Longitudinal Angular Momentum in Magneto-Mie Scattering 
Quantum Vacuum Correction to the Einstein-De Haas effect

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(Dated: May 4, 2021)

This work investigates the angular momentum of the electromagnetic quantum vacuum residing in a dielectric Mie sphere subject to the Faraday effect. Longitudinal electric modes are excited on its surface and are also created inside the sphere by the Faraday effect. The electromagnetic quantum vacuum creates a vortex of the Poynting vector that varies as $1/r$ to the center of the sphere and is associated with longitudinal angular momentum, connected to the vector potential $A(r,t)$. It emerges as a non-negligible quantum vacuum correction to the classical (diamagnetic) Einstein-De Haas effect in which an applied external magnetic field - via its action on microscopic magnetism - enforces macroscopic rotation.

Keywords: electromagnetic quantum vacuum, Mie scattering, magnetic dichroism

I. INTRODUCTION

Electromagnetic waves possess angular momentum. It often emerges as spin or orbital motion $[1,2]$, associated with the familiar electromagnetic waves with transverse electric and magnetic fields. "Longitudinal" angular momentum is associated with longitudinal electric waves. A classical example is when an electrical charge and a magnetic dipole are brought close together. Their combined fields induce a circulating Poynting vector field, that does not move energy around, but that carries a finite angular momentum $[3]$. In this work we describe how the electromagnetic quantum vacuum induces longitudinal angular momentum inside a dielectric “Mie” sphere, subject to the Faraday effect, and study the topology of the Poynting vector.

The interaction of the electromagnetic quantum vacuum with matter creates both “Casimir” energy $[4,5]$ and “Casimir” momentum $[2]$. For macroscopic objects both suffer often from power-law divergencies at high photon energies. Some are nonphysical or not observable and removed by dimensional regularization $[6]$, others require QED mass renormalization $[7,8]$. The classical electromagnetic “Abraham” momentum was observed first by Walker etal. $[9]$ and recently its predicted quantum vacuum corrections have been searched for $[10]$. 

Longitudinal angular momentum is well known to emerge when a particle (an electron) with charge $e$ and mass $m$ rotates around a homogeneous, time-dependent magnetic field ($z$-direction is along the vector $eB_0, e > 0$ henceforth). The Lenz law states that the sum of kinetic angular momentum $L_{\text{kin}} = (r \times mv)_z$ and enclosed magnetic flux $\Phi = B_0 A$ is conserved in time, 

$$ \frac{d}{dt}[L_{\text{kin},z} + e\frac{\Phi}{2\pi \epsilon_0}] = 0 $$

In the quantum-mechanical description, the electromagnetic angular momentum in Eq. (1) is manifestly longitudinal and gauge-variant. If no other electromagnetic fields are present the Hamiltonian is

$$ H = \frac{1}{2m}(p - \frac{e}{c_0}A(r,t))^2 $$

with $p$ the canonical momentum operator (i.e canonical to $r$ with $[r_i, p_j] = i\hbar \delta_{ij}$) and $A(r,t)$ the vector potential associated with the electromagnetic field. In the Coulomb gauge is $A(r,t) = B_0(t) \times r/2$ and $J_{\text{kin}} = J_{\text{can}} - J_{\text{long}}$ is identified as the kinetic angular momentum and $J_{\text{long}} = e r \times A/c_0$ as the longitudinal angular momentum. The canonical angular momentum $J_{\text{can},z} = (r \times p)_z$ commutes with $H$, and is thus conserved leading to Eq. (1). When the electromagnetic quantum vacuum is added to Eq. (2), spin and orbital motion associated with the transverse electromagnetic field enters the total angular momentum and provide the dominant QED correction $[11]$. The conservation of longitudinal angular momentum remains valid in the presence of a rotationally-symmetric confining potential for the charge $e$, and provided that no optical transitions occur that usually involve electromagnetic spin. This is true for the ground state of an isotropic harmonic oscillator with eigenfrequency $\omega_0$ for which we find that

$$ \langle J_{\text{long},z} \rangle = \frac{eB_0}{2c_0}(r^2 - z^2) = \hbar x \frac{\omega_c}{\omega_0} $$

with $\omega_c = eB_0/2mc_0$ the classical Larmor frequency. As a result, upon switching on a magnetic field, kinetic angular momentum is achieved by the matter. Since $\omega_c/\omega_0 \ll 1$ this diamagnetic effect is much smaller than in paramagnetism, where spins of order $\hbar$ are involved.

In this work we study the angular momentum of the quantum vacuum that emerges from its interaction with a macroscopic, dielectric Mie sphere $[12,13]$ subject to a magnetic field $[14]$. The sphere is described by a macroscopic complex-valued, frequency-dependent dielectric tensor assumed homogeneous for $r < a$. In a homogeneous medium the Faraday effect lifts the degeneracy of optical spin in the dispersion law. For a Mie
sphere this effect lifts the degeneracy of the magnetic quantum numbers in the eigenvalues of the Helmholtz equation, much like the Zeeman effect in atoms. Not less important, the Faraday effect couples longitudinal and transverse eigenmodes of the Helmholtz equation so that longitudinal electric fields reside not only at the surface but also inside the sphere. The connection between macroscopic and microscopic description is crucial to understand divergencies, and we shall model the Mie sphere by an assembly of classical harmonic, diamagnetic oscillators. In a simplified picture, their microscopic longitudinal angular momenta, given by Eq. (3), compensate inside the sphere and emerge as a macroscopic angular momentum at the surface of the Mie sphere. This is a manifestation of the classical Einstein-De Haas effect [12], recently also observed for a single molecule [13]. A torque will be exerted on the Mie sphere once the magnetic field is switched on. We will find that the electromagnetic quantum fluctuations of the field modes participate in the conservation of angular momentum, and emerge as a quantum vacuum correction to the classical, diamagnetic Einstein-De Haas effect. This correction depends explicitly on the constitutive optical constants of the sphere.

