Exact multi-soliton solutions in the four dimensional Skyrme model

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Exact analytic solutions of the four-dimensional Skyrme model defined on a simple spherically symmetric background (chosen to mimic finite volume effects) are presented. The static and spherically symmetric configurations have non-trivial winding number and finite soliton mass. These configurations possess an extra topological charge, allowing for a novel BPS bound which can actually be saturated, unlike what happens in the standard case. Such solutions include exact multi-Skyrmions of arbitrary (integer) winding number, composed by interacting elementary Skyrmions.

Skyrme’s theory \cite{1} is an important model in physics due to its wide range of applications. A remarkable property of the Skyrme term –introduced in order to support static finite energy soliton solutions known as Skyrmions– is that excitations around Skyrmion solitons may represent Fermionic degrees of freedom that could describe nucleons \cite{2 4}. Indeed, one of the most compelling results in this area was the identification of the winding number of the Skyrmion with the baryon number \cite{17}.

Despite these difficulties, many rigorous results about Skyrmion-BPS bound on the energy cannot be saturated \textsuperscript{\cite{16}}. In the present paper, exact spherically symmetric solitons of the Skyrme model with non-trivial spherically symmetric configurations have been derived, see for instance \textsuperscript{\cite{16} [23]}. The action of the \textit{SU}(2) Skyrme system in four dimensional spacetime is

$$S_{\text{Skyrme}} = \frac{K}{2} \int d^4x \sqrt{-g} \, \text{Tr} \left( \frac{1}{2} R^\mu_\nu R_\mu + \frac{\lambda}{16} F_{\mu\nu} F^{\mu\nu} \right),$$

where $R_\mu = U^{-1} \nabla_\mu U = R^t_i t_i$ and $F_{\mu\nu} := [R_\mu, R_\nu]$. Here $t_i$ are the \textit{SU}(2) generators and we set the units $\hbar = c = 1$. The coupling constants $K > 0$ and $\lambda > 0$ are fixed by comparison with experimental data \cite{2}. The presence of the first term of the Skyrme action \cite{1}, is mandatory to describe pions while the second is the only covariant term leading to second order field equations in time which supports the existence of Skyrmions in four dimensions.

In the present paper, exact spherically symmetric solutions of the Skyrme model with both a non-trivial winding number and a finite soliton mass (topological charge) are presented. Using the formalism introduced in \textsuperscript{\cite{24 25}}, it is shown that although the BPS bound in terms of the winding cannot be saturated, a new topological charge exists that can be saturated corresponding to a different BPS bound. The baryon number is the homotopy of the space into the group. The simplest choice would be to consider the curved background $S^3$ as physical space, as already considered in the pioneering papers \textsuperscript{\cite{24 25}}.

The second natural choice of special sections with integer homotopy into $SU(2)$ is $S^1 \times S^2$ (or $\mathbb{R} \times S^2$). This can be represented by a metric of the form

$$ds^2 = -dt^2 + dx^2 + R_0^2 (d\theta^2 + \sin^2 \theta \, d\phi^2).$$

In simple words, this geometry describes tridimensional cylinders whose sections are $S^2$ spheres. In principle, curvature effects can be as small as one may wish taking, e.g., $R_0$ much larger than the radius of the proton. However, the physical meaning of $R_0$ is that it takes into account finite volume effects. Indeed, one could put the Skyrme action into, say, a box. However, this way of proceeding often breaks symmetries. On the other hand, a spherical box of finite radius would lead to difficulties in requiring the Skyrmions approach the identity at the boundary. Therefore, it is much more convenient to choose a metric which at the same time takes into account finite volume effects and keeps the spherical symmetry. We are able construct exact Skyrmions in a finite volume but, instead of putting by hand a cut-off on the coordinates, we leave this task to the geometry. Besides, this geometry is such that the group of the isometries of \textsuperscript{\cite{2}} contains $SO(3)$ as a subgroup and so it includes the spherical symmetry of the Skyrmion in flat space. This fact allows examining how far is the BPS bound from being saturated and to construct an energy bound which can in fact be saturated. This could be of interest both in high energy and solid state physics whose features, after the papers \textsuperscript{\cite{24 25}}, have not been thoroughly investigated from the analytical viewpoint.

