Coupling the Sorkin-Johnston State to Gravity

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We consider the dynamics of the Sorkin-Johnston (SJ) state for a massless scalar field in two dimensions. We conduct a study of the renormalized stress-tensor by a subtraction procedure, and compare the results with those of the conformal vacuum, with an important contribution from correction term. We find a large trace anomaly and compute backreaction effects to two dimensional (Liouville) gravity. We find a natural interpretation for the mirror behavior of the SJ state described in previous works.

Keywords: Sorkin-Johnston State; Stress Tensor Renormalization, Trace Anomaly

I. INTRODUCTION

One of the fundamental problems of Quantum Field Theories in curved spacetime is the vacuum selection. Unlike maximally symmetric spaces – such as the usual Minkowski spacetime, for generic metrics there is no symmetry argument in order to single out a special, symmetric state. Questions about the meaning of particle states and decoherence of vacuum fluctuations, all too important to cosmological studies, depend upon the solution to this problem to be truly answered.

Recently, Afshordi et al. [1] made a proposal for a distinguished “vacuum” state of a scalar field for any globally hyperbolic spacetime. The physical idea behind the selection was based in the “causal set program” [2, 3], in which the vacuum state is associated not with a spacelike slice of a globally hyperbolic spacetime, but with a causal set of finite size, parametrized by $\ell$. The existence of a new parameter associated with the so-called Sorkin-Johnston (SJ) state codifies beautifully the physical fact that measurements of quantum states are inherently finite in space and time, and thus questions related to entanglement and (de)coherence can be studied from the SJ perspective. In particular, the entanglement properties of the SJ vacuum could shed new light to the Reeh-Schlieder theorem. In its original form the theorem states that the states constructed by local operators acting on the usual Poincaré invariant vacuum $|0\rangle$ is dense in the whole Hilbert space of the theory. It is an interesting question whether the SJ state satisfy this “highly entanglement” property. See [4, 5] and references therein for details in this direction.

From the dynamical perspective, though, the proposal meets some technical difficulties. The Green’s function on this state does not satisfy the Hadamard condition, and as such fluctuations and couplings are difficult to tame [6]. Backreaction properties should suffer severely from this fact, and the question on whether one could define interactions for the SJ state seem now uncertain. This letter strives to that direction, by studying the coupling of the SJ state to two dimensional gravity. In two dimensions, for massless scalar fields, one has the added structure of a conformal vacuum [7], with which one can easily compare the properties of the SJ state. We make use of this structure, as well as the SJ state constructed by [8] to study the stress-energy tensor and the backreaction to gravity via the conformal anomaly. This will allow us to conclude that i) the correction term [8] plays an important role in the renormalized stress-tensor, almost reducing severely the Casimir effect associated with the reflecting boundary conditions; ii) there is strong gravitational backreaction in the SJ state; and iii) once backreaction is taken into account, the mirror behavior arises naturally as reflective boundary conditions in the asymptotic induced geometry.

The letter is organized as follows. In Section II we review the SJ construction for the massless scalar field in two dimensions. In Section III we use point-splitting techniques to derive a renormalized stress tensor, and in Section IV we use the result to couple the field to gravity. We close with concluding remarks in Section V.
II. THE SORKIN-JOHNSTON PROPOSAL

Usual constructions of quantum fields in curved spaces rely heavily on the symmetries of the spacetime in order to select a maximally invariant (“vacuum”) state \([9]\). Even when it is applicable, Fock space – or highest weight – construction also suffers from the requirement of non-locality, since mode decomposition allows for arbitrarily delocalized eigenfunctions of the Laplacian.

