LINEAR QUASI-CATEGORIES AS TEMPLICIAL MODULES

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Abstract. We introduce a notion of enriched $\infty$-categories over a given monoidal category, in analogy with quasi-categories over the category of sets. We make use of certain colax monoidal functors, which we call templicial objects, as a replacement of simplicial objects that respects the monoidal structure. We relate the resulting enriched quasi-categories to nonassociative Frobenius monoidal functors, allowing us to prove that the free templicial module over an ordinary quasi-category is a linear quasi-category. To any dg-category we associate a linear quasi-category, the linear dg-nerve, which enhances the classical dg-nerve. Finally, we prove an equivalence between (homologically) non-negatively graded dg-categories on the one hand and templicial modules with a Frobenius structure on the other hand, indicating that nonassociative Frobenius templicial modules and linear quasi-categories can be seen as relaxations of dg-categories.

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1. Introduction

The present paper constitutes the first step in a project aimed at the study of linear $\infty$-topoi, which should stand to Grothendieck abelian categories as $\infty$-topoi stand to Grothendieck topoi. Based upon the distinctive flavour of the Gabriel-Popescu theorem when compared to Giraud’s theorem, one may expect the corresponding higher categorical theory to have distinctively “linear” features as well. In the present paper we establish a suitable underlying notion of linear $\infty$-category, which already brings up several novel issues.

A large part of the paper is developed in the context of more general enrichment in a suitable monoidal category $\mathcal{V}$ (which is supposed to be cocomplete and such that the tensor product preserves colimits in each variable). Note that models for enriched $\infty$-categories were already put forth in [Lur16] and [GH15]. However, for our purpose we decided it would be worthwhile to develop an elementary approach from scratch. The precise relation of our attempt with the higher works remains to be elucidated.

The theory of quasi-categories was initiated by Boardman and Vogt [BV73], extended by Joyal [Joy02], and more recently impressively furthered by Lurie [Lur09]. Our notion of enriched quasi-category, which is supposed to be an enriched analogue of a quasi-category, is inspired by what Leinster calls a homotopy monoid in [Lei00]. The leading motivating example is what we call the $\mathcal{V}$-enriched nerve $N_{\mathcal{V}}(\mathcal{C})$ of a small $\mathcal{V}$-category $\mathcal{C}$. In analogy with the classical nerve, we define for $n \geq 0$:

$$N_{\mathcal{V}}(\mathcal{C})_n = \coprod_{x_0, \ldots, x_n \in \mathcal{C}} \mathcal{C}(x_0, x_1) \otimes \cdots \otimes \mathcal{C}(x_{n-1}, x_n)$$

However, in general, this does not define a simplicial object. In the absence of projection maps to the factors of the tensor product in $\mathcal{V}$, outer face maps are not available, as was noted in [Sho16, Sho18] where homotopy monoids are used in the context of the generalised Deligne conjecture. Instead, keeping the endpoints $x_0$ and $x_n$ fixed, the data (1) can be organised into a strongly unital colax monoidal functor

$$N_{\mathcal{V}}(\mathcal{C}) : \Delta^{op} \to \text{Quiv}_{\text{Ob}(\mathcal{C})}(\mathcal{V})$$
where $\Delta_f \subseteq \Delta$ is the finite interval category as a subcategory of the simplex category and $\text{Quiv}_{\text{Ob}(C)}(\mathcal{V})$ is the category of $\mathcal{V}$-quivers with object set $\text{Ob}(C)$. More generally, we define (see Definition 2.4)

**Definition 1.1.** A tensor-simplicial or templicial $\mathcal{V}$-object with base set $S$ is a strongly unital colax monoidal functor $X : \Delta_f^{op} \to \text{Quiv}_S(\mathcal{V})$.

Templicial $\mathcal{V}$-objects (with varying bases $S$) can be organised into a category $S \otimes \mathcal{V}$ (Definition 2.9). In the appendix A, we discuss an alternative definition which avoids the use of quivers and makes use of colax monoidal functors landing in $\mathcal{V}$. Under appropriate conditions on $\mathcal{V}$, satisfied for sets and modules, both definitions are equivalent (see Definition A.10 and Theorem A.12). However, for our purposes, Definition 1.1 turned out more practical.

A $\mathcal{V}$-quasi-category is then defined as a templicial $\mathcal{V}$-object satisfying an analogue of the weak Kan condition (Definition 2.20). We denote the category of $\mathcal{V}$-quasi-categories by $\text{QCat}(\mathcal{V})$. The $\mathcal{V}$-enriched nerve gives rise to a fully faithful functor

$$N_\mathcal{V} : \text{Cat}(\mathcal{V}) \rightarrow \text{QCat}(\mathcal{V})$$

of which the essential image consists of the strongly monoidal functors (Proposition 2.28). There is a free-forget adjunction

$$\tilde{F} : \text{SSet} \rightleftarrows S \otimes \mathcal{V} : \tilde{U}$$

which facilitates the study of $\mathcal{V}$-quasi-categories to some extent. In particular, a templicial $\mathcal{V}$-object is a $\mathcal{V}$-quasi-category if and only if $\tilde{U}(X)$ is a quasi-category (Remark 2.21).

Remarkably, the question whether the free templicial object on a quasi-category is a $\mathcal{V}$-quasi-category turns out to be less straightforward, and motivates the introduction of nonassociative Frobenius ($\text{naF}$) structures in §3.1. As the name suggest, a $\text{naF}$-monoidal functor is a Frobenius monoidal functor in the sense of [DP08] in which associativity of the lax structure is dropped. From §4.1 on, we focus on the case $\mathcal{V} = \text{Mod}(k)$, and we show (see Propositions 3.5, 3.7, 4.13 and 4.14):

**Proposition 1.2.** The following statements hold:

(a) Let $X$ be a quasi-category, then $X$ has a $\text{naF}$-structure.

(b) The functor $\tilde{F}$ preserves $\text{naF}$-structures.

(c) Let $X$ be a templicial module with a $\text{naF}$-structure, then $X$ is a linear quasi-category.

Together, these statements imply that in the linear case, $\tilde{F}$ preserves quasi-categories.

Further, we show that for a linear quasi-category $X$, the homotopy category of $\tilde{U}(X)$ can be endowed with a linear structure, giving rise
to a linear homotopy category (Proposition 4.4). The situation can be summarised as follows (see Theorem 4.18):

**Theorem 1.3.** There is a diagram of adjunctions

\[
\begin{array}{ccc}
\text{Cat} & \xrightarrow{U} & \text{Cat}(k) \\
\downarrow N & \Leftarrow \Downarrow & \downarrow h \\
\text{QCat} & \xleftarrow{\tilde{U}} & \text{QCat}(k)
\end{array}
\]

which commutes in the sense that

\[
N_k \circ \mathcal{F} \simeq \tilde{F} \circ N \\
\mathcal{F} \circ h \simeq h_k \circ \tilde{F} \\
\tilde{U} \circ N_k \simeq N \circ U \\
h \circ \tilde{U} \simeq U \circ h_k
\]

Frobenius structures also play an important role in relating linear quasi-categories to dg-categories. We denote the category of templicial modules with an (associative) Frobenius structure by \( S_{\text{Frob}}^{\otimes} \text{Mod}(k) \).

Making use of an augmented, monoidal version of the Dold-Kan correspondence (Proposition 5.4), we construct a linear dg-nerve functor

\[
N_{dg}^{\text{dg}} : \text{dg Cat}(k) \to S_{\text{Frob}}^{\otimes} \text{Mod}(k)
\]

which enhances the classical dg-nerve \([\text{Lur16}]\) through \( \tilde{U} \). Finally, in Corollary 5.12 we show that \( N_{dg}^{\text{dg}} \) gives rise to an equivalence of categories

\[
\text{dg Cat}_{\geq 0}(k) \simeq S_{\text{Frob}}^{\otimes} \text{Mod}(k)
\]

This suggests that naF-templicial modules and linear quasi-categories can be seen as relaxations of dg-categories.

In work in progress, we intend to endow their categories with model structures and we will investigate the relation with \( A_\infty \)-structures.

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# 2. Templcial objects

Throughout the paper, we let \((\mathcal{V}, \otimes, I)\) denote a fixed cocomplete monoidal category such that the monoidal product \(- \otimes -\) preserves colimits in each variable.

In this section, we introduce a \(\mathcal{V}\)-enriched analogue of simplicial sets, which we call tensor simplical or templcial \(\mathcal{V}\)-objects (Definition 2.4). These are obtained from colax monoidal functors (§2.2) from the finite interval category to \(\mathcal{V}\)-quivers through a Grothendieck construction (§2.3), and underlie our enriched version of quasi-categories (Definition 2.20) in §2.5. In §2.4 we show that the category of templcial objects
is cocomplete. Finally, in §2.0 we construct the enriched analogue of the classical nerve functor. We start by recalling the relevant simplex categories in §2.1.

2.1. Simplex categories. We will make use of the simplex categories $\Delta_f \subseteq \Delta \subseteq \Delta_+$, where:

- $\Delta_+$ is the augmented simplex category. Its objects are the posets $[n] = \{0, ..., n\}$ with $n \geq -1$ (where $[-1] = \emptyset$), and its morphisms are the order morphisms $[m] \to [n]$.
- $\Delta$ is the ordinary simplex category, which is the full subcategory of $\Delta_+$ spanned by all $[n]$ with $n \geq 0$.
- $\Delta_f$ is the category of finite intervals, which is the subcategory of $\Delta$ consisting of all morphisms $f : [m] \to [n]$ that preserve the endpoints, that is $f(0) = 0$ and $f(m) = n$.

In contrast to the category $\Delta$, both the categories $\Delta_+$ and $\Delta_f$ are naturally endowed with monoidal structures.

The monoidal structure $(\star, [-1])$ on $\Delta_+$ is given by juxtaposition of posets and morphisms, as follows. For $m, n \geq -1$:

$$[m] \star [n] = [m + n + 1]$$

For morphisms $f : [m] \to [m']$ and $g : [n] \to [n']$ in $\Delta_+$:

$$(f \star g)(i) = \begin{cases} f(i) & \text{if } i \leq m \\ m' + 1 + g(i - m - 1) & \text{if } i > m \end{cases}$$

Similarly, the monoidal structure $(+, [0])$ on $\Delta_f$ is given by identifying respective top and bottom endpoints, as follows. For all $m, n \geq 0$:

$$[m] + [n] = [m + n]$$

For morphisms $f : [m] \to [m']$ and $[n] \to [n']$ in $\Delta_f$:

$$(f + g)(i) = \begin{cases} f(i) & \text{if } i \leq m \\ m' + g(i - m) & \text{if } i > m \end{cases}$$

There is a well-known monoidal equivalence $\Delta_+ \simeq \Delta_f^{op}$, the relevant functor in each direction being obtained by considering posets of morphisms into $[1]$ (see [Joy97]).

2.2. Colax monoidal functors. For an arbitrary category $\mathcal{C}$, one may consider the functor category $SC = \text{Fun}(\Delta^{op}, \mathcal{C})$ of simplicial $\mathcal{C}$-objects. For a monoidal category $(\mathcal{V}, \otimes, I)$, it makes sense to look for a variant of $S\mathcal{V}$ which is compatible with the monoidal structure of $\mathcal{V}$ to some extent. In order to do so, we will make use of the monoidal finite interval category $\Delta_f$ rather than the ordinary simplex category $\Delta$. In a first attempt, motivated by Proposition 2.1 below, we consider the category

$$\text{Colax}(\Delta_f^{op}, \mathcal{V})$$
of colax monoidal functors and monoidal natural transformations.

Note that $\Delta_f$ is generated by the inner coface maps $\delta_j : [n-1] \to [n]$ and codegeneracy maps $\sigma_i : [n+1] \to [n]$. That is, every morphism $f : [m] \to [n]$ can be uniquely expressed as

$$f = \delta_{i_1} \ldots \delta_{i_s} \sigma_{j_1} \ldots \sigma_{j_t}$$

with $0 < i_1 < \ldots < i_s < m$, $0 \leq j_1 < \ldots < j_t < n$ and $s, t \in \mathbb{N}$ such that $n - t + s = m$.

Explicitly, a colax monoidal functor $X : \Delta_{op} \to \mathcal{V}$ with comultiplication $\mu$ and counit $\epsilon$ corresponds to a sequence of $\mathcal{V}$-objects $(X_n)_{n \geq 0}$ endowed with inner face maps $d_j : X_n \to X_{n-1}$ for $0 < j < n$ and degeneracies $s_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$ satisfying the usual simplicial identities, as well as morphisms

$$\mu_{n,m} : X_{n+m} \to X_n \otimes X_m$$

and

$$\epsilon : X_0 \to I$$

satisfying naturality, coassociativity and counitality axioms (see []).

Since $\mu$ is coassociative, we have a well-defined map

$$\mu_{k_1,\ldots,k_n} : X_{k_1 + \ldots + k_n} \to X_{k_1} \otimes \ldots \otimes X_{k_n}$$

for all $n \geq 2$ and $k_1, \ldots, k_m \geq 0$. Moreover, we will set $\mu_{k_1,\ldots,k_n}$ to be the identity on $X_{k_1}$ if $n = 1$, and the counit $\epsilon$ if $n = 0$.

Recall that a monoidal category is cartesian if the tensor product is given by the categorical product.

**Proposition 2.1.** [Lei00, Proposition 3.1.7] Let $(\mathcal{V}, \times, 1)$ be a cartesian monoidal category. There is an isomorphism of categories

$$\text{Colax}(\Delta_{op}^{\mathbf{f}}, \mathcal{V}) \simeq S\mathcal{V}.$$ 

**Example 2.2.** For $\mathcal{V} = \text{Set}$, we denote the category of simplicial sets by $\text{SSet} = \text{SSet}$. According to Proposition 2.1, we have an isomorphism of categories $\text{Colax}(\Delta_{op}^{\mathbf{f}}, \text{Set}) \simeq \text{SSet}$.

Suppose $\mathcal{V}$ is cartesian as in Proposition 2.1. Explicitly, for a simplicial $\mathcal{V}$-object $X : \Delta_{op} \to \mathcal{V}$, we can consider its restriction to $\Delta_{op}^{\mathbf{f}}$ as a colax monoidal functor $\Delta_{op}^{\mathbf{f}} \to \mathcal{V}$ whose comultiplication is given by

$$\mu_{k,l} = (d_{k+1} \ldots d_{k+l}, d_0 \ldots d_0) : X_{k+l} \to X_k \times X_l$$

for all $k, l \geq 0$, and whose counit is given by the terminal map

$$\epsilon : X_0 \to 1.$$ 

Conversely, for a colax monoidal functor $(X : \Delta_{op}^{\mathbf{f}} \to \mathcal{V}, \mu, \epsilon)$ we obtain outer face maps $d_0$ and $d_n$ respectively as

$$X_{n+1} \xrightarrow{\mu_{1,n}} X_1 \times X_n \xrightarrow{p_2} X_n$$
and

\[ X_{n+1} \xrightarrow{\mu_{n,1}} X_n \times X_1 \xrightarrow{p_1} X_n \]

where we have made use of the projections \( p_1 \) and \( p_2 \) from the product to its factors.

If \( \mathcal{V} \) is not necessarily cartesian, these projections are not available in general and the comultiplication \( \mu \) of a colax monoidal functor can be considered as a kind of stand-in for the outer face maps.

2.3. Templicial objects. In order to use colax monoidal functors \( \Delta^{op} \rightarrow \mathcal{V} \) as a model for \( \mathcal{V} \)-enriched quasi-categories, we’d like them to have a set of objects. In this section, we will realise this by replacing \( \mathcal{V} \) by a category of \( \mathcal{V} \)-enriched quivers.

**Definition 2.3.** Given a set \( S \), a \( \mathcal{V} \)-enriched quiver (or \( \mathcal{V} \)-quiver) \( Q \) on \( S \) is a family \((Q(a,b))_{a,b \in S}\) of objects in \( \mathcal{V} \). A morphism \( f : Q \rightarrow P \) is a family of morphisms \((f_{a,b} : Q(a,b) \rightarrow P(a,b))_{a,b \in S}\) in \( \mathcal{V} \). We write

\[ \text{Quiv}_S(\mathcal{V}) = \text{Fun}(S \times S, \mathcal{V}) \]

for the category of all \( \mathcal{V} \)-quivers on \( S \) and morphisms between them.

The category \( \text{Quiv}_S(\mathcal{V}) \) has a monoidal structure \((\otimes_S, I_S)\) given by, for all \( Q, P \in \text{Quiv}_S(\mathcal{V}) \) and \( a, b \in S \):

\[
(Q \otimes_S P)(a,b) = \bigoplus_{c \in S} Q(a,c) \otimes P(c,b) \quad \text{and} \quad I_S(a,b) = \begin{cases} I & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}
\]

where 0 is the initial object of \( \mathcal{V} \). We will sometimes drop the subscript \( S \) from the notation when it is clear from context.

**Definition 2.4.** A pair \((X,S)\) with \( S \) a set and \( X : \Delta^{op} \rightarrow \text{Quiv}_S(\mathcal{V}) \) a colax monoidal functor is called a templicial \( \mathcal{V} \)-object if \( X \) is strongly unital, i.e. its counit is an isomorphism. In this case we call \( S \) the base of \( X \).

Note that in order to have the monoidal structure (2) available, it is essential to consider quivers with a fixed set of objects. We can now define the category of templicial \( \mathcal{V} \)-objects with varying sets of objects through a Grothendieck construction, as follows.

**Construction 2.5.** Let \( f : S \rightarrow T \) be a map between sets. We define

\[ f^* : \text{Quiv}_T(\mathcal{V}) \rightarrow \text{Quiv}_S(\mathcal{V}) \]

by setting for all \( \mathcal{V} \)-quivers \( Q \) on \( T \), and all \( a, b \in S \):

\[ f^*(Q)(a,b) = Q(f(a), f(b)) \]

and for any morphism \( g : Q \rightarrow P \) in \( \text{Quiv}_T(\mathcal{V}) \), and all \( a, b \in S \):

\[ f^*(g)_{a,b} = g_{f(a), f(b)} \]
Note that by identifying $\text{Quiv}_S(V) = \text{Fun}(S \times S, V)$, $f^*$ is in fact the precomposition functor $- \circ (f \times f)$. Consequently, it has a left adjoint given by the left Kan extension

$$f_! = \text{Lan}_{f \times f}(-) : \text{Quiv}_S(V) \to \text{Quiv}_T(V)$$

In this case, $f_!$ is easily seen to be given by,

$$f_!(Q)(x, y) = \prod_{a \in f^{-1}(x)} \prod_{b \in f^{-1}(y)} Q(a, b)$$

for all $Q \in \text{Quiv}_S(V)$ and $x, y \in T$.

**Lemma 2.6.** For any function $f : S \to T$, $f^*$ is a lax monoidal functor and $f_!$ is a colax monoidal functor.

**Proof.** Because of the adjunction $f_! \dashv f^*$, it suffices to show that $f^*$ is lax monoidal. Define the unit $u : I_S \to f^*(I_T)$ of $f^*$ by

$$u_{a,b} = \begin{cases} 
I & \text{if } a = b \\
0 \to I & \text{if } a \neq b, f(a) = f(b) \\
0 \to 0 & \text{if } f(a) \neq f(b)
\end{cases}$$

for all $a, b \in S$. Further, we have for any $Q, P \in \text{Quiv}_T(V)$ that

$$f^*(Q \otimes_T P)(a, b) = \prod_{x \in T} Q(f(a), x) \otimes Q(x, f(b))$$

and

$$(f^*(Q) \otimes_S f^*(P))(a, b) = \prod_{c \in S} Q(f(a), f(c)) \otimes P(f(c), f(b))$$

which gives a canonical map of quivers

$$m_{Q,P} : f^*(Q) \otimes_S f^*(P) \to f^*(Q \otimes_T P)$$

It is readily verified that $m_{Q,P}$ is natural in $Q$ and $P$, and that it is associative and counital with respect to $u$. \qed

In the next result, we consider $\text{Set}$ as a 2-category with discrete hom-categories. Further, we let $\text{Cat}$ denote the (large) strict 2-category of categories, functors and natural transformations and $\text{MonCat}$ the (large) strict 2-category of monoidal categories, colax monoidal functors and monoidal natural transformations. In particular, for monoidal categories $V, W$ we have $\text{MonCat}(V, W) = \text{Colax}(V, W)$.

**Proposition 2.7.** The assignments $S \mapsto \text{Quiv}_S(V)$ and $f \mapsto f_!$ define a pseudofunctor $(-)_! : \text{Set} \to \text{MonCat}$.

**Proof.** For any $V$-quiver $Q$ with set of objects $S$, we obviously have that $(\text{id}_S)_!(Q) \simeq Q$. Further, given maps of sets $f : R \to S$ and $g : S \to T$,
we have for all $Q \in \text{Quiv}_S(V)$ and $x, y \in T$:

$$
(g \circ f)_!(Q)(x, y) = \prod_{r \in (gf)^{-1}(x)} \prod_{s \in (gf)^{-1}(y)} Q(r, s)
$$

$$
g_!(f_!(Q))(x, y) = \prod_{a \in g^{-1}(x)} \prod_{b \in g^{-1}(y)} \prod_{r \in f^{-1}(a)} \prod_{s \in f^{-1}(b)} Q(r, s)
$$

So we have an isomorphism $(g \circ f)_!(Q) \simeq g_!(f_!(Q))$.

It follows from a direct verification that these isomorphisms make $(-)_!$ into a well-defined pseudofunctor. □

**Construction 2.8.** Consider the pseudofunctor

$$
\Phi_V = \text{Colax}(\Delta_*^{op}, (-)_!) : \text{Set} \to \text{Cat}
$$

sending a set $S$ to the category $\text{Colax}(\Delta_*^{op}, \text{Quiv}_S(V))$. A map of sets $f : S \to T$ is sent to the post-composition functor $f_! \circ -$.

Consider the Grothendieck construction $\int \Phi_V$ of $\Phi_V$. Explicitly, $\int \Phi_V$ is the category over Set whose objects are all pairs $(X, S)$ with $S$ a set and $X : \Delta_*^{op} \to \text{Quiv}_S(V)$ a colax monoidal functor. A morphism from $(X, S)$ to $(Y, T)$ is given by a pair $(\alpha, f)$ with $f : S \to T$ a map of sets and $\alpha : f_*X \to Y$ a monoidal natural transformation in $\Phi_V(T)$. The composition of two morphisms $(\alpha, f)$ and $(\beta, g)$ is given by

$$(\beta, g) \circ (\alpha, f) = (\beta \circ g \alpha \circ \varphi_{f,g}X, g \circ f)$$

where $\varphi_{f,g} : (g \circ f)_! \simeq g_! \circ f_!$ is the monoidal natural isomorphism given by the pseudofunctor $(-)_!$.

**Definition 2.9.** We denote the full subcategory of $\int \Phi_V$ spanned by all templicial $V$-objects by $S \otimes V$

We call the morphisms of $S \otimes V$ templicial morphisms.

**Construction 2.10.** Let $Y$ be a simplicial set. By Proposition 2.1, we may consider $Y$ as a colax monoidal functor $\Delta_*^{op} \to \text{Set}$ with comultiplication $\mu$ and counit $\epsilon$. Then define for all $n \geq 0$ and $a, b \in Y_0$

$$
\tilde{Y}_n(a, b) = \{ y \in Y_n \mid \mu_{0,...,0}(y) = (a, y, b) \}
$$

$$
= \{ y \in Y_n \mid d_1...d_n(y) = a, d_0...d_0(y) = b \}
$$

Given $f : [m] \to [n]$ in $\Delta_*$, it follows from the simplicial identities that $Y(f) : Y_n \to Y_m$ restricts to $Y(f)_{a,b} : \tilde{Y}_n(a, b) \to \tilde{Y}_m(a, b)$. Moreover, it is clear that for all $k, l \geq 0$ and $a, b \in Y_0$,

$$
\mu_{k,l} |_{\tilde{Y}_{k+l}(a, b)} : \tilde{Y}_{k+l}(a, b) \to \prod_{c \in Y_0} \tilde{Y}_k(a, c) \times \tilde{Y}_l(c, b)
$$
and

\[ \tilde{Y}_0(a, b) = \begin{cases} \{a\} & \text{if } a = b \\ \emptyset & \text{if } a \neq b \end{cases} \]

Consequently, \( \tilde{Y} \) is a strongly unital colax monoidal functor, whereby \((\tilde{Y}, Y_0)\) is a templicial object.

Conversely, if \((X, S)\) is a templicial object in Set, then we can define a simplicial set \( \overline{X} \) by setting for all \( n \geq 0 \):

\[ \overline{X}_n = \coprod_{a,b \in S} X_n(a, b) \]

It is readily verified that the assignments \( Y \mapsto \tilde{Y} \) and \( X \mapsto \overline{X} \) can be extended to mutually inverse equivalences between \( SSet \) and \( S \times Set \).

In Appendix A, we will present a more general comparison between templicial \( V \)-objects and colax monoidal functors \( \Delta_{op} \to V \) for suitable monoidal categories \( V \).

Remark 2.11. Let \((X, S)\) be a templicial \( V \)-object and consider \( a, b \in S \). The case \( V = Set \) suggests that \( X_n(a, b) \in V \) should be interpreted as the object of \( n \)-simplices with first vertex \( a \) and last vertex \( b \).

Construction 2.12. Consider another cocomplete monoidal category \( U \), whose monoidal product preserve colimits in each variable. Let \( H : U \to V \) be a strongly unital colax monoidal functor that preserves coproducts. Then for any set \( S \), \( H \) induces a colax monoidal functor

\[ H_S : \text{Quiv}_S(U) \to \text{Quiv}_S(V) \]

given by \( H_S(Q)(a, b) = H(Q(a, b)) \) for all \( a, b \in S \). If \( f : S \to T \) is a map of sets, then because \( H \) preserves coproducts, we have a monoidal natural isomorphism

\[ f_! \circ H_S \simeq H_T \circ f_! \]

and one can check that the functors \((H_S)_S\) form a pseudonatural transformation \( H_* \). Thus we have a pseudonatural transformation

\[ \text{Colax}(\Delta_{op}, H_*) : \Phi_U \to \Phi_V \]

Then the Grothendieck construction supplies us with a functor

\[ \tilde{H} : \int \Phi_U \to \int \Phi_V \]

Explicitly, a pair \((X, S)\) in \( \int \Phi_U \) is sent to the pair \((H_S \circ X, S)\) in \( \int \Phi_V \). Finally, as \( H \) is assumed to be strongly unital, each \( H_S \) is strongly unital as well and thus \( \tilde{H} \) restricts to a functor

\[ \tilde{H} : S \otimes U \to S \otimes V \]
2.4. Cocompleteness. In this section we show that the category $S \otimes V$ of templicial $V$-objects is cocomplete and we explicitly describe its colimits. We make use of the following well-known result (see for instance Corollary 3.3.7 of [Her93]).

**Proposition 2.13.** Let $C$ be a category and $\Psi : C \to \text{Cat}$ a pseudo-functor. Assume that

1. $C$ is cocomplete,
2. for every object $C$ of $C$, the category $\Psi(C)$ is cocomplete,
3. for every morphism $f$ in $C$, the functor $\Psi(f)$ preserves colimits.

