On D-spaces and Discrete Families of Sets

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Abstract. We prove several reflection theorems on D-spaces, which are Hausdorff topological spaces $X$ in which for every open neighbourhood assignment $U$ there is a closed discrete subspace $D$ such that

$$\bigcup\{U(x) : x \in D\} = X.$$ 

The upwards reflection theorems are obtained in the presence of a forcing axiom, while most of the downwards reflection results use large cardinal assumptions.

The combinatorial content of arguments showing that a given space is a D-space, can be formulated using the concept of discrete families. We note the connection between non-reflection arguments involving discrete families and the well known question of the existence of families allowing partial transversals without having a transversal themselves, and use it to give non-trivial instances of the incompactness phenomenon in the context of discretisations.

1. Introduction

We prove some reflection results about D-spaces, and note their combinatorial equivalent entitled discrete families. D-spaces were defined by E.K. van Douwen in 1978, and studied by van Douwen and W.K. Pfeffer [2], van Douwen and D. Lutzer [1] and W. Fleissner and A.M. Stanley [5], among others. To define D-spaces, recall that an open neighbourhood assignment (ONA) in a topological space $X$ is a function $U$ on $X$ such that for all $x \in X$ we have that $U(x)$ is an open neighbourhood of $x$. A space is said to be a D-space if for every ONA $U$ of $X$, there is a closed discrete $D \subseteq X$ such that $\bigcup\{U(x) : x \in D\} = X$. We equivalently say that a space is D or has property D.

It is easy to see that compact spaces, discrete spaces and metric spaces are D. A very puzzling open question about D-spaces is if all Lindelöf spaces are D. In an attempt to solve this question van Douwen and Pfeffer [2] studied the Sorgenfrey line $S$ and proved that all finite powers of $S$ are D, as well as introducing a larger class of spaces which are D. Continuing this line of research, the known results...
about $D$-spaces often concentrate on generalised metric spaces and linearly ordered spaces (LOTS) and their products. Much of this effort, including the question about Lindelöf spaces being $D$, can be viewed as focussing on the natural generalisations of the fact that compact spaces are $D$. We note that there is another natural line of generalisation of the basic facts about $D$-spaces, namely the observation that discrete spaces are $D$. In this vein, one should consider spaces which are locally small, that is, every point has a small neighbourhood. We prove for example (Corollary 2.17) that it is consistent with $CH$ and $2^{\aleph_1} > \aleph_2$ that every locally countable Hausdorff space of size $\leq \aleph_2$ in which every open subspace of size $\leq \aleph_1$ is $D$ in a strong sense, is $D$ itself. It is perhaps worth mentioning that there seems to be inherent difficulties in proving consistency results about $D$-spaces. In particular, the result we prove is to our knowledge the first such result, and it is still a reflection argument rather than an outright consistency result.

As the result just mentioned is an upward reflection result, it is natural to ask if there are also downward reflection results. With the help of large cardinal assumptions, we can get some such results. Namely, we prove that if $\kappa$ is a measurable cardinal and $\sigma < \kappa$, then every locally $\kappa$-$\sigma$-$D$ space of size $\kappa$ in which every point has a point-base of size $\kappa$, has open $\sigma$-$D$ subspaces of sizes arbitrarily close to but less than $\kappa$ (the notion of $\sigma$-$D$ spaces is defined below and is crucial for the upwards reflection results from the first section).

The same argument can be used with $D$ in place of $\sigma$-$D$, but we give instead an improvement due to W. Fleissner, where the large cardinal assumption is reduced to a strong inaccessible and the assumption of small character is not needed. The point in both arguments is to get open subspaces, as getting closed subspaces, for instance, is very easy since the property of being $D$ reflects downward to closed subspaces.

The paper finishes with a section on discrete families. When working with $D$-spaces, one quickly realises that there is a combinatorial argument repeatedly being used. Formalising the ingredients of this argument, we can abstract the combinatorial content of the $D$-space context, and arrive at the notion of discrete families. We give a short discussion of these, and note that reflection arguments about discrete families have a lot to do with the well studied problem of the existence of transversals. Then we use this observation and the known results about transversals to give non-trivial instances of the existence of discrete families.

Although many of the results mentioned here are still valid if we work with spaces which are only assumed to be $T_1$, we shall for simplicity only study Hausdorff topological spaces.

2. A consistency result on upwards reflection

In this section we prove the upwards reflection theorem announced in the introduction (Corollary 2.17), obtaining it as a consequence of the following more general Theorem.

**Theorem 2.1.** Suppose that $V$ is a universe in which

$$\aleph_0 \leq \lambda = \lambda^{+\lambda} < \lambda^+ = \kappa < \lambda^{++} = \mu$$

and $\mu^{+\lambda} = \mu$, while $2^\lambda = \kappa$ and $2^\kappa = \kappa^+$.

Then there is a cofinality and cardinality preserving forcing extension of $V$ in which no bounded subsets are added to $\kappa$ and the following hold:
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: (i) $2^\lambda = \lambda^+$ and $2^{\lambda^+} = \mu$.
: (ii) every locally $< \kappa$ topological space of size $\leq \kappa^+$ in which for every ONA there is a finer ONA with respect to which all open subspaces of size $\leq \kappa$ are $\kappa$-$D$, is $D$ itself.

Let us first recall the definitions of an ONA and a $D$-space, which were mentioned in the Introduction, give some background to the concepts needed for the proof, and most importantly, define what a $\kappa$-$D$-space is.

Definition 2.2. (1) An open neighbourhood assignment (ONA) in a topological space $(X, \tau)$ is a function $U : X \to \tau$ such that for all $x \in X$ we have that $x \in U(x)$. (2) A space is said to be a $D$-space iff for every ONA $U$ of $X$, there is a closed discrete $D \subseteq X$ such that $\bigcup \{U(x) : x \in D\} = X$. (3) For a cardinal $\kappa$, a topological space $X$ is said to be locally $< \kappa$ iff there is an ONA $U$ of $X$ such that $|U(x)| < \kappa$ for all $x \in X$. (4) If $U$ is an ONA of $X$ for which there is a closed discrete $D$ with $\bigcup U^+D \overset{\text{def}}{=} \bigcup \{U(x) : x \in D\} = X$, we say that $X$ is $D$ with respect to $U$.

Observation 2.3. If $U, U^*$ are ONA of $X$ such that $\forall x(U(x) \subseteq U^*(x))$, and $X$ is $D$ with respect to $U$, then $X$ is $D$ with respect to $U^*$.

Given a space $X$ and an ONA $U$ on it, if one tries to construct inductively or otherwise a subspace $D \subseteq X$ demonstrating that $X$ is a $D$-space with respect to $U$, there are two apparent difficulties that one may run into. One of these is that taking unions of infinitely many closed discrete subspaces does not necessarily give a closed subspace. This difficulty is resolved through the use of $U$-sticky sets, as introduced and studied by Fleissner and Stanley in [5].

Definition 2.4. Given an ONA $U$ of a topological space $X$:

(1) A subspace $D$ of $X$ is said to be $U$-sticky iff $D$ is closed discrete and satisfies

$$(\forall x \in X)[U(x) \cap D \neq \emptyset \implies x \in \bigcup U^+D].$$

(2) The partial order $\mathbb{P}_U = \mathbb{P}_U(X)$ is defined by letting

$\mathbb{P}_U \overset{\text{def}}{=} \{D \subseteq X : D \text{ is } U\text{-sticky}\}$,

ordered by letting $D \leq D'$ (where $D'$ is a stronger condition) iff $D \subseteq D'$ and $(D' \setminus D) \cap \bigcup U^+D = \emptyset$.

Fleissner and Stanley proved that $\mathbb{P}_U$ is well behaved with respect to the unions of $\leq$-increasing chains, see the following Theorem [2.5.1]. This gives hope that one could use $\mathbb{P}_U$ in inductive constructions, or as a forcing notion, but at least as much as the latter is concerned, this hope is slighted by a further result of Fleissner and Stanley. Namely, the second of the difficulties mentioned above, is a density problem: given a $U$-sticky $D$, and an $x \in X$, can we find an extension $D'$ of $D$ within $\mathbb{P}_U$ for which we have $x \in \bigcup U^+D'$? The second part of Theorem [2.5] shows that such a density condition is present iff $X$ is a $D$-space. While this gives a very
interesting characterisation of $D$-spaces, it also shows that $\mathbb{P}_U(X)$ cannot be used as a forcing notion to make $X$ into a $D$-space, if $X$ was not a $D$-space to start with.

**Theorem 2.5.** [Fleissner-Stanley] Given a topological space $X$ and an open neighbourhood assignment $U$ of $X$. Let $\mathbb{P}_U(X)$ be as defined above, in Definition 2.4(2). Then

1. If $D$ is a subset of $\mathbb{P}_U$ in which every $D, D'$ satisfy either $D \leq D'$ or $D' \leq D$, then $\bigcup D \in \mathbb{P}_U$. 
2. $X$ is a $D$-space iff for every ONA $U$ of $X$, every $D \in \mathbb{P}_U$, and every $x \in X$, there is a $D' \geq D$ in $\mathbb{P}_U$ such that $x \in \bigcup U^* D'$.