In order to construct the exact solution of the Skyrme model, the following standard parametrization of the $\mathfrak{su}(2)$-valued scalar $U(x^\mu)$ will be adopted,

$$U^{\pm 1}(x^\mu) = Y^0(x^\mu)1 \pm Y^i(x^\mu)t_i, \quad (Y^0)^2 + Y^i Y_i = 1. \quad (3)$$
Thus, the hedgehog ansatz describing a spherically symmetric Skyrmion in this background \cite{22} can be written in terms of the unit vectors

\[ Y^0 = \cos \alpha, \quad Y^i = \mathbf{n}^i \sin \alpha, \quad \alpha = \alpha(x), \quad (4) \]

and \( \mathbf{n}^1 = \sin \theta \cos \phi, \mathbf{n}^2 = \sin \theta \sin \phi \) and \( \mathbf{n}^3 = \cos \theta \). As it happens for the usual spherically symmetric Skyrmion in flat space, the energy-momentum tensor in the present case also corresponds to a spherically symmetric distribution, in spite of \( Y^0 \)'s and \( Y^i \)'s explicit angular dependence. The Skyrmie field equations of the action \( I \) are

\[ \nabla^\mu R_{\mu} + \frac{\lambda}{4} \nabla^\mu [R^\nu, F_{\mu\nu}] = 0. \quad (5) \]

Remarkably, for the ansatz \( \alpha \), the Skyrmie field equations reduce to one ordinary differential equation for the Skyrmion profile \( \alpha \):

\[ \left( 1 + \frac{2\lambda}{R^2_0} \sin^2 \alpha \right) \alpha'' - \frac{\sin(2\alpha)}{R^2_0} \left( 1 - \lambda \left( \frac{\alpha'}{R^2_0} \right)^2 - \sin^2 \alpha \right) = 0, \quad (6) \]

where \( \alpha' = \frac{d\alpha}{dx} \). This second order field equation \( (6) \) can be formally integrated as

\[ (\alpha')^2 = \left( I + \frac{1}{2} \right) + 2G(\alpha) \quad (7) \]

where \( I \) is an integration constant and we have defined

\[ F(\alpha) = 1 + \frac{2\lambda}{R^2_0} \sin^2 \alpha, \]

\[ G(\alpha) = \frac{\sin^2 \alpha}{R^2_0} \left( 1 + \frac{\lambda}{2R^2_0} \sin^2 \alpha \right). \quad (8) \]

Equation \( (7) \) is an exact analytic solution of the Skyrmie field equations in terms of a first integral. This expression is also appropriate to discuss the boundary conditions and, in particular, whether the Skyrmion has non-trivial winding number (see below). As will be shown next, this issue is closely related to the possibility of saturating certain BPS bound related to the energy.

The energy density is

\[ T_{00} = K \left[ F(\alpha)(\alpha')^2 + 2G(\alpha) \right] \quad (9) \]

\[ = \frac{KF}{2} \left[ \alpha' \pm \left( \frac{2G}{F} \right)^{1/2} \right]^2 \pm \sqrt{2K} |FG|^{1/2} \alpha', \]

from which an inequality for the total energy is found,

\[ E_{\text{tot}} = 4\pi R^2_0 \int_{T_{00}dx} \geq |Q|, \quad (10) \]

\[ Q = \sqrt{32\pi R^2_0 K} \int \left( |F(\alpha)G(\alpha)|^{1/2} \frac{d\alpha}{dx} \right) dx. \quad (11) \]

Clearly, \( Q \) is a boundary term and is therefore invariant under continuous deformations of the fields in the bulk, as suits a topological invariant. This inequality is saturated if \( \alpha \) satisfies the first-order differential equation

\[ \alpha' = \pm \left( \frac{2G(\alpha)}{F(\alpha)} \right)^{1/2}. \quad (12) \]

Remarkably, the saturation of the above bound coincides with equation \( (7) \) for \( I + \lambda^{-1} = 0 \).

