Uniqueness of Viscosity Solutions for Optimal Multi-Modes Switching Problem with Risk of default

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Abstract

In this paper we study the optimal $m$-states switching problem in finite horizon as well as infinite horizon with risk of default. We allow the switching cost functionals and cost of default to be of polynomial growth and arbitrary. We show uniqueness of a solution for a system of $m$ variational partial differential inequalities with inter-connected obstacles. This system is the deterministic version of the Verification Theorem of the Markovian optimal $m$-states switching problem with risk of default. This problem is connected with the valuation of a power plant in the energy market.

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1 Introduction

In this work we are concerned with the following systems of $m$ variational partial differential inequalities with inter-connected obstacles:

$$\begin{align*}
\min \{ v_i(t,x) - \left( \max_{j \in \mathcal{I}^{-i}} (-g_{ij}(t,x) + v_j(t,x)) \vee (-F_i(t,x)) \right), \\
-\partial_t v_i(t,x) - A v_i(t,x) - \psi_i(t,x) \} &= 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^k, \quad i \in \mathcal{I} = \{1, \ldots, m\}, \\
v_i(T,x) &= 0.
\end{align*}$$

(1.1)
\[
\begin{align*}
\min\{v_i(x) - \left(\max_{j \in \mathcal{I}^{-1}} (-g_{ij}(x) + v_j(x)) \lor (-F_i(x))\right), rv_i(x) - \mathcal{A}v_i(x) - \psi_i(x)\} = 0, \\
\forall x \in \mathbb{R}^k, \ i \in \mathcal{I} = \{1, ..., m\},
\end{align*}
\]

where \(g_{ij}, \psi_i\) and \(F_i\) are continuous functions, \(\mathcal{A}\) an infinitesimal generator associated with a diffusion process and finally \(\mathcal{I}^{-i} := \{1, ..., i - 1, i + 1, ..., m\}\).

These system is the deterministic version of the Verification Theorem of the optimal multi-modes switching problem in finite horizon and infinite horizon with risk of default. These problems, of real option type, can be introduced with the help of the following example:

Assume we have a power station/plant which produces electricity and which has several modes of production, e.g., the lower, the middle and the intensive modes. The price of electricity in the market, given by an adapted stochastic process \((X_t)_{t \leq T}\), fluctuates in reaction to many factors such as demand level, weather conditions, unexpected outages and so on. On the other hand, electricity is non-storable, once produced it should be almost immediately consumed. Therefore, as a consequence, the station produces electricity in its instantaneous most profitable mode known that when the plant is in mode \(i \in \mathcal{I}\), the yield per unit time is given by means of \(\psi_i\), switching the plant from the mode \(i\) to the mode \(j\) is not free and generates expenditures given by \(g_{ij}\) and, on the other hand, cost of default (definitely stop the production) in mode \(i \in \mathcal{I}\), is given by \(F_i\) and possibly by other factors in the energy market. So the manager of the power plant faces two main issues:

(i) when should he decide to switch the production from its current mode to another one?
(ii) to which mode the production has to be switched when the decision of switching is made?

In other words she faces the issue of finding the optimal strategy of management of the plant. This issue is in relation with the price of the power plant in the energy market.

Optimal switching problems for stochastic systems were studied by several authors (see e.g. [1, 2, 4, 5, 6, 11, 12, 13, 15, 16, 17, 19, 22, 30, 33] and the references therein). The motivations are mainly related to decision making in the economic sphere. Several variants of the problem we deal with here, including finite and infinite horizons, have been considered during the recent years. In order to tackle those problems, authors use mainly two approaches. Either a probabilistic one [12, 13, 16, 17, 22] or an approach which uses partial differential inequalities (PDIs for short) [1, 2, 4, 6, 15, 19, 33, 30].

The PDIs approach turns out to study and to solve, in some sense, the system of \(m\) PDIs with inter-connected obstacles (1.1) for finite horizon and (1.2) for infinite horizon. Recently El Asri and Hamadène [19] and El Asri [16] extended the work of Djehiche et al [13] in the finite
horizon and infinite horizon case but allowing general jumps. In all these works the existence of the value functions of optimal impulse control problem and uniqueness of viscosity solution are obtained assuming that the switching problems without risk of default.

Amongst the papers which consider the same problem as ours, and in the framework of viscosity solutions approach, the most elaborated works are certainly the ones by Djehiche and Hamadène [12], on the one hand, and by Arnarson et al. [1], on the other hand. In [12], the authors show existence of a solution for (1.1). Nevertheless the paper suffers from three facts: (i) the switching problem have only two modes; (ii) the switching cost functions $g_{ij}$ should not depend on $x$; (iii) the problem of switching is in finite horizon. The first issue of [12] has been treated by Arnarson et al. [1] since in their paper the authors show existence of the solution for (1.1) in the case when the growth of the functions $\psi_i$ is of arbitrary polynomial type. The second issue of has been treated by El Asri and Hamadène [19], since in their paper the authors show existence and uniqueness of the solution for (1.1) in the case when $F_i = -\infty$. The third issu of [12], i.e. considering the case of switching problem in infinite horizon with risk of default, was right now, according to our knowledge, an open problem. Note that in [1], the question of uniqueness is addressed but in the general case still remains open. Therefore the main objective of our work, and this is the novelty of the paper, is to show existence and uniqueness of a solution in viscosity sense for the systems (1.1) and (1.2) when the functions $\psi_i$, $g_{ij}$ and $F_i$ are continuous depending also on $x$ and satisfy an arbitrary polynomial growth condition. We show also that the solution is unique in the class of continuous functions with polynomial growth.

This paper is organized as follows:

In Section 2, we formulate the problem and we give the related definitions. In Section 3, we introduce the optimal switching problem in finite horizon and infinite horizon under consideration and give its probabilistic Verification Theorem. It is expressed by means of a Snell envelope of processes. Then we introduce the approximating scheme which enables to construct a solution for the Verification Theorem. Moreover we give some properties of that solution. Section 4, is devoted to the connection between the optimal switching problem in finite horizon, the Verification Theorem and the system of PDEs (1.1). This connection is made through backward stochastic differential equations with one reflecting obstacle in the case when randomness comes from a solution of a standard stochastic differential equation. We provide existence and uniqueness of viscosity solution of (1.1) in the class of continuous functions which satisfy a polynomial growth condition. Section 5, we show that the solution of (1.2) is unique in the class of continuous functions which satisfy a polynomial growth condition. □
2 Assumptions and formulation of the problem

In finite horizon

Throughout this paper $T$ (resp. $k$) is a fixed real (resp. integer) positive constant. Let us now consider the followings:

**H1:** $b : [0, T] \times \mathbb{R}^k \to \mathbb{R}^k$ and $\sigma : [0, T] \times \mathbb{R}^k \to \mathbb{R}^{k \times d}$ are two continuous functions for which there exists a constant $C \geq 0$ such that for any $t \in [0, T]$ and $x, x' \in \mathbb{R}^k$

$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|) \quad \text{and} \quad |\sigma(t, x) - \sigma(t, x')| + |b(t, x) - b(t, x')| \leq C|x - x'|$$ (2.1)

**H2:** for $i, j \in \mathcal{I} = \{1, \ldots, m\}$, $g_{ij} : [0, T] \times \mathbb{R}^k \to \mathbb{R}$, $F_i : [0, T] \times \mathbb{R}^k \to \mathbb{R}^+$ and $\psi_i : [0, T] \times \mathbb{R}^k \to \mathbb{R}$ are continuous functions and of polynomial growth, i.e., there exist some positive constants $C$ and $\mu$ such that for each $i, j \in \mathcal{I}$:

$$|\psi_i(t, x)| + |F_i(t, x)| + |g_{ij}(t, x)| \leq C(1 + |x|^\mu), \forall (t, x) \in [0, T] \times \mathbb{R}^k.$$ (2.2)

**H3:** Moreover we assume that there exists a constant $\alpha > 0$ such that for any $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\min\{g_{ij}(t, x), i, j \in \mathcal{I}, \ i \neq j\} \geq \alpha.$$ (2.3)

This condition means that switching from one mode to another one is not free and costs at least $\alpha > 0$.

We now consider the following system of $m$ variational inequalities with inter-connected obstacles: $\forall \ i \in \mathcal{I}$

$$\begin{cases}
\min\{v_i(t, x) - \left(\max_{j \in \mathcal{I}^i}(-g_{ij}(t, x) + v_j(t, x)) \lor (-F_i(t, x))\right), \\
-v_i(T, x) = 0,
\end{cases}$$

where $\mathcal{I}^i := \mathcal{I} - \{i\}$ and $\mathcal{A}$ is the following infinitesimal generator:

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1,k} (\sigma \sigma^*)_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1,k} b_i(t, x) \frac{\partial}{\partial x_i};$$ (2.5)

hereafter the superscript (*) stands for the transpose, $Tr$ is the trace operator and finally $<x, y>$ is the inner product of $x, y \in \mathbb{R}^k$.

