Random Distances Associated with Trapezoids

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Abstract

The distributions of the random distances associated with hexagons, rhombuses and triangles have been derived and verified in the existing work. All of these geometric shapes are related to each other and have various applications in wireless communications, transportation, etc. Hexagons are widely used to model the cells in cellular networks, while trapezoids can be utilized to model the edge users in a cellular network with a hexagonal tessellation. In this report, the distributions of the random distances associated with unit trapezoids are derived, when two random points are within a unit trapezoid or in two neighbor unit trapezoids. The mathematical expressions are verified through simulation. Further, we present the polynomial fit for the PDFs, which can be used to simplify the computation.

Index Terms

Random distances; distance distribution functions; trapezoids

I. THE PROBLEM

Denote a “unit trapezoid” as an isosceles trapezoid where the legs have the same length and equal to the length of the short base, 1, and the base angles are equal to \( \frac{\pi}{3} \). The length of the long base is 2. The distributions of the random distances between two points located within a unit trapezoid and in two neighbor unit trapezoids are of our interest. Several cases are considered depending on the arrangement of the trapezoids. As illustrated in Fig. 1, points A and B constitute the case where the two random points are located within a unit trapezoid, while CD, EF, and GH represent the cases where the two random points are located in two neighbor unit trapezoids. For CD, two neighbor unit trapezoids form a “unit hexagon” with side length
1. Two neighbor trapezoids share a leg for EF. In the last case, GH, two trapezoids share the short base, forming a concave polygon.

We derive the probability density functions (PDFs) and cumulative distribution functions (CDFs) of the 4 cases in the following section.

II. DISTANCE DISTRIBUTIONS ASSOCIATED WITH UNIT TRAPEZOIDS

A. PDF and CDF of the Random Distances within a Unit Trapezoid (Case |AB|)

A unit trapezoid in fact consists of three adjacent equilateral triangles. Since the distributions of the random distances within and between triangles are known [1], the PDF of the distances between two uniformly at random located points within a unit trapezoid can be derived using
the probabilistic sum, as follows.

\[
\begin{align*}
f_{D_{AB}}(d) &= 2d \\
&= \begin{cases} 
\frac{8}{9} \left( \frac{\pi}{9\sqrt{3}} + \frac{2}{3} \right) d^2 - \frac{80}{27} d + \frac{4\pi}{3\sqrt{3}} & 0 \leq d \leq \frac{\sqrt{3}}{2} \\
\frac{8}{3} \left( \frac{2}{9\sqrt{3}} d^2 + \frac{1}{\sqrt{3}} \right) \sin^{-1} \frac{\sqrt{3}}{2d} + \frac{16}{9} \left( \frac{1}{3} - \frac{\pi}{9\sqrt{3}} \right) d^2 \\
&\quad + \frac{28}{27} \sqrt{4d^2 - 3} - \frac{80}{27} d & \frac{\sqrt{3}}{2} \leq d \leq 1 \\
\frac{16}{9} \left( \frac{1}{3} d^2 + 2 \right) \sin^{-1} \frac{\sqrt{3}}{d} + \frac{4}{9} \left( \frac{1}{3} - \frac{4\pi}{9\sqrt{3}} \right) d^2 \\
&\quad + \frac{16}{9} \sqrt{d^2 - 3} - \frac{32}{27} d - \frac{32\pi}{27\sqrt{3}} & \sqrt{3} \leq d \leq 2 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

(1)

The corresponding CDF is

\[
\begin{align*}
F_{D_{AB}}(d) &= \begin{cases} 
0 & d < 0 \\
\frac{4}{9} \left( \frac{2}{3} + \frac{\pi}{9\sqrt{3}} \right) d^4 - \frac{160}{27} d^3 + \frac{4\pi}{3\sqrt{3}} d^2 & 0 \leq d \leq \frac{\sqrt{3}}{2} \\
\frac{8}{27\sqrt{3}} (d^2 + 9) d^2 \sin^{-1} \frac{\sqrt{3}}{2d} + \frac{8}{9} \left( \frac{1}{3} - \frac{\pi}{9\sqrt{3}} \right) d^4 \\
&\quad - \frac{160}{81} d^3 + \frac{58d^2+15}{81} \sqrt{4d^2 - 3} & \frac{\sqrt{3}}{2} \leq d \leq 1 \\
-\frac{8}{27\sqrt{3}} (d^2 - 6) d^2 \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{8\pi}{81\sqrt{3}} - \frac{2}{27} \right) d^4 \\
&\quad - \frac{64}{81} d^3 + \left( \frac{8\pi}{27\sqrt{3}} - \frac{4}{9} \right) d^2 \\
&\quad + \frac{22d^2+15}{81} \sqrt{4d^2 - 3} + 0.0735 & 1 \leq d \leq \sqrt{3} \\
\frac{8}{27\sqrt{3}} (d^2 + 12) d^2 \sin^{-1} \frac{\sqrt{3}}{d} + \frac{2}{9} \left( \frac{1}{3} - \frac{4\pi}{9\sqrt{3}} \right) d^4 \\
&\quad - \frac{64}{81} d^3 - \frac{32\pi}{27\sqrt{3}} d^2 + \frac{104d^2+48}{81} \sqrt{d^2 - 3} \\
&\quad + 0.4074 & \sqrt{3} \leq d \leq 2 \\
1 & d > 2
\end{cases}
\end{align*}
\]

