ON THE EXISTENCE OF DOUBLING MEASURES WITH CERTAIN REGULARITY PROPERTIES

PER BYLUND AND JAUME GUDAYOL

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ABSTRACT. Given a compact pseudo-metric space, we associate to it upper and lower dimensions, depending only on the pseudo-metric. Then we construct a doubling measure for which the measure of a dilated ball is closely related to these dimensions.

1. Introduction

Let \((X, \rho)\) be a compact metric space. Suppose that \((X, \rho)\) is homogeneous. This means that there exists a doubling measure \(\mu\) supported by \(X\); i.e. there is a constant \(c\) such that, for \(x \in X\) and \(R > 0\), one has \(0 < \mu(B(x, R)) < \infty\) and

\[
\mu(B(x, 2R)) \leq c\mu(B(x, R)).
\]

Dynkin proved in \([\text{Dyn}]\) that for certain subsets \(E\) of the unit sphere \(T \subset \mathbb{C}\) there exists a doubling measure on \(E\), and he conjectured that any compact \(E \subset \mathbb{R}^n\) is homogeneous. This conjecture was proved in \([\text{V-K}]\) by using a dimension first defined in \([\text{Lar}]\) called the uniform metric dimension, in this paper denoted by \(Y(E)\). More precisely, Volberg and Konyagin proved that \((X, \rho)\) is homogeneous if and only if there is some \(s < \infty\) such that any ball \(B(x, kR)\) contains at most \(Ck^s\) points separated from each other by a distance of at least \(R\). The uniform metric dimension \(Y(X) = Y(X, \rho)\) is then defined as the infimum of such \(s\). Furthermore, given \(s < \infty\) in the condition above Volberg and Konyagin proved that for any \(s' > s\) there exists a measure \(\mu\) such that, for \(0 < R \leq kR\),

\[
\mu(B(x, kR)) \leq Ck^{s'} \mu(B(x, R)).
\]

Clearly, any measure satisfying (2) is a doubling measure, and conversely, iterating (1) one gets (2) with \(s' = \log_2 c\). In particular, Volberg and Konyagin proved Dynkin’s conjecture by showing that on any compact \(E \subset \mathbb{R}^n\) there exists a measure \(\mu\) satisfying (2) with \(s = n\) (in the maximum metric).

In this paper we generalize their result by showing the existence of a measure \(\mu\) not only satisfying (2), but also the following analogous lower bound condition. Suppose there is a \(t \geq 0\) such that any ball \(B(x, kR)\) contains at least \(Ck^t\) points.
separated from each other by a distance of at least $R$. Then for any $t' < t$ there exists a measure $\mu$ such that, for $0 < R \leq kR$,

$$Ck' \mu(B(x, R)) \leq \mu(B(x, kR)).$$

(3)

In [J-W] Jonsson and Wallin studied function spaces on subsets of $\mathbb{R}^n$ supporting measures fulfilling both (2) and (3) in the special case $s' = t'$. Such sets are also called $s$-sets or Ahlfors-regular sets.

The general case $t' \leq s'$ was considered in [Jon].

The authors of this paper, independently of each other, also studied the general case $t' \leq s'$ in [Byl] and [Gud]. Each of these works contains the main result of this paper, in [Byl] formulated for Euclidean spaces and in [Gud] for metric spaces.

In this paper the result is stated in terms of pseudo-metric spaces.

2. Definitions and statements of results

Throughout we denote by $X = (X, d)$ a compact pseudo-metric space, where $d : X \times X \to [0, +\infty)$ is a pseudo-metric on $X$, i.e.

1. $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x), \ \forall x, y \in X$,
3. there is a constant $C_d$ such that $d(x, z) \leq C_d(d(x, y) + d(y, z)), \ \forall x, y, z \in X$.

On $X$ we consider the topology generated by the open pseudo-balls, and without loss of generality we assume that $\text{diam}(X) < 1$.

Given any ball $B(x, kR), x \in X$ and $0 < R \leq kR$, denote by $N(x, R, k)$ the maximum number of points in $B(x, kR)$ separated by a distance greater than or equal to $R$ from each other.

**Definition 1.** Define $X \in \Upsilon_s$ if there exists $C = C(s)$ such that, for $0 < R \leq kR$,

$$N(x, R, k) \leq Ck^s.$$

(\Upsilon_s)

The upper dimension $\Upsilon(X)$ is then defined as

$$\Upsilon(X) = \inf\{s \mid X \in \Upsilon_s\}.$$

$\Upsilon(X)$ was introduced in [Lar] called the uniform metric dimension.

