A numerical method for solution of ordinary differential equations of fractional order

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Abstract. In this paper we propose an algorithm for the numerical solution of arbitrary differential equations of fractional order. The algorithm is obtained by using the following decomposition of the differential equation into a system of differential equation of integer order connected with inverse forms of Abel-integral equations. The algorithm is used for solution of the linear and non-linear equations.

1 Introduction

In opposite to differential equations of integer order, in which derivatives depend only on the local behaviour of the function, fractional differential equations accumulate the whole information of the function in a weighted form. This is so called memory effect and has many applications in physics [10], chemistry [5], engineering [3], etc. For that reason we need a method for solving such equations which will be effective, easy-to-use and applied for the equations in general form. However, known methods used for solution of the equations have more disadvantages. Analytical methods, described in detail in [7, 12], do not work in the case of arbitrary real order. Another analytical method [9], which uses the multivariate Mittag-Leffler function and generalizes the previous results, can be used only for linear type of equations. On the other hand, for specific differential equations with oscillating and periodic solution there are some specific numerical methods [6, 13, 14]. Other numerical methods [2, 4] allow solution of the equations of arbitrary real order but they work properly only for relatively simple form of fractional equations.

Let us consider an initial value problem for the fractional differential equation

\[ a_1 \tau D_t^{\alpha_1} y(t) + a_2 \tau D_t^{\alpha_2} y(t) + \ldots + a_n \tau D_t^{\alpha_n} y(t) + a_{n+1} y(t) = f(t) \]  

connected with initial conditions

\[ y^{(k)}(\tau) = b_k, \]

where \( 0 < \tau \leq T < \infty, a_i \in \mathbb{R}, a_1 \neq 0, \alpha_i \in \mathbb{R}, i = 1, \ldots, n, m_i - 1 \leq \alpha_i < m_i, m_i \in \mathbb{N}, \alpha_l > \alpha_{l+1} \) for \( l = 1, \ldots, n - 1, b_k \in \mathbb{R}, k = 0, \ldots, m_1 - 1, f(t) \) is a given
function defined on the interval \([0, T]\), \(y(t)\) is the unknown function which is the solution of eqn. (1).

The fractional derivative operator \(D^{\alpha}\) is defined in the Riemann-Liouville sense [12]. We also have other definitions of the operator like Caputo [3], Grünwald-Letnikov [3], and Weyl-Marchaud [12]. Regarding to the Riemann-Liouville operator we can define it as the left-side operator

\[
(a D^{\alpha}_{t} y)(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^{m}}{dt^{m}} \int_{a}^{t} \frac{y(\xi)d\xi}{(t-\xi)^{\alpha-m+1}},
\]

where \(a < t \leq T < \infty\), \(m-1 < \alpha \leq m\), \(m \in \mathbb{N}\). In our study, we also use integral operators defined in the Riemann-Liouville sense [12]. When \(y(t) \in L_{1}(a, b)\) and \(\alpha > 0\) then left-side integral operator is defined as

\[
(a I^{\alpha}_{t} y)(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \frac{y(\xi)d\xi}{(t-\xi)^{1-\alpha}}, \quad t > a.
\]

More informations concerning the operators properties one can find in literature [3,12]. Applying definitions (3) and (4) we have the following property

\[
(a D^{\alpha}_{t} y)(t) = D^{m}(a I^{m-\alpha}_{t} y)(t),
\]

where \(D^{m}\) represents typical derivative operator of integer order \(m\). According to [3] in the fractional integrals the composition rule occurs

\[
a I^{\alpha}_{t}(a I^{\beta}_{t} y(t)) = a I^{\alpha+\beta}_{t} y(t) = a I^{\beta+\alpha}_{t} y(t) = a I^{\beta}_{t}(a I^{\alpha}_{t} y(t)).
\]

In the general case the Riemann-Liouville fractional derivatives do not commute

\[
(a D^{\alpha}_{t} y)(a D^{\beta}_{t} y))(t) \neq (a D^{\alpha+\beta}_{t})(t) \neq (a D^{\beta}_{t} a D^{\alpha}_{t} y))(t).
\]

Extending our considerations, the integer operator commutes with the fractional operator

\[
(D^{m}_{t} (a D^{\alpha}_{t} y))(t) = (a D^{m+\alpha}_{t})(t),
\]

but the opposite property is impossible. The mixed operators (derivatives and integrals) commute only in the following way

\[
(a D^{\alpha}_{t}(a D^{\beta}_{t} y))(t) = (a D^{\alpha-\beta}_{t})(t),
\]

but they do not commute in the opposite way.

