ON AUTOMORPHISMS OF CATEGORIES OF UNIVERSAL ALGEBRAS

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Abstract. Given a variety $\mathcal{V}$ of universal algebras, a new approach is suggested to characterize algebraically automorphisms of the category of free $\mathcal{V}$-algebras. It gives in many cases an answer to the problem set by the first of authors, if automorphisms of such a category are inner or not. This question is important for universal algebraic geometry [5, 9]. Most of results will actually be proved to hold for arbitrary categories with a represented forgetful functor.

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INTRODUCTION

It is a current opinion that the notions of an isomorphism and an automorphism of categories are not important. As far as the authors know there are no researches devoted to describing automorphisms of categories although this theme is very popular for the most of other algebraic structures. The first of authors set the problem to describe automorphisms of a category of free algebras of some given variety of universal algebras. It turns out that this problem is quite important for universal algebraic geometry [5, 9]. The most important case is, when all automorphisms of a category in question are inner or close to inner in a sense.

Recall that an automorphism $\Phi$ of a category $\mathcal{C}$ is called inner if it is isomorphic to the identity functor $Id_{\mathcal{C}}$ in the category of all endofunctors of $\mathcal{C}$. It means that for every object $A$ of the given category there exists an isomorphism $\sigma_A : A \to \Phi(A)$ such that for every morphism $\mu : A \to B$ we have

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$\Phi(\mu) = \sigma_B \circ \mu \circ \sigma_A^{-1}$. This fact explains the term “inner”. Thus if an automorphism $\Phi$ is inner the object $\Phi(A)$ is isomorphic to $A$ for every $\mathcal{C}$-object $A$.

Let $\mathcal{V}$ be a variety of universal algebras. Consider the category $\Theta(\mathcal{V})$ whose objects are all algebras from $\mathcal{V}$ and whose morphisms are all homomorphisms of them. Fix an infinite set $X_0$. Let $\Theta^0(\mathcal{V})$ be the full subcategory of $\Theta(\mathcal{V})$ defined by all free algebras from $\mathcal{V}$ over finite subsets of the set $X_0$. The group of automorphisms of the category $\Theta^0(\mathcal{V})$ is the subject of inquiry.

It is known, for example, that every automorphism of the category $\Theta^0(\mathcal{V})$ is inner if $\mathcal{V}$ is the variety of all groups [5]. But it is not so, for example, if $\mathcal{V}$ is the variety of all semigroups [3] or the variety of all Lie algebras [6]. In the last cases automorphisms are similar to inner but not inner exactly. What does it mean “similar to inner”? It means different things for different varieties. For instance, if $\mathcal{V}$ is the variety of all semigroups the maps $f_A : A \to \Phi(A)$ mentioned above are not necessarily isomorphisms but can be anti-isomorphisms too. If $\mathcal{V}$ is the variety of all modules over a ring $K$, these maps are so called semi-automorphisms, more exactly they are pairs of maps $(f, \sigma)$, where $f$ is an automorphism of the ring $K$, $\sigma : M \to N$ is an additive bijection satisfying the condition: $\sigma(ax) = f(a)\sigma(x)$ for all $a \in K$ and $x \in M$.

There exists a usual approach to the question if all automorphisms of the category $\Theta^0(\mathcal{V})$ are inner or semi-inner. It demands describing the group $\text{AUT END}(F)$ of all automorphisms of the semigroup of all endomorphisms of the free algebras $F$ of the given variety $\mathcal{V}$, and then applying the so called Reduction theorem (Theorem 5) due to B. Plotkin [1]. This theorem gives an opportunity to make a conclusion about automorphisms of the category $\Theta^0(\mathcal{V})$ (under some conditions) if all automorphisms of $\text{END}(F)$ are described, where finitely generated free algebra $F$ generates the variety $\mathcal{V}$. The original proof of this theorem is based on several special algebraic notions and constructions.

But it turns out that going over to a general case of categories supplied with a represented forgetful functor we can solve the mentioned problem as a

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[1] It should be mention that this theorem was first proved by Berzins [11] for the variety of commutative associative algebras over an infinite field.
whole and obtain very easily this theorem and even some more applications. And what is more a new approach can be suggested to answer the mentioned question. To describe the group $AUT\ END(F)$ for a given free algebra $F$ is a not trivial problem (see for example [3, 7]). Instead of this it is sufficient to find out how an automorphism of the given category acts on the set of all homomorphisms from a monogenic free algebra into a finitely generated free algebra. And the last problem is reduced to a purely algebraic problem to study some derivative (polynomial) operations in a free algebra, in most cases in a two-generated free algebra. This is the purpose of the present paper.

First of all a notion of potential-inner automorphisms is introduced (Section I). Let $C$ be a subcategory of a category $D$. An automorphism $\Phi$ of the category $C$ is called $D$–inner if for every object $A$ of the category $C$ there exists an $D$–isomorphism $f_A : A \to \Phi(A)$ such that $\Phi(\mu) = f_B \circ \mu \circ f_A^{-1}$ holds for every $C$-morphism $\mu : A \to B$. An automorphism $\Phi$ of a given category $C$ is called potential-inner if it is $D$-inner for some extension $D$ of $C$. This notion gives an opportunity to consider the problem from a new point of view. The necessary and sufficient condition found for an automorphism to be potential-inner (Theorem I) is satisfied in all important cases.

Thus the main problem can be formulated now in the following way: 1) what extension $D$ of the given category $C$ we have to construct in order to make all $C$-automorphisms to be $D$-inner and 2) when potential-inner automorphisms are in fact inner. These problems are solved in Section 3 for categories of free algebras using the derivative operation language (Lemma, Theorem). Roughly speaking, all potential-inner automorphisms are produced by isomorphisms of algebras onto derived algebras, a potential-inner automorphism is in fact inner, if there exist so called central isomorphisms of given algebras onto derived algebras (algebras on the same underlying sets with respect to derived operations).

Secondly, notions are introduced of left and right indicators in a category (Section 2). Let $Q : C \to Set$ be a forgetful functor. An object $A^0$ of the
category $\mathcal{C}$ is called a right indicator if for every two objects $A$ and $B$ and for every bijection $s : Q(A) \rightarrow Q(B)$ the following condition is satisfied:

if for every morphism $\nu : B \rightarrow A^0$ there exists a morphism $\mu : A \rightarrow A^0$ such that $Q(\mu) = Q(\nu) \circ s$, then there exists an isomorphism $\gamma : A \rightarrow B$ such that $Q(\gamma) = s$.

Dually the notion of a left indicator is defined.

For example, every free algebra generating a variety $\mathcal{V}$ is a right indicator in the category $\Theta(\mathcal{V})$ and every free algebra $H$ in $\mathcal{V}$ having not less free generators than arities of all operations is a left indicator. Using these notions, a generalized reduction theorem is proved (Theorem 2, Theorem 3) and it is shown in Section 3 how the original Reduction Theorem can be obtained from it (Theorem 5).

Further in this section, the method mentioned above is described and applied to categories of semigroups (Theorem 7) and inverse semigroups (Theorem 8) only to show how it works. In our next paper [13], we apply this method and characterize automorphisms of categories $\Theta^0(\mathcal{V})$ in the case $\mathcal{V}$ is the variety of all associative $K-$algebras, where $K$ is a infinite field, and in the case $\mathcal{V}$ is the variety of all group representations in unital $R-$modules, where $R$ is an associative commutative ring with unit.

In the last section (Section 4) some applications are given for categories of sets and semigroups of transformations (Theorem 9, Theorem 10, Theorem 11).

For the some notions and results of Category Theory and Universal Algebra we refer a reader to [4, 2]. Some part of results presented in this paper was published by the second author in [12].

