GAUGE-INvariant UNIQUnEss THEOREMS FOR
P-GRAPHS

ROBERT HUBEN, S. KALISZEWski, NADIA S. LARSEN, AND JOHN QUIGG

abstract. We prove a version of the result in the title that makes use of
maximal coactions in the context of discrete groups. Earlier Gauge-Invariant
Uniqueness theorems for $C^*$-algebras associated to $P$-graphs and similar $C^*$-
algebras exploited a property of coactions known as normality. In the present
paper, the view point is that maximal coactions provide a more natural starting
point to state and prove such uniqueness theorems. A byproduct of our ap-
proach consists of an abstract characterization of co-universal representations
for a Fell bundle over a discrete group.

1. INTRODUCTION

To put this paper into the context of the literature, we begin with the title: first
of all, by “uniqueness theorem” we mean here an answer to the following question:
given a $C^*$-algebra $A$, when is a homomorphism $\pi : A \rightarrow B$ faithful? More precisely,
what is a general enough sufficient condition to have faithfulness of $\pi : A \rightarrow B$ that
at the same time is verifiable in a suitably direct manner for specific classes of $C^*$-
algebras? Here $B$ denotes any $C^*$-algebra. Since we are dealing with $C^*$-algebras,
we can suppose that $\pi$ is surjective whenever we wish. In practice, $A$ typically arises
by applying some process to some given data such as a group, semigroup, graph, or
category, and the criterion should be in terms of the data. The “gauge-invariant”
phrase comes from an equivariance condition: if $A$ carries an action $\Gamma \curvearrowright A$ of a
group $\Gamma$, we can ask for the existence of an action $\Gamma \curvearrowright B$. Then a common type of
Gauge-Invariant Uniqueness Theorem also asks for a type of “partial” fidelity.

A simple example illustrates one of the challenges of proving a Gauge-Invariant
Uniqueness Theorem: for a group $G$, the full and reduced $C^*$-algebras of the group
arise out of its data in a natural way, there is a surjection of the former onto
the later which is faithful on the finite span of the generators, but the surjection
is an isomorphism if and only if the group is amenable. Every Gauge-Invariant
Uniqueness Theorem must avoid this potential counterexample, for instance by
assuming the group is amenable, enforcing normality of an associated coaction, or
considering classes of objects that do not give rise to the full and reduced $C^*$-
algebra of the group. Our innovation — which removes these restrictions — is
placed in the setting of $P$-graph algebras, and is phrased with the aid of maximal

Date: Revised draft, 25 July 2023.
2000 Mathematics Subject Classification. Primary 46L05.
Key words and phrases. $P$-graph, gauge-invariant uniqueness, gauge coaction, co-universal
algebra.

This research was partly funded by the Trond Mohn Foundation through the project “Pure
Mathematics in Norway”.

1
coactions, leveraging the machinery of maximal coactions to yield an efficient proof of a Gauge-Invariant Uniqueness Theorem.

In the context of this paper the “ur-” Gauge-Invariant Uniqueness Theorem (see [13 Theorem 2.3] and [4 Theorem 3.3]) involves a directed graph $E = (E^0, E^1, r, s)$. In this case $A$ is the Cuntz-Krieger algebra $C^*(E)$, with canonical (“gauge”) action $T \acts C^*(E)$. Then the Gauge-Invariant Uniqueness Theorem says that a representation (i.e., Cuntz-Krieger $E$-family) $\{P_v, S_e\}_{v \in E^0, e \in E^1}$ in $B$ is faithful on $C^*(E)$ if it is equivariant for an action $T \acts B$ and $P_v \neq 0$ for every vertex $v \in E^0$. More generally, for a $k$-graph $A$, we take $A$ to be the Cuntz-Krieger algebra $O(A)$, with canonical action $T^k \acts O(A)$, and then the Gauge-Invariant Uniqueness Theorem says that a representation $t : \Lambda \to B$ is faithful on $O(A)$ if it is equivariant for an action $T^k \acts B$ and is faithful on the fixed-point algebra $A^G$. The proof is immediate from the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\pi} & B \\
\downarrow & & \downarrow \\
A^G & \longrightarrow & B^G,
\end{array}
\]

where the vertical arrows are the canonical conditional expectations (integrating over $\Gamma$), which are faithful on positive elements, and of course the bottom arrow is the restriction of $\pi$. In the application to $k$-graphs, the hypothesis that $t$ is nonzero on the vertices implies that it is faithful on the fixed-point algebra.

To generalize the above Gauge-Invariant Uniqueness Theorems to other groups $\Gamma$, the thing to do is turn the action of an abelian group $\Gamma$ into a coaction of the dual group $G = \hat{\Gamma}$ (see [7 Example A.23]). When $\Gamma$ is compact, $G$ will be discrete, and hence (see [19]) coactions of $G$ are “essentially” equivalent to Fell bundles over $G$. More precisely, a coaction of $G$ on $A$ is a $*$-homomorphism $\delta : A \to M(A \otimes C^*(G))$ satisfying certain properties (see, e.g., [7 Definition A.21]), which in particular guarantee that $A$ is densely spanned by the spectral subspaces

$$A_g = \{a \in A : \delta(a) = a \otimes g\} \quad \text{for } g \in G.$$ 

In the coaction literature, the map $\delta$ itself, or the pair $(A, \delta)$, are interchangeably used to denote a coaction.

