NEW COPULAS AND THEIR APPLICATIONS TO SYMMETRIZATIONS OF BIVARIATE COPULAS

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Abstract
New copulas, based on perturbation theory, are introduced to clarify a symmetrization procedure for asymmetric copulas. We give also some properties of the symmetrized copula. Finally, we examine families of copulas with a prescribed symmetrized one. By the way, we study topologically, the set of all symmetric copulas and give some of its classical and new properties.

Introduction
The notion of non ex-changeability was first introduced by Hollander [5] to quantify the asymmetry of factors which describe a given phenomena. It was baptized asymmetry by Nelsen in [7], [8] and [9], Mesiar and Klement adopted the same nomenclature in [6], so all other researchers in this emergent field of copulas pursue how to detect, to measure and sometimes to avoid the asymmetry. In a general context, the ex-changeability relies on the asymmetry procedures with non commutative effect in different directions. In chemistry, for example, the dosage acid/base and reciprocally base/acid do not lead to same pH—aqueous solution. In economical point of view, the respective combination of C(k, l) of k and l units of capital and labor do not ensure the same utility as the C(l, k) ensures. Other examples abound in many fields to show the importance of asymmetry. A notion closely related to the topic is the radial symmetry as treated first in [8] and developed later in [11].

For statistical applications, multiple regression theory leads in a general framework to estimation of parameters as critical points of first order condition. For the simple case $Y = X\beta + \epsilon$ when some classical hypotheses are satisfied, an estimator of $\beta$ is given by $\hat{\beta} = (XX^t)^{-1}XY$. For a symmetric matrix $X$, the calculus become easier since, in adapted basis, $(XX^t)^{-1} = X^2$. The concept of symmetrization consists in, as it will be developed below, a classical and simple decomposition of a copula that governs the joint distribution of random variables. Copulas gained a great interest in recent decades because of their relatively simple use in statistics and probability. Their importance comes back to the historical Sklar’s theorem (see [1], [2] or the
unavoidable Nelsen’s book [8]). The Sklar result presents the copula as a natural bridge between margin distributions of random vector and its joint law. A recent and topological new proof of this important result was given by Durante et al. [10]. In the first section of the current paper, many results on copulas that we will need are given as preliminaries. We also definitely make precise notations and recall some recent developments on asymmetry. We recall some results on classical dependence parameters expressed in terms of copulas, see [13].

The second section is devoted to define the symmetrized and the radial symmetrized copulas. We give some properties linking a given copula and its symmetrized and radial symmetrized ones in the same spirit of [12].

In the last section, we study the inverse problem by giving new copulas as solutions of some functional equations. The new copulas are based on an asymmetric perturbation of the independence copula Π. More precise, for a given symmetric or radial symmetric copula $S$, we determine copulas for which $S$ is the prescribed symmetrized or the radial symmetrized part.

Along this paper, $I$ denotes the closed interval $[0, 1]$ and $\| \cdot \|$ the usual uniform norm on the set $C$ of all bivariate copulas. Let $S$ and $R$ denote respectively its subsets of all symmetric and radial symmetric copulas.

1. Preliminaries

Definition 1.1. A copula $C$ is a bifunction on $I^2$ into $I$ which satisfies the following conditions for all $u, v, u_1, v_1, u_2, v_2$ in $I$

1. Border conditions: $C(0, v) = C(u, 0) = 0$.
2. Uniform margins: $C(1, v) = v$ and $C(u, 1) = u$.
3. the C-volume property: $V_C(R) = C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$ for all rectangle $R = [u_1, v_1] \times [u_2, v_2] \subset I^2$ with $u_1 < u_2$ and $u_2 < v_2$.

Any element of $C$ is framed between Fréchet-Hoeffding bounds $W$ and $M$ given by $W(u, v) = \max(u + v - 1, 0)$ and $M(u, v) = \min(u, v)$. Precisely, we have

$$\forall (x, y) \in I^2: \quad W(x, y) \leq C(x, y) \leq M(x, y).$$

These bounds ($M$ and $W$) are also copulas but in higher dimensions, say for multivariate copulas, $W$ is not a copula.

