PRODUCT INNOVATION, PROCESS INNOVATION AND ADVERTISING-BASED GOODWILL: A DYNAMIC ANALYSIS IN A MONOPOLY

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ABSTRACT. In this paper, we develop a dynamic control model to investigate a monopolist’s investment strategies in product innovation, process innovation and advertising-based goodwill. The significant features of our study are: (i) considering the effect of product quality on goodwill; (ii) considering the instantaneous cost of producing a quality using machinery and/or skilled labour; (iii) the customers’ demand function depends on product quality, product price and goodwill in a separable multiplicative way between the state variables and control variables. Our results suggest that (i) the system admits unique saddle-point steady-state equilibrium under the monopolist optimum and the social optimum; (ii) and the monopolist will have an underinvestment problem as compared with the social planner; and (iii) although the product price is still determined by the monopolist under the social planner optimum, the product price is higher under the monopolist optimum than that under the social planner optimum.

1. Introduction. In this paper, we develop a dynamic control model to analyze a monopolist’s investments in product innovation, process innovation and advertising-based goodwill. In fact, our research relates to two streams of literature: The first stream focuses on the studies of the product and process innovation. In this stream of literature, the following authors’ works are closely related to ours, such as [8, 10, 16, 1, 14, 4, 2, 5, 6, 7, 15, 3] just to mention a few. Among them, it is worth mentioning a series of papers published by [5, 6, 7], where the authors analyse the optimal behaviour of the firms’ investments both in product and process innovation in a dynamic setting. The second stream focuses on the studies of advertising-based goodwill and the effect of product innovation on goodwill. In this stream of literature, the following authors’ papers are of particular importance, such as [11, 13, 4, 9]. Among them, [13] analyze the optimal advertising and quality

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improvement (product innovation) decisions by duopolistic firms competing in a dynamic setting; [4] views quality improvement (product innovation) as an important influential factor on the goodwill and production cost.

This paper bridges the aforementioned two streams of research and investigates optimal decision behaviour of a monopolist’s investments in product innovation, process innovation and advertising-based goodwill in a dynamic setting. In our work, the instantaneous cost function depends on the investments in advertisement, process and product innovation as well as the instantaneous cost of producing a quality level using machinery and/or skilled labour operating at decreasing returns; and the customers’ demand function depends on product quality, product price and goodwill in a separable multiplicative way between the state and control variables.

The remainder of this paper is organized as follows. In section 2 we develop a basic model. In sections 3 and 3 we investigate the steady state equilibrium under monopolist optimum and social optimum. In section 5 we provide a numerical comparison. Section 6 concludes the paper. All proofs are provided in the appendix.

2. The basic model. In this paper, we consider a dynamic control problem over continuous time \( t \in [0, \infty) \), where at any instant a single-product monopolist not only can invest in product and process innovation via instantaneous investments \( k(t) \) and \( h(t) \) to improve the product quality \( q(t) \) and lower the marginal production cost \( c(t) \), but also can invest in advertising via an instantaneous investment \( s(t) \) to increase goodwill \( g(t) \). According to [7, 12] the dynamics of product quality \( q(t) \) and marginal production cost \( c(t) \) as well as the goodwill \( g(t) \) are described by the following differential equations, respectively:

\[
\dot{q}(t) = k(t) - \delta q(t) \tag{1}
\]
\[
\dot{c}(t) = -h(t) + \sigma c(t) \tag{2}
\]
\[
\dot{g}(t) = s(t) - \mu g(t) \tag{3}
\]

where the parameter \( \delta \in [0, 1] \) represents the decay rate of product quality; the parameter \( \sigma \in [0, 1] \) indicates the obsolescence rate affecting production technology; the parameter \( \mu \in [0, 1] \) be the decay rate of goodwill due to customers’ forgetting. Accordingly, we assume that the instantaneous costs of investing in product and process innovation as well as in goodwill are given by \( \alpha k^2(t) \), \( \beta h^2(t) \), and \( \xi s^2(t) \) respectively, where \( \alpha, \beta, \) and \( \xi \) are positive parameters.

In reality, though the goodwill should increase with the monopolist’s advertising effort, product quality should also play an important role in accumulating the goodwill. As such, we assume that the product quality is a goodwill-building factor, in the sense that a product of higher quality should enjoy a higher reputation (conversely, a product with lower quality should damage its reputation). After introducing product quality as a goodwill-building factor, we modify the well-known Nerlove and Arrow model [12] to describe the dynamics of goodwill as follows:

\[
\dot{g}(t) = s(t) + \theta q(t) - \mu g(t) \tag{4}
\]

where the parameter \( \theta \in [0, 1] \) measures the marginal contribution of the product quality on goodwill.

Now, we consider the problem of customers’ demand function \( D(t) \). We assume that the customers’ demand function \( D(t) \) depends on product quality \( q(t) \), product price \( p(t) \), and goodwill \( g(t) \) in a separable multiplicative way between state and control variables, i.e., \( D(t) = f(q(t), g(t))u(p(t)) \), where \( f(q(t), g(t)) \) and
goodwill by investing in product innovation and advertising, respectively. Specifically, the quality-goodwill function is assumed to be linearly separable additive, i.e., $f(q(t), g(t)) = a_1 q(t) + a_2 g(t)$, where $a_1 q(t)$ denotes the improved quality arising from product innovation, $a_2 g(t)$ denotes the improved sales arising from the goodwill, $a_1$ and $a_2$ are positive constants. Besides, the price function is supposed to be linearly decreasing, i.e., $u(p(t)) = a_0 - a_3 p(t)$, where $a_0$ and $a_3$ are positive constants. Thus, the customers’ demand function $D(t)$ is expressed as

$$D(t) = [a_1 q(t) + a_2 g(t)][a_0 - a_3 p(t)]$$

(5)

In what follows, we analyze the problem of the monopolist’s total cost function. In this paper, we assume that there is no stock, and all the production is sold. Then, at time $t \in [0, \infty)$, the monopolist’s total cost function $C(t)$ can be written by the following form:

$$C(t) = c(t) D(t) + \alpha k^2(t) + \beta h^2(t) + \xi s^2(t) + \nu q^2(t)$$

$$= [a_1 q(t) + a_2 g(t)][a_0 - a_3 p(t)]c(t) + \alpha k^2(t) + \beta h^2(t) + \xi s^2(t) + \nu q^2(t)$$

(6)

In the expression (6), the term $\nu q^2(t)$ measures the instantaneous cost of producing a quality level $q(t)$ using machinery and/or skilled labour operating at decreasing returns at time $t$, where $\nu$ be a positive parameter [7].