**Definition 2.** A positive Borel measure $\mu \in U_s$ if there exists $C = C(s)$ such that, for $x \in X$ and $0 < R \leq kR$,

$$\mu(B(x, kR)) \leq Ck^s \mu(B(x, R)).$$

(U_s)

The dimension $U(X)$ is then defined as

$$U(X) = \inf\{s \mid U_s \neq \emptyset\}.$$

Note that by taking $k = 1/R$ in (U_s) one gets the weaker condition

$$\mu(B(x, R)) \geq CR^s, \ \ \ x \in X, \ \ 0 < R.$$

Also note that if $\mu \in U_s$, for some $s$, then $\text{supp}(\mu) = X$. As mentioned in the introduction, $\mu$ is doubling precisely when $\mu \in U_s$ for some $s < \infty$. We will write $\mathcal{U} = \bigcup_s U_s$ for the set of all doubling measures on $X$.

Volberg and Konyagin proved ([V-K]) that $\Upsilon(X) \leq U(X)$, and furthermore:

**Theorem 1 (Volberg-Konyagin).** Let $X$ be a compact metric space. If $X \in \Upsilon_s$, then for any $s' > s$ there exists a measure $\mu \in U_{s'}$. Consequently, $\Upsilon(X) = U(X)$. 
Our main result is Theorem 2 extending Theorem 1 to the analogue lower dimension. Note that Theorem 2 is stated for pseudo-metric spaces.

We start by defining the concept of the lower dimension.

**The lower dimension.**

**Definition 3.** Define $X \in \Lambda_t$ if there exists $C = C(t)$ such that, for $x \in X$ and $0 < R \leq kR$,

\[(\Lambda_t) \quad N(x, R, k) \geq Ck^t.\]

The lower dimension $\Lambda(X)$ is then defined as

\[\Lambda(X) = \sup \{t \mid X \in \Lambda_t\}.\]

$\Lambda(X)$ was introduced in (Lar) called the minimal dimension. Note that $X \in \Lambda_0$ is trivial.

**Definition 4.** A positive Borel measure $\mu \in L_t$ if there exists $C = C(t)$ such that, for $x \in X$ and $0 < R \leq kR$,

\[(L_t) \quad \mu(B(x, kR)) \geq Ck^t \mu(B(x, R)).\]

As before, by taking $k = 1/R$ in ($L_t$) one gets the weaker condition

\[(L_t') \quad \mu(B(x, R)) \leq CR^t, \quad x \in X, \quad 0 < R.\]

Now, observe that defining the lower dimension as

\[L(X) = \sup \{t \mid L_t \neq \emptyset\}\]

will not work since $\mu \in L_t$ does not imply $\text{supp}(\mu) = X$, so this will say nothing about $X \setminus \text{supp}(\mu)$. The appropriate definition is as follows.

**Definition 5.** Define the lower dimension $L(X)$ as

\[L(X) = \sup \{t \mid L_t \cap U \neq \emptyset\}.\]

Note that $L_0$ poses no restriction on $\mu \in U$.

**The main theorem.** We now state the main result of this paper. Note that in the special case $t = 0$ one can take $t' = t = 0$.

**Theorem 2.** Let $X \in \Upsilon_s \cap \Lambda_t$, $0 \leq t \leq s < +\infty$, be a compact pseudo-metric space. Then for any $s' > s$ and $t' < t$ there is a probability measure $\mu \in \Upsilon_{s'} \cap L_{t'}$.

From Theorem 2 and Propositions 4 and 5 below we then get

**Corollary 3.** If $\Upsilon(X) < +\infty$, then $\Upsilon(X) = U(X)$ and $\Lambda(X) = L(X)$.

3. **Proof of the theorem**

To prove Theorem 2 we construct a sequence of measures with certain properties and the desired measure $\mu$ will be a limit point of this sequence.

We start by proving the trivial inequalities $\Upsilon(X) \geq U(X)$ and $\Lambda(X) \geq L(X)$. 


3.1. The trivial inequalities.

Proposition 4.  If $\mu \in U_s$, then $X \in \mathcal{T}_s$, i.e. $\Upsilon(X) \leq U(X)$.

Proof. Let $\mu \in U_s$, fix any $x \in X$ and let $x_1, \ldots, x_N$ be points in $B(x, kR)$ with $d(x_i, x_j) \geq R$ for $i \neq j$. Since $\mu \in U_s$ and $B(x, 2C_d kR) \subset B(x, 4C_d^2 kR)$,

$$\mu(B(x, 2C_d kR)) \leq \mu(B(x, 4C_d^2 kR)) \leq C8s C_d^{3s} k^s \mu(B(x, \frac{R}{2C_d})).$$

Also, the balls $B(x_i, R/(2C_d))$ are disjoint and lie in $B(x, 2C_d kR)$, so

$$\mu(B(x, 2C_d kR)) \geq \sum_{i=1}^{N} \mu(B(x_i, \frac{R}{2C_d})) \geq N \mu(B(x, 2C_d kR)) C8s C_d^{3s} k^s.$$

Thus $N \leq Ck^s$, i.e. $\Upsilon(X) \leq U(X)$. 

Proposition 5.  If $\mu \in L_t \cap \mathcal{U}$, then $X \in \Lambda_t$, i.e. $\Lambda(X) \leq L(X)$.

