Hamiltonian reduction of Bianchi Cosmologies

Joachim Schirmer

Fakultät für Physik der Universität Freiburg
Hermann-Herder-Str. 3, 79104 Freiburg i.Br. / FRG

Abstract

It was noted recently that the ADM-diffeomorphism-constraint does not generate all observed symmetries for several Bianchi-models. We will suggest not to use the ADM-constraint restricted to homogeneous variables, but some equivalent which is derived from a restricted action principle. This will generate all homogeneity preserving diffeomorphisms, which will be shown to be automorphism generating vector fields, in class A and class B models. Following Dirac's constraint formalism one will naturally be restricted to the unimodular part of the automorphism group.

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1 3-dimensional Lie-groups

In this preliminary paragraph we quickly recapitulate the well known Bianchi-Behr classification of 3-dimensional Lie algebras [1, 2]. Up to isomorphy there exists exactly one simply-connected Lie-group associated with each type of algebra. The structure constant tensor is represented by a symmetric tensor-density and a covector:

\[ C^{a}_{bc} = n^{ad} \epsilon_{bcd} + a_{f} \delta^{fa}_{bc} \]  
\[ n^{ad} = \epsilon^{bc(d} C^{a)}_{bc} \] \[ a_{f} = \frac{1}{2} C^{a}_{fa} \]

The Jacobi-identity yields:

\[ a_{c} C^{c}_{ab} = 0 = a_{c} n^{cd} \]
Transforming the basis of the Lie algebra by $b_a \mapsto \bar{b}_{a'} = b_a A^{-1}a'$ the structure-coefficients change as follows:

$$C^a_{bc} \mapsto \bar{C}^{a'}_{b'c'} = A^{a'}_a C^a_{bc} A^{-1}b'A^{-1}c'$$

$$a_b \mapsto \bar{a}_{b'} = a_b A^{-1}b'$$

$$n^{ab} \mapsto \bar{n}^{a'b'} = \frac{1}{\det A} n^{ab} A^{a'}_a A^{b'}_b$$

Choosing a suitable basis we can diagonalize $n$ such that $n = \text{diag}(n^1, n^2, n^3)$ and assume that $a_0 = (0, 0, a)$. One distinguishes algebras of class A with $a = 0$ and B where $a \neq 0$. For class B algebras $\text{rg } n \leq 2$ holds because of (3). In the special case of $\text{rg } n = 2$, $a \neq 0$, $a$ cannot be normalized because of the determinant in the transformation law for $n$, so that these classes are 1-parameter-classes.

| cl | Bianchi-type | $a$ | $n^1$ | $n^2$ | $n^3$ | dim $\text{Aut}_g$ | dim $\text{Int}_g$ |
|----|--------------|----|------|------|------|-------------------|-------------------|
| A  | I            | 0  | 0    | 0    | 0    | 9                 | 0                 |
|    | II           | 0  | 1    | 0    | 0    | 6                 | 2                 |
|    | VI$_0$       | 0  | 1    | -1   | 0    | 4                 | 3                 |
|    | VII$_0$      | 0  | 1    | 1    | 0    | 4                 | 3                 |
|    | VIII         | 0  | 1    | 1    | -1   | 3                 | 3                 |
|    | IX           | 0  | 1    | 1    | 1    | 3                 | 3                 |
| B  | V            | 1  | 0    | 0    | 0    | 6                 | 3                 |
|    | IV           | 1  | 1    | 0    | 0    | 4                 | 3                 |
|    | III=VI$_1$   | 1  | 1    | -1   | 0    | 4                 | 2                 |
|    | VI$_a$       | a  | 1    | -1   | 0    | 4                 | 3                 |
|    | VII$_a$      | a  | 1    | 1    | 0    | 4                 | 3                 |

Considering the Lie algebras we can conclude that the associated groups of B-class-algebras are not compact. We recapitulate the proof given in [3]. Let $\chi_i = \frac{1}{2} \varepsilon_{ijk} b^j \wedge b^k$ be three linearly independent left invariant 2-forms. Assuming that $G$ is compact Stoke’s theorem leads to a contradiction:

$$0 = \int_{\partial G} \chi_i = \int_G d\chi_i = -\int_G \varepsilon_{ijk} C^j_{lm} b^l \wedge b^m \wedge b^k$$
$$= -2C^j_{ij} \text{vol}_b(G) = -4a_i \text{vol}_b(G) \implies a_i = 0$$

The dimension of the automorphism group of the Lie algebra is in all cases easily determined by considering the isotropy subgroup of the action of $Gl(3, \mathbb{R})$ on the structure constants. Every group automorphism induces an automorphism of the Lie algebra $g = T_e G$, but only if $G$ is simply-connected there is a 1:1-mapping between group and algebra automorphisms [4]. The group of algebra automorphisms as a group of linear endomorphisms is easily investigated, whereas the group of
group automorphisms is only known to be generally smaller unless in the simply-connected case. We will now give the characterizing property of automorphism vector fields, i.e. vector fields the associated flow of which consists of automorphisms.

