Three-point energy correlator in $\mathcal{N} = 4$ super-Yang Mills Theory

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An analytic formula is given for the three-point energy correlator (EEEC) at leading order (LO) in maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ sYM). This is the first analytic calculation of a three-parameter event shape observable, which provides valuable data for various studies ranging from conformal field theories to jet substructure. The associated class of functions define a new type of single-valued polylogarithms characterized by 16 alphabet letters, which manifest a $D_6 \times Z_2$ dihedral symmetry of the event shape. With the unexplored simplicity in the perturbative structure of EEEC, all kinematic regions including collinear, squeezed and coplanar limits are now available.

1. Introduction.

The energy correlator observable measures the energy deposited in multiple detectors as a function of angles between the detectors. From the phenomenological perspective, energy correlators probe the energy flow and can be used as jet observables \cite{2, 3} for precise tests of the standard model or new physics search. From the practical side, energy correlators is perhaps the simplest infrared safe event shape \cite{7, 8} to calculate analytically. This is due to the nature of quantum field theories \cite{9, 10, 11}, which provides valuable data for understanding the nature of quantum chromodynamics (QCD) \cite{13, 14}, and NNLO in quantum chromodynamics (QCD) \cite{13, 14, 26, 41–43} and NNLO in super-Yang Mills Theory ($\mathcal{N} = 4$ sYM). This is the first analytic calculation of multi-particle correlation involving the stress-energy tensors. In view of this property, previous studies in $\mathcal{N} = 4$ sYM theory employed various shortcut to obtaining the EEC by taking multiple discontinuities of Euclidean correlation functions and by exploiting the superconformal symmetries of the latter \cite{15, 16, 41–43}. It remains unknown whether such approaches are feasible for computing the higher-point energy correlators. In this letter we adopt an onshell approach to obtain the EEEC from the super form factor for protected scalar operator \cite{44, 45}, benefiting from the simplicity of matrix elements in $\mathcal{N} = 4$ sYM. We present the one-loop EEEC result for arbitrary angles, which is the first analytic calculation of multi-particle correlation observables with full shape dependence.

The energy correlator is an onshell observable that bares close relations to the offshell correlation functions involving the stress-energy tensors. In view of this property, previous studies in $\mathcal{N} = 4$ sYM theory employed various shortcut to obtaining the EEC by taking multiple discontinuities of Euclidean correlation functions and by exploiting the superconformal symmetries of the latter \cite{15, 16, 41–43}. It remains unknown whether such approaches are feasible for computing the higher-point energy correlators. In this letter we adopt an onshell approach to obtain the EEEC from the super form factor for protected scalar operator \cite{44, 45}, benefiting from the simplicity of matrix elements in $\mathcal{N} = 4$ sYM. We present the one-loop EEEC result for arbitrary angles, which is the first analytic calculation of multi-particle correlation observables with full shape dependence.

Our result encodes valuable information on the function space of the EEEC in perturbative quantum field theory: the classifications of symmetries, symbol alphabets and polylogarithms. These mathematical structures are studied much more thoroughly in the context of scattering amplitudes than finite observables in collider ex-

\begin{equation}
\text{EEEC}(\chi_1, \chi_2, \chi_3) = \int \prod_{i=1}^{3} \left[d\Omega_{\vec{n}_i} \delta(\vec{n}_i \cdot \vec{n}_{i+1})\right] \times \frac{\int d^2x e^{iqx} \langle 0 | O^\dagger(x) E(\vec{n}_1) E(\vec{n}_2) E(\vec{n}_3) O(0) | 0 \rangle}{\langle q \rangle^3 \int d^2x e^{iqx} \langle 0 | O^\dagger(x) O(0) | 0 \rangle}.
\end{equation}

