Does The Force From an Extra Dimension Contradict Physics in 4D?

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Abstract

We examine the question of whether violation of 4D physics is an inevitable consequence of existence of an extra non-compactified dimension. Recent investigations in membrane and Kaluza-Klein theory indicate that when the metric of the spacetime is allowed to depend on the extra coordinate, i.e., the cilindricity condition is dropped, the equation describing the trajectory of a particle in one lower dimension has an extra force with some abnormal properties. Among them, a force term parallel to the four-velocity of the particle and, what is perhaps more surprising, \( u_\mu f^\mu \neq u^\mu f_\mu \). These properties violate basic concepts in 4D physics. In this paper we argue that these abnormal properties are not consequence of the extra dimension, but result from the formalism used. We propose a new definition for the force, from the extra dimension, which is free of any contradictions and consistent with usual 4D physics. We show, using warp metrics, that this new definition is also more consistent with our physical intuition. The effects of this force could be detected observing objects moving with high speed, near black holes and/or in cosmological situations.

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1 INTRODUCTION

In “classical” versions of Kaluza-Klein theory the so-called cylinder condition is one of the basic assumptions. This condition basically states that metric coefficients do not depend on the fifth coordinate, in such a way that all derivatives with respect to this coordinate vanish.

In the last years there is a consensus in the physics community that cylinder condition is not required nor, in general, sustained. Indeed, it is now a common assumption that the metric tensor as well as other physical quantities depend on the fifth coordinate [1]-[5]. In multidimensional theories called “brane-world” models as well as in the space-time-matter theory in 5D, the implications of such an assumption are presently under intensive theoretical study.

In particular, the effects of extra dimensions on the trajectory of tests particles, as observed in 4D (one lower dimension) have been studied [4], [5]. Employing techniques similar to the ones used in classical Kaluza-Klein theory, a number of results have been obtained. For example, the dependence of the metric on the extra coordinate leads to, a new force term which presents two important properties, viz., (i) it is proportional to the first derivative of the metric with respect to the extra coordinate and (ii) it has a component which is parallel to the four-velocity of the particle.

The first property implies that the extra force cannot be implemented directly in brane-world models, in the RS2 scenario [7]. This is because these derivatives are discontinuous, and change sign, through the brane due to the δ-function singularity there. However, as it is discussed in [8], effective 4D equations can be obtained by taking mean values and applying Israel’s junction conditions through the brane. The resulting effective extra force depends on whether the brane universe is invariant or not under the $Z_2$ transformation. In a more realistic theory where the brane is not assumed to be infinitely thin, but has a finite width determined by the specifics of the theory, the extra force should be continuous, changing its sign, as one moves through the brane. Thus, for such “thick” branes there should be a region near the core of the brane where the force vanishes identically.

In this paper we deal with the second property mentioned above. Namely, that the extra force has a component parallel to the four-velocity of the particle. This is a violation of the laws of physics in 4D, where the 4-velocity $u^\mu$ and the 4-force are always orthogonal. Even more astonishing is the fact that $u_\mu f^\mu \neq u^\mu f_\mu$, which makes even harder the physical interpretation of this force. Due to these unusual properties, which cannot be explained by conventional 4D physics, such extra force has been called fifth force [1].

Does the force, from an extra dimension, necessarily violate physics in 4D? The current answer in the literature is positive [4], [5], [8]. This is an important question, from a theoretical and observational/experimental point of view. Therefore, it should
be thoroughly investigated, from different angles and perspectives.

The aim of this paper is to provide a less radical answer to this question. Namely, that the force from an extra dimension does not necessarily contradict 4D physics. Our interpretation is that the abnormal properties of the fifth force are consequence of the formalism used.

First, we will see that when the metric is allowed to depend on the extra coordinate, the formalism and definitions used in classical Kaluza-Klein theory are incompatible with the requirement of gauge invariance.

Second, we will show how to introduce a new definition for the 4D force, from an extra dimension, which is free of any contradictions and consistent with usual 4D physics.

