PRIMITIVE STABILITY AND BOWDITCH’S BQ-CONDITION ARE EQUIVALENT

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ABSTRACT. We prove the equivalence of two conditions on the primitive elements in an \( SL(2, \mathbb{C}) \) representation of the free group \( F_2 = \langle a, b \rangle \), which may hold even when the \( SL(2, \mathbb{C}) \) image of \( F_2 \) is not discrete. One is Minsky’s condition of primitive stability and the other is the \( BQ \)-condition introduced by Bowditch and generalised by Tan, Wong and Zhang.

Keywords: Free group on two generators, Kleinian group, non-discrete representation

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1. INTRODUCTION

In this paper we show the equivalence of two conditions on the primitive elements in an \( SL(2, \mathbb{C}) \) representation \( \rho \) of the free group \( F_2 = \langle a, b \rangle \) on two generators, which may hold even when the image \( \rho(F_2) \) is not discrete. One is the condition of primitive stability \( PS \) introduced by Minsky \([11]\) and the other is the so-called \( BQ \)-condition introduced by Bowditch \([3]\) and generalised by Tan, Wong and Zhang \([14]\). The same result has been proved independently by Jaejeong Lee and Binbin Xu \([9]\).

To facilitate the proof we introduce a third condition which we call the bounded intersection property \( BIP \) which as we will show is implied by but does not imply either of the other two.

We begin by explaining these three conditions one by one. Recall that an element \( u \in F_2 \) is called primitive if it forms one of a generating pair \( (u, v) \) for \( F_2 \). Let \( \mathcal{P} \) denote the set of primitive elements in \( F_2 \). It is well known that up to inverse and conjugacy, the primitive elements are enumerated by the rational numbers \( \hat{\mathbb{Q}} = \mathbb{Q} \cup \{ \infty \} \), see Section 2 for details.

1.1. The primitive stable condition \( PS \). The notion of primitive stability was introduced by Minsky in \([11]\) in order to construct an \( \text{Out}(F_2) \)-invariant subset of the \( SL(2, \mathbb{C}) \) character variety \( \chi(F_2) \) strictly larger than the set of discrete free representations.

Let \( d(P, Q) \) denote the hyperbolic distance between points \( P, Q \) in hyperbolic 3-space \( \mathbb{H}^3 \). Recall that a path \( t \mapsto \gamma(t) \subset \mathbb{H}^3 \) for \( t \in I \) (where \( I \) is a possibly infinite interval in \( \mathbb{R} \)) is called a \((K, \epsilon)\)-quasigeodesic if there exist constants \( K, \epsilon > 0 \) such that

\[
K^{-1} |s - t| - \epsilon \leq d(\gamma(s), \gamma(t)) \leq K |s - t| + \epsilon \quad \text{for all } s, t \in I.
\]

For a representation \( \rho: F_2 \to SL(2, \mathbb{C}) \), in general we will denote elements in \( F_2 \) by lower case letters and their images under \( \rho \) by the corresponding upper case, thus \( X = \rho(x) \) for \( x \in F_2 \). Thus if \( (u, v) \) is a generating pair for \( F_2 \) we write \( U = \rho(u), V = \rho(v) \).

Fix once and for all a base point \( O \in \mathbb{H}^3 \) and suppose that \( w = e_1 \ldots e_n, e_i \in \{ u^\pm, v^\pm \}, i = 1, \ldots, n \) is a cyclically shortest word in the generators \( (u, v) \). The broken geodesic \( \text{br}_\rho(w; (u, v)) \) of \( w \) with respect to \( (u, v) \) is the infinite path of geodesic segments joining vertices

\[
\ldots, E_{n-1}^{-1}E_n O, E_n^{-1} O, O, E_1 O, E_1 E_2 O, \ldots, E_1 E_2 \ldots E_n O, E_1 E_2 \ldots E_n E_1 O, \ldots
\]
where \( E_i = \rho(e_i) \).

**Definition 1.1.** Let \( (u, v) \) be a fixed generating pair for \( F_2 \). A representation \( \rho: F_2 \to SL(2, \mathbb{C}) \) is primitive stable, denoted \( PS \), if the broken geodesics \( br_\rho(w; (u, v)) \) for all cyclically shortest words \( w = e_1 \ldots e_n \in \mathcal{P}, e_k \in \{ u^\pm, v^\pm \}, k = 1 \ldots, n \), are uniformly \((K, \epsilon)\)-quasigeodesic for some fixed constants \((K, \epsilon)\).

Notice that this definition is independent of the choice of basepoint \( O \) and makes sense since the change from \( \mathbf{br}_\rho(w; (u, v)) \) to \( \mathbf{br}_\rho(w; (u', v')) \) for some other generator pair \((u', v')\) changes all the constants for all the quasigeodesics uniformly.

For \( g \in F_2 \) write \( ||g|| \) or more precisely \( ||g||_{u,v} \) for the word length of \( g \), that is the shortest representation of \( g \) as a product of generators \((u, v)\). It is easy to see that for fixed generators, the condition \( PS \) is equivalent to the existence of \( K, \epsilon > 0 \) such that

\[
K^{-1}||w|| - \epsilon \leq d(O, \rho(w)O) \leq K||w|| + \epsilon
\]

for all finite cyclically shortest words \( w \) which are subwords of the infinite reduced word \( \ldots e_1 \ldots e_n \ldots \).

Recall that an irreducible representation \( \rho: F_2 \to SL(2, \mathbb{C}) \) is determined up to conjugation by the traces of \( U = \rho(u), V = \rho(v) \) and \( UV = \rho(uv) \) where \((u, v)\) is a generator pair for \( F_2 \). More generally, if we take the GIT quotient of all (not necessarily irreducible) representations, then the resulting \( SL(2, \mathbb{C}) \) character variety of \( F_2 \) can be identified with \( \mathbb{C}^3 \) via these traces, see for example [7] and the references therein. (The only non-elementary (hence reducible) representation occurs when \( \text{Tr}[U, V] = 2 \). We exclude this from the discussion, see for example [13] Remark 2.1.)

**Proposition 1.2** ([11] Lemma 3.2). The set of primitive stable \( \rho: F_2 \to SL(2, \mathbb{C}) \) is open in the \( SL(2, \mathbb{C}) \) character variety of \( F_2 \).

Minsky showed that not all \( PS \) representations are discrete.

1.2. The Bowditch BQ-condition. The notion of primitive stability was introduced by Bowditch in [3] in order to give a purely combinatorial proof of McShane’s identity.

Again let \((u, v)\) be a generator pair for \( F_2 \) and let \( \rho: F_2 \to SL(2, \mathbb{C}) \).

**Definition 1.3.** Following [14], an irreducible representation \( \rho: F_2 \to SL(2, \mathbb{C}) \) is said to satisfy the BQ-condition if

\[
\text{Tr} \rho(g) \notin [-2, 2] \quad \forall g \in \mathcal{P} \quad \text{and}
\]

\[
\{ g \in \mathcal{P} : |\text{Tr} \rho(g)| \leq 2 \} \quad \text{is finite.}
\]

We denote the set of all representations satisfying the BQ-condition by \( \mathcal{B} \).

**Proposition 1.4** ([3] Theorem 3.16, [14] Theorem 3.2). The set \( \mathcal{B} \) is open in the \( SL(2, \mathbb{C}) \) character variety of \( F_2 \).

Bowditch’s original work [3] was on the case in which the commutator \( [X,Y] = XYX^{-1}Y^{-1} \) is parabolic and \( \text{Tr}[X,Y] = -2 \). He conjectured that all representations in \( \mathcal{B} \) of this type are quasifuchsian and hence discrete. While this question remains open, it is shown in [13] that without this restriction, there are definitely representations in \( \mathcal{B} \) which are not discrete.
The bounded intersection property BIP. Recall that a word \( w = e_1e_2 \ldots e_n \) in generators \((u, v)\) of \( F_2 \) is palindromic if it reads the same forwards and backwards, that is, if \( e_1e_2 \ldots e_n = e_ne_{n-1} \ldots e_1 \). Palindromic words have been studied by Gilman and Keen in [4, 5].

Suppose that \( \rho: F_2 \to SL(2, \mathbb{C}) \) and let \((u, v)\) be a generating pair. Denote the extended common perpendicular of the axes of \( U = \rho(u), V = \rho(v) \) by \( E(U, V) \). By applying the \( \pi \) rotation about \( E(U, V) \), it is not hard to see that a word \( w \) is palindromic in a generator pair \((u, v)\) if and only if the axis of \( W = \rho(w) \) intersects \( E(U, V) \) perpendicularly, see for example [1].

Fix generators \((a, b)\) for \( F_2 \). We call the pairs \((a, b), (a, ab)\) and \((b, ab)\) the basic generator pairs. (The order \( ab \) or \( ba \) is fixed but not important, see below.) Now given \( \rho: F_2 \to SL(2, \mathbb{C}) \) let \( A = \rho(a), B = \rho(b) \) and consider the three common perpendiculars \( E(A, B), E(A, AB) \) and \( E(B, AB) \). (We could equally well chose to use \( BA \) in place of \( AB \); the main point is that the choice is fixed once and for all.) We call these lines the special hyperelliptic axes.

Definition 1.5. Fix a basepoint \( O \in \mathbb{H}^3 \). A representation \( \rho: F_2 \to SL(2, \mathbb{C}) \) satisfies the bounded intersection property BIP if there exists \( D > 0 \) so that if a generator \( w \) is palindromic with respect to one of the three basic generators pairs, then its axis intersects the corresponding special hyperelliptic axis in a point at distance at most \( D \) from \( O \). Equivalently, the axes of all palindromic primitive elements intersect the appropriate hyperelliptic axes in bounded intervals.

Clearly this definition is independent of the choices of \((a, b)\) and \( O \).

A similar condition but related to all palindromic axes was used in [3] to give a condition for discreteness of geometrically finite groups.

In Proposition 2.1 we show that every generator is conjugate to one which is palindromic with respect to one of the three basic generator pairs. In fact each primitive element can be conjugated (in different ways) to be palindromic with respect to two out of the three possible basic pairs, for a more precise statement see below.

