It is shown that Randall-Sundrum model has the EDM term which violates the CP-symmetry. The comparison with the case of Kaluza-Klein theory is done. The chiral property, localization, anomaly phenomena are examined. We evaluate the bulk quantum effect using the method of the induced effective action. This is a new origin of the CP-violation.

1. Introduction
As the quantities to measure the extendedness of a particle, there are two important physical quantities: the magnetic dipole moment ($\mu$, MDM) and the electric dipole moment ($d$, EDM).

\[
O_1 = \frac{e}{2m} F_{ab} \bar{\psi} \sigma^{ab} \psi \sim -\mu \cdot B ,
\]
\[
O_2 = \frac{e}{2m} F_{ab} \bar{\psi} \gamma^5 \sigma^{ab} \psi \sim -d \cdot E ,
\]

where $B$ is the magnetic flux density vector and $E$ is the electric field vector. Both quantities, $O_1$ and $O_2$, have the physical dimension of $M^5$ (higher-dimensional operators). Hence they do not appear in the starting Lagrangian, and usually appear only through the quantum effect. In particular EDM term, $O_2$, violates the CP-symmetry. Because of this, the experimental efforts to measure EDM, as well as MDM, has been made vigorously. The latest result of the upper bounds are

\[
d_n < 2.9 \times 10^{-26} \text{ e} \cdot \text{cm} \quad (1)
\]
\[
d_e < 1.6 \times 10^{-27} \text{ e} \cdot \text{cm} \quad ,
\]

where $d_n$ is the neutron EDM and $d_e$ is the electron EDM. EDM term can appear in the standard model, but the estimated magnitude is $d_n = 10^{-32} \text{ e} \cdot \text{cm}$ [3,4]. This is far less than the present experimental bound. Hence the detection of EDM implies the new physics beyond the standard model. Thirring [5], very long time ago, showed EDM, as well as MDM, appear in a 4D model reduced from the 5 dimensional
model, Kaluza-Klein theory. The expected magnitude is far less than the experimental upper bound even at the present time. Although the result is numerically not interesting, it has some qualitatively-interesting features such as the dual relation. In the recent development of the unified theories using the extra dimension(s), Randall-Sundrum\textsuperscript{7,8} model has become one of the strong candidates for the realistic model. In the original papers by Randall and Sundrum, the comparison between the two models is made mainly in the mass hierarchy. Here we focus on another aspect, that is, the \textit{magnetic and electric dipole moment} terms.

The Kaluza-Klein theory has the long history.\textsuperscript{9,10} It is characterized by \textit{compactifying} the extra manifold. In this procedure the radius of the compact manifold, $1/\mu$, is introduced as the size parameter. On the other hand, in the Randall-Sundrum model\textsuperscript{7,8}, the \textit{localized} configuration in the extra space is utilized, instead of compactifying the extra space. In this procedure the size parameter, $1/k$, of the localization ("thickness" of the wall) is introduced. Both approaches accomplish the dimensional reduction by adjusting the size parameters.

The content of this paper is based on the result of Ref.\textsuperscript{11-12}.

2. Kaluza-Klein Theory

The 5D space-time manifold is described by the 4D coordinates $x^a$ ($a = 0, 1, 2, 3$) and an \textit{extra} coordinate $y$. We also use the notation $(X^m) = (x^a, y)$, $(m = 0, 1, 2, 3, 5)$. With the general 5D metric $\hat{g}_{mn}$,

$$
\hat{g}_{mn} = \hat{g}_{mn}(X) dX^m dX^n ,
$$

we assume the $S^1$ compactification condition for the extra space.

$$
\hat{g}_{mn}(x, y) = \hat{g}_{mn}(x, y + \frac{2\pi}{\mu}) ,
$$

where $\mu^{-1}$ is the \textit{radius} of the extra space circle. We specify the form of the metric as

$$
ds^2 = g_{ab}(x) dx^a dx^b + e^{2\sigma(x)} (dy - f A_a(x) dx^a)^2 ,
$$

where $g_{ab}(x), A_a(x)$ and $\sigma(x)$ are all 4D quantities, namely, the 4D metric, the U(1) gauge field and the dilaton (Weyl scale) field, respectively. $f$ is a coupling constant. This specification is based on the following additional assumptions.

1. $y$ is a \textit{space} (not time) coordinate.
2. The geometry is invariant under the U(1) symmetry: $y \rightarrow y + \lambda(x), \quad A_a(x) \rightarrow A_a(x) + \frac{1}{\lambda} \partial_a \lambda$.
3. We ignore the massive modes in the KK-expansion of $\hat{g}_{mn}(x, y)$.

