Combinatorial Modulus on Boundary of Right-Angled Hyperbolic Buildings

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Abstract

In this article, we discuss the quasiconformal structure of boundaries of right-angled hyperbolic buildings thanks to combinatorial tools. In particular we exhibit some examples of buildings of dimension 3 and 4 whose boundaries satisfy the combinatorial Loewner property. This property is a weak version of the Loewner property. In the case of buildings of dimension 2, many work have been done by M. Bourdon and H. Pajot. In particular, the Loewner property on the boundary permitted them to prove the quasi-isometry rigidity for some buildings.

Keywords: Boundary of hyperbolic space, building, combinatorial modulus, combinatorial Loewner property, quasi-conformal analysis.

1 Introduction

1.1 Starting point

The origin of the theory of modulus of curves in compact metric spaces must be found in the classical theory of quasiconformal maps in Euclidean spaces (see [Väi71] or [Vuo88]). The aim of the classical theory is to describe the regularity of quasiconformal maps in $\mathbb{R}^d$ and to exhibit invariants under these maps. The notion of abstract Loewner space, introduced by J. Heinonen and P. Koskela (see [HK98] or [Hei01]), intends to describe metric measured spaces whose quasiconformal maps have a behavior of Euclidean flavor.

Besides, since G.D. Mostow it is known that the quasiconformal structure of the boundary of a hyperbolic space controls the geometry of the space. Then it turns out that this idea extends to the setting of Gromov hyperbolic spaces and groups. Finding a Loewner space as the visual boundary of a Gromov hyperbolic group has been useful to establish
rigidity results about the group (see [Haï09b] for a survey on those results). The idea that one wants to use to prove rigidity is that the quasi-isometries of an hyperbolic space are given by the quasisymmetric homeomorphisms of the boundary. The Loewner property makes it possible because the classes of quasi-Moebius, quasisymmetric and quasiconformal maps are equal in a Loewner space.

Yet, to prove that the boundary of a hyperbolic group is a Loewner space is a difficult question. To do so, one needs to find a measure on the boundary that is optimal for the conformal dimension. This quasiconformal invariant has been introduced by P. Pansu in [Pan89]. Finding a measure that realizes the conformal dimension and even compute the conformal dimension, are very difficult questions that we can solve in few examples for the moment.

An interesting example of this kind is the work done by M. Bourdon and H. Pajot in of Fuchsian buildings. They proved that the boundary of these buildings are Loewner spaces and then used this structure to prove the quasi-isometry rigidity of these buildings (see [BP00]).

Buildings are singular spaces introduced by J. Tits to study exceptional Lie groups. Currently buildings became a topic of interest by themselves. Among them, right-angled buildings have been classified by F. Haglund and F. Paulin in [HP03]. They are equipped with a wall structure and with a simply transitive group action on the chambers that make them very regular objects. Fuchsian buildings are right-angled hyperbolic buildings of dimension 2. Regarding what did M. Bourdon and H. Pajot in dimension 2, we obtain the questions:

**Question 1.1.** Are higher dimensional right-angled hyperbolic buildings rigid? What are the quasiconformal properties of their boundaries?

The geometry of higher dimensional right-angled buildings is very close to the geometry of Fuchsian buildings. This gives hope that these questions may have interesting answers. Yet, the methods used in Fuchsian buildings are very specific to the dimension 2. Thus these question are not simple. In this article, we will use combinatorial modulus for a first approach of the conformal structure of higher dimensional right-angled buildings.

A major rigidity question that is discussed thanks to quasiconformal boundaries is the following conjecture due to J.W. Cannon.

**Conjecture 1.2 ([CS98, Conjecture 5.1.]).** If $\Gamma$ is a hyperbolic group and $\partial \Gamma$ is homeomorphic to $S^2$, then $\Gamma$ acts geometrically on $\mathbb{H}^3$.

In particular, this conjecture implies Thurston’s hyperbolization conjecture of 3-manifolds. Although Thurston’s conjecture is now a theorem by G. Perelman, Cannon’s conjecture remains very interesting as it is logically independent of Thurston’s conjecture.

The combinatorial modulus have been introduced by J.W. Cannon in [Can94] and by M. Bonk and B. Kleiner in [BK02] during the investigation of the quasiconformal structure.
of the 2-spheres to approach the conjecture and by P. Pansu in a more general context in [Pan89]. Combinatorial modulus gave birth to a weak version of the Loewner property: the Combinatorial Loewner Property (CLP). One of the feature of these modulus is that they can be used to characterize the conformal dimension as a critical exponent on the boundary.

Recently M. Bourdon and B. Kleiner (see [BK13]) gave examples of boundaries of Coxeter groups that satisfy the CLP but that are not known for satisfying the Loewner property. They used this property to give a new proof of Cannon’s conjecture for Coxeter groups. Some of the methods they used in Coxeter groups are transposable to right-angled buildings. This was a motivation to investigate higher dimensional right-angled buildings thanks to combinatorial modulus.

1.2 Main result

In this article, we use the combinatorial modulus to investigate the quasiconformal boundary of these buildings. Thanks to methods coming from [BK13], we obtain a control of the combinatorial modulus on the boundary by the curves contained in parabolic limit sets (see Section 6). Then we introduce a weighted modulus on the boundary of the apartments. This allows us to control the modulus in the building by a modulus in the apartment (see Section 8). For well chosen examples, the boundary of the apartment has a lot of symmetries that provide a strong control of the modulus. In particular, we exhibit some examples of dimension 3 and 4 buildings that satisfy the CLP.

**Theorem 1.3** (Corollary 10.3). Let $D$ be the right-angled dodecahedron in $\mathbb{H}^3$ or the right-angled 120-cell in $\mathbb{H}^4$. Let $W_D$ be the hyperbolic reflection group generated by the faces of $D$. For $q \geq 3$, let $\Delta$ be the right-angled building of constant thickness $q - 1$ and of Coxeter group $W_D$. Then $\partial \Delta$ satisfies the CLP.

Along with this result we also give in Theorem 9.1 a characterization of the conformal dimension of the building thanks to a critical exponent computed in an apartment.

1.3 Organization of the article

In Section 2 we introduce the combinatorial modulus of curves in the general setting of compact metric spaces. Then in Section 3 we restrict to the case of boundaries of hyperbolic spaces.

After these reminders, we give the main steps and ideas of the proof of Theorem 10.1 in Section 4. This section is essentially a summary of the article.

Then, in Section 5 we describe the geometry of locally finite right-angled hyperbolic buildings.

The key notion of parabolic limit sets is introduced in Section 6 where we study the modulus of curves in parabolic limit sets. This section is based on, the ideas used in Coxeter groups in [BK13, Section 5 and 6]. In particular, Theorem 6.13 is the first major step in
direction of the proof of Theorem 10.1. As a consequence of this theorem, we obtain a first application to the CLP (Theorem 6.14).

In Section 7, we describe the combinatorial metric on the boundary of the group thanks to the geometry of the building. This metric is convenient to use to compute combinatorial modulus. Then, in Section 8, we discuss how the modulus in the boundary of an apartment, may be related to a modulus in the boundary of the building. In particular, Theorem 8.9 is the second major step necessary to prove Theorem 10.1. We use this theorem to prove Theorem 9.1 that relates the conformal dimension of the boundary of the building to a critical exponent computed in the boundary of an apartment.

In Section 9, we add the assumption of constant thickness of the buildings which precises the results of the preceding section. In particular, we find that the conformal dimension of the boundary of the building is equal to a critical exponent computed in the boundary of an apartment (see Theorem 9.1). Eventually in Section 10, we gather these tools to obtain examples of right-angled-buildings of dimension 3 and 4 whose boundary satisfies the CLP (see Corollary 10.3).

1.4 Terminology and notations

All along this article, we will use the following conventions. The identity element in a group will always be designated by $e$. For a set $E$, the cardinality of $E$ is designated by $\#E$. A proper subset $F$ of $E$ is a subset $F \subsetneq E$.

If $G$ is a graph then $G^{(0)}$ is the set of vertices of $G$ and $G^{(1)}$ is the set of edges of $G$. For $v, w \in G^{(0)}$, we write $v \sim w$ if there exists an edge in $G$ whose extremities are $v$ and $w$. If $V \subset G^{(0)}$, the full subgraph generated by $V$ is the graph $G_V$ such that $G_V^{(0)} = V$ and an edge lies between to vertices $v, w$ if and only if there exists an edge between $v$ and $w$ in $G$. A full subgraph is called a circuit if it is a cyclic graph $C_n$ for $n \geq 3$. A graph is called a complete graph if for any pair of distinct vertices $v, w$ there exists an edge between $v$ and $w$.

A curve in a compact metric space $(Z, d)$ is a continuous map $\eta : [0, 1] \to Z$. Usually, we identify a curve with its image. If $\eta$ is a curve in $Z$, then $U_\epsilon(\eta)$ denotes the $\epsilon$-neighborhood of $\eta$ for the $C^0$-topology. This means that a curve $\eta' \in U_\epsilon(\eta)$ if and only if there exists $s : t \in [0, 1] \to [0, 1]$ a parametrization of $\eta$ such that for any $t \in [0, 1]$ one has $d(\eta(s(t)), \eta'(t)) < \epsilon$.

In a metric space $Z$, if $A \subset Z$ then $N_r(A)$ is the $r$-neighborhood of $A$. The closure of $A$ is designated by $\overline{A}$ and the interior of $A$ by $\text{Int}(A)$. If $B = B(x, R)$ is an open ball and $\lambda \in \mathbb{R}$ then $\lambda B$ is the ball of radius $\lambda R$ and of center $x$. A ball of radius $R$ is called an $R$-ball. The closed ball of center $x$ and radius $R$ is designated by $\overline{B}(x, R)$.

A geodesic line (resp. ray) in a metric space $(Z, d)$ is an isometry from $(\mathbb{R}, |\cdot - \cdot|)$ (resp. $([0, +\infty), |\cdot - \cdot|)$) to $(Z, d)$. The real hyperbolic space (resp. Euclidean space) of dimension $d$ is denoted $\mathbb{H}^d$ (resp. $\mathbb{E}^d$).
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2 Combinatorial modulus and the CLP

The combinatorial modulus are tools that have been developed to compute modulus of curves in a metric space without a natural measure. The idea is to approximate the metric space, with a sequence of thinner and thinner approximations. Then with these approximations we can construct discrete measures and compute combinatorial modulus. Finally, for well chosen examples we can check that this sequence of modulus has a good asymptotic behaviour.

In this first section, we present the general theory of combinatorial modulus in compact metric spaces. We also remind basic definitions and facts about abstract Loewner spaces as they inspired the theory of combinatorial modulus. Most of this section is reminder of [BK13, Section 2] to which we refer for details.

In this section \((Z,d)\) denotes a compact metric space.
2.1 General properties of combinatorial modulus of curves

For $k \geq 0$ and $\kappa > 1$, a $\kappa$-approximation of $Z$ on scale $k$ is a finite covering $G_k$ by open subsets such that for any $v \in G_k$ there exists $z_v \in v$ satisfying the following properties:

- $\forall v \in G_k$, $\exists z_v \in Z$ such that $B(z_v, \kappa^{-1/2} - k) \subset v \subset B(z_v, \kappa^{1/2} - k)$,
- $\forall v, w \in G_k$ with $v \neq w$ one has $B(z_v, \kappa^{-1/2} - k) \cap B(z_w, \kappa^{-1/2} - k) = \emptyset$.

A sequence $\{G_k\}_{k \geq 0}$ is called a $\kappa$-approximation of $Z$.

**Example 2.1.** For $k \geq 0$, a $2^{-k}$-separated subset of $Z$ is a subset $E$ such that $d(z, z') \geq 2^{-k}$ for any $z, z' \in E_k$. As $Z$ is compact any $2^{-k}$-separated subset of $Z$ is finite. Let $E_k$ be a $2^{-k}$-separated subset of $Z$ of maximal cardinality. Then $E_k$ satisfies the following property:

- for any $x \in Z$, there exists $z \in E_k$ such that $d(x, z) \leq 2^{-k}$.

The set $\{B(z, 2^{-k})\}_{z \in E_k}$ defines a $2$-approximation at scale $k$ of $Z$.

Now we fix the approximation $\{G_k\}_{k \geq 0}$. We construct a discrete measure based on each $G_k$ for $k \geq 0$. Let $\rho : G_k \rightarrow \mathbb{R}^+$ be a positive function and $\gamma$ be a curve in $Z$. The $\rho$-length of $\gamma$ is

$$L_\rho(\gamma) = \sum_{\gamma \cap v \neq \emptyset} \rho(v).$$

For $p \geq 1$, the $p$-mass of $\rho$ is

$$M_p(\rho) = \sum_{v \in G_k} \rho(v)^p.$$

Until the end of this subsection $p \geq 1$ is fixed. Let $F$ be a non-empty set of curves in $Z$. We say that the function $\rho$ is $F$-admissible if $L_\rho(\gamma) \geq 1$ for any curve $\gamma \in F$.

**Definition 2.2.** The $G_k$-combinatorial $p$-modulus of $F$ is

$$\text{Mod}_p(F, G_k) = \inf\{M_p(\rho)\}$$

where the infimum is taken over the set of $F$-admissible functions and with the convention $\text{Mod}_p(\emptyset, G_k) = 0$.

The following equality is an alternative definition of the modulus:

$$\text{Mod}_p(F, G_k) = \inf_{\rho} \frac{M_p(\rho)}{L_\rho(F)^p},$$

where the infimum is taken over the set of the positive functions on $G_k$ and with $L_\rho(F) = \inf_{\gamma \in F} L_\rho(\gamma)$.

The next proposition allows us to see the $G_k$-combinatorial $p$-modulus as a weak outer measure on the set of curves of $Z$. Usually, for an outer measure the subadditivity must holds over countable sets. This is useful to get intuition on these tools.
Proposition 2.3 ([BK13, Proposition 2.1.]).

1. Let $\mathcal{F}$ be a family of curves and $\mathcal{F}' \subset \mathcal{F}$. Then $\text{Mod}_p(\mathcal{F}', G_k) \leq \text{Mod}_p(\mathcal{F}, G_k)$.

2. Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be families of curves. Then $\text{Mod}_p(\bigcup_{i=1}^n \mathcal{F}_i, G_k) \leq \sum_{i=1}^n \text{Mod}_p(\mathcal{F}_i, G_k)$.

A function $\rho : G_k \to \mathbb{R}^+$ is called a minimal function for a set of curves $\mathcal{F}$ if $\text{Mod}_p(\mathcal{F}, G_k) = M_p(\rho)$. As we only compute finite sums, minimal functions always exist. Along with a convexity argument, this also provides an elementary control of the modulus as follow. For $\mathcal{F}$ a non-empty set of curves in $Z$ and $k \geq 0$

$$\frac{1}{(#G_k)^{p-1}} \leq \text{Mod}_p(\mathcal{F}, G_k) \leq #G_k.$$

In the sequel of this article we mainly discuss the curves of $Z$ of diameter larger than a fixed constant. For these curves the following basic property is useful.

Proposition 2.4. Let $\mathcal{F}$ be a non-empty set of curves in $Z$. Assume that there exists $d > 0$ such that $\text{diam} \gamma \geq d$ for any $\gamma \in \mathcal{F}$. Then for any $\epsilon > 0$, there exists $k_0 \geq 0$ such that for any $k \geq k_0$, there exists a minimal function $\rho : G_k \to \mathbb{R}^+$ such that $\rho(v) \leq \epsilon$ for any $v \in G_k$.

Proof. Let $\gamma \in \mathcal{F}$. We remind that $\kappa$ denotes the multiplicative constant of the approximation $\{G_k\}_{k \geq 0}$. For $k \geq \frac{\log(\kappa/d)}{\log \frac{1}{2}}$, as $\text{diam} \gamma > d$ the following inequality holds

$$\# \{v \in G_k : v \cap \gamma \neq \emptyset\} \geq \frac{d}{\kappa 2^{-k}}.$$ 

Hence the constant function $v \in G_k \rightarrow \frac{d}{\kappa 2^{-k}} \in \mathbb{R}^+$ is $\mathcal{F}$-admissible. This finishes the proof. \hfill $\square$

A metric space $Z$ is called doubling if there exists a uniform constant $N$ such that each ball $B$ of radius $r$ is covered by $N$ balls of radius $r/2$. In doubling spaces, the $G_k$-combinatorial $p$-modulus does not depend, up to a multiplicative constant, on the choice of the approximation.

Proposition 2.5 ([BK13 Proposition 2.2.]). Let $(Z, d)$ be a compact doubling metric space. For each $p \geq 1$, if $G_k$ and $G'_k$ are respectively $\kappa$ and $\kappa'$-approximations, there exists $D = D(\kappa, \kappa')$ such that for any $k \geq 0$

$$D^{-1} \cdot \text{Mod}_p(\mathcal{F}, G_k) \leq \text{Mod}_p(\mathcal{F}, G'_k) \leq D \cdot \text{Mod}_p(\mathcal{F}, G_k).$$

Usually, we work with $p \geq 1$ fixed and with approximately self-similar spaces (see Section 3). As these spaces are doubling, now we refer to the combinatorial modulus on scale $k$, omitting $p$ and the approximation.
2.2 Combinatorial Loewner property

In this subsection, we assume that \((Z,d)\) is a compact arcwise connected doubling metric space. Let \(\kappa > 1\) and let \(\{G_k\}_{k \geq 0}\) denote a \(\kappa\)-approximation of \(Z\). Moreover we fix \(p \geq 1\).

A compact and connected subset \(A \subset Z\) is called a **continuum**. Moreover, if \(A\) contains more than one point, \(A\) is called a **non-degenerate** continuum. The relative distance between two disjoint non-degenerate continua \(A, B \subset Z\) is

\[
\Delta(A, B) = \frac{\text{dist}(A, B)}{\min\{\text{diam} A, \text{diam} B\}}.
\]

If \(A\) and \(B\) are two such continua, \(F(A, B)\) denotes the set of curves in \(Z\) joining \(A\) and \(B\) and we write \(\text{Mod}_p(A, B, G_k) := \text{Mod}_p(F(A, B), G_k)\).

**Definition 2.6.** Let \(p > 1\). We say that \(Z\) satisfies the Combinatorial \(p\)-Loewner Property (CLP) if there exist two increasing functions \(\phi\) and \(\psi\) on \((0, +\infty)\) with \(\lim_{t \to 0} \psi(t) = 0\), such that for any pair of disjoint non-degenerate continua \(A\) and \(B\) in \(Z\) and for all \(k \geq 0\) with \(2^{-k} \leq \min\{\text{diam} A, \text{diam} B\}\) one has:

\[
\phi(\Delta(A, B)^{-1}) \leq \text{Mod}_p(A, B, G_k) \leq \psi(\Delta(A, B)^{-1}).
\]

As we assume that \(Z\) is doubling, thanks to Proposition 2.5, the CLP is independent of the choice of the approximation. As we noticed, the modulus on scale \(k\) is an outer measure (in a weak sense) over the set of curves in \(Z\). With the previous remarks we can say intuitively that

* A metric space satisfies the CLP if, the farer are the continua, the smaller is this amount of curves joining them.

We present examples and properties about the CLP in Subsection 2.4.

2.3 Loewner spaces

Now we define the notion of Loewner space. This notion introduced in \[HK98\] has inspired the definition of the CLP. Moreover, the proof of many basic properties of combinatorial modulus are directly inspired by the classical theory of modulus (see \[BK13\]).

Now we consider \((X, d, \mu)\) a metric measured space. For the simplicity, we assume that \(X\) is compact and \(Q\)-Ahlfors-regular (\(Q\)-AR or AR) for \(Q > 1\). This means that there exists a constant \(C > 1\) such that for any \(0 < R \leq \text{diam} X\) and any \(R\)-ball \(B \subset X\) one has

\[
C^{-1} \cdot R^Q \leq \mu(B) \leq C \cdot R^Q.
\]

Note that under this assumption the measure \(\mu\) is comparable to the Hausdorff measure \(\mathcal{H}_d\).
Let $\mathcal{F}$ be a set of curves in $X$. A measurable function $f : X \to \mathbb{R}^+$ is said to be $\mathcal{F}$-admissible if for any rectifiable curve $\gamma \in \mathcal{F}$

$$\int_{\gamma(t)} f(\gamma(t)) dt \geq 1.$$ 

Note that the notion of admissibility does not use the measure on $X$ but only the structure of metric space.

**Definition 2.7.** The $Q$-modulus of $\mathcal{F}$ is

$$\text{Mod}_Q(\mathcal{F}) = \inf \left\{ \int_X f^Q d\mu \right\}$$

where the infimum is taken over the set of $\mathcal{F}$-admissible functions and with the convention that $\text{Mod}_Q(\mathcal{F}) = 0$ if $\mathcal{F}$ does not contain rectifiable curves.

As before, if $A$ and $B$ are two disjoint non-degenerate continua, $\mathcal{F}(A,B)$ denotes the set of curves in $X$ joining $A$ and $B$. Moreover, we write $\text{Mod}_Q(A,B) := \text{Mod}_Q(\mathcal{F}(A,B))$. In the literature about quasiconformal maps the pair $(A,B)$ is called a condenser and the modulus (with respect to the Lebesgue measure) $\text{Mod}_Q(A,B)$ the capacity of $(A,B)$ (see [Vuot88]).

In $X$, the classical modulus are comparable to the combinatorial modulus in the following sense.

**Proposition 2.8 ([Hai09a, Prop B.2]).** Assume that $X$ is equipped with an approximation $\{G_k\}_{k \geq 0}$. For $d_0 > 0$, let $\mathcal{F}_0$ be the set of curves in $X$ of diameter larger than $d_0$. For $k$ large enough one has

$$\text{Mod}_Q(\mathcal{F}_0, G_k) \asymp \text{Mod}_Q(\mathcal{F}_0)$$

if $\text{Mod}_Q(\mathcal{F}_0) > 0$ and $\text{Mod}_Q(\mathcal{F}_0)$ goes to 0 when $k$ goes to infinity otherwise. Besides for any pair $A,B$ of non-degenerate disjoint continua and for $k$ large enough one has

$$\text{Mod}_Q(A,B, G_k) \asymp \text{Mod}_Q(A,B)$$

if $\text{Mod}_Q(A,B) > 0$ and $\text{Mod}_Q(A,B)$ goes to 0 when $k$ goes to infinity otherwise.

Note that this connection between combinatorial and classical modulus is only valid for the dimension $Q$.

Now we can define Loewner spaces.

**Definition 2.9.** We say that $(X,d,\mu)$ is a $Q$-Loewner space if there exists an increasing function $\phi : (0, +\infty) \to (0, +\infty)$ such that for any pair of non-degenerate disjoint continua $A$ and $B$ in $X$ one has:

$$\phi(\Delta(A,B)^{-1}) \leq \text{Mod}_Q(\mathcal{F}(A,B)).$$
We also say that $X$ satisfies the \textit{Loewner property} or the \textit{classical Loewner property} to avoid the confusion with the CLP.

The control of the modulus by above is not required in this definition because it is automatically provided by the structure of $Q$-AR space.

\textbf{Theorem 2.10} ([HK98, Lemma 3.14.]). There exists a constant $C > 0$ such that the following property holds. Let $A$ and $B$ be two non-degenerate disjoint continua. Let $0 < 2r < R$ and $x \in X$ be such that $A \subset B(x, r)$ and $B \subset x \setminus B(x, R)$. Then

$$\operatorname{Mod}_Q(A, B) \leq C \left( \log \frac{R}{r} \right)^{1-Q}.$$

As a consequence there exists an increasing function $\psi$ on $(0, +\infty)$ with $\lim_{t \to 0} \psi(t) = 0$, such that for any pair of disjoint non-degenerate continua $A$ and $B$

$$\operatorname{Mod}_Q(A, B) \leq \psi(\Delta^{-1}(A, B)).$$

More precisely, there exist some constants $K, C > 0$ such that $\psi(t) = K \left( \log(\frac{1}{t} + C) \right)^{1-Q}$ for any $t > 0$.

