Deletion in abstract Voronoi diagrams in expected linear time

Kolja Junginger
Faculty of Informatics, Università della Svizzera italiana
Lugano, Switzerland
kolja.junginger@usi.ch

Evanthia Papadopoulou
Faculty of Informatics, Università della Svizzera italiana
Lugano, Switzerland
evanthia.papadopoulou@usi.ch

Abstract
Updating an abstract Voronoi diagram in linear time, after deletion of one site, has been an open problem for a long time. Similarly for various concrete Voronoi diagrams of generalized sites, other than points. In this paper we present a simple, expected linear-time algorithm to update an abstract Voronoi diagram after deletion. We introduce the concept of a Voronoi-like diagram, a relaxed version of a Voronoi construct that has a structure similar to an abstract Voronoi diagram, without however being one. Voronoi-like diagrams serve as intermediate structures, which are considerably simpler to compute, thus, making an expected linear-time construction possible. We formalize the concept and prove that it is robust under an insertion operation, thus, enabling its use in incremental constructions.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases Abstract Voronoi diagram, linear-time algorithm, update after deletion, randomized incremental algorithm

Funding Supported in part by the Swiss National Science Foundation, DACH project SNF-200021E-154387.

1 Introduction

The Voronoi diagram of a set $S$ of $n$ simple geometric objects, called sites, is a well-known geometric partitioning structure that reveals proximity information for the input sites. Classic variants include the nearest-neighbor, the farthest-site, and the order-$k$ Voronoi diagram of $S$ ($1 \leq k < n$). Abstract Voronoi diagrams [11] offer a unifying framework for various concrete and well-known instances. Some classic Voronoi diagrams have been well investigated, with optimal construction algorithms available in many cases, see e.g., [2] for references and more information or [10] for numerous applications.

For certain tree-like Voronoi diagrams in the plane, linear-time construction algorithms have been well-known to exist, see e.g., [1],[7],[3],[5]. The first technique was introduced by Aggarwal et al. [1] for the Voronoi diagram of points in convex position, given the order of points along their convex hull. It can be used to derive linear-time algorithms for other fundamental problems: (1) updating a Voronoi diagram of points after deletion of one site in time linear to the number of the Voronoi neighbors of the deleted site; (2) computing the farthest Voronoi diagram of point-sites in linear time, after computing their convex hull; (3) computing the order-$(k+1)$ subdivision within an order-$k$ Voronoi region. There is also a much simpler randomized approach for the same problems introduced by Chew [7]. Klein
50:2 Deletion in abstract Voronoi diagrams in expected linear time

and Lingas [13] adapted the linear-time framework [1] to abstract Voronoi diagrams, under restrictions, showing that a Hamiltonian abstract Voronoi diagram can be computed in linear time, given the order of Voronoi regions along an unbounded simple curve, which visits each region exactly once and can intersect each bisector only once. This construction has been extended recently to include forest structures [4] under similar conditions, where no region can have multiple faces within the domain enclosed by a curve. The medial axis of a simple polygon is another well-known problem to admit a linear-time construction, shown by Chin et al. [8].

In this paper we consider the fundamental problem of updating a two-dimensional Voronoi diagram, after deletion of one site, and provide an expected linear-time algorithm to achieve this task. We consider the framework of abstract Voronoi diagrams to simultaneously address the various concrete instances under their umbrella. To the best of our knowledge, no linear-time construction algorithms are known for concrete diagrams of non-point sites, nor for abstract Voronoi diagrams. Related is our expected linear-time algorithm for the concrete farthest-segment Voronoi diagram [10], however, definitions are geometric, relying on star-shapeness and visibility properties of segment Voronoi regions, which do not extend to the abstract model. In this paper we consider a new formulation.

Abstract Voronoi diagrams. Abstract Voronoi diagrams (AVDs) were introduced by Klein [11]. Instead of sites and distance measures, they are defined in terms of bisecting curves that satisfy some simple combinatorial properties. Given a set $S$ of $n$ abstract sites, the bisector $J(p, q)$ of two sites $p, q \in S$ is an unbounded Jordan curve, homeomorphic to a line, that divides the plane into two open domains: the dominance region of $p$, $D(p, q)$ (having label $p$), and the dominance region of $q$, $D(q, p)$ (having label $q$), see Figure 1. The Voronoi region of $p$ is

$$
\text{VR}(p, S) = \bigcap_{q \in S \setminus \{p\}} D(p, q).
$$

The (nearest-neighbor) abstract Voronoi diagram of $S$ is $\mathcal{V}(S) = \mathbb{R}^2 \setminus \bigcup_{p \in S} \text{VR}(p, S)$.

Following the traditional model of abstract Voronoi diagrams (see e.g. [11, 5, 6, 12]) the system of bisectors is assumed to satisfy the following axioms, for every subset $S' \subseteq S$:

(A1) Each nearest Voronoi region $\text{VR}(p, S')$ is non-empty and pathwise connected.

(A2) Each point in the plane belongs to the closure of a nearest Voronoi region $\text{VR}(p, S')$.

(A3) After stereographic projection to the sphere, each bisector can be completed to a Jordan curve through the north pole.

(A4) Any two bisectors $J(p, q)$ and $J(r, t)$ intersect transversally and in a finite number of points. (It is possible to relax this axiom, see [12]).

$\mathcal{V}(S)$ is a plane graph of structural complexity $O(n)$ and its regions are simply-connected. It can be computed in time $O(n \log n)$, randomized [14] or deterministic [11]. To update $\mathcal{V}(S)$, after deleting one site $s \in S$, we compute $\mathcal{V}(S \setminus \{s\})$ within $\text{VR}(s, S)$. The sequence of site-occurrences along $\partial \text{VR}(s, S)$ forms a Davenport-Schinzel sequence of order 2 and this constitutes a major difference from the respective problem for points, where no repetition can occur. $\mathcal{V}(S \setminus \{s\}) \cap \text{VR}(s, S)$ contains disconnected Voronoi regions, which introduce

1 A preliminary version contains a gap when considering the linear-time framework of [1], thus, a linear-time construction for the farthest segment Voronoi diagram remains an open problem.
several complications. For example, \( \mathcal{V}(S') \cap \text{VR}(s, S' \cup \{s\}) \) for \( S' \subset S \setminus \{s\} \) may contain various faces that are not related to \( \mathcal{V}(S \setminus \{s\}) \cap \text{VR}(s, S) \), and conversely, an arbitrary sub-sequence of \( \partial \text{VR}(s, S) \) need not correspond to any Voronoi diagram. At first sight, a linear-time algorithm may seem infeasible.

**Our results.** In this paper we give a simple randomized algorithm to compute \( \mathcal{V}(S \setminus \{s\}) \) within \( \text{VR}(s, S) \) in expected time linear on the complexity of \( \partial \text{VR}(s, S) \). The algorithm is simple, not more complicated than its counterpart for points \( \mathcal{T} \), and this is achieved by computing simplified intermediate structures that are interesting in their own right. These are Voronoi-like diagrams, having a structure similar to an abstract Voronoi diagram, however, they are not Voronoi structures. Voronoi-like regions are supersets of real Voronoi regions, and their boundaries correspond to monotone paths in the relevant system of bisectors, rather than to an envelope in the same system as in a real Voronoi diagram (see Definition \( \mathcal{T} \)). We prove that Voronoi-like diagrams are well-defined, and also they are robust under an insertion operation, thus, making possible a randomized incremental construction for \( \mathcal{V}(S \setminus \{s\}) \cap \text{VR}(s, S) \) in linear time. We expect the concept to find uses in other Voronoi computations, where computing intermediate relaxed structures may simplify the entire computation. A first candidate in this direction is the linear-time framework of Aggarwal et al. \( \mathcal{T} \) that we plan to investigate next.

Our approach can be adapted (in fact, simplified) to compute in expected linear time the farthest abstract Voronoi diagram, after the sequence of its faces at infinity is known (see the appendix). The latter sequence can be computed in time \( O(n \log n) \). We also expect that our algorithm can be adapted to compute the order-\((k+1)\) subdivision within an order-\(k\) abstract Voronoi region in expected time linear on the complexity of the region boundary\(^2\). Our technique can be applied to concrete diagrams that may not strictly fall under the AVD model such as Voronoi diagrams of line segments that may intersect and of planar straight-line graphs (including simple and non-simple polygons).

### 2 Preliminaries

Let \( S \) be a set of \( n \) abstract sites (a set of indices) that define an admissible system of bisectors in the plane \( \mathcal{J} = \{J(p,q) : p \neq q \in S\} \), which fulfills axioms (A1)–(A4) for every \( S' \subseteq S \). The (nearest) Voronoi region of \( p \) is \( \text{VR}(p, S) = \bigcap_{q \in S \setminus \{p\}} D(p,q) \) and the Voronoi diagram of \( S \) is \( \mathcal{V}(S) = \mathbb{R}^2 \setminus \bigcup_{p \in S} \text{VR}(p, S) \), see, e.g., Figure 2.

Bisectors that have a site \( p \) in common are called \( p \)-related or simply related; related bisectors can intersect at most twice \( \mathcal{T} \) Lemma 3.5.2.5]. When two related bisectors \( J(p,q) \) and \( J(p,r) \) intersect, bisector \( J(q,r) \) also intersects with them at the same point(s) \( \mathcal{T} \).

---

\(^2\) The adaptation is non-trivial, thus, we only make a conjecture here and plan to consider details in subsequent work.
and these points are the Voronoi vertices of $\mathcal{V}(\{p, q, r\})$, see Figure 2. Since any two related bisectors in $\mathcal{J}$ intersect at most twice, the sequence of site occurrences along $\partial \mathcal{V}(p, S)$, $p \in S$, forms a Davenport-Schinzel sequence of order 2 (by [20, Theorem 5.7]).

To update $\mathcal{V}(S)$ after deleting one site $s \in S$, we compute $\mathcal{V}(S \setminus \{s\})$ within $\mathcal{V}(s, S)$, i.e., compute $\mathcal{V}(S \setminus \{s\}) \cap \mathcal{V}(s, S)$. Its structure is given in the following lemma. Figure 3 illustrates $\mathcal{V}(S \setminus \{s\}) \cap \mathcal{V}(s, S)$ (in red) for an unbounded region $\mathcal{V}(s, S)$, and Figure 9 illustrates the same for a bounded region, where the region’s boundary is shown in bold.

- **Lemma 1.** $\mathcal{V}(S \setminus \{s\}) \cap \mathcal{V}(s, S)$ is a forest having exactly one face for each Voronoi edge of $\partial \mathcal{V}(s, S)$. Its leaves are the Voronoi vertices of $\partial \mathcal{V}(s, S)$, and points at infinity if $\mathcal{V}(s, S)$ is unbounded. If $\mathcal{V}(s, S)$ is bounded then $\mathcal{V}(S \setminus \{s\}) \cap \mathcal{V}(s, S)$ is a tree.

**Proof.** Every face in $\mathcal{V}(S \setminus \{s\}) \cap \mathcal{V}(s, S)$ must touch the boundary $\partial \mathcal{V}(s, S)$ because Voronoi regions are non-empty and connected; this implies that the diagram is a forest. Every Voronoi edge $e \subseteq J(s, p)$ on $\partial \mathcal{V}(s, S)$ must be entirely in $\mathcal{V}(p, S \setminus \{s\})$. Thus, no leaf can lie in the interior of a Voronoi edge of $\partial \mathcal{V}(s, S)$. On the other hand, each Voronoi vertex of $\partial \mathcal{V}(s, S)$ must be a leaf of the diagram as its incident edges are induced by different sites.

