The geometry of finite topology
Bryant surfaces

By Pascal Collin, Laurent Hauswirth, and Harold Rosenberg

1. Introduction

In this paper we shall establish that properly embedded constant mean curvature one surfaces in $\mathbb{H}^3$ of finite topology are of finite total curvature and each end is regular. In particular, this implies the horosphere is the only simply connected such example, and the catenoid cousins the only annular examples of this nature. In general each annular end of such a surface is asymptotic to an end of a horosphere or an end of a catenoid cousin.

Robert Bryant discovered a holomorphic parametrization of (simply connected) mean curvature one surfaces in $\mathbb{H}^3$ which can be thought of as a generalization of the Weierstrass representation of minimal surfaces in $\mathbb{R}^3$ [2]. Each (simply connected) minimal surface in $\mathbb{R}^3$ is isometric to a mean curvature one surface in $\mathbb{H}^3$ (and vice versa); R. Bryant calls this the cousin of the minimal surface. This correspondence follows easily from Bonnet’s existence theorem for surfaces in the space forms. This may have been R. Bryant’s motivation to seek a meromorphic Weierstrass type representation of mean curvature one surfaces in $\mathbb{H}^3$.

Definition. A Bryant surface is a surface in $\mathbb{H}^3$ of constant mean curvature one.

The Weierstrass pair of a minimal (local) surface in $\mathbb{R}^3$ is a pair of meromorphic data $(g, \omega)$. The cousin in $\mathbb{H}^3$ has more local structure; in particular, one also has the hyperbolic Gauss map $G$. The surfaces are isometric so the metric is determined by $(g, \omega)$: $ds = |\omega| (1 + |g|^2)$. However, the Gauss map $G$ is fundamental to the geometry of the cousin in $\mathbb{H}^3$ ($G$ is also meromorphic on the minimal cousin in $\mathbb{R}^3$, but this seems never to have been considered).

An annular end of a finite total curvature minimal surface in $\mathbb{R}^3$ is conformally a punctured disk and the Gauss map $g$ extends meromorphically to the puncture. However, the Gauss map of an annular cousin in $\mathbb{H}^3$ may have an essential singularity at the puncture; R. Bryant observed this for Enneper’s minimal surface in $\mathbb{R}^3$ and its cousin in $\mathbb{H}^3$ [2]. An annular end in $\mathbb{H}^3$ is called regular if it is conformally a punctured disk and $G$ extends meromorphically to
the puncture. This notion was introduced and developed for Bryant surfaces by M. Umehara and K. Yamada [21]. The idea of regular ends originated in the paper of R. Schoen [19], where he introduced and studied regular minimal annular end hypersurfaces in \( \mathbb{R}^n \).

A properly embedded minimal annular end in \( \mathbb{R}^3 \) of finite total curvature is asymptotic to an end of a plane or catenoid. In \( \mathbb{H}^3 \), a properly embedded Bryant annular end, regular and of finite total curvature, is asymptotic to an end of a horosphere or a catenoid cousin [15].

We will prove that a properly embedded Bryant annular end in \( \mathbb{H}^3 \) is of finite total curvature and regular. This is the main result of our work and answers affirmatively a conjecture by M. Umehara and K. Yamada: there are no embedded irregular Bryant annular ends of finite total curvature [20].

When the annular end is part of a properly embedded Bryant surface, we prove it is asymptotic to a catenoid cousin end and not a horosphere end (unless \( M \) is equal to a horosphere). This is Theorem 12. This is quite different from properly embedded minimal surfaces in \( \mathbb{R}^3 \), where an annular end can be asymptotic to a catenoid or planar end, as in Costa’s surface.

The analogous theorem for minimal annular ends in \( \mathbb{R}^3 \) is not true: the helicoid has an annular end of infinite total curvature. The cousin of the helicoid in \( \mathbb{H}^3 \) is not embedded. In fact, our initial motivation was the search for a properly embedded simply connected Bryant surface in \( \mathbb{H}^3 \), other than a horosphere (the cousin of a plane in \( \mathbb{R}^3 \)). Now we know there is no such simply connected surface.

It is still unknown if the helicoid and plane are the only properly embedded minimal surfaces in \( \mathbb{R}^3 \) that are simply connected.

However, the geometry of properly embedded minimal \( M \) in \( \mathbb{R}^3 \), of finite topology and with at least two ends, is understood: \( M \) has finite total curvature; each annular end of \( M \) is asymptotic to a plane or catenoid end [3], [13].

There has been much important work done on the geometry of properly embedded annular \( H \)-ends in \( \mathbb{R}^3 \) and in \( \mathbb{H}^3 \) [9], [10], [12]. In \( \mathbb{R}^3 \), for \( H \neq 0 \), they prove such an end is asymptotic to a Delauney end [10]. Also it is proved that if \( H > 1 \) for such an end in \( \mathbb{H}^3 \) then it is also asymptotic to a Delauney end [9]. In fact, the linking number argument of our Theorem 9 is inspired by the linking number argument of [10]. However this argument needs to be adapted to our situation. Essentially, because of the noncompactness of horospheres, we cannot use them directly as barriers. So, we will construct stable surfaces with sufficiently known behavior at infinity and use them as comparison surfaces with horospheres.

There are examples of higher genus, mean curvature one surfaces in \( \mathbb{H}^3 \) of finite topology. Many such examples have been constructed by W. Rossman, M. Umehara and K. Yamada and computer images indicate many of
these surfaces may be embedded [18]. In $\mathbb{R}^3$, N. Kapouleas has constructed many properly immersed and embedded $H$-surfaces by desingularizing certain families of touching spheres. We hope that this may be done in $\mathbb{H}^3$, by desingularizing certain families of touching horospheres. For example, consider the three horospheres intersecting in three points as in Figure 1-a. One should be able to attach catenoid cousin necks near the three singular points and show there is a Bryant surface in a neighborhood of this new surface by Schauder fixed point techniques. This surface would have genus one and three ends, each asymptotic to a catenoid cousin end; see Figure 1-b.

The paper is organized as follows. In Section 2, we give a (brief) description of the Bryant representation; the interested reader may consult [2], [15] and [21] for a serious discussion.

In Section 3, we analyze the connected component of the intersection of a Bryant surface in $\mathbb{H}^3$ with its tangent horosphere at a point. There is more structure here than the trace of a minimal surface in $\mathbb{R}^3$ on its tangent plane at a point. We describe here this trace for properly embedded annular ends $E$ with $E \cap H(q)$ compact, $q \in E$, and $\partial E \cap H(q) = \emptyset$; $H(q)$ is the tangent horosphere at $q$.

In Section 4 we study properly embedded Bryant annular ends $E$, which are not dense at infinity. We first prove this end is regular: it is conformally the punctured disk and $G$ extends meromorphically to the puncture (Theorem 1). We then prove the asymptotic boundary of the end $E$ is precisely the limiting value of $G$ at the puncture (Theorem 2).

In Section 5 we continue the study of properly embedded annular ends $E$ assuming $E$ is regular. We prove $E$ then has finite total curvature. This is done by first proving such an end $E$ has finite total curvature if it is on the mean convex side of a catenoid cousin (Theorem 3). Then we prove this end can be placed on the mean convex side of a catenoid cousin (Theorem 4). This last result requires an analytic theorem concerning $H = 1$ graphs over noncompact domains (Theorem 5).
In Section 6 we prove the nondensity at infinity of finite topology properly
embedded Bryant surfaces. Using the trace on horospheres, we show that if
$M$ is dense at infinity there is a proper arc $\gamma$ on $M$ with $\partial_\infty \gamma$ two distinct
points at infinity. Using $M$ as a barrier we construct stable $H = 1$ surfaces
with boundary $\gamma$. Analyzing the asymptotic behavior of stable surfaces we see
that such stable surfaces cannot exist.

2. The Bryant representation

Let $L^4$ be Minkowski 4-space with the Lorentzian metric of signature
$(-,+,+,+)$. Hyperbolic 3-space can be represented as

$$\mathbb{H}^3 = \left\{ (t, x_1, x_2, x_3) \in L^4; \quad \sum_{i=1}^{3} x_i^2 - t^2 = -1, \; t > 0 \right\}$$

with the metric induced from $L^4$.

It is useful to identify $L^4$ with the space of $2 \times 2$ hermitian matrices: a
point $(t, x_1, x_2, x_3)$ corresponds to

$$\begin{pmatrix} t + x_3 & x_1 + ix_2 \\
(x_1 - ix_2) & t - x_3 \end{pmatrix}.$$  

Notice that $\mathbb{H}^3$ is the set of such matrices of determinant one, $t > 0$, and one
has $\mathbb{H}^3 = \{ aa^*; a \in \text{SL}(2, \mathbb{C}) \}$, where $a^* = ^t \bar{a}$.

Let $M$ be a simply connected Riemann surface and $F : M \to \text{SL}(2, \mathbb{C})$ a
holomorphic immersion satisfying:

$$dAdD - dBdC = 0,$$  

where $F = \begin{pmatrix} A & B \\
C & D \end{pmatrix}$.

Then $f = FF^* : M \to \mathbb{H}^3$ is a conformal immersion of mean curvature-
one. If $H \in \text{SU}(2)$ then $f = F_1F_1^*$ where $F_1 = FH$.

Conversely any mean curvature one surface in $\mathbb{H}^3$ is given locally by such
an $F$. The reader should consult [2] and [21] for the details.

In the upper half-space model of $\mathbb{H}^3$, one can express the immersion in
terms of $F$.

$$(x_1 + ix_2)(z) = \frac{A\bar{C} + B\bar{D}}{|C|^2 + |D|^2}(z),$$  

$$x_3(z) = \frac{1}{|C|^2 + |D|^2}(z).$$

The Weierstrass data of the minimal cousin in $\mathbb{R}^3$ are given by:

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\
1 & -g \end{pmatrix} \omega.$$
Then one obtains

\[ g = -\frac{B'}{A'}, \quad \omega = AC' - A'C, \quad \text{and} \quad G = \frac{A'}{C'}. \]

The metric induced on \( M \) is \( ds = |\omega| \left( 1 + |g|^2 \right) \).

3. The tangent horosphere

For \( M \) an immersed surface in \( \mathbb{H}^3 \) and \( q \in M \), the tangent horosphere \( H(q) \) of \( M \) at \( q \) is the horosphere tangent to \( M \) at \( q \) whose mean curvature vector at \( q \) has the same direction as that of \( M \) at \( q \). This horosphere is unique when the mean curvature of \( M \) at \( q \) is nonzero.

Note that \( H(q) \) separates \( \mathbb{H}^3 \) into two components. We let \( H(q)^+ \) denote the mean convex component bounded by \( H(q) \) and we call it the inside of \( H(q) \). The surfaces at a constant distance \( t \) from \( H(q) \) are also horospheres with the same point at infinity as \( H(q) \) and they foliate \( \mathbb{H}^3 \). We denote this equidistant horosphere by \( H_t(q) \), and for \( t > 0 \), \( H_t(q) \) will be inside \( H(q) \) and outside \( H(q) \) for \( t < 0 \).

Now suppose \( M \) is a Bryant surface properly embedded in \( \mathbb{H}^3 \). We allow \( M \) to have a compact boundary since many of our results concern the ends of such surfaces. We also assume \( M \) is not a part of a horosphere.

For \( q \in \text{int}(M) \), the intersection of \( M \) and \( H(q) \) is an analytic curve near \( q \) with isolated singularities, and at the singularity \( q \), there are \( 2k + 2 \) smooth branches meeting at equal angles where \( k \) is an integer at least one. In fact, \( k \) is the same as the order of contact of the cousin minimal surface in \( \mathbb{R}^3 \) with its tangent plane, and \( k - 1 \) is the order of \( q \) as a branch point of the Gauss map \( G \).

When \( \partial M = \emptyset \), \( M \) separates \( \mathbb{H}^3 \) into two connected components since \( M \) is properly embedded. We let \( W \) denote the mean convex component bounded by \( M \). When \( \partial M \neq \emptyset \) and is compact, we introduce a mean convex component \( W \) as follows. It is not hard to see that there is an embedded compact orientable surface \( \Sigma \) such that \( \partial \Sigma = \partial M \) and \( \Sigma \cap \text{int}(M) = \emptyset \) (take a large ball \( B \) of \( \mathbb{H}^3 \), containing \( \partial M \), such that \( M \) is transverse to \( \partial B \). Let \( M_0 = M \cap B \) so that \( \partial M_0 = \partial M \cup \Gamma \), where \( \Gamma \) is a one-dimensional submanifold of \( \partial B \). Now \( \Gamma \) bounds a compact domain \( D \subset \partial B \) such that \( \overline{\Gamma} \), the mean curvature vector of \( M \), points towards \( D \) along \( \Gamma \). Then \( \Sigma \) can be obtained by smoothing \( M_0 \cup D \) along \( \Gamma \) and displacing this slightly, in the direction of \( \overline{\Gamma} \), keeping \( \partial M \) fixed). Now \( \Sigma \cup M \) separates \( \mathbb{H}^3 \) into two components and we call \( W \) the component into which \( \overline{\Gamma} \) points along \( M \). We will use \( W \) far from \( \partial M \) so that the choice of \( \Sigma \) is not important.
We will now derive properties of $E \cap H(q)$ where $E$ is a properly embedded Bryant annular end (homeomorphic to $S^1 \times [1, +\infty]$). We henceforth assume $\partial E \cap H(q) = \emptyset$ and $E$ is topologically the unit disk punctured at the origin.

**Lemma 1.** Let $E_1$ be a connected component of $E - H(q)$ that is outside $H(q)$. Then $E_1$ is not compact or $\partial E_1 \subset E_1$.

