A further analysis of the energy-momentum for a system of point particles and for a perfect fluid in finite volume

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Abstract. A simple analysis of the total energy-momentum of a system of non-interacting point particles shows that it represents a well defined 4-vector. A generalization for interacting charged particles including the self-forces is developed. A comparison between a system of point particles in an infinite volume and the perfect fluid in a finite volume is done. The finite volume case is analyzed in order to define an energy-momentum depending on the notion of instantaneity for a perfect fluid. The relativistic transformations of thermodynamic quantities are defined in a covariant form leading to a redefined relativistic Thermodynamics.

1. Introduction
In the early times of Special Relativity, Einstein and Planck developed the theory of relativistic transformations of the thermodynamic quantities [1], [2]. However, due to the doubts that had Einstein of his own theory [3], different theories appeared in the beginning of the middle of the last century [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16]. Foregrounding the covariance of the theory, Gamba [17] improved the discussions about the correctness of the transformation laws. At the end of the last century, Sieniutycz [18] said that the controversy was still alive and the three following different proposals remained the most acceptable: the Ott [4], the Einstein-Planck [1], [2] and the Rohrlich light proposals [8]. Nevertheless, in the first years of this century, Landsberg and Matsas proposed that it is not possible to define a universal definition of the temperature [6], [19], [20], [21], [22]. Based on Fermi ideas of instantaneity and a formal definition of a 4-vector [23], [24], [25], [26], Ares de Parga et al developed a new theory which is known as the Redefined Relativistic Thermodynamics [27], [28], [29], [30], [31], [32], [33], [34].

In this paper, the concept of the 4-vector total energy-momentum is developed for a system of point particles in an infinite volume and for the perfect fluid in a finite volume [35] which permits to understand that the Einstein-Planck theory, with some corrections, represents a covariant formal theory which is called the Redefined Relativistic Thermodynamics [30]. In the course of the article, the no interaction theorem [36] is proven for classical particles and the case of a system of interacting point charged particles is analyzed including the reaction force.

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2. The Energy-Momentum Vector and Tensor for a System of Point Particles

The definition of a 4-vector total energy-momentum presents many difficulties. Before going into the definition by means of an energy-momentum tensor, it is convenient to give the simplest definition (we take $c = 1$).

2.1. The total energy-momentum vector for a system of point particles

In an inertial reference frame $K_1$, regardless whether or not there is interaction, one would be tempted to define the total 4-vector total energy-momentum of a system of particles as

$$P^\alpha_{K_1} = \sum_{i=1}^{n} p^\alpha_i(x_i(t)), \quad (1)$$

where $p^\alpha_i$ represents the 4-vector energy-momentum of the $i$th particle measured in reference frame $K_1$. As will be shown, $P^\alpha_{K_1}$ does not represent a 4-vector in general because it is the sum of many 4-vectors in different events ($x_i(t) \neq x_j(t)$ for $i \neq j$). However, when there is no interaction, free particles, $p^\alpha_i$ are constant and the sum of them represents a 4-vector. Let us prove this assertion. Consider a reference frame $K_2$ which velocity with respect to $K_1$ is $\vec{u}$. Infinite volumes $V_1$ and $V_2$ are defined as 3-hypersurface in the 4-space being instantaneous in each reference frame $K_1$ and $K_2$. The times in $K_1$ and $K_2$ are denoted as $t$ and $t'$, respectively. If the $i$th-particle trajectory is analyzed in reference frame $K_1$, it is clear that it crosses the 3-hypersurface $V_2$ in a time $t_i$ (see Figure 1). Since the 4-vector energy-momentum of each particle is constant, we have

$$p_i^\alpha(x_i(t)) = p_i^\alpha(x_i(t_i)). \quad (2)$$

![Figure 1](image-url)
Therefore,

\[ P_\alpha^{K_1} = \sum_{i=1}^{n} p_\alpha^i(x_i(t)) \]

\[ = \sum_{i=1}^{n} p_\alpha^i(x_i(t_i)). \]  

(3)

Because each \( p_\alpha^i(x_i(t_i)) \) can be expressed as

\[ p_\alpha^i(x_i(t_i)) = \Lambda_{\alpha \beta}^{K_2 \rightarrow K_1} p_\beta^i(x_i(t'_i)), \]  

(4)

where \( \Lambda_{\alpha \beta}^{K_2 \rightarrow K_1} \) represents the Lorentz transformation from \( K_2 \) to \( K_1 \) and \( x_i(t'_i) \) corresponds to the position of the \( i \)-particle in \( K_2 \). It has to be noticed that \( t'_i \) is the common time for all the particles in \( K_2 \) because the time \( t_i \) corresponds to the time in \( K_1 \) where the \( i \)-particle crosses the 3-hypersurface \( V_2 \) at a common time \( t'_i \) in \( K_2 \). Therefore, Eq. (4) can be used in Eq. (3), obtaining

\[ P_\alpha^{K_1} = \sum_{i=1}^{n} \Lambda_{\alpha \beta}^{K_2 \rightarrow K_1} \sum_{i=1}^{n} \Lambda_{\beta \gamma}^{K_1 \rightarrow K_2} p_\gamma^i(x_i(t'_i)) \]

\[ = \Lambda_{\alpha \beta}^{K_2 \rightarrow K_1} \sum_{i=1}^{n} p_\beta^i(x_i(t'_i)). \]

\[ = \Lambda_{\alpha \beta}^{K_2 \rightarrow K_1} P_\beta^{K_2}, \]  

(5)

which shows that \( P_\alpha^{K_1} \) is a 4-vector. It has to be noticed that since the 4-vector energy-momentum of the particles are constant, \( P_\alpha^{K_1} \) and \( P_\alpha^{K_2} \) are also time independent. Finally, it can be concluded that \( P_\alpha^{K_1} \) is a well defined constant 4-vector total energy-momentum when the system of particles is composed by non interacting particles.

Let us now consider a system of interacting particles. In this case, the particles cross the 3-hypersurface \( K_2 \) at different times \( t_i^{int} \). Therefore, since the momentum of each particle is not anymore constant,

\[ p_\alpha^i(x_i(t)) \neq p_\alpha^i(x_i(t_i^{int})), \]  

(6)

the above calculation cannot be repeated. Consequently, we cannot define a 4-vector total energy-momentum as in Eq. (1) (see Figure 2). This is equivalent to the no interaction theorem [36] for classical particles because it has been proven that a 4-vector total energy-momentum cannot be defined for a system of interacting particles. Indeed, this represents Pryce's major idea [37] which is equivalent to the no interaction theorem. Moreover, since in the case of free particles, the 4-vector total energy-momentum in Eq. (1) is a 4-vector, it is possible to find a reference frame \( K_o \) where the 4-vector total energy-momentum may be written as

\[ P_\alpha^{K_o} = (P_0^{K_o}, \vec{0}), \]  

(7)

with \( P_0^{K_o} \) constant. That is: a rest frame \( K_o \) exists. It has to be noticed that with this simple calculation the no interaction theorem has been proven without using Lie algebras, Lorentz groups or any sophisticated mathematical tools [36].
Figure 2. A system of interacting particles in an infinite volume. $V_2$ represents the infinite volume which is instantaneous at the reference frame $K_2$ and $V_1$ represents the infinite volume which is instantaneous at the reference frame $K_1$.

2.2. The energy-momentum tensor for a system of particles

The energy-momentum tensor is normally introduced for a continuous system. However, it can defined for a system of point particles in the following manner [35].

$$ T^{\alpha\beta}_{K} = \sum_{i=1}^{n} p_{i}^{\alpha} K dx_{\beta} i dt \delta^{3} (- \vec{x} - \vec{x}_{i}(t)). \tag{8} $$

Apparently, these quantities do not constitute a 4–tensor of rank 2 because they are not covariantly defined and they are composed of a sum of 4–vectors that are multiplied by quantities which are not 4–vectors in different points of the space-time. However, it can be expressed in a covariant form by using the four dimension delta function; that is:

$$ T^{\alpha\beta}_{K} = \sum_{i=1}^{n} \int p_{i}^{\alpha} K dx_{\beta} i d\tau_{i} \delta^{4}(x - x_{i}(\tau_{i})) d\tau_{i}. \tag{9} $$

Since $T^{\alpha\beta}_{K}$ has a covariant expression and the delta function enables it to be seen as a sum of quantities in $(x, t)$, it can be affirmed that the energy-momentum tensor is a 4–tensor of rank 2.

