INVARINAT OF NONCOMMUTATIVE PROJECTIVE SCHEMES

GONÇALO TABUADA

Abstract. In this note we compute several invariants (e.g. algebraic K-theory, cyclic homology and topological Hochschild homology) of the noncommutative projective schemes associated to Koszul algebras of finite global dimension.

1. Introduction

Noncommutative projective schemes. Let \( k \) be a field and \( A = \bigoplus_{n \geq 0} A_n \) a \( \mathbb{N} \)-graded Noetherian \( k \)-algebra. Throughout the note, we will always assume that \( A \) is connected, i.e. \( A_0 = k \), and locally finite-dimensional, i.e. \( \operatorname{dim}_k(A_n) < \infty \) for every \( n \). Following Manin \cite{12}, Gabriel \cite{6}, Artin-Zhang \cite{1}, and others, the noncommutative projective scheme \( \mathfrak{qgr}(A) \) associated to \( A \) is defined as the quotient category \( \operatorname{gr}(A)/\operatorname{tors}(A) \), where \( \operatorname{gr}(A) \) stands for the abelian category of finitely generated \( \mathbb{Z} \)-graded (right) \( A \)-modules and \( \operatorname{tors}(A) \) for the Serre subcategory of torsion \( A \)-modules. This definition was motivated by Serre's celebrated result \cite[19, Prop. 7.8]{19}, which asserts that in the particular case where \( A \) is commutative and generated by elements of degree 1 the quotient category \( \mathfrak{qgr}(A) \) is equivalent to the abelian category of coherent \( \mathcal{O}_{\operatorname{Proj}(A)} \)-modules \( \operatorname{coh}(\operatorname{Proj}(A)) \). For example, when \( A \) is the polynomial \( k \)-algebra \( k[x_1, \ldots, x_d] \), with \( \deg(x_i) = 1 \), we have the following equivalence \( \mathfrak{qgr}(k[x_1, \ldots, x_d]) \simeq \operatorname{coh}(\mathbb{P}^{d-1}) \).

Invariants of dg categories. A dg category \( \mathcal{A} \) is a category enriched over complexes of \( k \)-vector spaces; consult Keller's survey \cite{9}. Every (dg) \( k \)-algebra \( B \) gives naturally rise to a dg category with a single object. Another source of examples is provided by exact categories since the bounded derived category \( D^b(\mathcal{E}) \) of every exact category \( \mathcal{E} \) admits a canonical dg enhancement \( D^b_{\operatorname{dg}}(\mathcal{E}) \); see \cite[§4.4]{9}. In what follows, we will denote by \( \operatorname{dgcat}(k) \) the category of dg categories and dg functors. A functor \( \mathcal{E} : \operatorname{dgcat}(k) \to T \), with values in a triangulated category, is called:

(i) Morita invariant if it inverts the Morita equivalences; see \cite[§4.6]{9}.

(ii) Localizing if it sends short exact sequences of dg categories, in the sense of Drinfeld/Keller (see \cite[3][9, §4.6]), to distinguished triangles:

\[
\begin{array}{c}
0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0
\rightarrow E(A) \longrightarrow E(B) \longrightarrow E(C) \longrightarrow \Sigma E(A)
\end{array}
\]

(iii) Co-continuous if it preserves sequential (homotopy) colimits.

Examples of functors satisfying the conditions (i)-(iii) include nonconnective algebraic K-theory \( K \), homotopy K-theory \( KH \), étale K-theory \( K^{\text{et}} \), the mixed complex \( C \), Hochschild homology \( HH \), cyclic homology \( HC \), and topological Hochschild homology.

\textit{Date:} November 5, 2018.

2010 \textit{Mathematics Subject Classification.} 14A22, 14N05, 16S37, 19D55, 19E08.

\textit{Key words and phrases.} Noncommutative algebraic geometry, projective geometry, Koszul duality, algebraic K-theory, cyclic homology and its variants, topological Hochschild homology.

The author was partially supported by a NSF CAREER Award.
homology $THH$; see [22, §8.2]. Some other functors such as periodic cyclic homology $HP$ and negative cyclic homology $HN$ only satisfy conditions (i)-(ii). When applied to $B$, resp. to $D^b_{dg}(E)$, all the preceding invariants of dg categories reduce to the corresponding invariants of the (dg) $k$-algebra $B$, resp. of the exact category $E$.

**Notation 1.1.** Given a functor $E: dgcat(k) \to T$, an object $o \in T$, an integer $m \in \mathbb{Z}$, and a dg category $\mathcal{A}$, let us write $E^m_o(\mathcal{A}) := \text{Hom}_T(\Sigma^m(o), E(\mathcal{A}))$. Whenever $T$ is symmetric monoidal with $\otimes$-unit $1$, we will write $E_m(\mathcal{A})$ instead of $E^1_m(\mathcal{A})$.

**Statement of results.** Let $k$ be a field and $A = \bigoplus_{n \geq 0} A_n$ a $\mathbb{N}$-graded Noetherian $k$-algebra. Assume that $A$ is Koszul and has finite global dimension $d$. Under these assumptions, the Hilbert series $h_A(t) := \sum_{n \geq 0} \dim_k(A_n)t^n \in \mathbb{Z}[t]$ is invertible and its inverse $h_A(t)^{-1}$ is a polynomial $1 - \beta_1 t + \beta_2 t^2 - \cdots + (-1)^d \beta_d t^d$ of degree $d$, where $\beta_i$ stands for the dimension of the $k$-vector space $\text{Tor}^A_i(k, k)$ (or $\text{Ext}^i_A(k, k)$).

In what follows, we write $\beta := \beta_d$. Our main result is the following computation:

**Theorem 1.2.** Let $A$ be a $k$-algebra as above and $E: dgcat(k) \to T$ a functor satisfying conditions (i)-(iii). Assume that $T$ is $R$-linear for a commutative ring $R$.

