Euclidean super Yang–Mills theory
on a hyper–Kähler eightfold

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Abstract

The construction of a $N_T = 3$ cohomological gauge theory on the hyper–Kähler eight–fold, whose group theoretical description was given previously by Blau and Thompson \cite{1}, is performed explicitly.

1. Introduction

In recent years the study of cohomological gauge theories on manifolds of special holonomy in various dimensions have attracted a lot of interest \cite{2,3,4,5}. In the context of string theory and M–theory special holonomy plays a prominent role especially because the simplest vacua preserving part of the supersymmetry are compactifications on manifolds of special holonomy. So far, Calabi–Yau three–folds with SU(3) holonomy have received the most intensive study in connection with heterotic string compactifications and also due to the miraculous mirror symmetry of type II strings on such manifolds. Recently, the $G_2$– and Spin(7)–holonomy Joyce seven– and eight–folds, respectively, have received considerable attention as well, since they may provide the simplest way to compactify M–theory to four dimensions and to understand the dynamics of $N = 1$ supersymmetric field theories.

Moreover, it has been shown \cite{3,4} that the ideas underlying topological theories can be extended to dimensions higher than four. These cohomological theories, which do not require for a topological twist, acquire many of the characteristics of a topological theory. Nevertheless, such theories, which have a rather intriguing structure, are not fully topological, since they are only invariant under those metric variations which do not change the reduced holonomy structure. At present, the physical status of such theories in $D > 4$ has not been entirely understood. But, since it is widely believed that the effective world volume theory of the D–brane is the dimensional reduction of $N = 1, D = 10$ super Yang–Mills theory (SYM) \cite{6}, such cohomological gauge theories could arise naturally in the study of wrapped Euclidean D–branes in string theory (see, e.g., \cite{7}).

Examples of such theories in $D = 8$ are the SYM on Spin(7) holonomy Joyce eight–folds \cite{3,4} — a $N_T = 1$ theory which is the eight–dimensional analogue of Donaldson–Witten theory — and the SYM on a Calabi–Yau four–folds with Spin(6) $\sim SU(4)$ holonomy \cite{3} — a $N_T = 2$ theory which is a holomorphic analogue of Donaldson–Witten theory. The only other possible Ricci–flat manifolds admitting covariant constant (parallel) spinors are hyper–Kähler eight–folds with Spin(5) $\sim Sp(4)$ holonomy which could be interesting as well \cite{8}. In \cite{1}, making use of a previous work by Ward \cite{9}, Blau and Thompson gave a group theoretical description of the SYM on that manifolds. The aim of the present paper is to construct this theory explicitly. In view of

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possible generalizations, it might be instructive to start with the somewhat more involved case of a quaternionic Kähler manifold with $Sp(4) \otimes Sp(2)$ holonomy and to consider the particular case of a hyper–Kähler eight–fold only in the final analysis.

The paper is organized as follows. In Sect. 2, we first derive, in accordance with \[9\], the generalized $Sp(4) \otimes Sp(2)$ instanton equations. Then, we give the action and the whole set of transformation rules of the Euclidean $N = 2, D = 8$ SYM in flat space with the original full $SO(8)$ rotation invariance reduced to $Sp(4) \otimes Sp(2)$. In Sect. 3, we formulate this theory on the hyper–Kähler eight–fold, i.e, for the particular case when the curvature of the $Sp(2)$ spin connections vanishes. Otherwise, i.e., for the general case of a quaternionic Kähler manifold, which is a Einstein space and not Ricci–flat, there are no parallel spinors. In the Appendix we describe in some detail the derivation of the cohomological $N_T = 3, D = 8$ SYM.

2. The $Sp(4) \otimes Sp(2)$–invariant, $N_T = 3$ Euclidean super Yang–Mills theory in eight dimensions

Let us first consider the eight–dimensional Euclidean space $\mathbb{R}^8$ when the full symmetry group $SO(8)$ is reduced to its subgroup $Sp(4) \otimes Sp(2)$. Then, the Euclidean coordinates $x^M (M = 1, \ldots, 8)$ can be expressed through complex coordinates $z_{\bar{A}A}$, where $A = 1, 2, 3, 4$ and $a = 1, 2$ are $Sp(4)$ and $Sp(2)$ indices, respectively, thereby preserving the Euclidean metric,

$$ds^2 = dx^M dx_M = \epsilon^{AB} e^{ab} dz_{\bar{A}a} dz_B^a, \quad x_M = e^{\bar{A}A}_M z_{\bar{A}A}.$$  

Here, $e^{\bar{A}A}_M$ defines a (non–singular) map from the 8–dimensional Euclidean space to the 4–dimensional complex space $\mathbb{C}^4$,

$$e^{\bar{A}A}_M e_{N\bar{A}a} = \delta_{MN}, \quad e^{\bar{A}A}_M e_{M\bar{A}b} = \epsilon^{AB} \epsilon^{ab}.$$  

The indices $A$ and $a$ are raised and lowered as follows, $z^{\bar{A}A} = \epsilon^{AB} e^{ab} z_{\bar{B}b}$ and $z_{Bb} = z^{\bar{A}A} \epsilon_{AB} e_{ab}$, with $\epsilon_{AC} \epsilon^{BC} = \delta_A^B$ and $\epsilon_{ac} \epsilon^{bc} = \delta_a^b$, where $\epsilon_{AB}$ and $\epsilon_{ab}$ are the invariant symplectic tensors of the group $Sp(4)$ and $Sp(2)$, respectively, $\epsilon_{12} = -\epsilon_{21} = \epsilon_{34} = -\epsilon_{43} = 1$. Explicitly, for the components of $z_{\bar{A}A} = e^{\bar{A}A}_M x_M$ we choose

$$z_{11} = x_1 + ix_2, \quad z_{21} = -x_3 + ix_4, \quad z_{31} = x_5 + ix_6, \quad z_{41} = -x_7 + ix_8,$$

$$z_{12} = x_3 + ix_4, \quad z_{22} = x_1 + ix_2, \quad z_{32} = x_7 + ix_8, \quad z_{42} = x_5 - ix_6,$$

which will be grouped into the complex conjugated coordinates $z_A \equiv (z_{11}, z_{12}, z_{31}, z_{32})$ and $\bar{z}_A \equiv (z_{22}, -z_{21}, -z_{12}, z_{41})$.

Then, on $\mathbb{C}^4$ an irreducible, $Sp(4) \otimes Sp(2)$–invariant action of the gauge field $A^a_A$, being in the adjoint representation of some compact gauge group $G$, is given by

$$S_{YM} = \int d^4z \int d^4\bar{z} \text{tr} \left\{ \frac{1}{4} F^{AB}_{ab} F_{AB}^{a b} \right\},$$  

(1)

where $F_{AB}^{a b} = \partial_A A_B^a - [A_A^a, A_B^b]$ is the corresponding field strength.