Then the Grothendieck construction $\int \Psi$ is cocomplete and a colimit of objects $(X_i, C_i)$ with $C_i \in C$ and $X_i \in \Psi(C_i)$ is obtained as

$$\text{colim}_i (X_i, C_i) = (\text{colim}_i \Psi(\iota^i)(X_i), \text{colim}_i C_i)$$

for the canonical morphisms $\iota^i : C_i \to \text{colim}_i C_i$ in $C$.

Let $\Phi_V$ be as in Construction 2.8. In order to apply Theorem 2.13, we further make use of the following general result, which is not hard to prove:

**Proposition 2.14.** Consider monoidal categories $C$ and $D$ with $C$ small. If $D$ is cocomplete, then so is the category $\text{Colax}(C, D)$, and the forgetful functor $\text{Colax}(C, D) \to \text{Fun}(C, D)$ preserves colimits.

**Corollary 2.15.** The category $\int \Phi_V$ is cocomplete.

**Proof.** Since $V$ is cocomplete, so is $\text{Quiv}_S(V) = V^{S \times S}$ for every set $S$. Therefore, also $\Phi_V(S) = \text{Colax}(\Delta_f^{op}, \text{Quiv}_S(V))$ is cocomplete by Proposition 2.14. Moreover, if $f$ is a map of sets, then $f_!$ is left adjoint to $f^*$ and thus preserves colimits. It follows that $\Phi_V(f)$ preserves colimits as well. Thus by Theorem 2.13, the category $\int \Phi_V$ is cocomplete. □

Let us explicitly describe the colimits of $\int \Psi$ from Theorem 2.13 for $\Psi = \Phi_V$. Consider a diagram

$$D : J \to \int \Phi_V$$

Write $D(j) = (X^j, S^j)$ for every $j \in J$ and $D(t) = (\alpha^t, f^t) : D(i) \to D(j)$ for every $t : i \to j$ in $J$. Then the colimit of $D$ is given by

$$(\text{colim} \hat{D}, S)$$

where $S = \text{colim}_{j \in J} S^j$ in Set with canonical maps $\iota^j : S^j \to S$, and

$$\hat{D} : J \to \text{Colax}(\Delta_f^{op}, \text{Quiv}_S(V))$$

is defined by for all $i, j \in J$ and $t : i \to j$ in $J$:

$$\hat{D}(j) = \iota^j X^j \quad \text{and} \quad \hat{D}(t) : \iota^i X^i \simeq \iota^i f^t_! X^i \xrightarrow{\iota^t \alpha^t} \iota^j X^j$$
where the isomorphism $\iota^i_! X^i \simeq \iota^i_! f^i_! X^i$ is given by the fact that $\iota^i f = \iota^i$.

Next we turn to templicial objects.

**Proposition 2.16.** The category $S \otimes V$ is cocomplete

*Proof.* We check that the subcategory $S \otimes V$ is closed under colimits in $\int \Phi_V$. So let $J$ be a small category and $D : J \to S \otimes V \subseteq \int \Phi_V$ a diagram. With notations as above, the colimit of $D$ in $\int \Phi_V$ is the pair $(\text{colim} \tilde{D}, S)$. For every $j \in J$, write $\epsilon^X_j$ and $\epsilon^{\iota_j}_!$ for the counits of $X^j$ and $\iota^j_!$ respectively.

Boiling down the definitions, we see that the counit $(\text{colim} \tilde{D})_0 \to I_S$ of $\text{colim} \tilde{D}$ is the composition

$$\text{colim}_{j \in J} \iota^j_!(X^j_0) \xrightarrow{\text{colim}_{j \in J} \iota^j_!(\epsilon^X_j)} \text{colim}_{j \in J} \iota^j_!(I_{S_j}) \xrightarrow{\text{colim}_{j \in J} \iota^j_!(\epsilon^{\iota_j}_!)} \text{colim}_{j \in J} I_S \xrightarrow{\nabla} I_S$$

in $\text{Quiv}_S(V)$, where $\nabla$ is the codiagonal. Now for any $x, y \in S$,

$$(\text{colim}_{j \in J} \iota^j_!(I_{S_j}))(x, y) \simeq \begin{cases} \text{colim}_{j \in J} \bigsqcup_{a \in (\iota^j_!)^{-1}(x)} I \simeq I & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

A careful verication shows that this isomorphism is precisely the composition $\nabla \circ \text{colim}_{j \in J} \epsilon^{\iota_j}_!$. Since $\epsilon^X_j$ is assumed to be an isomorphism, we conclude that $\text{colim} \tilde{D}$ is strongly counital and thus that $\text{colim} D$ is a templicial object. □

**Lemma 2.17.** Let $H : U \to V$ be a strongly unital colax monoidal functor between cocomplete monoidal categories whose monoidal products preserve colimits in each variable. Assume that $H$ preserves colimits. Then the induced functor of Construction 2.12

$$\tilde{H} : S \otimes U \to S \otimes V$$

preserves colimits.

*Proof.* Let $J$ be a small category and $D : J \to S \otimes U$ a diagram. With notations as above, we have a monoidal natural isomorphism

$$H_S \circ \text{colim} \tilde{D} = H_S \circ \text{colim}_{j \in J} \iota^j_! X^j \simeq \text{colim}_{j \in J} \iota^j_! H_S X^j$$

because $H$ preserves colimits and $H_T f \simeq f H_S$ for every map of sets $f : S \to T$. It follows that $\tilde{H}$ preserves colimits. □

2.5. **Enriched quasi-categories.** Consider the category $\text{Set}$ with the cartesian monoidal structure. Up to natural isomorphism, we have a unique colimit-preserving strong monoidal functor

$$F : \text{Set} \to V : S \mapsto \bigsqcup_{a \in S} I$$

It has a lax monoidal right adjoint

$$U = V(I, -) : V \to \text{Set}$$
Given \( S \in \text{Set} \) and \( V \in \mathcal{V} \), we view \( F(S) \) as the free object on \( S \) in \( \mathcal{V} \) and \( U(V) \) as the underlying set of \( V \). These notations will remain fixed for the rest of this paper.

As \( F \) is strong monoidal and preserves colimits, it induces a functor

\[
\tilde{F} : \text{SSet} \simeq S \times \text{Set} \to S \otimes \mathcal{V}
\]

by Construction 2.12. Since by Lemma 2.17, \( \tilde{F} \) preserves colimits, the following result is immediate.

**Proposition 2.18.** The functor \( \tilde{F} : \text{SSet} \to S \otimes \mathcal{V} \) has a right adjoint

\[
\tilde{U} : S \otimes \mathcal{V} \to \text{SSet}
\]

that is given by, for all templicial objects \( X \) and \( n \geq 0 \),

\[
\tilde{U}(X)_n = S \otimes \mathcal{V}(\tilde{F}(\Delta^n), X)
\]

**Definition 2.19.** We refer to the functors \( \tilde{F} \) and \( \tilde{U} \) as the *free templicial object functor* and the *underlying simplicial set functor* respectively.

We are now ready to define our model for \( \mathcal{V} \)-enriched quasi-categories. Analogously to ordinary quasi-categories, we require that a templicial object satisfies the weak Kan condition, i.e. it fills all inner horns.

**Definition 2.20.** Let \( (X, S) \) be a templicial \( \mathcal{V} \)-object. We call \( (X, S) \) a \( \mathcal{V} \)-enriched quasi-category or \( \mathcal{V} \)-quasi-category if for all \( 0 < k < n \), every diagram of solid arrows in \( S \otimes \mathcal{V} \)

\[
\begin{array}{ccc}
\tilde{F}(\Lambda^n_k) & \rightarrow & X \\
\downarrow & & \downarrow \\
\tilde{F}(\Delta^n) & \leftarrow & \\
\end{array}
\]

has a lift represented by the dotted arrow. In this case, we call the elements of \( S \) the *objects* of \( X \). We denote the full subcategory of \( S \otimes \mathcal{V} \) spanned by all \( \mathcal{V} \)-quasi-categories by \( \text{QCat}(\mathcal{V}) \).

**Remark 2.21.** By the adjunction \( \tilde{F} \dashv \tilde{U} \), it is immediately clear that a templicial object \( X \) is an enriched quasi-category if and only if its underlying simplicial set \( \tilde{U}(X) \) is an ordinary quasi-category.

Let \( Y \) be a simplicial set and \( (X, S) \) a templicial \( \mathcal{V} \)-object. Then since for all \( n \geq 0 \) and \( a, b \in Y_0 \):

\[
\tilde{F}(Y)_n(a, b) = F(Y_n(a, b)) = \coprod_{y \in Y_n(a, b)} I
\]

a templicial morphism \( (\alpha, f) : \tilde{F}(Y) \to X \) is equivalent to a family

\[
\left( \alpha_y \in U(X_n(f(a), f(b))) \right)_{a, b \in Y_n, n > 0}
\]
along with the map $f : Y_0 \to S$, satisfying
\[
\begin{align*}
  d_j(\alpha_y) &= \alpha_{d_j(y)} & \forall 0 < j < n \\
  s_i(\alpha_y) &= \alpha_{s_i(y)} & \forall 0 \leq i \leq n \\
  \mu_{k,n-k}(\alpha_y) &= \alpha_{d_{k+1} \ldots d_n(y)} \otimes \alpha_{d_0 \ldots d_0(y)} & \forall 0 < k < n 
\end{align*}
\]
where we also denoted $d_j$, $s_i$ and $\mu_{k,n-k}$ for the underlying maps of sets.

Note that this family is completely determined by the entries for which $y \in Y_n(a, b)$ is non-degenerate and not a face of another simplex.

Taking $Y = \Delta^n$ for some $n \geq 0$, we obtain:

**Proposition 2.22.** Consider a templicial $\mathcal{V}$-object $(X, S)$ with underlying simplicial set $\hat{U}(X)$. An $n$-simplex of $\hat{U}(X)$ is equivalent to a pair
\[
\begin{align*}
  \left( (\alpha_i \in S)_{0 \leq i \leq n}, \left( \alpha_{i,j} \in U(X_{j-i}(\alpha_i, \alpha_j)) \right)_{0 \leq i < j \leq n} \right)
\end{align*}
\]
such that for all $0 \leq i < k < j \leq n$, we have
\[
\mu_{k-i,j-k}(\alpha_{i,j}) = \alpha_{i,k} \otimes \alpha_{k,j}
\]
In particular, we have $\hat{U}(X)_0 \simeq S$.

We will write a pair (3) compactly as $(\alpha_{i,j})_{0 \leq i \leq j \leq n}$, with $\alpha_{i,i} = \alpha_i$.

2.6. The enriched nerve. We finish this section by introducing the $\mathcal{V}$-enriched nerve functor and discussing some properties.

Note that a $\mathcal{V}$-category $\mathcal{C}$ with object set $S = \text{Ob}(\mathcal{C})$ can be identified with a monoid in $\text{Quiv}_S(\mathcal{V})$. We will often write $\mathcal{C}$ for the underlying $\mathcal{V}$-quiver as well, and write $m_\mathcal{C} : \mathcal{C} \otimes S \mathcal{C} \to \mathcal{C}$ and $u_\mathcal{C} : I_S \to \mathcal{C}$ for its composition and unit respectively.

Given a map $f : S \to T$ of sets and a $\mathcal{V}$-category $\mathcal{D}$ with $\text{Ob}(\mathcal{D}) = T$, we get a $\mathcal{V}$-category $f^*(\mathcal{D})$ with object set $S$ because $f^*$ is lax monoidal. Then a $\mathcal{V}$-enriched functor $H : \mathcal{C} \to \mathcal{D}$ can be identified with a map $f : S \to T$ on objects along with a map $\mathcal{C} \to f^*(\mathcal{D})$ of monoids in $\text{Quiv}_S(\mathcal{V})$, which we also denote by $H$.

**Construction 2.23.** Let $\mathcal{C}$ be a small $\mathcal{V}$-enriched category. For all $n \geq 0$, define the $\mathcal{V}$-quiver
\[
N_\mathcal{V}(\mathcal{C})_n = \mathcal{C}^{\otimes n}
\]
and for all $i \in [n]$ and $j \in \{1, \ldots, n-1\}$, define
\[
\begin{align*}
  d_j &= \text{id}_{\mathcal{C}^{\otimes i}} \otimes m_\mathcal{C} \otimes \text{id}_{\mathcal{C}^{\otimes n-i-1}} : \mathcal{C}^{\otimes n} \to \mathcal{C}^{\otimes n-1} \\
  s_i &= \text{id}_{\mathcal{C}^{\otimes i}} \otimes u_\mathcal{C} \otimes \mathcal{C}^{\otimes n-i} : \mathcal{C}^{\otimes n} \to \mathcal{C}^{\otimes n+1}
\end{align*}
\]
By the associativity and unitality conditions on $\mathcal{C}$, we have a functor
\[
N_\mathcal{V}(\mathcal{C}) : \Delta^\text{op}_n \to \text{Quiv}_{\text{Ob}(\mathcal{C})}(\mathcal{V})
\]
Further, for any \( k, l \geq 0 \) we let

\[
\mu_{k,l} : C^{\otimes k} \otimes C^{\otimes l} \to C^{\otimes k+l} \quad \text{and} \quad \epsilon : C^{\otimes 0} \to I_{\text{Ob}(C)}
\]

be the canonical isomorphisms in Quiv_{\text{Ob}(C)}(\mathcal{V})}. It immediately follows that this defines a colax (even strong) monoidal structure on \( N_\mathcal{V}(\mathcal{C}) \).

We conclude that

\[
(N_\mathcal{V}(\mathcal{C}), \text{Ob}(\mathcal{C}))
\]

is a templicial \( \mathcal{V} \)-object, called the \( (\mathcal{V}-\text{enriched}) \) nerve of \( \mathcal{C} \).

Note that explicitly, for all \( n \geq 0 \) and \( A, B \in \text{Ob}(\mathcal{C}) \) we have

\[
N_\mathcal{V}(\mathcal{C})_n(A, B) = \prod_{A_0 = A, A_n = B} \mathcal{C}(A_0, A_1) \otimes \ldots \otimes \mathcal{C}(A_{n-1}, A_n)
\]

**Lemma 2.24.** Let \( (X, \mu, \epsilon) \) be a templicial object with base \( S \), \( \mathcal{C} \) a small \( \mathcal{V} \)-enriched category and \( f : S \to \text{Ob}(\mathcal{C}) \) a map of sets. Then we have a bijection between monoidal natural transformations \( f_!: X \to N_\mathcal{V}(\mathcal{C}) \) and quiver morphisms \( H : X_1 \to f^*(\mathcal{C}) \) such that the diagrams (5)

\[
\begin{array}{c}
X_1 \otimes^2 \\ \downarrow \mu_{1,1} \\
X_2 \\
\downarrow \sigma_0 \\
X_1 \\
\downarrow H \\
X_0 \\
\downarrow f^*(\epsilon) \\
\end{array}
\]

commute.

**Proof.** Given a monoidal natural transformation \( \alpha : f_!: X \to N_\mathcal{V}(\mathcal{C}) \), define \( H_\alpha \) to be the adjoint of \( \alpha_1 : f_!(X_1) \to \mathcal{C} \). It follows from the monoidality of \( \alpha \) that for all \( n \geq 0 \), \( \alpha_n \) is the composite

\[
f_!(X_n) \xrightarrow{f_!(\mu_{1,1})} f_!(X_1 \otimes^2) \xrightarrow{f^*(\mu_{1,1})} f_!(X_1) \otimes f_!(X_1) \xrightarrow{\alpha_1 \otimes \alpha_1} C \otimes C
\]

So the assignment \( \alpha \mapsto H_\alpha \) is injective. Moreover, it then follows from the naturality of \( \alpha \) that \( H_\alpha \) satisfies (5).

Conversely, if \( H : X_1 \to f^*(\mathcal{C}) \) satisfies (5), then defining \( \alpha_1 \) as adjoint to \( H \) and \( \alpha_n \) as above, it follows that \( \alpha : f_!X \to N_\mathcal{V}(\mathcal{C}) \) is a natural transformation. It is immediate that \( \alpha \) is also monoidal. \( \square \)

**Remark 2.25.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be small \( \mathcal{V} \)-enriched categories, \( f : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{D}) \) a map of sets and \( H : \mathcal{C} \to f^*(\mathcal{D}) \) a morphism in Quiv_{\text{Ob}(\mathcal{C})}(\mathcal{V})}. Then the diagrams (5) with \( X = N_\mathcal{V}(\mathcal{C}) \) precisely express that \( (H, f) \) is a \( \mathcal{V} \)-enriched functor \( \mathcal{C} \to \mathcal{D} \).

**Construction 2.26.** Let \( (H, f) : \mathcal{C} \to \mathcal{D} \) be a \( \mathcal{V} \)-enriched functor between small \( \mathcal{V} \)-enriched categories. By Lemma 2.24 there exists a unique templicial morphism

\[
N_\mathcal{V}(H) : N_\mathcal{V}(\mathcal{C}) \to N_\mathcal{V}(\mathcal{D})
\]
such that the quiver morphism $N_V(H)_1 : f_!(C) \to D$ corresponds to $H : C \to f^*(D)$ by adjunction. A careful verification shows that this defines a functor

$$N_V : \text{Cat}(V) \to S \otimes V,$$

the \textit{(V-enriched) nerve functor}, where $\text{Cat}(V)$ denotes the category of small $V$-enriched categories.

Note that explicitly, for all $n \geq 0$ and $A,B \in \text{Ob}(C)$ we have

$$(N_V(H)_n)_{A,B} = \bigsqcup_{A_0,\ldots,A_n \in \text{Ob}(C)} H_{A_0,A_1} \otimes \ldots \otimes H_{A_{n-1},A_n}$$

Proposition 2.27. The nerve functor $N_V$ is fully faithful.

\textit{Proof.} This follows from Lemma 2.24 and Remark 2.25. \hfill \Box

Proposition 2.28. Let $(X, S) \in S \otimes V$. The following are equivalent:

1. $X$ is strong monoidal,
2. $(X, S)$ is isomorphic to the nerve of a small $V$-category.

\textit{Proof.} The implication (2) $\Rightarrow$ (1) is clear by definition of the nerve. Conversely, suppose $X$ is strong monoidal, i.e. its comultiplication $\mu$ is an isomorphism. Then we have for all $n \geq 0$:

$$\mu_{1,\ldots,1} : X_n \xrightarrow{\sim} X_1 \otimes \ldots \otimes X_1$$

in Quiv$_S(V)$. Through these isomorphisms, the face $d_1 : X_2 \to X_1$ and degeneracy $s_0 : X_0 \to X_1$, give us quiver maps

$$m : X_1 \otimes S X_1 \to X_1 \quad \text{and} \quad u : I_S \to X_1$$

It follows by the simplicial identities and the naturality, associativity and counitality of $\mu$ that these maps define an associative monoid structure on $X_1$ in Quiv$_S(V)$. That is, $C = (X_1, m, u)$ is a $V$-category with set of objects $S$.

Finally, again by the naturality of $\mu$, the maps $\mu_{1,\ldots,1}$ combine to give an isomorphism $X \simeq N_V(C)$ between functors $\Delta_f^{op} \to \text{Quiv}_S(V)$. This natural isomorphism is monoidal by the associativity of $\mu$, showing that $(X, S)$ is isomorphic to $(N_V(C), S)$ in $S \otimes V$. \hfill \Box

We end this section with some compatibility results. The adjunction $F \dashv U$ between Set and $V$ also induces an adjunction between small ordinary and small $V$-enriched categories respectively, which we will denote by

$$\begin{array}{ccc}
\text{Cat} & \xleftarrow{\tilde{F}} & \text{Cat}(V) \\
\xrightarrow{U} & & \\
\text{Cat} & \xrightarrow{\mathcal{F}} & \text{Cat}(V)
\end{array}$$

Let $N : \text{Cat} \to \text{SSet}$ denote the classical nerve functor.

Proposition 2.29. We have a natural isomorphism

$$N_V \circ \mathcal{F} \simeq \tilde{F} \circ N$$
Proof. Let \( C \) be a category with set of objects \( S \). Consider its nerve \( N(C) \) as a templicial set. Then we have an isomorphism of \( \mathcal{V} \)-quivers
\[
\alpha_1 : \tilde{F}(N(C))_1 = F(N(C)_1) \simto \mathcal{F}(C)
\]
It immediately follows from the definitions that this isomorphism satisfies the diagrams 5 and thus there exists a unique templicial morphism
\[
(\alpha, \text{id}_{\text{Ob}(C)}) : \tilde{F}(N(C)) \to N_{\mathcal{V}}(\mathcal{F}(C))
\]
extending \( \alpha_1 \). Moreover, since \( F \) and \( N(C) : \Delta^\text{op} \to \text{Quiv}_{\text{Ob}(C)}(\text{Set}) \) are strong monoidal, we find that \( \alpha \) is an isomorphism of templicial objects.

Finally, it is easily verified that this isomorphism is natural in \( C \). \( \Box \)

**Proposition 2.30.** We have a natural isomorphism
\[
\tilde{U} \circ N_{\mathcal{V}} \simeq N \circ U
\]

**Proof.** Let \( C \) be a small \( \mathcal{V} \)-category and \( n \geq 0 \). Then we have the following isomorphisms, natural in \( n \) and \( C \):
\[
\tilde{U}(N_{\mathcal{V}}(C))_n = S \otimes \mathcal{V}(\tilde{F}(\Delta^n), N_{\mathcal{V}}(C)) \simeq S \otimes \mathcal{V}(N_{\mathcal{V}}(\mathcal{F}(\lfloor n \rfloor)), N_{\mathcal{V}}(C))
\]
\[
\simeq \mathcal{V} \text{-Fun}(\mathcal{F}(\lfloor n \rfloor), C) \simeq \text{Fun}(\lfloor n \rfloor, U(C)) \simeq N(U(C))_n
\]
where we subsequently used the isomorphism \( \Delta^n \simeq N(\lfloor n \rfloor) \), Proposition 2.29, the fact that \( N_{\mathcal{V}} \) is fully faithful, and the adjunction \( \mathcal{F} \dashv U \). \( \Box \)

**Corollary 2.31.** For any small \( \mathcal{V} \)-enriched category \( C \), the nerve \( N_{\mathcal{V}}(C) \) is a \( \mathcal{V} \)-enriched quasi-category.

**Proof.** This is immediate from Proposition 2.30 and Remark 2.21. \( \Box \)

3. Nonassociative Frobenius structures

In this section we introduce a weakening of the classical notion of a Frobenius monoidal functor [DP08], where we no longer require the lax monoidal structure to be associative (§3.1). This may seem like a bit of a detour. However, these nonassociative Frobenius structures come up quite naturally as a kind of rigidification of the composition in ordinary quasi-categories (see Proposition 3.5). They will also play an essential role when enriching over modules in §4.

First, in §3.2 we establish some elementary notations and properties regarding nonassociative Frobenius structures. Next, §3.3 is devoted to the proof of Theorem 3.14, which gives an equivalence between ordinary Frobenius monoidal functors \( \Delta^\text{op}_f \to \mathcal{V} \) on the one hand and lax monoidal functors \( \Delta^\text{op}_f \to \mathcal{V} \) on the other hand, and which will be important when we define the linear dg-nerve in §5.
3.1. Nonassociative Frobenius structures.

**Definition 3.1.** Let $H : \mathcal{U} \to \mathcal{V}$ be a functor between monoidal categories with a colax monoidal structure $(\mu, \epsilon)$. A nonassociative Frobenius (naF) structure on $H$ is a pair $(Z, \eta)$ with $\eta : I \to H(I)$ a morphism in $\mathcal{V}$, called the unit, and

$$Z : H(-) \otimes H(-) \to H(- \otimes -)$$

a natural transformation, called the multiplication, such that the following diagrams commute for all $A, B, C \in \mathcal{U}$:

(7) $$H(A \otimes B) \otimes H(C) \xrightarrow{\mu_{A,B,C} \otimes \text{id}} H(A) \otimes H(B) \otimes H(C)$$

$$Z_{A \otimes B, C} \downarrow \quad \downarrow \text{id} \otimes Z_{B, C}$$

$$H(A \otimes B \otimes C) \xrightarrow{\mu_{A,B,C}} H(A) \otimes H(B \otimes C)$$

(8) $$H(A) \otimes H(B \otimes C) \xrightarrow{\text{id} \otimes \mu_{B,C}} H(A) \otimes H(B) \otimes H(C)$$

$$Z_{A, B \otimes C} \downarrow \quad \downarrow Z_{A,B} \otimes \text{id}$$

$$H(A \otimes B \otimes C) \xrightarrow{\mu_{A,B,C}} H(A \otimes B) \otimes H(C)$$

and

$$H(A) \otimes H(I) \xrightarrow{Z_{A,I}} H(A \otimes I)$$

$$H(A) \otimes I \xrightarrow{\lambda_H(A)} H(A) \quad H(I) \otimes H(A) \xrightarrow{Z_{I,A}} H(I \otimes A)$$

$$H(A \otimes I) \xrightarrow{\eta(H(A))} H(I) \otimes H(A) \xrightarrow{\lambda_{H(A)}} H(A) \quad \eta \circ H(A) \xrightarrow{\lambda_{H(a)}} H(A) \otimes H(A) \xrightarrow{\rho_{H(a)}} H(A)$$

where $\lambda$ and $\rho$ denote the left and right unit isomorphisms respectively.

For the purposes of this paper, we will always assume that a naF-structure is strongly unital. That is, $\epsilon$ is invertible and

$$\eta = \epsilon^{-1}$$

We call a colax monoidal functor equipped with a naF-structure a naF-monoidal functor.

**Example 3.2.** Let $H$ be a naF-monoidal functor as above whose multiplication $Z$ is associative, that is for all $A, B, C \in \mathcal{U}$:

$$Z_{A \otimes B, C}(Z_{A,B} \otimes \text{id}_C) = Z_{A,B \otimes C}(\text{id}_A \otimes Z_{B,C})$$

Then $H$ is a precisely a Frobenius monoidal functor of [DP08] whose unit and counit are each others inverses. In this case, we refer to $H$ as an $F$-monoidal functor.
3.2. Nonassociative Frobenius templicial objects. A templicial $\mathcal{V}$-object is in particular a colax monoidal functor $X : \Delta^{op}_f \to \text{Quiv}_S(\mathcal{V})$ for a given set $S$. Let $(Z, \eta)$ be a naF-structure on $X$ with comultiplication $\mu$ and counit $\epsilon$. Then diagrams (7) and (8) become

\[
\mu_{k,l}(Z^p,q) = \begin{cases} (Z^{p,k-p} \otimes \text{id}_{X_p})((\text{id}_{X_p} \otimes \mu_{k-p,l}) & \text{if } p \leq k \\ (\text{id}_{X_p} \otimes Z^{p-k,q})(\mu_{k,p-k} \otimes \text{id}_{X_q}) & \text{if } p \geq k \end{cases}
\]

for all $k, l, p, q \geq 0$ such that $k + l = p + q$.