We take the Cartan formalism to introduce Dirac fermions and to compute the geometric quantities such as the connection and the Riemann curvature.\textsuperscript{6} The
fünf-bein $\hat{e}^\mu_m$ is given by

$$\langle \hat{e}^\mu_m \rangle = \begin{pmatrix} e_\alpha^\mu & 0 \\ -f e^\sigma A_\alpha e^\sigma \end{pmatrix}$$

(6)

The connection is given as

$$\hat{\omega}^5_5 = -\frac{1}{2} \partial_a \sigma \, dx^a \ , \ \hat{\omega}^5_\alpha = -\frac{1}{2} \partial_a \sigma \, e^a_\alpha \hat{\theta}^5 ,$$

$$\hat{\omega}^\alpha_\beta = \omega^\alpha_\beta + \frac{f}{2} e^a F^a_{\beta} \hat{\theta}^5 , \ F_{ab} = \partial_a A_b - \partial_b A_a \ , \ F_{\alpha\beta} \equiv e^a_{\alpha} e^b_{\beta} F_{ab} ,$$

(7)

where $\omega^\alpha_\beta$ is the 4D connection. The 5D Riemann scalar curvature $\hat{R}_{\mu \nu}$ can be decomposed as

$$\hat{R} = R + \frac{f^2}{4} e^{2\sigma} F_{\alpha\beta} F_{\alpha\beta} + \frac{1}{2} \partial_a \sigma \partial_b \sigma g^{ab} + \frac{1}{2} \nabla^2 \sigma \ ,$$

(8)

which shows the theory of gravity, electro-magnetism and the dilaton in the 4D world.

We consider the simple case $\sigma = 0$ for the present purpose.

**3. Fermions in Kaluza-Klein Theory**

The 5D Dirac equation is generally given by

$$\begin{cases} \gamma^\mu \hat{e}_\mu^m \frac{\partial}{\partial X^m} + \frac{1}{8} (\hat{\omega}^\sigma)_{\mu \nu} \gamma^\sigma [\gamma^\mu, \gamma^\nu] + \hat{m} \end{cases} \hat{\psi} = 0 \ .$$

(9)

$m$ is the mass parameter of the 5D fermion ($-\infty < \hat{m} < \infty$). $(\hat{\omega}^\sigma)_{\mu \nu}$ is the spin connection. For simplicity we switch off the 4D gravity: $e^a_\alpha \to \delta^a_\alpha$, $\omega^\alpha_\beta \to 0$.

$$\begin{cases} \gamma^a (\partial_a + f A_a \partial_5) + \gamma^5 \partial_5 - \frac{f}{16} F_{ab} \gamma^5 [\gamma^a, \gamma^b] + \hat{m} \end{cases} \hat{\psi} = 0 \ ,$$

(10)

where $\partial_5 = \partial/\partial y$.

(A) Charged Fermion

Corresponding to a mass mode in the 4D reduction, we consider the following form for a charged fermion.

$$\hat{\psi}(x,y) = e^{i(\phi \gamma^5 + \mu y)} \psi(x) \ .$$

(11)

Here we regard the charged fermion as a KK-massive mode. The phase parameter $\phi$ is chosen as

$$(i) \gamma^5 \mu + \hat{m} = \sqrt{\hat{m}^2 + \mu^2} \equiv M \ .$$

$$\begin{cases} \hat{m} \neq 0 : \ \tan 2\phi = -\frac{\mu}{\hat{m}} , \\
-\frac{\pi}{2} < 2\phi \leq 0 \ for \ \hat{m} > 0; \ \pi \leq 2\phi < \frac{3\pi}{2} \ for \ \hat{m} < 0 , \\
\hat{m} = 0 : \ 2\phi = -\frac{\pi}{2} \ .
\end{cases}$$

(12)
Fig. 1. The angle parameter $\phi$ which defines the 4D charged fermion in the KK dimensional reduction (11). $m$ is the 5D fermion mass parameter, $\mu^{-1}$ is the size of the extra compact manifold. $-\pi/2 < 2\phi < 0$ for $\hat{m} > 0$; $\pi < 2\phi < 3\pi/2$ for $\hat{m} < 0$; $2\phi = -\pi/2$ for $\hat{m} = 0$. The upper half region gives the same 4D fermion action as the lower half region by the transformation $\psi \rightarrow \gamma^5 \psi$.

See the lower half region of Fig.1. Then (10) reduces to

$$\left\{ \gamma^a (\partial_a + ieA_a) + M - \frac{1}{16\hat{m}} (\hat{m} e - ie\gamma^5) F_{ab} \gamma^5 [\gamma^a, \gamma^b] \right\} \psi = 0 ,$$

where $e \equiv f\mu$ is the electric coupling constant. $M$ is identified as a 4D fermion mass. We notice, in this expression, the EDM and the MDM naturally appear.

We consider the following two limits:

(i) CP-preserved limit [Small radius limit, 4D limit]

$$\hat{m}/\mu \rightarrow \pm 0, 2\phi \rightarrow -(\pi/2 - 0) \ [3\pi/2 - 0] \ \text{for upper [lower] sign;}$$

(ii) CP-extremely-violated limit [Large radius limit, 5D limit]

$$\hat{m}/\mu \rightarrow \pm \infty, 2\phi \rightarrow -0 \ [\pi + 0] \ \text{for upper [lower] sign.}$$

From these results, we see CP-violation comes from the presence of the connection term in (9) (5 dimensionality itself), while the introduction of the phase parameter $\phi$ in (11) makes MDM appear. (cf. Kobayashi-Maskawa’s CP-phase.)

