Monge-Kantorovitch Measure Transportation and Monge-Ampère Equation on Wiener Space

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Abstract: Let \((W, \mu, H)\) be an abstract Wiener space assume two \(\nu_i, i = 1, 2\) probabilities on \((W, \mathcal{B}(W))\). We give some conditions for the Wasserstein distance between \(\nu_1\) and \(\nu_2\) with respect to the Cameron-Martin space

\[
d_H(\nu_1, \nu_2) = \sqrt{\inf_{\beta} \int_{W \times W} |x - y|^2_H d\beta(x, y)}
\]

to be finite, where the infimum is taken on the set of probability measures \(\beta\) on \(W \times W\) whose first and second marginals are \(\nu_1\) and \(\nu_2\). In this latter situation we prove the existence of a unique (cyclically monotone) map \(T = I_W + \xi\), with \(\xi : W \to H\), such that \(T\) maps \(\nu_1\) to \(\nu_2\). Besides, if \(\nu_2 \ll \mu^2\), then \(T\) is stochastically invertible, i.e., there exists \(S : W \to W\) such that \(S \circ T = I_W \nu_1\) a.s. and \(T \circ S = I_W \nu_2\) a.s. If \(\nu_1 = \mu\), then there exists a 1-convex function \(\phi\) in the Gaussian Sobolev space \(\mathbb{D}^2_{2, 1}\), such that \(\xi = \nabla \phi\). These results imply that the quasi-invariant transformations of the Wiener space with finite Wasserstein distance from \(\mu\) can be written as the composition of a transport map \(T\) and a rotation, i.e., a measure preserving map. We give also 1-convex sub-solutions and Ito-type solutions of the Monge-Ampère equation on \(W\).

1 Introduction

In 1781, Gaspard Monge has published his celebrated memoire about the most economical way of earth-moving [22]. The configurations of excavated earth and remblai were modeled as two measures of equal mass, say \(\rho\) and \(\nu\), that Monge had supposed absolutely continuous with respect to the volume measure. Later Ampère has studied an analogous question about the electricity current in a media with varying conductivity. In modern language of measure theory we can express the problem in the following terms: let \(W\) be a Polish space on which are given two positive measures \(\rho\) and \(\nu\), of finite, equal mass. Let \(c(x, y)\) be a cost function on \(W \times W\), which is, usually, assumed positive. Does there exist a map \(T : W \to W\) such that \(T \rho = \nu\) and \(T\) minimizes the integral

\[
\int_W c(x, T(x)) d\rho(x)
\]

1cf. Theorem 6.1 for the precise hypothesis about \(\nu_1\) and \(\nu_2\).
2In fact this hypothesis is too strong, cf. Theorem 6.1
between all such maps? The problem has been further studied by Appell \[3, 4\] and by Kantorovitch \[18\]. Kantarovitch has succeeded to transform this highly nonlinear problem of Monge into a linear problem by replacing the search for $T$ with the search of a measure $\gamma$ on $W \times W$ with marginals $\rho$ and $\nu$ such that the integral

$$\int_{W \times W} c(x, y) d\gamma(x, y)$$

is the minimum of all the integrals

$$\int_{W \times W} c(x, y) d\beta(x, y)$$

where $\beta$ runs in the set of measures on $W \times W$ whose marginals are $\rho$ and $\nu$. Since then the problem addressed above is called the Monge problem and the quest of the optimal measure is called the Monge-Kantorovitch problem.

In this paper we study the Monge-Kantorovitch and the Monge problem in the frame of an abstract Wiener space with a singular cost. In other words, let $W$ be a separable Fréchet space with its Borel sigma algebra $B(W)$ and assume that there is a separable Hilbert space $H$ which is injected densely and continuously into $W$, hence in general the topology of $H$ is stronger than the topology induced by $W$. The cost function $c : W \times W \to \mathbb{R}_+ \cup \{\infty\}$ is defined as

$$c(x, y) = |x - y|_H^2,$$

we suppose that $c(x, y) = \infty$ if $x - y$ does not belong to $H$. Clearly, this choice of the function $c$ is not arbitrary, in fact it is closely related to Ito Calculus, hence also to the problems originating from Physics, quantum chemistry, large deviations, etc. Since for all the interesting measures on $W$, the Cameron-Martin space is a negligible set, the cost function will be infinity very frequently. Let $\Sigma(\rho, \nu)$ denote the set of probability measures on $W \times W$ with given marginals $\rho$ and $\nu$. It is a convex, compact set under the weak topology $\sigma(\Sigma, C_b(W \times W))$. As explained above, the problem of Monge consists of finding a measurable map $T : W \to W$, called the optimal transport of $\rho$ to $\nu$, i.e., $T \rho = \nu^3$ which minimizes the cost

$$U \to \int_W \|x - U(x)\|_H^2 d\rho(x),$$

between all the maps $U : W \to W$ such that $U \rho = \nu$. The Monge-Kantorovitch problem will consist of finding a measure on $W \times W$, which minimizes the function $\theta \to J(\theta)$, defined by

$$J(\theta) = \int_{W \times W} |x - y|_H^2 d\theta(x, y),$$

(1.1)

where $\theta$ runs in $\Sigma(\rho, \nu)$. Note that $\inf\{J(\theta) : \theta \in \Sigma(\rho, \nu)\}$ is the square of Wasserstein metric $d_H(\rho, \nu)$ with respect to the Cameron-Martin space $H$.

\[3\]We denote the push-forward of $\rho$ by $T$, i.e., the image of $\rho$ under $T$, by $T \rho$. 

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Any solution \( \gamma \) of the Monge-Kantorovitch problem will give a solution to the Monge problem provided that its support is included in the graph of a map. Hence our work consists of realizing this program. Although in the finite dimensional case this problem is well-studied in the path-breaking papers of Brenier [6] and McCann [21] the things do not come up easily in our setting and the difficulty is due to the fact that the cost function is not continuous with respect to the Fréchet topology of \( W \), for instance the weak convergence of the probability measures does not imply the convergence of the integrals of the cost function. In other words the function \( |x - y|^2_H \) takes the value plus infinity “very often”. On the other hand the results we obtain seem to have important applications to several problems of stochastic analysis that we shall explain while enumerating the contents of the paper.

In Section 2 we explain some basic results about the functional analysis constructed on the Wiener space (cf., for instance [13, 28]) and the probabilistic theory of convex functions recently developped in [14]. Section 3 is devoted to the derivation of some inequalities which control the Wasserstein distance. In particular, with the help of the Girsanov theorem, we give a very simple proof of an inequality, initially discovered by Talagrand ([25]); this facility gives already an idea about the efficiency of the infinite dimensional techniques for the Monge-Kantorovitch problem\(^4\). We indicate some simple consequences of this inequality to control the measures of subsets of the Wiener space with respect to second moments of their gauge functionals defined with the Cameron-Martin distance. These inequalities are quite useful in the theory of large deviations. Using a different representation of the target measure, namely by constructing a flow of diffeomorphisms of the Wiener space (cf. Chapter V of [29]) which maps the Wiener measure to the target measure, we obtain also a new control of the Kantorovitch-Rubinstein metric of order one. The method we employ for this inequality generalizes directly to a more general class of measures, namely those for which one can define a reasonable divergence operator.

In Section 4 we solve directly the original problem of Monge when the first measure is the Wiener measure and the second one is given with a density, in such a way that the Wasserstein distance between these two measures is finite. We prove the existence and the uniqueness of a transformation of \( W \) of the form \( T = I_W + \nabla \phi \), where \( \phi \) is a 1-convex function in the Gaussian Sobolev space \( \mathbb{D}_{2,1} \) such that the measure \( \gamma = (I_W \times T)\mu \) is the unique solution of the problem of Monge-Kantorovitch. This result gives a new insight to the question of representing an integrable, positive random variable whose expectation is unity, as the Radon-Nikodym derivative of the image of the Wiener measure under a map which is a perturbation of identity, a problem which has been studied by X. Fernique and by one of us with M. Zakai (cf., [11, 12, 29]). In [29], Chapter II, it is shown that such random variables are dense in \( L^1_{\text{lower}}(\mu) \) (the lower index 1 means that the expectations are equal to one), here we prove that this set of random variables contains the random variables who are at finite Wasserstein distance from the Wiener measure. In fact even if this distance is infinite, we show that there is a solution to this problem if we enlarge \( W \) slightly by taking \( \mathbb{N} \times W \).

Section 5 is devoted to the immediate implications of the existence and the unique-

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\(^4\)In Section 7 we shall see another illustration of this phenomena.
ness of the solutions of Monge-Kantorovitch and Monge problems constructed in Section 4. Indeed the uniqueness implies at once that the absolutely continuous transformations of the Wiener space, at finite (Wasserstein) distance, have a unique decomposition in the sense that they can be written as the composition of a measure preserving map in the form of the perturbation of identity with another one which is the perturbation of identity with the Sobolev derivative of a 1-convex function. This means in particular that the class of 1-convex functions is as basic as the class of adapted processes in the setting of Wiener space.

In Section 6 we prove the existence and the uniqueness of solutions of the Monge-Kantorovitch and Monge problems for the measures which are at finite Wasserstein distance from each other. The fundamental hypothesis we use is that the regular conditional probabilities which are obtained by the disintegration of one of the measures along the orthogonals of a sequence of regular, finite dimensional projections vanish on the sets of co-dimension one. In particular, this hypothesis is satisfied if the measure under question is absolutely continuous with respect to the Wiener measure. The method we use in this section is totally different from the one of Section 4; it is based on the notion of cyclic monotonicity of the supports of the regular conditional probabilities obtained through some specific disintegrations of the optimal measures. The importance of cyclic monotonicity has first been remarked by McCann and used abundantly in [20] and in [16] for the finite dimensional case. Here the things are much more complicated due to the singularity of the cost function, in particular, contrary to the finite dimensional case, the cyclic monotonicity is not compatible with the weak convergence of probability measures. A curious reader may ask why we did not treat first the general case and then attack the subject of Section 4. The answer is twofold: even if we had done so, we would have needed similar calculations as in Section 4 in order to show the Sobolev regularity of the transport map, hence concerning the volume, the order that we have chosen does not change anything. Secondly, the construction used in Section 4 has an interest by itself since it explains interesting relations between the transport map and its inverse and the optimal measure in a more detectable situation, in this sense this construction is rather complementary to the material of Section 6.

Section 7 studies the Monge-Ampère equation for the measures which are absolutely continuous with respect to the Wiener measure. First we briefly indicate the notion of second order Alexandroff derivative and the Alexandroff version of the Ornstein-Uhlenbeck operator applied to a 1-convex function in the finite dimensional case. With the help of these observations, we write the corresponding Jacobian using the modified Carleman-Fredholm determinant which is natural in the infinite dimensional case (cf., [29]). Afterwards we attack the infinite dimensional case by proving that the absolutely continuous part of the Ornstein-Uhlenbeck operator applied to the finite rank conditional expectations of the transport function is a submartingale which converges almost surely. Hence the only difficulty lies in the calculation of the limit of the Carleman-Fredholm determinants. Here we have a major difficulty which originates from the pathology of the Radon-Nikodym derivatives of the vector measures with respect to a scalar measure as explained in [26]: in fact even if the second order Sobolev derivative of a Wiener function is a vector measure with values in the space of Hilbert-Schmidt operators, its absolutely continuous part has no reason to be Hilbert-Schmidt. Hence
the Carleman-Fredholm determinant may not exist, however due to the 1-convexity, the determinants of the approximating sequence are all with values in the interval $[0, 1]$. Consequently we can construct the subsolutions with the help of the Fatou lemma.

Last but not the least, in section 7.1 we prove that all these difficulties can be overcome thanks to the natural renormalization of the Ito stochastic calculus. In fact using the Ito representation theorem and the Wiener space analysis extended to the distributions, cf. [27], we can give the explicit solution of the Monge-Ampère equation. This is a remarkable result in the sense that such techniques do not exist in the finite dimensional case.

