Semiseparable integral operators and explicit solution of an inverse problem for the skew-self-adjoint Dirac-type system

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Abstract
Inverse problem to recover the skew-self-adjoint Dirac-type system from the generalized Weyl matrix function is treated in the paper. Sufficient conditions under which the unique solution of the inverse problem exists, are formulated in terms of the Weyl function and a procedure to solve the inverse problem is given. The case of the generalized Weyl functions of the form \( \phi(\lambda) \exp\{-2i\lambda D\} \), where \( \phi \) is a strictly proper rational matrix function and \( D = D^* \geq 0 \) is a diagonal matrix, is treated in greater detail. Explicit formulas for the inversion of the corresponding semiseparable integral operators and recovery of the Dirac-type system are obtained for this case.

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1 Introduction

The skew-self-adjoint Dirac-type system

\[
\frac{d}{dx} u(x, \lambda) = \left( i\lambda j + jV(x) \right) u(x, \lambda), \quad x \geq 0,
\]

where
\[
j = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix},
\]
$I_p$ is the $p \times p$ identity matrix, and $v$ is a $p \times p$ matrix function, is actively studied in analysis and soliton theory (see, for instance, [11, 12] and the references therein). System (1.1) is an auxiliary linear system for the focusing matrix NLSE, sine-Gordon and other important integrable equations.

The inverse problem to recover a self-adjoint Dirac type system from its Weyl or spectral function is closely related to the inversion of the integral operators with difference kernels, see [9, 26, 32, 36, 37] and various references. For the discrete analogues of Dirac systems, Toeplitz matrices appear instead of the operators with difference kernels [7, 10, 15, 38]. (Various results on Toeplitz matrices and related $j$-theory one can find, for instance, in [5, 8, 13, 14].)

When the Weyl functions of the self-adjoint Dirac type system are rational, one can solve the inverse problem explicitly. One of the approaches to solve the inverse problem explicitly is connected with a version of the Bäcklund-Darboux transformation and some notions from system theory [20, 22]. (See also [15, 16, 21, 27] for this approach, and see [39] and the references therein for explicit formulas for the radial Dirac equation.) Another method is to apply the general theory. It proves [2] that for the case of rational Weyl functions the corresponding operators with difference kernels can be inverted explicitly by formulas from [4].

The case of the skew-self-adjoint Dirac type system with the rational Weyl function was treated in [21]. It was shown that any strictly proper rational $p \times p$ matrix function is the Weyl function of a skew-self-adjoint Dirac type system on semi-axis and the solution of the inverse problem was constructed explicitly similar to the self-adjoint case treated in [20].

The analogues of the operators with difference kernel for the skew-self-adjoint system (1.1) are bounded operators $S_l$ in $L^2_p(0, l)$ ($0 < l < \infty$), which have the form [30, 33]

$$S_l f = S f = f(x) + \frac{1}{2} \int_0^l \int_{|x-t|}^{x+t} k\left(\frac{r+x-t}{2}\right) k\left(\frac{r+t-x}{2}\right)^* dr f(t) dt, \quad (1.3)$$

where $\sup_{0<x<l} \|k(x)\| < \infty$. The kernel of the operator $S_l$ is denoted by $K$:

$$K(x,t) = \frac{1}{2} \int_{|x-t|}^{x+t} k\left(\frac{r+x-t}{2}\right) k\left(\frac{r+t-x}{2}\right)^* dr. \quad (1.4)$$
In this paper we show that for a Weyl function of the form
\[ \varphi(\lambda) = \phi(\lambda) \exp\{-2i\lambda D\} R, \quad D \geq 0, \] (1.5)
where \( \phi \) is a strictly proper rational \( p \times p \) matrix function, \( D \) is a \( p \times p \) diagonal matrix, and \( R \) is a \( p \times p \) unitary matrix, the corresponding operator \( S \) is semiseparable. Using results on the inversion of the semiseparable operators, the inverse problem to recover the system from \( \varphi \) is solved explicitly. Putting \( D = 0 \), we get the subcase of rational Weyl functions. Some definitions and results for the general type (non-explicit) case of inverse problem including Theorem 2.3 and the important formula (3.9) are also new. The semiseparable matrices and operators are actively studied (see, for instance, [11, 19, 18, 40]), and their application to inverse problems is of interest, too.

Various definitions and results on a general type inverse problem for the skew-self-adjoint Dirac type system and on explicit solutions of the inverse problem, when the Weyl functions are proper rational, are given in Section 2. Some properties of the operator \( S_t \) of the form (1.3) are studied in Section 3. The explicit solution of the inverse problem for the Weyl functions of the form (1.5) is contained in Section 4.

We denote by \( \mathbb{R} \) the real axis, by \( \mathbb{R}_+ \) the positive semi-axis, by \( \mathbb{C} \) the complex plane, and by \( \mathbb{C}_+ \) the open upper halfplane \( \Im \lambda > 0 \). The class of bounded linear operators acting from \( H_1 \) into \( H_2 \) is denoted by \( \{H_1, H_2\} \), the identity operators are denoted by \( I \), and spectrum is denoted by \( \sigma \).

2 Inverse problem. Preliminaries

First, normalize the fundamental solution \( u(x, \lambda) \) of system (1.1) by the initial condition
\[ u(0, \lambda) = I_{2p}. \] (2.1)
If
\[ \sup_{0<x<\infty} \|v(x)\| \leq M, \] (2.2)
the unique \( p \times p \) Weyl matrix function \( \varphi(\lambda) \) of the skew-self-adjoint Dirac type system (1.1) on the semi-axis \( [0, \infty) \) can be defined [29] (see also [6, 21, 33])
by the inequality
\[
\int_0^\infty \left[ \begin{array}{cc} \varphi(\lambda)^* & I_p \end{array} \right] u(x,\lambda)^* u(x,\lambda) \left[ \begin{array}{c} \varphi(\lambda) \\ I_p \end{array} \right] dx < \infty,
\]
which holds for all \(\lambda\) in the halfplane \(\Im \lambda < -M < 0\). Under condition (2.2) such a Weyl function always exists.

