Non-recurrent parameter rays of the Mandelbrot set*

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Abstract

In this paper, we prove that any parameter ray at a non-recurrent angle $\theta$ lands at a non-recurrent parameter $c$ with $\theta$ a characteristic angle of $f_c$; and conversely, every non-recurrent parameter $c$ is the landing point of one or two parameter rays at non-recurrent angles, and these angles are exactly the characteristic angles of $f_c$.

1 Introduction

The quadratic family $\{f_c : z \mapsto z^2 + c\}$ exhibits rich dynamics, when iterated. The Mandelbrot set

$$\mathcal{M} := \{c \in \mathbb{C} | f_c^n(c) \not\to \infty \text{ as } n \to \infty\}$$

organizes the space of quadratic polynomials up to conjugacy and has a beautiful structure. It has been a very active area of research in the past few decades. The importance of the Mandelbrot set is due to the fact that it is the simplest non-trivial parameter space of analytic families of iterated holomorphic maps, and because of its universality as explained in [DH1, Mc].

Much of the topological and combinatorial structures of the Mandelbrot set has been discovered by the work of Douady and Hubbard [DH2]. A fundamental result in [DH2] is to describe the landing behavior of the rational parameter rays. One can see also [Mil2, Sch, PR] for alternative approaches.

In this article, we study the landing property of irrational, precisely the non-recurrent, parameter rays.

Set $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and $\tau : \mathbb{T} \to \mathbb{T}$ the angle doubling map, i.e., $\tau(\theta) = 2\theta \mod \mathbb{Z}, \theta \in \mathbb{T}$. By abuse of notations, we identify $\mathbb{T}$ with the unit circle under the correspondence $t \mapsto e^{2\pi it}$.

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An angle $\theta \in \mathbb{T}$ is called non-recurrent if $\tau^n(\theta) \neq \tau^m(\theta)$ for any $0 \leq m < n$, and there exists $\delta > 0$ such that $|\theta - \tau^n(\theta)| > \delta$ for all $n \geq 1$. A quadratic polynomial $f_c$, or the parameter $c$, is called non-recurrent if $c \in \mathcal{M}$, all periodic points of $f_c$ are repelling and $|f^n_c(0)| > \delta$ for all $n \geq 1$ and a positive constant $\delta$.

It is known that the Julia sets of non-recurrent quadratic polynomials are connected and locally-connected. The angle of an external ray landing at the critical value $c$ is said to be a characteristic angle of $f_c$. Refer to Section 2 for the definitions of external rays and parameter rays. The following is our main theorem.

**Theorem 1.1.** Any parameter ray at a non-recurrent angle lands at a non-recurrent parameter $c$ such that $\theta$ is a characteristic angle of $f_c$. Conversely, every non-recurrent parameter $c$ is the landing point of one or two parameter rays at non-recurrent angles, and these angles are exactly the characteristic angles of $f_c$.

Our proof is based on Kiwi’s Combinatorial Continuity Theorem [K2, Theorem 1] and Yoccoz Rigidity Theorem [Hu, Theorem III] (or [Ze, Theorem 4.1]). We will review some background of polynomial dynamics and fix notations in Section 2, and introduce the concept of real lamination in Section 3. In Section 4, we verify two combinatorial results used in the proof of the main theorem, and the proof of Theorem 1.1 is left in Section 5.

## 2 Polynomial dynamics

One can refer to [DH2] for the details of the content in this section.

Let $f_c(z) = z^2 + c$, $z \in \mathbb{C}$ be a quadratic polynomial. The set of all points which remain bounded under all iterations of $f_c$ is called the Filled-in Julia set $K_c$. The boundary of the Filled-in Julia set is defined to be the Julia set $J_c$ and the complement of the Julia set is defined to be its Fatou set $F_c$.

