We show that the one-dimensional projection of Chern-Simons gauged Nonlinear Schrödinger model is equivalent to an Abelian gauge field theory of continuum Heisenberg spin chain. In such a theory, the matter field has geometrical meaning of coordinates in tangent plane to the spin phase space, while the $U(1)$ gauge symmetry relates to rotation in the plane. This allows us to construct the infinite hierarchy of integrable gauge theories and related magnetic models. To each of them a $U(1)$ invariant gauge fixing constraint of non-Abelian BF theory is derived. The corresponding moving frames hierarchy is obtained and the spectral parameter is interpreted as a constant-valued statistical gauge potential constrained by the 1-cocycle condition.
I. INTRODUCTION

The Nonlinear Schrödinger equation (NLS) in two space dimensions interacting with Abelian Chern-Simons gauge field (the Jackiw-Pi model) has attracted much attention recently due to the beautiful structure of the static limit, admitting N-soliton solution\(^1\). After quantization these solitons become quasiparticles with an arbitrary statistics, called anyons, while the Chern-Simons gauge field is interpreted as the "statistical" one. The anyons have an interesting application to the planar physical phenomena as the Quantum Hall Effect\(^2,3\). Very recently attempts to study the generalized statistics of many particle systems in 1+1 dimensions and the relation with 2+1 dimensional anyons were done\(^4–6\). For configurations depending of the one space direction (the lineal theory), the reduced theory describes 1+1 dimensional NLS, interacting with an Abelian BF gauge field. Due to the pure gauge form it is possible to exclude this field from the gauge invariant description. However, since in one dimension NLS is an integrable system, an interesting problem of the relation between a long-range structure of the statistical gauge field in two and integrability in one dimensions arises.

In the present paper we show that the "trivial" gauge field plays the crucial role for integrability of the NLS model. Namely, the homogeneous statistical gauge field is identical to the spectral parameter for the NLS linear problem. Our construction is based on a mapping of the reduced Jackiw-Pi model to non-Abelian BF gauge theory. This mapping has to appear as the gauge fixing condition corresponding to the classical Heisenberg spin model. Then, the matter field has geometrical meaning of coordinates in tangent to the spin phase space plane, while \(U(1)\) gauge symmetry relates to the rotation in the plane.

In section I we consider dimensional reduction of the Jackiw-Pi model. We find representation of the model as a \(U(1)\) invariant gauge fixing condition of non-Abelian BF theory. This allows us to construct an infinite hierarchy of associated integrable models and the related gauge fixing constraints. In Sec.II we show that the model has natural interpretation as the gauge theory of continuum Heisenberg model defined on the sphere \(S^2\) or hyperboloid \(S^{1,1}\), according to the sign of nonlinearity. Then, we derive the associated Heisenberg and moving frame hierarchies. Sec.III is devoted to \(U(1)\) gauge invariant description of integrable time hierarchy in the BF theory context. We show that \(B\) field plays the role of squared eigenfunctions for the Zakharov-Shabat linear problem. In terms of the recursion operator eigenvalue problem, the hierarchy of constraints is related to the evolution hierarchy and higher Poisson structures. In Conclusion we discuss the chiral solitons for odd members of the hierarchy and some
open problems.

II. THE JACKIW-PI MODEL IN 1+1 DIMENSIONS

The Lagrangian of 2+1 dimensional Nonlinear Schrödinger model interacting with Chern-Simons gauge field is

\[ L = \frac{i}{2} (\bar{\psi} D_0 \psi - \bar{D}_0 \psi \bar{\psi}) - \bar{D}_1 \bar{\psi} D_1 \psi - \bar{D}_2 \bar{\psi} D_2 \psi + g |\psi|^4 + \frac{k}{4} \epsilon^{\mu \nu \lambda} A_\mu \partial_\nu A_\lambda, \quad (2.1) \]

where \( \psi = \psi(x_1, x_2, t) \) is complex matter and \( A_\mu = A_\mu(x_1, x_2, t), (\mu = 0, 1, 2) \), is a \( U(1) \) Abelian (statistical) gauge field. Here, \( D_\mu = \partial_\mu - \frac{i}{2} A_\mu \), denotes the covariant derivative, \( g \) and \( k \) are the self-interaction and the Chern-Simons coupling constants respectively. The related Euler-Lagrange equations of motion are,

\[ i D_0 \psi + (D_1^2 + D_2^2) \psi + 2g |\psi|^2 \psi = 0, \quad (2.2a) \]

\[ \epsilon^{ij} \partial_i A_j = - \frac{1}{k} |\psi|^2, \quad (2.2b) \]

\[ \partial_0 A_i - \partial_i A_0 = - \frac{i}{k} \epsilon^{ij} (\bar{D}_j \bar{\psi} \psi - \bar{\psi} D_j \psi). \quad (2.2c, d) \]

In the special case when the coupling constants connected by the relation \( g = \frac{1}{k} \), the static configurations of this model are subject to the self-dual Chern-Simons equations. The last one admits the linear representation and has been related to the Liouville model with N- vortex/soliton solutions. However, there is no evidence that dynamics of these solitons according to eqs.(2.1) is also integrable. Recently, the integrable dynamics of the Chern-Simons solitons has been described by the Davey-Stewartson equation, being considered as the 2+1 dimensional extension of the NLS. Furthermore, it is shown below that model (2.1) reduced to 1+1 dimension is also integrable. The corresponding soliton dynamics subject to the NLS.

Let us consider the boundary as a rectangle on the plane. Then, the boundary dynamics of (2.1) is described by the 1+1 reduced model. For \( x_2 \) independent part we obtain the Lagrangian,

\[ \mathcal{L} = \frac{i}{2} (\bar{\psi} D_0 \psi - \bar{D}_0 \psi \bar{\psi}) - \bar{D}_1 \bar{\psi} D_1 \psi - \frac{1}{4} B^2 \bar{\psi} \psi + g |\psi|^4 + \frac{k}{4} B \epsilon^{\mu \nu} F_{\mu \nu}, \quad (2.3) \]

where \( B \equiv A_2 \) is the gauge field component in the compactified direction, and \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, (\mu = 0, 1) \). For the vanishing matter field \( \psi = 0 \), this Lagrangian reduces to a pure Abelian BF theory and the model (2.3) can be considered as the 1+1 dimensional BF gauged NLS. Due to the non-propagating character, the gauge field can
be eliminated from (2.3). Nevertheless, as we can see below, (2.3) is equivalent to the classical spin model and the statistical gauge potential carries an important information about spectrum of the model. Moreover, when $\psi \neq 0$, the model (2.3) still is the BF theory, however for a non-Abelian gauge group in a special gauge.

To proceed first we redefine gauge potentials (and the related covariant derivatives) as,

$$W_0 \equiv A_0 - \frac{1}{2} B^2, \quad W_1 \equiv A_1.$$  \hspace{1cm} (2.4)

Then, the Lagrangian (2.3) becomes,

$$\mathcal{L} = \frac{i}{2} (\bar{\psi} D_0 \psi - \bar{D}_0 \psi \bar{\psi}) - \bar{D}_1 \psi D_1 \psi + g|\psi|^4 + \frac{k}{4} B \epsilon^{\mu \nu} F_{\mu \nu} + \Delta \mathcal{L},$$ \hspace{1cm} (2.5)

where,

$$\Delta \mathcal{L} = - \frac{k}{6} \partial_1 B^3,$$ \hspace{1cm} (2.6)

is the total derivative (which becomes nontrivial at the boundaries) and does not influence equations of motion. The Euler-Lagrange equations for (2.5) are given by the system,

$$iD_0 \psi + D_1^2 \psi + 2g|\psi|^2 \psi = 0,$$ \hspace{0.5cm} (2.7a)

$$\partial_0 W_1 - \partial_1 W_0 = 0,$$ \hspace{0.5cm} (2.7b)

and

$$\partial_1 B = - \frac{1}{k} |\psi|^2,$$ \hspace{0.5cm} (2.8a)

$$\partial_0 B = \frac{i}{k} (\bar{D}_1 \psi \psi - \bar{\psi} D_1 \psi),$$ \hspace{0.5cm} (2.8b)

where $D_\mu = \partial_\mu - i/2W_\mu$. Eqs.(2.7) are exactly gauged NLS model (or the Heisenberg model in the tangent space form), while (2.8) are the defining relations for $B$ (the trace of higher dimension) in terms of the charge conservation law. However, Eq.(2.8a) - the reduced Chern-Simons Gauss law - is more fundamental than (2.8b). Indeed, Eqs.(2.7) imply the charge conservation law,

$$\partial_0 |\psi|^2 + i\partial_1 ((\bar{D}_1 \psi \psi - \bar{\psi} D_1 \psi) = 0.$$ \hspace{1cm} (2.9)

Then, Eq. (2.8b) arises from the Gauss law (2.8a) and (2.9) in a similar way to the Chern-Simons theory\(^1\). In other words, Eqs.(2.8) guarantee the existence of first conservation law for (2.7). In fact, the dual current $C_\mu = \epsilon_{\mu \nu} \partial_\nu B$, due to (2.8) $C_\mu$ is conserved.
The above systems (2.7) and (2.8) are invariant under $U(1)$ gauge transformations,

$$\psi \to \psi e^{i\alpha}, \quad A_0 \to A_0 + 2\partial_0 \alpha, \quad A_1 \to A_1 + 2\partial_1 \alpha,$$

preserving $B$ field: $B \to B$. From Lagrangian (2.5) we recognize that $B$ field plays the role of Lagrangian multiplier for the zero strength constraint (2.7b). Moreover, according to the one dimensional Chern-Simons (or better to say the BF) Gauss law (2.8a), creation of a particle at a point on the line, where $|\psi|^2 \neq 0$, is accompanied with creation of the local $B$ field gradient. This gradient represents one-dimensional analog of the statistical magnetic field. For vanishing $\psi$ the $B$ field is homogeneous, and defined up to the additive constant. Integrating (2.8a) along the $x_1$ line, then we obtain,

$$B(+\infty) - B(-\infty) = -\frac{1}{k} \int_{-\infty}^{+\infty} |\psi|^2 dx_1.$$

(2.11)

For nontrivial charged configurations (solitons) the r.h.s do not vanish and the related $B$ field takes the shift on the boundaries.

