Asymptotic approximation of the likelihood of stationary determinantal point processes

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Abstract
Continuous determinantal point processes (DPPs) are a class of repulsive point processes on $\mathbb{R}^d$ with many statistical applications. Although an explicit expression of their density is known, it is too complicated to be used directly for maximum likelihood estimation. In the stationary case, an approximation using Fourier series has been suggested, but it is limited to rectangular observation windows and no theoretical results support it. In this contribution, we investigate a different way to approximate the likelihood by looking at its asymptotic behavior when the observation window grows toward $\mathbb{R}^d$. This new approximation is not limited to rectangular windows, is faster to compute than the previous one, does not require any tuning parameter, and some theoretical justifications are provided. It moreover provides an explicit formula for estimating the asymptotic variance of the associated estimator. The performances are assessed in a simulation study on standard parametric models on $\mathbb{R}^d$ and compare favorably to common alternative estimation methods for continuous DPPs.

KEYWORDS
determinantal point process, maximum likelihood inference, repulsive
1 | INTRODUCTION

Determinantal point processes (DPPs for short) are a type of repulsive point processes with statistical applications ranging from machine learning Kulesza and Taskar (2012) to telecommunications (Deng et al., 2015; Gomez et al., 2015; Miyoshi & Shirai, 2014), biology (Affandi et al., 2014), forestry (Lavancier et al., 2015), signal processing (Bardenet et al., 2020), and computational statistics (Bardenet & Hardy 2020). In this paper, we focus on likelihood estimation of parametric families of stationary DPPs on $\mathbb{R}^d$, but we will also include in our study stationary DPPs defined on $\mathbb{Z}^d$. From a theoretical point of view, we are specifically interested in an increasing domain setting, meaning that we assume to observe only one realization of the DPP on a bounded window $W \subset \mathbb{R}^d$, and our asymptotic results will concern the case where $W$ grows toward $\mathbb{R}^d$, making the cardinality of the observed DPP tend to infinity. From this perspective, the likelihood is just the density of the DPP.

For a DPP on $\mathbb{R}^d$ with kernel $K$, the expression of its density on any compact set $W$ (with respect to the unit rate Poisson point process) is known since the seminal paper of Macchi (1975). But this expression is hardly tractable. It requires the knowledge of another kernel, usually called $L$, that can only be obtained from $K$ by solving an integral equation or by knowing the spectral representation of the integral operator associated to $K$ on $W$. Some approximations are then needed. In the stationary case and when $W$ is a rectangular window, an approximation of the density has been proposed in Lavancier et al. (2015) by considering a Fourier series approximation of $K$. This approximation has the pleasant feature to be explicit, but is restricted to rectangular windows and lacks theoretical justifications.

Our contribution is an (increasing domain) asymptotic approximation of the density as well as a way to correct the edge effects arising as a consequence of this approximation. This approach is not restricted to rectangular windows $W$, does not depend on any tuning parameter, and is faster to compute than the Fourier series approximation of Lavancier et al. (2015). Moreover, unlike the previous one, our approximation is generally smooth in the parameter of the model and we can compute explicitly its derivatives with respect to the parameter. This allows us to approximate the Fisher information matrix, and thus to estimate the asymptotic variance of the maximum likelihood estimator by an explicit formula.

The density of a DPP depends on the log-determinant of a random kernel matrix whose behaviour is difficult to control from a theoretical point of view, making challenging a theoretical study of our approximation. The situation simplifies slightly for stationary DPPs defined on a regular grid, typically $\mathbb{Z}^d$. We prove in this case that our approximation has the same asymptotic behaviour as the true density, under mild assumptions. The proof relies on an asymptotic control of the $L$ kernel when $W$ grows to $\mathbb{Z}^d$ and on concentration inequalities for DPPs established in Pemantle and Peres (2014). For DPPs defined on $\mathbb{R}^d$, getting the same kind of results remains an open problem. However we prove that any DPP on $\mathbb{R}^d$ is arbitrarily close to a DPP defined on a small enough regular grid, the density approximation of which is consistent from the previous result.

Likelihood estimation of DPPs has been considered in other settings. For DPPs defined on a finite space, getting the expression of the density from $K$ is not an issue (providing the space dimension is not too large), as it only requires the eigendecomposition of the kernel $K$, which reduces to a matrix in this case. Likelihood estimation in this setting, based on the observation of $n$ i.i.d. discrete DPPs, has been studied in Brunel et al. (2017), who investigate asymptotic properties when $n$ tends to infinity. In the continuous case, likelihood estimation based on $n$ i.i.d.
observations is considered in Bardenet and Aueb (2015). In this contribution, the DPP is directly defined through the kernel $L$, not $K$, avoiding the need to approximate its density from $K$. However, this comes at the cost of a loss of interpretability of the parameters, and more importantly, this approach does not allow to consider increasing domain asymptotic. Indeed, as detailed in Section 2, it is extremely difficult to relate the kernel $L_{|W|$ associated integral operator on $R^d$ with the kernel $L_{|W|$ for $W \subset W'$. For this reason, it is difficult to suggest a parametric family of kernels $L_{|W|$ indexed by $W$. In contrast the kernel $K$ of the DPP on any set $W$ is just the restriction of $K$ to $W$, and it suffices to define $K$ on $R^d$ in order to automatically get a consistent family of kernels on any subset $W$.

The remainder of the paper is organised as follows. We introduce our notations and basic definitions in Section 2. Our asymptotic approximation of the likelihood is presented in Section 3, along with some theoretical justifications. We show in Section 4 how this approximation applies to standard parametric families of DPPs in $R^d$. Section 5 is devoted to a simulation study demonstrating the performances of our approach. Some concluding remarks are given in Section 6. Finally Section 7 includes the proof of our theoretical results, while some technical lemmas are gathered in the appendix.

2 DEFINITIONS AND NOTATION

We consider point processes on $(\mathcal{X}, B(\mathcal{X}), \nu)$ where $\mathcal{X}$ is either $R^d$ or $Z^d$ and the corresponding measure $\nu$ is either the Lebesgue measure on $R^d$ or the counting measure on $Z^d$, respectively. For any point process $X$ and $\nu$-measurable set $W \subset \mathcal{X}$ we write $N(W)$ for the number of points of $X \cap W$ and $|W|$ for the volume of $W$, that is, $|W| = \nu(W)$ is either the Lebesgue measure of $W$ if $\mathcal{X} = R^d$ or its cardinality if $\mathcal{X} = Z^d$. Moreover, for any finite set $X \subset \mathcal{X}$ and any function $F : \mathcal{X}^2 \to R$, we write $F[X]$ for the matrix $(F(x,y))_{x,y \in X}$ where all $x \in X$ are arbitrarily ordered. We write $F_0(y-x) := F(x,y)$ if $F$ is invariant by translation, in which case $F_0[X]$ will refer to the matrix $F[X]$, and we write $F_{\text{rad}}(||y-x||) := F(x,y)$ if $F$ is a radial function. Here $||.||$ denotes the euclidean norm on $\mathcal{X}$ but we will also use the notation $||.||$ for the operator norm when applied to a linear operator, without ambiguity. We denote by $\hat{f}$ the Fourier transform of any function $f : \mathcal{X} \mapsto R$, defined for any $x \in \mathcal{X}^*$ by

$$
\hat{f}(x) := \int_\mathcal{X} f(t) \exp(-2i\pi \langle t,x \rangle) d\nu(t),
$$

where $\mathcal{X}^* = R^d$ if $\mathcal{X} = R^d$, $\mathcal{X}^* = [0,1]^d$ if $\mathcal{X} = Z^d$ and $\langle .., .. \rangle$ denotes the usual scalar product on $\mathcal{X}$. For any Hermitian matrix $M$ we write $\lambda_{\text{max}}(M)$ and $\lambda_{\text{min}}(M)$ for the largest and the lowest eigenvalue of $M$, respectively, and, for any two Hermitian matrices (or operators on a Hilbert space) $M$ and $M'$, we use the Loewner order notation $M \leq M'$ when $M' - M$ is positive definite.

Let us recall some important properties of operator theory. We refer the reader to Simon (2005) for more information. Let $K : \mathcal{X}^2 \to R$ be a locally square integrable, Hermitian function. Its associated integral operator on $L^2(\mathcal{X}, \nu)$ is defined by

$$
K : f \mapsto \left( Kf : x \mapsto \int_\mathcal{X} K(x,y)f(y)d\nu(y) \right).
$$
For any compact set \( W \) of \( \mathcal{X} \), we write \( \mathcal{K}_W \) for the projection of \( \mathcal{K} \) on \( L^2(W, \nu) \) defined for all \( f \in L^2(W, \nu) \) by \( \mathcal{K}_W f(x) = \mathcal{K} f(x) I_W(x) \). If \( \mathcal{K}_W \) is a compact operator then it can be written by Mercer’s theorem as

\[
K_W(x,y) = \sum_i \lambda_i^W \phi_i^W(x)\overline{\phi_i^W}(y), \quad \forall x,y \in W, \tag{1}
\]

where the \( \lambda_i^W \) are the eigenvalues of \( \mathcal{K}_W \) and the \( \phi_i^W \) are the corresponding family of orthonormal eigenfunctions (see Hough et al., 2009 for more details). If \( \sum_i |\lambda_i^W| < \infty \) for all compact sets \( W \subset \mathcal{X} \) then \( \mathcal{K} \) is said to be locally of trace class. Finally, we write \( I \) for the identity operator on \( L^2(\mathcal{X}, \nu) \) and \( I_W \) for its restriction on \( L^2(W, \nu) \) for any \( W \subset \mathcal{X} \).

DPPs are commonly defined through their joint intensity functions.

**Definition 1.** Let \( X \) be a point process on \((\mathcal{X}, \nu)\) and \( n \geq 1 \) be an integer. If there exists a nonnegative function \( \rho_n : \mathcal{X}^n \to \mathbb{R} \) such that

\[
\mathbb{E} \left[ \sum_{x_1, \ldots, x_n \in X} f(x_1, \ldots, x_n) \right] = \int_{\mathcal{X}^n} f(x_1, \ldots, x_n) \rho_n(x_1, \ldots, x_n) \nu(dx_1) \cdots \nu(dx_n),
\]

for all locally integrable functions \( f : \mathcal{X}^n \to \mathbb{R} \), where the symbol \( \neq \) means that the sum is done for distinct \( x_i \), then \( \rho_n \) is called the \( n \)th-order joint intensity function of \( X \).

DPPs are then defined the following way.

**Definition 2.** Let \( K : \mathcal{X}^2 \to \mathbb{R} \) be a locally square integrable, Hermitian function such that its associated integral operator on \( L^2(\mathcal{X}, \nu) \) is locally of trace class with eigenvalues in \([0,1]\). \( X \) is said to be a DPP on \((\mathcal{X}, \mathcal{B}(\mathcal{X}), \nu)\) with kernel \( K \) if its joint intensity functions exist and satisfy

\[
\rho_n(x_1, \ldots, x_n) = \det(K[x]), \tag{2}
\]

for all integer \( n \) and for all \( x = (x_1, \ldots, x_n) \in \mathcal{X}^n \).

When \( \mathcal{X} = \mathbb{R}^d \) the DPP is said to be continuous and when \( \mathcal{X} = \mathbb{Z}^d \) then the DPP is said to be discrete. In the latter case, the integral operator \( \mathcal{K} \) can be seen as the infinite matrix \( K[\mathbb{Z}^d] := (K(x,y))_{x,y \in \mathbb{Z}^d} \). Moreover, when \( K \) is translation invariant then the distribution of the associated DPP is also translation invariant and the DPP is said to be stationary. Similarly, when \( K \) is a radial function then the distribution of the associated DPP is invariant by isometries and the DPP is said to be isotropic.

Let \( X \) be a DPP on \( \mathcal{X} \) with kernel \( K \) and associated integral operator \( \mathcal{K} \). If \( \| \mathcal{K} \| < 1 \), then \( X \) admits on any compact set \( W \subset \mathcal{X} \) a density with respect to the unit rate homogenous Poisson point process on \( W \), as described now. We define the operator \( \mathcal{L} = \mathcal{K}(I - \mathcal{K})^{-1} \) and denote by \( L \) its kernel. Similarly, we define the operator \( \mathcal{L}_{|W|} = \mathcal{K}_W(I_W - \mathcal{K}_W)^{-1} \) and denote by \( L_{|W|} \) its kernel. If \( \mathcal{K}_W \) has a spectral decomposition of the form \( \mathcal{L}_{|W|} \) reads

\[
L_{|W|}(x,y) := \sum_i \frac{\lambda_i^W}{1 - \lambda_i^W} \phi_i^W(x)\overline{\phi_i^W}(y). \tag{3}
\]

Note that contrary to \( \mathcal{K}_W \) with \( \mathcal{K} \), the operator \( \mathcal{L}_{|W|} \) does not correspond to the restriction of \( \mathcal{L} \) to \( L^2(W, \nu) \). Another difference between \( \mathcal{L}_{|W|} \) and \( \mathcal{L} \) is that when \( X \) is a stationary (resp.
isotropic) DPP, \( L(x,y) \) only depends on \( y-x \) (resp. \( ||y-x|| \)) but this is not necessarily true for \( L_{[W]} \).

