STABILITY FOR THE MODIFIED AND FOURTH-ORDER BENJAMIN-BONA-MAHONY EQUATIONS

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(Communicated by Jerry Bona)

Abstract. In this work we establish new results about the existence of smooth, explicit families of periodic traveling waves for the modified and fourth-order Benjamin-Bona-Mahony equations. We also prove, under certain conditions, that these families are nonlinearly stable in the energy space. The techniques employed may be of further use in the study of the stability of periodic traveling-wave solutions of other nonlinear evolution equations.

1. Introduction. Nonlinearity is a prevalent feature of mathematical models of natural phenomena. One active area where nonlinearity is centrally important is the theory of nonlinear dispersive evolution equations. In particular the well-known traveling-wave solutions (solitary wave, cnoidal waves) owe their existence to a balance between nonlinearity and dispersion. The study of existence and stability/instability of these special states of motion turn out to be very important to the understanding of phenomena observed in various scientific fields such as fluid mechanics, plasma physics and nonlinear optics. In the present essay, interest will focus upon periodic traveling-wave solutions of certain nonlinear, dispersive wave equations to be introduced presently. The first work about the existence and orbital stability of periodic traveling waves was that of Benjamin [9], a study of periodic waves of cnoidal type for the Korteweg-de Vries equation. Certain lacunae in the stability theory developed in [9] were recently addressed by Angulo, Bona and Scialom [8]. In the last years, several papers about periodic traveling-wave solutions have appeared in the literature dealing with existence, properties, nonlinear stability/instability and spectral stability of solutions. The spectral stability is quite an interesting aspect of this kind of solution, see for instance [2, 3, 4, 6, 7, 8, 17, 18, 19, 23].

2000 Mathematics Subject Classification. Primary: 35Q53; Secondary: 35B35, 35B10.

Key words and phrases. Periodic traveling waves, Nonlinear stability, Modified BBM equation, Fourth BBM equation.
In this paper, we are interested in studying special cases of the generalized Benjamin-Bona-Mahony equation (gBBM equation henceforth)

\[ u_t + u_x + (p+1)u^pu_x - u_{xxt} = 0, \tag{1} \]

where \( p \) is a positive integer and \( u \) is a real-valued function. The case \( p = 1 \) corresponds to the Benjamin-Bona-Mahony (BBM) equation, which has been derived as a model to describe water waves in the long-wave regime, see [11, 25, 26]. When \( p = 2 \), equation (1) is known as the modified BBM equation, which describes wave propagation in a one-dimensional nonlinear lattice [29, 30]. Thus the generalization considered here is not only of mathematical interest.

In the present work, we develop a stability theory for the periodic traveling-wave solutions of equation (1), in the cases \( p = 2 \) (mBBM) and \( p = 4 \) (4-BBM). The other possible cases, \( p = 3, 5, 6, 7, \ldots \), are not considered here because the methods used to obtain solutions present some difficulties. For instance, if odd \( p \geq 3 \), we do not know the explicit Fourier transform of the solitary wave solution associated to the gBBM equation (see (4) below), which is needed in order to apply the Poisson Summation Theorem. Using the quadrature method, we believe that there are chances of getting periodic waves in the cases not studied here. We will present these results in a future work.

The traveling-wave solutions to be considered here will be of the general form

\[ u(x,t) = \phi_c(x - ct), \]

where \( \phi_c : \mathbb{R} \to \mathbb{R} \) is a smooth periodic function (of period \( L \)) and \( c > 1 \). Substituting this type of solution in (1) and integrating once, we obtain

\[ c\phi''_c - (c-1)\phi'_c + \phi^{p+1}_c = d, \tag{3} \]

where \( d \) is a constant of integration. Now, since our theory of stability is based on the ideas of Benjamin [10] and Weinstein [31], we will consider the solutions \( \phi_c \) as critical points for the functional \( \mathcal{E} + (c-1)\mathcal{F} \), where \( \mathcal{E} \) and \( \mathcal{F} \) represent conserved quantities with respect to (1), given by

\[ \mathcal{F}(u) = \frac{1}{2} \int u^2 + u_x^2 dx \quad \text{and} \quad \mathcal{E}(u) = \frac{1}{2} \int_0^L u_x^2 - \frac{2}{p+2} u^{p+2} dx. \tag{2} \]

Assuming that the function \( \phi_c \) is a critical point of \( \mathcal{E} + (c-1)\mathcal{F} \) we conclude that the constant of integration \( d \) must be zero and thus \( \phi_c \) satisfies

\[ c\phi''_c - (c-1)\phi'_c + \phi^{p+1}_c = 0. \tag{3} \]

For \( c > 1 \), the gBBM equation considered in \( \mathbb{R} \) has solitary-wave solutions of the form

\[ \varphi_c(x,t) = \left[ (p+2)(c-1) \right]^{1/p} \text{sech}^{2/p} \left( \frac{p}{2} \sqrt{\frac{c-1}{c}} (x - ct) \right). \tag{4} \]

These solitary waves are orbitally stable for all speeds \( c > 1 \) if \( 1 \leq p \leq 4 \). For \( p \geq 5 \), there exists a critical speed \( c_p > 1 \) such that the solitary waves are nonlinearly stable if \( c > c_p \), and nonlinearly unstable when \( c \in (1, c_p) \), see [27, 31]. In the particular case of the BBM equation it was proved [16, 22] that the solitary waves are asymptotically stable. Zeng [33] proved the existence and stability of solitary-wave solutions for a family of equations of BBM type using variational methods. Recently, in the periodic setting, Hărăguş [19] considered the spectral stability of periodic waves of the gBBM equations, which are small perturbations of the constant solution \( u = (c-1)^{1/p} \) in \( L^2(\mathbb{R}) \) or \( C_b(\mathbb{R}) \). If \( 1 \leq p \leq 2 \), Hărăguş obtained
spectral stability for all $c > 1$. For $p = 3$ there exists $c_3 > 1$ such that the waves are stable for $c > c_3$ and unstable in $(1, c_3)$, and in the case $p > 3$ there exists a critical speed $c_p$ such that the periodic waves are spectrally stable for $c \in (c_p, \frac{p}{p-3})$, and unstable for $c \in (1, c_p) \cup \left( \frac{p}{p-3}, \infty \right)$.

Angulo, Banquet and Scialom in [5] considered the following general model

$$u_t + u_x + u^p u_x + Hu_t = 0,$$  \hspace{1cm} (5)

where $p$ is a positive integer and $H$ is a differential or pseudo-differential operator. In the context of periodic functions, they obtained sufficient conditions for the orbital stability of traveling waves associated with equation (5). In particular, they proved the existence and nonlinear stability of a family of periodic traveling-wave solutions of cnoidal type for the BBM equation ($H = -\partial_x^2$) with minimal period $L > \frac{2\pi}{\sqrt{2}}$ and wave speeds $c > L^2 - \frac{2\pi}{2}$.

In this work we establish, for the mBBM equation, via the Poisson Summation Theorem, the existence of a smooth curve of periodic traveling-wave solutions of dnoidal type of (3) with minimal period $L > \sqrt{2}\pi$ and $c > \frac{L^2}{L^2 - 2\pi^2}$, namely the profile is

$$\phi_c(x) = \eta \text{dn} \left( \frac{\eta x}{\sqrt{2c}}; k \right),$$  \hspace{1cm} (6)

where $\eta$ and $k$ are positive smooth functions depending on the wave speed $c$. For the 4-BBM equation, following the ideas in Angulo and Natali [7], we prove the existence of an explicit family of periodic traveling-wave solutions of (3), with minimal period $L > \pi$ and $c > \frac{L^2}{L^2 - \pi^2}$. The smooth curve of periodic solutions is given by

$$\phi_c(\xi) = \sqrt{\eta_3} \frac{\text{dn} \left( \frac{2g\sqrt{3c}}{2\sqrt{3c}} \xi; k \right)}{\sqrt{1 - \alpha^2 \text{sn}^2 \left( \frac{2g\sqrt{3c}}{2\sqrt{3c}} \xi; k \right)}},$$  \hspace{1cm} (7)

where the parameters $\eta_3, g, \alpha$ and $k$ depend smoothly on the wave speed $c$. It is worth noting that all our stability results hold for any fixed period $L > 0$ such that $pL^2 > 4\pi^2$ and not only for large periods.

To obtain our result of stability for the mBBM and the 4-BBM equation we use the ideas developed by Benjamin [10], Bona [12], Weinstein [32] and Angulo and Natali [6] combined with the recent theory developed by Angulo, Banquet and Scialom in [5]. In [5], sufficient conditions were obtained that guarantee certain spectral properties of the linear operator

$$\mathcal{L}_p = -cH + (c - 1) - \phi_p^p,$$  \hspace{1cm} (8)

which, in turn, are required to prove stability. Here $H$ is defined as $\hat{H}u(n) = \alpha(n)\hat{u}(n)$, for all $n \in \mathbb{Z}$. The symbol $\alpha$ is assumed to be a real, measurable, locally bounded, even function on $\mathbb{R}$, satisfying the conditions

$$A_1 |n|^{m_1} \leq \alpha(n) \leq A_2 (1 + |n|)^{m_2},$$

where $1 \leq m_1 \leq m_2$, $|n| > n_0$, $\alpha(n) > b$ for all $n \in \mathbb{Z}$, with $b$ a real constant and $A_i > 0$, for $i = 1, 2$. Actually, we confirmed that the following set of conditions
which guarantee the stability of $\phi_c$ in $H_{per}^1([0, L])$ are valid in our case too.

