Unitarization of elastic amplitude on $SO_\mu(2.1)$ group

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Abstract

We obtain the solution of the unitarity equation for the elastic processes in terms of the expansion coefficients of the amplitude as a function on the $SO_\mu(2.1)$ group. This approach is a generalization of the eikonal representation to the case of small impact parameters and large transverse momenta. We show how the unitarity relation is modified when the contributions of the backward scattering are taken into account. We discuss the simplest models of the profile functions in the following cases: full reflection, full absorption and the combination of these two cases.

Keywords: Unitarity, scattering amplitude, analytic properties

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I. INTRODUCTION

Experimental data about elastic, inelastic and total cross-sections of $pp$-scattering at energy $\sqrt{s} = 14$ TeV will be obtained at the LHC. The most interesting objects of study in this energy region are the asymptotic behaviour of cross-sections at high energy, approaching of the Froissart limit and the behaviour of the ratio $\sigma_{el}/\sigma_{tot}$ at $s \to \infty$. Also the mechanism of a possible deviation of that ratio from the value $\sigma_{el}/\sigma_{tot} = 1/2$ (black disk model) is of interest.

The theoretical considerations to these problems are closely related to the unitarization of the elastic scattering amplitude [1], [2]. As a rule, for the unitarization of an amplitude one uses the eikonal approximation. It is well known that the eikonal approximation complies with the unitarity equation for the elastic amplitude only in a region of small scattering angles and large impact parameters.

In this paper, we obtain a solution of the unitarity equation in terms of expansion coefficients of the elastic amplitude as a function on the $SO_{\mu}(2)$ group, introduced in [3]-[5]. This approach allows us to generalize the eikonal formalism of the impact parameter to the large angle region and the small phase system volume of collided particles.

II. UNITARITY EQUATION AND ITS SOLUTION

At first, we note the main points of solution of the unitarity equation in terms of the partial waves. Let us consider two particle elastic process $A + B \to A + B$, FIG. 1. The unitarity equation is a consequence of $S$ matrix unitarity – $S + S = 1$:

$$i(F^+ - F) = F^+ F, \quad iF = S - I . \quad (2.1)$$

Using the expansion of the unity operator in Fock space and the translation invariance of the operator $F$, as a result of standard operations [6] one obtains the equation on the elastic amplitude $f^\pm$: $$2Imf^{(+)}(q_\perp; p) = p^2 \lambda(p) \sum_{\epsilon = \pm 1} \int d\Omega_{\vec{k}} f^{(+)}(k_\perp; q) f^{(+)}(k_\perp; p) + A_{inel}(q, p) , \quad (2.2)$$

here

$$A_{inel}(q, p) = \sum_{s} \int \prod_{i=1}^{s} d\vec{k}_i \delta^4(P_m - \sum_{i} k_i) \langle \{ \vec{k}_i \} | A | q; -\vec{q} \rangle^* \langle \{ \vec{k}_i \} | A | p; -\vec{p} \rangle ,$$

$$f^{(+)}(k_\perp; q) \equiv \langle \vec{k}_\perp, \epsilon \sqrt{q^2 - k^2_\perp}; k_1 = -\vec{k}| A | q; -\vec{q} \rangle,$$

$$d\Omega_{q} = \frac{1}{q \sqrt{q^2 - q^2_\perp}} dq_\perp .$$

$\epsilon$ is the sign of the projection of scattered particle momenta on the z-axis, $\epsilon = \pm 1$. The first term on the righthand side of equation (2.2) describes a two-particle contribution to the spectral density of the elastic amplitude, FIG. 1. The second term corresponds to inelastic intermediate states. For derivation of (2.2) we use an integral relation which is valid for an arbitrary function $\Phi(q)$:

$$\int dq \Phi(q) = \int q^2 dq d\Omega \Phi(q) = \sum_{\epsilon = \pm 1} \int q^2 dq d\Omega_{q} \Phi(q, \epsilon \sqrt{q^2 - q^2_\perp}) . \quad (2.4)$$
Due to the momentum conservation we have \( p = |\vec{k}| = |\vec{q}| = |\vec{p}| \). The matrix element of the \( A \)-operator is related to the matrix element of the \( F \)-operator in the following way:

\[
<f|F|in> = \delta^4(P_{in} - P_f) <f|A|in> .
\]  