**Theorem 1** ([Macchi, 1975; Soshnikov, 2000]). Let \( X \) be a DPP on \((\mathcal{X},\nu)\) with kernel \( K \) whose eigenvalues lie in \([0,1]\) and let \( W \) be a compact set of \( \mathcal{X} \). Then \( X \cap W \) is absolutely continuous with respect to the homogeneous Poisson point process on \( W \) with intensity 1 and has density

\[
    f(x) = \exp(||W||) \det(I_W - \mathcal{K}_W) \det(L_{[W]}[x]),
\]

for all \( x \in \bigcup_n W^n \).

In the above expression, the first determinant corresponds to the Fredholm determinant of the operator \( I_W - \mathcal{K}_W \), which is equal to \( \prod_i (1 - \lambda_i W) \), while the second determinant is the standard matrix determinant.

### 3 | LIKELIHOOD OF DPPS

#### 3.1 | Likelihood estimation

Let \( X \) be a DPP on \((\mathcal{X},\nu)\) with kernel \( K^\theta \) belonging to a parametric family \( \{K^\theta, \theta \in \Theta\} \), where \( \Theta \) is the space of parameters. We consider the likelihood estimation of \( \theta^* \), as described below, from a unique observation of \( X \cap W \) where \( W \) is a bounded subset of \( \mathcal{X} \). We furthermore consider an increasing domain asymptotic framework, meaning that our asymptotic properties stand when \( n \to \infty \) and \( W = W_n \) is an increasing sequence of subsets of \( \mathcal{X} \).

For the standard parametric families of continuous DPPs in \( \mathbb{R}^d \), as those presented in Section 4.1, the parameter space \( \Theta \) is a subset of \( \mathbb{R}^p \) for some integer \( p \geq 1 \). This specification is useful when it comes to control the likelihood approximation we develop below uniformly over \( \Theta \), as in Proposition 2. However, our approximation makes sense whatever \( \Theta \) is, provided the associated DPP kernel is translation invariant. In particular, it can be used when \( \mathcal{X} = \mathbb{Z}^d \) and the aim is to estimate the whole matrix \( K_W \) from a realization of \( X \cap W \) as considered in image analysis in Launay et al. (2021).

From Theorem 1, we get that the (normalized) log-likelihood of \( X \cap W \) for any parametric family of DPPs reads:

\[
    \ell(\theta|X) = 1 + \frac{1}{|W|} \left( \log \det(I_W - K^\theta_W) + \log \det(L^\theta_{[W]}[X \cap W]) \right), \tag{4}
\]

where \( K^\theta \) is the integral operator associated to \( K^\theta \) and \( L^\theta_{[W]} \) is given by (3), the eigenvalues and eigenvectors then depending on \( \theta \). The maximum likelihood estimate of \( \theta \) is then

\[
    \hat{\theta} \in \arg \max_{\theta \in \Theta} \ell(\theta|X).
\]

Computing the log-likelihood (4) requires knowing the spectral decomposition of \( K^\theta_W \) for all \( \theta \). This is possible in the case of DPPs on a finite space whose kernels are finite matrices, provided the dimension of the space is not too large, but this spectral decomposition is usually not known for continuous DPPs. This motivates the following approximations.
3.2 Approximation of the likelihood for stationary DPPs

When \( \mathcal{X} = \mathbb{R}^d \) and the observation window \( W \) is rectangular, an approximation of (4) for stationary kernels is proposed in Lavancier et al. (2015), using a truncated Fourier series. For example, if \( W = [-l_1/2, l_1/2] \times \cdots \times [-l_d/2, l_d/2] \) for some \( l_1, \ldots, l_d > 0 \), denoting by \( \Delta \) the diagonal matrix with diagonal entries \( l_1, \ldots, l_d \) (so that \( \det(\Delta) = |W| \)), this relies on the following approximation of the kernel:

\[
K^0(x, y) = K^0_0(x - y) \approx \sum_{k \in \mathbb{Z}^d} \frac{c_k}{|W|} e^{i2\pi (k, \Delta^{-1}(y - x))},
\]

where

\[
c_k := \int_W K^0_0(t) e^{-i2\pi (k, \Delta^{-1}t)} dt \approx \hat{K}^0_0(\Delta^{-1}k),
\]

for some truncation constant \( N \). Note that (5) is an equality if \( (x - y) \in W \) and \( N = \infty \), while (6) is an equality when \( K^0_0 \) vanishes outside \( W \). Since the eigenvalues and eigenvectors of this kernel approximation are, respectively, \( \hat{K}^0_0(\Delta^{-1}k) \) and \( x \mapsto |W|^{-1/2} e^{i2\pi (k, \Delta^{-1}x)} \), the log-likelihood (4) is then approximated in Lavancier et al. (2015) by

\[
1 + \frac{1}{|W|} \left( \sum_{k \in \mathbb{Z}^d, |k| < N} \log(1 - \hat{K}^0_0(\Delta^{-1}k)) + \log \det(t^\theta_{\text{app}}[X \cap W]) \right),
\]

where

\[
L^\theta_{\text{app}}(x, y) := \frac{1}{|W|} \sum_{k \in \mathbb{Z}^d, |k| < N} \frac{\hat{K}^0_0(\Delta^{-1}k)}{1 - \hat{K}^0_0(\Delta^{-1}k)} e^{i2\pi (k, \Delta^{-1}(y - x))}.
\]

The same kind of approximations can be carried out when \( \mathcal{X} = \mathbb{Z}^d \), still for rectangular windows \( W \), in which case \( \hat{K}^0_0 \) in (7) has to be replaced by the discrete Fourier transform of \( K_0(x), x \in W \), and no truncation is needed since the series become a finite sum. This approximation in \( \mathbb{Z}^d \) amounts to consider a periodic extension of the stationary DPP outside \( W \), see Launay et al. (2021) for details.

Our new approximation is based on a different expression of (4) in terms of the self-convolution products of the function \( (x, y) \mapsto \mathbb{1}_W(x)K^\theta(x, y)\mathbb{1}_W(y) \) through the following identities (see Shirai & Takahashi, 2003 for example). For all \( W \subset \mathcal{X} \),

\[
\log \det(I_W - \mathcal{K}_W^\theta) = -\sum_{k=1}^\infty \frac{1}{k} \int_{W^k} K^\theta(x_1, x_2) \cdots K^\theta(x_{k-1}, x_k)K^\theta(x_k, x_1) d\nu^k(x),
\]

and for all \( x, y \in W \),

\[
L^\theta_{|W|}(x, y) = K^\theta(x, y) + \sum_{k=1}^\infty \int_{W^k} K^\theta(x, z_1)K^\theta(z_1, z_2) \cdots K^\theta(z_{k-1}, z_k)K^\theta(z_k, y) d\nu^k(z),
\]
\[ L^\theta(x, y) = K^\theta(x, y) + \sum_{k=1}^{\infty} \int_{\mathcal{X}^c} K^\theta(x, z_1)K^\theta(z_1, z_2) \cdots K^\theta(z_{k-1}, z_k)K^\theta(z_k, y) d\nu^k(z). \] (11)

These convolution products are too difficult to be computed in the general case, but for stationary DPPs satisfying \( \|K^\theta\| < 1 \) then \( L^\theta \) is a translation-invariant function of the form

\[
L^\theta(x, y) = L^\theta_0(y - x) \quad \text{and} \quad \hat{K}^\theta_0 = \hat{K}^\theta_0 / (1 - \hat{K}^\theta_0) \tag{12}
\]

as a consequence of (11). Accordingly, as justified later in Proposition 1, an asymptotic approximation when the observation window \( W \) is large enough gives

\[
L^\theta_W(x, y) \approx L^\theta_0(y - x) = \int_{\mathcal{X}^c} \frac{\hat{K}^\theta_0(t)}{1 - \hat{K}^\theta_0(t)} \exp(2i\pi \langle t, y - x \rangle) dt.
\]

\[
\frac{1}{|W|} \log \det(I_W - K^\theta_W) \approx \int_{\mathcal{X}^c} \log(1 - \hat{K}^\theta_0(x)) dx.
\] (13)

This motivates our following approximation of the log-likelihood:

\[
\tilde{\ell}(\theta|X) := 1 + \int_{\mathcal{X}^c} \log(1 - \hat{K}^\theta_0(x)) dx + \frac{1}{|W|} \log \det(L^\theta_0[X \cap W]),
\] (14)

where \( L^\theta_0 \) is given in (12). This approximation, like (7), can be computed whenever we know the expression of \( \hat{K}^\theta_0 \), which is the case for all classical families of stationary DPPs built from covariance functions, as those presented in Section 4.1. The main advantage of (14) compared to the Fourier approximation (7) is that it is not limited to rectangular windows \( W \) but can be used with any window shape. It has also the advantage of not requiring any tuning parameter of any kind compared to the choice of \( N \) in (7) or alternative moment methods (Biscio & Lavancier, 2017; Lavancier et al., 2021).

The idea to use a convolution approximation was actually briefly suggested in (Lavancier et al., 2015, appendix L) but the associated approximation was given under a more restrictive form that required knowing an exact expression of the iterative self-convolution products of \( K^\theta_0 \) for all \( \theta \). Moreover, an important drawback was pointed out in Lavancier et al. (2015) concerning the presence of possible edge effects, which may affect the quality of estimation of strongly repulsive DPPs. As shown in Section 5, this problem also occurs with our approximation: while it works really well with DPPs with low repulsion, and therefore minimal edge effects, some edge corrections are needed for more repulsive DPPs. The next section deals with this aspect.

### 3.3 Periodic edge-corrections

In order to alleviate the possible edge-effects mentioned above, we suggest to introduce a periodic approximation. We assume in this section that the observation window \( W \subset \mathcal{X} \) is rectangular. Without loss of generality, we set \( W = ([-l_1/2, l_1/2] \times \cdots \times [-l_d/2, l_d/2]) \cap \mathcal{X} \). Using a periodic approximation amounts to consider the observation window as the flat torus \( T_W := \mathcal{X} \setminus l_1\mathbb{Z} \times \cdots \times \mathcal{X} \setminus l_d\mathbb{Z} \). This way, points close to the border of the window \( W \) are brought close to each other in order to compensate edge effects.
More precisely, we replace all instances of \( L_0^\theta(y - x) \) in the stochastic part \( L_0^\theta[ X \cap W ] \) of (14) by

\[
L_0^{\theta, T}(y - x) := L_0^\theta \left( \begin{array}{c}
y_1 - x_1 \mod(l_1) \\
\vdots \\
y_d - x_d \mod(l_d)
\end{array} \right).
\]

This is equivalent to replacing \( L_0^\theta \) by a periodic version of itself on \( W \). The approximate likelihood then reads for any parameter \( \theta \):

\[
\tilde{\ell}(\theta|X) := 1 + \int_{X} \log(1 - \hat{K}_0^\theta(x))dx + \frac{1}{|W|} \log \det \left( L_0^{\theta, T}[ X \cap W ] \right).
\]

Note that since we consider a periodic version of \( L_0^\theta \) on \( W \) then it can be approximated by its Fourier series, which corresponds to the idea of the approximation (7) of Lavancier et al. (2015). This is why both (15) and (7) are nearly equal, see Figure 1 for an example. But approximating \( L_0^\theta[ W ] \) as in (15) instead of using a truncation of its Fourier series leads to a smoother likelihood and overall slightly better results, as well as a more computationally efficient method. Indeed, as explained in Lavancier et al. (2015), the Fourier approximation (8) of \( L_0^\theta[ W ] \) is a sum of \( (2N)^d \) terms where the truncation parameter \( N \) is chosen such that

\[
\sum_{n \in \mathbb{Z}^d \cap [-N,N]} \hat{K}_0^\theta(n) > 0.99 \sum_{n \in \mathbb{Z}^d} K_0^\theta(n).
\]

For important parametric models, including the Whittle–Matern and the Bessel families (see Section 4.1), \( \hat{K}_0^\theta(n) \) has a polynomial decay with respect to \( n \), leading to a large choice of \( N \) in (8). In comparison, as detailed in Section 4.2, depending on the parametric model, we either have an analytic expression of \( L_0^\theta \) or, when the self convolution products of \( K_0^\theta \) are known, we can express \( L_0^\theta \) as the infinite sum

\[
L_0^\theta(x) = \sum_{n \geq 1} (K_0^\theta)^{*n}(x),
\]

where

\[
|(K_0^\theta)^{*n}(x)| = \left| \int_{\mathbb{R}^d} (\hat{K}_0^\theta)^n(t) e^{-i2\pi(x,t)} dt \right| \leq K_0^\theta(0)||\hat{K}_0^\theta||_{\infty}^{n-1},
\]

has an exponential decay with respect to \( n \). The approximation of \( L_0^\theta[ W ] \) by (16) will then require much fewer terms than the approximation by (8).

Despite the appealing properties of the approximation (15), there is one possible issue in that the determinant of \( L_0^{\theta, T}[ X \cap W ] \) is not necessarily positive. Remember that this positivity is guaranteed for any \( X \cap W \) whenever the kernel \( L_0^{\theta, T} \) is positive, or equivalently whenever its associated integral operator has positive eigenvalues. But due to the periodicity of \( L_0^{\theta, T} \), these eigenvalues correspond to the coefficients of its Fourier series that read for any \( k = (k_1, \ldots, k_d) \)

\[
\frac{1}{|W|} \int_W L_0^\theta(x) \exp \left( -2i\pi \sum_{i=1}^{d} \frac{k_i x_i}{l_i} \right) dv(x).
\]
Comparison between the two approximations (7) (solid line) and (15) (dashed line) of \( \alpha \rightarrow \mathcal{L}^\alpha (\rho^*, a|X) \) where \( X \) is a realization of a determinantal point process with Gaussian-type kernel (see Table 1) with true parameters \( \rho^* = 100 \) and \( a^* = 0.05 \) on the window \( W = [0, 1]^2 \).

