TRANSLATING SOLUTIONS FOR A CLASS OF QUASILINEAR PARABOLIC INITIAL BOUNDARY VALUE PROBLEMS IN LORENTZ-MINKOWSKI PLANE $\mathbb{R}_1^2$

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Abstract. In this paper, we investigate the evolution of spacelike curves in Lorentz-Minkowski plane $\mathbb{R}_1^2$ along prescribed geometric flows (including the classical curve shortening flow or mean curvature flow as a special case), which correspond to a class of quasilinear parabolic initial boundary value problems, and can prove that this flow exists for all time. Moreover, we can also show that the evolving spacelike curves converge to a spacelike straight line or a spacelike Grim Reaper curve as time tends to infinity.

Keywords: Mean curvature flow, spacelike curves, Lorentz-Minkowski plane, Neumann boundary condition.

MSC 2020: 35K20, 53B30.

1. Introduction

To our knowledge, the start of the study of mean curvature flow (MCF for short) maybe is due to Brakke [3] where he used the geometric measure theory to investigate the motion of surface by its mean curvature, while Huisken [11] (for higher dimensional case), Gage-Hamilton [6] and Grayson [8] (for lower dimensional case) gave pioneering contributions to this theory, and after that many interesting related conclusions (or improvements) have been obtained. BTW, we would like to refer books [5, 7, 13] and references therein such that readers can have a relatively comprehensive understanding about the fundamental theory and some important improvements of the MCF or the CSF.

In order to explain our motivation of writing this paper clearly, we prefer to give a brief introduction to several results on MCF or CSF first. They are:

- (I) Graphic curves (defined over an interval) in Euclidean 2-space $\mathbb{R}^2$ satisfying a class of quasilinear parabolic initial boundary value problems (IBVPs for short) have been investigated, and authors therein have proven that the class of quasilinear parabolic IBVPs considered therein has smooth solution for $t \in [0, \infty)$ (i.e., has the long-time existence) and its solution converges as $t \to \infty$ to a solution moving by translation with a constant speed. Especially, if two contact angles are equal, the graphic curve determined by the solution of the class of quasilinear parabolic IBVPs converges to a straight line as $t \to \infty$. It is easy to check that the standard heat flow equation and the MCF (or CSF) equation are covered by the class of quasilinear parabolic IBVPs considered in [1] as special cases. Therefore, as direct consequences, the authors therein

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1 The curve shortening flow (CSF for short) is the MCF in a prescribed ambient space of dimension 2, that is, the CSF is essentially the lower dimensional case of MCF.

2 In [1], the boundary value condition of the class of quasilinear parabolic boundary problems (1.2) considered therein is actually an inhomogeneous Neumann boundary condition (NBC for short), which at each endpoint can be described as the tangent value of the contact angle between the graphic curve and the parabolic boundary. BTW, the admissible range of the contact angle therein is $(-\pi/2, \pi/2)$, which implies that the NBC considered in (1.2) of [1] is not necessary to be zero.
showed that along the heat flow (resp., the MCF), graphic curves (defined over an interval) in $\mathbb{R}^2$ converges as $t \to \infty$ to a uniquely determined portion of a parabola or a straight line (resp., the Grim Reaper or a straight line).

- \([\[1\]]\) For a given 3-dimensional Lorentz manifold $M^2 \times \mathbb{R}$ with the metric $\sum_{i,j=1}^{2} \sigma_{ij} dw^i \otimes dw^j - ds \otimes ds$, where $M^2$ is a 2-dimensional complete Riemannian manifold with the metric $\sum_{i,j=1}^{2} \sigma_{ij} dw^i \otimes dw^j$ and nonnegative Gaussian curvature, the authors therein investigated the evolution of spacelike graphs (defined over compact, strictly convex domains in $M^2$) along the nonparametric MCF with prescribed nonzero NBC, and proved that this flow exists for all the time and its solutions converge to ones moving only by translation. This interesting conclusion somehow extends, for instance, the following results: (Huisken \([\[2\]]\)) Graphs defined over bounded domains (with $C^{2,\alpha}$ boundary) in $\mathbb{R}^n$ ($n \geq 2$), which are evolving by the MCF with vertical contact angle boundary condition (i.e., vanishing NBC), have been investigated, and it was proven that this evolution exists for all the time and the evolving graphs converge to a constant function as time tends to infinity (i.e., $t \to \infty$); (Altschuler-Wu \([\[2\]]\)) Graphs, defined over strictly convex compact domains in $\mathbb{R}^2$, evolved by the non-parametric MCF with prescribed contact angle (not necessary to be vertical, i.e., the NBC is not necessary to be zero), converge to translating surfaces as $t \to \infty$.

