Extensions of Stern’s congruence for Euler numbers

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Abstract

For a nonzero integer \(a\) let \(E^{(a)}_n\) be given by
\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} a^{2k} E^{(a)}_{n-2k} = (1-a)^n \quad (n = 0, 1, 2, \ldots),
\]
where \(\lfloor x \rfloor\) is the greatest integer not exceeding \(x\). As \(E^{(1)}_n = E_n\) is the Euler number, \(E^{(a)}_n\) can be viewed as a generalization of Euler numbers. Let \(k\) and \(m\) be positive integers, and let \(b\) be a nonnegative integer. In this paper, we determine \(E^{(a)}_{2^m k+b}\) modulo \(2^{m+10}\) for \(m \geq 5\). For \(m \geq 5\) we also establish congruences for \(U_{k \varphi(5^m)+b}\), \(E_{k \varphi(5^m)+b}\), \(S_{k \varphi(5^m)+b}\) (mod \(5^{m+5}\)) and \(S_{k \varphi(3^m)+b}\) (mod \(3^{m+5}\)), where \(U_{2n} = E^{(3/2)}_{2n}\), \(S_n = E^{(2)}_n\) and \(\varphi(n)\) is Euler’s function.

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1 Introduction

Let \(\mathbb{Z}\) and \(\mathbb{N}\) be the set of integers and the set of positive integers, respectively. The Euler numbers \(\{E_n\}\) are given by
\[
E_0 = 1, \quad E_{2n-1} = 0, \quad \sum_{r=0}^{n} \binom{2n}{2r} E_{2r} = 0 \quad (n = 1, 2, 3, \ldots).
\]

For \(k, m \in \mathbb{N}\) and \(b \in \{0, 2, 4, \ldots\}\), in 1875 Stern [9] proved the following congruence, which is now known as Stern’s congruence:
\[
E_{2^m k+b} \equiv E_b + 2^m k \pmod{2^{m+1}}.
\] (1.1)

There are many modern proofs of (1.1). See for example [1,3,8,10].

Let \(b \in \{0, 2, 4, \ldots\}\) and \(k, m \in \mathbb{N}\). In [7] the first author and L.L. Wang showed that
\[
E_{2^m k+b} \equiv E_b + 2^m k(7(b+1)^2 - 18 + 2^m k(7-b)) \pmod{2^{m+7}} \quad \text{for} \quad m \geq 3.
\] (1.2)
In [4], the first author introduced the sequence \( \{E_n^{(a)}\} \) as a generalization of Euler numbers. For \( a \neq 0 \), \( \{E_n^{(a)}\} \) is given by

\[
\sum_{k=0}^{[n/2]} \binom{n}{2k} a^{2k} E^{(a)}_{n-2k} = (1-a)^n \quad (n = 0, 1, 2, \ldots).
\] (1.3)

Clearly \( E_n = E_n^{(1)} \). The first few values of \( E_n^{(a)} \) are shown below:

\[
\begin{align*}
E_0^{(a)} &= 1, & E_1^{(a)} &= 1 - a, & E_2^{(a)} &= 1 - 2a, & E_3^{(a)} &= 1 - 3a + 2a^3, \\
E_4^{(a)} &= 1 - 4a + 8a^3, & E_5^{(a)} &= 1 - 5a + 20a^3 - 16a^5, \\
E_6^{(a)} &= 1 - 6a + 40a^3 - 96a^5, \\
E_7^{(a)} &= 1 - 7a + 70a^3 - 336a^5 + 272a^7, \\
E_8^{(a)} &= 1 - 8a + 112a^3 - 896a^5 + 2176a^7.
\end{align*}
\] (1.4)

For a prime \( p \) and nonzero integer \( n \) let \( \text{ord}_p n \) denote the unique nonnegative integer \( \alpha \) such that \( p^\alpha \mid n \) and \( p^{\alpha+1} \nmid n \). In [4], Z.H. Sun showed that for given nonzero integer \( a \), nonnegative integer \( b \) and positive integers \( k \) and \( m \),

\[
E_{2^m k+b}^{(a)} - E_b^{(a)} \equiv \begin{cases} 
2^m k(a^3((b-1)^2+5) - a + 2^m ka^3(b-1)) \pmod{2^{m+4+3\text{ord}_2 a}} & \text{if } 2 \mid a, \\
2^m ka((b+1)^2 + 4 - 2^m k(b+1)) \pmod{2^{m+4}} & \text{if } 2 \nmid a \text{ and } 2 \mid b, \\
2^m k(a^2 - 1) \pmod{2^{m+4}} & \text{if } 2 \nmid ab.
\end{cases}
\] (1.5)

In this paper we determine \( E_{2^m k+b}^{(a)} - E_b^{(a)} \pmod{2^{m+10}} \) for \( m \geq 4 \). See Theorems 2.1-2.3.

For a real number \( x \) let \( [x] \) be the greatest integer not exceeding \( x \). Let \( \{U_n\} \) and \( \{S_n\} \) be given by

\[
U_0 = 1, \quad U_n = -2 \sum_{k=1}^{[n/2]} \binom{n}{2k} U_{n-2k} \quad (n \geq 1),
\]

\[
S_0 = 1, \quad S_n = 1 - \sum_{k=0}^{n-1} \binom{n}{k} 2^{2n-2k-1} S_k \quad (n \geq 1).
\]

From [4,5] we know that

\[
U_{2n} = E_n^{(3/2)} \quad \text{and} \quad S_n = E_n^{(2)}.
\] (1.6)

Let \( \varphi(n) \) denote Euler’s totient function. In [6] the first author established congruences for \( E_{k\varphi(3^n)+b} \pmod{3^{n+4}} \) and \( U_{k\varphi(3^n)+b} \pmod{3^{n+4}} \).

For \( m \in \mathbb{N} \), let \( \mathbb{Z}_m \) be the set of rational numbers whose denominator is coprime to \( m \). In [2], the first author introduced the notion of \( p\)-regular functions. If \( f(k) \in \mathbb{Z}_p \) for any nonnegative integers \( k \) and \( \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) \equiv 0 \pmod{p^n} \) for all \( n \in \mathbb{N} \), then \( f \) is called a \( p \)-regular function.

Let \((\frac{a}{p})\) be the Legendre symbol. In [5] and [3], the first author proved that for any odd prime \( p \), both \( f_1(k) = (1-(\frac{a}{p})p^{k(p-1)+b})U_{k(p-1)+b} \) and \( f_2(k) = (1-(-1)^{\frac{k(p-3)+1}{2}} p^{k(p-1)+b})E_{k(p-1)+b} \). 


are \(p\)-regular functions. In Sections 3-6, using \(p\)-regular function and the binomial inversion formula, we obtain congruences for

\[
U_{k\varphi(5^m)+b}, \ E_{k\varphi(5^m)+b}, \ S_{k\varphi(5^m)+b} \pmod{5^{m+5}} \quad \text{and} \quad S_{k\varphi(3^m)+b} \pmod{3^{m+5}},
\]

where \(k, m \in \mathbb{N}, m \geq 5\) and \(b\) is a nonnegative integer. See Theorems 3.1, 4.1, 5.1 and 6.1.