When $X$ is a Loewner space, the asymptotic behavior of $\phi$ is described in [HK98, Theorem 3.6.]. For $t$ small enough $\phi(t) \approx \log \frac{1}{t}$, for $t$ large enough $\phi(t) \approx (\log t)^{1-Q}$.

As we will see in the sequel an essential difference between the combinatorial and the classical modulus property is the importance of the dimension $Q$ in the discussions about classical modulus.

\section{2.4 First properties and examples}

In this section $Z$ is a compact arcwise connected doubling metric space and $X$ is a compact $Q$-AR metric space ($Q > 1$). First we remind a theorem and a conjecture that compare the CLP and the classical Loewner property.

\textbf{Theorem 2.11} ([BK13, Theorem 2.6.]). If $X$ is a compact $Q$-AR and Loewner metric space, then $X$ satisfies the combinatorial $Q$-Loewner property.

The next conjecture is a motivation for looking to boundaries of group that satisfy the CLP.

\textbf{Conjecture 2.12} ([Kle06, Conjecture 7.5.]). Assume that $Z$ is quasi-Moebius homeomorphic to the boundary of a hyperbolic group. If $Z$ satisfies the CLP then it is quasi-Moebius homeomorphic to a Loewner space.

As announced we want to find and use the CLP on boundaries of hyperbolic groups. Quasi-isometries between hyperbolic spaces extends in quasi-Moebius homeomorphisms between the boundaries, so it is fundamental to know how these properties behave under
quasi-Möbius maps. These maps have been introduced by J. Väisälä in [Väi85]. We remind that in a metric space \((Z, d)\) the cross-ratio of four distinct points \(a, b, c, d \in Z\) is

\[
[a : b : c : d] = \frac{d(a, b)}{d(a, c)} \cdot \frac{d(c, d)}{d(b, d)}.
\]

For \(Z, Z'\) two metric spaces, an homeomorphism \(f : Z \to Z'\) is quasi-Möbius if there exists an homeomorphism \(\phi : \mathbb{R}^+ \to \mathbb{R}^+\) such that for any quadruple of distinct points \(a, b, c, d \in Z\)

\[
[f(a) : f(b) : f(c) : f(d)] \leq \phi([a : b : c : d]).
\]

If \(f\) is quasi-Möbius, as \([a : c : b : d] = [a : b : c : d]^{-1}, f^{-1} : Z' \to Z\) is also quasi-Möbius.

**Theorem 2.13** ([BK13, Theorem 2.6.]). If \(Z'\) is quasi-Möbius homeomorphic to a compact space \(Z\) satisfying the CLP, then \(Z'\) also satisfies the CLP (with the same exponent).

The Loewner property does not behave so well under quasi-Möbius maps. In particular, it is perturbed by a change of dimension whereas the CLP is not.

**Theorem 2.14** ([Tys98]). Let \(X\) and \(X'\) be respectively \(Q\)-Loewner and \(Q'\)-AR compact metric spaces. Assume that \(X'\) is quasi-Möbius homeomorphic to \(X\). Then \(X'\) is a Loewner space if and only if \(Q = Q'\).

If we apply to \(X\) a snowflake transformation \(f_\epsilon : (X, d) \to (X, d')\), \(0 < \epsilon < 1\) then \(\dim_H(X, d') = \frac{1}{\epsilon} \dim_H(Z, d)\). Such a transformation is a quasi-Möbius homeomorphism and along with the previous theorem we get the following fact.

**Fact 2.15.** The Loewner property is not invariant by quasi-Möbius homeomorphisms.

Yet quasi-Möbius maps are the appropriate homeomorphisms to discuss between Loewner spaces.

**Theorem 2.16** ([HK98]). Let \(X\) and \(X'\) be compact \(Q\)-regular Loewner spaces and let \(f : X \to X'\) be a homeomorphism. The following are equivalent

1. \(f\) is quasi-Möbius,
2. there exists \(C > 1\) such that

\[
C^{-1} \cdot \text{Mod}_Q(\mathcal{F}) \leq \text{Mod}_Q(f(\mathcal{F})) \leq C \cdot \text{Mod}_Q(\mathcal{F})
\]

for any set of curves \(\mathcal{F}\) in \(X\).

The next proposition gives examples of spaces that do not satisfy the CLP.

**Proposition 2.17** ([HK98] or [BK13, Theorem 2.6.]). Assume that \(Z\) satisfies the CLP then it has no local cut point, i.e no connected open subset is disconnected by removing a point.
Along with the theorem of Bowditch (see [Bow98]) this proposition says that the boundary of a one-ended hyperbolic group who splits along a two-ended subgroup does not satisfy the CLP.

The first examples of spaces that satisfy the CLP are provided by Theorem 2.11 and by the examples of formerly known Loewner spaces. The next examples are provided by [BK13].

Example 2.18.

1. The following spaces are Loewner spaces
   i) the Euclidean space \( \mathbb{R}^d \) for \( d \geq 2 \), this result is due to C. Loewner for \( d \geq 3 \) (see [Loe59]),
   ii) any Riemannian compact manifold modeled by \( \mathbb{R}^d \) for \( d \geq 2 \) (see [HK98]),
   iii) visual boundaries of right-angled Fuchsian buildings (see [BP99]).

2. The following spaces verify the CLP (see [BK13])
   i) the Sierpinski carpet and the \( n \)-dimensional Menger sponge embedded in the Euclidean space,
   ii) boundaries of Coxeter groups of various type: simplex groups, some prism groups, some highly symmetric groups and some groups with planar boundary.

For Examples 2.18.2, we do not know if they are Loewner spaces. This provides a first kind of interesting questions.

Question 2.19. Is one of the Examples 2.18.2 quasi-Möbius homeomorphic to a Loewner space?

Among these cases the one of the Sierpinski carpet is the first that should be discussed as it should be the easiest one.

Note that Example 2.18.1(ii) provides many examples of hyperbolic groups whose boundaries are Loewner spaces. Indeed consider a group \( \Gamma \) that is acting geometrically (see Subsection 3.1) on the standard hyperbolic space \( \mathbb{H}^d \) for \( d \geq 3 \). Then \( \partial \Gamma \) is quasi-Möbius equivalent to \( S^{d-1} \) equipped with the standard spherical metric. Hence with Example 2.18.1(ii), \( \partial \Gamma \) is a Loewner space.

Actually the Loewner property and the CLP have been interesting subject of discussion on boundaries of groups to approach Cannon’s conjecture (see Conjecture 1.2) that is a reverse problem. By a theorem of D. Sullivan in [Sul82], Cannon’s conjecture is equivalent to the following in which the quasiconformal structure of the boundary is the main point.

Conjecture 2.20 ([BK02, Conjecture 1.3.]). If \( \Gamma \) is a hyperbolic group and \( \partial \Gamma \) is homeomorphic to \( S^2 \), then it is quasi-Möbius homeomorphic to the standard 2-sphere.
The notion of Loewner space is then interesting to find on boundaries of groups as quasi-Moebius maps are the good morphisms to consider between Loewner spaces. If Conjecture 2.12 is true, the CLP would provide many interesting examples of Loewner spaces. This motivates this second question.

**Question 2.21.** Can we find new examples of compact metric spaces satisfying the CLP?

In this article, we find examples of boundaries of right-angled buildings of dimension 3 and 4 that satisfy the CLP. The discussion about these buildings is suggested by Examples 2.18.1.iii) and 2.18.2.ii). The strategy applied is to lift some tools used in [BK13] to study the Coxeter groups thanks to some ideas used [BP00] to study buildings of dimension 2.

The examples of buildings that we obtain, have been studied by J. Dymara and D. Osajda who described the topology of the boundary.

**Theorem 2.22 ([DO07]).** Let $\Delta$ be a right-angled thick building whose associated Coxeter group is a cocompact reflection group in $\mathbb{H}^d$. Then $\partial \Delta$ is homeomorphic to the universal $(d - 1)$-dimensional Menger space $\mu^{d-1}$.

### 3 Combinatorial modulus of curves on boundaries of hyperbolic groups

Boundaries of hyperbolic groups are naturally endowed with metric structures that satisfy a property of self-similarity. This permits to rescale the sets of curves of the boundary with a controlled modulus. Which is very useful to prove the CLP.

In this section, we present how the boundary of a hyperbolic group can be seen as approximately self-similar spaces. Then, we remind the connection between the combinatorial modulus and the conformal dimension of the boundary. Eventually, we give a sufficient condition for the boundary to satisfy the CLP.

Most of this section is a reminder of [BK13] Section 3 and 4 to which we refer for details.

#### 3.1 Boundaries of hyperbolic groups and approximately self-similar spaces

For details concerning hyperbolic groups and spaces, we refer to [CDP90], [GdlH90] or [KB02]. Here $(X,d)$ is a proper geodesic metric space. We say that a finitely generated group $\Gamma$ acts geometrically on $X$, if:

- $\Gamma < \text{Isom}(X)$,
- $\Gamma$ acts cocompactly,
- $\Gamma$ acts properly discontinuously.
We say that \( X \) is hyperbolic (in the sense of Gromov) if there exists a constant \( \delta > 0 \) such that for every geodesic triangle \([x,y] \cup [y,z] \cup [z,x] \subset X\) and every \( p \in [x,y] \), one has
\[
\text{dist}(p,[y,z] \cup [z,x]) \leq \delta.
\]
A finitely generated group that acts geometrically on a hyperbolic space \( X \) is called a hyperbolic group. In this case, the Cayley graph of a hyperbolic group is a hyperbolic space.

From now, \( X \) is a hyperbolic space with a fixed base point \( x_0 \) and \( \Gamma \) is a hyperbolic group acting geometrically on \( X \). Let \( \partial X \) be the quotient space defined by the set of half-geodesics and by the equivalence relation:
\[
\gamma, \gamma' : [0, +\infty) \to X \text{ are equivalent if and only if there exists } K > 0 \text{ such that } d(\gamma(t), \gamma'(t)) \leq K \text{ for any } t \in [0, +\infty).
\]
Thanks to the hyperbolicity condition, we can restrict to the set of half-geodesics starting from \( x_0 \). We can equip \( \partial X \) and \( X \cup \partial X \) with topologies which make them compact sets. In this setting, \( X \), is dense in \( X \cup \partial X \) and \( \partial X \) is called the boundary of \( X \). Moreover we can equip \( \partial X \) with a family of visual metric. A metric \( \delta(\cdot, \cdot) \) is visual if there exist two constants \( A \geq 1 \) and \( \alpha > 0 \) such that for all \( \xi, \xi' \in \partial X \):
\[
A^{-1}e^{-\alpha \ell} \leq \delta(\xi, \xi') \leq Ae^{-\alpha \ell},
\]
where \( \ell \) is the distance from \( x_0 \) to a geodesic line \((\xi, \xi')\). In such a situation we also write
\[
\delta(\xi, \xi') \asymp e^{-\alpha \ell}.
\]

The action of \( \Gamma \) on \( X \) extends naturally on \((\partial X, \delta)\) and elements of \( \Gamma \) are uniform quasi-Möbius homeomorphisms of the boundary. Moreover, if \( \partial \Gamma \) is also equipped with a visual metric, the homeomorphism \( \partial \Gamma \to \partial X \) induced by the orbit map \( g \in \Gamma \to gx_0 \in X \) is quasi-Möbius.

The following definition is a generalization of the classical notion of self-similar space.

**Definition 3.1.** A compact metric space \((Z, d)\) is called approximately self-similar if there exists a constant \( L \geq 1 \) such that for every \( r \)-ball \( B \) with \( 0 < r < \text{diam } Z \), there exists an open subset \( U \subset Z \) which is \( L \)-bi-Lipschitz homeomorphic to the rescaled ball \((B, \frac{1}{L}d)\).

The property that follows shows that the notion of approximately self-similar space fits with the metric structure of the boundary of a hyperbolic group and with the action of the group on its boundary.

**Definition 3.2.** Let \( \Gamma \) be a hyperbolic group. A metric \( d \) on \( \partial \Gamma \) is called a self-similar metric if there exists a hyperbolic space \( X \) on which \( \Gamma \) acts geometrically, such that \( d \) is the preimage of a visual metric on \( \partial X \) by the canonical quasi-Möbius homeomorphism \( \partial \Gamma \to \partial X \).

**Proposition 3.3** ([BK13, Proposition 3.3.]). The space \( \partial \Gamma \) equipped with a self-similar metric is doubling and is an approximately self-similar space, the partial bi-Lipschitz maps being restrictions of group elements. Moreover, \( \Gamma \) acts on \((\partial \Gamma, d)\) by (non-uniformly) bi-Lipschitz homeomorphisms.
3.2 Combinatorial modulus and conformal dimension

Here we present the connection between combinatorial modulus and the conformal dimension in approximately self-similar spaces.

Let $Z$ be an arcwise connected approximately self-similar metric space. In practice $Z$ will be the boundary of a hyperbolic group. Let $\{G_k\}_{k \geq 0}$ be a $\kappa$-approximation of $Z$ and $d_0$ be a small constant compared with $\text{diam} Z$ and with the constant of self-similarity.

Let $F_0$ denote the family of curves in $Z$ of diameter larger than $d_0$. In [BK13], it is proved that the properties of the combinatorial modulus are contained in the asymptotic behavior of $\text{Mod}_p(F_0, G_k)$. This point is explained in Subsection 3.3. In particular, to describe this behavior the following submultiplicative inequality is used. Here we write $M_k := \text{Mod}_p(F_0, G_k)$.

**Proposition 3.4 ([BK13 Proposition 3.6.])**. Let $p \geq 1$. There exists a constant $C > 0$ such that for every pair of integers $k, \ell \in \mathbb{N}$ one has

$$M_{k+\ell} \leq C \cdot M_k \cdot M_\ell.$$ 

Furthermore, when $p$ belongs to a compact subset of $[1, +\infty)$, the constant $C$ may be chosen independent of $p$.

The following proposition allows to define a critical exponent that is related to the conformal dimension of $Z$.

**Proposition 3.5.** There exists $p_0 \geq 1$ such that for $p \geq p_0$ the modulus $M_k$ goes to zero as $k$ goes to infinity.

**Proof.** Let $C$ be the constant provided by Proposition 3.4. According to Proposition 2.4 there exists $\ell_0 \geq 1$ such that there exists a $F_0$-minimal function $\rho : G_{\ell_0} \to \mathbb{R}^+$ that satisfies $\rho(v) < 1/2$ for any $v \in G_{\ell_0}$. Now we fix $p_0 \geq 1$ such that

$$C \cdot M_{\ell_0} < 1.$$ 

Let $q \geq 0$ and $0 \leq r < \ell_0$ be the results of the Euclidean division of $k$ by $\ell_0$ ($k = q\ell_0 + r$). Applying $q$ times Proposition 3.4 we obtain

$$M_k \leq C^q \cdot M_{\ell_0}^q \cdot M_r.$$ 

As $r < \ell_0$ it comes that $M_r$ is bounded by above, thus there exists $K > 0$ such that

$$M_k \leq K \cdot (CM_{\ell_0})^q.$$ 

This proves that $M_k$ goes to zero as $k$ goes to infinity. Eventually we notice that for $k \geq \ell_0$ fixed the function $p \mapsto \text{Mod}_p(F_0, G_k)$ is non-increasing. This finishes the proof. \qed
According to this proposition the next definition makes sense.

**Definition 3.6.** The critical exponent $Q$ associated with the curve family $F_0$ is defined as follow

$$Q = \inf\{p \in [1, +\infty) : \lim_{k \to +\infty} \text{Mod}_p(F_0, G_k) = 0\}.$$  

This critical exponent is related to the conformal dimension, which provides another motivation to study combinatorial modulus. The conformal dimension has been introduced by P. Pansu in [Pan89]. It is a key point in the conformal structure of the boundary of a hyperbolic group. In particular, it is invariant under quasi-Möbius maps.

In the following, $\mathcal{H}_d(\cdot)$ and $\dim_H(Z, d)$ respectively denote the Hausdorff measure and the Hausdorff dimension of $Z$ equipped with $d$. The *Ahlfors-regular conformal gauge* of $(Z, d)$ is defined as follow

$$\mathcal{J}_c(Z, d) := \{(Z', \delta) : (Z', \delta) \text{ is AR and quasi-moebius homeomorphic to } (Z, d)\}.$$  

**Definition 3.7.** Let $(Z, d)$ be a compact metric space. The *Ahlfors-regular conformal dimension* of $(Z, d)$ is

$$\text{Confdim}(Z, d) := \inf\{\dim_H(Z', \delta) : (Z', \delta) \in \mathcal{J}_c(Z, d)\}.$$  

For the simplicity, in the following of the article we will refer to the conformal dimension for the Ahlfors-regular conformal dimension.

In [KK], S. Keith and B. Kleiner proved that the combinatorial modulus are related to the conformal dimension. The proof of the following theorem may be found in [Car11].

**Theorem 3.8** ([KK] or [Car11, Theorem 4.5.]). The critical exponent $Q$ (see Definition 3.6) is equal to $\text{Confdim}(Z, d)$.

The definition of the conformal dimension, along with basic topology give the following inequalities:

$$\dim_T(Z) \leq \text{Confdim}(Z, d) \leq \dim_H(Z, d),$$

where $\dim_T(Z)$ is the topological dimension of $Z$.

The following theorem due to J. Tyson makes a connection between the conformal dimension and the Loewner property.

**Theorem 3.9** ([MT10, Corollary 4.2.2.]). If $X$ is a $Q$-regular and $Q$-Loewner space, then $\text{Confdim}(X) = Q$.

**Example 3.10.** It has been proved independently by B. Kleiner and in [KL04] that the Euclidean metric on the Sierpinski carpet does not realize the conformal dimension. As a consequence the Sierpinski carpet equipped with this metric does not satisfy the Loewner property. Yet it satisfies the CLP (see Example 2.18).
Again, Cannon’s conjecture has been an important motivation to discuss the conformal dimension of the boundary of a hyperbolic group. In particular in [BK05] it is proved that Conjecture [1.2] is equivalent to the following.

**Conjecture 3.11.** If $\Gamma$ is a hyperbolic group and $\partial \Gamma$ is homeomorphic to $\mathbb{S}^2$, then $\text{Confdim}(\partial \Gamma)$ is attained by a metric in $\mathcal{J}_c(\partial \Gamma)$.

### 3.3 How to prove the CLP

Now we give the sufficient condition that will be used in this article to exhibit some examples of groups with a boundary that satisfy the CLP.

Let $Z$ be an arcwise connected approximately self-similar metric space and let $\{G_k\}_{k \geq 0}$ be a $\kappa$-approximation of $Z$.

The approximately self-similar structure and Proposition 2.5, allow to deform any set of small curves to a set of large curves with a control of the modulus. This is what we prove in the following proposition.

**Proposition 3.12.** Let $B$ be a ball in $\partial \Gamma$ such that $diam B < 1$. Let $g \in \Gamma$ be the local $L$-bi-Lipschitz homeomorphism that rescales $B$ (given by Definition 3.3). Let $\mathcal{F}$ be a set of curves contained in $\lambda B$ for $\lambda < 1$. Then there exist $\ell \in \mathbb{N}$, and $D > 1$ depending only on $L$ and on the doubling constant of $\partial \Gamma$ such that the following property holds.

If $k \geq 0$ is large enough so that

$$\{v \in G_k : \gamma \cap v \neq \emptyset \text{ for some } \gamma \in \mathcal{F}\} \subset \{v \in G_k : v \subset B\},$$

then

$$D^{-1} \cdot \text{Mod}_p(g \mathcal{F}, G_k) \leq \text{Mod}_p(\mathcal{F}, G_{k+\ell}) \leq D \cdot \text{Mod}_p(g \mathcal{F}, G_k),$$

where $g \mathcal{F} = \{g \gamma : \gamma \in \mathcal{F}\}$.

**Proof.** Let $k \geq 0$ be large enough so that, if $\gamma \cap v \neq \emptyset$ with $\gamma \in \mathcal{F}$ and $v \in G_k$, then $v \subset B$. Let $d = diam B$ and let $\ell \in \mathbb{N}$ denote the integer verifying $2^{-(\ell+1)} < d \leq 2^{-\ell}$. Let $v \in G_{k+\ell}$ such that $v \subset B$ and assume

$$B(\xi, \kappa^{-1}2^{-(k+\ell)}) \subset v \subset B(\xi, \kappa 2^{-(k+\ell)}).$$

Then

$$B(g\xi, (L\kappa)^{-1}2^{-k}) \subset gv \subset B(g\xi, 2L\kappa 2^{-k}).$$

Thus the set $\{gv : v \in G_{k+\ell}, v \subset B\}$ is a $\kappa'$-approximation of $gB$ on scale $k$, with $\kappa'$ depending only on $\kappa$ and $L$. Thanks to Proposition 2.5 we obtain the desired inequalities. \[\square\]
Now we fix $d_0 > 0$ a small constant compared with $\text{diam} \ Z$ and with the constant of self-similarity. More precisely it must be small enough so that any non-constant curve in $Z$ may be rescaled to a curve of diameter larger than $d_0$ by self-similarity. In practice, $Z$ will by the boundary of a hyperbolic group and $d_0$ will depend on the hyperbolicity constant.

Let $F_0$ denote the family of curves in $Z$ of diameter larger than $d_0$. This rescaling property explains that the properties of the combinatorial modulus are contained in the asymptotic behavior of $\text{Mod}_p(F_0, G_k)$.

**Proposition 3.13 ([BK13 Proposition 4.5.]).** Let $Z$ be an arcwise connected approximately self-similar metric space. Let $\{G_k\}_{k \geq 0}$ be a $\kappa$-approximation of $Z$ and $d_0$ be a small constant compared with $\text{diam} \ Z$ and with the constant of self-similarity. Let $F_0$ denote the family of curves in $Z$ of diameter larger than $d_0$.

For $p = 1$, we assume that $\text{Mod}_p(F_0, G_k)$ is unbounded. For $p \geq 1$, we assume that for every non-constant curve $\eta \subset Z$ and every $\epsilon > 0$, there exists $C = C(p, \eta, \epsilon)$ such that for every $k \in \mathbb{N}$:

$$\text{Mod}_p(F_0, G_k) \leq C \cdot \text{Mod}_p(U_\epsilon(\eta), G_k).$$

Suppose furthermore when $p$ belongs to a compact subset of $[1, +\infty)$ the constant $C$ may be chosen independent of $p$. Then $Z$ satisfies the combinatorial $Q$-Loewner property.

### 4 Steps of the proof of Theorem 10.1

Before going into details about boundaries of right-angled buildings, we give a sketch of proof of the main theorem. In this section, $D$ is the right-angled dodecahedron in $\mathbb{H}^3$ or the right-angled 120-cell in $\mathbb{H}^4$. We write $W_D$ for the hyperbolic reflection group generated by the faces of $D$. The main theorem of this article may be stated as follow.

**Theorem 4.1** (Theorem 10.1). Let $q \geq 3$ and let $\Gamma$ be the a graph product of constant thickness $q - 1$ and of Coxeter group $W_D$. Then $\partial \Gamma$ equipped with a visual metric satisfies the CLP.

As announced, we will verify that $\partial \Gamma$ satisfies the hypothesis of Proposition 3.13. To prove that $\text{Mod}_1(F_0, G_k)$ is unbounded, it is enough to see that for every $N \in \mathbb{N}$ there exist $N$ disjoint curves in $\partial \Gamma$. Indeed, this implies that for $k \geq 0$ large enough $\text{Mod}_1(F_0, G_k) > N$.

**To follow curves to control the modulus.** For $p > 1$, we want to prove that the curves of $\partial \Gamma$ satisfy a property of the following form.