Here the detector operator that measures the energy flux in the direction $\vec{n}$ is given by an integrated stress-energy tensor $T_{\mu\nu}$ \cite{9, 17, 30}, $E(\vec{n}) = \int_{-\infty}^{\infty} dt \lim_{r \to \infty} r^n T_{\mu\nu}(t = \tau + r, r \vec{n})$. The operators $O$ (source) and $O^\dagger$ (sink) create the final state, whose particles are detected by the two calorimeters. The choice of the local operator $O$ depends on the physical problem. For $e^+e^-$ annihilation, $O$ is given by an electromagnetic current.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{three-point-energy-correlator.png}
\caption{Graphical representation of the three-point energy correlator: particles produced out of the vacuum by the source are captured by the three detectors located at spatial infinity in the directions of the unit vectors $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$. They can be mapped onto three points located on a circle with radius $|y| = \tan \frac{\theta}{2}$ on the celestial sphere. The three angles are parametrized by $(\sin \frac{\theta}{2}, \sin \frac{\phi}{2}, \sin \frac{\chi}{2}) = \sin \theta (\sin \frac{\theta}{2}, \sin \frac{\phi}{2}, \sin \frac{\chi}{2})$.}
\end{figure}
periments. We are strongly motivated to initiate the discussion on these topics for the energy correlator observable, starting from $\mathcal{N} = 4$ sYM. As they provide powerful tools and experience for QCD $[46]$, which is phenomenologically relevant for the cutting edge studies at LHC.

2. EEEC from four point form factor.

In $\mathcal{N} = 4$ sYM, we may choose the source and sink to be scalar operators that are the bottom component of the supermultiplet of conserved currents. As such, they are natural analogs of the electromagnetic current, and have fixed conformal weight two. The matrix element for producing a given onshell super-state from the vacuum defines the so-called form factor $[47, 51]$

$$\int d^4 x e^{i q \cdot x} \langle X|O(x)|0\rangle \equiv (2\pi)^4 \delta^4(q - p_X) F_X$$  \hspace{1cm} (2)

In perturbative theories EEEC can be obtained from the squared form factor by performing a weighted sum over the onshell external states. For convenience we normalize the event shape by a volume factor of the phase space, thus defining a function $H$ through $\text{EEEC}(x_1, x_2, x_3) \equiv (8\pi^2) \times ||\vec{n}_1 \wedge \vec{n}_2 \wedge \vec{n}_3||^{-1} H(\vec{n}_1, \vec{n}_2, \vec{n}_3)$, which we can evaluate by carrying out the onshell phase-space integration while fixing the directions of three particles in the final states

$$H(\vec{n}_1, \vec{n}_2, \vec{n}_3) = \frac{1}{\sigma_{\text{tot}}} \sum_{(i,j,k) \in X} \int d\Pi_X \delta^2(\vec{n}_1 - \vec{p}_i) \delta^2(\vec{n}_2 - \vec{p}_j) \delta^2(\vec{n}_3 - \vec{p}_k) \frac{E_i E_j E_k}{(q^0)^3} |F_X|^2$$ \hspace{1cm} (3)

where $i, j$ and $k$ run over all final-state particles. $H$ has the perturbative expansion $\sum_{k \geq 1} a^k H^{(k)}$ in the ‘t Hooft coupling. The born level event shape is a delta function due to 3-kineamatic constraints, $H_{\text{Born}} = \delta(||\vec{n}_1 \wedge \vec{n}_2 \wedge \vec{n}_3||) \csc^2 \frac{\theta_1}{2} \csc^2 \frac{\theta_2}{2} \csc^2 \frac{\theta_3}{2}$. The leading order that has nontrivial three-angle dependence is $O(\alpha^2)$, where the complication comes from the tree-level four-point NMHV super matrix elements $|F_4|^2$, for details see $[52, 53]$. After summing over super states and symmetrization over the final-state momenta, the squared four-point form factor can be organized into a concise form involving dual conformal cross ratios,

$$|F_{\text{sym}}|^2 = \frac{q^4}{s_{12}s_{23}s_{34}s_{41}} \left[ \frac{1}{4} + \frac{2s_{23}s_{41}}{s_{31}s_{412}} + \frac{s_{23}s_{34}}{s_{12}s_{341}} \right] + \text{perm}(1, 2, 3, 4).$$ \hspace{1cm} (4)

where the second and third term in the bracket correspond to the NMHV contribution.