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1. Inner and potential-inner automorphisms

We consider such categories $\mathcal{C}$ that are represented in the category $\text{Set}$ (the category of all sets and maps), that is, there exists a faithful functor $Q : \mathcal{C} \rightarrow \text{Set}$. Such a functor is called a forgetful functor. If $\mathcal{C}$ is a category of
universal algebras, then the forgetful functor is usually the natural forgetful functor, which assigns to every algebra \( A \) the underlying set \(|A|\) and to every homomorphism itself as a mapping, but not only this case.

We assume that a forgetful functor for every category we consider is fixed, and we say that a category \( \mathcal{C} \) is a subcategory of a category \( \mathcal{D} \) having in the mind that the forgetful functor for the category \( \mathcal{D} \) is an extension of the forgetful functor for the category \( \mathcal{C} \).

If \( \mathcal{C} \) is a subcategory of a category \( \mathcal{D} \) then \( \text{Id}_\mathcal{D}^\mathcal{C} : \mathcal{C} \to \mathcal{D} \) denotes the natural embedding functor, that is the restriction of identity functor \( \text{Id}_\mathcal{D} \) to \( \mathcal{C} \).

**Definition 1.** Let \( \mathcal{C} \) be a subcategory of a category \( \mathcal{D} \). A functor \( \Phi : \mathcal{C} \to \mathcal{D} \) is said to be *inner* if it is isomorphic in the sense of category of functors to the functor \( \text{Id}_\mathcal{D}^\mathcal{C} \).

It is clear that every inner functor is faithful.

**Definition 2.** Let \( \mathcal{C} \) be a subcategory of a category \( \mathcal{D} \). An automorphism \( \Phi \) of the category \( \mathcal{C} \) is called \( \mathcal{D} \)-inner if the functor \( \text{Id}_\mathcal{D}^\mathcal{C} \circ \Phi : \mathcal{C} \to \mathcal{D} \) is inner. That is for every object \( A \) of the category \( \mathcal{C} \) there exists a \( \mathcal{D} \)-isomorphism \( f_A : A \to \Phi(A) \) such that for every \( \mathcal{C} \)-morphism \( \mu : A \to B \) we have \( \Phi(\mu) = f_B \circ \mu \circ f_A^{-1} \). That is, the following diagram is commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{f_A} & \Phi(A) \\
\downarrow_{\mu} & & \downarrow_{\Phi(\mu)} \\
B & \xrightarrow{f_B} & \Phi(B)
\end{array}
\]

A \( \mathcal{C} \)-inner automorphism of a category \( \mathcal{C} \) is called an *inner* automorphism. A category is called *perfect* if all its automorphisms are inner.

In other words, an automorphism of the category \( \mathcal{C} \) is \( \mathcal{D} \)-inner if it can be extended to an inner automorphism of the category \( \mathcal{D} \).

If an automorphism \( \Phi \) of \( \mathcal{C} \) is \( \mathcal{D} \)-inner, a family \( (f_A : A \to \Phi(A) \mid A \in \text{Ob} \mathcal{C}) \) exists with the condition above. But this family is not only one possible. For example, the identity automorphism is of course inner, because it is determined by the family of identities, but it may be there exists another family of \( \mathcal{D} \)-morphisms which determines it.
**Definition 3.** A function $c$ which assigns a permutation $c_A: Q(A) \rightarrow Q(A)$ to every object $A$ of $C$ is called a central function if it determines the identity automorphism of $C$, that is, $c_B \circ Q(\mu) \circ c_A^{-1} = Q(\mu)$ for every $C$–morphism $\mu: A \rightarrow B$.

It is obvious that two families $(f_A: A \rightarrow \Phi(A)|A \in ObC)$ and $(g_A: A \rightarrow \Phi(A)|A \in ObC)$ of $\mathcal{D}$–isomorphisms determine the same automorphism $\Phi$ of the category $C$ if and only if the corresponding maps are equal up to a central function: $Q(f_A) = Q(g_A) \circ c_A$ for all objects $A$.

**Definition 4.** Let $C$ be a category with a forgetful functor $Q: C \rightarrow Set$. An automorphism $\Phi$ of the category $C$ is said to be potential-inner if it is $\mathcal{D}$-inner for some category $\mathcal{D}$ such that $C$ is a subcategory of $\mathcal{D}$ with the same objects.

We illustrate the last definition with the following examples.

**Examples.**

1. Let $C$ be the category of semigroups and homomorphisms, in this case $\mathcal{D}$ can be the category of semigroups and homomorphisms and anti-homomorphisms, the functor $Q$ is the natural forgetful functor. Thus in the definition above $f_A$ can be either an isomorphism or an anti-isomorphism.

2. Let $C$ be a category of unitary modules over a ring $K$. The functor $Q$ is the natural forgetful functor. The category $\mathcal{D}$ has the same objects but its morphisms are additives maps $\sigma$ of modules $M$ to $N$ satisfying the condition: $\sigma(ax) = f(a)\sigma(x)$ for all $a \in K$ and $x \in M$, where $f$ is an automorphisms of the ring $K$. The category $C$ can be identified with the subcategory of $\mathcal{D}$ for which $f = 1_K$.

3. (See [8]). Let $C$ be an arbitrary category of algebras and $G$ be a fixed non-trivial algebra in $C$. Denote by $C^G$ the category which objects are $C$-monomorphisms with the same domain $G$ and morphisms of which are commutative diagrams:

\[
\begin{array}{ccc}
G & \xrightarrow{h} & H \\
\downarrow^{1_G} & & \downarrow^{\mu} \\
G & \xrightarrow{h'} & H',
\end{array}
\]
where \( h, h' \) are monomorphisms and \( \mu \) is a morphism in \( C \). The objects of this category can be considered as \( G \)-algebras \( H \), that is, the algebras with fixed algebra of constants \( G \). In this way, to elements \( g \in G \) correspond constants, i.e., nullary operations in \( H \).

The category \( C^G \) is a subcategory of the category \( C(G) \) that has the same objects and the morphisms of which are the commutative diagrams:

\[
\begin{array}{ccc}
G & \xrightarrow{h} & H \\
\downarrow{\sigma} & & \downarrow{\mu} \\
G & \xrightarrow{h'} & H'
\end{array}
\]

where \( \sigma \) is now an automorphism of \( G \).

Define a functor \( Q \) in the same way as in the previous example: \( Q(h : G \to H) = |H| \) and \( Q \) assigns to every commutative diagram above the map \( \mu : |H| \to |H'| \). Let the category \( D \) in Definition 2 be \( C(G) \), then \( D \)-inner automorphisms of the category \( C^G \) are semi-inner automorphisms of this category introduced in [8].

If all automorphisms of a category \( C \) are potential inner the problem is to find the smallest its extension \( D \) such that all automorphisms of \( C \) are \( D \)-inner.

We start with several very simple facts.

**Lemma 1.** Let \( C \) be a subcategory of a category \( D \) and \( \Phi : C \to D \) be a functor. Let \( E \) be a subcategory of \( C \) such the restriction of \( \Phi \) to \( E \) is a \( D \)-inner automorphism of \( E \). Then \( \Phi \) is a composition of two functors \( \Phi = \Psi \circ \Gamma \), where \( \Gamma : C \to D \) is an identity on \( E \) (preserves all objects and all morphisms of \( E \)), and the functor \( \Psi \) is an inner automorphism of the category \( D \).

**Proof.** Let the family \( (s_A : A \to \Phi(A) \mid A \in ObE) \) be an isomorphism of functors \( Id^D_E \to \Phi|_E \), that is, \( \Phi(\nu) = s_B \circ \Phi(\nu) \circ s_A^{-1} \) for every \( E \)-morphism \( \nu : A \to B \). We construct an inner automorphism \( \Psi \) of \( D \) in the following way. For every \( D \)-object \( X \), we set \( \Psi(X) = X \) if \( X \) is not an object of \( E \), and \( \Psi(A) = \Phi(A) \) for every object \( A \) of \( E \). Further, we define \( D \)-isomorphism \( u_X : X \to \Psi(X) \) in the following way: \( u_X = 1_X \) if \( X \) is not an object of \( E \), and \( u_A = s_A \), where \( A \in ObE \). For every \( D \)-morphism \( f : X \to Y \), let
\[ \Psi(f) = u_Y \circ f \circ u_X^{-1}. \]
According to the given construction, \( \Psi \) is an inner automorphism of \( \mathcal{D} \).