1.4. The main result. The main theorem of this paper is

**Theorem A.** The conditions BQ and PS are equivalent. Both imply, but are not implied by, the condition BIP.

In the case of real representations, Damiano Lupi [10] showed by case by case analysis following [6] that the conditions BQ and PS are equivalent.

To see that BIP does not imply the other conditions, first note that conditions PS and BQ both imply that no element in \( \rho(P) \) is elliptic or parabolic. The condition BIP rules out parabolicity (consider the fixed point of a palindromic parabolic element to be a degenerate axis which clearly meets the relevant hyperelliptic axis at infinity). However the condition does not obviously rule out elliptic elements in \( \rho(P) \). In particular, consider any \( SO(3) \) representation, discrete or otherwise. Here all axes are elliptic and all pass through a central fixed point which is also at the intersection of all three hyperelliptic axes. Such a representation clearly satisfies BIP.

The plan of the paper is as follows. In Section 2 we present background on the Farey tree and prove an important result on palindromic representation of primitive elements, Proposition 2.1. We also introduce Bowditch’s condition of Fibonacci growth. In Section 3 we summarise Bowditch’s method of assigning an orientation to the edges of the Farey tree which results in
a characterisation of the BQ-condition in terms of the existence of a certain finite attracting subtree.

Using the condition of Fibonacci growth (see Definition 2.2), it is not hard to show that PS implies BQ. This was proved in [10] and is sketched in Section 4. Developing the results of Section 3, we then make estimates which prove (Theorem 4.3) that BQ implies BIP.

Finally in Section 5 we use Theorem 4.3 to prove that BQ implies PS. The advantage of the condition BIP over BQ is that it ties down the location of axes. After some preliminary work on quasigeodesics, which heavily relies on the condition BIP, we obtain further results which will eventually allow us to control broken geodesic paths for all but finitely many generator pairs. The proof of Theorem A is completed by Theorem 5.11 which shows that BQ implies PS.

We would like to thank Tan Ser Peow and Yasushi Yamashita for initial discussions about this paper. The work involved in Lupi’s thesis [10] also made a significant contribution. The idea of introducing the condition BIP arose while trying to interpret some very interesting computer graphics involving non-discrete groups made by Yamashita. We hope to return to this topic elsewhere.

2. Primitive elements, the Farey tree and Fibonacci growth

The Farey diagram $\mathcal{F}$ as shown in Figures 1 and 2 consists of the images of the ideal triangle with vertices at 1/0, 0/1 and 1/1 under the action of $SL(2, \mathbb{Z})$ on the upper half plane, suitably conjugated to the position shown in the disk. The label $p/q$ in the disk is just the conjugated image of the actual point $p/q \in \mathbb{R}$.

![Figure 1](image1.png)  

**Figure 1.** The Farey diagram, showing the arrangement of rational numbers on the left with the corresponding primitive words on the right.

Since the rational points in $\hat{\mathbb{Q}} = \mathbb{Q} \cup \infty$ are precisely the images of $\infty$ under $SL(2, \mathbb{Z})$, they correspond bijectively to the vertices of $\mathcal{F}$. A pair $p/q, r/s \in \hat{\mathbb{Q}}$ are the endpoints of an edge if and only if $pr - qs = \pm 1$; such pairs are called neighbours. A triple of points in $\hat{\mathbb{Q}}$ are the vertices of a triangle precisely when they are the images of the vertices of the initial triangle (1/0, 0/1, 1/1); such triples are always of the form $(p/q, r/s, (p + r)/(q + s))$ where $p/q, r/s$ are neighbours. In other words, if $p/q, r/s$ are the endpoints of an edge, then the vertex of the triangle on the side away from the centre of the disk is found by ‘Farey addition’ to be $(p+r)/(q+s)$. Starting from $1/0 = -1/0 = \infty$ and 0/1, all points in $\hat{\mathbb{Q}}$ are obtained recursively.
in this way. (Note we need to start with \(-1/0 = \infty\) to get the negative fractions on the left side of the left hand diagram in Figure [1])

As noted in the introduction, up to inverse and conjugation, the primitive elements in \(F_2\) are enumerated by \(\hat{Q}\). Formally, we set \(\overline{P}\) to be the set of equivalence classes of primitive elements under the relation \(u \sim v\) if and only if either \(v = gug^{-1}\) or \(v = gu^{-1}g^{-1}, g \in F_2\). We call the equivalence classes, extended conjugacy classes. In particular, the set of all cyclic permutations of a given word are in the same extended class. Since we are working in the free group, a word is cyclically shortest if it, together with all its cyclic permutations, is reduced, that is, contains no occurrences of \(x\) followed by \(x^{-1}\), \(x \in \{a^\pm, b^\pm\}\).

The right hand picture in Figure [1] shows an enumeration of representative elements from \(\overline{P}\), starting with initial triple \((a, b, ab)\). Each vertex is labelled by a certain cyclically shortest generator \(w_{p/q}\). Corresponding to the process of Farey addition, the words \(w_{p/q}\) can be found by juxtaposition as indicated on the diagram. Note that for this to work it is important to preserve the order: if \(u, v\) are the endpoints of an edge with \(u\) before \(v\) in the anti-clockwise order round the circle, the correct concatenation is \(uv\). Note also that the words on the left side of the diagram involve \(b^{-1}a\) corresponding to starting with \(\infty = -1/0\). It is not hard to see that pairs of primitive elements form a generating pair if and only if they are at the endpoints of an edge, while the words at the vertices of a triangle correspond to a generator triple of the form \((u, v, uv)\).

The word \(w_{p/q}\) is a representative of the extended conjugacy class identified with \(p/q \in \hat{Q}\). We denote this class by \([p/q]\). It is easy to see that \(e_a(w_{p/q})/e_b(w_{p/q}) = p/q\), where \(e_a(w_{p/q}), e_b(w_{p/q})\) are the sum of the exponents in \(w_{p/q}\) of \(a, b\) respectively. All other words in \([p/q]\) are cyclic permutations of \(w_{p/q}\) or its inverse. In what follows we largely focus on different representatives of the class which are palindromic with respect to one of the basic generator pairs, see Proposition [2.1] below.

### 2.1. Generators and palindromicity.

Let \(E = \{0/1, 1/0, 1/1\}\) and define a map \(\beta : \hat{Q} \to E\) by \(\beta(p/q) = \bar{p}/\bar{q}\), where \(\bar{p}, \bar{q}\) are the mod 2 representatives of \(p, q\) in \(\{0, 1\}\). We refer to \(\beta(p/q)\) as the mod 2 equivalence class of \(p/q\). Say \(p/q \in \hat{Q}\) is of type \(\eta \in E\) if \(\beta(p/q) = \eta\). Say a generator \(u \in \bar{F}_2\) is of type \(\eta\) if \(u \in [p/q]\) and \(p/q\) is of type \(\eta\); likewise a generator pair \((u, v)\) is type \((\eta, \eta')\) if \(u, v\) are of types \(\eta, \eta'\) respectively. As in Section [1.3], we fix once and for all a generator pair \((a, b)\) and identify \(a\) with \(0/1\), \(b\) with \(1/0\) and \(ab\) with \(1/1\). The basic generator pairs are the three (unordered) generator pairs \((a, b), (a, ab)\) and \((b, ab)\) corresponding to \((0/1, 1/0), (0/1, 1/1)\) and \((1/0, 1/1)\) respectively. (Here the order \(ba\) or \(ab\) is not important but fixed.) For \(\eta, \eta' \in E\) we say \(u\) is palindromic with respect to \((\eta, \eta')\) if it is palindromic when rewritten in terms of the basic pair of generators corresponding to \((\eta, \eta')\); equally we say that a generator pair \((u, v)\) is cyclically shortest (respectively palindromic with respect to the pair \((\eta, \eta')\)) if each of \(u, v\) have the same property. We refer to a generator pair \((u, v)\) which is palindromic with respect to some pair of generators, as a palindromic pair. Finally, say a generator pair \((u, v)\) is conjugate to a pair \((u', v')\) is there exists \(g \in F_2\) such that \(gug^{-1} = u'\) and \(gvg^{-1} = v'\).

**Proposition 2.1.** If \(u \in P\) is of type \(\eta \in E\), then, for each \(\eta' \neq \eta\), up to inverses there is exactly one conjugate generator \(u'\) which is cyclically shortest and palindromic with respect to \((\eta, \eta')\).
If \((u, v)\) is a generator pair of type \((\eta, \eta')\), then up to inverses, there is exactly one conjugate generator pair \((u', v')\) which is cyclically shortest and palindromic with respect to \((\eta, \eta')\).

Proof. We begin by proving the existence part of the second statement. Observe that the edges of the Farey tree \(\mathcal{T}\) may be divided into three classes, depending on the mod two equivalence classes of the generators labelling the neighbouring regions. In this way we may assign colours \(r, g, b\) to the pairs \((0/1, 1/0); (0/1, 1/1); (1/0, 1/1)\) respectively and extend to a map \(\text{col}\) from edges to \(\{r, g, b\}\), see Figure 2. Note that no two edges of the same colour are adjacent, and that the colours round the boundary of each complementary region alternate.

Let \(e_0\) be the edge of \(\mathcal{T}\) with adjacent regions labelled by \((a, b)\) and let \(q^+(e_0)\) and \(q^-(e_0)\) denote the vertices at the two ends of \(e_0\), chosen so that the neighbouring regions are \((a, b, ab)\) and \((a, b, ab^{-1})\) respectively. Removing either of these two vertices disconnects \(\mathcal{T}\). We deal first with the subtree \(\mathcal{T}^+\) consisting of the connected component of \(\mathcal{T} \setminus \{q^-(e_0)\}\) which contains \(q^+(e_0)\). Note that the regions adjacent to all edges of \(\mathcal{T}^+\) correspond to non-negative fractions.