### Table 1 Examples of parametric kernels \( K_0 \) on \( \mathbb{R}^d \), along with their Fourier transform \( \hat{K}_0 \)

| \( K_0(x) \) | \( \hat{K}_0(x) \) | \( \rho_{\text{max}} \) |
|---|---|---|
| Gauss | \( \rho \exp \left( -\frac{||x||^2}{a^2} \right) \) | \( \rho (\sqrt{\pi} a)^d \exp(- \pi \alpha x||)^2 \) | \( (\sqrt{\pi} a)^{-d} \)
| Bessel | \( \rho 2^{d/2} \Gamma(d/2 + 1) J_{d/2} (\sqrt{2a ||x||})/\alpha \) | \( \rho \frac{\Gamma(d/2 + 1)}{\Gamma(d/2 + 1/2)} \frac{\Gamma(d/2 + 1)}{\Gamma(d/2 + 1/2) \sigma} \) | \( \rho_{\text{max}} \) |
| Cauchy | \( \rho \left( 1 + \frac{\sigma}{||x||} \right)^{d+1/2} \) | \( \frac{\rho (\sqrt{\pi} a)^d \sqrt{\pi} e^{-||2ax||}}{\Gamma(d+1/2)} \) | \( \frac{\Gamma(d+1/2)}{\Gamma(d+1/2) \sigma} \)
| WM | \( \frac{\rho_{\text{max}}}{\Gamma(\sigma)} \frac{\Gamma(d+1/2)}{\Gamma(d+1/2) \sigma} \) | \( \frac{\Gamma(d+1/2)}{\Gamma(d+1/2) \sigma} \) | \( \rho_{\text{max}} \)

Notes: For each family, the intensity is \( \rho \) and the range parameter is \( a \). The existence condition \( \hat{K}_0 \leq 1 \) is equivalent to \( \rho \leq \rho_{\text{max}} \) where \( \rho_{\text{max}} \) is given in the last column. The Whittle–Matérn model (WM) also contains a shape parameter \( \sigma > 0 \). Here \( J_{d/2} \) denotes the Bessel function of the first kind and \( K_\sigma \) the modified Bessel function of the second kind.

When \( W \) is large, the above integral is approximately equal to \( f_0^\theta (k_1/1, \cdots, k_d/1) \) which is positive. This shows that we can expect the determinant of \( L^{\theta,X} \) to be positive when \( W \) is large enough. In our simulations displayed in Section 5, this determinant was positive in all runs, except a few times with the Bessel-type kernel associated to high values of the repulsion parameter \( \alpha \).

Finally, note that extending the above edge correction to non rectangular windows is not straightforward and we do not provide a general solution. We however introduce in the simulation example of Section 5.3 a procedure that can be adapted to any isotropic DPP model.

### 3.4 Theoretical results

In order to verify the theoretical soundness of the asymptotic log-likelihood approximation (14) we want to show that \( \frac{\tilde{\ell}(\theta|X) - \ell(\theta|X)}{\text{d.s.}} \xrightarrow{} 0 \) uniformly for all \( \theta \in \Theta \) when the observation window \( W \) grows toward \( \mathcal{L} \). For this purpose, we consider a sequence of increasing observation windows \( W_n \) satisfying the following assumptions.

---

**Figure 1** Comparison between the two approximations (7) (solid line) and (15) (dashed line) of \( \alpha \rightarrow \mathcal{L}^\alpha (\rho^*, a|X) \) where \( X \) is a realization of a determinantal point process with Gaussian-type kernel (see Table 1) with true parameters \( \rho^* = 100 \) and \( a^* = 0.05 \) on the window \( W = [0, 1]^2 \).
Condition \((W)\): \(W_n\) is an increasing sequence of compact subsets of \(\mathcal{X}\) such that \(\bigcup_{n \geq 0} W_n = \mathcal{X}\) and there exists an increasing nonnegative sequence \(r_n \in \mathbb{R}^+\) such that \(\lim_{n \to \infty} r_n = \infty\) and

\[
|(\partial W_n \oplus r_n) \cap W_n| = o(|W_n|),
\]

(18)

where, by a small abuse of notation, we write \(\partial W_n \oplus r_n\) for the Minkowski sum of \(\partial W_n\), the boundary of \(W_n\), and a centered ball with radius \(r_n\), which corresponds to the set of points whose distance to the boundary of \(W_n\) is lower than \(r_n\). Moreover,

\[
\forall \delta > 0, \sum_{n \geq 0} \exp(-\delta|W_n|) < \infty.
\]

(19)

The first assumption (18) means that the boundary of \(W_n\) must not be too irregular. This is not an issue in most practical applications. For example, if \(\mathcal{X} = \mathbb{R}^d\) and \((W_n)_{n \geq 0}\) is a sequence of spheres with radius \(R_n \to \infty\), then (18) is satisfied with \(r_n = \sqrt{R_n}\). As another example, assume that \((W_n)_{n \geq 0}\) is a sequence of rectangular windows \([-l_{1,n}/2, l_{1,n}/2] \times \cdots \times [-l_{d,n}/2, l_{d,n}/2]\) such that \(l_{i,n} \to \infty\) for each \(i\), then

\[
(\partial W_n \oplus r_n) \cap W_n \subset \left(\bigcup_{j=1}^{d} [-l_{j,n}/2 - l_{j,n}/2 + r_n, l_{j,n}/2] \times \cdots \times [-l_{d,n}/2 - l_{d,n}/2 + r_n, l_{d,n}/2]\right).
\]

hence

\[
\frac{|(\partial W_n \oplus r_n) \cap W_n|}{|W_n|} \leq \prod_{i=1}^{d} \left(\frac{2r_n}{l_{i,n}}\right),
\]

which vanishes when \(n\) goes to infinity with the choice \(r_n = \sqrt{\min_{i} l_{i,n}}\). The second hypothesis (19) is a technical assumption needed to get the almost sure convergence in Proposition 2. Without this assumption, the convergence remains true but in probability instead of almost surely.

We first consider the uniform convergence of the deterministic part of (4), which is the Fredholm log-determinant. Its asymptotic behavior given below is justified in Section 7.1 and was already proved in a slightly different setting in (Shirai & Takahashi, 2003, proposition 5.9).

**Proposition 1.** Let \(\{K_0^\theta : \mathcal{X} \to \mathbb{R}, \theta \in \Theta\}\) be a family of functions in \(L^1(\mathcal{X}, \nu)\) with integrable Fourier transforms \(\hat{K}_0^\theta\) taking values in \([0, M]\) for some \(M < 1\) and let \((W_n)_{n \geq 0}\) satisfy Condition \((W)\). Additionally, we assume that \(\sup_{\theta \in \Theta} K_0^\theta(0) < \infty\) and that the function \(x \mapsto \sup_{\theta \in \Theta} |K_0^\theta(x)|\) is integrable on \((\mathcal{X}, \nu)\). We denote by \(K_{W_n}^\theta\) the projection on \(L^2(W_n)\) of the integral operator associated with the kernel \((x, y) \mapsto K_0^\theta(x - y)\). Then,

\[
\sup_{\theta \in \Theta} \left| \frac{1}{|W_n|} \log \det(L_{W_n} - K_{W_n}^\theta) - \int_{\mathcal{X}} \log(1 - \hat{K}_0^\theta(\lambda))d\lambda \right| \to 0.
\]

Concerning the stochastic part of the log-likelihood (4), that is \(\log \det(L_{W}^\theta[X \cap W])\), its behaviour is much more difficult to control in general. The main issue is that the determinant vanishes when two points of \(X \cap W\) gets arbitrarily close to each other, but no relationship between how close these points are from each other and the value of the determinant is known, making the likelihood difficult to control. To our knowledge, the only related result is that, in most cases,
the lowest eigenvalue of $L_\theta^{\psi}[X]$ is nonzero iff $\inf_{x,y \in X} ||y - x|| > 0$ Bachoc and Furrer (2016). The latter condition is automatically satisfied if $X$ is supported on a lattice but not when $\mathcal{X} = \mathbb{R}^d$. The next result focuses on the first case.

**Proposition 2.** Let $(W_n)_{n \in \mathbb{N}}$ satisfy Condition (W) and let $\{K^\theta, \theta \in \Theta\}$ be a family of translation-invariant DPP kernels on $\mathbb{Z}^d$ such that $\Theta$ is a compact set of $\mathbb{R}^p$ for some integer $p \geq 1$ and the function $(\theta, x) \mapsto \hat{K}^\theta(x)$ is continuous on $\Theta \times \mathbb{R}^d$. Additionally, assume there exists constants $A, \tau > 0$ and $M < 1$ satisfying

$$\forall \theta \in \Theta, \forall x \in \mathbb{R}^d, \quad |K_0^\theta(x)| \leq \frac{A}{1 + ||x||^{d + \tau}} \quad \text{and} \quad 0 < \hat{K}_0^\theta(x) \leq M.$$

Let $X$ be the realization of a DPP on $\mathbb{Z}^d$ with kernel $K^\psi^\star$, $\psi^\star \in \Theta$. Then, for all $\theta \in \Theta$,

$$\sup_{\theta \in \Theta} \frac{1}{|W_n|} \left| \log \det(L_0^\psi[X \cap W_n]) - \log \det(L_\theta^{\psi}[X \cap W_n]) \right| \xrightarrow{a.s.} 0.$$

The only restrictive assumptions in Proposition 2 is the need for $K_0^\theta$ to decay faster than $||x||^{-d}$ and the fact that $\hat{K}_0^\theta$ never vanishes. In the usual setting where the kernels are parametric covariance functions (see Proposition 4), these assumptions are generally satisfied. That includes the Gaussian, Cauchy, and Whittle–Matern kernels. The only exception amongst standard kernels is the Bessel-type kernel, that will be examined by simulations in Section 5.2. Based on Propositions 1 and 2 and noticing that the assumptions of Proposition 2 imply the assumptions of Proposition 1, we thus obtain the consistency of the likelihood approximation (14) when $\mathcal{X} = \mathbb{Z}^d$.

**Corollary 1.** Let $\{K^\theta, \theta \in \Theta\}$ be a family of translation-invariant DPP kernels on $\mathbb{Z}^d$ satisfying the assumptions of Proposition 2, then $\sup_{\theta \in \Theta} |\hat{\ell}(\theta)[X] - \ell(\theta)[X]| \xrightarrow{a.s.} 0$.

Getting the same result for DPPs on $\mathbb{R}^d$ is still an open problem. However the next proposition shows that a DPP on $\mathbb{R}^d$ can be approximated by a discrete DPP on an arbitrarily small regular grid of $\mathbb{R}^d$, for which Corollary 1 applies. Note that the assumptions on $\hat{K}_0$ below are satisfied for all standard parametric families, see Section 4.1.

**Proposition 3.** Let $X$ be a stationary DPP on $\mathbb{R}^d$ with kernel $K(x,y) = K_0(y - x)$, where $K_0$ is a square integrable function such that $\hat{K}_0$ takes values in $[0,1]$ and

$$\forall x \in \mathbb{R}^d, \quad 0 \leq \hat{K}_0(x) \leq \frac{A}{1 + ||x||^{d + \tau}},$$

for some constant $A, \tau > 0$. For all $\epsilon > 0$, define $X_\epsilon$ as the DPP on $\mathbb{Z}^d$ with kernel $K_\epsilon(x,y) := \epsilon^d K_0(\epsilon(y - x))$. Then, $X_\epsilon$ is well-defined for small enough $\epsilon$ and the distribution of $\epsilon X_\epsilon$, the DPP $X_\epsilon$ rescaled by a factor $\epsilon$, weakly converges to the distribution of $X$ when $\epsilon$ tends to 0.

In the end, Corollary 1 tells us that the asymptotic approximation of the log-likelihood (14) is theoretically sounded for most classical parametric families of stationary DPPs on $\mathbb{Z}^d$ and, as a consequence of Proposition 3, also theoretically sounded for any discrete approximation of continuous DPPs on an arbitrarily small regular grid of $\mathbb{R}^d$. 


TABLE 2 Expression of \( L_0 \) defined in (12) for the parametric kernels given in Table 1

| \( \) | \( L_0(x) \) |
|---|---|
| Gauss | \( \sum_{n \geq 1} \rho^n \left( \frac{\sqrt{n} \| x \|}{\pi n^{\alpha}} \right) \) |
| Bessel | \( \sum_{n \geq 1} \rho^n \left( \frac{\sqrt{n} \| x \|}{\pi n^{\alpha}} \right) \) |
| Cauchy | \( \sum_{n \geq 1} \rho^n \left( \frac{\sqrt{n} \| x \|}{\pi n^{\alpha}} \right) \) |
| Whittle-Mat'ern | \( \sum_{n \geq 1} \rho^n \left( \frac{\sqrt{n} \| x \|}{\pi n^{\alpha}} \right) \) |

4 APPLICATION TO STANDARD PARAMETRIC FAMILIES

4.1 Classical parametric families of stationary DPPs

A classical way of generating parametric families of stationary DPPs is the following result.