If $c \in \mathcal{M}$, the filled-in Julia set $K_c$ is simply-connected, i.e., $\mathbb{C} \setminus K_c$ is connected. There is then a unique biholomorphic map $\phi_c$ from $\mathbb{C} \setminus K_c$ onto $\mathbb{C} \setminus \overline{D}$, called the Böttcher coordinate, such that $\lim_{z \to \infty} \phi_c(z)/z = 1$ and $\phi_c \circ f_c(z) = \phi_c(z)^2$ for $z \in \mathbb{C} \setminus K_f$. The preimage of $(1, \infty)e^{2\pi i t}$ under $\phi_c$, denoted $R_c(t)$, is called the external ray at angle $t$. We say that the external ray $R_c(t)$ lands if $\overline{R_c(t)} \cap K_c$ is a singleton, and this point, denoted by $\gamma_c(t)$, is called the landing point of $R_c(t)$.

In the parameter plane, the Mandelbrot set $\mathcal{M}$ is simply-connected, and there is a biholomorphic map $\Phi$ from $\mathbb{C} \setminus \mathcal{M}$ onto $\mathbb{C} \setminus \overline{D}$. The parameter ray at angle $\theta$ is defined as the set $R_{\mathcal{M}}(\theta) := \{ \Phi^{-1}(re^{2\pi i \theta}) , r > 1 \}$. Similarly, if $\overline{R_{\mathcal{M}}(\theta)} \cap \mathcal{M}$ is a singleton, we say that $R_{\mathcal{M}}(\theta)$ lands.
3 The impression and real lamination of polynomials

Let $f_c$ be a quadratic polynomial with connected Julia set. If $J_c$ is locally connected, the map $\phi_c^{-1}$ can be continuously extended to $\mathbb{C} \setminus \mathbb{D}$ and each external ray $R_c(t)$ lands at a Julia point. The map $\gamma_c : \mathbb{T} \to J_f$, $t \mapsto \gamma_c(t)$, is continuous and surjective. In this case, the landing pattern of external rays for $f_c$ induces an equivalence relation $\lambda(c)$ on $\mathbb{T}$ such that $t \sim (c) s$ if and only if $\gamma_c(t) = \gamma_c(s)$.

Kiwi [Ki2] generalized the definition of such an equivalence relation to a class of non locally-connected case, with the concept impression instead of the landing point in the locally-connected case.

We still assume that $f_c$ has the connected Julia set. Consider an argument $t \in \mathbb{T}$. We say that $z \in J_f$ belongs to the impression of $t$, written $\text{Imp}_c(t)$, if and only if there exists a sequence $\{z_n \in R_c(t_n)\}$ converging to $z$, with $\{t_n\} \subset \mathbb{T}$ converging to $t$. Note that $f_c(\text{Imp}_c(t)) = \text{Imp}_c(\tau(t))$, and $\text{Imp}_c(t) = \gamma_c(t)$ for each $t \in \mathbb{T}$ if $J_c$ is locally connected.

Similar to the locally-connected case, we have the following two facts about the impressions, which will be used in the proof of Proposition 5.1

Lemma 3.1. 1. If there exists a sequence $\{z_n \in \text{Imp}_c(t_n)\}$ with $z_n \to z$ and $t_n \to t$ as $n \to \infty$, then $z \in \text{Imp}_c(t)$.

2. For any $z \in J_c$, there exists an argument $t \in \mathbb{T}$ with $z \in \text{Imp}_c(t)$.

Proof. 1. For each $n$, since $z_n \in \text{Imp}_c(t_n)$, we can choose an argument $s_n$ and a point $w_n \in R_c(s_n)$ such that $|z_n - w_n| < |z_n - z|$ and $|t_n - s_n| < |t_n - t|$. Then we have $|w_n - z| < 2|z_n - z| \to 0$ and $|s_n - t| < 2|t_n - t| \to 0$ as $n \to \infty$. It means that $w_n \in R_c(s_n)$ converges to $z$ and $s_n$ converges to $t$, hence $z \in \text{Imp}_c(t)$.

