Multipolar Expansions for Closed and Open Systems of Relativistic Particles.

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Abstract

Dixon’s multipoles for a system of N relativistic positive-energy scalar particles are evaluated in the rest-frame instant form of dynamics. The Wigner hyper-planes (intrinsic rest frame of the isolated system) turn out to be the natural framework for describing multipole kinematics. Classical concepts like the barycentric tensor of inertia turn out to be extensible to special relativity only by means of the quadrupole moments of the isolated system. Two new applications of the multipole technique are worked out for systems of interacting particles and fields. In the rest-frame of the isolated system of either free or interacting positive energy particles it is possible to define a unique world-line which embodies the properties of the most relevant centroids introduced in the literature as candidates for the collective motion of the system. This is no longer true, however, in the case of open subsystems of the isolated system. While effective mass, 3-momentum and angular momentum in the rest frame can be calculated from the definition of the subsystem energy-momentum tensor, the definitions of effective center of motion and effective intrinsic spin of the subsystem are not unique. Actually, each of the previously considered centroids corresponds to a different world-line in the case of open systems. The pole-dipole description of open subsystems is compared to their description as effective extended objects. Hopefully, the technique developed here could be instrumental for the relativistic treatment of binary star systems in metric gravity.

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I. INTRODUCTION.

An important area of research is nowadays the construction of templates for gravitational waves emitted by binary systems. Analytically, this can be done within the framework of PN approximations by means of essentially non-relativistic multipole expansions for compact bodies. On the other hand, since the main emission are supposed to take place in a region where the PN approximation fails, it would be desirable to have at disposal a relativistic treatment of multipolar expansions as a preliminary kinematical tool for dealing with open gravitating systems. This paper focuses just on the construction of a suitable relativistic kinematical background by an N-body system as a tool. A preliminary extension of the results of this paper to relativistic fluids is given in [1].

Our starting point will be the result recently obtained[2] concerning a complete treatment of the kinematics of the relativistic N-body problem in the rest-frame instant form of dynamics [3, 4, 5, 6]. This program required the re-formulation of the theory of isolated relativistic systems on arbitrary space-like hyper-surfaces [7], and has been based essentially upon Dirac’s reformulation[8] of classical field theory (suitably extended to particles) on arbitrary space-like hyper-surfaces (equal time and Cauchy surfaces). For each isolated system (containing any combination of particles, strings and fields) one obtains in this way a re-formulation of the standard theory as a parametrized Minkowski theory[3]. This program shows the extra bonus of being naturally prepared for the coupling to gravity in its ADM formulation. The price to be paid is that the functions $z^\mu(\tau, \vec{\sigma})$ describing the embedding of the space-like hyper-surface in Minkowski space-time become configuration variables in the action principle. Since the action is invariant under separate $\tau$-reparametrizations and space-diffeomorphisms, first class constraints emerge ensuring the independence of the description from the choice of the 3+1 splitting. The embedding configuration variables $z^\mu(\tau, \vec{\sigma})$ are thus the gauge variables associated with this particular kind of general covariance.

We summarize here the main results that are necessary to understand all the subsequent technical developments. First of all, recall that since the intersection of a time-like world-line with a space-like hyper-surface corresponding to a value $\tau$ of the time parameter is identified by 3 numbers $\vec{\sigma} = \vec{\eta}(\tau)$ instead of four, in parametrized Minkowski theories each particle must have a well defined sign of the energy: therefore the two topologically disjoint branches of the mass hyperboloid cannot be described simultaneously as in the standard manifestly Lorentz-covariant theory, and there are no more mass-shell constraints. Each particle with a definite sign of the energy is described by the canonical coordinates $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ while the 4-position of the particles are given by $x^\mu_i(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau))$. The following 4-momenta $p^\mu_i(\tau)$ are $\vec{\kappa}_i$-dependent solutions of $p^2_i - m^2_i = 0$ with the chosen sign of the energy.