Now we show that no two edges of $\partial \mathcal{V}(s, S)$ can be incident to the same face of $\mathcal{V}(S \setminus \{s\}) \cap \mathcal{V}(s, S)$. Consider two edges on $\partial \mathcal{V}(s, S)$ induced by the same site $p \in S \setminus \{s\}$. Then there exists an edge between them, induced by a site $q \neq p$, such that the bisector $J(s, q)$ has exactly two intersections with $J(p, s)$. As shown in Figure 3, the bisector $J(p, q)$ intersects them at the same two points. Since the bisector system is admissible, and thus $\mathcal{V}(p, \{s, p, q\})$ is connected, $J(p, q)$ connects these endpoints through $D(p, s) \cap D(q, s)$ as shown in Figure 4. Thus, $J(p, q) \cap \mathcal{V}(s, \{p, s, q\})$ consists of two unbounded connected components. This implies that $D(p, q) \cap \mathcal{V}(s, S)$ must have two disjoint faces, each of which is incident to exactly one of the two edges of $p$. Thus $\mathcal{V}(p, S \setminus \{s\}) \cap \mathcal{V}(s, S)$ cannot be connected and the two edges of $p$ must be incident to different faces of $\mathcal{V}(S \setminus \{s\}) \cap \mathcal{V}(s, S)$.

If $\mathcal{V}(s, S)$ is unbounded, two consecutive edges of $\partial \mathcal{V}(s, S)$ can extend to infinity, in which case there is at least one edge of $\mathcal{V}(S \setminus \{s\}) \cap \mathcal{V}(s, S)$ extending to infinity between them; thus, leaves can be points at infinity. If $\mathcal{V}(s, S)$ is bounded, all leaves of $\mathcal{V}(S \setminus \{s\}) \cap \mathcal{V}(s, S)$ must lie on $\partial \mathcal{V}(s, S)$. Since no face is incident to more than one edge of $\partial \mathcal{V}(s, S)$, in this case $\mathcal{V}(S \setminus \{s\}) \cap \mathcal{V}(s, S)$ cannot be disconnected, and thus is a tree.

![Figure 3](image1.png) $\mathcal{V}(S \setminus \{s\}) \cap \mathcal{V}(s, S)$ in red, where $\mathcal{V}(s, S)$ is unbounded and its boundary is shown in black bold.

![Figure 4](image2.png) $\mathcal{V}(p, S \setminus \{s\}) \cap \mathcal{V}(s, S)$ cannot be connected because of $J(p, q)$.

Let $\Gamma$ be a closed Jordan curve in the plane large enough to enclose all the intersections of bisectors in $\mathcal{J}$, and such that each bisector crosses $\Gamma$ exactly twice and transversally. Without loss of generality, we restrict all computations within $\Gamma$. The curve $\Gamma$ can be interpreted
Figure 5 The domain $D_s = \text{VR}(s, S) \cap D_r$.

Figure 6 (a) A $p$-inverse cycle. (b) A $p$-cycle.

as $J(p, s_\infty)$, for all $p \in S$, where $s_\infty$ is an additional site at infinity. Let the interior of $\Gamma$ be denoted as $D_r$. Our domain of computation is $D_s = \text{VR}(s, S) \cap D_r$, see Figure 5 we compute $V(S \setminus \{S\}) \cap D_s$.

The following lemmas are used as tools in our proofs. Let $C_p$ be a cycle of $p$-related bisectors in the arrangement of bisectors $J \cup \Gamma$. If for every edge in $C_p$ the label $p$ appears on the outside of the cycle then $C_p$ is called $p$-inverse, see Figure 6(a). If the label $p$ appears only inside $C_p$ then $C_p$ is called a $p$-cycle, see Figure 6(b). A $p$-inverse cycle cannot contain pieces of $\Gamma$.

Lemma 2. In an admissible bisector system there is no $p$-inverse cycle.

Proof. The farthest Voronoi region of $p$ is $FVR(p, S) = \bigcap_{q \in S \setminus \{p\}} D(q, p)$. By its definition, $FVR(p, S)$ must be enclosed in any $p$-inverse cycle $C_p$. But farthest Voronoi regions must be unbounded deriving a contradiction.

The following transitivity lemma is a consequence of transitivity of dominance regions [3, Lemma 2] and the fact that bisectors $J(p, q), J(q, r), J(p, r)$ intersect at the same point(s).

Lemma 3. Let $z \in \mathbb{R}^2$ and $p, q, r \in S$. If $z \in D(p, q)$ and $z \in \overline{D(q, r)}$, then $z \in D(p, r)$.

We make a general position assumption that no three $p$-related bisectors intersect at the same point. This implies that Voronoi vertices have degree 3.

3 Problem formulation and definitions

Let $S$ denote the sequence of Voronoi edges along $\partial \text{VR}(s, S)$, i.e., $S = \partial \text{VR}(s, S) \cap D_r$. We consider $S$ as a cyclically ordered set of arcs, where each arc is a Voronoi edge of $\partial \text{VR}(s, S)$. Each arc $\alpha \in S$ is induced by a site $s_\alpha \in S \setminus \{s\}$ such that $\alpha \subseteq J(s, s_\alpha)$. A site $p$ may induce several arcs on $\partial \text{VR}(s, S)$; recall, that the sequence of site occurrences along $\partial \text{VR}(s, S)$ is a Davenport-Schinzel sequence of order 2.

We can interpret the arcs in $S$ as sites that induce a Voronoi diagram $\mathcal{V}(S)$, where $\mathcal{V}(S) = \mathcal{V}(S \setminus \{s\}) \cap D_s$ and $D_s = \text{VR}(s, S) \cap D_r$. Figure 7(a) illustrates $S$ and $\mathcal{V}(S)$ in black (bold) and red, respectively. By Lemma 1 each face of $\mathcal{V}(S \setminus \{s\}) \cap D_s$ is incident to exactly one arc in $S$. In this respect, each arc $\alpha$ in $S$ has a Voronoi region, $\mathcal{V}(\alpha, S)$, which is the face of $\mathcal{V}(S \setminus \{s\}) \cap D_s$ incident to $\alpha$.

For a site $p \in S$ and $S' \subseteq S$, let $J_{p, S'} = \{J(p, q) \mid q \in S', q \neq p\}$ denote the set of all $p$-related bisectors involving sites in $S'$. The arrangement of a bisector set $J$ is denoted by $\mathcal{A}(J)$, $\mathcal{A}(J_{p, S'})$ may consist of more than one connected components.

Definition 4. A path $P$ in $J_{p, S'}$ is a connected sequence of alternating edges and vertices of the arrangement $\mathcal{A}(J_{p, S'})$. An arc $\alpha$ of $P$ is a maximally connected set of consecutive edges and vertices of the arrangement along $P$, which belong to the same bisector. The common endpoint of two consecutive arcs of $P$ is a vertex of $P$. An arc of $P$ is also called an edge.
Deletion in abstract Voronoi diagrams in expected linear time

Definition 5. A path \( P \) in \( \mathcal{J}_{p,s'} \) is called \( p \)-monotone if any two consecutive arcs \( \alpha, \beta \in P \), where \( \alpha \subseteq J(p,s_a) \) and \( \beta \subseteq J(p,s_b) \), induce the Voronoi edges of \( \partial VR(p, \{p, s_a, s_b\}) \), which are incident to the common endpoint of \( \alpha, \beta \) (see Figure 7).

Definition 6. The envelope of \( \mathcal{J}_{p,s'} \), with respect to site \( p \), is \( env(\mathcal{J}_{p,s'}) = \partial VR(p, S' \cup \{p\}) \), called a \( p \)-envelope (see Figure 8(a)).

Figure 8 illustrates two \( p \)-monotone paths, where (a) is a \( p \)-envelope. Notice, \( S \) is the envelope of the \( s \)-related bisectors in \( \mathcal{J} \). \( S = env(\mathcal{J}_{s,s \setminus \{s\}}) \cap D_r \). A \( p \)-monotone path that is not a \( p \)-envelope can be a Davenport-Schinzel sequence of order \( \geq 2 \), with respect to site occurrences in \( S \setminus \{s\} \).

The system of bisectors \( \mathcal{J}_{p,s'} \) may consist of several connected components. For convenience, in order to unify the various connected components of \( A(\mathcal{J}_{p,s'}) \) and to consider its \( p \)-monotone paths as single curves, we include the curve \( \Gamma \) in the corresponding system of bisectors. Then \( env(\mathcal{J}_{p,s'} \cup \Gamma) \) is a closed \( p \)-monotone path, whose connected components in \( \mathcal{J}_{p,s'} \) are interleaved by arcs of \( \Gamma \).

Definition 7. Consider \( S' \subseteq S \) and let \( S' = \{s_a \in S \mid \alpha \in S' \subseteq S \setminus \{s\}\} \) be its corresponding set of sites. A closed \( s \)-monotone path in \( \mathcal{J}_{s,s'} \cup \Gamma \) that contains all arcs in \( S' \) is called a boundary curve for \( S' \). The part of the plane enclosed in a boundary curve \( \mathcal{P} \) is called the domain of \( \mathcal{P} \), and it is denoted by \( D_{\mathcal{P}} \). Given \( \mathcal{P} \), we also use notation \( S_{\mathcal{P}} \) to denote \( S' \).

A set of arcs \( S' \subseteq S \) can admit several different boundary curves. One such boundary curve is its envelope \( \mathcal{E} = env(\mathcal{J}_{s,s'} \cup \Gamma) \). Figure 9(b) illustrates a boundary curve for \( S' \subseteq S \), where \( S \) is the set of arcs in Figure 9(a).

A boundary curve \( \mathcal{P} \) in \( \mathcal{J}_{s,s'} \cup \Gamma \) consists of pieces of bisectors in \( \mathcal{J}_{s,s'} \), called boundary arcs, and pieces of \( \Gamma \), called \( \Gamma \)-arcs. \( \Gamma \)-arcs correspond to openings of the domain \( D_{\mathcal{P}} \) to infinity. Among the boundary arcs, those that contain an arc of \( S' \) are called original and others are called auxiliary arcs. Original boundary arcs are expanded versions of the arcs in \( S' \). To distinguish between them, we call the elements of \( S \) core arcs and use an * in their notation. In Figure 9, the core arcs are illustrated in bold.

For a set of arcs \( S' \subseteq S \), we define the Voronoi diagram of \( S' \subseteq S \) as \( V(S') = V(S') \cap D_E \), where \( E \) is the envelope \( env(\mathcal{J}_{s,s'} \cup \Gamma) \). \( V(S') \) can be regarded as the Voronoi diagram of the \( s \)-envelope \( E \), thus, it can also be denoted \( V(E) \). The face of \( V(S') \) incident to an arc \( \alpha \in E \) is called the Voronoi region of \( \alpha \) and is denoted by \( VR(\alpha, S') \). We would like to extend the definition of \( V(S') \) to any boundary curve stemming out of \( S' \). To this goal we define a Voronoi-like diagram for any boundary curve \( \mathcal{P} \) of \( S' \). Notice, \( D_s \subseteq D_E \subseteq D_{\mathcal{P}} \).

Definition 8. Given a boundary curve \( \mathcal{P} \) in \( \mathcal{J}_{s,s'} \cup \Gamma \), a Voronoi-like diagram of \( \mathcal{P} \) is a plane graph on \( \mathcal{J}(S') = \{J(p,q) \in \mathcal{J} \mid p, q \in S'\} \) inducing a subdivision on the domain \( D_{\mathcal{P}} \) as follows (see Figure 11(b)):
1. There is exactly one face $R(\alpha)$ for each boundary arc $\alpha$ of $\mathcal{P}$, and $\partial R(\alpha)$ consists of the arc $\alpha$ plus an $s_\alpha$-monotone path in $\mathcal{J}_{s_\alpha,S'} \cup \Gamma$.

