Strongly-resonant \( p \)-wave superfluids

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We study theoretically a dilute gas of identical fermions interacting via a \( p \)-wave resonance. We show that, depending on the microscopic physics, there are two distinct regimes of \( p \)-wave resonant superfluids, which we term “weak” and “strong”. Although expected naively to form a BCS-BEC superfluid, a strongly-resonant \( p \)-wave superfluid is in fact unstable towards the formation of a gas of fermionic triplets. We examine this instability and estimate the lifetime of the \( p \)-wave molecules due to the collisional relaxation into triplets. We discuss consequences for the experimental achievement of \( p \)-wave superfluids in both weakly- and strongly-resonant regimes.

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Recently there has been considerable interest in trying to create a \( p \)-wave resonant superfluid experimentally. The BCS and BEC regimes for such superfluids are not just different aspects of the same phase, as they are for the \( s \)-wave resonant superfluids, but rather are different phases. Thus the tuning from the BCS to BEC regime involves a phase transition (or sometimes a sequence of phase transitions) \(^1\)\(^2\). Such a transition can even be topological in some cases \(^1\)\(^2\)\(^4\)\(^5\)\(^6\). If the superfluid is confined to two dimensions, the BCS phase will be topological and will support vortices with non-Abelian excitations \(^1\)\(^2\).

In this paper we show that resonant \( p \)-wave superfluids must be classified as two distinct types, with weak or strong Feshbach resonances (to be defined precisely later). The existing mean field theory of \( p \)-wave superfluids, worked out in \(^1\)\(^2\)\(^7\), applies only to the case of weak Feshbach resonances. However, as we shall establish below, the \( p \)-wave resonance used in ongoing experiments on \(^{40}\text{K}\) \(^2\)\(^3\)\(^4\) is a strong resonance. It is therefore important to determine the properties of strongly-resonant \( p \)-wave superfluids.

The full theory of strong \( p \)-wave resonances is yet to be constructed. Here we investigate an effect first noticed by Y. Castin and collaborators \(^10\): in the regime of strong \( p \)-wave resonances the fermions form triplet states with angular momentum (spin) 1. Superficially similar to Efimov states \(^11\), these triplets are quite unusual. They are very strongly bound, with a binding energy largely independent of detuning from the resonance, as long as the detuning is not too large (but dependent on the strength of the resonance). Correspondingly, their size is of the order of the closed channel bound molecular state, far smaller than the average interparticle separation. We find the critical value of the resonance’s strength at which the triplets first appear, and calculate their binding energy as a function of the resonance strength.

Thus if a BEC condensate of strongly-resonant \( p \)-wave molecules is created, one of its main channels of decay will be by molecular inelastic collisions, with two molecules turning into one atom and one triplet. We estimate the molecular lifetime due to this process and compare this with experimental observations\(^9\). We discuss limitations on the achievement of \( p \)-wave superfluids in both weak and strong resonances, arising from this and other inelastic decay processes.

The theory developed here can be used to investigate the true ground state of a strongly-resonant \( p \)-wave condensate. This is likely to be a gas of fermionic spin 1 triplets (or possibly of larger composite particles).

We consider a \( p \)-wave resonantly coupled superfluid, whose Hamiltonian is given by \(^1\)\(^2\)\(^3\)\(^4\):

\[
H = \sum_p \frac{p^2}{2m} \hat{\alpha}_p^{\dagger} \hat{\alpha}_p + \sum_{\mathbf{q},\mu} \left( \epsilon_0 + \frac{q^2}{4m} \right) \hat{b}_{\mu \mathbf{q}}^{\dagger} \hat{b}_{\mu \mathbf{q}}
\]

\[
+ \sum_{\mathbf{q},\mathbf{p},\mu} \frac{g(|\mathbf{p}|)}{\sqrt{V}} \left( \hat{b}_{\mu \mathbf{q}} p_\mu \hat{\alpha}^{\dagger}_{\frac{\mathbf{p}+\mathbf{q}}{2}} \hat{\alpha}^{\dagger}_{\frac{\mathbf{p}+\mathbf{q}}{2}} + h.c. \right).
\]