In the following discussion we shall be assuming that $X$ is a given topological space and $U$ an ONA on $X$. We shall work with a variant of $U$-stickiness which will be used as a forcing notion. For a given regular cardinal $\kappa \geq \aleph_1$ we define the partial order $\mathbb{P}_U^\kappa$ as follows:

**Definition 2.6.** The partial order $\mathbb{P}_U^\kappa = \mathbb{P}_U^\kappa(X)$ is defined by letting

$$
\mathbb{P}_U^\kappa \overset{\text{def}}{=} \{D \subseteq X : D \text{ is } U\text{-sticky } \& \text{ } |D| < \kappa\},
$$

ordered by letting $D \leq D'$ (where $D'$ is a stronger condition) iff $D \subseteq D'$ and $(D' \setminus D) \cap \bigcup U^* D = \emptyset$.

We start the discussion of $\mathbb{P}_U^\kappa$ by a slight generalisation of Theorem 2.5(1), proved in a manner similar to the one used for the proof of that theorem.

**Observation 2.7.** (1) If $D$ is a subset of $\mathbb{P}_U^\kappa$ with $|D| < \kappa$ and such that for each $D, D' \in D$ there is a $D'' \in D$ with $D'' \geq D, D'$, then $\bigcup D$ is an element of $\mathbb{P}_U^\kappa$. (2) If $D$ is a directed subset of $\mathbb{P}_U$, then $\bigcup D$ is an element of $\mathbb{P}_U^\kappa$.

**Proof.** (1) Let $D^* \overset{\text{def}}{=} \bigcup D$, and let $x \in X$. If $U(x) \cap D^* = \emptyset$, then certainly $x \notin D^* \setminus \{x\}$. Otherwise, there is $D \in D$ such that $U(x) \cap D \neq \emptyset$. Since $D \in \mathbb{P}_U^\kappa$, we have that $x \in \bigcup U^* D \subseteq \bigcup U^* D^*$. This demonstrates the second part of the definition of being $U$-sticky.

Fix an $x \in X$ again. We shall use it to show that $D^*$ is closed and discrete. Suppose that $U(x) \cap D \neq \emptyset$ for some $D \in D$. Since $D$ is closed discrete, there is an open neighbourhood $V$ of $x$ with $V \cap D$ finite. As $U(x) \cap D \neq \emptyset$ we have that $x \in \bigcup U^* D$, so we can assume that $V \subseteq U(x) \cap \bigcup U^* D$. Given $D' \in D$, let $D''$ be a common extension of $D$ and $D'$ in $\mathbb{P}_U^\kappa$. Hence

$$(D' \setminus D) \cap \bigcup U^* D \subseteq (D'' \setminus D) \cap \bigcup U^* D = \emptyset,$$

so $V \cap D' \subseteq \bigcup U^* D \cap D'$ and $V \cap D' \subseteq V \cap D$. In conclusion, $V \cap D' = V \cap D$ is finite, and so $x \notin D^* \setminus \{x\}$, by the Hausdorff property of $X$. If $U(x) \cap D = \emptyset$ for all $D \in D$, then $U(x) \cap D^* = \emptyset$, so clearly $x \notin D^* \setminus \{x\}$. This argument demonstrates that $\bigcup D$ is closed discrete. As $|D| < \kappa$, we have $|D^*| < \kappa$.

(2) The same argument as above, omitting the last sentence. \textit{\#2.7}

The order $\mathbb{P}_U^\kappa$ can be used to define what is meant by a $\kappa$-$D$-space, keeping until a further notice the convention that $\kappa$ is a chosen uncountable regular cardinal.
Definition 2.8. (1) $X$ is said to be $\kappa$-$D$ with respect to an ONA $U$ of $X$ iff for every $D \in \mathbb{P}_U^\kappa$, and every $x \in X$, there is $D' \succeq D$ in $\mathbb{P}_U^\kappa$ such that $x \in \bigcup U \cdot D'$.

(2) Given two ONA $U$ and $V$ of $X$, we say that $V$ is finer than $U$ iff $V(x) \subseteq U(x)$ for all $x \in X$.

(3) $X$ is $\kappa$-$D$ iff for every ONA $U$ of $X$ there is a finer ONA $V$ such that $X$ is $\kappa$-$D$ with respect to $V$.

We use the terminology “strongly-$D$” in place of “$\aleph_1$-$D$”.

Note. The definition of $X$ being $\kappa$-$D$ does not require $X$ to be $\kappa$-$D$ with respect to every ONA $U$ of $X$. Note also that $X$ is $D$ iff $X$ is $|X|^+-$D.

The choice of our terminology can be explained by the following

Observation 2.9. Suppose that $\kappa \geq \aleph_1$ is regular. Then every $\kappa$-$D$-space of size $\leq \kappa$ is $D$.

Proof. Suppose that $X = \{x_\alpha : \alpha < \alpha^+ \leq \kappa\}$ is a given $\kappa$-$D$-space and that $U^*$ is a given ONA of $X$. Let $U$ be a finer ONA such that $X$ is $\kappa$-$D$ with respect to $U$. By induction on $\alpha$ we define $\langle D_\alpha : \alpha \leq \alpha^* \rangle$, a continuous increasing chain of elements of $\mathbb{P}_U^\kappa$, with $D_0 = \emptyset$ and $x_\alpha \in \bigcup U \cdot D_{\alpha+1}$, while $\bigcup \alpha \leq \alpha^*, U \cdot D_\alpha = X$. The induction at successor stages uses the assumption of $\kappa$-$D$-ness, and at limit stages, Observation 2.7(1). Using the same Observation, we can see that taking $D = \bigcup \alpha \leq \alpha^*, D_\alpha$ demonstrates that $X$ is $D$ with respect to $U$, hence $X$ is $D$ with respect to $U^*$ (Observation 2.3).

Further discussion about the relationship between being $D$ and $\kappa$-$D$ can be found at the end of this section. We intend to use $\mathbb{P}_U^\kappa$ in a universe of set theory in which a certain version of Martin’s axiom for $\kappa^+\kappa$ holds. For this we shall need a theorem of S. Shelah from 10 (there proved with $\kappa = \aleph_1$, but, as is well known to the author of 10 and has been used in many of his results, the same proof gives the more general result here quoted as Theorem 2.11). To introduce this theorem, we need a definition.

Definition 2.10. We say that a forcing notion $\mathbb{P}$ satisfies $(*)_{\kappa}$ iff the following conditions (a)-(c) hold:

: (a) if $p, q$ are compatible in $\mathbb{P}$, then they have a least upper bound (lub) in $\mathbb{P}$,

: (b) if $\langle p_\alpha : \alpha < \alpha^+ < \kappa \rangle$ is an increasing sequence of conditions in $\mathbb{P}$, then the sequence has the lub in $\mathbb{P}$,

: (c) if $\langle p_i : i < \kappa^+ \rangle$ is a set of conditions in $\mathbb{P}$, then there is a club $C \subseteq \kappa^+$ and a regressive function $f$ on $C$ such that whenever $i, j \in C$ are of cofinality $\kappa$ and $f(i) = f(j)$, then $p_i$ and $p_j$ are compatible.

Theorem 2.11. [Shelah] Suppose that $\mathbf{V}$ is a universe of set theory in which

$\aleph_0 \leq \lambda = \lambda^\kappa < \lambda^+ = \kappa < \lambda^{++} < \mu$

and $\mu^\kappa = \mu$, while $2^\lambda = \kappa$ and $2^\kappa = \kappa^+$.

Then there is a cofinality and cardinality preserving forcing extension of $\mathbf{V}$ in which no bounded subsets are added to $\kappa$ and the following hold:

: (i) $2^\lambda = \lambda^+$ and $2^\kappa = \mu$,
of conditions in
Proof.
By Observation 2.7(1) we have
to show that
condition. W e can without loss of generality assume that
Let us prove the nontrivial part of this Observation, so assu me that
Proof.
Observation 2.7 allows for an easy proof of the following
Observation 2.12. Suppose that \( \langle D_\alpha : \alpha < \alpha^+ \rangle \) is an increasing sequence
of conditions in \( \mathbb{P}_U^\kappa \). Then \( \bigcup_{\alpha < \alpha^+} D_\alpha \) is the lub of
the sequence.
Proof. By Observation 2.7(1) we have
\[ D \overset{\text{def}}{=} \bigcup_{\alpha < \alpha^+} D_\alpha \in \mathbb{P}_U^\kappa, \]
so we only need to show that \( D \) is an extension of each \( D_\alpha \), and
that is actually the least such condition. We can without loss of generality assume that \( \alpha^+ \)
is a limit ordinal.
Let \( \alpha < \alpha^+ \). If \( x \in D \setminus D_\alpha \), then for some \( \beta \in (\alpha, \alpha^+) \) we have
\( x \in D_\beta \), so, since
\[ D_\alpha \subseteq D_\beta, \] we have that \( x \notin \bigcup U^+D_\alpha \). Hence \( D_\alpha \subseteq D \).
Suppose that \( D' \geq D_\alpha \) for all \( \alpha \). In particular, \( D' \supseteq D \). If \( x \in D' \setminus D \)
but \( x \in \bigcup U^+D \), then there is \( \alpha < \alpha^+ \) such that \( x \in \bigcup U^+D_\alpha \),
contradicting that \( D_\alpha \subseteq D \) and \( D_\alpha \leq D' \). Hence \( x \in D \setminus D \implies x \notin \bigcup U^+D \),
so \( D \leq D' \).