- Up to rescalings and isometries of the Lorentz-Minkowski plane $\mathbb{R}^2_1$, Halldorsson \([\[10\]]\) successfully gave a classification of all self-similar solutions to the MCF of spacelike curves in $\mathbb{R}^2_1$, which is a continuation of his previous work \([\[9\]]\) about the classification of all self-similar solutions to the CSF of immersed curves in the plane $\mathbb{R}^2$. As explained in \([\[10\]]\) Sect. 1], for the MCF in $\mathbb{R}^2$ and in $\mathbb{R}^2_1$, there are some notable differences: (I) in $\mathbb{R}^2$, it is not hard to know that the length of the evolving simple closed curves is non-increasing along the MCF, and actually that is the reason why the MCF in $\mathbb{R}^2$ is called the CSF. However, in $\mathbb{R}^2_1$, since the curvature of curves blows up at lightlike points, the MCF for simple closed curves would not be considered. One can define the MCF for spacelike (or timelike) curves with finite Minkowski-length (without having endpoints) in $\mathbb{R}^2_1$. For the evolution of spacelike curves (in $\mathbb{R}^2_1$) along the MCF, the length of evolving spacelike curves has the possibility of decreasing or increasing – see, e.g., \([\[10\]]\) Figure 16] for an intuitive explanation. (II) Halldorsson \([\[10\]]\) have shown some examples of curves in $\mathbb{R}^2_1$ which are initially disjoint but then intersect under the MCF, and have also shown examples of non-uniqueness of the MCF in $\mathbb{R}^2_1$. This behavior is totally different from that of the curves arose in the classification of self-similar solutions to the MCF in $\mathbb{R}^2$, since those curves in $\mathbb{R}^2$ have bounded curvature.

Our successful experience on the MCF with nonzero NBC in the Lorentz 3-manifold $M^2 \times \mathbb{R}$ (see \([\[4\]]\)) and Halldorsson’s works \([\[9\]]\) \([\[10\]]\) motivate us to consider the evolution of spacelike curves in $\mathbb{R}^2_1$ and try to extend Altschuler-Wu’s result \([\[1\]]\) for curves in $\mathbb{R}^2$.

Denote by $\mathbb{R}^2_1$ the Lorentz-Minkowski plane with the Lorentzian metric

$$\langle \cdot , \cdot \rangle_L = dx^2 - dy^2.$$ 

Let $I_d := [-d, d]$ be a closed interval on the $x$-axis, $d \in \mathbb{R}^+$, and let $\Omega_d := I_d \times (0, \infty)$ be the rectangular region in $\mathbb{R}^2_1$. Clearly, $\partial I_d = \{(-d,0)\} \cup \{(d,0)\}$. For a one-parameter family of spacelike graphic curves $G_t := (x, u(x,t))$ defined over $I_d$, it is not hard to know that its tangent vector, the future-directed timelike unit normal vector and the curvature are given by

$$\begin{align*}
  \vec{e} &= (1, u_x), \\
  \vec{v} &= \frac{(u_x, 1)}{\sqrt{1 - u_x^2}}, \\
  k &= \frac{u_{xx}}{(1 - u_x^2)^{\frac{3}{2}}},
\end{align*}$$
where the notations \( u_x = \frac{\partial u}{\partial x}, u_{xx} = \frac{\partial^2 u}{\partial x^2} \) have been used, and the spacelike assumption implies \(|u_x| < 1\). We investigate the evolution of \( G_t \) along the MCF in \( \mathbb{R}^2 \), and can prove the following conclusion.

**Theorem 1.1.** Let \( X_0 : I_d \to \mathbb{R}^2 \) such that \( G_0 := X_0(I_d) \) can be written as a graph defined over the interval \( I_d \). Assume further that

\[
G_0 = \text{graph}_{I_d} u_0
\]

is a spacelike graph over \( I_d \) for a positive function \( u_0 : I_d \to \mathbb{R} \) satisfying

\[
(1.2) \quad (u_0)_x(-d) = \theta_{-d}, \quad (u_0)_x(d) = \theta_d, \quad \theta_i \in (-1, 1), \quad i = -d, d.
\]

Then the following IBVP

\[
(1.3) \quad \begin{cases}
    u_t = \frac{u_{xx}}{1 - u_x^2} & \text{in } I_d \times [0, \infty), \\
    u_x(i, t) = \theta_i & t \in [0, \infty), \quad i = -d, d \\
    u(\cdot, 0) = u_0 & u_0 \in C^\infty(I_d)
\end{cases}
\]

converges as \( t \to \infty \) to a solution moving by translation with speed \( A(\theta_{-d}, \theta_d, d) \) given by

\[
(1.4) \quad A(\theta_{-d}, \theta_d, d) = \frac{\text{artanh}(\theta_d) - \text{artanh}(\theta_{-d})}{2d}.
\]

Moreover, the leaves \( G_t \) are spacelike graphs defined over \( I_d \), i.e.,

\[
G_t := \text{graph}_{I_d} u(\cdot, t),
\]

and these leaves converge as \( t \to \infty \) to a spacelike straight line or a spacelike Grim Reaper curve.