Actually, Eqs.(2.7) is the 1+1 dimensional analog of the Ginzburg-Landau equation for superconductor in the vanishing electric field, $E = \partial_0 A_1 - \partial_1 A_0 = 0$. An important consequence of the gauge invariance (2.10) is that (2.7) is independent of the local gauge transformations parameter $\alpha = \alpha(x,t)$. This property can be considered as a generalisation of the well known isospectrality condition for integrable systems. Indeed, the second equation (2.7) allows one to exclude the potentials $A_0$ and $A_1$ by $U(1)$ gauge transformation. Due to (2.7b) a real function $\phi(x,t)$ exists such that $A_\mu = 2\partial_\mu \phi$. Then, we define new $\Psi = \psi e^{i\phi}$, subject to the Nonlinear Schrödinger equation,

$$i\partial_0 \Psi + \partial_1^2 \Psi + 2g|\Psi|^2 \Psi = 0.$$

(2.12)

This equation admits an infinite number of conservation laws. The conservation law (2.9) in terms of $\Psi$ is the first member of this hierarchy, having physical meaning of the charge (number of particles) conservation.

In traditional approach the NLS integrability follows from the Lax pair or the Zero Curvature representation with a constant spectral parameter, of which the NLS is independent (isospectrality)$^9$. This fact can be explained now as the consequence of a $U(1)$ gauge invariance of the gauged NLS (2.7).

The integrability of (2.7) becomes transparent if we represent the self interaction term in a pure geometrical way. Thus, for redefined fields,

$$V_0 = W_0 + 4g|\psi|^2, \quad V_1 = W_1,$$

(2.13)
Eqs. (2.7) have the form,
\[ iD_0 \psi + D_1^2 \psi = 0, \quad (2.14a) \]
\[ \partial_0 V_1 - \partial_1 V_0 = -4g \partial_1 |\psi|^2, \quad (2.14b) \]
of the linear Schrödinger equation (the quantum mechanics) in the electric field proportional to the gradient of the local particles density. The system (2.14) is equivalent to the following one,
\[ D_0 \psi = D_1 \psi_0, \quad (2.15a) \]
\[ [D_0, D_1] = 2g (\bar{\psi}\psi_0 - \bar{\psi}_0 \psi). \quad (2.15b) \]
with the constraint,
\[ \psi_0 = iD_1 \psi. \quad (2.16) \]

The above Eqs. (2.15) define constant curvature surface with the scalar curvature equal to coupling constant \( g \). In fact, we can show that if \( \psi \) and \( \psi_0 \) are the zweibein fields, then the first equation (2.15a) is the torsionless condition for definition of the spin connection, identified with the Abelian gauge field \( V_\mu \). Then, the the second equation is just the constant curvature condition,
\[ R = g, \quad (2.17) \]
where \( R \) is the scalar curvature, written in terms of zweibeins and the spin connection. The model (2.17) is known as the Jackiw-Teitelboim lineal gravity\(^{10}\), while (2.15) as the BF gauge theoretical formulation of it in terms of the Einstein-Cartan variables\(^{11}\). The system (2.15) appears as the Euler-Lagrange equations for the BF action (4.1), discussed in Sec.IV. Depending on the sign of \( g \) the corresponding BF theory has non-Abelian \( SU(2), (g = 1) \), or \( SU(1, 1), (g = -1) \), gauge group. Thus, the reduced Jackiw-Pi model (2.14) is defined by gauge condition (2.16) of Euclidean BF gravity characterizing a surface of the Jackiw-Teitelboim model.

The mapping (2.15), (2.16) allows us to construct an infinite hierarchy of gauge constraints, compatible with (2.16), and describing the hierarchy of corresponding equations of motion. Eq.(2.15b) can be written in the form,
\[ \partial_0 V_1 - \partial_1 (V_0 + 4ig \int^x (\bar{\psi}\psi_0 - \bar{\psi}_0 \psi)(x')dx') = 0, \quad (2.18) \]
Then, as in the above procedure from (2.7) to (2.14), but going in the opposite direction, we introduce the flat Abelian gauge field,
\[ W_0 = V_0 + 4ig \int^x (\bar{\psi}\psi_0 - \bar{\psi}_0 \psi)(x')dx', \quad W_1 = V_1. \quad (2.18) \]
Then, in terms of redefined covariant derivatives, (2.15) becomes,

\[ D_0 \psi = D_1 \psi_0 + 2g \psi \int^x (\bar{\psi} \psi_0 - \bar{\psi}_0 \psi) dx' , \]  

or

\[ \partial_0 W_1 - \partial_1 W_0 = 0. \]  

(2.19a)

(2.19b)

Of course, for nonsingular gauge configurations we can always exclude field \( A_\mu \). But, we find it is important to keep this gauge field, playing the role similar to the spectral parameter in the usual Inverse Scattering approach. Then, model (2.12) written in terms of \( \Psi \), is explicitly invariant under the local gauge transformations (2.10) and the gauge invariance plays the role of generalized isospectrality. Below we show how far this analogy can be continued. It is convenient to write Eq.(2.19a) and its complex conjugate in the matrix form,

\[
\begin{pmatrix}
D_0 \psi \\
\bar{D}_0 \bar{\psi}
\end{pmatrix} =
\begin{pmatrix}
D_1 + 2g \psi \int^x \bar{\psi} & -2g \psi \int^x \bar{\psi} \\
-2g \bar{\psi} \int^x \bar{\psi} & \bar{D}_1 + 2g \bar{\psi} \int^x \bar{\psi}
\end{pmatrix}
\begin{pmatrix}
\psi_0 \\
\bar{\psi}_0
\end{pmatrix},
\]  

(2.20)

Then, the constraint (2.16) can be written as,

\[
\begin{pmatrix}
\psi_0^{(1)} \\
\bar{\psi}_0^{(1)}
\end{pmatrix} = i \sigma_3 \begin{pmatrix}
D_1 \psi \\
D_1 \bar{\psi}
\end{pmatrix} = \Lambda \begin{pmatrix}
\psi \\
\bar{\psi}
\end{pmatrix}
\]  

(2.21)

where we introduced the integro-differential operator

\[
\Lambda = i \sigma_3 \begin{pmatrix}
D_1 + 2g \psi \int^x \bar{\psi} & -2g \psi \int^x \bar{\psi} \\
-2g \bar{\psi} \int^x \bar{\psi} & \bar{D}_1 + 2g \bar{\psi} \int^x \bar{\psi}
\end{pmatrix}.
\]  

(2.22)

For skew-symmetric linear operator (see Sec.IV) we have to consider the integral part defined by the symmetric boundaries

\[
\int^x f(y) dy = \frac{1}{2} \left( \int_{-\infty}^x f(y) dy + \int_{\infty}^x f(y) dy \right).
\]  

(2.22a)

The above operator \( \Lambda \) represents the \( U(1) \) gauge covariant extension of the AKNS operator \( \mathcal{L} \). In fact,

\[
\Lambda = \mathcal{L} + \frac{1}{2} W_1 I,
\]  

(2.23a)

where

\[
\mathcal{L} = i \sigma_3 \begin{pmatrix}
\partial_1 + 2g \psi \int^x \bar{\psi} & -2g \psi \int^x \bar{\psi} \\
-2g \bar{\psi} \int^x \bar{\psi} & \partial_1 + 2g \bar{\psi} \int^x \bar{\psi}
\end{pmatrix}.
\]  

(2.23b)

Here \( I \) is the identity matrix, and \( \Lambda \) is covariant under \( U(1) \) gauge transformations (2.10). Then, gauged NLS (2.7) has the form,

\[
i \sigma_3 \begin{pmatrix}
D_0 \psi \\
\bar{D}_0 \bar{\psi}
\end{pmatrix} = \Lambda^2 \begin{pmatrix}
\psi \\
\bar{\psi}
\end{pmatrix}.
\]  

(2.24)
This representation suggests how to create the whole hierarchy of equations associated with gauged NLS (2.24). We define the set of constraints,

\[
\begin{pmatrix}
\psi_0^{(n)} \\
\bar{\psi}_0^{(n)}
\end{pmatrix} = \Lambda^n \begin{pmatrix}
\psi \\
\bar{\psi}
\end{pmatrix},
\]

labeled with an integer \( n \). As mentioned above, (2.15) describes a classical motion for the BF theory, with \( \psi_0 \) playing the role of Lagrange multipliers. The arbitrariness of \( \psi_0 \) guarantees the general time reparametrization invariance of the theory, while a specific choice for \( \psi_0 \) defines the corresponding evolution. The hierarchy (2.25) can be considered as the hierarchy of gauge fixing constraints of BF theory. Then, every constraint is related to the nonlinear spin model (Heisenberg hierarchy), being just the tangent space representation for the model.