Let \(e\) be a given edge of \(\mathcal{T}^+\) and let \(q^+(e)\) denote the vertex of \(e\) furthest from \(q^-(e_0)\). Let \(\gamma = \gamma(e)\) be the unique shortest edge path joining \(q^+(e)\) to \(q^-(e_0)\), hence including both \(e\) and \(e_0\). The coloured level of \(e\), denoted \(\text{col.lev}(e)\), is the number of edges \(e'\) including \(e\) itself in \(\gamma(e)\) with \(\text{col}(e') = \text{col}(e)\). Note that \(\gamma(e)\) necessarily includes \(e_0\), and, provided \(e \neq e_0\), one or other of the two edges emanating from \(q^+(e_0)\) other than \(e_0\). Thus \(\text{col.lev}(e) = 1\) for all three edges meeting \(q^+(e_0)\) while for all other edges of \(\mathcal{T}^+\) we have \(\text{col.lev}(e) > 1\).

Now suppose that \(e\) is the edge of \(\mathcal{T}^+\) whose neighbouring regions are labelled by the given generator pair \((u, v)\). The proof will be by induction on \(\text{col.lev}(e)\).

Suppose first \(\text{col.lev}(e) = 1\). If \(e = e_0\) the result is clearly true, since the pair \((a, b)\) is palindromic with respect to itself. The other two edges emanating from \(q^+(e_0)\) have neighbouring
adjacent generators are of type \((\eta,\eta')\). Suppose that \(\text{col}(e) = c\) and let \(e'\) be the next edge of \(\gamma\) with \(\text{col}(e') = c\) along the path \(\gamma(e)\) from \(q^+(e)\) to \(q^-(e_0)\). (Note that such \(e'\) always exists since \(k + 1 \geq 2\).) By the induction hypothesis there is a pair of generators \((u, v)\) adjacent to \(e'\) which is palindromic with respect to the same basic generator pair \((\eta, \eta')\).

Let \(q^+(e')\) be the vertex of \(e'\) closest to \(e\), so that the subpath path \(\gamma'\) of \(\gamma\) from \(q^+(e')\) to \(q^-(e)\) contains no other edges of colour \(c\), where \(q^-(e)\) is the vertex of \(e\) other than \(q^+(e)\). Since there cannot be two adjacent edges of the same colour, the edges of \(\gamma'\) must alternate between the two other colours. This implies (see Figure 2) that \(\gamma'\) forms part of the boundary of a complementary region \(R\) of \(\mathcal{T}^+\). Moreover the third edge at each vertex along \(\partial R\) (that is, the one which is not contained in \(\partial R\)), is coloured \(c\).

Denote the generator associated to \(R\) by \(w\). Since the regions adjacent to \(R\) and not in \(R\) include the one labelled \(\partial R\), the labels of the regions around \(\partial R\) can be written, in order, in the form \(\ldots, w^{-2}v, w^{-1}v, v, wv, w^2v, \ldots\). Since \(u\) is adjacent to \(v\) in this list, either \(u = w^{-1}v\) or \(u = wv\). Since \(e\) is coloured \(c\) it points out of \(\partial R\) so that the regions adjacent to \(e\) also appear in this list so are of the form \((w^n v, w^{n+1} v)\) for some \(n\).

Suppose \(u = w^{-1}v\). Then \((w^n v, w^{n+1} v) = ((vu^{-1})^n v, (vu^{-1})^{n+1} v)\). If \(n \geq 0\) then this pair is clearly palindromic with respect to \((u, v)\). Since \((u, v)\) is palindromic with respect to \((\eta, \eta')\), it follows then that so is \((w^n v, w^{n+1} v)\). If \(n < 0\) then noting that \((vu^{-1})^n v = (w^{-1})^{-n} v\) we see that \((w^n v, w^{n+1} v)\) is again palindromic in \((u, v)\) and hence with respect to \((\eta, \eta')\). The argument in case \(u = wv\) is similar.

By the same argument for the tree \(\mathcal{T}^-\) consisting of the connected component of \(\mathcal{T} \setminus \{q^+(e_0)\}\) which contains \(q^-(e_0)\) we arrive at the statement that the generators associated to each edge of \(\mathcal{T}^-\) can be written in a form which is palindromic with respect to one of the three generator pairs associated to the edges emanating from \(q^- (e_0)\), that is, \((ab^{-1}, (a, ab^{-1})\) or \((ab^{-1}, b^{-1})\). The first pair is obviously palindromic with respect to \((a, b^{-1})\). Noting that \(ab^{-1} = a(b^{-1}a^{-1})a\) which is palindromic with respect to \((a, ab)\), the result follows.

Now we prove the existence part of the first claim. Suppose that \(u \in \mathcal{P}\) is of type \(\eta \in \mathbb{E}\) and that \(\eta' \neq \eta\). Choose a generator \(v\) of type \(\eta'\) so that \((u, v)\) is a generator pair. By the above there is a conjugate pair \((u', v')\) palindromic with respect to \((\eta, \eta')\) and \(u'\) is a generator as required.

To see that \(u'\) is unique, suppose that cyclically shortest primitive elements \(u\) and \(u'\) are in the same extended conjugacy class and are both palindromic with respect to the same pair of generators, which we may as well take to be \(\{0/1, 1/0\}\). Notice that \(u\) necessarily has odd length, for otherwise the exponents of \(a\) and \(b\) are both even.

Let \(u = e_r \ldots e_1 fe_1 \ldots e_r\) and suppose that \(f' = e_k\) is the centre point about which \(u'\) is palindromic for some \(1 \leq k \leq r\). Then \(\ldots uu\ldots\) is periodic with minimal period of length \(2r + 1\) and contains the subword

\[
e_r \ldots e_1 fe_1 \ldots e_{k-1} f' e_{k-1} \ldots e_1 fe_1 \ldots e_r
\]

so after \(fe_1 \ldots e_{k-1} f' e_{k-1} \ldots e_1\) the sequence repeats. Since this subword has length \(2k < 2r + 1\) this contradiction proves the result.
The claimed uniqueness of generator pairs follows immediately. \[\square\]

2.2. Fibonacci growth. Since all words in an extended conjugacy class have the same length, and since representative of the extended conjugacy class corresponding to \(p/q \in \hat{\mathbb{Q}}\) can be found by concatenation starting from the initial generators \((a, b)\), it follows that \(|w|_{(a,b)} = p + q\) for all \(w \in [p/q]\). This leads to the following definition from [3]:

**Definition 2.2.** A representation \(\rho: F_2 \to SL(2, \mathbb{C})\) has Fibonacci growth if there exists \(c > 0\) such that for all cyclically reduced words \(w \in \mathcal{P}\) we have \(\log^+ |\operatorname{Tr} \rho(w)| < c|w|_{(a,b)}\) and moreover \(\log^+ |\operatorname{Tr} \rho(w)| > |w|_{(a,b)}/c\) for all but finitely many cyclically reduced \(w \in \mathcal{P}\) where \(\log^+ x = \max\{0, \log |x|\}\).

Notice that although the definition is made relative to a fixed pair of generators for \(F_2\), it is in fact independent of this choice.

The following result is fundamental:

**Proposition 2.3** ([3] Proof of Theorem 2, [14] Theorem 3.3). If \(\rho: F_2 \to SL(2, \mathbb{C})\) satisfies the BQ-condition then \(\rho\) has Fibonacci growth.

3. More on the Bowditch condition

In this section we explain some further background to the BQ-condition. For more detail see [3] and [14], and for a quick summary [13]. As above, \(\mathcal{P}\) is identified \(\hat{\mathbb{Q}}\) and hence with the set \(\Omega\) of complementary regions of the Farey tree \(\mathcal{T}\). We denote the region associated to a generator \(u\) by \(u\), thus \(u' = u\) for all \(u' \sim u\). For a given representation \(\rho: F_2 \to SL(2, \mathbb{C})\), note that \(\operatorname{Tr}[U,V]\) and hence \(\mu = \operatorname{Tr}[A,B] + 2\) is independent of the choice of generators of \(F_2\), where as usual \(U = \operatorname{Tr} \rho(u)\) and so on. Since \(\operatorname{Tr} U\) is constant on extended equivalence classes of generators, for \(u \in \Omega\) we can define \(\phi(u) = \phi_\rho(u) = \operatorname{Tr} U\) for any \(u \in \Omega\). For notational convenience we will sometimes write \(\hat{u}\) in place of \(\phi(u)\).

For matrices \(X, Y \in SL(2, \mathbb{C})\) set \(x = \operatorname{Tr} X, y = \operatorname{Tr} Y, z = \operatorname{Tr} XY\). Recall the trace relations:

\begin{align*}
\operatorname{Tr} XY^{-1} &= xy - z \\
(4) \\
(5) \\
x^2 + y^2 + z^2 &= xyz + \operatorname{Tr}[X,Y] + 2.
\end{align*}

Setting \(\mu = \operatorname{Tr}[X,Y] + 2\), this last equation takes the form

\[x^2 + y^2 + z^2 - xyz = \mu.\]

As is well known and can be proven by applying the above trace relations inductively, if \(u, v, w\) is a triple of regions round a vertex of \(\mathcal{T}\), then \(\hat{u}, \hat{v}, \hat{w}\) satisfy [5]. (In particular, \(\operatorname{Tr}[U,V]\) is independent of choice of generators.) Likewise if \(e\) is an edge of \(\mathcal{T}\) with adjacent regions \(u, v\) and if \(w, z\) are the third regions at either end of \(e\), then \(\hat{u}, \hat{v}, \hat{w}, \hat{z}\) satisfy [4]. (A map \(\phi: \Omega \to \mathbb{C}\) with this property is called a Markoff map in [3].)

Given \(\rho: F_2 \to SL(2, \mathbb{C})\), define \(\Omega_\rho\) to be the tree whose complementary regions are labelled by the function \(\phi = \phi_\rho\). Following Bowditch [3], we orient the edges of \(\Omega_\rho\) as follows. Suppose that labels of the regions adjacent to some edge \(e\) are \(\hat{u}, \hat{v}\) and the labels of the two remaining regions at the two end vertices are \(\hat{w}, \hat{z}\) so that \(\hat{z} = \hat{u}\hat{v} - \hat{w}\). Orient \(e\) by putting an arrow from
\( \vec{z} \) to \( \vec{w} \) whenever \( |\vec{z}| > |\vec{w}| \) and vice versa. If both moduli are equal, make either choice; if the inequality is strict, say that the edge is oriented decisively.