The next item needed for the proof of Theorem 2.1 is the existence of a lub
of two conditions compatible in \( \mathbb{P}_U^\kappa \), and the proof of this is another elementary
argument of the sort used to prove the previous Observations.

Observation 2.13. If \( D, D' \in \mathbb{P}_U^\kappa \), then \( D \) and \( D' \) are compatible iff their
union \( D \cup D' \) is their common upper bound, in which case \( D \cup D' \) is the lub of \( D \)
and \( D' \) in \( \mathbb{P}_U^\kappa \).
Proof. Let us prove the nontrivial part of this Observation, so assume that \( D'' \)
is a common upper bound of \( D \) and \( D' \). Clearly, \( D \cup D' \) is closed and discrete, and
has size \( \leq \kappa \). If \( x \) is an element of \( (D \cup D') \setminus D \), then \( x \in D'' \setminus D \), and in particular
\( x \notin \bigcup U^+D \). By symmetry, the same argument can be applied with \( D' \)
in place of \( D \). Next, if \( x \) is such that \( U(x) \cap (D \cup D') \neq \emptyset \), then either
\( U(x) \cap D \neq \emptyset \), hence \( x \in \bigcup U^+D \), or similarly, \( x \in \bigcup U^+D' \).
Finally, we have to show that \( D'' \geq D \cup D' \), which can be done by similar
elementary arguments.

Now we are ready to prove
Lemma 2.14. Let \( X \) be a topological space whose points are among the ordinals
\( \leq \kappa^+ \). Suppose that \( U \) is an ONA of \( X \) that has the property
\( |U(x)| < \kappa \ (\forall x \in X) \), and assume that \( \kappa^+ < \kappa = \kappa \). Then \( \mathbb{P}_U^\kappa \)
satisfies \((\kappa^+)\).
Proof. Observations 2.13 and 2.12 provide us with the properties (a) and (b) from
Definition of \((\kappa^+)\). We shall now prove the required chain condition. Suppose that
we are given \( \{D_\alpha : \alpha < \kappa^+ \} \) from \( \mathbb{P}_U^\kappa \). \( \kappa^+ < \kappa \), we also have \( (\kappa^+) < \kappa = \kappa^+ \), so
we can fix a bijection \( F \) from \( ([\kappa^+] < \kappa)^2 \) onto \( \kappa^+ \). We now define several subsets of
\( \kappa^+ \):
\[
C_0 \overset{\text{def}}{=} \{ \alpha < \kappa^+ : (\forall \beta < \alpha)(\forall A, B \in [\beta] < \kappa) F(A, B) < \alpha \}, \nC_1 \overset{\text{def}}{=} \{ \alpha < \kappa^+ : (\forall \beta < \alpha)D_\beta \subseteq \alpha \}
\]
and
\[
C_2 \overset{\text{def}}{=} \{ \alpha < \kappa^+ : (\forall \beta < \alpha)U(\beta) \subseteq \alpha \}.
\]
Let \( C \) def \( \{C_0 \cap C_1 \cap C_2 \} \setminus \{0\} \). Standard arguments show that \( C \) is a club of \( \kappa^+ \). This will be the club demonstrating the required condition. In order to finish the demonstration, we also need to define a regressive function \( f \). To motivate its definition, let us first prove

**Sublemma 2.15.** Suppose \( \alpha < \beta < \kappa^+ \) are such that

\[
D_\beta \cap \bigcup_{\gamma < \beta} U^\gamma(\bigcup_{\gamma < \alpha} D_\gamma) = D_\alpha \cap \bigcup_{\gamma < \alpha} U^\gamma(\bigcup_{\gamma < \alpha} D_\gamma)
\]

and

\[
\bigcup_{\gamma < \beta} U^\gamma D_\alpha \cap \bigcup_{\gamma < \alpha} D_\gamma = \bigcup_{\gamma < \alpha} U^\gamma D_\alpha \cap \bigcup_{\gamma < \alpha} D_\gamma.
\]

Then \( D_\alpha \) and \( D_\beta \) are compatible.

**Proof of the Sublemma.** Let \( \alpha \) and \( \beta \) be as claimed. We shall show that \( D \defeq D_\alpha \cup D_\beta \) is a common upper bound of \( D_\alpha \) and \( D_\beta \). \( D \) is clearly a closed and discrete superset of \( D_\alpha \) and \( D_\beta \), and has size \( < \kappa \). Suppose that \( x \in X \) and \( U(x) \cap D \neq \emptyset \), then \( U(x) \cap D \neq \emptyset \) for some \( l \in \{\alpha, \beta\} \). In any case, \( x \in \bigcup_{\alpha \in D} \cup D_\alpha \). Hence, the intersection between \( D \alpha \) and \( D_\beta \) is empty. As we have assumed \( \alpha < \beta \), this argument does not automatically yield the analogous conclusion with \( \beta \) in place of \( \alpha \), but the rest of our assumptions about \( \alpha \) and \( \beta \) can be used now, as follows.

If \( x \in D \setminus D_\alpha \), then \( x \in D_\alpha \setminus D_\beta \). Supposing that also \( x \in \bigcup_{\alpha \in D_\beta} \cup D_\alpha \), we have \( x \in \bigcup_{\alpha \in D_\beta} \cup D_\alpha \cap \bigcup_{\gamma < \alpha} D_\gamma \), which is the same as \( \bigcup_{\alpha \in D_\beta} \cup D_\alpha \cap \bigcup_{\gamma < \alpha} D_\gamma \). Hence

\[
x \in D_\alpha \cap \bigcup_{\gamma < \alpha} D_\gamma \subseteq D_\alpha \cap \bigcup_{\gamma < \alpha} \cup U^\gamma D_\alpha \cap \bigcup_{\gamma < \alpha} D_\gamma = D_\beta \cap \bigcup_{\gamma < \alpha} \cup U^\gamma D_\alpha \cap \bigcup_{\gamma < \alpha} D_\gamma,
\]

contradicting the assumption that \( x \notin D_\beta \). \( \star \)

**Proof of Lemma 2.14 continued.** We define \( f \) on \( C \) by letting for \( \alpha \in C \) with \( cf(\alpha) = \kappa \)

\[
f(\alpha) \defeq f(D_\alpha \cap \bigcup_{\gamma < \alpha} \cup U^\gamma(\bigcup_{\gamma < \alpha} D_\gamma), \bigcup_{\gamma < \alpha} U^\gamma D_\alpha \cap \bigcup_{\gamma < \alpha} D_\gamma),
\]

and letting \( f(\alpha) = 0 \) otherwise. Notice that the cardinal assumptions on \( U \) and \( D_\alpha \)'s guarantee that \( f \) is well defined. Let us show that it is regressive on \( C \). For the nontrivial part of this, let \( \alpha \in C \) be of cofinality \( \kappa \). By the choice of \( C_1 \) and \( C_2 \) we have that

\[
\bigcup_{\gamma < \alpha} D_\gamma \subseteq \alpha \text{ and } \bigcup_{\gamma < \alpha} \cup U^\gamma D_\gamma \subseteq \alpha.
\]

Hence \( A \defeq D_\alpha \cap \bigcup \cup U^\gamma(\bigcup_{\gamma < \alpha} D_\gamma) \) and \( B \defeq (\bigcup_{\gamma < \alpha} D_\gamma) \cap \bigcup \cup U^\gamma D_\alpha \) are both subsets of \( \alpha \) of size \( < \kappa \), hence bounded. By the choice of \( C_0 \) we have \( f(\alpha) = F(A, B) < \alpha \).

Now let us see that \( C \) and \( f \) work: suppose that \( \alpha < \beta \) are in \( C \), have cofinality \( \kappa \) and \( f(\alpha) = f(\beta) \). By Sublemma 2.15 we have that \( D_\alpha \) and \( D_\beta \) are compatible. \( \star \)
Proof of Theorem 2.1. Starting with \( V \) and the cardinals as in the statement of the Theorem, using Shelah’s Theorem we pass to a universe \( W \) in which the conclusions of that theorem hold. From now on, let us work in \( W \). Let \( (X, \tau) \) be a given locally \( < \kappa \)-space of size \( \leq \kappa^+ \), such that for every ONA \( U^* \) of \( X \) there is a finer ONA \( U \) with the property that every open subspace of size \( \leq \kappa \) of \( X \) is \( \kappa \)-D with respect to \( U \). We shall show that \( X \) is a \( D \)-space. By Observation 2.9, we can assume that the size of \( X \) is \( \kappa^+ \). Without loss of generality the points of \( X \) are the ordinals \( < \kappa^+ \).