**Proposition:**

\[ Z \in \text{aut} G \quad \text{if and only if} \quad Z(p \cdot q) = TL_p Z(q) + TR_q Z(p) \quad \forall p, q \in G. \]

**Proof:**

\[ Z \in \text{aut} G \iff \Phi^Z_t \in \text{Aut} G \iff \Phi^Z_t(pq) = \Phi^Z_t(p) \Phi^Z_t(q) \iff Z(\Phi^Z_t(pq)) = TL_{\Phi^Z_t(p)} Z(\Phi^Z_t(q)) + TR_{\Phi^Z_t(q)} Z(\Phi^Z_t(p)) \]

Whereas there is only a 1:1-mapping between group and algebra automorphisms in the simply-connected case there exists a 1:1-mapping always for the case of inner automorphisms: \( \text{Int}_G \approx G/C \approx \text{Int}_G \), where \( C \) is the center of the group. So it is natural to ask for the automorphism vector field \( (\in \text{int} G) \) given by the element \( X \in \mathfrak{g} \) of the Lie algebra, which directly defines \( \text{ad}_X \in \text{int} G \).

**Proposition:**

The inner derivation \( \text{ad}_X \) associated to the element \( X \in T_e G \) of the Lie algebra defines a vector field of \( \text{int} G \) given by

\[ Z(p) = X^R(p) - X^L(p) := TR_p X - TL_p X \quad (4) \]

**Proof:**

\( Z \) generates inner automorphisms since right and left invariant vector fields commute and the left multiplication, regarded as an operation of the group on itself, has right invariant fundamental vector fields and vice versa:

\[ \Phi_t^Z = \lim_{n \to \infty} \left( \Phi_{t/n}^{X^R} \circ \Phi_{t/n}^{X^L} \right)^n = \Phi_t^{X^R} \Phi_t^{X^L} = L_{\exp tX} R_{\exp -tX} \]

\[ T \Phi_t^Z = T L_{\exp tX} R_{\exp -tX} = \text{Ad}_{\exp tX} = e^{t \text{ad}_X} \]

For the convenience of the reader we note that the Christoffel-symbols of the Levi-Civita-connection with respect to a left invariant basis \( \{ b_a \} \) expressed by structure constants take the form:

\[ \Gamma^a_{bc} = \frac{1}{2}(C^a_{bc} + C^a_{b} + C^a_{c} b) \quad (5) \]

The connection form is as usual

\[ \omega^a_b = \Gamma^a_{cb} b^c \quad . \quad (6) \]
2 Kinematical Preliminaries

In this paragraph we will set the kinematical stage. We will carefully analyze the question of what the admissible shift vector fields are.

**Definition:** A homogeneous cosmological model is a Lorentz-manifold \((M, g)\) on which a 3-dimensional isometry group acts simply transitively on spatial hypersurfaces.

This definition is chosen for simplicity. One could imagine to define more generally that an isometry group acts forming 3-dimensional spacelike orbits. In this case these orbits are homogeneous Riemannian spaces. One can classify the simply-connected 3-dimensional homogeneous Riemannian spaces according to the dimension of the isometry group and it turns out that apart from the Kantowski-Sachs-case there is a three-dimensional subgroup acting simply-transitive on the orbits [5]. Discarding the Kantowski-Sachs-case the simply-connected models in this more general definition are contained in our definition.

A homogeneous cosmological model is (time-locally) isometric to \((\mathbb{R} \times G, -ds \otimes ds + q_{ab}(s)a^a \otimes a^b)\), where \(\{a^a\}_{a=1,2,3}\) is a left invariant basis of \(T^*(G)\). We sketch the proof \([3, 4]\):

Consider the normalized geodesic at a point \(p\), normal to the orbit of \(G\). The orbit at the point \(q\) reached by the geodesic at a certain parameter value is parallel to the original orbit, i.e. taking the normalized geodesic at another point \(p'\) of the original orbit in normal direction one reaches the orbit of \(q\) at the same parameter value.

