Abstract

Color superconductivity of QCD at finite density, temperature and flavor asymmetry is studied within an approximation which the interaction is modeled upon four–fermion interactions. We calculate the thermodynamic potential in the so-called NJL-type model and study the phase structure at finite density, temperature and flavor asymmetry. We find that a mixed phase appears at sufficiently large flavor asymmetry and low temperature. A tricritical point in the $T$-$\mu_B$-$\mu_I$ plane is also found.

1 INTRODUCTION

Quantum Chromodynamics (QCD) describes the interaction of quarks and gluons. Since QCD is asymptotically free, the coupling constant becomes small when either temperature or the chemical potential, namely the Fermi momentum, is large. By Cooper theorem, Fermi surface has instability when an arbitrarily small attractive interaction between quarks presents. For quark–quark scattering, where the quarks are in the antitriplet $\bar{3}$, one gluon exchange interaction is attractive and in the $6$, it is repulsive. Thus we expect that, in the case of two quark flavors, the quark–quark condensate in $\bar{3}$ channel is formed and the color superconductivity, spontaneous breakdown of the color gauge symmetry $SU(3)_C \to SU(2)_C$, occurs [1]. This phenomenon can be understood in the sense that the effective dimensional reduction $(3 + 1) \to (1 + 1)$ occurs with the large chemical potential as well as the case of QED$_4$ in a constant magnetic field [4].

There has been many studies of the color superconductivity in cold and dense QCD using the so-called Nambu–Jona-Lasinio (NJL) type models [3, 4], the instanton liquid model [3] and the Schwinger-Dyson approach to QCD in weak coupling limit [3], in other words, in $\mu \to \infty$ limit. The first two models have shown that the superconducting gap could be large as 100 MeV. On the other hand, due to the exchange of the magnetic gluons, the gap from QCD in weak coupling limit could be large as compared to that obtained under the condition which the magnetic gluons acquire static screening mass. If we extrapolate the gap down to the intermediate region of the chemical potential, it increases up to 100 MeV. However, as temperature grows, the color superconducting
gap is reduced, then, the phase undergoes phase transition. The order of the transition seems to be second order in either case of with or without gauge fields. However, with gauge fields, there is a possibility of first order or a smooth crossover.

We consider the color superconductivity under the situation where a flavor asymmetry takes place. The color superconductivity at finite flavor asymmetry has been studied in Ref. [7, 8]. Especially, in Ref. [8], it has been argued that a mixed phase of the normal and the superconducting phase exists with a sufficiently large flavor asymmetry. In this paper, we study at finite temperature and discuss what extent does the mixed phase persist. Such an investigation has importance in studies of neutron stars and, of course, high-energy heavy-ion collisions at Alternating Gradient Synchrotron (AGS) and CERN Super Proton Synchrotron (SPS).

The study of the color superconductivity in QCD is nontrivial and, therefore, we choose a tractable model to describe the quark–quark condensate. The most familiar of such models is known as the NJL (-type) model. Particularly, in the leading order of the $1/N_c$ expansion, the resultant gap function is independent of momentum and it makes our analysis simple one. Moreover, the structure of the color superconducting phase depends on the number of the quark flavors. At extremely high density, where we can neglect the mass of $s$ quark, the color-flavor locked phase appears [9]. However we expect, at relatively low density, there is a window of two-flavor color superconducting phase. Such being the case, we use the two–flavor, three–color NJL-type model.

This paper is organized as follows. In Sec. II, we present our model and derive the thermodynamic potential at finite density, temperature and flavor asymmetry. Then, in Sec. III, we present numerical results for the thermodynamic potential. Section IV is devoted to the summary and discussion. Our unit are $\hbar = k_B = c = 1$.

## 2 MODEL

Our aim is to explore the phase diagram of two–flavor QCD. The $SU(2)_L \times SU(2)_R$ chially symmetric Lagrangian [11]

\[
\mathcal{L}_{NJL} = \bar{\psi}_i \gamma \cdot \partial \psi + \mathcal{L}_{int},
\]

\[
\mathcal{L}_{int} = G_1 \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \vec{\tau} \psi)^2 \right],
\]

is our starting point. The quark field $\psi$ is assigned to the fundamental representation of the color $SU(N_c)$ group and carries the flavor index $i$ and the color index $a$ as $\psi^i_a$. The Pauli matrices $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ act in the flavor space. In Eq. (2), we include two–fermion interactions, namely, the scalar isosinglet ($\sigma$) and pseudo-scalar isotriplet ($\pi$). However, we do not abandon the possibility of adding other four–fermion interactions; e.g., the scalar isotriplet ($a_0$) and the pseudo-scalar isosinglet ($\eta'$).