It is clear that \( \Phi = \Psi \circ \Psi^{-1} \circ \Phi \). Let \( \Gamma = \Psi^{-1} \circ \Phi \). According to this definition, we have that \( \Gamma(A) = (\Psi^{-1} \circ \Phi)(A) = A \) for all \( A \in \text{Ob} \mathcal{E} \) and \( \Gamma(\nu) = (\Psi^{-1} \circ \Phi)(\nu) = s_B^{-1} \circ \Phi(\nu) \circ s_A = \nu \) for all \( \mathcal{E} \)-morphisms \( \nu : A \to B \).

Assuming in the above lemma a category \( \mathcal{E} \) to be discrete, that is, it contains no arrows besides identities, we obtain as a consequence the following result (see also [6]).

**Lemma 2.** Let \( \Phi \) be an automorphisms of a category \( \mathcal{C} \). Suppose further that for some class \( \mathcal{E} \) of \( \mathcal{C} \)-objects, \( \Phi(A) \) is isomorphic to \( A \) for every \( A \in \mathcal{E} \). Then \( \Phi \) is a composition of two \( \mathcal{C} \)-automorphisms \( \Phi = \Psi \circ \Gamma \), where \( \Gamma \) leaves fixed all objects from \( \mathcal{E} \) and \( \Psi \) is an inner automorphism.

**Proof.** If the class \( \mathcal{E} \) is closed under \( \Phi \) and \( \Phi^{-1} \), apply the previous lemma to the subcategory \( \mathcal{E} \) whose class of objects is \( \mathcal{E} \) and whose morphisms are identity morphisms only (discrete subcategory) and obtain the required result where \( \mathcal{D} = \mathcal{C} \). If \( \mathcal{E} \) is not closed we can consider its \( \Phi \)- and \( \Phi^{-1} \)-closure that has clearly the same property.

The next result will give us a necessary condition for an automorphism to be potential inner.

**Lemma 3.** Let \( F \) be a free object in a category \( \mathcal{C} \) over a set \( X \). If an automorphism \( \Phi \) of \( \mathcal{C} \) is potential inner then \( \Phi(F) \) is also a free object over \( X \) and hence it is isomorphic to \( F \).

**Proof.** Under hypothesis, there is a map \( m : X \to \mathcal{Q}(F) \) such that for every \( \mathcal{C} \)-object \( A \) and for every map \( f : X \to \mathcal{Q}(A) \), there exists an unique morphism \( \bar{f} : F \to A \) such that \( f = \mathcal{Q}(\bar{f}) \circ m \). If the given automorphism \( \Phi \) of \( \mathcal{C} \) is potential inner, that is, it is \( \mathcal{D} \)-inner for some extension \( \mathcal{D} \) of the category \( \mathcal{C} \), then there exist a family of \( \mathcal{D} \)-isomorphisms \( s_A : A \to \Phi(A) \) such that \( \Phi(\nu) = s_B \circ \nu \circ s_A^{-1} \) for every \( \mathcal{C} \)-morphism \( \nu : A \to B \). Set \( \bar{m} = \mathcal{Q}(s_F) \circ m \). It is a map from \( X \) to \( \mathcal{Q}(\Phi(F)) \). Let \( A \) be an object and \( f : X \to \mathcal{Q}(A) \) a map.
Consider the object $B = \Phi^{-1}(A)$ and the map $g = Q(s_B^{-1}) \circ f$ from $X$ to $B$. Then we have a unique morphism $\tilde{g} : F \to B$ such that $g = Q(\tilde{g}) \circ m$. Hence we have a morphism $\nu = \Phi(\tilde{g})$ from $\Phi(F)$ to $A$ with the following condition:

$$Q(\nu) \circ \tilde{m} = Q(s_B \circ \tilde{g} \circ s_f^{-1}) \circ Q(s_F) \circ m = Q(s_B \circ \tilde{g} \circ s_f^{-1} \circ s_F) \circ m = Q(s_B) \circ Q(\tilde{g}) \circ m = = Q(s_B) \circ g = Q(s_B) \circ Q(s_{B}^{-1}) \circ f = f.$$ 

The uniqueness if the morphism $\nu$ with the condition $Q(\nu) \circ \tilde{m} = f$ is clear. Thus $\Phi(F)$ is also a free object over $X$ with respect to the map $\tilde{m} : X \to Q(\Phi(F))$. □

We consider categories $\mathcal{C}$ such that the forgetful functor $Q : \mathcal{C} \to \text{Set}$ is represented by a pair $(A_0, x_0)$ where $A_0$ is an object of $\mathcal{C}$ and $x_0 \in Q(A_0)$. It means that for every element $a \in Q(A)$ for some object $A$ there exists an unique $\mathcal{C}$-morphism $\alpha^A_a : A_0 \to A$ such that

$$Q(\alpha^A_a)(x_0) = a,$$ 

(1.1)

in other words $A_0$ is a free object over set $\{x_0\}$. In the case the object $A$ is known, we turn down the letter ”$A$” in the designation $\alpha^A_a$.

If an automorphism $\Phi$ of $\mathcal{C}$ is potential inner, then, according to Lemma 3, $\Phi(A_0)$ is isomorphic to $A_0$. If $\Phi(A_0)$ is isomorphic to $A_0$, then (according to Lemma 2) $\Phi$ is a composition of two automorphisms $\Phi = \Psi \circ \Gamma$, where $\Gamma$ preserves $A_0$, and the functor $\Psi$ is a $\mathcal{D}$-inner automorphism of the category $\mathcal{C}$. Therefore we can restrict ourself to considering automorphisms that preserve the object $A_0$.

Let $\Phi$ be such an automorphism. Consider an arbitrary object $A$. It was mentioned above that there is a bijection $a \mapsto \alpha^A_a$ between sets $Q(A)$ and $\text{Hom}(A_0, A)$ defined by (1.1). Define a map $s^\Phi_A : Q(A) \to Q(\Phi(A))$ setting for every $a \in Q(A)$:

$$s^\Phi_A(a) = \bar{a} \iff \Phi(\alpha^A_a) = \alpha^\Phi_A(a)$$ 

(1.2)

or

$$s^\Phi_A(a) = Q(\Phi(\alpha^A_a))(x_0).$$ 

(1.3)
It is very simple to verify that the function \( s^\Phi : A \mapsto s^\Phi_A \) has the following properties:

\[ s^\Phi_{Id} = 1_A, \quad (1.4) \]

\[ s^\Phi^{-1} = (s^\Phi)^{-1}, \quad (1.5) \]

and for two automorphisms \( \Phi \) and \( \Psi \)

\[ s^{\Psi \circ \Phi} = s^\Psi \circ s^\Phi. \quad (1.6) \]

If an automorphism \( \Phi \) is fixed, we do not write the superscript \( \Phi \). For every morphism \( \nu : A \to B \) and for every \( a \in Q(A) \), we have \( \nu \circ \alpha^A_a = \alpha^{Q(\nu)(a)}_a \). Applying to this equation the automorphism \( \Phi \) we obtain: \( \Phi(\nu) \circ \alpha^A_{\Phi(A)} = \alpha^{\Phi(B)}_{s_B(Q(\nu)(a))} \). Now apply the functor \( Q \) and take the common value of corresponding maps in the point \( x_0 \). We have \( Q(\Phi(\nu)) \circ s_A(a) = s_B(Q(\nu)(a)) \).