For any \( m \geq 0 \) and \( \rho: F_2 \to SL(2, \mathbb{C}) \) define \( \Omega_\rho(m) = \{ u \in \Omega | |\phi_\rho(u)| \leq m \} \).

Now we collect up some important results from [14] which generalise those of [3].

**Lemma 3.1** ([14, Lemma 3.7]). Suppose \( u, v, w \in \Omega \) meet at a vertex \( q \) of \( T \) with the arrows on both the edges adjacent to \( u \) pointing away from \( q \). Then either \( |\phi(u)| \leq 2 \) or \( \phi(v) = \phi(w) = 0 \). In particular, if \( \rho \in \mathcal{B} \) then \( |\phi(u)| \leq 2 \).

**Lemma 3.2** ([14, Lemma 3.11] and following comment). Suppose \( \beta \) is an infinite ray consisting of a sequence of edges of \( T_\rho \) all of whose arrows point away from the initial vertex. Then \( \beta \) meets at least one region \( u \in \Omega \) with \( |\phi(u)| < 2 \). Furthermore, if the ray does not follow the boundary of a single region, it meets infinitely many regions with this property.

**Theorem 3.3** ([14, Theorem 3.1(2)]). For any \( m \geq 2 \), the set \( \Omega_\rho(m) \) is connected. Moreover \( |\Omega_\rho(m)| < \infty \) if and only if \( \rho \in \mathcal{B} \).

**Proof.** The first statement is [14] Theorem 3.1(2). The statement on finiteness of \( \Omega_\rho(m) \) follows from Lemma 3.2 and finiteness of \( \Omega_\rho(2) \).

Let \( y_i, i \in \mathbb{Z} \) be the regions in order around the boundary \( \partial u \) of a single region \( u \in \Omega \). It is easy to see (see the proof of Proposition 2.1) that the values \( \phi(y_i) \) satisfy a simple recurrence relation and hence grow exponentially unless \( \phi(u) \) is in the exceptional set \( E = [-2, 2] \cup \{ \pm \sqrt{\mu} \} \subset \mathbb{C} \). If \( \rho \in \mathcal{B} \) then by definition \( \phi(u) \notin [-2, 2] \), while if \( \phi(u) = \pm \sqrt{\mu} \) the values approach zero in one direction round \( \partial u \) (see [14] Lemma 3.10) and hence \( \rho \notin \mathcal{B} \) since condition [3] is not satisfied. Thus we find:

**Lemma 3.4** ([14, Lemma 3.20]). Suppose that \( \rho \in \mathcal{B} \) and \( u \in \Omega \) and consider the regions \( y_i, i \in \mathbb{Z} \) adjacent to \( u \) in order round \( \partial u \). Then away from a finite subset, the values \( |\phi_\rho(y_i)| \) are increasing and approach infinity as \( i \to \infty \) in both directions. Moreover there exists a finite segment of \( \partial u \) such that the edges adjacent to \( u \) and not in this segment are directed towards this segment.

Let \( \vec{e} \) be a directed edge. Its head and tail are the two ends of \( e \), chosen so that the arrow on \( \vec{e} \) points towards its head. Note that \( T \setminus \{ \vec{e} \} \) has two components. We define the wake of \( \vec{e} \), denoted \( W(\vec{e}) \), to be the set of regions whose boundaries are contained in the component of \( T \setminus \{ \vec{e} \} \) which contains the tail of \( \vec{e} \), together with the two regions adjacent to \( \vec{e} \). (This is the subset of \( \Omega \) denoted \( \Omega^0(\vec{e}) \) in [3] and [14].)

For \( u \in W(\vec{e}) \) let \( d(u) \) be the number of edges in the shortest path from \( u \) to the head of \( \vec{e} \). Following [14] P.777, define a function \( F_\vec{e} \) on \( W(\vec{e}) \) as follows: \( F_\vec{e}(w) = 1 \) if \( w \) is adjacent to \( \vec{e} \) and \( F_\vec{e}(u) = F_\vec{e}(v) + F_\vec{e}(w) \) otherwise, where \( v, w \) are the two regions meeting \( u \) and closer to \( \vec{e} \) than \( u \), that is, with \( d(v) < d(u), d(w) < d(u) \).

We need the following refinement of Proposition 2.3.

**Lemma 3.5.** Suppose that \( \rho \in \mathcal{B} \) and that \( \vec{e} \) is a directed edge such at most one of the adjacent regions is in \( \Omega(2) \), and suppose that the arrows on the edges of \( W(\vec{e}) \) are all directed towards \( \vec{e} \). Then there exist \( c > 0, n_0 \in \mathbb{N} \), independent of \( \vec{e} \) (but depending on \( \rho \)), so that \( \log |\phi_\rho(u)| \geq cF_\vec{e}(u) \) for all but at most \( n_0 \) regions \( u \in W(\vec{e}) \).
Proof. This essentially Lemmas 3.17 and 3.19 of [14], see also Corollary 3.6 of [3]. We only need
to see that the constants \( c, n_0 \) are independent of \( \vec{e} \). By Lemma 3.17 in [14], if neither adjacent
region to \( \vec{e} \) is in \( \Omega(2) \), then it suffices to take \( c = m - \log 2 \) where \( m = \min\{\log 3, \inf\{\log |\phi(u)| : u \notin \Omega(2)\}\} \) and \( n_0 = 1 \). Since the sets \( \Omega(3) \) and \( \Omega(2) \) are finite for any \( M \), and since either
\( \Omega(3) \setminus \Omega(2) = \emptyset \) or the infimum is a minimum, we have \( m - \log 2 > 0 \) and the result follows.

Equally, if one of the adjacent regions to \( \vec{e} \) is in \( \Omega(2) \) then the constant \( c \) in Lemma 3.19 and
the number \( n_0 \) for which the inequality fails depends on the unique region \( x_0 \in \Omega(2) \) adjacent
to \( \vec{e} \). Since \( \Omega(2) \) is finite once again these bounds are uniform independent of \( \vec{e} \). \( \square \)

Finally, we will need the sink tree defined in the course of the proof of Theorem 3.3 in [14]
and explained in more detail in Theorem 2.7 of [13].

Proposition 3.6. There is a finite connected non-empty subtree tree \( T_F \) of \( T_\rho \) so that every
path of strictly decreasing arrows eventually lands on an edge of \( T_F \). Moreover \( T_F \) contains all
sink vertices and all edges abutting on any sink vertex. There is a constant \( M_0 \geq 2 \) so that if
regions \( u, v \) are adjacent to an edge of \( T_F \), then \( |\text{Tr} U|, |\text{Tr} V| \leq M_0 \). In particular, if \( u \) is a
region touching a sink vertex then \( |\text{Tr} U| \leq M_0 \).

Proof. Most of the assertions are proved on p. 782 of [14], see also Corollary 3.12 of [3]. The
assertion that \( T_F \) contains all sink vertices is included in Theorem 2.7 of [13]; this follows since
\( T_F \) is connected and the arrow on each edge not in \( T_F \) points towards \( T_F \). Finally, to include all
edges adjacent to any sink vertex we note that \( T_F \) can always be enlarged, possibly increasing
the constant \( M_0 \), to a larger finite tree with the same properties and which strictly contains
the original one, see the proofs of Theorem 3.2 of [14] and Theorem 3.16 of [3]. \( \square \)

4. The Bowditch condition implies Bounded Intersection

In this section we prove some implications among the three basic concepts. The first two
results are easy:

Proposition 4.1. If a representation \( \rho : F_2 \to SL(2, \mathbb{C}) \) is primitive stable then it satisfies
BIP.

Proof. The broken geodesic corresponding to any primitive element by definition passes through
the basepoint \( O \). The broken geodesics \( \{ \text{br}_\rho(u; (a,b)) \}, u \in \mathcal{P} \) are by definition uniformly
quasigeodesic, so each is at uniformly bounded distance to its corresponding axis. Hence all
the axes are at uniformly bounded distance to \( O \) and so in particular axes corresponding
to primitive palindromic elements cut the three corresponding special hyperelliptic axes in
bounded intervals. \( \square \)

Proposition 4.2. The condition PS implies the Bowditch BQ-condition.

Proof. This is not hard, see for example [10]. From primitive stability, uniformity of constants
in [2] implies Fibonacci growth, which in turn implies that only finitely many elements have
lengths and therefore traces less than a give bound (see Lemma 4.4 below). \( \square \)

The main result of this section is:

Theorem 4.3. The BQ-condition implies the bounded intersection property BIP.
The idea of the proof is the following. Suppose that \((u, v)\) is a palindromic pair of generators, so that their axes intersect one of the three special hyperelliptic axes \(E\) perpendicularly. The hyperbolic cosine formula expresses the perpendicular distance \(d\) between \(AxU\) and \(AxV\) (which is measured along \(E\)) in terms of the translation lengths of \(U, V\) and \(UV^{-1}\). Provided these lengths are sufficiently long and that \(|\operatorname{Tr} U| \geq |\operatorname{Tr} UV^{-2}|\), we get an estimate showing \(d\) is exponentially small in the minimum of \(\ell(U)\) and \(\ell(V)\) (Proposition 4.6). We then move stepwise along \(E\) from its intersection point of with \(AxU\) to its intersection point with one of the axes in \(\Omega(M)\) for suitable \(M\) using intermediate intervals whose end points are the intersection points with \(E\) of axes corresponding to generator pairs, all of which are palindromic with respect to the same basic generator pair as \((u, v)\) (Proposition 4.8). The estimates of Fibonacci growth as in Lemma 3.5 show that the sum of lengths of these intervals is finite proving the result. The details follow.

