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Existence theory for sequential fractional differential equations with anti-periodic type boundary conditions

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Abstract: We develop the existence theory for sequential fractional differential equations involving Liouville-Caputo fractional derivative equipped with anti-periodic type (non-separated) and nonlocal integral boundary conditions. Several existence criteria depending on the nonlinearity involved in the problems are presented by means of a variety of tools of the fixed point theory. The applicability of the results is shown with the aid of examples. Our results are not only new in the given configuration but also yield some new special cases for specific choices of parameters involved in the problems.

Keywords: Sequential fractional differential equations, Liouville-Caputo, Anti-periodic, Nonlocal, Existence, Fixed point

MSC: 34A08, 34B10, 34B15

1 Introduction

Recently, there has been an utterly great interest in developing theoretical analysis for boundary value problems of nonlinear fractional-order differential equations supplemented with a variety of boundary conditions. It has been mainly due to nonlocal nature of fractional-order differential operators which take into account memory and hereditary properties of some important and useful materials and processes. Fractional calculus has played a key role in improving the mathematical modelling of several phenomena occurring in engineering and scientific disciplines, such as blood flow problems, control theory, aerodynamics, nonlinear oscillation of earthquake, the fluid-dynamic traffic model, polymer rheology, regular variation in thermodynamics etc. For more details and explanation, see, for instance [1–3]. Some recent results on fractional-order boundary value problem can be found in a series of papers [4–12] and the references cited therein. Sequential fractional differential equations have also received considerable attention, for instance see [13–17].

Anti-periodic boundary conditions are found to be quite significant and important in the mathematical modeling of certain physical processes and phenomena, for example trigonometric polynomials in the study of interpolation.
problems, wavelets, physics etc. For more details, see [18] and the references cited therein. For some recent works on fractional-order anti-periodic boundary value problems, we refer the reader to [19–23]. However, the study of sequential fractional differential equations equipped with anti-periodic boundary conditions is yet to be initiated.

In this paper, we study new boundary value problems of Liouville-Caputo type sequential fractional differential equation:

\[ (C^\alpha D + k C^\alpha D^{-1})u(t) = f(t, u(t)), \quad 1 < \alpha \leq 2, \quad 0 < t < T, \quad T > 0, \]

subject to anti-periodic type (non-separated) boundary conditions of the form:

\[ \alpha_1 u(0) + \rho_1 u(T) = \beta_1, \quad \alpha_2 u'(0) + \rho_2 u'(T) = \beta_2, \]

and anti-periodic type (non-separated) nonlocal integral boundary conditions:

\[ \alpha_1 u(0) + \rho_1 u(T) = \lambda_1 \int_0^T u(s)ds + \lambda_2, \quad \alpha_2 u'(0) + \rho_2 u'(T) = \mu_1 \int_0^T u(s)ds + \mu_2, \]

where \( C^\alpha D \) denotes the Liouville-Caputo fractional derivative of order \( \alpha, \quad k \in \mathbb{R}^+, \quad 0 < \eta < \xi < T, \quad \alpha_1, \alpha_2, \rho_1, \rho_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \mu_1 \mu_2 \in \mathbb{R} \) with \( \alpha_1 + \rho_1 \neq 0, \alpha_2 + \rho_2 e^{-kT} \neq 0, \) and \( f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) is a given continuous function. Instead of writing the so-called “Caputo” derivative, we will call it “Liouville-Caputo” derivative as it was introduced by Liouville many decades ago.

The rest of the paper is organized as follows. In Section 2, we recall some basic concepts of fractional calculus and obtain the integral solution for the linear variants of the given problems. Section 3 contains the existence results for problem (1)-(2) obtained by applying Schaefer’s fixed point theorem, Leray-Schauder’s nonlinear alternative, Leray-Schauder’s degree theory, Banach’s contraction mapping principle and Krasnoselskii’s fixed point theorem. In Section 4, we provide the outline for the existence results of problem (1)-(3).

**2 Preliminaries and auxiliary results**

This section is devoted to some basic definitions of fractional calculus [1, 2] and auxiliary lemmas.

**Definition 2.1.** The fractional integral of order \( q \) with the lower limit zero for a function \( f \) is defined as

\[ I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad t > 0, \quad q > 0, \]

provided the right hand-side is point-wise defined on \([0, \infty)\), where \( \Gamma(\cdot) \) is the gamma function, which is defined by \( \Gamma(q) = \int_0^\infty t^{q-1} e^{-t} dt \).

**Definition 2.2.** The Riemann-Liouville fractional derivative of order \( q > 0, \quad n-1 < q < n, \quad n \in \mathbb{N} \), is defined as

\[ D_{0+}^q f(t) = \frac{1}{\Gamma(n-q)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-q-1} f(s)ds, \]

where the function \( f(t) \) has absolutely continuous derivative up to order \( n-1 \).

**Definition 2.3.** The Liouville-Caputo derivative of order \( q \) for a function \( f : [0, \infty) \rightarrow \mathbb{R} \) can be written as

\[ C^q D f(t) = D^q \left( f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n-1 < q < n. \]
Remark 2.4. If \( f(t) \in C^n[0, \infty) \), then
\[
^c D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^q+1-n} ds = I^{n-q} f^{(n)}(t), \quad t > 0, \quad n-1 < q < n.
\]

To define the fixed point problems associated with problems (1)-(2) and (1)-(3), we consider the following lemmas dealing with the linear variant of equation (1).