Let \( \{ V(x) : x \in X \} \) be an open neighbourhood assignment on \( X \) which demonstrates that \( X \) is locally \( < \kappa \), hence \( |V(x)| < \kappa \) for every \( x \in X \). Note that in order to show that \( X \) is \( D \), we may concentrate on those ONA \( U^* \) of \( X \) for which we have \( U^*(x) \subseteq V(x) \) for all \( x \in X \). Let \( U^* \) be such an ONA and let \( U \) be a finer ONA with respect to which all open subspaces of size \( \leq \kappa \) are \( \kappa \)-D. It suffices to show that \( X \) is \( D \) with respect to \( U \).

Let \( P = P_\kappa^U \). As we have assumed that \( \kappa^{< \kappa} = \kappa \) in \( V \), and no bounded subsets of \( \kappa \) are added when \( W \) is formed, we have that \( \kappa^{< \kappa} = \kappa \) holds in \( W \). Therefore, Lemma 2.14 applies, and we conclude that \( P \) satisfies \((*)_\kappa\). The main part of the rest of the proof is a density argument.

Claim 2.16. For every \( x \in X \), the set
\[
D_x \overset{\text{def}}{=} \{ D \in P : x \in \bigcup U^*D \}
\]
is dense in \( P \).

Note. One may wonder if the fact that Claim 2.16 holds and the part (2) of Fleissner-Stanley Theorem, do not automatically imply that \( X \) is \( D \), without a reference to forcing. This is not necessarily the case, as Claim 2.16 only refers to \( U \)-sticky sets of size \( < \kappa \).

Proof of the Claim. Let us fix a large enough regular cardinal \( \chi \).

Given \( D \in P \) and \( x \in X \). We choose \( N \prec (H(\chi), \in, \prec^*_\chi) \) of size \( \kappa \) and such that
\[
: (i) \quad X, \tau, D, x, U \in N,
: (ii) \quad \kappa, N \subseteq N \text{ and } \kappa + 1 \subseteq N.
\]
Such a choice is possible, as \( \kappa^{< \kappa} = \kappa \). We shall look for a \( D' \) which is a required extension of \( D \) in \( D_x \).

Consider first the subset \( N \cap X \) of \( X \) in the subspace topology. If \( y \in N \cap X \), then \( U(y) \subseteq N \). As \( H(\chi) \models \{ |U(y)| < \kappa \} \), there is an \( \alpha < \kappa \) and a function \( f \) from \( \alpha \) onto \( U(y) \). By elementarity, there is such a function in \( N \), and since \( \alpha \subseteq N \), we have that \( U(y) \subseteq N \). This shows that \( N \cap X \) is an open subspace of \( X \), and hence, by our assumptions, \( N \cap X \) is \( \kappa \)-D with respect to \( U \). In order to use this fact, we shall show that \( D \in P_\kappa^{U^*(N \cap X)}(N \cap X) \). Here \( U^N \upharpoonright (N \cap X) \) stands for the function assigning \( U(x) \cap N (= U(x)) \) to each \( x \in N \cap X \). Note again that we are considering \( N \cap X \) in the subspace topology, not the topology induced by elementarity, so in particular \( U^N \upharpoonright (N \cap X) \) is an ONA of \( N \cap X \). Let \( \mathbb{R} \overset{\text{def}}{=} P_\kappa^{U^N \upharpoonright (N \cap X)}(N \cap X) \).

An argument similar to the one showing that \( N \cap X \) is open, shows that \( D \) is a subset of \( N \cap X \). Clearly, \( D \) is closed and discrete in \( N \cap X \), and has size \( < \kappa \). Suppose that \( y \in N \cap X \) is such that \( U(y) \cap D \cap (N \cap X) \neq \emptyset \), then by the fact that \( D \in P \), we have that \( y \in \bigcup U^*D \), so \( y \in \bigcup \{ U(z) \cap N : z \in D \} \). This demonstrates that \( D \in \mathbb{R} \). Hence, by our assumptions (as \( x \in N \)), there is \( D' \in \mathbb{R} \).
with $D \leq_{\mathbb{R}} D'$ and $x \in \bigcup\{U(y) \cap N : y \in D'\}$. In particular, $D \subseteq D'$ and $|D'| < \kappa$, and $x \in \bigcup U^aD'$. As $D' \in [\kappa]^{<\kappa}$, we have that $D' \in N$. It is easily seen that, since $N \cap X$ is open, the fact that $D'$ is discrete in $N \cap X$, implies that $D'$ is discrete in $X$.

To show that $D'$ is closed in $X$, we shall have to use elementarity. If $y \in (N \cap X) \setminus D'$, we have that for some open $O$ containing $y$ we have $O \cap (N \cap X) \cap D' = \emptyset$, as $D'$ is closed in $N \cap X$. By $X$ being locally $< \kappa$, we can assume that $|O| < \kappa$. Letting $V = O \cap (N \cap X)$, we get that $y \in V \in \tau$ and $V$ is a subset of $N$ of size $< \kappa$, hence $V \in N$. Consequently,

$$N \models \forall y \in (X \setminus D') \exists \tau \in \tau (y \in V \& V \cap D' = \emptyset),$$

so the same is true in $\mathcal{H}(\chi)$, demonstrating that $D'$ is closed in $X$.

Note that if $y \in D' \setminus D$, then, using that $D' \subseteq N$ and $D \leq_{\mathbb{R}} D'$, we have that $y \notin \bigcup\{U(z) \cap N : z \in D\}$, so $y \notin \bigcup U^aD$. For the rest of the proof, suppose that $z \in X$ is such $U(z) \cap D' \neq \emptyset$, yet $z \notin \bigcup U^aD'$. By elementarity, there is $z' \in N$ such that $U(z') \cap D' \neq \emptyset$ and $z' \notin \bigcup\{U(y) \cap N : y \in D' \cap N\}$. As $D' \subseteq N$, we have $z' \notin \bigcup\{U(y) \cap N : y \in D'\}$, contradicting the fact that $D' \in \mathbb{R}$. Hence $D'$ is as required. $\star$.

**Proof of Theorem 2.1 continued.** By the choice of $W$, we can find a $\mathbb{P}$-filter $G$ such that $G \cap D_x \neq \emptyset$ for every $x \in X$. In particular, as $G$ is a directed subset of $\mathbb{P}_U$, by Observation 2.1(2) we have that $D \overset{\text{def}}{=} \bigcup G$ is an element of $\mathbb{P}_U$. Then $D$ is closed and discrete, while the choice of the dense sets intersected by $G$ guarantees that $\bigcup U^aD = X$. $\star$.

**Corollary 2.17.** It is consistent with $ZFC$ that $CH$ holds, $2^{\aleph_1} > \aleph_2$ and every locally countable space of size $\leq \aleph_2$ in which for every ONA $U^*$ there is a finer ONA with respect to which every open subspace of size $\leq \aleph_1$ is strongly $D$, is $D$ itself.

**Proof.** We apply Theorem 2.1 with $\lambda = \aleph_0$. $\star$.

Having finished the proof of Theorem 2.1, there are several questions that come to mind. Firstly, is there a difference between spaces which are $\kappa$-$D$ and those which are simply $D$, and what does the assumption of being locally $< \kappa$ contribute to this difference? The simplest instance of this question would be:

**Question 2.18.** Is there a locally countable $D$-space of size $\aleph_1$ which is not strongly $D$?

A simple argument shows that the simplest example of a locally countable $D$-space of size $\aleph_1$ is strongly $D$, namely

**Claim 2.19.** Suppose that $X$ is a non-stationary subset of $\omega_1$, with the order topology. Then $X$ is strongly $D$.

**Note.** By the van Douwen-Lutzer [11] characterisation of linearly ordered $D$-spaces, such an $X$ is necessarily $D$.

**Proof.** Since $X$ is non-stationary, there is a club $C$ of $\omega_1$ with $C \cap X = \emptyset$. We can assume that $\min(C) < \min(X)$. For $\alpha \in X$ a limit ordinal, define $\beta_\alpha \overset{\text{def}}{=} \sup(C \cap \alpha)$,
hence $\beta_\alpha < \alpha$ for every such $\alpha$. Observe that, since $C$ is unbounded, there is for any $\beta < \omega_1$ a $\delta = \delta(\beta)$ such that for all $\alpha \geq \delta$, if $\beta_\alpha$ is defined, then $\beta_\alpha > \beta$.

Suppose that $U^*$ is an ONA of $X$. We choose a finer ONA such that $U(\alpha) \in (\beta_\alpha, \alpha + 1)$ for limit ordinals $\alpha \in X$, and $U(\alpha) = \{ \alpha \}$ otherwise. Let $D \in P_{U^*}(X)$ and $x \notin \bigcup U^* D$. As $X$ is $D$, by the Fleissner-Stanley Theorem, there is $D' \geq D$ in $P_U(X)$, with $x \in \bigcup U^* D'$. Let $\gamma_0 \in D'$ be such that $x \in U(\gamma_0)$, and then define by induction an increasing sequence $\langle \gamma_n : n < \omega \rangle$ of countable ordinals such that $\gamma_{n+1} \geq \delta(\gamma_n)$, and $(\gamma_n, \gamma_{n+1}) \cap C \neq \emptyset$. Also require that $D \subseteq \gamma_1$. Now let $\gamma \overset{\text{def}}{=} \sup \gamma_n$ and $D'' = D' \cap \gamma$. As $\gamma \in C$, we have that $\gamma \notin X$, hence $D''$ is closed in $X$.

