NOTES ON THE CONTACT OZSVÁTH–SZABÓ INVARIANTS

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Abstract. In this paper we prove various results on contact structures obtained by contact surgery on a single Legendrian knot in the standard contact three–sphere. Our main tool are the contact Ozsváth–Szabó invariants.

1. Introduction

According to a recent result of Ding and Geiges [3] any closed contact 3–manifold is obtained by contact surgery along a Legendrian link \( L \) in the standard contact 3–sphere \((S^3, \xi_{st})\), where the surgery coefficients on the individual components of \( L \) can be chosen to be \( \pm 1 \) relative to the contact framing. (For additional discussion on this theorem see [4].) It is an intriguing question how to establish interesting properties of a contact structure from one of its surgery presentations. More precisely, we would like to find a way to determine whether the result of a certain contact surgery is tight or fillable. Recall that contact \((-1)\)–surgery (also called Legendrian surgery) on a Legendrian link \( L \) produces a Stein fillable, hence tight contact 3–manifold.

Given a Legendrian knot \( K \subset (S^3, \xi_{st}) \), we shall denote the result of contact \((+1)\)–surgery along \( K \) by \((Y_K, \xi_K)\). A first result, which has an elementary proof, is the following.

Theorem 1.1. Let \( K \) be a Legendrian knot in the standard contact three–sphere. Assume that, for some orientation of \( K \), a front projection of \( K \) contains the configuration of Figure 1, with an odd number of cusps between the strands \( U \) and \( U' \). Then, \((Y_K, \xi_K)\) is overtwisted.

\[ \text{Figure 1. Configuration producing an overtwisted disk} \]

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Corollary 1.2. Let $K$ be a Legendrian knot in the standard contact three–sphere. If $K$ is smoothly isotopic to a negative torus knot then $(Y_K, \xi_K)$ is overtwisted.

Notice the contrast: when the Legendrian knot $K$ satisfies $\text{tb}(K) = 2g_s(K) - 1$ (where $g_s(K)$ denotes its slice genus) then $(Y_{\tilde{K}}, \xi_K)$ is tight \[13\]. The tightness question for contact structures can be fruitfully attacked with the use of the contact Ozsváth–Szabó invariants \[21\]. In fact, the nonvanishing of these invariants implies tightness, while their computation can sometimes be performed (see e.g. \[13, 14\]) using a contact surgery presentation in conjunction with the surgery exact triangle established in Heegaard Floer theory by Peter Ozsváth and Zoltán Szabó \[19\]. Such ideas can be used to prove the following.

Theorem 1.3. Let $K \subset S^3$ be a smooth knot. Suppose that, for some integer $n > 0$, the 3–manifold $S^3_n(K)$ is a lens space. Let $L \subset (S^3, \xi_{st})$ be a Legendrian knot smoothly isotopic to $K$. Then, $L$ has Thurston–Bennequin invariant not greater than $n$.

In the proof of Theorem 1.3 we will only assume that $S^3_n(K)$ is an $L$–space, a weaker condition specified in Section 2 and known to be satisfied by lens spaces.

In our investigations we prove tightness by establishing the nonvanishing of the appropriate contact Ozsváth–Szabó invariant. Therefore, we are interested in cases when this invariant vanishes, although overtwistedness does not obviously hold.

Proposition 1.4. Let $L_1, L_2 \subset (S^3, \xi_{st})$ be two smoothly isotopic Legendrian knots whose Thurston–Bennequin invariants satisfy

$$\text{tb}(L_1) < \text{tb}(L_2).$$

Then, the result of contact $(+1)$–surgery along $L_1$ has vanishing contact Ozsváth–Szabó invariant. If $\text{tb}(L) \leq -2$ then the contact Ozsváth–Szabó invariant $c^+(Y_L, \xi_L)$ vanishes.

Remark 1.5. The hypotheses of Proposition 1.4 do not imply that either $L_1$ or $L_2$ be stabilizations of other Legendrian knots. In fact, examples of Legendrian knots $L_1$ and $L_2$ satisfying the assumptions of Proposition 1.4 without being stabilizations were found by Etnyre and Honda \[8\].

In many cases the contact invariants can be explicitly computed. We will perform such computations for a subfamily of Legendrian knots called Chekanov–Eliashberg knots, cf. \[5\]. These knots are of particular interest because they have equal “classical invariants” (i.e., knot type, Thurston–Bennequin invariant and rotation number) but are not Legendrian isotopic. Our computation shows that, at least when combined with the particular surgery approach we adopt here, the contact Ozsváth–Szabó invariant is not strong enough to distinguish these knots up to Legendrian isotopy. For the precise formulation of this fact see Section 4.

As a further application, we present examples where the contact Ozsváth–Szabó invariants distinguish contact structures defined on a fixed 3–manifold. In particular, by a simple calculation we recover the main result of \[12\]:

Theorem 1.6 (\[12\]). The Brieskorn integral homology sphere $-\Sigma(2, 3, 6n - 1)$ admits at least $(n - 1)$ nonisotopic tight contact structures.