II. PERTURBATIONAL APPROACH TO ANGULAR MOMENTUM

Quantum vacuum fluctuations of the electromagnetic fields are described by the Fluctuation-Dissipation Theorem. When applied to a polarizable medium it states that the field correlation of the electric field is given by

\[
\langle \tilde{E}_i(r_1, \omega) \tilde{E}_j(r_2, \omega') \rangle = \frac{4\omega Q(\omega)}{-2i\epsilon_0} \delta(\omega - \omega') \delta_{ij} (r_1, r_2, \omega + i0)\]  

(4)

with the anti-hermitian operator \(\Delta G \equiv \mathbf{G} - \mathbf{G}^\dagger\) in terms of the classical (retarded) Green’s tensor \(\mathbf{G}\) of the Helmholtz wave equation for the electric field. The quantum vacuum is described by the energy \(Q(\omega) = \hbar \omega\) for positive frequencies and \(Q(\omega) = 0\) for negative frequencies. We use the symbols \(\dagger\) and \(-\) for full hermitian conjugate and complex conjugation, respectively. The rigorously conserved quantity in classical Maxwell-Lorentz theory is \(J_{\text{kin}} + J\) [14], with

\[
\mathbf{J} = \int d^3r \mathbf{r} \times \mathbf{K} + \text{momentum density} \ \mathbf{K} = \tilde{\mathbf{E}} \times \mathbf{B}/4\pi \epsilon_0. \]

Under very general circumstances, we can establish from Eq. (4) that [14],

\[
\nabla \cdot \langle \tilde{\mathbf{E}}(\mathbf{r}) \times \mathbf{H}(\mathbf{r}) \rangle = 0 \\
\int d^3r \langle \tilde{\mathbf{E}}(\mathbf{r}) \times \mathbf{H}(\mathbf{r}) \rangle = 0
\]

(5)

This expression states that electromagnetic energy of the quantum vacuum does not move energy and that the spatial average of the Poynting vector vanishes. It does not imply that the Poynting vector itself vanishes, for instance, the curl of some other vector field satisfies both equations [14]. In this work we concentrate on the dielectric response of the matter at optical frequencies and set \(\mu(\omega) = 1\). Hence \(\mathbf{H} = B\) so that the total average electromagnetic momentum, \(\int d^3r (\mathbf{K}(\mathbf{r}))\) vanishes. As a result, the average electromagnetic angular momentum is independent on the choice of the origin. The quantum expectation value of the electromagnetic angular momentum is split up into three terms [15] according to (from now on we drop the brackets denoting quantum average)

\[
J_z = -\frac{1}{2\pi i\epsilon_0} \int_{-\infty}^{\infty} d\omega Q(\omega) \mathbf{TR} J_z \cdot [\mathbf{G}(\omega) - \mathbf{G}^\dagger(\omega)]
\]

(6)

with the angular momentum operator \(J_z = L_z + S_z + J_{\text{long}, z}\) featuring orbital momentum \((L_z)_{mn} = (\mathbf{r} \times \mathbf{p})_{\delta_{mn}}, \) spin \((S_z)_{mn} = -i\epsilon_{mn};\) and longitudinal angular momentum \((J_{\text{long}, z})_{mn} = -\epsilon_{mn}\mathbf{p}/\hbar;\) TR stands for the full operator trace, including both spin components and mechanical degrees of freedom. The last term only acts on longitudinal electric fields and can be shown to be equivalent to the longitudinal angular momentum \(\sum_i q_i \mathbf{r}_i \times \mathbf{A}(\mathbf{r}_i)/\epsilon_0\) of the microscopic charges.

The Helmholtz Green function is given by the operator,

\[
\mathbf{G}(\mathbf{r}, \mathbf{p}, z) = \frac{1}{\epsilon(\mathbf{r}, z) z^2/c_0^2 - p^2 \Delta_p}
\]

(7)

with the transverse operator \(p^2(\Delta_p)_{nm} = p^2\delta_{nm} - p_n p_m;\) and is - by causality- analytic for all complex frequencies \(z\) with \(\text{Im } z > 0\). For a magnetic field in the \(z\) direction, the dielectric tensor is

\[
\epsilon_{nm}(\mathbf{r}, z) = m^2(r, z) \delta_{ij} - W(r, z) \frac{c_0}{z} (S_z)_{mn}
\]

(8)

in terms of the hermitian spin operator defined above. In a simplified description, the Faraday effect is due to a uniform Zeeman splitting of all involved optical transitions, \(\epsilon(\omega) = m^2(\omega + \omega_0, S_z)\), we obtain the well-known Becquerel relation \(W = -\omega_0/c_0 \omega d\mu_3/d\omega\). It is associated with the so-called “A-term” in the microscopic theory for molecular magneto-dichroism, often with an extra constant of order unity in front [15, 20].

We seek an angular momentum linear in the external magnetic field. If we treat the Faraday effect as a linear perturbation to the standard Mie problem, Eq. (7) is expanded as,

\[
\mathbf{G}(z) = \mathbf{G}_M(z) + \frac{z}{c_0} W(r, z) S_z \cdot \mathbf{G}_M(z)
\]

(9)

in terms of the Green’s function of the Mie sphere \(\mathbf{G}_M\) without magnetic field.

III. SINGLE ELECTRIC DIPOLE

The classical model of an harmonically bound electron interacting with an electromagnetic wave of frequency \(\omega\) and a (quasi-) static external magnetic field \(\mathbf{B}_0\) is described by the well-known equation

\[
\frac{d^2\mathbf{r}}{dt^2} = -\omega_0^2 \mathbf{r} + e \mathbf{E} + \frac{e}{mc_0} \frac{d\mathbf{r}}{dt} \times \mathbf{B}_0 + \tau R \frac{d^2\mathbf{r}}{dt^2}
\]

(10)
The last term describes the radiation loss with time-scale \( \tau_R \sim r_c/c_0 \) \[22\]. For an harmonic field \( \mathbf{E}(\omega) \exp(-i \omega t) \), the electric dipole moment \( \mathbf{d} = e \mathbf{r} \) satisfies \( \mathbf{d}(\omega) = \alpha(\omega) \cdot \mathbf{E}(\omega) \) with the polarizability given by

\[
\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - (\omega + \omega_0) S_z^2 - i \tau \omega^2} \quad (11)
\]

The radiation reaction induces a non-causal and large “run-away” pole \( \omega \approx i/\tau_R \) in the upper complex plane \[22\] that is often avoided by putting \( i\tau_R \omega^3 \rightarrow i \omega \) and by identifying the radiation life time \( \Lambda = \tau_R \omega_0 \).

