In this work, the mass of the neutral pion is investigated in the presence of background magnetic fields in the framework of the Nambu–Jona-Lasinio model. Taking into account the anisotropic four-fermion interactions, a tensor current arises in the magnetized QCD system, which forms an anomalous magnetic moment (AMM) coupling in the Dirac equation for the quarks. By solving the gap equations, we find that the sign of the dynamically generated AMM is opposite to the sign of the quark’s charge and its magnitude is definitely smaller than the constituent mass. We construct two generalized Nambu-Goldstone pions, which emerge as combinations of the quantum fluctuations around the conventional scalar and the emergent tensor chiral condensates. We analytically demonstrate that the Goldstone nature has been spoiled by the dimensional reduction in the two-particle state and the corresponding decreasing mass of the lighter generalized pionic mode is a remnant of the infrared dynamics.

I. INTRODUCTION

Arising as a powerful probe in the study of vacuum properties and phase structure under the influence of the external magnetic field, the phase diagram of Quantum Chromodynamics (QCD) matter is extensively explored due to its relevance in the context of lattice gauge theories \[1–4\], off-central heavy-ion collisions \[5, 7\] and the merging process of neutron stars \[8\]. As one of the fundamental properties, the spectra of hadrons are used to describe the confined, chirally broken QCD phase and to construct the low-energy strong interactions. Among these quasi-particles, the pseudo-Goldstone meson plays an essential role since it is the degree of freedom carrying the chiral effective theory \[9\]. Unlike thermal QCD systems, the presence of the magnetic field breaks chiral symmetry from SU(2) to U(1), thus, the identified Nambu-Goldstone (NG) boson is reduced from the pseudo-scalar triplet to the individual neutral pion \[10\]. The magnetized masses of neutral and/or charged pions have been calculated by a variety of model approaches, found in \[11–22\].

In the present paper, the study of the energy dispersion of \(\pi_0\) is motivated by the emergence of the tensor polarization of the chiral condensates in magnetized QCD matter \[12\]. It is known that a spontaneous symmetry breaking appears when the Lagrangian of a system is invariant under the symmetry transformation, but the ground state is not. A more precise description is that if several ground states simultaneously break the same global symmetries, the corresponding number of NG excitations is unchanged, even though more gap equations are necessary to characterize the one-point particle state of the system. It is observed that, while the scalar vacuum expectation value (VEV) generates a dynamical fermion mass, the developed VEV of the tensor current gives rise to a dynamical anomalous magnetic moment (AMM) for the fermions \[23–25\]. We stress that our purpose is not to claim that previous model calculations with fixed AMM are wrong. Indeed, the \(eB\)-field is not the only source responding to the spin-dependent anomaly, but also several microscopic mechanisms offer a momentum-dependent AMM for strongly interacting fermions \[26–31\]. The effects of quark AMM on the phase structure as well as mesonic properties are found in the works of \[32–40\].

Although the infrared dynamics of quark condensation is catalyzed by the influence of dimensional reduction in a strong magnetic field, it does not affect the motion of the neutral NG mode, as explained in \[41, 42\]. On the other hand, the Nambu–Jona-Lasinio (NJL) model calculation numerically shows that the AMM effect corresponds to a monotonic decrease in the spectra of the neutral pion as the strength of magnetic field grows \[39\], which is consistent with lattice QCD simulations \[11–43\]. Once the magnitude of AMM reaches a critical value, the energy of neutral pions vanishes and their condensation suggests a newly possible superfluid state in the QCD vacuum. These infrared phenomena are not in accordance with Goldstone’s theorem. Also, the calculations in the low-energy effective theories observed that, taking into account the single scalar chiral condensate, the properties of the NG meson remain valid \[43–44\]. Hence, one shall move towards an analysis of how the unusual infrared behavior of \(\pi_0\) is enhanced by the multiple ground states. In the present paper, we will further study the scenario of the dimensional reduction, not restricted to the one-point, but also occurring in the two-point correlators. The relation among the current quark mass, quark condensates and the mass of low-lying meson is examined, as well.

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In the framework of the NJL model, mesons are treated as correlated quark-antiquark states in the random phase approximation (RPA). However, corresponding to simultaneous fluctuations of two VEVs, meson modes must be formed in terms of the excitation of $\langle \bar{\psi} \gamma^\mu \psi \rangle$ and $\langle \bar{\psi} \sigma^{\mu \nu} \psi \rangle$. While the chiral mesons have been studied as quantum fluctuations of the scalar order parameter, a quantitative representation of pseudo-scalar modes including the tensor state of $\langle \bar{\psi} \sigma^{\mu \nu} \psi \rangle$ is still lacking. Another aim of the present article is to study the behavior of the generalized pions, contributing to the understanding of the proper degrees of freedom in the presence of external magnetic fields [45,46].

This paper is organized as follows: in Sec. II we introduce the NJL model Lagrangian with tensor coupling and compute the quark propagator as well as the gap equations, with two order parameters. Then, we determine the sign and the strength of $\langle \bar{\psi} \psi \rangle$ and $\langle \bar{\psi} \sigma^{\mu \nu} \psi \rangle$ for the four-fermion coupling constants $G_S = G_T$. Next, we identify the collective modes in detail in Sec. III where two pseudo-scalar pionic modes are presented due to the scalar-tensor mixing. We discuss the dimensional reduction appearing in the meson kernel in Sec. IV and investigate the spectrum of pions under the influence of the AMM. Finally, we discuss the results in Sec. V.

II. MODEL AND FORMALISM

Integrating out the degrees of freedom of gluons and large quark fluctuations, whose momenta are larger than $\Lambda_{QCD}$, the NJL model utilizes the simple four-fermion point interactions to describe spontaneous chiral symmetry breaking in QCD, which is a successful tool studied in many previous works. We will apply it to investigate the dynamics of strong interactions at low energies in a constant and homogeneous magnetic fields, without including the phenomenon of confinement, for simplicity.

A. Formalism of the quark propagator

The Lagrangian density of the NJL two-flavor model in the presence of an external magnetic field is given by

$$\mathcal{L} = \bar{\psi} (i \partial - m) \psi + G_S \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} \gamma_5 \bar{\tau} \psi)^2 \right],$$

(1)

where the covariant derivative $D_\mu = -i \partial_\mu - q_\mu A_\mu$; $q_\mu = \text{diag}(q_u, q_d)$; $\psi$ is the quark spinor with Dirac, color and flavor indices; $A_\mu = (0, 0, Bx, 0)$ for $\mu = 0, 1, 2, 3$ and the particular constant and homogeneous magnetic field is pointing in the $x_3$-direction. As customary, we assume from the very beginning $m_u = m_d$ for the bare quark mass matrix $m$. $\bar{\tau}$ is a vector of Pauli matrices in flavor space. The conventional four-fermion scalar and pseudo-scalar channels, shown in the bracket, with coupling strength $G_S$ are employed.