## 2 Congruences for \(E_{2m^k+b}^{(a)} \pmod{2^{m+10}}\)

For \(a \neq 0\) let \(\{E_n^{(a)}\}\) be given by (1.3), and let

\[
e_s(a, b) = 2^{-s} \sum_{r=0}^{s} \binom{s}{r} (-1)^r E_{2r+b-2[\frac{r}{2}]}^{(a)} \tag{2.1}
\]

for \(s \in \mathbb{N}\). Then the first few \(e_s(a, b)\) are as follows:

\[
e_1(a, b) = \begin{cases} \frac{a}{b} & \text{if } 2 \mid b, \\ \frac{a - a^3}{2} & \text{if } 2 \nmid b. \end{cases}
\]

\[
e_2(a, b) = \begin{cases} 2a^3 & \text{if } 2 \mid b, \\ 4a^3(1 - a^2) & \text{if } 2 \nmid b, \end{cases}
\]

\[
e_3(a, b) = \begin{cases} 2a^3(6a^2 - 1), & \text{if } 2 \mid b, \\ 2a^3(-17a^4 + 18a^2 - 1) & \text{if } 2 \nmid b, \end{cases}
\]

\[
e_4(a, b) = \begin{cases} 8a^5(17a^2 - 4), & \text{if } 2 \mid b, \\ -48a^5 + 544a^7 - 496a^9 & \text{if } 2 \nmid b, \end{cases}
\]

\[
e_5(a, b) = \begin{cases} 16a^5 - 680a^7 + 2480a^9 & \text{if } 2 \mid b, \\ 16a^5 - 1360a^7 + 12400a^9 - 11056a^{11} & \text{if } 2 \nmid b, \end{cases}
\]

\[
e_6(a, b) = \begin{cases} 816a^7 - 19840a^9 + 66336a^{11} & \text{if } 2 \mid b, \\ 1088a^7 - 49600a^9 + 398016a^{11} - 349504a^{13} & \text{if } 2 \nmid b, \end{cases}
\]

\[
e_7(a, b) = \begin{cases} -272a^7 + 41664a^9 - 773920a^{11} + 2446528a^{13} & \text{if } 2 \mid b, \\ -272a^7 + 6940a^9 - 2321760a^{11} + 1712596a^{13} - 14873104a^{15} & \text{if } 2 \nmid b. \end{cases}
\]

**Lemma 2.1.** (See [4, Theorem 2.1]) Let \(n\) be a nonnegative integer and \(a \neq 0\). Then

\[
E_n^{(a)} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (1 - a)^{n-2k} a^{2k} E_{2k}. 
\]

**Lemma 2.2.** (See [4, Theorem 3.1]) Let \(a\) be a nonzero integer, \(n \in \mathbb{N}\) and let \(b\) be a nonnegative integer. Suppose that \(\alpha_n\) is a nonnegative integer given by \(2^{\alpha_n - 1} < n < 2^{\alpha_n} - 1\).

(i) We have

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E_{2k}^{(a)} \equiv \begin{cases} 0 \pmod{2^{(n+1)\ord_2 a - \ord_2 n + 2n}} & \text{if } 2 \mid n, \\ 0 \pmod{2^{\ord_2 a + 2n - \ord_2 n}} & \text{if } 2 \nmid n. \end{cases}
\]

and

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E_{2k+1}^{(a)} \equiv \begin{cases} 0 \pmod{2^{(n+1)\ord_2 a + 2n}} & \text{if } 2 \mid n, \\ 0 \pmod{2^{\ord_2 a + 2n - \ord_2 (n+1)}} & \text{if } 2 \nmid n. \end{cases}
\]

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(ii) We have
\[ \sum_{k=0}^{n} \binom{n}{k} (-1)^k E_{2k+b}^{(a)} \equiv 0 \pmod{2^{(2+\text{ord}_2 a)n-\alpha_a}}. \]

Lemma 2.3. (See [4, (3.4)]) Let \( n \in \mathbb{N}, a \in \mathbb{Z}, a \neq 0, b \in \{0, 1, 2, \ldots\} \). Then
\[ \sum_{r=0}^{n} \binom{n}{r} (-1)^r E_{2r+b}^{(a)} = \frac{1}{2} \left( \left[ \frac{1}{2} \right] \right) (-1)^r \sum_{s=0}^{r+n} \binom{r+n}{s} (-1)^s E_{2s+b-2\left[\frac{s}{2}\right]}^{(a)}. \]

Lemma 2.4. Let \( a \) be a nonzero integer, \( k, m \in \mathbb{N} \) and \( b \in \{0, 1, 2, \ldots\} \). Then
\[ E_{2^m k+b}^{(a)} - E_b^{(a)} = 2^{m-1} k \sum_{r=1}^{8} \binom{2^{m-1} k - 1}{r - 1} \frac{(-2)^r}{r} A_r(a, b) \pmod{2^{m+13+9\text{ord}_2 a}}, \]
where
\[ A_r(a, b) = 2^{-r} \sum_{i=0}^{r} \binom{r}{i} (-1)^i E_{2i+b}^{(a)}. \]

Proof. Let \( e_s(a, b) \) and \( A_s(a, b) \) be given by (2.1) and (2.3). Then \( E_b^{(a)} = A_0(a, b) \). From the binomial inversion formula, we get
\[ \sum_{r=0}^{s} \binom{s}{r} \cdot (-1)^r \cdot A_r(a, b) \cdot 2^r = E_{2s+b}^{(a)}. \]
Hence
\[ E_{2^m k+b}^{(a)} - E_b^{(a)} = \sum_{r=0}^{2^{m-1} k} \binom{2^{m-1} k}{r} \cdot (-1)^r \cdot 2^r \cdot A_r(a, b) - A_0(a, b) \]
\[ = \sum_{r=1}^{2^{m-1} k} \binom{2^{m-1} k}{r} \cdot (-1)^r \cdot 2^r \cdot A_r(a, b) \]
\[ = 2^{m-1} k \sum_{r=1}^{2^{m-1} k} \binom{2^{m-1} k - 1}{r - 1} \cdot (-1)^r \cdot \frac{2^r}{r} A_r(a, b). \]

From Lemma 2.2(ii) we have
\[ 2^{(2+\text{ord}_2 a)r-\alpha_r} \mid \sum_{i=0}^{r} \binom{r}{i} (-1)^i E_{2i+b}^{(a)}. \]
Thus \( 2^{(1+\text{ord}_2 a)r-\alpha_r} \mid A_r(a, b) \).