The first main objective of this paper is to focus on the uniqueness of the solution in viscosity sense of (2.4). To proceed we will precise the notion of a viscosity solution of the system (2.4). It will be done in terms of subjets and superjets.
Definition 1 Let $v \in C((0,T) \times \mathbb{R}^k)$, $(t,x)$ an element of $(0,T) \times \mathbb{R}^k$ and finally $S_k$ the set of $k \times k$ symmetric matrices. We denote by $J^{2,+}v(t,x)$ (resp. $J^{2,-}v(t,x)$), the superjets (resp. the subjets) of $v$ at $(t,x)$, the set of triples $(p,q,X) \in \mathbb{R} \times \mathbb{R}^k \times S_k$ such that:

$$v(s,y) \leq v(t,x) + p(s-t) + \langle q,y-x \rangle + \frac{1}{2}(X(y-x),y-x) + o(|s-t| + |y-x|^2)$$

(resp. $v(s,y) \geq v(t,x) + p(s-t) + \langle q,y-x \rangle + \frac{1}{2}(X(y-x),y-x) + o(|s-t| + |y-x|^2)$). □

Note that if $\varphi - v$ has a local maximum (resp. minimum) at $(t,x)$, then we obviously have:

$$(D_i \varphi(t,x), D_x \varphi(t,x), D^2_{xx} \varphi(t,x)) \in J^{2,-}v(t,x) \text{ (resp. } J^{2,+}v(t,x)).\square$$

We now give the definition of a viscosity solution for the system of PDE equations with risk of default in finite horizon $[2,4]$.

Definition 2 Let $(v_1,\ldots,v_m)$ be a $m$-uplet of continuous functions defined on $[0,T] \times \mathbb{R}^k$, $\mathbb{R}$-valued and such that $(v_1,\ldots,v_m)(T,x) = 0$ for any $x \in \mathbb{R}^k$. The $m$-uplet $(v_1,\ldots,v_m)$ is called a viscosity supersolution (resp. subsolution) of $[2,4]$ if for any $i \in \mathcal{I}$, $(t,x) \in (0,T) \times \mathbb{R}^k$ and $(p,q,X) \in J^{2,-}v_i(t,x)$ (resp. $J^{2,+}v_i(t,x)$),

$$\min \left\{ v_i(t,x) - \left( \max_{j \in \mathcal{I}^i} (-g_{ij}(t,x) + v_j(t,x)) \vee (-F_i(t,x)) \right), -p - \frac{1}{2}Tr[\sigma^*X\sigma] - \langle b,q \rangle - \psi_i(t,x) \right\} \geq 0 \text{ (resp. } \leq 0).$$

It is called a viscosity solution it is both a viscosity subsolution and supersolution. □

As pointed out previously we will show that system $[2,4]$ has a unique solution in viscosity sense. A particular case of this system is the deterministic version of the optimal $m$-states switching problem in finite horizon with risk of default which is well documented e.g. in $[1,12]$ and which we will describe in the next section.

In infinite horizon

Let us now consider the followings assumption:

H4: $b : \mathbb{R}^k \rightarrow \mathbb{R}^k$ and $\sigma : \mathbb{R}^k \rightarrow \mathbb{R}^{k \times d}$ are two continuous functions for which there exists a constant $C \geq 0$ such that for any $x,x' \in \mathbb{R}^k$

$$|b(x)| + |\sigma(x)| \leq C(1 + |x|) \quad \text{and} \quad |\sigma(x) - \sigma(x')| + |b(x) - b(x')| \leq C|x - x'| \quad (2.6)$$

H5: for $i,j \in \mathcal{I} = \{1,\ldots,m\}$, $g_{ij} : \mathbb{R}^k \rightarrow \mathbb{R}$ is a continuous function. Moreover we assume that there exists a constant $\alpha > 0$ such that for any $x \in \mathbb{R}^k$,

$$\frac{1}{\alpha} \leq g_{ij}(x) \leq \alpha, \quad \forall i,j \in \mathcal{I}, \quad i \neq j. \quad (2.7)$$
**H6**: for $i \in I$ $\psi_i : \mathbb{R}^k \to \mathbb{R}$ and $F_i : \mathbb{R}^k \to \mathbb{R}^+$ are continuous function of polynomial growth, i.e., there exist a constant $C$ and $\mu$ such that for each $i \in I$:

$$|\psi_i(x)| + |F_i(x)| \leq C(1 + |x|^\mu), \ \forall x \in \mathbb{R}^k. \quad (2.8)$$

We now consider the following system of $m$ variational inequalities with inter-connected obstacles: $\forall i \in I$

$$\min \left\{ v_i(x) - \left( \max_{j \in I^{-i}} (-g_{ij}(x) + v_j(x)) \lor (-F_i(x)) \right), rv_i(x) - \mathcal{A}v_i(x) - \psi_i(x) \right\} = 0, \quad (2.9)$$

where $I^{-i} := I \setminus \{i\}$, $r$ is a positive discount factor and $\mathcal{A}$ is the following infinitesimal generator:

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1,k} (\sigma^*)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1,k} b_i(x) \frac{\partial}{\partial x_i}. \quad (2.10)$$

The second main objective of this paper is to focus on the uniqueness of the solution in viscosity sense of (2.9). We now give the definition of a viscosity solution of the elliptic system with inter-connected obstacles (2.9).

**Definition 3** Let $(v_1, ..., v_m)$ be a $m$-uplet of continuous real-valued functions defined on $\mathbb{R}^k$. The $m$-uplet $(v_1, ..., v_m)$ is called a viscosity supersolution (resp. subsolution) of (2.9) if for any $i \in I$, $x \in \mathbb{R}^k$ and $(q, X) \in J^2-v_i(t, x)$ (resp. $J^{2+}v_i(t, x)$),

$$\min \left\{ v_i(x) - \left( \max_{j \in I^{-i}} (-g_{ij}(x) + v_j(x)) \lor (-F_i(x)) \right), rv_i(x) - \mathcal{A}v_i(x) - \psi_i(x) \right\} \geq 0 \ (\text{resp.} \leq 0).$$

It is called a viscosity solution if it is both a viscosity subsolution and supersolution. $\square$

As pointed out previously we will show that system (2.9) has a unique solution in viscosity sense. This system is the deterministic version of the optimal $m$-states switching problem in infinite horizon with default risk which is well documented in [12, 13, 17] and which we will describe briefly in the next section.

### 3 The optimal $m$-states switching problem

#### 3.1 In finite horizon with risk of default

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space on which is defined a standard $d$-dimensional Brownian motion $B = (B_t)_{0 \leq t \leq T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{0 \leq t \leq T}$. Let $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$
be the completed filtration of \((\mathcal{F}_t^0)_{0 \leq t \leq T}\) with the \(P\)-null sets of \(\mathcal{F}\), hence \((\mathcal{F}_t)_{0 \leq t \leq T}\) satisfies the usual conditions, \textit{i.e.}, it is right continuous and complete. Furthermore, let:

- \(\mathcal{P}\) be the \(\sigma\)-algebra on \([0, T] \times \Omega\) of \(\mathcal{F}\)-progressively measurable sets;
- \(\mathcal{M}^{2,k}\) be the set of \(\mathcal{P}\)-measurable and \(\mathbb{R}^{k}\)-valued processes \(w = (w_t)_{t \leq T}\) such that \(E\left[\int_0^T |w_s|^2 ds\right] < \infty\) and \(\mathcal{S}^2\) be the set of \(\mathcal{P}\)-measurable, continuous processes \(w = (w_t)_{t \leq T}\) such that \(E[\sup_{t \leq T} |w_t|^2] < \infty\);
- for any stopping time \(\tau \in [0, T]\), \(\mathcal{T}_\tau\) denotes the set of all stopping times \(\theta\) such that \(\tau \leq \theta \leq T\).

The problem of multiple switching can be described through an example as follows. Assume we have a plant which produces a commodity, \textit{e.g.} a power station which produces electricity. The production activity have \(m\) modes, or "definitely closed/defaulting" indicated by \(\dagger\). A management strategy of the plant consists, on the one hand, of the choice of a sequence of nondecreasing stopping times \((\tau_n)_{n \geq 1}\) (\(i.e.\) \(\tau_n \leq \tau_{n+1}\) and \(\tau_0 = 0\)) and the stopping time \(\gamma\) where the manager decides to switch the activity from its current mode to another one or definitely stop the production. On the other hand, it consists of the choice of the mode \(\xi_n\), a \(\mathcal{F}_{\tau_n}\)-measurable with values in \(\mathcal{I}\), to which the production is switched at \(\tau_n\) from its current mode. Therefore the admissible management strategies of the plant are the pairs \((\delta, \xi) := ((\tau_n)_{n \geq 1}, \gamma, (\xi_n)_{n \geq 1})\) and the set of these strategies is denoted by \(\mathcal{D}\).