Let us define a 4–quantity $D^{\alpha}_{K}$ that may coincide with the 4–vector total energy-momentum for free particles:

$$ D^{\alpha}_{K} = \int T^{\alpha0}_{K} dV_{K} = \int \left[ \sum_{i=1}^{n} p_{i}^{\alpha} K dx_{0} i dt \delta^{3} (- \vec{x} - \vec{x}_{i}(t)) \right] dV_{K} = \int \left[ \sum_{i=1}^{n} p_{i}^{\alpha} \delta^{3}(x - \vec{x}_{i}(t)) \right] dV_{K} $$
\[ \sum_{i=1}^{n} p_i^\alpha(t) = P^\alpha_K, \]  

(10)

where \( V_K \) represents the infinite volume in reference frame \( K \). Therefore, \( D^\alpha_K = P^\alpha_K \). However, even if we know that this 4–vector total energy-momentum just represents a 4–vector for a system of non interacting particles, it is necessary to express it in a covariant form in order to try to develop a new expression leading to a well defined 4–vector regardless of the existence or not of interaction.

2.2.1. Conservation theorem  Let us consider \( \Theta^{\alpha\beta} \), a 4–tensor of rank 2 with the following properties:

A- \( \Theta^{\alpha\beta} \) must be of type \( C^1 \).

B- Its 4–divergence is null,

\[ \frac{\partial \Theta^{\alpha\beta}}{\partial x^\beta} = 0 \]  

(11)

C- \( \Theta^{\alpha\beta} \) vanishes for large distances (\( \Theta^{\alpha\beta} \rightarrow 0 \) for \( |\vec{r}| \rightarrow \infty \)).

A 4–volume space-time \( V^4 \) is bounded by two space-like 3–hypersurface planes \( V_1 \) and \( V_2 \) and a time-like 3–hypersurface \( \sigma \) which will be considered at infinite large distance. Integrating the 4–divergence of \( \Theta^{\alpha\beta} \) over such a 4–volume \( V^4 \) and by using the generalized Gauss theorem to four dimensions (see Figure 3), obtains

\[ \int_{V^4} \frac{\partial \Theta^{\alpha\beta}}{\partial x^\beta} d^4 x = 0 \]  

(12)

Figure 3. Conservation integral. \( V_2 \) represents the infinite volume which is instantaneous at the reference frame \( K_2 \) and \( V_1 \) represents the infinite volume which is instantaneous at the reference frame \( K_1 \).
The unitary 4−vectors \( n_{1\beta} \) and \( n_{2\beta} \) are measured in \( K_1 \) and are perpendicular to the hypersurfaces \( V_1 \) and \( V_2 \), respectively. \( n_{\sigma\beta} \) is perpendicular to the hypersurface \( \sigma \) and \( \epsilon_1 \), \( \epsilon_\sigma \) and \( \epsilon_2 \) take the values 1 or −1 depending on the hypersurface orientations. It is necessary to remember that all the quantities are measured in \( K_1 \) and when \( \sigma \to \infty \) the tensor \( \Theta^{\alpha\beta} \) vanishes. Consequently, we arrive at

\[
\int_{V_1} \Theta^{\alpha\beta}_{K_1} n_{1\beta} \, dV_1 = \int_{V_2} \Theta^{\alpha\beta}_{K_1} n_{2\beta} \, dV_2. \tag{13}
\]

Notice that all the events that belong to \( V_1 \) are instantaneous in \( K_1 \) and all the events that belong to \( V_2 \) are instantaneous in \( K_2 \). We have

\[
n_{1\beta} = (1, 0, 0, 0) \text{ and } n_{2\beta} = (\gamma, \gamma \overrightarrow{u}), \tag{14}
\]

where \( \gamma = (1 - u^2)^{-1/2} \) and \( \overrightarrow{u} \) represents the relative motion between the frame \( K_1 \) and \( K_2 \). However, in the right side of Eq. (13), \( \Theta^{\alpha\beta}_{K_1} n_{2\beta} \) is measured in \( K_1 \) and the integral must be done at the 3−hypersurface \( V_2 \). Then, it is convenient to express all the quantities in \( K_2 \). Let us call \( S^\alpha_{K_i} \) as

\[
S^\alpha_{K_i} = \int_{V_i} \Theta^{\alpha\beta}_{K_i} n_{i\beta} \, dV_i, \tag{15}
\]

where \( n_{i\beta} = (1, 0, 0, 0) \) in \( K_i \). That is,

\[
S^\alpha_{K_i} = \int_{V_i} \Theta^{\alpha0}_{K_i} \, dV_i, \tag{16}
\]

It must be noticed that \( S^\alpha_{K_i} \) is expressed in a covariant form by using Eq. (15) and in a not apparently covariant form by using Eq. (16). Finally, it is easy to see from Eq. (13) that

\[
S^\alpha_{K_1} = \int_{V_2} \Theta^{\alpha\beta}_{K_1} n_{2\beta} \, dV_2 = \int_{V_2} \Lambda^\alpha_\lambda_{K_2\to K_1} \Theta^{\lambda\beta}_{K_2} \, dV_2 = \Lambda^\alpha_\lambda_{K_2\to K_1} \int_{V_2} \Theta^{\lambda\beta}_{K_2} \, dV_2, \tag{17}
\]

Since, \( n_{2\beta} = (1, 0, 0, 0) \), one obtains

\[
S^\alpha_{K_1} = \Lambda^\alpha_\lambda_{K_2\to K_1} \int_{V_2} \Theta^{\lambda0}_{K_2} \, dV_2 = \Lambda^\alpha_\lambda_{K_2\to K_1} S^\lambda_{K_2}. \tag{18}
\]

Therefore, \( S^\alpha_{K_1} \) is a 4−vector.

Since in this case Eq. (11) is satisfied, a conservation property can be deduced. Indeed, Eq. (11) can be expressed as

\[
\frac{\partial \Theta^{\alpha0}_{K_1}}{\partial x^0} + \frac{\partial \Theta^{\alpha j}_{K_1}}{\partial x^j} = 0, \tag{19}
\]
where \( j = 1, 2, 3 \), it implies that

\[
\frac{dS_\alpha^{K_i}}{dt} = \int_{V_i} \frac{\partial \Theta_\alpha^{0}}{\partial t} dV_i = - \int_{V_i} \frac{\partial \Theta_\alpha^{j}}{\partial x^j} dV_i
\]

\[
= - \int_{A_i} \Theta_\alpha^{j} d\vec{A}_{ij}, \tag{20}
\]

where \( A_i \) and \( d\vec{A}_{ij} \) represent the surface of the volume \( V_i \) and the \( j \)-component of its surface vector, respectively. The 3-dimensional Gauss theorem has been used. Since \( \Theta_\alpha^{0} = 0 \) when \( |\vec{x}| \to \infty \) and \( V_1 \) represents all the 3-dimensional space at a time \( t \) in \( K_i \).

\[
\frac{dS_\alpha^{K_i}}{dt} = 0. \tag{21}
\]

The conservation property is that \( S_\alpha^{K_i} \) is a constant.

