ON THE SQUARED EIGENFUNCTION SYMMETRY OF THE TODA LATTICE HIERARCHY

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Abstract. The squared eigenfunction symmetry for the Toda lattice hierarchy is explicitly constructed in the form of the Kronecker product of the vector eigenfunction and the vector adjoint eigenfunction, which can be viewed as the generating function for the additional symmetries when the eigenfunction and the adjoint eigenfunction are the wave function and the adjoint wave function respectively. Then after the Fay-like identities and some important relations about the wave functions are investigated, the action of the squared eigenfunction related to the additional symmetry on the tau function is derived, which is equivalent to the Adler-Shiota-van Moerbeke (ASvM) formulas.

Keywords: squared eigenfunction symmetry, the Toda lattice hierarchy, additional symmetry, Fay-like identities, ASvM formula.

1. Introduction

The squared eigenfunction symmetry [1–4], also called “ghost” symmetry [5], is a kind of symmetry generated by eigenfunctions and adjoint eigenfunctions in the integrable system. One important application lies in that by identifying the squared eigenfunction symmetry with the usual flow of the integrable hierarchy, one can get the corresponding symmetry constraint [2, 4, 6–11]. The other is its connection with the additional symmetry [5, 12, 13], which is the symmetry depending explicitly on the space and time variables [14–22]. By now, much work has done in the field of the squared eigenfunction symmetry. For examples, 1) the squared eigenfunction symmetry for the KP hierarchy is studied in its connection with the corresponding additional symmetry in [5]; 2) by using the squared eigenfunction symmetry to construct the new flow, the extended integrable systems [23, 24] are developed, which contain the integrable equations with self-consistent sources; 3) the squared eigenfunction symmetries for the BKP hierarchy and the discrete KP hierarchy are systematically developed in [12] and [13] respectively.

The Toda lattice equation [25], as an important integrable system, describes the motion of one-dimensional particles with exponential interaction of neighbors, which plays significant role in physics. The Toda lattice hierarchy was first introduced by Ueno and Takasaki [26] to generalize the Toda lattice equations along the theory about the KP hierarchy [27], which is investigated in many aspects:

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such as interface growth \cite{28}, connection with Laplace-Darboux transformation for general second order partial differential equations \cite{29}, connection with infinite dimensional Lie algebras \cite{26,30}, and its two extensions: extended Toda hierarchy \cite{31} and extended bigraded Toda hierarchy \cite{32–34}. However, the squared eigenfunction symmetry of the Toda lattice hierarchy is not given explicitly in literature.

In this paper, a handy form of the squared eigenfunction symmetry of the Toda lattice hierarchy is given in the form of the Kronecker product of the vector eigenfunctions and the vector adjoint eigenfunctions. Then the relation with the additional symmetry of the Toda lattice hierarchy is investigated: the particular squared eigenfunction symmetry generated by the wave function and the adjoint wave function can be viewed as the generating functions of the additional symmetries for the Toda lattice hierarchy \cite{17,19,35}. Next, in order to show the action of the particular squared eigenfunction symmetry on the tau function, which is actually the so-called Adler-Shiota-van Moerbeke (ASvM) formulas \cite{17,18,36}, the Fay-like identities and some important relations about the wave functions are studied. The Fay-like identities for the Toda lattice hierarchy shows the algebraic properties for the tau functions \cite{37–39}. With the help of the Fay-like identities, some relations about the wave functions are derived in this paper. At last, upon the preparation above, the action of the squared eigenfunction symmetry on the tau function is obtained. By considering the connection between the squared eigenfunction symmetry and the additional symmetry, another proof of the ASvM formula is in fact showed.

This paper is organized in the following way. In Section 2, some basic knowledge about the Toda lattice hierarchy is reviewed. Then, the squared eigenfunction symmetry of the Toda lattice hierarchy is constructed in Section 3. Next, in Section 4, the Fay-like identities and some relations about the wave functions are given. The action of the squared eigenfunction symmetry on the tau function is showed in Section 5. At last, in section 6, some conclusions and discussions are given.

### 2. the Toda lattice hierarchy

The Toda lattice hierarchy \cite{17,26} is defined in the Lax forms as

\[
\partial_{x_n} L = [(L^n_1, 0)_+, L] \quad \text{and} \quad \partial_{y_n} L = [(0, L^n_2)_+, L], \quad n = 1, 2, \cdots
\]

with \( L \) be a pair of infinite matrices given by

\[
L = (L_1, L_2) = \left( \sum_{-\infty<i\leq1} \text{diag}[a^{(1)}_i(s)] \Lambda^i, \sum_{-1\leq i<\infty} \text{diag}[a^{(2)}_i(s)] \Lambda^i \right) \in \mathcal{D},
\]

where \( \Lambda = (\delta_{j-i,1})_{i,j \in \mathbb{Z}} \), and \( a^{(1)}_i(s) \) and \( a^{(2)}_i(s) \) are the functions of \( x = (x_1, x_2, \cdots) \) and \( y = (y_1, y_2, \cdots) \), such that

\[
a^{(1)}_1(s) = 1 \quad \text{and} \quad a^{(2)}_1(s) \neq 0 \quad \forall s,
\]
and the algebra
\[ \mathcal{D} = \{(P_1, P_2) \in \text{gl}(\mathbb{R}) \times \text{gl}(\mathbb{R}) \mid (P_1)_{ij} = 0 \text{ for } j - i \gg 0, \ (P_2)_{ij} = 0 \text{ for } i - j \gg 0\} \]
has the following splitting:
\[ \mathcal{D} = \mathcal{D}_+ + \mathcal{D}_-, \]
\[ \mathcal{D}_+ = \{(P, P) \in \mathcal{D} \mid (P)_{ij} = 0 \text{ for } |i - j| \gg 0\} = \{(P_1, P_2) \in \mathcal{D} \mid P_1 = P_2\}, \]
\[ \mathcal{D}_- = \{(P_1, P_2) \in \mathcal{D} \mid (P_1)_{ij} = 0 \text{ for } j \geq i, \ (P_2)_{ij} = 0 \text{ for } i > j\}, \]
thus \((P_1, P_2) = (P_1, P_2)_+ + (P_1, P_2)_-\) can be given by
\[ (P_1, P_2)_+ = (P_{1u} + P_{2l}, P_{1u} + P_{2l}), \]
\[ (P_1, P_2)_- = (P_{1l} - P_{2l}, P_{2u} - P_{1u}), \]
where for a matrix \(P\), \(P_u\) and \(P_l\) denote the upper (including diagonal) and strictly lower triangular parts of \(P\), respectively.