2. The Voronoi-like diagram of $\mathcal{P}$ is $\mathcal{V}(\mathcal{P}) = D_\mathcal{P} \setminus \bigcup_{\alpha \in \mathcal{P}} R(\alpha)$.

Voronoi-like regions in $\mathcal{V}(\mathcal{P})$ are related to real Voronoi regions in $\mathcal{V}(S')$ as supersets as shown in the following lemma. In Figure 9(b) the Voronoi-like region $R(\eta)$ is a superset of its corresponding Voronoi region $VR(\eta,S)$ in (a); similarly for e.g., $R(\alpha)$. Note that not every boundary curve of $S' \subset S$ needs to admit a Voronoi-like diagram.

**Lemma 9.** Let $\alpha$ be a boundary arc in a boundary curve $\mathcal{P}$ of $S'$ such that a portion $\tilde{\alpha} \subseteq \alpha$ appears on the $s$-envelope $\mathcal{E}$ of $S'$, $\mathcal{E} = \text{env}(\mathcal{J}_{s_\alpha,S'} \cup \Gamma)$. Given $\mathcal{V}(\mathcal{P})$, $R(\alpha) \supseteq VR(\tilde{\alpha},S')$. If $\alpha$ is original, then $R(\alpha) \supseteq VR(\tilde{\alpha},S') \supseteq VR(\alpha^*,S)$.

**Proof.** By the definition of a Voronoi region, no piece of a bisector $J(s_\alpha,\cdot)$ can appear in the interior of $VR(\tilde{\alpha},S')$, where $\tilde{\alpha} \in \mathcal{E}$ (recall that $V(S') = \mathcal{V}(\mathcal{E})$). Since in addition $\alpha \supseteq \tilde{\alpha}$, the claim follows. For an original arc $\alpha$, since $S' \subseteq S$, by the monotonicity property of Voronoi regions, we also have $VR(\tilde{\alpha},S') \supseteq VR(\alpha^*,S)$.

As a corollary to Lemma 9, the adjacencies of the real Voronoi diagram $\mathcal{V}(S')$ are preserved in $\mathcal{V}(\mathcal{P})$, for all arcs that are common to the envelope $\mathcal{E}$ and the boundary curve $\mathcal{P}$. In addition, $\mathcal{V}(\mathcal{E})$ coincides with the real Voronoi diagram $\mathcal{V}(S')$.

**Corollary 10.** $\mathcal{V}(\mathcal{E}) = \mathcal{V}(S')$. This also implies $\mathcal{V}(S) = \mathcal{V}(S')$.

The following Lemma 12 gives a basic property of Voronoi-like regions that is essential for subsequent proofs. To establish it we first need the following observation.

**Lemma 11.** $\overline{D_\mathcal{P}}$ cannot contain a $p$-cycle of $\mathcal{J}(S_\mathcal{P}) \cup \Gamma$, for any $p \in S_\mathcal{P}$.

**Proof.** Let $p \in S_\mathcal{P}$ define an original arc along $\mathcal{P}$. This arc is bounding $VR(p,S_\mathcal{P} \cup \{s\})$, thus, it must have a portion within $VR(p,S_\mathcal{P})$. Hence, $VR(p,S_\mathcal{P})$ has a non-empty intersection with $\mathbb{R}^2 \setminus \overline{D_\mathcal{P}}$. But $VR(p,S_\mathcal{P})$ must be enclosed within any $p$-cycle of $\mathcal{J}(S_\mathcal{P}) \cup \Gamma$, by its definition. Thus, no such $p$-cycle can be contained in $\overline{D_\mathcal{P}}$. Refer to Figure 10.

**Lemma 12.** Suppose bisector $J(s_\alpha,s_\beta)$ appears within $R(\alpha)$ (see Figure 11). For any connected component $e$ of $J(s_\alpha,s_\beta) \cap R(\alpha)$ that is not intersecting $\alpha$, the label $s_\alpha$ must appear on the same side of $e$ as $\alpha$. Let $\partial R_\alpha(\alpha)$ denote the portion of $\partial R(\alpha)$ cut out by such a component $e$, at opposite side from $\alpha$. Then $\partial R_\alpha(\alpha) \subseteq D(s_\beta,s_\alpha)$.

By Lemma 12, any components of $J(s_\alpha,s_\beta) \cap R(\alpha)$ must appear sequentially along $\partial R(\alpha)$. Note that $\partial R_\alpha(\alpha)$ may as well contain $\Gamma$-arcs.
Proof. Suppose for the sake of contradiction that there is such a component \( e \subseteq J(s_\alpha, s_\beta) \cap R(\alpha) \) with the label \( s_\alpha \) appearing at opposite side of \( e \) as \( \alpha \) (see Figure 12). Then \( e \) and \( \partial R(\alpha) \) form an \( s_\alpha \)-cycle \( C \) within \( \overline{D_P} \), contradicting Lemma \[11\]. Suppose now that \( \partial R(\alpha) \) lies only partially in \( D(s_\beta, s_\alpha) \). Then \( J(s_\beta, s_\alpha) \) would have to re-enter \( R(\alpha) \) at \( \partial R(\alpha) \), resulting in another component of \( J(s_\beta, s_\alpha) \cap R(\alpha) \) with an invalid labeling. ▶

The following lemma extends Lemma \[12\] when a component \( e \) of \( J(s_\alpha, s_\beta) \cap R(\alpha) \) intersects arc in \( \alpha \). If \( J(s_\alpha, s_\beta) \) intersects \( \alpha \), then there is also a component \( \tilde{\beta} \) of \( J(s_\beta, s_\alpha) \cap R(\alpha) \) intersecting \( \alpha \) at the same point as \( e \). If \( \tilde{\beta} \) has only one endpoint on \( \alpha \) let \( \partial R(\alpha) \) denote the portion of \( \partial R(\alpha) \) that is cut out by \( e \), at the side of its \( s_\beta \)-label (see Figure \[13\] (a)). If both endpoints of \( \tilde{\beta} \) are on \( \alpha \) then there are two components of \( J(s_\beta, s_\alpha) \cap R(\alpha) \) incident to \( \alpha \) (see Figure \[13\] (b)); let \( \partial R(\alpha) \) denote the portion of \( \partial R(\alpha) \) between these two components.

\[ \text{Lemma 13.} \text{ Let } e \text{ be a component of } J(s_\alpha, s_\beta) \cap R(\alpha). \text{ Then } \partial R(\alpha) \subseteq D(s_\beta, s_\alpha). \]

**Proof.** Suppose that following \( e \), bisector \( J(s_\alpha, s_\beta) \) re-enters \( R(\alpha) \) through \( \partial R(\alpha) \), inducing another component \( f \) of \( J(s_\alpha, s_\beta) \cap R(\alpha) \). Then \( f \) cannot intersect \( \alpha \), because \( J(s_\alpha, s_\beta), J(s, s_\alpha), J(s, s_\beta) \) intersect at most twice and in the same point(s). This implies that \( f \) has a reverse labeling, contradicting Lemma \[12\]. Thus \( \partial R(\alpha) \subseteq D(s_\beta, s_\alpha). \) ▶

Using the basic property of Lemma \[12\] and its extension, we show that if there is any non-empty component of \( J(s_\alpha, s_\beta) \cap R(\alpha) \), then \( (s, s_\beta) \) must also intersect \( D_P \), i.e., there exists a non-empty component of \( J(s, s_\beta) \cap D_P \) that is missing from \( \mathcal{P} \). Using this property and Theorem \[18\] of the next section, we obtain the following theorem. Its proof is deferred to Section \[23\].

\[ \text{Theorem 14.} \text{ Given a boundary curve } \mathcal{P} \text{ of } S' \subseteq S, \mathcal{V}_l(\mathcal{P}) \text{ (if it exists) is unique.} \]

The complexity of \( \mathcal{V}_l(\mathcal{P}) \) is \( O(|\mathcal{P}|) \), where \( |\mathcal{P}| \) denotes the number of boundary arcs in \( \mathcal{P} \), as it is a planar graph with exactly one face per boundary arc and vertices of degree 3 (or 1).

## 4 Insertion in a Voronoi-like diagram

Consider a boundary curve \( \mathcal{P} \) for \( S' \subseteq S \) and its Voronoi-like diagram \( \mathcal{V}_l(\mathcal{P}) \). Let \( \beta^* \) be an arc in \( S \setminus S' \), thus, \( \beta^* \) is contained in the closure of the domain \( \overline{D_P} \).

\[ \text{Figure 13 Illustrations for Lemma \[13\]. The bold red parts } \partial R(\alpha) \text{ belong to } D(s_\beta, s_\alpha). \]

\[ \text{Figure 14 } \mathcal{P}_\beta = \mathcal{P} \oplus \beta, \text{ core arc } \beta^* \text{ is bold, black. Endpoints of } \beta \text{ are } x,y. \]
We define arc $\beta \supseteq \beta^*$ as the connected component of $J(s, s_\beta) \cap D^{\overline{P}}$ that contains $\beta^*$ (see Figure 14). We also define an insertion operation $\oplus$, which inserts arc $\beta$ in $P$ deriving a new boundary curve $P_\beta = P \oplus \beta$, and also inserts $R(\beta)$ in $V_l(P)$ deriving the Voronoi-like diagram $V_l(P_\beta) = V_l(P) \oplus \beta$. $P_\beta$ is the boundary curve obtained by deleting the portion of $P$ between the endpoints of $\beta$, which lies in $D(s_\beta, s)$, and substituting it with $\beta$.

Figure 15 enumerates the possible cases of inserting arc $\beta$ in $P$ and is summarized in the following observation.

\[\text{Figure 15 Insertion cases for an arc } \beta.\]

\[\text{Observation 15. Possible cases of inserting arc } \beta \text{ in } P \text{ (see Figure 15). } D_{P_\beta} \subseteq D_P.\]

\[\begin{align*}
(a) & \text{ } \beta \text{ straddles the endpoint of two consecutive boundary arcs; no arcs in } P \text{ are deleted.} \\
(b) & \text{ } \text{Auxiliary arcs in } P \text{ are deleted by } \beta; \text{ their regions are also deleted from } V_l(P_\beta). \\
(c) & \text{ } \text{An arc } \alpha \in P \text{ is split into two arcs by } \beta; R(\alpha) \text{ in } V_l(P) \text{ will also be split.} \\
(d) & \text{ } \text{A } \Gamma \text{-arc is split in two by } \beta; V_l(P_\beta) \text{ may switch from being a tree to being a forest.} \\
(e) & \text{ } \text{A } \Gamma \text{-arc is deleted or shrunk by inserting } \beta. \text{ } V_l(P_\beta) \text{ may become a tree.} \\
(f) & \text{ } P \text{ already contains a boundary arc } \tilde{\beta} \supseteq \beta^*; \text{ then } \tilde{\beta} = \beta \text{ and } P_\beta = P.
\end{align*}\]

Note that $P_\beta$ may contain fewer, the same number, or even one extra auxiliary arc compared to $P$.

\[\text{Lemma 16. The curve } P_\beta = P \oplus \beta \text{ is a boundary curve for } S' \cup \{\beta^*\}.\]

\[\text{Proof. Since } P \text{ is a (closed) } s\text{-monotone path in } J_{s; S'} \cup \Gamma, \text{ } P_\beta \text{ is also such a path in } J_{s; S' \cup \{s_\beta\}} \cup \Gamma, \text{ by construction. No original arc in } P \text{ can be deleted by the insertion of } \beta, \text{ because every core arc in } S \text{ appears on the envelope env}(J_{s; \Gamma}); \text{ thus, such an arc cannot be cut out by the insertion of } \beta \text{ on } P. \text{ Hence, } P_\beta \text{ contains all arcs in } S' \cup \{\beta^*\}.\]