**Proposition 4.** Let \( K_0 : \mathcal{X} \to \mathbb{R} \) be a bounded square integrable symmetric function on \( \mathbb{R}^d \) such that its Fourier transform \( \hat{K}_0 \) takes values in \([0, 1]\). Then, the function \( K(x, y) := K_0(y - x) \) is a DPP kernel on \((\mathcal{X}, \nu)\).

This proposition is proved in Lavancier et al. (2015) in the case \( \mathcal{X} = \mathbb{R}^d \). Since symmetric functions \( K_0 \) with nonnegative Fourier transform are covariance functions, this result implies that we can consider as many parametric families of DPPs as there are parametric families of covariance functions. The assumption that \( \hat{K}_0 \leq 1 \) simply adds a bound on the parameters of the family. Various examples are presented and studied in (Biscio & Lavancier, 2016; Lavancier et al., 2015). We provide in Table 1 some examples in \( \mathbb{R}^d \). Note that for simplification, we call in this table Bessel kernel the particular case of the Bessel kernel in Biscio and Lavancier (2016) where the shape parameter is \( \sigma = 0 \), and Cauchy kernel the particular case in Lavancier et al. (2015) where the shape parameter is \( 1/2 \). If the shape parameter is different for these models, then closed formulas are available for \( K_0 \) and \( \hat{K}_0 \), but not for \( L_0 \) (see the next section and Table 2).

4.2 Expressions of \( L_0 \)

When computing the approximate log-likelihood \( \tilde{\ell}(\theta | X) \) in (14) or its edge-corrected version (15), one has to compute \( L_0(y - x) \) for each pair of points \((x, y) \in (X \cap W)^2\). It is thus important to find faster ways to compute values of \( L_0 \) than the \( d \)-dimensional integral (12). An important example arises when \( K_0 \) is a radial function, denoted by \( K_{\text{rad}} \). In this case, the corresponding DPP is isotropic and \( L_0 \) is also a radial function, denoted by \( L_{\text{rad}} \). The Fourier transform can then be expressed by a Hankel transform which gives

\[
\hat{K}_{\text{rad}}(r) = \frac{2\pi}{rd/2-1} \int_0^\infty s^{d/2} K_{\text{rad}}(s) J_{d/2-1}(2\pi rs) ds,
\]

and

\[
L_{\text{rad}}(r) = \frac{2\pi}{rd/2-1} \int_0^\infty s^{d/2} \frac{\hat{K}_{\text{rad}}(s)}{1 - \hat{K}_{\text{rad}}(s)} J_{d/2-1}(2\pi rs) ds.
\]
The expression of $L_0$ therefore simplifies into a unidimensional integral.

Moreover, we may exploit the relation $\hat{L}_0 = \hat{K}_0/(1 - \hat{K}_0) = \sum_{n \geq 1} (\hat{K}_0)^n$ and try to compute the inverse Fourier transform to express $L_0$ as a series with exponentially decreasing coefficients (see the discussion in Section 3.3) or even get an analytic expression. This strategy leads to closed-form formulas of $L_0$ for the classical parametric families displayed in Table 1. The results, obtained after straightforward calculus, are given in Table 2.

4.3 | Estimation of the intensity by MLE

Assume that the parametric DPP kernel reads for some parameters $\rho$ and $\theta$

$$K^{\rho,\theta}(x,y) = \rho \tilde{K}^{\theta}(x,y),$$

where $\tilde{K}^{\theta}(x,x) = 1$ for all $x$. The parameter $\rho$ corresponds here to the intensity of the DPP and $\theta$ to the other parameters of the model. This is the setting of all standard parametric models, including those presented in Table 1.

When jointly estimating $(\rho, \theta)$ from a realization of the DPP $X$ on $W$ by the approximate MLE, simulations usually show that the estimate of $\rho$ appears to be very close to $N(W)/|W|$. One explanation given in Lavancier et al. (2015) is that, by doing a first-order convolution approximation in (9) and (11), we get

$$\ell'(\rho, \theta|X) \approx 1 - \rho + \log(\rho) \frac{N(W)}{|W|} + \frac{1}{|W|} \log \det(\tilde{K}^{\theta}_W[X \cap W]),$$

and the maximum point of this approximation is $\hat{\rho} = N(W)/|W|$. We even show in Proposition 5 that, in the case of Bessel-type DPP kernels with parameters $(\rho, \alpha)$ as presented in Table 1, $\hat{\rho} = N(W)/|W|$ is always the maximum point of $\rho \mapsto \ell'(\rho, \alpha|X)$ for any $\alpha$. This result suggests that, instead of jointly estimating $\rho$ and $\theta$, it is more computationally efficient to directly estimate $\rho$ by $\hat{\rho} = N(W)/|W|$ and then $\theta$ by an argument of the maximum of $\theta \mapsto \tilde{\ell}(\hat{\rho}, \theta|X)$.

4.4 | Estimation of the MLE SEs

For most statistical models, the MLE is expected to have an asymptotic variance equal to the inverse Fisher information matrix. Even if this property is not theoretically proved for DPPs’ models, it is a natural conjecture to make. To estimate this variance, it is common (and even advocated in Efron and Hinkley (1978)) to use the observed information, which is the matrix with entries $-[W]|\partial_{\theta_i} \partial_{\theta_j} \ell'(\theta|X)$, whose expectation defines the genuine Fisher information matrix.

Since our approximation (14) is generally smooth in the parameters (see below), we may consider the following approximation of the observed information matrix:

$$\tilde{I}(\theta) := -[W] \left( \partial_{\theta_i} \partial_{\theta_j} \tilde{\ell}(\theta|X) \right)_{1 \leq i, j \leq p}.$$ 

If $\hat{\theta}$ is the approximated MLE based on (14), we can thus estimate its variance by $\tilde{I}(\hat{\theta})^{-1}$. 
This estimation is possible whenever \( \theta \mapsto K_0^\theta(x) \) and \( \theta \mapsto L_0^\theta(x) \) are twice differentiable on \( \Theta \subset \mathbb{R}^p \) for all \( x \in X \). Then so is \( \theta \mapsto \tilde{\ell}(\theta|X) \) and we obtain

\[
\begin{align*}
\partial_{\theta_i} \partial_{\theta_j} \tilde{\ell}(\theta|X) &= \int_{X^*} \frac{-(\partial_{\theta_i} \partial_{\theta_j} K_0^\theta(x))(1 - K_0^\theta(x)) - \partial_{\theta_i} K_0^\theta(x) \partial_{\theta_j} K_0^\theta(x)}{(1 - K_0^\theta(x))^2} \, dx \\
&\quad + \frac{1}{|W|} \text{Tr} \left( (\partial_{\theta_i} \partial_{\theta_j} L_0^\theta(L_0^0)^{-1} - (\partial_{\theta_i} L_0^\theta)(L_0^0)^{-1} (\partial_{\theta_j} L_0^\theta)(L_0^0)^{-1} \right),
\end{align*}
\]

(22)

where we have written \( L_0^\theta \) for \( L_0^\theta[X \cap W] \). Each derivative in this expression can easily be deduced from Tables 1 and 2 for the parametric models discussed before.

Note that such variance estimation is not possible for the Fourier series approximation (7) because this approximation is not differentiable in general, as illustrated in Figure 1 for the scale parameter of the Gaussian kernel, so that the observed information is not well defined in this case. Moreover, concerning the alternative minimum contrast estimators of a parametric DPP model considered in (Biscio & Lavancier, 2017; Lavancier et al., 2015), no tractable formulas are available for their asymptotic variance. For these estimation methods, the only way to approximate the associated standard errors is parametric bootstrap, a very time consuming procedure.

## 5 SIMULATION STUDY

In this section we perform a simulation study to investigate the performance of our approximate MLE, with and without edge effect correction, and compare it to minimum contrast estimators (MCE for short) based on Ripley’s \( K \) function and on the pair correlation function (pcf for short), both being common second-order moment estimators used in spatial statistics. We refer to Biscio and Lavancier (2017) for more detailed information on these MCEs applied to DPPs. At the exception of the special case of Bessel-type DPPs considered in Section 5.2, we chose not to compare our estimators to the Fourier approximation (7) of Lavancier et al. (2015) since, as explained in Section 3.3, this estimator yields nearly the same results as our corrected MLE, which we confirmed in our tests, with the notable difference of the Fourier approximation being about 10 times longer to compute in our examples.

### 5.1 Whittle-Matérn, Cauchy, and Gaussian-type DPPs

We consider in this section the parametric models in Table 1 that are covered by our theoretical assumptions in Section 3.4, that are the Whittle–Matérn, Cauchy and Gaussian-type DPPs. From this perspective, these are favorable models for our likelihood approximation approach. All these models are of the form (21), then following Section 4.3, we estimate \( \rho \) by \( \hat{\rho} = N(W)/|W| \) for all methods, and the performances are evaluated on the estimation of \( \alpha \) only. Note that for the Whittle–Matérn model, we do not consider the estimation of the shape parameter \( \sigma \), which was assumed to be known. The joint estimation of \( (\alpha, \sigma) \) for this model is known to be a poorly identifiable problem and it is customary to choose the best \( \sigma \) from a small finite grid by profile likelihood (see Lavancier et al., 2015). For the estimation of \( \alpha \), we have performed the same kind of simulations for the three models in \( \mathbb{R}^2 \). The results and conclusions are similar. In the
We consider realizations of the Gaussian-type DPP with true parameters \( \rho^* = 100 \) and \( \alpha^* \in \{0.01, 0.03, 0.05\} \), when the observation window \( W \) is either \([0,1]^2\), \([0,2]^2\) or \([0,3]^2\). When \( \rho^* = 100 \), \( \alpha \) can take values in \( ]0, (10\sqrt{\pi})^{-1} \approx 0.056[ \) since the process exists if and only if \( \pi \rho \alpha^2 \leq 1 \). Therefore, \( \alpha^* = 0.01 \) corresponds to a weakly repulsive point process, close to a Poisson point process, while \( \alpha^* = 0.03 \) corresponds to a mildly repulsive DPP and \( \alpha^* = 0.05 \) corresponds to a strongly repulsive DPP. Examples of realizations are shown in Figure 2. We estimate \( \alpha^* \) by the approximate MLE defined in (14) and compare it to its edge-corrected version defined in (15) as well as MCEs based on the pcf or Ripley’s \( K \) function. As mentioned before, \( \rho \) is replaced by \( \hat{\rho} = N(W)/|W| \) in (14) and (15). Moreover we truncate the series defining \( L_0 \) (see Table 2) to the minimal value of \( n \leq 50 \) such that all remainder terms in the series become less than \( 10^{-4} \) times the first term. This choice leads to \( n \leq 10 \) for most values of \( \alpha \) and to \( n = 50 \) only for \( \alpha > 0.9\alpha_{\max} \), where \( \alpha_{\max} = 1/\sqrt{\pi \hat{\rho}} \). All realizations have been generated in R (R Core Team, 2017) using the \textit{spatstat} Baddeley et al. (2015) package and both MCEs were computed by the function \textit{dppm} of the same package. The tuning parameters for these MCEs were \( r_{\min} = 0.01, \ r_{\max} \) being one quarter of the side length of the window and \( q = 0.5 \) as recommended in Diggle (2003). Boxplots of the difference between the four considered estimators and the true value \( \alpha^* \) for 500 runs in all different cases are displayed in Figure 3 and the corresponding mean square errors are given in Table 3.

\[\begin{array}{ccccccccc}
\text{Window} & |0,1|^2 & |0,2|^2 & |0,3|^2 \\
\hline
\alpha^* & 0.01 & 0.03 & 0.05 & 0.01 & 0.03 & 0.05 & 0.01 & 0.03 & 0.05 \\
MLE based on \( \tilde{L}^T \) & 0.83 & 0.81 & 0.41 & 0.21 & 0.18 & 0.088 & 0.090 & 0.079 & 0.051 \\
MLE based on \( \tilde{L} \) & 1.25 & 1.75 & 0.54 & 0.24 & 0.23 & 0.28 & 0.095 & 0.10 & 0.20 \\
MCE (pcf) & 0.86 & 0.77 & 0.74 & 0.31 & 0.27 & 0.23 & 0.17 & 0.17 & 0.19 \\
MCE (K) & 1.81 & 1.17 & 0.51 & 0.74 & 0.46 & 0.21 & 0.48 & 0.23 & 0.12 \\
\end{array}\]
From these results, we remark that when $\alpha^* = 0.01$ and $\alpha^* = 0.03$, inference based on the approximate likelihood $\tilde{L}(\hat{\rho}, \alpha|X)$ outperforms moment-based inference for the window $[0, 2]^2$ and larger ones. This is expected from maximum likelihood-based inference and shows that hundreds of points are enough for $\tilde{L}(\hat{\rho}, \alpha|X)$ to be a good approximation of the true likelihood when the underlying DPP is not too repulsive. When $\alpha^* = 0.05$, that is when the negative dependence of the DPP is very strong, then $\tilde{L}(\hat{\rho}, \alpha|X)$ suffers from edge effects and is heavily biased. In fact, as can be seen in Figure 4, $\tilde{L}(\hat{\rho}, \alpha|X)$ is an increasing function of $\alpha$ in this case and the estimate is often the highest possible value for $\alpha$, which is $1/\sqrt{\pi\hat{\rho}}$. The correction $\tilde{L}^T$ introduced in (15) gives more accurate values of the likelihood for high values of $\alpha$, as shown in Figure 4. Finally this estimator outperforms the other ones in nearly every cases and especially the most repulsive ones.