We begin with two easy results. For a loxodromic element \(X \in \text{SL}(2, \mathbb{C})\) let \(\ell(X) > 0\) denote the (real) translation length and let \(\lambda(X) = (\ell(X) + i\vartheta(X))/2\) be half the complex length, so that \(\operatorname{Tr} X = \pm 2 \cosh \lambda(X)\).

**Lemma 4.4.** There exists \(L_0 > 0\) so that if \(\xi + i\eta \in \mathbb{C}\) with \(\xi > L_0\) then \(\xi - \log 3 \leq \log |\cosh(\xi + i\eta)|\) \(\leq \xi\). In particular, for \(X \in \text{SL}(2, \mathbb{C})\) we have \(e^{\ell(X)}/3 \leq |\operatorname{Tr} X|/2 \leq e^{\ell(X)}\) whenever \(\ell(X) > L_0\). Also \(|\sinh(\xi + i\eta)| \geq \xi/3\).

*Proof.* For the right hand inequality, since \(|\cosh(\xi + i\eta)| = e^{\xi}|(1 + e^{-2\xi - 2i\eta})|/2\) we have
\[
\log |\cosh(\xi + i\eta)| = \xi + \log |(1 + e^{-2\xi - 2i\eta})|/2 \leq \xi
\]
since \(|(1 + e^{-2\xi - 2i\eta})|/2 \leq 1\).

For the left hand inequality, since \(\xi > L_0\) we have, choosing \(L_0\) large enough, \(|(1 + e^{-2\xi - 2i\eta})|/2 \geq 1/3\) so that \(\log |(1 + e^{-2\xi - 2i\eta})|/2 \geq -\log 3\) and hence \(\log |\cosh(\xi + i\eta)| \geq \xi - \log 3\).

The estimate on \(|\sinh(\xi + i\eta)|\) follows similarly. \(\square\)

**Lemma 4.5.** Suppose that \(\rho : F_2 \to \text{SL}(2, \mathbb{C})\) and that \((u, v)\) are generators such that \(|\operatorname{Tr} UV| \geq |\operatorname{Tr} UV^{-1}|\). If \(\ell(U), \ell(V) > L_0\) with \(L_0\) as in Lemma 4.4, then \(\ell(U) + \ell(V) - 2\log 3 \leq \ell(UV)\).

*Proof.* Let \(\hat{u} = \operatorname{Tr} U, \hat{v} = \operatorname{Tr} V, \hat{z} = \operatorname{Tr} UV, \hat{w} = \operatorname{Tr} UV^{-1}\), so that by assumption \(|\hat{z}| \geq |\hat{w}|\). Since \(\hat{u}\hat{v} = \hat{z} + \hat{w}\) this gives \(|\hat{u}\hat{v}| \leq 2|\hat{z}|\) and hence \(\log |\hat{u}| + \log |\hat{v}| \leq \log |\hat{z}| + \log |2|\). Since \(\hat{u} = 2 \cosh(\ell(U))\) and so on, we have
\[
2 \log 2 + \log |\cosh l(U)| + \log |\cosh l(V)| \leq 2 \log 2 + \log |\cosh l(UV)|.
\]
Together with Lemma 4.4 this gives
\[
\ell(U) + \ell(V) - 2\log 3 \leq \ell(UV)
\]
as required. \(\square\)

We start the proof of Theorem 4.3 with an estimate of the perpendicular distance between the axes of ‘long’ pairs of generators. For \(t \in \mathbb{R}, f : \mathbb{R} \to \mathbb{R}\) write \(|f(t)| \leq O(t)\) to mean there exists \(c > 0\), depending only on the representation \(\rho\), such that \(|f(t)| \leq ct\).

**Proposition 4.6.** Suppose that \((u, v)\) is a pair of generators palindromic with respect to one of the three basic generator pairs, and suppose that \(|\operatorname{Tr} U| \geq |\operatorname{Tr} UV^{-2}|\), where as usual \(U = \rho(u), V = \rho(v)\). Then with \(L_0 > 0\) as in Lemma 4.4, if \(\ell(U), \ell(V) > L_0\) then \(d(AxU, AxV) \leq O(e^{-m})\) where \(m = \min\{\ell(U), \ell(V)\}\), with constants depending only on \((a, b)\) and \(O\).
Proof. Consider the right angled hexagon $H$ whose alternate sides follow the axes of $U, U^{-1}$ and $V^{-1}$. Orienting the sides consistently round $H$, we may define the complex distances $\sigma_1, \ldots, \sigma_6$ between the sides in such a way that, assuming that $\Re \lambda(U) > 0$ and so on, we have $\sigma_1 = \lambda(U)$ or $\lambda(U) + i\pi$; $\sigma_3 = \lambda(UV^{-1})$ or $\lambda(UV^{-1}) + i\pi$ and $\sigma_5 = \lambda(V^{-1})$ or $\lambda(V^{-1}) + i\pi$. Moreover $\sigma_6 = \pm (d + i\theta)$ where $=d + i\theta$ is the complex distance between the correctly oriented axes of $U$ and $V$ and where we take $d \geq 0$. (See for example [12] for a discussion of complex length and hyperbolic right angled hexagons, although as we shall see shortly such detail is not needed here.)

The hexagon formula in $H$ gives:

\begin{equation}
\cosh \sigma_6 = \frac{\cosh \sigma_3 - \cosh \sigma_1 \cosh \sigma_5}{\sinh \sigma_1 \sinh \sigma_5}
\end{equation}

Thus

\begin{equation}
| \cosh \sigma_6 + 1 | \leq \left| \frac{\cosh \sigma_3}{\sinh \sigma_1 \sinh \sigma_5} \right| + |1 - \coth \sigma_1 \coth \sigma_5|
\end{equation}

which in view of the remarks above can be rewritten

\begin{equation}
| \cosh +1 | \leq \left| \frac{\cosh \lambda(UV^{-1})}{\sinh \lambda(U) \sinh \lambda(V)} \right| + |1 - \coth \lambda(U) \coth \lambda(V)|
\end{equation}

Now assume that $\ell(U), \ell(V), \ell(UV^{-1}) > L_0$ with $L_0$ as in Lemma 4.4.

For $z \in \mathbb{C}$ we have $\coth z = (1 + e^{-2z})/(1 - e^{-2z})$, hence for large enough $|z|$ we have $\coth z = 1 + O(e^{-2z})$ so that

$$\coth \lambda(U) \coth \lambda(V) = 1 + O(e^{-2m}),$$

where we use estimates as in Lemma 4.4: $| \cosh(\xi + i\eta) | \leq O(e^\xi)$ and $| \sinh(\xi + i\eta) | \geq O(e^\xi)$ for $\xi > L_0$.

Similar estimates also give

\begin{equation}
\left| \frac{\cosh \lambda(UV^{-1})}{\sinh \lambda(U) \sinh \lambda(V)} \right| \leq c \exp(\ell(UV^{-1}) - \ell(U) - \ell(V))/2.
\end{equation}

By Lemma 4.5 applied to the generator pair $(V, UV^{-1})$, since by hypothesis $|\text{Tr } U| \geq |\text{Tr } UV^{-2}|$, we have $\ell(U) \geq \ell(UV^{-1}) + \ell(V) - 2 \log 3$ so that $-2\ell(V) \geq \ell(UV^{-1}) - \ell(V) - \ell(U) - 2 \log 3$ and hence

$$\left| \frac{\cosh \lambda(UV^{-1})}{\sinh \lambda(U) \sinh \lambda(V)} \right| \leq O(e^{-m}).$$

Now suppose we only have $\ell(U), \ell(V) > L_0$ while $\ell(UV^{-1}) \leq L_0$. Then instead of the estimate in (10) we get

\begin{equation}
\left| \frac{\cosh \lambda(UV^{-1})}{\sinh \lambda(U) \sinh \lambda(V)} \right| \leq O(e^{-m})
\end{equation}

with a bound independent of $(U, V)$. (In fact in this case the estimate improves to $O(\exp(-(\ell(U) + \ell(V)))$ but we won’t need this here.)

In either case we have

$$| \cosh(d + i\pi) - 1 | \leq O(e^{-m})$$

and hence $d = \Re \delta = O(e^{-m})$. □
We now proceed to estimate the distance between arbitrary pairs of palindromic axes. Assume that the representation \( \rho \in \mathcal{B} \). Choose \( M_0 \geq 2 \) and a finite connected non-empty subtree tree \( T_F \) of \( \mathcal{T} \) as in Proposition 3.6.

**Lemma 4.7.** Let \( M \geq M_0 \) and suppose that \( u \in \Omega \setminus \Omega_\rho(M) \). Then there is an oriented edge \( \varepsilon \) pointing out of \( u \) so that \( \varepsilon \) is not contained in \( T_F \).

**Proof.** Label the regions adjacent to \( u \) consecutively round \( \partial u \) by \( y_n, n \in \mathbb{Z} \) and let \( e_n \) denote the edge between \( y_n, u \). By Lemma 3.4, for large enough \( |n| \) the arrows on the edges round \( \partial u \) point in the direction of decreasing \( |n| \). Thus there is at least one \( r \in \mathbb{Z} \) so that the heads of \( e_r \) and \( e_{r+1} \) meet at a common vertex \( q \in \partial u \). The remaining arrow at \( q \) must point out of \( u \) for otherwise \( q \) is a sink vertex and hence by Proposition 3.6 all the edges meeting at \( q \) are in \( T_F \), so that \( u \in \Omega(M_0) \subset \Omega(M) \) contrary to assumption. \( \square \)

We call such a vertex a **plughole** of \( u \).

**Proposition 4.8.** Let \( M \geq M_0 \) and suppose that the representation \( \rho \in \mathcal{B} \) and that the generator \( u \) is palindromic of type \( \eta \in \mathcal{E} \). Suppose also that \( u \notin \Omega(M) \). Pick \( \eta' \neq \eta \). Then there is a sequence of generators \( u_0 = u, u_1, \ldots, u_k \in \mathcal{P} \) such that for \( i = 0, \ldots, k - 1 \):

1. \( (u_i, u_{i+1}) \) are neighbours.
2. \( (u_i, u_{i+1}) \) are palindromic with respect to \( (\eta, \eta') \).
3. \( |\text{Tr}(U_i)| \geq |\text{Tr}(U_iU_{i+1}^{-1})| \).
4. \( u_i \in \Omega(M) \) but \( u_i \notin \Omega(M), 0 \leq i < k \).