Consider the case of the so called pseudo-exponential potentials [21], which are denoted by the acronym PE. A potential \(v \in \text{PE}\) is determined by three parameter matrices, that is, by the \(n \times n\) matrix \(\alpha\) \((n > 0)\) and two \(p \times n\) matrices \(\theta_1\) and \(\theta_2\), which satisfy the identity
\[
\alpha - \alpha^* = i(\theta_1 \theta_1^* + \theta_2 \theta_2^*),
\]
(2.4)
The pseudo-exponential potential has the form
\[
v(x) = 2\theta_1^* e^{ix\alpha} \Sigma(x)^{-1} e^{ix\alpha} \theta_2,
\]
(2.5)
where
\[
\Sigma(x) = I_n + \int_0^x \Lambda(t) j \Lambda(t)^* dt, \quad \Lambda(x) = \left[ \begin{array}{cc} e^{-ix\alpha} \theta_1 & e^{ix\alpha} \theta_2 \end{array} \right].
\]
(2.6)
By Proposition 1.4 in [21] the pseudoexponential potential \(v\), i.e., the potential given by (2.5) is bounded on the semi-axis. The Weyl function of the system (1.1) with \(v \in \text{PE}\) is a rational matrix function, which is also expressed in terms of the parameter matrices [21]:
\[
\varphi(\lambda) = i \theta_1^* (\lambda I_n - \beta)^{-1} \theta_2, \quad \beta = \alpha - i \theta_2 \theta_2^*.
\]
(2.7)
In spite of the requirement
\[
\beta - \beta^* = i(\theta_1 \theta_1^* - \theta_2 \theta_2^*),
\]
(2.8)
which is implied by the equalities (2.4) and \(\beta = \alpha - i \theta_2 \theta_2^*\), any strictly proper rational matrix function can be presented in the form (2.7). The inverse problem to recover \(v\) from the strictly proper rational matrix function \(\varphi\) is solved explicitly in [21], using a minimal realization of \(\varphi\) and formula (2.5).
When (2.2) is true, inequality (2.3) implies other inequalities:

\[
\sup_{x \leq l, \Im \lambda < -M} \left\| e^{ix\lambda} u(x, \lambda) \left[ \begin{array}{c} \phi(\lambda) \\ I_p \end{array} \right] \right\| < \infty \quad \text{for all } 0 < l < \infty,
\]

(2.9)

which can be treated as a more general definition of the Weyl function.

**Definition 2.1** Let the system (1.1) be given on the semi-axis \([0, \infty)\). Then a \(p \times p\) matrix function \(\varphi(\lambda)\) analytic in some halfplane \(\Im \lambda < -M < 0\) is called a Weyl function of this system, if inequalities (2.9) hold.

If

\[
\sup_{0 < x < l} \|v(x)\| < \infty \quad \text{for all } 0 < l < \infty,
\]

(2.10)

then there is at most one Weyl function.

**Definition 2.2** The inverse spectral problem (ISP) for system (1.1) on the semi-axis is the problem to recover \(v(x)\) satisfying (2.9) and (2.10) from the Weyl function \(\varphi\).

For an analytic matrix function \(\varphi\) satisfying the condition

\[
\sup_{\Im \lambda < -M} \|\lambda^2 (\varphi(\lambda) - \alpha/\lambda)\| < \infty,
\]

(2.11)

where \(\alpha\) is some \(p \times p\) matrix, the solution of the inverse problem always exists (see Lemma 1 [30] for the scalar version of this result and the matrix case can be proved quite similar).

The general (non-explicit) procedure to solve ISP is described in [28, 29, 30, 33]. Fix a positive value \(l\) \((0 < l < \infty)\). The first step to solve ISP is to recover a \(p \times p\) matrix function \(s(x)\) with the entries from \(L^2(0, l)\) \((l < \infty)\), i.e., \(s(x) \in L^2_{p \times p}(0, l)\) via the Fourier transform. That is, we put

\[
s(x) = \frac{i}{2\pi} e^{-\eta x} \text{l.i.m.}_{a \to -\infty} \int_{-a}^{a} e^{i\xi x} \lambda^{-1} \varphi(\lambda/2) d\xi \quad (\lambda = \xi + i\eta, \quad \eta < -2M),
\]

(2.12)

the limit l.i.m. being the limit in \(L^2(0, l)\). As (2.12) has sense for any \(l < \infty\) the matrix function \(s(x)\) is defined on the non-negative real semi-axis \(x \geq 0\). Moreover, it is easily checked that \(s\) is absolutely continuous, it does not
depend on the choice of $\eta < -2M$, $s'$ is bounded on any finite interval, and $s(0) = 0$. To define the operator $S_l$ we substitute $k(x) = s'(x)$ into (1.3).

Next, denote the $p \times 2p$ block rows of $u$ by $\omega_1$ and $\omega_2$:

$$\omega_1(x) = [I_p \ 0]u(x,0), \quad \omega_2(x) = [0 \ I_p]u(x,0).$$  \hspace{1cm} (2.13)

It follows from (1.1) that $u(x,0)^*u(x,0) = I_{2p}$. Hence, by (1.1) and (2.13) we have

$$v(x) = \omega_1'(x)\omega_2(x)^*, \hspace{1cm} (2.14)$$

and $\omega_1, \omega_2$ satisfy the equalities

$$\omega_1(0) = [I_p \ 0], \quad \omega_1\omega_1^* \equiv I_p, \quad \omega_1'\omega_1^* \equiv 0, \quad \omega_1\omega_2^* \equiv 0. \hspace{1cm} (2.15)$$

It is immediate that $\omega_1$ is uniquely recovered from $\omega_2$ using (2.15).

Finally, we obtain $\omega_2$ via the formula

$$\omega_2(l) = [0 \ I_p] - \int_0^l \left(S_l^{-1}s'(x)\right)^*[I_p \ s(x)]dx \quad (0 < l < \infty), \hspace{1cm} (2.16)$$

where $S_l^{-1}$ is applied to $s'$ columnwise.

