Higher-order approximate confidence intervals

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Abstract

We derive accurate confidence intervals based on higher-order approximate quantiles of
the score function. The coverage approximation error is $O(n^{-3/2})$ while the approximation
error of confidence intervals based on the asymptotic normality of MLEs is $O(n^{-1/2})$.
Monte Carlo simulations confirm theoretical findings. An implementation for regression
models and real data applications are provided.

Key words: Accurate confidence intervals; Maximum likelihood estimates; Modified score
equations; Nuisance parameters; Regression models.

1 Introduction

In applied statistics it is often the case that the practitioner wishes to estimate the parameters
of a statistical model fitted to a dataset. A point estimate is a single realization from the
distribution of the possible values of the chosen estimator and does not carry information on
its uncertainty. An uncertainty description is provided by confidence intervals.

Confidence intervals can, in principle, be constructed from the distribution of the estimator,
but the exact distribution is unknown except in particular situations. The most common ap-
proach for estimating parameters of a parametric family of distributions is the use of maximum
likelihood estimates (MLEs) coupled with the corresponding first-order asymptotic normal ap-
proximation. For large sample sizes, this approach is reasonably reliable and leads to virtually
unbiased estimates and confidence intervals with approximately correct coverage. In small and
moderate sized samples, the distribution of the MLEs may be far from normal, exhibiting bias,
skewness, and also kurtosis that are not compatible with the normal distribution.

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A vast literature on bias adjustments for MLEs is available; see for instance Cordeiro & Cribari-Neto (2014), Ferrari & Cribari-Neto (1998), Firth (1993), Pagui et al. (2017), and references therein. Additionally, effort has been made to correct the coverage of approximate confidence intervals in limited samples. Bartlett (1953) proposed an approximate confidence interval based on a skewness correction to the score function. An alternative possible approach is the use of computer intensive methods such as the bootstrap procedure (DiCiccio & Efron, 1996).

This paper proposes accurate approximate confidence intervals for a scalar parameter of interest possibly in the presence of a vector of nuisance parameters in general parametric families. We construct the proposed confidence intervals from a third-order approximation to the quantiles of the score function. Some noticeable features of the proposed confidence intervals are: first, the coverage approximation error is $O(n^{-3/2})$ while the approximation error of confidence intervals based on the asymptotic normality of MLEs is $O(n^{-1/2})$; second, they are equivariant under interest-respecting reparameterizations; third, they account for skewness and kurtosis of the distribution of the score function; fourth, they do not require computer intensive calculations; fifth, the confidence limits are simply computed from modified score equations.

In Section 2, we deal with the case where no nuisance parameter is present. In Section 3, we extend the results to the nuisance parameter situation. An implementation of accurate approximate confidence intervals for regression models is presented in Section 4. Monte Carlo evidence on the performance of the modified confidence intervals is presented in Section 5. Applications are presented and discussed in Section 6. The paper ends with concluding remarks in Section 7. Some technical details are left for two appendices.

2 No nuisance parameters

Let $y$ be the data, for which we consider a regular parametric model indexed by a parameter $\theta \in \Theta \subset \mathbb{R}$. Let $\ell(\theta)$ be the log-likelihood function and $U(\theta) = \partial \ell(\theta) / \partial \theta$ be the score function. The first four cumulants of $U(\theta)$ are $\kappa_1 = \mathbb{E}_\theta(U(\theta)) = 0$, $\kappa_2 = \mathbb{E}_\theta(U(\theta)^2) = \text{var}_\theta(U(\theta)) = i(\theta)$, $\kappa_3 = \mathbb{E}_\theta(U(\theta)^3)$, $\kappa_4 = \mathbb{E}_\theta(U(\theta)^4) - 3\kappa_2^2$, where $i(\theta)$ is the Fisher information. We assume that all $\kappa$’s are of order $O(n)$, where $n$ is the sample size.

The maximum likelihood estimator of $\theta$, $\hat{\theta}$, comes from the estimating equation $U(\theta) = 0$, and has a normal distribution with mean $\theta$ and variance $\kappa_2^{-1} = i(\theta)^{-1}$ asymptotically. Let $u_\alpha$ be the $\alpha$-quantile of a standard normal distribution and let $\hat{\theta}_\alpha = \hat{\theta} - u_\alpha / \sqrt{i(\hat{\theta})}$. For $\alpha < 0.5$, $(-\infty, \hat{\theta}_\alpha]$ and $[\hat{\theta}_{1-\alpha}, +\infty)$ are the usual Wald-type one-sided confidence intervals for $\theta$ with approximate confidence level $\gamma = 1 - \alpha$. Analogously, $[\hat{\theta}_{1-\alpha/2}, \hat{\theta}_{\alpha/2}]$ is the usual Wald-type two-sided confidence interval for $\theta$ with approximate confidence level $\gamma = 1 - \alpha$. These confidence intervals are first-order accurate; the approximation error is $O(n^{-1/2})$. 

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The Cornish-Fisher expansion of the $\alpha$-quantile of the score function is

$$q_\alpha(U(\theta)) = u_\alpha \sqrt{\kappa_2} + \frac{1}{6} \frac{\kappa_3}{\kappa_2} (u_\alpha^2 - 1) + \frac{1}{24} \frac{\kappa_4}{\kappa_2^3} (u_\alpha^3 - 3u_\alpha) - \frac{1}{36} \frac{\kappa_5^2}{\kappa_2^5/2} (2u_\alpha^3 - 5u_\alpha) + O(n^{-3/2})$$

(Pace & Salvan, 1997, Chap. 10, eq. (10.19)). We then define the $\alpha$-quantile modified score

$$\tilde{U}_\alpha(\theta) = U(\theta) - u_\alpha \sqrt{\kappa_2} - \frac{1}{6} \frac{\kappa_3}{\kappa_2} (u_\alpha^2 - 1) - \frac{1}{24} \frac{\kappa_4}{\kappa_2^3} (u_\alpha^3 - 3u_\alpha) + \frac{1}{36} \frac{\kappa_5^2}{\kappa_2^5/2} (2u_\alpha^3 - 5u_\alpha).$$

(1)

We have that $q_\alpha(\tilde{U}_\alpha(\theta)) = O(n^{-3/2})$ and

$$P_{\theta}(\tilde{U}_\alpha(\theta) \leq 0) = \alpha + O(n^{-3/2});$$

(2)

see Appendix A. Let $\tilde{\theta}_\alpha$ be the solution of the $\alpha$-quantile modified score equation

$$\tilde{U}_\alpha(\theta) = 0.$$  

(3)

If $\tilde{U}_\alpha(\theta)$ is monotonically decreasing in $\theta$, the events $\tilde{U}_\alpha(\theta) \leq 0$ and $\tilde{\theta}_\alpha \leq \theta$ are equivalent and

$$P_{\theta}(\tilde{\theta}_\alpha \leq \theta) = \alpha + O(n^{-3/2}).$$

(4)

Hence, $\theta$ is approximately the $\alpha$-quantile of the distribution of $\tilde{\theta}_\alpha$, and $\tilde{\theta}_{0.5}$ is a median bias reduced estimator of $\theta$ (Pagui et al., 2017); the approximation error is of order $n^{-3/2}$.

Equation (4) can be used to obtain confidence limits for $\theta$. For $\alpha < 0.5$, we have that $(-\infty, \tilde{\theta}_\alpha]$ and $[\tilde{\theta}_{1-\alpha}, +\infty)$ are one-sided confidence intervals for $\theta$ with approximate confidence level $\gamma = 1 - \alpha$. Analogously, $[\tilde{\theta}_{1-\alpha/2}, \tilde{\theta}_{\alpha/2}]$ is a two-sided confidence interval for $\theta$ with approximate confidence level $\gamma = 1 - \alpha$. These confidence intervals are third-order accurate, i.e. the approximation error is $O(n^{-3/2})$. This is an improvement over the $O(n^{-1/2})$ approximation error of confidence intervals based on the asymptotic normality of MLEs.