**Proof.** If this were not so, then $E_1$ would be compact and $\partial E_1 \subset H(q)$. Consider a “large” horosphere $H_t(q)$ outside $H(q)$ such that $E_1 \subset H_t(q)^+$ (so $t$ is near $-\infty$ in our notation). Then increase $t$: since $E_1$ is compact there will be a largest $t_0$ such that $H_{t_0}(q)$ touches $E_1$ at a point $x \in \text{int}(E_1)$. But then $E = H_{t_0}(q)$ by the maximum principle, a contradiction; cf. Figure 2-a. □

**Lemma 2.** Let $E_1$ be a compact connected component of $E - H(q)$ with $\partial E_1 \subset H(q)$ and $E_1$ inside $H(q)^+$. Let $D_1 \subset H(q)$ be a compact domain, $\partial D_1 = \partial E_1$, and $D_1 \cup E_1 = \partial Q_1$, $Q_1$ a compact domain in $H(q)^+$. Then $Q_1$ is mean convex along $E_1$.

**Proof.** Consider a “small” horosphere $H_t(q)$ contained in $H(q)^+ - Q_1$ (so $t$ is near $+\infty$). When $t$ decreases, there will be a positive $t_0$ where $H_{t_0}(q)$ touches $Q_1$ for the first time. The point $x$ where they touch is in $E_1$ and by the maximum principle, the mean curvature vector of $E_1$ at $x$ is the negative of that of $H_{t_0}(q)$ at $x$. So this mean curvature vector points into $Q_1$ and $Q_1$ is mean convex; see Figure 2-b. □

**Lemma 3.** There is at most one compact component at $q$ of $E - H(q)$, whose boundary is in $H(q)$; by “at $q$” is meant a connected component of $E - H(q)$ containing $q$ in its closure.
Proof. Suppose this fails. Then at least two components $E_1, E_2$ at $q$ are compact and $(\partial E_1 \cup \partial E_2) \subset H(q)$. By Lemma 1, we know that $E_1 \cup E_2 \subset H(q)^+$. For $i = 1, 2$, let $D_i \subset H(q)$ be compact domains, $\partial D_i = \partial E_i$ and $E_1 \cup D_i = \partial Q_i$, where $Q_i$ is a compact domain in $H(q)^+$.

Since $E$ is a graph over $H(q)$ near $q$, the mean curvature vectors (which we denote by $\vec{H}$) of $E_1$ and $E_2$ point into the same connected component $C$ of $H(q)^+ - (E_1 \cup E_2)$ near $q$. So $\vec{H}(E_1)$ and $\vec{H}(E_2)$ point into $C$ along $E_1 \cup E_2$.

If $C$ is compact, then $C$ is contained in $Q_1$ or $Q_2$, say $Q_1$. So $\vec{H}(E_2)$ points into $C \cap Q_1$. Now $E_2 \subset Q_1$ hence $Q_2 \subset Q_1$ so along $E_2$, $\vec{H}(E_2)$ points to the noncompact component of $H(q)^+ - E_2$; this contradicts Lemma 2; see Figure 3.

If $C$ is noncompact, then $\vec{H}(q)$ points into $C$, so along $E_1$, $\vec{H}$ points into $C$ as well. But $\vec{H}$ points into $Q_1$ along $E_1$ by Lemma 2. This proves Lemma 3.

\[ \text{Figure 3} \]

Lemma 4. The number of connected components of $E - H(q)$ at $q$ is at least three. If equality holds then the order of contact $k$ of $E$ with $H(q)$ is one.

Proof. Let $2k + 2$ denote the number of branches of $E \cap H(q)$ at $q$. Let $f : D \to E$ be a local parametrization of a neighborhood of $q$ on $E$ by a disk of $\mathbb{C}$ so that $f(0) = q$ and lines passing through $0$ of slope an integral multiple of $\frac{\pi}{k+1}$ are sent to $E \cap H(q)$. Let $A_i$ be the sector of $D$ defined by $\{\frac{(i-1)\pi}{k+1} < \arg(z) < \frac{i\pi}{k+1}\}$ and $B_i = f(A_i)$. We know that the $B_i$ are alternatively in $H(q)^+$ and $H(q)^-$ around $q$.

If $B_1$ and $B_3$ are not in the same component of $E - H(q)$, then by the observation above, $B_2$ yields a third component so that Lemma 4 is true.

If $B_1$ and $B_3$ are in the same component of $E - H(q)$, we can construct a cycle $\alpha_{13}$ on $E$ as follows: let $a_1, a_3$ be two points of $A_1, A_3$ respectively, and $\beta$ a path in $E - H(q)$ from $f(a_1)$ to $f(a_3)$; $\alpha_{13} = \beta \cup \beta'$ where $\beta'$ is the image by $f$ of the line segment from $a_1$ to 0, followed by the line segment from 0 to $a_3$. Now $\alpha_{13}$ meets $H(q)$ exactly at $q$.
If $B_2$ and $B_4$ were in the same component, we could find a cycle $\alpha_{24}$ on $E$ which meets $\alpha_{13}$ in a single point, which is impossible since the genus of $E$ is zero. Thus we get at least three components in this case as well.

Now we study the case of equality. In this case there is only one component at $q$ in either $H(q)^+$ or $H(q)^-$. Then we can assume that all the $B_i$ for $i$ odd are in the same global component of $E - H(q)$. If $k \geq 2$ this means that we can construct cycles $\alpha_{15}$ (with $A_1$ and $A_5$), and $\alpha_{35}$ (with $A_3$ and $A_5$) exactly as we constructed $\alpha_{13}$. As before, these three cycles separate the components of $B_2$, $B_4$ and $B_6$ in $E - H(q)$. Hence we obtain at least four components at $q$ in this case. This completes the proof of Lemma 4.

We define $\Sigma(q)$ to be the connected component of $q$ in $E \cap H(q)$ and we assume $\Sigma(q)$ is compact in the rest of this section.

**Lemma 5.** $E - H(q)$ has exactly three components at $q$: $B$, $E_1$ and $E_2$. The first, $B$, is compact and contains $\partial E$; $E_1$ is compact with boundary in $H(q)$; $E_2$ is noncompact (recall the assumption that $\partial E \cap H(q) = \emptyset$ and $\Sigma(q)$ is compact).

**Proof.** First we prove $E - H(q)$ has at most one noncompact component at $q$. This is immediate if $E \cap H(q)$ is compact (in fact $E \cap H(q)$ will always be compact until we arrive at Theorem 7 of this paper. There we will need to work with the weaker hypothesis: $\Sigma(q)$ is compact). Let $F$ be the noncompact component of $E - \Sigma(q)$. For $\varepsilon > 0$, $\varepsilon$ small, the points of $F$ a distance $\varepsilon$ from $\Sigma(q)$ form a compact curve $C$ disjoint from $H(q)$. Since $E$ is an annulus, $C$ is in fact connected. This Jordan curve $C$ separates $\Sigma(q)$ from the puncture.

With the notation of Lemma 4, if two $B_i$ and $B_j$ ($i \neq j$) are in noncompact components, then a curve in $E - H(q)$ from $q$ to the puncture starting in $B_i$ (and $B_j$) meets $C$. So $B_i$ and $B_j$ are in the same global component.

Then by Lemma 4, there are at least two components at $q$ of $E - H(q)$ that are compact. At most one may contain $\partial E$, and the others are compact with boundary in $H(q)$ (and there is at least one). But such a component is in $H(q)^+$ by Lemma 1 and is unique by Lemma 3. We call $E_1$ this unique component. Then the other compact component necessarily contains $\partial E$ -- we call it $B_-$, and the third component ($E_2$, say) is noncompact.

Finally $E - H(q)$ has exactly three components whose closure contains $q$ and by Lemma 4, the order of contact $k$ of $E$ with $H(q)$ is one. \hfill $\square$

**Lemma 6.** There are two possibilities:

- $\partial E_1 = C_1$ is a Jordan curve on $H(q)$,

or

- $\partial E_1 = C_1 \cup C_2$ is a figure eight; the union of two Jordan curves $C_1$, $C_2$ on $H(q)$ meeting at $q$. 

Lemma 1. This proves each cycle in $\partial E$ is an embedded curve.

Proof. First notice that $\partial E_1$ contains no cycle $c$ disjoint from $q$. To see this, let $\gamma$ be a path in $E_2 \cup B \cup \{q\}$ going from $\partial E$ to the puncture and meeting $H(q)$ exactly at $q$. The cycle $c$ does not meet $\gamma$ so $c$ bounds a compact domain $D$ in $E$, $D \cap \partial E = \emptyset$ and $D \neq E_1$ (since $q \notin c = \partial D$). But $D$ would contain a compact domain outside $H(q)$ with boundary in $H(q)$ contradicting Lemma 1. This proves each cycle in $\partial E_1$ meets $q$, $E_1$ is a disk and $\partial E_1 - \{q\}$ is an embedded curve.

We know that we have locally at $q$ exactly four components $B_1$, $B_2$, $B_3$, $B_4$; $B_1$ and $B_3$ in $H(q)^+$. Assume $B_1$ is in $E_1$ and $B_3$ is not in $E_1$. Then $q$ is not a double point of $\partial E_1$ and $\partial E_1 = C_1$ is a Jordan curve on $H(q)$. On the contrary if $B_3$ is also in $E_1$, then $\partial E_1 = C_1 \cup C_2$; $C_1$ and $C_2$ Jordan curves meeting exactly at $q$; i.e., $\partial E_1$ is a figure eight.

\[ \square \]

Let $D \subset H(q)$, $Q_1 \subset H(q)^+$ be such that $\partial Q_1 = E_1 \cup D$, where $Q_1$ is compact.

Proposition 1. With the notation of Lemma 5:

If $\partial E_1 = C_1$, then $\partial E \subset Q_1$ and every divergent path starting at $x \in \partial E$ must intersect $E_1 \cup H(q)$ at a point other than $x$.

If $\partial E_1 = C_1 \cup C_2$, then every path starting at $\partial E$, staying in $W$, and diverging in $\mathbb{H}^3$ must intersect $H(q)$; $Q_1$ separates $W$. Moreover $\partial E$ is outside $H(q)$, and $C_1$ and $C_2$ are each homologous to $\partial E$ on $E$.

Proof. The two possibilities are given by Lemma 6. If $\partial E_1 = C_1$, then $D$ is the disk of $H(q)$ bounded by $C_1$ and $D \cup E_1 = \partial Q_1$. At $q$, $\overline{H}(q)$ must point into $Q_1$ (since it does so at points of $E_1$ near $q$) by Lemma 2 so that $E_2$ or $B$ is inside $Q_1$ near $q$. It cannot be $E_2$ since $E_2$ is noncompact (the puncture is in $E_2$) and $E_2$ is properly embedded. Thus $B$ is inside $Q_1$ near $q$. Thus, $B \subset Q_1$ and, in particular, $\partial E \subset Q_1$. Now any path starting at a point of $\partial E$ and diverging in $\mathbb{H}^3$, must intersect $\partial Q_1 = E_1 \cup D$; see Figure 4. This proves the first assertion of the proposition.

Now suppose $\partial E_1 = C_1 \cup C_2$. Again $\overline{H}(q)$ points into $Q_1$ and $Q_1$ is mean convex along $E_1$. By Lemma 1, it is clear that $C_1$ and $C_2$ are not homologous to zero in $E$, hence each $C_1$, $C_2$ is homologous to $\partial E$ in $E$. Let $D_1$ be the disk of $H(q)$ bounded by $C_1$, $D_2$ bounded by $C_2$. We have $\text{int}(D_1) \cap \text{int}(D_2) = \emptyset$ or one disk is contained in the other. This latter case is impossible. For if $D_2 \subset D_1$, we have $\partial Q_1 \subset D_1$. Locally at $q$, $E_1$ is a graph over two opposite sectors of $H(q)$ (the projection of the $B_1$ and $B_3$ of Lemma 6). By the local structure of $E_1$ near $q$, the two complementary sectors are in $\partial Q_1$, hence in $D_1$. As $D_1$ is a disk on $H(q)$, at least one projection of $B_1$ or $B_3$, $B_3$ say, must be in $D_1$. Now $E_1$ is a disk with two points on the boundary identified at $q$. 


Then $\partial E_1 - \{q\} = C_1 \cup C_2$ hence one of the two boundary arcs of $B_3 - \{q\}$ (near $q$) is in $C_1$ and the other in $C_2$. But then, points of $C_1$ are in the interior of $D_1$, which is a contradiction.

Then $\partial Q_1 = E_1 \cup D_1 \cup D_2$ so $Q_1$ separates $\partial E$ from infinity in $W$: any path starting at $\partial E$, in $W$ and diverging in $\mathbb{H}^3$, must pass through $Q_1$. Moreover $B$ and $E_2$ are outside $H(q)$, in particular $\partial E$ is outside $H(q)$; see Figure 5. 

**Remark 1.** Assume $\Sigma(q)$ is compact and $E$ is transverse to $\Sigma(q) - \{q\}$. Then $\Sigma(q)$ is a figure eight, the union of two Jordan curves $C_1$, $C_2$ meeting at $q$.
By the transversality hypothesis and the local structure at \( q \), \( \Sigma(q) \) consists of two analytic curves meeting at equal angles at \( q \). It is then a figure eight. Note that in the first case of Proposition 1, \( C_2 \) together with \( \partial E \) bounds \( B \) (other curves in \( \partial B \) would give rise to a compact component outside \( H(q) \)).

**Corollary 1.** Let \( E \) be a properly embedded Bryant annular end, \( q \in E \) with \( E \cap H(q) \) compact and disjoint from \( \partial E \). Then if \( \partial E \subset H(q)^+ \) every divergent path starting at \( x \in \partial E \) must intersect \( E \cup H(q) \) at a point other than \( x \); if \( \partial E \subset H(q)^- \) then every divergent path starting at \( x \in \partial E \) and staying in \( W \), must intersect \( H(q) \).