Proof. Let $(x_1, \ldots, x_N)$ be a maximal set of points in $B(x, kR)$ separated by a distance greater than or equal to $R$. Fix any $\mu \in L_t \cap \mathcal{U}$. Then, since $\mu$ is doubling and $B(x_i, kR) \subset B(x, 2C_d kR)$ for all $i$,

$$C \mu(B(x, kR)) \geq \mu(B(x, 2C_d kR)) \geq \mu(B(x_i, kR)) \geq Ck^s \mu(B(x_i, R)).$$

Also, $B(x, kR) \subset \bigcup_{i=1}^{N} B(x_i, R)$, since $(x_1, \ldots, x_N)$ is maximal, i.e.

$$\mu(B(x, kR)) \leq \sum_{i=1}^{N} \mu(B(x_i, R)) \leq \frac{N}{Ck^s} \mu(B(x, kR)).$$

Thus, $N \geq Ck^t$, i.e. $X \in \Lambda_t$. 

3.2. The main lemma. Assume that $X \in \Lambda_t \cap \mathcal{T}_s$. Let $C_d$ be the constant associated to the pseudo-metric $d$, $C_t$ the constant appearing in $\Lambda_t$ and $C_s$ the one in $\mathcal{T}_s$. Given $t' < t$ and $s' > s$, choose $A \geq 16C_d^4$ large enough such that $A^{s'-s} > C_s$ and $A^{t'-t} > 4C_d^2 e C_t^{-1}$. For each non-negative integer $j$, let $S_j$ be a maximal set of points in $X$ separated by a distance greater than or equal to $A^{-j}$.

Define mappings $\mathcal{E} = \mathcal{E}_m : S_{m+1} \rightarrow S_m$ for $m \geq 0$ as follows. For $g \in S_{m+1}$ choose one of the points $e \in S_m$ for which $d(g, e) = d(g, S_m)$, and denote it by $e = \mathcal{E}(g)$. Then for $e \in S_m$ let

$$S_{c, m+1} = \{ g \in S_{m+1}, e = \mathcal{E}(g) \}.$$

It is easy to see that $\{ S_{c, m+1} \ | \ e \in S_m \}$ form a partition of $S_{m+1}$.

The desired measure $\mu$ will be a limit of measures $\mu_m$ supported by $S_m$. Lemma 6 below will allow us to perform the inductive step that constructs $\mu_{m+1}$ from $\mu_m$. First though we need the following prepratory lemma.

Lemma 6.  Let $e \in S_m$. Then

$$A^{t'} \leq \#(S_{e, m+1}) \leq A^{s'},$$

where $\#$ denotes the cardinality of a set.

Proof. Fix any $e \in S_m$. Clearly $S_{e, m+1} \subset B(e, A^{-m})$ since $S_m$ is maximal. Therefore, and since $X \in \mathcal{T}_s$ and $A^{s'-s} > C_s$,

$$\#(S_{e, m+1}) \leq \#(S_{m+1} \cap B(e, A^{-m})) \leq N(e, A^{-m-1}, A) \leq C_s A^s \leq A^{s'},$$

which proves the right inequality of the lemma.
For the left inequality, we first note that there exists \( g \in S_{m+1} \) for which \( d(g, e) < A^{-m-1} \), and as \( A > 2C_d \) it is clear that \( e = E(g) \) for such \( g \).

Also, for \( e' \neq e'' \) we have \( B(e', A^{-m}/2C_d) \cap B(e'', A^{-m}/2C_d) = \emptyset \). Thus,

\[
S_{m+1} \cap B(e, A^{-m}/(2C_d)) \subset S_{e,m+1}.
\]

Next, for \( \{g_i\}_{i=1}^n = S_{m+1} \cap B(e, A^{-m}/2C_d) \) we have

\[
(*) \quad n \geq N(e, A^{-m-1}, A/2C_d^2 - 1).
\]

To check it, suppose the contrary, that is, suppose that \( n < N(e, A^{-m-1}, A/2C_d^2 - 1) = n_1 \).

Then there would exist points \( x_1, \ldots, x_{n_1} \) in \( B(e, A/(2C_d^2 - 1)A^{-m-1}) \) separated from each other by a distance greater than or equal to \( A^{-m-1} \).

But, for \( g \in S_{m+1} \setminus \left( S_{m+1} \cap B(e, A^{-m}/2C_d) \right) \) we have

\[
d(g, x_i) \geq \frac{1}{C_d} d(g, e) - d(e, x_i) \geq \frac{A}{2C_d} A^{-m-1} - \left( \frac{A}{2C_d} - 1 \right) A^{-m-1} = A^{-m-1},
\]

which means that the set

\[
S'_{m+1} = \left( \{x_i\}_{i=1}^{n_1} \cup S_{m+1} \right) \setminus \left( S_{m+1} \cap B(e, A^{-m}/2C_d) \right)
\]

fulfills \( \#(S'_{m+1}) > \#(S_{m+1}) \), a contradiction to the maximality of \( S_{m+1} \). Thus, from \((*)\), \((\dagger)\), the choice of \( A \) and the fact that \( X \in A_\ast \), we conclude

\[
\#(S_{e,m+1}) \geq \#(S_{m+1} \cap B(e, A^{-m}/2C_d)) \geq N(e, A^{-m-1}, A/2C_d^2 - 1)
\]

\[
\geq C_t \left( A/2C_d^2 - 1 \right)^t \geq C_t A'(4C_d^2)^{-t} \geq A'.
\]

\[ \square \]