These problems motivate the Sorkin-Johnston construction for a state which can pass as “vacuum” for arbitrary (globally hyperbolic) spacetimes. We will limit the discussion to scalar Klein-Gordon fields \(\hat{\phi}(x)\). The SJ state \(|SJ\rangle\) is a covariant and uniquely assigned state defined for a real scalar field in a bounded region of a globally hyperbolic spacetime \([1]\). Much like the vacuum state in Poincaré invariant is the highest weight state for annihilation operators associated with the Klein-Gordon Hamiltonian, the SJ state is the highest weight state associated with the mode decomposition of a non-local linear operator:

\[
\hat{S}f(x) = \int_M i\Delta(x,y)f(y)dV_y,
\]

where the kernel \(i\Delta(x,y)\) is the Pauli-Jordan function, \(i.e.,\) a causal distributional solution to the classical Klein-Gordon equation. The SJ vacuum is defined in such a way that upon quantization, the relation

\[
\langle SJ|\hat{\phi}(x)\hat{\phi}(y)|SJ\rangle = \int_M i\Delta(x,y)f(y)dV_y,
\]

is a Green’s function for the Klein-Gordon equation which vanishes outside the causal cone. The SJ proposal promotes this lack of ambiguity of the Pauli-Jordan function to a guiding principle behind the definition of “positive” and “negative” frequencies.

It is proven in \([1]\) that, for any finite region \(M\), the operator \(\hat{S}\) is self-adjoint. One can then obtain eigenmodes \(T_k(x)\) such that:

\[
\int_M i\Delta(x,y)T_k(y)dV_y = \lambda_k T_k(x).
\]

With this at hand one decomposes the spectrum into “positive” \((\lambda_k > 0)\) and “negative” \((\lambda_k < 0)\) frequency modes \(T^\pm_k(x)\), and define the SJ Wightman function as:

\[
W(x,y) := \langle SJ|\hat{\phi}(x)\hat{\phi}(y)|SJ\rangle = \sum_{k=1}^{\infty} \lambda_k T^+_k(x)T^+_k(y)^* \quad \lambda_k > 0.
\]

Given that the operator is self-adjoint, one can also impose the reality condition on the eigenmodes \([T^+_k]^* = T^-_k\).

The SJ Wightman function \(W(x,y)\) thus defined satisfies these three conditions \([3]\):

- Commutator: \(i\Delta(x,y) = W(x,y) - W^*(x,y)\)
- Positivity: \(\int_\mathcal{M} dV_x \int_\mathcal{M} dV_y f^*(x)W(x,y)f(y) \geq 0\)
- Orthogonal supports: \(\int_\mathcal{M} dV_y W(x,y)W^*(y,z) = 0\)

The first two conditions are satisfied by the two-point function of any state, while the third acts as the ground state condition which can be interpreted as the requirement that the Wightman function will be the “positive frequency part” of the Pauli-Jordan function, thought as an \((c\text{-number})\) operator on the Hilbert space of square integrable functions \(L^2(\mathcal{M},dV)\). The splitting of the Hilbert space between positive and negative frequency solutions is the Grassmannian structure at the core of any quantum field theory. The spirit of defining a quantum field theory from its correlation functions, rather than the Fock space construction, was anticipated by the old axiomatic approach, found an explicit representation in the conformal bootstrap \([10]\) and has been incorporated into recent formulations of quantum field theory. See \([11]\) and references therein for a discussion on the advantages and difficulties.

The construction thus defines a unique state \(|SJ\rangle\) if \(\mathcal{S}^2\) has a unique square root. This is supposed to hold as long as \(\hat{S}\) is self-adjoint and there are no zero modes. This condition holds for any bounded region of a globally hyperbolic four-dimensional spacetime \([1]\), while for a bounded region in a two dimensional spacetime the integral operator \(\hat{S}\) is self-adjoint and Hilbert-Schmidt, with finite trace squared \([8]\).
A. SJ state for the massless field in $1 + 1$ space-time

We will specialize to a single massless scalar field in $1 + 1$ dimensions, and chose for $M$ a causal diamond defined by lightcone coordinates $\mathcal{M} : \{-\ell \leq u \leq \ell, -\ell \leq v \leq \ell\}$. The metric and the Klein-Gordon equation is simply:

$$ds^2 = -2du dv, \quad \partial_u \partial_v \phi(u, v) = 0.$$  \hfill (4)