**4. The regularity and asymptotic boundary of an annular end not dense at infinity**

**Theorem 1.** Let \( E \) be a properly embedded Bryant annular end. If \( \partial_\infty E \neq S_\infty \), then \( E \) is conformally a punctured disk and the hyperbolic Gauss map \( G \) extends meromorphically to the puncture (i.e., \( E \) is regular).

*Proof. *We will now work in the upper half-space model of \( \mathbb{H}^3 \) with \( S_\infty = \{ x_3 = 0 \} \cup \{ \infty \} \). Since the asymptotic boundary of \( E \) is closed and not \( S_\infty \), we can assume \( E \subset B_R = \{ x_1^2 + x_2^2 + x_3^2 < R^2, x_3 > 0 \} \).

First we will show that \( G \) is bounded on some subend of \( E \). If not, then for some \( q_n \in E \), diverging on \( E \), we would have \( |G(q_n)| \to \infty \). Since \( \partial E \) is compact, we have \( x_3|_{\partial E} \geq \delta > 0 \) for some \( \delta \). Choose \( n \) sufficiently large so that \( H(q_n) \cap B_R \) is below \( x_3 = \delta \). This is possible since \( x_3(q_n) \to 0 \); cf. Figure 6. However, \( \partial E \subset H(q_n)^+ \) and we can find a path from \( \partial E \) to \( G(q_n) \) which does not intersect \( E \cup H(q_n) \) except at its endpoint (choose a path from a point of \( \partial E \) to a point of \( \partial B_R \), not meeting \( E \); then choose \( n \) big enough so that \( H(q_n) \) is below this path, and then continue to \( G(q_n) \)).

This contradicts Corollary 1, and so \( G \) is bounded.

![Figure 6](image_url)
To prove Theorem 1, it suffices to prove that $E$ is conformally the punctured disk.

We will prove this by constructing a complete metric on $E$ of the form $d\sigma = \lambda|dz|$ where $\lambda$ is the module of a holomorphic function on $E$ (R. Osserman [14]).

Let $\tilde{E}$ be the universal cover of $E$, so that $F : \tilde{E} \to \text{SL}(2, \mathbb{C})$ is holomorphic, $F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega$, where $(g, \omega)$ are the Weierstrass data and $\Psi : \tilde{E} \to \mathbb{H}^3$, $\Psi = F^*\mathcal{T}$, defines the immersion of $E$ in $\mathbb{H}^3$. The metric $ds = |\omega|(1 + |g|^2)$, and the meromorphic map $G$ are well-defined on $E$.

There is a dual immersion (with $H = 1$) $F^\# : \tilde{E} \to \text{SL}(2, \mathbb{C})$ defined by $F^{-1} : \tilde{E} \to \text{SL}(2, \mathbb{C})$, introduced by M. Umehara and K. Yamada [20].

The Weierstrass data $(g^\#, \omega^\#) = (G, -G^*\omega)$, and $\Psi^\# : \tilde{E} \to \mathbb{H}^3$ is $(F^{-1})^*(F^{-1})$. This immersion need not define an immersion of $E$ in $\mathbb{H}^3$ but the metric $ds^\#$ is well-defined and nonsingular since $\Psi^\#$ is an immersion:

$$ds^\# = \frac{(1 + |G|^2)}{|G'|}|g'||\omega|.$$ 

In particular, $g'|\omega/G'$ is a nonvanishing holomorphic form.

Since $|G|$ is bounded, the metric

$$d\sigma = \frac{|g'|}{|G'|}|\omega|$$

will be complete if $ds^\#$ is complete. Thus it suffices to prove $ds^\#$ is complete on $E$.

Let $\gamma$ be a divergent path on $E$, which is proper so that $\gamma$ diverges in $\mathbb{H}^3$. Now in the Lorentzian model of

$$\mathbb{H}^3 = \left\{(x_1, x_2, x_3, t) \in \mathcal{L}^4; x_1^2 + x_2^2 + x_3^2 - t^2 = -1, t > 0 \right\},$$

the path $\gamma$ diverges so that $t(\gamma) \to \infty$.

Writing $F : \tilde{E} \to \text{SL}(2, \mathbb{C})$, $F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we have $2t = |A|^2 + |B|^2 + |C|^2 + |D|^2$. The dual immersion $F^\# = F^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$ so that $t^\# = t$.

In particular $t^\#(\gamma) \to \infty$ as well, and $\gamma$ diverges in $\mathbb{H}^3$ on the dual surface. Since $ds^\#$ is the induced metric on the dual surface from its immersion in $\mathbb{H}^3$, the $ds^#$ length of $\gamma$ is infinite. This proves Theorem 1.

\textbf{Remark 2.} The metric $ds^\#$ gives information on values of the Gauss map $G$. Zu-Huan Yu has proved $G$ is constant if $G$ misses more than four points [22]; he proves more generally that $ds^\#$ is complete. We have proved $G$ can miss at most three points when $M$ has finite total curvature (and $M$ is not a horosphere) [4].
Theorem 2. Let $E$ be a properly embedded Bryant annular end. If $E$ is conformally the punctured disk $D^*$ and $G$ extends meromorphically to the puncture, then $\partial_\infty E = G(0)$ (the value of $G$ at the puncture).

Proof. As in the proof of Theorem 1, we work in the upper half-space model and assume $G(0)$ is the point at infinity. First observe that $G(0) \in \partial_\infty E$. For otherwise – since $E$ is proper and $\partial_\infty E$ is closed on $S_\infty$ –, $E$ would be contained in some half space (a complement of a neighborhood of $\infty$) $B_R = \{x_1^2 + x_2^2 + x_3^2 < R^2, x_3 > 0\}$. Then (as in the proof of Theorem 1) $G$ must be bounded; a contradiction.

Now suppose $E$ accumulates at another point at infinity which we may assume $\sigma = (0,0,0)$. Let $C_T$ be the cylinder $\{x_1^2 + x_2^2 \leq T^2, 0 < x_3 < T\}$. There are points of $E$ in $C_T$ for all $T > 0$. As $q$ diverges on $E$, towards $\sigma$, $G(q)$ tends to infinity. The geodesic normal to $E$ at $q$ is a half circle meeting $x_3 = 0$ at two points, one point close to $\sigma$ (close in the metric $dx_1^2 + dx_2^2$) and the other point $G(q)$ that is “far” from $\sigma$. Thus the mean curvature vector $\overline{H}(q)$ of $E$ at $q$, tends to a vertical vector pointing up. As $x_3(q)$ decreases this vector becomes more vertical.

Now choose $T$ sufficiently small that $q \in C_T$ implies the angle between $\overline{H}(q)$ and $\overline{e}_3 = (0,0,1)$ is less than $\pi/8$.

Then for $q \in E \cap C_T$, the vertical segment going down from $q$ to $S_\infty$ does not meet $E$ again, since $E$ bounds a mean convex domain $W$ (this makes sense since $C_T$ can be chosen far from $\partial E$). So $E \cap C_T$ is a vertical graph $u$ over a (possibly disconnected) planar domain.

Now we prove that for $T$ sufficiently small, $E \cap C_T$ is a vertical graph over the whole base of $C_T$: $x_1^2 + x_2^2 < T^2$. Since there can be no points of $E$ below this graph, this contradicts $\sigma \in \partial_\infty E$.

We now make useful gradient estimates for this graph $u$ at $q \in E \cap C_T$ in the Euclidean metric.

Consider the vertical plane $Q$ containing the unit normal vector $\overline{n}$ to $E$ at $q$. $G(q)$ is also in this plane and we have Figure 7 in the plane $Q$.

Figure 7
Here $O$ is the center of $H(q)$ and $R$ is the radius of $H(q)$. Then
\[ \vec{n} = \frac{1}{W}(-u_{x_1}, -u_{x_2}, 1), \quad W = \sqrt{1 + |\nabla u|^2}. \]

We have $a = \left| \frac{R}{W}(-u_{x_1}, -u_{x_2}) \right| = \frac{|\nabla u|}{W} R$, and $a^2 = R^2 - (R - u)^2 = u(2R - u)$. Hence
\[ \frac{a^2}{R^2} = \frac{|\nabla u|^2}{W^2} = \frac{u(2R - u)}{R^2}, \]
\[ \frac{|\nabla u|^2}{uW^2} = \frac{2R - u}{R^2}. \]

Thus the horizontal component of $\vec{n}$ has length $l = \frac{2au}{a^2 + u^2}$, and the vertical component length $t = \frac{a^2 - u^2}{a^2 + u^2}$.

Now $G(0) = \infty$ and so for any large $b > 0$ we can assure that $a > b$ in $C_T$ for $T$ small enough and
\[ |\nabla u| = \frac{l}{t} = \frac{2 \left( \frac{u}{a} \right)}{1 - \left( \frac{u}{a} \right)^2} \leq 4 \left( \frac{u}{b} \right). \]

Then the auxiliary function $v = \ln u$ has bounded gradient.

Starting with $x_3(q)$ small with respect to $T$, we have $v \leq \ln(T/2)$ on the base of $C_T$, and $E$ does not leave $C_T$ at the top $\{x_3 = T\}$. Moreover $v$ is never $-\infty$ hence $E$ never reaches $\{x_3 = 0\}$ in $C_T$. Thus $E \cap C_T$ is a graph over the base of $C_T$ and Theorem 2 is proved.

Theorems 1 and 2 immediately imply:

**Corollary 2.** Let $E$ be a properly embedded Bryant annular end. If $\partial_\infty E \neq S_\infty$, then $E$ is regular and $\partial_\infty E$ is the limiting value of $G$ on $E$.

5. Finite total curvature of nondense annular ends

**Theorem 3.** Let $E$ be a properly embedded Bryant annular end. If $E$ is on the mean convex side of a catenoid cousin end, then $E$ has finite total curvature.

**Proof.** First we make precise “the mean convex side.” The ends of the family of catenoid cousins can be written as graphs (in the upper half-space model) over domains at infinity: $x_1^2 + x_2^2 \geq r_0^2$, $x_3 = 0$. These ends are asymptotically $\frac{1}{r^2}$, $\alpha > -1$ and $r^2 = x_1^2 + x_2^2$. Let $C_\alpha$ be such a catenoid cousin end and extend $C_\alpha$ to an embedded surface with no boundary by attaching the horizontal disk along $\partial C_\alpha$. The mean convex side of $C_\alpha$ is then the component
to which $\overline{H}$ points along $C_\alpha$ (here $\overline{H}$ is pointing up). So our hypothesis on $E$ is that $E$ is contained in this mean convex side of $C_\alpha$. Clearly the catenoidal ends are ordered by $\alpha$ and we can assume $\alpha > 0$.

Since $\partial_\infty E$ is the point at infinity, Theorem 1 applies and we know $E$ is conformally a punctured disk $D^* = \{0 < |z| \leq 1\}$, and $G$ extends meromorphically to 0. Parametrize so that $G(z) = \frac{1}{z^p}$ for some integer $p \geq 1$.

The end $E$ is determined by $F : \overline{E} \to \text{SL}(2, \mathbb{C})$, $F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with $C = z^\nu f$, $f$ holomorphic in $D^*$, and similar representations for $A$, $B$ and $D$ (this is proved in Lemma 7, following the present proof).

We know that $x_3 = \frac{1}{|C|^2 + |D|^2}$. Suppose $C = z^\nu f$ and $f$ has an essential singularity at 0. Then for some sequence $z_n \to 0$, we have

$$|C(z_n)|^2 \geq \frac{1}{|z_n|^{(p+1)\alpha}}.$$ 

Let $q_n$ be the point on $E$ corresponding to $z_n$. Since $E$ is above the catenoid cousin $C_\alpha$:

$$r(q_n)^\alpha \geq \frac{1}{x_3(q_n)},$$

so by the previous inequality for $C(z_n)$, we conclude $r(q_n) \geq \frac{1}{|z_n|^{(p+1)\alpha}}$, and for any integer $k > 1$, and $n$ sufficiently large:

$$r(q_n) > \frac{k}{|z_n|^p}.$$ 

That is, the horizontal (Euclidean) distance from the point $q_n$ to the point $s = (0, 0, x_3(q_n))$ is at least $\frac{2}{|z_n|^p}$. Observe that $d(q_n, G(q_n))$ is at least $d(q_n, s) - d(G(q_n), s)$ where $d$ denotes the horizontal Euclidean distance.

Let $l$ be the horizontal disk of diameter $\frac{2}{|z_n|^p}$, centered at the point $p = (G(q_n), x_3(q_n))$. Since the horizontal distance from $G(q_n)$ to $(0, 0)$ is $\frac{1}{|z_n|^p}$, the disk $l$ is in the interior of $H(q_n)^+$; see Figure 8.

Now the origin is under one of the boundary points of $l$. Observe that the catenoid cousin $C_\alpha$ is above the segment $[p, s]$ on $l$, since the height of $C_\alpha$ at $G(q_n)$ is asymptotically $\frac{1}{|C(q_n)|} = |z_n|^{p\alpha}$ and $x_3(q_n) \leq |z_n|^{(p+1)\alpha}$. Since the graph of $C_\alpha$ is monotone decreasing with $r$, the segment $[p, s]$ is below $C_\alpha$. Also, $E$ is above $C_\alpha$ so that $[p, s]$ is disjoint from $E$.

Moreover let $N$ be a compact embedded surface with boundary the boundary of $E$ so that $N \cup E$ is an embedded surface. $N$ can be chosen above the union of the catenoid cousin $C_\alpha$ and the flat disk capping off $C_\alpha$. Then exactly as in Section 3, $N \cup E$ separates the ambient space so one can find a path $\gamma$ from $s$ to $\partial E$ which meets the $N \cup E$ only at the endpoint (first vertical, then a fixed path). The $k$ of the above inequality can be chosen large enough so that this path, together with the boundary of $E$, is inside $H(q_n)$. 

But \( \gamma \) together with \([p, s]\) can be extended to a divergent path disjoint from \( E \cup H(q_n) \), by going down vertically to \( G(q_n) \) from \( p \). This divergent path from \( \partial E \subset H(q_n)^+ \) does not meet \( H(q_n) \cup E \) again, which contradicts Corollary 1.