The metric \((-ds \otimes ds + q_{ab}a^a \otimes a^b)\) is often used as the starting point to evaluate Einstein’s equations for homogeneous models. We notice that our space-time is naturally foliated in space and time and that our metric is represented in a lapse-1-shift-0-representation. For conceptional reasons it is necessary to check if this is choice of lapse and shift is the only possible, in which the model is described by a time-dependent left invariant metric. In the normal ADM-framework the secondary constraints associated with lapse and shift reflect the gauge freedom and are equivalent to the equations \(G(n, n) = 0\) and \(G(n, a_a) = 0\). If our model required a certain gauge these constraints are satisfied identically. The question of possible shift vector fields is also of practical importance since a suitable choice might simplify the matrix \(q_{ab}\) and help in the search of solutions. A shift vector field generates a diffeomorphism of the foliating spacelike surface. If this diffeomorphism shall not destroy spatial homogeneity it has to leave left invariant metrics left invariant.
Lemma: If a diffeomorphism $\psi$ maps every left invariant metric of a connected Lie-group onto a left invariant metric, it has the form $\psi = L_{\psi(e)} \circ \varphi$, where $\varphi$ is an automorphism.

Proof:
Let $\varphi := L_{\psi(e)}^{-1} \circ \psi$ and $\xi^L$, $\eta^L$ the left invariant vector fields associated with the vectors $\xi$, $\eta \in T_eG$ and $q$ a left invariant metric.

$$\varphi^* q(g)(\xi^L(g), \eta^L(g)) = q(\varphi(g))(T \varphi TL_g \xi, T \varphi TL_g \eta)$$

$$= q(e)(T L^{-1}_{\varphi(g)} T \varphi TL_g \xi, T L^{-1}_{\varphi(g)} T \varphi TL_g \eta)$$

$$\varphi^* q(g)(\xi^L(g), \eta^L(g)) = \varphi^* q(e)(\xi, \eta) = q(e)(T \varphi \xi, T \varphi \eta)$$

Since this holds for arbitrary metrics, one concludes

$$T_g \varphi T_e L_g = T_e L_{\varphi(g)} T_e \varphi \quad \forall g \in G$$

and hence $(T_e \varphi)^L(\varphi(g)) = T_g \varphi(\xi^L(g))$. Then $\varphi(g \exp t \xi)$ and $\varphi(g) \varphi(\exp t \xi)$ are integral curves of the vector field $(T_e \varphi \xi)^L$ and thus they agree. Since a connected group is generated by a neighbourhood of the unit element, $\varphi$ is an automorphism and the lemma is proved.

Now we can write down the most general time vector field preserving spatial homogeneity.

$$\frac{\partial}{\partial t} = N(t)n + \vec{N} = N(t)\frac{\partial}{\partial s} + \vec{N} \in \text{aut}G$$

The lapse-function can only depend on time, since the normal vector field is a natural vector field $n = \frac{\partial}{\partial s}$. The dual relation of $s$ and $t$ is $n^s = ds = N dt$ and application of the exterior derivative on both sides shows that the lapse function must not depend on space-coordinates. The interpretation is simple: If the lapse-function also depended on space-coordinates the slices of simultaneity would not coincide with the group orbits. The admissible shift vector field must be a sum of a rightinvariant vector field and an element of $\text{aut}G$, the Lie algebra of the group of automorphisms of $G$. The right invariant vector field generates only lefttranslations which leave the metric unchanged. So it has a trivial action on the metric and will be disregarded. For Bianchi-type VIII and IX all automorphisms are inner. According to our proposition of the preceding paragraph a vector field generating inner automorphisms has the form $\vec{N} = \xi^R - \xi^L$ for a certain $\xi \in g = T_eG$. Neglecting the right invariant field one can choose a left invariant shiftvector field $\vec{N} \in g = \mathfrak{X}^L = \{\text{leftinv. vector fields}\}$ without loss of generality. But this is only correct for these two Bianchi-types whereas in all other cases the automorphism group is larger than the inner automorphism group. For the Bianchi-type I, II and
III the associated groups even have a nontrivial center. Thus we stick to the more general assumption that the shift is an element of $\text{aut}G$ and note that therefore the coefficients of these fields with respect to a left invariant basis need not to be constant. This must be taken into account when one asks for the degrees of freedom in the reduced phase space.

We first consider the Bianchi-IX-case. There exists a simply-connected compact group, $SU(2)$, with group manifold $S^3$. There is a 1:1-relation between group- and algebra-automorphisms and all automorphisms are inner automorphisms. Even in the non-simply-connected case of $SO(3)$ with underlying manifold $\mathbb{RP}^3$ every automorphism of the Lie algebra defines an automorphism of the group, since every automorphism of $SU(2)$ induces an automorphism of $SO(3)$. In all other Bianchi-types the universal covering group is not compact. In order to obtain a compact group one must divide by a compact normal subgroup. This procedure reduces the automorphism group since identified points must be mapped onto identified points. The effect can easily be seen in the Bianchi-I-case. The universal covering group is $(\mathbb{R}^3, +)$, the automorphism group is $Gl(3, \mathbb{R})$. After compactification to $(T^3, +)$ only a discrete subgroup of $Gl(3, \mathbb{R})$ is left, $SL(3, \mathbb{Z})$. This means that we cannot introduce a shift vector field without destroying manifest homogeneity, the gauge is fixed. Of course it is still possible to use a right invariant shift, which has no effect to the metric.

