A FAMILY OF ANISOTROPIC INTEGRAL OPERATORS AND BEHAVIOUR OF ITS MAXIMAL EIGENVALUE

B.S. MITYAGIN, A.V. SOBOLEV

Abstract. We study the family of compact integral operators $K_{\beta}$ in $L^2(\mathbb{R})$ with the kernel

$$K_{\beta}(x, y) = \frac{1}{\pi} \frac{1}{1 + (x - y)^2 + \beta^2 \Theta(x, y)},$$

depending on the parameter $\beta > 0$, where $\Theta(x, y)$ is a symmetric non-negative homogeneous function of degree $\gamma \geq 1$. The main result is the following asymptotic formula for the maximal eigenvalue $M_{\beta}$ of $K_{\beta}$:

$$M_{\beta} = 1 - \lambda_1 \beta^{\frac{2}{\gamma + 1}} + o(\beta^{\frac{2}{\gamma + 1}}), \beta \to 0,$$

where $\lambda_1$ is the lowest eigenvalue of the operator $A = |d/dx| + \frac{1}{2} \Theta(x, x)$. A central role in the proof is played by the fact that $K_{\beta}, \beta > 0$, is positivity improving. The case $\Theta(x, y) = (x^2 + y^2)^2$ has been studied earlier in the literature as a simplified model of high-temperature superconductivity.

1. Introduction and the main result

1.1. Introduction. The object of the study is the following family of integral operators on $L^2(\mathbb{R})$:

$$(1) \quad K_{\beta}u(x) = \int K_{\beta}(x, y)u(y)dy,$$

(here and below we omit the domain of integration if it is the entire real line $\mathbb{R}$) with the kernel

$$(2) \quad K_{\beta}(x, y) = \frac{1}{\pi} \frac{1}{1 + (x - y)^2 + \beta^2 \Theta(x, y)},$$

where $\beta > 0$ is a small parameter, and the function $\Theta = \Theta(x, y)$ is a homogeneous non-negative function of $x$ and $y$ such that

$$(3) \quad \Theta(tx, ty) = t^{\gamma} \Theta(x, y), \; \gamma > 0,$$

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for all \( x, y \in \mathbb{R} \) and \( t > 0 \), and the following conditions are satisfied:

\[
\begin{cases}
  c \leq \Theta(x, y) \leq C, & |x|^2 + |y|^2 = 1, \\
  \Theta(x, y) = \Theta(y, x), & x, y \in \mathbb{R}.
\end{cases}
\]

By \( C \) or \( c \) (with or without indices) we denote various positive constants whose value is of no importance. The conditions (3) and (4) guarantee that the operator \( K_\beta \) is self-adjoint and compact.

Such an operator, with \( \Theta(x, y) = (x^2 + y^2)^2 \) was suggested by P. Krotkov and A. Chubukov in [6] and [7] as a simplified model of high-temperature superconductivity. The analysis in [6], [7] reduces to the asymptotics of the top eigenvalue \( M_\beta \) of the operator \( K_\beta \) as \( \beta \to 0 \). Heuristics in [6] and [7] suggest that \( M_\beta \) should behave as \( 1 - w \beta^2 + o(\beta^2) \) with some positive constant \( w \). A mathematically rigorous argument given by B. S. Mityagin in [9] produced a two-sided bound supporting this formula. The aim of the present paper is to find and justify an appropriate two-term asymptotic formula for \( M_\beta \) as \( \beta \to 0 \) for a homogeneous function \( \Theta \) satisfying (3), (4) and some additional smoothness conditions (see (8)).

As \( \beta \to 0 \), the operator \( K_\beta \) converges strongly to the positive-definite operator \( K_0 \), which is no longer compact. The norm of \( K_0 \) is easily found using the Fourier transform

\[
\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} f(x) dx,
\]

which is unitary on \( L^2(\mathbb{R}) \). Then one checks directly that

\[
\text{(5) the Fourier transform of } m_t(x) = \frac{1}{\pi t^2 + x^2}, \quad t > 0, \quad \text{equals } \hat{m}_t(\xi) = \frac{1}{\sqrt{2\pi}} e^{-|\xi|},
\]

and hence the operator \( K_0 \) is unitarily equivalent to the multiplication by the function \( e^{-|\xi|} \), which means that \( \|K_0\| = 1 \).

1.2. The main result. For the maximal eigenvalue \( M_\beta \) of the operator \( K_\beta \) denote by \( \Psi_\beta \) the corresponding normalized eigenfunction. Note that the operator \( K_\beta \) is positivity improving, i.e. for any non-negative non-zero function \( u \) the function \( K_\beta u \) is positive a.a. \( x \in \mathbb{R} \) (see [12], Chapter XIII.12). Thus, by [12], Theorem XIII.43 (or by [3], Theorem 13.3.6), the eigenvalue \( M_\beta \) is non-degenerate and the eigenfunction \( \Psi_\beta \) can be assumed to be positive a.a. \( x \in \mathbb{R} \). From now on we always choose \( \Psi_\beta \) in this way. The behaviour of \( M_\beta \) as \( \beta \to 0 \), is governed by the model operator

\[
(Au)(x) = |D_x|u(x) + 2^{-1}\theta(x)u(x),
\]

where

\[
\theta(x) = \Theta(x, x) = \begin{cases} 
  |x|^\gamma \Theta(1, 1), & x \geq 0; \\
  |x|^\gamma \Theta(-1, -1), & x < 0.
\end{cases}
\]
This operator is understood as the pseudo-differential operator $\text{Op}(a)$ with the symbol
\begin{equation}
a(x, \xi) = |\xi| + 2^{-1} \theta(x).
\end{equation}

For the sake of completeness recall that $P = \text{Op}(p)$ is a pseudo-differential operator with the symbol $p = p(x, \xi)$ if
\begin{equation}
(Pu)(x) = \frac{1}{2\pi} \int \int e^{i(x-y)\xi} p(x, \xi) u(y) dy d\xi
\end{equation}
for any Schwartz class function $u$. The operator $A$ is essentially self-adjoint on $C_0(\mathbb{R})$, and has a purely discrete spectrum (see e.g. [14], Theorems 26.2, 26.3). Using the von Neumann Theorem (see e.g. [11], Theorem X.25), one can see that $A$ is self-adjoint on $D(A) = D(|D_x|^\gamma) \cap D(|x|^{2\gamma})$, i.e. $D(A) = H^1(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^{2\gamma})$. Denote by $\lambda_{l} > 0$, $l = 1, 2, \ldots$ the eigenvalues of $A$ arranged in ascending order, and by $\phi_{l}$ a set of corresponding normalized eigenfunctions. As shown in Lemma 2, the lowest eigenvalue $\lambda_{1}$ is non-degenerate and its eigenfunction $\phi_{1}$ can be chosen to be non-negative a.a. $x \in \mathbb{R}$. From now on we always choose $\phi_{1}$ in this way.