Applying the Fierz transformations to these color singlet interactions, one can transform them into color $3$ and $6$ channels. In this paper, we consider the following scalar isoscalar color $3$ diquark condensation which breaks $SU(3)_C$ to $SU(2)_C$; i.e., taking a third direction as a preferred one in color space,

\[
\Delta = \Delta_3 = \Delta^*_3 \sim \epsilon^{ij} \epsilon_{akl} \langle (\psi^i_a)^T C \gamma_5 \psi^j_b \rangle,
\]
where $C$ is a charge conjugation matrix, defined by $C\gamma_\mu C^{-1} = -\gamma^T_\mu$ and $C = -C^{-1} = -C^T$ and $\Delta$ is independent of momentum in our approximation (mean field approximation or, equivalently, leading order in the $1/N_c$ expansion). The interaction Lagrangian which leads to the diquark condensation is written as

$$\mathcal{L}_{\text{diag}} = G_2 \left[ -\left( \bar{\psi}\gamma_5 \tau^2 \lambda^2 \psi C \right) \left( \bar{\psi}^C \gamma_5 \tau^2 \lambda^2 \psi \right) + \ldots \right],$$

(4)

where $\tau^2$ is the antisymmetric Pauli matrix, $\lambda^2$ is the antisymmetric (color $\bar{3}$) color generator, and the ellipsis denotes irrelevant diquark channels. Since the NJL model is not renormalizable, one cannot define the model until a regularization scheme has been specified. In this paper, we calculate the thermodynamic potential with a 3D sharp cutoff in momentum space. For a mesonic sector, we use a set of parameters: a coupling constant $G_1 = 5.01$ GeV$^{-2}$ and a 3D ultraviolet cutoff $\Lambda = 0.65$ GeV $[4]$. These parameters reproduce the value of $f_\pi = 93$ MeV and the quark condensate $\langle \bar{q}q \rangle = (-225\text{MeV})^{-3}$. The coupling constant $G_2$, mediating an attractive four–fermion diquark interaction, is related to $G_1$ through the Fierz transformations. In other words, the strength of $G_2$ is fixed by the Fierz transformations. The Lagrangian $[2]$ gives $G_2 = G_1/4$. However, as mentioned before, one can change the ratio $G_2/G_1$ by adding other color singlet channel to Eq. $[2]$. Therefore we take $G_2$ independent on $G_1$ and study the $G_2$ dependence of $\Delta$ and a phase structure.

We start our study with an investigation of the thermodynamic potential as a function of $\Delta$ at nonzero temperature, “averaged” chemical potential and “isospin” chemical potential. The averaged chemical potential $\mu_B$ and isospin chemical potential $\mu_I$ are given as follows

$$\mu_B = \frac{\mu_u + \mu_d}{2}, \quad \mu_I = \frac{\mu_u - \mu_d}{2},$$

(5)

where $\mu_u(d)$ is a chemical potential of the $u(d)$ quark. In order to evaluate the thermodynamic potential, one can use auxiliary field method with the saddle point approximation or the Cornwall-Jackiw-Tomboulis effective action $[11]$ up to two-loop level with the Nambu-Gorkov formalism. These two methods are correspond to the evaluation at the leading order of the $1/N_c$ expansion and consistent with each other. The thermodynamic potential $V(\Delta; \mu_B, \mu_I)$ in the leading order of the $1/N_c$ expansion is obtained as follows

$$V(\Delta; \mu_B, \mu_I) = \frac{\Delta^2}{4G_2} + i \int \frac{d^4 q}{(2\pi)^4} \left\{ \ln \left[ \left( (\omega + \mu_I)^2 - q^2 - \mu_B^2 \right)^2 - 4\mu_B^2 q^2 \right] \right\}$$

$$+ \ln \left[ \left( (\omega - \mu_I)^2 - q^2 - \mu_B^2 \right)^2 - 4\mu_B^2 q^2 \right]$$

$$+ 2 \ln \left[ \left( (\omega + \mu_I)^2 - q^2 - \mu_B^2 - \Delta^2 \right)^2 - 4\mu_B^2 q^2 \right]$$

$$+ 2 \ln \left[ \left( (\omega - \mu_I)^2 - q^2 - \mu_B^2 - \Delta^2 \right)^2 - 4\mu_B^2 q^2 \right] \right\}.$$  (6)