Here \( \hat{\alpha}_p \), \( \hat{\alpha}^{\dagger}_p \) are the creation and annihilation operators of a spinless fermion (atom) with mass \( m \), and \( \hat{b}^{\dagger}_{\mu \mathbf{q}} \), \( \hat{b}_{\mu \mathbf{q}} \) are the creation and annihilation operators of a bosonic diatomic molecule of spin 1 (the 3D vector index \( \mu \) represents the projection of spin). This superfluid is controlled by four parameters. The first two are the detuning \( \epsilon_0 \) and the overall particle number \( N \), an expectation value of the operator \( \hat{N} = \sum_p \hat{\alpha}_p^{\dagger} \hat{\alpha}_p + 2 \sum_{\mu,\mathbf{q}} \hat{b}^{\dagger}_{\mu \mathbf{q}} \hat{b}_{\mu \mathbf{q}} \). It is more convenient to work with the energy equivalent of \( N \), the Fermi energy \( \epsilon_F = \left( 6\pi^2 h^2 N/V \right)^{2/3} / (2m) \). The other two are contained in the coupling constant \( g(|\mathbf{p}|) \). The physical origin of the dependence of \( g \) on \( |\mathbf{p}| \) lies in the fact that the molecules have finite size. This can be captured by choosing \( g \) to remain constant as long as \( |\mathbf{p}| \ll \Lambda \) (which we denote simply by \( g \)) and quickly drop to zero if \( |\mathbf{p}| \gg \Lambda \). Here \( R_c \sim h/\Lambda \) is the physical (closed-channel) size of the molecules. The knowledge of exactly how \( g \) drops to zero at large momenta may be important. In this paper we adopt the “hard momentum cutoff” approach \( g(|\mathbf{p}|) = g(\Theta(\Lambda - |\mathbf{p}|)) \) (\( \Theta \) is equal to 1 or
0 depending on whether its argument is positive or negative. We have studied other types of cutoff, and find these do not change the main conclusions of this paper.

Two dimensionless parameters can be constructed out of $g$, $\epsilon_F$, and $\Lambda$, namely

$$\gamma = \frac{m^2 g^2 \sqrt{\epsilon_F}}{\hbar^3}, \quad c_2 = \frac{m^2 g^2 \Lambda}{3\pi^2 \hbar^3}. \quad (2)$$

Notice that in order to observe universal (short distance physics independent) behavior, the interparticle separation ($\sim \hbar/\sqrt{m\epsilon_F}$) must be kept much bigger than $R_c$, thus $\gamma \ll c_2$.

Both of these parameters control the perturbative expansion of $\Pi$ in powers of the coupling $g$. It is customary, when analyzing Eq. (1), to apply a mean field approximation whose validity is based on the smallness of $g$. Strictly speaking, both $\gamma$ and $c_2$ must be small in order for the mean field approximation employed in the original publications investigating Eq. (1) [1, 2] to be valid. $\gamma$ depends on the interparticle separation and can be made small simply by reducing the particle density. $c_2$ however depends solely on the physics of the Feshbach resonance which led to Eq. (1): its value, which can be small or large, is fixed by the atomic type and Feshbach resonance involved, so it cannot be continuously controlled.

One terms the superfluids with $\gamma \ll 1$ as those with narrow Feshbach resonances, while the ones with $\gamma \gg 1$ are the broad Feshbach resonance superfluids [1, 5, 12]. Likewise, we will term the $c_2 \gg 1$ resonances as the strong $p$-wave Feshbach resonances, while those with $c_2 \ll 1$ are weak resonances. The $p$-wave resonances are typically narrow because, even if they are not, they can be made narrow by reducing the particle density.

The narrow and weak $p$-wave resonances have been thoroughly investigated in prior publications. It is therefore imperative to consider the narrow and strong resonances. The main idea behind the analysis is based on the fact that fluctuational corrections to the mean field come from two distinct regions in momentum space, $p$ of order $\hbar/l$ where $l$ is interparticle spacing, and $p$ of order $\Lambda$. The former capture the many-body physics of Eq. (1) and are small as long as $\gamma$ is small. The latter come from high momenta and energies at which no real particles propagate. Thus this contribution, controlled by $c_2$, is essentially few-body, equivalent to solving some few-body Schrödinger equation, which although difficult, is not an impossible task.