**Lemma 7.** Let \( f_0 \) be a measure on \( S_m \) such that for any \( e, e' \in S_m \) we have

\[
f_0(e') \leq C_1 f_0(e)
\]

whenever \( d(e, e') \leq C_2 A^{-m} \), with \( C_1 = A^{s'-t'} \), and \( C_2 = 8C_d^3 \). Then there is a measure \( f_1 \) on \( S_{m+1} \) with the following properties:

\[(a) \quad f_1(g') \leq C_1 f_1(g) \text{ for any } g, g' \in S_{m+1} \text{ with } d(g, g') \leq C_2 A^{-m-1}.
\]

\[(b) \quad \text{If } g \in S_{e,m+1}, \text{ then } A^{-s'} f_0(e) \leq f_1(g) \leq A^{-t'} f_0(e).
\]

\[(c) \quad f_0(X) = f_1(X).
\]

\[(d) \quad \text{The construction of the measure } f_1 \text{ from the measure } f_0 \text{ can be regarded as}
\]
\[ \text{a transfer of mass from the points in } S_m \text{ to those of } S_{m+1}, \text{ with no mass}
\]
\[ \text{transferred over a distance greater than } 2C_d A^{-m} \text{. This means that if } g \in S_{m+1}
\]
\[ \text{receives mass from } e \in S_m, \text{ then } d(g, e) \leq 2C_d A^{-m}.
\]

\[ \text{Proof of the lemma.} \] Let \( f_{00} \) be the measure obtained by homogeneously distributing the mass of each \( e \in S_m \) on the points in \( S_{e,m+1} \). By doing so, we obtain a measure satisfying \((b)\) (because of Lemma \( \square \), \( (c) \)) and \( (d) \). If \( f_{00} \) satisfies \((a)\), then let \( f_1 = f_{00} \) and we are done.

Assume that \( f_{00} \) does not satisfy \((a)\). Let \( \{g_i, g''_i\}_{i=1}^T \) be all the pairs of points in \( S_{m+1} \) with \( d(g_i, g''_i) \leq C_2 A^{-m-1} \). We will construct a finite sequence of measures \( \{f_{0j}, j = 1, \ldots, T\} \), such that \( f_{0j} \) will satisfy \((a)\) for all the pairs \( \{(g'_i, g''_i)\}_{i=1}^T \), and as we will see \( f_1 = f_{0T} \) is the desired measure.
The construction of \( f_{0j+1} \) from \( f_{0j} \) is as follows:

If \( C_1^{-1} f_{0j}(g'_{j+1}) \leq f_{0j}(g''_{j+1}) \leq C_1 f_{0j}(g''_{j+1}) \), then let \( f_{0j+1} = f_{0j} \). Otherwise, only one of these inequalities can fail, and without loss of generality we may assume that \( f_{0j}(g'_{j+1}) > C_1 f_{0j}(g''_{j+1}) \). Then we move mass from \( g'_{j+1} \) to \( g''_{j+1} \) by defining \( f_{0j+1} \) as

\[
\begin{align*}
    f_{0j+1}(g'_{j+1}) &= f_{0j}(g'_{j+1}) - \frac{f_{0j}(g'_{j+1}) - C_1 f_{0j}(g''_{j+1})}{C_1 + 1}, \\
    f_{0j+1}(g''_{j+1}) &= f_{0j}(g''_{j+1}) + \frac{f_{0j}(g'_{j+1}) - C_1 f_{0j}(g''_{j+1})}{C_1 + 1}, \\
    f_{0j+1}(g) &= f_{0j}(g) \quad \text{if} \ g \notin \{g'_{j+1}, g''_{j+1}\}.
\end{align*}
\]

With this definition \( f_{0j+1}(g'_{j+1}) = C_1 f_{0j+1}(g''_{j+1}) \), which means that \( \text{(a)} \) is true for \( f_{0j+1} \) with respect to \( (g'_{j+1}, g''_{j+1}) \). In particular, note that \( \text{(a)} \) is true for \( f_{0j} \) with respect to \( (g'_j, g''_j) \).