**Remark.** $\tilde{\theta}_\alpha$ is equivariant under reparameterization. Let $\omega(\theta)$ be a smooth reparameterization with inverse $\theta(\omega)$. In the parameterization $\omega$, the score function is $U(\theta(\omega)) \theta'(\omega)$, and its first four cumulants are $\kappa_r \theta'(\omega)^r$, $r = 1, 2, 3, 4$, where $\theta'(\omega) = d\theta(\omega)/d\omega$. Hence, the $\alpha$-quantile modified score in the parameterization $\omega$ is $\tilde{U}_\alpha(\theta(\omega)) \theta'(\omega)$, and the solution of $\tilde{U}_\alpha(\theta(\omega)) \theta'(\omega) = 0$ is $\tilde{\omega}_\alpha = \omega(\tilde{\theta}_\alpha)$. Then confidence sets for $\omega$ are constructed as in the previous paragraph with $\tilde{\theta}_\alpha$ replaced by $\omega(\tilde{\theta}_\alpha)$. If $\omega(\theta)$ is monotonically increasing, $P_{\theta}(\tilde{\omega}_\alpha \leq \omega) = P_{\theta}(\tilde{\theta}_\alpha \leq \theta)$. Otherwise, $P_{\theta}(\tilde{\omega}_\alpha \geq \omega) = P_{\theta}(\tilde{\theta}_\alpha \leq \theta)$.

**Example 1.** *One parameter exponential family.* For a random sample $y_1, \ldots, y_n$ of a one parameter exponential family with canonical parameter $\theta$ and pdf

$$f(y; \theta) = \exp\{\theta T(y) - A(\theta)\} h(y),$$

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we have $U(\theta) = \sum_{i=1}^{n} T(y_i) - n \, dA(\theta)/d\theta$, $\kappa_r = n \, d^r A(\theta)/d\theta^r$, $r = 2, 3, 4$. The modified score equation is given as in (3) by plugging these quantities in (1).

For a random sample of an exponential distribution with mean $1/\theta$, we have $T(y) = -y$, $A(\theta) = -\log(\theta)$, $U(\theta) = n/\theta - \sum_{i=1}^{n} y_i$, $\hat{\theta} = n/\sum_{i=1}^{n} y_i$, $\kappa_2 = n/\theta^2$, $\kappa_3 = -2n/\theta^3$, $\kappa_4 = 6n/\theta^4$, $\hat{\theta}_\alpha = \hat{\theta}(1 - u_\alpha/\sqrt{n})$, and $\tilde{\theta}_\alpha = \hat{\theta}(1 - c_{n,\alpha}/\sqrt{n})$, where $c_{n,\alpha} = u_\alpha - (u_\alpha^2 - 1)/(3\sqrt{n}) + (u_\alpha^3 - 7u_\alpha)/(36n)$. A third-order median bias reduced estimator of $\theta$ is $\tilde{\theta}_{0.5} = \hat{\theta}(1 - 1/(3\sqrt{n}))$.

The difference between the lengths of the two-sided confidence sets $[\tilde{\theta}_{1-\alpha/2}, \tilde{\theta}_{\alpha/2}]$ and $[\hat{\theta}_{1-\alpha/2}, \hat{\theta}_{\alpha/2}]$ with nominal level $\gamma = 1 - \alpha$, for $\alpha < 0.5$, is $\hat{\theta} u_{1-\alpha/2}(u_{1-\alpha/2}^2 - 7)/(18n^{3/2})$. Hence the modified confidence interval we propose is shorter than the confidence interval based on the asymptotic normality of $\hat{\theta}$ whenever $\gamma < 1 - 2(1 - \Phi(\sqrt{7})) \approx 0.9918$, where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

Table 1 presents approximate 95% confidence intervals for $\theta$ based on samples with mean equal to 1, i.e. the MLE of $\theta$ is $\hat{\theta} = 1$, for different small sample sizes, namely $n = 3, 5, 7$. The approximate confidence limits considered are the usual Wald-type confidence limits and an adjusted version that replaces $\hat{\theta}$ by $\tilde{\theta}_{0.5}$, and the confidence limits proposed here. Like the usual Wald-type confidence limits, the adjusted version is first-order accurate, while those proposed in this paper are third-order accurate. The table also gives the exact $1 - \alpha$ confidence intervals $[\chi^2_{\alpha/2,2n}/(2\sum_{i=1}^{n} y_i), \chi^2_{1-\alpha/2,2n}/(2\sum_{i=1}^{n} y_i)]$, where $\chi^2_{\alpha,n}$ is the $\alpha$-quantile of a chi-squared distribution with $n$ degrees of freedom. The figures in Table 1 reveal that the third-order approximate confidence limits proposed here are remarkably accurate even in very small samples. They lead to a considerable improvement over both first-order approximate confidence intervals. Both Wald-type confidence intervals may even include negative numbers, i.e. values outside the parameter space. Figure 1 shows plots of the score and modified score functions for illustrative purposes.

![Figure 1: Plots of the score and modified score functions with $1 - \alpha = 90\%$ (left), $95\%$ (middle), and $99\%$ (right); exponential distribution, $n = 5$.](image-url)
Table 1: Approximate and exact $1 - \alpha$ confidence intervals for $\theta$ for different sample sizes $n$

| $n$   | 90%       | 95%        | 99%        |
|-------|-----------|------------|------------|
| 3     | [0.05, 1.95] | [-0.13, 2.13] | [-0.49, 2.49] |
| 5     | [0.26, 1.74] | [0.12, 1.88] | [-0.15, 2.15] |
| 7     | [0.04, 1.73] | [0.22, 1.84] | [0.14, 3.11] |

3  Presence of nuisance parameters

Let $\theta = (\psi, \lambda)$, where $\psi$ is the scalar parameter of interest and $\lambda = (\lambda_1, \ldots, \lambda_p)$ is the nuisance parameter. Let the subscript $\psi$ refer to the parameter $\psi$ and the indices $a, b, c, \ldots$ refer to the components of $\lambda$, so that the log-likelihood derivatives are $U_\psi = U_\psi(\psi, \lambda) = \partial \ell(\psi, \lambda)/\partial \psi$, $U_a = U_a(\psi, \lambda) = \partial \ell(\psi, \lambda)/\partial \lambda_a$, $U_{ab} = \partial^2 \ell(\psi, \lambda)/\partial \lambda_a \partial \lambda_b$, $U_{n\psi a} = \partial^2 \ell(\psi, \lambda)/\partial \psi \partial \lambda_a$, etc., $a = 1, \ldots, p$. Consider the tensorial notation for the joint cumulants of derivatives of the log-likelihood function (Lawley, 1956; Hayakawa, 1977; McCullagh, 1984, 1987): $\kappa_\psi = E(U_\psi) = 0$, $\kappa_a = E(U_a) = 0$, $\kappa_{ab} = E(U_{ab})$, $\kappa_{abc} = E(U_{abc})$, $\kappa_{abcd} = E(U_{abcd})$, $\kappa_{a,b} = E(U_a U_b)$, $\kappa_{a,b,c} = E(U_a U_b U_c)$, $\kappa_{a,b,c,d} = E(U_a U_b U_c U_d)$, etc. All $\kappa$’s refer to a total over the sample and are, in general, of order $n$. In addition, $\kappa^{a,b}$ denotes the element $(a, b)$ of the inverse covariance matrix of $U_\lambda$. In the sequel, the Einstein convention of sum of repeated indices is used, i.e. if an index occurs both as a superscript and as a subscript in a single term then summation over that index is understood.

Let $\hat{\theta} = (\hat{\psi}, \hat{\lambda})$ be the MLE of $\theta$ and let $\hat{\lambda}_\psi$ be the MLE of $\lambda$ for fixed $\psi$. The profile log-likelihood function for $\psi$ is

$$\ell_P(\psi) = \ell(\psi, \hat{\lambda}_\psi),$$

and the score function derived from the profile log-likelihood function is

$$U_P(\psi) = \frac{\partial \ell_P(\psi)}{\partial \psi} = U_\psi(\psi, \hat{\lambda}_\psi).$$
The leading term of the expansion of $U_P(\psi)$ is the efficient score function for $\psi$, i.e. $\mathbf{U}_\psi = U_\psi - \beta_\psi^a U_a$, where $\beta_\psi^a = \kappa_{\psi,a} \kappa_{\psi,a}$. The ordinary score function minus its orthogonal projection onto the closed linear span of the score function for the nuisance parameter (Murphy & van der Vaart, 2000, eq 4). Approximate expressions for the first four cumulants of $\mathbf{U}_\psi$ are

$$
\kappa_1 = -\frac{1}{2} \kappa_{\psi,a}^a \left[ \kappa_{\psi,a,b} + \kappa_{\psi,a,b} - \beta_\psi^a (\kappa_{\psi,a} + \kappa_{\psi,a}) \right],
$$

$$
\kappa_2 = \kappa_{\psi,a} - \beta_\psi^a \kappa_{\psi,a},
$$

$$
\kappa_3 = \kappa_{\psi,a} \psi - 3 \beta_\psi^a \kappa_{\psi,a} \psi^2 + 3 \beta_\psi^a \beta_\psi^b \kappa_{\psi,a} \psi^2 - \beta_\psi^a \beta_\psi^b \kappa_{\psi,a} \psi^2 \kappa_{\psi,a},
$$