The finite temperature thermodynamic potential $V(\Delta; T, \mu_B, \mu_I)$ is obtained by using the formula:

$$\int \frac{d^4 q}{(2\pi)^4} F(q) \rightarrow \frac{-1}{4\pi} \int \frac{d^3 q}{(2\pi)^3} \int d\omega \tanh \frac{\beta \omega}{2} F(\omega, q),$$

(7)
where $\beta = 1/T$ and the $\omega$-integration is to be understood that it is round an anti-clockwise contour including the pole of $F(\omega, q)$. Thus, we obtain the thermodynamic potential as follows

$$V(\Delta; T, \mu_B, \mu_I) = \frac{\Delta^2}{4G_2} - 2 \int_{-\Lambda}^{\Lambda} \frac{q^2 dq}{2\pi^2} \left\{ \epsilon_-(q) + \epsilon_+(q) \right\}$$

$$+ T \ln (1 + \exp [-\beta(\epsilon_-(q) - \mu_I)]) + T \ln (1 + \exp [-\beta(\epsilon_-(q) + \mu_I)])$$

$$+ T \ln (1 + \exp [-\beta(\epsilon_+(q) - \mu_I)]) + T \ln (1 + \exp [-\beta(\epsilon_+(q) + \mu_I)])$$

$$- 4 \int_{-\Lambda}^{\Lambda} \frac{q^2 dq}{2\pi^2} \left\{ E_-(q) + E_+(q) \right\}$$

$$+ T \ln (1 + \exp [-\beta(E_-(q) - \mu_I)]) + T \ln (1 + \exp [-\beta(E_-(q) + \mu_I)])$$

$$+ T \ln (1 + \exp [-\beta(E_+(q) - \mu_I)]) + T \ln (1 + \exp [-\beta(E_+(q) + \mu_I)])$$

where

$$\epsilon_\pm(q) = |q| \pm \mu_B \quad (9)$$

are the gapless (color 3) quasi-particle energies relative to the (averaged) Fermi surface and

$$E_\pm(q) = (|q| \pm \mu_B) \sqrt{1 + \frac{\Delta^2}{(|q| \pm \mu_B)^2}} \quad (10)$$

are the gapful (color 1, 2) quasi-particle energies and 3D cutoff $\Lambda$ is introduced to regularize the divergent integrals.

The extremum condition for $V(\Delta; T, \mu_B, \mu_I)$ with respect to $\Delta$ lead to the Schwinger-Dyson equation (SDE) for $\Delta$ at nonzero temperature, density, and flavor asymmetry. The Cooper pairs condensate $\langle \psi \psi \rangle$ is related $\Delta_0$ the solution of the SDE as

$$\langle \psi \psi \rangle = \Delta_0/(2G_2). \quad (11)$$

The “baryon number density” and the “isospin density” are obtained by

$$n_B \equiv n_u + n_d = -\frac{\partial}{\partial \mu_B} V(\Delta; T, \mu_B, \mu_I)$$

$$n_I \equiv n_u - n_d = -\frac{\partial}{\partial \mu_I} V(\Delta; T, \mu_B, \mu_I). \quad (12)$$

3 COLOR SUPERCONDUCTIVITY WITH ASYMMETRIC MATTER

Figure 1 shows the $G_2$ and $\mu_B$ dependence of $\Delta$ at zero temperature and zero flavor asymmetry. The value $G_2 = 3G_1/4$ was taken by Berges and Rajagopal [3], where $G_1$ is the coupling constant in $\sigma$ (scalar isosinglet) channel. Note that we use the $G_1$ different from Berges and Rajagopal and 3D sharp cutoff instead of introducing a smooth form.
factor $F(q) = \Lambda^2/(q^2 + \Lambda^2)$. Then, the behavior of $\Delta$ slightly different from theirs. The value $G_2 = G_1/2$ and $G_2 = G_1/4$ are introduced by the instanton vertex of two–flavor QCD and the $SU(2)_L \times SU(2)_R$ symmetric NJL model, respectively. As we can see, $\Delta$ is quite sensitive to $G_2$. For $G_2 = 3G_1/4$, $\Delta$ could be as large as 100 MeV. However, it is about 50 MeV for $G_2 = G_1/2$ and negligible small for $G_2 = G_1/4$. In the following, we use $G_2 = 3G_1/4$ to get sizeable gap.