These combine to form a Fell bundle $A = \{A_g\}_{g \in G}$, and there is a canonical surjection $C^*(A) \to A$. The fixed-point algebra $A^k$ coincides with the unit fibre $A_e$ (where $e$ is the identity element of $G$). In the case $A = O(A)$ for a $k$-graph $A$, the “degree functor” $d : A \to \mathbb{N}^k$ determines a Fell-bundle structure over $G = \mathbb{Z}^k$.

In the current paper we consider Gauge-Invariant Uniqueness Theorems for $P$-graphs (see Section 3 for the precise definition). Roughly speaking, a $P$-graph involves a pair $(G, P)$ consisting of a subsemigroup $P$ of a group $G$. For $k$-graphs the pair is $(\mathbb{Z}^k, \mathbb{N}^k)$. $P$-graphs were introduced in [1] for abelian $P$. A Gauge-Invariant Uniqueness Theorem for abelian $P$ is proved in [2] Proposition 2.7], and in [12 Theorem 4.32] the first author proved a Gauge-Invariant Uniqueness
Theorem for nonabelian \( P \). We remark that the idea of reducing the verification of faithfulness to an amenable group has been employed previously, see e.g. Proposition 6.6 in [16]. To describe the GIUT of [12], we start with a (not very satisfying) abstract Gauge-Invariant Uniqueness Theorem for nonabelian groups: if \((A, \delta)\) is a normal coaction of a discrete group \( G \), then a *-homomorphism \( \pi : A \to B \) is faithful if it is equivariant for a coaction of \( G \) on \( B \) and is faithful on the unit fibre \( A_e \). The appropriate version of diagram (1) is

\[
\begin{array}{ccc}
A & \xrightarrow{\pi} & B \\
\downarrow & & \downarrow \\
A_e & \longrightarrow & B_e,
\end{array}
\]

where the vertical arrows are again the canonical conditional expectations. Since the coaction \( \delta \) is assumed to be normal, the left-hand conditional expectation is faithful — it is useful to note that normality is automatically satisfied when \( G \) is amenable. We characterize this uniqueness theorem as unsatisfying because it only applies to normal coactions \((A, \delta)\). Although [12] does not appeal to the above abstract Gauge-Invariant Uniqueness Theorem, [12, Proposition 4.31] gives a sufficient condition ("reducing to an amenable ordered group") for the canonical coaction on the \( P \)-graph algebra to be normal. We quote from [12]: "What we mean by a gauge invariant uniqueness theorem for \( P \)-graphs is a similar statement: for any \( P \)-graph, there is exactly one \( \Lambda \)-faithful, tight, gauge coacting representation up to canonical isomorphism. We have already seen that in generalizing from \( N \)-graphs to \( P \)-graphs, we needed an additional hypothesis (finite alignment). We will see in Lemma 4.24 that such a gauge invariant uniqueness theorem need not be true in general, so another hypothesis is needed. We spend Chapter 3 developing this hypothesis on \((G, P)\), and then in Chapter 4 prove a gauge invariant uniqueness theorem for \( P \)-graphs that satisfy this new hypothesis."

In the current paper we prove a Gauge-Invariant Uniqueness Theorem [3,7] that applies to all finitely aligned \( P \)-graphs. Our strategy is to take full advantage of the theory of maximal coactions. Since \( G \) is discrete, a coaction \((A, \delta)\) is maximal if and only if the canonical surjection \( C^*(A) \to A \) is faithful, by [5, Proposition 4.2].

In [14, Theorems 3.1 and 3.3], three of us used the same strategy for a similar purpose, in the context of product systems over \( P \). In Theorem 3.6 of that same paper we appeal to the theory of normalizations of coactions to deduce a uniqueness theorem for normal coactions. Actually, the uniqueness theorem [3, Corollary 4.9] applies essentially the same techniques. Corollary 4.11 of that same paper is a Gauge-Invariant Uniqueness Theorem proven by giving sufficient conditions for the canonical coaction to be normal (similarly to [12]).

In the final Section 4 we show how the co-universal \( P \)-graph algebra can be quickly produced using normalizations of coactions.

This paper is to a large extent intended to illustrate that the use of maximal coactions allows for a more direct and elementary approach, which we hope will be more widely applicable.

We would also like to remark that it may be possible to prove a GIUT through the machinery of product systems. One would show that \( O(\Lambda) \) admits a realization as the \( C^* \)-algebra associated to a product system of correspondences over \( P \). Our results eschew a product system realization due to the ground work done by the first author in [12]. While this realization is outside the scope of the present article, it
would be worthwhile to see it materialized in future work. Moreover, this realization would add nicely to the list of uniqueness results obtained recently by Dor-On, Kakariadis, Katsoulis, Laca, Li [5] and Sehnem [20, 21].

We thank the anonymous referee, whose comments lead to significant improvements of this paper.

2. Preliminaries

Throughout, $G$ will be a discrete group. We need to exploit the strong connections between Fell bundles over $G$ and coactions of $G$. For Fell bundles, we refer to [8, 10, 17, and 19], and for coactions to [17 Appendix A], [19, 6, 8, 14 Appendix A], [17], and [15]. It will be convenient to use some elementary categorical language. Since $G$ is discrete, coactions and Fell bundles are “essentially” the same thing (see [19, 17]); however, we will take different approaches to the categories: we take the category of all coactions, but then we take the category of all faithful coaction-compatible representations of a fixed Fell bundle, as we explain shortly.

