Lorentz-invariant mass and entanglement of biphoton states

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Received 17 April 2019
Accepted for publication 26 April 2019
Published 13 May 2019

Abstract

The concept of the Lorentz-invariant mass of a group of particles is shown to be applicable to biphoton states formed in the process of spontaneous parametric down conversion. The conditions are found when the Lorentz-invariant mass is related directly with (proportional to) the Schmidt parameter determining a high degree of entanglement of a biphoton state with respect to transverse wave vectors of emitted photons.

Keywords: entanglement, biphoton states, Lorentz-invariant mass

1. Introduction

As known [1, 2], in the relativistic physics the mass \( m \) of a group of particles is determined by the ‘energy-mass-momentum’ interrelation

\[
m^2 c^4 = \left( \sum_i \varepsilon_i \right)^2 - \left( \sum_i \vec{p}_i \right)^2 = \varepsilon_{tot}^2 - c^2 \vec{P}_{tot}^2
\]

where \( i \) numerates particles, \( \varepsilon_{tot} \) and \( \vec{P}_{tot} \) are the total energy and momentum of the group, and \( \vec{P}_{tot} \) is the total 4-momentum. Clearly, defined in this way, the mass \( m \) is Lorentz-invariant, as the expression in the second line of equation (1) is proportional to the squared ‘length’ of the 4-momentum of the group, which is invariant with respect to rotations in the 4-dimensional Minkowski space. Moreover, as argued by Okun [2], equation (1) provides the only reasonable definition of a mass ‘compatible with the standard language of relativity theory’.

The definition of equation (1) is valid both for particles with masses and for groups of massless particles, photons. For a single photon with the energy \( \hbar \omega \) and momentum \( \hbar \vec{k} = \hbar \omega / c \), equation (1) gives immediately \( m_{\text{single photon}} = 0 \), as it has to be. But for groups of photons the mass of a group can be different from zero if the absolute value of the vectorial sum of momenta \( \hbar \sum \vec{k}_i \) is less than \( \hbar \sum \omega_i / c \). The simplest example of this kind is a pair of noncollinear photons with equal frequencies \( \omega \) and some angle \( \theta \) between directions of their wave vectors (figure 1), for which equation (1) yields

\[
m(\theta) = \frac{2 \hbar \omega}{c^2} \sin \left( \frac{\theta}{2} \right).
\]

At \( \theta = 0 \) (the case \( b \)) propagation of photons is collinear and the mass of the group equals zero. The mass \( m(\theta) \) is maximal at \( \theta = \pi \) (the case \( c \)) and

\[
m_{\text{max}} = m(\pi) = \frac{2 \hbar \omega}{c^2} = \frac{4 \pi \hbar}{c \lambda},
\]

where \( \lambda \) is the photon wavelength. Numerically, the maximal mass of two photons is rather small. E.g. at \( \lambda = 1 \mu m \) equation (3) gives \( m_{\text{max}} \approx 4 \times 10^{-33} \text{g} \), which is about 5–6 orders of magnitude smaller than the electron mass \( \sim 10^{-27} \text{g} \). However in a system of \( N \) photon pairs with the same wave vectors \( \vec{k}_1 \) and \( \vec{k}_2 \), the mass of the group becomes \( N \) times larger and can become comparable with or even larger than the electron mass.

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Existence of a nonzero mass means immediately that the system under consideration moves as a whole with a speed \( \vec{v} \) smaller than the speed of light \( c \). In a general case of any group of particles its mean velocity of motion is given by:

\[
\vec{v} = c^2 p_{\text{tot}} / \varepsilon_{\text{tot}} = c \left( 1 - m^2 / c^2 \right)^{1/2}.
\]  

(4)

Evidently, if \( m \neq 0 \) the mean propagation speed of a group of particles is smaller than the speed of light, \( \vec{v} < c \). For two-photon states, decreasing of the mean velocity owing to noncollinearity of photon propagation was seen experimentally [3].

