Convergence Analysis of a Cooperative Diffusion Gauss-Newton Strategy
Mou Wu, Naixue Xiong, Liansheng Tan

Abstract—In this paper, we investigate the convergence performance of a cooperative diffusion Gauss-Newton (GN) method, which is widely used to solve the nonlinear least squares problems (NLLS) due to the low computation cost compared with Newton’s method. This diffusion GN collects the diversity of temporal-spatial information over the network, which is used on local updates. In order to address the challenges on convergence analysis, we firstly consider to form a global recursion relation over spatial and temporal scales since the traditional GN is a time iterative method and the network-wide NLLS need to be solved. Secondly, the derived recursion related to the network-wide deviation between the successive two iterations is ambiguous due to the uncertainty of descent discrepancy in GN update step between two versions of cooperation and non-cooperation. Thus, an important work is to derive the boundedness conditions of this discrepancy. Finally, based on the temporal-spatial recursion relation and the steady-state equilibria theory for discrete dynamical systems, we obtain the sufficient conditions for algorithm convergence, which require the good initial guesses, reasonable step size values and network connectivity. Such analysis provides a guideline for the applications based on this diffusion GN method.

Index Terms—Gauss-Newton method, diffusion algorithm, distributed estimation, adaptive networks, Nonlinear least squares.

I. INTRODUCTION

Gauss-Newton method has found wide applications, such as deep learning in artificial intelligence and neural network [1], [2], and parameter estimate in a networked system [3]–[5]. Deriving from Newton’s method, GN algorithm discards the second-order terms in the computation of Hessian for small residual NLLS problems, thereby resulting in saving in computation. Such the amount of computations can be further reduced via the mathematical process. In order to compute easily the first derivative of objective function, the perturbed GN method is proposed in [6], where a perturbed derivative version substitutes the original one. The truncated GN method [7] is proposed to implement the inexact update instead of exact one. The truncated-perturbed GN method [7] integrates the above two advantages into the update step.

Many scenarios can be modeled as the NLLS problem depended on the performance of GN, such as computer vision [8], image alignment and reconstruction [9], [10], network-based localization [11], [12], signal processing for direction-of-arrival estimation and frequency estimation [13], logistic regression [14] and power system state estimation [4], [15].

Despite the widespread utility, it is difficult for exploiting the original GN method as a fully cooperative scheme for a distributed network, since its iteration rule involves the matrix inverse operator, which is ideally suited to be implemented in a centralized way. However, for the well known advantages such as load balancing and robustness, distributed algorithm with the improvement of performance is preferred.

The purpose of this work is to analyze the convergence of a cooperative diffusion GN strategy over a distributed network, where every node sense the temporal data that is variable over the spatial domain. Several diffusion GN methods [17], [18] are proposed for solving the localization problem in wireless sensor networks. However, they are centralized in nature and implemented in a non-cooperative way, in which the local intermediate estimates are not shared over the diffusion network.

Notation: The operator $\cdot^T$ denotes the transpose for matrix or vector, the operator $(\cdot)^{-1}$ denotes the inverse of a non-singular matrix. The capital letters are used when the matrices are denoted, while the small letters are used when the vectors or scalars are denoted. The Euclidean norm of a vector $x$ is written as $\|x\|$, 2-norm and Frobenius norm of a matrix $G$ is denoted by $\|G\|$ and $\|G\|_F$, respectively. $I_N$ and $1_N$ denote the $N \times N$ identity matrix and $N \times 1$ vector whose every entry is 1, respectively. We will use subscripts $k$, $l$, $u$ and $t$ to denote node, and superscript $j$, $i$ to denote time.

II. DESCRIPTION OF COOPERATIVE DIFFUSION GAUSS-NEWTON SOLUTION

A. Centralized solution

For an adaptive network represented by a set $\mathcal{N} = \{1, \cdots, N\}$, we would like to estimate a $M \times 1$ unknown parameter vector $x = [x_1, \cdots, x_M]^T$ belonging to a closed convex set $\mathcal{X}$. Let $f(x) = [f_1(x), \cdots, f_N(x)]^T : \mathbb{R}^M \rightarrow \mathbb{R}^N$ be a continuous and differentiable global cost function throughout the network, where $f_k(x) : \mathbb{R}^M \rightarrow \mathbb{R}$ is the individual cost function associated with node $k \in \mathcal{N}$ by collecting the measurements from the related events. The estimation problem can be formulated as

$$\min_x \| f(x) \|^2, \quad (1)$$

By rewriting $\|f(x)\|^2 = \sum_{k=1}^N |f_k(x)|^2$, the object of each node in the network is to seek a $M \times 1$ vector $x$ that solve the following Non-Linear Least Squares (NLLS) problem with the form

$$\min_{x} \sum_{k=1}^{N} |f_k(x)|^2. \quad (2)$$

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The GN method is well recognized for solving NLLS problems. Let us consider a fusion center (FC) that can communicate with all nodes in the network. Given an initial good guess \( x^0 \), a centralized scheme can be implemented on FC based on the GN update rule in an iterative way

\[
x^{i+1} = x^i - \alpha^i d^i,
\]

where \( x^i \) is the estimation of \( x \) at iteration \( i \), \( d^i \) denotes a descent direction of GN, and \( \alpha^i \) is the step size parameter that ensure \( x^{i+1} \) is nearer a stationary point than \( x^i \).

In this paper, we adopt the following assumption for the above optimization problem.

**Assumption 1.**

1. The stationary points \( x^* \in \mathbb{R}^M \) that satisfy
   \[
   \nabla f(x^*) = 2F^T(x^*)f(x^*) = 0
   \]
   always exist, where \( F(x) \) is the Jacobian of \( f(x) \) with the size \( N \times M \) and the entries \( F(x)_{k,i} = \partial f_k(x)/\partial x_i \), \( 1 \leq k \leq N, 1 \leq l \leq M \).
   
   2. The notations \( \lambda_{\min}(\cdot) \) and \( \lambda_{\max}(\cdot) \) are denoted as the minimum and maximum eigenvalues. For all \( x \in \mathcal{X} \) and \( k \in \mathcal{N} \), let
   \[
   \Sigma_{\min} = \min \{ \lambda_{\min}(F^T(x)F(x)) \},
   \]
   and
   \[
   \Sigma_{\max} = \max \{ \lambda_{\max}(F^T(x)F(x)) \},
   \]
   where \( 0 < \Sigma_{\min} < \Sigma_{\max} < \infty \).

Under Assumption 1, the approximate Hessian \( F^T(x)F(x) \) of \( f(x) \) is positive definite. Thereby, a local minimizer of \( f(x) \) denoted by \( x^* \) that belongs to the set of stationary points always exist [19], [20]. Thus, the descent direction of GN update is written as

\[
d^i = [F^T(x^i)F(x^i)]^{-1}F^T(x^i)f(x^i).
\]

By rewriting

\[
F(x) = \text{col}\{\partial f_1(x)/\partial x, \partial f_2(x)/\partial x, \ldots, \partial f_M(x)/\partial x\} \quad (N \times M)
\]

and defining

\[
F_k(x) \triangleq \partial f_k(x)/\partial x, \quad (1 \times M)
\]

we get

\[
d^i = \sum_{k=1}^{N} F_k^T(x^i)F_k(x^i)^{-1}\sum_{k=1}^{N} F_k^T(x^i)f_k(x^i).
\]

Therefore, we have the following GN iteration update

\[
x^{i+1} = x^i - \alpha^i \sum_{k=1}^{N} F_k^T(x^i)F_k(x^i)^{-1}\sum_{k=1}^{N} F_k^T(x^i)f_k(x^i).
\]

To successfully implement [3] in a centralized way, we assume that the FC can communicate with all nodes over network and the same initial estimate is given by \( x^0 = x^0, k \in \mathcal{N} \). In the centralized GN algorithm, the computation results of \( F_k^T(x^i)f_k(x^i) \) and \( F_k^T(x^i)F_k(x^i) \) from each node \( k \) are aggregated by the FC to obtain the new estimate \( x^{i+1} \) based on [8]. Then the estimate \( x^{i+1} \) is returned to all nodes until an appropriate termination condition is satisfied, for example \( \|x^{i+1} - x^i\| \leq \varepsilon \) or \( i = I \), where \( \varepsilon \) and \( I \) are the predefined minimum norm decline and the maximum number of iterations, respectively. Thus, the centralized GN includes actually a step of diffusion for new estimate \( x^{i+1} \) form FC to individual nodes.