The consequences for the Hamiltonian description can be anticipated. In the case of the torus $(T^3, +)$ one can use the normal ADM-diffeomorphism-constraint, since integration by parts is possible for this compact model. But after restriction to homogeneous metrics and the evaluation of the covariant derivative in terms of structure constants the constraint vanishes identically, so it does not generate any gauge. In the case of $(\mathbb{R}, +)$ one should not use the ADM-constraint, but some analogon which is found by restricting the action principle to left invariant metrics and avoiding integration by parts. This analogon generates all automorphisms. For the $SU(2)$-model both descriptions coincide.

After introducing a shift vector field it is necessary to change the left invariant basis, since it is no longer time-independent $\frac{d}{dt}a_{a} = [\frac{\partial}{\partial t}, a_{a}] = 0$, whereas one had $[\frac{\partial}{\partial s}, a_{a}] = 0$. By applying a time-dependent Lie-algebra-automorphism $b_{b} := a_{a}S^{-1a}_{b}(t)$,

$$S : \mathbb{R} \rightarrow \text{Aut}g,$$

the condition $[\frac{\partial}{\partial t}, b_{b}] = 0$ which is needed for the ADM-formalism can be satisfied:

$$b_{b} = a_{a}S^{-1a}_{b}, \quad S : \mathbb{R} \rightarrow \text{Aut}g$$

$$0 = [\frac{\partial}{\partial t}, b_{b}] = a_{a}S^{-1a}_{b} + [N \frac{\partial}{\partial s} + \vec{N}, a_{a}]S^{-1a}_{b}$$

$$= b_{c}S^{c}_{a}S^{-1a}_{b} + L_{\vec{N}}b_{b} = 0$$

$$b_{c}S^{c}_{a}S^{-1a}_{b} + L_{\vec{N}}b_{b} = 0$$
\[ \begin{align*}
\iff \quad \dot{S}^c_a S^{-1}_b &= b^c(L_{\vec{N}} b_b) =: A(\vec{N})^c_b, \quad A : \mathbb{R} \longrightarrow \text{aut} \mathfrak{g}
\end{align*} \]

Substitution of the basis \( \{ds, a^a\} \) by \( \{dt, b^b\} \) leads to a change of the metric:

\[ \begin{align*}
ds &= N dt \\
a^a &= S^{-1}_b (N^b dt + b^b) \\
N^b &:= b^b(\vec{N})
\end{align*} \]

\[ g = -N^2 dt \otimes dt + S^{-1}_c a q_{cd} S^{-1}_d b (N^a dt + b^a) \otimes (N^b dt + b^b) \]

The infinitesimal automorphism is generally not constant because the automorphism field \( A \in \text{aut} \mathfrak{g} \) may be time-dependent. It was suggested in [8] that it has to be constant and because of a localizability requirement only inner automorphism are acceptable.

We can read-off the effect of the automorphism vector field to the spatial metric components:

\[ q_{ab}(t) \mapsto S^{-1}_c a(t) q_{cd}(t) S^{-1}_b(t) \tag{8} \]

Using the well known equation for the extrinsic curvature one can easily check that the extrinsic curvature remains left invariant after introduction of the shift vector field so that the dynamics of the homogeneous models can be regarded as a finite-dimensional mechanical system.

\[ \begin{align*}
\dot{q} &\quad = 2NK + L_{\vec{N}}q \\
K_{ab} b^a \otimes b^b &\quad = \frac{1}{2N} \left[ \dot{q}_{ab} b^a \otimes b^b - L_{\vec{N}} \left( q_{ab} b^a \otimes b^b \right) \right] \\
&\quad = \frac{1}{2N} \left[ \dot{q}_{ab} + q_{ac} A(\vec{N})^c_b + A(\vec{N})^c_a q_{cb} \right] b^a \otimes b^b \tag{10}
\end{align*} \]

### 3 Hamiltonian formulation of homogeneous models

Usually one specializes the ADM-constraint and the equations of motion to left invariant metrics in order to gain a Hamiltonian description of homogeneous models. But this ”restriction procedure” is normally not allowed since left invariant metrics and automorphism shifts are often not contained in the mathematical spaces for which the ADM-framework was derived. This is the reason why the usual diffeomorphism-constraint fails to generate all automorphisms. Instead we will restrict the 3:1-split action-principle to left invariant metrics, automorphism-shifts and spatially constant lapse-functions. Then we can directly derive equivalents of the constraints and as well equations of motion. According to a mathematical argument by Palais [9] both descriptions coincide in the case of compact groups, but for
class B algebras there are no compact groups and for the compact Bianchi-I-model there are no nontrivial shift vector fields.