(P) : For $\epsilon > 0$, there exists a finite set $F$ of bi-Lipschitz maps $f : \partial \Gamma \rightarrow \partial \Gamma$ such that for any curve $\gamma \in F_0$ and any curve $\eta$ in $\partial \Gamma$, the set $\bigcup_{f \in F} f(\gamma)$ contains a curve that belongs to $U_\epsilon(\eta)$.
Where \( \mathcal{U}(\eta) \) denotes the \( \epsilon \)-neighborhood of \( \eta \) for the \( C^0 \) distance (see Subsection 1.4 for details). Intuitively, \((P)\) holds if from any curve \( \gamma \) we can follow any other curve \( \eta \) thanks to bi-Lipschitz maps. The following computation shows that the property \((P)\) implies the desired inequality.

**Proposition 4.2.** If \( \text{Mod}_1(\mathcal{F}_0, G_k) \) is unbounded, then property \((P)\) implies the CLP.

**Proof.** Let \( \eta \) be a curve in \( \partial \Gamma \) and \( \epsilon > 0 \). Fix \( \rho \) a \( \mathcal{U}(\eta) \)-admissible function. The inequality required by the hypothesis of Proposition 3.13 is obtained if we find a constant \( K > 0 \) independent of the scale \( k \) and an \( \mathcal{F}_0 \)-admissible function \( \rho' \) such that \( M_p(\rho') \leq K \cdot M_p(\rho) \).

Let \( F \) be the set of bi-Lipschitz maps given by the property \((P)\). We assume, without a loss of generality that \( F \) is symmetric. We define the function \( \rho' : G_k \rightarrow \mathbb{R}^+ \) by:

\[
(*) \quad \rho'(v) = \sum_{f \in F} \sum_{w \cap v \neq \emptyset} \rho(w).
\]

Let \( \gamma \in \mathcal{F}_0 \) and \( \theta \subset \bigcup_{f \in F} f \gamma \) such that \( \theta \in \mathcal{U}(\eta) \). Then

\[
L_{\rho'}(\gamma) = \sum_{f \in F} \sum_{w \cap f \gamma \neq \emptyset} \rho(w) \geq \sum_{f \in F} \sum_{w \cap f \gamma \neq \emptyset} \rho(w).
\]

Yet

\[
L_{\rho}(\theta) \leq \sum_{f \in F} L_{\rho}(f \gamma) = \sum_{f \in F} \sum_{w \cap f \gamma \neq \emptyset} \rho(w).
\]

Hence \( L_{\rho'}(\gamma) \geq L_{\rho}(\theta) \) and \( \rho' \) is \( \mathcal{F}_0 \)-admissible.

Then the number or terms in the right-hand side of the definition \((*)\) is bounded by a constant \( N \) depending on \#\( F \), the bi-Lipschitz constants of the elements of \( F \), and the doubling constant of \( \partial \Gamma \). Therefore by convexity

\[
M_p(\rho') = \sum_{v \in G_k} \left( \sum_{f \in F} \sum_{w \cap v \neq \emptyset} \rho(w) \right)^p \leq N^{p-1} \cdot \sum_{v \in G_k} \sum_{f \in F} \sum_{w \cap v \neq \emptyset} \rho(w)^p \leq N^p \cdot \#F \cdot M_p(\rho).
\]

\[\square\]

Note that this idea of following curves may be used to obtain an inequality between any pair of sets of curves.

**The issue of parabolic limit sets.** As \( \Gamma \) acts on \( \partial \Gamma \) by bi-Lipschitz homeomorphisms, it is natural to intend to follow curves thanks to elements of \( \Gamma \). Yet, in right-angled buildings some curves may be contained in parabolic limit sets (boundaries of residues). As we will see in Example 6.9 these curves are an obstacle to prove the property \((P)\) with the elements of \( \Gamma \).
To solve this problem we start by showing that $\text{Mod}_p(\mathcal{F}_0, G_k)$ is determined by the combinatorial modulus of the sets of curves of the form $\mathcal{F}_{\delta,r}(\partial P)$ as defined at the beginning Subsection 6.2. This is what is done at the beginning of the proof of Theorem 8.12. As a first approximation, it is enough to see $\mathcal{F}_{\delta,r}(\partial P)$ as the set of curves contained in a parabolic limit set $\partial P$.

**Following curves inside parabolic limit sets.** Then inside the parabolic limit set $\partial P$ it is possible to follow curves. An analogous of property $(P)$ inside the parabolic limit sets is proved in Proposition 6.12. From this property we can obtain Theorem 6.13. This theorem is the first major step toward the proof. Essentially it says that the combinatorial modulus of $\mathcal{F}_{\delta,r}(\partial P)$ is controlled by any curve contained in $\partial P$.

**Controlling the modulus in $\partial \Gamma$ by the modulus in the boundary of an apartment.** The second major step in the proof is to use the building structure to reduce the problem in the $\partial \Gamma$ to a problem in the boundary of an apartment $i.e$ in $\partial W_D$. This is what allows Theorem 8.9. Essentially, this theorem says that the modulus of a curve in $\partial \Gamma$ is controlled by a weighted modulus defined in the boundary of an apartment. The idea used to prove this is that, from the point of view of the modulus, $\partial \Gamma$ can be seen as the direct product of the boundary of an apartment by a finite set whose cardinality depends on the scale.

**Conclusion of the proof thanks to the symmetries of $D$.** Now, thanks to Theorems 6.13 and 8.9 we arrive at a point where the modulus of $\mathcal{F}_{\delta,r}(\partial P)$, and thus of $\mathcal{F}_0$, is controlled by some modulus of the parabolic limit sets of $W_D$. The idea we use to conclude is that the symmetries of $D$ extends to the boundary of $W_D$. Along with the elements of the groups, these symmetries permit us to follow curves in $\partial W_D$. As a consequence we obtain a strong control of the modulus of the parabolic limit sets in $\partial W_D$ and we can complete the proof.

## 5 Locally finite right-angled hyperbolic buildings

The aim of this article is to discuss combinatorial modulus on boundaries of hyperbolic buildings. Here we set up the context about hyperbolic buildings that will be used until the end of this article. In particular, we discuss the geometry of locally finite right-angled hyperbolic buildings.

For details concerning the notions reminded in this section, we refer to [Tit74], [Ron89], or [AB08]. Concerning the Davis realization, we can refer to [Dav08 Chapter 8] or to [Mei96] for an example of the Davis construction along with suggestive pictures. Most of the omitted details can be also found in [Cla].

Here $S = \{s_1, \ldots, s_n\}$ is a fixed finite set.
5.1 Chamber system

Following the definition of J. Tits, a chamber system $X$ over $S$ is a set endowed with a family of partitions indexed by $S$. The elements of $X$ are called chambers.

Hereafter $X$ is a chamber system over $S$. For $s \in S$, two chambers $c, c' \in X$ are said to be $s$-adjacent if they belong to the same subset of $X$ in the partition associated with $s$. Then we write $c \sim_s c'$. Usually, omitting the type of adjacency we refer to adjacent chambers and we write $c \sim c'$. Note that any chamber is adjacent to itself.

A morphism $f : X \to X'$ between two chamber systems $X, X'$ over $S$ is a map that preserves the adjacency relations. A bijection of $X$ that preserves the adjacency relations is called an automorphism and we designate by $\text{Aut}(X)$ the group of automorphisms of $X$. A subsystem of chamber $Y$ of $X$ is a subset $Y \subset X$ such that the inclusion map is a morphism of chamber systems.

We call gallery, a finite sequence $\{c_k\}_{k=1}^{\ell}$ of chambers such that $c_k \sim c_{k+1}$ for $k = 1, \ldots, \ell - 1$. The galleries induce a metric on $X$.

**Definition 5.1.** The distance between two chambers $x$ and $y$ is the length of the shortest gallery connecting $x$ to $y$.

We use the notation $d_P(\cdot, \cdot)$ for this metric over $X$. A shortest gallery between two chambers is called minimal.

Let $I \subset S$. A subset $C$ of $X$ is said to be $I$-connected if for any pair of chambers $c, c' \in C$ there exists a gallery $c = c_1 \sim \cdots \sim c_\ell = c'$ such that for any $k = 1, \ldots, \ell - 1$, the chambers $c_k$ and $c_{k+1}$ are $i_k$-adjacent for some $i_k \in I$.

**Definition 5.2.** The $I$-connected components are called the $I$-residues or the residues of type $I$. The cardinality of $I$ is called the rank of the residues of type $I$. The residues of rank 1 are called panels.

The following notion of convexity is used in chamber systems.

**Definition 5.3.** A subset $C$ of $X$ is called convex if every minimal gallery whose extremities belong to $C$ is entirely contained in $C$.

The convexity is stable by intersection and for $A \subset X$, the convex hull of $A$ is the smallest convex subset containing $A$. In particular, convex subsets of $X$ are subsystems. The following example is crucial because it will be used to equip Coxeter groups and graph products with structures of chamber systems (see Definition 5.7 and Theorem 5.10).

**Example 5.4.** Let $G$ be a group, $B$ a subgroup and $\{H_i\}_{i \in I}$ a family of subgroups of $G$ containing $B$. The set of left cosets of $H_i/B$ defines a partition of $G/B$. We denote by $C(G, B, \{H_i\}_{i \in I})$ this chamber system over $I$. This chamber system comes with a natural action of $G$. The group $G$ acts by automorphisms and transitively on the set of chambers.
5.2 Coxeter systems

A Coxeter matrix over $S$ is a symmetric matrix $M = \{m_{r,s}\}_{r,s \in S}$ whose entries are elements of $\mathbb{N} \cup \{\infty\}$ such that $m_{s,s} = 1$ for any $s \in S$ and $\{m_{r,s}\} \geq 2$ for any $r, s \in S$ distinct. Let $M$ be a Coxeter matrix. The Coxeter group of type $M$ is the group given by the following presentation

$$W = \langle s \in S | (rs)^{m_{r,s}} = 1 \text{ for any } r, s \in S \rangle.$$

We call special subgroup a subgroup of $W$ of the form $W_I = \langle s \in I | (rs)^{m_{r,s}} = 1 \text{ for any } r, s \in I \rangle$ with $I \subset S$.

Definition 5.5. We call parabolic subgroup a subgroup of $W$ of the form $wW_Iw^{-1}$ where $w \in W$ and $I \subset S$. An involution of the form $wsw^{-1}$ for $w \in W$ and $s \in S$ is called a reflection.

Example 5.6. Let $\mathcal{X}^d = S^d, E^d$ or $H^d$. A Coxeter polytope is a convex polytope of $\mathcal{X}^d$ such that any dihedral angle is of the form $\pi k$ with $k$ not necessarily constant. Let $D$ be a Coxeter polytope and let $\sigma_1, \ldots, \sigma_n$ be the codimension 1 faces of $D$. We set $M = \{m_{i,j}\}_{i,j=1,\ldots,n}$ the matrix defined by $m_{i,i} = 1$, if $\sigma_i$ and $\sigma_j$ do not meet in a codimension 2 face $m_{i,j} = \infty$, and if $\sigma_i$ and $\sigma_j$ meet in a codimension 2 face $\pi_{m_{i,j}}$ is the dihedral angle between $\sigma_i$ and $\sigma_j$.

Then a theorem of H. Poincaré (see [GP01, Theorem 1.2.]) says that the reflection group of $\mathcal{X}^d$ generated by the codimension 1 faces of $D$ is a discrete subgroup of $\text{Isom}(\mathcal{X}^d)$ and is isomorphic to the Coxeter group of type $M$.

Definition 5.7. With the notation introduced in Example 5.4, the Coxeter system associated with $W$ is the chamber system over $S$ given by $C(W, \{e\}, \{W_s\}_{s \in S})$. We use the notation $(W, S)$ to designate this chamber system.

The chambers of $(W, S)$ are the elements of $W$ and two distinct chambers $w, w' \in W$ are $s$-adjacent if and only if $w = w's$. For $I \subset S$, notice that for any $I$-residue $R$ in $(W, S)$ there exists $w \in W$ such that, as a set $R = wW_I$. Again $W$ is a group of automorphisms of $(W, S)$ that acts transitively on the set of chambers.

Hereafter $(W, S)$ is fixed Coxeter system.

Example 5.8. In the case of Example 5.6, the chamber system associated with $W$ is realized geometrically by the tiling of $\mathcal{X}^d$ by copies of the polytope $D$. Two chambers are adjacent in $(W, S)$ if and only if the corresponding copies of $D$ in $\mathcal{X}^d$ share a codimension 1 face.

5.3 The Davis chamber of $(W, S)$

The geometric realization of M.W. Davis can be seen as a generalization to all the Coxeter groups of the Example 5.8. The first step to describe the geometric realization of $W$ is to construct its Davis chamber. We remind that $S = \{s_1, \ldots, s_n\}$ is a set of generators of $W$. Let $S_{\neq S}$ be the set of subsets of $S$ different from $S$. We denote by
Following [Dav08, Appendix A], a poset admits a geometric realization which is a simplicial complex. This complex is such that the inclusion relations between cells represent the partial order. We denote by $D$ the Davis chamber which is the geometric realization of the poset $S_f$. In the following we give details of this construction.

Let $\Delta^{n-1}$ be the standard $(n-1)$-simplex and label the codimension 1 faces of $\Delta^{n-1}$ with distinct elements of $S$. Then $\sigma$ a codimension $k$ face of $\Delta^{n-1}$ is associated with a type i.e a subset $I \subset S$ of cardinality $k$. In this setting, we write $\sigma_I$ for the face of type $I$. Equivalently, we can say that each vertex of the barycentric subdivision of $\Delta^{n-1}$ is associated with a subset of $S$. Adding the fact that the empty set is associated with the barycenter of the whole simplex, we get a bijection between the vertices of the barycentric subdivision and $S \neq S$. Hence a vertex in the barycentric subdivision is designated by $(s_i)_{i \in K}$ for $K \subset \{1, \ldots, n\}$. Using this identification, let $T$ be the subgraph of the 1-skeleton of the barycentric subdivision of $\Delta^{n-1}$ obtained as follows:

- $T^{(0)} = S \neq S$,
- the vertices $(s_i)_{i \in I}$ and $(s_j)_{j \in J}$, with $\# J \geq \# I$, are adjacent if and only if $I \subset J$ and $\# I = \# J - 1$.

In the following definition, for $k \geq 1$ we call a $k$-cube, a CW-complex that is isomorphic, as a cellular complex, to the Euclidean $k$-cube $[0,1]^k$. In particular, it is not necessary to equip these cubes with a metric for the purpose of this chapter.

**Definition 5.9.** The 1-skeleton of the Davis chamber $D^{(1)}$ is the full subgraph of $T$ generated by the elements of $S_f$. The Davis chamber is obtained from $D^{(1)}$ by attaching a $k$-cube inside any full subgraph generated by $2^k$ vertices that is the 1-skeleton of a $k$-cube.

By construction, $D \subset \Delta^{n-1}$. We call maximal faces of $D$ the subsets of the form $\sigma \cap D$ where $\sigma$ is a codimension 1 face of $\Delta^{n-1}$. Likewise, for $I \subset S$, the face of $D$ of type $I$ is $D \cap \sigma_I$. Note that that the faces of $D$ are made of branching of cubes of various dimensions.

**Example 5.10.** In the case of Example 5.6, the Davis chamber is combinatorially identified with the Coxeter polytope. Then if we equip $D$ with the appropriate metric (Euclidean, spherical, or hyperbolic) we recover the Coxeter polytope.

### 5.4 Buildings

Buildings are singular spaces defined by J. Tits. At first sight, we can understand them as higher dimensional trees. Hereafter $(W,S)$ is a fixed Coxeter system.

**Definition 5.11 ([Tit74, Definition 3.1])**. A chamber system $\Delta$ over $S$ is a building of type $(W,S)$ if it admits a maximal family $\text{Ap}(\Delta)$ of subsystems isomorphic to $(W,S)$, called apartments, such that
• any two chambers lie in a common apartment,

• for any pair of apartments $A$ and $B$, there exists an isomorphism from $A$ to $B$ fixing $A \cap B$.

For $A \in Ap(\Delta)$ we write $H(A)$ for the set of half-spaces and $M(A)$ for the set of walls of $A$.

A straightforward application of this definition is the existence of retraction maps of the building over apartments.

**Definition 5.12.** Let $x \in \Delta$ and $A \in Ap(\Delta)$. Assume that $x$ is contained in $A$. We call retraction onto $A$ centered $x$ the map $\pi_{A,x} : \Delta \to A$ defined by the following property.

For $c \in \Delta$, there exists a chamber $\pi_{A,x}(c) \in A$ such that for any apartment $A'$ containing $x$ and $c$, for any isomorphism $f : A \to A'$ that fixes $A \cap A'$, then $f(c) = \pi_{A,x}(c)$

Hereafter, $\Delta$ is a fixed building of type $(W,S)$. The building $\Delta$ is called a thin (resp. thick) building if any panel contains exactly two (resp. at least three) chambers. Note that thin buildings are Coxeter systems.

### 5.5 Graph products and right-angled buildings

Let $\mathcal{G}$ denote a **finite simplicial graph** i.e $\mathcal{G}^{(0)}$ is finite, each edge has two different vertices, and no edge is double. We denote by $\mathcal{G}^{(0)} = \{v_1, \ldots, v_n\}$ the vertices of $\mathcal{G}$. If for $i \neq j$, the corresponding vertices $v_i, v_j$ are connected by an edge, we write $v_i \sim v_j$. A finite cyclic group $G_i = \langle s_i \rangle$ of order $q_i \geq 2$ is associated with each $v_i \in \mathcal{G}^{(0)}$ and we set $S = \{s_1, \ldots, s_n\}$. Along this article, we assume that $n \geq 2$ and that $\mathcal{G}$ has at least one edge.

**Definition 5.13.** The graph product given by $(\mathcal{G}, \{G_i\}_{i=1,\ldots,n})$ is the group defined by the following presentation

$$\Gamma = \langle s_i \in S | s_i^{q_i} = 1, s_is_j = s_js_i \text{ if } v_i \sim v_j \rangle.$$ 

**Example 5.14.** If the graph $\mathcal{G}$ is fixed and the orders $\{q_i\}_{i=1,\ldots,n}$ go from 2 to $+\infty$, graph products are groups between right-angled Coxeter groups (see [Dav08]) and right-angled Artin groups (see [Cha07]). If the integers $\{q_i\}_{i=1,\ldots,n}$ are fixed and we add edges to the graph starting from a graph with no edge, those groups are groups between free products and direct products of cyclic groups.

From now, we fix a graph product $\Gamma$ associated with the pair $(\mathcal{G}, \{G_i\}_{i=1,\ldots,n})$. By analogy with Definition 5.3, we define parabolic subgroups in $\Gamma$.

**Definition 5.15.** The subgroup of $\Gamma$ generated by a subset $I \subset S$ is denoted by $\Gamma_I$ and a subgroup of the form $g\Gamma_Ig^{-1}$, with $g \in \Gamma$, is called a parabolic subgroup.
Let \( W \) be the graph product defined by the pair \((G, \{\mathbb{Z}/2\mathbb{Z}\}_{i=1,...,n})\). This graph product is isomorphic to the right-angled Coxeter group of type \( M = \{m_{i,j}\}_{i,j=1,...,n} \) defined by:
\[
m_{i,j} = 2 \text{ if } v_i \sim v_j \text{ and } m_{i,j} = \infty \text{ if } v_i \not\sim v_j.
\]

All along this article, \( W \) denotes this Coxeter group canonically associated with \( \Gamma \) and \((W, S)\) is the Coxeter system associated with \( W \).

**Theorem 5.16** ([Day98, Theorem 5.1.]). Let \( \Delta \) be the chamber system \( C(\Gamma, \{e\}, \{\Gamma_s\}_{s \in S}) \) (see Example 5.4). Then \( \Delta \) is a building of type \((W, S)\).

Hereafter, \( \Delta \) denotes the right-angled building associated with \( \Gamma \) by the previous theorem. In the following, we describe the Davis complex associated with this building.

Now we assume first that \( \Gamma \) is infinite. This happens if and only if there exists two distinct vertices \( v_i, v_j \) such that \( v_i \not\sim v_j \). Then, we assume that \( \Gamma \) is hyperbolic. A criterion of M. Gromov allows J. Meier to prove that \( \Gamma \) is a hyperbolic group if and only if in \( G \) any circuit of length 4 contains a chord (see [Mei96]). Eventually, we assume that \( \partial \Gamma \) is arcwise connected. In [DM02], a necessary and sufficient condition on \( G \) is given for \( \partial \Gamma \) to be arcwise connected (see Theorem 6.17 above).

### 5.6 The Davis complex associated with \( \Gamma \)

Now we introduce the Davis complex associated with \( \Gamma \). This complex is analogous to the Davis complex of a Coxeter system. Again we refer to [Mei96] for an example along with suggestive pictures.

Let \( D \) be the Davis chamber associated with \( W \) as in Subsection 5.3. Again a face of \( D \) is associated with a type \( I \subset S \). For \( x \in D \), if \( I \) is the type of the face containing \( x \) in its interior, we set \( \Gamma_x := \Gamma_I \). To the interior points of \( D \) we associate the trivial group \( \Gamma_\emptyset \).

Now we can define the *Davis complex* : \( \Sigma = D \times \Gamma / \sim \) with
\[
(x, g) \sim (y, g') \text{ if and only if } x = y \text{ and } g^{-1}g' \in \Gamma_x.
\]

We study the building \( \Delta \) through it geometric realization \( \Sigma \) and we briefly remind what this mean.

A chamber of \( \Sigma \) is a subset of the form \([D \times \{g\}]\) with \( g \in \Gamma \). Two chambers are adjacent if and only if they share a maximal face. For a subset \( E \subset \Sigma \) we designate by \( \text{Ch}(E) \) the set of chambers contained in \( E \). Equipped with this chamber system structure, \( \Sigma \) is isomorphic to \( \Delta \). In particular, the set of apartments in \( \Sigma \) is designated by \( \text{Ap}(\Sigma) \). Then the left action of \( \Gamma \) on itself induces an action on \( \Sigma \). For \( \gamma \in \Gamma \) and \([x, g] \in \Sigma \) we set \( \gamma[(x, g)] := [(x, \gamma g)] \). Moreover this action induces a simply transitive action of \( \Gamma \) on \( \text{Ch}(\Sigma) \). Naturally this action is also isometric for \( d_c(\cdot, \cdot) \).

**Example 5.17.** In the case of Example 5.6, if we equip \( D \) with the appropriate metric we recover that the Davis complex is realized by the tiling of \( \mathbb{X}^d \) by \( D \).
5.7 Building-walls and residues in the Davis complex

We call base chamber of \( \Sigma \), denoted by \( x_0 \), the chamber \([D \times \{e\}]\). For \( g \in \Gamma \), as \([D \times \{g\}]\) is the image of \( x_0 \) under \( g \), we designate this chamber by \( gx_0 \). Here we present some basic tools used to describe the structure of \( \Sigma \). In particular, we extend to \( \Sigma \) some definitions and properties that have been used in Coxeter systems.

The notion of walls in a Coxeter system extends to right-angled buildings.

**Definition 5.18.**

1. We call building-wall in \( \Sigma \) the subcomplex \( M \) stabilized by a non-trivial isometry \( r = gs^\alpha g^{-1} \) with \( g \in \Gamma \), \( s \in S \), \( \alpha \in \mathbb{Z} \) and \( s^\alpha \neq e \). The isometry \( r \) is called a rotation around \( M \). We denote by \( M(\Sigma) \) the set of building-walls of \( \Sigma \).

2. Let \( M \) be a building-wall associated with a rotation \( r \in \Gamma \). For \( x \in \text{Ch}(\Sigma) \) we say that \( M \) is along \( x \) if \( r(x) \) is adjacent to \( x \).

