Flashing Motor at High Transition Rate

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Abstract

The movement of a Brownian particle in a fluctuating two-state periodic potential is investigated. At high transition rate, we use a perturbation method to obtain the analytical solution of the model. It is found that the net current is a peaked function of thermal noise, barrier height and the fluctuation ratio between the two states. The thermal noise may facilitate the directed motion at a finite intensity. The asymmetry parameter of the potential is sensitive to the direction of the net current.
I. INTRODUCTION

Much of the interest in non-equilibrium-induced transport processes has been on the stochastically driven ratchets [1, 2, 3, 4, 5, 6]. The noise-induced ratchet has recently attracted considerable attentions. It is related to the symmetry breaking and comes from the desire for an explanation of directional transport in biological systems. Several models have been, for example, proposed to describe the muscle contraction [7, 8] and the asymmetric polymerization of action filaments responsible for cell mobility [9].

The focus of research has been on the noise-induced unidirectional motion over the last decade. In these systems directed-Brownian-motion of particles is generated by nonequilibrium noise in the absence of any net macroscopic forces and potential gradients. Ratchets have been proposed to model the unidirectional motion due to the zero-mean nonequilibrium fluctuation. Typical examples are rocking ratchets [10, 11, 12, 13], flashing ratchets [14], diffusion ratchets [15] and correlation ratchets [16]. Ghosh et. al. [17, 18] developed some analytical solutions of the current variation with the noise in the ratchet. In all these studies the potential is taken to be asymmetric in space. It has also been shown that a unidirectional current can appear for spatially symmetric potentials. For the case of spatially symmetric potential, an external random force should be either temporally asymmetric or spatially-dependent.

The previous works are limited to case of single potential. The present work extends the analysis to the case of two potentials in flashing thermal ratchet: one constant potential and one periodic in space and constant in time. No external driving forces are required to induce unidirectional current in these ratchets of two potentials. The emphasis is on the current as the function of noise and other system parameters. This is achieved by using a perturbation method to solve two coupled Simoluchowsky equations.

II. FLASHING MOTOR

Consider the flashing motor (flashing ratchet) model aiming for describing the spatially unidirectional motion along x-direction in Fig. 1 of a Brownian particle due to the two potentials (states): one constant and the other spatially-periodic. This model was initially
proposed in an attempt of describing molecular motor in biological systems [19].

The rate of fluctuation between the two potential states is governed by two rate constants $k_1$ and $k_2$, respectively. Here the former is the rate from State 1 to State 2 and the latter is the rate from State 2 to State 1. While the particle diffuses freely at State 2, it is localized near a local minimum at State 1. The particle motion satisfies the dimensionless equation of motion

$$\frac{dx}{dt} = -\frac{V_i(x)}{dx} + f_{Bi}(t), \quad i = 1, 2,$$

where $f_{Bi}(t)$ is the Brownian random force, $x$ the position of the particle, $t$ the time, $V_i(x)$ the potential, the subscript $i$ stands for the state which can take value of 1 or 2. The probability densities $P_i(x, t)(i = 1, 2)$ of state 1 and 2 are governed by two coupled Smoluchowsky equations [2]:

$$\frac{\partial P_1(x, t)}{\partial t} = D \frac{\partial^2 P_1(x, t)}{\partial x^2} + k_1 P_1(x, t) - k_2 P_2(x, t) = -\frac{\partial J_1}{\partial x},$$

$$\frac{\partial P_2(x, t)}{\partial t} = D \frac{\partial^2 P_2(x, t)}{\partial x^2} + k_2 P_1(x, t) - k_1 P_2(x, t) = -\frac{\partial J_2}{\partial x},$$

where $f(x) = -V_1'(x)$, the prime stands for the derivative with respect to $x$, $J_1, J_2$ are probability densities of current, $D$ the diffusivity, $k_B$ Boltzmann constant, $T$ the absolute temperature. Here $x, t, k_1, k_2, D, k_B T$ are all dimensionless. Also,