(B) Neutral Fermion

We regard the neutral fermion as the zero mode of the KK-expansion.

$$\psi(x,y) = \psi(x) ,$$

$$\left\{ \gamma^a \partial_a - \frac{f}{16} F_{ab} \gamma^5 [\gamma^a, \gamma^b] + \hat{m} \right\} \psi(x) = 0 .$$

Only the EDM term appears. Although the fermion has no charge, the dipole moment appears. This case is similar to the limit (ii).
Let us do the order estimation. From the reduction
\[ S = \frac{1}{2G_5} \int d^5X \sqrt{-g} \hat{R} = -\frac{1}{2G_5} \frac{2\pi}{\mu} \int d^4x \sqrt{-g} (R + \frac{f^2}{4} F_{\alpha\beta} F_{\alpha\beta} + \cdots) , \]
we know
\[ \frac{1}{G_5\mu} \sim \frac{1}{G} , \quad \frac{f^2}{G_5\mu} \sim 1 , \]
where \( G \) is the (4D) gravitational constant. This gives \( f \sim \sqrt{G} = 10^{-19}\text{GeV}^{-1} \). On the other hand, we know \( e = \mu f \sim 10^{-1} \). Hence we obtain \( \mu \sim 10^{-1} f^{-1} \sim 10^{18} \text{GeV} \).

We originally have four parameters, \( \hat{m}, \mu, G_5 \) and \( f \). Among them we have the two relations \(^{16}\) from the observation. We have another restriction among them from the present theoretical knowledge. The most natural interpretation of the parameters is that \( \mu^{-1} \) is the infrared regularization, \( \hat{m} \) is the energy scale of this 5D KK system. Then the validity of the 5D classical treatment requires that the 5D Planck mass \( \gg |\hat{m}| \):\[ \frac{1}{\sqrt{G_5}} \sim \sqrt{100\mu} \gg |\hat{m}| , \]
and this reduces to, through the previous parameter relations and values, \( |\hat{m}| \ll 10^{19} \text{GeV} \).

Now we consider the following three cases in order to evaluate the EDM and MDM couplings. (Note that the Planck length \( \sim 10^{-32} \text{cm} \).)

(i) \( |\hat{m}| \sim \mu \) We can estimate the electric and magnetic couplings as
\[ \frac{|\hat{m}|}{M\mu} e \sim 10^{-32} \text{ e cm} , \quad \frac{e}{M} \sim 10^{-32} \text{ cm e}/\mu_0 , \]
where \( \mu_0 \) is the permeability constant. Both electric and magnetic moment terms equally appear. In this case, however, the theoretical restriction \(^{17}\) is not so well satisfied.

(ii) \( |\hat{m}| \ll \mu \) (small radius, 4D limit), CP-preserved limit We can estimate as
\[ \frac{\hat{m}}{M\mu} e \sim \frac{\hat{m}}{\mu} \times 10^{-32} \text{ e cm} , \quad \frac{e}{M} \sim \frac{e}{\mu} \sim 10^{-32} \text{ cm e}/\mu_0 . \]

In this case the EDM coupling is suppressed by the factor of the mass parameter ratio \( \frac{m}{\mu} \ll 1 \). The theoretical restriction \(^{17}\) is satisfied, hence this parameter region is well controlled theoretically.

(iii) \( |\hat{m}| \gg \mu \) (large radius, 5D limit), CP-extremely-violated limit We can estimate as
\[ \frac{|\hat{m}|}{M\mu} e \sim \frac{e}{\mu} \sim 10^{-32} \text{ e cm} , \quad \frac{e}{M} \sim \frac{\mu}{|\hat{m}|} \times 10^{-32} \text{ cm e}/\mu_0 . \]

\(^a\) Bohr magneton \( e\hbar/2m_e = 1.7 \times 10^{-12} (\text{e cm}/\hbar) \).
In this case the MDM coupling is suppressed by the factor \( \mu |\hat{m}| \ll 1 \). The theoretical restriction \( (17) \), however, is not satisfied. This implies the 5D quantum effect can not be negligible in this parameter region.

We note that the ratio of the two massive parameters, (the radius of the extra space)\(^{-1} \mu \) and the 5D fermion mass \( \hat{m} \), controls the dual aspect (electric versus magnetic) of the theory. This point will be compared with the RS case later.

As for the EDM, all cases are far below the experimental upper bound described in \((2)\). As for the MDM, we know, (from the formula: \( e/2m \)), the order of the observed values are \( 10^{-11} \text{ cm} \left( \frac{e}{\mu_0} \right) \) for the electron, \( 10^{-14} \text{ cm} \left( \frac{e}{\mu_0} \right) \) for the proton, \( 10^{-16} \text{ cm} \left( \frac{e}{\mu_0} \right) \) for the top quark. The prediction of 5D KK theory is superweak compared with these values. Hence the present model is viable but quantitatively not so attractive. We are, at present, content with the qualitatively interesting points.

Thirring\(^{6}\) showed that the CP-violating term (EDM term) appears not because the discrete symmetries (charge conjugation, parity and time reversal) do not exist in the Dirac equation \((13)\), but because they appear in the form which differs from the ordinary one.

4. Fermions in Randall-Sundrum Theory

We consider the following 5D space-time geometry\(^{7,8}\)

\[
ds^2 = e^{-2\sigma(y)} \eta_{ab} dx^a dx^b + dy^2 = \hat{g}_{mn} dX^m dX^n ,
-\infty < y < +\infty , \quad -\infty < x^a < +\infty ,
\]

where \( \sigma(y) \) is a "scale factor" field. \((\eta_{ab}) = \text{diag}(-1, 1, 1, 1)\). When the geometry is AdS, \( \sigma(y) = c|y|, c > 0 \).