Lemma 2.5. Let \( h \in C([0, T], \mathbb{R}) \). Then the problem consisting of the equation
\[ (^c D^\alpha + k \cdot D^{\alpha-1}) u(t) = h(t), \quad 1 < \alpha \leq 2, \quad 0 < t < T, \quad T > 0, \] (4)
and the boundary conditions (2) is equivalent to the integral equation
\[
u(t) = v_1(t) + \int_0^t e^{-k(t-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} h(x) dx \right) ds + v_2(t) \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s) ds + v_3(t) \int_0^T e^{-k(T-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} h(x) dx \right) ds,
\] (5)
where
\[
v_1(t) = \frac{\beta_1}{(\alpha_1 + \rho_1)} + \frac{(\alpha_1 + \rho_1 e^{-kT}) - (\alpha_1 + \rho_1 e^{-k1})}{k(\alpha_1 + \rho_1)(\alpha_2 + \rho_2 e^{-kT})} \beta_2,
\]
\[
v_2(t) = \frac{\rho_2(\alpha_1 + \rho_1 e^{-k1} - (\alpha_1 + \rho_1 e^{-k1}))}{k(\alpha_1 + \rho_1)(\alpha_2 + \rho_2 e^{-kT})},
\]
\[
v_3(t) = \frac{\alpha_1 \rho_2 - \alpha_2 \rho_1 - \rho_2(\alpha_1 + \rho_1) e^{-k1}}{(\alpha_1 + \rho_1)(\alpha_2 + \rho_2 e^{-kT})}.
\] (6)

Proof. As argued in [13], the general solution of (4) can be written as
\[
u(t) = A_0 e^{-kt} + A_1 + \int_0^t e^{-k(t-s)} I^{\alpha-1} h(s) ds,
\] (7)
where \( A_0 \) and \( A_1 \) are arbitrary constants and
\[ I^{\alpha-1} h(t) = \int_0^t \frac{(t-x)^{\alpha-2}}{\Gamma(\alpha-1)} h(x) dx.
\]
Differentiating (7) with respect to \( t \), we obtain
\[
u'(t) = -k A_0 e^{-kt} - k \int_0^t e^{-k(t-s)} I^{\alpha-1} h(s) ds + I^{\alpha-1} h(t).
\] (8)
Using the boundary conditions (2) in (7) and (8), we get
\[ A_0(\alpha_1 + \rho_1 e^{-kT}) + A_1(\alpha_1 + \rho_1) + \rho_1 \int_0^T e^{-k(T-s)} I^{\alpha-1} h(s) ds = \beta_1, \] (9)
\(-kA_0(\alpha_2 + \rho_2 e^{-kT}) - k\rho_2 \int_0^T e^{-k(T-s)} t^{\alpha-1} h(s) ds + \rho_2 t^{\alpha-1} h(T) = \beta_2.\)  

(10)

Solving the system (9) and (10) for \(A_0\) and \(A_1\), we get

\[
A_0 = \frac{-\beta_2}{k(\alpha_2 + \rho_2 e^{-kT})} + \frac{\rho_2}{k(\alpha_2 + \rho_2 e^{-kT})} \left\{ t^{\alpha-1} h(T) - k \int_0^T e^{-k(T-s)} \left( \int_0^s (s-x)^{\alpha-2} h(x) dx \right) ds \right\},
\]

\[
A_1 = \frac{\beta_1}{\alpha_1 + \rho_1} + \frac{\rho_2(\alpha_1 + \rho_1 e^{-kT})}{(\alpha_1 + \rho_1)(\alpha_2 + \rho_2 e^{-kT})} \int_0^T (T-s)^{\alpha-2} h(s) ds
- \frac{\rho_2(\alpha_1 + \rho_1 e^{-kT})}{(\alpha_1 + \rho_1)(\alpha_2 + \rho_2 e^{-kT})} \int_0^T e^{-k(T-s)} \left( \int_0^s (s-x)^{\alpha-2} h(x) dx \right) ds.
\]

Substituting the values of \(A_0\) and \(A_1\) in (7) yields the solution (5). Conversely, by direct computation, it can be established that (5) satisfies the equation (4) and boundary conditions (2). This completes the proof.

Lemma 2.6. Let \(h \in C([0, T], \mathbb{R})\). Then the problem consisting of linear equation (4) equipped with boundary conditions (3) is equivalent to the integral equation

\[
u(t) = B_1(t) \left\{ \lambda_1 \int_0^s \left( \int_0^x e^{-k(s-x)} t^{\alpha-1} h(x) dx \right) ds + \mu_1 \int_0^T e^{-k(T-s)} t^{\alpha-1} h(s) ds + \lambda_2 \right\}
+ B_2(t) \left\{ \mu_1 \int_0^T e^{-k(T-s)} t^{\alpha-1} h(s) ds + k\mu_2 \int_0^T e^{-k(T-s)} t^{\alpha-1} h(s) ds
- \rho_2 t^{\alpha-1} h(T) + \mu_2 \right\} + \int_0^T e^{-k(t-s)} t^{\alpha-1} h(s) ds,
\]

(11)

where

\[
B_1(t) = \frac{(\epsilon_2 e^{-kT} + \delta_2)}{\Delta}, \quad B_2(t) = \frac{(\epsilon_1 e^{-kT} - \delta_1)}{\Delta}, \quad \Delta = \delta_1 \epsilon_2 + \delta_2 \epsilon_1,
\]

\[
\delta_1 = \alpha_1 + \rho_1 e^{-kT} + \lambda_1 (e^{-kT} - 1), \quad \epsilon_1 = (\alpha_1 + \rho_1 - \lambda_1 \eta),
\]

\[
\delta_2 = -k\alpha_2 - k\rho_2 e^{-kT} + \mu_1 (e^{-kT} - e^{-kT}), \quad \epsilon_2 = \mu_1 (T - \xi).
\]

Proof. Since the proof is similar to that of Lemma 2.5, we omit it.

