Quantum cosmology with vector torsion

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Abstract – We extend the treatment of quantum cosmology to a manifold with torsion. We adopt a model of Einstein-Cartan-Sciama-Kibble compatible with the cosmological principle. The universe wave function is shown to be subject to a $\mathcal{PT}$-symmetric Hamiltonian. With a vanishing energy-momentum tensor, the universe evolution in the semiclassical and classical regimes is shown to suggest a two-stage inflationary process induced by torsion.

Introduction. – In the theory of quantum cosmology, the universe is described by a wave function subject to the quantized Hamiltonian of Einstein’s General Relativity (GR) [1]; this is known as Wheeler-DeWitt (WDW) equation ($\hat{H}\Psi = 0$). In the ensemble interpretation, the WDW wave function indicates the possibility of the universe being spontaneously created from a certain initial quantum state. Hence it circumvents the classical singularity associated with the beginning of the universe.

On another note, overwhelming experimental results have plagued the standard model of cosmology based on GR with conventionalist stratagems [2–5]. Theories with torsion have recently gained ground as a promising extension to GR that would be in line with all the observationally successful results of GR in the regime of a vanishing torsion [6–11].

Einstein-Cartan-Sciama-Kibble theory (ECSK) is a theory on Einstein-Cartan manifold which is noteworthy for being the simplest gauge theory for gravity. ECSK has the flavor of GR with an additional ingredient of an antisymmetric part to the connection defined via torsion. At low energies, torsion effects are suppressed by the square of the gravitational constant ($k^2$) so its effect is insignificant for the late universe [12] and the theory is expected to take over at high energies as an intermediate between GR and a more fundamental theory of quantum gravity.

Torsion is linked to the canonical spin tensor of the matter action and it is usually modeled in cosmology as a macroscopically spinning fluid. However such description fails to capture the essence of the theory mainly because of its incompatibility with the cosmological principle and the shaky assumption of a classical matter at such high energies where torsion dominates. Torsion emerges in other different contexts, for instance it may arise as a gradient of a scalar field [13] or as Kalb-Ramond field in string theory [14,15]. It is also worth noting that supergravity is a natural framework where torsion, curvature, and matter fields are treated in analogous manner [16].

We believe a successful model should obey the cosmological principle and be studied in a quantum setting where torsion effects are expected to manifest. A study for gravity on a Riemann-Cartan manifold $U_4$ in the framework of quantum cosmology has been lacking in the literature. In this work, we consider the case for ECSK with vector torsion in a minimal setup. The corresponding WDW equation and its solutions are discussed and interpreted with emphasis on the Hamiltonian being $\mathcal{PT}$-symmetric which defines a physical theory with real spectrum and unitary evolution. We find that in the absence of any other matter field, torsion can drive a period of accelerated expansion within a semiclassical region and it continues in the classical region.

Setup. – A Riemann-Cartan space-time $U_4$ in general has 40 independent degrees of freedom: 16 of which are encoded in the metric and 24 in the torsion tensor [12,17]. The spatial-maximal symmetry of the manifold reduces the metric to the FRW metric. In the ADM formalism, one splits spacetime into space-like hypersurfaces $\Sigma$’s separated by a time lapse $N(t)dt$. After the $3 + 1$ split and a standard choice of coordinates, FRW metric can be written as [18,19]

$$ds^2 = -N^2 dt^2 + a(t)^2 \left( \frac{1}{1 - kr^2} dr^2 + r^2 d\Omega^2 \right),$$  \hspace{1cm} (1)

where $a(t)$ is the cosmic scale factor and the curvature constant $k = -1, 0, 1$ corresponds to open, flat, and closed
universes, respectively. On the other hand, for torsion to be compatible with the same symmetries, its traceless part must vanish and the vector and axial-vector parts must conform to the following form [20]:

\begin{align}
S_{\theta_0} &= -S^i_{\theta_0} = N\phi(t), \\
S_{[ijk]} &= \chi(t),
\end{align}

where \( \phi(t) \) or \( \chi(t) \) are scalar functions of time. The choice of axial vector eq. (3) minimally couples with gravity through the Lorentz-invariant measure \( \sqrt{-g}\,d^4x \). In the following analysis we adopt the more interesting choice of the vector form eq. (2) which couples as well to the metric derivative as we shall see. Our aim is to canonically quantize the action 

\[ L \equiv \int_M d^4x \sqrt{-g}R(g_{\mu\nu}, S_{\mu\nu\lambda}), \]