(C0) There is a nontrivial smooth curve of periodic solutions for (3) of the form $c \in I \subset \mathbb{R} \rightarrow \phi_c \in H_{per}^1([0, L])$.

(C1) $L_p$ has a unique negative eigenvalue and it is simple.

(C2) The eigenvalue zero is simple.

(C3) $\frac{d}{dc} \int_0^L (\phi'_c)^2 + \phi_c^2 \, dx > 0$.

Concerning the Cauchy problem associated to equation (1) in the periodic setting, the global well-posedness in the Sobolev space $H^s_{per}([0, L])$ with $s \geq 1$ is good enough for our purposes. This result follows in a straightforward fashion from the result of Albert [1], where it was obtained for the continuous case. It is worth to note that, in the non-periodic case of the BBM equation, there exists a better result given by Bona and Tzvetkov in [13], where they proved that the initial value problem associated to the BBM equation is globally well-posed in $H^s(\mathbb{R})$ if $s \geq 0$. This result is sharp in the sense that the BBM equation cannot be solved by iteration of a bounded mapping in $H^s(\mathbb{R})$ for $s < 0$. Bona and Chen in [14] have additional results on the well-posedness of a family of equations which includes the gBBM equation.

In the last part of this paper, the stability theory of constant solutions for the gBBM is considered. We use the approach given in [8] to obtain the nonlinear stability of the family of solutions of (3) in the form $\phi_0(\xi) = (c - 1)^{1/p}$. Note that the restrictions on the period $L$ and the speed $c$ given by $L > \frac{2\pi}{\sqrt{p}}$ and $c > \frac{\pi L^2 - 4}{4\pi^2}$, always come up when we prove stability in the case of non-constant periodic solutions for the BBM, mBBM and 4-BBM equations. If we impose the same restriction on $L$ and assume $1 < c < \frac{\pi L^2}{pL^2 - 4\pi^2}$, the constant solutions of the gBBM equation are stable, but if we suppose that $0 < L \leq \frac{2\pi}{\sqrt{p}}$, we obtain orbital stability using only the assumption $c > 1$, which is a necessary condition for the existence of solutions.

The outline of this paper is as follows. We introduce notation to be used throughout the whole article and the well-posedness results in Section 2. Section 3 is devoted to showing the existence and stability of periodic traveling waves for the mBBM equation. In Section 4, we deal with the existence and nonlinear stability of periodic solutions for the 4-BBM equation. Finally, in Section 5, the stability of constant solutions is established.

2. Notation and preliminaries. The $L^2$-based Sobolev spaces of periodic functions are defined as follows (for further details see Iorio and Iorio [20]). Let $P = C^\infty_{per}$ denote the collection of all functions $f : \mathbb{R} \rightarrow \mathbb{C}$ which are $C^\infty$ and periodic with period $L > 0$. The collection $P'$ of all continuous linear functionals from $P$ into $\mathbb{C}$ is the set of periodic distributions. If $\Psi \in P'$ then we denote the value of $\Psi$ at $\varphi$ by $\Psi(\varphi) = (\Psi, \varphi)$. Define the functions $\Theta_k(x) = \exp(2\pi ikx/L)$, $k \in \mathbb{Z}$, $x \in \mathbb{R}$. The Fourier transform of $\Psi$ is the function $\hat{\Psi} : \mathbb{Z} \rightarrow \mathbb{C}$ defined by the formula $\hat{\Psi}(k) = \frac{1}{L} (\Psi, \Theta_{-k})$, $k \in \mathbb{Z}$. So, if $\Psi$ is a periodic function with period $L$, we have

$$\hat{\Psi}(k) = \frac{1}{L} \int_0^L \Psi(x) e^{-\frac{2\pi ikx}{L}} \, dx.$$
For \( s \in \mathbb{R} \), the periodic Sobolev space of order \( s \), with period \([0, L]\) is simply denoted by \( H^s_{\text{per}} \). It is the set of all \( f \in \mathcal{P} \) such that 
\[
(1 + |k|^2)^s \hat{f}(k) \in L^2(\mathbb{Z}),
\]
with norm
\[
||f||^2_{H^s_{\text{per}}} = L \sum_{k = -\infty}^{\infty} (1 + |k|^2)^s |\hat{f}(k)|^2.
\]
We also note that \( H^s_{\text{per}} \) is a Hilbert space with respect to the inner product
\[
(f|g)_s = L \sum_{n = -\infty}^{\infty} (1 + |k|^2)^s \hat{f}(k) \overline{\hat{g}(k)}.
\]
In the case \( s = 0 \), \( H^0_{\text{per}} \) is denoted by \( L^2_{\text{per}} \), 
\[
(f|g)_0 = (f, g) = \int_0^L f g \, dx.
\]
and its norm by \( || \cdot ||_{L^2_{\text{per}}} \).

For the sake of completeness we present the Poisson Summation Theorem. It will be used in Section 3 to find the periodic traveling-wave solutions for the mBBM equation.

Theorem 2.1. Let \( \hat{f}^R(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx \) and \( f(x) = \int_{-\infty}^{\infty} \hat{f}^R(\xi) e^{2\pi i x \xi} \, d\xi \) satisfy
\[
|f(x)| \leq A(1 + |x|)^{-1-\delta} \quad \text{and} \quad |\hat{f}^R(\xi)| \leq A(1 + |\xi|)^{-1-\delta},
\]
where \( A > 0 \) and \( \delta > 0 \) (so \( f \) and \( \hat{f} \) can be assumed to be continuous functions). Then, for any \( L > 0 \)
\[
\sum_{n = -\infty}^{\infty} f(x + Ln) = \frac{1}{L} \sum_{n = -\infty}^{\infty} \hat{f}^R \left( \frac{n}{L} \right) e^{2\pi i x n/L}.
\]
The two series above converge absolutely.

Proof. See Stein and Weiss [28], for example.

Well-posedness result: Using a fixed point argument combined with the fact that \( H^s_{\text{per}} \) with \( s > 1/2 \) is a Banach algebra we obtain the next result.

Theorem 2.2. Suppose \( s > 1/2 \), then the Cauchy problem associated to the gBBM equation (1) is locally well-posed in \( H^s_{\text{per}} \).

Since the \( H^1_{\text{per}} \)-norm is conserved by the flow of the gBBM equation a global result is immediately obtained.

Corollary 1. The Cauchy problem associated to the gBBM equation (1) is globally well-posed in \( H^1_{\text{per}} \).

We finish this section presenting the definition of the class \( PF(2) \) discrete and one of its properties which will be used repeatedly.
Definition 2.3. We say that a sequence \( \alpha = (\alpha_n)_{n \in \mathbb{Z}} \subset \mathbb{R} \) is in the class \( PF(2) \) discrete if

\begin{enumerate}[(i)]  \item \( \alpha_n > 0 \), for all \( n \in \mathbb{Z} \),  
  \item \( \alpha_{n_1-m_1} \alpha_{n_2-m_2} - \alpha_{n_1-m_2} \alpha_{n_2-m_1} > 0 \) for \( n_1 < n_2 \) and \( m_1 < m_2 \).
\end{enumerate}

Theorem 2.4. Let \( \alpha \) and \( \beta \) be two even sequences in the class \( PF(2) \) discrete, then the convolution \( \alpha \ast \beta \) is in \( PF(2) \) discrete (if the convolution makes sense).

Proof. See Chapter 8 in Karlin [21]. See also Angulo and Natali [7]. \( \square \)

3. Existence and stability of periodic traveling-wave solutions for the mBBM equation. In this section we establish the existence of a smooth curve of periodic traveling-wave solutions for the mBBM equation

\[ u_t + u_x + 3u^2 u_x - uu_{xxt} = 0. \tag{10} \]

The equation which determines the periodic traveling-wave solution is

\[ c\phi''_c - (c - 1)\phi_c + \phi_c^3 = d, \]

where \( d \) is a constant of integration. As \( \phi_c \) is a critical point of the functional \( E + (c - 1)F \), where \( E \) and \( F \) are given in (2), we have that \( d = 0 \), therefore

\[ c\phi''_c - (c - 1)\phi_c + \phi_c^3 = 0. \tag{11} \]

The solitary wave solutions for the mBBM equation are given by

\[ \varphi_w(x) = \sqrt{2(w - 1)} \sech\left( \sqrt{\frac{w-1}{w}} x \right), \quad w > 1, \quad x \in \mathbb{R}, \tag{12} \]

with Fourier transform \( \hat{\varphi}_w(\xi) = \sqrt{2w\pi} \sech\left( \frac{\pi \xi}{2} \sqrt{\frac{w}{w-1}} \right) \). Then, from Theorem 2.1 we obtain the following periodic function

\[ \psi_w(x) := \frac{\sqrt{2w\pi}}{L} + \frac{2\sqrt{2w\pi}}{L} \sum_{n=1}^{\infty} \sech\left( \frac{\pi n}{2L} \sqrt{\frac{w}{w-1}} \right) \cos\left( \frac{2\pi nx}{L} \right), \tag{13} \]

where \( w > 1 \) will be chosen later for \( \psi_w \) to be a periodic traveling-wave solution of (10).