(i) For every compact object $o \in T$, we have $R$-module isomorphisms

$$E^m_o(D^b_{dg}(\text{qgr}(A))) \simeq R[t]/\langle h'_A(t)^{-1} \rangle \otimes_R E^m_o(k), \quad m \in \mathbb{Z},$$

where $h'_A(t)^{-1} = 1 - \beta'_1 t + \beta'_2 t^2 - \cdots + (-1)^{d'} \beta'_d t^{d'}$ stands for the image of the polynomial $h_A(t)^{-1}$ in $R[t]$.

(ii) Assume moreover that $1/\beta' \in R$ and that $T$ is compactly generated. Under these assumptions, we have an isomorphism $E(D^b_{dg}(\text{qgr}(A))) \simeq E(k)^{\otimes d'}$.

**Remark 1.4.** (i) If $\beta = 1$, then $\beta' = \beta$ and $d' = d$. As proved in [21, Cor. 0.2], in the particular case where $d = 3$, we always have $h_A(t)^{-1} = (1 - t)^3$.

(ii) If $R$ is a field, then $1/\beta' \in R$. Moreover, $\beta' = \beta$ and $d' = d$ if and only if the characteristic of $R$ does not divide $\beta$.

**Corollary 1.5.** Let $A$ be a $k$-algebra as above and $E: dgcat(k) \to T$ a functor satisfying conditions (i)-(iii). Assume moreover that $T$ is compactly generated.

Under these assumptions, we have an isomorphism $E(D^b_{dg}(\text{qgr}(A)))_{1/\beta'} \simeq E(k)^{\otimes d'}$ in the $\mathbb{Z}[1/\beta']$-linearized triangulated category $T_{1/\beta'}$.

**Proof.** By construction, the triangulated category $T_{1/\beta'}$ is compactly generated and $R[1/\beta']$-linear. Moreover, the $\mathbb{Z}[1/\beta']$-linearization functor $(-)_{1/\beta'}: T \to T_{1/\beta'}$ is triangulated and preserves arbitrary direct sums. Therefore, the proof follows from Theorem 1.2(ii) applied to $E = E(-)_{1/\beta'}$ (with $R = R[1/\beta']$).

**Example 1.6** (Algebraic $K$-theory). Nonconnective algebraic $K$-theory gives rise to a functor $\mathcal{K}: dgcat(k) \to \text{Ho}(\text{Spt})$, with values in the homotopy category of spectra, satisfying conditions (i)-(iii); see [22, §8.2.1]. Therefore, by applying Theorem 1.2(i) to $E = \mathcal{K}$ (with $R = \mathbb{Z}$) and to the sphere spectrum $o = S$, we obtain isomorphisms

$$(1.7) \quad \mathcal{K}_m(\text{qgr}(A)) \simeq \mathbb{Z}[t]/\langle h_A(t)^{-1} \rangle \otimes_{\mathbb{Z}} \mathcal{K}_m(k), \quad m \in \mathbb{Z}.$$

1Let $\mathcal{G}$ be a set of compact generators of $T$. Recall that $T_{1/\beta'}$ may be defined as the Verdier quotient of $T$ by the smallest localizing (=closed under arbitrary direct sums) triangulated subcategory containing the objects $\{\text{cone}(\beta \cdot o)\} | o \in \mathcal{G}$.

2In the particular case where $m = 0$, the isomorphism (1.7) was originally established by Mori-Smith in [14, Thm. 2.3].
Moreover, since the triangulated category Ho(Spt) is compactly generated, Corollary 1.5 implies that $K(k)_{01/2} \simeq K(k)_{1/2}$. All the above holds mutatis mutandis with $K$ replaced by $KH$ or $K^a$.

**Example 1.8 (Mixed complex).** Following Kassel [8], a mixed complex is a (right) dg module over the $k$-algebra of dual numbers $\Lambda := k[e]/e^2$ with deg$(e) = -1$ and $d(e) = 0$. The mixed complex gives rise to a functor $C\colon dgcat(k) \to D(\Lambda)$, with values in the derived category of $\Lambda$, satisfying conditions (i)-(iii); see [22, §8.2.4]. Therefore, since the category $D(\Lambda)$ is compactly generated, by applying Theorem 1.2(ii) to $E = C$ (with $R = k$), we obtain an isomorphism $C(qgr(A)) \simeq C(k)_{0d'}$.

**Example 1.9 (Cyclic homology and its variants).** As explained by Keller in [11, §2.2], Hochschild homology $HH$, cyclic homology $HC$, periodic cyclic homology $HP$, and negative cyclic homology $HN$, can be recovered from the mixed complex $C$. Therefore, making use of Example 1.8, we conclude that

$$HH(qgr(A)) \simeq HH(k)_{0d'} \quad HC(qgr(A)) \simeq HC(k)_{0d'}$$

$$HP(qgr(A)) \simeq HP(k)_{0d'} \quad HN(qgr(A)) \simeq HN(k)_{0d'}.$$

**Example 1.10 (Topological Hochschild homology).** Topological Hochschild homology gives rise to a (lax symmetric monoidal) functor $THH\colon dgcat(k) \to Ho(Spt)$ satisfying conditions (i)-(iii); see [22, §8.2.8]. Since the “inclusion of the 0th skeleton” yields a ring homomorphism $k \to THH_0(k)$, the abelian groups $THH_*$ are then naturally equipped with a $k$-linear structure. Therefore, using the fact that the triangulated category $Ho(Spt)$ is (compactly) generated by the sphere spectrum $\mathbb{S}$, an argument similar to the one used in the proof of Theorem 1.2(ii) allows us to conclude that $THH(qgr(A)) \simeq THH(k)_{0d'}$. For example, in the particular where $k = \mathbb{F}_p$, with $p$ a prime number, we have the following isomorphisms:

$$THH_m(qgr(A)) \simeq \begin{cases} (\mathbb{F}_p)_{0d'} & \text{if } m \geq 0 \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively speaking, Theorem 1.2 (as well as Corollary 1.5 and Examples 1.6-1.10) shows that all the different invariants of a noncommutative projective scheme $qgr(A)$ are completely determined by the Hilbert series $h_A(t)$ of $A$.