Thus, once product innovation, process innovation and advertisement take place, the monopolist’s instantaneous profit function $\pi(t)$ is given by:

$$\pi(t) = [p(t) - c(t)][a_1 q(t) + a_2 g(t)][a_0 - a_3 p(t)]$$

$$- [\alpha k^2(t) + \beta h^2(t) + \xi s^2(t) + \nu q^2(t)]$$

(7)

Next sections, we will investigate the optimal conditions and steady state equilibrium under monopolist optimum and social optimum over continuous time $t \in [0, \infty)$, respectively.

3. The monopolist optimum. Under the monopolist optimum, the monopolist’s problem is to determine the optimal levels of $p(t)$, $k(t)$, $h(t)$ and $s(t)$, such that the discounted profit flow $\Pi$ is maximized over continuous time $t \in [0, \infty)$, which can be written as follows:

$$\Pi = \max_{p,k,h,s} \int_0^\infty e^{-rt} \{[p(t) - c(t)][a_1 q(t) + a_2 g(t)][a_0 - a_3 p(t)]$$

$$- [\alpha k^2(t) + \beta h^2(t) + \xi s^2(t) + \nu q^2(t)]\} dt$$

$$S.t \begin{cases} 
\dot{q}(t) = k(t) - \delta q(t) \\
\dot{c}(t) = -h(t) + \sigma c(t) \\
\dot{g}(t) = s(t) + \theta q(t) - \mu g(t)
\end{cases}$$

(8)

where $r$ be the risk-free discount rate, and the initial conditions are $q(0) = q_0$, $c(0) = c_0$, and $g(0) = g_0$. 

u(p(t)) reflect the effects of product quality $q(t)$ and goodwill $g(t)$, and product price $p(t)$ on the customers’ demand function $D(t)$, respectively. This demand function states that the monopolist not only can affect demand function by controlling product price, but also can affect demand function through product quality and goodwill by investing in product innovation and advertising, respectively.
Accordingly, we can write the current value Hamiltonian function $H^m$ for the optimal control problem (8) as follows:

$$H^m = [p(t) - c(t)][a_1 q(t) + a_2 g(t)][a_0 - a_3 p(t)] - [\alpha k^2(t) + \beta h^2(t) + \xi s^2(t) + \nu q^2(t)]$$

$$+ \lambda_1(t)[k(t) - \delta q(t)] + \lambda_2(t)[-h(t) + \sigma c(t)] + \lambda_3(t)[s(t) + \theta q(t) - \mu g(t)]$$

(9)

where $\lambda_i(t)$ be the dynamic co-state variables associated with their state equations $\dot{q}(t)$, $\dot{c}(t)$, and $\dot{g}(t)$, respectively, $i = 1, 2, 3$.

From the monopolist’s Hamiltonian function (9), one can obtain the following first-order conditions and dynamics of co-state equations, namely,

$$\frac{\partial H}{\partial k(t)} = \lambda_1(t) - 2\alpha k(t) = 0$$

(10)

$$\frac{\partial H}{\partial h(t)} = -\lambda_2(t) - 2\beta h(t) = 0$$

(11)

$$\frac{\partial H}{\partial s(t)} = \lambda_3(t) - 2\xi s(t) = 0$$

(12)

$$\frac{\partial H}{\partial p(t)} = [a_1 q(t) + a_2 g(t)][a_0 - a_3 p(t)] - a_3[p(t) - c(t)][a_1 q(t) + a_2 g(t)] = 0$$

(13)

$$\dot{\lambda}_1(t) = r \lambda_1(t) - \frac{\partial H}{\partial q(t)} = (r + \delta) \lambda_1(t) - \theta \lambda_3(t) + 2\nu q(t) - a_1[p(t) - c(t)][a_0 - a_3 p(t)]$$

(14)

$$\dot{\lambda}_2(t) = r \lambda_2(t) - \frac{\partial H}{\partial c(t)} = (r - \sigma) \lambda_3(t) + [a_1 q(t) + a_2 g(t)][a_0 - a_3 p(t)]$$

(15)

$$\dot{\lambda}_3(t) = r \lambda_3(t) - \frac{\partial H}{\partial g(t)} = (r + \mu) \lambda_3(t) - a_2[p(t) - c(t)][a_0 - a_3 p(t)]$$

(16)

where the associated transversality conditions are $\lim_{t \to \infty} \lambda_1(t) q(t) e^{-rt} = 0$, $\lim_{t \to \infty} \lambda_2(t) q(t) e^{-rt} = 0$, and $\lim_{t \to \infty} \lambda_3(t) q(t) e^{-rt} = 0$.

From the expression (13), we have the following product pricing condition:

$$p(t) = \frac{a_0 + 3a_3 c(t)}{2a_3}$$

(17)

In the static setting, expression (17) is a well-known product pricing condition. This condition has to hold at every point in time $t$, so that at the optimum, variation in marginal revenue must equal the corresponding variation in marginal cost.