We turn our attention to the composition rule in two following ways. First of all we found in literature [10,11] the fact, that authors neglect the general property of fractional derivatives given by eqn. (7). Regarding to solution of fractional differential equations we will apply above properties in the next sections.
2 Mathematical background

In this section we concentrate on describing the method which is decomposition of fractional differential equation into a system of one ordinary differential equation and inverse forms of Abel-integral equations. We focus on decomposition ways of arbitrary fractional differential equation (1) with initial conditions (2). In our considerations we classify the fractional differential equations in dependence on the fractional derivative occurrence as:

- one-term equations, in which $D^\alpha$ occurs only one,
- multi-term equations, in which $D^\alpha$ occurs much more times.

Corollary 1. An arbitrary one-term equation

$$a_1 \tau D^\alpha t \frac{d^m y(t)}{d t^m} + a_2 y(t) = f(t), \quad 0 \leq m - 1 \leq \alpha < m, \quad m \in \mathbb{N}, \quad (10)$$

in which $a_1 \neq 0$ and initial conditions $y^{(k)}(\tau) = b_k, \quad (k = 0, \ldots, m - 1)$ should be taken into account, can decomposed into the following system

$$\begin{align*}
D^m t z(t) &= \frac{a_2}{a_1} y(t) + \frac{1}{a_1} f(t) \\
y^{(0)}(t) &= \sum_{k=0}^{m-1} b_k \frac{(t-\tau)^k}{k!} + \tau D^m t - \alpha z(t) \\
y^{(1)}(t) &= \sum_{k=1}^{m-1} b_k \frac{(t-\tau)^k}{k!} + \tau D^m t - \alpha + 1 z(t) \\
&\quad \vdots \\
y^{(m-1)}(t) &= b_{m-1} \frac{(t-\tau)^{m-1}}{(m-1)!} + \tau D^{2m} t - \alpha z(t) 
\end{align*} \quad (11)$$

Proof. We use the property (5) and we introduce a new variable $z(t)$ which we call a temporary function. For such introduction we have

$$\tau D^\alpha t y(t) = D^m t z(t) \quad \text{and} \quad z(t) = (\tau I_t^m - \alpha y(t)).$$

The new variable represents the Abel integral equation which solution, found in literature [12], is the left inverse operator

$$y(t) = (\tau D^\alpha t z(t)).$$

In dependence on the integer order $m$ we introduce initial conditions $b_k$ which are multiplied by a term $\frac{(t-\tau)^k}{k!}$. The term issues from a kernel of the left-side Riemman-Liouville operator (3). \hfill \Box

In the system of eqns. (11), the first equation is the differential equation of $m$ integer order and the next equations represent inverse forms of Abel-integral equations. We also found in literature [2,4] different or similar approaches for solution of the eqn. (10).

In mathematical point of view, more interesting is the second class called multi-term equations. Let us consider the eqn. (1) with initial conditions (2). As in previous class we use the rule (3). Following the previous corollary we apply
a system of new variables $z_i(t) = (\tau I_t^{m_i - \alpha_i}y)(t)$ and we decompose eqn. (1) into a system of the following equations

$$
\begin{align*}
\begin{cases}
    a_1 D_t^{m_1} z_1(t) + a_2 D_t^{m_2} z_2(t) + \ldots + a_n D_t^{m_n} z_n(t) + \\
    + a_{n+1} y(t) = f(t) \\
    z_1(t) = (\tau I_t^{m_1 - \alpha_1}y)(t) \\
    z_2(t) = (\tau I_t^{m_2 - \alpha_2}y)(t) \\
    \ldots \\
    z_n(t) = (\tau I_t^{m_n - \alpha_n}y)(t)
\end{cases}
\end{align*}
$$

(12)

Such system is not a final form and requires additional transformations. In multiterm class we distinguish two sub-classes depending on $\alpha_i$ or follows the decomposition on $m_i$:

- independent subclass, in which $m_i$ are different derivatives orders of integer type $m_i \neq m_{i+1}$ for $i = 1, \ldots, n - 1$,
- dependent subclass, in which one can find some dependence between the derivatives order of integer type, e.g. $m_1 = m_2 = m_3$ or $m_1 = m_4$, etc.

