Continuity of composition operators in Sobolev spaces

Gérard Bourdaud & Madani Moussai

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Abstract

We prove that all the composition operators $T_f(g) := f \circ g$, which take the Adams-Frazier space $W^m_p \cap \dot{W}^{1/p}_m(R^n)$ to itself, are continuous mappings from $W^m_p \cap \dot{W}^{1/p}_m(R^n)$ to itself, for every integer $m \geq 2$ and every real number $1 \leq p < +\infty$. The same automatic continuity property holds for Sobolev spaces $W^m_p(R^n)$ for $m \geq 2$ and $1 \leq p < +\infty$.

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1 Introduction

We want to establish the so-called automatic continuity property for composition operators in classical Sobolev spaces, i.e. the following statement:

Theorem 1 Let us consider an integer $m > 0$, and $1 \leq p < +\infty$. If $f : \mathbb{R} \to \mathbb{R}$ is a function s.t. the composition operator $T_f(g) := f \circ g$ takes $W^m_p(R^n)$ to itself, then $T_f$ is a continuous mapping from $W^m_p(R^n)$ to itself.

This theorem has been proved

- for $m = 1$, see [5] in case $p = 2$, and [22] in the general case,
- for $m > n/p$, $m > 1$ and $p > 1$ [14].

It holds also trivially in the case of Dahlberg degeneracy, i.e. $1 + (1/p) < m < n/p$, see [19]. It does not hold in case $m = 0$, see Section 2 below. Thus it remains to be proved in the following cases:

- $m = 2$, $p = 1$, and $n \geq 3$.
- $m = n/p > 1$ and $p > 1$.
- $m \geq \max(n, 2)$ and $p = 1$.

If we except the space $W^2_1(R^n)$, all the Sobolev spaces under consideration are particular cases of the Adams-Frazier spaces, or of the Sobolev algebras. We will prove the automatic continuity for those spaces, and for their homogeneous counterparts, conveniently realized. Contrarily to the case $m = 1$, where the proof of continuity of $T_f$ is much more difficult for $p = 1$, see [22, p. 219], our proof in case $m \geq 2$ will cover all values of $p \geq 1$. 
Plan - Notation

In Section 2 we recall the classical result on the continuity of $T_f$ in $L_p$ spaces. We take this opportunity to correct some erroneous statement in the literature. In Section 3 we recall the characterization of composition operators acting in inhomogenous and homogeneous Adams-Frazier spaces, and in Sobolev algebras. In Section 4 we explain the specific difficulties concerning the continuity of $T_f$ in homogeneous spaces, which can be partially overcome by using realizations. Section 5 is devoted to the proof of the continuity of $T_f$.

We denote by $N$ the set of all positive integers, including 0. All functions occurring in the paper are assumed to be real valued. We denote by $P_k$ the set of polynomials on $\mathbb{R}^n$, of degree less or equal to $k$. If $f$ is a function on $\mathbb{R}^n$, we denote by $[f]_k$ its equivalence class modulo $P_k$. We consider a classical mollifiers sequence $\theta_{\nu}(x) := \nu^n \theta(\nu x)$, $\nu \geq 1$, where $\theta \in \mathcal{D}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \theta(x) \, dx = 1$. For all $N \in \mathbb{N}$, we denote by $C^N_b(\mathbb{R}^n)$ the space of functions $f : \mathbb{R} \to \mathbb{R}$, with continuous bounded derivatives up to order $N$. In all the paper, $m$ is an integer $> 1$ and the real number $p$ satisfies $1 \leq p < +\infty$. $W_p^m(\mathbb{R}^n)$ and $\dot{W}_p^m(\mathbb{R}^n)$ are the classical inhomogeneous and homogeneous Sobolev spaces, endowed with the norms and seminorms

$$
\|g\|_{W_p^m} := \sum_{|\alpha| \leq m} \|g^{(\alpha)}\|_p, \quad \|g\|_{\dot{W}_p^m} := \sum_{|\alpha| = m} \|g^{(\alpha)}\|_p,
$$

respectively. For topological spaces $E, F$, the symbol $E \hookrightarrow F$ means an imbedding, i.e.: $E \subseteq F$ and the natural mapping $E \to F$ is continuous.

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2 The case of $L_p$

In a recent survey paper on composition operator, the first author said that all the composition operators acting in $L_p(\mathbb{R}^n)$ are continuous (see [16], in particular the first line of the tabular at page 123). This assertion is erroneous. Indeed $T_f$ takes $L_p(\mathbb{R}^n)$ to itself iff $|f(t)| \leq c |t|$ for some constant $c$, see [6 thm. 3.1]. Clearly this property does not imply the continuity of $f$ outside of 0. Instead we have the following:

**Proposition 1** Let $(X, \mu)$ be a measure space. Let $f : \mathbb{R} \to \mathbb{R}$ be s.t. $T_f$ takes $L_p(X, \mu)$ to itself. Then $T_f$ is continuous from $L_p(X, \mu)$ to itself iff $f$ is continuous.

**Proof.** Let us assume that $T_f$ is continuous on $L_p$. Without loss of generality, assume that $f(0) = 0$. Let $A$ be a measurable set in $X$ s.t. $0 < \mu(A) < +\infty$. For all real numbers $u, v$, it holds

$$
\|f \circ u\chi_A - f \circ v\chi_A\|_p = |f(u) - f(v)| \mu(A)^{1/p}.
$$

(1)

Clearly $v \to u$ in $\mathbb{R}$ implies $v\chi_A \to u\chi_A$ in $L_p$. By the continuity of $T_f$, and by (1), we obtain the continuity of $f$. For the reverse implication, we refer to [6 thm. 3.7]. We can also use the following statement:

**Proposition 2** Assume $q \in [1, +\infty[$. Let $(X, \mu)$ be a measure space. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function s.t. for some constant $c > 0$, it holds $|f(t)| \leq c |t|^{p/q}$, for all $t \in \mathbb{R}$. Then $T_f$ is a continuous mapping from $L_p(X, \mu)$ to $L_q(X, \mu)$.

2
The inhomogeneous and homogeneous Adams-Frazier spaces are defined as follows:

3.1 Function spaces

Adams-Frazier spaces and related spaces

By Dominated Convergence Theorem, we conclude that a classical measure theoretic result

By the continuity of \( f \), it holds \( f \circ g_{v_k} \to f \circ g \) a.e.. By assumption on \( f \), it holds

By Dominated Convergence Theorem, we conclude that \( \|f \circ g_{v_k} - f \circ g\|_q \) tends to 0.

3 Adams-Frazier spaces and related spaces

3.1 Function spaces

The inhomogeneous and homogeneous Adams-Frazier spaces are defined as follows:

\[
A^m_p(\mathbb{R}^n) := W^m_p \cap \dot{W}^1_{mp}(\mathbb{R}^n), \quad A^m_p(\mathbb{R}^n) := \dot{W}^m_p \cap \dot{W}^{1}_{mp}(\mathbb{R}^n).
\]

Both spaces are endowed with their natural norm and seminorm:

\[
\|f\|_{A^m_p} := \|f\|_{W^m_p} + \|f\|_{\dot{W}^1_{mp}}, \quad \|f\|_{\dot{A}^m_p} := \|f\|_{\dot{W}^m_p} + \|f\|_{\dot{W}^{1}_{mp}}.
\]

The pertinency of those spaces w.r.t. composition operators was first noticed in [1], see also the Introduction of [11]. By Sobolev imbedding, it holds \( W^m_p(\mathbb{R}^n) \hookrightarrow W^1_{mp}(\mathbb{R}^n) \), in case \( m \geq n/p \), hence

\[ m \geq n/p \quad \Rightarrow \quad A^m_p(\mathbb{R}^n) = W^m_p(\mathbb{R}^n). \quad (2) \]

In particular the critical Sobolev spaces \( W^m_p(\mathbb{R}^n) \), \( m = n/p \), are Adams-Frazier spaces.