For coactions, there is a subtlety: we must choose what sort of morphisms we want. For this paper, since $G$ is discrete, the easiest choice is the following: a morphism $\phi: (A, \delta) \to (B, \varepsilon)$ between coactions is just a $\delta - \varepsilon$ equivariant *-homomorphism $\phi: A \to B$. The category of coactions has maximal objects: a coaction $(A, \delta)$ is maximal if whenever $\pi: (B, \varepsilon) \to (A, \delta)$ is a surjective morphism that is faithful on the fixed-point algebra $B^\varepsilon$, then $\pi$ is an isomorphism (and $\varepsilon$ is maximal). This is not the original definition of maximal, but is a characterization that is recorded in [14 Proposition A.1]. Reformulating [14 Corollary A.2] slightly, we get an abstract GIUT:

**Theorem 2.1** (Abstract GIUT, [9, 14]). If $(A, \delta)$ is a maximal coaction, then a surjective *-homomorphism $\pi: A \to B$ is faithful if and only if it is faithful on the fixed-point algebra $A^\delta$ and equivariant for $\delta$ and a maximal coaction $\varepsilon$ on $B$.

[14 Proposition A.1], in turn, is based upon [15 Proposition 3.1]. Given any coaction $(A, \delta)$, there are a maximal coaction $(A^m, \delta^m)$ and a surjective morphism $\psi_\delta: (A^m, \delta^m) \to (A, \delta)$ that is faithful on the fixed-point algebra, and moreover the assignment $(A, \delta) \mapsto (A^m, \delta^m)$ is functorial. Any such $\psi_\delta$, or the coaction $(A^m, \delta^m)$, is a maximalization of $(A, \delta)$, and is unique up to isomorphism. The existence of maximalizations was first proved in [9 Theorem 3.3], although the construction was nonfunctorial; then [11 Theorem 6.4] gave a functorial construction of maximalizations. The category also has minimal objects: a coaction $(A, \delta)$ is normal if whenever $\pi: (A, \delta) \to (B, \varepsilon)$ is a surjective morphism that is faithful on $A^\delta$, then $\pi$ is an isomorphism (and $\varepsilon$ is normal). Again, this is not the original definition of normal, but is a characterization recorded in [14 Proposition A.1]. Given any coaction $(A, \delta)$ there are a normal coaction $(A^n, \delta^n)$ and a surjective morphism $\eta_\delta: (A, \delta)\to (A^n, \delta^n)$ that is faithful on the fixed-point algebra, and moreover the assignment $(A, \delta) \mapsto (A^n, \delta^n)$ is functorial. Any such $\eta_\delta$, or the coaction $(A^n, \delta^n)$, is a normalization of $(A, \delta)$, and is unique up to isomorphism.

On the other hand, a representation of a given Fell bundle $A = \{A_g\}_{g \in G}$ in a $C^*$-algebra $B$ is a map $\pi: A \to B$ that is linear on the fibres, is multiplicative and involutive, and in particular is a *-homomorphism on $A_e$. Then $\text{span}(\pi(A))$ is a $C^*$-subalgebra denoted by $C^*(\pi)$. There is a category of representations of $A$ in which a morphism $\phi: \pi \to \sigma$ of representations is a homomorphism $\phi: C^*(\pi) \to C^*(\sigma)$
such that $\sigma = \phi \circ \pi$. A universal representation of $\mathcal{A}$ is an initial object $j$ in this category — by abstract nonsense all such are isomorphic, but as usual we imagine that we have picked one, and we call it the universal representation. The $C^*$-algebra $C^*(j)$ is denoted by $C^*(\mathcal{A})$ and called the $C^*$-algebra of $\mathcal{A}$. Thus, we have a universal property: for every representation $\pi$ of $\mathcal{A}$ there is a unique $*$-homomorphism $\phi : C^*(\mathcal{A}) \to C^*(\pi)$, called the integrated form of $\pi$, such that $\phi \circ j = \pi$. A representation $\pi$ is faithful if it is injective on the fibres $A_g$ (and it suffices to check this on the unit fibre $A_e$).

We say that a representation $\pi$ is coaction-compatible if there is a (necessarily unique) coaction $\delta_\pi$ on $C^*(\pi)$ such that $\delta_\pi(\pi(a)) = \pi(a_g) \otimes s$ for $a_g \in A_g$. The universal representation $j$ is coaction-compatible [19 Proposition 3.3], and we write $\delta_\pi$ for $\delta_j$, and call it the canonical coaction on $C^*(\mathcal{A})$. The faithful coaction-compatible representations of $\mathcal{A}$ form a subcategory of all representations, in which $j$ is still an initial object. Moreover, the assignment $\pi \mapsto (C^*(\pi), \delta_\pi)$ is functorial from representations to coactions. A co-universal representation of $\mathcal{A}$ is a final object $\omega$ in this subcategory, so that for every faithful representation $\pi$ of $\mathcal{A}$ there is a unique morphism $\phi : \pi \to \omega$. By abstract nonsense, $\omega$ is unique up to (unique) isomorphism. It follows from [19 Corollary 3.7] (see also [9 Proposition 3.7]) that in fact the regular representation $\lambda_\mathcal{A}$ is a suitable choice: $C^*(\lambda_\mathcal{A})$ is denoted by $C^*_r(\mathcal{A})$ and called the reduced $C^*$-algebra of $\mathcal{A}$. The canonical coaction $\delta_\lambda_\mathcal{A}$ is actually the normalization $\delta^*_\mathcal{A}$ of $\delta_\mathcal{A}$, and more generally for every faithful coaction-compatible representation $\pi$ the unique morphism $\phi : \pi \to \lambda_\mathcal{A}$ is the normalization of $\delta_\pi$.