On the other hand, we have the Fourier expansion of the dnoidal Jacobi elliptic function of period \( L \) (see Oberhettinger [24])

\[ \frac{2K}{L} \text{dn}\left( \frac{2K\xi}{L}; k \right) = \frac{\pi}{L} + \frac{2\pi}{L} \sum_{n=1}^{\infty} \sech\left( \frac{n\pi K'}{K} \right) \cos\left( \frac{2n\pi \xi}{L} \right), \tag{14} \]

where \( K = K(k) \) is the complete elliptic integral of the first kind and \( K'(k) = K(\sqrt{1-k^2}) \).

Because of the shape of the series that determines \( \psi_w \), we consider \( \phi_c \) given in (6) with \( \eta > 0 \) and \( k \in (0, 1) \) fixed, a periodic solution (with period \( L \)) for the equation in (11). Then, substituting this type of solution in (11) and using the fact that the fundamental period of the dnoidal function is \( 2K \), we obtain the following compatibility relations

\[ c - 1 = \frac{\eta^2}{2} (2 - k^2) \quad \text{and} \quad \eta = \frac{2\sqrt{2cK(k)}}{L}. \tag{15} \]
Thus, for $k \in (0, 1)$ we should have that $\eta \in (\sqrt{c-1}, \sqrt{2(c-1)})$ and $\frac{c-1}{c} > \frac{2\pi^2}{L^2}$.

Combining the two equations given in (15) it is easy to see that
\[
L^2 = \frac{4c(2-k^2)K^2(k)}{c-1},
\]
and since $c > 1$ we obtain the a priori estimate $L > \sqrt{2\pi}$. The compatibility relations in (15) also imply that
\[
c = \frac{L^2}{L^2 - 4(2-k^2)K^2(k)} := f_L(k).
\]

Using again the assumption $c > 1$ we should have that there exists $k_L \in (0, 1)$ such that
\[
L^2 - 4(2-k^2)K^2(k) > 0, \quad \text{for all } k \in (0, k_L).
\]

For example, for $L = 5$, using Maple program we get $k_L = 0.9041841218$ (see Figure 1).

Fig. 1: Graph of $c(k)$ with $L = 5$

Fig. 2: Graph of $c(k)$ with $L = 4$

If we consider $\phi_c$ given by (6) with period $T_{\phi_c}$, we obtain from (15) that the fundamental period of $\phi_c$ is a function of $\eta$ given by

\[
T_{\phi_c}(\eta) = 2K(k)\sqrt{2-k^2}\sqrt{\frac{c}{c-1}},
\]

with $k^2 = 2 - \frac{2(c-1)}{\eta^2}$. If $\eta \to \sqrt{c-1}$, then $k \to 0^+$, therefore $T_{\phi_c}(\eta) \to \sqrt{2\pi}\sqrt{\frac{c}{c-1}}$.

If $\eta \to \sqrt{2(c-1)}$, then $k \to 1^-$, therefore $T_{\phi_c}(\eta) \to +\infty$. Since $\eta \to T_{\phi_c}(\eta)$ is a strictly increasing function (we prove it later), we obtain that $T_{\phi_c}(\eta) > \sqrt{2\pi}\sqrt{\frac{c}{c-1}}$.

**Remark 1.** Note that if $\eta \to \sqrt{2(c-1)}$, we have that
\[
\phi_c(x) = \sqrt{2(c-1)}\text{sech}\left(\sqrt{\frac{c-1}{c}}x\right), \quad x \in \mathbb{R}, \ c > 1,
\]

which is the solitary wave solution for the mBBM. If $\eta \to \sqrt{c-1}$, we obtain the constant solution for the mBBM $\phi_c(x) = \sqrt{c-1}$.

Next, we construct a family of dnovidal waves solutions with period $L > \sqrt{2\pi}$ for the mBBM equation. Let $k_L \in (0, 1)$ satisfying (17) and consider $c_0 > \frac{L^2}{2\pi^2}$. Choose the unique $k_0 \in (0, k_L)$ such that $c_0 = f_L(k_0)$, where $f_L$ is defined in (16).
Then there exists a unique \( \eta_0 = \eta(c_0) \) such that \( \eta_0 \in \left( \sqrt{c_0 - 1}, \sqrt{2(c_0 - 1)} \right) \) and the fundamental period of \( \phi_{c_0} \) is \( T_{\phi_{c_0}}(\eta_0) = L \).

**Theorem 3.1.** Let \( L > \sqrt{2\pi} \) fixed and \( k_L \in (0,1) \) satisfying (17). Consider \( c_0 > \frac{L^2}{L^2 - 2\pi^2} \) and the unique \( \eta_0 = \eta(c_0) \) such that \( T_{\phi_{c_0}}(\eta_0) = L \), then

(i) there exist an interval \( I(c_0) \) around \( c_0 \), an interval \( J(\eta_0) \) around \( \eta_0 \) and a unique smooth function \( \Lambda : I(c_0) \to J(\eta_0) \) such that \( \Lambda(c_0) = \eta_0 \) and

\[
2\sqrt{2 - k^2} K(k) \sqrt{\frac{c}{c-1}} = L,
\]

(ii) The dnoidal wave solution \( \phi_c \) given by (6), determined by \( \eta = \eta(c) \), has fundamental period \( L \) and satisfies (11). Furthermore, the mapping

\[
c \in I(c_0) \mapsto \phi_c \in H^n_{\text{per}}([0,L]),
\]

is smooth, for all \( n \in \mathbb{N} \).

(iii) \( I(c_0) \) can be chosen as \( \mathcal{I} = \left( \frac{L^2}{L^2 - 2\pi^2}, +\infty \right) \).

**Proof.** Since the arguments are standard, we only make a sketch of the proof. For this purpose, define

\[
\Omega = \left\{ (\eta, c) \in \mathbb{R}^2 : c > \frac{L^2}{L^2 - 2\pi^2} \text{ and } \eta \in \left( \sqrt{c-1}, \sqrt{2(c-1)} \right) \right\}
\]

and \( \Gamma : \Omega \to \mathbb{R} \) given by

\[
\Gamma(\eta, c) = 2K(k(\eta, c)) \sqrt{2 - k^2(\eta, c)} \sqrt{\frac{c}{c-1}},
\]

where \( k^2(\eta, c) = 2 - \frac{2(c-1)}{\eta^2} \). Note that \( \Gamma(\eta_0, c_0) = L \). We prove that \( \Gamma_\eta(\eta, c) > 0 \) for all \( c > \frac{L^2}{L^2 - 2\pi^2} \). In fact, writing \( k^2 = \sqrt{1 - k^2} \) and using the relation \( E(k) = \frac{dk}{dk} + (k')^2 K(k) \), we obtain

\[
\Gamma_\eta(\eta, c) = \frac{2}{k^2 \sqrt{2 - k^2}} \sqrt{\frac{c}{c-1}} \frac{dk}{dk} \left[ (2 - k^2) E - 2k^2 K \right].
\]

Since \( \frac{dk}{d\eta} > 0 \) and \( k \mapsto (2 - k^2) E(k) - 2k^2 K(k) \) is a positive function in \((0,1)\), we obtain \( \Gamma_\eta(\eta, c) > 0 \). Therefore, the Implicit Function Theorem implies that, there exists a unique smooth function \( \Lambda \) defined on a neighborhood \( I(c_0) \) of \( c_0 \) with \( \Gamma(\Lambda(c), c) = L \) for all \( c \in I(c_0) \). Since \( c_0 \) was arbitrarily fixed in the interval \( \mathcal{I} = \left( \frac{L^2}{L^2 - 2\pi^2}, +\infty \right) \), we obtain from the uniqueness of \( \Lambda \), that we can extend \( I(c_0) \) to \( \mathcal{I} \).

Now, note that (18) is equivalent to \( \frac{1}{c} = \frac{L^2 - 4(2 - k^2(c)) K^2(k(c))}{L^2} \), therefore we have that \( k(c) \in (0, k_L) \) for all \( c \in \mathcal{I} \). The rest of the proof follows from the smoothness of the functions involved and using the fact that \( k^2(c) = 2 - \frac{2(c-1)}{\eta^2(c)} \) and (18) imply the compatibility relations given in (15).

\[ \square \]

In the next corollary we choose the speed \( w = w(c) \) in such way that \( \psi_w \) in (13) becomes a periodic traveling wave with dnoidal profile.
Corollary 2. Define

$$w(c) = \frac{16c(2 - k^2)K'^2}{16c(2 - k^2)K'^2 - c + 1},$$

where $k = k(c) \in (0, k_L)$ and $c > \frac{k^2}{2\pi^2}$. Then $\phi_c = \sqrt{\frac{c}{w(c)}} \psi_{w(c)}$.