The Pauli-Jordan function in these coordinates is given by:

$$i\Delta(u, v; u'v') = -\frac{i}{2}\left[\theta(u-u') + \theta(v-v') - 1\right]$$  \hfill (5)

such that eigenfunctions for positive eigenvalues of the integral operator split into two families $[11]$:

$$f_k(u, v) := e^{-iku} - e^{-ikv}, \quad \text{with} \quad k = \frac{n\pi}{\ell}, n = 1, 2, \ldots$$  \hfill (6)

$$g_k(u, v) := e^{-iku} + e^{-ikv} - 2 \cos(k\ell), \quad \text{with} \quad k_n \in \mathcal{K} = \{k \in \mathbb{R} | \tan(k\ell) = 2k\ell \text{ and } k > 0\}$$  \hfill (7)

with eigenvalues $\lambda_k = \ell/k$ real. The sum of the squared eigenvalues is finite ($2\ell^4$), as befitting to an operator of the Hilbert-Schmidt class. Summing over the two sets of modes, one finds from (3) the Wightman function $[12]$:

$$W_{\text{SJ}}(u, v; u'v') = \frac{1}{4\pi} \left\{-\log \left[1 - e^{-\frac{\pi(u-u')}{2\ell}}\right] - \log \left[1 - e^{-\frac{\pi(v-v')}{2\ell}}\right] + \log \left[1 + e^{-\frac{\pi(u-v')}{2\ell}}\right]ight\} + \epsilon(u, v; u', v').$$  \hfill (8)

The $\epsilon$ term is a “correction term”, stemming from the fact that the values of $k \in \mathcal{K}$ are approximately given by odd multiples of $\pi/2$, $(2n-1)\pi/2$, for sufficiently large $k$:

$$\epsilon(u, v; u', v') = \sum_{k_n \in \mathcal{K}} g_{k_n}(u, v)g_{k_n}^*(u', v') - \sum_{n=1}^{\infty} \frac{g_n^0(u, v)(g_n^0)^*(u', v')}{4\pi(2n-1)},$$  \hfill (9)

with $g_n^0 = \exp[-(2k-1)i\pi n \frac{\pi}{2\ell}] + \exp[-(2k-1)i\pi n \frac{\pi}{2\ell}]$ is the limit of $g_k(u, v)$ as $k \to (2n-1)\pi/2\ell$. This term contributes comparatively little for the SJ function, but will contribute to the stress-energy tensor. It does have a well-defined coincidence limit $(u, v) \to (u', v')$ and hence it can be studied independently of the divergent part. If we disregard the correction term, the expression for the SJ Wightman function becomes

$$W_{\text{SJ, box}}(u, v; u', v') = \frac{1}{4\pi} \log \left[\frac{\cos \left(\frac{\pi}{2\ell}(u-v')\right) \cos \left(\frac{\pi}{2\ell}(v-u')\right)}{\sin \left(\frac{\pi}{2\ell}(v-u)\right) \sin \left(\frac{\pi}{2\ell}(v-v')\right)}\right],$$  \hfill (10)

exactly the expression for the two point function of a massless scalar field in a box with reflecting boundaries at $x = \pm \sqrt{2\ell}$ in a conformal vacuum.

In the large $\ell$ limit, the SJ Wightman function approximates the usual Minkowski expression, up to an additive constant, for values of $u$ and $v$ close to the center of the causal diamond. For values close to the corner, however, there is an interesting “mirror behavior”, where the reflecting boundaries become important. The consideration of this behavior, as well as a thorough numerical study of the correction term $\epsilon(u, v, u', v')$, is performed in $[8]$.