From the considerations in [29, 30] (see also similar constructions in [31], where the Weyl theory for the linear system auxiliary to the nonlinear optics equation is treated) it follows that one can solve ISP under requirements on $\varphi$ and $s(x)$ weaker than (2.11). Namely, we assume

$$\sup_{\Im \lambda < -M} \|\varphi(\lambda)\| < \infty, \hspace{1cm} (2.17)$$

$$\varphi(\lambda) \in L^2_{p \times p}(-\infty, \infty), \quad \lambda = \xi + i\eta \ (-\infty < \xi < \infty) \text{ for all } \eta < -M, \hspace{1cm} (2.18)$$

$$s(0) = 0, \quad \sup_{0 < x < l} \|k(x)\| < \infty \quad \text{for all } 0 < l < \infty, \quad k(x) := s'(x), \hspace{1cm} (2.19)$$

$$\int_0^\infty e^{-cx}\|k(x)\|dx < \infty \hspace{1cm} (2.20)$$

for some $c > 0$.

**Theorem 2.3** Let the matrix function $\varphi$ be analytic in the halfplane $\Im \lambda < -M$ and satisfy the relations (2.17) and (2.18). Let also the matrix function $s(x)$ defined via $\varphi$ by formula (2.12) be absolutely continuous and satisfy (2.19) and (2.20). Then ISP has a unique solution, which is given by formulas (2.14)-(2.16), where $S_l \geq I$ has the form (1.3) with $k = s'$. 

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3 Factorization of $S$ and operator identity

Consider again the operator $S = S_I$. It is easy to see that functions, which are bounded on the interval, can be approximated in the $L^1$-norm by the continuous functions. As $k = s'$ is bounded on the finite intervals, one can see that the kernel $K$ of $S$, which is given by (1.4), is continuous with respect to $x$ and $t$. Hence, the kernel of $S_I^{-1}$ is continuous with respect to $x$, $t$, and $l$ ([23], p. 185). Therefore, $S_I^{-1}k$ has the form $(S_I^{-1}k)(x) = k(x) + k_1(x)$, where $k_1$ is continuous, and the matrix function $(S_I^{-1}k)(l)$ is well-defined:

$$
(S_I^{-1}k)(l) = k(l) + k_1(l) = k(l) + \lim_{x \to l-0} \left( (S_I^{-1}k)(x) - k(x) \right) \quad (0 < l < \infty).
$$

To express $v$ in terms of $(S_I^{-1}k)(l)$ we need some preparations. According to [34] there are triangular operators $\hat{V}_l \in \{L^2_p(0, l), L^2_p(0, l)\}$, such that

$$
(\hat{V}_l f)(x) = f(x) + \int_0^x \hat{V}_-(x, t) f(t) \, dt, \quad \hat{V}_l A \hat{V}_l^{-1} = i\omega_1(x) \int_0^x \omega_1(t)^* \cdot dt,
$$

where $\hat{V}_-(x, t)$ does not depend on $l$, and the operators $\hat{V}_l$ and $\hat{V}_l^{-1}$ map functions with bounded derivatives into functions with bounded derivatives. Moreover, as bounded functions on an interval can be approximated in the $L^1$-norm by the continuous functions, it follows from the construction in [34] that $\hat{V}(x, t)$ ($x \geq t$) is continuous with respect to $x$ and $t$.

Next, introduce the operator

$$
(\tilde{V}_l f)(x) = f(x) + \int_0^x \tilde{V}_-(x - t) f(t) \, dt, \quad \tilde{V}_-(x) := \frac{d}{dx} \left( \tilde{V}_l^{-1} \omega_{11} \right)(x),
$$

where $\omega_{11}$ is the first $p \times p$ block of $\omega_1$, and put

$$
V_l := \tilde{V}_l \tilde{V}_l = I + \int_0^x V_-(x, t) \cdot dt.
$$

It is easy to see that $\tilde{V}_l A = A \tilde{V}_l$, and so the second equality in (3.2) yields

$$
V_l A V_l^{-1} = i\omega_1(x) \int_0^x \omega_1(t)^* \cdot dt.
$$
By (2.13) we see that \( \omega_{11}(0) = I_p \). Hence, using definition (3.3) one gets

\[
\tilde{V}_l I_p = I_p + \int_0^x \left( \frac{d}{dx} (\tilde{V}_l^{-1} \omega_{11}) \right) (x-t) dt = \int_0^x \left( \frac{d}{dx} (\tilde{V}_l^{-1} \omega_{11}) \right) t dt = (\tilde{V}_l^{-1} \omega_{11})(x).
\]

Formula (3.6) implies \( V_l^{-1} \omega_{11} = I_p \). Moreover, from [29, 33] it follows that under the conditions of Theorem 2.3 the equalities

\[
(V_l^{-1} \omega_1)(x) = [I_p \ s(x)]
\]

and

\[
S_l^{-1} = V_l^* V_l
\]

are also true.

**Remark 3.1** Under the conditions of Theorem 2.3 we have

\[
v(l) = (S_l^{-1} s')(l).
\]

Indeed, using (3.8) and changing variables \( l \) and \( x \) into \( x \) and \( t \), correspondingly, we rewrite (2.16) in the form

\[
\omega_2(x) = [0 \ I_p] - \int_0^x (V_x s')(t) (V_x [I_p s(t)]) dt.
\]

As \( V_\_ \) does not depend on \( l \) we have \( (V_x s')(t) = (V_l s')(t) \) for \( t \leq x \leq l \). Thus, according to (3.7) and (3.10), we get

\[
\omega_2'(x) = - (V_l s')(x) * \omega_1(x).
\]

Multiplying both sides of (3.11) by \( \omega_1^* \) from the right and taking into account (2.14) and (2.13), one derives \(-v(x)^* = -(V_l s')(x)^*\), i.e., the equality

\[
v(x) = (V_l s')(x)
\]

is true. As \( \hat{V}(x,t) \) is continuous, taking into account (3.3) and (3.4) we see that \((V_l s')(x) - s'(x)\) is continuous. It is also immediate from (3.4) that

\[
(V_l^* f)(x) = f(x) + \int_x^l V_\_ (t,x) f(t) dt.
\]
Hence, according to (3.1), (3.8), and (3.13) we have

\[(S_l^{-1}s')(l) = (V_is')(l). \tag{3.14}\]

Finally, formula (3.9) follows from (3.12) and (3.14).