For any object with \( m \neq 0 \) and \( \vec{v} < c \) there is a frame where its mean velocity turns zero, i.e. the rest frame (r.f.). Of course, for groups of particles in the rest frame particles can move but only in such a way that the vectorial sums of their velocities and momenta compensate each other and give zeros in the sums, i.e.

\[
\vec{p}_{\text{tot}}^f = 0, \quad \vec{v}_{\text{tot}}^f = 0.
\]  

(5)

This clarifies the physical meaning of the Lorentz-invariant mass of a group of particles: multiplied by \( c^2 \), the mass \( m \) coincides with the total energy of a group in its rest frame

\[
m = \varepsilon_{\text{tot}} / c^2.
\]  

(6)

An example of the rest frame for a pair of photons is that shown in figure 1(c).

As classical light fields consist of photons, the definition of the Lorentz-invariant mass is applicable also to light pulses considered as relativistic objects [4, 5]. In this work we will consider manifolds of photons produced in nonlinear birefringent crystals under the action of a classical pump in the process of spontaneous parametric down-conversion (SPDC). As known, such photons can be entangled. The key question to be addressed below is whether there is any connection between the Lorentz-invariant mass of SPDC photons and the degree of their entanglement.

As for the pump, its Lorentz-invariant mass has to be found too, at least as the benchmark for comparison with that of SPDC photons. For the pump we can use the result of our previous work [5]: in equation (3) of [5] the mean propagation speed of a diverging Gaussian pulse was found to be given by

\[
\vec{v} = c \left( 1 - \frac{\lambda_p^2}{8 \pi^2 w_p^2} \right),
\]  

(7)

where \( w_p \) is the pump waist. By comparing this result with that of equation (4) at \( m \ll \varepsilon_{\text{tot}} \) we find the pump-pulse Lorentz-invariant mass

\[
m_{\text{p tot}} = \varepsilon_{\text{tot}} \frac{\lambda_p}{2 \pi c w_p^2}.
\]  

(8)

By assuming that \( \varepsilon_{\text{tot}} = N \hbar \omega_p \), where \( N \) is the number of photons in the pump pulse, we can find from equation (8) the Lorentz-invariant mass per one photon

\[
m_p = \frac{\hbar}{c w_p}.
\]  

(9)

Let us consider a rather simple case of the collinear frequency-degenerate regime of SPDC with the type-I phase matching. This means that the pump propagates in a nonlinear crystal as extraordinary and both emitted photons as ordinary waves. If the pump is vertically polarized, both emitted photons have the same horizontal polarization. Frequencies of the pump and of both emitted photons are assumed to be equal, correspondingly, to \( \omega_p \) and \( \omega_p / 2 \). Let \( \Omega \) be the central pump-propagation direction, and distribution of the pump in the transverse directions \( (\perp \Omega) \) be Gaussian with the width \( w_p \):

\[
E_p \propto \exp \left( -\vec{k}_{\perp p}^2 / 2 w_p^2 \right).
\]

The momentum-representation wave function of emitted photons depends on the transverse momenta of emitted photon \( \vec{k}_{\perp 1} \) and \( \vec{k}_{\perp 2} \) and is known [6] to be given by

\[
\Psi(\vec{k}_{\perp 1}, \vec{k}_{\perp 2}) = N \exp \left[ -\left( \vec{k}_{\perp 1} + \vec{k}_{\perp 2} \right)^2 / 2 w_p^2 \right] \times \text{sinc} \left[ \frac{L \lambda_p}{8 \pi n_o} \left( \vec{k}_{\perp 1} - \vec{k}_{\perp 2} \right)^2 \right],
\]  

(10)

where sinc = \( \sin(z) / z \), \( L \) is the length of a crystal (along the \( z \)-axis), \( n_o \) is the refractive index of the ordinary wave in the crystal, \( N \) is the normalization factor.