**Proof.** Suppose that \( u \notin \Omega(M) \) and let \( \varepsilon \) be an oriented edge pointing out of some plughole of \( u \). Of the two regions adjacent to \( \varepsilon \), one, \( u' \) say, is of type \( \eta' \). Set \( u_0 = u, u_1 = u' \) and arrange by cyclic permutation if necessary that \( (u_0, u_1) \) is palindromic with respect to \( (\eta, \eta') \). Then the other region adjacent to \( \varepsilon \) can be chosen to be \( uu'^{-1} \) and the region at the head of \( \varepsilon \) is \( uu'^{-2} \). Thus \( |\text{Tr}(U)| \geq |\text{Tr}(UU'^{-2})| \).

If \( u_1 \in \Omega(M) \) conditions (1)-(4) are satisfied with \( k = 1 \). Otherwise we repeat the argument. The process terminates because by Proposition 3.6 every descending path of arrows eventually meets \( T_F \), and both regions adjacent to an edge in \( T_F \) are in \( \Omega(M_0) \subset \Omega(M) \). \( \square \)

**Proof of Theorem 4.3** Suppose the generator \( u = u_0 \) is palindromic with respect \( \eta \) and that \( \eta' \neq \eta \). Let \( \mathcal{E} \) be the corresponding special hyperelliptic axis. Choose \( L > L_0 \) as in Lemma 4.4 so that \( |\text{Tr}(U)| > 2e^L \) implies \( \ell(U) > L \). With \( M_0 \) as in Proposition 3.6 choose \( M = \max\{M_0, 2e^L\} \). Let \( \Xi \) denote the set of axes corresponding to elements in \( \mathcal{V} \subset \Omega(M) \) which are of types either \( \eta \) or \( \eta' \). It is sufficient to see that \( \text{Ax}U \) meets \( \mathcal{E} \) at a uniformly bounded distance to one of the finitely many axes in \( \Xi \).

Let \( u_0 = u, u_1, \ldots, u_k \) be the sequence of Proposition 4.8. If \( k = 0 \) there is nothing to prove since \( \text{Ax}U_k \in \Xi \).

Suppose \( k > 0 \). Let \( \varepsilon \) be the edge emanating from the plughole of \( u_i, 0 \leq i < k \) and consider the two adjacent regions \( u_{i+1}, u_{i+1}^{-1} \), choosing the numbering so that \( u_{i+1} \) is of type \( \eta' \), while \( u_iu_{i+1}^{-1} \) is of the third type \( \eta'' \). Since \( u_i \notin \Omega(M) \) we have \( \ell(U_i) > L \) by choice of \( M \). If in addition \( \ell(U_{i+1}) > L \) then the pair \( (u_i, u_{i+1}) \) satisfy the condition of Proposition 4.6 so that \( d(\text{Ax}U_i, \text{Ax}U_{i+1}) \leq O(e^{-m_i}), m_i = \min\{\ell(U_i), \ell(U_{i+1})\} \). Otherwise, \( \ell(U_{i+1}) \leq L \) so that \( u_{i+1} \in \Omega(M) \subset \Xi \) and therefore \( k = i + 1 \) and the process terminates.
Lemma 5.1. Given any angle \( \psi > 0 \), there exists \( L = L(\psi) > 0 \), depending only on \( \psi \), such that if \( \gamma \) is any broken geodesic whose segments are of at least \( L \) length at each bending point, and such that the interior bending angle at each bending point is at least \( \psi \), then \( \gamma \) is a quasigeodesic with constants depending only on \( \psi \).

Proof. Let \( L_1 \) be length of the finite side of a triangle with angles \( 0, \pi/2, \psi/2 \). Suppose that lines \( QP, Q'P \) make an angle of \( \alpha > \psi \) at \( P \) and also that \( |QP|, |Q'P| > L_1 \). Let \( \Lambda, \Lambda' \) be the lines through \( Q, Q' \) and orthogonal to \( QP, Q'P \) respectively. Then \( \Lambda, \Lambda' \) do not meet.

Now pick \( L > L_1 \) and consider a broken geodesic with segments \( s_1, \ldots, s_n \) of lengths \( \ell_1, \ell_2, \ldots, \ell_n \) with \( \ell_i > 3L \) and such that the interior angles between segments \( s_i, s_{i+1} \) are at least \( \psi \) for all \( i \). Let \( H^-_i, H'^+_i \) be half planes orthogonal to \( s_i \) at distance \( L \) from the initial and final points \( s^-_i, s'^+_i \) of \( s_i \) respectively. Clearly \( H^-_i \cap H'^+_i = \emptyset \) and \( d(H'^+_i, H^-_i) \geq \ell_i - 2L \). Let \( \Pi \) be the plane containing \( s_i \) and \( s_{i+1} \). Then the lines \( s_i, s_{i+1} \) together with the lines \( H'^+_i \cap \Pi, H^-_{i+1} \cap \Pi \) are exactly in the configuration described in the first paragraph, and hence \( H'^+_i \cap H^-_{i+1} = \emptyset \).

This shows that the half planes \( H^-_1, H'^+_1, \ldots, H^-_n, H'^+_n \) are nested and that

\[
d(s^-_1, s'^+_n) \geq \sum_{i=1}^n (\ell_i - 2L) > \sum_{i=1}^n \ell_i / 3
\]

which proves the result.

5. Bounds on broken geodesics

In this section we prove our main result Theorem 5.1. We begin by collecting some basic results on quasigeodesics.

By a broken geodesic we mean a path composed of a sequence of geodesic segments \( \ldots, s_i, s_{i+1}, \ldots \) meeting at their endpoints \( P_i, P_{i+1} \), where \( P_i \) is the meeting of the end point of \( s_i \) with the initial point of \( s_{i+1} \). We call the \( P_i \) bending points and define the exterior bending angle \( \phi_i \) at \( P_i \) to be the angle between the extension of \( s_i \) through \( P_i \) and \( s_{i+1} \). Thus \( s_i, s_{i+1} \) combine to form a single longer geodesic segment iff \( \phi_i = 0 \). The interior bending angle is \( \pi - \phi_i \).

The following three lemmas are well known, but for the reader’s convenience we provide proofs.

Lemma 5.1. Given any angle \( \psi > 0 \), there exists \( L = L(\psi) > 0 \), depending only on \( \psi \), such that if \( \gamma \) is any broken geodesic whose segments are of at length at least \( L \), and such that the interior bending angle at each bending point is at least \( \psi \), then \( \gamma \) is a quasigeodesic with constants depending only on \( \psi \).

Proof. Let \( L_1 \) be length of the finite side of a triangle with angles \( 0, \pi/2, \psi/2 \). Suppose that lines \( QP, Q'P \) make an angle of \( \alpha > \psi \) at \( P \) and also that \( |QP|, |Q'P| > L_1 \). Let \( \Lambda, \Lambda' \) be the lines through \( Q, Q' \) and orthogonal to \( QP, Q'P \) respectively. Then \( \Lambda, \Lambda' \) do not meet.

Now pick \( L > L_1 \) and consider a broken geodesic with segments \( s_1, \ldots, s_n \) of lengths \( \ell_1, \ell_2, \ldots, \ell_n \) with \( \ell_i > 3L \) and such that the interior angles between segments \( s_i, s_{i+1} \) are at least \( \psi \) for all \( i \). Let \( H^-_i, H'^+_i \) be half planes orthogonal to \( s_i \) at distance \( L \) from the initial and final points \( s^-_i, s'^+_i \) of \( s_i \) respectively. Clearly \( H^-_i \cap H'^+_i = \emptyset \) and \( d(H'^+_i, H^-_i) \geq \ell_i - 2L \). Let \( \Pi \) be the plane containing \( s_i \) and \( s_{i+1} \). Then the lines \( s_i, s_{i+1} \) together with the lines \( H'^+_i \cap \Pi, H^-_{i+1} \cap \Pi \) are exactly in the configuration described in the first paragraph, and hence \( H'^+_i \cap H^-_{i+1} = \emptyset \).

This shows that the half planes \( H^-_1, H'^+_1, \ldots, H^-_n, H'^+_n \) are nested and that

\[
d(s^-_1, s'^+_n) \geq \sum_{i=1}^n (\ell_i - 2L) > \sum_{i=1}^n \ell_i / 3
\]

which proves the result.

\[\square\]
Lemma 5.2. Suppose given a hyperbolic triangle $\Delta$ with side lengths $a, b, c$ opposite vertices $A, B, C$ and angle $\psi$ at vertex $C$. Given $k > 0$ there exist $L, \epsilon > 0$ such that if $c \geq a + b - k$ then $\psi > \epsilon$ whenever $a, b > L$.

Proof. The formula
\[
\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma
\]
rearranges to
\[
\cos \gamma - 1 = \frac{\cosh a \cosh b}{\sinh a \sinh b} - 1 - \frac{\cosh c}{\sinh a \sinh b}.
\]
Since
\[
\frac{\cosh a \cosh b}{\sinh a \sinh b} \rightarrow 1 \text{ and } \frac{\cosh c}{\sinh a \sinh b} \geq \frac{4e^{a+b-k}}{2e^a e^b} \geq 2e^{-k}
\]
as $a, b \rightarrow \infty$ we see that $\cos \gamma - 1$ is bounded away from 0 giving the required bound. □

Lemma 5.3. Let $w$ be a cyclically shortest word in $F_2$ and let $\rho \colon F_2 \rightarrow SL(2, \mathbb{C})$. Suppose that the image $W = \rho(w)$ is loxodromic. Suppose also that the generators $(u, v)$ have images $U = \rho(u), V = \rho(v)$. Then the broken geodesic $br_{\rho}(w; (u, v))$ is quasigeodesic with constants depending only on $\rho, w,$ and $(u, v)$.