2.2.2. 4-vector total energy-momentum conservation

Let us prove that the energy-momentum tensor satisfies Eq (11). Since, the energy-momentum tensor is well defined in any reference frame, we abandon for the moment the subscript \( K \). Then,

\[
\frac{\partial T^{\alpha0}}{\partial x^0} = \frac{\partial}{\partial t} \left[ \sum_{i=1}^{n} p_\alpha^{i}(t) \delta^3(\vec{r} - \vec{r}_i(t)) \right]
\]

\[
= \sum_{i=1}^{n} \frac{dp_\alpha^{i}(t)}{dt} \delta^3(\vec{r} - \vec{r}_i(t)) + p_\alpha^{i}(t) \frac{\partial}{\partial t} \left[ \delta^3(\vec{r} - \vec{r}_i(t)) \right]. \tag{22}
\]

On the other hand, by using the properties of the delta function \( (j = 1, 2, 3) \)

\[
\frac{\partial T^{\alpha j}}{\partial x^j} = \sum_{i=1}^{n} p_\alpha^{i}(t) \frac{dx^j}{dt} \frac{\partial}{\partial x^j} \left[ \delta^3(\vec{r} - \vec{r}_i(t)) \right]
\]

\[
= - \sum_{i=1}^{n} p_\alpha^{i}(t) \frac{\partial}{\partial t} \left[ \delta^3(\vec{r} - \vec{r}_i(t)) \right]. \tag{23}
\]

From Eqs. (22) and (23), It is clear that

\[
\frac{\partial T^{\alpha \beta}}{\partial x^\beta} = \sum_{i=1}^{n} \frac{dp_\alpha^{i}(t)}{dt} \delta^3(\vec{r} - \vec{r}_i(t)) = G^\alpha, \tag{24}
\]

where \( G^\alpha \) represents the force density. Therefore, if there is no interaction \( dp_\alpha^{i}(t)/dt = 0 \), \( T^{\alpha \beta} \) satisfies Eq. (11) and consequently \( P^\alpha \) represents a 4-vector and is constant. In fact, as it has been noted above, this is equivalent to the no interaction theorem [36] for classical particles. That is, a 4-vector total energy-momentum can be well defined when there is no interaction and a rest reference frame \( K_0 \) exists (see Eq. (7)). However, in the general case, the interaction exists and it is needed to add an extra tensor to the energy-momentum tensor in order to obtain a null-divergence of the energy-momentum tensor. Let us consider a system of charged particles where the electromagnetic interaction has to be considered and accordingly \( G^\alpha \) does not vanish. It is well known that the electromagnetic energy-momentum field tensor \( T^{\alpha \beta}_{em} \) can be expressed as

\[
T^{\alpha \beta}_{em} = F^{\alpha \lambda} F^{\beta \lambda} - \frac{1}{4} \epsilon^{\alpha \beta \lambda \epsilon} F_{\lambda \epsilon} F^{\lambda \epsilon}, \tag{25}
\]
where $F^{\alpha\beta}$ and $\eta^{\alpha\beta}$ represent the stress electromagnetic tensor and the Minkowskian metric tensor, respectively. We have

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = -F^{\alpha\lambda}J_\lambda,$$

(26)

where

$$J_\alpha = \sum_{i=1}^n q_i \frac{dx_i^\alpha(t)}{dt} \delta^3(\mathbf{r} - \mathbf{r}_i(t)),$$

(27)

being $q_i$ the charge of the $i-$particle. Then,

$$F^{\alpha\lambda}J_\lambda = \sum_{i=1}^n q_i F^{\alpha\lambda} \frac{dx_i^\lambda(t)}{dt} \delta^3(\mathbf{r} - \mathbf{r}_i(t)).$$

(28)

Lorentz equation permits us to arrive at

$$F^{\alpha\lambda}J_\lambda = \sum_{i=1}^n q_i \frac{dp_i^\alpha}{dt} \delta^3(\mathbf{r} - \mathbf{r}_i(t)) = G^\alpha.$$  

(29)

Then, by adding to the energy-momentum tensor the electromagnetic energy-momentum field tensor, a total energy-momentum tensor can be defined as

$$T^{\alpha\beta} = T^{\alpha\beta}_{\text{em}} + T^{\alpha\beta}_{\text{em}},$$

(30)

with the interesting property deduced from Eqs. (26) and (29),

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0.$$  

(31)

Thus a 4-vector total energy-momentum can be defined as

$$P_{\alpha K_i}^\alpha = \int T_{\alpha K_i}^{\alpha\beta} dV_i,$$

(32)

and a rest reference frame $K_o$ such that

$$P_{\alpha K_o}^\alpha = (p^0, 0, 0, 0).$$

(33)

2.2.3. Divergences and reaction force  The 4-vector total energy-momentum and the energy-momentum tensor posses some inconveniences. Indeed, Eq. (25) shows that the electromagnetic field must be evaluated at the point particles which leads to divergences in both vector and tensor. Some renormalization technique has been developed in order to avoid such infinities in the deduction of the reaction force [38]; even more in General Relativity, similar techniques were developed to find the expression of the reaction force [39], [40], [41], [42]. Although, there exist different points of view as the perfect absorber universe developed by Wheeler and Feynman [43], [44] and similar theories where action at a distance theories are used, Dirac’s result [38] is the most accepted method to deduce the reaction force. However, due to some physical inconsistencies as the runaway’s solutions and the preaccelerations, corrections have been made for the Lorentz-Dirac equation leading principally to two equations, Eliezer equation [45] and Landau-Lifshitz equation [46], which are mathematically different but physically equivalent since their differences cannot be detected within Classical Electrodynamics [47], [48], [49], [50] (the
differences are out of Shen’s zone \[51\] where quantum effects are important). Therefore, Eq. (28) can be rewritten as

\[ F^{\alpha\lambda}_{J^\lambda} = \sum_{i=1}^{n} q_i F^{\alpha\lambda} \frac{dx^\lambda_i(t)}{dt} \delta^3(\vec{r} - \vec{r}_i(t)) \]

\[ = \sum_{k=1}^{n} \sum_{i=1}^{n} q_i F^{\alpha\lambda}_k \frac{dx^\lambda_i(t)}{dt} \delta^3(\vec{r} - \vec{r}_i(t)) \]

\[ = \sum_{k=1}^{n} \sum_{i=1}^{n} q_i F^{\alpha\lambda}_k \frac{dx^\lambda_i(t)}{dt} \delta^3(\vec{r} - \vec{r}_i(t)) \]

\[ + \sum_{i=1}^{n} q_i F^{\alpha\lambda}_{\text{self}} \frac{dx^\lambda_i(t)}{dt} \delta^3(\vec{r} - \vec{r}_i(t)). \] (34)

The term \( q_i F^{\alpha\lambda}_k (dx^\lambda_i(t)/d\tau_i) \) represents the force exerted by the \( k \)-particle on the \( i \)-particle. In particular, \( q_i F^{\alpha\lambda}_{\text{self}} (dx^\lambda_i(t)/d\tau_i) \) is the self-force and may be expressed as the radiation force \( f_{i-rad}^\mu \), such that

\[ \frac{dp^\alpha_i}{d\tau_i} = \sum_{i \neq k} q_i F^{\alpha\lambda}_k \frac{dx^\lambda_i(t)}{d\tau_i} + f_{i-rad}^\mu \]

\[ = q_i F^{\alpha\lambda}_i \frac{dx^\lambda_i(t)}{d\tau_i} + f_{i-rad}^\mu; \] (35)

where \( F^{\alpha\lambda} = \sum_{i \neq k} q_i F^{\alpha\lambda}_k \). Therefore, if we substitute this result on Eq. (35), we obtain

\[ F^{\alpha\lambda}_{J^\lambda} = \sum_{i=1}^{n} \left[ q_i F^{\alpha\lambda}_i \frac{dx^\lambda_i(t)}{dt} + f_{i-rad}^\mu \frac{d\tau_i}{dt} \right] \delta^3(\vec{r} - \vec{r}_i(t)) \]

\[ = \sum_{i=1}^{n} \frac{dp^\alpha_i}{d\tau_i} \delta^3(\vec{r} - \vec{r}_i(t)). \] (36)

Finally, Eq. (36) shows that even if we consider the reaction force (or self-force), Eq. (11) is satisfied. However, the problem is not closed since there are many discussions about the validity of the renormalization process and the expression for the reaction force. Moreover, the total tensor also possesses divergences which complicate the calculation of the the total energy-momentum tensor. The principal problem is due to the divergence of the electromagnetic field in the point particle itself (\( T^{\alpha\beta}_{\text{em}} \) is not a \( C^1 \) type). Frequently, a system of charged particles can be represented with a high degree of approximation by continuous density and current. In such a cases, the divergences disappear and the total energy-momentum is well defined.