The Lax operator of the Toda lattice hierarchy (11) can be expressed in terms of wave matrices \(W = (W_1, W_2)\):
\[ W_1(x, y) = S_1(x, y) e^{\xi(x, \Lambda)}, \quad W_2(x, y) = S_2(x, y) e^{\xi(y, \Lambda^{-1})} \]
as follows
\[ L = W(\Lambda, \Lambda^{-1}) W^{-1} = S(\Lambda, \Lambda^{-1}) S^{-1}, \]
where \(S = (S_1, S_2)\) has the forms below
\[ S_1(x, y) = \sum_{i \geq 0} \text{diag}[c_i(s; x, y)] \Lambda^{-i}, \quad S_2(x, y) = \sum_{i \geq 0} \text{diag}[c'_i(s; x, y)] \Lambda^i, \]
with \(c_0(s; x, y) = 1\) and \(c'_0(s; x, y) \neq 0\) for any \(s\), and \(\xi(x, \Lambda^\pm) = \sum_{n \geq 1} x_n \Lambda^\pm n\). The wave matrices evolve according to
\[ \partial_{x_n} S = -(L_1^n, 0)_- S, \quad \partial_{y_n} S = -(0, L_2^n)_- S, \]
\[ \partial_{x_n} W = (L_1^n, 0)_+ W, \quad \partial_{y_n} W = (0, L_2^n)_+ W. \]
The vector wave functions \(\Psi = (\Psi_1, \Psi_2)\) are defined in the following way
\[ \Psi_i(x, y; z) = (\Psi_i(n; x, y; z))_{n \in \mathbb{Z}} = W_i(x, y) \chi(z), \]
where \(\chi(z) = (z^i)_{i \in \mathbb{Z}}\), while the adjoint wave functions are defined by
\[ \Psi_i^*(x, y; z) = (\Psi_i^*(n; x, y; z))_{n \in \mathbb{Z}} := (W_i(x, y)^{-1})^T \chi^*(z) \]
with \(\chi^*(z) = \chi(z^{-1})\) and \(T\) refers to the matrix transpose. Thus from (14), we know
\[ L \Psi = (z, z^{-1}) \Psi. \]
And further the wave functions satisfy the following equations

\[ \begin{align*}
\partial_{x_1}\Psi &= (L_1^0, 0)_+\Psi, & \partial_{y_1}\Psi &= (0, L_2^0)_+\Psi.
\end{align*} \]  \(11\)

If vector functions \(q = (q(n; x, y))_{n \in \mathbb{Z}}\) and \(r = (r(n; x, y))_{n \in \mathbb{Z}}\) satisfy

\[ \begin{align*}
\partial_{x_1}q &= (L_1^n)_uq, & \partial_{y_1}q &= (L_2^n)_lq, \\
\partial_{x_1}r &= -(L_1^n)^T_uq, & \partial_{y_1}r &= -(L_2^n)^T_l r,
\end{align*} \]  \(12\)

we call them vector eigenfunction and vector adjoint eigenfunction for the Toda lattice hierarchy respectively. Obviously, the wave functions \(\Psi_1\) and \(\Psi_2\) are eigenfunctions, and the adjoint wave functions \(\Psi^*_1\) and \(\Psi^*_2\) are the adjoint eigenfunctions.

From (7), the bilinear relation for the Toda lattice hierarchy can be easily got

\[ W_1(x, y)W_1(x', y')^{-1} = W_2(x, y)W_2(x', y')^{-1} \]  \(13\)

for any \(x, x'\) and \(y, y'\). In order to rewrite (13) in terms of the (adjoint) wave functions, the following lemma [17] is needed.

**Lemma 1.** Given two operators \(U = (U_1, U_2), V = (V_1, V_2) \in \mathcal{D}\) depending on \(x\) and \(y\), one has

\[ \begin{align*}
U_1V_1 &= \text{res}_z \frac{1}{z} (U_1\chi(z)) \otimes (V_1^T\chi^*(z)), \\
U_2V_2 &= \text{res}_z \frac{1}{z} (U_2\chi(z^{-1})) \otimes (V_2^T\chi^*(z^{-1})),
\end{align*} \]  \(14\) and \(15\)

where \(\text{res}_z \sum a_iz^i = a_{-1}\) and \((A \otimes B)_{ij} = A_iB_j\).

Therefore, according to (8) [9], the bilinear relation (13) is equivalent to the following residue formula,

\[ \text{res}_z \frac{1}{z} \Psi_1(x, y; z) \otimes \Psi^*_1(x', y'; z) = \text{res}_z \frac{1}{z} \Psi_2(x, y; z^{-1}) \otimes \Psi^*_2(x', y'; z^{-1}), \]  \(16\)

which further can be showed as

\[ \text{res}_z \frac{1}{z} \Psi_1(s; x, y; z) \Psi^*_1(s'; x', y'; z) = \text{res}_z \frac{1}{z} \Psi_2(s; x, y; z^{-1}) \Psi^*_2(s'; x', y'; z^{-1}). \]  \(17\)

Using the bilinear identity (17) above, Ueno and Takasaki [26] proved that there exists a tau function \(\tau(s, x, y)\) such that

\[ \begin{align*}
\Psi_1(s; x, y; z) &= \frac{\tau(s; x - [z^{-1}], y)}{\tau(s, x, y)} e^{\xi(x, z)} z^s, \\
\Psi_2(s; x, y; z) &= \frac{\tau(s + 1; x, y - [z])}{\tau(s, x, y)} e^{\xi(y, z^{-1})} z^{-s}, \\
\Psi^*_1(s; x, y; z) &= \frac{\tau(s + 1; x + [z^{-1}], y)}{\tau(s + 1, x, y)} e^{-\xi(x, z)} z^{-s}, \\
\Psi^*_2(s; x, y; z) &= \frac{\tau(s; x, y + [z])}{\tau(s + 1; x, y)} e^{-\xi(y, z^{-1})} z^{-s},
\end{align*} \]  \(18\)
where \([z] = (z, \frac{1}{2}z^2, \frac{1}{3}z^3, \cdots)\).