Concerning the computation time, even if our MLE approximation is much faster than the Fourier approximation (7), it can be heavy due to the need to optimize a function defined as the log-determinant of an $n \times n$ matrix, where $n$ is the number of observed points. For comparison, each MCE took less than 1 s on a regular laptop in each case considered in Figure 3, while each computation of the approximate MLE took between 1 and 2 seconds when $W = [0, 2]^2$ and about 10 s when $W = [0, 3]^2$. 

**Figure 3** Boxplots of $\hat{\alpha} - \alpha^*$ generated from 500 simulations of Gaussian-type determinantal point processes with true parameters $\rho^* = 100$ and, from top to bottom, $\alpha^* = 0.01, 0.03$ and 0.05. Each row shows the behavior of the following four estimators when the simulation window is, from left to right in each box, $W = [0, 1]^2, [0, 2]^2$ and $[0, 3]^2$: the approximate MLE with edge-corrections based on $\tilde{L}^T(\hat{\rho}, \alpha|X)$, the approximate MLE based on $\tilde{L}(\hat{\rho}, \alpha|X)$, the minimum contrast estimator (MCE) based on the pair correlation function and the MCE based on the Ripley’s $K$ function.
FIGURE 4 Comparison between $\tilde{l}(100, a|X)$ (solid lines) and $\tilde{l}_T(100, a|X)$ (dashed lines) with respect to $a$ where $X$ is one realization of a determinantal point process on $[0, 1]^2$ with a Gaussian-type kernel with true parameters $\rho^* = 100$ and, from left to right, $\alpha^* = 0.01, 0.03$ and $0.05$.

TABLE 4 Estimated mean square errors ($\times 10^4$) of $\hat{\alpha}$ for Bessel-type determinantal point processes on different windows and with different values of $\alpha$, each computed from 500 simulations.

| Window  | $[0, 1]^2$ | $[0, 2]^2$ | $[0, 3]^2$ |
|---------|-----------|-----------|-----------|
| $\alpha^*$ | 0.01     | 0.03     | 0.05     | 0.01     | 0.03     | 0.05     | 0.01     | 0.03     | 0.05     |
| MLE based on $\tilde{l}_T$ | 0.56     | 0.49     | **0.04** | 0.12     | 0.08     | **0.01** | 0.05     | **0.03** | 0.01     |
| Fourier approx. MLE | **0.47** | **0.32** | 0.09     | **0.11** | **0.06** | 0.02     | **0.05** | **0.03** | **0.01** |
| Minimum contrast estimators (MCE) (pcf) | 0.50     | 0.39     | 0.33     | 0.21     | 0.14     | 0.11     | 0.10     | 0.11     | 0.07     |
| MCE ($K$) | 0.95     | 0.46     | 0.19     | 0.41     | 0.15     | 0.04     | 0.27     | 0.10     | 0.02     |

5.2 Performance for Bessel-type DPPs

In order to evaluate the possible limitations of our approach, we consider in this section the estimation of Bessel-type DPPs, see Table 1, whose kernels do not satisfy the theoretical assumptions in Section 3.4. As in the previous section, we set $\rho^* = 100$, $\alpha^* = 0.01, 0.03, 0.05$, corresponding to weak, medium and strong repulsiveness, and the observation window is $[0, 1]^2$, $[0, 2]^2$, and $[0, 3]^2$. The results on 500 runs in each situation are shown in Figure 5 and in Table 4. We compare our edge-correction approximate MLE, the Fourier series approximation (7), the MCE based on the pair correlation function and the MCE based on the Ripley’s $K$ function. The performances are globally in line with the observations made in the previous section, showing that the approximate MLE outperforms MCEs, especially when the observation window is large enough. Note that we have added the Fourier series approximation for comparison, because contrary to the models considered in the previous section, its behavior slightly differs from our edge-correction approximation for Bessel-type DPPs, as discussed in the following.

Despite the decent results of our approximation for Bessel-type DPPs, some issues appear with this model in the most repulsive case $\alpha^* = 0.05$. As noticed in Section 3.3, the determinant in
FIGURE 5  Boxplots of $\hat{\alpha} - \alpha^*$ generated from 500 simulations of Bessel-type determinantal point processes with true parameters $\rho^* = 100$ and, from top to bottom, $\alpha^* = 0.01, 0.03,$ and $0.05$. Each row shows the behavior of the following four estimators when the simulation window is, from left to right in each box, $W = [0, 1]^2$, $[0, 2]^2$ and $[0, 3]^2$: the approximate MLE with edge-corrections based on $\tilde{\xi}^T(\hat{\rho}, \alpha | X)$, the Fourier series approximate MLE (7), the minimum contrast estimators (MCE) based on the pair correlation function and the MCE based on the Ripley’s $K$ function.

(15) may be negative for high values of $\alpha$, making the computation of the approximate likelihood impossible. This problem is illustrated in the rightmost plot of Figure 6, that shows an example of an approximated likelihood function as in (15) from one realization of a Bessel-type DPP on $W = [0, 3]^2$ with $\rho^* = 100$ and $\alpha^* = 0.05$. The cross-type points on the right of this plot indicate the values of $\alpha$ where the determinant was negative. More generally, for the highest values of $\alpha$, the approximate likelihood is clearly not trustable. Fortunately, the optimization procedure was not affected by this phenomena and succeeded to return a local maximum in the vicinity of $\alpha^*$. However, another peculiar behavior occurs in this situation, which is the small M-shape of the approximate likelihood in this vicinity. This feature was common to most of the approximate likelihoods in our simulations on $W = [0, 3]^2$ with $\rho^* = 100$ and $\alpha^* = 0.05$, but we are not able to provide a clear explanation of this phenomena. The consequence is that the optimizer chooses one of the two local maxima from this M-shape, resulting in a bi-modal distribution of $\hat{\alpha}$ in this case, as showed in the leftmost plot of Figure 6. This also explains the shape of the boxplot associated to this case in Figure 5. In front of such peculiar M-shape of the contrast function, it might be natural to choose as the optimum the average of the two local maxima instead of one of them. Adopting this strategy decreases the estimation mean square error from 1 to 0.25 ($\times 10^{-6}$).

It is interesting to note that for Bessel-type DPPs, unlike the DPP models of Section 5.1, the Fourier series approximation (7) of the MLE has a more significative difference in behaviour than
Figure 6  Left: distribution of $\hat{\alpha}$ obtained by the approximate MLE $\hat{\ell}_T^\alpha$, based on 500 simulations of a Bessel-type determinantal point process when $\rho^* = 100$, $\alpha^* = 0.05$ (represented by the vertical line) and $W = [0, 3]^2$. Right: $\alpha \mapsto \hat{\ell}_T^\alpha(\hat{\rho}, \alpha|X)$ (circles) and Fourier series approximation (7) of the log-likelihood (black squares) from one realization $X$ as before, where the vertical line shows the true parameter $\alpha^* = 0.05$ and the red cross-type points indicate the values of $\alpha$ for which the determinant in $\hat{\ell}_T^\alpha(\hat{\rho}, \alpha|X)$ was negative.

Our approximate MLE with edge correction (15). As shown in Figure 6, it does not have undefined values and it does not follow a chaotic behavior for large values of $\alpha$. Moreover, because the Fourier transform of the Bessel kernel only takes two different values (see Table 1), the terms in the Fourier approximation (7) when $d = 2$ simplify as:

$$\sum_{k \in \mathbb{Z}^2 \setminus \{|k| < N\}} \log(1 - \hat{K}_0^\theta(k)) = \log \left(1 - \rho \pi \alpha^2\right) \sum_{k \in \{-N, \ldots, N\}} \left(2 \left\lfloor \frac{1}{\pi^2 \alpha^2 - k^2} \right\rfloor + 1\right),$$

and

$$L_{\text{app}}^\theta(x, y) = \frac{\rho \pi \alpha^2}{1 - \rho \pi \alpha^2} \sum_{k \in \{-N, \ldots, N\}} \frac{\cos(2\pi kx) \sin(2\pi y \left\lfloor \frac{1}{\pi^2 \alpha^2 - k^2} \right\rfloor + 1)}{\sin(\pi y)},$$

where the truncation constant is $N = \left\lfloor \frac{1}{\pi \alpha} \right\rfloor$. This simplification makes it easier to compute than in the general case, and results in a more competitive computation time, similar to our approximation (15). As a result, we observe in Table 4 and Figure 5 that for $\alpha^* = 0.01$ and 0.03, the Fourier approximation method has very similar performances to our approximation (15). When $\alpha^* = 0.05$, the Fourier approximation estimator has also a similar quadratic error, but the distribution of the estimator is more regular, for the reasons noticed above.

Finally, despite the fact that Bessel-type DPPs are not covered by our theory and the peculiar behaviour of $\hat{\ell}_T^\alpha$ for some values of $\alpha$, our approach still remains competitive in this case and outperforms standard MCE methods. Nevertheless, because the Fourier approximation (7) simplifies nicely in this setting and does not show the same chaotic behaviour as (15) for large values of $\alpha$, it seems to be a slightly better choice for Bessel-type DPPs. However, we recall that this approach is limited to rectangular observation windows only.
5.3 Simulations on a nonrectangular window

We consider in this section the estimation of a Gaussian-type DPP on the (nonrectangular) R-shape window as in the simulations of Figure 7. The underlying parameters are $\rho^* = 100$, resulting in 370 points on average, and $\alpha^* = 0.01, 0.03$ and 0.05. The estimation of $\alpha^*$ is carried out by the MLE approximation (14) (without edge-corrections), the edge-corrected version described below, and the MCEs based on the pcf and the Ripley’s $K$-function. Note that in this situation, the Fourier approximation (7) is not feasible.

We handle the edge-effects for this non-rectangular window in the following way. Note that the number of replacements in this edge-correction procedure is limited: they only concern the border points of $X$, then

\[ r_{\text{max}} = \arg \max_{r_{ij}} L_{\text{rad}}^+(r_{ij}) > 0.001 L_{\text{rad}}^+(0), \]

where $L_{\text{rad}}^+ = L_{\text{rad}}^\theta$ for $\theta = (\hat{\rho}, 0.9 \alpha_{\text{max}}), \hat{\rho} = N(W)/|W|$, $\alpha_{\text{max}} = \sqrt{1/(\pi \hat{\rho})}$ and $L_{\text{rad}}^+(0)$ is the maximal possible value of $L_{\text{rad}}^+$. This choice guarantees that for any $r > r_{\text{max}}$ and any $\theta = (\hat{\rho}, \alpha)$ with $\alpha < 0.9 \alpha_{\text{max}}$, $L_{\text{rad}}^+(r)$ can be considered to be negligible.

1. We start by setting a maximal range of interaction $r_{\text{max}}$. In our example we choose

\[ r_{\text{max}} = \arg \max_{r_{ij}} \{ L_{\text{rad}}^+(r_{ij}) > 0.001 L_{\text{rad}}^+(0) \}, \]

where $L_{\text{rad}}^+ = L_{\text{rad}}^\theta$ for $\theta = (\hat{\rho}, 0.9 \alpha_{\text{max}}), \hat{\rho} = N(W)/|W|$, $\alpha_{\text{max}} = \sqrt{1/(\pi \hat{\rho})}$ and $L_{\text{rad}}^+(0)$ is the maximal possible value of $L_{\text{rad}}^+$. This choice guarantees that for any $r > r_{\text{max}}$ and any $\theta = (\hat{\rho}, \alpha)$ with $\alpha < 0.9 \alpha_{\text{max}}$, $L_{\text{rad}}^+(r)$ can be considered to be negligible.

2. For $i = 1, \ldots, n$, we denote by $d_i$ the Euclidean distance from $x_i$ to $\partial W$, and by $n_i = \text{card} \{ j, r_{ij} < r_{\text{max}} \}$ the number of neighbors of $x_i$ in $X$. We further denote by $B = \{ x_i \in X, d_i < r_{\text{max}} \}$ the set of “border” points of $X$ in $W$ and by $\bar{B} = X \setminus B$ the set of “interior” points of $X$ in $W$. Finally, we consider $R_{\bar{B}} = \{ r_{ij}, x_i \in \bar{B} \}$ the set of observed pairwise distances for the interior points of $X$, and $N_{\bar{B}} = \{ n_i, x_i \in \bar{B} \}$ the set of numbers of neighbors of the interior points.

3. For all $x_i \in B$, we randomly pick out $\tilde{n}_i$ in $N_{\bar{B}}$ and compare it to $n_i$. If $n_i \geq \tilde{n}_i$, we do nothing. Else, for $j = (i + 1), \ldots, (i + \tilde{n}_i - n_i) \wedge n$ and if $r_{ij} > r_{\text{max}}$, we randomly pick out $\tilde{r}_{ij}$ in $R_{\bar{B}} \cap \{ r_{ij} > d_i \}$ and we replace $r_{ij}$ and $r_{ji}$ by $\tilde{r}_{ij}$.

Note that the number of replacements in this edge-correction procedure is limited: they only concern the border points of $X$, there are a maximum of $\tilde{n}_i - n_i$ of them for each border point $x_i$, and the replaced value $\tilde{r}_{ij}$ of $r_{ij} > r_{\text{max}}$ is necessarily greater than $d_i$, which in many cases (especially if $\alpha$ is small) entails $L_{\text{rad}}^\theta(\tilde{r}_{ij}) \approx 0$ and does not affect the initial value $L_{\text{rad}}^\theta(r_{ij}) \approx 0$. With the resulting new matrix $R$, there is no guarantee that $L_{\text{rad}}^\theta(R)$ is positive, a common issue with the periodic edge corrections of Section 3.3, but the restricted number of replacements limits the risk to encounter such a problem. In our experience, this happened only for very high values of $\alpha$ and did not affect the optimization procedure.