2 Line Element in Kaluza-Klein Theory

To facilitate the discussion and set the notation, we start with a brief summary of the Kaluza-Klein equations. We consider a five-dimensional manifold with coordinates $\xi^A (A = 0, 1, 2, 3, 4)$ and metric tensor $\gamma_{AB}$. The 5D interval is then given by

$$dS^2 = \gamma_{AB} d\xi^A d\xi^B.$$  \hfill (1)

It is a popular choice to consider that the first four coordinates $\xi^\mu$ are the coordinates of the spacetime $x^\mu (\mu = 0, 1, 2, 3)$, while $\xi^4$ is the extra dimension, which we will denote $y$, viz,

$$x^\mu = \xi^\mu$$
$$y = \xi^4.$$  \hfill (2)

Now setting $\gamma_{44} = \gamma_{44} A_\mu$ and $\gamma_{44} = \epsilon \Phi^2$, the general line element (1), without any loss of generality, can be written as

$$dS^2 = ds^2 + \epsilon \Phi^2 (dy + A_\mu dx^\mu)^2,$$ \hfill (3)

where $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is the spacetime interval with metric $g_{\mu\nu} = (\gamma_{\mu\nu} - \epsilon \Phi^2 A_\mu A_\nu)$. The quantities $\Phi$ and $A_\mu$ are called the scalar and vector potentials, respectively. The factor $\epsilon$ is taken to be +1 or −1 depending on whether the extra dimension is timelike or spacelike, respectively. The above separation is invariant under the set of transformations

$$x^\mu = \bar{x}^\mu,$$
$$y = \bar{y} + f(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3),$$ \hfill (4)
which in $5D$ reflect the freedom in the choice of origin for $y$, while in $4D$ correspond to the usual gauge freedom of the potentials

$$\tilde{A}_\mu = A_\mu + \frac{\partial f}{\partial \bar{x}_\mu} = A_\mu + f_\mu.$$  

(5)

The basic postulate, regarding the question discussed here, is that the equations of motion for test particles are obtained by minimizing interval (1), or (3) in more familiar notation. This postulate, which means that the motion of test particles is geodesic, as well as equations (1)-(5), are accepted in both, compactified and non-compactified Kaluza-Klein theories.

### 3 Test Particles in Kaluza-Klein Theory

In this section we critically review the notions that lead to a fifth force. We compare the formalism in the compactified and non-compactified versions of the theory. We show that when the definition of force, used in the compactified version, is extended to the non-compactified version we obtain a force which is not gauge invariant. We then discuss the properties of the fifth force.

#### 3.1 Compactified extra Dimension

This is the classical Kaluza-Klein theory where physical quantities are allowed to depend on $x^\mu$ but not on $y$ (cylinder condition). The geodesic equation splits up in two sets of equations. The first one, corresponds to the motion in spacetime, and provides a definition for the “extra” force (per unit mass), namely,

$$\frac{D u^\mu}{ds} = \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = f^\mu,$$  

(6)

where

$$f^\mu = (\Phi u^{(4)}) F^{\mu\rho} u_\rho + \frac{\epsilon (u^{(4)})^2}{\Phi} [\Phi^\mu - u^\rho \Phi_{\rho\mu}],$$  

(7)

$$\Gamma^\mu_{\alpha\beta}$$ are the usual Christoffel symbols constructed from $g_{\mu\nu}$, $F_{\mu\nu}$ is the antisymmetric tensor $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ and $u^{(4)}$ is

$$u^{(4)} = \epsilon \Phi \left[ \frac{dy}{ds} + A_\mu u^\mu \right].$$  

(8)

The evolution of this quantity is provided by the remaining component of the geodesic equation. It is

$$\frac{1}{1 + \epsilon (u^{(4)})^2} \frac{du^{(4)}}{ds} = -\frac{\Phi}{\Phi} u^\mu u^{(4)}.$$  

(9)
All these equations are invariant under the set of gauge transformations (4). In particular, the force (6), (7) is gauge invariant and orthogonal to the four-velocity \( u^\mu \), i.e.,
\[
  u_\mu f^\mu = u^\nu f_\nu = 0 \\
  f_\nu = g_{\nu\mu}f^\mu \\
  Dg_{\mu\nu} = 0
\]  

(10)

3.2 Non-compactified Extra Dimension

This is the typical scenario in membrane theory and Kaluza-Klein gravity. Here the spacetime metric and the other quantities are allowed to be functions of \( y \). As before, the 5D geodesic equation separates into a 4D part and an extra part.

Again the 4D part (3) is used to define the extra force. However, now this definition is not gauge invariant. This is a consequence of the non-invariance of Christoffel symbols under transformations (4), viz.,
\[
  \bar{\Gamma}^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} + \frac{1}{2}g^{\lambda\rho}(g_{\rho\alpha,y}f_{,\beta} + g_{\rho\beta,y}f_{,\alpha} - g_{\alpha\beta,y}f_{,\rho}),
\]

(11)

which follows from the fact that \( \bar{g}_{\mu\nu,\lambda} = g_{\mu\nu,\lambda} + g_{\mu\nu,y}f_{,\lambda} \).