Theorem 1.2 (as well as Corollary 1.5) may be applied to the following algebras:

**Example 1.11 (Quantum polynomial algebras).** Choose constant elements $q_{ij} \in k^x$ with $1 \leq i < j \leq d$. The following $\mathbb{N}$-graded Noetherian $k$-algebra

$$A := k\langle x_1, \ldots, x_d \rangle / \langle x_j x_i - q_{ij} x_i x_j \mid 1 \leq i < j \leq d \rangle,$$

with deg$(x_i) = 1$, is called the quantum polynomial algebra associated to $q_{ij}$. This algebra is Koszul, has global dimension $d$, and $h_A(t)^{-1} = (1-t)^d$; see [13, §1].

**Example 1.12 (Quantum matrix algebras).** Choose a $q \in k^x$. The $\mathbb{N}$-graded Noetherian $k$-algebra $A$ defined as the quotient of $k\langle x_1, x_2, x_3, x_4 \rangle$ by the relations

$$x_1 x_2 = q x_2 x_1 \quad x_1 x_3 = q x_3 x_1 \quad x_1 x_4 - x_4 x_1 = (q - q^{-1}) x_2 x_3$$

$$x_2 x_3 = x_3 x_2 \quad x_2 x_4 = q x_4 x_2 \quad x_3 x_4 = q x_4 x_3,$$

with deg$(x_i) = 1$, is called the quantum matrix algebra associated to $q$. This algebra is Koszul, has global dimension 4, and $h_A(t)^{-1} = (1-t)^4$; see [13, §1].
Example 1.13 (Sklyanin algebras). Let $C$ be a smooth elliptic curve, $\sigma \in \text{Aut}(C)$ an automorphism given by translation under the group law, and $\mathcal{L}$ a line bundle on $C$ of degree $d \geq 3$. We write $\Gamma_\sigma \subset C \times C$ for the graph of $\sigma$ and $V$ for the $d$-dimensional $k$-vector space $H^0(C, \mathcal{L})$. The $\mathbb{N}$-graded Noetherian $k$-algebra $A := T(V)/R$, where

$$R := H^0(C \times C, (\mathcal{L} \boxtimes \mathcal{L})(-\Gamma_\sigma)) \subset H^0(C \times C, \mathcal{L} \boxtimes \mathcal{L}) = V \otimes V,$$

is called the Sklyanin algebra associated to the triple $(C, \sigma, \mathcal{L})$. This algebra is Koszul, has global dimension $d$, and $h_A(t)^{-1} = (1 - t)^d$; see [4][24, §1].

Example 1.14 (Homogenized enveloping algebras). Let $g$ be a finite dimensional Lie algebra. The following $\mathbb{N}$-graded Noetherian $k$-algebra (z is a new variable)

$$A := T(g \oplus k z)/(\{z \otimes x - x \otimes z \mid x \in g\} \cup \{x \otimes y - y \otimes x - [x, y] \otimes z \mid x, y \in g\}),$$

is called the homogenized enveloping algebra of $g$. This algebra is Koszul, has global dimension $d := \dim(g) + 1$, and $h_A(t)^{-1} = (1 - t)^d$; see [20, §12].

Example 1.15. Let $k$ be an uncountable algebraically closed field. Choose a pair of elements $(\theta, \rho)$ of $k^\times$ which are algebraically independent over the prime field of $k$ and write $\Theta := \frac{\theta}{\rho}$ and $\Delta := \frac{\theta^4}{\rho^5}$. Under these assumptions and notations, the $\mathbb{N}$-graded Noetherian $k$-algebra $A := k\langle x_1, x_2, x_3, x_4 \rangle / \langle f_1, \ldots, f_6 \rangle$, where

$$f_1 := x_1(\Theta x_1 - x_3) + x_3(x_1 - \Theta x_3), \quad f_2 := x_1(\Theta x_2 - x_4) + x_3(x_2 - \Theta x_4),$$

$$f_3 := x_2(\Theta x_1 - x_3) + x_4(x_1 - \Theta x_3), \quad f_4 := x_2(\Theta x_2 - x_4) + x_4(x_2 - \Theta x_4),$$

$$f_5 := x_1(\Delta x_1 - x_2) + x_4(x_1 - \Delta x_2), \quad f_6 := x_1(\Delta x_3 - x_4) + x_4(x_3 - \Delta x_4),$$

is Koszul, has global dimension 4, and $h_A(t)^{-1} = (1 - t)^4$; see [18, Thm. 3.5].

Gorenstein algebras. Recall that a $\mathbb{N}$-graded Noetherian $k$-algebra $A = \bigoplus_{n \geq 0} A_n$ is called Gorenstein, with Gorenstein parameter $l$, if it has finite injective dimension $m$ and $\text{RHom}_A(k, A) \simeq \Sigma^{-m}k(l)$, where $k(l)$ stands for the $\mathbb{Z}$-graded (right) $A$-module $k(l)_n := k_{n+l}$. Let us assume moreover that $A$ has finite global dimension $d$; this implies that $d = m$. Under these assumptions, a remarkable result of Orlov (see [15, Cor. 2.7]) asserts that the bounded derived category $\mathcal{D}^b(\text{qgr}(A))$ admits a full exceptional collection of length $l$. This leads naturally to the following result:

Theorem 1.16. Let $A$ be a $\mathbb{N}$-graded Noetherian $k$-algebra and $E$ a functor satisfying conditions (i)-(ii). Assume that $A$ is Gorenstein, with Gorenstein parameter $l$, and has finite global dimension $d$. Under these assumptions, we have an isomorphism $E(\mathcal{D}^b(\text{qgr}(A))) \simeq E(k)^{\oplus l}$.

Proof. As explained in [22, §2.4.2 and §8.4.5], every functor $E$ satisfying conditions (i)-(ii) sends a full exceptional collections of length $l$ to the direct sum $E(k)^{\oplus l}$. □

Remark 1.17. (i) Since $A$ is connected and has finite global dimension, the Hilbert series $h_A(t)$ is invertible and its inverse $h_A(t)^{-1}$ is a polynomial. Moreover, the Gorenstein condition implies that $h_A(t)^{-1}$ is monic and has degree $l$.

(ii) As proved in [16, Chap. 2 Thm. 2.5], $A$ is moreover Koszul if and only if $d = l$.