**Remark 1.1.**

1. The precondition \((1.2)\) is actually the compatibility condition of the IBVP \((1.3)\), which can be used to make sure the regularity (or smoothness) of the solution \( u(\cdot, t) \) to the IBVP \((1.3)\).
2. In fact, the spacelike assumption for \( G_0 \) implies that \((u_0)_x \in (-1, 1)\) holds not only at endpoints \(-d, d\) but also on the whole interval \( I_d \).
3. For a one-parameter family of spacelike graphic curves \( G_t = (x, u(x, t)) \) defined over \( I_d \) given by the mapping \( X : I_d \times [0, T) \to \mathbb{R}^2 \) for some \( T > 0 \), by using \((1.1)\), it is easy to know that the evolution of \( G_t \) along the MCF (with nonzero NBC) in \( \mathbb{R}^2 \) can be described by the IBVP \((1.3)\). Then Theorem \((1.1)\) tells us that this evolution exists for all the time (i.e., \( T = \infty \)) and moreover very nice asymptotical behavior of the evolving curves can be obtained.
4. To make sure that the RHS of the evolution equation in \((1.3)\) does not degenerate, one needs to show \(|u_x(x, t)| < 1\) during the evolving process (or equivalently, the spacelike property is preserved under the flow), which will be given by the gradient estimate of Section \((2)\).
5. Clearly, by the property of the inverse hyperbolic tangent function, it is easy to know that \( A(\theta_{-d}, \theta_d, d) = 0 \) if and only if \( \theta_{-d} = \theta_d \).
6. The IBVP \((1.3)\) has smooth solution on \( I_d \times [0, \infty) \) — see Subsection \((2.2)\) for this fact.

After we have got the main conclusion of Theorem \((1.1)\) we find that by using analysis techniques in Section \((2)\) here and Altschuler-Wu’s in \((1)\), a more general result can be obtained. In fact, we have:

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\(^3\)This convention implies that \( u_t = \frac{\partial u}{\partial t}, u_{xt} = \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial}{\partial x} \frac{\partial u}{\partial t}, \) and so on. Of course, here we require that the graphic function \( u(x, t) \) has enough regularity.

\(^4\)Obviously, this mapping \( X \) satisfies \( X(\cdot, 0) = X_0(\cdot) \).
Theorem 1.2. Let $X_0 : I_d \rightarrow \mathbb{R}^2_1$ such that $G_0 := X_0(I_d)$ can be written as a graph defined over the interval $I_d$. Assume further that

$$G_0 = \text{graph}_{I_d}u_0$$

is a spacelike graph over $I_d$ for a positive function $u_0 : I_d \rightarrow \mathbb{R}$ satisfying

$$(u_0)_x(-d) = \theta_{-d}, \quad (u_0)_x(d) = \theta_d, \quad \theta_i \in (-1, 1), \quad i = -d, d.$$ 

Then the following IBVP

$$
\begin{cases}
  u_t - (v(u_x))_x = 0 & \text{in } I_d \times [0, \infty), \\
  u_x(i, t) = \theta_i & t \in [0, \infty), \quad i = -d, d \\
  u(\cdot, 0) = u_0 & u_0 \in C^\infty(I_d)
\end{cases}
$$

converges as $t \to \infty$ to a solution moving by translation with speed $\overline{A}(\theta_{-d}, \theta_d, d)$ given by

$$\overline{A}(\theta_{-d}, \theta_d, d) = \frac{v'(d) - v'(\theta_d)}{2d},$$

where $v \in C^\infty((-1, 1))$ with its derivative function satisfying $v' > 0$. Moreover, the leaves $G_t$ are spacelike graphs defined over $I_d$, i.e.,

$$G_t := \text{graph}_{I_d}u(\cdot, t),$$

and these leaves converge as $t \to \infty$ to a spacelike straight line or a spacelike Grim Reaper curve.

Remark 1.2. (1) If $v(u_x) = \text{artanh}(u_x)$, then Theorem 1.2 degenerates into Theorem 1.1 directly and completely. The reason of retaining Theorem 1.1 here is that we prefer to show the origin of our idea (on thinking this topic) to readers clearly.

(2) (The heat flow in $\mathbb{R}^2_1$) For $v(u_x) = u_x$, the spacelike graph of $u(x, t)$ converges as $t \to \infty$ to a uniquely determined portion of a spacelike parabola or a spacelike straight line. Moreover, in this setting, the translating speed is $\overline{A}(\theta_{-d}, \theta_d, d) = (\theta_d - \theta_{-d})/2d$.

2. The special case: spacelike MCF

We devote to give the proof of Theorem 1.2 in this section.

