | Electric Power  |
| $u_4$       | 9503     | Kansai Electric Power      | Electric Power  |
| $u_4$       | 9504     | Chuugoku Electric Power    | Electric Power  |
| $u_4$       | 9506     | Tohoku Electric Power      | Electric Power  |
TABLE 2  S&P issues with $Z_i \geq \alpha_{th}$

| Eigenvector | Ticker | Company Name                  | Industries                      |
|-------------|--------|--------------------------------|---------------------------------|
| $u_2$       | AEP    | American Electric Power       | Electric Power                  |
| $u_2$       | DUK    | Duke Energy Corporation       | Electric Power, Natural Gas     |
| $u_3$       | APC    | Anadarko Petroleum Corp.      | Oil, Gas                        |
| $u_3$       | BHI    | Baker Hughes Inc.             | Oil Related                     |
| $u_3$       | XOM    | Exxon Mobil Corporation      | Oil, Coal, Copper               |
| $u_3$       | HAL    | Halliburton Company           | Oil, Gas                        |
| $u_3$       | RD     | Royal Dutch Petroleum Co.     | Oil, Gas, Chemical              |
| $u_3$       | SLB    | Schlumberger Ltd.             | Oil                             |
| $u_3$       | UCL    | Unocal Corporation            | Oil, Gas                        |
| $u_4$       | GP     | Georgia-Pacific Group         | Paper Manufacture, Pulp         |
| $u_4$       | IP     | International Paper Co.       | Paper Manufacture               |
| $u_4$       | MEA    | Mead Corporation              | Paper Manufacture, Pulp, Gum    |
| $u_4$       | WY     | Weyerhaeuser Company          | Paper Manufacture, Pulp, Forestry, Wooden Goods |
In S&P, Electric Power sector and, Oil and Gas related sectors play major parts in the correlations. In TSE, Electric Products sector and Construction sector play major parts.

In S&P, each eigenvector corresponds to an industrial sector. This means that each industrial sector forms a strongly correlated group. On the other hand, in TSE, there are eigenvectors whose participants are from different industrial sectors, which may indicate a more complicated correlation structure of the market. Thus it seems that TSE and S&P (NYSE) have some differences in the structure of the correlations, while the “random” part is well described by the universal theory in the both markets. It would be interesting to find the origin of the difference. This might be useful to give some insights into the difference of the economic structures of the two countries.

As far as our data samples are concerned, we may conclude that the method which we propose utilizing small eigenvectors and their IPR effectively distinguishes strongly correlated groups in the markets.

We noticed that Giada et al. investigated the grouping of S&P data in Ref. [14] based on a model considered by Noh [15]. The method proposed in Refs. [14, 15] has the advantage of directly giving the “noise-undressed” correlation matrix. However, the basic assumption of their method is that each issue belongs to only one cluster of correlated issues. This assumption is apparently not quite true according to our analysis. For example, “Tohoku Electric Power” appears in three different groups in TABLE 1. Therefore we believe that more analysis based on conservative assumptions should be made before applying the estimated “true” correlation to the portfolio management.
VIII. CONCLUSIONS

We analyzed the eigenvalues and the eigenvectors of the cross correlation matrices of TSE and NYSE (S&P500) stock market data. We found that results of Refs. [4, 5, 6] reported for NYSE are also valid for TSE. The eigenvalue distribution obeys the RMT prediction in the bulk but there are some deviations at the large eigenvalues. We also examined the nearest neighbor spacing, the next-nearest neighbor spacing and the rigidity of the eigenvalues and found that they follow the universality of GOE. These are consistent with Refs. [4, 5, 6] and imply that the large eigenvalues are due to the existence of correlations while the eigenvalues distributed in the bulk are due to randomness. We also examined IPR of the eigenvectors of the correlation matrices. In the bulk, IPR distribution follows the prediction of GOE, but there are deviations outside the RMT bounds. Plerou et al. [5, 6] argued that deviations at the lower edge are due to strong correlations. We found that this reasoning is qualitatively valid, but quantitatively it cannot explain the fact that small eigenvalues with large IPR concentrate at the lower edge and the observed eigenvalues are smaller than the expected values.

To explain this phenomenon, we studied RMT with deterministic correlations. We found that each eigenvalue from deterministic correlations is observed at values which are repelled from the bulk distribution. We interpreted this repulsion as a reminiscent of the effect of randomness, known as “level repulsion”. This effect is shown to be deduced by solving the Dyson-type equation numerically.

We also proposed a method to distinguish strongly correlated groups of issues based on IPR. It reduces the accidental appearance of uncorrelated issues. Applying this method, we found that issues of S&P are grouped according to the industrial sectors. On the other hand, issues of TSE are grouped in more complicated ways, suggesting some differences in the structure of the markets.

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