Remark 2.2. Warning: a faithful representation $\pi$ of $\mathcal{A}$ is not necessarily coaction-compatible. To see this, first note that $C^*(\pi)$ is $G$-graded in the sense of [9 Definition 3.1]. If there is a bounded linear map on $C^*(\pi)$ that is the identity on $\pi(A_e)$ and zero on $\pi(A_g)$ for all $g \neq e$, then Exel would call $C^*(\pi)$ topologically graded. The discussion following [10 Proposition 19.3] gives an example of a faithful representation $\pi$ for which $C^*(\pi)$ is graded but not topologically graded. Perhaps more surprising is that $C^*(\pi)$ can be topologically graded without $\pi$ being coaction-compatible (see [8 Remark 2.2]).

For every coaction $(\mathcal{A}, \delta)$, the fibres $A_g$ give a Fell bundle $\mathcal{A}$, and there is a unique (automatically faithful) representation $\pi$ of $\mathcal{A}$ such that $A = C^*(\pi)$. The coaction $\delta$ is maximal if and only if the representation $\pi$ is universal, and is normal if and only if $\pi$ is co-universal. Reformulating the latter slightly gives an abstract co-universality result:

**Theorem 2.3** (Abstract co-universal algebra, [9 14]). A faithful representation $\pi$ of a Fell bundle $\mathcal{A}$ is co-universal if and only if $\delta_\pi$ is normal, and then for every faithful representation $\sigma$ the unique morphism $\phi : \sigma \to \pi$ is a normalization of the coaction $\delta_\pi$.

3. $P$-graphs

An ordered group is a pair $(G, P)$, where $G$ is a group and $P$ a submonoid such that $P \cap P^{-1} = \{e\}$. We always give $G$ the partial order defined by $a \leq b$ if $a^{-1}b \in P$. A weakly quasi-lattice ordered (WQLO) group (a term coined by Exel [10 Definition 32.1 (v)]) is an ordered group $(G, P)$ such that for all $p, q \in P$, if $\{p, q\}$ is bounded above then it has a least upper bound in $P$, denoted by $p \vee q$. Note that this property is intrinsic to the monoid $P$, i.e., is independent of the
particular group $G$. Throughout this paper, $P$ will always refer to part of a WQLO group $(G, P)$.

A $P$-graph is a countable category $\Lambda$ together with a functor $d : \Lambda \to P$ satisfying the factorization property: for all $\alpha \in \Lambda$ and $p, q \in P$ such that $d(\alpha) = pq$, there exist unique $\beta, \gamma \in \Lambda$ such that $\alpha = \beta \gamma$ and $d(\beta) = p$ (and hence $d(\gamma) = q$). For $p \in P$ we write $\Lambda^p = d^{-1}(p)$. We call elements of $\Lambda$ paths, we identify the objects with the identity morphisms, which we call vertices, and we write $\Lambda^0$ for the set of vertices. It is easy to see that $\Lambda^0 = 0^\chi$ (where we remind the reader that we write $e$ for the unit of $G$), and that $\Lambda$ is a category of paths in the sense of Spielberg [22].

We give $\Lambda$ the partial order $\alpha \leq \beta$ if $\beta \in \alpha \Lambda$. We refer to an upper bound of $S \subseteq \Lambda$, i.e., an element of $\bigcap_{s \in S} \gamma \Lambda$, as a common extension of $S$, and a least upper bound as a minimal common extension. We write $\vee S$ for the set of minimal common extensions of $S$. If $S = \{\alpha, \beta\}$ we write $\alpha \vee \beta = \vee S$. We write $\alpha \sqcap \beta$ if $\alpha \Lambda \cap \beta \Lambda \neq \emptyset$, and $\alpha \perp \beta$ otherwise.

The first author proved in [12] Lemma 2.38 that if $S \subseteq \Lambda$ is finite then

$$\bigvee S = \left\{ \mu \in \bigcap_{\gamma \in S} \gamma \Lambda : d(\mu) = \bigvee d(S) \right\}$$

$$\bigcap_{\gamma \in S} \gamma \Lambda = \bigcup_{\mu \in \bigvee S} \mu \Lambda.$$  

Actually, [12] proves it for $S = \{\alpha, \beta\}$, but it generalizes routinely to finite sets.

We say that $\Lambda$ is finitely aligned if $\bigvee S$ is finite for all finite $S \subseteq \Lambda$. Note that it suffices to check this for $S$ of the form $\{\alpha, \beta\}$. If $\alpha \in \Lambda$ we say $E \subseteq \alpha \Lambda$ is exhaustive if for all $\beta \in \alpha \Lambda$ there exists $\gamma \in E$ such that $\beta \sqcap \gamma$.

A representation of a finitely aligned $P$-graph $\Lambda$ in a $C^*$-algebra $B$ is a map $t : \Lambda \to B$ such that for all $\alpha, \beta \in \Lambda$ we have

- (R1) $t_\alpha t_\beta = t_{\alpha \beta}$ whenever $s(\alpha) = r(\beta)$;
- (R2) $t_\alpha^* t_\alpha = t_{s(\alpha)}$;
- (R3) $t_\alpha t_\beta t_\gamma = \sum_{\gamma \in \alpha \wedge \beta} t_\gamma t_\alpha^* t_\beta^*$.