In this paper, we adopt the constant step size \( \alpha_k = \alpha \in (0, 1] \) for the subsequent development and analysis.

### B. Diffusion Gauss-Newton

Consider the adaptive network \( \mathcal{N} \), where any node \( k \) at time \( i \) receives a set of estimates \( \{x^i_k\}_{l \in \mathcal{N}_k} \) from all its 1-hop neighbors \( \mathcal{N}_k \) including itself. Thus, the local estimates \( \{x^i_k\}_{l \in \mathcal{N}_k} \) is combined in a weighted combination way denoted by

\[
\mathcal{X}^i_k = \sum_{l \in \mathcal{N}_k} c_{kl} x^i_l,
\]

where \( c_{kl} \) is the weighted coefficient between node \( k \) and \( l \in \mathcal{N}_k \).

and the conditions

\[
\sum_{l \in \mathcal{N}_k} c_{kl} = 1 \text{ and } c_{kl} \in [0, 1] \text{ for } l \in \mathcal{N}_k
\]

is satisfied.

Once the aggregate estimate \( \mathcal{X}^i_k \) is obtained as the local weighted estimate, any node \( k \) in the network can implement the GN update step as follows:

\[
x^{i+1}_k = \mathcal{X}^i_k - \alpha \{Q_k(\mathcal{X})\}^{-1}q^i_k(\mathcal{X}),
\]

where we define

\[
Q_k^i(\mathcal{X}) \triangleq \sum_{l \in \mathcal{N}_k} F^T_l(\mathcal{X}^i_l)f_l(\mathcal{X}^i_l)
\]

and

\[
q^i_k(\mathcal{X}) \triangleq \sum_{l \in \mathcal{N}_k} F^T_l(\mathcal{X}^i_l)f_l(\mathcal{X}^i_l).
\]

Removing the aggregate step of diffusion GN algorithm, we obtain a non-cooperative diffusion GN algorithm, where each node in the network acts as the FC to implement the centralized GN by communicating with all immediate neighbors. Its GN update step is given by

\[
x^{i+1}_k = x^i_k - \alpha \{Q_k(x^i_k)\}^{-1}q^i_k(x^i_k),
\]

where we define

\[
Q_k^i(x^i_k) \triangleq \sum_{l \in \mathcal{N}_k} F^T_l(x^i_l)f_l(x^i_l)
\]

and

\[
q^i_k(x^i_k) \triangleq \sum_{l \in \mathcal{N}_k} F^T_l(x^i_l)f_l(x^i_k).
\]

Note that the expression on arguments in [12] [13] [15] [16] shows the main difference between cooperative and non-cooperative algorithms.

The question that remains is how well does the diffusion GN algorithm perform in terms of its expected convergence behavior. First, what are the sufficient conditions of convergence for
the diffusion GN algorithm? Second, is better the diffusion GN algorithm on convergence, compared with its non-cooperative counterpart? In other words, what are the benefits of cooperation? The following analysis and simulations will answer the above questions.

III. CONVERGENCE ANALYSIS

A. Assumptions and data model

To proceed the analysis, several reasonable assumptions need to be given as is commonly done in the literature [4], [21].

**Assumption 2.**

(1) $f_i \in \mathcal{N}_k(x_k^i)$ is bounded for all $x_k^i \in \mathcal{X} \subset \mathbb{R}^M$ near $x^*$, and satisfies

$$
\|f_i(x_k^i)\| \leq \epsilon_{\text{max}}
$$

and

$$
\|f_i(x_k^i)\| = \epsilon_{\text{min}},
$$

where $\|f_i(x_k^i)\|$ denotes the minimum value of $\|f_i(x_k^i)\|$ when evaluated at $x_k^i = x^*$.

(2) For all $x \in \mathcal{X}$ and $k = 1, \ldots, N$, let

$$
\sigma_{\text{min}} = \min \sqrt{\lambda_{\text{min}}(F_k^T(x)F_k(x))}
$$

and

$$
\sigma_{\text{max}} = \max \sqrt{\lambda_{\text{max}}(F_k^T(x)F_k(x))},
$$

where $0 < \sigma_{\text{min}} < \sigma_{\text{max}} < \infty$.

(3) Both $f_i(x_k^i)$ and $F_k(x)$ are Lipschitz continuous on $\mathcal{X}$ with Lipschitz constant $\omega > 0$ such that

$$
\|f_i(x_k^i) - f_j(x_{k,j}^i)\| \leq \omega\|x - y\|
$$

and

$$
\|F_k(x) - F_j(x)\| \leq \omega\|x - y\|
$$

for all $x, y \in \mathcal{X}$. Furthermore, we have the following results [22]

$$
\|F_k^T(x)f_k(x) - F_k^T(y)f_k(y)\| \leq \gamma_f\|x - y\|
$$

and

$$
\|F_k^T(x)F_k(x) - F_k^T(y)F_k(y)\| \leq \gamma_F\|x - y\|,
$$

where $\gamma_f \geq \omega(\epsilon_{\text{max}} + \Sigma_{\text{max}})$ and $\gamma_F \geq 2\Sigma_{\text{max}}\omega$ are the corresponding Lipschitz constants.

In addition, the studying of the local convergence behavior need to be considered from the global view of network, since the performance of individual node depends on the whole network including cooperation rule and network topology. Thus, we introduce the global quantities

$$
x^i_G \triangleq \{x^i_1, \ldots, x^i_N\}, \quad (NM \times 1)
$$

$$
X^i_G \triangleq \{x^i_1, \ldots, x^i_N\}, \quad (NM \times 1)
$$

$$
\overline{x}^i \triangleq \{x^i, \ldots, x^i\}, \quad (NM \times 1)
$$

$$
D^i_G \triangleq \{D^i_1, \ldots, D^i_N\}, \quad (NM \times 1)
$$

$$
d^i_G \triangleq \{d^i_1, \ldots, d^i_N\}, \quad (NM \times 1)
$$

where

$$
D^i_k \triangleq [Q_k(x^i)]^{-1}q^i_k(x^i), \quad k \in \mathcal{N},
$$

and

$$
d^i_k \triangleq [Q_k(x^i)]^{-1}q^i_k(x^i), \quad k \in \mathcal{N},
$$

$$
A(x^i_G) \triangleq \text{diag}\{F_i \in \mathcal{N}_i(x^i_1), \ldots, F_i \in \mathcal{N}_i(x^i_N)\}, \quad (NN \times NM)
$$

$$
A(\overline{x}) \triangleq \text{diag}\{F_i \in \mathcal{N}_i(x^i), \ldots, F_i \in \mathcal{N}_i(x^i)\}, \quad (NN \times NM)
$$

$$
b(x^i_G) \triangleq \{f_i \in \mathcal{N}_i(x^i), \ldots, f_i \in \mathcal{N}_i(x^i)\}, \quad (NN \times 1)
$$

$$
b(\overline{x}) \triangleq \{f_i \in \mathcal{N}_i(x^i), \ldots, f_i \in \mathcal{N}_i(x^i)\}, \quad (NN \times 1)
$$

where $\text{diag}(\cdot)$ is a block diagonal matrix whose entries are those of the column vector $\{\cdot\}$.