2.1. The gradient estimate. Let $0 < \alpha < 1$ and $T^*$ be the maximal time such that there exists some

$$u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(I_d \times [0, T^*)) \cap C^\infty(I_d \times (0, T^*))$$

which solves (1.3). Next, we shall prove a priori estimates for those admissible solutions on $[0, T^*)$ with $T < T^*$. First, we have the $C^0$ estimate as follows:

Lemma 2.1. Let $u$ be a solution of (1.3). Then we have

$$\inf_{I_d} u(x, 0) \leq u(x, t) \leq \sup_{I_d} u(x, 0), \quad \forall x \in I_d, \quad t \in [0, T].$$

Proof. Let $u(x, t) = u(t)$ (independent of $x$) be a solution of (1.3) with $u(0) = C$. In this case, the first equation in (1.3) reduces to an ordinary differential equation (ODE for short)

$$\frac{d}{dt}u = 0,$$

Therefore

$$u(t) = C.$$
Using the maximum principle, we can obtain that
\[ \inf_{I_d} u(x, 0) \leq u(x, t) \leq \sup_{I_d} u(x, 0). \]

This completes the proof. \(\square\)

Then we can obtain the \(u_t\) estimate:

**Lemma 2.2.** Let \(u\) be a solution of (1.3), we have
\[ \inf_{I_d} u_t(x, 0) \leq u_t(x, t) \leq \sup_{I_d} u_t(x, 0), \quad \forall x \in I_d, \ t \in [0, T]. \]

**Proof.** Set
\[ \Phi(x, t) = u_t(x, t). \]

Differentiating both sides of the first evolution equation of (1.3), it is easy to get that
\[
\begin{cases}
\frac{\partial \Phi}{\partial t} = \frac{1}{1 - u_x^2} \Phi_{xx} + \frac{2u_x u_{xx}}{(1 - u_x^2)^2} \Phi_x & \text{in } I_d \times [0, T], \\
\Phi_x(i, t) = 0 & t \in [0, T], \ i = -d, d \\
\Phi(\cdot, 0) = \Phi_0 & u_0 \in C^\infty(I_d).
\end{cases}
\]

Using the maximum principle and Hopf’s Lemma, we have
\[ \inf_{I_d} u_t(x, 0) \leq u_t(x, t) \leq \sup_{I_d} u_t(x, 0), \quad \forall x \in I_d, \ t \in [0, T], \]
which finishes the proof. \(\square\)

The gradient estimate can be obtained as follows:

**Lemma 2.3.** Let \(u\) be a solution of (1.3), we have
\[ |u_x|(x, t) \leq \sup_{I_d} |u_x|(x, 0) < 1 \quad \forall x \in I_d, \ t \in [0, T]. \]

**Proof.** Set \(\varphi = \frac{|u_x|^2}{2}\). By differentiating \(\varphi\), we have
\[
\frac{\partial \varphi}{\partial t} = u_x u_{xt} = u_x \left( \frac{u_{xxx}}{1 - u_x^2} + \frac{2u_x u_{xx}^2}{(1 - u_x^2)^2} \right)
= \frac{u_x u_{xxx}}{1 - u_x^2} + \frac{2u_x^2 u_{xx}^2}{(1 - u_x^2)^2}.
\]

Since
\[ \varphi_x = u_x u_{xx}, \quad \varphi_{xx} = u_{xx}^2 + 2u_x u_{xxx}, \]
one has \(u_x u_{xxx} = \varphi_{xx} - u_{xx}^2\), and then
\[
\frac{\partial \varphi}{\partial t} = \frac{1}{1 - u_x^2} \varphi_{xx} - \frac{u_{xx}^2}{1 - u_x^2} + \frac{2u_x^2 u_{xx}^2}{(1 - u_x^2)^2}
\leq \frac{1}{1 - u_x^2} \varphi_{xx} + \frac{2u_x u_{xx}}{(1 - u_x^2)^2} \varphi_x.
\]
Then we have
\[
\begin{aligned}
\frac{\partial \varphi}{\partial t} &\leq \frac{1}{1-u_x^2} \varphi_{xx} + \frac{2u_x u_{xx}}{(1-u_x^2)^2} \varphi_x \\
\varphi(i,t) &= \frac{\theta_i^2}{2} \quad t \in [0,T], \ i = -d,d \\
\varphi(\cdot,0) &= \frac{|u_x(\cdot,0)|^2}{2} \quad u_0 \in C^\infty(I_d).
\end{aligned}
\]

Using the maximum principle to (2.1), we have
\[
|\varphi(x,t)| \leq \sup_{I_d}|\varphi(x,0)|, \quad \forall x \in I_d, \ t \in [0,T].
\]
So,
\[
|u_x(x,t)| \leq \sup_{I_d}|u_x(x,0)|, \quad \forall x \in I_d, \ t \in [0,T].
\]
Since \(G_0 := \{(x,u(x,0))|x \in I_d}\) is a spacelike graph of \(\mathbb{R}^2_1\), one has
\[
|u_x(x,t)| \leq \sup_{I_d}|u_x(\cdot,0)| < 1, \quad \forall x \in I_d, \ t \in [0,T].
\]
Our proof is finished.