$$
\kappa_4 = \kappa_{\psi,a} \psi - 4 \beta_\psi^a \kappa_{\psi,a} \psi^2 + 6 \beta_\psi^a \beta_\psi^b \kappa_{\psi,a} \psi^2 + 4 \beta_\psi^a \beta_\psi^b \kappa_{\psi,a} \psi^2 \beta_\psi^d \kappa_{\psi,a} \psi^d;
$$

see Pagui et al. (2017, eq. (7)) for $\kappa_1$, $\kappa_2$, and $\kappa_3$, and Barndorff-Nielsen & Cox (1989, Chap. 7) for $\kappa_4$. If $\psi$ and $\lambda$ are orthogonal parameters, we have $\kappa_{\psi,a} = 0$ and $\beta_\psi^a = 0$, and hence the equations in (5) reduce to

$$
\kappa_1 = -\frac{1}{2} \kappa_{\psi,a}^a (\kappa_{\psi,a} + \kappa_{\psi,a}), \quad \kappa_2 = \kappa_{\psi,a}, \quad \kappa_3 = \kappa_{\psi,a,\psi}, \quad \kappa_4 = \kappa_{\psi,a,\psi,\psi}.
$$

The Cornish-Fisher expansion of the $\alpha$-quantile of $\mathbf{U}_\psi$ is

$$
g_\alpha(\mathbf{U}_\psi) = \kappa_1 + u_\alpha \sqrt{\kappa_2} + \frac{1}{6} \kappa_3 u_\alpha^2 - \frac{1}{24} \kappa_4 u_\alpha^3 - \frac{1}{36} \kappa_5 u_\alpha^5 + O(n^{-3/2});
$$

see Pace & Salvan (1997, eq. (10.19)). Since $\mathbf{U}_\psi$ is the leading term of the asymptotic expansion of $U_P(\psi)$, we define the $\alpha$-quantile modified profile score

$$
\tilde{U}_\alpha(\psi) = U_P(\psi) + M_{\psi,\alpha},
$$

where

$$
M_{\psi,\alpha} = -\kappa_1 - u_\alpha \sqrt{\kappa_2} - \frac{1}{6} \kappa_3 u_\alpha^2 - \frac{1}{24} \kappa_4 u_\alpha^3 - \frac{1}{36} \kappa_5 u_\alpha^5 + O(n^{-3/2});
$$

and denote by $\tilde{\psi}_\alpha$ the estimator that is solution of the $\alpha$-quantile modified score equation

$$
\tilde{U}_\alpha(\psi) = 0
$$

with $\lambda$ replaced by $\tilde{\lambda}_\psi$. To numerically obtain $\tilde{\psi}_\alpha$, one can solve a system of estimating equations consisting of the modified score equation for $\psi$, $U_\psi(\psi, \lambda) + M_{\psi,\alpha} = 0$, and the score equations $U_\alpha(\psi, \lambda) = 0$, for $a = 1, \ldots, p$, for the components of the nuisance parameter $\lambda$.

The $\alpha$-quantile modified profile score has $\alpha$-quantile zero with error of order $O(n^{-3/2})$, i.e., $P_\alpha(\tilde{U}_\alpha(\psi) \leq 0) = \alpha + O(n^{-3/2})$. If $\tilde{U}_\alpha(\psi)$ is monotonically decreasing in $\psi$, the events $\tilde{U}_\alpha(\psi) \leq 0$ and $\tilde{\psi}_\alpha \leq \psi$ are equivalent so that

$$
P_\alpha(\tilde{\psi}_\alpha \leq \psi) = \alpha + O(n^{-3/2}).
$$

Similarly to the one-parameter case, $\tilde{\psi}_\alpha$ can be used to obtain approximate confidence limits for $\psi$ with approximation error $O(n^{-3/2})$ in place of the $O(n^{-1/2})$ approximation error of the confidence limits obtained from the asymptotic normality of MLEs.
Remark. As in the one-parameter case, \( \tilde{\psi}_\alpha \) is equivariant under interest-respecting reparameterization. Let \( \omega(\phi, \tau) \) be a smooth reparameterization with \( \phi = \phi(\psi) \) and \( \tau = \tau(\lambda) \) being one-to-one functions with inverse \( \psi(\phi) \) and \( \lambda(\tau) \), respectively. In the parameterization \( \omega \), the score function for \( \phi \) is \( U_\phi(\psi(\phi), \lambda(\tau))\psi'(\phi) \) and the corresponding four cumulants in (5) are \( \kappa_{\phi, ab} = \kappa_{\psi, ab} \psi'(\phi) \), where \( \psi'(\phi) = d\psi(\phi)/d\phi \). Then, \( M_{\phi, \alpha} = M_{\psi, \alpha} \psi'(\phi) \); note, for instance, that \( \kappa_{\phi, ab} = \kappa_{\psi, ab} \psi'(\phi) \) and \( \beta_\phi^a = \beta_\psi^a \psi'(\phi) \). It follows that the \( \alpha \)-quantile modified score for \( \phi \) is \( U_\phi(\psi(\phi), \tau(\lambda)) + M_{\phi, \alpha} = (U_\psi(\psi(\phi), \lambda(\tau)) + M_{\psi, \alpha})\psi'(\phi) \). Hence, \( \tilde{\psi}_\alpha = \phi(\tilde{\psi}_\alpha) \). If \( \phi(\psi) \) is monotonically increasing, \( P_\theta(\tilde{\psi}_\alpha \leq \phi) = P_\theta(\tilde{\psi}_\alpha \leq \psi) \). Otherwise, \( P_\theta(\tilde{\psi}_\alpha \geq \phi) = P_\theta(\tilde{\psi}_\alpha \leq \psi) \).

Example 2. Gamma distribution with mean \( \mu \) and coefficient of variation \( \phi^{-1/2} \), \( \text{Gamma}(\mu, \phi) \). For a random sample \( y_1, \ldots, y_n \) of a \( \text{Gamma}(\mu, \phi) \) distribution with pdf

\[
f(y; \mu, \phi) = \frac{1}{\Gamma(\phi) \mu^\phi} y^{\phi-1} \exp\left(-\frac{\phi}{\mu}y\right), \quad y > 0, \quad \mu > 0, \quad \phi > 0,
\]

the score function for \( \mu \) and \( \phi \) are, respectively,

\[
U_\mu(\mu, \phi) = n\frac{\phi}{\mu^2}(\overline{y} - \mu)
\]

and

\[
U_\phi(\mu, \phi) = -n\Psi(\phi) + n + n \log \frac{\mu}{\phi} + \sum_{i=1}^{n} \log(y_i) - \frac{n}{\mu},
\]

where \( \Gamma(\cdot) \) is the gamma function, \( \Psi(\phi) = \Gamma'(\phi)/\Gamma(\phi) \), and \( \overline{y} = \sum_{i=1}^{n} y_i/n \). Here, \( \mu \) and \( \phi \) are orthogonal parameters and it comes from (6) that \( \kappa_{1\mu} = 0 \), \( \kappa_{2\mu} = n\phi/\mu^2 \), \( \kappa_{3\mu} = 2n\phi/\mu^3 \), and \( \kappa_{4\mu} = 6n\phi/\mu^4 \). Additionally, \( \kappa_{1\phi} = n/(2\phi) \), \( \kappa_{2\phi} = n(-1/\phi + \Psi^{(1)}(\phi)) \), \( \kappa_{3\phi} = n\left(1/\phi^2 + \Psi^{(2)}(\phi)\right) \), and \( \kappa_{4\phi} = n\left(-2/\phi^3 + \Psi^{(3)}(\phi)\right) \), where \( \Psi^{(r)}(\phi) = d^r \Psi(\phi)/d\phi^r \). These cumulants may be plugged in (7)-(8) to compute confidence sets for \( \mu \) and \( \phi \).