The fünf-bein \( \hat{e}_\mu^m \) is given by

\[
(\hat{e}_\mu^m) = \left( \begin{array}{cc}
  e^{-\sigma} \eta_{a}^a & 0 \\
  0 & 1
\end{array} \right).
\]

We obtain the connection 1-form \( \hat{\omega}_\mu^m \), as

\[
\hat{\omega}_5^5 = 0 \quad , \quad \hat{\omega}_a^b = -\hat{\omega}_b^a = -\sigma' \theta^a \quad , \quad \hat{\omega}_5^a = -\hat{\omega}_a^5 = \sigma' \theta_a \quad , \quad \hat{\omega}_a^a = 0 ,
\]

where \( \sigma' = \frac{d\sigma}{dy} \).

The 5D Dirac Lagrangian in the RS theory is given, from \((9)\) and the results of Sec.4, as

\[
\sqrt{-\hat{g}}(\mathcal{L}^{\text{Dirac}} + i\hat{m}(y)\bar{\psi}\gamma^5 \psi) = ie^{-\frac{\sigma}{2}}\bar{\psi}\left\{ \gamma^\alpha \partial_\alpha - 2e^{-\sigma} \left( \frac{1}{4} \sigma' - \frac{1}{2} \partial_y \right) \gamma^5 + \hat{m}(y) e^{-\sigma} \right\}(\bar{\psi}e^{-\frac{\sigma}{2}} \gamma^5 \psi).
\]

where \( \hat{m} \) is the 5D fermion mass \( -\infty < \hat{m} < +\infty \). For the later use, we here allow \( \hat{m} \) to have the \( y \)-dependence : \( \hat{m} = \hat{m}(y) \).
Let us do the dimensional reduction from 5D to 4D \[ \tilde{\psi}(x, y) = \sum_n (\tilde{\psi}^n_L(x)\xi_n(y) + \tilde{\psi}^n_R(x)\eta_n(y)), \]

\[ \gamma^5\tilde{\psi}_L(x) = -\tilde{\psi}_L(x), \quad \gamma^5\tilde{\psi}_R(x) = +\tilde{\psi}_R(x), \] (25)

where \( \{\xi_n(y), \eta_n(y)\} \) is a complete set of some eigenfunctions to be determined. For simplicity, we consider the "5D-parity" even case for \( \tilde{\psi}(x, y) \).

\[ \gamma^5\tilde{\psi}(x, -y) = +\tilde{\psi}(x, y). \] (26)

This requires \( \xi_n(y) \) to be an odd function and \( \eta_n(y) \) to be an even function with respect to the \( Z_2 \) transformation: \( y \leftrightarrow -y \).

\[ \xi_n(-y) = -\xi_n(y), \quad \eta_n(-y) = +\eta_n(y). \] (27)

From these we get the following important boundary conditions,

\[ \xi_n(0) = 0 \quad (\text{Dirichlet}), \quad \eta_y|_{y=0} = 0 \quad (\text{Neumann}), \] (28)

when \( \xi_n(y) \) and \( \eta_y|_{y=0} \) are continuous at \( y = 0 \). The Lagrangian reduces to

\[ \sqrt{-g}(\mathcal{L}^{\text{Dirac}} + i\tilde{m}(y)\tilde{\psi}) = i \sum_m (\tilde{\psi}^m_L\tilde{\xi}_m(y) + \tilde{\psi}^m_R\tilde{\eta}_m(y)) \times \]

\[ \sum_n \{\gamma^a\partial_a\psi^n_L\tilde{\xi}_n - e^{-\sigma}(\sigma' - \partial_y)(-\psi^n_L)\tilde{\xi}_n + \gamma^a\partial_a\psi^n_R\tilde{\eta}_n - e^{-\sigma}(\sigma' - \partial_y)\psi^n_R\tilde{\eta}_n \]

\[ + e^{-\sigma}\tilde{m}(y)(\psi^n_L\tilde{\xi}_n + \psi^n_R\tilde{\eta}_n)\}, \] (29)

where we define \( \tilde{\xi}_n \equiv e^{-\frac{\sigma}{2}}\xi_n \) and \( \tilde{\eta}_n \equiv e^{-\frac{\sigma}{2}}\eta_n \).

We now take the set of eigenfunctions \( \{\tilde{\xi}_n, \tilde{\eta}_n\} \) as

\[ e^{-\sigma}(\sigma' - \partial_y)\tilde{\xi}_n + e^{-\sigma}\tilde{m}(y)\tilde{\xi}_n = m_n\tilde{\eta}_n, \]

\[ -e^{-\sigma}(\sigma' - \partial_y)\tilde{\eta}_n + e^{-\sigma}\tilde{m}(y)\tilde{\eta}_n = m_n\tilde{\xi}_n, \] (30)

which are orthonormalized as

\[ \int_{-\infty}^{\infty} dy \tilde{\xi}_n(y)\tilde{\xi}_m(y) = \int_{-\infty}^{\infty} dy \tilde{\eta}_n(y)\tilde{\eta}_m(y) = \delta_{nm}, \int_{-\infty}^{\infty} dy \tilde{\xi}_n(y)\tilde{\eta}_m(y) = 0 \] (31)

Then the 5D action (24) finally reduces to the sum of 4D free fermions.