Starting from the 3:1-split action-principle we recapitulate the derivation of the ADM-Constraints. Let

\[ L(q, \dot{q}, N, \vec{N}) = \int_\Sigma \eta N (K_{ab} K^{ab} - K^2 + R) \]  

where \( q \) is the spatial 3-metric, \( K \) the extrinsic curvature, \( R \) the Ricci-scalar of the spatial 3-metric, \( N \) lapse and \( \vec{N} \) the shift vector field.

\[ p_{ab} = \frac{\delta L}{\delta \dot{q}_{ab}} = (K^{ab} - K q^{ab}) \eta \]

\[ K^{ab} = * p^{ab} - \frac{1}{2} p q_{ab} \]

\[ p := * p^{ab} q_{ab} = -2K \]

\( p^{ab} \) is the momentum, a tensor valued 3-form. We can perform the Legendre-transformation and obtain the Hamiltonian:

\[ H = \int_\Sigma \dot{q}_{ab} p^{ab} - L = \int_\Sigma p^{ab} L_{\vec{N}} q_{ab} + \int_\Sigma \eta N \left[ (p^{ab}|p_{ab}) - \frac{1}{2} p^2 - \hat{R} \right] \]

Here \( (\cdot|\cdot) \) denotes the metric product of 3-forms. Since the time-derivatives of lapse and shift do not appear in the Lagrangian one obtains primary constraints

\[ p_N = \delta L / \delta \dot{N} \approx 0 \quad p_{\vec{N}} = \delta L / \delta \vec{N} \approx 0 \]

and subsequently secondary constraints (which are equivalent to the Euler-Lagrange-equations associated to lapse and shift).

\[ C_H = \{ H, p_N \} = \eta \left( (p^{ab}|p_{ab}) - \frac{1}{2} p^2 - \hat{R} \right) \]

\[ C_{Dc} = \{ H, p_{N^c} \} = -2q_{ac} D_i b p^{ab} \]

where \( D \) is the exterior covariant derivative. These secondary constraints must be regarded as functionals on a certain space of lapse functions or shift vector fields. Especially in the case of the diffeomorphism-constraint a boundary integral was neglected:

\[ C_{Dc} = \{ H, p_{N^c} \} = \delta / \delta N^c \int_\Sigma p^{ab} L_{\vec{N}} q_{ab} \]

\[ \int_\Sigma p^{ab} L_{\vec{N}} q_{ab} = 2 \int_\Sigma p^{ab} N_{a:b} = 2 \int_\Sigma p^{ab} i_b D_{\Sigma} \]

\[ = 2 \int_\Sigma i_b p^{ab} \wedge D_{\Sigma} N_{a} = 2 \int_\Sigma N_{a} i_b p^{ab} - 2 \int_\Sigma N^c q_{ca} D_i b p^{ab} \]
So $H$ is only functionally differentiable if the boundary term vanishes and this is a restriction to the space of admissible shift vector fields.

It is useful to remember the link between the 3:1-split Hamiltonian and the space-time-description. Having chosen a foliation there is an adapted tetrad $\{n, b_a\}_{a=1,2,3}$, where $n$ is normal, $b_a$ parallel to the spacelike hypersurfaces. Then the Hamiltonian constraint $C_H = 0$ is equivalent to the Einstein-equation $G(n,n) = 0$, the diffeomorphism constraint $C_{Dc} = 0$ reflects the equation $G(n, b_a) = 0$ and the equations of motion represent the equations $G(b_a, b_b) = 0$.

When one restricts this ADM-framework to left invariant metrics, momentum forms, automorphism-shifts and spatially constant lapse functions, and substitutes $\Sigma$ by $G$ and uses the formula (5) for the Levi-Civita-connection with respect to a left invariant basis one obtains in case of the diffeomorphism constraint:

$$C_{Dc}_{li} = -2q^{ac}Di_b p^{ab} = -2(q^{ac}Di_b p^{ab} + q^{ac}\omega^a_d \wedge i_b p^{db})$$

$$= -2q^{ac}C^{cd}p^{db} + 4q^{ac}p^{ab}a_b = -2tr qk_c p + 4(\tilde{a}pq)_c$$

where $a$ is as in (2) and thus the second term vanishes for class A algebras. If this restricted constraint is a useful analogon in the homogeneous case it should generate all automorphisms of the group. Since in the homogeneous case we deal with a finite-dimensional mechanical system we can easily determine the momentum mapping for the action of the automorphism group and compare it with this constraint.