Let now \(X := (X_t)_{0 \leq t \leq T}\) be an \(\mathcal{P}\)-measurable, \(\mathbb{R}^k\)-valued continuous stochastic process which stands for the market price of \(k\) factors which determine the market price of the commodity. On the other hand, assuming that the production activity is in mode 1 at the initial time \(t = 0\), let \((u_t)_{t \leq T}\) denote the indicator of the production activity’s mode at time \(t \in [0, T]\):

\[
    u_t = \mathbb{1}_{[0,\tau_1]}(t) + \sum_{n \geq 1} \xi_n \mathbb{1}_{(\tau_n,\tau_{n+1}]}(t).
\]

(3.1)

Then for any \(t \leq T\), the state of the whole economic system related to the project at time \(t\) is represented by the vector:

\[
    \begin{align*}
        & (t, X_t, u_t), & \text{if } \tau_n < t \leq \tau_{n+1}; \\
        & (\gamma, X_\gamma), & \text{if in mode } \dagger.
    \end{align*}
\]

(3.2)

Finally, let \(\psi_i(t, X_t)\) be the instantaneous profit when the system is in state \((t, X_t, i)\), for \(i, j \in \mathcal{I} \quad i \neq j\), let \(g_{ij}(t, X_t)\) denote the switching cost of the production at time \(t\) from current mode \(i\) to another mode \(j\) and let \(F_i(\gamma, X_\gamma)\) denote the cost of default (definitely stop the production) at time \(\gamma\), when in mode \(i\) and denote \(F_i(\gamma, X_\gamma) = F(\gamma, X_\gamma, u_\gamma)\) when \(u_\gamma = i\). Then if the plant is run under the strategy \((\delta, \xi) = ((\tau_n)_{n \geq 1}, \gamma, (\xi_n)_{n \geq 1})\) the expected total profit is
given by:

\[ J(\delta, \xi) = E\left[ \int_0^\gamma \psi_u(s, X_s) ds - \sum_{n \geq 1} g_{u_{\tau_n}}(\tau_n, X_{\tau_n}) \mathbb{1}_{[\tau_n < \gamma]} - F(\gamma, X_{\gamma}, u_{\gamma}) \mathbb{1}_{[\gamma < T]} \right]. \]

Therefore the problem we are interested in is to find an optimal strategy, i.e., a strategy \((\delta^*, u^*) = ((\tau_n^*)_{n \geq 1}, \gamma^*), (\xi^*_{n})\) such that \(J(\delta^*, \xi^*) \geq J(\delta, \xi)\) for any \((\delta, \xi) \in \mathcal{D}\).

Note that in order that the quantity \(J(\delta, \xi)\) makes sense we assume throughout this paper that for any \(i, j \in I\) the processes \((F_i(t, X_t))_{t \leq T}\), \((g_{ij}(t, X_t))_{t \leq T}\) (resp. \((\psi_i(t, X_t))_{t \leq T}\)) belong to \(S^2\) (resp. \(M^{2-1}\)). On the other hand there is a bijective correspondence between the pairs \((\delta, \xi)\) and the pairs \((\delta, u)\). Therefore throughout this paper one refers indifferently to \((\delta, \xi)\) or \((\delta, u)\).

The verification Theorem for the \(m\)-states optimal switching with risk of default problem is the following:

**Theorem 1** Assume that there exist \(m\) processes \((Y^i := (Y^i_t)_0 \leq t \leq T, i = 1, \ldots, m)\) of \(S^2\) such that:

\[ \forall t \leq T, \quad Y^i_t = \text{ess sup}_{\tau \geq t} E\left[ \int_t^\tau \psi_i(s, X_s) ds + \max_{j \in I \setminus i} (-g_{ij}(\tau, X_{\tau}) + Y^j_{\tau}) \right] \vee F_i(\tau, X_{\tau})1_{[\tau < T]}[F_t], \]

\[ Y^i_T = 0. \]

Then:

(i) \(Y^i_0 = \sup_{(\delta, u) \in D} J(\delta, u)\).

(ii) Define the sequence of \(F\)-stopping times \(\delta = (\tau^i_t, \gamma), i = 1, 2, \ldots, m, 0 \leq t \leq T\) as follows:

\[ \tau^i_t = \sigma^i_t \land \overline{\sigma}^i_t \land T, \]

where:

- The first time the activity defaults while in mode \(i\) is given by

\[ \sigma^i_t := \inf\{s \geq t, Y^i_s = F_i(s, X_s)\} \land T, \quad i = 1, \ldots, m. \]

- The first time the activity is switched from mode \(i\) to any of the other modes \(j \neq i\) is given by

\[ \overline{\sigma}^i_t := \inf\{s \geq t, Y^i_s = \max_{j \neq i} (-g_{ij}(s, X_s) + Y^j_{s})\} \land T \]

Finally, let \(\gamma = \sup_{0 \leq t \leq T} \tau^i_t\). Then, the strategy \((\delta = ((\tau^i_t)_{t \geq 0}, \gamma), u^*)\) is optimal. \(\square\)
Proof. The arguments of proof are standard, based on the properties the Snell envelope and is proved in \[13\], Theorem 1. □

The issue of existence of the processes \(Y^1, ..., Y^m\) which satisfy (3.10) is also addressed in \[13\]. Also for \(n \geq 0\) let us define the processes \((Y^{1,n}, ..., Y^{m,n})\) recursively as follows: for \(i \in \mathcal{I}\) we set,
\[
Y^{i,0}_t = \operatorname{ess sup}_{\tau \geq t} E\left[ \int_t^\tau \psi_i(s, X_s) \, ds + F_i(\tau, X_\tau) 1_{[\tau < T]} |\mathcal{F}_\tau \right], \quad 0 \leq t \leq T,
\]
and for \(n \geq 1\),
\[
Y^{i,n}_t = \operatorname{ess sup}_{\tau \geq t} E\left[ \int_t^\tau \psi_i(s, X_s) \, ds + \max_{k \in I^{-i}} (-g_{ik}(\tau, X_\tau) + Y^{k,n-1}_\tau) \vee F_i(\tau, X_\tau) 1_{[\tau < T]} |\mathcal{F}_\tau \right], \quad 0 \leq t \leq T.
\]
Then the sequence of processes \(((Y^{1,n}, ..., Y^{m,n}))_{n \geq 0}\) have the following properties:

**Proposition 1** (\[13\], Pro.3 and Th.2)

(i) for any \(i \in \mathcal{I}\) and \(n \geq 0\), the processes \(Y^{1,n}, ..., Y^{m,n}\) are well-posed, continuous and belong to \(\mathcal{S}^2\), and verify
\[
\forall t \leq T, \quad Y^{i,n}_t \leq Y^{i,n+1}_t \leq E\left[ \int_t^T \max_{i=1,m} |\psi_i(s, X_s)| \, ds |\mathcal{F}_t \right]; \quad (3.6)
\]

(ii) there exist \(m\) processes \(Y^1, ..., Y^m\) of \(\mathcal{S}^2\) such that for any \(i \in \mathcal{I}\):

(a) \(\forall t \leq T, \quad Y^i_t = \lim_{n \to \infty} Y^{i,n}_t \) and
\[
E\left[ \sup_{s \leq T} |Y^{i,n}_s - Y^i_s|^2 \right] \to 0 \quad \text{as} \quad n \to +\infty
\]

(b) \(\forall t \leq T, \)
\[
Y^i_t = \operatorname{ess sup}_{\tau \geq t} E\left[ \int_t^\tau \psi_i(s, X_s) \, ds + \max_{k \in I^{-i}} (-g_{ik}(\tau, X_\tau) + Y^{k}_\tau) \vee F_i(\tau, X_\tau) 1_{[\tau < T]} |\mathcal{F}_\tau \right] \quad (3.7)
\]
i.e. \(Y^1, ..., Y^m\) satisfy the Verification Theorem □

**Remark 1** Note that the characterization (3.7) implies that the processes \(Y^1, ..., Y^m\) of \(\mathcal{S}^2\) which satisfy the Verification Theorem are unique.

### 3.2 In infinite horizon with risk of default

Let \((\Omega, \mathcal{F}, P)\) be a fixed probability space on which is defined a standard \(d\)-dimensional Brownian motion \(B = (B_t)_{t \geq 0}\) whose natural filtration is \((\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{t \geq 0}\). Let \(\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}\) be
the completed filtration of \((\mathcal{F}_t^0)_{t \geq 0}\) with the \(P\)-null sets of \(\mathcal{F}\), hence \((\mathcal{F}_t)_{t \geq 0}\) satisfies the usual conditions, i.e., it is right continuous and complete. Furthermore, let:

- \(\mathcal{P}\) be the \(\sigma\)-algebra on \([0, +\infty) \times \Omega\) of \(\mathbb{F}\)-progressively measurable sets;
- \(\mathcal{M}^{2,k}\) be the set of \(\mathcal{P}\)-measurable and \(\mathbb{R}^k\)-valued processes \(w = (w_t)_{t \geq 0}\) such that 
\[
E[\int_0^{+\infty} |w_s|^2 ds] < \infty \text{ and } S^2 \text{ be the set of } \mathcal{P}\text{-measurable, continuous processes } w = (w_t)_{t \geq 0}\text{ such that } E[\sup_{t \geq 0} |w_t|^2] < \infty;
\]
- for any stopping time \(\tau \in \mathbb{R}^+\), \(\mathcal{T}_\tau\) denotes the set of all stopping times \(\theta\) such that \(\tau \leq \theta\);
- for any stopping time \(\tau\), \(\mathcal{F}_\tau\) is the \(\sigma\)-algebra on \(\Omega\) which contains the sets \(A\) of \(\mathcal{F}\) such that \(A \cap \{\tau \leq t\} \in \mathcal{F}_t\) for every \(t \geq 0\).

A decision (strategy) of the problem of multiple switching, on the one hand, consists of the choice of a sequence of nondecreasing stopping times \((\tau_n)_{n \geq 1}\) (i.e. \(\tau_n \leq \tau_{n+1}\) and \(\tau_0 = 0\)) and the stopping time \(\gamma\) where the manager decides to switch the activity from its current mode to another one or definitely stop the production. On the other hand, it consists of the choice of the mode \(\xi_n\), a r.v. \(\mathcal{F}_{\tau_n}\)-measurable with values in \(\mathcal{I}\), to which the production is switched at \(\tau_n\) from its current mode. Therefore the admissible management strategies of the plant are the pairs \((\delta, \xi) := ((\tau_n)_{n \geq 1}, \gamma, (\xi_n)_{n \geq 1})\) and the set of these strategies is denoted by \(\mathcal{D}\).