(2.5)

The first and the second arguments of \( f^{(\mp)}(\vec{k}_\perp; \vec{q}) \) correspond to the final and the initial momenta of the scattered particle in the center-of-mass system (c.m.s.) respectively. The factor \( \lambda(p) \) is related to the relative velocity in c.m.s. by a relation:

\[
|\vec{u}| = \frac{p(E_A + E_B)}{E_A \cdot E_B} = \frac{1}{\lambda(p)} ,
\]  

(2.6)

where \( E_A, E_B \) are the energies of colliding particles in c.m.s.

Assuming \( \vec{q}_\perp = 0 \) in the initial equation (2.2), it is easy to obtain the optical theorem:

\[
Im f^{(+)}(\vec{q}_\perp = 0; \vec{p}) = \frac{1}{8\pi^{2}\lambda(p)}(\sigma_{el} + \sigma_{inel}) ,
\]  

(2.7)

where

\[
\sigma_{el}^{(\mp)}(p) = (2\pi)^2 p^2 \lambda^2(p) \int d\Omega_{\vec{k}} \left| f^{(\mp)}(\vec{k}_\perp, \vec{p}) \right|^2 ,
\]  

(2.8)

\[
\sigma_{inel} = (2\pi)^2 \lambda(p) \sum s \int \prod_{i=1}^s d\vec{k}_i \delta^4(P_{in} - \sum_i \vec{k}_i) |<\{\vec{k}_i\}|A|\vec{p}; -\vec{p} >|^2 .
\]  

(2.9)

Presently, only one consistent method of solving equation (2.2) is known. This method is based on the expansion of the amplitude \( f^{(\circ)}(\vec{k}_\perp; \vec{q}) \) over the representation of the group \( O(3) \), which is realized by Legendre polynomials \( P_l(z) \):

\[
f(\vec{k}; \vec{q}) = \sum_{l=0}^{\infty} (2l + 1) a_l(p) P_l \left( \frac{\vec{k} \cdot \vec{q}}{p^2} \right) .
\]  

(2.10)

Substituting expansion (2.10) into the unitarity equation (2.2), one obtains that \( a_l(p) \) satisfies the following

\[
Im a_l(p) = 2K |a_l(p)|^2 + g_{inel}(l, p) ,
\]  

(2.11)

here

\[
K = \pi p^2 \lambda(p) ,
\]  

(2.12)
\[ g_{\text{inel}}(l, p) = \frac{1}{4} \int_{-1}^{1} dc \, P_l(c) \, A_{\text{inel}}(\vec{q}, \vec{p}) , \quad c = \frac{\vec{q} \cdot \vec{p}}{p^2} . \quad (2.13) \]

Solving equation (2.11) one obtains the following representation for the partial amplitude \( a_l(p) \):
\[ a_l(p) = \eta e^{i\delta_l} - \frac{1}{4iK} , \quad \text{with} \quad g_{\text{inel}}(l, p) = \frac{1 - \eta_l^2}{8K} , \]

where \( \delta_l \) is an arbitrary real phase.

This is a well known result of the elastic amplitude unitarization in terms of coefficients of its expansion on the \( O(3) \) group. Such description of scattering in terms of partial waves \( a_l(p) \) is effective in low energy region, since the amplitude \( f(\vec{k}; \vec{q}) \) is saturated by the lowest terms of expansion (2.10). But, in a high energy region all terms in the sum (2.10) become important, and the use of integration instead of summation is needed. The procedure of changeover to the integration corresponds to the eikonal quasi-classic approach:

\[ l \approx bp , \quad P_l \left( \frac{\vec{k} \cdot \vec{q}}{p^2} \right) \approx J_0(\vec{k}_\perp) , \quad \text{at} \ \theta \sim 0 , \quad bp \gg 1 , \]

where \( b \) has to be understood as an impact parameter, \( k_\perp = p \sin \theta \) and \( \cos \theta = \vec{k} \cdot \vec{q}/p^2 \).