Remark 1.7. The same result was obtained in a more general form by O. Plamenevskaya \[22\].
Section 2 is devoted to the necessary (and brief) recollection of background information about contact surgery and Ozsváth–Szabó invariants. Proofs of most of the statements announced in the Introduction are given in Section 3. Section 4 is devoted to the Legendrian Chekanov–Eliashberg knots. In Section 5 we prove Theorem 1.6.

2. Preliminaries

For the basics of contact geometry and topology we refer the reader to [6, 9].

Contact surgery. Let \((Y, \xi)\) be a closed, contact 3–manifold and \(L \subset (Y, \xi)\) a Legendrian knot. The contact structure \(\xi\) can be extended from the complement of a neighborhood of \(L\) to the 3–manifold obtained by \((\pm 1)\)–surgery along \(L\) (with respect to the contact framing). In fact, by the classification of tight contact structures on the solid torus \(S^1 \times D^2\), such an extension is uniquely specified by requiring that its restriction to the surgered solid torus be tight. The same uniqueness property holds for all surgery coefficients of the form \(\frac{1}{k}\) with \(k \in \mathbb{Z}\). For a general nonzero rational surgery coefficient, there is a finite number of choices for the extension. Consequently, a Legendrian knot \(L \subset (S^3, \xi_{st})\) decorated with \(+1\) or \(-1\) gives rise to a well–defined contact 3–manifold, which we shall denote by \((Y_L, \xi_L)\) and \((Y_{\tilde{L}}, \xi_{\tilde{L}})\), respectively. For a more extensive discussion on contact surgery see [3].

Heegaard Floer theory. In this subsection we recall the basics of the Ozsváth–Szabó homology groups. For a more detailed treatment see [16, 17, 18].

According [16], to a closed, oriented spin\(^c\) 3–manifold \((Y, t)\) one can associate a finitely generated Abelian group \(\widehat{HF}(Y, t)\) and a finitely generated \(\mathbb{Z}[U]\)–module \(HF^+(Y, t)\). A spin\(^c\) cobordism \((W, s)\) between \((Y_1, t_1)\) and \((Y_2, t_2)\) gives rise to homomorphisms \(\widehat{F}_{W,s}: \widehat{HF}(Y_1, t_1) \rightarrow \widehat{HF}(Y_2, t_2)\) and \(F^+_{W,s}: HF^+(Y_1, t_1) \rightarrow HF^+(Y_2, t_2)\), with \(F^+_{W,s}\) \(U\)–equivariant.

Let \(Y\) be a closed, oriented 3–manifold and \(K \subset Y\) a framed knot with framing \(f\). Let \(Y(K)\) denote the 3–manifold given by surgery along \(K \subset Y\) with respect to the framing \(f\). The surgery can be viewed at the 4–manifold level as a 2–handle addition. The resulting cobordism \(X\) induces a homomorphism

\[\widehat{F}_X := \sum_{s \in Spin^c(X)} \widehat{F}_{X,s}: \widehat{HF}(Y) \rightarrow \widehat{HF}(Y(K))\],

where \(\widehat{HF}(Y) := \oplus_{t \in Spin^c(Y)} \widehat{HF}(Y, t)\). Similarly, there is a cobordism \(Z\) defined by adding a 2–handle to \(Y(K)\) along a normal circle \(N\) to \(K\) with framing \(-1\) with respect to a normal disk to \(K\). The boundary components of \(Z\) are \(Y(K)\) and the 3–manifold \(Y'(K)\) obtained from \(Y\) by a surgery along \(K\) with framing \(f + 1\). As before, \(Z\) induces a homomorphism

\[\widehat{F}_Z: \widehat{HF}(Y(K)) \rightarrow \widehat{HF}(Y'(K))\].

The above construction can be repeated starting with \(Y(K)\) and \(N \subset Y(K)\) equipped with the framing specified above: we get \(Z\) (playing the role previously played by \(X\)) and a new cobordism \(W\) starting from \(Y'(K)\), given by attaching a 4–dimensional 2–handle along a normal circle \(C\) to \(N\) with framing \(-1\) with respect to a normal disk. It is easy to check that this last operation yields \(Y\) at the 3–manifold level.
Theorem 2.1 ([17], Theorem 9.16). The homomorphisms \( \hat{F}_X, \hat{F}_Z \) and \( \hat{F}_W \) fit into an exact triangle

\[
\begin{array}{ccc}
\hat{F}_W & \rightarrow & \hat{F}_X \\
\downarrow & & \downarrow \\
\hat{F}(Y(K)) & \rightarrow & \hat{F}(Y) \\
\end{array}
\]

For a torsion spin\( ^c \) structure (i.e. a spin\( ^c \) structure whose first Chern class is torsion) the homology theories \( \hat{HF} \) and \( HF^+ \) come with a relative \( \mathbb{Z} \)–grading which admits a lift to an absolute \( \mathbb{Q} \)–grading [19]. The action of \( U \) shifts this degree by \(-2\).