Under the influence of a uniform magnetic field, the tensor structure of the gluon propagator separates into longitudinal and transverse parts and so does the Lagrangian density of NJL, based on the effective one-gluon exchange-type interaction, given as:

$$\mathcal{L}_{\text{int}} = g_{\mu}^2 \left( \bar{\psi} \gamma_\mu \psi \right)^2 + g_{\tau}^2 \left( \bar{\psi} \gamma_\tau \psi \right)^2.$$  \hspace{1cm} (2)

To take into account the fact that the rotation symmetry has been reduced from $O(3)$ to $O(2)$, presented in Ref. [34], anisotropic Fierz identities have to be applied as:

$$\left( \gamma^\mu_\mu \right)_{il} \left( \gamma^\nu_\nu \right)_{jk} = \frac{1}{2} \left\{ (1)_{il} (1)_{jk} + (i \gamma_3)_{il} (i \gamma_3)_{jk} + \frac{1}{2} (\sigma^{\mu \nu}_{\perp})_{il} (\sigma^{\nu \mu}_{\perp})_{jk} - (\sigma^{03}_{\perp})_{il} (\sigma^{03}_{\perp})_{jk} + \cdots \right\},$$

(3)

$$\left( \gamma^\mu_\mu \right)_{il} \left( \gamma^\nu_{\tau} \right)_{jk} = \frac{1}{2} \left\{ (1)_{il} (1)_{jk} + (i \gamma_3)_{il} (i \gamma_3)_{jk} - \frac{1}{2} (\sigma^{\mu \nu}_{\perp})_{il} (\sigma^{\nu \mu}_{\perp})_{jk} + (\sigma^{03}_{\perp})_{il} (\sigma^{03}_{\perp})_{jk} + \cdots \right\}.$$  \hspace{1cm} (3)

We note here that $\perp, \parallel$ carry the Lorentz indices of (0, 3) and (1, 2), respectively, regarding the direction of the magnetic field. It obviously shows that the difference between $g_{\nu, \perp}$ will manifest themselves through the frozen four-fermion interactions in the tensor channels rather than the usual interactions of scalar and pseudo-scalar couplings in the NJL model. We conclude that

$$\mathcal{L}_{\text{int}} = G_S \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} \gamma_5 \bar{\tau} \psi)^2 \right] + G_T \sum_{a=0,3} \left[ (\bar{\psi} \sigma^{12}_{\tau_a} \tau_a \psi)^2 + (\bar{\psi} \gamma_5 \sigma^{12}_{\tau_a} \tau_a \psi)^2 \right].$$  \hspace{1cm} (4)

$G_T \leq G_S$ since $G_S \sim g_{\perp}^2 + g_{\parallel}^2$ and $G_T \sim g_{\perp}^2 - g_{\parallel}^2$ [34]. $\tau_a = (I_2, \bar{\tau}_a, \tau_a)$ and $\bar{\tau} = \tau_{1,2,3}$ are Pauli matrices. In a magnetic environment, the vacuum state must be neutral to maintain stability. Therefore, we have omitted the non-diagonal components of the condensates in flavor space. Consequently, terms in the summation over $a$ are limited to $a = 0, 3$. The transverse index $\sigma^{12}_{\tau_a}$ is selected with respect to the magnetic field in the $x_3$-direction. As discussed in Ref. [34], the positive definiteness of $G_T$ arises from the
dominance of longitudinal contributions from one-gluon exchange over transverse ones due to the emergence of a dimension reduction effect caused by the magnetic field; i.e., $G_s \sim G_T$ for negligible $g_\perp$ in strong magnetic fields.

In the presence of an external magnetic field, the SU(2) $\times$ SU(2) chiral symmetry of the two-flavor NJL model is explicitly broken down to U(1)$_{\perp}$ $\times$ U(1)$_{\parallel}$. Both a chiral condensate and a tensor condensate break the invariant Lagrangian to U(1)$_{\perp}$ $\times$ U(1)$_{\parallel}$. The chiral condensate creates a mass gap for the quarks. The tensor condensate generates an anomalous magnetic moment for the quarks. We examine the phase structure of the model based on these two condensates and their dependence on the quark charge. Written as:

$$
\Sigma_f = -G_S \langle \bar{\psi}_f \psi_f \rangle, \quad \kappa_f = -G_T \langle \bar{\psi}_f \gamma^0 \gamma^5 \psi_f \rangle,
$$

for $f = u, d$. We adopt the notation that $\Sigma = \Sigma_u + \Sigma_d$ and assume that the chiral condensate $\Sigma_u = \Sigma_d$ for maximal flavor symmetry. As we mentioned before, a non-trivial coupling constant $\kappa_f$ of the anomalous magnetic moment is produced through several microscopic mechanisms. The coefficient of quark AMM is not uniquely adopted in many previous works [32–40], which is proportional to either $q_f$, or $q_f^2$, or charge independent if it is created via a compensation of the color-AMM. The main point in the present investigation is that we will dynamically determine and extract it from the gap equations in the following.

Performing the Hubbard-Stratanovich transformation in the Lagrangian density of Eq. (4) and plugging into the Eq. (5), we continue to derive the magnetized quark propagator with the AMM coupling. Hence, the fermionic Lagrangian density in the mean-field approximation is rewritten as:

$$
\mathcal{L}_{\text{eff}} = \bar{\psi} \left( D - M + \vec{k} \cdot \tau_a (\sigma^{\mu\nu} F_{\mu\nu}) \right) \psi,
$$

where $M = \Sigma + m$, $\sigma^{\mu\nu} = i \left[ \gamma^\mu, \gamma^\nu \right] / 2$. By summing over $a = 0, 3$, we obtain the two-vector $\vec{k} = \frac{1}{2} (\kappa_u + \kappa_d, \kappa_u - \kappa_d)$ that represents $\vec{k} \cdot \tau_a = \text{diag}(\kappa_u, \kappa_d)$ in flavor space. $F_{\mu\nu} = F_{\mu\nu} / ||F||$ is the dimensionless electromagnetic (EM) tensor. Note here that we let the energy scale of $\kappa_f$ be the same as the mass, which was scaled to a dimensionless quantity in some works.