It should be noted that, in principle, anisotropy of the refractive index of the extraordinary pump wave, \( n_p \), can give rise to the additional term in the argument of the sinc-function, \( f_{\text{anisot}}(\vec{k}_{\perp 1}, \vec{k}_{\perp 1}) = L n_p'(k_{1z} + k_{2z}) / 4 \pi \), where \( n_p' \) is the derivative of the refractive index \( n_p \) over the angle \( \theta \) between the pump wave vector \( \vec{k}_p \) and the optical axis of a crystal \( OA \), with \( OA \) assumed to be located in the \( (x, z) \)-plane. If \( f_{\text{anisot}} \) is not small (compared to 1), it plays a very important role.
by providing anomalously high degree of angular entangle-
ment of SPDC photons [7, 8]. But in this work we assume that the
term \( f_{\text{anisec}} \) is small and can be ignored. Specifically, the
condition justifying this assumption has the form \( L | p_p | \ll w_p \),
where typically \( | p_p | \sim 0.1 \). In the opposite case of strongly
pronounced anisotropy, entanglement of SPDC photons was
investigated in the works [7, 8], and the arising in this case
peculiarities of derivation of the Lorentz-invariant mass will be
considered elsewhere separately.

The wave function (10) can be used for finding mean val-
ues of the biphonon momentum \( h(\vec{k}_1 + \vec{k}_2) \) and, finally,
the Lorentz-invariant mass \( m_{\text{biph}} \). Because of the axial sym-
metry of the expression (10), mean transversal components of
wave vectors are equal zero, \( \langle \vec{k}_{\perp 1} + \vec{k}_{\perp 2} \rangle = 0 \), whereas for
the mean sum of longitudinal momenta in the paraxial approx-
imation we get

\[
\langle \vec{k}_{z 1} + \vec{k}_{z 2} \rangle = \frac{\omega_p}{c} \frac{c}{2 \omega_p} \left[ \langle \vec{k}_{z 1} + \vec{k}_{z 2} \rangle^2 + \langle \vec{k}_{z 1} - \vec{k}_{z 2} \rangle^2 \right],
\]

(11)

where \( \vec{q}_{\pm} = \vec{k}_{\pm 1} \pm \vec{k}_{\pm 2} \).

In terms of two 2D vectors \( \vec{q}_+ \) \ and \( \vec{q}_- \), averagings
are understood as \( \langle \ldots \rangle = \frac{1}{2} \int d\vec{q}_+ d\vec{q}_- |\Psi|^2 (\ldots) \), with \( \frac{1}{2} \) being the
transition Jacobian from variables \( \vec{k}_{\pm 1}, \vec{k}_{\pm 2} \) to \( q_+, q_- \), and
with the squared wave function (10) taking the form

\[
|\Psi|^2 = N^2 \exp \left( -\vec{q}_p^2 w_p^2 \right) \frac{c}{\lambda_p} \left( \frac{L_{\lambda_p}}{8 \pi \eta_0} \vec{q}_p^2 \right).
\]

(12)

In this expression terms depending on \( \vec{q}_+ \) \ and \( \vec{q}_- \) \ are
factorized, which is very convenient for averagings.

The first step of calculations consists in finding the nor-
malization factor from the condition

\[
\frac{1}{2} \int d\vec{q}_+ \int d\vec{q}_- |\Psi|^2 = 1.
\]

(13)

Both integrals over \( \vec{q}_+ \) \ and \( \vec{q}_- \) \ in (12) are easily calculated sepa-
rately to give, correspondingly, \( \pi / w_p^2 \) and \( \lambda_p^2 \times \pi / x \int_0^\infty dx \sin^2 (x) \)
with the last integral over \( x \) equal to \( \pi / 2 \). Combined together,
these results give finally

\[
N = \frac{w_p}{\pi^2} \sqrt{\frac{L_{\lambda_p}}{\eta_0}}.
\]

(14)

The next steps are finding \( \langle \vec{q}_+^2 \rangle \) \ and \( \langle \vec{q}_-^2 \rangle \) \ in equation (11).

