Equivalence of Ensembles Under Inhomogeneous Conditioning and Its Applications to Random Young Diagrams

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Received: 7 May 2013 / Accepted: 24 August 2013 / Published online: 13 September 2013
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Abstract We prove the equivalence of ensembles or a realization of the local equilibrium for Bernoulli measures on $\mathbb{Z}$ conditioned on two conserved quantities under the situation that one of them is spatially inhomogeneous. For the proof, we extend the classical local limit theorem for a sum of Bernoulli independent sequences to those multiplied by linearly growing weights. The motivation comes from the study of random Young diagrams and their evolitional models, which were originally suggested by Herbert Spohn. We discuss the relation between our result and the so-called Vershik curve which appears in a scaling limit for height functions of two-dimensional Young diagrams. We also discuss a related random dynamics.

Keywords Equivalence of ensembles · Local equilibrium · Local limit theorem · Young diagram · Vershik curve

1 Introduction

The equivalence of ensembles, that is the asymptotic equivalence of canonical and grand canonical ensembles for large systems, plays a fundamental role in equilibrium statistical physics [4, 7, 12]. It is mostly discussed for canonical ensembles obtained under conditioning on spatially homogeneous physical quantities. In our problem, the grand canonical ensembles are simply Bernoulli measures on $\mathbb{Z}$, but the system has two conservation laws so that the corresponding (micro) canonical ensembles are defined through conditioning on a quantity which is not translation-invariant. As a result, the macroscopic profile for the grand canonical ensemble turns out to be spatially dependent. Such situation is called the local
equilibrium in statistical physics. We will show that this profile has a connection to the so-called Vershik curve which appears in a scaling limit for two-dimensional Young diagrams, \cite{1, 2, 9, 14, 15}.

The height function $\psi_q$ of a two-dimensional Young diagram, which is associated with a distinct partition $q = \{q_1 > q_2 > \cdots > q_K \geq 1\}$ of a positive integer $M$ by positive integers $\{q_i\}_{i=1}^K$ (i.e. $M = \sum_{i=1}^K q_i$), is given by a right continuous non-increasing step function

$$\psi_q(u) = \sum_{i=1}^K 1_{\{u < q_i\}}, \quad u \geq 0. \tag{1.1}$$

Its (negative) height difference $\eta = \{\eta_k\}_{k \in \mathbb{N}}, \mathbb{N} = \{1, 2, \ldots\}$, is defined by

$$\eta_k = \psi_q(k-1) - \psi_q(k) \in \{0, 1\} \quad \text{or} \quad \eta_k = 1 \iff k \in \{q_i\}_{i=1}^K. \tag{1.2}$$

Then, in terms of $\eta$, the height $K$ of the Young diagram at $u = 0$ is represented as

$$\psi_q(0) = \sum_{k \in \mathbb{N}} \eta_k$$

and the area $M$ as

$$\int_0^\infty \psi_q(u) du = \sum_{k \in \mathbb{N}} k \eta_k.$$

The sequence $\eta = \{\eta_k\}_{k \in \mathbb{N}}$ can be interpreted as defining a configuration of particles on $\mathbb{N}$ in the sense that the site $k$ is occupied by a particle if $\eta_k = 1$ and vacant if $\eta_k = 0$ and, from the viewpoint of random Young diagrams, it is natural to restrict the space of configurations $\eta$ to those satisfying two conditions:

$$\sum_{k \in \mathbb{N}} \eta_k = K, \quad \sum_{k \in \mathbb{N}} k \eta_k = M, \tag{1.3}$$

given two constants $K$ and $M$. The first condition is equivalent to assign the total number of particles in the system, while the second one assigns the sum of coordinates of occupied sites and involves a non-translation-invariant linearly growing weight $k$. The simplest random structure can be introduced to the set of Young diagrams with height $K$ at $u = 0$ and area $M$ by means of a uniform probability measure $\nu_{K,M}$ on the space of configurations $\eta$ with these two constraints. In fact, such random structure is the subject of our study.

We formulate our problem and main results in Sect. 2. For the proof, we need to extend the local limit theorem for a sum of independent random variables. If the random variables satisfy certain proper moment conditions (see, e.g., Theorems VII.4 and VII.12 of \cite{10}), the classical theory applies, but in our case, such conditions do not hold since the random variables have unbounded linearly growing weights. This is discussed in Sect. 3 relying on the fact that the weights are linear by analyzing the behavior of characteristic functions. Sections 4 and 5 are devoted to the proofs of main results. We also show that the limit curves obtained in \cite{1} and here are identical, although the random structures imposed on the set of Young diagrams are different.

The original motivation of this paper comes from the study of an area-preserving random dynamics on two-dimensional Young diagrams with height differences restricted to be 0 or 1. Such dynamics is related to a surface diffusion model studied in physics \cite{11}. As we have seen above, it can be transformed into an equivalent particle system on $\mathbb{N}$ with two
conservation laws. In Sect. 6, we construct a dynamics which is reversible under \( v_{K,M} \) and give some discussions on it.

## 2 Main Results

Let us state our problem precisely. We consider a particle system on \( \mathbb{Z} \) rather than on \( \mathbb{N} \) following the usual setting in statistical physics. Each site is occupied by at most one particle. Therefore the particle configuration on \( \mathbb{Z} \) is represented by \( \eta = \{ \eta_k \}_{k \in \mathbb{Z}} \in \Sigma := \{0, 1\}^\mathbb{Z} \), where \( \eta_k = 1 \) means that the site \( k \) is occupied by a particle and \( \eta_k = 0 \) means that \( k \) is vacant. For a finite set \( \Lambda \) in \( \mathbb{Z} \) and \( \eta \in \Sigma \), we define

\[
K_\Lambda(\eta) := \sum_{k \in \Lambda} \eta_k, \quad M_\Lambda(\eta) := \sum_{k \in \Lambda} k \eta_k.
\]

For given \( K, M \) and \( \ell \in \mathbb{N} \), let \( v_{\Lambda,\ell,K,M} \) be the uniform probability measure on the configuration space \( \Sigma_{\Lambda,\ell,K,M} = \{ \eta \in \Sigma_{\Lambda,\ell}; K_{\Lambda,\ell}(\eta) = K, M_{\Lambda,\ell}(\eta) = M \} \), where \( \Lambda_\ell = \{ k \in \mathbb{Z}; |k| \leq \ell \} \) and \( \Sigma_A = \{0, 1\}^A \). For \( \alpha \in (0, 1) \) and a finite set \( \Lambda \) in \( \mathbb{Z} \), let \( v_\alpha^A \) be the Bernoulli measure on \( \Sigma_A \) with mean \( \alpha \), that is,

\[
v_\alpha^A(\eta) = (1 - \alpha)\sharp \{ k \in A; \eta_k = 0 \} \alpha\sharp \{ k \in A; \eta_k = 1 \}
\]

for each \( \eta \in \Sigma_A \).

Then, the measure \( v_{\Lambda,\ell,K,M} \) is identical to the conditional measure of \( v_\alpha^A \) or \( v_\alpha^A \) with \( \Lambda_\ell \subset \Lambda \) on \( \Sigma_{\Lambda,\ell,K,M} \):

\[
\nu_{\Lambda,\ell,K,M}(\xi) = \nu_{\Lambda,\ell}(\xi|\Sigma_{\Lambda,\ell,K,M}) = \nu_{\Lambda}(\eta_{\Lambda,\ell}|\Sigma_{\Lambda,\ell,K,M} \times \{ \eta_{\Lambda,\ell} \})
\]

independently of the choice of the outer conditions \( \eta_{\Lambda,\ell} \); see also Lemma 4.1 below. We have denoted by \( \eta_{\Lambda,\ell} \) and \( \eta_{\Lambda,\ell} \) the configurations \( \eta \) restricted on these sets.

We first observe the possible values of \( K = K_{\Lambda,\ell}(\eta) \) and \( M = M_{\Lambda,\ell}(\eta) \) for \( \eta \in \Sigma_{\ell} \). It is easy to see that \( 0 \leq K \leq 2\ell + 1 \), while the values of \( M \) range as

\[
|M| \leq \frac{1}{2} \left( K(2\ell + 1) - K^2 \right)
\]

under the condition that \( K_{\Lambda,\ell}(\eta) = K \). Indeed the extreme values of \( M \) given \( K \) are attained when all \( K \) particles are closely packed at the right or left most edges on \( \Lambda_\ell \). Accordingly, we have that

\[
0 \leq \frac{K}{2\ell + 1} \leq 1, \quad \left| \frac{M}{(2\ell + 1)^2} \right| \leq \frac{1}{2} \left( \frac{K}{2\ell + 1} - \left( \frac{K}{2\ell + 1} \right)^2 \right).
\]

We fix \( p \in \mathbb{N} \) and assume that, for \( \ell \in \mathbb{N} \) and \( 1 \leq j \leq p \), sequences \( K = K_\ell \), \( M = M_\ell \) and \( k_j = k_{j,\ell} \) are given and satisfy

\[
\lim_{\ell \to \infty} \frac{K}{2\ell + 1} = \rho \in (0, 1), \quad (2.1)
\]

\[
\lim_{\ell \to \infty} \frac{M}{(2\ell + 1)^2} = m \in (-v/2, v/2), \quad (2.2)
\]

\[
\lim_{\ell \to \infty} \frac{k_j}{\ell} = x_j \in (-1, 1), \quad (2.3)
\]
respectively, with distinct limits \( \{x_j\}_{j=1}^p \), where \( v = \rho(1 - \rho) \). A function \( f \) on \( \Sigma \) is called local if it depends only on finitely many coordinates in \( \eta \). The shift operators \( \tau_i \) are defined by \( \tau_i f(\eta) = f(\tau_i \eta) \) and \( (\tau_i \eta)_k = \eta_{k+i} \) for \( i, k \in \mathbb{Z} \) and \( \eta \in \Sigma \). We are now at the position to formulate our main theorem.