The main result of this paper is contained in the next theorem.

**Theorem 1.** Let $K_{\beta}$ be an integral operator defined by (1) with $\gamma \geq 1$. Suppose that the function $\Theta$ satisfies conditions (3), (4) and the following Lipshitz conditions:
\begin{equation}
\begin{cases}
|\Theta(t, 1) - \Theta(1, 1)| \leq C|t - 1|, & t \in (1 - \epsilon, 1 + \epsilon),
|\Theta(t, -1) - \Theta(-1, -1)| \leq C|t + 1|, & t \in (-1 - \epsilon, -1 + \epsilon),
\end{cases}
\end{equation}
with some $\epsilon > 0$. Let $M_{\beta}$ be the largest eigenvalue of the operator $K_{\beta}$ and $\Psi_{\beta}$ be the corresponding eigenfunction. Then
\begin{equation}
\lim_{\beta \to 0} \beta^{-\frac{2}{2\sigma+1}} (1 - M_{\beta}) = \lambda_{1}.
\end{equation}
Moreover, the rescaled eigenfunctions $\alpha^{-\frac{1}{2}} \Psi_{\beta}(\alpha^{-1} \cdot)$, $\alpha = \beta^{\frac{1}{2\sigma+1}}$, converge in norm to $\phi_{1}$ as $\beta \to 0$.

The top eigenvalue of $K_{\beta}$ was studied by B. Mityagin in [9] for $\Theta(x, y) = (x^2 + y^2)\sigma$, $\sigma > 0$. It was conjectured that $\lim_{\beta \to 0} \beta^{-\frac{2}{2\sigma+1}} (1 - M_{\beta}) = L$ with some $L > 0$, but only the two-sided bound
\begin{equation}
c\beta^{\frac{2}{2\sigma+1}} \leq 1 - M_{\beta} \leq C\beta^{\frac{2}{2\sigma+1}},
\end{equation}
with some constants $0 < c \leq C$ was proved. It was also conjectured that in the case $\sigma = 2$ the constant $L$ should coincide with the lowest eigenvalue of the operator $|D_x| + 4x^4$. Note that for this case the corresponding operator (6) is in fact $|D_x| + 2x^4$. J. Adduci found an approximate numerical value $\lambda_{1} = 0.978\ldots$ in this case, see [1].

Similar eigenvalue asymptotics were investigated by H. Widom in [15] for integral operators with difference kernels. Some ideas of this paper are used in the proof of Theorem 1.

Let us now establish the non-degeneracy of the eigenvalue $\lambda_{1}$.
Lemma 2. Let $A$ be as defined in (6). Then

1. The semigroup $e^{-tA}$ is positivity improving for all $t > 0$,
2. The lowest eigenvalue $\lambda_1$ is non-degenerate, and the corresponding eigenfunction $\phi_1$ can be chosen to be positive a.a. $x \in \mathbb{R}$.

Proof. The non-degeneracy of $\lambda_1$ and positivity of the eigenfunction $\phi_1$ would follow from the fact that $e^{-tA}$ is positivity improving for all $t > 0$, see [12], Theorem XIII.44.

The proof of this fact is done by comparing the semigroups for the operators $A$ and $A_0 = |D_x|$. Using (5) it is straightforward to find the integral kernel of $e^{-tA_0}$:

$$m_t(x - y) = \frac{1}{\pi t^2 + (x - y)^2}, t > 0,$$

which shows that $e^{-tA_0}$ is positivity improving. To extend the same conclusion to $e^{-tA}$ let

$$V_\alpha(x) = \begin{cases} 
2^{-1} \theta(x), & |x| \leq n, \\
2^{-1} \theta(\pm n), & \pm x > n, 
\end{cases} \quad n = 1, 2, \ldots.$$

Since $(A_0 + V_n)f \to Af$ and $(A - V_n)f \to A_0f$ as $n \to \infty$ for any $f \in C_0^\infty(\mathbb{R})$, by [10], Theorem VIII.25a the operators $A_0 + V_n$ and $A - V_n$ converge to $A$ and $A_0$ resp. in the strong resolvent sense as $n \to \infty$. Thus by [12], Theorem XIII.45, the semigroup $e^{-tA}$ is also positivity improving for all $t > 0$, as required. \hfill \Box

1.3. Rescaling. As a rule, instead of $K_\beta$ it is more convenient to work with the operator obtained by rescaling $x \to \alpha^{-1}x$ with $\alpha > 0$. Precisely, let $U_\alpha$ be the unitary operator on $L^2(\mathbb{R})$ defined as $(U_\alpha f)(x) = \alpha^{-\frac{1}{2}} f(\alpha^{-1}x)$. Then $U_\alpha K_\beta U_\alpha^*$ is the integral operator with the kernel

$$B_\alpha(x, y) = \frac{1}{\pi \alpha^2 + (x - y)^2 + 2 \alpha^{-\gamma} \Theta(x, y)}.$$

Under the assumption $\beta^2 = \alpha^{\gamma + 1}$, this kernel becomes

$$B_\alpha(x, y) = \frac{\alpha}{\pi \alpha^2 + (x - y)^2 + \alpha^{3\gamma} \Theta(x, y)}.$$

Thus, denoting the corresponding integral operator by $B_\alpha$, we get

$$K_\beta = U_\alpha^* B_\alpha U_\alpha, \quad \alpha = \beta^{\frac{1}{\gamma + 1}}.$$

Henceforth the value of $\alpha$ is always chosen as in this formula.