Our first conclusion, therefore, is that the definition for the force (3) is inappropriate, for the general 5D metric (3). Indeed, a more detailed analysis indicates that, invariance of 4D physics under transformations in 5D requires changing the usual definition of various quantities, including Christoffel symbols and the electromagnetic tensor \( F_{\mu\nu} \). The appropriate definitions are provided in Ref. [9].

Inspection of (11) shows that the non-invariance of Christoffel symbols is a result of the inclusion of electromagnetic potentials \( A_\mu \). These symbols would be gauge invariant if \( A_\mu \) were zero. This leads to the question of whether the force definition (3) would still work for the simplified metric
\[
  dS^2 = g_{\mu\nu}(x^\rho, y)dx^\mu dx^\nu + \epsilon \Phi^2(x^\rho, y)dy^2,
\]

(12)

In this case we obtain
\[
  \frac{Du^\sigma}{ds} = \epsilon \Phi \left( \frac{dy}{ds} \right)^2 [\Phi^\sigma - u^\sigma \Phi u^\rho] + \left( \frac{1}{2}u^\sigma u^\lambda - g^{\sigma\lambda} \right) u^\rho \frac{\partial g_{\lambda\rho}}{\partial y} \frac{dy}{ds}.
\]

(13)

The first term, representing the force associated with the scalar field \( \Phi \), is identical to the one in (7) and satisfies all the appropriate requirements. Therefore, in what follows we will set \( \Phi = 1 \) and concentrate our attention in the other terms.
The second term in (13) behaves like a 4D vector under transformations \( x^\mu = \bar{x}^\mu(x^\lambda), \ y = \bar{y} \) which leave the separation (12) invariant. This vectorial behavior, apart from (6), is probably the motivation to identify this term with the force (per unit mass) associated with the existence of a non-compactified extra dimension, viz.,

\[
\frac{Du^\mu}{ds} = f_{(lit)}^\mu = \left( \frac{1}{2} u^\mu u^\lambda - g^{\mu\lambda} \right) u^\rho \frac{\partial g_{\rho\lambda}}{\partial y} \frac{dy}{ds}. \quad (14)
\]

Here \( f_{(lit)}^\mu \) stands for: force as defined in the literature. This is the so-called fifth force (per unit inertial mass) typical of membrane theory and Kaluza-Klein theory [2], [6].

### 3.2.1 Properties of The Fifth Force

In order to isolate some of the properties of \( f_{(lit)}^\mu \), we evaluate \( Du_\mu/ds \) in an independent way. Omitting intermediate calculations, we obtain

\[
\frac{Du_\sigma}{ds} = f_{(lit)}^\sigma = \frac{1}{2} u_\sigma u^\lambda u^\rho \frac{\partial g_{\rho\lambda}}{\partial y} \frac{dy}{ds}, \quad (15)
\]

where, for the same reasons as above, we have made the identification with the covariant component of the force.

The unique properties of \( f_{(lit)}^\mu \) are immediately obvious. First, it not only has a component parallel to the 4-velocity of the particle, but also \( u_\mu f_{(lit)}^\mu \neq u^\nu f_{(lit)}^\nu \), namely,

\[
u_\sigma f_{(lit)}^\sigma = -\frac{1}{2} u^\sigma u^\lambda \frac{\partial g_{\rho\lambda}}{\partial y} \frac{dy}{ds}, \quad (16)\]

\[
u^\sigma f_{(lit)}^\sigma = +\frac{1}{2} u^\sigma u^\lambda \frac{\partial g_{\rho\lambda}}{\partial y} \frac{dy}{ds}. \quad (17)\]

Also,

\[
f_{(lit)}^\mu = g_{\mu\sigma} f_{(lit)}^\sigma + u^\rho \frac{\partial g_{\mu\rho}}{\partial y} \frac{dy}{ds}. \quad (18)\]

These expressions indicate that, although \( f_{(lit)}^\mu \) transforms like a four-vector, it is not a “regular” 4-vector. Indeed, the above relations are inconsistent with what we usually understand as a 4-vector. In addition, (16) and (17) seem to contradict each other. Only when there is no dependence on \( y \) we recover total self-consistency as in (10).

The current interpretation is that the abnormal properties of this force, which violate the laws of 4D physics, are consequence of the existence of extra non-compactified dimensions [2], [6]. It is therefore suggested that the Kaluza-Klein scenario can be tested by detecting inconsistencies with 4D physics.
4 New Approach. No Contradictions With 4D Physics

In this Section we propose an alternative, less radical, point of view. Our proposal consists of two parts.