Note that with the notations of the definition, \( M \) is the building-wall fixed by any rotation \( gs^\alpha g^{-1} \) with \( s^\alpha \neq e \). Besides, because of the right-angles, a building-wall is associated with a type. Which is not true for walls in a generic Coxeter system.

We say that the building-wall \( M \) is non-trivial if it contains more than one point. A non-trivial building-wall \( M \) may be equipped with a building structure. Indeed, if \( s_i \) is the type of \( M \), associated with \( v_i \in G(0) \), we write \( I = \{ j : v_j \sim v_i, v_j \neq v_i \} \) and \( V = \{ v_j \in G(0) : j \in I \} \). Then if \( G_V \) is the full subgraph generated by \( V \), we can check that, \( M \) is isomorphic to the geometric realization of the graph product \((G_V, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i \in I})\). The Davis chamber of this geometric realization is the face of type \( s_i \) of \( D \). Moreover building-walls also divide \( \Sigma \) in isomorphic connected components. In the case of infinite dimension 2 buildings, the building-walls are trees and thus they have been called trees-walls by M. Bourdon and H. Pajot in [BP00]. These explain our terminology.

**Definition 5.19.** Let \( M \) be a building-wall of type \( s \) and let \( r \in \Gamma \) be a rotation around \( M \). We call dial of building bounded by \( M \) the closure in \( \Sigma \) of the connected components of \( \Sigma \setminus M \).

This definition implies the following fact.

**Fact 5.20.** Let \( M \) be a building-wall of type \( s \). Assume that \( s \) is of finite order \( q \). Then \( \Sigma \setminus M \) consists of \( q \) connected components. We designate by \( D_0(M), D_1(M), \ldots, D_{q-1}(M) \) these dials of building, with the convention that \( x_0 \subset D_0(M) \). In this setting, for any \( i = 1, \ldots, q-1 \), if \( y \in \text{Ch}(D_i(M)) \) then

\[
\text{Ch}(D_i(M)) = \{ x \in (W,S) : d_c(y,x) < d_c(y,rx) \}.
\]

Eventually \( r \) permutes \( D_0(M), D_1(M), \ldots, D_{q-1}(M) \).

For a building-wall associated with a type \( s \in S \) of infinite order, the analogous property holds.
In the following, we write $D(\Sigma)$ for the set of all the dials of building of $\Sigma$. Naturally, if we consider thin right-angled buildings, the definitions of building-wall and dial of building coincide with the definition of wall and half-space. It is also important to notice that the structure of building-walls in $\Sigma$ comes from the wall structures in the apartments.

**Fact 5.21.** For any $M \in M(\Sigma)$ (resp. $D \in D(\Sigma)$), any $A \in Ap(\Sigma)$, and any retraction $\pi : \Sigma \rightarrow A$, there exists a wall $m$ (resp. a half-space $H$) in $A$ such that $M$ (resp. $\text{Int}(D)$) is one of the connected component of $\pi^{-1}(m)$ (resp. $\pi^{-1}(\text{Int}(H))$).

The following terminology is frequently used to describe the building-walls relatively to the chambers.

**Definition 5.22.** Let $M \in M(\Sigma)$ and $E,F \subset \Sigma$.

i) We say that $M$ crosses $E$ if $E \setminus M$ has several connected components.

ii) We say that $M$ the building-wall separates $E$ and $F$ if their interior are entirely contained in two distinct connected components of $\Sigma \setminus M$.

The distance over the chambers is determined by the building-wall structure (see [Cla] for a proof).

**Proposition 5.23.** Let $x_1$ and $x_2$ be two chambers. If we denote $d_c(\cdot,\cdot)$ the metric on the chamber system, then

$$d_c(x_1,x_2) = \# \{ M \in M(\Sigma) : M \text{ separates } x_1 \text{ and } x_2 \}.$$  

In a right-angled building it appears that two distinct building-walls are either parallel or orthogonal. This explains the following notations.

**Notation.** Let $M$ and $M'$ be two distinct building-walls.

i) if $M \cap M' \neq \emptyset$ we write $M \perp M'$ and we say that $M$ is orthogonal to $M'$,

ii) if $M \cap M' = \emptyset$ we write $M \parallel M'$ and we say that $M$ is parallel to $M'$.

If $M \perp M'$, then $D \cap D'$ contains a chamber for any pair $D, D'$ of dials of building bounded by $M$ and $M'$. If $M \parallel M'$ then there exists $D$ bounded by $M$ and $D'$ bounded $M'$ such that $M' \subset D$ and $M \subset D'$.

The proof of the following lemma can be found in [Cla].

**Lemma 5.24.** Let $M$ and $M'$ be two building-walls such that $M \perp M'$. Let $r \in \Gamma$ be a rotation around $M$ and $D'$ be a dial of building bounded by $M'$ then $r(D') = D'$.

In right-angled buildings, we can define projection maps not only on residues but also on dials of building. This will be useful in Section [7] to understand the metric on the boundary in the hyperbolic case. A proof of this proposition can be found in [Cla].
Proposition 5.25. Let $D$ be a residue or a dial of building and $C = \text{Ch}(D)$. Then for any $x \in \text{Ch}(\Sigma)$ there exists a unique chamber $\text{proj}_C(x) \in C$ such that

$$d_c(x, \text{proj}_C(x)) = \text{dist}(x, C).$$

Moreover, for any chamber $y \in C$ there exists a minimal gallery from $x$ to $y$ passing through $\text{proj}_C(x)$.

For the simplicity, if $D \in \mathcal{D}(\Sigma)$ we write $\text{proj}_D(\cdot)$ instead of $\text{proj}_{\text{Ch}(D)}(\cdot)$.

Lemma 5.26. Let $D$ and $D'$ be two dials of building such that $\text{Ch}(D \cap D') \neq \emptyset$. If $x \in \text{Ch}(D)$ then $\text{proj}_{D'}(x) \in \text{Ch}(D \cap D')$.

Applying inductively the projection maps on dials of building, we define projection maps on finite intersections of dials of building. Again the proof can be found in [Cla].

Proposition 5.27. Let $D_1, \ldots, D_k \in \mathcal{D}(\Sigma)$ and $C = \text{Ch}(D_1 \cap \cdots \cap D_k)$. Assume that $C \neq \emptyset$. Then for any $x \in \text{Ch}(\Sigma)$ there exists a unique chamber $\text{proj}_C(x) \in C$ such that

$$d_c(x, \text{proj}_C(x)) = \text{dist}(x, C).$$

Moreover, for any chamber $y \in C$ there exists a minimal gallery from $x$ to $y$ passing through $\text{proj}_C(x)$. Eventually $\text{proj}_C(x) = \text{proj}_{D_k} \circ \cdots \circ \text{proj}_{D_1}(x)$.

For the simplicity, if $D_1, \ldots, D_k$ is a collection of dials of building such that $C = D_1 \cap \cdots \cap D_k$ contains a chamber, we write $\text{proj}_C(\cdot)$ instead of $\text{proj}_{\text{Ch}(C)}(\cdot)$.

Notice that it is not always possible to define a projection on a convex set of chambers. For instance, if $\Sigma$ is a thick building there exist pairs of adjacent chambers $x$ and $y$ with $d_c(x_0, x) = d_c(x_0, y)$.

5.8 Geometric characterization of parabolic subgroups

Now we discuss the notion of residues in $\Sigma$ as it is done at the end of Subsection 5.2.

Notation. For $I \subset S$ and $g \in \Gamma$, let $g\Sigma_I$ denote the union of the chambers of the $I$-residue containing $gx_0$.

Notice that $g\Sigma_I = g\Gamma_I x_0$ and $\text{Ch}(g\Sigma_I) = g\Gamma_I$. For the simplicity, in the following we also call a subset $g\Sigma_I \subset \Sigma$ a residue. Notice that a rotation around a building-wall that crosses $g\Sigma_I$ is of the form $g\gamma s^a \gamma^{-1} g^{-1}$ with $s \in S$, $s^a \neq e$ and $\gamma \in \Gamma_I$. Along with the definitions of the action and the residues we obtain the following fact.

Fact 5.28. Let $R = g\Sigma_I$ be a residue. Then
• $R$ is stabilized by the rotations around the building-walls that cross it,

• $\text{Stab}_\Gamma(R) = g\Gamma_1 g^{-1}$ is generated by these rotations,

• the type $I$ is given by the type of the building-walls that cross $R$ along $gx_0$.

The following result gives a converse to the previous fact.

**Theorem 5.29** (see [Cla]). Let $R \subset \Sigma$ be the union of a convex set of chambers and let $P_R$ denote the group generated by the rotations around the building-walls that cross $R$. If $P_R$ stabilizes $R$, then $R$ is a residue in $\Sigma$.

In particular, this last theorem is used in Subsection 6.1 where we discuss the boundaries of the residues in the hyperbolic case.

Eventually in $\Gamma$ (resp. in $W$) intersections of parabolic subgroups are parabolic.

## 5.9 Geodesic metric on $\Sigma$

A natural geodesic metric on $\Sigma$ is obtained as follows. We designate by $D$ the Davis chamber of $\Gamma$. We remind that $D$ is obtained from $D^{(1)}$ by attaching a $k$-cube inside any full subgraph generated by $2^k$ vertices that is the 1-skeleton of a $k$-cube. Now, for any $k$, we equip each $k$-cube of $D$ with the Euclidean metric of the $[0, 1]^k$.

The polyhedral metric $d(\cdot, \cdot)$ induced on $\Sigma$ by this construction is geodesic and complete. Moreover, any automorphism of $\Delta$ is an isometry of $(\Sigma, d)$. In particular, $\Gamma$ acts geometrically on $(\Sigma, d)$. Thus $(\Sigma, d)$ is a hyperbolic metric space.

In $(\Sigma, d)$ the building-walls are convex and connected subsets. More precisely, let $M \in \mathcal{M}(\Sigma)$ of type $s$ and let $x \in \text{Ch}(\Sigma)$ such that $M$ is along $x$. Then $M$ coincides with the geodesic extensions of all the segments contained in the maximal face of type $s$ of $x$.

Yet in the case where $W$ is a reflection group of the hyperbolic space $\mathbb{H}^d$ it seems more natural to equip $D$ with the hyperbolic metric. Then $D$ is isometric to the Coxeter polytop provided by $W$. We designate by $d'(\cdot, \cdot)$ the piecewise hyperbolic metric on $\Sigma$ induced by this construction. This metric satisfies the same properties stated above (geodesic, complete, hyperbolic and a geometric action of $\Gamma$).

The two metrics $d$ and $d'$ are quasi-isometric. As our purpose is to study $\partial \Gamma$, essentially it makes no difference to consider $(\Sigma, d)$ or $(\Sigma, d')$. Yet, the arguments presented in Sections 6, 7, 8, and 9 hold in the generic case so we consider $(\Sigma, d)$ in these sections. For Section 10 it will be more convenient to consider $(\Sigma, d')$.

## 5.10 Boundary of the building

Here we describe basic properties of $\partial \Gamma$. In the sequel, we use the geometric action of $\Gamma$ on $(\Sigma, d)$ to identify $\partial \Gamma$ and $\partial \Sigma$. Now we remind that in a hyperbolic group $G$ if $H < G$ there is two possible cases:
\[ \begin{align*}
&\text{either } \partial H \simeq \partial G, \\
&\text{either } \text{Int}(\partial H) \text{ is empty in } \partial G.
\end{align*} \]

Now consider \( M \) a building-wall of type \( s \). Under the identification \( \partial \Sigma \simeq \partial \Gamma \) we get \( \partial M \simeq \partial \text{Stab}_\Gamma(M) \). Therefore, thanks to the previous reminder, to describe \( \partial M \) we essentially need to distinguish two cases.

In the first case, \( s \) commutes with any generator \( r \in S \). In the Davis complex this means that all the other building-walls are orthogonal to \( M \). Then \( \text{Stab}_\Gamma(M) = \Gamma \). In this case the boundary of \( M \) as well as the boundary of any dial of building bounded by \( M \) is \( \partial \Gamma \).

In the second case, there exists \( r \in S \) that does not commute with \( s \). In the Davis complex this means that there exists a building-wall \( M' \) parallel to \( M \). This implies that \( \partial M \subseteq \partial \Gamma \). In this case, \( \partial \Gamma \setminus \partial M \) is the disjoint union \( \text{Int}(\partial D_0(M)) \sqcup \cdots \sqcup \text{Int}(\partial D_{q-1}(M)) \) where \( D_0(M), \ldots, D_{q-1}(M) \) are the dihedral of building bounded by \( M \). Naturally a rotation around \( M \) extends to the boundary as an homeomorphism that permutes \( \partial D_0(M), \ldots, \partial D_{q-1}(M) \) and fixes \( \partial M \). Moreover \( \text{Int}(\partial M) = \emptyset \), \( \text{Int}(\partial D_i(M)) \neq \emptyset \), and the topological boundary of \( \partial D_i(M) \) in \( \partial \Gamma \) is \( \partial M \) for any \( i = 1, \ldots, q - 1 \).

We summarize the last two paragraphs by the following fact.

**Fact 5.30.** Let \( D \) be a dial of building bounded by the building-wall \( M \). Then

\[ \begin{align*}
&\text{either } \partial M = \partial D = \partial \Gamma, \\
&\text{either the topological boundary of } \partial D \text{ in } \partial \Gamma \text{ is } \partial M. \text{ In this case, } \text{Int}(\partial D_i(M)) \neq \emptyset \text{ and } \text{Int}(\partial M) = \emptyset.
\end{align*} \]

Besides, the hyperbolicity condition gives the following fact describing the asymptotic position of two distinct building-walls.

**Fact 5.31.** Let \( M \) and \( M' \) be two distinct building-walls. If \( M \parallel M' \) then \( \partial M \cap \partial M' = \emptyset \).

From this fact we can construct many parallel building-walls.

**Lemma 5.32.** Let \( M_1, \ldots, M_k \) be a collection of building-walls such that any \( M_i \) admits a parallel building-wall. Assume that \( M_i \perp M_j \) for any \( i \neq j \). Then there exists \( M \in \mathcal{M}(\Sigma) \) such that \( M \parallel M_i \) for any \( i \).

**Proof.** We prove the proposition by induction on \( k \). For \( k = 1 \) there is nothing to prove. For \( k \geq 1 \), pick \( M_1, \ldots, M_k \) a collection of building-walls verifying the hypothesis of the lemma. Assume that there exists \( M \) a building-wall such that \( M \parallel M_1, \ldots, M_k \parallel M_k \). Let \( M_{k+1} \in \mathcal{M}(\Sigma) \) be such that \( M_{k+1} \perp M_1, \ldots, M_{k+1} \perp M_k \).

If \( M \) is parallel to \( M_{k+1} \) there is nothing more to say. Now we assume \( M \perp M_{k+1} \) and we pick a wall \( M' \) parallel to \( M_{k+1} \). If \( M' \) is parallel to \( M_1, \ldots, M_k \) there is nothing more to say. Now we assume that there exists \( 1 \leq i \leq k \) such that, up to a reordering
First we consider the case \( M' \perp M \). With

- \( M \perp M_{k+1}, M_{k+1} \perp M_1, M_1 \perp M' \)
- and \( M' \parallel M_{k+1}, M \parallel M_1 \),

we obtain that the building-walls \( M', M, M_{k+1}, \) and \( M_1 \) form a right-angled rectangle. Which is a contradiction with the hyperbolicity of \( \Sigma \).

Secondly we consider the case \( M' \parallel M \). Let \( r' \in \Gamma \) be a rotation around \( M' \). Then \( r'(M) \) is such that

\[
r'(M) \parallel M_1, \ldots, r'(M) \parallel M_i.
\]

Indeed for \( 1 \leq j \leq i \), as \( M \parallel M_j \) it comes that \( r'(M) \parallel r'(M_j) \). Besides \( M' \perp M_j \) and according to Lemma 5.24, \( r'(M_j) = M_j \). Then \( r'(M) \parallel M_j \).

Moreover \( r'(M) \) is such that

\[
r'(M) \parallel M_{i+1}, \ldots, r'(M) \parallel M_{k+1}.
\]

Indeed \( M_{i+1} \cap \cdots \cap M_{k+1} \neq \emptyset \). As \( M' \parallel M_j \) for \( i + 1 \leq j \leq k + 1 \), this means that the building-walls \( M_{i+1}, \ldots, M_{k+1} \) are entirely contained in the same connected component of \( \Sigma \setminus M' \). Let \( C \) be this connected component. As \( M \parallel M' \) and \( M \cap M_{k+1} \neq \emptyset \) it comes that \( M \) is also contained in \( C \). Thus \( r'(M) \) is not contained in \( C \) and \( r'(M) \parallel M_j \) for \( i + 1 \leq j \leq k + 1 \).

In [BK13, Proposition 5.2.], it is proved that the boundary of half-spaces in a hyperbolic Coxeter group generates that visual topology. In the case of right-angled building the analogous statement holds.

**Fact 5.33.** The topology generated by \( \{ \partial D : D \in \mathcal{D}(\Sigma) \} \) is equivalent to the topology induced by a visual metric on \( \partial \Gamma \).

Eventually, consider an apartment \( A \) containing the base chamber \( x_0 \) and the retraction map \( \pi_{A,x_0} : \Sigma \longrightarrow A \). This retraction maps any geodesic ray of \( \Sigma \) starting from a based point \( p_0 \in x_0 \) to a geodesic ray in \( A \) starting from \( p_0 \). Hence \( \pi_{A,x_0} \) extends naturally to the boundaries and we keep the notation \( \pi_{A,x_0} : \partial \Sigma \longrightarrow \partial A \) for this extension.

**Remark 5.34.** In [DM02], M.W. Davis and J. Meier described how properties of connectedness of \( \partial \Gamma \) are encoded in the combinatorial structure of \( \mathcal{G} \). We use a corollary of their result in Subsection 6.2.

**Remark 5.35.** A classification of F. Haglund and F. Paulin states that the construction presented in Subsection 5.5 describes all the right-angled buildings in the following sense.
Theorem 5.36 ([HP03, Proposition 5.1]). Let $\Gamma$ be the graph product given by the pair $(G, \{G_i\}_{i=1,...,n})$ as in Definition 5.13. Let $\Delta$ be the building of type $(W, S)$ associated with $\Gamma$. Assume that $\Delta'$ is a building of type $(W, S)$ such that for any $s_i \in S$ the $\{s_i\}$-residues of $\Delta'$ are of cardinality $\#G_i$. Then $\Delta$ and $\Delta'$ are isomorphic.

6 Curves in connected parabolic limit sets

As we will see with Example 6.9 parabolic limit sets (i.e. boundaries of residues of the building) play a key role in the proof of the CLP in boundaries of graph product.

In this section, we use the ideas of [BK13, Section 5 and 6] to prove Theorem 6.13 that is a first major step to prove the main result of this article (Theorem 10.1). The idea of this theorem is to control the modulus of the curves of a parabolic limit set by the neighborhood of a single curve. Then we apply this theorem to recover a result about boundaries of right-angled Fuchsian buildings.

We use the notations and the setting of Section 5. In particular $\Gamma$ is a fixed graph product given by the pair $(G, \{\mathbb{Z}/q\mathbb{Z}\}_{i=1,...,n})$. We identify the building $\Delta$ with its Davis complex $\Sigma$ equipped with the piecewise Euclidean metric. The base chamber is $x_0$. We assume that $\Gamma$ and $\Sigma$ are hyperbolic and that $\partial \Gamma$ is connected. The metric on $\text{Ch}(\Sigma)$ is denoted $d_c(\cdot, \cdot)$. Moreover, in this section we equip $\partial \Gamma$ with a self-similar metric that comes from the action of $\Gamma$ on $\Sigma$.

6.1 Parabolic limit sets in $\partial \Gamma$

Here we give basic properties of boundaries of parabolic subgroups. At the end of this subsection we will see that these subsets of the boundary are an issue to prove the CLP.

Definition 6.1. Let $P = g \Gamma g^{-1}$ be a parabolic subgroup of $\Gamma$. If the limit set of $P$ in $\partial \Gamma$ is non-empty, we call it a parabolic limit set. If moreover $\partial P \neq \partial \Gamma$ the parabolic limit set is called a proper parabolic limit set.

Equivalently we could have say that a subset $F \subset \partial \Gamma$ is a parabolic limit set if there exists a residue $g\Sigma_I$ such that $F$ is homeomorphic to $\partial(g\Sigma_I)$ under the canonical homeomorphism between $\partial \Gamma$ and $\partial \Sigma$. In the following we will frequently use this point of view about parabolic limit sets.

The following convex hull of a subset of the boundary will be used in the sequel of this section.

Definition 6.2. Let $F$ be a subset of $\partial \Gamma$ containing more than one point and such that $\overline{F} \neq \partial \Gamma$. Let 
\[ \mathcal{D}^c(F) = \{D \in \mathcal{D}(\Sigma) : F \subset \partial \Gamma \setminus \partial D\}. \]

Then we call convex hull of $F$ in $\Sigma$ 
\[ \text{Conv}(F) = \Sigma \setminus \bigcup_{D \in \mathcal{D}^c(F)} D. \]
If $\mathbf{F} = \partial \Gamma$ then we set $\text{Conv}(F) = \Sigma$.

We extend the notion of minimal gallery to infinite galleries.

**Definition 6.3.** An infinite gallery $x_0 \sim x_1 \sim \cdots$ (resp. a bi-infinite gallery $\cdots \sim x_{-1} \sim x_0 \sim x_1 \sim \cdots$) is minimal if for any $k \in \mathbb{N}$ (resp. $k \in \mathbb{Z}$) and $\ell \in \mathbb{N}$ the gallery $x_k \sim \cdots \sim x_{k+\ell}$ is minimal.

Naturally, $\text{Ch}(\Sigma \setminus D)$ is a convex set of chambers for any $D \in \mathcal{D}(\Sigma)$. Thus if $F$ contains at least two points, $\text{Ch}(\text{Conv}(F))$ is a convex set of chambers. We can see $\text{Conv}(F)$ as the maximal convex whose boundary is inside $\mathbf{F}$. Indeed, let $C$ be the union of a convex set of chambers with $\partial C \subset \mathbf{F}$, then $C \subset \text{Conv}(F)$.

In particular, $\text{Conv}(F)$ contains any bi-infinite minimal gallery whose end points are in $F$. Besides note that $\partial \text{Conv}(F) = \mathbf{F}$.

**Example 6.4.** Let $\partial P$ be a parabolic limit set and assume that $P = \Gamma_I$. Then we can verify that $\text{Conv}(\partial P) = \Sigma_J$ where $J = \{ s_j \in S : s_j s_i = s_i s_j \text{ for any } s_i \in I \}$.

In particular, if $M \in \mathcal{M}(\Sigma)$ then $\text{Conv}(\partial M)$ is the the union of all the chambers along $M$.

In the following definition $\overline{\Sigma} = \Sigma \cup \partial \Sigma$ and if $M$ is a building-wall $\overline{M} = M \cup \partial M$.

**Definition 6.5.**

i) Let $F$ be a subset of $\partial \Sigma$. We say that a building-wall $M$ cuts $F$ if there exist two distinct indices $i$ and $j$ such that $F$ meets both $\partial D_i(M)$ and $\partial D_j(M)$.

ii) If $E_1 \subset \partial \Gamma$ and $E_2 \subset \Sigma$ (resp. $E_2 \subset \partial \Gamma$) we say that the building-wall separates $E_1$ and $E_2$ if they are entirely contained in two distinct connected components of $\Sigma \setminus \overline{M}$.

The proof of the following fact is identical to the proof of [BK13, Lemma 5.7].

**Fact 6.6.** Let $F$ be a subset of $\partial \Sigma$. The building-wall $M$ cuts $F$ if and only if $M$ crosses $\text{Conv}(F)$ (see Definition 5.22).

Along with Example 6.4, this fact implies that the characterization of the residues thanks to rotations (Theorem 5.29) extends to the boundary into a characterization of the parabolic limit sets.