$$V_1(x) = \begin{cases} \frac{V_0}{\lambda}(x - mL), & mL < x \leq mL + \lambda; \\ \frac{V_0}{\lambda}[-x + (m + 1)L], & mL + \lambda < x \leq (m + 1)L, \end{cases}$$

where $m = 0, 1, 2, \ldots$. At steady state such that $\frac{\partial P_1}{\partial t} = 0$ and $\frac{\partial P_2}{\partial t} = 0$, Eqs (2) and (3) lead to the net current

$$J = -D \frac{\partial P(x)}{\partial x} + D_1 f(x) P_1(x),$$

$$D[P''(x) - P_1''(x)] + k[(1 + \mu)P_1(x) - \mu P(x)] = 0,$$

where

$$J = J_1 + J_2, P(x) = P_1(x) + P_2(x), D_1 = \frac{D}{k_B T},$$

and

$$k = k_1, \mu = k_2/k.$$
FIG. 1: Two-state model with regular ratchets: $V_1(x)$ is spatially-periodic sawtooth of period $L$ and barrier height $V_0$. $\lambda$ is an asymmetry parameter ($V_1(x)$ is symmetric when $\lambda = L/2$); $V_2(x)$ is a constant potential.

III. ANALYTICAL SOLUTION

When the fluctuation is at high rate such that $k \gg 1$, we can expand $P(x), P_1(x)$ and $J$ in power series of a small parameters $k^{-1}$,

$$P(x) = \sum_{n=0}^{\infty} k^{-n} p_n(x), P_1(x) = \sum_{n=0}^{\infty} k^{-n} p_{1n}(x), J = \sum_{n=0}^{\infty} k^{-n} j_n.$$  \hspace{1cm} (9)

The coefficients $p_n, p_{1n}$ and $j_n$ can be obtained by substituting Eq.(9) into Eqs (5) and (6) and equating coefficients of $k^{-n}$,

$$p_0'(x) - \frac{\mu D_1}{(1 + \mu) D} f(x)p_0(x) = -\frac{j_0}{D}, \hspace{1cm} \text{(10)}$$

$$p_{10}(x) = \frac{\mu}{1 + \mu} p_0(x), \hspace{1cm} \text{(11)}$$

$$- D p_n'(x) + \frac{\mu D_1}{1 + \mu} f(x)p_n(x) = j_n + G_{n-1}(x), n = 1, 2, 3, ..., \hspace{1cm} \text{(12)}$$

$$G_n(x) = \frac{DD_1}{1 + \mu} f(x)[p_n''(x) - p_{1n}''(x)], n = 0, 1, 2, .... \hspace{1cm} \text{(13)}$$
Under the periodicity conditions

\[ p_n(x + L) = p_n(x), n = 0, 1, 2, \ldots \quad (14) \]

and the normalization of the distribution \( p(x) \) over the period \( L \),

\[ \int_0^L p_n(x) \, dx = \delta_{0n}, n = 0, 1, 2, \ldots, \quad (15) \]

we can obtain all coefficients of \( p_n, p_{1n} \) and \( j_n \). Since our attention is mainly on the current \( J \), we only list \( j_n(x) \) here

\[ j_0 = 0 \]

\[ j_n = -\frac{\int_0^L G_{n-1}(x) U^{-1}(x) \, dx}{\int_0^L U^{-1}(x) \, dx}, \quad n = 1, 2, \ldots \quad (16) \]

where

\[ U(x) = \exp[-\frac{\mu D_1}{(1 + \mu)D} V_1(x)]. \quad (17) \]

In particular,

\[ j_1 = -\frac{\mu^2 D_1^2}{(1 + \mu)^2 D} \frac{\int_0^L f^3(x) \, dx}{\int_0^L U(x) \, dx \int_0^L U^{-1}(x) \, dx}. \quad (18) \]