Example 3. Symmetric distributions. Let \( y_1, \ldots, y_n \) be a random sample of a continuous symmetric distribution \( \text{S}(\mu, \phi) \) with location parameter \( \mu \in \mathbb{R} \) and scale parameter \( \phi > 0 \), and pdf

\[
f(y; \mu, \phi) = \frac{1}{\phi} v\left(\left(\frac{y - \mu}{\phi}\right)^2\right), \quad y \in \mathbb{R},
\]

for some function \( v \), called the density generating function (dgf), such that \( v(u) \geq 0 \), for all \( u \geq 0 \), and \( \int_{0}^{\infty} u^{-1/2}v(u)du = 1 \). Some members of the symmetric class of distributions are the normal, Student-\( t \), type I logistic, type II logistic, and power exponential, with corresponding dgf: \( v(u) = (2\pi)^{-1/2}e^{-u^2/2} \), \( v(u) = \nu^{\nu/2}B(1/2, \nu/2)^{-1}(\nu + u)^{-\nu - 1/2} \), with \( \nu > 0 \) and \( B(\cdot, \cdot) \) being the beta function, \( v(u) = ce^{-u}(1 + e^{-u})^{-2} \), with \( c \cong 1.4843 \) being the normalizing constant, \( v(u) = e^{-\sqrt{u}}(1 + e^{-\sqrt{u}})^{-2} \), and \( v(u) = (1/C(\nu)) \exp\{-\frac{1}{2}u^{1/(1+\nu)}\} \) with \(-1 < \nu \leq 1 \) and \( C(\nu) = \)
\[ l(\theta) = -n \log(\phi) + \sum_{i=1}^{n} \log v(x_i^2), \]

\[ U_\mu(\mu, \phi) = \phi^{-1} \sum_{i=1}^{n} w_i \epsilon_i, \quad U_\phi(\mu, \phi) = -n \phi^{-1} + \phi^{-1} \sum_{i=1}^{n} w_i \epsilon_i^2, \]

where \( \epsilon_i = (y_i - \mu)/\phi \) and \( w_i = -2d \log v(u)/du|_{u=\epsilon_i^2} \). The parameters \( \mu \) and \( \phi \) are orthogonal and we have from (6) that \( \kappa_{1\mu} = \kappa_{3\mu} = 0, \quad \kappa_{2\mu} = n \delta_{20000}/\phi^2, \) and \( \kappa_{4\mu} = n(\delta_{40000} - 3\delta_{20000}^2)/\phi^4 \).

Additionally,

\[ \kappa_{1\phi} = -\frac{1}{2\phi} \frac{\delta_{00101} + 2\delta_{11001}}{\delta_{20000}}, \quad \kappa_{2\phi} = \frac{n}{\phi^2} \left( \delta_{20002} - 1 \right), \quad \kappa_{3\phi} = \frac{2n}{\phi^3} \left( 1 + \delta_{11003} \right), \]

\[ \kappa_{4\phi} = \frac{n}{\phi^4} \left( \delta_{40004} + 4\delta_{30003} + 12\delta_{20002} - 3\delta_{20002}^2 - 6 \right). \]

Here, \( \delta_{abced} = \mathbb{E}(s^a s^b s^c s^d s^e), \) for \( a, b, c, d, e \in \{0, 1, 2, 3, 4\}, \) \( s^{(r)} = d^r s(\epsilon)/d\epsilon^r \) with \( s(\epsilon) = \log v(\epsilon^2) \) and \( \epsilon \sim S(0, 1). \) For the symmetric distributions listed above the \( \delta \)'s are given in Uribe-Opazo et al. (2008). These cumulants may be plugged in (7)–(8) to compute confidence sets for \( \mu \) and \( \phi \).

4 An implementation for regression models

Let \( f(y; \mu, \phi) \) be the pdf of a parametric distribution with two parameters, \( \mu \) and \( \phi \), and let \( y_1, \ldots, y_n \) be independent random variables, where each \( y_i \) has pdf \( f(y; \mu_i, \phi_i) \). Consider the regression model that specifies \( \mu_i \) and \( \phi_i \) as

\[ g(\mu_i) = x_i^T \beta, \quad h(\phi_i) = z_i^T \gamma. \]

Here, \( \beta = (\beta_1, \ldots, \beta_q)^T \) and \( \gamma = (\gamma_1, \ldots, \gamma_m)^T \) are vectors of unknown parameters \( (\beta \in \mathbb{R}^q, \gamma \in \mathbb{R}^m, q + m = p < n) \), and \( x_i = (x_{i1}, \ldots, x_{iq})^T \) and \( z_i = (z_{i1}, \ldots, z_{im})^T \) collect observations on covariates, which are assumed fixed and known. The link functions, \( g(\cdot) \) and \( h(\cdot) \), are strictly monotonic and twice differentiable, and map, respectively, the range of \( \mu_i \) and \( \phi_i \), into \( \mathbb{R} \). We assume that the model matrices \( X \) and \( Z \), with element \( (i, k) \) given by \( x_{ik} \) and \( z_{ik} \), respectively, are of full rank, i.e. \( \text{rank}(X) = q \) and \( \text{rank}(Z) = m \).

Let

\[
U_{\mu} = \frac{\partial \log f(y_i; \mu_i, \phi_i)}{\partial \mu_i}, \quad U_{\phi} = \frac{\partial \log f(y_i; \mu_i, \phi_i)}{\partial \phi_i},
\]

\[
U_{\mu_\mu} = \frac{\partial^2 \log f(y_i; \mu_i, \phi_i)}{\partial \mu_i \partial \mu_i}, \quad U_{\mu_\phi} = \frac{\partial^2 \log f(y_i; \mu_i, \phi_i)}{\partial \mu_i \partial \phi_i}, \quad U_{\phi_\phi} = \frac{\partial^2 \log f(y_i; \mu_i, \phi_i)}{\partial \phi_i \partial \phi_i}.
\]
Let $\ell(\theta) = \sum_{i=1}^{n} \log f(y_i; \mu_i, \phi_i)$ be the log-likelihood function for $\theta = (\beta^\top, \gamma^\top)^\top$. The first and second order log-likelihood derivatives with respect to the unknown parameters are

$$
U_{\beta_r} = \sum_{i=1}^{n} x_{ir} \frac{1}{g'(\mu_i)} U_{\mu_i},
U_{\gamma_r} = \sum_{i=1}^{n} z_{ir} \frac{1}{h'(\phi_i)} U_{\phi_i},
$$

$$
U_{\beta_s\beta_t} = \sum_{i=1}^{n} \left\{ x_{is} x_{it} \frac{1}{g'(\mu_i)^2} U_{\mu_i \mu_t} - x_{is} x_{it} \frac{g''(\mu_i)}{g'(\mu_i)^3} U_{\mu_i} \right\},
$$

$$
U_{\beta_s\gamma_t} = \sum_{i=1}^{n} \left\{ x_{is} z_{it} \frac{1}{h'(\phi_i)} U_{\mu_i \phi_t} \right\},
$$

$$
U_{\gamma_s\gamma_t} = \sum_{i=1}^{n} \left\{ z_{is} z_{it} \frac{1}{h'(\phi_i)^2} U_{\phi_i \phi_t} - z_{is} z_{it} \frac{h''(\phi_i)}{h'(\phi_i)^3} U_{\phi_i} \right\}.
$$

Cumulants of the log-likelihood derivatives may be obtained from cumulants of derivatives of $\log f(y_i; \mu_i, \phi_i)$. For instance,

$$
\kappa_{\beta_s,\beta_t} = E(U_{\beta_s} U_{\beta_t}) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{is} x_{it} \frac{1}{g'(\mu_i)^2} E(U_{\mu_i} U_{\mu_j}) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{is} x_{it} \frac{1}{g'(\mu_i)^2} \kappa_{\mu_i,\mu_j},
$$

where $\kappa_{\mu_i,\mu_j} = E(U_{\mu_i}^2)$, since $E(U_{\mu_i} U_{\mu_j}) = 0$, for $i \neq j$.

To computationally implement the cumulants of the profile score given in (5), it is useful to collect them in arrays. The notation is as follows. Let $\kappa_{\mu,\mu}$, $\kappa_{\mu,\phi}$, etc., be joint cumulants of derivatives of $\log f(y; \mu, \phi)$, and let $\text{diag}\{u\}$ represent an $n \times n$ diagonal matrix with diagonal elements $u_1, \ldots, u_n$. Let $K_{\mu,\mu} = \text{diag}\{\kappa_{\mu,\mu}\}$, $K_{\mu,\phi} = \text{diag}\{\kappa_{\mu,\phi}\}$, $K_{\phi,\phi} = \text{diag}\{\kappa_{\phi,\phi}\}$, $K_{\mu,\mu,\mu} = \text{diag}\{\kappa_{\mu,\mu,\mu}\}$, $K_{\mu,\phi,\mu} = \text{diag}\{\kappa_{\mu,\phi,\mu}\}$, $K_{\phi,\mu,\mu} = \text{diag}\{\kappa_{\phi,\mu,\mu}\}$, etc. Let $T = \text{diag}\{1/g'(\mu)\}$, $H = \text{diag}\{1/h'(\phi)\}$, $S = \text{diag}\{-g''(\mu)/g'(\mu)^2\}$, and $Q = \text{diag}\{-h''(\phi)/h'(\phi)^2\}$. Let $D_X$ be the $n \times n \times q$ array with element $(i, j, k)$ given by $D_X[i, j, k] = x_{ik}$, if $i = j$, and $= 0$, if $i \neq j$, i.e. $D_X$ is formed by joining the $n \times n$ diagonal matrices with diagonal elements $x_{1k}, \ldots, x_{nk}$, for $k = 1, \ldots, q$. Analogously, $D_Z$ denotes the $n \times n \times m$ array formed by joining the $n \times n$ diagonal matrices with diagonal elements $z_{1k}, \ldots, z_{nk}$, for $k = 1, \ldots, m$.