Thus \( C \) and \( D \) are meromorphic at 0. We have

\[
F^{-1} dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega
\]

so that \( dC = (Cg + D)i \omega \) and \( dD = -g(Cg + D)i \omega \). Consequently \( g = -\frac{dD}{dC} \) is also meromorphic at the puncture and this proves \( E \) has finite total curvature.

\[
\text{Lemma 7.} \quad \text{Let } A, B, C, D \text{ be the holomorphic (multivalued) data on } D^* \text{ parametrizing the end } E \text{ of Theorem 3. Then } A(z) = z^\beta f(z) \text{ for some real } \beta \text{ and } f \text{ holomorphic on } D^*. \text{ Also, } B, C \text{ and } D \text{ have similar representations.}
\]

\[
\text{Proof.} \quad \text{Let } \tilde{D}^* = \{ y \in \mathbb{C}; \text{Re } y \leq 0 \} \text{ and } e^y = z \in D^* \text{ be the covering map. We have } F(y) = \begin{pmatrix} A(y) & B(y) \\ C(y) & D(y) \end{pmatrix} \text{ in SL}(2, \mathbb{C}), \text{ and } F(y + 2\pi i) = F(y) H, \text{ where } H \in \text{SU}(2) \text{ by Section 2.}
\]

Let \( P \in \text{SU}(2) \) diagonalize \( H \), \( PHP^{-1} = \Delta = \begin{pmatrix} e^{i\beta 2\pi} & 0 \\ 0 & e^{-i\beta 2\pi} \end{pmatrix} \). Then \( F_1 = FP^{-1} \) defines the same end \( E \) and

\[
F_1(y + 2\pi i) = F(y + 2\pi i)P^{-1} = F(y)HP^{-1} = F(y)P^{-1} \Delta = F_1(y)\Delta.
\]

Thus \( A_1(y + 2\pi i) = A_1(y)e^{i\beta 2\pi} \) and similarly for \( B_1, C_1 \) and \( D_1 \).

Now define \( f(y) = e^{-y\beta}A_1(y) \), so that

\[
f(y + 2\pi i) = e^{-(y+2\pi i)\beta}A_1(y + 2\pi i) = f(y),
\]

and \( f \) defines a holomorphic map \( f(z) \) on \( D^* \), by \( f(z) = f(y), e^y = z \).

Then \( e^{y\beta}f(z) = A_1(y) \), so that the (multi-valued) \( A_1(z) \) on \( D^* \) satisfies \( A_1(z) = z^\beta f(z) \).

\[
\text{Theorem 4.} \quad \text{Let } E \text{ be a properly embedded Bryant annular end. If } \partial_{\infty}E \text{ is not } S_\infty \text{ then } E \text{ has finite total curvature.}
\]
Corollary 3. Let $E$ be a properly embedded Bryant annular end. If $\partial_{\infty}E \neq S_\infty$, then $E$ is regular and the total curvature of $E$ is finite. $E$ is asymptotic to a catenoid cousin or horosphere end.

Proof of the corollary. Theorem 4 and Corollary 2 yield the facts that $E$ is regular, $\partial_{\infty}E$ is one point and the total curvature of $E$ is finite. Then the theorem of E. Toubiana and R. Sa Earp yields the asymptotic behavior [15].

Proof of the theorem. We know from Theorem 3, that $E$ will have finite total curvature if we can find a catenoid cousin $C_\alpha$ with $E$ on the mean convex side of $C_\alpha$; we will find such a $C_\alpha$ to prove Theorem 4. By Theorems 1 and 2 we know that $\partial_{\infty}E$ is one point, which we take to be infinity in the upper half-space model of $\mathbb{H}^3$.

Let $B$ be a ball in $\mathbb{H}^3$, whose interior contains $\partial E$ and where $E$ is transverse to $\partial B$. Let $E_1$ denote the noncompact component of $E-B$, and let $W$ denote the mean convex domain (along $E_1$) bounded by $E_1$ and a compact domain on $\partial B$.

If $x_3 \geq c > 0$ on $E$ then $C_\alpha$ can be constructed using a catenoid cousin end below height $c$ which is a graph over an exterior domain $x_1^2 + x_2^2 > r_0^2$, asymptotic to the plane $x_3 = 0$ at infinity. So we can assume there is a sequence $q_n \in E_1$ with $x_3(q_n) \to 0$. Since $\partial_{\infty}E = \infty$, we have $r(q_n) = \sqrt{x_1(q_n)^2 + x_2(q_n)^2} \to \infty$.

For $q \in E_1$, let $\gamma$ be the minimizing geodesic of $\mathbb{H}^3$ joining $q$ to a point of $\partial B$. We will be working with $q$ lower than $B$. Assume $B = \{x_1^2 + x_2^2 + (x_3-4)^2 = 1\}$ for convenience, and $x_3(q) \leq 1$, $r(q) > 6$. Parametrize $\gamma$ by arc length so that $\gamma(0)$ is the highest point of $\gamma$ (which is not on $B$ by our choice of constants), and $\gamma(t_0) = q$ with $t_0 < 0$.

Let $P(t)$ be the family of (hyperbolic) planes orthogonal to $\gamma$ at $\gamma(t)$. For $t$ very negative, $P(t)$ is disjoint from $E_1$ since $\partial_{\infty}E_1 = \infty$, and $E_1$ is proper so that there is a first $t_1 \leq t_0$ (as $t$ increases) such that $P(t_1)$ touches $E_1$ at a point $q_1$.

We do Alexandrov reflection of $E_1$ with the planes $P(t)$ as $t$ increases from $t_1$ to 0. Let $S(t)$ be symmetry of $\mathbb{H}^3$ through $P(t)$, $E_1(t)^+$ the part of $E_1$ on the side of $P(t)$ not containing $B$, and $E_1(t)^* = S(t)(E_1(t)^+)$.

For $t$ slightly larger than $t_1$, $E_1(t)^+$ is a graph over (part of) $P(t)$, int $(E_1(t)^*) \subset W$, and the angle between $P(t)$ and $E_1(t)^+$ is never $\pi/2$ along $\partial E_1(t)^+$. These properties continue to hold until the first $t$ ($t_2$ say) such that $E_1(t_2)^*$ touches $\partial B$, for if one of these properties failed to hold at some earlier $t$, $P(t)$ would be a plane of symmetry of $E$. Then $E$ is part of a properly embedded, mean curvature one, compact surface $M$, with $\partial M = \emptyset$. This is impossible.
Clearly $t_2 < 0$ since $q$ is lower than $B$, and so the symmetry of $q$ through some plane $P(t)$, $t < 0$ meets $B$. Thus there is some point $\bar{q} \in E_1(t_2)^+ such that $S_{t_2}(\bar{q}) \in B$.

Let $\delta_1 = \text{dist}(\bar{q}, \gamma)$, and $q_t = S_t(\bar{q})$. Since $\gamma$ is invariant by $S_t$, we have $\text{dist}(q_t, \gamma) = \delta_1$ as well. For $t = t_2$, $q_t$ is on $\partial B$, so that $\text{dist}(q_t, \gamma) \leq \text{diam}(B) = \delta$. The curve $q_t$ joining $\bar{q}$ to $\partial B$, as $t$ varies from $t_1$ to $t_2$, is an equidistant curve $\beta$ whose distance from $\gamma$ is less than $\delta$, and this equidistant curve is contained in $W$. We emphasize that this discussion is valid for any $q \in E_1$ with $x_3(q) < 1$, $r(q) > 6$.

In particular, consider the sequence $q_n \in E_1$, satisfying $x_3(q_n) \to 0$, $r(q_n) \to \infty$. Then a subsequence of the geodesics $\gamma_n$ joining $q_n$ to $B$ converges to a vertical geodesic over $B$ and the equidistant curves $\beta_n$ from $\tilde{q}_n$ to $B$ are in $W$ and a distance at most $\delta$ from $\gamma_n$. So the equidistant curves $\beta_n$ are in the tubular neighborhood of $\gamma_n$ of radius $\delta$. As $n \to \infty$, the tubular neighborhoods converge to a vertical cone of hyperbolic width $\delta$. Let $C(\delta)$ denote this cone; for simplicity we can assume the base of $C(\delta)$ is the origin.

Now we can prove that $E_1 \cap A$ is a graph where $A = \{x_3 < 1, r \geq 6\}$.

Suppose this were not true. Let $N$ be the Euclidean unit normal to $E$, $\nabla H > 0$ and suppose that $N_3 \leq 0$ at some point $q \in E_1 \cap A$. Then the horosphere tangent to $E$ at $q$, $H(q)$, is at most of (Euclidean) radius 1 and $\partial E \subset H(q)^-$.

Then by Corollary 1, $H(q)$ separates $W$ into three connected components. One is compact and contains part of $\partial B$. One is noncompact, and contains the points $\tilde{q}_n$, $n$ large. And the third is compact and inside $H(q)^+$. But the equidistant curves $\beta_n$ are in $W$ and disjoint from $H(q)$ for $n$ large; this is impossible since the $\beta_n$ go to $\partial B$ in $W$. This proves $E_1 \cap A$ is a graph.

In fact the above argument proves much more: for $q \in A$, $H(q)$ must intersect $C(\delta)$; otherwise the equidistant curves $\beta_n$ would be disjoint from $H(q)$ for $n$ large; cf. Figure 9.

For $q \in E_1 \cap A$, let $R$ be the Euclidean radius of $H(q)$ and let $d$ be the Euclidean distance of $q$ to $C(\delta)$. Then (since $C(\delta)$ is invariant by homothety from $\sigma$ and $C(\delta) \cap H(q) \neq \emptyset$) there is a $\lambda > 0$ such that

$$2R \geq d \geq 2\lambda r(q),$$

and $\lambda$ depends only on $C(\delta)$. In particular $R \to \infty$ when $r(q) \to \infty$.

Now we shall prove that $E$ is below some horosphere $x_3 = \text{constant}$.

We know that $E_1 \cap A$ is the graph of a function $u$ and in Theorem 2, we derived the formula:

$$\frac{|\nabla u|^2}{u(1 + |\nabla u|^2)} = \frac{2R - u}{R^2} \leq \frac{2}{R}. $$
Since $u \leq 1$ this implies
\[ |\nabla u|^2 \leq \frac{2u}{R-2} \leq \frac{2u}{\lambda r(q) - 2}. \]
In particular, at the point $q_n \in E_1$, where $x_3(q_n) \to 0$, $r(q_n) \to \infty$, we obtain
\[ |\nabla u(q_n)| \leq \varepsilon_n^2, \]
for a sequence $\varepsilon_n \to 0$.

Now recall our discussion of Alexandrov reflection by planes orthogonal to the geodesics $\gamma_n$ joining $q_n$ to $\partial B$. We found a point $\tilde{q}_n$ in $E_1$, associated to the first accident of Alexandrov reflection, and we showed the equidistant curve $\beta_n$ from $\tilde{q}_n$ to $\partial B$ was in $W$. We have $|r(\tilde{q}_n) - r(q_n)| < 1$ by construction, so at $\tilde{q}_n$ we also have an estimate
\[ |\nabla u(\tilde{q}_n)| \leq \varepsilon_n^2, \]
for $\varepsilon_n \to 0$, $\varepsilon_n \sim \frac{1}{r(\tilde{q}_n)^{1/4}}$.

Then the maximum oscillation of $u$ on the horizontal (Euclidean) disk $D$ of radius $\frac{x_3(\tilde{q}_n)}{\varepsilon_n}$, centered at $\tilde{q}_n$, is $2\varepsilon_n x_3(\tilde{q}_n)$.

To check this, notice that the most $|\nabla u|$ can be is $r(\overline{q})^{-1/2}$, where $\overline{q}$ is a point of $D$ closest to the origin. Thus,
\[ r(\overline{q}) = r(\tilde{q}_n) - \frac{x_3(\tilde{q}_n)}{\varepsilon_n} \geq \frac{r(\tilde{q}_n)}{2}. \]

Then $r(\overline{q})^{-1/2} \leq \sqrt{2}r(\tilde{q}_n)^{-1/2}$ and the oscillation on $D$ is at most
\[ |\nabla u(\overline{q})| \frac{x_3(\tilde{q}_n)}{\varepsilon_n} \leq \sqrt{2} r(\tilde{q}_n)^{-1/2} x_3(\tilde{q}_n) r(\tilde{q}_n)^{1/4} \leq 2 x_3(\tilde{q}_n) \varepsilon_n. \]
Define \( D_n = D + (0, 0, x_3(\tilde{q}_n)) \); \( D_n \) is a horizontal disk above the graph of \( u \) over \( D \) so that \( D_n \subset W \) and the hyperbolic radius of \( D_n \) tends to infinity (it is \( 1/2\varepsilon_n \)). Also the hyperbolic distance between \( D_n \) and the graph of \( u \) over \( D \) is bounded by \( \ln(2) \).

Let \( t_n < 0 \) denote the first time that \( S(t_n)(\tilde{q}_n) \) touches \( \partial B \) (the first accident when we do Alexandrov reflection with the planes orthogonal to \( \gamma_n \)). We have \( F_n = S(t_n)(D_n) \subset W \) and the distance of \( F_n \) to \( \partial B \) is at most \( \ln(2) \). As \( n \to \infty \), \( F_n \) converges to a horizontal horosphere \( F \) which must be in \( W \). Thus \( E \) is below \( F \).