**Corollary 6.7.** Let $F$ be a subset of $\partial \Sigma$ containing at least two distinct points and $P_F$ denotes the group generated by the rotations around the building-walls that cut $F$. If $P_F$ stabilizes $F$, then $F$ is a parabolic limit set.

This characterization gives the following corollary concerning the connectedness of the parabolic limit sets.

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Corollary 6.8. Let $\partial P$ be a parabolic limit set. Then any connected component $F$ of $\partial P$ in $\partial \Gamma$ is a parabolic limit set.

Proof. Let $M$ be a building-wall that cuts $F$. As $M$ cuts $\partial P$ a rotation $r \in \Gamma$ around $M$ stabilize $\partial P$ so in particular it sends a connected component of $\partial P$ on another connected component of $\partial P$. With $r(M \cap F) = M \cap F$ we deduce that $r(F) = F$ and so $F$ is a parabolic limit set.

Eventually the next example is crucial in the sense that it illustrates the issue of the parabolic limit sets to prove the CLP.

Example 6.9. Let $M \in \mathcal{M}(\Sigma)$ be a building-wall of type $s$ along the base chamber $x_0$. Let $P = \text{Stab}_\Gamma(M)$. The group $P$ is the parabolic subgroup that is generated by the generators $r \in S \setminus \{s\}$ such that $rs = sr$. Moreover, as we reminded in Subsection 5.10, $\partial P \simeq \partial M$. Now we assume that $\partial P$ is a proper parabolic limit set and we pick $g \in \Gamma$. Then we verify that

- either $g\partial P = \partial P$,
- either $\partial P \cap g\partial P = \emptyset$.

Indeed if $M' \perp M$ then $M$ and $M'$ are of distinct types. As $M$ and $gM$ are of the same types it comes that $M \cap gM \neq \emptyset$ implies $M = gM$ and $g\partial P = \partial P$. Then thanks to Fact 5.31, if $M \cap gM = \emptyset$ we obtain $\partial P \cap g\partial P = \emptyset$.

Eventually the set $\bigcup_{g \in \Gamma} g\partial M$ is made of countably many disjoint copies of $\partial M$. In the introduction we reminded that an efficient way to prove the CLP is to follow curves thanks to bi-Lipschitz maps (see Section 4). As $\Gamma$ acts by bi-Lipschitz homeomorphisms on its boundary, the first idea is to use $\Gamma$ to follow curves. Yet if a non-constant curve $\eta$ is contained in $\partial M$ we cannot hope to follow the curves of $\partial \Gamma$ thanks to $\eta$ and $\Gamma$.

6.2 Modulus of curves in connected parabolic limit set

Here we apply the ideas of [BK13, Section 5 and 6] to $\Gamma$. In this subsection, as in Subsection 5.2, $d_0$ denotes a small constant compared with $\text{diam} \, \partial \Gamma$ and with the constant of approximate self-similarity. Then $\mathcal{F}_0$ is the set of curves of diameter larger than $d_0$. Here we prove that, from the point of view of the modulus, curves in a parabolic limit set are all the same (see consequences of Theorem 6.13).

Until the end of this article, we use the following notations:

Notation. Let $\partial P$ be a connected parabolic limit set in $\partial \Gamma$. For $\delta, r > 0$, let $\mathcal{F}_{\delta,r}(\partial P)$ denote the set of curves in $\partial \Gamma$ such that:

- $\text{diam} \, \gamma \geq d_0$, 

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• $\gamma \subset N_\delta(\partial P)$,
• $\gamma \nsubseteq N_r(\partial Q)$ for any connected parabolic limit set $\partial Q \subsetneq \partial P$.

As we saw in Example 6.9, the curves contained in a parabolic limit set $\partial P$ are an issue to follow other curves. Nevertheless, here we prove that these curves can be used to follow the curves in $\partial P$ (Proposition 6.12). Then we deduce from this property a control of the modulus of the curves in parabolic limit sets (Theorem 6.13). To this purpose we use the following notion.

**Definition 6.10.** Let $L \geq 0$ and $I$ a non-empty subset of $S$. A curve $\gamma$ in $\partial \Gamma$ is called a $(L,I)$-curve if

- $x_0 \subset \text{Conv}(\gamma)$,
- for all $s \in I$, there exists a panel $\sigma_s$ of type $s$ inside $\text{Conv}(\gamma)$ with $\text{dist}(x_0,\sigma_s) \leq L$.

As we see, curves in parabolic limit sets are $(L,I)$-curves.

**Proposition 6.11.** Let $I \subset S$ and $P = h\Gamma_1h^{-1}$. Then for all $r > 0$, there exist $L > 0$ and $\delta > 0$ such that if $x_0 \subset \text{Conv}(\gamma)$ and $\gamma \in \mathcal{F}_{\delta,r}(\partial P)$, then $\gamma$ is a $(L,I)$-curve.

**Proof.** We fix $r > 0$ and we assume that for every integer $n \geq 1$, there exists a curve $\gamma_n$ such that:

- $x_0 \subset \text{Conv}(\gamma_n)$,
- $\gamma_n \in \mathcal{F}_{1/n,r}(\partial P)$,
- $\gamma$ is not a $(n,I)$-curve.

For $n \geq 1$, we designate the ball of center $x_0$ and of radius $n$ for the distance over the chambers by

$$B_c(x_0,n) = \{x \in \text{Ch}(\Sigma) : d_c(x_0,x) \leq n\}.$$ 

For the simplicity we also designate by $B_c(x_0,n)$ the union of its chambers. Up to a subsequence we can suppose that for a fixed $s \in I$, there is no panel of type $s$ in $B_c(x_0,n) \cap \text{Conv}(\gamma_n)$ for $n \geq 1$.

We want to reveal a contradiction thanks to the sequences $\gamma_n$ and $\text{Conv}(\gamma_n)$. According to [Mun75, p. 281] the set of non-degenerate continua in a compact space is a compact set for the Hausdorff distance. Therefore, up to a subsequence, we can suppose that $\gamma_n$ tends to a non-degenerate continuum $L \subset \partial P$.

As $x_0 \subset \text{Conv}(\gamma_n)$, using a diagonal argument we can also assume that, up to a subsequence, $\text{Conv}(\gamma_n+k) \cap B_c(x_0,n)$ is non-empty and constant for $k \geq 0$. Then we denote $C := \bigcup_{n \geq 1} \text{Conv}(\gamma_n) \cap B_c(x_0,n)$.
With $\gamma_n \subset \partial\text{Conv}(\gamma_n)$, it comes that $L \subset \partial C$. As $C$ does not contain any panel of type $s$, $L$ is contained in the limit set of a parabolic subgroup of the form $g\Gamma_Jg^{-1}$ with $s \notin J$. Then, intersections of parabolic subgroups are parabolic subgroups (see [Cla]). Therefore $L$ is contained in the limit set of a proper parabolic subgroup of $P$. Let $\partial Q$ be the connected component of this parabolic limit set that contains $L$. Thanks to Corollary 6.8, $\partial Q$ is a parabolic limit set. Now we see that for $n \geq 0$ large enough $\gamma_n \subset N_r(\partial Q)$ which is a contradiction with $\gamma_n \in F_{1/n,r}(\partial P)$.

\[ \square \]

An interesting feature of $(L,I)$-curves is that these curves are crossed by building-walls of type in $I$. Which means that from a $(L,I)$-curve, we can follow curves using rotations around building-walls of type in $I$.

**Proposition 6.12.** Let $\epsilon > 0$, $L > 0$ and $I$ be a non-empty subset of $S$. For $P = h\Gamma_Ih^{-1}$, let $\eta$ denote a curve contained in $\partial P$. Then there exists a finite subset $F \subset \Gamma$ such that for any $(L,I)$-curve $\gamma$ the set $\bigcup_{g \in F} g\gamma$ contains a curve that belongs to $U(\eta)$.

**Proof.** We divide the proof in four steps. In this proof $M_s$ denotes the building-wall of type $s \in S$ along $x_0$.

i) First, we can suppose without loss of generality that $P = \Gamma_I$. Indeed, as $h \in \Gamma$ is a bi-Lipschitz homeomorphism of $(\partial \Gamma, d)$, then if the property holds for $\Gamma_I$ it holds for $h\Gamma_Ih^{-1}$.

ii) Now we prove that the following property holds.

The set $\bigcup_{g \in F_L} g\gamma$ contains a curve passing through $\partial M_s$ for every $s \in I$.

Where $F_L = \{g \in \Gamma : |g| \leq L\}$ and $|g|d_c(x_0, gx_0)$.

As $\gamma$ is a $(L,I)$-curve, there exist $s \in I, \alpha \in \mathbb{Z}$ and $g \in F_L$ such that $gx_0$ and $gs^\alpha x_0$ belongs to $\text{Conv}(\gamma)$. Let $x_0 \sim g_1 x_0 \sim \cdots \sim g_k x_0$ be a gallery contained in $\text{Conv}(\gamma)$ with $g_{i-1} = gs^\alpha$ and $g_\ell = g$. For any $i = 0, \ldots, \ell - 1$, let $M_i$ denote the building-wall separating $g_i x_0$ and $g_{i+1} x_0$. In particular, $\partial M_i$ cuts $\gamma$ for any $i = 0, \ldots, \ell - 1$. This means that if $M_i$ is of type $s_i$ then $\gamma \cap g_{i} s_i^\alpha g_{i-1}^{-1} \gamma \neq \emptyset$ for any $\alpha \in \mathbb{Z}$. In particular, if $\alpha_i$ is such that $g_{i+1} = g_i s_i^{-\alpha_i}$, then $\gamma \cap g_i s_i^{-1} \gamma \neq \emptyset$ for any $i = 0, \ldots, \ell - 1$. Hence the set $\gamma \cup g_1^{-1} \gamma \cup \cdots \cup g_\ell^{-1} \gamma$ is arcwise connected and $g_\ell^{-1} \gamma$ intersects $\partial M_s$. Thus the property is verified.

iii) Let $\Sigma_I \subset \Sigma$ be the residue associated with $\Gamma_I$. We remind that this means $\Sigma_I = \Gamma_I x_0$. Let $D_1, \ldots, D_k$ be a collection of dials of building bounded by the building-walls $M_1, \ldots, M_k$. Then we assume that each $D_i$ intersects $\Sigma_I$ properly (i.e. $\Sigma_I \cap D_i \neq \emptyset$ and $\Sigma_I \cap D_i \neq \Sigma_I$). In particular, this means that the building-walls $M_1, \ldots, M_k$ have their types contained in $I$.

Now we prove that the following property holds.

There exists a finite subset $F_0 \subset \Gamma$ such that for every $(L,I)$-curve $\gamma$ the set $\bigcup_{g \in F_0} g\gamma$ contains a curve passing through $\partial D_1, \ldots, \partial D_k$.

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For \( i = 1, \ldots, k \) pick \( h_i \in \Gamma_f \) such that \( M_i \) is along \( h_i x_0 \in \Sigma_f \). In particular, for any \( i \), we can write \( M_i = h_i(M_s) \) where \( s \in I \) is the type of \( M_i \). Let \( g_1 x_0 = h_1 x_0 \sim g_2 x_0 \sim \cdots \sim g_k x_0 = h_k x_0 \) be a gallery in \( \Sigma_f \) passing through \( h_1 x_0, \ldots, h_k x_0 \) in this order.

Applying the previous step of the proof, there exists a finite set \( \eta \) of \( i \) for any \( D \) with \( \theta \). Let \( \partial D \) be the boundary of \( D \). We use the two previous propositions to obtain a control of \( \partial D_{1, \ldots, k} \).

We set \( F_0 = \{ g \in \Gamma : |g| \leq L, 1 \leq i \leq \ell \} \), and it is enough to check that \( \bigcup_{g \in F_0} g \gamma \) is arcwise connected. For any \( i = 1, \ldots, \ell - 1 \) let \( s_i \in I \) and \( \alpha_i \in \mathbb{Z} \) be such that \( g_{i+1} = g_i s_i^{\alpha_i} \). Then \( g_i = (s_i^{\alpha_i} g_i^{-1}) g_i \). As \( \theta \) intersects any \( \partial M_{s_i} \) then \( \theta \cap g_i(\partial M_{s_i}) \neq \emptyset \) and this intersection is fixed by \( g_i s_i^{\alpha_i} g_i^{-1} \). Thus \( g_i \theta \cap g_{i+1} \theta \neq \emptyset \).

We use the two previous propositions to obtain a control of \( \text{Mod}_{\partial P}(F_{\delta, r}(\partial P)) \).

**Theorem 6.13.** There exists an increasing function \( \delta_0 : (0, +\infty) \rightarrow (0, +\infty) \) verifying the following property. Let \( p \geq 1 \), let \( \eta \in F_0 \), and let \( \partial P \) be the smallest parabolic limit set containing \( \eta \). Let \( r > 0 \) be small enough so that \( \eta \notin \overline{N_r}(\partial Q) \) for any connected parabolic limit set \( \partial Q \subset \partial P \). Let \( \delta < \delta_0(r) \) and let \( \epsilon > 0 \) be small enough so that \( U_\epsilon(\eta) \subset F_{\delta, r}(\partial P) \). Then there exists a constant \( C = C(d_0, p, \eta, r, \epsilon) \) such that

\[
\text{Mod}_p(U_\epsilon(\eta), G_k) \leq \text{Mod}_p(F_{\delta, r}(\partial P), G_k) \leq C \cdot \text{Mod}_p(U_\epsilon(\eta), G_k).
\]

Furthermore when \( p \) belongs to a compact subset of \([1, +\infty)\) the constant \( C \) may be chosen independent of \( p \).

Before the proof, we need to explain the statement of the theorem. Indeed, it is not clear that for \( \epsilon > 0 \) small enough \( U_\epsilon(\eta) \subset F_{\delta, r}(\partial P) \). In particular, if \( \text{diam} \eta = d_0 \) then \( U_\epsilon(\eta) \) may not be made of curves of diameter larger than \( d_0 \). Nevertheless thanks to Proposition 3.12 we can rescale \( U_\epsilon(\eta) \) to a set of larger curves with a control of the modulus. Hence we can say that, up to rescaling, for \( \epsilon > 0 \) small enough, \( U_\epsilon(\eta) \subset F_{\delta, r}(\partial P) \). This gives sense to the statement of the theorem and left-hand side inequality is now trivial by Proposition 2.3 (1).
Proof of [6.13] Let $P = h \Gamma_I h^{-1}$, let $\eta$ be a curve in $\partial P$, $r > 0$ and $\epsilon > 0$ as in the hypothesis of the theorem.

To prove the right-hand side inequality, thanks to Proposition 6.12, we can assume without loss of generality that if $\gamma \in F_0$ then $x_0 \in \text{Conv}(\gamma)$. Indeed, there exists an upper bound $N$ depending on $d_0$ such that if $\gamma \in F_0$ then $\text{dist}(x_0, \text{Conv}(\gamma)) \leq N$. So there exists only a finite set $E$ of elements of $\Gamma$ such that if $g \in E$, there exists $\gamma \in F_0$ with $\text{dist}(x_0, \text{Conv}(\gamma)) = d_e(x_0, gx_0)$.

With this assumption we can apply Proposition 6.11 and set $L > 0$ and $\delta > 0$ such that the curves of $F_{\delta,r}(\partial P)$ are $(L, I)$-curves. Let $F \subset \Gamma$ be the finite set given by Proposition 6.12 and let $\rho : G_k \rightarrow \mathbb{R}^+$ be a $\mathcal{U}_\epsilon(\eta)$-admissible function. We define $\rho' : G_k \rightarrow \mathbb{R}^+$ by:

$$(\ast) \quad \rho'(v) = \sum_{g \in F} \sum_{w \cap g \gamma \neq \emptyset} \rho(w).$$

Let $\gamma \in F_{\delta,r}(\partial P)$ and $\theta \subset \bigcup_{g \in F} g \gamma$ such that $\theta \in \mathcal{U}_\epsilon(\eta)$. Then

$$L_{\rho'}(\gamma) = \sum_{g \in F} \sum_{v \cap g \gamma \neq \emptyset} \sum_{w \cap g \gamma \neq \emptyset} \rho(w) \geq \sum_{g \in F} \sum_{w \cap g \gamma \neq \emptyset} \rho(w).$$

Yet

$$L_{\rho}(\theta) \leq \sum_{g \in F} L_{\rho}(g \gamma) = \sum_{g \in F} \sum_{w \cap g \gamma \neq \emptyset} \rho(w).$$

Thus $L_{\rho'}(\gamma) \geq L_{\rho}(\theta)$ and $\rho'$ is $F_{\delta,r}(\partial P)$-admissible.

Then the number or terms in the right-hand side of the definition (\ast) is bounded by a constant $N$ depending on $\#F$, the bi-Lipschitz constants of the elements of $F$, and the doubling constant of $\partial T$. Therefore by convexity

$$M_{\rho}(\rho') = \sum_{v \in G_k} \left( \sum_{g \in F} \sum_{w \cap g \gamma \neq \emptyset} \rho(w) \right)^p \leq N^{p-1} \sum_{v \in G_k} \sum_{g \in F} \sum_{w \cap g \gamma \neq \emptyset} \rho(w) \leq N^{p} \cdot \#F \cdot \sum_{w \in G_k} \rho(w)^p.$$ 

Which prove the inequality.

As a straightforward application, we notice that under the assumptions of the theorem, $\text{Mod}_{\rho}(\mathcal{U}_\epsilon(\eta), G_k)$ does not depend, up to multiplicative constants, on the choice of $\eta$ and $\epsilon$. Indeed, for $r > 0$ fixed and $\delta < \delta_0(r)$, if $\eta, \eta' \subset \partial P$ and $\epsilon, \epsilon' > 0$ such that the hypothesis of the theorem are satisfied. Then there exist $C = C(\eta, r, \epsilon)$ and $C' = C'(\eta', r, \epsilon')$ such that

$$C^{-1} \cdot \text{Mod}_{\rho}(\mathcal{U}_\epsilon(\eta'), G_k) \leq \text{Mod}_{\rho}(\mathcal{U}_\epsilon(\eta), G_k) \leq C' \cdot \text{Mod}_{\rho}(\mathcal{U}_\epsilon(\eta'), G_k).$$

Of course, if $\eta = \eta'$ and $\epsilon' < \epsilon$ we can choose $C = 1$. 

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6.3 Application to Fuchsian buildings

Another consequence of the theorem is that if the boundary of a graph product does not contain connected parabolic limit sets, then it satisfies the CLP.

**Theorem 6.14.** Let \( \Gamma \) be a thick hyperbolic graph product such that \( \partial \Gamma \) is connected and any proper parabolic limit set is disconnected. Then \( \partial \Gamma \) equipped with a visual metric satisfies the CLP.

**Proof.** We check the hypothesis of Proposition 3.13. To prove that \( \text{Mod}_1(F_0, G_k) \) is unbounded, it is enough to see that for every \( N \in \mathbb{N} \) there exist \( N \) disjoint curves in \( \partial \Gamma \). Indeed, this implies that for \( k \geq 0 \) large enough \( \text{Mod}_1(F_0, G_k) > N \).

Let \( \eta \) be a non-constant curve in \( \partial \Gamma \). Up to a change of scale, by Proposition 3.12, we can assume \( \eta \in F_0 \). Then as \( \partial \Gamma \) is the only parabolic limit set containing \( \eta \), it is enough to apply Theorem 6.13 to verify the second hypothesis of Proposition 3.13.

In particular, we can apply this result to the case of right-angled Fuchsian buildings.

In the following, we call *right-angled Fuchsian building* a building associated with a graph product \( (C_n, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,...,n}) \) where \( C_n \) is the cyclic graph with \( n \geq 5 \) vertices and \( q_1, \ldots, q_n \) is a family of integers larger or equal than 3.

**Corollary 6.15.** For \( n \geq 5 \), let \( C_n \) be the cyclic graph with \( n \) vertices and let \( q_1, \ldots, q_n \) be a family of integers larger or equal than 3. Let \( \Gamma \) be the graph product given by the pair \( (C_n, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,...,n}) \). Then \( \partial \Gamma \) equipped with a visual metric satisfies the CLP.

This result was known since boundaries of right-angled Fuchsian buildings are Loewner spaces (see [BP00, Proposition 2.3.4.]). Yet, here we give a direct proof of this result.

Besides, we can prove that these thick graph products are the only one to satisfy the hypothesis of Theorem 6.14. To verify this we need to introduce the following simplicial complex.

**Definition 6.16.** Let \( \Gamma = (\mathcal{G}, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,...,n}) \) be a graph product, the nerve of \( \Gamma \) is the simplicial complex \( L = L(\mathcal{G}) \) such that:

- the 1-skeleton of \( L \) is \( \mathcal{G} \),
- \( k \) vertices of \( \mathcal{G} \) span a \((k-1)\)-simplex in \( L \) a corresponding parabolic subgroup in \( \Gamma \) is finite.

The following theorem is a special case of [DM02, Corollary 5.14.].

**Theorem 6.17.** The boundary of \( \Gamma \) is connected if and only if \( L \setminus \sigma \) is connected for any simplex \( \sigma \subset L \).

Now we can prove the following.
Proposition 6.18. Let $\Gamma = (G, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\ldots,n})$ be a hyperbolic graph product. Assume that $\partial \Gamma$ is connected and that any proper parabolic limit set $\partial P$ is disconnected, then the building associated with $\Gamma$ is a right-angled Fuchsian building.

Proof. Here we only need to prove that $G$ contains a circuit of length $n \geq 5$. According to Corollary 6.8 if any proper parabolic limit set in $\partial \Gamma$ is disconnected then any proper parabolic limit set in $\partial \Gamma$ is discrete. Moreover, $\partial \Gamma$ contains at least one proper parabolic limit set of the form $\partial \Gamma_I$ with $\# I = n - 1$ otherwise $\partial \Gamma = \emptyset$. The subgroup $\Gamma_I$ is a graph product associated with the graph $G_I$. This graph is obtained from $G$ to which we remove a vertex $p$ and all the edges adjacent to $p$. Then if $L_I$ is the nerve associated with $\Gamma_I$, we get $L_I$ from $L$ to which we remove the interior of any simplex containing $p$.

Now, thanks to Theorem 6.17, we know that there exists a simplex $\sigma \subset L_I$ such that $L_I \setminus \sigma$ is disconnected. Let $C_1$ and $C_2$ be two connected components of $L_I \setminus \sigma$. Up to a subsimplex, we can assume that any vertex of $\sigma$ is connected to $C_1$ or to $C_2$ by an edge. Yet, if we consider the simplex $\sigma$ in $L$, it comes that $L \setminus \sigma$ is connected because $\partial \Gamma$ is connected. Therefore there exist two edges in $L$ attaching $p$ respectively to $C_1$ and to $C_2$.

We set $V = \{v_1, \ldots, v_k\}$ the vertices of $\sigma$ that are not connected to $p$ by an edge and $V' = \{v'_1, \ldots, v'_{k'}\}$ the rest of the vertices of $\sigma$. At this point, we assume by contradiction, that $G$ contains no circuit of length $n \geq 4$. We can check that under this assumption the following situations does not occur

- $V'$ is empty,
- there exists $v \in V$ such that $v$ is adjacent to both $C_1$ and $C_2$,
- there exist $v, w \in V$ such that $v$ is adjacent to $C_1$ and $w$ is adjacent to $C_2$.

Hence, for instance, the vertices in $V$ are all adjacent to $C_1$. As a consequence, if $\sigma'$ designates the simplex in $L$ spanned by $V' \cup \{p\}$ then $L \setminus \sigma'$ is not connected. Which is not possible because $\partial \Gamma$ is connected.

Therefore $G$ contains a circuit of length $n \geq 4$, but as $\Gamma$ is hyperbolic it contains no circuit of length 4. This conclude the proof.