For \( a \in \mathbb{Q} \), define \( T^+_a := \oplus_b (T^+_a)_b \) as the graded \( \mathbb{Z}[U] \)–module such that, for every \( b \in \mathbb{Q} \),

\[
(T^+_a)_b = \begin{cases} 
\mathbb{Z} & \text{for } b \geq a \text{ and } b - a \in 2\mathbb{Z}, \\
0 & \text{otherwise,}
\end{cases}
\]

and the \( U \)–action \( (T^+_a)_b \rightarrow (T^+_a)_{b-2} \) is an isomorphism for every \( b \neq a \). The following proposition can be extracted from [17, Theorem 10.1] and [19, Propositions 4.2 and 4.10].

Proposition 2.2 ([17] [19]). Let \( Y \) be a rational homology sphere. Then, for each \( t \in Spin^c(Y) \)

\[
HF^+(Y, t) = T^+_a \oplus A(Y),
\]

where \( a \in \mathbb{Q} \) and \( A(Y) = \oplus_d A_d(Y) \) is a graded, finitely generated Abelian group. Moreover,

\[
HF^+(Y, -t) = T^-_a \oplus A(-Y),
\]

with \( A_d(-Y) \cong A_{-d-1}(Y) \). If \( b_1(Y) = 1 \) and \( t \in Spin^c(Y) \) is torsion then

\[
HF^+(Y, t) = T^+_a \oplus T^+_{a'} \oplus A'(Y),
\]

where \( a - a' \) is an odd integer and \( A'(Y) = \oplus_d A'_d(Y) \) is a graded, finitely generated Abelian group. Moreover,

\[
HF^+(Y, -t) = T^-_a \oplus T^-_{a'} \oplus A'(Y),
\]

with \( A'_d(-Y) \cong A'_{-d-1}(Y) \).

The two theories \( \hat{HF} \) and \( HF^+ \) are related by a long exact sequence, which takes the following form for a torsion spin\( ^c \) structure \( t \)

\[
\ldots \rightarrow \hat{HF}_a(Y, t) \xrightarrow{f} HF^+_a(Y, t) \xrightarrow{U} HF^+_{a-2}(Y, t) \rightarrow \hat{HF}_{a-1}(Y, t) \rightarrow \ldots
\]

where \( U \) denotes “multiplication by \( U \)”. All the gradings appearing in the sequence can be worked out from the definitions and the construction of the exact sequence (cf. [19, Section 2]).

Corollary 2.3. Let \( Y \) be a rational homology 3–sphere. Then, \( HF^+(Y, t) \cong T^+_a \) if and only if \( \hat{HF}(Y, t) \cong \mathbb{Z} \). If \( b_1(Y) = 1 \) and \( t \) is a torsion spin\( ^c \) structure, then \( HF^+(Y, t) \cong T^+_a \oplus T^+_a \) if and only if \( \hat{HF}(Y, t) \cong \mathbb{Z}^2 \).
Proof. We sketch the proof of the statement for \( b_1(Y) = 0 \), the other case can be proved by similar arguments. Clearly, if \( HF^+(Y, t) \cong T_a^+ \) then it follows immediately from Exact Sequence (2.1) that \( \hat{HF}(Y, t) \cong \hat{H}_a(Y, t) \cong \mathbb{Z} \). Conversely, if \( \hat{HF}(Y, t) \cong \mathbb{Z} \) then Exact Sequence (2.1) and Proposition 2.2 imply \( HF^+(Y, t) \cong T_a^+ \). □

Observe that, in view of Corollary 2.3 if \( Y \) is a rational homology 3–sphere, the following two conditions are equivalent:

1. For each spin\(^c\) structure \( t \in Spin^c(Y) \), \( HF^+(Y, t) \cong T_a^+ \) for some \( a \);
2. For each spin\(^c\) structure \( t \in Spin^c(Y) \), \( \hat{HF}(Y, t) \cong \mathbb{Z} \).

Definition 2.4. A rational homology 3–sphere satisfying any of the above equivalent conditions is called an \( L \)–space.

It follows from Proposition 2.2 that an oriented rational homology 3–sphere \( Y \) is an \( L \)–space if and only if \( -Y \) is an \( L \)–space. Moreover, lens spaces are \( L \)–spaces [17, Section 3].

We will use the following fact regarding the maps connecting the Ozsváth–Szabó homology groups. Suppose that \( W \) is a cobordism defined by a single 2–handle attachment.

Proposition 2.5 ([13]). Let \( W \) be a cobordism containing a smooth, closed, oriented surface \( \Sigma \) of genus \( g \), with \( \Sigma \cdot \Sigma > 2g - 2 \). Then, the induced maps \( \hat{F}_W, s \) and \( F_W^+ \) vanish for every spin\(^c\) structures \( s \) on \( W \).