To study the behavior of $\vec{k}$, we examine its dependence on the quark charge in the one-flavor model and neglect the vector symbol. It is important to note that $\kappa_f$ is a flavour dependent parameter in both the one and two flavor models. This is pointed out throughout the manuscript. Following Schwinger’s proper time method, we obtain the quark propagator as:

$$
G = \frac{1}{D - M + \kappa \sigma \vec{F}} = \frac{1}{D - M + \kappa \sigma \vec{F}} = \frac{D + M + \kappa \sigma \vec{F}}{D + M + \kappa \sigma \vec{F}} = i \left( D + M + \kappa \sigma \vec{F} \right) \int ds e^{i(D - M + \kappa \sigma \vec{F}) s},
$$

where $D^2 = D^2 - q_f^2 F / 2$. The formula is given in matrix notation, e.g. $F_{\mu\nu} = (F)_{\mu\nu}$, $\sigma F = \sigma^{\mu\nu} F_{\mu\nu}$. The additional AMM term gives

$$
\Omega = -2i \kappa \left( \gamma^1 \gamma^5 \partial_0 - \gamma^0 \gamma^5 \partial_3 \right).
$$

$\Omega$ commutes with $D^2$ with $\Omega \parallel 1, 2$, since $[(\sigma^{\mu\nu})_\parallel, \gamma^{0,3}) \gamma_5] = 0$. At this point, it allows for an expansion of the exponential [47]

$$
e^{i\Omega s} = \cosh (i\Omega s) + \frac{\Omega}{\Omega} \sinh (i\Omega s),
$$

with constant matrix $\Omega = ||\Omega|| = 2|\kappa| \sqrt{-\partial_0^2 + \partial_3^2}$. Finally, the Green’s function takes the form:

$$
G(x, y) = \frac{\phi(x, y)}{4\pi^2} \sum_{\pm} \int \frac{ds}{s^2} e^{i[(D^2 - M^2 + \kappa^2 s^2) - L(s)]} \left[ \frac{1}{2} \gamma^\mu \left( \{s + q_f \sigma F \}_{\mu\nu}(x - y)^\nu + M + \kappa \sigma \vec{F} \right) \right] \left[ 1 \pm \frac{\Omega}{\Omega} \right],
$$

where $\phi(x, y)$ is the well-known phase factor [48][49] and

$$
\Pi^2 = \frac{1}{4s} (x - y) f(s)(x - y) + \frac{q_f \sigma \vec{F}}{2}; \quad f(s) = q_f F \coth \left( q_f F s \right); \quad L(s) = \frac{1}{2} \text{tr} \ln \left( \frac{q_f F s}{q_f} \right).
$$

The position dependence of $G(x, y)$ has been attributed to the Schwinger phase factor $\phi(x, y)$ and the left term in Eq. (10) is translation invariant. It is convenient to transform it to momentum space and further decompose over the Landau pole, representing it as

$$
G(q_f, k) = \exp \left[ -\frac{k_\perp^2 |q_f| B}{2} \sum_{n=0}^{\infty} \frac{m^2}{k_\perp^2 - 2n|q_f| B - M^2 + k^2 \pm 2|q_f| k_{\perp}} \right]
$$

(12)
with
\[ \Lambda_{\pm} = \frac{1}{2} \pm \frac{\gamma^3 y^5 k_0 - y^0 y^5 k_3}{2|k_i|} \text{ sign}(\kappa), \]  
(13)
and
\[ D_n(q_f, k) = \left( k_i + M + \kappa \sigma F \right) \left[ P_{-} L_n \left( 2z_f \right) - P_{+} L_{n-1} \left( 2z_f \right) \right] + 4k_i L_{n-1}^1 \left( 2z_f \right). \]  
(14)
We note here that \( P_{\pm} = 1 \pm iy^{1/2} \text{ sign}(q_f) \), \( z_f = k_i^2 / (q_f |B|) \), \( k_i = (k_0, k_3) \), \( k_{\perp} = (K_{22}, K_{12}) \), \( \gamma_0 = (\gamma_0, \gamma_1) \) and \( \gamma_{\perp} = (\gamma_1, \gamma_2) \) as usual.

We derive the gap equations with respect to the order parameters \( \Sigma \) and \( \kappa \) for a fixed electrical charge \( q_f \), the dynamical solutions are given as:
\[ \frac{M - m}{2iG_T} = N_c \text{ tr } G; \]  
(15)
\[ \frac{\kappa}{2iG_T} = N_c \text{ tr } \left[ \sigma^{12} G \right], \]  
(16)
The notation of \( (\text{tr}) \) runs in Dirac and coordinate spaces, one has
\[ \text{tr } G(k) = \sum_{n=0}^{\infty} (-1)^n \int \frac{d^2 k}{8\pi^2} \frac{M (L_n - L_{n-1}) - \xi |k| (L_n + L_{n-1}) \mp \xi |k| (L_n + L_{n-1})}{(|k_i| \pm |k|)^2 - M_n^2}, \]  
(17)
where \( |k_i| = \sqrt{k_0^2 - k_3^2} \), \( M_n^2 = M^2 + 2nq_f |B| \) and \( \xi = \text{ sign}(\kappa \cdot q_f) \). Since the role of \( \kappa \)'s sign has been attributed to a function of \( \xi \), from now on, we abbreviate \( |k| \) to \( \kappa \). It is known that \( L_{n-1} \) vanishes for \( n = 0 \) and
\[ \int \frac{d^2 k_{\perp}}{4\pi^2} e^{-i\xi (-1)^n L_n} = \frac{|q_f| B}{4\pi}; \quad \int \frac{d^2 k_{\perp}}{4\pi^2} e^{-i\xi (-1)^n L_{n-1}} = \frac{|q_f| B}{4\pi}. \]  
(18)
After integrating over the transverse momentum space, one has
\[ \text{tr } G(k) = \frac{|q_f| B}{8\pi^3} \sum \int d^2 k_{\perp} \left\{ \frac{M - \xi \kappa \mp \xi |k|}{(|k_i| \pm \kappa)^2 - M_n^2} + \sum_{n=1}^{\infty} \frac{2M}{(|k_i| \pm \kappa)^2 - M_n^2} \right\}. \]  
(19)

**B. Sign of the AMM**

While \( \kappa < M \), the double degenerate roots of the denominators \( (|k_i| \pm \kappa)^2 - M_n^2 \) are \( k_0 = a \sqrt{k_3^2 + (M_n \mp \kappa)^2} - iae \) for \( a = \pm 1 \). For the real roots of \( f(x) \) located at \( x_0 \), one has to apply the Jacobian feature of the Dirac-delta function
\[ \delta \left[ f(x) \right] = \frac{\delta(x - x_0)}{|f'(x_0)|}. \]  
(20)
Then, we close the contour of the semicircle in the upper half plane to complete the integral of Eq. 19 with respect to \( k_0 \), shown as
\[ \text{tr } G(k) = - \frac{|q_f| B}{4\pi^2} \int dk_3 \left\{ \frac{M + \xi \kappa}{\sqrt{k_3^2 + (M + \xi \kappa)^2}} + \sum_{n=1}^{\infty} \frac{M}{M_n} \frac{M_n \pm \kappa}{\sqrt{k_3^2 + (M_n \pm \kappa)^2}} \right\}. \]  
(21)
It is noticed that only one of the Zeeman splitting states, \( \Lambda_{\pm} \), has been survived in the Lowest Landau Level (LLL), which is not determined by the charge of quarks but with the product of \( \xi \) instead. From the right-hand side of the above equation, we can see that \( (i \text{ tr } G) \) is positive when \( M > \kappa \).