The intersections \( \dot{W}^m_p \cap L_{\infty}(\mathbb{R}^n) \) and \( W^m_p \cap L_{\infty}(\mathbb{R}^n) \) are subalgebras of \( L_{\infty}(\mathbb{R}^n) \) for the usual pointwise product, see Remark 2 below. We call them the homogeneous and inhomogeneous Sobolev algebras, and we endow them with their natural norms. Let us notice that the second is a proper subspace of the first, since the nonzero constant functions do not belong to \( W^m_p(\mathbb{R}^n) \). By the Gagliardo-Nirenberg inequalities, see e.g. [11] (6), p. 6108], we have the imbeddings

\[ \dot{W}^m_p \cap L_{\infty}(\mathbb{R}^n) \hookrightarrow A^m_p(\mathbb{R}^n), \quad W^m_p \cap L_{\infty}(\mathbb{R}^n) \hookrightarrow A^m_p(\mathbb{R}^n). \quad (3) \]

In particular \( W^m_p(\mathbb{R}^n) \) coincides with the corresponding Sobolev algebra if \( m > n/p \), or \( m = n \) and \( p = 1 \). The following statement characterizes the Adams-Frazier spaces which coincide with the corresponding Sobolev algebras:

**Proposition 3** 1- The inclusion \( A^m_p(\mathbb{R}^n) \subset L_{\infty}(\mathbb{R}^n) \) holds iff \( m > n/p \), or \( m = n \) and \( p = 1 \).

2- The inclusion \( A^m_p(\mathbb{R}^n) \subset L_{\infty}(\mathbb{R}^n) \) holds iff \( m = n \) and \( p = 1 \).

\[ ^1 \text{See the proof of the completeness of } L_p, \text{ e.g. [25] thm. 3.11, p. 67} \]
Proof. 1- In case $m \geq n/p$, it suffices to apply [2] and the known properties of Sobolev spaces. Now assume $m < n/p$. Define $f_\lambda(x) := |x|^{\lambda} \varphi(x)$, where $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi(x) = 1$ near 0. Then $f_\lambda \in W^m_p(\mathbb{R}^n)$ iff $\lambda > m - (n/p)$. If we take $1 - \frac{n}{mp} < \lambda < 0$, we obtain $f_\lambda \in A^m_p(\mathbb{R}^n)$ but $f_\lambda \notin L_\infty(\mathbb{R}^n)$.

2- In case $m < n/p$, or $m = n/p$ and $p > 1$, the first part implies a fortiori $\dot{A}^m_p(\mathbb{R}^n) \subseteq L_\infty(\mathbb{R}^n)$.

Now assume $m > n/p$. Define $f_\lambda(x) := |x|^{\lambda} (1 - \varphi(x))$, with the same function $\varphi$ as above. Then $f_\lambda \in \dot{W}^m_p(\mathbb{R}^n)$ iff $\lambda < m - (n/p)$. If we take $0 < \lambda < 1 - \frac{n}{mp}$, we obtain $f_\lambda \in \dot{A}^m_p(\mathbb{R}^n)$ but $f_\lambda \notin L_\infty(\mathbb{R}^n)$.

If $f \in \dot{W}^1_n(\mathbb{R}^n)$, there exists $g \in C_0(\mathbb{R}^n)$ s.t. $f - g \in \mathcal{P}_{n-1}$, see [12] thm. 3. If moreover $f \in \dot{W}^1_n(\mathbb{R}^n)$, one proves easily that $f - g$ is a constant. Thus we obtain the inclusion $\dot{A}^1_1(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$.

Remark 1 The above proof shows also that $W^2_1(\mathbb{R}^n)$ does not coincide with $A^2_1(\mathbb{R}^n)$ in case $n \geq 3$: if we consider the function $f_\lambda$ of the above first part, with $2 - n < \lambda < 1 - \frac{n}{2}$, then $f_\lambda \in W^2_1(\mathbb{R}^n)$, but $f_\lambda \notin \dot{W}^2_1(\mathbb{R}^n)$.

3.2 Uniform localization

Let us introduce a function $\psi \in \mathcal{D}(\mathbb{R})$, positive, s.t.

$$\sum_{\ell \in \mathbb{Z}} \psi(x - \ell) = 1 \quad \text{for all } x \in \mathbb{R}.$$  

If $f$ is any distribution in $\mathbb{R}$, we set $f_\ell(x) := f(x) \psi(x - \ell)$, and we call the decomposition $f = \sum_{\ell \in \mathbb{Z}} f_\ell$ a locally uniform decomposition (abbreviated as LU-decomposition) of $f$. If $E$ is a normed space of distributions in $\mathbb{R}$, we denote by $E_{lu}$ the set of distributions $f$ such that

$$\|f\|_{E_{lu}} := \sup_{\ell \in \mathbb{Z}} \|f_\ell\|_E < +\infty.$$  

Under standard assumptions on $E$, it is known that the space $E_{lu}$ does not depend on the choice of the function $\psi$. We refer to [9, 2, 3] for details. In case $E = L_p(\mathbb{R})$, $E_{lu}$ is denoted by $L_{p,lu}(\mathbb{R})$. For $N \in \mathbb{N}$, let us denote by $W^N_{L_{p,lu}}(\mathbb{R})$ the homogeneous Sobolev space based upon $L_{p,lu}(\mathbb{R})$, i.e. the set of functions $f$ s.t. $f^{(N)} \in L_{p,lu}(\mathbb{R})$. The inhomogeneous Sobolev space $W^N_{L_{p,lu}}(\mathbb{R})$ is defined similarly. Both spaces are endowed with their natural norm and seminorm:

$$\|f\|_{W^N_{L_{p,lu}}} := \|f^{(N)}\|_{L_{p,lu}}, \quad \|f\|_{W^N_{L_{p,lu}}} := \sum_{j=0}^N \|f^{(j)}\|_{L_{p,lu}}.$$  

Let us recall that $W^N_{L_{p,lu}}(\mathbb{R})$ coincides with $(W^N_p(\mathbb{R}))_{lu}$, with equivalent norms. As a consequence, we have the following property:
Proposition 4 Let $N \in \mathbb{N}$. For all compact interval $I$ of $\mathbb{R}$, there exists a constant $c = c(N, p, I) > 0$ s.t.

$$\left\| \sum_{\ell \in \mathbb{Z}} g_\ell \right\|_{W^N_{p, I, I}} \leq c \sup_{\ell \in \mathbb{Z}} \| g_\ell \|_{W^N_p} ,$$

for all sequence $(g_\ell)_{\ell \in \mathbb{Z}}$ s.t. the function $g_\ell$ is supported by the translated interval $I + \ell$, for all $\ell \in \mathbb{Z}$.

Proof. See [9 lem.1].

Proposition 5 For any $N \in \mathbb{N}$, $C^\infty \cap W^N_{L, p, I_u} (\mathbb{R})$ is a dense subspace of $W^N_{L, p, I_u} (\mathbb{R})$.