Denote by $\mu_\alpha$ the maximal eigenvalue of the operator $B_\alpha$, and by $\psi_\alpha$ – the corresponding normalized eigenfunction. By the same token as for the operator $K_\beta$, the eigenvalue $\mu_\alpha$ is non-degenerate and the choice of the corresponding eigenfunction $\psi_\alpha$ is determined uniquely by the requirement that $\psi_\alpha > 0$ a.e.. Moreover,

$$\mu_\alpha = M_\beta, \quad \psi_\alpha(x) = (U_\alpha \Psi_\beta)(x) = \alpha^{-\frac{1}{2}} \Psi_\beta(\alpha^{-1}x), \quad \alpha = \beta^{\frac{1}{\gamma + 1}}.$$

This rescaling allows one to rewrite Theorem 1 in a somewhat more compact form:
Theorem 3. Let $\gamma \geq 1$ and suppose that the function $\Theta$ satisfies conditions (3), (4) and (8). Then
\[
\lim_{\alpha \to 0} \alpha^{-1}(1 - \mu_\alpha) = \lambda_1.
\]
Moreover, the eigenfunctions $\psi_\alpha$, converge in norm to $\phi_1$ as $\alpha \to 0$.

The rest of the paper is devoted to the proof of Theorem 3, which immediately implies Theorem 1.

2. “De-symmetrization” of $K_\beta$ and $B_\alpha$

First we de-symmetrize the operator $K_\beta$. Denote
\[
K_\beta^{(l)} u(x) = \int K_\beta^{(l)}(x, y) u(y) dy,
\]
with the kernel
\[
K_\beta^{(l)}(x, y) = \frac{1}{\pi} \frac{1}{1 + (x-y)^2 + \beta^2 \theta(x)}.
\]

Lemma 4. Let $\beta \leq 1$ and $\gamma \geq 1$. Suppose that the conditions (3), (4) and (8) are satisfied. Then
\[
\|K_\beta^{(l)} - K_\beta\| \leq C q \beta^2.
\]

Proof. Due to (3) and (4),
\[
c(|t| + 1)^\gamma \leq \Theta(t, \pm 1) \leq C(|t| + 1)^\gamma, \quad t \in \mathbb{R}.
\]
Also,
\[
\begin{cases}
|\Theta(t, 1) - \Theta(1, 1)| \leq C(|t| + 1)^{\gamma-1}|t - 1|, \\
|\Theta(t, -1) - \Theta(-1, -1)| \leq C(|t| + 1)^{\gamma-1}|t + 1|,
\end{cases}
\]
for all $t \in \mathbb{R}$. Indeed, (8) leads to the first inequality (14) for $|t - 1| < \epsilon$. For $|t - 1| \geq \epsilon$ it follows from (13) that
\[
|\Theta(t, 1) - \Theta(1, 1)| \leq C(|t| + 1)^\gamma \leq C' \epsilon^{-1}(|t| + 1)^{\gamma-1}|t - 1|.
\]
The second bound in (14) is checked similarly.

Now we can estimate the difference of the kernels
\[
K_\beta(x, y) - K_\beta^{(l)}(x, y)
\]
\[
= \frac{1}{\pi} \frac{\beta^2(\Theta(x, x) - \Theta(x, y))}{(1 + (x-y)^2 + \beta^2 \Theta(x, x))(1 + (x-y)^2 + \beta^2 \Theta(x, x))}.
\]
It follows from (14) with $t = y|x|^{-1}$ that
\[
|\Theta(x, x) - \Theta(y, x)| \leq C(|x| + |y|)^{\gamma-1}|x - y|.
\]
Substituting into (15), we get

\[ |K_\beta(x, y) - K_\beta^{(l)}(x, y)| \leq C \frac{|x - y|}{(1 + (x - y)^2)^{2-\delta}} \frac{\beta^2(|x| + |y|)^{\gamma-1}}{(1 + \beta^2(|x| + |y|)^{\gamma})^{\delta}}, \]

for any \( \delta \in (0, 1) \). The second factor on the right-hand side does not exceed

\[ \beta^2 \max_{t \geq 0} \frac{t^{\gamma-1}}{(1 + t^2)^{\delta}}, \]

which is bounded by \( C \beta^{2/\gamma} \) under the assumption that \( \delta \geq 1 - \gamma^{-1} \). Therefore

\[ |K_\beta(x, y) - K_\beta^{(l)}(x, y)| \leq C \beta^2 \frac{|x - y|}{(1 + (x - y)^2)^{2-\delta}}. \]

For any \( \delta \in (0, 1) \) the right hand side is integrable in \( x \) (or \( y \)). Now, estimating the norm using the standard Schur Test, see Proposition 15, we conclude that

\[ \|K_\beta - K_\beta^{(l)}\| \leq C \beta^2 \int \frac{|t|}{(1 + t^2)^{2-\delta}} dt \leq C' \beta^2, \]

which is the required bound. \( \square \)

Similarly to the operator \( K_\beta \), it is readily checked by scaling that the operator \( K_\beta^{(l)} \) is unitarily equivalent to the operator \( B_\alpha^{(l)} \) with the kernel

\[ B_\alpha^{(l)}(x, y) = \frac{\alpha}{\pi \alpha^2 + (x - y)^2 + \alpha^2 \theta(x)}. \]

Thus the bound (12) ensures that

\[ \|B_\alpha - B_\alpha^{(l)}\| = \|K_\beta - K_\beta^{(l)}\| \leq C \alpha^{1+\frac{1}{\gamma}}, \alpha \leq 1, \]

see (10) for the definition of \( \alpha \).

3. Approximation for \( B_\alpha^{(l)} \)

3.1. Symbol of \( B_\alpha^{(l)} \). Now our aim is to show that the operator \( I - \alpha A \) is an approximation of the operator \( B_\alpha^{(l)} \), defined above. To this end we need to represent \( B_\alpha^{(l)} \) as a pseudo-differential operator. Rewriting the kernel (16) as

\[ B_\alpha^{(l)}(x, y) = t^{-1} m_\alpha t(x - y), \ t = g_\alpha(x), \]

with

\[ g_\alpha(x) = \sqrt{1 + \alpha \theta(x)}, \]

and using (5), we can write for any Schwartz class function \( u \):

\[ (B_\alpha^{(l)}u)(x) = \frac{1}{2\pi} \int \int e^{i(x-y)\xi} b_\alpha^{(l)}(x, \xi) u(y) dy d\xi, \]
where
\[ b^{(l)}_\alpha(x, \xi) = \frac{1}{g_\alpha(x)} e^{-\alpha|\xi|g_\alpha(x)}. \]
Thus \( B^{(l)}_\alpha = \text{Op}(b^{(l)}_\alpha) \).