By (3.8) the equality

\[A S - S A^* = V_l^{-1}(V_l AV_l^{-1} - (V_l AV_l^{-1})^*)(V_l^{-1})^*\]

is valid for \(S = S_l\). Therefore, taking into account (3.5) and (3.7) one can see that \(S\) satisfies the operator identity

\[A S - S A^* = i\Pi \Pi^*, \quad \Pi = [\Phi_1 \Phi_2], \quad \Phi_1 g \equiv g, \quad \Phi_2 g = s(x)g. \tag{3.15}\]

Here \(\Phi_k \in \{\mathbb{C}^p, L^p_0(0, l)\} \quad (k = 1, 2)\) and \(\mathbb{C}\) denotes the complex plane. This identity differs from the identity \(A S - S A^* = i(\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*)\) \(\text{[35, 36]}\) for an operator with difference kernel. Matrices satisfying a discrete analogue of (3.15) were treated in \(\text{[17]}\). The operator identity (3.15) for the case, when \(k\) in (1.3) is a vector, was studied in \(\text{[25]}\). It could be useful also to prove (3.15) directly. In fact, we prove below a somewhat more general identity.

**Proposition 3.2** Let the operator \(S\) in \(L^p_0(0, l) \quad (0 < l < \infty)\) be defined by

\[Sf = f(x) + \frac{1}{2} \int_0^l \int_{|x-t|}^{x+t} k\left(\frac{r+x-t}{2}\right)\tilde{k}\left(\frac{r+t-x}{2}\right) dr df(t) dt, \tag{3.16}\]

where \(\sup_{0 < x < t} \left(\|k(x)\| + \|	ilde{k}(x)\|\right) < \infty\). Then \(S\) satisfies the operator identity

\[A S - S A^* = i \int_0^l (I_p + \psi(x)\tilde{\psi}(t)) \cdot dt, \tag{3.17}\]

where \(\psi(x) = \int_0^x k(t) dt, \tilde{\psi}(x) = \int_0^x \tilde{k}(t) dt\).

**Proof.** Using (3.16) and changing the order of integration we have

\[A S f = Af + i \int_0^l \gamma_1(x, t) f(t) dt, \tag{3.18}\]

\[\gamma_1(x, t) := \frac{1}{2} \int_0^x \int_{|y-t|}^{y+t} k\left(\frac{r+y-t}{2}\right)\tilde{k}\left(\frac{r+t-y}{2}\right) dr dy. \tag{3.19}\]
Taking into account that for the scalar product $(\cdot,\cdot)_l$ in $L^2_\phi(0,l)$ we have $(A^*f,g)_l=(f,Ag)_l$, rewrite $SA^*$ in the form

$$SA^*f = A^*f - i \int_0^l \gamma_2(x,t)f(t)dt,$$

(3.20)

$$\gamma_2(x,t) := \frac{1}{2} \int_0^t \int_{|x-y|}^{x+y} k\left(\frac{r+x-y}{2}\right) \tilde{k}\left(\frac{r+y-x}{2}\right) drdy. \quad (3.21)$$

First, consider the case $t \geq x$. From (3.19), after changes of variables $\xi = (r+y-t)/2$ and $\eta = t-y+\xi$, we get

$$\gamma_1(x,t) = \frac{1}{2} \int_0^x \int_{t-y}^{y+t} k\left(\frac{r+y-t}{2}\right) \tilde{k}\left(\frac{r+t-y}{2}\right) drdy \quad (3.22)$$

$$= \int_0^x \int_0^y k(\xi) \tilde{k}(t-y+\xi) d\xi dy = \int_0^x \int_0^x k(\xi) \tilde{k}(t-y+\xi) dy d\xi$$

$$= \int_0^x \int_{t-x+\xi}^{t} k(\xi) \tilde{k}(\eta) d\eta d\xi.$$

Next, calculate $\gamma_2(x,t)$ ($t \geq x$). From (3.21) it follows that

$$\gamma_2(x,t) = \gamma_{21}(x,t) + \gamma_{22}(x,t),$$

(3.23)

where

$$\gamma_{21}(x,t) := \frac{1}{2} \int_0^x \int_{x-y}^{x+y} k\left(\frac{r+x-y}{2}\right) \tilde{k}\left(\frac{r+y-x}{2}\right) drdy, \quad (3.24)$$

$$\gamma_{22}(x,t) := \frac{1}{2} \int_0^t \int_{y-x}^{y+x} k\left(\frac{r+x-y}{2}\right) \tilde{k}\left(\frac{r+y-x}{2}\right) drdy. \quad (3.25)$$

Replace the variable $r$ by $\eta = (r+y-x)/2$ in (3.24), then change the order of integration, and after that put $\xi = x-y+\eta$ and change the order of integration again to obtain

$$\gamma_{21}(x,t) = \int_0^x \int_0^\xi k(\xi) \tilde{k}(\eta) d\eta d\xi. \quad (3.26)$$

In (3.25), replace $r$ by $\xi = (r+x-y)/2$, change the order of integration and put $\eta = y-x+\xi$. We get

$$\gamma_{22}(x,t) = \int_0^x \int_0^{t-x+\xi} k(\xi) \tilde{k}(\eta) d\eta d\xi. \quad (3.27)$$
By (3.22), (3.23), (3.26), and (3.27) the equality

\[ \gamma_1(x,t) + \gamma_2(x,t) = \int_0^x \int_0^t k(\xi) \tilde{k}(\eta) d\eta d\xi = \psi(x) \tilde{\psi}(t) \quad (3.28) \]

is true for \( t \geq x \). Using similar calculations one can show that (3.28) holds also for \( x \geq t \), i.e., (3.28) is true for all \( 0 \leq x, t \leq l \). Finally, formulas (3.18), (3.20), and (3.28) yield (3.17).

\[ \square \]

4 ISP and semiseparable operators \( S_l \)

In this section we consider matrix functions of the form

\[ \varphi(\lambda) = i\theta_1^*(\lambda I_n - \beta)^{-1} \theta_2 e^{-2i\lambda D} R, \quad (4.1) \]

\[ D = \text{diag}\{d_1,\ldots,d_p\}, \quad d_{k_1} \geq d_{k_2} \geq 0 \quad \text{for} \quad k_1 > k_2, \quad (4.2) \]

where \( \theta_1 (j = 1, 2) \) is an \( n \times p \) matrix with the \( m \)-th column denoted by \( \theta_{j,m} \), \( \beta \) is an \( n \times n \) matrix, \( R \) is a \( p \times p \) matrix, and \( D \) is a \( p \times p \) diagonal matrix.