Besides of (1), one can construct also a first–stage reducible, $Sp(4) \otimes Sp(2)$–invariant action, namely

$$S_T = \int d^4z \int d^4\bar{z} \text{tr} \left\{ \frac{1}{12} \left( \epsilon^{a b} F^{a b}_{ab} \epsilon^{c d} F_{AB cd} - \epsilon_{AB} F^{A b} e_{CD} F_{CD}^{a b} - F^{a b}_{ab} F_{AB}^{b a} \right) \right\},$$  

(2)

which can be recast into the form

$$S_T = \int d^4z d^4\bar{z} \text{tr} \left\{ \frac{1}{12} \epsilon^{ABCD}_{abcd} F_{AB}^{a b} F_{CD}^{c d} \right\},$$  

2
where the (fourth rank) tensor $\epsilon_{ABCD}^{abcd}$ is totally skew–symmetric in the index pairs $(Aa)$, $(Bb)$, $(Cc)$ and $(Dd)$,

$$\epsilon_{ABCD}^{abcd} = \epsilon_{AB} \epsilon_{CD} \Omega^{abcd} + \epsilon_{BC} \epsilon_{AD} \Omega^{abcd} + \epsilon_{CA} \epsilon_{BD} \Omega^{abcd},$$

thereby, for later use, we have introduced the following projection operator,

$$\Omega_{abcd} = \epsilon_{ac} \epsilon_{bd} + \epsilon_{ad} \epsilon_{bc}, \quad \frac{1}{2} \Omega_{abef} \epsilon_{ef}^{cd} = \Omega_{abcd}.$$  \hspace{1cm} (4)

The tensor $\Omega$ defines a linear map of the space of field strengths, $F_{AB}^{ab}$, onto itself, namely

$$\frac{1}{2} \epsilon_{ABCD}^{abcd} F^{CD} = \lambda F_{AB}^{ab}.$$  \hspace{1cm} (5)

Thus, one can look for its eigenvalues $\lambda$. In accordance with $Sp(4) \otimes Sp(2)$ decomposition of the adjoint representation of $SO(8)$, $28 \rightarrow (4, 2) \otimes (4, 2) \rightarrow (1, 3) \oplus (5, 3) \oplus (10, 1)$, it is straightforward to prove that the 3 irreducible subspaces of $\{ F_{AB}^{ab} \}$ are characterized by the eigenvalues $\lambda = -5, -1, 3$. Accordingly, under the branching $SO(8) \rightarrow Sp(4) \otimes Sp(2)$ the field strength decomposes into

$$F_{MN} \rightarrow F_{AB}^{ab} = \frac{1}{4} \epsilon_{AB} \epsilon_{CD} F^{CDab} + \frac{1}{4} \Omega_{ABCD} F^{CDab} + \frac{1}{2} \epsilon_{ab}^{cd} F_{AB}^{cd},$$

where, besides $\Omega$, we still have introduced another projection operator,

$$\Omega_{ABCD} = 2(\epsilon_{AC} \epsilon_{BD} - \epsilon_{AD} \epsilon_{BC}) - \epsilon_{AB} \epsilon_{CD}, \quad \frac{1}{4} \Omega_{ABEF} \Omega_{CDF} = \Omega_{ABCD},$$

Thereby, $\epsilon_{CD} F_{CDab}$ and $\epsilon_{ab}^{cd} F_{ab}^{cd}$ in the first and third term on the r.h.s. of (5), by virtue of $F_{AB}^{ab} = -F_{BA}^{-ba}$, are symmetric in $(ab)$ and $(AB)$, respectively, and $\Omega_{ABCD} F_{CD}^{cd}$ in the second term is symmetric in $(ab)$ and skew–symmetric and trace free in $(AB)$.

On the other hand, the action $\Pi$ can be rewritten as

$$SYM = S_T + \int d^4 z d^4 \bar{z} \text{tr} \left\{ \frac{1}{12} \Omega_{ABCD} F_{AB}^{ab} F^{CDab} + \frac{1}{6} \epsilon_{AB} F_{AB}^{ab} \epsilon_{CD} F^{CDab} \right\}. $$

Hence, if we, formally, just as in case of the $Spin(3)$ instantons in $D = 4$, impose $SYM = S_T$, by virtue of (5), it follows that the $Sp(4) \otimes Sp(2)$ instanton equations are characterized by the eigenvalue $\lambda = 3$,

$$\frac{1}{6} \epsilon_{ABCD}^{abcd} F_{CD}^{cd} = F_{AB}^{ab} \quad \Leftrightarrow \quad \Omega_{ABCD} F_{CD}^{cd} = 0, \quad \epsilon_{AB} F_{AB}^{ab} = 0. $$

This result is in accordance with $\Pi$, where the $Sp(4) \otimes Sp(2)$–invariant integrable equations for the gauge field $A_A^a$ are discussed, and where the 18 equations (5) are obtained as the integrability conditions $\pi_a \pi_b F_{AB}^{ab} = 0$ of the linear equations $\pi_a (\partial A^a + A_A^a) \psi = 0$; here, $\pi_a$ are the homogeneous coordinates on the complex projective space, $\mathbb{C}P^1$.

Now, we like to construct a cohomological theory, whose action localizes onto the moduli space of the generalized self–duality equations (5), in flat space. This theory can be obtained from the Euclidean $N = 2$, $D = 8$ SYM by reducing the group $SO(8)$ to $Sp(4) \otimes Sp(2)$.

The gauge multiplet of the Euclidean $N = 2$, $D = 8$ SYM consists of the vector field $A_M$, a chiral and anti–chiral Weyl spinor, $\lambda$ and $\bar{\lambda}$, and the scalar fields $\phi$ and $\bar{\phi}$. The vector, the chiral spinor and the anti–chiral spinor representation, which are all eight–dimensional, will be denoted by $8_v$, $8_s$ and $8_c$, respectively. Under the branching $SO(8) \rightarrow Sp(4) \otimes Sp(2)$ these representations decompose as follows $\Pi$,

$$8_v \rightarrow (4, 2), \quad A_M \rightarrow A_A^a,$$

$$8_s \rightarrow (5, 1) \oplus (1, 3), \quad \lambda \rightarrow \chi_{AB}, \eta^{ab},$$

$$8_c \rightarrow (4, 2), \quad \bar{\lambda} \rightarrow \bar{\psi}_A^a.$$  \hspace{1cm} (9)
i.e., the vector and the anti–chiral representation remain irreducible whereas the the spinor representation decomposes into $\chi_{AB}$ which is skew–symmetric and traceless, $\epsilon^{AB}\chi_{AB} = 0$, and $\eta^{ab}$ which is symmetric.

After reducing $SO(8)$ to $Sp(4) \otimes Sp(2)$ one ends up with the following $Sp(4) \otimes Sp(2)$–invariant action with an extended, $N_T = 3$, on–shell equivariantly nilpotent topological supersymmetry (for details, see, Appendix),

\[
S^{(N_T=3)}_{Sp(4) \otimes Sp(2) \subset SO(8)} = \int d^4z \, d^4\bar{z} \left\{ \frac{1}{4} F^{AB}_{ab} F^{ab}_{AB} - 2 D^A \phi D_A \phi - 2 \chi_{AB} D^A \psi^B_a + 2 \eta^{ab} D^A \psi^A_a \right\},
\]

where $D^A = \partial^A + [A^A, \cdot \cdot \cdot]$. All the fields are in the adjoint representation and take their values in the Lie algebra $\text{Lie}(G)$ of the gauge group $G$.