Our configuration space is the space of all positive definite symmetric $(0,2)$-tensors over the Lie algebra $g = T_e G$. The tangent bundle is the space of all symmetric $(0,2)$-tensors over $g$, the cotangent-bundle $T^*Q$ is the space of all symmetric $(2,0)$-tensors over the basis $Q$ and the natural product is given by contraction. Using a basis $b_a \in T_e G$ one identifies $TQ$ and $T^*Q$ with symmetric matrices over the base manifold of all positive definite matrices, and the contraction is given by the trace of the matrices’ product. By the choice of the basis the automorphism group of $g$ is identified with a subgroup of $Gl(3, IR)$. We now construct the momentum mapping of the lifted group action on $T^*Q$:

$$\Phi : Aut g \times Q \rightarrow Q \quad \Phi^{T^*} : Aut g \times T^*Q \rightarrow T^*Q$$

$$\Phi(S,q) = S^{-1}T q S^{-1}$$

$$\Phi^{T^*}(S,\pi) = S \pi S^T$$

$$P(A)(\pi_q) = \pi \left( \frac{d}{dt} \bigg|_{t=0} \Phi(e^{tA}, q) \right) = \pi(-A^T q - q A) = -2tr q A \pi$$

(21)
Introducing a basis of the automorphism group of the Lie-algebra we can easily compare the diffeomorphism constraint and the momentum mapping:

\[ \{r_\mu\}_{\mu=0,\ldots,\text{dim aut } g-1} \quad \text{basis of aut } g \]
\[ \{r_i\}_{i=1,\ldots,\text{dim aut } g-1} \quad \text{basis saut } g := \{ A \in \text{aut } g \mid \text{tr } A = 0 \} \]

\[ P(A) = -2\text{tr } qA\pi = -2A^\mu \text{tr } qr_\mu \pi \quad (22) \]

So the diffeomorphism-constraint (20) contains two different problems:

1. The second summand which is only present for class B algebras has no meaning.
2. Discarding the second summand – restricting to class A algebras – the diffeomorphism-constraint seems to generate only inner automorphisms.

After the discussion of the preceding paragraph we are able to guess the reasons of these shortcomings:

1. Class B Lie groups are not compact. The boundary integral which appears in the derivation of the constraint has to be defined as a limit over compact subsets for which Stoke’s theorem is valid), and this sequence does not tend to zero, since an automorphism field cannot satisfy any asymptotic fall-off conditions.
2. Automorphism fields which do not generate inner automorphisms cannot have constant components with respect to a left invariant basis. In order to generate outer automorphisms it is necessary to integrate the constraint with the appropriate shift vector field. Thus the restricted constraint is not suitable for the finite dimensional model.

If one instead derives the constraints from an action principle that is restricted to left invariant metrics, they generate all automorphisms regardless of class A or B algebras. Unfortunately the action integral is not defined in the case of non-compact groups, but since the variational principle holds locally we can integrate over a nonempty open subset \( U \) with compact closure.

\[ L(q, \dot{q}, N, A) = \int_U N(K_{ab}^\alpha K^a_{\alpha} - K^a_{b\alpha} + R)\eta = NV(K_{ab}^\alpha K^a_{\alpha} - K^a_{b\alpha} + R) \quad (23) \]

\[ V := \int_U \eta \quad N = N(t) \quad \tilde{N} \in \text{aut } G \]

\[ K = \frac{1}{2N}(\dot{q} - L_{\tilde{N}}q) = \frac{1}{2N}(\dot{q} + A^T q + qA) \]
The momentum is:

\[ \tilde{p}^{ab} = V(K^{ab} - K^c_q^{ab}) = \int_U \rho^{ab} \quad K^{ab} = \frac{1}{V}(\tilde{p}^{ab} - \tilde{p}^c_q^{ab}) \]  

(24)

Performing the Legendre transformation one obtains

\[ H = -2A^\mu \text{tr} q_{\mu} \tilde{p} + N \left[ \frac{1}{V} (\text{tr} (\tilde{p}q))^2 - \frac{1}{2} \text{tr}^2 \tilde{p}q \right] - VR \]

which agrees with the restriction of (16) to homogeneous variables using \( \tilde{p}^{ab} = \int_U \rho^{ab} \). Since the time derivatives of \( A \) and \( N \) do not appear in the Lagrangian one finds the primary constraints

\[ p_{A^\mu} \approx 0 \quad p_N \approx 0 \]

(26)

and the associated secondary constraints

\[ C_D{\mu} = \{ H, p_{A^\mu} \} = -2 \text{tr} q_{\mu} \tilde{p} \approx 0 \]

(27)

\[ C_H = \{ H, p_N \} = \frac{1}{V} (\text{tr} (\tilde{p}q))^2 - \frac{1}{2} \text{tr}^2 \tilde{p}q \} - VR \approx 0 \]

(28)

Thus the Hamiltonian is, as usual, a sum of the constraints.

\[ H(q, \tilde{p}, A, N) = -2A^\mu C_{D\mu} + NC_H =: H_D(A) + H_H(N) \approx 0 \]

(29)