We will refer to a templicial object equipped with a naF-structure as a naF-templicial object.

Remark 3.3. By the strong unitality, it follows that $\mu_{k,l}Z^{k,l} = \text{id}_{X_k \otimes X_l}$ for all $k, l \geq 0$.

Example 3.4. Let $C$ be a small $\mathcal{V}$-enriched category. Its nerve $N\mathcal{V}(C)$ is a strong monoidal functor $\Delta^{op}_f \to \text{Quiv}_{\text{Ob}(C)}(\mathcal{V})$. In particular, $N\mathcal{V}(C)$ is a naF-templicial object whose multiplication is given by the inverses of the comultiplication maps $\mu_{k,l}$:

\[
N\mathcal{V}(C) \cong \bigoplus_{n \geq 0} N\mathcal{V}(A)_n.
\]

For example, let $A$ be a $k$-algebra considered as a one-object $k$-linear category. Then $N_k(A)_n = A^\otimes n$ for $n \geq 0$ and the multiplication and comultiplication of $N_k(A)$ as a Frobenius functor correspond to the usual multiplication (concatenating tensors) and comultiplication (separating tensors) on $T(A) = \oplus_{n \geq 0} N_k(A)_n$.

An example of a naF-templicial object that is not Frobenius in the classical sense comes from ordinary quasi-categories. Recall that by Construction 2.10, we may identify simplicial and templicial sets.

Proposition 3.5. Let $X$ be an ordinary quasi-category. Then, $X$ has a naF-structure.

Proof. We define a map of quivers $Z^{p,q} : X_p \times X_q \to X_{p+q}$ by induction on $p, q \geq 0$. Take 0-simplices $a$ and $b$ of $X$. Then

\[
(X_p \times X_q)(a, b) = \coprod_{c \in X_0} X_p(a, c) \times X_q(c, b)
\]

So let $x \in X_p(a, c)$ and $y \in X_q(c, b)$ for some basis element $b$, and assume that $x$ and $y$ are non-degenerate.

In case $p = 0$ or $q = 0$, respectively define

\[
Z^{0,q}(a, y) = y \quad \text{and} \quad Z^{p,0}(x, b) = x
\]

Now if $p > 0$ and $q > 0$, choose $Z^{p,q}(x, y)$ to be a $(p + q)$-simplex of $X$ such that for all $i \in [p + q]$

\[
d_i(Z^{p,q}(x, y)) = \begin{cases} Z^{p-1,q}(d_i(x), y) & \text{if } i < p \\ Z^{p,q-1}(x, d_{i-p}(y)) & \text{if } i > p \end{cases}
\]
This simplex exists because \( X \) is a quasi-category and the described faces form a horn \( \Lambda_{p+q}^+ \to X \). Indeed, it follows by induction that for all \( i < j \) in \([p+q] \setminus \{p\}\):

- If \( j < p \):
  \[
  d_i(Z^{p-1,q}(d_j(x), y)) = Z^{p-2,q}(d_id_j(x), y) = Z^{p-2,q}(d_{j-1}d_i(x), y) = d_{j-1}(Z^{p-1,q}(d_i(x), y))
  \]

- If \( i < p < j \):
  \[
  d_i(Z^{p,q-1}(x, d_{j-p}(y)) = Z^{p,q-1}(d_i(x), d_{j-p}(y)) = d_{j-1}(Z^{p-1,q}(d_i(x), y))
  \]

- If \( p < i \):
  \[
  d_i(Z^{p,q-1}(x, d_{j-p}(y)) = Z^{p,q-1}(d_id_j(x), y) = Z^{p-2,q}(d_{j-1}d_i(x), y) = d_{j-1}(Z^{p-1,q}(d_i(x), y))
  \]

Further, if \( x = s_i(x') \) or \( y = s_i(y') \) for some \( x' \in X_{p-1}, y' \in X_{q-1} \), set

\[
Z^{p,q}(s_i(x'), y) = s_i(Z^{p-1,q}(x', y))
\]

\[
Z^{p,q}(x, s_i(y)) = s_i+p(Z^{p,q-1}(x, y'))
\]

It follows that \( Z^{p,q} \) is natural in \( p \) and \( q \).

We verify equation (2). Let \( k, l \geq 0 \) with \( k + l = p + q \), then if \( k \leq p \):

\[
\mu_{k,l}(Z^{p,q}(x, y)) = (d_{k+1}...d_{k+l}(Z^{p,q}(x, y)), d_0...d_0(Z^{p,q}(x, y)) = (d_{k+1}...d_p(Z^{p,q}(x, d_1...d_q(y)), Z^{p-k,q}(d_0...d_0(x), y)) = (d_{k+1}...d_p(x), Z^{p-k,q}(d_0...d_0(x), y)) = (id_{X_k} \times Z^{p-k,q}(\mu_{k,p-k} \times id_{X_q}))(x, y)
\]

and similarly if \( p \geq k \).

The converse to Proposition 3.5 is false, as the following example shows.

**Example 3.6.** Let \( X \) be the simplicial set defined as the colimit of

\[
\Lambda^3_3 \rightarrow \Lambda^3_0 \rightarrow \Delta^3
\]

It is the standard 3-simplex \( \Delta^3 \), whose simplices we will represent by their vertices \([i_0, ..., i_n]\), with two non-degenerate 3-simplices \( x \) and \( y \) glued on. We have

\[
\forall i \in \{0, 1, 2\} : d_i(x) = [0, ..., i, ..., 3] \quad \text{but} \quad d_3(x) \neq [0, 1, 2]
\]

\[
\forall j \in \{1, 2, 3\} : d_j(y) = [0, ..., j, ..., 3] \quad \text{but} \quad d_0(y) \neq [1, 2, 3]
\]
In $X$, not all horns can be filled. Indeed, since
\[ \begin{align*}
d_0d_3(x) &= d_2([1, 2, 3]) = [1, 2] = d_0([0, 1, 2]) = d_2d_0(y), \\
d_2d_3(x) &= d_2([0, 1, 3]) = [0, 1] \quad \text{and} \quad d_1d_0(y) = d_0([0, 1, 3]) = [1, 3]
\end{align*} \]
the faces $d_3(x)$, $d_0(y)$ and $[0, 1, 3]$ form a horn $\Lambda^3_1$ in $X$. But there is no 3-simplex in $X$ with these faces.

However, $X$ does have a naF-structure. It suffices to define $Z^{pq}(a, b)$ on non-degenerate simplices $a$ and $b$. For those contained in $\Delta^3$, define
\[ Z^{pq}([i_0, \ldots, i_p], [i_p, \ldots, i_{p,q}]) = [i_0, \ldots, i_{p+q}] \]
note that this includes all edges of $X$. Further, set
\[ Z^{2,1}(d_3(x), [2, 3]) = x, \quad \text{and} \quad Z^{1,2}([0, 1], d_0(y)) = y \]
It is easy to check that this satisfies equation (9).

**Proposition 3.7.** Let $H : \mathcal{U} \to \mathcal{V}$ be a strong monoidal functor between cocomplete monoidal categories whose monoidal product preserves colimits in each variable. Assume $H$ preserves coproducts. If $X$ is a templicial $\mathcal{U}$-object with base $S$ and naF-structure $Z$, then the $k$-quiver morphisms
\[ \left( Z^{pq}_{\tilde{H}(X)} : H_S(X_p) \otimes H_S(X_q) \xrightarrow{\sim} H_S(X_p \otimes X_q) \xrightarrow{H_S(Z^{pq})} H_S(X_{p+q}) \right)_{p,q \geq 0} \]
define a naF-structure on $\tilde{H}(X)$, with $\tilde{H}$ as in Construction 2.12.

**Proof.** Write $\mu$ and $\epsilon$ for the comultiplication and counit of $X$ respectively. Given $p, q \geq 0$, denote by $\nu_{p,q}$ the isomorphism of $k$-quivers
\[ H_S(X_p \otimes S X_q) \xrightarrow{\sim} H_S(X_p) \otimes S H_S(X_q) \]
Then by definition, we have for all $p, q \geq 0$:
\[ Z^{pq}_{\tilde{H}(X)} = H_S(Z^{pq}) \circ \nu_{p,q}^{-1} \]
while the comultiplication of $\tilde{H}(X)$ is given by, for all $k, l \geq 0$:
\[ \mu_{k,l}^{\tilde{H}(X)} = \nu_{k,l} \circ H_S(\mu_{k,l}) \]
It follows that $(Z^{pq}_{\tilde{H}(X)})_{p,q \geq 0}$ is a naF-structure on $\tilde{H}(X)$.

We introduce some notation.

**Definition 3.8.** Let $f : [m] \to [n]$ be a morphism of $\Delta_f$. If $f$ is injective, we call $f$ a partition of $n$. Writing $\Delta_f^{\text{inj}}$ for the subcategory of $\Delta_f$ of all monomorphisms, we set
\[ \mathcal{P}_n = \coprod_{m \geq 0} \Delta_f^{\text{inj}}([m], [n]) \]
to be the set of all partitions of $n$.

For any morphism $f : [m] \to [n]$, we write $\ell(f) = m$ and call it the length of $f$. 
Remark 3.9. Note that for all \(0 \leq i \leq j\), we have a bijection
\[
P_{j-i} \simeq \{I \subseteq \{i, i+1, \ldots, j\} \mid i, j \in I\}
\]
identifying a partition \(f : [m] \to [j - i]\) with \(\{f(k) + i \mid k \in [m]\}\). We will implicitly use this identification and describe partitions as sets, in which case we will often denote them by capital letters \(I, J, K, \ldots\) to emphasize this interpretation.

Given \(I \in \mathcal{P}_n\), we may interpret \(I\) as a subset of \(\{i, i+1, \ldots, i+n\}\) for any \(i \geq 0\). In practice it should be clear from context which starting point \(i\) we consider.

The monoidal structure \((+, [0])\) on \(\Delta_f\) induces an operation on subsets as follows. Given \(I \in \mathcal{P}_m\) and \(J \in \mathcal{P}_n\) with starting points \(i\) and \(j\) respectively, we have
\[
I + J = I \cup (J + (m - j)) \in \mathcal{P}_{m+n}
\]
Finally, note that for \(I \in \mathcal{P}_n\), \(\ell(I) = |I| - 1\) and \(I \simeq [\ell(I)]\) as posets.

For the rest of this subsection, let \(X\) be a templicial \(\mathcal{V}\)-object with comultiplication \(\mu\) and counit \(\epsilon\). Assume that \(X\) has a naF-structure with multiplication \(Z\).

Given \(f : [m] \to [n]\) in \(\Delta_f\), we denote \(\mu_{f(1), f(2)-f(1), \ldots, n-f(m-1)}\) by
\[
\mu_f : X_n \to X_{f(1)} \otimes X_{f(2)-f(1)} \otimes \ldots \otimes X_{n-f(m-1)}
\]
We’d like to similarly define a map
\[
Z^f : X_{f(1)} \otimes \ldots \otimes X_{n-f(m-1)} \to X_n
\]
However, since \(Z\) is not assumed to be associative, this will depend on how we compose the two-variable maps \(Z^{p,q}\). Nevertheless, making an arbitrary choice, we define
\[
Z^{p_1, \ldots, p_m} = Z^{p_1, p_2 + \ldots + p_m} (\text{id}_{X_{p_1}} \otimes Z^{p_2, \ldots, p_m})
\]
inductively on \(m \geq 2\), for all \(p_1, \ldots, p_m \geq 0\), and subsequently set
\[
Z^f = \begin{cases} 
\epsilon^{-1} & \text{if } m = 0 \\
\text{id}_{X_n} & \text{if } m = 1 \\
Z^{f(1), \ldots, n-f(m-1)} & \text{if } m \geq 2 
\end{cases}
\]
It will turn out that this suffices for the following couple of results.

Finally, define \(X_f = X_{f(1)} \otimes \ldots \otimes X_{n-f(m-1)}\) so that
\[
\mu_f : X_n \to X_f \quad \text{and} \quad Z^f : X_f \to X_n
\]
Often, we will consider the case where \(f\) is a partition of \(n\).

Remark 3.10. Consider a partition \(I = \{0 = i_0 < \ldots < i_m = n\}\) of \(n\). It follows from the naturality of \(Z\) that for all \(i, j \in [n] \setminus I:\)
\[
d_jZ^f = Z^\sigma^{-1}(j)(\text{id} \otimes \ldots \otimes \text{id} \otimes d_{j-i_{p-1}} \otimes \text{id} \otimes \ldots \otimes \text{id})
\]
\[
s_iZ^f = Z^\sigma_{i-j}(1)(\text{id} \otimes \ldots \otimes \text{id} \otimes s_{i-i_{p-1}} \otimes \text{id} \otimes \ldots \otimes \text{id})
\]
where \( p \in \{1, \ldots, m\} \) is minimal such that \( i < i_p \) or \( j < i_p \) respectively.

On the other hand, if \( i \in I \), then

\[
s_i Z^I = Z^{s_i I}(id \otimes \ldots \otimes id \otimes s_0 e^{-1} \otimes id \otimes \ldots \otimes id)
\]

However, if \( 0 < j < n \) and \( j \in I \) the naturality of \( Z \) doesn’t supply us with a formula to pass the face map \( d_j \) through \( Z \).

**Definition 3.11.** Let \( I \in \mathcal{P}_n \) with \( n \geq 0 \) and starting point \( i \geq 0 \). For any \( i \leq s \leq i + n \), set

\[
I^{\leq s} = (I \cap \{i, \ldots, s-1\}) \cup \{s\}
\]

\[
I^{\geq s} = \{s\} \cup (I \cap \{s+1, \ldots, i+n\})
\]

Then \( I^{\leq s} + I^{\geq s} = I \cup \{s\} \).

Given \( f: [m] \to [j-i] \) in \( \Delta_f \), the splitting of \( I \) over \( f \) is the \( m \)-tuple

\[
(I_1, \ldots, I_m)
\]

where for all \( k \in \{1, \ldots, m\} \), \( I_k = (I^{\leq f(k)+i})^{\geq f(k-1)+i} \).

**Proposition 3.12.** Let \( n \geq 0 \) and \( I, J \in \mathcal{P}_n \). Then

\[
\mu I Z^J = (Z^{J_1} \otimes \ldots \otimes Z^{J_k})(\mu_{I_1} \otimes \ldots \otimes \mu_{I_l})
\]

where \( (J_1, \ldots, J_k) \) is the splitting of \( J \) over \( I \) and \( (I_1, \ldots, I_l) \) is the splitting of \( I \) over \( J \).

**Proof.** We use induction on \( k = \ell(I) \) and \( l = \ell(J) \). If either \( k = 0 \) or \( l = 0 \), then both are zero and (10) is trivially true. For \( k = 1 \), both sides of (10) reduce to \( Z^I \). A similar argument proves the case \( l = 1 \).

Assume further that \( k, l \geq 2 \). Let \( i \in I \) and \( j \in J \) be minimal such that \( 0 < i \) and \( 0 < j \). Then

\[
\mu I Z^J = (id_{X_i} \otimes \mu_{I_{\geq 1}})(\mu_{i,n-i} Z^{j,n-i-j}(id_{X_j} \otimes Z^{J_{\leq j}}))
\]

If \( i \leq j \), then \( \mu_{i,n-i} Z^{j,n-i-j} = (id_{X_i} \otimes Z^{j-i,n-i-j})(\mu_{j-i} \otimes id_{X_{n-j}}) \) by (9). So, by the induction hypothesis, we have

\[
\mu I Z^J = (id_{X_i} \otimes \mu_{I_{\geq 1}} Z^{j-i,n-i-j})(\mu_{i,j-i} \otimes Z^{J_{\leq j}})
\]

\[
= (id_{X_i} \otimes \mu_{I_{\geq 1}} Z^{J_{\leq j}})(\mu_{i,j-i} \otimes id_{X_{i,j-i}})
\]

\[
= (id_{X_i} \otimes Z^{j_2} \otimes \ldots \otimes Z^{j_k})(id_{X_i} \otimes \mu_{I_{\geq 1}} Z^{J_{\leq j}})(\mu_{i,j-i} \otimes \mu_{I_{\geq 1} \otimes \mu I_2 \otimes \ldots \otimes \mu I_l})
\]

where we used that \( J_1 = \{0 < i\} \) since \( i \leq j \). A similar argument shows the case \( i \geq j \). \( \square \)

**Corollary 3.13.** Let \( n \geq 0 \) and \( I, J \in \mathcal{P}_n \).

(a) If \( I \subseteq J \), and \( (J_1, \ldots, J_k) \) is the splitting of \( J \) over \( I \), then

\[
\mu I Z^J = Z^{J_1} \otimes \ldots \otimes Z^{J_k}
\]
(b) If \( J \subseteq I \), and \((I_1, ..., I_l)\) is the splitting of \( I \) over \( J \), then
\[
\mu_I Z^J = \mu^{I_1} \otimes ... \otimes \mu^{I_l}
\]

(c) We have \( \mu_I Z^J \mu_J = \mu_I Z^I \mu_J \).

Proof. Consider the splittings \((J_1, ..., J_k)\) and \((I_1, ..., I_l)\) of \( I \cup J \) over \( I \) and \( J \) respectively.

If \( I \subseteq J \), then \((I_1, ..., I_l)\) is the splitting of \( J \) over itself. Therefore, \( \ell(I_i) = 1 \) and thus \( \mu_{I_i} \) is the identity for all \( i \in \{1, ..., l\} \). This shows (a) and a similar argument shows (b).

To prove (c), note that \( I \subseteq I \cup J \) and thus by (a),
\[
\mu_I Z^{I \cup J} \mu_{I \cup J} = (Z^{I_1} \otimes ... \otimes Z^{I_l}) \mu_{I_1, ..., I_l} = (Z^{I_1} \otimes ... \otimes Z^{I_k}) \mu_{J_1, ..., J_k}
\]
where we used the coassociativity of \( \mu \).

3.3. Frobenius and lax functors over Ab-categories. For the remainder of this section, we assume that \( \mathcal{V} \) is moreover enriched over abelian groups and has kernels. Note that under our standing assumptions, it follows that \( \mathcal{V} \) is additive and has all finite limits. This implies in particular that the monoidal product is additive in both variables, since it preserves coproducts.

Let \( \text{Frob}_{\text{su}}(\Delta^{op}_f, \mathcal{V}) \) be the category of (strongly unital) F-monoidal functors \( \Delta^{op}_f \to \mathcal{V} \) endowed with bimonoidal natural transformations, that is, natural transformations that are monoidal with respect to both the lax and colax structures. Let \( \text{Lax}(\Delta^{op}_+, \mathcal{V}) \) be the category of lax monoidal functors \( \Delta^{op}_+ \to \mathcal{V} \). The main goal in this subsection is to prove the following theorem.

**Theorem 3.14.** There is an adjoint equivalence
\[
\text{Frob}_{\text{su}}(\Delta^{op}_f, \mathcal{V}) \overset{\text{K}}{\rightleftarrows} \text{Lax}(\Delta^{op}_+, \mathcal{V}).
\]

We will explicitly construct both functors. To make this slightly easier, we first replace \( \Delta_+ \) by an isomorphic category.

**Definition 3.15.** We denote by \( \Delta_- \) the subcategory of \( \Delta \) whose objects are all \([n]\) with \( n > 0 \) and whose morphisms are all \( f : [m] \to [n] \) such that \( f^{-1}(\{0\}) = \{0\} \) and \( f^{-1}(\{n\}) = \{m\} \). We call functors \( \Delta_-^{op} \to \mathcal{V} \) narrow simplicial objects of \( \mathcal{V} \).

It is easy to show that \( \Delta_- \) is generated by the inner coface and inner codegeneracy maps. More precisely, every morphism \( f : [m] \to [n] \) in \( \Delta_- \) has a unique representation
\[
f = \delta_{j_1} ... \delta_{j_k} \sigma_{i_1} ... \sigma_{i_t}
\]
with \( n > j_1 > ... > j_k > 0 \) and \( 0 < i_1 < ... < i_l < m - 1 \) and \( m - l + k = n \). Note that we have inclusions of categories:

\[
\Delta_- \subseteq \Delta_f \subseteq \Delta \subseteq \Delta_+
\]

Recall from [27] the monoidal structure \((\ast, [-1])\) on \(\Delta_+\).

**Lemma 3.16.** The functor

\[
[0] \ast - \ast [0] : \Delta_+ \to \Delta_- : [n] \mapsto [n + 2]
\]

is an isomorphism of categories.

**Proof.** For any \( f : [m] \to [n] \) in \(\Delta_+\), we have

\[
(id_{[0]} \ast f \ast id_{[0]})(i) = \begin{cases} 0 & \text{if } i = 0 \\ f(i - 1) + 1 & \text{if } 0 < i < m + 2 \\ n + 2 & \text{if } i = m + 2 \end{cases}
\]

so that \(id_{[0]} \ast f \ast id_{[0]}\) is indeed a morphism of \(\Delta_-\). Thus, \([0] \ast - \ast [0]\) is a well-defined functor. Note that it is bijective on objects.

Further, any morphism \( g : [m] \to [n] \) in \(\Delta_-\) is reached by a unique morphism \( f \) of \(\Delta_+\), given by

\[
f : [m - 2] \to [n - 2] : i \mapsto g(i + 1) - 1
\]

Consequently, \(\Delta_-\) inherits a monoidal structure from \(\Delta_+\), which we will denote by \((\diamond, [1])\). Explicitly, it is given on objects by:

\[
[m] \circ [n] = [m + n - 1]
\]

for all \( m, n \geq 1 \), and on morphisms by

\[
(f \circ g)(i) = \begin{cases} (f \circ g)(i) = f(i) & \text{if } i < m \\ (f \circ g)(i) = g(i - m + 1) + p - 1 & \text{if } i \geq m \end{cases}
\]

for all \( i \in [m + n - 1] \) and \( f : [m] \to [p] \), \( g : [n] \to [q] \) in \(\Delta_-\).

We thus have an isomorphism of categories

\[
Lax(\Delta_-^{op}, \mathcal{V}) \simeq Lax(\Delta_+^{op}, \mathcal{V})
\]

Let \((m, u)\) be a lax monoidal structure on a narrow simplicial object \( A : \Delta_-^{op} \to \mathcal{V} \). Then explicitly, \( u : I \to A_1 \) is a map and \( m \) consists of a family of maps

\[
(m_{p,q} : A_p \otimes A_q \to A_{p+q-1})_{p,q \geq 1}
\]

that is natural in \( p \) and \( q \), and such that \( m \) and \( u \) satisfy the relevant associativity and unitality conditions.

By the associativity of \( m \), we have an unambiguously defined map

\[
m_{p_1, \ldots, p_k} : A_{p_1} \otimes \ldots \otimes A_{p_k} \to A_{p_1+\ldots+p_k-k+1}
\]
for all \( p_1, \ldots, p_k \geq 1 \) and \( k \geq 2 \). Then set, for all \( I = \{i_0 < \ldots < i_m\} \):

\[
m_I = \begin{cases} 
  u & \text{if } m = 0 \\
  \text{id}_{X_{i_1-i_0}} & \text{if } m = 1 \\
  m_{n_1-i_0, \ldots, i_m-i_{m-1}} & \text{if } m \geq 2 
\end{cases}
\]

Note that in all cases, \( m_I : A_I \to A_{n-i(I)+1} \).

**Construction 3.17.** Suppose \( X \in \text{Colax}(\Delta_\mathcal{V}, \mathcal{V}) \) with comultiplication \( \mu \). We define a narrow simplicial object \( K(X) \) as follows. Set

\[
K(X)_n = \ker \left( (\mu_{k,n-k})_{k=1}^{n-1} : X_n \to \bigoplus_{k=1}^{n-1} X_k \otimes X_{n-k} \right)
\]

for all \( n \geq 1 \). It follows from the naturality of \( \mu \) that the inner face maps and inner degeneracy maps of \( X \) restrict to the objects \( K(X)_n \), defining a functor \( K(X) : \Delta_\mathcal{V} \to \mathcal{V} \).

Consider an additional \( Y \in \text{Colax}(\Delta_\mathcal{V}, \mathcal{V}) \) and let \( \alpha : X \to Y \) be a monoidal natural transformation. Because \( \alpha \) respects the comultiplications, \( \alpha_n \) restricts to a morphism \( K(X)_n \to K(Y)_n \) for all \( n \geq 1 \). This yields a natural transformation \( K(\alpha) : K(X) \to K(Y) \).

These assignments clearly define a functor

\[
K : \text{Colax}(\Delta_\mathcal{V}, \mathcal{V}) \to \text{Fun}(\Delta_\mathcal{V}, \mathcal{V})
\]

**Remark 3.18.** Note that \( K(X) \) does not inherit any outer degeneracy maps \( s_0, s_n : K(X)_n \to K(X)_{n+1} \) from \( X \) in the same way. For example, \( s_0 : X_1 \to X_2 \) does not restrict to \( K(X)_1 = X_1 \to K(X)_2 = \ker \mu_{1,1} \).

**Lemma 3.19.** For all \( f : [k] \to [p] \) and \( g : [k] \to [q] \) in \( \Delta_- \), we have

\[
\delta_p \circ (f \circ g) = (f + g) \circ \delta_k
\]

**Proof.** We have for all \( i \in [k + l - 1] \):

\[
\delta_p(f \circ g)(i) = \begin{cases} 
  \delta_p f(i) & \text{if } i < k \\
  \delta_p(g(i - k + 1) + p - 1) & \text{if } i \geq k 
\end{cases}
\]

\[
= \begin{cases} 
  f(\delta_k(i)) & \text{if } i < k \\
  g(\delta_k(i) - k) + p & \text{if } i \geq k
\end{cases} = (f + g)\delta_k(i)
\]

\( \square \)

**Proposition 3.20.** Let \( X : \Delta_\mathcal{V} \to \mathcal{V} \) be an \( F \)-monoidal functor with comultiplication \( \mu \), counit \( \epsilon \) and multiplication \( \delta \). Then there is a lax monoidal structure \( (m, u) \) on \( K(X) \) given by

\[
m_{p,q} = d_p Z^{p,q}|_{K(X)_p \otimes K(X)_q} : K(X)_p \otimes K(X)_q \to K(X)_{p+q-1}
\]

for all \( p, q \geq 1 \), and \( u = s_0\epsilon^{-1} : I \to K(X)_1 \).
Proof. Take \( p, q \geq 1 \) and set \( n = p + q \). Note that for \( 0 < k < n - 1 \):
\[
\mu_{k,n-k-1}d_pZ^{p,q} = \begin{cases} 
(d_p \otimes \text{id}_{n-k-1})\mu_{k+1,n-k-1}Z^{p,q} & \text{if } p \leq k \\
(\text{id}_k \otimes d_{p-k})\mu_{k,n-k}Z^{p,q} & \text{if } p > k 
\end{cases}
\]
which implies that the map \( d_pZ^{p,q} : X_p \otimes X_q \to X_{p+q-1} \) factors through \( K(X) \) as \( m_{p,q} : K(X)_p \otimes K(X)_q \to K(X)_{p+q-1} \).