To compute the EEEC, we first apply topology identification to the squared matrix elements. With the EEEC measurement functions, they can be decomposed into four topologies. To proceed, we parameterize the kinematic invariants by the energy fractions of three detected final-state particles as well as the three angles $(\theta, \phi_1, \phi_2)$ as depicted in Fig. 1. In particular, we switch to the following set of angle parameters

$$s = \tan^2 \frac{\theta}{2}, \quad \tau_1 = e^{i\phi_1}, \quad \tau_2 = e^{i\phi_2}$$ \hspace{1cm} (5)

such that both the matrix elements and phase space simplify down. Integrating the four-particle phase space $[54]$ against the measurement functions, we are left with a set of two-fold integrals which are linearly reducible $[55, 56]$, which allow us to compute directly in HyperInt $[57]$.

3. Symbol alphabets

The EEEC can be expressed in a frame independent manner as a function of three conformally invariant variables

$$\zeta_{ij} = \frac{q_i^2 (p_i \cdot p_j)}{2(q \cdot p_i)(q \cdot p_j)} = \langle p_i p_j \rangle \langle \xi_i \xi_j \rangle / \langle p_i \xi_i \rangle \langle p_j \xi_j \rangle, \quad |\xi_j| \equiv q|j|$$ \hspace{1cm} (6)

From the the results of function $H$ we read off 16 symbol alphabets, which contains two types of algebraic roots

$$|\Delta_1| \equiv ||\vec{n}_1 \wedge \vec{n}_2 \wedge \vec{n}_3|| = |(1 - u_1 - u_2 - u_3)^2 - 4u_1 u_2 u_3| \quad |\Delta_2| \equiv ||\vec{n}_1 \wedge \vec{n}_2 + \vec{n}_2 \wedge \vec{n}_3 + \vec{n}_3 \wedge \vec{n}_1|| = |\sqrt{\lambda(\zeta_{12}, \zeta_{23}, \zeta_{31})}|$$

where $\{u_i\} \equiv \{1 - \zeta_{ij}\}$, and $\lambda(a, b, c) \equiv a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$ is the Källen function.

The representation of variables in Eq. (6) foreshadows a $D_6$ dihedral symmetry of the EEEC, which can be better visualized as we embed the kinematic data $|p_i| \equiv |2i - 1|, |\xi_i| \equiv |2i + 2|$ in a $2 \times 6$ matrix $Z$: $Z_a = |a| \in CP^1$,

$$Z \equiv \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \end{pmatrix}, \quad \eta_a - \eta_b = \frac{\langle ab \rangle}{\langle a \infty \rangle \langle b \infty \rangle}$$ \hspace{1cm} (7)

where $|\infty\rangle = (0, 1)^T$.

The geometric interpretation of the above formalism is clear in the center of mass frame where $q = (q^0, 0, 0, 0)$.
so that under stereographic projection the three vectors \( \vec{p}_i \) are mapped onto a triangle \( (y_i, \vec{y}_i) = (\eta_i, -1/\eta_i) \) on the celestial sphere. Let us further introduce a special point \( (y_f, \vec{y}_f) = (\eta_f, -1/\eta_f) \) representing the center of the triangle, whose location determined by the equations

\[
\frac{\langle 1I \rangle \langle 4\bar{I} \rangle}{\langle 14 \rangle \langle 2I \rangle \langle 5\bar{I} \rangle} = \frac{\langle 3I \rangle \langle 6\bar{I} \rangle}{\langle 36 \rangle \langle 14 \rangle \langle 4\bar{I} \rangle} = 1
\]  
(8)

We may impose \( \langle I \rangle = (0, 1)^T, \langle \bar{I} \rangle = (1, 0)^T \) and \( \langle I1 \rangle = \langle aI \rangle = 1 \), and expand the kinematic space to include \( I \) (or equivalently \( \bar{I} \)). Thus we fix the gauge under which

\[
Z|I = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -s_1 \tau_2 & \tau_1 & -s & \tau_1 \tau_2 & -s_1 \tau_2 \end{pmatrix}
\]  
(9)

The kinematic space is mapped onto two similar triangles whose circumcenter sit at the origin, which we display as a hexagon located on a unit circle FIG. 2.