We do not suppose here that \( \theta_1 \) and \( \beta \) satisfy the identity (2.8).

**Proposition 4.1** Let matrix function \( \varphi \) be given by (4.1). Then, the matrix function \( s \), which is defined via \( \varphi \) by (2.12), has the form \( s = CR \), where \( C = [c_1 \ c_2 \ \ldots \ c_p] \), the columns \( c_m (p \geq m \geq 1) \) being given by the formulas

\[ c_m(x) = 0 \quad \text{for} \quad 0 \leq x \leq d_m, \quad (4.3) \]

\[ c_m(x) = 2\theta_1^* \int_0^{x-d_m} \exp\{2it\beta\} dt \theta_{2,m} \quad \text{for} \quad x \geq d_m, \quad (4.4) \]

and the function \( \varphi \) is the Weyl function of system (1.1) with potential \( v \) satisfying (2.10).

**Proof.** First, choose \( M > 0 \) such that \( \sigma(\beta + iMI_n) \subset \mathbb{C}_+ \), where \( \sigma \) means spectrum and \( \mathbb{C}_+ \) is the open upper halfplane. According to (4.1) \( \varphi(\lambda) \) is analytic and the function \( \lambda \varphi(\lambda) \) is bounded in the halfplane \( \Im \lambda < -M \). So, the conditions (2.17) and (2.18) on \( \varphi \) are fulfilled. The fact that \( s \) is absolutely continuous and satisfies conditions (2.19) and (2.20) is immediate.
from (4.3) and (4.4). Therefore, after we have proved (4.3) and (4.4), it will follow from Theorem 2.3 that \( \varphi \) is the Weyl function of system (1.1) with potential \( v \) satisfying (2.10).

Now, let us prove (4.3) and (4.4). As \( \lambda \varphi(\lambda) \) is bounded, one can rewrite (2.12) as a pointwise limit:

\[
\begin{align*}
\theta^*_{1}\left[\begin{array}{c}
c_1 \ c_2 \ \ldots \ c_p
\end{array}\right] R, & \quad c_m(x) = -\frac{1}{\pi}\theta^*_{1}\int_{-\infty}^{\infty} e^{i\lambda(x-d_m)} \\
\times \lambda^{-1}(\lambda I_n - 2\beta)^{-1}d\xi\theta_{2,m} & \quad (\lambda = \xi + i\eta, \ \eta < -2M).
\end{align*}
\]

Introduce the counterclockwise oriented contours, where \( \xi \) may take complex values:

\[
\begin{align*}
\Gamma^+_a & = [-a, a] \cup \{ \xi : |\xi| = a, \Im \xi > 0 \}, \\
\Gamma^-_a & = [-a, a] \cup \{ \xi : |\xi| = a, \Im \xi < 0 \}.
\end{align*}
\]

For \( \lambda = \xi + i\eta \) and for the fixed values of \( \eta < -2M \), it follows from (4.5) that

\[
c_m(x) = -\frac{1}{\pi}\theta^*_{1}\lim_{a \to \infty} \int_{\Gamma^+_a} e^{i\lambda(x-d_m)}\lambda^{-1}(\lambda I_n - 2\beta)^{-1}d\xi\theta_{2,m}
\]

in the case \( x \geq d_m \), and

\[
c_m(x) = \frac{1}{\pi}\theta^*_{1}\lim_{a \to \infty} \int_{\Gamma^-_a} e^{i\lambda(x-d_m)}\lambda^{-1}(\lambda I_n - 2\beta)^{-1}d\xi\theta_{2,m}
\]

in the case \( x \leq d_m \). As \( e^{i\lambda(x-d_m)}\lambda^{-1}(\lambda I_n - 2\beta)^{-1} \) is analytic with respect to \( \xi \) inside \( \Gamma^-_a \) and on the contour itself, equality (4.3) is immediate from (4.7).

Next, consider the case \( x \geq d_m \). For sufficiently large \( a \) all the poles of \( (\lambda I_n - 2\beta)^{-1} \) (and the pole \( \xi = -i\eta \) of \( \lambda^{-1} \)) are contained inside \( \Gamma^+_a \) and taking into account (4.6) we have

\[
c_m(x) = -\frac{1}{\pi}\theta^*_{1}\int_{\Gamma^+_a} e^{i\lambda(x-d_m)}\lambda^{-1}(\lambda I_n - 2\beta)^{-1}d\xi\theta_{2,m}.
\]

Let us approximate \( \beta \) by matrices \( \beta_\varepsilon \) such that \( ||\beta - \beta_\varepsilon|| < \varepsilon \) and \( \det \beta_\varepsilon \neq 0 \) (if \( \det \beta \neq 0 \) we put \( \beta = \beta_\varepsilon \)). It is easy to see that

\[
\lambda^{-1}(\lambda I_n - 2\beta)^{-1} = (2\beta_\varepsilon)^{-1}((\lambda I_n - 2\beta_\varepsilon)^{-1} - \lambda^{-1}I_n).
\]
For sufficiently small \( \varepsilon \) all the poles of \( (\lambda I_n - 2\beta \varepsilon)^{-1} \) are contained inside \( \Gamma_+^\varepsilon \) and we have
\[
\frac{1}{2\pi i} \int_{\Gamma_+^\varepsilon} e^{i\xi x}(\lambda I_n - 2\beta \varepsilon)^{-1}d\xi = e^{ix\beta\varepsilon} \quad (x \geq 0). \tag{4.10}
\]
Finally, using (4.8)-(4.10) we get
\[
c_m(x) = \lim_{\varepsilon \to 0} 2\theta_1^* \int_0^{x-d_m} \exp(2it\beta\varepsilon) dt \theta_{2,m}.
\]
Hence, formula (4.4) is immediate. ■

**Remark 4.2** Note that the matrix functions \( \varphi \) of the form (4.1) in general position do not satisfy (2.11) and so they do not satisfy in a scalar case conditions of Lemma 1 [30], but the conditions of Theorem 2.3 are fulfilled.