In order to exploit the separate spatial and time reparametrization invariances of parametrized Minkowski theories, we can first of all restrict the foliation to space-like hyper-planes as leaves. For each configuration of the isolated system with time-like 4-momentum, we further restrict to the special leaves defined by hyper-planes orthogonal to the conserved system 4-momentum (Wigner hyper-planes). This foliation is fully determined by the configuration of the isolated system. One gets in this way[3] the definition of the Wigner-covariant rest-frame instant form of dynamics for any isolated system whose configurations have well defined and finite Poincaré generators with time-like total 4-momentum (see Ref.[9] for the traditional forms of dynamics). Finally, this formulation casts some light on the
long standing problem of defining a relativistic center of mass. As well known, no definition of this concept can enjoy all the properties of its non-relativistic counterpart. See Refs. [10, 11, 13, 14, 15] for a partial bibliography of all the existing attempts.

As shown in Appendix A of Ref. [2] only four first class constraints survive in the rest-frame instant form on Wigner hyper-planes. The original configuration variables \( z^\mu(\tau, \vec{\sigma}) \), \( \vec{\eta}(\tau) \) and their conjugate momenta \( \rho_\mu(\tau, \vec{\sigma}) \), \( \vec{\kappa}(\tau) \) are reduced to:

i) a **decoupled particle** \( \tilde{x}_s^\mu(\tau), p_s^\mu \) (the only remnant of the space-like hyper-surface) with a positive mass \( \epsilon_s = \sqrt{p_s^2} \) determined by the first class constraint \( \epsilon_s - M_{sys} \approx 0 \), \( M_{sys} \) being the invariant mass of the isolated system. As a gauge fixing to the constraint, the rest-frame Lorentz scalar time \( T_s = \tilde{x}_s^\mu \epsilon_s^{-1} \) is put equal to the mathematical time: \( T_s - \tau \approx 0 \). Here, \( \tilde{x}_s^\mu(\tau) \) is a **non-covariant canonical** variable for the \( 4 \)-center of mass. After the elimination of \( T_s \) and \( \epsilon_s \) by means of such pair of second class constraints, we are left with a decoupled free point (point particle clock) of mass \( M_{sys} \) and canonical 3-coordinates \( \tilde{z}_s = \epsilon_s(\tilde{x}_s - \frac{p_s}{\mu_s^2} \tilde{x}^0) \), \( \tilde{k}_s = \frac{\tilde{p}_s}{\epsilon_s} \) [16]. The non-covariant canonical \( \tilde{x}_s^\mu(\tau) \) must not be confused with the 4-vector \( x_s^\mu(\tau) = z^\mu(\tau, \vec{\sigma} = 0) \) identifying the origin of the 3-coordinates \( \vec{\sigma} \) inside the Wigner hyper-planes. The world-line \( x_s^\mu(\tau) \) is arbitrary because it depends on \( x_s^\mu(0) \) while its 4-velocity \( \tilde{x}_s^\mu(\tau) \) depends on the Dirac multipliers associated with the remaining 4 (or 3 after having imposed \( T_s - \tau \approx 0 \)) first class constraints (see Section II). Correspondingly, this worldline may be considered as an arbitrary **centroid** for the isolated system. The unit time-like 4-vector \( u^\mu(p_s) = p_s^\mu/\epsilon_s \) (= \( \tilde{x}_s^\mu(\tau) \) after having imposed \( T_s - \tau \approx 0 \)) is orthogonal to the Wigner hyper-planes and describes their orientation in the chosen inertial frame.

ii) the **particle canonical variables** \( \vec{\eta}(\tau), \vec{\kappa}(\tau) \) inside the Wigner hyper-planes. They are Wigner spin-1 3-vectors, like the coordinates \( \vec{\sigma} \). They are restricted by the three first class constraints (the rest-frame conditions) \( \vec{k}_+ = \sum_{i=1}^{N} \vec{\kappa}_i \approx 0 \). Since the role of the relativistic decoupled 4-center of mass is taken by \( \tilde{x}_s^\mu(\tau) \) (or, after the gauge fixing \( T_s - \tau \approx 0 \), by an external 3-center of mass \( \tilde{z}_s \), defined in terms of \( \tilde{x}_s^\mu \) and \( p_s^\mu \) [2]), the rest-frame conditions imply that the internal canonical 3-center of mass \( \vec{q}_+ = \tilde{\vec{\sigma}}_{com} \) is a gauge variable that can be eliminated through a gauge fixing [17]. This amounts in turn to a definite choice of the world-line \( x_s^\mu(\tau) \) of the centroid.