The analysis carried out in Ref. [7] showed that the $p$-wave Feshbach resonance in $^{40}$K used in Refs. [8, 9] was strong. The derivation relied on the scattering amplitude of two atoms calculated in Ref. [1, 7]

$$f(k) = \frac{k^2}{\frac{k^2}{2} + i k_0 k^2 - i k^3}. \quad (3)$$

where

$$v = -\frac{mg^2}{6\pi \hbar (1 + c_2) \omega_0}, \quad k_0 = -\frac{4\Lambda (1 + c_2)}{\pi c_2}. \quad (4)$$

Here $v$ is the so-called effective volume, controlled by the physical detuning $\omega_0$ [related to $c_0$ by $\omega_0 = (c_0 - mg^2 \Lambda^3/(9\pi^2 \hbar^2))/1 + c_2)]$ and $k_0$ is a parameter similar to the effective range of $s$-wave scattering (having, however, the dimensions of inverse length). If $k_0$ and $\Lambda$ are known (numerically or experimentally), $c_2$ can be found from (1).

We also remark that had we considered a one-channel model of identical fermions interacting via a short range $p$-wave potential, such as $V(r) = \lambda \partial^2 \phi(r) \partial^2 \phi$, we would have obtained Eq. (3) with $k_0 \sim -\Lambda$ [7]. In other words, such a model automatically describes strong resonances.

We now turn our attention to the physical consequences of strong resonances. Its main consequence is the existence of a bound state of three atoms when $c_2$ exceeds a certain threshold. To show this, we calculate the scattering amplitude of one atom and one molecule. This is given by a sequence of diagrams depicted on Fig. 1. These diagrams identical to the ones studied in the context of the $s$-wave BCS-BEC crossover [13, 14].

Here the atoms propagate with the free propagator $G(p, \omega) = 1/ \left( \omega - \frac{p^2}{2m} + i\delta \right)$, while to find the molecular propagator one needs to calculate its self-energy [7]

$$D_{\mu\nu}(q, \omega) = \delta_{\mu\nu} \left[ (1 + c_2) \left( \omega - \frac{q^2}{4m} - \omega_0 + i\delta \right) + c_2 \sqrt{m} \frac{4\Lambda}{q^2 - 4m\omega - i\delta} \right], \quad (5)$$

(In these, and subsequent, expressions we set $\hbar = 1$ for clarity.) Each loop in the diagrams on Fig. 1 is linearly divergent. It is this divergence, occurring at momenta $p \sim \Lambda$ and controlled by $c_2$, which would like to capture. To do so, we study the atom-dimer scattering problem with the following kinematics; a boson of spin $\mu$ and $4$-momentum $(0, \kappa + E_3)$ scatters off a fermion with $4$-momentum $(0, 0)$. The outgoing particles are a boson with spin $\nu$ and $4$-momentum $(q, q_0 + \kappa + E_3)$ and a fermion with $(-q, -q_0)$. Here $\kappa(\omega_0)$ is an implicit function of the detuning such that the bosonic propagator $D(q, q_0 + \kappa)$ has a pole as $q, q_0 \rightarrow 0$. $E_3 \leq 0$ is the energy at which we are looking for a bound state. The scattering $T$-matrix has the following general form

$$T_{\mu\nu}(p, p_0) = T_1(p, p_0) \delta_{\mu\nu} + T_2(p, p_0) p_\mu p_\nu / p^2. \quad (6)$$

FIG. 1: The diagrams whose sum gives the scattering amplitude between an atom and a molecule.
and the scattering length $a_{\text{dd}}$ is related to $T_1(0,0)$ (evaluated at $E_3 = 0$) as $a_{\text{dd}} = \frac{2}{3\pi} T_1(0,0)$.

The integral equation for the $T$-matrix is derived analogously to the $s$-wave problem [13, 14], and is

$$T_{\mu \nu}(p, p_0) = -\frac{2}{1 + c_2} G(p, p_0 + \kappa + E_3) p_{\mu} p_{\nu} g(|p|/2) - 4i \int \frac{d^4 q}{(2\pi)^4} T_{\mu \alpha}(q, q_0) D(q, q_0 + \kappa + E_3) G(-q, -q_0) \times G(p + q, p_0 + q_0 + \kappa + E_3)(p + q/2)_{\alpha} g(|p + q/2|) g(|q + p/2|)$$  \hspace{1cm} (7)