**Proposition 1.** Taking into consideration the independent subclass it is possible to formulate the unique solution of eqn. (12) in the following form

$$
\begin{align*}
\begin{cases}
    D_t^{m_1} z_1(t) = -\frac{1}{\alpha_1} (a_2 D_t^{m_2} z_2(t) + \ldots + a_n D_t^{m_n} z_n(t)) + \\
    + \frac{1}{\alpha_1} (-a_{n+1} y(t) + f(t)) \\
    z_2(t) = a D_t^{m_1 - m_2 - \alpha_1 + \alpha_2} z_1(t) \\
    z_3(t) = a D_t^{m_1 - m_3 - \alpha_1 + \alpha_3} z_1(t) \\
    \ldots \\
    z_n(t) = a D_t^{m_1 - m_n - \alpha_1 + \alpha_n} z_1(t) \\
    y^{(0)}(t) = \sum_{k=0}^{m_1 - 1} b_k \frac{(t-\tau)^k}{k!} + a D_t^{m_1 - \alpha_1} z_1(t) \\
    y^{(1)}(t) = \sum_{k=1}^{m_1 - 1} b_k \frac{(t-\tau)^k}{k!} + a D_t^{m_1 - \alpha_1 + 1} z_1(t) \\
    \ldots \\
    y^{(m_1 - 1)}(t) = b_{m_1 - 1} \frac{(t-\tau)^{m_1 - 1}}{(m_1 - 1)!} + a D_t^{m_1 - \alpha_1 - 1} z_1(t)
\end{cases}
\end{align*}
$$

(13)

**Proof.** In this proof we try to show a dependence of temporary functions $z_i(t)$ ($i = 2, \ldots, n$) on the temporary function $z_1(t)$. We assume, that initial conditions $b_k = 0$. In such way we can easily construct the proof. We observe that the Abel integral operators in (12) have the order $0 < m_i - \alpha_i < 1$. We consider the inverse form of such operators and for two operators we have

$$
\tau D_t^{\beta} z_2(t) = \tau D_t^{\beta} z_1(t),
$$

(14)

where $0 < \beta < 1$ and $0 < \gamma < 1$ respectively. Using the property (3) we obtain

$$
D_t^{\beta}(\tau I_t^{1-\beta} z_2(t)) = D_t^{\beta}(\tau I_t^{1-\gamma} z_1(t)).
$$

Neglecting the operator $D_t^{\beta}$ we can obtain the following formula

$$
\tau I_t^{1-\beta} z_2(t) = \tau I_t^{1-\gamma} z_1(t) + C_{1,2},
$$

(15)
where $C_{1,2}$ is an arbitrary constant. Additionally, we multiply the operator $\tau I_t^\beta$ for eqn. (15) and applying the property (8) we obtain
\[
\tau I_t^\beta z_2(t) = \tau I_t^{1-\gamma+\beta} z_1(t) + \tau I_t^{\beta} C_{1,2}.
\]  
(16)
Differentiating both sides by the operator $D_t^\beta$ and applying the property (8) (see literature [7,9]) we obtain the final dependence
\[
z_2(t) = \tau D_t^{\gamma-\beta} z_1(t) + C_{1,2} \cdot \frac{\beta}{\Gamma(1+\beta) \cdot t^{1+\beta}}.
\]  
(17)
In general way we state that $z_i(t) \ (i = 2, \ldots, n)$ depends on $z_1(t)$ together with additional term represented by the constants $C_{1,i}$. \hfill \square

**Proposition 2.** All constants $C_{1,i} = 0 \ (i = 2, \ldots, n)$ in eqn. (12) are equaled to zero for $\tau = 0$.

**Proof.** Taking into consideration the formula (17), we can calculate the constants
\[
C_{1,i} = \frac{\Gamma(1+\beta) \cdot t^{1+\beta}}{\beta} \cdot \left( z_i(t) - \tau D_t^{\gamma-\beta} z_1(t) \right).
\]
Additionally, putting the initial condition $t = \tau = 0$ we obtain all $C_{1,i} = 0$. \hfill \square

**Remark 1.** According to previous considerations, the initial conditions for temporary functions $z_i(t)$ satisfy the following dependence
\[
z_i(\tau = 0) = 0, \ i = 1, \ldots, n.
\]  
(18)
In practical point of view given by formula (18), we can solve ordinary differential equation of integer type with variable coefficients under zeros initial conditions of temporary functions $z_i(t)$.