Since \( a \) is an arbitrary element of \( Q(A) \) we obtain finally: \( Q(\Phi(\nu)) \circ s_A = s_B \circ Q(\nu) \) and hence

\[ Q(\Phi(\nu)) = s_B \circ Q(\nu) \circ s_A^{-1}. \quad (1.7) \]

Notice that the map \( s_{A_0} \) satisfies some special condition: \( s_{A_0}(x_0) = x_0 \). We formulate the obtained result in the following lemma.

**Lemma 4.** Let \( \mathcal{C} \) be a category with a forgetful functor \( Q : \mathcal{C} \to \text{Set} \) that is represented by a pair \((A_0, x_0)\). If \( \Phi \) is an automorphism of the category \( \mathcal{C} \) that leaves fixed the object \( A_0 \) then there exists a family of bijections \((s_A : Q(A) \to Q(\Phi(A)), \; |A| \in \text{Ob}\mathcal{C})\), such that for every \( \mathcal{C} \)-morphism \( \nu : A \to B \) we have:

\[ Q(\Phi(\nu)) = s_B \circ Q(\nu) \circ s_A^{-1} \]

and \( s_{A_0}(x_0) = x_0 \).

Further, if \( \mathcal{D} \) is an extension of \( \mathcal{C} \) and the functor \( Q \) can be extended to a functor from \( \mathcal{D} \) to \( \text{Set} \), such that for every \( \nu : A \to B \)

\[ \Phi(\nu) = \sigma_B \circ \nu \circ \sigma_A^{-1}, \]

where \( \sigma_A : A \to \Phi(A) \) are \( \mathcal{D} \)-isomorphisms and \( Q(\sigma_{A_0})(x_0) = x_0 \), then

\[ Q(\sigma_A) = s_A \]
for every \( C \)-object \( A \).

\textbf{Proof.} Let the pair \((A_0, x_0)\) represent the functor \( Q \). The first statement of the lemma is proved above. The second one follows from the hypotheses immediately:

\[ s_A(a) = Q(\Phi(\alpha_a))(x_0) = Q(\sigma_A) \circ Q(\alpha_a)(x_0) = Q(\sigma_A)(a) \]

for every \( a \in Q(A) \), that is, \( s_A = Q(\sigma_A) \).

\( \square \)

\textbf{Theorem 1.} Let \( C \) be a category with a forgetful functor \( Q : C \to \text{Set} \) that is represented by an object \( A_0 \). An automorphism \( \Phi \) of the category \( C \) is potential-inner if and only if the \( A_0 \) and \( \Phi(A_0) \) are isomorphic.

\textbf{Proof.} The necessity of that condition follows from Lemma 3. We prove that it is sufficient. Under hypothesis, the category \( C \) is isomorphic to a subcategory of the category \( \text{Set} \), and we can assume that \( C \) is a subcategory of \( \text{Set} \) with the trivial forgetful functor \( \text{Id}_{\text{Set}} \). Now we can assume that the automorphism \( \Phi \) preserves \( A_0 \). According to Lemma 1 there exists a family of bijections \( (s_A : A \to \Phi(A), |A \in \text{Ob}C) \), such that for every \( C \)-morphism \( \nu : A \to B \) we have:

\[ \Phi(\nu) = s_B \circ \nu \circ s_A^{-1}. \]

Adding these bijections \( s_A \) and their inverses \( s_A^{-1} \) to the category \( C \) we obtain a new category of sets \( D \) containing \( C \) as a subcategory with the same objects. Under definition the automorphism \( \Phi \) is \( D \)-inner.

\( \square \)

We assume now that automorphisms of the category \( C \) we consider leave fixed the object \( A_0 \). According to Lemma 1 describing of such automorphisms is reduced to the problem of finding out what maps \( s_A \) are. The following simple facts are very useful. First we obtain a corollary from Lemma 1.

\textbf{Corollary 1.} Let \( A \) be an object. Suppose that \( \Phi(A) = A \) and for some subset \( X \subseteq Q(A) \) there exists an automorphism \( \sigma \) of \( A \) such that \( \Phi(\alpha^A_x) = \alpha^A_{\sigma(x)} \) for every \( x \in X \) (in other words, \( s^{\Phi}_A(x) = \sigma(x) \)). Then \( \Phi \) is a composition of two automorphisms \( \Phi = \Psi \circ \Gamma \), where \( \Psi \) is an inner automorphism and \( \Gamma(\alpha^A_x) = \alpha^A_x \) for all \( x \in X \), that is \( s^{\Gamma}_A(x) = x \).
Proof. Consider a subcategory $E$ of $C$ whose objects are $A_0$ and $A$ only and whose morphisms are identities on these two objects and all morphisms $\alpha_x^A$ for $x \in X$ only. Under hypotheses, for every $x \in X$ we have: $\Phi(\alpha_x^A) = \sigma \circ \alpha_x^A \circ 1_{A_0}$. It means that $\Phi$ induces an inner automorphism of $E$. According to Lemma 1, $\Phi$ is a composition of two automorphisms $\Phi = \Psi \circ \Gamma$, where $\Psi$ is an inner automorphism and $\Gamma$ is an identity on $E$, that is, $\Gamma(\alpha_x^A) = \alpha_x^A$ for all $x \in X$. □

Now we apply this general fact to a special situation.

**Corollary 2.** Suppose that $A = F(X)$ is a free object in $C$ over a set $X$ and $m : X \to Q(A)$ the corresponding map. Suppose that $\Phi(A) = A$. Then $\Phi$ is a composition of two automorphisms $\Phi = \Psi \circ \Gamma$, where $\Psi$ is an inner automorphism and $\Gamma(\alpha_m^A(x)) = \alpha_m^A(x)$ for all $x \in X$, that is $s_A^f(m(x)) = m(x)$.

Proof. Denote for simplicity $s = s_A^\Phi$. Consider two maps $s \circ m : X \to Q(A)$ and $s^{-1} \circ m : X \to Q(A)$. They give us two morphisms $\sigma : A \to A$ and $\tau : A \to A$ such that $Q(\sigma) \circ m = s \circ m$ and $Q(\tau) \circ m = s^{-1} \circ m$. These both conditions can be written in the following way:

$$\sigma \circ \alpha_m(x) = \alpha_{s(m(x))}, \quad \tau \circ \alpha_m(x) = \alpha_{s^{-1}(m(x))} \quad (1.8)$$

for all $x \in X$. Applying $\Phi^{-1}$ to the first one, we obtain $\Phi^{-1}(\sigma) \circ \alpha_{s^{-1}(m(x))} = \alpha_m(x)$ and hence $Q(\Phi^{-1}(\sigma))(s^{-1}(m(x))) = m(x)$ for all $x \in X$. It means that

$$\Phi^{-1}(\sigma) \circ \tau = 1_A$$

and hence

$$\sigma \circ \Phi(\tau) = 1_A.$$

Applying $\Phi$ to the second condition in $1.8$ we obtain in the same way that

$$\Phi(\tau) \circ \sigma = 1_A.$$

These equalities give the fact that $\sigma$ and $\tau$ are inverse, that is they are automorphisms of $A$. Applying Corollary 1 we obtain the required result. □

The last result gives us opportunity to restrict our consideration to the case of automorphisms $\Phi$ such that $s_A^\Phi(m(x)) = m(x)$ for a given free object $A$. The next result shows how it simplifies the situation. Denote by $\theta_f$ the unique
endomorphism of the object \( A \) such that \( Q(\theta_f)(m(x)) = f(x) \) for all \( x \in X \),
where \( f : X \to Q(A) \) is a given map.