Hence, the expression for the amplitude (2.10) changes to the eikonal representation:
\[ f(\vec{k}; \vec{q}) = 2p^2 \int_0^\infty a(b, p) \, J_0(\vec{k}_\perp) \, b db , \quad (2.14) \]

here
\[ a(b, p) = \eta(b)e^{i\delta(b)} - \frac{1}{4iK} . \quad (2.15) \]

In that case (2.14), \( a(b, p) \) would satisfy condition (2.11).

However, here one has a problem: the eikonal amplitude (2.14) with the profile function (2.15) does not satisfy the initial unitarity equation (2.2). The reason for that is that Bessel functions \( J_0(\vec{k}_\perp) \) do not form an orthogonal system in the physical interval of values \( 0 \leq k_\perp \leq p \) i.e.
\[ \int_0^p J_0(\vec{k}_1) \, J_0(\vec{k}_2) \, k_\perp^2 dk_\perp \neq \delta(\vec{k}_1 - \vec{k}_2) . \]

In this paper, we attempt to solve the above mentioned problem in the context of the theoretical-group generalization of the impact parameter and obtain the unitarity equation in terms of profile functions \( u_p(b) \). In our formalism, the impact parameter \( b \) is interpreted as a vector of the maximal approach between two particles in c.m.s.

Let us point the main steps of our formalism. First, we construct a quantum analog of the impact parameter. As a basis we take the expression for components of the vector of maximal approach [3]:
\[ d_i = \frac{1}{q^2} \varepsilon_{ijk} q_j L_k , \quad (2.16) \]

where \( q_j \) is a relative momentum of two particles, \( L_k \) is a relative orbital moment in c.m.s.
We take the expression (2.16) as a basis for the construction of a quantum analog of the
impact parameter. The canonical commutation relation between a relative momentum and
a relative coordinate lies in the basis of quantization (here and later $\hbar = 1$):

$$[\xi_i, q_j] = i\delta_{ij}. $$

Then we perform the transition to the quantum description that consists in the standard
procedure of the substitution of the $C$-numbers by the corresponding operators in the relation
(2.16) and in the reduction of it to hermitian form. This leads us to the following system of
the commutation relations:

$$d_i = \frac{1}{q^2}(\varepsilon_{ijk}q_jL_k - iq_i),$$
$$d_i = (d_i)^+, \quad [d_i, q^2] = 0, \quad [L_i d_j] = i\varepsilon_{ijk}d_k,$$
$$[L_i L_j] = i\varepsilon_{ijk}L_k, \quad [d_i d_j] = -\frac{i}{q^2}\varepsilon_{ijk}L_k.$$

(2.17)

The additinal term in $d_i$ operator appears due to the requirement of hermiticity. The relations
(2.17) show that operators $d_i$ and $L_j$ form an algebra $SO(3.1)$ on the sphere $q^2 = const.$
The Casimir operator of this algebra is a pure number:

$$\hat{C} = d^2 - \frac{1}{q^2}L^2 \equiv \frac{1}{q^2},$$

therefore we cannot use the full algebra (2.17) for the construction of a wave function with
defined impact parameter.

One way to build a wave function with defined impact parameter consists in a selection
of a nontrivial subalgebra of (2.17) [4]. The natural choice is $d_1, d_2, L_3$ operators. They
form the following algebra:

$$[d_1 d_2] = -\frac{i}{q^2}L_3,$$
$$[d_1 L_3] = -id_2,$$
$$[d_2 L_3] = id_1.$$

(2.18)

This is the algebra of $SO(2.1)$ group. The properties of the $SO(2.1)$ group are studied
in detail in the monograph of N.J.Vilenkin [7], and also in [8]. This group is noncompact
and has the basic continued family of the unitary representation and the discrete finite-
dimensional family of the nonunitary representation in the space of quadratically integrable
function. The Casimir operator of algebra (2.18) is

$$\hat{K} = d^2_1 - \frac{1}{q^2}L^2_3, \quad d^2_\perp = d^2_1 + d^2_2, \quad [d_1, \hat{K}] = 0, \quad [d_3, \hat{K}] \neq 0.$$