III. STRESS TENSOR RENORMALIZATION

In the Poincaré vacuum, the expectation value for the stress tensor is infinite, and one usually deals with this by assuming normal ordering. Physically, one could think of the normal ordering as imposing the constraint that the vacuum should have zero energy and momentum. When one considers gravity, however, the question of the vacuum energy is no longer so simple. The stress tensor sources gravity, and one has to resort to other means to be rid of the infinities. One such method is by adding a cosmological constant term $[7]$, which sets some fiducial, usually associated to the “vacuum” in curved space time, stress tensor $T^0_{ab}$, to zero. The physical stress tensor will then be:

$$T^\text{ren}_{ab}(x) = T_{ab}(x) - T^0_{ab}(x).$$  \hfill (11)
Of course, once one could change the fiduciary metric or some other property of the spacetime, \( T^0_{0b} \) becomes observable. Such effects would correspond, for instance, to the Casimir effect or the dependence of the Cosmological constant to some physical parameter. The procedure to define \( T^0_{0b} \) is as follows. One starts with the symmetric Green’s function, or the Hadamard elementary function

\[
G^{(1)}(x, x’) = \langle \Omega| \{\phi(x), \phi(x’)\}|\Omega\rangle, \tag{12}
\]

for a generic “vacuum” state \(|\Omega\rangle\), and takes the derivatives involved in computing the stress energy tensor before taking the coincidence limit:

\[
T^0_{ab} = \langle T_{ab}(x)\rangle_{\Omega} = \lim_{x’ \rightarrow x} \partial_{x’} \partial_{x} \mathcal{D}(x, x’) G^{(1)}(x, x’); \quad \mathcal{D}(x, x’) = \frac{1}{2} [\nabla_a \nabla_{b’} + \nabla_{a’} \nabla_b]. \tag{13}
\]

In this way one bypasses the ambiguities involved in defining the product of local operators at coincident points, and take a symmetric view on the arguments. In our application, the vacuum state is \(|\Omega\rangle = |S J\rangle\), and the differential operator \( \mathcal{D}(x, x’) \) will have the flat space expression. We will implement the renormalization by inspecting how \( T^0_{ab} \) changes with the parameter \( \ell \). Because of scale invariance, this may mean changing the size of the causal diamond or changing the cosmological constant constant “counterterm”. We will delay the discussion about backreaction to the next Section.

To compute \( T^0_{ab} \) with Hadamard’s elementary function associated with \( |\Omega\rangle \), we first check how the expression changes with \( \ell \) [7]:

\[
\frac{\partial}{\partial \ell} T^0_{ab}(x) = \lim_{x’ \rightarrow x} \frac{\partial}{\partial \ell} \mathcal{D}(x, x’) G^{(1)}(x, x’), \tag{14}
\]

which can be split into the contributions coming from \( W_{S J, box} \) and the correction term, or \( T^0_{ab} = T^0_{ab}^{box} + T^0_{ab}^{e} \). We can check that:

\[
\partial_u \partial_{u’} (W_{S J, box}(u, v; u’, v’) + W_{S J, box}(u’, v’; u, v)) = \partial_u \partial_{u’} (W_{S J, box}(u, v; u’, v’) + W_{S J, box}(u’, v’; u, v)) = -\frac{\pi}{32 \ell^2 \sin^2 \left( \frac{\pi (u-u’)}{4 \ell} \right)}. \tag{15}
\]

With this expression, the limit \( u \rightarrow u’ \) diverges, but the derivative of the stress energy tensor is finite:

\[
\frac{\partial}{\partial \ell} T^{box}_{uu}(u, v) = \frac{\pi}{48 \ell^3}. \tag{16}
\]

Thus the finite part of \( T^{box}_{uu} \) should be:

\[
T^{box}_{uu}(u, v) = -\frac{\pi}{96 \ell^2}. \tag{17}
\]

It can be checked that \( \langle T^{box}_{uu} \rangle \) has exactly the same value. The calculation is entirely similar, but with the roles of \( u \) and \( v \) interchanged. This non-zero value is nothing but the Casimir energy associated with the fact that the SJ Wightman function \( W_{S J, box} \) behaves as if one has reflecting boundary conditions at \( x = \pm \sqrt{2} \ell \).