(2)
B. PDF and CDF of the Distances between Two Uniformly at Random Located Points in Neighbor Unit Trapezoids (Case $|CD|$)

In this case, the two neighbor trapezoids form a hexagon. Since the distributions of the random distances within a unit hexagon have been derived and verified in [2], the PDF of the distances between two uniformly at random located nodes in two neighbor unit trapezoids can be derived as

$$f_{D_{CD}}(d) = 2d \begin{cases} 
-\frac{4}{27} \left(1 + \frac{5\pi}{3\sqrt{3}}\right) d^2 + \frac{32}{27}d & 0 \leq d \leq \frac{\sqrt{3}}{2} \\
-\frac{1}{3\sqrt{3}} \left(\frac{16}{9}d^2 + 8\right) \sin^{-1} \frac{\sqrt{3}}{2d} + \frac{4}{27} \left(\frac{\pi}{3\sqrt{3}} - 1\right) d^2 & \frac{\sqrt{3}}{2} \leq d \leq 1 \\
- \frac{28}{27} \sqrt{4d^2 - 3} + \frac{32}{27}d + \frac{4\pi}{3\sqrt{3}} & 1 \leq d \leq \sqrt{3} \\
-\frac{4}{9\sqrt{3}} \left(\frac{8}{3}d^2 + 10\right) \sin^{-1} \frac{\sqrt{3}}{2d} + \frac{4}{27} \left(\frac{5\pi}{3\sqrt{3}} + 1\right) d^2 & \sqrt{3} \leq d \leq 2 \\
\frac{8}{9\sqrt{3}} \left(\frac{1}{3}d^2 + 8\right) \sin^{-1} \frac{\sqrt{3}}{d} - \frac{8}{27} \left(\frac{\pi}{3\sqrt{3}} + 2\right) d^2 & \sqrt{3} \leq d \leq 2 \\
+ \frac{8}{3} \sqrt{d^2 - 3} + \frac{32}{27}d - \frac{64\pi}{27\sqrt{3}} - \frac{8}{3} & \sqrt{3} \leq d \leq 2 \\
0 & \text{otherwise}
\end{cases}$$

(3)
The corresponding CDF is

\[
F_{D_{CD}}(d) = \begin{cases} 
0 & d < 0 \\
\frac{2}{27} \left(\frac{5\pi}{3\sqrt{3}} + 1\right) d^4 + \frac{64}{81} d^3 & 0 \leq d \leq \frac{\sqrt{3}}{2} \\
-\frac{8}{27\sqrt{3}} (d^2 + 9)d^2 \sin^{-1} \frac{\sqrt{3}}{d} + \frac{2}{27} \left(\frac{\pi}{3\sqrt{3}} - 1\right) d^4 & \frac{\sqrt{3}}{2} \leq d \leq 1 \\
\frac{8}{27\sqrt{3}} (2d^2 + 15)d^2 \sin^{-1} \frac{\sqrt{3}}{2d} + \frac{2}{27} \left(\frac{5\pi}{3\sqrt{3}} + 1\right) d^4 & 1 \leq d \leq \sqrt{3} \\
\frac{4}{27\sqrt{3}} (d^2 + 48)d^2 \sin^{-1} \frac{\sqrt{3}}{d} - \frac{4}{27} \left(\frac{\pi}{3\sqrt{3}} + 2\right) d^4 & \sqrt{3} \leq d \leq 2 \\
1 & d > 2 
\end{cases}
\]

\[(4)\]