### 3. Discrete Density Representation of the Energy-Momentum Tensor of a system of particles

As we mentioned above, in many cases a continuous description of a system of particles is adequate. However, this does not mean that densities have to be continuous. Indeed, we can describe a system of particles by using a discrete density for a free particle system. First, consider that a rest frame \( K_o \) exists as in Eq. (7); then, we can define a discrete energy density \( \rho_d \),

\[ \rho_d = T^{00}_{K_o} = \sum_{i=1}^{n} \rho_i^0 \delta^3(\vec{r} - \vec{r}_i(t)), \] (37)
and a discrete pressure $p_d$,

$$p_d = \frac{1}{3} \sum_{j=1}^{3} T^{jj}_{K_o} = \frac{1}{3} \sum_{j=1}^{3} \sum_{i=1}^{n} p^j_i \frac{dx^j_i}{dt} \delta^3(\vec{x} - \vec{x}^i(t)).$$

(38)

Being the other components,

$$T^{oj}_{K_o} = \sum_{i=1}^{n} p^j_i \frac{dx^j_i}{dt} \delta^3(\vec{x} - \vec{x}^i(t))$$

$$= \sum_{i=1}^{n} p^j_i \delta^3(\vec{x} - \vec{x}^i(t)) = T^{j0}_{K_o}$$

(39)

and for $j \neq k$

$$T^{jk}_{K_o} = \sum_{i=1}^{n} p^j_i \frac{dx^k_i}{dt} \delta^3(\vec{x} - \vec{x}^i(t)).$$

(40)

The energy-momentum is calculated as

$$P^0_{K_o} = \int \rho_d dV_o$$

$$= \int \sum_{i=1}^{n} p^0_i \delta^3(\vec{x} - \vec{x}^i(t)) dV_o$$

$$= \sum_{i=1}^{n} p^0_i .$$

(41)

and

$$P^j_{K_o} = \int \sum_{i=1}^{n} p^j_i \delta^3(\vec{x} - \vec{x}^i(t)) dV_o$$

$$= \sum_{i=1}^{n} p^j_i = 0,$$

(42)

as expected since $K_o$ is the rest frame. It is important to notice that $T^{oj}_{K_o} \neq 0$, Eq. (39), but for the purpose of calculating the spatial components of the 4-vector total energy-momentum in the rest frame $K_o$, it does not contribute, Eq. (42). However, strictly $T^{0j}_{K_o} \neq 0$ and the other components $T^{jk} (j \neq k)$ do not vanish too. Although in this calculation we have just repeated the results of section 2, the question that arises is whether a system of free particles can actually be described by a 4-vector total energy-momentum and by a energy-momentum tensor with the same structure than the one used in the continuous cases. The answer is contained in the following subsections and sections.

3.1. Three cases of free particle systems

In order to posteriori compare discrete free particle systems with different types of fluids which are continuous, it is necessary to impose some constraints on the energy-momentum tensor.

3.1.1. The imperfect discrete free particle system This is the case of a regular discrete free particle system with the characteristic of not possessing any constraint; that is: $T^{\alpha\beta}$ do not vanish as the general case which is just described above in Eqs. (37), (38), (39) and (40).
3.1.2. The dust discrete free particle system

In this case, the particles of the system are at rest in the rest reference frame $K_o$; that is:
\[ p^j_i = 0 \quad \text{for} \quad j = 1, 2, 3. \]  \hfill (43)

This means that
\[ \rho_d = \sum_{i=1}^{n} p^0_i \delta^3(\vec{x} - \vec{x}_i(t)). \]  \hfill (44)

Consequently,
\[ \begin{cases} T^{00}_{K_o} = \rho_d \\ T^{\alpha\beta}_{K_o} = 0 \quad \text{Otherwise} \end{cases} \]  \hfill (45)

In a subsequent section, it will be shown that this representation of the energy-momentum tensor will generate a 4-vector total energy-momentum similar to the case of the continuous dust fluid in a finite volume. Notice that the pressure vanishes in this case because all the particles are at rest in the rest reference frame $K_o$.

3.1.3. The perfect discrete free particle system

In this case we will impose to a discrete free particle system the following constraints:
\[ \begin{cases} T^{jk}_{K_o} = p_d \delta_{jk} \quad \text{for} \quad j, k = 1, 2, 3 \\ T^{k0}_{K_o} = 0 \quad \text{for} \quad k = 1, 2, 3 \\ T^{00}_{K_o} = \rho_d \end{cases} \]  \hfill (46)

where $\rho_d$ and $p_d$ can be described by Eqs. (37) and (38). However, these constraints impose symmetries on the free particle system. For example, in order to accomplish Eq. (46), in each trajectory of a particle with spatial components of the momentum $p^j_i$ there must exist a set $A_i$ of other particles $k (k \in A_i, i \in A_i)$ with the same trajectories and with spatial components of the momentum $p^j_k$ such that the sum of all the momenta of the particles in $A$ vanishes; that is:
\[ \sum_{k \in A_i} p^j_k = 0. \]  \hfill (47)

Therefore,
\[ \sum_{k \in A_i} p^j_k \delta(\vec{x} - \vec{x}_k) = \delta(\vec{x} - \vec{x}_i) \sum_{k \in A_i} p^j_k \quad \text{with} \quad i \in A_i \]  \hfill (48)

The union of all sets $A_i$ being the set of all the particles of the system; that is: $\cup A_i = \{1, 2, 3, ..., n\}$. Therefore,
\[ T^{j0} = \sum_{i=1}^{n} p^j_i \delta^3(\vec{x} - \vec{x}_i(t)) = \sum_{\cup A_i} \sum_{k \in A_i} p^j_k \delta(\vec{x} - \vec{x}_k) = 0. \]  \hfill (49)

This cannot be accomplished unless we consider that the differential of the volume $dV_o$ contains a number of particles with this kind of properties; that is: if we consider a continuous fluid. Finally, the energy-momentum tensor of a discrete set of particles cannot satisfy Eq. (46) unless it is considered as a continuous fluid. Moreover, we will see that the representation of the energy-momentum tensor in Eq. (46) does not satisfy Eq. (11).
3.2. Relativistic transformations with the discrete energy-momentum tensor

In order to know how useful are the discrete energy-momentum tensors just described above, let us analyze their relativistic transformations and the 4–vector energy-momenta obtained from them.

3.2.1. Transformation of the total dust discrete energy-momentum

The discrete dust energy-momentum in a reference frame \( K \) is obtained by applying Lorentz transformations to Eq. (45),

\[
T_{\alpha\beta}^K = \rho_d u^\alpha u^\beta,
\]

where \( u^\alpha \) represents the 4–vector which describes the motion between \( K \) and \( K_0 \) and \( \rho_d \) is measured in the rest reference frame \( K_0 \). We have to remember that in the definition of the dust discrete free particle system, Eq. (45), all the particles are at rest in a particular reference frame which is considered as the rest frame \( K_0 \). Therefore, the existence of the rest frame is assured by definition in the case of the dust discrete free particle system. Let us calculate the total 4–vector energy-momentum. Let us see that the total 4–vector energy-momentum is a 4–vector. We have,

\[
P^\alpha_K = \int \rho_d u^\alpha u^0 dV = \int \rho_d u^\alpha u^0 \gamma dV_o = u^\alpha \int \rho_d dV_o = u^\alpha T_{\beta}^0_{K_0},
\]

as expected. But this result must be consistent with the conservation theorem. If a 4–tensor vanishes in a particular reference frame, it vanishes in all the reference frame. Therefore, we can analyze the conservation theorem in the rest frame \( K_0 \). Since in such a frame \( u^\alpha = (1, 0, 0, 0) \), we need just to analyze the component “0”; that is:

\[
\frac{\partial T_{\beta}^0_{K_0}}{\partial x^\beta} = \frac{\partial \rho_d}{\partial t} = \frac{\partial}{\partial t} \left[ \sum_i n_i p_{i}^0 \delta^3(\mathbf{x} - \mathbf{x}_i) \right]
\]

\[
= \sum_i \frac{dp_{i}^0}{dt} \delta^3(\mathbf{x} - \mathbf{x}_i) - p_{i}^0 \frac{dx_i}{dt} \frac{\partial}{\partial x^\beta} \delta^3(\mathbf{x} - \mathbf{x}_i) \]

\[
= 0.
\]

Since we know that in the case of a dust discrete free particle system both \( \frac{dp_{i}^0}{dt} \) and \( \frac{dx_i}{dt} \) vanish (remember that all the particles are in rest in \( K_0 \)). Therefore, the conservation theorem is accomplished. We will see that for a dust continuous fluid, the result holds for a finite volume.