The winding number for the generalized hedgehog ansatz in \( (4) \) is

\[ W = -\frac{1}{24\pi^2} \int \epsilon^{ijk} T_{tr}(U^{-1}\partial_i U)(U^{-1}\partial_j U)(U^{-1}\partial_k U) \]

\[ = \frac{2}{\pi} \int (\alpha' \sin^2 \alpha)dx = \frac{2}{\pi} \int_{\alpha(x_1)}^{\alpha(x_2)} \sin^2 \alpha d\alpha, \quad (13) \]

where \( (x_1, x_2) \) correspond to the limits in the range of spatial direction, that can be taken as \((-L/2, L/2)\) or \((-\infty, \infty)\). This winding number takes integer values \( n \), for boundary conditions

\[ \alpha(x_2) - \alpha(x_1) = n\pi, \quad n \in \mathbb{Z}. \quad (14) \]

These boundary conditions are unique in that they ensure \( U(x_2) = (-1)^n U(x_1) \), which correspond to bosonic and fermion states for even and odd \( n \), respectively. Moreover, it can be seen that the topological charge \( Q \) is bounded from below, \(|Q| > |W|\), as it should be \( [3] \).

We will prove that smooth solutions exists for any \( n \) satisfying the above boundary conditions for a finite range, e.g., \((-L/2, L/2)\). Such solutions will depend on the value of the integration constant \( I \) in \( (7) \).

When the bound is saturated, the profile of \( \alpha \) and its corresponding energy density \( T_{00} \), are depicted in Fig. 1. In this case, the \( x \) coordinate ranges from \(-\infty\) to \( \infty \) and the winding number is \( n = \pm 1 \), so that the ground state is Fermionic.

![FIG. 1: Profile of \( \alpha \) (solid black line) and \( T_{00} \) (dashed red line) for the ground state \( I = -1/\lambda \). The energy density is rescaled.](image_url)
dependence of the integration constant $I$ on the winding number $n$ is determined by the following condition

$$\frac{L}{n} = \pm \int_0^{\pi} \sqrt{\frac{F(\alpha)}{I + 1/\lambda + 2G(\alpha)}} d\alpha \quad (15)$$

Thus, it can be seen that this equation for $I$ has always a solution (in particular when the winding number is very large $I$ is proportional to $n^2$). The nature of the multi-Skyrmions solutions presented there is depicted in Fig. 2. It is worth to emphasize that such solutions do not saturate the bound. The multi-Skyrmion are not a simple linear superposition of single Skyrmions, since they include strong repulsive interactions among the elementary Skyrmions. This can be seen from the expression for the total energy as a function of $n$. In particular, in the limit of large $n$ and fixed $L$, the energy of the multi-Skyrmions grows as $n^2$, as shown in Fig. 3. An intriguing feature of the present multi-Skyrmionic configurations is that the elementary Skyrmions composing them are self-arranged. Namely, they are placed equidistantly along the tube (as is represented in Fig. 2). Hence, in the case in which the tube is closed the configurations look like necklaces of elementary Skyrmions. The tendency of the Skyrmions to follow ordered pattern has been investigated numerically in refs. [28]. Recently, the same observation was done analytically in [23] for configurations with trivial winding number. To the best of our knowledge, the results presented here are the first exact multi-solitons of the four-dimensional Skyrme model with these features.