We are now going to check that when a pair satisfies \( \text{(b)} \) for \( f_{0j+1} \). To do so, suppose that \( \text{(b)} \) holds for \( f_{0j} \), i.e. suppose that

\[ A^{-s'} f_0(e) \leq f_1(g) \leq A^{-t'} f_0(e), \quad g \in S_{e,m+1}. \]

If \( f_{0j+1} = f_{0j} \) or \( g \notin \{g'_{j+1}, g''_{j+1}\} \), then there is nothing to check. Otherwise, as before we can assume that \( f_{0j}(g'_{j+1}) > C_1 f_{0j}(g''_{j+1}) \). Let \( e' = \mathcal{E}(g'_{j+1}) \) and \( e'' = \mathcal{E}(g''_{j+1}) \). It is clearly enough to prove that \( f_{0j+1}(g'_{j+1}) \geq A^{-s'} f_0(e') \) and \( f_{0j+1}(g''_{j+1}) \leq A^{-t'} f_0(e'') \) (because \( f_{0j+1}(g'_{j+1}) < f_{0j}(g'_{j+1}) \leq A^{-t'} f_0(e') \) and \( f_{0j+1}(g''_{j+1}) > f_{0j}(g''_{j+1}) \geq A^{-s'} f_0(e'') \)). Now

\[
\begin{align*}
    d(e', e'') &\leq C_d d(e', g'_{j+1}) + C_d^2 d(g'_{j+1}, g''_{j+1}) + C_d^2 d(g''_{j+1}, e'') \\
    &\leq C_d A^{-m} + C_d^2 C_d A^{-m-1} + C_d^2 A^{-m} \\
    &\leq C_d^2 A^{-m},
\end{align*}
\]

so \( f_0(e') \leq C_1 f_0(e'') \). Therefore

\[
\begin{align*}
    f_{0j+1}(g''_{j+1}) &= C_1^{-1} f_{0j+1}(g'_{j+1}) \leq C_1^{-1} f_{0j}(g'_{j+1}) \\
    &\leq C_1^{-1} A^{-t'} f_0(e') \leq A^{-t'} f_0(e'').
\end{align*}
\]

Analogously, \( f_0(e'') \geq C_1^{-1} f_0(e') \). Thus,

\[
\begin{align*}
    f_{0j+1}(g'_{j+1}) &= C_1 f_{0j+1}(g'_j) + C_1 f_{0j}(g''_{j+1}) \\
    &\geq C_1 A^{-s'} f_0(e'') \geq A^{-s'} f_0(e').
\end{align*}
\]

Consequently, since \( \text{(b)} \) holds for \( f_{00} \) it is clear that it holds for \( f_1 = f_{0T} \) as well.

We are now going to check that when a pair satisfies \( \text{(a)} \) with respect to \( f_{0j} \), it also does with respect to \( f_{0j+1} \). To this end, pick any pair \( (g_1, g_2) \), \( d(g_1, g_2) \leq C_d A^{-m-1} \), for which

\[ C_1^{-1} f_{0j}(g_1) \leq f_{0j}(g_2) \leq C_1 f_{0j}(g_1). \]

If \( (g_1, g_2) \) and \( (g'_{j+1}, g''_{j+1}) \) have no point in common or if \( f_{0j+1} = f_{0j} \), then we are done. Otherwise, \( f_{0j+1} \neq f_{0j} \) and \( f_{0j}(g'_{j+1}) > C_1 f_{0j}(g''_{j+1}) \). Then the two pairs have only one point in common, say \( g_1 \). In this case \( f_{0j+1}(g_2) = f_{0j}(g_2) \).
We have two possible cases to consider, either \( g_1 = g_{j+1}' \) or \( g_1 = g_{j+1}'' \):

1. If \( g_1 = g_{j+1}' \), then \( f_{0j+1}(g_1) > f_{0j}(g_1) \). Thus, in this case it is enough to prove that \( f_{0j+1}(g_1) \leq C_1 f_{0j+1}(g_2) \). Let \( e' = E(g_{j+1}') \) and \( e_2 = E(g_2) \). Then

\[
d(e', e_2) \leq C_d d(e', g_{j+1}') + C_d^3 d(g_{j+1}', g_1) + C_d^3 d(g_1, g_2) + C_d^3 d(g_2, e_2)
\]

\[
\leq C_d A^{-m} + 2C_d^3 C_2 A^{-m-1} + C_d^3 A^{-m} \leq C_2 A^{-m},
\]

so \( f_0(e') \leq C_1 f_0(e_2) \). Also, since we already know that \( (b) \) is true, we have \( f_0(e_2) \leq A^{-t'} f_0(g_2) \) and \( f_{0j+1}(g_{j+1}') \leq A^{-t'} f_0(e') \). Thus,

\[
f_{0j+1}(g_1) = f_{0j+1}(g_{j+1}') = C_1^{-1} f_{0j+1}(g_{j+1}') \leq C_1^{-1} A^{-t'} f_0(e') \leq A^{-t'} f_0(g_2) = A^{-t'} f_0(g_2) = C_1 f_{0j+1}(g_2).
\]

2. Otherwise, if \( g_1 = g_{j+1}'' \), then \( f_{0j+1}(g_1) < f_{0j}(g_1) \). Thus, it is enough to check that \( f_{0j+1}(g_1) \geq C_1^{-1} f_{0j+1}(g_2) \). But, for \( e'' = E(g_{j+1}'') \), then as in \((4)\),

\[
d(e'', e_2) \leq C_2 A^{-m}.
\]

Also, \( f_{0j+1}(g_1) = C_1 f_{0j+1}(g_{j+1}'') \). Thus, from \((b)\) we then get

\[
f_{0j+1}(g_1) = C_1 f_{0j+1}(g_{j+1}'') \geq C_1 A^{-s'} f_0(e'') \geq A^{-s'} f_0(e_2)
\]

\[
\geq A^{-s'} f_0(g_2) = C_1^{-1} f_{0j+1}(g_2).
\]

This concludes the proof that \((a)\) is true for \( f_1 \).