We can compare this outcome to scattering theory, according to which the Helmholtz Green function \[17\] is related to the \( T \)-matrix of the dielectric object according to,

\[
\mathbf{G}(\mathbf{k}, \mathbf{k}', \omega) = \mathbf{G}_0(\mathbf{k}, \omega) \delta_{\mathbf{k} \mathbf{k}'} + \mathbf{G}_0(\mathbf{k}, \omega) \cdot \mathbf{T}_{\mathbf{k} \mathbf{k}'}(\omega) \cdot \mathbf{G}_0(\mathbf{k}', \omega)
\]

with \( \mathbf{G}_0 = \mathbf{k} k_0^2/\omega^2 + \Delta_k/[\omega^2/\omega_0^2 - k_0^2] \) the one of empty space, and \( z = \omega + i \theta \) for the retarded Green’s function. Given an incident plane wave with wave number \( \mathbf{k} \), frequency \( \omega \), and transverse polarization \( \mathbf{p}_\mathbf{k} \), the induced dipole moment of the object to given by \( \mathbf{d}(\omega) = -\mathbf{T}_{\mathbf{0} \mathbf{k}}(\omega) \cdot \mathbf{p}_\mathbf{k} \omega_0^2/\omega^2 \) \[21\]. For a dipole small compared to the wavelength, its \( T \)-matrix should not depend on wave vectors, so that \( t(\omega) = -\alpha(\omega) \omega^2/\omega_0^2 \). We can now evaluate the trace in Eq. \((10)\) in Fourier space (with \( \sum_k \equiv \int d^3 \mathbf{k}/(2\pi)^3 \))

\[
dL_z(\omega) + dS_z(\omega) = \frac{\hbar \omega d\omega}{2\pi c_0} \sum_{\mathbf{z}nm} \frac{1}{\omega^2/\omega_0^2 - k_0^2} [\Delta_k \cdot t(\omega)]_{nm} - \text{h.c.}
\]

and

\[
dJ_{\text{long},z}(\omega) = -\frac{\hbar \omega d\omega}{2\pi c_0} \sum_{\mathbf{z}nm} \frac{1}{\omega^2/\omega_0^2 - k_0^2} [t(\omega) \cdot \Delta_k]_{nm} - \text{h.c.}
\]

The first integral over \( \mathbf{k} \) for the transverse angular momentum converges for all frequencies, whereas the longitudinal angular momentum suffers from the divergence of \( \sum_k k^3/\omega_0^2 \). In Appendix A we show that the relation \( t(\omega) = -\alpha(\omega) \omega^2/\omega_0^2 \) is consistent with a regularization of this integral as \( 3/2r_c \), with \( r_c \) the classical electron radius. Upon extracting the divergence of the longitudinal angular momentum we get,

\[
J_{\text{long},z} = -\frac{\epsilon_{\text{znm}}}{2\pi c_0^2 r_c} \int_0^\infty d\omega \omega \left( \alpha_{\text{nm}}(\omega) - \alpha_{\text{nm}}^4(\omega) \right) = \hbar \times \frac{\omega}{\omega_0} \quad (12)
\]

to leading order in \( \omega_c/\omega_0 \). The remainder, including the transverse angular momentum, has a finite \( k \)-integral proportional to \( i\tau_0/\omega \) and we find

\[
J_{\text{rem},z} \sim \frac{\hbar \omega r_c}{c_0} \Re \int_0^\infty \frac{d\omega}{\omega^3} \left[ \frac{\omega^3}{\omega_0^2 - (\omega + iA/2)^2} \right]^2 
\]

In this case the frequency integral diverges logarithmically, much like the non-relativistic derivation of the Lamb-shift \[4\]. We also simplified the radiation reaction which would here have provided an upper cut-off \( \omega \sim 1/\tau_R \). We will attribute this modest singularity at high-frequencies to the failure of the present model, and assume that it is eliminated by QED \[7\]. Replacing the frequency integral by a number of order \(-\log \omega_0 \tau_R \sim 17\) at most, we see that \( J_{\text{rem},z} \) still differs from \( J_{\text{long},z} \) by a factor \( 17 \times \omega_0 r_c/c_0 \sim 10^{-6} \).

We conclude that the angular momentum induced by the interaction of the quantum vacuum with the classical electric dipole is entirely of longitudinal nature, associated with the longitudinal electromagnetic field, and equal to \( \hbar \omega_c/\omega_0 \). This coincides with the outcome \[13\] for a quantum mechanical oscillator, for which this angular momentum is “built in”. In Ref. \[2\] this somewhat surprising notion is referred to as “the necessity for the vacuum field”, and is linked to the fluctuation-dissipation relation. Similarly, for the single dipole a Casimir energy \( E = \frac{\lambda}{2} \hbar \omega_0 \) can be calculated from the fluctuation-dissipation theorem, which is exactly the energy of the quantum-mechanical ground state. In the present context, the equivalence of the two different pictures learns us how to regularize the diverging \( k \)-integral.

### IV. DIELECTRIC MIE SPHERE

In this section we will express the angular momentum of a dielectric sphere with homogeneous but frequency-dependent index of refraction inside, in terms of the Mie eigenfunctions. A standard procedure in treatments of Casimir energy in macroscopic media is to transform to Matsubara frequencies \[8\], facilitated by the complex analyticity of \( G(z) \). Since \( W(z) \), like \( G(z) \), typically decays as \( 1/z^2 \) at large frequencies, the frequency integral of \( G(\omega + i0) \) can be moved to the positive imaginary axis, and the one for \( G^1(\omega + i0) = G(\omega - i0) \) to the negative imaginary axis. One has to be careful to miss neither longitudinal poles in this expression at \( \omega = 0 \) nor possible diverging contributions as \( \omega \to \infty \). With this in mind \[22\],

\[
J_z = \frac{kT}{c_0} \sum_{n=-\infty}^{\infty} s_n^2 W(is_n) TR \quad J_z \cdot G_M(is_n) \cdot \theta_a S_z \cdot G_M(is_n) \quad (14)
\]

with the Matsubara frequencies \( s_n = 2\pi n kT/\hbar \) and \( \theta_a = 1 \) inside the sphere and zero outside. The advantage of this formalism is that \( G_M(is_n) \) has become an hermitian operator, and that both \( W(is_n) \) and \( m(is_n) \) have become real-valued. The Helmholtz operator for the classical Mie problem without magnetic field is