where \( g_{\mu\nu} \) is the FRW metric eq. (1) and the non-vanishing components of \( S_{\mu\nu\lambda} \) are those of vector torsion eq. (2). In order to have a well-posed variational principle, one has to define boundary conditions on the manifold \( M \) which amounts to adding a surface term to the action. While the choice of boundary does not affect the classical equations of motion, it directly alters their quantum counterparts owing to the differential nature of the momenta operators. Hence we add the Gibbons-Hawking-York (GHY) boundary term \( \int_{\partial M} 2\sqrt{H}Kd^3x \) to have the relevant action for quantum cosmology [21,22]:

\[ \mathcal{L}_{ECSK} = \sqrt{-g}(3)R + K_{ij}K^{ij} - K^2, \]

where the surface \( \Sigma \) has an intrinsic curvature (Ricci scalar) \( K_{ij} \) and an extrinsic curvature tensor \( K_{ij} \equiv -N\Gamma_{ij}, \). The intrinsic curvature and the non-vanishing components of the extrinsic curvature can be calculated explicitly and substituted into eq. (5) to get the reduced Lagrangian:

\begin{align}
(3)R &= 6\frac{k}{a^2}, \\
K_{ij} &= -g_{ij}N^{-1}(2N\phi + \frac{\dot{a}}{a}), \\
\mathcal{L}_{ECSK} &= NKa - N^{-1}a\dot{a}^2 - 4\phi a\dot{a}^2 - 4N\phi a^2.
\end{align}

We now construct the full Lagrangian by adding the matter part and a possible higher-order Lagrangian of invariant torsion contractions, schematically we write it as \( \mathcal{L}_T \equiv \mathcal{L}_T(V,S^2) \). Similar to quadratic curvature \( R^2 \), contributions of kinetic terms \( \mathcal{L}_T \) are only relevant at high energies so it is suppressed by a mass scale \( M_\ast \simeq M_{pl} \):

\[ \mathcal{L} = \mathcal{L}_{ECSK} + \frac{2}{M_\ast^2}\mathcal{L}_T; \]

the purpose of the second term is to show that the canonical momentum of \( \phi \) does not necessarily vanish as we shall see in the next section. However in our subsequent analysis we will be interested only in the semiclassical region so constructing such higher-order term explicitly is not on our agenda.

**Modified Wheeler-DeWitt equation.** We now wish to find a modified version of WDW by quantizing the Lagrangian eq. (9). In order to find the Hamiltonian of the system, we first compute its conjugate momenta:

\[ P_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = -2N^{-1}a\dot{a} - 4\alpha a^2 \]

\[ \Rightarrow \dot{a} = -2N\phi a - N\frac{P_a}{2a}, \]

\[ P_\phi = \frac{2}{M_\ast^2} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \]

\[ P_N = \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0, \]

the canonical Hamiltonian defined by \( \mathcal{H}_\ast(P,q) \equiv \sum_i P_i \dot{q}_i - \mathcal{L} \) can be computed from the above momenta:

\[ \mathcal{H}_\ast(P,q) = N \left(-\frac{P_a^2}{4a} - 2\phi a P_a - Ka - P_\phi \phi \right) + a^3 \left(\rho_M + \rho_{vac} + \frac{1}{M_\ast^2} \mathcal{L}_T(P_\phi, \phi)\right); \]

we note that \( \mathcal{L}_T \) minimally couple to vilenkin through the \( \sqrt{-g} \) factor so it does not contribute to the momentum of \( a \). Also we note the lapse function \( N \) is a Lagrange multiplier and its equation of motion \( \frac{4P_a}{a^2} = -\frac{\partial \mathcal{H}_\ast}{\partial a} \) leads to the Hamiltonian constraint:

\[ -\frac{P_a^2}{4a} - 2\phi a P_a - Ka - P_\phi \phi + 2a^3 \left(\rho + \rho_{vac}\right) + \frac{1}{M_\ast^2} \mathcal{L}_T(P_\phi, \phi) = 0. \]