For \( r \geq 9 \) we have \( \text{ord}_2 A_r(a, b) \geq 9(1 + \text{ord}_2 a) - \alpha_9 = 9(1 + \text{ord}_2 a) - 4 = 5 + 9\text{ord}_2 a \), and \( 2^r \equiv 0 \pmod{2^9} \). So
\[ E_{2^m k+b}^{(a)} - E_b^{(a)} = 2^{m-1} k \sum_{r=1}^{2^{m-1} k} \binom{2^{m-1} k - 1}{r - 1} \cdot (-1)^r \cdot \frac{2^r}{r} A_r(a, b) \]
\[ \equiv 2^{m-1} k \sum_{r=1}^{8} \binom{2^{m-1} k - 1}{r - 1} \cdot (-1)^r \cdot \frac{2^r}{r} A_r(a, b) \pmod{2^{m+13+9\text{ord}_2 a}}. \]
This yields the result. \( \square \)
Theorem 2.1. Let $a$ be a nonzero integer, $k, m \in \mathbb{N}$, $m \geq 4$ and $b \in \{0, 1, 2, \ldots\}$. If $2 \mid a$, then
\[
E_{2m,k+b}^{(a)} - E_{b}^{(a)} \equiv \begin{cases} 
2^m k \{9a^3 b^2 - (2a^3 - 128a)b + 86a^3 - 257a \\
+ 2^m ka^3 (b - 1) \pmod{2^{m+10}} & \text{if } 2 \mid b, \\
2^m k \{(a^3 - 2a^5)b^2 + (30a^3 + 128a)b - 2a^5 + 86a^3 + 127a \\
+ 2^m ka^3 (b - 1) \pmod{2^{m+10}} & \text{if } 2 \nmid b.
\end{cases}
\]

Proof. From Lemma 2.2(i) we have
\[
\sum_{s=0}^{r+8} \binom{r+8}{s} (-1)^s E_{2s+b-2[\frac{r}{2}]}^{(a)} \equiv 0 \pmod{2^{14+9\text{ord}_2 a}} \quad \text{for } r \in \{0, 1, 2, \ldots\}. \]

Thus,
\[
A_8(a,b) = \frac{1}{2^8} \sum_{r=0}^{r+8} \binom{\left[\frac{r}{2}\right]}{r} (-1)^r \sum_{s=0}^{r+8} \binom{r+8}{s} (-1)^s E_{2s+b-2[\frac{r}{2}]}^{(a)} \equiv 0 \pmod{2^{6+9\text{ord}_2 a}}.
\]

From the above and Lemma 2.3 we see that for $n \leq 7$,
\[
A_n(a,b) = \frac{1}{2^n} \sum_{r=0}^{r+n} \binom{\left[\frac{r}{2}\right]}{r} (-1)^r \sum_{s=0}^{r+n} \binom{r+n}{s} (-1)^s E_{2s+b-2[\frac{r}{2}]}^{(a)}
\equiv \frac{1}{2^n} \sum_{r=0}^{7-n} \binom{\left[\frac{r}{2}\right]}{r} (-1)^r \sum_{s=0}^{r+n} \binom{r+n}{s} (-1)^s E_{2s+b-2[\frac{r}{2}]}^{(a)} \pmod{2^{14-n+9\text{ord}_2 a}}.
\]

That is,
\[
A_n(a,b) \equiv \sum_{r=0}^{7-n} \binom{\left[\frac{r}{2}\right]}{r} (-2)^r c_{r+n}(a,b) \pmod{2^{14-n+9\text{ord}_2 a}}. \quad (2.4)
\]

Hence, using (2.2) we see that
\[
A_7(a,b) \equiv e_7(a,b) \equiv 0 \pmod{2^4},
\]
\[
A_6(a,b) \equiv e_6(a,b) - 2\left[\frac{b}{2}\right] e_7(a,b) \equiv \begin{cases} 
0 & \text{if } 2 \mid a, \\
16(3a + b + 2) & \text{if } 2 \mid a \text{ and } 2 \nmid b, \\
0 & \text{if } 2 \nmid ab.
\end{cases} \equiv 0 \pmod{2^6}
\]

Similarly,
\[
A_5(a,b) \equiv e_5(a,b) - 2\left[\frac{b}{2}\right] e_6(a,b) + 4\left[\frac{b}{2}\right] e_7(a,b) \equiv \begin{cases} 
0 & \text{if } 2 \mid a, \\
24a - 8b^2 & \text{if } 2 \nmid a \text{ and } 2 \mid b, \\
0 & \text{if } 2 \nmid ab,
\end{cases} \equiv 0 \pmod{2^6}
\]
\[
A_4(a,b) \equiv e_4(a,b) - 2\left[\frac{b}{2}\right] e_5(a,b) + 4\left[\frac{b}{2}\right] e_6(a,b) - 8\left[\frac{b}{2}\right] e_7(a,b) \equiv 0 \pmod{2^9} \quad \text{for even } a,
\]
\[
A_3(a,b) \equiv e_3(a,b) - 2\left[\frac{b}{2}\right] e_4(a,b) + 4\left[\frac{b}{2}\right] e_5(a,b) - 8\left[\frac{b}{2}\right] e_6(a,b) + 16\left[\frac{b}{4}\right] e_7(a,b) \equiv -2a^3 + 4a^5 \pmod{2^8} \quad \text{for even } a
\]
and
\[
A_2(a, b) \equiv e_2(a, b) - 2\left(\frac{b}{2}\right)e_3(a, b) + 4\left(\frac{b}{7}\right)e_4(a, b) - 8\left(\frac{b}{3}\right)e_5(a, b) + 16\left(\frac{b}{4}\right)e_6(a, b) - 32\left(\frac{b}{5}\right)e_7(a, b)
\]
\[
\equiv \begin{cases} 
2a^3 - \frac{4}{3}a^5b + 2a^3b \pmod{2^{10}} & \text{if } 2 \mid a \text{ and } 2 \nmid b, \\
2a^3 - 4a^5b + 2a^3b - 6a^7 + 6a^7b \pmod{2^{10}} & \text{if } 2 \mid a \text{ and } 2 \nmid b.
\end{cases}
\]

When \(2 \mid a\) and \(2 \nmid b\), we have
\[
- 2A_1(a, b) \equiv \frac{8}{3}a^5b - \frac{4}{3}a^5b^2 + 2a^3b^2 - 2a \equiv 24a^5b - 12a^5b^2 + 2a^3b^2 - 2a \pmod{2^{11}},
\]
\[
\frac{2^2}{2} \cdot A_2(a, b) = 2A_2(a, b) \equiv (4a^3 - 24a^5b + 4a^3b) \pmod{2^{11}},
\]
\[
- \frac{2^3}{3} A_3(a, b) \equiv -\frac{8}{3}(-2a^3 + 4a^5) \equiv 176a^3 - 32a^5 \pmod{2^{11}}.
\]