3.2.2. Transformation of the total energy-momentum for a perfect discrete free particle system

In the case of a perfect discrete free particle system, by using Eq. (46) and the Lorentz transformations, it is easy to see that the energy-momentum tensor may be written in any reference frame \( K \) as

\[
T_{\alpha\beta}^K = (\rho_d + p_d) u^\alpha u^\beta - p_d n_{\alpha\beta},
\]

where \( \rho_d \) and \( p_d \) must be measured in the rest frame \( K_0 \) and \( n_{\alpha\beta} \) is the Minkowskian metric. In the case of infinite volume, let us see if this formalism leads to the same result obtained in Eq. (10).

\[
P^\alpha_K = \int [(\rho_d + p_d) u^\alpha u^0 - p_d n^{\alpha}{}^0] dV.
\]
For $\alpha = 0$, considering $u^\alpha = (\gamma, \gamma u)$, we have

$$P^0_K = \int T^{00} dV = \int \left[ (\rho_d + p_d)u^0u^0 - p_d n^{00} \right] dV$$

$$= \int \frac{[(\rho_d + p_d)\gamma \gamma - p_d]}{\gamma} dV_o$$

$$= \gamma P^0_{K_o} + \gamma (\mathbf{\varpi})^2 \frac{1}{3} \sum_{j=1}^{n} \sum_{i=1}^{3} p^j_i \frac{dx^j_i}{dt}. \hspace{1cm} (55)$$

We can conclude that it does not coincide with the transformation described in Eq. (10). For the corresponding spatial coordinates, we have, $j = 1, 2, 3$

$$P^j_K = \int \left[ (\rho_d + p_d)u^j u^0 - p_d n^{j0} \right] dV$$

$$= \int \left[ (\rho_d + p_d)\gamma^2 u^j \right] dV$$

$$= \int \left[ (\rho_d + p_d)\gamma^2 u^j \right] dV_o$$

$$= \gamma u^j \left( P^0_{K_o} + \frac{1}{3} \sum_{k=1}^{n} \sum_{i=1}^{3} p^k_i \frac{dx^k_i}{dt} \right), \hspace{1cm} (56)$$

where $u_x$ represents the $j$–component of $\mathbf{\varpi}$. We notice that this 4–vector energy-momentum does not transform as the 4–vector in Eq.(10). For the perfect discrete system of free particles, the 4–divergence of the energy-momentum defined in Eq. (53) does not vanish. Indeed, as we noticed above, if a 4–tensor vanishes in a reference frame, it will vanish in all other reference frames. Therefore, we just need to show that the energy-momentum tensor in the rest frame, Eq. (46), does not satisfy Eq. (11); that is:

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} \neq 0. \hspace{1cm} (57)$$

Let us proved it. We calculate

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = \frac{\partial}{\partial x^\beta} \left[ (\rho_d + p_d) u^\alpha u^\beta - p_d n^{\alpha\beta} \right]. \hspace{1cm} (58)$$

In this case $u^\alpha$ represents the 4–velocity between the rest frame $K_o$ and itself. Consequently, $u^\alpha$ is equal to $u^\alpha = (1, 0)$. Therefore,

$$\frac{\partial T^{0\beta}}{\partial x^\beta} = \left[ \frac{\partial}{\partial x^\beta} (\rho_d + p_d) \right] u^\beta - \left[ \frac{\partial}{\partial x^\beta} p_d \right] n^{0\beta}. \hspace{1cm} (59)$$

Let us consider $\alpha = 0$,

$$\frac{\partial T^{0\beta}}{\partial x^\beta} = \left[ \frac{\partial}{\partial x^\beta} (\rho_d + p_d) \right] u^\beta - \left[ \frac{\partial}{\partial x^\beta} p_d \right] n^{0\beta}$$

$$= \frac{\partial}{\partial t} (\rho_d + p_d) - \frac{\partial}{\partial t} p_d$$
\[
\begin{align*}
\frac{\partial}{\partial t} \rho_d &= \frac{\partial}{\partial t} \left[ \sum_{i=1}^{n} p_i^0 \delta^3(x - x_i(t)) \right] \\
&= \sum_{i=1}^{n} \left[ \frac{dp_i^0}{dt} \delta^3(x - x_i(t)) - p_i^0 \frac{dx_i^j}{dt} \frac{\partial}{\partial x_i^j} \delta^3(x - x_i) \right] \\
&= \sum_{i=1}^{n} \left[ \frac{dp_i^0}{dt} \delta^3(x - x_i(t)) + \frac{\partial}{\partial x^j} \left[ p_i^j \delta^3(x - x_i) \right] \right].
\end{align*}
\]

(60)

Since we are considering that there is no interaction, \( \frac{dp_i^0}{dt} = 0 \),

\[
\frac{\partial T^{0\beta}}{\partial x^\beta} = \frac{\partial}{\partial x^j} \left[ \sum_{i=1}^{n} \left[ p_i^j \delta^3(x - x_i) \right] \right] = \frac{\partial}{\partial x^j} T^{0j},
\]

(61)

which does not vanish in general. However, in order to have \( T^{00} = 0 \), as in Eq. (46), Eq. (49) must be satisfied which will imply that the divergence in Eq. (61) vanishes. As we noted above, this cannot be accomplished unless a continuous fluid is considered instead of a system of free particles. The reason is because in Eq. (23), it is clear that the term \( T^{00} \) contributes to nullify the divergence but in the case of this representation the non-diagonal elements are not considered.

Therefore, in this scheme, the conservation theorem is not assured and \( P^\alpha \) is not defined as a regular 4-vector. Even more, the existence of the rest frame is not assured. However, the formalism can be used to define a 4-vector but based in defining the vector on the concept of instantaneity. That is, defining the vector on a reference frame where the quantity is measured in certain reference frame \( K \). The existence of the rest frame \( K_o \) is assured when the system of particles or the fluid is constrained to a physical volume. Moreover, this will permit using a finite volume which is an important application in Relativistic Thermodynamics.