\[\blacksquare\]

**Corollary 2.4.** Let \(u\) be a solution of (1.3), we have:

1. \(0 < c_1 \leq \frac{1}{1-u_x^2} \leq c_2\) holds for some positive constants \(c_1, c_2\) depending only on \(u_x(x,0)\);
2. \(u_{xx}\) has uniform bounds depending on \(u_x(x,0)\) and \(u_t(x,0)\).

**Proof.** This corollary can be easily proven by Lemma 2.2, Lemma 2.3, and the evolution equation \(u_t = \frac{u_{xx}}{1-u_x^2}\).

\[\blacksquare\]

### 2.2. The long-time existence.

We can get an integral estimate for \(u_{xt}\) as follows:

**Lemma 2.5.** For all \(\varepsilon \in \mathbb{R}^+\), there exists a time \(T\) such that for \(t \geq T\), the integral \(\int u_{xt}^2 dx(t)\) satisfies \(\int u_{xt}^2 dx(t) \leq \varepsilon\).

**Proof.** In fact, by differentiating \(\int u_{xt}^2 dx\), we have
\[
\begin{aligned}
\frac{d}{dt} \int u_{xt}^2 dx &= 2 \int u_t u_{xt} dx \\
&= 2 \int u_t \left( \frac{u_{xx}}{1-u_x^2} \right)_t dx \\
&= -2 \int \frac{1}{1-u_x^2} u_x^2 dx < 0.
\end{aligned}
\]

That is, there exists a positive constant \(c_3\) (independent of \(t\)) such that \(\int_0^\infty \int u_{xt}^2 dx dt \leq c_3 < \infty\).

It is easy to know that \(c_3\) depends only on \(c_1, c_2\), and moreover, depends essentially on \(u_x(x,0)\). Next, we show that the integral cannot have arbitrarily small spikes in time. By differentiating
\[ \int u_t^2 \, dx, \] we have
\[ \frac{d}{dt} \int u_t^2 \, dx = 2 \int u_{tx} u_{tx} \, dx \]
\[ = -2 \int u_{tx} u_{tx} \, dx \]
\[ = -2 \int \frac{1}{1-u_x^2} \left( \frac{u_{tx}^2 + 2u_{xx} u_{tx} u_{tx}}{1-u_x^2} \right) \, dx \]
\[ \leq -2 \int \frac{1}{1-u_x^2} \left( u_{xx}^2 + u_{xx} u_{tx}^2 \right) \, dx \]
\[ \leq 2 \int \frac{1}{1-u_x^2} u_{xx}^2 u_{tx}^2 \, dx \]
\[ \leq c_4 \int u_t^2 \, dx, \]
where \( c_4 \) is a positive constant depending only on \( u_x(x, 0) \) and \( \sup_{I_d} u_t(x, 0) \). So, from (2.2), we have \( \int u_t^2 \, dx \to 0 \) as \( t \to \infty \).

By Lemmas 2.2 and 2.5, we know that \( u_t = \frac{u_{xx}}{1-u_x^2} \) is a uniformly parabolic equation with Hölder continuous coefficients. Therefore, by the linear theory of second-order parabolic PDEs (see, e.g., [14, Chap. 4]), there exist some \( 0 < \beta < 1 \) and some constant \( C > 0 \) such that
\[ ||u||_{C^{2+\beta,1+\beta/2}(I_d \times [0,T])} \leq C (||u_0||_{C^{2+\alpha,1+\alpha}(I_d)}, \beta, I_d). \]

By the Arzelà-Ascoli theorem, we know that \( u_{T^*} := u(\cdot, T^*) \) is also the solution of (1.3). So under the hypothesis of Theorem 1.1 we conclude \( T^* = +\infty \). Besides, we can further improve the regularity of \( u \) to \( C^\infty \) (i.e., from the Hölder regularity to the smooth regularity) – see Lemma 3.4 with choosing \( v(\cdot) = \text{artanh}(\cdot) \) for the proof.