An $N \times N$ aggregate matrix $C$ can be given with non-negative real entries $\{c_{kl}\}$ that is redefined with the following conditions

$$
c_{kl} = 0 \quad \text{if} \quad l \notin \mathcal{N}_k \quad \text{and} \quad \sum_{l=1}^{N} c_{kl} = 1, c_{kl} \geq 0. \quad (17)
$$

Conditions (17) indicate that the sum of all entries on each row of the matrix $C$ is one, while the entry $c_{kl}$ of $C$ shows the degree of closeness between nodes $k$ and $l$. We will see the influence of selecting $\{c_{kl}\}$ on the performance of the resulting algorithms in later simulations.

Similarly, we introduce an $N \times N$ adjacency matrix $\Phi$ with the element $\varphi_{kl} \in \{0, 1\}$, in which $\varphi_{kl} = 1$ if node $k$ is linked with node $l$; otherwise $0$.

We also introduce the extended aggregate matrix $G$

$$
G \triangleq C \otimes I_M, \quad (NM \times NM)
$$

where $\otimes$ is the Kronecker product operation and $I_M$ is the $M \times M$ identity matrix.

B. Temporal-spatial recursion relation

The temporal-spatial relation across network need to be considered as a starting point of convergence analysis. First, the diffusion strategy leads to the frequent spatial interaction between the neighborhoods, thereby each node $k$ is influenced by both local information such as $f_k$ and spatial information from neighbours $l \in \mathcal{N}_k$ such as $\{f_l, x_l\}$. Second, the iteration way decides that the estimates and the local collected information on each node $k$ are time-variant, i.e., $\{f_k^i, x_k^i\}$.

To begin with (9), we have

$$
\hat{X}_G^i = Gx_G^i. \quad (18)
$$

Using (18), we rewrite the local diffusion GN update step (11) as a global representation

$$
x_G^{i+1} = Gx_G^i - \alpha D_G^i. \quad (19)
$$

Accordingly, we get the global non-cooperative GN update step

$$
x_G^{i+1} = x_G^i - \alpha d_G^i. \quad (20)
$$

Subtracting $\overline{x}$ on both sides of the equation (19) and embedding the equation (20), we get

$$
x_G^{i+1} - \overline{x} = (Gx_G^i - x_G^i) + (x_G^i - \overline{x} - \alpha d_G^i) + \alpha(d_G^i - D_G^i). \quad (21)
$$

Using the triangle inequality for vectors, we get the following recursion

$$
\|x_G^{i+1} - \overline{x}\| \leq \|Gx_G^i - x_G^i\| + \|x_G^i - \overline{x} - \alpha d_G^i\| + \|\alpha(d_G^i - D_G^i)\|.
$$

...
\[ \|x_G^{i+1} - \bar{x}\| \leq \|Gx_G^i - x_G^i\| + \|x_G^i - \bar{x}\| - \alpha d\| \tag{22} \]

The inequality (22) can be regarded as a temporal-spatial recursion relation, where the superscript \(i\) and the subscript \(G\) reflect the evolution of diffusion GN algorithm from temporal and spatial dimensions, respectively. And we establish the relation between diffusion GN and non-cooperative diffusion algorithms from the global perspective.

For the first term of the right side of (22), we have
\[
\|Gx_G^i - x_G^i\| = \|Gx_G^i - G\bar{x} + (\bar{x} - x_G^i)\| \\
\leq \|Gx_G^i - G\bar{x}\| + \|x_G^i - \bar{x}\| \\
= \left(\|G\|_F + 1\right)\|x_G^i - \bar{x}\| \\
= \left(\|G\|_F + 1\right)\|x_G^i - \bar{x}\|, \\
\tag{23}
\]

where we use \(G\bar{x} = \bar{x}\) based on the property of \(G\).

For the second term of the right side of (22), we have the following conclusion.

**Lemma 1.** Let Assumptions 1 and 2 hold. The norm of global vector \(x_G^i - \bar{x}\) satisfies the following recursion
\[
\|x_G^i - \bar{x}\| \leq t_1\|x_G^i - \bar{x}\|^2 + t_2\|x_G^i - \bar{x}\|, \\
\tag{24}
\]

where
\[
t_1 \triangleq \frac{\omega}{2\Sigma_{\min}}, \quad t_2 \triangleq \frac{(1 - \alpha)\Sigma_{\max}}{\Sigma_{\min}} + \frac{\sqrt{2N\omega e_{\min}}}{\Sigma_{\min}^2} \\
\tag{25}
\]

**Proof:** See Appendix A.

Given (23) and (24), we rewrite the temporal-spatial recursion (22) as
\[
\|x_G^{i+1} - \bar{x}\| \leq t_1\|x_G^i - \bar{x}\|^2 + \|x_G^i - \bar{x}\| + \alpha\|D_G^i - d_G^i\|. \\
\tag{26}
\]

Given the above, the left side of (26) is the network deviation at time \(i + 1\), while the right side of (26) will be related to the network deviation \(\|x_G^i - \bar{x}\|\) at time \(i\) if we can confirm that \(\|D_G^i - d_G^i\|\) shares the same character or is bounded by a given constant \(\xi\). Then, we can establish the relation of the network deviation between the successive two times in diffusion GN.

C. Boundness of descent discrepancy

\(D_G^i - d_G^i\) denotes the GN descent discrepancy over network between two modes of cooperative and non-cooperative. To decide the boundness of the discrepancy, we first evaluate the entry of \(D_G^i - d_G^i\), i.e., \(D_k^i - d_k^i\).

To begin the process, we write the entry as
\[
D_k^i - d_k^i = |Q_k(\mathcal{X})|^{-1}q_k(\mathcal{X}) - |Q_k(x_k^i)|^{-1}q_k(x_k^i), \quad k \in \mathcal{N}. \\
\tag{27}
\]

Because of the matrix inverse operator, we introduce two quantities
\[
S_k^i \triangleq Q_k(\mathcal{X}) - Q_k(x_k^i) \\
s_k^i \triangleq q_k(\mathcal{X}) - q_k(x_k^i). \\
\tag{28}
\]

And in order to lower the impact of inverse operator for our analysis, the known matrix expansion formula (21) will be used frequently in our analysis. That is
\[
(Z + \delta Z)^{-1} = \sum_{u=0}^{\infty} (-1)^u(Z^{-1}\delta Z)^uZ^{-1} \\
\tag{29}
\]

for any matrix \(Z\) and \(\delta Z\) if \(\|Z^{-1}\delta Z\| < 1\).

From (9), \(S_k^i\) is a convex combination of \(\{x_k^i\}\) for \(l \in \mathcal{N}_k\). Thus, Assumptions 1 and 2 hold for \(X_k^i\).