**Corollary 3.** \( \Phi(\theta_f) = \theta_{s_A}^* f \).

*Proof.* Under definition, we have \( \theta_f \circ \alpha_{m(x)} = \alpha_f(x) \). Apply \( \Phi \) and obtain \( \Phi(\theta_f) \circ \alpha_{m(x)} = \Phi(\alpha_f(x)) = \alpha_{s_A(f(x))} \). Hence \( Q(\Phi(\theta_f))(m(x)) = s_A(f(x)) \) for all \( x \in X \). \( \square \)

It turns out that the most interesting categories satisfy the condition we used above, namely, their automorphisms take a free object over an one-element set to an isomorphic one. In such a case all automorphisms are potential-inner and problem is to find a suitable extension for such a category. How to do it we show in the next sections.

## 2. Generalized Reduction theorem

In this section we consider a category \( \mathcal{C} \), its extension \( \mathcal{D} \) with the same objects and a faithful functor \( Q \) from \( \mathcal{D} \) to the category of sets. Therefore \( \mathcal{D} - \)morphisms (and of course \( \mathcal{C} - \)morphisms) can be regarded as a maps.

**Definition 5.** An object \( A^0 \) of the category \( \mathcal{C} \) is called a right indicator respectively \( \mathcal{D} \) if for every two objects \( A \) and \( B \) and for every bijection \( s : Q(A) \to Q(B) \) the following condition is satisfied:

if for every \( \mathcal{C} - \)morphism \( \nu : B \to A^0 \) there exists a \( \mathcal{D} - \)morphism \( \mu : A \to A^0 \) such that \( Q(\mu) = Q(\nu) \circ s \), then there exists a \( \mathcal{D} - \)isomorphism \( \gamma : A \to B \) such that \( Q(\gamma) = s \).

Dually

**Definition 6.** An object \( A^0 \) of the category \( \mathcal{C} \) is called a left indicator respectively \( \mathcal{D} \) if for every two objects \( A \) and \( B \) and for every bijection \( s : Q(A) \to Q(B) \) the following condition is satisfied:

if for every \( \mathcal{C} - \)morphism \( \nu : A^0 \to A \) there exists a \( \mathcal{D} - \)morphism \( \mu : A^0 \to B \) such that \( Q(\mu) = s \circ Q(\nu) \), then there exists a \( \mathcal{D} - \)isomorphism \( \gamma : A \to B \) such that \( Q(\gamma) = s \).
Roughly speaking, the both conditions are the following ones: if composition of \( s \) and a \( C \)-morphism is a \( D \)-morphism then \( s \) is a \( D \)-isomorphism. In the case \( D = C \) we use the term *indicator*. We give some important examples of indicators in the varieties of universal algebras.

**Examples.** Let \( C \) be a category of universal algebras and all their homomorphisms. We choose in this case the natural forgetful functor in the capacity of \( Q \).

1. Let \( A^0 \) be an algebra such that for every \( C \)-object \( A \) and every two its different elements \( a_1 \) and \( a_2 \) there exists a homomorphism \( \nu : A \to A^0 \) with \( \nu(a_1) \neq \nu(a_2) \). Then \( A^0 \) is a right indicator in \( C \).

   Indeed, let \( s : A \to B \) be a bijection. Suppose that for every homomorphism \( \nu : B \to A^0 \) the composition \( \nu \circ s \) is a homomorphism too. Consider an \( n \)-ary operation symbol \( \omega \) and \( n \) elements \( a_1, \ldots, a_n \in A \). Let \( a = \omega(a_1, \ldots, a_n) \) and \( b = \omega(s(a_1), \ldots, s(a_n)) \). We have to show that \( s(a) = b \). Suppose the contrary, that is, \( s(a) \neq b \). Under hypothesis there exists a homomorphism \( \nu : B \to A^0 \) with \( \nu(s(a)) \neq \nu(b) \). We have: \( (\nu \circ s)(a) = \omega(\nu \circ s(a_1), \ldots, \nu \circ s(a_n)) = \omega(\nu(s(a_1)), \ldots, \nu(s(a_n))) = \nu(\omega(s(a_1), \ldots, s(a_n))) = \nu(b) \).

   That means in contradiction to assumption that \( \nu(s(a)) = \nu(b) \). Hence \( s \) is a homomorphism and therefore \( A^0 \) is a right indicator.

2. Let \( C \) be a full subcategory of the category \( \Theta^0(V) \) for some variety \( V \) and let \( F^0 \) be a free algebra generating the variety \( V \). It is easy to see that the algebra \( A^0 = F^0 \) satisfies the conditions of the previous example. Thus if \( F^0 \) is an object of \( C \) it is a right indicator in this category.

3. Let \( A^0 \) be a such object of \( C \) that for every \( C \)-object \( A \) and every finite subset \( X \) of \( A \) having as many elements as arity of an operation, there exists a homomorphism \( \nu : A^0 \to A \) with \( X \subset \nu(A^0) \). Then \( A^0 \) is a left indicator in \( C \).

   Indeed, let \( s : A \to B \) be a bijection. Suppose that for every homomorphism \( \nu : A^0 \to A \) the composition \( s \circ \nu \) is a homomorphism too. Consider an \( n \)-ary operation symbol \( \omega \) and \( n \) elements \( a_1, \ldots, a_n \in A \). Let \( a = \omega(a_1, \ldots, a_n) \) and \( b = \omega(s(a_1), \ldots, s(a_n)) \). We have to show that \( s(a) = b \). Under hypothesis
there exists a homomorphism \( \nu : A^0 \to A \) with \( a_1, \ldots, a_n \in \nu(A^0) \). It means that there exist \( n \) elements \( w_1, \ldots, w_n \in A^0 \) such that \( a_i = \nu(w_i) \) for all \( i = 1, \ldots, n \). We have: \( s(a) = s(\omega(a_1, \ldots, a_n)) = s(\omega(\nu(w_1), \ldots, \nu(w_n))) = s \circ \nu(\omega(w_1, \ldots, w_n)) = \omega(s(\nu(w_1)), \ldots, s(\nu(w_n))) \). That means \( s : A \to B \) is a homomorphism. And we make conclusion that \( A^0 \) is a left indicator in \( C \).

4. As a corollary from the previous example, we obtain that every free algebra \( H \) in \( C \) such that a set of free generators of \( H \) has not less elements than arities of all operations is a left indicator. Particularly, every free algebra with two generators is a left indicator in a category of algebras with binary, unary and nullary operations only.

The following result shows that if an automorphism of a category is indeed inner it is possible to detect this considering only two objects.

**Theorem 2.** Let \( C \) be a category with a forgetful functor \( Q : C \to \text{Set} \) such that:

1) functor \( Q \) is represented by an object \( A_0 \); 
2) there is a right (or left) indicator \( A^0 \) in \( C \).

If \( \Phi : C \to C \) is an automorphism of the category \( C \) that does not change the objects \( A_0 \) and \( A^0 \) and induces the identity map on \( \text{Hom}(A_0, A^0) \) then \( \Phi \) is an inner automorphism.

**Proof.** Let us fix the objects \( A^0 \) and \( A_0 \) existing under hypotheses. Let \( \Phi \) be an isomorphism of the category \( C \) satisfying required conditions. According to Lemma 4, we define by 1.2 the family of bijections \( (s_A : Q(A) \to Q(\Phi(A)), |A \in \text{Ob}C) \), such that 1.7 is satisfied. Under hypothesis, \( \Phi \) lives fixed all morphisms from \( A_0 \) to \( A^0 \), hence we have \( s_{A^0} = 1_{Q(A^0)} \).