(2.19)

As the next step, let us consider the following system of equations

$$\hat{K}\Psi = b^2\Psi, \quad L_3\Psi = m\Psi.$$

(2.20)

In this equation we interpret the eigenvalue of the Casimir operator $b^2$ as a squared impact
parameter or as a squared transverse component of the maximal approach vector of two
colliding particles.
Using the explicit form of operators $d_i$, $L_3$ in the momentum space we obtain the particular solution at $m = 0$:

$$\Psi_\mu(\vec{q}_\perp) = \frac{q}{\sqrt{q^2 - q_\perp^2}} P_{-1/2+i\mu} \left( \frac{q}{\sqrt{q^2 - q_\perp^2}} \right) = \frac{1}{|\cos \theta|} P_{-1/2+i\mu} \left( \frac{1}{|\cos \theta|} \right),$$

(2.21)

where $\theta$ is an azimuth angle of momentum $q$, $0 \leq \theta \leq \pi$, $q_\perp = q \sin \theta$, and

$$\mu = (q^2 b^2 - 1/4)^{1/2}, \quad b^2 \geq \frac{1}{4q^2}.$$  

So, $\Psi_\mu(\vec{q}_\perp)$ represents the wave function of two-particle state in the momentum space with the defined value of the squared impact parameter $b^2$. The spectrum restriction $q^2 b^2 \geq 1/4$ reflects the uncertainty relation for the phase space of two-particle system. It limits small $b^2$ region at fixed value of momentum $q$.

The connection between $\Psi_\mu(\vec{q}_\perp)$ and the eikonal representation kernel (2.10) follows from the Fock expansion of the cone function [9]:

$$P_{-1/2+i\mu} \left( \frac{q}{\sqrt{q^2 - q_\perp^2}} \right) = J_0(bq_{\perp}) + O \left( \frac{q_{\perp}}{q} \right) + O \left( \frac{1}{bq} \right).$$

(2.23)

As it was shown in [4], [5], the system of functions $\Psi_\mu(\vec{q}_\perp)$ forms an orthogonal and complete basis independently for each of the two parts of the momentum space $q_3 > 0$ and $q_3 < 0$ (so called forward and backward semi-spheres):

$$\int_0^{\pi/2} \Psi_\mu(\vec{q}_\perp) \Psi_\nu(\vec{q}_\perp) \sin \theta d\theta = \frac{1}{\mu \text{th}(\pi \mu)} \delta(\mu - \nu),$$

(2.24)

$$\int_0^{\infty} d\mu \ \mu \text{th}(\pi \mu) \Psi_\mu(\vec{q}_\perp) \Psi_\mu(\vec{q}_\perp') = \delta(|\cos \theta| - |\cos \theta'|).$$

(2.25)

The analog of the Legendre expansion (2.10) is the expansion of amplitude $f^{(e)}(\vec{k}_\perp; \vec{\kappa})$ as a function on the group $SO_\mu(2.1)$ over the basis of $\{\Psi_\mu(\vec{q}_\perp)\}$:

$$f^{(e)}(\vec{k}_\perp; \vec{\kappa}) = \int_0^\infty (v_0 u'_0) \ P_{-1/2+i\mu} (vu') \ u^{(e)}(\mu) \ \mu \text{th}(\pi \mu) d\mu,$$

(2.26)

where we introduce the cone variables:

$$v = (v_0, \vec{v}) = \left( \frac{\kappa}{\kappa_3}, \frac{\vec{k}_\perp}{\kappa_3} \right), \quad v^2 = v_0^2 - \vec{v}^2 = 1, \quad \kappa_3 > 0$$

$$u' = (u'_0, \vec{u}') = \left( \frac{k}{|k_3|}, \frac{\vec{k}_\perp}{|k_3|} \right), \quad u'^2 = u'_0^2 - \vec{u}'^2 = 1, \quad |\kappa| = |k| = \rho$$