2.3. The asymptotical behavior.

Lemma 2.6.
\[ -\int |u_{xx}| \, dx + A(\theta_d, \theta_d, d) \leq u_t \leq \int |u_{xx}| \, dx + A(\theta_d, \theta_d, d). \]

Proof. The inequality (2.3) follows from the estimate
\[ \inf g - \sup g + \frac{\int g \, dx}{\int dx} \leq g \leq \sup g - \inf g + \frac{\int g \, dx}{\int dx}, \]
Replacing \( g \) by \( u_t \) in (2.4) yields
\[ -\int |u_{xx}| \, dx + \frac{\int u_t \, dx}{\int dx} \leq u_t \leq \int |u_{xx}| \, dx + \frac{\int u_t \, dx}{\int dx}. \]
Since \( u_t = \frac{u_{xx}}{1-u_x^2} = (\text{artanh}(u_x))_x \), we have
\[ \int u_t \, dx = \text{artanh}(u_x(d,t)) - \text{artanh}(u_x(-d,t)) = \text{artanh}(\theta_d) - \text{artanh}(\theta_d), \]
and then
\[ -\int |u_{xx}| \, dx + A(\theta_d, \theta_d, d) \leq u_t \leq \int |u_{xx}| \, dx + A(\theta_d, \theta_d, d). \]

\[ ^5 \text{For convenience, we will abuse the notation } C \text{ for constants, and different notations will also be used if necessary.} \]
This completes the proof. \hfill \Box

At the end, we will discuss the asymptotical behavior of the solution to IBVP (1.3) in two cases. In fact, by Lemma 2.5 and formula (2.3), we know that \( u_t \to A(\theta_{-d}, \theta_d, d) \) and \( A(\theta_{-d}, \theta_d, d) \cdot C x^2 + B x + D \leq u \leq A(\theta_{-d}, \theta_d, d) \cdot x^2 + B x + D \) as \( t \to \infty \) (i.e., the limiting curve \( u(\cdot, \infty) := \lim_{t \to \infty} u(\cdot, t) \) should be pinched by two spacelike parabolas), where \( C \) is a constant depending only on \( \sup_{I_d} |u_x|(\cdot, 0) \), and \( B, D \) are constants. Then:

**Case 1.** Assume that \( A(\theta_{-d}, \theta_d, d) = 0 \), i.e., \( \theta_{-d} = \theta_d \). Clearly, the graph of \( u(x, t) \) converges as \( t \to \infty \) to a uniquely determined portion of a spacelike straight line. Besides, if \( \theta_{-d} = \theta_d = 0 \), then it is a horizontal line; \( \theta_{-d} = \theta_d \neq 0 \), it is a straight line with a certain slope.

**Case 2.** Assume that \( A(\theta_{-d}, \theta_d, d) \neq 0 \), i.e., \( \theta_{-d} \neq \theta_d \). Clearly, the graph of \( u(x, t) \) converges as \( t \to \infty \) to a uniquely determined portion of the spacelike Grim Reaper curve.

3. **A general case: A class of quasilinear parabolic initial boundary value problems**

In this section, the proof of Theorem 1.2 will be shown in details.

3.1. **The gradient estimate.** Denote by \( v = v(u_x) \) and \( v' = v'(u_x) = \frac{d}{du_x} v(u_x) \). From \( u_t = (v)_x = v' u_{xxx} \), we have

\[
\frac{\partial}{\partial t} u_x = v' u_{xxx} + v'' u_{xx}^2,
\]

and

\[
\frac{\partial}{\partial t} v = v' v_{xx}.
\]

Hence, the NBC on \( u_x \) and the above calculations yield

\[
\begin{align*}
&v(u_x(i, t)) = v(\theta_i), & \text{for } i = -d, d \\
&v_t(u_x(i, t)) = v_{ttt}(u_x(i, t)) = \ldots = 0 & \text{for } i = -d, d.
\end{align*}
\]

As before, let \( 0 < \alpha < 1 \) and \( T^* \) be the maximal time such that there exists some

\[
u \in C^{2+\alpha,1+\frac{\alpha}{2}}(I_d \times [0, T^*)) \cap C^{\infty}(I_d \times (0, T^*))
\]

which solves (1.5). Next, we shall prove a priori estimates for those admissible solutions on \( [0, T] \), with \( T < T^* \). First, we can obtain the following gradient estimate:

**Lemma 3.1.** Let \( u \) be a solution of (1.3), we have

\[
|u_x|(x, t) \leq \sup_{I_d} |u_x|(x, 0) < 1 \quad \forall x \in I_d, \ t \in [0, T].
\]

**Proof.** Set \( \varphi = \frac{|u_x|^2}{2} \). By differentiating \( \varphi \), we have

\[
\frac{\partial \varphi}{\partial t} = u_x u_{xt}
\]

\[
= u_x (v' u_{xxx} + v'' u_{xx}^2)
\]

\[
= v'' u_{xx}^2 u_x + v' u_{xxx} u_x.
\]