Contact Ozsváth–Szabó invariants. Let \( (Y, \xi) \) be a closed, contact 3–manifold. Then, the contact Ozsváth–Szabó invariants

\[
\hat{c}(Y, \xi) \in \hat{HF}(-Y, t_\xi)/\langle \pm 1 \rangle \quad \text{and} \quad c^+(Y, \xi) \in HF^+(-Y, t_\xi)/\langle \pm 1 \rangle
\]

are defined [21], with \( f(\hat{c}(Y, \xi)) = c^+(Y, \xi) \), where \( f \) is the homomorphism appearing in Exact Sequence (2.1) and \( t_\xi \) is the spin\(^c\) structure induced by the contact structure \( \xi \).

To simplify notation, throughout the paper we ignore the sign ambiguity in the definition of the contact invariants, and treat them as honest elements of the appropriate homology groups rather than equivalence classes. The reader should have no problem checking that there is no loss in making this abuse of notation. Alternatively, one could work with \( \mathbb{Z}/2\mathbb{Z} \) coefficients to make the sign ambiguity disappear altogether. The properties of \( \hat{c} \) and \( c^+ \) which will be relevant for us can be summarized as follows.

Theorem 2.6 ([21]). Let \( (Y, \xi) \) be a closed, contact 3–manifold, and denote by \( c(Y, \xi) \) either one of the contact invariants \( \hat{c}(Y, \xi) \) and \( c^+(Y, \xi) \). Then,

1. The class \( c(Y, \xi) \) is an invariant of the isotopy class of the contact structure \( \xi \) on \( Y \).
2. If \( (Y, \xi) \) is overtwisted then \( c(Y, \xi) = 0 \), while if \( (Y, \xi) \) is Stein fillable then \( c(Y, \xi) \neq 0 \).
3. Suppose that \( (Y_2, \xi_2) \) is obtained from \( (Y_1, \xi_1) \) by a contact \((+1)\)–surgery. Then we have
   \[
   F_{-X}(c(Y_1, \xi_1)) = c(Y_2, \xi_2),
   \]
   where \( -X \) is the cobordism induced by the surgery with orientation reversed and \( F_{-X} \) is the sum of \( F_{-X, s} \) over all spin\(^c\) structures \( s \) extending the spin\(^c\) structures induced on \(-Y_i\) by \( \xi_i \), \( i = 1, 2 \). In particular, if \( c(Y_2, \xi_2) \neq 0 \) then \( (Y_1, \xi_1) \) is tight.
Suppose that $t_\xi$ is torsion. Then $c(Y, \xi)$ is a homogeneous element of degree $-h(\xi) \in \mathbb{Q}$, where $h(\xi)$ is the Hopf–invariant of the 2–plane field defined by the contact structure $\xi$. □

Remark 2.7. The Hopf–invariant can be easily determined for a contact structure defined by a contact $(\pm 1)$–surgery diagram along the Legendrian link $L \subset (S^3, \xi_{st})$ [4]. In fact, fix an orientation of $L$ and consider the 4–manifold $X$ defined by the Kirby diagram specified by the surgery [10]. Let $c \in H^2(X; \mathbb{Z})$ denote the cohomology class which evaluates as $\text{rot}(L)$ on the homology class determined by a component $L$ of the link $L$. If $t_\xi$ is torsion, then $c^2 \in \mathbb{Q}$ is defined, and $h(\xi)$ is equal to $1/4(c^2 - 3\sigma(X) - 2\chi(X) + 2) + q$, where $q$ is the number of $(+1)$–surgeries made along $L$ to get $(Y, \xi)$.

3. Proofs

Now we can turn to the proofs of the statements announced in Section 1.

Proof of Theorem 1.1. Consider the Legendrian push–off $K'$ of $K$ drawn as a dotted line in the left–hand side of Figure 2. The obvious annulus between $K$ and $K'$ induces framing $tb(K)$ on both $K$ and $K'$. Consider the modification $K''$ of $K'$ illustrated in the right–hand side of Figure 2. The obvious surface $S$ between $K''$ and $K$ is oriented because of the hypotheses on the cusps of the front projection, it has genus 1 and it induces framing $tb(K) + 1$ on $K$. In particular, $S$ extends to a meridian disk $D$ inside the surgered solid torus. Since $S$ induces framing $tb(K) + 1$ on $K''$, while $tb(K'') = tb(K') + 3 = tb(K) + 3$, we have $tb_{S\cup D}(K'') = 2$, i.e. the Legendrian knot $K'' = \partial(S\cup D)$ violates the Bennequin–Eliashberg inequality with respect to $S\cup D$. We conclude that $(Y_K, \xi_K)$ is overtwisted. □

To prove Theorem 1.3, Corollary 1.2 and Proposition 1.4 we shall need the following lemma (for a different proof of a more general result see [15]).