Another important object is the vertex operators \([17]\) for the Toda lattice hierarchy, which are defined in the following way

\[
X(x, \lambda, \mu) := \left(\left(\frac{\nu}{\lambda}\right)_n X(x, \lambda, \mu)\right)_{n \in \mathbb{Z}}, \quad \Bar{X}(y, \lambda, \mu) := \left(\left(\frac{\lambda}{\nu}\right)_n X(y, \lambda, \mu)\right)_{n \in \mathbb{Z}},
\]

where

\[
X(x, \lambda, \mu) = \frac{1}{\lambda} e^{\theta(\lambda)} \colon e^{\theta(\mu)} := \frac{1}{\lambda} e^{\xi(x+[\lambda^{-1}], \mu)-\xi(x, \lambda)} e^{\sum_{l=1}^{\infty} \frac{1}{l} (\lambda^{-l}-\mu^{-l}) \frac{\partial}{\partial x_l}},
\]

\[
\frac{1}{\lambda} e^{\xi(x, \mu)-\xi(x-[\mu^{-1}], \lambda)} e^{\sum_{l=1}^{\infty} \frac{1}{l} (\lambda^{-l}-\mu^{-l}) \frac{\partial}{\partial x_l}} + \delta(\lambda, \mu),
\]

and

\[
\theta(\lambda) = -\sum_{l=1}^{\infty} \lambda^l x_l + \sum_{l=1}^{\infty} \frac{1}{l} \lambda^{-l} \frac{\partial}{\partial x_l},
\]

the columns \(\ldots\) indicate Wick normal ordering with respect to the creation/annihilation “modes” \(x_l\) and \(\frac{\partial}{\partial x_l}\), respectively. And the delta-function is defined as

\[
\delta(\lambda, \mu) = \sum_{n=0}^{\infty} \frac{\mu^n}{\lambda^{n+1}} = \frac{1}{\lambda 1 - \frac{\mu}{\lambda}} + \frac{1}{\mu 1 - \frac{\lambda}{\mu}}.
\]

According to \([18]\), one can find

\[
\frac{X(x, \lambda, \mu)\tau}{\tau} = \left(\frac{1}{\lambda^2} \Psi_1(n; x + [\lambda^{-1}], y; \mu) \Psi_1^*(n-1; x, y; \lambda)\right)_{n \in \mathbb{Z}}
\]

\[
= \left(\frac{-1}{\lambda \mu} \Psi_1(n; x, y; \mu) \Psi_1^*(n-1; x - [\mu^{-1}], y; \lambda) + \delta(\lambda, \mu)\right)_{n \in \mathbb{Z}},
\]

\[
\frac{\Bar{X}(y, \lambda, \mu)\tau}{\tau} = \left(\frac{1}{\mu} \Psi_2(n-1; x, y + [\lambda^{-1}]; \mu^{-1}) \Psi_2^*(n-1; x, y; \lambda^{-1})\right)_{n \in \mathbb{Z}}
\]

\[
= \left(\frac{-1}{\mu} \Psi_2(n; x, y; \mu^{-1}) \Psi_2^*(n; x, y - [\mu^{-1}]; \lambda^{-1}) + \delta(\lambda, \mu)\right)_{n \in \mathbb{Z}}.
\]

The additional symmetry for the Toda lattice hierarchy \([17, 19, 35]\) can be expressed in terms of the Orlov-Shulman operators \([16]\), which is defined as

\[
M \equiv (M_1, M_2) = W(\varepsilon, \varepsilon^*) W^{-1},
\]

where

\[
\varepsilon = \text{diag}[s] \Lambda^{-1}, \quad \varepsilon^* = -\varepsilon^T + \Lambda,
\]

satisfying

\[
M \Psi = (\partial_z, \partial_{z^{-1}}) \Psi, \quad [L, M] = (1, 1),
\]

\[
\partial_{x_n} M = [(L_1^n, 0)_+, M], \quad \partial_{y_n} M = [(0, L_2^n)_+, M].
\]
By introducing additional independent variables \(x_{m,l}^*\) and \(y_{m,l}^*\), the actions of the additional symmetry on the wave matrices are showed as

\[
\partial_{x_{m,l}^*} W = -(M_1^n L_1^M, 0)_- W, \quad \partial_{y_{m,l}^*} W = -(0, M_2^n L_2^M)_- W.
\]  

One can further find

\[
\partial_{x_{m,l}^*} \Psi = -(M_1^n L_1^M, 0)_- \Psi, \quad \partial_{y_{m,l}^*} \Psi = -(0, M_2^n L_2^M)_- \Psi,
\]

\[
\partial_{x_{m,l}^*} L = -(M_1^n L_1^M, 0)_-, \quad \partial_{y_{m,l}^*} L = -(0, M_2^n L_2^M)_- L,
\]

\[
\partial_{x_{m,l}^*} M = -(M_1^n L_1^M, 0)_-, \quad \partial_{y_{m,l}^*} M = -(0, M_2^n L_2^M)_- M,
\]

and by acting on the space of the wave matrices, \(\partial_{x_{m,l}^*}\) and \(\partial_{y_{m,l}^*}\) forms into Lie algebra \(w_\infty \times w_\infty\).

3. The squared eigenfunction symmetry for the Toda lattice hierarchy

Given a couple of vector (adjoint) eigenfunctions \(q\) and \(r\), the squared eigenfunction flow of the Toda lattice hierarchy can be defined by its actions on the wave operators,

\[
\partial_a W_1 = (q \otimes r)_t W_1, \quad \partial_a W_2 = -(q \otimes r)_u W_2,
\]  

or

\[
\partial_a W = (q \otimes r, 0)_- W = -(0, q \otimes r)_- W.
\]

Note that for a matrix \(P\), \(P_u\) and \(P_l\) denote the upper (including diagonal) and strictly lower triangular parts of \(P\) respectively, and \((A \otimes B)_{ij} = A_i B_j\).

According to (4), one can further have the squared eigenfunction flow on the Lax operator

\[
\partial_a L_1 = [(q \otimes r)_t, L_1], \quad \partial_a L_2 = -[(q \otimes r)_u, L_2],
\]

or

\[
\partial_a L = [(q \otimes r, 0)_-, L] = -[(0, q \otimes r)_-, L].
\]

The proposition below shows that this squared eigenfunction flow is indeed a kind of symmetry, and thus is called the squared eigenfunction symmetry

**Proposition 2.**

\[
[\partial_a, \partial_{x_n}] = [\partial_a, \partial_{y_n}] = 0.
\]  

**Proof.** In fact, according to (1), (7), (29) and (31)

\[
[\partial_a, \partial_{x_n}] W_1 = \partial_a \left((L_1^n)_{u} W_1\right) - \partial_{x_n} \left((q \otimes r)_t W_1\right)
\]

\[
= \left[(q \otimes r)_t, L_1^n\right]_u W_1 + (L_1^n)_u (q \otimes r)_t W_1
\]

\[
- (L_1^n)_u q \otimes r)_t W_1 + (q \otimes (L_1^n)^T_r) W_1 - (q \otimes r)_t (L_1^n)_u W_1
\]

\[
= \left[(q \otimes r)_t, L_1^n\right]_u W_1 + [(L_1^n)_u, (q \otimes r)_t] W_1
\]
Firstly, according to Lemma 1, one has

\[ -\left((L^n_1)_u(q \otimes r)\right)_l W_1 + \left((q \otimes r)(L^n_1)_u\right)_l W_1 \]

\[ = [(q \otimes r)_l, (L^n_1)_u] W_1 + [(L^n_1)_u, (q \otimes r)_l] W_1 - [(L^n_1)_u, (q \otimes r)_l] W_1 \]

\[ = [(q \otimes r)_l, (L^n_1)_u] W_1 + [(L^n_1)_u, (q \otimes r)_l] W_1 \]

\[ = [(q \otimes r)_l, (L^n_1)_u] W_1 + [(L^n_1)_u, (q \otimes r)_l] W_1 = 0. \]

Note that \((L^n_1)_u q \otimes r = (L^n_1)_u(q \otimes r)\), and \(q \otimes (L^n_1)_u^T r = (q \otimes r)(L^n_1)_u\) are used in the third identity. While \([A_u, B_u]_l = [A_l, B_l]_u = 0\) is used in the fifth identity.