2 Preliminaries and notations

Let $W$ be a separable Fréchet space equipped with a Gaussian measure $\mu$ of zero mean whose support is the whole space. The corresponding Cameron-Martin space is denoted by $H$. Recall that the injection $H \Rightarrow W$ is compact and its adjoint is the natural injection $W^* \Rightarrow H^* \subset L^2(\mu)$. The triple $(W, \mu, H)$ is called an abstract Wiener space. Recall that $W = H$ if and only if $W$ is finite dimensional. A subspace $F$ of $H$ is called regular if the corresponding orthogonal projection has a continuous extension to $W$ and in $F$. Let $\sigma(\pi_{F_n})$ be the $\sigma$-algebra generated by $\pi_{F_n}$, then for any $f \in L^p(\mu)$, the martingale sequence $(E[f|\sigma(\pi_{F_n})], n \geq 1)$ converges to $f$ (strongly if $p < \infty$) in $L^p(\mu)$. Observe that the function $f_n = E[f|\sigma(\pi_{F_n})]$ can be identified with a function on the finite dimensional abstract Wiener space $(F_n, \mu_n, F_n)$, where $\mu_n = \pi_n \mu$.

Since the translations of $\mu$ with the elements of $H$ induce measures equivalent to $\mu$, the Gâteaux derivative in $H$ direction of the random variables is a closable operator on $L^p(\mu)$-spaces and this closure will be denoted by $\nabla$ cf., for example [13, 28]. The corresponding Sobolev spaces (the equivalence classes) of the real random variables will be denoted as $\mathbb{D}_{p,k}$, where $k \in \mathbb{N}$ is the order of differentiability and $p > 1$ is the order of integrability. If the random variables are with values in some separable Hilbert space, say $\Phi$, then we shall define similarly the corresponding Sobolev spaces and they are denoted as $\mathbb{D}_{p,k}(\Phi)$, $p > 1$, $k \in \mathbb{N}$. Since $\nabla : \mathbb{D}_{p,k} \rightarrow \mathbb{D}_{p,k-1}(H)$ is a continuous linear operator its adjoint is a well-defined operator which we represent by $\delta$. In the case of classical Wiener space, i.e., when $W = C(\mathbb{R}_+, \mathbb{R}^d)$, then $\delta$ coincides with the Ito integral of the Lebesgue density of the adapted elements of $\mathbb{D}_{p,k}(H)$ (cf.[28]).

For any $t \geq 0$ and measurable $f : W \rightarrow \mathbb{R}_+$, we note by

$$P_t f(x) = \int_W f \left( e^{-t} x + \sqrt{1 - e^{-2t}} y \right) \mu(dy),$$

it is well-known that $(P_t, t \in \mathbb{R}_+)$ is a hypercontractive semigroup on $L^p(\mu), p > 1$, which is called the Ornstein-Uhlenbeck semigroup (cf.[13, 28]). Its infinitesimal generator is denoted by $-\mathcal{L}$ and we call $\mathcal{L}$ the Ornstein-Uhlenbeck operator (sometimes for the notational simplicity, in the sequel we shall denote it by $\pi_{F_n}$.
called the number operator by the physicists). The norms defined by

\[ \|\phi\|_{p,k} = \|(I + \mathcal{L})^{k/2}\phi\|_{L^p(\mu)} \]  \tag{2.2} 

are equivalent to the norms defined by the iterates of the Sobolev derivative \(\nabla\). This observation permits us to identify the duals of the space \(\mathcal{D}_{p,k}(\Phi); p > 1, k \in \mathbb{N}\) by \(\mathcal{D}_{q,-k}(\Phi')\), with \(q^{-1} = 1 - p^{-1}\), where the latter space is defined by replacing \(k\) in \(2.2\) by \(-k\), this gives us the distribution spaces on the Wiener space \(W\) (in fact we can take as \(k\) any real number). An easy calculation shows that, formally, \(\delta \circ \nabla = \mathcal{L}\), and this permits us to extend the divergence and the derivative operators to the distributions as linear, continuous operators. In fact \(\delta : \mathcal{D}_{q,k}(H \otimes \Phi) \to \mathcal{D}_{q,k-1}(\Phi)\) and \(\nabla : \mathcal{D}_{q,k}(\Phi) \to \mathcal{D}_{q,k-1}(H \otimes \Phi)\) continuously, for any \(q > 1\) and \(k \in \mathbb{R}\), where \(H \otimes \Phi\) denotes the completed Hilbert-Schmidt tensor product (cf., for instance \([28]\)).

Let us recall some facts from the convex analysis. Let \(K\) be a Hilbert space, a subset \(S\) of \(K \times K\) is called cyclically monotone if any finite subset \(\{(x_1, y_1), \ldots, (x_N, y_N)\}\) of \(S\) satisfies the following algebraic condition:

\[ \sum_{i=1}^{N} (y_i, x_{\sigma(i)} - x_i) \leq 0, \]

where \(\langle \cdot, \cdot \rangle\) denotes the inner product of \(K\). It turns out that \(S\) is cyclically monotone if and only if

\[ \sum_{i=1}^{N} (y_i, x_{\sigma(i)} - x_i) \leq 0, \]

for any permutation \(\sigma\) of \(\{1, \ldots, N\}\) and for any finite subset \(\{(x_i, y_i) : i = 1, \ldots, N\}\) of \(S\). Note that \(S\) is cyclically monotone if and only if any translate of it is cyclically monotone. By a theorem of Rockafellar, any cyclically monotone set is contained in the graph of the subdifferential of a convex function in the sense of convex analysis \([23]\) and even if the function may not be unique its subdifferential is unique.

Let now \((W, \mu, H)\) be an abstract Wiener space; a measurable function \(f : W \to \mathbb{R} \cup \{\infty\}\) is called 1-convex if

\[ h \to f(x + h) + \frac{1}{2} |h|^2_H = F(x, h) \]

is convex on the Cameron-Martin space \(H\) with values in \(L^0(\mu)\). Note that this notion is compatible with the \(\mu\)-equivalence classes of random variables thanks to the Cameron-Martin theorem. It is proven in \([14]\) that this definition is equivalent the following condition: Let \((\pi_n, n \geq 1)\) be a sequence of regular, finite dimensional, orthogonal projections of \(H\), increasing to the identity map \(I_H\). Denote also by \(\pi_n\) its continuous extension to \(W\) and define \(\pi_n^{-1} = I_W - \pi_n\). For \(x \in W\), let \(x_n = \pi_n x\) and \(x_n^{-1} = \pi_n^{-1} x\). Then \(f\) is 1-convex if and only if

\[ x_n \to \frac{1}{2} |x_n|^2_H + f(x_n + x_n^{-1}) \]

is \(\pi_n^{-1}\)\(\mu\)-almost surely convex.
3 Some Inequalities

Definition 3.1 Let $\xi$ and $\eta$ be two probabilities on $(W, \mathcal{B}(W))$. We say that a probability $\gamma$ on $(W \times W, \mathcal{B}(W \times W))$ is a solution of the Monge-Kantorovitch problem associated to the couple $(\xi, \eta)$ if the first marginal of $\gamma$ is $\xi$, the second one is $\eta$ and if

$$J(\gamma) = \int_{W \times W} |x - y|^2_H d\gamma(x, y) = \inf \left\{ \int_{W \times W} |x - y|^2_H d\beta(x, y) : \beta \in \Sigma(\xi, \eta) \right\},$$

where $\Sigma(\xi, \eta)$ denotes the set of all the probability measures on $W \times W$ whose first and second marginals are respectively $\xi$ and $\eta$. We shall denote the Wasserstein distance between $\xi$ and $\eta$, which is the positive square-root of this infimum, with $d_H(\xi, \eta)$.

Remark: Since the set of probability measures on $W \times W$ is weakly compact and since the integrand in the definition is lower semi-continuous and strictly convex, the infimum in the definition is always attained even if the functional $J$ is identically infinity.

The following result is an extension of an inequality due to Talagrand \[25\] and it gives a sufficient condition for the Wasserstein distance to be finite:

Theorem 3.1 Let $L \in \mathcal{L} \log \mathcal{L}(\mu)$ be a positive random variable with $E[L] = 1$ and let $\nu$ be the measure $d\nu = Ld\mu$. We then have

$$d_H^2(\nu, \mu) \leq 2E[L \log L]. \quad (3.3)$$

Proof: Without loss of generality, we may suppose that $W$ is equipped with a filtration of sigma algebras in such a way that it becomes a classical Wiener space as $W = C_0(\mathbb{R}_+, \mathbb{R}^d)$. Assume first that $L$ is a strictly positive and bounded random variable. We can represent it as

$$L = \exp \left[ -\int_0^\infty (\dot{u}_s, dW_s) - \frac{1}{2} |u|_H^2 \right],$$

where $u = \int_0^1 \dot{u}_s ds$ is an $H$-valued, adapted random variable. Define $\tau_n$ as

$$\tau_n(x) = \inf \left\{ t \in \mathbb{R}_+ : \int_0^t |\dot{u}_s(x)|^2 ds > n \right\}.$$

$\tau_n$ is a stopping time with respect to the canonical filtration $(\mathcal{F}_t, t \in \mathbb{R}_+)$ of the Wiener process $(W_t, t \in \mathbb{R}_+)$ and $\lim_n \tau_n = \infty$ almost surely. Define $u^n$ as

$$u^n(t, x) = \int_0^t 1_{[0, \tau_n(x)]}(s) \dot{u}_s(x) ds.$$

Let $U_n : W \to W$ be the map $U_n(x) = x + u^n(x)$, then the Girsanov theorem says that $(t, x) \to U_n(x)(t) = x(t) + \int_0^t \dot{u}_s^n ds$ is a Wiener process under the measure $L_n d\mu$, where $L_n = E[L | \mathcal{F}_{\tau_n}]$. Therefore

$$E[L_n \log L_n] = E \left[ L_n \left\{ -\int_0^\infty (\dot{u}_s^n, dW_s) - \frac{1}{2} |u^n|_H^2 \right\} \right]$$

$$= \frac{1}{2} E[L_n | u^n|_H^2]$$

$$= \frac{1}{2} E[L | u^n|_H^2].$$
Define now the measure $\beta_n$ on $W \times W$ as

$$\int_{W \times W} f(x,y) d\beta_n(x,y) = \int_W f(U_n(x), x)L_n(x)d\mu(x).$$

Then the first marginal of $\beta_n$ is $\mu$ and the second one is $L_n.\mu$. Consequently

$$\inf \left\{ \int_{W \times W} |x - y|^2_H d\theta : \pi_1 \theta = \mu, \pi_2 \theta = L_n.\mu \right\} \leq \int_W |U_n(x) - x|^2_H L_n d\mu = 2E[L_n \log L_n].$$

Hence we obtain

$$d^2_H(L_n.\mu, \mu, \mu) = J(\gamma_n) \leq 2E[L_n \log L_n],$$

where $\gamma_n$ is a solution of the Monge-Kantorovitch problem in $\Sigma(L_n.\mu, \mu)$. Let now $\gamma$ be any cluster point of the sequence $(\gamma_n, n \geq 1)$, since $\gamma \to J(\gamma)$ is lower semi-continuous with respect to the weak topology of probability measures, we have

$$J(\gamma) \leq \lim inf_n J(\gamma_n) \leq \sup_n 2E[L_n \log L_n] \leq 2E[L \log L],$$

since $\gamma \in \Sigma(L.\mu, \mu)$, it follows that

$$d^2_H(L.\mu, \mu) \leq 2E[L \log L].$$

For the general case we stop the martingale $E[L|\mathcal{F}_t]$ appropriately to obtain a bounded density $L_n$, then replace it by $P_{1/n}L_n$ to improve the positivity, where $(P_t, t \geq 0)$ denotes the Ornstein-Uhlenbeck semigroup. Then, from the Jensen inequality,

$$E[P_{1/n}L_n \log P_{1/n}L_n] \leq E[L \log L],$$

therefore, using the same reasoning as above

$$d^2_H(L.\mu, \mu) \leq \lim inf_n d^2_H(P_{1/n}L_n.\mu, \mu) \leq 2E[L \log L],$$

and this completes the proof.