As in [22] Remark 6.4, $\{t_\alpha\}_{\alpha \in \Lambda}$ consists of pairwise orthogonal projections, and $t_\alpha t_\alpha^* \leq t_{r(\alpha)}$ for all $\alpha \in \Lambda$. Regarding item (R3) note that for distinct $\gamma, \nu \in \alpha \vee \beta$ we have $\gamma \perp \nu$, so $\gamma \vee \nu = \emptyset$, and therefore the right-hand side is a sum of pairwise orthogonal projections. We write $C^*(t)$ for the $C^*$-subalgebra of $B$ generated by $\{t_\alpha\}_{\alpha \in \Lambda}$. The first author proved in [12] Lemma 2.42 that $C^*(t) = \mathbf{sp} \{t_\alpha t_\beta^* : \alpha, \beta \in \Lambda\}$.

Remark 3.1. Throughout this paper, we only consider representations of finitely-aligned $P$-graphs, due to (R3). If a $P$-graph is not finitely aligned, i.e., there are $\alpha, \beta$ for which $\alpha \vee \beta$ is infinite, then (R3) becomes an infinite sum of pairwise orthogonal projections, which will not converge in a $C^*$-algebra.

A representation $t$ of a finitely aligned $P$-graph $\Lambda$ is tight if for every $v \in \Lambda^0$ and every finite exhaustive set $E \subseteq v \Lambda$ we have

- (R4) $t_v = \bigvee_{\alpha \in E} t_\alpha t_\alpha^*$.

Given two representations $s$ and $t$ of the same $P$-graph $\Lambda$, we say $s$ covers $t$ if there is a *-homomorphism $\phi_t^* : C^*(s) \to C^*(t)$ satisfying $\phi_t^*(s_\alpha) = t_\alpha$ for all $\alpha \in \Lambda$. 


That is, \( s \) covers \( t \) if there is a map \( \phi^s_t \) making the following diagram commute:

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{s} & C^*(s) \\
\downarrow{t} & & \downarrow{\phi^s_t} \\
C^*(t) & & 
\end{array}
\]

Note that if such a map \( \phi^s_t \) exists, it is unique since its values on the generating set are fixed. In this way we get a category of representations of \( \Lambda \), whose morphisms are the above \( \phi^s_t \). See also [12, Chapter 2].

For a representation \( t \) of a finitely aligned \( P \)-graph \( \Lambda \), we say that a gauge coaction is a coaction \( \delta : C^*(t) \to C^*(t) \otimes C^*(G) \) satisfying

\[
\delta(\alpha t) = \alpha \otimes d(\alpha) \quad \text{for } \alpha \in \Lambda,
\]

and if such a \( \delta \) exists we say that \( t \) is gauge-compatible. If furthermore the coaction \( \delta \) is maximal, we say that \( t \) has a maximal gauge coaction.

One can show that any *-homomorphism \( \delta \) satisfying (3) is automatically a coaction (see [12, Proposition 2.50(3)]), and that for graphs and higher-rank graphs this gauge coaction corresponds to the usual gauge action.

The following lemma shows that in the context of representations with gauge coactions, covering maps automatically provide equivariance.

**Lemma 3.2.** Let \( s \) and \( t \) be representations of a \( P \)-graph \( \Lambda \). Suppose that \( s \) covers \( t \), with associated surjection \( \phi^s_t : C^*(s) \to C^*(t) \). Further suppose that we have coactions \( \delta \) on \( C^*(s) \) and \( \varepsilon \) on \( C^*(t) \). If \( \delta \) is a gauge coaction, then \( \phi^s_t \) is \( \delta - \varepsilon \) equivariant if and only if \( \varepsilon \) is a gauge coaction.

**Proof.** Suppose \( \phi^s_t \) is \( \delta - \varepsilon \) equivariant. Since \( \delta \) is a gauge coaction, then for all \( \lambda \in \Lambda \), \( \delta(s_\lambda) = s_\lambda \otimes d(\lambda) \). Then we have

\[
\varepsilon(t_\lambda) = \varepsilon(\phi^s_t(s_\lambda)) = (\phi^s_t \otimes \text{id}_G)(\delta(s_\lambda)) = (\phi^s_t \otimes \text{id}_G)(s_\lambda \otimes d(\lambda)) = t_\lambda \otimes d(\lambda)
\]

showing that \( \varepsilon \) is a gauge coaction, as desired.

Conversely, if \( \varepsilon \) is a gauge coaction, then

\[
\varepsilon(\phi^s_t(s_\lambda)) = \varepsilon(t_\lambda) = t_\lambda \otimes d(\lambda) = (\phi^s_t \otimes \text{id}_G)(s_\lambda \otimes d(\lambda)) = (\phi^s_t \otimes \text{id}_G)(\delta(s_\lambda))
\]

That is, \( \varepsilon \circ \phi^s_t = (\phi^s_t \otimes \text{id}_G) \circ \delta \) on each \( s_\lambda \). But since the set of \( s_\lambda \)'s generate \( C^*(s) \), this identity holds on all of \( C^*(s) \), which is to say \( \phi^s_t \) is \( \delta - \varepsilon \) equivariant. \( \square \)

Let \( \Lambda \) be a finitely aligned \( P \)-graph. A Cuntz-Krieger algebra of \( \Lambda \) is a pair \( (\mathcal{O}(\Lambda), s) \), where \( \mathcal{O}(\Lambda) \) is a \( C^* \)-algebra and \( s : \Lambda \to \mathcal{O}(\Lambda) \) is a tight representation that is universal in the sense that it covers every tight representation. Such a universal algebra exists by the typical arguments (see [12]), and is unique up to isomorphism.
Proposition 3.3. The Cuntz-Krieger algebra \( \mathcal{O}(\Lambda) \) has a maximal gauge coaction \( \delta \).