The coincident limit of the \( u, v \) derivative tells us how the two-point function, and hence the effective action \( \langle (\partial \phi)^2 \rangle \) depend on a scale transformation \( u \rightarrow \lambda u, v \rightarrow \lambda v \). By general considerations [13] this is the expectation value of the trace of the stress-energy tensor. The coincident limit of the \( u, v \) derivative of the Hadamard function is:

\[
\langle T^{box}_{uu}(u, v) \rangle = \lim_{(u’, v’)-(u, v)} \partial_u \partial_{u’} G^{(1)}(u, v; u’, v’) = \frac{\pi}{32 \ell^2 \cos^2 \left( \frac{\pi (u-v)}{4 \ell} \right)}. \tag{18}
\]

We note that it depends explicitly on the coordinates. Again, this should not be a surprise given that the choice of the causal diamond explicitly breaks Poincaré invariance. It diverges at the positions of the “mirrors” \( x = \pm \sqrt{2} \ell \), a fact we will turn back to in the next Section.

The \( \epsilon \) term [10] has a finite coincidence limit and its contribution to the \( uu \) and \( vv \) component of the stress-energy
polynomial on (23) will be felt near the divergent points \( x \) expansion, and then the contribution of the correction term to the trace will be subdominant with respect to (20). This term also diverges for \( x \) whose contribution for states constructed from \(| \psi \rangle\) with \(| \psi \rangle\) whose contribution for states constructed from \(| \psi \rangle\) with \(| \psi \rangle\). This term also diverges for \( x \). We will call the total contribution a Casimir energy.

The trace correction due to the epsilon term is also expressed as a sum over the roots of \( \tan k = 2k \):

\[
T_{\nu}^\nu = T_{\nu}^\nu = \frac{1}{4\ell^2} \sum_{n=1}^{\infty} \left[ k_n \frac{4k_n^2 + 1}{4k_n^2 - 1} \cos \left( k_n \frac{u-v}{\ell} \right) - (2n-1) \frac{\pi}{2} \cos \left( (2n-1) \frac{\pi(u-v)}{2\ell} \right) \right].
\] (20)

Using that \( k_n \approx (2n-1)\pi/2 - 1/(2n-1)\pi + \mathcal{O}(n^{-3}) \), the sum can be expressed in terms of polylogarithms. Let us consider the first two terms of the summand:

\[
\frac{u-v}{2\ell} \sin \left( (2n-1) \frac{\pi(u-v)}{2\ell} \right) - \frac{(u-v)^2}{4(2n-1)\pi^2} \cos \left( (2n-1) \frac{\pi(u-v)}{2\ell} \right)
\]

The potentially dangerous first term gives a sum of delta functions:

\[
\langle T_{\nu}^\nu \rangle^{(0)} = -i \pi \frac{u-v}{8\ell^3} \left[ \delta \left( \frac{u-v}{\ell} \right) - \delta \left( -\frac{u-v}{\ell} \right) \right],
\] (22)

whose contribution for states constructed from \(| SJ \rangle\) with smooth test functions is zero. The next order term can be readily computed from the mode sum, using the Taylor expansion of \( \log(1 + e^{iz}) \):

\[
\langle T_{\nu}^\nu \rangle^{(1)} = \frac{(u-v)^2}{8\pi^3} \log \tan^2 \left( \frac{\pi(u-v)}{4\ell} \right),
\] (23)

which also depends on the coordinates. This term also diverges for \( x = \pm \sqrt{2} \ell \), albeit logarithmically, even though the correction term is supposed to be small. Terms of higher order will also display this behavior. Using the identities:

\[
\sum_{k \text{ odd}} \frac{\cos k\theta}{k^n} k_n = \frac{1}{2} \left( \text{Li}_n(e^{i\theta}) + \text{Li}_n(e^{-i\theta}) - \text{Li}_n(-e^{i\theta}) - \text{Li}_n(-e^{-i\theta}) \right)
\]