Proof. We use the mollifiers sequence $(\theta_\nu)_{\nu \geq 1}$ introduced in Notation, with $n = 1$. Let $f \in W^N_{L, p, I_u} (\mathbb{R})$ and $\varepsilon > 0$. Let $f = \sum_{\ell \in \mathbb{Z}} f_{\ell}$ be the LU-decomposition of $f$. Let us set

$$v(x) := \sum_{\ell \in \mathbb{Z}} \theta_{j_\ell} * f_{\ell} ,$$

for some convenient sequence $(j_\ell)_{\ell \in \mathbb{Z}}$. First we observe that the sum which defines the function $v$ is locally finite. Indeed $\theta_{j_\ell} * f_{\ell}$ is supported by $I + \ell$, for some fixed compact interval $I$. As a consequence, $v$ is a $C^\infty$ function. By condition $p < \infty$ and by classical properties of Sobolev spaces, we can choose $j_\ell$ s.t. $\| \theta_{j_\ell} * f_{\ell} - f_{\ell} \|_{W^N_p} \leq \varepsilon$. By Proposition 4 we deduce

$$\| v - f \|_{W^N_{L, p, I_u}} \leq c \sup_{\ell \in \mathbb{Z}} \| \theta_{j_\ell} * f_{\ell} - f_{\ell} \|_{W^N_p} \leq c \varepsilon .$$

This ends up the proof.

Proposition 6 For any integer $N > 0$, it holds $W^N_{L, p, I_u} (\mathbb{R}) = \dot{W}^N_{L, p, I_u} \cap L_\infty (\mathbb{R}) \hookrightarrow C^{N-1}_b (\mathbb{R})$.

Proof. 1- Let $f = \sum_{\ell \in \mathbb{Z}} f_{\ell}$ be the LU-decomposition of $f \in W^N_{L, p, I_u} (\mathbb{R})$. By condition $N \geq 1$, we have the Sobolev imbedding $W^N_p (\mathbb{R}) \hookrightarrow C_b (\mathbb{R})$, hence

$$\sup_{\ell \in \mathbb{Z}} \| f_{\ell} \|_\infty \leq c_1 \sup_{\ell \in \mathbb{Z}} \| f_{\ell} \|_{W^N_p} \leq c_2 \| f \|_{W^N_{L, p, I_u}} .$$

By considering the support of $f_{\ell}$, we conclude that $f \in C_b (\mathbb{R})$.

2- Let $f \in \dot{W}^N_{L, p, I_u} \cap L_\infty (\mathbb{R})$. By [10 lem. 1], it holds $f \in W^{N-1}_\infty (\mathbb{R})$. A fortiori, it holds $f^{(j)} \in L_{p, I_u} (\mathbb{R})$ for all $j = 0, \ldots, N - 1$. Hence $f \in W^N_{L, p, I_u} (\mathbb{R})$.

3- If $f \in W^N_{L, p, I_u} (\mathbb{R})$, then $f \in W^N_p (\mathbb{R})_{\text{loc}}$. By Sobolev imbedding, it follows that $f \in C^{N-1}_b (\mathbb{R})$.

3.3 Composition operators in Adams-Frazier spaces

Proposition 7 The composition operator $T_f$ takes $A^m_p (\mathbb{R}^n)$ to itself if $f' \in W^{m-1}_{L, p, I_u} (\mathbb{R})$ and $f(0) = 0$. Moreover, for some constant $c = c(n, m, p) > 0$, it holds

$$\| f \circ g \|_{A^m_p} \leq c \| f' \|_{W^{m-1}_{L, p, I_u}} (1 + \| g \|_{A^m_p})^m ,$$

for all $g \in A^m_p (\mathbb{R}^n)$.
**Proposition 8** The composition operator \( T_f \) takes \( \dot{A}_p^n(R^n) \) to itself if \( f' \in W^{m-1}_{L_p,iu}(R^n) \). Moreover, the estimation (4) holds true with \( A \) replaced by \( \dot{A} \).

We refer to [11 thm. 1, p. 6107], [16 thm. 25] and to Proposition 8. The condition \( f' \in W^{m-1}_{L_p,iu}(R^n) \) is not only sufficient, but also necessary, for all Adams-Frazier spaces which are not included into \( L_\infty(R^n) \). For such spaces, Propositions 7 and 8 constitute characterizations of the acting composition operators. For spaces imbedded into \( L_\infty(R^n) \), we have the following alternative result:

**Theorem 2** The operator \( T_f \) takes \( \dot{W}^m_p \cap L_\infty(R^n) \) to itself iff \( f \in W^m_p(R^n)_{loc} \), and \( W^m_p \cap L_\infty(R^n) \) to itself iff \( f \in W^m_p(R^n)_{loc} \) and \( f(0) = 0 \).

**Proof.** This statement is classical, see [10] and [16 thm. 2]. We recall here the part of the proof which will be useful to prove the continuity of \( T_f \). For every \( r > 0 \), we denote by \( B_r \) the ball of center 0 and radius \( r \) in \( \dot{W}^m_p \cap L_\infty(R^n) \). It will suffice to prove that \( T_f \) takes \( B_r \) to \( \dot{W}^m_p \cap L_\infty(R^n) \), for all \( r > 0 \). We introduce a family of auxiliary functions \( \omega_r \in \mathcal{D}(R) \) s.t. \( \omega_r(t) = 1 \) for \( |t| \leq r \). Then

\[
\forall g \in B_r, \quad T_f(g) = T_{f\omega_r}(g). \tag{5}
\]

If \( f \in W^m_p(R^n)_{loc} \), it is clear that \( (f \omega_r)' \in W^{m-1}_{L_p,iu}(R^n) \). We can apply Proposition 8 to \( f \omega_r \). By (3), we conclude that \( T_{f\omega_r} \) takes \( \dot{W}^m_p \cap L_\infty(R^n) \) to \( \dot{W}^m_p(R^n) \). By condition \( m \geq 1 \), the Sobolev imbedding yields \( f \omega_r \in C_b(R) \), and we obtain that \( T_{f\omega_r} \) takes \( \dot{W}^m_p \cap L_\infty(R^n) \) to \( L_\infty(R^n) \). This ends up the proof.

**Remark 2** In particular, any function of class \( C^m \) acts on \( \dot{W}^m_p \cap L_\infty(R^n) \) by composition. Applying this to the function \( f(t) := t^2 \), we derive immediately the algebra property.

For \( n = 1, 2 \), the space \( W^2_1(R^n) \) is a Sobolev algebra, for which the acting composition operators are described in Theorem 2. In the other cases, we have the following result (see [10]):

**Theorem 3** Assume that \( n \geq 3 \). Then \( T_f \) takes \( W^2_1(R^n) \) to itself iff \( f(0) = 0 \) and \( f'' \in L_1(R) \). Moreover there exists \( c = c(n) > 0 \) s.t.

\[
\| f \circ g \|_{W^2_1} \leq c (\| f'(0) \| + \| f'' \|_1) \| g \|_{W^2_1}, \tag{6}
\]

for all such \( f \)'s and all \( g \in W^2_1(R^n) \).