Statistically speaking, Fréchet-Hoeffding bounds $M$ and $W$ model respectively the co-monotonicity and anti-monotonicity of empirical variables $X$ and $Y$. The copula $\Pi: \{x, y\} \in I^2 \mapsto xy$ characterizes the total independence between the two variables.

On the other hand, some derived copulas from a given one will serve as efficient tool to treat the asymmetry questions mainly those of transpose copula, survival one.

Definition 1.2. The survival copula of a given copula $C$ is the function $\hat{C}$ defined by the formula

$$\forall u, v \in I: \quad \hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$$

we denote by $Q$ ([8], pages 159-182) the difference of two probabilities $Q = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0)$ for a pair of continuous random vectors $(X_1, X_2)$ and
(Y_1, Y_2). If the corresponding copulas are C_1 and C_2, then we have

\[ Q(C_1, C_2) = 4 \int I_2 C_1(u, v) \, dC_2(u, v) - 1 \]

The usual four measures of concordance may be defined in terms of the concordance function \( Q \).

Kendall’s tau of \( C \) is defined by

\[ \tau(C) = Q(C, C) \]

Spearman’s rho by

\[ \rho(C) = 3Q(C, \Pi) \]

Gini’s gamma by

\[ \gamma(C) = Q(C, M) + Q(C, W) \]

Blomqvist’s Beta by

\[ \beta(C) = 4C \left( \frac{1}{2}, \frac{1}{2} \right) - 1. \]

**Theorem 1.3.** Let \( X \) and \( Y \) be continuous random variables with distribution functions \( F \) and \( G \), respectively, and let \( C \) be the copula of \( X \) and \( Y \). If these limits exist, then the upper and lower tail dependence are given as:

\[ \lambda_U = 2 - \lim_{t \rightarrow 1^-} \frac{1 - C(t, t)}{1 - t} \quad \text{and} \quad \lambda_L = \lim_{t \rightarrow 0^+} \frac{C(t, t)}{t}. \]

2. Asymmetry and radial asymmetry of bivariate copulas

**Definition 2.4.** A bivariate copula \( C \) is said:

- **Symmetric if** it satisfies
  \[ \forall u, v \in I : \quad C(u, v) = C(v, u). \]
- **Radial symmetric if** it satisfies
  \[ \hat{C} = C. \]

We denote \( C^T \) the transpose of \( C \) defined by \( C^T(u, v) = C(v, u) \).

**Example 2.5.**
1. The bounds \( M \) and \( W \) of Fréchet-Hoeffding and the independence copula \( \Pi \) are symmetric and also radial symmetric.
2. Let \( \alpha \) and \( \beta \) be in \((0, 1)\). The Marshall Olkin family of copulas

\[ C_{\alpha,\beta}(u, v) = \min(u^{1-\alpha}v, uv^{1-\beta}) \]

is asymmetric if \( \alpha \neq \beta \).

**Proposition 2.6.** The subsets \( S \) and \( R \) are both convex and closed in \( C \) with respect to the uniform convergence.

**Proof.** Let \((C_n)\) be a sequence of symmetric copulas which converges uniformly to a copula \( C \). We show that \( C \) is also symmetric. For all \( u,v \in I \), we have:

\[ C(v, u) = \lim_{n \rightarrow +\infty} C_n(v, u) = \lim_{n \rightarrow +\infty} C_n(u, v) = C(u, v) \]
Let now \((C_n)\) be a sequence in \(\mathcal{R}\) which converges uniformly to a copula \(C\).

\[
C(u,v) = \lim_{n \to +\infty} C_n(u,v) = \lim_{n \to +\infty} \hat{C}_n(u,v) = \hat{C}(u,v)
\]

If \(C_1, C_2 \in \mathcal{S}\) and \(\lambda\) is a real number in \(I\), then easily \(\lambda C_1 + (1 - \lambda)C_2\) belongs to \(\mathcal{S}\).