\[
\sum_{k \text{ odd}} \frac{\sin k\theta}{k^n} k_n = \frac{1}{2i} \left( \text{Li}_n(e^{i\theta}) - \text{Li}_n(e^{-i\theta}) - \text{Li}_n(-e^{i\theta}) + \text{Li}_n(-e^{-i\theta}) \right),
\] (24)

one can write the generic term as a polynomial times the combination of polylogarithms \( \text{Li}_n(z) \). The singular behavior of the trace stems from the expansion of the polylogarithm near \( z = 1 \) [14]:

\[
\text{Li}_n(z) = -\frac{(z-1)^{n-1}}{(n-1)!} \log(1-z) + f(z) + (1-z)g(z) \log(1-z),
\] (25)

with \( f(z) \) and \( g(z) \) analytic at \( z = 1 \). One should expect an logarithm divergence at \( x = \pm \sqrt{2} \ell \) for all terms in the expansion, and then the contribution of the correction term to the trace will be subdominant with respect to [20]. Its effect on (23) will be felt near the divergent points \( x = \pm \sqrt{2} \ell \) by changin the \((u-v)^2\) term in front to a generic polynomial on \( (u-v) \). We will disregard the contribution of these subdominant terms from now on.

With this provision, let us now sum the different contributions. In terms of coordinates \( t \) and \( x \), \( T_{\mu\nu}(t, x) \) is given
by:

\[
\langle T_{tt}(t, x) \rangle = -\frac{(1 - \sigma)\pi}{96\ell^2} + \frac{\pi}{32\ell^2 \cos^2 \left(\frac{\pi x}{2\sqrt{2}\ell}\right)} + \frac{x^2}{4\pi\ell^4} \log \tan^2 \left(\frac{\pi x}{2\sqrt{2}\ell}\right),
\]

\[
\langle T_{xx}(t, x) \rangle = -\frac{(1 - \sigma)\pi}{96\ell^2} - \frac{\pi}{32\ell^2 \cos^2 \left(\frac{\pi x}{2\sqrt{2}\ell}\right)} - \frac{x^2}{4\pi\ell^4} \log \tan^2 \left(\frac{\pi x}{2\sqrt{2}\ell}\right).
\]  

(26)

The off-diagonal term \( \langle T_{tx} \rangle \) is zero.

The result can be written in the form:

\[
\langle T_{ab}(t, x) \rangle = -\frac{(1 - \sigma)\pi}{96\ell^2} (\eta_{ab} + 2u_a u_b) - \frac{\pi}{32\ell^2 \cos^2 \left(\frac{\pi x}{2\sqrt{2}\ell}\right)} + \frac{x^2}{4\pi\ell^4} \log \tan^2 \left(\frac{\pi x}{2\sqrt{2}\ell}\right) \eta_{ab},
\]

(27)

where \( \eta_{ab} \) is the Minkowski metric and \( u^a = (\partial / \partial t)^a \) a constant time-like vector. The result displays not only the Casimir energy, but also the contribution from the trace anomaly. Its effect, even for small values of \( x \), overwhelms completely the Casimir term. The divergence means that we can no longer disregard the coupling to gravity, at least for finite \( \ell \). As we take \( \ell \to \infty \), the contribution vanishes as one should expect from the Minkowski vacuum.

### IV. COUPLING TO GRAVITY

The result (27) shows that the expectation value for the stress energy in the SJ state diverges for finite \( \ell \) at the positions \( x = \pm \sqrt{2}\ell \). This divergence comes from the non-zero value for the expectation value of the trace of the stress-energy tensor. Classically, the trace should vanish because of the scale invariance of the massless scalar field in two dimensions. The choice of SJ state breaks this scale invariance, and thus we should have a coupling between the expectation value of the field and the metric. In other words, we should have induced gravity.