The first of these two quantities is determined by the integral of
the exponential function in equation (12) giving

\[
\langle \vec{q}_+^2 \rangle = \frac{1}{w_p^2},
\]

(15)

As for the second term, \( \langle \vec{q}_-^2 \rangle \), it can be reduced in a similar
way to the following integral form

\[
\langle \vec{q}_-^2 \rangle = \frac{8 \pi \eta_0}{L_{\lambda_p}} \int_0^\infty x \sin^2 (x) dx.
\]

(16)

where \( x \) is the integration variable equal to the argument of the
sinc-function in equation (12). In principle, formally, the
remaining integral in equation (16) diverges logarithmically at
\( x \rightarrow \infty \). But this divergence is related to the used above para-
axial approximation. In fact, this approximation is valid only as
long as \( |k_{\pm 1, 2}| \ll \frac{\omega_p}{c}, |q_-| \ll \frac{\omega_p}{c} \), and \( x < x_{\text{max}} = \frac{L_{\lambda_p}}{\eta_0} \left( \frac{\omega_p}{c} \right)^2 \).

For this reason, the integral in equation (16) can be estimated
as \( \int_{x_{\text{max}}}^{\infty} \sin^2 (x) dx / x \approx \frac{1}{2} \ln (x_{\text{max}}) \) to give

\[
\langle \vec{q}_-^2 \rangle = \frac{4 \pi \eta_0}{L_{\lambda_p}} \ln \left( \frac{\pi L_{\lambda_p}}{2 \eta_0 \lambda_p} \right).
\]

(17)

Altogether equations (11), (15) and (17) give the following expres-
sion for the mean momentum of biphonon pair (multi-
plied by the speed of light \( c \))

\[
\langle c (p_{z 1} + p_{z 2}) \rangle = \hbar \omega_p - c \delta p,
\]

(18)

where

\[
\delta p = \frac{h c}{2 \omega_p} \left[ \frac{1}{w_p^2} + \frac{4 \pi \eta_0}{L_{\lambda_p}} \ln \left( \frac{\pi L_{\lambda_p}}{2 \eta_0 \lambda_p} \right) \right].
\]

(19)

As always \( \lambda_p \ll \{ w_p, L \} \), in any case \( c \delta p \ll \hbar \omega_p \) and, as
\( \epsilon_{\text{biph}} \equiv (\hbar \omega_p)^2 \), the difference \( \epsilon_{\text{biph}} - \langle c (p_{z 1} + p_{z 2}) \rangle \) \( \hbar \omega_p \)
equals approximately \( 2 \hbar \omega_p c \delta p \). In this approximation the biphonon
Lorentz-invariant mass per one pair of SPDC photons takes the
form

\[
m_{\text{biph}} = \frac{\hbar}{c} \left[ \frac{1}{w_p^2} + \frac{4 \pi \eta_0}{L_{\lambda_p}} \ln \left( \frac{\pi L_{\lambda_p}}{2 \eta_0 \lambda_p} \right) \right]^{1/2}.
\]

(20)

3. Entanglement

Transverse momenta of SPDC photons \( \vec{k}_{\perp 1} \) \ and \( \vec{k}_{\perp 2} \) \ are 2D
continuous variables. As known [9], the degree of entangle-
ment in such variables can be evaluated by the Schmidt
entanglement parameter \( K \) defined as the inverse trace of the
squared reduced density matrix. Calculation of this parameter is
rather simple and straightforward for the so called double-
Gaussian wave functions, having the form of a product of two
Gaussian functions, one of which depends on the sum of
variables and the other one—on their difference. For non-
double-Gaussian wave functions functions calculation of the
parameter \( K \) is a problem, and even more difficult problem is its
direct experimental measurement.

Another and much easier measurable entanglement param-
eter \( R \) was suggested for the first time in the work [10]. This
parameter is defined mathematically as the ratio of widths of
the unconditional to conditional probability densities depend-
ing on the variable of one of two particles (e.g. photons).