Proof. From the definition of $w$ and (16) it is easy to see that $\sqrt{\frac{w}{w-1}} = \frac{2K'L}{K}$. Then, using (13) and (14) we obtain

$$\psi_{w}(\xi) = \frac{2\sqrt{2}wK}{L} \text{dn} \left( \frac{2K\xi}{L}; k \right) = \sqrt{\frac{w}{c}} \phi_c,$$

where in the second identity we used the relation $\frac{c}{2\sqrt{2}K} = \frac{2K}{L}$, which is obtained from (15). This finishes the proof of the corollary. □

Corollary 3. Consider the mapping $\Lambda : I(c_0) \to J(\eta_0)$ determined by the last theorem. Then, $\Lambda$ is a strictly increasing function on $I(c_0)$.

Proof. We have that $\Gamma(A(c), c) = L$, for all $c \in I(c_0)$. By the Implicit Function Theorem we get $\frac{d\Lambda(c)}{dc} = -\frac{\Gamma_c}{\Gamma_{\eta}}$. Since $\Gamma_{\eta} > 0$, we just have to prove that $\Gamma_c < 0$. In fact, since

$$\Gamma_c(\eta, c) = \sqrt{c - 1} \left\{ \frac{2c}{c - 1} \eta^2 k^2 k'^2 \sqrt{2 - k^2} \right\}$$

and $2k'^2 K(k) - (1 + k^2) E(k) < 0$, for all $k \in (0, 1)$, we obtain $\Gamma_c < 0$, which finishes the proof of the corollary. □

Corollary 4. Consider $w : \left( \frac{k^2}{2\pi^2}, +\infty \right) \to \mathbb{R}$, where $w$ is given by (19). Then $\frac{dw}{dc} > 0$.

Proof. For $K'(k) = K(\sqrt{1 - k^2})$ and $E'(k) = E(\sqrt{1 - k^2})$ we have that

$$\frac{dK'}{dk} = -\left( \frac{E' - k^2 K'}{kk'^2} \right).$$

So, it follows that

$$\frac{dw}{dc} = \frac{16(2 - k^2)K'^2}{16c(2 - k^2)K'^2 - c + 1} \left[ \frac{32c(1 - c)K'}{k^2 k'^2} \right] \frac{dK'}{dk} \left[ (2 - k^2)E' - k^2 K' \right].$$

Using the inequality $E' > K'$, we get that $(2 - k^2)E'(k) - k^2 K'(k) > 0$, for all $k \in (0, 1)$. Then, to finish the proof we just have to show that $\frac{dK'}{dk} > 0$. It is enough to study the sign of $2(c - 1) \frac{dk}{dc} - \eta$ in

$$\frac{dk}{dc} = \frac{1}{kk'^2} \left[ 2(c - 1) \frac{d\eta}{dc} - \eta \right].$$

Corollary 3 implies $\frac{d\eta}{dc} = -\frac{\Gamma_c}{\Gamma_{\eta}}$. Note that

$$\Gamma_{\eta}(\eta, c) = \frac{4(c - 1)}{kk'^2 k^2 \sqrt{1 + k^2} \sqrt{c - 1}} \left[ (1 + k^2) E - 2k'^2 K \right].$$
Let \( \chi = L \). Theorem 3.3. Banquet and Scialom \[5\], see also \[10, 12, 32\].
\[
\left( \eta > 0 \right)
\]

\( \text{in fact, Corollary 2 implies } \phi \) is unstable in \( H^1_{\text{per}} \).

Before proving the principal result of this section we define the type of stability we are considering.

**Definition 3.2.** Let \( \phi_c \) be a periodic traveling-wave solution of \((3)\) with period \( L \).

We define the orbit generated by \( \phi_c \) as
\[
\mathcal{O}_{\phi_c} := \{ f \in H^1_{\text{per}} : f = \phi_c(\cdot + r) \text{ for some } r \in \mathbb{R} \},
\]
and, for \( \gamma > 0 \)
\[
U_\gamma := \{ f \in H^1_{\text{per}} : \inf_{g \in \mathcal{O}_{\phi_c}} \| f - g \|_{H^1_{\text{per}}} < \gamma \}.
\]

With this terminology, we say that \( \phi_c \) is (orbitally) stable in \( H^1_{\text{per}} \) by the flow generated by \((1)\) if the following conditions hold:

(i) there exists \( \delta > 0 \) such that \( H^1_{\text{per}} \subset H^1_{\text{per}} \) and the initial value problem associated to \((1)\) is globally well-posed in \( H^1_{\text{per}} \).

(ii) for every \( \epsilon > 0 \), there is \( \delta(\epsilon) > 0 \) such that, for all \( u_0 \in U_\delta \cap H^1_{\text{per}} \), the solution \( u \) of \((1)\) with \( u(0, x) = u_0(x) \) satisfies \( u(t) \in U_\epsilon \) for all \( t > 0 \).

Otherwise, we say that \( \phi_c \) is unstable in \( H^1_{\text{per}} \).

The proof of the next stability theorem is obtained following the ideas in Angulo, Banquet and Scialom \[5\], see also \[10, 12, 32\].

**Theorem 3.3.** Let \( \phi_c \) be a periodic traveling-wave solution of \((3)\), and suppose that part \( (i) \) of the definition of stability holds. Suppose also that the operator \( \mathcal{L}_p = -c\partial_x^2 + (c - 1) - (p + 1)\phi_c^p \) has properties \((C_1)\) and \((C_2)\) in \((9)\). Choose \( \chi \in L^2_{\text{per}} \) such that \( \mathcal{L}_p \chi = \phi_c - \phi_c'' \), and define \( I = (\chi, \phi_c - \phi_c'')_{L^2_{\text{per}}} \). If \( I < 0 \) then \( \phi_c \) is stable.

Next, we present the main result of this section, the stability theorem for the mBBM equation.

**Theorem 3.4.** Let \( L > \sqrt{2\pi} \) and \( c > \frac{L^2}{2\sqrt{2\pi}} \). Then the dnodal traveling-wave solution \( \phi_c \) constructed in Theorem 3.1 is stable in \( H^1_{\text{per}}([0, L]) \) by the flow of the mBBM equation.

**Proof.** First we prove that \((C_1)\) and \((C_2)\) given in \((9)\) hold for the operator \( \mathcal{L}_2 = -c\partial_x^2 - 1 + c - 3\phi_c^2 \). Since \( \phi_c \) is positive and even, from Theorem 8.1 in Angulo, Banquet and Scialom \[5\], we just have to show that \( \widehat{\phi}_c > 0 \) and \( \widehat{\phi}_c^2 \) belongs to \( PF(2) \) discrete. In fact, Corollary 2 implies \( \widehat{\phi}_c(n) = \frac{c}{w} \widehat{\psi}_w(n) \), for all \( n \in \mathbb{Z} \). By the Poisson Summation Theorem we obtain \( \widehat{\psi}_w(n) = \frac{1}{L} \widehat{\varphi}_w(n) \), where \( \varphi_w \) is the solitary wave given in \((12)\).

Therefore
\[
\widehat{\phi}_c(n) = \frac{\sqrt{2c}}{L}\pi \text{ sech}\left( \frac{\pi n}{2L} \sqrt{\frac{w(c)}{w(c) - 1}} \right), \quad n \in \mathbb{Z}.
\]
We conclude that \( \widehat{\phi}_c \in PF(2) \) discrete, because \( f(x) = \text{Asech}(Bx) \in PF(2) \) continuous with \( A > 0 \) and \( B \in \mathbb{R} \setminus \{0\} \). From Theorem 2.4, we also obtain \( \widehat{\phi}_c^2 = \widehat{\phi}_c \ast \widehat{\phi}_c \in PF(2) \) discrete. Since \( \widehat{\phi}_c > 0 \), it is easy to see that \( \widehat{\phi}_c^2 > 0 \). Therefore
(C1) and (C2) hold for \( \mathcal{L} \).

Now, we prove that (C3) given in (9) holds. Since the mapping \( c \mapsto \phi_c \in H^1_{per}([0, L]) \) is smooth, we have that \( \mathcal{L} \left(-\frac{d}{dx}\phi_c\right) = \phi_c - \phi''_c \). Therefore, using Parseval identity we obtain
\[
I = -\frac{1}{2} \frac{d}{d\phi} \|\phi\|_{H^1_{per}}^2 = -\frac{L}{2} \frac{d}{d\phi} \|(1 + |\cdot|^2)^{\frac{1}{2}}\hat{\phi}_c\|_{L^2}^2.
\]
But
\[
|(1 + |\cdot|^2)^{\frac{1}{2}}\hat{\phi}_c|_n^2 = \frac{2\pi^2}{L^2} \sum_{n=-\infty}^{\infty} (1 + |n|^2) \text{sech}^2 \left( \frac{\pi n}{L} \sqrt{\frac{w}{w-1}} \right).
\]
Hence,
\[
\frac{d}{d\phi} \|(1 + |\cdot|^2)^{\frac{1}{2}}\hat{\phi}_c\|_{L^2}^2 = \frac{2\pi^2}{L^2} \sum_{n=-\infty}^{\infty} (1 + |n|^2) \text{sech}^2 \left( \frac{\pi n}{L} \sqrt{\frac{w}{w-1}} \right)
+ \frac{C}{(w-1)^{\frac{1}{2}}} \frac{w-1}{w} \frac{dw}{d\phi} \sum_{n=-\infty}^{\infty} (1 + |n|^2)n \text{ sech}^2 \left( \frac{\pi n}{L} \sqrt{\frac{w}{w-1}} \right) \tan \left( \frac{\pi n}{L} \sqrt{\frac{w}{w-1}} \right),
\]
where the constant \( C = C(c, L) > 0 \). Since the sequence \( \left\{ n \tan \left( \frac{\pi n}{L} \sqrt{\frac{w}{w-1}} \right) \right\} \) is positive, using Corollary 4 we get that \( \frac{d}{d\phi} \|(1 + |\cdot|^2)^{\frac{1}{2}}\hat{\phi}_c\|_{L^2}^2 > 0 \), which proves the stability of \( \phi_c \).