To quantize the system, we turn the classical observables into operators: \( a \rightarrow \hat{a}, P_a \rightarrow -i\frac{\partial}{\partial a}, \) and \( \phi \rightarrow \hat{\phi} \). Then we introduce the so-called wave function of the universe \( \Psi(\hat{a}, \hat{\phi}) \) which is subject to the Hamiltonian constraint \( \widehat{\mathcal{H}}\Psi = 0 \). We take into consideration the operator ordering ambiguity by taking \( \hat{P}_a = -ia\hat{\phi} \frac{\partial}{\partial \hat{a}} - a^{-q} \) and \( \hat{P}_a^2 = -a^{-p} \frac{\partial}{\partial a} \left(a^p \frac{\partial}{\partial \hat{a}} \right). \) Now we obtain the wave equation for \( \Psi \):

\[ a^{-p} \frac{\partial}{\partial a} \left(a^p \frac{\partial}{\partial \hat{a}} \Psi \right) + 8i\phi a^2 \frac{\partial}{\partial a} \left(a^{-q} \Psi \right) - 4Ka^2 \Psi = 0, \]

\[ -4aP_\phi \Psi + 8a^4(\rho_M + \rho_{vac} + \frac{1}{M_\ast^2} \mathcal{L}_T(P_\phi, \phi)) \Psi = 0, \]

\[ \left(\frac{\partial^2}{\partial a^2} + \left(\frac{p}{a} + 8i\phi a^2\right) \frac{\partial}{\partial a} - 4Ka^2 - 4P_\phi \right) + 8a^4(\rho_M + \rho_{vac} + \frac{1}{M_\ast^2} \mathcal{L}_T(P_\phi, \phi)) \Psi = 0. \]

On physical grounds, we set \( q = 0 \) in the last line, otherwise the equation will be plagued by an imaginary effective potential and the Hamiltonian would be non-conservative. Having done that, we are still left with a non-Hermitian operator, however it is now \( \mathcal{T} \)-symmetric (at least in the regime of a suppressed kinematical action). For a Hamiltonian endowed with such symmetry, one can safely define a physical theory with real spectrum, unitary time
evolution, and states of positive norm under a \( CPT \) inner product [23]:

\[
(f \mid g) = \int_C \mathrm{d}x [\mathcal{CPT} f(x)] g(x).
\]

In the context of quantum cosmology, we are only interested in the ground state and not the time evolution or the rest of the energy spectrum as WDW dictates \( \dot{H} \Psi = 0 \).

**Probability current and emergent time.** We follow the semiclassical approach in tackling the infamous problem of time in quantum cosmology [24]. We rewrite the wave function in the WKB ansatz as \( \Psi(a, \phi) = R(a, \phi)e^{iS(a, \phi)} \), where \( R(a, \phi) \) and \( S(a, \phi) \) are real functions. Eq. (16) can be cast into

\[
\frac{\partial}{\partial a} \Psi = \Psi H^+ \Psi^*,
\]

from the conservation equation (21) and by substitution from the guidance equation (20), one gets an equation relating the probability density to the time evolution of the scale factor

\[
R^2 = \frac{a^{-p}}{2a a_0} C_0(\phi)^2 = \frac{C_0(\phi)^2}{2a^{p+2} H(a)},
\]

where \( C_0(\phi) \) is an arbitrary function and we write it squared for convenience. We can now find the evolution of the scale factor in this semiclassical regime by solving the WDW for the time-independent probability density \( \dot{R}(a, \phi) \) and then solving eq. (23) for the evolution of \( a(t) \).

In this picture, the quantity in eq. (23) which we conventionally call “probability density” has a different interpretation than in quantum mechanics. Now it signifies the universe evolution from the state \( a_1 \) to \( a_2 \) as \( \int_{a_1}^{a_2} \rho da \sim \int_{t_1}^{t_2} \mathrm{d}t \) [25]. In order to have a finite probability density and Hubble parameter in the early universe \( a \rightarrow 0 \), the ordering parameter has to be \( p = -2 \). In the following analysis, we adopt this particularly intriguing choice.

**Solution in flat space.** Now, we attempt to examine a minisuperspace model with an ordering parameter \( p = -2 \) and in flat space \( K = 0 \) so WDW equation reads

\[
\left( \frac{\partial^2}{\partial a^2} + \left( 8a^2 - \frac{2}{a} \right) \frac{\partial}{\partial a} \right) \Psi(a, \phi) = 0,
\]

even with vanishing conventional forms of energy, the torsion guarantees a non-trivial solution to the wave function, namely

\[
\Psi(a, \phi) = C_1(\phi) e^{-\frac{2}{\phi} a^3 \phi} - 1 + C_2(\phi).
\]

The particular form of \( C_1(\phi) \) and \( C_2(\phi) \) should be found in principle by matching the solution with that in the region where the contribution of a dynamical torsion \( \mathcal{L}_T \) is significant. We demand that the universe has zero probability of emerging in a singular state; this is equivalent to setting a boundary condition \( \Psi(0, \phi) = 0 \Rightarrow C_2(\phi) = 0 \).

\[
\Psi(a, \phi) = \frac{C_1(\phi)}{\phi} (e^{-\frac{2}{\phi} a^3 \phi} - 1).
\]

The quantum regime is defined for a small scale factor \( a < 1 \), and a large field \( \phi \approx 1 \) (a Planckian field). In this regime, the wave function represents an ensemble of universes that would require an external observer to make sense of any expectation value. However when the universe is large enough and the field is very small (\( a > 1, \phi \ll 1 \)), it can interact with itself and evolves in a classical trajectory.