C. PDF and CDF of the Distances between Two Uniformly at Random Located Points in Neighbor Unit Trapezoids (Case $|EF|$)

Knowing the PDF of the random distances between rhombuses [3] and within a unit trapezoid, the PDF of the random distances between two uniformly distributed points, one in each of the neighbor unit trapezoids in this case, is
\[
f_{DEF}(d) = \begin{cases} 
- \frac{4}{27} \left( \frac{2\pi}{3\sqrt{3}} + 1 \right) d^2 + \frac{16}{27} d & 0 \leq d \leq \frac{\sqrt{3}}{2} \\
- \frac{8}{9\sqrt{3}} \left( \frac{2}{3} d^2 + 1 \right) \sin^{-1} \frac{\sqrt{3}}{2d} + \frac{4}{27} \left( \frac{4\pi}{3\sqrt{3}} - 1 \right) d^2 + \frac{16}{27} d + \frac{4\pi}{9\sqrt{3}} - \frac{4}{9} \sqrt{3} \sin^{-1} \frac{\sqrt{3}}{d} - \frac{4}{9} \sqrt{3} d^2 - 3 & \frac{\sqrt{3}}{2} \leq d \leq 1 \\
\frac{8}{9\sqrt{3}} \left( \frac{1}{3} d^2 + 1 \right) \sin^{-1} \frac{\sqrt{3}}{2d} - \frac{4}{27} \left( \frac{2\pi}{3\sqrt{3}} + 1 \right) d^2 + \frac{10}{27} \sqrt{4d^2 - 3} - \frac{4\pi}{27\sqrt{3}} - \frac{2}{9} & 1 \leq d \leq \sqrt{3} \\
\frac{8}{9\sqrt{3}} \left( \frac{1}{3} d^2 - 2 \right) \sin^{-1} \frac{\sqrt{3}}{2d} - \frac{8}{9\sqrt{3}} \left( \frac{1}{3} d^2 + 2 \right) \sin^{-1} \frac{\sqrt{3}}{d} + \frac{4}{27} \left( \frac{\pi}{3\sqrt{3}} + 2 \right) d^2 - \frac{8}{9} \sqrt{d^2 - 3} - \frac{14}{27} \sqrt{4d^2 - 3} + \frac{32\pi}{27\sqrt{3}} + \frac{10}{9} & \sqrt{3} \leq d \leq 2 \\
\frac{8}{9\sqrt{3}} \left( \frac{1}{3} d^2 - 2 \right) \sin^{-1} \frac{\sqrt{3}}{2d} + \frac{16}{27} \sin^{-1} \frac{\sqrt{3}}{d} - \frac{4}{27} \left( \frac{\pi}{3\sqrt{3}} - 1 \right) d^2 - \frac{14}{27} \sqrt{4d^2 - 3} + \frac{16}{27} \sqrt{d^2 - 3} + \frac{2}{9} & 2 \leq d \leq \sqrt{7} \\
\frac{8}{9\sqrt{3}} \left( -\frac{1}{3} d^2 + 4 \right) \sin^{-1} \frac{\sqrt{3}}{d} + \frac{4}{27} \left( \frac{\pi}{3\sqrt{3}} - 1 \right) d^2 + \frac{8}{9} \sqrt{d^2 - 3} - \frac{16\pi}{27\sqrt{3}} - \frac{8}{9} & \sqrt{7} \leq d \leq 2\sqrt{3} \\
0 & \text{otherwise} 
\end{cases}
\]
The corresponding CDF is

\[
F_{D_EF}(d) = \begin{cases} 
0 & d < 0 \\
-\frac{2}{27} \left( \frac{2\pi}{3\sqrt{3}} + 1 \right) d^4 + \frac{32}{81} d^3 & 0 \leq d \leq \frac{\sqrt{3}}{2} \\
-\frac{8}{27\sqrt{3}} (d^2 + 3) d^2 \sin^{-1} \frac{\sqrt{3}}{2d} + \frac{2}{27} \left( \frac{4\pi}{3\sqrt{3}} - 1 \right) d^4 & \frac{\sqrt{3}}{2} < d \leq 1 \\
\frac{4}{27\sqrt{3}} (d^2 + 6) d^2 \sin^{-1} \frac{\sqrt{3}}{2d} - \frac{2}{27} \left( \frac{2\pi}{3\sqrt{3}} + 1 \right) d^4 & 1 \leq d \leq \sqrt{3} \\
\frac{4}{27\sqrt{3}} d^2 \left( (d^2 - 12) \sin^{-1} \frac{\sqrt{3}}{2d} -(d^2 + 12) \sin^{-1} \frac{\sqrt{3}}{d} \right) & \sqrt{3} \leq d \leq 2 \\
-\frac{2}{27} \left( \frac{\pi}{3\sqrt{3}} - 1 \right) d^4 + \frac{2}{9} d^2 + \frac{32d^2+48}{162} \sqrt{d^2 - 3} - 0.3528 & 2 \leq d \leq \sqrt{7} \\
-\frac{4}{27\sqrt{3}} (d^2 - 24) d^2 \sin^{-1} \frac{\sqrt{3}}{d} + \frac{2}{27} \left( \frac{\pi}{3\sqrt{3}} - 1 \right) d^4 & \sqrt{7} \leq d \leq 2\sqrt{3} \\
-\frac{8}{9} \left( \frac{2\pi}{3\sqrt{3}} + 1 \right) d^2 + \frac{44d^2+120}{81} \sqrt{d^2 - 3} - 1.6677 & d > 2\sqrt{3} \\
1 & \end{cases}
\]
D. PDF and CDF of the Distances between Two Uniformly at Random Located Points in Neighbor Unit Trapezoids (Case \(|GH|\))