Similarly, \([\partial_\alpha, \partial_{x_n}] W_2 = [\partial_\alpha, \partial_{y_n}] W_1 = [\partial_\alpha, \partial_{y_n}] W_2 = 0\) can be proved.

Define the following double expansions

\[ Y_1(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} M_1^m L_1^{m+l}, \]

\[ Y_2(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} M_2^m L_2^{m+l}, \]

which can be viewed as the generator of the additional symmetries for the Toda lattice hierarchy. This double expansions can be related with the (adjoint) eigenfunctions in the following way,

**Proposition 3.**

\[ Y_1(\lambda, \mu) = \lambda^{-1} \Psi_1(x, y; \mu) \otimes \Psi_1^*(x, y; \lambda), \quad (34) \]

\[ Y_2(\lambda, \mu) = \lambda^{-1} \Psi_2(x, y; \mu^{-1}) \otimes \Psi_2^*(x, y; \lambda^{-1}). \quad (35) \]

**Proof.** Firstly, according to Lemma 1, one has

\[ M_1^m L_1^{m+l} = M_1^m W_1 L_1^m W_1^{-1} \]

\[ = \text{res}_z z^{-1} \left(M_1^m W_1 L_1^m \chi(z)\right) \otimes \left((W_1^{-1})^T \chi^*(z)\right) \]

\[ = \text{res}_z \left(z^{-1+m+l} \partial_z^m \Psi_1(x, y; z)\right) \otimes \Psi_1^*(x, y; z), \]

then,

\[ Y_1(\lambda, \mu) = \text{res}_z \sum_{m=0}^{\infty} \sum_{l=-\infty}^{\infty} \frac{z^{m+l}}{\lambda^{m+l+1}} \frac{(\mu - \lambda)^m}{m!} \left(z^{-1} \partial_z^m \Psi_1(x, y; z)\right) \otimes \Psi_1^*(x, y; z) \]

\[ = \text{res}_z \delta(\lambda, z) z^{-1} \partial_z \Psi_1(x, y; z) \otimes \Psi_1^*(x, y; z) \]

\[ = \lambda^{-1} \partial_z \Psi_1(x, y; \lambda) \otimes \Psi_1^*(x, y; \lambda) \]

\[ = \lambda^{-1} \Psi_1(x, y; \mu) \otimes \Psi_1^*(x, y; \lambda), \]

where \(\text{res}_z(\delta(\lambda, z) f(z)) = f(\lambda)\) is used.

Thus (34) are proved.
Similarly for (35),
\[
M_2^m L_2^{-m+l} = M_2^m W_2 \Delta^{-m-l} W_2^{-1}
\]
\[
= \text{res}_z z^{-1} (M_2^m W_2 \Delta^{-m-l} \chi(z^{-1})) \otimes ((W_2^{-1})^T \chi^*(z^{-1}))
\]
\[
= \text{res}_z (z^{-1+m+l} \partial_z^m \Psi_2(x, y; z^{-1})) \otimes \Psi_2^*(x, y; z^{-1}),
\]
then
\[
Y_2(\lambda, \mu) = \text{res}_z \sum_{m=0}^{\infty} \sum_{l=-\infty}^{\infty} \frac{z^{m+l} (\mu - \lambda)^m}{\lambda^{l+m+1}} (z^{-1} \partial_z^m \Psi_2(x, y; z^{-1})) \otimes \Psi_2^*(x, y; z^{-1})
\]
\[
= \text{res}_z \delta(\lambda, z) z^{-1} e^{(\mu-\lambda)\partial_z} \Psi_2(x, y; z^{-1}) \otimes \Psi_2^*(x, y; z^{-1})
\]
\[
= \lambda^{-1} e^{(\mu-\lambda)\partial_z} \Psi_2(x, y; \lambda^{-1}) \otimes \Psi_2^*(x, y; \lambda^{-1})
\]
\[
= \lambda^{-1} \Psi_2(x, y; \mu^{-1}) \otimes \Psi_2^*(x, y; \lambda^{-1}).
\]
\[\square\]

If define \(\partial_{\alpha_1}\) and \(\partial_{\alpha_2}\) flows,
\[
\partial_{\alpha_1} W_1 = (\lambda^{-1} \Psi_1(x, y; \mu) \otimes \Psi_1^*(x, y; \lambda)) W_1, \quad (36)
\]
\[
\partial_{\alpha_1} W_2 = (\lambda^{-1} \Psi_1(x, y; \mu) \otimes \Psi_1^*(x, y; \lambda)) W_2, \quad (37)
\]
and
\[
\partial_{\alpha_2} W_1 = (\lambda^{-1} \Psi_2(x, y; \mu^{-1}) \otimes \Psi_2^*(x, y; \lambda^{-1})) W_1, \quad (38)
\]
\[
\partial_{\alpha_2} W_2 = (\lambda^{-1} \Psi_2(x, y; \mu^{-1}) \otimes \Psi_2^*(x, y; \lambda^{-1})) W_2, \quad (39)
\]

one can find that \(\partial_{\alpha_1}\) and \(\partial_{\alpha_2}\) are the squared eigenfunction symmetries generated by the pair of \(\Psi_1(x, y; \mu) \& \lambda^{-1} \Psi_1^*(x, y; \lambda)\) and the pair of \(\Psi_2(x, y; \mu^{-1}) \& \lambda^{-1} \Psi_2^*(x, y; \lambda^{-1})\) respectively.

Further from (34) and (35), it is can be known that the squared eigenfunction symmetries \(\partial_{\alpha_1}\) and \(\partial_{\alpha_2}\) are the generators of the additional symmetries for the Toda lattice hierarchy, that is,
\[
\partial_{\alpha_1} = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{k=-\infty}^{\infty} \lambda^{-k-m-1} \partial_{x_{m,m+k}}^*,
\]
\[
\partial_{\alpha_2} = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{k=-\infty}^{\infty} \lambda^{-k-m-1} \partial_{y_{m,m+k}}^*.
\]

In the rest of this paper, we will mainly study this two particular squared eigenfunction symmetries \(\partial_{\alpha_1}\) and \(\partial_{\alpha_2}\), and show their actions on the tau function of the Toda lattice hierarchy.
4. Fay-like Identities and Some Important Relations about the Wave Functions

In this section, the Fay-like identities for the Toda lattice hierarchy are reviewed and some important relations about the wave functions are derived, which is helpful for the research on the squared eigenfunction symmetries.