4. Existence and stability of periodic traveling-wave solutions for the 4-BBM equation. We start looking for periodic traveling-wave solutions \( \phi_c \) of the 4-BBM equation
\[
u_t + u_x + 5u^4u_x - u_{xxt} = 0.
\]
In this case \( \phi_c \) satisfies
\[
c\phi''_c - (c-1)\phi_c + \phi_c^5 = d,
\]
where \( d = 0 \) as in (11). Therefore
\[
c\phi''_c - (c-1)\phi_c + \phi_c^5 = 0.
\]
Multiplying (22) by \( \phi'_c \) and integrating we arrive at
\[
(\phi'_c)^2 = \frac{1}{3c} \left[-\phi_c^6 + 3(c-1)\phi^2 + 6A_{\phi_c}\right],
\]
where \( A_{\phi_c} \) is constant. Since we are interested in a positive solution (to apply the theory established in Angulo, Banquet and Scialom [5]) we may suppose that \( \phi_c = \sqrt{\psi_c} \), then (23) becomes
\[
(\psi'_c)^2 = \frac{4}{3c} \left[-\psi_c^4 + 3(c-1)\psi_c^2 + 6A_{\psi_c}\right].
\]
Let \( \eta_1, \eta_2 \) and \( \eta_3 \) be the nonzero roots of the polynomial \( F(t) = -t^4 + 3(c-1)t^2 + 6At \). From the last identity we obtain
\[
(\psi'_c)^2 = \frac{4}{3c} F(\psi_c) = \frac{4}{3c} \psi_c(\psi_c - \eta_1)(\psi_c - \eta_2)(\psi_c - \eta_3),
\]
and
\[
\begin{cases}
\eta_1 + \eta_2 + \eta_3 = 0 \\
\eta_1 \eta_2 + \eta_2 \eta_3 + \eta_1 \eta_3 = -3(c-1) \\
\eta_1 \eta_2 \eta_3 = 6A_{\psi_c}.
\end{cases}
\]
Assuming $\eta_1 < 0 < \eta_2 < \eta_3$, then $\eta_2 \leq \psi_c \leq \eta_3$. Other possibilities for $\eta_1, \eta_2$ and $\eta_3$ will be discussed later.

Now, consider $c > 1$ arbitrarily fixed. We are seeking a non-constant periodic solution $\psi_c$ such that the maximum and minimum values in $[0, L]$ are given by $\psi_c(0) = \eta_3$ and $\psi_c(\xi_0) = \eta_2$, for some $\xi_0 \in (0, L)$, respectively. From (24) and using the Leibnitz rule we conclude from the last assumptions that

$$\int_{\eta_1}^{\eta_3} \frac{dt}{\sqrt{(t - \eta_3)(t - \eta_2)(t - \eta_1)}} = \frac{2}{g \sqrt{3c}} (\xi + M_c),$$

where $M_c$ is a constant of integration and $g$ is defined below. Thus, formula 257.00 in Byrd and Friedman [15] leads to

$$\psi_c(\xi) = \frac{\eta_3(\eta_2 - \eta_1) + \eta_1(\eta_3 - \eta_2)\sin^2 \left(\frac{2}{g \sqrt{3c}} \xi; k\right)}{(\eta_2 - \eta_1) + (\eta_3 - \eta_2)\sin^2 \left(\frac{2}{g \sqrt{3c}} \xi; k\right)},$$

where $g = \frac{2}{\sqrt{\eta_3(\eta_2 - \eta_1)}}$ and $k^2 = \frac{\eta_2(\eta_3 - \eta_2)}{\eta_3(\eta_2 - \eta_1)}$. We rewrite $\psi_c$ as

$$\psi_c(\xi) = \eta_3 \left[ \frac{\sin^2 \left(\frac{2}{g \sqrt{3c}} \xi; k\right)}{1 - \alpha^2 \sin^2 \left(\frac{2}{g \sqrt{3c}} \xi; k\right)} \right], \quad (26)$$

where $\alpha^2 = \frac{\eta_2}{\eta_3} k^2 < 0$. Since we want $\psi_c$ with minimal period $L > 0$, we obtain that $\frac{2}{g \sqrt{3c}} = \frac{2K}{L}$.

From (25) we have that $\eta_2$ and $\eta_3$ satisfy

$$\eta_2 + \eta_3 + \eta_2 \eta_3 = 3(c - 1), \quad (27)$$

which implies $\eta_2 < \sqrt{c - 1} < \eta_3 < \sqrt{3(c - 1)}$ and

$$L = \frac{2\sqrt{3c} K(k)}{\sqrt{a}}, \quad (28)$$

where $a := 12(c - 1) \eta_3^2 - 3 \eta_3^2$. Using (25) and (27) it easy to see that

$$\eta_2 = \frac{-\eta_3 + \sqrt{12(c - 1) - 3 \eta_3^2}}{2} \quad \text{and} \quad \eta_1 = \frac{-\eta_3 - \sqrt{12(c - 1) - 3 \eta_3^2}}{2}. \quad (29)$$

Therefore, we can express $k$ as a function of $\eta_3$

$$k^2 = \frac{3 \eta_3^2 + \sqrt{a} - 6(c - 1)}{2 \sqrt{a}}. \quad (30)$$

Combining (28) and (30) we obtain the system

$$\begin{cases} \eta_3^2 L^2 - 2(c - 1)L^2 = 4c(2k^2 - 1)K^2, \\ 4(c - 1) \eta_3^2 L^4 - \eta_3^2 L^4 = 48c^2 K^4. \end{cases} \quad (31)$$

Substituting $\eta_3^2 L^2$ obtained in the first equation of (31) in the second one, we arrive at

$$(c - 1)^2 L^4 - 16c^2 (k^4 - k^2 + 1) K^4 = 0. \quad (32)$$
Thus, for $k \in (0, 1)$ we get from the asymptotic properties of $K$ that $c > \frac{L^2}{1 + \pi^2}$. From (32) we also obtain

$$L^2 = \frac{4c\sqrt{k^4 - k^2 + 1}K^2(k)}{c - 1}. $$

Since we assumed $c > 1$, we have the a priori estimate $L > \pi$. Note that (32) is equivalent to

$$c = \frac{L^2}{L^2 - r(k)} := g_L(k),$$

where $r(k) := 4\sqrt{k^4 - k^2 + 1}K^2(k)$. Therefore, we should have that there exists $k_L \in (0, 1)$ such that

$$L^2 - 4\sqrt{k^4 - k^2 + 1}K^2(k) > 0, \quad \text{for all } k \in (0, k_L).$$

For example, for $L = 4$ and $L = 7$ using the Maple program we obtain that $k_4 = 0.8515822215$ and $k_7 = 0.9927289494$ (see Figure 2).

If we consider $\psi_c$ given in (26) with fundamental period $T_{\psi_c}$, we get from (25) that

$$T_{\psi_c}(\eta_3) = \frac{2\sqrt{3cK(k(\eta_3))}}{[12(c - 1)\eta_2^2 - 3\eta_3^4]^{1/4}}.$$ 

Assume from now that $\eta_3 \in (\sqrt{c - 1}, \sqrt{3(c - 1)}) \rightarrow T_{\psi_c}(\eta_3)$ is a strictly increasing function (we prove it later), then we obtain the following a priori estimate

$$T_{\psi_c}(\eta_3) > \pi \sqrt{\frac{c}{c - 1}}.$$ 

**Remark 2.** (i) Note that, if $\eta_3 \rightarrow \sqrt{c - 1}^+$, then $k \rightarrow 0^+$ and $\psi_c \rightarrow \sqrt{c - 1}$. Hence we obtain the constant solution $\phi_c(\xi) = \sqrt{c - 1}$. On the other hand, if $\eta_3 \rightarrow \sqrt{3(c - 1)}^-$, then $k \rightarrow 1^-$ and $\psi_c(\xi) \rightarrow \sqrt{3(c - 1)} \text{sech} \left(2\sqrt{c - 1}\xi \right)$ which is the square of the solitary wave solution of the 4-BBM equation.