4. 4–Vector Total Energy-Momentum in a Finite Volume

Until now we have shown that for a system of free particles it is possible to define a 4–vector total energy-momentum by using an infinite volume. A representation of a perfect discrete free particle system does not gives a universal definition of a 4–vector total energy-momentum due to the invalidity of the conservation theorem unless the acceptance of symmetries that implies a continuous description of the system of particles, Eq. (61). When a finite volume (not physical) is considered with a system of particles, the particles will abandon the volume and the energy-momentum of the system contained in the volume will change or if the volume represents a physical volume due to the interaction of the particles with the walls of the volume, the conservation theorem will not apply. However, for finite volume and perfect fluid, it is possible to define a 4–vector total energy-momentum as long as it is related to the instantaneity in a particular reference frame. Similar results have been obtained by Callen and Horwitz [52] by defining a 4-vector enthalpy-momentum. Moreover, as we will see, the result obtained with the perfect fluid in a finite volume corresponds to the one deduced with the discrete free particle system in an infinite volume provided certain terms are substituted as follows [53]:

\[
P^{0}_{K_o} = \int \rho_o dV_o \rightarrow E_o \quad \text{and} \quad \frac{1}{3} \sum_{j=1}^{3} \sum_{i=1}^{n} p_i^j \frac{dx_i^j}{dt} \rightarrow pV_o.
\]

(62)
Let us begin by considering a perfect fluid which expression for the energy-momentum tensor similar to Eq. (53), but with the difference that the density $\rho$ and the pressure $p$ correspond to a continuous uniform system which is in thermal equilibrium. $\rho$ and $p$ are constant and they are measured in a reference frame $K_o$ where the fluid is contained in a finite volume at rest in $K_o$; that is:

$$T^{\alpha\beta}_K = (\rho + p)u^\alpha u^\beta - pn^{\alpha\beta}. \quad (63)$$

Since $\rho$ and $p$ are constant, the conservation theorem is satisfied; that is,

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta}_K = 0. \quad (64)$$

Remember that in this scheme $u^\alpha$ represents the velocity between the frames $K$ and $K_o$ and it is constant. By using Eq. (64), it can be thought out that a 4-vector total energy-momentum could be defined as in Eq. (18) but first it has to be remembered that we are dealing with a finite volume. Accordingly, the energy-momentum on the finite time-like 3-hypersurface $\sigma$ (see Eq. (12)) will not vanish and therefore a universal 4-vector energy-momentum cannot be obtained. This can be corrected by forcing $\rho$ to vanish in the finite time-like 3-hypersurface $\sigma$ which corresponds to having a non constant $\rho$ in the of surface of the volume. This implies that Eq. (64) is not valid. Physically, this represents a fluid contained in a volume and there must exist interaction between the particles of the fluid and the walls of the volume. Therefore, in order to define a 4-quantity $P^\alpha_{K,K}$ in a reference frame $K$, it is necessary to resort to the concept of instantaneity defined in the projected volume $V_K$ of $V_o$ (see Figure 4).

![Figure 4. Finite Volume](image-url)

Hence, the instantaneity along $V_K$ is described by $n_{K,\alpha} = (1, 0, 0, 0)$ and $P^\alpha_{K,K}$ by

$$P^\alpha_{K,K} = \int_{V_K} \left[ (\rho + p)u^\alpha u^0 - pn^\alpha 0 \right] dV_K$$

$$= \int_{V_K} \left[ (\rho + p)u^\alpha u^\beta - pn^\alpha_\beta \right] n_{K,\beta} dV_K$$
\[
\alpha \lambda 
\int_{V_o} \left[ (\rho + p) u^\lambda - p n_{K_o}^\lambda \right] dV_o, \\
\]
with
\[
n_{K_o}^{\alpha} = (\gamma, \gamma \bar{u}) \quad \text{and} \quad u^\alpha = (1, 0, 0, 0).
\]

Since in the parenthesis of Eq. (65), \(u^\alpha\) represents the relative motion of \(K_o\) with itself. Of course \(n_{K_o}^{\alpha}\) represents the instantaneity in \(K\) measured in \(K_o\). Therefore,

\[
P^\alpha_{K,K} = \Lambda^\alpha_{K_o \rightarrow K} \int_{V_o} \left[ (\rho + p) u^\lambda \gamma - p n_{K_o}^\lambda \right] dV_o
\]

Finally,

\[
P^0_{K,K} = \Lambda^0_{K_o \rightarrow K} \int_{V_o} \left[ (\rho + p) u^0 \gamma - p n_{K_o}^0 \right] dV_o \\
= \Lambda^0_{K_o \rightarrow K} \int_{V_o} [(\rho + p) \gamma - p \gamma] dV_o \\
+ \Lambda^0_{K_o \rightarrow K} \int_{V_o} [(\rho + p) 0 \gamma - p \gamma u_{x_j}] dV_o \\
= \gamma \left( E_o + (\bar{u})^2 p V_o \right),
\]

where
\[
E_o = \int_{V_o} \rho dV_o.
\]

For the spatial component
\[
P^j_{K,K} = \Lambda^j_{K_o \rightarrow K} \int_{V_o} \left[ (\rho + p) u^j \gamma - p n_{K_o}^j \right] dV_o \\
= \Lambda^j_{K_o \rightarrow K} \int_{V_o} [(\rho + p) u^j \gamma - p n_{K_o}^j] dV_o \\
+ \Lambda^j_{K_o \rightarrow K} \int_{V_o} [(\rho + p) u^k \gamma - p n_{K_o}^k] dV_o \\
= \gamma u_{x_j} \int_{V_o} [(\rho + p) \gamma - p \gamma] dV_o \\
+ \gamma \int_{V_o} (-p \gamma u_{x_j}) dV_o \\
= \gamma u_{x_j} (E_o + p V_o).
\]

Eqs. (68) and (70) does not correspond to what it was expected. Indeed, the vector in \(K_o\) is not \((E_o, 0)\) but \((E_o, p \bar{u} V_o)\). We were expecting to obtain
\[
P^\alpha_{K,K} = \Lambda^\alpha_{K_o \rightarrow K} (E_o, 0) = \Lambda^\alpha_{K_o \rightarrow K_o} P^\alpha_{K_o, K_o}.
\]

The reason why for a finite volume the energy-momentum does not behave as in Eq. (71), consists of noticing that the existence of a volume implies an interaction with the walls of the volume and consequently Eq. (11) is not anymore valid. Therefore,

\[
P^\alpha_{K,K} = \left\{ \begin{array}{l}
\gamma \left( P^0_{K_o} + u^2 p V_o \right) \\
\gamma u_{x_j} \left( P^0_{K_o} + p V_o \right)
\end{array} \right\},
\]

(72)
corresponds to the Lorentz transformation of the $4-$quantity $(E_o, p \overrightarrow{u} V_o)$. Landau and Lifshitz [54] does not check the $0-$component of the vector and simply affirmed that $pV_o$ must be considered as part of the mass of the system. This can be considered for low velocities. However, for relativistic velocities, the case we are concerned, the $0-$component possesses a $(\overrightarrow{u})^2$ term in the parenthesis. A similar result happens when the electromagnetic moment is calculated as Gamba [17], [24] has noticed by considering a universal definition of the electromagnetic energy-momentum vector.

Before continuing, let us make two comments:

A- First, it has to be noticed that if we substitute in Eq. (72) the corresponding changes in Eq. (62), we will obtain Eqs. (55) and (56), and vice versa. This shows that both technique are based on defining a $4-$vector depending on which reference frame the instantaneity is considered.

B- Second, when continuous systems are considered, each part of the fluid possesses a velocity; that is, $u^\alpha$ represents the velocity of an infinitesimal part of the fluid contained in $dV$ at a point $\overrightarrow{r}$ and a time $t$ with respect a reference frame $K$. This means that there are comoving frames for each part of the fluid and $u^\alpha = u^\alpha(\overrightarrow{r}, t)$. Consequently, their derivatives do not have to vanish. $u^\alpha$ is a function of the position and time. Therefore, some constraints can be used for forcing the conservation theorem. That is:

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0 = \frac{\partial}{\partial x^\beta} \left[ (\rho + p) u^\alpha u^\beta - p n^{\alpha\beta} \right]$$

$$= \left[ \frac{\partial}{\partial x^\beta} (\rho + p) u^\alpha u^\beta \right] - \left[ \frac{\partial}{\partial x^\beta} p \right] n^{\alpha\beta}$$

$$+ (\rho + p) \frac{\partial}{\partial x^\beta} \left( u^\alpha u^\beta \right).$$

(73)

This equation allows to conserve the energy-momentum for a continuous perfect fluid by imposing some constraint to the velocities of each element of the fluid (see Weinberg [35]). Notice that a generalization of the energy-momentum tensor to General Relativity includes the derivation of $g^{\alpha\beta}$ ($n^{\alpha\beta} \rightarrow g^{\alpha\beta}$) complicating the equations. Nevertheless, Relativistic Hydrodynamics will imply interaction with the walls of the volume and the conservation theorem will not apply too and it will lead to a non universal definition of the $4-$vector momentum.