This proposition extends [12] Proposition 4.23, item 4], which showed the existence of \( \delta \), by showing that \( (\mathcal{O}(\Lambda), \delta) \) is maximal.

Proof. Routine calculations verify that \( t_\alpha := s_\alpha \otimes d(\alpha) \in \mathcal{O}(\Lambda) \otimes C^*(G) \) satisfies the axioms (R1)–(R4) (see the proof of [12] Proposition 2.50] for details). Thus by the universal property of \( \mathcal{O}(\Lambda) \) there is a \(*\)-homomorphism \( \delta : \mathcal{O}(\Lambda) \to \mathcal{O}(\Lambda) \otimes C^*(G) \) satisfying \( s_\alpha \mapsto s_\alpha \otimes d(\alpha) \), i.e., a gauge coaction.

For the maximality, we follow a strategy similar to [18] Theorem 7.1 (iv)): let \( A_\Lambda \) be the Fell bundle over \( G \) associated to the coaction \( \delta \), with universal algebra \( C^*(A_\Lambda) \) and maximal coaction \( (C^*(A_\Lambda), \delta_{A_\Lambda}) \). First of all, by the general theory of coactions of discrete groups we know that there is a unique \( \delta_{A_\Lambda} = \delta \) equivariant surjection \( \psi : C^*(A_\Lambda) \to \mathcal{O}(\Lambda) \), and that \( \psi \) restricts to the inclusion map on each fibre \( A_\beta \) of \( A_\Lambda \). Thus it suffices to show that \( \psi \) has a left inverse. This comes down to the universal property of \( \mathcal{O}(\Lambda) \) and the fact that the relations (R1)–(R4) are graded. Explicitly, there is a map \( \rho_0 : \Lambda \to A_\Lambda \) defined by \( \rho_0(\alpha) = s_\alpha \). Moreover, \( \rho_0 \) is a tight representation, because the required identities are satisfied in \( \Gamma_C(A_\Lambda) \). Thus \( \rho_0 \) extends uniquely to a \(*\)-homomorphism \( \rho : \mathcal{O}(\Lambda) \to C^*(A_\Lambda) \). For \( \alpha \in \Lambda \) we have
\[
\rho \circ \psi(\rho_0(\alpha)) = \rho(s_\alpha) = \rho_0(\alpha).
\]
Since the spectral subspaces of the coaction \( \delta \), and hence the Fell bundle \( C^*-\)algebra of \( A_\Lambda \), are generated by the image of \( \rho_0 \), it follows that \( \rho \circ \psi = \text{id} \), as desired. \( \square \)

Now we appeal to [13] Corollary A.2] to (begin to) get a GIUT for \( \mathcal{O}(\Lambda) \):

Theorem 3.4. If \( \pi : \mathcal{O}(\Lambda) \to B \) is a surjection that is equivariant for \( \delta \) and a maximal coaction \( \varepsilon \) on \( B \), then \( \pi \) is faithful if and only if \( \pi \) is faithful on the fixed-point algebra \( \mathcal{O}(\Lambda)^\delta \).

This should only be regarded as a first step toward our GIUT, though, because it is hard to check fidelity on all of \( \mathcal{O}(\Lambda)^\delta \). Rather, we would like to know that it suffices to check that \( \pi \) is “\( \Lambda \)-faithful” in the sense that \( \pi(s_\alpha) \neq 0 \) for all \( \alpha \in \Lambda \). This will require an appeal to [12] Lemma 4.16].

First, a preliminary lemma, which is essentially borrowed from [12] Lemma 4.13], and is included for completeness:

Lemma 3.5. Let \( t : \Lambda \to B \) be a representation. Let \( \alpha \in \Lambda v \), and let \( E \subseteq \alpha \Lambda \) be finite exhaustive. Then there is a unique finite exhaustive set \( F \subseteq v \Lambda \) such that \( E = \alpha F \). Moreover,
\[
\prod_{\gamma \in E} (t_\alpha t^*_\gamma - t_\gamma t^*_\alpha) = 0
\]
if and only if
\[
t_v = \bigvee_{\beta \in F} t_\beta t^*_\beta.
\]

Proof. For all \( \gamma \in E \), there is a unique \( \beta \in v \Lambda \) such that \( \gamma = \alpha \beta \). Let \( F \) be the set of such \( \beta \). Then \( F \subseteq v \Lambda \) is finite, and we will verify that it is exhaustive. Let \( \mu \in v \Lambda \). Then \( \alpha \mu \in \alpha \Lambda \), so because \( E \) is exhaustive we can choose \( \gamma \in E \) such that \( \alpha \mu \Lambda \cap \gamma \Lambda \neq \emptyset \). Writing \( \gamma = \alpha \beta \), we have \( \beta \in F \) and \( \mu \Lambda \cap \beta \Lambda \neq \emptyset \) since \( \Lambda \) is left cancellative.
For the other part, we compute
\[
\prod_{\gamma \in E} (t_\alpha t_\alpha^* - t_\gamma t_\gamma^*) = \prod_{\beta \in F} (t_\alpha t_\alpha^* - t_\alpha t_\beta t_\beta^* t_\alpha^*)
\]
\[
= \prod_{\beta \in F} t_\alpha (t_v - t_\beta t_\beta^*) t_\alpha^*
\]
\[
= t_\alpha \left( \prod_{\beta \in F} (t_v - t_\beta t_\beta^*) \right) t_\alpha^*,
\]
which is zero if and only if
\[
\prod_{\beta \in F} (t_v - t_\beta t_\beta^*) = 0.
\]