Clearly \( f_{0j+1}(X) = f_{0j}(X) \), so \((c)\) is also true for \( f_1 \).

It remains to check \((d)\). When passing from \( f_0 \) to \( f_{00} \) no mass is moved over a distance exceeding \( A^{-m} \), because \( S_{e, m+1} \subset B(e, A^{-m}) \), and when going from \( f_{0j} \) to \( f_{0j+1} \) no mass is moved over a distance exceeding \( C_2 A^{-m-1} \), and \( C_2 / A < 1 \). It therefore remains to prove that in the construction of \( f_1 \) from \( f_0 \) there are no pairs \( (g_1, g_2) \) and \( (g_2, g_3) \) in \( S_{m+1} \) for which mass is first moved from \( g_1 \) to \( g_2 \) and then at a subsequent step from \( g_2 \) to \( g_3 \). To prove this, assume the opposite. Then

\[
f_{00}(g_1) > C_1 f_{00}(g_2) \quad \text{and} \quad f_{00}(g_2) > C_1 f_{00}(g_3).
\]

But, if \( e_1 = E(g_1) \) and \( e_3 = E(g_3) \), then as in \((4)\),

\[
d(e_1, e_3) \leq C_2 A^{-m},
\]

so by the hypothesis \( C_1^{-1} f_0(e_1) \leq f_0(e_3) \leq C_1 f_0(e_1) \).

Also,

\[
A^{-s'} f_0(e_i) \leq f_{00}(g_i) \leq A^{-t'} f_0(e_i),
\]

for \( i = 1 \) and \( i = 3 \). Adding these two inequalities, we would then get

\[
f_0(e_1) \geq A^{t'} f_{00}(g_1) > C_1 A^{t'} f_{00}(g_2) > C_2 A^{t'} f_{00}(g_3) \geq C_2 A^{t'} f_{00}(e_3),
\]

contradicting \( f_0(e_1) \leq C_1 f_0(e_3) \), as \( d(e_1, e_3) \leq C_2 A^{-m} \) and \( C_1 = A^{s'-t'} \).

\[\square\]

3.3. Proof of the theorem. We will now use Lemma \[\text{[7]}\] to construct a sequence of probability measures and prove that any limit point of this sequence belongs to \( L_{\nu'} \cap U_{\nu'} \).

We start by defining a probability measure \( \mu_0 \) on \( S_0 \) (note that \( S_0 \) consists of one point only, by the assumption \( \text{diam}(X) < 1 \)). Obviously \( \mu_0 \) satisfies the hypothesis of Lemma \[\text{[7]}\]. By using Lemma \[\text{[7]}\] to construct \( \mu_{j+1} = f_1 \) on \( S_{j+1} \) from \( \mu_j = f_0 \), \( j \geq 0 \), we then get a sequence \( \{\mu_j\}_{j=0}^{\infty} \) of probability measures. This sequence belongs to the unit ball of the dual of the Banach space \( C(X) \), and thus has at least one weak limit point. Let \( \mu \) be any limit point of this sequence. In the proof we will frequently use the following proposition, based on \((d)\) of Lemma \[\text{[7]}\].
Proposition 8. Let $j \in \mathbb{N}$, $r \geq 0$, $x \in X$ and put $C_3 = 2C_d^2/(1 - C_d/A)$. Then
\[
\mu_j(B(x, r)) \leq \mu(B(x, r + C_3A^{-j}))
\]
and
\[
\mu(B(x, r)) \leq \mu_j(B(x, r + C_3A^{-j})).
\]

Proof. According to (d) of Lemma 1, no mass is moved at a distance exceeding $2C_dA^{-j}$ when constructing $\mu_{j+1}$ from $\mu_j$. Thus, when passing from $\mu_j$ to $\mu_{j+k}$, $k \geq 1$, no mass is moved at a distance exceeding
\[
2C_d^2A^{-j} \sum_{n=0}^{k-1} (C_d/A)^n < \frac{2C_d^2}{1 - C_d/A}A^{-j} = C_3A^{-j},
\]
which means that there is no mass transfer from $B(x, r)$ into the complement of $B(x, r + C_3A^{-j})$, and vice versa. Thus,
\[
\mu_j(B(x, R)) \leq \mu_{j+k}(B(x, r + C_3A^{-j}))
\]
and
\[
\mu_{j+k}(B(x, r)) \leq \mu_j(B(x, r + C_3A^{-j})).
\]
Now, as $\mu$ is a weak limit point of $\{\mu_{j+k}\}$, the same is true for $\mu$ as well. \qed

We will now prove that $\mu \in L^1 \cap U_{\nu}$. To this end, fix $x \in X$ and some $R$ and $k$ for which $0 < R \leq kR$. Then choose integers $m$ and $M$ such that
\[
kR \leq A^{-m} < AkR \quad \text{and} \quad \frac{R}{A} \leq A^{-M} < R.
\]
Denote by $e_{M+1}$ one of the points in $S_{M+1} \cap U_x$ (there may be several) and for $j = 0, \ldots, M - m$ define $e_{M-j} = E(e_{M-j+1}) \in S_{M-j}$.