Then by Lemma 3.19 we find that
\[
K(X)(f \circ g)m_{p,q} = X(f \circ g)d_pZ^{p,q}|_{K(X)_p \otimes K(X)_q} = d_kX(f + g)Z^{p,q}|_{K(X)_p \otimes K(X)_q} = d_kZ^{k,l}(X(f) \otimes X(g))|_{K(X)_p \otimes K(X)_q} = m_{k,l}(K(X)(f) \otimes K(X)(g))
\]
So, the maps \( (m_{p,q})_{p,q \geq 1} \) are natural in \( p \) and \( q \).

Further, \( m \) is associative since for all \( p, q, r \geq 1 \):
\[
m_{p+q-1,r}(m_{p,q} \otimes \text{id}_{K(X)_r}) = d_{p+q-1}Z^{p+q-1,r}((d_pZ^{p,q} \otimes \text{id}_{X_r})|_{K(X)_{p+q,r}}
\]
\[
= d_{p+q-1}d_pZ^{p+q+r}(Z^{p,q} \otimes \text{id}_{X_r})|_{K(X)_{p,q,r}}
\]
\[
= d_pZ^{p,q+r}(\text{id}_{X_p} \otimes d_qZ^{q,r})|_{K(X)_{p,q,r}} = m_{p+q+r-1}(\text{id}_{K(X)_p} \otimes m_{q,r})
\]
where \( K(X)_{p,q,r} = K(X)_p \otimes K(X)_q \otimes K(X)_r \).

Finally, \( m \) is unital because for all \( p \geq 1 \):
\[
m_{p,1}(\text{id}_{K(X)_1} \otimes u) = d_pZ^{p,1}(\text{id}_{X_p} \otimes s_0\epsilon^{-1})|_{K(X)_p \otimes k}
\]
\[
= d_ps_pZ^{p,0}(\text{id}_{X_p} \otimes \epsilon^{-1})|_{K(X)_p \otimes k} = \lambda_{X_p}|_{K(X)_p \otimes k} = \lambda_{K(X)_p}
\]
and similarly for the right unit. \( \square \)

If \( \alpha : X \to Y \) is a bimonoidal natural transformation between F-monoidal functors, then it immediately follows that \( K(\alpha) : K(X) \to K(Y) \) is a monoidal natural transformation. Consequently, \( K \) can be upgraded to a functor
\[
K : \text{Frob}_{su}(\Delta^{op}_f, \mathcal{V}) \to \text{Lax}(\Delta^{op}_f, \mathcal{V})
\]

Next, we construct the functor in the opposite direction.

Construction 3.21. Let \( A = (A_n)_{n \geq 1} \) be a graded object of \( \mathcal{V} \). Set
\[
\overline{A} = \bigoplus_{m \geq 0} A^{\otimes m}
\]
This is a non-negatively graded object of \( \mathcal{V} \) with for all \( n \geq 0 \):
\[
\overline{A}_n = \bigoplus_{m \geq 0} (A^{\otimes m})_n = \bigoplus_{m \geq 0} \bigoplus_{\sum_{k_i \geq 1} k_1, \ldots, k_m = n} A_{k_1} \otimes \ldots \otimes A_{k_m} \simeq \bigoplus_{I \in \mathcal{P}_n} \mathcal{A}_I
\]
For any $I \in \mathcal{P}_n$, we write $p_I : \overline{A}_n \to A_I$ and $t_I : A_I \to \overline{A}_n$ for the canonical projections and coprojections. Note that $A_I \otimes A_J \simeq A_{I+J}$ for all $I \in \mathcal{P}_m$ and $J \in \mathcal{P}_n$. We further write

$$p_{I,J} = p_I \otimes p_J : \overline{A}_m \otimes \overline{A}_n \to A_{I+J}$$

$$t_{I,J} = t_I \otimes t_J : A_{I+J} \to \overline{A}_m \otimes \overline{A}_n$$

Define maps $\mu_{m,n} : \overline{A}_{m+n} \to \overline{A}_m \otimes \overline{A}_n$ and $Z^{m,n} : \overline{A}_m \otimes \overline{A}_n \to \overline{A}_{m+n}$ by

$$p_{I,J} \circ \mu_{m,n} = p_{I+J} \quad \text{and} \quad Z^{m,n} \circ t_{I,J} = t_{I+J}$$

for all $I \in \mathcal{P}_m$ and $J \in \mathcal{P}_n$. This defines graded maps

$$\mu : \overline{A} \to \overline{A} \otimes \overline{A} \quad \text{and} \quad Z : \overline{A} \otimes \overline{A} \to \overline{A}$$

Finally, note that $\overline{A}_0$ is the monoidal unit of $\mathcal{V}$. So $\overline{A}$ has a unit and counit map, given by the identity.

**Proposition 3.22.** Let $A = (A_n)_{n \geq 1}$ be a graded object of $\mathcal{V}$. The graded maps $\mu$ and $Z$ of Construction 3.21 are coassociative and counital, respectively associative and unital, and they satisfy equation (9).

In other words, $\overline{A}$ is an F-monoidal functor $\mathbb{N} \to \mathcal{V}$.

**Proof.** The coassociativity of $\mu$ follows from the fact that

$$(p_I \otimes p_J \otimes p_K)(\mu_{k,l} \otimes \text{id}_{\overline{A}_n})\mu_{k+l,m} = (p_{I+J} \otimes p_K)\mu_{k+l,m} = p_{I+J+K}$$

for all $k, l, m \geq 0$ and $I \in \mathcal{P}_k$, $J \in \mathcal{P}_l$ and $K \in \mathcal{P}_m$. Further, $\mu$ is counital since $p_{0}, i \circ \mu_{0,n} = p_I$ for all $I \in \mathcal{P}_n$ and thus $\mu_{0,n} = \nu^{-1}_{A_n}$. Similarly, $\mu_{n,0} = \lambda^{-1}_{A_n}$. Dually, $Z$ is associative and unital.

To check the equation (9), take $k, l, p, q \geq 0$ such that $k + l = p + q$. Further take $I \in \mathcal{P}_k$, $J \in \mathcal{P}_l$, $S \in \mathcal{P}_p$ and $T \in \mathcal{P}_q$. Then on one hand,

$$p_{I,J} \mu_{k,l} Z^{p,q}_{S,T} p_{I+J,S+T} = \begin{cases} 
\text{id}_{A_{I+J}} & \text{if } I + J = S + T \\
0 & \text{if } I + J \neq S + T
\end{cases}$$

On the other hand, suppose $k \leq p$. Then we have

$$p_{I,J} (\text{id}_{\overline{A}_k} \otimes Z^{p-k,q}) (\mu_{k,p-k} \otimes \text{id}_{\overline{A}_q}) t_{S,T}$$

$$= \sum_{S' \in \mathcal{P}_{p-k}, T' \in \mathcal{P}_q} \sum_{I' \in \mathcal{P}_k, J' \in \mathcal{P}_{p-k}} (p_I \otimes p_{S'} \otimes p_{T'}) (t_{I'} \otimes t_{J'} \otimes t_{T'})$$

$$= \sum_{U \in \mathcal{P}_p} \text{id}_{A_{I+J}}$$

Now, if $U \in \mathcal{P}_{p-k}$ such that $U + T = J$ and $I + U = S$, then $I + J = I + U + T = S + T$. Note that such a set $U$ is necessarily unique. Conversely, if $I + J = S + T$, it follows from $k \leq p$ that $I \subseteq S$ and thus $k \in S$. Splitting $S$ over $\{0, k, p\}$ we get $S = I' + U$. Then since
Definition 3.23. Let \( I, I' \in P_k \) and \( I + J = I' + U + T \), we get \( I = I' \) and \( J = U + T \). We conclude that
\[
p_{I,J}(\mu_{k,p} Z_{p,q} t_{S,T}) = p_{I,J}(\mu_{k,p} \otimes \text{id}_{\mathcal{T}_q})(\mu_{k,p-k} \otimes \text{id}_{\mathcal{T}_q}) t_{S,T}
\]
The case where \( k \geq p \) is proven similarly. \( \square \)

**Definition 3.23.** Let \( n \geq 0 \) and \( I = \{0 = i_0 < \ldots < i_m = n\} \) a partition of \( n \). We define the complement of \( I \) by
\[
I^c = \begin{cases} \delta_{m-1} \ldots \delta_i & \text{if } m > 0 \\ \sigma_0 & \text{if } m = 0 \end{cases}
\]

Note that in both cases, \( \ell(I^c) = n - \ell(I) + 1 \). If \( m > 0 \), then \( I^c \) can also be described as the set
\[
I^c = \{0\} \cup (\{1, 2, \ldots, n-1\} \setminus I) \cup \{n\} \in P_n
\]

Further, for \( i \leq s \leq j \), we write
\[
p_I(s) = \min\{p \in [\ell(I)] \mid s \leq i_p\}
\]

Note that
\[
\ell(I^{\leq s}) = p_I(s) \quad \text{and} \quad \ell(I^{\geq s}) = \begin{cases} \ell(I) - p_I(s) & \text{if } s \in I \\ \ell(I) - p_I(s) + 1 & \text{if } s \not\in I \end{cases}
\]

**Remark 3.24.** Given \( f : [m] \to [n] \) in \( \Delta_f \) and \( I \in P_n \), set \( J = f^{-1}(I) \in P_m \). Then there is a unique \( f_I \) that makes the following diagram commute:

\[
\begin{array}{ccc}
\ell(I^c) & \overset{J^c}{\longrightarrow} & [m] \\
\downarrow f_I & & \downarrow f \\
\ell(I^c) & \overset{I^c}{\longrightarrow} & [n]
\end{array}
\]

When \( n > 0 \), the uniqueness holds because \( I^c \) and \( J^c \) are monomorphisms. We can construct \( f_I \) as the restriction \( f|_{J^c} : J^c \to I^c \) through the morphisms \( I^c \simeq [\ell(I^c)] \) and \( J^c \simeq [\ell(J^c)] \). If \( n = 0 \), then \( I = \{0\} \) and \( f_I \) is the identity on \( [1] \).

Note that for \( j \in J^c \) with \( 0 < j < m \), we have \( f(j) \not\in I \) and thus \( f(j) \neq 0, n \). Thus the map \( f_I \) in fact lies in \( \Delta_\_ \).

**Lemma 3.25.** Let \( f : [k] \to [p] \) and \( g : [l] \to [q] \) be morphisms in \( \Delta_f \), and \( I \in P_p, J \in P_q \). Then \( (f + g)_{I+J} = f_I \circ g_J \).

**Proof.** Denote \( K = f^{-1}(I) \) and \( L = g^{-1}(J) \), and set \( r = \ell(I^c) \) and \( s = \ell(K^c) \). We have
\[
(I^c + J^c)\delta_r = (I + J)^c \quad \text{and} \quad (K^c + L^c)\delta_s = (K + L)^c
\]

Thus it follows from Lemma \ref{lem:lem3.19} that
\[
(I + J)^c(f_I \circ g_J) = (I^c + J^c)\delta_r(f_I \circ g_J) = (I^c + J^c)(f_I + g_J)\delta_r = (f + g)(K^c + L^c)\delta_s = (f + g)(K + L)^c = (I + J)^c(f + g)_{I+J}
\]
Construction 3.26. Let \( A : \Delta^{op} \to \mathcal{V} \) be a narrow simplicial object with a lax monoidal structure \((m,u)\). Given \( f : [m] \to [n] \in \Delta_f \), we define a map

\[
\overline{A}(f) : \overline{A}_n \to \overline{A}_m
\]
as follows. Take \( I \in \mathcal{P}_n \) and set \( J = f^{-1}(I) = \{0 = j_1 < \ldots < j_p = m\} \).

Let \((I_1, \ldots, I_p)\) be the splitting of \( I \) over \([p] \xrightarrow{I} [m] \xrightarrow{f} [n]\), that is \( I_i = (I\leq \ell(j_i)) \geq f(j_{i-1}) \) in \( \mathcal{P}_{f(\ell(j_i)),f(j_i)} \) for all \( i \in \{1, \ldots, p\} \).

Further, let \( f_i : [j_i-j_{i-1}] \to [f(j_i)-f(j_{i-1})] \) be the unique morphisms in \( \Delta_f \) such that \( f_1+\ldots+f_p = f \). For all \( i \in \{1, \ldots, p\} \), denote \( f_{i\ell} = (f_i)_{\ell_i} \).

Finally we define

\[
\overline{A}(f)_I = A(f_{i1})m_{i1} \otimes \ldots \otimes A(f_{ip})m_{ip} : A_I \to A_J
\]

and \( \overline{A}(f) = \sum_{I\in \mathcal{P}_n} t_{f^{-1}(I)} \overline{A}(f)_I p_I : \overline{A}_n \to \overline{A}_m \).

Remark 3.27. In the situation of Construction 3.26 note that since \( I = I_1 + \ldots + I_p \) we have by Lemma 3.25 that

\[
f_I = f_{i1} \circ \ldots \circ f_{ip}
\]

Further, note that for all \( i \in \{1, \ldots, p\} \), \( f^{-1}(I_i) = \{j_{i-1} < j_i\} \) and thus \( f^{-1}(I_i)^c \) is the identity on \([j_i-j_{i-1}]\). Hence, we have

\[
I_i^c \circ f_{i\ell} = f_i
\]

Example 3.28. Let \( n \geq 0 \) and \( I = \{0 = i_0 < \ldots < i_m = n\} \) a partition of \( n \). Consider \( \delta_j : [n-1] \to [n] \) with \( 0 < j < n \). Then

\[
\overline{A}(\delta_j)_I = \begin{cases} 
\id_{A_{i1}} \otimes \ldots \otimes \id_{A_{i-p-1}} \otimes \ldots \otimes \id_{A_{n-i-1}} & \text{if } j \notin I \\
\id_{A_{i1}} \otimes \ldots \otimes \id_{m_{i-p-1}} \otimes \ldots \otimes \id_{A_{n-i-1}} & \text{if } j \in I
\end{cases}
\]

where \( p = p_I(j) \). Similarly, for \( \sigma_i : [n+1] \to [n] \) with \( 0 \leq i \leq n \),

\[
\overline{A}(\sigma_i)_I = \begin{cases} 
\id_{A_{i1}} \otimes \ldots \otimes \id_{A_{n-i-1}} \otimes \ldots \otimes \id_{A_{n-i-1}} & \text{if } i \notin I \\
\id_{A_{i1}} \otimes \ldots \otimes \id_{A_{n-i-1}} \otimes \ldots \otimes \id_{A_{n-i-1}} & \text{if } i \in I
\end{cases}
\]

where \( p = p_I(i) \), and \( u \) is interposed between \( A_{i-p-1} \) and \( A_{p+1-i} \).

Lemma 3.29. Let \( A \) be a narrow simplicial object of \( \mathcal{V} \) with a lax monoidal structure \((m,u)\). Then the assignments \( f \mapsto \overline{A}(f) \) of Construction 3.26 make \( \overline{A} \) into a functor \( \Delta^{op} \to \mathcal{V} \).

Proof. We use the same notations as in Construction 3.26. Let \( n \geq 0 \) and \( I \in \mathcal{P}_n \). Then \( \overline{A}(\id_n)_I = A(f_{i1})m_{i1} \otimes \ldots \otimes A(f_{ip})m_{ip} \) is the identity on \( A_I \) because \((I_1, \ldots, I_n)\) is the splitting of \( I \) over itself whereby \( \ell(I_i) = 1 \) and thus \( f_{i\ell} = m_{i1} = \id \) for all \( i \in \{1, \ldots, n\} \).
Next, we prove that $\overline{A}$ preserves compositions. Take morphisms $g : [k] \to [m]$ and $f : [m] \to [n]$ in $\Delta_f$ and $I \in \mathcal{P}_n$. Set $J = f^{-1}(I)$. We may assume that the length of $g^{-1}(J)$ is 1. In that case, we have:

$$A(g_J)m_J(A(f_{I_1})m_{I_1} \otimes \ldots \otimes A(f_{I_p})m_{I_p})$$

$$= A((f_{I_1} \circ \ldots \circ f_{I_p})g_J)m_J(m_{I_1} \otimes \ldots \otimes m_{I_p})$$

$$= A(f_{Ig_J})m_{I_1+\ldots+I_p} = A((fg)_I)m_I$$

where we used the naturality of $\mu$.

**Proposition 3.30.** Let $A$ be a narrow simplicial object of $\mathcal{V}$ with a lax monoidal structure $((m,u), (\alpha, \beta))$. Then the graded maps $\mu$ and $Z$ of Construction 3.21 form natural transformations

$$\mu : \overline{A}(- + -) \to \overline{A}(-) \otimes \overline{A}(-) \quad \text{and} \quad Z : \overline{A}(-) \otimes \overline{A}(-) \to \overline{A}(- + -)$$

between functors $\Delta^\text{op}_f \times \Delta^\text{op}_f \to \mathcal{V}$.

In other words, $\overline{A}$ is an $F$-monoidal functor $\Delta^\text{op}_f \to \mathcal{V}$.

**Proof.** Take $f : [k] \to [p]$ and $g : [l] \to [q]$ in $\Delta_f$, and $I \in \mathcal{P}_p$, $J \in \mathcal{P}_q$. It follows immediately from the definition that

$$\overline{A}(f + g)_{I + J} = \overline{A}(f)_I \otimes \overline{A}(g)_J$$

Consequently, $Z$ is natural as

$$Z^{k,l}(\overline{A}(f) \otimes \overline{A}(g)) = \sum_{I \in \mathcal{P}_p, J \in \mathcal{P}_q} t_{f^{-1}(I) + g^{-1}(J)}(\overline{A}(f)_I \otimes \overline{A}(g)_J)_{PI,J}$$

$$= \sum_{H \in \mathcal{P}_{p+q}} t_{(f+g)^{-1}(H)} \overline{A}(f + g)_{HPH}Z^{p,q} = \overline{A}(f + g)Z^{p,q}$$

and $\mu$ is natural because

$$(\overline{A}(f) \otimes \overline{A}(g))_{\mu_{p,q}} = \sum_{I \in \mathcal{P}_p, J \in \mathcal{P}_q} t_{f^{-1}(I)g^{-1}(J)}(\overline{A}(f)_I \otimes \overline{A}(g)_J)_{PI+J}$$

$$= \mu_{k,l} \sum_{H \in \mathcal{P}_{p+q}} t_{(f+g)^{-1}(H)} \overline{A}(f + g)_{HPH} = \mu_{k,l} \overline{A}(f + g)$$

where the second equality holds because for any $K \in \mathcal{P}_k$ and $L \in \mathcal{P}_l$ such that $K + L = (f + g)^{-1}(H)$, there exist unique $I \in \mathcal{P}_p$ and $J \in \mathcal{P}_q$ such that $H = I + J$ and $K = f^{-1}(I)$ and $L = g^{-1}(J)$. Indeed, since $k \in K + L$ also $p \in H$ and thus we can set $(I,J)$ to be the splitting of $H$ over $\{0 < p < p + q\}$.

**Construction 3.31.** Let $A$ and $B$ be narrow simplicial objects of $\mathcal{V}$ with respective lax monoidal structures $(m^A, u^A)$ and $(m^B, u^B)$. Given $\alpha : A \to B$ a monoidal natural transformation, we define $\overline{\alpha} : \overline{A} \to \overline{B}$ as follows. For all $n \geq 0$, set

$$\overline{\alpha}_n = \bigoplus_{I \in \mathcal{P}_n} \alpha_I : \bigoplus_{I \in \mathcal{P}_n} A_I \to \bigoplus_{I \in \mathcal{P}_n} B_I$$
where
\[
\alpha_I = \alpha_{i_1} \otimes \alpha_{i_2-i_1} \otimes ... \otimes \alpha_{n-i_{m-1}} : A_I \to B_I
\]
for any partition \( I = \{0 = i_0 < i_1 < ... < i_m = n\} \) of \( n \).

**Lemma 3.32.** \( \overline{\alpha} : \overline{A} \to \overline{B} \) is a bimonoidal natural transformation.

**Proof.** It follows immediately from the naturality and monoidality of \( \alpha \) that \( \overline{\alpha} \) is a natural transformation. Since by definition, \( \overline{\alpha}_0 \) is the identity on the monoidal unit of \( \mathcal{V} \), \( \overline{\alpha} \) is clearly unital and counital. Finally, since \( \overline{\alpha}_{I+J} = \overline{\alpha}_I \otimes \overline{\alpha}_J \) for all partitions \( I \) and \( J \), \( \overline{\alpha} \) is also bimonoidal. □

It is clear that the construction \( A \mapsto \overline{A} \) defines a functor
\[
(-) : \text{Lax} (\mathcal{D}^{op}, \mathcal{V}) \to \text{Frob}_{su} (\mathcal{D}^{op}, \mathcal{V})
\]

We conclude this section by proving that the functors \( K \) and \( (-) \) are inverse to each other.

**Lemma 3.33.** Let \( n \geq 0 \) and \( I, K \in \mathcal{P}_n \) with \( I \subset K \), then
\[
\sum_{J \in \mathcal{P}_n} (-1)^{\ell(J)} = 0
\]

**Proof.** Choose \( k \in K \setminus I \), then \( J \mapsto J \setminus \{k\} \) defines a bijection
\[
\{J \in \mathcal{P}_n \mid I \subseteq J \subseteq K, k \in J\} \sim \{J \in \mathcal{P}_n \mid I \subseteq J \subseteq K, k \not\in J\}
\]
Moreover, if \( k \in J \), then \( \ell(J \setminus \{k\}) = \ell(J) - 1 \). The result follows. □

**Proposition 3.34.** Let \( X : \Delta^{op}_I \to \mathcal{V} \) be a naF-monoidal functor with comultiplication \( \mu \) and multiplication \( Z \). Then for all \( n \geq 2 \) and \( I \in \mathcal{P}_n \) with \( \ell(I) \geq 2 \), we have
\[
\mu_I \left( \sum_{J \in \mathcal{P}_n} (-1)^{\ell(J)} Z^J \mu_J \right) = 0
\]

**Proof.** By Corollary 3.13, the left hand side equals:
\[
\sum_{K \in \mathcal{P}_n} \sum_{J \in \mathcal{P}_n} (-1)^{\ell(J)} \mu_I Z^K \mu_K = \sum_{K \in \mathcal{P}_n} \left( \sum_{J \in \mathcal{P}_n} (-1)^{\ell(J)} \mu_I Z^K \mu_K \right)
\]
which is zero by Lemma 3.33. □

**Construction 3.35.** Let \( X : \Delta^{op}_I \to \mathcal{V} \) be an F-monoidal functor with comultiplication \( \mu \) and multiplication \( Z \). For \( n \geq 0 \), define
\[
\psi_n = (Z^I |_{K(X)_I})_{I \in \mathcal{P}_n} : K(X)_n = \bigoplus_{I \in \mathcal{P}_n} K(X)_I \to X_n
\]
Its inverse is constructed as follows. By Proposition 3.34 we have
\[
\xi_n = \sum_{I \in \mathcal{P}_n} (-1)^{\ell(I)+1} Z^I \mu_I : X_n \to K(X)_n
\]
Setting $\xi_J = \xi_{j_1} \otimes \xi_{j_2-j_1} \otimes \ldots \otimes \xi_{n-j_{m-1}} : X_J \to K(X)_J$ for any partition $J = \{0 = j_1 < j_2 < \ldots < j_m = n\}$ of $n$, we define

$$\varphi_n = (\xi_J \circ \mu_J)_{J \in \mathcal{P}_n} : X_n \to \bigoplus_{J \in \mathcal{P}_n} K(X)_J = \overline{K(X)}_n$$

**Lemma 3.36.** Let $X : \Delta^\text{op}_I \to \mathcal{V}$ be an $F$-monoidal functor with co-multiplication $\mu$ and multiplication $\mu$. Then for every $n \geq 0$, the maps $\psi_n$ and $\varphi_n$ of Construction 3.35 are inverse to each other.

**Proof.** Since $Z$ is associative, we can prove that $\xi_m \circ Z^J = 0$ for any $m \geq 1$ and $J \in \mathcal{P}_m$ with $\ell(J) \geq 2$, completely dually to equation (12).

Now take $I, J \in \mathcal{P}_n$ and write $I = \{0 = i_0 < i_1 < \ldots < i_p = n\}$ and $q = \ell(J)$. Then by Proposition 3.12,

$$\xi_{i_1} \mu_I Z^J|_{K(X)_J} = (\xi_{i_1} Z^J \otimes \ldots \otimes \xi_n \otimes \mu_q)_|_{K(X)_J}$$

where $(J_1, \ldots, J_p)$ and $(I_1, \ldots, I_q)$ are the splittings of $J$ over $I$ and $I$ over $J$ respectively. The right hand side unless the length of every $J_i$ and $I_j$ is 1. In the latter case, we find that $I = J$ and

$$\xi_{i_1} \mu_I Z^J|_{K(X)_J} = (\xi_{i_1} |_{K(X)_{i_1}} \otimes \ldots \otimes \xi_n |_{K(X)_{n-p-1}}) = \text{id}_{K(X)_J}$$

It follows that $\psi_n$ is a right inverse of $\varphi_n$.

To prove that $\psi_n$ is also a left inverse, let $I = \{0 = i_0 < \ldots < i_p = n\}$ be a partition of $n$ and consider

$$Z^I \xi_{i_1} \mu_I = \sum_{j=1}^{p} \sum_{J_j \in \mathcal{P}_{i_j-j_{j-1}}} (-1)^{\ell(J_1)+\ldots+\ell(J_p)+p} Z^I \mu_J (Z^J \mu_{J_1} \otimes \ldots \otimes Z^p \mu_{J_p}) \mu_I$$

$$= \sum_{J \in \mathcal{P}_n} (-1)^{\ell(J)+\ell(I)} Z^I \mu_J$$

Thus we find that

$$\psi_n \circ \varphi_n = \sum_{I \in \mathcal{P}_n} Z^I \xi_{i_1} \mu_I = \sum_{J \in \mathcal{P}_n} (-1)^{\ell(J)} \left( \sum_{I \in \mathcal{P}_n} (-1)^{\ell(I)} \right) Z^I \mu_J$$

For each $J \neq \{0 < n\}$, the corresponding term is zero by Lemma 3.33. If $J = \{0 < n\}$, then also $I = \{0 < n\}$ and thus $\psi_n \circ \varphi_n = \text{id}_{X_n}$. \(\square\)

**Proposition 3.37.** There is a natural isomorphism

$$\psi : (-) \circ K \xrightarrow{\sim} \text{id}$$
Proof. Let $X : \Delta^\op_f \to \mathcal{V}$ be an F-monoidal functor with comultiplication $\mu$ and multiplication $Z$. Let $n \geq 0$ and $I \in \mathcal{P}_n$. Then note that the lax monoidal structure on $K(X)$ is given by

$$m_I = X(I^c) Z^I$$

Now, consider the isomorphism $\psi_n : \overline{K(X)}_n \to X_n$ of Construction 3.38. Then for any $f : [m] \to [n]$ in $\Delta_f$ and $I \in \mathcal{P}_n$, set $J = f^{-1}(I)$ and $p = \ell(J)$. Then

$$\psi_m \circ \overline{K(X)}(f) \circ \iota_I = Z^J(X(f_I)m_{I_1} \otimes \ldots \otimes X(f_{I_p})m_{I_p})$$

$$= Z^J(X(I^c f_{I_1}) Z^{I_1}) \otimes \ldots \otimes X(I^c f_{I_p}) Z^{I_p})$$

$$= X(f_1 + \ldots + f_p) Z^{f \circ J}(Z^{I_1} \otimes \ldots \otimes Z^{I_p})$$

$$= X(f) \circ Z^I = X(f) \circ \psi_n \circ \iota_I$$

showing that $\psi_n$ is natural in $n$.