Given $V_l(P)$ and arc $\beta$, where $\beta^* \in S \setminus S'$, we define a merge curve $J(\beta)$, within $V_l(P)$, which delimits the boundary of $R(\beta)$ in $V_l(P_\beta)$. We define $J(\beta)$ incrementally, starting at an endpoint of $\beta$. Let $x$ and $y$ denote the endpoints of $\beta$, where $x, y, g$ are in counterclockwise order around $P_\beta$; refer to Figure 16.

\[\text{Definition 17. Given } V_l(P) \text{ and arc } \beta \subset J(s, s_\beta), \text{ the merge curve } J(\beta) \text{ is a path } (v_1, \ldots, v_m) \text{ in the arrangement of } s_\beta\text{-related bisectors, } J_{s_\beta, s_\beta} \cup \Gamma, \text{ connecting the endpoints of } \beta, v_1 = x \text{ and } v_m = y. \text{ Each edge } e_i = (v_i, v_{i+1}) \text{ is an arc of a bisector } J(s_\beta, \cdot), \text{ called an ordinary edge, or an arc on } \Gamma. \text{ For } i = 1: \text{ if } x \in J(s_\beta, s_\alpha), \text{ then } e_1 \subseteq J(s_\beta, s_\alpha); \text{ if } x \in \Gamma, \text{ then } e_1 \subseteq \Gamma. \text{ Given } v_i, \text{ vertex } v_{i+1} \text{ and edge } e_{i+1} \text{ are defined as follows (see Figure 16). Wlog we assume a clockwise ordering of } J(\beta).\]

1. If $e_i \subseteq J(s_\beta, s_\alpha)$, let $v_{i+1}$ be the other endpoint of the component $J(s_\beta, s_\alpha) \cap R(\alpha)$ incident to $v_i$. If $v_{i+1} \in J(s_\beta, \cdot) \cap J(s_\beta, s_\alpha)$, then $e_{i+1} \subseteq J(s_\beta, \cdot)$. If $v_{i+1} \in \Gamma$, then $e_{i+1} \subseteq \Gamma$. (In Figure 16 see $e_i = e', v_i = z, v_{i+1} = z'.)\]

2. If $e_i \subseteq \Gamma$, let $g$ be the $\Gamma$-arc incident to $v_i$. Let $e_{i+1} \subseteq J(s_\beta, s_\gamma)$, where $R(\gamma)$ is the first region, incident to $g$ clockwise from $v_i$, such that $J(s_\beta, s_\gamma)$ intersects $g \cap R(\gamma)$; let $v_{i+1}$ be this intersection point. (In Figure 16 see $v_i = v$ and $v_{i+1} = w$.)
A vertex $v$ along $J(\beta)$ is called valid if $v$ is a vertex in the arrangement $\mathcal{A}(\mathcal{J}_{s_3, S_3} \cup \Gamma)$ or $v$ is an endpoint of $\beta$. The following theorem shows that $J(\beta)$ is well defined, given $V_l(P)$, and that it forms an $s_3$-monotone path. We defer its proof to the end of this section.

**Theorem 18.** $J(\beta)$ is a unique $s_3$-monotone path in the arrangement of $s_3$-related bisectors $\mathcal{J}_{s_3, S_3} \cup \Gamma$ connecting the endpoints of $\beta$. $J(\beta)$ can contain at most one ordinary edge per region of $V_l(P)$, with the exception of $e_1$ and $e_{m-1}$, when $v_1$ and $v_m$ are incident to the same face in $V_l(P)$. $J(\beta)$ cannot intersect the interior of arc $\beta$.

We define $R(\beta)$ as the area enclosed by $\beta \cup J(\beta)$. Let $V_l(P) \oplus \beta$ be the subdivision of $D_{P, \beta}$ obtained by inserting $J(\beta)$ in $V_l(P)$ and deleting any portion of $V_l(P)$ enclosed by $J(\beta)$, $V_l(P) \oplus \beta = (V_l(P) \setminus R(\beta)) \cup J(\beta)) \cap D_{P, \beta}$. We prove that $V_l(P) \oplus \beta$ is a Voronoi-like diagram. To this goal we need an additional property of $J(\beta)$.

**Lemma 19.** If the insertion of $\beta$ splits an arc $\alpha \in P$ (Observation 17), then $J(\beta)$ also splits $R(\alpha)$ and $J(\beta) \not\subset R(\alpha)$. In no other case can $J(\beta)$ split a region $R(\alpha)$ in $V_l(P)$.

**Proof.** Suppose for the sake of contradiction that $\beta$ splits arc $\alpha$ and $J(\beta) \subset R(\alpha)$, as shown in Figure 17. Then $J(\beta) = J(s_\alpha, s_\beta) \cup R(\alpha)$ and the bisector $J(s_\alpha, s_\beta)$ together with the arc $\alpha$ form a forbidden $s_\alpha$-inverse cycle, deriving a contradiction to Lemma 2. Thus, $J(\beta)$ must intersect $\partial R(\alpha)$ in $V_l(P)$ and therefore $J(\beta) \not\subset R(\alpha)$. By Theorem 18, $J(\beta)$ can only enter some other region at most once. Thus, $J(\beta)$ cannot split any other region.

**Theorem 20.** $V_l(P) \oplus \beta$ is a Voronoi-like diagram for $P_\beta = P \oplus \beta$, denoted $V_l(P_\beta)$.

**Proof.** By Theorem 18, $R(\beta)$ fulfills the properties of a Voronoi-like region. Moreover, the updated boundary of any other region $R(\alpha)$ in $V_l(P)$, which is truncated by $J(\beta)$, remains an $s_\alpha$-monotone path. By Lemma 19, $J(\beta)$ cannot split a region $R(\alpha)$ in $V_l(P)$, and thus, it cannot create a face that is not incident to $\alpha$. Therefore, $V_l(P) \oplus \beta$ fulfills all properties of Definition 8.

The tracing of $J(\beta)$ within $V_l(P)$, given the endpoints of $\beta$, can be done similarly to any ordinary Voronoi diagram, see e.g., [11, 2] Ch. 7.5.3 for AVDs, or [21, Ch. 7.4] [19, Ch. 5.5.2.1] for concrete diagrams. For a Voronoi-like diagram this can be established due to the basic property of Lemma 12. In particular, when computing $J(\beta)$ and entering some region...
Figure 19 Scanning $\partial R(\gamma)$ from $v_i$ counterclockwise, Lemma 12 assures that $v_{i+1}$ is the first encountered intersection of $J(s_\beta, s_\gamma)$ with $\partial R(\gamma)$.

$R(\gamma)$ at a point $v_i$, we scan $\partial R(\gamma)$ counterclockwise for the first intersection with $J(s_\beta, s_\gamma)$ to determine $v_{i+1}$, see Figure 19. The first such intersection is indeed the endpoint of edge $e_i$. Lemma 12 assures that no other intersection of $J(s_\beta, s_\gamma)$ with $\partial R(\gamma)$, before $v_{i+1}$, is possible, because such an intersection would yield a component of $J(s_\beta, s_\gamma) \cap R(\gamma)$ with a labeling contradicting Lemma 12, see Figure 19.

Special care is required in cases (e), (f), and (g) of Observation 15, in order to identify the first edge of $J(\beta)$; in these cases, $\beta$ may not overlap with any feature of $\mathcal{V}_i(P)$, thus, a starting point for tracing $J(\beta)$ is not readily available. In case (g), we trace a portion of $\partial R(\beta)$, which does not get deleted afterwards, thus it adds to the time complexity of the operation $\mathcal{V}_i(P) \oplus \beta$ (see Lemma 21). In cases (f) and (g), we show that if no feature of $\mathcal{V}_i(P)$ overlaps $\beta$, then either there is a leaf of $\mathcal{V}_i(P)$ in the neighboring $\Gamma$-arc or $J(\beta) \subseteq \overline{R(\alpha)}$. In either case a starting point for $J(\beta)$ can be identified in $O(1)$ time. Notice, if $J(\beta) \subseteq \overline{R(\alpha)}$, then it consists of a single bisector $J(s_\beta, s_\gamma)$ (and one or two $\Gamma$-arcs).

The following lemma gives the time complexity to compute $J(\beta)$ and update $\mathcal{V}_i(\mathcal{P}_\beta)$. The statement of the lemma is an adaptation from [10], however, the proof contains cases that do not appear in a farthest segment Voronoi diagram. $|\cdot|$ denotes complexity.

Let $\mathcal{P}$ denote a finer version of $\mathcal{P}$, where a $\Gamma$-arc between two consecutive boundary arcs in $\mathcal{P}$ is partitioned into smaller $\Gamma$-arcs as defined by the incident faces of $\mathcal{V}_i(\mathcal{P})$. Since $|\mathcal{V}_i(P)|$ is $O(|\mathcal{P}|)$, $|\mathcal{P}|$ is also $O(|\mathcal{P}|)$.

Lemma 21. Let $\alpha$ and $\gamma$ be the first original arcs on $\mathcal{P}_\beta$ occurring before and after $\beta$. Let $d(\beta)$ be the number of arcs in $\mathcal{P}$ between $\alpha$ and $\gamma$ (both boundary and $\Gamma$-arcs). Given $\alpha$, $\gamma$, and $\mathcal{V}_i(\mathcal{P})$, in all cases of Observation 15, except (g), the merge curve $J(\beta)$ and the diagram $\mathcal{V}_i(\mathcal{P}_\beta)$ can be computed in time $O(|R(\beta)| + d(\beta))$. In case (g), where an arc is split and a new arc $\omega$ is created by the insertion of $\beta$, the time is $O(|\partial R(\beta)| + |\partial R(\omega)| + d(\beta))$.

Proof. First, to determine $\beta$ (i.e., to determine the endpoints of $\beta$ on $\mathcal{P} \oplus \beta^*$) we trace the arcs between $\alpha$ and $\gamma$ in $\mathcal{P}$ in time $O(d(\beta))$.

Let $T(\beta)$ denote the portion of $\mathcal{V}_i(\mathcal{P})$ enclosed between $J(\beta)$ and $\mathcal{P}$, which gets deleted in $\mathcal{V}_i(\mathcal{P}_\beta)$. $|T(\beta)|$ is a forest of complexity $O(|J(\beta)| + d(\beta))$ since the number of faces of $T(\beta)$ is proportional to the number of edges of $J(\beta)$ plus the number of auxiliary arcs that get deleted. To determine $J(\beta)$ we essentially trace $T(\beta)$ in time $O(|T(\beta)|)$, similarly to any ordinary Voronoi diagram. However, we first need to determine a leaf of $T(\beta)$.

A leaf of $T(\beta)$ is readily available (after determining $\beta$) in cases (a) and (b) of Observation 15. Thus, we can trace $J(\beta)$ in total time $O(|J(\beta)| + d(\beta))$.

In cases (f) and (g) of Observation 15, $T(\beta)$ may or may not have a leaf on $\mathcal{P}$. If it has a leaf, then it can be found in $O(1)$ time, because it either lies between $x$ and $y$ or it is incident to a neighboring $\Gamma$-arc of $\mathcal{P}$. If there is no leaf, then we show that $J(\beta)$ has length $m = 3$ or $m = 4$ and can be computed in $O(1)$ time. First observe that if $T(\beta)$ has no leaf on $\mathcal{P}$, $x,y$
Suppose both appear sequentially along P. We conclude that Lemma 23.