Furthermore, denoting the 16 (real) supercharges by $Q^{ab}$, $Q_{AB}$ and $\bar{Q}_a A$, where $Q^{ab}$ is symmetric, $Q^{ab} = \frac{1}{2} \Omega^{abcd} Q_{cd}$, and $Q_{AB}$ is skew–symmetric and traceless, $Q_{AB} = \frac{1}{4} \Omega_{ABCD} Q^{CD}$, for the on–shell $Sp(4)$ scalar, tensor and vector supersymmetry transformations one gets

\[
Q^{ab} A^c_A = \Omega^{abcd} \psi^d_A,
\]
\[
Q^{ab} \psi^c_A = \Omega^{abcd} D^d \phi,
\]
\[
Q^{ab} \phi = 0,
\]
\[
Q^{ab} \bar{\phi} = \eta^{ab},
\]
\[
Q^{ab} \eta^{cd} = -\frac{1}{4} \Omega^{abce} \Omega^{cdef} \epsilon_{AB} F_{AB}^{ef} + \Omega^{abcd} [\phi, \bar{\phi}],
\]
\[
Q^{ab} \chi_{AB} = \frac{1}{4} \Omega^{abcd} \Omega_{ABCD} F_{CD}^{ab},
\]
\[
Q_{AB} A^c_A = \Omega_{ABCD} \psi^d_A,
\]
\[
Q_{AB} \psi^c_A = -\Omega_{ABCD} D^d \phi,
\]
\[
Q_{AB} \phi = 0,
\]
\[
Q_{AB} \bar{\phi} = \chi_{AB},
\]
\[
Q_{AB} \eta^{cd} = -\frac{1}{4} \Omega_{ABCD} \Omega^{abcd} F_{CD}^{ab},
\]
\[
Q_{AB} \chi_{CD} = -\frac{1}{4} \Omega_{ABEG} \Omega_{CDF}^{G} \epsilon^{ab} F^{EF}_{ab} + \Omega_{ABCD} [\phi, \bar{\phi}],
\]

and

\[
\bar{Q}_a A^b_B = \epsilon_{AB} \eta^{ab} + \epsilon^{ab} \chi_{AB},
\]
\[
\bar{Q}_a \psi^b_B = F^{ab}_{AB} - \frac{1}{4} \Omega_{ABCD} \Omega^{abcd} f_{CD}^{ab} + \epsilon_{AB} \epsilon^{ab} [\phi, \bar{\phi}],
\]
\[
\bar{Q}_a \phi = -\psi^a_A,
\]
\[
\bar{Q}_a \bar{\phi} = 0,
\]
\[
\bar{Q}_a \eta^{cd} = -\Omega^{abcd} D_{AB} \bar{\phi},
\]
\[
\bar{Q}_a \chi_{CD} = \Omega_{ABCD} D_{Ba} \bar{\phi},
\]

respectively, where the projection operators $\Omega^{abcd}$ and $\Omega_{ABCD}$ have been introduced in (4) and (6).
The on–shell algebraic relations among the supercharges \(Q^{ab}, Q_{AB}\) and \(\bar{Q}^a_A\) are

\[
\{Q^{ab}, Q^{cd}\} = -2\Omega^{abcd}\delta_G(\phi), \quad \{Q_{AB}, Q_{CD}\} = 2\Omega_{ABCD}\delta_G(\phi),
\]
\[
\{Q^{ab}, Q_{AB}\} = 0,
\]
\[
\{Q^{ab}, \bar{Q}^c_A\} = -\Omega^{abcd}(\partial_{Ad} + \delta_G(A_{Ad})), \quad \{Q_{AB}, \bar{Q}^a_A\} = \Omega_{ABCD}(\partial^{Da} + \delta_G(A^{Da})),
\]
\[
\{\bar{Q}^a_A, \bar{Q}^b_B\} = 2\epsilon_{AB}\epsilon^{ab}\delta_G(\phi),
\]

where \(\equiv\) indicates that the corresponding equations hold on–shell, and \(\delta_G(\phi)\) denotes a gauge transformation depending on the fields \(\phi = (A_A^a, \phi, \bar{\phi})\), being defined by \(\delta_G(\phi)A_A^a = -D_A^a(A)\phi\) and \(\delta_G(\phi)X = [\phi, X]\) for all the other fields.

In order to verify that the the action (10) is invariant under the transformations (11)–(13) one has to use the following basic relations for the projection operators,

\[
\Omega_{abef}\Omega_{cdg}^f + \Omega_{abfg}\Omega_{cdg}^e = \Omega_{abce}\epsilon_{df} + \Omega_{abde}\epsilon_{cf} - \Omega_{cdbe}\epsilon_{af} + \Omega_{cdbe}\epsilon_{af},
\]
\[
\Omega_{abef}\Omega_{cdg}^f - \Omega_{abfg}\Omega_{cdg}^e = 2\Omega_{abed}\epsilon_{ef},
\]

and

\[
\Omega_{ABCD}\Omega_{CDF}^G + \Omega_{ABFG}\Omega_{CDE}^G = \Omega_{ABCE}\epsilon_{DF} - \Omega_{ABDE}\epsilon_{CF} + \Omega_{ABCF}\epsilon_{DE} - \Omega_{ABDF}\epsilon_{CE} - \Omega_{CDFE}\epsilon_{BE} + \Omega_{CDFE}\epsilon_{AE},
\]
\[
\Omega_{ABCD}\Omega_{CDF}^G - \Omega_{ABFG}\Omega_{CDE}^G = 2\Omega_{ABCD}\epsilon_{EF},
\]

thereby we used the following equalities,

\[
\epsilon_{ac}\epsilon_{bd} + \text{cyclic (a, b, c)} = 0,
\]
\[
\epsilon_{AC}\epsilon_{BD}\epsilon_{EF} - \epsilon_{AC}(\epsilon_{BE}\epsilon_{DF} - \epsilon_{BF}\epsilon_{DE}) - \epsilon_{BD}(\epsilon_{AE}\epsilon_{CF} - \epsilon_{AF}\epsilon_{CE}) + \text{cyclic (A, B, C)} = 0.
\]

Finally, in order to obtain a cohomological action we still have to split off from (10) the first–stage reducible action (2),

\[
S^{(N_T=3)} = S_{Sp(4)\otimes Sp(2)\subset SO(8)}^{(N_T=3)} - S_T,
\]

which, by virtue of (7), yields

\[
S^{(N_T=3)} = \int d^4z d^4\bar{z} \text{tr}\left\{\frac{1}{12}\Omega_{ABCD}F^{AB}_{\ abc}F^{CDab} + \frac{1}{6}\epsilon^{ABC}F^{AB}_{\ cde}F^{CDab}
- 2\chi_{AB}D^A\psi^a + 2\eta^{ab}D^A\psi_{Ab} - 2\bar{\phi}\psi^a_A - 2\bar{\phi}\phi_A^a - \frac{1}{2}\phi\chi_{AB} - \psi^{\ a}\eta_{ab} - 2D^A\bar{\phi}D_A\phi - 2(\bar{\phi}, \phi)^2\right\}.
\]

Then, on–shell, upon using the equations of motion of \(\chi_{AB}\) and \(\eta^{ab}\), the action (14) can be cast into the \(Q^{ab}\)–exact form

\[
S^{(N_T=3)} = Q^{ab}\Psi_{ab},
\]

with the gauge fermion

\[
\Psi_{ab} = \Omega_{abcd}\int d^4z d^4\bar{z} \text{tr}\left\{\frac{1}{12}\chi_{AB}F^{ABcd} + \frac{1}{12}\epsilon^{ef}\epsilon_{AB}F^{ABde} + \frac{1}{6}\eta^{cd}(\phi, \bar{\phi}) - \frac{1}{3}\phi D^c\psi^a_A\right\}.
\]
3. Euclidean super Yang–Mills theory on the hyper–Kähler eightfold

So far we have assumed that the space is flat, but we shall now formulate the cohomological theory on a hyper–Kähler eight–fold. In view of possible generalizations it may be convenient to study the more involved case of a quaternionic Kähler manifold. For that purpose, we shall begin by considering an eight–dimensional Riemannian manifold with $Sp(4) \otimes Sp(2)$ holonomy, endowed with hermitean metric $g_{\mu \nu}$ and $Sp(4)$ and $Sp(2)$ spin connections $\omega^A_{\mu B}$ and $\omega^a_{\mu b}$, respectively. The curved coordinates will be indicated as $x^\mu$ ($\mu = 1, \ldots, 8$) and the complex coordinates will again be denoted by $z_{Aa}$,

$$ds^2 = g^{\mu \nu} dx_\mu dx_\nu,$$
where, locally, $e^A_\mu$ is the invertible 8–bein on the quaternionic Kähler manifold,

$$e^A_\mu A_\alpha = g_{\mu \nu}, \quad e^A_\mu e^{B b}_\nu = e^{AB c}_{\nu a}.$$