Since

\[
\varphi_x = u_x u_{xx}, \ \varphi_{xx} = u_{xx}^2 + u_x u_{xxx},
\]

\[
\varphi = u_x u_{xxx} + u_{xxx} u_x,
\]

\[
\varphi_{xxx} = u_x u_{xxxx} + u_{xxxx} u_x.
\]

\[
\frac{\partial \varphi}{\partial t} = u_x u_{xxx} + u_{xxx} u_x.
\]
one has \( u_x u_{xxx} = \varphi_{xx} - u_{xx}^2 \), and then
\[
\frac{\partial \varphi}{\partial t} = v'' u_{xx} \varphi_x + v' (\varphi_{xx} - u_{xx}^2) \\
= v' \varphi_{xx} + v'' u_{xx} \varphi_x - v' u_{xx}^2 \\
\leq v' \varphi_{xx} + v'' u_{xx} \varphi_x.
\]

Therefore, we can obtain
\[
\begin{cases}
\frac{\partial \varphi}{\partial t} \leq v' \varphi_{xx} + v'' u_{xx} \varphi_x & \text{in } I_d \times [0, T], \\
\varphi(i, t) = \frac{\theta_i^2}{2} & t \in [0, T], \ i = -d, d \\
\varphi(\cdot, 0) = \frac{|u_x(\cdot, 0)|^2}{2} & u_0 \in C^\infty(I_d)
\end{cases}
\]

Using the maximum principle to the above system, one has
\[
|\varphi|(x, t) \leq \sup_{I_d} |\varphi|(x, 0), \quad \forall x \in I_d, \ t \in [0, T].
\]

So
\[
|u_x|(x, t) \leq \sup_{I_d} |u_x|(x, 0), \quad \forall x \in I_d, \ t \in [0, T].
\]

Since \( G_0 = \{(x, u(x, 0)) | x \in I_d\} \) is a spacelike graph in \( \mathbb{R}_1^2 \), it follows that
\[
|u_x|(x, t) \leq \sup_{I_d} |u_x|(\cdot, 0) < 1, \quad \forall x \in I_d, \ t \in [0, T],
\]
which completes the proof.

Applying the above gradient estimate and the IBVP (1.5), it is not hard to get the following estimates.

**Corollary 3.2.** Let \( u \) be a solution of (1.5). Then we have:
1. \( 0 < c_5 \leq v'(t) \leq c_6 \) holds for some positive \( c_5, c_6 \) only depending on \( u_x(x, 0) \);
2. \( v, v', v'', v''' \) and all higher derivatives have uniform bounds depending only on \( u_x(x, 0) \).

### 3.2. The long-time existence.

Combing the system of IBVP (1.5) and an almost same argument to that of [1, Lemma 2.2], we have:

**Lemma 3.3.** For all \( \varepsilon \in \mathbb{R}^+ \), there exists a time \( T \) such that for \( t \geq T \), the integral \( \int v_x^2 dx(t) \) satisfies \( \int v_x^2 dx(t) \leq \varepsilon \).

Furthermore, the following integral estimates for high-order derivatives can also be obtained.

**Lemma 3.4.** We have
\[
(3.3) \quad - \int |v_{xx}| dx + \tilde{A}(\theta_{-d}, \theta_d, d) \leq v_x \leq \int |v_{xx}| dx + \tilde{A}(\theta_{-d}, \theta_d, d),
\]
\[
(3.4) \quad \sup \left( \frac{\partial^k v}{\partial t^k} \right)^2 \leq 2d \int \left| \frac{\partial^k v_x}{\partial t^k} \right|^2 dx \quad \text{for} \ k \geq 1,
\]
and
\[
(3.5) \quad \sup \left( \frac{\partial^k v_x}{\partial t^k} \right)^2 \leq 2d \int \left| \frac{\partial^k v_{xx}}{\partial t^k} \right|^2 dx \quad \text{for} \ k \geq 1.
\]
Proof. Similar to Lemma 2.6, the first inequality (3.3) follows from the estimate
\[ \inf g - \sup g + \int \frac{g}{dx} \leq g \leq \sup g - \inf g + \int \frac{g}{dx}, \]
and the replacement of \( v_x \) to \( g \) in the above estimate. Of course, in this process the NBC in (1.3) has been used.

Now, take \( h = \frac{\partial u}{\partial t} \) and \( h(x_0) = \max_{I_d} h \). In order to show (3.4) and (3.5), we need to prove
\[ h^2(x_0) \leq 2d \int h_x^2 dx, \]
that is,
\[ |h(x_0)| \leq (2d)^{\frac{1}{2}} \left( \int h_x^2 dx \right)^{\frac{1}{2}}. \]
By (3.2), one has
\[ |h(x_0)| = |h(x_0) - h(-d)| \]
\[ = \int_{-d}^{x_0} h_x dx \]
\[ \leq \int_{-d}^{d} h_x dx \]
\[ = ||h_x||_{L^1}. \]
Together with the fact
\[ ||h_x||_{L^1} \leq ||1||_{L^2} ||h_x||_{L^2}, \]
it follows that
\[ |h(x_0)| \leq ||1||_{L^2} ||h_x||_{L^2}, \]
which is (3.4) exactly. Taking \( s = \frac{\partial u}{\partial t} \) and using a similar argument, the estimate (3.5) follows without any difficulty.