2. Let $z \in J_c$. Since $J_c$ is the boundary of the basin $\mathbb{C} \setminus K_c$, there exists a sequence $\{z_n\} \subset \mathbb{C} \setminus K_c$ with $z_n$ converging to $z$. Each $z_n$ belongs to an external ray of argument $t_n$. By picking a subsequence, we assume $t_n \to t$ as $n \to \infty$. It follows from the definition that $z \in \text{Imp}_c(t)$. \hfill \n
Let $f_c$ be a quadratic polynomial with connected Julia set and without irrational neutral cycles. Following Kiwi (see [Ki2] Definition 2.2), the real lamination of $f_c$ is the smallest equivalence relation $\lambda(c)$ in $\mathbb{T}$ which identities $s$ and $t$ whenever $\text{Imp}_c(s) \cap \text{Imp}_c(t) \neq \emptyset$. For a $\lambda(c)$-class $A$, we denote $\text{Imp}_c(A)$ the union of the impressions $\text{Imp}_c(t)$ for all $t \in A$.

4 The equivalence relation generated by angles

For any angle $\theta \in \mathbb{T}$, its preimages $\{\theta/2, (\theta + 1)/2\}$ under $\tau$ divide $\mathbb{T}$ into two closed half circles, which are denoted by $L_0, L_1$ respectively. Then we can endow each angle $t \in \mathbb{T}$
two itineraries \( \iota^+_\theta(t) \) with respect to \( \theta \) such that \( \iota^+_\theta(t) = i_0 i_1 \ldots \) if, for each \( n \geq 0 \), there exists \( \epsilon > 0 \) with \( (\tau^n(t), \tau^n(t) \pm \epsilon) \subset L_{i_n} \).

From the definition, we can see that if \( t \) is not an iterated preimage of \( \theta \), then \( \iota^+_\theta(t) = \iota^-_\theta(t) \). In particular, we have \( \iota^+_\theta(\theta) = \iota^-_\theta(\theta) \) for any non-periodic \( \theta \). In this case, the sequence \( \iota^+_\theta(\theta) = \iota^-_\theta(\theta) \) is called the kneading sequence of \( \theta \), written \( \nu(\theta) \). It is called aperiodic if it is not a periodic symbol sequence under the shift map.

By Kiwi [Ki2, Definition 4.5], the equivalence relation generated by \( \theta \in \mathbb{T} \), denoted by \( \lambda(\theta) \), is defined as the smallest equivalence relation such that if \( \iota^+_\theta(\theta) = \iota^-_\theta(\theta) \), then \( s \) and \( t \) are equivalent.

From now on, we always assume that \( \theta \in \mathbb{T} \) is non-recurrent. The following is a key Lemma in our proof.

**Lemma 4.1.** If \( \theta \) is non-recurrent, then \( \nu(\theta) \) is aperiodic.

**Proof.** On the contrary, we assume that \( \nu(\theta) \) is periodic of period \( p \geq 1 \). For each integer \( 0 \leq k \leq p \), set

\[
B_k := \{ \tau^{kn+p}(\theta) \mid n \geq 0 \}.
\]

Then we have \( \tau(B_k) \subset B_{k+1} \), \( k \in \{0, \ldots, p-1\} \), and \( B_0 \subset B_0 \). Note that all elements of \( B_k \) have a common itinerary, then each \( B_k \) is contained in a component of \( \mathbb{T} \setminus \{ \theta/2, (\theta+1)/2 \} \). Moreover, by the non-recurrent property, the closures \( B_k \) are disjoint from \( \{ \theta/2, (\theta+1)/2 \} \). It follows that for each \( k \in \{0, \ldots, p-1\} \), the map \( \tau : B_k \rightarrow B_{k+1} \) is injective, and hence \( \tau^p : B_0 \rightarrow B_0 \) is injective. According to [Mil1, Lemma 18.8], the set \( B_0 \) is finite, a contradiction. \( \Box \)