Lemma 3.1. Let $K$ be a Legendrian knot in the standard contact three–sphere. If $K$ is the stabilization of another Legendrian knot then $(Y_K, \xi_K)$ is overtwisted.

Proof. By assumption, $K$ admits a front projection containing one of the configurations of Figure 3. Without loss, we may assume that we are in the situation of the left–hand side of Figure 3. Consider the Legendrian push–off $K'$ of $K$ drawn as a dotted line in the left–hand side of Figure 4. The obvious annulus between $K$ and $K'$ induces framing...
Figure 3. The two possible “zig–zags”

tb(K) on both K and K’. Consider the modification K'' of K' illustrated in the right–hand side of Figure 4. There still is an obvious annulus A between K'' and K, except

Figure 4. The modification of the Legendrian push–off

that now it induces framing tb(K'') = tb(K) + 1 on K and K''. Since we perform contact (+1)–surgery on K, the annulus A extends to a meridian disk D inside the surgered solid torus. Therefore, D ∪ A is an overtwisted disk in (Y_K, ξ_K).

□

The proof of Lemma 3.1 clearly applies to establish the following slight generalization:

Proposition 3.2. Suppose that the Legendrian link L ⊂ (S^3, ξ_{st}) is obtained by stabilizing some components of another Legendrian link. Let (Y_L, ξ_L) be the result of contact (±1)–surgeries along the components of L. If the surgery coefficient on one of the stabilized components is (+1), then (Y_L, ξ_L) is overtwisted.

□

Proof of Corollary 1.2. Examining [7, Figure 8], it is easy to check that any Legendrian negative torus knot K with maximal Thurston–Bennequin invariant contains the configuration of Figure 1 with an odd number of cusps between the two strands U and U’. Therefore, by Theorem 1.1 (Y_K, ξ_K) is overtwisted. On the other hand, according to the results of [7], any Legendrian negative torus knot K’ with non–maximal Thurston–Bennequin invariant is isotopic to the stabilization of one with maximal Thurston–Bennequin invariant. Thus, by Lemma 3.1 (Y_{K’}, ξ_{K’}) is overtwisted.

□

Proof of Theorem 1.3. By contradiction, suppose that S^3_n(K) is an L–space (recall that lens spaces are L–spaces) and L_1 ⊂ (S^3, ξ_{st}) is a Legendrian knot smoothly isotopic to K with tb(L_1) > n. Let L be obtained by stabilizing L_1 tb(L_1)–n times, so that tb(L) = n.
Denote by \((Y_L, \xi_L)\) the result of contact \((+1)\)–surgery along \(L\). By Lemma 3.1 \((Y_L, \xi_L)\) is overtwisted, hence \(\hat{c}(Y_L, \xi_L) = 0\). On the other hand, we can compute \(\hat{c}(Y_L, \xi_L)\) using Theorem 2.6, getting \(\hat{c}(Y_L, \xi_L) = \hat{F}_{-X}(c(S^3, \xi_{st}))\) where \(X\) is the appropriate cobordism. The map \(\hat{F}_{-X}\) fits into the exact triangle

\[
\begin{array}{ccc}
\hat{HF}(S^3) & \xrightarrow{\hat{F}_{-X}} & \hat{HF}(S^3_{n-1}(K)) \\
\hat{F}_W & & \hat{HF}(S^3_n(K)) \\
& & \end{array}
\]

where \(\overline{K}\) is the mirror image of \(K\) and \(S^3(K)\) denotes the result of \(r\)–surgery along \(K\). Since \(S^3_{-n}(\overline{K}) = -S^3_n(K)\) is an \(L\)–space, we have

\[
\text{rk} \, \hat{HF}(S^3_{-n}(\overline{K})) = |H_1(S^3_{-n}(\overline{K}))| = n,
\]

while by Proposition 2.2

\[
\text{rk} \, \hat{HF}(S^3_{-n-1}(\overline{K})) \geq |H_1(S^3_{-n-1}(\overline{K}))| = n + 1.
\]

Exactness of the triangle immediately implies \(\hat{F}_W = 0\), therefore \(\hat{F}_{-X}\) must be injective. Since \(\hat{c}(S^3, \xi_{st}) \neq 0\), this shows \(\hat{c}(Y_L, \xi_L) \neq 0\), which contradicts the fact that \((Y_L, \xi_L)\) is overtwisted. \(\square\)

**Proof of Proposition 1.4.** Consider a Legendrian knot \(L'\) obtained by stabilizing \(L_2\) until \(tb(L_1) = tb(L')\). Since \(L'\) and \(L_1\) are smoothly isotopic and have the same contact framing, the cobordisms associated to the contact \((+1)\)–surgeries along \(L_1\) and \(L'\) can be identified. Since \(c(Y_{L_1}, \xi_{L_1})\) and \(c(Y_{L'\prime}, \xi_{L'})\) are images of \(c(S^3, \xi_{st})\) under the same map, \(c(Y_{L_1}, \xi_{L_1}) = 0\) if and only if \(c(Y_{L'}, \xi_{L'}) = 0\). Lemma 3.1 gives \(c(Y_{L'}, \xi_{L'}) = 0\), and the first statement follows.