Figure 2 shows the zero temperature and $\mu_B = 400$ MeV thermodynamic potential as a function of $\Delta$ for several $\mu_I$. For $\Delta$ larger than $\mu_I$, all curves are coincident with each other. This is because, for $\Delta > \mu_I$, the flavor asymmetry does not affect the thermodynamic potential. As $\mu_I$ grows, the potential in the trivial state ($\Delta = 0$) decreases. It has two degenerate minima and the superconducting phase and the normal phase coexist at some value of $\mu_I$. For more large values of $\mu_I$, the normal phase is favored. As a result, $\Delta$ behaves constant up to a certain critical value of $\mu_I$ and, then, jumps to zero. We have a strong first order phase transition. We can see the same behavior of the thermodynamic potential and the order parameter in chiral symmetry restoration at zero temperature and finite baryon density.

In Fig. 3, we present the phase diagram as a function of $\mu_B$ and $\mu_I$ at zero temperature. Along the horizontal axis, color superconducting phase exists for arbitrary nonzero $\mu_B$ because the BCS instability exists for arbitrary small coupling strength. On the other hand, as mentioned before, we know that it undergoes a phase transition of first order with increasing flavor asymmetry. Then we get the phase boundary such as shown in Fig. 3. This boundary reflect the $\mu_B$ dependence of $\Delta$ at $\mu_I = 0$ (see Fig. 1). It is also interesting to see the phase diagram as a function of $\mu_B$ and $\mu_I$. Figure 4 shows the phase diagram in the $n_B-n_I$ plane. For the flavor asymmetry $n_I/n_B$ larger than about 15% the system remains in the mixed phase for arbitrarily high densities. This nature is almost the same as shown in Ref. [8]. For the flavor asymmetry larger than about 40% the phase transition is complete and the system undergoes a phase transition into the normal phase. This point is differ from the results of Ref. [8] and the difference arises from the choice of the parameters.

We now turn to physics at finite temperature. Figure 5 shows the $\mu_I$ dependence of $V$ for $T = 20$ MeV and $\mu_B = 400$ MeV. At relatively low temperature, a first order phase transition occurs. On the other hand, at relatively high temperature, the phase transition changes to second order. From the behavior of the thermodynamic potential, the existence of a tricritical point in the $T$-$\mu_I$ plane is expected. In Fig. 7, we present the typical phase diagram as a function of $T$ and $\mu_I$ for $\mu_B$ fixed to 400/500 MeV. It is known that, in four–fermion interaction model which neglects the gauge fields, we have a second order phase transition for $\mu_B \neq 0$ and $\mu_I = 0$ with increasing temperature (BCS theory predicts the critical temperature $T_c$ is given by $T_c \sim 0.57\Delta_{T=0}$). On the other hand, we have a first order transition for $T = 0$ with increasing flavor asymmetry $\mu_I$. Therefore the existence of a tricritical point in the $T$-$\mu_I$ plane, where the second order phase transition becomes that of the first order, is expected. In fact, we found

\begin{align*}
T &\sim 35 \text{ MeV, } \mu_I \sim 65 \text{ MeV for } \mu_B = 400 \text{ MeV,} \\
T &\sim 42 \text{ MeV, } \mu_I \sim 69 \text{ MeV for } \mu_B = 500 \text{ MeV,}
\end{align*}
from the condition

$$a_2(T, \mu_I) = a_4(T, \mu_I) = 0, \quad (13)$$

where we expand the thermodynamic potential, in the vicinity of the tricritical point, as a power series in $\Delta$:

$$V(\Delta; T, \mu_I) = V(0, T, \mu_I) + a_2(T, \mu_I)\Delta^2 + a_4(T, \mu_I)\Delta^4 + \cdots \quad (14)$$

Therefore, there is a possibility of a first order phase transition as temperature grows, with sufficiently large flavor asymmetry.