Now combining Lemma 2.4 with the above we obtain
\[
E_{2^m k+b}^{(a)} - E_b^{(a)}
\]
\[
\equiv 2^{m-1}k \sum_{r=1}^{8} \left(\frac{2^{m-1}k - 1}{r - 1}\right) \cdot (-1)^r \cdot \frac{2^r}{r} \cdot A_r(a, b).
\]
\[
\equiv 2^{m-1}k\{(24a^5b - 12a^5b^2 + 2a^3b^2 - 2a) + (2^{m-1}k - 1)(4a^3 - 24a^5b + 4a^3b) + \left(\frac{2^{m-1}k - 1}{2}\right)(176a^3 - 32a^5)\}
\]
\[
\equiv 2^{m-1}k\{48a^5b - 12a^5b^2 + 2a^3b^2 - 2a - 4a^3 - 4a^3b + 176a^3 - 32a^5
\]
\[
+ 2^{m+1}k(b-1)a^3\} \pmod{2^{m+10}}.
\]

To see the result, we note that
\[
32a^5 = 2^{10}\left(\frac{a}{2}\right)^5 \equiv 2^{10}\left(\frac{a}{2}\right) = 512a \pmod{2^{11}}, \quad 48a^5b = 3 \cdot 2^{10}\left(\frac{a}{2}\right)^5b \equiv 256ab \pmod{2^{11}},
\]
\[
-12a^5b^2 = -3 \cdot 2^9\left(\frac{a}{2}\right)^5\left(\frac{b}{2}\right)^2 \equiv 16a^3b^2 \pmod{2^{11}}.
\]

When \(2 \mid a\) and \(2 \nmid b\), we have
\[
- 2A_1(a, b) \equiv 8a^5b - 4a^5 - 4a^5b^2 + 2a^3b^2 - 2a \pmod{2^{11}},
\]
\[
\frac{2^2}{2} A_2(a, b) = 2A_2(a, b) \equiv 4a^3(b+1) - 8a^5b + 12a^7(b-1) \pmod{2^{11}},
\]
\[
- \frac{2^3}{3} A_3(a, b) \equiv -\frac{8}{3}(-2a^3 + 4a^5) \equiv 176a^3 - 32a^5 \pmod{2^{11}}.
\]

Combining Lemma 2.4 with the above we get
\[
E_{2^m k+b}^{(a)} - E_b^{(a)}
\]
This completes the proof.

To see the result, we note that

\[16a^5b = 2^9\left(\frac{a}{2}\right)^5b \equiv 2^9\left(\frac{a}{2}\right)^3b = 64a^3b \pmod{2^{11}},\]
\[-12a^7(b - 1) = -3 \cdot 2^{10}\left(\frac{a}{2}\right)^7\frac{b - 1}{2} \equiv 256ab - 256a \pmod{2^{11}}.\]

This completes the proof.

**Theorem 2.2.** Let \(a\) be an odd integer, \(k, m \in \mathbb{N}, m \geq 5\) and \(b \in \{0, 2, 4, \ldots\}\). Then

\[
E_{2^m k + b}^{(a)} - E_b^{(a)} \equiv 2^m k(7a b^6 - 6 a b^5 + (3 a^3 - 14 a) b^4 + (4 a^3 + 56 a) b^3
\]
\[\quad - (6 a^4 + 35 a^3 - 12 a^2 + 106 a - 122) b^2
\]
\[\quad + (38 a^3 - 8 a - 256) b + (70 a^3 + 64 a^2 - 81 a + 448)
\]
\[\quad + 2^m k(b^4 + 2b^3 + 2ab^2 + (a^3 + 2a)b + 16 - a^3) \pmod{2^{m+10}}.
\]

**Proof.** By (2.4) we have

\[A_1(a, b) \equiv \sum_{r=0}^{6} \binom{\left\lceil \frac{b}{2} \right\rceil}{r} (-2)^r e_{r+1}(a, b)
\]
\[\equiv a - 2 a^3 b + a^3 b(b - 2)(6a^2 - 1) - \frac{4}{3} b(b - 2)(b - 4)(19a^3 - 6a)
\]
\[\quad + b(b - 2)(b - 4)(b - 6)(15a^5 + a^3 + 17a) + b(b - 2)(b - 4)(b - 6)(b - 8)(28a - 16)
\]
\[\quad - b(b - 2)(b - 4)(b - 6)(b - 8)(b - 10)(-5a^3 + 14a) \pmod{2^{10}},
\]

\[A_2(a, b) \equiv \sum_{r=0}^{5} \binom{\left\lceil \frac{b}{2} \right\rceil}{r} (-2)^r e_{r+2}(a, b)
\]
\[\equiv 2a^3 - 2a^3 b(6a^2 - 1) + 4 a^5 b(b - 2)(17a^2 - 4) - b(b - 2)(b - 4)(-40a - 84a^3)
\]
\[\quad + b(b - 2)(b - 4)(b - 6)(50a^3 - 20a)
\]
\[\quad - b(b - 2)(b - 4)(b - 6)(b - 8)(-30a^3 - 12a) \pmod{2^{10}}.
\]

\[A_3(a, b) \equiv \sum_{r=0}^{4} \binom{\left\lceil \frac{b}{2} \right\rceil}{r} (-2)^r e_{r+3}(a, b)
\]
\[\equiv 2a^3(6a^2 - 1) - 8ab(17a^2 - 4) + b(b - 2)(32a - 20a^3) - b(b - 2)(b - 4)(8a + 16)
\]
\[\quad + b(b - 2)(b - 4)(b - 6)(-2a) \pmod{2^{8}}
\]

and

\[A_4(a, b) \equiv \sum_{r=0}^{3} \binom{\left\lceil \frac{b}{2} \right\rceil}{r} (-2)^r e_{r+4}(a, b)
\]
\[ \equiv 8(13a^5 + 17a^3 - 17a) - b(16a + 8a^3) + b(b - 2)(24a^3 + 16a - 64) + 8ab(b - 2)(b - 4) \pmod{2^6}. \]

By the proof of Theorem 2.1,

\[ A_5(a, b) \equiv 24a - 8b^2 \pmod{2^6}, \]
\[ A_6(a, b) \equiv 16(3a + b + 2) \pmod{2^6}, \quad A_7(a, b) \equiv 0 \pmod{2^4}. \]