Similarly, we obtained the related trace of \( \kappa \) as
\[ \text{tr } \left[ \sigma^{12} G(k) \right] = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \int \frac{d^2 k}{8\pi^2} \frac{(\kappa \pm |k|) (L_n - L_{n-1}) \text{ sign}(\kappa) - M (L_n + L_{n-1}) \text{ sign}(q_f)}{(|k_i| \pm |k|)^2 - M_n^2} \]  
\[ = \frac{|q_f| B}{8\pi^2} \sum \int d^2 k_{\perp} \left\{ \frac{(\kappa \pm |k|) \text{ sign}(\kappa) - M \text{ sign}(q_f)}{(|k_i| \pm |k|)^2 - M_n^2} + \sum_{n=1}^{\infty} \frac{2(\kappa \pm |k|) \text{ sign}(\kappa)}{(|k_i| \pm |k|)^2 - M_n^2} \right\}. \]  
(22)
Adopting the sign function $\text{sign}(x)$, which satisfies $\text{sign}(x) \cdot x = \text{Abs}(x)$ for $x \neq 0$, we can determine the sign of $\kappa$ as follows:

$$
\text{sign}(\kappa) \text{tr} \left[ i\sigma^{12} G(k) \right] = \frac{|q|B}{8\pi^3} \int d^2k \left\{ \frac{(\kappa \pm |k|)M}{(\kappa \pm |k|)^2 - M^2} + \sum_{n=1}^{\infty} \frac{2(\kappa \pm |k|)}{(\kappa \pm |k|)^2 - M_n^2} \right\},
$$

$$
= \frac{|q|B}{4\pi^2} \int d^3k \left\{ \frac{-\kappa - \xi M}{\sqrt{k_x^2 + (M + \xi k)^2}} + \sum_{n=1}^{\infty} \frac{\mp M_n - \kappa}{\sqrt{k_x^2 + (M_n + \kappa)^2}} \right\}. \tag{23}
$$

It is observed that the l.h.s. of Eq. (23) is positive definite, thus, one needs $\xi = -1$ to get a nontrivial solution of $\kappa$. Using the LLL approximation and taking the chiral limit $m \to 0$, we recover the result that $M/\kappa = G_5/G_T \tag{34}$. Moreover, the contribution from finite Landau levels (i.e., the second term in the above bracket) is negative since the absolute values coming from $(M_n + \kappa)$ are larger than those from $(M_n - \kappa)$.

In the second case of $M^2_f < \kappa^2 < M^2_{f+1}$, there is no root in the denominators $(|k| + \kappa)^2 - M_n^2$ for $n \leq l$, on the contrary, the root is four-fold degenerate in the term of $(|k| - \kappa)^2 - M_n^2$, known as $k_0 = \pm \sqrt{k_x^2 + (\kappa + aM_n)^2} \mp ia \varepsilon$ with $a = \pm 1$. Hence, two poles, $-\sqrt{k_x^2 + (\kappa + M_n)^2}$ and $\sqrt{k_x^2 + (\kappa - M_n)^2}$, will contribute while taking the Cauchy integral in the upper half plane $\tag{50}$. For $n \geq l + 1$, the root behaviors of $(|k| \pm \kappa)^2 - M_n^2$ reduce to double degenerate states as usual. Without loss of generality, let $l = 1$, $\text{sign}(\xi) = -1$, and then,

$$
\mathbb{S}_x = \text{tr} G(k) = -i \frac{|q|B}{4\pi^2} \int d^3k \left\{ \frac{-\kappa - M}{\sqrt{k_x^2 + (M - \kappa)^2}} + \sum_{M} \frac{\kappa \pm M_1}{\sqrt{k_x^2 + (\kappa \pm M_1)^2}} + \sum_{n=2}^{\infty} \frac{M}{M_n} \frac{M_n \pm \kappa}{\sqrt{k_x^2 + (M_n \pm \kappa)^2}} \right\}, \tag{24}
$$

$$
\mathbb{S}_\kappa = \text{sign}(\kappa) \text{tr} \left[ \sigma^{12} G(k) \right] = -i \frac{|q|B}{4\pi^2} \int d^3k \left\{ \frac{-\kappa - M}{\sqrt{k_x^2 + (M - \kappa)^2}} + \sum_{\pm} \frac{-\kappa - M_1}{\sqrt{k_x^2 + (\kappa \pm M_1)^2}} + \sum_{n=2}^{\infty} \frac{M_n - \kappa}{\sqrt{k_x^2 + (M_n \pm \kappa)^2}} \right\}. \tag{25}
$$

Since

$$
f(x) = \int_{-\Lambda}^{\Lambda} dk \frac{x}{\sqrt{k_x^2 + x^2}} = x \log \frac{\sqrt{\Lambda^2 + x^2} + \Lambda}{\sqrt{\Lambda^2 + x^2} - \Lambda}, \tag{26}
$$

one notices that $f(x)$ is increasing as $x$ is growing and then $\mathbb{S}_x < \mathbb{S}_\kappa$. Therefore, it means that no solution exists in the second case for $G_5 \geq G_T$ after comparing between the dynamical solutions of $M$ and $\kappa$. The same conclusion can be drawn if we let $\xi = 1$.