The first part, is that when the condition of cilindricity is dropped, (6) does not constitute a consistent definition of force neither for the general metric (3) nor for the simplified one (12). Only in the classical Kaluza-Klein theory, with cilindricity, (6) provides a consistent definition of force per unit mass. Therefore, the abnormal properties of the extra force discussed above are not a consequence of the extra dimension, but a result of an incorrect definition of force in 4D.

The second part is a constructive one. We show how to introduce a new definition for the 4D force, which is mathematically correct, and leads to an extra force, from the extra dimension, which is free of any contradictions and consistent with usual 4D physics.

It is not difficult to see that the source of inconsistencies (from 4D viewpoint) in (16)-(18) is that now $Dg_{\mu\nu} \neq 0$, instead of $Dg_{\mu\nu} = 0$ as in (10).

$$Dg_{\mu\nu} = \left[g_{\mu\nu,\rho} - \left(\Gamma^{\lambda}_{\mu\rho}g_{\lambda\nu} + \Gamma^{\lambda}_{\nu\rho}g_{\lambda\mu}\right)\right] dx^\rho + \frac{\partial g_{\mu\nu}}{\partial y} dy. \quad (19)$$

The first term is the absolute differential in 4D, which we will denote as $D^{(4)}$. For which $D^{(4)}g_{\mu\nu} = 0$. For an arbitrary vector $V_\alpha$

$$DV_\alpha = \left(V_{\alpha,\rho} - \Gamma^{\lambda}_{\alpha\rho}V_\lambda\right) dx^\rho + \frac{\partial V_\alpha}{\partial y} dy = D^{(4)}V_\alpha + \frac{\partial V_\alpha}{\partial y} dy, \quad (20)$$

where $D^{(4)}V_\alpha$ represents the absolute differential of $V_\alpha$ in 4D. Obviously, for any object we can define its four-dimensional absolute derivative as

$$D^{(4)}(\cdots) = D(\cdots) - \frac{\partial (\cdots)}{\partial y} dy. \quad (21)$$

This definition is invariant under the set of transformations that keep unchanged the 4 + 1 separation provided by (12). For the case of more general metrics, $D^{(4)}$ can also be defined, but this requires the introduction of the appropriate projectors [9].

Physical quantities defined in 4D should be appropriately separated from their 5D counterparts. In particular, the 4D force (per unit mass) should be defined through $D^{(4)}u^\mu$ instead of $Du^\mu$, namely,

$$f^\mu = \frac{D^{(4)}u^\mu}{ds}, \quad \bar{f}_\mu = \frac{D^{(4)}u_\mu}{ds}. \quad (22)$$
Since $D^{(4)}g_{\mu\nu} = 0$, we have $f_\sigma = g_{\sigma\mu}f^\mu$, as desired.

Let us now find the contravariant components, $f^\mu$. Following (21), $D^{(4)}u^\mu = Du^\mu - (\partial u^\mu / \partial y)dy$. Thus, we need to evaluate $(\partial u^\mu / \partial y)$.

$$du^\mu = \frac{d(dx^\mu)}{ds} - \frac{dx^\mu}{(ds)^2} d\left(\sqrt{g_{\alpha\beta}dx^\alpha dx^\beta}\right).$$

Taking derivatives and rearranging terms we get

$$\frac{\partial u^\mu}{\partial y} = -\frac{1}{2}u^\mu \frac{\partial g_{\alpha\beta}}{\partial y} u^\alpha u^\beta.$$ \hfill (24)

For the covariant components $f_\mu$ we need $(\partial u_\mu / \partial y)$. This can be obtained from above and $u_\mu = g_{\mu\nu}u^\nu$, as

$$\frac{\partial u_\mu}{\partial y} = \partial g_{\mu\lambda} u^\lambda - \frac{1}{2} u_\mu \frac{\partial g_{\alpha\beta}}{\partial y} u^\alpha u^\beta.$$ \hfill (25)

Collecting results, we finally have

$$\frac{D^{(4)}u^{(\sigma)}}{ds} = f^\sigma = \left[u^\sigma u^\lambda - g^{\sigma\lambda}\right] u^\rho \frac{\partial g_{\rho\lambda}}{\partial y} \frac{dy}{ds}. $$ \hfill (26)

Also,

$$\frac{D^{(4)}u_\mu}{ds} = f_\mu = \left[u_\mu u^\rho - \delta^\rho_\mu\right] u^\lambda \frac{\partial g_{\rho\lambda}}{\partial y} \frac{dy}{ds}. $$ \hfill (27)

It follows that, with this new definition, the contravariant and covariant components of the force satisfy the usual requirements for four-vectors \( \text{(10)}. \) In particular, this force is orthogonal to the four-velocity of the particle.