The diffeomorphism part in the Hamiltonian directly equals the momentum mapping of the action of \( \text{Aut} \mathfrak{g} \) on the configuration space, if one identifies \( \pi \) and \( \tilde{p} \). An important property of the momentum mapping can be checked easily:

\[ \{ H_D(A), H_D(B) \} = H_D([A, B]) \]

(30)

Thus the constraints \( C_{D\mu} \) commute weakly:

\[ \{ C_{D\mu}, C_{D\nu} \} = \alpha^\rho_{\mu\nu} C_{D\rho} \quad [r^\mu, r^\nu] =: \alpha^\rho_{\mu\nu} r^\rho \]

(31)

The identification of \( \pi \) and \( \tilde{p} \) leads to another difficulty: \( p \) is a tensor valued 3-form, \( \tilde{p} \) a tensor valued volume, and so it transforms differently. This is the reason why the diffeomorphism constraint does not generate the Lie derivative of \( \tilde{p} \), as one expects according to ADM-theory.

\[ \tilde{p} \in T^2_{e_{sym}} G \otimes \Lambda^3(T_e G) =: W \quad \Phi^W : \text{Aut} \mathfrak{g} \times W \longrightarrow W \]
The ADM-equations of motion are

\[ \Phi(S, \bar{p}) = \det S^{-1} S \bar{p} S^T \]  

\[ \frac{d}{dt} \Phi(e^{tA}, \bar{p}) = -\tr A \bar{p} + A \bar{p} + \bar{p} A^T = L_N \bar{p} \]  

This would also be the restriction of the diffeomorphism part of the ADM-equations of motion to homogeneous variables. The first term is not generated by \( H_D(A) = -2\tr q A \bar{p} \), but it vanishes if one only considers the unimodular subgroup of \( \text{SAut}_q \). The restriction to the unimodular subgroup is a natural consequence of a tertiary constraint which has no equivalent in ADM-theory:

\[ \{ H, C_H \} = A^\mu \{ C_{D \mu}, C_H \} = -A^\mu \frac{d}{dt} \bigg|_{t=0} C_H \circ \Phi^T \left( \exp tr_{\mu}(q, \bar{p}) \right) \]

\[ = -A^0 \tr r_0 \left( \frac{1}{2} \left( \tr ( \bar{p} q )^2 - \frac{1}{2} \tr ( \bar{p} q )^2 \right) + VR \right) \approx 0 \]

\[ \implies A^0 \approx 0 \]

Here we used the fact that \( C_{D \mu} \) acts as a momentum mapping for the action of \( \text{Aut}_q \), where \( \bar{p} \) is treated as a symmetric tensor not as a tensor valued volume. The group operation leaves the traces and the scalar \( R \) invariant. So \( A^0 \approx 0 \) is a tertiary constraint and of course the pair \((A^0, p_{A^0})\) is second-class. Hence one can discard this degree of freedom and restrict to the unimodular subgroup of the automorphism group.

Finally we derive the equations of motion and compare them with the restriction of the ADM-equations to homogeneous variables.

\[ \dot{q}_{ab} = \frac{\partial H}{\partial \bar{p}^{ab}} = -2q_{c(a} A^{c b)} + \frac{2N}{V} \left( \bar{p}_{ab} - \frac{1}{2} \bar{p} q_{ab} \right) \]

\[ \dot{p}_{ab} = -\frac{\delta H}{\delta q_{ab}} = 2A^{(a c} \bar{p}^{b) c} + \frac{2N}{V} \left( \bar{p}^{a c} \bar{p}_{c b} - \frac{1}{2} \bar{p}^{a b} \bar{p} \right) + \frac{N}{2V} \left( \bar{p}^{a d} \bar{p}_{c d} - \frac{1}{2} \bar{p}^2 \right) q_{a b} \]

\[ -NV \left( R_{a b} - \frac{1}{2} R q_{a b} \right) + 2NV \left( 2a^a b^b - a^c C_{a b}^{(a c)} \right) \]  

The ADM-equations of motion are

\[ \dot{q}_{ab} = \frac{\delta H}{\delta \bar{p}^{ab}} = L_N q_{ab} + 2N \left( *p_{a b} - \frac{1}{2} \bar{p} q_{a b} \right) \]

\[ \dot{p}_{ab} = -\frac{\delta H}{\delta q_{ab}} = L_N \bar{p}^{ab} - 2N \left( *p^{a c} p_{b}^{\ b} - \frac{1}{2} \bar{p} \bar{p}^{a b} \right) + \frac{N}{2} \eta q^{a b} \left( \bar{p}^{c d} | p_{c d} - \frac{1}{2} \bar{p}^2 \right) \]

\[ -N \eta \left( R_{a b} - \frac{1}{2} R q_{a b} \right) + \left( \nabla^a \nabla^b N - q_{a b} \nabla^c \nabla_c N \right) \eta \]  