In this way we identify all 16 EEEC alphabets with products of plicity variables \( \langle ab \rangle, \langle aI \rangle \) as well as certain homogeneous polynomials in the form

\[
d_{(ab)(cd)(ef)} \equiv \langle ad \rangle \langle cb \rangle \langle cf \rangle - \langle af \rangle \langle cb \rangle \langle ed \rangle
\]  
(10)

Switching variables into 3 conformally invariant ratios,

\[
y = \frac{\langle 31 \rangle \langle 5I \rangle}{\langle 15 \rangle \langle 13 \rangle}, \quad z = \frac{\langle 31 \rangle \langle 5I \rangle}{\langle 15 \rangle \langle 56 \rangle}, \quad w = \frac{\langle 51 \rangle \langle 62 \rangle}{\langle 35 \rangle \langle 16 \rangle}.
\]

we could transform the alphabets into 16 polynomials with all positive signs, which read

\[
\{ w, 1 + w, y, 1 + y, z, 1 + z, w + z, 1 + w + z + yz, y + z + yz, w + y + z + yz, 1 + w + z + yz, 1 + w + y + 2z + yz, y + w + y + 2z + yz, 1 + w + y + 2z + yz, 1 + w + y + wy + 2z + yz, 1 + w + wy + y^2 + 2z + yz + y^2z, 1 + w + wy + y^2 + 2z + yz + y^2z \}
\]  
(11)

4. Symmetries and functional basis.

Within our embedding formalism, the EEEC exhibits a set of discrete symmetries. First we have \( D_6 \) dihedral symmetries acting on the hexagon coordinates \( Z_a (a = 6 \mapsto a) \), which are generated by dihedral flip \( \tau \): \( a \xrightarrow{\tau} 4 - a \); cyclic permutation \( \sigma \): \( a \xrightarrow{\sigma} a + 2 \); as well as parity conjugation \( P \): \( a \xrightarrow{P} a + 3 \). In addition, there is a residual \( Z_3 \) symmetry corresponding to the exchange between two solutions to Eq. [5]. It is generated by an operation which we call reflection \( R \): \( (a) \xrightarrow{R} (a + 6) \). \( P \) and \( R \) flip the signs of \( \Delta_1 \) and \( \Delta_2 \), respectively, such that \( \Delta_1 \xrightarrow{P} -\Delta_1, \Delta_1 \xrightarrow{R} \Delta_1 \), while \( \Delta_2 \xrightarrow{P} \Delta_2, \Delta_2 \xrightarrow{R} -\Delta_2 \).

In light of these properties, we are ready to lift the one-loop symbols into polylogarithmic functions. To start, we shall identify a set of variables \( S \) which is closed under \( \{ \tau, \sigma, P, R \} \), such that \{ \( S, 1 + S \) \} factorize into polynomials that cover the EEEC alphabet letters in Eq. [11].

Let us introduce these variables. For clarity, we switch to a different gauge by performing a \( GL(2) \times GL(1)^6 \) transformation on Eq. [9],

\[
Z|I = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -s_1 \tau_2 & \tau_1 & -s & \tau_1 \tau_2 & -s_1 \tau_2 \end{pmatrix}
\]  
(12)

thus introducing four parameters \( (x_1, x_2, x_3, x_4) \), three of which being independent.

Notice that the hexagon \( Z \) corresponds to the Grassmannian \( Gr(2, 6)/GL(1)^5 \), which can be associated with an \( A_3 \) cluster algebra, with a quiver being \( (x_1, x_2, x_3) \). Given this observation, we introduce 15 conformally invariant ratios to cover the full set of \( X \)-coordinates [60, 61], namely

\[
w_3 = x_1 = \frac{23}{14}, \quad z_3 = x_2 = \frac{34}{15}, \quad x_2 = \frac{1 + x_3}{x_2 x_3} = \frac{46}{16}, \quad w_3 \xrightarrow{P} \bar{w}_3 = \frac{14}{45}, \quad z_3 \xrightarrow{P} \bar{z}_3 = \frac{16}{46}.
\]  
(14)

as well as their images under cyclic permutations.