4.1 Proving Theorem 18

We first establish that J(β) cannot intersect arc β, other than its endpoints, using the following Lemma.

► Lemma 22. Given V1(P), for any arc α ∈ P, R(α) ⊆ D(s, sα).

Proof. The contrary would yield an sα-inverse cycle defined by J(s, sα) and ∂R(α).

Lemma 22 implies that bisector J(sβ, sα) cannot intersect J(s, sα) within any region R(α) of V1(P): if it did, J(s, sα) would also pass through the same point in R(α) contradicting that R(α) ⊆ D(s, sα). Thus J(β) cannot intersect arc β in its interior.

The following lemma is a property that is used in several proofs. It describes how a bisector J(s,·) can intersect P.

► Lemma 23. D(s,·) ∩ DP is always connected. Thus, any components of J(s,·) ∩ DP must appear sequentially along P.

Proof. If we assume the contrary we obtain an s-inverse cycle defined by J(s,·) and P.

To prove Theorem 18 we use a bi-directional induction on the vertices of J(β). Let Jx = (v1, v2, ..., vi), 1 ≤ i < m, be the subpath of J(β) starting at v1 = x up to vertex vi, including a small neighborhood of ei incident to vi, see Figure 18. Note that vertex vi uniquely determines ei, however, its other endpoint is not yet specified. Similarly, let Jy = (vm, vm−1, ..., vm−j+1), 1 ≤ j < m, denote the subpath of J(β), starting at vm up to
vertex \( v_{m-j+1} \), including a small neighborhood of edge \( e_{m-j} \). Recall that we refer to the edges of \( J(\beta) \) that are not \( \Gamma \)-arcs as ordinary. For any ordinary edge \( e_\ell \in J(\beta) \), let \( o_\ell \) denote the boundary arc that \( e_\ell \) induces on, i.e., \( e_\ell \subseteq J(s_{o_\ell}, s_{\beta}) \cap R(o_\ell) \).

**Induction hypothesis:** Suppose \( J_x^i \) and \( J_y^j \), \( i, j \geq 1 \), are disjoint \( s_\beta \)-monotone paths. Suppose further that each ordinary edge of \( J_x^i \) and of \( J_y^j \) passes through a distinct region of \( V_l(P) \): \( o_\ell \) is distinct for \( \ell, 1 \leq \ell \leq i \) and \( m-j \leq \ell < m \), except possibly \( o_i = o_{m-j} \) and \( o_1 = o_{m-1} \).

**Induction step:** Assuming that \( i + j < m \), we prove that at least one of \( J_x^i \) or \( J_y^j \) can respectively grow to \( J_x^{i+1} \) or \( J_y^{j+1} \) at a valid vertex (Lemmas 24, 25), and it enters a new region of \( V_l(P) \) that has not been visited so far (Lemma 26). A finish condition when \( i + j = m \) is given in Lemma 27. The base case for \( i = j = 1 \) is trivially true.

Suppose that \( e_i \subseteq J(s_{o_i}, s_\beta) \) and \( v_i \in \partial R(o_1) \). To show that \( v_{i+1} \) is a valid vertex it is enough to show that (1) \( v_{i+1} \) cannot be on \( \alpha_i \), and (2) if \( v_1 \) is on a \( \Gamma \)-arc then \( v_{i+1} \) can be determined on the same \( \Gamma \)-arc. However, we cannot easily derive these conclusions directly. Instead we show that if \( v_{i+1} \) is not valid then \( v_{m-j} \) will have to be valid.

In the following lemmas we assume that the induction hypothesis holds.

**Lemma 24.** Suppose \( e_i \subseteq J(s_{o_i}, s_\beta) \) but \( v_{i+1} \notin \alpha_i \), i.e., it is not a valid vertex because \( e_i \) hits \( \alpha_i \). Then vertex \( v_{m-j} \) must be a valid vertex in \( A(J_{s_\beta}, s_{m-j}) \), and \( v_{m-j} \) can not be on \( P \).

**Proof.** Suppose vertex \( v_{i+1} \) of \( e_i \) lies on \( \alpha_i \) as shown in Figure 22(a). Vertex \( v_{i+1} \) is the intersection point of related bisectors \( J(s_{o_i}, s_\beta) \) and \( J(s_{o_i}, s_{\beta}) \) and thus also of \( J(s_{o_i}, s_\beta) \). Thus, \( v_1, v_m, v_{i+1} \in J(s, s_\beta) \). First note that vertices \( v_1, v_{i+1}, v_m \) appear on \( P \) in clockwise order, because \( J_1^{i+1} \) cannot intersect \( \beta \). Observe that arc \( \beta \) partitions \( J(s, s_\beta) \) in two parts: \( J_1 \) incident to \( v_1 \) and \( J_2 \) incident to \( v_m \). We claim that \( v_{i+1} \) lies on \( J_2 \). Suppose otherwise, i.e., \( v_{i+1} \) lies on \( J_1 \), then \( J_2^{i+1} \) and \( J_1 \) form a cycle. Since \( J_2^{i+1} \subseteq D_\beta \), Lemma 23 implies that this cycle must be a forbidden \( s_\beta \)-inverse cycle, see the dashed black and the green solid curve in Figure 22(a), contradicting Lemma 2. Thus, \( v_{i+1} \) lies on \( J_2 \). Further, by Lemma 23 the components of \( J_2 \cap D_\beta \) appear on \( P \) clockwise after \( v_{i+1} \) and before \( v_m \), as shown in Figure 22(b) illustrating \( J(s, s_\beta) \) as a black dashed curve.

Now consider \( J_y^j \). We show that \( v_{m-j} \) cannot be on \( P \). First observe that \( v_{m-j} \) can not lie on \( P \), clockwise after \( v_m \) and before \( v_1 \), since \( J_y^{j+1} \) cannot cross \( \beta \). Now we prove that \( v_{m-j} \) cannot lie on \( P \) clockwise after \( v_1 \) and before \( v_{i+1} \). To see that, note that edge \( e_{m-j} \) cannot cross any non-\( \Gamma \) edge of \( J_x^{i+1} \), because by the induction hypothesis, \( o_{m-j} \) is distinct from all \( o_\ell, \ell \leq i \). In addition, by the definition of a \( \Gamma \)-arc, \( v_{m-j} \) cannot lie on any \( \Gamma \)-arc of \( J_x^i \). Finally, we show that \( v_{m-j} \) cannot lie on \( P \) clockwise after \( v_{i+1} \) and before \( v_m \). If \( v_{m-j} \) lay on the boundary arc \( o_{m-j} \) then we would have \( v_{m-j} \in J(s, s_\beta) \). This would define an \( s_\beta \)-inverse cycle \( C_\beta \), formed by \( J_y^{j+1} \) and \( J(s, s_\beta) \), see Figure 22(b). If \( v_{m-j} \) lay on a \( \Gamma \)-arc then there would also be a forbidden \( s_\beta \)-inverse cycle formed by \( J_y^{j+1} \) and \( J(s, s_\beta) \) because
The assumption that \( v_i \in \Gamma \) and \( v_{i+1} \) of the merge curve \( J'_j \) cannot be determined as in Lemma 25.

in order to reach \( \Gamma \) edge \( e \), must cross \( J(s, s_\beta) \). See the dashed black and the green curve in Figure 23(c). Thus \( v_{m-j} \notin \mathcal{P} \).

Since \( v_{m-j} \in \partial R(\alpha_{i+1}) \) but \( v_{m-j} \notin \mathcal{P} \), it must be a vertex of \( \mathcal{A}(J_{s_\beta}, \mathcal{S}_\mathcal{P}) \).

The proof for the following lemma is similar.

Lemma 25. Suppose vertex \( v_i \) is on a \( \Gamma \)-arc \( g \) but \( v_{i+1} \) cannot be determined because no bisector \( J(s_i, s_j) \) intersects \( R(\gamma) \cap g \), clockwise from \( v_i \). Then vertex \( v_{m-j} \) must be a valid vertex in \( \mathcal{A}(J_{s_\beta}, \mathcal{S}_\mathcal{P}) \) and \( v_{m-j} \) can not be on \( \mathcal{P} \).

Proof. We truncate the \( \Gamma \)-arc \( g \) to its portion clockwise from \( v_i \), let \( w \) be the endpoint of \( g \) clockwise from \( v_i \), see Figure 23(a). If no \( J(s_\beta, s_i) \cap \partial R(\gamma) \) intersects \( g \), as we assume in this lemma, then \( \partial R(\gamma) \cap g \subseteq D(s_\beta, s_i) \), for any region \( R(\gamma) \) incident to \( g \). Thus, \( w \in D(s_\beta, s) \). However, \( v_i \in D(s, s_\beta) \), since, by Lemma 22, \( R(\alpha_{i-1}) \subseteq D(s, s_{\alpha_{i-1}}) \) and \( v_i \) is incident to \( J(s_\beta, s_{\alpha_{i-1}}) \). Thus, \( J(s, s_\beta) \) must intersect \( g \) at some point \( z \) clockwise from \( v_i \).

Arc \( \beta \) partitions \( J(s, s_\beta) \) in two parts: \( J_1 \) incident to \( v_1 \) and \( J_2 \) incident to \( v_m \). Lemma 24 implies that all components of \( J_2 \cap \mathcal{D}_\mathcal{P} \) appear on \( \mathcal{P} \) clockwise after \( v_i \) and before \( v_m \), as shown by the black dashed curve in Figure 23(a); also \( z \) lies on \( J_2 \).

Now we can show that vertex \( v_{m-j} \) of \( J^y_\beta \) cannot be on \( \mathcal{P} \) analogously to the proof of Lemma 23. The only difference is that we must additionally show that \( v_{m-j} \) cannot lie on \( \mathcal{P} \) clockwise after \( v_i \) and before \( w \). But this holds already by the assumption of this lemma. Refer to Figures 23(b) and (c).

We conclude that \( v_{m-j} \) cannot lie on \( \mathcal{P} \) and it is a valid vertex of \( \mathcal{A}(J_{s_\beta}, \mathcal{S}_\mathcal{P}) \).

Lemma 26. Let \( \alpha \in \mathcal{P} \) but \( \alpha \notin \mathcal{P}_\beta \). Then \( \partial R(\alpha) \subseteq D(s_\beta, s_\alpha) \).

Proof. By Lemma 22 it holds \( R(\alpha) \subseteq D(s_\beta, s_\alpha) \). Let \( R_\alpha = R(\alpha) \cap D(s_\beta, s_\beta) \) and \( R_\beta = R(\alpha) \cap D(s_\beta, s) \). By transitivity of dominance regions we have \( R_\beta \subseteq D(s_\beta, s_\alpha) \). By Lemma 23 \( R_\alpha \) is not incident to \( \alpha \). Thus if \( J(s_\beta, s_\alpha) \) intersected \( R_\alpha \) then it would create a forbidden \( s_\alpha \)-cycle \( C \) and contradict Lemma 11 see the dashed gray line in Figure 24. This implies that also \( R_\alpha \subseteq D(s_\beta, s_\alpha) \). Thus \( R(\alpha) = R_\alpha \cup R_\beta \subseteq D(s_\beta, s_\alpha) \) which implies \( \partial R(\alpha) \subseteq D(s_\beta, s_\alpha) \).
Lemma 27. Suppose \( i + j > 2 \) and either (1) or (2) holds: (1) \( \alpha_i = \alpha_{m-j} \), i.e., \( v_i \) and \( v_{m-j+1} \) are incident to a common region \( R(\alpha_i) \) and \( e_i, e_{m-j} \subseteq J(s_\beta, s_\alpha) \); or (2) \( v_i \) and \( v_{m-j+1} \) are on a common \( \Gamma \)-arc \( g \) of \( \mathcal{P} \) and \( e_i, e_{m-j} \subseteq \Gamma \). Then \( v_{i+1} = v_{m-j+1}, \ v_{m-j} = v_i, \) and \( m = i+j \).