Therefore, to the first-order approximation,

\[ J \simeq j_0 + k^{-1} j_1 = -\frac{\mu^2 D_1^2}{k(1 + \mu)^4 D} \frac{\int_0^L f^3(x) \, dx}{\int_0^L U(x) \, dx \int_0^L U^{-1}(x) \, dx}. \quad (19) \]

After substituting \( f(x) \) and \( U(x) \), we have

\[ J \simeq -\frac{\mu^4 D^2 V_0^5 (2\lambda - L)}{(1 + \mu)^6 \beta^5 k L \lambda^2 (L - \lambda)^2 (e^{\mu + \mu \beta V_0} + e^{-\mu + \mu \beta V_0} - 2)}, \quad (20) \]

where \( \beta = k_B T \). By letting \( \frac{\partial J}{\partial \gamma} = 0 \) with \( \gamma = \frac{\beta}{V_0} \), we have

\[ (5\gamma - \frac{\mu}{1 + \mu}) \exp\left[\frac{\mu}{(1 + \mu)\gamma}\right] + (5\gamma + \frac{\mu}{1 + \mu}) \exp\left[-\frac{\mu}{(1 + \mu)\gamma}\right] - 10\gamma = 0, \quad (21) \]

which leads to the optimum \( \gamma \) for the maximum \( J \) \( (J_{\text{max}, \gamma}) \).

By letting \( \frac{\partial J}{\partial \mu} = 0 \), similarly, we have the optimum \( \mu \) for the maximum \( J \) \( (J_{\text{max}, \mu}) \),

\[ [-2\mu^2 + (2 - \frac{1}{\gamma})\mu + 4] \exp\left[\frac{\mu}{(1 + \mu)\gamma}\right] + [-2\mu^2 + (2 + \frac{1}{\gamma})\mu + 4] \exp\left[-\frac{\mu}{(1 + \mu)\gamma}\right] + 4(\mu^2 - \mu - 2) = 0. \quad (22) \]
FIG. 2: Plot of the optimum $\gamma$ vs the optimum $\mu$ for the maximum $J$ ($V_0 = 5.0$, $D = 1.0$, $L = 2.0$, $k = 100.0$, $\lambda = 0.5$).

FIG. 3: Plot of the maximum $J$ shown in Eqns. (21) and (22). (a) Plot of the maximum $J$ with the optimum $\gamma$ vs $\mu$. (b) Plot of the maximum $J$ with the optimum $\mu$ vs $\gamma$ ($V_0 = 5.0$, $D = 1.0$, $L = 2.0$, $k = 100.0$, $\lambda = 0.5$).

IV. RESULTS AND DISCUSSION

Equation (20) indicates that the direction of the net current is determined by the asymmetry parameter $\lambda$. When $0 < \lambda < 1$, the current is positive, the current is negative when
1 < \lambda < 2. There is no current at \lambda = 1.

Figure 2 shows the solution of the Eqns (21) and (22). The dot line gives the optimum \gamma for the maximum \( J \) (Eq. (21)) and the solid line gives the optimum \( \mu \) for the maximum \( J \) (Eq. (22)). It is easy to find that the dot line meets the solid line at the point A \((\mu_{opt},\gamma_{opt})\), which indicates that one can obtain the maximum \( J \) for both the optimum \( \gamma \) and the optimum \( \mu \) at the same time. The corresponding \( J_{\max,\gamma} \) vs \( \mu \) and \( J_{\max,\mu} \) vs \( \gamma \) are shown in Fig. 3a and Fig. 3b, respectively. From Fig. 3a, we can find \( J_{\max,\gamma} \) as the function of \( \mu \) have a maximum value, at which the \( \mu \) and \( \gamma \) are optimal, namely, the solid line will meet the dot line as shown in Fig. 2. The similar results can also be obtained in Fig. 3b.

FIG. 4: Dimensionless probability current \( J \) vs thermal noise strength \( k_B T \) for different values of asymmetric parameters \((V_0 = 5.0, D = 1.0, L = 2.0, k = 100.0, \mu = 1.0)\).