For the second statement consider the exact triangle in the \(HF^+\)–theory provided by the surgery along \(L\). (The Thurston–Bennequin invariant \(tb(L)\) is denoted by \(t\).) After reversing orientation the triangle takes the shape

\[
\begin{array}{ccc}
HF^+(S^3) & \xrightarrow{F^+_W} & HF^+(S^3_{t-1}(\overline{L})) \\
\hat{HF}(S^3_t(\overline{L})) & & \end{array}
\]

Now the assumption \(t < -1\) implies that \(-t - 1 > 0\), hence the cobordism \(-W\) inducing the first map is positive definite. It is known that the map \(F^+_W\) on the \(HF^\infty\)–theory vanishes if \(b^+_2(-W) > 0\) \cite{17}. Since for \(S^3\) the natural map \(HF^\infty(S^3) \to HF^+(S^3)\) is onto, this implies that \(F^+_W = 0\). Since

\[
c^+(Y_L, \xi_L) = F^+_W(c^+(S^3, \xi_{st})),
\]

the vanishing of the contact invariant \(c^+(Y_L, \xi_L)\) follows. \(\square\)

**4. Examples**

Given a Legendrian knot \(L \subset (S^3, \xi_{st})\), we shall denote by \((Y_L, \xi_L)\), respectively \((Y^L, \xi^L)\), the contact 3–manifold obtained by contact \((+1)\)–, respectively \((-1)\)–surgery.
Let $L_i = L_i(n)$, $i = 1, \ldots, n - 1$, be the Legendrian knot given by Figure 5(b). The knots $L_i(n)$ ($n$ fixed and $\geq 2$) were considered in [5]. They are all smoothly isotopic to the $n$-twist knot of Figure 5(a) (having $n$ negative half-twists). The knots $L_i$ were the first examples of smoothly isotopic Legendrian knots having equal classical invariants (i.e. Thurston–Bennequin invariants and rotation numbers) but not Legendrian isotopic [1, 5].

The reader should be aware that our convention for representing a Legendrian knot via its front projection differs from the one used in [5]. In fact, we use the contact structure given by the 1–form $dz + xdy$ rather than the 1–form $-dz + ydx$, used in [5]. However, the contactomorphism between the two contact structures given by sending $(x, y, z)$ to $(y, -x, z)$ induces a one–to–one correspondence between the corresponding front projections, and under this correspondence Figure 1 from [5] is sent to our Figure 5(b).

**Proposition 4.1.** For every $1 \leq i, j \leq n - 1$ we have
\[ c(Y_{L_i}, \xi_{L_i}) = c(Y_{L_j}, \xi_{L_j}). \]

**Proof.** The statement follows easily from basic properties of the contact invariant: by the surgery formula for contact $(+1)$–surgeries, we have $c(Y_{L_i}, \xi_{L_i}) = F_X(c(S^3, \xi_{st}))$, where $X$ is the cobordism induced by the 4–dimensional handle attachment dictated by the surgery. Since $X$ depends only on the smooth isotopy class of the Legendrian knot and its Thurston–Bennequin invariant, and is therefore independent of $i$, the claim trivially follows.

According to the main result of this section, Theorem 4.2, the same equality holds if we perform Legendrian surgeries along $L_i(n)$, that is, the contact Ozsváth–Szabó invariants of the results of contact $(\pm 1)$–surgeries do not distinguish the Chekanov–Eliashberg knots.
Theorem 4.2. Let \( n \geq 2 \) be an even integer, and let \( 1 \leq i, j \leq n - 1 \) be both odd. Then,
\[
\hat{c}(Y^{L_i}, \xi^{L_i}) = \hat{c}(Y^{L_j}, \xi^{L_j}).
\]

The proof of Theorem 4.2 rests on the following two lemmas.

Lemma 4.3 (20). Let \( n \geq 2 \) be an even integer, and denote by \( L(n) \) the mirror image of \( L(n) \). Then,
\[
HF^+(S^3_0(L(n))) \cong \mathcal{T}_n^{+1} \oplus \mathcal{T}_n^{+2} \oplus \mathbb{Z}_2^{n-1}.
\]

Proof. Let \( k = \frac{n}{2} \). Choosing a suitable oriented basis for an obvious Seifert surface for \( L(n) \) one can easily compute the Seifert matrix
\[
\begin{pmatrix}
-k & k - 1 \\
 0 & -k
\end{pmatrix},
\]
with eigenvalues \(-1\) and \(1 - 4k\). This immediately gives signature \( \sigma(L(n)) = -2 \) and Alexander polynomial
\[
\Delta_{L(n)}(t) = kt^{-1} - (2k - 1) + kt.
\]