Since $\psi_0 = \epsilon^{-1}$ where $\epsilon$ is the counit of $X$, $\psi$ respects the counits. Write $\overline{\mu}$ for the comultiplication of $\overline{K(X)}$. Take $k, l \geq 0$ and $I \in \mathcal{P}_k$, $J \in \mathcal{P}_l$, then by Lemma 3.36

$$(p_I \otimes p_J) \circ \overline{\mu}_{k,l} \circ \psi^{-1}_{k+l} = p_{I+l} \circ \psi^{-1}_{k+l} = \xi_{l+J} \circ \mu_{l+J}$$

$$= (\xi_{l} \mu_{l} \otimes \xi_{j} \mu_{j}) \circ \mu_{k,l} = (p_{I} \otimes p_{J}) \circ (\psi^{-1}_{k} \otimes \psi^{-1}_{l}) \circ \mu_{k,l}$$

Write $\overline{Z}$ for the Z-map of $\overline{K(X)}$. Then we have

$$Z^{k,l} \circ (\psi_k \otimes \psi_l) \circ (\iota_I \otimes \iota_J) = Z^{k,l}(Z^I \otimes Z^J)|_{K(X)_I \otimes K(X)_J}$$

$$= Z^{I+J}|_{K(X)_I \otimes J} = \psi_{k+l} \circ \iota_{I+l} = \psi_{k+l} \circ \overline{Z}^{k,l} \circ (\iota_I \otimes \iota_J)$$

Hence, $\psi$ is an isomorphism of F-monoidal functors $\overline{K(X)} \simeq X$.

Finally, it quickly follows from the definitions that this isomorphism is natural in $X$. $\square$

Construction 3.38. Suppose we are given a narrow simplicial module $A$ with a lax monoidal $(m, u)$. Consider for all $n \geq 1$, the coprojection $\iota_{\{0<n\}} : A_n \to \overline{A}_n$. Let $\mu$ denote the comultiplication of $\overline{A}$. Then for any $0 < k < n$ and $I \in \mathcal{P}_k$ and $J \in \mathcal{P}_{n-k}$, we have

$$(p_I \otimes p_J) \circ \mu_{k,n-k} \circ \iota_{\{0<n\}} = p_{I+J} \circ \iota_{\{0<n\}} = 0$$

Consequently, $\iota_{\{0<n\}}$ factors through $i : K(\overline{A})_n \to \overline{A}_n$ as

$$\phi_n : A_n \to K(\overline{A})_n$$

Proposition 3.39. There is a natural isomorphism

$$\phi : \text{id} \sim K \circ (-)$$

Proof. Let $A$ be a narrow simplicial object with a lax monoidal structure $(m, u)$. Take $n \geq 1$ and $I \in \mathcal{P}_n$. If there is a $k \in I \setminus \{0, n\}$,
then \( I = I_1 + I_2 \) for some \( I_1 \in \mathcal{P}_k \) and \( I_2 \in \mathcal{P}_{n-k} \). In that case, \( p_{1}i = (p_{1} \otimes p_{1})\mu_{k,n-k}i = 0 \). Thus
\[
i = \sum_{i \in \mathcal{P}_n} \iota_ip_{1}i = \iota_{\{0<n\}}p_{\{0<n\}}i = \iota_{\{0<n\}}\phi_{\{0<n\}}i
\]
Since clearly also \( p_{\{0<n\}}\phi_{\{0<n\}} = \text{id}_{A_{n}} \), we have that \( \phi_{n} \) is an isomorphism with inverse \( p_{\{0<n\}}i \).

Take \( f : [m] \rightarrow [n] \) in \( \Delta_- \). Then \( f^{-1}(\{0 < n\}) = \{0 < m\} \) and thus
\[
iK(\overline{A})(f)\phi_{n} = \overline{A}(f)\iota_{\{0<n\}} = \iota_{\{0<m\}}A(f)m_{\{0<n\}} = i\phi_{m}A(f)
\]
showing that \( \phi \) is natural in \( n \).

Write \( Z \) for the multiplication of \( \overline{A} \) and \( \overline{m} \) for the induced multiplication on \( K(\overline{A}) \). Then we have for all \( p,q \geq 1 \):
\[
is_{p,q}(\phi_{p} \otimes \phi_{q}) = \overline{A}(\delta_{p})Z^{p,q}(\iota_{\{0<p\}} \otimes \iota_{\{0<q\}}) = \overline{A}(\delta_{p})\iota_{\{0<p+q\}}
\]
\[
= \iota_{\{0<p+q-1\}}A(id_{[p+q-1]})m_{p,q} = i\phi_{p+q-1}m_{p,q}
\]
Moreover, the unit of \( K(\overline{A}) \) is just the degeneracy map \( s_0 : k \rightarrow \overline{A}_0 = \overline{A}_1 \), which in turn is the unit \( u \) of \( A \). Hence, \( \phi \) is an isomorphism of narrow simplicial objects \( A \simeq K(\overline{A}) \).

Finally, it quickly follows from the definitions that this isomorphism is natural in \( A \). \( \square \)

Combining Lemma 3.16 and Propositions 3.37 and 3.39 we can now put the final touches to the proof of Theorem 3.14.

Proof of Theorem 3.14. It suffices to check the triangle identities:
\[
\psi_{\overline{A}} \circ \phi_{\overline{A}} = \text{id}_{\overline{A}} \quad \text{and} \quad K(\psi_{X}) \circ \phi_{K(X)} = \text{id}_{K(X)}
\]
for all \( X \in \text{Frob}_{sa}(\Delta^{op}, \mathcal{V}) \) and \( A \in \text{Lax}(\Delta^{op}, \mathcal{V}) \).

For all \( n \geq 0 \) and \( I = \{0 = i_0 < ... < i_m = n\} \), we have
\[
\psi_{\overline{A}_n} \circ (\phi_{\overline{A}})_n \circ i_{I}^{A} = \psi_{\overline{A}_n} \circ i_{K(\overline{A})} \circ \phi_{A_I}
\]
\[
= Z^{I} \circ (\iota_{\{0<i_1\}} \otimes \cdots \otimes \iota_{\{i_m-1<n\}}) = i_{I}^{A}
\]
Further, for all \( n \geq 1 \), we have
\[
K(\psi_{X})_{n} \circ \phi_{K(X)_{n}} = i \circ \psi_{X} \circ i_{\{0<n\}}^{K(X)} = i \circ Z^{\{0<n\}}|_{K(X)_{n}} = \text{id}_{K(X)_{n}}
\]
where \( i \) is the embedding \( K(X)_{n} \rightarrow X_{n} \). \( \square \)

4. Templicial modules

Fix a commutative unital ring \( k \). In this section, we restrict to the case \( \mathcal{V} = \text{Mod}(k) \) with the tensor product over \( k \). We will replace \( \text{Mod}(k) \) by \( k \) in some of the notations where we use \( \mathcal{V} \) in the general case. For example, we call \( \text{Mod}(k) \)-quivers simply \( k \text{-quivers} \), we write \( N_{k} \) for the \( \text{Mod}(k) \)-enriched nerve and we denote the category of \( \text{Mod}(k) \)-enriched quasi-categories by \( \text{QCat}(k) \). We will also call its objects \( (k-) \text{-linear quasi-categories} \).
The results in this section make use of the free templicial module functor $\tilde{F} : SSet \to S \otimes \text{Mod}(k)$ and the underlying simplicial set functor $\tilde{U} : S \otimes \text{Mod}(k) \to SSet$ (see Definition 2.19). In §4.1, we show that for a linear quasi-category $X$, the classical homotopy category of $\tilde{U}(X)$ can be endowed with a $k$-linear structure (Proposition 4.4), thus giving rise to the linear homotopy category of $X$. In §4.2, making use of nonassociative Frobenius structures, we show that the free templicial module $\tilde{F}(Y)$ of an ordinary quasi-category $Y$ is a linear quasi-category (Corollary 4.16).

4.1. Homotopy category of a linear quasi-category. Recall the classical homotopy functor $h : \text{QCat} \to \text{Cat}$ associating to a quasi-category its homotopy category, by identifying homotopic 1-simplices. We now introduce the linear analogue.

Let $X$ be a templicial module with base $S$. Recall from Proposition 2.22 that an $n$-simplex of the underlying simplicial set $\tilde{U}(X)$ corresponds to a pair $\alpha = (((\alpha_i)_{i = 0}^n, (\alpha_{i,j})_{0 \leq i < j \leq n})$ with $\alpha_i \in S$ and $\alpha_{i,j} \in X_{j-i}(\alpha_i, \alpha_j)$ such that for all $i < k < j$,

$$\mu_{k-i,j-k}(\alpha_{i,j}) = \alpha_{i,k} \otimes \alpha_{k,j}$$

Note that pointwise operations do not define a $k$-module structure on $\tilde{U}(X)_n$. Instead, we consider the following.

Definition 4.1. Given $m, n \geq 0$ and $\alpha \in \tilde{U}(X)_m$, $\beta \in \tilde{U}(X)_n$, we define $\text{Hom}_X(\alpha, \beta)$ to be the set

$$\{ \gamma \in \tilde{U}(X)_{m+n+1} \mid d_{m+1}...d_{m+n+1}(\gamma) = \alpha, d_0...d_0(\gamma) = \beta \}$$

where the face maps $d_i$ are to be considered in the simplicial set $\tilde{U}(X)$.

Note that a simplex $\gamma \in \tilde{U}(X)_{m+n+1}$ lies in $\text{Hom}_X(\alpha, \beta)$ if and only if we have for all $0 \leq i < j \leq n$ that

$$\gamma_i = \begin{cases} \alpha_i & \text{if } i \leq m \\ \beta_{i-m-1} & \text{if } i > m \end{cases} \quad \text{and} \quad \gamma_{i,j} = \begin{cases} \alpha_{i,j} & \text{if } j \leq m \\ \beta_{i-m-1,j-m-1} & \text{if } i > m \end{cases}$$

It follows that the set $\text{Hom}_X(\alpha, \beta)$ has a $k$-module structure given by pointwise operations on all $\gamma_{i,j}$ with $0 \leq m < j$.

Example 4.2. Recall from Proposition 2.22 that $\tilde{U}(X)_0 \simeq S$ is the base of a templicial module $X$. It immediately follows from the definition that for all $a, b \in S$, $\text{Hom}_X(a, b) \simeq X_1(a, b)$ as $k$-modules.

Let $f$ and $g$ be two 1-simplices of a simplicial set with $d_1(f) = d_1(g) = a$ and $d_0(f) = d_0(g) = b$. Recall that a (left) homotopy from $f$ to $g$ is a 2-simplex $w$ such that $d_0(w) = f$, $d_1(w) = g$ and $d_2(w) = s_0(a)$. We say $f$ is (left) homotopic to $g$ if there exists a left homotopy from
$f$ to $g$. If the simplicial set is a quasi-category, this is an equivalence relation and we write $f \sim g$. The homotopy class of $f$ is denoted $[f]$.

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (0.5,-0.5) {$c$};
\draw (a) to node[auto] {$f$} (b);
\draw (c) to node[auto] {$g$} (b);
\end{tikzpicture}
\end{array}
\]

Applied to $\hat{U}(X)$, we see that $\text{Hom}_X(s_0(a), b)$ is the $k$-module of left homotopies between edges $a \to b$. We have bijections $\hat{U}(X)_1(a, b) \simeq X_1(a, b)$ for all $a, b \in S$. So if $X$ is a linear quasi-category, we can transfer the homotopy relation on edges from $\hat{U}(X)$ to $X$.

**Lemma 4.3.** Assume $X$ is a linear quasi-category. Then for all objects $a$ and $b$ of $X$, we have:

(a) for all $f, g \in X_1(a, b)$:

$$f \sim g \iff f - g \sim 0,$$

(b) the set $N_{a,b} = \{f \in X_1(a, b) \mid f \sim 0\}$ is a submodule of $X_1(a, b)$.

**Proof.** (a) Since $s_0(g) \in \text{Hom}_X(s_0(a), b)$, the map $w \mapsto w + s_0(g)$ defines a bijection between homotopies from $f-g$ to 0 and homotopies from $f$ to $g$.

(b) It follows from (a) that $N_{a,b}$ is an abelian subgroup of $X_1(a, b)$.

Let $w$ be a homotopy from $f$ to 0. Then for all $\lambda \in k$, $\lambda w \in \text{Hom}_X(s_0(a), b)$ is a homotopy from $\lambda f$ to 0.

**Proposition 4.4.** Assume $(X, S)$ is a linear quasi-category, then the homotopy category of $\hat{U}(X)$ carries the structure of a $k$-linear category.

Moreover, if $\alpha : X \to Y$ is a templicial map between linear quasi-categories. Then the functor $h\hat{U}(\alpha) : h\hat{U}(X) \to h\hat{U}(Y)$ is $k$-linear with respect to these structures.

**Proof.** We have $\text{Ob}(h\hat{U}(X)) = \hat{U}(X)_0 \simeq S$. Note that by Lemma 4.3 the hom-sets of $h\hat{U}(X)$ are precisely $X_1(a, b)/N_{a,b}$ for all $a, b \in S$ and thus carry an induced $k$-module structure.

It suffices to show that the composition map in $h\hat{U}(X)$ is bilinear. Take $a, b, c \in S$, $\lambda \in k$, $f, f' \in X_1(a, b)$ and $g \in X_1(b, c)$. Assume that $h, h' \in X_1(a, c)$ are compositions of $(f, g)$ and $(f', g)$ respectively, i.e. there exist $w, w' \in \hat{U}(X)_2$ such that

\[
\begin{array}{c}
w_{0,1} = f, \quad w'_{0,1} = f', \quad w_{1,2} = w'_{1,2} = g \\
d_1(w_{0,2}) = h, \quad d_1(w'_{0,2}) = h'
\end{array}
\]

So $w, w' \in \text{Hom}_X(a, g)$ and thus also $\lambda w + w' \in \text{Hom}_X(a, g)$ with

\[
\begin{array}{c}
(\lambda w + w')_{1,2} = g, \quad (\lambda w + w')_{0,1} = \lambda f + f' \\
d_1((\lambda w + w')_{0,2}) = \lambda h + h',
\end{array}
\]
Thus $\lambda h + h'$ is a composition of $(\lambda f + f', g)$. Hence
\[ [g] \circ (\lambda[f] + [f']) = \lambda([g] \circ [f]) + [g] \circ [f'] \]
Similarly, the composition is linear in the other component.

Now consider the templicial map $(\alpha, f) : (X, S) \rightarrow (Y, T)$. The object map of the functor $h\tilde{U}(\alpha)$ is precisely $f : S \rightarrow T$ and for any $a, b \in S$, $\tilde{U}(\alpha)$ is given on morphisms by
\[
\frac{X_1(a, b)}{N_{a,b}} \rightarrow \frac{Y_1(f(a), f(b))}{N_{f(a), f(b)}} : [x] \mapsto [\alpha_1(x)]
\]
which is $k$-linear because $(\alpha_1)_{a,b} : X_1(a, b) \rightarrow Y_1(f(a), f(b))$ is $k$-linear.

**Construction 4.5.** It is clear from Proposition 4.4 that the functor $h : \text{QCat} \rightarrow \text{Cat}$ can be upgraded to a functor $h_k : \text{QCat}(k) \rightarrow \text{Cat}(k)$ such that there is a natural isomorphism $\mathcal{U} \circ h_k \simeq h \circ \tilde{U}$.

**Definition 4.6.** We call the functor $h_k$ defined in Construction 4.5 the **linear homotopy functor**. Given a linear quasi-category $X$, we call $h_k X$ the **linear homotopy category** of $X$.

**Lemma 4.7.** Assume $(X, S)$ is a linear quasi-category with objects $a$ and $c$ and $w \in X_2(a, c)$. If
\[
\mu_{1,1}(w) = \sum_{i=1}^{n} f_i \otimes g_i
\]
for some $n \geq 0$ and $b_i \in S$, $f_i \in X_1(a, b_i)$ and $g_i \in X_1(b_i, c)$, for $i \in \{1, \ldots, n\}$. Then
\[
[d_1(w)] = \sum_{i=1}^{N} [g_i] \circ [f_i]
\]
in $h_k X(a, c)$.

**Proof.** Since $X$ is a linear quasi-category, we can find for any $i \in \{1, \ldots, n\}$ a $w_i \in X_2(a, c)$ such that $\mu_{1,1}(w_i) = f_i \otimes g_i$. Then $w' = w - \sum_{i=1}^{n} w_i$ defines a 2-simplex $\gamma$ of $\tilde{U}(X)$ with $\gamma_{0,1} = 0$ and $\gamma_{1,2} = 0$ and thus $[d_1(w')] = [0] \circ [0] = 0$ in $h_k X(a, b)$. Consequently,
\[
[d_1(w)] = [d_1(w')] + \sum_{i=1}^{n} [d_1(w_i)] = \sum_{i=1}^{n} [g_i] \circ [f_i]
\]

**Proposition 4.8.** We have $h_k \vdash N_k$. 
Proof. Let $X$ be a linear quasi-category and let $\nu : X_1 \to h_k X$ be the quotient $k$-quiver map. By Lemma 2.24, $\nu$ satisfies the diagrams (3) of Lemma 2.24. So we have a unique templicial map $\eta_X : X \to N_k(h_k X)$ such that $\eta_X \sim \nu$. We claim that $\eta_X$ is the unit of an adjunction $h_k \dashv N_k$.

Let $C$ be a small $\mathcal{V}$-category and $(\alpha, f) : X \to N_k(C)$ a templicial map. Suppose $\nu \in N_{a,b}$ for objects $a, b$ of $X$. Then clearly $(\alpha_1)_{a,b}(h) \sim 0$ in $N_k(C)$. But since $\tilde{U}(N_k(C)) \simeq N(\mathcal{U}(C))$, two edges in $N_k(C)$ are homotopic if and only if they are equal. Thus $N_{a,b} \subseteq \ker((\alpha_1)_{a,b})$.

Write $\alpha'_1 : X_1 \to f^*(C)$ for the quiver map corresponding to $\alpha_1$ via the adjunction $f_1 \vdash f^*$. Then there exists a unique quiver map $H : h_k X \to f^*(C)$ such that $H \circ \nu = \alpha'_1$ and it follows by Lemma 2.24 that $(H, f)$ is a $\mathcal{V}$-enriched functor $h_k X \to C$. Again by Lemma 2.24, $H$ is unique such that $N_k(H) \circ \eta_X = \alpha$. \hfill $\square$

**Corollary 4.9.** We have a natural isomorphism $h_k \circ N_k \simeq \text{id}_{\text{Cat}(k)}$.

**Proof.** This immediately follows from Propositions 4.8 and the fact that the linear nerve functor is fully faithful. \hfill $\square$

**4.2. The free functor preserves quasi-categories.** Consider the free templicial module functor $\tilde{F} : SSet \to S_{\otimes} \text{Mod}(k)$ (see Definition 2.19). We will show that given a quasi-category $X$, the templicial module $\tilde{F}(X)$ is a linear quasi-category. Recall by Proposition 3.7 that $\tilde{F}$ preserves nonassociative Frobenius structures. It turns out that in the linear setting, having a nonassociative Frobenius structure is sufficient in order to satisfy the weak Kan condition.

**Definition 4.10.** Let $n \geq 2$. We write $W^n$ for the simplicial subset of $\Delta^n$ defined by, for all $m \geq 0$:

$$W^n([m]) = \{ f : [m] \to [n] \mid f(m) \leq n - 1 \text{ or } f(0) \geq 1 \}$$

and call it the $n$th wedge. It consists of the 0th and $n$th face of $\Delta^n$.

We say a simplicial set $X$ lifts wedges if every simplicial map $W^n \to X$ extends to $\Delta^n$ for all $n \geq 2$.

**Lemma 4.11.** The inclusion $W^n \to \Delta^n$ is inner anodyne for all $n \geq 2$.

**Proof.** Since $W^2 = \Lambda^2_1$, we may assume that $n > 2$. Let $\star$ denote the join of simplicial sets. Note that the wedge inclusion $W^n \to \Delta^n$ is precisely the pushout-product

$$(\Delta^{n-3} \star \Delta^1) \coprod_{\Delta^{n-3} \star \{0\}} (\Delta^{n-2} \star \{0\}) \to \Delta^{n-2} \star \Delta^1$$

where $\Delta^{n-3} \to \Delta^{n-2}$ includes the 0th face and $\{0\} \to \Delta^1$ is the horn inclusion $\Lambda^0_0 \to \Delta^1$. It is well-known that the pushout-product with respect to $\star$ of a monomorphism with a left anodyne map is inner anodyne (see for example [Lur09, Lemma 2.1.2.3]). \hfill $\square$
As an immediate consequence, we have:

**Proposition 4.12.** Every quasi-category lifts wedges.

**Proposition 4.13.** Let $X$ be a $naF$-templicial module. Then $\tilde{U}(X)$ lifts wedges.

**Proof.** Let $n \geq 2$. A simplicial map $\alpha : W^n \to \tilde{U}(X)$ is corresponds to the data of objects $\alpha_0, ..., \alpha_n \in S$ along with elements $\alpha_{i,j} \in X_{j-i}(\alpha_i, \alpha_j)$ for all $i < j$ in $[n]$ with $(i, j) \neq (0, n)$ satisfying the compatibility relation (4). To extend $\alpha$ to $\Delta^n$, we must find an element $\alpha_{0,n} \in X_n(\alpha_0, \alpha_n)$ such that $(\alpha_{i,j})_{0 \leq i \leq j \leq n}$ lies in $\tilde{U}(X)_n$.

Define for all $I = \{0 = i_0 < i_1 < ... < i_m = n\}$ with $m \geq 2$:

$$\alpha_I = \alpha_{0,i_1} \otimes \alpha_{i_1,i_2} \otimes ... \otimes \alpha_{i_{m-1},n}$$

Then we can prove completely analogously to Corollary 3.13 that

$$\mu_I Z^J(\alpha_J) = \mu_I Z^{I\cup J}(\alpha_{I\cup J})$$

for all $I, J \in \mathcal{P}_n$. We employ a similar trick as in Proposition 3.34,

$$\mu_I \sum_{J \in \mathcal{P}_n, \ell(J) \geq 2} (-1)^{\ell(J)} Z^J(\alpha_J) = \sum_{K \in \mathcal{P}_n, J \in \mathcal{P}_n, \ell(J) \geq 2} (-1)^{\ell(J)} \mu_I Z^K(\alpha_K)$$

$$= \sum_{K \in \mathcal{P}_n, J \in \mathcal{P}_n, \ell(J) \geq 2} (-1)^{\ell(J)} \mu_I Z^K(\alpha_K) = \mu_I Z^I(\alpha_I) = \alpha_I$$

where we used Lemma 3.33 and the observation that if $\ell(J) = 1$ and $K \cap I^c \subseteq J \subseteq K$, we must have $K = I$. Hence, it suffices to set

$$\alpha_{0,n} = \sum_{J \in \mathcal{P}_n, \ell(J) \geq 2} (-1)^{\ell(J)} Z^J \alpha_J$$

**Proposition 4.14.** Let $X$ be a templicial module. Then $\tilde{U}(X)$ lifts wedges if and only if $X$ is a linear quasi-category.

**Proof.** If $X$ is a linear quasi-category, then $\tilde{U}(X)$ lifts wedges by Proposition 4.12. Conversely, take $0 < k < n$ and $\alpha : \Lambda^k_n \to \tilde{U}(X)$ in $SSet$. If $y$ is an $m$-simplex of $\Lambda^k_n$, given by vertices $0 \leq i_0 \leq ... \leq i_m \leq n$, we write $\alpha_{i_0,...,i_m} \in X_m(\alpha_{i_0}, \alpha_{i_m})$ for the image of $y$ under $\alpha$.

Let us start by noting that if $x \in X_n$ satisfies

$$\mu_{p,q}(x) = \alpha_{0,...,p} \otimes \alpha_{p,...,n} \quad (\forall p, q \geq 1 \text{ with } p + q = n)$$

then we have for all $p, q \geq 1$ that

$$\mu_{p,q}(s_j(\alpha_{0,...,j,...,n} - d_j(x))) = 0 \quad \text{(for } 0 < j < k)$$

$$\mu_{p,q}(s_{j-1}(\alpha_{0,...,j,...,n} - d_j(x))) = 0 \quad \text{(for } k < j < n)$$
Indeed, for the first equation, there are three cases:

\[ \mu_{p,q}(s_j(\alpha_{0,...,j,...,n} - d_j(x))) = \begin{cases} 
(s_j \otimes \text{id})(\alpha_{0,...,j,...,p} \otimes \alpha_{p,...,n} - (d_j \otimes \text{id})\mu_{p,q}(x)) & \text{if } j < p \\
(d \otimes s_0)(\alpha_{0,...,p+1} \otimes \alpha_{p+1,...,n} - (d_p \otimes \text{id})\mu_{p+1,q-1}(x)) & \text{if } j = p \\
(d \otimes s_{j-p})(\alpha_{0,...,p} \otimes \alpha_{p,...,j,...,n} - (d \otimes d_{j-p})\mu_{p,q}(x)) & \text{if } j > p 
\end{cases} 
\]

= 0

Note that when \( j = p \), we have \( q-1 \geq 1 \) because \( p+1 = j+1 \leq k < n \). The second equation follows similarly.

Now restrict \( \alpha \) to \( W^n \). By hypothesis, this extends to \( \beta : \Delta^n \rightarrow X \). Let \( x^0 = \beta_{0,...,n} \in X_n \). Then \( x^0 \) satisfies (13). Define for all \( 0 < j < k \):

\[ x^j = x^{j-1} + s_j(\alpha_{0,...,j,...,n} - d_j(x^{j-1})) \]

By the previous remarks, each \( x^j \) satisfies (13). We then prove by induction on \( j \) that for all \( 0 < p \leq j \):

\[ d_p(x^j) = \alpha_{0,...,\bar{p},...,n} \]

Indeed, \( x^0 \) satisfies this trivially and if \( j > 0 \), we have:

\[ d_p(x^j) = \begin{cases} 
\alpha_{0,...,\bar{p},...,n} - s_{j-1}(d_p(\alpha_{0,...,j,...,n}) - d_{j-1}(\alpha_{0,...,\bar{p},...,n})) & \text{if } p < j \\
\quad & \text{if } p = j \\
\quad & \text{if } p > j 
\end{cases} 
\]

Finally, set \( x^n = x^{k-1} \) and define for all \( k < j < n \):

\[ x^j = x^{j+1} + s_{j-1}(\alpha_{0,...,\bar{j},...,n} - d_j(x^{j+1})) \]

Then again \( x^j \) satisfies (13) for all \( k < j < n \). We prove by induction on \( j \) that for all \( p \in \{1,...,k-1\} \cup \{j,...,n-1\} \):

\[ d_p(x^j) = \alpha_{0,...,\bar{p},...,n} \]

Again, \( x^n \) satisfies this trivially and if \( j < n \), we have

\[ d_p(x^j) = \begin{cases} 
\alpha_{0,...,\bar{p},...,n} - s_{j-2}(d_p(\alpha_{0,...,j,...,n}) - d_{j-1}(\alpha_{0,...,\bar{p},...,n})) & \text{if } p < k \\
\quad & \text{if } p = j \\
\quad & \text{if } p > j 
\end{cases} 
\]

Setting \( \alpha_{0,...,n} = x^{k+1} \), we conclude that \( (\alpha_{i,...,j})_{0 \leq i \leq j \leq n} \in \tilde{U}(X)_n \), and that this defines an extension of \( \alpha \) to \( \Delta^n \). \( \square \)

The previous proposition does not hold for ordinary simplicial sets, as the following example shows.