Taking into account to our earlier-reached conclusion in the first case of $\kappa < M$, it requires that $\xi = -1$. Returning to a two-flavor quark state consisting of up and down quarks, we find that the allowed non-trivial solution has a generalized form of diag $(-\kappa, -\kappa, \kappa, \kappa)$, which is equivalent to $\kappa \cdot \tau_3$, where $\kappa$ and $\tau_3$ are defined as non-negative. Here we restrict ourselves to keeping the maximum chiral condensate; namely, $\kappa_0 = \kappa_d = \kappa$. With $q_f = \text{diag}(e/3, -2e/3)$, we obtain the solution as $\kappa_0 = (0, -\kappa)$ where the only non-zero component of the Pauli matrix is $\tau_3$. It also allows us to convert the gap equation [16] to a absolutely definite $\kappa$, presented as

$$
\frac{|\kappa|}{2iG_T} = -\text{Tr} \left[ \tau_3 \sigma^{12} G(k) \right], \tag{27}
$$

where the notation of capital trace ($\text{Tr} = N_c \sum_{a,b} \text{Tr}$) runs in color, flavor, Dirac and coordinate spaces. Hence, the first conclusion in the present work is that the sign of the dynamically generated AMM is not arbitrary, and it must be opposite to the sign of the quark charge. Besides, the magnitude of $\kappa$ is smaller than the dynamically generated quark mass, if no other sources are taken into account in the current two-flavor NJL model approach.

### III. ASPECTS OF THE GENERALIZED PIONS

In this Section, we show the essential features of the mixing of the generalized pseudo-Goldstone modes in the description of the NJL model.
Following the discussions of Refs. [45, 46], while the low-energy effective Lagrangian is written in terms of the two order parameter fields, its associated collective modes are presented by two condensates as a model-independent consequence. While scalar and tensor condensates break chiral symmetries, in this context, it is also instructive to describe the pion as the generalized one, which is the excitation of the simultaneous fluctuations on account of \( \langle \bar{\psi} \psi \rangle \) and \( \langle \bar{\psi} \sigma \gamma^5 \psi \rangle \). Iterating the vertex of the four-fermion interactions, the meson is defined as the solution to the Bethe-Salpeter equation for the bound states. The equation reads
\[
1 - 2G_S \Pi_{ps} (m_0^2) = 0, \tag{28}
\]
where \( \Pi_{ps} \) is the ordinary quark-antiquark polarization tensor for (pseudo)-scalar. While the tensor condensate exists, another meson correlation of arises through the (pseudo)-tensor channels [45, 46]. As a result, we have two sets of pion triplets. Besides, \( \Pi \)

When the scalar-tensor mixing vanishes, its properties can be calculated one-by-one, which is exactly the situation of \( \kappa = 0 \). However, considerable differences are caused when nonvanishing and one has to calculate the two-by-two matrix of the polarization tensor to correctly describe the NG modes.

Now, the NG mesons are superpositions of ordinary quark-antiquark fluctuations \( \Pi_{SS} \), plus the fluctuations of tensor quark-antiquark \( \Pi_{TT} \) [22]. Note that \( S \) and \( T \) label the Lorentz index. As demonstrated by Fig. [1], in terms of two fields \( (\pi) \equiv (\pi, \bar{\pi})^T \), the T-matrix in the random phase approximation is extended as
\[
\frac{1}{i} \Pi_{ps} = \frac{1}{i} \begin{pmatrix} \Pi_{SS}^{ST} & \Pi_{ST} \end{pmatrix}, \tag{29}
\]
where
\[
\frac{1}{i} \Pi_{AB}^{AB} = -N_c \sum \Psi_i \left[ iG(p) i\gamma_5 \Gamma^A_{\alpha} iG(q) i\gamma_5 \Gamma^B_{\alpha} \right], \tag{30}
\]
for \( A, B = S \) and \( T \). Here, \( G \) is the fermion propagator and \( \Gamma^{(A,B)}_{\alpha} = \left( l_4 \tau_\alpha, \sigma^{12 \tau_\alpha} \right) \), respectively. For \( \alpha = 0, \pm 1, 3 \), the quark bubble corresponds to the meson polarization function of \( \sigma \) and pion triplet \( (\pi_\alpha, \pi_\alpha) \). As we demonstrate below, mixing makes one of the two pions heavier while the other becomes lighter. We write down the mass spectra of \( \bar{\pi}_n \) and \( \pi_n \), which are described as the two eigenvalues corresponding to the transformation of,
\[
\left( \pi, \bar{\pi} \right) = \mathcal{F} \left( \tilde{\pi}, \tilde{\bar{\pi}} \right), \tag{31}
\]
where the rotation matrix \( \mathcal{F}^{-1} \) is applied to diagonalize the T-matrix of \( \Pi_{ps} \).

**IV. DIMENSIONAL REDUCTION IN THE TWO-PARTICLE STATE**

In this section, we complete the calculation of the T-matrix of Eq. (29) and demonstrate the dimensional reduction in the NG meson kernel. In this section, we computed the neutral pion and \( \tau_\alpha = \tau_3 \). The subscript \( \alpha \) of \( \Pi \) is omitted.

**A. Polarization tensor**

Since \( \xi = -1 \), one has
\[
\Pi_{SS} = N_c \sum_{q_i = \pm} \frac{|q_i| B}{16 \pi^3} \int d^2k \left\{ \left( 1 - \frac{p_0 q_0 - p_3 q_3}{|p_i q_i|} \right) \left( M + \kappa \pm |p_i| \right) \left( M + \kappa \pm |q_i| \right) \left( |q_i| \pm \kappa \right)^2 - M_0^2 \right\}, \tag{32a}
\]
\[
+ \left( 1 + \frac{p_0 q_0 - p_3 q_3}{|p_i q_i|} \right) \left( M + \kappa \pm |p_i| \right) \left( M + \kappa \pm |q_i| \right) \left( |q_i| \pm \kappa \right)^2 - M_0^2 \tag{32b}
\]
\[
+ 2 \sum_{n=1}^{\infty} \left( 1 + \frac{p_0 q_0 - p_3 q_3}{|p_i q_i|} \right) \frac{M_n^2 + (|p_i| \pm \kappa) (|q_i| \pm \kappa)}{(|q_i| \pm \kappa)^2 - M_0^2} \tag{32c}
\]
\[
+ 2 \sum_{n=1}^{\infty} \left( 1 + \frac{p_0 q_0 - p_3 q_3}{|p_i q_i|} \right) \frac{M_n^2 - (|p_i| \pm \kappa) (|q_i| \pm \kappa)}{(|q_i| \pm \kappa)^2 - M_0^2} \tag{32d}
\]
The term of \( I \) where brackets as:

Before doing a full computation, we show the resulting expressions and conclusions for zero \( I \) and the blue dot vertex denotes \( \bar{\psi} i \gamma_5 \tau_3 \psi ^2 \). The four-fermion coupling \( \bar{\psi} i \gamma_5 \tau_3 \psi \). The case of \( \bar{\psi} i \gamma_5 \tau_3 \psi \) and its conjugate term are absent in the leading order (i.e., the off-diagonal elements in the numerator of the right above corner).