Corollary 3.1 Assume that $\nu_i (i = 1, 2)$ have Radon-Nikodym densities $L_i (i = 1, 2)$ with respect to the Wiener measure $\mu$ which are in $\mathbb{L} \log \mathbb{L}$. Then

$$d_H(\nu_1, \nu_2) < \infty.$$
Proof: This is a simple consequence of the triangle inequality (cf. [5]):

\[ d_H(\nu_1, \nu_2) \leq d_H(\nu_1, \mu) + d_H(\nu_2, \mu). \]

Let us give a simple application of the above result in the lines of [19]:

**Corollary 3.2** Assume that \( A \in \mathcal{B}(W) \) is any set of positive Wiener measure. Define the \( H \)-gauge function of \( A \) as

\[ q_A(x) = \inf(|h|_H : h \in (A - x) \cap H). \]

Then we have

\[ E[q_A^2] \leq 2 \log \frac{1}{\mu(A)}, \]

in other words

\[ \mu(A) \leq \exp \left\{ -\frac{E[q_A^2]}{2} \right\}. \]

Similarly if \( A \) and \( B \) are \( H \)-separated, i.e., if \( A_\varepsilon \cap B = \emptyset \), for some \( \varepsilon > 0 \), where \( A_\varepsilon = \{ x \in W : q_A(x) \leq \varepsilon \} \), then

\[ \mu(A_\varepsilon) \leq \frac{1}{\mu(A)} e^{-\varepsilon^2/4} \]

and consequently

\[ \mu(A) \mu(B) \leq \exp \left( -\frac{\varepsilon^2}{4} \right). \]

**Remark:** We already know that, from the 0–1-law, \( q_A \) is almost surely finite, besides it satisfies \(|q_A(x + h) - q_A(x)| \leq |h|_H\), hence \( E[\exp \lambda q_A^2] < \infty \) for any \( \lambda < 1/2 \) (cf. [29]). In fact all these assertions can also be proved with the technique used below.

**Proof:** Let \( \nu_A \) be the measure defined by

\[ d\nu_A = \frac{1}{\mu(A)} 1_A d\mu. \]

Let \( \gamma_A \) be the solution of the Monge-Kantorovitch problem, it is easy to see that the support of \( \gamma_A \) is included in \( W \times A \), hence

\[ |x - y|_H \geq \inf \{|x - z|_H : z \in A\} = q_A(x), \]

\( \gamma_A \)-almost surely. This implies in particular that \( q_A \) is almost surely finite. It follows now from the inequality \( E[q_A^2] \leq -2 \log \mu(A), \)

hence the proof of the first inequality follows. For the second let \( B = A_\varepsilon \) and let \( \gamma_{AB} \) be the solution of the Monge-Kantorovitch problem corresponding to \( \nu_A, \nu_B \). Then we have from the Corollary 3.1

\[ d_H^2(\nu_A, \nu_B) \leq -4 \log \mu(A) \mu(B). \]
Besides the support of the measure $\gamma_{AB}$ is in $A \times B$, hence $\gamma_{AB}$-almost surely $|x-y|_H \geq \epsilon$ and the proof follows.

For the distance defined by

$$d_1(\nu, \mu) = \inf \left\{ \int_{W \times W} |x-y|_H d\theta : \pi_1 \theta = \mu, \pi_2 \theta = \nu \right\}$$

we have the following control:

**Theorem 3.2** Let $L \in \mathbb{L}_1^+ (\mu)$ with $E[L] = 1$. Then we have

$$d_1(L, \mu, \mu) \leq E \left[ \left\| (I + \mathcal{L})^{-1} \nabla L \right\|_H \right]. \tag{3.4}$$

**Proof:** To prove the theorem we shall use a technique developed in [8]. Using the conditioning with respect to the sigma algebra $\mathcal{V}_n = \sigma(\delta e_1, \ldots, \delta e_n)$, where $(e_i, i \geq 1)$ is a complete, orthonormal basis of $H$, we reduce the problem to the finite dimensional case. Moreover, we can assume that $L$ is a smooth, strictly positive function on $\mathbb{R}^n$.

Define now $\sigma = (I + \mathcal{L})^{-1} \nabla L$ and

$$\sigma_t(x) = \frac{\sigma(x)}{t + (1-t)L},$$

for $t \in [0,1]$. Let $(\phi_{s,t}(x), s \leq t \in [0,1])$ be the flow of diffeomorphisms defined by the following differential equation:

$$\phi_{s,t}(x) = x - \int_s^t \sigma_\tau (\phi_{s,\tau}(x)) d\tau.$$

From the standart results (cf. [29], Chapter V), it follows that $x \rightarrow \phi_{s,t}(x)$ is Gaussian under the probability $\Lambda_{s,t,\mu}$, where

$$\Lambda_{s,t} = \exp \int_s^t (\delta \sigma_\tau)(\phi_{s,\tau}(x)) d\tau$$

is the Radon-Nikodym density of $\phi_{s,t}^{-1}\mu$ with respect to $\mu$. Define

$$H_s(t, x) = \Lambda_{s,t}(x) \left\{ t + (1-t)L \circ \phi_{s,t}(x) \right\}.$$

It is easy to see that

$$\frac{d}{dt} H_s(t, x) = 0$$

for $t \in (s, 1)$. Hence the map $t \rightarrow H_s(t, x)$ is a constant, this implies that

$$\Lambda_{s,1}(x) = s + (1-s)L(x).$$
We have, as in the proof of Theorem 3.1,
\[
d_1(L\mu,\mu) \leq E[|\phi_{0,1}(x) - x|_H]\Lambda_{0,1}
\leq E\left[\int_0^1 |\sigma_t(\phi_{0,t}(x))|_H dt\right]
= E\left[\int_0^1 |\sigma_t(\phi_{0,1}(x))|_H dt\right]
= E\left[\int_0^1 |\sigma_t(\phi_{1,1}(x))|_H dt\right]
= E\left[\int_0^1 |\sigma_t|_H dt\right]
= E[|\sigma|_H],
\]
and the general case follows via the usual approximation procedure.

\[\square\]

4 Construction of the transport map

In this section we give the construction of the transport map in the Gaussian case. We begin with the following lemma:

**Lemma 4.1** Let \((W,\mu,H)\) be an abstract Wiener space, assume that \(f: W \to \mathbb{R}\) is a measurable function such that it is Gâteaux differentiable in the direction of the Cameron-Martin space \(H\), i.e., there exists some \(\nabla f: W \to H\) such that
\[
f(x + h) = f(x) + \int_0^1 (\nabla f(x + \tau h), h)_H d\tau,
\]
\(\mu\)-almost surely, for any \(h \in H\). If \(|\nabla f|_H \in L^2(\mu)\), then \(f\) belongs to the Sobolev space \(H^1(\mu)\).

**Proof:** Since \(|\nabla f|_H \leq |\nabla f|_H\), we can assume that \(f\) is positive. Moreover, for any \(n \in \mathbb{N}\), the function \(f_n = \min(f, n)\) has also a Gâteaux derivative such that \(|\nabla f_n|_H \leq |\nabla f|_H\) \(\mu\)-almost surely. It follows from the Poincaré inequality that the sequence \((f_n - E[f_n], n \geq 1)\) is bounded in \(L^2(\mu)\), hence it is also bounded in \(L^0(\mu)\). Since \(f\) is almost surely finite, the sequence \((f_n, n \geq 1)\) is bounded in \(L^0(\mu)\), consequently the deterministic sequence \((E[f_n], n \geq 1)\) is also bounded in \(L^0(\mu)\). This means that \(\sup_n E[f_n] < \infty\), hence the monotone convergence theorem implies that \(E[f] < \infty\) and the proof is completed.

\[\square\]

**Theorem 4.1** Let \(\nu\) be the measure \(d\nu = Ld\mu\), where \(L\) is a positive random variable, with \(E[L] = 1\). Assume that \(d_H(\mu,\nu) < \infty\) (for instance \(L \in L \log L\)). Then there exists a 1-convex function \(\phi \in H^1(\mu)\), unique up to a constant, such that the map \(T =\)
$I_W + ∇φ$ is the unique solution of the original problem of Monge. Moreover, its graph supports the unique solution of the Monge-Kantorovitch problem $γ$. Consequently

$$(I_W × T)μ = γ$$

In particular $T$ maps $μ$ to $ν$ and $T$ is almost surely invertible, i.e., there exists some $T^{-1}$ such that $T^{-1}ν = μ$ and that

$$1 = μ \{ x : T^{-1} ∘ T(x) = x \} = ν \{ y ∈ W : T ∘ T^{-1}(y) = y \}.$$  

Proof: Let $(\pi_n, n ≥ 1)$ be a sequence of regular, finite dimensional orthogonal projections of $H$ increasing to $I_H$. Denote their continuous extensions to $W$ by the same letters. For $x ∈ W$, we define $π_n^⊥ x := x_n^⊥ = x - π_n x$. Let $ν_n$ be the measure $π_n ν$. Since $ν$ is absolutely continuous with respect to $μ$, $ν_n$ is absolutely continuous with respect to $μ_n := π_n μ$ and

$$\frac{dν_n}{dμ_n} ∘ π_n = E[L|V_n] =: L_n,$$

where $V_n$ is the sigma algebra $σ(π_n)$ and the conditional expectation is taken with respect to $μ$. On the space $H_n$, the Monge-Kantorovitch problem, which consists of finding the probability measure which realizes the following infimum

$$d^2_H(μ_n, ν_n) = \inf \{ J(β) : β ∈ M_1(H_n × H_n), p_1β = μ_n, p_2β = ν_n \}$$

where

$$J(β) = \int_{H_n × H_n} |x - y|^2 dβ(x, y),$$

has a unique solution $γ_n$, where $p_i, i = 1, 2$ denote the projections $(x_1, x_2) → x_i, i = 1, 2$ from $H_n × H_n$ to $H_n$ and $M_1(H_n × H_n)$ denotes the set of probability measures on $H_n × H_n$. The measure $γ_n$ may be regarded as a measure on $W × W$, by taking its image under the injection $H_n × H_n ↪ W × W$ which we shall denote again by $γ_n$. It results from the finite dimensional results of Brenier and of McCann (6, 20) that there are two convex continuous functions (hence almost everywhere differentiable) $Φ_n$ and $Ψ_n$ on $H_n$ such that

$$Φ_n(x) + Ψ_n(y) ≥ (x, y)_{H_n}$$

for all $x, y ∈ H_n$ and that

$$Φ_n(x) + Ψ_n(y) = (x, y)_{H_n}$$

$γ_n$-almost everywhere. Hence the support of $γ_n$ is included in the graph of the derivative $∇Φ_n$ of $Φ_n$, hence $∇Φ_n μ_n = ν_n$ and the inverse of $∇Φ_n$ is equal to $∇Ψ_n$. Let

$$φ_n(x) = Φ_n(x) - \frac{1}{2} |x|_{H_n}^2$$

$$ψ_n(y) = Ψ_n(y) - \frac{1}{2} |y|_{H_n}^2.$$
Then $\phi_n$ and $\psi_n$ are 1-convex functions and they satisfy the following relations:

$$\phi_n(x) + \psi_n(y) + \frac{1}{2}|x - y|_H^2 \geq 0,$$

for all $x, y \in H_n$ and

$$\phi_n(x) + \psi_n(y) + \frac{1}{2}|x - y|_H^2 = 0,$$

$\gamma_n$-almost everywhere. From what we have said above, it follows that $\gamma_n$-almost surely

$$y = x + \nabla \phi_n(x),$$

consequently

$$J(\gamma_n) = E[|\nabla \phi_n|_H^2].$$

(4.7)

Let $q_n : W \times W \to H_n \times H_n$ be defined as $q_n(x, y) = (\pi_n x, \pi_n y)$. If $\gamma$ is any solution of the Monge-Kantorovitch problem, then $q_n \gamma \in \Sigma(\mu_n, \nu_n)$, hence

$$J(\gamma_n) \leq J(q_n \gamma) \leq J(\gamma) = d_H^2(\mu, \nu).$$

(4.8)

Combining the relation (4.7) with the inequality (4.8), we obtain the following bound

$$\sup_n J(\gamma_n) = \sup_n d_H^2(\mu_n, \nu_n)$$

$$= \sup_n E[|\nabla \phi_n|_H^2]$$

$$\leq d_H^2(\mu, \nu) = J(\gamma).$$

(4.9)

For $m \leq n$, $q_m \gamma_n \in \Sigma(\mu_m, \nu_m)$, hence we should have

$$J(\gamma_m) = \int_{W \times W} |\pi_m x - \pi_m y|_H^2 d\gamma_m(x, y)$$

$$\leq \int_{W \times W} |\pi_n x - \pi_n y|_H^2 d\gamma_n(x, y)$$

$$\leq \int_{W \times W} |\pi_n x - \pi_n y|_H^2 d\gamma_n(x, y)$$

$$= \int_{W \times W} |x - y|_H^2 d\gamma_n(x, y)$$

$$= J(\gamma_n),$$

where the third equality follows from the fact that we have denoted the $\gamma_n$ on $H_n \times H_n$ and its image in $W \times W$ by the same letter. Let now $\gamma$ be a weak cluster point of the sequence of measures $(\gamma_n, n \geq 1)$, where the word “weak”\(^6\) refers to the weak convergence of measures on $W \times W$. Since $(x, y) \to |x - y|_H$ is lower semi-continuous,

\(^6\)To prevent the reader against the trivial errors let us emphasize that $\gamma_n$ is not the projection of $\gamma$ on $W_n \times W_n$. 