Note that Theorem 1.2 does not follows from Theorem 1.16 because, in general, Koszulness does not implies Gorensteinness. For instance, the algebras $A$ of Example 1.15 are Koszul but not Gorenstein; see [18, Thm. 3.5]. In this latter example,

More generally, condition (ii) can be replaced by additivity in the sense of [22, Def. 2.1].

In the particular case where $d = 3$, Koszulness indeed implies Gorensteinness; see [21, Cor. 0.2].
we have moreover $\dim_k(\operatorname{Ext}_i^j(k, A)) = \infty$ for $i = 2, 3, 4$; see [18, Prop. 5.11]. Consequently, the $k$-linear triangulated categories $\mathcal{D}^b(\operatorname{qgr}(A))$ are not even Ext-finite.

2. Proof of Theorem 1.2

Recall from Quillen [17, §2] that an exact category $\mathcal{E}$ is an additive category equipped with a family of short exact sequences satisfying some standard conditions. In order to simplify the exposition, given an exact functor $F: \mathcal{E} \to \mathcal{E}'$, we will still denote by $F: \mathcal{D}^b_{\operatorname{dg}}(\mathcal{E}) \to \mathcal{D}^b_{\operatorname{dg}}(\mathcal{E}')$ the induced dg functor. We start with the following general result of independent interest:

**Proposition 2.1.** Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be a short exact sequence of exact functors $F_1, F_2, F_3: \mathcal{E} \to \mathcal{E}'$. Given any localizing functor $E: \operatorname{dgc}(k) \to \mathcal{T}$, we have the following equality $E(F_2) = E(F_1) + E(F_3)$.

**Proof.** Let $\operatorname{Ex}(\mathcal{E}')$ be the category of short exact sequences $\varepsilon = (a \to b \to c)$ in $\mathcal{E}'$; this is also an exact category with short exact sequence defined componentwise. By construction, $\operatorname{Ex}(\mathcal{E}')$ comes equipped with the following exact functors

$$
\iota_1: \mathcal{E}' \longrightarrow \operatorname{Ex}(\mathcal{E}') \quad a \mapsto (a \to a \to 0)
$$

$$
\iota_2: \mathcal{E}' \longrightarrow \operatorname{Ex}(\mathcal{E}') \quad a \mapsto (0 \to a \to a)
$$

$$
\pi_1: \operatorname{Ex}(\mathcal{E}') \xrightarrow{\varepsilon \mapsto a} \mathcal{E}' \quad \pi_2: \operatorname{Ex}(\mathcal{E}') \xrightarrow{\varepsilon \mapsto b} \mathcal{E}' \quad \pi_3: \operatorname{Ex}(\mathcal{E}') \xrightarrow{\varepsilon \mapsto c} \mathcal{E}'
$$

satisfying the equalities $\pi_1 \circ \iota_1 = \pi_2 \circ \iota_1 = \text{id}$, $\pi_3 \circ \iota_1 = \pi_1 \circ \iota_2 = 0$, and $\pi_2 \circ \iota_2 = \pi_3 \circ \iota_2 = \text{id}$. Moreover, we have the following short exact sequence of dg categories

$$
0 \longrightarrow \mathcal{D}^b_{\operatorname{dg}}(\mathcal{E}') \xrightarrow{\iota_1} \mathcal{D}^b_{\operatorname{dg}}(\operatorname{Ex}(\mathcal{E}')) \xrightarrow{\pi_3} \mathcal{D}^b_{\operatorname{dg}}(\mathcal{E}') \longrightarrow 0
$$

and consequently the following distinguished triangle

$$
E(\mathcal{D}^b_{\operatorname{dg}}(\mathcal{E}')) \xrightarrow{E(\iota_1)} E(\mathcal{D}^b_{\operatorname{dg}}(\operatorname{Ex}(\mathcal{E}'))) \xrightarrow{E(\pi_3)} E(\mathcal{D}^b_{\operatorname{dg}}(\mathcal{E}')) \xrightarrow{\partial} \Sigma E(\mathcal{D}^b_{\operatorname{dg}}(\mathcal{E}')).
$$

Since $\pi_3 \circ \iota_2 = \text{id}$, the preceding triangle splits and induces an isomorphism

$$
E(\mathcal{D}^b_{\operatorname{dg}}(\mathcal{E}')) \xrightarrow{E(\iota_1)} E(\mathcal{D}^b_{\operatorname{dg}}(\operatorname{Ex}(\mathcal{E}'))) \oplus E(\mathcal{D}^b_{\operatorname{dg}}(\mathcal{E}')) \xrightarrow{\Sigma} E(\mathcal{D}^b_{\operatorname{dg}}(\operatorname{Ex}(\mathcal{E}'))).
$$

Note that a short exact sequence of exact functors $0 \to F_1 \to F_2 \to F_3 \to 0$ is the same data as an exact functor $F: \mathcal{E} \to \operatorname{Ex}(\mathcal{E}')$. Therefore, by combining the equalities $E(\pi_2) \circ [E(\iota_1) \circ E(\iota_2) \circ \text{id}] = [\text{id} \circ \text{id}]$ and $E(\pi_2) \circ [E(\iota_1) \circ E(\iota_2)] = [\text{id} \circ \text{id}]$, we conclude that $E(\pi_2) = E(\pi_1) + E(\pi_3)$. The proof follows now from the equalities $\pi_1 \circ F = F_1$, $\pi_2 \circ F = F_2$, and $\pi_3 \circ F = F_3$. □

Let $B = \bigoplus_{n \geq 0} B_n$ be a $\mathbb{N}$-graded $k$-algebra and $\operatorname{grproj}(B)$ the exact category of finitely generated projective $\mathbb{Z}$-graded (right) $B$-modules. The following general computation is also of independent interest:

**Proposition 2.3.** We have an isomorphism $E(\mathcal{D}^b_{\operatorname{dg}}(\operatorname{grproj}(B))) \simeq \bigoplus_{n \geq 0} E(B_n)$.