(ii) If we suppose that $\psi_c(0) = \eta_2$ and $\psi_c(\xi_0) = \eta_3$ for some $\xi_0 \in (0, L)$, we obtain from formula 256.00 in [15] that

$$\psi_c(\xi) = \frac{\sqrt{\eta_2\eta_3}}{\sqrt{\eta_3 - (\eta_3 - \eta_2)\text{sn}^2 \left(\frac{2\xi}{g\sqrt{2c}}k\right)}}.$$ 

This positive solution converges to zero when $k \rightarrow 1^-$. In a future work the stability theory for this family of solutions will be addressed.

(iii) Other possible choices for $\eta_1$, $\eta_2$ and $\eta_3$ always produce a solution that is either negative or zero at some point. We do not have a theory for solutions which can be negative.

Now, we prove the existence of a smooth curve of periodic solutions with fixed minimal period $L > \pi$ for the equation (21). Since the square root is a smooth curve on $(0, +\infty)$ and the fundamental periods of $\phi_c$ and $\psi_c$ are the same, we will study the curve

$$c \in \left(\frac{L^2}{L^2 - \pi^2}, +\infty\right) \rightarrow \psi_c \in H^1_{\text{per}}([0, L]).$$
Let $L > \pi$ be fixed and $k_L \in (0,1)$ such that (34) is satisfied. Consider $c_0 > \frac{L^2}{2L - \pi}$ and choose the unique $k_0 \in (0, k_L)$ such that $g_L(k_0) = c_0$, where $g_L$ is defined in (33). Then there exists $\eta_{3,0} = \eta_3(c_0)$ such that $\eta_{3,0} \in (\sqrt{c_0 - 1}, \sqrt{3(c_0 - 1)})$ and the fundamental period of $\phi_{c_0}$ is given by $T_{\psi_{c_0}}(\eta_{3,0}) = L$. Therefore from (29) and (30) we get an explicit solution of (24) in the form (26) and depending only on $\eta = \eta_3$.

The next theorem shows that the choice $c \mapsto \eta(c)$ is smooth.

**Theorem 4.1.** Let $L > \pi$ fixed and $k_L \in (0,1)$ satisfying (34). Consider $c_0 > \frac{L^2}{2L - \pi}$ and $\eta_{3,0}$ such that $T_{\psi_{c_0}}(\eta_{3,0}) = L$. Then,

(i) there exist intervals $I(c_0)$ around $c_0$, $J(\eta_{3,0})$ around $\eta_{3,0}$ and a unique function $\Lambda : I(c_0) \to J(\eta_{3,0})$ such that $\Lambda(c_0) = \eta_{3,0}$ and

$$2\sqrt{3c}K \frac{12(c - 1)\eta^2 - 3\eta^4}{[12(c - 1)\eta^2 - 3\eta^4]^{1/4}} = L,$$

where $c \in I(c_0)$, $\eta = \Lambda(c)$ and $k = k(c) \in (0, k_L)$ is given by (30).

(ii) The function $\phi_c = \sqrt{\psi_c}$, where $\psi$ is given in (26), determine by $\eta_3$ has fundamental period $L$ and satisfies (22). Furthermore

$$c \in I(c_0) \mapsto \psi_c \in H^\alpha_{per}([0, L])$$

is smooth for all $n \in \mathbb{N}$

(iii) $I(c_0)$ can be chosen as $I = \left(\frac{L^2}{2L - \pi}, +\infty\right)$.

**Proof.** The proof follows the ideas of Theorem 4.2 in Angulo and Natali [7]. For this reason we only present a sketch.

Define $\Omega = \{(\eta, c) \in \mathbb{R}^2 : c > \frac{L^2}{2L - \pi}$ and $\eta \in \left(\sqrt{c - 1}, \sqrt{3(c - 1)}\right)\}$ and

$$\Gamma : \Omega \to \mathbb{R}$$

as

$$\Gamma(\eta, c) = \frac{2\sqrt{3c}K(k(\eta, c))}{[12(c - 1)\eta^2 - 3\eta^4]^{1/4}},$$

where $k^2(\eta, c)$ is given by (30). Denote $a := 12(c - 1)\eta^2 - 3\eta^4$, then

$$\Gamma_\eta(\eta, c) = \frac{2\sqrt{3c}}{\sqrt{a^3}} \left( \frac{dK}{dk} \frac{a - 3\eta [2(c - 1) - \eta^2]}{\eta^2} K \right).$$

Using (30) we obtain $\frac{dK}{d\eta} > 0$. We consider two cases.

Case 1: $\eta \in \left(\sqrt{2(c - 1)}, \sqrt{3(c - 1)}\right)$. In this case $2(c - 1) - \eta^2 \geq 0$, therefore $\frac{dK}{d\eta} > 0$.

Case 2: $\eta \in \left(\sqrt{c - 1}, \sqrt{2(c - 1)}\right)$. In this case it is easy to see that

$$a[2(c - 1) - \eta^2]^2 < 9(c - 1)^4.$$

Hence by (36)

$$\frac{\sqrt{a^3} \Gamma_\eta}{6\eta^{3/2} \sqrt{3c}} = \frac{1}{\sqrt{a}} \left\{ \frac{1}{k} \frac{6(c - 1)^2}{k} \frac{dK}{dk} - \sqrt{a} [2(c - 1) - \eta^2] K \right\},$$

$$> \frac{1}{\sqrt{a}} \left\{ \frac{6(c - 1)^2}{k^2} \frac{dK}{dk} - 3(c - 1)^2 K \right\} = \frac{3(c - 1)^2}{k^2 k'^2 \sqrt{a}} \left\{ 2E - (2 + k^2) k'^2 K \right\}.$$
Now, note that (35) is equivalent to \( \frac{1}{c} = \frac{L^2 - r(k(c))}{L^2} \), therefore we obtain \( k(c) \in (0, k_L) \), for all \( c \in I \). The rest of the proof follows from the smoothness of the functions involved.

**Corollary 5.** Consider \( \Lambda : I(c_0) \rightarrow J(\eta_{3,0}) \) determined by the last theorem. Then \( \Lambda \) is strictly increasing.

**Proof.** By Theorem 4.1, we have that \( \Gamma(\Lambda(c), c) = L \) for all \( c \in I(c_0) \). Then using the Implicit Function Theorem, we obtain

\[
\frac{d\Lambda(c)}{dc} = -\frac{\Gamma_c}{\Gamma_\eta}.
\]

We already showed that \( \Gamma_\eta > 0 \), then we only have to prove that \( \Gamma_c < 0 \). In fact,

\[
\Gamma_c(\eta, c) = a^{-5/4}/3c \left[ \frac{a}{c} K + 2a \frac{dK}{dc} \frac{dk}{dc} - 6\eta^2 K \right].
\]

Since \( \frac{dk}{dc} = -\frac{9(c-1)\eta^2}{k\sqrt{a}} \), we obtain

\[
\Gamma_c(\eta, c) = \eta^2 a^{-7/4}/3c \left[ \sqrt{a} \left( \frac{a}{c^2} - 6 \right) K - \frac{18(c-1) dK}{k} \right].
\]

We consider two cases. If \( \eta^2 \geq 2(c-1) - 2 \), then \( \frac{a}{c^2} - 6 < 0 \), therefore \( \Gamma_c < 0 \). If \( \eta^2 < 2(c-1) - 2 \), we have that \( c > 3 \) since \( \eta^2 > c - 1 \). Using that \( \eta \in (\sqrt{c-1}, \sqrt{2(c-1) - 2}) \) it easy to see that \( \sqrt{a} \left( \frac{a}{c^2} - 6 \right) < 9(c-1) \left( \frac{c-2}{c} \right) \).

Therefore,

\[
\Gamma_c < 9\eta^2(2(c-1)\sqrt{3} a^{-7/4}k^{-2}k^{-2} \left[ k^2(2 + k^2)K - 2E \right].
\]

Since \( k^2(2 + k^2)K - 2E < 0 \), we have \( \Gamma_c < 0 \), which finishes the proof of the corollary.

**Corollary 6.** Consider \( L > \pi \), \( c > \frac{L^2}{L^2 - \pi^2} \), \( \eta_3(c) = \Lambda(c) \) and the modulus function

\[
k = \left( \frac{3\eta_3^3 - 6(c-1) + \sqrt{12(c-1)\eta_3^2 - 3\eta_3^4}}{2\sqrt{12(c-1)\eta_3^2 - 3\eta_3^4}} \right)^{1/2}.
\]

Then \( \frac{dk}{dc} > 0 \).