Clearly, $D''$ is discrete, countable, satisfies $D \subseteq D''$ and $x \in \bigcup U^* D''$. To show $D'' \in P_U$, suppose $y \in X$ is such that $U(y) \cap D'' \neq \emptyset$. Then $y \in \bigcup U^* D'$, by the choice of $D'$. Suppose $y \notin \bigcup U^* D''$, and let $\alpha \in D'$ be the minimal such that $y \in U(\alpha)$. Hence $\alpha > \gamma$. If $\alpha$ is a successor, then $y = \alpha$, so $U(y) = \{ \alpha \}$, contradicting the assumption $U(y) \cap D'' \neq \emptyset$. Hence, $\alpha$ is a limit and $y \in U(\alpha) \subseteq (\beta_\alpha, \alpha + 1)$.

As $\gamma \in C$, we have that $\beta_\gamma \geq \gamma$. Either $y$ is a successor ordinal in $(\beta_\alpha, \alpha + 1)$, contradicting $U(y) \cap D'' \neq \emptyset$, or $y$ is a limit ordinal. In the latter case, $\beta_y \geq \beta_\alpha$, again contradicting $U(y) \cap D'' \neq \emptyset$.

Finally, $D \subseteq D''$ because $D \subseteq D'$.  \[2.19\]

In fact, much more is true: by analysing the proof of Theorem 3.3. of Fleissner-Stanley’s paper \[5\], we can see that

**FACT 2.20.** A linearly ordered topological space is $D$ iff it is $\kappa$-$D$ for all regular uncountable $\kappa$.

W. Fleissner proved that the Cantor tree (see \[11\] for details) is $D$ and not strongly $D$. The size of this space is $2^{\aleph_0}$.

Another question that might be worth asking is if the assumptions of Theorem \[2.1\] are necessary. If one considers a non-reflecting stationary subset $S$ of $\omega_2$ in the order topology, one has a space all of whose subspaces of size $\leq \aleph_1$ are strongly $D$, yet the space itself is not $D$. If the points in $S$ have countable cofinality, this space is even locally countable. However, for a given ONA $U^*$ of $X$ and a subspace $Y$ of $X$ with $|Y| \leq \aleph_1$, the finer ONA $U$ with respect to which $Y$ is strongly $D$, depends on $Y$, i.e. cannot be chosen uniformly for all $Y$, as in the assumptions of Theorem \[2.1\]. This indicates that some assumption additional to small open subspaces being strongly $D$ is necessary in the statement of Corollary \[2.17\] and similarly in that of Theorem \[2.1\].

A tension between the existence of small open neighbourhoods and a certain amount of compactness is a well studied subject, see for example I. Juhasz, S. Shelah and L. Soukup’s \[8\]. Along these lines, one may ask when there are locally countable non-discrete $D$-spaces of large cardinality, although the fact that the relationship between being $D$ and other versions of compactness is not entirely clear, may mean that such a question is premature.

We end the section by discussing the possibility of strengthening Observation \[2.1\].

**CLAIM 2.21.** Suppose that $X$ is a $\kappa$-$D$ space and $Y$ is a closed subspace of $X$. Then $Y$ is $\kappa$-$D$. 

Proof of the Claim. Let $\kappa, X, Y$ be as in the statement of the Claim, and let $U^*$ be an ONA of $Y$. For $x \in X \setminus Y$, let $U^*(x) = X \setminus Y$, hence $U^*$ has been extended to an ONA of $X$. Let $U$ be an ONA of $X$ finer than $U^*$ such that $X$ is $\kappa$-D with respect to $U$. We claim that $Y$ is $\kappa$-D with respect to $U \upharpoonright Y$. In this direction, let $D \subseteq Y$ of size $< \kappa$ be $U \upharpoonright Y$-sticky and let $y \in Y \setminus \bigcup U^* D$. As $U$ is finer than $U^*$, we have that $D$ is $U$-sticky, and hence there is $D' \geq_U D$ with $|D'| < \kappa$ and $y \in \bigcup U^* D'$. Now $D'' = D' \cap Y$ demonstrates that $Y$ is $\kappa$-D. \hfill \star_2.21$

Note. The choice of $U$ above depends on $Y$. This and previous observations motivate the following definition:

Definition 2.22. We say that $X$ is uniformly $\kappa$-D iff for every ONA $U^*$ of $X$, there is ONA $U$ finer than $U^*$ such that every closed subspace of $X$ is $\kappa$-D with respect to $U$.

The following argument is due to W. Fleissner.

Claim 2.23. Suppose that $\kappa$ is a regular uncountable cardinal and $X$ is a uniformly $\kappa$-D space. Then $X$ is $D$.

Proof of the Claim. Let $U^*$ be an ONA of $X$ and let $U$ be a finer ONA demonstrating that $X$ is uniformly $\kappa$-D. We shall show that $X$ is $D$ with respect to $U$.

Given a $U$-sticky $D$ and $x \notin \bigcup U^* D$. Let $Y \overset{\text{def}}{=} X \setminus \bigcup U^* D$, so $Y$ is closed and $x \in Y$. As $\emptyset$ is $U$-sticky, we can find $D'$ with $|D'| < \kappa$ and $x \in \bigcup U^* D'$, such that $D$ is $U \upharpoonright Y$-sticky. Let $D'' \overset{\text{def}}{=} D \cup D'$, then $D'' \geq_U D$ is as required. \hfill \star_2.23

It is perhaps instructive to contrast Claim 2.23 with the Conclusion of Theorem 2.21.

3. On downwards reflection

We prove several theorems which give conditions on a $D$-space to have proper $D$-subspaces with specified properties. The first theorem is an easy remark using the downward reflection of property $D$ on closed subspaces, while the others are more involved and use a large cardinal assumption. As is often the case with such large cardinal downward reflection arguments in topology, for one of the latter theorems an additional assumption has to be made on the space in question in order to make the reflection argument work. We concentrate on spaces with a small character, for a detailed discussion of other possible assumptions, the reader may consult \[3\] and \[4\].

Let us first note that it is easy to obtain reflection results involving closed $D$-subspaces of a given $D$-space, because of the following

Observation 3.1. If $X$ is a $D$-space and $Y \subseteq X$ is a closed subspace of $X$, then $Y$ is a $D$-space.

Proof. The same proof as that of Claim 2.21. \hfill \star_3.1

The following argument is due to W. Fleissner.

Claim 2.23. Suppose that $\kappa$ is a regular uncountable cardinal and $X$ is a uniformly $\kappa$-D space. Then $X$ is $D$.

Proof of the Claim. Let $U^*$ be an ONA of $X$ and let $U$ be a finer ONA demonstrating that $X$ is uniformly $\kappa$-D. We shall show that $X$ is $D$ with respect to $U$.

Given a $U$-sticky $D$ and $x \notin \bigcup U^* D$. Let $Y \overset{\text{def}}{=} X \setminus \bigcup U^* D$, so $Y$ is closed and $x \in Y$. As $\emptyset$ is $U$-sticky, we can find $D'$ with $|D'| < \kappa$ and $x \in \bigcup U^* D'$, such that $D$ is $U \upharpoonright Y$-sticky. Let $D'' \overset{\text{def}}{=} D \cup D'$, then $D'' \geq_U D$ is as required. \hfill \star_2.23

It is perhaps instructive to contrast Claim 2.23 with the Conclusion of Theorem 2.21.

3. On downwards reflection

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Let us first note that it is easy to obtain reflection results involving closed $D$-subspaces of a given $D$-space, because of the following

Observation 3.1. If $X$ is a $D$-space and $Y \subseteq X$ is a closed subspace of $X$, then $Y$ is a $D$-space.

Proof. The same proof as that of Claim 2.21. \hfill \star_3.1
Theorem 3.2. Suppose that $\kappa$ is a strong limit and $X$ is a space of size $\kappa$.

(1) If $X$ is $D$, then for every $\theta < \kappa$ and a subspace $Z$ of $X$ with $|Z| < \kappa$, there is a closed $D$-subspace $Y$ of $X$ with $\theta < |Y| < \kappa$ and $Z \subseteq Y$.

(2) If $X$ is $\sigma$-$D$ for some $\sigma < \kappa$, then for every $\theta < \kappa$ and a subspace $Z$ of $X$ with $|Z| < \kappa$, there is a closed $\sigma$-$D$ subspace of $X$ with $\theta < |Y| < \kappa$ and $Z \subseteq Y$.