Particularly, 1.7 gives for every morphism \( \nu : A \to A^0 \) that \( Q(\Phi(\nu)) = s_{A^0} \circ Q(\nu) \circ s_A^{-1} = Q(\nu) \circ s_A^{-1} \). Under hypotheses for the category \( C \), there exists an isomorphism \( \sigma_A : A \to \Phi(A) \) such that \( s_A = Q(\sigma_A) \). The dual case gives the same conclusion. And finally we obtain that for every \( \nu : A \to B \): \( \Phi(\nu) = \sigma_B \circ \nu \circ \sigma_A^{-1} \), that ends the proof. \( \square \)
Now we combine together Lemma 1, Lemma 2 and Theorem 2 to get a general result which reduces the problem if an automorphism of a category is inner to more simple question.

**Theorem 3.** Let functor $Q$ be represented by an object $A_0$ and $A^0$ be a left (or right) indicator in $C$. Then an automorphism $\Phi$ of $C$ is inner if and only if there are two isomorphisms $\sigma : A_0 \to \Phi(A^0)$, $\tau : A^0 \to \Phi(A^0)$ such that for every $C$-morphism $\nu : A_0 \to A^0$ we have $\Phi(\nu) = \tau \circ \nu \circ \sigma^{-1}$.

**Proof.** The necessity of the given condition is obvious. We show that it is sufficient. Let $\Phi : C \to C$ is an automorphism such that there are two isomorphisms $\sigma : A_0 \to \Phi(A^0)$, $\tau : A^0 \to \Phi(A^0)$ satisfying the mentioned conditions. According to Lemma 2, we can assume that $\Phi$ preserves both mentioned objects. If $A_0 = A^0$ then $\sigma = \tau$. Thus if objects $A_0$ and $A^0$ are different and if they coincide we can apply Lemma 1 and obtain that $\Phi$ is a composition of two automorphisms $\Phi = \Psi \circ \Gamma$, where $\Gamma$ preserves objects $A_0$ and $A^0$ and all morphisms $\nu : A_0 \to A^0$, and the functor $\Psi$ is an inner automorphism of the category $C$. According to Theorem 2 $\Gamma$ is an inner automorphism of the category $C$. \qed

The similar arguments lead to a sufficient conditions for an automorphism to be $D-$inner for a given extension $D$ of the category $C$.

**Theorem 4.** Let functor $Q$ be represented by an object $A_0$ and $A^0$ be a left (or right) indicator in $C$ respectively $D$. Let an automorphism $\Phi$ of $C$ leave fixed these two objects. Then $\Phi$ is $D-$inner if if the bijection $s_{A_0}^\Phi$ is the $Q-$value of a $D-$isomorphism.

**Proof.** Let $A^0$ be a left indicator in $C$ respectively $D$. Since $Q(\Phi(\nu)) = s_A \circ Q(\nu) \circ s_{A^0}^{-1}$ for every $C-$morphism $\nu : A^0 \to A$, we have $s_A \circ Q(\nu) = Q(\Phi(\nu)) \circ s_{A^0}$ for every $C-$morphism $\nu : A^0 \to A$. We see that the right side of this equation is the $Q-$value of a $D-$morphism, because under hypotheses $s_{A^0}$ is the $Q-$value od a $D-$isomorphism. Under Definition 5 $s_A$ is the $Q-$value of a $D-$isomorphism $\sigma_A$ for every object $A$. Thus $Q(\Phi(\nu)) = Q(\sigma_B \circ \nu \circ \sigma_A^{-1})$ for
every $C$-morphism $\nu : A \to B$. Since $Q$ is faithful, $\Phi$ is $D$-inner. The same conclusion is clearly true in the case $A^0$ is a right indicator in $C$ respectively $D$.

3. CATEGORIES OF UNIVERSAL ALGEBRAS

In this section, we consider categories of universal algebras only. Given a variety $\mathcal{V}$ of universal algebras of a type $\Xi$, $\Theta(\mathcal{V})$ denotes a the category of all $\mathcal{V}$-algebras and their homomorphisms. We apply our results to a full subcategory $C$ of the category $\Theta(\mathcal{V})$. In this case, a functor $Q$ is the natural forgetful functor. Therefore we denote homomorphisms and corresponding maps with the same letter, that is, we write $\nu$ instead of $Q(\nu)$ for every homomorphism $\nu : A \to B$. To apply Theorem 3 it is necessary to find two objects $A_0$ and $A^0$ satisfying conditions (1) and (2) respectively. If a monogenic free algebra $A_0$ in $\Theta(\mathcal{V})$ is an object of $C$ the condition (1) is realized. The condition (2) is realized if our category contains an algebra $A^0$ satisfying at least one of the conditions given in examples in the previous section.

First we show how to obtain the original Reduction Theorem [6]. Consider the category $C = \Theta^0$ described in Introduction, i.e. $\Theta^0$ is the full subcategory of $\Theta(\mathcal{V})$ defined by all free algebras from $\mathcal{V}$ over finite subsets of an infinite fixed set $X_0$. Let $F_0$ be a free algebra in $\Theta^0$ over an one-element set $\{x_0\}$, $x_0 \in X_0$.

**Theorem 5.** [6] Assume that

(R1) every object of $\Theta^0$ is a hopfian algebra and

(R2) there exists an object $F^0 = F(X)$ in $\Theta^0$ generating the whole variety $\mathcal{V}$.

Let $\nu_0 : F^0 \to F_0$ be the homomorphism induced by the constant map $X \to \{x_0\}$, i.e. $\nu_0(x) = x_0$ for all $x \in X$. Under these conditions, if $\Phi : \Theta^0 \to \Theta^0$ is an automorphism such that 1) it does not change objects, 2) it induces the identity automorphism on the semigroup $\text{END}(F^0)$ and 3) it preserves $\nu_0 : \Phi(\nu_0) = \nu_0$, then $\Phi$ is an inner automorphism.

**Proof.** We apply Theorem 2. In the present case, $C = \Theta^0$ and $Q : \Theta^0 \to \text{Set}$ is the natural forgetful functor. Then the functor $Q$ is represented by the object $F_0$ and the condition (1) in Theorem 2 for the category $\Theta^0$ is satisfied.
According to the Example 2 in Section 2, the algebra $F^0 = F(X^0)$ is a right indicator in $\Theta^0$, and therefore the condition (2) is also satisfied.

Consider an automorphism $\Phi : \Theta^0 \to \Theta^0$. Let $\mu : F_0 \to F^0$ be an arbitrary homomorphism and let $\nu = \mu \circ \nu_0 : F^0 \to F^0$. Under hypotheses, we have:

$$\nu = \Phi(\nu) = \Phi(\mu) \circ \Phi(\nu_0) = \Phi(\mu) \circ \nu_0.$$ 

Hence $\mu \circ \nu_0 = \Phi(\mu) \circ \nu_0$ and therefore $\Phi(\mu) = \mu$. Thus the automorphism $\Phi$ satisfies the hypothesis of Theorem 2 and therefore it is an inner automorphism.

□

Remark 1. The condition (R1) has not been used in the proof given above. Thus it is not necessary for the Theorem 5. However this condition is used in [6] to prove the fact that every automorphism of the category $\Theta^0$ takes every object to an isomorphic one. But we have seen that it is also not necessary, it is sufficient to assume this condition only for monogenic free algebra.

Now we begin to characterize automorphisms which are not inner.

In most cases, every automorphism of the category $\Theta(V)$ takes the monogenic free algebras to isomorphic algebras. This is an argument to assume further that an automorphism $\Phi$ of $\mathcal{C}$ preserves $A_0$. According to [17] for $a \in A$: $\Phi(\alpha_a) = s_A \circ \alpha_a \circ s_{A_0}^{-1}$, where $s_{A_0}$ is a permutation of $A_0$. Under definitions of maps $s_A$ above we have $s_{A_0}(x_0) = x_0$ and consequently for every $\alpha : A_0 \to A$:

$$\Phi(\alpha)(x_0) = (s_A \circ \alpha)(x_0). \quad (3.1)$$

The bijections $s_A : A \to \Phi(A)$ can be used to define a new algebraic structure on the underlying set of the algebra $\Phi(A)$. We denote this new algebra by $A^*$, hence $s_A : A \to A^*$ is an isomorphism. Of course, $A^*$ need not be an object of $\mathcal{C}$.