Since \( L(n) \) is an alternating knot with genus \( g(L(n)) = 1 \), applying [20 Theorem 1.4] we get
\[
\begin{cases}
HF^+(S^3_0(L(n)), s) \cong \mathcal{T}_n^{+1} \oplus \mathcal{T}_n^{+2} \oplus \mathbb{Z}_2^{n-1} & \text{if } c_1(s) = 0, \\
HF^+(S^3_0(L(n)), s) = 0 & \text{if } c_1(s) \neq 0.
\end{cases}
\]

By Proposition 2.2 this implies the result. \( \square \)

Lemma 4.4. Let \( k \geq 0 \) be an integer, and let \( V(k) \) be the oriented 3–manifold defined by the surgery diagram of Figure 6. Then,
\[
\hat{HF}(V(k)) \cong \mathbb{Z}^{2k+2} \quad \text{and} \quad HF^+(V(k)) = \bigoplus_{i=1}^{2k+2} \mathcal{T}_i^{+a_i} \quad \text{for some } a_i \in \mathbb{Q}.
\]

Figure 6. Surgery diagram for \( V(k) \)
Proof. In order to compute $\widehat{HF}(V(k))$ we will use the exact triangle defined by the $(k+1)$–framed unknot of Figure 6. It is easy to see that the unknot of Figure 6 bounds a punctured torus smoothly embedded in the complement of the knot $K$. Thus, the cobordism we get by attaching this last 2–handle contains a torus with self–intersection $(k+1)$, and the induced map in the surgery triangle vanishes by Proposition 2.5. Consequently, the surgery triangle is actually a short exact sequence. Notice that $K$ is the (left–handed) trefoil knot, hence $\widehat{HF}(S^3_0(K)) = \mathbb{Z}^2$ [20, Theorem 1.4]. Arguing by induction we get

$$\widehat{HF}(V(k+1)) \cong \widehat{HF}(V(k)) \oplus \mathbb{Z}^2$$

for every $k \geq 0$. On the other hand, for $k = 0$ the unknot can be blown down, showing that $V(0) \cong S^1 \times S^2$. This fact immediately implies

$$\widehat{HF}(V(k)) \cong \mathbb{Z}^{2k+2}$$

for every $k \geq 0$. Using the surgery presentation of Figure 6 it is easy to check that

$$H_1(V(k); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/(k+1)\mathbb{Z},$$

therefore $V(k)$ admits $(k+1)$ different torsion spin$^c$ structures. By Proposition 2.2 and Exact Sequence 2.1 we have

$$\text{rk } \widehat{HF}(V(k), t) \geq 2$$

if $t$ is a torsion spin$^c$ structure. Therefore, using (4.1), we see that $\widehat{HF}(V(k), t) \cong \mathbb{Z}^2$ for each torsion spin$^c$ structure $t$ and

$$\widehat{HF}(V(k), t) = 0$$

if $t$ is not torsion. The statement now follows from Proposition 2.2 and Corollary 2.3. \qed

Proof of Theorem 4.2. The idea of the proof is the following: First we will find a contact 3–manifold $(Y, \xi)$ such that contact $(+1)$–surgery along some Legendrian knot $K \subset (Y, \xi)$ gives $(Y_{Li}, \xi_{Li})$ and $A(Y) \subset HF^+(Y, t_{Li})$ (as it is defined in Proposition 2.2) vanishes. Therefore $c^+(Y, \xi)$ is an element of some $T^+_a$. The $U$–equivariance of the map induced by the surgery will then show that $c^+(Y_{Li}, \xi_{Li}) \in T^+_a \subset HF^+(Y_{Li}, t_{Li})$, from which the conclusion will easily follow.

To this end, consider the contact structure $\eta_i(n)$ defined by Legendrian surgery along the 2–component link of Figure 7. Notice that one of the knots in Figure 7 is topologically the unknot, while the other one is $L_i(n)$. According to the Kirby moves indicated in Figure 8, it follows that this contact structure lives on the 3–manifold $Y(n) := -V(\frac{n}{2})$, where $V(k)$ is defined by Figure 6. According to [2], the effect of a contact $(\pm 1)$–surgery along a Legendrian knot can be cancelled by contact $(\mp 1)$–surgery along a Legendrian push–off of the knot. Therefore, doing contact $(+1)$–surgery along the push–off of the unknot in Figure 7 we get $(Y_{Li}, \xi_{Li})$. On the other hand, denoting by $X_n$ the cobordism induced by the contact $(+1)$–surgery, we have

$$\widehat{F}_{-X_n}(\hat{c}(Y(n), \eta_i(n))) = \hat{c}(Y_{Li}, \xi_{Li}).$$

A simple computation shows that $h(\xi_{Li}) = -\frac{1}{2}$, therefore by Theorem 2.6(4) we have