Proof. (1) Let $\lambda \in (\theta, \kappa)$ be a cardinal and let $Z$ be a given subspace of $X$ of size $\lambda$, with an additional arbitrarily chosen set of $\lambda$ points of $X$. Let $Y = \bar{Z}$. By Observation 3.1, it suffices to show that $|Y| < \kappa$. This follows because $|Y| \leq 2^{\mathfrak{c}} < \kappa$, as it is well known that for Hausdorff spaces $W$ we have $|\bar{W}| \leq 2^{\mathfrak{c}}$, see [7] II, 2.4.

(2) The same proof. \[3.2\]

We thank W. Fleissner for simplifying the above argument and providing the gist of the following.

Theorem 3.3. (W. Fleissner, private communication) Suppose that $\kappa$ is strongly inaccessible, and that $X$ is a locally $\kappa$-$D$-space of size $\kappa$.

Then for every $\theta < \kappa$ and subspace $Z$ of $X$ with $|Z| < \kappa$, there is an open subspace $Y$ of $X$ with $\theta < |Y| < \kappa$ which is $D$, and which contains $Z$ as a subspace.

Proof. For any subspace $Z$ of $X$ with $|Z| < \kappa$, we shall inductively choose sequences $\langle Y_n = Y_n(Z) : n < \omega \rangle$ and $\langle Z_n = Z_n(Z) : n < \omega \rangle$ as follows:

- $Y_0 = Z = Z_0$,
- $Z_{n+1} = Y_n$,
- $Y_{n+1}$ is an open subspace of $X$ with $Y_{n+1} \supseteq Z_{n+1}$ and $|Y_{n+1}| < \kappa$.

Let $Y^* = Y^*(Z) = \bigcup_{n<\omega} Y_n = \bigcup_{n<\omega} Z_n$. It suffices to show that $Y^*(Z)$ can always be chosen as required, and that it is an open $D$ subspace of $X$ of size $\kappa$. Let $Z$ with $|Z| < \kappa$ be given.

It is clear that the sequence of $Z_n$s can be chosen as required, so we show by induction on $n < \omega$ that we can choose $Y_n$s as well. A part of the inductive hypothesis is that $|Y_n| < \kappa$. Coming to $Y_{n+1}$, we have $|Z_{n+1}| \leq 2^{\mathfrak{c}}|Y_n| < \kappa$, as $\kappa$ is a strong limit. Then to choose $Y_{n+1}$, we pick for every $y \in Z_{n+1}$ an open neighbourhood of size $< \kappa$ and let $Y_{n+1}$ be the union of all these. As $\kappa$ is regular, we have $|Y_{n+1}| < \kappa$.

It is clear that $|Y^*| < \kappa$ and that $Y^*$ is open. We show that $Y^*$ is $D$, so let $U$ be a given ONA of $Y^*$.

For $y \in Y^*$ let $n(y)$ be the first $n$ such that $y \in Z_n$ and let

$$V(y) = U(y) \setminus \bigcup_{k < n(y)} Z_k.$$

Hence $V(y)$ is open and $V(y) \subseteq U(y)$, while $y \in V(y)$. We shall find a $V$-sticky $D^*$ such that $\bigcup V^*D^* = Y^*$, which is clearly sufficient. Note that by the choice of $V$ we have that if $n < k$ and $D \subseteq Z_n$ is $V \upharpoonright Z_n$-sticky, then $D$ is $V \upharpoonright Z_k$-sticky, and also $D$ is $V$-sticky.

Now we build by recursion on $n < \omega$ an increasing sequence $\langle D_n : n < \omega \rangle$ of elements of $\mathbb{P}_V$ such that each $D_n$ is $V \upharpoonright Z_n$-sticky and $\bigcup D_n \supseteq Z_n$. This can be done because each $Z_n$ is closed, using Theorem 2.5. At the end let $D^* = \bigcup_{n<\omega} D_n$. \[3.3\]
This argument greatly simplifies the one we had originally, which used a measurable cardinal. Note that the conclusion of the theorem clearly implies that the assumption of $X$ being locally $< \kappa$ is necessary. The same method works for uniformly $\sigma$-D-spaces, but does not seem to work if uniformity is not assumed. In order to obtain somewhat of an an analogue for $\sigma$-D-spaces, we need a stronger large cardinal assumption, and an additional assumption on $X$.

**Theorem 3.4.** Suppose that $\kappa$ is measurable and $X$ is a locally $< \kappa$ space of size $\kappa$, such that every point in $X$ has a point-base of size $< \kappa$, and suppose that $X$ is $\sigma$-D for some $\sigma < \kappa$.

Then for every $\theta < \kappa$ there is $Y \subseteq X$ which is an open $D$-subspace of $X$ and satisfies

$$\theta < |Y| < \kappa.$$ 

**Proof.** Let $j : V \rightarrow M$ be an embedding witnessing that $\kappa$ is measurable, so in particular $\kappa$ is the critical point of $j$ and $^\kappa M \subseteq M$.

Suppose that $(X, \tau)$ is a given space with the properties as listed above. The idea of the proof is, as one would imagine, that $j^*X$ is a subspace of $j(X)$ that has the properties as required of $Y$ when translated by $j$, so that by elementarity $X$ must have a subspace $Y$ as required. However, topological reflections arguments are not so simple as many notions involved are highly non-first order, and in particular, an assumption has to be used to guarantee that $j^*X$ is actually a subspace of $j(X)$. In fact, we first have to clarify which topologies we have in mind when discussing $j(X)$ and $j^*X$.

For simplicity we shall assume, without loss of generality, that $X$ as a set is $\kappa$. By the assumption on the character of $X$, we can fix a sequence $B = \langle B_\alpha : \alpha < \kappa \rangle$ such that for every $\alpha < \kappa$ we have that $B_\alpha = \langle B^\alpha_\zeta : \zeta < \zeta_\alpha < \kappa \rangle$ is a point base for $\alpha \in X$. As $X$ is locally $< \kappa$, we may assume that $|B^\alpha_\zeta| < \kappa$ for all $\alpha$ and $\zeta$. Let

$$B = \{ B^\alpha_\zeta : \alpha < \kappa, \zeta < \zeta_\alpha \},$$

so $B$ is a basis for the topology $\tau$ on $X$. By elementarity, $j(B)$ is a basis for a topological space in $M$ whose set of points is $j(\kappa)$. We shall abbreviate this space as $j(X)$. For every $\alpha < \kappa$, a point-base at $j(\alpha) = \alpha$ in $j(X)$ is given by $\langle j(B^\alpha_\zeta) : \zeta < \zeta_\alpha \rangle$. As for every $\zeta < \zeta_\alpha$ we have that $j(B^\alpha_\zeta) \cap j^*X (= \kappa) = j^*B^\alpha_\zeta = B^\alpha_\zeta$, we have that $B = j^*B$ generates the subspace topology on $j^*X$. Clearly the original topology of $X$ is at least as fine as this topology, but in fact our assumptions guarantee that these two topologies are the same. For if $Y \subseteq X$ is open, then it is the union of a sequence of $\leq \kappa$ elements of $B$, hence this sequence is a member of $M$ and so $Y$ is open in $M$.

**Observation 3.5.** In $M$, $X$ is an open subspace of $j(X)$.

**Proof of the Observation.** Let $x \in X$, and let $\zeta < \zeta_x$ be arbitrary. Then $|B^x_\zeta| < \kappa$ and $x \in j(B^x_\zeta) = j^*B^x_\zeta = B^x_\zeta \subseteq X$, hence $X$ is open. ★

**Observation 3.6.** In $M$ we have that $X$ is a $\sigma$-$D$-subspace of $j(X)$.

**Proof of the Observation.** In $M$, let $U$ be an ONA of $X$. Back in $V$, as $X$ is $\sigma$-$D$, there is a finer ONA $W$ of $X$ such that $X$ is $\sigma$-$D$ with respect to $W$. Remembering that the topologies of $X$ in $M$ and $V$ are the same and that $^\kappa M \subseteq M$, we have that
$W \in M$ and clearly $W$ is a finer ONA of $X$ then $U$ is. We claim that $M$ satisfies that $X$ is $\sigma$-$D$ with respect to $W$.