**Example 4.15.** Consider the simplicial set \( X = \Delta^3 \amalg \partial \Delta^2 \Delta^2 \), gluing an extra 2nd face to the standard 3-simplex. Formally, it is the pushout of the inclusion \( \partial \Delta^2 \subseteq \Delta^2 \) along the map \( \partial \Delta^2 \rightarrow \Delta^3 \) sending vertices \( 0 \mapsto 0 \), \( 1 \mapsto 1 \) and \( 2 \mapsto 3 \). Denote the simplices of \( \Delta^3 \) by ordered
sequences \([i_0, \ldots, i_m]\) and denote the extra face by \(x \in X_2\). We then have \(d_0(x) = [1, 3], d_1(x) = [0, 3]\) and \(d_2(x) = [0, 1]\), but \(x \neq [0, 1, 2]\).

Then \(X\) is certainly not a quasi-category as there exists no 3-simplex \(z\) with \(d_0(z) = [1, 2, 3], d_2(z) = x\) and \(d_3(z) = [0, 1, 2]\).

However, all wedges in \(X\) can be filled. Indeed, a map \(\alpha: W_n \to X\) is uniquely determined by simplices \(y, z \in X_{n-1}\) such that \(d_0(y) = d_{n-1}(z)\). If either \(y\) or \(z\) is degenerate, \(\alpha\) extends trivially to \(\Delta^n\). Assuming they are both non-degenerate, we have either \(n = 2\) or \(n = 3\). As \(W^2 = \Lambda^2_1\) and the quasi-category \(\Delta^3\) contains all edges of \(X\), the case \(n = 2\) is covered. If \(n = 3\), we must have \(y = [0, 1, 2]\) and \(z = [1, 2, 3]\), which can be filled by \([0, 1, 2, 3]\) itself.

**Corollary 4.16.** Let \(X\) be an ordinary quasi-category. Then \(\tilde{F}(X)\) is a linear quasi-category.

**Proof.** This follows from Propositions 3.5, 3.7, 4.13 and 4.14. \(\square\)

**Corollary 4.17.** There is a natural isomorphism \(h_k \circ \tilde{F} \cong F \circ h\)

**Proof.** This follows from the uniqueness of left-adjoints since \(h_k \circ \tilde{F} \dashv \tilde{U} \circ N_k, F \circ h \dashv N \circ U\) and \(N \circ U \cong \tilde{U} \circ N_R\). \(\square\)

We end this section by collecting some previous results in the following theorem.

**Theorem 4.18.** There is a diagram of adjunctions

\[
\begin{array}{ccc}
\text{Cat} & \xrightarrow{\tilde{U}} & \text{Cat}(k) \\
\downarrow N & & \downarrow h_k \\
\text{QCat} & \xrightarrow{\tilde{F}} & \text{QCat}(k)
\end{array}
\]

commutes in the sense that

\[
\begin{align*}
N_k \circ F & \simeq \tilde{F} \circ N \\
F \circ h & \simeq h_k \circ \tilde{F} \\
\tilde{U} \circ N_k & \simeq N \circ U \\
h \circ \tilde{U} & \simeq U \circ h_k
\end{align*}
\]

5. The linear dg-nerve

In this final section, we introduce a linear analogue (Definition 5.13) of the classical dg-nerve functor [Lur16]. More precisely, in §5.2 we prove Theorem 5.11 which states that templicial modules with an associative Frobenius structure are equivalent to small non-negatively graded dg-categories. This goes in two steps. The first step requires the general Theorem 3.14 and the second step uses an augmented version of the classical Dold-Kan correspondence, see Corollary 5.7 from §5.1. Finally, in §5.3 and §5.4 we respectively show that the linear dg-nerve behaves well with respect to the homotopy category functor and with respect to the classical dg-nerve.
5.1. The augmented Dold-Kan correspondence. Recall the classical Dold-Kan correspondence. We have an equivalence

\[ S \text{Mod}(k) \xrightarrow{\sim} \text{Ch}_{\geq 0}(k) \]

between the categories of simplicial modules and non-negatively graded chain complexes over \( k \). Here, \( N_* \) is the normalized chain complex functor and \( \Gamma \) is its right adjoint. We will consider \( N_* \) to be defined as

\[ N_n(A) = \frac{A_n}{\sum_{i=0}^{n-1} s_i(A_{n-1})} \]

for any simplicial module \( A \) and \( n \geq 0 \). Writing \( \overline{d}_i \) for the map \( N_n(A) \to N_{n-1}(A) \) induced by the \( i \)th face map \( d_i \), the differential is given by, for all \( n \geq 1 \):

\[ \partial_n = \sum_{i=0}^{n} (-1)^i \overline{d}_i : N_n(A) \to N_{n-1}(A) \]

Given a simplicial set \( X \), we write \( N_*(X; k) \) for the normalized chain complex of the free simplicial module on \( X \), that is \( N_*(F \circ X) \).

Then \( \Gamma \) can be obtained by a nerve construction. That is, for all \( n \geq 0 \) and \( C_* \) any chain complex:

\[ \Gamma(C_*)_n = \text{Ch}(k)(N_*(\Delta^n; k), C_*) \]

Explicitly, \( \Gamma(C_*)_n \) is the module consisting of all families \((a_I)_{\emptyset \neq I \subseteq [n]} \) with \( a_I \in C_{|I|-1} \) that satisfy, for all \( \emptyset \neq I = \{i_0 < \ldots < i_m\} \subseteq [n] \):

\[ \partial(a_I) = \sum_{j=0}^{m} (-1)^j a_{I \setminus \{i_j\}} \quad \text{if} \quad |I| \geq 2 \quad \text{and} \quad \partial(a_{\{i\}}) = 0 \quad \text{for all} \quad i \in [n] \]

This description becomes slightly more elegant if we consider augmented simplicial modules instead, the category of which we denote by \( S_+ \text{Mod}(k) = \text{Fun}(\Delta^{op}_+, \text{Mod}(k)) \).

**Construction 5.1.** Given an augmented simplicial module \( A \), define \( \tilde{N}_*(A) \) as the non-negatively graded chain complex given by

\[ \tilde{N}_n(A) = \frac{A_{n-1}}{\sum_{i=0}^{n-2} s_i(A_{n-2})} \]

for all \( n \geq 0 \). So in low degrees: \( \tilde{N}_0(A) = A_{-1} \), \( \tilde{N}_1(A) = A_0 \) and \( \tilde{N}_2(A) = A_1/s_0(A_0) \). The differential is given by, for all \( n \geq 0 \):

\[ \partial_{n+1} = \sum_{i=0}^{n} (-1)^i \overline{d}_i : \tilde{N}_{n+1}(A) \to \tilde{N}_n(A) \]

By the simplicial identities, this is well-defined and squares to zero.
Given an augmented simplicial map $\alpha : A \to B$, set $\tilde{N}_n(\alpha) = \overline{\alpha}_{n-1}$ to be the map $\tilde{N}_n(A) \to \tilde{N}_n(B)$ induced by $\alpha_{n-1}$. This defines a chain map

$$\tilde{N}_\bullet(\alpha) : \tilde{N}_\bullet(A) \to \tilde{N}_\bullet(A)$$

by the naturality of $\alpha$. It is clear that we get a functor

$$\tilde{N}_\bullet : S_+ \text{Mod}(k) \to \text{Ch}_{\geq 0}(k)$$

Given an augmented simplicial set $X$, we will also write $\tilde{N}_\bullet(X; k)$ for $\tilde{N}_\bullet(F \circ X)$, analogously to the classical nominalized chain complex.

**Remark 5.2.** We have a functor $(-)_{\geq 0} : S_+ \text{Mod}(k) \to \text{SMod}(k)$ sending every augmented simplicial module $A$ to its underlying simplicial module $A_{\geq 0}$ by forgetting $A_{-1}$ and the face map $d_0 : A_0 \to A_{-1}$.

Further, we have a functor $s : \text{Ch}_{\geq 0}(k) \to \text{Ch}_{\geq 0}(k)$, which sends a non-negatively graded chain complex $C_\bullet$ to a shifted chain complex $sC_\bullet$, given by $C_n = C_{n+1}$ if $n \geq 0$, and simply $\partial_n^{sC} = \partial_{n+1}^C$ for all $n > 0$.

Note that by construction, we have an equality of functors

$$s \circ \tilde{N}_\bullet = \tilde{N}_\bullet \circ (-)_{\geq 0}$$

**Construction 5.3.** It follows from the fact that $N_\bullet$ preserves colimits, that the same holds for $\tilde{N}_\bullet$. Consequently, the nerve construction yields a right adjoint $\tilde{\Gamma} : \text{Ch}(k) \to S_+ \text{Mod}(k)$ to $\tilde{N}$ given by, for every chain complex $C_\bullet$ and $n \geq -1$:

$$\tilde{\Gamma}(C_\bullet)_n = \text{Ch}(k)(\tilde{N}_\bullet(\Delta^+_n; k), C_\bullet)$$

Here, $\Delta^+_n$ is the augmented simplicial set $\Delta_+(-, [n])$. That is, it has a unique $(-1)$-simplex and its underlying simplicial set is just $\Delta^n$.

Explicitly, $\tilde{\Gamma}(C_\bullet)$ is the submodule of $\bigoplus_{I \subseteq [n]} C_{|I|}$ consisting of all families $(a_I)_I$ that satisfy

$$\partial(a_I) = \sum_{j=0}^k (-1)^j a_{I \setminus \{i_j\}}$$

for all $I = \{i_0 < \ldots < i_k\} \subseteq [n]$. For $f : [m] \to [n]$ in $\Delta_+$, the map

$$\tilde{\Gamma}(C_\bullet)(f) : \tilde{\Gamma}(C_\bullet)_n \to \tilde{\Gamma}(C_\bullet)_m : (a_I)_{I \subseteq [n]} \mapsto (b_J)_{J \subseteq [m]}$$

is given by $b_J = a_f(J)$ if $f|_J$ is injective and $b_J = 0$ otherwise.

Further, if $f : C_\bullet \to D_\bullet$ is a chain map, then

$$\tilde{\Gamma}(f)_n : \tilde{\Gamma}(C_\bullet)_n \to \tilde{\Gamma}(D_\bullet)_n : (a_I)_{I \subseteq [n]} \mapsto (f(a_I))_{I \subseteq [n]}$$

for all $n \geq -1$.

**Proposition 5.4.** The functors $\tilde{N}$ and $\tilde{\Gamma}$ form an adjoint equivalence

$$S_+ \text{Mod}(k) \xrightarrow{\tilde{N}} \text{Ch}_{\geq 0}(k) \xleftarrow{\tilde{\Gamma}}$$
Proof. Since \( s \circ \tilde{N}_\ast = N \circ (\cdot)_{\geq 0} \) we have the following isomorphisms
\[
\text{Ch}(k)(\tilde{N}_\ast(A), \tilde{N}_\ast(B)) \\
\simeq \text{Ch}(k)(N_\ast(A_{\geq 0}), N_\ast(B_{\geq 0})) \times \text{Mod}(k)(A_{-1}, B_{-1}) \\
\simeq S_+ \text{Mod}(k)(A_{\geq 0}, B_{\geq 0}) \times \text{Mod}(k)(A_{-1}, B_{-1}) \\
\simeq S_+ \text{Mod}(k)(A, B)
\]
This proves that \( \tilde{N}_\ast : S_+ \text{Mod}(k) \to \text{Ch}(k) \) is fully faithful. Further, \( C_\ast \) is a non-negatively graded chain complex, consider the simplicial module \( A_{\geq 0} = \Gamma(sC_\ast) \) so that \( N_\ast(A_{\geq 0}) \simeq sC_\ast \). Note that \( A_0 = C_1 \), so we can promote \( A_{\geq 0} \) to an augmented simplicial module \( A \) by setting \( A_{-1} = C_0 \) and \( d_0 = \partial : A_0 \to A_{-1} \). It follows that \( \tilde{N}_\ast(A) \simeq C_\ast \). Thus \( \tilde{N}_\ast \) is essentially surjective as well.

We can endow the category \( S_+ \text{Mod}(k) \) with the monoidal structure of the Day convolution. This is also known as the join of augmented simplicial objects. Explicitly, the tensor product of two augmented simplicial modules \( A \) and \( B \) is given by
\[
(A \otimes B)_n = \bigoplus_{k+l+1=n} A_k \otimes B_l
\]
for all \( n \geq -1 \). Given \( f : [m] \to [n] \), and \( k, l \geq -1 \) such that \( k+l+1 = n \), there exist unique \( f^k_1 : [p] \to [k] \) and \( f^k_2 : [q] \to [l] \) with \( p+q+1 = m \) and \( f^k_1 \ast f^k_2 = f \). With these notations, we have
\[
(A \otimes B)(f) = \sum_{k+l+1=n} A(f^k_1) \otimes B(f^k_2)
\]
The monoidal unit \( I \) is given by \( I_{-1} = k \) and \( \forall n \geq 0 : I_n = 0 \).

Lemma 5.5. The functor \( \tilde{N} \) is strong monoidal.

Proof. Let \( A \) and \( B \) be augmented simplicial modules and \( n \geq 1 \). For all \( i \in [n - 1] \), the degeneracy map
\[
s_i : \bigoplus_{k+l+1=n-2} (A_k \otimes B_l) = (A \otimes B)_{n-2} \to (A \otimes B)_{n-1}
\]
is given by, for all \( k, l \geq -1 \) such that \( k+l+1 = n-2 \):
\[
s_i|_{A_k \otimes B_l} = \begin{cases} 
A^k_i \otimes \text{id}_{B_l} & \text{if } i \leq k \\
\text{id}_{A_k} \otimes s^B_{i-k-1} & \text{if } i > k
\end{cases}
\]
It follows that we have an equality of submodules of \( (A \otimes B)_{n-1} \):
\[
\sum_{i=0}^{n-2} s_i((A \otimes B)_{n-2}) = \bigoplus_{p+q=n} \left( \sum_{i=0}^{p-2} s^A_i(A_{p-2}) \otimes B_{q-1} + \sum_{i=0}^{q-2} A_{p-1} \otimes s^B_i(B_{q-2}) \right)
\]
Consequently, we have an isomorphism
\[
\tilde{N}_n(A \otimes B) \simeq \bigoplus_{p+q=n} (\tilde{N}_p(A) \otimes \tilde{N}_q(B)) = (\tilde{N}_\ast(A) \otimes \tilde{N}_\ast(B))_n
\]
Moreover this isomorphism is a chain map. This follows from the fact that for all $n \geq 0$ and $i \in [0]$, the face map

$$d_i : \bigoplus_{k+l+1=n} (A_k \otimes B_l) = (A \otimes B)_n \to (A \otimes B)_{n-1}$$

is given by, for all $k, l \geq -1$ such that $k + l + 1 = n$:

$$d_i|_{A_k \otimes B_l} = \begin{cases} d^A_i \otimes \text{id}_{B_l} & \text{if } i \leq k \\
\text{id}_{A_k} \otimes d^B_{i-k-1} & \text{if } i > k \end{cases}$$

So, we get an isomorphism

$$\mu_{A,B} : \tilde{\mathcal{N}}(A \otimes B) \cong \tilde{\mathcal{N}}(A) \otimes \tilde{\mathcal{N}}(B)$$

It is a direct verification that this isomorphism is natural in $A$ and $B$, and associative.

We clearly have an isomorphism $\epsilon : \tilde{\mathcal{N}}(I) \cong k$ and it follows easily that $\mu$ is counital with respect to $\epsilon$. □

**Proposition 5.6.** The functors $\tilde{\mathcal{N}}$ and $\tilde{\Gamma}$ form a monoidal equivalence

$$S_+ \text{Mod}(k) \xrightarrow{\tilde{\mathcal{N}}} \text{Ch}_{\geq 0}(k) \xleftarrow{\tilde{\Gamma}}$$

**Proof.** Through the adjoint equivalence of Proposition 5.4, the strong monoidal structure on $\tilde{\mathcal{N}}$ induces a strong monoidal structure on $\tilde{\Gamma}$ that makes the unit and counit of the adjunction into monoidal natural transformations. □

**Corollary 5.7.** We have an adjoint equivalence

$$\text{Cat}(S_+ \text{Mod}(k)) \xrightarrow{\tilde{\mathcal{N}}} \text{dgCat}_{\geq 0}(k)$$

between the categories of small $S_+ \text{Mod}(k)$-enriched categories and of small non-negatively graded dg-categories over $k$.

Let us analyse the functor $\tilde{\Gamma}$ a bit further. Given any small dg-category $\mathcal{C}_\bullet$ over $k$ (not-necessarily bounded), we obtain a $S_+ \text{Mod}(k)$-enriched category $\tilde{\Gamma}(\mathcal{C}_\bullet)$, by applying $\tilde{\Gamma} : \text{Ch}(k) \to S_+ \text{Mod}(k)$ to its hom-complexes. If $m : \mathcal{C}_\bullet(x, y) \otimes \mathcal{C}_\bullet(y, z) \to \mathcal{C}_\bullet(x, z)$ is the composition of $\mathcal{C}_\bullet$, then the composition of $\tilde{\Gamma}(\mathcal{C}_\bullet)$

$$\tilde{m}_{p,q} : \tilde{\Gamma}(\mathcal{C}_\bullet(x, y))_p \otimes \tilde{\Gamma}(\mathcal{C}_\bullet(y, z))_q \to \tilde{\Gamma}(\mathcal{C}_\bullet(x, z))_{p+q+1}$$

for $p, q \geq -1$, is given by

$$\tilde{m}_{p,q}((a_I)_{I \subseteq [p]} \otimes (b_I)_{I \subseteq [q]}) = (m(a_{J_1} \otimes b_{J_2}))_{J \subseteq [p+q+1]}$$

where $J_1 = \{ j \in J \mid j \leq p \}$ and $J_2 = \{ j - p - 1 \mid j \in J, j > p \}$. Further, the identity $\text{id}_x$ on $x \in \text{Ob}(\mathcal{C}_\bullet)$ lives in degree $-1$ since $\mathcal{C}_0(x, x) \simeq \tilde{\Gamma}(\mathcal{C}_\bullet(x, x))_{-1}$. 
5.2. Frobenius templicial modules and the linear dg-nerve.

With Theorem 3.14 and the augmented Dold-Kan correspondence, we will construct an equivalence between non-negatively graded dg-categories over \( k \) and templicial modules with a Frobenius structure, which we will refer to as Frobenius templicial modules. This will allow us to define the linear dg-nerve.

**Definition 5.8.** We refer to a templicial module with a Frobenius structure as a **Frobenius templicial module** or **F-templicial module**. We call a templicial map \((\alpha, f) : (X, S) \rightarrow (Y, T)\) between F-templicial modules an **F-templicial map** if the adjoint \( X \rightarrow f^*Y \) of \( f : X \rightarrow Y \) is a monoidal natural transformation with respect to the lax monoidal structures on \( X \) and \( f^*Y \). This is equivalent to

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 f_!(X_n) \quad \alpha_n \\
 \downarrow \quad \downarrow f_!(Z^{k,n-k}) \\
 f_!(X_k \otimes X_{n-k}) \\
 \end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 Y_n \\
 \leftarrow f_!(Z^{k,n-k}) \\
 f_!(X_k) \otimes f_!(X_{n-k}) \quad \alpha_k \otimes \alpha_{n-k} \\
 \end{array}
\end{array}
\end{array}
\]

being a commutative diagram in \( \text{Quiv}_T(V) \).

It follows that the composition of two F-templicial maps, considered as morphisms of \( S_\otimes \text{Mod}(k) \), is again an F-templicial map. We denote the category of all F-templicial modules and F-templicial maps by

\[ S^{Frob}_\otimes \text{Mod}(k) \]

We have an obvious forgetful functor \( S^{Frob}_\otimes \text{Mod}(k) \rightarrow S_\otimes \text{Mod}(k) \), which factors through \( \text{QCat}(k) \) by Propositions 4.13 and 4.14.

**Construction 5.9.** Let \( S \) be a set. Applying Construction 3.17 to the \( k \)-monoidal category \( \text{Quiv}_S(k) \), we get a functor

\[ K_S : S_\otimes \text{Quiv}_S(k) \rightarrow \text{Fun}(\Delta^{op}, \text{Quiv}_S(k)) \]

Take a map of sets \( f : S \rightarrow T \), \( X \) a templicial module with base \( S \) and \( n \geq 0 \). Then \( K_T(f_!X)_n \) is the kernel of the composite

\[
f_!(X_n) \xrightarrow{f_!(\mu_{k,n-k})}_k f_! \left( \bigoplus_{k=1}^{n-1} X_k \otimes X_{n-k} \right) \rightarrow \bigoplus_{k=1}^{n-1} f_!(X_k) \otimes f_!(X_{n-k})
\]

in \( \text{Quiv}_T(k) \). Let \( e_{X_n} : K_S(X)_n \rightarrow X_n \) denote the canonical inclusion. Since the comultiplication of \( f_! \) is a monomorphism, it follows that \( f_!(e_{X_n}) \) factors uniquely as

\[
f_!(K_S(X)_n) \xrightarrow{\psi_{f,X_n}} K_T(f_!X)_n \xrightarrow{e_{f,X_n}} f_!(X_n)
\]

where \( \psi_{f,X_n} \) is an isomorphism. It quickly follows this defines a natural isomorphism \( \psi_{f,X} : f_! K_S(X) \xrightarrow{\sim} K_T(f_!X) \) of functors \( \Delta^{op} \rightarrow \text{Quiv}_T(k) \), which is moreover natural in \( X \).
Lemma 5.10. Let \( f : R \to S \) and \( g : S \to T \) be maps of sets and \( X \) a templicial module with base \( S \). Then the following diagram commutes:

\[
\begin{array}{c}
(gf)_! K_R(X) \xrightarrow{\psi_{gf,X}} K_T((gf)_! X) \\
\sim \\
g_! f_! K_R(X) \xrightarrow{g_!(\psi_{f,X})} g_! K_S(f_! X) \xrightarrow{\psi_{g,f,X}} K_T(g_! f_! X)
\end{array}
\]

Proof. Take \( n \geq 0 \) and compose both sides of the diagram with the canonical inclusion \( e_{g_! f_! X_n} : K_T(g_! f_! X)_n \to g_! f_! (X_n) \). Then the result follows from Construction 5.9, the naturality of the isomorphism \((gf)_! \simeq g_! f_!\) and the definition of \( K_T \). \( \square \)

Let \( S^{Frob}_\Delta \text{Mod}(k)_S \) be the subcategory of \( S^{Frob}_\Delta \text{Mod}(k) \) spanned by all -templicial modules with base \( S \). Further, we write \( S^{\Delta}_- \text{Mod}(k)_S \) for the category of narrow simplicial modules and \( \text{Cat}(S_- \text{Mod}(k))_S \) for the category of \( S_- \text{Mod}(k) \)-enriched categories with set of objects \( S \). Now note that we have isomorphisms

\[
S^{Frob}_\Delta \text{Mod}(k)_S \simeq \text{Frob}_{\text{nu}}(\Delta^{op}_\Delta, \text{Quiv}_S(k)) \quad \text{Cat}(S_- \text{Mod}(k))_S \simeq \text{Lax}(\Delta^{op}_\Delta, \text{Quiv}_S(k))
\]

Hence Theorem 3.14 states that \( K_S \) can be upgraded to an equivalence

\[
S^{Frob}_\Delta \text{Mod}(k)_S \xrightarrow{K_S} \text{Cat}(S_- \text{Mod}(k))_S
\]

Theorem 5.11. There is an adjoint equivalence of categories

\[
S^{Frob}_\Delta \text{Mod}(k)_S \xrightarrow{K} \text{Cat}(S_- \text{Mod}(k))
\]

Proof. Take an \( F \)-templicial map \( (\alpha, f) : (X, S) \to (Y, T) \) between \( F \)-templicial modules. Then by Construction 3.17, we have a natural transformation \( K_T(\alpha) : K_T(f_! X) \to K_T(Y) \). Using Construction 5.9 and the adjunction \( f_! \dashv f^* \), we get a natural transformation, which we denote

\[
K(\alpha) : K_S(X) \to f^* K_T(Y)
\]

As the adjoint \( X \to f^* Y \) of \( \alpha \) is monoidal, it follows that \( K(\alpha) \) is monoidal as well. Hence, \( (K(\alpha), f) \) is a \( S_- \text{Mod}(k) \)-enriched functor \( K(X) \to K(Y) \). It follows from Lemma 5.10 that this defines a functor

\[
K : S^{Frob}_\Delta \text{Mod}(k)_S \to \text{Cat}(S_- \text{Mod}(k))
\]

which restricts to \( K_S \) on \( S^{Frob}_\Delta \text{Mod}(k)_S \) for each set \( S \).

Finally, the isomorphisms of Construction 5.9 provide isomorphisms \( f^*(A)_T \simeq (f^* A)_S \) for every map of sets \( f : S \to T \) and \( S_- \text{Mod}(k) \)-enriched category \( A \) with set of objects \( T \). A similar argument then
proves that the functors \((-\_)_S\) combine to give a functor
\[
(-) : \text{Cat}(S_\_ \text{Mod}(k)) \to S^{\text{Frob}}_\otimes \text{Mod}(k)
\]
which is inverse and right adjoint to \(K\).

\[\□\]

**Corollary 5.12.** There is an adjoint equivalence of categories
\[
S^{\text{Frob}}_\otimes \text{Mod}(k) \cong dg \text{Cat}_{\geq 0}(k)
\]

\[\text{Proof.}\] Combine Theorem 5.11, Lemma 3.16 and Corollary 5.7 \[\□\]

**Definition 5.13.** Let \((-_\#)_S : \text{Cat}(S_+ \text{Mod}(k)) \sim \text{Cat}(S_- \text{Mod}(k))\) be the isomorphism obtained by Lemma 3.16. We call the composite
\[
N_{dg}^k : dg \text{Cat}(k) \xrightarrow{\Gamma(-)_{\#}} S^{\text{Frob}}_\otimes \text{Mod}(k) \to \text{QCat}(k)
\]
the \((k-)\text{linear dg-nerve functor}\).