3.2. **Approximation for \( B^{(l)}_\alpha \).** Let the operator \( A \) and the symbol \( a(x, \xi) \) be as defined in (6) and (7). Our first objective is to check that the error
\[ r^{(l)}_\alpha(x, \xi) := b^{(l)}_\alpha(x, \xi) - (1 - \alpha a(x, \xi)) \]
is small in a certain sense. The condition \( \gamma \geq 1 \) will allow us to use standard norm estimates for pseudo-differential operators. Using the formula
\[ e^{-\alpha y} = 1 - \alpha y + \alpha \int_0^y (1 - e^{-\alpha t}) dt, \quad y > 0, \]
we can split the error as follows:
\[ r^{(l)}_\alpha(x, \xi) = r^{(1)}_\alpha(x) + r^{(2)}_\alpha(x, \xi), \]
\[ r^{(1)}_\alpha(x) = \frac{1}{g(x)} + \alpha 2^{-1} \theta(x) - 1, \]
\[ r^{(2)}_\alpha(x, \xi) = \frac{\alpha}{g(x)} \int_0^{||\xi||g(x)} (1 - e^{-\alpha t}) dt, \]
where we have used the notation \( g(x) = g_\alpha(x) \) with \( g_\alpha \) defined in (18). Since \( \gamma \geq 1 \), we have
\[ |g'(x)| \leq C g(x), \quad C = C(\gamma), \quad x \neq 0, \]
for all \( \alpha \leq 1 \). Introduce also the function \( \zeta \in C^\infty(\mathbb{R}_+) \) such that
\[ \zeta'(x) \geq 0, \quad \zeta(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 2, & x \geq 2. \end{cases} \]
Note that
\[ \zeta(x_1 x_2) \leq 2 \zeta(x_1) x_2, \quad x_1 \geq 0, x_2 \geq 1. \]
We study the above components \( r^{(1)}, r^{(2)} \) separately and introduce the function
\[ e^{(1)}_\alpha(x) = \frac{1}{\langle x \rangle^{\gamma} (\alpha \langle x \rangle)^{\gamma}} r^{(1)}_\alpha(x), \]
and the symbol
\[ e^{(2)}_\alpha(x, \xi) = g_\alpha(x)^{-\kappa} \left( \zeta(\langle \alpha \langle \xi \rangle \rangle^{\kappa}) \langle \xi \rangle \right)^{-1} r^{(2)}_\alpha(x, \xi), \]
where \( \kappa \in (0, 1] \) is a fixed number. To avoid cumbersome notation the dependence of \( e^{(2)}_\alpha \) on \( \kappa \) is not reflected in the notation. We denote the operators \( \text{Op}(r_\alpha) \) and \( \text{Op}(e_\alpha) \) by \( R_\alpha \) and \( E_\alpha \) respectively (with or without superscripts).
Lemma 5. Let $\gamma \geq 1$. Then for all $\alpha > 0$,
$$\|e^{(1)}_\alpha\|_{L_\infty} \leq C\alpha.$$ 

Proof. Estimate the function $r^{(1)}_\alpha$:
$$|r^{(1)}_\alpha(x)| \leq \begin{cases} \frac{C\alpha^2|x|^{2\gamma}}{\alpha\theta(x) \leq 1/2}, \\
C\alpha|x|^{\gamma}, \alpha\theta(x) > 1/2, \end{cases}$$
with a constant $C$ independent of $x$. The second estimate is immediate, and the first one follows from the Taylor’s formula
$$\frac{1}{\sqrt{1 + t}} = 1 - \frac{t}{2} + O(t^2), \ 0 \leq t \leq \frac{1}{2}.$$ 
Thus
$$|r^{(1)}_\alpha(x)| \leq C\alpha|x|^{\gamma}(\alpha|x|^{\gamma}).$$ 
This leads to the proclaimed estimate for $e^{(1)}_\alpha$. \qed

Lemma 6. Let $\gamma \geq 1$. Then for all $\alpha > 0$ and any $\kappa \in (0,1]$, 
$$\|E^{(2)}_\alpha\| \leq C_\kappa\alpha.$$ 

Proof. To estimate the norm of $\text{Op}(e^{(2)}_\alpha)$ we use Proposition 16. It is clear that the distributional derivatives $\partial_x, \partial_\xi, \partial_x\partial_\xi$ of the symbol $e^{(2)}_\alpha(x,\xi)$ exist and are given by
$$\partial_x r^{(2)}_\alpha(x,\xi) = -\frac{\alpha}{g^2} g' \int_0^{\xi|g|} (1 - e^{-\alpha t}) dt + \frac{\alpha}{g} |\xi| g'(1 - e^{-\alpha|\xi|g}),$$
$$\partial_\xi r^{(2)}_\alpha(x,\xi) = \alpha \text{ sign } \xi (1 - e^{-\alpha|\xi|g}),$$
$$\partial_x\partial_\xi r^{(2)}_\alpha(x,\xi) = \alpha^2 \xi e^{-\alpha|\xi|g},$$ 
for all $x \neq 0, \xi \neq 0$. For any $\kappa \in (0,1]$ the elementary bounds hold:
$$\int_0^{\xi|g|} (1 - e^{-\alpha t}) dt \leq |\xi| g \zeta((\alpha|\xi|g)^\kappa) \leq 2|\xi| g^{1+\kappa} \zeta((\alpha|\xi|g)^\kappa),$$
$$|1 - e^{-\alpha|\xi|g}| \leq \zeta((\alpha|\xi|g)^\kappa) \leq 2g^{\kappa} \zeta((\alpha|\xi|g)^\kappa),$$
$$\alpha|\xi| g e^{-\alpha|\xi|g} \leq \zeta((\alpha|\xi|g)^\kappa) \leq 2g^{\kappa} \zeta((\alpha|\xi|g)^\kappa).$$
Here we have used (20). Thus, in view of (19),
$$|r^{(2)}_\alpha(x,\xi)| + |\partial_\xi r^{(2)}_\alpha(x,\xi)| + |\partial_x r^{(2)}_\alpha(x,\xi)| \leq C\alpha \xi g \zeta((\alpha|\xi|g)^\kappa).$$ 
Also,
$$|\partial_x\partial_\xi r^{(2)}_\alpha(x,\xi)| \leq \alpha \frac{|g'|}{g} (\alpha|\xi| g e^{-\alpha|\xi|g}) \leq C\alpha |g|^{\kappa} \zeta((\alpha|\xi|g)^\kappa).$$
Now estimate the derivatives of the weights:

\[ |\partial_x g^{-\varkappa}| = \varkappa g^{-\varkappa - 1} g' \leq C g^{-\varkappa}, \quad x \neq 0, \]

\[ |\partial_\xi (\langle \xi \rangle \zeta ((\alpha \langle \xi \rangle)^\varkappa))^{-1}| \leq C \frac{1}{\langle \xi \rangle^2 \zeta ((\alpha \langle \xi \rangle)^\varkappa)}, \xi \in \mathbb{R}. \]