Since \( m \geq 5 \) we have

\[ 2(2^{m-1}k - 1)A_2(a, b) \equiv -2(2a^3 - 2a^3 b(6a^2 - 1) + 4a^5 b(b - 2)(17a^2 - 4) - b(b - 2)(b - 4)(-40a - 84a^3) + b(b - 2)(b - 4)(b - 6)(50a^3 - 20a) - b(b - 2)(b - 4)(b - 6)(-30a^3 - 12a)) \]
\[ + 2^m k(2a^3 - 2b(6a - a^3) + 52ab(b - 2) - 4b(b - 2)(b - 4) + 2b(b - 2)(b - 4)(b - 6)) \pmod{2^{11}}, \]
\[ \left( \frac{2^{m-1}k - 1}{2} \right) \cdot (-1)^3 \cdot \frac{2^3}{3} \cdot A_3(a, b) \]
\[ = \frac{-4}{3}(2^{m-1}k - 1)(2^{m-1}k - 2)A_3(a, b) \]
\[ \equiv \left( \frac{2}{3}(a^3 - 2a^3 b(6a^2 - 1) - 8ab(17a^2 - 4) + b(b - 2)(32a - 20a^3) - b(b - 2)(b - 4)(8a + 16) + b(b - 2)(b - 4)(b - 6)(-2a)) + 2^{m+1} k(12a - 2a^3 - 4b^2) \pmod{2^{11}}, \right. \]
\[ \left. \left( \frac{2^{m-1}k - 1}{3} \right) \cdot (-1)^4 \cdot \frac{2^4}{4} \cdot A_4(a, b) \right. \]
\[ = \frac{2}{3}(2^{m-1}k - 1)(2^{m-1}k - 2)(2^{m-1}k - 3)A_4(a, b) \equiv \frac{11}{3}2^m k A_4(a, b) - 4A_4(a, b) \]
\[ \equiv -4(104a^5 + 136a^3 - 136a - b(16a + 8a^3) + b(b - 2)(24a^3 + 16a - 64) + 8ab(b - 2)(b - 4)) \]
\[ + \frac{11}{3}2^m k(40a + 8b^2 - 8ab) \pmod{2^{11}}, \]
\[ \left( \frac{2^{m-1}k - 1}{4} \right) \cdot (-1)^5 \cdot \frac{2^5}{5} \cdot A_5(a, b) \]
\[ = \frac{4}{15}(2^{m-1}k - 1)(2^{m-1}k - 2)(2^{m-1}k - 3)(2^{m-1}k - 4)A_5(a, b) \]
\[ \equiv \frac{5}{3}2^{m+2}k A_5(a, b) - \frac{32}{5}A_5(a, b) \equiv \frac{5}{3}2^{m+5}k - \frac{32}{5}(24a - 8b^2) \pmod{2^{11}} \]

and

\[ \left( \frac{2^{m-1}k - 1}{5} \right) \cdot (-1)^6 \cdot \frac{2^6}{6} \cdot A_6(a, b) \]
\[ = \frac{4}{45}(2^{m-1}k - 1)A_6(a, b) \equiv \frac{137}{45}2^{m+2}k A_6(a, b) - \frac{32}{3}A_6(a, b) \]
\[ \equiv -\frac{32}{3}(48a + 16b + 32) \equiv -512(a + b + 2) \pmod{2^{11}}. \]

Now combining the above congruences with Lemma 2.4 we deduce that

\[ E_{2^m k + b}^{(a)} - E_b^{(a)} \]
\[
\equiv 2^{m-1}k\{-1024 + 286a + 400a^5b - 516a^3b - 96ab - 248ab^2 + 378a^3b^2 + 24ab^3 \\
- 8a^2b^3 + 6ab^4 + 40ab^5 + 10a^3b^6 + 4ab^6 - 136a^7b^2 - 276a^5b^2 + 272a^7b \\
+ 140a^3 - 448a^5 + 256b^2 - 32b^5 - 30a^5b^4 + 12a^3b^5 + 2a^3b^4 + 104a^5b^3 \\
+ 2^{m+1}k(16 + 2ab - a^3 + 2b^3 + b^4 + a^3b + 2ab^2)\} \pmod{2^{m+10}}.
\]

By simplifying the above congruence we obtain the result. \hfill \Box

**Corollary 2.1.** Let \( k, m \in \mathbb{N}, m \geq 5 \) and \( b \in \{0, 2, 4, \ldots\} \). Then
\[
E_{2^m k + b} - E_b \equiv 2^m k\{7b^6 - 6b^5 - 11b^4 + 60b^3 - 13b^2 - 226b + 501 \\
+ 2^m k(b^4 + 2b^3 + 2b^2 + 3b + 15)\} \pmod{2^{m+10}}
\]

**Proof.** Putting \( a = 1 \) in Theorem 2.2 we deduce the result. \hfill \Box

**Theorem 2.3.** Let \( a \) be an odd integer, \( k, m \in \mathbb{N}, m \geq 5 \) and \( b \in \{1, 3, 5, \ldots\} \). Then
\[
E_{2^m k + b}^{(a)} - E_b^{(a)} \equiv 2^m k\{(17a^7 + 162a^5 + 153a^3 + 64a^2 + 180a + 192)b^2 \\
- (102a^7 + 216a^5 + 386a^3 - 704a)b - (211a^7 - 10a^5 - 32a^4 + 66a^3 - 267a - 224) \\
+ 2^{m+1}(a^2 - 1)\} \pmod{2^{m+10}}.
\]

**Proof.** By (2.4) we have
\[
A_7(a, b) \equiv 0 \pmod{2^4}, \quad A_6(a, b) \equiv 0 \pmod{2^6}, \quad A_5(a, b) \equiv 0 \pmod{2^6}, \\
A_4(a, b) \equiv 0 \pmod{2^9}, \quad A_3(a, b) \equiv -34a^7 + 36a^5 - 2a^3 \pmod{2^9}.
\]

Also,
\[
A_2(a, b) \equiv \sum_{r=0}^{5} \binom{b^r}{r}(-2)^r e_{r+2}(a, b) \\
= e_2(a, b) - 2a^2 e_3(a, b) + 4\left(\binom{b^3}{2}\right)e_4(a, b) - 8\left(\binom{b^4}{3}\right)e_5(a, b) + 16\left(\binom{b^5}{4}\right)e_6(a, b) \\
- 32\left(\binom{b^6}{5}\right)e_7(a, b)
\]
\[
\equiv -2a^3(b-1)(-17a^4 + 18a^2 - 1) + (b-1)(b-3)(-24a^5 + 48a^3 - 24a) \\
+ 4a^3(1 - a^2) \pmod{2^{10}}
\]

and
\[
A_1(a, b) \equiv \sum_{r=0}^{6} \binom{b^r}{r}(-2)^r e_{r+1}(a, b) \\
= e_1(a, b) - 2a b e_2(a, b) + 4\left(\binom{b^2}{2}\right)e_3(a, b) - 8\left(\binom{b^3}{3}\right)e_4(a, b) + 16\left(\binom{b^4}{4}\right)e_5(a, b) \\
- 32\left(\binom{b^5}{5}\right)e_6(a, b) + 64\left(\binom{b^6}{6}\right)e_7(a, b)
\]
Thus,
\[
\binom{2^{m-1}k - 1}{1} \cdot 2 \cdot A_2(a, b) = 2^m k A_2(a, b) - 2 A_2(a, b)
\]
\[
\equiv -2(4a^3(1 - a^2) - 2a^3(b - 1)(-17a^4 + 18a^2 - 1) + (b - 1)(b - 3)(-24a^5 + 48a^3 - 24a)) + 2^{m+2}k a(1 - a^2) \pmod{2^9}
\]
and
\[
\binom{2^{m-1}k - 1}{2} \cdot (-1)^3 \cdot \frac{2^3}{3} \cdot A_3(a, b)
\]
\[
= -\frac{4}{3}(2^{m-1}k - 1)(2^{m-1}k - 2) A_3(a, b) \equiv 2^{m+1}k A_3(a, b) - \frac{8}{3} A_3(a, b)
\]
\[
\equiv -\frac{8}{3}(-34a^7 + 36a^5 - 2a^3) \pmod{2^{11}}.
\]