If \(C_1, C_2 \in \mathcal{R}\) and \(\lambda\) is a real number in \(I\), then \(\lambda C_1 + (1 - \lambda)C_2 \in \mathcal{R}\).

Proposition 2.7.

- The projection of any bivariate copula \(C\) respect to \(\mathcal{S}\) is the symmetric copula \(C_S = \frac{C + C^T}{2}\).
- The projection of any bivariate copula \(C\) respect to \(\mathcal{R}\) is the radial symmetric copula \(C_R = \frac{C + \hat{C}}{2}\).

Proof. We show that \(\frac{C + C^T}{2}\) is the closest symmetric copula to \(C\).

Let \(S\) be any symmetric copula. Since

\[
\forall (u,v) \in I^2, \quad |C_1(u,v) - S(u,v)| = |C^T(v,u) - S^T(v,u)| = |C^T(v,u) - S(v,u)|.
\]

we have \(\|C - S\| = \|S - C^T\|\). And this equality gives

\[
\|C - C^T\| \leq \|C - S\| + \|S - C^T\| \leq 2\|C - S\|
\]

so

\[
\|\frac{C - C^T}{2}\| \leq \|C - S\|
\]

A similar proof can be given to the second point.

Remark 2.8. A measures \(\mu\) of asymmetry (\(\nu\) of radial asymmetry) of a copula \(C\) can be defined naturally as the distance between \(C\) and \(\mathcal{S}\) (\(C\) and \(\mathcal{R}\)). So

\[
\mu(C) = \|\frac{C - C^T}{2}\|
\]

and respectively

\[
\nu(C) = \|\frac{C - \hat{C}}{2}\|
\]

In the sequel, we give some obvious but important consequences of this splitting procedure of copulas to (radial) symmetric and (radial) asymmetric parts.

Proposition 2.9.

1. Let \(C\) be a copula. We have

\[
(C^T)_S = C_S, \quad (\hat{C})_R = C_R, \quad \hat{C}_S = \left(\hat{C}\right)_S, \quad (C_R)^T = (C^T)_R
\]

2. The copulas \(C\) and \(C_S\) have same Spearman \(\rho\), same Gini’s gamma and same Blomqvist’s beta. But not necessarily same Kendhal’s tau.

3. The copulas \(C\) and \(C_R\) have same Spearman \(\rho\), same Gini’s gamma and same Blomqvist’s beta. But not necessarily same Kendhal’s tau.

4. If a copulas \(C\) have tail dependence parameters, then \(C_S\) have same upper and lower tails dependence parameter than \(C\). Meaning:

\[
\lambda_L(C) = \lambda_L(C_S) \quad \text{and} \quad \lambda_U(C) = \lambda_U(C_S)
\]
(5) If a copulas \( C \) have tail dependence parameters, then also for the radial symmetrized \( C_R \) and we have

\[
\lambda_L(C_R) = \lambda_U(C_R) = \frac{1}{2} (\lambda_L(C) + \lambda_U(C))
\]

Proof. (1) These equalities are an immediate consequences of definitions.

(2) •

\[
2 \int_{I^2} C_S(u, v) \, dudv = \int_{I^2} (C + C^T)(u, v) \, dudv = \int_{I^2} C(u, v) \, dudv + \int_{I^2} C^T(u, v) \, dudv
\]

And this last calculation leads to \( \rho(C_S) = \rho(C) \).

• By use of symmetry of \( M \) and \( W \), we have also \( \mathcal{Q}(C, M) = \mathcal{Q}(C^T, M) \) and \( \mathcal{Q}(C_S, W) = \mathcal{Q}(C^T, W) \) so \( \gamma(C_S) = \gamma(C) \).

• From the equality \( C_S \left( \frac{1}{2}, \frac{1}{2} \right) = C \left( \frac{1}{2}, \frac{1}{2} \right) \) we conclude that \( \beta(C_S) = \beta(C) \).