**Theorem 2.1** Let \( f_j = f_j(\eta) \), \( 1 \leq j \leq p \) be local functions on \( \Sigma \). Then, under the conditions (2.1)–(2.3), we have that

\[
\lim_{\ell \to \infty} E_{v, \Lambda_{\ell,K,M}} \left[ \prod_{j=1}^p \tau_{k_j} f_j \right] = \prod_{j=1}^p \mathbb{E}_{\nu_{\beta(x)}} \left[ f_j \right],
\]

where \( v, \alpha \in (0, 1) \) denotes the Bernoulli measure on \( \Sigma \) with mean \( \alpha \) and

\[
\beta(x) \equiv \beta(x; a, b) = \frac{e^{bx}a}{e^{bx}a + (1 - a)}, \quad x \in [-1, 1],
\]

with two parameters \( a \in (0, 1) \) and \( b \in \mathbb{R} \) determined from \( \rho \) and \( m \) by the relations:

\[
\int_{-1}^{1} \beta(x)dx = 2\rho, \quad \int_{-1}^{1} x\beta(x)dx = 4m.
\]

The convergence is uniform in \((K, M)\) in the region that \( \varepsilon \leq K/(2\ell + 1) \leq 1 - \varepsilon \) and \( M/(2\ell + 1)^2 \in (-v/2 + \varepsilon, v/2 - \varepsilon) \) for every \( \varepsilon > 0 \).

For every \( \rho \) and \( m \), one can find \( a \) and \( b \) uniquely as is shown in Lemma 4.2 below. This theorem asserts that, as \( \ell \to \infty \) under the canonical ensemble \( \nu_{\Lambda_{\ell,K,M}} \), the limit distributions are asymptotically independent for microscopic regions which are macroscopically separated, and the microscopic limit distribution around the macroscopic point \( x \in (-1, 1) \) is the grand canonical ensemble \( \nu_{\beta(x)} \) with macroscopically dependent profile \( \beta(x) \). Such situation is called the local equilibrium in statistical physics. It will be useful to have another expression of \( \beta \):

\[
\beta(x) = \frac{1}{b} \frac{g'(x)}{g(x)} = \frac{1}{b} (\log g(x))',
\]

where \( g(x) = e^{bx}a + (1 - a) \) and we denote \( g'(x) = d g/d x \). In particular, we have that

\[
\int_{-1}^{1} \beta(x)dx = \frac{1}{b} \log \frac{e^{b}a + (1 - a)}{e^{-b}a + (1 - a)},
\]

\[
\int_{-1}^{1} x\beta(x)dx = \frac{1}{b} \log \frac{(e^{b}a + (1 - a))(e^{-b}a + (1 - a))}{(1 - a)^2}
\]

\[
+ \frac{1}{b^2} \left\{ L_2 \left( \frac{-ae^b}{1 - a} \right) - L_2 \left( \frac{-ae^{-b}}{1 - a} \right) \right\},
\]

where \( L_2(z) := -\int_0^z \frac{1}{t} \log(1 - t)dt \), \( z < 1 \) is the Euler dilogarithm, see [5], p. 642.

**Remark 2.1**

(1) Theorem 2.1 gives the equivalence of ensembles under the situation that the system has two conservation laws, especially, one of them is not translation-invariant.
(2) The uniformity of the convergence near the boundary values of \( K/(2\ell + 1) \) and \( M/(2\ell + 1)^2 \) can not be shown, because the cumulant \( \lambda_3 \), which controls the error estimate in the local limit theorem, diverges near the boundary values.

(3) The macroscopic mean \( \beta(x) \) of the limit measure is not translation-invariant, but the distribution of the microscopic configuration near each point \( x \) is a Bernoulli measure so that it is translation-invariant.

(4) As we will see in the proof of Lemma 4.2, \( a \) is increasing in \( \rho \) and \( b \) is increasing in \( m \), respectively. In particular, the constant \( m \) measures the extent of the bias for the particles, that is, a larger \( m \) implies more particles on the right side.

(5) The function \( \beta(x) \) satisfies the differential equation: \( \beta' = b\beta(1 - \beta) \) on \([-1, 1]\). In other words, it is a stationary solution of the viscous Burgers’ equation: \( \partial_t \beta = \beta'' - b(\beta(1 - \beta))' \) for \( \beta = \beta(t, x) \), cf. (5.5) in [2].

(6) The function \( \beta(x) \) appears in other context: The asymmetric simple exclusion process on \( \mathbb{Z} \) with jump rates \( p \) to the right and \( q \) to the left satisfying \( 0 < p < 1, p + q = 1 \), has \( \nu_{\beta(;a,b)} \) with \( b = \log p/q \) as its invariant measure for every \( a \in (0, 1) \), where \( \nu_{\beta(\cdot)} \) denotes the product measure on \( \Sigma \) such that \( E_{\nu_{\beta(\cdot)}}[\eta_k] = \beta(k) \), \( k \in \mathbb{Z} \), and the function \( \beta(\cdot; a, b) \) is defined by (2.4) for all \( x \in \mathbb{R} \) (or \( x \in \mathbb{Z} \)), see [8], p. 382.

The macroscopic profile \( \beta(x) \) has a connection to the Vershik curve which appears in a scaling limit for random Young diagrams distributed under the restricted uniform (Fermi) statistics. In fact, one can associate the height function \( \psi^\ell(u) \), \( u \in [-\ell - 1, \ell] \) of the Young diagram with the particle configuration \( \eta \in \Sigma \) by

\[
\psi^\ell(u) = \sum_{k \in \Lambda_\ell; k > u} \eta_k, \quad u \in [-\ell - 1, \ell].
\]  

Note that \( \psi^\ell \) is a right continuous non-increasing step function and satisfies

\[
\psi^\ell(-\ell - 1) = K_{\Lambda_\ell}(\eta), \quad \psi^\ell(\ell) = 0,
\]  

with the area

\[
\int_{-\ell - 1}^\ell \psi^\ell(u) du = (\ell + 1)K_{\Lambda_\ell}(\eta) + M_{\Lambda_\ell}(\eta).
\]  

Under the distribution \( \nu_{\Lambda_\ell, K, M} \), we consider the macroscopically scaled height function defined by

\[
\tilde{\psi}^\ell(x) := \frac{1}{\ell} \psi^\ell(\ell x), \quad x \in [-1, 1].
\]  

**Corollary 2.2** Under the conditions (2.1) and (2.2), \( \tilde{\psi}^\ell \) converges as \( \ell \to \infty \) to \( \psi \) in probability in the following sense:

\[
\lim_{\ell \to \infty} \nu_{\Lambda_\ell, K, M} \left( \sup_{x \in [-1, 1]} \left| \tilde{\psi}^\ell(x) - \psi(x) \right| > \delta \right) = 0,
\]

for every \( \delta > 0 \). The limit \( \psi \) is defined by \( \psi(x) = \int_x^1 \beta(y) dy, x \in [-1, 1] \) with \( \beta(x) \) determined in Theorem 2.1.
For the proof of Theorem 2.1, we need to establish the local limit theorem jointly for the
3 Local Limit Theorem for Inhomogeneous Bernoulli Sequence with Unbounded
sum of inhomogeneous Bernoulli sequence and the sum with unbounded linearly growing
the asymptotic behaviors of $E_n, F_n, U_n$ in Theorem 2.2 of [2], the Vershik curve
be given. Let $c = -\psi$ that the limit
and the ordinary differential equation
\[ \psi'' + c\psi' (1 + \psi') = 0, \tag{2.13} \]
where $c = \pi / \sqrt{12}$. Here we consider in a rectangular box. We see from Remark 2.1-(5)
that the limit \( \psi \) in Corollary 2.2 satisfies the same ordinary differential equation (2.13) with
\( c = -b \). Further discussions on the Vershik curves will be held in Sect. 5.

3 Local Limit Theorem for Inhomogeneous Bernoulli Sequence with Unbounded
Weight

For the proof of Theorem 2.1, we need to establish the local limit theorem jointly for the
sum of inhomogeneous Bernoulli sequence and the sum with unbounded linearly growing
weights $k$ having some defects in them. Let a continuous function $\alpha = \alpha(\cdot) : [0, 1] \to (0, 1)$
and a sequence $\{a_k^n \in (0, 1)\}_{k=1}^n$ satisfying the condition
\[ \lim_{n \to \infty} \max_{1 \leq k \leq n} |a_k^n - \alpha(k/n)| = 0, \tag{3.1} \]
be given. Let $\{X_k\}_{k=1}^n = \{X_k^n\}_{k=1}^n$ be $\{0, 1\}$-valued independent random variables with mean
$E[X_k] = a_k^n$ for $1 \leq k \leq n$ and $n \in \mathbb{N}$. Note that $\alpha_- \leq \alpha(x), \alpha_k^n \leq \alpha_+, x \in [0, 1], 1 \leq k \leq n,$
$n \in \mathbb{N}$ holds with some $0 < \alpha_- < \alpha_+ < 1$. We assume that a subset $\Gamma_n$ of $\{1, 2, \ldots, n\}$ is
given for each $n \in \mathbb{N}$ and the size $|\Gamma_n|$ is uniformly bounded in $n$: $|\Gamma_n| \leq C$ for all $n$. Under
these settings, we consider the sums
\[ S_n = S_n(X) := \sum_{k \in \Gamma_n} X_k, \quad T_n = T_n(X) := \sum_{k \in \Gamma_n^c} kX_k, \tag{3.2} \]
for $n \in \mathbb{N}$, where $\Gamma_n^c = \{1, 2, \ldots, n\} \setminus \Gamma_n$; $\Gamma_n$ are defects in these sums. Then, it is easy to
see that the (joint) central limit theorem holds for $(S_n, T_n)$. Indeed, we define
\[ \bar{S}_n := \frac{1}{\sqrt{U_n}} (S_n - E_n), \quad \bar{T}_n := \frac{1}{\sqrt{V_n}} (T_n - F_n), \]
with
\[ E_n = \sum_{k \in \Gamma_n} \alpha_k^n, \quad F_n = \sum_{k \in \Gamma_n^c} k\alpha_k^n, \quad U_n = \sum_{k \in \Gamma_n} \nu_k^n, \quad V_n = \sum_{k \in \Gamma_n^c} k^2 \nu_k^n, \]
where $\nu_k^n := \alpha_k^n (1 - \alpha_k^n)$. Recalling that $\alpha(\cdot) \in C([0, 1])$, the condition (3.1) and $|\Gamma_n| \leq C$,
the asymptotic behaviors of $E_n, F_n, U_n$ and $V_n$ as $n \to \infty$ are given by
\[ E_n = n(\bar{\alpha} + o(1)), \quad F_n = n^2 (\bar{\alpha} + o(1)), \quad U_n = n(\bar{\nu} + o(1)), \quad V_n = n^3 (\bar{\nu} + o(1)), \tag{3.3} \]
where $\bar{\alpha}, \tilde{\alpha}, \bar{v}, \tilde{v}$ are positive constants defined by

$$
\bar{\alpha} = \int_0^1 \alpha(x)dx, \quad \tilde{\alpha} = \int_0^1 x\alpha(x)dx,
$$

$$
\bar{v} = \int_0^1 \alpha(x)(1 - \alpha(x))dx, \quad \tilde{v} = \int_0^1 x^2\alpha(x)(1 - \alpha(x))dx,
$$

respectively. Then $(\tilde{S}_n, \tilde{T}_n)$ weakly converges to $(Y_1, Y_2)$ as $n \to \infty$, where $Y = (Y_1, Y_2)$ is an $\mathbb{R}^2$-valued Gaussian random variable with mean 0 and $E[Y_1^2] = E[Y_2^2] = 1$, $E[Y_1Y_2] = \lambda$, where

$$
\lambda = \frac{1}{\sqrt{\tilde{v}\bar{v}}} \int_0^1 x\alpha(x)(1 - \alpha(x))dx.
$$