Now combining the above congruences with Lemma 2.4, we deduce that
\[
F_{r(2^{m+k})}^a - E_{r(2^m)}^a \equiv 2^{m-1}k(-382a - 44ab^4(a^4 + 1) + 640a(a^2b^3 - a^4 - b) + 704ab^2(a^4 + 1)
+ 360ab^2 - 604a^3 - 204a^7b - 432a^5b - 772a^3b + 324a^5b^2 + 1330a^3b^2
+ 34a^7b^2 + 88a^3b^4 - 422a^7 + 2^{m+2}ka(1 - a^2)) \pmod{2^{m+10}}.
\]

To see the result, we note that \(a^4 \equiv b^4 \equiv 1 \pmod{16},
\]
\[
5a(a^2b^3 - a^4 - b) \equiv 5a(a^2b^3 - 1 - b) = 5ab(a^2b^2 - 1) - 5a \equiv a^2b^2 - 1 - 5a \pmod{16}
\]
and
\[
ab^3(a^4 + 1) = ab^3(a^4 - 1 + 2) \equiv a^4 - 1 + 2ab^3 \equiv a^4 + 2ab + 2b^2 - 3 \pmod{32}.
\]

\[
\square
\]

3 A congruence for \(U_{k, \varphi(5^m)+b} \pmod{5^{m+5}}\)

For \(n \in \mathbb{N}\) and \(i \in \{0, 1, \ldots, n\}\) let \(s(n, i)\) be the Stirling number of the first kind given by
\[
x(x - 1) \cdots (x - n + 1) = \sum_{i=0}^{n} (-1)^{n-i} s(n, i) x^i.
\]

**Lemma 3.1.** ([2, p.197]) Let \(p\) be an odd prime, \(n \in \mathbb{N}\) and let \(f(k)\) be a \(p\)-regular function. Let \(A_m = p^{-m} \sum_{r=0}^{m} \binom{m}{r} (-1)^r f(r), a_0 = A_0\) and
\[
a_i = (-1)^i \sum_{r=1}^{n-1} s(r, i) \frac{p^r}{r!} A_r \quad \text{for} \quad i = 1, 2, 3, \ldots.
\]
Then
\[
f(k) \equiv \sum_{i=0}^{n-1} a_i k^i \pmod{p^n}.
\]
Lemma 3.2. Let $m \in \{5, 6, 7, \ldots \}$, $p \in \{3, 5\}$ and $s \in \{1, 2, 3, 4, 5, 6\}$, and let $f(k)$ be a $p$-regular function. Then

$$f(p^{m-1}k) \equiv f(0) - p^{m-1}k \sum_{s=1}^{6} \frac{1}{s} \sum_{r=0}^{s} \binom{s}{r} (-1)^{s-r} f(r) \pmod{p^{m+5}}.$$ 

Proof. As $m \geq 5$, we have $m \geq 7 - s + \text{ord}_p s!$ and so $m - 1 - \text{ord}_p s! \geq 6 - s$ for $s \geq 2$. Thus, for $s \geq 2$ we have $\frac{p^{m-1}}{s!} \equiv 0 \pmod{p^{6-s}}$ and so

$$p^s \binom{p^{m-1}k}{s} = \frac{p^{m-1+k} \binom{p^{m-1}k-1}{(p^{m-1}k-2) \cdots (p^{m-1}k-s+1)}}{s!} \equiv p^{m-1+s}k \frac{(-1)(-2) \cdots (-s+1)}{s!} = (-1)^s p^{m-1+s}k \pmod{p^{m+5}}.$$ 

Therefore, for $s = 1, 2, 3, 4, 5, 6$ we have

$$\binom{p^{m-1}k}{s} \sum_{r=0}^{s} \binom{s}{r} (-1)^{s-r} f(r) = -p^{m-1+k} \frac{1}{s} \sum_{r=0}^{s} \binom{s}{r} (-1)^r f(r) = -p^{m-1+k} \frac{1}{s} \sum_{r=0}^{s} \binom{s}{r} (-1)^r f(r) \pmod{p^{m+5}}.$$ 

Hence,

$$\sum_{s=1}^{6} \binom{p^{m-1}k}{s} \sum_{r=0}^{s} \binom{s}{r} (-1)^{s-r} f(r) \equiv -p^{m-1+k} \sum_{s=1}^{6} \frac{1}{s} \sum_{r=0}^{s} \binom{s}{r} (-1)^r f(r) \pmod{p^{m+5}}. \quad (3.1)$$

As $p^s/7 \in \mathbb{Z}_p$ for $s \geq 7$, we see that

$$\frac{1}{s} \sum_{r=0}^{s} \binom{s}{r} (-1)^{s-r} f(s) \equiv 0 \pmod{p^7} \quad \text{for} \quad s \geq 7.$$ 

Using the binomial inversion formula, we see that

$$f(p^{m-1}k) = \sum_{s=0}^{p^{m-1}k} \binom{p^{m-1}k}{s} (-1)^s \sum_{r=0}^{s} \binom{s}{r} (-1)^r f(r) = f(0) + p^{m-1+k} \sum_{s=1}^{p^{m-1}k} \binom{p^{m-1}k-1}{s-1} \frac{1}{s} \sum_{r=0}^{s} \binom{s}{r} (-1)^{s-r} f(r) \equiv f(0) + \sum_{s=1}^{6} \binom{p^{m-1}k}{s} \sum_{r=0}^{s} \binom{s}{r} (-1)^{s-r} f(r) \pmod{p^{m+6}}.$$ 

This together with (3.1) yields the result. \hfill \Box

Lemma 3.3. Let $k$ be a nonnegative integer. Then

$$(1 + 5^k) U_{4k} \equiv 6250k^6 + 50625k^5 + 51250k^4 + 59875k^3 + 72600k^2 + 7545k + 2 \pmod{5^7},$$

$$(1 + 5^k) U_{4k+2} \equiv 59375k^6 + 40625k^5 + 10625k^4 + 48875k^3 + 5575k^2 + 3750k - 52 \pmod{5^7}.$$
Proof. Set \( f(k) = (1+5^{4k+b})U_{4k+b} \). From [5, Theorem 2.1] we know that \( f(k) \) is a 5–regular function. Thus \( A_m = 5^{-m} \sum_{r=0}^{m} \binom{m}{r} (-1)^r f(r) \in \mathbb{Z}_5 \) for \( m \in \{0, 1, 2, \ldots \} \). It is easy to check that for \( b = 0 \),

\[
A_0 = 2, \quad A_1 = 75371 \pmod{5^7}, \quad A_2 = 31378 \pmod{5^7}, \quad A_3 = 73991 \pmod{5^7},
\]
and that for \( b = 2 \),

\[
A_0 = 78073, \quad A_1 = 6360 \pmod{5^7}, \quad A_2 = 26626 \pmod{5^7}, \quad A_3 = 22469 \pmod{5^7}, \quad A_4 = 55958 \pmod{5^7}, \quad A_5 = 28490 \pmod{5^7}, \quad A_6 = 28961 \pmod{5^7}.
\]