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we have

\[ J(\gamma) = \int_{W \times W} |x - y|^2_H d\gamma(x, y) \]
\[ \leq \liminf_n \int_{W \times W} |x - y|^2_H d\gamma_n(x, y) \]
\[ = \liminf_n J(\gamma_n) \]
\[ \leq \sup_n J(\gamma_n) \]
\[ \leq J(\gamma) = d_H^2(\mu, \nu), \]

from the relation (4.9). Consequently

\[ J(\gamma) = \lim_n J(\gamma_n). \] (4.10)

Again from (4.9), if we replace \( \phi_n \) with \( \phi_n - E[\phi_n] \) and \( \psi_n \) with \( \psi_n + E[\phi_n] \) we obtain a bounded sequence \( (\phi_n, n \geq 1) \) in \( W \times W \), in particular it is bounded in the space \( L^2(\gamma) \) if we inject it into latter by \( \phi_n(x) \to \phi_n(x) \otimes 1(y) \). Consider now the sequence of the positive, lower semi-continuous functions \( (F_n, n \geq 1) \) defined on \( W \times W \) as

\[ F_n(x, y) = \phi_n(x) + \psi_n(y) + \frac{1}{2} |x - y|^2_H. \]

We have, from the relation (4.6)

\[ \int_{W \times W} F_n(x, y) d\gamma(x, y) = \int_W \phi_n d\mu + \int_W \psi_n(y) d\nu + \frac{1}{2} J(\gamma) \]
\[ = \frac{1}{2} (J(\gamma) - J(\gamma_n)) \to 0. \]

Consequently the sequence \( (F_n, n \geq 1) \) converges to zero in \( L^1(\gamma) \), therefore it is uniformly integrable. Since \( (\phi_n, n \geq 1) \) is uniformly integrable as explained above and since \( |x - y|^2 \) has a finite expectation with respect to \( \gamma \), it follows that \( (\psi_n, n \geq 1) \) is also uniformly integrable in \( L^1(\gamma) \) hence also in \( L^1(\nu) \). Let \( \phi' \) be a weak cluster point of \( (\phi_n, n \geq 1) \), then there exists a sequence \( (\phi'_n, n \geq 1) \) whose elements are the convex combinations of some elements of \( (\phi_k, k \geq n) \) such that \( (\phi'_n, n \geq 1) \) converges in the norm topology of \( W \times W \) and \( \mu \)-almost everywhere. Therefore the sequence \( (\psi'_n, n \geq 1) \), constructed from \( (\psi_k, k \geq n) \), converges in \( L^1(\nu) \) and \( \nu \)-almost surely. Define \( \phi \) and \( \psi \) as

\[ \phi(x) = \limsup_n \phi'_n(x) \]
\[ \psi(y) = \limsup_n \psi'_n(y), \]

hence we have

\[ G(x, y) = \phi(x) + \psi(y) + \frac{1}{2} |x - y|^2_H \geq 0 \]
for all \((x, y) \in W \times W\), also the equality holds \(\gamma\)-almost everywhere. Let now \(h\) be any element of \(H\), since \(x - y\) is in \(H\) for \(\gamma\)-almost all \((x, y) \in W \times W\), we have
\[
|x + h - y|_H^2 = |x - y|_H^2 + |h|_H^2 + 2\langle h, x - y \rangle
\]
\(\gamma\)-almost surely. Consequently
\[
\phi(x + h) - \phi(x) \geq -\langle h, x - y \rangle - \frac{1}{2} |h|_H^2
\]
\(\gamma\)-almost surely and this implies that
\[
y = x + \nabla \phi(x)
\]
\(\gamma\)-almost everywhere. Define now the map \(T : W \to W\) as \(T(x) = x + \nabla \phi(x)\), then
\[
\int_{W \times W} f(x, y) d\gamma(x, y) = \int_{W \times W} f(x, T(x)) d\gamma(x, y)
\]
\[
= \int_{W} f(x, T(x)) d\mu(x),
\]
for any \(f \in C_b(W \times W)\), consequently \((I_W \times T)\mu = \gamma\), in particular \(T \mu = \nu\).

Let us notice that any weak cluster point of \((\phi_n, n \geq 1)\), say \(\tilde{\phi}\), satisfies
\[
\nabla \tilde{\phi}(x) = y - x
\]
\(\gamma\)-almost surely, hence \(\mu\)-almost surely we have \(\tilde{\phi} = \phi\). This implies that \((\phi_n, n \geq 1)\) has a unique cluster point \(\phi\), consequently the sequence \((\phi_n, n \geq 1)\) converges weakly in \(\mathbb{D}_{2,1}\) to \(\phi\). Besides we have
\[
\lim_n \int_W |\nabla \phi_n|_H^2 d\mu = \lim_n J(\gamma_n)
\]
\[
= J(\gamma)
\]
\[
= \int_{W \times W} |x - y|_H^2 d\gamma(x, y)
\]
\[
= \int_{W} |\nabla \phi|_H^2 d\mu,
\]

hence \((\phi_n, n \geq 1)\) converges to \(\phi\) in the norm topology of \(\mathbb{D}_{2,1}\). Let us recapitulate what we have done till here: we have taken an arbitrary optimal \(\gamma \in \Sigma(\mu, \nu)\) and an arbitrary cluster point \(\phi\) of \((\phi_n, n \geq 1)\) and we have proved that \(\gamma\) is carried by the graph of \(T = I_W + \nabla \phi\). This implies that \(\gamma\) and \(\phi\) are unique and that the sequence \((\gamma_n, n \geq 1)\) has a unique cluster point \(\gamma\).

Certainly \((\psi_n, \geq 1)\) converges also in the norm topology of \(L^1(\nu)\). Moreover, from the finite dimensional situation, we have \(\nabla \psi_n(x) + \nabla \psi_n(y) = 0 \gamma_n\)-almost everywhere. Hence
\[
E_\nu[|\nabla \psi_n|_H^2] = E[|\nabla \phi_n|_H^2]
\]
this implies the boundedness of \((\nabla \psi_n, n \geq 1)\) in \(L^2(\nu, H)\) (i.e., \(H\)-valued functions). To complete the proof we have to show that, for some measurable, \(H\)-valued map, say \(\eta\), it
holds that $x = y + \eta(y) \gamma$-almost surely. For this let $F$ be a finite dimensional, regular subspace of $H$ and denote by $\pi_F$ the projection operator onto $F$ which is continuously extended to $W$, put $\pi_F = I_W - \pi_F$. We have $W = F \oplus F^\perp$, with $F^\perp = \ker \pi_F = \pi_F(W)$. Define the measures $\nu_F = \pi_F(\nu)$ and $\nu_F^\perp = \pi_F^\perp(\nu)$. From the construction of $\psi$, we know that, for any $v \in F^\perp$, the partial map $u \rightarrow \psi(u + v)$ is 1-convex on $F$. Let also $A = \{ y \in W : \psi(y) < \infty \}$, then $A$ is a Borel set with $\nu(A) = 1$ and it is easy to see that, for $\nu_F^\perp$-almost all $v \in F^\perp$, one has

$$
\nu(A|\pi_F^\perp = v) > 0.
$$

It then follows from Lemma 3.4 of [13], and from the fact that the regular conditional probability $\nu(\cdot | \pi_F^\perp = v)$ is absolutely continuous with respect to the Lebesgue measure of $F$, that $u \rightarrow \psi(u + v)$ is $\nu(\cdot | \pi_F^\perp = v)$-almost everywhere differentiable on $F$ for $\nu_F^\perp$-almost all $v \in F^\perp$. It then follows that, $\nu$-almost surely, $\psi$ is differentiable in the directions of $F$, i.e., there exists $\nabla_F \psi \in F \nu$-almost surely. Since we also have

$$
\psi(y + k) - \psi(y) \geq (x - y, k)_H - \frac{1}{2}k^2_H,
$$

we obtain, $\gamma$-almost surely

$$
(\nabla_F \psi(y), k)_H = (x - y, k)_H,
$$

for any $k \in F$. Consequently

$$
\nabla_F \psi(y) = \pi_F(x - y)
$$

$\gamma$-almost surely. Let now $(F_n, n \geq 1)$ be a total, increasing sequence of regular subspaces of $H$, we have a sequence $(\nabla_n \psi, n \geq 1)$ bounded in $L^2(\nu)$ hence also bounded in $L^2(\gamma)$. Besides $\nabla_n \psi(y) = \pi_n x - \pi_n y \gamma$-almost surely. Since $(\pi_n(x - y), n \geq 1)$ converges in $L^2(\gamma, H)$, $(\nabla_n \psi, n \geq 1)$ converges in the norm topology of $L^2(\gamma, H)$. Let us denote this limit by $\eta$, then we have $x = y + \eta(y) \gamma$-almost surely. Note that, since $\pi_n \eta = \nabla_n \psi$, we can even write in a weak sense that $\eta = \nabla \psi$. If we define $T^{-1}(y) = y + \eta(y)$, we see that

$$
1 = \gamma\{ (x, y) \in W \times W : T \circ T^{-1}(y) = y \} = \gamma\{ (x, y) \in W \times W : T^{-1} \circ T(x) = x \},
$$

and this completes the proof of the theorem.

\[\square\]

**Remark 4.1** Assume that the operator $\nabla$ is closable with respect to $\nu$, then we have $\eta = \nabla \psi$. In particular, if $\nu$ and $\mu$ are equivalent, then we have

$$
T^{-1} = I_W + \nabla \psi,
$$

where is $\psi$ is a 1-convex function.
Remark 4.2 Assume that $L \in \mathbb{L}^1_+(\mu)$, with $E[L] = 1$ and let $(D_k, k \in \mathbb{N})$ be a measurable partition of $W$ such that on each $D_k$, $L$ is bounded. Define $d\nu = Ld\mu$ and $\nu_k = \nu(\cdot | D_k)$. It follows from Theorem 3.1 that $d_H(\mu, \nu_k) < \infty$. Let then $T_k$ be the map constructed in Theorem 4.1 satisfying $T_k\mu = \nu_k$. Define $n(dk)$ as the probability distribution on $\mathcal{N}$ given by $n(\{k\}) = \nu(D_k)$, $k \in \mathbb{N}$. Then we have

$$\int_W f(y)d\nu(y) = \int_{W \times \mathbb{N}} f(T_k(x))\mu(dx)n(dk).$$

A similar result is given in [12], the difference with that above lies in the fact that we have a more precise information about the probability space on which $T$ is defined.