3 Existence results for the problem (1)-(2)

In view of Lemma 2.5, we introduce a fixed point problem associated with the problem (1)-(2) as follows:

\(u = \mathcal{H}u.\)  

(13)
where the operator $\mathcal{H} : \mathcal{E} \to \mathcal{E}$ is
\[
(\mathcal{H}u)(t) = v_1(t) + \int_0^t e^{-k(t-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} f(x, u(x)) \, dx \right) \, ds + v_2(t) \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, u(s)) \, ds
\]
\[+ v_3(t) \int_0^T e^{-k(T-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} f(x, u(x)) \, dx \right) \, ds.
\]
(14)

Here $\mathcal{E} = C([0, T], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0, T] \to \mathbb{R}$ endowed with the norm defined by $\|u\| = \sup \{ |u(t)| : t \in [0, T] \}$.

Observe that that problem (1)-(2) has solutions if the operator equation (13) has fixed points. For computational convenience, we set the notation:
\[
Q = \sup_{t \in [0, T]} \left\{ \frac{1 - e^{-kt}}{k\Gamma(\alpha)^{1-\alpha}} + \frac{|v_2(t)|}{\Gamma(\alpha)T^{1-\alpha}} + \frac{|v_3(t)|(1 - e^{-kT})}{k\Gamma(\alpha)T^{1-\alpha}} \right\}.
\]
(15)

Now we are in a position to present our main results for the problem (1)-(2). The first one deals with Schaefer’s fixed point theorem [24].

**Lemma 3.1.** Let $X$ be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $Y = \{ u \in X \mid u = \mu Tu, 0 < \mu < 1 \}$ is bounded. Then $T$ has a fixed point in $X$.

**Theorem 3.2.** Assume that there exists a positive constant $L_1$ such that $|f(t, u(t))| \leq L_1$ for $t \in [0, T]$, $u \in \mathbb{R}$. Then the boundary value problem (1)-(2) has at least one solution on $[0, T]$.

**Proof.** In the first step, we show that the operator $\mathcal{H}$ defined by (14) is completely continuous. Observe that continuity of $\mathcal{H}$ follows from the continuity of $f$. For a positive constant $r$, let $B_r = \{ u \in \mathcal{E} : \|u\| \leq r \}$ be a bounded ball in $\mathcal{E}$. Then for $t \in [0, T]$, we have
\[
|(\mathcal{H}u)(t)| \leq |v_1(t)| + \int_0^t e^{-k(t-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} |f(x, u(x))| \, dx \right) \, ds + |v_2(t)| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, u(s))| \, ds
\]
\[+ |v_3(t)| \int_0^T e^{-k(T-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} |f(x, u(x))| \, dx \right) \, ds.
\]
\[
\leq |v_1(t)| + L_1 \int_0^t e^{-k(t-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} \, dx \right) \, ds + |v_2(t)| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \, ds
\]
\[+ |v_3(t)| \int_0^T e^{-k(T-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} \, dx \right) \, ds \leq L_1 Q + \|v_1\|,
\]
which consequently implies that
\[
\| (\mathcal{H}u) \| \leq L_1 Q + \|v_1\|.
\]

where $Q$ is defined by (15).

Next we show that the operator $\mathcal{H}$ maps bounded sets into equicontinuous sets of $\mathcal{E}$. Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$ and $u \in B_r$. Then we have
\[
|(\mathcal{H}u)(\tau_2) - (\mathcal{H}u)(\tau_1)| \leq |v_1(\tau_2) - v_1(\tau_1)|
\]
Then, for sets (balls) into bounded sets in \( E \)

**Proof.**

As \( k \rightarrow 0 \), the right-hand side of the above inequality tends to zero independently of \( u \in B_r \). Therefore, by the Arzelà-Ascoli theorem, the operator \( \mathcal{H} \) is completely continuous.

Finally, we consider the set \( V = \{ u \in E : u = \mu \mathcal{H}u, \ 0 < \mu < 1 \} \) and show that \( V \) is bounded. For \( u \in V \) and \( t \in [0, T] \), we get

\[
\| u \| \leq L_1 Q + \| v_1 \|.
\]

Therefore, \( V \) is bounded. Hence, by Lemma 3.1, the problem (1)-(2) has at least one solution on \([0, T]\).

Our next existence result is based on Leray-Schauder’s nonlinear alternative.