where \( p = k + \frac{M_f}{2} \), \( q = k - \frac{M_f}{2} \) and \( M_f = (m_f, 0, 0, 0) \) in the center-of-mass frame. The influence of the flavor matrix \( \tau_3 \) is trivial in constructing the neutral pion. After a tedious but straightforward calculation, one gets

where

The term of \( I_1 / (M - \kappa) \) is derived from Eq. (A3), and further explanation is provided below. Here we have introduced the brackets as:

$$
\langle X(k_i, m_{\pi}) \rangle_0 = N_c \sum_{q_f} \frac{|q_f| B}{16 \pi^3} \int \frac{d^2 k_i}{|k_i|} X(k_i, m_{\pi});
$$

$$
\langle X(n, k_i, m_{\pi}) \rangle_0 = N_c \sum_{q_f} \frac{|q_f| B}{16 \pi^3} \sum_{n_1} \int \frac{d^2 k_i}{2|p_n q_f|} X(n, k_i, m_{\pi}),
$$

(35)

to denote the associated summation and integration in \( k \)-space. The detail forms of \( J, K_{11} \) are shown in the Appendix.

Before doing a full computation, we show the resulting expressions and conclusions for zero \( \kappa \) and zero \( M \). Firstly, it is easily found that \( \text{Tr} \ G \sim (I_1 + I_2) \), read off Eq. [19]. Here, the term of \( I_1 \) is rewritten as,

$$
I_1 (M, \kappa) = 2 (M - \kappa) N_c \sum_{q_f} \frac{|q_f| B}{8 \pi^3} \int \frac{d^{D-2} k_i}{k_i^8 - (M - \kappa)^2},
$$

(36)

which is the pole contribution in the fermion propagator from the famous lowest-landau-level. In \( 2 + 1 \) dimensions, \( I_1 \) remains finite in the limit of \( (M, \kappa) \to 0 \). This suggests that the infrared dynamics appearing in the one-point correlator is in response to enhancing fermion masses by a strong magnetic field in \( 3 + 1 \) dimensions. It also means that the motion of charged fermions is restricted in the lower dimensions, i.e., \( D \to D - 2 \). Moreover, \( \pi_0 \) is determined by the original polarization tensor \( \Pi^{55} \) in Eq. (33) while \( \kappa = 0 \). Continuing to solve the Bethe-Salpeter equation, one observes that \( \pi_0 \) becomes massless in the chiral limit, as presented in Ref. [44]. We emphasize here that it is consistent with the conclusions drawn in Ref. [42], that the dimensional reduction catalyzes the condensate \( \langle \bar{\psi} \psi \rangle \), but does not affect the dynamics of the neutral meson excitation since it is the same.
infrared term that arose in the one- and two-point correlators. We can attribute the IR-divergent $I_1$ in meson kernels to the dynamical quark mass $M$. Therefore, the Goldstone nature of $\pi_0$ is preserved and the propagators of neutral hadrons are well-behaved in $D$-dimension.

Returning to our procedure with non-vanishing $\kappa$, the other analogous expressions of polarization tensors are presented as:

$$\Pi^{TT} = \frac{I_1}{M - \kappa} - \frac{I_2}{M} - m_n^2 \langle J_0 \rangle - m_n^2 \langle K_{22} \rangle_n,$$  \hspace{1cm} (37)

and

$$\Pi^{ST} = \Pi^{TS} = \frac{I_1}{M - \kappa} - m_n^2 \langle J_0 \rangle - m_n^2 \langle K_{12} \rangle_n.$$  \hspace{1cm} (38)

Again, the detailed forms of $K_{12,22}$ can be found in the Appendix.

B. Matrix of the meson kernel

Regarding the fluctuations of two mean fields, the BS equation of meson modes converts to

$$\begin{pmatrix} 1 - 2g\Pi_{SS} & -2g\Pi_{ST} \\ -2g\Pi_{TS} & 1 - 2g\Pi_{TT} \end{pmatrix} = A + m_n^2 B$$  \hspace{1cm} (39)

where

$$A = 1 - 2g\Pi_{p} \big|_{\omega^2 = 0} = \begin{pmatrix} \eta & 0 \\ 0 & 2 - \eta \end{pmatrix} - \frac{2i\hbar I_1}{M} \frac{1}{1 - \zeta} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$  \hspace{1cm} (40)

$$B = 2i\hbar \begin{pmatrix} \langle J_0 \rangle + \langle K_{11} \rangle_n & \langle J_0 \rangle + \langle K_{12} \rangle_n \\ \langle J_0 \rangle + \langle K_{12} \rangle_n & \langle J_0 \rangle + \langle K_{22} \rangle_n \end{pmatrix},$$  \hspace{1cm} (41)

$G_S = G_T = g$, $\eta = \frac{m}{\pi}$ and $\zeta = \frac{m}{\kappa}$.

In the polarization tensor $\Pi_{ps}$, all elements acquire the pole contribution of the LLL, seen Eq. (A3), which take the form

$$\tilde{I}_1(M, \kappa) = N_c \sum_{q_f} \frac{|q_f| B}{8\pi^3} \int \frac{d^{D-2}k_1}{|k_1|(|k_1| - M + \kappa)}$$

$$= N_c \sum_{q_f} \frac{|q_f| B}{4\pi^3} \int \frac{d^{D-2}k_1}{k_1^2 - (M - \kappa)^2}.$$  \hspace{1cm} (42)

For $D = 2 + 1$, $\tilde{I}_1$ is governed by the diverging integrand $dk/k^2$ in the limits of $\langle M, \kappa \rangle \to 0$, while it retains logarithmic singularity in IR limits for $D = 3 + 1$, demonstrating the emergent dimensional reduction in the neutral mesonic excitations. Under the LLL approximation, it is observed that $M = \kappa + m$ and $I_1 \sim m \ln m^2$. Thus, the infrared dynamics has a strong hierarchy of meson and quark sectors, where $I_1 \sim \ln m^2$ differs from $I_1$ with $m \to 0$. The NG bosons are formed in the infrared region, which cannot be washed out by the dynamical quark mass and results in a remarkably lighter meson mass. Since $M - \kappa > m$ in regard to the contributions from the finite Landau levels, as we used above, we have a weaker infrared expression $\tilde{I}_1 = I_1 / (M - \kappa)$ in the present work.