**5.3. Comparison with homotopy categories.** Any small \(k\)-linear category can be considered as a small dg-category concentrated in degree 0. Let us denote this embedding by \(\iota : \text{Cat}(k) \to dg \text{Cat}(k)\). Conversely, we can apply the 0th homology functor to all hom-complexes of a small dg-category \(C_*\) over \(k\) to get a small \(k\)-linear category \(H_0(C_*)\). In this section, we show that under the linear dg-nerve functor \(N_{dg}^k\), the adjunction \(H_0 \dashv \iota\) corresponds to the adjunction \(h_k \dashv N_k\) in the sense that the diagram
\[
\begin{array}{ccc}
\text{QCat}(k) & \xleftarrow{N_{dg}^k} & dg \text{Cat}_{\geq 0}(k) \\
N_k & \searrow & H_0 \\
& \iota & \nearrow
\end{array}
\]
commutes up to natural isomorphism in both directions.

**Proposition 5.14.** We have natural isomorphisms
\[
N_{dg}^k \circ \iota \simeq N_k \quad \text{and} \quad h_k \circ N_{dg}^k \simeq H_0
\]

\[\text{Proof.}\] Let’s denote the functor from left to right in the equivalence of Corollary 5.12 by \(C_* : S^{\text{Frob}}_\otimes \text{Mod}(k) \to dg \text{Cat}_{\geq 0}(k)\).

Let \(C\) be a small \(k\)-linear category. Then by Example 3.4, \(N_k(C)\) has a unique Frobenius structure. Since the comultiplication maps of \(N_k(C)\) are invertible, we have that \(K(N_k(C))\) is concentrated in degree 1 and thus \(C_*(N_k(C))\) is concentrated in degree 0. It follows that \(C_* \circ N_k\) is naturally isomorphic to \(\iota\) and therefore \(N_{dg}^k \circ \iota \simeq N_k\).

Let \(X\) be an \(F\)-templicial module with base \(S\). Boiling down the definitions, we see that the set of objects of \(C_*(X)\) is \(S\) as well and that for every \(x \in S\), the degenerate 1-simplex \(s_0(x)\) represents the identity in both \(hX\) and \(H_0(C_*(X))\). Take \(x,y,z \in S\). Then the differential \(\partial : C_1(X)(x,z) \to C_0(X)(x,z)\) is just \(-d_1 : \ker(\mu_{1,1})(x,z) \to X_1(x,z)\).
Hence, for any three \( f \in X_1(x, y), \ g \in X_1(y, z) \) and \( h \in X_1(x, z) \), the composition \( gf \) is homologous to \( h \) in \( C_\bullet(X) \) if and only if there exists a \( w \in \ker(\mu_{1,1})(x, z) \) such that \( d_1(w) = gf - h \). This is equivalent to the existence of a templicial map \( \alpha : \Delta^2 \to X \) with \( \alpha_{0,1} = 0 \), \( \alpha_{1,2} = s_0(x) \) and \( \alpha_{0,2} = gf - h \). In other words, \( gf - h \) is homotopic to \( 0 \) in \( X \), i.e. 
\[
[g] \circ [f] = [h] \text{ in } hX.
\]
Specializing to the case \( f = s_0(x) \), we find that \( [g] = [h] \) in \( H_0(C_\bullet(X)) \) if and only if \( [g] = [f] \) in \( hX \). This shows that \( [f] \mapsto [f] \) defines an isomorphism of \( k \)-linear categories
\[
h_kX \simeq H_0(C_\bullet(X))
\]
It follows easily that this isomorphism is natural in \( X \). We conclude that also \( h_k \circ N^{dg}_k \simeq H_0 \). \( \square \)

**5.4. Comparison with the dg-nerve.** Consider the classical dg-nerve, which assigns to every small dg-category \( C_\bullet \) over \( k \) a quasi-category \( N^{dg}(C_\bullet) \) \cite{Lur18}. The incarnation of the functor \( N^{dg} \) we will be using is as follows. For every \( n \geq 0 \), \( N^{dg}(C_\bullet)_n \) is the set of all pairs
\[
((x_i)_{i=0}^n, (a_{i,j}))
\]
where for all \( i \in [n] \), \( x_i \) is an object of \( C_\bullet \) and for every subset \( I = \{i_0, \ldots, i_m\} \subseteq [n] \) with \( \ell(I) = m \geq 1 \), \( a_I \in \mathcal{C}_{m-1}(x_{i_0}, x_{i_m}) \) satisfying
\[
\partial(a_I) = \sum_{j=1}^{m-1} (-1)^j a_I(i,j) + (-1)^{m+1} a_{\{i_j, \ldots, i_m\}} \circ a_{\{i_0, \ldots, i_{j-1}\}}
\]
as it appears in \cite{Lur18} \text{Tag 00PL}.

We finish this section by showing that the linear dg-nerve can indeed be considered as a linear enhancement of \( N^{dg} \), in the sense that the following diagram commutes up to natural isomorphism:

\[
\begin{array}{ccc}
\text{QCat}(k) & \xrightarrow{N^{dg}_k} & \text{dg Cat}_{\geq 0}(k) \\
\downarrow U & & \downarrow N^{dg} \\
\text{QCat} & & \text{QCat}
\end{array}
\]

**Definition 5.15.** Let \( \mathcal{A} \) be a small \( S_\bullet \text{-Mod}(k) \)-enriched category, with composition \( m_{p,q} : \mathcal{A}_p(x, y) \otimes \mathcal{A}_q(y, z) \to \mathcal{A}_{p+q-1}(x, z) \). We define a simplicial set \( S(\mathcal{A}) \) as follows. For \( n \geq 0 \), let \( S(\mathcal{A})_n \) denote the set of all pairs
\[
((x_i)_{i=0}^n, (a_{i,j}))_{0 \leq i < j \leq n}
\]
with \( x_0, \ldots, x_n \) objects of \( \mathcal{A} \) and \( a_{i,j} \in \mathcal{A}_{i-j}(x_i, x_j) \) for all \( 0 \leq i < j \leq n \).

For \( 0 \leq l \leq n \), the face map \( d_l : S(\mathcal{A})_n \to S(\mathcal{A})_{n-1} \) sends a pair \((x_i)_l, (a_{i,j})_{i,j})\) to \((y_i)_l, (b_{i,j})_{i,j})\), with \( y_i = x_{b_i(i)} \) for all \( 0 \leq i \leq n \), and
\[
b_{i,j} = \begin{cases} a_{i+1,j+1} & \text{if } l \leq i \\ d_{l-1}^A(a_{i,j}) + m_{l-i-j+1}(a_{i,l} \otimes a_{l,j+1}) & \text{if } i < l \leq j \\ a_{i,j} & \text{if } j < l \end{cases}
\]
Similarly, the degeneracy map \( s_i : \mathcal{S}(\mathcal{A})_n \to \mathcal{S}(\mathcal{A})_{n+1} \) sends a pair \((x_i), (a_{i,j})_{i,j}\) to \((y_i), (b_{i,j})_{i,j}\), with \( y_i = x_{\sigma_i(i)} \) for all \( 0 \leq i \leq n \), and

\[
 b_{i,j} = \begin{cases} 
 a_{i-1,j-1} & \text{if } l < i \\
 s_{i-1}^A(a_{i,j}) & \text{if } i < l < j - 1 \\
 a_{i,j} & \text{if } j \leq l \\
 \text{id}_{x_i} & \text{if } l = i = j - 1 \\
 0 & \text{otherwise}
\end{cases}
\]

Given a \( S_\ast \text{-Mod}(k) \)-enriched functor \( H : \mathcal{A} \to \mathcal{B} \), we define a simplicial map \( \mathcal{S}(H) : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{B}) \) by setting for all \( n \geq 0 \):

\[
 \mathcal{S}(H)_n : \mathcal{S}(\mathcal{A})_n \to \mathcal{S}(\mathcal{B})_n : ((x_i), (a_{i,j})_{i,j}) \mapsto ((H(x_i))_i, (H_{j-l}(a_{i,j}))_{i,j})
\]

This defines a functor \( \mathcal{S} : \text{Cat}(S_\ast \text{-Mod}(k)) \to \text{SSet} \).

**Proposition 5.16.** There is a natural isomorphism

\[
 \tilde{U} \circ (-) \simeq \mathcal{S}
\]

where we suppressed the forgetful functor \( S_{\otimes}^{\text{Frob}} \text{-Mod}(k) \to \text{QCat}(k) \) in the notation.

**Proof.** Let \( \mathcal{A} \) be a small \( S_\ast \text{-Mod}(k) \)-enriched category and \( n \geq 0 \). Write \( \mu \) for the comultiplication of \( \mathcal{A} \). Then take \( x_0, \ldots, x_n \in \text{Ob}(\mathcal{A}) \) and an arbitrary family \( (\alpha_{i,j})_{0 \leq i < j \leq n} \) with

\[
 \alpha_{i,j} = (a_{i,j}^I)_{I \in \mathcal{P}_n} \in \mathcal{A}_{j-1}(x_i, x_j) = \bigoplus_{I \in \mathcal{P}_n} \mathcal{A}_I(x_i, x_j)
\]

and \( a_{i,j}^I \in \mathcal{A}_{i-1}(x_i, x_i) \otimes \cdots \otimes \mathcal{A}_{j-m-1}(x_{i_{m-1}}, x_{j}) \subseteq \mathcal{A}_I(x_i, x_j) \), for all \( I = \{i = i_0 < \cdots < i_{m-1} < i_m = j\} \). Then for any \( 0 \leq i < k < j \leq n \),

\[
 \mu_{k-i-j-k}(\alpha_{i,j}) = \alpha_{i,k} \otimes \alpha_{k,j} \iff \forall I \in \mathcal{P}_{k-i}, \forall J \in \mathcal{P}_{j-k} : a_{i,j}^{I+J} = a_{i,k}^I \otimes a_{k,j}^J
\]

Consequently, the family \( (\alpha_{i,j})_{i,j} \) is completely determined by \( (a_{i,j}^{i,j})_{i,j} \), whereby we have a bijection

\[
 \tilde{U}(\mathcal{A})_n \to \mathcal{S}(\mathcal{A})_n : ((x_i)_i^n, (\alpha_{i,j})_{i,j}) \mapsto ((x_i)_i^n, (a_{i,j}^{i,j})_{i,j})
\]

Next, we check that this bijection is natural in \( n \). Take \( 0 \leq l \leq n + 1 \) and an element \( ((x_i)_i^n, (\alpha_{i,j})_{i,j}) \) of \( \tilde{U}(\mathcal{A})_{n+1} \) as above. Then the face map \( d_l : \tilde{U}(\mathcal{A})_{n+1} \to \tilde{U}(\mathcal{A})_n \) sends this element to the pair \( ((x_{\delta_l(i)})_i^n, (\beta_{l,j})_{0 \leq i \leq j \leq n}) \), where

\[
 \beta_{i,j} = \begin{cases} 
 \alpha_{i+1,j+1} & \text{if } l \leq i \\
 d_{l-1}^a(\alpha_{i,j+1}) & \text{if } i < l \leq j \\
 \alpha_{i,j} & \text{if } j < l
\end{cases}
\]

So, applying the bijection above to \( \beta_{i,j} \) gives precisely the \( l \)th face of \( ((x_i)_i, (a_{i,j}^{i,j})_{i,j}) \in \mathcal{S}(\mathcal{A})_{n+1} \). The case for the degeneracy maps is proven similarly.
Finally, the naturality in $\mathcal{A}$ follows immediately from the definitions.

Next, we compare $N^\mathcal{dg}(\mathcal{C}_\bullet)$ with $\mathcal{S}(\tilde{\Gamma}(\mathcal{C}_\bullet)^F)$ for a given small dg-category $\mathcal{C}_\bullet$ over $k$. Fix $n \geq 0$ and $x_0, \ldots, x_n \in \text{Ob}(\mathcal{C}_\bullet)$. Then an $n$-simplex of $\mathcal{S}(\tilde{\Gamma}(\mathcal{C}_\bullet)^F)$ is a pair $((x_i)_{i=0}^n, (\alpha_{i,j})_{0 \leq i < j \leq n})$ where $\alpha_{i,j} \in \tilde{\Gamma}(\mathcal{C}_\bullet(x_i, x_j))_{j-i-2}$. In particular, $\alpha_{i,j}$ is a family $(a_{i,j}^t)_{t \in [j-i-2]}$ with $a_{i,j}^t \in \mathcal{C}_{j}(x_i, x_j)$. By considering $I = \{i\} \cup (J + i + 1) \cup \{j\}$, we can equivalently describe $(\alpha_{i,j})_{i,j}$ as a family

\[(a_I)_I \in \bigoplus_{I=\{i_0 \ldots i_m\} \subseteq [n]} \mathcal{C}_{m-1}(x_{i_0}, x_{i_m})\]

Note that an $n$-simplex of $N^\mathcal{dg}(\mathcal{C}_\bullet)$ is just as well described by a pair $((x_i)_{i=0}^n, (a_I)_I)$ with $(a_I)_I$ a family of type $\{10\}$. Setting $a_{i} = \text{id}_{x_i} \in \mathcal{C}_0(x_i, x_i)$ for all $i \in [n]$, we can consider these families $(a_I)_I$ as ranging over all non-empty $I \subseteq [n]$.

**Remark 5.17.** For the following statements and proofs, recall Definitions $3.11$ and $3.23$. Further, we will write $m \equiv n$ to indicate that two integers $m$ and $n$ are equal modulo $2$.

**Lemma 5.18.** Let $\mathcal{C}_\bullet$ be a small dg-category over $k$ and $n \geq 0$. Fix objects $x_0, \ldots, x_n$ of $\mathcal{C}_\bullet$. Assume that for all $I \subseteq [n]$ with endpoints $i < j$, we are given elements $a_I, b_I \in \mathcal{C}_{\ell(I)-1}(x_i, x_j)$ that satisfy

\[\sum_{s \in I^c} (-1)^{\epsilon(s,I)} a_{I^s} b_{I^s} = 0\]

where $\epsilon(s, I) = \ell(I)p_I(s)+\ell(I)+p_I(s)$ for all $s \in I^c \setminus \{i\}$ and $\epsilon(i, I) \equiv 0$.

Then the families $(a_I)_I$ and $(b_I)_I$ completely determine each other, and moreover

\[((x_i)_i, (a_I)_I) \in N^\mathcal{dg}(\mathcal{A}_\bullet)_n \iff ((x_i)_i, (b_I)_I) \in \mathcal{S}(\tilde{\Gamma}(\mathcal{A}_\bullet)^F)_n\]

**Proof.** For any $I \subseteq [n]$ with endpoints $i < j$, set $d(I) = j-i$. We prove by induction on $d \geq 1$ that the families $(a_I)_{I, d(I) \leq N}$ and $(b_I)_{I, d(I) \leq N}$ completely determine each other, and that the following statements are equivalent:

\[(a_d)\] For all $\emptyset \neq I \subseteq [n]$ with $d(I) \leq d$, we have

\[\partial(a_I) = \sum_{t \in I \setminus \{i,j\}} (-1)^{\ell(t)} a_{I \setminus \{t\}} - (-1)^{\ell(I)(\ell(t)+1)} a_{I^s} a_{I^s} \]

\[(b_d)\] For all $\emptyset \neq I \subseteq [n]$ with $d(I) \leq d$, we have

\[\partial(b_I) = \sum_{t \in I \setminus \{i,j\}} (-1)^{\ell(t)+1} b_{I \setminus \{t\}}\]
For $d = 1$, we have $a_I = b_I$ by (17) and the fact that $a_{\{i\}} = b_{\{i\}} = \text{id}_{x_i}$. This further implies that $\partial(a_I) = 0$ if and only if $\partial(b_I) = 0$ and thus $(a_I)$ is equivalent to $(b_I)$.

Let now $d > 1$. Then since $a_{\{i\}} = b_{\{i\}} = \text{id}_{x_i}$, (17) implies that $a_I$ can be written in terms of $(b_J)_I$ and elements $a_J$ with $d(J) < d(I)$. It follows by the induction hypothesis that $a_I$ can be written only in terms of $(b_J)_I$. Similarly, $b_I$ is completely determined by $(a_I)_I$.

Take $I \subseteq [n]$ with endpoints $i < j$ and $d(I) = j - i \leq d$. For the rest of the proof, we will write $p = p_I$ and $m = \ell(I) \geq 1$. If either $(a_d)$ or $(b_d)$ holds, then both $(a_{d-1})$ and $(b_{d-1})$ hold by the induction hypothesis. Take $s \in I^c \setminus \{i, j\}$ and apply $(a_{d-1})$ to $I^{\geq s}$ and $(b_{d-1})$ to $I^{\leq s}$. This gives

$$
\partial(a_I^{\geq s}) = \sum_{t \in I^{\geq s} \setminus \{s, J\}} (-1)^{p_{I^{\geq s}(t)} + (t)} a_{I^{\geq s}(t)} + (-1)^{\ell(I^{\geq s})(p_{I^{\geq s}(t)} + (t) + 1)} a_{I^{\geq s}(t)} \geq a_{I^{\geq s}(t)} \leq t
$$

$$
\partial(b_I^{\leq s}) = \sum_{t \in I^{\leq s} \setminus \{s, J\}} (-1)^{p_{I^{\leq s}(t)} + (t)} b_{I^{\leq s}(t)} = \sum_{t \in I^{\leq s} \setminus \{s, J\}} (-1)^{p_{I^{\leq s}(t)} + (t)} b_{I^{\leq s}(t)}
$$

Hence, applying $\partial$ to equation (17) gives

$$
\partial(a_I) + (-1)^m \partial(b_I) + \sum_{s \in I^c, t \in I} (-1)^{mp(s) + m + p(t)} a_{I^{\geq s}(t)} \geq a_{I^{\leq s}(t)} \leq s
$$

(18) 

$$
+ \sum_{s \in I^c, t \in I} (-1)^{mp(s) + p(t)(p(t) + m + p(s) + p(t)} a_{I^{\geq s}(t)} \geq a_{I^{\leq s}(t)} \leq s
$$

$$
+ \sum_{s \in I^c, t \in I} (-1)^{mp(s) + p(t)} a_{I^{\geq s}(t)} \geq a_{I^{\leq s}(t)} \leq s = 0
$$

where we used the graded Leibniz rule and the fact that $a_{I^{\geq s}}$ has degree $\ell(I^{\geq s}) - 1 = m - p(s)$. We can simplify this equation by using (17).

Let $t \in I \setminus \{i, j\}$ be arbitrary. Then for $s \in (I \setminus \{t\})^c = I^c \cup \{t\}$, we have $p_{I \setminus \{t\}}(s) = p(s)$ if $s \leq t$ and $p_{I \setminus \{t\}}(s) = p(s) - 1$ if $s > t$. Since also $\ell(I \setminus \{t\}) = m - 1$, it follows that

$$
\epsilon(s, I \setminus \{t\}) \equiv \begin{cases} 
mp(s) + m + 1 & \text{if } s \leq t \\
mp(s) + 1 & \text{if } s > t
\end{cases}
$$
Hence, equation (18) reduces to
\[ \sum_{s \leq t, i \leq s} (-1)^{mp(s)+m+1} a_{(I \setminus \{t\}) \leq i} b_{I \leq s} + \sum_{s \leq t, i < j} (-1)^{mp(s)+1} a_{I \leq i} b_{(I \setminus \{t\}) \leq s} = -a_{I \setminus \{t\}} + (-1)^{mp(t)+m} a_{I \geq t} b_{I \leq t} + (-1)^m b_{I \setminus \{t\}} \]

Thus applying equation (17) to \( I \setminus \{t\} \), we get
\[ \sum_{s \leq t, i < t} (-1)^{mp(s)+m+1} a_{(I \setminus \{t\}) \leq i} b_{I \leq s} + \sum_{s \leq t, i < s} (-1)^{mp(s)+1} a_{I \leq i} b_{(I \setminus \{t\}) \leq s} \]

Hence, equation (18) reduces to
\[
\begin{align*}
\partial(a_I) + (-1)^n \partial(b_I) + 
& \sum_{i \in I \setminus \{i,j\}} (-1)^{p(i)} \left( -a_{I \setminus \{i\}} + (-1)^{mp(t)+m} a_{I \geq t} b_{I \leq t} + (-1)^m b_{I \setminus \{t\}} \right) \\
& + \sum_{s \leq t, i < s < t} (-1)^{mp(s)+p(t)+mp(s)+t} a_{I \geq s} a_{I \leq t} b_{I \leq s} = 0
\end{align*}
\]

Again for \( t \in I \setminus \{i,j\} \) arbitrary, we now consider \( I^{=t} \). Note that \( \ell(I^{=t}) = p(t) \) and for all \( s \in (I^{=t})^c = (I')^{=t} \), we have \( p_{I^{=t}}(s) = p(s) \). Consequently, applying equation (17) to \( I^{=t} \) yields
\[
\sum_{i \leq s < t} (-1)^{p(s)p(t)} a_{(I^{=t}) \leq s} b_{I \leq s} + (-1)^{p(t)} b_{I \leq t} = -a_{I \leq t}
\]

So equation (19) further reduces to
\[
\begin{align*}
\partial(a_I) - 
& \sum_{i \in I \setminus \{i,j\}} \left( (-1)^{p(i)} a_{I \setminus \{i\}} + (-1)^{mp(t)+1} a_{I \geq t} a_{I \leq t} \right) \\
& = (-1)^{n+1} \left( \partial(b_I) - \sum_{i \in I \setminus \{i,j\}} (-1)^{p(t)+1} b_{I \setminus \{t\}} \right)
\end{align*}
\]

from which conclude that \((a_d)\) is equivalent to \((b_d)\). \( \square \)

**Proposition 5.19.** There is a natural isomorphism
\[
\mathcal{S} \circ \sharp \circ \tilde{\Gamma} \simeq N^{dg}
\]

**Proof.** Let \( \mathcal{C}_\bullet \) be a small dg-category over \( k \). Lemma 5.18 provides a bijection between \( N^{dg}(\mathcal{C}_\bullet)_n \) and \( \mathcal{S}(\tilde{\Gamma}(\mathcal{C}_\bullet))_n \) for every \( n \geq 0 \). We prove that it is natural in \( n \). Take \( n \geq 0 \) and \( t \in [n+1] \). Then the face map \( d_t : N^{dg}(\mathcal{C}_\bullet)_{n+1} \to N^{dg}(\mathcal{C}_\bullet)_n \) is given by
\[
d_t \left( \left( x_i \right)_{i=0}^{n+1}, (a_j)_{j \leq [n+1]} \right) = \left( \left( x_{\delta(i)} \right)_{i=0}^{n}, (a'_{I})_{I \leq [n]} \right)
\]
with \( a'_I = a_{\delta(I)} \) for all \( I \subseteq [n] \) with \( \ell(I) \geq 1 \). Further, from the definitions we see that \( d_t : \mathcal{S}(\tilde{\Gamma}(\mathcal{C}_\bullet))_{n+1} \to \mathcal{S}(\tilde{\Gamma}(\mathcal{C}_\bullet))_n \) is given by
\[
d_t \left( \left( x_i \right)_{i=0}^{n+1}, (b_j)_{j \leq [n+1]} \right) = \left( \left( x_{\delta(i)} \right)_{i=0}^{n}, (b'_I)_{I \leq [n]} \right)
\]
where, for all \( I \subseteq [n] \) with endpoints \( i < j \):
\[
b'_I = \begin{cases} 
  b_{\delta(I)} + (-1)^{\ell(I)(p(I)+1)} b_{(\delta(I)) \geq I} & \text{if } i < l \leq j \\
  b_{\delta(I)} & \text{otherwise}
\end{cases}
\]
Now for all \( s \leq l \) and \( p_{\ell_1}(s) = p_1(s) \) if \( s \leq l \) and \( p_{\ell_1}(s) = p_1(s-1) \) if \( s > l \). Consequently if \( i < l \leq j \), then \( \ell(\delta I)^{\leq l} = \ell(I) \) and \( \ell(\delta I)^{\geq l} = \ell(I) - p_1(l) + 1 \). The sign in the expression for \( b'_I \) is present because

\[
m_{c^*}(b_{(\delta I)^{\leq l}} \otimes b_{(\delta I)^{\geq l}}) = (-1)^{((\delta I)^{\leq l})-1)(\ell((\delta I)^{\geq l})-1)} b_{(\delta I)^{\geq l}} b_{(\delta I)^{\leq l}} = (-1)^{\ell(l)(p_1(l)-1)} b_{(\delta I)^{\geq l}} b_{(\delta I)^{\leq l}}.
\]

To prove that the bijections of Lemma 5.18 respect the face maps, we must show that if \( (a_J)_{J \subseteq [n+1]} \) and \( (b_J)_{J \subseteq [n+1]} \) satisfy equation (17), then so do \( (a'_I)_{I \subseteq [n]} \) and \( (b'_I)_{I \subseteq [n]} \).

Fix \( I \subseteq [n] \) with endpoints \( i < j \) and put \( p = p_I \) and \( m = \ell(I) \geq 1 \). We may assume that \( i < l \leq j \), as otherwise the conclusion is clear.

Consider

\[
\sum_{s \in I^c} (-1)^{\epsilon(s, I)} a_{(\delta I)^{\geq s}} b'_{(\delta I)^{\leq s}} = \sum_{s \in (\delta I)^{c_i} \cap [s]} (-1)^{\epsilon(s, I)} a_{(\delta I)^{\geq s}} b_{(\delta I)^{\leq s}} + \sum_{s \in (\delta I)^{c_i} \cap [s]} (-1)^{\epsilon(s, I)} a_{(\delta I)^{\geq s}} b_{(\delta I)^{\leq s}} + \sum_{s \in (\delta I)^{c_i} \cap [s]} (-1)^{\epsilon(s, I) + p(s-1)(p(l)-1)} a_{(\delta I)^{\geq s}} b_{(\delta I)^{\leq s}} \]

where we used that \( \ell(I^{\leq s-1}) = p(s-1) \) and \( p_{(\delta I)^{\leq s-1}}(l) = p(l) \) if \( s > l \).