**Lemma 3.3** (Nonlinear alternative for single valued maps [2–5]). Let \( E \) be a Banach space, \( C \) a closed, convex subset of \( E \), \( U \) an open subset of \( C \) and \( 0 \in U \). Suppose that \( \mathcal{A} : U \rightarrow C \) is a continuous, compact (that is, \( \mathcal{A}(U) \) is a relatively compact subset of \( C \)) map. Then either

(i) \( \mathcal{A} \) has a fixed point in \( U \), or

(ii) there is an \( x \in \partial U \) (the boundary of \( U \) in \( C \)) and \( \lambda \in (0, 1) \) with \( x = \lambda \mathcal{A}(x) \).

**Theorem 3.4.** Assume that

\( (E_1) \) there exists a continuous nondecreasing function \( \chi : [0, \infty) \rightarrow (0, \infty) \) and a function \( p \in C([0, T], \mathbb{R}^+) \) such that

\[
|f(t, u)| \leq p(t) \chi(\| u \|) \text{ for each } (t, u) \in [0, T] \times \mathbb{R};
\]

\( (E_2) \) there exists a constant \( N > 0 \) such that

\[
\frac{N}{\chi(N) \| p \| Q + \| v_1 \|} > 1,
\]

where \( Q \) is given by (15).

Then the boundary value problem (1)-(2) has at least one solution on \([0, T]\).

**Proof.** We complete the proof in different steps. We first show that the operator \( \mathcal{H} \) defined by (14) maps bounded sets (balls) into bounded sets in \( E \). For a positive constant \( r \), let \( B_r = \{ u \in E : \| u \| \leq r \} \) be a bounded ball in \( E \). Then, for \( t \in [0, T] \), we have

\[
|(\mathcal{H}u)(t)| \leq |v_1(t)| + \chi(\| u \|) \| p \| \left( \int_0^t e^{-k(t-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} \, dx \right) \, ds + |v_2(t)| \int_0^T (T-s)^{\alpha-2} \Gamma(\alpha-1) \, ds \right)
\]

\[
+ |v_3(t)| \int_0^T e^{-k(T-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} \, dx \right) \, ds \right) \leq |v_1| + \chi(\| u \|) \| p \| Q,
\]

which implies that \( \| (\mathcal{H}u) \| \leq |v_1| + \chi(r) \| p \| Q \).

In the second step, we establish that the operator \( \mathcal{H} \) maps bounded sets into equicontinuous sets of \( E \). As in the proof of the previous result, for \( t_1, t_2 \in [0, T] \) with \( t_1 < t_2 \) and \( u \in B_r \), we can have

\[
|(\mathcal{H}u)(t_2) - (\mathcal{H}u)(t_1)| \rightarrow 0 \text{ as } t_2 - t_1 \rightarrow 0,
\]
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independently of \( u \in B_R \). Therefore, it follows by the Arzelá-Ascoli theorem that the operator \( \mathcal{H} : \mathcal{E} \to \mathcal{E} \) is completely continuous.

Let \( u \) be a solution. Then, for \( t \in [0, T] \), we have that

\[
\|u\| \leq \chi(\|u\|)\|p\| Q + \|v_1\|.
\]

In view of (\( E_2 \)), there exists \( N \) such that \( \|u\| \neq N \). Let us set

\[
\mathcal{U} = \{u \in \mathcal{E} : \|u\| < N\}.
\]

We see that the operator \( \mathcal{H} : \mathcal{U} \to \mathcal{E} \) is continuous and completely continuous. From the choice of \( \mathcal{U} \), there is no \( u \in \partial \mathcal{U} \) such that \( u = \theta \mathcal{H} u \) for some \( \theta \in (0, 1) \). Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.3), we deduce that \( \mathcal{H} \) has a fixed point \( u \in \overline{\mathcal{U}} \) which is a solution of the problem (1)-(2). This completes the proof.

The next existence result is based on Leray-Schauder’s degree theory [25].

**Theorem 3.5.** Let \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) be a continuous function. Suppose that

(\( E_3 \)) there exist constants \( 0 \leq \omega < Q^{-1} \), and \( M_1 > 0 \) such that

\[
|f(t, u)| \leq \omega |u| + M_1 \quad \text{for all} \quad (t, u) \in [0, T] \times \mathbb{R},
\]

where \( Q \) is given by (15).

Then the boundary value problem (1)-(2) has at least one solution on \([0, T]\).

**Proof.** We have to show the existence of at least one solution \( u \in \mathcal{E} \) satisfying the fixed point problem

\[
u = \mathcal{H},\]

where the operator \( \mathcal{H} : \mathcal{E} \to \mathcal{E} \) is defined by (14). Introduce a ball \( B_R \subset \mathcal{E} \) as

\[
B_R = \{u \in \mathcal{E} : \|u\| < R\},
\]

with a constant radius \( R > 0 \). Hence, we will show that the operator \( \mathcal{H} : \overline{B_R} \to \mathcal{E} \) satisfies the condition

\[
u \neq \theta \mathcal{H} u, \quad \forall u \in \partial B_R, \quad \forall \theta \in [0, 1]. \tag{18}
\]

Set

\[
V(\theta, u) = \theta \mathcal{H} u, \quad u \in \mathcal{E}, \quad \theta \in [0, 1].
\]