### 4 Homogeneous spaces and their realizations

Usually, an homogeneous function space \( F \), such as \( \dot{W}^m_p(R^n) \), is only a seminormed space, with \( \| f \| = 0 \) iff \( f \in \mathcal{P}_k \), for some \( k \in \mathbb{N} \) depending on \( F \). The presence of polynomials, with a seminorm equal to 0, has some pathological effects on composition operators. Recall, for instance, the following (see [11 prop. 11]):

**Proposition 9** If \( m > 1 \) and \( n > 1 \), the only functions \( f \), for which \( T_f \) takes \( \dot{W}^m_p(R^n) \) to itself, are the affine ones.
This degeneracy phenomenon does not occur in homogeneous Adams-Frazier spaces, see Proposition 8. However, the continuity of $T_f$ is a tricky question. The statement: “$T_f$ is continuous as a mapping of the seminormed space $\dot{A}_p^m(\mathbb{R}^n)$ to itself” makes sense, but it has no chance to be true. Assume that, for a sequence $(g_\nu)$ tending to $g$ in $\dot{A}_p^m(\mathbb{R}^n)$, the sequence $(f \circ g_\nu)$ tends to $f \circ g$ in $\dot{A}_p^m(\mathbb{R}^n)$. Then, for a sequence $(c_\nu)$ of real numbers, the sequence $(g_\nu + c_\nu)$ tends also to $g$ in $\dot{A}_p^m(\mathbb{R}^n)$. But the sequence $(f \circ g_\nu + c_\nu) - f \circ g)$ cannot tend to $0$ in $\dot{A}_p^m(\mathbb{R}^n)$, whatever be the sequence $(c_\nu)$. If, for instance, $f(t) := \sin t$, $g$ is a nonzero function in $\mathcal{D}(\mathbb{R}^n)$, and $g_\nu := g$ for all $\nu \in \mathbb{N}$, then $f \circ (g + \pi) - f \circ g = -2f \circ g$, a function which is not constant.

In order to avoid the disturbing effect of polynomials, two ideas seems available. The first one would be to consider the factor space $F/\mathcal{P}_k$. But that does not work. Indeed, if $g_1$ and $g_2$ differ by a polynomial, the same does not hold for $f \circ g_1$ and $f \circ g_2$, hence we cannot extend the operator $T_f$ to the factor space. The second one consists in restricting $T_f$ to a vector subspace $E$ s.t. $F = E \oplus \mathcal{P}_k$. We will exploit this idea in case of Adams-Frazier spaces $\dot{A}_p^m(\mathbb{R}^n)$, and the space $\dot{W}_1^2(\mathbb{R}^n)$.

### 4.1 Realizations of homogeneous Adams-Frazier spaces

Let us begin with a remark:

**Proposition 10**

1. The factor space $\dot{A}_p^m(\mathbb{R}^n)/\mathcal{P}_0$, endowed with the norm $\| - \|_{\dot{A}_p^m}$, is a Banach space.

2. Any subspace $E$ of $\dot{A}_p^m(\mathbb{R}^n)$ s.t.

$$\dot{A}_p^m(\mathbb{R}^n) = E \oplus \mathcal{P}_0$$

is a Banach space for the norm $\| - \|_{\dot{A}_p^m}$.

**Proof.** 1- It is well known that $\dot{W}_1^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ is a Banach space if endowed with the norm $\| - \|_{\dot{W}_1^m}$; see [23] 1.1.13, thm. 1. If $([g_\nu])$ is a Cauchy sequence in $\dot{A}_p^m(\mathbb{R}^n)/\mathcal{P}_0$, then there exist $u \in \dot{W}_1^m(\mathbb{R}^n)$ and $v \in \dot{W}_1^{mp}(\mathbb{R}^n)$ s.t. $(g_\nu)$ tends to $u$ in $\dot{W}_1^m(\mathbb{R}^n)$ and to $v$ in $\dot{W}_1^{mp}(\mathbb{R}^n)$. By [23] 1.1.13, thm. 2], there exist a sequence $(r_\nu)$ in $\mathcal{P}_{m-1}$ and a sequence $(c_\nu)$ in $\mathcal{P}_0$, s.t. $(g_\nu - r_\nu)$ tends to $u$, and $(g_\nu - c_\nu)$ to $v$, in $L_1(\mathbb{R}^n)_{loc}$. Thus $(r_\nu - c_\nu)$ is a sequence in $\mathcal{P}_{m-1}$, which tends to $v - u$ in $L_1(\mathbb{R}^n)_{loc}$. We conclude that $v - u \in \mathcal{P}_{m-1}$, and that $([g_\nu])$ tends to $[v]_0$ in $\dot{A}_p^m(\mathbb{R}^n)/\mathcal{P}_0$.

2- If $E$ is a subspace of $\dot{A}_p^m(\mathbb{R}^n)$ s.t. $[\mathbb{I}]$, the linear map $f \mapsto [f]_0$ is an isometry from $E$ onto $\dot{A}_p^m(\mathbb{R}^n)/\mathcal{P}_0$. The completeness of $E$ follows by the first part. This ends up the proof.

A subspace $E$ satisfying $[\mathbb{I}]$ will be of interest only if it is a Banach space of distributions. This motivates the following definition:

**Definition 1** A subspace $E$ of $\dot{A}_p^m(\mathbb{R}^n)$ s.t. $[\mathbb{I}]$ is called a realization of $\dot{A}_p^m(\mathbb{R}^n)$ if one of the following equivalent properties holds:

1. the inclusion mapping $E \to \mathcal{S}'(\mathbb{R}^n)$ is continuous;
2. the inclusion mapping $E \rightarrow L_1(\mathbb{R}^n)_{\text{loc}}$ is continuous;

3. for every sequence $(g_\nu)$ tending to $g$ in $E$, there exists a subsequence $(g_{\nu_k})$ s.t. $g_{\nu_k} \rightarrow g$ a.e..

The equivalence between the three properties follows easily by the Closed Graph Theorem.

**Remark 3** In [11], we used a slightly weaker definition for a realization of $\dot{W}^{m}_p(\mathbb{R}^n)$. We said that a subspace $E$ of $\dot{W}^{m}_p(\mathbb{R}^n)$ is a realization if

$$
\dot{W}^{m}_p(\mathbb{R}^n) = E \oplus \mathcal{P}_{m-1}.
$$

If (8) holds we obtain a linear mapping $\sigma : \dot{W}^{m}_p(\mathbb{R}^n)/\mathcal{P}_{m-1} \rightarrow \mathcal{S}'(\mathbb{R}^n)$ s.t.

$$
\forall u \in \dot{W}^{m}_p(\mathbb{R}^n)/\mathcal{P}_{m-1} \quad [\sigma(u)]_{m-1} = u,
$$

and whose range is $E$. Then $\sigma$ is a realization, in the sense of [8, 12, 13], if $\sigma$ is a continuous mapping from $\dot{W}^{m}_p(\mathbb{R}^n)/\mathcal{P}_{m-1}$ to $\mathcal{S}'(\mathbb{R}^n)$: this is precisely what means Definition 1.

Now we turn to the description of the usual realizations of $\dot{A}^{m}_p(\mathbb{R}^n)$. Except in case $m = n$, $p = 1$, it will suffice to realize $\dot{W}^{1}_{mp}(\mathbb{R}^n)$, then restrict to $\dot{A}^{m}_p(\mathbb{R}^n)$. The most natural realizations are those which retain the invariance properties of $\dot{A}^{m}_p(\mathbb{R}^n)$ w.r.t. translations or dilations. It is classically known that such realizations do not always exist, see [8, 12, 13].

1- Case $m < n/p$. Let us set

$$
\frac{1}{q} := \frac{1}{mp} - \frac{1}{n}.
$$

Then $L_q \cap \dot{W}^{1}_{mp}(\mathbb{R}^n)$ is a realization of $\dot{W}^{1}_{mp}(\mathbb{R}^n)$, see [11 prop. 14]. Hence $L_q \cap \dot{A}^{m}_p(\mathbb{R}^n)$ is a realization of $\dot{A}^{m}_p(\mathbb{R}^n)$. Clearly it is invariant w.r.t. translations and dilations.