\[ \int d^5X \sqrt{-g}(\mathcal{L}^{\text{Dirac}} + i\tilde{m}(y)\tilde{\psi}) = \]

\[ i \int d^4x \sum_n \{\tilde{\psi}^n_L(\gamma^a\partial_a\psi^n_L + m_n\psi^n_R) + \tilde{\psi}^n_R(\gamma^a\partial_a\psi^n_R + m_n\psi^n_L)\}. \] (32)

The information of this fermion dynamics is now in the set of the eigen values \( \{m_n\} \) determined by (30).
From the coupled equation (30) with respect to $\tilde{\xi}_n$ and $\tilde{\eta}_n$, we get the differential equation for $\tilde{\xi}_n$ as
\[
e^{-2\sigma'[y]} \left( \frac{\sigma''}{2} - \frac{3}{4} \sigma'^2 + 2\sigma' \partial_y - \tilde{m}(y) \sigma' + \tilde{m}(y)^2 - \partial_y^2 + \tilde{m}(y) \right) \tilde{\xi}_n = m_n^2 \tilde{\xi}_n \quad . \tag{33}
\]
For simplicity, we consider the thin wall limit:
\[
\sigma(y) = \omega |y| \quad , \quad \sigma'(y) = \omega \epsilon(y) \quad , \quad \sigma''(y) = 2\omega \delta(y) \quad , \\
\tilde{m}(y) = \tilde{m}\epsilon(y) \quad , \quad \tilde{m}'(y) = 2\tilde{m}\delta(y) \quad , \tag{34}
\]
where $\omega(>0)$ and $\tilde{m}(>0)$ are some constants. $\epsilon(y)$ is the sign function: $\epsilon(y) = 1$ for $y > 0$ and $\epsilon(y) = -1$ for $y < 0$ ($\epsilon'(y) = 2\delta(y)$). In this limit, the equation (33) can be explicitly solved.
\[
e^{-2\omega|y|/\omega} \left( (\omega + 2\tilde{m}) \delta(y) - \frac{3}{4} \omega^2 + 2\omega \epsilon(y) \partial_y - \tilde{m}\omega + \tilde{m}^2 - \partial_y^2 \right) \tilde{\xi}_n = m_n^2 \tilde{\xi}_n \quad , \tag{35}
\]
where $\epsilon(y)^2 = 1$ is used. The presence of $\delta(y)$ indicates a singularity of the solution at $y = 0$. First let us see the solution in the region $y > 0$. In terms of a new coordinate $z \equiv \frac{1}{\omega} e^{\omega y}$, the above equation reduces to the Bessel differential equation.
\[
\left\{ \partial_z^2 - \frac{1}{z} \partial_z + \frac{1 - \nu^2}{z^2} + m_n^2 \right\} \tilde{\xi}_n = 0 \quad , \quad \nu = \left| \frac{\tilde{m}}{\omega} - \frac{1}{2} \right| \quad . \tag{36}
\]
The solution $\tilde{\xi}_n$ is obtained as
\[
\tilde{\xi}_n(y) = \frac{1}{(\omega z)^{3/2}} \xi_n(z) = z \{ J_\nu(m_n z) + c_n N_\nu(m_n z) \} \quad , \quad c_n = - \frac{J_\nu(m_n/\omega)}{N_\nu(m_n/\omega)} \tag{37}
\]
where $c_n$ is determined by the Dirichlet boundary condition (28). $J_\nu(z)$ and $N_\nu(z)$ are two independent Bessel functions. As for the solution valid for the full region $-\infty < y < \infty$, we can obtain, taking into account the singularity at $y = 0$ and the odd property (27), as
\[
\tilde{\xi}_n(y) = e^{-3\omega|y|/2} \xi_n(y) = \frac{\epsilon(y)}{\omega} e^{\omega|y|/\omega} \{ J_3(m_n/\omega) e^{\omega|y|} + c_n N_3(m_n/\omega) \} \quad , \\
c_n = - \frac{J_3(m_n/\omega)}{N_3(m_n/\omega)} \quad , \tag{38}
\]
where the parameter $\nu$ in (37) is fixed to be 3. ($\tilde{m} = -\frac{3}{2} \omega$) With the above explicit solution of $\tilde{\xi}_n$, we can obtain $\tilde{\eta}_n$ using the first equation of (30). Another boundary condition (Neumann) on $\eta_n$ (28) gives us the set $\{ m_n \}$ as the zeros of some combination of Bessel functions.

The importance of the 5D mass "function" $\tilde{m}(y)$ is now clear. In the next section, we explain its origin in the bulk field theory. Nature requires the Yukawa interaction between the 5D fermion and the 5D Higgs [16].
5. Bulk Higgs Mechanism and Massless Chiral Fermion Localization

One of the most important characters of the brane world model is the massless chiral fermion localization. It is phenomenologically attractive because the smallness of the quark and lepton masses, compared with the Planck mass, could be naturally explained. Theoretically it is also necessary as the dimensional reduction mechanism. The feature comes from the $Z_2 (y \leftrightarrow -y)$ properties of the system. The most natural way to introduce the properties is to use the Higgs mechanism in the bulk world.\(^b\)

Let us examine the case the fermion system has the Yukawa coupling with the bulk Higgs field.