Now for all \( s \in (\delta I)^{c_i} \) we have, \( \epsilon(s, \delta I) \equiv \epsilon(s, I) \) if \( s < l \) and \( \epsilon(s, \delta I) \equiv \epsilon(s-1, I) \) if \( s > l \). Therefore, the above expression becomes

\[
\sum_{s \in (\delta I)^{c_i} \cap [s]} (-1)^{\epsilon(s, \delta I)} a_{(\delta I)^{\geq s}} b_{(\delta I)^{\leq s}} + \sum_{s \in (\delta I)^{c_i} \cap [s]} (-1)^{\epsilon(s, \delta I) + p(s-1)(p(l)-1) + m} a_{(\delta I)^{\geq s}} b_{(\delta I)^{\leq s}} \]

Now, using equation (17) applied to \( \delta I \), this is equal to

\[
= (-1)^{\ell(l, \delta I)} a_{(\delta I)^{\geq s}} b_{(\delta I)^{\leq s}} + \sum_{s \in (\delta I)^{c_i} \cap [s]} (-1)^{mp(s-1)+p(s-1)p(l)+mp(l)} a_{(\delta I)^{\geq s}} b_{(\delta I)^{\leq s}} \]

where we used that \( \epsilon(l, \delta I) + 1 = (m + 1)(p(l) + 1) \) and \( \epsilon(s, (\delta I)^{\geq l}) = mp(s-1) + p(s-1)p(l) + mp(l) + p(l) + 1 \) for all \( s > l \). In the last equality, we used equation (17) applied to \( (\delta I)^{\geq l} \).
Further, the degeneracy map given by
\[
\sigma \mapsto a = \sum_{\sigma \in \mathcal{C}} \varepsilon_{\sigma} a_{\sigma,l}
\]
where, for all \( I \subseteq [n+1] \) with endpoints \( i < j \):
\[
a'_l = \begin{cases} 
  a_{\sigma,l} & \text{if } \sigma|_I \text{ is injective} \\
  u & \text{if } l = i = j - 1 \\
  0 & \text{otherwise}
\end{cases}
\]

Further, the degeneracy map \( s_l : \mathcal{S}(\tilde{\Gamma}(\mathcal{C}_\bullet)^{\otimes})_n \to \mathcal{S}(\tilde{\Gamma}(\mathcal{C}_\bullet)^{\otimes})_{n+1} \) is given by
\[
s_l \left( (x)_l \right) = (x_{\sigma(i)})_{I \subseteq [n+1]} = \left( (x_{\sigma(i)})_{I \subseteq [n]} , (a'_l)_{I \subseteq [n+1]} \right)
\]
where, for all \( I \subseteq [n+1] \) with endpoints \( i < j \):
\[
b'_l = \begin{cases} 
  b_{\sigma,l} & \text{if } \sigma|_I \text{ is injective and } l \neq j - 1 \\
  u & \text{if } l = i = j - 1 \\
  0 & \text{otherwise}
\end{cases}
\]

To prove that the bijections of Lemma (5.18) respect the degeneracy maps, it suffices to prove that if \((a_j)_{J \subseteq [n]} \) and \((b_j)_{J \subseteq [n]} \) satisfy equation (17), then so do \((a'_I)_{I \subseteq [n+1]} \) and \((b'_I)_{I \subseteq [n+1]} \).

Fix \( I \subseteq [n+1] \) with endpoints \( i < j \). We may assume that either \( l = i = j - 1 \) or \( i < l = j - 1 \) as otherwise the conclusion is clear. In the former case, \( I = \{i, i+1\} \) and \( a'_l - b'_l = u - u = 0 \) as desired. Assume now that \( i < l = j - 1 \). Then since \( b'_l = 0 \) and \((\sigma_{j-1}|_I)^c = I \setminus \{j\}\),
\[
\sum_{s \in I^c} (-1)^{\epsilon(s,I)}a'_{l \geq s, b'_{l \leq s}} = \sum_{s \in (\sigma_l)^c} (-1)^{\epsilon(s,I)}a'_{l \geq s, b'_{l \leq s}}
\]

Now if \( j - 1 \in I \), then \( \sigma_{j-1} \) is not injective on \( I \) and thus \( a'_{l \geq s} = 0 \) for all \( s \in (\sigma_{j-1}|_I)^c \). On the other hand, if \( j - 1 \not\in I \), then \( \sigma_{j-1} \) is injective on \( I \). Moreover, it follows that \( \epsilon(s,I) \equiv \epsilon(s,\sigma_{j-1}|_I) \) for all \( s \in (\sigma_{j-1}|_I)^c \) and thus the above sum becomes
\[
\sum_{s \in I'} (-1)^{\epsilon(s,I)}a_{\sigma_{j-1}(I \geq s)}b_{\sigma_{j-1}(I \leq s)} = \sum_{s \in (\sigma_l)^c} (-1)^{\epsilon(s,\sigma_{j-1}|_I)}a_{\sigma_{j-1}(I \geq s)}b_{\sigma_{j-1}(I \leq s)}
\]
which is seen to be 0 by applying equation (17) to \( \sigma_{j-1}|_I \).

Thus we have constructed an isomorphism \( N^{dg}(\mathcal{C}_\bullet) \simeq \mathcal{S}(\tilde{\Gamma}(\mathcal{C}_\bullet)^{\otimes}) \) of simplicial sets. Take a dg-functor \( f : \mathcal{C}_\bullet \to \mathcal{D}_\bullet \). Then it immediately follows that for any \( n \geq 0 \) and \((a_I)_{I \subseteq [n]}, (b_I)_{I \subseteq [n]} \) satisfying equation (17), the families \((f_{I \cup \{i\}}(a_I))_{I \subseteq [n]} \) and \((f_{I \cup \{i\}}(b_I))_{I \subseteq [n]} \) satisfy it as well. Consequently, the constructed isomorphism is natural in \( \mathcal{C}_\bullet \).

\[ \square \]

**Theorem 5.20.** We have a natural isomorphism
\[ \tilde{U} \circ N^{dg}_{k} \simeq N^{dg} \]
Proof. This is now a direct consequence of Propositions 5.16 and 5.19.

\[ \square \]

Appendix A. Alternative Definition of Templicial Objects

In this appendix, we discuss an alternative definition of templicial objects. It is simpler than the one given in Definition 2.4 and doesn’t rely on quivers, but for our purposes it turned out to be less practical. We show that under mild conditions on the monoidal category $\mathcal{V}$, both definitions coincide (see Definition A.10 and Theorem A.12). As usual, $\mathcal{V}$ is assumed to be cocomplete such that the monoidal product $- \otimes -$ preserves colimits in each variable.

A.1. cc-functors.

Definition A.1. As the free functor $F : \text{Set} \to \mathcal{V}$ is strong monoidal, we have an induced functor between categories of comonoids:

$$\text{Set} \simeq \text{Comon}(\text{Set}) \to \text{Comon}(\mathcal{V})$$

Further, for any colax monoidal functor $X : \Delta^\text{op}_f \to \mathcal{V}$, $X_0$ has the structure of a comonoid in $\mathcal{V}$. We obtain a functor

$$(-)_0 : \text{Colax}(\Delta^\text{op}_f, \mathcal{V}) \to \text{Comon}(\mathcal{V})$$

We define the category $\text{Colax}_c(\Delta^\text{op}_f, \mathcal{V})$ of canonical colax monoidal functors or cc-functors by the 2-pullback

$$\text{Colax}_c(\Delta^\text{op}_f, \mathcal{V}) \twoheadrightarrow \text{Colax}(\Delta^\text{op}_f, \mathcal{V})$$

$$\downarrow$$

$$\text{Set} \twoheadrightarrow \text{Comon}(\mathcal{V})$$

$$\downarrow (-)_0$$

Explicitly, a cc-functor is a colax monoidal functor $(X, \mu, \epsilon) : \Delta^\text{op}_f \to \mathcal{V}$ equipped with an isomorphism $X_0 \simeq F(S)$ for some set $S$ such that through this isomorphism, $\mu_{0,0}$ and $\epsilon$ are induced by the diagonal $S \to S \times S$ and the terminal map $S \to \{\ast\}$ respectively. We call $S$ the base of $X$. A morphism of cc-functors $X \to Y$ with respective bases $S$ and $T$ is a monoidal natural transformation $\alpha$ such that through the above isomorphisms, $\alpha_0$ is induced by some map of sets $f : S \to T$.

We now describe a comparison functor from templicial objects to cc-functors. In the next subsection, we will give sufficient conditions on $\mathcal{V}$ for this functor to be an equivalence.

Construction A.2. Consider the natural transformation $t : 1_{\text{Set}} \to *$ given by the terminal map $t_S : S \to \{\ast\}$ for every set $S$. This induces a pseudonatural transformation

$$\Phi_\mathcal{V}t : \Phi_\mathcal{V} \to \Phi_\mathcal{V} \circ *$$
between pseudofunctors \( \text{Set} \to \text{Cat} \), where \( \Phi_V = \text{Colax}(\Delta^{op}, (-)!) \) as in Construction 2.8. Through the Grothendieck construction, we obtain a functor

\[
\mathcal{c} : \int \Phi_V \to \int \Phi_V \circ * \simeq \text{Colax}(\Delta^{op}, V) \times \text{Set}
\]

Explicitly, this functor sends a pair \((X, S)\) with \( S \) a set and \( X : \Delta^{op} \to \text{Quiv}_S(\mathcal{V}) \) colax monoidal to the pair \( (\mathcal{c}X, S) \), where

\[
\mathcal{c}X_n = (t_S)_!(X_n) = \coprod_{a,b \in S} X_n(a,b)
\]

for all \( n \geq 0 \). The comultiplication and counit are induced by those of \( X \). Moreover, a templicial morphism \((\alpha, f) : (X, S) \to (Y, T)\) is sent to the pair \( (\mathcal{c}\alpha, f) \), where for every \( n \geq 0 \),

\[
\mathcal{c}\alpha_n : \coprod_{a,b \in S} X_n(a,b) \to \coprod_{x,y \in T} Y_n(x,y)
\]

factors through \((\alpha_n)_{a,b} : X_n(a,b) \to Y_n(f(a), f(b))\) for all \( a, b \in S \).

Note that, up to equivalence, we may consider \( \text{Colax}_c(\Delta^{op}, \mathcal{V}) \) as a subcategory of \( \text{Colax}(\Delta^{op}, \mathcal{V}) \times \text{Set} \).

**Proposition A.3.** The functor \( \mathcal{c} : \int \Phi_V \to \text{Colax}(\Delta^{op}, \mathcal{V}) \times \text{Set} \) of Construction A.2 restricts to a functor

\[
\mathcal{c} : S_S \mathcal{V} \to \text{Colax}_c(\Delta^{op}, \mathcal{V})
\]

**Proof.** Note that for any set \( S \), \((t_S)_!(I_S) \simeq \coprod_{x \in S} I = F(S)\). Take an object \((X, S)\) of \( \int \Phi_V \), then the counit \( \epsilon : X_0 \to I_S \) induces a morphism

\[
\varphi_{(X,S)} : \mathcal{c}X_0 = (t_S)_!(X_0) \to F(S)
\]

in \( \mathcal{V} \). It easily follows that \( \varphi_{(X,S)} \) is a comonoid morphism which is natural in \((X, S)\). Moreover, if \((X, S)\) is a templicial object, then \( \epsilon \) and thus \( \varphi_{(X,S)} \) is an isomorphism. \( \square \)

**A.2. Decomposing monoidal categories.** We now describe how to invert the comparison functor \( \mathcal{c} : S_S \mathcal{V} \to \text{Colax}_c(\Delta^{op}, \mathcal{V}) \). For this we need to “pull apart” the objects \( X_n \in \mathcal{V} \) of a cc-functor to form a quiver. This goes as follows.

**Construction A.4.** Let \( X : \Delta^{op}_f \to \mathcal{V} \) be a cc-functor with comultiplication \( \mu \) and base \( S \). Via the isomorphism \( X_0 \simeq \coprod_{a \in S} I \), we have for every \( n \geq 0 \), a morphism

\[
\mu_{0,n,0} : X_n \to X_0 \otimes X_n \otimes X_0 \simeq \coprod_{a,b \in S} X_n
\]

which assemble into a natural transformation \( \mu_{0,-,0} : X \to \coprod_{a,b} X \).
Assume \( V \) has equalizers, then define \( X(a, b) \) as the equalizer

\[
\begin{array}{ccc}
X(a, b) & \xrightarrow{c_{a,b}} & X \xleftarrow{\prod_{a,b\in S}} \coprod_{a,b\in S} X
\end{array}
\]

in \( \text{Fun}(\Delta^\text{op}_f, V) \), where \( c_{a,b} \) is the \((a,b)\)th coprojection.

**Lemma A.5.** Let \( X \) be a cc-funtor with comultiplication \( \mu \) and base \( S \). Let \( \nabla : \coprod_{a,b\in S} X \to X \) be the codiagonal. Then

\[
\left( \prod_{a,b\in S} \mu_{0,-,0} \right) \mu_{0,-,0} = \left( \prod_{a,b\in S} c_{a,b} \right) \mu_{0,-,0} \quad \text{and} \quad \nabla \mu_{0,-,0} = \text{id}_X
\]

**Proof.** Note that through the isomorphism \( X_0 \simeq \coprod_{a\in S} I \), the counit \( \epsilon : X_0 \to I \) becomes the codiagonal. Moreover, for all \( n \geq 0 \), the morphisms \( \text{id}_{X_0} \otimes \mu_{0,n,0} \otimes \text{id}_{X_0} \) and \( \mu_{0,0} \otimes \text{id}_{X_n} \otimes \mu_{0,0} \) become \( \coprod_{a,b} \mu_{0,n,0} \) and \( \coprod_{a,b} c_{a,b} \) respectively. Thus the result follows from the counitality and coassociativity of \( \mu \) and \( \epsilon \). \( \square \)

The previous lemma leads us to define the following.

**Definition A.6.** Let \( C \) be a category with coproducts. Let \( I \) be a set and \( A \in C \). We denote \( \iota_j : A \to \coprod_{i\in I} A \) for the \( j \)th coprojection and \( \nabla : \coprod_{i\in I} A \to A \) for the codiagonal. A morphism \( f : A \to \coprod_{i\in I} A \) is called **decomposing** if

\[
\left( \prod_{i\in I} f \right) f = \left( \prod_{i\in I} \iota_i \right) f \quad \text{and} \quad \nabla f = \text{id}_A
\]

A **decomposing equalizer** is the equalizer of a decomposing morphism with a coprojection \( \iota_j \).

Recall that a diagram

\[
E \xrightarrow{e} A \xleftarrow{f} B \xrightarrow{g}
\]

in a category \( C \) such that \( fe = ge \) is called a **split equalizer** if there exist morphisms \( p : B \to A \) and \( s : A \to E \) in \( C \) such that

\[
se = \text{id}_E \quad pg = \text{id}_A \quad pf = es
\]

The diagram is called a **coreflexive equalizer** if \( e \) is the equalizer of \( f \) and \( g \) and there exists a morphism \( p : B \to A \) such that \( pf = pg = \text{id}_A \).

A split equalizer is always an equalizer. Moreover, it is an absolute limit, meaning that it is preserved by every functor with domain \( C \).

**Remark A.7.** Any coprojection \( \iota_j : A \to \coprod_i A \) is itself decomposing.

Further note that because of the condition \( \nabla f = \text{id}_A \), a decomposing equalizer is always coreflexive.
Lemma A.8. Let $C$ be a category with coproducts and consider a decomposing morphism $f : A \to \coprod_{i \in I} A$. Then

$$A \xrightarrow{f} \coprod_{i \in I} A \xrightarrow{\bigoplus_i f} \coprod_{i \in I} \coprod_{j \in I} A$$

is a split equalizer.

Proof. Let $\nabla : \coprod_i \coprod_j A \to \coprod_j A$ denote the codiagonal which collapses the outer coproduct. Then it immediately follows that $\nabla \coprod_i f = f \nabla$ and $\nabla \coprod_i \epsilon_i = \text{id}$. By hypothesis, we also have $\nabla f = \text{id}_A$. □

Proposition A.9. Suppose that $\mathcal{V}$ has equalizers and that coproducts commute with decomposing equalizers in $\mathcal{V}$. Let $X$ be a cc-functor with base $S$, comultiplication $\mu$ and counit $\epsilon$. Then:

(a) The canonical natural transformation $(e_{a,b})_{a,b} : \coprod_{a,b \in S} X(a,b) \to X$

is an isomorphism.

(b) If coproducts are disjoint in $\mathcal{V}$, then for all $a,b \in S$, the composition

$$X_0(a,a) \xrightarrow{e_{a,a}} X_0 \xrightarrow{\epsilon} I$$

is an isomorphism, and $X_0(a,b) \simeq 0$ if $a \neq b$.

(c) If the monoidal product $\otimes$ of $\mathcal{V}$ preserves decomposing equalizers in each variable, then for all $k,l \geq 0$ and $a,b \in S$, the map $\mu_{k,l} e_{a,b}$ factorizes uniquely as

$$X_{k+l+1}(a,b) \xrightarrow{\mu_{k,l}^a} \coprod_{c \in S} X_k(a,c) \otimes X_l(c,b) \xrightarrow{(e_{a,c} \otimes e_{c,b})_c} X_k \otimes X_l$$

Proof. (a) By Lemma A.5, $\coprod_{a,b \in S} X(a,b)$ is the equalizer of $\coprod_{a,b} \mu_{0,-,0}$ and $\coprod_{a,b} e_{a,b}$. Hence by Lemma A.8, it is isomorphic to $X$. More precisely, for the isomorphism $\varphi : \coprod_{a,b} X(a,b) \xrightarrow{\simeq} X$ we have $\coprod_{a,b} e_{a,b} = \mu_{0,-,0} \varphi$ and thus as $\epsilon$ coincides with the codiagonal $\nabla : \coprod_a I \to I$, we get $\varphi = (e_{a,b})_{a,b}$.

(b) As coproducts are disjoint we have an equalizer diagram

$$I_{a,x,b} \xrightarrow{\epsilon_{a,x}} I \xrightarrow{\epsilon_{a,b}} \coprod_{y,z \in S} I$$

where $I_{a,x,b} = I$ if $a = b = x$ and $I_{a,x,b} = 0$ otherwise. Taking the coproduct of this diagram over all $x \in S$, we find an equalizer

$$I_{a,b} \xrightarrow{\bigoplus_{z \in S} \epsilon_{x,z,x}} \coprod_{x \in S} I \xrightarrow{\bigoplus_{y,z \in S} \epsilon_{y,x,z}} \coprod_{x,z \in S} I$$

where $I_{a,b} = I$ if $a = b$ and $I_{a,b} = 0$ if $a \neq b$. Now via the isomorphism $X_0 \simeq \coprod_x I$, $\mu_{0,0,0}$ becomes $\coprod_x \epsilon_{x,x}$ and thus we have
an isomorphism $\varphi : X_0(a, b) \to I_{a, b}$ such that $\iota_{a, b} \varphi = e_{a, b}$. As $\epsilon$ coincides with the codiagonal $\nabla$, we find that $\varphi = \epsilon e_{a, b}$.

(c) Note that since decomposing equalizers are coreflexive, and they are preserved by $- \otimes -$ in each variable, they are also preserved in both variables together (use Lemma 4.2 of [BW05] for example).

It then follows from Lemma A.5 that the morphism

$$\coprod_{c \in S} X_k(a, c) \otimes X_l(c, b) \xrightarrow{\coprod_{c \in S} e_{a, c} \otimes e_{c, b}} \coprod_{c \in S} X_k \otimes X_l$$

is the equalizer of $\coprod_{c} \mu_{0, k, 0} \otimes \mu_{0, l, 0}$ and $\coprod_{c} c_{a, c} \otimes c_{c, b}$. Using the isomorphism $X_0 \simeq \coprod_{a,b} I$, we see that this is equivalently the equalizer of $\mu_{0, k, 0} \otimes \id_{X_0} \otimes \mu_{0, l, 0}$ and $c_{a, *} \otimes \mu_{0, 0, 0} \otimes c_{b, *}$, where

$$c_{a, *} : X_k \simeq I \otimes X_k \xrightarrow{\iota_{a} \otimes \id_{X_k}} \coprod_{a \in S} I \otimes X_k \simeq X_0 \otimes X_k$$

and similarly for $c_{b, *}$.

Now note that for the maps $c_{a, b} : X_{k+1} \to X_0 \otimes X_{k+1} \otimes X_0$ and $e_{a, b} : X_{k+1}(a, b) \to X_{k+1}$, we have

$$(\mu_{0, k, 0} \otimes \id_{X_0} \otimes \mu_{0, l, 0}) e_{a, b} = (\id_{X_0} \otimes \mu_{k, 0, 0, 0} \otimes \id_{X_0}) \mu_{0, k, 0} e_{a, b}$$

$$= (\id_{X_0} \otimes \mu_{k, 0, 0, 0} \otimes \id_{X_0}) (c_{a, 0} \otimes \mu_{0, 0, 0} \otimes c_{b, *}) \mu_{0, l, 0} e_{a, b}$$

Thus there is a unique $\mu_{a, b} : X_{k+1}(a, b) \to \coprod_{c \in S} X_k(a, c) \otimes X_l(c, b)$ such that $(\coprod_{c} e_{a, c} \otimes e_{c, b}) \mu_{a, b} = \mu_{0, 0} e_{a, b}$. Composing this equality with the codiagonal $\coprod_{c} X_k \otimes X_l \to X_k \otimes X_l$, the result follows.

\begin{definition}
Suppose $\mathcal{V}$ has equalizers. We call $\mathcal{V}$ decomposing if it satisfies the hypotheses of Proposition A.9(a)-(c), that is:

- coproducts commute with decomposing equalizers in $\mathcal{V}$,
- coproducts are disjoint in $\mathcal{V}$,
- the monoidal product $- \otimes -$ of $\mathcal{V}$ preserves decomposing equalizers in each variable.
\end{definition}

\begin{construction}
Let $\mathcal{V}$ be decomposing. We construct a functor $\mathfrak{d} : \text{Colax}_c(\Delta^{op}_f, \mathcal{V}) \to S \otimes \mathcal{V}$

Take a cc-functor $X$ of $\mathcal{V}$ with base $S$, comultiplication $\mu$ and counit $\epsilon$. From Construction A.4 we have a collection of functors $(X(a, b) : \Delta^{op}_f \to \mathcal{V})_{a, b \in S}$, which we can regard as a functor

$$\tilde{X} : \Delta^{op}_f \to \text{Quiv}_S(\mathcal{V})$$

By Proposition A.9(b), we have a quiver isomorphism $\tilde{\epsilon} : \tilde{X}_0 \tilde{\simeq} I_S$, and the maps $\mu_{a, b}^{a, b}$ of Proposition A.9(c) combine to give a quiver morphism

$$\tilde{\mu}_{a, b} : \tilde{X}_{k+1} \to \tilde{X}_k \otimes_S \tilde{X}_l$$
It follows from the coassociativity and counitality of $\mu$ and $\epsilon$ that $\tilde{\mu}$ and $\tilde{\epsilon}$ define a strongly unital colax monoidal structure on $\tilde{X}$ and thus $(\tilde{X}, S)$ is a templicial object in $\mathcal{V}$.

Next, let $X$ and $Y$ be cc-functors of $\mathcal{V}$ with respective bases $S$ and $T$. Let $\alpha : X \to Y$ be a morphism of cc-functors. As $\alpha$ is a monoidal natural transformation, there exist unique $\alpha^{a,b} : X(a, b) \to Y(f(a), f(b))$ such that $e_{f(a), f(b)} \alpha^{a,b} = \alpha e_{a,b}$, for all $a, b \in S$. This defines a natural transformation $\tilde{X} \to f^* \tilde{Y}$. It further follows from the monoidality of $\alpha$ that the corresponding natural transformation $\tilde{\alpha} : f^! \tilde{X} \to \tilde{Y}$ is monoidal. Hence, $(\tilde{\alpha}, f)$ is a morphism of templicial objects $\tilde{X} \to \tilde{Y}$.

If further $\beta : Y \to Z$ is a morphism of cc-functors, then by uniqueness, $(\beta \circ \alpha)^{a,b} = \beta f(a), f(b) \circ \alpha^{a,b}$ for all $a, b \in S$. It follows that the assignments $X \mapsto (\tilde{X}, S)$ and $\alpha \mapsto (\tilde{\alpha}, f)$ define a functor.

**Theorem A.12.** Suppose $\mathcal{V}$ is decomposing. Then we have an adjoint equivalence of categories

$$ S \otimes \mathcal{V} \xrightarrow{\varepsilon_S} \text{Colax}_c(\Delta^{op}_f, \mathcal{V}) $$

**Proof.** The isomorphism of Proposition A.9(a) is monoidal by (b) and (c). Moreover, it is directly seen to be natural in $\mathcal{V}$. Thus $\varepsilon_\mathcal{V} \circ \delta \simeq \text{id}$.

Let $(X, S)$ be a templicial object of $\mathcal{V}$. For every $a, b \in S$, we have a functor $X(a, b) : \Delta^{op}_f \to \mathcal{V}$. As coproducts are disjoint in $\mathcal{V}$, the equalizer of $\iota_{a,b}, \iota_{c,d} : X(c, d) \to \bigsqcup_{x,y \in S} X(c, d)$ is $X(a, b)$ if $(c, d) = (a, b)$ and 0 otherwise. Because coproducts commute with decomposing equalizers, we get an equalizer diagram

$$ X(a, b) \xrightarrow{\iota_{a,b}} \bigsqcup_{c,d \in S} X(c, d) \xrightarrow{\coprod_{c,d \in S} \iota_{c,d}} \bigsqcup_{c,d \in S} \bigsqcup_{x,y \in S} X(c, d) $$

Now $\bigsqcup_{c,d} X(c, d)$ is the functor underlying $c(X, S)$ and the morphisms $\coprod_{c,d} \iota_{c,d}$ and $\coprod_{c,d} \iota_{a,b}$ correspond to the induced maps $\mu_{0,-}, 0$ and $c_{a,b}$ on $c(X, S)$ respectively. Consequently, we have an isomorphism between the underlying functors of $(X, S)$ and $\mathcal{D} c(X, S)$. It follows from the definitions that this isomorphism is monoidal and that it is natural in $(X, S)$. Therefore $\delta \circ \varepsilon \simeq \text{id}$.

Finally, the triangle identities are easily verified. \qed

We finish this section by giving some examples of monoidal categories that are decomposing, and thus for which Theorem A.12 is applicable.

**Example A.13.** In a cartesian category $\mathcal{V}$, the product $- \times -$ commutes with all equalizers. So if we assume that coproducts are disjoint and commute with equalizers, then $\mathcal{V}$ is decomposing.

This is the case for Set, Top, Cat and Poset for example.

**Lemma A.14.** Let $\mathcal{C}$ be a category enriched over abelian groups. Then any decomposing equalizer in $\mathcal{C}$ is split.
Proof. Let $f : A \to \bigoplus_{i \in I} A$ be a decomposing morphism in $C$ and fix $j \in I$. Consider the equalizer $e : E \to A$ of $f$ and $\iota_j$. Then for the $j$th projection $p : \bigoplus_{i \in I} A \to A$ we have $p\iota_j = \text{id}_A$ and

$$fpf = p' \left( \bigoplus_{i \in I} f \right) f = p' \left( \bigoplus_{i \in I} \iota_i \right) f = \iota_j pf$$

where $p' : \bigoplus_{i,k} A \to \bigoplus_k A$ is the projection onto the component $i = j$. So there exists a unique $s : A \to E$ such that $es = pf$. Then, $ese = pfe = p\iota_j e = e$ and thus $se = \text{id}_E$ because $e$ is a monomorphism. □

Proposition A.15. If $\mathcal{V}$ is enriched over abelian groups and has kernels, then $\mathcal{V}$ is decomposing.

Proof. By Lemma A.14, decomposing equalizers in $\mathcal{V}$ are split equalizers and are thus preserved by all functors. In particular, both the coproduct functor $\mathcal{V}^I \to \mathcal{V}$ and the monoidal product $- \otimes -$ preserve decomposing equalizers. Further, in an Ab-enriched category, coproducts are always disjoint. □

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