2- Case $m > n/p$. By condition $1 > \frac{n}{mp}$, $\dot{W}^{1}_{mp}(\mathbb{R}^n)$ is a subset of $C(\mathbb{R}^n)$. Then the subspace $\{f \in \dot{A}^{m}_p(\mathbb{R}^n) : f(0) = 0\}$ is a dilation invariant realization of $\dot{A}^{m}_p(\mathbb{R}^n)$.

3- Case $m = n$ and $p = 1$. As observed in the proof of Proposition 3, $C_0 \cap \dot{A}^{1}_1(\mathbb{R}^n)$ is a realization of $\dot{A}^{1}_1(\mathbb{R}^n)$, clearly invariant w.r.t. translations and dilations.

4- Case $m = n/p$ and $p > 1$. In such a case, $\dot{A}^{m}_p(\mathbb{R}^n)$ does not admit invariant realizations. This can be deduced from [13 thms. 5.4, 5.7].

In all cases we can use “rough” realizations described as follows. Let us recall that $\dot{A}^{m}_p(\mathbb{R}^n)$ is a subset of $L_{pm}(\mathbb{R}^n)_{\text{loc}}$, see [23 1.1.2]. Let us define $q$ by

$$
\frac{1}{q} := 1 - \frac{1}{mp}.
$$

Let $\varphi$ be a compactly supported measurable function in $L_q(\mathbb{R}^n)$, s.t.

$$
\int_{\mathbb{R}^n} \varphi(x) \, dx = 1.
$$
We can define a linear functional on \( \dot{A}_p^m(\mathbb{R}^n) \) by setting

\[
\Lambda(g) := \int_{\mathbb{R}^n} \varphi(x)g(x) \, dx.
\]

Then the kernel of \( \Lambda \) is a realization of \( \dot{A}_p^m(\mathbb{R}^n) \), with no invariance property.

### 4.2 Realizations of \( \dot{W}_1^2(\mathbb{R}^n) \)

According to Proposition\(^9\), there is no nontrivial composition operator which takes \( \dot{W}_1^2(\mathbb{R}^n) \) to itself if \( n > 1 \). In such a case, we are forced to introduce realizations, i.e. subspaces \( E \) s.t. \( \dot{W}_1^2(\mathbb{R}^n) \triangleq E \oplus P_1 \), and satisfying the equivalent properties of Definition\(^1\). Let us recall the known results concerning invariant realizations, and composition operators acting on them, see \[12\] and \[11\] prop. 18 for details.

1- Case \( n = 1 \). \( T_f \) takes \( \dot{W}_1^2(\mathbb{R}) \) to itself iff \( f \in \dot{W}_1^2(\mathbb{R}) \), and, for a such \( f \), it holds

\[
\| (f \circ g)'' \|_1 \leq c \left( \| f'' \|_1 + |f'(0)| \right) \left( \|g''\|_1 + |g'(0)| \right), \tag{9}
\]

for every \( g \in \dot{W}_1^2(\mathbb{R}) \).

**Lemma 1** For all \( \alpha := (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \) s.t. \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \), the subspace

\[
E_\alpha = \{ g \in \dot{W}_1^2(\mathbb{R}) : g(0) = 0, \; \alpha_1 g'(-\infty) + \alpha_2 g'(0) + \alpha_3 g'(+\infty) = 0 \} \tag{10}
\]

is a dilation invariant realization of \( \dot{W}_1^2(\mathbb{R}) \).

**Proof.** Let us recall that every function in \( \dot{W}_1^2(\mathbb{R}) \) is continuous, with finite limits at \( \pm \infty \). Thus the definition of \( E_\alpha \) makes sense. It is easily seen that \( \dot{W}_1^2(\mathbb{R}) = E_\alpha \oplus P_1 \). If \( g \in \dot{W}_1^2(\mathbb{R}) \) and \( g(0) = 0 \), it holds \( \| g' \|_\infty \leq |g'(-\infty)| + \| g'' \|_1 \), hence \( |g(x)| \leq |x| \left( |g'(-\infty)| + \| g'' \|_1 \right) \) for all \( x \in \mathbb{R} \). It follows that \( E_\alpha \) is imbedded into \( L_1(\mathbb{R}^n) \).

**Remark 4** It can be proved that any dilation invariant realization of \( \dot{W}_1^2(\mathbb{R}) \) is necessarily equal to \( E_\alpha \) for some \( \alpha \), see \[12\] prop. 11.

2- Case \( n = 2 \). According to \[12\] thm. 3,

\[
E := C_0 \cap \dot{W}_1^2(\mathbb{R}^2) \tag{11}
\]

is a realization of \( \dot{W}_1^2(\mathbb{R}^2) \). Indeed, it is the unique translation invariant realization, see \[12\] thm. 6]. By Theorem\(^2\), \( T_f \) takes \( E \) to \( \dot{W}_1^2(\mathbb{R}^2) \) iff \( f \in \dot{W}_1^2(\mathbb{R}) \).

3- Case \( n \geq 3 \). According to \[12\] thm. 2], if \( \frac{1}{q} := 1 - \frac{2}{n} \), then

\[
E := L_q \cap \dot{W}_1^2(\mathbb{R}^n) \tag{12}
\]

is a realization of \( \dot{W}_1^2(\mathbb{R}^n) \). Indeed, it is the unique translation invariant realization, and the unique dilation invariant realization, of \( \dot{W}_1^2(\mathbb{R}^n) \), see \[12\] thm. 6, prop. 11]. \( T_f \) takes \( E \) to \( \dot{W}_1^2(\mathbb{R}) \) iff \( f \in \dot{W}_1^2(\mathbb{R}) \). The estimation

\[
\| f \circ g \|_{\dot{W}_1^2} \leq c \left( |f'(0)| + \| f \|_{\dot{W}_1^2} \right) \| g \|_{\dot{W}_1^2} \tag{13}
\]

holds for all \( f \in \dot{W}_1^2(\mathbb{R}) \) and all \( g \in E \).
5 Continuity theorems

Theorem 4 Let \( f : \mathbb{R} \to \mathbb{R} \) be s.t. \( f' \in W^{m-1}_p(\mathbb{R}) \). Let \( E \) be a realization of \( \dot{A}_p^m(\mathbb{R}^n) \). Then \( T_f \) is continuous from \( E \) to \( \dot{A}_p^m(\mathbb{R}^n) \). If moreover \( f(0) = 0 \), then \( T_f \) is continuous from \( A_p^m(\mathbb{R}^n) \) to itself.

Under the stronger assumption \( f' \in C_b^{m-1}(\mathbb{R}) \), the continuity of \( T_f \) on \( A_p^m(\mathbb{R}^n) \) is a classical result, seemingly with no reference in the literature; in their article on composition operators in fractional Sobolev spaces [17], Brezis and Mironescu said only that the proof is “very easy via the standard Gagliardo-Nirenberg inequality”. This “folkloric” proof will be recalled below, see the proof of Lemma 3: what we do for terms with \( s < m \) works as well for \( s = m \), in case \( f' \in C_b^{m-1}(\mathbb{R}) \).

Theorem 5 Let \( f \in W^m_p(\mathbb{R})_{\text{loc}} \). Then \( T_f \) is continuous from \( \dot{W}^m_p \cap L_\infty(\mathbb{R}^n) \) to itself. If moreover \( f(0) = 0 \), then \( T_f \) is continuous from \( W^m_p \cap L_\infty(\mathbb{R}^n) \) to itself.