Now, we introduce a product of multidimensional arrays, denoted here by “$\circ$”, as follows. If $A$ is an $r \times c \times d$ array and $B$ is a $c \times e$ array (matrix) then $C = A \circ B$ is the $r \times e \times d$ array such that $C[i, \cdot, \cdot] = A[\cdot, i, \cdot] \times B[\cdot, \cdot]$, where “$\times$” is the usual matrix product. If $A$ is an $r \times c \times d$ array and $B$ is a $c \times e \times f$ array then $C = A \circ B$ is the $r \times e \times d \times f$ array such that $C[i, \cdot, j, \cdot] = A[\cdot, i, \cdot] \times B[\cdot, j, \cdot]$. If $A$ is an $r \times c$ array and $B$ is a $c \times e \times f$ array, then $C = A \circ B$ is the $r \times e \times f$ array such that $C[i, \cdot, \cdot] = A[i, \cdot] \times B[\cdot, \cdot]$. An R language (R Core Team, 2018) implementation of the proposed multidimensional arrays product “$\circ$” is given in function %m%, available with some examples at https://github.com/elianecpinheiro/MultiDimensionalArrayProduct.

The second, third, and fourth cumulants of the log-likelihood function are written as multidimensional arrays in equations (B.1)–(B.4) in Appendix B.1.
The profile cumulants (5) when \( \psi = \beta_k \) or \( \psi = \gamma_k \) can be obtained from (B.1)–(B.4). The multidimensional arrays (B.1)–(B.4), the profile cumulants (5) when \( \psi = \beta_k \), and when \( \psi = \gamma_k \), and the modified confidence intervals are implemented in the functions `cumulants`, `cumulantsbeta`, `cumulantsgamma`, and `hoaci`, respectively, in the R language (R Core Team, 2018). The functions are available at https://github.com/elianecpinheiro/HOACI.

We shall note that not all the cumulants of log-likelihood derivatives in (B.1)–(B.4) are necessary for computing the modified confidence intervals when \( \beta \) and \( \gamma \) are orthogonal parameters. In this case, \( \kappa_{\mu,\phi} = 0 \) and it comes from (B.1)–(B.4) and (5) that the terms that involve \( \kappa_{\mu,\mu,\phi}, \kappa_{\mu,\mu,\phi,\phi}, \) and \( \kappa_{\mu,\phi,\phi,\phi} \) are zero; hence these cumulants are not needed and may be replaced by zero for computational purposes.

The formulas for the arrays of cumulants in (B.1)–(B.4) can be extended to nonlinear regression models. Let (9) be replaced by

\[
g(\mu_i) = \eta_i = \eta(x_i, \beta) \quad \text{and} \quad h(\phi_i) = \delta_i = \delta(z_i, \gamma),
\]

where \( x_i \) and \( z_i \) are known fixed vectors of dimensions \( q' \) and \( m' \) respectively, and \( \eta(\cdot, \cdot) \) and \( \delta(\cdot, \cdot) \) are allowed to be nonlinear functions in the second argument. Let \( \mathcal{X} \) be the derivative matrix of \( \eta = (\eta_1, \ldots, \eta_n)^\top \) with respect to \( \beta^\top \). Analogously, let \( \mathcal{Z} \) be the derivative matrix of \( \delta = (\delta_1, \ldots, \delta_n)^\top \) with respect to \( \gamma^\top \). In the linear case (9), \( \mathcal{X} = X \) and \( \mathcal{Z} = Z \). We assume that \( \text{rank}(\mathcal{X}) = q \) and \( \text{rank}(\mathcal{Z}) = m \) for all \( \beta \) and \( \gamma \). The arrays containing the needed cumulants of log-likelihood derivatives in the nonlinear case coincide with those in (B.1)–(B.4), with \( X \) and \( Z \) replaced by \( \mathcal{X} \) and \( \mathcal{Z} \), respectively.

Example 4. Symmetric and log-symmetric linear regression. Consider a heteroskedastic symmetric linear regression model as follows. Let \( y_1, \ldots, y_n \) be independent random variables, each \( y_i \) having a symmetric distribution \( S(\mu_i, \phi_i) \) with \( \mu_i \) and \( \phi_i \) as in (9); see Example 3. The link functions are taken as \( g(\mu_i) = \mu_i \) and \( h(\phi_i) = \log(\phi_i) \). For this choice of link functions we have \( g'(\mu) = 1, g''(\mu) = 0 \), \( h'(\phi) = 1/\phi \) and \( h''(\mu) = -1/\phi^2 \). The needed quantities for evaluating the profile cumulants in (5) when \( \psi = \beta_k \) or \( \psi = \gamma_k \) are given in (B.1)–(B.4) with the cumulants presented in Appendix B.2.

A class of log-symmetric linear regression models for positive continuous responses is defined by assuming that \( t_1, \ldots, t_n \) are such that \( t_i = \exp(x_i^\top \beta)\xi_i^{\phi_i} \), where the \( \xi_i \)'s are independent and have a standard log-symmetric distribution with pdf \( \xi^{-1}\nu(\xi^2), \xi > 0 \) (Vanegas & Paula, 2015, 2016). An interesting feature of these models is that \( \exp(x_i^\top \beta) \) is the median of \( t_i \) and \( \phi_i \) is interpreted as a skewness parameter. Since \( y_i = \log t_i \sim S(x_i^\top \beta, \phi_i) \), the results above are also applicable for inference regarding the parameters of the log-symmetric linear regression models; see Medeiros & Ferrari (2017, Sect. 4).

Example 5. Beta regression. Beta regression models are widely applicable when the response variable is a continuous proportion (Ferrari & Cribari-Neto, 2004; Ferrari, 2017). We consider
the beta regression model that assumes that \( y_i \) has a beta distribution with mean \( \mu_i \) \((0 < \mu_i < 1)\) and precision parameter \( \phi_i \) \((\phi_i > 0)\), and pdf

\[
f(y_i; \mu_i, \phi_i) = \frac{\Gamma(\phi_i)}{\Gamma(\mu_i \phi_i) \Gamma((1 - \mu_i) \phi_i)} y_i^{\mu_i \phi_i - 1} (1 - y_i)^{(1 - \mu_i) \phi_i - 1}, \quad 0 < y_i < 1,
\]

with \( \mu_i \) and \( \phi_i \) as in (9) (Smithson & Verkuilen, 2006). The link functions are taken as the logit link for the mean, \( g(\mu_i) = \log(\mu_i/(1 - \mu_i)) \), and the log link for the precision parameter, \( h(\phi_i) = \log(\phi_i) \). For this choice of link functions we have \( g'(\mu) = 1/[\mu(1 - \mu)] \), \( g''(\mu) = (2\mu - 1)/[\mu^2(1 - \mu)^2] \), \( h'(\phi) = 1/\phi \) and \( h''(\mu) = -1/\phi^2 \).

The needed quantities for evaluating the profile cumulants in (5) when \( \psi = \beta_k \) or \( \psi = \gamma_k \) are given in (B.1)–(B.4) with the cumulants presented in Appendix B.3.

5 Monte Carlo simulation

We present Monte Carlo simulation results to evaluate the finite-sample performance of confidence intervals based on the \( \alpha \)-quantile modified score (modified CIs). For comparison, we included results for Wald-type CIs, i.e. CIs that use the asymptotic normality of MLE (usual CIs), and those that are constructed similarly to Wald-type CIs with median bias reduced estimates of all the parameters in place of MLEs (adjusted CIs). The simulations were implemented in \( \mathbb{R} \) language (R Core Team, 2018). The number of Monte Carlo replicates is 100,000. As initial guesses for the modified confidence limits, we used the maximum likelihood estimates.