$$\sqrt{-\hat{g}} \mathcal{L} = \sqrt{-\hat{g}} (\mathcal{L}^\text{Dirac} + \mathcal{L}^Y) , \quad \mathcal{L}^Y = ig_Y \bar{\psi} \tilde{\psi} \Phi ,$$

where the Higgs field $\Phi$ is the 5D(bulk) scalar field and $g_Y$ is the Yukawa coupling. We assume that the Higgs field, besides the "scale factor" field $\sigma(y)$, is some background given by the (classical) field equation of the 5D gravity-Higgs system.

$$\sqrt{-\hat{g}} (\mathcal{L}^\text{grav} + \mathcal{L}^S) , \quad \mathcal{L}^\text{grav} = -\frac{1}{2G_5} \hat{R} , \quad \mathcal{L}^S = -\frac{1}{2} \nabla^m \Phi \nabla_m \Phi - V(\Phi) ,$$

where $V(\Phi)$ is the ordinary Higgs potential. In ref\(^{17,18}\), it is shown that the above gravity-Higgs system has a stable kink (domain wall) solution for the case $\Phi = \Phi(y)$. In the IR asymptotic region far from the wall, $\sigma'(y)$ and $\Phi(y)$ behave as

$$\sigma'(y) = \begin{cases} +\omega , & ky \to +\infty \\ -\omega , & ky \to -\infty \end{cases} , \quad \Phi(y) = \begin{cases} +v_0 , & ky \to +\infty \\ -v_0 , & ky \to -\infty \end{cases} ,$$

where $k$ (the inverse of thickness), $\omega$ (brane tension) and $v_0$ (5D Higgs vacuum expectation value) are some positive constants expressed by a free parameter, the vacuum parameters and the 5D gravitational constant. Near the origin of the extra axis ($k|y| \ll 1$), they behave as

$$\sigma'(y) = \omega \tanh(ky) , \quad \Phi(y) = v_0 \tanh(ky) .$$

The dimensional reduction to 4D is performed by taking the thin wall limit $k \to \infty$, which is precisely defined as

$$k \gg \frac{1}{r_c} .$$

where $r_c$ is the infrared cutoff of the extra axis ($-r_c < y < r_c$). (See ref\(^{17}\).) In this limit, above quantities behave as $\sigma'(y) \to \omega \epsilon(y) , \quad \Phi(y) \to v_0 \epsilon(y)$. All dimensional parameters are a) $G_5^{-1/3}$: 5D Planck mass; b) $|\tilde{m}| = g_Y v_0$: 5D fermion mass; c) $k^{-1}$: thickness of the domain; d) $r_c$: Infrared regularization of the extra axis. Among them

\(^b\)In the case of the flat space-time, Rubakov and Shaposhnikov\(^{19}\) proposed a domain wall model caused by the bulk Higgs potential.
there exists a theoretical restriction from the requirement: 5D classical treatment works well.

\[ \frac{1}{\sqrt{G_5}} \gg k , \]  

(44)

The 5D Dirac equation of (39) is given by (cf. eq. (24)),

\[ i e^\sigma \left( \gamma^a \partial_a - 2e^{-\sigma}(\sigma' - \frac{1}{2}\partial_y)\gamma^5 + g_Y e^{-\sigma}\Phi \right) \hat{\psi} = 0 . \]  

(45)

This is just the lagragian of (24) with \( \hat{m}(y) = g_Y \Phi(y) \). Let us examine a solution of the left chirality zero mode.

\[ \hat{\psi}(x,y) = \psi^0_L(x)\eta(y) , \gamma^5 \psi^0_L = -\psi^0_L , \gamma^a \partial_a \psi^0_L = 0 . \]  

(46)

The equation (45) reduces to

\[ \partial_y \eta = (2\sigma' + g_Y \Phi(y))\eta . \]  

(47)

In the IR asymptotic region \( k|y| \gg 1 \), the solution behaves as

\[ \eta(y) = \text{const} \times e^{(g_Y v^0_0 + 2\omega)|y|} , \]  

(48)

which shows the exponentially damping for the case: \( g_Y v^0_0 + 2\omega < 16 \). This is called massless chiral fermion localization. Near the origin of the extra axis \( k|y| \ll 1 \), \( \eta(y) \) behaves as

\[ \eta(y) = \text{const} \times e^{\frac{k}{2}(g_Y v^0_0 + 2\omega)y^2} , \]  

(49)

which shows the Gaussian damping. The behavior is shown in Fig.2. It shows the regular property of the solution. For the right chirality zero-mode, we can show the same behavior using the anti-kink solution instead of the kink solution (41).
6. Five Dimensional QED and Bulk Quantum Effect

Let us examine the 5D QED, $\mathcal{L}^{QED} = -\bar{\psi} \gamma^\mu \hat{e}_\mu \psi A_m$, with the Yukawa interaction in RS geometry.

$$\sqrt{-\hat{g}}(\mathcal{L}^{Dirac} + \mathcal{L}^{QED} + \mathcal{L}^Y) = \sqrt{-\hat{g}} \left[ i \bar{\psi} \left\{ \gamma^\mu \hat{e}_\mu (\partial_\mu + i e A_\mu) + \frac{1}{8} (\hat{\omega}^\sigma)_{\mu\nu} \gamma_\sigma [\gamma^\mu, \gamma^\nu] \right\} \hat{\psi} + i g_Y \bar{\psi} \psi \Phi \right].$$  \hspace{1cm} (50)