Let now \(X := (X_t)_{t \geq 0}\) be an \(\mathcal{P}\)-measurable, \(\mathbb{R}^k\)-valued continuous stochastic process which stands for the market price of \(k\) factors which determine the market price of the commodity. On the other hand, assuming that the production activity is in mode 1 at the initial time \(t = 0\), let \((u_t)_{t \geq 0}\) denote the indicator of the production activity’s mode at time \(t \in \mathbb{R}^+\):

\[
u_t = \mathbb{1}_{[0,\tau_1]}(t) + \sum_{n \geq 1} \xi_n \mathbb{1}_{(\tau_n, \tau_{n+1}]}(t).
\]

Then for any \(t \geq 0\), the state of the whole economic system related to the project at time \(t\) is given by the vector:

\[
(t, X_t, \nu_t), \quad \text{if } \tau_n < t \leq \tau_{n+1};
\]

\[
(\gamma, X_\gamma), \quad \text{if in mode } \dagger.
\]

Finally, let \(\psi_i(X_t)\) be the instantaneous profit when the system is in state \((t, X_t, i)\), for \(i, j \in \mathcal{I} \ i \neq j\), let \(g_{ij}(X_t)\) denote the switching cost of the production at time \(t\) from the current mode \(i\) to another mode \(j\) and let \(F_i(\gamma, X_\gamma)\) denote the cost of default (definitely stop the production) at time \(\gamma\); when in mode \(i\) and denote \(F_i(X_\gamma) = F(X_\gamma, u_\gamma)\) when \(u_\gamma = i\). When the plant is run under the strategy \((\delta, \xi) = ((\tau_n)_{n \geq 1}, \gamma, (\xi_n)_{n \geq 1})\) the expected total profit is given by:

\[
J(\delta, \xi) = E[\int_0^\gamma e^{-rs} \psi_{u_\gamma}(X_s) ds - \sum_{n \geq 1} e^{-r\tau_n} g_{u_{\tau_n-1} u_{\tau_n}}(X_{\tau_n}) \mathbb{1}_{[\tau_n < \gamma]} - e^{-r\gamma} F(X_\gamma, u_\gamma)].
\]
Then the problem we are interested in is to find an optimal strategy, i.e., a strategy $(\delta^*, \xi^*)$ such that $J(\delta^*, \xi^*) \geq J(\delta, \xi)$ for any $(\delta, \xi) \in D$.

Note that in order that the quantity $J(\delta, \xi)$ makes sense we assume throughout this paper that for any $i \in I$ the processes $(e^{-rt}\psi_i(X_t))_{t \geq 0}$ and $(e^{-rt}F_i(X_t))_{t \geq 0}$ belong to $\mathcal{M}^{2,1}$ and $\mathcal{S}^2$ respectively.

The Verification Theorem for the $m$-states optimal switching with risk of default problem in infinite horizon is the following:

**Theorem 2.** Assume that there exist $m$ processes $(Y^i := (Y^i_t)_{t \geq 0}, i = 1, \ldots, m)$ of $\mathcal{S}^2$ such that:

\[
\forall t \geq 0, \quad e^{-rt}Y^i_t = \text{ess sup}_{\tau \geq t} E\left[\int_t^\tau e^{-rs}\psi_i(X_s)ds + e^{-\tau r} \max_{j \in I - i} (-g_{ij}(X_\tau) + Y^j_\tau) \lor e^{-\tau r}F_i(X_\tau)|\mathcal{F}_t\right],
\]

\[
\lim_{t \to +\infty} (e^{-rt}Y^i_t) = 0.
\]

Then:

(i) $Y^1_0 = \sup_{(\delta, \xi) \in D} J(\delta, \xi)$.

(ii) Define the sequence of $\mathcal{F}$-stopping times $\delta = (\tau^i_t, \gamma)$, $i = 1, 2, \ldots, m$, $t \geq 0$. as follows:

\[
\tau^i_t = \sigma^i_t \land \tilde{\sigma}^i_t,
\]

where:

- The first time the activity defaults while in mode $i$ is given by

\[
\sigma^i_t := \inf\{s \geq t, Y^i_s = F_i(X_s)\}, \quad i = 1, \ldots, m.
\]

- The first time the activity is switched from mode $i$ to any of the other modes $j \neq i$ is given by

\[
\tilde{\sigma}^i_t := \inf\{s \geq t, Y^i_s = \max_{j \neq i} (-g_{ij}(s, X_s) + Y^j_s)\}
\]

Finally, let $\gamma = \sup_{t \geq 0} \tau^i_t$. Then, the strategy $\delta = ((\tau^i_t)_{t \geq 0}, \gamma)$ is optimal. □

**Proof.** The arguments of proof are standard, based on the properties the Snell envelope and is proved in [16, Theorem.1]. □

The issue of existence of the processes $Y^1, \ldots, Y^m$ which satisfy (3.10) is also addressed in [13]. For $n \geq 0$ let us define the processes $(Y^{n,1}, \ldots, Y^{n,m})$ recursively as follows: for $i \in I$ we set,

\[
e^{-rt}Y^{0,i}_t = \text{ess sup}_{\tau \geq t} E\left[\int_t^\tau e^{-rs}\psi_i(X_s)ds + e^{-\tau r}F_i(X_\tau)|\mathcal{F}_t\right], \quad t \geq 0,
\]

(3.11)
and for $n \geq 1$,
\[
e^{-rt}Y^n_t = \text{ess sup}_{r \geq t} E\left[ \int_t^r e^{-rs} \psi(X_s) ds + e^{-rt} \max_{k \in \mathcal{I}^{-i}} (-g_{ik}(X_r) + Y^n_{t-1,k}) \vee e^{-rt} F_i(X_r)|\mathcal{F}_t \right], \quad t \geq 0.
\] (3.12)

Then the sequence of processes $((Y^n_1, \ldots, Y^n_m))_{n \geq 0}$ have the following properties:

**Proposition 2** ([13], Pro.3 and Th.2)

(i) for any $i \in \mathcal{I}$ and $n \geq 0$, the processes $Y^n_1, \ldots, Y^n_m$ are well-posed, continuous and belong to $\mathcal{S}^2$, and verify
\[
\forall t \geq 0, \quad e^{-rt}Y^n_t \leq e^{-rt}Y^n_{t+1} \leq E\left[ \int_t^{\infty} e^{-rs} \max_{i=1}^{m} |\psi_1(X_s)| ds |\mathcal{F}_t \right]; \quad (3.13)
\]

(ii) there exist $m$ processes $Y^1, \ldots, Y^m$ of $\mathcal{S}^2$ such that for any $i \in \mathcal{I}$:

(a) $\forall t \geq 0, \quad Y^i_t = \lim_{n \to \infty} Y^n_t$

(b) $\forall t \geq 0,$
\[
e^{-rt}Y^i_t = \text{ess sup}_{r \geq t} E\left[ \int_t^r e^{-rs} \psi(X_s) ds + e^{-rt} \max_{k \in \mathcal{I}^{-i}} (-g_{ik}(X_r) + Y^k_{t-1}) \vee e^{-rt} F_i(X_r)|\mathcal{F}_t \right]
\]
\[
\text{i.e. } Y^1, \ldots, Y^m \text{ satisfy the Verification Theorem} \square
\] (3.14)

**Remark 2** The characterization ([3], [14]) implies that the processes $Y^1, \ldots, Y^m$ of $\mathcal{S}^2$ which satisfy the Verification Theorem are unique. \square

4 Uniqueness of the viscosity solution in finite horizon

Let $(t, x) \in [0, T] \times \mathbb{R}^k$ and let $(X_s^{tx})_{s \leq T}$ be the solution of the following standard SDE:
\[
dX_s^{tx} = b(s, X_s^{tx}) ds + \sigma(s, X_s^{tx}) dB_s \quad \text{for } t \leq s \leq T \text{ and } X_s^{tx} = x \text{ for } s \leq t
\] (4.1)

where the functions $b$ and $\sigma$ are the ones of (2.1). These properties of $\sigma$ and $b$ imply in particular that the process $(X_s^{tx})_{0 \leq s \leq T}$ solution of the standard SDE (4.1) exists and is unique, for any $t \in [0, T]$ and $x \in \mathbb{R}^k$.

The operator $A$ that is appearing in (2.5) is the infinitesimal generator associated with $X^{t,x}$. 

12
**Theorem 3** There are deterministic functions $v^1, \ldots, v^m : [0, T] \times \mathbb{R}^k \to \mathbb{R}$ such that:

$$
\forall (t, x) \in [0, T] \times \mathbb{R}^k, \forall s \in [t, T], Y_{s}^{i,tx} = v^i(s, X_{s}^{tx}), \ i = 1, \ldots, m.
$$

Moreover the functions $(v^1, \ldots, v^m) : [0, T] \times \mathbb{R}^k \to \mathbb{R}$ are continuous, solution in viscosity sense of the system of variational inequalities with inter-connected obstacles (2.4) and of polynomial growth. $\square$

**Proof:** The continuity of functions $v^1, \ldots, v^m$ follows from the dynamic programming principle and is proved in [19]. $\square$

Now we give an equivalent of quasi-variational inequality (2.4). In this section, we consider the new function $\Gamma_i$ given by the classical change of variable $\Gamma_i(t, x) = \exp(t)v^i(t, x)$, for any $t \in [0, T]$ and $x \in \mathbb{R}^k$. Of course, the function $\Gamma_i$ is continuous and of polynomial growth with respect to its arguments.