The simulation results are shown in plots of ‘non-coverage discrepancy’ of one-sided and two-sided confidence intervals, and the mean length of two-sided intervals. The non-coverage discrepancy of an interval is defined as the ratio of the non-coverage probability (evaluated via simulation) and 1 minus the nominal level. The non-coverage discrepancy is plotted against 1 minus the nominal level, i.e. the nominal non-coverage probability. Intervals with non-coverage discrepancy close to (greater than) 1 are those with coverage probability close to (smaller than) the nominal level. Since the intervals may not have the correct coverage probability, the mean length is plotted against the coverage probability and not the nominal level. de Peretti & Siani (2010) suggest the use of mean length curves constructed this way to compare the ‘effectiveness’ of different confidence regions.

We now list the scenarios for the simulation experiments.

Example 1 (cont.). One parameter exponential family. We generated random samples of size \( n = 5 \) from an exponential distribution with unit mean. Results are shown in Figure 2.

Example 2 (cont.). Gamma distribution with mean \( \mu \) and coefficient of variation \( \phi^{-1/2} \), \( \text{Gamma}(\mu, \phi) \). We generated random samples of size \( n = 15 \) from a gamma distribution with \( \mu = 10 \) and \( \phi = 3 \). Results are given in Figure 3.
Example 5 (cont.). Beta regression. We generated data from a beta regression model with mean and precision parameters defined, respectively, as \( \log\left(\frac{\mu_i}{1 - \mu_i}\right) = \beta_0 + \beta_1 x_{i1} \), \( \log(\phi_i) = \gamma_0 + \gamma_1 z_{i1}, i = 1, \ldots, n \), with \( n = 25, \beta_0 = \beta_1 = 1, \gamma_0 = 1, \) and \( \gamma_1 = 2 \). The values of the covariates \( x \) and \( z \) were drawn from uniform distributions in the intervals \((-1/2, 1/2)\) and \((1, 2)\), respectively. Results are seen in Figure 4.

![Figure 2](image2.png)

Figure 2: Plots of non-coverage discrepancy of the lower and upper one-sided confidence intervals (first and second plots) and non-coverage discrepancy and mean length of two-sided confidence intervals (third and fourth plots); exponential distribution.

![Figure 3](image3.png)

Figure 3: Plots of non-coverage discrepancy of the lower and upper one-sided confidence intervals (first and second plots) and non-coverage discrepancy and mean length of two-sided confidence intervals (third and fourth plots); gamma distribution.

In Figures 2-4, the red, blue and green curves refer to modified CIs (i.e. the CIs derived in this paper), usual CIs, and adjusted CIs, respectively. From these figures the following conclusions may be drawn. First, the non-coverage discrepancy tends to be much closer to 1.
Figure 4: Plots of non-coverage discrepancy of the lower and upper one-sided confidence intervals (first and second plots) and non-coverage discrepancy and mean length of two-sided confidence intervals (third and fourth plots); beta regression.

for the modified CIs, particularly for high nominal confidence levels, when compared with the usual CIs. Second, in some cases the adjusted CIs partially correct the coverage probability of the usual CIs but are clearly outperformed by the modified CIs proposed in this paper. Third, in some cases the lower and upper one-sided usual CIs behave in opposite directions. For instance, for the gamma distribution (Figure 3) the lower one-sided usual CI for $\mu$ is conservative while the corresponding upper CI has coverage probability smaller than the nominal level. This is expected because the asymptotic normality of MLEs does not account for the skewness of the MLEs in finite samples. This undesirable behavior does not occur when the modified CIs are
employed. Finally, for each fixed coverage probability, the mean length of the modified two-sided confidence intervals tends to be similar to or smaller than that of the usual two-sided confidence intervals. Overall, the simulations suggest that the method we propose performs considerably better than the usual approach, that employs the asymptotic normality of MLE when constructing confidence sets.

6 Applications

We now present two applications using function hoaci. In the output ML, MBR, and QBR denote the usual Wald-type CIs, i.e. CIs that use the asymptotic normality of MLE, the adjusted Wald-type CIs (MLEs are replaced by median bias reduced estimates), and the modified CIs proposed in this paper, respectively.

6.1 Orange data

The application considers the data set presented in Table 1 of Mirhosseini & Tan (2010). The data consist on 20 observations collected to investigate the effect of emulsion components on orange beverage emulsion properties. The response variable is the emulsion density, measured in $\text{mg/cm}^3$, and the independent variables considered here are the amount of arabic gum and of orange oil, both measured in $\text{g/10g}$. Medeiros & Ferrari (2017, Eq. (12)) fitted a Student-t regression model with 3 degrees of freedom,

$$
\mu = \beta_0 + \beta_1 \text{arabic gum} + \beta_2 \text{orange oil},
$$

and constant dispersion parameter $\phi$. The point and interval estimates are computed using function hoaci as follows.

```r
> hoaci(density~arabic_gum + orange_oil, data = Orange, type="Student",
       link.mu="identity", link.phi="log" , CL=0.95, nu = 3)
```

| Point estimates | ML | MBR |
|-----------------|----|-----|
| (Intercept)     | 1017.533 | 1017.533 |
| arabic_gum      | 26.765   | 26.765 |
| orange_oil      | -22.524  | -22.524 |
| log(\phi)       | 0.723    | 0.841 |

| Confidence limits | 97.5 % CL - One-sided and 95 % CL - Two-sided |
|-------------------|-----------------------------------------------|
| ML                | MBR | QBR |
| (Intercept)       | 1007.498 | 1027.568 | 1006.234 | 1028.832 | 1003.047 | 1035.699 |
| arabic_gum        | 22.975   | 30.554 | 22.498   | 31.031   | 21.811   | 31.622   |
| orange_oil        | -29.101  | -15.947 | -29.929  | -15.119  | -31.688  | -16.216  |
| log(\phi)         | 0.285    | 1.161  | 0.403    | 1.280    | 0.335    | 1.402    |

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Figure (5) shows the CIs for different confidence levels. The modified CIs tend to be larger than the others. Unlike the usual and adjusted CIs, they account for skewness in the distribution of the MLEs.

![Graph showing CIs for different confidence levels](image)

**Figure 5:** Plots of confidence intervals (blue square: usual Wald-type CI, green circle: adjusted Wald-type CI, red triangle: modified CI); Orange data.

### 6.2 Reading skills

The application considers the data set for assessing the contribution of non-verbal IQ to reading skills of dyslexic and non-dyslexic children (Smithson & Verkuilen, 2006). The data set comprises 44 observations and is included in the package `betareg` (Cribari-Neto & Zeileis, 2010). The independent variables are `dyslexia` (−1 and 1 for the control and dyslexic groups, respectively) and the nonverbal intelligent quotient (`iq`, converted to z scores), and the response variable is `accuracy` (score on a test of reading accuracy). Smithson & Verkuilen (2006) proposed the following beta regression model:

\[
\log\left[\frac{\mu}{1 - \mu}\right] = \beta_0 + \beta_1 \text{dyslexia} + \beta_2 \text{iq} + \beta_3 \text{dyslexia} \times \text{iq},
\]

\[
\log(\phi) = \gamma_0 + \gamma_1 \text{dyslexia} + \gamma_2 \text{iq}.
\] \hspace{1cm} (11)

The point and interval estimates are computed using function `hoaci` as follows.

```r
> hoaci(accuracy ~ dyslexia * iq | dyslexia + iq, data = ReadingSkills, type="beta", CL=0.95)
```

Coefficients (mean model with logit link):

|               | ML   | MBR  |
|---------------|------|------|
| (Intercept)   | 1.123| 1.109|
| dyslexia     | -0.742| -0.728|
| iq            | 0.486| 0.472|
| dyslexia:iq  | -0.581| -0.565|

Coefficients (mean model with logit link):

|                           | ML   | MBR  |
|---------------------------|------|------|
| 97.5 % CL - One-sided     |      |      |
| 95 % CL - Two-sided       |      |      |
The modified CIs (QBR) are quite different from the others in some cases. For instance, the 95% CIs for the coefficient of iq in the precision model are (0.705, 1.753) (ML), (0.533, 1.595) (MBR), and (-0.333, 2.380) (QBR). Hence, the modified CI does not provide evidence of effect of non-verbal IQ in the variability of reading accuracy unlike the other CIs. Figure (6) shows the CIs for different confidence levels. Considerable differences in the length and shape (asymmetry) of intervals are observed, suggesting that the normal approximation for the MLEs is not accurate.

Figure 6: Plots of confidence intervals (blue square: usual Wald-type CI, green circle: adjusted Wald-type CI, red triangle: modified CI); Reading skills data.
a simulation experiment using the beta regression model (11) with \( n = 44 \) and the values of the parameters taken as the MLEs computed with the readings skills data set. The coverage of CIs at nominal levels 90\%, 95\% and 99\% are presented in Table 2. It is clear from the table that the usual Wald-type CIs are anti-conservative (undercover) and modified CIs proposed here are slightly conservative but cover the true values of the parameters with probability much closer to the nominal levels than usual CIs.