In case \( p > 1 \), Theorem 5 has been proved in [14, cor. 2], as a particular case of a continuity theorem for composition in Lizorkin-Triebel spaces. F.Isaia has also proved it for \( W^m_p(\mathbb{R}^n) \), in case \( m > n/p \) and \( p \geq 1 \), but with a stronger condition on \( f \), namely \( f \in W^m_\infty(\mathbb{R})_{\text{loc}} \), see [20, thm. 2.1, (iii)].

Theorem 6 Let \( E \) be the realization of \( \dot{W}^2_1(\mathbb{R}^n) \) defined by (10) or (11) or (12) according to the value of \( n \). Let \( f : \mathbb{R} \to \mathbb{R} \) be s.t. \( T_f \) takes \( E \) to \( \dot{W}^2_1(\mathbb{R}^n) \). Then \( T_f \) is continuous from \( E \) to \( \dot{W}^2_1(\mathbb{R}^n) \). If moreover \( f(0) = 0 \), then \( T_f \) is continuous from \( W^2_1(\mathbb{R}^n) \) to itself.

Let us notice that Theorem 6 is less general than Theorem 4 since we do not consider all the realizations, but only the invariant ones.

5.1 Main tools

Four propositions will be useful, where the first is elementary, the second follows by Hölder inequality and the third is classical, see e.g. [7, I §8.2, prop. 2, p. 100].

Proposition 11 If \( \alpha > 1 \), the functions \( t \mapsto |t|^{\alpha} \) and \( t \mapsto \text{sgn}(t)|t|^{\alpha} \) are of class \( C^1 \) on \( \mathbb{R} \).

Proposition 12 Let \( p_1, \ldots, p_s \in [1, +\infty[ \) s.t.

\[
\sum_{j=1}^{s} \frac{1}{p_j} = \frac{1}{p}.
\]

Then the mapping \( (\varphi_1, \ldots, \varphi_s) \mapsto \varphi_1 \cdots \varphi_s \) is continuous from \( L_{p_1} \times \cdots \times L_{p_s} \) to \( L_p \).

Proposition 13 Let us denote \( I := \{1, \ldots, m\} \). Then, for every \( (x_1, \ldots, x_m) \in \mathbb{R}^m \), it holds

\[
x_1x_2 \cdots x_m = \frac{(-1)^m}{m!} \sum_{H \subseteq I} (-1)^{|H|} \left( \sum_{k \in H} x_k \right)^m,
\]

where the sum runs over all the subsets of \( I \), and \(|H|\) denotes the cardinal of \( H \).
Proposition 14 Let $E$ be a Banach space of distributions in $\mathbb{R}^n$ s.t. $E \hookrightarrow L_1(\mathbb{R}^n)_{loc}$. Let $T$ be a continuous mapping from $E$ to $L_p(\mathbb{R}^n)$. Let $\Phi \in C_0(\mathbb{R})$. Define the mapping $V : E \to L_p(\mathbb{R}^n)$ by $V(g) := (\Phi \circ g) T(g)$. Then $V$ is continuous from $E$ to $L_p(\mathbb{R}^n)$.

Proof. Let $(g_\nu)$ be a sequence s.t. $\lim g_\nu = g$ in $E$. By the imbedding $E \hookrightarrow L_1(\mathbb{R}^n)_{loc}$, we can assume that $\lim g_\nu = g$ a.e., up to replacement by some subsequence. It holds

$$
\|V(g_\nu) - V(g)\|_p \leq \|\Phi\|_\infty \|T(g_\nu) - T(g)\|_p + \left(\int_{\mathbb{R}^n} |\Phi \circ g_\nu - \Phi \circ g|^p |T(g)|^p \, dx\right)^{1/p}.
$$

The second term of the above r.h.s. tends to 0 by Dominated Convergence Theorem (Convergence a.e. follows by continuity of $\Phi$, domination by boundedness of $\Phi$ and the fact that $T(g) \in L_p(\mathbb{R}^n)$).

5.2 Proof of Theorem 4

In all the proof, we denote by $E$ the space $A^m_p(\mathbb{R}^n)$, or a realization of $\hat{A}^m_p(\mathbb{R}^n)$. Without loss of generality, we assume that $f$ is a smooth function s.t. $f' \in W^{m-1}_{L_p,lu}(\mathbb{R})$, see Proposition 3 and the estimation (4).

5.2.1 Continuity of $T_f$ from $E$ to $\dot{W}^1_{mp}$ and to $L_p$

By assumption $m \geq 2$, and by Proposition 6, we have $f' \in C_0(\mathbb{R})$. For $j = 1, \ldots, n$, it holds $\partial_j(f \circ g) = (f' \circ g) \partial_j g$. Thus we can apply Proposition 14 and conclude that $g \mapsto \partial_j(f \circ g)$ is a continuous mapping from $E$ to $L_{mp}(\mathbb{R}^n)$. We have also $\|f \circ g_1 - f \circ g_2\|_p \leq \|f'\|_\infty \|g_1 - g_2\|_p$, hence the continuity from $A^m_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$, in case $f(0) = 0$.

5.2.2 Continuity to $\dot{W}^m_p$: heart of the proof

Let us consider the nonlinear operator

$$
S_D(g) := (f^{(m)} \circ g)(Dg)^m,
$$

where $D$ is any first order differential operator with constant coefficients, say $D := \sum_{j=1}^n c_j \partial_j$, for some real numbers $c_j$. The fact that $S_D$ takes $\hat{A}^m_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ is the heart of the proof of Proposition 8 see [11, par. 2.3]. Thus, we can expect that the following result will be the main argument for proving Theorem 4

Lemma 2 The nonlinear operator $S_D$ is continuous from $E$ to $L_p(\mathbb{R}^n)$.

Proof. Let $u \in \mathcal{D}(\mathbb{R})$ be such that $u \geq 0$ and

$$
\forall y \in \mathbb{R}, \quad \sum_{\ell \in \mathbb{Z}} u^2(y - \ell) = 1.
$$

Let us define $\Phi_\ell(y) := u(y - \ell) |f^{(m)}(y)|^p$, for all $y \in \mathbb{R}$. Since $f^{(m)} \in L_{p,lu} \cap C^\infty(\mathbb{R})$, it holds

$$
\Phi_\ell \in C(\mathbb{R}) \quad \text{and} \quad \sup_{\ell \in \mathbb{Z}} \|\Phi_\ell\|_1 < +\infty.
$$
We define $\Psi_\ell(y) := \int_y^{+\infty} \Phi_\ell(t) \, dt$. Then

$$\Psi_\ell \in C^1(\mathbb{R}) \quad \text{and} \quad \sup_{\ell \in \mathbb{Z}} \|\Psi_\ell\|_\infty < +\infty. \quad (15)$$

Now we assume that $mp > 2$ (the exceptional case, $m = 2$ and $p = 1$, needs a minor change, see below). We can use Proposition 11 with $\alpha := mp - 1$. It holds

$$\|SD(g)\|_p^p = \sum_{\ell \in \mathbb{Z}} \left( \int_{\mathbb{R}^n} f^{(m)} \circ g |Dg|^{mp} (u^2 \circ (g - \ell)) \, dx \right)$$

$$= \sum_{\ell \in \mathbb{Z}} \left( \int_{\mathbb{R}^n} (\Phi_\ell \circ g) (Dg) (\text{sgn}(Dg) |Dg|^{mp-1} (u \circ (g - \ell))) \, dx \right),$$

hence

$$\|SD(g)\|_p^p = \sum_{j=1}^{n} c_j \sum_{\ell \in \mathbb{Z}} V_{j,\ell},$$

with

$$V_{j,\ell} := \int_{\mathbb{R}^n} (\Phi_\ell \circ g) \partial_j g \left( \text{sgn}(Dg) |Dg|^{mp-1} (u \circ (g - \ell)) \right) \, dx.$$