As argued in Theorem 3.2, the operator \( \mathcal{H} \) is continuous, uniformly bounded and equicontinuous. Thus, by the Arzelá-Ascoli theorem, a continuous map \( h_\theta \) defined by \( h_\theta(u) = u - V(\theta, u) = u - \theta \mathcal{H} u \) is completely continuous. If (18) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

\[
\deg(h_\theta, B_R, 0) = \deg(I - \theta \mathcal{H}, B_R, 0) = \deg(h_1, B_R, 0)
\]

\[
= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_R,
\]

where \( I \) denotes the unit operator. By the nonzero property of Leray-Schauder degree, we have \( h_1(u) = u - \mathcal{H} u = 0 \) for at least one \( u \in B_R \). Let us assume that \( u = \theta \mathcal{H} u \) for some \( \theta \in [0, 1] \) and for all \( t \in [0, T] \). Then, using the assumption (\( E_3 \)), it is easy to find that

\[
|u(t)| = |\theta \mathcal{H} u(t)| \leq (\omega \|u\| + M_1) Q + \|v_1\|,
\]

which implies that

\[
\|u\| \leq \frac{M_1 Q + \|v_1\|}{1 - \omega Q} + 1.
\]

If \( R = \frac{M_1 Q + \|v_1\|}{1 - \omega Q} + 1 \), the inequality (18) holds. This completes the proof. \( \square \)
Next we show the existence of a unique solution of the problem (1)-(2) by applying Banach’s contraction mapping principle.

**Theorem 3.6.** Assume that \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) is a continuous function satisfying the Lipschitz condition:

\((E_4)\) there exists a positive number \( \ell \) such that \( |f(t, u) - f(t, v)| \leq \ell |u - v|, \forall t \in [0, T], u, v \in \mathbb{R}. \)

Then the boundary value problem (1)-(2) has a unique solution on \([0, T]\) if \( Q < 1/\ell \), where \( Q \) is given by (15).

**Proof.** Consider a set \( B_r = \{ u \in \mathcal{E} : \|u\| \leq r \} \) with \( r \geq \frac{QM + \|v_1\|}{1 - \ell Q} \), where \( M = \sup_{t \in [0, T]} |f(t, 0)| \) and \( Q \) is given by (15). In the first step, we show that \( \mathcal{H}B_r \subset B_r \), where the operator \( \mathcal{H} \) is defined by (14). For any \( u \in B_r, t \in [0, T] \), observe that

\[
|f(t, u(t))| = |f(t, u(t)) - f(t, 0) + f(t, 0)| \leq |f(t, u(t)) - f(t, 0)| + |f(t, 0)| \leq \ell \|u\| + M \leq \ell r + M,
\]

where we have used the assumption \((E_4)\). Then, for \( u \in B_r \), we obtain

\[
\| (\mathcal{H}u) \| \leq \sup_{t \in [0, T]} \left\{ |v_1(t)| + \int_0^T e^{-\ell (t-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} |f(x, u(x))| \, dx \right) \, ds \right\} + |v_2(t)| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, u(s))| \, ds
\]

\[
+ |v_3(t)| \int_0^T e^{-\ell (T-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} |f(x, u(x))| \, dx \right) \, ds
\]

\[
\leq (\ell r + M) \sup_{t \in [0, T]} \left\{ |v_1(t)| + \int_0^T e^{-\ell (t-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} |f(x, u(x))| \, dx \right) \, ds \right\} + |v_2(t)| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \, ds + |v_3(t)| \int_0^T e^{-\ell (T-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} \, dx \right) \, ds
\]

\[
\leq (\ell r + M) Q + \|v_1\| \leq r.
\]

which implies that \( \mathcal{H}u \in B_r \). Thus \( \mathcal{H}B_r \subset B_r \). Next we show that the operator \( \mathcal{H} \) is a contraction. Using the assumption \((E_4)\) and (15), we get

\[
\| \mathcal{H}u - \mathcal{H}v \| \leq \sup_{t \in [0, T]} \left\{ \int_0^T e^{-\ell (t-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} |f(x, u(x)) - f(x, v(x))| \, dx \right) \, ds \right\} + |v_2(t)| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, u(s)) - f(s, v(s))| \, ds
\]

\[
+ |v_3(t)| \int_0^T e^{-\ell (T-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} |f(x, u(x)) - f(x, v(x))| \, dx \right) \, ds
\]

\[
\leq \ell \| u - v \| \sup_{t \in [0, T]} \left\{ \int_0^T e^{-\ell (t-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} \, dx \right) \, ds \right\}
\]
In view of the assumption: $Q < 1/\ell$, it follows that the operator $H$ is a contraction. Thus, by Banach’s contraction mapping principle, we deduce that the operator $H$ has a fixed point, which in turn implies that there exists a unique solution for the problem (1)-(2) on $[0, T]$.

In the following theorem, we show the existence of solutions for the problem (1)-(2) by applying Krasnoselskii’s fixed point theorem.

**Lemma 3.7** (Krasnoselskii’s fixed point theorem [24]). Let $Y$ be a closed bounded, convex and nonempty subset of a Banach space $X$. Let $B_1, B_2$ be the operators such that (i) $B_1 y_1 + B_2 y_2 \in Y$ whenever $y_1, y_2 \in Y$; (ii) $B_1$ is compact and continuous and (iii) $B_2$ is a contraction mapping. Then there exists $z \in Y$ such that $z = B_1 z + B_2 z$.