**Proof.** Consider $B_0$ as an $\mathbb{N}$-graded $k$-algebra concentrated in degree zero. The canonical inclusion $B_0 \to B$ and projection $B \to B_0$ of $\mathbb{N}$-graded $k$-algebras give rise to the following base-change exact functors:

$$
\varphi: \operatorname{grproj}(B_0) \longrightarrow \operatorname{grproj}(B) \quad P \mapsto P \otimes B_0 B
$$

$$
\psi: \operatorname{grproj}(B) \longrightarrow \operatorname{grproj}(B_0) \quad P \mapsto P \otimes_B B_0.
$$

Since $\psi \circ \varphi = \text{id}$, it follows from Lemma 2.5 below that $\varphi$ and $\psi$ give rise to inverse isomorphisms between $E(\mathcal{D}^b_{\operatorname{dg}}(\operatorname{grproj}(B)))$ and $E(\mathcal{D}^b_{\operatorname{dg}}(\operatorname{grproj}(B_0)))$. 
Now, note that we have the following canonical equivalence of exact categories

\[(2.4) \quad \text{grproj}(B_0) \xrightarrow{\sim} \Pi_{n \in \mathbb{Z}} \text{proj}(B_0) \quad P \mapsto \{P_n\}_{n \in \mathbb{Z}},\]

where \(\text{proj}(B_0)\) stands for the exact category of finitely generated projective (right) \(B_0\)-modules. Since the dg category \(\mathcal{D}_{dg}^b(\text{proj}(B_0))\) is Morita equivalent to the \(k\)-algebra \(B_0\) and the functor \(E\) is co-continuous, we then conclude from the equivalence \((2.4)\) that \(E(\mathcal{D}_{dg}^b(\text{grproj}(B_0))) \cong \oplus_{n \in \mathbb{Z}} E(B_0)\). This finishes the proof. \(\square\)

**Lemma 2.5.** The following endomorphism is equal to the identity

\[E(\varphi \circ \psi) : E(\mathcal{D}_{dg}^b(\text{grproj}(B))) \longrightarrow E(\mathcal{D}_{dg}^b(\text{grproj}(B))).\]

**Proof.** Let \(P \in \text{grproj}(B)\). Note first that the exact endofunctor \(\varphi \circ \psi\) of \(\text{grproj}(B)\) is given by \(P \mapsto \bigoplus_{n \in \mathbb{Z}} \psi(P)_n \otimes_{B_0} B(-n)\). Since the functor \(E\) is co-continuous, this yields the following equality

\[(2.6) \quad E(\varphi \circ \psi) = \sum_{n \in \mathbb{Z}} E(\psi(-)_n \otimes_{B_0} B(-n)).\]

Given a finitely generated projective \(\mathbb{Z}\)-graded (right) \(B\)-module \(P\) and an integer \(m \in \mathbb{Z}\), let us write \(F_m(P)\) for the \(\mathbb{Z}\)-graded (right) \(B\)-module \(P\) generated by the elements \(\cup_{n \leq m} P_n\). In the same vein, given an integer \(q \geq 0\), let us denote by \(\text{grproj}_q(B)\) the full subcategory of \(\text{grproj}(B)\) consisting of those \(\mathbb{Z}\)-graded (right) \(B\)-module \(P\) such that \(F_{-(q+1)}(P) = 0\) and \(F_q(P) = P\). Note that by definition we have an exhaustive increasing filtration

\[(2.7) \quad \text{grproj}_0(B) \subset \text{grproj}_1(B) \subset \cdots \subset \text{grproj}_q(B) \subset \cdots \subset \text{grproj}(B).\]

As explained by Quillen in [17, pages 99-100], for every \(m \in \mathbb{Z}\), the assignment \(P \mapsto F_m(P)/F_{m-1}(P)\) is an exact endofunctor of \(\text{grproj}(B)\). Moreover, we have a canonical isomorphism of exact functors between \(\psi(-)_m \otimes_{B_0} B(-m)\) and \(F_m(-)/F_{m-1}(-)\). Consequently, we obtain the following equality

\[(2.8) \quad \sum_{n \in \mathbb{Z}} E(\psi(-)_n \otimes_{B_0} B(-n)) = \sum_{n \in \mathbb{Z}} E(F_n(-)/F_{n-1}(-)).\]

Now, note that every \(\mathbb{Z}\)-graded (right) \(B\)-module \(P \in \text{grproj}_q(B)\) admits a canonical filtration \(0 = F_{-(q+1)}(P) \subset \cdots \subset F_q(P) = P\). This yields a sequence \(0 = F_{-(q+1)}(-) \rightarrow \cdots \rightarrow F_q(-) = \text{id}\) of exact endofunctors of \(\text{grproj}_q(B)\). Consequently, an inductive argument using the above general Proposition 2.1 implies that the sum \(\sum_{n=-q}^q E(F_n(-)/F_{n-1}(-))\) is equal to the identity of \(E(\mathcal{D}_{dg}^b(\text{grproj}_q(B)))\). Finally, using the fact that the above filtration \((2.7)\) of \(\text{grproj}(B)\) is exhaustive and that the functor \(E\) is co-continuous, we hence conclude that

\[(2.9) \quad \sum_{n \in \mathbb{Z}} E(F_n(-)/F_{n-1}(-)) = \text{id}.\]

The proof follows now from the combination of \((2.6)\) with \((2.8)-(2.9)\). \(\square\)

Recall that \(A\) is a (connected and locally finite-dimensional) \(\mathbb{N}\)-graded Noetherian \(k\)-algebra, which we assume to be Koszul and of finite global dimension \(d\).