By Proposition 4.1 the matrix function \( k \) in the expression (1.4) for the kernel of the operator \( S_l \), generated by the Weyl function \( \varphi \) of the form (4.1), is given by the formula
\[
k(x) = s'(x) = 2\theta_1^* e^{2ix\beta} \nu \chi(x) R, \quad \nu := \{\exp(-2id_{m,\beta} \theta_{2,m})\}_{m=1}^p, \tag{4.11}
\]
\[
\chi(x) = \text{diag}\{\chi_1(x), \chi_2(x), \ldots, \chi_k(x)\}, \quad \chi_m(x) := \begin{cases} 0, & 0 \leq x < d_m, \\ 1, & x > d_m. \end{cases}
\]
According to (1.4) and (4.11) we have
\[
K(x,t) = 2\theta_1^* \int_{|x-t|}^{x+t} \exp\left(i(r + x - t)\beta\right) Q(r,x,t) \exp\left(-i(r + t - x)\beta^*\right) dr \theta_1,
\]
where
\[
Q(r,x,t) = \nu \chi\left(\frac{r + x - t}{2}\right) RR^* \chi\left(\frac{r + t - x}{2}\right) \nu^*. \tag{4.13}
\]
The matrix function \( Q(r,x,t) \) is piecewise constant with respect to \( r \) and without loss of generality we assume \( Q(0,x,t) = 0 \). It is easy to see that \( Q(r,x,t) \) has only a finite number of jumps \( \{Q_j\} \). Moreover, if \( \sigma(\beta) \cap \sigma(\beta^*) = \emptyset \), the matrix identity \( i(\beta X_j - X_j \beta^*) = Q_j \) always has the solution \( X_j \). Therefore we have
\[
e^{ir\beta} Q_j e^{-ir\beta^*} = \frac{d}{dr}\left(e^{ir\beta} X_j e^{-ir\beta^*}\right). \tag{4.14}
\]
Hence, according to (4.12) and (4.14) we can express the kernel $K(x,t)$ of $S$ explicitly in terms of matrix exponents and $\{X_j\}$. It follows also from (4.3) that

$$K(x,t) = K(t,x)^*, \quad (4.15)$$

and so we need to simplify (4.12) only for $x > t$.

Another approach to the presentation of $K$ in terms of matrix exponents is given in the following lemma.

**Lemma 4.3** Put

$$g_j(r) := [0 \ e^{ir\beta}]e^rE_j \begin{bmatrix} I_p \\ 0 \end{bmatrix}, \quad E_j := \begin{bmatrix} -i\beta^* & 0 \\ Q_j & -i\beta \end{bmatrix}. \quad (4.16)$$

Then we have

$$\left(\frac{d}{dr}g_j\right)(r) = e^{ir\beta}Q_je^{-ir\beta^*}. \quad (4.17)$$

**Proof.** By (4.16) we have

$$\frac{d}{dr}g_j = i\beta g_j + [0 \ e^{ir\beta}]E_je^rE_j \begin{bmatrix} I_p \\ 0 \end{bmatrix} = i\beta g_j + e^{ir\beta}Q_je^{-ir\beta^*} - i\beta g_j,$$

and (4.17) is immediate. ■

Recall [18] that the operator $S$ is called semiseparable, when $K$ admits representation

$$K(x,t) = F_1(x)G_1(t) \quad \text{for} \ x > t, \quad K(x,t) = F_2(x)G_2(t) \quad \text{for} \ x < t, \quad (4.18)$$

where $F_1$ and $F_2$ are $p \times \tilde{p}$ matrix functions and $G_1$ and $G_2$ are $\tilde{p} \times p$ matrix functions. For the operator $S$ to be semiseparable, assume

$$RR^* = I_p. \quad (4.19)$$

Then the matrix function $Q$ has the form

$$Q(r,x,t) = \nu \chi \left(\frac{r + t - x}{2}\right)^* \nu^* \quad \text{for} \ x > t. \quad (4.20)$$
Rewrite (4.2) as

\[ D = \text{diag}\{\tilde{d}_1 I_{p_1}, \ldots, \tilde{d}_k I_{p_k}\}, \quad p_1 + \ldots + p_k = p, \quad \tilde{d}_k > \tilde{d}_2 \geq 0 \text{ for } k_1 > k_2, \]  
(4.21)

and put

\[ Q_j = \nu P_j \nu^*, \quad P_j = \text{diag}\{0, \ldots, 0, I_{p_j}, 0, \ldots, 0\}. \]  
(4.22)

**Remark 4.4** Some notations. Further we consider \( K(x, t) \) \((x > t)\) on the intervals \( \tilde{d}_m < t < \min(x, \tilde{d}_{m+1}) \), where we choose such \( m \) that the inequalities \( \tilde{d}_m < x \) hold. If \( \tilde{d}_1 > 0 \) we put \( \tilde{d}_0 = 0 \) and include the interval \( \tilde{d}_0 < t < \min(x, \tilde{d}_1) \) into consideration. If \( x > \tilde{d}_k \), we include the interval \( \tilde{d}_k < t < \tilde{d}_{k+1} \) (\( \tilde{d}_{k+1} = x \)). Some matrix functions, like \( B(t) \) and \( C(t) \), will be considered on the intervals as above, but with \( x = l \). In the following, in all such cases we simply write \( \tilde{d}_m < t < \tilde{d}_{m+1} \). We also assume that \( \sum_{j=1}^m \ldots = 0 \), when \( m = 0 \).

**Proposition 4.5** Let the matrix function \( \varphi \) be given by (4.1), where \( D \) satisfies (4.21) and \( R \) is unitary. Assume also that the matrix identities

\[ i(\beta X_j - X_j \beta^*) = Q_j, \]  
(4.23)

where \( Q_j \) are given by (4.22), have solutions \( X_j \). Then the operator \( S \), which is defined via \( \varphi \) by formulas (1.3), \( k = s' \) and (2.12), is semiseparable, and its kernel \( K(x, t) \) \((0 < x, t < l)\) is given by relation (4.15) and by the equalities

\[ K(x, t) = 2\theta_1^*(e^{2i\beta x} Z_m e^{-2i\beta^*} - e^{2i(x-t)\beta} \tilde{Z}_m) \theta_1 \quad (\tilde{d}_m < t < \tilde{d}_{m+1}) \]  
(4.24)

for \( t < x < l \). Here

\[ Z_m = \sum_{j=1}^m X_j, \quad \tilde{Z}_m = \sum_{j=1}^m \left( \exp\left(2i\tilde{d}_j \beta\right)\right) X_j \exp\left(-2i\tilde{d}_j \beta^*\right). \]  
(4.25)

Moreover, there are self-adjoint solutions of (4.23) and we suppose \( X_j = X_j^* \) in (4.23) and (4.25).
Proof. First, note that we can choose \( X_j = X_j^* \) because the adjoint of each solution of (4.23) also satisfies (4.23).