In order to break down the structure group $GL(4, \mathbb{C}) \otimes GL(2, \mathbb{C})$ to $Sp(4) \otimes Sp(2)$, we have to require covariant constancy of the symplectic tensors $e^{AB}$ and $e^{ab}$,

$$\nabla^A_\mu e^{AB} \equiv \partial_\mu e^{AB} + \omega^A_\mu C e^{CB} + \omega^B_\mu C e^{AC} = 0,$$
$$\nabla^a_\mu e^{ab} \equiv \partial_\mu e^{ab} + \omega^a_\mu c e^{cb} + \omega^b_\mu c e^{ac} = 0,$$

where $\nabla_\mu$ denotes the metric covariant derivative. In addition, the integrability conditions $[\nabla_\mu, \nabla_\nu](e^{AB}, e^{ab}) = 0$ imply that $e^{AB}$ and $e^{ab}$ are constant and that the antisymmetric part of the spin connections can be chosen equal to zero.

Furthermore, in order to ensure covariant constancy of the metric, we impose

$$\nabla^A_\mu e^{Aa}_\nu \equiv \partial_\mu e^{Aa}_\nu - \Gamma^A_{\mu \nu} \lambda e^{Aa}_\lambda + \omega^A_\mu B e^{Ba}_\nu + \omega^a_\nu b e^{Ab}_\nu = 0,$$

where $\Gamma^A_{\mu \nu \lambda}$ is the affine connection.

The crucial ingredient of a quaternionic Kähler manifold is a triplet of complex structures,

$$(J^\alpha)_{\mu \nu} = -ie^A_\mu A_\alpha b e^{\nu}_{Ab}, \quad \alpha = 1, 2, 3,$$

where $(\sigma^\alpha)_{a b}$ are the $Sp(2)$ generators, $(\sigma^\alpha)_{a c}(\sigma^\beta)_{c b} = ie^\alpha_{\alpha \beta}(\sigma^\gamma)_{a b} + \delta^\alpha_{\beta} \delta^c_{d}$, which obey the algebra of the quaternions,

$$(J^\alpha)_{\mu}^{\phantom{\mu} \rho}(J^\beta)_{\rho \nu} = e^\alpha_{\beta \gamma}(J^\gamma)_{\mu}^{\phantom{\mu} \nu} - \delta^\alpha_{\mu} \delta^\beta_{\nu}.$$

Since the metric $g_{\mu \nu}$ is preserved and hermitean, it holds

$$(J^\alpha)_{\mu}^{\phantom{\mu} \rho} g_{\rho \nu} + (J^\alpha)_{\nu}^{\phantom{\nu} \rho} g_{\rho \mu} = 0.$$

Now, for any choice of $(J_\alpha)_{\mu \nu}$, we can associate a triplet of Kähler two–forms $\rho_\alpha$ via

$$\rho_\alpha \equiv \frac{1}{2} i (J_\alpha)_{\mu}^{\phantom{\mu} \rho} g_{\rho \nu} dx^\mu \wedge dx^\nu = \frac{1}{2} e_{AB}(\sigma^\alpha)_{a b} dz^A_a \wedge dz^B_b.$$

Then, by making use of the completeness relation, $(\sigma^\alpha)_{a c}(\sigma^\beta)_{c b} = \epsilon_{a b c d} - \delta^a_{d} \delta^c_{b}$, we can define the four–form

$$\Omega \equiv \frac{1}{2} \rho_\alpha \wedge \rho^\alpha = \frac{1}{2} e_{ABCD} \epsilon_{abcd} dz^A_a \wedge dz^B_b \wedge dz^C_c \wedge dz^D_d,$$

which reveals the geometrical origin of the fourth rank tensor $e_{ABCD}^{abcd}$ introduced in [8].
Furthermore, on a quaternionic Kähler manifold the complex structures \( \omega^\alpha \) are required to be covariant constant as well,

\[
\nabla_\mu (J^\alpha)_\nu^\lambda \equiv \partial_\mu (J^\alpha)_\nu^\lambda - \Gamma^\mu_{\nu\rho} (J^\alpha)_\rho^\lambda + \Gamma^\mu_\rho \lambda (J^\alpha)_\nu^\rho + 2\epsilon^{\alpha\beta\gamma} \omega^\beta \omega^\gamma (J_\gamma)_\nu^\lambda = 0, \tag{17}
\]

where in place of \( \omega^{ab} \) we have used the triplet representation \( \omega^\alpha = \frac{1}{2} i (\sigma^\alpha)^{ab} \omega^{ab} \). Since a quaternionic Kähler manifold — in contrast to a hyper–Kähler eight–fold — does not have a vanishing Nijenhuis tensor, the affine connection and the \( Sp(2) \) spin connection can not be uniquely defined without additional requirements. ³

In order to define in (17) the \( Sp(2) \) spin connection \( \omega_{\mu}^\alpha \) we adopt the two requirements proposed in [10],

\[
(J^\alpha)_{\mu}^\nu \omega_{\nu\alpha} = 0, \quad N_{\mu\nu}^\lambda = -\frac{1}{2} (J^\alpha)_{\mu}^\lambda \omega_{\nu\alpha}, \tag{18}
\]

where \( N_{\mu\nu}^\lambda \) is the Nijenhuis tensor (in the normalization of [10])

\[
N_{\mu\nu}^\lambda \equiv \frac{1}{12} (J^\alpha)_{\mu}^\rho \partial_{[\nu} (J_\alpha)_{\rho]}^\lambda - \frac{1}{12} (J^\alpha)_{\nu}^\rho \partial_{[\mu} (J_\alpha)_{\rho]}^\lambda = -N_{\nu\mu}^\lambda.
\]

The second condition in (18), which ensures that \( \Gamma^\mu_\rho \lambda \) is actually the (torsionless) Levi–Civita connection, can be easily solved for \( \omega^\alpha_{\mu} \). One finds

\[
\omega^\alpha_{\mu} = -\frac{1}{3} N^\mu_{\nu\rho} (J^\alpha)_\lambda = \omega^\mu \alpha.
\]

Then, one can show that the Levi–Civita connection in (17) is equal to the Oproiu connection [11],

\[
\Gamma^\mu_\rho \lambda = -\frac{1}{6} \partial_{(\mu} (J^\alpha)_\nu)_{\rho]}^\lambda - \frac{1}{12} \epsilon^{\alpha\beta\gamma} (J_\beta)_{(\mu} \partial_{[\nu]} (J_\gamma)_{\rho]} \sigma (J_\sigma)_{\rho]}^\lambda - \frac{1}{2} (J^\alpha)_{(\mu} \omega_{\nu\alpha)}. \tag{19}
\]

It differs from the Obata connection [12] — the Levi–Civita connection in the case of a hyper–Kähler eight–fold — by the \( \omega^\alpha_{\mu} \)-dependent term. With that choice for \( \omega^{ab} = i (\sigma^\alpha)^{ab} \omega^\alpha \) and \( \Gamma^\mu_\rho \lambda \) the \( Sp(4) \) spin connection \( \omega_{\mu}^{AB} \) can be immediately obtained from the condition (15).