Then we have
\[
\|S_k^i\| = \| \sum_{l \in \mathcal{N}_k} [F_l^T(\mathcal{X}_l^i)F_l(\mathcal{X}_l^i) - F_l^T(x_k^i)F_l(x_k^i)] \| \\
\leq \sum_{l \in \mathcal{N}_k} \gamma F\|X_l^i - x_k^i\| \\
\leq \sum_{l \in \mathcal{N}_k} \gamma F\|X_l - x_k\|. \\
\tag{30}
\]

and
\[
\|s_k^i\| = \| \sum_{l \in \mathcal{N}_k} [F_l^T(\mathcal{X}_l^i)f_l(\mathcal{X}_l^i) - F_l^T(x_k^i)f_l(x_k^i)] \| \\
\leq \sum_{l \in \mathcal{N}_k} \gamma f\|X_l^i - x_k^i\| \\
\leq \sum_{l \in \mathcal{N}_k} \gamma f\|X_l - x_k\|. \\
\tag{31}
\]

From (30) and (31), both \(S_k^i\) and \(s_k^i\) depend on \(\|X_l^i - x_k^i\|\). We now study the boundness of \(X_l^i - x_k^i\). Before that, we define a \(1 \times N\) vector
\[
c_l \triangleq \text{row}\{c_1, c_2, \ldots, c_N\}, \quad l \in \mathcal{N}
\]

which is the \(l\) row of matrix \(C\).

Evaluating the norm of \(X_l^i - x_k^i\), we get
\[
\|X_l^i - x_k^i\| = \|c_l x_l^i - \alpha 1_N x_k^i\| \\
\leq \|c_l\|\|x_l^i - 1_N x_k^i\| \\
\leq \|x_l^i - 1_N x_k^i\|. \\
\tag{32}
\]

The block quantity \(x_l^i - 1_N x_k^i\) represents the estimate difference across the network at time \(i\) and is written by
\[
x_l^i - 1_N x_k^i = \text{col}\{x_1^i - x_k^i, x_2^i - x_k^i, \ldots, x_N^i - x_k^i\}
\]

whose individual entry is a \(M \times 1\) vector.

For the norms of \(X_l^i - x_k^i\) and \(X_l - x_k\), \(l, k \in \mathcal{N}\) and \(i \geq 1\), we have the following Lemmas.

**Lemma 2.** Let Assumptions 1 and 2 hold. The estimate difference between nodes \(l\) and \(k\) through the non-cooperative GN update (14) is bounded by
\[
\|x_l^i - x_k^i\| \leq \Pi^i, \quad i \geq 1 \\
\tag{33}
\]

where
\[
\Pi^i \triangleq a_2 \sum_{j=1}^{i} (a_1)^{j-1}, \\
a_1 \triangleq 1 + \alpha n_{kl} + 2\alpha n_{kl} \gamma f, \\
a_2 \triangleq \frac{(n_k + 3n_{kl} + 3n_{kl})\alpha \sigma_{e_{\max}}}{2n_{kl}^2}, \\
\tag{34}
\]

\(n_{kl}\) denotes the number of nodes that are both in \(N_k\) and \(N_l\), \(n_{kl}\) denotes the number of nodes that are in \(N_k\) but not in \(N_l\).

**Proof:** See Appendix B.

**Lemma 3.** Let Assumptions 1 and 2 hold. The estimate difference between nodes \(l\) and \(k\) through the diffusion GN update (11) is bounded by
\[
\|X_l^i - x_k^i\| \leq N\Pi^i, \quad i \geq 1 \\
\tag{37}
\]

and
\[
\|\{Q_k(x_k^i)\}^{-1}S_k^i\| < 1 \\
\tag{38}
\]
always holds under the sufficient condition
\[ n_{kl} > 0, \quad (39) \]
where \( a_1, a_2, Q_k^i(x_k^i) \) and \( S_k^i \) are assigned by \([55, 56, 13]\) and \([38]\), respectively.

**Proof:** See Appendix D.

The condition \((39)\) means that any two nodes \( k \) and \( l \) in the network have at least one common neighboring node, which is more likely to be achieved by a small and dense network. However, the condition can be relaxed in practice by allowing that all nodes are linked over single-hop or multi-hops so that it holds for the large scale networks. Thus, it is reasonable that the sufficient condition \(\| Q_k^i(x_k^i)^{-1} S_k^i \| < 1\) for applying the expansion formula in \((27)\) always holds underLemma 2.

Thus, we use the expansion formula \((29)\) and the norm operator on \((27)\) as follows
\[
\begin{align*}
&\| D_k^i - d_k^i \| \\
= &\sum_{u=0}^{\infty} \left( |Q_k^i(x_k^i) - 1| S_k^i \right)^u \| [Q_k^i(x_k^i)^{-1} q_k^i(x_k^i)] \| \\
= &\sum_{u=0}^{\infty} (1 - \| Q_k^i(x_k^i)^{-1} S_k^i \|)^u \| [Q_k^i(x_k^i)^{-1} q_k^i(x_k^i) + s_k^i] \\
+ &\sum_{u=0}^{\infty} (1 - \| Q_k^i(x_k^i)^{-1} S_k^i \|)^u \| [Q_k^i(x_k^i)^{-1} q_k^i(x_k^i) + s_k^i] \\
&\leq \| Q_k^i(x_k^i)^{-1} S_k^i \| + \| Q_k^i(x_k^i)^{-1} q_k^i(x_k^i) \| + \| s_k^i \| \\
&\leq \left[ \sum_{u=1}^{\infty} (1 - \| Q_k^i(x_k^i)^{-1} S_k^i \|)^u \right] \| Q_k^i(x_k^i)^{-1} q_k^i(x_k^i) \| + \| s_k^i \| \\
&\leq \frac{N \gamma_f \Pi}{\sigma_{\text{min}}} \left( \frac{\alpha \xi}{\sigma_{\text{min}}^2} + \frac{\sigma_{\text{max}}^2 \varepsilon_{\text{max}} + N \gamma_f \Pi}{\sigma_{\text{min}}^2} \right),
\end{align*}
\]
\((40)\)

where the last equality comes from the obtained results including \([105, 107, 112, 113]\) and the definitions \([34, 114]\) of \( \Pi \) and \( \zeta_i \in (0, 1) \) (see Appendixes C and D). From \((96)\), we know that \( \Pi \) is a bounded quantity that depends on the network topology.

Finally, we obtain the bounding conclusion as follows
\[
\begin{align*}
&\| D_k^i - d_k^i \| \leq N \| D_k^i - d_k^i \| \\
&\leq \frac{N^2 \gamma_f \Pi}{\sigma_{\text{min}}} \left( \frac{\alpha \xi}{\sigma_{\text{min}}^2} + \frac{\sigma_{\text{max}}^2 \varepsilon_{\text{max}} + N \gamma_f \Pi}{\sigma_{\text{min}}^2} \right) \equiv \xi.
\end{align*}
\]
\((41)\)

**D. Sufficient conditions for system convergence**

Giving the constant \( \xi \) that satisfies \((41)\), we rewrite the global recursion relation \((26)\) as
\[
\begin{align*}
&\| x_{G_k}^{t+1} - x_{G_k}^t \| \\
&\leq t_1 \| x_{G_k}^t - x_{G_k}^t \|^2 + (t_2 + \| G \| F + 1) \| x_{G_k}^t - x_{G_k}^t \| + \alpha \xi,
\end{align*}
\]
\((42)\)

which can be regarded as a nonlinear discrete dynamical system. Let \( y^t \equiv \| x_{G_k}^t - x_{G_k}^t \| \), we will simplify notation of \((42)\) with the general form
\[
y^{t+1} \leq t_1 (y^t)^2 + (t_2 + \| G \| F + 1) y^t + \alpha \xi,
\]
\((43)\)

whose steady-state equilibrium is a level \([23]\) that solves
\[
y = \phi(y) = t_1 y^2 + (t_2 + \| G \| F + 1) y + \alpha \xi.
\]
\((44)\)

With this expression it is easy to know that the global error is determined by the dynamical system \( y^{t+1} = \phi(y^t) \) since \( y > 0 \). Thus, guaranteeing the stability of system \( y^{t+1} = \phi(y^t) \) will be needed.