In addition we introduce \( r_1 = 1/x_4 = -\frac{14}{34}, \) whose images under \( D_6 \) transformations form a set of 12 cross ratios:

\[
r_a = \frac{(a + 3)(a + 2)}{(a + 2)(a + 3)}, \quad \bar{r}_a = \frac{(a + 5a + 2)(a + 3)}{(a + 5a + 3)(a + 2)}.
\]  
(15)
satisfying \( r_a \xrightarrow{P} r_{a+3}, r_a \xrightarrow{R} \tilde{r}_a, a + 6 \preceq a \).

In terms of these ingredients, we can define \( S = \{ r_1, w_1, z_1, v_1, -w_1/\tilde{w}_1, -z_1^2 \} \) plus their \( D_6 \)-images and conclude that it has the desired properties, i.e., \( L_{i,2}(S) \) (modulo products of logarithms) account for the one-loop symbol letters for the EEEC.

Next we investigate the structures of physical singularities, leading to a set of first-entry conditions that further constrain the function space

- Single-valueness in the physical domain away from the coplanar limit: For \( |r_1| = |r_2| = 1, |s| < 1 \) (or \( |s| < 1 \)), the function must be free from ambiguity in the principal values of azimuthal angles, i.e., invariant as \( r_1 \rightarrow e^{\pm 2\pi i}r_1, r_2 \rightarrow e^{\pm 2\pi i}r_2 \).

- Near triple-collinear limit: the function is free from logarithmic singularity in the triple-collinear limit as \( s \rightarrow 0 \) or \( \infty \).

As a consequence, only 6 independent letters drawn from the set \( \{ \frac{\tilde{w}_1}{w_1} = \prod_{i=1}^{3} \frac{1 + w_i}{1 + w_i}, |z_i|^2 = \frac{v_i}{v_i}, 1 + v_i \} \) can appear in the first entry. In particular, a parity odd letter \( \frac{\tilde{w}_1}{w_1} = \frac{v_i}{v_i} \) is allowed, which distinguishes the EEEC function space from the standard single-valued polylogarithms [22].

In conclusion, the one-loop EEEC function space comprises of classical polylogarithms whose arguments drawn from the set \( \{ -S, 1 + S \} \) satisfying the first-entry conditions. We observe that the final answer can be decomposed onto 14 such functions as well as their cyclic permutations. Hence the one-loop EEEC in \( N = 4 \) super Yang-Mills can be written in the a form that has manifest \( D_6 \) symmetry,

\[
H_{N=4}(\vec{n}_1, \vec{n}_2, \vec{n}_3) = \sum_{i=1}^{14} b_i F_i + \text{perm}(\vec{n}_1, \vec{n}_2, \vec{n}_3) \tag{16} \]

where \( \vec{b} \) is a set of rational functions of \( (s, r_1, r_2) \). \( \vec{F} \) contains weight-1 and weight-2 polylogarithmic functions. More explicitly,

\[
\vec{F} \equiv \{ f_1, f_2, f_3, g_1, \ldots, g_{11} \} \tag{17}.
\]

Each member of \( \vec{F} \) has a distinct signature under the operation \( \tau \) and \( P \), where \( \{ f_1, f_2, g_2-4, g_8-11 \} \) and \( \{ f_1, g_3, g_4, g_7, g_8 \} \) are odd under \( \tau \) and \( P \), respectively. The first three members \( f_{1,2,3} \) are weight-1 functions,

\[
f_1 = \ln \frac{\tilde{w}_1}{w_1}, \quad f_2 = \ln |z_2|^2, \quad f_3 = \ln (1 + v_2) \tag{18}
\]

The rest, \( g_{1-11} \) are weight-2 functions, among which \( g_{1-8} \) are characterized by the 9 \( A_3 \)-cluster alphabets, depending only on \( \{ w_1, z_1, v_1 \} \) and their parity conjugation.