The kinetic (propagator) part for the electromagnetic, gravitational and Higgs fields is provided by

$$\sqrt{-\hat{g}}(\mathcal{L}^{EM} + \mathcal{L}^{grav} + \mathcal{L}^S), \quad \mathcal{L}^{EM} = -\frac{1}{4} \hat{g}^{mn} \hat{g}^{kl} F_{mk} F_{nl},$$  \hspace{1cm} (51)

where $\mathcal{L}^{grav}$ and $\mathcal{L}^S$ are given in [10]. We assume, as in Sect.4 and 5, $\hat{g}^{mn}$ and $\Phi$ are the brane background fields obtained as the stable solution of the system $\mathcal{L}^{grav} + \mathcal{L}^S$.

We have introduced the Yukawa coupling in order to localize the fermion on the wall. This model, however, is still unsatisfactory in that the vector (gauge) field is not localized [10]. One resolution is to take 6D model [20]. Here we are content only with the fermion part and do not pursue this problem.

Let us examine the bulk quantum effect. It induces the 5D effective action $S_{eff}$ which reduces to a 4D action in the thin wall limit. From the diagram of Fig.3, we expect

$$\frac{\delta S_{eff}^{(1)}}{\delta A_\mu(X)} \equiv < J_\mu > \sim e^2 g_Y \epsilon_{\mu\nu\lambda\sigma} \Phi F^{\nu\lambda} F^{\sigma\tau}. \hspace{1cm} (52)$$

Then the effective action is integrated as

$$S_{eff}^{(1)} \sim e^2 g_Y \int d^5 X \epsilon_{\mu\nu\lambda\sigma} \Phi A_\mu F^{\nu\lambda} F^{\sigma\tau}. \hspace{1cm} (53)$$

In the thin wall limit we may approximate as $\Phi = \Phi(y) \sim \nu_0 \epsilon(y)$ where $\epsilon(y)$ is the step function. (See the description below [12].) Under the U(1) gauge transforma-
tion $\delta A^\mu = \partial^\mu \Lambda$, $S^{(1)}_{\text{eff}}$ changes as

$$\delta \Lambda S^{(1)}_{\text{eff}} \sim e^2 g_\nu v_0 \int d^5 X \epsilon_{\nu \mu \lambda \sigma \tau} \epsilon(y) \partial^\mu \Lambda F^{\nu \lambda} F^{\sigma \tau}$$

$$= e^2 g_\nu v_0 \int d^5 X \{ \partial^\mu (\epsilon_{\nu \mu \lambda \sigma \tau} \epsilon(y) \Lambda F^{\nu \lambda} F^{\sigma \tau}) - \epsilon_{5 \nu \lambda \sigma \tau} \delta(y) \Lambda F^{\nu \lambda} F^{\sigma \tau} \}$$

$$= -e^2 g_\nu v_0 \int d^4 x \Lambda(x) F^{\alpha \beta} \tilde{F}_{\alpha \beta}^\gamma , \quad (54)$$

where $\tilde{F}_{\alpha \beta}^\gamma \equiv \epsilon_{\alpha \beta \gamma \delta} F^{\gamma \delta}$. In the above we assume that the boundary term vanishes. Callan and Harvey interpreted this result as the "anomaly flow" between the boundary (our 4D world) and the bulk. Through the analysis of the induced action in the bulk, we can see the dual aspect of the 4D QED.

Another interesting bulk quantum effect is given by Fig.4. The induced effective action $S^{(2)}_{\text{eff}}$ is expected to satisfy

$$\frac{\delta S^{(2)}_{\text{eff}}}{\delta F^{\mu \nu}} = \langle J_{\mu \nu} \rangle \sim e^2 g_\nu v_0 \epsilon_{\mu \nu \lambda \sigma \tau} \partial^\lambda \bar{\psi} \tilde{\psi} \psi \Sigma^{\sigma \tau} \text{.} \quad (55)$$

Then $S^{(2)}_{\text{eff}}$ is obtained as, in the thin wall limit,

$$S^{(2)}_{\text{eff}} \sim e^2 g_\nu v_0 \epsilon_{\mu \nu \lambda \sigma \tau} \int d^5 X \bar{\psi} \tilde{\psi} \psi \Sigma^{\sigma \tau} \text{.}$$

$$= e^2 g_\nu v_0 \epsilon_{\alpha \beta \gamma \delta} \int d^4 x F^{\alpha \beta} \tilde{\bar{\psi}} \gamma^\gamma \psi = -ie^2 g_\nu v_0 \int d^4 x F^{\alpha \beta} \tilde{\bar{\psi}} \gamma^5 \sigma_{\alpha \beta} \psi \text{.} \quad (56)$$

This term is the EDM term of (13). The coupling depends on the vacuum expectation value of $\Phi$, $v_0 = \langle \Phi \rangle$ which are, at present, not known. In order to estimate the magnitude of the coupling, it is necessary to apply this model to the quark-lepton (electro-weak) theory and fix the value. We expect the magnitude could be sufficiently large so that the result can be tested by present or near-future experiments.