Indeed, the convergence of the corresponding characteristic functions is shown by Lemma 3.2 below. Note that $\lambda \in (0, 1)$ by Schwarz’s inequality. The joint distribution density function of $Y$ is given by

$$
q_0(y) = \frac{1}{2\pi \sqrt{1 - \lambda^2}} \exp\left\{ -\frac{y_1^2 - 2\lambda y_1 y_2 + y_2^2}{2(1 - \lambda^2)} \right\}, \quad y = (y_1, y_2) \in \mathbb{R}^2.
$$

(3.4)

We now state the corresponding local limit theorem. The set of all possible values of $(S_n, T_n)$ is denoted by

$$
\mathcal{P}_n := \{(K, L) \in \mathbb{Z}_+ \times \mathbb{Z}_+; P(S_n = K, T_n = L) > 0\},
$$

where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

**Proposition 3.1** We have that

$$
\lim_{n \to \infty} \sup_{(K, L) \in \mathcal{P}_n} \left| \sqrt{U_nV_n} P(S_n = K, T_n = L) - q_0(y_1, y_2) \right| = 0,
$$

where

$$
y_1 = \frac{1}{\sqrt{U_n}}(K - E_n), \quad y_2 = \frac{1}{\sqrt{V_n}}(L - F_n).
$$

(3.5)

**Remark 3.1** If $\alpha(\cdot) \in C^1([0, 1])$ and $\alpha_n^k$ are given by $\alpha_n^k = \alpha(k/n)$, then the convergence in Proposition 3.1 takes place with speed $O(1/\sqrt{n})$:

$$
\sup_{(K, L) \in \mathcal{P}_n} \left| \sqrt{U_nV_n} P(S_n = K, T_n = L) - q_0(y_1, y_2) \right| \leq \frac{C}{\sqrt{n}}
$$

with some $C > 0$. However, in Sect. 4, we are forced to consider more general $\alpha_n^k$ satisfying the condition (3.1). See Remark 4.1 below.

The local limit theorem for $T_n$ was studied by [6] in homogeneous Bernoulli case without defects and we extend it to the joint variables $(S_n, T_n)$ in inhomogeneous case with defects. The rest of this section is devoted to the proof of Proposition 3.1. We essentially follow the arguments in [10], but because of the unboundedness of the weights $k$ for $T_n$, a phenomenon
different from the classical situation arises in the Fourier mode especially for the term \( I_2 \) introduced below.

Let \( f(s, t; S, T), s, t \in \mathbb{R}, \) be the characteristic function of \( \mathbb{R}^2 \)-valued random variable \((S, T)\) in general:

\[
f(s, t; S, T) = E[e^{isS + itT}],
\]

where \( i = \sqrt{-1}. \) Then, for all \((K, L) \in \mathcal{P}_n, \) we have that

\[
(2\pi)^2 P(S_n = K, T_n = L) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-i(sK + tL)} f(s, t; S_n, T_n) dsdt
\]

\[
= \frac{1}{\sqrt{U_n V_n}} \int_{-\sqrt{U_n} \pi}^{\sqrt{U_n} \pi} \int_{-\sqrt{V_n} \pi}^{\sqrt{V_n} \pi} e^{-i(sy_1 + ty_2)} f(s, t; \tilde{S}_n, \tilde{T}_n) dsdt,
\]

where \( y_1, y_2 \) are defined by (3.5). The second equality is due to a change of variables noting that

\[
f(s, t; S_n, T_n) = \exp\{i(sE_n + tF_n)\} f(\sqrt{U_n} s, \sqrt{V_n} t; \tilde{S}_n, \tilde{T}_n).
\]

Thus, if we define the error term by

\[
R_n(K, L) = (2\pi)^2 \left\{ \sqrt{U_n} V_n P(S_n = K, T_n = L) - q_0(y_1, y_2) \right\},
\]

it can be rewritten as

\[
R_n(K, L) = \int_{-\sqrt{U_n} \pi}^{\sqrt{U_n} \pi} \int_{-\sqrt{V_n} \pi}^{\sqrt{V_n} \pi} e^{-i(sy_1 + ty_2)} f(s, t; \tilde{S}_n, \tilde{T}_n) dsdt
\]

\[
- \int_{\mathbb{R}^2} e^{-i(sy_1 + ty_2)} e^{-(s^2 + 2\lambda st + t^2)/2} dsdt,
\]

since \((2\pi)^2 q_0(y_1, y_2)\) is given by the second integral in the above expression. We divide \( R_n(K, L) \) into the sum of three integrals:

\[
R_n(K, L) = I_1 + I_2 + I_3,
\]

where

\[
I_1 = \int_{D_{1,n}} e^{-i(sy_1 + ty_2)} \left\{ f(s, t; \tilde{S}_n, \tilde{T}_n) - e^{-(s^2 + 2\lambda st + t^2)/2} \right\} dsdt,
\]

\[
I_2 = \int_{D_{2,n}} e^{-i(sy_1 + ty_2)} f(s, t; \tilde{S}_n, \tilde{T}_n) dsdt,
\]

\[
I_3 = \int_{D_{3,n}} e^{-i(sy_1 + ty_2)} e^{-(s^2 + 2\lambda st + t^2)/2} dsdt,
\]

respectively. Three domains are defined by

\[
D_{1,n} = \left\{ (s, t) \in \mathbb{R}^2; |s|, |t| \leq c \sqrt{n} \right\},
\]

\[
D_{2,n} = \left\{ (s, t) \in \mathbb{R}^2; c \sqrt{n} < |s| \leq \sqrt{U_n \pi} \text{ or } c \sqrt{n} < |t| \leq \sqrt{V_n \pi} \right\},
\]

\[
D_{3,n} = \left\{ (s, t) \in \mathbb{R}^2; |s| \geq c \sqrt{n} \text{ or } |t| \geq c \sqrt{n} \right\},
\]

respectively, with a small enough \( c \in (0, \sqrt{\pi}) \) chosen later.
The estimate on $I_3 \equiv I_{3,n}(K, L)$ is easy. In fact, noting that $s^2 + 2\lambda st + t^2 \geq (1 - \lambda)(s^2 + t^2)$ and $\lambda < 1$, we have a uniform bound on $I_3$: There exist $C_1, c_1 > 0$ such that
\[
\sup_{K, L} |I_3| \leq C_1 e^{-c_1 n}. \tag{3.7}
\]

To give the estimate on $I_1$, we prepare a lemma which is a two-dimensional version of Lemma 1 in Chapter V of [10], p. 109.

Lemma 3.2 For every $\delta > 0$, there exist $n_0 \in \mathbb{N}, c_2, c_3 > 0$ such that, if $n \geq n_0$,
\[
|f(s, t; \tilde{S}_n, \tilde{T}_n) - e^{-(s^2+2\lambda st+t^2)/2}| \leq \delta(|s|^3 + |t|^3 + s^2 + t^2)e^{-c_2(s^2+t^2)}
\]
holds for every $(s, t) \in \mathbb{R}^2$ satisfying $|s|, |t| \leq c_3 \sqrt{n}$.