If we are considering a system of particles which are concentrated in a volume (not physical), these particles eventually will abandon the finite volume and there will be no conservation relation and the result of Sec. 1 will not apply. For a finite time, the volume can be considered enough big and the theory of Sec. 1 will be valid but by using the particle energy-momentum tensor, Eq. (8), and considering that at a moment the particle will abandon the volume.

Once we know that the discrete density model does not permit to universally define a $4-$vector total energy-momentum and that the regular energy-momentum tensor, Eq. (18) does not match with the continuous energy-momentum tensor, Eq. (53), we can abandon the discrete representation and put our attention just in the continuous case. The density and pressure of a thermodynamic system are constant and the relative velocity between two inertial reference frame, $\overrightarrow{w}$, is constant. It is clear that Eq. (11) is accomplished for such constant energy-momentum tensor. However, the existence of a physical volume implies some interaction between the particles and the walls of the volume. When we obligate the particles to live in a finite volume, the problem of defining a $4-$vector total energy-momentum persists. Indeed, a finite volume implies that the interaction between the walls of the physical volume and the particles exists. Consequently, the conservation equation, Eq. (11) is not anymore valid. Therefore, we need to develop a theory
where a 4-vector total energy-momentum can be defined for finite volume even if it depends on a particular reference frame where the instantaneity is considered.

Let us begin by defining a 4-quantity in a reference frame where the instantaneity is taken. The hypersurface is represented by a 4-vector \( n_{K}^{\alpha} = (1,0,0,0) \) which represents the volume where the physical quantities will be measured. Then

\[
P_{\alpha}^{K,K} = \int_{V} T_{\alpha \beta}^{\gamma} n_{K}^{\beta} dV_{K}. \tag{74}
\]

This represents a 4-vector measured in a reference frame \( K \) where the instantaneity is in \( K \).

Without losing generality, just two dimensions are considered in the following. Let us calculate it: \( u^{\alpha} = (\gamma, \gamma u) \) in \( K \) and \( n_{K}^{\alpha} = (1,0) \) in \( K \).

\[
P_{\alpha}^{K,K} = \int_{V_{o}} \left[ (\rho + p) u^{\alpha} \gamma - p \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] dV_{o} \gamma
\]

\[
= \int_{V_{o}} \left[ (\rho + p) \left( \frac{\gamma}{\gamma u} \right) - p \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] dV_{o} \gamma
\]

\[
= \left( \frac{\gamma}{\gamma u} \int_{V_{o}} (\rho + p) dV_{o} \right) \gamma u f_{V_{o}}(\rho + p) dV_{o}
\]

\[
= \left( \frac{\gamma u}{\gamma u} \int_{V_{o}} \gamma u (\rho + p) dV_{o} \right)
\]

\[
= \frac{\gamma}{\gamma u} (E_{o} + u^{2} pV_{o})
\]

\[
= \frac{\gamma u(E_{o} + pV_{o})}{\gamma u(\gamma u + pV_{o})} \tag{75}
\]

Eq. (75) represents the Einstein-Planck-Tolman transformation of the 4-vector energy-momentum. It has to be noted that the 4-quantity \( P_{\alpha}^{K,K} \) must transform into the 4-quantity in Eq. (76) by a Lorentz transformation. Let us confirm this assertion.

\[
P_{\alpha}^{K,K_{o}} = \Lambda_{K \rightarrow K_{o}}^{\alpha} \Lambda_{\alpha}^{\lambda} K.K_{K_{o}}^{\lambda}
\]

\[
= \int_{V_{o}} T_{\alpha \beta}^{\gamma} n_{K}^{\beta} dV_{K}
\]

\[
= \int_{V_{o}} T_{\alpha \beta}^{\gamma} n_{K}^{\beta} dV_{o}
\]

\[
= \int_{V_{o}} T_{\alpha \beta}^{\gamma} n_{K}^{\beta} dV_{o}
\]

where it has to be remembered that even if \( n_{K_{o}}^{\beta} \) is measured in \( K_{o} \) and the instantaneity is taken on \( K \); that is:

\[
n_{K_{o}}^{\beta} = (\gamma, \gamma \beta), \tag{77}
\]
because \( n_{K\beta} \) is the unit time-like 4-vector which represents the reference frame \( K \) observed from the reference frame \( K_o \). We have continuously been using the contraction of the volume, \( dV_o = \gamma dV_K \) and we can generalized it by introducing the 4-velocity \( u^{\alpha}_{K\alpha} = (\gamma, \gamma^\beta) \), \( u^{\alpha}_{K\alpha} = (1, 0) \). Therefore, in Eq. (77), \( \gamma \) can be substituted by

\[
\gamma = u^{\lambda}_{K\lambda} n_{K\lambda}.
\] (78)

We arrive at:

\[
P^\alpha_{K,K_o} = \int_{V_o} T^{\alpha\beta}_{K_o} n_{K\beta} \frac{dV_o}{u^\lambda n_{K\lambda}}.
\] (79)

Therefore, by using the continuous definition of the energy-momentum tensor for a perfect fluid, we have:

\[
P^\alpha_{K,K_o} = \int_{V_o} [(\rho + p) u^{\alpha}_{K\alpha} - p n^{\alpha\beta}] n_{K\beta} \frac{dV_o}{u^\lambda n_{K\lambda}}.
\] (80)

Finally, substituting the values of \( u^{\alpha}_{K\alpha} \) and \( n_{K\beta} \), we obtain

\[
P^\alpha_{K,K_o} = \int_{V_o} \left[(\rho + p) \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} - p \begin{pmatrix} \gamma u \\ \gamma u \end{pmatrix}\right] dV_o \\
= \left( \int_{V_o} \rho dV_o \right) \left( \begin{pmatrix} E_o \\ upV_o \end{pmatrix} \right)
\] (81)

which corresponds to

\[
P^\alpha_{K,K} = \Lambda^\alpha_{K_o,K} P^\alpha_{K_o,K},
\] (82)

or term by term

\[
\begin{pmatrix} P^0_{K,K} \\ P^j_{K,K} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma u \\ \gamma u & \gamma \end{pmatrix} \begin{pmatrix} E_o \\ upV_o \end{pmatrix} \\
= \begin{pmatrix} \gamma (E_o + u^2 pV_o) \\ \gamma u (E_o + pV_o) \end{pmatrix},
\] (83)

which shows that the representation of a 4-vector energy-momentum for a finite volume must be done by defining the instantaneity of the measurement.

If the theory is consistent, we must be able to calculate any 4-energy-momentum in a reference frame \( K_i \) considering the instantaneity in \( K_j \). That is:

\[
P^\alpha_{K_j,K_i} = \int_{V_o} [(\rho + p) u^{\alpha}_{K_j} - p n^{\alpha\beta}] n_{K_j\beta} \frac{dV_o}{u^\lambda n_{K_j\lambda}}.
\] (84)

where \( K_j \) and \( K_i \) may be \( K \) or \( K_o \). We recall that in the notation of the volume vector

\[
n^{\beta}_{K_j},
\] (85)
where \( K_j \) denotes the reference where the instantaneity is considered and \( K_i \) represents the reference frame where the volume vector is expressed. Therefore, for \( K_j = K_o \) and \( K_i = K_o \), we have

\[
P^\alpha_{K_o,K_o} = \int_{V_o} \left( (\rho + p) \, u_{K_oK_o}^\alpha u_{K_oK_o}^\beta - p n_{K_oK_o}^{\alpha\beta} \right) n_{K_oK_o} \, \frac{dV_o}{K_o} \tag{86}
\]

where \( u_{K_oK_o}^\alpha = (1, 0) \) and \( n_{K_oK_o}^{\alpha\beta} = (1, 0) \). We have

\[
P^\alpha_{K_o,K_o} = \frac{\int_{V_o} (\rho + p) \, u_{K_oK_o}^\alpha u_{K_oK_o}^\beta - p n_{K_oK_o}^{\alpha\beta} \, dV_o}{\int_{V_o} \rho \, dV_o} = \left( \frac{E_o}{0} \right) \tag{87}
\]

as expected. However, to be consistent we need to prove that

\[
P^\alpha_{K_o,K_o} = \Lambda^\alpha_{K_oK_o} \, P^\alpha_{K_oK_o} \tag{88}
\]

using the definition of \( P^\alpha_{K_o,K_o} \). That is:

\[
P^\alpha_{K_o,K} = \int_{V_o} \left( (\rho + p) \, u_{K_oK}^\alpha u_{K_oK}^\beta - p n_{K_oK}^{\alpha\beta} \right) n_{K_oK} \, \frac{dV_o}{K_o} \tag{89}
\]

where

\[
u_{KK_o}^\alpha = \left( \begin{array}{c} \gamma \\ \gamma u \end{array} \right) \quad \text{and} \quad n_{KK_o}^{\alpha\beta} = (\gamma, \gamma u). \tag{90}
\]