Lemma 5. An automorphism $\Phi$ of $\mathcal{C}$ is inner if and only if there exists a central function $A \mapsto c_A$, $A \in Ob\mathcal{C}$ such that $c_{\Phi(A)}$ is an isomorphism of $\Phi(A)$ onto $A^*$.
Proof. If $\Phi$ is inner, then there exists a function $A \mapsto \tau_A$, $A \in ObC$ such that $\tau_A$ is an isomorphism of $A$ onto $\Phi(A)$ and for all $\nu : A \to B$ we have $\Phi(\nu) = \tau_B \circ \nu \circ \tau_A^{-1}$. Consider $c_{\Phi(A)} = s_A \circ \tau_A^{-1}$. Clearly, $c_{\Phi(A)}$ is an isomorphism of $\Phi(A)$ onto $A^*$. On the other hand, for every homomorphism $\nu : \Phi(A) \to \Phi(B)$ we have $c_{\Phi(B)} \circ \nu \circ c_{\Phi(A)}^{-1} = s_B \circ (\tau_B^{-1} \circ \nu \circ \tau_A) \circ s_A^{-1} = s_B \circ \Phi^{-1}(\nu) \circ s_A^{-1} = \nu$. Hence the function $A \mapsto c_A$, $A \in ObC$ is central.

Inversely, if there exists such central function $A \mapsto c_A$, $A \in ObC$, that $c_{\Phi(A)}$ is an isomorphism of $\Phi(A)$ onto $A^*$, then we set $\tau_A = c_{\Phi(A)}^{-1} \circ s_A$. Clearly that $\tau_A$ is an isomorphism of $A$ onto $\Phi(A)$ and $\tau_B \circ \nu \circ \tau_A^{-1} = c_{\Phi(B)}^{-1} \circ s_B \circ \nu \circ s_A^{-1} \circ c_{\Phi(A)} = c_{\Phi(B)}^{-1} \circ \Phi(\nu) \circ c_{\Phi(A)} = \Phi(\nu)$ for all $\nu : A \to B$. Hence $\Phi$ is inner. □

The crucial fact is that in the case $A$ is a free algebra with enough number of free generators the structure $A^*$ can be found and therefore the map $s_A$ can be described.

Denote by $r_\Xi$ the maximal arity of operations in $\Xi$. Let now $A$ be a free algebra which set of free generators is $X = \{x_1, \ldots, x_n\}$ where $n \geq r_\Xi$. Every element of $A$ being a term in corresponding language determines derivative (or polynomial, in other words) operations, which arities depend on definition but are not greater than $n$. Particularly, suppose that $\Phi(A) = A$ and $\Phi(\alpha_x) = \alpha_x$ for all $x \in X$. Let $\omega$ be a $k$-ary signature operation in $A$, $k \leq n$. Let $u = \omega(x_1, \ldots, x_k)$ and $v$ be an element of $A$ defined by the equation: $\Phi(\alpha_u) = \alpha_v$, that is, $v = s_A^\Phi(u)$. Then according to mentioned above $v$ determines a polynomial $k$-ary operation $\omega^\Phi$ which can be expressed in the following way:

**Definition 7.**

$$\omega^\Phi(x_1, \ldots, x_k) = s_A^\Phi(\omega(x_1, \ldots, x_k))$$ (3.2)

and for every elements $a_1, \ldots, a_k$

$$\omega^\Phi(a_1, \ldots, a_k) = \theta_f(\omega^\Phi(x_1, \ldots, x_k)),$$ (3.3)

where $f(x_1) = a_1, \ldots, f(x_k) = a_k, f(x_{k+1}) = x_{k+1}, \ldots, f(x_n) = x_n$.

Thus the automorphism $\Phi$ determines a new algebra $A^\Phi$ of the same type and with the same underlying set that the algebra $A$. We call this algebra a
Φ-derivative of the algebra $A$. It is clear that along with Φ-derivative we have Φ$^{-1}$-derivative of the algebra.

**Theorem 6.** Let $\mathcal{C}$ be a full subcategory of the category $\Theta(\mathcal{V})$ for some variety $\mathcal{V}$. Let $\mathcal{C}$ contain a monogenic free algebra $A_0$ and $A$ be a free finitely generated algebra in $\mathcal{C}$ with the set of free generators $X = \{x_1, \ldots, x_n\}$ where $n \geq r_\Xi$. If $\Phi$ is an automorphism of $\mathcal{C}$ that preserves $A_0$ and $A$, and $\Phi(\alpha_x) = \alpha_x$ for all $x \in X$, then $A^* = A^\Phi$.

**Proof.** Denote $s = s_A$. Let $\omega$ be a $k$-ary signature operation of the algebra $A$ and $u = \omega(x_1, \ldots, x_k)$. Let $a_1, \ldots, a_k$ be elements of this algebra and $f(x_1) = a_1, \ldots, f(x_k) = a_k, f(x_{k+1}) = x_{k+1}, \ldots, f(x_n) = x_n$. Then $\alpha_{\omega(s^{-1}(a_1), \ldots, s^{-1}(a_k))} = \theta_{s^{-1} \circ f} \circ \alpha_u$. Applying $\Phi$ to this equation and using the definition $\omega^*(a_1, \ldots, a_k) = s(\omega(s^{-1}(a_1), \ldots, s^{-1}(a_k)))$, we obtain:

$$\alpha_{\omega^*(a_1, \ldots, a_k)} = \theta_f \circ \alpha_{s(u)}$$

or

$$\omega^*(a_1, \ldots, a_k) = \omega^\Phi(a_1, \ldots, a_k).$$

□

This result gives us an opportunity to reduce describing automorphisms of subcategories of $\Theta(\mathcal{V})$ to studying derivative operations (polynomial operations) on free algebras of the variety $\mathcal{V}$. We have seen that $A^*$ is derivative algebra of the same type than $A$ and is isomorphic to $A$, hence $A$ is derivative algebra with respect $A^*$. It means that for every basic $k$-ary operation $\omega$, the value $\omega^*(x_1, \ldots, x_k)$ is a polynomial $w$ in $A$ and $\omega(x_1, \ldots, x_k)$ is a polynomial $w^*$ in $A^*$. Replacing in $w^*$ all operations of $A^*$ by its expressions as polynomials in $A$, we obtain an identity $\omega(x_1, \ldots, x_k) = \overline{w^*}$ which must be satisfied in our variety.

We show how these reasons simplify the problem, namely, we show how simple is to obtain some generalizations of known results [5, 7] using our general approach.
Theorem 7. Let $C$ be a full subcategory of of the category of all semigroups containing a free monogenic semigroup $A_0 = W(x_0)$ and a free semigroup $A^0 = W(x, y)$ with two free generators $x, y$. Let the category $D$ be the extension of $C$ by adding anti-homomorphisms. If an automorphism $\Phi$ of $C$ takes $A_0$ and $A^0$ to isomorphic to them semigroups, then it is $D$-inner.

Proof. According to Lemma 2 we can assume that $\Phi$ preserves $A_0$ and $A^0$. Let $s = s_{A^0}$. According to Theorem 6 we look for a derivative binary operation and corresponding semigroup $(A^0)^\ast$ with the same underlying set such that $s$ is an isomorphism preserving $x$ and $y$. Since the unique identity of the kind $xy = w$ in variety of all semigroups is $xy = xy$, we conclude that there exist only two derivative operations: $x \cdot y = xy$ and $x \ast y = yx$. Therefore $s(xy) = xy$ or $s(xy) = yx$. In the first case, $s$ is the identity mapping. In the second one, $s$ maps every word $u$ to the ”indirect” word $\bar{u}$, that is all letters are written in reverse order. According to Theorem 4 $\Phi$ is $D$-inner. \hfill \Box

It is proved in [7] that the category of all free inverse semigroups is perfect. The proof uses a description of $AUT END(F)$ for free inverse semigroups $F$. The next theorem generalizes this result using Theorem 6.