The computation of $V_{j,\ell}$ relies upon an integration by parts w.r.t the $j$-th coordinate. This I.P. is justified by a classical theorem. Any function which belongs locally to $W_p^1(\mathbb{R}^n)$ can be modified on a negligible set of $\mathbb{R}^n$, in such a way that the resulting function has the following property: its restriction, on almost every line parallel to a coordinate axis, is locally absolutely continuous. This theorem originates to Calkin [18]; see [23, 1.1.3, thm.1] for the precise statement. Here, this theorem can be applied to the function $\Psi_\ell \circ g$, which belongs locally to $W_{mp}^1(\mathbb{R}^n)$, and to the function

$$w := \text{sgn}(Dg) |Dg|^{mp-1} (u \circ (g - \ell)),$$

which satisfies

$$w \in \dot{W}_1^1 \cap L_q(\mathbb{R}^n), \quad q := \frac{mp}{mp - 1}. \quad (16)$$

Proof of (16). Let $k = 1, \ldots, n$. By the assumption $g \in E$, and by the Gagliardo-Nirenberg inequality

$$\|g\|_{\dot{W}_p^{2mp/2}} \leq c \|g\|_{\dot{W}_1^{mp}} \|g\|_{\dot{W}_p^{mp}}$$

(see [11] (7), p. 6108), it holds $\partial_k Dg \in L_{mp/2}(\mathbb{R}^n)$. On the other hand, since

$$|\partial_k w| \leq c_1 |\partial_k Dg| |Dg|^{mp-2} + c_2 |Dg|^{mp-1} |\partial_k g|,$$

the Hölder inequality applied to the r.h.s. (with exponents $\frac{mp}{2}$ and $\frac{mp}{mp-2}$ in the first term, $\frac{mp}{mp-1}$ and $mp$ in the second) gives $\partial_k w \in L_1(\mathbb{R}^n)$.

The property (16) implies the following:
For almost every \((x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \mathbb{R}^{n-1}\), the function
\[
t \mapsto w(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)
\]
is absolutely continuous on \(\mathbb{R}\), with limit 0 at \(\pm \infty\).

Thus we obtain
\[
V_{j, \ell} = \int_{\mathbb{R}^n} (\Psi_{\ell} \circ g) \partial_j \left\{ \text{sgn}(Dg) |Dg|^{mp-1} (u \circ (g - \ell)) \right\} \, dx,
\]

hence
\[
\|SD(g)\|_p^p = \int_{\mathbb{R}^n} (F \circ g) T_1(g) \, dx + \int_{\mathbb{R}^n} (G \circ g) T_2(g) \, dx,
\]

(17)

where
\[
F(y) := \sum_{\ell \in \mathbb{Z}} \Psi_{\ell}(y) u(y - \ell), \quad G(y) := \sum_{\ell \in \mathbb{Z}} \Psi_{\ell}(y) u'(y - \ell),
\]

and
\[
T_1(g) := (mp - 1) (D^2g) |Dg|^{mp-2}, \quad T_2(g) := |Dg|^{mp}.
\]

In case \(m = 2\) and \(p = 1\), a similar computation starting with
\[
\|SD(g)\|_1 = \int_{\mathbb{R}^n} |f'' \circ g| (Dg)^2 \, dx,
\]

shows that (17) is also valid.

By Gagliardo-Nirenberg and by the definition of \(E\), the linear mappings \(g \mapsto D^2g, g \mapsto Dg\) are continuous from \(E\) to \(L_{mp/2}(\mathbb{R}^n)\) and \(L_{mp}(\mathbb{R}^n)\) respectively. Thus, by using Proposition 2 the mappings
\[
g \mapsto D^2g, \quad g \mapsto |Dg|^{mp-2}, \quad g \mapsto |Dg|^{mp}
\]
are continuous from \(E\) to \(L_{mp}(\mathbb{R}^n)\), \(L_{mp-2}(\mathbb{R}^n)\) and \(L_{1}(\mathbb{R}^n)\) respectively. Since
\[
\frac{2}{mp} + \frac{mp - 2}{mp} = 1,
\]
we can use Proposition 12 and conclude that \(T_1\) and \(T_2\) are nonlinear continuous mappings from \(E\) to \(L_1(\mathbb{R}^n)\). By the continuity of \(\Psi_{\ell}\), and by (15), it follows that \(F\) and \(G\) belong to \(C_b(\mathbb{R})\). From (17) and Proposition 14 we deduce the following property:
\[
g \mapsto \|SD(g)\|_p\] is a continuous nonlinear functional on \(E\).
\]

(18)

Now consider a sequence \((g_\nu)\) which converges to \(g\) in \(E\). By the continuity of \(f^{(m)}\), it holds
\[
\lim SD(g_\nu) = SD(g) \quad \text{a.e.,}
\]

up to replacement by a subsequence. By (18), it holds
\[
\lim \|SD(g_\nu)\|_p = \|SD(g)\|_p.
\]

Then we can apply the Theorem of Scheffé 27 and conclude that \(\lim SD(g_\nu) = SD(g)\) in \(L_p(\mathbb{R}^n)\).

\footnote{Usually attributed to Scheffé, but first proved by F. Riesz 24, see the survey of N. Kusolitsch 21.}
5.2.3 Continuity to $\dot{W}_p^m$: end of the proof

We have to prove the continuity of $g \mapsto (f \circ g)^{(\alpha)}$, from $E$ to $L_p(\mathbb{R}^n)$, for all $\alpha \in \mathbb{N}^n$ s.t. $|\alpha| = m$. We use the classical formula

$$(f \circ g)^{(\alpha)} = \sum c_{\alpha,s,\gamma}(f^{(s)} \circ g)^{\gamma_1} \cdots g^{\gamma_s},$$

(19)

where the parameters satisfy the conditions

$$s = 1, \ldots, m, \ |\gamma_r| > 0 \ (r = 1, \ldots, s), \ \sum_{r=1}^s \gamma_r = \alpha,$$

(20)

and the $c_{\alpha,s,\gamma}$’s are some combinatorial constants. Formula (19) holds true for any smooth function $g$. We need to prove it for any function $g \in E$. We associate with $g$ the sequence $g^\nu := \theta^\nu \ast g, \ \nu \geq 1$. It is easily seen that $(f \circ g^\nu)^{(\alpha)} \to (f \circ g)^{(\alpha)}$ in the sense of distributions.

Now we prove that the r.h.s. of (19) behaves similarly.

Case 1: terms with $s < m$. Let us consider

$$S(g) := (f^{(s)} \circ g)^{\gamma_1} \cdots g^{\gamma_s},$$

(21)

for a set of parameters satisfying (20), and $s < m$.

**Lemma 3** The nonlinear operator $S$, given by (21), with $s < m$, is well defined and continuous from $E$ to $L_p(\mathbb{R}^n)$.