Proof. Suppose that $||w||_{(u,v)} = k$ and number the vertices of $br_{\rho}(w; (u, v))$ in order as $g_n O, r \in \mathbb{Z}$. We have to show that there exist constants $K, \epsilon > 0$ so that if $n < m$ then
\[
(m - n)/K - \epsilon \leq d(g_n O, g_m O) \leq K(m - n) + \epsilon.
\]
Pick $c > 0$ so that $d(O, hO) \leq c$ for $h \in \{u, v\}$. Clearly $d(g_n O, g_m O) \leq c(m - n)$. For the lower bound, write $m - n = rk + k_1$ for $r \geq 0, 0 \leq k_1 < k$. Then for some cyclic permutation of $w$, say $w'$, we have, setting $W' = \rho(w'), W'^r(g_n O) = g_{n+rk}(O)$ so that $d(g_n O, g_{n+rk} O) \geq r\ell(W)$. Thus
\[
d(g_n O, g_m O) \geq d(g_n O, g_{n+rk} O) - d(g_{n+rk} O, g_m O) \geq (m - n)\ell(W)/k - kc - \ell(W)/k.
\]
□

From now on, we assume that $\rho \in B$ so that by Theorem 4.3 $\rho$ satisfies BIP. The following simple consequence of BIP is critical:

Lemma 5.4. There exists $D > 0$ so that for any $u \in P$ which palindromic with respect to one of the three basic generator pairs, we have $d(U^r O, Ax U) < D$ for any $r \in \mathbb{Z}$.

Proof. By BIP, we may assume that $Ax U$ intersects one of the three special hyperelliptic axes at bounded distance at most $D$ to $O$, where $D$ is independent of $u$. Since $Ax U$ is invariant under $U$ we have also $d(U^r O, Ax U) < D$ for any $r \in \mathbb{Z}$.

Lemma 5.5. Let $u \in P$ be palindromic with respect to one of the three basic generators pairs. Let $\psi$ be the angle at vertex $O$ in the triangle with vertices $O, U^{-1}(O), U(O)$. Then there are $L_1, \epsilon_1 > 0$ so that if $\ell(U) > L_1$ then $\psi > \epsilon_1$.

Proof. Let $a, b, c$ denote the lengths of the sides $OU^{-1}(O), OU(O), U^{-1}(O)U(O)$ respectively. By Lemma 5.4 we have $a \leq \ell(U) + 2D, b \leq \ell(U) + 2D$ with $D$ as in that lemma. Clearly $c \geq \ell(U^2) = 2\ell(U)$. Thus $c \geq a + b - 4D$ and the conclusion follows from Lemma 5.2 □
Lemma 5.6. Suppose that \((u,v)\) is a generator pair palindromic with respect to one of the three basic generators pairs, and so that \(|\text{Tr } UV| \geq |\text{Tr } UV^{-1}|\). Let \(\psi\) be the angle at vertex \(O\) in the triangle with vertices \(O,V^{-1}(O),U(O)\). Then there exist \(L_2, \epsilon_2 > 0\) so that if \(\ell(U), \ell(V) > L_2\) then \(\psi > \epsilon_2\).

**Proof.** The proof is similar to that of the preceding lemma with \(a = d(O,V^{-1}(O)), b = d(O,U(O)), c = d(V^{-1}(O),U(O))\). As before \(a \leq \ell(V) + 2D, b \leq \ell(U) + 2D\) while clearly \(c \geq \ell(UV)\). From Lemma 4.5 it follows that for large enough \(L_2\),

\[
\ell(UV) \geq \ell(U) + \ell(V) - 2\log 3.
\]

Applying Lemma 5.2 gives the result. \(\square\)

Thus we have proved:

**Proposition 5.7.** There exists \(L_3 > 0\) with the following property. Suppose that for a palindromic generator pair \((u,v)\) we have \(\ell(U), \ell(V) > L_3\) and that \(|\text{Tr } UV| \geq |\text{Tr } UV^{-1}|\). Let \(C(u,v)\) denote the set of all cyclically shortest words in positive powers of \(u\) and \(v\). Then the collection of broken geodesics \(\{\text{br}_\rho(w;(u,v)), w \in C(u,v)\}\) is uniformly quasigeodesic, with constants depending only on \((u,v)\).

**Proof.** Choose \(L_1, L_2\) as in Lemmas 5.5 and 5.6 and then choose \(\psi = \min\{\epsilon_1, \epsilon_2\}\). Then choose \(L_3 = L(\psi)\) as in Lemma 5.1. \(\square\)

Given a generator pair \((u,v)\), we now apply the above results to generator pairs of the form \((u^nv, u^{N+1}v)\), \(N \in \mathbb{Z}\).

**Corollary 5.8.** Let \((u,v)\) be a generator pair \((u,v)\) such that \((u^{-1}v, v)\) is palindromic and let \(N \in \mathbb{Z}\). Then there exist \(m_0 \geq 2, \epsilon_2 > 0\), depending on \(u, v\) but not on \(N\), with the following property. Suppose that \(m \geq m_0\) and that \(u \in \Omega(m)\) while \(u^Nv, u^{N+1}v \notin \Omega(m)\). Then the interior angle at \(O\) in the triangle with vertices \((U^NV)^{-1}O, O, U^{N+1}VO\) is at least \(\epsilon_2\).

**Proof.** This is Lemma 5.6 applied to the generator pair \((u^nv, u^{N+1}v)\). Choose \(L_2\) as in Lemma 5.6 and then choose \(m_0\) so that \(|\text{Tr } X| > m_0\) implies \(\ell(X) > L_0\) for \(X \in SL(2, \mathbb{C})\) (use \(L_0\) as in Lemma 4.4). Note that, for any \(m \geq 2\), since \(\Omega(m)\) is connected and \(u^Nv, u^{N+1}v \notin \Omega(m)\) while \(u \in \Omega(m)\) it follows that \(u^Nvu^{N+1}v \notin \Omega(m)\) so that \(|\text{Tr } U^NVU^{N+1}V| \geq |\text{Tr } U|\).

Set \(x = u^{-1}v\) so that by the hypothesis \((x,v)\) is a palindromic pair. Then, as in the proof of Proposition 2.1, \((u^nv, u^{N+1}v) = ((vx^{-1})^Nv, (vx^{-1})^{N+1}v)\) is also palindromic and the result follows with \(\epsilon_2\) as in Lemma 5.6. \(\square\)

**Lemma 5.9.** For any palindromic generator pair \((u,v)\), there exist \(n_0 \in \mathbb{N}\) and \(\alpha > 0\), depending only on \(U\) and \(V\), so that \(d(O, U^NVO) \geq \alpha |N|\) whenever \(|N| > n_0\).

**Proof.** With suitable choice of \(n_0\) and \(\alpha\) we have

\[
(12) \quad d(O, U^NVO) \geq d(O, U^NO) - d(O, VO) \geq |N|\ell(U) - \ell(V) - 2D \geq \alpha |N| \quad \text{for} \quad |N| \geq n_0.
\]

Our next result allows us to deal simultaneously with all words in generator pairs \((u^nv, u^{N+1}v)\), \(N \in \mathbb{Z}\).
Proposition 5.10. Let \((u, v)\) be a generator pair such that the pair \((u^{-1}v, v)\) is palindromic. Suppose that \(\hat{w} = \hat{w}(u^nv, u^{N+1}v)\) for \(N \in \mathbb{Z}\) is a word written in positive powers of the generators \((u^Nv, u^{N+1}v)\) and let \(w\) be the same element of \(F_2\) written in terms of \((u, v)\). Suppose that \(u, v, m\) satisfy the conditions of Corollary 5.8. Then there exists \(n_1 \in \mathbb{N}\) so that \(br_{\rho}(w; (u, v))\) is quasigeodesic with constants depending on \((u, v)\) but not on \(N\) for any \(|N| \geq n_1\).

Proof. We have to show that the distance between any two vertices of \(br_{\rho}(w; (u, v))\) is bounded below by their distance in the word \(w\). For simplicity we may assume that \(N > 0\); the other case is similar. We first deal with subsegments of the form \(x_1x_2 \ldots x_M\) where each \(x_i\) is either \(u^Nv\) or \(u^{N+1}v\), and handle initial and final segments later. For simplicity we write \(\hat{N}\) to indicate either \(N\) or \(N + 1\) as the case may be.

Label the vertices of the broken geodesic \(br_{\rho}(x_1x_2 \ldots x_M; (u^Nv, u^{N+1}v))\) as \(P_1, \ldots, P_{M+1}\) so that, after translation if needed, \(P_1 = O, P_2 = X_1O, P_3 = X_1X_2O, \ldots, P_{M+1} = X_1X_2 \ldots X_MO\).

Choose \(\epsilon_2 > 0\) as in Corollary 5.8. Since \(|\text{Tr} U^N V| \to \infty\) as \(|N| \to \infty\) we can choose \(n_1\) so that \(|N| > n_1\) implies \(\ell(U^N V) > L\) where \(L = L(\epsilon_2)\) is as in Lemma 5.1. Then by Lemma 5.1, the broken geodesic consisting of geodesic segments joining \(P_0, P_1, \ldots, P_{M+1}\) is quasigeodesic, in particular, there exists \(c > 0\) so that

\[
|P_1P_{M+1}| \geq c\left(\sum_{i=1}^{M} |P_iP_{i+1}|\right) - c. \tag{13}
\]

By Lemma 5.9 there exists \(\alpha > 0\) so that \(|P_iP_{i+1}| \geq \alpha \hat{N}\) giving (after slight adjustment of constants since \(||x_i||_{(u,v)} = \hat{N} + 1\) not \(\hat{N}\)) an inequality of the form

\[
|d(O, X_1X_2 \ldots X_MO)| \geq c' ||x_1x_2 \ldots x_M||_{(u,v)} - c. \tag{14}
\]

It remains to deal with possible initial and final segments of \(w\) which are not full periods of \(u^Nv\). This means we have to consider words of the form \(yx_1x_2 \ldots x_Mz\) where \(x_i\) are as above and \(y\) and \(z\) are respectively initial and final segments of the form \(u^h v\) and \(u^k v\) for \(h, k' \in \{0, \ldots, \hat{N} - 1\}\).