Now, for each \( \beta \in F \) the range projection \( t_\beta t_\beta^* \) is less than or equal to the vertex projection \( t_v \), so \( \prod_{\beta \in F} (t_v - t_\beta t_\beta^*) = 0 \) if and only if \( t_v = \bigvee_{\beta \in F} t_\beta t_\beta^* \). \( \square \)

**Proposition 3.6.** Let \( \Lambda \) be a finitely aligned \( P \)-graph, and let \( t : \Lambda \to B \) be a tight representation that is \( \Lambda \)-faithful, i.e., such that \( t_\alpha \neq 0 \) for all \( \alpha \in \Lambda \).

Then the associated *-homomorphism \( \pi : \mathcal{O}(\Lambda) \to B \) is faithful on the fixed-point algebra \( \mathcal{O}(\Lambda)^\delta \).

**Proof.** Note that \( \pi \) is the homomorphism \( \phi^s_t \) from (2) with \( s \) taken to be the universal tight representation of \( \Lambda \). With the aid of Lemma 3.5 above, this follows immediately from item 2 of [12, Lemma 4.16]. \( \square \)

Proposition 3.6 should be compared with the sufficiency condition in [20, Lemma 3.6]. Note that [20] is set in a general product-system context.

We are now ready for the announced GIUT.

**Theorem 3.7** (Gauge-Invariant Uniqueness Theorem for Tight Representations and Maximal Coactions). Let \( \Lambda \) be a finitely aligned \( P \)-graph, and let \( t : \Lambda \to B \) be a \( \Lambda \)-faithful tight representation with a maximal gauge coaction \( \varepsilon \). Then the associated *-homomorphism \( \pi : \mathcal{O}(\Lambda) \to B \) is faithful.

**Proof.** By Proposition 3.6, \( \pi \) is faithful on the fixed-point algebra \( \mathcal{O}(\Lambda)^\delta \), and since both \( \delta \) and \( \varepsilon \) are gauge coactions, by Lemma 3.2 the *-homomorphism \( \pi \) is \( \delta - \varepsilon \) equivariant. Thus the result follows from Theorem 3.4. \( \square \)

That is, we have shown that the universal tight representation \( s : \Lambda \to \mathcal{O}(\Lambda) \) is the unique \( \Lambda \)-faithful tight representation with a maximal gauge coaction.

**Theorem 3.7** should be compared with [20, Theorem 3.10].

---

1Note that there is a minor typo in the quoted lemma: \( s \) and \( t \) are representations of \( \Lambda \), not of \( (G, F) \).
4. Co-universal $P$-graph algebra

Recall our discussion of co-universal representations at the end of Section 2, and in particular Theorem 2.3.

Let $\Lambda$ be a finitely aligned $P$-graph, and let $\delta$ be the canonical coaction on the Cuntz-Krieger algebra $O(\Lambda)$. Further let $\mathcal{A}_\Lambda$ be the associated Fell bundle. We find it convenient to identify $O(\Lambda)$ with the Fell bundle algebra $C^*_\mathcal{A}(\Lambda)$, so that terminology can be freely passed between the two. In particular, the regular representation $s_\Lambda$ of $\Lambda$, by definition, corresponds to the regular representation $\lambda_{\mathcal{A}(\Lambda)}$ of $\mathcal{A}_\Lambda$, and the reduced $C^*$-algebra of $\Lambda$ is defined to be $O_r(\Lambda) = C^*(s_\Lambda)$. Recall from [2] that for a representation $t$ of $\Lambda$ the integrated form is the surjection $\phi^t_s : O(\Lambda) \to C^*_t$, where $s : \Lambda \to O(\Lambda)$ is the universal tight representation. The following lemma now follows from the above and Theorem 2.3.

Lemma 4.1. The regular representation $s_\Lambda$ is a normalization of $\delta$.

Recall that a representation $t$ of $\Lambda$ is called $\Lambda$-faithful if $t_{t\alpha} \neq 0$ for all $\alpha \in \Lambda$, and this implies that the associated representation of the Fell bundle $\mathcal{A}_\Lambda$ is faithful as well. The $\Lambda$-faithful gauge-compatible representations give a full subcategory of the representations of $\Lambda$.

Definition 4.2. A representation $t$ of $\Lambda$ is co-universal if it is terminal in the category of $\Lambda$-faithful gauge-compatible representations, in which case $C^*_t$ is a co-universal $C^*$-algebra of $\Lambda$.

Now Theorem 2.3 quickly implies the following result, which recovers [12, Theorem 4.22] (and $O_r(\Lambda)$ is there denoted $C^*_{\text{min}}(\Lambda)$).

Theorem 4.3. A $\Lambda$-faithful gauge-compatible representation $t$ of $\Lambda$ is co-universal if and only if $\delta_t$ is normal. In particular, the regular representation of $\Lambda$ is co-universal, and $O_r(\Lambda)$ is a co-universal $C^*$-algebra of $\Lambda$.