7 Combinatorial metric on boundaries of right-angled hyperbolic buildings

Here we describe how, the geometry of the boundary is determined by the boundaries of the walls.

In this section, we first start by discussing boundaries of intersections of dials of building. Afterwards, we describe a combinatorial and self-similar metric on $\partial \Gamma$ in terms of dials of
building. Then, we build an approximation of $\partial \Gamma$ that will be convenient to use in Section 8.

Here we use the notations and the setting of Section 5 and 6. In particular, $\Gamma$ is a fixed graph product given by the pair $(\mathcal{G}, \{\mathbb{Z}/q_i \mathbb{Z}\}_{i=1,\ldots,n})$ and acting on the building $\Sigma$. The base chamber is $x_0$, and $W$ is the right-angled Coxeter group associated with $\Gamma$. We assume that $\Gamma$ is hyperbolic and $\partial \Gamma$ is connected.

### 7.1 Shadows on $\partial \Gamma$

The following tools are used in the sequel of this article to describe the topology and the metric on $\partial \Gamma$. We remind that the boundary of $\Gamma$ is canonically identified with the boundary of $\Sigma$.

**Definition 7.1.** Let $x \in \text{Ch}(\Sigma)$. We call cone of chambers of base $x$ and we denote $C_x \subset \Sigma$, the union of the set of chambers $y \in \text{Ch}(\Sigma)$ such that there exists a minimal gallery from $x_0$ to $y$ passing through $x$.

Thanks to projection maps, we characterize the cones in terms of dials of building.

**Proposition 7.2.** Let $D_1, \ldots, D_k \in \mathcal{D}(\Sigma)$ and $C = D_1 \cap \cdots \cap D_k$. Assume that $C$ contains a chamber and that $x_0 \not\in D_i$ for $i = 1, \ldots, k$. If we set $x = \text{proj}_C(x_0)$ then $C_x = C$.

**Proof.** According to Definition 7.1 and to Proposition 5.27, $C \subset C_x$. Now let $y \in \text{Ch}(C_x)$ and for $i = 1, \ldots, k$ let $M_i$ be the building-wall that bound $D_i$. If $x_0 \sim \cdots \sim x \sim \cdots \sim y$ is a minimal gallery, then the subgallery $x \sim \cdots \sim y$ does not cross any building-wall $M_i$ and $y \subset C$.

Reciprocally cones are intersections of dials of building.

**Proposition 7.3.** Let $x \in \text{Ch}(\Sigma)$ and let $D_1, \ldots, D_k$ denote the family of dials of building such that for any $i = 1, \ldots, k$

\[ x_0 \not\in D_i \text{ and } x \subset D_i. \]

Then $C_x = D_1 \cap \cdots \cap D_k$.

**Proof.** Let $C = D_1 \cap \cdots \cap D_k$. According to Proposition 7.2, it is enough to prove that $\text{proj}_C(x_0) = x$. If we write $x' = \text{proj}_C(x_0)$, with Proposition 7.2, $C = C_{x'}$. Hence there exists a minimal gallery

\[ x_0 \sim \cdots \sim x' \sim \cdots \sim x. \]

Now assume that $x' \neq x$, this means that there exists a building-wall $M$ that separates $x$ and $x'$. As the gallery is minimal, the dial of building $D$ bounded by $M$ that contains $x$ does not contain $x'$ and $x_0$. Thus $D$ is one of the $D_1, \ldots, D_k$ and $x' \not\in C$ which is a contradiction. \qed
In particular, it makes sense to consider projection maps over cones of chambers with \( \text{proj}_{C_x}(x_0) = x \). Thanks to the previous proposition, we obtain the following fact that describes how dials of building intersect.

**Fact 7.4.** Let \( D_1, \ldots, D_k \) be a family of distinct dials of building such that \( D_i \notin D_j \) for any \( i \neq j \). Assume that \( x_0 \notin D_i \) for any \( i \). For any \( i \), set \( M_i \) the building-wall that bound \( D_i \) and let \( C = D_1 \cap \cdots \cap D_k \). Then exactly one of these propositions holds.

- There exists \( i, j \) such that \( M_i \parallel M_j \) and \( C = \emptyset \).
- For any \( i, j \), \( M_i \cap M_j \neq \emptyset \) and there exists \( i \neq j \) such that \( M_i = M_j \). In this case \( C \) is contained in \( M_i \).
- \( M_i \bot M_j \) for any \( i \neq j \). In this case \( C \) is a cone.

This fact, up to a translation and up to a subfamily, describes how a finite family of dials intersects. The following lemma precise the case when the intersection is a cone.

**Lemma 7.5.** Let \( D_1, \ldots, D_k \) be a family of distinct dials of building bounded by the building-walls \( M_1, \ldots, M_k \). Let \( C = D_1 \cap \cdots \cap D_k \). Assume that \( x_0 \notin D_i \) for any \( i \) and that for any \( i = 1, \ldots, k \), \( x \subset D_i \) and \( M_i \) is along \( x \). Then

\[
\text{proj}_{D_k} \circ \cdots \circ \text{proj}_{D_1}(x_0) \notin \text{Ch}(D_k).
\]

Hence \( \text{proj}_{C_x}(x_0) = \text{proj}_{D_k}(\text{proj}_{D_{k-1}}(\circ \cdots \circ \text{proj}_{D_1}(x_0))) \) is along \( M_k \). Changing the order of the family of building-walls, the same argument applies to \( M_1, \ldots, M_{k-1} \) and the prove is finished.

Eventually we obtain the following characterization of cones.

**Proposition 7.6.** Let \( x \in \text{Ch}(\Sigma) \) and let \( D_1, \ldots, D_k \) be the family of dials of building bounded by \( M_1, \ldots, M_k \) such that for any \( i = 1, \ldots, k \),

\[
x_0 \notin D_i, \quad x \subset D_i, \quad \text{and} \quad M_i \text{ is along } x.
\]

Then \( C_x = D_1 \cap \cdots \cap D_k \).

**Proof.** Let \( D'_1, \ldots, D'_\ell \) be the family of dials of building such that \( x_0 \notin D_i \) and \( x \subset D_i \) for any \( i = 1, \ldots, \ell \). Then

\[
\{D_1, \ldots, D_k\} \subseteq \{D'_1, \ldots, D'_\ell\}.
\]

According to Proposition \( 7.3 \), \( C_x = D'_1 \cap \cdots \cap D'_\ell \), thus \( C_x \subset D_1 \cap \cdots \cap D_k \). For any \( i \), let \( M'_i \) be the wall that bounds \( D'_i \). Up to a subfamily, we can assume that \( C_x = D'_1 \cap \cdots \cap D'_\ell \).
and for any $i \neq j$, $M'_i \perp M'_j$. In this case, according to Lemma 7.5 any building-wall $M'_i$ is along $x$. Indeed $\text{proj}_{C_x}(x_0) = x$. Eventually, we get

$$\{D'_1, \ldots, D'_\ell\} \subseteq \{D_1, \ldots, D_k\}$$

and $D_1 \cap \cdots \cap D_k \subset C_x$. \hfill \Box

In the sequel of this article, we use boundaries of cones as a base of the topology of $\partial \Gamma$ and to construct approximations.

**Proposition 7.7.** Let $x \in \text{Ch}(\Sigma)$ and $C_x$ be the cone based on $x$. Then $\partial C_x$ is of non-empty interior in $\partial \Gamma$.

**Proof.** According to Proposition 7.3 and Fact 7.4 we can write that

$$\partial C_x = \partial D_1 \cap \cdots \cap \partial D_k,$$

where $D_1, \ldots, D_k$ is a collection of dials of building bounded by $M_1, \ldots, M_k \in \mathcal{M}(\Sigma)$ with $M_i \perp M_j$ for any $i \neq j$. By the rotations around $M_1, \ldots, M_k$, all the connected components of $\Sigma \setminus (M_1 \cup \cdots \cup M_k)$ are isomorphic. Hence, thanks to Lemma 5.32 there exists $M \in \mathcal{M}(\Sigma)$ such that $M \parallel M_i$ for any $i = 1, \ldots, k$ and $M \subset D_1 \cap \cdots \cap D_k$.

In particular, there exists $D \in \mathcal{D}(\Sigma)$ bounded by $M$ such that $D \subset D_1 \cap \cdots \cap D_k$. As $\partial D$ is of non-empty interior, we obtain that $\partial C_x$ is of non-empty interior. \hfill \Box

**Definition/Notation 7.8.** Let $x \in \text{Ch}(\Sigma)$ and $C_x$ be the corresponding cone of chambers. We call shadow of $x$ the boundary of $C_x$ in $\partial \Gamma$ and we write $v_x = \partial C_x$.

### 7.2 Combinatorial metric on $\partial \Gamma$

Until now we have been considering on $\partial \Gamma$ the visual metric coming from the geometric action of $\Gamma$ on $\Sigma$. Now we use infinite minimal galleries to describe a combinatorial metric on $\partial \Gamma$ that will be more convenient to use in the sequel.

Let $\mathcal{D}(\Sigma)$ denote the *dual graph* of $\Sigma$. This graph is defined by:

- The set of vertices $\mathcal{D}(\Sigma)^{(0)}$ is given by $\text{Ch}(\Sigma)$ the set of chambers in $\Sigma$. If $v \in \mathcal{D}(\Sigma)^{(0)}$ then $c_v$ denotes the corresponding chamber in $\text{Ch}(\Sigma)$.

- There exists an edge between two vertices $v_1$ and $v_2$ if and only if $c_{v_1}$ is adjacent to $c_{v_2}$ in $\Sigma$.

- Each edge is isometric to the segment $[0,1]$. 

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Naturally, $\mathcal{DG}(\Sigma)$ is a proper geodesic and hyperbolic space. It is quasi-isometric to $\Sigma$ and the action of $\Gamma$ on $\mathcal{DG}(\Sigma)$ is geometric. Therefore we identify

$$\partial \mathcal{DG}(\Sigma) \simeq \partial \Gamma.$$ 

**Example 7.9.** If for any $i = 1, \ldots, n$, $q_i = 2$ or $3$ then $\mathcal{DG}(\Sigma)$ is identified with Cay($\Gamma$) the Cayley graph of $\Gamma$. Otherwise, if you consider a generator $s \in S$ of $\Gamma$ of order $q \geq 4$ then in $\mathcal{DG}(\Sigma)$ the full sub-graph generated by the vertices associated with $e, s, \ldots, s^{q-1}$ is a complete graph. In Cay($\Gamma$) the full sub-graph generated by the vertices associated with $e, s, \ldots, s^{q-1}$ is a cyclic graph of length $q$. Nevertheless $\mathcal{DG}(\Sigma)$ and Cay($\Gamma$) are always quasi-isometric.

With Definition 6.3, infinite minimal galleries are identified with geodesic rays in $\mathcal{DG}(\Sigma)$ starting from a vertex. Therefore we can write $\partial \Gamma$ as a quotient of the set of infinite minimal galleries as follow

$$\partial \Gamma \simeq \{ x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots : x_i \in \text{Ch}(\Sigma) \} / \mathcal{R}$$

where the equivalence relation $\mathcal{R}$ is defined by

$$[x_0 \sim x_1 \sim \cdots] = [y_0 = x_0 \sim y_1 \sim \cdots]$$

if and only if there exists $K > 0$ such that $d_c(x_i, y_i) < K$ for all $i \in \mathbb{N}$.

**Example 7.10.** Here again we consider only minimal galleries. Let $x \in \text{Ch}(\Sigma)$ with $d_c(x_0, x) = k \geq 1$. Then we can describe the shadow $v_x$ as follow

$$v_x \simeq \{ x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots : x_i \in \text{Ch}(\Sigma) \text{ and } x_k = x \} / \mathcal{R}.$$ 

Likewise, if $\partial P$ is a parabolic limit set associated with the residue $g \Sigma_I$. Let $x := \text{proj}_{g \Sigma_I}(x_0)$ and assume that $d_c(x_0, x) = k \geq 1$. Then we can describe $\partial P$ as follow

$$\partial P \simeq \{ x_0 \sim x_1 \sim \cdots : x_k = x \text{ and } x_{k+i} \sim s_i \ x_{k+i+1} \text{ with } s_i \in I \text{ for any } i \geq 0 \} / \mathcal{R}.$$ 

Now we use the following notation.

**Notation.** If $x_0 \sim x_1 \sim \cdots$ is a minimal infinite gallery that goes asymptotically to $\xi \in \partial \Gamma$, then we write

$$\xi = [x_0 \sim x_1 \sim \cdots].$$

**Definition 7.11.** Let $\xi, \xi'$ be two distinct points in $\partial \Gamma$, let $\{\xi|\xi'\}x_0$ denote the largest integer $\ell$ such that there exist two infinite minimal galleries representing $\xi$ and $\xi'$

$$\xi = [x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots] \text{ and } \xi' = [x_0 \sim x_1' \sim \cdots \sim x_i' \sim \cdots]$$

with

$$x_i = x_i' \text{ for } i \leq \ell \text{ and } x_{\ell+1} \neq x_{\ell+1}'.$$
In terms of shadows, \( \{\xi|\xi'\}_{x_0} \) is the largest integer such that there exists a shadow \( v_x \), with \( d_c(x_0, x) = \{\xi|\xi'\}_{x_0} \), that contains both \( \xi \) and \( \xi' \). The following proposition gives a characterization of this quantity in terms of building-walls. We remind that \( D_0(M) \) designates the dial of building bounded by \( M \) and containing \( x_0 \).

**Proposition 7.12.** Let \( \xi, \xi' \) be two distinct points in \( \partial \Gamma \). Then

\[
\{\xi|\xi'\}_{x_0} = \#\{M \in \mathcal{M}(\Sigma) : \text{there exists } \alpha \neq 0 \text{ with } \{\xi, \xi'\} \subset \partial D_\alpha(M)\}.
\]

**Proof.** Let \( M_1, \ldots, M_k \) be the set of building-walls such that there exists \( \alpha \neq 0 \) with \( \{\xi, \xi'\} \subset \partial D_\alpha(M) \). Let \( \ell = \{\xi|\xi'\}_{x_0} \). We prove that \( k = \ell \).

For each \( M_i \), pick \( D_i \) such that \( \{\xi, \xi'\} \subset \partial D_i \) and \( x_0 \not\in D_i \). We set \( C = D_1 \cap \cdots \cap D_k \). As the building-walls are distinct and \( \partial C \neq \emptyset \) it follows from Fact 7.4 that \( C \) is a cone. Let \( x = \text{proj}_C(x_0) \). As \( \{\xi, \xi'\} \subset \partial C \), there exists an infinite minimal gallery starting from \( x_0 \) going asymptotically to \( \xi \) (resp. \( \xi' \)) passing through \( x \). Eventually we obtain \( \ell \geq d_c(x_0, x) \geq k \).

Consider \( x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots = \text{min} \) (resp. \( x_0 \sim \cdot \sim \sim x_i \sim \cdots = \text{min} \)) a minimal infinite gallery representing \( \xi \) (resp. \( \xi' \)) in \( \partial \Gamma \). Assume that

\[
x_i = x'_i \text{ for } i \leq \ell \text{ and } x_{\ell+1} \neq x'_{\ell+1}.
\]

For any \( i = 1, \ldots, \ell \) let \( D'_i \) be the dial of building such that \( x_{i-1} \not\in D'_i \) and \( x_i \subset D'_i \). By minimality of the galleries, we get that \( \{\xi, \xi'\} \subset \partial D'_i \) for any index \( i \). Therefore \( \ell \leq k \) and the proof is finished.

\[\square\]

In the following, we prove that \( \{\cdot|\cdot\}_{x_0} \) coincides with a Gromov product in \( \partial \Gamma \) and thus controls a visual metric on \( \partial \Gamma \).

**Proposition 7.13.** Let \( \xi, \xi' \) be two distinct points in \( \partial \Gamma \). Then there exists a bi-infinite minimal gallery between \( \xi \) and \( \xi' \) that lies at a distance smaller than \( \{\xi|\xi'\}_{x_0} + 1 \) of \( x_0 \).

**Proof.** Let \( \ell = \{\xi|\xi'\}_{x_0} \) and assume that \( \xi = [x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots] \) and \( \xi' = [x_0 \sim x'_1 \sim \cdots \sim x'_i \sim \cdots] \) with

\[
x_i = x'_i \text{ for } i \leq \ell \text{ and } x_{\ell+1} \neq x'_{\ell+1}.
\]

We consider two cases. Either \( x_{\ell+1} \) is adjacent to \( x'_{\ell+1} \), either \( x_{\ell+1} \) is not adjacent to \( x'_{\ell+1} \).

In the first case, the bi-infinite gallery

\[
\cdots \sim x_{\ell+2} \sim x_{\ell+1} \sim x'_{\ell+1} \sim x'_{\ell+2} \sim \cdots
\]

is minimal. Indeed, thanks to Proposition 7.12, it only crosses once the building-walls that separate \( \xi \) and \( \xi' \). In the second case, we apply the same reasoning to the bi-infinite gallery

\[
\cdots \sim x_{\ell+2} \sim x_{\ell+1} \sim x_{\ell} \sim x'_{\ell+1} \sim x'_{\ell+2} \sim \cdots.
\]
Eventually \( \{\xi|\xi'|_{x_0} \) or \( \{\xi|\xi'|_{x_0} + 1 \) is the distance between \( x_0 \) and a bi-infinite minimal gallery between \( \xi \) and \( \xi' \).

**Notation.** Let \( d(\cdot,\cdot) \) be the self-similar metric on \( \partial \Gamma \) coming from the geometric action of \( \Gamma \) on \( DG(\Sigma) \) (see Definition 3.2).

With Proposition 7.13, there exist two constants \( A \geq 1 \) and \( \alpha > 0 \) such that for any \( \xi,\xi' \in \partial \Gamma \):

\[
A^{-1} e^{-\alpha \{\xi|\xi'|}_{x_0} \leq d(\xi,\xi') \leq A e^{-\alpha \{\xi|\xi'|}_{x_0}.
\]

In the sequel we also write

\[
d(\xi,\xi') \asymp e^{-\alpha \{\xi|\xi'|}_{x_0}.
\]

Which means that, \( d(\xi,\xi') \) is, up to a multiplicative constant, equal to \( e^{-\alpha \{\xi|\xi'|}_{x_0} \).

Applications of this description of this visual metric on \( \partial \Gamma \) is the following proposition.

**Proposition 7.14.** For every \( \epsilon > 0 \), there exists only a finite set of parabolic limit sets of diameter larger than \( \epsilon \).

**Proof.** Let \( \partial P \) be a parabolic limit set. Let \( g'S_I \) be a residue in \( \Sigma \) such that \( \partial P \simeq \partial(g'S_I) \). According to Proposition 7.25, there exists a unique chamber \( x \in g'S_I \) such that for every chamber \( y \in g'S_I \) there exists a minimal gallery from \( x_0 \) to \( y \) passing through \( x \). Let \( g \in \Gamma \) such that \( x = gx_0 \). Then the diameter of \( \partial P \) is controlled by \( e^{-\alpha |g|} \) with \( |g| = d_c(x_0,gx_0) \). As there exists only a finite number of \( g \in \Gamma \) such that \( |g| \) is smaller than a fixed constant, the proposition is proved.

### 7.3 Shadows and balls of the boundary

Here we discuss how shadows control the balls of \( \partial \Gamma \).

**Lemma 7.15.** Let \( x \in \text{Ch}(\Sigma) \) and \( C_x \) be the cone of chambers in \( \Sigma \) based on \( x \). Then there exist \( g \in \Gamma \) and \( C \subset \Sigma \) of the form \( C = D_0(M_1) \cap \cdots \cap D_0(M_k) \) with

- for any \( i,j \) distinct: \( M_i \perp M_j \),
- for any \( i \): \( M_i \) is along \( x_0 \),

such that \( g(C) = C_x \).

**Proof.** Let \( D_1,\ldots,D_k \) designate the family of dials of building bounded by \( M_1,\ldots,M_k \) such that for any \( i = 1,\ldots,k \)

\[
x_0 \notin D_i, \; x \subset D_i, \; \text{and} \; M_i \text{ is along } x.
\]

According to Proposition 7.6, \( C_x = D_1 \cap \cdots \cap D_k \). Now if we choose \( g \in \Gamma \) such that \( gx_0 = x \) and set \( C = g^{-1}(C_x) \), we obtain the desired property.

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Proposition 7.16. There exists $\lambda > 1$ such that for any $x \in \text{Ch}(\Sigma)$ with $d_c(x_0, x) = k$ there exists $z \in v_x$ with

$$B(z, \lambda^{-1} e^{-\alpha k}) \subset v_x \subset B(z, \lambda e^{-\alpha k}).$$

Proof. To prove the right-hand side inclusion it is enough to notice that $\text{diam } v_x \leq A e^{-\alpha k}$. Let $C_x$ be the cone based on $x$. Let $C = D_0(M_1) \cap \cdots \cap D_0(M_k)$ and $g \in \Gamma$ such that $g(C) = C_x$, provided by Lemma 7.15. Now we remind that $g^{-1}$ is a bi-Lipchitz homeomorphism. Restricted to $v_x$, it essentially rescales the metric by a factor $e^{\alpha k}$. Now, according to Proposition 7.7, there exist $r > 0$ and $z \in \partial C$ such that $B(z, r) \subset \partial G$. As there is only a finite number of possible $C$, this achieves the proof. \qed

Adapting the proof of [BK13, Proposition 5.2.], to the case of right-angled hyperbolic buildings, we obtain the following proposition.

Proposition 7.17. There exists $\lambda \geq 1$ depending only on the geometry of $\Sigma$, such that for every $\xi \in \partial \Gamma$ and every $0 < r \leq \text{diam } \partial \Gamma$ there exists $D \in D(\Sigma)$ associated with such that:

$$B(\xi, \lambda^{-1} r) \subset \partial D \subset B(\xi, \lambda r).$$

7.4 Approximation of $\partial \Gamma$ with shadows

Let $x \in \text{Ch}(\Sigma)$ and $v_x$ be the associated shadow as in Definition 7.8. Thanks to Proposition 7.13, if $d_c(x_0, x) = k$ then $\text{diam } v_x \asymp e^{-\alpha k}$. We use this property to build an approximation of $\partial \Gamma$ made of shadows.

For an integer $k \geq 0$ we set

$$S_k = \{ x \in \text{Ch}(\Sigma) : d_c(x_0, x) = k \}.$$

The set $\{ v_x : x \in S_k \}$ is a finite covering of $\partial \Gamma$. Now let $S'_k$ be a subset of $S_k$ such that $\{ v_x : x \in S'_k \}$ defines a minimal covering of $\partial \Gamma$. This means that for every $x \in S'_k$ there exists $z \in v_x$ such that $z \not\in v_y$ for any $y \in S'_k \setminus \{ x \}$. Finally we set

$$G_k = \{ v_x : x \in S'_k \}$$

and, in the following, we prove that the sequence $\{ G_k \}_{k \geq 0}$ defines an approximation of $\partial \Gamma$.

Proposition 7.18. For $k \geq 0$, let $S'_k$ be the set of chambers previously defined and $G_k$ be the minimal covering of $\partial \Gamma$ associated with $S'_k$. There exists $\kappa > 1$ such that for any $x \in S'_k$, there exists $\xi_x \in v_x$ such that:

- $\forall x \in S'_k : B(\xi_x, \kappa^{-1} e^{-\alpha k}) \subset v_x \subset B(\xi_x, \kappa e^{-\alpha k})$,  
- $\forall x, y \in S'_k$ with $x \neq y : B(\xi_x, \kappa^{-1} e^{-\alpha k}) \cap B(\xi_y, \kappa^{-1} e^{-\alpha k}) = \emptyset$.  