5 Polar factorization of the absolutely continuous transformations of the Wiener space

Assume that $V = I_W + v : W \to W$ be an absolutely continuous transformation and let $L \in \mathbb{L}^1_+(\mu)$ be the Radon-Nikodym derivative of $V\mu$ with respect to $\mu$. Let $T = I_W + \nabla \phi$ be the transport map such that $T\mu = L\mu$. Then it is easy to see that the map $s = T^{-1} \circ V$ is a rotation, i.e., $s\mu = \mu$ (cf. [29]) and it can be represented as $s = I_W + \alpha$. In particular we have

$$\alpha + \nabla \phi \circ s = v. \tag{5.11}$$

Since $\phi$ is a 1-convex map, we have $h \to \frac{1}{2}|h|_H^2 + \phi(x + h)$ is almost surely convex (cf. [14]). Let $s' = I_W + \alpha'$ be another rotation with $\alpha' : W \to H$. By the 1-convexity of $\phi$, we have

$$\frac{1}{2}|\alpha'|_H^2 + \phi \circ s' \geq \frac{1}{2}|\alpha|_H^2 + \phi \circ s + \left(\alpha + \nabla \phi \circ s, \alpha' - \alpha\right)_H,$$

$\mu$-almost surely. Taking the expectation of both sides, using the fact that $s$ and $s'$ preserve the Wiener measure $\mu$ and the identity (5.11), we obtain

$$E \left[\frac{1}{2}|\alpha|_H^2 - (v, \alpha)_H\right] \leq E \left[\frac{1}{2}|\alpha'|_H^2 - (v, \alpha')_H\right].$$

Hence we have proven the existence part of the following

**Proposition 5.1** Let $\mathcal{R}_2$ denote the subset of $L^2(\mu, H)$ whose elements are defined by the property that $x \to x + \eta(x)$ is a rotation, i.e., it preserves the Wiener measure. Then $\alpha$ is the unique element of $\mathcal{R}_2$ which minimizes the functional

$$\eta \to M_v(\eta) = E \left[\frac{1}{2}|\eta|_H^2 - (v, \eta)_H\right].$$

**Proof:** To show the uniqueness, assume that $\eta \in \mathcal{R}_2$ be another map minimizing $J_v$. Let $\beta$ be the measure on $W \times W$, defined as

$$\int_{W \times W} f(x,y)d\beta(x,y) = \int_W f(x + \eta(x), V(x))d\mu.$$
Then the first marginal of $\beta$ is $\mu$ and the second marginal is $L\mu$. Since $\gamma = (I_W \times T)\mu$ is the unique solution of the Monge-Kantorovitch problem, we should have
\[
\int |x - y|^2_H d\beta(x, y) > \int |x - y|^2_H d\gamma(x, y) = E[|\nabla \phi|^2_H].
\]
However we have
\[
\int_{W \times W} |x - y|^2_H d\beta(x, y) = E[|v - \eta|^2_H] \\
= E[|v|^2_H] + 2M_v(\eta) \\
= E[|v|^2_H] + 2M_v(\alpha) \\
= E[|v - \alpha|^2_H] \\
= E[|\nabla \phi \circ s|^2_H] \\
= E[|\nabla \phi|^2_H] \\
= \int_{W \times W} |x - y|^2_H d\gamma(x, y) \\
= J(\gamma)
\]
and this gives a contradiction to the uniqueness of $\gamma$.

The following theorem, whose proof is rather easy, gives a better understanding of the structure of absolutely continuous transformations of the Wiener measure:

**Theorem 5.1** Assume that $U : W \to W$ be a measurable map and $L \in \mathbb{L} \log \mathbb{L}$ a positive random variable with $E[L] = 1$. Assume that the measure $\nu = L \cdot \mu$ is a Girsanov measure for $U$, i.e., that one has
\[E[f \circ UL] = E[f],\]
for any $f \in C_0(W)$. Then there exists a unique map $T = I_W + \nabla \phi$ with $\phi \in \mathbb{D}_{2,1}$ is 1-convex, and a measure preserving transformation $R : W \to W$ such that $U \circ T = R \mu$-almost surely and $U = R \circ T^{-1} \nu$-almost surely.

**Proof:** By Theorem 4.1 there is a unique map $T = I_W + \nabla \phi$, with $\phi \in \mathbb{D}_{2,1}$, 1-convex such that $T$ transports $\mu$ to $\nu$. Since $U\nu = \mu$, we have
\[E[f \circ UL] = E[f],\]
Therefore $x \to U \circ T(x)$ preserves the measure $\mu$. The rest is obvious since $T^{-1}$ exists $\nu$-almost surely.

Another version of Theorem 5.1 can be announced as follows:
Theorem 5.2 Assume that $Z : W \to W$ is a measurable map such that $Z\mu \ll \mu$, with $d_H(Z\mu,\mu) < \infty$. Then $Z$ can be decomposed as

$$Z = T \circ s,$$

where $T$ is the unique transport map of the Monge-Kantorovitch problem for $\Sigma(\mu, Z\mu)$ and $s$ is a rotation.

Proof: Let $L$ be the Radon-Nikodym derivative of $Z\mu$ with respect to $\mu$. We have, from Theorem 4.1,

$$E[f] = E[f \circ T^{-1} \circ T] = E[f \circ T^{-1} L] = E[f \circ T^{-1} \circ Z],$$

for any $f \in C_b(W)$. Hence $T^{-1} \circ Z = s$ is a rotation. Since $T$ is uniquely defined, $s$ is also uniquely defined. □

Although the following result is a translation of the results of this section, it is interesting from the point of view of stochastic differential equations:

Theorem 5.3 Let $(W, \mu, H)$ be the standard Wiener space on $\mathbb{R}^d$, i.e., $W = C(\mathbb{R}_+, \mathbb{R}^d)$. Assume that there exists a probability $P \ll \mu$ which is the weak solution of the stochastic differential equation

$$dy_t = dW_t + b(t, y)dt,$$

such that $d_H(P, \mu) < \infty$. Then there exists a process $(T_t, t \in \mathbb{R}_+)$ which is a pathwise solution of some stochastic differential equation whose law is equal to $P$.

Proof: Let $T$ be the transport map constructed in Theorem 4.1 corresponding to $dP/d\mu$. Then it has an inverse $T^{-1}$ such that $\mu\{T^{-1} \circ T(x) = x\} = 1$. Let $\phi$ be the 1-convex function such that $T = I_W + \nabla \phi$ and denote by $(D_s \phi, s \in \mathbb{R}_+)$ the representation of $\nabla \phi$ in $L^2(\mathbb{R}_+, ds)$. Define $T_t(x)$ as the trajectory $T(x)$ evaluated at $t \in \mathbb{R}_+$. Then it is easy to see that $(T_t, t \in \mathbb{R}_+)$ satisfies the stochastic differential equation

$$T_t(x) = W_t(x) + \int_0^t l(s, T(s))ds , t \in \mathbb{R}_+ ,$$

where $W_t(x) = x(t)$ and $l(s, x) = D_s \phi \circ T^{-1}(x)$.

6 Construction and uniqueness of the transport map in the general case

In this section we call optimal every probability measure\footnote{In fact the results of this section are essentially true for the bounded, positive measures.} $\gamma$ on $W \times W$ such that $J(\gamma) < \infty$ and that $J(\gamma) \leq J(\theta)$ for every other probability $\theta$ having the same marginals.
as those of $\gamma$. We recall that a finite dimensional subspace $F$ of $W$ is called regular if the corresponding projection is continuous. Similarly a finite dimensional projection of $H$ is called regular if it has a continuous extension to $W$.

We begin with the following lemma which answers all kind of questions of measurability that we may encounter in the sequel:

**Lemma 6.1** Consider two uncountable Polish spaces $X$ and $T$. Let $t \to \gamma_t$ be a Borel family of probabilities on $X$ and let $\mathcal{F}$ be a separable sub-$\sigma$-algebra of the Borel $\sigma$-algebra $\mathcal{B}$ of $X$. Then there exists a Borel kernel

$$N_t f(x) = \int_X f(y) N_t(x, dy),$$

such that, for any bounded Borel function $f$ on $X$, the following properties hold true:

i) $(t,x) \to N_t f(x)$ is Borel measurable on $T \times X$.

ii) For any $t \in T$, $N_t f$ is an $\mathcal{F}$-measurable version of the conditional expectation $E_{\gamma_t}[f | \mathcal{F}]$.

**Proof:** Assume first that $\mathcal{F}$ is finite, hence it is generated by a finite partition $\{A_1, \ldots, A_k\}$. In this case it suffices to take

$$N_t f(x) = \sum_{i=1}^k \frac{1}{\gamma_t(A_i)} \left( \int_{A_i} f d\gamma_t \right) 1_{A_i}(x) \quad \text{(with } 0 = 0).$$

For the general case, take an increasing sequence $(\mathcal{F}_n, n \geq 1)$ of finite sub-$\sigma$-algebras whose union generates $\mathcal{F}$. Without loss of generality we can assume that $(X, \mathcal{B})$ is the Cantor set (Kuratowski Theorem, cf., [9]). Then for every clopen set (i.e., a set which is closed and open at the same time) $G$ and any $t \in T$, the sequence $(N_t^n 1_G, n \geq 1)$ converges $\gamma_t$-almost everywhere. Define

$$H_G(t, x) = \limsup_{m,n \to \infty} |N_t^n 1_G(x) - N_t^m 1_G(x)|.$$

$H_G$ is a Borel function on $T \times X$ which vanishes $\gamma_t$-almost all $x \in X$, besides, for any $t \in T$, $x \to H_G(t, x)$ is $\mathcal{F}$-measurable. As there exist only countably many clopen sets in $X$, the function

$$H(t, x) = \sup_G H_G(t, x)$$

inherits all the measurability properties. Let $\theta$ be any probability on $X$, for any clopen $G$, define

$$N_t 1_G(x) = \lim_n N_t^n 1_G(x) \quad \text{if } H(t, x) = 0,$$

$$= \theta(G) \quad \text{if } H(t, x) > 0.$$  

Hence, for any $t \in T$, we get an additive measure on the Boolean algebra of clopen sets of $X$. Since such a measure is $\sigma$-additive and extends uniquely as a $\sigma$-additive measure on $\mathcal{B}$, the proof is completed. \[\square\]
Lemma 6.2 Let $\rho$ and $\nu$ be two probability measures on $W$ such that
\[ d_H(\rho, \nu) < \infty \]
and let $\gamma \in \Sigma(\rho, \nu)$ be an optimal measure, i.e., $J(\gamma) = d_H^2(\rho, \nu)$, where $J$ is given by (1.1). Assume that $F$ is a regular finite dimensional subspace of $W$ with the corresponding projection $\pi_F$ from $W$ to $F$ and let $\pi_F^\perp = I_W - \pi_F$. Define $p_F$ as the projection from $W \times W$ onto $F$ with $p_F(x, y) = \pi_F x$ and let $p_F^\perp(x, y) = \pi_F^\perp x$. Consider the Borel disintegration
\[ \gamma(\cdot) = \int_{F^\perp \times W} \gamma(\cdot | x^\perp) \gamma^\perp(dz^\perp) \]
along the projection of $W \times W$ on $F^\perp$, where $\rho^\perp$ is the measure $\pi_F^\perp \rho$, $\gamma(\cdot | x^\perp)$ denotes the regular conditional probability $\gamma(\cdot | p_F^\perp x = x^\perp)$ and $\gamma^\perp$ is the measure $p_F^\perp \gamma$. Then, $\rho^\perp$ and $\gamma^\perp$-almost surely $\gamma(\cdot | x^\perp)$ is optimal on $(x^\perp + F) \times W$.