\[
g_1 = L_{i,2}(-v_2) \tag{19}
\]

\[
g_2 = L_{i,2}(1 + w_1) + L_{i,2}(1 + v_3) + 2L_{i,2}(-v_1) - L_{i,2}(1 + w_1) - L_{i,2}(1 + v_1) - 2L_{i,2}(-v_1) \]

\[
g_3 = L_{i,2}(-z_2) - L_{i,2}(-z_2) + \frac{1}{2} \ln |z_2|^2 \ln 1 + z_2 \]

\[
g_4 = L_{i,2}(1 + w_1) - L_{i,2}(1 + v_1) + L_{i,2}(1 + w_2) - L_{i,2}(1 + \tilde{w}_2) + L_{i,2}(1 + \tilde{w}_3) - L_{i,2}(1 + \tilde{w}_3) \]

\[
g_5 = \pi^2 \]

\[
g_6 = \ln^2 \frac{\tilde{w}_1}{w_1} \]

\[
g_7 = \ln \frac{\tilde{w}_1}{w_1} \ln |z_2|^2 \]

\[
g_8 = \ln (1 + v_3) \ln |z_2|^2 - \ln (1 + v_1) \ln |z_3|^2 \]

\[
g_{11} \text{ is the only member depending on } \{ r_i \}, \text{ and the only one exhibiting an odd } Z_2 \text{-signature: } g_{11} \xrightarrow{R} -g_{11}. \tag{20}
\]

The last two members \( g_{9,10} \) are responsible for the homogeneous polynomials that appear as alphabet letters: \( (12) \langle 34 \rangle (56) - (23) \langle 45 \rangle (61), (54) \langle 12 \rangle (36) - (41) \langle 23 \rangle (65) \).

\[
g_9 = \frac{1}{2} L_{i,2}(1 - \tilde{w}_1) - \frac{1}{2} L_{i,2}(1 - \frac{w_1}{w_1}) \tag{21}
\]

\[
g_{10} = L_{i,2}(1 - |z_2|^2) + \frac{1}{2} \ln |z_2|^2 \ln |1 - z_2|^2 \tag{22}
\]

In the ancillary files, we provide the explicit expressions for the coefficients \( \vec{b} \), as well as the full analytic expression for \( H_{N=4}^0 \) in terms of the \( \zeta_{ij} \)-variables. In FIG. we display the function \( H \) in various kinematic regions.

5. Special kinematics. In the EEEC, physical singularities emerge on the surfaces in the three-dimensional parameter space displayed in FIG. [4]. Our rational parametrization Eq. [5] makes it easy to access the three types of singular regions where the EEEC is enhanced, namely the limit where three detectors are collinear (\( s \rightarrow \infty \)), two of them are collinear (\( r_2 \rightarrow 1 \)), or the three detectors are coplanar (\( s \rightarrow 1 \)). We extract analytically the leading power asymptotic behaviour in all these limits.

**Triplet-collinear limit** describes single jet events where the all three angles are small \( \zeta_{ij} \sim 0 \). Taking \( s \rightarrow 0 \), we verify that it is a regular limit free from logarithmic enhancement, such that \( H \sim \frac{1}{2} G(\tau_1, \tau_2) \). Our expression for the function \( G \) agrees with [39], upon setting \( z = \frac{1 - \tau_1}{1 - \tau_2}, \xi = \frac{1 - 1/\tau_1}{1 - 1/\tau_2} \).

**Squeezed limit** corresponds to the regime where we put two detectors on top of each other, \( \zeta_{12} \sim 0, \zeta_{13} \sim \zeta_{23} \sim \zeta \). We can access this limit by taking \( r_2 \rightarrow 1 \) keeping \( s, \tau_1 \) fixed, and we find that the leading-power contributions can be grouped into a simple form

\[
H_{\zeta_{ij} \rightarrow 0} \left[ \frac{6}{\zeta_{12}} \left( \zeta + \ln(1 - \zeta) \right) \left( -1 + \zeta \right) \zeta^4 \right] \tag{23}
\]
where the coefficient of the leading pole depends on a single variable $\zeta = -\frac{s(1 - \tau_1)^2}{(1 + s)^2 \tau_1}$.