The boxplots displayed in Figure 7 along with the corresponding mean square errors in Table 5 show that the above edge-correction version of (14), that we still abusively denote $\tilde{\rho}_\gamma$, overall provides the best results. They also confirm that this edge-correction is only necessary for the most...
FIGURE 7  Top: Examples of realizations of Gaussian-type determinantal point processes with parameters $\rho^* = 100$ and $\alpha^* = 0.01, 0.03, 0.05$ (from left to right) on a R-shape window. Bottom: distribution of $\hat{\alpha} - \alpha^*$ from 100 simulations for each value of $\alpha^*$ and for the following estimators (from left to right in each plot): the approximate MLE (14), its edge-corrected version $\hat{\ell}^T$ as detailed in the text, and the minimum contrast estimators based on the pair correlation function and the Ripley’s $K$ function.

repulsive DPPs, that is, $\alpha^* = 0.05$ here, otherwise the approximation (14) and its edge-corrected version perform pretty similarly.

5.4  Estimation of the SEs

In order to numerically assess the quality of estimation of the SEs, as described in Section 4.4, we consider the estimation of $\alpha^*$ for Gaussian, Cauchy and Bessel families of DPPs when the
observation window $W$ is either $[0,1]^2$, $[0,2]^2$, or $[0,3]^2$, the intensity is $\rho^* = 100$ and for three different values of $\alpha^*$. These values of $\alpha^*$ correspond to low, mild and strong repulsion, specifically $\alpha^* \in \{0.01, 0.03, 0.05\}$ for the Gaussian and Bessel models and $\alpha^* \in \{0.005, 0.02, 0.035\}$ for the Cauchy model.

In these cases, we have $\theta = (\rho, \alpha)$ and the observed information matrix is estimated by

$$\widetilde{I}(\hat{\rho}, \hat{\alpha}) = -|W| \begin{bmatrix} \frac{\partial^2 \tilde{E}^T(\hat{\rho}, \hat{\alpha}|X)}{\partial \rho^2} & \frac{\partial^2 \tilde{E}^T(\hat{\rho}, \hat{\alpha}|X)}{\partial \rho \partial \alpha} \\ \frac{\partial^2 \tilde{E}^T(\hat{\rho}, \hat{\alpha}|X)}{\partial \alpha^2} & \frac{\partial \tilde{E}^T(\hat{\rho}, \hat{\alpha}|X)}{\partial \alpha} \end{bmatrix},$$

where $\hat{\rho} = N(W)/|W|$, $\hat{\alpha} \in \arg\min_{\alpha} \tilde{E}^T(\hat{\rho}, \alpha|X)$, $\tilde{E}^T$ is given by (15), and the derivatives are obtained using (22), Tables 1 and 2.

Following Section 4.4, the variance of $(\hat{\rho}, \hat{\alpha})$ is estimated by $\tilde{I}(\hat{\rho}, \hat{\alpha})^{-1}$ and an approximated 95% confidence interval for $\alpha^*$ is then

$$\hat{\alpha} \pm \frac{1.96}{\sqrt{|W|}} \left( \frac{\partial^2 \tilde{E}^T(\hat{\rho}, \hat{\alpha}|X)}{\partial \alpha^2} \right)^{1/2} (\hat{\rho}, \hat{\alpha}|X) - \frac{\partial \tilde{E}^T(\hat{\rho}, \hat{\alpha}|X)}{\partial \alpha} \right)^{-1/2}. \quad (23)$$

In Table 6, we report, for each DPP family, each choice of window $W$ and each value of $\alpha^*$, the proportion of times $\alpha^*$ falls in that interval, out of 500 simulations. Note that it might happen that (23) is not well-defined, which occurs when $\tilde{I}(\hat{\rho}, \hat{\alpha})$ is not positive definite. We report in parenthesis in Table 6 the proportion of times this issue arose for each case.

We observe from Table 6 that the approximated confidence interval (23) is inaccurate for the Bessel family, which is in line with our findings of Section 5.2. In particular, the peculiar behavior of $\alpha \mapsto \tilde{E}^T(\hat{\rho}, \alpha|X)$ in the vicinity of $\alpha^*$ when $\alpha^* = 0.05$ and $W = [0,3]^2$, as shown in Figure 6, makes irrelevant any estimation of its derivatives at $\alpha^*$, which certainly explains the low coverage rate of the interval (23) observed in this case. For the Gaussian and Cauchy DPPs families, the results are all the more satisfying when the window is large. For these families, the approximated confidence interval (23) seems to be trustable whenever there are more than 400 points (corresponding to the case $W = [0,2]^2$), even if it tends to seemingly underestimate the SE of $\hat{\alpha}$.

We finally made the same kind of simulations for the (nonrectangular) R-shaped window of Section 5.3 using the the same Gaussian-type DPP we used in that section. Note that the setting is comparable with the Gaussian-type DPP when $W = [0,2]^2$ in Table 6, except for the shape of the window, since the mean number of points are similar and the kernel and the values of $\alpha^*$ are the same. The coverage rate for the R-shaped window was 87.4% for $\alpha^* = 0.01$ (low repulsion), 90% for $\alpha^* = 0.03$ (mild repulsion) and 63.2% for $\alpha^* = 0.05$ (strong repulsion). These results are

| $\alpha^*$ | 0.01 | 0.03 | 0.05 |
|-----------|------|------|------|
| MLE based on $\tilde{E}$ | 0.19 | 0.37 | 0.37 |
| MLE based on $\tilde{E}^T$ | 0.18 | 0.33 | 0.08 |
| Minimum contrast estimators (MCE) (pair correlation function) | 0.29 | 0.33 | 0.32 |
| MCE ($K$) | 0.77 | 0.44 | 0.21 |
TABLE 6 |  Coverage rate (×100), out of 500 simulations, of the approximated 95% confidence interval (23) of $\alpha^*$, for the Gaussian, Cauchy, and Bessel determinantal point processes’ models, three different observation windows and three values of $\alpha^*$ corresponding to low, mild and strong repulsion (see the text for the exact values)

| Window | \([0, 1]^2\) | \([0, 2]^2\) | \([0, 3]^2\) |
|---|---|---|---|
| $\alpha^*$ | Low | Mild | High | Low | Mild | High | Low | Mild | High |
| Gauss | 88.2 | 89.6 | 92 | 88.6 | 94.2 | 92.6 | 93.2 | 93.2 | 92.8 |
| Cauchy | 89.8 | 88.4 | 72 | 92.4 | 92.2 | 83.6 | 91.4 | 95 | 87.2 |
| Bessel | 66 | 76.6 | 56 | 77.2 | 78.8 | 82.2 | 81 | 74.4 | 12 |

Note: Values in parenthesis indicate the proportion of simulations (×100) when the interval (23) was not well-defined.

of the same order as in Table 6 for low and mild repulsion, but worse for $\alpha^* = 0.05$. This last result is probably due to edge-effects in this case, and shows that there is still an avenue to improve edge-corrections for non-rectangular windows.

6 | CONCLUSION

In this paper, we have introduced an asymptotic approximation (14) of the log-likelihood of stationary DPPs on $\mathbb{R}^d$ and $\mathbb{Z}^d$. While the true likelihood is not numerically tractable, this approximation can be computed for stationary parametric families of DPPs based on correlation functions with a known Fourier transform, as the classical ones presented in Table 1. Compared to the Fourier approximation of Lavancier et al. (2015) that only works for rectangular windows, our approximation can be computed for windows of any shape and provides an estimation of the asymptotic variance of the resulting maximum likelihood estimator. However, due to edge effects, the resulting estimator gets heavily biased for strongly repulsive DPPs, as shown in Figure 3. We have proposed to use the periodic correction (15) to fix this issue in the case of rectangular windows and showed that the resulting approximation is very close to the one in Lavancier et al. (2015) (see Figure 1) but overall easier to compute. The idea to use a periodic correction has been detailed for rectangular windows, but a similar idea can be applied for a window with a different shape, as exemplified in Section 5.3. We showed in the simulation study of Section 5 that for standard parametric families of DPPs, the resulting approximate MLE outperforms classical moment methods based on the pair correlation function and Ripley’s $K$ function.

Finally, we proved in Propositions 1 and 2 that the difference between the true log-likelihood and our approximated log-likelihood converges almost surely towards 0 for classical parametric families of stationary DPPs on $\mathbb{Z}^d$. We also showed in Proposition 3 that DPPs on $\mathbb{R}^d$ can be arbitrarily approached by DPPs on a regular grid, which suggests that our approximation should also converge for DPPs on $\mathbb{R}^d$. A formal proof of such result is still a seemingly difficult open problem. Beyond the approximation of the likelihood, as proposed in this paper, a natural theoretical concern is the consistency of the maximum likelihood estimator, either based on the true likelihood or on the approximated one. This question is challenging and is not addressed in the present contribution. We however think that our findings are a step in the right direction toward such
a result, because they allow to replace the true likelihood by an easier expression to deal with mathematically.

7 | PROOFS OF SECTION 3

7.1 | Proof of Proposition 1

In the case where \( \mathcal{X} = \mathbb{R}^d \) and \( W_n \) is of the form \( n \times W \) for some compact set \( W \), then the convergence of \( \frac{1}{|W_n|} \log \det(I_{W_n} - \mathcal{K}_W^\theta) \) for any fixed value of \( \theta \) corresponds to (Shirai & Takahashi, 2003, proposition 5.9) with \( \alpha = -1 \) and \( f(x) = \infty \times 1_w \). Our proof follows a similar idea.

Since all eigenvalues of \( \mathcal{K}_W^\theta \) are in \([0, 1]\) then the logarithm of the Fredholm determinant of \( I_{W_n} - \mathcal{K}_W^\theta \) can be expanded into

\[
\log \det(I_{W_n} - \mathcal{K}_W^\theta) = -\sum_{k \geq 1} \frac{\text{Tr}((\mathcal{K}_W^\theta)^k)}{k} = -\sum_{k \geq 1} \frac{1}{k} \int_{W_n} \mathcal{K}_W^\theta(x_2 - x_1) \cdots \mathcal{K}_W^\theta(x_1 - x_k) d\nu^k(x).
\]

Now, we first assume that \( \mathcal{K}_W^\theta_0 \) and \( \mathcal{K}_W^\theta_0 \) are integrable. Then, for any \( x_1 \in \mathcal{X} \) the function

\[
f_{x_1}^\theta : (x_2, \cdots, x_k) \mapsto \mathcal{K}_W^\theta_0(x_2 - x_1) \cdots \mathcal{K}_W^\theta_0(x_1 - x_k),
\]

is integrable and its integral is equal to \((\mathcal{K}_W^\theta_0)^{\ast k}(0)\) where \((\mathcal{K}_W^\theta_0)^{\ast k}(0)\) is the \( k \)th times self-convolution of \( \mathcal{K}_W^\theta_0 \). Since we assumed that \((W_n)_{n \geq 0}\) satisfy (18), then by Lemma 3 we get that

\[
\left| \frac{1}{|W_n|} \int_{W_n} \mathcal{K}_W^\theta_0(x_2 - x_1) \cdots \mathcal{K}_W^\theta_0(x_1 - x_k) d\nu^k(x) - (\mathcal{K}_W^\theta_0)^{\ast k}(0) \right| \leq \int_{(B(0, r_n)^c)^{k-1}} |f_{x_1}^\theta(y)| d\nu^{k-1}(y) + \frac{|\partial W_n \Theta r_n \cap W_n|}{|W_n|} \| f_{x_1}^\theta \|_{L^1},
\]

for any positive sequence \((r_n)_{n \in \mathbb{N}}\). Now, since we assumed that \( x \mapsto \sup_{\theta \in \Theta} \mathcal{K}_W^\theta_0(x) \) is integrable then \( x \mapsto \sup_{\theta \in \Theta} f_{x_1}^\theta(x) \) is also integrable. Moreover, the sequence \((W_n)_{n \in \mathbb{N}}\) satisfies condition \((W)\), hence

\[
\sup_{\theta \in \Theta} \left| \frac{1}{|W_n|} \int_{W_n} \mathcal{K}_W^\theta_0(x_2 - x_1) \cdots \mathcal{K}_W^\theta_0(x_1 - x_k) d\nu^k(x) - (\mathcal{K}_W^\theta_0)^{\ast k}(0) \right| \xrightarrow{n \to \infty} 0.
\]

Finally, since

\[
\frac{|\text{Tr}((\mathcal{K}_W^\theta_0)^k)|}{k |W_n|} \leq \frac{|\mathcal{K}_W^\theta|^k}{k} \times \frac{\text{Tr}(\mathcal{K}_W^\theta_0^k)}{|W_n|} \leq \frac{M^k}{k} \sup_{\theta \in \Theta} \mathcal{K}_W^\theta_0(0),
\]

which is summable with respect to \( k \) and does not depend on \( n \) and \( \theta \), then we can conclude by the dominated convergence theorem that
\[
\sup_{\theta \in \Theta} \left| \frac{1}{|W_n|} \log \det(I_{W_n} - \mathcal{K}_n^\theta) + \sum_{k \geq 1} \frac{\mathcal{K}_n^\theta k}{k} \right| \xrightarrow{n \to \infty} 0.
\]

The proof is completed by using the relation
\[
- \sum_{k \geq 1} \left( \frac{\mathcal{K}_n^\theta k}{k} \right) = \int_{\mathcal{X}} \log(1 - \hat{\mathcal{K}}_n^\theta(x)) dx.
\]

### 7.2 Proof of Proposition 2

For this proof, we consider \( X \) to be a DPP on \( \mathbb{Z}^d \). Let \( \theta \in \Theta \), we denote by \( \lambda_n^\theta \) the lowest eigenvalue of \( \mathcal{K}^\theta[\mathbb{Z}^d] \) and we define \( \lambda_n^0 := \inf_{\theta \in \Theta} \lambda_n^\theta \). It is important to note that \( \lambda_n^0 > 0 \) as a consequence of (Bachoc & Furrer, 2016, theorem 5) and the assumptions on \( \mathcal{K}^\theta \). We begin by proving the following lemma allowing us to control \( \mathcal{L}_n^\theta(X \cap W) - \mathcal{L}_n^\theta(W \cap X) \) for any \( W \subset \mathbb{Z}^d \) by controlling the difference between their associated operators.