³From the integrability condition \( [\nabla_\mu, \nabla_\nu] e_\rho^A = 0 \) of (16) it follows that the Riemannian curvature \( R_{\mu\rho}^\sigma = \partial_\mu \Gamma_{\nu\rho}^\sigma + \Gamma_{\lambda(\mu} \Gamma_{\nu\rho)}^\lambda \) decomposes into

\[
R_{\mu\rho}^\sigma = -(J_\alpha)_\rho^\sigma R_{\mu\nu}^\alpha - i \epsilon_{ab} e_\rho^a e_\tau^b R_{\mu\nu}^{AB},
\]

where \( R_{\mu\nu}^\alpha = \frac{1}{2} i (\sigma^\alpha)_{ab} R_{\mu\nu}^{ab} \) and \( R_{\mu\nu}^{AB} \) are the \( Sp(2) \) and \( Sp(4) \) curvatures, respectively,

\[
R_{\mu\nu}^\alpha = \partial_\mu \omega^\nu = 2 \epsilon^{\alpha\beta\gamma} \omega^\beta \omega^\gamma, \quad R_{\mu\nu}^{AB} = \partial_\mu \omega_{\nu}^{AB} + \omega_{[\mu} A_{\nu]} \omega^B C.
\]

Furthermore, from the integrability condition \( [\nabla_\mu, \nabla_\nu] (J^\alpha)_\rho^\sigma = 0 \) of (17), namely

\[
R_{\mu\nu}^\sigma (J^\alpha)_\rho^\lambda = R_{\mu\rho}^\lambda (J^\alpha)_\rho^\sigma + 2 \epsilon^{\alpha\beta\gamma} R_{\mu\rho\beta} (J_\gamma)_\rho^\sigma = 0, \tag{20}
\]

together with the relation \( \frac{1}{8} R_{\mu\nu}^\sigma (J^\alpha)_\rho^\sigma = R_{\mu\nu}^\alpha \), it can be shown that the Ricci tensor \( R_{\mu\nu} = R_{\lambda\mu\nu} \) is proportional to the curvature scalar, \( R_{\mu\nu} = \frac{1}{8} g_{\mu\nu} R \), and that the \( Sp(2) \) curvature is proportional to the complex structures, \( R_{\mu\nu} = \frac{1}{3} R (J^\alpha)_{\mu\nu} [10] \), i.e., the quaternionic Kähler manifold is Einstein in contrast to the hyper–Kähler eight–fold which is Ricci–flat.

³The covariant constancy condition (17) is left invariant when one performs simultaneously the replacements \( \Gamma^\mu_{\nu\rho} \rightarrow \Gamma^\mu_{\nu\rho} + (\delta^\mu_\alpha \delta^\rho_\beta - (J^\gamma)_{(\mu} \epsilon^\alpha_\gamma \epsilon^\beta_\nu) \xi_\rho \) and \( \omega^\nu = \omega^\nu_{\alpha} \rightarrow \omega^\nu_{\alpha} + (J^\alpha)_{(\mu} \epsilon^\nu_\rho \xi_\rho \) where \( \xi_\rho \) is an arbitrary vector (see, Appendix B of [10]). Notice, that in our convention (anti)symmetrization are defined by \( (X, Y) = XY + YX \) resp. \( [X, Y] = XY - YX \), whereas in [10] the corresponding definitions include an additional factor \( 1/2 \).
After having specified the data of a Riemannian manifold with \( Sp(4) \otimes Sp(2) \) holonomy let us now discuss whether or not the cohomological \( N_T = 3 \), \( D = 8 \) SYM can be coupled to such a background. In the case of a hyper–Kähler eight–fold, i.e., for a Ricci–flat manifold with vanishing \( Sp(2) \) curvature, \( R_{\mu\nu} = -\frac{8}{7}(J_{\alpha})_{\mu}^{\rho}R_{\rho\nu}{}^{\alpha} = 0 \), there is no problem. As usual, in order to put a flat space gauge invariant theory on a curved space one has to covariantize the action via \( \delta_{MN} \rightarrow g_{\mu\nu}, d^4z d^4\bar{z} \rightarrow d^4z d^4\bar{z} \sqrt{g} \) and \( D_Aa \rightarrow D_A^{\text{cov}}a = e_{\alpha}^{Aa}\nabla_{\mu} + [c^\alpha_a, \cdot] \), where \( D_A^{\text{cov}}a \) denotes the gauge and metric covariant derivative. Furthermore, it that case there is a triplet of parallel spinors \( \zeta_\alpha \) obeying

\[
\nabla_\mu \zeta_\alpha = 0, \quad \nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ABab}\gamma_{ABab}.
\]

(21)

Thereby, \( \gamma_{ABab} = \frac{1}{2}[\gamma_{Aa}, \gamma_{Bb}] \) are the generators of the holonomy group (with \( \gamma_{Aa} \) being the \( Sp(4) \otimes Sp(2) \) Dirac matrices) and

\[
\omega_\mu{}^{ABab} = -e^{\nu Aa}(\partial_\mu e_\nu{}^Bb - \Gamma_{\mu\nu}{}^\lambda Aa\gamma_{\nu\lambda}).
\]

(22)

is the affine spin connection (with \( \Gamma_{\mu\nu}{}^\lambda \) being the Obata connection). The spinors \( \zeta_\alpha \) can be identified with the parameters \( \zeta^{ab} \), via \( \zeta^{ab} = i(\sigma^a)^{ab}\zeta_\alpha \), of the scalar supersymmetries (11), which are already in a covariantized form.

Obviously, in the case of a quaternionic Kähler manifold, i.e., if \( \Gamma_{\mu\nu}{}^\lambda \) in (22) agrees with the Oproiu connection (19), the integrability condition \( [\nabla_\mu, \nabla_\nu]\zeta_\alpha = 0 \) only allows for the trivial solution \( \zeta_\alpha = 0 \), since this manifold is Einstein, \( R \neq 0 \). But, in the particular case when the curvature scalar \( R = \frac{7}{2}\lambda_0^2 \) is constant (the pre-factor is dimension dependent), the integrability condition does not forbid the existence of a triplet of (conformal) Killing spinors \( \zeta_\alpha \) satisfying

\[
\nabla_\mu \zeta_\alpha = \frac{1}{8}\lambda_0 \gamma_\mu \zeta_\alpha, \quad \gamma_\mu = e_\mu{}^{Aa}\gamma_{Aa}.
\]

(23)

Indeed, acting on (23) with \( \nabla_\nu \) and antisymmetrising, by virtue of (22), one obtains

\[
e^{\rho Aa}e_\sigma{}^{Bb}R_{\mu\nu\rho}{}^\sigma\gamma_{ABab}\zeta_\alpha = -\frac{1}{8}\lambda_0^2\gamma_{\mu\nu}\zeta_\alpha, \quad \gamma_{\mu\nu} = e_\mu{}^{Aa}e_\nu{}^{Bb}\gamma_{ABab},
\]

(24)

where \( R_{\mu\nu\rho}{}^\sigma \) is the Riemannian curvature tensor being introduced above. Since \( \Gamma_{\mu\nu}{}^\lambda \) is torsionless and since \( g_{\mu\nu} \) is preserved, this curvature obeys the cyclicity property \( R_{\mu\nu\rho}{}^\sigma = R_{\rho\mu\nu}{}^\sigma \) and the exchange property \( R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \). Then, multiplying (24) by \( \gamma^{\nu} \) and taking into account these properties one arrives at

\[
R_{\mu\nu\rho}{}^{\rho\gamma\nu\sigma} = R_{\mu\nu}{}^{\gamma\nu}\zeta_\alpha = \frac{7}{16}\lambda_0^2\gamma_\mu \zeta_\alpha.
\]

(25)

where \( R_{\mu\nu} = \frac{1}{8}g_{\mu\nu}R \). This condition admits a non–trivial solution for \( R = \frac{7}{2}\lambda_0^2 \). Obviously, \( \lambda_0 \rightarrow 0 \) brings us back to (21).