**Coplanar limit** corresponds to the surface where $\Delta_1$ vanishes. In this regime, a soft particle recoils against three coplanar hard particles, which are scattered into two different half-planes. We extract the leading singular behaviour by expanding the function $H$ around $s = 1$, recalling $\tau_1 = e^{i\phi_1}, \tau_2 = e^{i\phi_2}$,

$$H \sim 18\pi \frac{\theta(-\cos \frac{\phi_1}{2} \cos \frac{\phi_2}{2} \cos \frac{\phi_1 + \phi_2}{2})}{\sin \frac{\phi_1}{2} \sin \frac{\phi_2}{2} \sin \frac{\phi_1 + \phi_2}{2}} \times \frac{1}{|1 - s|} \ln \left[ (1 - s)^2 \tan^2 \frac{\phi_1}{2} \tan^2 \frac{\phi_2}{2} \tan^2 \frac{\phi_1 + \phi_2}{2} \right] \quad (24)$$

The pole at $s = 1$ comes solely from the discontinuity of the $g_s$ function in the region where $w_i$ sit on the negative real axis. The physical origin of the leading logarithms is soft-collinear singularity. Lifting the result to negative real axis. The physical origin of the leading soft-collinear singularity. Lifting the result to negative real axis. The physical origin of the leading log.

**FIG. 4.** Kinematic regions for the EEEC and its singular regimes.

6. **Outlook.** Our work opens the way for several applications and further studies. Our one-loop formula Eq. (16) constitutes the first analytic result for an event-shape observable living in a three-dimensional parameter space. Its symbol defines a set of 16 rational alphabet letters describing a finite physical observable. By embedding the kinematic space in a hexagon located on a unit circle, we identify the symmetry properties and first-entry conditions that provide key information on the function space. Further studies on these mathematical structures and analytic properties will be crucial for bootstrapping the observable at higher perturbative order in supersymmetric gauge theories or in QCD.

In addition to the leading asymptotic behaviours we provide, our result contains information about subleading powers as well. The data in the triple-collinear, squeezed and coplanar limit will shed new light on corresponding OPE limits of the light-ray operators [64], thus making it possible to understand these limits at arbitrary coupling [65, 66]. In the meantime, the analysis in each aforementioned kinematic limits can be generalized to QCD, providing rich content for theoretical and phenomenological studies, as major progress has been achieved in the triple-collinear limit [67, 70].

Our approach to compute the EEEC in $N = 4$ sYM benefits from the simplicity of the squared super form factor, which allows an integral representation in a concise form. As novel research ideas emerged in recent studies on the form factors by means of harmonic superspace formalism [71], [73], modern amplitude techniques [74–77] and integrability descriptions [78–80], they open the way to probing the energy correlator observable at higher loop order or finite coupling, as well as relevant generalization of the event shape in quantum field theories [11, 81–84].

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SUPPLEMENTAL MATERIALS

In the supplemental materials, we present the full analytic expression for three-point energy correlator EEEC in \( N = 4 \) sYM.

**Analytic results for the \( b_i \) coefficients**

The EEEC in \( N = 4 \) sYM is completely described by the function space in Eq. \( [10] \), together with 14 rational coefficients. Recall that,

\[
\text{EEEC}(\chi_1, \chi_2, \chi_3) \equiv (8\pi^2) \times \Delta_i^{-1} H(\vec{n}_1, \vec{n}_2, \vec{n}_3) = (8\pi^2) \times \Delta_i^{-1} \left( \sum_{i=1}^{14} b_i F_i + \text{perm}(\vec{n}_1, \vec{n}_2, \vec{n}_3) \right)
\]

where there are two square roots in the expression, as shown in the letter:

\[
\Delta_1 = \sqrt{\zeta_{12}^2 + \zeta_{13}^2 + \zeta_{23}^2 - 2\zeta_{12}\zeta_{13} - 2\zeta_{12}\zeta_{23} - 2\zeta_{13}\zeta_{23} + 4\zeta_{12}\zeta_{13}\zeta_{23}} \\
\Delta_2 = \sqrt{\zeta_{12}^2 + \zeta_{13}^2 + \zeta_{23}^2 - 2\zeta_{12}\zeta_{13} - 2\zeta_{12}\zeta_{23} - 2\zeta_{13}\zeta_{23}}
\]