Proof: Let $p_1, p_2$ be the projections of $W \times W$ defined as $p_1(x, y) = \pi_F(x)$ and $p_2(x, y) = \pi_F(y)$. Note first the following obvious identity:
\[ p_1 \gamma(\cdot | x^\perp) = \rho(\cdot | x^\perp), \]
$\rho^\perp$ and $\gamma^\perp$-almost surely. Define the sets $B \subset F^\perp \times \mathcal{M}_1(F \times F)$ and $C$ as
\[ B = \{(x^\perp, \theta) : \theta \in \Sigma(p_1 \gamma(\cdot | x^\perp), p_2 \gamma(\cdot | x^\perp))\} \]
\[ C = \{(x^\perp, \theta) \in B : J(\theta) < J(\gamma(\cdot | x^\perp))\}, \]
where $\mathcal{M}_1(F \times F)$ denotes the set of probability measures on $F \times F$. Let $K$ be the projection of $C$ on $F^\perp$. Since $B$ and $C$ are Borel measurable, $K$ is a Souslin set, hence it is $\rho^\perp$-measurable. The selection theorem (cf. [9]) implies the existence of a measurable map
\[ x^\perp \rightarrow \theta_{x^\perp} \]
from $K$ to $\mathcal{M}_1(F \times F)$ such that, $\rho^\perp$-almost surely, $(x^\perp, \theta_{x^\perp}) \in C$. Define
\[ \theta(\cdot) = \int_K \theta_{x^\perp}(\cdot) d\rho^\perp(x^\perp) + \int_{K^c} \gamma(\cdot | x^\perp) d\rho^\perp(x^\perp). \]
Then \( \theta \in \Sigma(\rho, \nu) \) and we have
\[
J(\theta) = \int_K J(\theta_{x\perp})d\rho(x\perp) + \int_{K^c} J(\gamma(\cdot|x\perp))d\rho(x\perp)
\]
\[
< \int_K J(\gamma(\cdot|x\perp))d\rho(x\perp) + \int_{K^c} J(\gamma(\cdot|x\perp))d\rho(x\perp)
\]
\[
= J(\gamma),
\]
hence we obtain \( J(\theta) < J(\gamma) \) which is a contradiction to the optimality of \( \gamma \). 

\[\square\]

**Lemma 6.3** Assume that the hypothesis of Lemma 6.2 holds and let \( F \) be any regular finite dimensional subspace of \( W \). Denote by \( \pi_F \) the projection operator associated to it and let \( \pi_F^\perp = I_W - \pi_F \). If \( \pi_F^\perp \rho \)-almost surely, the regular conditional probability \( \rho(\cdot|\pi_F^\perp = x\perp) \) vanishes on the subsets of \( x\perp + F \) whose Hausdorff dimension are at most equal to \( \dim(F) - 1 \), then there exists a map \( T_F : F \times F^\perp \to F \) such that
\[
\gamma \left( \{ (x, y) \in W \times W : \pi_Fy = T_F(\pi_Fx, \pi_F^\perp x) \} \right) = 1.
\]

**Proof:** Let \( C_{x\perp} \) be the support of the regular conditional probability \( \gamma(\cdot|x\perp) \) in \( (x\perp + F) \times W \). We know from Lemma 6.2 that the measure \( \gamma(\cdot|x\perp) \) is optimal in \( \Sigma(\pi_1\gamma(\cdot|x\perp), \pi_2\gamma(\cdot|x\perp)) \), with \( J(\gamma(\cdot|x\perp)) < \infty \) for \( \rho^\perp \)-almost everywhere \( x\perp \). From Theorem 2.3 of [16] and from [1], the set \( C_{x\perp} \) is cyclically monotone, moreover, \( C_{x\perp} \) is a subset of \( (x\perp + F) \times H \), hence the cyclic monotonicity of it implies that the set \( K_{x\perp} \subset F \times F \), defined as
\[
K_{x\perp} = \{ (u, \pi_Fv) \in F \times F : (x\perp + u, v) \in C_{x\perp} \}
\]
is cyclically monotone in \( F \times F \). Therefore \( K_{x\perp} \) is included in the subdifferential of a convex function defined on \( F \). Since, by hypothesis, the first marginal of \( \gamma(\cdot|x\perp) \), i.e., \( \rho(\cdot|x\perp) \) vanishes on the subsets of \( x\perp + F \) of co-dimension one, the subdifferential under question, denoted as \( U_F(u, x\perp) \) is \( \rho(\cdot|x\perp) \)-almost surely univalent (cf. [2], [20]). This implies that
\[
\gamma(\cdot|x\perp) \left( \{ (u, v) \in C_{x\perp} : \pi_Fv = U_F(u, x\perp) \} \right) = 1,
\]
\( \rho^\perp \)-almost surely. Let
\[
K_{x\perp,u} = \{ v \in W : (u, v) \in K_{x\perp} \}.
\]
Then \( K_{x\perp,u} \) consists of a single point for almost all \( u \) with respect to \( \rho(\cdot|x\perp) \). Let
\[
N = \{ (u, x\perp) \in F \times F^\perp : \text{Card}(K_{x\perp,u}) > 1 \},
\]
note that \( N \) is a Souslin set, hence it is universally measurable. Let \( \sigma \) be the measure which is defined as the image of \( \rho \) under the projection \( x \to (\pi_Fx, \pi_F^\perp x) \). We then have
\[
\sigma(N) = \int_{F^\perp} \rho^\perp(dx\perp) \int_F 1_N(u, x\perp)\rho(du|x\perp)
\]
\[
= 0.
\]
Hence \((u, x^\perp) \mapsto K_{x^\perp u} = \{y\}\) is \(\rho\) and \(\gamma\)-almost surely well-defined and it suffices to denote this map by \(T_F\) to achieve the proof.

**Theorem 6.1** Suppose that \(\rho\) and \(\nu\) are two probability measures on \(W\) such that

\[d_H(\rho, \nu) < \infty.\]

Let \((\pi_n, n \geq 1)\) be a total increasing sequence of regular projections (of \(H\), converging to the identity map of \(H\)). Suppose that, for any \(n \geq 1\), the regular conditional probabilities \(\rho(\cdot | x_n^\perp = x^\perp)\) vanish \(\pi_n^\perp\)-almost surely on the subsets of \((\pi_n^\perp)^{-1}(W)\) with Hausdorff dimension \(n - 1\). Then there exists a unique solution of the Monge-Kantorovitch problem, denoted by \(\gamma \in \Sigma(\rho, \nu)\) and \(\gamma\) is supported by the graph of a Borel map \(T\) which is the solution of the Monge problem. \(T : W \to W\) is of the form \(T = I_W + \xi\), where \(\xi \in H\) almost surely. Besides we have

\[
d_H^2(\rho, \nu) = \int_{W \times W} |T(x) - x|_H^2 d\gamma(x, y)
= \int_W |T(x) - x|_H^2 d\rho(x),
\]

and for \(\pi_n^\perp\)-almost almost all \(x_n^\perp\), the map \(u \to \xi(u + x_n^\perp)\) is cyclically monotone on \((\pi_n^\perp)^{-1}\{x_n^\perp\}\), in the sense that

\[
\sum_{i=1}^{N} \left(\xi(x_n^\perp + u_i), u_{i+1} - u_i\right)_H \leq 0
\]

\(\pi_n^\perp\)-almost surely, for any cyclic sequence \(\{u_1, \ldots, u_N, u_{N+1} = u_1\}\) from \(\pi_n(W)\). Finally, if, for any \(n \geq 1\), \(\pi_n^\perp\)-almost surely, \(\nu(\cdot | x_n^\perp = y^\perp)\) also vanishes on the \(n - 1\)-Hausdorff dimensional subsets of \((\pi_n^\perp)^{-1}(W)\), then \(T\) is invertible, i.e., there exists \(S : W \to W\) of the form \(S = I_W + \eta\) such that \(\eta \in H\) satisfies a similar cyclic monotonicity property as \(\xi\) and that

\[
1 = \gamma \{ (x, y) \in W \times W : T \circ S(y) = y \}
= \gamma \{ (x, y) \in W \times W : S \circ T(x) = x \}.
\]

In particular we have

\[
d_H^2(\rho, \nu) = \int_{W \times W} |S(y) - y|_H^2 d\gamma(x, y)
= \int_W |S(y) - y|_H^2 d\nu(y).
\]

**Remark 6.2** In particular, for all the measures \(\rho\) which are absolutely continuous with respect to the Wiener measure \(\mu\), the second hypothesis is satisfied, i.e., the measure \(\rho(\cdot | x_n^\perp = x^\perp)\) vanishes on the sets of Hausdorff dimension \(n - 1\).
Proof: Let \((F_n, n \geq 1)\) be the increasing sequence of regular subspaces associated to \((\pi_n, n \geq 1)\), whose union is dense in \(W\). From Lemma 6.3, for any \(F_n\), there exists a map \(T_n\), such that \(\pi_n y = T_n(\pi_n x, \pi_n^\perp x)\) for \(\gamma\)-almost all \((x, y)\), where \(\pi_n^\perp = I_W - \pi_n\). Write \(T_n\) as \(I_n + \xi_n\), where \(I_n\) denotes the identity map on \(F_n\). Then we have the following representation:
\[
\pi_n y = \pi_n x + \xi_n(\pi_n x, \pi_n^\perp x),
\]
\(\gamma\)-almost surely. Since
\[
\pi_n y - \pi_n x = \pi_n (y - x) = \xi_n(\pi_n x, \pi_n^\perp x)
\]
and since \(y - x \in H\) \(\gamma\)-almost surely, \((\pi_n y - \pi_n x, n \geq 1)\) converges \(\gamma\)-almost surely. Consequently \((\xi_n, n \geq 1)\) converges \(\gamma\), hence \(\rho\) almost surely to a measurable \(\xi\). Consequently we obtain
\[
\gamma \left( \{ (x, y) \in W \times W : y = x + \xi(x) \} \right) = 1.
\]
Since \(J(\gamma) < \infty\), \(\xi\) takes its values almost surely in the Cameron-Martin space \(H\). The cyclic monotonicity of \(\xi\) is obvious. To prove the uniqueness, assume that we have two optimal solutions \(\gamma_1\) and \(\gamma_2\) with the same marginals and \(J(\gamma_1) = J(\gamma_2)\). Since \(\beta \mapsto J(\beta)\) is linear, the measure defined as \(\gamma = \frac{1}{2}(\gamma_1 + \gamma_2)\) is also optimal and it has also the same marginals \(\rho\) and \(\nu\). Consequently, it is also supported by the graph of a map \(T\). Note that \(\gamma_1\) and \(\gamma_2\) are absolutely continuous with respect to \(\gamma\), let \(L_1(x, y)\) be the Radon-Nikodym density of \(\gamma_1\) with respect to \(\gamma\). For any \(f \in C_b(W)\), we then have
\[
\int_W f d\rho = \int_{W \times W} f(x) d\gamma_1(x, y) = \int_{W \times W} f(x) L_1(x, y) d\gamma(x, y) = \int_W f(x) L_1(x, T(x)) d\rho(x).
\]
Therefore we should have \(\rho\)-almost surely, \(L_1(x, T(x)) = 1\), hence also \(L_1 = 1\) almost everywhere \(\gamma\) and this implies that \(\gamma = \gamma_1 = \gamma_2\). The second part about the invertibility of \(T\) is totally symmetric, hence its proof follows along the same lines as the proof for \(T\). \(\square\)

Corollary 6.1 Assume that \(\rho\) is equivalent to the Wiener measure \(\mu\), then for any \(h_1, \ldots, h_N \in H\) and for any permutation \(\tau\) of \(\{1, \ldots, N\}\), we have, with the notations of Theorem 6.1,
\[
\sum_{i=1}^N (h_i + \xi(x + h_i), h_{\tau(i)} - h_i)_H \leq 0
\]
\(\rho\)-almost surely.
Proof: Again with the notations of the theorem, \( \rho_k^\perp \)-almost surely, the graph of the map \( x_k \rightarrow x_k + \xi_k(x_k, x_k^\perp) \) is cyclically monotone on \( F_k \). Hence, for the case \( h_i \in F_n \) for all \( i = 1, \ldots, N \) and \( n \leq k \), we have

\[
\sum_{i=1}^{N} \left( h_i + x_k + \xi_k(x_k + h_i, x_k^\perp), h_{\tau(i)} - h_i \right)_H \leq 0.
\]

Since \( \sum_{i}(x_k, h_{\tau(i)} - h_i)_H = 0 \), we also have

\[
\sum_{i=1}^{N} \left( h_i + \xi_k(x_k + h_i, x_k^\perp), h_{\tau(i)} - h_i \right)_H \leq 0.
\]

We know that \( \xi_k(x_k + h_i, x_k^\perp) \) converges to \( \xi(x + h) \) \( \rho \)-almost surely. Moreover \( h \rightarrow \xi(x + h) \) is continuous from \( H \) to \( L^0(\rho) \) and the proof follows. \( \square \)

7 The Monge-Ampère equation

Assume that \( W = I_{\mathbb{R}^n} \) and take a density \( L \in I_{\mathbb{L} \log I_{\mathbb{L}}} \). Let \( \phi \in D_{2,1} \) be the 1-convex function such that \( T = I + \nabla \phi \) maps \( \mu \) to \( L \cdot \mu \). Let \( S = I + \nabla \psi \) be its inverse with \( \psi \in D_{2,1} \). Let now \( \nabla^2 \phi \) be the second Alexandrov derivative of \( \phi \), i.e., the Radon-Nikodym derivative of the absolutely continuous part of the vector measure \( \nabla^2 \phi \) with respect to the Gaussian measure \( \mu \) on \( \mathbb{R}^n \). Since \( \phi \) is 1-convex, it follows that \( \nabla^2 \phi \geq -I_{\mathbb{R}^n} \) in the sense of the distributions, consequently \( \nabla^2 \phi \geq -I_{\mathbb{R}^n} \mu \)-almost surely. Define also the Alexandrov version \( L_a \phi \) of \( L \phi \) as the Radon-Nikodym derivative of the absolutely continuous part of the distribution \( L \phi \). Since we are in finite dimensional situation, we have the explicit expression for \( L_a \phi \) as

\[
L_a \phi(x) = (\nabla \phi(x), x)_{\mathbb{R}^n} - \text{trace} \left( \nabla^2 \phi \right).
\]

Let \( \Lambda \) be the Gaussian Jacobian

\[
\Lambda = \det_2 \left( I_{\mathbb{R}^n} + \nabla^2 \phi \right) \exp \left\{ -L_a \phi - \frac{1}{2} |\nabla \phi|^2_{\mathbb{R}^n} \right\}.
\]

Remark 7.1 In this expression as well as in the sequel, the notation \( \det_2 (I_H + A) \) denotes the modified Carleman-Fredholm determinant of the operator \( I_H + A \) on a Hilbert space \( H \). If \( A \) is an operator of finite rank, then it is defined as

\[
\det_2 (I_H + A) = \prod_{i=1}^{n} (1 + l_i) e^{-l_i},
\]

where \( (l_i, i \leq n) \) denotes the eigenvalues of \( A \) counted with respect to their multiplicity.