**Theorem 3.8.** Let $f: [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition $(E_4)$ and that $|f(t, x)| \leq g(t)$, $\forall (t, x) \in [0, T] \times \mathbb{R}$ with $g \in C([0, T], \mathbb{R}^+)$ and $\sup_{t \in [0, T]} |g(t)| = \|g\|$. In addition, it is assumed that $Q < 1/\ell$, where

$$Q_1 = \sup_{t \in [0, T]} \left\{ \frac{|v_2(t)|}{(\Gamma(\alpha)^{1-\alpha})} + \frac{|v_3(t)|}{k \Gamma(\alpha)^{1-\alpha}} \right\}. \quad (19)$$

Then problem (1)-(2) has at least one solution on $[0, T]$.

**Proof.** Consider $B_a = \{u \in \mathcal{E} : \|u\| \leq a\}$, where $a \geq Q\|g\| + \|v_1\|$ with $Q$ given by (15). We define the operators $H_1$ and $H_2$ on $B_a$ as

$$(H_1 u)(t) = \int_0^t e^{-k(t-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} f(x, u(x)) dx \right) ds$$

$$(H_2 u)(t) = v_1(t) + v_2(t) \int_0^t \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, u(s)) ds$$

$$+ v_3(t) \int_0^t e^{-k(T-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} f(x, u(x)) dx \right) ds.$$  

For $u, v \in B_a$, it is easy to verify that $\|H_1 u + H_2 v\| \leq Q\|g\| + \|v_1\|$, where $Q$ is given by (15). Thus, $H_1 u + H_2 v \in B_a$. Using the assumption $(E_4)$ and (19), we can get $\|H_2 u - H_2 v\| \leq \ell Q_1 \|u - v\|$, which implies that $H_2$ is a contraction in view of the given condition: $Q_1 < 1/\ell$.

Notice that continuity of $f$ implies that the operator $H_1$ is continuous. Also, $H_1$ is uniformly bounded on $B_a$ as

$$\|H_1 u\| \leq \frac{(1 - e^{-kT}) T^{\alpha-1}}{k \Gamma(\alpha)} \|g\|. $$

Next, it will be shown that the operator $H_1$ is compact. Fixing $\sup_{(t, u) \in [0, T] \times B_a} |f(t, u)| = f_a$ and for $t_1, t_2 \in [0, T]$ ($t_1 < t_2$), consider

$$\|(H_1 u)(t_2) - (H_1 u)(t_1)\|$$

$$\leq f_a \left( e^{-k t_2} - e^{-k t_1} \right) \int_0^{t_1} e^{k s} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} dx \right) ds + \int_{t_1}^{t_2} e^{-k(t_2-s)} \left( \int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} dx \right) ds$$

$$\to 0 \text{ as } t_2 - t_1 \to 0.$$
independent of $u$. This implies that $\mathcal{H}_1$ is relatively compact on $B_2$. Hence, by the Arzelá-Ascoli Theorem, the operator $\mathcal{H}_1$ is compact on $B_2$. Thus all the assumptions of Lemma (3.7) are satisfied. In consequence, by the conclusion of Lemma (3.7), the problem (1)-(2) has at least one solution on $[0, T]$.

Example 3.9. Consider the following anti-periodic fractional boundary value problem:

$$\begin{align*}
\left\{ \begin{array}{ll}
(c D^{3/2} + 1.5c D^{1/2})u(t) &= f(t, u(t)), \\ 1.25u(0) + 4u(2) &= -1, 0.5u'(0) - 2u'(2) &= 2.5.
\end{array} \right.
\end{align*}$$

(20)

Here $T = 2, k = 1.5, \alpha_1 = 1.25, \rho_1 = 4, \beta_1 = -1, \alpha_2 = 0.5, \rho_2 = -2, \beta_2 = 2.5$. With the given values, we find that $Q \approx 7.742915$ ($Q$ is given by (15)).

(a) Let

$$f(t, u) = e^{-t} u^2 \left( \frac{\cos^2(3u + 2)}{\sqrt{t^2 + 9}} + \frac{t \sin t}{1 + u^2} + \frac{2}{t + 3} \right).$$

(21)

Clearly $|f(t, u(t))| \leq 3 = L_1$ for all $t \in [0, 2], u \in \mathbb{R}$. Thus, by Theorem 3.2, the problem (20) with $f(t, u)$ given by (21) has at least one solution on $[0, 2]$.

(b) Letting

$$f(t, u) = e^{-t} \left( \frac{|u|^3}{1 + |u|^3} + \frac{|u|}{1 + |u|} + \frac{1}{t + 1} \right).$$

(22)

we have $|f(t, u)| \leq e^{-t} / 9 = p(t) \chi(||u||)$. Selecting $\chi(||u||) = 1$ and $p(t) = e^{-t} / 9 (\|p\| = 1/9)$, we find that the assumption $(E_2)$ holds true for $N > 4.064141$. As all the conditions of Theorem 3.4 are satisfied, there exists at least one solution of the problem (20) with $f(t, u)$ given by (22) on $[0, 2]$.

(c) Let us take

$$f(t, u) = \frac{1}{\sqrt{t^2 + 100}} \sin u + \frac{1}{t + 2}.$$ 

(23)

Then $|f(t, u)| \leq (1/10)u + 1/2$ implies that $\omega = 1/10, M_1 = 1/2$. Clearly $\omega < 1/Q$ ($Q \approx 7.742915$). In consequence, the conclusion of Theorem 3.5 applies and the problem (20) with $f(t, u)$ given by (23) has a solution on $[0, 2]$.