Next, using (4.12), (4.20), and (4.22) we get equalities

\[
K(x, t) = 2\theta_1 \left( \sum_{j=1}^m \int_{x-t+2\tilde{a}_j}^{x+t} \left( \exp i(r + x - t)\beta \right) Q_j \right.
\]

\[
\times \exp \left( -i(r + t - x)\beta^* \right) dr \right) \theta_1
\]

(4.26)

for \( t < x \) and \( \tilde{a}_m < t < \tilde{a}_{m+1} \). From (4.14) and (4.26) it follows that (4.24) holds. Formula (4.15) was derived earlier. ■

Remark 4.6 By (2.14)-(2.16) and (4.11) the equality \( v(x) = 0 \) is valid for \( 0 < x < d_1 \) in the case \( d_1 > 0 \). This fact corresponds to the inequality

\[
\sup_{x \leq d_1, 3\lambda < -M} \left\| e^{ix\lambda} e^{ix\lambda} \left[ \begin{array}{c} \phi(\lambda)e^{-2\lambda D} \\ I_p \end{array} \right] \right\| < \infty,
\]

(4.27)

which can be easily checked directly and is implied also by (2.9).

When the operator \( S = I + \int_0^t K(x, t) \cdot dt \) is semiseparable and its kernel \( K \) is given by (4.18), the kernel of the operator \( T = S^{-1} \) is expressed in terms of the \( 2\tilde{p} \times 2\tilde{p} \) solution \( U \) of the differential equation

\[
\left( \frac{d}{dx} U \right)(x) = H(x)U(x), \quad x \geq 0, \quad U(0) = I_{2\tilde{p}},
\]

(4.28)

where

\[
H(x) := B(x)C(x), \quad B(x) = \left[ \begin{array}{c} -G_1(x) \\ G_2(x) \end{array} \right], \quad C(x) = \left[ \begin{array}{cc} F_1(x) & F_2(x) \end{array} \right].
\]

(4.29)

Namely, we have (see, for instance, [18])

\[
T = S^{-1} = I + \int_0^t T(x, t) \cdot dt,
\]

(4.30)

\[
T(x, t) = \begin{cases} 
C(x)U(x)(I_{2\tilde{p}} - P^x)U(t)^{-1}B(t), & x > t, \\
-C(x)U(x)P^xU(t)^{-1}B(t), & x < t.
\end{cases}
\]

(4.31)
Here \( P^\times \) is given in terms of the \( \bar{\rho} \times \bar{\rho} \) blocks \( U_{21}(l) \) and \( U_{22}(l) \) of \( U(l) \):

\[
P^\times = \begin{bmatrix} 0 & 0 \\ U_{22}(l)^{-1}U_{21}(l) & I_{\bar{\rho}} \end{bmatrix},
\]

and the invertibility of \( U_{22}(l) \) is a necessary and sufficient condition for the invertibility of \( S \).

If \( K \) admits the representation

\[
K(x,t) = \begin{cases} 
Ce^xA(I_{2\bar{\rho}} - P)e^{-tA}B, & x > t, \\
-Ce^xAPe^{-tA}B, & x < t,
\end{cases}
\]

where \( A \), \( B \), and \( C \) are constant matrices, then \( U \) is calculated explicitly [19]. In our case a representation

\[
K(x,t) = \begin{cases} 
C_me^xA(I_{2\bar{\rho}} - \bar{P}m)e^{-tA}B_m, & t < x < \tilde{d}_m, \tilde{d}_m < t < \tilde{d}_{m+1}, \\
-C_m e^xAP_m e^{-tA}B_m, & x < t < \tilde{d}_m, \tilde{d}_m < x < \tilde{d}_{m+1},
\end{cases}
\]

where \( \tilde{\rho} = n \) and

\[
A = 2i \begin{bmatrix} \beta & 0 \\ 0 & \beta^* \end{bmatrix},
\]

easily follows from (4.15) and (4.24). However, (4.34) is insufficient for the explicit construction of \( U \) and we shall construct \( U \) and \( T \) explicitly, using more general formulas (4.28)-(4.32). For this purpose we introduce \( B(x) \) and \( C(x) \) \((0 < x < l)\) by the equalities

\[
B(x) = \sqrt{2} \begin{bmatrix} e^{-2ix\beta} \tilde{Z}_m - Z_m e^{-2ix\beta^*} \\ e^{-2ix\beta^*} \end{bmatrix} \theta_1 (\tilde{d}_m < x < \tilde{d}_{m+1}),
\]

\[
C(x) = \sqrt{2} \begin{bmatrix} e^{2ix\beta} Z_m - \tilde{Z}_m e^{2ix\beta^*} \\ e^{2ix\beta^*} \end{bmatrix} \theta_1^* (\tilde{d}_m < x < \tilde{d}_{m+1}),
\]

where \( Z_m = Z_m^* \) and \( \tilde{Z}_m = \tilde{Z}_m^* \) are defined in (4.25).