The PDF of the random distances between two nodes uniformly deployed in two neighbor unit trapezoids (case \(|GH|\)), is

\[
f_{DGH}(d) = 2d \begin{cases} 
\frac{4}{27} \left( \frac{\pi}{3\sqrt{3}} - 1 \right) d^2 + \frac{16}{27} d & 0 \leq d \leq \frac{\sqrt{3}}{2} \\
\frac{8}{9\sqrt{3}} \left( \frac{2}{3} d^2 - 1 \right) \sin^{-1} \frac{\sqrt{3}}{2d} - \frac{4}{27} \left( \frac{5\pi}{3\sqrt{3}} + 1 \right) d^2 & \frac{\sqrt{3}}{2} \leq d \leq 1 \\
- \frac{4}{27} \sqrt{4d^2 - 3} + \frac{16}{27} d + \frac{4\pi}{9\sqrt{3}} & 1 \leq d \leq \sqrt{3} \ . \ \\
\frac{8}{9\sqrt{3}} \left( \frac{1}{3} d^2 - 2 \right) \sin^{-1} \frac{\sqrt{3}}{2d} - \frac{8}{9\sqrt{3}} \left( \frac{1}{3} d^2 - 2 \right) \sin^{-1} \frac{\sqrt{3}}{d} & \sqrt{3} \leq d \leq \sqrt{7} \\
+ \frac{8}{27} \left( \frac{\pi}{3\sqrt{3}} - 1 \right) d^2 + \frac{14}{27} \sqrt{4d^2 - 3} + \frac{8}{27} \sqrt{d^2 - 3} \ \\
- \frac{16\pi}{27\sqrt{3}} - \frac{10}{9} & \sqrt{3} \leq d \leq \sqrt{7} \\
0 & \text{otherwise}
\end{cases}
\]
The corresponding CDF is

\[ F_{D\perp}(d) = \begin{cases} 
0 & d < 0 \\
\frac{2}{27} \left(\frac{\pi}{3\sqrt{3}} - 1\right) d^4 + \frac{32}{81} d^3 & 0 \leq d \leq \frac{\sqrt{7}}{2} \\
\frac{8}{27\sqrt{3}} (d^2 - 3) d^2 \sin^{-1} \frac{\sqrt{3}}{2d} - \frac{2}{27} \left(\frac{5\pi}{3\sqrt{3}} + 1\right) d^4 + \frac{32}{81} d^3 + \frac{4\pi}{9\sqrt{3}} d^2 - \frac{26d^3}{27} \sqrt{4d^2 - 3} & \frac{\sqrt{7}}{2} \leq d \leq 1 \\
\frac{4}{27\sqrt{3}} (d^2 - 18) d^2 \sin^{-1} \frac{\sqrt{3}}{2d} - \frac{2}{9} \left(\frac{\pi}{3\sqrt{3}} - 1\right) d^4 + \frac{2}{3} \left(\frac{14\pi}{9\sqrt{3}} + 1\right) d^2 - \frac{86d^2 + 39}{162} \sqrt{4d^2 - 3} + 0.0176 & 1 \leq d \leq \sqrt{3} \\
-\frac{4}{27\sqrt{3}} (d^2 - 12) d^2 \left(\sin^{-1} \frac{\sqrt{7}}{2d} + \sin^{-1} \frac{\sqrt{3}}{d}\right) + \frac{4}{27} \left(\frac{\pi}{3\sqrt{3}} - 1\right) d^4 - \frac{2}{9} \left(\frac{8\pi}{3\sqrt{3}} + 5\right) d^2 + \frac{54d^2 + 27}{162} \sqrt{4d^2 - 3} + \frac{12d^2 + 72}{81} \sqrt{d^2 - 3} & \sqrt{3} \leq d \leq \sqrt{7} \\
1 & d > \sqrt{7}
\end{cases} \]