**Proposition 2.10.** We have a short exact sequence of dg categories

\[(2.11) \quad 0 \longrightarrow \mathcal{D}_{dg}^b(\text{tors}(A)) \longrightarrow \mathcal{D}_{dg}^b(\text{gr}(A)) \longrightarrow \mathcal{D}_{dg}^b(\text{qgr}(A)) \longrightarrow 0.\]
Proof. As explained by Keller in [9, Thm. 4.11], (2.11) is a short exact sequence of dg categories if and only if the associated sequence of triangulated categories

\[ D^b(\text{tors}(A)) \longrightarrow D^b(\text{gr}(A)) \longrightarrow D^b(\text{qgr}(A)) \]

is exact sequence in the sense of Verdier. By definition, we have a short exact sequence of abelian categories \( 0 \to \text{tors}(A) \to \text{gr}(A) \to \text{qgr}(A) \to 0 \). Therefore, thanks to [10, Lem. 1.15] (consult also [7]), in order to show that (2.12) is exact in the sense of Verdier, it suffices to prove the following condition: given a short exact sequence \( 0 \to L \to M \to N \to 0 \) in the abelian category \( \text{gr}(A) \), with \( L \in \text{tors}(A) \), there exists a morphism of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & L' & \longrightarrow & L'' & \longrightarrow & 0 & & \\
\end{array}
\]

with \( L' \) and \( L'' \) belonging to \( \text{tors}(A) \). Recall that the category \( \text{tors}(A) \) of torsion \( A \)-modules is defined as the full subcategory of \( \text{gr}(A) \) consisting of those \( \mathbb{Z} \)-graded (right) \( A \)-modules which are (globally) finite-dimensional over \( k \). Given a \( \mathbb{Z} \)-graded (right) \( A \)-module \( M \) and an integer \( m \in \mathbb{Z} \), let us write \( M_{\geq m} \) for the submodule \( \bigoplus_{n \geq m} M_n \) of \( M \). Since by assumption \( L \) is torsion and \( M \) is finitely generated, there exists an integer \( m \gg 0 \) such that \( L \cap M_{\geq m} = 0 \). Consequently, we can construct the following morphism of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & L & \longrightarrow & M/\langle M_{\geq m} \rangle & \longrightarrow & M/\langle M_{\geq m} \rangle + L & \longrightarrow & 0. \\
\end{array}
\]

The proof follows now from the fact that, by construction, the \( \mathbb{Z} \)-graded (right) \( A \)-modules \( M/\langle M_{\geq m} \rangle \) and \( M/\langle M_{\geq m} + L \rangle \) belong to \( \text{tors}(A) \). \( \square \)

**Remark 2.13.** By assumption, the functor \( E \) is localizing. Therefore, the short exact sequence of dg categories (2.11) gives rise to a distinguished triangle:

\[
E(D^b_{\text{dg}}(\text{tors}(A))) \longrightarrow E(D^b_{\text{dg}}(\text{gr}(A))) \longrightarrow E(D^b_{\text{dg}}(\text{qgr}(A))) \xrightarrow{\partial} \Sigma E(D^b_{\text{dg}}(\text{tors}(A))).
\]

Since \( A \) has finite global dimension, the inclusion of categories \( \text{grproj}(A) \subset \text{gr}(A) \) induces a Morita equivalence \( D^b_{\text{dg}}(\text{grproj}(A)) \to D^b_{\text{dg}}(\text{gr}(A)) \). Therefore, by first using the general Proposition 2.3 (with \( B = A \)) and then by applying the functor \( E \) to the preceding Morita equivalence, we obtain an induced isomorphism

\[
(2.14) \quad \oplus_{-\infty}^{+\infty} E(k) \simeq E(D^b_{\text{dg}}(\text{grproj}(A))) \xrightarrow{\sim} E(D^b_{\text{dg}}(\text{gr}(A))).
\]

**Proposition 2.15.** We have a Morita equivalence

\[
(2.16) \quad D^b_{\text{dg}}(\text{tors}(A)) \longrightarrow D^b_{\text{dg}}(\text{grproj}(A)),
\]

where \( A' \) stands for the Koszul dual \( k \)-algebra of \( A \).

**Proof.** Given a \( \mathbb{N} \)-graded \( k \)-algebra \( B = \bigoplus_{n \geq 0} B_n \), let us denote by \( \text{Gr}(B) \) the category of all \( \mathbb{Z} \)-graded (right) \( B \)-modules and by \( D(\text{Gr}(B)) \) the associated (unbounded) derived category. Following Beilinson-Ginzburg-Soergel [2, §2.12], let \( D^i(\text{Gr}(B)) \), resp. \( D^i(\text{Gr}(B)) \), be the full subcategory of \( D(\text{Gr}(B)) \) consisting of
those cochain complexes of $\mathbb{Z}$-graded (right) $B$-modules $M$ such that for some integer $m \gg 0$ we have $M_n^m \neq 0 \Rightarrow (q \geq -m$ or $q + n \leq m$), resp. $M_n^m \neq 0 \Rightarrow (q \leq -m$ or $q + n \geq -m$). These categories admit canonical dg enhancements $\mathcal{D}_{\mathrm{dg}}(\text{Gr}(B)), \mathcal{D}_{\mathrm{dg}}^+(\text{Gr}(B))$, and $\mathcal{D}_{\mathrm{dg}}^+(\text{Gr}(B))$. Now, recall from [2, Thm. 2.12.1] (consult also [5, §2]) the construction of the Koszul duality dg functor $\mathcal{D}_{\mathrm{dg}}(\text{Gr}(A)) \to \mathcal{D}_{\mathrm{dg}}(\text{Gr}(A^!))$. As proved in loc. cit., this dg functor restricts to a Morita equivalence

$$\mathcal{D}_{\mathrm{dg}}^+(\text{Gr}(A)) \to \mathcal{D}_{\mathrm{dg}}^+(\text{Gr}(A^!))$$

(2.17)

which sends the $\mathbb{Z}$-graded (right) $A$-modules $k(i), i \in \mathbb{Z}$, to the $\mathbb{Z}$-graded (right) $A^!$-modules $\Sigma^{-i}A^!(i), i \in \mathbb{Z}$. Therefore, making use of the general Lemma 2.18 below (with $B = A$ and $B = A^!$), we conclude that (2.17) restricts furthermore to the above Morita equivalence (2.16).

Lemma 2.18. Let $B = \bigoplus_{n \geq 0} B_n$ be a (connected and locally finite-dimensional) N-graded Noetherian $k$-algebra. The smallest thick triangulated subcategory of $\mathcal{D}^b(\text{gr}(B))$ containing the $\mathbb{Z}$-graded (right) $B$-modules $\{k(i) \mid i \in \mathbb{Z}\}$, resp. $\{B(i) \mid i \in \mathbb{Z}\}$, agrees with $\mathcal{D}^b(\text{tors}(B))$, resp. $\mathcal{D}^b(\text{grproj}(B))$.