The first few constraints from (2.25) are given by,

\[
\begin{pmatrix}
\psi_0^{(0)} \\
\bar{\psi}_0^{(0)}
\end{pmatrix} = \begin{pmatrix}
\psi \\
\bar{\psi}
\end{pmatrix},
\]

(2.26)

\[
\begin{pmatrix}
\psi_0^{(1)} \\
\bar{\psi}_0^{(1)}
\end{pmatrix} = i\sigma_3 \begin{pmatrix}
D_1 \psi \\
\bar{D}_1 \bar{\psi}
\end{pmatrix},
\]

(2.27)

\[
\begin{pmatrix}
\psi_0^{(2)} \\
\bar{\psi}_0^{(2)}
\end{pmatrix} = -\begin{pmatrix}
D_1^2 \psi + 2g|\psi|^2 \psi \\
\bar{D}_1^2 \bar{\psi} + 2g|\psi|^2 \bar{\psi}
\end{pmatrix},
\]

(2.28)

\[
\begin{pmatrix}
\psi_0^{(3)} \\
\bar{\psi}_0^{(3)}
\end{pmatrix} = -i\sigma_3 \begin{pmatrix}
D_1^3 \psi + 6g|\psi|^2 D_1 \psi \\
\bar{D}_1^3 \bar{\psi} + 6g|\psi|^2 \bar{D}_1 \bar{\psi}
\end{pmatrix}.
\]

(2.29)

Thus, for every \( n \), the corresponding \( \psi_0^{(n)} \) defines the evolution equation,

\[
i\sigma_3 \begin{pmatrix}
D_{0,n} \psi \\
\bar{D}_{0,n} \bar{\psi}
\end{pmatrix} = \Lambda^n \begin{pmatrix}
\psi \\
\bar{\psi}
\end{pmatrix} = \begin{pmatrix}
\psi_0^{(n)} \\
\bar{\psi}_0^{(n)}
\end{pmatrix},
\]

(2.30a)

\[
[D_{0,n}, D_1] = 0.
\]

(2.30b)

The first members of the hierarchy, \( n = 1, 2, 3 \), are,

\[
D_0 \psi - D_1 \psi = 0,
\]

(2.31)

\[
iD_0 \psi + D_1^2 \psi + 2g|\psi|^2 \psi = 0,
\]

(2.32)

\[
D_0 \psi + D_1^3 \psi + 6g|\psi|^2 D_1 \psi = 0.
\]

(2.33)

This \( U(1) \) gauged hierarchy has the flat Abelian connection (2.30b). It means that every equation (2.30) in terms of the gauge invariant variables, likes (2.12), reduces to
the form of the usual NLS hierarchy. Indeed, due to (2.30b) for any \( n \) there exists a real function \( \alpha_n = \alpha_n(x,t_n) \) such that,

\[
W_1 = \partial_1 \alpha_n, \quad W_{0_n} = \partial_{0_n} \alpha_n.
\] (2.34)

Therefore, for the gauge invariant fields \( \Psi = \psi e^{i\alpha_n} \) the hierarchy (2.30) reduces to the NLS one,

\[
\dot{\sigma}_3 \left( \frac{\partial_{0_n} \Psi}{\partial_{0_n} \bar{\Psi}} \right) = \mathcal{L}^n \left( \frac{\Psi}{\bar{\Psi}} \right) = \left( \frac{\Psi^{(n)}_0}{\bar{\Psi}^{(n)}_0} \right).
\] (2.35)

However, the gauged hierarchy (2.30) contains much more information in the addition. As we show in the next section, restricted on the subclass of constant gauge potentials, Eqs.(2.30) provide the linear problem for any of Eqs.(2.35). Moreover, every equation of the hierarchy (2.30) can be reduced dimensionally from the related 2+1 dimensional Chern - Simons gauged theory.

### III. ABELIAN GAUGE THEORY FOR MOVING FRAMES HIERARCHY

The hierarchy of Abelian gauge theories described in the previous section by (2.30), is equivalent to the system (2.15) with the hierarchy of U(1) gauge invariant constraints (2.25). But system (2.15) is the constant curvature surface equation written in the Einstein-Cartan zweibein formalism. As we see below, every constraint (2.25) supplied to (2.15), defines an evolution equation for three dimensional unit vector \( \mathbf{s} \), and can be considered as nonlinear \( \sigma \) model on the sphere \( S^2 \) or pseudosphere \( S^{1,1} \).

Let us consider the Lie group \( G \) with element \( g \), generated by \( \tau_i (i = 1, 2, 3) \), satisfying to the relations,

\[
\tau_i \tau_j = h_{ij} + ic_{ijk} \tau_k,
\] (3.1)

where \( h_{ij} \) and \( c_{ijk} \) are the Killing metric and the structure constants of the algebra respectively. We define an orthonormal trihedral set of unit vectors \( \mathbf{n}_i \) in the adjoint representation,

\[
(n_i, \tau) = n^k_i \tau_k = h_{kl} n^k_i \tau^l = g \tau_i g^{-1},
\] (3.2)

The Killing metric \( h_{ij} \) and structure constants \( c_{ijk} = -c_{jik} \) defines correspondingly the inner and the cross product between three-vectors, transforming in the adjoint representation:

\[
(n_i, n_j) = h_{ij},
\] (3.3a)

\[
\mathbf{n}_i \wedge \mathbf{n}_j = c_{ijk} \mathbf{n}_k.
\] (3.3b)
Given smooth vector fields \( n_i = n_i(x,t) \) define at each space \( x \) and time \( t \) three vectors \((n_1(x,t), n_2(x,t), n_3(x,t))\) forming an orthonormal basis, called the moving frame. The chiral current,

\[
J_\mu = g^{-1} \partial_\mu g,
\]

in the adjoint representation defines rotation of the moving frame by equations,

\[
\partial_\mu n_i = (J^{ad}_\mu)_{ik} n_k,
\]

where,

\[
(J^{ad}_\mu)_{ik} = -ic_{ijk}(J_\mu)_j = i(J_\mu)_j c_{ijk},
\]

and \( J_\mu = \sum(J_\mu)_j (1/2) \tau_j \). Matrices \( J^{ad}_\mu \) have the symmetry, \((J^{ad}_\mu)_{ij} h_{jj} = -(J^{ad}_\mu)_{ji} h_{ii},\) and are antisymmetric for \( SU(2) \) case \( h_{ij} = \delta_{ij} , c_{ijk} = \epsilon_{ijk}.\)

The current (3.4) satisfies the zero curvature equations,

\[
\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0,
\]

as compatibility conditions for the system (3.5). We decompose matrix \( J_\mu \) to the diagonal and off diagonal parts, \( J_\mu = J^{(0)}_\mu + J^{(1)}_\mu \), parametrized in the form\(^{13}\),

\[
J^{(0)}_\mu = i/4\sigma_3 V_\mu , \quad J^{(1)}_\mu = \begin{pmatrix} 0 & -g\bar{\psi}_\mu \\ \psi_\mu & 0 \end{pmatrix},
\]

where \( g = +1 \) for \( su(2) \), and \( g = -1 \), for \( su(1,1) \) cases. Then, in the adjoint representation (3.6) we have the form,

\[
(J^{ad}_\mu) = \frac{1}{2} \begin{pmatrix} 0 & V_\mu & 4g\Re(\psi_\mu) \\ -V_\mu & 0 & 4g\Im(\psi_\mu) \\ -4\Re(\psi_\mu) & -4\Im(\psi_\mu) & 0 \end{pmatrix}.
\]

The moving frame rotates according to equations,

\[
\partial_\mu n_i = -1/2V_\mu c_{ij} n_j - gU_{i\mu} s,
\]

\[
\partial_\mu s = U_{i\mu} n_i,
\]

where \( U_\mu \equiv 2(\Re(\psi_\mu), \Im(\psi_\mu)) \). For vector \( s \equiv n_3 \) the constraint, \((s(x,t), s(x,t)) = h_{33},\) is valid, where \( h_{33} = 1, \) which means that it belongs to two-dimensional sphere \( S^2 \) or pseudosphere \( S^{1,1} \), correspondingly. Two vector fields \((n_1(x), n_2(x))\) at each \((x,t)\) form a basis in the tangent plane to the corresponding manifold for \( s(x) \). However, by
eq. (2.3) vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are not determined uniquely. If we choose the other pair $\mathbf{n}_1', \mathbf{n}_2'$, as the rotated basis,

$$
\mathbf{n}_1' = \cos \alpha \mathbf{n}_1 - \sin \alpha \mathbf{n}_2, \mathbf{n}_2' = \cos \alpha \mathbf{n}_2 + \sin \alpha \mathbf{n}_1,
$$