Proof A simple computation leads to
\[
f(s, t; \tilde{S}_n, \tilde{T}_n) = \prod_{k \in \Gamma_n} \{\alpha_k e^{i\gamma_k(1-\alpha_k)} + (1 - \alpha_k)e^{-i\gamma_k \alpha_k}\}, \tag{3.8}
\]
where $\gamma_k \equiv \gamma^n_k(s, t) := s/\sqrt{U_n} + tk/\sqrt{V_n}$ and we simply write $\alpha_k$ instead of $\alpha^n_k$. If $|s|, |t| \leq c_3 \sqrt{n}$ with $c_3 > 0$ chosen later, from (3.3), $\gamma_k$ can be estimated as
\[
|\gamma_k| \leq \tilde{c} c_3, \tag{3.9}
\]
with some $\tilde{c} > 0$. Therefore, since Taylor’s formula implies
\[
|e^z - \left(1 + z + \frac{1}{2} z^2\right)| \leq C_4 |z|^3,
\]
with some $C_4 > 0$ for all $z \in \mathbb{C} : |z| \leq \tilde{c} c_3$, we have that
\[
\alpha_k e^{i\gamma_k(1-\alpha_k)} + (1 - \alpha_k)e^{-i\gamma_k \alpha_k}
= \alpha_k \left(1 + i \gamma_k(1 - \alpha_k) - \frac{1}{2} \gamma_k^2 (1 - \alpha_k)^2\right) + (1 - \alpha_k)\left(1 - i \gamma_k \alpha_k - \frac{1}{2} \gamma_k^2 \alpha_k^2\right) + R_k
= 1 - \frac{v_k}{2} \gamma_k^2 + R_k, \tag{3.10}
\]
with an error term $R_k \in \mathbb{C}$ having an estimate:
\[
|R_k| \leq C_4 (\alpha_k(1 - \alpha_k)^3|\gamma_k|^3 + (1 - \alpha_k)\alpha_k^3|\gamma_k|^3) \leq C_5 |\gamma_k|^3, \tag{3.11}
\]
where we write $v_k$ for $v^n_k$. Let $\log(1 + z)$ be the principal value defined for $z \in \mathbb{C} \setminus \{z = x \in \mathbb{R}; x \leq -1\}$. Then,
\[
|\log(1 + z) - z| \leq C_6 |z|^2
\]
holds for $|z| < 1/2$ with some $C_6 > 0$, and therefore, from (3.8) and (3.10), we have that
\[
\left|\log f(s, t; \tilde{S}_n, \tilde{T}_n) - \sum_{k \in \Gamma_n} \left(-\frac{v_k}{2} \gamma_k^2 + R_k\right)\right| \leq C_6 \sum_{k \in \Gamma_n} \left|\frac{v_k}{2} \gamma_k^2 + R_k\right|, \tag{3.12}
\]
if \(|−v_kγ_k^2/2 + R_k| < 1/2\) for all \(1 \leq k \leq n\). Note that, from (3.9) and (3.11), the last condition holds if we choose \(c_3 > 0\) sufficiently small. However, we see that
\[
\sum_{k \in \Gamma_n^c} v_kγ_k^2 = (s^2 + 2λ_st + t^2)(1 + o(1)),
\] (3.13)
by using (3.3) and recalling \(|Γ_n| \leq C\). We thus obtain that
\[
\log f(s, t; \tilde{S}_n, \tilde{T}_n) = -\frac{1}{2}(s^2 + 2λ_st + t^2) + R,
\] (3.14)
with an error term \(R\) having a bound:
\[
|R| \leq \sum_{k=1}^n |R_k| + 2C_6\left(\frac{1}{\sqrt{n}} \sum_{k=1}^n v_k^2γ_k |γ_k|^4 + \sum_{k=1}^n |R_k|^2\right) + o(1)(s^2 + t^2).
\]
The sum over \(Γ_n^c\) is bounded by that over all \(1 \leq k \leq n\). However, from (3.11) and (3.3), we have that
\[
\sum_{k=1}^n |R_k| \leq C_5 \sum_{k=1}^n |γ_k|^3 \leq \frac{C_7}{\sqrt{n}}(|s|^3 + |t|^3),
\] (3.15)
with some \(C_7 > 0\). For the sum of \(v_k^2γ_k |γ_k|^4\), since \(v_k^2 ≤ (1/4)^2\), we have that
\[
\sum_{k=1}^n v_k^2γ_k |γ_k|^4 ≤ C_8 \sum_{k=1}^n \left(\frac{|s|^4}{U_n^2} + \frac{|t|^4k}{V_n^2}\right) ≤ \frac{C_9}{n}(|s|^4 + |t|^4) ≤ \frac{C_9c_3}{\sqrt{n}}(|s|^3 + |t|^3),
\] (3.16)
with some \(C_8, C_9 > 0\), if \(|s|, |t| ≤ c_3 \sqrt{n}\). Since (3.9) and (3.11) show that \(|R_k|\) is bounded, (3.15) implies that
\[
\sum_{k=1}^n |R_k|^2 ≤ \frac{C_{10}}{\sqrt{n}}(|s|^3 + |t|^3).
\] (3.17)
Therefore, (3.15)–(3.17) can be summarized into
\[
|R| \leq \frac{C_{11}}{\sqrt{n}}(|s|^3 + |t|^3) + o(1)(s^2 + t^2),
\]
if \(|s|, |t| ≤ c_3 \sqrt{n}\). Coming back to (3.14), we have that
\[
|f(s, t; \tilde{S}_n, \tilde{T}_n) − e^{−(s^2+2λ_st+t^2)/2}| = e^{−(s^2+2λ_st+t^2)/2} |e^R − 1|
\]
\[
\leq e^{−(1−λ)(s^2+t^2)/2} |R| |e^{|R|}|
\]
However, if \(|s|, |t| ≤ c_3 \sqrt{n}\), \(|R| ≤ (C_{11}c_3 + o(1))(s^2 + t^2)\) and therefore, by choosing \(c_3 > 0\) sufficiently small and \(n_0\) sufficiently large, we have that
\[
e^{−(1−λ)(s^2+t^2)/2} |e^{|R|}| \leq C_{12}e^{−c_2(s^2+t^2)},
\]
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with some $c_2, C_{12} > 0$ for every $n \geq n_0$. We have thus completed the proof of the lemma by changing the choice of $n_0$ to bound $|R|$ by means of the given $\delta > 0$, if necessary. \hfill \Box

Lemma 3.2 gives a uniform estimate on $I_1 \equiv I_{1,n}(K, L)$ under the choice of $c = c_3$: For every $\delta > 0$,

$$
\sup_{K, L} |I_1| \leq \delta \int_{\mathbb{R}^2} \left( |s|^3 + |t|^3 + s^2 + t^2 \right) e^{-c_2(s^2 + t^2)} \, ds \, dt,
$$

(3.18)

holds if $n \geq n_0$; note that the last integral is converging.

Finally, we give an estimate on $I_2 \equiv I_{2,n}(K, L)$. Using the relation (3.6), $I_2$ can be rewritten and estimated as

$$
|I_2| \leq \sqrt{U_n V_n} \int_{E_{2,n}} |f(s, t; S_n, T_n)| \, ds \, dt,
$$

(3.19)

where $E_{2,n} = \{(s, t); c_4 \leq |s| \leq \pi \text{ or } c_5/n \leq |t| \leq \pi \}$ and $c_4 = c_{\inf} \sqrt{n/U_n}, c_5 = c_{\inf} \sqrt{n^3/V_n} > 0$. Once we can show that there exist sufficiently small $\theta > 0$ and $\kappa \in (0, 1)$ such that

$$
\exists \left\{ k; 1 \leq k \leq n, k \notin \Gamma_n, s + kt \notin [-\theta, \theta] \mod 2\pi \right\} \geq \kappa n - 1 - |\Gamma_n|,
$$

(3.20)

for all $(s, t) \in E_{2,n}$, then we have that

$$
|f(s, t; S_n, T_n)| = \left| \prod_{k \in \Gamma_n^c} \left\{ \alpha_k e^{i(kt + t)} \right\} \right| \leq \sqrt{\kappa n - 1 - C},
$$

where $\alpha_k = \alpha_k^0$ and

$$
r := \max_{\nu \in [0, \pi]} \max_{k \in \Gamma_n^c} |\alpha_k e^{i\nu} + (1 - \alpha_k)| < 1,
$$

by recalling $0 < \alpha_- \leq \alpha_k \leq \alpha_+ < 1$. This together with (3.19) and (3.3) proves

$$
\sup_{K, L} |I_2| \leq (2\pi)^2 \sqrt{U_n V_n} r^{\kappa n - 1 - C}
$$

$$
\leq C' n^2 r^{\kappa n - 1 - C}.
$$

(3.21)

Three uniform estimates (3.7), (3.18) and (3.21) conclude the proof of Proposition 3.1.

The final task is to show (3.20) for all $(s, t) \in E_{2,n}$. To this end, we may assume $t \geq 0$ by symmetry. We may also assume $\Gamma_n = \emptyset$ and prove (3.20) without $-|\Gamma_n|$ in the right hand side. We rewrite the region $E_{2,n} \cap [t \geq 0]$ into a union of three regions: $E_{2,n} \cap [t \geq 0] = E_{2,n}^{(1)} \cup E_{2,n}^{(2)} \cup E_{2,n}^{(3)}$, where

$$
E_{2,n}^{(1)} = \left\{ (s, t); 2\theta \leq t \leq \pi, |s| \leq \pi \right\},
$$

$$
E_{2,n}^{(2)} = \left\{ (s, t); \frac{c_5}{n} \leq t < 2\theta, |s| \leq \pi \right\},
$$

$$
E_{2,n}^{(3)} = \left\{ (s, t); 0 \leq t < \frac{c_5}{n}, c_4 \leq |s| \leq \pi \right\}.
$$

Note that $c (= c_3)$ and therefore $c_5$ was chosen sufficiently small so that we may assume $0 < c_5 < 2\pi$. For $(s, t) \in E_{2,n}^{(1)}$, since "$s + kt \in [-\theta, \theta] \mod 2\pi$" implies "$s + (k + 1)t \notin [\theta - 2\pi, \theta + 2\pi] \mod 2\pi$" we have

$$
\frac{c_5}{n} \leq t < \frac{c_5}{n} + \frac{2\pi}{\theta}.
$$

Therefore, for $(s, t) \in E_{2,n}^{(2)}$ we have

$$
\frac{c_5}{n} \leq t < \frac{c_5}{n} + \frac{2\pi}{\theta}.
$$

This proves (3.20) for $(s, t) \in E_{2,n}^{(1)} \cup E_{2,n}^{(2)}$. By symmetry, this proves (3.20) for $(s, t) \in E_{2,n}^{(3)}$. This completes the proof of Proposition 3.1.
Thus we obtain (3.20) for \( \Gamma_n = \emptyset \) and \( \kappa = 1/2 \). Now we take \((s, t) \in E_{2n}^{(2)}\). The \( n \) points \( \{kt\}_{k=1}^{n} \) are arranged on \( \mathbb{R} \) in an equal distance and the interval \([t, nt]\) containing all these points are covered by at most \( m := \left\{ (n - 1)t/(2\pi) \right\} + 1 \) disjoint intervals of length \( 2\pi \), where \([ \cdot \] means the integer part. However, for an arbitrary interval \( I \subset \mathbb{R} \) of length \( 2\pi \),
\[
\sharp \{ k; 1 \leq k \leq n, kt \in I, s + kt \in [\theta, \pi \mod 2\pi] \} \leq \frac{2\theta}{t} + 1.
\]

Thus, since \( c_5/n \leq t < 2\theta \) for \((s, t) \in E_{2n}^{(2)}\), we have that
\[
\sharp \{ k; 1 \leq k \leq n, s + kt \in [\theta, \pi \mod 2\pi] \} \leq m \left( \frac{2\theta}{t} + 1 \right) \leq \left( \frac{nt}{2\pi} + 1 \right) \left( \frac{2\theta}{t} + 1 \right) = \frac{\theta}{n} + \frac{n}{2\pi} + 1 \leq \frac{2\theta}{n} + \frac{2\theta}{c_5} + \frac{2\theta}{n} + 1 \leq \frac{\theta}{2} + 1,
\]
by choosing \( \theta : 0 < \theta < (1/\pi + 1/c_5)^{-1}/4 \), and this proves (3.20) for \( \Gamma_n = \emptyset \) and \( \kappa = 1/2 \).
Finally we take \((s, t) \in E_{2n}^{(3)}\). Then, since \( 0 < c_5 < 2\pi \) and \( 0 \leq t < c_5/n \), \( n \) points \( \{kt\}_{k=1}^{n} \) are all located in the interval \([0, 2\pi]\). Now choose \( \theta : 0 < \theta < c_4/8 \). Then, recalling that \( c_4 \leq |s| \leq \pi \), for example in the case that \( -\pi \leq s \leq -c_4 \),
\[
\{ k; 1 \leq k \leq n, s + kt \in [\theta, \pi \mod 2\pi] \} = \{ k; 1 \leq k \leq n, s + kt \in [\theta, \theta]\}
\]
\[
\subset \{ k; 1 \leq k \leq n, kt \geq -\theta - s \} \subset \{ k; 1 \leq k \leq n, k \geq \frac{7c_4}{8c_5}n \}.
\]

Thus we obtain (3.20) for \( \Gamma_n = \emptyset \) by choosing \( \kappa = (7c_4/(8c_5)) \wedge (1/2) > 0 \). The case that \( c_4 \leq s \leq \pi \) can be discussed in a similar way. The proof of (3.20) is completed for all \((s, t) \in E_{2n}\).