Hence, it is interesting to enquire whether the existence of parallel spinors on a hyper–Kähler eight–fold, which is irreducible and Ricci–flat, implies self–duality of the spin connection (and conversely), whereas the existence of Killing spinors on a quaternionic Kähler manifold, which is a Einstein space, is equivalent to certain generalized self–duality conditions for the spin connections. With another words, it is possible — in the case of a quaternionic Kähler manifold with constant \( R = \frac{7}{2}\lambda_0^2 \) — to add certain \( R \)-dependent terms to the covariantized action (14) in such a way that the Killing spinors \( \zeta_\alpha \) may be identified with the parameters \( \zeta^{ab} \), via \( \zeta^{ab} = i(\sigma^a)^{ab}\zeta_\alpha \), of the supersymmetries (11). Unfortunately, we have not so far been successful in finding such \( R \)-dependent terms. Nevertheless, we believe that this problem is worthy of further investigations.
Appendix

In order to derive from the Euclidean $N = 2$, $D = 8$ SYM a cohomological theory where the $SO(8)$ invariance is broken down to $Sp(4) \otimes Sp(2)$ we proceed as follows: First we choose the standard embedding of $Sp(4) \otimes Sp(2)$ into $SO(8)$, which is defined as the stability subgroup $Sp(4) \otimes Sp(2) \subset SO(8)$ of the vector representation according to which we have the decomposition

\[
\begin{align*}
8_v & \rightarrow (5, 1) \oplus (1, 3), \quad A_M \rightarrow G_i, V_\alpha, \\
8_s & \rightarrow (4, 2), \quad \lambda \rightarrow \lambda_{Aa}, \\
8_e & \rightarrow (4, 2), \quad \bar{\lambda} \rightarrow \bar{\lambda}_{Aa}.
\end{align*}
\]

Usually, one discards this representation since it does not allow for the existence of a $Sp(4) \otimes Sp(2)$–invariant, totally skew–symmetric (fourth rank) tensor. But, in the present case, after having established the $N = 2$, $D = 8$ SYM in that representation one can simply deduce the structure of the cohomological $N_T = 3$, $D = 8$ SYM according to the decomposition $[10].$

To begin with, let us first consider the full $SO(8)$–invariant action of the Euclidean $N = 2$, $D = 8$ SYM, which is obtained from the Minkowskian $N = 1$, $D = 10$ SYM $[13]$ by ordinary dimensional reduction and performing a Wick rotation into the Euclidean space. It is built up from an anti–hermitean vector field $A_M$, 16–component chiral and anti–chiral Weyl spinors, $\lambda$ and $\bar{\lambda}$, respectively, and the scalar fields $\phi$ and $\bar{\phi}$. All the fields take their values in the Lie algebra $Lie(G)$ of some compact gauge group $G$. As a result, for the dimensionally reduced Euclidean action one obtains

\[
S^{(N=2)} = \int_E d^8x \text{tr} \left\{ \frac{1}{4} F^{MN} F_{MN} + 2 \lambda \Gamma^M D_M \lambda - 2 D_M \bar{\phi} D^M \phi \\
+ 2 \lambda^T C_8^{-1} [\phi, \lambda] - 2 \bar{\lambda} C_8 [\bar{\phi}, \bar{\lambda}^T] - 2 [\bar{\phi}, \phi]^2 \right\},
\]

where $F_{MN} = \partial_{[M} A_{N]} + [A_M, A_N]$ and $D_M = \partial_M + [A_M, \cdots]$. $C_8$ is the charge conjugation matrix, $C_8^{-1} \Gamma_M C_8 = - \Gamma_M^T$, which can be chosen to be symmetric. The Dirac matrices $\Gamma_M$ for the $SO(8)$ spinor representation will be specified below, $\frac{1}{2} \{\Gamma_M, \Gamma_N\} = \delta_{MN} I_{16}$.

The action (A.1) is invariant under the following supersymmetry transformations

\[
\begin{align*}
\delta A_M &= \tilde{\zeta} \Gamma_M \lambda - \bar{\lambda} \Gamma_M \zeta, \\
\delta \phi &= \tilde{\zeta} C_8 \lambda^T, \\
\delta \lambda &= -\frac{1}{4} \Gamma^{MN} \zeta F_{MN} + \Gamma^M C_8 \bar{\phi} D_M \phi - \bar{\zeta} [\phi, \bar{\phi}], \\
\delta \bar{\phi} &= \lambda^T C_8^{-1} \zeta, \\
\delta \bar{\lambda} &= -\frac{1}{4} \Gamma^{MN} F_{MN} - \bar{\zeta}^T C_8^{-1} \Gamma^M D_M \phi - \bar{\zeta} [\bar{\phi}, \phi],
\end{align*}
\]

with $\zeta$ and $\bar{\zeta}$ being constant chiral and anti–chiral Weyl spinors, respectively ($\Gamma_9 \zeta = - \zeta$ with $\Gamma_9 = \Gamma_1 \cdots \Gamma_8$), and where $\Gamma_{MN} = \frac{1}{2} \{\Gamma_M, \Gamma_N\}$ are the $SO(8)$ generators.

In order to reduce in the action (A.1) the $SO(8)$ invariance to $Sp(4) \otimes Sp(2)$, we replace the $SO(8)$ matrices by the standard embedding $\Gamma_M = (\Gamma_i, \Gamma_{5+\alpha})$ ($i = 1, \ldots, 5$, $\alpha = 1, 2, 3$) of $Sp(4) \otimes Sp(2)$ into $SO(8)$. In this representation $\Gamma_i, \Gamma_{5+\alpha}$ and the chirality matrix $\Gamma_9$ are given
\[
\begin{align*}
\Gamma_i &= \begin{pmatrix}
  0 & (\gamma_i)_A^B \delta_a^b \\
  (\gamma_i)_A^B \delta_a^b & 0
\end{pmatrix}, \\
\Gamma_{5+\alpha} &= \begin{pmatrix}
  0 & -i\delta_A^B (\sigma_\alpha)_a^b \\
  i\delta_A^B (\sigma_\alpha)_a^b & 0
\end{pmatrix}, \\
\Gamma_9 &= \Gamma_1 \ldots \Gamma_8 = \begin{pmatrix}
  -\delta_A^B \delta_a^b & 0 \\
  0 & \delta_A^B \delta_a^b
\end{pmatrix},
\end{align*}
\]
(A.3)