The general Fay-like identities are investigated in [37]. In this paper, only some particular cases are needed. The starting point of Fay-like identities is the bilinear identity of the Toda lattice hierarchy in terms of the tau function, which can be derived by substituting (18) into the bilinear identity (17)

\[
\begin{align*}
\text{res}_z \left( \frac{1}{(1-zs_1)(1-zs_2)} \right) &= \left( \frac{1}{(1-zs_2)} - \frac{1}{(1-zs_1)} \right) \frac{1}{z(s_2-s_1)}, \\
\text{res}_z \left( \sum_{n=-\infty}^{\infty} a_n(\z) z^{-n} \right) \frac{1}{1-z/\zeta} &= \zeta \left( \sum_{n=1}^{\infty} a_n(\z) z^{-n} \right) \bigg|_{z=\zeta}.
\end{align*}
\]

Lemma 4.

\[
\begin{align*}
(1) \quad & s_1 \tau(n; x - [s_1], y) \tau(n + 1; x - [s_2], y) - s_2 \tau(n; x - [s_2], y) \tau(n + 1; x - [s_1], y) \\
& \quad = (s_1 - s_2) \tau(n + 1; x, y) \tau(n; x - [s_1] - [s_2], y), \\
(2) \quad & s_1 \tau(n + 1; x, y - [s_1]) \tau(n; x, y - [s_2]) - s_2 \tau(n + 1; x, y - [s_2]) \tau(n; x, y - [s_1]) \\
& \quad = (s_1 - s_2) \tau(n; x, y) \tau(n + 1; x, y - [s_1] - [s_2]), \\
(3) \quad & \tau(n; x - [s_1], y) \tau(n; x, y - [s_2]) - \tau(n; x, y) \tau(n; x - [s_1], y - [s_2]) \\
& \quad = s_1 s_2 \tau(n - 1; x - [s_1], y) \tau(n + 1; x, y - [s_2]).
\end{align*}
\]

Another group of particular Fay-like identities can also be obtained starting from (40). In fact, by considering the following four cases of (40):

Case I: \( s' = s = n - 1, x' = x - [s_1] - [s_2] - [s_3], y' = y, \)
Case II: \( s' = s = n, x' = x - [s_1] - [s_2], y' = y - [s_3], \)
Case III: \( s' = s + 1 = n + 1, x' = x - [s_3], y' = y - [s_1] - [s_2], \)
Case IV: \( s' = s + 2 = n + 1, x' = x, y' = y - [s_1] - [s_2] - [s_3], \)

and using (41) and (42), one can obtain the following lemma.

Lemma 5.

\[
\begin{align*}
(1) \quad & s_1(s_2 - s_3) \tau(n; x - [s_1], y) \tau(n; x - [s_2] - [s_3], y) \\
& \quad + s_2(s_3 - s_1) \tau(n; x - [s_2], y) \tau(n; x - [s_3] - [s_1], y)
\end{align*}
\]
\[ + s_3(s_1 - s_2)\tau(n; x - [s_3], y)\tau(n; x - [s_1] - [s_2], y) = 0, \quad (46) \]

\[ (2) \quad s_1\tau(n; x - [s_1], y)\tau(n + 1; x - [s_2], y - [s_3]) \]

\[ - s_2\tau(n; x - [s_2], y)\tau(n + 1; x - [s_1], y - [s_3]) \]

\[ = (s_1 - s_2)\tau(n; x - [s_1] - [s_2], y)\tau(n + 1; x, y - [s_3]), \quad (47) \]

\[ (3) \quad (s_1 - s_2)s_3\tau(n + 1; x, y - [s_1] - [s_2])\tau(n - 1; x - [s_3], y) \]

\[ = \tau(n; x, y - [s_1])\tau(n; x - [s_3], y - [s_2]) - \tau(n; x, y - [s_2])\tau(n; x - [s_3], y - [s_1]), \quad (48) \]

\[ (4) \quad (s_2 - s_3)\tau(n; x, y - [s_1])\tau(n + 1; x, y - [s_2] - [s_3]) \]

\[ + (s_3 - s_1)\tau(n; x, y - [s_2])\tau(n + 1; x, y - [s_3] - [s_1]) \]

\[ + (s_1 - s_2)\tau(n; x, y - [s_3])\tau(n + 1; x, y - [s_1] - [s_2]) = 0. \quad (49) \]

Next with the help of the two groups of particular Fay-like identities, one can get the following relations about the wave functions for the Toda lattice hierarchy.

**Lemma 6.**

\[ \Psi_1^*(n; x, y; \lambda)\Psi_1(n; x, y; z) = z^{-1}(\Psi_1(n + 1; x, y; z)\Psi_1^*(n; x - [z^{-1}], y; \lambda) \]

\[ - \Psi_1(n; x, y; z)\Psi_1^*(n - 1; x - [z^{-1}], y; \lambda)), \quad (50) \]

\[ \Psi_1^*(n; x, y; \lambda)\Psi_2(n; x, y; z) = \Psi_2(n; x, y; z)\Psi_1^*(n; x, y - [z]; \lambda) \]

\[ - \Psi_2(n + 1; x, y; z)\Psi_1^*(n + 1; x, y - [z]; \lambda), \quad (51) \]

\[ \Psi_2^*(n; x, y; \lambda^{-1})\Psi_1(n; x, y; z) = z^{-1}(\Psi_1(n + 1; x, y; z)\Psi_2^*(n; x - [z^{-1}], y; \lambda^{-1}) \]

\[ - \Psi_1(n; x, y; z)\Psi_2^*(n - 1; x - [z^{-1}], y; \lambda^{-1})), \quad (52) \]

\[ \Psi_2^*(n; x, y; \lambda^{-1})\Psi_2(n; x, y; z) = \Psi_2(n; x, y; z)\Psi_2^*(n; x, y - [z]; \lambda^{-1}) \]

\[ - \Psi_2(n + 1; x, y; z)\Psi_2^*(n + 1; x, y - [z]; \lambda^{-1}). \quad (53) \]