A second property is given by the

**Proposition 3** $v_i$ is a viscosity solution of (2.4) if and only if $\Gamma_i$ is a viscosity solution to the following quasi-variational inequality in $[0, T] \times \mathbb{R}^k$:

$$
\begin{align*}
\min \{ & \Gamma_i(t, x) - \max_{j \in I \setminus i} (-e^t g_{ij}(t, x) + \Gamma_j(t, x)) \vee (-e^t F_i(t, x)) \\
& \Gamma_i(t, x) - \partial_t \Gamma_i(t, x) - A \Gamma_i(t, x) - e^t \psi_i(t, x) \} = 0,
\end{align*}
$$

(4.2)

We are going now to address the question of uniqueness of the viscosity solution of the system (2.4). We have the following:

**Theorem 4** The solution in viscosity sense of the system of variational inequalities with interconnected obstacles (2.4) is unique in the space of continuous functions on $[0, T] \times \mathbb{R}^k$ which satisfy a polynomial growth condition, i.e., in the space

$$
C := \{ \varphi : [0, T] \times \mathbb{R}^k \to \mathbb{R}, \ continuous \ and \ for \ any \\
(t, x), |\varphi(t, x)| \leq C(1 + |x|^\mu) \ for \ some \ constants \ C \ and \ \mu \}.
$$

**Proof.** We will show by contradiction that if $u_1, \ldots, u_m$ and $w_1, \ldots, w_m$ are a subsolution and a supersolution respectively for (4.2) then for any $i = 1, \ldots, m$, $u_i \leq w_i$. Therefore if we have two solutions of (4.2) then they are obviously equal. Actually for some $R > 0$ suppose there exists $(\overline{t}, \overline{x}, \overline{t}) \in (0, T) \times B_R \times I$ ($B_R := \{ x \in \mathbb{R}^k; |x| < R \}$) such that:

$$
\max_{t, x, i} \left( u_i(t, x) - w_i(t, x) \right) = u_{\overline{t}}(\overline{t}, \overline{x}) - w_{\overline{t}}(\overline{t}, \overline{x}) = \eta > 0.
$$

(4.3)
Let us take $\theta, \lambda$ and $\beta \in (0, 1]$ small enough, so that the following holds:

$$
\begin{cases}
\beta T < \frac{\nu}{4} \\
-\lambda w^i(t, x) < \frac{\nu}{4} \\
\lambda \gamma < \frac{\nu}{4}.
\end{cases}
$$

(4.4)

Here $\gamma$ is the growth exponent of the functions which w.l.o.g we assume integer and $\geq 2$. Then, for a small $\epsilon > 0$, let us define:

$$
\Phi^i(\varepsilon, t, x, y) = u_i(t, x) - (1 - \lambda)w_i(t, y) - \frac{1}{2\epsilon}|x - y|^{2\gamma} - \theta(|x - \tilde{\omega}|^{2\gamma} + |y - \tilde{\omega}|^{2\gamma}) - \beta(t - \tilde{\omega})^2 - \frac{\lambda}{t}.
$$

(4.5)

By the growth assumption on $u_i$ and $w_i$, there exists a $(t_0, x_0, y_0, i_0) \in (0, T] \times B_R \times B_R \times I$, such that:

$$
\Phi^i(t_0, x_0, y_0) = \max_{(t, x, y, i)} \Phi^i(t, x, y).
$$

On the other hand, from $2\Phi^i(t_0, x_0, y_0) \geq \Phi^i(t_0, x_0, x_0) + \Phi^i(t_0, y_0, y_0)$, we have

$$
\frac{1}{2\epsilon}|x_0 - y_0|^{2\gamma} \leq (u_{i_0}(t_0, x_0) - w_{i_0}(t_0, y_0)) + (1 - \lambda)(w_{i_0}(t_0, x_0) - w_{i_0}(t_0, y_0)),
$$

(4.6)

and consequently $\frac{1}{2\epsilon}|x_0 - y_0|^{2\gamma}$ is bounded, and as $\epsilon \to 0$, $|x_0 - y_0| \to 0$. Since $u_{i_0}$ and $w_{i_0}$ are uniformly continuous on $[0, T] \times \overline{B_R}$, then $\frac{1}{2\gamma}|x_0 - y_0|^{2\gamma} \to 0$ as $\epsilon \to 0$.

Since

$$
(1 - \lambda)u^i(t_0, x_0) - w^i(t_0, y_0) - \frac{\lambda}{t} \leq \Phi^i(t_0, x_0, y_0) \leq (1 - \lambda)u^i(t_0, x_0) - w^i(t_0, y_0) - \frac{\lambda}{t_0},
$$

it follows as $\lambda \to 0$ and the continuity of $u$ and $w$ that, up to a subsequence,

$$
(t_0, x_0, y_0, i_0) \to (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{i}).
$$

(4.7)

Next let us show that $t_0 < T$. Actually if $t_0 = T$ then,

$$
\Phi^i(\tilde{t}, \tilde{x}, \tilde{y}) \leq \Phi^i(T, x_0, y_0),
$$

and,

$$
u (\tilde{t}, \tilde{x}) - (1 - \lambda)\nu (\tilde{t}, \tilde{x}) - \frac{\lambda}{t} \leq -\beta(T - \tilde{t})^2 - \frac{\lambda}{t},
$$

since $u_{i_0}(T, x_0) = w_{i_0}(T, y_0) = 0$. Then thanks to (4.3) we have,

$$
\eta \leq -\lambda w^i(\tilde{t}, \tilde{x}) + \beta T + \frac{\lambda}{t},
$$

$$
\eta \geq \frac{3}{4}\eta.
$$

which yields a contradiction and we have $t_0 \in (0, T)$. We now claim that:

$$
u u_{i_0}(t_0, x_0) - \left(\max_{j \in I^i_{-}} \{-e^{t_0}g_{i_0j}(t_0, x_0) + u_{j}(t_0, x_0)\} \lor (-e^{t_0}F_{i_0}(t_0, x_0))\right) > 0.
$$

(4.8)
Indeed if
\[ u_{i_0}(t_0, x_0) - \left( \max_{j \in I^{-i_0}} \{-e^{t_0}g_{i_0j}(t_0, x_0) + u_j(t_0, x_0)\} \lor (-e^{t_0}F_{i_0}(t_0, x_0)) \right) \leq 0. \]

**Case 1**: \[ \max_{j \in I^{-i_0}} \{-e^{t_0}g_{i_0j}(t_0, x_0) + u_j(t_0, x_0)\} \lor (-e^{t_0}F_{i_0}(t_0, x_0)) = -e^{t_0}F_{i_0}(t_0, x_0). \]

Then
\[ u_{i_0}(t_0, x_0) \leq -e^{t_0}F_{i_0}(t_0, x_0). \]

From the supersolution property of \( w_{i_0}(t_0, y_0) \), we have
\[ w_{i_0}(t_0, y_0) \geq \left( \max_{j \in I^{-i_0}} \{-e^{t_0}g_{i_0j}(t_0, y_0) + w_j(t_0, y_0)\} \lor (-e^{t_0}F_{i_0}(t_0, y_0)) \right), \]
then
\[ w_{i_0}(t_0, y_0) \geq -e^{t_0}F_{i_0}(t_0, y_0). \]

It follows that:
\[ u_{i_0}(t_0, x_0) - w_{i_0}(t_0, y_0) \leq -e^{t_0}(F_{i_0}(t_0, x_0) - F_{i_0}(t_0, y_0)). \]

But this contradicts the definition of (4.3), since \( F, u, w \) is uniformly continuous on \([0, T] \times \overline{B_R} \) and sending \( \lambda \to 0 \) and the claim (4.8) holds.

**Case 2**: \[ \max_{j \in I^{-i_0}} \{-e^{t_0}g_{i_0j}(t_0, x_0) + u_j(t_0, x_0)\} \lor (-e^{t_0}F_{i_0}(t_0, x_0)) = \max_{j \in I^{-i_0}} \{-e^{t_0}g_{i_0j}(t_0, x_0) + u_j(t_0, x_0)\}. \]

Then there exists \( k \in I^{-i_0} \) such that:
\[ u_{i_0}(t_0, x_0) \leq -e^{t_0}g_{i_0k}(t_0, x_0) + u_k(t_0, x_0). \]

From the supersolution property of \( w_{i_0}(t_0, y_0) \), we have
\[ w_{i_0}(t_0, y_0) \geq \left( \max_{j \in I^{-i_0}} \{-e^{t_0}g_{i_0j}(t_0, y_0) + w_j(t_0, y_0)\} \lor (-e^{t_0}F_{i_0}(t_0, y_0)) \right), \]
then
\[ w_{i_0}(t_0, y_0) \geq -e^{t_0}g_{i_0k}(t_0, y_0) + w_k(t_0, y_0). \]

It follows that:
\[ u_{i_0}(t_0, x_0) - (1 - \lambda)w_{i_0}(t_0, y_0) - (u_k(t_0, x_0) - (1 - \lambda)w_k(t_0, y_0)) \leq (1 - \lambda)e^{t_0}g_{i_0k}(t_0, y_0) - e^{t_0}g_{i_0k}(t_0, x_0). \]

Now since \( g_{ij} \geq \alpha > 0 \), for every \( i \neq j \), and taking into account of (4.5) to obtain:
\[ \Phi_{i_0}^{(t_0, x_0, y_0)} - \Phi_{i_0}^{(t_0, x_0, y_0)} < -\alpha \lambda e^{t_0} + e^{t_0}g_{i_0k}(t_0, y_0) - e^{t_0}g_{i_0k}(t_0, x_0). \]