Solving \((44)\), we get two steady-state equilibrium points as follows
\[
y_{\text{max}} = \frac{-t_2 - \| G \| F + \sqrt{(t_2 + \| G \| F)^2 - 4t_1 \alpha \xi}}{2t_1}
\]
\((45)\)

and
\[
y_{\text{min}} = \frac{-t_2 - \| G \| F - \sqrt{(t_2 + \| G \| F)^2 - 4t_1 \alpha \xi}}{2t_1}
\]
\((46)\)

with the condition
\[
(t_2 + \| G \| F)^2 - 4t_1 \alpha \xi \geq 0.
\]
\((47)\)

The equilibrium points \( y_{\text{max}} \) and \( y_{\text{min}} \) of the dynamical system \((43)\) is stable if and only if \([23]\)
\[
\left| \frac{d(\phi(y))}{y} \right| < 1,
\]
\((48)\)

where \( \frac{d(\phi(y))}{y} \) is the first order derivative of \( \phi(y) \).

Thus, we know that \( y_{\text{max}} \) is unstable since
\[
\left| \frac{d(\phi(y))}{y} \right|_{y_{\text{max}}} = \left| 1 + \sqrt{(t_2 + \| G \| F)^2 - 4t_1 \alpha \xi} \right| > 1,
\]
\((49)\)

while \( y_{\text{min}} \) can be stable if
\[
\left| \frac{d(\phi(y))}{y} \right|_{y_{\text{min}}} = \left| 1 - \sqrt{(t_2 + \| G \| F)^2 - 4t_1 \alpha \xi} \right| < 1
\]
\((50)\)

holds.

Because of
\[
\| G \| F = \sqrt{\sum_{k=1}^{N} \sum_{l=1}^{N} M(c_{kl})^2} \leq \sqrt{M \sum_{k=1}^{N} \sum_{l=1}^{N} c_{kl}^2} = \sqrt{MN},
\]
\((51)\)

from \((50)\), we get the following constraints
\[
2\sqrt{t_1 \alpha \xi} - t_2 < \| G \| F < \min\{ \sqrt{4 + 4t_1 \alpha \xi} - t_2, \sqrt{MN} \}
\]
\((52)\)

and
\[
\max\{ \frac{(t_2 + \| G \| F)^2 - 4}{4t_1 \xi} , 0 \} < \alpha < \min\{ \frac{(t_2 + \| G \| F)^2}{4t_1 \xi} , 1 \},
\]
\((53)\)

where \( t_1 \) and \( t_2 \) are given by \([23]\).

According to the locally stable theory for a steady-state equilibria in discrete dynamical systems, under the conditions \([52, 53]\), as long as the initial global error \( \| x_{G_k}^0 - x_{G_k}^0 \| > y_{\text{max}} \), the diffusion GN algorithm converges asymptotically to the unique steady-state equilibrium point \( y_{\text{min}} \).

Conversely, the initial global error \( \| x_{G_k}^0 - x_{G_k}^0 \| > y_{\text{max}} \) will lead to the instability of algorithm and the growing global error level. Namely,
\[
\lim_{i \to \infty} \| x_{G_k}^i - x_{G_k}^i \| = y_{\text{min}}, \text{ if } y_0 < y_{\text{max}}.
\]
\((54)\)
E. Comparison on convergence rate

In this section, we try to compare the convergence rate between cooperative and non-cooperative algorithms. Starting from the global diffusion GN update (19) and non-cooperative GN update (20), and subtracting \( \mathbf{r}^* \) on both sides of (19) and (20), we get

\[
x_G^{i+1} - \mathbf{r}^* = G x_G^i - \mathbf{r}^* - \alpha D_G^i
= G x_G^i - \alpha D_G^i + G D_G^i,
\]
and

\[
x_G^{i+1} - \mathbf{r}^* = G x_G^i - \mathbf{r}^* - \alpha d_G^i
= G x_G^i + G d_G^i,
\]
respectively, where we denote \( D_G^i \triangleq \text{col}\{D_G^1, D_G^2, \ldots, D_G^N\} \) and \( d_G^i \triangleq \text{col}\{d_1^i, d_2^i, \ldots, d_N^i\} \).

Applying the Triangle Inequality on the norm of (55) and (56), we have

\[
\|x_G^{i+1} - \mathbf{r}^*\| \leq \|G x_G^i - G \mathbf{r}^*\| + \|\alpha D_G - D_G^i\|
= \|G x_G^i - G \mathbf{r}^*\| + \|\alpha D_G^i\|,
\]
and

\[
\|x_G^{i+1} - \mathbf{r}^*\| \leq \|x_G^i - \mathbf{r}^*\| + \|\alpha d_G - d_G^i\|
= \|x_G^i - \mathbf{r}^*\| + \|\alpha d_G^i\|,
\]
where we introduce the matrices

\[
\Lambda_D \triangleq \text{diag}\{[Q_1^1(x^*)]^{-1}, \ldots, [Q_N^1(x^*)]^{-1}\}, \ (N M \times N M),
\]
and the vectors

\[
\rho_D \triangleq \text{col}\{q_1^1(x^*), \ldots, q_N^1(x^*)\}, \ (N M \times 1),
\]

Then, \( \|\rho_D\| \) and \( \|\rho_d\| \) are bounded as

\[
\|\rho_D\| = \left\| \sum_{k=1}^{N} \sum_{l \in N_k} (F_l^T(x_l^i) f_l(x_l^i) - F_l^T(x^*) f_l(x^*)) \right\|^{1/2}
\leq \left\| \sum_{k=1}^{N} \sum_{l \in N_k} (F_l^T(x_l^i) f_l(x_l^i) - F_l^T(x^*) f_l(x^*)) \right\|^{1/2}
\leq \gamma_f \sum_{k=1}^{N} \sum_{l \in N_k} \|x_l^i - x^*\|^{1/2}
= \gamma_f \|\Omega(x_G^i - \mathbf{r}^*)\|,
\]
and

\[
\|\rho_d\| = \left\| \sum_{k=1}^{N} \sum_{l \in N_k} (F_l^T(x_l^i) f_l(x_l^i) - F_l^T(x^*) f_l(x^*)) \right\|^{1/2}
\leq \left\| \sum_{k=1}^{N} \sum_{l \in N_k} (F_l^T(x_l^i) f_l(x_l^i) - F_l^T(x^*) f_l(x^*)) \right\|^{1/2}
\leq \gamma_f \sum_{k=1}^{N} \sum_{l \in N_k} \|x_l^i - x^*\|^{1/2}
= \gamma_f \|\Omega(x_G^i - \mathbf{r}^*)\|,
\]
respectively, where \( \Omega \) is a \( N M \times N M \) matrix that can be written as a \( N M \times N M \) matrix whose \( k \)-th entry is the diagonal matrix diag\( \{\varphi_{k1}, \ldots, \varphi_{kN}\} \).