**Proof.** Firstly, according to (48) and (53), one can get

\[ \Psi_1^*(n; x, y; \lambda)\Psi_1(n; x, y; z) \]

\[ = e^{\xi(x, z) - \xi(x, \lambda)} \left( \frac{z}{\lambda} \right)^n \frac{\tau(n + 1; x + [\lambda^{-1}], y)\tau(n; x - [z^{-1}], y)}{\tau(n + 1; x, y)\tau(n; x, y)} \]

\[ = e^{\xi(x, z) - \xi(x, \lambda)} \left( \frac{z}{\lambda} \right)^n \left( \frac{z^{-1} - \lambda^{-1}}{z^{-1} + \lambda^{-1}} \right) \frac{\tau(n; x + [\lambda^{-1}] - [z^{-1}], y)}{\tau(n; x, y)} \]

\[ = z^{-1}(\Psi_1(n + 1; x, y; z)\Psi_1^*(n; x - [z^{-1}], y; \lambda) - \Psi_1(n; x, y; z)\Psi_1^*(n - 1; x - [z^{-1}], y; \lambda)), \]

and thus (50) is obtained.
Then, similarly (18) and (45) leads to (51) and (52), that is,

\[
\Psi_1^*(n; x, y; \lambda)\Psi_2(n; x, y; z) = \left(\frac{z}{\lambda}\right)^n e^{-\xi(x,\lambda)+\xi(y,\lambda^{-1})} \frac{\tau(n + 1; x + \lfloor \lambda^{-1} \rfloor, y)\tau(n + 1; x, y - \lfloor z \rfloor)}{\tau(n + 1; x, y)}
\]

\[
= \left(\frac{z}{\lambda}\right)^n e^{-\xi(x,\lambda)+\xi(y,\lambda^{-1})} \frac{\tau(n + 1; x + \lfloor \lambda^{-1} \rfloor, y - \lfloor z \rfloor)}{\tau(n; x, y)} - z \frac{\tau(n + 1; x + \lfloor \lambda^{-1} \rfloor, y - \lfloor z \rfloor)}{\tau(n + 1; x, y)}
\]

\[
= \Psi_2(n; x, y; z)\Psi_1^*(n; x, y - \lfloor z \rfloor; \lambda) - \Psi_2(n + 1; x, y; z)\Psi_1^*(n + 1; x, y - \lfloor z \rfloor; \lambda),
\]

and

\[
\Psi_2^*(n; x, y; \lambda^{-1})\Psi_1(n; x, y; z) = e^{\xi(x,\lambda^{-1})-\xi(y,\lambda)}(\lambda z)^n \frac{\tau(n; x, y + \lfloor \lambda^{-1} \rfloor)\tau(n + 1; x, y - \lfloor z \rfloor)}{\tau(n + 1; x, y)}
\]

\[
= e^{\xi(x,\lambda^{-1})-\xi(y,\lambda)}(\lambda z)^n \frac{\tau(n; x - \lfloor z \rfloor, y + \lfloor \lambda^{-1} \rfloor)}{\tau(n + 1; x, y)} - (\lambda z)^{-1} e^{\xi(x,\lambda^{-1})-\xi(y,\lambda)}(\lambda z)^n \frac{\tau(n; x - \lfloor z \rfloor, y + \lfloor \lambda^{-1} \rfloor)}{\tau(n + 1; x, y)}
\]

\[
= z^{-1}(\Psi_1(n + 1; x, y; z)\Psi_1^*(n; x - \lfloor z \rfloor; \lambda) - \Psi_2(n + 1; x, y; z)\Psi_2^*(n - 1; x - \lfloor z \rfloor; y; \lambda)).
\]

At last, by using (15) and (44), one can obtain (53).

\[
\Psi_2^*(n; x, y; \lambda^{-1})\Psi_2(n; x, y; z) = e^{\xi(x,\lambda^{-1})-\xi(y,\lambda)}(\lambda z)^n \frac{\tau(n; x, y + \lfloor \lambda^{-1} \rfloor)\tau(n + 1; x, y - \lfloor z \rfloor)}{\tau(n + 1; x, y)}
\]

\[
= \lambda^{-1}(\lambda^{-1} - z)^{-1} e^{\xi(x,\lambda^{-1})-\xi(y,\lambda)}(\lambda z)^n \frac{\tau(n; x, y + \lfloor \lambda^{-1} \rfloor - \lfloor z \rfloor)}{\tau(n + 1; x, y)} - (\lambda z)^{-1} e^{\xi(x,\lambda^{-1})-\xi(y,\lambda)}(\lambda z)^n \frac{\tau(n + 1; x, y + \lfloor \lambda^{-1} \rfloor - \lfloor z \rfloor)}{\tau(n + 1; x, y)}
\]

\[
= \Psi_2(n; x, y; z)\Psi_2^*(n; x, y - \lfloor z \rfloor; \lambda^{-1}) - \Psi_2(n + 1; x, y; z)\Psi_2^*(n + 1; x, y - \lfloor z \rfloor; \lambda^{-1}).
\]

\[
\square
\]

From (50)-(53), one can further get

\[
\sum_{k<0} \Psi_1^*(n + k; x, y; \lambda)\Psi_1(n + k; x, y; z) = z^{-1}\Psi_1^*(n - 1; x - \lfloor z \rfloor, y; \lambda)\Psi_1(n; x, y; z),
\]

\[
\sum_{k\geq 0} \Psi_1^*(n + k; x, y; \lambda)\Psi_2(n + k; x, y; z) = \Psi_2(n; x, y; z)\Psi_1^*(n; x, y - \lfloor z \rfloor; \lambda),
\]

\[
\sum_{k<0} \Psi_2^*(n + k; x, y; \lambda^{-1})\Psi_1(n + k; x, y; z) = z^{-1}\Psi_1(n; x, y; z)\Psi_2^*(n - 1; x - \lfloor z \rfloor, y; \lambda^{-1}),
\]

\[
\sum_{k\geq 0} \Psi_2^*(n + k; x, y; \lambda^{-1})\Psi_2(n + k; x, y; z) = \Psi_2(n; x, y; z)\Psi_2^*(n; x, y - \lfloor z \rfloor; \lambda^{-1}).
\]

If introduce the following two notations,

\[
G_1(\xi)f(x, y; z) = f(x - \lfloor \xi^{-1} \rfloor, y; z), \quad G_2(\xi)f(x, y; z) = f(x, y - \lfloor \xi \rfloor; z),
\]

one can further have another group of relations about the wave functions, which is the lemma below.
Lemma 7.

\[
(G_1(z) - 1)\Psi_1(n; x, y; \mu)\Psi^*_1(n - 1; x - [\mu^{-1}], y; \lambda) \\
= -\frac{\mu}{z}\Psi_1(n; x, y; \mu)\Psi^*_1(n - 1; x - [\mu^{-1}], y; \lambda),
\]