Thus the symbol \( e^{(2)}(x, \xi) \) as well as its derivatives \( \partial_x, \partial_\xi, \partial_x \partial_\xi \) are bounded by \( C\alpha \) for all \( \alpha > 0 \) uniformly in \( x, \xi \). Now the required estimate follows from Proposition 16. \( \square \)

We make a useful observation:

**Corollary 7.** Let \( \gamma \geq 1 \) and \( \varkappa \in (0, 1] \). Then for any function \( f \in D(A) \),

\[ \alpha^{-1} \| R^{(1)}_\alpha f \| \to 0, \quad \alpha \to 0, \quad (23) \]

\[ \alpha^{-1} \| E^{(2)}_\alpha (D_x) \zeta ((\alpha \langle D_x \rangle)^\varkappa) f \| \to 0, \quad \alpha \to 0. \quad (24) \]

**Proof.** Rewrite:

\[ \| R^{(1)}_\alpha f \| = \| E^{(1)}_\alpha \langle x \rangle^{\gamma \zeta (\alpha \langle x \rangle)^\gamma} f \| \leq \| E^{(1)}_\alpha \| \| \langle x \rangle^{\gamma \zeta (\alpha \langle x \rangle)^\gamma} f \|. \]

By Lemma 5 the norm of \( E^{(1)}_\alpha \) on the right-hand side is bounded by \( C\alpha \). The function \( \langle x \rangle^{\gamma \zeta (\alpha \langle x \rangle)^\gamma} f \) tends to zero as \( \alpha \to 0 \) a.a. \( x \in \mathbb{R} \), and it is uniformly bounded by the function \( \langle x \rangle^{\gamma} |f| \), which belongs to \( L^2 \), since \( f \in D(A) \). Thus the second factor in (25) tends to zero as \( \alpha \to 0 \) by the Dominated Convergence Theorem. This proves (23).

Proof of (24). Estimate:

\[ \| E^{(2)}_\alpha (D_x) \zeta ((\alpha \langle D_x \rangle)^\varkappa) f \| \leq \| E^{(2)}_\alpha \| \| \langle \xi \rangle^{\zeta (\alpha \langle \xi \rangle)^\gamma} \hat{f} \|. \]

By Lemma 6 the norm of the first factor on the right-hand side is bounded by \( C\alpha \). The second factor tends to zero as \( \alpha \to 0 \) for the same reason as in the proof of (23). \( \square \)

**4. Norm-convergence of the extremal eigenfunction**

Recall that the maximal positive eigenvalue \( \mu_\alpha \) of the operator \( B_\alpha \) is non-degenerate, and the corresponding (normalized) eigenfunction \( \psi_\alpha \) is positive a.a. \( x \in \mathbb{R} \).

The principal goal of this section is to prove that any infinite subset of the family \( \psi_\alpha, \alpha \leq 1 \) contains a norm-convergent sequence. We begin with an upper bound for \( 1 - \mu_\alpha \) which will be crucial for our argument.

**Lemma 8.** If \( \gamma \geq 1 \), then

\[ \limsup_{\alpha \to 0} \alpha^{-1}(1 - \mu_\alpha) \leq \lambda_1. \quad (26) \]
Proof. Denote $\phi := \phi_1$. By a straightforward variational argument it follows that

$$\mu_\alpha \geq (B_\alpha \phi, \phi) \geq |(B^{(l)}_\alpha \phi, \phi)| - \|B_\alpha - B^{(l)}_\alpha\|$$

$$\geq ((I - \alpha A)\phi, \phi) - |(R_\alpha \phi, \phi)| + o(\alpha)$$

$$= 1 - \alpha \lambda_1 - |(R_\alpha \phi, \phi)| + o(\alpha),$$

where we have also used (17). By definitions (21) and (22),

$$|(R_\alpha \phi, \phi)| \leq \|R^{(1)}_\alpha \phi\| + \|E^{(2)}_\alpha (D_x)\zeta((\alpha(D_x))^\varkappa)\phi\| \|g^{\varkappa}_\alpha \phi\|,$$

where $\varkappa \in (0, 1]$. It is clear that $g^{\varkappa}_\alpha \phi \in L^2$ and its norm is bounded uniformly in $\alpha \leq 1$. The remaining terms on the right-hand side are of order $o(\alpha)$ due to Corollary 7. This leads to (26). \(^\square\)

The established upper bound leads to the following property.

**Lemma 9.** For any $\varkappa \in (0, 1)$,

$$\|g^{\varkappa}_\alpha \psi_\alpha\| \leq C$$

uniformly in $\alpha \leq 1$.

**Proof.** By definition of $\psi_\alpha$,

$$g^{\varkappa}_\alpha \psi_\alpha = \mu_\alpha^{-1} g^{\varkappa}_\alpha B_\alpha \psi_\alpha.$$

In view of (4), by definition (18) we have $\Theta(x, y) \geq C|x|^\gamma \geq c\theta(x)$, so that the kernel $B_\alpha(x, y)$ is bounded from above by

$$B_\alpha(x, y) \leq \frac{\alpha}{\pi} \frac{C}{(x - y)^2 + \alpha^2 g_\alpha(x^2)},$$

and thus the kernel $\tilde{B}_\alpha(x, y) = g_\alpha(x)^\varkappa B_\alpha(x, y)$ satisfies the estimate

$$\tilde{B}_\alpha(x, y) \leq \frac{C}{\pi \alpha} \frac{1}{(1 + \alpha^{-2}(x - y)^2)^{1-\frac{\varkappa}{2}}}.$$

Since $\varkappa < 1$, by Proposition 15 this kernel defines a bounded operator with the norm uniformly bounded in $\alpha > 0$. Thus

$$\|g^{\varkappa}_\alpha \psi_\alpha\| \leq C \mu_\alpha^{-1} \|\psi_\alpha\| \leq C \mu_\alpha^{-1}.$$