Substituting (63) and (64) into (57) and (58), respectively, we obtain the error recursions for diffusion GN algorithm and non-cooperative algorithm as follows

\[
\|x_G^{i+1} - \mathbf{r}^*\| \leq \|G x_G^i - G \mathbf{r}^*\| + \|\alpha f\| \|\Lambda_D\| F \|\Omega(x_G^i - \mathbf{r}^*)\|
\leq \|G x_G^i - G \mathbf{r}^*\| + \|\alpha f\| \|\Lambda_D\| F \|\Omega\| F \|G x_G^i - G \mathbf{r}^*\|
\leq (1 + \|\alpha f\| \|\Lambda_D\| F \|\Omega\| F) \|G x_G^i - G \mathbf{r}^*\|
\]
and

\[
\|x_G^{i+1} - \mathbf{r}^*\| \leq \|x_G^i - \mathbf{r}^*\| + \|\alpha d\| \|\Lambda_d\| F \|\Omega(x_G^i - \mathbf{r}^*)\|
\leq (1 + \|\alpha d\| \|\Lambda_d\| F \|\Omega\| F) \|x_G^i - \mathbf{r}^*\|.
\]

From (6), we know that \( X_k^i \) is a convex combination of \( \{x_k^i\} \) for \( k \in N_k \). Thus, Assumption 1 holds for \( X_k^i \). Under Assumption 1(1), we have

\[
\|[Q_k^1(X)]^{-1}\| = \left\| \sum_{l \in N_k} F_l^T(X_l^i) F_l(X_l^i) \right\|^{-1} \leq \frac{1}{n_k \sigma_{\text{min}}}
\]
and

\[
\|[Q_k^1(x)]^{-1}\| \leq \frac{1}{n_k \sigma_{\text{min}}}
\]

Because of

\[
\|\Lambda_d\| F = \sum_{k=1}^{N} \|\Omega(X_k^i - \mathbf{r}^*)\| \leq \frac{\sqrt{N}}{n_k \sigma_{\text{min}}},
\]
and

\[
\|\Lambda_d\| F = \sum_{k=1}^{N} \|\Omega(X_k^i - \mathbf{r}^*)\| \leq \frac{\sqrt{N}}{n_k \sigma_{\text{min}}},
\]
we know that \( 1 + \|\alpha f\| \|\Lambda_D\| F \|\Omega\| F \) and \( 1 + \|\alpha d\| \|\Lambda_d\| F \|\Omega\| F \) are upper bounded by a small common constant when the small step size is selected.

The recursions (63) and (64) describe how the global error evolve over time for diffusion and non-cooperative GN algorithms, respectively. It can be known for non-cooperative version that the global error \( \|x_G^{i+1} - \mathbf{r}^*\| \) asymptotically become smaller as the iteration goes on, since \( 1 + \|\alpha f\| \|\Lambda_D\| F \|\Omega\| F \) is bounded. The observation leads to the error curve becomes less steep with the increase of iteration till the system reaches the stability. A similar conclusion can be obtained for diffusion GN algorithm.

Another observation is that the diffusion version will outperform the non-cooperative version in the sense of convergence rate, since it provides a \( \|G\| F \) times error reduction room
than the counterpart from the perspective of global error norm when \( \|A_0\|_F \) and \( \|A_d\|_F \) reach their same upper bound based on (69) and (70), as long as \( \|G\|_F > 1 \). Thus, it can be known that the global estimate \( x_G^t \) in diffusion GN will be closer to the global minimizer \( \bar{x} \) at every iteration than that in non-cooperative GN. The above conclusions confirm the cooperative role in GN algorithm for improvements on convergence.

### IV. Conclusions

In this paper, we analyze the convergence of a cooperative diffusion GN paradigm for nonlinear least squares problems in a distributed networked system. By the presented theoretical results, we show that cooperation strategy can obtain a room for improvement in term of convergence and guarantee algorithm’s convergence when the derived sufficient conditions are satisfied, i.e., the good initial guesses, reasonable step size values and network connectivity. In order to avoid data incest and double counting, future works will focus on optimal estimate fusion, by which the network can adaptively adjust the weights or select the good nodes participating in the estimation.

### APPENDIX A

**Proof of Lemma 1**

The vector \( x_G^t - \bar{x} - \alpha d_G^t \) is the global representation of \( x_k^t - x^* - \alpha d_k^t \), which is written by

\[
x_k^t - x^* - \alpha d_k^t = x_k^t - x^* - \alpha \sum_{i \in N_k} f_i(x_k^t) f_i(x_k^t),
\]

where \( F^+(\cdot) \) denotes the generalized inverse of matrix \( F(\cdot) \) and we use \( [F_i(x^t)]^T f_i(x_k^t) = 0 \) according to Assumption 1.

According to the Assumption 1 and 2, we have the following inferences

\[
\| A(x_G^t) \| \leq \Sigma_{max}, \quad \| A(x_G^t)^+ \| \leq \frac{1}{\Sigma_{min}} \tag{72}
\]

and

\[
\| A(x_G^t) - A(\bar{x}) \| \leq \omega \| x_G^t - \bar{x} \|. \tag{73}
\]

Thus, we can write the global representation as

\[
x_G^t - \bar{x} - \alpha d_G^t = [A(x_G^t)]^+ A(x_G^t)(x_G^t - \bar{x}) - \alpha [A(x_G^t)]^+ b(x_G^t) + \alpha \omega \| x_G^t - \bar{x} \|^2.
\]

Using the mean-value theorem and (72) (73), we have

\[
\| ab(\bar{x}^t) - ab(x_G^t) - A(x_G^t)(\bar{x}^t - x_G^t) \|
\]

\[
= \| \alpha \int_0^1 [A(x_G^t + u(\bar{x}^t - x_G^t))(\bar{x}^t - x_G^t)]du
\]

\[
= \| \alpha \int_0^1 [A(x_G^t + u(x^t - x_G^t))(x^t - x_G^t)]du
\]

\[
- A(x_G^t)(\bar{x}^t - x_G^t)]du
\]

\[
= \| \alpha \int_0^1 [A(x_G^t + u(x^t - x_G^t)) - A(x_G^t)(x^t - x_G^t)]du
\]

\[
- (1 - \alpha)A(x_G^t)(\bar{x}^t - x_G^t)]du
\]

\[
\leq \alpha \int_0^1 \| A(x_G^t + u(x^t - x_G^t)) - A(x_G^t) \| \| x^t - x_G^t \| \| x_G^t - \bar{x}^t \|
\]

\[
+ (1 - \alpha)\Sigma_{max} \| x_G^t - \bar{x}^t \|
\]

\[
\leq \frac{\alpha \omega}{2} \| x_G^t - \bar{x}^t \|^2 + (1 - \alpha)\Sigma_{max} \| x_G^t - \bar{x}^t \|.
\]

Applying the Lemma 1[56, Lemma 1], we have

\[
\| (A(x_G^t))^+ - (A(\bar{x}))^+ \| \leq \sqrt{2} \| (A(\bar{x}^t))^+ \| \| (A(x_G^t))^+ \| \| A(x_G^t) - A(\bar{x}^t) \|
\]

\[
\leq \sqrt{2} \| x_G^t - \bar{x}^t \|.
\]

and

\[
\| b(\bar{x}^t) \| \leq \sum_{k=1}^N \| f_i(\bar{x}^t) \| = N\epsilon_{min}.
\]

Therefore, substituting (75) (76) and (77) into (74) leads to (24).