Here $I_2$, which is related to the contributions from the finite landau levels, is canceled out by the valence quark mass $M$. For simplicity, we rewrite the expression of

$$B = 2i\hbar \tilde{J} \begin{pmatrix} 1 & 1 - \alpha \\ 1 - \alpha & 1 - \beta \end{pmatrix},$$  \hspace{1cm} (43)

where $\tilde{J} = \langle J_0 \rangle + \langle K_{11} \rangle_n$, $\alpha, \beta$ are functions of $\langle J_0 \rangle$ and $\langle K_{(12,22)} \rangle_n$ and $\alpha, \beta \ll 1$ if the magnitude of $\langle J_0 \rangle$ from the LLL is much larger than $\langle K_{(11,12,22)} \rangle_n$ for $n \geq 1$.

We obtain the roots of two pionic modes, $\tilde{\xi}_0$ and $\tilde{\bar{\xi}}_0$, in the approximation of $(\eta, \zeta, \alpha, \beta) \ll 1$. Their perturbed masses take the forms of

$$m^2_{\tilde{\xi}} = \frac{1}{-2i\hbar \tilde{J}} \frac{m + \kappa + igI_1 I_1}{M \tilde{J}} + \mathcal{O}(\alpha, \beta);$$

$$m^2_{\tilde{\bar{\xi}}} = \frac{1}{-ig(2\alpha - \beta) \tilde{J}} + \mathcal{O}(\alpha^0, \beta^0).$$  \hspace{1cm} (44, 45)
Here $iI_1$ and $i\tilde{I}$ are positive and negative definite, respectively, since they originate from the loop integrations with one- and two-quark propagators. For the lighter pionic mode $\tilde{\pi}$, its leading structure $m_2^2 = m/(-2igJM)$ clearly reflects Goldstone’s theorem, as found earlier \[42\] \[44\]. According to the second term of the r.h.s. of Eq. \[44\], the spectra of $\tilde{\pi}$ is dramatically lowered by $\kappa$, which is in accordance with the numerical calculations in the work of \[39\]. It is interesting to remark that such mode behavior raises an interesting possibility of Bose condensation when the critical value

$$\kappa_c \approx \frac{mM}{2igI_1} - igI_1 - m,$$  \hspace{1cm} (46)

is reached. It is well known that in pion superfluidity the critical isospin chemical potential is equal to the pion mass at zero temperature and chemical potential \[51\]. Therefore, in the limit $\mu_2^I = 0^+$, where the pion mass disappears due to massless quarks, the system immediately enters a Bose-Einstein Condensate (BEC) pion state. Similarly, regardless of how small $\kappa$ is, the conventional hadronic gas state becomes unstable and a new phase emerges in the chiral limit at $eB \neq 0$. Since several possible states have been proposed for strongly magnetized QCD matter \[52\] \[53\], a sophisticated investigation of the vacuum state with more model parameters will be explored elsewhere.

Due to the finite $\kappa$, the neutral pion fails to manifest itself as the Nambu-Goldstone boson. Such special nature is broken by a two-fold aspect. Firstly, the zero- and two-form pions couple to each other through the pseudo scalar-tensor bubbles, induced by the additional Dirac structures in the modified fermion propagator. Secondly, a newly developed infrared dynamics forms, as Eq. \[42\] in the meson kernels. As $\kappa \to M$, the related LLL term is strongly enhanced, which reveals that the dimensional reduction is not restricted to the quark condensates, but is also present in the motion of the neutral excitation. As a remnant of the infrared dynamics, it is reasonable to recognize that a very small $\kappa$ is sufficient to reduce the mass of $\pi_0$ to zero. Of course, presented by the lattice simulations \[1\] \[4\], the Bose condensation of neutral pion does not occur until the ultra-strong limit is reached, where $eB \sim 3.5 \text{ GeV}^2$. Other additional factors are under exploration for a stronger strength of the anomalous coupling in a magnetized environment.

\section{V. CONCLUSION}

In the present work, we have employed the two-flavor NJL model to examine the properties of the generalized neutral pion in a constant background magnetic field. Taking into account the back-reaction from the gluon sector, it allows us to decouple the longitudinal and transverse space and introduce the extra tensor-like four-fermion interactions in the model construction. The novelty of the employed framework lies in that there are two order parameters emerging in the vacuum, described by $\langle \bar{\psi} \psi \rangle \sim M$ and $\langle \bar{\psi} \sigma_{12} \psi \rangle \sim \kappa$. Here $\kappa$ plays a role similar to the anomalous magnetic moment in the quark Dirac equation. As a spin dependent coupling, it is no wonder that $\kappa$ is not degenerate under the operation of charge conjugation like mass is. We prove its allowed sign is opposite to the sign of the quark’s electric charge. Restricting to the model parameters where the coupling constants $G_S = G_T$, we examine that the magnitude of $\kappa$ is smaller than the dynamical solution of the quark mass, as well.

Secondly, we revisit the qualitative description of the neutral pion under the influence of the AMM coupling. The key observation is that the ordinary meson is no longer the collective excitation of the system with multiple order parameters. A simultaneous treatment of fluctuation equations has to be implemented to realize the degrees of freedom of the meson modes. To the best of our knowledge, such a generalized neutral pion has not been examined in strong magnetic fields before. Properly including the pseudo-tensor vertex in RPA loop calculations, the spectra of the two pionic meson modes are presented after diagonalization. It is found that the familiar Goldstone nature is corrupted for the lightest chiral meson $\pi_0$. Moreover, $\kappa$ strongly reduces its mass on two sides. On the one hand, the existence of the mixing always lowers one eigenvalue, but enhances another massive mode. On the other hand, we observe that a unique infrared dynamics arises in the meson correlator, which is cannot be labeled as the catalyzed dynamical quark mass. Hence, we point out that the treatment of the infrared cutoff will be very sensitive in the case of the nonrenormalized model calculations. Under a simple assumption, where the reduction from $D \to D - 2$ affects only the charged channels, it is implied that the neutral $\pi_0$ is free to move in the original 3 + 1 dimension and acts as an NG boson \[42\]. However, as the system behaves like a 1 + 1 dimension described by both one- and two-point correlators, it is likely that an inhomogeneous phase emerges, such as a chiral density wave state \[52\] or a chiral soliton lattice \[53\]. A full numerical simulation is required to determine the phase state under the dimension reduction effect of the AMM coupling and will be calculated in the future.

According to our results, the generalized pion continuously becomes lighter while $\kappa$ increases. Eventually, we expect that the interesting BEC of the generalized pion occurs when $\kappa$ is strong enough. Such exotic phase may be realized in a more complicated magnetized system \[54\] \[55\]. We will discuss such a possibility in a future publication.