Next we observe that \( E_2 = E \cap (\Omega \times \mathbb{R}^+) \) is a vertical graph, where \( \Omega = \{x_1^2 + x_2^2 > a^2\} \), for some \( a > 0 \). To see this, remark that \( x_3(q) \leq c_0 \) for some constant \( c_0 \) and so if \( \overline{H}(q) \) does not point up then \( q \) is in the upper hemisphere of its tangent horosphere so \( x_3(q) \geq R = \text{the Euclidean radius of } H(q) \). Hence \( R \leq c_0 \) and \( H(q) \) will be disjoint from the cone \( C(\delta) \) for \( r(q) \) larger than some fixed \( a \). As before, this is impossible since the equidistant curves \( \beta_n \), for \( n \) large, will not intersect \( H(q) \).

Now on the domain \( \Omega \times \mathbb{R}^+ \) where the subend \( E_2 \) is a graph, we consider the family of catenoid cousin ends \( C(t) \) with each \( C(t) \) a graph over \( \Omega \times \mathbb{R}^+ \), tangent to the vertical cylinder \( \partial \Omega \times \mathbb{R}^+ \) and \( \partial C(t) \) is at height \( t \) on \( \partial \Omega \times R^+ \). These surfaces are described in [7].

For \( t > c_0 \), \( \partial C(t) \) is above \( E_2 \). If \( C(t) \) intersects \( E_2 \), then by Theorem 5, \( \Gamma = C(t) \cap E_2 \) is compact. Note that \( \Gamma \) is not homologous to zero on \( E_2 \) (nor is any subcycle of \( \Gamma \)) since this would yield a compact domain \( N \) on \( E_2 \) whose boundary is in \( C(t) \). Now vary \( t \) to obtain a last point of contact of \( C(t) \) with \( N \); then \( C(t) = E_2 \) by the maximum principle. It follows that \( \Gamma \) is a Jordan curve on \( E_2 \) that generates \( \Pi_1(E_2) \). On \( C(t) \), \( \Gamma \) bounds a catenoid cousin end that is below \( E_2 \) and Theorem 4 is clear by Theorem 3.

Now, we can assume \( C(t) \cap E_2 = \emptyset \) for \( t > c_0 \), and then decrease \( t \) to 0. There is some largest \( t \) where \( C(t) \) is disjoint from \( E_2 \) and \( C(s) \cap E_2 \neq \emptyset \), for \( s < t \). Since \( C(t) \) is vertical along \( \partial \Omega \times \mathbb{R}^+ \) and \( E_2 \) is a graph (not vertical) there, \( \partial C(t) \) is always above \( E_2 \). Thus we are in the previous situation where \( C(s) \cap E_2 \neq \emptyset \) and \( \partial C(s) \) is above \( E_2 \) and Theorem 4 is proved.

**Theorem 5.** Let \( \Omega \) be a noncompact domain in the plane \((x_1, x_2)\) with at least one component of \( \partial \Omega \) noncompact. Let \( u_1, u_2 \) be defined on \( \Omega \) with their graphs solutions of the mean curvature equation \( H = 1 \) in \( \mathbb{H}^3 \). Suppose the following conditions are satisfied:

a) \( u_2 \leq u_1 \leq 1 \) on \( \Omega \), \( u_1 = u_2 \) on \( \partial \Omega \),

b) \( \frac{C_1}{r^\alpha} \leq u_2 \leq \frac{C_2}{r^\alpha} \), for some positive constants \( C_1, C_2, \alpha \) (\( u_2 \) is the graph of a catenoid cousin),
c) $\frac{|\nabla u_1|^2}{u_1} \leq \frac{C}{r^2}$, for some $C > 0$, $r^2 = x_1^2 + x_2^2$.

It then follows that $u_1 = u_2$ on $\Omega$.

Remark 3. In order to apply this theorem to prove Theorem 4, we need to verify that the graph $u = u_1$ of $E_2$ in Theorem 4 satisfies the conditions a, b, and c. The conditions a and b are satisfied by construction; the condition c needs some discussion.

In the proof of Theorem 2 we derived the gradient bound for $u$:

$$\frac{|\nabla u|^2}{u(1 + |\nabla u|^2)} \leq \frac{2}{R},$$

where $R$ is the Euclidean radius of the horosphere $H(q)$. Since $u \leq 1$,

$$\frac{|\nabla u|^2}{u} \leq \frac{2}{R - 2}.$$

So we need to know $R$ is of order $r^2$ for the graph $u$, to satisfy condition c.

We see this by considering $H(q)$, $q$ on the graph of $u$. Let $E$ denote the graph of $u$ (this is the $E_2$ in the proof of Theorem 4), and let $C$ be the vertical compact cylinder joining $\partial E$ to the plane $x_3 = 0$. Observe that for $q \in E$, $H(q)$ must intersect $C$. For if $H(q)$ passes over $C$, then $\partial E \subset H(q)^-$ so the figure eight in $H(q) \cap E$, would contain a Jordan curve $C_1$ that is homological to $\partial E$ on $E$ (Proposition 1). However, $u$ takes its maximum value on $\partial E$ ($u$ has no interior maximum since the graph of $u$ would touch a horizontal horosphere at a local maximum and have the same mean curvature vector). Thus $C_1$ would be lower than $\partial E$ and link the cylinder $C$. Hence $H_q$ must intersect $C$. We want to estimate $1/R$ from above, so that for $q \in E$, we can assume $H(q)$ intersects the vertical segment over the origin at a point $p$ at height $x_3(p)$ less than some fixed $b > 0$. Now for $x_3(q) < b$, the horosphere $H(q)$ passing through $p$, intersects the plane $x_3 = 0$ at the point $G(q)$; see Figure 10.

![Figure 10](image-url)
Then \( t^2 + (R - x_3(p))^2 = R^2 \), so that \( R = \frac{x_3^2}{x_3(p)} + \frac{x_3(p)}{2} \geq \frac{x_3^2}{2} \). Apply the same (Pythagorean) calculation with \( p \) replaced by \( q \) to obtain \( R \geq \frac{x_3^2}{2} \) where \( \tau \) is the horizontal distance from \( q \) to \( G(q) \). Since \( \tau + t \geq r \), \( \tau \) or \( t \) is at least \( \frac{r}{2} \) so that \( R \geq \frac{r^2}{8} \) as desired.

Before proving Theorem 5, we need some lemmas.

**Lemma 8.** Let \( u \) be a solution of the equation \( H = 1 \) on \( \Omega \). Then \( v = \ln u \) satisfies:

\[
\text{div} \left( \frac{\nabla v}{W} \right) = \frac{-|\nabla u|^4}{u^2W(1+W)^2}, \text{ where } W^2 = 1 + |\nabla u|^2.
\]

*Proof.* We have \( \text{div} \left( \frac{\nabla u}{W} \right) = \frac{2}{u} \left( 1 - \frac{1}{W} \right) \), \( (H = 1) \), hence

\[
\text{div} \left( \frac{\nabla v}{W} \right) = \text{div} \left( \frac{\nabla u}{uW} \right) = \frac{-|\nabla u|^2}{u^2W} + \frac{2}{u^2W} \left( 1 - \frac{1}{W} \right)
= \frac{-|\nabla u|^2}{u^2W} + \frac{2}{u^2W} \left( 1 - \frac{1}{W} \right) = \frac{|\nabla u|^2}{u^2W} \left( \frac{2}{1+W} - 1 \right)
= \frac{|\nabla u|^2}{u^2W} \left( 1 - \frac{1}{1+W} \right) = \frac{|\nabla u|^2}{u^2W} \left( 1 - \frac{1}{1+W} \right) = \frac{-|\nabla u|^4}{u^2W(1+W)^2}.
\]

**Lemma 9.** Let \( \Omega(r) = \{ x \in \Omega; |x| \leq r \} \) and \( C(r) = \Omega(r) \cap \{|x| = r\} \). Define \( v = \ln u_1 - \ln u_2 \) \( (u_1, u_2 \text{ as in Theorem 5}) \), and \( M(r) = \sup \{ |v(x)|; |x| = r \} \). Then if \( v \neq 0 \) there is a \( \beta < \alpha \) such that \( M(r) \geq (\alpha - \beta) \ln r \).

*Proof.* Consider a family of catenoid cousin graphs \( u_{\tau}(x) \), with \( u_{\tau} \) strictly above \( u_2 \) on \( \partial \Omega \) and \( u_{\tau} \) comes down to \( u_2 \) as \( \tau \to \alpha \), with \( u_{\tau} = u_2 \) for \( \tau = \alpha \). Parametrize so that the growth of \( u_{\tau} \) is \( 1/r^\tau \), \( \tau < \alpha \). As \( \tau \to \alpha \), one cannot have \( u_{\tau} \) above \( u_1 \) for all \( \tau \) (otherwise \( u_1 = u_2 \)). Hence the graph of some \( u_\beta \), \( \beta < \alpha \), intersects the graph of \( u_1 \). As usual, we know the intersection cannot be homologous to zero on the graph (vary \( \tau \) to get a last point of contact), and the intersection is not one compact cycle (otherwise there is a catenoid cousin below \( u_1 \) and Lemma 9 is proved) so that the intersection is not compact and \( u_1 \) is above \( u_\beta \) on a noncompact domain. Thus \( M(r) \geq (\alpha - \beta) \ln r \).

*Proof of Theorem 5.* We study \( v = \ln u_1 - \ln u_2 = v_1 - v_2 \). Clearly \( v \geq 0 \), \( v = 0 \) on \( \partial \Omega \) and \( v \leq \gamma \ln r \) for some positive \( \gamma \). We will show that if \( v \) is not identically zero, then for some integer \( k > 1 \), \( M(r) \) grows faster than \( (\ln r)^k \). This latter growth is impossible and so \( v \equiv 0 \).
By Stokes’ theorem,
\[
\int_{\Omega(R)} \text{div} \left( v \frac{\nabla v_1}{W_1} \right) - \text{div} \left( v \frac{\nabla v_2}{W_2} \right) = \int_{\partial \Omega(R)} v (\frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2}, N),
\]
where \( N \) is the outer conormal along \( \partial \Omega(R) \). Apply this equation to \( v = v_1 - v_2 \),

\[
(1) \quad \int_{\Omega(R)} (\nabla v_1 - \nabla v_2) \left( \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right) + \int_{\Omega(R)} v \text{ div} \left( \frac{\nabla v_1}{W_1} \right) - \text{div} \left( \frac{\nabla v_2}{W_2} \right) = \int_{\partial \Omega(R)} v (\frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2}, N).
\]

By Lemma 8, and the estimates \( \frac{|\nabla u_i|^4}{u_i^2} \leq \frac{C_i}{r^4}, \quad i = 1, 2, \) and \( v \leq \gamma \ln r \),
\[
\left| \int_{R_0} \int_{C(r)} v \text{ div} \left( \frac{\nabla v_i}{W_i} \right) \right| \leq \left| \int_{R_0} \int_{C(r)} v \frac{|\nabla u_i|^4}{u_i^2 W_i (1 + W_i)^2} \right| \\
\leq C_i \left| \int_{R_0} \int_{C(r)} \frac{\ln r}{r^4 W_i (1 + W_i)^2} \right|.
\]

Since the last integral converges we have \( \left| \int_{\Omega(R)} v \text{ div} \left( \frac{\nabla v_i}{W_i} \right) \right| \leq a_i \), for some constants \( a_1, a_2 \). Then equation (1) yields

\[
(2) \quad a_3 + \int_{\Omega(R)} (\nabla v_1 - \nabla v_2) \left( \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right) \leq \int_{\partial \Omega(R)} v (\frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2}, N).
\]

For \( R_1 > 0 \), define \( \mu(R_1) = \int_{\Omega(R_1)} (\nabla v_1 - \nabla v_2) \left( \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right) \). We have
\[
(\nabla v_1 - \nabla v_2) \left( \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right) = W_1 \left( \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right)^2 \\
+ (W_1 - W_2) \frac{\nabla v_2}{W_2} \left( \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right).
\]

Also
\[
\left| (W_1 - W_2) \frac{\nabla v_2}{W_2} \left( \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right) \right| \leq \frac{\nabla u_2}{u_2 W_2} \left| \nabla u_1 \right|^2 \left| \nabla u_2 \right|^2 \left| \nabla v_1 - \nabla v_2 \right| \\
\leq C_1 \frac{1}{r^3} \left| \nabla v_1 - \nabla v_2 \right|.
\]

Then (2) implies

\[
(3) \quad a_3 + \mu(R_1) + \int_{R_1} \int_{C(r)} W_1 \left( \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right)^2 - \int_{R_1} \int_{C(r)} C_1 \frac{1}{r^3} \left| \nabla v_1 - \nabla v_2 \right| \\
\leq \int_{C(r)} v (\frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2}, N).
\]
Define } \eta(r) = \int_{C(r)} \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|. \text{ Now,}

\frac{\eta^2(r)}{2\pi r} \leq \int_{C(r)} \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|^2 \leq \int_{C(r)} W_1 \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|^2.

Inequality (3) then implies

\begin{equation}
\eta^2(r) \leq \int_{C(r)} \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|^2 \leq \int_{C(r)} W_1 \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|^2.
\end{equation}

Next

\left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right| \geq \frac{1}{W_1} |\nabla v_1 - \nabla v_2| - |\nabla v_2| \left| \frac{1}{W_2} - \frac{1}{W_1} \right|

and \frac{1}{W_1} \geq c_2 > 0, |\nabla v_2| \leq \frac{\alpha}{r}, \left| \frac{1}{W_2} - \frac{1}{W_1} \right| \leq 2, \text{ so that}

\eta(r) \geq c_2 M(r) - 4\pi \alpha.

By Lemma 9 we conclude } \eta(r) \rightarrow \infty, \text{ as } r \rightarrow \infty, \text{ unless } v \equiv 0. \text{ Then there is a constant } c_3 > 0 \text{ and } R_0 \geq 0 \text{ such that for } r \geq R_0,

\frac{\eta^2(r)}{2\pi r} - \frac{c_1 \eta(r)}{r^3} \geq \frac{c_3 \eta^2(r)}{r}.