$$(vu') = v_0 u'_0 - (\vec{v} \cdot \vec{u}').$$
We call the expansion coefficient $u^{(ε)}_p(µ)$ in (2.26) the profile function on the group $SO_µ(2.1)$. Substituting the expansion (2.26) into the initial equation (2.2), we obtain the unitarity condition for the profile functions $u^{(ε)}_p(µ)$ on the group $SO_µ(2.1)$:

$$Im\ u^{(+)}_p(µ) = K \sum_{ε=±1} |u^{(ε)}_p(µ)|^2 + G^{(+)}_{inel}(µ, p),$$

where

$$G^{(+)}_{inel}(µ, p) = \frac{1}{4π} \int dΩ_q Ψ_µ(q_⊥) A_{inel}(\{q_⊥, q_3 = +\sqrt{q^2 - q_⊥^2}\}, p).$$

The equation (2.28) is the main result of this article. It differs from the unitarity relation for $a_λ(p)$ (2.11) by a presence of the signature $ε$. The appearance of signature $ε$ allows us to consider scattering amplitudes independently in forward and backward semi-sphere.

In course of the derivation of (2.28) we used the summation theorem for spherical Legendre functions [10], [11]:

$$P_{−1/2+iµ} \{zz'−(z^2 − 1)^{1/2}(z'^2 − 1)^{1/2}\cos φ\} = P_{−1/2+iµ}(z) P_{−1/2+iµ}(z') + 2 \sum_{m=1}^{∞} (-1)^m \frac{Γ(−1/2 + iµ + m + 1)}{Γ(−1/2 + iµ + m)} \frac{P^{−1/2+iµ}(z)}{P^{−1/2+iµ}(z')} \cos mφ.$$  

(2.30)

For convinience we rewrite $G^{(+)}_{inel}(µ, p)$ from (2.29) in the following parametric form:

$$G^{(+)}_{inel}(µ, p) = \frac{1 − η^2_{inel}(µ)}{4K}.$$  

(2.31)

The parameter $η_{inel}(µ)$ plays the role of a standard inelastic coefficient and it varies within bounds $0 ≤ η_{inel}(µ) ≤ 1$. The expression for cross-sections in terms of the profile functions $u^{(±)}_p(µ)$ follows from the optical theorem (2.7) and the relation (2.28).

$$σ_{el} = σ^{(+)}_{el} + σ^{(−)}_{el},$$

$$σ^{(±)}_{el} = (2π)^3 p^2 λ^2(p) \int_0^{∞} (|u^{(±)}_p(µ)|^2) dΩ_µ,$$  

(2.32)

$$σ_{inel} = \frac{2π}{p^2} \int_0^{∞} (1 − η^2_{inel}(µ))dΩ_µ.$$

III. SIMPLE MODELS

As an application we discuss the simplest phenomenological models for the profile function $u^{(±)}_p(µ)$, which is consistent with unitarity condition (2.28).

Let us divide the interval of the impact parameter $b$ values in three parts:

- $0 ≤ b ≤ R_0(p) = 1/2p$ - the interval of forbidden $b$-values due to the condition $b^2p^2 ≥ 1/4$,
- $R_0(p) ≤ b ≤ R_{refl}(p)$ - the interval of full reflection,
- $R_{refl}(p) ≤ b ≤ R(p)$ - the interval of full absorption.
FIG. 2: The structure of the scattering area

A. Full absorption

First we assume that \( R_{refl}(p) = R_0(p) \), which corresponds to the full absorption target. In that case we have

\[
|u_p^{(-)}(\mu)|^2 \equiv 0, \quad \eta_{inel} = 0,
\]

in the whole area of the disk \( R_0(p) \leq b \leq R(p) \). The unitarity condition (2.28) takes the form

\[
\text{Im} \ u_p^{(+)}(\mu) = K |u_p^{(+)}(\mu)|^2 + \frac{1}{4K}.
\] (3.1)

The solution of the equation (3.1) is

\[
u_p^{(+)}(\mu) = \frac{i}{2K}, \quad R_0(p) \leq b \leq R(p).
\] (3.2)