Table 2: Coverage of confidence intervals; reading skills data scenario

|       | \( \beta_0 \) | \( \beta_1 \) | \( \beta_2 \) | \( \beta_3 \) | \( \gamma_0 \) | \( \gamma_1 \) | \( \gamma_2 \) |
|-------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 90\%  | ML | 86.9 | 87.0 | 85.0 | 84.8 | 75.8 | 82.2 | 78.7 |
|       | QBR | 90.6 | 90.9 | 90.5 | 90.3 | 93.0 | 90.4 | 92.6 |
| 95\%  | ML | 92.5 | 92.5 | 91.2 | 91.0 | 83.5 | 88.9 | 86.6 |
|       | QBR | 96.2 | 96.1 | 95.8 | 95.6 | 96.9 | 96.1 | 97.1 |
| 99\%  | ML | 97.7 | 97.5 | 97.4 | 97.3 | 93.1 | 96.2 | 94.4 |
|       | QBR | 99.4 | 99.4 | 99.4 | 99.4 | 99.4 | 99.7 | 99.7 |

7 Concluding remarks

We derived highly accurate confidence intervals for a scalar parameter of interest possibly in the presence of a vector of nuisance parameters in general parametric families. The proposed confidence limits are computed from modified score equations and possess desirable properties: they are equivariant under interest-respecting reparameterizations, they account for skewness and kurtosis of the score function, and they are simple to compute, not requiring computer intensive methods. We provided an implementation for regression models and presented two real data applications. Monte Carlo simulations evidenced that the usual confidence sets may have actual coverage probability far from the chosen nominal level in small samples, while the proposed confidence intervals remain accurate.

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Appendices

A Proof of (2)

The Edgeworth expansion for the cumulative distribution function (cdf) of a standardized sum of independent random variables, say \( S_n^* \), is

\[
F_{S_n^*}(x) = \Phi(x) - \phi(x) h(x) + O(n^{-3/2})
\]
where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cdf and the pdf of the standard normal distribution and

$$h(x) = \frac{1}{6} \frac{\kappa_3}{\kappa_2^{3/2}} (x^2 - 1) + \frac{1}{24} \frac{\kappa_4}{\kappa_2^2} (x^3 - 3x) + \frac{1}{72} \frac{\kappa_5^2}{\kappa_2^2} (x^5 - 10x^3 + 15x);$$

Pace & Salvan (1997, eq. (10.14)). Let

$$h_1(x) = \frac{1}{6} \frac{\kappa_3}{\kappa_2^{3/2}} (x^2 - 1), \quad h_2(x) = \frac{1}{24} \frac{\kappa_4}{\kappa_2^2} (x^3 - 3x) - \frac{1}{72} \frac{\kappa_5^2}{\kappa_2^2} (4x^3 - 10x), \quad h_3(x) = \frac{1}{72} \frac{\kappa_5^2}{\kappa_2^2} (x^5 - 6x^3 + 5x).$$

Note that $h(x) = h_1(x) + h_2(x) + h_3(x)$; $h_1(x)$ is of order $O(n^{-1/2})$, and $h_2(x)$ and $h_3(x)$ are of order $O(n^{-1})$.

From (1) and applying the Edgeworth expansion given above to the cdf of $U(\theta)/\sqrt{\kappa_2}$ we have

$$P_{\theta} \left( \frac{U(\theta)}{\sqrt{\kappa_2}} \leq u_\alpha + h_1(u_\alpha) + h_2(u_\alpha) \right) = \Phi(u_\alpha + h_1(u_\alpha) + h_2(u_\alpha)) h(u_\alpha + h_1(u_\alpha) + h_2(u_\alpha)) + O(n^{-3/2}).$$

Now, applying a Taylor series expansion and using the fact that $\phi'(u_\alpha) = -u_\alpha \phi(u_\alpha)$, we have after some algebra

$$P_{\theta} \left( \frac{U(\theta)}{\sqrt{\kappa_2}} \leq u_\alpha + h_1(u_\alpha) + h_2(u_\alpha) \right) = \Phi(u_\alpha - \phi(u_\alpha) h(u_\alpha)) + \left( \Phi(u_\alpha) - \phi(u_\alpha) h(u_\alpha) \right) \phi'(u_\alpha) h_1(u_\alpha) h_2(u_\alpha) + O(n^{-3/2})$$

$$= \alpha - \phi(u_\alpha) h_3(u_\alpha) - \phi'(u_\alpha) h_1(u_\alpha) - \phi(u_\alpha) h_1(u_\alpha) h_1(u_\alpha) + \frac{1}{2} \phi'(u_\alpha)^2 h_1(u_\alpha)^2 + O(n^{-3/2})$$

$$= \alpha + O(n^{-3/2}).$$

**B Regression models: cumulants**

**B.1 Multidimensional arrays of cumulants**

$$K_{2(p \times p)} = \begin{pmatrix} X^\top T^2 K_{\mu,\mu} X & X^\top THK_{\mu,\mu} Z \\ Z^\top HTK_{\phi,\mu} X & Z^\top H^2 K_{\phi,\phi} Z \end{pmatrix};$$

$$K_{3(p \times p \times p)} = \begin{pmatrix} K_{3(p \times p \times p)}^{(2)} & K_{3(p \times p \times p)}^{(3)} \end{pmatrix},$$

where

$$K_{3}^{(2)} = \begin{pmatrix} X^\top ((D_X \otimes (T^2 K_{\mu,\mu})) \otimes Z) & X^\top ((D_X \otimes (T^2 H K_{\mu,\mu} \phi)) \otimes Z) \\ Z^\top ((D_X \otimes (T^2 K_{\mu,\mu} \phi)) \otimes X) & Z^\top ((D_X \otimes (T^2 H K_{\mu,\mu} \phi)) \otimes X) \end{pmatrix},$$

$$K_{3}^{(3)} = \begin{pmatrix} X^\top ((D_Z \otimes (T^2 H K_{\mu,\mu} \phi)) \otimes X) & X^\top ((D_Z \otimes (T H^2 K_{\mu,\mu} \phi)) \otimes Z) \\ Z^\top ((D_Z \otimes (T^2 H K_{\mu,\mu} \phi)) \otimes X) & Z^\top ((D_Z \otimes (T H^2 K_{\mu,\mu} \phi)) \otimes Z) \end{pmatrix};$$
\[ K_{4(p \times p \times p \times m)}^{\beta} = \begin{pmatrix} K_{4(p \times p \times p \times q)}^{\beta} & K_{4(p \times p \times q \times m)}^{\beta} \\ K_{4(p \times p \times q \times m)}^{\beta} & K_{4(p \times p \times m \times m)}^{\beta} \end{pmatrix}, \]

where

\[ K_{4}^{\beta} = \begin{pmatrix} X^\top \circ ((D_X \circ (D_X \circ (T^4K_{\mu,\mu,\mu,\mu})) \circ X) & X^\top \circ ((D_X \circ (D_X \circ (T^3HK_{\mu,\mu,\mu,\phi})) \circ Z) \\ Z^\top \circ ((D_X \circ (D_X \circ (T^3HK_{\mu,\mu,\mu,\phi})) \circ X) & Z^\top \circ ((D_X \circ (D_X \circ (T^2HK^2_{\mu,\mu,\phi,\phi})) \circ Z) \end{pmatrix}, \]

\[ K_{4}^{\beta\gamma} = \begin{pmatrix} X^\top \circ ((D_Z \circ (D_X \circ (T^3HK_{\mu,\mu,\mu,\phi})) \circ X) & X^\top \circ ((D_Z \circ (D_X \circ (T^2HK^2_{\mu,\mu,\phi,\phi})) \circ Z) \\ Z^\top \circ ((D_Z \circ (D_X \circ (T^3HK_{\mu,\mu,\mu,\phi})) \circ X) & Z^\top \circ ((D_Z \circ (D_X \circ (T^3HK_{\mu,\mu,\mu,\phi})) \circ Z) \end{pmatrix}, \]

\[ K_{4}^{\gamma\beta} = \begin{pmatrix} X^\top \circ ((D_Z \circ (D_Z \circ (T^2HK_{\mu,\mu,\phi,\phi})) \circ X) & X^\top \circ ((D_Z \circ (D_Z \circ (T^3HK_{\mu,\mu,\phi,\phi})) \circ Z) \\ Z^\top \circ ((D_Z \circ (D_Z \circ (T^2HK_{\mu,\mu,\phi,\phi})) \circ X) & Z^\top \circ ((D_Z \circ (D_Z \circ (T^3HK_{\mu,\mu,\phi,\phi})) \circ Z) \end{pmatrix}; \]