Remark 3.2 Our argument relies on the specific form \( k \) of the weights. It can be extended to linearly growing weights, but not for general weights such as a power of \( k \).

4 Proof of Theorem 2.1

For a function \( \beta = \beta(\cdot) : [-1, 1] \to (0, 1) \), we denote by \( v_{\beta(\cdot)}^{\Lambda_{\ell}} \) the distribution on \( \Sigma_{\Lambda_{\ell}} \) of \([0, 1]\)-valued independent sequences \( \{ \eta_k \}_{k \in \Lambda_{\ell}} \) such that \( E[\eta_k] = \beta(k/\ell) \), \( k \in \Lambda_{\ell} \). The next lemma explains the reason that the functions of the form (2.4) appear in the limit.

Lemma 4.1 For a function \( \beta(\cdot) \equiv \beta(\cdot; a, b) \) of the form (2.4), the conditional measure of \( v_{\beta(\cdot)}^{\Lambda_{\ell}} \) on \( \Sigma_{\Lambda_{\ell}, K, M} \) is a uniform probability measure for every \( a \in (0, 1) \) and \( b \in \mathbb{R} \):
\[
v_{\beta(\cdot)}^{\Lambda_{\ell}}(\cdot | \Sigma_{\Lambda_{\ell}, K, M}) = v_{\Lambda_{\ell}, K, M}(\cdot).
\]

Proof For \( a \in (0, 1) \), let \( \mu_a \) be the probability measure on \([0, 1]\) defined by \( \mu_a(1) = a \).
Then, if \( a = e^b a/(e^b a + (1 - a)) \) for some \( a \in (0, 1) \) and \( b \in \mathbb{R} \), it holds that
\[
\mu_a(\xi) = \frac{1}{z_{a,b}} e^{b \xi} \mu_a(\xi),
\]
for $\xi = 0, 1$ with a constant $z_{a,b} = e^b a + (1 - a)$. Therefore, for $v_{\beta}^{A_t}$ on $\Sigma_{A_t}$ with $\beta(\cdot)$ of the form (2.4), we have that

$$v_{\beta(\cdot)}^{A_t}(\eta) = \prod_{k \in A_t} \mu_{\beta(k/\ell)}(\eta_k) = \prod_{k \in A_t} z_{a,bk/\ell}^{-1} e^{b k \eta_k / \ell} \mu_a(\eta_k)$$

$$= Z^{-1} e^{b \sum_{k \in A_t} k \eta_k / \ell} v_a^A(\eta),$$

for $\eta \in \Sigma_{A_t}$ with a normalizing constant $Z \equiv \prod_{k \in A_t} z_{a,bk/\ell} = E_{v_a^A} [e^{b \sum_{k \in A_t} k \eta_k / \ell}]$. This implies that

$$v_{\beta(\cdot)}^{A_t}(\cdot | \Sigma_{A_t,K,M}) = v_a^{A_t}(\cdot | \Sigma_{A_t,K,M}),$$

since $\sum_{k \in A_t} k \eta_k = M$ on $\Sigma_{A_t,K,M}$. However, $v_a^{A_t}(\cdot | \Sigma_{A_t,K} = K)$ is a uniform measure on $\{\eta \in \Sigma_{A_t}; K_{A_t}(\eta) = K\}$ so that $v_a^{A_t}(\cdot | \Sigma_{A_t,K,M})$ is also a uniform measure on $\Sigma_{A_t,K,M}$, that is, $v_{A_t,K,M}$.

We next establish the one-to-one correspondence between $(\rho, m)$ and $(a, b)$ defined by (2.5). In particular, one can uniquely find $(a, b)$ for every $(\rho, m)$ in Theorem 2.1. Consider the map $\Phi$ for $(a, b) \in (0, 1) \times \mathbb{R}$ to $(\rho, m) = (F(a, b), G(a, b)) \in D := ((\rho, m); \rho \in (0, 1), m \in (-v/2, v/2))$ with $v = \rho (1 - \rho)$ defined by

$$F(a, b) = \frac{1}{2} \int_{-1}^{1} \beta(x; a, b) dx, \quad G(a, b) = \frac{1}{4} \int_{-1}^{1} x \beta(x; a, b) dx.$$

**Lemma 4.2** The map $\Phi$ is a diffeomorphism from $(0, 1) \times \mathbb{R}$ onto the domain $D$.

**Proof** Recalling that $\beta(x; a, b)$ is given by $\beta(x; a, b) = e^{b x} a / g(x; a, b)$ with $g(x) = g(x; a, b) = e^{b x} a + (1 - a)$, we can easily compute the derivatives of $F$ and $G$:

$$\frac{\partial F}{\partial a} = \frac{1}{2} \int_{-1}^{1} e^{b x} g(x)^2 dx, \quad \frac{\partial F}{\partial b} = \frac{a(1 - a)}{2} \int_{-1}^{1} e^{b x} g(x)^2 dx,$$

$$\frac{\partial G}{\partial a} = \frac{1}{4} \int_{-1}^{1} x^{2} e^{b x} g(x)^2 dx, \quad \frac{\partial G}{\partial b} = \frac{a(1 - a)}{4} \int_{-1}^{1} x^{2} e^{b x} g(x)^2 dx.$$

This, with the help of Schwarz’s inequality, implies that the Jacobian of the map $\Phi$ is positive everywhere, that is, $J = \frac{\partial F}{\partial a} \frac{\partial G}{\partial b} - \frac{\partial F}{\partial b} \frac{\partial G}{\partial a} > 0$. In particular, the map $\Phi$ is a local diffeomorphism.

For every $b \in \mathbb{R}$, set $C_b := \{(F(a, b), G(a, b)) \in D; a \in (0, 1)\}$. Then, $C_b$ is a Jordan arc in $D$ connecting two points $(0, 0)$ and $(1, 0)$. Indeed, from $\partial F / \partial a > 0$, $C_b$ has no double points so that it is a Jordan arc. As $a \downarrow 0$, we have $\beta(x; a, b) \to 0$ so that $(F(a, b), G(a, b)) \to (0, 0)$, while $(F(a, b), G(a, b)) \to (1, 0)$ as $a \uparrow 1$, since $\beta(x; a, b) \to 1$. Especially, $C_0 = \{(a, 0); a \in (0, 1)\}$ is a line segment connecting $(0, 0)$ and $(1, 0)$. From the fact that $J > 0$, $\partial F / \partial a > 0$ and $\partial G / \partial b > 0$, we see that the arc $C_{b_1}$ is located above $C_{b_2}$ in the $\rho - m$ plane if $b_1 > b_2$. In fact, let $b_2 = b$ and $b_1 = b + \varepsilon$ with $b \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$. Then, since the tangent line at $(F(a, b), G(a, b))$ to the arc $C_b$ on $\rho - m$ plane is given by

$$m = \frac{\partial G}{\partial a}(\rho - F(a, b)) + G(a, b),$$

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the point \((F(a, b + \varepsilon), G(a, b + \varepsilon))\) on the arc \(C_{b+\varepsilon}\), which asymptotically behaves as 
\((F(a, b), G(a, b)) + \varepsilon(\partial F/\partial b(a, b), \partial G/\partial b(a, b)) + O(\varepsilon^2)\) as \(\varepsilon \downarrow 0\), is located above \(C_b\) if
\[
\varepsilon \frac{\partial G}{\partial b} > \frac{\partial G}{\partial a} \frac{\varepsilon}{\partial F/\partial a}.
\]

However, this follows from \(J > 0\) and therefore proves that the map \(\Phi\) is one to one.

To show the onto property of \(\Phi\), we consider \((a, b)\) satisfying \(F(a, b) = \rho\) for a fixed \(\rho \in (0, 1)\). From (2.7), this condition can be rewritten as
\[
a = \frac{e^{2\rho b} - 1}{e^b + e^{2\rho b} - e^{b(2\rho - 1)} - 1}
\]

and therefore
\[
\beta(x; a, b) = \frac{e^{bx} (e^{2\rho b} - 1)}{e^{bx} (e^{2\rho b} - 1) + (e^b - e^{b(2\rho - 1)})},
\]

which behaves as
\[
\beta(x; a, b) \to \begin{cases} 1 & \text{if } x + 2\rho > 1, \\ 0 & \text{if } x + 2\rho < 1, \end{cases}
\]
as \(b \to \infty\). Thus we have
\[
\lim_{b \to \infty} G(a, b) = \frac{1}{4} \int_{1-2\rho}^1 x \, dx = \frac{1}{2} \rho (1 - \rho),
\]
under the condition \(F(a, b) = \rho\). In a similar way, we can show that
\[
\lim_{b \to -\infty} G(a, b) = -\frac{1}{2} \rho (1 - \rho).
\]

This proves the onto property of the map \(\Phi\). \(\Box\)

We are now at the position to complete the proof of Theorem 2.1. Our method is standard in the sense that we apply the local limit theorem to compute the conditional distributions; see, e.g., [7], p. 353 and also [15], whose method is similar to ours and relies on characteristic functions. When applying Proposition 3.1, the term \(q_0(y_1, y_2)\) needs to be uniformly positive to dominate the error term and this restricts our choice of the underlying measures in such a way that both \(y_1\) and \(y_2\) are sufficiently close to 0. For instance, in [7], it was only required to adjust the density parameter so that the macroscopic profile was constant over the space, but here the problem involves two parameters \(K\) and \(M\) or \(\rho\) and \(m\). However, once we take the inhomogeneous Bernoulli measures with the functions \(\beta(\cdot; a, b)\) as macroscopic profiles, our choice allows two parameters \(a\) and \(b\). This suits our purpose and we can realize the situation that both \(y_1\) and \(y_2\) are close to 0 at the same time under a suitable choice. The local limit theorem with defects is prepared to treat the numerator \(B_\ell\) introduced later.