Since \( \nu(\theta) \) is aperiodic, by [Ki2, Proposition 4.7], we get that

**Proposition 4.2.** The equivalence relation \( \lambda(\theta) \) is closed and satisfies that

1. each \( \lambda(\theta) \)-class is a finite subset of \( \mathbb{T} \);
2. if \( A \) is a \( \lambda(\theta) \)-class, then \( \tau(A) \) is a \( \lambda(\theta) \)-class;
3. for any two different \( \lambda(\theta) \)-classes \( A, B \), the convex hulls of \( A \) and \( B \) are disjoint.

Combining the fact that \( \theta \) is non-recurrent, we can obtain more information about \( \lambda(\theta) \). Since \( \theta \) is not periodic, then \( \iota^+_\theta(\theta/2) = \iota^-_\theta((\theta+1)/2) \), and hence \( \theta/2 \) and \( (\theta+1)/2 \) are \( \lambda(\theta) \)-equivalent. We call the \( \lambda(\theta) \)-class containing \( \{ \theta/2, (\theta+1)/2 \} \) the critical class, denoted by \( C_\theta \); and the one containing \( \theta \) the characteristic class, denoted by \( A_\theta \).

**Lemma 4.3.** Let \( A_\theta \) be the characteristic class of \( \lambda(\theta) \). Then we have

1. \( A_\theta \) is wandering, i.e., \( \tau^m(A_\theta) \cap \tau^n(A_\theta) = \emptyset \) for each \( 0 \leq m < n \);
2. \( A_\theta \) contains at most two angles;
3. \( A_\theta \) is aperiodic.

\( \Box \)
3. $A_\theta$ is non-recurrent, i.e., $\exists \delta > 0$ s.t $\text{dist}(A_\theta, \tau^n(A_\theta)) > \delta$ for all $n \geq 1$.

Proof. 1. On the contrary, without loss of generality, we assume that $A_\theta$ is periodic. Then the fact of $\#A_\theta < \infty$ implies that $\theta$ is eventually periodic, a contradiction.

2. Since $A_\theta$ is wandering, its orbit does not contain $C_\theta$. Note that each $\lambda(\theta)$-class except $C_\theta$ is contained in one component of $\mathbb{T} \setminus \{\theta/2, (\theta + 1)/2\}$ (by 3 of Proposition [4.2]), it follows that $\#A_\theta = \#\tau^n(A_\theta)$ for all $n \geq 0$. Using Thurston’s No Wandering Polygon Theorem ([Thur Theorem II.5.2]), the conclusion holds.

3. If $A_\theta$ contains one angle, since $\theta$ is non-recurrent, the set $A_\theta$ is naturally non-recurrent. So, by assentation (2), we just need to prove the case that $\#A_\theta = 2$.

For any $\alpha \neq \beta \in \mathbb{T}$, we define the arc $(\alpha, \beta) \subset \mathbb{T}$ as the closure of the connected component of $\mathbb{T} \setminus \{\alpha, \beta\}$ that consists of the angles we traverse if we move on $\mathbb{T}$ in the counterclockwise direction from $\alpha$ to $\beta$. The length of an arc $S \subset \mathbb{T}$ is denoted by $|S|$. We define a map $\sigma$ on all arcs in $\mathbb{T}$ such that $\sigma(\alpha, \beta) = (\tau(\alpha), \tau(\beta))$ if $\tau(\alpha) \neq \tau(\beta)$, and equals to $\tau(\alpha)$ otherwise. It is apparent that $|\sigma(S)| = 2|S|$ if $|S| < 1/2$ and $|\sigma(S)| = 2|S| - 1$ otherwise.