\[ K_{12}^{\beta} = \begin{pmatrix} K_{12(p \times p \times q)}^{\beta} & K_{12(p \times p \times m)}^{\gamma} \end{pmatrix}, \]

where

\[ K_{12}^{\beta} = \begin{pmatrix} X^\top \circ ((D_X \circ (T^3K_{\mu,\mu} + T^2SK_{\mu,\mu})) \circ X) & X^\top \circ ((D_X \circ (T^2HK_{\mu,\mu,\phi})) \circ Z) \\ Z^\top \circ ((D_X \circ (T^2HK_{\mu,\mu,\phi})) \circ X) & Z^\top \circ ((D_X \circ (TH^2K_{\mu,\phi,\phi} + THQK_{\phi,\phi})) \circ Z) \end{pmatrix}, \]

\[ K_{12}^{\gamma} = \begin{pmatrix} X^\top \circ ((D_Z \circ (T^2HK_{\phi,\phi,\mu,\phi})) \circ X) & X^\top \circ ((D_Z \circ (TH^2K_{\phi,\phi,\mu})) \circ Z) \\ Z^\top \circ ((D_Z \circ (T^2HK_{\phi,\phi,\mu})) \circ X) & Z^\top \circ ((D_Z \circ (TH^2K_{\phi,\phi,\mu}) + H^2QK_{\phi,\phi})) \circ Z) \end{pmatrix}. \]

### B.2 Symmetric regression: cumulants

The first and second order derivatives of \( \log f(y; \mu, \phi) \) are given by \( U_\mu = -\phi^{-1} s^{(1)} \), \( U_\phi = -\phi^{-1}(1 + s^{(1)} \epsilon) \), \( U_{\mu \mu} = \phi^{-3} s^{(2)} \), \( U_{\phi \phi} = \phi^{-2}(1 + 2s^{(1)} \epsilon + s^{(2)} \epsilon^2) \), and \( U_{\mu \phi} = \phi^{-2}(s^{(1)} + s^{(2)} \epsilon) \), \( \frac{d^r s(e)}{d \epsilon^r} \) with \( s(\epsilon) = \log v(\epsilon^2) \) and \( \epsilon = (y - \mu)/\phi \). From these derivatives, we obtain the following cumulants:

\[
\begin{align*}
\kappa_{\mu,\mu} &= \delta_{20000}/\phi^2, \quad \kappa_{\phi,\phi} = (\delta_{20002} - 1)/\phi^2, \quad \kappa_{\mu,\mu,\phi} = 2\delta_{11001}/\phi^3, \quad \kappa_{\phi,\phi,\phi} = 2(\delta_{11003} + 1)/\phi^3, \\
\kappa_{\mu,\mu,\mu,\mu} &= (\delta_{30000} - 3\delta_{20000})/\phi^4, \quad \kappa_{\mu,\mu,\mu,\phi} = (\delta_{30000} + \delta_{10001})/\phi^4, \\
\kappa_{\mu,\mu,\phi,\phi} &= (2\delta_{30001} + 4\delta_{30002} - 6\delta_{20002} + 3\delta_{20001})/\phi^4, \quad \kappa_{\mu,\mu,\phi,\phi} = (\delta_{40001} + 3\delta_{30001} + 3\delta_{20001})/\phi^4, \\
\kappa_{\phi,\phi,\phi,\phi} &= (\delta_{40004} + 4\delta_{30003} + 12\delta_{20002} - 6\delta_{20001} + 3\delta_{20001})/\phi^4, \quad \kappa_{\mu,\mu,\phi} = (\delta_{11001} - 6\delta_{10001})/\phi^3, \\
\kappa_{\phi,\phi,\phi} &= (4\delta_{10002} + \delta_{10003} - 2)/\phi^3, \quad \kappa_{\mu,\mu,\mu} = \delta_{10001}/\phi^3, \\
\kappa_{\mu,\phi} &= \kappa_{\mu,\mu,\mu} = \kappa_{\mu,\phi,\phi} = \kappa_{\mu,\phi} = \kappa_{\phi,\mu} = 0. 
\end{align*}
\]

### B.3 Beta regression: cumulants

The first and second order derivatives of \( \log f(y; \mu, \phi) \) are given by

\[
\begin{align*}
U_\mu &= \phi(y^* - \mu^*), \quad U_\phi = \mu(y^* - \mu^*) + (y^t - \mu^t), \\
U_{\mu \mu} &= -\phi^2 \left( \Psi^{(1)}(\mu \phi) + \Psi^{(1)}((1 - \mu) \phi) \right), \quad U_{\mu \phi} = -\mu^2 \Psi^{(1)}(\mu \phi) + \Psi^{(1)}(\phi), \\
U_{\mu \phi} &= y^* - \mu^* - \phi(\mu \Psi^{(1)}(\mu \phi) - (1 - \mu) \Psi^{(1)}((1 - \mu) \phi)).
\end{align*}
\]
$y^* = \log(y/(1-y))$, $y^\dagger = \log(1-y)$, $\mu^* = \Psi^{(0)}(\mu\phi) - \Psi^{(0)}((1-\mu)\phi)$, and $\mu^\dagger = \Psi^{(0)}((1-\mu)\phi) - \Psi^{(0)}(\phi)$. From these derivatives, we obtain the following cumulants:

$$
\begin{align*}
\kappa_{\mu,\mu} &= \phi^2(\Psi^{(1)}(\mu\phi) + \Psi^{(1)}((1-\mu)\phi)), \\
\kappa_{\mu,\phi} &= \phi(\mu\Psi^{(1)}(\mu\phi) - (1-\mu)\Psi^{(1)}((1-\mu)\phi)), \\
\kappa_{\phi,\phi} &= \mu^2\Psi^{(1)}(\mu\phi) + (1-\mu)^2\Psi^{(1)}((1-\mu)\phi) - \Psi^{(1)}(\phi), \\
\kappa_{\mu,\mu,\mu} &= \phi^3(\Psi^{(2)}(\mu\phi) - \Psi^{(2)}((1-\mu)\phi)), \\
\kappa_{\mu,\mu,\phi} &= \phi^2(\mu\Psi^{(2)}(\mu\phi) + (1-\mu)\Psi^{(2)}((1-\mu)\phi)), \\
\kappa_{\mu,\phi,\phi} &= \phi(\mu^2\Psi^{(2)}(\mu\phi) - (1-\mu)^2\Psi^{(2)}((1-\mu)\phi)), \\
\kappa_{\phi,\phi,\phi} &= \mu^3\Psi^{(2)}(\mu\phi) + (1-\mu)^3\Psi^{(2)}((1-\mu)\phi) - \Psi^{(2)}(\phi), \\
\kappa_{\mu,\mu,\mu,\mu} &= \phi^4(\Psi^{(3)}(\mu\phi) + \Psi^{(3)}((1-\mu)\phi)), \\
\kappa_{\mu,\mu,\mu,\phi} &= \phi^3(\mu\Psi^{(3)}(\mu\phi) - (1-\mu)\Psi^{(3)}((1-\mu)\phi)), \\
\kappa_{\mu,\mu,\phi,\phi} &= \phi^2(\mu^2\Psi^{(3)}(\mu\phi) + (1-\mu)^2\Psi^{(3)}((1-\mu)\phi)), \\
\kappa_{\mu,\phi,\phi,\phi} &= \phi(\mu^3\Psi^{(3)}(\mu\phi) - (1-\mu)^3\Psi^{(3)}((1-\mu)\phi)), \\
\kappa_{\phi,\phi,\phi,\phi} &= \mu^4\Psi^{(3)}(\mu\phi) + (1-\mu)^4\Psi^{(3)}((1-\mu)\phi) - \Psi^{(3)}(\phi), \\
\kappa_{\phi,\mu,\mu} &= \phi(\Psi^{(1)}(\mu\phi) + \Psi^{(1)}((1-\mu)\phi)), \\
\kappa_{\phi,\phi,\mu} &= \mu\Psi^{(1)}(\mu\phi) - (1-\mu)\Psi^{(1)}((1-\mu)\phi), \\
\kappa_{\mu,\mu,\mu} &= \kappa_{\phi,\mu,\mu} = \kappa_{\mu,\phi,\phi} = \kappa_{\phi,\phi,\mu} = 0.
\end{align*}
$$

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