The wave function in that limit would be \( \Psi \sim a^3 \) so it has the form of WKB ansatz with amplitude \( R = a^3 \) and
phase \( S = 0 \). Then one can get the evolution immediately by using eq. (20) with \( S' = 0 \),
\[
\frac{\dot{a}}{a} = -2\phi \Rightarrow a = a_0 e^{-2\int_0^t \phi(t) dt}; \tag{26}
\]
this result admits a stage of accelerated expansion in agreement with the result from the classical analysis in the literature [26]. In an earlier intermediate stage the universe can enter the classical domain \( a > 1 \) with a large value for the torsion field \( \phi \approx 1 \), and the probability density \( \rho \equiv |\Psi|^2 = C_1(\phi)^2 \sin^2(\frac{\phi}{2}a^3) \) should now be understood according to the semiclassical interpretation.

We note from fig. 1 that there is a zero probability of finding the universe at certain states \( a_n = (\frac{\phi}{2}n \pi)^{1/3} \) for integer \( n \). And despite the non-renormalizability of the wave function, the quantity \( I \equiv \frac{1}{\Delta a} \int_0^{a+\Delta a} \rho da \) as depicted from fig. 1 remains regular and would eventually converge to a particular value as \( a \) increases and \( \Delta a \) tends to zero,
\[
\rho(a \gg 1) \equiv \lim_{\Delta a \rightarrow 1} \frac{\int_0^{a+\Delta a} \rho da}{\Delta a} = \frac{C_1(\phi)^2}{2\phi^2};
\]
in this regime the probability density only depends on \( \phi \) and using eq. (18), we can get the classical evolution of the scale factor \( a \),
\[
H(a \gg 1, \phi \sim 1) = \frac{C_0}{C_1} \phi^2 \Rightarrow a(t) = a_0 e^{\frac{1}{c^2 t^{3/2}} \phi^2 dt}, \tag{27}
\]
this solution also admits a stage of accelerated expansion prior to the classical eq. (26). Unlike the classical evolution, Hubble parameter eq. (27) depends on the functions \( C_1(\phi) \) and \( C_0(\phi) \) which can be found by explicitly identifying \( \mathcal{L}_T \), solving its dynamical equation of \( \phi \) and matching the solutions at the boundary.

In order for inflation to continue past this phase, the inflationary parameter \( \epsilon \equiv \frac{\dot{\phi}^2}{2\phi^2} \) must be less than unity implying \( \frac{\dot{\phi}}{\phi} \ll 1 \). In the sub-Planckian region, inflation will continue as long as \( \epsilon = \frac{\dot{\phi}}{\phi} < 1 \). This condition will eventually fail as \( \phi \ll 1 \) marking the exit of inflation at \( \epsilon \sim 1 \).

**Conclusions.** – We have investigated a universe with a non-vanishing torsion in its early stages. The corresponding WDW equation was found and shown to be \( \mathcal{PT} \)-symmetric. The ordering factor of \(-2 \) was adopted and we showed the universe undergoes a two-stage inflationary process under the influence of torsion in the absence of conventional matter fields.

Our analysis is applied to a class of theories with effective ECSK action. To study any of these theories one needs to identify the Lagrangian \( \mathcal{L}_T(\nabla S, S^2) \) responsible for the kinetic terms of torsion and possibly a contribution to the potential. In constructing such action, we envisage that one should be guided by conserving the \( \mathcal{PT} \)-symmetry to avoid problems associated with non-unitarity that are commonplace in quantum cosmology.

In recent studies, possible Dirac and Maxwell fields allowing forms of torsion compatible with the cosmological constant have been discussed [27]. Also we have in a previous work considered the triad field in classical regime and it was shown to derive an exponential inflation [28]. It is an interesting avenue of research to explore the evolution of such models in the semiclassical regime and its influence on structure formation.

**Data availability statement:** All data that support the finding of this study are included within the article (and any supplementary files).

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