Rich phenomena have been reported in the present QCD×QED environment. For example, the debate on the superconducting QCD vacuum \[56\] \[57\], the puzzle of magnetic susceptibility \[42\] \[39\] \[58\], the understanding of the role of the pion mass in first-principles’ simulations \[59\] \[60\] and the strange metal phase of QCD in 1 + 1 dimensions \[61\]. Our approach including the effect of the AMM coupling has a potential to shed light on these discussions. We leave these projects to future works.
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Appendix A: Appendix: Technical Details of the Meson Kernel Matrix Element Calculation

Considering the allowed kinematic regions of the pole, the first line \(32a\) in the bracket simplifies to

\[
32a = \left( 1 - \frac{p_0q_0 - p_3q_3}{[p_r,q_3]} \right) \frac{1}{(|p_r| - M + \kappa) (|q_3| - M + \kappa)} - \frac{1}{(|p_r| - |q_3|)^2 + m_\pi^2} \frac{1}{|p_r| - |q_3|} \left( \frac{1}{|q_3| - M + \kappa} - \frac{1}{|p_r| - M + \kappa} \right). \tag{A1}
\]

Similarly, one has

\[
32b = \sum_{\pm} \left( 1 + \frac{p_0q_0 - p_3q_3}{[p_r,q_3]} \right) \frac{1}{(|p_r| \mp (M - \kappa)) (|q_3| \pm (M - \kappa))} - \frac{1}{(|p_r| + |q_3|)^2 - m_\pi^2} \frac{1}{|p_r| + |q_3|} \left( \frac{1}{|q_3| \pm (M - \kappa)} + \frac{1}{|p_r| \pm (M - \kappa)} \right). \tag{A2}
\]

To sum the two terms together, one has

\[
32a + 32b = \frac{1}{|p_r| (|p_r| - M + \kappa)} + \frac{1}{|q_3| (|q_3| - M + \kappa)} + \frac{m_\pi^2}{2|p_r,q_3|} J \tag{A3}
\]

where

\[
J = \left[ p_r^2 - (M - \kappa)^2 \right] \left[ q_3^2 - (M - \kappa)^2 \right]. \tag{A4}
\]

Stepping to the terms of finite Landau levels, it contains

\[
32c = \sum_{\pm} \sum_{n=1}^\infty \left( 1 - \frac{p_0q_0 - p_3q_3}{[p_r,q_3]} \right) \frac{1}{(|p_r| \pm \kappa)^2 - M_n^2} + \frac{1}{(|q_3| \pm \kappa)^2 - M_n^2}
\]

\[
+ \sum_{\pm} \sum_{n=1}^\infty \frac{m_\pi^2}{2|p_r,q_3|} \left( 1 - \frac{4k_0^2}{(|p_r| + |q_3|)^2} \right) \frac{1}{(|p_r| \pm \kappa)^2 - M_n^2} \left[ (|q_3| \pm \kappa)^2 - M_n^2 \right] \tag{A5}
\]

and from the last line of

\[
32d = \sum_{\pm} \sum_{n=1}^\infty \left( 1 + \frac{p_0q_0 - p_3q_3}{|p_r,q_3|} \right) \frac{1}{(|p_r| \pm \kappa)^2 - M_n^2} + \frac{1}{(|q_3| \pm \kappa)^2 - M_n^2}
\]

\[
+ \sum_{\pm} \sum_{n=1}^\infty \frac{m_\pi^2}{2|p_r,q_3|} \left( \frac{4k_0^2}{(|p_r| - |q_3|)^2} - 1 \right) \frac{1}{(|p_r| \pm \kappa)^2 - M_n^2} \left[ (|q_3| \pm \kappa)^2 - M_n^2 \right] \tag{A6}
\]

Combining them together, one obtains that

\[
32c + 32d = \sum_{\pm} \sum_{n=1}^\infty \left[ \frac{-2}{(|p_r| \pm \kappa)^2 - M_n^2} + \frac{-2}{(|q_3| \pm \kappa)^2 - M_n^2} + \frac{m_\pi^2}{2|p_r,q_3|} K_{11} \right], \tag{A7}
\]

where

\[
K_{11} = \left( 1 - \frac{4k_0^2}{(|p_r| + |q_3|)^2} \right) \frac{1}{(|p_r| \pm \kappa)^2 - M_n^2} \left[ (|q_3| \pm \kappa)^2 - M_n^2 \right] + \left( \frac{4k_0^2}{(|p_r| - |q_3|)^2} - 1 \right) \frac{1}{(|p_r| \pm \kappa)^2 - M_n^2} \left[ (|q_3| \pm \kappa)^2 - M_n^2 \right]. \tag{A8}
\]
Tracing in Dirac space, it is easy to get the mixed meson-meson correlators in the mixture of $i\gamma^5\sigma_{12} \otimes i\gamma^5\sigma_{12}$ and $i\gamma^5\sigma_{12} \otimes i\gamma_5$, described in terms of

$$K_{22} = \left( 1 - \frac{4k_0^2}{(|p_n| + |q_n|)^2} \right) + \frac{4M^2 - (|p_n| - |q_n|)^2}{(|p_n| + \kappa)^2 - M_n^2} \frac{1}{(|q_n| + \kappa)^2 - M_n^2} \left( \frac{1}{(|p_n| - |q_n|)^2} - 1 \right) \left( \frac{1}{(|p_n| + \kappa)^2 - M_n^2} + \frac{1}{(|q_n| + \kappa)^2 - M_n^2} \right) - \frac{4k_0^2}{(|p_n| - |q_n|)^2}$$ (A9)

and

$$K_{12} = \left( 1 - \frac{4k_0^2}{(|p_n| + |q_n|)^2} \right) \frac{\pm 2M (|p_n| + |q_n| \pm 2\kappa)}{(|p_n| + \kappa)^2 - M_n^2} \frac{1}{(|q_n| + \kappa)^2 - M_n^2} \left( \frac{1}{(|p_n| - |q_n|)^2} - 1 \right) \left( \frac{1}{(|p_n| + \kappa)^2 - M_n^2} + \frac{1}{(|q_n| + \kappa)^2 - M_n^2} \right)$$ (A10)

We note here that the term $\frac{1}{(|p_n| - |q_n|)^2}$ in the bracket will not lead to a new discussion of regularization for mesons, since its poles are located at i) $k_0 = 0$ for any $m_\pi$, which is not contributing due to the $k_0^2$ in the nominator; ii) $m_\pi = 0$ for any $k_0$. In the latter case, the massless property of the neutral pion is guaranteed by the chiral quark and zero $\kappa$, hence, the explicit value of $K$ is not important at all.

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