The factor $1 + c_2$ is the inverse residue of the bosonic propagator. $T_{\mu \nu}(q, q_0)$ is analytic in the upper halfplane of $q_0$ and thus we may integrate out $q_0$, setting $q_0 \to -q^2/2m$. To solve the integral equation we then let $p_0 \to -p^2/2m$. For simplicity define $T_3(p, -p^2/2) \equiv T_3(p)$. Measuring momenta in units of the cutoff, energies in units of $\Lambda^2/m$, and the $T$-matrix itself in units of $1/(m\Lambda)$ we find the integral equation

$$T_3(p) = 6\pi^2 \frac{c_2}{1 + c_2} p^2 \frac{p^2}{\kappa + E_3 - p^2 - q^2} \Theta(1 - p) \delta_{2j} - 3c_2 \int q^2 dq D(q, -q^2/2 + \kappa + E_3)a_{ji}(p, q)T_i(q)$$  \hspace{1cm} (8)

The coefficients $a_{ji}(p, q)$ are given by an integration over directions of $q$

$$a_{ji}(p, q) = \int \frac{d\Omega_q}{4\pi} \frac{1}{\kappa + E_3 - p^2 - q^2 - p \cdot q} \delta_{ji} \left( \frac{2p \cdot p_0}{p^2} \right)_k \left( \frac{2q \cdot q_0}{q^2} \right)_k \times \frac{1}{g^2} (p + q/2)_{\alpha} (q + p/2)_{\nu} g(|p + q/2|) g(|q + p/2|)$$  \hspace{1cm} (9)

The scattering length $a_{\text{dd}}$ is found by solving Eq. 8 at $E_3 = 0$. The binding energy of the triplet corresponds to a pole in the $T$-matrix and thus to a solution of the homogeneous integral equation at a specific value of $E_3$. Fig. 2(a) shows how the scattering length is negative for a weak Feshbach resonance, becoming more negative and diverging at $c_2 \approx 3.3$. This is the strength of the resonance at which the bound triplet appears, as illustrated in Fig. 2(b). As $c_2 \to \infty$ the scattering length saturates at $a_{\text{dd}} \approx 1.9/\Lambda$ and the binding energy at $E_3 \approx -0.11\Lambda^2/m$. The existence of a finite $c_2 \to \infty$ limit can also be seen by observing that each of the diagrams depicted on Fig. 3 has a finite $c_2 \to \infty$ limit. It should be noted that whereas the presence of the bound state and the general features of the scattering length and binding energy do not depend on the method of cutoff, the exact numerical values will in general depend on the method chosen.

The existence of the bound trimer state for large $c_2$ raises the possibility of an inelastic decay channel in which two dimers collide to leave a trimer and an unbound atom (with large relative velocity). (Henceforth we use the term “dimer” to refer to a molecule of two atoms, to distinguish this clearly from a triplet, or trimer.) In a non-degenerate gas of dimers, these inelastic losses will cause the density of dimers $n_{\text{dd}}$ to decay as

$$\frac{dn_{\text{dd}}}{dt} = -\alpha_{\text{dd}} n_{\text{dd}}^2$$  \hspace{1cm} (10)

with $\alpha_{\text{dd}} = 2\frac{\hbar}{m}(k_i \sigma_i(k_i))$ where the average is over the relative momenta of the incident dimers, $k_i$, and $\sigma_i(k_i) = \int |f_{\text{in}}|^2 d\Omega_{k_i}$ with $f_{\text{in}}$ the inelastic scattering amplitude into a final momentum $k_f$. Arguments similar to the ones presented above for dimer-atom scattering show that $|f_{\text{in}}|^2 \sim R_i^2$. Thus, for large $c_2$, such that the trimer binding energy is $-E_3 \sim \hbar^2/(mR_2^2)$ and is large compared to the incident kinetic energy, one finds

$$\alpha_{\text{dd}} \sim \frac{\hbar}{m} R_e .$$  \hspace{1cm} (11)

It is instructive to compare this result with the inelastic decay constants into deep bound states for $s$-wave dimers, formed from (two-component) fermions or from bosons with $s$-wave scattering length $a$. Close to the $s$-wave resonance, $a \gg R_e$, the decay constant (11) is much smaller than that expected for bosons, $\alpha_{\text{dd}}^{\text{boson}} \sim h/ma$. 