Now applying Lemma 3.1 we deduce the result. \( \square \)

**Theorem 3.1.** Let \( k, m \in \mathbb{N} \), \( m \geq 5 \) and let \( b \in \{0, 2, 4, \ldots \} \). Then

\[
U_{k\varphi(5^m)+b} - (1 + 5^b)U_b \equiv \begin{cases} 
5^{m-1}k(7545 + 5050b - 5375b^2 + 1250b^3 + 3125b^4 + 9375b^5) \pmod{5^{m+5}} & \text{if } b \equiv 0 \pmod{4}, \\
5^{m-1}k(-1575 + 3535b + 2250b^2 + 7500b^3 + 3125b^5) \pmod{5^{m+5}} & \text{if } b \equiv 2 \pmod{4}.
\end{cases}
\]

Proof. From [5, Theorem 4.2] we know that \( f(k) = (1-(\frac{1}{5})5^{4k+b})U_{4k+b} = (1+5^{4k+b})U_{4k+b} \) is a 5-regular function. Let \( r \in \{0, 1, 2, 3, 4, 5, 6\} \). From Lemma 3.3 we see that for \( b \equiv 0 \pmod{4} \),

\[
f(r) = (1 + 5^{4r+b})U_{4r+b} \equiv 6250 \left( r + \frac{b}{4} \right)^6 + 50625 \left( r + \frac{b}{4} \right)^5 + 51250 \left( r + \frac{b}{4} \right)^4 + 59875 \left( r + \frac{b}{4} \right)^3 + 72600 \left( r + \frac{b}{4} \right)^2 + 7545 \left( r + \frac{b}{4} \right) + 2 \pmod{5^7},
\]
and that for \( b \equiv 2 \pmod{4} \),

\[
f(r) = (1 + 5^{4r+b})U_{4r+b} \equiv 59375 \left( r + \frac{b-2}{4} \right)^6 + 40625 \left( r + \frac{b-2}{4} \right)^5 + 10625 \left( r + \frac{b-2}{4} \right)^4 + 48875 \left( r + \frac{b-2}{4} \right)^3 + 5575 \left( r + \frac{b-2}{4} \right)^2 + 37500 \left( r + \frac{b-2}{4} \right) - 52 \pmod{5^7}.
\]

Now combining the above with Lemma 3.2 gives the result. \( \square \)

4 A congruence for \( E_{k\varphi(5^m)+b} \pmod{5^{m+5}} \)

**Lemma 4.1.** Let \( k \) be a nonnegative integer. Then

\[
(1 - 5^k)E_{4k} \equiv 31250k^6 + 11875k^5 + 1875k^4 + 64875k^3 + 54500k^2 + 50005k \pmod{5^7},
\]

\[
(1 - 5^{k+2})E_{4k+2} \equiv 31250k^6 + 10625k^5 + 68750k^4 + 60375k^3 + 4625k^2 + 74290k + 24 \pmod{5^7}.
\]
By Lemma 3.2, Proof. From [3, Lemma 7.1] we know that \( f(k) = (1 - 5^{4k+b})E_{4k+b} \) is a 5-regular function. Now one can prove the result by using Lemma 3.1 and doing some calculations. \( \square \)

**Theorem 4.1.** Let \( k, m \in \mathbb{N}, m \geq 5 \) and \( b \in \{0, 2, 4, \ldots\} \). Then

\[
E_{k\varphi(5^m)+b} - (1 - 5^{b})E_b \\
\equiv \begin{cases} 
5^{m-1}k(3130 - 4000b + 3375b^2 + 3125b^3 - 3125b^4) \pmod{5^{m+5}} & \text{if } b \equiv 0 \pmod{4}, \\
5^{m-1}k(4790 + 1750b^2 + 3125b^3 + 6250b^4) \pmod{5^{m+5}} & \text{if } b \equiv 2 \pmod{4}.
\end{cases}
\]

Proof. From [3, Lemma 7.1] we know that \( f(k) = (1 - 5^{4k+b})E_{4k+b} \) is a 5-regular function. By Lemma 3.2,

\[
E_{k\varphi(5^m)+b} - (1 - 5^{b})E_b \equiv -5^{m-1}k \sum_{s=1}^{6} \frac{1}{s} \sum_{r=0}^{s} \binom{s}{r}(-1)^{s-r}f(r) \pmod{5^{m+5}}. \tag{4.1}
\]

Let \( r \in \{0, 1, 2, 3, 4, 5, 6\} \). From Lemma 4.1 we see that for \( b \equiv 0 \pmod{4} \),

\[
f(r) \equiv 50005\left(r + \frac{b}{4}\right) + 54500\left(r + \frac{b}{4}\right)^2 + 64875\left(r + \frac{b}{4}\right)^3 + 18750\left(r + \frac{b}{4}\right)^4 + 11875\left(r + \frac{b}{4}\right)^5 + 31250\left(r + \frac{b}{4}\right)^6 \pmod{5^7}.
\]

and that for \( b \equiv 2 \pmod{4} \),

\[
f(r) \equiv 24 + 74290\left(r + \frac{b-2}{4}\right) + 4625\left(r + \frac{b-2}{4}\right)^2 + 60375\left(r + \frac{b-2}{4}\right)^3 + 68750\left(r + \frac{b-2}{4}\right)^4 + 10625\left(r + \frac{b-2}{4}\right)^5 + 31250\left(r + \frac{b-2}{4}\right)^6 \pmod{5^7}.
\]

Now combining the above with (4.1) gives the result. \( \square \)

## 5 Congruences for \( S_{k\varphi(3^m)+b} \pmod{3^{m+5}} \)

**Lemma 5.1.** Let \( k \) be a nonnegative integer. Then

\[
(1 - 3^{2k})S_{2k} \equiv 1620k^6 + 1620k^5 + 621k^4 + 981k^3 + 1179k^2 + 564k \pmod{3^7},
\]

\[
(1 + 3^{2k+1})S_{2k+1} \equiv 324k^6 + 2106k^5 + 1809k^4 + 1197k^3 + 549k^2 + 888k + 2183 \pmod{3^7}.
\]

Proof. From [4, Theorem 4.1] we know that \( f(k) = (1 - (-1)^b3^{2k+b})S_{2k+b} \) is a 3-regular function. Now one can prove the result by using Lemma 3.1 and doing some calculations. \( \square \)