Proof. Consult the proof of [15, Lem. 2.3]. \(\square\)

Recall that since $A$ is connected, its Koszul dual algebra $A^!$ is also connected. Therefore, by first applying the functor $E$ to (2.16) and then by using the above general Proposition 2.3 (with $B = A^!$), we obtain an induced isomorphism

$$E(\mathcal{D}_{\mathrm{dg}}^b(\text{tors}(A))) \xrightarrow{\cong} E(\mathcal{D}_{\mathrm{dg}}^b(\text{grproj}(A^!))) \simeq \oplus_{i \geq -\infty} E(k).$$

(2.19)

Since $A$ is Koszul and of finite global dimension $d$, we have a linear free resolution

$$0 \to A(-d)^{\oplus \beta_d} \to \cdots \to A(-2)^{\oplus \beta_2} \to A(-1)^{\oplus \beta_1} \to A \to k \to 0$$

of the $\mathbb{Z}$-graded (right) $A$-module $k$. As mentioned in §1, the integer $\beta_i$ agrees with the dimension of the $k$-vector space $\text{Tor}_i^A(k, k)$ (or $\text{Ext}_A^i(k, k)$).

Proposition 2.21. Under the above isomorphisms (2.14) and (2.19), the distinguished triangle of Remark 2.13 identifies with

$$\oplus_{i \geq -\infty} E(k) \xrightarrow{\partial} \oplus_{i \geq -\infty} E(k) \to E(\mathcal{D}_{\mathrm{dg}}^b(\text{gr}(A))) \xrightarrow{\partial} \oplus_{i \geq -\infty} \Sigma E(k),$$

(2.22)

where $M'$ stands for the (infinite) matrix $M'_{ij} := (-1)^j(-1)^{(i-j)}\beta_{i-j}$.

Proof. Let $\text{NMot}(k)$ be the category of noncommutative motives constructed in [22, §8.2]; denoted by $\text{NMot}(k)_{\text{loc}}$ in loc. cit. By construction, this triangulated category comes equipped with a functor $U: \text{dgcat}(k) \to \text{NMot}(k)$ which is initial among all the functors satisfying conditions (i)-(iii). Concretely, given a functor $E: \text{dgcat}(k) \to \mathcal{T}$ satisfying conditions (i)-(iii), there exists a (unique) triangulated functor $\mathcal{E}: \text{NMot}(k) \to \mathcal{T}$ such that $\mathcal{E} \circ U \simeq E$. Moreover, $\mathcal{E}$ preserves arbitrary direct sums; see [22, Thm. 8.5]. This implies that in order to prove Theorem 2.21, it suffices to show that the triangle of Remark 2.13 (with $E = U$) identifies with

$$\oplus_{i \geq -\infty} U(k) \xrightarrow{\partial} \oplus_{i \geq -\infty} U(k) \to U(\mathcal{D}_{\mathrm{dg}}^b(\text{gr}(A))) \xrightarrow{\partial} \oplus_{i \geq -\infty} \Sigma U(k),$$

(2.23)

where $M$ stands for the (infinite) matrix $M_{ij} := (-1)^j(-1)^{(i-j)}\beta_{i-j}$. Recall from [22, §8.6] that, for every dg category $\mathcal{A}$, we have a natural isomorphism

$$\text{Hom}_{\text{NMot}(k)}(U(k), U(A)) \simeq K_0(A).$$
Moreover, \( U(k) \) is a compact object of the triangulated category \( \text{NMot}(k) \). Therefore, since \( K_0(U(k)) \cong \mathbb{Z} \), an endomorphism of \( \oplus_{-\infty}^{+\infty} U(k) \) corresponds to an infinite matrix with integer coefficients in which every column has solely a finite number of non-zero entries. Let us denote by \( M \) the matrix corresponding to \( U(D^b_{\text{dg}}(\text{tors}(A))) \to U(D^b_{\text{dg}}(\text{gr}(A))) \) under the isomorphisms \((2.14)\) and \((2.19)\) (with \( E = U \)). By applying the functor \( \text{Hom}_{\text{NMot}(k)}(U(k), -) \) to the isomorphisms \((2.14)\) and \((2.19)\) (with \( E = U \)), we obtain induced abelian group isomorphisms

\[
\begin{align*}
\oplus_{-\infty}^{+\infty} \mathbb{Z} & \cong K_0(D^b(\text{grproj}(A))) \\
& \cong K_0(D^b(\text{gr}(A)))
\end{align*}
\]

\((2.24)\)

\[
K_0(D^b(\text{tors}(A))) \cong K_0(D^b(\text{grproj}(A'))) \cong \oplus_{-\infty}^{+\infty} \mathbb{Z}.
\]

\((2.25)\)

The element \( 1 \in \mathbb{Z} \), placed at the \( j \)-th component of the direct sum \( \oplus_{-\infty}^{+\infty} \mathbb{Z} \), corresponds under \((2.25)\) to the Grothendieck class \( [\Sigma^{-j} k(-(j))] = (-1)^j[k(-(j))] \in K_0(D^b(\text{tors}(A))). \) In the same vein, the element \( 1 \in \mathbb{Z} \), placed at the \( i \)-th component of the direct sum \( \oplus_{-\infty}^{+\infty} \mathbb{Z} \), corresponds under \((2.24)\) to the Grothendieck class \( [A(\text{(-i)})] \in K_0(D^b(\text{gr}(A))). \) Thanks to the above linear free resolution \((2.20)\), we have moreover the following equality \( [k(\text{(-j)})] = \sum_{i=0}^{d} (-1)^i \beta_i \text{[A(\text{-i-j})]} \) in the Grothendieck group \( K_0(D^b(\text{gr}(A))). \) The above considerations allow us to conclude that the \((i,j)\)-th entry of the matrix \( M \) is given by the integer \((-1)^j(-1)^{(i-j)}\beta_{i,j} \). This finishes the proof.