**Proof.** According to Gagliardo-Nirenberg (see e.g. [11, (7), p. 6108]), we have the imbeddings

$$\dot{A}_p^m(\mathbb{R}^n) \hookrightarrow \dot{W}_p^{m/N},$$

for all $N = 1, \ldots, m - 1$. Applying this to $N = |\gamma_r|$, for $r = 1, \ldots, s$, and Proposition 12, we obtain the continuity of the mapping $g \mapsto g^{\gamma_1} \cdots g^{\gamma_s}$ from $E$ to $L_p(\mathbb{R}^n)$. By Proposition 6 we have $f^{(s)} \in C_b(\mathbb{R})$. Hence we can apply Proposition 14 and conclude that $S$ is continuous. This ends up the proof of Lemma 3.

As a consequence, it holds

$$\lim_{\nu \to \infty} (f^{(s)} \circ g^\nu)^{\gamma_1} \cdots g^{\gamma_s} = (f^{(s)} \circ g)^{\gamma_1} \cdots g^{\gamma_s},$$

in $L_p(\mathbb{R}^n)$.

Case 2: terms with $s = m$. Let us consider the operator

$$S(g) := (f^{(m)} \circ g) \partial_{j_1}g \cdots \partial_{j_m}g,$$

with $j_r \in \{1, \ldots, n\}$ for $r = 1, \ldots, m$. For every subset $H$ of $\{1, \ldots, m\}$, let us denote

$$D_H := \sum_{k \in H} \partial_{j_k}.$$
According to Proposition 13 it holds

\[ S = \frac{(-1)^m}{m!} \sum_H (-1)^{|H|} S_{DH}, \]

see (14) for the definition of \( S_D \). By Lemma 2 it holds \( \lim_{r \to \infty} S(g_r) = S(g) \) in \( L_p(\mathbb{R}^n) \).

As a consequence of the above cases 1 and 2, the formula (19) holds for all \( g \in E \). This ends up the proof.

5.3 Proofs of Theorems 5 and 6

**Proof of Theorem 5.** We refer to the proof of Theorem 4. By Theorem 4 and by the first imbedding given in (3), \( T_{f_\omega} \) is a continuous mapping from \( \dot{W}^m_p \cap L_\infty(\mathbb{R}^n) \) to \( \dot{W}^m_p(\mathbb{R}^n) \). Since \( (f_\omega) \in L_\infty(\mathbb{R}), T_{f_\omega} \) is also continuous from \( \dot{W}^m_p \cap L_\infty(\mathbb{R}^n) \) to \( L_\infty(\mathbb{R}^n) \). We conclude that \( T_{f_\omega} \) is continuous from \( \dot{W}^m_p \cap L_\infty(\mathbb{R}^n) \) to itself for every \( r > 0 \). The continuity of \( T_f \) follows by (5).

**Proof of Theorem 6.** It is similar to that of Theorem 4. We assume \( n \neq 2 \), since the case \( n = 2 \) is covered by Theorem 5. By the estimations (6), (9), (13), and by the density of \( \mathcal{D}(\mathbb{R}) \) into \( L_1(\mathbb{R}) \), we can restrict ourselves to the case \( f'' \in \mathcal{D}(\mathbb{R}) \). Notice that such a property implies \( f' \in C^\infty(\mathbb{R}) \), hence \( \partial_j \partial_k (f \circ g) = U^{j,k}(g) + S^{j,k}(g) \), where \( U^{j,k}(g) := (f' \circ g) \partial_j \partial_k g \) and \( S^{j,k}(g) := (f'' \circ g)(\partial_j g)(\partial_k g) \), see [11] step 2, p. 6111, and the proof of prop. 18, p. 6128]. The continuity of \( U^{j,k} : E \to L_1(\mathbb{R}^n) \) follows by Proposition 14. To prove the continuity of \( S^{j,k} \), we introduce

\[ S_D(g) := (f'' \circ g)(Dg)^2, \]

where \( D \) is a first order differential operators with constant coefficients, and we set

\[ h(x) := \int_x^{+\infty} |f''(t)| \, dt. \]

Then we must discuss according to \( n \).

**Case \( n > 2 \).** For all \( g \in E \), and \( j = 1, \ldots, n \), it holds \( \partial_j g \in \dot{W}^1_1 \cap L_r(\mathbb{R}^n) \), with \( \frac{1}{r} := 1 - \frac{1}{n} \), see [11] prop. 15. This implies the following property: for almost every \( (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \mathbb{R}^{n-1}, the function

\[ t \mapsto \partial_j g(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n) \]

is absolutely continuous on \( \mathbb{R} \), with limit 0 at \( \pm \infty \). Thus we can make integrations by parts w.r.t. each coordinate, obtaining

\[ \|S_D(g)\|_1 = \int_{\mathbb{R}^n} (h \circ g)(D^2 g) \, dx, \quad (22) \]

for all \( g \in E \). The continuity of \( S^{j,k} : E \to L_1(\mathbb{R}^n) \) follows, exactly as in the proof Theorem 4. By Sobolev’s Theorem, the inhomogeneous space \( W^2_1(\mathbb{R}^n) \) is imbedded into \( E \). The continuity of \( T_f \) on \( W^2_1(\mathbb{R}^n) \) follows at once if \( f(0) = 0 \).
Case n = 1. Now \( D = d/dx \) and the formula (22) becomes

\[
\| S_D(g) \|_1 = \int_{\mathbb{R}} (h \circ g) g'' \, dx - h(g(+\infty)) \, g'(+\infty) + h(g(-\infty)) \, g'(-\infty).
\]

If \( \lim g_\nu = g \) in \( E \), then, by the proof of Lemma 1, \( \lim g_\nu(x) = g(x) \) for every \( x \in \mathbb{R} \), and also for \( x = \pm \infty \). Since \( h \) is continuous on \( \mathbb{R} \), we conclude that \( \lim \| S_D(g_\nu) \|_1 = \| S_D(g) \|_1 \). The rest of the proof is unchanged.

**Conclusion**

Let us mention possible continuations of the present work:

1- **Generalization of the Theorem 4 to Sobolev spaces with fractional order of smoothness.** The automatic continuity is known to hold in the following cases:

- Besov spaces \( B^s_{p,q}(\mathbb{R}^n) \) with \( 0 < s < 1 \), by interpolating between \( L_p \) and \( W^1_p \), see [26, 5.5.2, thm. 3].

- Besov spaces \( B^s_{p,q}(\mathbb{R}) \) and Lizorkin-Triebel spaces \( F^s_{p,q}(\mathbb{R}) \) with \( s > 1 + (1/p) \), \( p,q \in [1, +\infty[ \), see [14, cor. 2] and [16, thm. 8] (in those papers, they are some restrictions in case of Besov spaces, which have been removed in [15]).

The extension to the spaces on \( \mathbb{R}^n \), for \( n > 1 \) and \( s > 1 \) noninteger, is completely open: the first difficulty is that we have not even a full characterization of functions which act by composition.

2- **Proof of the higher-order chain rule.**

In the proof of Theorem 4 we have established the formula (19) for all \( g \in E \), but only for smooth functions \( f \). Could we generalize it to any \( f \) s.t. \( f' \in W^{m-1}_{L_p,L^1}(\mathbb{R}) \)? In this respect, we can refer to the partial results of F. Isaia [20].

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Gérard Bourdaud  
Université Paris Diderot, I.M.J. - P.R.G (UMR 7586)  
Case 7012  
75205 Paris Cedex 13  
bourdaud@math.univ-paris-diderot.fr

Madani Moussai  
Laboratory of Functional Analysis and Geometry of Spaces,  
M. Boudiaf University of M'Sila,  
28000 M'Sila, Algeria  
moussai@yahoo.fr