(26)

The variables we introduce in section 4 can be written in terms of the standard \( \zeta_{ij} - \)variables, more explicitly,

\[
w_1 = \frac{-2 - \Delta_1 + \zeta_{12} + \zeta_{13} + \zeta_{23}}{2(-1 + \zeta_{12})(-1 + \zeta_{13})}, \quad z_1 = \frac{-\Delta_1 + \zeta_{12} + \zeta_{13} - 2\zeta_{12}\zeta_{13} - \zeta_{23}}{2\zeta_{12}(-1 + \zeta_{13})}, \\
v_i = \frac{\zeta_{23}}{1 - \zeta_{23}}, \quad r_1 = \frac{\Delta_2 - \zeta_{12} + \zeta_{13} - \zeta_{23}}{\Delta_1 + \Delta_2}
\]

(27)

as well as their images under cyclic shift: \( i \rightarrow i + 1, (i = i + 3) \), Parity conjugation: \( \Delta_1 \rightarrow -\Delta_1 \) and \( R \)-conjugation: \( \Delta_2 \rightarrow -\Delta_2 \).

Finally the \( b_i \) coefficients are given below.

\[
b_1 = -\frac{\Delta_1 (\zeta_{12} + \zeta_{13} + \zeta_{23})}{3\zeta_{12}\zeta_{13}\zeta_{23}} \quad b_2 = -\frac{2(\zeta_{12} - \zeta_{23})}{\zeta_{12}\zeta_{13}\zeta_{23}} \quad b_3 = -\frac{\zeta_{12} - 2\zeta_{13}\zeta_{12} - 2\zeta_{23}\zeta_{12} + \zeta_{13}^2 + \zeta_{23}^2 - 2\zeta_{13}\zeta_{23}}{\zeta_{12}\zeta_{13}\zeta_{23}}
\]

(28)

\[
b_4 = \frac{\zeta_{12} - 1 \zeta_{13} - 1 \zeta_{23} + \zeta_{12} + \zeta_{13} - 1 \zeta_{23} - 1 \zeta_{13} + \zeta_{23} - 1}{(\zeta_{12} - 1)(\zeta_{13} - 1)(\zeta_{23} - 1)} \left( -\frac{2\zeta_{12}\zeta_{13} + 2\zeta_{23}\zeta_{13} \zeta_{23}}{3\zeta_{12}\zeta_{13}\zeta_{23}} \right)
\]

\[
b_5 = \frac{\zeta_{12} - 1 \zeta_{13} - 1 \zeta_{23} + \zeta_{12} + \zeta_{13} - 1 \zeta_{23} - 1 \zeta_{13} + \zeta_{23} - 1}{(\zeta_{12} - 1)(\zeta_{13} - 1)(\zeta_{23} - 1)} \left( \sum_{i=1}^{14} b_i F_i + \text{perm}(\vec{n}_1, \vec{n}_2, \vec{n}_3) \right)
\]

\[
b_6 = \frac{\zeta_{12} - 1 \zeta_{13} - 1 \zeta_{23} + \zeta_{12} + \zeta_{13} - 1 \zeta_{23} - 1 \zeta_{13} + \zeta_{23} - 1}{(\zeta_{12} - 1)(\zeta_{13} - 1)(\zeta_{23} - 1)} \left( \sum_{i=1}^{14} b_i F_i + \text{perm}(\vec{n}_1, \vec{n}_2, \vec{n}_3) \right)
\]
In the ancillary file, we provide these coefficients and the transcendental function space in terms of the $\zeta_{ij}$ variables. In particular, to meet the definition of logarithms' branch cuts in Mathematica, we slightly reorganize some of the polylogarithmic functions. As a check, we also set up the one-fold numerical integration in the file, which is in good agreement with our analytic result.