Proof. Let \( \alpha = \alpha_i \). Suppose (1) holds, then \( e_i, e_{m-j} \subseteq J(s_\beta, s_\alpha), \) see Figure 25(a). The boundary \( \partial R(\alpha_i) \) is partitioned in four parts, using a counterclockwise traversal starting at \( \alpha_i \); \( \partial R_1 \), from the endpoint of arc \( \alpha_i \) to \( v_i \); \( \partial R_2 \), from \( v_i \) to \( v_{m-j+1} \); \( \partial R_3 \), from \( v_{m-j+1} \) to the next endpoint of \( \alpha_i \); and \( \partial R_4 \). We show that \( e_i \) and \( e_{m-j} \) cannot hit any of these parts; thus, \( e_i = e_{m-j} \).

1. Edge \( e_i \) cannot hit \( \partial R_1 \) and edge \( e_{m-j} \) cannot hit \( \partial R_3 \) by the basic property of Lemma 12.
2. We prove that edge \( e_i \) cannot hit \( \partial R_2 \). Analogously for edge \( e_{m-j} \). Let \( \rho \) be any edge on \( \partial R_2 \). (If \( v_i \in \rho \) or \( v_{m-j+1} \in \rho \), assume that \( \rho \) is truncated with endpoint \( v_i \) or \( v_{m-j+1} \) respectively).
   a. Suppose that \( \rho \) is an ordinary edge, \( \rho \subseteq J(s_\beta, s_\alpha) \), see Figure 25(a). Then at least one of \( J_1, J_2 \), or \( \beta \) must pass through \( R(\gamma) \). Suppose that \( J_1 \) does, as shown in Figure 25(a). Then by the basic property of Lemma 12 (or by Lemma 13 if this is the first edge of \( J_1 \)), \( \rho \subseteq D(s_\beta, s_\gamma) \). By transitivity (Lemma 3), it also holds that \( \rho \subseteq D(s_\gamma, s_\alpha) \). Thus, \( e_i \) cannot hit \( \rho \). Symmetrically for \( J_2 \). If only \( \beta \) passes through \( R(\gamma) \), then we can use Lemma 26 to derive that \( \rho \not\subseteq D(s_\beta, s_\gamma) \); the rest follows.
   b. Suppose that \( \rho \not\subseteq \Gamma \). Then either \( \rho \) itself is part of an edge of \( J_1 \) or of \( J_2 \), or \( \beta \) passes through \( R(\alpha) \) and \( \rho \) is at opposite side of it than \( \alpha \). In the former case, \( \rho \subseteq D(s_\beta, s_\alpha) \) by the definition of a \( \Gamma \)-edge in the merge curve. In the latter case, the same is derived by Lemma 22 and transitivity (Lemma 3). Thus, \( e_i \) cannot hit \( \rho \).
3. Edge \( e_i \) (resp. \( e_{m-j} \)) cannot hit \( \partial R_3 \) because if it did, \( e_i \) and \( e_{m-j} \) would not appear sequentially on \( R(\alpha) \) contradicting Lemma 12.
4. It remains to show that \( e_i \) and \( e_{m-j} \) cannot both hit \( \alpha_i \). But this is already shown in Lemma 23.

Now suppose (2) holds, see Figure 25(b). Let \( R(\gamma) \) be a region in \( \mathcal{V}_1(\mathcal{P}) \) incident to \( g \) at a \( \Gamma \)-arc \( \rho \subseteq g \), between \( v_i \) and \( v_{m-j+1} \). At least one of \( J_1 \) or \( J_2 \) or \( \beta \) must pass through \( R(\gamma) \). By the exact same arguments as before, \( \rho \subseteq D(s_\beta, s_\gamma) \). We infer that there is no bisector \( J(s_\beta, s_\gamma) \) in \( R(\gamma) \), for any region \( R(\gamma) \) incident to \( g \) between \( v_i \) and \( v_{m-j+1} \). Thus, \( v_{i+1} = v_{m-j+1}, \ v_{m-j} = v_i, \) and \( m = i+j \). \( J(\beta) \) is the concatenation of \( J_1 \) and \( J_2 \) with \( e_{i+1} = e_{m-j+1} \). \( \blacksquare \)
\[ \partial R_\ell (\alpha_\ell) \]

**Lemma 28.** Suppose vertex \( v_{i+1} \) is valid and \( e_{i+1} \subseteq J(s_\beta, s_{\alpha_{i+1}}) \). Then \( R(\alpha_{i+1}) \) has not been visited by \( J_x^j \) nor \( J_y^j \), i.e., \( \alpha_{i+1} \neq \alpha_\ell \) for \( \ell \leq i \) and for \( m - j < \ell \).

**Proof.** Let \( e_k, k \leq i \), be an ordinary edge of \( J_x^j \). Denote by \( \partial R_\ell(k) \) the portion of \( \partial R(\alpha_\ell) \) from \( \alpha_\ell \) to \( v_k \) in a counterclockwise traversal, see the bold red part \( \partial R_1^1 \) in Figure 26. Analogously for an ordinary edge \( e_{m-j} \) of \( J_y^1 \), where \( \partial R_{m-j}^1 \) is defined in a clockwise traversal of \( \partial R(\alpha_{m-j}) \).

Recall that \( \partial R_{e_k}(\alpha_\ell) \), denotes the portion of \( \partial R(\alpha_\ell) \) cut out by edge \( e_k \), at opposite side from \( \alpha_\ell \).

The basic property of Lemma 12 implies that \( v_{i+1} \) cannot be on \( \partial R_{e_k}(\alpha_\ell) \) for any \( \ell, \ell < i \) and \( m - j < \ell \) and that \( v_{i+1} \) cannot be on \( \partial R_\ell(k) \). This implies that \( v_{i+1} \) cannot be on \( \partial R_\ell(k) \) for any \( \ell < i \), because we have a plane graph in \( D_P \) and by its layout \( \partial R_\ell(k) \) is not reachable from \( e_i \) without first hitting \( \partial R_{e_k}(\alpha_\ell) \) or \( \partial R_\ell(k) \). See Figure 26. Thus, \( v_{i+1} \) can not be on \( \partial R(\alpha_\ell) \), \( \ell < i \). By Lemma 27, \( v_{i+1} \) cannot be on \( \partial R_{m-j}(k) \). This implies, again by the layout, that \( v_{i+1} \) cannot be on \( \partial R_\ell(k) \) for all \( \ell > m - j \). Thus, \( v_{i+1} \) can not be on \( \partial R(\alpha_\ell) \), for any \( \ell > m - j \). This implies that \( \alpha_{i+1} \neq \alpha_\ell \), for any \( \ell, \ell \leq i \) or \( \ell > m - j \). \( \blacksquare \)

By Lemma 28, \( J_x^{i+1} \) and \( J_y^{j+1} \) always enter a new region of \( V_l(P) \) that has not been visited yet; thus, conditions (1) or (2) of Lemma 27 must be fulfilled at some point of the induction. Hence, the proof of Theorem 18 is complete. Completing the induction establishes also that the conditions of Lemmas 24 and 25 can never be met, thus, no vertex of \( J(\beta) \), except its endpoints, can be on a boundary arc of \( P \).

5 \( V_l(P) \) is unique

In this section we establish that \( V_l(P) \) is unique, proving Theorem 14 from Section 5. Notation \( \beta \) refers to some arc that is not included in \( P \).

\[ \triangleright \text{Lemma 29.} \] Suppose there is a non-empty component \( e \) of \( J(s_\alpha, \cdot) \) intersecting \( R(\alpha) \) in \( V_l(P) \). Then \( J(s, \cdot) \) must also intersect \( D_P \). Further, there exists a component of \( J(s, \cdot) \cap D_P \), denoted as \( \beta \), such that the merge curve \( J(\beta) \) in \( V_l(P) \) contains \( e \).

We say that boundary arc \( \beta \) is missing from \( P \).

**Proof.** Suppose \( J(s, s_\beta) \cap D_P = \emptyset \), however, there is a non-empty component \( e \) of \( J(s_\alpha, s_\beta) \cap R(\alpha) \). Since \( J(s, \beta) \cap D_P = \emptyset \), for any arc \( \chi \in P, \chi \subseteq D(s_\chi, s_\beta) \). Let \( \partial R_e \) denote the portion of \( \partial R(\alpha) \) cut out by \( e \) (at opposite side from \( \alpha \)). By Lemma 12, \( \partial R_e \subseteq D(s_\beta, s_\alpha) \). Consider an endpoint \( v \) of \( e \). We distinguish two cases for \( v \):

1. If \( v \) is on an edge \( \rho \) of \( \partial R_e \), incident to a region \( R(\gamma) \), then \( J(s_\beta, s_\gamma) \) intersects \( R(\gamma) \) by an edge \( e_\rho \), incident to \( v \), leaving \( \rho \) and \( \gamma \) at opposite sides, because \( \gamma \subseteq D(s_\gamma, s_\beta) \). See Figure 27.
2. If $v$ is on a $\Gamma$-arc $g$, let $R(\gamma)$ be the first region after $v$ (towards $D(s_{\beta}, s_{\alpha})$) with $J(s_{\beta}, s_{\alpha})$ intersecting $g \cap \overline{R(\gamma)}$ at point $u$ (see Figure 28). There exists such $R(\gamma)$ because for all boundary arcs $\chi \in \mathcal{P}$, $\chi \subseteq D(s_{\gamma}, s_{\beta})$, and this includes the boundary arc that is incident to $g$. Let $e_g$ be the component of $J(s_{\beta}, s_{\alpha}) \cap R(\gamma)$ incident to $u$.

Thus, given $e$ and $v$, we derive an edge $e'$, either $e' = e_\rho$ or $e' = e_g$, with the same properties as $e$, in another region of $V_i(\mathcal{P})$. This process repeats and there is no way to break it because for any arc $\chi \in \mathcal{P}$, $\chi \subseteq D(s_{\gamma}, s_{\beta})$. Thus, we create a closed curve on $V_i(\mathcal{P})$ consisting of consecutive pieces of $J(s_{\beta}, s_{\alpha})$, possibly interleaved with $\Gamma$-arcs, which has the label $s_{\beta}$ in its interior. No two edges of this curve can intersect because otherwise the bisector corresponding to such intersecting edges would not be a Jordan curve. Furthermore, by their definition, no two $\Gamma$-arcs of the curve can intersect. By our general position assumption that no three $s_{\beta}$-related bisectors can intersect at the same point, no vertex of this curve can repeat. Thus, the closed curve must be an $s_{\beta}$-cycle $C$ as shown in Figure 27. But $C$ is contained in $D_P$, contradicting Lemma 11.

Thus, our assumption was wrong and there must exist some arc $\chi_0 \in \mathcal{P}$ such that $\chi_0 \notin D(s_{\gamma}, s_{\beta})$.

Let $\chi_0$ be the first such arc encountered in the process described above. Since $\chi_0 \notin D(s_{\gamma}, s_{\beta})$ there is a component $\beta$ of $J(s, s_{\beta}) \cap D_P$, incident to $\chi_0$, see Figure 29. Let $J(\beta)$ denote the sequence of edges $e_\rho$ starting with the initial edge $e$ and ending on the arc $\chi_0$ incident to $\beta$. Observe that from the vertex incident to $\beta$ until the edge $e$, the path $J(\beta)$ fulfills the definition of the merge curve $J(\beta)$ (Definition 17). Since by Theorem 18 the merge curve $J(\beta)$ on $V_i(\mathcal{P})$ is unique, it follows that $J(\beta)$ includes $J(\beta)$, and thus it includes edge $e$.