By Lemmas 3.3, 3.4, and Corollary 3.2, using a similar argument to [1, Lemma 2.4], we can obtain:

**Lemma 3.5.** There exist constants \( c_7, c_8 \in \mathbb{R}^+ \) such that
\[ \int v_t^2 dx(t) \leq c_7 e^{-c_8 t} \int v_t^2 dx(0), \]
with \( c_7, c_8 \) depending only on \( u_x(x, 0) \).

By Lemmas 3.3, 3.5, we know that \( u_t = v^t u_{xx} \) is a uniformly parabolic equation with Hölder continuous coefficients. So under the hypothesis of Theorem 1.2 we conclude \( T^* = +\infty \) and moreover the solution to the IBVP (1.5) is smooth.

### 3.3. The asymptotical behavior.

The asymptotical behavior will be discussed in two cases. By the fact \( \sup(v_t) \leq 2d \int |v_x|^2 dx \) (i.e., the formula (3.2) with \( k = 1 \) and Lemma 3.3) we know that \( v_t \to 0 \) exponentially as \( t \to \infty \). Therefore, by formula (3.3), \( v_x \to \bar{A}(\theta_d, \theta_d, d) \) and \( u_t = v_x \to \bar{A}(\theta_d, \theta_d, d) \) exponentially as \( t \to \infty \), \( \bar{A}(\theta_d, \theta_d, d) \cdot c_9 x^2 + Bx + D \leq u \leq \bar{A}(\theta_d, \theta_d, d) \cdot c_{10} x^2 + Bx + D \) as \( t \to \infty \) (i.e., the limiting curve \( u(\cdot, \infty) := \lim_{t \to \infty} u(\cdot, t) \) should be pinched by two spacelike parabolas), where \( c_9 := (\lim_{s \to 1^-} v'(s))^{-1} \), \( c_{10} := (\lim_{s \to (1)+} v'(s))^{-1} \), and \( B, D \) are constants. Then:

**Case 1.** Assume that \( \bar{A}(\theta_d, \theta_d, d) = 0 \), i.e., \( \theta_d = \theta_d \). Clearly, the graph of \( u(x, t) \) converges as
$t \to \infty$ to a uniquely determined portion of a spacelike straight line. Besides, if $\theta_{-d} = \theta_d = 0$, then it is a horizontal line; if $\theta_{-d} = \theta_d \neq 0$, then it is a straight line with a certain slope.

**Case 2.** Assume that $A(\theta_{-d}, \theta_d, d) \neq 0$, i.e., $\theta_{-d} \neq \theta_d$. Clearly, the graph of $u(x, t)$ converges as $t \to \infty$ to a uniquely determined portion of a spacelike Grim Reaper curve.

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**References**

[1] S. J. Altschuler, L. F. Wu, *Convergence to translating solutions for a class of quasilinear parabolic boundary problems*, Math. Ann. **295** (1993) 761–765.

[2] S. J. Altschuler, L. F. Wu, *Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle*, Calc. Var. Partial Differential Equations **2** (1994) 101–111.

[3] K. A. Brakke, *The Motion of a Surface by Its Mean Curvature*, Math. Notes, vol. 20, Princeton University Press, Princeton, 1978.

[4] L. Chen, D. D. Hu, J. Mao, N. Xiang, *Translating surfaces of the non-parametric mean curvature flow in Lorentz manifold $M^2 \times \mathbb{R}$*, Chinese Ann. Math., Ser. B **42**(2) (2021) 297–310.

[5] K. S. Chou, X. P. Zhu, *The Curve Shortening Problem*, Chapman & Hall/CRC, Boca Raton, 2001.

[6] M. Gage, R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom. **23** (1986) 69–96.

[7] Y. Giga, *Surface Evolution Equations – A Level Set Approach*, In: Monographs in Mathematics, vol. 99, Birkhäuser Verlag, 2006.

[8] M. A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom. **26** (1987) 285–314.

[9] Hoeskuldur P. Halldorsson, *Self-similar solutions to the curve shortening flow*, Trans. Amer. Math. Soc. **364**(10) (2012) 5285–5309.

[10] Hoeskuldur P. Halldorsson, *Self-similar solutions to the mean curvature flow in the Minkowski plane $\mathbb{R}^{1,1}$*, J. reine angew. Math. **704** (2015) 209–243.

[11] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984) 237–266.

[12] G. Huisken, *Non-parametric mean curvature evolution with boundary conditions*, J. Differential Equat. **77** (1989) 369–378.

[13] C. Mantegazza, *Lecture Notes on Mean Curvature Flow*, In: Progress in Mathematics, vol. 290, Birkhäuser Verlag, 2011.

[14] G. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific Publishing Co., 1996.

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