$$\hat{c}(Y_{Li}, \xi_{Li}) \in \widehat{HF}_{-\frac{1}{2}}(-Y_{Li}).$$

Moreover, $\hat{c}(Y_{Li}, \xi_{Li})$ is primitive [22]. Thus, to prove the statement it will be enough to verify that there is a rank–1 subgroup of $\widehat{HF}_{-\frac{1}{2}}(-Y_{Li})$ containing $\widehat{F}_{-X_n}(\hat{c}(Y(n), \eta_i(n)))$.
for every $i$. An easy computation shows that (since we assumed $n$ to be even) the Thurston–Bennequin numbers of the knots $L_i(n)$ are all equal to 1, cf. [5], hence each of the 3–manifolds $Y^{L_i}$ is diffeomorphic to $S^3_0(L(n))$. By Lemma 4.3

\[ HF^+(-S^3_0(L(n))) \cong \mathcal{T}^+_\frac{1}{2} \oplus \mathcal{T}^+_\frac{3}{2} \oplus A, \]

where $A$ is a finitely generated abelian group, while by Lemma 4.4 we have

\[ HF^+(-Y(n)) = \oplus_{i=1}^{n+2} \mathcal{T}^+_{a_i}, \]

for some $a_i \in \mathbb{Q}$. Since $F^+_{X_n}$ is $U$–equivariant and for sufficiently large $h$ the action of $U^h$ vanishes on $A$, we have

\[ \text{Im}(F^+_{X_n}) \subseteq \mathcal{T}^+_{\frac{1}{2}} \oplus \mathcal{T}^+_{\frac{3}{2}} \subseteq HF^+(-S^3_0(L(n))). \]

Therefore, up to sign, there is a unique primitive element in $\text{Im}(F^+_{X_n})$ of degree $\frac{1}{2}$, implying that $c^+(Y^{L_i}, \xi^{L_i}) = c^+(Y^{L_j}, \xi^{L_j})$ for $i, j$ as in the statement. Since

\[ HF^+_{-\frac{1}{2}}(-S^3_0(L(n))) = 0, \]

it follows that the homomorphism

\[ f: \widehat{HF}^+_{\frac{1}{2}}(-S^3_0(L(n))) \to HF^+_{\frac{1}{2}}(-S^3_0(L(n))) \]

from Exact Sequence (2.1) is injective. Since

\[ f(\hat{c}(Y^{L_i}, \xi^{L_i})) = c^+(Y^{L_i}, \xi^{L_i}) \in \text{Im}(F^+_{-X_n}) \]

for every $i$, this concludes the proof. \[ \square \]
5. Distinguishing tight contact structures

**Definition 5.1.** Let $\xi_i$, for $i = 1, \ldots, n-1$, denote the contact structure on the Brieskorn sphere $-\Sigma(2,3,6n-1)$ defined by the contact surgery specified by Figure 9.

**Theorem 5.2.** The contact invariants $c^+(\xi_1), \ldots, c^+(\xi_{n-1})$ are linearly independent over $\mathbb{Z}$.

**Proof.** Consider the Legendrian push-off $\tilde{K}_1$ of the Legendrian trefoil $K_1$ of Figure 9. Attach a 4-dimensional 2-handle along $\tilde{K}_1$ to $-\Sigma(2,3,6n-1)$ with framing equal to the contact framing $+1$. Since contact $(+1)$-surgery along a Legendrian push-off cancels contact $(-1)$-surgery, we get a cobordism $W$ such that $F_W(c^+(\xi_i)) = c^+(\eta_i)$, where $\eta_i$ is the contact structure on $L(n,1)$ defined by Figure 10. The contact invariants $c^+(\eta_i)$ are linearly independent because they belong to groups corresponding to different $spin^c$ structures on the same lens space $L(n,1)$. Therefore, the invariants $c^+(\xi_i)$ are also linearly independent, concluding the proof.

**Corollary 5.3.** The contact structures $\xi_1, \ldots, \xi_{n-1}$ are pairwise non-isotopic.

Corollary 5.3 was first proved by Lisca and Matić [12] using Seiberg–Witten theory. For a different Heegaard Floer theoretic proof (of a more general statement) see [22].
Remark 5.4. It is known [19] that $HF^+(-\Sigma(2, 3, 6n - 1)) = \mathcal{T}^+_{-2} \oplus \mathbb{Z}^{n-1}_{(2)}$, therefore by Proposition 2.2 $HF^+(\Sigma(2, 3, 6n - 1)) = \mathcal{T}^+_{2} \oplus \mathbb{Z}^{n-1}_{(1)}$. It follows from Theorem 5.2 that the elements $c^+(\xi_i)$ ($i = 1, \ldots, n - 1$) span $HF^1(\Sigma(2, 3, 6n - 1))$.

Notice that if the trefoil knot of Figure 9 is replaced by any Legendrian knot $L$, the statement of Theorem 5.2 holds with the same proof. If $tb(L) = 1$ and $rot(L) = 0$, then the contact resulting structures $\xi_1, \ldots, \xi_{n-1}$ are all homotopic as 2–plane fields.
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