• The equality between the Kendal tau of \( C \) and \( C_S \) is not generally satisfied. Consider the Marshal Olkin copula \( C_{\frac{1}{2}, \frac{1}{2}} \) for a counter example.

\[
C(u, v) = u^{\frac{1}{4}} v^{\frac{1}{4}} I_{u \leq \frac{1}{2}} (u, v) + uv^{\frac{1}{4}} I_{u > \frac{1}{2}} (u, v) \quad \tau(C) = \frac{1}{5} \text{ See [9, 165]}
\]

\[
\int_{I^2} C^T \, dC = \int_{v \leq u^2} C^T \, dC + \int_{u^2 \leq v \leq \sqrt{u}} C^T \, dC + \int_{\sqrt{u} \leq v} C^T \, dC
\]

\[
= \frac{3}{4} \int_{u^2} u^{\frac{1}{4}} v^{\frac{1}{4}} \, dudv + \frac{3}{4} \int_{u^2} u^{\frac{1}{4}} (uv) \, dudv + \frac{1}{2} \int_{u^2} u^{\frac{1}{4}} \, dudv
\]

\[
= \frac{3}{4} \int_{0}^{1} u \left( \int_{0}^{u^2} v^{\frac{1}{4}} \, dv \right) \, du + \frac{3}{4} \int_{0}^{1} u^{\frac{1}{4}} \left( \int_{0}^{\sqrt{u}} v^{\frac{1}{4}} \, dv \right) \, du + \frac{1}{2} \int_{0}^{1} u^{\frac{1}{4}} \left( \int_{\sqrt{u}}^{1} v \, dv \right) \, du
\]

Hence

\[
\int_{I^2} C_S(u, v) \, dC_S = \frac{1}{4} \int_{I^2} (C + C^T)(dC + dC^T))
\]

\[
= \frac{1}{2} \int_{I^2} C \, dC + \frac{1}{2} \int_{I^2} C^T \, dC
\]

\[
= \frac{1}{2} \left( 1 + \tau_C \right) + \frac{1}{2} \int_{I^2} C^T \, dC \neq \frac{1}{5}
\]

(3) It suffices to see that

\[
\mathcal{Q} (\hat{C}, \Pi) = \mathcal{Q} (C, \Pi), \quad \mathcal{Q} (\hat{C}, M) = \mathcal{Q} (C, M) \quad \text{and} \quad \mathcal{Q} (\hat{C}, W) = \mathcal{Q} (C, W)
\]

(4) It suffices to remark that \( \forall t \in I: \ C(t, t) = C_S(t, t) \).

(5) An obvious calculus leads to the result.
As a natural consequence, all copulas in the convex envelope of \(C, C_S\) and \(C_R\) have the same Spearman’s coefficient \(\rho\), the same Blomqvist’s Beta and the same Gini’s Gamma.

A natural question is the inverse problem in the sense that given a symmetric copula \(S\), does it exist a copula \(C\) other than \(S\) itself such that \(C_S = S\)? In the following section, we give an answer to this problem for some remarkable copulas mainly the comprehensive ones: \(M, W\) and \(\Pi\).

3. New copulas and inverse problem

As mentioned above, the sequel is devoted to the inverse problem as explained at the end of the last section. To do so, we begin by defining a new copula based on perturbation theory:

**Theorem 3.10.** For all \((u, v) \in I^2\), we put: \(P(u, v) = (u - v) \min(u, v) \min(1 - u, 1 - v)\). Then, the mapping

\[
C(u, v) = \Pi(u, v) + P(u, v) = uv + (u - v) \min(u, v) \min(1 - u, 1 - v)
\]

defines a copula.