If \( \bar{N} \in \text{saut}_G \), \( N = N(t) \) and one integrates the second equation over \( U \subset G \) the equations differ only in the term \( 2NV \left( 2a^a b^b - a^c C_{a b}^{(a c)} \right) \), which vanishes for class A algebras. This term originates from the variation of the Ricci-tensor:

\[ N \delta R_{a b} q^{a b} = N \left( q^{a c} q^{b d} - q^{a b} q^{c d} \right) \nabla_a \nabla_b \delta q_{c d} \]  

(36)
Usually integration by parts leads to the term \((\nabla^a \nabla^b N - q^{ab} \nabla^c \nabla_c N) \eta\), but for left invariant variations one can evaluate \(\nabla_a \nabla_b \delta q_{cd}\) using (6) for the Christoffel symbols. This leads to:

\[
N \delta R_{ab} q^{ab} = N \delta q_{ab} \left(4 a^a b^b - 2 a^c C^{(a}_{c} b^{b})\right) =: -N Q^{ab} \delta q_{ab} \quad (37)
\]

In the ADM-theory the equation for \(\dot{\rho}^{ab}\) is equivalent to Einstein’s equation \(G(b_a, b_b) = 0\). Thus equation (34) is usually corrected \([11]\) by a term that cancels the additional contributions derived from a left invariant action principle:

\[
\dot{q}_{ab} = \{q_{ab}, H\}, \quad \dot{\rho}^{ab} = \{\rho^{ab}, H\} + NV Q^{ab} - \rho^{ab} \text{tr} A \quad \iff \quad G^{ab} = 0 \quad \iff \quad (34)
\]

As we have shown the correction by the second term would not be necessary, since \(\text{tr} A\) vanishes in the physical part of the phase space. Since \(Q^{ab} dq_{ab}\) – the exterior derivative refers to the configuration space \(Q\) where \(q_{ab}\) are coordinates – is not exact, one cannot substitute \(H\) by \(H' = H + U\), s.t. \(\{\rho^{ab}, H\} + NV = \{\rho^{ab}, H'\}\). So \(Q^{ab}\) plays the role of a non-conservative force in this approach \([12]\). But one could also suppose that Einstein’s equation \(G(b_a, b_b) = 0\) does not hold for class B algebras and accept equation (34). As we have seen the ADM-diffeomorphism-constraint equivalent to \(G(n, b_a) = 0\) does not generate automorphisms in case of class B algebras. If one regards Einstein’s equations as a consequence of the action principle one could accept to reject the Einstein-equations in the homogeneous case and to work with the equivalents we have given. One should also take into account that the assumptions of the model are non-covariant, since covariance and the existence of a natural foliation are incompatible. It is the question what is meant by reducing Einstein’s theory to homogeneous cosmologies: Reducing Einstein’s equations or reducing the action principle from which it can be derived.

**References**

[1] R. T. Jantzen *The Dynamical Degrees of Freedom in Spatially Homogeneous Cosmology*, Commun. Math. Phys., 64, 211-232, 1979

[2] F. B. Estabrook, H. D. Wahlquist, C. G. Behr J. Math. Phys. 9, 497 (1968)

[3] A. Ashtekar, J. Samuel *Bianchi cosmologies: The role of spatial topology*, Class. Quantum Grav. 8, 2191-2215, 1991

[4] F. W. Warner *Foundations of Differentiable Manifolds and Lie-Groups* Springer New York Berlin 1983
[5] C. B. Collins *Global structure of the “Kantowski-Sachs” cosmological models*, J. Math. Phys, Vol. 18, No. 11, 1977

[6] M. A. H. MacCallum *Anisotropic and inhomogeneous relativistic Cosmologies* in *An Einstein Centenary Survey* Ed. S. W. Hawking, W. Israel, Cambridge University Press, Cambridge 1979

[7] N. Straumann *Allgemeine Relativittstheorie und relativistische Astrophysik* Springer Berlin, Heidelberg 1988

[8] O. Coussaert, M. Henneaux, *Bianchi Cosmological Models and Gauge Symmetries*, Class. Quantum Grav. 10 1993, 1607-1617

[9] R. S. Palais, *The Principle of Symmetric Criticality*, Commun. Math. Phys. 69, 19-30, 1979

[10] R. Abraham, J. E. Marsden *Foundations of Mechanics*, Benjamin/ Cummings, Reading Massachusetts 1978

[11] R. T. Jantzen *Variational Principles in Cosmology*, Il Nuovo Cimento, Vol. 55 B, N. 2, 161-171, 1980

[12] G. E. Sneddon *Hamiltonian cosmology: a further investigation*, J. Phys. A, Vol 9, No. 2, 1976, 229-238

[13] P. A. M. Dirac *Lectures of Quantum Mechanics* Yeshiva University Press, New York 1964