Let us consider the next subclass of a fractional differential equation. We assume, that some integer orders $m_i$ are the same. This may happen in the case when some $\alpha_i$ and $\alpha_{i+1}$, $(i = 1, \ldots, n-1)$ belong to the same integer values $m_i = m_{i+1}$. We introduce a parameter $r$ which denotes a number of temporary functions having the same derivative operator $D_t^{r}$ of integer type. If $m_1 = m_2 = \ldots = m_r$ for different temporary functions $z_i(t)$, $(i = 1, \ldots, r)$ then it can express the functions $z_i(t)$, $(i = 2, \ldots, r)$ through the $z_1(t)$ function. It may observe a relationship in eqn. (13) for $z_2(t)$ and $z_1(t)$ functions respectively $z_2(t) = \tau D_t^{m_1-m_2-\alpha_i+\alpha_1} z_1(t)$. We assume an equalities of the integer orders $m_1 = m_2$. The previous relationship becomes $z_2(t) = \tau D_t^{-\alpha_i+\alpha_1} z_1(t)$. Following the property $(\tau D_t^{-\alpha} y)(t) = (\tau D_t^\alpha y)(t)$ we have $z_i(t) = \tau I_t^{\alpha_i-\alpha} z_1(t), \ (i = 2, \ldots, r)$. In the ordinary differential equation presented in the system (13) we change all $z_i(t), \ (i = 2, \ldots, r)$ which depends on $z_1(t)$. Moreover, we introduce a new temporary function in the following form
\[
w(t) = z_1(t) + \frac{a_2}{a_1} \tau I_t^{\alpha_i-\alpha_2} z_1(t) + \ldots + \frac{a_r}{a_1} \tau I_t^{\alpha_i-\alpha_r} z_1(t).
\]  
(19)
We need to find an inverse form of eqn. (13) as

\[ z_1(t) = (1 + \frac{a_2}{a_1} I_t^{\alpha_2} + \ldots + \frac{a_r}{a_1} I_t^{\alpha_r})^{-1} w(t), \]  

where \((1 + \frac{a_2}{a_1} I_t^{\alpha_2} + \ldots + \frac{a_r}{a_1} I_t^{\alpha_r})^{-1}\) denotes the left inverse operator to the operator \((1 + \frac{a_2}{a_1} I_t^{\alpha_2} + \ldots + \frac{a_r}{a_1} I_t^{\alpha_r})\). We can apply the known Laplace transform for eqn. (19) that to find the left inverse operator in expression (20). The formula (20) is not suitable for practical purposes and computations.

**Proposition 3.** Connecting expressions (14) and (15) to (20), the solution of the second subclass of the fractional differential equation is presented as follows:

\[
\begin{align*}
D_t^{m_1} w(t) &= \frac{1}{\alpha_1} (a_{r+1} D_t^{m_{r+1}} z_{r+1}(t) + \ldots +
+ a_n D_t^{m_n} z_n(t) + a_{n+1} y(t)) + \frac{1}{\alpha_1} f(t),
\end{align*}
\]

\[
\begin{align*}
z_1(t) &= (1 + \frac{a_2}{a_1} I_t^{\alpha_2} + \ldots + \frac{a_r}{a_1} I_t^{\alpha_r})^{-1} w(t),

z_{r+1}(t) &= a D_t^{m_1-m_{r+1}-\alpha_1+\alpha_{r+1}} z_1(t)
\]
\]

\[
\begin{align*}
\vdots
\end{align*}
\]

\[
\begin{align*}
z_n(t) &= a D_t^{m_1-m_n-\alpha_1+\alpha_n} z_1(t)

y^{(0)}(t) &= \sum_{k=0}^{m_1-1} b_k \frac{(t-r)^k}{k!} + a D_t^{m_1-\alpha_1} z_1(t)

y^{(1)}(t) &= \sum_{k=1}^{m_1-1} b_k \frac{(t-r)^k}{k!} + a D_t^{m_1-\alpha_1+1} z_1(t)
\]
\]

\[
\begin{align*}
\vdots
\end{align*}
\]

\[
\begin{align*}
y^{(m_1-1)}(t) &= b_{m_1-1} \frac{(t-r)^{m_1-1}}{(m_1-1)!} + a D_t^{2m_1-\alpha_1-1} z_1(t)
\end{align*}
\]

We do not need to proof the above proposition because it was done in the previous subclass of the multi-term class.