### APPENDIX B

**Proof of Lemma 2**

To use the expansion formula (29), we define

\[
E_{kl}^t \triangleq Q_k^l(x_k^t) - Q_l^l(x_l^t)
\]

and

\[
c_{kl}^t \triangleq q_k^l(x_k^t) - q_l^l(x_l^t).
\]

Using (18), we can rewrite (78) and (79) as

\[
E_{kl}^t = \sum_{u \in N_k} F_{u}^T(x_k^t) F_u(x_k^t) - \sum_{t \in N_l} F_{t}^T(x_l^t) F_t(x_l^t)
\]

and

\[
c_{kl}^t = \sum_{u \in N_k} F_{u}^T(x_k^t) f_u(x_k^t) - \sum_{t \in N_l} F_{t}^T(x_l^t) f_t(x_l^t).
\]

Now we use the mathematical induction to obtain the results of Lemma 2.

**A. Initial Case: i = 1**

Given by the same initial estimate \( x^0_k = x_k^0 = x_l^0, k, l \in \mathcal{N} \) for all nodes in the network and (14), we have

\[
x_l^1 - x_k^1 = \alpha (Q_k^l(x_k^0))^{-1} q_k^l(x_k^0) - \alpha (Q_l^l(x_l^0))^{-1} q_l^l(x_l^0)
\]

\[
= \alpha (Q_k^l(x_k^0) + E_{kl}^0)^{-1} q_k^l(x_k^0) + c_{kl}^0 - \alpha (Q_l^l(x_l^0))^{-1} q_l^l(x_l^0).
\]

We now consider the conditions (the below Corollary 1) for applying the expansion formula (29) when we let \( Z = Q_k^l(x_k^0) \) and \( \delta Z = E_{kl}^0 \).
Corollary 1. Let Assumptions 1 and 2 hold. The following recursion can be obtained
\[ \|E^k_{kl}\| \leq n_{kl}\gamma_F\|x_k^l - x_k^l\| + (n_{kl} + n_{lk})\sigma_{max}^2, \] (83)
and \[ \|Q^l(x^0_l)\|^{-1}E^k_{kl}\| < 1 \] holds when the following condition
\[ \frac{n_{kl} + n_{lk}}{n_l} < \frac{\sigma_{min}^2}{\sigma_{max}^2} < 1 \] (84)
is satisfied.

Proof: See Appendix C.

From (83), \[ \|Q^l(x^0_l)\|^{-1}E^k_{kl}\| < 1 \] holds for a reasonable large denominator \( n_l \) and a reasonable small numerator \( n_{kl} + n_{lk} \). In other words, a high connectivity for the network is helpful for diffusion GN algorithm.

Under Corollary 1, we use the the expansion formula (29) as follows
\[ [(Q_l^0(x^0_l) + E^0_{kl})^{-1}q_l^0(x^0_l)]^n[(Q_l^0(x^0_l) + E^0_{kl})^{-1}q_l^0(x^0_l)] + e_{kl} \]
\[ = \sum_{u=0}^{\infty}(-1)^u[(Q_l^0(x^0_l) + E^0_{kl})^{-1}]^u[q_l^0(x^0_l) + e_{kl}] \]
\[ = \sum_{u=1}^{\infty}(-1)^u[(Q_l^0(x^0_l) + E^0_{kl})^{-1}]^u[q_l^0(x^0_l) + e_{kl}] \]
\[ + \lim_{u \to \infty}[(Q_l^0(x^0_l) + E^0_{kl})^{-1}]^u[q_l^0(x^0_l) + e_{kl}] \]
Substituting (83) into (82), for convenience of notation, we define
\[ x^1_k - x^1_k \triangleq \alpha(p^0_1 + p^0_2) \] (86)
where \( p^0_1 \) and \( p^0_2 \) correspond sequentially to the last two terms of (83), respectively.

Now we evaluate the norm of the vectors \( p^0_1 \) and \( p^0_2 \). According to the CBS inequality and the triangle inequality, we have
\[ \|p^0_1\| \leq \sum_{u=1}^{\infty}[(Q_l^0(x^0_l) + E^0_{kl})^{-1}]^u[q_l^0(x^0_l) + e_{kl}] \]
\[ \leq \frac{(n_l + n_{kl} + n_{lk})\sigma_{max}^2}{n_l\sigma_{min}^2} \sum_{u=1}^{\infty}[(n_l + n_{kl} + n_{lk})\sigma_{max}^2]u \]
where we use (105), (106), (107) for \( i = 0 \) and (109).

We set
\[ \frac{(n_l + n_{kl} + n_{lk})\sigma_{max}^2}{n_l\sigma_{min}^2} = \mu_{kl}^0 \in (0,1), \] (87)
can be rewritten as
\[ \|p^0_1\| \leq \frac{(n_l + n_{kl} + n_{lk})\sigma_{max}^2\sigma_{max}^2\mu_{kl}^0}{n_l\sigma_{min}^2(1 + \mu_{kl}^0)} \]
\[ \leq \frac{(n_l + n_{kl} + n_{lk})\sigma_{max}^2\sigma_{max}^2}{2n_l\sigma_{min}^2} \] (88)

For the norm of \( p^0_2 \), we have
\[ \|p^0_2\| \leq \frac{(n_l + n_{kl} + n_{lk})\sigma_{max}^2\sigma_{max}^2}{n_l\sigma_{min}^2} . \] (89)

Therefore,
\[ \|x^1_k - x^1_k\| \leq \alpha(\|p^0_1\| + \|p^0_2\|), \] (90)
where \( \|p^0_1\| \) and \( \|p^0_2\| \) are given by (83) and (84), respectively. Therefore, we have \( \|x^1_k - x^1_k\| \leq a_2 \).

B. Induction: \( i = I \) and \( i = I + 1 \)

For \( i = I \) and any \( l \neq k \), let
\[ \|x^I_l - x^I_k\| \leq a_2 \sum_{j=1}^{l} (a_1)^{j-1} \] (91)
holds, where \( a_1 \) and \( a_2 \) are given by (55) and (56), respectively. Then for \( i = I + 1 \), we have
\[ x^{I+1}_l - x^{I+1}_k = x^I_l - x^I_k + \alpha(Q^I_k(x^I_l))^{-1}q^I_k(x^I_k) - \alpha(Q^I_k(x^I_l))^{-1}q^I_k(x^I_l) \]
\[ = x^I_l - x^I_k + \alpha(Q^I_k(x^I_l) + E^I_{kl})^{-1}(q^I_k(x^I_l) + e_{kl}) \]
\[ - \alpha(Q^I_k(x^I_l))^{-1}q^I_k(x^I_l) \] (92)

To apply the the expansion formula (29) here, substituting (91) into (108) for \( i = I \), the following condition
\[ \|Q^I_k(x^I_l)\|^{-1}E^I_{kl}\| \]
\[ = \frac{n_{kl}\gamma_F a_2}{n_l\sigma_{min}^2} \sum_{j=1}^{l} (a_1)^{j-1} + (n_{kl} + n_{lk})\sigma_{max}^2 \] (93)
\[ \leq \frac{n_l\sigma_{min}^2}{n_{kl}\gamma_F} \]
\[ \triangleq \mu_{kl} < 1 \]

need to be satisfied. Substituting (55) and (56) into the above inequality, we get
\[ \alpha[(1 + \alpha\theta)^{I+1} - (1 + \alpha\theta)] \]
\[ < \frac{n_l\sigma_{min}^2}{n_{kl}\gamma_F} \]
\[ < \frac{n_l\sigma_{min}^2}{n_{kl}\gamma_F(n_l + 3n_{kl} + 3n_{lk})\sigma_{max}^2\sigma_{max}^2} \] (94)
where
\[ \theta \triangleq \frac{n_{kl} + 2n_{kl}\gamma_f}{2n_l\sigma_{min}^2} \] (95)