We work in $M$ and let $x \in X$ and $D \subseteq X$ be such that $|D| < \sigma$ and $D$ is $W$-sticky. Then in $V$ we have that $D$ is $W$-sticky, so there is $D' \supseteq D$ with $x \in D'$ and $|D'| < \sigma$. It is easy to verify that $D' \in M$ satisfies the same requirements. \hfill $\blacklozenge$

Now let us finish the proof of the Theorem. Let $\theta < \kappa$ be given. As $X \in M$, in $M$ we have that there is $Y = X \subseteq j(X)$ of size a cardinal in $(j(\theta) = \theta, j(\kappa))$, such that for every $U$ which is a function from $Y$ to $\{O \cap Y : O \in j(B)\}$ with the property $y \in U(y)$ for all $y$, there is a function $W$ from $Y$ to $\{O \cap Y : O \in j(B)\}$ with the following properties:

- (i) $\forall y \in Y)[W(y) \subseteq U(y) \& y \in W(y)]$,
- (ii) Let $\varphi(D; Y, j(B), W)$ stand for

$$|D| < \sigma \& \langle \forall x \in Y \rangle [W(y) \subseteq U(y) \& y \in W(y)],$$

and let

$$\forall y \in D)(\exists B \in j(Y)\langle B \cap D = \{y\} \rangle \& \langle \forall y \in Y \rangle [W(y) \cap D \neq \emptyset \implies (\exists z \in D)y \in W(z)].$$

Then

$$\forall x \in Y)(\forall D \subseteq Y)[\varphi(D; Y, j(B), W) \implies (\exists D' \supseteq D)$$

$$(x \in D' \& \varphi(D'; Y, j(B), W) \& (\forall z \in D' \setminus D)(W(z) \cap D = \emptyset)).$$

By elementarity, in $V$ we can find a subset $Y$ of $X$ with $\theta < |Y| < \kappa$ such that for every $U$ which is a function from $Y$ to $\{O \cap Y : O \in B\}$ with the property $y \in U(y)$ for all $y$, there is $W$ from $Y$ to $\{O \cap Y : O \in B\}$ with the following properties:

- (i) as above
- (ii)

$$\forall x \in Y)(\forall D \subseteq Y)[\varphi(D; Y, B, W) \implies (\exists D' \supseteq D)(x \in D' \& \varphi(D'; Y, B, W) \& (\forall z \in D' \setminus D)(W(z) \cap D = \emptyset),$$

which is as required. \hfill $\blacklozenge$

At this point it is natural to ask if we can obtain similar downward transfer properties between $\aleph_2$ and $\aleph_1$ by applying the technique of generic embeddings, for example by a Lévy collapse of a large cardinal to $\aleph_2$, or using a huge cardinal embedding in the fashion of M. Foreman and R. Laver in [6]. Although some weak partial results can be easily obtained, difficulties with the transfer of property $D$ make it unclear whether the exact analogue of any of our downward reflection principles can be obtained. Let us also note that in Theorem 3.4 one could perhaps relax the various assumptions made, but we feel that the version presented is convenient as a contrast to the theorems from Section 3.1, and for the simplicity of reflection arguments, while the details of the consistency strength and topological strength investigation might be premature before we understand more about the relationship between $D$, $\sigma$-$D$ and uniformly $\sigma$-$D$. 

\hfill $\blacklozenge$
4. Discrete families of sets

The $D$-space problem can be formulated as a purely combinatorial statement involving discrete families of sets, as will be shown below, where we shall also exhibit some basic properties of the families in question. We commence by a definition.

**Definition 4.1.** (1) A non-empty family $\mathcal{F}$ of non-empty sets is said to be **discrete** iff there is a choice function $f$ on $\mathcal{F}$ such that

$$F_0 \neq F_1 \in \mathcal{F} \implies f(F_0) \notin F_1.$$  

A function $f$ as above is called a **discretisation** of $\mathcal{F}$.

(2) Let $\mathcal{F}$ be as above, and $\mathcal{G} \subseteq \mathcal{P}(\bigcup \mathcal{F})$. We say that $\mathcal{F}$ is $\mathcal{G}$-discrete iff there is a $D \subseteq \mathcal{F}$ and a discretisation $f$ of $D$ such that

: (i) $\bigcup D = \bigcup \mathcal{F}$
: (ii) $\{f(D) : D \in D\} \in \mathcal{G}$.

In such a case, the pair $(D, f)$ is called a **$\mathcal{G}$-discretisation** of $\mathcal{F}$.

Hence a (Hausdorff) topological space $X$ is $D$ iff every ONA $U$ of $X$, there is a closed discretisation of $\{U(x) : x \in X\}$. The definition of discretisation is similar to the definition of a transversal, which is a one-to-one choice function. However, one should note that the requirement for a function to be a discretisation is stronger than just being one-to-one. Of course, the two notions coincide when the family $\mathcal{F}$ consists of pairwise disjoint elements, but we are mainly interested in the cases where the existence of a transversal is not an obvious consequence of the axiom of choice. There is a body of work about transversals, cf. Shelah’s book [9], often concentrating on the incompactness properties, that is, families in which every smaller subfamily has a transversal, but the whole family does not have it. Such problems are known to be equivalent to the existence of certain families of functions, see II 6.2 in [9]. A similar argument can be used to characterise the existence of a non-discrete family of sets in which every smaller subfamily is discrete, as expressed by the following Theorem 4.3. Although the proof is very much the same as that of the corresponding one in the case of transversals, in [9], as the details there are not fully explained and as we need them for later use, we have decided to spell out the proof here. Following this theorem we shall obtain as a corollary a connection between families whose small subfamilies have transversals and such families that in addition satisfy a discreteness requirement (see Theorem 4.4). Let us first make

**Observation 4.2.** If $\lambda$ is a cardinal, there is no family of $> \lambda$ subsets of $\lambda$ that has a transversal.

**Proof.** Suppose that $\mathcal{A} = \{A_\alpha : \alpha < \alpha^*\}$ is such a family with $|\alpha^*| > \lambda$. Let $f$ be a transversal of $\mathcal{A}$. Then $\{f(A_\alpha) : \alpha < \alpha^*\}$ is a subset of $\lambda$ of size $> \lambda$, a contradiction. \(	extcircled{*}\)

**Theorem 4.3.** Suppose that $\mu > \lambda \geq \theta \geq \kappa$ are infinite cardinals. Then the following are equivalent:

(A) There is a family $\mathcal{P}^*$ of $\mu$ subsets of $\lambda$, each of power $\leq \kappa$ (none of whose subfamilies of size $\mu$ has a transversal, but) whose every subfamily of size $< \theta$ is discrete, and
There is a regular ideal $J$ on $\kappa$, and a family $F^*$ of $\mu$ many functions from $\kappa$ to $\lambda$ such that for every subfamily $F$ of $F^*$ with $|F| < \theta$, there is a sequence $\langle s_f : f \in F \rangle$ of sets in $J$ such that

$$i \in \kappa \setminus (s_f \cap s_g) \implies f(i) \neq g(i),$$

but there is no such sequence for any subfamily of $F^*$ which has size $\mu$. Moreover, if $F \subseteq F^*$ is of size $\geq \lambda^+$ there is no sequence $\langle s_f : f \in F \rangle$ of sets in $J$ such that

$$i \in \kappa \setminus (s_f \cup s_g) \implies f(i) \neq g(i).$$

**Note.** Claim II 6.2 of [9] gives a similar characterisation, in which the existence of a discretisation is replaced by the existence of a transversal, and “$s_f \cap s_g$” in (B) above is replaced by “$s_f \cup s_g$”.

**Proof.** (A) $\implies$ (B). Let us enumerate $[\kappa]^{<\aleph_0}$ as $\{ w_i : i < \kappa \}$, and let $F$ be a bijection from $[\lambda]^{<\aleph_0}$ onto $\lambda$. We define

$$J \overset{\text{def}}{=} \{ A \subseteq \kappa : (\exists i < \kappa)(\forall j \in A) w_i \not\subseteq w_j \}. $$

It is clear that $J$ is a proper ideal on $\kappa$. If $A \subseteq \kappa$ is bounded, with $\text{sup}(A) = \alpha < \kappa$, then $| \bigcup \{ w_j : j \leq \alpha \}| < \kappa$, so there must be an $i < \kappa$ such that $w_i$ is not a subset of $w_j$ for any $j \in A$. Hence, $J$ is regular.

Let $P^*$ be a family as in the assumptions of (A). For $X \in P^*$, let us enumerate $X = \{ \alpha^X_j : \zeta < \chi \leq \kappa \}$. For $i < \kappa$, let

$$f_X(i) \overset{\text{def}}{=} F(\{ \alpha^X_j : j \in w_i \cap \zeta \chi \}),$$

hence each $f_X$ is a function from $\kappa$ into $\lambda$. Let $F^* \overset{\text{def}}{=} \{ f_X : X \in P^* \}$, and let us claim that $F^*$ is as required. First note that $X_0 \neq X_1 \implies f_{X_0} \neq f_{X_1}$, so the size of $F^*$ is $\mu$.

Let $F \subseteq F^*$ be of size $< \theta$, so $F = \{ f_X : X \in P \}$ for some $P \subseteq P^*$ with $|P| < \theta$. Hence $P$ is discrete, and we can fix a discretisation $h$ of $P$. In particular notice that $h(X) \in X$ for all $X \in P$. We define

$$s_{f_X} \overset{\text{def}}{=} \{ i < \kappa : h(X) \notin \{ \alpha^X_j : j \in w_i \} \},$$

for $X \in P$. Notice that each $s_{f_X} \in J$, as one can take $i < \kappa$ such that $h(X) = \alpha^X_{\zeta'}$ for some $\zeta'$ and $w_i = \{ \zeta' \}$. Then if $j \in s_{f_X}$, we have that $\{ h(X) \}$ is not contained in $\{ \alpha^X_{\zeta} : \zeta \in w_j \}$, so $\zeta' \notin w_j$, and hence $w_i \not\subseteq w_j$.