Now, we investigate the dynamic equations with respect to $k(t)$, $h(t)$, and $s(t)$. Substituting the expressions (10)-(12) into (14)-(16) respectively, using expression (17), and rearranging, one can obtain

$$\dot{k}(t) = (r + \delta)k(t) - \frac{\xi q(t)}{\alpha} s(t) + \frac{\mu}{\alpha} q(t) - \frac{a_1}{8a_3} [a_0 - a_3 c(t)]^2$$

(18)

$$\dot{h}(t) = (r - \sigma)h(t) - \frac{1}{4\beta} [a_1 q(t) + a_2 g(t)][a_0 - a_3 c(t)]$$

(19)

$$\dot{s}(t) = (r + \mu)s(t) - \frac{a_2}{8\xi a_3} [a_0 - a_3 c(t)]^2$$

(20)

From the dynamic differential equations (18)-(20), we have the following Proposition 1:
Proposition 1. In the case of monopolist optimum, all things being equal, \( \forall t \in [0, \infty) \), we have: (i) \( \frac{\partial k(t)}{\partial h(t)} < 0 \); (ii) \( \frac{\partial s(t)}{\partial s(t)} < 0 \); (iii) \( \frac{\partial h(t)}{\partial k(t)} < 0 \); (iv) \( \frac{\partial h(t)}{\partial s(t)} < 0 \); (v) \( \frac{\partial h(t)}{\partial h(t)} = 0 \); (vi) \( \frac{\partial s(t)}{\partial h(t)} < 0 \).

Proof. See Appendix A.

From proposition 1-(i) and (ii) we see that the change rate of investment of a monopolist’s product innovation \( \dot{k}(t) \) is decrease with its investments of process innovation \( h(t) \) and advertising \( s(t) \), respectively; from proposition 1-(iii) and (iv) we find that the change rate of investment of a monopolist’s product innovation \( \dot{h}(t) \) is decrease with its investments of product innovation \( k(t) \) and advertising \( s(t) \), respectively; from proposition 1-(v) and (vi) we find that the change rate of investment of a monopolist’s advertising \( \dot{s}(t) \) is decrease with its investment of process innovation \( h(t) \), while the investment of product innovation \( k(t) \) has no effect on the change rate of investment of advertising \( \dot{s}(t) \).

Now, we investigate the complementarity (substitutability) relationship among the monopolist’s investments in product innovation, process innovation, and advertisement at any time \( t \in [0, \infty) \), besides, we examine the effects of \( \theta \) (marginal contribution of the product quality on goodwill) on the complementarity (substitutability) relationship among the investments of a monopolist’s product innovation, process innovation, and advertisement (in this paper, the marginal contribution of the product quality on goodwill \( \theta \) is a key parameter, with which one can analyze the effects of product quality on goodwill). To this effect, one can define following auxiliary variables \( \Lambda_1 = k(t)h(t), \Lambda_2 = k(t)s(t), \) and \( \Lambda_3 = h(t)s(t) \) as indicators of the substitutability \( (\Lambda_1 < 0, \Lambda_2 < 0, \text{and } \Lambda_3 < 0) \) or complementary \( (\Lambda_1 > 0, \Lambda_2 > 0, \text{and } \Lambda_3 > 0) \) relationship among the investments of a monopolist’s product innovation, process innovation, and advertisement at any time \( t \in [0, \infty) \), where \( \dot{k}(t), \dot{h}(t), \text{and } \dot{s}(t) \) are given by the dynamic equations (18)-(20), respectively. Accordingly, we have the following Proposition 2:

Proposition 2. In the case of monopolist optimum, all things being equal, \( \forall t \in [0, \infty) \), as the marginal contribution of the product quality on goodwill \( \theta \) rises, (i) the investments of a monopolist’s product innovation and process innovation tend to be substitutable (or complementary) if and only if \( \dot{h}(t) > 0 \) (or \( \dot{h}(t) < 0 \)); (ii) the investments of a monopolist’s product innovation and advertising tend to be substitutable (or complementary) if and only if \( \dot{s}(t) > 0 \) (or \( \dot{s}(t) < 0 \)); (iii) the investments of a monopolist’s process innovation and advertising tend to be substitutable (or complementary) if and only if \( \dot{s}(t) > 0 \) (or \( \dot{s}(t) < 0 \)).

Proof. See Appendix B.

In this paper, our main issue is to determine whether the monopolist’s investment strategies lead to steady-state equilibrium, and further evaluate the stability properties of the dynamic system. One can prove the following Proposition 3:

Proposition 3. In the case of monopolist optimum, there exist admissible parameter constellations such that the steady-state equilibrium \( \{k^m, h^m, s^m, q^m, c^m, g^m\} \) is an unique saddle-point equilibrium, in which, \( q^m = \frac{k^m}{\delta} = M[a_0 - a_3c^m]^2, \quad c^m = \frac{h^m}{\sigma} = \frac{1}{\delta^3}[a_0 - \frac{3}{2} \sqrt{\frac{B}{7}} + \frac{2}{\sqrt{\frac{B}{7}}} + (\frac{A}{3})^3] - \frac{3}{2} \sqrt{\frac{B}{7}} - 2(\frac{B}{7})^2 + (\frac{A}{3})^3], \quad g^m = \frac{1}{\mu}(s^m + \theta q^m) = \)
\[
\frac{1}{\rho}[N(a_0 - a_3c^m)^2 + \theta q^m], \quad M = \frac{a_2 \xi \theta + a_1 \xi (r+\mu)}{a_2 (a_1 \mu + \theta) \xi (r+\mu) + a_2 (r+\mu) - \sigma \xi}, \quad N = \frac{a_2}{a_2 (a_1 \mu + \theta) \xi (r+\mu) + a_2 (r+\mu) - \sigma \xi}, \quad A = \frac{a_2}{a_2 (a_1 \mu + \theta) \xi (r+\mu) + a_2 (r+\mu) - \sigma \xi}, \quad B = \frac{a_2}{a_2 (a_1 \mu + \theta) \xi (r+\mu) + a_2 (r+\mu) - \sigma \xi}.
\]

Proof. See Appendix C.