(3.11)

the related $V'_\mu$ and $\psi'_\mu$ are the $U(1)$ gauge transformed fields,

$$
V'_\mu = V_\mu + 2 \partial_\mu \alpha, \psi'_\mu = e^{i\alpha} \psi_\mu
$$

(3.12)

The expression (3.11) suggests us to introduce a complex basis $\mathbf{n}_+ = \mathbf{n}_1 + i\mathbf{n}_2, \mathbf{n}_- = \mathbf{n}_1 - i\mathbf{n}_2$, satisfying the following relations,

$$
(n_\pm, n_\pm) = 0, (n_+, n_-) = 2g,
$$

(3.12a)

$$
n_+ \times \mathbf{s} = i\mathbf{n}_+, n_- \times \mathbf{s} = -i\mathbf{n}_-, n_- \times n_+ = 2ig\mathbf{s}.
$$

(3.12b)

Then, we get,

$$
\psi_\mu = 1/2\kappa^2(\partial_\mu \mathbf{s}, \mathbf{n}_+), \bar{\psi}_\mu = 1/2\kappa^2(\partial_\mu \mathbf{s}, \mathbf{n}_-).
$$

(3.13)

In terms of (3.12) the moving frame system (3.10) becomes,

$$
D_\mu \mathbf{n}_+ = -2g\psi_\mu \mathbf{s},
$$

(3.13a)

$$
\partial_\mu \mathbf{s} = \psi_\mu \mathbf{n}_- + \bar{\psi}_\mu \mathbf{n}_+,
$$

(3.13b)

where $D_\mu \equiv \partial_\mu - i/2V_\mu$, is $U(1)$ covariant derivative. This form is explicitly invariant under the local $U(1)$ gauge transformations

$$
\mathbf{n}_+ \rightarrow e^{i\alpha} \mathbf{n}_+, \mathbf{n}_- \rightarrow e^{-i\alpha} \mathbf{n}_-, V_\mu \rightarrow V_\mu + 2 \partial_\mu \alpha, \mathbf{s} \rightarrow \mathbf{s},
$$

(3.14)

that are just the local rotations in tangent to the vector $\mathbf{s}$ plane. From (3.13) follows that $V_\mu$ and $\psi_\mu$ fields subject to the system (the integrability condition),

$$
D_0 \psi = D_1 \psi_0, \quad [D_0, D_1] = 2g(\bar{\psi}\psi_0 - \bar{\psi}_0\psi),
$$

(3.15a, b)

where we skip the index for $\psi_1$ field. This system coincides with (2.15). Moreover, the reduced Jackiw-Pi model (2.7) corresponds to the constraint (2.16). Under this constraint, the moving frame evolution (3.13) describes,

$$
D_0 \mathbf{n}_+ = -2giD_1 \psi \mathbf{s}, D_1 \mathbf{n}_+ = -2g\psi \mathbf{s},
$$

(3.16a, b)
\[ \partial_0 \mathbf{s} = iD_1 \psi_- - i\bar{D}_1 \bar{\psi}_+ , \quad \partial_1 \mathbf{s} = \psi_- + \bar{\psi}_+ , \]

(3.16c, d)

Differentiating (3.16d),

\[ \partial_1^2 \mathbf{s} = D_1 \psi_- + \bar{D}_1 \bar{\psi}_+ - 4g|q_1|^2 \mathbf{s} , \]

(3.17)

we immediately see that vector \( \mathbf{s} \) satisfies the Landau-Lifshitz equation for continuum isotropical Heisenberg spin chain model,

\[ \partial_0 \mathbf{s} = \mathbf{s} \times \partial_1^2 \mathbf{s} , \]

(3.18)

where \( \mathbf{s} \) belongs to the 2-dimensional sphere \( S^2 \) (\( g = 1 \))\(^{14} \), or pseudosphere \( S^{1,1} \) (\( g = -1 \))\(^{15} \). The above results illuminate the role of the Chern-Simons statistical gauge field in the Heisenberg model. Two components, \( A_0, A_1 \) are just the gauge degrees of freedom of rotation in the tangent plane, while the \( B = A_2 \) component is defined by relations

\[ \partial_1 B = -\frac{1}{4gk} (\partial_1 \mathbf{s})^2 , \]

(3.19a)

\[ \partial_0 B = \frac{1}{k} \partial_1 \mathbf{s} (\partial_1^2 \mathbf{s} \wedge \mathbf{s}) , \]

(3.19b)

The right hand side of Eq.(3.19a) is the energy density of the model (3.18). As in Sec.II, (see Eq.(2.8a)), the local magnetic energy is always accompanied with the gradient of the \( B \) field. From above Eqs.(3.19) it is evident that the total magnetic energy is conserved quantity, defining the asymptotic jump

\[ B(+\infty) - B(-\infty) = -\frac{1}{4gk} \int_{-\infty}^{+\infty} (\partial_1 \mathbf{s})^2 dx. \]

(3.20)

As shown in Sec.II, the gauge constraint (2.16) is the second member of the infinite hierarchy of the gauge constraints (2.25). To show that to every constraint from this hierarchy corresponds the moving frame system and the spin model we can proceed analogously to the case (2.18-19). After redefinition (2.18) the system (3.13) becomes,

\[
\begin{pmatrix}
D_0 \mathbf{n}_+ \\
D_0 \mathbf{n}_-
\end{pmatrix} = -2gs \begin{pmatrix}
\psi_0 \\
\bar{\psi}_0
\end{pmatrix} + 2g \begin{pmatrix}
\mathbf{n}_+ \\
-\mathbf{n}_-
\end{pmatrix} \int^x (\bar{\psi}\psi - \bar{\psi}_0\psi) ,
\]

(3.21)

\[
\begin{pmatrix}
D_1 \mathbf{n}_+ \\
D_1 \mathbf{n}_-
\end{pmatrix} = -2gs \begin{pmatrix}
\psi \\
\bar{\psi}
\end{pmatrix} ,
\]

(2.22)

\[ \partial_0 \mathbf{s} = \psi_0 \mathbf{n}_- + \bar{\psi}_0 \mathbf{n}_+ , \quad \partial_1 \mathbf{s} = \psi_- + \bar{\psi}_+ . \]

(2.23)
The tangent space vectors evolution (3.21) is convenient to be rewritten as,

$$
\begin{align*}
\left( \frac{D_0 n_+}{D_0 n_-} \right) &= -2g \left( \begin{array}{c}
\mathbf{s} - n_+ \int^x \bar{\psi} \\
n_- \int^x \bar{\psi} \\
\mathbf{s} - n_- \int^x \psi
\end{array} \right) \left( \begin{array}{c}
\psi_0 \\
\bar{\psi}_0
\end{array} \right),
\end{align*}
$$

(3.24)

Then, we note that compatibility condition for system (3.22 -24) is given by (2.20). Moreover, the hierarchy of the gauge fixing constraints (2.25) produces the hierarchy of the moving frame evolutions (3.24) and the related higher order analogs of the NLS (2.30). This moving frame hierarchy generates also the higher order analogs of the Heisenberg model (3.18). To follow this direction we need to extract information only in terms of the spin vector \( \mathbf{s} \). First we note, that the integrand of (2.21)

$$
2ig(\bar{\psi}\psi_0 - \bar{\psi}_0\psi) = s(\partial_0 s \wedge \partial_1 s),
$$

(3.25)

is the topological charge density for the spin configuration on the space-time worldsheet, or the volume element on the spin phase space. Then, we obtain the recursion relations for the evolution equations,

$$
\partial_0 s = [s \wedge \partial_1 - \partial_1 s \int^x s(\partial_1 s \wedge \cdot)]\partial_0 s_n s,
$$

(3.26)

where integer \( n \) describes the \( n \)-th member of the hierarchy. The first few evolutions are given by,

$$
\partial_0 s = \partial_1 s, \\
\partial_0 s = s \wedge \partial^2_1 s, \\
\partial_0 s = s \wedge s \wedge \partial^3_1 s - \frac{3}{2}(\partial_1 s \partial_1 s)\partial_1 s.
$$

(3.27-29)