\((\gamma_i)_A^B \ (A = 1, 2, 3, 4)\) and \((\sigma_\alpha)_a^b \ (a = 1, 2)\) being the \(Sp(4)\) and \(Sp(2)\) generators, respectively. The \(Sp(2)\) matrices obey

\[
(\sigma_\alpha)_a^c (\sigma_\beta)_c^b = i\epsilon_{a\beta\gamma} (\sigma_\gamma)_a^b + \delta^{\alpha\beta} \delta_a^b,
\]
(A.4)

and are symmetric \((\sigma_\alpha)_a^b = (\sigma_\alpha)_b^a\). For the \(Sp(4)\) matrices we take the particular representation

\[
(\gamma_\alpha)_A^B = \begin{pmatrix}
  0 & -i(\sigma_\alpha)_a^b \\
i(\sigma_\alpha)_a^b & 0
\end{pmatrix}, \quad (\gamma_4)_A^B = \begin{pmatrix}
  0 & \delta_a^b \\
\delta_a^b & 0
\end{pmatrix}, \quad (\gamma_5)_A^B = \begin{pmatrix}
  \delta_a^b & 0 \\
0 & -\delta_a^b
\end{pmatrix},
\]

where \(\alpha\) runs over 1, 2, 3 (recalling that \(Sp(4) \sim Spin(5)\) is the covering group of \(SO(5)\)). They satisfy the relations

\[
\begin{align*}
(\gamma_i)_A^C (\gamma_j)_C^B &= (\gamma_{ij})_A^B + \delta_{ij} \delta_A^B, \\
(\gamma_i)_A^C (\gamma_{mn})_C^B &= \delta_{i[m} (\gamma_{n]}_A^B - \frac{1}{2} \epsilon_{ijkmn} (\gamma^k)_A^B, \\
(\gamma_{ij})_A^C (\gamma_{mn})_C^B &= \delta_{i[m} (\gamma_{n]}_A^B + \epsilon_{ijkmn} (\gamma^k)_A^B - \delta_{i[m} \delta_{n]} \delta_A^B,
\end{align*}
\]
(A.5)

where \(\epsilon_{ijkmn}\) is a totally antisymmetric unit tensor. Here, the 5 matrices \((\gamma_{ij})_{AB}\) are skew-symmetric and traceless, \((\gamma_{ij})_{AB} = -(\gamma_{ij})_{BA}\) and \(\epsilon^{AB} (\gamma_{ij})_{AB} = 0\), whereas the 10 generators \((\gamma_{ij})_{AB}\) of the \(Sp(4)\) rotations are symmetric, \((\gamma_{ij})_{AB} = (\gamma_{ij})_{BA}\).

Then, for the 64 (antisymmetric) generators \(\Gamma_{MN} = (\Gamma_{ij}, \Gamma_{i,5+\alpha}, \Gamma_{5+\alpha, 5+\beta})\) from (A.3) one obtains

\[
\begin{align*}
\Gamma_{ij} &= \begin{pmatrix}
  (\gamma_{ij})_A^B \delta_a^b & 0 \\
0 & (\gamma_{ij})_A^B \delta_a^b
\end{pmatrix}, \\
\Gamma_{i,5+\alpha} &= \begin{pmatrix}
  i(\gamma_{i})_A^B (\sigma_\alpha)_a^b & 0 \\
0 & -i(\gamma_{i})_A^B (\sigma_\alpha)_a^b
\end{pmatrix}, \\
\Gamma_{5+\alpha, 5+\beta} &= \begin{pmatrix}
  i\delta_A^B \epsilon_{\alpha\beta\gamma} (\sigma_\gamma)_a^b & 0 \\
0 & i\delta_A^B \epsilon_{\alpha\beta\gamma} (\sigma_\gamma)_a^b
\end{pmatrix}.
\end{align*}
\]
(A.6)

A particular and crucial feature of the action (A.1) is that the charge conjugation matrix \(C_8\) can be chosen to be symmetric,

\[
C_8 = \begin{pmatrix}
  \epsilon_{ab} \epsilon_{AB} & 0 \\
0 & -\epsilon_{ab} \epsilon_{AB}
\end{pmatrix}, \quad C_8^{-1} \Gamma_M C_8 = -\Gamma_M^T.
\]

After having specified the representation of the matrices \(\Gamma_M\) let us now determine the action (A.1) and the transformation rules (A.2) of the \(N = 2, D = 8\) SYM with the \(SO(8)\) rotation invariance reduced to \(Sp(4) \otimes Sp(2)\). To begin with, we write the Weyl spinors \(\lambda\) and \(\bar{\lambda}\) as

\[
\lambda = -\Gamma_9 \lambda = \begin{pmatrix}
  \lambda_{Aa} \\
0
\end{pmatrix}, \quad \bar{\lambda} = \bar{\lambda} \Gamma_9 = (0, \bar{\lambda}^{Aa}),
\]
(A.7)
which, just as in the Minkowski space, are related by hermitean conjugation, \( \bar{\lambda} = \lambda^\dagger \), with

\[
E_8 = \begin{pmatrix} 0 & e^{ab}e^{AB} \\ e^{ab}e^{AB} & 0 \end{pmatrix}, \quad E_8 \Gamma_M E_8^{-1} = \Gamma^M_M,
\]

where we have renamed \( \lambda^\dagger_{AA} = \bar{\lambda}_{AA} \).

Then, substituting into (A.1) for \( \Gamma_M \) the matrices (A.3) and splitting the gauge field into \( A_M = (G_i, V_a) \), one ends up with the following \( Sp(4) \otimes Sp(2) \)-invariant action with an underlying \( N = 2 \) supersymmetry,

\[
S^{(N=2)} \mid_{Sp(4) \otimes Sp(2) \subset SO(8)} = \int d^4x d^4\bar{\mathcal{z}} \text{tr} \left\{ \frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} F^{i\alpha} F_{i\alpha} + \frac{1}{4} F_{\alpha\beta} F_{\alpha\beta} + 2\bar{\chi}^{\alpha} (\gamma^j)_A B_i D_{\alpha} \lambda_{Ba} - 2 D_i \phi D_i \phi + 2i\bar{\lambda}^{\alpha} (\sigma^\alpha)_{a} b D_{\alpha} \lambda_{Ab} - 2 D_a \phi D_a \phi - 2 \phi \{ \lambda_{AA}, \lambda^{\dagger} \} - 2 \bar{\phi} \{ \bar{\lambda}^{\dagger} \lambda_{AA} \} - 2 [\bar{\phi}, \phi]^2 \right\}. \tag{A.8}
\]

For the sake of simplicity, the various field strength tensors and covariant derivatives are denoted by \( F_{ij} = \partial_i G_j + [G_i, G_j], F_{i\alpha} = \partial_i V_{\alpha} - \partial_{\alpha} G_i + [G_i, V_{\alpha}], F_{\alpha\beta} = \partial_{[\alpha} V_{\beta]} + [V_{\alpha}, V_{\beta}] \) and \( D_i = \partial_i + [G_i, \cdot], D_{\alpha} = \partial_{\alpha} + [V_{\alpha}, \cdot] \), respectively.

Denoting the 16 (real) supercharges with \( Q_{Aa} \) and \( \bar{Q}^{Aa} \), and decomposing the supersymmetry transformations according to \( \delta = \bar{\zeta}^{Aa} \bar{Q}_{Aa} - \zeta_{Aa} Q^{Aa} \), from (A.2) one obtains

\[
Q^{Aa} G_i = (\gamma_i)_A^B \bar{\lambda}_{Ba}, \quad Q^{Aa} V_{\alpha} = -i (\sigma_{\alpha})_b^a \bar{\lambda}^{ab}, \quad Q^{Aa} \phi = 0, \quad Q^{Aa} \bar{\phi} = \lambda^{Aa}, \quad Q^{Aa} \bar{\lambda}_{Bb} = (\gamma_i)_A^B \epsilon^{ab} D_i \phi - i \epsilon^{AB} (\sigma_\alpha)^{ab} D^\alpha \phi, \tag{A.9}
\]

and

\[
\bar{Q}^{Aa} G_i = (\gamma_i)_A^B \lambda_{Ba}, \quad \bar{Q}^{Aa} V_{\alpha} = i (\sigma_{\alpha})_b^a \lambda_{Ab}, \quad \bar{Q}^{Aa} \phi = 0, \quad \bar{Q}^{Aa} \bar{\phi} = -\bar{\lambda}_{Aa}, \quad \bar{Q}^{Aa} \bar{\lambda}_{Bb} = - (\gamma_i)_A^B \epsilon_{ab} D^i \bar{\phi} - i \epsilon^{AB} (\sigma_{\alpha})_{ab} D^\alpha \bar{\phi}, \tag{A.10}
\]