(59)

\[
(\Delta G_2(z) - 1)\Psi_1(n; x, y; \mu)\Psi^*_1(n - 1; x - [\mu^{-1}], y; \lambda) \\
= \mu\Psi_1(n; x, y; \mu)\Psi^*_1(n; x, y - [z]; \lambda),
\]

(60)

\[
(G_1(z) - 1)\Psi_2(n; x, y; \mu^{-1})\Psi^*_2(n; x, y - [\mu^{-1}]; \lambda^{-1}) \\
= z^{-1}\Psi_2(n; x, y; \mu^{-1})\Psi^*_2(n - 1; x - [z^{-1}], y; \lambda^{-1}),
\]

(61)

\[
(\Delta G_2(z) - 1)\Psi_2(n; x, y; \mu^{-1})\Psi^*_2(n; x, y - [\mu^{-1}]; \lambda^{-1}) \\
= -\Psi_2(n; x, y; \mu^{-1})\Psi^*_2(n; x, y - [z]; \lambda^{-1}).
\]

(62)

Proof. Firstly for (59), with the help of (18) and (58), one has

\[
(G_1(z) - 1)\Psi_1(n; x, y; \mu)\Psi^*_1(n - 1; x - [\mu^{-1}], y; \lambda) \\
= \lambda\left(1 - \frac{\lambda}{\mu}\right)^{-1}\left(\frac{\mu}{\lambda}\right)^n e^{\xi(x, \mu) - \xi(x, \lambda)} z^{-1} \left(1 - \frac{\lambda}{z}\right)^{-1} \frac{1}{\tau(n; x, y)\tau(n; x - [z^{-1}], y)} \\
\times ((z - \mu}\tau(n; x, y)\tau(n; x + [\lambda^{-1}] - [\mu^{-1}] - [z^{-1}], y) \\
- (z - \lambda)\tau(n; x - [z^{-1}], y)\tau(n; x + [\lambda^{-1}] - [\mu^{-1}], y)),
\]

and further according to (60) with \(s_1 = \lambda^{-1}, s_2 = \mu^{-1},\) and \(s_3 = z^{-1},\)

\[
(G_1(z) - 1)\Psi_1(n; x, y; \mu)\Psi^*_1(n - 1; x - [\mu^{-1}], y; \lambda) \\
= -\frac{\lambda\mu}{z}\left(\frac{\mu}{\lambda}\right)^n e^{\xi(x, \mu) - \xi(x, \lambda)} \left(1 - \frac{\lambda}{z}\right)^{-1} \frac{1}{\tau(n; x, y)\tau(n; x - [z^{-1}], y)} \\
\times ((\mu\tau(n; x, y)\tau(n + 1; x + [\lambda^{-1}] - [\mu^{-1}], y - [z])) \\
- \lambda\tau(n + 1; x, y - [z])\tau(n; x + [\lambda^{-1}] - [\mu^{-1}], y)) \\
= -\frac{\mu}{z}\Psi_1(n; x, y; \mu)\Psi^*_1(n - 1; x - [z^{-1}], y; \lambda).
\]

Then by (18), (58) and (47) with \(s_1 = \lambda^{-1}, s_2 = \mu^{-1},\) and \(s_3 = z,\) (60) is obtained.

\[
(\Delta G_2(z) - 1)\Psi_1(n; x, y; \mu)\Psi^*_1(n - 1; x - [\mu^{-1}], y; \lambda) \\
= \left(1 - \frac{\lambda}{\mu}\right)^{-1}\left(\frac{\mu}{\lambda}\right)^n e^{\xi(x, \mu) - \xi(x, \lambda)} \frac{1}{\tau(n; x, y)\tau(n + 1; x, y - [z])} \\
\times ((\mu\tau(n; x, y)\tau(n + 1; x + [\lambda^{-1}] - [\mu^{-1}], y - [z])) \\
- \lambda\tau(n + 1; x, y - [z])\tau(n; x + [\lambda^{-1}] - [\mu^{-1}], y)) \\
= \mu\Psi_1(n; x, y; \mu)\Psi^*_1(n; x, y - [z]; \lambda).
\]

Similarly, (18), (58) and (47) for \(s_1 = \lambda^{-1}, s_2 = \mu^{-1}, s_3 = z^{-1}\) lead to (61)

\[
(G_1(z) - 1)\Psi_2(n; x, y; \mu^{-1})\Psi^*_2(n; x, y - [\mu^{-1}]; \lambda^{-1})
\]
\[ e^{\xi(y,\mu)-\xi(y,\lambda)} \left( 1 - \frac{\lambda}{\mu} \right)^{-1} \left( \frac{\lambda}{\mu} \right)^n \frac{1}{(n+1; x, y; x - [z^{-1}], y) \times (n; x, y) \tau(n; x, y) \tau(n; x - [z^{-1}], y)} \]

\[ = \lambda^{-1} z^{-1} e^{\xi(y,\mu)-\xi(y,\lambda)} \left( \frac{\lambda}{\mu} \right)^n \frac{\tau(n+1; x, y; x - [z^{-1}], y) \times (n-1; x - [z^{-1}], y)}{\tau(n; x, y) \tau(n; x - [z^{-1}], y)} \]

\[ = z^{-1}\Psi_2(n; x, y; \mu^{-1})\Psi_2(n-1; x - [z^{-1}], y; \lambda^{-1}). \]

At last, (62) can be got by according to (49) with \( s_1 = \lambda^{-1}, s_2 = \mu^{-1}, s_3 = z \),

\[ (\Lambda G_2(z) - 1)\Psi_2(n; x, y; \mu^{-1})\Psi_2(n; x, y - [\mu^{-1}; \lambda^{-1}]) \]

\[ = \left( 1 - \frac{\lambda}{\mu} \right)^{-1} \left( \frac{\lambda}{\mu} \right)^n e^{\xi(y,\mu)-\xi(y,\lambda)} \frac{(\lambda^{-1} - z)^{-1}}{\tau(n; x, y) \tau(n+1; x, y - [z]) \times (n-1; x, y) \tau(n; x, y)} \times (n+1; x, y + [\lambda^{-1} - [z]) \tau(n; x, y)

\[ = -\lambda^{-1}(\lambda^{-1} - z)^{-1} \left( \frac{\lambda}{\mu} \right)^n e^{\xi(y,\mu)-\xi(y,\lambda)} \frac{\tau(n+1; x, y - [\mu^{-1}] \times (n; x, y + [\lambda^{-1} - [z]) \tau(n+1; x, y - [z])}{\tau(n; x, y) \tau(n+1; x, y - [z])} \]

\[ = -\Psi_2(n; x, y; \mu^{-1})\Psi_2(n; x, y - [z]; \lambda^{-1}). \]

\[ \square \]

5. The Actions of the Squared Eigenfunction Symmetries on the Tau Function

Based upon the preparation above, we now give the actions of the squared eigenfunction symmetry on the wave functions, and further on the tau function.