\[
H_M(s) = \varepsilon(r, is)^{-1/2} s^2 \Delta_p \varepsilon(r, is)^{-1/2} \quad (15)
\]
Since $s$ is real-valued, the operator $\mathbf{H}_M$ is hermitian for all $s$. For all given $s$ it has orthonormal eigenvectors written as $|\Psi_{\nu J M k}(s)\rangle$ ($J$ and $M$ are the usual eigenvalues of the operators $J^2$ and $J_z$, $\nu = e, m, \ell$ is a polarization index with prescribed parity, and $k$ is a wave number associated with radial decay). They have eigenvalues $\tilde{k}_{\nu}^2 = k^2 \geq 0$ for the two transverse polarizations $\nu = e, m$ and a highly degenerated eigenvalue $k_{el} = 0$ for the longitudinal eigenfunctions $\nu = \ell$ and all $k > 0$. For any real-valued $s$, the wave functions $\{|\Psi_{\nu J M k}(s)\rangle\}$, including the longitudinal states, constitute a complete orthonormal set that spans the electromagnetic Hilbert space, equipped with a scalar product associated with electromagnetic energy. The Helmholtz Green’s function $G_M$ in Eq. (17) applied to the Mie sphere can now be decomposed as
\[
G_M(is) = e^{-1/2} \sum_{\nu J M k} \frac{|\Psi_{\nu J M k}\rangle \langle \Psi_{\nu J M k}|}{-s^2/c_0^2 - k_{\nu}^2} \epsilon^{-1/2} = \sum_{J M k} \frac{|E_{\nu J M k}\rangle \langle E_{\nu J M k}|}{-s^2/c_0^2 - k^2} \epsilon^{-1/2},
\]
The electric field modes $|E\rangle = \epsilon(r, is)^{-1/2}|\Psi_{\nu J M k}\rangle$ are not necessarily orthonormal for different $k$. For the transverse modes $\nu = e, m$ with finite spin the sum over $J$ starts at 1. For a Mie sphere with size $a$ and index of refraction $m$ we write $\epsilon(r, is) = 1 + (m^2(is) - 1)\theta(a - r)$. For an ideal Mie sphere, $\theta(a - r)$ is the perfect step function on its surface, but in the longitudinal discontinuities that accumulate near the surface we shall assume it to be “smooth but rapidly varying”. One crucial property of the index of refraction is its decay as $m^2(is) \sim 1/s^2$ at high frequencies, associated with free electron behavior. We choose the standard Lorenz-Lorentz model,
\[
m^2(is) = 1 + \frac{\rho_a(is)}{1 + \frac{\rho_a(is)}{s^2}} \tag{16}
\]
with $\rho_a(is) = \omega_p^2/\omega_a^2 + s^2$. This models the sphere as a collection of independent harmonic oscillators with uniform volume density $\rho$ and plasma frequency $\omega_p^2 = \rho_a(is)\omega_a^2$, with local field correction. It should be a reliable model away from elastic resonances, which is the case on the imaginary frequency axis. It is straightforward to add more microscopic resonances to this model but this is beyond the scope of this work. If the atomic resonance $\omega_a$ is located in the visible regime, the effects of finite temperature due to the discrete Matsubara sum happen beyond several thousand degrees and will not be discussed.

### A. Transverse angular momentum

The sum of orbital and spin momentum is governed by transverse modes so that in the following we consider $\nu = e, m$ only. Because $|\Psi_{\nu J M k}(s)\rangle$ is an eigenfunction of the angular momentum operator $L_z + S_z$ with eigenvalue $M$,
\[
L_z + S_z = \frac{2kT}{c_0^2} \sum_{n=1}^{\infty} W(is_n)s_n^2 \sum_{J M k} M \times \langle E_{\nu J M k}| \epsilon(r)^{-1}|\Psi_{\nu J M k}\rangle \langle \Psi_{\nu J M k}| \epsilon^{-1} \theta S_z |E_{\nu J M k}\rangle \frac{(s_n^2/c_0^2 + k^2)^2}{(s_n^2/c_0^2 + k^2)^2}
\]
Using $\epsilon(r)^{-1} = 1 + (\epsilon(r)^{-1} - 1)\hat{\rho}a$, we split this expression up into one term that counts angular momentum everywhere in space, and a second term that exists only inside the sphere. The first simplifies using the orthonormality of the eigenfunctions $|\Psi_{\nu J M k}\rangle$,
\[
L_z + S_z = \frac{2kT}{c_0^2} \sum_{n=1}^{\infty} W(is_n)s_n^2 \sum_{\nu J M k} M \times \langle E_{\nu J M k}| \theta a S_z |E_{\nu J M k}\rangle \frac{(s_n^2/c_0^2 + k^2)^2}{(s_n^2/c_0^2 + k^2)^2}
\]
The matrix element involving the spin operator $S_z$ is equal to $M/J(J + 1) \times \epsilon_{iJ}(a)$ with
\[
\epsilon_{iJ}(a) \equiv \int_{|r| < a} d^3r |E_{iJ}(r)|^2 \tag{17}
\]
proportional to the electric energy of the mode $|\nu J M k\rangle$ inside the Mie sphere, independent of $M$. It can be large near specific resonant values for $k$ but this is of no relevance here since the integral over $k$ is not sensitive to narrow, large spectral peaks. With $\sum_{M} M^2 = (2J + 1)J(J + 1)/3$ this gives the expression,
\[
L_z + S_z = \frac{2kT}{3c_0^2} \sum_{n=1}^{\infty} W(is_n)s_n^2 \sum_{k,J} (2J + 1)\epsilon_{iJ}(a) \frac{(s_n^2/c_0^2 + k^2)^2}{(s_n^2/c_0^2 + k^2)^2}
\]
This Matsubara sum diverges logarithmically when $W(is) \sim 1/s^2$ for large frequencies. This divergence is more modest than the quadratic divergence of Casimir energy [3] of a dielectric sphere or the Casimir momentum density in magneto-electric media [4], but still it is diverging. The divergence is caused by the vacuum modes that interact only with the Faraday effect $W$ inside the sphere, without sensing the rest of the sphere (i.e. the Born approximation). The un-scattered transverse modes are,
\[
E_{0kJM}(r) = \frac{4\pi i^{J+1}}{(2\pi)^{3/2}} k J j_J kr \hat{X}_{J M}(\hat{r}) \tag{18}
\]
\[
E_{0kJM}(r) = \frac{4\pi i^{J+1}}{(2\pi)^{3/2}} \left[ \sqrt{J(J + 1)} j_J kr \hat{N}_{J M}(\hat{r}) \right. \\
\left. + (k r j_J kr')^J \hat{Z}_{J M}(\hat{r}) \right]
\]
in terms of the three orthonormal vector harmonics $\hat{X}_{J M}$, $\hat{N}_{J M}$ and $\hat{Z}_{J M}$ [18]. Their electric energy will be denoted by $\epsilon_{0(J)}(a)$. Using $\sum_{J=1}^{\infty} (2J + 1)j_J^2(x) = 1 - j_0^2(x) + \sum_{J=1}^{\infty} J(J + 1)(2J + 1)j_J^2(kr) = 2r^2/3$ we see that the $k$-integral associated with both equals $Vc_0/2s$, with $V$
the volume of the Mie sphere. With $W \sim 1/s^2$ the sum diverges logarithmically. The remaining angular momentum, associated with the stored energy $E_{\nu Jk}(a) - E_{\nu Jk}^{(0)}(a)$ of scattered vacuum radiation, is finite and depends smoothly on the two parameters $\omega_p/\omega_0$ and $\omega_0a/c_0$. The same is true for the term proportional to $(\varepsilon^{-1}(is) - 1)$ split off earlier, and that vanishes at large $s$ so that the integral over $s$ converges. This is summarized by

$$L_z + S_z = \frac{2hN_\omega r_e}{3\pi c_0} \int_0^\infty \frac{ds}{s} + f_{LS} \left( \frac{\omega_p}{\omega_0}, \frac{\omega_0a}{c_0} \right)$$

with $f_{LS}$ a smooth function of order unity and $N$ the number of dipoles in the sphere. The logarithmic divergence at high frequencies stems from the one already encountered in Eq. (13) for a single oscillator, and was already argued to be irrelevant. Therefore, the electromagnetic angular momentum due to orbit and spin induced by the quantum vacuum is of order $N \times h \times \omega_0 r_e/c_0$. The last factor $\approx 10^{-16}$ makes the transverse angular momentum entirely negligible.