In order to find this metric, let us recall that diffeomorphism invariance requires that the stress-energy tensor should be conserved \( \nabla^a T_{ab} = 0 \). But as we can see with a direct calculation:

\[
\partial_x \langle T_{xx}(t, x) \rangle = -\frac{\pi^2 \sin \left(\frac{\pi x}{2\sqrt{2}\ell}\right)}{32\sqrt{2}\ell^3 \cos^3 \left(\frac{\pi x}{2\sqrt{2}\ell}\right)} - \frac{x^2}{2\pi\ell^4} \log \tan^2 \left(\frac{\pi x}{2\sqrt{2}\ell}\right) - \frac{x^2}{2\sqrt{2}\ell^5 \sin \left(\frac{\pi x}{2\sqrt{2}\ell}\right)},
\]

(28)

with the flat metric expression. The way around it is to impose the constraint that, in the induced metric \( g_{ab} \) we will have conservation, with \( \nabla_a \) the covariant derivative associated with \( g_{ab} \).

Any two dimensional metric is conformally flat, so we can write the Ansatz for \( g_{ab} \):

\[
ds^2 = \exp(2\varphi)(-dt^2 + dx^2),
\]

(29)

where \( \varphi(t, x) \) is the Liouville field, and will set the induced scale. With \( T_{ab} \) as in (27), the following equation ensues from \( \nabla^a T_{ab} = 0 \):

\[
(T_{xx} - T_{tt}) \partial_x \varphi = 0,
\]

\[
(T_{xx} - T_{tt}) \partial_t \varphi = \partial_x T_{xx}.
\]

(30)

The first equation above states that \( \varphi \) doesn’t depend on \( t \) as expected, while the second can be readily integrated to:

\[
\exp(2\varphi) = \frac{\pi}{32\ell^2 \cos^2 \left(\frac{\pi x}{2\sqrt{2}\ell}\right)} + \frac{x^2}{4\pi\ell^4} \log \tan^2 \left(\frac{\pi x}{2\sqrt{2}\ell}\right).
\]

(31)

One should note that the \( \ell \to \infty \) limit is taken by a scaling procedure, where the line element \( ds^2 = \ell^2 ds^2 \) goes to the proper Minkowskian limit when one takes the size of the box to infinity. The Riemann scalar \( R = -2e^{-2\varphi} \partial_x^2 \varphi \) is a complicated function of \( x \), but it asymptotes a constant curvature for \( x = \pm \sqrt{2}\ell \):

\[
R = -\frac{8\pi}{\ell^2} + \frac{12\pi}{\ell^4} (x \pm \sqrt{2}\ell)^2 + \ldots
\]

(32)
One can see that the induced metric is the consistency condition for the two dimensional trace anomaly [13]:

\[ \langle T^a_a \rangle = \frac{c}{24\pi} R, \]  

(33)

where \( c = 1 \) for the massless scalar field.

The geometry of the induced metric is asymptotically that of two dimensional anti-de Sitter space (AdS\(_2\)). The patch of the causal diamond covers a part of AdS\(_2\), see Fig. 1. The geometry has a conformal boundary at \( x = \pm \sqrt{2}\ell \), which is spatial infinity. Because of the nature of spatial infinity, light rays take a finite time, as counted by the observer sitting at \( x = 0 \) to reach it. Placing reflecting boundary conditions is natural from a broad perspective, since it allows for an unitary bulk and boundary dynamics [15, 16]. It is not by any means the unique choice [17]. The mirror behavior found in [8] is at any rate deeply tied with the induced metric. It would not be compatible with any other type of asymptotic geometry.

\[ x = -\sqrt{2}\ell \quad x = \sqrt{2}\ell \]

\[ t = \sqrt{2}\ell \quad t = -\sqrt{2}\ell \]

**FIG. 1.** The causal structure of the induced metric. The causal diamond is shaded. The conformal boundary sits at \( x = \pm \sqrt{2}\ell \) and a light ray takes a finite interval of \( t \) to cross from one boundary to the other.