We consider an inverse semigroup $A$ as an algebra with two operations, a binary multiplication $\cdot$ and a unary inversion $^{-1}$ (here $a^{-1}$ is the inverse of an element $a$). The class of all inverse semigroups forms a variety $V$ defined by the identities:

$$(xy)z = x(yz), \quad (xy)^{-1} = y^{-1}x^{-1}, \quad (x^{-1})^{-1} = x,$$

$xx^{-1}x = x, \quad x^{-1}xy^{-1}y = y^{-1}yx^{-1}x.$$

Theorem 8. Let $C$ be a full subcategory of of the category of inverse semigroups containing a free monogenic inverse semigroup $A_0 = W(x_0)$ and a free inverse semigroup $A^0 = W(x, y)$ with two free generators $x, y$. If an automorphism $\Phi$ of $C$ takes $A_0$ and $A^0$ to isomorphic to them semigroups, then it is inner.

Proof. The first step is the same that in Theorem 7, that is, we have a permutation $s$ of $A^0$ and we look for a derivative inverse semigroup $(A^0)^\ast$ with the
same underlying set such that \( s \) is an isomorphism of \( A^0 \) onto \((A^0)^*\), preserving \( x \) and \( y \). First of all, if a term \( u \) determines an involution it does not contain \( y \) and thus \( u = u(x) \). Hence we have the identity \( x = u(u(x)) \). It is known that such identity is fulfilled in the variety of all inverse semigroups if and only if the word on the right has the form \((xx^{-1})^k x \) or the form \( x(x^{-1}x)^k \). Every part of these words is equal to \( x \) or to \( xx^{-1} \) or to \( x^{-1}x \) or to \( x^{-1} \). The first three words can not determine involutions. Thus \( u \equiv x^{-1} \) and \( s(x^{-1}) = x^{-1} \).

Finally\(^2\), let a term \( w(x, y) \) determine a binary operation and consider the following system of three equations: \( w(x, x^{-1}) = xx^{-1}, w(x, w(x^{-1}x)) = x, w(w(xx^{-1}), x) = x \). The only two terms satisfying this system are \( w = xy \) and \( w = yx \). Thus \( s(xy) = xy \) or \( s(xy) = yx \). In the first case, \( s \) is the identity mapping and \((A^0)^* \) coincides with \( A^0 \). In the second one, \((A^0)^* \) is the dual inverse semigroup to \( A^0 \). Because the involution \( c_A : a \mapsto a^{-1} \) is an isomorphism is an isomorphism of every inverse semigroup \( A \) onto is dual inverse semigroup \( A^* \) and the function \( A \mapsto c_A \) is a central, we conclude, according to Lemma \( 5 \) that \( \Phi \) is an inner automorphism.

\( \square \)

The similar method can be applied to categories of groups, modules, linear algebras and so on. In our next paper \( 13 \), we apply this method and characterize automorphisms of categories \( \Theta^0(\mathcal{V}) \) where \( \mathcal{V} \) is the variety of all associative \( K \)-algebras and where \( \mathcal{V} \) is the variety of all free group representations.

4. Some other applications

We should mention that potential-inner automorphisms in fact are inner in the case of full subcategories of the category \( Set \). Thus the following result is a trivial corollary of Theorem \( 1 \).

\textbf{Theorem 9.} Every full subcategory of the category \( Set \) containing one-element set is perfect.

\(^2\)The idea of the next step comes from G. Mashevitzky.
The following old classical result immediately follows from Theorem 1 in other words, Theorem 1 can be regarded as a generalization of this result to categories.

**Theorem 10.** [10] Every automorphism of the semigroup $T_X$ of all transformations of some set $X$ is inner.

*Proof.* Consider the category with one object $X$ and the set $S_X$ as the set of all its morphisms. Let functor $Q$ be $Hom(X, -)$. It is faithful and representable under definition. According to Lemma 4 all automorphisms of this category are inner. \[\Box\]

The next also known old result follows from Theorem 9.

**Theorem 11.** [11] Every automorphism of the semigroup $F_X$ of all partial transformations of a set $X$ is inner.

*Proof.* Consider the category $P(X)$ of all subsets of $X$ and their mappings. According to Theorem 9 it is perfect. Denote by $\Delta_A$ the identity relation on the set $A \subseteq X$: $\Delta_A = \{(a, a) | a \in A\}$. Let $\Phi$ be an automorphism of the semigroup $F_X$. It is known that $\Phi(\Delta_A) = \Delta_B$ for some $B \subseteq X$. Thus we define $\Phi(A) = B \iff \Phi(\Delta_A) = \Delta_B$ and consider $\Phi$ as an automorphism of the category $P(X)$. Since it is inner there is a family of bijections $(s_A : A \to \Phi(A) | A \subseteq X)$ such that for every partial transformation $f$ considered as a map $f : A \to B$ the following equality takes place: $\Phi(f) = s_B \circ f \circ s_A^{-1}$. Particularly for $f = \Delta_A : A \to X$, we have $\Delta_{\Phi(A)} = \Phi(\Delta_A) = s_X \circ \Delta_A \circ s_A^{-1} = s_X \circ s_A^{-1}$. The last means that $s_A$ is the restriction of $s_X$ on $A$. Thus $\Phi(f) = s_X \circ f \circ s_X^{-1}$ for every $f \in F_X$. \[\Box\]

The same reasons are valid for the inverse semigroup of all one-to-one partial transformations of a given set $X$.

**References**

[1] A. Berzins, B. Plotkin, E. Plotkin, *Algebraic geometry in varieties of algebras with the given algebra of constants*, Journal of Math. Sci. **102**(3) (2000), 4039-4070.
[2] G. Grätzer, *Universal Algebra*. D. Van Nostrand Company, Inc., 1968.

[3] E. Formanek, *A question of B. Plotkin about semigroup of endomorphisms of a free group*, Proc. Amer. Math. Soc. **130** (2002), 935-937.

[4] S. MacLane, *Categories for the working mathematicians*, Springer, 1971.

[5] G. Mashevitzky, B. Plotkin, E. Plotkin, *Automorphisms of categories of free algebras of varieties*, Electronic Research Announcements of AMS. **8** (2002), 1-10.

[6] G. Mashevitzky, B. Plotkin, E. Plotkin, *Automorphisms of categories of free Lie algebras*. Preprint, arXiv:math.GM/0210187.

[7] G. Mashevitzky, Boris M. Schein, and Grigori I. Zhitomorski, *Automorphisms of the semigroup of endomorphisms of free inverse semigroups*. (to appear).

[8] B. Plotkin, *Seven lectures on the universal algebraic geometry*. Preprint, (2002), Arxiv:math.GM/0204245, 87pp.

[9] B. Plotkin, *Algebras with the same (algebraic) geometry*, Proceedings of the Steklov Institute of Mathematics. **242** (2003), 165-196.

[10] J. Schreier, *Über Abbildungen einer abstrakten Menge auf ihre Teilmengen*. Fundamenta Mathematica **28** (1936), 261-264.

[11] É.G. Shutov, *Homomorphisms of the semigroup of all partial transformations*, Izvestiya Vyssh. Ucebn. Zaved. (Matematika), no. 3 (1961), 178-184.

[12] G. Zhitomirski, *A generalization and a new proof of Plotkin’s Reduction theorem*. Preprint, arXiv:math.CT/0309018.

[13] B. Plotkin, G. Zhitomirski, *On automorphisms of categories of free algebras of some varieties*. Preprint.

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