Theorem 4.3 should be compared with [5, Theorems 4.9 and 5.3] and [21, Theorem 5.1]. Note that [5, 21] are set in a general product-system context.

References

[1] Nathan Brownlowe, Aidan Sims, and Sean T. Vittadello, Co-universal $C^*$-algebras associated to generalised graphs, Israel J. Math. 193 (2013), no. 1, 399–440. MR 3038557
[2] Toke Meier Carlsen, Sooran Kang, Jacob Shotwell, and Aidan Sims, The primitive ideals of the Cuntz-Krieger algebra of a row-finite higher-rank graph with no sources, J. Funct. Anal. 266 (2014), no. 4, 2570–2589. MR 3150171
[3] Toke M. Carlsen, Nadia S. Larsen, Aidan Sims, and Sean T. Vittadello, Co-universal algebras associated to product systems, and gauge-invariant uniqueness theorems, Proc. Lond. Math. Soc. (3) 103 (2011), no. 4, 563–600. MR 2837016
[4] Sergio Doplicher, Claudia Pinzari, and Rita Zuccante, The $C^*$-algebra of a Hilbert bimodule, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (1998), no. 2, 263–281. MR 1638139
[5] A. Dor-On, E. T. A. Kakariadis, E. Katsoulis, M. Laca, and X. Li, $C^*$-envelopes for operator algebras with a coaction and co-universal $C^*$-algebras for product systems, Adv. Math. 400 (2022), Paper No. 108286, 40. MR 4387241
[6] Siegfried Echterhoff, S. Kaliszewski, and John Quigg, Maximal coactions, Internat. J. Math. 15 (2004), no. 1, 47–61. MR 2039211
[7] Siegfried Echterhoff, S. Kaliszewski, John Quigg, and Iain Raeburn, A categorical approach to imprimitivity theorems for $C^*$-dynamical systems, Mem. Amer. Math. Soc. 180 (2006), no. 850, viii+169. MR 2203990
Siegfried Echterhoff and John Quigg, *Induced coactions of discrete groups on $C^*$-algebras*, Canad. J. Math. 51 (1999), no. 4, 745–770. MR 1701340

Ruy Exel, *Amenability for Fell bundles*, J. Reine Angew. Math. 492 (1997), 41–73. MR 1488064

[8] Siegfried Echterhoff and John Quigg, *Induced coactions of discrete groups on $C^*$-algebras*, Canad. J. Math. 51 (1999), no. 4, 745–770. MR 1701340

Ruy Exel, *Amenability for Fell bundles*, J. Reine Angew. Math. 492 (1997), 41–73. MR 1488064

[9] Ruy Exel, *Amenability for Fell bundles*, J. Reine Angew. Math. 492 (1997), 41–73. MR 1488064

Robert Fischer, *Maximal coactions of quantum groups*, Preprint no. 350, SFB 478 Geometrische Strukturen in der Mathematik, 2004.

[10] Robert Huben, *Gauge-invariant uniqueness and reductions of ordered groups*, arXiv:2103.08792 [math.OA].

[11] Ruy Exel, *Amenability for Fell bundles*, J. Reine Angew. Math. 492 (1997), 41–73. MR 1488064

[12] Robert Huben, *Gauge-invariant uniqueness and reductions of ordered groups*, arXiv:2103.08792 [math.OA].

[13] Astrid an Huef and Iain Raeburn, *The ideal structure of Cuntz-Krieger algebras*, Ergodic Theory Dynam. Systems 17 (1997), no. 3, 611–624. MR 1452183

[14] S. Kaliszewski, Nadia S. Larsen, and John Quigg, *Inner coactions, Fell bundles, and abstract uniqueness theorems*, Münster J. Math. 5 (2012), 209–231. MR 3047633

[15] S. Kaliszewski and John Quigg, *Categorical Landstad duality for actions*, Indiana Univ. Math. J. 58 (2009), no. 1, 415–441. MR 2504419

[16] Marcelo Laca and Iain Raeburn, *Semigroup crossed products and the Toeplitz algebras of nonabelian groups*, J. Funct. Anal. 139 (1996), no. 2, 415–440. MR 1402771

[17] Chi-Keung Ng, *Discrete coactions on $C^*$-algebras*, J. Austral. Math. Soc. Ser. A 60 (1996), no. 1, 118–127. MR 1364557

[18] David Pask, John Quigg, and Iain Raeburn, *Coverings of k-graphs*, J. Algebra 289 (2005), no. 1, 161–191. MR 2139097

[19] John C. Quigg, *Discrete $C^*$-coactions and $C^*$-algebraic bundles*, J. Austral. Math. Soc. Ser. A 60 (1996), no. 2, 204–221. MR 1375586

[20] Camila F. Sehnem, *On $C^*$-algebras associated to product systems*, J. Funct. Anal. 277 (2019), no. 2, 558–593. MR 3952163

[21] Camila F. Sehnem, *C*-envelopes of tensor algebras of product systems*, J. Funct. Anal. 283 (2022), no. 12, Paper No. 109707, 31. MR 4488124

[22] Jack Spielberg, *Groupoids and $C^*$-algebras for categories of paths*, Trans. Amer. Math. Soc. 366 (2014), no. 11, 5771–5819. MR 3256184

Somerville, MA

Email address: rvhuben@gmail.com

School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287

Email address: kaliszewski@asu.edu

Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N-0316 Oslo, Norway

Email address: nadiasl@math.uio.no

School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287

Email address: quigg@asu.edu