**V. DISCUSSION**

In this letter we conducted a study of the stress tensor renormalization for the Sorkin-Johnston state of a massless scalar field in two dimensions associated with a causal diamond. We have found that for the purposes of renormalization of the stress-tensor energy one can no longer disregard the correction function arising from finite size effect to the wave number of the modes. The correction function counteracts the Casimir term, reducing severely its strength. Moreover, one also finds a trace anomaly term, which diverges at the spatial coordinates at the tip of the causal diamond. Subsequent backreaction shows that the induced metric is that of an asymptotically anti-de Sitter space with a conformal boundary sitting at the tip of the causal diamond. This provides another view on the mirror behavior found previously, corresponding to reflective boundary conditions for the field in spatial infinity of AdS space.

From this study one should expect a completely different behavior for non-zero mass fields: in this case, even classically, the trace is not expected to be zero, the value is proportional to the mass. It is not reasonable then to expect the same kind of mirror behavior, as renormalization effects can be absorbed by a mass redefinition. Also, questions have been raised about the stability of the linear perturbations in anti-de Sitter space with respect to backreaction (see [18] for a recent discussion). Translated to the SJ construction, these could mean instability of the SJ state itself. Perhaps this is the reason behind the large contribution of the correction-term, dominating over the “box” Casimir energy. This should have implications even to the original purpose of defining the SJ state as a “vacuum” which is associated with finite-time measurements.

In two dimensions, many problematic aspects of the SJ state do not appear, which actually makes the whole treatment of this letter possible. For instance, the treatment in Section [11] guarantees that the SJ state considered...
here satisfies the Hadamard condition. For more generic settings, like the addition of a mass term, one may not be so lucky. At any rate, the coupling of massive fields will also have a crucial contribution from the correction term [9], but the analysis can be forced to include the modification term proposed by [19]. The renormalized stress tensor will surely play a big impact on the understanding of the thermodynamics of the causal diamond, along with the entanglement entropy. We hope to address these points in future work.

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[1] N. Afshordi, S. Aslanbeigi, and R. D. Sorkin, JHEP 1208, 137 (2012), 1205.1296
[2] S. Johnston, Phys. Rev. Lett. 103, 180401 (2009) [arXiv:0909.0944 [hep-th]]
[3] R. D. Sorkin, J. Phys. Conf. Ser. 306:012017, 2011 (2011), 1107.0698
[4] A. Balachandran, A. de Queiroz, and S. Vaidya, Eur. Phys. J. Plus 128, 112 (2013) [arXiv:1212.1239]
[5] R. Clifton and H. Halvorson, Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 32, 1 (2001).
[6] C. J. Fewster and R. Verch, Class. Quant. Grav. 29, 205017 (2012) [arXiv:1206.1562 [math-ph]]
[7] N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space (Cambridge University Press, 1982).
[8] N. Afshordi, M. Buck, F. Dowker, D. Rideout, R. D. Sorkin, et al., JHEP 1210, 088 (2012), 1207.7101
[9] R. M. Wald, (2009), arXiv:0907.0416 [gr-qc]
[10] A. Belavin, A. M. Polyakov, and A. Zamolodchikov, Nucl.Phys. B241, 333 (1984)
[11] S. P. Johnston, (2010), arXiv:1010.5514 [hep-th]
[12] This expression corrects two sign typos in [8].
[13] F. Di Francesco, P. Mathieu, and D. Sénéchal, Conformal Field Theory (Springer-Verlag New York, 1997).
[14] I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products, seventh ed. (Elsevier/Academic Press, Amsterdam, 2007).
[15] S. Avis, C. Isham, and D. Storey, Phys.Rev. D18, 3565 (1978)
[16] R. M. Wald, J.Math.Phys. 21, 2802 (1980)
[17] A. Ishibashi and R. M. Wald, Class.Quant.Grav. 21, 2981 (2004) [arXiv:hep-th/0402184 [hep-th]]
[18] H. Friedrich, Class.Quant.Grav. 31, 105001 (2014) [arXiv:1401.7172 [gr-qc]]
[19] M. Brum and K. Fredenhagen, Class.Quant.Grav. 31, 025024 (2014) [arXiv:1307.0482 [gr-qc]]