In fact this determinant has an analytic extension to the space of Hilbert-Schmidt operators on a separable Hilbert space, cf. \[10\] and Appendix A.2 of \[29\]. As explained
in [29], the modified determinant exists for the Hilbert-Schmidt operators while the ordinary determinant does not, since the latter requires the existence of the trace of \( A \). Hence the modified Carleman-Fredholm determinant is particularly useful when one studies the absolute continuity properties of the image of a Gaussian measure under non-linear transformations in the setting of infinite dimensional Banach spaces (cf., [29] for further information).

It follows from the change of variables formula given in Corollary 4.3 of [21], that, for any \( f \in C_b(\mathbb{R}^n) \),

\[
E[f \circ T \Lambda] = E\left[f 1_{\partial \Phi(M)}\right],
\]

where \( M \) is the set of non-degeneracy of \( I_{\mathbb{R}^n} + \nabla^2 a \),

\[
\Phi(x) = \frac{1}{2}|x|^2 + \phi(x)
\]

and \( \partial \Phi \) denotes the subdifferential of the convex function \( \Phi \). Let us note that, in case \( L > 0 \) almost surely, \( T \) has a global inverse \( S \), i.e., \( S \circ T = T \circ S = I_{\mathbb{R}^n} \) \( \mu \)-almost surely and \( \mu(\partial \Phi(M)) = \mu(S^{-1}(M)) \). Assume now that \( \Lambda > 0 \) almost surely, i.e., that \( \mu(M) = 1 \). Then, for any \( f \in C_b(\mathbb{R}^n) \), we have

\[
E[f \circ T] = E\left[f \circ T \frac{\Lambda}{\Lambda \circ T^{-1} \circ T}\right] = E\left[f \frac{1}{\Lambda \circ T^{-1}} 1_{\partial \Phi(M)}\right] = E[f L],
\]

where \( T^{-1} \) denotes the left inverse of \( T \) whose existence is guaranteed by Theorem 4.1. Since \( T(x) \in \partial \Phi(M) \) almost surely, it follows from the above calculations

\[
\frac{1}{\Lambda} = L \circ T,
\]

almost surely. Take now any \( t \in [0, 1) \), the map \( x \rightarrow \frac{1}{2}|x|^2_H + t\phi(x) = \Phi_t(x) \) is strictly convex and a simple calculation implies that the mapping \( T_t = I + t\nabla \phi \) is \( (1-t) \)-monotone (cf. [29], Chapter 6), consequently it has a left inverse denoted by \( S_t \). Let us denote by \( \Psi_t \) the Legendre transformation of \( \Phi_t \):

\[
\Psi_t(y) = \sup_{x \in \mathbb{R}^n} \{(x, y) - \Phi_t(x)\}.
\]

A simple calculation shows that

\[
\Psi_t(y) = \sup_x \left[ (1-t) \left\{(x, y) - \frac{|x|^2}{2}\right\} + t \left\{(x, y) - \frac{|x|^2}{2} - \phi(x)\right\} \right] \leq (1-t) \frac{|y|^2}{2} + t \Psi_1(y).
\]

Since \( \Psi_1 \) is the Legendre transformation of \( \Phi_1(x) = |x|^2/2 + \phi(x) \) and since \( L \in \mathbb{L} \log \mathbb{L} \), it is finite on a convex set of full measure, hence it is finite everywhere. Consequently
\( \Psi_t(y) < \infty \) for any \( y \in \mathbb{R}^n \). Since a finite, convex function is almost everywhere differentiable, \( \nabla \Psi_t \) exists almost everywhere on and it is equal almost everywhere on \( T_t(M_t) \) to the left inverse \( T_t^{-1} \), where \( M_t \) is the set of non-degeneracy of \( I_t + t \nabla_a^2 \phi \). Note that \( \mu(M_t) = 1 \). The strict convexity implies that \( T_t^{-1} \) is Lipschitz with a Lipschitz constant \( \frac{1}{1-t} \). Let now \( \Lambda_t \) be the Gaussian Jacobian

\[
\Lambda_t = \det_2 \left( I_{\mathbb{R}^n} + t \nabla^2_a \phi \right) \exp \left\{ -t \mathcal{L}_a \phi - \frac{t^2}{2} |\nabla \phi|^2_{\mathbb{R}^n} \right\}.
\]

Since the domain of \( \phi \) is the whole space \( \mathbb{R}^n \), \( \Lambda_t > 0 \) almost surely, hence, as we have explained above, it follows from the change of variables formula of [21] that \( T_t \mu \) is absolutely continuous with respect to \( \mu \) and that

\[
\frac{1}{\Lambda_t} = L_t \circ T_t ,
\]

\( \mu \)-almost surely.

Let us come back to the infinite dimensional case: we first give an inequality which may be useful.

**Theorem 7.1** Assume that \((W, \mu, H)\) is an abstract Wiener space, assume that \( K, L \in I_{L^1}^1(\mu) \) with \( K > 0 \) almost surely and denote by \( T : W \to W \) the transfer map \( T = I_{W} + \nabla \phi \), which maps the measure \( K \, d\mu \) to the measure \( L \, d\mu \). Then the following inequality holds:

\[
\frac{1}{2} \, E[|\nabla \phi|^2_H] \leq E[-\log K + \log L \circ T] .
\]

**Proof:** Let us define \( k \) as \( k = K \circ T^{-1} \), then for any \( f \in C_b(W) \), we have

\[
\int_W f(y) L(y) \, d\mu(y) = \int_W f \circ T(x) K(x) \, d\mu(x) = \int_W f \circ T(x) k \circ T(x) \, d\mu(x) ,
\]

hence

\[
T \mu = L_k \cdot \mu .
\]

It then follows from the inequality \[3\] that

\[
\frac{1}{2} \, E[|\nabla \phi|^2_H] \leq E \left[ \frac{L}{k} \log \frac{L}{k} \right] = E \left[ \log \frac{L \circ T}{k \circ T} \right] = E[- \log K + \log L \circ T] .
\]

Suppose that \( \phi \in \mathcal{D}_{2,1} \) is a 1-convex Wiener functional. Let \( V_n \) be the sigma algebra generated by \( \{ \delta e_1, \ldots, \delta e_n \} \), where \( (e_n, n \geq 1) \) is an orthonormal basis of the Cameron-Martin space \( H \). Then \( \phi_n = E[\phi|V_n] \) is again 1-convex (cf. [13]), hence \( \mathcal{L} \phi_n \) is a measure

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as it can be easily verified. However the sequence \((L\phi_n, n \geq 1)\) converges to \(L\phi\) only in \(\mathcal{D}'\). Consequently, there is no reason for the limit \(L\phi\) to be a measure. In case this happens, we shall denote the Radon-Nikodym density with respect to \(\mu\), of the absolutely continuous part of this measure by \(L_a\).

**Lemma 7.1** Let \(\phi \in \mathcal{D}_{2,1}\) be 1-convex and let \(V_n\) be defined as above and define \(F_n = E[\phi|V_n]\). Then the sequence \((L_a F_n, n \geq 1)\) is a submartingale, where \(L_a F_n\) denotes the \(\mu\)-absolutely continuous part of the measure \(L F_n\).

**Proof:** Note that, due to the 1-convexity, we have \(L_a F_n \geq L F_n\) for any \(n \in \mathbb{N}\). Let \(X_n = L_a F_n\) and \(f \in \mathcal{D}\) be a positive, \(V_n\)-measurable test function. Since \(LE[\phi|V_n] = E[L\phi|V_n]\), we have

\[
E[X_{n+1} f] \geq \langle LF_{n+1}, f \rangle = \langle LF_n, f \rangle,
\]

where \(\langle \cdot, \cdot \rangle\) denotes the duality bracket for the dual pair \((\mathcal{D}', \mathcal{D})\). Consequently

\[
E[f E[X_{n+1}|V_n]] \geq \langle LF_n, f \rangle,
\]

for any positive, \(V_n\)-measurable test function \(f\), it follows that the absolutely continuous part of \(LF_n\) is also dominated by the same conditional expectation and this proves the submartingale property.

**Lemma 7.2** Assume that \(L \in \mathcal{L}\log \mathcal{L}\) is a positive random variable whose expectation is one. Assume further that it is lower bounded by a constant \(a > 0\). Let \(T = I_W + \nabla \phi\) be the transport map such that \(T \mu = L \mu\) and let \(T^{-1} = I_W + \nabla \psi\). Then \(L \psi\) is a Radon measure on \((W, \mathcal{B}(W))\). If \(L\) is upper bounded by \(b > 0\), then \(L \phi\) is also a Radon measure on \((W, \mathcal{B}(W))\).

**Proof:** Let \(L_n = E[L|V_n]\), then \(L_n \geq a\) almost surely. Let \(T_n = I_W + \nabla \phi_n\) be the transport map which satisfies \(T_n \mu = L_n \mu\) and let \(T^{-1}_n = I_W + \nabla \psi_n\) be its inverse. We have

\[
L_n = \det_2 (I_H + \nabla^2 \psi_n) \exp \left[ -\mathcal{L}_a \psi_n - \frac{1}{2} |\nabla \psi_n|^2_H \right].
\]

By the hypothesis \(-\log L_n \leq -\log a\). Since \(\psi_n\) is 1-convex, it follows from the finite dimensional results that \(\det_2 (I_H + \nabla^2 \psi_n) \in [0, 1]\) almost surely. Therefore we have

\[
\mathcal{L}_a \psi_n \leq -\log a,
\]

besides \(\mathcal{L} \psi_n \leq \mathcal{L}_a \psi_n\) as distributions, consequently

\[
\mathcal{L} \psi_n \leq -\log a
\]
as distributions, for any \(n \geq 1\). Since \(\lim_n \mathcal{L} \psi_n = \mathcal{L} \psi\) in \(\mathcal{D}'\), we obtain \(\mathcal{L} \psi \leq -\log a\), hence \(-\log a - \mathcal{L} \psi \geq 0\) as a distribution, hence \(\mathcal{L} \psi\) is a Radon measure on \(W\), c.f., [13], [28]. This proves the first claim. Note that whenever \(L\) is upper bounded, \(\Lambda = 1/L \circ T\) is lower bounded, hence the proof of the second claim is similar to that of the first one. \(\square\)
Theorem 7.2 Assume that $L$ is a strictly positive bounded random variable with $E[L] = 1$. Let $\phi \in \mathbb{D}_{2,1}$ be the 1-convex Wiener functional such that

$$T = I_W + \nabla \phi$$

is the transport map realizing the measure $L.\mu$ and let $S = I_W + \nabla \psi$ be its inverse. Define $F_n = E[\phi|V_n]$, then the submartingale $(\mathcal{L}_n F_n, n \geq 1)$ converges almost surely to $\mathcal{L}_a \phi$. Let $\lambda(\phi)$ be the random variable defined as

$$\lambda(\phi) = \liminf_{n \to \infty} A_n$$

$$= \left( \liminf_n \det_2 (I_H + \nabla^2 F_n) \right) \exp \left\{ -\mathcal{L}_a \phi - \frac{1}{2} |\nabla \phi|^2_H \right\}$$

where

$$A_n = \det_2 (I_H + \nabla^2 F_n) \exp \left\{ -\mathcal{L}_a F_n - \frac{1}{2} |\nabla F_n|^2_H \right\}.$$  