We denote the supports of the local functions \(f_j\) in Theorem 2.1 by \(\Gamma_j \subset \mathbb{Z}\) and set \(\Gamma(= \Gamma^{(\ell)}) = \bigcup_{j=1}^{\rho} \tau_k \Gamma_j\), where \(\tau_k \Gamma_j = \Gamma_j + k\) and recall \(k_j = k_j \ell\). Note that \(\Gamma \subset \Lambda_\ell\) and \(\{\tau_k \Gamma_j\}_{j=1}^{\rho}\) are disjoint if \(\ell\) is sufficiently large, since \(k_j\) asymptotically behave as \(x_j \ell\) and...
Lemma 4.2, respectively, to indicate the defects clearly. Then, we have that
\[
\text{The numerator of the fractional expression in the right hand side of (4.1) is equal to}
\[
= \sum_{\xi \in (0,1)^\ell} \left( \prod_{j=1}^{p} f_j(\xi_j) - \prod_{j=1}^{p} E_{v_{\beta(\xi)}} [f_j] \right) v_{\beta(\xi)}^{A_\ell}(\eta|_\Gamma = \xi, K_{A_\ell}(\eta) = K, M_{A_\ell}(\eta) = M) v_{\beta(\xi)}^{A_\ell}(K_{A_\ell}(\eta) = K, M_{A_\ell}(\eta) = M),
\]
(4.1)
where \( \beta(\cdot) \) is the function determined in Theorem 2.1, \( \eta|_\Gamma \) stands for the restriction of \( \eta \) to \( \Gamma \) and we denote by \( \xi_j = \xi|_{\Gamma_j} \). For given \( K = K_\ell \) and \( M = M_\ell \), our special choice of \( \beta_\ell(\cdot) \) will be the function \( \beta_\ell(\cdot) = \beta(\cdot; a_\ell, b_\ell) \) of the form (2.4) with the solution \((a_\ell, b_\ell)\) of two equations (2.5) for \( \rho = K/(2\ell + 1) \) and \( m = M/(2\ell + 1)^2 \).

To apply Proposition 3.1 for \( \{\eta_k\}_{k=1}^n \), we need to shift it and consider \( X = \{X_k\}_{k=1}^n \) with \( n = 2\ell + 1 \) determined by \( X_k = \eta_{k-\ell-1} = (\tau_{\ell+1} \eta)_k \) for \( k = 1, 2, \ldots, n \). For such \( X \) and defects \( \Gamma_n \subset \{1, 2, \ldots, n\} \), we denote two sums \( S_n \) and \( T_n \) in (3.2) by \( S_n^\Gamma(X) \) and \( T_n^\Gamma(X) \), respectively, to indicate the defects clearly. Then, we have that
\[
S_n^\Gamma(X) = K_{A_\ell}(\eta) \quad \text{and} \quad T_n^\Gamma(X) = M_{A_\ell}(\eta) + (\ell + 1)K_{A_\ell}(\eta).
\]
(4.2)
The numerator of the fractional expression in the right hand side of (4.1) is equal to
\[
P_{\alpha^n}(X|_{\tau_{\ell+1} \Gamma}) = \tau_{\ell+1}^{-1} \xi, \quad S_n^\Gamma(X) = K, \quad T_n^\Gamma(X) = M + (\ell + 1)K
\]
\[
= \nu_{\beta(\cdot)}^{\Gamma}(\xi) P_{\alpha^n} \left( S_n^{(\tau_{\ell+1} \Gamma)}(X) = K - S_n^{(\tau_{\ell+1} \Gamma)}(\tau_{\ell+1}^{-1} \xi), \quad T_n^{(\tau_{\ell+1} \Gamma)}(X) = M + (\ell + 1)K - T_n^{(\tau_{\ell+1} \Gamma)}(\tau_{\ell+1}^{-1} \xi) \right),
\]
where \( \alpha^n = \{\alpha_k^n\}_{k=1}^n := \{\beta_k((k - \ell - 1)/\ell)\}_{k=1}^n \). We have denoted by \( P_{\alpha^n} \) the distribution of \( X \) such that \( E[X_k] = \alpha_k^n \) for \( 1 \leq k \leq n \), by \( \nu_{\beta(\cdot)}^\Gamma \) the measure \( \nu_{\beta(\cdot)}^{\Gamma}\) restricted on \( \Gamma \), and \( (\tau_{\ell+1} \Gamma)^c = \{1, 2, \ldots, n\} \setminus \tau_{\ell+1} \Gamma \). However, under the scaling conditions (2.1) and (2.2), by Lemma 4.2, \( \beta_\ell(\cdot) \) converges to \( \beta(\cdot) \) uniformly:
\[
\lim_{\ell \to \infty} \max_{x \in [-1,1]} |\beta_\ell(x) - \beta(x)| = 0.
\]
(4.3)
Hence, we have that
\[
\sum_{\xi \in (0,1)^\ell} \left( \prod_{j=1}^{p} f_j(\xi_j) - \prod_{j=1}^{p} E_{v_{\beta(\xi)}} [f_j] \right) \nu_{\beta(\cdot)}^{\Gamma}(\xi) = o(1),
\]
as \( \ell \to \infty \). Therefore, (4.1) can be rewritten as
\[
= \sum_{\xi \in (0,1)^\ell} \left( \prod_{j=1}^{p} f_j(\xi_j) - \prod_{j=1}^{p} E_{v_{\beta(\xi)}} [f_j] \right) \nu_{\beta(\cdot)}^{\Gamma}(\xi) \times \left\{ \frac{B_\ell}{A_\ell} - 1 \right\} + o(1),
\]
(4.4)
where
\[ A_\ell := P_{\omega^n}(S^0_n(X) = K, T^n_\ell(X) = M + (\ell + 1)K), \]
\[ B_\ell := P_{\omega^n}(S^\ell_\ell(X) = K - S^{(\ell+1)\Gamma}_\ell(\tau^{-1}_{\ell+1}\xi), \]
\[ T^{(\ell+1)\Gamma}_n(X) = M + (\ell + 1)K - T^{(\ell+1)\Gamma}_n(\tau^{-1}_{\ell+1}\xi). \]

We now apply Proposition 3.1 to compute the asymptotic behaviors of \( A_\ell \) and \( B_\ell \). Note that, from (4.3), \( \alpha^n = \{\alpha^n_k\}_{k=1}^n \) satisfies the condition (3.1) with \( \alpha(x) := \beta(2x - 1), x \in [0, 1] \).

By the choice of \((a_\ell, b_\ell)\), we can show that \( y_1 = y_2 = O(1/\sqrt{n}) \) as \( \ell \to \infty \) (or equivalently \( n \to \infty \)) for \( y_1 \) and \( y_2 \) defined by (3.5) for both \( A_\ell \) and \( B_\ell \). Indeed, for \( A_\ell \),

\[ y_1 = \frac{1}{\sqrt{U_n}}(K - E_n) = \frac{1}{\sqrt{U_n}} \left\{ K - \sum_{k=1}^n \beta_\ell \left( \frac{k - \ell - 1}{\ell} \right) \right\} \]
\[ = \frac{1}{\sqrt{U_n}} \left[ K - \ell \left\{ \int_{-1}^1 \beta_\ell(x)dx + O\left(\frac{1}{\ell}\right) \right\} \right] \]
\[ = \frac{1}{\sqrt{U_n}} \left[ K - \ell \left\{ \frac{2K}{2\ell + 1} + O\left(\frac{1}{\ell}\right) \right\} \right] = O(1/\sqrt{n}). \]

Here, for the third equality, we have used the uniform bound: \( \sup_{\ell} \max_{x \in [-1, 1]} |\beta_\ell'(x)| < \infty \), recall the choice of \( \beta_\ell(\cdot) \) for the fourth and (3.3) for the last. Similarly,

\[ y_2 = \frac{1}{\sqrt{V_n}}(L - F_n) = \frac{1}{\sqrt{V_n}} \left\{ M + (\ell + 1)K - \sum_{k=1}^n k\beta_\ell \left( \frac{k - \ell - 1}{\ell} \right) \right\} \]
\[ = \frac{1}{\sqrt{V_n}} \left[ M + (\ell + 1)K - \ell^2 \left\{ \int_{-1}^1 (x + 1)\beta_\ell(x)dx + O\left(\frac{1}{\ell}\right) \right\} \right] \]
\[ = O(1/\sqrt{n}). \]

For \( B_\ell \), compared with \( A_\ell \), there are additional terms

\[ -\frac{1}{\sqrt{U_n}} \left\{ S^{(\ell+1)\Gamma}_n(\tau^{-1}_{\ell+1}\xi) - \sum_{k \in \tau_{\ell+1}\Gamma} \beta_\ell \left( \frac{k - \ell - 1}{\ell} \right) \right\} \]

in \( y_1 \) and

\[ -\frac{1}{\sqrt{V_n}} \left\{ T^{(\ell+1)\Gamma}_n(\tau^{-1}_{\ell+1}\xi) - \sum_{k \in \tau_{\ell+1}\Gamma} k\beta_\ell \left( \frac{k - \ell - 1}{\ell} \right) \right\} \]

in \( y_2 \), but both these terms behave as \( O(1/\sqrt{n}) \). Thus, from Proposition 3.1, two terms \( A_\ell \) and \( B_\ell \) both behave as

\[ A_\ell, B_\ell = \frac{1}{\sqrt{U_n V_n}} \left\{ \frac{1}{2\pi \sqrt{1 - \lambda^2}} + o(1) \right\}, \]

as \( \ell \to \infty \) (or equivalently \( n \to \infty \)). Since \( 1 - \lambda^2 > 0 \), this shows that \( A_\ell/B_\ell \to 1 \) and completes the proof of Theorem 2.1 from (4.4).
Remark 4.1 If we take $\beta(\cdot)$ itself instead of $\beta_s(\cdot)$, the terms $y_1$ and $y_2$ in the exponential of $q_0$ behave as $o(\sqrt{n})$ so that $q_0$ in general converges to 0 and the local limit theorem turns out to be useless.