Let $A_\theta = \{\theta, \eta\}$. Then it divides $\mathbb{T}$ into two closed arcs. We denote the shorter one by $S_1^+$ and the longer one by $S_1^-$. For $n \geq 1$, set

$$A_n := \tau^{n-1}(A_\theta), \quad S_n^+ := \sigma^{n-1}(S_1^+), \quad S_n^- := \sigma^{n-1}(S_1^-).$$

According to the proof of assentation 2, each $A_n$ is a $\lambda(\theta)$-class containing two angles. And it divides $\mathbb{T}$ into $S_n^+$ and $S_n^-$. By 2 of Proposition [4.2] the critical class $C_\theta$ is equal to $\{\theta/2, \theta/2 + 1/2\}$. It divides $\mathbb{T}$ into four arcs. We denote the two shorter ones by $S_0^+, -S_0^-$, and the two longer ones by $S_0^-, -S_0^-$. It is clear that $|S_0^+| = |S_0^-| < |S_0^1| = |-S_0^-| < 1/2$ and $\sigma(\pm S_0^1) = S_0^1$ for $\delta \in \{+, -\}$.

We claim that in the set $\{S_n^\pm | n \geq 1\}$, the arc $S_1^+$ has the shortest length. If not, suppose that $k \geq 2$ is the first integer such that $S_k^+$ or $S_k^-$, say $S_k^+$, has a shorter length than $S_1^+$. Then $|S_{k+1}^-| > 1/2$. It follows from 3 of Proposition [4.2] that the arc $S_{k+1}^+$ contains $C_\theta$, and hence contains three of the arcs $\pm S_0^\pm, \pm S_0^-$. Its image $\sigma(S_{k+1}^+) = S_k^+$ therefore contains either $\tau(S_0^+) = S_1^+$ or $\tau(S_0^-) = S_1^-$. It implies $|S_k^+| \geq \min\{|S_1^+|, |S_1^-|\} = |S_1^+|$, a contradiction to the assumption that $|S_k^+| < |S_1^+|$.

Set $\theta_n := \tau^n(\theta)$ and $\eta_n := \tau^n(\eta)$ for all $n \geq 1$. We now start to prove point 3 by contradiction. We can assume that $\text{dist}(A_{n_k}, A_\theta) \to 0$, $\theta_{n_k} \to \theta'$, and $\eta_{n_k} \to \eta'$ as $k \to \infty$ by passing to a subsequence if necessary. Since $\lambda(\theta)$ is closed, $\eta', \theta'$ are in the same $\lambda(\theta)$-class. Notice that $|S_{n_k}^\pm| \geq |S_1^+|$ as explained in the claim above. Then $\eta' \neq \theta'$. Since $\text{dist}(A_{n_k}, A_\theta) \to 0$ and $\#A_\theta = 2$, we have $\theta' = \eta$ and $\eta' = \theta$.

Hence for any $\epsilon > 0$, there exist $k_1$ and $k_2$ such that $|\eta_{n_{k_1}} - \theta| < \epsilon/2$ and

$$2^{n_{k_1}}|\theta_{n_{k_2}} - \eta| = |\tau^{n_{k_1}}(\theta_{n_{k_2}}) - \tau^{n_{k_1}}(\eta)| < \epsilon/2.$$

It follows that $|\tau^{n_{k_1}+n_{k_2}}(\theta) - \theta| < \epsilon$ for any $\epsilon > 0$, which is impossible. \qed
5 Proof of Theorem 5.1

The proof of Theorem 5.1 is based on the following two propositions.

**Proposition 5.1.** Given a non-recurrent angle \( \theta \), the parameter ray \( R_M(\theta) \) lands at a non-recurrent parameter \( c \) such that \( f_c \) takes \( \theta \) as a characteristic angle.

**Proof.** We set \( \text{Acc}_M(\theta) \) the accumulation set of \( R_M(\theta) \) on \( M \). By Lemma 4.1 and [Ki2, Theorem 1], we have that all cycles of \( f_c \) are repelling and \( \lambda(c) = \lambda(\theta) \) for all \( c \in \text{Acc}_M(\theta) \). Let \( c \in \text{Acc}_M(\theta) \). We will show that \( f_c \) is non-recurrent with a characteristic angle \( \theta \).