**Lemma 1.** Let \( \mathcal{K} \) be an integral operator on \( L^2(\mathcal{X}, \nu) \) with kernel \( \mathcal{K} \) such that \( \| \mathcal{K} \| < 1 \). For any Borel set \( W \subset \mathcal{X} \), we denote by \( P_W \) the projection on \( L^2(W) \) and we define the operators \( \mathcal{K}_W := P_W \mathcal{K} P_W \) on \( L^2(W) \), \( \mathcal{L}_W := \mathcal{K}_W^\theta(I_W - \mathcal{K}_W)^{-1} \) on \( L^2(W) \) and \( \mathcal{L} := \mathcal{K}(I - \mathcal{K})^{-1} \) on \( L^2(\mathcal{X}) \). We denote by \( L \) the kernel of \( L \) and finally we define the operator \( \mathcal{N}_W := P_W L P_W \mathcal{L} P_W \) on \( L^2(W) \) with kernel
\[
N_W(x, y) = \int_{W^c} L(x, z)L(z, y) \nu(z) \quad \forall x, y \in W.
\]

(24) Then,
\[
0 \leq P_W L P_W - \mathcal{L}_W \leq \mathcal{N}_W.
\]

**Proof.** We consider the following decomposition of the linear operators \( I - \mathcal{K} \) and \( (I - \mathcal{K})^{-1} \) on \( L^2(W) \oplus L^2(W^c) \):

\[
I - \mathcal{K} = \begin{pmatrix}
I_W - \mathcal{K}_W & -P_W \mathcal{K} P_W^c \\
-P_W^c \mathcal{K} P_W & I_{W^c} - \mathcal{K}_{W^c}
\end{pmatrix} = \begin{pmatrix}
\mathcal{L}_W + I_W & -P_W \mathcal{K} P_W^c \\
-P_W^c \mathcal{K} P_W & \mathcal{L}_{W^c} + I_{W^c}
\end{pmatrix},
\]

and

\[
(I - \mathcal{K})^{-1} = I + L = \begin{pmatrix}
P_W L P_W + I_W & P_W L P_W^c \\
P_W^c L P_W & P_W^c L P_W^c + I_{W^c}
\end{pmatrix}.
\]

A well-known result is that the (1, 1) block of \( I - \mathcal{K} \) is equal to the inverse of the Schur complement of \( (I - \mathcal{K})^{-1} \) relative to its (2, 2) block. This property is proved for \( 2 \times 2 \) block matrices in (Puntanen & Zhang, 2005, theorem 1.2), and since the proof does not use any finite dimensionality argument, it works all the same for nonsingular operators on a Hilbert space, see Fujimoto
et al. (2004) for example. As a consequence, we get

$$(L_{W} + I_{W})^{-1} = (P_{W}L_{P_{W}} + I_{W} - P_{W}L_{P_{WC}}(P_{WC}L_{P_{WC}} + I_{W})^{-1}P_{WC}L_{P_{W}})^{-1},$$

hence

$$P_{W}L_{P_{W}} - L_{W} = P_{W}L_{P_{WC}}(P_{WC}L_{P_{WC}} + I_{W})^{-1}P_{WC}L_{P_{W}} \geq 0.$$  

Finally, since $(P_{WC}L_{P_{WC}} + I_{W})^{-1} \leq I_{WC}$ this concludes the lemma.

Now, we rewrite

$$\frac{1}{|W_n|} \left| \log \det(L^\theta[X \cap W_n]) - \log \det(L^\theta_{W_n}[X \cap W_n]) \right|,$$

as

$$\frac{1}{|W_n|} \left| \log \det(Id + (L^\theta[X \cap W_n] - L^\theta_{W_n}[X \cap W_n])L^\theta_{W_n}[X \cap W_n]^{-1}) \right|.$$  

By Lemma 1, we know that

$$0 \leq L^\theta[X \cap W_n] - L^\theta_{W_n}[X \cap W_n] \leq N^\theta_{W_n}[X \cap W_n],$$

where $N^\theta_{W_n}$ is defined as in (24). Therefore, using Lemma 2 we obtain the bound

$$0 \leq \log \det(L^\theta[X \cap W_n]) - \log \det(L^\theta_{W_n}[X \cap W_n]) \leq \text{Tr}(N^\theta_{W_n}[X \cap W_n]L^\theta_{W_n}[X \cap W_n]^{-1}).$$

Now, since $L^\theta_{W_n} \geq K^\theta_{W_n}$ by definition, then $\lambda_{\min}(L^\theta_{W_n}[X \cap W_n]) \geq \lambda_{\min}(K^\theta[X \cap W_n]) \geq \lambda^\theta_m \geq \lambda^\theta_0$ where the second to last inequality is a consequence of $K^\theta[X \cap W_n]$ being a submatrix of $K^\theta[\mathbb{Z}^d]$. Therefore,

$$\text{Tr}(N^\theta_{W_n}[X \cap W_n]L^\theta_{W_n}[X \cap W_n]^{-1}) \leq (\lambda^\theta_0)^{-1} \sum_{x \in X \cap W_n} N^\theta_{W_n}(x, x).$$

The function $X \mapsto |W_n|^{-1} \sup_{\theta \in \Theta} \sum_{x \in X} N^\theta_{W_n}(x, x)$ is Lipschitz continuous on $\bigcup_{k \geq 0} W_n^k$ with constant $\sup_{\theta \in \Theta} \|N^\theta_{W_n}\|_{\infty}/|W_n|$. Since $N^\theta_{W_n}$ is the kernel of a nonnegative integral operator then $\text{det}(N^\theta_{W_n}[X]) \geq 0$ for any point configuration $X \subset W$. Applying this inequality to $X = \{x, y\}$ for any $x, y \in W$ gives

$$N^\theta(x, y)^2 \leq N^\theta(x, x)N^\theta(y, y) = \int_{W_C} L^\theta_{0}(z - x)^2 d\nu(z) \int_{W_C} L^\theta_{0}(z - y)^2 d\nu(z) \leq \|L^\theta_{0}\|_2^4,$$

hence

$$\sup_{\theta \in \Theta} \|N^\theta_{W_n}\|_{\infty} \leq \sup_{\theta \in \Theta} \|L^\theta_{0}\|_2^2 = \sup_{\theta \in \Theta} \left\| \frac{K^\theta_{0}}{1 - K^\theta_{0}} \right\|_2^2 \leq \sup_{\theta \in \Theta} \|K^\theta_{0}\|_2^2 \frac{1}{1 - M} = \frac{\sup_{\theta \in \Theta} \|K^\theta_{0}\|_2^2}{1 - M}.$$
This expression is finite since we assumed (20). By (Pemantle & Peres, 2014, theorem 3.5), we then get for all \( a \in \mathbb{R}_+ \)

\[
\mathbb{P}_{\theta^*} \left( \frac{1}{|W_n|} \left| \sup_{\theta \in \Theta} \sum_{x \in X \cap W_n} N_{W_n}^\theta(x, x) - \mathbb{E}_{\theta^*} \left[ \sup_{\theta \in \Theta} \sum_{x \in X \cap W_n} N_{W_n}^\theta(x, x) \right] \right| > a \right) 
\leq 5 \exp \left( - \frac{a^2 |W_n|^2 / \sup_{\theta \in \Theta} \|N_{W_n}^\theta\|_2^2}{16(a|W_n|/\sup_{\theta \in \Theta} \|N_{W_n}^\theta\|_\infty + 2\mathbb{E}_{\theta^*}[N(W_n)])} \right), \tag{25}
\]

where \( \mathbb{E}_{\theta^*}[N(W_n)] = |W_n|K_{\theta^*}^\theta(0) \) and

\[
\frac{1}{|W_n|} \mathbb{E}_{\theta^*} \left[ \sup_{\theta \in \Theta} \sum_{x \in X \cap W_n} N_{W_n}^\theta(x, x) \right] 
\leq \frac{1}{|W_n|} \mathbb{E}_{\theta^*} \left[ \sum_{x \in X \cap W_n} \sup_{\theta \in \Theta} N_{W_n}^\theta(x, x) \right] 
= \frac{K_{\theta^*}^\theta(0)}{|W_n|} \int_{W_n} \sup_{\theta \in \Theta} \int_{W_n^C} L_0^\theta(y - x)^2 \nu(x) \nu(dy) 
\leq \frac{K_{\theta^*}^\theta(0)}{|W_n|} \int_{W_n} \sup_{\theta \in \Theta} \int_{W_n^C} L_0^\theta(y - x)^2 \nu(x) \nu(dy) 
= K_{\theta^*}^\theta(0) \left( \int_{\mathbb{Z}^d} \sup_{\theta \in \Theta} L_0^\theta(y - x)^2 \nu(dy) - \frac{1}{|W_n|} \int_{W_n} \sup_{\theta \in \Theta} L_0^\theta(y - x)^2 \nu(x) \nu(dy) \right) 
= K_{\theta^*}^\theta(0) \left( \int_{\mathbb{Z}^d} \sup_{\theta \in \Theta} L_0^\theta(y)^2 \nu(dy) - \frac{1}{|W_n|} \int_{W_n} \sup_{\theta \in \Theta} L_0^\theta(y - x)^2 \nu(x) \nu(dy) \right) .
\]

But, as a consequence of Lemma 3, we have

\[
\frac{1}{|W_n|} \int_{W_n} \sup_{\theta \in \Theta} L_0^\theta(y - x)^2 \nu(dy) \xrightarrow{n \to \infty} \int_{\mathbb{Z}^d} \sup_{\theta \in \Theta} L_0^\theta(y)^2 \nu(dy)
\]

hence

\[
\frac{1}{|W_n|} \mathbb{E}_{\theta^*} \left[ \sup_{\theta \in \Theta} \sum_{x \in X \cap W_n} N_{W_n}^\theta(x, x) \right] \xrightarrow{n \to \infty} 0.
\]

Finally, by (25) and the inequality \( \|N_{W_n}^\theta\|_\infty \leq \|L_0^\theta\|_2^2 \), we get that for all \( a \in \mathbb{R}_+ \),

\[
\mathbb{P}_{\theta^*} \left( \frac{1}{|W_n|} \sup_{\theta \in \Theta} \sum_{x \in X \cap W_n} N_{W_n}^\theta(x, x) > a \right) 
= O \left( \exp \left( - \frac{a^2 |W_n|^2}{16 \sup_{\theta \in \Theta} \|L_0^\theta\|_2^2 (a + 2K_{\theta^*}^\theta(0) \sup_{\theta \in \Theta} \|L_0^\theta\|_2^2))} \right) \right).
\]
Since we assumed (19), then by the Borel–Cantelli Lemma,
\[
\frac{1}{|W_n|} \sup_{\theta \in \Theta} \sum_{x \in \mathcal{X} \cap W_n} K_\theta(x, x) \xrightarrow{a.s.} 0,
\]
and therefore
\[
\frac{1}{|W_n|} \sup_{\theta \in \Theta} \log \det(L^\theta[X \cap W_n]) - \log \det(L^\theta_W[X \cap W_n]) \xrightarrow{a.s.} 0.
\]