Hence, for the cross-sections \( \sigma_{el} \) and \( \sigma_{inel} \) we obtain:

\[
\sigma_{el} = \sigma_{el}^{(+)} = \sigma_{inel} \approx \frac{2\pi}{p^2} \sqrt{R^2p^2 - 1/4} \int_0^{\mu_{th}(\pi \mu)} \mu \, d\mu.
\] (3.3)

For the case of \( p^2R^2 \gg 1/4 \) we find

\[
\sigma_{el} = \sigma_{el}^{(+)} = \sigma_{inel} \approx \pi R^2,
\]

\[
\sigma_{tot} = \sigma_{el} + \sigma_{inel} \approx 2\pi R^2,
\]

\[
\sigma_{el}/\sigma_{tot} = 1/2.
\] (3.4)

Thus, using (2.28) we obtain well known results for the black disk model.

B. Full reflection

Now we consider the model of the full reflection disk, or in other words let \( R_{refl}(p) \) be equal \( R(p) \). In this case we have

\[
G_{inel} = 0, \quad R_0(p) \leq b \leq R(p).
\]

The unitarity relation (2.28) takes the form

\[
\text{Im} \ u_p^{(+)}(\mu) = K (|u_p^{(+)}(\mu)|^2 + |u_p^{(-)}(\mu)|^2),
\] (3.5)
We rewrite this relation as follows

\[ Im \ u_p^{(+)}(\mu) = K |u_p^{(+)}(\mu)|^2 + \frac{1 - \eta_p^2(\mu)}{4K}, \]  

(3.6)

where

\[ \eta_p(\mu) = \sqrt{1 - 2K u_p^{(-)}(\mu)^2}, \quad 0 \leq \eta_p(\mu) \leq 1. \]  

(3.7)

The solution of equation (3.6) is

\[ u_p^{(+)}(\mu) = \frac{\eta_p(\mu) e^{2i\delta(\mu,p)} - 1}{2iK}. \]  

(3.8)

where \( \delta(\mu,p) \) is an arbitrary real phase. So, we see that the amplitude of scattering into a backward semi-sphere \( u_p^{(-)}(\mu) \) plays a role of the effective absorption coefficient for the scattering amplitude into forward semi-sphere \( u_p^{(+)}(\mu) \).

Let us define the absolute elastic scattering on a disk with radius \( R \) as the scattering with parameters

\[ \eta_p(\mu) = 0, \quad \text{if} \quad R_0(p) \leq b \leq R(p), \]  

\[ \eta_p(\mu) = 1, \quad \delta(\mu,p) = 0 \quad \text{if} \quad b \geq R(p). \]  

(3.9)

In this case we have:

\[ |u_p^{(-)}(\mu)|^2 = \frac{1}{4K^2}, \quad u_p^{(+)}(\mu) = \frac{i}{2K}, \quad \text{if} \quad R_0(p) \leq b \leq R(p), \]  

\[ u_p^{(-)}(\mu) = u_p^{(+)}(\mu) = 0, \quad \text{if} \quad b \geq R(p). \]  

(3.10)

Then for the elastic scattering we obtain

\[ \sigma_{el} = \sigma_{el}^{(+)} + \sigma_{el}^{(-)} = 4\pi \frac{\sqrt{R^2p^2 - 1/4}}{p^2} \int_0^\mu \frac{\mu}{\theta(\pi \mu)} d\mu \simeq 2\pi R^2. \]  

(3.11)

And for the cross-section ratio we have

\[ \sigma_{el}/\sigma_{tot} \simeq 1. \]

C. Combining model of reflection and absorption

The integration interval for \( \mu \) in cross-sections \( \sigma_{el} \) and \( \sigma_{inel} \) is divided into the interval of full reflection \((0 \leq \mu \leq \mu_{refl})\) and the interval of full absorption \((\mu_{refl} \leq \mu \leq \mu_R)\) as illustrated in FIG. 2. Here

\[ \mu^2 = 0 = p^2R_0^2(p) - 1/4, \]  

\[ \mu_{refl}^2 = p^2R_{refl}^2(p) - 1/4, \]  

\[ \mu_R^2 = p^2R^2(p) - 1/4. \]  