All this leads to a **doubling of viewpoints and concepts**:

1) The **external** viewpoint, taken by an arbitrary inertial Lorentz observer, who describes the Wigner hyper-planes determined by the time-like configurations of the isolated system. A change of inertial observer by means of a Lorentz transformation rotates the Wigner hyper-planes and induces a Wigner rotation of the 3-vectors inside each Wigner hyperplane. Every such hyperplane inherits an induced internal Euclidean structure while an external realization of the Poincaré group induces an internal Euclidean action.

Then, three **external** concepts of 4-center of mass can be defined by using the external realization of the Poincaré algebra (each one corresponding to a different 3-location inside the Wigner hyper-planes):

a) the **external** non-covariant canonical **4-center of mass** (also named 4-center of spin) \( \tilde{x}_s^\mu \) (with 3-location \( \tilde{\vec{\sigma}} \)),

b) the **external** non-covariant non-canonical Møller 4-center of energy \( R_s^\mu \) (with 3-location \( \vec{\sigma}_R \)),

\[ \tilde{x}_s^\mu(\tau) = \epsilon_s(\tilde{x}_s - \frac{p_s}{\mu_s^2} \tilde{x}^0) \]

\[ \vec{k}_s = \frac{\vec{p}_s}{\epsilon_s} \]
c) the external covariant non-canonical Fokker-Pryce 4-center of inertia $Y_s^\mu$ (with 3-location $\vec{s}_\gamma$).

Only the canonical non-covariant center of mass $\vec{x}_s^\mu(\tau)$ is relevant to the Hamiltonian treatment with Dirac constraints, while only the Fokker-Pryce $Y_s^\mu$ is a 4-vector by construction. See Ref.[2] for the construction of the 4-centers starting from the corresponding 3-centers (3-center of spin[14], 3-center of energy [12], 3-center of inertia[13, 14]), which are group-theoretically defined in terms of generators of the external Poincaré group.

2) The internal viewpoint, taken by an observer inside the Wigner hyper-planes. This viewpoint is associated to a unfaithful internal realization of the Poincaré algebra: the total internal 3-momentum of the isolated system vanishes due to the rest-frame conditions. The internal energy and angular momentum are the invariant mass $M_{sys}$ and the spin (the angular momentum with respect to $\vec{x}_s^\mu(\tau)$) of the isolated system, respectively. By means of the internal realization of the Poincaré algebra we can define three internal 3-centers of mass: the internal canonical 3-center of mass (or 3-center of spin) $\vec{q}_+$, the internal Møller 3-center of energy $\vec{R}_+$ and the internal Fokker-Pryce 3-center of inertia $\vec{y}_+$. However, due to the rest-frame condition $\vec{κ}_+ \approx 0$, they all coincide: $\vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+$. As a natural gauge fixing to the rest-frame conditions, we can add the vanishing of the internal Lorentz boosts $\vec{K}$ (recall that they are equal to $-\vec{R}_+/M_{sys}$). This is equivalent to locate the internal canonical 3-center of mass $\vec{q}_+$ in $\vec{σ} = 0$, i.e. in the origin $x_s^\mu(\tau) = z^\mu(\tau, 0)$. Upon such gauge fixings and with $T_s - \tau \approx 0$, the world-line $x_s^\mu(\tau)$ becomes uniquely determined except for the arbitrariness in the choice of $x_s^\mu(0) \ [u^\mu(p_s) = p_s^\mu/\epsilon_s]$

$$x_s^\mu(\tau) = x_s^\mu(0) + u^\mu(p_s)T_s,$$  \hspace{1cm} (1.1)

and, modulo $x_s^\mu(0)$, it coincides with the external covariant non-canonical Fokker-Pryce 4-center of inertia, $x_s^\mu(\tau) = x_s^\mu(0) + Y_s^\mu[2]$.