Equations (26)-(27) show that the force from an extra non-compactified dimension does not necessarily contradict physics in 4D. We propose these equations, instead of the abnormal force \( \text{(14)}-(\text{18)} \), as the correct expressions for the force from a non-compactified extra dimension.

Finally, for completeness, we provide the equation for \( (dy/ds) \). It is given by the fourth component of the geodesic equation as

$$\frac{d^2 y}{ds^2} = \frac{\epsilon}{2} \left[1 + \epsilon\left(\frac{dy}{ds}\right)^2\right] \frac{\partial g_{\mu\nu}}{\partial y} u^\mu u^\nu.$$ \hfill (28)

We notice that $\epsilon$ does not appear explicitly in (28)-(27). However, the character of the extra dimension influences the 4D force via \( (dy/ds) \), namely

$$\epsilon = -1, \quad \frac{dy}{ds} = \text{Tanh}[\frac{1}{2}(w_0 - w)]$$

$$\epsilon = +1, \quad \frac{dy}{ds} = \tan[\frac{1}{2}(w - w_0)],$$ \hfill (29)

where $w = \int (\partial g_{\mu\nu}/\partial y)u^\mu u^\nu ds$, and $w_0$ is a constant of integration.
5 Discussion and Conclusions

The purpose of this work has been to show that the existence of an extra non-compactified dimension does not violate 4D laws of particle mechanics. With this aim, we have formulated a new definition for the force from a non-compactified extra dimension, which is compatible with what we know in 4D physics (Eqs. (26)-(27)).

In order to get another perspective in the discussion, let us consider the so-called warp metrics. These are

\[ dS^2 = \Omega(y)\tilde{g}_{\mu\nu}(x^\rho, y)dx^\mu dx^\nu + \epsilon dy^2, \]  

(30)

where the conformal factor \( \Omega \) is called warp factor, and \( \tilde{g}_{\mu\nu}(x^\rho, y) \) is interpreted as the physical metric on the embedded hypersurface of one lower dimension. These metrics are popular in “brane” theory and space-time-matter theory \[10\]. In the case where \( \tilde{g}_{\mu\nu} \) is not a function of \( y \), the spacetime metric is essentially that of compactified Kaluza-Klein theory and we would not expect any force from the extra dimension. However, a simple calculation from (14) gives

\[ f_{(\text{lit})}^\mu = -\frac{u^\mu}{2\Omega} \frac{d\Omega}{dy} d\tilde{s}, \]  

(31)

where \( ds = \sqrt{\Omega} d\tilde{s} \) and now \( u^\mu = dx^\mu / d\tilde{s} \). On the other hand, the calculation from (26) gives

\[ f^\mu = 0, \]  

(32)

which is more acceptable from a physical point of view. Indeed, we would expect the force from the extra dimension should come from the dependence of the physical metric on \( y \) and not from the conformal factor, as in (31). While this case is very simple, and more complicated metrics can be considered, it clearly illustrates our point. Namely that (26) is more consistent than (14) not only with the usual physics in 4D, but also with our physical intuition.

Predicting some effects of this new force will require some specific model. For astrophysical and cosmological observations/experiments, we can consider a line element with spherical symmetry

\[ dS^2 = e^\nu(t,r,y)(dt)^2 - e^\lambda(t,r,y)[(dr)^2 + r^2 (d\Omega)^2] + \epsilon \Phi^2(t,r,y)(dy)^2, \]  

(33)

where \( (d\Omega)^2 = (d\theta)^2 + \sin^2 \theta (d\phi)^2 \), and the metric coefficients are some solution of the field equations. It is not difficult to see that the spatial part of \( f^\mu \), in (26), is collinear with the three-velocity of the particle. In short \( f = \alpha v \), where \( \alpha \) depends on \((\partial v / \partial y)\) and \((\partial \lambda / \partial y)\). Therefore the particle will move under the influence of two forces; the gravitational one (which roughly is proportional to \((\partial v / \partial r)\) and does not depend on the velocity) and the extra force which does depend on the velocity.
One can imagine a scenario, of particles at high speed, where the extra force could be comparable and even prevail over the gravitational one. The effects from this force could in principle be detected in ultra-relativistic particles in the vicinity of black holes and/or cosmological situations as the peculiar motions of galaxies [11]-[12].

The implications of this force for astrophysics and cosmology is a topic worth of future investigation. This should give one the opportunity to test different models experimentally for their compatibility with observational data.

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