Therefore,

\[
P^\alpha_{K_o,K} = \int_{V_o} \left( (\rho + p) \, \left( \begin{array}{c} \gamma \\ \gamma u \end{array} \right) - p \left( \begin{array}{c} \gamma \\ \gamma u \end{array} \right) \right) \, dV_o = \left( \gamma \right) \int_{V_o} \rho \, dV_o = \left( \gamma \right) E_o, \tag{91}
\]

which shows the validity of Eq. (88).

If we choose to measure a thermodynamic system which is at rest in \( K_o \) but we are looking at it from reference frame \( K \), and the instantaneity is taken on \( K \), that is, we are measuring \( P^\alpha_{K,K} \). We can compare it with \( P^\alpha_{K,K_o} \) which represents the transformed \( P^\alpha_{K,K} \) to \( K_o \). This is the Planck-Einstein-Tolman point of view [30]. If we take the simultaneity in \( K_o \), we will look for \( P^\alpha_{K_o,K} \) and its transformation to reference frame \( K \) is \( P^\alpha_{K_o,K} \) which represents one of the options described by Rohrlich [30]. We are now in a position to develop a Relativistic Thermodynamics which includes a relativistic moving system.
5. Relativistic Redefined Thermodynamics

We are interested to reproduce the laws of Thermodynamics in a moving frame with respect to the reference frame where the volume of the thermodynamic system is at rest. We have now developed the basis to express such laws in a covariant form. Let us constrain our point of view just to two systems, a lab frame \( K \) and the rest frame \( K_0 \). Since we are interested in expressing the laws in the lab frame \( K \), the instantaneous is considered in the lab frame itself and the notation can be simplified by just putting

\[
\alpha_K = w^\alpha. \tag{92}
\]

It is clear that for calculating the 4-vector total energy-momentum, it is necessary just to use Eq. (84). The transformation laws of thermodynamics are reached by defining the redefined thermodynamic quantities in the following way [30]:

A- In order to reproduce the first law of Thermodynamics, it has to be noticed that for constant \( \rho \) and \( p \), the 4-vector energy-momentum may be expressed as:

\[
P^\mu = (E_o + pV_o) u^\mu - pV_o \frac{w^\mu}{w_\lambda u^\lambda}, \tag{93}
\]

with \( E_o = \rho V_o \). It is necessary to define the 4-vector \( G^\mu \) as:

\[
G^\mu = [(\rho + p) u^\mu u^\nu - (\rho + p) n^{\mu\nu}] \frac{w_\nu}{w_\lambda u^\lambda} V_o
= (E_o + pV_o) u^\mu - (E_o + pV_o) \frac{w^\mu}{w_\lambda u^\lambda}. \tag{94}
\]

Then, we define the 4-vector redefined energy-momentum by

\[
\xi^\mu = P^\mu - G^\mu
= (E_o + pV_o) u^\mu - pV_o \frac{w^\mu}{w_\lambda u^\lambda}
- (E_o + pV_o) u^\mu + (E_o + pV_o) \frac{w^\mu}{w_\lambda u^\lambda}
= \frac{u^\mu}{w_\lambda u^\lambda} E_o. \tag{95}
\]

It is easy to show that with the redefined energy-momentum, we arrive at

\[
d\xi^\mu = d\Theta^\mu - dW^\mu, \tag{96}
\]

where

\[
d\Theta^\mu = w^\mu \frac{dQ_o}{u^\lambda w_\lambda} \quad \text{and} \quad dW^\mu = w^\mu \frac{pdV_o}{u^\lambda w_\lambda}. \tag{97}
\]

Eq. (96) represents the first law of Thermodynamics but in a covariant form. However, a rapid inspection shows that it represents just the first law multiplied by the factor \( w^\mu / u^\lambda w_\lambda \). However, when the entropy is written in this formalism, the theory shows its applicability.

B- To obtain the second law, it is just necessary to define the entropy and show that it represents an invariant. Let us begin by defining the 4-vector temperature and inverse 4-vector temperature as [11]

\[
T^\mu = \frac{u^\mu}{u^\lambda w_\lambda} \quad \text{and} \quad \beta_\mu = \frac{u^\mu}{kT_0}. \tag{98}
\]
The entropy may be written as

\[ dS = \beta_{\mu} V_{\nu} \frac{w_{\nu}}{w_{\lambda}} dT_{\mu\nu} + \beta_{\mu} p dV^{\mu} \]

\[ = \frac{u_{\mu}}{kT_0} V_{\nu} \frac{w_{\nu}}{w_{\lambda}} dT_{\mu\nu} + \frac{u_{\mu}}{kT_0} p \frac{w_{\nu}}{w_{\lambda}} dV_{\nu} \]

\[ = \frac{u_{\mu}}{kT_0} \frac{V_{\nu}}{w_{\lambda}} d \left[ (\rho + p) u^{\mu} u^{\nu} - p \eta_{\mu\nu} \right] + \frac{p dV_{\nu}}{kT_0} \]

\[ = \frac{1}{kT_0} dE_{\nu} + \frac{p}{kT_0} dV_{\nu}. \quad (99) \]

Consequently, we arrive to an invariant definition of the entropy which is compatible with the formalism used for the first law of Thermodynamics. The third law of thermodynamics follows immediately from Eq. (99). This theory has been used to analyze the black body radiation from a moving frame [27], [28], to solve the Thermodynamics of the mixture of two black bodies [30] including the conservation of the heat in any reference frame, to propose a relativistic thermometer [30] [31], to obtain a Statistical Relativistic Mechanics [32], to deduce the Jüttner distribution [32] and finally to show the invariance of the equations of states [33].

6. Conclusion

The following results have been obtained:

1- An alternative and simple proof of the no interaction theorem.

2- The inclusion of the self-force gives a conservation of the total energy-momentum but the divergences still exist.

3- Using the concept of instantaneity, a 4-vector total energy-momentum can be defined for a thermodynamic system (continuous) or a discrete system.

4- The perfect discrete energy-momentum tensor \( T^{\alpha\beta} \) does not satisfy the conservation theorem. Although it is different from the energy-momentum tensor for a perfect fluid \((\rho + p) u^{\alpha} u^{\beta} - \eta^{\alpha\beta} p\), a 4-vector total energy-momentum can be defined using an specific instantaneous system for both infinite volume in the case of the discrete free particles system and finite volume for the perfect fluid. The results are similar.

5- A Relativistic Redefined Thermodynamics is derived.

Some applications for Astrophysical Jets and for TOKAMAKS will be showed in future works. Moreover, in the last years, Gill and Zachary have developed the theory of proper time [56] and by using their formalism Ares de Parga et al [57] obtained an equation of motion which includes an special reaction force for each particle of a system of charged particles. The interaction consists of an action at a distance with retarded potentials which gives a constant total momentum. Therefore, a rest frame can be defined and the above theory for a system of interacting charged particles may be used to calculate the 4-vector total energy-momentum for each frame where instantaneity is considered.

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