(d) Let us choose

$$f(t, u) = \frac{1}{10} \tan^{-1} u(t) + \cos t.$$ 

(24)

Clearly $\ell = 1/10$ as $|f(t, u) - f(t, v)| \leq 1/10|u - v|$ and $\ell Q \approx 0.774292 < 1$. Thus all the conditions of Theorem 3.6 are satisfied. Hence we deduce by the conclusion of Theorem 3.6 that there exists a unique solution for the problem (20) with $f(t, u)$ given by (24).

For the applicability of Theorem 3.8, we find that $|f(t, u)| \leq g(t) = \pi/20 + \cos t$ with $\|g\| = (20 + \pi)/20$ and $Q_1 \approx 6.732035$ ($Q_1$ is given by (19)). Obviously $\ell Q_1 \approx 0.673203 < 1$. Thus all the conditions of Theorem 3.8 are satisfied. Hence the conclusion of Theorem 3.8 implies that the problem (20) with $f(t, u)$ given by (24) has at least one solution on $[0, 2]$.

Remark 3.10. By fixing the parameters involved in the boundary conditions (2), we can obtain some new special results for different problems arising from the problem (1)-(2). For instance, for $\alpha_1 = \alpha_2 = \rho_1 = \rho_2 = 1, \beta_1 = \beta_2 = 0$, we obtain the existence results for sequential fractional differential equation (1) with anti-periodic boundary condition: $u(0) + u(T) = 0, u'(0) + u'(T) = 0$. Our results correspond to the ones obtained in [14] for $u(0) = a, u'(0) = u'(1)$ by taking $\alpha_1 = 1 = \alpha_2, \rho_1 = 0, \rho_2 = -1, \beta_1 = a, \beta_2 = 0$. 

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4 Existence results for the problem (1)-(3)

In this section, we present some existence results for the problem (1)-(3). We omit the proofs as the method of proof is similar to the one employed in the previous section. First of all, by Lemma 2.6, we define a fixed point operator \( \mathcal{G} : \mathcal{E} \to \mathcal{E} \) associated with the problem (1)-(3) as

\[
(\mathcal{G}u)(t) = B_1(t) \left\{ \lambda_1 \int_0^T e^{-k(T-s)} t^{\alpha-1} h(s) ds + \rho_1 \int_0^T e^{-k(T-s)} t^{\alpha-1} h(s) ds + \lambda_2 \right\} + B_2(t) \left\{ \mu_1 \int_0^T e^{-k(T-s)} t^{\alpha-1} h(s) ds \right\} + \frac{\rho_2 t^{\alpha-1} e^{-k(T-t)} - \rho_2 t^{\alpha-1} e^{-kT}}{k} \] (25)

where \( B_1(t) \) and \( B_2(t) \) are given by (12).

Using the operator (25) and the method of proof for the results obtained in the last section, we can establish the following results for the problem (1)-(3).

**Theorem 4.1.** Let \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) be a continuous function satisfying the assumption (E4). Then the boundary value problem (1)-(3) has a unique solution on \([0, T]\) if \( \bar{Q} < 1/\ell \), where

\[
\bar{Q} = \sup_{t \in [0, T]} \left\{ |B_1(t)\left[ \frac{\lambda_1}{k\Gamma(\alpha)} \left( t^{\alpha-1} - \frac{t^{\alpha-1} (e^{-kT} - e^{-k\xi})}{k} \right) - \frac{\rho_1 t^{\alpha-1} (1 - e^{-kT})}{k\Gamma(\alpha)} \right] + B_2(t) \left[ \frac{\mu_1}{k\Gamma(\alpha)} \left( t^{\alpha-1} - \frac{t^{\alpha-1} (e^{-kT} - e^{-k\xi})}{k} \right) - \frac{\rho_2 t^{\alpha-1} (1 - e^{-kT})}{k\Gamma(\alpha)} \right] \right\}. 
\] (26)

**Theorem 4.2.** Let \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) be a continuous functions satisfying the condition (E4) and that \( |f(t, x)| \leq \bar{g}(t) \) \( \forall (t, x) \in [0, T] \times \mathbb{R} \) with \( \bar{g} \in C([0, T], \mathbb{R}^+) \). In addition, it is assumed that \( \bar{Q}_1 < 1/\ell \), where

\[
\bar{Q}_1 = \sup_{t \in [0, T]} \left\{ |B_1(t)\left[ \frac{\lambda_1}{k\Gamma(\alpha)} \left( t^{\alpha-1} - \frac{t^{\alpha-1} (e^{-kT} - e^{-k\xi})}{k} \right) - \frac{\rho_1 t^{\alpha-1} (1 - e^{-kT})}{k\Gamma(\alpha)} \right] + B_2(t) \left[ \frac{\mu_1}{k\Gamma(\alpha)} \left( t^{\alpha-1} - \frac{t^{\alpha-1} (e^{-kT} - e^{-k\xi})}{k} \right) - \frac{\rho_2 t^{\alpha-1} (1 - e^{-kT})}{k\Gamma(\alpha)} \right] \right\}. 
\] (27)

Then problem (1)-(3) has at least one solution on \([0, T]\).