### B. Longitudinal angular momentum

The longitudinal angular momentum is determined by the longitudinal electric field of the quantum vacuum. In the absence of magnetic field, the longitudinal modes exist only at the surface $r = a$ of the sphere. However, the spin operator $S_z$ associated with the magneto-optical perturbation (9) couples transverse electric modes ($\nu = e$) to longitudinal modes inside the sphere.

Any longitudinal Mie eigenfunction of $H_M$ defined in Eq. (15) can be written as $|\Psi_{\nu \epsilon f, J M k}\rangle = -k^{-1}\varepsilon(r)^{1/2}p|\Phi_{kJM}\rangle$, with $\Phi(r)$ essentially the scalar potential field, the factor $k$ introduced for reasons of normalization and dimension. Since $[L_z + S_z, p_l] = 0$ this is an eigenfunction of $L_z + S_z$ provided that $|\Phi_{kJM}\rangle$ is an eigenfunction of $L_z$. To have normalized longitudinal eigenfunctions, we require that

$$\langle \Psi_{kJM} | \Phi_{kJM} \rangle = \delta(k - k')\delta_{MM'}\delta_{JJ'}$$

This is satisfied if the scalar potentials are the orthonormal solutions of the eigenvalue equation,

$$p \cdot \varepsilon(r)p|\Phi_{kJM}\rangle = k^2|\Phi_{kJM}\rangle$$

normalized such that they behave far outside the sphere as $(4\pi/(2\pi)^{3/2})\sin(kr - \phi_J)/r$ [23]. The longitudinal angular momentum becomes ($k := \{kJM\}$),

$$J_{\text{long}, z} = -\frac{2kT}{c_0} \sum_{n=1}^\infty W(is_n) \sum_k \sum_{\nu = e, m} \sum_{\epsilon = +, -} \langle E_{\nu \epsilon} \rangle (\epsilon \cdot r) \frac{p^2|\Phi_{kJM}\rangle | \Phi_{kJM} \rangle \cdot p \cdot \theta_a S_z \cdot | E_{\nu \epsilon} \rangle}{k^2(s_n^2/c_0^2 + k^2)}$$

We inserted $|E_{\nu \epsilon}\rangle = \varepsilon(r, is)^{-1/2}|\Psi_{\nu \epsilon}\rangle$ for the transverse eigenfunctions. The sum over $\nu$ runs only over the two transverse modes since longitudinal modes alone cannot generate angular momentum. We will first ignore a surface contribution and use Eq. (20) away from $r = a$ where $p^2|\Phi_{kJM}\rangle = k^2\varepsilon^{-1}|\Phi_{kJM}\rangle$. The factor $k^2$ cancels, and using the closure relation $\Sigma_\nu |\Phi_{kJM}\rangle \langle \Phi_{kJM}| = 1$,

$$J_{\text{long}, z}^{(1)} = -\frac{2kT}{c_0} \sum_{n=1}^\infty W(is_n) \times \sum_{\kappa, \nu = e/m} \frac{(E_{\nu \epsilon n} \cdot (\epsilon \cdot r)) \varepsilon^{-1} \cdot p \cdot \theta_a S_z \cdot | E_{\nu \epsilon n} \rangle}{s_n^2/c_0^2 + k^2}$$

This expression can be further simplified by averaging over the orientation of the $z$-axis and by using $p\theta_a(r) = -i\mathbf{r} \cdot \partial / \partial \mathbf{r}$. To treat discontinuities across the surface in the term involving $\partial / \partial a$, we imagine the surface profile $\varepsilon(r) = 1 + (m^2 - 1)\theta_a$ to be a smooth function that decays from 1 to 0 over a small range $2\mu$ around $r = a$ and use

$$\int_a^{a+w} dr f(\theta_a) \frac{d\theta_a}{dr} = F(\theta_a = 0) - F(\theta_a = 1)$$

with $F$ the primitive function of $f$. Because the electric field $\mathbf{r} \cdot \mathbf{E}$ directed along the surface is continuous for a perfectly sharp surface, it should be approximately continuous in the thin layer around $r = a$. Thus, $1/\varepsilon(\theta_a) = 1/(1 + (m^2 - 1)\theta_a)$ is the only factor in $J_{\text{long}, z}^{(1)}$ that varies significantly across the surface. With Eq. (21) giving $-(\log m^2)/(m^2 - 1)$, the surface contribution thus becomes

$$J_{\text{long}, z}^{(1)} = \frac{2kT a^3}{c_0} \sum_{n=1}^\infty W(is_n) \sum_{\kappa, \nu = e/m} \frac{1}{s_n^2/c_0^2 + k^2} \times \log m^2$$

$$\frac{m^2 - 1}{m^2} \int \frac{d^2 \mathbf{r}}{m^2} |E_{\nu \epsilon n}(a \mathbf{r})|^2$$

with $E_{\nu \epsilon}^T = \mathbf{r} \times \mathbf{E}$ the electric field parallel to the surface. The term involving $\theta_a(S_z)_{nm} p_m E_{\nu \epsilon n}$ can be treated similarly, with the normal displacement vector $\varepsilon(r)E_{\nu \epsilon}^L \cdot (\varepsilon(r)\mathbf{r}) \cdot \mathbf{E}$ continuous across the surface. The sum is

$$J_{\text{long}, z}^{(1)} = \frac{2kT a^3}{c_0} \sum_{n=1}^\infty W(is_n) \sum_{\kappa, \nu = e/m} \frac{1}{s_n^2/c_0^2 + k^2} \times$$

$$\int \frac{d^2 \mathbf{r}}{m^2 a^3} \int dr r^2 |E_{\nu \epsilon n}(r)|^2$$

where all the fields are defined inside the sphere, and

$$T(m) = \frac{\log m^2}{m^2 - 1} - \frac{2m^2}{2m^2}$$

$$L(m) = \frac{3m^2}{2} - (m^2 - 1) \frac{1}{2m^2} - \frac{(m^2 - 1)(m^2 + 2)}{6m^2}$$
determine the weight of the transverse and normal components of the eigenfunctions at the surface. Note that first and third term in Eq. (22) are strictly positive, but that the normal components $E^p(a)$ at the surface can make a negative contribution when $m$ is large enough.