In closing, the following comments are in order:

1. Within the family of solutions in eq. (7) obtained here, the case $I = -1/(4\lambda)$ is exceptional. For this value the profile obeys

$$(\alpha')^2 = \frac{3}{4\lambda^2} + \frac{2}{R_0^2} \sin^2 \alpha + \frac{\lambda}{R_0^2} \sin^4 \alpha \quad (16)$$

which reduces exactly to a sine-Gordon equation after differentiation, $\alpha'' = -2R_0^{-1} \sin 2\alpha = 0$. Thus, the solution can be written as

$$\alpha(x) = \text{am} \left( \frac{\sqrt{2}}{\sqrt{\lambda}} x, i \sqrt{\frac{2\lambda}{3R_0^2}} \right)$$

where the Jacobi amplitude function $\text{am}(u, k)$ is related with the Jacobi elliptic functions $\text{sn}(x, k)$ and $\text{cn}(x, k)$ by the relations $\text{sn}(x, k) = \sin \text{am}(x, k)$ and $\text{cn}(x, m) = \cos \text{am}(x, k)$, where the modulus parameter $k$ is defined in the range $0 < k < 1$. The parameters of the solution are related through relation (15) which $L = \frac{2\sqrt{\lambda}}\sqrt{\alpha} F \left( n\pi, i \sqrt{\frac{2\lambda}{3R_0^2}} \right)$. Here $F(\phi, k)$ is the incomplete elliptic integral of the first kind which inverse is the Jacobi amplitude function. In this case the group element $U$ takes the form

$$U^{-1} = \text{cn} \left( \frac{\sqrt{2}}{\sqrt{\lambda}} x, i \sqrt{\frac{2\lambda}{3R_0^2}} \right) \text{sn} \left( \frac{\sqrt{2}}{\sqrt{\lambda}} x, i \sqrt{\frac{2\lambda}{3R_0^2}} \right)^{-1}$$

Note that as required by Eq. (16), the Jacobi elliptic functions satisfy the relation $\text{cn}^2(x, k) + \text{sn}^2(x, k) = 1$.

2. It is clear that the same construction can also be carried out if one includes a mass term in the Skyrme action. The only change in eq. (7), is the replacement of the function $G(\alpha) \rightarrow G(\alpha) + m^2 \cos \alpha$.

3. These multi-Skyrmions live on a curved background which allows to explore finite-size effects. This is an important technical point in order to compute thermodynamic quantities such as the compressibility modulus, for which the explicit dependence of the total energy of the multi-Skyrmions of winding number $n$ on the volume is needed. In the present case, this can be obtained by using eq. (7), which allows to changes the integration variable in (10) for $E_{tot}$ from the $dx$ to $d\alpha$, so that the total energy reads

$$E_{tot} = 2\pi k R_0^2 \left( (I + \lambda^{-1}) L + 4n \int_0^{\pi} \frac{GF^{1/2}}{\sqrt{I + \lambda^{-1} + 2G}} d\alpha \right),$$
which determines its dependence both on the volume and on the winding number.

4. Another interesting application is to consider the collective coordinate approach in order to identify quantum ground state, following the ideas in ref. [7]. The eigenvalues of the corresponding quantum Hamiltonian take the form

$$E_{\ell} = M + \frac{1}{8\Lambda} \ell (\ell + 2),$$

where $\ell$ are positive integer numbers and

$$M = 2\pi K R_0^2 \int_{-\infty}^{\infty} \left[ \alpha'^2 + \frac{2\sin^2 \alpha}{R_0^2} + \lambda \frac{\sin^2 \alpha}{R_0^2} \left( 2\alpha'^2 + \frac{\sin^2 \alpha}{R_0^2} \right) \right] dx,$$

$$\Lambda = \frac{8\pi K R_0^2}{3} \int_{-\infty}^{\infty} \sin^2 \alpha \left\{ 1 + \lambda \left( \alpha'^2 + \frac{\sin^2 \alpha}{R_0^2} \right) \right\} dx.$$  

In this way, it is possible to fit the neutron ($M_N = 939$ MeV) and delta ($M_\Delta = 1232$ MeV) masses. Following the conventions of [7] one finds,

$$F_\pi = 141 \text{ MeV}, \quad e = 5.45, \quad R_0 = 0.65 \text{ fm}, \quad (17)$$

where $F_\pi$ and $e$ are related to the constant of the action $\mathcal{A}$ as

$$K = \frac{1}{4} F_\pi^2, \quad K\lambda = \frac{1}{e^2}. \quad (18)$$

The comparison with experiments as well as other phenomenological applications will be discussed in greater depth in [31].

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