5 Proof of Corollary 2.2 and Relations to Vershik Curves

For the proof of Corollary 2.2, it suffices to show the weaker convergence
\[
\lim_{\ell \to \infty} \nu_{\Lambda_{\ell}, K, M} \left( \left| \langle \tilde{\psi}^\ell, \varphi \rangle - \langle \psi, \varphi \rangle \right| > \delta \right) = 0,
\]
for every $\delta > 0$ and $\varphi \in C([-1, 1])$, where $\langle \psi, \varphi \rangle = \int_{-1}^1 \psi(x) \varphi(x) dx$. In fact, (5.1) implies the stronger convergence result stated in Corollary 2.2 due to the monotonicity of $\tilde{\psi}^\ell$, see Remark 2.5 in [2]. For proving (5.1), it is enough to show that $\langle \tilde{\psi}^\ell, \varphi \rangle$ converges to $\langle \psi, \varphi \rangle$ in $L^2$-sense. However, recalling (2.11) and (2.8), a simple computation leads to
\[
\langle \tilde{\psi}^\ell, \varphi \rangle = \frac{1}{\ell} \sum_{k=-\ell+1}^{\ell} \eta_k \tilde{\varphi} \left( \frac{k}{\ell} \right),
\]
where $\tilde{\varphi}(x) = \int_{-1}^x \varphi(y) dy$. Therefore, once we can show that
\[
\lim_{\ell \to \infty} E_{\nu_{\Lambda_{\ell}, K, M}} \left[ \left\{ \int_{-1}^1 \eta_{\ell x_1+1} \tilde{\varphi}(x) dx \right\}^2 \right] = \langle \beta, \tilde{\varphi} \rangle^2,
\]
the $L^2$-convergence follows by noting that $\langle \beta, \tilde{\varphi} \rangle = \langle \psi, \varphi \rangle$. We only give the proof of (5.3), since (5.2) is similar and easier. To this end, from the estimate
\[
\left| \langle \tilde{\psi}^\ell, \varphi \rangle - \int_{-1}^1 \eta_{\ell x_1+1} \tilde{\varphi}(x) dx \right| \leq 2 \sup_{|x_1 - x_2| \leq \frac{1}{\ell}} |\tilde{\varphi}(x_1) - \tilde{\varphi}(x_2)|,
\]
it is enough to prove
\[
\lim_{\ell \to \infty} E_{\nu_{\Lambda_{\ell}, K, M}} \left[ \left\{ \int_{-1}^1 \eta_{\ell x_1} \tilde{\varphi}(x) dx \right\}^2 \right] = \langle \beta, \tilde{\varphi} \rangle^2,
\]
where $[\ell x]$ stands for the integer part of $\ell x$. However, the expectation in (5.4) is expanded as
\[
\int_{-1}^1 \int_{-1}^1 E_{\nu_{\Lambda_{\ell}, K, M}} \left[ \eta_{\ell x_1} \eta_{\ell x_2} \right] \tilde{\varphi}(x_1) \tilde{\varphi}(x_2) dx_1 dx_2,
\]
and Theorem 2.1 applied for $p = 2$ and $f_1(\eta) = f_2(\eta) = \eta_0$ implies that
\[
\lim_{\ell \to \infty} E_{\nu_{\Lambda_{\ell}, K, M}} \left[ \eta_{\ell x_1} \eta_{\ell x_2} \right] = \beta(x_1) \beta(x_2).
\]
Thus, Lebesgue’s convergence theorem shows (5.4) and this completes the proof of Corollary 2.2.

We next compare our result in Corollary 2.2 with those in [1]. In [1] (and [9]), the grand canonical ensembles (and canonical ensembles, respectively) in a rectangular box for the uniform (Bose) statistics are dealt based on combinatorial methods, while we have discussed the canonical ensembles in a rectangular box for the restricted uniform (Fermi) statistics due to a probabilistic approach. In [15], the limit curves together with fluctuations are studied under the grand canonical and canonical ensembles for the uniform statistics.

Theorem 1 of [1] shows, by rotating the plane coordinates by 45 degree, that the limit curve
t \in [0, 1] \rightarrow L(t)
in the box with the ratio of height/width given by \( \bar{\rho}/(1 - \bar{\rho}) \), \( \bar{\rho} \in (0, 1) \) is determined by
\[
L(t) \equiv L(\bar{\rho}, \bar{c}(t)) = \frac{1}{\bar{c}} \log \frac{h(t)}{h(0)},
\]
where
\[
h(t) = e^{-\bar{c}t} - e^{\bar{c}(t - 2\bar{\rho} - t)} - e^{-\bar{c}(t - 2\bar{\rho})}, \quad t \in [0, 1],
\]
and \( \bar{c} \in \mathbb{R} \) is a parameter which controls the area. By rotating back to the original coordinates, this curve is transformed to the curve \( v = \phi(u) \) in \( u-v \) plane given implicitly by
\[
u = \frac{1}{\sqrt{2}}(t + L(t)) \in [0, (1 - \bar{\rho})/\sqrt{2}], \quad v = \frac{1}{\sqrt{2}}(-t + L(t)) \in [-\sqrt{2}\bar{\rho}, 0]. \tag{5.5}
\]
However, as seen in Proposition 4.4 of [2], the curve \( v = \phi(u) \) appearing in the uniform statistics can be related to the curve \( y = \tilde{\psi}(x) \) in \( x-y \) plane appearing in the restricted uniform statistics by
\[
\bar{\beta}(x) \equiv -\tilde{\psi}'(x) = \frac{-\phi'(G^{-1}_{\phi}(x))}{1 - \phi'(G^{-1}_{\phi}(x))},
\]
where \( G_{\phi}(u) = u - \phi(u) \) and \( G^{-1}_{\phi} \) is its inverse function. But, (5.5) shows that \( G_{\phi}(u) = \sqrt{2}t \) and thus, setting \( x = G_{\phi}(u) \in [0, \sqrt{2}] \), we have that
\[
\bar{\beta}(\sqrt{2}t) = \frac{-\phi'(u)}{1 - \phi'(u)} = \frac{1 - L'(t)}{2},
\]
or equivalently
\[
\bar{\beta}(x) = \frac{1 - L'(x/\sqrt{2})}{2}.
\]
A simple computation leads to \( L'(t) = h'(t)/(\bar{c}h(t)) \) and \( L''(t) = \bar{c}(1 - L'(t)^2) \), since \( h''(t) = \bar{c}^2 h(t) \). This proves that
\[
\bar{\beta}'(x) = -\frac{1}{2\sqrt{2}}L''(x/\sqrt{2}) = \frac{\bar{c}}{2\sqrt{2}}(L'(x/\sqrt{2})^2 - 1) = -\sqrt{2}\bar{c} \bar{\beta}(x)(1 - \bar{\beta}(x)),
\]
so that we arrive at the ordinary differential equation for \( \tilde{\psi} \):
\[
\tilde{\psi}''(x) + \sqrt{2}\bar{c} \tilde{\psi}'(x)(1 + \tilde{\psi}'(x)) = 0, \quad x \in [0, \sqrt{2}]. \tag{5.6}
\]
Moreover, noting that \( L(0) = 0 \) and \( L(1) = 1 - 2\tilde{\rho} \), the height difference of \( \bar{\psi} \) at two boundary points is given by

\[
\bar{\psi}(0) - \bar{\psi}(\sqrt{2}) = \int_0^{\sqrt{2}} \bar{\beta}(x) dx = \sqrt{2}\tilde{\rho},
\]

or, by normalizing \( \bar{\psi}(\sqrt{2}) = 0 \), we have that \( \bar{\psi}(0) = \sqrt{2}\tilde{\rho} \).

To compare \( \bar{\psi} \) with \( \psi \) in Corollary 2.2, note that \( \psi \) is defined for \( x \in [-1, 1] \). The shift in \( x \) does not change the form of the equation. To consider on the interval of same length, we introduce the scaling for \( \psi \) defined on \([0, \sqrt{2}]\) by

\[
\tilde{\psi}(x) := \frac{1}{\gamma} \psi(\gamma x), \quad x \in [0, \sqrt{2}/\gamma],
\]

for \( \gamma > 0 \). Then, if \( \tilde{\psi} \) satisfies \( \tilde{\psi}'' + \kappa \tilde{\psi}'(1 + \tilde{\psi}') = 0 \), \( \tilde{\psi} \) satisfies the equation \((\tilde{\psi}''') + \kappa \tilde{\psi}''(1 + (\tilde{\psi}')) = 0\). Applying this for Eq. (5.6) with \( \gamma = 1/\sqrt{2} \) and \( \kappa = \sqrt{2}\bar{c} \), we can derive the equation for \( \tilde{\psi}^{1/\sqrt{2}} \):

\[
(\tilde{\psi}^{1/\sqrt{2}})'' + \bar{c}(\tilde{\psi}^{1/\sqrt{2}})'(1 + (\tilde{\psi}^{1/\sqrt{2}})') = 0, \quad x \in [0, 2],
\]

with \( \tilde{\psi}^{1/\sqrt{2}}(0) = 2\tilde{\rho} \) and \( \tilde{\psi}^{1/\sqrt{2}}(2) = 0 \). This coincides with the ordinary differential equation (2.13) for the Vershik curve with \( c = \bar{c} \) and \( \rho = \tilde{\rho} \). Thus, we can relate the limit curves in a rectangular box for grand canonical ensembles in the uniform statistics and for canonical ensembles in the restricted uniform statistics, namely \( \tilde{\psi}^{1/\sqrt{2}} = \psi \) (except the shift in \( x \) by 1), if the relations \( \bar{c} = -b \) and \( \tilde{\rho} = \rho \) hold.

6 Related Dynamics

So far we have discussed static problems. Here we introduce a related dynamics. Let \( Q_{K,M} \) be the set of all distinct partitions \( q = \{q_1 > q_2 > \cdots > q_K \geq 1\} \) of \( M \in \mathbb{N} \) considered in Sect. 1. Recall that, by (1.1), a partition \( q \in Q_{K,M} \) is identified with the height function \( \psi_q \) of a two-dimensional Young diagram satisfying that \( \psi_q(0) = K \) and \( \int_0^\infty \psi_q(u) du = M \). We denote by \( \mu_{K,M} \) the uniform probability measure on \( Q_{K,M} \).