The mapping \(C\) is a natural perturbation of independence copula \(\Pi\). One may see \(^{[13]}\) for more detail on perturbation theory of copulas. The novelty in this new copula is, for the best of our knowledge, its asymmetry, contrary to the most known perturbed copulas in the literature. Let us prove that \(C\) is actually a copula of Theorem (3.12). \(P\) satisfies for all \(u, v \in I\)

\[
P(u, u) = -P(u, v), P(u, 0) = P(u, 1) = P(0, v) = P(1, v) = 0
\]

Thus the bifunction \(C = \Pi + P\) satisfies the copula boundary conditions. Let \(R = [u_1, u_2] \times [v_1, v_2]\) be a rectangle in \(I^2\). We denote

\[
\Delta = \{(x, y) \in I^2 \mid y = x\}, \quad \Delta^- = \{(x, y) \in I^2 \mid y \leq x\} \text{ and } \Delta^+ = \{(x, y) \in I^2 \mid y \geq x\}.
\]

For geometrical considerations, it suffices to discuss the following cases:

- If \((v_1, v_2)\) and \((u_1, u_2)\) are both in the mean diagonal \(\Delta\), meaning \(u_1 = v_1\) and \(u_2 = v_2\) then clearly \(V_\Pi(R) = 0\) holds
- If \(R \subset \Delta^-\)

\[
V_\Pi(R) = \frac{v_2(1 - u_2)(u_2 - v_2) + v_1(1 - u_1)(u_1 - v_1) - v_2(1 - u_1)(u_1 - v_2) - v_1(1 - u_2)(u_2 - v_1)}{(v_2 - v_1)(u_2 - u_1)(1 - u_2 - u_1 + v_1 + v_2)} = V_\Pi(R)(1 - u_2 - u_1 + v_1 + v_2)
\]

Thus \(V_C(R) = V_\Pi(R)(2 - u_2 - u_1 + v_1 + v_2) \geq 0\)

- If \(R \subset \Delta^+\) then

\[
V_\Pi(R) = u_2(1 - v_2)(u_2 - v_2) + u_1(1 - v_1)(u_1 - v_1) - u_2(1 - v_1)(u_1 - v_2) - u_1(1 - v_2)(u_2 - v_1)
\]

It remains to prove that for all \(u, v \in I\) we have \(C(u, v) \in I\). The partial derivatives of \(C\) can be written as follows:

\[
\begin{cases}
\frac{\partial C}{\partial u}(u, v) = v(2 - 2u + v) \\
\frac{\partial C}{\partial v}(u, v) = 2u - 2uv + v^2
\end{cases}
\]
Thus the extremal values of $C$ and $R$ are taken in the borders of $\Delta^-$ and $\Delta^+$. □

This new copula will allow to establish easily the third point of the following proposition

**Proposition 3.11.**

1. $M$ is the unique copula $C$ satisfying $C_S = M$.
2. $W$ is the unique copula $C$ satisfying $C_S = W$.
3. There are many copulas $C$ satisfying $C_S = \Pi$.

**Proof.**

1. Let $C$ be a copula such $C_S = M$. Meaning $C - M = M - C_S$. $C - M$ is non positive bifunction and equal to a non negative one $M - C_S$ thus $C - M = 0$
2. Let $C$ be a copula such $C_S = W$. Meaning $C - W = W - C_S$. $C - W$ is non negative bifunction and equal to a non positive one $W - C_S$ thus $C = W$
3. It is an immediate consequence of the theorem (3.12) above.

**Theorem 3.12.** For all $(u, v) \in \mathbb{I}^2$, we put: $Q(u, v) = (u + v - 1) \min(1 - u, v) \min(u, 1 - v)$. Then, the mapping

$$C(u, v) = \Pi(u, v) + Q(u, v) = w + (u + v - 1) \min(1 - u, v) \min(u, 1 - v)$$

defines a copula.

**Proof.** $Q$ satisfies for all $u, v \in \mathbb{I}$

$$Q(1 - u, 1 - v) = - Q(u, v), Q(u, 0) = Q(u, 1) = Q(0, v) = Q(1, v) = 0$$

We denote

$$\Omega = \{(x, y) \in \mathbb{I}^2 \, y = 1 - x\}, \quad \Omega^- = \{(x, y) \in \mathbb{I}^2 \, y \leq 1 - x\}, \quad \text{and} \quad \Omega^+ = \{(x, y) \in \mathbb{I}^2 \, y \geq 1 - x\}$$

We have to show that the independence perturbed copula $C = \Pi + Q$ is a copula.