Besides, under steady-state equilibrium, one can obtain the price
\[p^m = \frac{a_0 + a_3c^m}{2a_3}\] (21)

Next section, we will take into account the social welfare associated with the monopoly equilibrium.

4. The social planner optimum. In the remainder of the analysis clarified in this section, we will consider a situation in which, a benevolent social planner is in charge of choosing optimal investment paths for product innovation, process innovation and advertising to maximise the discounted social welfare flow at the same discount factor \(r\) used by the monopolist. It is worth noting that, in the case of social planner optimum, we assume that the price is still determined by the monopolist, namely, the price \(p(t)\) is still given by the expression (17). Now, we define the social planner’s instantaneous social welfare function as \(sw(t) = \pi(t) + cs(t)\), where \(\pi(t)\) be the monopolist’s instantaneous profit at continuous time \(t \in [0, \infty)\), which is given by the expression (7), while \(cs(t)\) measures the consumer surplus at continuous time \(t \in [0, \infty)\), which is given by
\[
cs(t) = \int_p^{a_0/a_3} \{[a_1 q(t) + a_2 g(t)][a_0 - a_3 z(t)]\} dz
= \frac{1}{2a_3} [a_1 q(t) + a_2 g(t)][a_0 - a_3 p(t)]^2
\] (22)

According to monopolist’s instantaneous profit function (7) and above consumer surplus function (22), and using expression (17), one can yield the social planner’s instantaneous welfare function \(sw(t)\), namely
\[
sw(t) = \frac{3}{8a_3}[a_1 q(t) + a_2 g(t)][a_0 - a_3 c(t)]^2
- [\alpha k^2(t) + \beta h^2(t) + \xi s^2(t) + \nu q^2(t)]
\] (23)

Now, the social planner’s problem is to determine the optimal levels of \(k(t)\), \(h(t)\), and \(s(t)\), such that the discounted social welfare flow \(SW\) is maximized over continuous time \(t \in [0, \infty)\), that is
\[
SW = \max_{k,h,s} \int_0^\infty e^{-rt} \{\frac{3}{8a_3}[a_1 q(t) + a_2 g(t)][a_0 - a_3 c(t)]^2
- [\alpha k^2(t) + \beta h^2(t) + \xi s^2(t) + \nu q^2(t)]\} dt
\]
\[
S.t \begin{cases}
q(t) = k(t) - \delta q(t) \\
n(t) = \sigma c(t) \\
g(t) = s(t) + \theta q(t) - \mu g(t)
\end{cases}
\] (24)

where \(r\) be the risk-free discount rate, and the initial conditions are \(q(0) = q_0, c(0) = c_0,\) and \(g(0) = g_0\).
The current value Hamiltonian function $H^p$ for the optimal control problem (24) can be written as
\[
H^p = \frac{3}{8a_3} [a_1 q(t) + a_2 g(t)] [a_0 - a_3 c(t)]^2 - [\alpha k^2(t) + \beta h^2(t) + \xi s^2(t) + \nu q^2(t)] + \vartheta_1(t)[k(t) - \delta q(t)] + \vartheta_2(t)[-h(t) + \sigma c(t)] + \vartheta_3(t)[s(t) + \theta q(t) - \mu g(t)]
\]
(25)
where $\vartheta_i(t)$ be the co-state variables associated with their state equations $\dot{q}(t), c(t)$, and $\dot{s}(t)$, respectively, $i = 1, 2, 3$.

From the current value Hamiltonian function (25), one can obtain the following first-order conditions and dynamics of co-state equations:
\[
\frac{\partial H}{\partial k(t)} = \vartheta_1(t) - 2\alpha k(t) = 0 \tag{26}
\]
\[
\frac{\partial H}{\partial h(t)} = -\vartheta_2(t) - 2\beta h(t) = 0 \tag{27}
\]
\[
\frac{\partial H}{\partial s(t)} = \vartheta_3(t) - 2\xi s(t) = 0 \tag{28}
\]
\[
\dot{\vartheta}_1(t) = r\vartheta_1(t) - \frac{\partial H}{\partial q(t)} = (r + \delta)\vartheta_1(t) - \theta\vartheta_3(t) + 2\nu q(t) - \frac{3a_1}{8a_3} [a_0 - a_3 c(t)]^2 \tag{29}
\]
\[
\dot{\vartheta}_2(t) = r\vartheta_2(t) - \frac{\partial H}{\partial c(t)} = (r - \sigma)\vartheta_2(t) + \frac{3}{4} [a_1 q(t) + a_2 g(t)] [a_0 - a_3 c(t)] \tag{30}
\]
\[
\dot{\vartheta}_3(t) = r\vartheta_3(t) - \frac{\partial H}{\partial g(t)} = (r + \mu)\vartheta_3(t) - \frac{3a_2}{8a_3} [a_0 - a_3 c(t)]^2 \tag{31}
\]
where the associated transversality conditions are $\lim_{t \to \infty} \vartheta_1(t) q(t) e^{-rt} = 0$, $\lim_{t \to \infty} \vartheta_2(t) q(t) e^{-rt} = 0$, and $\lim_{t \to \infty} \vartheta_3(t) q(t) e^{-rt} = 0$. Substituting the expressions (26)-(28) into (29)-(31), respectively, using product pricing condition (17), and rearranging, one can obtain the following dynamic equations with respect to $k(t), h(t)$, and $s(t)$, namely,
\[
\dot{k}(t) = (r + \delta)k(t) - \frac{\xi \theta}{\alpha} s(t) + \frac{\nu}{\alpha} q(t) - \frac{3a_1}{8a_3} [a_0 - a_3 c(t)]^2 \tag{32}
\]
\[
\dot{h}(t) = (r - \sigma)h(t) - \frac{3}{4} [a_1 q(t) + a_2 g(t)] [a_0 - a_3 c(t)] \tag{33}
\]
\[
\dot{s}(t) = (r + \mu)s(t) - \frac{3a_2}{8\xi a_3} [a_0 - a_3 c(t)]^2 \tag{34}
\]
From differential equations (32)-(34), we have the following Proposition 4:

**Proposition 4.** In the case of social optimum, all things being equal, $\forall t \in [0, \infty)$, we have: (i) $\frac{\partial k(t)}{\partial t} < 0$; (ii) $\frac{\partial h(t)}{\partial t} < 0$; (iii) $\frac{\partial h(t)}{\partial t} < 0$; (iv) $\frac{\partial s(t)}{\partial t} < 0$; (v) $\frac{\partial s(t)}{\partial t} > 0$.