Defined in this way the parameter \( R \) was found to coincide
exactly with the Schmidt parameter \( K \) [11], and in other cases

\[\rightarrow K \] 12. In experiment one has to split the
The original beam of biphonons for two channels and perform two
kinds of measurements. At first, photons can be counted by a
single scanning detector in only one of two channels to plot
the single-particle distribution and to find its single-particle
width. In the second series of measurements one has to use two detectors located in different channels, one of them scanning and another one kept at a constant position, and only coinciding signals have to be registered at the computer. In this way one gets the coincidence (or conditional) distribution and measures its width. The ratio of found widths is the parameter $R$.

It is known also that if the wave function has the form of a product of two terms, one of which depends on the sum and the other one—on the difference of variables and if the widths of these distributions are $a$ and $b$, then the coincidence and single-particle widths are, correspondingly, $\min\{a, b\}$ and $\max\{a, \}$ and, consequently, $R = \max\{a, b\}/\min\{a, b\}$.

Applied to the wave function of the form (10) and variables $k_{1,w}, k_{2,v}$, these definitions give $a = 1/w_p$, $b = 2\pi\sqrt{n_o/L\lambda_p}$ and $R \sim \frac{\Delta k^{(i)}_{1,w}}{\Delta k^{(e)}_{1,v}} = \max\left\{\frac{1}{w_p}, 2\pi\sqrt{n_o/L\lambda_p}\right\}/\min\left\{\frac{1}{w_p}, 2\pi\sqrt{n_o/L\lambda_p}\right\}$. (21)

This derivation and the derived expression confirm the known result [7, 8] that the degree of entanglement is high either if $w_p \ll \sqrt{L\lambda_p}$ or if $w_p \gg \sqrt{L\lambda_p}$. In the second of these two limiting cases the crystal is assumed to be short compared to the diffraction length of the pump, $L \ll w_p^2/\lambda_p \equiv L_d$. Under this condition

$$K \sim R_{\text{short}} \sim \frac{2\pi w_p/\sqrt{L\lambda_p}}{\sqrt{L\lambda_p}} \gg 1. \quad (22)$$

With this expression for the entanglement parameters $K$ and $R$, the derived above formula (20) for the Lorentz-invariant mass of biphoton pairs can be rewritten as

$$m_{\text{biph}} = \frac{\hbar}{c w_p} \left[ 1 + \frac{K^2}{\pi} \ln \left( \frac{\pi L}{2n_o \lambda_p} \right) \right]^{1/2},$$

$$\approx \frac{\hbar K}{2c w_p} \left[ \frac{1}{\pi} \ln \left( \frac{\pi L}{2n_o \lambda_p} \right) \right] \gg \frac{\hbar}{2c w_p}, \quad (23)$$

This result shows that in the case $L \ll w_p^2/\lambda_p \equiv L_d$ both the Lorentz-invariant mass of the biphoton state (10) and its degree of entanglement are high and they are related to each other: the Lorentz-invariant mass is proportional to the Schmidt entanglement parameter $K$.

It is true, however, that this connection is not universal. In the case of very strong focusing of the pump $L \gg L_d = w_p^2/\lambda_p$, opposite to that considered above, the degree of entanglement is high too, $K \sim \sqrt{L/L_d} \gg 1$, but the Lorentz-invariant mass $m_{\text{biph}} \propto K \gg 1$ (23).

4. Conclusion

The main result of the presented analysis concerns demonstration that there are conditions when biphotons states are highly entangled and this high entanglement is directly related to the relatively high Lorentz-invariant mass, $m_{\text{biph}} \propto K \gg 1$ (23). We believe that this is a fundamentally important new knowledge. Though we realize that it may be difficult to imagine any ways of measuring the Lorentz-invariant mass of biphotons directly and independently of measuring coincidence and single-particle widths of momentum-distributions, finding the entanglement parameter $R \sim K$ and then using if for finding $m_{\text{biph}}$ from equation (23).

Acknowledgment

The work of S V Vintskevich was supported by the Foundation for the Advancement of Theoretical Physics BASIS (PhD Student Grant No. 17-15-603-1). D A Grigoriev and M V Fedorov acknowledge support of the Russian Foundation for Basic Researches, grant RFBR 18-02-00634.

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