(3.12)
FIG. 3: The dependence of the ratio $\sigma_{el}/\sigma_{tot}$ from $R_{refl}$

Using obtained in the sections III.A and III.B expressions for $u^{(+)}_p(\mu)$, $u^{(-)}_p(\mu)$, $\eta_{inel}$ and the expression for cross-section (2.32), it is easy to find that

$$\sigma^{(+)}_{el} = \frac{2\pi}{p^2} \int_0^{\mu_R} d\Omega_\mu , \quad \sigma^{(-)}_{el} = \frac{2\pi}{p^2} \int_0^{\mu_{refl}} d\Omega_\mu , \quad \sigma_{inel} = \frac{2\pi}{p^2} \int_{\mu_{refl}}^{\mu_R} d\Omega_\mu . \quad (3.13)$$

Thus, for the cross-section ratio we obtain

$$\sigma_{el}/\sigma_{tot} = 1 - \Delta , \quad (3.14)$$

where

$$\Delta = \frac{1}{2} \int_0^{\mu_{refl}} \frac{d\Omega_\mu}{\mu_R} . \quad (3.15)$$

The values of $\Delta$ lie within the range $0 \leq \Delta \leq 1/2$. The limit value $\Delta = 0$ is reached for $R_{refl} \to R$, which corresponds to the case of the full reflection. The limit value $\Delta = 1/2$ is reached for $R_{refl} \to R_0$, which corresponds to the case of the full absorption. In the region of the phase volume $R^2 p^2$, $R_{refl}^2 p^2 \gg 1/4$, we can obtain:

$$\sigma_{el}/\sigma_{tot} \simeq \frac{1}{2} \left( 1 + \frac{R_{refl}^2}{R^2} \right) , \quad R_{refl} \leq R . \quad (3.16)$$

The results of the numerical calculations of integrals (3.14) at $\sqrt{s} = 14$ TeV and $R = 1 F$ are shown in FIG. 3. One can see that the deviation of the ratio $\sigma_{el}/\sigma_{tot}$ from the value of 1/2 starts in the region, where $R_{refl}$ is of order $R$. In the region of $R_{refl} \sim R$ the value of $\sigma_{el}/\sigma_{tot}$ changes sharply and converges to 1. Thus, the unitarity saturation takes place on an object with a very narrow full absorption periphery, $\Delta R(p)/R \ll 1$. In other words, the
unitarity saturation results from the processes of scattering to backward semi-sphere (also called Reflective Scattering).

There is a prediction for the ratio of cross-section \( \sigma_{el}/\sigma_{tot} \) reaching the value of 0.67 at LHC energies \( \sqrt{s} = 14 \text{ TeV} \). For these values, from relation (3.14) we have \( R_{refl} \approx 0.58F \). Thus, we conclude that, for the energies \( \sqrt{s} = 14 \text{ TeV} \), the size of the full reflection region is almost of the same order as the size of the full absorption region.

IV. CONCLUSION

In the present paper, we have shown that the unitarity equation for the elastic processes can be rewritten in terms of the profile functions \( u_{\pm}^{\pm}(b) \), where \( b \) is the generalization of the impact parameter in the context of the group-theoretical approach [4]. In such terms the unitarity equation is local for the whole interval of \( b \), \( 1/2p \leq b \leq \infty \), where \( p \) is the momentum in c.m.s. The signature \( \epsilon = \pm \) of the scattering into the forward and the backward semi-sphere appears in the unitarity equation. The appearance of signature \( \epsilon \) allows us to consider scattering amplitudes independently in forward and backward semi-sphere.

The analysis of the dependence of the ratio \( \sigma_{el}/\sigma_{tot} \) on the radius of the full reflection \( R_{refl}(p) \) and the diffraction radius \( R(p) \) is given within the limits of the simplest models for the profile function \( u_{\pm}(\mu) \). It is shown that the limit of the unitarity saturation \( \sigma_{el}/\sigma_{tot} \rightarrow 1 \) is achieved at the limit case of full reflection, \( R_{refl}(p) \rightarrow R(p) \).

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