We can now prove Theorem 14 from Section 3.

\begin{theorem}
Given a boundary curve $\mathcal{P}$ of $S' \subseteq S$, $V_i(\mathcal{P})$ (if it exists) is unique.
\end{theorem}

\begin{proof}
Let $\mathcal{P}$ be a boundary curve for $S' \subseteq S$ such that $\mathcal{P}$ admits a Voronoi-like diagram $V_i(\mathcal{P})$. Suppose there exist two different Voronoi-like diagrams of $\mathcal{P}$, $V_j^1 \neq V_j^2$. Then there must be an edge $e^{(1)}$ of $V_j^1$ bounding regions $R^{(1)}(\alpha)$ and $R^{(1)}(\beta)$ of $V_j^1$, where $\alpha, \beta \in \mathcal{P}$, such that $e^{(1)}$ intersects region $R^{(2)}(\alpha)$ of $V_j^2$, since $\alpha$ is common to both $R^{(1)}(\alpha)$ and $R^{(2)}(\alpha)$.

Let edge $e \subseteq J(s_{\beta}, s_{\alpha})$ be the component of $R^{(2)}(\alpha) \cap J(s_{\beta}, s_{\alpha})$ overlapping with $e^{(1)}$, see Figure 29. From Lemma 29 it follows that there is a non-empty component $\beta_0$ of $J(s_{\beta}, s_{\alpha}) \cap D_P$ such that $J(\beta_0)$ on $V_j^2$ contains edge $e$. Since $J(\beta_0)$ and $\partial R^{(1)}(\beta)$ have an overlapping portion ($e \cap e^{(1)}$) and they bound the regions of two different arcs $\beta_0 \neq \beta$ of site $s_{\beta}$, they form an $s_{\beta}$-cycle $C$ as shown in Figure 29. But $C$ is contained in $D_P$, deriving a contradiction to Lemma 11.
\end{proof}
6 A randomized incremental algorithm

Consider a random permutation of the set of arcs \( S, o = (\alpha_1, \ldots, \alpha_h) \). For \( 1 \leq i \leq h \) define \( S_i = \{\alpha_1, \ldots, \alpha_i\} \subseteq S \) to be the subset of the first \( i \) arcs in \( o \). Given \( S_i \), let \( \mathcal{P}_i \) denote a boundary curve for \( S_i \), which induces a domain \( D_i = D_{\mathcal{P}_i} \).

The randomized algorithm is inspired by the randomized, two-phase, approach of Chew [4] for the Voronoi diagram of points in convex position; however, it constructs Voronoi-like diagrams of boundary curves \( \mathcal{P}_i \) within a series of shrinking domains \( D_i \supseteq D_{i+1} \). The boundary curves are obtained by the insertion operation, starting with \( J(s, s, o) \), thus, they always admit a Voronoi-like diagram. In phase 1, the arcs in \( S \) get deleted one by one in reverse order of \( o \), while recording the neighbors of each deleted arc at the time of its deletion. Let \( \mathcal{P}_1 = \partial(D(s, s, o) \cap D_T) \) and \( D_1 = D(s, s, o) \cap D_T \). Let \( R(o_1) = D_1 \). \( V_i(\mathcal{P}_i) = \emptyset \) is the Voronoi-like diagram for \( \mathcal{P}_i \).

In phase 2, we start with \( V_i(\mathcal{P}_i) \) and incrementally compute \( V_i(\mathcal{P}_{i+1}), i = 1, \ldots, h-1 \), by inserting arc \( \alpha_{i+1} \) in \( V_i(\mathcal{P}_i) \), where \( \mathcal{P}_{i+1} = \mathcal{P}_i \oplus \alpha_{i+1} \) and \( V_i(\mathcal{P}_{i+1}) = V_i(\mathcal{P}_i) \cup \alpha_{i+1} \). At the end we obtain \( V_i(\mathcal{P}_h) \), where \( \mathcal{P}_h = S \).

We have already established that \( V_i(S) = V(S) \) (Corollary 10), and \( \mathcal{P}_h = S \), thus, the algorithm is correct. Given the analysis and the properties of Voronoi-like diagrams established in Sections 3 and 4, as well as Lemma 21, the time analysis becomes similar to the one for the farthest-segment Voronoi diagram [10].

\textbf{Lemma 30.} \( \mathcal{P}_i \) contains at most \( 2i \) arcs; thus, the complexity of \( V_i(\mathcal{P}_i) \) is \( O(i) \).

\textbf{Proof.} \( |\mathcal{P}_i| = 2 \). At each step of phase 2, one original arc is inserted and at most one additional arc is created by a split. Thus, the total number of arcs in \( \mathcal{P}_i \) is at most \( 2i \). The complexity of \( V_i(\mathcal{P}_i) \) is \( O(|\mathcal{P}_i|) \), thus, it is \( O(i) \). \( \blacksquare \)

\textbf{Lemma 31.} The expected number of arcs in \( \tilde{\mathcal{P}}_i \) (auxiliary boundary arcs and fine \( \Gamma \)-arcs) that are visited while inserting \( \alpha_{i+1} \) is \( O(1) \).

\textbf{Proof.} To insert arc \( \alpha_{i+1} \) at one step of phase 2, we may trace a number of arcs in \( \tilde{\mathcal{P}}_i \) that may be auxiliary arcs and/or fine \( \Gamma \)-arcs between the pair of consecutive original arcs that has been stored with \( \alpha_{i+1} \) in phase 1. Since every element of \( S_{i+1} \) is equally likely to be \( \alpha_{i+1} \), each pair of consecutive original arcs in \( \mathcal{P}_{i+1} \) has probability \( 1/i \) to be considered at step \( i \). Let \( n_j \) be the number of arcs between the \( j \)th pair of original arcs in \( \tilde{\mathcal{P}}_i \), \( 1 \leq j \leq i \); \( \sum_{j=1}^i n_j = |\tilde{\mathcal{P}}_i| \) which is \( O(i) \). The expected number of arcs that are traced is then \( \sum_{j=1}^i n_j/i \in O(1) \). \( \blacksquare \)

Using the same backwards analysis as in [10], we conclude with the following theorem. We include its proof for completeness.

\textbf{Theorem 32.} Given an abstract Voronoi diagram \( V(S) \), \( V(S \setminus \{s\}) \cap VR(s, S) \) can be computed in expected \( O(h) \) time, where \( h \) is the complexity of \( \partial VR(s, S) \). Thus, \( V(S \setminus \{s\}) \) can also be computed in expected time \( O(h) \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure30.png}
\caption{Illustrations for Theorem 14}
\end{figure}
Proof. We use backwards analysis, going from $V_l(P_{i+1})$ to $V_l(P_i)$. By Lemma 21 inserting $\alpha_{i+1}$ in $V_l(P_i)$ takes $O(|\partial R(\alpha_{i+1})| + |\partial R(\omega)| + d(\alpha_{i+1}))$ time, where $d(\alpha_{i+1})$ is the number of arcs in $P_i$ (auxiliary boundary arcs and fine $\Gamma$-arcs) between the two core arcs that have been stored with $\alpha_{i+1}$ in phase 1. By Lemma 31 the expected number $d(\alpha_{i+1})$ of these visited arcs is constant. The addition $|\partial R(\omega)|$ reflects the complexity of the neighboring region of $\alpha_{i+1}$, in case $\omega$ was created because $\alpha_{i+1}$ split an arc when inserted in $P_i$ (case (c) of Observation 15). The latter addition to the complexity represents a difference from the corresponding argument in the case of points. Since $o$ is a random permutation of the arcs in $S$, the expected time complexity of inserting $\alpha_{i+1}$ in $V_l(P_i)$ is equivalent to the expected complexity of a randomly selected region in $V_l(P_i)$, plus the expected complexity of its immediate neighbor. Since $V_l(P_i)$ has size $O(i)$ and it consists of $O(i)$ faces, the expected complexity of a randomly selected region is constant. The same holds for one neighbor of the randomly selected region. Thus, the expected time spent to insert $R(\alpha_{i+1})$ in $V_l(P_i)$ is constant. Since the total number of arcs is $h$, the claim follows.

Concluding remarks

Updating an abstract Voronoi diagram, after deletion of one site, in deterministic linear time remains an open problem. We plan to investigate the applicability of Voronoi-like diagrams in the linear-time framework of Aggarwal et al. in subsequent research.

The algorithms and the results in this paper (Theorem 32) are also applicable to concrete Voronoi diagrams of line segments and planar straight-line graphs (including simple and non-simple polygons) even though they do not strictly fall under the AVD model unless segments are disjoint. For intersecting line segments, $\partial VR(s, S)$ is a Davenport-Schinzel sequence of order 4 but this does not affect the complexity of the algorithm, which remains linear.

Examples of concrete diagrams that fall under the AVD umbrella and thus can benefit from our approach include: disjoint line segments and disjoint convex polygons of constant size in the $L_p$ norms, or under the Hausdorff metric; point sites in any convex distance metric or the Karlsruhe metric; additively weighted points that have non-enclosing circles; power diagrams with non-enclosing circles.
References

1. A. Aggarwal, L. Guibas, J. Saxe, and P. Shor. A linear-time algorithm for computing the Voronoi diagram of a convex polygon. *Discrete & Computational Geometry*, 4:591–604, 1989.

2. F. Aurenhammer, R. Klein, and D.-T. Lee. *Voronoi Diagrams and Delaunay Triangulations*. World Scientific, 2013.

3. C. Bohler, P. Cheilaris, R. Klein, C. H. Liu, E. Papadopoulou, and M. Zavershynskyi. On the complexity of higher order abstract Voronoi diagrams. *Computational Geometry: Theory and Applications*, 48(8):539–551, 2015.

4. C. Bohler, R. Klein, and C. Liu. Forest-like abstract Voronoi diagrams in linear time. In *Proc. 26th Canadian Conference on Computational Geometry (CCCG)*, 2014.

5. C. Bohler, R. Klein, and C.-H. Liu. An Efficient Randomized Algorithm for Higher-Order Abstract Voronoi Diagrams. In *32nd International Symposium on Computational Geometry (SoCG)*, volume 51, pages 21:1–21:15, Dagstuhl, Germany, 2016.

6. C. Bohler, C. H. Liu, E. Papadopoulou, and M. Zavershynskyi. A randomized divide and conquer algorithm for higher-order abstract Voronoi diagrams. *Computational Geometry: Theory and Applications*, 59(C):26–38, 2016.

7. L. P. Chew. Building Voronoi diagrams for convex polygons in linear expected time. Technical report, Dartmouth College, Hanover, USA, 1990.

8. F. Chin, J. Snoeyink, and C. A. Wang. Finding the medial axis of a simple polygon in linear time. *Discrete & Computational Geometry*, 21(3):405–420, 1999.

9. M. de Berg, O. Schwarzkopf, M. van Kreveld, and M. Overmars. *Computational Geometry: Algorithms and Applications*. Springer-Verlag, 2nd edition, 2000.

10. E. Khramtcova and E. Papadopoulou. An expected linear-time algorithm for the farthest-segment Voronoi diagram. arXiv:1411.2816v3 [cs.CG], 2017. Preliminary version in *Proc. 26th Int. Symp. on Algorithms and Computation (ISAAC)*, LNCS 9472, 404–414, 2015.