But this contradicts the definition of \( i_0 \), since \( g_{i_0k} \) is uniformly continuous on \([0, T] \times \overline{B_R} \) and the claim (4.8) holds.
Next let us denote
\[ \varphi_\epsilon(t, x, y) = \frac{1}{2\epsilon} |x - y|^{2\gamma} + \theta(|x - \overline{x}|^{2\gamma} + |y - \overline{y}|^{2\gamma}) + \beta(t - \overline{t})^2 + \frac{\lambda}{t}, \]
(4.9)

Then we have:
\[
\begin{cases}
D_t \varphi_\epsilon(t, x, y) = 2\beta(t - \overline{t}) - \frac{\lambda}{t^2}, \\
D_x \varphi_\epsilon(t, x, y) = 2\epsilon(x - y)|x - y|^{2\gamma - 2} + \theta(2\gamma + 2)(x - \overline{x})|x - \overline{x}|^{2\gamma}, \\
D_y \varphi_\epsilon(t, x, y) = -2\epsilon(x - y)|x - y|^{2\gamma - 2} + \theta(2\gamma + 2)(y - \overline{y})|y - \overline{y}|^{2\gamma}, \\
B(t, x, y) = D^2_{x,y} \varphi_\epsilon(t, x, y) = \frac{1}{\epsilon} \begin{pmatrix} a_1(x, y) & -a_1(x, y) \\ -a_1(x, y) & a_1(x, y) \end{pmatrix} + \begin{pmatrix} a_2(x) & 0 \\ 0 & a_2(y) \end{pmatrix}
\end{cases}
\]
(4.10)

with
\[ a_1(x, y) = \gamma|x - y|^{2\gamma - 2}I + \gamma(2\gamma - 2)(x - y)(x - y)^*|x - y|^{2\gamma - 4} \]
and
\[ a_2(x) = \theta(2\gamma + 2)|x - \overline{x}|^{2\gamma}I + 2\theta(2\gamma + 2)(x - \overline{x})(x - \overline{x})^*|x - \overline{x}|^{2\gamma - 2}. \]

Taking into account (4.8) then applying the result by Crandall et al. (Theorem 8.3, [7]) to the function
\[ u_{i_0}(t, x) - (1 - \lambda)w_{i_0}(t, y) - \varphi_\epsilon(t, x, y) \]
at the point \((t_0, x_0, y_0)\), for any \(\epsilon_1 > 0\), we can find \(c, d \in \mathbb{R}\) and \(X, Y \in S_k\), such that:
\[
\begin{cases}
(c, \frac{2}{\epsilon}(x_0 - y_0)|x_0 - y_0|^{2\gamma - 2} + \theta(2\gamma + 2)(x_0 - \overline{x})|x_0 - \overline{x}|^{2\gamma}, X) \in J^{2,+}(u_{i_0}(t_0, x_0)), \\
(-d, \frac{2}{\epsilon}(x_0 - y_0)|x_0 - y_0|^{2\gamma - 2} - \theta(2\gamma + 2)(y_0 - \overline{y})|y_0 - \overline{y}|^{2\gamma}, Y) \in J^{2,-}(1 - \lambda)w_{i_0}(t_0, y_0)), \\
c + d = D_t \varphi_\epsilon(t_0, x_0, y_0) = 2\beta(t_0 - \overline{t}) - \frac{\lambda}{t_0} \text{ and finally} \\
-(\frac{1}{\epsilon_1} + ||B(t_0, x_0, y_0)||)I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq B(t_0, x_0, y_0) + \epsilon_1 B(t_0, x_0, y_0)^2.
\end{cases}
\]
(4.11)

Taking now into account (4.8) and the definition of viscosity solution, we get:
\[
-c + u_{i_0}(t_0, x_0) - \frac{1}{2}Tr[\sigma^*(t_0, x_0)X\sigma(t_0, x_0)] - \frac{2}{\epsilon}(x_0 - y_0)|x_0 - y_0|^{2\gamma - 2} + \theta(2\gamma + 2)(x_0 - \overline{x})|x_0 - \overline{x}|^{2\gamma}, b(t_0, x_0) \leq 0 \text{ and} \\
d + (1 - \lambda)w_{i_0}(t_0, y_0) - \frac{1}{2}Tr[\sigma^*(t_0, y_0)Y\sigma(t_0, y_0)] - \frac{2}{\epsilon}(x_0 - y_0)|x_0 - y_0|^{2\gamma - 2} - \theta(2\gamma + 2)(y_0 - \overline{y})|y_0 - \overline{y}|^{2\gamma}, b(t_0, y_0) - (1 - \lambda)e^{\theta}w_{i_0}(t_0, y_0) \geq 0
\]
which implies that:

\[-c - d + u_{io}(t_0, x_0) - (1 - \lambda)w_{io}(t_0, y_0) \leq \frac{1}{2} Tr [\sigma^*(t_0, x_0)X\sigma(t_0, x_0) - \sigma^*(t_0, y_0)Y\sigma(t_0, y_0)]
\]

\[+ \langle 2(x_0 - y_0)|x_0 - y_0|^{2\gamma - 2}, b(t_0, x_0) - b(t_0, y_0) \rangle\]

\[+ \langle \theta(2\gamma + 2)(x_0 - x)|x_0 - x|^{2\gamma}, b(t_0, x_0) \rangle\]

\[+ \langle \theta(2\gamma + 2)(y_0 - x)|y_0 - x|^{2\gamma}, b(t_0, y_0) \rangle\]

\[+ e^{\lambda o}\psi_{io}(t_0, x_0) - (1 - \lambda)e^{\lambda o}\psi_{io}(t_0, y_0).\]

(4.12)

But from (4.10) there exist two constants $C$ and $C_1$ such that:

\[||a_1(x_0, y_0)|| \leq C|x_0 - y_0|^{2\gamma - 2} \text{ and } (||a_2(x_0)|| \lor ||a_2(y_0)||) \leq C_1 \theta.\]

As

\[B = B(t_0, x_0, y_0) = \frac{1}{\epsilon} \begin{pmatrix} a_1(x_0, y_0) & -a_1(x_0, y_0) \\ -a_1(x_0, y_0) & a_1(x_0, y_0) \end{pmatrix} + \begin{pmatrix} a_2(x_0) & 0 \\ 0 & a_2(y_0) \end{pmatrix}\]

then

\[B \leq \frac{C}{\epsilon} |x_0 - y_0|^{2\gamma - 2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1 \theta I.\]

It follows that:

\[B + \epsilon_1 B^2 \leq C(\frac{1}{\epsilon} |x_0 - y_0|^{2\gamma - 2} + \frac{\epsilon_1}{\epsilon^2} |x_0 - y_0|^{4\gamma - 4}) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1 \theta I \]

(4.13)

where $C$ and $C_1$ which hereafter may change from line to line. Choosing now $\epsilon_1 = \epsilon$, yields the relation

\[B + \epsilon B^2 \leq \frac{C}{\epsilon} (|x_0 - y_0|^{2\gamma - 2} + |x_0 - y_0|^{4\gamma - 4}) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1 \theta I. \]

(4.14)

Now, from (2.1), (1.11) and (4.14) we get:

\[\frac{1}{2} Tr [\sigma^*(t_0, x_0)X\sigma(t_0, x_0) - \sigma^*(t_0, y_0)Y\sigma(t_0, y_0)] \leq \frac{C}{\epsilon} (|x_0 - y_0|^{2\gamma} + |x_0 - y_0|^{4\gamma - 2} + C_1 \theta (1 + |x_0|^2 + |y_0|^2)).\]

Next

\[\langle \frac{\gamma}{\epsilon} (x_0 - y_0)|x_0 - y_0|^{2\gamma - 2}, b(t_0, x_0) - b(t_0, y_0) \rangle \leq \frac{C^2}{\epsilon} |x_0 - y_0|^{2\gamma}\]

and finally,

\[\langle \theta(2\gamma + 2)(x_0 - x)|x_0 - x|^{2\gamma}, b(t_0, x_0) \rangle + \langle \theta(2\gamma + 2)(y_0 - x)|y_0 - x|^{2\gamma}, b(t_0, y_0) \rangle \leq \theta C(1 + |x_0||x_0 - x|^{2\gamma + 1} + |y_0||y_0 - x|^{2\gamma + 1}).\]
So that by plugging into (4.12) and note that \( \lambda > 0 \) we obtain:

\[
-2\beta(t_0 - t) + \frac{\lambda}{\lambda_0} + u_{i_0}(t_0, x_0) - (1 - \lambda)w_{i_0}(t_0, y_0) \leq C_\epsilon(|x_0 - y_0|^{2\gamma} + |x_0 - y_0|^{4\gamma - 2}) + C_1 \theta(1 + |x_0|^2 + |y_0|^2) + C_2|\epsilon|^2|x_0 - y_0|^{2\gamma} + \theta C(1 + |x_0||x_0 - \overline{\Xi}|^{2\gamma + 1} + |y_0||y_0 - \overline{\Xi}|^{2\gamma + 1}) + e^{\psi_{i_0}(t_0, x_0)} - (1 - \lambda)e^{\psi_{i_0}(t_0, y_0)}.
\]

By sending \( \epsilon \to 0 \), \( \lambda \to 0 \), \( \theta \to 0 \) and taking into account of the continuity of \( \psi_{i_0} \) and \( \gamma \geq 2 \), we obtain \( \eta \leq 0 \) which is a contradiction. The proof of Theorem 4 is now complete. \( \Box \)