7.3 Proof of Proposition 3

First, we need to show that \( X_\epsilon \) is a well defined DPP for small enough \( \epsilon \) by showing that its kernel, the infinite matrix \( \epsilon^d K(\epsilon \mathbb{Z}^d) \), is Hermitian with eigenvalues in \([0, 1]\). Everything is trivial except for showing that the eigenvalues become lower or equal to 1 as \( \epsilon \) vanishes. For every \( v = (v_j)_{j \in \mathbb{Z}^d} \) such that \( \sum_j |v_j|^2 = 1 \), we define the function
\[
\phi(t) = \sum_{j \in \mathbb{Z}^d} v_je^{2\pi \langle j, t \rangle},
\]
such that the integral of \( |\phi|^2 \) on any unit cube is equal to 1. Therefore, we can write
\[
\langle v, \epsilon^d K(\epsilon \mathbb{Z}^d) v \rangle = \sum_{j,k \in \mathbb{Z}^d} \epsilon^d v_jv_kK_0(\epsilon|k - j|)
\]
\[
= \sum_{j,k \in \mathbb{Z}^d} v_jv_k \int_{\mathbb{R}^d} \hat{K}_0(t/\epsilon)e^{2\pi \langle k - j, t/\epsilon \rangle} dt
\]
\[
= \int_{\mathbb{R}^d} \hat{K}_0(t/\epsilon)|\phi(t)|^2 dt
\]
\[
\leq \sum_{i \in \mathbb{Z}^d} \sup_{x \in \mathbb{C}_i} \hat{K}_0(x/\epsilon),
\]
where \( C_i \) is the unit cube defined as \([i_1 - 1/2, i_1 + 1/2] \times \cdots \times [i_d - 1/2, i_d + 1/2] \) for all \( i = (i_1, \cdots, i_d) \in \mathbb{Z}^d \). By our assumptions on \( \hat{K}_0 \), we have \( \sup_{x \in \mathbb{C}_0} \hat{K}_0(x/\epsilon) \leq ||\hat{K}_0||_\infty < 1 \) and
\[
\sup_{x \in \mathbb{C}_i} \hat{K}_0(x/\epsilon) \leq \sup_{v_j, x_j \in [i_j - 1/2, i_j + 1/2]} A \frac{1 + \epsilon^{-(d+r)}(\sum_{j=1}^d x_j^2)^{d+r}/2}{1 + \epsilon^{-(d+r)}(\sum_{j=1}^d (|i_j| - 1/2)^2)^{d+r}/2},
\]
hence, the sum of all \( \sup_{x \in \mathbb{C}_i} \hat{K}_0(x/\epsilon) \) for \( i \) of the form \((i_1, \cdots, i_k, 0, \cdots, 0)\) where \( i_1, \cdots, i_k \in \mathbb{Z} \setminus \{0\} \) and \( k \in \{1, \cdots, d\} \) is bounded by
\[
\sum_{i_1, \cdots, i_k \in (\mathbb{Z} \setminus \{0\})^k} A \frac{1 + \epsilon^{-(d+r)}(\sum_{j=1}^k (|i_j| - 1/2)^2)^{d+r}/2}{1 + \epsilon^{-(d+r)}(\sum_{j=1}^d (|i_j| - 1/2)^2)^{d+r}/2}.
\]
\[ \sum_{i_1, \ldots, i_d \in \{0\}^d} A \left( \sum_{j=1}^k |i_j|^2 \right)^{(d+\tau)/2} \xrightarrow{\epsilon \to 0} 0. \]

By symmetry, this is also true for the sum of all \( \sup_{x \in C_i} \hat{K}_0(x/\epsilon) \) for \( i \) with any \( k \) non-zero components and \( d - k \) zero components when \( k \in \{1, \ldots, d\} \). This shows that

\[ \sum_{i_1, \ldots, i_d \in Z^d} \sup_{i \in \{0, \ldots, d\}} \hat{K}_0(x/\epsilon) \xrightarrow{\epsilon \to 0} 0, \]

and therefore

\[ 0 \leq \sup_{v: \sum_{j} |v_j| = 1} \langle v, \epsilon^d K[\epsilon^d Z^d] v \rangle \leq 1, \]

for small enough values of \( \epsilon \), and in this case the DPP \( X_\epsilon \) is then well defined.

Now, we prove the weak convergence of the discrete DPPs to the continuous one by showing the pointwise convergence of their Laplace functionals (see Daley & Vere-Jones, 2007, proposition 11.1.VIII). We recall that the Laplace functional of a point process \( Y \) is defined as

\[ L_Y(f) := \mathbb{E}_Y \left[ \exp \left( -\sum_{x \in Y} f(x) \right) \right], \]

for all nonnegative continuous function \( f \) vanishing outside a bounded set. Let \( D \) be a compact set of \( \mathbb{R}^d \) and \( f : \mathbb{R}^d \to \mathbb{R} \) be a continuous function vanishing outside \( D \). We define the kernel

\[ K_f : (x, y) \mapsto \sqrt{1 - e^{-f(x)} K_0(y - x)} \sqrt{1 - e^{-f(y)}}, \]

and call \( K_f \) its associated integral operator. Then, the Laplace transform of the continuous DPP \( X \) reads (see Shirai & Takahashi, 2003)

\[ L_X(f) = \det(I - K_f) = \exp \left( -\sum_{n \geq 1} \frac{1}{n} \text{Tr}(K^n_f) \right), \]

and for all \( \epsilon \), the rescaled DPP \( \epsilon X_\epsilon \) has the same distribution as a DPP on \( \epsilon^d Z^d \) with kernel \( \epsilon^d K_0(y - x) \) hence its Laplace transform reads

\[ L_{\epsilon X_\epsilon}(f) = \det \left( I - \epsilon^d K_f[\epsilon^d Z^d] \right) \]

\[ = \exp \left( -\sum_{n \geq 1} \frac{\epsilon^{dn}}{n} \text{Tr} \left( K_f[D \cap \epsilon^d Z^d]^n \right) \right) \]

\[ = \exp \left( -\sum_{n \geq 1} \frac{1}{n} \left( \epsilon^{dn} \sum_{x_1, \ldots, x_n \in D \cap \epsilon Z^d} K_f(x_1, x_2) \cdots K_f(x_{n-1}, x_n) K_f(x_n, x_1) \right) \right). \]

For all \( n \geq 1 \), we have the convergence of the following Riemann sum on the compact sets \( D^n \):
\[ \epsilon \frac{d}{dn} \sum_{x_1, \ldots, x_n \in D \cap \epsilon \mathbb{Z}^d} K_f(x_1, x_2) \cdots K_f(x_{n-1}, x_n) K_f(x_n, x_1) \xrightarrow{\epsilon \to 0} \int_{D^n} K_f(x_1, x_2) \cdots K_f(x_{n-1}, x_n) K_f(x_n, x_1) \, dx = \text{Tr}(\mathcal{K}_f^n). \]

Moreover, we have
\[ \text{Tr} \left( \left( \epsilon^d K_f \left[ D \cap \epsilon \mathbb{Z}^d \right] \right)^n \right) \leq \lambda_{\max} \left( \epsilon^d K_f \left[ D \cap \epsilon \mathbb{Z}^d \right] \right)^{n-1} \text{Tr} \left( \epsilon^d K_f \left[ D \cap \epsilon \mathbb{Z}^d \right] \right), \]

and since \( \mathcal{K}_f \leq \mathcal{K} \) then \( \lambda_{\max}(\epsilon^d K_f[D \cap \epsilon \mathbb{Z}^d]) \leq \lambda_{\max}(\epsilon^d K[D \cap \epsilon \mathbb{Z}^d]) \leq \lambda_{\max}(\epsilon^d K[\epsilon \mathbb{Z}^d]) \) which we showed was arbitrary close to \( \|K_0\|_\infty < 1 \) for small enough \( \epsilon \), then by the dominated convergence theorem we get that
\[ L_{X_\epsilon}(f) \xrightarrow{\epsilon \to 0} L_X(f), \]

which proves the weak convergence of the distributions of \( \epsilon X_\epsilon \) towards the distribution of \( X \) when \( \epsilon \) goes towards 0.

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### APPENDIX. TECHNICAL LEMMAS

**Lemma 2.** Let $n \in \mathbb{N}$ and $A, B$ be two $n \times n$ positive semi-definite matrices. Then,

$$0 \leq \log \det(I + AB) \leq \text{Tr}(AB).$$

**Proof.** We first assume that $B$ is the identity matrix. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues (with multiplicity) of $A$. Then,

$$0 \leq \log \det(I + A) = \sum_{i=1}^{n} \log(1 + \lambda_i) \leq \sum_{i=1}^{n} \lambda_i = \text{Tr}(A).$$

In the general case, Sylvester’s determinant identity gives us
Lemma 3. Let \( f : \mathcal{X}^k \mapsto \mathbb{R} \) be a translation invariant function such that

\[
(x_2, \ldots, x_k) \mapsto f(0, x_2, \ldots, x_k) \in L^1 \left( \mathcal{X}^{k-1}, \nu^{k-1} \right),
\]

and let \( W_n \) be a sequence of increasing compact subsets of \( \mathcal{X} \). Then, for any \( r > 0 \), we have

\[
\left| \int_{\mathcal{X}^{k-1}} f(0, x_2, \ldots, x_k)\,d\nu(x_2)\ldots d\nu(x_k) - \frac{1}{|W_n|} \int_{W_n^k} f(x)\,d\nu^k(x) \right| \\
\leq \int_{(B(0, r)^C)^{k-1}} |f(0, y)|\,d\nu^{k-1}(y) + \frac{1}{|W_n|} \left| (\partial W_n \oplus r) \cap W_n \right| \|f(0, \cdot)\|_{L^1},
\]

where \( B(0, r)^C \) is the complement of the euclidian ball centered at the origin with radius \( r \).

In particular, if there exists a sequence \( (r_n)_{n \geq 0} \) satisfying

\[
|\partial W_n \oplus r_n \cap W_n| = o(|W_n|), \quad \text{A1}
\]

then

\[
\frac{1}{|W_n|} \int_{W_n \times \mathcal{X}^{k-1}} f(x)\,d\nu^k(x) \xrightarrow{n \to \infty} \int_{\mathcal{X}^{k-1}} f(0, x_2, \ldots, x_k)\,d\nu(x_2)\ldots d\nu(x_k). \quad \text{A2}
\]

Proof. We write \( W_n \ominus r \) for the set \( W_n \setminus (\partial W_n \oplus r) \) of points in \( W_n \) at distance at least \( r \) from the boundary of \( W_n \). Since \( f \) is translation invariant then the right term in (A2) is equal to

\[
\frac{1}{|W_n|} \int_{W_n \times \mathcal{X}^{k-1}} f(x)\,d\nu^k(x).
\]

As a consequence,

\[
\left| \int_{\mathcal{X}^{k-1}} f(0, x_2, \ldots, x_k)\,d\nu(x_2)\ldots d\nu(x_k) - \frac{1}{|W_n|} \int_{W_n^k} f(x)\,d\nu^k(x) \right| \\
= \frac{1}{|W_n|} \left| \int_{W_n \times (\mathcal{X}^{k-1} \setminus W_n^{k-1})} f(x)\,d\nu^k(x) \right| \\
= \frac{1}{|W_n|} \left| \int_{W_n \ominus r} \left( \int_{\mathcal{X}^{k-1} \setminus W_n^{k-1}} f(x)\,d\nu(x_2)\ldots d\nu(x_k) \right)\,d\nu(x_1) \right| \\
\hspace{2cm} + \frac{1}{|W_n|} \left( \int_{(\partial W_n \oplus r) \cap W_n} \int_{\mathcal{X}^{k-1} \setminus W_n^{k-1}} f(x)\,d\nu(x_2)\ldots d\nu(x_k) \right)\,d\nu(x_1) \\
\leq \frac{1}{|W_n|} \int_{W_n \ominus r} \int_{\mathcal{X}^{k-1}} |f(0, y)| \mathbb{1}_{\{v_i, \|y\| > r\}} \,d\nu^{k-1}(y)\,d\nu(x) \\
\hspace{2cm} + \frac{1}{|W_n|} \int_{(\partial W_n \oplus r) \cap W_n} \int_{\mathcal{X}^{k-1}} |f(0, y)|\,d\nu^{k-1}(y)\,d\nu(x)
\]
\[ \leq \int_{(\mathbb{R}(0,r)^{c})^{d-1}} |f(0,y)|d\nu^{k-1}(y) + \frac{|(\partial W_n \oplus r) \cap W_n|}{|W_n|}||f(0,\cdot)||_{L^1}. \Box \]

**Proposition 5.** Let \( X \) be a DPP with Bessel-type kernel \( K_{0}^{\rho,\alpha} \), as defined in Table 1, observed on a window \( W \subset \mathbb{R}^d \). Recall that \( \rho_{\text{max}} \), given in Table 1, is the upper bound of \( \rho \) for which \( X \) is well-defined. Then, for all \( \alpha > 0 \) such that \( N(W)/|W| \leq \rho_{\text{max}} \),

\[
\arg \max_{0 \leq \rho \leq \rho_{\text{max}}} \tilde{\ell}(\rho, \alpha|X) = \left\{ \frac{N(W)}{|W|} \right\}. \tag{A3}
\]

**Proof.** By noticing that \( \rho_{\text{max}} \) is the volume of the \( d \)-dimensional ball with radius \( \sqrt{d/(2\pi^2 \alpha^2)} \), we get from the expression of \( \hat{K}_{0}^{\rho,\alpha} \) in Table 1 that

\[
\int_{\mathbb{R}^d} \log(1 - \hat{K}_{0}^{\rho,\alpha}(x))dx = \rho_{\text{max}} \log(1 - \rho/\rho_{\text{max}}).
\]

Moreover, \( L_{0}^{\rho,\alpha} \) can be written as \( \rho F^{\alpha}/(1 - \rho/\rho_{\text{max}}) \), where \( F^{\alpha} \) is a function not depending on \( \rho \) (see Table 2). Therefore, \( \log \det(L_{0}^{\rho,\alpha}[X \cap W]) \) can be expressed as the sum of

\[
N(W) \log \left( \frac{\rho}{1 - \rho/\rho_{\text{max}}} \right),
\]

and an expression not depending on \( \rho \). As a consequence, \( \tilde{\ell}(\rho, \alpha|X) \) is twice differentiable with respect to \( \rho \) with derivative

\[
\frac{-1}{1 - \rho/\rho_{\text{max}}} + \frac{N(W)}{|W|\rho(1 - \rho/\rho_{\text{max}})}.
\]

It is easy to see that this expression vanishes only when \( \rho = N(W)/|W| \) with the second derivative being negative at this point, concluding the proof. \( \Box \)