**Proposition 8.** The actions of the squared eigenfunction symmetry on the wave functions is given as follows

\[ \frac{\partial_{\alpha_1}}{\Psi} \Psi = \left( (G_1(z) - 1)\Xi(x, \lambda, \mu)\tau, (\Lambda G_2(z) - 1)\Xi(x, \lambda, \mu)\tau \right), \quad (63) \]

\[ \frac{\partial_{\alpha_2}}{\Psi} \Psi = \frac{\mu}{\lambda} \left( (G_1(z) - 1)\Xi(y, \lambda, \mu)\tau, (\Lambda G_2(z) - 1)\Xi(y, \lambda, \mu)\tau \right). \quad (64) \]

**Proof.** Firstly by (52), (59) and (23),

\[ \partial_{\alpha_1} \Psi_1 = (\lambda^{-1} \Psi_1(x, y; \mu) \otimes \Psi_1^*(x, y; \lambda))_1 \Psi_1(x, y; z) \]

\[ = \left( \lambda^{-1} \Psi_1(n; x, y; \mu) \sum_{k<0}^{\infty} \Psi_1^*(n+k; x, y; \lambda) \Psi_1(n+k; x, y; z) \right)_{n \in \mathbb{Z}} \]

\[ = (\lambda^{-1} z^{-1} \Psi_1(n; x, y; \mu) \Psi_1(n-1; x - [z^{-1}], y; \lambda) \Psi_1(n; x, y; z))_{n \in \mathbb{Z}} \]

\[ = (\lambda^{-1} z^{-1} \Psi_1(n; x, y; \mu) \Psi_1(n-1; x - [\mu^{-1}], y; \lambda))_{n \in \mathbb{Z}} \]

\[ = \Psi_1(G_1(z) - 1)\frac{\Xi(x, \lambda, \mu)\tau}{\tau}. \]
Then with the help of (55), (60) and (24),

\[
\partial_{\alpha_1} \Psi_2 = -(\lambda^{-1} \Psi_1(x, y; \mu) \otimes \Psi_1^*(x, y; \lambda)) \Psi_2(x, y; z)
\]

\[
= - \left( \lambda^{-1} \Psi_1(n; x, y; \mu) \sum_{k \geq 0} \Psi_1^*(n + k; x, y; \lambda) \Psi_2(n + k; x, y; z) \right)_{n \in \mathbb{Z}}
\]

\[
= - (\lambda^{-1} \Psi_1(n; x, y; \mu) \Psi_1^*(n; x, y - [z]; \lambda) \Psi_2(n; x, y; z))_{n \in \mathbb{Z}}
\]

\[
= (\mu \Psi_2(\Lambda G_2(z) - 1) \Psi_1(n; x, y; \mu) \Psi_1^*(n - [\mu^{-1}]; \lambda, \lambda))_{n \in \mathbb{Z}}
\]

and (55), (61) and (24) lead to

\[
\partial_{\alpha_2} \Psi_1 = -(\lambda^{-1} \Psi_2(x, y; \mu^{-1}) \otimes \Psi_2^*(x, y; \lambda^{-1})) \Psi_1(x, y; z)
\]

\[
= - \left( \lambda^{-1} \Psi_2(n; x, y; \mu^{-1}) \sum_{k < 0} \Psi_2^*(n + k; x, y; \lambda^{-1}) \Psi_1(n + k; x, y; z) \right)_{n \in \mathbb{Z}}
\]

\[
= - (\lambda^{-1} \Psi_2(n; x, y; \mu^{-1}) \Psi_2^*(n - 1; x - [z^{-1}]; \lambda^{-1}) \Psi_1(n; x, y; z))_{n \in \mathbb{Z}}
\]

\[
= (\mu \Psi_1(\Lambda^{-1} G_1(z) - 1) \Psi_2(n; x, y; \mu^{-1}) \Psi_2^*(n; x, y - [\mu^{-1}]; \lambda^{-1}))_{n \in \mathbb{Z}}
\]

At last, according to (57), (62) and (24)

\[
\partial_{\alpha_2} \Psi_2 = (\lambda^{-1} \Psi_2(x, y; \mu^{-1}) \otimes \Psi_2^*(x, y; \lambda^{-1})) \Psi_2(x, y; z)
\]

\[
= \left( \lambda^{-1} \Psi_2(n; x, y; \mu^{-1}) \sum_{k \geq 0} \Psi_2^*(n + k; x, y; \lambda^{-1}) \Psi_2(n + k; x, y; z) \right)_{n \in \mathbb{Z}}
\]

\[
= (\lambda^{-1} \Psi_2(n; x, y; \mu^{-1}) \Psi_2^*(n - 1; x, y - [z]; \lambda^{-1}) \Psi_2(n; x, y; z))_{n \in \mathbb{Z}}
\]

\[
= (\mu \Psi_2(\Lambda G_2(z) - 1) \Psi_2(n; x, y; \mu^{-1}) \Psi_2^*(n; x, y - [\mu^{-1}]; \lambda^{-1}))_{n \in \mathbb{Z}}
\]

\[
= \frac{\mu}{\lambda} \Psi_2(\Lambda G_2(z) - 1) \frac{\tilde{X}(y, \mu, \lambda)}{\tau}.
\]

At last according to the relations (18) between the wave functions and the tau functions, we get the following proposition.

**Proposition 9.** The actions of the squared eigenfunction symmetries on the tau functions are given as follows

\[
\partial_{\alpha_1} \tau = \tilde{X}(x, \lambda, \mu) \tau, \quad \partial_{\alpha_2} \tau = \frac{\mu}{\lambda} \tilde{X}(y, \lambda, \mu) \tau.
\]
6. CONCLUSIONS AND DISCUSSIONS

The squared eigenfunction symmetry of the Toda lattice hierarchy is constructed explicitly in the form of the Kronecker product of the vector eigenfunction and the vector adjoint eigenfunction. And the relation with the additional symmetry is also investigated, that is, the squared eigenfunction symmetry can be viewed as the generating function of the additional symmetries when the eigenfunction and the adjoint eigenfunction are the wave function and the adjoint wave function respectively. The action of the particular squared eigenfunction symmetry on the tau function of the Toda lattice hierarchy is obtained with the help of the Fay-like identities and the relations about the wave functions. And thus another proof of the ASvM formulas for the Toda lattice hierarchy is given. The squared eigenfunction symmetry here is expected to be applied in the study of the symmetry constraints for the Toda lattice hierarchy. And the spectral representation for the eigenfunction of the Toda lattice hierarchy, similar to KP hierarchy case [5], is also expected to be set up. We will focus on these questions later in the future paper.

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