It remains to observe that by Lemma 8 the eigenvalue $\mu_\alpha$ is separated from zero uniformly in $\alpha \leq 1$. \(^\square\)

Now we obtain more delicate estimates for $\psi_\alpha$. For a number $h \geq 0$ introduce the function

$$S_\alpha(t; h) = \frac{\alpha}{\pi} \frac{1}{\alpha^2 + t^2 + h^2}, t \in \mathbb{R},$$

and denote by $S_\alpha(h)$ the integral operator with the kernel $S_\alpha(x - y; h)$. Along with $S_\alpha(h)$ we also consider the operator

$$T_\alpha(h) = S_\alpha(0) - S_\alpha(h).$$
Due to (5) the Fourier transform of $S_\alpha(t; h)$ is
\begin{equation}
\hat{S}_\alpha(\xi; h) = \frac{\alpha}{\sqrt{2\pi\sqrt{\alpha^2 + h}}} e^{-|\xi|\sqrt{\alpha^2 + h}}, \xi \in \mathbb{R},
\end{equation}
so that
\begin{equation}
\|S_\alpha(h)\| = \frac{\alpha}{\sqrt{\alpha^2 + h}}, \|T_\alpha(h)\| = 1 - \frac{\alpha}{\sqrt{\alpha^2 + h}}.
\end{equation}
Denote by $\chi_R$ the characteristic function of the interval $(-R, R)$.

**Lemma 10.** For sufficiently small $\alpha > 0$ and $\alpha R \leq 1$,
\begin{equation}
\|\hat{\psi}_\alpha \chi_R\|^2 \geq 1 - \frac{4\lambda_1}{R}.
\end{equation}

*Proof.* Since $B_\alpha(x, y) < S_\alpha(x - y; 0)$ (see (9) and (27)) and $\psi_\alpha \geq 0$, we can write, using (28):
\begin{align*}
\mu_\alpha &= (B_\alpha \psi_\alpha, \psi_\alpha) < \int_{\mathbb{R}} \int_{\mathbb{R}} S_\alpha(x - y; 0) \psi_\alpha(x) \psi_\alpha(y) dxdy = \int_{\mathbb{R}} e^{-\alpha|\xi|} |\hat{\psi}_\alpha(\xi)|^2 d\xi \\
&\leq \int_{|\xi| \leq R} |\hat{\psi}_\alpha(\xi)|^2 d\xi + e^{-\alpha R} \int_{|\xi| > R} |\hat{\psi}_\alpha(\xi)|^2 d\xi \\
&= (1 - e^{-\alpha R}) \int_{|\xi| \leq R} |\hat{\psi}_\alpha(\xi)|^2 d\xi + e^{-\alpha R}.
\end{align*}
Due to (26), $\mu_\alpha \geq 1 - 2\alpha \lambda_1$ for sufficiently small $\alpha$, so
\[1 - e^{-\alpha R} - 2\alpha \lambda_1 \leq (1 - e^{-\alpha R}) \|\hat{\psi}_\alpha \chi_R\|^2,
\] which implies that
\[\|\hat{\psi}_\alpha \chi_R\|^2 \geq 1 - \frac{2\alpha \lambda_1}{1 - e^{-\alpha R}}.
\] Since $e^{-s} \leq (1 + s)^{-1}$ for all $s \geq 0$, we get $(1 - e^{-s})^{-1} \leq 2s^{-1}$ for $0 < s \leq 1$, which entails (30) for $\alpha R \leq 1$. \qed

**Lemma 11.** For sufficiently small $\alpha > 0$ and any $R > 0$,
\begin{equation}
\|\psi_\alpha \chi_R\| \geq 1 - 4\alpha \lambda_1 - \frac{C}{R^7},
\end{equation}
with some constant $C > 0$ independent of $\alpha$ and $R$.

*Proof.* It follows from (4) that $\Theta(x, y) \geq c|x|^\gamma$, so that the kernel $B_\alpha(x, y)$ satisfies the bound
\[B_\alpha(x, y) \leq S_\alpha(x - y; c\alpha^3 R^7), \text{ for } |x| \geq R > 0.
\]
Since $\psi_\alpha \geq 0$,
\[
\mu_\alpha = (B_\alpha \psi_\alpha, \psi_\alpha) \leq (S_\alpha(0) \psi_\alpha, \psi_\alpha) + (S_\alpha(\alpha^3 R^\gamma) \psi_\alpha, \psi_\alpha(1 - \chi_R))
\]
\[
= (T_\alpha(\alpha^3 R^\gamma) \psi_\alpha, \psi_\alpha) + (S_\alpha(\alpha^3 R^\gamma) \psi_\alpha, \psi_\alpha).
\]
In view of (29),
\[
\mu_\alpha \leq \|T_\alpha(\alpha^3 R^\gamma)\| \|\psi_\alpha \chi_R\| + \|S_\alpha(\alpha^3 R^\gamma)\|.
\]
Using, as in the proof of the previous lemma, the bound (26), we obtain that
\[
1 - \frac{1}{\sqrt{1 + \alpha R^\gamma}} - 2\alpha \lambda_1 \leq \left(1 - \frac{1}{\sqrt{1 + \alpha R^\gamma}}\right) \|\psi_\alpha \chi_R\|
\]
so
\[
1 - \frac{4\lambda_1 (1 + \alpha R^\gamma)}{c R^\gamma} \leq \|\psi_\alpha \chi_R\|.
\]
This entails (31). $\Box$

Now we show that any sequence from the family $\psi_\alpha$ contains a norm-convergent subsequence. The proof is inspired by [15], Lemma 7. We precede it with the following elementary result.

**Lemma 12.** Let $f_j \in L^2(\mathbb{R})$ be a sequence such that $\|f_j\| \leq C$ uniformly in $j = 1, 2, \ldots$, and $f_j(x) = 0$ for all $|x| \geq \rho > 0$ and all $j = 1, 2, \ldots$. Suppose that $f_j$ converges weakly to $f \in L^2(\mathbb{R})$ as $j \to \infty$, and that for some constant $A > 0$, and all $R \geq R_0 > 0$,
\[
\|\hat{f}_\chi_R\| \geq A - CR^{-\kappa}, \quad \kappa > 0,
\]
uniformly in $j$. Then $\|f\| \geq A$.