**Proof.** See Appendix D. □

The interpretation of Proposition 4 is just analogous to the Proposition 1. Similar to the case of monopolist optimum, under social optimum, we also can define auxiliary variables $\Omega_1 = \dot{k}(t)\dot{h}(t), \Omega_2 = \dot{k}(t)\dot{s}(t), \Omega_3 = \dot{h}(t)\dot{s}(t)$ to investigate the effects of $\theta$ (marginal contribution of the product quality on goodwill) on the complementarity (substitutability) relationship among the investments of a monopolist’s
product innovation, process innovation, and advertisement. We have the following Proposition 5:

**Proposition 5.** In the case of social optimum, ∀t ∈ [0, ∞), as the marginal contribution of the product quality on goodwill θ rises, (i) the investments of a monopolist’s product innovation and process innovation tend to be substitutable (complementary) only when h(t) > 0 (h(t) < 0); (ii) the investments of a monopolist’s product innovation and advertising tend to be substitutable (complementary) only when s(t) > 0 (s(t) < 0); (iii) the investments of a monopolist’s process innovation and advertising tend to be substitutable (complementary) if and only if s(t) > 0 (s(t) < 0).

**Proof.** See Appendix E.

Now, we analyse the steady-state equilibrium and the features of steady-state equilibrium in the setting that the social planner maximizes its own social welfare. Given the same initial and transversality conditions as in the previous section. The outcome is summarised by the following proposition 6:

**Proposition 6.** In the case of social optimum, the steady-state equilibrium \{k^p, h^p, s^p, q^p, c^p, g^p\} is an unique saddle-point equilibrium, in which \( q^p = \frac{1}{\sigma}h^p = \frac{1}{a_3}[a_0 - \sqrt{-\frac{U}{2} + \frac{1}{2}(\frac{U}{2})^2 + (\frac{\sigma}{2})^2}] \), \( g^p = \frac{1}{\mu}(s^p + \theta q^p) = \frac{1}{\mu}[N(a_0 - a_3 c^p)^2 + \theta q^p] \), where \( M = \frac{a_2(\beta + \alpha \xi)}{a_3(\beta + \alpha \xi)(r + \mu)} \) .

**Proof.** See Appendix F.

Besides, the equilibrium price under social planner optimum is given by

\[
p^p = \frac{a_0 + a_3 c^p}{2a_3}
\]

(35)

So far, our investigation focuses on the steady-state equilibrium and the saddle-point property of the dynamic control system under optimal monopolist behavior and optimal social planner behavior, respectively. However, according to the steady-state equilibrium presented in Proposition 3 and Proposition 6, it is difficult to determine whether the investments of a monopolist’s product innovation, process innovation, and advertisement could be lower under the monopolist optimum than that under the social planner optimum. Next section, we will provide a numerical analysis between steady state equilibrium under monopolist optimum and that under social planner optimum.

5. **Comparative analysis.** Substituting the numerical values of parameters in the table 1 in Appendix into expressions of Proposition 3 and Proposition 6, respectively, one can obtain

\[
\{k^m, h^m, s^m, q^m, c^m, g^m\} \approx \{5.424, 3.833, 3.292, 4.622, 2.355, 4.424\}
\]

(36)

\[
\{k^p, h^p, s^p, q^p, c^p, g^p\} \approx \{5.986, 4.021, 3.668, 5.456, 2.724, 5.212\}
\]

(37)

From above expressions (36) and (37) we find that \( c^m > c^p \), hen according to expressions (21) and (35), it is easy to yield that \( p^m > p^p \). Further, we have the following Observation 1:
Observation 1. (i) the investments of a monopolist’s product innovation, process innovation and advertising are lower \( (k^m < k^p, h^m < h^p, s^m < s^p) \) under the monopolist optimum than that under the social optimum; (ii) accordingly, the product quality and goodwill are lower \( (q^m < q^p, g^m < g^p) \) under the monopolist optimum than that under the social optimum; while the marginal production cost is higher \( (c^m > c^p) \) under the monopolist optimum than that under the social optimum; and (iii) although the product price is still determined by the monopolist under the social planner optimum, the product price is higher under the monopolist optimum than that under the social planner optimum \( (p^m > p^p) \).

6. Conclusions. The problem of simultaneous decision-making of a monopolist’s product innovation, process innovation and advertising-based goodwill is crucial for manufacturing and high-tech industries. In this paper, we develop an optimal control model to analyze a monopolist’s investment strategies in process innovation, product innovation and advertising-based goodwill. Our paper can be viewed as an extension and continuation of [7] work. In our work, we not only consider the effect of product quality on goodwill, but also assume that the customers’ demand function depends on product quality, product price, and goodwill in a separable multiplicative way between state and control variables. Our main results can be summarized as follows: (i) the system admits saddle-point steady-state equilibrium under the monopolist optimum and the social optimum, respectively; (ii) the effect of product innovation on advertising-based goodwill can affect substitutability/complementary relationship between product innovation and process innovation; (iii) under the social incentive, the investments in product innovation, process innovation, and advertising are always larger than the private incentive characterising the profit-seeking monopolist; and (iv) although the price is still determined by the monopolist under the social planner optimum, the price is higher under the monopolist optimum than that under the social planner optimum.