As a by-product we have the following corollary:

**Corollary 1** Let \( (v^1, ..., v^m) \) be a viscosity solution of (2.4) which satisfies a polynomial growth condition then for \( i = 1, ..., m \) and \( (t, x) \in [0, T] \times \mathbb{R}^k \),

\[
v^i(t, x) = \sup_{(\delta, \xi) \in D^i_t} E\left[ \int_t^\gamma \psi_{i_\tau}(s, X^{tx}_\tau) ds - \sum_{n=1}^\tau g_{u_{r_{n-1}}u_{r_n}}(\tau, X^{tx}_{r_n}) \mathbb{I}_{[\tau_n < \gamma]} - F(\gamma, X^{tx}_\tau, u_\gamma) \mathbb{I}_{[\gamma < T]} \right]. \quad \Box
\]

## 5 Uniqueness of the viscosity solution in infinite horizon

Let \( x \in \mathbb{R}^k \) and let \( X^x \) be the solution of the following standard SDE:

\[
dX^x_t = b(X^x_t)dt + \sigma(X^x_t)dB_t, \quad X^x_0 = x
\]

where the functions \( b \) and \( \sigma \) are the ones of \( H_4 \). These properties of \( \sigma \) and \( b \) imply in particular that \( X^x \) solution of the standard SDE (5.1) exists and is unique in \( \mathbb{R}^k \). The operator \( A \) defined in (2.10) is the infinitesimal generator associated with \( X^x \).

**Theorem 5** There are deterministic functions \( v^1, ..., v^m : \mathbb{R}^k \to \mathbb{R} \) such that:

\[
\forall x \in \mathbb{R}^k, Y^{i,x}_0 = v^i(x), \quad i = 1, ..., m.
\]

Moreover the functions \( (v^1, ..., v^m) : \mathbb{R}^k \to \mathbb{R} \) are continuous, solution in viscosity sense of the system of variational inequalities with inter-connected obstacles (2.9) and of polynomial growth.

**Proof:** The continuity of functions \( v^1, ..., v^m \) follows from the dynamic programming principle and is proved in [16]. \( \Box \)

We are going now to address the question of uniqueness of the viscosity solution of the system (2.9). We have the following:

**Theorem 6** The solution in viscosity sense of the system of variational inequalities with inter-connected obstacles (2.9) is unique in the space of continuous functions on \( \mathbb{R}^k \) which satisfy a polynomial growth condition, i.e., in the space

\[
C := \{ \varphi : \mathbb{R}^k \to \mathbb{R}, \text{ continuous and for any } x, |\varphi(x)| \leq C(1 + |x|^\mu) \text{ for some constants } C \text{ and } \mu \}.
\]
Proof: We will show by contradiction that if \( u_1, ..., u_m \) and \( w_1, ..., w_m \) are a subsolution and a supersolution respectively for (2.9) then for any \( i = 1, ..., m, u_i \leq w_i \). Therefore if we have two solutions of (2.9) then they are obviously equal. Actually for some \( R > 0 \) suppose there exists \((x_0, i_0) \in B_R \times I \) \( (B_R := \{ x \in \mathbb{R}^k; |x| < R \}) \) such that:

\[
\max_{(x,i)} (u_i(x) - w_i(x)) = u_{i_0}(x_0) - w_{i_0}(x_0) = \eta > 0. \tag{5.2}
\]

Then, for a small \( \epsilon > 0 \), and \( \theta, \lambda \in (0, 1) \) small enough, let us define:

\[
\Phi^i_\epsilon(x, y) = u_i(x) - (1 - \lambda)w_i(y) - \frac{1}{2\epsilon} |x - y|^{2\gamma} - \theta(|x - x_0|^{2\gamma+2} + |y - x_0|^{2\gamma+2}). \tag{5.3}
\]

By the polynomial growth assumption on \( u_i \) and \( w_i \), there exists a \((x_\epsilon, y_\epsilon, i_\epsilon) \in B_R \times B_R \times I\), such that:

\[
\Phi^i_\epsilon(x_\epsilon, y_\epsilon) = \max_{(x,y,i)} \Phi^i_\epsilon(x, y).
\]

On the other hand, from \( 2\Phi^i_\epsilon(x_\epsilon, y_\epsilon) \geq \Phi^i_\epsilon(x_\epsilon, x_\epsilon) + \Phi^i_\epsilon(y_\epsilon, y_\epsilon) \), we have

\[
\frac{1}{2\epsilon} |x_\epsilon - y_\epsilon|^{2\gamma} \leq (u_{i_\epsilon}(x_\epsilon) - u_{i_\epsilon}(y_\epsilon)) + (1 - \lambda)(w_{i_\epsilon}(x_\epsilon) - w_{i_\epsilon}(y_\epsilon)) \\
\leq \sum_{i \in I} |u_i(x_\epsilon) - u_i(y_\epsilon)| + (1 - \lambda) \sum_{i \in I} |w_i(x_\epsilon) - w_i(y_\epsilon)| \tag{5.4}
\]

and consequently \( \frac{1}{2\epsilon} |x_\epsilon - y_\epsilon|^{2\gamma} \) is bounded, and as \( \epsilon \to 0 \), \( |x_\epsilon - y_\epsilon| \to 0 \). Since \( u_i \) and \( w_i \) are uniformly continuous on \( B_R \), then \( \frac{1}{2\epsilon} |x_\epsilon - y_\epsilon|^{2\gamma} \to 0 \) as \( \epsilon \to 0 \).

Since

\[
u_{i_0}(x_0) - (1 - \lambda)w_{i_0}(x_0) \leq \Phi^i_\epsilon(x_\epsilon, y_\epsilon) \leq u_{i_\epsilon}(x_\epsilon) - (1 - \lambda)w_{i_\epsilon}(y_\epsilon),
\]

it follow as \( \lambda \to 0 \) and the continuity of \( u_i \) and \( w_i \) that, up to a subsequence,

\[
(x_\epsilon, y_\epsilon, i_\epsilon) \to (x_0, x_0, i_0). \tag{5.5}
\]

We now claim that:

\[
u_{i_\epsilon}(x_\epsilon) - \left( \max_{j \in I - i_\epsilon} \{-g_{i,j}(x_\epsilon) + u_j(x_\epsilon)\} \right) > 0. \tag{5.6}
\]

Indeed if

\[
u_{i_\epsilon}(x_\epsilon) - \left( \max_{j \in I - i_\epsilon} \{-g_{i,j}(x_\epsilon) + u_j(x_\epsilon)\} \right) \leq 0.
\]

Case 1:

\[
\left( \max_{j \in I - i_\epsilon} \{-g_{i,j}(x_\epsilon) + u_j(x_\epsilon)\} \right) = -F_{i_\epsilon}(x_\epsilon).
\]

Then

\[
u_{i_\epsilon}(x_\epsilon) \leq -F_{i_\epsilon}(x_\epsilon).
\]
From the supersolution property of \( w_{i_{\epsilon}}(y_{\epsilon}) \), we have

\[
w_{i_{\epsilon}}(y_{\epsilon}) \geq \left( \max_{j \in I^{-i_{\epsilon}}} \{-g_{i,j}(y_{\epsilon}) + w_{j}(y_{\epsilon})\} \vee (-F_{i_{\epsilon}}(y_{\epsilon})) \right),
\]

then

\[
w_{i_{\epsilon}}(y_{\epsilon}) \geq -F_{i_{\epsilon}}(y_{\epsilon}).
\]

It follows that:

\[
u_{i_{\epsilon}}(x_{\epsilon}) - w_{i_{\epsilon}}(y_{\epsilon}) \leq -(F_{i_{\epsilon}}(x_{\epsilon}) - F_{i_{\epsilon}}(y_{\epsilon})).
\]

But this contradicts the definition of (5.2), since \( F, u, w \) is uniformly continuous on \( \overline{B}_R \) and sending \( \lambda \to 0 \) and the claim (5.6) holds.

**Case 2:**

\[
\max_{j \in I^{-i_{\epsilon}}} \{-g_{i,j}(x_{\epsilon}) + u_{j}(x_{\epsilon})\} = \max_{j \in I^{-i_{\epsilon}}} \{-g_{i,j}(x_{\epsilon}) + u_{j}(x_{\epsilon})\}.
\]

then there exists \( k \in I^{-i_{\epsilon}} \) such that:

\[
u_{i_{\epsilon}}(x_{\epsilon}) \leq -g_{i,k}(x_{\epsilon}) + u_{k}(x_{\epsilon}).
\]

From the supersolution property of \( w_{i_{\epsilon}}(y_{\epsilon}) \), we have

\[
w_{i_{\epsilon}}(y_{\epsilon}) \geq \left( \max_{j \in I^{-i_{\epsilon}}} \{-g_{i,j}(y_{\epsilon}) + w_{j}(y_{\epsilon})\} \vee (-F_{i_{\epsilon}}(y_{\epsilon})) \right),
\]

then

\[
w_{i_{\epsilon}}(y_{\epsilon}) \geq -g_{i,k}(y_{\epsilon}) + w_{k}(y_{\epsilon}).
\]

It follows that:

\[
u_{i_{\epsilon}}(x_{\epsilon}) - (1 - \lambda)w_{i_{\epsilon}}(y_{\epsilon}) - (u_{k}(x_{\epsilon}) - (1 - \lambda)w_{k}(y_{\epsilon})) \leq (1 - \lambda)g_{i,k}(y_{\epsilon}) - g_{i,k}(x_{\epsilon}).
\]

Now since \( g_{ij} \geq \alpha > 0 \), for every \( i \neq j \), and taking into account of (5.3) to obtain:

\[
\Phi_{i_{\epsilon}}(x_{\epsilon}, y_{\epsilon}) - \Phi_{k_{\epsilon}}^{t}(x_{\epsilon}, y_{\epsilon}) < -\alpha \lambda + g_{i,k}(y_{\epsilon}) - g_{i,k}(x_{\epsilon})
\]

But this contradicts the definition of \( i_{\epsilon} \), since \( g_{i,k} \) is uniformly continuous on \( \overline{B}_R \) and the claim (5.6) holds.