### 3 Numerical treatment and examples of calculations

In this section we propose an explicit numerical scheme. For numerical solution of the differential equation of integer order we apply one-step Euler’s method [5]. There is no limits to extend above approach to the Runge-Kutta method. According to results presented by Oldham and Spanier [5] we use numerical scheme for integral operator as

\[
0 D_t^\alpha z_i = \frac{h^\alpha}{\Gamma(1+\alpha)} \left[ z_0 (i^{\alpha} - (i-1)^{\alpha}) + z_i + \sum_{j=1}^{i-1} z_{i-j} ((j+1)^{\alpha} - (j-1)^{\alpha}) \right],
\]

which is valid for arbitrary \(\alpha > 0\). We also use an algorithm given by Oldham and Spanier [5] for numerical differentiation. The algorithm depends on \(\alpha\) range. For \(0 \leq \alpha < 1\) we have

\[
0 D_t^\alpha z_i = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \left[ (1-\alpha) z_i + \frac{i-1}{\Gamma(\alpha)} \sum_{j=0}^{i-1} (z_{i-j} - z_{i-(j+1)}) ((j+1)^{1-\alpha} - j^{1-\alpha}) \right].
\]
In literature one may find the algorithm for highest values.

For limited practical solution of eqn. (20) we found in literature Babenko’s method. In general form of eqn. (20) we cannot consider the left inverse operator. For that, we found only a simplified formula limited to the two fractional orders $\alpha_1, \alpha_2$, which binomial expansion we can show as

$$z_{11} = \sum_{j=0}^{\infty} (-1)^j \left(\frac{\alpha_2}{\alpha_1}\right)^j \cdot I^{(\alpha_1-\alpha_2)}(\alpha_1) w_i.$$  

(24)

For some comparison we select the most popular fractional differential equation in literature

$$a \cdot 0D_t^2 y(t) + b \cdot 0D_t^{1.5} y(t) + c \cdot y(t) = f(t)$$  

(25)

called the Bagley-Torvik equation. Fig. a presents the qualitative comparison of the numerical solution generated by our method with the analytical solution. We assume: $a = 1, b = 0.5, c = 0.5$, the function $f(t) = \begin{cases} 8 & \text{for } 0 \leq t < 1 \\ 0 & \text{for } 1 < t < \infty \end{cases}$ and initial conditions $y(0) = 0, y'(0) = 0$ respectively. The analytical solution one may find in Podlubny’s work. We also found a simple way of solution created in Podlubny’s work. Fig. b certifies that our numerical results behave good in comparison to the analytical solution. The great advantage would be the apply of our approach to the non-linear form of fractional differential equation. Let us consider the non-linear form of the fractional differential equation

$$0D_t^2 y(t) + 0.5 \cdot 0D_t^{1.5} y(t) + 0.5 \cdot y^3(t) = f(t)$$  

(26)

where $f(t) = \begin{cases} 8 & \text{for } 0 \leq t < 1 \\ 0 & \text{for } 1 < t < \infty \end{cases}$ with initial conditions $y(0) = 0, y'(0) = 0$. This is the Bagley-Torvik equation where non-linear term $y^3(t)$ is introduced. Fig. b.
shows a behaviour of the numerical solution of eqn. (26). We can see that the step \( h \) has strong influence to the solution \( y(t) \). Non-linear fractional differential equations need some computational tests that to choose right value of the step.

4 Conclusions

In this paper we propose a new method for the numerical solution of arbitrary differential equations of fractional order. Following to Blank’s results [2] the main advantage of the method is a decomposition of fractional differential equation into a system composed with one ordinary differential equation of integer order and the left inverse equations of the Abel-integral operator. We distinguish two classes of such system: one-term fractional derivative and multi-term fractional derivatives. The comparison certifies that our method gives quite good results. Summarizing these results, we can say that the decomposition method in its general form gives a reasonable calculations, is the effective method and easy to use and is applied for the fractional differential equations in general form.

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