The left side of (94) is an increasing exponential function of the iteration time \( I \). It is a reasonable assumption that the inequality (94) holds at any time \( I \) when we set a sufficiently small step size parameter \( \alpha \). When (93) is satisfied, we obtain the following useful conclusion
\[ \|x^I_l - x^I_k\| \leq \frac{n_l\sigma_{min}^2 - (n_{kl} + n_{lk})\sigma_{max}^2}{n_{kl}\gamma_F} \] (96)

For (92), we can bound \( \|x^{I+1}_l - x^{I+1}_k\| \) by using the expansion formula (29)
\[ \|x^{I+1}_l - x^{I+1}_k\| \leq \|x^I_l - x^I_k\| + \alpha(\|p^0_1\| + \|p^0_2\|), \] (97)
where we define \( p^I_1 \) and \( p^I_2 \) that are similar to (83), and
\[ \|p^I_1\| \leq \sum_{u=1}^{\infty}[(Q^I_l(x^I_l))^{-1}E^I_{kl}\|]u||Q^I_l(x^I_l))^{-1}||q^I_l(x^I_l) + e_{kl}|| \]
\[ \leq \frac{(n_l + n_{kl} + n_{lk})\sigma_{max}^2 + n_{kl}\|x^I_l - x^I_k\|}{n_l\sigma_{min}^2} \sum_{u=1}^{\infty}\|p^I_1\|^u \]
\[ \leq \frac{(n_l + n_{kl} + n_{lk})\sigma_{max}^2 + n_{kl}\|x^I_l - x^I_k\|}{n_l\sigma_{min}^2} \mu_{kl}^I \] (98)
and
\[ \|p^I_2\| \leq \frac{n_{kl}\gamma_f\|x^I_l - x^I_k\|}{n_l\sigma_{min}^2} + \frac{(n_{kl} + n_{lk})\sigma_{max}^2}{n_l\sigma_{min}^2} \] (99)
Substituting (98) (99) into (97) and rearranging them, we obtain
\[ \|x^{l+1}_i - x^{l+1}_k\| \leq a_1\|x^l_i - x^l_k\| + a_2. \] (100)
Substituting (91) into (100), we can obtain
\[ \|x^{l+1}_i - x^{l+1}_k\| \leq a_1a_2 \sum_{j=1}^{l} (a_1)^{-j} + a_2 \]
\[ = a_2 (a_1) \sum_{j=1}^{l} (a_1)^{-j} + 1 \]
\[ = a_2 \sum_{j=1}^{l+1} (a_1)^{-j} - 1. \] (101)
\[ \text{APPENDIX C} \]
\[ \text{PROOF OF COROLLARY 1} \]

Starting from (80), we get
\[ E^i_{kl} = \sum_{u \in N_k} F^T_u(x^i_k)F_u(x^i_k) - \sum_{t \in N_i} F^T_t(x^i_t)F_t(x^i_t) \]
\[ = \sum_{u \in N_k} [F^T_u(x^i_k)F_u(x^i_k) - F^T_u(x^i_t)F_u(x^i_t)] \]
\[ + \sum_{r \in N_{kl}} F^T_r(x^i_k)F_r(x^i_k) - \sum_{t \in N_{rk}} F^T_t(x^i_t)F_t(x^i_t) \] (102)
and
\[ e^i_{kl} = \sum_{u \in N_k} F^T_u(x^i_k)f_u(x^i_k) - \sum_{t \in N_i} F^T_t(x^i_t)f_t(x^i_t) \]
\[ = \sum_{u \in N_k} [F^T_u(x^i_k)f_u(x^i_k) - F^T_u(x^i_t)f_u(x^i_t)] \]
\[ + \sum_{r \in N_{kl}} F^T_r(x^i_k)f_r(x^i_k) - \sum_{t \in N_{rk}} F^T_t(x^i_t)f_t(x^i_t) \] (103)

Then from Assumption 2 and the CBS Inequality, we get
\[ \|E^i_{kl}\| \leq nk \gamma F \|x^i_k - x^i_t\| + (nk_l + n_{l|k}) \sigma_{max}^2 \] (104)
and
\[ \|Q^i(x^i_k)\|^{-1} \leq \frac{1}{nk \sigma_{min}^2}. \] (105)

Based on Assumption 2, \( \|f_u(x^i_k)\|^2 \leq \epsilon_{max}^2 \) and \( \|f_u(x^i_k)\|^2 \geq \epsilon_{min}^2 \), thus \( f_u(x^i_k) \) has the upper and lower bounds for all \( k \in N \) and \( x^i_k \in \tilde{X} \). For convenience, we let \( \epsilon_{min} \leq \|f_u(x^i_k)\| \leq \epsilon_{max} \). Thus, we have \( \sigma_{min} \leq \|F^T_u(x^i_k)f_u(x^i_k)\| \leq \sigma_{max} \) and get
\[ \|e^i_{kl}\| \leq nk \gamma f \|x^i_k - x^i_t\| + (nk_l + n_{l|k}) \sigma_{max} \epsilon_{max}, \] (106)
and
\[ \|Q^i(x^i_k)\|^{-1} \leq \frac{1}{nk \sigma_{min} \epsilon_{max}}. \] (107)

Thus, we have
\[ \|Q^i(x^i_k)\|^{-1} E^i_{kl} \leq \frac{nk \gamma F \|x^i_k - x^i_t\| + (nk_l + n_{l|k}) \sigma_{max}^2}{nk \sigma_{min}^2}. \] (108)
To ensure \( \|Q^i(x^i_k)\|^{-1} E^i_{kl} < 1 \) when given \( x^0_k = x^0_t \), we obtain the following condition
\[ \|Q^i(x^i_k)\|^{-1} E^i_{kl} \leq \frac{(nk_l + n_{l|k}) \sigma_{max}^2}{nk \sigma_{min}^2} < 1. \] (109)
To further rewrite (109), we get (84).

\[ \text{APPENDIX D} \]
\[ \text{PROOF OF LEMMA 3} \]

Under Lemma 2 and the inequality (12), we know that \( \|x^i_k - x^i_t\| \) is also bounded by
\[ \|x^i_k - x^i_t\| \leq N \|x^i_k - x^i_t\|. \] (110)
Thus, (12) can be obtained.
Thus, we can replace (30) (31) as
\[ \|S^i_k\| \leq Nnk \gamma F \Pi^i \] (111)
and
\[ \|s^i_t\| \leq Nnk \gamma F \Pi^i \] (112)
Setting \( Z = Q^i_k(x^i_k) \) and \( \delta Z = S^i_k \), we have
\[ \|Q^i_k(x^i_k)\|^{-1} \|S^i_k\| \leq \|Q^i_k(x^i_k)\|^{-1} \|S^i_k\| \leq \frac{Nnk \gamma F \Pi^i}{\sigma_{min}^2}. \] (113)

Now defining \( \frac{Nnk \gamma F \Pi^i}{\sigma_{min}^2} \) as \( \zeta_i \), (114) for subsequent use, and solving \( \zeta_i < 1 \) and combining (96), an inequality is obtained as
\[ (Nnk_{l|k} + n_{l|k}) \sigma_{min}^2 < (Nnk_k + n_{k|l}) \sigma_{max}^2. \] (115)

By rewriting (84) as follows
\[ (nk_{l|k} + n_{l|k}) \sigma_{max}^2 < nk \sigma_{min}^2, \] (116)
and substituting it into (115) gives
\[ nk > 0. \] (117)

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