**Proof.** In Corollary 5 we proved that \( c \in \left( \frac{L^2}{L^2 - \pi^2}, +\infty \right) \mapsto \eta_3(c) \) is differentiable. So using (30) we get

\[
\frac{dk}{dc} = \frac{9(c-1) \left[ 2(c-1)\eta_3' - \eta_3 \right] \eta_3}{ka^{3/2}},
\]

where \( a := 12(c-1)\eta_3^2 - 3\eta_3^4 \). From (36), (37) and (38) we obtain

\[
2(c-1)\eta_3' = \eta \left[ \frac{18(c-1)^2}{k} \frac{dk}{dc} - \sqrt{a}(c-1) \left( \frac{a}{c^2} - 6 \right) K \right].
\]

Hence

\[
2(c-1)\eta_3' - \eta_3 = \eta^{-1} \frac{a^2 K}{\eta_3^2} \left[ 1 - \frac{c-1}{x} \right] \left[ \frac{18(c-1)^2}{k} \frac{dk}{dc} - 3\sqrt{a} \left[ 2(c-1) - \eta_3^2 \right] K \right].
\]

Since \( c > 1 \) and the denominator in (39) is positive (we proved it in Theorem 4.1), we get that \( 2(c-1)\eta_3' - \eta_3 > 0 \). This proves the desired result.
Fourier expansion of $\psi_c$: Using a very similar analysis as in Angulo and Natali [7] we obtain the following Fourier expansion of the periodic traveling-wave solution given in (26)

$$
\psi_c(\xi) = C(\eta_3, \alpha, k) \left[ \Lambda_0(w, k) + 2 L \sum_{n=1}^{\infty} \frac{\csch(n\pi w)}{K} \sinh\left(\frac{\pi F(w, k') n}{K}\right) \cos\left(\frac{2\pi n \xi}{L}\right) \right],
$$

where $\Lambda_0$ is the Heuman’s Lambda function, see Byrd and Friedman [15].

Since it will be necessary to establish our result of stability for the 4-BBM equation we express $\eta_1, \eta_2, \eta_3$ as functions of the modulus $k \in (0, k_L)$. Replacing $c$ given in (33) on the first equation of (31) we obtain

$$
\eta_3 = \frac{2\sqrt{c} K}{L} \sqrt{2\sqrt{k^2 - 1} + 2k^2 - 1}.
$$

Then, from the last identity and (29) we have that

$$
\eta_1 = -\frac{\sqrt{c} K}{L} \left[ \sqrt{3 \sqrt{2\sqrt{k^4 - k^2 + 1} - (2k^2 - 1)} - \sqrt{2\sqrt{k^4 - k^2 + 1} + 2k^2 - 1}} \right]
$$

$$
\eta_2 = \frac{\sqrt{c} K}{L} \left[ \sqrt{3 \sqrt{2\sqrt{k^4 - k^2 + 1} - (2k^2 - 1)} + \sqrt{2\sqrt{k^4 - k^2 + 1} + 2k^2 - 1}} \right].
$$

Next, we present our theorem of stability for the 4-BBM equation. It is worth to note that the next result differs from that one obtained by Angulo and Natali in [7] for periodic traveling-wave solutions of the critical KdV equation. Indeed, the authors in [7] proved the existence of a critical speed $c_L$ such that the periodic traveling-wave solutions $\varphi_c$ (of period L), associated to the critical KdV, are orbitally stable if $c < c_L$ and orbitally unstable if $c > c_L$. In our case we do not have such a critical speed.

**Theorem 4.2.** Let $L > \pi$. Consider $c > \frac{L^2}{2\pi - \pi^2}$, then the periodic traveling-wave solution $\phi_c$ given in (7) is stable in $H^1_{\text{per}}([0, L])$ by the periodic flow of the 4-BBM equation.
Thus, from (33) we get that

\[ L \chi = \phi_c - \phi'_c. \]

Since the mapping \( c \in \left( \frac{L^2}{4\pi}, +\infty \right) \mapsto \phi_c \in H^1_{per}(0, L) \) is smooth we choose \( \chi = -\frac{d}{dx} \phi_c. \) Then \( I = -\frac{1}{2} \frac{d}{dx} \| \phi_c \|^2_{H^1_{per}} = -\frac{1}{2} \frac{d}{dx} \left( \int_0^L \phi_c^2 + (\phi'_c)^2 \, d\xi \right). \) Since \( 0 < -\alpha^2 < +\infty, \) Formula 410.04 in [15] implies for \( w = \sin^{-1} \sqrt{\frac{c^2}{\lambda^2-k^2}} \) that

\[
\frac{d^2}{dx^2} \int_0^K \frac{d^2(x; k)}{1-\alpha^2 \sin^2(x; k)} \, dx = \frac{\pi (k^2 - \alpha^2) \Lambda_0(w; k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.
\]

Consequently, using (26) we arrive at

\[
\int_0^L \psi_c(\xi) d\xi = \frac{2\sqrt{\frac{c}{2\sqrt{k^4-k^2+1+2k^2-1}} \sqrt{1+f(k)} G(w; k)}}{\sqrt{f(k)[1+k^2f(k)]}},
\]

where \( G(w; k) := \frac{\pi}{2} \Lambda_0(w, k) \) and

\[
f(k) = -\frac{\eta_3}{\eta_1} = \frac{2\sqrt{2\sqrt{k^4-k^2+1+2k^2-1}}}{\sqrt{2\sqrt{k^4-k^2+1+2k^2-1}+\sqrt{3}2\sqrt{k^4-k^2+1}-(2k^2-1)}}.
\]

Note that we have \( w = \sin^{-1} \sqrt{\frac{f(k)}{1+f(k)}}. \) If we define the function \( m \) (which does not depend on the period \( L \)) by

\[
m(k) = \frac{2\sqrt{2\sqrt{k^4-k^2+1+2k^2-1}\sqrt{1+f(k)} G(w; k)}}{\sqrt{f(k)[1+k^2f(k)]}},
\]

we obtain that

\[
\int_0^L \phi_c^2(\xi) d\xi = \int_0^L \psi_c(\xi) d\xi = \sqrt{c} m(k). \tag{40}
\]

On the other hand, multiplying (22) by \( \phi_c \) and integrating from 0 to \( L \) we get

\[
-c \int_0^L (\phi'_c)^2 d\xi - (c-1) \int_0^L \phi_c^2 d\xi + \int_0^L \phi_c^6 d\xi = 0. \tag{41}
\]

Integrating (23) from 0 to \( L \) yields

\[
3c \int_0^L (\phi'_c)^2 d\xi + \int_0^L \phi_c^6 d\xi - 3(c-1) \int_0^L \phi_c^2 = 6A_{\phi_c} L . \tag{42}
\]

Combining (41) and (42), it is easy to see that

\[
\int_0^L (\phi'_c)^2 d\xi = \frac{c-1}{2c} \int_0^L \phi_c^2 d\xi + \frac{6A_{\phi_c} L }{4c}.
\]

From the last identity, (40), (25) and the expression given for \( \eta_1, \eta_2 \) and \( \eta_3 \), we have that

\[
\int_0^L \phi_c^2 + (\phi'_c)^2 d\xi = \sqrt{c} \left\{ m(k) + \frac{c-1}{2c} [n(k) + m(k)] \right\},
\]

where

\[
n(k) = \left( \frac{2k^2 - 1 - \sqrt{k^4-k^2+1}}{\sqrt{2\sqrt{k^4-k^2+1}+2k^2-1}} \right) \frac{\sqrt{2\sqrt{k^4-k^2+1}+2k^2-1}}{K(k)}.
\]

Thus, from (33) we get that

\[
\int_0^L \phi_c^2 + (\phi'_c)^2 d\xi = \frac{L}{\sqrt{L^2-r(k)} \left\{ [m(k) + \frac{r(k)}{2L} [n(k) + m(k)]] \right\}} := P_L(k).
\]
Therefore,
\[
\frac{d}{dc} \left( \int_0^L \phi_c^2 + (\phi'_c)^2 \, d\xi \right) = \frac{dP_L(k)}{dk} \frac{dk}{dc}.
\]
From Corollary 6 and the fact that for any \( L > \pi \), the function \( P_L \) is strictly increasing in \((0, k_L)\) (see Figure 3 and 4), we obtain that \( \frac{d}{dc} \left( \int_0^L \phi_c^2 + (\phi'_c)^2 \, d\xi \right) > 0 \) in \( \left( \frac{L^2}{k^2 - \pi^2}, +\infty \right) \).

Finally, using a very similar analysis as in Angulo and Natali [7] we prove that \( \hat{\phi} \in PF(2) \) and \( \hat{\phi}(n) > 0 \), for all \( n \in \mathbb{Z} \), which finishes the proof of the theorem.

5. Stability of constant solutions for the gBBM equation. In this section we prove the nonlinear stability of constant solutions for the gBBM equation. The proof follows the ideas established by Angulo, Bona and Scialom in [8]. For \( c > 1 \), the periodic traveling wave solution for (1) satisfy (3) with constant solutions given by \( \phi_0(x) = (c-1)^{1/p} \).

The proof of our result of stability is based on the conserved quantities given in (2). The next functional will be useful to obtain our result of stability for the constant solutions
\[
L_0 := \mathcal{E}''(\phi_0) + (c-1)\mathcal{F}''(\phi_0) = -c\partial_x^2 - p(c-1).
\]

Let us start studying the spectral properties of \( L_0 \).

**Proposition 1.** Let \( L > 0 \), \( c > 1 \) and consider the operator \( L_0 : D(L_0) \to L^2_{\text{per}} \), defined above with domain \( D(L_0) = H^2_{\text{per}}([0,L]) \). Then,

(i) If \( L > \frac{2\pi \sqrt{p}}{c} \) and \( c < \frac{pL^2}{pL^2 - 4\pi^2} \), the operator \( L_0 \) has its first eigenvalue negative and the rest of the eigenvalues are double and positive.