First claim.
\[
m_{m+2}(e_{m+2}) \leq \mu(B(x, kR)) \leq C_3A^3(1 + C_3)^jC_1\mu_m(e_m).
\]

Proof. By the definition of $e_{M-j}$ and property 3 of the pseudo-metric $d$, we have
\[
d(x, e_{m+2}) \leq C_dA^{-m-2} \sum_{j=0}^{\infty} (C_d/A)^j = \frac{C_d}{1 - C_d/A}A^{-m-2}.
\]
Let $y \in B(e_{m+2}, C_3A^{-m-2})$. Then, by (3),
\[
d(y, x) \leq C_dA^{-m-2} + \frac{C_d^2}{1 - C_d/A}A^{-m-2} \leq A^{-m-1} < kR,
\]
i.e. $B(e_{m+2}, C_3A^{-m-2}) \subset B(x, kR)$. From Proposition 8 we then get
\[
m_{m+2}(e_{m+2}) \leq \mu(B(e_{m+2}, C_3A^{-m-2})) \leq \mu(B(x, kR)),
\]
proving the left inequality in (6). To prove the right inequality, note that (5) and Proposition 8 imply
\[
\mu(B(x, kR)) \leq \mu_m(B(x, kR + C_3A^{-m})) \leq \mu_m(B(x, (1 + C_3)A^{-m})).
\]
But, $d(x, e_m) \leq \frac{C_d}{1 - C_d/A}A^{-m}$. Thus, if $e \in S_m \cap B(x, (1 + C_3)A^{-m})$, then
\[
d(e, e_m) \leq C_d(1 + C_3)A^{-m} + \frac{C_d^2}{1 - C_d/A}A^{-m} \leq C_2A^{-m},
\]
so from Lemma 7 it follows that $\mu_m(e) \leq C_1 \mu_m(e_m)$. Now,

$$\# \left( S_m \cap B(x, (1 + C_3)A^{-m}) \right) \leq C_s(1 + C_3)^s,$$

so from Proposition 8 and the fact that $kR \leq A^{-m}$, we get

$$\mu(B(x, kR)) \leq \mu_m(B(x, (1 + C_3)A^{-m})) \leq C_s(1 + C_3)^s C_1 \mu_m(e_{x, m}),$$

which concludes the proof of the first claim.

Second claim.

(7) \[ \mu_{M+1}(e_{M+1}) \leq \mu(B(x, R)) \leq C_s(1 + C_3)^s' A^{2s'} C_1 \mu_{M+1}(e_{M+1}). \]

Proof. By the definition of $e_{M+1}$,

$$d(e_{M+1}, x) = d(x, S_{M+1}) \leq A^{-M-1} < R/A.$$ 

Thus, for $y \in B(e_{x, M+1}, C_3A^{-M-1})$,

$$d(y, x) \leq C_d C_3 A^{-M-1} + C_d A^{-M-1} \leq A^{-M} < R.$$

Again by Proposition 8,

$$\mu_{M+1}(e_{M+1}) \leq \mu(B(e_{M+1}, C_3A^{-M-1})) \leq \mu(B(x, R)),$$

proving the left inequality in (7). To prove the right inequality, note that from Proposition 8 and the fact that $R \leq A^{-M+1}$, by the choice of $M$,

$$\mu(B(x, R)) \leq \mu_{M-1}(B(x, R + C_3 A^{-M+1})) \leq \mu_{M-1}(B(x, (1 + C_3)A^{-M+1})).$$

Also, for $g \in B(x, R + C_3 A^{-M+1}) \cap S_{M-1}$,

$$d(g, e_{M-1}) \leq C_d d(g, x) + C^3_d d(x, e_{M+1}) + C^3_d d(e_{M+1}, e_{M}) + C^2_d d(e_{M}, e_{M-1})$$

$$\leq C_d(1 + C_3)A^{-M+1} + C^3_d A^{-M-1} + C^3_d A^{-M} + C^2_d A^{-M+1} \leq C_2 A^{-M+1}.$$

Thus, from (a) and (b) of Lemma 7 we get (recalling $e_{M-j} = E(e_{M-j+1})$),

$$\mu_{M-1}(g) \leq C_1 \mu_{M-1}(e_{M-1}) \leq C_1 A^{2s'} \mu_{M+1}(e_{M+1}).$$