![FIG. 2: (a) Scattering length $a_{\text{dd}}$ in units of $\hbar/\Lambda$ and (b) binding energy $E_3$ of the triplet in units of $\Lambda^2/m$, both as functions of $c_2$. Here, detuning has been set to zero. The large $c_2$ limit is indicated.](image-url)
but is larger than that for s-wave dimers of fermions, \( \alpha_{dd}^{\text{fermion}} \sim \frac{\hbar^2}{m^2} (R_e/a)^{2.55} \) [13]. The suppressed decay of s-wave dimers of fermions is explained in Ref. [13] as an effect of the Pauli principle, reducing the probability to find three atoms within a lengthscale \( R_e \). In a p-wave dimer the two atoms have a probability of order unity to be inside the centrifugal barrier, at a separation of order \( R_e \). Taking this feature of the p-wave dimers into account, simple estimates lead to \( \alpha_{dd} \sim \hbar R_e/m \) for decay into trimers, consistent with the result from the T-matrix calculation. In addition to this channel, there are inelastic channels – active for both weak and strong resonances – involving decay into deep dimer states. Applying the same simple estimates, one finds that the inelastic decay constants for dimer-dimer and dimer-atom scattering are also \( \alpha_{dd} \sim \alpha_{da} \sim \hbar R_e/m \).

In recent experimental work [9] a gas of p-wave Feshbach dimers was created in \(^{40}\text{K}\). Unfortunately the lifetime of the dimers was observed to be quite short, about 2 ms. While \(^{40}\text{K}\) can suffer losses through dipolar relaxation (an effect expected to be absent for p-wave resonances in other fermionic systems, for example \(^{6}\text{Li}\)), Ref. [9] found that the lifetime was shorter than that predicted for dipolar relaxation alone. Additional losses could arise from inelastic collisions of the dimers. This mechanism would imply a density dependence of the decay rate; this dependence has not, as yet, been established experimentally.

Within the above considerations, we expect the decay rate of dimers via relaxation into deep trimers or dimers under inelastic collisions (with other dimers or with unbound atoms) to be of order \( \Gamma_{\text{in}} \sim \frac{\hbar R_e}{m} n \), where \( n \) is the density of atoms or dimers with which a given dimer can collide. Taking \( n \approx 7 \times 10^{12} \text{cm}^{-3} \) (the atomic density in the experiments of Ref. [9]) we find \( \Gamma_{\text{in}} \sim 10 \text{Hz} \). This estimate is more than one order of magnitude smaller than the additional decay rate required to account for the observations of Ref. [9]. However, we note that the prefactor to the estimate is uncertain. In view of this uncertainty, and in view of the lack of clear evidence of a density dependence in the experiment, it remains an open issue whether the dimer lifetime in Ref. [9] is limited by inelastic collisions.

Our analysis has important consequences for possibilities to achieve superfluid phases close to a p-wave resonance. On the BEC side of the resonance, our calculations show that the elastic dimer-dimer scattering amplitude is \( f_{dd} \sim R_e \). Consequently, the elastic scattering rate is of order \( \Gamma_{\text{el}} \sim \frac{\hbar R_e}{m} n_{\text{d}} (k R_e) \), which is typically much smaller than the inelastic decay rate, \( \Gamma_{\text{in}} \sim \frac{\hbar R_e}{m} n_{\text{d}} \). (For a BEC of dimers, \( k_i \) is small compared to the inverse particle spacing, \( 1/l \), so \( k_i R_e \ll R_e/l \ll 1 \)). It is therefore unlikely that a BEC of dimers can undergo sufficient elastic collisions to thermalize before inelastic losses deplete the gas. On the other hand, on the BCS side of the resonance, thermalization can proceed at a much faster rate, and will be limited by the rate of hybridisation of the dimers with the unbound atoms (this is the rate at which pairs of atoms can exchange their relative momentum). Assuming the densities of dimers and atoms to be comparable, \( n_{\text{d}} \sim n_{\text{a}} \equiv n \), one finds that the hybridisation rate, as set by the width of the resonance [3], is \( \Gamma_{\text{hyb}} \sim \frac{7}{r_e^2} \frac{\hbar}{m} \). Using [2], we find that \( \Gamma_{\text{hyb}}/\Gamma_{\text{in}} \sim \frac{n}{r_e^2} \). Thus, provided the resonance is not very weak (\( r_e \) very small), the rate of hybridisation is parametrically the same as \( \Gamma_{\text{in}} \), and the system may thermalize before inelastic losses deplete the gas. Thus our results show that it is on the BCS side of a strong resonance that one has the best opportunity to attain a thermalized p-wave superfluid phase. Finally, we note that the limitations we have described in this paragraph, arising from decay into deep bound states, could be eliminated in an “optical Feshbach” scheme in which the particles are coupled to a deep closed-channel molecule. In this case, it is important that the resonance should be sufficiently weak in order also to eliminate inelastic decay processes into the triplet states that always exist for strong resonances.

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