Then it holds true that

$$E[f \circ T \lambda(\phi)] \leq E[f]$$  

(7.13)

for any $f \in C^+_b(W)$, in particular $\lambda(\phi) \leq \frac{1}{L \circ T}$ almost surely. If $E[\lambda(\phi)] = 1$, then the inequality in (7.13) becomes an equality and we also have

$$\lambda(\phi) = \frac{1}{L \circ T}.$$  

Proof: Let us remark that, due to the 1-convexity, $0 \leq \det_2 (I_H + \nabla^2 F_n) \leq 1$, hence the lim inf exists. Now, Lemma 7.2 implies that $\mathcal{L} \phi$ is a Radon measure. Let $F_n = E[\phi|V_n]$, then we know from Lemma 7.1 that $(\mathcal{L}_n F_n, n \geq 1)$ is a submartingale. Let $\mathcal{L}_a \phi$ denote the positive part of the measure $\mathcal{L} \phi$. Since $\mathcal{L}_a \phi \geq \mathcal{L} \phi$, we have also $E[\mathcal{L}_a \phi|V_n] \geq E[\mathcal{L} \phi|V_n] = \mathcal{L} F_n$. This implies that $E[\mathcal{L}_a \phi|V_n] \geq \mathcal{L}_a^+ F_n$. Hence we find that

$$\sup_n E[\mathcal{L}_a^+ F_n] < \infty$$

and this condition implies that the submartingale $(\mathcal{L}_n F_n, n \geq 1)$ converges almost surely. We shall now identify the limit of this submartingale. Let $\mathcal{L}_s G$ be the singular part of the measure $\mathcal{L} G$ for a Wiener function $G$ such that $\mathcal{L} G$ is a measure. We have

$$E[\mathcal{L} \phi|V_n] = E[\mathcal{L}_a \phi|V_n] + E[\mathcal{L}_s \phi|V_n]$$

$$= \mathcal{L}_a F_n + \mathcal{L}_s F_n,$$

hence

$$\mathcal{L}_a F_n = E[\mathcal{L}_a \phi|V_n] + E[\mathcal{L}_a \phi|V_n]_a$$

almost surely, where $E[\mathcal{L}_a \phi|V_n]_a$ denotes the absolutely continuous part of the measure $E[\mathcal{L}_a \phi|V_n]$. Note that, from the Theorem of Jessen (cf., for example Theorem 1.2.1 of [29]), $\lim_n E[\mathcal{L}_a^+ \phi|V_n]_a = 0$ and $\lim_n E[\mathcal{L}_s \phi|V_n]_a = 0$ almost surely, hence we have

$$\lim_n \mathcal{L}_a F_n = \mathcal{L}_a \phi,$$
\( \mu \)-almost surely. To complete the proof, an application of the Fatou lemma implies that

\[
E[f \circ T \lambda(\phi)] \leq E[f] = E\left[f \circ T \frac{1}{L \circ T}\right],
\]

for any \( f \in C^+_b(W) \). Since \( T \) is invertible, it follows that

\[
\lambda(\phi) \leq \frac{1}{L \circ T}
\]

almost surely. Therefore, in case \( E[\lambda(\phi)] = 1 \), we have

\[
\lambda(\phi) = \frac{1}{L \circ T},
\]

and this completes the proof. \( \square \)

**Corollary 7.1** Assume that \( K, L \) are two positive random variables with values in a bounded interval \([a, b] \subset (0, \infty)\) such that \( E[K] = E[L] = 1 \). Let \( T = I_W + \nabla \phi \), \( \phi \in \mathbb{D}_{2,1} \), be the transport map pushing \( Kd\mu \) to \( Ld\mu \), i.e., \( T(Kd\mu) = Ld\mu \). We then have

\[
L \circ T \lambda(\phi) \leq K,
\]

\( \mu \)-almost surely. In particular, if \( E[\lambda(\phi)] = 1 \), then \( T \) is the solution of the Monge-Ampère equation.

**Proof:** Since \( a > 0 \),

\[
\frac{dT\mu}{d\mu} = \frac{L}{K \circ T} \leq \frac{b}{a}.
\]

Hence, Theorem 7.13 implies that

\[
E[f \circ TL \circ T \lambda(\phi)] \leq E[f L] = E[f \circ TK],
\]

consequently

\[
L \circ T \lambda(\phi) \leq K,
\]

the rest of the claim is now obvious. \( \square \)

For later use we give also the following result:

**Theorem 7.3** Assume that \( L \) is a positive random variable of class \( \mathbb{L}\log \mathbb{L} \) such that \( E[L] = 1 \). Let \( \phi \in \mathbb{D}_{2,1} \) be the 1-convex function corresponding to the transport map \( T = I_W + \nabla \phi \). Define \( T_t = I_W + t\nabla \phi \), where \( t \in [0, 1] \). Then, for any \( t \in [0, 1] \), \( T_t\mu \) is absolutely continuous with respect to the Wiener measure \( \mu \).
Proof: Let $\phi_n$ be defined as the transport map corresponding to $L_n = E[P_{1/n} L_n | V_n]$ and define $T_n$ as $I_W + \nabla \phi_n$. For $t \in [0, 1)$, let $T_{n,t} = I_W + t \nabla \phi_n$. It follows from the finite dimensional results which are summarized in the beginning of this section, that $T_{n,t, \mu}$ is absolutely continuous with respect to $\mu$. Let $L_{n,t}$ be the corresponding Radon-Nikodym density and define $\Lambda_{n,t}$ as

$$
\Lambda_{n,t} = \det_2 \left( I_H + t \nabla^2 \phi_n \right) \exp \left\{ -t \mathcal{L}_a \phi_n - \frac{t^2}{2} |\nabla \phi_n|^2_H \right\}.
$$

Besides, for any $t \in [0, 1)$,

$$
\left((I_H + t \nabla^2 \phi_n) h, h \right)_H > 0,
$$

(7.14)

$\mu$-almost surely for any $0 \neq h \in H$. Since $\phi_n$ is of finite rank, (7.14) implies that $\Lambda_{n,t} > 0$ $\mu$-almost surely and we have shown at the beginning of this section

$$
\Lambda_{n,t} = \frac{1}{L_{n,t} \circ T_{n,t}}
$$

$\mu$-almost surely. An easy calculation shows that $t \rightarrow \log \det_2(I + t \nabla^2 \phi_n)$ is a non-increasing function. Since $\mathcal{L}_a \phi_n \geq \mathcal{L} \phi_n$, we have $E[\mathcal{L}_a \phi_n] \geq 0$. Consequently

$$
E[L_{t,n} \log L_{t,n}] = E[\log L_{n,t} \circ T_{n,t}]
= -E[\log \Lambda_{t,n}]
= E \left[ -\log \det_2 \left( I_H + t \nabla^2 \phi_n \right) + t \mathcal{L}_a \phi_n + \frac{t^2}{2} |\nabla \phi_n|^2_H \right]
\leq E \left[ -\log \det_2 \left( I_H + \nabla^2 \phi_n \right) + \mathcal{L}_a \phi_n + \frac{1}{2} |\nabla \phi_n|^2_H \right]
= E[L_{t,n} \log L_{t,n}]
\leq E[L \log L],
$$

by the Jensen inequality. Therefore

$$
\sup_n E[L_{n,t} \log L_{n,t}] < \infty
$$

and this implies that the sequence $(L_{n,t}, n \geq 1)$ is uniformly integrable for any $t \in [0, 1]$. Consequently it has a subsequence which converges weakly in $L^1(\mu)$ to some $L_t$. Since, from Theorem 4.1, $\lim_{n} \phi_n = \phi$ in $\mathcal{D}_{2,1}$, where $\phi$ is the transport map associated to $L$, for any $f \in C_b(W)$, we have

$$
E[f \circ T_t] = \lim_k E[f \circ T_{n_k,t}]
= \lim_k E[f L_{n_k,t}]
= E[f L_t],
$$

hence the theorem is proved. \qed
7.1 The solution of the Monge-Ampère equation via Ito-renormalization

We can interpret the Monge-Ampère equation as follows: given two probability densities $K$ and $L$, find a map $T : W \to W$ such that

$$L \circ T J(T) = K$$

almost surely, where $J(T)$ is a kind of Jacobian to be written in terms of $T$. In Corollary 7.1 we have shown the existence of some $\lambda(\phi)$ which gives an inequality instead of the equality. Although in the finite dimensional case there are some regularity results about the transport map (cf., [7]), in the infinite dimensional case such techniques do not work. All these difficulties can be circumvented using the miraculous renormalization of the Ito calculus. In fact assume that $K$ and $L$ satisfy the hypothesis of the corollary. First let us indicate that we can assume $W = C_0([0,1], \mathbb{R})$ (cf., [29], Chapter II, to see how one can pass from an abstract Wiener space to the standard one) and in this case the Cameron-Martin space $H$ becomes $H^1([0,1])$, which is the space of absolutely continuous functions on $[0,1]$, with a square integrable Sobolev derivative. Let now

$$\Lambda = \frac{K}{L \circ T},$$

where $T$ is as constructed above. Then $\Lambda, \mu$ is a Girsanov measure for the map $T$. This means that the law of the stochastic process $(t,x) \to T_t(x)$ under $\Lambda, \mu$ is equal to the Wiener measure, where $T_t(x)$ is defined as the evaluation of the trajectory $T(x)$ at $t \in [0,1]$. In other words the process $(t,x) \to T_t(x)$ is a Brownian motion under the probability $\Lambda, \mu$. Let $(\mathcal{F}^T_t, t \in [0,1])$ be its filtration, the invertibility of $T$ implies that

$$\bigvee_{t \in [0,1]} \mathcal{F}^T_t = \mathcal{B}(W).$$

$\Lambda$ is upper and lower bounded $\mu$-almost surely, hence also $\Lambda, \mu$-almost surely. The Ito representation theorem implies that it can be represented as

$$\Lambda = E[\Lambda^2] \exp \left\{ - \int_0^1 \dot{\alpha}_s dT_s - \frac{1}{2} \int_0^1 |\dot{\alpha}_s|^2 ds \right\},$$

where $\alpha(\cdot) = \int_0^{\cdot} \dot{\alpha}_s ds$ is an $H$-valued random variable. In fact $\alpha$ can be calculated explicitly using the Ito-Clark representation theorem (cf., [28]), and it is given as

$$\dot{\alpha}_t = \frac{E_{\Lambda}[D_t \Lambda | \mathcal{F}^T_t]}{E_{\Lambda}[\Lambda | \mathcal{F}^T_t]}. \quad (7.15)$$

dt \times d\mu$-almost surely, where $E_{\Lambda}$ denotes the expectation operator with respect to $\Lambda, \mu$ and $D_t \Lambda$ is the Lebesgue density of the absolutely continuous map $t \to \nabla \Lambda(t,x)$. From the relation (7.15), it follows that $\alpha$ is a function of $T$, hence we have obtained the strong solution of the Monge-Ampère equation. Let us announce all this as
Theorem 7.4 Assume that $K$ and $L$ are upper and lower bounded densities, let $T$ be the transport map constructed in Theorem 6.1. Then $T$ is also the strong solution of the Monge-Ampère equation in the Ito sense, namely

$$E[\Lambda^2] L \circ T \exp \left\{ - \int_0^1 \dot{\alpha}_s dT_s - \frac{1}{2} \int_0^1 |\dot{\alpha}_s|^2 ds \right\} = K,$$

$\mu$-almost surely, where $\alpha$ is given with (7.15).

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