**Proof.** Since $f_j$ are uniformly compactly supported, the Fourier transforms $\hat{f}_j(\xi)$ converge to $\hat{f}(\xi)$ a.a. $\xi \in \mathbb{R}^d$ as $j \to \infty$. Moreover, the sequence $\hat{f}_j(\xi)$ is uniformly bounded, so $\hat{f}_j \chi_R \to \hat{f} \chi_R$, $j \to \infty$ in $L^2(\mathbb{R})$ for any $R > 0$. Therefore (32) implies that
\[
\|\hat{f} \chi_R\| \geq A - CR^{-\kappa}.
\]
Since $R$ is arbitrary, we have $\|f\| = \|\hat{f}\| \geq A$, as claimed. $\Box$

**Lemma 13.** For any sequence $\alpha_n \to 0, n \to \infty$, there exists a subsequence $\alpha_{n_k} \to 0, k \to \infty$, such that the eigenfunctions $\psi_{\alpha_{n_k}}$ converge in norm as $k \to \infty$.

**Proof.** Since the functions $\psi_\alpha, \alpha \geq 0$ are normalized, there is a subsequence $\psi_{\alpha_{n_k}}$ which converges weakly. Denote the limit by $\psi$. From now on we write $\psi_k$ instead of $\psi_{\alpha_{n_k}}$ to avoid cumbersome notation. In view of the relations
\[
\|\psi_k - \psi\|^2 = 1 + \|\psi\|^2 - 2 \text{Re}(\psi_k, \psi) \to 1 - \|\psi\|^2, k \to \infty,
\]
it suffices to show that \( \| \psi \| = 1 \).

Fix a number \( \rho > 0 \), and split \( \psi_k \) in the following way:

\[
\psi_k(x) = \psi_{k,\rho}^{(1)}(x) + \psi_{k,\rho}^{(2)}(x), \quad \psi_{k,\rho}^{(1)}(x) = \psi_k(x)\chi_{\rho}(x).
\]

Clearly, \( \psi_{k,\rho}^{(1)} \) converges weakly to \( \psi_\rho = \psi\chi_\rho \) as \( k \to \infty \). Assume that \( \alpha_{n_k} \leq \rho^{-\gamma} \), so that by (31),

\[
\| \psi_{k,\rho}^{(1)} \|_2^2 \geq 1 - C\rho^{-\gamma}, \quad \| \psi_{k,\rho}^{(2)} \|_2^2 \leq C\rho^{-\gamma}.
\]

Therefore, for any \( R > 0 \),

\[
\| \hat{\psi}_{k,\rho}^{(1)} \chi_R \| \geq \| \hat{\psi}_k \chi_R \| - \| \psi_{k,\rho}^{(2)} \| \geq 1 - 4\lambda_1 R^{-1} - C\rho^{-\frac{2}{\gamma}},
\]

where we have used (30). By Lemma 12,

\[
\| \psi_\rho \| \geq 1 - C\rho^{-\frac{2}{\gamma}}.
\]

Since \( \rho \) is arbitrary, \( \| \psi \| \geq 1 \), and hence \( \| \psi \| = 1 \). As a consequence, the sequence \( \psi_k \) converges in norm, as claimed.

\[\square\]

5. Asymptotics of \( \mu_\alpha, \alpha \to 0 \): proof of Theorem 1

As before, by \( \lambda_l, l = 1, 2, \ldots \) we denote the eigenvalues of \( A \) arranged in ascending order, and by \( \phi_l \) a set of corresponding normalized eigenfunctions. Recall that the lowest eigenvalue \( \lambda_1 \) of the model operator \( A \) is non-degenerate and its (normalized) eigenfunction \( \phi_1 \) is chosen to be positive a.e. \( x \in \mathbb{R} \). We begin with proving Theorem 3.

**Proof of Theorem 3.** The proof essentially follows the plan of [15]. It suffices to show that for any sequence \( \alpha_n \to 0, n \to \infty \), one can find a subsequence \( \alpha_{n_k} \to 0, k \to \infty \) such that

\[
\lim_{k \to \infty} \alpha_{n_k}^{-1}(1 - \mu_{\alpha_{n_k}}) = \lambda_1,
\]

and \( \psi_{\alpha_{n_k}} \) converges in norm to \( \phi_1 \) as \( k \to \infty \). By Lemma 13 one can pick a subsequence \( \alpha_{n_k} \) such that \( \psi_{\alpha_{n_k}} \) converges in norm as \( k \to \infty \). As in the proof of Lemma 13 denote by \( \psi \) the limit, so \( \| \psi \| = 1 \) and \( \psi \geq 0 \) a.e.. For simplicity we write \( \psi_\alpha \) instead of \( \psi_{\alpha_{n_k}} \).

For an arbitrary function \( f \in D(A) \) write

\[
\mu_\alpha(\psi_\alpha, f) = (B_\alpha \psi_\alpha, f) = (\psi_\alpha, B_\alpha^{(l)} f) + (\psi_\alpha, (B_\alpha - B_\alpha^{(l)}) f)
\]

\[
= (\psi_\alpha, f) - \alpha(\psi_\alpha, A f) + (\psi_\alpha, R_\alpha f) + (\psi_\alpha, (B_\alpha - B_\alpha^{(l)}) f).
\]

This implies that

\[
\alpha^{-1}(1 - \mu_\alpha)(\psi_\alpha, f) = (\psi_\alpha, A f) - \alpha^{-1}(\psi_\alpha, R_\alpha f) - \alpha^{-1}(\psi_\alpha, (B_\alpha - B_\alpha^{(l)}) f).
\]

(33)
In view of (17) the last term on the right-hand side tends to zero as \( \alpha \to 0 \). The first term trivially tends to \((\psi, Af)\). Consider the second term:

\[
|\langle \psi_\alpha, R_\alpha f \rangle| = \langle \psi_\alpha, R_\alpha^{(1)} f \rangle + \langle g_\alpha^\kappa \psi_\alpha, E_\alpha^{(2)} \langle D_x \rangle \zeta(\langle \alpha \langle D_x \rangle \rangle^\kappa) f \rangle \leq \|R_\alpha^{(1)} f\| + \|g_\alpha^\kappa \psi_\alpha\| \|E_\alpha^{(2)} \langle D_x \rangle \zeta(\langle \alpha \langle D_x \rangle \rangle^\kappa) f\|.
\]