(ii) If \( L > \frac{2\pi \sqrt{p}}{c} \) and \( c = \frac{pL^2}{pL^2 - 4\pi^2} \), the operator \( L_0 \) has its first eigenvalue negative, zero is a double eigenvalue and the rest of the eigenvalues are double and positive.

(iii) If \( L \leq \frac{2\pi \sqrt{p}}{c} \), we obtain the same result in (i) for all \( c > 1 \).

**Proof.** We are interested in studying the eigenvalue problem
\[
\begin{align*}
L_0 f &= \lambda f \\
 f(0) &= f(L), \quad f'(0) = f'(L).
\end{align*}
\]

The last problem is equivalent to
\[
\begin{align*}
- f'' &= \sigma f \\
 f(0) &= f(L), \quad f'(0) = f'(L),
\end{align*}
\]
where \( \sigma = \frac{\lambda + p(c-1)}{c} \). It is well known that \( \sigma_0 = 0 \) and \( \sigma_{2m+1} = \sigma_{2m+2} = \frac{4(m+1)^2\pi^2}{L^2} \), for \( m \geq 0 \), with eigenfunctions given by \( f_0 = 1, \quad f^-_m(x) = \sin \left( \frac{2(m+1)\pi x}{L} \right) \) and \( f^+_m(x) = \cos \left( \frac{2(m+1)\pi x}{L} \right) \). Then \( \lambda_0 = -p(c-1) < 0 \) and
\[
\lambda_{2m+1} = \lambda_{2m+2} = \frac{4c(m+1)^2\pi^2}{L^2} - p(c-1).
\]
Hence \( \lambda_1 = \lambda_2 = \frac{4\pi^2}{L^2} - p(c - 1) = 0 \Leftrightarrow \frac{c-1}{c} = \frac{4\pi^2}{pL^2} \), and we obtain (ii). It is easy to see that \( \lambda_{2m+1} = \lambda_{2m+2} > 0 \Leftrightarrow \frac{c-1}{c} < \frac{4\pi^2}{pL^2} \), which proves (i). Finally, note that if \( pL^2 \leq 4\pi^2 \), then \( \frac{4\pi^2}{pL^2} \geq 1 > \frac{c-1}{c} \) for all \( c > 1 \). Thus \( \lambda_{2m+1} = \lambda_{2m+2} > 0 \), for all \( m \geq 0 \), and we get (iii).

Next we present our theorem of stability for the constant solutions.

**Theorem 5.1.** Let \( L > \frac{2\pi}{\sqrt{p}} \). Then for all \( c \) satisfying \( 1 < c < \frac{pL^2}{4\pi^2} \), the constant solution \( \phi_0(x) = (c-1)^{1/p} \) is nonlinearly stable by the periodic flow of the gBBM equation.

**Proof.** Define the functionals \( \mathcal{E}, \mathcal{F} : H^1_{\text{per}}([0, L]) \to \mathbb{R} \) given by (2), \( \mathcal{E} \) and \( \mathcal{F} \) are well-defined in \( H^1_{\text{per}} \) and are continuous. Consider \( v := u - \phi_0 \in H^1_{\text{per}} \) and let \( u_0 \) be the initial data associated to the periodic problem

\[
\begin{align*}
\frac{\partial u}{\partial t} + u_x + (p+1)u^p u_x - u_{xxt} &= 0, \quad t \geq 0, \quad x \in \mathbb{R} \\
u(x,0) &= u_0(x).
\end{align*}
\]

Define \( B := \mathcal{E} - (c-1)\mathcal{F} \). Then using the immersion \( H^1_{\text{per}} \to L^q_{\text{per}} \), for all \( q \geq 2 \), it is easy to see that

\[
\Delta B(t) = B(u_0) - B(\phi_0) = B(v(t) + \phi_0) - B(\phi_0) = \frac{1}{2} \left( \mathcal{L}_0 v, v \right) - \frac{1}{p+2} \sum_{k=0}^{p-1} \left( \frac{k}{p+2} \right) \int_0^L v^{p+2-k} \phi_0^k dx \
\geq \frac{1}{2} \left( \mathcal{L}_0 v, v \right) - C(c,p) \sum_{k=0}^{p-1} \|v\|^{p+2-k}_{H^s_{\text{per}}}. \tag{43}
\]

Assume that \( \mathcal{F}(u_0) = \mathcal{F}(\phi_0) \), then \( \mathcal{F}(u(t)) = \mathcal{F}(\phi_0) \). Since \( v = u - \phi_0 \) we obtain

\[
-2(v, \phi_0) = \|v\|_{H^1_{\text{per}}}. \tag{44}
\]

Define \( v^\perp := v - \bar{v} \), where \( \bar{v} = \frac{1}{L} \int_0^L v dx \). Note that \( \int_0^L v^\perp dx = 0 \). So, it follows from the Poincaré’s inequality that

\[
\int_0^L (v^\perp)^2 dx \geq \frac{4\pi^2}{L^2} \int_0^L (v^\perp)^2 dx.
\]

Using the last inequality, we have that

\[
(\mathcal{L}_0 v^\perp, v^\perp) = \int_0^L c(v^\perp)^2 - p(c-1)(v^\perp)^2 dx \geq \frac{c4\pi^2}{L^2} - p(c-1) \int_0^L (v^\perp)^2 dx.
\]

Since \( 0 < \frac{c-1}{c} < \frac{4\pi^2}{L^2} \), we obtain \( \bar{\beta}_1 = \frac{4\pi^2}{L^2} - p(c-1) > 0 \), and therefore

\[
(\mathcal{L}_0 v^\perp, v^\perp) \geq \bar{\beta}_1 \|v^\perp\|^2_{L^2_{\text{per}}}. \tag{45}
\]

On the other hand, using (44) we get that

\[
\bar{v} = \frac{1}{L} \int_0^L v dx = \frac{1}{L\phi_0}(v, \phi_0) = -\frac{1}{2L\phi_0} \|v\|_{H^1_{\text{per}}}.
\]

Therefore \( \|v\|^2_{H^s_{\text{per}}} = \|v^\perp\|^2_{L^2_{\text{per}}} + \|ar{v}\|^2_{L^2_{\text{per}}} = \|v^\perp\|^2_{L^2_{\text{per}}} + C(p, L, c) \|v\|^4_{H^s_{\text{per}}} \). From (45) we get

\[
(\mathcal{L}_0 v^\perp, v^\perp) \geq \bar{\beta}_1 \|v^\perp\|^2_{L^2_{\text{per}}} - \bar{\beta}_2 \|v\|^4_{H^s_{\text{per}}}. \tag{46}
\]
Note that
\[ \mathcal{L}_0 \mathcal{P}(\mathcal{P}) = -p(c-1)\|\mathcal{P}\|^2 L^2_{per} = -\beta_3\|v\|^4 H^1_{per}. \] (47)
Using the Cauchy-Schwarz inequality, we obtain
\[ (\mathcal{L}_0 \mathcal{P}, v^+) \geq -C(p, L, c)\|\mathcal{P}\| L^2_{per} \|v^+\| L^2_{per} \geq -\beta_4\|v\|^4 H^1_{per} - \beta_5\|v\|^4 H^1_{per}. \] (48)
Therefore, from (46), (47), (48) and the specific form of \( \mathcal{L}_0 \) we conclude that
\[ (\mathcal{L}_0 v, v) \geq \beta_0\|v\|^2 H^1_{per} - \beta_1\|\mathcal{P}\| H^1_{per} - \beta_2\|v\|^2 H^1_{per}. \] (49)
Hence, by (43) and (49) we arrive at \( \Delta B(t) \geq g \left( \|v\|_{H^1_{per}} \right) \), for all \( t \geq 0 \), where \( g(s) = s^2 - \sum_{k=0}^{p-1} \alpha_k s^{p+2-k} \) and \( \alpha_k = \alpha_k(c, p, L) > 0 \). Since
\[ d_t(u(t), \phi_0) := \inf_{y \in \mathbb{R}} \|u(t) - \phi_0(y)\|_{H^1_{per}} = \|v\|_{H^1_{per}}, \quad \forall \ t \geq 0 \]
we obtain \( \Delta B(t) \geq g(d_t(u(t), \phi_0)) \), for all \( t \geq 0 \). This finishes the proof of the theorem. \( \square \)

**Remark 3.** In particular, if \( L \leq \frac{2\pi}{\sqrt{p}} \) Theorem 5.1 holds for all \( c > 1 \).

**Acknowledgments.** J. Angulo was partially supported by Grant CNPq/Brazil and by Edital Universal MCT/CNPq, 14/2009. C. Banquet was supported by CNPq/Brazil doctoral fellowship at State University of Campinas/ Brazil. C. Banquet is also grateful to the University of Córdoba, Montería, Colombia for its financial support. The authors are grateful to the reviewers for their fruitful remarks.

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Received November 2009; revised December 2010.

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