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This property is enough to construct an approximation of $\partial \Gamma$. Indeed the visual constant $\alpha$ can be chosen such that $1/2 \leq e^{-\alpha} < 1$. In this case we can extract from $\{G_k\}_{k \geq 0}$ a subsequence that is an approximation of $\partial \Gamma$ as defined in Subsection 2.1.

**Proof of Proposition 7.18** Let $x \in S'_k$, and let $\xi_x \in v_x$. With $\text{diam } v_x \propto e^{-\alpha k}$, there exists $\kappa > 1$ such that for all $x \in S'_k$: $v_x \subset B(\xi_x, \kappa e^{-\alpha k})$.

We remind that the hyperbolicity, provides a constant $N \geq 1$, depending only on the hyperbolicity parameter, such that for $x, x' \in \text{Ch}(\Sigma)$ with $d_c(x_0, x) = d_c(x_0, x')$ if $d_c(x, x') \geq N$ then $v_x \cap v_{x'} = \emptyset$.

For any $x \in S'_k$, we pick $z_x \in v_x$ such that $z_x \notin v_y$ for any $y \in S'_k \setminus \{x\}$. Let $x, y \in S'_k$, $x \neq y$ and let $c \in \text{Ch}(\Sigma)$ be such that $d_c(x_0, c) = \{z_x | z_y\}_{x_0}$ and $\{z_x, z_y\} \subset v_c$. In this setting we can write that $z_x$ and $z_y$ are represented by infinite minimal galleries of the form:

- $z_x = [x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots \sim x_k \sim x_{k+1} \sim \cdots]$
- $z_y = [x_0 \sim y_1 \sim \cdots \sim y_i \sim \cdots \sim y_k \sim y_{k+1} \sim \cdots]$

with

- $x_i = c$ and $y_i = c$ for one $i \in \{1, \ldots, k - 1\}$,
- $x_k = x$ and $y_k = y$.

Then, as we saw in the proof of Proposition 7.13 one of the following galleries:

- $\cdots \sim x_{k+1} \sim x_k \sim \cdots \sim x_{i+1} \sim c \sim y_{i+1} \sim \cdots \sim y_k \sim y_{k+1} \sim \cdots$
- $\cdots \sim x_{k+1} \sim x_k \sim \cdots \sim x_{i+1} \sim y_{i+1} \sim \cdots \sim y_k \sim y_{k+1} \sim \cdots$

is a bi-infinite minimal gallery from $z_x$ to $z_y$. In particular $d_c(x_{k+N}, y_{k+N}) > N$ and the corresponding shadows do not intersect:

$$v_{x_{k+N}} \cap v_{y_{k+N}} = \emptyset.$$ 

Now according to Proposition 7.16 there exist $\xi_x \in v_{x_{k+N}}$ and $\xi_y \in v_{y_{k+N}}$ such that

$$B(\xi_x, \lambda^{-1} e^{-\alpha(k+N)}) \subset v_{x_{k+N}}$$
and
$$B(\xi_y, \lambda^{-1} e^{-\alpha(k+N)}) \subset v_{y_{k+N}}.$$ 

With $v_{x_{k+N}} \cap v_{y_{k+N}} = \emptyset$, $v_{x_{k+N}} \subset v_x$, and $v_{y_{k+N}} \subset v_y$ we obtain the desired property.

$\square$
8 Modulus in the boundary of a building and in the boundary of an apartment

The boundary of an apartment is, in a well chosen case, much more easier to understand than the boundary of the building. This is why we want to compare the modulus in the boundary of the building with some modulus in the boundary of an apartment.

In this section, we start by defining a convenient approximations on $\partial \Gamma$ and on the boundaries of the apartments thanks to shadows and retraction maps. Afterwards, we introduce the weighted modulus on the boundary of an apartment. Then, we prove Theorem 8.9. This theorem is, after Theorem 6.13, the second major step to prove the main theorem (Theorem 10.1). Theorem 8.9 states that weighted modulus are comparable to the modulus in $\partial \Gamma$. Eventually, using the ideas of Subsection 3.2, we reveal a connection between the conformal dimension of $\partial \Gamma$ and a critical exponent computed in the boundary of an apartment.

We use the notations and the setting of Section 5, 6 and 7. In particular $p \geq 1$ is fixed constant. We fix $\Gamma$ the graph product associated with the pair $(G, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,...,n})$. The self-similar metric $d(\cdot, \cdot)$ on $\partial \Gamma$ is defined as in Subsection 7.2. The visual exponent of $d(\cdot, \cdot)$ is $\alpha$. As in Section 3, $d_0$ denotes a small constant compared with $\text{diam} \partial \Gamma$ and with the constant of approximate self-similarity. Then $F_0$ is the set of curves of diameter larger than $d_0$.

8.1 Notations and conventions in $\partial A$ and in $\partial \Gamma$

In the sequel of this article we fix an apartment $A$ containing the base chamber $x_0$. Then we try to connect the geometry and the modulus in $\partial A$ and in $\partial \Gamma$. Naturally we will use in $\partial A$ and in $\partial \Gamma$ the same concepts, this is why we summarize some of the notations used in the following to avoid confusion. Fist we write

$$\mathcal{A}p_0(\Sigma) = \{B \in \mathcal{A}p(\Sigma) : x_0 \subset B\}.$$ 

For the simplicity, $\pi$ denotes the retraction $\pi_{A,x_0} : \Sigma \rightarrow A$. We also denote by $\pi$ the extension of the retraction on the boundary. The notations $d(\cdot, \cdot)$ and $\alpha$ are also used to describe the metric on $\partial B$ for any $B \in \mathcal{A}p_0(\Sigma)$.

An apartment is a thin building, so we can use in $\partial A$ the tools presented in Subsections 7.1 and 7.2. First, we define on $\partial A$ a combinatorial self-similar metric as in Subsection 7.2. Yet $x_0 \subset A$, so for $\xi, \xi' \in \partial A$, the quantity $\{\xi|\xi'\}_{x_0}$ is the same if we compute it in $A$ or in $\Sigma$. Hence, if we choose the same visual exponents for the visual metric in $\partial \Gamma$ and the visual metric in $\partial A$, then the metrics coincide up to a multiplicative constant. For this reason we use the same notation in $\partial A$ and in $\partial \Gamma$ for the metric $d(\cdot, \cdot)$ and for the visual constants $\alpha$ and $A$.

Eventually, it makes sense to talk about cones of chambers in $A$ and shadows $\partial A$. In $\partial A$ the results of 7.1 holds.
Notation.

- $D(\Sigma)$ (resp. $H(A)$) designates the set of dials of building in $\Sigma$ (resp. half-spaces in $A$),
- $M(\Sigma)$ (resp. $M(A)$) designates the set of building-walls in $\Sigma$ (resp. walls in $A$),
- for $\xi \in \partial A$ and $r > 0$ we write $B(\xi, r) \subset \partial \Gamma$ (resp. $B^A(\xi, r) \subset \partial A$) for the open ball of radius $r$ and center $\xi$,
- for $x \in \text{Ch}(A)$ we write $C_x$ (resp. $C^A_x$) for the cone of chambers based on $x$ in $\Sigma$ (resp. in $A$),
- for $x \in \text{Ch}(A)$ we write $v_x$ (resp. $w_x$) for the shadow of $x$ in $\partial \Gamma$ (resp. $\partial A$).

Usually we will use the following conventions.

- $v$ (resp. $w$) designates an open subset of $\partial \Gamma$ (resp. of $\partial A$),
- $\partial P$ (resp. $\partial Q$) designates a parabolic limit set in $\partial \Gamma$ (resp. in $\partial A$),
- $D$ (resp. $H$) designates a dial of building in $\Sigma$ (resp. a half-space in $A$).

8.2 Choice of approximations

The following lemma says that shadows have a nice behavior under retraction maps.

**Lemma 8.1.** Let $A \in A_{P0}(\Sigma)$ and let $x \in \text{Ch}(\Sigma)$ and $v_x$ be the associated shadow in $\partial \Gamma$ as defined in Definition 7.8. Then

- either $x \notin \text{Ch}(A)$ and $\text{Int}(v_x) \cap \partial A = \emptyset$,
- either $x \in \text{Ch}(A)$ and $v_x \cap \partial A$ is a shadow in $\partial A$.

In the second case $v_x \cap \partial A = \pi(v_x)$.

**Proof.** Let $C_x$ be the cone based on $x$. If $\text{Int}(v_x) \cap \partial A \neq \emptyset$ then there exists a chamber $c$ in $A \cap C_x$. By convexity, a minimal gallery from $x_0$ to $c$ that passes through $x$ is included in $A$ and $x \subset A$. Therefore $v_x \cap \partial A$ is the shadow in $\partial A$ associated with $x$. 

We fix $\{G^A_k\}_{k \geq 0}$ an approximation of $\partial A$ based on shadows as constructed in Subsection 7.3.

**Notation.** For $k \geq 0$ we set

$$G_k := \{v_y \subset \partial \Gamma : \pi(v_y) \in G_k^A\}.$$
We remind that we chose the same visual exponents for the metrics in \( \partial \Gamma \) and \( \partial A \). Moreover for \( v, v' \in G_k \), we observe that \( \text{Int}(v \cap v') \neq \emptyset \) if and only if \( \pi(v) \cap \pi(v') \neq \emptyset \) and \( \pi(v) \neq \pi(v') \). As a consequence and with Lemma 8.1 we get the following fact.

**Fact 8.2.** There exists \( \kappa > 1 \) such that \( \{G_k^A\}_{k \geq 0} \) and \( \{G_k\}_{k \geq 0} \) are \( \kappa \)-approximations. Moreover, for \( w \in G_k^A \) there exists a unique \( \tilde{w} \in G_k \) such that \( \tilde{w} \cap \partial A \neq \emptyset \) and \( \pi(\tilde{w}) = w \).

Hereafter, \( \{G_k\}_{k \geq 0} \) designates the approximation of \( \partial \Gamma \) obtained from \( \{G_k^A\}_{k \geq 0} \) thanks to the previous fact. This approximation of \( \partial \Gamma \) is canonically associated with \( \{G_k^A\}_{k \geq 0} \) in the following sense: from \( \{G_k\}_{k \geq 0} \) we can equip any \( B \in A_{p_0}(\Sigma) \) with an approximation isometric to \( \{G_k^A\}_{k \geq 0} \). Indeed if \( B \in A_{p_0}(\Sigma) \), for \( k \geq 0 \) we set

\[
G^B_k := \{w = \partial B \cap v : v \in G_k\}.
\]

Now let \( B \in A_{p_0}(\Sigma) \) and \( f : B \to A \) be the type preserving isometry that fixes \( x_0 \). The map \( f \) is realized by the restriction to \( B \) of the retraction \( \pi \) and we get the following fact.

**Fact 8.3.** \( G^A_k = \{f(v)\}_{v \in G^B_k} \).

Now that an approximation \( \{G_k\}_{k \geq 0} \) is fixed the results we will obtain on the combinatorial modulus in \( \partial \Gamma \) will be valid, up to multiplicative constants, for any approximation thanks to Proposition 2.5.

### 8.3 Weighted modulus in \( \partial A \)

On scale \( k \geq 0 \), to compare the modulus in the building with the modulus in the apartment means to compare the cardinality of \( G_k \) with the cardinality of \( G_k^A \). If the building is thick these quantities differ by an exponential factor in \( k \). This is the reason why we attach a weight to the elements of \( G_k^A \).

**Definition 8.4.** Let \( w \in G_k^A \), we set \( q(w) = \#\{v \in G_k : \pi(v) = w\} \).

Let \( k \geq 0 \) and let \( F^A \) be a set of curves contained in \( \partial A \). As in Subsection 2.1 a positive function \( \rho : G_k^A \to \mathbb{R}^+ \) is said to be \( F^A \)-admissible if for any \( \gamma \in F^A \)

\[
\sum_{\gamma \cap w \neq \emptyset} \rho(w) \geq 1.
\]

The weighted \( p \)-mass of \( \rho \) in \( \partial A \) is

\[
WM^A_p(\rho) = \sum_{w \in G_k^A} q(w)\rho(w)^p.
\]

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Definition 8.5. Let \( k \geq 0 \) and let \( \mathcal{F}^A \) be a set of curves contained in \( \partial A \), we define the weighted \( G_k^A \)-combinatorial \( p \)-modulus of \( \mathcal{F}^A \) by
\[
\operatorname{Mod}^p_A(\mathcal{F}^A, G_k^A) := \inf \{ WM^p_A(\rho) \}.
\]
Where the infimum is taken over the set of \( \mathcal{F}^A \)-admissible functions and with the convention \( \operatorname{Mod}^p_A(\emptyset, G_k^A) = 0 \). For the simplicity, we usually use the terminology weighted modulus.

We can check that, Proposition 2.3 holds for weighted modulus as well and the proof is identical to the one for the usual combinatorial modulus.

Yet, this definition of the weighted modulus is strongly depending on the choice we have made for the approximation. In particular, it does not permit to compute the weighted modulus relatively to a generic approximation of \( \partial A \). As a consequence, an analogous to Proposition 2.5 would make no sense here. This is a huge restriction on the use we can do of the weighted modulus. Indeed, this proposition is essential to prove Proposition 3.12 or Theorem 6.13 for the usual combinatorial modulus.

Moreover, as we are interested in the modulus in \( \partial \Gamma \) we can use weighted modulus computed in a precise approximation and deduce, up to multiplicative constant, generalities about modulus in \( \partial \Gamma \).

The weights are given by the types of the building-walls crossed by a minimal galleries.

Proposition 8.6. Let \( w \in G_k^A \) be such that \( w \) is a shadow \( w = w_x \) for \( x \in \text{Ch}(A) \). Let \( x_0 \sim s_1 x_1 \sim s_2 \cdots \sim s_{k-1} x_{k-1} \sim s_k x \) be a minimal gallery where \( s_1, \ldots, s_k \) is the family of types of the walls crossed by this gallery. If \( q_1, \ldots, q_k \) are the orders of these generators of \( \Gamma \), then
\[
q(w) = \prod_{i=1}^{k} q_i - 1.
\]

Proof. Let \( w \in G_k^A \) and \( x \in \text{Ch}(A) \) be such that \( w = w_x \) in \( \partial A \). Then we observe that \( \{ v \subset \partial \Gamma : \pi(v) = w \} = \{ v_y \subset \partial \Gamma : \pi(y) = x \} \). As a consequence, we obtain
\[
q(w) = \# \pi^{-1}(x).
\]

Now consider the gallery \( x_0 \sim s_1 x_1 \sim s_2 \cdots \sim s_{k-1} x_{k-1} \sim s_k x \) given in the statement of proposition. As \( \pi \) preserves the types, \( y \in \text{Ch}(\Sigma) \) is in \( \pi^{-1}(x) \) if and only if there exists a minimal gallery from \( x_0 \) to \( y \) in \( \Sigma \) of the form \( x_0 \sim s_1 y_1 \sim s_2 \cdots \sim s_{k-1} y_{k-1} \sim s_k y \). Eventually, we obtain \( q(w) = \prod_{i=1}^{k} q_i - 1 \).

Thanks to the choices we have made, the weighted modulus is invariant up to a change of apartment in the following sense. For \( B \in \mathcal{A}_{\pi}(\Sigma) \) consider the approximation \( G_k^B \) given by Fact 8.3. To any element \( w \in G_k^B \) we attach a weight and define a weighted \( G_k^B \)-combinatorial \( p \)-modulus as it is done in \( \partial A \). Now let \( f : B \to A \) be a type preserving isometry that fixes \( x_0 \) and denote \( f : \partial B \to \partial A \) the extension of this map to the boundary. The map \( f \) is realized by the restriction of the retraction \( \pi \) to \( B \). Thus \( f \) preserves the weights. Then the following fact is a straightforward consequence of Fact 8.3.
Fact 8.7. Let $B \in \mathcal{A}_p(\Sigma)$. Then for any $k \geq 0$ and any set of curves $\mathcal{F}^B$ contained in $\partial B$ one has
\[
\text{Mod}_p^B(\mathcal{F}^B, G_k^B) = \text{Mod}_p^A(f(\mathcal{F}^B), G_k^A).
\]

Note that, for any $k \geq 0$ and any $w \in G_k^A$ one has
\[
1 \leq q(w) \leq (q - 1)^k \text{ with } q := \max\{q_1, \ldots, q_n\}.
\]

Therefore for any set of curves $\mathcal{F}^A$ contained in $\partial A$, the following inequalities come directly from the definition
\[
\text{mod}_p^A(\mathcal{F}^A, G_k^A) \leq \text{Mod}_p^A(\mathcal{F}^A, G_k^A) \leq (q - 1)^k \text{mod}_p^A(\mathcal{F}^A, G_k^A).
\]

Where the modulus in small letters designates the usual modulus computed in $\partial A$. In particular if $\Gamma$ is of constant thickness $q - 1 \geq 2$ then
\[
\text{Mod}_p^A(\mathcal{F}^A, G_k^A) = (q - 1)^k \text{mod}_p^A(\mathcal{F}^A, G_k^A).
\]

As a consequence, at fixed scale $k \geq 0$, the weighted modulus depends only on the boundary of an apartment. We will discuss this particular case in Sections 9 and 10.

The following proposition is a major motivation of the definition of the weighted modulus.

Proposition 8.8. Let $\mathcal{F}$ be a set of curves in $\partial \Gamma$ and let $\mathcal{F}^A$ be a set of curves in $\partial A$ such that $\pi(\mathcal{F}) \subset \mathcal{F}^A$. Then
\[
\text{Mod}_p(\mathcal{F}, G_k) \leq \text{Mod}_p^A(\mathcal{F}^A, G_k^A).
\]

Proof. Let $\rho_A$ be a $\mathcal{F}^A$-admissible function. We set $\rho : G_k \rightarrow \mathbb{R}_+$ defined by
\[
\rho(v) = \rho_A \circ \pi(v).
\]

If $\gamma \in \mathcal{F}$, let $\gamma_A := \pi \circ \gamma$. Then, as $\gamma_A \in \mathcal{F}^A$
\[
L_\rho(\gamma) = \sum_{v \cap \gamma \neq \emptyset} \rho_A \circ \pi(v) \geq \sum_{w \cap \gamma_A \neq \emptyset} \rho_A(w) \geq 1,
\]

thus $\rho$ is $\mathcal{F}$-admissible. Furthermore, one has:
\[
M_\rho(\rho) = \sum_{v \in G_k} \rho_A \circ \pi(v)^p = \sum_{w \in G_k^A} q(w) \cdot \rho_A(w)^p = WM_p^A(\rho_A).
\]

With the first point it follows that $\text{Mod}_p(\mathcal{F}, G_k) \leq \text{Mod}_p^A(\mathcal{F}^A, G_k^A)$. \qed
8.4 Modulus in $\partial \Gamma$ compared with weighted modulus in $\partial A$

We remind that the apartment $A \in A_{\rho_0}(\Sigma)$ is fixed. Yet, thanks to Fact 8.7 the following results holds for any apartment containing $x_0$.

Here we keep considering the approximations $G_k$ and $G_k^A$ defined in the begin of Subsection 8.2. We remind that if $\eta$ is a non-constant curve of $\partial \Gamma$, the notation $U_\epsilon(\eta)$ designates the $\epsilon$-neighborhood of $\eta$ relative to the $C^0$ topology. If $\eta$ is a non-constant curve contained in $\partial A$, we use the notation

$U_\epsilon^A(\eta) := \{ \gamma \in U_\epsilon(\eta) : \gamma \subset \partial A \}$.

The next theorem proves that in this case, the modulus of $U_\epsilon(\eta)$ in the boundary of the building is controlled by the weighted modulus of $U_\epsilon^A(\eta)$ in the boundary of the apartment. It is a key point to prove Theorem 10.1.

**Theorem 8.9.** Let $p \geq 1$, let $\eta \in F_0$ and assume $\eta \subset \partial A$. For $\epsilon > 0$ small enough so that the hypothesis of Theorem 6.13 hold in $\partial \Gamma$, there exists a positive constant $C = C(d_0, p, \eta, \epsilon)$ independent of $k$ such that for $k \geq 0$ large enough

$$\text{Mod}_p(U_\epsilon(\eta), G_k) \leq \text{Mod}_p^A(U_\epsilon^A(\eta), G_k^A) \leq C \cdot \text{Mod}_p(U_\epsilon(\eta), G_k).$$

Furthermore, when $p$ belongs to a compact subset of $[1, +\infty)$ the constant $C$ may be chosen independent of $p$.

For the rest of the subsection, we fix $\eta \in F_0$ and $\epsilon > 0$ as in the hypothesis of Theorem 8.13. Moreover, we assume $\eta \subset \partial A$. To prove the theorem we need to introduce the following notations:

- $\text{Aut}_\Sigma$ is the full group of type preserving isometries of $\Sigma$.
- For $n \geq 0$, $B_n \subset \text{Ch}(\Sigma)$ is the ball of center $x_0$ and of radius $n$ for the distance over the chambers $d_\epsilon(\cdot, \cdot)$.
- For $n \geq 0$, $K_n < \text{Aut}_\Sigma$ is the fixed point stabilizer of $B_n$ under the action of $\text{Aut}_\Sigma$.
- $F_n := \{ g \gamma \subset \partial \Gamma : g \in K_n \text{ and } \gamma \in U_\epsilon^A(\eta) \}$.

The main step to prove the theorem is to show that $F_n$ is an intermediate set of curves between $U_\epsilon^A(\eta)$ and $U_\epsilon(\eta)$.

**Lemma 8.10.** Let $p \geq 1$. For $n \geq 0$ large enough, there exist two positive constants $C_1$, $C_2$ depending on $d_0$, $p$, $\eta$, $\epsilon$, and independent of $k$ such that for $k > n$:

$$\text{Mod}_p^A(U_\epsilon^A(\eta), G_k^A) \leq C_1 \cdot \text{Mod}_p(F_n, G_k) \leq C_2 \cdot \text{Mod}_p(U_\epsilon(\eta), G_k).$$

Furthermore, when $p$ belongs to a compact subset of $[1, +\infty)$ the constants may be chosen independent of $p$.  