The moving frame approach allows us to introduce also the traditional zero curvature description. To proceed, we need to find an appropriate reduction for (3.7-8). We note that the NLS (2.12) can be considered as a quantum mechanics with the potential function proportional to the probability density, \( U(x) = \Psi(x)\Psi(x) \). But, as well known\(^{16} \), the Galilean boosts representation in the quantum mechanics makes use of 1-cocycle \( \omega_1 \):

$$
U(v)\Psi(x) = e^{-2\pi i \omega_1(x; v)}\Psi(x - vt),
$$

(3.30)

with the following condition to satisfy,

$$
\omega_1(x - u; v) - \omega_1(x; u + v) + \omega_1(x; u) = 0, (mod \ Z),
$$

(3.31)

where \( U(v) \) is the unitary operator shifting the coordinates as

$$
U(v)xU^{-1}(v) = x - vt.
$$

(3.32)
Then, condition (3.31) ensures the group multiplication for \( U(v) \) operators:

\[
U(v)U(u) = U(u + v).
\] (3.33)

The quantity \( \omega_1(x; v) \) is given by

\[
2\pi\omega_1(x; v) = -\frac{v^2}{4}t + \frac{v}{2}x,
\] (3.34)

and provides the local phase transformation for the wave function. Thus, the NLS model (2.12) is invariant under the Galileo transformations

\[
x \to x' = x - vt, \quad t \to t' = t,
\] (3.35a)

\[
\Psi \to \Psi' \exp i\left(\frac{v^2}{4}t' + \frac{v}{2}x'\right).
\] (3.35b)

This invariance can be interpreted in two ways. First, that the space-time transformation to constant velocity moving frame should be accompanied with the phase transformation for the ”wave function” \( \Psi \). This is the 1-cocycle representation as given above. In the second interpretation we can say, that any one-parameter group \((v = \text{const})\) of the phase transformations (3.35b) becomes the symmetry of the model (2.12) if we transform coordinates to the moving frame with velocity \( v \) according to (3.35a). The second interpretation has the flavor of the \( U(1) \) Abelian local gauge theory with the gauge potentials transforming on the constant value. In fact, the linear in space and time phase transformations (3.35b) satisfying (3.31) generate a subgroup of the Abelian gauge transformations, shifting \( W_1 \) potential (2.4) on the constant velocity. This is why it is of interest to consider the restricted class of the constant gauge potentials. As shown below they are associated with the spectral parameter in the linear problem for the NLS. Therefore, in our approach the whole structure of the linear problem with the spectral parameter has the origin from the \( U(1) \) gauge potential \( W_1 \), while the related phase is the linear function of space and time.

Our idea has been suggested by some known facts from the theory of superconductivity\(^{17}\) and the field theory with curved momentum space\(^{18}\). For the superconductor in the ground state under the external electromagnetic action the generalised canonical momentum \( p = mv + e/cA \) vanishes. This fact provides the relation between superconducting electrons velocity and the vector potential,

\[
v = -e/cA,
\] (3.36)

restricting Abelian gauge transformations to the London gauge\(^{17}\),

\[
div A = 0.
\] (3.37)
In one space dimension this gauge leads to the constant gauge potential \( A \). Furthermore, for constant potentials \( A_\mu \), transition to the new momentum,

\[
p_\mu \rightarrow p_\mu - c_\mu, \tag{3.38}
\]
generates the group of parallel transitions in the momentum space. For the non-relativistic theory it is just the Galileo transformation. Finally, as was shown recently\(^1\), the spectral parameter arises as the constant valued statistical gauge field of compactified 2+1 dimensional theory. All results mentioned above suggest that the spectral parameter is deeply connected with Abelian gauge field.

Let us consider the moving frame equations (3.16),

\[
D_0 n_+ = -2g i D_1 \psi s + 2ig |\psi|^2 n_+, \quad D_1 n_+ = -2g \psi s, \tag{3.39a, b}
\]

\[
\partial_0 s = i D_1 \psi n_- - i \bar{D}_1 \bar{\psi} n_+, \quad \partial_1 s = \psi n_- + \bar{\psi} n_+, \tag{3.39c, d}
\]

written in terms of the zero strength gauge potential \( W_\mu \), (2.13),

\[
\partial_0 W_1 - \partial_1 W_0 = 0. \tag{3.39e}
\]

The general solution of the last equation is of the pure gauge form given in terms of the real function,

\[
W_\mu = \partial_\mu \alpha. \tag{3.40}
\]

For a restricted class of functions, linear in \( x \) and \( t \), we put

\[
W_1 = 2\lambda = \text{const}. \tag{3.41}
\]

In this case the NLS (2.7a) becomes,

\[
i \partial_0 \psi - 2i \lambda \partial_1 \psi + \partial_1^2 \psi + 2g |\psi|^2 \psi = 0, \tag{3.42}
\]

where we have to choose \( W_0 = 1/2W_1^2 = 2\lambda^2 \). In the constant velocity frame, \( v = -2\lambda \), this equation acquires the normal form (2.7a),

\[
i \partial_0 \psi + \partial_1^2 \psi + 2g |\psi|^2 \psi = 0, \tag{3.43}
\]

and any \( \lambda \) dependence disappears. However, it remains in the corresponding moving frame system (3.39) having the form,

\[
\partial_1' \begin{pmatrix} n_+ \\ n_- \\ s \end{pmatrix} = \begin{pmatrix} i\lambda & 0 & -2g \psi \\ 0 & -i\lambda & -2g \bar{\psi} \\ \bar{\psi} & \psi & 0 \end{pmatrix} \begin{pmatrix} n_+ \\ n_- \\ s \end{pmatrix}, \tag{3.44a}
\]
theoretical structure of it. The frame method is going deeply in to the problem and naturally illuminates the gauge doesn’t seem so useful. In opposite, we find that a more general setting - the moving curvature which is proportional to the nonlinearity and can be negative this analogy for the NLS (3.43).

Let us first rewrite the hierarchy (2.30) for the constant potentials equations (2.35) we need to modify the above Galileo transformation approach (3.42-45) as a moving curve equations

\[ \partial_0' \left( \begin{array}{c} n_+ \\ n_- \\ s \end{array} \right) = \left( \begin{array}{ccc} -i\lambda^2 + 2ig|\psi|^2 & 0 & -2ig\partial_1'\psi + 2g\lambda\psi \\ 0 & i\lambda^2 - 2ig|\psi|^2 & 2ig\partial_1'\bar{\psi} + 2g\lambda\bar{\psi} \\ -(i\partial_1'\bar{\psi} + \lambda\bar{\psi}) & i\partial_1'\psi - \lambda\psi & 0 \end{array} \right) \left( \begin{array}{c} n_+ \\ n_- \\ s \end{array} \right), \quad (3.44b) \]

In the normal basis \( n_1, n_2, s \), we have equations

\[ \partial_{1'} \left( \begin{array}{c} n_1 \\ n_2 \\ s \end{array} \right) = \left( \begin{array}{ccc} 0 & -\lambda & -gU_1 \\ \lambda & 0 & -gU_2 \\ U_1 & U_2 & 0 \end{array} \right) \left( \begin{array}{c} n_1 \\ n_2 \\ s \end{array} \right), \quad (3.45a) \]

\[ \partial_0' \left( \begin{array}{c} n_1 \\ n_2 \\ s \end{array} \right) = \left( \begin{array}{ccc} 0 & \lambda^2 - 2g|\psi|^2 & g\partial_1'U_2 + g\lambda U_1 \\ -\lambda^2 + 2g|\psi|^2 & 0 & -g\partial_1'U_2 + g\lambda U_2 \\ -\partial_1'U_2 - \lambda U_1 & \partial_1'U_1 - \lambda U_2 & 0 \end{array} \right) \left( \begin{array}{c} n_1 \\ n_2 \\ s \end{array} \right), \quad (3.45b) \]

where we defined (see eqs. (3.10)) \( U_1 = 2\Re(\psi), U_2 = 2\Im(\psi) \). The first system (3.45a) in terms of the gauge invariant fields as in (2.12),\( (\lambda = 0) \), has the form of the Frenet equations for the curve. This suggests the idea to interpret the gauge invariant form of (3.45) as a moving curve equations\(^\dagger\). But due to the essential role of the space-time curvature which is proportional to the nonlinearity and can be negative this analogy doesn’t seem so useful. In opposite, we find that a more general setting - the moving frame method is going deeply in to the problem and naturally illuminates the gauge theoretical structure of it.

Note that the potentials choice,

\[ W_1 = 2\lambda, W_0 = 2\lambda^2, \quad (3.46) \]

which allowed us to remove the \( \lambda \) dependence from equations of motion (3.42) is exactly the 1-cocycle (3.35b). To reproduce the usual Lax type representation we use the spinor representation of system (3.45), and according (3.6) we have \( 2 \times 2 \) Zakharov-Shabat spectral problem

\[ J_1 = +\frac{i}{2}\lambda\sigma_3 + \left( \begin{array}{cc} 0 & -g\bar{\psi} \\ \psi & 0 \end{array} \right), \quad (3.47a) \]

\[ J_0 = i\sigma_3[-\frac{\lambda^2}{2} + g|\psi|^2] + \left( \begin{array}{cc} 0 & g(i\partial_1 + \lambda)\bar{\psi} \\ (i\partial_1 - \lambda)\psi & 0 \end{array} \right). \quad (3.47b) \]

for the NLS (3.43).