Consider a word of type \(yx_1x_2 \ldots x_M; \) the other cases are similar. If \(h \leq r_0\) (for some \(r_0\) to be chosen below) then the total length of the broken geodesic \(br_{\rho}(y; (u, v))\) is at most \(r_0d(O, UO) + d(O, VO) \leq a(r_0\ell(U) + \ell(V)) < K\) say. Concatenation with such segments at the beginning of \(br_{\rho}(x_1x_2 \ldots x_M; (u, v))\) adds at most a bounded constant and the desired inequality follows from (14).

Otherwise adjoin before the initial segment \(br_{\rho}(y; (u, v))\) extra images of \(O\) under \(U\) so that, again after translation, we have a full period \(O, UO, \ldots, U^{N-h}O, U^{N-h+1}O, \ldots, U^NO, U^NO\), with \(v\) corresponding to the segment \(U^{N-h+1}O, \ldots, U^NO, U^NO\). Let \(P_0 = O, Q = U^{N-h}O, P_1 = U^NO\) and let \(P_2, P_3, \ldots, P_{M+1}\) be the remaining vertices of \(br_{\hat{w}}(u^Nv, u^{N+1}v))\), so that \(P_1, P_2, \ldots, P_{M+1}\) is the same broken geodesic as before, translated by \(U^NV\). We need a lower bound of the form \(d(Q, P_{M+1}) \geq c(h + ||x_1x_2 \ldots x_M||_{(u,v)}) - c\).

As above, the broken geodesic joining points \(P_0P_1P_2 \ldots P_{M+1}\) is uniformly quasigeodesic (with constants depending on \(U, V\) but not \(M, N\)). Let \(Q\) be the foot of the perpendicular from \(Q\) onto \(P_0P_1\). Lemma 5.9 implies that the broken arc joining \(P_0 = O, UO, \ldots, U^NO, U^NO = P_1\) is quasigeodesic with constants depending only on \((U, V)\). Hence it is within uniformly bounded Hausdorff distance of the arc \(P_0P_1, \) (II III H Theorem 1.7) so that \(|QQ'| < D_0\) for some fixed
$D_0$. Thus we have comparisons

$$|Q'P_i| - D_0 \leq |QP_i| \leq |Q'P_i| + D_0, \quad i = 1, \ldots, M + 1.$$  

By Lemma 5.9 again we have $|QP_1| > \alpha h$, so that for $h > r_0$ for suitable $r_0$ we have $|Q'P_1| > L$ with $L = L(e_2)$ as in Lemma 5.1. Hence the broken arc $Q'_1P_2 \ldots P_{M+1}$ is uniformly quasigeodesic and so

$$d(Q, P_{M+1}) \geq d(Q', P_{M+1}) - D_0 \geq c(|Q'P_1| + \sum_{i=1}^{M} |P_iP_{i+1}|) - c'$$

for suitable $c, c' > 0$. Using Lemma 5.9 once more together with (13) and (15) gives the result. \(\square\)

We are finally ready to prove our main result:

**Theorem 5.11.** If a representation $\rho: F_2 \to SL(2, \mathbb{C})$ satisfies the BQ-condition, then $\rho$ is primitive stable.

**Proof.** Suppose that $\rho \in \mathcal{B}$ and let $m \geq 2$. For each $u \in \Omega(m)$ choose $u \in \mathcal{U}$ and fix some neighbour $\hat{u}$ of $u$, chosen so that $(u^{-1}\hat{u}, \hat{u})$ is a palindromic pair. Note that the regions adjacent to $u$ are of the form $u^*\hat{u}$ for $r \in \mathbb{Z}$. Let $V$ denote the vertices of the tree $T$. For $q \in V$, denote by $\mathcal{N}(q)$ the three regions abutting at $q$. We make the following definitions:

- $Int^V\Omega(m)$ is the set of $q \in V$ for which $|\phi_\rho(u)| \leq m$ for all $u \in \mathcal{N}(q)$.
- $\partial^V\Omega(m)$ is the set of $q \in V$ for which $|\phi_\rho(u)| \leq m$ for some $u \in \mathcal{N}(q)$ and $|\phi_\rho(u')| > m$ for some $u' \in \mathcal{N}(q)$.
- $\partial^V_0\Omega(m)$ is the set of $q \in \partial^V\Omega(m)$ for which only one $u \in \mathcal{N}(q)$ is in $\Omega(m)$.
- For $N_0 \in \mathbb{N}$, $\partial^V_{N_0}\Omega(m)$ is the set of $q \in \partial^V\Omega(m)$ for which $\mathcal{N}(q) = \{u, u^N\hat{u}, u^{N+1}\hat{u}\}$ with $u \in \Omega(m)$ and $|N| \geq N_0$.

Choose $L_3$ as in Proposition 5.7 and choose $m > 0$ so that $|\phi(X)| > m$ implies $\ell(X) > L_3, X \in SL(2, \mathbb{C})$. There are only finitely many regions $u$ with $|\phi(u)| \leq m$ so $Int^V\Omega(m)$ is finite. By Lemma 3.4 traces increase round $\partial u$ for any $u \in \Omega(m)$, hence $\partial^V\Omega(m) \setminus \partial^V_0\Omega(m)$ is finite. Finally, choose $N_0$ large enough for Lemma 5.9 and so that $\ell(U^N\hat{U}) > L_3$ whenever $|N| > N_0$. It follows that $\partial^V_{N_0}\Omega(m) \subset \partial^V_0\Omega(m)$ and that $\partial^V\Omega(m) \setminus \partial^V_{N_0}\Omega(m)$ is finite.

**Lemma 5.12.** Suppose that $q \notin Int^V\Omega(m) \cup \partial^V\Omega(m)$. Then $q$ is connected by a finite descending path to a vertex in $\partial^V\Omega(m)$. Moreover the first such point along this path is in $\partial^V_0\Omega(m)$.

**Proof.** By Lemma 3.2 there is a finite descending path to an edge one of whose neighbouring regions is in $\Omega(2)$. So there must be a first vertex $q_*$ exactly one of whose neighbours is in $\Omega(m)$, so that $q_* \in \partial^V_0\Omega(m)$. \(\square\)

Given a vertex $q \in \partial^V\Omega(m)$, there is a unique oriented edge $\bar{e}$ whose head is $q$ and which is not contained in $\partial^V\Omega(m)$. We define the wake of $q$ to be the wake of $\bar{e}$.

Suppose that a region $v$ is not in $\Omega(m)$. If $v$ has a vertex in common with $\Omega(m)$, then it has an edge $e$ say in common with $\Omega(m)$. Let $z \in \Omega(m)$ be the other region adjacent to $e$, and let $y, y'$ denote the other regions adjacent to $v$ at the two ends of $e$. Note there are at most finitely many regions $v$ for which at least one of $y, y'$ is in $\Omega(m)$; for all other $v$ the two end vertices of $e$ are in $\partial^V_0\Omega(m)$.
Otherwise $v$ has no vertex in common with $\Omega(m)$. Pick any oriented edge $\vec{e}$ of $\partial v$. It follows from Lemma 5.12 that $\vec{e}$ is connected by a finite descending path which first meets $\partial^V \Omega(m)$ in some vertex $q_*(v) \in \partial^V \Omega(m)$, and $v$ is in the wake of $q_*(v)$. Moreover since $\partial^V \Omega(m) \setminus \partial^V_{N_0} \Omega(m)$ is finite, all but finitely many of these wakes land in $\partial^V_{N_0} \Omega(m)$.

In summary we have shown that any region is either in $\Omega(m)$; or has an edge in common with $\Omega(m)$ with at least one other adjacent region in $\Omega(m)$; or is in the wake of a descending path which lands at some point $q_* \in \partial^V \Omega(m) \setminus \partial^V_{N_0} \Omega(m)$; or finally is in the wake of a descending path which lands at $q_* \in \partial^V_{N_0} \Omega(m)$.

By Lemma 5.3, the broken geodesic $br_\rho(w; (a, b))$ constructed from each of the finitely many regions $u$ of the first two types is quasigeodesic with constants depending on $u$ for any $w \sim u$.

Suppose that $z$ in the wake of a vertex in $\partial^V \Omega(m) \setminus \partial^V_{N_0} \Omega(m)$. Let $(x, y)$ be the palindromic pair of generators adjacent to the edge $e$ whose head is $q_*(z)$. (These will be of the form $(u^n \hat{u}, u^{n+1} \hat{u})$ for some $n \in \mathbb{Z}$.) Since $\ell(X), \ell(Y) > L$, since the arrow on $e$ points into $\Omega(m)$, and since $z$ is, up to cyclic permutation, a product of positive powers of $x, y$, by Proposition 5.7 the collection of such broken geodesics $br_\rho(z; (x, y))$ is uniformly quasigeodesic with constants depending on $(x, y)$.

Finally suppose that $z$ is in the wake of a descending path which lands at $q_*(z) \in \partial^V_{N_0} \Omega(m)$. The neighbours of the edge whose head is $q_*(z)$ are of the form $(u^n \hat{u}, u^{n+1} \hat{u})$ where $|n| \geq N_0$. Note that, up to cyclic permutation, any $z \in \mathbb{Z}$ can be written as a product of positive powers of $(u^n \hat{u}, u^{n+1} \hat{u})$. Hence by Proposition 5.10 the collection of such broken geodesics $br_\rho(z; (u, \hat{u}))$ is uniformly quasigeodesic with constants depending only on $(u, \hat{u})$.

Putting all this together, there is a finite set of generator pairs $S$, such that any $w \in F_2$ can be expressed as a word in some $(s, s') \in S$ in such a way that $br_\rho(w; (s, s'))$ is quasigeodesic with constants depending only on $(s, s')$. The quasigeodesic $br_\rho(w; (s, s'))$ can be replaced by a broken geodesic $br_\rho(w; (a, b))$ which is also quasigeodesic with a change of constants. The total number of replacements required involves only finitely many constants and the result follows.

\[ \square \]

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