The results presented above are for the “unit” trapezoids. However, the results can be easily extended for scaled trapezoids. Assume each side of the unit trapezoid is scaled by a non-negative parameter \( s \), then,

\[ F_{sD}(d) = P(sD \leq d) = P(D \leq \frac{d}{s}) = F_{D\perp}(\frac{d}{s}). \]

Therefore,

\[ f_{sD}(d) = F'_{D\perp}(\frac{d}{s}) = \frac{1}{s} f_{D\perp}(\frac{d}{s}). \]

III. Verification

Figure 2 demonstrates the PDF of the random distances associated with trapezoids, including the random distances within a unit trapezoid and between neighbor unit trapezoids.

Figure 3 shows the comparison between the simulation and analytical results of the CDFs. The simulation is done in Matlab, generating 10,000 pairs of random points. As the figure suggests, there is a close match between the mathematical and simulation results, which verifies the accuracy of our analytical results.
Figure 2 shows the PDFs of the random distances within a unit trapezoid as well as between two neighbor unit trapezoids, along with the fitted polynomial functions. The polynomial coefficients for the PDFs of the random distances associated with trapezoids, along with their norm of residual, are presented in Table I. In Fig. 4, the approximated polynomial functions are shown with dots over the original PDF curves. One can observe that the polynomial functions
TABLE I: Coefficients of the Polynomial Fit and the Norm of Residuals (NormR)

| PDF     | Polynomial Coefficients                                      | NormR |
|---------|-------------------------------------------------------------|-------|
| $f_{D_{AB}}(d)$ | $[-6.3419 \ 72.4436 \ -356.68 \ 988.25 \ -1686.9 \ 1824.8 \ -1239.4 \ 504.30 \ -109.17 \ 9.6850 \ -5.1833 \ 4.7245 \ 0.0021]$ | 0.1537 |
| $f_{D_{CD}}(d)$ | $[10.55774 \ -124.22 \ 632.96 \ -1829.6 \ 3302.5 \ -3864.6 \ 2953.0 \ -1452.6 \ 443.86 \ -80.1282 \ 9.6421 \ -0.27620.0024]$ | 0.1692 |
| $f_{D_{EF}}(d)$ | $[0.0196 \ -0.3853 \ 3.2716 \ -15.6480 \ 46.2235 \ -87.0513104.33e \ -77.1478324560 \ -6.2059 \ 0.6906 \ -0.0154]$ | 0.2405 |
| $f_{D_{GH}}(d)$ | $[0.6286 \ -16.1105 \ 1.9174 \times 10^4 \ -1.4062 \times 10^4 \ 7.1067 \times 10^3 \ -2.6205 \times 10^4 \ 7.2760 \times 10^4 \ -1.5469 \times 10^5 \ 2.5338 \times 10^5 \ -3.1899 \times 10^5 \ 3.0528 \times 10^5 \ -2.1736 \times 10^5 \ 1.1081 \times 10^5 \ -3.7549 \times 10^4 \ 6.9028 \times 10^3 \ 41.6579 \ -3.2855 \times 10^2 \ 69.7237 \ -4.9233 \ 0.2052 \ -0.0016]$ | 0.1622 |

demonstrate a good match with the original PDFs. The polynomial functions are accurate and can be used instead of the original PDFs to reduce the computational complexity.

V. CONCLUSIONS

In this report, the closed-form expressions for the random distances within a unit trapezoid and between two neighbor unit trapezoids are given. The analytical results are verified through simulation. In addition, the polynomial fits for the PDFs are obtained. These polynomials offer a good fit and can be used instead of the original PDFs to reduce the computational complexity.

ACKNOWLEDGMENT

This work is supported in part by the NSERC, CFI and BCKDF. The authors would like to thank Lei Zhang, Tianming Wei and Fei Tong for their help.

REFERENCES

[1] Y. Zhuang and J. Pan, “Random Distances Associated with Equilateral Triangles,” arXiv:1207.1511, 2012.
[2] Y. Zhuang and J. Pan, “Random Distances Associated with Hexagons,” arXiv:1106.2200, 2011.
[3] Y. Zhuang and J. Pan, “Random Distances Associated with Rhombuses,” arXiv:1106.1257, 2011.
Fig. 4: Polynomial Fits.