Finally, we have to consider the surface term ignored earlier, identified as

$$p^2 - \frac{1}{\varepsilon(r)}p \cdot \varepsilon(r)p = \frac{1}{\varepsilon(r)}(m^2 - 1) (i\partial_r\Phi_s)^2 \partial_r$$

Since the transverse electric field as well as the normal, longitudinal displacement vector $-\varepsilon(r)\partial_r\Phi_s$ should vary slowly over the thin layer, we identify the rapidly varying function $f = -(m^2 - 1)/\varepsilon(r)^2$ in the first matrix element of the expression for $J_{\text{long},z}$. Equation (21) produces $1 - 1/m^2$ for the integral around the surface, and one obtains,

$$J_{\text{long},z}^{(2)} = -\frac{2kT}{c_0} \sum_{n=1}^{\infty} W(is_n) \sum_{n'k} \sum_{n''} (m^2 - 1) s_n^*/c_0^2 + k^2 \times$$

$$\int_{r=a} \int d^3\mathbf{r} (iE^P_{\text{long}}) z E_{\text{long},r} \times \int d^3\mathbf{r} (E_{\text{long},\mathbf{r}} \cdot S_{\mathbf{z}} \cdot E_{\text{long},\mathbf{r}})$$

with all fields again to be evaluated inside the sphere, and where $\nabla \Phi_s = ikE_{\text{long}}$ has been substituted. Because of parity, only the transverse electric mode $\nu = c$ couples to the longitudinal mode.

We infer from Eqs. (22) and (23) that a significant amount of angular momentum can be expressed as an integral over the surface of the sphere. This is to be expected because that is where the longitudinal electric fields and the surface charges of the Mie sphere reside. Nevertheless, Eq. (22) clearly contains a bulk term as well, induced by longitudinal electric waves created by the Faraday effect. The angular momentum $J_{\text{long},z}^{(2)}$ resides at the surface as well but originates from a mode conversion of transverse electric to longitudinal electric field modes by the Faraday effect inside the sphere, as can be inferred from the second, bulk integral of Eq. (23).

Since $W(is) \sim 1/s^2$ for large $s$, no divergencies show up in the frequency integral of the longitudinal angular momentum and we have typically $s \sim \omega_0$. However, $J_{\text{long},z}^{(1)}$ and $J_{\text{long},z}^{(2)}$ share the divergence of the $k$-integral for all frequencies and for all $J$, that is not due to the singular quantum vacuum. This UV catastrophe would even exist for a Mie sphere exposed to a bandwidth-limited classical noise. It is caused by the breakdown of the description of the Mie sphere at the smallest length scale associated with Coulomb charges, which is the electron radius. We postulate that this divergence is regularized as was done in section III for a single oscillator.

To extract the divergence we focus on the Mie problem for fixed $s \sim \omega_0$ and large $k$. The transverse Mie eigenfunctions inside the sphere are given by

$$E_{\text{mk},JM}(r) = \frac{4\pi i^J}{(2\pi)^{3/2}} A_{nk,j}^{m,k} j_{kk}(mr) \hat{X}_{JM}(r)$$

$$E_{\text{mk},JM}(r) = \frac{4\pi i^J}{(2\pi)^{3/2}} A_{nk,j}^{m,k} \left[ \sqrt{J(J+1)}j_{kk}(mr) \hat{N}_{JM}(r) + (mrj_{kk}(mr))' \hat{Z}_{JM}(r) \right]$$

(24)

They involve complex coefficients $A_{nk,j}^{m,k}(is)$ that depend on $m$, $s$, and the size parameter $ka$ of the sphere. For $ka \to \infty$ we easily obtain from the exact expressions (12, 13)

$$|A_{nk,j}^{m,k}|^2 \to \frac{2}{m^2 + 1 + (m^2 - 1)\cos(2ka - J\pi)}$$

If we ignore the oscillating function in the integral over $k$, these asymptotic expressions no longer depend on $J$ and $k$. If we also restrict to zero temperature, the contribution of the transverse magnetic mode ($\nu = m$) to Eq. (22) is

$$J_{\text{long},z}^{(1,m)} = \frac{4\pi a^3}{3c_0} \int_0^{\infty} ds W(is) \frac{2}{m^2 + 1} \sum_{k,j} (2J + 1)$$

$$\times \left( T(m)j_{kk}^2(mka) + \frac{1}{2m^2a^2} \int_0^a \int d^3\mathbf{r} j_{kk}^2(mkr) \right)$$

We insert $\sum_{j=1}(2J + 1)j_{kk}^2(y) = 1 - j_{kk}^2(y)$ and ignore the oscillating term $j_{kk}^2$ that produces a finite and negligible longitudinal angular momentum proportional to the surface $a^2$ of the sphere. What remains is,

$$J_{\text{long},z}^{(1,m)} = \frac{4\pi a^3}{3c_0} \frac{3}{2r_e} \frac{1}{\int_0^{\infty} ds W(is) m^2 + 1} \left( T(m) + \frac{1}{6m^2} \right)$$

where we have regularized, as in section III, $4\pi/(2\pi)^3 f dk = 3/2r_e$. The transverse electric wave can be treated similarly, this time using the sums $\sum_{j=1}(J(J + 1))j_{kk}^2(y) = 2y^2/3$ and $\sum_{j=1}(2J + 1)[j_{kk}j_{kk}']^2 = y^2/3 + \sin^2 y$. The result is,

$$J_{\text{long},z}^{(1,c)} = \frac{4\pi a^3}{3c_0} \frac{3}{2r_e} \frac{1}{\int_0^{\infty} ds W(is) m^2 + 1} \times$$

$$\left( T(m) + 2L(m) \frac{1}{3} + \frac{1}{6m^2} \right)$$

Only when $m = 1$ is $J_{\text{long},z}^{(1,c)} = J_{\text{long},z}^{(1,m)}$. For the simple model (16) the Becquerel relation gives

$$W(is) = -\frac{2\omega_0 c_s d^2 m^2}{c_0} = \omega_c \left( m^2 + \frac{2}{3} \right)^2 2\rho_r c_0 s^2 (\omega_0^2 + s^2)^2$$

Upon substituting $x = s/\omega_0$,

$$J_{\text{long},z}^{(1)} = N\omega_0 \frac{1}{\omega_0^2} \int_0^{\infty} dx \frac{x^2}{(x^2 + 1)^2} \left( m^2 + \frac{2}{3} \right)^2 J_1(m)$$

(25)
Clearly, the electromagnetic vacuum at 0 is a function of \( \omega_p/\omega_0 \). The dashed curves \( J_1 \) and \( J_2 \) denote the angular momentum given by Eqs. (24) and (26), the solid black curve is the total longitudinal angular momentum, all expressed in units of the classical Einstein-De Haas effect. The difference is interpreted as the electromagnetic vacuum contribution to the classical Einstein-De Haas effect induced by the induced quantum vacuum at \( 0 \). Here \( J \) denotes the integral in Eq. (23) for the curling Poynting vector inside the Mie sphere, and is approximately equal to 1/3 for all \( \omega_p/\omega_0 \).