Further research can be made on the investments in the product innovation, process innovation and advertising-based goodwill by using data from firm-level evidence. In addition, possible extension can also be made on the investment strategies of the product and process innovation as well as advertising-based goodwill, in which investment activities for the product and process innovation generate a positive spillover for the rival in a in a dynamic duopoly.

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Thus we have that
\[ \frac{\partial \Lambda(t)}{\partial \theta} = -\frac{\xi s(t)}{\alpha} \hat{h}(t), \]
which implies that \( \frac{\partial \Lambda_1}{\partial \theta} \propto -\hat{h}(t) \). Recall that \( \theta \) is the marginal contribution of product quality on goodwill. Thus, one can deduce that as \( \theta \) rises, the investments of a monopolist’s
product innovation and process innovation tend to be substitutes (or complements) if and only if \( \dot{h}(t) > 0 \) (or \( \dot{h}(t) < 0 \)).

(ii) differentiating the auxiliary variable \( \Lambda_2 = \dot{k}(t)\dot{s}(t) \) with respect to \( \theta \), it is easy to verify that \( \frac{\partial \Lambda_2}{\partial \theta} = -\frac{\xi(t)}{\alpha} \dot{s}(t) \), which implies that \( \frac{\partial \Lambda_2}{\partial \theta} \propto -\dot{s}(t) \). Thus, we can obtain that as \( \theta \) rises, the investments of a monopolist’s product innovation and advertising tend to be substitutes (or complements) if and only if \( \dot{s}(t) > 0 \) (or \( \dot{s}(t) < 0 \)).

(iii) similarly, differentiating the auxiliary variable \( \Lambda_3 = \dot{h}(t)\dot{s}(t) \) with respect to \( \theta \), we can verify that \( \frac{\partial \Lambda_3}{\partial \theta} = -\frac{\alpha_2 g(t)}{4\beta} [a_0 - a_3 c(t)] \dot{s}(t) \), which implies that \( \frac{\partial \Lambda_3}{\partial \theta} \propto -\dot{s}(t) \). Thus, we can yield that as \( \theta \) rises, the investments of a monopolist’s process innovation and advertising tend to be substitutes (or complements) if and only if \( \dot{s}(t) > 0 \) (or \( \dot{s}(t) < 0 \)).

Finishing the proof.

**Appendix C. Proof of Proposition 3:** From the differential equations (1)-(3) and (18)-(20), one can obtain the following system of differential equations:

\[
\begin{align*}
\dot{k}(t) &= (r + \delta)k(t) - \frac{\xi \theta}{\alpha} s(t) + \frac{\nu}{\alpha} q(t) - \frac{a_1}{8\alpha a_3} [a_0 - a_3 c(t)]^2 \\
\dot{h}(t) &= (r - \sigma)h(t) - \frac{\alpha_1 q(t) + \alpha_2 g(t)}{4\beta} [a_0 - a_3 c(t)] \\
\dot{s}(t) &= (r + \mu)s(t) - \frac{a_2}{8\xi a_3} [a_0 - a_3 c(t)]^2 \\
\dot{q}(t) &= k(t) - \delta q(t) \\
\dot{c}(t) &= -h(t) + \sigma c(t) \\
\dot{\gamma}(t) &= s(t) + \theta q(t) - \mu g(t)
\end{align*}
\]

Solving the differential equations system (38) under steady-state conditions \( \dot{k}(t) = \dot{h}(t) = \dot{s}(t) = \dot{q}(t) = \dot{c}(t) = \dot{\gamma}(t) = 0 \) gives steady-state equilibrium \( \{k^m, h^m, s^m, q^m, c^m, g^m\} \), where \( q^m = \frac{\alpha^m}{\alpha} = M[a_0 - a_3 c^m]^2, c^m = \frac{\alpha^m}{\alpha} = \frac{1}{\alpha_3} [a_0 - \sqrt{\frac{\beta}{2} + \sqrt{(\frac{\beta}{2})^2 + \frac{4}{3} - \sqrt{(\frac{\beta}{2})^2 + 2(\frac{\beta}{2})}]} \right] \), \( g^m = \frac{1}{\mu} (s^m + \theta q^m) = \frac{1}{\mu} [N(a_0 - a_3 c^m)^2 + \theta q^m] \), where \( M = \frac{a_2 \theta + a_1 (r + \mu)}{8\xi a_3 (r + \mu)}, N = \frac{2a_2 \theta}{32\beta \xi a_2 (r + \mu) ([r + \delta] \alpha \delta + \nu [r - \sigma] \sigma)}, A = \frac{32 \beta \xi a_2 (r + \mu) ([r + \delta] \alpha \delta + \nu [r - \sigma] \sigma)}{32 \beta \xi a_2 (r + \mu) ([r + \delta] \alpha \delta + \nu [r - \sigma] \sigma)}, B = \frac{a_3 (a_1 \mu + \theta) [a_2 \theta + a_1 \xi (r + \mu)] + a_2^2 [(r + \delta) \alpha \delta + \nu]}{a_3 (a_1 \mu + \theta) [a_2 \theta + a_1 \xi (r + \mu)] + a_2^2 [(r + \delta) \alpha \delta + \nu]} \right] \).