**Theorem 5.1.** For \( k, m \in \mathbb{N}, m \geq 5 \) and \( b \in \{0, 1, 2, \ldots\} \) we have

\[
S_{k\varphi(3^m)+b} - (1 - (-3)^b)S_b \equiv \begin{cases} 
3^{m-2}k(-495 + 1350b + 567b^2 - 162b^3 + 972b^4 - 729b^5) \pmod{3^{m+5}} & \text{if } 2 \mid b, \\
3^{m-2}k(1422 + 1242b - 891b^2 - 81b^3 + 243b^4 + 729b^5) \pmod{3^{m+5}} & \text{if } 2 \nmid b.
\end{cases}
\]
Proof. From [4, Theorem 4.1(i)] we know that \( f(k) = (1 - (-1)^b 3^{2k+b})S_{2k+b} \) is a 3-regular function. By Lemma 3.2,

\[
S_{k\varphi(3^m)+b} - (1 - (-3)^b)S_b \equiv -3^{m-1}k \sum_{s=1}^{6} \frac{1}{s} \sum_{r=0}^{s} \binom{s}{r} (-1)^{s-r} f(r) \pmod{3^{m+5}}. \tag{5.1}
\]

Let \( r \in \{0, 1, 2, 3, 4, 5, 6\} \). From Lemma 5.1 we see that for \( b \equiv 0 \pmod{2} \),

\[
f(r) \equiv 1620\left(r + \frac{b}{2}\right)^6 + 324\left(r + \frac{b}{2}\right)^4 + 549\left(r + \frac{b}{2}\right)^2 + 564\left(r + \frac{b}{2}\right) \pmod{3^7}
\]

and that for \( b \equiv 1 \pmod{2} \),

\[
f(r) \equiv 1620\left(r + \frac{b-1}{2}\right)^6 + 324\left(r + \frac{b-1}{2}\right)^4 + 549\left(r + \frac{b-1}{2}\right)^2 + 564\left(r + \frac{b-1}{2}\right) \pmod{3^7}
\]

Combining the above with (5.1) yields the result. \( \Box \)

6 Congruences for \( S_{k\varphi(5^m)+b} \pmod{5^{m+5}} \)

Lemma 6.1. Let \( k \) and \( b \) be nonnegative integers. Then

\[
(1 + 5^{4k+b})S_{4k+b} \equiv \begin{cases} 
33750k^5 - 6250k^4 - 14250k^3 + 21500k^2 + 930k + 2 \pmod{5^7}, & \text{if } b \equiv 0 \pmod{4}, \\
9375k^5 + 22500k^4 + 18750k^3 + 40000k^2 + 23525k^2 + 7370k - 6 \pmod{5^7}, & \text{if } b \equiv 1 \pmod{4}, \\
25000k^5 + 41875k^5 + 56250k^4 + 30875k^3 + 64650k^2 + 44290k - 78 \pmod{5^7}, & \text{if } b \equiv 2 \pmod{4}, \\
-3125k^5 + 40625k^5 - 9375k^4 + 67250k^3 - 8550k^2 + 14525k + 1386 \pmod{5^7}, & \text{if } b \equiv 3 \pmod{4}.
\end{cases}
\]

Proof. From [4, Theorem 4.1(i)] we know that \( f(k) = (1 + 5^{4k+b})S_{4k+b} \) is a 5-regular function. Now one can prove the result by using Lemma 3.1 and doing some calculations. \( \Box \)

Theorem 6.1. For \( b, k, m \in \mathbb{N} \) and \( m \geq 5 \) we have

\[
S_{k\varphi(5^m)+b} - (1 + 5^b)S_b \\
\equiv \begin{cases} 
5^{m-1}k(930 - 4875b - 4625b^2 - 6250b^3 - 3125b^4) \pmod{5^{m+5}}, & \text{if } 4 \mid b, \\
5^{m-1}k(4670 + 1450b + 1250b^2 + 6250b^3 + 3125b^4 + 6250b^5) \pmod{5^{m+5}}, & \text{if } 4 \mid b - 1, \\
5^{m-1}k(-4235 + 3700b - 3000b^2 + 6250b^3 - 9375b^4 + 6250b^5) \pmod{5^{m+5}}, & \text{if } 4 \mid b - 2, \\
5^{m-1}k(3725 - 1025b - 2625b^2 + 6250b^3 + 3125b^5) \pmod{5^{m+5}}, & \text{if } 4 \mid b - 3.
\end{cases}
\]
Proof. Set \( f(k) = (1 + 5^{4k+b})S_{4k+b} \). By Lemma 3.2,
\[
S_{k\varphi(5^m)+b} - (1 + 5^b)S_b \equiv -5^{m-1}k \sum_{s=1}^{6} \sum_{r=0}^{s} \binom{s}{r}(-1)^{s-r}f(r) \pmod{5^{m+5}}. 
\] (6.1)

Let \( r \in \{0, 1, 2, 3, 4, 5, 6\} \). From Lemma 6.1 we see that for \( b \equiv 0 \pmod{4} \),
\[
f(r) \equiv 33750\left(r + \frac{b}{4}\right)^5 - 6250\left(r + \frac{b}{4}\right)^4 - 14250\left(r + \frac{b}{4}\right)^3 + 21500\left(r + \frac{b}{4}\right)^2 \\
+ 930\left(r + \frac{b}{4}\right) + 2 \pmod{5^7};
\]
for \( b \equiv 1 \pmod{4} \),
\[
f(r) \equiv 9375\left(r + \frac{b-1}{4}\right)^6 + 22500\left(r + \frac{b-1}{4}\right)^5 + 18750\left(r + \frac{b-1}{4}\right)^4 + 40000\left(r + \frac{b-1}{4}\right)^3 \\
+ 23525\left(r + \frac{b-1}{4}\right)^2 + 7370\left(r + \frac{b-1}{4}\right) - 6 \pmod{5^7};
\]
for \( b \equiv 2 \pmod{4} \),
\[
f(r) \equiv 25000\left(r + \frac{b-2}{4}\right)^6 + 41875\left(r + \frac{b-2}{4}\right)^5 + 56250\left(r + \frac{b-2}{4}\right)^4 + 30875\left(r + \frac{b-2}{4}\right)^3 \\
+ 64650\left(r + \frac{b-2}{4}\right)^2 + 44290\left(r + \frac{b-2}{4}\right) - 78 \pmod{5^7};
\]
and for \( b \equiv 3 \pmod{4} \),
\[
f(r) \equiv -3125\left(r + \frac{b-3}{4}\right)^6 + 40625\left(r + \frac{b-3}{4}\right)^5 - 9375\left(r + \frac{b-3}{4}\right)^4 + 67250\left(r + \frac{b-3}{4}\right)^3 \\
- 8550\left(r + \frac{b-3}{4}\right)^2 + 14525\left(r + \frac{b-3}{4}\right) + 1386 \pmod{5^7}.
\]
Combining the above with (6.1) we obtain the result. \( \Box \)

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