\[
\square
\]

We now have all the ingredients necessary for the conclusion of the proof of Theorem 1.2(i). Let \( a \in \mathbb{T} \) be a compact object. By applying the functor \( \text{Hom}_T(a, -) \) to the triangle \((2.22)\), we obtain an induced long exact sequence of \( R \)-modules:

\[
\cdots \to \oplus_{-\infty}^{+\infty} E^0_m(k) \xrightarrow{M^t} \oplus_{-\infty}^{+\infty} E^1_m(k) \to E^2_m(D^b_{\text{dg}}(\text{gr}(A))) \xrightarrow{\partial} \oplus_{-\infty}^{+\infty} E^0_{m-1}(k) \to \cdots
\]

\((2.26)\)

Since \( M'^t = (-1)^i(-1)^{(i-j)}\beta'_{i,j} \), with \( \beta'_{0,0} = 1 \) and \( \beta'_{0,r} = 0 \) whenever \( r \notin \{0, \ldots, d'\} \), a simple matrix computation shows that the preceding homomorphism \( M' \) of \( R \)-modules is injective. Consequently, the long exact sequence breaks-up into short exact sequences of \( R \)-modules:

\[
\begin{align*}
0 & \to \oplus_{-\infty}^{+\infty} E^0_m(k) \to \oplus_{-\infty}^{+\infty} E^1_m(k) \to E^2_m(D^b_{\text{dg}}(\text{gr}(A))) \to 0.
\end{align*}
\]

Thanks to Lemma 2.28 below and to the definition of the homomorphism \( \phi \) (see below), we also have the following short exact sequences of \( R \)-modules:

\[
0 \to R[t, t^{-1}] \otimes_R E^0_m(k) \xrightarrow{\phi \otimes \text{id}} R[t, t^{-1}] \otimes_R E^0_m(k) \to R[t]/(h'_A(t)^{-1}) \otimes_R E^0_m(k) \to 0.
\]

Now, consider the Poincaré polynomial \( p_A(t) := \sum_{i=0}^{d} (-1)^i \beta_i t^i \) (and \( p'_A(t) := \sum_{i=0}^{d} (-1)^i \beta'_i t^i \)). Thanks to the linear free resolution \((2.20)\), we have \( h_A(t)^{-1} = p_A(t) \) (and \( h'_A(t)^{-1} = p'_A(t) \)). This implies that under the canonical isomorphism between \( \oplus_{-\infty}^{+\infty} E^0_m(k) \) and \( R[t, t^{-1}] \otimes_R E^0_m(k) \), the matrix \( M' \) corresponds to the homomorphism \( \phi \otimes \text{id} \). Consequently, we obtain induced \( R \)-module isomorphisms

\[
E^2_m(D^b_{\text{dg}}(\text{gr}(A))) \cong R[t]/(h'_A(t)^{-1}) \otimes_R E^0_m(k)
\]

\((2.27)\)

This concludes the proof of Theorem 1.2(i).

**Lemma 2.28.** We have the following short exact sequence of \( R \)-modules

\[
0 \to R[t, t^{-1}] \xrightarrow{\phi} R[t, t^{-1}] \to R[t]/(h'_A(t)^{-1}) \to 0,
\]

where \( \phi \) stands for the homomorphism \( p(t) \mapsto p(-t) \cdot h'_A(t)^{-1} \).
Proof: Since $h_A'(0)^{-1} = 1$, the homomorphism $\phi$ is injective. Moreover, we have the following natural isomorphisms
\[
\text{coker}(\phi) = R[t, t^{-1}]/\text{Im}(\phi) \cong R[t, t^{-1}]/(h_A'(t)^{-1}) \cong R[t]/(h_A'(t)^{-1}),
\]
where (a) follows from the fact that the homomorphisms $\phi$ and $-h_A'(t)^{-1}$ have the same image, and (b) from the fact that the polynomial $t$ is invertible in $R[t]/(h_A'(t)^{-1})$ (this follows from the fact that $h_A'(0)^{-1} = 1$). This concludes the proof. \(\square\)

We now have all the ingredients necessary for the conclusion of the proof of Theorem 1.2(ii). Consider the following composition
\[
\bigoplus_{n=0}^{d-1} E(k) \longrightarrow \bigoplus_{m=1}^{\infty} E(k) \longrightarrow E(D^b_{dg}(\text{qgr}(A))).
\]
By assumption, the triangulated category $\mathcal{T}$ is compactly generated. Therefore, the morphism (2.29) is invertible if and only if for every compact object $o \in \mathcal{T}$ the induced $R$-module homomorphisms
\[
\bigoplus_{n=0}^{d-1} E_m^o(k) \longrightarrow E_m^o(D^b_{dg}(\text{qgr}(A))) \quad m \in \mathbb{Z}
\]
are invertible. Under the canonical identification $\bigoplus_{n=0}^{d-1} R \otimes_R E_m^o(k) \simeq \bigoplus_{n=1}^{d-1} E_m^o(k)$, the composition of (2.30) with (2.27) corresponds to the $R$-module homomorphisms:
\[
\left((1, t, \ldots, t^{d-1}) : \bigoplus_{n=0}^{d-1} R \longrightarrow R[t]/(h_A'(t)^{-1})\right) \otimes_R E_m^o(k) \quad m \in \mathbb{Z}.
\]
By assumption, we have $1/\beta' \in R$. Therefore, the factorization algorithm for polynomials applied to $R[t]$ allows us to conclude that the $R$-module homomorphism $(1, t, \ldots, t^{d-1})$ is invertible. This implies that the induced $R$-module homomorphisms (2.30) are also invertible, and so the proof of Theorem 1.2(ii) is finished.

Acknowledgments: The author is grateful to Michael Artin for useful discussions concerning noncommutative projective schemes and also to Theo Raedschelders for important comments on a previous version of this note.

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Gonçalo Tabuada, Department of Mathematics, MIT, Cambridge, MA 02139, USA

E-mail address: tabuada@math.mit.edu

URL: http://math.mit.edu/~tabuada