Now, we analyze the stability properties of the steady-state equilibrium \( \{k^m, h^m, s^m, q^m, c^m, g^m\} \). Denoting by \( \Sigma^m = \frac{\partial (k, h, s, q, c, g)}{\partial (k, h, s, q, c, g)} \) the Jacobian matrix of the system (38) at the steady-state \( \{k^m, h^m, s^m, q^m, c^m, g^m\} \), namely,

\[
\Sigma^m = \begin{bmatrix}
\frac{r + \delta}{\alpha} & 0 & -\frac{\alpha_2}{\alpha} & \frac{\alpha_1}{\alpha} & \frac{a_1}{\alpha} & \frac{\alpha_1}{\alpha} \\
0 & \frac{r - \sigma}{\alpha} & 0 & \frac{a_1}{\alpha} & \frac{a_1}{\alpha} & \frac{a_2}{\alpha} \\
0 & 0 & \frac{r + \mu}{\alpha} & 0 & \frac{\alpha_1}{\alpha} & \frac{a_1}{\alpha} \\
1 & 0 & 0 & -\delta & 0 & 0 \\
0 & 0 & 0 & \sigma & 0 & 0 \\
0 & 0 & 0 & \theta & 0 & -\mu
\end{bmatrix}
\]

Let \( \{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6\} \) and \( E \) denote the eigenvalues and unit matrix of the Jacobian matrix (39), respectively. Since the characteristic equation \( |\Sigma^m - \xi E| = 0 \)
(i = 1, 2, 3, 4, 5, 6) yields six eigenvalues, and unfortunately, assessing analytically the sign of the six eigenvalues of the characteristic equation is not feasible as their expressions are cumbersome. However, we may resort to numerical calculations. The numerical values of parameters used in the numerical calculations are presented in table 1.

Table 1. The parameters used in the numerical examples

| r  | a₀  | a₁  | a₂  | a₃  | µ   | δ   | α   | β   | γ₀  | θ   | σ   | q₀  | c₀  | ξ     | ν  |
|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-------|----|
| 0.06| 10  | 0.5 | 0.3 | 0.2 | 0.2 | 0.03| 7.2 | 5.3 | 20  | 0.05| 0.02| 3   | 15  | 3.9   | 9.2|

Solving the characteristic equation \(|\Sigma^m - \xi_i E| = 0\) by using the numerical values of parameters in the table 1, one can obtain

\(\{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6\} = \{0.982, -0.915, -0.016 + 0.244i, -0.016 - 0.244i, 0.233, -0.148\} (40)\)

From expression (40) we see that \(\xi_2\) and \(\xi_6\) are negative, \(\xi_1\) and \(\xi_5\) are positive, and \(\xi_3\) and \(\xi_4\) are complex numbers with negative real parts (Re(\(\xi_i\)) < 0 and Re(\(\xi_i\)) < 0). Consequently, it appears that the steady state equilibrium \(\{k^m, h^m, s^m, g^m, c^m, g^m\}\) is stable in the saddle point sense given the parameter range where the equilibrium values of states and controls are indeed economically acceptable. Accordingly, we can obtain that, in the case of monopolist optimum, there exist admissible parameter constellations such that the steady state equilibrium \(\{k^m, h^m, s^m, g^m, c^m, g^m\}\) is a saddle point equilibrium.

Finishing the proof.

Appendix D. Proof of Proposition 4: (i) differentiating the dynamic equation (32) with respect to \(h(t)\), we have \(\frac{\partial k(t)}{\partial h(t)} = \frac{3a_1a_3}{4a_3}[a_0 - a_3c(t)]\frac{\partial h(t)}{\partial h(t)}\). Using dynamic equation (2), one can get \(\frac{\partial c(t)}{\partial h(t)} = \frac{\partial}{\partial h(t)}\int_0^t \dot{q}_i(s)ds = -t < 0\). Since \(\frac{3a_1a_3}{4a_3}[a_0 - a_3c(t)] > 0\), one can immediately derive \(\frac{\partial k(t)}{\partial h(t)} < 0\).

(ii) differentiating the dynamic equation (32) with respect to \(s(t)\), one can immediately obtain \(\frac{\partial k(t)}{\partial s(t)} = -\frac{E^0}{2t} < 0\).

(iii) differentiating the dynamic equation (33) with respect to \(k(t)\), one can yield \(\frac{\partial h(t)}{\partial k(t)} = -\frac{1}{2t}[a_1 \frac{\partial q(t)}{\partial k(t)} + a_2 \frac{\partial q(t)}{\partial k(t)}] [a_0 - a_3c(t)]\). Using dynamic equations (1) and (3), one obtains: \(\frac{\partial q(t)}{\partial k(t)} = \frac{\partial}{\partial k(t)} \int_0^t \dot{q}_i(s)ds = t > 0\), \(\frac{\partial q(t)}{\partial k(t)} = \frac{\partial}{\partial q(t)} \int_0^t \dot{q}_i(s)ds = \frac{\partial}{\partial q(t)} \frac{\partial q(t)}{\partial k(t)} = \frac{\partial}{\partial q(t)} \frac{\partial}{\partial q(t)} \frac{\partial}{\partial k(t)} = \beta t > 0\), respectively. Thus we have \(\frac{\partial h(t)}{\partial k(t)} < 0\).

(iv) we can write: \(\frac{\partial h(t)}{\partial s(t)} = \frac{\partial h(t)}{\partial s(t)} \frac{\partial s(t)}{\partial t} = t > 0\). From dynamic equation (33), one yields \(\frac{\partial h(t)}{\partial q(t)} = -\frac{3a_2}{4t} [a_0 - a_3c(t)] < 0\), so we have \(\frac{\partial h(t)}{\partial q(t)} < 0\).

(v) differentiating the dynamic equation (34) with respect to \(k(t)\), one can immediately derive \(\frac{\partial s(t)}{\partial k(t)} = 0\).

(vi) we can write \(\frac{\partial h(t)}{\partial s(t)} = \frac{\partial}{\partial s(t)} \frac{\partial h(t)}{\partial s(t)} = \frac{\partial}{\partial s(t)} \frac{\partial h(t)}{\partial s(t)} = -t < 0\). From dynamic equation (34), one yields \(\frac{\partial h(t)}{\partial q(t)} = \frac{3a_2a_3}{4a_3}[a_0 - a_3c(t)] > 0\), so we have \(\frac{\partial h(t)}{\partial q(t)} < 0\).