Thus (4) may be replaced by (5) for } R_1 \geq R_0:

\begin{equation}
a_3 + \mu(R_1) + c_3 \int_{R_1}^{R} \frac{\eta^2(r)}{r} \leq M(R) \eta(R).
\end{equation}

Now we will show that for } R_1 \text{ greater than or equal to some (other) } R_0, \text{ we have } \tilde{\mu}(R_1) = a_3 + \mu(R_1) > 0, \text{ for}

\tilde{\mu}(R_1) = a_3 + \int_{\Omega(R_1)} W_1 \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|^2 + \int_{\Omega(R_1)} (W_1 - W_2) \nabla v_2 \left( \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right).

The module of the second integral is at most

\int_{R_0}^{R_1} \frac{c_1}{r^3} \eta(r),

and \, W_1 \geq 1 \text{ so that } \tilde{\mu}(R_1) \geq a_3 + \int_{R_0}^{R_1} \left( \frac{\eta(r)^2}{2\pi r^3} - \frac{c_1 \eta(r)}{r^3} \right), \text{ which diverges since } \eta(r) \rightarrow \infty.

Now } \tilde{\mu}(R_1) \geq \tilde{\mu}(R_0) + c_3 \int_{R_0}^{R_1} \frac{\eta^2(r)}{r}. \text{ By Lemma 9 and the comparison between } \eta(r) \text{ and } M(r) \text{ we conclude } \tilde{\mu}(R_1) \text{ grows at least as fast as } \ln^3(R_1).
We write equation (5) as:

\[
\tilde{\mu}(R_1) + c_3 \int_{R_1}^R \frac{\eta^2(r)}{r} \leq A\eta(R),
\]

for \( R \in [R_1, R_2] \), and \( A = \sup\{M(R); R_1 \leq R \leq R_2\} \).

Let \( \xi \) be the function defined on the interval \( J = [R_1, R_1 \exp(\frac{2A^2}{c_3\tilde{\mu}(R_1)})] \) by

\[
c_3 \frac{A}{\tilde{\mu}(R_1)} \ln \left( \frac{R}{R_1} \right) = \frac{2A}{\tilde{\mu}(R_1)} - \frac{1}{\xi(R)}.
\]

On \( J \), \( \xi \) satisfies the equation:

\[
\frac{\tilde{\mu}(R_1)}{2} + c_3 \int_{R_1}^R \frac{\xi^2(r)}{r} = A\xi(R).
\]

The connected component of \( \{ R \in J \cap [R_1, R_2]; \xi(R) < \eta(R) \} \) that contains \( R_1 \), is open by construction and closed by equation (6). Thus it is the interval \( J \cap [R_1, R_2] \). Since \( \xi(r) \to \infty \) when \( r \) converges (\( r \) increasing) to the right end point of \( J \), and \( \eta \) is bounded on \( [R_1, R_2] \), we conclude \( R_2 \in J \). Thus

\[
R_2 \leq R_1 \exp \left( \frac{2A^2}{c_3\tilde{\mu}(R_1)} \right).
\]

Since \( A \leq \gamma \ln(R_2) \), we have

\[
\left[ \frac{c_3\tilde{\mu}(R_1)}{2} \ln \left( \frac{R_2}{R_1} \right) \right]^{\frac{1}{2}} \leq \gamma \ln(R_2),
\]

for \( R_0 \leq R_1 \leq R_2 \). However this contradicts our estimate for the growth of \( \tilde{\mu}(R_1) \) (take \( R_2 = R_1^2 \)). This completes the proof of Theorem 5.

\[ \square \]

6. Nondensity at infinity of finite topology surfaces

Let \( M \) be a properly embedded Bryant surface with \( \partial M \) perhaps not compact. Assume a properly embedded surface \( \Sigma \) exists with \( \partial \Sigma = \partial M \) and \( \Sigma \cup M = \partial W \) with \( M \) mean convex along \( W \). Let \( P \) be a (hyperbolic) plane with \( \mathbb{H}^3 - P = P^+ \cup P^- \), the connected components of the complement. Assume \( \Sigma \subset P^- \). Let \( M^+ = M \cap P^+ \).

**Theorem 6.** There is a constant \( c > 0 \) (independent of \( M \)) such that if \( |K(q)| < c \) for \( q \in M^+ \), then int \( (\partial_\infty M^+) = \emptyset \); i.e., \( M \) cannot be asymptotic to an open set at infinity in \( P^+ \). In the half-space model, for \( q \in M^+ \) and \( x_3(q) \) sufficiently small, \( M^+ \) is a vertical graph near \( q \), no point of \( M \) is below this local graph, and the angle between \( \overrightarrow{H}(q) \) and \( e_3^q \) is at most \( \pi/4 \).
Proof. We work in the upper half-space model. At each \( q \in M \), \( M \) is locally a graph over \( H(q) \), in geodesic coordinates orthogonal to \( H(q) \). If \( |K| \) is small on \( M \) then the second fundamental form of \( M \) is close to that of \( H(q) \), since \( H = 1 \) and \( K \) small implies the principal curvatures of \( M \) are close to 1. Hence there is a \( c > 0 \) such that if \( q \in M \) and \( |K(q)| < c \), then \( M \) is a graph over the disk \( D(q) \) of radius 3 in \( H(q) \), centered at \( q \), and the maximum distance of the graph to this disk \( D(q) \) is one-half. We will see that this \( c \) works in Theorem 6.

We now suppose \( |K(q)| < c \) for \( q \in M^+ \). Let \( q \in M^+ \) and suppose \( \overline{H}(q) \cdot e^3_3 \leq 0 \) (i.e., \( \overline{H}(q) \) points down). Then \( H(q) \) is a Euclidean sphere tangent to \( S_\infty \) at one point. The upper hemisphere of \( H(q) \) has (hyperbolic) diameter 2 and \( q \) is in this upper hemisphere so that \( D(q) \) contains this hemisphere. Hence \( M \) is a graph over the upper hemisphere. We call this graph \( \text{Cap}(q) \).

The graph is at most a distance one-half from the hemisphere so that \( x_3 \) has a maximum at an interior point \( p \in \text{Cap}(q) \). At \( p \), \( \overline{H}(p) \) has the direction of \(-e^3_3 \) (by comparison with the horizontal horosphere \( \{x_3 = x_3(p)\} \)) and a simple calculation of the Euclidean Gaussian curvature at a point of \( M \) with \( \overline{H} \) parallel to \(-e^3_3 \) shows \( M \) is strictly Euclidean convex at \( p \). So the planes \( x_3 = \text{constant} \) meet \( \text{Cap}(q) \) in convex compact curves at heights a little below \( x_3(p) \).

We can assume \( P^+ = \{x_1^2 + x_2^2 + x_3^2 \leq 9, x_3 > 0\} \) and the origin \( \sigma \) is in \( \partial_\infty M \). We will prove that if \( q \in M^+ \) and \( q \) is sufficiently close to \( \sigma \) (in the Euclidean metric) then \( M \) is a vertical graph in a neighborhood of \( q \), over a domain \( \Omega \subset \{x_3 = 0\} \), and in \( \Omega \times \mathbb{R}^+ \), there is no point of \( M \) below this graph of \( M \) near \( q \). Thus \( \Omega \cap \partial_\infty M = \emptyset \) and \( \text{int}(\partial_\infty M^+) = \emptyset \).

Define \( \text{Cyl}(r) = \{x_1^2 + x_2^2 < r^2, x_3 > 0\} \) and suppose \( q \in M^+ \cap \text{Cyl}(1/4) \). If \( \overline{H}(q) \cdot e^3_3 > 0 \) for each such \( q \) with \( x_3(q) \) sufficiently small then \( M \) is a graph over a domain \( \Omega \) and if \( M_1 \) denotes this part of \( M \) near \( q \) where \( M \) is a vertical graph, then for \( p \in M_1 \), the vertical segment from \( p \) to \( \{x_3 = 0\} \) cannot meet \( M \) again since at the first point where this segment again meets \( M \), the vector \( \overline{H} \) would necessarily point into \( W \), hence it would have to point down, a contradiction. Thus it suffices to prove \( \overline{H}(q) \cdot e^3_3 > 0 \) for \( x_3(q) \) sufficiently small.

Suppose the contrary, \( \overline{H}(q) \cdot e^3_3 \leq 0 \), for \( q \) arbitrarily low. For \( q \in \text{Cyl}(1/4) \) and \( x_3(q) \leq 1/8 \), we know \( H(q) \) has at most (Euclidean) radius 1/8 so \( H(q) \subset \text{Cyl}(1) \), and \( \text{Cap}(q) \subset \text{Cyl}(1) \). Let \( p \in \text{Cap}(q) \) be a point where \( x_3(p) \) is a local maximum and the level curves of \( \text{Cap}(q) \) near \( p \) are compact Jordan curves \( C(t) \) in the planes \( x_3 = \text{constant} \).

Consider the evolution of these level curves \( C(t) \) as \( x_3 \) decreases from \( x_3(p) \). For values near \( x_3(p) \), there is no other part of \( M \) inside the disk \( D(t) \) of \( \{x_3 = t\} \) bounded by \( C(t) \). As long as \( C(t) \) stays compact and nonsingular, there is no other part of \( M \) in \( D(t) \), since the part would bound a compact
domain above $x_3 = t$, and under $\{C(\tau); t \leq \tau \leq x_3(p)\}$ and at the highest point of this compact part of $M$, $\overrightarrow{H}$ is parallel to $\overrightarrow{e_3}$ so that $M$ would equal a horosphere $x_3 = \text{constant}$; cf. Figure 11-a.

Also notice that $C(t)$ cannot acquire a singularity (i.e., a point where $\nabla x_3 = 0$) as long as $C(t)$ stays compact. For if a singularity occurs at a point $q_1 \in C(t)$ then $\overrightarrow{H}(q_1)$ is vertical. It cannot point up, since then $D(s)$ would contain other parts of $M$ for $s > t$, $s$ near $t$, and this is impossible by the previous paragraph; see Figure 11-b.

But $\overrightarrow{H}(q_1)$ cannot point down either since $M$ would then be strictly locally (Euclidean) convex near $q_1$ and $x_3$ would have a local maximum at $q_1$, not a critical point of negative index.

Thus as long as $C(t)$ stays inside Cyl(2), it is a smooth Jordan curve.

As $t$ decreases to zero, $C(t)$ must leave Cyl(2) since $\text{Cap}(q)$ must connect to the rest of $M$. Thus there are values of $t$ where $C(t)$ traverses $\partial \text{Cyl}(3/2)$.

Now if $x_3(q)$ is small and $q \in \text{Cyl}(1/8)$, there will be points $\tilde{q}$ of $C(t)$ in $\partial \text{Cyl}(3/2)$ where $\overrightarrow{H}(\tilde{q}), \overrightarrow{e_3} \leq 0$ and $x_3(\tilde{q}) \leq 1/8$. Then $\text{Cap}(\tilde{q}) \subset \text{Cyl}(2) - \text{Cyl}(1)$ and $\text{Cap}(\tilde{q})$ has a local maximum of $x_3$ near $\tilde{q}$. So the curves $C(t)$ are not connected before leaving Cyl(2), a contradiction.

It remains to obtain the gradient bound for the graph. For any horosphere of (Euclidean) radius $R$ in $\mathbb{H}^3$, the part of the horosphere where the mean curvature vector makes an angle greater than $\pi/4$ with $\overrightarrow{e_3}$ is of hyperbolic diameter at most 5. So if $q \in M^+$ and $|K|$ is sufficiently small on $M^+$, then $M$ will be a graph over a geodesic disk in $H(q)$ that contains the northern hemisphere of $H(q)$, if the angle between $\overrightarrow{H}(q)$ and $\overrightarrow{e_3}$ is greater than $\pi/4$. Now the same argument as before (with $\text{Cap}(q)$ and the $C(t)$) leads to a contradiction. This proves Theorem 6.

**Theorem 7.** Let $E$ be a properly embedded Bryant annular end. If $\partial_\infty E = \partial_\infty S_\infty$ then there is a proper arc $\gamma$ on $E$ with $\partial_\infty \{\gamma(t); t \geq 0\} = p_1$, $\partial_\infty \{\gamma(t); t \leq 0\} = p_2$ and $p_1 \neq p_2$. 

![Figure 11-a](image1.png) ![Figure 11-b](image2.png)
Proof. On any subend of $E$, there must be points $q$, with $2x_3(q) < \inf (x_3\partial E)$ and $\overline{H}(q), \overline{\delta}_3 \leq 0$; otherwise the subend would be a graph near $x_3 = 0$, and so could not be dense at infinity.

At such a point $q$, $\partial E \subset H(q)^-$ since $H(q)$ is a sphere of Euclidean radius less than $2x_3(q)$. If the connected component $\Sigma(q)$ of $E \cap H(q)$ is not compact then there is an arc $\gamma_q$ on $H(q)$ in $\Sigma(q)$ joining $q$ to $G(q)$; i.e., $\gamma_q$ is asymptotic to $G(q)$ at infinity. We will show next that such a $q$ can be found so that $\Sigma(q)$ is not compact.

Suppose $\Sigma(q)$ is compact. Then Proposition 1 gives $E_1$ and a figure eight $C_1 \cup C_2, C_1 \cup C_2 = \partial E_1, E_1 \subset H(q)^+, E_1$ is compact, and $E_1$ separates $E$. Assume $C_1$ is the Jordan curve homologous to $\partial E$ in $E - E_1$; cf. Figure 5.

Let $E'$ be the subannulus of $E$ bounded by $C_1$. We can assume there are points $q' \in E'$, $q' \neq q$, with $G(q') = G(q)$, for we can consider $q$ near $q$ on $E$; if we could not find $q'$ on $E'$ with $G(q') = G(q)$ then $\bigcap_{E'}$ would miss an open set $\Omega$ (the open set being the image by $G$ of an open set about $q$ on $E$) in $S_\infty, \Omega$ a neighborhood of $G(q)$. However $E'$ is dense at infinity so there are points $y$ of $E'$ converging to $\Omega$ with $\overline{H}(y), \overline{\delta}_3 \leq 0$ (otherwise $E'$ would be a graph near $\Omega$, and then $G(y) \in \Omega$, for $x_3(y)$ small.