Here

\[
\mathcal{J}_1(m) = \frac{(3 + m^2)T(m) + \frac{1}{2}m^2 + 2m^2L(m) + \frac{1}{2}}{2(m^2 + 1)}
\]

Since \( m = m(x, \omega_p/\omega_0) \) in our simple model, the integral depends only on the ratio \( \omega_p/\omega_0 \) of plasma and resonant frequency. This angular momentum is proportional to the one of a single dipole, and the number \( N = 4\pi a^3 \rho/3 \) of the dipoles.

The extraction of the divergence in \( J^{(2)}_{\text{long, z}} \) is harder because of the double integral over \( k \) and \( k' \). We refer to Appendix B for details,

\[
J^{(2)}_{\text{long, z}} \approx N \times \hbar \frac{\omega_e}{\omega_0} \frac{1}{2\pi} \int_0^{\infty} dx \frac{x^2}{(x^2 + 1)^2} \left( \frac{m^2 + 2}{3} \right)^2 \mathcal{J}_2(m)
\]

with \( \mathcal{J}_2(m) = 4m^2(m^2 - 1)/(m^2 + 1)^2 \).

The calculations are summarized in Figure 1 where \( J^{(1)}_{\text{long, z}} \) and \( J^{(2)}_{\text{long, z}} \) are shown as a function of \( \omega_p/\omega_0 \). Clearly \( J^{(1)}_{\text{long, z}} \) dominates throughout. For a dilute sphere \( \omega_p/\omega_0 \ll 1 \), \( J_{\text{long, z}} = N \times \hbar \omega_e/\omega_0 \), recognized as the classical Einstein-De Haas effect induced by the induced angular momentum of the individual dipoles. The presence of the Mie sphere lowers the total angular momentum. The deviation is interpreted as the electromagnetic quantum vacuum contribution to the classical Einstein-De-Haas effect. This difference becomes significant once \( \omega_p/\omega_0 \approx 1 \). For larger values, the total angular momentum has even a sign opposite to the classical diamagnetic effect.

\[\text{V. A Poynting Vortex}\]

A finite electromagnetic angular momentum along the external magnetic field \( B_0 \) implies the existence of an electromagnetic momentum density \( \mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})/4\pi \epsilon_0 \) circulating around the external magnetic field \( B_0 \). According to Eq. (5), such a curl does neither move optical energy around, nor does it possess a net momentum. In the previous section we have seen that a part of the angular momentum resides in the bulk, because the Faraday effect induces a coupling between longitudinal and transverse electric modes.

Let us first address this issue for the harmonic, diamagnetic oscillator interacting described by Eq. (11) with the electromagnetic vacuum at \( 0 \). At any distance from the dipole we can calculate the expectation value of \( \mathbf{K}(\mathbf{r}) = \mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})/4\pi \epsilon_0 \) using the fluctuation-dissipation theorem [1]. We find a Casimir-Polder type formula,

\[
\mathbf{K}(\mathbf{r}) = -\hbar \omega_e \frac{r_e}{\omega_0 (4\pi)^2 r^3} \mathbf{r} \times \mathbf{B}_0
\]

with \( F(x) = \exp(2ix)(1 - ix)^2/x^3 \). In the static regime \( r \ll \omega_0/\epsilon_0 \), \( F = 1/x^3 \) so that

\[
\mathbf{K}(\mathbf{r}) = -\hbar \omega_e \frac{r_e}{\omega_0 (4\pi)^2 r^3} \mathbf{r} \times \mathbf{B}_0
\]

We see that \( K \) decays rapidly with distance so that it is constrained close to the dipole. The total angular momentum between \( r \) and \( r + dr \) is \( dJ_z \sim +\hbar(\omega_e/\omega_0)r_e dr/4\pi r^2 \). The angular momentum resides at length scales as small as \( r \sim r_e/4\pi \), consistent with Eq. (11). This conclusion is qualitatively consistent with the quantum mechanical picture of longitudinal angular momentum being confined to the charge carriers.

We next calculate the momentum density \( \mathbf{K}(\mathbf{r}) \) inside the Mie sphere, but stay away from the singular boundary. Spherical symmetry imposes that \( \mathbf{K}(\mathbf{r}) = (\hat{\mathbf{r}} \times \mathbf{B}_0) ) \cdot \mathbf{n} \), where \( \mathbf{n} \) is harder at length scales as small as \( r \sim r_e/4\pi \). Clearly \( J^{(1)}_{\text{long, z}} \) dominates throughout. For a dilute sphere \( \omega_p/\omega_0 \ll 1 \), \( J_{\text{long, z}} = N \times \hbar \omega_e/\omega_0 \), recognized as the classical Einstein-De Haas effect induced by the induced angular momentum of the individual dipoles. The presence of the Mie sphere lowers the total angular momentum. The deviation is interpreted as the electromagnetic quantum vacuum contribution to the classical Einstein-De-Haas effect. This difference becomes significant once \( \omega_p/\omega_0 \approx 1 \). For larger values, the total angular momentum has even a sign opposite to the classical diamagnetic effect.

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This clearly identifies surface and bulk contributions to explain why the sign for the angular momentum is identified as either bulk or surface contribution. This is shown in Figure \( \omega \). The angular momentum is induced by the Faraday effect, paramagnetic materials in particular, complicated and more realistic implementations of the Faraday effect on the same footing and provides an interesting opportunity to test it experimentally. Our major postulate of this work is also consistent with this viewpoint; the regularization of a macroscopic wave number integral on scales as small as the inverse electron radius is justified because macroscopic longitudinal angular momentum should still be associated with the microscopic charge carriers. The explicit temperature dependence of the quantum vacuum comes in when \( kT \sim \hbar \omega_0 \). Our formalism acknowledges the temperature dependence of the quantum vacuum, but as long as the atomic resonance \( \omega_0 \) is located in the visible regime of the electromagnetic spectrum, temperature dependence is negligible, as usual in diamagnetism.