To derive the hierarchy of moving frames, providing the linear problem for any of equations (2.35) we need to modify the above Galileo transformation approach (3.42-43). Let us first rewrite the hierarchy (2.30) for the constant potentials \( W_\mu \), denoting \( W_1 = 2\lambda \):

\[ i\partial_{0\alpha} \left( \begin{array}{c} \psi \\ -\bar{\psi} \end{array} \right) + \frac{1}{2} W_{0\alpha} \left( \begin{array}{c} \psi \\ \bar{\psi} \end{array} \right) = (\mathcal{L} + \lambda I)^n \left( \begin{array}{c} \psi \\ \bar{\psi} \end{array} \right) = \sum_{k=0}^{n} \binom{k}{n} \lambda^{n-k} \mathcal{L}^k \left( \begin{array}{c} \psi \\ \bar{\psi} \end{array} \right) = \]
\[
\sum_{k=0}^{n} \binom{k}{n} \lambda^{n-k} \left( \frac{\psi^{(k)}}{\bar{\psi}_0^{(k)}} \right) = \lambda^n \left( \frac{\psi}{\bar{\psi}} \right) + \sum_{k=1}^{n-1} \binom{k}{n} \lambda^{n-k} \left( \frac{\psi^{(k)}}{\bar{\psi}_0^{(k)}} \right) + \left( \frac{\psi^{(n)}}{\bar{\psi}_0^{(n)}} \right) = \\
\lambda^n \left( \frac{\psi}{\bar{\psi}} \right) + \sum_{k=1}^{n-1} \binom{k}{n} \lambda^{n-k} i \sigma_3 \partial_{\theta_k} \left( \frac{\psi}{\bar{\psi}} \right) + \mathcal{L}^n \left( \frac{\psi}{\bar{\psi}} \right),
\]

(3.48)

where \([\mathcal{L}, \lambda I] = 0\) is used. Choosing the gauge potentials \(W_{0n} = 2\lambda^n\), and collecting all terms with the time hierarchy derivatives in the left hand side we get,

\[
i\{\partial_{\theta_n} - \sum_{k=1}^{n-1} \binom{k}{n} \lambda^{n-k} \partial_{\theta_k} \} \left( \frac{\psi}{\bar{\psi}} \right) = \mathcal{L}^n \left( \frac{\psi}{\bar{\psi}} \right).
\]

(3.49)

Defining the new time hierarchy \(\{t_0', t_0'', ..., t_0', ...\}\), such that,

\[
\partial_{\theta_n} = \partial_{\theta_n} - \sum_{k=1}^{n-1} \binom{k}{n} \lambda^{n-k} \partial_{\theta_k},
\]

(3.50)

we obtain for (3.49) the usual NLS hierarchy form (2.35):

\[
i \sigma_3 \partial_{\theta_n} \left( \frac{\psi}{\bar{\psi}} \right) = \mathcal{L}^n \left( \frac{\psi}{\bar{\psi}} \right).
\]

(3.51)

The matrix structure of the above time hierarchy transformation can be realized by the upper triangular matrix

\[
\begin{pmatrix}
 t_1' \\
t_2' \\
t_3' \\
\vdots \\
t_n'
\end{pmatrix} = 
\begin{pmatrix}
 1 & a_{12} & a_{13} & \ldots & a_{1n} \\
 0 & 1 & a_{23} & \ldots & a_{2n} \\
 0 & 0 & 1 & \ldots & a_{3n} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
 t_1 \\
t_2 \\
t_3 \\
\vdots \\
t_n
\end{pmatrix},
\]

(3.52)

while the time derivatives transform by the lower triangular matrix

\[
\begin{pmatrix}
 \partial_{\theta_1} \\
\partial_{\theta_2} \\
\partial_{\theta_3} \\
\vdots \\
\partial_{\theta_n}
\end{pmatrix} = 
\begin{pmatrix}
 1 & 0 & 0 & \ldots & 0 \\
 a_{12} & 1 & 0 & \ldots & 0 \\
 a_{13} & a_{23} & 1 & \ldots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 a_{1n} & a_{2n} & a_{3n} & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
 \partial_{\theta_1'} \\
\partial_{\theta_2'} \\
\partial_{\theta_3'} \\
\vdots \\
\partial_{\theta_n'}
\end{pmatrix},
\]

(3.53)

with coefficients chosen as an appropriate polynomial of the parameter \(\lambda\).

By using the moving frame formulas (3.21-24) we can construct the hierarchy of moving frames, which provides also the linear problem for any member of the hierarchy (3.51). For the space part we have,

\[
\begin{pmatrix}
 D_1 n_+ \\
 D_1 n_- \\
\partial_1 s
\end{pmatrix} = 
\begin{pmatrix}
 0 & 0 & -2g\bar{\psi} \\
 0 & 0 & -2g\psi \\
\bar{\psi} & \psi & 0
\end{pmatrix}
\begin{pmatrix}
 n_+ \\
n_- \\
s
\end{pmatrix},
\]

(3.54)
in which $W_1 = 2\lambda$ gives us the space part of the moving frame system and is universal for the whole hierarchy

$$\partial_1 \left( \begin{array}{c} n_+ \\ n_- \\ s \end{array} \right) = \left( \begin{array}{ccc} i\lambda & 0 & -2g\psi \\ 0 & -i\lambda & -2g\bar{\psi} \\ \bar{\psi} & \psi & 0 \end{array} \right) \left( \begin{array}{c} n_+ \\ n_- \\ s \end{array} \right)$$

(3.53)

For the time evolution we start from the system for the $t_n$ time:

$$\left( \begin{array}{c} D_{0_n} n_+ \\ D_{0_n} n_- \\ \partial_{0_n} s \end{array} \right) = -2g \left( \begin{array}{ccc} s - n_+ \int \bar{\psi} \\ n_- \int \bar{\psi} \\ -s - n_- \int \bar{\psi} \end{array} \right) \left( \begin{array}{c} n_+ \int \psi \\ n_- \int \psi \end{array} \right) \Lambda^{(n-1)} \left( \begin{array}{c} \psi_0(n-1) \\ \bar{\psi}_0(n-1) \end{array} \right)$$

(3.54)

After substituting the value $W_{0_n} = 2\lambda^n$ and (2.25), we have the expression

$$\partial_{0_n} \left( \begin{array}{c} n_+ \\ n_- \\ s \end{array} \right) = i\lambda^n \left( \begin{array}{c} n_+ \\ n_- \\ 0 \end{array} \right) - 2g \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c} n_+ \\ n_- \\ s \end{array} \right)$$

(3.55)

Then, for the transformed time hierarchy using (3.50) and (3.55), we get

$$\partial_{0_n} \left( \begin{array}{c} n_+ \\ n_- \\ s \end{array} \right) = i\lambda^n (3 - 2^n) \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} n_+ \\ n_- \\ s \end{array} \right) - 2g \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c} n_+ \\ n_- \\ s \end{array} \right)$$

$$\left( \begin{array}{c} s - n_+ \int \bar{\psi} \\ n_- \int \bar{\psi} \\ -s - n_- \int \bar{\psi} \end{array} \right) \left( \begin{array}{c} n_+ \int \psi \\ n_- \int \psi \end{array} \right) \Lambda^{n-1} \sum_{k=1}^{n-1} \binom{k}{n} \lambda^{n-k} \Lambda^{k-1} \left( \begin{array}{c} \psi \\ \bar{\psi} \end{array} \right)$$

(3.56)

The system (3.53), (3.56) is the moving frame system for the hierarchy (3.51). The space part is linear in the spectral parameter $\lambda$, while the time part contains the $n$-th degree polynomial for the $n$-th member of the hierarchy. To reproduce this dependence in the explicit form, according to (2.23a), we decompose the gauge covariant operator $\Lambda = \mathcal{L} + 2I$. As a result we have the moving frame evolution in terms of the usual AKNS operator $\mathcal{L}$,

$$\partial_{0_n} \left( \begin{array}{c} n_+ \\ n_- \\ s \end{array} \right) = i\lambda^n (3 - 2^n) \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} n_+ \\ n_- \\ s \end{array} \right) - 2g \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c} n_+ \\ n_- \\ s \end{array} \right)$$

$$\left( \begin{array}{c} s - n_+ \int \bar{\psi} \\ n_- \int \bar{\psi} \\ -s - n_- \int \bar{\psi} \end{array} \right) \left( \begin{array}{c} n_+ \int \psi \\ n_- \int \psi \end{array} \right) \left( \sum_{l=0}^{n-1} \binom{l}{n-1} \right) \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \binom{k}{n} \binom{l}{k-1} \lambda^{n-1-l} \mathcal{L}^l \left( \begin{array}{c} \psi \\ \bar{\psi} \end{array} \right)$$

(3.57)
For \( n = 2 \) these moving frame system coincides with the one (3.44) obtained by the Galileo transformations. This suggests one to construct the hierarchy of the Galileo transformations with the properly defined time hierarchy for any of equations (3.51) and the 1-cocycle conditions.