The appearance of the EDM term corresponds to the CP-extremely-violated case (ii) $|\hat{n}| \gg \mu$ of Sec.3. We compare the parameter relation with the thin wall
relation \( k \gg 1/r_c \). Because the parameter \( \mu \) in KK case corresponds to \( 1/r_c \) in RS case, we notice the 5D fermion mass in KK case (\(|\hat{m}|\)) corresponds to the inverse of the thickness in RS case (\( k \)).

The more fascinating view on the correspondence is that the RS approach and the KK one are "dual" each other. We compare the above thin wall limit with the small radius limit (i) \(|\hat{m}| \ll \mu \) of Sec.3. The thin wall limit, which is regarded as the dimensional reduction, can be consistently taken in RS model and EDM term naturally appears there. CP is not preserved. The theoretical treatment is justified as far as the relations (43) and (44) are satisfied: \( 1/\sqrt{G_5} \gg k \gg 1/r_c \). While, in the KK case, the dimensional reduction takes place in the case (i) of Sec.3, which can be controlled theoretically as far as the relation (17) \( 1/\sqrt{G_5} = \sqrt{\frac{100}{10}} \mu \geq \mu \gg |\hat{m}| \) is satisfied. MDM term naturally appears and CP is preserved here.

In the effective action evaluation, the bulk Higgs field \( \Phi(y) \) plays an important role. It serves as a bridge between the bulk world and the 4D world.

7. Discussion and Conclusion

Since the appearance of EDM in the Randall-Sundrum model was pointed out in Ref. \[11,12\], several years have passed. The difficulty of the analysis is due to the lack of the proper treatment of the bulk quantum effects. (We do not have the renormalizable higher dimensional theory.) Practically, however, an effective approach called "spurion analysis" has been developed. In Ref. \[22,23\], the neutron EDM is estimated, in the RSII model, using the approach. They find the estimated value is 20 times larger than the current experimental bound and it reveals a "CP problem".

The present approach to this bulk quantum theory is different from the above one. We take the induced effective action method. It has been used in relation to the chiral and Weyl anomalies. Famous successful ones are 2D WZNW model derived from the 2D QCD \[24\] and 2D induced quantum gravity \[25\]. We have examined mainly the thin wall limit. In order to examine the configuration off the limit, we can take some numerical approach.

We have comparatively examined the KK model and the RS model. Both have attractive features as the higher dimensional unification models. The periodic functions appear in the former case, while the Bessel functions characteristically appear in the latter case. The dual property is controlled by two scale parameters, \( \hat{m} \) and \( \mu \) in the KK case whereas \( k \) and \( r_c \) in the RS case. In particular we stress that, as was pointed out by Thirring for the KK case, the CP-violation term naturally appears also in the RS model.

Finally we list the correspondence in Table 1.
### Table 1: Comparison of KK model and RS model.

|                        | Kaluza-Klein                                                                 | Randall-Sundrum                                                                 |
|------------------------|-------------------------------------------------------------------------------|--------------------------------------------------------------------------------|
| electric charge        | $e = f\mu$, $f$: free para.                                                   | $e$: free para.                                                               |
| $U(1)$ sym.            | $y \rightarrow y + A(x)$, transl. in $\{y\}$                                | $A_m \rightarrow A_m + \partial_m A$, internal sym.                          |
|                        | $A_a(x) \rightarrow A_a(x) + \frac{1}{f} \partial_a A$                      | $(X^m) = (x^a, y)$: fixed                                                    |
| asympt. geometry       | $S^1 \times M_4$                                                             | $AdS_5$ ($\omega, \tilde{m} = g_Y v_0$)                                     |
| vacuum                 | a massive KK mode for                                                         | 5D bulk Higgs vacuum                                                         |
|                        | charged fermion $\psi$, 0-th KK modes for $g_{ab}, A_a, \sigma$ and         | $<\Phi> = \pm v_0$, $y \rightarrow \pm \infty$                             |
|                        | neutral fermion                                                              | kink sol., $Z_2$-symmetry                                                   |
| 4D fermion mass        | $\sqrt{\tilde{m}^2 + \mu^2}$                                                | $g_Y v_0 \times$ overlap-int.                                               |
| physical scale         | $\tilde{m}$: 5D fermion mass                                                 | $k$: (thickness)$^{-1}$                                                     |
| global size            | $\mu^{-1}$: radius of extra $S^1$                                             | $r_c$: IR cutoff                                                            |
|                        | $y \rightarrow y + 2\pi \mu^{-1}$, periodic                                 | $-r_c \leq y \leq r_c$                                                     |
| dimensional            |                                                                              |                                                                              |
| reduction cond.        | $\mu \gg \tilde{m}$                                                          | $k \gg 1/r_c$                                                                |
|                        | small radius limit                                                           | thin-wall limit                                                              |
| 5D classical condition | $1/\sqrt{G_5} \sim 100\mu \gg \tilde{m}$                                   | $1/\sqrt{G_5} \gg k$                                                       |
| mode functions         | $e^{ik\mu y}, k \in \mathbb{Z}$                                             | $J_\nu(m_k z), N_\nu(m_k z), \nu = \left|\frac{1}{2} - \tilde{m}/\omega\right|$ |
| in extra space         | periodic func.                                                                | $z\omega = e^{\omega y}, k \in \mathbb{Z} $ Bessel func.                   |
| CP property            | MDM in small radius limit, CP-preserved                                       | EDM in thin wall, limit, CP-violated                                         |

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