Next let us denote

\[
\varphi(x, y) = \frac{1}{2\epsilon}|x - y|^{2\gamma} + \theta(|x - x_0|^{2\gamma+2} + |y - x_0|^{2\gamma+2}).
\]
Then we have:

\[
\begin{align*}
D_x\varphi_\epsilon(t,x,y) &= \frac{2}{\epsilon}(x-y)|x-y|^{2\gamma-2} + \theta(2\gamma+2)(x-x_0)|x-x_0|^{2\gamma}, \\
D_y\varphi_\epsilon(t,x,y) &= -\frac{2}{\epsilon}(x-y)|x-y|^{2\gamma-2} + \theta(2\gamma+2)(y-y_0)|y-y_0|^{2\gamma}, \\
B(t,x,y) &= D_{x,y}^2\varphi_\epsilon(t,x,y) = \frac{1}{\epsilon} \begin{pmatrix} a_1(x,y) & -a_1(x,y) \\ -a_1(x,y) & a_1(x,y) \end{pmatrix} + \begin{pmatrix} a_2(x) & 0 \\ 0 & a_2(y) \end{pmatrix}
\end{align*}
\] (5.8)

with \( a_1(x,y) = \gamma|x-y|^{2\gamma-2}I + \gamma(2\gamma-2)(x-y)(y-y_0)|x-y|^{2\gamma-4} \) and

\( a_2(x) = \theta(2\gamma+2)|x-x_0|^{2\gamma}I + 2\theta(2\gamma+2)(x-x_0)(x-x_0)^*|x-x_0|^{2\gamma-2} \)

where \( I \) stands for the identity matrix of dimension \( k \). Taking into account (5.6) then applying the result by Crandall et al. (Theorem 3.2, [7]) to the function

\[ u_i(x) - (1 - \lambda)w_i(y) - \varphi_\epsilon(x,y) \]

at the point \((x_\epsilon, y_\epsilon)\), for any \( \epsilon_1 > 0 \), we can find \( X, Y \in S_k \), such that:

\[
\begin{align*}
&\left\{ \begin{align*}
\frac{2}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^{2\gamma-2} + \theta(2\gamma+2)(x_\epsilon - x_0)|x_\epsilon - x_0|^{2\gamma}, X \end{align*} \right\} \in J^{2,+}(u_i(x_\epsilon)), \\
&\left\{ \begin{align*}
\frac{2}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^{2\gamma-2} - \theta(2\gamma+2)(y_\epsilon - y_0)|y_\epsilon - y_0|^{2\gamma}, Y \end{align*} \right\} \in J^{2,-}((1 - \lambda)w_i(y_\epsilon)), \\
&-(\frac{1}{\epsilon_1} + ||B(x_\epsilon, y_\epsilon)||) \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq B(x_\epsilon, y_\epsilon) + \epsilon_1 B(x_\epsilon, y_\epsilon)^2.
\end{align*}
\] (5.9)

Taking now into account (5.6), and the definition of viscosity solution, we get:

\[
ru_i(x_\epsilon) - \frac{1}{2}Tr[\sigma^*(x_\epsilon)X\sigma(x_\epsilon)] - \frac{1}{2}Tr[\sigma^*(x_\epsilon)Y\sigma(y_\epsilon)] - (\frac{2}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^{2\gamma-2} + \theta(2\gamma+2)(x_\epsilon - x_0)|x_\epsilon - x_0|^{2\gamma}, b(x_\epsilon)) - \psi_i(x_\epsilon) \leq 0 \text{ and}
\]

\[
r(1 - \lambda)w_i(y_\epsilon) - \frac{1}{2}Tr[\sigma^*(y_\epsilon)Y\sigma(y_\epsilon)] - (\frac{2}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^{2\gamma-2} - \theta(2\gamma+2)(y_\epsilon - x_0)|y_\epsilon - x_0|^{2\gamma}, b(y_\epsilon)) - (1 - \lambda)\psi_i(y_\epsilon) \geq 0
\]

which implies that:

\[
ru_i(x_\epsilon) - r(1 - \lambda)w_i(y_\epsilon) \leq \frac{1}{2}Tr[\sigma^*(x_\epsilon)X\sigma(x_\epsilon) - \sigma^*(y_\epsilon)Y\sigma(y_\epsilon)] + (\frac{2}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^{2\gamma-2}, b(x_\epsilon) - b(y_\epsilon)) + (\theta(2\gamma+2)(x_\epsilon - x_0)|x_\epsilon - x_0|^{2\gamma}, b(x_\epsilon)) + (\theta(2\gamma+2)(y_\epsilon - x_0)|y_\epsilon - x_0|^{2\gamma}, b(y_\epsilon)) + \psi_i(x_\epsilon) - (1 - \lambda)\psi_i(y_\epsilon).
\] (5.10)

But from (5.8) there exist two constants \( C \) and \( C_1 \) such that:

\[
||a_1(x_\epsilon, y_\epsilon)|| \leq C|x_\epsilon - y_\epsilon|^{2\gamma-2} \text{ and } (||a_2(x_\epsilon)|| \lor ||a_2(y_\epsilon)||) \leq C_1 \theta.
\]
As
\[ B = B(x, y) = \frac{1}{\epsilon} \begin{pmatrix} a_1(x, y) & -a_1(x, y) \\ -a_1(x, y) & a_1(x, y) \end{pmatrix} + \begin{pmatrix} a_2(x) & 0 \\ 0 & a_2(y) \end{pmatrix} \]
then
\[ B \leq \frac{C}{\epsilon} |x - y|^{2\gamma - 2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1 \theta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \]
It follows that:
\[ B + \epsilon_1 B^2 \leq C \left( \frac{1}{\epsilon} |x - y|^{2\gamma - 2} + \frac{\epsilon_1}{\epsilon^2} |x - y|^{4\gamma - 4} \right) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1 \theta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \]
where \( C \) and \( C_1 \) which hereafter may change from line to line. Choosing now \( \epsilon_1 = \epsilon \), yields the relation
\[ B + \epsilon B^2 \leq \frac{C}{\epsilon} (|x - y|^{2\gamma - 2} + |x - y|^{4\gamma - 4}) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1 \theta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \]
Now, from \( \textbf{H4}, \text{(5.9)} \) and \( \text{(5.12)} \) we get:
\[ \frac{1}{2} Tr[\sigma^*(x)X\sigma(x) - \sigma^*(y)Y\sigma(y)] \leq \frac{C}{\epsilon} (|x - y|^{2\gamma} + |x - y|^{4\gamma - 2}) + C_1 \theta (1 + |x|^2 + |y|^2). \]
Next
\[ \langle \frac{\gamma}{\epsilon} (x - y), |x - y|^{2\gamma - 2}, b(x) - b(y) \rangle \leq \frac{C^2}{\epsilon} |x - y|^{2\gamma} \]
and finally,
\[ \langle \theta(2\gamma + 2)(x - x_0)|x - x_0|^{2\gamma}, b(x) \rangle \leq \theta C (1 + |x|^2) |x - x_0|^{2\gamma + 1} \]
\[ \langle \theta(2\gamma + 2)(y - x_0)|y - x_0|^{2\gamma}, b(y) \rangle \leq \theta C (1 + |y|^2) |y - x_0|^{2\gamma + 1}. \]
So that by plugging into \( \text{(5.10)} \) we obtain:
\[ ru_i(x) - r(1 - \lambda)w_i(y) \leq \frac{C}{\epsilon} (|x - y|^{2\gamma} + |x - y|^{4\gamma - 2}) + C_1 \theta (1 + |x|^2 + |y|^2) + \frac{C^2}{\epsilon} |x - y|^{2\gamma} + \theta C (1 + |x|^2) |x - x_0|^{2\gamma + 1} + \theta C (1 + |y|^2) |y - x_0|^{2\gamma + 1} + \psi_i(x) - (1 - \lambda)\psi_i(y). \]
By sending \( \epsilon \to 0, \lambda \to 0, \theta \to 0 \) and taking into account of the continuity of \( \psi_i \), we obtain \( u_i(0) - w_i(0) < 0 \) which is a contradiction. The proof of Theorem 6 is now complete. \( \Box \)
As a by-product we have the following result:

**Corollary 2** Let \( (v^1, ..., v^m) \) be a viscosity solution of \( \text{(2.9)} \) which satisfies a polynomial growth condition. Then for \( i = 1, ..., m \) and \( (t, x) \in \mathbb{R}^k \),
\[ v^i(x) = \sup_{(\delta, \xi) \in D_0^*} E \int_0^\gamma \int_{D_0^*} e^{-rs} \psi_s(X^{x}^s)ds - \sum_{n \geq 1} e^{-r \tau_n} g_{u_{r_{n-1}}} \langle X^{x}_{r_n} \rangle \mathbb{1}_{[\tau_n < r]} - e^{-r \gamma} F(X^{x}, u_{x}) \].
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