**IV. BF GAUGE THEORY AND THE TIME HIERARCHY**

In the present section we map the integrable hierarchy from Sec.III to the non-Abelian BF gauge theory. The BF theory is described by the action

\[
S = \frac{kL}{4\pi} \int_{\Sigma_2} Tr(BF),
\]

(4.1)

where, \( F = e^{\mu\nu} F_{\mu\nu} \), is the gauge curvature tensor in the adjoint representation of the appropriately chosen group, while \( B \) is the world scalar Lagrange multipliers, transforming according to the coadjoint representation\(^{11}\). This model, formulated in terms of 2-form \( F \), and the zero-form \( B \) for any 2-dimensional manifold \( \Sigma_2 \) defines the topological field theory\(^{20}\). For the non-compact groups \( SO(2,1) \) and \( ISO(1,1) \) it provides a gauge theoretical formulation of the Jackiw-Teitelboim lineal gravity theories\(^{11,21,22}\). This model, being the constant curvature theory, includes the world scalar Lagrangian multiplier field together with the Riemann scalar \( R \) and is a type of scalar-tensor theories or the "dilaton" gravity. In the gauge formulation of the gravity theory one introduces instead of the metric tensor \( g_{\mu\nu} \), the Einstein-Cartan zweibeins and the spin-connection viewed as independent variables. Then, the diffeomorphism-invariance of the gravity theory becomes a part of the local gauge invariance. The matter \( \psi \) and the gauge field \( V \) from the Sec.II are just the zweibeins and the spin-connection fields\(^{23}\), defining the moving frames equations. The \( SU(2) \) and \( SU(1,1) \) with the compact Abelian \( U(1) \) subgroup in our case, correspond to the Euclidean gravity, with the local \( O(2) \) rotations in the tangent plane, insted of the \( O(1,1) \) Lorentz rotations.

To describe the time evolution we suppose that \( \Sigma_2 = R \times T \). Then the action (4.1) is,

\[
S = \frac{kL}{2\pi} \int TrB(\partial_0 J_1 - \partial_1 J_0 + [J_0, J_1])d^2x = \frac{kL}{2\pi} \int Tr(B\partial_0 J_1 + J_0(\partial_1 B + [J_1, B]))d^2x,
\]

(4.2)

where we integrated the second term by parts. In terms of (3.8) parametrization and with \( B \) field given by \( B = i\phi_0 \sigma_3 + g\phi \sigma_- - \bar{\phi} \sigma_+ \) the action becomes,

\[
S = \gamma \int (\bar{\phi} \partial_0 \psi + \phi \partial_0 \bar{\psi} + \frac{1}{2} \phi_0 \partial_0 V_1 - \mathcal{H})d^2x,
\]

(4.3)
where the Hamiltonian density is given by,
\[ H = -\{ \bar{\psi}_0 (D_1 \phi + 2i g \phi_0 \bar{\psi}) + \psi_0 (\bar{D}_1 \bar{\phi} - 2i g \bar{\phi}_0 \bar{\psi}) + V_0 \frac{1}{2} (\partial_1 \phi_0 + i \phi \bar{\psi} - i \bar{\phi} \psi) \}, \quad (4.4) \]
and \( \gamma \) is the renormalized coupling constant. Since the action is already in the first order form we can immediately write the canonical brackets\(^{24} \):
\[ \{ \psi(x), \bar{\phi}(y) \} = \frac{1}{\gamma} \delta(x - y), \quad \{ \bar{\psi}(x), \phi(y) \} = \frac{1}{\gamma} \delta(x - y), \]
\[ \{ V_1(x), \phi_0(y) \} = \frac{2}{\gamma} \delta(x - y), \quad (4.5) \]
The nondynamical Lagrange multipliers \( \psi_0, \bar{\psi}_0 \) and \( V_0 \) lead to the BF “Gauss-law” constraints
\[ D_1 \phi + 2i g \bar{\phi}_0 = 0, \quad \bar{D}_1 \bar{\phi} - 2i g \bar{\phi}_0 = 0, \quad (4.6) \]
\[ \partial_1 \phi_0 + i \bar{\phi} \bar{\psi} - i \bar{\phi} \psi = 0, \quad (4.7) \]
Due to the canonical brackets (4.5) the Gauss’s law constraints generate the \( SU(2) \) (\( g = 1 \)), or \( SU(1,1) \) (\( g = -1 \)), algebra of the gauge transformations in the dynamical fields \( \psi, \bar{\psi} \) and \( V_1 \). Since Hamiltonian (4.4) is proportional to the constraints (4.6), (4.7) it is weakly vanishing \( H \approx 0 \). The above properties characterise any reparametrization invariant theory.

We can partially solve constraints (4.6), (4.7) by integration of the last one
\[ \phi_0 = -i \int^x (\bar{\psi} \phi - \psi \bar{\phi}) \quad (4.8) \]
Then, substituting to the first two (4.6), we get
\[ M_\psi(\phi) \equiv D_1 \phi + 2g \psi \int^x (\bar{\psi} \phi - \psi \bar{\phi})(y)dy = 0, \quad (4.9a) \]
\[ M_\bar{\psi}(\bar{\phi}) \equiv \bar{D}_1 \bar{\phi} - 2g \bar{\psi} \int^x (\bar{\psi} \phi - \psi \bar{\phi})(y)dy = 0, \quad (4.9b) \]
where we defined the integro-differential operators \( M \). Combined these two relations can be written in terms of \( U(1) \) gauge covariant \( \Lambda \) operator (2.22):
\[ i \sigma_3 \left( \frac{M_\psi(\phi)}{M_\bar{\psi}(\bar{\phi})} \right) = \Lambda \left( \begin{array}{c} \phi \\ \bar{\phi} \end{array} \right) = 0 \quad (4.10) \]
This means that the constraint surface of the BF theory is defined by zero modes of the covariant \( \Lambda \) operator. In terms of the usual \( \mathcal{L} \) operator (2.23) we have the relation
\[ \mathcal{L} \left( \begin{array}{c} \phi \\ \bar{\phi} \end{array} \right) = -\frac{1}{2} V_1 \left( \begin{array}{c} \phi \\ \bar{\phi} \end{array} \right), \quad (4.11) \]
which for the constant valued gauge potential, like in (3.36), $V_1 = -2\lambda$, becomes the eigenvalue problem for the operator $\mathcal{L}$:

$$\mathcal{L} \left( \frac{\phi}{\partial x} \right) = \lambda \left( \frac{\phi}{\partial x} \right)$$

(4.12)

Comparing with the theory of solitons$^{12}$ show that our world scalars $\phi$ and $\bar{\phi}$ are very similar to the squared eigenfunctions for the Zhakharov-Shabat problem. Moreover, the constraints surface system (4.6),(4.7) is related to the Maxwell-Bloch equations$^{25}$, describing the propagation of pulses through an inhomogeneously broadened and resonant medium, where $\psi$ is the electric field envelope, $\phi$ is the polarization and $\phi_0$ is the excitation number density. It would be interesting to illuminate the relation between last problem and the BF theory.

We can express the action (4.3) in terms of the bilinear form. For any two complex functions $\phi$ and $\chi$ vanishing at infinities we define:

$$<\phi, \chi> = \int_{-\infty}^{\infty} (\bar{\phi}\chi + \phi\bar{\chi}) dx = 2\Re \int_{-\infty}^{\infty} \bar{\phi}\chi dx,$$

(4.13)

Then, the operator $\mathcal{M}_\psi(\phi)$ from (4.9) is skew symmetric with respect to this bilinear form

$$<\phi, \mathcal{M}\chi> = -<\mathcal{M}\phi, \chi>$$

(4.14)

if for the integral part of $\mathcal{M}$ we use the proper definition (2.22a). Another skew symmetric operator is the immaginary unit $i = \sqrt{-1}$,

$$<\phi, i\chi> = -<i\phi, \chi>$$

(4.15)

In spite of the skew symmetry of these two operators, the product

$$\mathcal{R} = i\mathcal{M},$$

(4.16)

being the recursion operator, is not hermitian. This is due to the noncommutativity of the operators,

$$[i, \mathcal{M}](\phi) = -4ig\psi \int_{x}^{\infty} \bar{\phi}\psi$$

(4.17)

To derive the Hamiltonian dynamics for action (4.3) we introduce the $U(1)$ gauge invariant variables. Representing potential $V_1$ in a pure gauge form,

$$V_1 = 2\partial_1 \alpha(x),$$

(4.18)
for the gauge invariant $B(x)$ (the magnetic field analog),

$$\phi_0(x) = \int^x B(y)dy,$$

(4.19)

the Poisson bracket (4.5) results from the following canonical bracket,

$$\{\alpha(x), B(y)\} = -\frac{1}{\gamma}\delta(x - y).$$

(4.20)

Defining the $U(1)$ gauge invariant fields,

$$\psi = \Psi e^{i\alpha}, \quad \phi = \Phi e^{i\alpha},$$

(4.21)

for action (4.3) we get,

$$S = \gamma \int [\bar{\Phi} \partial_0 \Psi + \Phi \partial_0 \bar{\Psi} + \{\bar{\psi}_0 e^{i\alpha}(\partial_1 \Phi + 2ig\Psi \int^x B) + c.c.\} + \frac{1}{2}(V_0 - 2\partial_0\alpha)\{B + i\bar{\Psi} \Phi - i\bar{\Phi} \Psi\}].$$