If $X \neq Y \in P$, then we have that $h(X) \notin Y$, so clearly $h(X) \notin \{ \alpha^Y_j : j \in w_i \}$ for any $i$. But if $i \notin s_{f_X}$, we have $h(X) \in \{ \alpha^X_j : j \in w_i \}$, so

$$\{ \alpha^X_j : j \in w_i \cap \zeta_X \} \neq \{ \alpha^Y_j : j \in w_i \cap \zeta_Y \},$$

and in particular the images of these sets under $F$ are distinct. Hence, for each such $i$ we have $f_X(i) \neq f_Y(i)$, and by symmetry the same is true for $i \notin s_{f_X}$.

Now let us prove the last claim of (B). Suppose that $F_0 \subseteq F^*$ has size $\geq \lambda^+$ and that $(s_{f_{X}} : X \in F_0)$ can be defined as required. For each $X \in F_0$, we can find $i_X \in (\kappa \setminus s_{f_{X}})$, which is possible as $J$ is proper. Then there is $F_1 \subseteq F_0$ of size $\geq \lambda^+$ and $i^* < \kappa$ such that for all $X \in F_1$ we have $i_X = i^*$. But then $f_X(i^*)$ for $X \in F_1$ are $\geq \lambda^+$ distinct elements of $\lambda$, a contradiction.
(B) \implies (A). This direction is easier: fix a bijection $F$ between $\kappa \times \lambda$ and $\lambda$. Starting with $\mathcal{F}^*$ as in (B), for $f \in \mathcal{F}^*$ define $X_f = \{F(i, f(i)) : i < \kappa\}$. Let $\mathcal{P}^* \overset{\text{def}}{=} \{X_f : f \in \mathcal{F}^*\}$, it is easy to check that $\mathcal{P}^*$ is as required.

\textbf{Note.} A non-trivial $\Delta$-system is an example of a discrete family. Hence, for the Theorem \ref{thm:delta-system} to be interesting, we need at least to be in a situation in which $\Delta$-system Lemma between $\mu$ and $\kappa^+$ does not hold, so we should have $\sigma^\kappa \geq \mu$ for some $\sigma < \mu$.

The following theorem establishes a simple direct relationship between transversals and discretisations.

\textbf{Theorem 4.4.} Suppose that $\mu > \lambda \geq \theta > \kappa$ are infinite cardinals, and suppose that $\mathcal{P}$ is a family of $\mu$ elements of $[\lambda]^{\leq \kappa}$ such that every subfamily of $\mathcal{P}$ of size $< \theta$ has a transversal. Then there is such a family $\mathcal{P}$ such that in addition, every subfamily of $\mathcal{P}$ whose size $\sigma$ satisfies $\sigma < \theta$ and $\text{cf}(\sigma) > \kappa$, has a subfamily of size $\text{cf}(\sigma)$ that has a discretisation.

\textbf{Proof.} Let $\mathcal{P}$ satisfy the assumptions of the theorem. Then Shelah’s result from \cite{Shelah} II 6.2 is that the proof of Theorem \ref{thm:delta-system} with the ideal $J$ defined as there but using $\mathcal{P}$ in place of $\mathcal{P}^*$, yields a family $\mathcal{F}^*$ of $\mu$ many functions from $\kappa$ to $\lambda$ such that for every subfamily $\mathcal{F}$ of $\mathcal{F}^*$ of size $< \theta$ there is a sequence $\langle s_f : f \in \mathcal{F} \rangle$ of elements of $J$ such that

\text{(1) } |\{g \in \mathcal{F} : (\exists i \in \kappa \setminus (s_f \cup s_g)) f(i) = g(i)\}| \leq \kappa.

We shall show that $\mathcal{F}^*$ has the property that for every subfamily $\mathcal{F}$ of $\mathcal{F}^*$ of size $\sigma < \theta$ with $\text{cf}(\sigma) > \kappa$ there is a subfamily $\mathcal{F}'$ of $\mathcal{F}$ of size $\text{cf}(\sigma)$ for which (1) holds with “$s_f \cup s_g$” replaced by “$s_f \cap s_g$” and “$\leq \kappa^+$” replaced by “$\leq 1$”.

Suppose that $\mathcal{F}$ is a subfamily of $\mathcal{F}^*$ of size $\sigma < \theta$ with $\text{cf}(\sigma) > \kappa$. Let $\langle s_f : f \in \mathcal{F} \rangle$ be as guaranteed by (1). Note that the sets

$A_i \overset{\text{def}}{=} \{j < \kappa : w_i \not\subseteq w_j\}$

generate the ideal $J$, in the sense that every element of $J$ is a subset of an $A_i$. Here the sets $w_i$ are as in the proof of Theorem \ref{thm:delta-system}. Since the conclusion of (1) does not change if we replace each $s_f$ by a set that is larger than $s_f$ but still in $J$, we can assume that for each $f$ there is $i(f) < \kappa$ such that $s_f = A_{i(f)}$. Hence there is $i^* < \kappa$ such that for $\text{cf}(\sigma)$ many $f$ we have $i(f) = i^*$. For such $f$ let $s^* = s_f$. Let $\mathcal{F}_0 = \{f \in \mathcal{F} : s_f = s^*\}$, hence $|\mathcal{F}_0| = \text{cf}(\sigma)$ and we can enumerate $\mathcal{F}_0 = \{f_\zeta : \zeta < \text{cf}(\sigma)\}$.

Given $\zeta < \text{cf}(\sigma)$, we notice that

$d(\zeta) \overset{\text{def}}{=} \sup\{\xi < \text{cf}(\sigma) : (\exists i \in \kappa \setminus s^*) f_\xi(i) = f_\xi(i)\} < \text{cf}(\sigma)$.

Hence, by induction on $\alpha < \text{cf}(\sigma)$ we can define $\zeta_\alpha$ as follows:

Let $\zeta_0 = 0$. Given $\zeta_\alpha$ let $\zeta_{\alpha+1} = d(\zeta_\alpha) + 1$. For $\alpha$ a limit ordinal $< \text{cf}(\sigma)$ let $\zeta_\alpha \overset{\text{def}}{=} \sup\{\zeta_\beta : \beta < \alpha\}$. Then we can let $\mathcal{F}' = \{f_\zeta : \zeta < \text{cf}(\sigma)\}$.

Having shown this property of $\mathcal{F}'$, we proceed to define $\mathcal{P}'$ as in the proof of (B) \implies (A) of Theorem \ref{thm:delta-system}. It is easy to check that $\mathcal{P}'$ is as required. ★
Shelah uses the following notation for the situation in the assumptions of the Theorem 4.4

**Definition 4.5.** Suppose that $\mu \geq \lambda \geq \theta_1 \geq \theta_2 + \kappa$ and $J$ is an ideal on $\kappa$. Then $\text{NPT}_J(\mu, \lambda, \theta_1, \theta_2, \kappa)$ means that there is a family $F^*$ of $\mu$ functions from $\kappa$ to $\lambda$ such that

: (a) for any subfamily $F$ of $F^*$ with $|F| < \theta_1$, there is a sequence $\langle s_f : f \in F \rangle$ of elements of $J$ such that for each $f \in F$

$$|\{g \in F : (\exists i \in \kappa \setminus (s_f \cup s_g)) f(i) = g(i)\}| < \theta_2,$$

: (b) the analogue of (a) with $F^*$ in place of $F$ fails.

In conjunction with the following Theorem 4.6 of Shelah from [9] II 6.3, Theorem 4.4 can be used to read off non-trivial instances of families with discretisations. The function $\text{cov}$ is discussed in detail in [9], but an instance of a situation in which the assumptions of Theorem 4.6 hold is

$$\lambda^{\aleph_0} > \lambda^+ \& (\forall \theta < \lambda)(\theta^{\aleph_0} < \lambda).$$

**Theorem 4.6. (Shelah)** Suppose that $\lambda > \text{cf}(\lambda) = \aleph_0$ and $\text{cov}(\lambda, \lambda, \aleph_1, 2) > \lambda^+$. Then $\text{NPT}_{\text{Fin}}(\lambda^+, \lambda, \lambda^+, 2, \aleph_0)$ holds.

In the above $\text{Fin}$ stands for the ideal of finite subsets of $\omega$.

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**5. Concluding Remarks**

We investigated reflection phenomena that arise in connection with van Douwen’s notion of $D$-spaces. In the first two sections we concentrated on the topological aspects of this problem, studying both upwards and downwards reflection. The last section shows that there is a purely combinatorial aspect of the problem, in the sense that one can define a generalisation of $D$-property that is formulated in terms of a covering of one family of sets by another. Then one can talk about discreteness properties of such covers and obtain the original topological formulation of $D$-spaces as a particular instance of this more general setting. In tune with the rest of the paper, we concentrated again on reflection properties of such covers and showed that such properties of discrete families of sets have a strong connection with the well studied combinatorial problem of the existence of transversals. This indicates that it would be of interest to study discrete families of sets from the purely combinatorial point of view, an investigation that is outside of the scope of this paper. One could then however hope that families of sets with given discreteness properties could be topologised in order to give examples of topological spaces of some relevance to the $D$-space problem.
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