**Proposition 4.7** Let the conditions of Proposition 4.5 be fulfilled and let \( S \) be defined via \( \varphi \) by formulas (1.3), \( k = s' \) and (2.12). Then the operator \( T = S^{-1} \) is given by formulas (4.30)-(4.32), (4.30), (4.31), and

\[
U(x) = \Omega_m e^{-xA^*}e^{x\tilde{A}_m}e^{-x\bar{A}_m}U(\tilde{d}_m) (\tilde{d}_m \leq x \leq \tilde{d}_{m+1}), \quad U(0) = I_{2n},
\]

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where \( A \) is defined by (4.35) and
\[
\mathcal{A}_m^\times := A + 2Y_m, \quad Y_m := \begin{bmatrix} \tilde{Z}_m & \theta_1 \theta_1^* \theta_1^* & I_n & -\tilde{Z}_m \end{bmatrix},
\]
(4.39)

\[
\Omega_m := \begin{bmatrix} I_n & -Z_m & 0 \end{bmatrix}, \quad 
\Xi_m := \Omega_m e^{-\tilde{d}_m A_{\pm} e^{-\tilde{d}_m A_{\pm}}},
\]
(4.40)

Moreover, we have
\[
U(x)^* JU(x) = J, \quad U(x)^{-1} = JU(x)^* J^*, \quad J := \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.
\]
(4.41)

**Proof.** Recall that \( B \) and \( C \) are recovered from \( K \) by formulas (4.18) and (4.29). In view of (4.15) and (4.24) we have
\[
F_1(x) = \sqrt{2} \theta_1^* e^{2ix\beta}, \quad F_2(x) = G_1(x)^*,
\]
(4.42)

\[
G_1(x) = \sqrt{2} \left( Z_m e^{-2ix\beta^*} - e^{-2ix\beta} \tilde{Z}_m \right) \theta_1 (\tilde{d}_m < x < \tilde{d}_m + 1),
\]
(4.43)

\[
G_2(x) = F_1(x)^*.
\]
(4.44)

Therefore, formulas (4.29) and (4.42)-(4.44) imply that \( B \) and \( C \) corresponding to \( S \) are given by (4.36) and (4.37). It follows from (4.29) and (4.35)-(4.37) that
\[
H(x) = 2\Omega_m e^{-xA} Y_m e^{xA} \Omega_m^{-1} (\tilde{d}_m < x < \tilde{d}_m + 1),
\]
(4.45)

where \( Y_m \) is given in (4.39), \( \Omega_m \) is given in (4.40), and
\[
\Omega_m^{-1} = \begin{bmatrix} I_n & Z_m \\ 0 & I_n \end{bmatrix}.
\]
(4.46)

According to (4.38), (4.39), and (4.45) we get
\[
\left( \frac{d}{dx} U \right)(x) = \Omega_m e^{-xA} \left( \mathcal{A}_m^\times - A \right) e^{xA} \Xi_m^{-1} U(\tilde{d}_m) = H(x)U(x)
\]
for \( \tilde{d}_m < x < \tilde{d}_m + 1 \), and so \( U \) of the form (4.38) satisfies (4.28). In other words, formulas (4.36)-(4.38) define explicitly \( B, C \) and \( U \), which are used in the expressions (4.31) and (4.32) to construct the kernel of \( T = S^{-1} \).
It remains to prove (4.41). Note that
\[ JA^* J^* = -A, \quad J\Omega_m^* J^* = \Omega_m^{-1}, \quad JY_m^* J^* = -Y_m. \] (4.47)
Hence, we have \( JH^* J^* = -H \), i.e.,
\[ \frac{d}{dx}(U(x)^* JU(x)) \equiv 0. \] (4.48)
Formula (4.41) follows from (4.48) and from \( U(0) = I_{2n} \). ■

Taking into account (4.36)-(4.38) we get
\[ \tilde{F}(x) := C(x)U(x) = \sqrt{2}\theta_1^* [I_n - \tilde{Z}_m] e^{x A_n^*} \Xi_m^{-1} U(\tilde{d}_m), \] (4.49)
\[ \tilde{G}(t) := U(t)^{-1} B(t) = \sqrt{2} U(\tilde{d}_m)^{-1} \Xi_m e^{-t A_n^*} \begin{bmatrix} \tilde{Z}_m \\ I_n \end{bmatrix} \theta_1, \] (4.50)
\[ \tilde{d}_m < x < \tilde{d}_{m+1}, \quad \tilde{d}_m < t < \tilde{d}_{m+1}. \]

Corollary 4.8 Let the conditions of Proposition 4.5 be fulfilled. Then the kernel \( T(x,t) \) of the operator \( T = S_t^{-1} \) has the form
\[ T(x,t) = \begin{cases} 
\tilde{F}(x)(I_{2n} - P^x) \tilde{G}(t), & x > t, \\
-\tilde{F}(x) P^x \tilde{G}(t), & x < t,
\end{cases} \] (4.51)
where \( \tilde{F} \) and \( \tilde{G} \) are given by (4.49) and (4.50).

By (4.39), (4.50) for \( \tilde{d}_m < t < \tilde{d}_{m+1} \) we get
\[ \tilde{G}(t)k(t) = \sqrt{2} U(\tilde{d}_m)^{-1} \Xi_m e^{-t A_n^*} (2Y_m) e^{t A} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \nu \sum_{j=1}^{m} P_j R \\
= -\sqrt{2} U(\tilde{d}_m)^{-1} \Xi_m \frac{d}{dt} \left( e^{-t A_n^*} e^{t A} \right) \begin{bmatrix} I_n \\ 0 \end{bmatrix} \nu \sum_{j=1}^{m} P_j R. \] (4.52)

From Remark 3.1 and formulas (4.30), (4.51), and (4.52) the explicit solution of the inverse problem is immediate.
Theorem 4.9 Let the Weyl matrix function \( \varphi \) be given by (4.1), where \( D \) satisfies (4.21) and \( R \) is unitary. Assume also that the matrix identities (4.23), where \( Q_j \) are given by (4.22), have solutions \( X_j = X_j^* \). Then the ISP solution \( v \) is given by the formula

\[
v(l) = k(l) + \tilde{F}(l)(I_{2n} - P^\times) \sum_{m=1}^{N} \sqrt{2} U(\tilde{d}_m)^{-1} \Xi_m \nu \sum_{j=1}^{m} P_j R, \tag{4.53}
\]

where \( k \) is given by (4.11), \( U \) is given by (4.38), \( P^\times \) is given by (4.32), and \( \Xi_m \) is given by (4.40). The number \( N \) in the sum is chosen in the following way: if \( l < \tilde{d}_1 \) then \( N = 0 \); if \( \tilde{d}_j < l < \tilde{d}_{j+1} \) then \( N = j \); if \( l > \tilde{d}_k \) then \( N = k \). We put \( \tilde{d}_m = \tilde{d}_m \) for \( m \leq N \) and \( \tilde{d}_{N+1} = l \).

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