The bifunction $C$ satisfies the border conditions.

Let $R = [u_1, u_2] \times [v_1, v_2]$ be a rectangle in $\mathbb{I}^2$. As above, the geometrical considerations lead to discuss only the following cases:

- **If** $(u_1, v_2)$ and $(u_2, v_1)$ are in $\Omega$, then
  $$V_Q(R) = Q(u_1, v_1) + Q(u_2, v_2) - Q(u_1, 1 - u_1) - Q(u_2, 1 - u_2) = 0$$

- **If** $R \subset \Omega^-$. Let $R'$ be the rectangle $[1 - u_2, 1 - u_1] \times [1 - v_2, 1 - v_1]$ then
  $$V_Q(R) = V_{\Pi}(R)(u_1 + u_2 + v_1 + v_2) \geq 0$$

- **If** $R \subset \Omega^+$. Let $R'$ be the rectangle $[1 - u_2, 1 - u_1] \times [1 - v_2, 1 - v_1]$ then
  $$V_Q(R) = - V_{\Pi}(R')(1 - u_1 + 1 - u_2 + 1 - v_1 + 1 - v_2 - 1)$$

Thus

$$V_C(R) = V_{\Pi}(R)(u_1 + u_2 + v_1 + v_2 - 2) \geq 0 \quad \text{because} \quad u_1 + v_1 \geq 1 \quad \text{and} \quad u_2 + v_2 \geq 1$$
It remains to prove that for all \( u, v \in \mathbb{I} \) we have \( C(u, v) \in \mathbb{I} \). The partial derivatives of \( C \) are given by

\[
\begin{align*}
(u, v) \in \Omega^- & \quad \frac{\partial C}{\partial u}(u, v) = v(2u + v) \\
& \quad \frac{\partial C}{\partial v}(u, v) = u(u + 2v)
\end{align*}
\]

and

\[
\begin{align*}
(u, v) \in \Omega^+ & \quad \frac{\partial C}{\partial u}(u, v) = 2 - 2u - 2v + 2uv + v^2 \\
& \quad \frac{\partial C}{\partial v}(u, v) = 2 - 2u - 2v + 2uv + u^2
\end{align*}
\]

Thus the extremal values of \( C \) are taken on the borders of \( \Omega^- \) and \( \Omega^+ \).}

As just done for a symmetric copula, given a radial symmetric copula \( R \). Does it exist a copula \( C \) other than \( R \) itself such that \( C_R = R \)? We try to answer for the same tree copulas as done above.

**Proposition 3.13.**

1. \( M \) is the only copula \( C \) satisfying \( C_R = M \).
2. \( W \) is the only copula \( C \) satisfying \( C_R = W \).
3. There are many copulas \( C \) satisfying \( C_R = \Pi \).

**Proof.**

1. Let \( C \) be a copula such \( C_R = M \). Meaning \( C - M = M - \hat{C} \). \( C - M \) is non positive bifunction and equal to a non negative one \( M - \hat{C} \) thus \( C = M \).
2. Let \( C \) be a copula such \( C_R = W \). Meaning \( C - W = W - \hat{C} \). \( C - W \) is non negative bifunction and equal to a non positive one \( W - \hat{C} \) thus \( C = W \).

**Remark 3.14.** For any given \( \theta \in [-1, 1] \),

- The bifunction \( \Pi + \theta P \) is a copula whose symmetrized is \( \Pi \).
- The bifunction \( \Pi + \theta Q \) is a copula whose radial symmetrized is \( \Pi \).

**4. Conclusion**

The notion of symmetrized and radial symmetrized copula will be probably a key to study the phenomena which are not symmetric by approximating with a symmetric distribution. We have cited some properties preserved by the symmetrization procedure and we hope to investigate statistically the loss of such operation and characterize copulas for which the symmetrized one is archimedean.

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