(4.22)

After redefinition of the Lagrange multipliers,

$$V_0 \equiv V_0 - 2\partial_0\alpha, \quad \bar{\Psi}_0 \equiv \bar{\psi}_0 e^{i\alpha}, \quad \Psi_0 \equiv \psi_0 e^{-i\alpha}$$

(4.23)

the action is written only in terms of the gauge invariant dynamical variables $\Psi, \bar{\Psi}, \Phi, \bar{\Phi}$:

$$S = \gamma \int [\bar{\Phi} \partial_0 \Psi + \Phi \partial_0 \bar{\Psi} + \{\bar{\psi}_0 e^{i\alpha}(\partial_1 \Phi + 2ig\Psi \int^x B) + c.c.\} + \frac{1}{2}V_0\{B + i\bar{\Psi} \Phi - i\bar{\Phi} \Psi\}].$$

(4.24)

We can solve the Gauss law constraint explicitly,

$$B = -i(\bar{\Psi} \Phi - \Psi \bar{\Phi}).$$

(4.25)

Then the action is,

$$S = \gamma \int (\bar{\Phi} \partial_0 \Psi + \Phi \partial_0 \bar{\Psi}) - H,$$

(4.26a)

where the Hamiltonian in terms of the bilinear form (4.13) becomes,

$$H = -\gamma <\Psi_0, M\Phi> = \gamma <M\Psi_0, \Phi>$$

(4.26b)

Here, the skew symmetric operator $M$ is the noncovariant form of the $\mathcal{M}$ operator (4.9):

$$M_\Psi(\Phi) \equiv \partial_1 \Phi + 2g\Psi \int^x (\bar{\Psi} \Phi - \Psi \bar{\Phi})(y)dy = 0,$$

(4.27)
and coincides with the one introduced by Magri. The Poisson brackets for $U(1)$ gauge invariant dynamical variables,

$$\{\Psi(x), \Phi(y)\} = \frac{1}{\gamma} \delta(x-y), \quad \{\Psi(x), \tilde{\Phi}(y)\} = \frac{1}{\gamma} \delta(x-y)$$

(4.28)

generate the Hamiltonian equations given by,

$$\partial_0 \Psi = \{\Psi, H\} = \frac{1}{\gamma} \frac{\delta H}{\delta \Phi} = M_\Psi(\Psi_0),$$

(4.29a)

$$\partial_0 \tilde{\Psi} = \{\tilde{\Psi}, H\} = -\frac{1}{\gamma} \frac{\delta H}{\delta \Phi} = M_\Psi(\Psi_0),$$

(4.29b)

or in terms of the recursion operator, $R = iM$,

$$i\partial_0 \Psi = R_\Psi(\Psi_0), \quad -i\partial_0 \tilde{\Psi} = \overline{R_\Psi(\Psi_0)}$$

(4.30)

Thus, we can see the effect of introducing the $\Phi$ field, as the canonical momentum conjugate to the dynamical field $\Psi$, is to eliminate the nondynamical Lagrange multipliers $\Psi_0$. Due to the arbitrariness of these multipliers, by the recursion operator $R$, we can construct the whole hierarchy of the multipliers,

$$\Psi_0^{(n)} = R_\Psi(\Psi_0^{(n-1)}) = ... = R_\Psi^n(\Psi_0^{(0)}),$$

(4.31)

Then, the related actions,

$$S_{n+1} = \gamma \int (\tilde{\Phi} \partial_{0_{n+1}} \Psi + \Phi \partial_{0_{n+1}} \tilde{\Psi}) - H_{n+1},$$

(4.32a)

with the Hamiltonian,

$$H_{n+1} = -\gamma <\Psi_0^{(n)}, M_\Psi \Phi > = \gamma < M_\Psi \Psi_0^{(n)}, \Phi >,$$

(4.32b)

define the evolution hierarchy

$$i\partial_{0_{n+1}} \Psi = R_{\Psi}^{n+1}(\Psi_0^{(0)}), \quad -i\partial_{0_{n+1}} \tilde{\Psi} = \overline{R_{\Psi}^{n+1}(\Psi_0^{(0)})}.$$  

(4.33)

By skew symmetry property of the $M$ operator we have the relations,

$$H_{n+1} = -\gamma < R^{(n)}(\Psi_0^{(0)}), M(\Phi) > = -\gamma < R^{(n-1)}(\Psi_0^{(0)}), MR(\Phi) > =$$

$$-\gamma < R^{(n-2)}(\Psi_0^{(0)}), MR^2(\Phi) > = ... = -\gamma < \Psi_0^{(0)}, MR^n(\Phi) >$$

(4.34)

or,

$$H_{n+1} = \gamma < iR^{(n+1)}(\Psi_0^{(0)}), \Phi > = \gamma < i\Psi_0^{(n+1)}, \Phi^{(0)} > =$$
\[ \gamma < \Psi_0^{(0)}, i R^{n+1}(\Phi) > = \gamma < \Psi_0^{(0)}, i \Phi^{(n+1)} >, \quad (4.35) \]

where,

\[ \Phi^{(n+1)} = R^{n+1}(\Phi^{(0)}), \quad \Phi^{(0)} \equiv \Phi. \quad (4.36) \]

If the constraint surface is defined by the relation

\[ R(\Phi^{(0)}) = 0, \quad (4.37) \]

then it satisfies to the constraints,

\[ R^2(\Phi^{(0)}) = R^3(\Phi^{(0)}) = ... = 0, \quad (4.38) \]

for any degree \( n \). Moreover, the surface (4.37) defines evolution,

\[ i \partial_0 \Psi = R(\Psi^0), \quad (4.39) \]

while the higher degree surface (4.38) generates the higher evolution,

\[ i \partial_n \Psi = R^n(\Psi^0). \quad (4.40) \]

This just confirms the evident fact, that the first member of the hierarchy (4.39) completely defines the higher evolutions.

Hence, in the Hamiltonian interpretation, the canonical bracket (4.28) for the first member of hierarchy (4.39):

\[ \{ \Psi(x), \Phi^{(0)}(y) \} = \frac{1}{\gamma} \delta(x - y), \quad \{ \overline{\Psi}(x), \Phi^{(0)}(y) \} = \frac{1}{\gamma} \delta(x - y), \quad (4.41) \]

defines the higher Poisson brackets

\[ \{ \Psi(x), \Phi^{(n)}(y) \} = \{ \Psi(x), R^n \Phi^{(0)}(y) \} \frac{1}{\gamma} \delta(x - y), \]

\[ \{ \overline{\Psi}(x), \Phi^{(n)}(y) \} = \{ \overline{\Psi}(x), R^n \Phi^{(0)}(y) \} \frac{1}{\gamma} \delta(x - y). \quad (4.42) \]

In addition, every Hamiltonian (4.35) corresponds to the higher integrals of motion for the model. Thus, the BF formulation provides a natural description of the whole integrable hierarchy.

The above procedure can be extended to the case of spectral parameter dependence, by reduction of the \( U(1) \) connection to the constant value, as in Sec.II and III. The details of calculation would be published soon.
V. CONCLUSION

The main ingredient of our consideration is the moving frame method. Historically, the moving frame idea has origin from mechanics where studying the rigid body motion, one gets a one-parameter frames family depending on time and completely characterizing the rigid motion. G.Darboux and E.Cotton has generalized one-parameter frame to several parameters. The richness of the theory was demonstrated by E. Cartan (la méthode du repère mobile) who successfully applied it to research in differential geometry. We find that this deep method is conceptually simple to study integrable models and the underlying gauge theoretical structure.

In the recent study of the dimensionally reduced Jackiw-Pi model, to produce dynamics for the B field, an extension of the Lagrangian was given and the chiral solitons, propogating only in one direction was constructed. From our consideration it is clear that the chiral solitons can be created by the odd members of the hierarchy with the odd dispersion. This fact is valid even for the linear approximation. Thus, for the NLS (2.32) the linear wave with the wave number $k$ is propagating with velocity $v = k$, while the wave for $-k$ is propagating in the opposite direction with the same velocity. On the contrary, for the next member of the hierarchy, Eq.(2.33), both waves with $+k$ and $-k$ are propogating in the same direction with velocity $v = k^2$. The similar structure is valid also for solitons. This shows that the chiral solitons in our approach belong to the odd members of the NLS hierarchy.

In conclusion, as shown in the present paper, the 1+1 dimensional projection of Chern-Simons gauge interacting theory contains the hidden hierarchy of integrable systems. The statistical gauge field, after projection, still plays the crucial role defining spectral characteristics of the system. The key point for the construction is the mapping to non-Abelian BF theory, which covers all the rich structure of integrable system. The commutativity of spectral flows naturally provides the hierarchy of the gauge fixing constraints and the corresponding time hierarchy. We hope that our procedure could clarify some difficult questions of the diffeomorphism invariance breaking in the theories like the gravity, and the time hierarchy creation with the physical degrees of freedom from the reparametrization invariance.

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