So we can assume there is $q' \in E'$, $q' \neq q$ and $G(q') = G(q)$. If $\Sigma(q')$ is not compact then the arc $\gamma_{q'}$ joining $q'$ to $G(q')$ exists on $H(q')$. Thus, we suppose $\Sigma(q')$ compact.

There are two possibilities.

- Case 1. $\partial E' \subset H(q')^+$. In this case, we have a compact component $E'_1 \subset (E' \cap H(q')^+)$ with $\partial E'_1 = C'_1$. And there is a compact disk $D' \subset H(q')$ with $D' \cup E'_1 = \partial Q'_1, Q'_1$ a compact domain in $H(q')^+$ (Proposition 1 and Figure 4). Also $\partial E' \subset Q'_1$.

Now $Q'_1 \cap H(q)^+$ contains a connected compact component $Q$ with $\partial E' \subset Q$; cf. Figure 5. $Q$ is mean convex and $E_1 \subset Q$ so that $Q_1 \subset Q$. Also, $Q_1$ is mean convex along $E_1$. Since $E_1$ and $E_2 = (\partial Q) \cap \text{int} (H(q)^+)$ are on $E$, there must be another component $F$ of $E$ in $Q - Q_1$ that separates $E_1$ and $E_2$ ($W$ is mean convex along $E$). Then $\overline{H}$ points into the noncompact component of $H(q)^+ - F$, along $F$, and this contradicts Lemma 2.

- Case 2. $\partial E' \subset H(q')^-$. In this case, a Jordan curve of $H(q')$, $C'_1$ say, together with $C_2$ bounds a compact annulus $N \subset E$. Near $C_2, N$ is outside $H(q)$, so that $N \cap H((q)^-)$ is a compact domain on $E$ with boundary on $H(q)$ and outside $H(q)$. This contradicts Lemma 1. Thus we can construct a proper arc $\gamma_q$ from $q$ to $G(q)$ on $E$.

Now do the same construction at a point $q_1 \in E$ with $G(q) \neq G(q_1)$. Join $q$ to $q_1$ by a path $\delta$ on $E$. Then the arc $\gamma = \gamma_q \cup \gamma_{q_1} \cup \delta$ works to prove the theorem. \[\square\]
Theorem 8. Let $M$ be a properly embedded Bryant surface. Suppose $\gamma$ is a proper arc on $M$ that separates $M$ into two components $M_1, M_2$. There exist two properly embedded Bryant surfaces $\Sigma_1, \Sigma_2$ satisfying:

a) $\Sigma_1$ and $\Sigma_2$ are stable, $\partial \Sigma_1 = \partial \Sigma_2 = \gamma$, $\Sigma_1 \cap \Sigma_2 = \gamma$;

b) $\Sigma_1 \cup \Sigma_2$ bounds a domain $R$ contained in the mean convex component $W$ of $\mathbb{H}^3 - M$;

c) $R$ is mean convex;

d) $\Sigma_1 \cup M_1$ separates $\mathbb{H}^3$ and $\Sigma_2 \cup M_2$ as well.

Proof. Fix a point $p \in \gamma$ and let $B = B_R$ denote the ball of $\mathbb{H}^3$ centered at $p$ of radius $R$. Let $M_R$ be the connected component of $M \cap B$ containing $p$. The connected component of $\gamma \cap M_R$ containing $p$, separates $M_R$ into two components; denoted $M_1(R)$ and $M_2(R)$.

$M_R$ together with a compact domain on $\partial B \cup M_0$, $M_0$ the part of $M - M_R$ in $B$, bound a mean convex domain $Q$; $Q \subset W$. The part of $\partial Q$ on $\partial B$ has mean curvature greater than 1.

Let $D_1 \subset Q$ be a least area embedded minimal surface with $\partial D_1 = \Gamma_1 = \partial M_1(R)$, and let $Q_1$ be the compact domain bounded by $D_1 \cup M_1(R)$. $D_1$ is a barrier for the Plateau problem so we can find a least area minimal surface $D_2 \subset Q - Q_1$ with $\partial D_2 = \Gamma_2 = \partial M_2(R)$. Let $Q_2$ be the compact domain bounded by $D_2 \cup M_2(R)$. We have $Q_1 \cup Q_2 \subset Q \subset W$ and $\text{int}(Q_1) \cap \text{int}(Q_2) = \emptyset$.

Now consider domains $\tilde{Q} \subset Q$ with $\partial \tilde{Q} = \Gamma_1 \cup \Sigma$, $\Sigma$ a surface with $\partial \Sigma = \partial M_1(R) = \Gamma_1$. The functional on $(\tilde{Q}, \partial \tilde{Q})$:

$$(\tilde{Q}, \partial \tilde{Q}) \mapsto \text{area}(\Sigma) + 2\text{Vol}(\tilde{Q})$$

has a minimum and at such a $\tilde{Q}$, the smooth points of $\Sigma$ have mean curvature-one. This is proved in [1] when the mean curvature of $\partial Q$ is strictly greater than one; the only difference is that the minimum may now touch $M_1(R)$, in which case $\Sigma = M_1(R)$ and $M_1(R)$ is stable in $Q$.

So let $\Sigma_1$ be a minimum, $\partial \Sigma_1 = \Gamma_1$, $\Sigma_1 \cup M_1(R) = \partial \tilde{Q}$, and the mean curvature of $\Sigma_1$ is one.

Observe that $\Sigma_1 \subset Q_1$ (this is proved in [1]) since, if $\tilde{Q}$ went outside $Q_1$, one could remove the part of $\tilde{Q}$ outside of $D_1$ and reduce the functional.

Notice also that the mean curvature vector of $\Sigma_1$ points outside of $\tilde{Q}$. Otherwise $\tilde{Q}$ would be mean convex so that one could find a least area minimal surface $\tilde{D} \subset \tilde{Q}$, $\partial \tilde{D} = \Gamma_1$. Then the functional is smaller on the domain bounded by $\tilde{D} \cup M_1(R)$; a contradiction.
Now working with $\Gamma_2 = \partial M_2(R)$ and $Q_2$, one finds a mean curvature-one, stable surface $\Sigma_2 \subset Q_2$, $\partial \Sigma_2 = \Gamma_2$ and the mean curvature vector of $\Sigma_2$ points outside of the domain bounded by $\Sigma_2 \cup M_2(R)$. Thus the domain of $W \cap B$ bounded by $\Sigma_1 \cup \Sigma_2$ (and a part of $\partial B$) is mean convex; cf. Figure 12.

For $R > r > 0$, one has uniform area and curvature bounds of $\Sigma_1$ and $\Sigma_2$ on balls of radius $r$ a fixed distance from $\partial \Sigma_1$ and $\partial \Sigma_2$. Then (as in [1]), one can find a convergent subsequence of $\Sigma_1$ and $\Sigma_2$, as $R \to \infty$, which yield the $\Sigma_1$ and $\Sigma_2$ of Theorem 8.

In the case $\Sigma_1$ (or $\Sigma_2$) $\subset M$ then this part of $M$ is stable in $W$ but this easily implies stability in $\mathbb{H}^3$ (look at an unstable domain $D$ corresponding to a first eigenvalue $\lambda_1 < 0$). This proves Theorem 8.

**Theorem 9.** Let $M$ be a properly embedded Bryant surface of finite topology. Then $\partial \infty M \neq S_\infty$.

**Proof.** Assume the contrary; $M$ is dense at infinity. Then by Corollary 2 for some annular end $E$ of $M$, $\partial \infty E = S_\infty$, and so Theorem 7 applies: there is a proper arc $\gamma$ on $E$ and $\partial \infty \gamma$ equals two distinct points $p_1$, $p_2$. Also, $E$ has genus zero so that $\gamma$ separates $E$, hence $M$ as well. Theorem 8 then yields stable surfaces $\Sigma_1$, $\Sigma_2$ satisfying the conditions a through d of Theorem 8.

Let $\Gamma \subset S_\infty = \{x_3 = 0\} \cup \{\infty\}$, be a circle separating $p_1$ and $p_2$. Note that $\Sigma_1$ and $\Sigma_2$ are stable so their curvature is small far from $\gamma$. In particular, when $c$ is the constant of Theorem 6, there is a $c_0 > 0$ such that $|K(q)| < c$ for $q \in \Sigma = \Sigma_1 \cup \Sigma_2$, $\text{dist}(q, \gamma) \geq c_0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure12.png}
\caption{Figure 12}
\end{figure}
Let $T$ be those points of $\mathbb{H}^3$ whose Euclidean distance to $\Gamma$ is at most $c_1 > 0$. Then for $c_1$ sufficiently small, $\Sigma_1 \cap T$ and $\Sigma_2 \cap T$ are vertical graphs over domains $\Omega_1$ and $\Omega_2 \subset \{x_3 = 0\}$. We know $\overline{H} \cdot \overline{e}_3 > 0$ on $G = (\Sigma_1 \cap T) \cup (\Sigma_2 \cap T)$ and $\Sigma_1 \cup \Sigma_2$ bounds a mean convex domain $R$ by Theorem 8, so $\Omega_1 \cap \Omega_2 = \emptyset$. Also we can assume the angle between $\overline{H}(q)$ and $\overline{e}_3$ is less than $\pi/4$ on $G$. Then for $q \in G$, and $t = x_3(q)$ sufficiently small, $G$, near $q$, is a vertical graph over a horizontal disk $D(q)$, centered at $q$, of Euclidean radius $t$.

Let $\tau > 0$ and $\Gamma_\tau = \Gamma + \tau \overline{e}_3$. Choose $\tau$ small so that $\Gamma_\tau \subset T$ and $\Gamma_\tau$ is transverse to $\Sigma$. The linking number of $\Gamma_\tau$ and $\gamma$ is one so that $\Gamma_\tau \cap \Sigma_1$ consists of an odd number of points. Now, $\Sigma = \Sigma_1 \cup \Sigma_2$ bounds the mean convex domain $R$ so that there is an arc of $\Gamma_\tau$, which we denote $(q_1, q_2)$, joining a point $q_1 \in \Sigma_1$ to $q_2 \in \Sigma_2$ and the interior of the arc is in the interior of $R$.

For $q$ on the arc $(q_1, q_2)$, let $J(q)$ be the disk $D(q)$ together with the lower hemisphere of the horosphere that contains $\partial D(q)$ and is vertical along $\partial D(q)$. Note that $J(q)$ has a corner along $\partial D(q)$.

For $q = q_2$, $J(q) \subset \Omega_2 \times \mathbb{R}^+$ by our gradient bound on the graph $G$. Now move $q$ on the arc $(q_1, q_2)$ from $q_2$ to $q_1$. We know that $\Omega_1 \cap \Omega_2 = \emptyset$ so that $J(q_2) \cap \Sigma_1 = \emptyset$. There will be a first $\tilde{q}$ on the arc where $J(q)$ touches $\Sigma_1$. We will next see that $J(q)$ touches $\Sigma_1$ at infinity.

Suppose $J(q)$ first touches $\Sigma_1$ at a smooth point $p$ on the horosphere in $J(q)$. The mean curvature vector of the horosphere points up at $p$, and the mean curvature vector of $\Sigma_1$ points up at $p$ too. So the vectors are equal and $\Sigma_1$ is a horosphere. This is impossible because the proper arc $\gamma$ is on $\Sigma_1$ and $\gamma$ has two points at infinity, $p_1$ and $p_2$; the horosphere has one point at infinity.

Next suppose the first point $p$ where $J(q)$ touches $\Sigma_1$ is on $\partial D(q)$. We know that the horizontal segment in $D(q)$, joining $p$ to $q$ (which we call $[p, q]$) meets $\Sigma_1$ only at $p$. Also this segment does not meet $\Sigma_2$ because our gradient bound implies $[p, q] \subset \Omega_1 \times \mathbb{R}^+$.

Thus the segment $[p, q]$ is contained in $R$. The (Euclidean) tangent plane to $\Sigma_1$ at $p$ is a support plane of $J(q)$ and $\overline{H}(p)$ points up at $p$. This contradicts the fact that $R$ is mean convex: $\overline{H}(p)$ points into $R$, and $[p, q] \subset R$ is on the other side of the tangent plane than $\overline{H}(p)$.

Thus there is a point $q$ on the arc where $J(q)$ touches $\Sigma_1$ for the first time at a point $q_\infty \in \Gamma$; see Figure 13.

![Figure 13](image-url)
Now consider \( q' \) on the arc \((q, q_2)\) at Euclidean distance less than \( \tau \) from \( q \), such that the point of \( \Gamma \) below \( q' \) is not in \( \partial_\infty \Sigma_1 \) but \( q_\infty \) is below \( D(q') \). By Lemma 10, there exists a one-parameter family of vertical graphs \( C(t), 0 < t \leq 1 \), such that \( C(1) \) is the original horosphere of \( J(q') \), and \( C(t) \) \((t < 1)\) is a catenoid cousin end; each \( C(t) \) is vertical along \( \partial C(t) \) and \( \partial C(t) \) is contained in the vertical cylinder containing \( \partial D(q') \). As \( t \to 0 \), \( x_3|_{C(t)} \to 0 \). Since \( \Sigma_1 \) is a graph in this cylinder, \( C(t) \) cannot meet \( \Sigma_1 \) for the first time at a point of \( \partial C(t) \) (where \( C(t) \) is vertical). Also, \( C(t) \) cannot touch \( \Sigma_1 \) at an interior point by the maximum principle, nor at infinity. So \( C(t) \) never touches \( \Sigma_1 \) and \( q_\infty \) cannot be in the asymptotic boundary of \( \Sigma_1 \). This proves Theorem 9.