4 SUMMARY AND DISCUSSION

In summary, we studied the phase structure of the color superconductivity at finite density, temperature and flavor asymmetry by investigating the thermodynamic potential in the NJL model.

Since specifying the ratio $G_2/G_1$ by the Fierz transformations in the mean-field approximation is not unique, we examined the $G_2$ dependence of $\Delta$ and found that $\Delta$ is quite sensitive to $G_2$.

First, let us see $\mu_I$ dependence of $\Delta$. At zero temperature, as $\mu_I$ grows, the superconducting gap $\Delta$ remains constant until $\mu_I$ is increased to its critical value and, then, vanishes. This behavior seems to be the same as obtained by Bedaque [8]. Following Alford et al., let us consider a concrete example; a gold nucleus at a baryon density eight times in nuclear matter. This matter, the number of $d$ quark exceed that of $u$ quark, corresponds to $\mu_B = 535$ MeV and $\mu_I = 12$ MeV (up to sign). They have found that the gap is a little (about 18%) decreased. Our resultant gap does not changed in this condition. The distinction maybe arise from the difference in the coupling constant and the regularization scheme.

Second, concerning the phase structure, we found the similar phase structure in the $n_B-n_I/n_B$ plane to that reported by Bedaque [8]. Moreover, we found that the structure of the color superconducting phase is remarkably rich:

1. mixed phase appears at sufficiently large flavor asymmetry with temperature less than 40 MeV,
2. tricritical point exists in the $T-\mu_B-\mu_I$ phase diagram.

The most favorable condition for the color superconductivity, cold and dense quark matter, are at the core of neutron stars. However, the values of $T$, $\mu_B$ and $\mu_I$ accomplished in heavy-ion collisions at, for example, AGS and SPS, maybe reach into the mixed phase, since it lies up until $T \simeq 40$ MeV (see Fig. 7). Then, if the charged bubbles is formed, it may lead to observable effects, e.g., an obvious change of the $\pi^-/\pi^+$ ratio.

Our results strongly depend on the regularization scheme, although qualitative features do not changes. In addition, it seems to be necessary to take into account next-to-leading order in the $1/N_c$ expansion, because the ratio of $G_2/G_1$ has a order $O(1/N_c)$. Also, the dynamical chiral symmetry breaking and the pion condensation should be included in further study. The pion condensed phase has been confirmed by the Monte Carlo simulation of two–color QCD [12]. We also plan to study the effective potential
within QCD in weak coupling limit and the three flavor case where the color flavor locked phase takes place.

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Figure 1: Superconducting gap $\Delta$ as a function of $\mu_B$. The curves correspond to $G_2/G_1 = 3/4, 1/2, 1/4$ (top to bottom).

Figure 2: The zero temperature thermodynamic potential as a function of $\Delta$ for several $\mu_I$. The curves correspond to $\mu_I = 0, 50, 77.3, 100$ MeV. The value $\mu_I = 77.3$ MeV corresponds to a critical isospin chemical potential. Note that we neglect the contribution of color 3 quark to $V$ which is independent of $\Delta$. The same neglect will be done in Fig. 5 and 6.
Figure 3: Phase diagram in the $\mu_B$-$\mu_I$ plane. The (dotted) line indicates the first order phase transition.

Figure 4: Phase diagram in the $n_B$-$n_I/n_B$ plane (in unit of $10^6\text{MeV}^3$).
Figure 5: The thermodynamic potential for $T = 20$ MeV, $\mu_B = 400$ MeV and several $\mu_I$. The curves correspond to $\mu_I = 0, 71, 80$ MeV (top to bottom).

Figure 6: The thermodynamic potential for $T = 50$ MeV, $\mu_B = 400$ MeV and several $\mu_I$. The curves correspond to $\mu_I = 0, 49, 80$ MeV (top to bottom).
Figure 7: Phase boundaries in the $T$-$\mu_I$ plane for $\mu_B = 500$ MeV (upper line) and 400 MeV (lower line). The solid lines describe the transition of second order and the dashed line describe that of first order. Points indicate tricritical points.