**Theorem 4.3.** Let \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) be a jointly continuous function. Assume that \( (E_5) \) there exists a continuous nondecreasing function \( \psi : [0, \infty) \to (0, \infty) \) and a function \( \varphi \in C([0, T], \mathbb{R}^+) \) such that

\[
|f(t, u)| \leq \psi(t) \psi(||u||) \quad \text{for each} \quad (t, u) \in [0, T] \times \mathbb{R};
\]

\( (E_6) \) there exists a constant \( N_1 > 0 \) such that

\[
\frac{N_1}{\psi(N_1)\psi(\|u\|)} \geq 1.
\] (28)

Then the boundary value problem (1)-(3) has at least one solution on \([0, T]\).
Theorem 4.4. Let \( f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function. Suppose that
\[
|f(t, u)| \leq \omega_1 |u| + M_2 \text{ for all } (t, u) \in [0, T] \times \mathbb{R},
\]
where \( \Omega \) is given by (26).

Then the boundary value problem (1)-(3) has at least one solution on \([0, T]\).

Theorem 4.5. Let \( f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function. Assume that there exists a positive constant \( L_2 \) such that \( |f(t, u(t))| \leq L_2 \) for \( t \in [0, T] \), \( u \in \mathbb{R} \). Then the boundary value problem (1)-(3) has at least one solution on \([0, T]\).

Example 4.6. Consider the following anti-periodic fractional boundary value problem:
\[
\begin{cases}
D^{3/2} + 1.5D^{1/2}u(t) = \frac{1}{\sqrt{t+25}} \left( \frac{(5-t)\sin u}{30} + e^{-t} \cos t \right), & t \in [0, 2], \\
1.25u(0) + 4u(2) = -\int_0^1 u(s)ds - 1, & 0.5u'(0) - 2u'(2) = 2 \int_0^{1/25} u(s)ds + 2.5,
\end{cases}
\tag{30}
\]
where \( f(t, u(t)) = \frac{1}{\sqrt{t+25}} \left( \frac{(5-t)\sin u}{15} + e^{-t} \cos t \right), T = 2, \eta = 1, \xi = 1.25, k = 1.5, \alpha_1 = 1.25, \rho_1 = 4, \lambda_1 = -1, \lambda_2 = -1, \alpha_2 = 0.5, \rho_2 = -2, \mu_1 = 2, \mu_2 = 2.5.

With the given values, we find that \( \Omega \approx 14.422595 \) and \( \ell = 1/30 \) as \( |f(t,u) - f(t,v)| \leq \frac{1}{30}|u-v|. Clearly, \( \Omega < 1/\ell \). Thus all the conditions of Theorem 4.1 are satisfied. Hence we deduce by the conclusion of Theorem 4.1 that there exists a unique solution for the problem (30).

For the applicability of Theorem 4.2, we find that \( |f(t,u)| \leq \mathcal{R}(t) = \frac{1}{\sqrt{t+25}} \left( \frac{5-t}{30} + 1 \right) \) with \( \|g\| = \frac{7}{30} \) and \( \mathcal{Q}_1 \approx 13.411715 \) (\( \mathcal{Q}_1 \) is given by (27)). Obviously \( \mathcal{Q}_1 < 1/\ell \). Thus all the conditions of Theorem 4.2 are satisfied. Hence the conclusion of Theorem 4.2 applies to the problem (30).

To illustrate Theorem 4.3, we take \( |f(t,u)| \leq \varphi(t)\psi(\|u\|) = \frac{1}{\sqrt{t+25}} \left( \frac{5-t}{30} + 1 \right) \), \( \psi(\|u\|) = 1 \) with \( \|\varphi\| = \frac{7}{30} \) and \( N > \psi(N,\|\varphi\|\mathcal{Q} + \mathcal{Q}) = 9.333636 \) (\( \mathcal{Q} \) and \( \mathcal{Q} \) are given by (26) and (29) respectively). Thus all the conditions of Theorem 4.3 are satisfied. Hence the conclusion of Theorem 4.3 implies that the problem (30) has at least one solution on \([0, 2]\).

Remark 4.7. Several special cases of the existence results for the problem (1)-(3) follow by fixing the values of the parameters involved in the problem. For example, by taking \( \alpha_1 = 1, \alpha_2, \rho_1 = 0, \rho_2, \lambda_1 = 1, \mu_1, \lambda_2 = 0 = \mu_2 \), the results of this section correspond to the conditions: \( u(0) = \int_0^T u(s)ds, u'(0) = \int_T^T u(s)ds \). In case we take \( \alpha_1 = 0 = \alpha_2, \rho_1 = 1 = \rho_2, \lambda_1 = 1 = \mu_1, \lambda_2 = 0 = \mu_2 \), we obtain the results for terminal-point conditions: \( u(T) = \int_0^T u(s)ds, u'(T) = \int_T^T u(s)ds \). Letting \( \alpha_1 = 1 = \alpha_2, \rho_1 = 1 = \rho_2, \lambda_1 = 1/\ell \), \( \mu_1 = 1/(T - \xi), \lambda_2 = 0 = \mu_2 \), we get the results for the average valued (integral) conditions: \( u(0) + u(T) = (1/\ell) \int_0^T u(s)ds, u'(0) + u'(T) = 1/(T - \xi) \int_T^T u(s)ds \). By taking \( \alpha = 2, \) our results correspond to the equation: \( (D^2 + k\lambda)u(t) = f(t, u(t)), 0 < t < T, T > 0 \), which are also new.

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