A significant part of the vacuum-induced angular momentum resides at the surface of the sphere. Inside the sphere angular momentum emerges as a vortex in the electromagnetic Poynting vector that decays as \( 1/r \) from the center of the sphere.

A future challenge is to apply this approach to more complicated and more realistic implementations of the Faraday effect, paramagnetic materials in particular, with typical spins of order \( \hbar \) and significant dependence on temperature \( T \).

The author would like to thank Geert Rikken for useful advice.

Appendix A: regularization of divergence

The Helmholtz equation for a single electric dipole can be written as \( p^2 \Delta - V(r) \), with matter-light interaction \( V(r) = -d(r) \alpha(0) \times \omega^2/c^2 \) if we consider the electric dipole to be a point-like particle. In the limit \( \omega \to 0 \) the Born expansion applies and \( t(\omega) \).
\[ \alpha(\omega) = \frac{e^2}{mc_0^3} \left( 1 + \omega^2 \frac{\alpha_0^2}{\omega_0^2} + i \tau_R \omega^4 \right) \]

\[ = \alpha(0) - \frac{\omega^2}{\omega_0^2} \alpha(0) \cdot G_0(r = 0, \omega) + \cdots \]

The first line follows from the expansion of Eq. (10). The real part of \( G_0(r = 0, \omega) \) in the second line diverges. The longitudinal divergence generates a frequency-independent term and is physically due to a local field correction to the static polarizability \( \alpha(0) \). The transverse divergence in \( G_0(0) \) is independent on frequency and negative. We identify

\[ \text{Re} G_\ell(r = 0) = \frac{2}{3} \sum_k \frac{1}{k^2} \to \frac{1}{r_e} \] (A1)

The imaginary part does not diverge and \( \text{Im} G_0(r = 0) = -i\omega/6\pi\epsilon_0 \). This identifies \( \tau_R = r_e/6\pi\epsilon_0 \).

**Appendix B: Bulk-induced longitudinal angular momentum at surface**

We provide an approximate evaluation of the contribution to the longitudinal angular momentum. The longitudinal eigenfunctions inside the sphere obey Eq. (20). With normal displacement \( \varepsilon E^L \) and potential \( \Phi \) continuous at the boundary \( r = a \), and with proper normalization of \( \Phi \) far outside the sphere, the longitudinal modes inside the sphere are

\[ E_{k,J,M}(r) = \frac{4\pi i}{(2\pi)^{3/2}} A_{k,J}^L \frac{k}{r_m} \left[ j_j(kr/m')N_{J,M}(r) + \sqrt{J(J+1)} \frac{k}{kr/m} Z_{J,M}(r) \right] \]

By inserting the eigenfunction into the expression for the longitudinal angular momentum \( J_{\text{long},z}^{(2)} \) and by carrying out the angular integrations over \( \Phi \), using general properties of the vector spherical harmonics [14, 18], we get the still rigorous expression,

\[ J_{\text{long},z}^{(2)} = -\frac{ha^2}{3\pi\epsilon_0 (2\pi)^{3/2}} \int_0^\infty ds W(is) \frac{m^2 - 1}{m} \sum_k \sum J \times (2J + 1) |A_{k,J}^L|^2 \frac{k^2}{s^2/r_0^2 + k^2} (y j_j') (m ka)/(k'a/m) \]

\[ \times \int_0^\infty dr \theta \left\{ \frac{j_j(x)}{x} (y j_j') + (J(J+1) j_j x j_j(y) + j_j'(x)(y j_j')) \right\} \]

with \( x = k'r/m \) and \( y = mkr \). This expression diverges for large wave numbers but since two \( k \)-integrals exist the extraction of the divergence is more complicated. Our approximation will be that we consider this expression for both large \( k \) and \( k' \) in which case

\[ |A_{k,J}^L|^2 \to \frac{2m^2}{m^2 + 1 + (m^2 - 1) \cos(2ka/m - J\pi)} \]

We ignore all functions that oscillate with \( k \) and that produce finite corrections for the longitudinal momentum, proportional to the surface. To perform the integral involving \( k' \), we use

\[ \sum_k k^2 j'_j (ka) j_j(kr) = \frac{\pi}{2} \left( \frac{\delta(r-a)}{a^2} - \frac{J(J+1) r^{J-1}}{2J+1} \right) \]

with neglect of terms of measure \( 0 \) at \( r = a \). The result is,

\[ J^{(2)}_{\text{long},z} \approx \frac{ha}{6\pi\epsilon_0 (2\pi)^{3/2}} \int_0^\infty ds W(is) \frac{4m^2 (m^2 - 1)}{(m^2 + 1)^2} \times \]

\[ \left( - \sum_{J=1} (J+1)^2 \partial_\theta [a I_J(r,a)] + a \right) \]

\[ + \int dr \theta \delta(r-a) \sum_{J=1}^\infty (2J+1)(\partial_\theta a)(\partial_r r) I_J(r,a) \]

with the definition

\[ I_J(r,a) = \sum_k \frac{j'_j(kr) j_j(ka)}{k^2} \]

This function can be expressed using the hypergeometric function \( F(J - \frac{1}{2}, J + 1, 2, z) \) [23]. From this we can show that \( \partial_\theta [I_J(r,a)] \sim a/J^4 \) for large \( J \). Hence the first term in \( J^{(2)}_{\text{long},z} \) is finite and of order \( h(\omega_0/\omega_0) \times r_e a^2 \), which is negligible with respect to what was found in Eq. (25). Using a sum rule for spherical Bessel functions, the second term follows from

\[ \sum_{J=1}^\infty (2J+1)(\partial_\theta a)(\partial_r r) I_J(r,a) = \]

\[ \sum_k \frac{1}{k^2} (\partial_\theta a)(\partial_r r) \left( \sin ka / k(a-r) \right) \left( \sin ka / k \right) \]

\[ = \sum_k \left( \frac{\sin^2 ka}{k^2} - \frac{a^2}{3} \right) \text{ for } r = a \]

Regularizing as in Appendix A \( 4\pi/(2\pi)^3 \int dk = 3/2 r_e \) and from Eq. (21) \( \int dr \theta \delta(r-a) = 1/2 \), we find

\[ J^{(2)}_{\text{long},z} \approx \frac{ha^3}{3\pi r_e} \int_0^\infty ds W(is) \frac{4m^2 (m^2 - 1)}{(m^2 + 1)^2} \] (B1)
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