Assume now that \( \kappa < 1 \). By Corollary 7 and Lemma 9, the right-hand side is of order \( o(\alpha) \), and hence, if \((\psi, f) \neq 0\), then passing to the limit in (33) we get

\[
\lim_{\alpha \to 0} \alpha^{-1}(1 - \mu_\alpha) = \frac{\langle \psi, Af \rangle}{\langle \psi, f \rangle}.
\]

Let \( f = \phi_l \) with some \( l \), so that \((\psi, Af) = \lambda_l(\psi, \phi_l)\). Suppose that \((\psi, \phi_l) \neq 0\), so that

\[
\lim_{\alpha \to 0} \alpha^{-1}(1 - \mu_\alpha) = \lambda_l.
\]

By the uniqueness of the above limit, \((\psi, \phi_j) = 0\) for all \( j \)'s such that \( \lambda_j \neq \lambda_k \). Thus, by completeness of the system \( \{\phi_k\} \), the function \( \psi \) is an eigenfunction of \( A \) with the eigenvalue \( \lambda_l \). In view of (26), \( \lambda_l \leq \lambda_1 \). Since the eigenvalues \( \lambda_j \) are labeled in ascending order we conclude that \( \lambda_l = \lambda_1 \). As this eigenvalue is non-degenerate and the corresponding eigenfunction \( \phi_1 \) is positive a.e., we observe that \( \psi = \phi_1 \). \( \square \)

**Proof of Theorem 1.** Theorem 1 follows from Theorem 3 due to the relations (11). \( \square \)

### 6. Miscellaneous

In this short section we collect some open questions related to the spectrum of the operator (1).

#### 6.1. Theorems 1 and 3 give information on the largest eigenvalue \( M_\beta \) of the operator \( K_\beta \) defined in (1), (2). Let

\[
M_\beta \equiv M_{1,\beta} \geq M_{2,\beta} \geq \ldots
\]

be the sequence of all positive eigenvalues of \( K_\beta \) arranged in descending order. The following conjecture is a natural extension of Theorem 1.

**Conjecture 14.** For any \( j = 1, 2, \ldots \)

\[
\lim_{\beta \to 0} \beta^{-\frac{2}{\gamma+1}}(1 - M_{j,\beta}) = \lambda_j,
\]

where \( \lambda_1 < \lambda_2 \leq \ldots \) are eigenvalues of the operator \( A \) defined in (6), arranged in ascending order.

For the case \( \Theta(x, y) = (x^2 + y^2)^2 \) the formula (35) was conjectured in [9], Section 7.1, but without specifying what the values \( \lambda_j \) are. As in [9], the formula (35) is prompted by the paper [15] where asymptotics of the form (35) were found for an integral operator with a difference kernel.
6.2. Although the operator $K_\beta$ converges strongly to the positive-definite operator $K_0$ as $\beta \to 0$, we can’t say whether or not $K_\beta$, $\beta > 0$, has negative eigenvalues.

6.3. Suppose that the function $\Theta(x, y)$ in (2) is even, i.e. $\Theta(-x, -y) = \Theta(x, y)$, $x, y \in \mathbb{R}$. Then the subspaces $H^e$ and $H^o$ in $L^2(\mathbb{R})$ of even and odd functions are invariant for $K = K_\beta$. Consider restriction operators $K^e = K \upharpoonright H^e$ and $K^o = K \upharpoonright H^o$ and their positive eigenvalues $\lambda^e_j$ and $\lambda^o_j$, $j = 1, 2, \ldots$, arranged in descending order. Remembering that the top eigenvalue of $K$ is non-degenerate and its eigenfunction is positive a.e., one easily concludes that $\lambda^e_1 > \lambda^o_1$. Are there similar inequalities for the pairs $\lambda^e_j, \lambda^o_j$ with $j > 1$?

7. Appendix. Boundedness of integral and pseudo-differential operators

In this Appendix, for the reader’s convenience we remind (without proofs) simple tests of boundedness for integral and pseudo-differential operators acting on $L^2(\mathbb{R}^d)$, $d \geq 1$. Consider the integral operator

$$(36) \quad (Ku)(x) = \int_{\mathbb{R}^d} K(x, y)u(y)dy,$$

with the kernel $K(x, y)$, and the pseudo-differential operator

$$(37) \quad (Op(a)u)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot \xi}a(x, \xi)u(y)dyd\xi,$$

with the symbol $a(x, \xi)$.

The following classical result is known as the Schur Test and it can be found, even in a more general form, in [4], Theorem 5.2.

**Proposition 15.** Suppose that the kernel $K$ satisfies the conditions

$$M_1 = \sup_x \int_{\mathbb{R}^d} |K(x, y)|dy < \infty, \quad M_2 = \sup_y \int_{\mathbb{R}^d} |K(x, y)|dx < \infty.$$

Then the operator (36) is bounded on $L^2(\mathbb{R}^d)$ and $\|K\| \leq \sqrt{M_1 M_2}$.

For pseudo-differential operators on $L^2(\mathbb{R}^d)$ we use the test of boundedness found by H.O.Cordes in [2], Theorem $B'_1$.

**Proposition 16.** Let $a(x, \xi), x, \xi \in \mathbb{R}^d, d \geq 1$, be a function such that its distributional derivatives of the form $\nabla^n_x \nabla^m_\xi a$ are $L^\infty$-functions for all $0 \leq n, m \leq r$, where

$$r = \left\lceil \frac{d}{2} \right\rceil + 1.$$

Then the operator (37) is bounded on $L^2(\mathbb{R}^d)$ and

$$\|Op(a)\| \leq C \max_{0 \leq n, m \leq r} \|\nabla^n_x \nabla^m_\xi a\|_{L^\infty},$$

with a constant $C$ depending only on $d$. 

It is important for us that for \( d = 1 \) the above test requires the boundedness of derivatives \( \partial_x^n \partial_\xi^m a \) with \( n, m \in \{0, 1\} \) only. This result is extended to arbitrary dimensions by M. Ruzhansky and M. Sugimoto, see [13] Corollary 2.4. Recall that the classical Calderón-Vaillancourt theorem needs more derivatives with respect to each variable, see [2] and [13] for discussion. A short prove of Proposition 16 was given by I.L. Hwang in [5], Theorem 2 (see also [8], Lemma 2.3.2 for a somewhat simplified version).

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