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Proof. First we prove the right-hand side inequality. According to Proposition 7.13 for any \( g \in K_n \) and any \( \xi \in \partial \Gamma \), \( d(\xi, g\xi) \sim e^{-\alpha n} \). Then for \( n \geq 0 \) large enough, by triangular inequality, \( \mathcal{F}_n \subset U_2(\eta) \). As a consequence of Theorem 6.13, \( \text{Mod}_p(\mathcal{U}_k(\eta), G_k) \) does not depend up to a multiplicative constant on \( \epsilon \). Hence, with Proposition 2.3 (1), there exists \( C = C(p, d_0, \epsilon, \eta) \) such that

\[
\text{Mod}_p(\mathcal{F}_n, G_k) \leq C \cdot \text{Mod}_p(\mathcal{U}_k(\eta), G_k).
\]

Now we fix an integer \( n \geq 0 \) large enough so that the first point of the proof holds. We use the notation \( K := K_n \) for the simplicity. Moreover we assume that \( k > n \). Let \( \rho : G_k \to \mathbb{R}^+ \) be a minimal \( \mathcal{F}_n \)-admissible function and set \( \rho_A : G_k^A \to \mathbb{R}^+ \) the function defined by:

\[
\rho_A(w) = \int_K \rho(g\bar{w})d\mu(g),
\]

where \( \mu \) denotes the Haar probability measure over \( K \) and where the function : \( w \in G_k^A \to \bar{w} \in G_k \) is given by Fact 8.2. Let \( w \in G_k^A \) and let \( x \in \text{Ch}(\Sigma) \) be such that \( v_x = \bar{w} \), then \( d_c(x_0, x) = k \). As in Proposition 8.6 let \( x_0 \sim s_1 x_1 \sim s_2 \cdots \sim s_n x_n \sim s_{n+1} \cdots \sim s_{k-1} x_{k-1} \sim s_k x \)

be a minimal gallery where \( s_1, \ldots, s_k \) is the family of types of the building-walls crossed by this gallery. Then we set

\[
q(w, n) = \prod_{i=n+1, \ldots, k} q_i - 1.
\]

We notice that for any \( g \in K \) the translated \( g\bar{w} = gv_x \) is the shadow \( v_{gx} \). Then, if we write \( K.\bar{w} = \{g\bar{w} \}_{g \in K} \) one has \( K.\bar{w} = \{v_{gx} \}_{g \in K} \). Therefore \( \#K.\bar{w} = q(w, n) \) and we can write:

\[
\rho_A(w) = \frac{1}{q(w, n)} \cdot \sum_{v \in K.\bar{w}} \rho(v).
\]

Let \( \gamma \in \mathcal{U}_k(\eta) \):

\[
L_{\rho_A}(\gamma) = \sum_{w \cap \gamma \neq \emptyset} \int_K \rho(g\bar{w})d\mu(g) = \int_K \sum_{w \cap \gamma \neq \emptyset} \rho(g\bar{w})d\mu(g) = \int_K \sum_{v \cap g(\gamma) \neq \emptyset} \rho(v)d\mu(g).
\]

Yet \( g(\gamma) \in \mathcal{F}_n \), thus \( \sum_{v \cap g(\gamma) \neq \emptyset} \rho(v) \geq 1 \) and \( \rho_A \) is \( \mathcal{F}_A \)-admissible. Then, thanks to Jensen’s inequality, for \( p \geq 1 \) one has:

\[
W_{M_p}^A(\rho_A) \leq \sum_{w \in G_k^A} q(w) \int_K \rho(g\bar{w})^p d\mu(g) = \sum_{w \in G_k^A} q(w) q(w, n) \cdot \sum_{v \in K.\bar{w}} \rho(v)^p \leq (q - 1)^n M_p(\rho),
\]

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where \( q := \max\{q_1, \ldots, q_n\} \). Eventually we get:

\[
\Mod_p^A(\mathcal{U}^A(\eta), G^A_k) \leq (q - 1)^n \Mod_p(F, G_k).
\]

\( \square \)

Proof of Theorem 8.9. As \( \pi(U(\eta)) \subset U^A(\eta) \), Proposition 8.8 and Lemma 8.10 prove the theorem.

\( \square \)

8.5 Consequences

Here we keep considering the approximations \( G_k \) and \( G^A_k \) defined in Subsection 8.2. For \( \eta \) a non-constant curve in \( \partial A \), \( \partial Q \) a parabolic limit set in \( \partial A \), and \( \delta, r, \epsilon > 0 \), we use the following notations:

- \( \mathcal{F}^A_0 = \{ \gamma \in \mathcal{F}_0 : \gamma \subset \partial A \} \),
- \( \mathcal{F}^A_{\delta,r}(\partial Q) \) is the subset of \( \mathcal{F}^A_0 \) of the curves \( \gamma \) such that:
  - \( \gamma \subset N_\delta(\partial Q) \),
  - \( \gamma \not\subset N_r(\partial Q') \) for any connected parabolic limit set \( \partial Q' \subset \subset \partial Q \),
- \( \delta_0(\cdot) \) refers to the increasing function define in Theorem 6.13.

We remind that the apartment \( A \in \mathcal{A}_p(\Sigma) \) is fixed. Yet, thanks to Fact 8.7 the following results holds for any apartment containing \( x_0 \).

Lemma 8.11. Let \( p \geq 1 \) and \( A \in \mathcal{A}_p(\Sigma) \). Let \( \partial P \) be a parabolic limit set in \( \partial \Gamma \) and assume that \( x_0 \subset \operatorname{Conv}(\partial P) \). Let \( \gamma \) be a non-constant curve in \( \partial Q = \partial P \cap \partial A \) such that \( \partial Q \) is the smallest parabolic limit set of \( \partial A \) containing \( \gamma \). Let \( r > 0 \) be small enough so that \( \gamma \not\subset \overline{N_\epsilon(\partial Q')} \) for any connected parabolic limit set \( \partial Q' \subset \partial Q \). Let \( \delta < \delta_0(r) \) and \( \epsilon > 0 \) be small enough so that \( \mathcal{U}^A(\gamma) \subset \mathcal{F}^A_{\delta,r}(\partial Q) \). Then there exists a constant \( C = C(d_0, p, \gamma, r, \epsilon) \) such that

\[
\Mod_p(\mathcal{F}^A_{\delta,r}(\partial P), G_k) \leq C \cdot \Mod_p(\mathcal{U}(\gamma), G_k) \leq C \cdot \Mod_p^A(\mathcal{F}^A_{\delta,r}(\partial Q), G^A_k).
\]

In particular

\[
\Mod_p(\mathcal{F}^A_{\delta,r}(\partial P), G_k) \leq C \cdot \Mod_p^A(F^A_0, G^A_k).
\]

Furthermore, when \( p \) belongs to a compact subset of \([1, +\infty)\) the constant \( C \) may be chosen independent of \( p \).

Before the proof, we do the same remark as we did at Theorem 6.13. Up to rescaling, the assumption for \( \epsilon > 0 \) small enough \( \mathcal{U}^A(\gamma) \subset \mathcal{F}^A_{\delta,r}(\partial Q) \) makes sense.
Proof. With the assumption on $\epsilon$ and Proposition 2.3(1) we obtain:
$$\text{Mod}^A_p(\mathcal{U}^A(\gamma),G_k^A) \leq \text{Mod}^A_p(\mathcal{F}^A_{\delta,r}(\partial Q),G_k^A) \leq \text{Mod}^A_p(\mathcal{F}^A_0,G_k^A).$$
As $\pi(\mathcal{U}(\gamma)) \subset \mathcal{U}^A(\gamma)$, with Proposition 8.8 one has
$$\text{Mod}_p(\mathcal{U}(\gamma),G_k) \leq \text{Mod}^A_p(\mathcal{U}^A(\gamma),G_k^A).$$
Finally thanks to Theorem 6.13 there exists $C = C(d_0,p,\gamma,\epsilon,r)$ such that:
$$\text{Mod}_p(\mathcal{F}_{\delta,r}(\partial P),G_k) \leq C \cdot \text{Mod}_p(\mathcal{U}(\gamma),G_k).$$

Now we have enough to prove the following theorem that is used in the proof of Theorem 10.1.

**Theorem 8.12.** For any $p \geq 1$, there exists a constant $D = D(p,d_0)$ such that for $k \geq 0$
$$\text{Mod}_p(\mathcal{F}_0,G_k) \leq D \cdot \text{Mod}^A_p(\mathcal{F}^A_0,G_k^A).$$

**Proof.** First, as it is done in [BK13, as a remark of Corollary 6.2.] in boundaries of Coxeter groups, we observe that $\mathcal{F}_0$ splits in a finite disjoint union
$$\mathcal{F}_0 = \mathcal{F}_{\delta_1, r_1}(\partial P_1) \sqcup \cdots \sqcup \mathcal{F}_{\delta_N, r_N}(\partial P_N)$$
with $\delta_i < \delta_0(r_i)$. To see this we remind that for $\delta > 0$ small enough compared with $d_0$, according to Proposition 7.14 there exists only a finite number of parabolic limit sets $\partial P$ such that $\mathcal{F}_{\delta,r}(\partial P) \neq \emptyset$. Then we call the height of a parabolic limit set $\partial P$ the maximal length of a sequence of parabolic limit sets included in $\partial P$ of the form:
$$\partial Q_0 \subsetneq \partial Q_1 \subsetneq \cdots \subsetneq \partial Q_k = \partial P.$$
If the parameters $\delta_k$ and $r_k$ are given for the parabolic limit sets of height $k$, then we set $r_{k+1} = \delta_k$ and $\delta_{k+1} < \delta_0(\delta_k)$. Starting with $\delta_0$ small enough, we obtain the desired decomposition by induction on the height.

Let $\partial P$ be one of the parabolic limit sets involved in the previous decomposition of $\mathcal{F}_0$ and $\delta, r > 0$ be the corresponding constants. Applying the same argument as in the begin of the proof of Theorem 6.13 we can assume that $x_0 \subset \text{Conv}(\partial P)$. Pick $B \in \mathcal{A}_0(\Sigma)$ such that $\partial B \cap \partial P \neq \emptyset$. With $C$ the constant provided by the Lemma 8.11 we get
$$\text{Mod}_p(\mathcal{F}_{\delta,r}(\partial P),G_k) \leq C \cdot \text{Mod}^A_p(\mathcal{F}^A_0,G_k^A).$$
Moreover, with Fact 8.7 the weighted modulus on the right-hand side of the inequality is independent of $\partial P$. Eventually, with the Proposition 2.3(2), there exists a constant $D = D(p,d_0)$ such that
$$\text{Mod}_p(\mathcal{F}_0,G_k) \leq D \cdot \text{Mod}^A_p(\mathcal{F}^A_0,G_k^A).$$
Note that for the moment we cannot prove a converse inequality between the modulus. Indeed, in the proof of the Lemma 8.11 the use of Theorem 6.13 in $\partial \Gamma$ is a key point. As we said before, we cannot prove an analogous of Theorem 6.13 for the weighted modulus.

Nevertheless, we can define a critical exponent in connection with the weighted modulus as it is done in the reminders of Subsection 3.2. Then Theorem 8.12 helps us to understand this new critical exponent.

**Proposition 8.13.** There exists $p_0 \geq 1$ such that for $p \geq p_0$ the weighted modulus $\operatorname{Mod}^A_p(F^A_0, G^A_k)$ goes to zero as $k$ goes to infinity.

**Proof.** Let $C$ be the constant provided by Proposition 3.4 applied to the usual combinatorial modulus in $\partial A$. For any pair of integer $k, \ell \geq 0$ we obtain

$$
(*) \quad \operatorname{Mod}^A_p(F^A_0, G^A_{k+\ell}) \leq q^{k+\ell} \operatorname{mod}^A_p(F^A_0, G^A_k) \leq C \cdot q^\ell \operatorname{mod}^A_p(F^A_0, G^A_k) \cdot q^k \operatorname{mod}^A_p(F^A_0, G^A_k),
$$

where $q := \max\{q_1, \ldots, q_n\}$.

According to Proposition 2.4 there exists $\ell_0 \geq 0$ such that there exists a $F^A_0$-minimal function for $\operatorname{mod}^A_p(\cdot, G^A_{\ell_0})$ of the form

$$
\rho : G^A_{\ell_0} \to \mathbb{R}^+ \text{ with } \rho(v) < 1/2 \text{ for any } v \in G^A_{\ell_0}.
$$

Now fix $p_0 \geq 1$ such that

$$
C \cdot q^{\ell_0} \operatorname{mod}^A_p(F^A_0, G^A_{\ell_0}) < 1.
$$

Let $a \geq 0$ and $0 \leq b < \ell_0$ be the results of the Euclidean division of $k$ by $\ell_0$ ($k = a\ell_0 + b$). Applying $a$ times the inequality $(*)$ we obtain

$$
\operatorname{Mod}^A_p(F^A_0, G^A_k) \leq (C \cdot q^{\ell_0} \operatorname{mod}^A_p(F^A_0, G^A_{\ell_0}))^a \cdot q^b \operatorname{mod}^A_p(F^A_0, G^A_{\ell_0}).
$$

As $b < \ell_0$ it comes that $q^b \operatorname{mod}^A_p(F^A_0, G^A_{\ell_0})$ is bounded by above, thus there exists $K > 0$ such that

$$
\operatorname{Mod}^A_p(F^A_0, G^A_k) \leq K \cdot (C \cdot q^{\ell_0} \operatorname{mod}^A_p(F^A_0, G^A_{\ell_0}))^a.
$$

This proves that $\operatorname{Mod}^A_p(F^A_0, G^A_k)$ goes to zero as $k$ goes to infinity. Eventually we notice that for $k \geq \ell_0$ fixed the function $p \mapsto \operatorname{Mod}^A_p(F_0, G_k)$ is non-increasing. This completes the proof.

It is now natural to define a critical exponent for the weighted modulus in the apartment.

**Definition 8.14.** The critical exponent $Q_W$ of the weighted modulus in $\partial A$ is defined as follow

$$
Q_W = \inf\{p \in [1, +\infty) : \lim_{k \to +\infty} \operatorname{Mod}^A_p(F^A_0, G^A_k) = 0\}.
$$

To avoid confusion, we use the following notations
• $Q$ for the critical exponent associated with the usual modulus $\text{Mod}_p(\cdot, G_k)$ in $\partial \Gamma$,
• $Q_A$ for the critical exponent associated with the usual modulus $\text{mod}_p^A(\cdot, G^A_k)$ in $\partial A$,
• $Q_W$ for the critical exponent associated with the weighted modulus $\text{Mod}_p^A(\cdot, G^A_k)$ in $\partial A$.

We remind that $Q$ and $Q_A$ are respectively the conformal dimension of $\partial \Gamma$ and of $\partial A \simeq \partial W$ (see Theorem 3.8). The inequalities between the different modulus provide the next corollary.

**Corollary 8.15.** The following inequalities hold

$$Q_A \leq Q \leq Q_W.$$  

**Proof.** With Proposition 2.3 (1) and Theorem 8.12 one has

$$\text{mod}_p^A(\mathcal{F}_0^A, G^A_k) \leq \text{Mod}_p(\mathcal{F}_0, G_k) \leq D \cdot \text{Mod}_p^A(\mathcal{F}_0^A, G^A_k).$$

The inequalities between the critical exponents follow.  

9 Application to buildings of constant thickness

Here we use the notations and the setting of the previous section. In particular, the self-similar metric on $\partial \Gamma$ is $d(\cdot, \cdot)$. We fix $d_0$ a small constant compared with $\text{diam} \partial \Gamma$ and with the constant of approximate self-similarity. Then $\mathcal{F}_0$ is the set of curves of diameter larger than $d_0$. The notation $\delta_0(\cdot)$ still refers to the increasing function define in Theorem 6.13.

As before we fix an apartment $A \in A p_0(\Sigma)$ and $\mathcal{F}_0^A$ is the set of curves in $\partial A$ of diameter larger than $d_0$.

We assume that $\Gamma$ is of constant thickness $q - 1 \geq 2$. This means that $\Gamma$ is the graph product given by the pair $(G, \{\mathbb{Z}/q\mathbb{Z}\}_{i=1,...,n})$. As before $\{G^A_k\}_{k \geq 0}$ and $\{G_k\}_{k \geq 0}$ are the approximations of $\partial A$ and $\partial \Gamma$ provided by Fact 8.2. We already noticed that, with the constant thickness assumption, we obtain for $k \geq 0$ and $\mathcal{F}^A$ a set of curves contained in $\partial A$

$$\text{Mod}_p^A(\mathcal{F}^A, G^A_k) = q^k \text{mod}_p(\mathcal{F}^A, G^A_k),$$

where the modulus in small letters designates the usual modulus computed in $\partial A$. In particular, this means that from Theorem 6.13 applied to $\text{mod}_p^A(\cdot, G^A_k)$ we can obtain analogous inequalities for $\text{Mod}_p^A(\cdot, G^A_k)$.

Along with the results of Subsection 8.5 this lead to find a control of $\text{Mod}_p(\mathcal{F}_0, G_k)$ by $\text{Mod}_p^A(\mathcal{F}_0^A, G^A_k)$. 

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Theorem 9.1. For any $p \geq 1$, there exists a constant $D = D(p,d_0)$ such that for $k \geq 0$

$$D^{-1} \cdot \text{Mod}_p^A(F_0^A, G_k^A) \leq \text{Mod}_p(F_0, G_k) \leq D \cdot \text{Mod}_p^A(F_0^A, G_k^A).$$

In particular $Q_W = Q$.

Proof. The right-hand side inequality is given by Theorem 8.12. The proof is almost the same for the left-hand side inequality. Indeed, $F_0^A$ admits a decomposition analogous to the decomposition used in the beginning of the proof of Theorem 8.12. Hence, with Proposition 2.3 (2), it is sufficient to prove that for any parabolic limit set $\partial Q \subset \partial A$ and any $\delta, r > 0$ with $\delta < \delta_0(r)$, there exists a constant $C = C(p,d_0,\partial Q,\delta,r)$ such that

$$\text{Mod}_p^A(F_{\delta,r}^A(\partial Q), G_k^A) \leq C \cdot \text{Mod}_p(F_0^A, G_k^A).$$

To this purpose, pick $\eta$ a non-constant curve in $\partial Q$ and $\varepsilon > 0$ such that the hypothesis of Theorem 6.13 in $\partial A$ and of Theorem 8.9 are satisfied. Then there exist two constants $K$ and $K'$ independent of $k$ such that

$$\text{Mod}_p^A(F_{\delta,r}^A(\partial Q), G_k^A) \leq K \cdot \text{Mod}_p^A(U_\varepsilon^A(\eta), G_k^A) \leq K' \cdot \text{Mod}_p(U_\varepsilon(\eta), G_k).$$

Eventually, with the hypothesis on $\eta$ and $\varepsilon$, one has, up to rescaling, $U_\varepsilon(\eta) \subset F_0$. This achieves the proof.

The equality of the critical exponents is a straightforward consequence of the inequalities between the modulus.

Remark 9.2. In the case where $\Sigma$ is a right-angled Fuchsian building of constant thickness, M. Bourdon gave the explicit value of the conformal dimension of $\partial \Gamma$.

Theorem 9.3 (Bou97). Let $\Gamma$ be the graph product associated with a pair $(C_n, \{\mathbb{Z}/q\mathbb{Z}\}_{i=1,...,n})$ where $C_n$ is a cyclic graph of length $n \geq 5$ and $q \geq 2$, then

$$\text{Confdim}(\partial \Gamma) = 1 + \frac{\log(q - 1)}{\text{Arg cosh} \frac{n-2}{2}}.$$

10 Dimension 3 and 4 right-angled buildings with boundary satisfying the CLP

In a well chosen case, the symmetries of the Davis chamber, that extend to the boundary of an apartment, provide a strong control of the weighted modulus. This lead to the proof of the main theorem of this article.

Here we still assume that $\Gamma$ is of constant thickness $q - 1 \geq 2$. As usual, $W$ is the Coxeter group, associated with $\Gamma$. As before $\{G_k^A\}_{k \geq 0}$ and $\{G_k\}_{k \geq 0}$ are the approximations of $\partial A$ and $\partial \Gamma$ provided by Fact 8.2.
In this subsection, we assume that $W$ is the group generated by the reflections along the faces of a compact right-angled polytope $D \subset \mathbb{H}^d$. Now, under some assumptions on the regularity of $D$, we prove that $\partial \Gamma$ satisfies the CLP.

**Theorem 10.1.** Let $\Gamma$ be a graph product of constant thickness $q - 1 \geq 2$. Assume that $W$ is the group generated by the reflections along the faces of a compact right-angled polytope $D \subset \mathbb{H}^d$ and let $R_{\text{ef}}(D)$ be the finite group of the hyperbolic reflections that preserve $D$. Moreover, assume that the quotient of $D$ by $R_{\text{ef}}(D)$ is a simplex in $\mathbb{H}^d$. Then $\partial \Gamma$ verifies the CLP.

Now we assume that the hypothesis of the previous theorem hold and we use the following notations.

**Notation.**
- $T$ is the hyperbolic simplex in $\mathbb{H}^d$ isometric to $D/R_{\text{ef}}(D)$.
- $W_T$ is the hyperbolic reflection group generated by the reflections along the codimension 1 faces of $T$.

We notice that $W$ is a finite index subgroup of $W_T$. Indeed, $W$ is a subgroup of $W_T$ and both are acting discretely on $\mathbb{H}^d$ with finite co-volume. Then $W_T$ acts by polyhedral isometries on an apartment of $\Sigma$. Indeed, a reflection along a face of $T$ either preserves $D$, either is a reflection along a face of $D$. In particular, it preserves the tiling of $\mathbb{H}^d$ by $D$.

Thanks to the results of the previous section and of the constant thickness, we essentially need to study the usual combinatorial modulus in the apartment to prove the theorem.

**Lemma 10.2.** Let $p \geq 1$ and let $A \in A_{p0}(\Sigma)$. Let $\eta$ be a non-constant curve in $\partial A$. There exists a constant $C = C(p, \eta, \epsilon)$ such that
\[
\operatorname{mod}^A_p(F^A_{0}, G^A_k) \leq C \cdot \operatorname{mod}^A_p(U^A_\epsilon(\eta), G^A_k).
\]
Furthermore, when $p$ belongs to a compact subset of $[1, +\infty)$ the constant $C$ may be chosen independent of $p$.

**Proof.** To prove this lemma, we use the fact that $\partial W_T$ is identified with $\partial A$ and that $\partial W_T$ contains no proper parabolic limit set. The group $W_T$ acts geometrically on $A$, so the combinatorial visual metric on $\partial A$ defines a self-similar metric $d_{W_T}$ on $\partial W_T$. Then, a $\kappa$-approximation $\{G^A_k\}_{k \geq 0}$ induces a $\kappa$-approximation on $\partial W_T$ with same modulus.

Now, a proper parabolic limit set $\partial P$ in $\partial A$, is not a parabolic limit set in $\partial W_T$. Indeed, in $W_T$ all the proper parabolic subgroups are finite. In particular, for any non-constant curve $\eta \subset \partial W_T$, the smallest parabolic subset containing $\eta$ is $\partial W_T$. Thus, according to Theorem 6.13 in $W_T$, we get that for every $\epsilon > 0$, there exists $C = C(p, \eta, \epsilon)$ such that
\[
\operatorname{mod}^A_p(F^A_{0}, G^A_k) \leq C \cdot \operatorname{mod}^A_p(U^A_\epsilon(\eta), G^A_k).
\]
\[\square\]
Proof of Theorem 10.1. We check that the hypothesis of Proposition 3.13 are satisfied. To prove that \( \text{Mod}_1(\mathcal{F}_0, G_k) \) we do the same as at the beginning of the proof of Theorem 6.14.

Now we set \( p \geq 1, \eta \) a non-constant curve in \( \partial \Gamma \), and \( \epsilon > 0 \). Thanks to Proposition 3.12 we can assume that \( \eta \in \mathcal{F}_0 \).

Then we can assume that there exists an apartment \( A \) containing \( \eta \). Indeed, let \( \partial P \) be the smallest parabolic limit set containing \( \eta \) and let \( \eta' \) be a non-constant curve in \( \partial A \cap \partial P \) such that \( \partial P \) is the smallest parabolic limit set of \( \partial A \) containing \( \eta' \). Then, as a consequence of Theorem 6.13 the modulus of \( \mathcal{U}^A(\eta) \) and \( \mathcal{U}^A(\eta') \) are essentially the same.

Using the same argument as in the beginning of the proof of Theorem 6.13 we can also assume that \( x_0 \in \text{Ch}(A) \).

Then, because of the constant thickness, the inequality of the Lemma 10.2 becomes

\[
\text{Mod}_p^A(\mathcal{F}^A_0, G^A_k) \leq C \cdot \text{Mod}_p^A(\mathcal{U}^A(\eta), G^A_k).
\]

Eventually, it is enough to apply Theorem 8.12 to the left-hand term and Theorem 8.9 to the right-hand term to complete the proof.

\[\square\]

Corollary 10.3. Let \( q \) be a positive integer \( q \geq 3 \). Let \( \Sigma \) be a building of constant thickness \( q \). Assume that the Coxeter group of \( \Sigma \) is the reflection group of the right-angled dodecahedron in \( \mathbb{H}^3 \) or the reflection group of the right-angled 120-cells in \( \mathbb{H}^4 \), then \( \partial \Sigma \) verifies the CLP.

Remark 10.4. The hyperbolic 120-cell has been described by H.S.M. Coxeter in [Cox73] (see also [Dav08, Appendix B.2.]). It has been used by M.W. Davis to build a compact hyperbolic 4-manifold in [Dav85].
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