In order to infer from (A.8)–(A.10) the structure of the cohomological \( N_T = 3, D = 8 \) SYM according to the decomposition \( [9] \) we have to replace \( (G_i, V_{\alpha}) \) and \( \lambda_{Aa} \) through \( A_{Aa} \) and \( (\chi_i, \eta_a) \), respectively, in an appropriate manner. To this end, we view \( \lambda_{Aa} \) and \( \bar{\lambda}_{Aa} \) in the Euclidean space as two independent 8–component spinors, so that they are no longer related by hermitean conjugation. Hence, hermiticity is abandoned — which is not a problem here since hermiticity is primarily necessary for the unitarity of the theory, and unitarity only makes sense for theories in Minkowski space.
Next, we express \( F_{ij}, F_{i\alpha} \) and \( F_{\alpha\beta} \) in terms of the field strength tensor \( F_{ABab} \) according to

\[
\begin{align*}
F_{ij} &= \frac{1}{4}(\gamma_{ij})_{AB} \epsilon_{ab} F^{ABab}, \\
F_{i\alpha} &= \frac{1}{4}(\gamma_i)_{AB}(i\sigma_{\alpha})_{ab} F^{ABab}, \\
F_{\alpha\beta} &= \frac{1}{4} \epsilon_{AB\epsilon_{\alpha\beta\gamma}(i\sigma_{\gamma})_{ab} F^{ABab}}, 
\end{align*}
\]

so that, by making use of the completeness relations

\[
(i\sigma_{\alpha})_{ab}(i\sigma_{\alpha})_{cd} = \Omega_{abcd}, \quad (\gamma_i)_{AB}(i^{\gamma})_{CD} = \Omega_{ABCD},
\]

where \( \Omega_{abcd} \) and \( \Omega_{ABCD} \) have been introduced in (4) and (6), respectively, we can split \( F_{ABab} \) into

\[
F_{ABab} = \frac{1}{4}(\gamma_{ij})_{AB} \epsilon_{ab} F^{ij} + \frac{1}{4}(\gamma_i)_{AB}(i\sigma_{\alpha})_{ab} (F^{i\alpha} - F^{\alpha i}) + \frac{1}{4} \epsilon_{AB\epsilon_{\alpha\beta\gamma}(i\sigma_{\gamma})_{ab} F^{\alpha\beta}} \tag{A.13}
\]

which corresponds to the \( Sp(4) \otimes Sp(2) \) decomposition \( 28 = (1, 3) \oplus (5, 3) \oplus (10, 1) \) of the adjoint representation of \( SO(8) \). Notice that the relative factors in (A.11) and (A.13) are the same. The relationships (A.11) and (A.13) immediately suggests how one has to carry out the above mentioned replacements.

Namely, from (A.8) we deduce that the cohomological action, with an underlying \( N_T = 3 \) equivariant shift symmetry, should be of the form

\[
S^{(N_T = 3)} \Big|_{Sp(4) \otimes Sp(2) \subset SO(8)} = \int d^4z d^4\bar{z} tr \left\{ \frac{1}{4} F^{ABab} F_{ABab} + 2 D^A \bar{\phi} D_A \phi - 2 \bar{\phi} \phi \right\} + 2 \bar{\lambda}^A (\gamma^i)_A^B D_B \chi_i + 2 i \bar{\lambda}^A (\sigma^a)_a^b D_B \eta_a \\
- 2 \bar{\phi} \{ \chi^i, \chi_i \} - 2 \bar{\phi} \{ \eta^a, \eta_a \} - 2 \bar{\phi} \{ \bar{\lambda}^A, \bar{\lambda}_A \} \right\} \tag{A.14}
\]

Moreover, splitting \( Q_{Aa} \) into \( (Q_i, Q_\alpha) \) from (A.9) and (A.10) we deduce that the supersymmetry transformations generated by \( Q_i, Q_\alpha \) and \( \bar{Q}_{Aa} \) should be taken as

\[
\begin{align*}
Q_\alpha A_a &= i(\sigma_{\alpha})_a^b \bar{\lambda}_{ab}, \\
Q_\alpha \bar{\lambda}_{Aa} &= i(\sigma_{\alpha})_a^b D_{Ab} \phi, \\
Q_\alpha \phi &= 0, \\
Q_\alpha \bar{\phi} &= \eta_\alpha, \\
Q_\alpha \eta_\beta &= -\frac{1}{4} i \epsilon_{\alpha\beta\gamma}(\sigma^{\gamma})_{ab} \epsilon_{AB} F^{AB}_{ab} + \delta_{\alpha\beta} \{ \phi, \bar{\phi} \}, \\
Q_\alpha \chi_i &= \frac{1}{4} i (\sigma_{\alpha})_{ab} (\gamma_i)_{AB} F^{AB}_{ab}, \tag{A.15}
\end{align*}
\]

\[
\begin{align*}
Q_i A_a &= -(\gamma_i)_A^B \bar{\lambda}_{Ba}, \\
Q_i \bar{\lambda}_{Aa} &= (\gamma_i)_A^B D_{Ba} \phi, \\
Q_i \phi &= 0, \\
Q_i \bar{\phi} &= \chi_i, \\
Q_i \chi_j &= -\frac{1}{4} (\gamma_{ij})_{AB} \epsilon_{AB} F^{AB}_{ab} + \delta_{ij} \{ \phi, \bar{\phi} \}, \\
Q_i \eta_\alpha &= -\frac{1}{4} i (\gamma_i)_{AB}(\sigma_{\alpha})_{ab} F^{AB}_{ab}, \tag{A.16}
\end{align*}
\]
and

\[ \begin{align*}
\tilde{Q}_{Aa} A_{Bb} & = i \epsilon_{AB} (\sigma_\alpha)_{ab} \eta^\alpha + (\gamma^i)_{AB} \epsilon_{ab} \chi_i, \\
\tilde{Q}_{Aa} \bar{\lambda}_{Bb} & = F_{ABab} + \frac{1}{4} (\gamma_i)_{AB} (\sigma_\alpha)_{ab} (\gamma^j)_{CD} (\sigma^\alpha)_{cd} F^{CDcd} - \epsilon_{AB} \epsilon_{ab} [\bar{\varphi}, \phi], \\
\tilde{Q}_{Aa} \phi & = -\bar{\lambda}_{Aa}, \\
\tilde{Q}_{Aa} \bar{\phi} & = 0, \\
\tilde{Q}_{Aa} \eta_a & = -i (\sigma_\alpha)_{ab} D_{Ab} \bar{\phi}, \\
\tilde{Q}_{Aa} \chi_i & = -i (\gamma_i)_{AB} \phi, \\
\end{align*} \]

respectively. After a straightforward, but tedious calculation one proves that, in fact, the above transformations leave the action (A.14) invariant.

Finally, we rename \( \lambda_{Aa} = \psi_{Aa} \) and introduce the fields \( \chi_{AB} \) and \( \eta_{ab} \) via \( \chi_{AB} = (\gamma_i)_{AB} \chi^i \) and \( \eta_{ab} = i (\sigma_\alpha)_{ab} \eta^\alpha \), respectively. Similarly, instead of \( Q_i \) and \( Q_\alpha \), we introduce the supercharges \( Q_{AB} \) and \( Q_{ab} \) via \( Q_{AB} = (\gamma_i)_{AB} Q^i \) and \( Q_{ab} = i (\sigma_\alpha)_{ab} Q^\alpha \), respectively.

In this manner, by virtue of (A.12), from (A.14) and (A.15)–(A.17) we arrive exactly at the action (10) together with the supersymmetry transformations (11)–(13).

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