**Lemma 10.** Let \( C \) be a circle in \( \{x_3 = 0\} \) with center \( q_\infty = (0,0) \). There is a one-parameter family of catenoid cousin (and horosphere) ends \( C(t), 0 < t \leq 1, \) satisfying:

a) each \( C(t) \) is a vertical graph over \( \{0 < x^2 + y^2 < A^2\} \), \( A \) the radius of \( C \),

b) \( C(t) \) is vertical over \( \{x^2 + y^2 = A^2\} \),

c) \( x_3(\partial C(1)) = A, C(1) \) is a horosphere,

d) \( q_\infty = \partial_\infty C(t), \) for each \( t \), and

e) \( x_3(C(t)) \to 0 \) as \( t \to 0 \),

f) \( \bar{H}(C(t)).e_3 \geq 0 \).

**Proof.** J.M. Gomes has proved that a family of this nature exists as graphs over the exterior domain of \( C \) [7]. To get the \( C(t) \) of the lemma, one does inversion of this family through a plane \( P \) with \( \partial_\infty P = C \), followed by a homothety from \( q_\infty \); cf. Figure 14; the homothety takes \( B \) to \( A \). In the appendix we show how these surfaces can be obtained.\[\square\]
Theorem 10. Let $E$ be a properly embedded Bryant annular end. Then $E$ is not dense at infinity, has finite total curvature and is regular. Hence (by Corollary 3) $E$ is asymptotic to a catenoid cousin end or to a horosphere end.

Proof. We remark that the proof of Theorem 9 proves Theorem 10 when $E$ is part of a properly embedded surface $M$ as in Theorem 9. Here is the argument in general.

Let $\Sigma$ be a compact embedded surface such that $\partial \Sigma = \partial E$ and $M = \Sigma \cup E$ is an embedded surface (not necessarily smooth along $\partial E$). Change the metric of $\mathbb{H}^3$ in a compact neighborhood of $\Sigma$ so that $M$ has mean curvature greater than 1 near $\Sigma$. Now prove Theorem 8 with $M$ in this new metric. The $\Sigma_1, \Sigma_2$ one obtains will satisfy all the conditions necessary to do the argument of Theorem 9. What matters is the structure of $\Sigma_1, \Sigma_2$ near infinity. The same argument as in the proof of Theorem 9 then shows $E$ cannot be dense at infinity. Thus, Corollary 3 yields Theorem 10.

Theorem 11. Let $M$ be a properly embedded finite topology, Bryant surface. If $M$ is simply connected (more generally if $M$ has only one end), $M$ is a horosphere. If $M$ has two ends then $M$ is a catenoid cousin. If $M$ has three ends, then $M$ is a bigraph over a plane $P$; i.e., $M$ is invariant by symmetry in $P$ and each component of $M - P$ is a geodesic graph over $P$.

Proof. When $M$ is simply connected, $\partial_\infty M$ is one point by Theorem 10. Then M. do Carmo and B. Lawson [5] proved $M$ is a horosphere. When $M$ has two ends, $\partial_\infty M$ is two points and $M$ is invariant by rotations about the geodesic joining the two points [11]. Thus $M$ is a catenoid cousin. When $M$ has three ends, $\partial_\infty M$ consists of three points so that $\partial_\infty M$ is contained in a circle of $S_\infty$. The conclusion is then proved in [11].

Theorem 12. Let $M$ be a properly embedded Bryant surface, $M$ not a horosphere. Then each annular end of $M$ is asymptotic to a catenoid cousin end.

Proof. We know by Theorem 10, that each annular end $E$ is asymptotic to a catenoid end or to a horosphere end. We will assume $E$ is asymptotic to a horosphere end and obtain a contradiction.

We work in the upper half-space model of $\mathbb{H}^3$, $\{x_3 > 0\}$, and assume $E$ is asymptotic to a horosphere $x_3 = c > 0$. In particular the mean curvature vector of $E$ points up outside of some compact set of $E$. There are no ends of $M$ above $E$ since their mean curvature vector would also point up (each such end is asymptotic to a horizontal horosphere or a catenoid cousin end whose limiting normal points vertically up) and $M$ separates $\mathbb{H}^3$ into two connected components so that no such end is above $E$. 
Then for \( \varepsilon > 0 \), the part \( A \) of \( M \) above \( c + \varepsilon \) is compact. At the highest point of \( A \) (if \( A \) is not empty) the mean curvature vector of \( M \) points down. But this highest point can be joined by an arc in \( \mathbb{H}^3 - M \) to a point of \( E \) where the mean curvature vector points up. Thus \( M \) is completely below \( x_3 = c \).

Let \( \varepsilon > 0 \) and let \( C \) be a small circle in the plane \( x_3 = c - \varepsilon \) so that \( C \) is above \( M \). Just as in the proof of the half-space theorem for properly immersed minimal surfaces in \( \mathbb{H}^3 \) [16], one can take a family of catenoid cousin ends \( C(\lambda), \partial C(1) = C \) with \( C(1) \) above \( M \), where \( C(\lambda) \) converges to the plane \( x_3 = c - \varepsilon \) as \( \lambda \to 0 \). Then some \( C(\lambda) \) touches \( M \) at a point \( q \in M \) and the maximum principle would yield \( M \) equals this catenoid cousin. Thus each end of \( M \) is asymptotic to a catenoid cousin.

**Appendix: The family of graphs of Lemma 10**

Consider the family of vertical catenoids in \( \mathbb{R}^3 \) whose waist circle is of length \( |\lambda| \) and in the \( \{x_3 = 0\} \) plane. Orient by the inner pointing normal. The Weierstrass data on the simply connected covering space \( \mathbb{C} \) are given by \( g(z) = e^z, \omega(z) = |\lambda|e^{-z}dz \), and the metric is \( ds = |\omega| (1 + |g|^2) = 2|\lambda|cosh(x)|dz|, \) \( z = x + iy \).

The cousins of these catenoids (as \( \lambda \) varies) have second fundamental form \( \tilde{II} = II + ds^2 \), and \( II \) is the second fundamental form of the catenoid in \( \mathbb{R}^3 \). The second fundamental form of the catenoid is calculated with respect to the inner pointing normal if \( \lambda > 0 \) and the outer normal for \( \lambda < 0 \).

One can explicitly find the cousins by solving for \( F \) in

\[
F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega.
\]

This is done in [21] and [17], and one obtains in the upper half-space model:

\[
(x_1 + ix_2)(z) = \left[ \frac{\left(\frac{1}{2} - \alpha^2\right)(e^x + e^{-x}) e^{2ax}}{\left(\frac{1}{2} - \alpha\right) e^{-x} + \left(\frac{1}{2} + \alpha\right)^2 e^x} \right] e^{2i\alpha y}
\]

\[
x_3(z) = \frac{2\alpha e^{2ax}}{\left(\frac{1}{2} - \alpha\right)^2 e^{-x} + \left(\frac{1}{2} + \alpha\right)^2 e^x}
\]

where \( \alpha^2 = \frac{1}{4} + \lambda \). This is a surface of revolution for \( \lambda > -\frac{1}{4} \), embedded for \( \lambda > 0 \) and immersed for \( -\frac{1}{4} < \lambda < 0 \).

Let \( a = \frac{1}{2} + \alpha, b = \frac{1}{2} - \alpha \). The generatrix \( \Gamma \) in the \( (x_1, x_3) \) plane of these surfaces of revolution is then
\[ x_1(t) = \frac{ab(e^t + e^{-t})e^{2\alpha t}}{b^2e^{-t} + a^2e^t}, \]
\[ x_3(t) = \frac{2\alpha e^{2\alpha t}}{b^2e^{-t} + a^2e^t}. \]

The points of \( \Gamma \) with vertical tangents are the solutions of \( \frac{\partial x_1}{\partial t} = 0 \) and are the solutions of

\[ a^2e^{2t} - (2ab + 1) + b^2e^{-2t} = 0. \]

The discriminant is then \( \delta^2 = 2(1 - 2\alpha^2) \), so for \( 0 < \alpha < \frac{1}{\sqrt{2}} \) there are two distinct roots \( e^{2\tau}, e^{2\tau'} \) and \( e^{2\tau}e^{2\tau'} = \frac{b^2}{a^2} < 1 \). We take \( \tau < \tau' \), so that \( e^{2\tau} = \frac{2ab + 1 - \delta}{2\alpha^2} \), and \( \tau < 0 \).

For \( \frac{1}{2} < \alpha < \frac{1}{\sqrt{2}} \), one obtains an embedded surface and \( \tau = \tau' \) for \( \alpha = \frac{1}{\sqrt{2}} \); see Figures 15-a and 15-b. For \( 0 < \alpha < \frac{1}{2} \) (\( \lambda < 0 \)), one obtains an immersed surface; cf. Figure 15-c.
We are interested in the case \(0 < \alpha < \frac{1}{2}\). For \(-\infty < t < \tau\), \(\Gamma\) is a graph over an interval \((0, x_1(\tau))\). Since we want a graph over a fixed interval \((0, A)\), we renormalize by a hyperbolic isometry which is homothety from the origin.

More precisely, let \(G_\alpha\) be the graph over \((0, A)\), defined for \(-\infty < t < \tau\). We have on \(G_\alpha\):

\[
\begin{align*}
    x_1(t) &= \frac{A}{x_1(\tau)} \frac{ab (e^t + e^{-t}) e^{2\alpha t}}{b^2 e^{-t} + a^2 e^t}, \\
    x_3(t) &= \frac{A}{x_1(\tau)} \frac{2\alpha e^{2\alpha t}}{b^2 e^{-t} + a^2 e^t}.
\end{align*}
\]

Hence

\[
\frac{x_3(t)}{x_1(t)} = \frac{\alpha}{ab \cosh(t)} \leq \frac{\alpha}{ab \cosh(\tau)},
\]

since \(t < \tau < 0\). It is easy to see that \(\lim_{\alpha \to 0} \left( \frac{\alpha}{ab \cosh(\tau)} \right) = 0\); hence the graphs limit to \((0, A) \times \{0\}\) as \(\alpha \to 0\), as desired.

---

**References**

[1] H. ALENCAR and H. ROSENBERG, Some remarks on the existence of hypersurfaces of constant mean curvature with a given boundary, or asymptotic boundary, in hyperbolic space, *Bull. Sci. Math.* **121** (1997), 61–69.

[2] R. BRYANT, Surfaces of mean curvature one in hyperbolic space, *Astérisque* **154–155** (1987), Soc. Math de France, 321–347.

[3] P. COLLIN, Topologie et courbure des surfaces minimales proprement plongées de \(\mathbb{R}^3\), *Ann. of Math.* **145** (1997), 1–31.

[4] P. COLLIN, L. HAUSWIRTH, and H. ROSENBERG, The gaussian image of mean curvature-one surfaces in \(\mathbb{H}^3\) of finite total curvature, to appear.

[5] M. DO CARMO and H. B. LAWSON Jr., On Alexander-Bernstein theorems in hyperbolic space, *Duke Math. J.* **50** (1983), 995–1003.

[6] H. FUJIMOTO, On the number of exceptional values of the Gauss maps of minimal surfaces, *J. Math. Soc. Japan* **40** (1988), 235–247.

[7] J. M. GOMES, Spherical surfaces with constant mean curvature in hyperbolic space, *Bol. Soc. Brasil. Mat.* **18** (1987), 49–73.

[8] N. KAPOULEAS, Complete constant mean curvature surfaces in Euclidean three-space, *Ann. of Math.* **131** (1990), 239–330.

[9] N. KOREVAAR, R. KUSNER, W. H. MEeks III, and B. SOLOMON, Constant mean curvature surfaces in hyperbolic space, *Amer. J. Math.* **114** (1992), 1–43.
[10] N. Korevaar, R. Kusner, and B. Solomon, The structure of complete embedded surfaces with constant mean curvature, *J. Differential Geom.* **30** (1989), 465–503.

[11] G. Levitt and H. Rosenberg, Symmetry of constant mean curvature hypersurfaces in hyperbolic space, *Duke Math. J.* **52** (1985), 53–59.

[12] W. H. Meeks III, The topology and geometry of embedded surfaces of constant mean curvature, *J. Differential Geom.* **27** (1988), 539–552.

[13] W. H. Meeks III and H. Rosenberg, The geometry and conformal structure of properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$, *Invent. Math.* **114** (1993), 625–639.

[14] R. Osserman, *A Survey of Minimal Surfaces*, Van Nostrand Reinhold Co., N.Y., 1969, second edition, Dover Publ. Inc., N.Y., 1986.

[15] R. Sa Earp and E. Toubiana, Remarks on the geometry of constant mean curvature-one surfaces in hyperbolic space, *Illinois J. Math.*, to appear.

[16] L. Rodriguez and H. Rosenberg, Half-space theorems for mean curvature one surfaces in hyperbolic space, *Proc. A.M.S.* **126** (1998), 2755–2762.

[17] H. Rosenberg, Bryant surfaces, to appear.

[18] W. Rossman, M. Umehara, and K. Yamada, Irreducible constant mean curvature-1 surfaces in hyperbolic space with positive genus, *Tohoku Math. J.* **49** (1997), 449–484.

[19] R. Schoen, Uniqueness, symmetry, and embeddedness of minimal surfaces, *J. Differential Geom.* **18** (1983), 791–809.

[20] M. Umehara and K. Yamada, A duality on CMC-1 surfaces in hyperbolic space and a hyperbolic analogue of the Osserman inequality, *Tsukuba J. Math.* **21** (1997), 229–237.

[21] M. Umehara and K. Yamada, Complete surfaces of constant mean curvature 1 in the hyperbolic 3-space, *Ann. of Math.* **137** (1993), 611–638.

[22] Zu-Huan Yu, The value distribution of the hyperbolic Gauss map, *Proc. A.M.S.* **125** (1997), 2997–3001.

(Received February 10, 1999)

(Revised August 25, 2000)