We construct a random dynamics \( q(t) \) on \( Q_{K,M} \), which is reversible under \( \mu_{K,M} \). To describe such dynamics, we adopt a particles’ picture. Let \( X_{K,M} \) be the set of all \( \eta \in \{0,1\}^\mathbb{N} \) satisfying two conditions in (1.3) and let \( \nu_{K,M} \) be the uniform probability measure on \( X_{K,M} \). Recall that one can identify two spaces \( Q_{K,M} \) and \( X_{K,M} \) under the map \( \psi_q \in Q_{K,M} \mapsto \eta \in X_{K,M} \) determined by (1.2) and \( \nu_{K,M} \) is the image measure of \( \mu_{K,M} \) under this map. In Sects. 2–5 we worked on \( \mathbb{Z} \), but here we return to the original setting on \( \mathbb{N} \) as in Sect. 1.

For \( \eta \in X \) and \( i, j \in \mathbb{N} \) such that \( i \neq j \), we define a new configuration \( \eta^{i,j} \in X \) by interchanging the states at the sites \( i \) and \( j \), that is,

\[
(\eta^{i,j})_k = \begin{cases} 
\eta_j, & k = i, \\
\eta_i, & k = j, \\
\eta_j, & k \neq i, j,
\end{cases}
\]
and for $i, j \in \mathbb{N}$ such that $2 \leq i < j$, we define

$$\sigma^{(i,j)} \eta \equiv \sigma^{i-1,i;j,j+1} \eta := \left(\eta^{i-1,j}\right)^{j+1},$$

the configuration obtained by interchanging the states at $i - 1$ and $i$, and those at $j$ and $j + 1$ simultaneously. For functions $f : \mathcal{X} \to \mathbb{R}$, define

$$\pi^{(i,j)} f(\eta) \equiv \pi^{i-1,i;j,j+1} f(\eta) := f(\sigma^{(i,j)} \eta) - f(\eta).$$

For $r = 1, 2, \ldots$, define four functions $F_r^\pm$, $G_r^\pm$ on $\tilde{X} := \{0, 1\}^Z$ by

$$F_r^+ (\eta) = 1_{\{\eta_{(r-1)}=0, \eta_0=\cdots=\eta_r=1, \eta_{r+1}=0\}},$$

$$F_r^- (\eta) \equiv F_r^+(\sigma^{(0,r)} \eta) = 1_{\{\eta_{(r-1)}=1, \eta_0=0, \eta_1=\cdots=\eta_{r-1}=1, \eta_r=0, \eta_{r+1}=1\}},$$

$$G_r^+ (\eta) \equiv F_r^-(\hat{\eta}) = 1_{\{\eta_{(r-1)}=0, \eta_0=1, \eta_1=\cdots=\eta_{r-1}=0, \eta_r=1, \eta_{r+1}=0\}},$$

$$G_r^- (\eta) \equiv F_r^+(\hat{\eta}) = 1_{\{\eta_{(r-1)}=1, \eta_0=0, \eta_1=\cdots=\eta_{r-1}=0, \eta_r=0, \eta_{r+1}=1\}},$$

where $\sigma^{(i,j)} \eta$ is defined similarly as above for $\eta \in \tilde{X}$ and $i, j \in \mathbb{Z}$ such that $i < j$, and $\hat{\eta} \in \tilde{X}$ is the configuration determined from $\eta \in \tilde{X}$ by $\hat{\eta}_i = 1 - \eta_i$ for all $i \in \mathbb{Z}$. The shift operators $\tau_i : \tilde{X} \to \tilde{X}$ are defined by $(\tau_i \eta)_k := \eta_{k+i}$ for $\eta \in \tilde{X}$, $i, k \in \mathbb{Z}$, and they induce operators acting on functions $F : \tilde{X} \to \mathbb{R}$ as $\tau_i F(\eta) := F(\tau_i \eta)$.

Let non-negative bounded constants $(c_r^F)_{r \geq 1}$ and $(c_r^G)_{r \geq 1}$ be given and consider the operator $L$ acting on functions $f : \mathcal{X} \to \mathbb{R}$ as

$$Lf(\eta) = \sum_{2 \leq i < j} c_{(i,j)}(\eta) \pi^{(i,j)} f(\eta),$$

where

$$c_{(i,j)}(\eta) = c_{j-i}^F \tau_i \left(F_{j-i}^+ + F_{j-i}^-\right) + c_{j-i}^G \tau_i \left(G_{j-i}^+ + G_{j-i}^-\right).$$

We may assume $c_0^G = 0$, so $F_1^\pm \equiv G_1^\pm$ so that the transitions under $F_1^\pm$ are the same as those under $G_1^\pm$. Note that, for $i, j \in \mathbb{N}$ such that $2 \leq i < j$, $\tau_i F_{j-i}^\pm$ and $\tau_i G_{j-i}^\pm$ are functions on $\mathcal{X}$. For example, we have that

$$\tau_i F_{j-i}^+(\eta) = 1_{\{\eta_{(j-1)}=0, \eta_j=\cdots=\eta_{j+i}=1, \eta_{j+i+1}=0\}}.$$

The operator $L$ determines a Markov process $\eta(t)$ on $\mathcal{X}$. This is a kind of exclusion process on $\mathbb{N}$, which has a special character that the jumps are made simultaneously by two different particles to their neighboring sites in such a manner that, if one particle jumps to the right, the other moves to the left. More precisely, the time evolution of particles is governed by the following two rules:

1. two particles at the ends of consecutively lined $r + 1$ particles jump simultaneously to outward directions at rate $c_r^F$ if two visiting sites are vacant, and its reversed transition occurs at the same rate $c_r^F$ and
2. two particles at the ends of consecutive $r + 1$ vacant sites jump simultaneously to inward directions at rate $c_r^G$, and its reversed transition occurs at the same rate $c_r^G$.

Under these transitions, two quantities in (1.3) are always preserved so that $\eta(0) \in X_{K,M}$ implies $\eta(t) \in X_{K,M}$ for all $t > 0$. Under the map $\eta \in X_{K,M} \mapsto \psi_q \in Q_{K,M}$, the Markov
process $\eta(t)$ defines an area-preserving random dynamics of Young diagrams $\psi_q(t)$ or $q(t)$ in $Q_{K,M}$.

In a physical picture, the surface of the Young diagrams is considered to describe a decreasing interface which separates two phases and each unit square corresponds to an atom. In our dynamics, the transitions are local in the sense that only transitions in nearest possible pairs of atoms are allowed. This character of dynamics is quite natural from the physical point of view. The atoms move on the surface until they find the nearest stable position. Mathematically, this local character is adequate to discuss the hydrodynamic behavior of the model.

The operator $L$ is reversible under $\nu_{K,M}$ on $X_{K,M}$ and also under the Bernoulli measures $\nu_{\rho}$ on $X$ for all $\rho \in [0,1]$, which represents the mean of $\nu_{\rho}$. In fact, for $\nu_{K,M}$, the corresponding Dirichlet form is given by

$$E_{K,M}(f,g) := -E_{\nu_{K,M}}[Lf \cdot g] = \frac{1}{2} \sum_{2 \leq i < j} E_{\nu_{K,M}}[c(i,j) \pi(i,j) f \pi(i,j) g],$$

for $f, g : X_{K,M} \to \mathbb{R}$. Note that the sum can be limited to $2 \leq i < j \leq M - 1$, since $\eta \in X_{K,M}$ implies $\eta_j = 0$ for all $j \geq M + 1$ so that $c(i,j)(\eta) = 0$ if $j \geq M + 1$.

One of the interesting problems to study is the hydrodynamic behavior of the Markov process $\eta(t)$ generated by $L$ or the corresponding time-evolution of Young diagrams. Taking the spatial size $N \in \mathbb{N}$ of the system as a scaling parameter, (2.1) and (2.2) suggest that $K$ and $M$ should be scaled as $K = c_1 N$ and $M = c_2 N^2 \in \mathbb{N}$ with some fixed $c_1, c_2 > 0$. Furthermore, the scaling in time is expected to be $N^4$ so that we consider the process $\eta_N(t)$ on $X_{c_1 N, c_2 N^2}$ with generator $L^N = N^4 L$. The limit hydrodynamic equation would be a fourth order non-linear partial differential equation of parabolic type, which might be a variant of Cahn-Hilliard equation with further non-linearity. In a derivation of such equation, we compute $L^N \eta_k$ as usual and a diverging factor $N^4$ appears due to the time change. In fact, the factor $N^2$ can be extinct freely, but for an additional $N^2$, we need to replace a seemingly diverging term of order $N^2$ describing currents with a better converging term of order 1. In the theory of the hydrodynamic limit in deriving second order parabolic equations, such replacement is called the gradient replacement, see [7, 13]. However, in our setting, the gradient replacement is not enough and instead its elaboration, the Laplacian replacement, would be required.

The equivalence of ensembles is required as one of the steps in the proof of the hydrodynamic limit. Another important step is to prove the spectral gap of order $O(N^{-4})$ for our generator $L$ defined on $X_{c_1 N, c_2 N^2}$ under certain conditions on the jump rates $c^{F}_r$ and $c^{G}_r$. This remains open at this moment.

**Remark 6.1**

(1) It is natural to assume that each atom located on the surface of the Young diagrams moves over the surface like a Brownian particle. Then, the jump rates $c^{F}_r$ and $c^{G}_r$ would behave as $C/r^2$ with $C > 0$ as $r$ grows.

(2) Different dynamics of two-dimensional Young diagrams, which are associated with the grand canonical ensembles for uniform or restricted uniform statistics, were studied by [2, 3]. Under the hydrodynamic space-time scaling, second order non-linear partial differential equations of parabolic type were derived and the Vershik curves were characterized as their unique stationary solutions. The creation and annihilation of atoms
located at the surface of the Young diagrams take place simultaneously in the model introduced in this paper, while the dynamics studied by [2, 3] does not equip this property and the creation and annihilation occur independently.

Acknowledgements The author thanks Herbert Spohn for a lot of stimulating discussions for many years including the problems discussed in this paper. He also thanks Yoshiki Otobe for informing him the reference [5].

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