But,

$$\# \left( B(x, (1 + C_3)A^{-M+1}) \cap S_{M-1} \right) \leq C_s(1 + C_3)^s,$$

so

$$\mu(B(x, R)) \leq \mu_{M-1}(B(x, (1 + C_3)A^{-M+1})) \leq C_s(1 + C_3)^s A^{2s'} C_1 \mu_{M+1}(e_{M+1}),$$

proving the second claim. To conclude the proof, note that

$$\mu(e_m) \leq A^{s'(M+1-m)} \mu_{M+1}(e_{M+1}) \quad \text{and} \quad \mu_{m+2}(e_{m+2}) \geq A^{s'(M-m-1)} \mu_{M+1}(e_{M+1})$$

by (b) in Lemma 7. Also note that $k < A^{M-m} \leq A^2 k$, by the choice of $m$ and $M$.

Thus, from the two claims it follows that

$$\mu(B(x, kR)) \leq C \mu_m(e_m) \leq C A^{s'(M-m)} \mu_{M+1}(e_{M+1}) \leq C k^{s'} \mu(B(x, R)),$$

and similarly,

$$\mu(B(x, kR)) \geq \mu_{m+2}(e_{m+2}) \geq C A^{s'(M-m)} \mu_{M+1}(e_{M+1}) \geq C k^{s'} \mu(B(x, R)),$$

i.e. $\mu \in \Lambda_{s'} \cap \Upsilon_{s'}$. 


Note that the final constants $C$ depend only on the given constants $C_d$, $C_s$, $C_t$ and the choice of $A$, $s'$ and $t'$. Also note that the last inequality depends on the fact that $\Upsilon(X) < +\infty$. 

4. THE NON-COMPACT CASE

In [L-S] Theorem 1 was generalized to a non-compact complete metric space $X$. It is easy to see that their proof holds for a pseudo-metric, too. We conclude this paper by showing that Theorem 2 combined with their proof gives the analogue generalization of Theorem 2 as well. Before that we just briefly sketch their proof, and refer to [L-S] for details:

Let $s' > s$ and cover $X \in \Upsilon_s$ with a countable collection of compact balls $X_n = B(x_0, n)$, $n \in \mathbb{N}$, $x_0 \in X$. Every $X_n$ carries a $\mu_n \in U_{s'}$, by Theorem 1.

By using the weak-* compactness of the unit ball of $C(X_n)$ and a Cantor’s diagonal process they show the existence of a subsequence $\{\mu_n^j\}$ of $\{\mu_n\}$ such that, for every continuous $f \geq 0$ with compact support on $X$, $\int_{X_n} fd\mu_n^j$ converges to $\int_X fd\mu$ for some $\mu \in U_{s'}$.

**Theorem 9.** Let $X \in \Upsilon_s \cap \Lambda_t$ be any complete pseudo-metric space. Then there exists a $\mu \in U_{s'} \cap L_\mu'$ for every $t' < t$ and $s' > s$.

**Proof.** We use the notation above. It remains to prove $\mu \in L_{\mu'}$. From Theorem 2 it is clear that $\mu_n^j \in U_{s'} \cap L_{\mu'}$, where the constant $C$ in $L_{\mu'}$ is the same for all $j$. Let $x \in X$, $r > 0$, $k > 1$. Let $0 < \varepsilon < (k - 1)/(k + 1)$ and pick continuous functions $0 \leq f, g \leq 1$ such that $f = 1$ on $B(x, (1 - \varepsilon)kr)$ and $g = 1$ on $B(x, r)$, and such that $f$ and $g$ have compact support on $B(x, kr)$ and $B(x, (1 + \varepsilon)r)$, respectively. Put $c^{-1} = Ck^{t'}((1 - \varepsilon)/(1 + \varepsilon))^{t'}$, choose $p$ such that $B(x, kr) \subset X_p$ and choose $j$ large enough that $|\int_{X_p} \mu(B(x, r) + \varepsilon) - \int_{X_p} f d\mu| < \varepsilon$ for $h = f, g$. Then

$$\mu(B(x, r)) \leq \int_{X_p} g d\mu \leq \int_{X_p} g d\mu^j + \varepsilon \leq \mu^j(B(x, (1 + \varepsilon)r)) + \varepsilon$$

$$\leq c \mu^j(B(x, (1 - \varepsilon)kr)) + \varepsilon \leq c \int_{X_p} f d\mu^j + \varepsilon$$

$$\leq c \int_{X_p} f d\mu + c\varepsilon + \varepsilon \leq c\mu(B(x, kr)) + c\varepsilon + \varepsilon$$

Letting $\varepsilon \to 0$ gives $Ck^{t'} \mu(B(x, r)) \leq \mu(B(x, kr))$, i.e. $\mu \in L_{\mu'}$. 

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Department of Mathematics, University of Umeå, S-90187 Umeå, Sweden
E-mail address: Per.Bylund@math.umu.se

Departament de Matèmàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain
E-mail address: gudayol@mat.ub.es