Figure 4 shows the variation of the net current \( J \) with the thermal noise intensity \( k_B T \). The curve is observed to be bell-shaped, a feature of resonance. The current reversal appears at \( \lambda = 1 \) at which the potential \( V_1(x) \) is symmetry. When \( k_B T \to 0 \), \( J \) tends to zero for all values of \( \lambda \). Therefore, there are no transitions out of the wells when the thermal noise vanishes. When \( k_B T \to \infty \) so that the thermal noise is very large, the ratchet effect also disappear. The current \( |J| \) has a maximum value for fixed value \( \lambda \) at certain value of \( k_B T \). By Eq. (21), the optimized value is \( k_B T = 0.5073(\gamma = 0.1015, V_0 = 5.0) \). The maximum
$J_{\text{max}} = 0.0949$ at $\lambda = 0.5$. Therefore, certain thermal noise can induce a large current $|J|$, while the thermal noise blocks the unidirectional motion in general.

Figure 5 shows the net current as the function of barrier height $V_0$. We again observe the current reversals at $\lambda = 1.0$. When the barrier height $V_0$ is small, the effect of the ratchet is also small; the thermal noise effect is dominant so that the net current disappear. When the barrier height $V_0$ is large, on the other hand, the particle can not pass the barrier. It can only diffuse at State 1 so that the net current is also very small. Therefore, there is an optimized value of $V_0$ (4.9281) at which $J$ takes its maximum value ($J_{\text{max}} = 0.0949$), for example, $\lambda = 0.5$.

Figure 6 shows the current as the function of $\mu$. When $\mu \to 0$, $k_2$ tends to zero. The attraction by State 2 is too small such that the particle is always staying at State 1. The ratchet reduces to one-state ratchet without external force. Therefore, no current exits. When $\mu \gg 1$, the attraction by State 1 becomes too small so that the particle can only diffuse at State 2. The net current becomes to zero again. Hence, there exits an optimized value of $\mu$ (0.3991) at which $J$ takes its maximum value ($J_{\text{max}} = 0.1954$) as shown in Eq. (22).
FIG. 6: Dimensionless probability current $J$ vs the ratio of the two state transition rate $\mu$ for different values of $\lambda$ ($k_B T = 0.5$, $V_0 = 5.0$, $D = 1.0$, $L = 2.0$, $k = 100.0$).

V. CONCLUDING REMARKS

Two coupled Simoluchowsky equations are solved by a perturbation method to obtain the net current. The current is peaked function of thermal noise $k_B T$, barrier height $V_0$ and the fluctuation ratio $\mu$ between the two states. It is positive for $0 < \lambda < 1$ and negative for $1 < \lambda < 2$. Therefore, the current reverses its direction at $\lambda = 1.0$ (symmetric potential). When the thermal noise is small, the particle can not pass the barrier such that the current $J$ tends to zero. When the thermal noise is too large, the ratchet effect disappear so that $J$ tends to zero, also. There is an optimized value of thermal noise at which $J$ takes its maximum value. For the case of $V_0 \to 0$, the thermal noise is dormant and the current disappears. When $V_0 \to \infty$, on the other hand, the particle can not pass the barrier. When $\mu \to 0$, the attraction from State 2 is too small and the particle is always at State 1. The ratchet reduces to one-state ratchet without external force. Therefore, no currents occur. When $\mu$ is very large, similarly, the attraction from State 1 is too small, the particle can only stay at State 2, and the current also tends to zero. There exits optimized values of $k_B T, V_0$ and $\mu$ at which the current takes its maximum value.

Here, the thermal noise can facilitate the directed motion of the Brownian particles.
This differs from the one prevalent in the literature that the thermal noise always destroyed the directed motion. The noise-induced transport is associated with the breaking of either reflection symmetry of spatially periodic system or statistical symmetry of temporal nonequilibrium fluctuations characterized by multitime correlation functions. The symmetry-breaking driven transport can be used to explain the directed motion of macromolecules in biological cell and to construct well-controlled devices of high resolution for separation of macro-particles and micro-particles [21].

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