Finishing the proof.
Appendix E. Proof of Proposition 5: using the auxiliary variables $\Omega_1 = \dot{k}(t)\dot{h}(t)$, $\Omega_2 = \dot{k}(t)\dot{s}(t)$ and $\Omega_3 = \dot{h}(t)\dot{s}(t)$, we have

(i) $\frac{\partial \Omega_1}{\partial \theta} = -\frac{\xi}{\alpha} \dot{s}(t)$, which implies that $\frac{\partial \Omega_1}{\partial \theta} \propto -\dot{h}(t)$. Thus, one can deduce that as $\theta$ rises, product innovation and process innovation tend to be substitutes (complements) if and only if $\dot{h}(t) > 0$ ($\dot{h}(t) < 0$).

(ii) $\frac{\partial \Omega_2}{\partial \theta} = \frac{\xi}{\alpha} \dot{s}(t)$, which implies that $\frac{\partial \Omega_2}{\partial \theta} \propto -\dot{s}(t)$. Thus, we can obtain that as $\theta$ rises, product innovation and advertising tend to be substitutes (complements) if and only if $\dot{s}(t) > 0$ ($\dot{s}(t) < 0$).

(iii) $\frac{\partial \Omega_3}{\partial \theta} = -\frac{3\alpha}{4\beta} \dot{q}(t)[a_0 - a_3c(t)]\dot{s}(t)$, implies that $\frac{\partial \Omega_3}{\partial \theta} \propto -\dot{s}(t)$. Thus, we can yield that as $\theta$ rises, process innovation and advertising tend to be substitutes (complements) if and only if $\dot{s}(t) > 0$ ($\dot{s}(t) < 0$).

Finishing the proof.

Appendix F. Proof of Proposition 6: From differential equations (1)-(3) and (32)-(34), one can obtain the following system of differential equations:

\[
\begin{align*}
\dot{h}(t) &= (r-\sigma)h(t) - \frac{1}{4\beta}[a_1q(t) + a_2g(t)][a_0 - a_3c(t)] \\
\dot{s}(t) &= (r+\mu)s(t) - \frac{3\alpha}{8\Omega_3}[a_0 - a_3c(t)]^2 \\
\dot{q}(t) &= -h(t) + \sigma c(t) \\
\dot{g}(t) &= s(t) + \theta q(t) - \mu g(t)
\end{align*}
\]  

(41)

Solving the differential equations system (41) under steady-state conditions $\dot{k}(t) = \dot{h}(t) = \dot{s}(t) = \dot{q}(t) = \dot{c}(t) = \dot{g}(t) = 0$, one can get the steady state equilibrium $\{k^*, h^*, s^*, g^*, c^*, p^*, q^*, c^*\}$, where, $k^* = \delta q^*$, $h^* = \sigma c^*$, $s^* = N(a_0 - a_3c^2)^2$, $c^* = \frac{1}{a_0}[a_0 - \frac{3}{2} \left(-\frac{U}{2} + \sqrt{\frac{Z}{3}} \right)^2 + \frac{Z}{3} - \frac{3}{2} \left(-\frac{U}{2} - \sqrt{\frac{Z}{3}} \right)^2 + \left(\frac{U}{2} + \sqrt{\frac{Z}{3}} \right)^2 + \frac{Z}{3}], g^* = \frac{1}{\mu} (s^* + \theta g^*), M = \frac{a_2(g^* + a_1(\xi + \frac{\sigma}{\mu})))}{8\Omega_3(r+\mu)}[(r+\sigma)/(r+\mu)], N = \frac{a_2}{8\Omega_3(r+\mu)}$, $Z = \frac{2\Omega_3(a_2(r+\sigma))}{a_2r^3(a_0 - a_3c^2)^2}$.

Now, we analyze the stability properties of the steady-state equilibrium $\{k^*, h^*, s^*, q^*, c^*, g^*\}$. Denoting by $\Sigma^m = \frac{\partial(k, h, s, q, c, g)}{\partial(k, h, s, q, c, g)}$ the Jacobian matrix of the system (41) at the steady-state $\{k^*, h^*, s^*, q^*, c^*, g^*\}$, namely,

\[
\Sigma^m = \begin{bmatrix}
 r + \delta & 0 & -\frac{\xi}{\alpha} & 0 & -\frac{3\alpha}{4 \Omega_3} (a_0 - a_3c^2) & 0 \\
 0 & r - \sigma & 0 & -\frac{3\alpha}{4 \Omega_3} (a_0 - a_3c^2) & 0 & -\frac{3\alpha}{4 \Omega_3} (a_0 - a_3c^2) \\
 0 & 0 & r + \mu & -\frac{\xi}{\alpha} & 0 & -\frac{3\alpha}{4 \Omega_3} (a_0 - a_3c^2) \\
 0 & 0 & 0 & 0 & -\delta & 0 \\
 0 & -\mu & 0 & \theta & 0 & \sigma \\
 0 & 0 & 1 & \theta & 0 & -\mu
\end{bmatrix}
\]  

(42)

Let $\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6\}$ and $E$ denote the eigenvalues and unit matrix of the Jacobian matrix (42), respectively. Since the characteristic equation $|\Sigma^m - \phi_i E| = 0$ by using the numerical values of parameters in the table 1, one can obtain

$\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6\} = \{0.933, -0.921, -0.019 + 0.253i, -0.019 - 0.253i, 0.242, -0.155\}$  

(43)
From expression (43) we see that $\phi_2$ and $\phi_6$ are negative, $\phi_1$ and $\phi_5$ are positive, and $\phi_3$ and $\phi_4$ are complex numbers with negative real parts ($\text{Re}(\phi_3) < 0$ and $\text{Re}(\phi_4) < 0$). Accordingly, we can obtain that, in the case of monopolist optimum, there exist admissible parameter constellations such that the steady state equilibrium $\{k^p, h^p, s^p, q^p, c^p, g^p\}$ is a saddle point equilibrium.

Finishing the proof.

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