11. R. Klein. *Concrete and Abstract Voronoi Diagrams*, volume 400 of *Lecture Notes in Computer Science*. Springer-Verlag, 1989.

12. R. Klein, E. Langetepe, and Z. Nilforoushan. Abstract Voronoi diagrams revisited. *Computational Geometry: Theory and Applications*, 42(9):885–902, 2009.

13. R. Klein and A. Lingas. Hamiltonian abstract Voronoi diagrams in linear time. In *Algorithms and Computation, 5th International Symposium, (ISAAC)*, volume 834 of *Lecture Notes in Computer Science*, pages 11–19, 1994.

14. R. Klein, K. Mehlhorn, and S. Meiser. Randomized incremental construction of abstract Voronoi diagrams. *Computational geometry: Theory and Applications*, 3:157–184, 1993.

15. K. Mehlhorn, S. Meiser, and R. Rasch. Further site abstract Voronoi diagrams. *International Journal of Computational Geometry and Applications*, 11(6):583–616, 2001.

16. A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*. John Wiley, second edition, 2000.

17. E. Papadopoulou and S. K. Dey. On the farthest line-segment Voronoi diagram. *International Journal of Computational Geometry and Applications*, 23(6):443–459, 2013.

18. E. Papadopoulou and M. Zavershynskyi. The higher-order Voronoi diagram of line segments. *Algorithmica*, 74(1):415–439, 2016.

19. S. M. Preparata F.P. *Computational Geometry*. Texts and Monographs in Computer Science. Springer, New York, NY, 1985.

20. M. Sharir and P. K. Agarwal. *Davenport-Schinzel sequences and their geometric applications*. Cambridge university press, 1995.
A The farthest abstract Voronoi diagram

The farthest Voronoi region of a site \( p \in S \) is \( \text{FVR}(p, S) = \bigcap_{q \in S \setminus \{p\}} D(q, p) \) and the farthest abstract Voronoi diagram of \( S \) is \( \text{FVD}(S) = R^2 \setminus \bigcup_{p \in S} \text{FVR}(p, S) \). \( \text{FVD}(S) \) is a tree of complexity \( O(n) \), however, regions may be disconnected and a farthest Voronoi region may consist of \( \Theta(n) \) disjoint faces \[15\]. Let \( D^*(p, q) = D(q, p) \); then \( \text{FVR}(p, S) = \bigcap_{q \in S \setminus \{p\}} D^*(p, q) \).

Unless otherwise noted, we adopt the following convention: we reverse the labels of bisectors and use \( D^*(\cdot, \cdot) \) in the place of \( D(\cdot, \cdot) \) in most definitions and constructs of Sections \[3\] and \[4\]. Under this convention the definition of an e.g., \( p \)-monotone path remains the same but uses \( \partial \text{FVR}(p, \cdot) \) in the place of \( \partial \text{VR}(p, \cdot) \). The corresponding arrangement of \( p \)-related bisectors \( J_{p, S} \), \( S' \subseteq S \), is considered with the labels of bisectors and their dominance regions reversed from the original system \( J \).

Consider the enclosing curve \( \Gamma \) as defined in Section \[2\] and let \( S \) be the sequence of arcs on \( \Gamma \) derived by \( \Gamma \cap \text{FVD}(S) \). \( S \) represents the sequence of farthest Voronoi faces at infinity. The domain of computation is \( D_{\Gamma} \). For an arc \( \alpha \) of \( S \) let \( s_\alpha \) denote the site in \( S \) for which \( \alpha \subseteq \text{FVR}(s_\alpha, S) \). With respect to site occurrences, \( S \) is a Davenport-Schinzel sequence of order 2. \( S \) can be computed in time \( O(n \log n) \) in a divide and conquer fashion, similarly to computing the hull of a farthest segment Voronoi diagram, see e.g., \[17\].

We treat the arcs in \( S \) as sites and compute \( \mathcal{V}(S) = \text{FVD}(S) \cap D_{\Gamma} \). Let \( \text{VR}(\alpha, S) \) denote the face of \( \text{FVD}(S) \cap D_{\Gamma} \) incident to \( \alpha \in S \), see Figure \[31\]. \( \mathcal{V}(S) \) is a tree whose leaves are the endpoints of the arcs in \( S \).

For \( S' \subseteq S \), let \( S' \subseteq S \setminus \{s\} \) be the set of sites that define the arcs in \( S' \). Let \( \mathcal{J}(S') = \{J(p, q) \in \mathcal{J} \mid p, q \in S', p \neq q\} \).

"Definition 33. A boundary curve \( P \) for \( S' \) is a partitioning of \( \Gamma \) into arcs whose endpoints are in \( \Gamma \cap \mathcal{J}(S') \) such that any two consecutive arcs \( \alpha, \beta \in P \) are incident to \( J(s_\alpha, s_\beta) \in \mathcal{J}(S') \), having consistent labels, and \( P \) contains an arc \( \alpha \supseteq \alpha^* \), for every core arc \( \alpha^* \in S' \). We say that the labels of \( \alpha, \beta \) are consistent, if there is a neighborhood \( \bar{\alpha} \subseteq \alpha \) incident to the common endpoint of \( \alpha \) and \( \beta \) such that \( \bar{\alpha} \in D^*(s_\alpha, s_\beta) \), and respectively for \( \beta \).

There can be several different boundary curves for \( S' \), where one such curve is \( E = \Gamma \cap \text{FVD}(S') \). The arcs in \( P \) that contain a core arc in \( S \) are called original and any remaining arcs are called auxiliary. The arcs in \( P \) are all boundary arcs and none is considered a \( \Gamma \)-arc in the sense of the previous sections. The endpoint \( J(s_\alpha, s_\beta) \cap \Gamma \) on \( P \) separating two consecutive arcs \( \alpha, \beta \) is denoted by \( \nu(\alpha, \beta) \).

![Figure 31](image-url) The farthest Voronoi diagram \( \mathcal{V}(S) = \text{FVD}(S) \cap D_{\Gamma} \) and the Voronoi region \( \text{VR}(\alpha, S) \). Bisector labels are shown in the farthest (reversed) sense.
Deletion in abstract Voronoi diagrams in expected linear time

The Voronoi-like diagram of a boundary curve $\mathcal{P}$ is defined analogously to Definition 8. Since $\mathcal{P}$ consists only of boundary arcs, $\mathcal{V}_l(\mathcal{P})$ is a tree whose leaves are the vertices of $\mathcal{P}$. The properties of a Voronoi-like diagram in Section 3 remain the same (under the conventions of this section).

Given $\mathcal{V}_l(\mathcal{P})$ for a boundary curve $\mathcal{P}$ of $S' \subset S$, we can insert a core arc $\beta^* \in S \setminus S'$ and obtain $\mathcal{V}_l(\mathcal{P} \oplus \beta)$. Let $\beta \supseteq \beta^*$ with endpoints $x, y$ defined as follows: let $\delta$ be the first arc on $\mathcal{P}$ counterclockwise (resp. clockwise) from $\beta^*$ such that $J(s_\beta, s_\delta) \cap \delta \neq \emptyset$; let $x = \nu(\delta, \beta)$ (resp. $y = \nu(\beta, \delta)$). Let $\mathcal{P}_\beta = \mathcal{P} \oplus \beta$ be the boundary curve obtained from $\mathcal{P}$ by substituting with $\beta$ its overlapping piece. No original arc of $\mathcal{P}$ can be deleted in $\mathcal{P}_\beta$. Observation 15 remains the same, except cases (d),(e) that do not exist.

The merge curve $J(\beta)$, given $\mathcal{V}_l(\mathcal{P})$, is defined analogously to Definition 17; it is only simpler as it does not contain $\Gamma$-arcs. Theorem 18 remains valid, i.e., $J(\beta)$ is an $s_\beta$-monotone path in $J_{s_\beta} \mathcal{S}$ connecting the endpoints of $\beta$. The proof structure is the same as for Theorem 18; however, Lemma 24 requires a different proof, which we give in the sequel. Lemma 25 is not relevant; Lemma 27 and Lemma 28 are analogous.

In the following lemma we restore the labeling of bisectors to the original.

\textbf{Lemma 34.} In an admissible bisector system $\mathcal{J}$ (resp. $\mathcal{J} \cup \Gamma$) there cannot be two $p$-cycles, $p \in S$, with disjoint interior.

\textbf{Proof.} By its definition, the nearest Voronoi region $\text{VR}(p, S)$ (resp. $\text{VR}(p, S) \cap D_\Gamma$) must be enclosed in the interior of any $p$-cycle of the admissible bisector system $\mathcal{J}$ (resp. $\mathcal{J} \cup \Gamma$). But $\text{VR}(p, S)$ (resp. $\text{VR}(p, S) \cap D_\Gamma$) is connected (by axiom (A1)), thus, there cannot be two different $p$-cycles with disjoint interior. \hfill $\blacksquare$

\textbf{Lemma 35.} Suppose $v_{i+1}$ is not a valid vertex because $v_{i+1} \in \alpha_i$, i.e., $e_i$ hits arc $\alpha_i$. Then vertex $v_{m-j}$ can not be on $\mathcal{P}$.

\textbf{Proof.} Suppose otherwise, i.e., vertex $v_{m-j}$ is on the boundary arc $\alpha_{m-j}$. Then $J_x^i$ and $J_y^j$ partition $D_\Gamma$ in three parts: a middle part incident to $\beta$, and two parts $C_1$ and $C_2$ at either side of $J_x^i$ and $J_y^j$ respectively, whose closures are disjoint, see Figure 32. But the boundaries of $C_1$ and $C_2$ are $s_\beta$-cycles in the admissible bisector system $\mathcal{J} \cup \Gamma$ contradicting Lemma 34. Note that here we use the original labels of bisectors, including $\Gamma = J(s_\beta, s_\infty)$. \hfill $\blacksquare$

The diagram $\mathcal{V}_l(\mathcal{P}) \oplus \beta$ is defined analogously. To prove Theorem 20 we need a new proof for Lemma 19; however, the lemma statement remains identical.

\textbf{Lemma 36.} If the insertion of $\beta$ splits an arc $\alpha \in \mathcal{P}$ (Figure 33) then $J(\beta)$ must split $R(\alpha)$ and $J(\beta) \nsubseteq R(\alpha)$. In no other case can $J(\beta)$ split a region $R(\alpha)$ in $\mathcal{V}_l(\mathcal{P})$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure32.png}
\caption{Figure 32 Illustration for Lemma 35. Nearest labels are shown.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure33.png}
\caption{Figure 33 Illustration for Lemma 36. Nearest labels are shown.}
\end{figure}
Proof. Suppose for the sake of contradiction that $\beta$ splits arc $\alpha$ but $J(\beta) \subseteq R(\alpha)$, see Figure 33. Then $J(\beta) = J(s_\alpha, s_\beta) \cap D_\Gamma$, and $D(s_\alpha, s_\beta) \cap D_\Gamma$ is the interior of an $s_\alpha$-cycle entirely enclosed in $R(\alpha)$. However, $\partial(D_\Gamma \setminus R(\alpha))$ is another $s_\alpha$-cycle, with disjoint interior, contradicting Lemma 34. The rest of the proof remains the same as in Lemma 19. ▶

The randomized algorithm for computing $V(S) = \text{FVD}(S) \cap D_\Gamma$ is the same as in Section 6. Thus, we obtain the following result.

▶ Theorem 37. Given the sequence of its faces at infinity (i.e., $S$), $\text{FVD}(S)$ can be computed in expected $O(h)$ time, where $h \in O(n)$ is the number of faces of $\text{FVD}(S)$ ($h = |S|$).