By Lemma 3.1 (2), we choose an angle \( t_0 \in \mathbb{T} \) with \( 0 \in \text{Imp}_c(t_0) \). Note that the Böttcher coordinate \( \phi_c \) satisfies that \( \phi_c(-z) = -\phi_c(z) \) for \( z \in \mathbb{C} \setminus K_c \), then the sets \( \text{Imp}_c(t_0) \) and \( \text{Imp}_c(t_0 + 1/2) \) are symmetric about the origin. It follows that \( 0 \in \text{Imp}_c(t_0) \cap \text{Imp}_c(t_0 + 1/2) \), and hence \( t_0, t_0 + 1/2 \) are in a common \( \lambda(\theta) \)-class (because \( \lambda(c) = \lambda(\theta) \)).

By Proposition 4.2 (3), we know that \( t_0, t_0 + 1/2 \) are contained in the critical class \( C_\theta \). Then the impression of the characteristic class \( \text{Imp}_c(A_\theta) \) contains the critical value \( c \).

Set \( c_n := f_c^{n-1}(c) \) and recall that \( A_n := \tau^{n-1}(A_\theta) \). Then \( c_n \) belongs to \( \text{Imp}_c(A_n) \) for each \( n \geq 1 \). The non-recurrent property of \( f_c \) is equivalent to that the accumulation set of \( \{c_n, n \geq 1\} \) is disjoint from the critical value \( c \). We continue the argument by contradiction and assume that the sequence \( \{\text{Imp}_c(t_{n_k})\} \) with \( t_{n_k} \in A_{n_k} \) converges to \( c \) as \( n \to \infty \).

By passing to a subsequence if necessary, we assume that \( t_{n_k} \to t \) as \( k \to \infty \). From Lemma 3.1 (1), we see that \( c \in \text{Imp}_c(t) \), and hence \( t \in A_\theta \). This contradicts Lemma 4.3 (3). Thus \( f_c \) is non-recurrent. Since the Julia sets of non-recurrent quadratic polynomials are locally connected, the set \( \text{Imp}_c(A_\theta) \) reduces to one point \( c \). So \( R_c(\theta) \) lands at \( c \).

We have seen that all \( f_c \) with \( c \in \text{Acc}_M(\theta) \) are non-recurrent and have a common real lamination \( \lambda(\theta) \). Due to Yoccoz Rigidity Theorem [Hu, Theorem III] or [Ze, Theorem 4.1], we have that \( \text{Acc}_M(\theta) \) reduces to one point. So the parameter ray \( R_M(\theta) \) lands at \( c \).

**Proposition 5.2.** Let \( f_c \) be a non-recurrent quadratic polynomial. Then it has at most two characteristic angles, and the parameter rays at these angles land at \( c \).

**Proof.** Let \( f_c \) be a non-recurrent quadratic polynomial. Since \( J_c \) is locally connected, there exists a characteristic angle \( \theta \) with \( R_c(\theta) \) landing at \( c \). Clearly \( \theta \) is non-recurrent. By Proposition 5.1, the parameter ray \( R_M(\theta) \) lands at a non-recurrent parameter \( c' \) so that \( f_{c'} \) has a characteristic angle \( \theta \). We then have that \( \theta/2 \) and \( (\theta+1)/2 \) are contained in both a \( \lambda(c) \)-class and a \( \lambda(c') \)-class. By [Ki2, Proposition 4.10], we get \( \lambda(c) = \lambda(c') = \lambda(\theta) \). Using again the Yoccoz Rigidity Theorem, it follows that \( c = c' \). By Lemma 4.3 (2), the cardinality of the \( \lambda(c) \)-class that contains \( \theta \) is at most two. Hence \( f_c \) has at most two characteristic angles.

**Proof of Theorem 1.1.** It follows directly from Propositions 5.1 and 5.2.
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