The doubling of concepts, the external non-covariant canonical 4-center of mass $\vec{x}_s^\mu(\tau)$ (or the external 3-center of mass $\vec{z}_s$ when $T_s - \tau \approx 0$) and the internal canonical 3-center of mass $\vec{q}_+ \approx 0$ replaces the separation of the non-relativistic 3-center of mass due to the Abelian translation symmetry. Correspondingly, the non-relativistic conserved 3-momentum is replaced by the external $\vec{p}_s = \epsilon_s \vec{K}_s$, while, as already said, the internal 3-momentum vanishes, $\vec{κ}_+ \approx 0$, as a definition of rest frame.

In the gauge where $\epsilon_s \equiv M_{sys}$, $T_s \equiv \tau$, the canonical basis $\vec{z}_s$, $\vec{K}_s$, $\vec{y}_s$, $\vec{κ}_i$ is restricted by the three pairs of second class constraints $\vec{κ}_+ = \sum_{i=1}^N \vec{κ}_i \approx 0$, $\vec{q}_+ \approx 0$, so that 6N canonical variables describe the N particles like in the non-relativistic case. We still need a canonical transformation $\vec{η}_i, \vec{κ}_i \rightarrow \vec{q}_+[\approx 0], \vec{κ}_+[\approx 0], \vec{p}_a, \vec{π}_a [a = 1, ..., N-1]$ identifying a set of relative canonical variables. The final 6N-dimensional canonical basis is $\vec{z}_s$, $\vec{K}_s$, $\vec{p}_a$, $\vec{π}_a$. To get this result we need a highly non-linear canonical transformation[2], which can be obtained by exploiting the Gartenhaus-Schwartz singular transformation[18].

At the end we obtain the Hamiltonian for the relative motions as a sum of N square roots, each one containing a squared mass and a quadratic form in the relative momenta, which goes into the non-relativistic Hamiltonian for relative motions in the limit $c \rightarrow \infty$. This fact has the following implications:
a) if one tries to make the inverse Legendre transformation to find the associated Lagrangian, it turns out that, due to the presence of square roots, the Lagrangian is a hyperelliptic function of $\dot{\vec{\rho}}_a$ already in the free case. A closed form exists only for $N=2$, $m_1 = m_2 = m$: $L = -m \sqrt{4 - \dot{\rho}^2}$. This exceptional case already shows that the existence of the limiting velocity $c$ forbids a linear relation between the spin (center-of-mass angular momentum) and the angular velocity.

b) the $N$ quadratic forms in the relative momenta appearing in the relative Hamiltonian cannot be simultaneously diagonalized. In any case the Hamiltonian is a sum of square roots, so that concepts like reduced masses, Jacobi normal relative coordinates and tensor of inertia cannot be extended to special relativity. As a consequence, for example, a relativistic static orientation-shape SO(3) principal bundle approach [19, 20] can be implemented only by using non-Jacobi relative coordinates.

c) the best way of studying rotational kinematics, viz the non-Abelian rotational symmetry associated with the conserved internal spin, is based on the canonical spin bases with the associated concepts of spin frames and dynamical body frames introduced in Ref.[2]. It is remarkable that they can be build as in the non-relativistic case [21] starting from the canonical basis $\vec{\rho}_a$, $\vec{\pi}_a$.

Let us clarify this important point. In the non-relativistic N-body problem it is easy to make the separation of the absolute translational motion of the center of mass from the relative motions, due to the Abelian nature of the translation symmetry group. This implies that the associated Noether constants of motion (the conserved total 3-momentum) are in involution, so that the center-of-mass degrees of freedom decouple. Moreover, the fact that the non-relativistic kinetic energy of the relative motions is a quadratic form in the relative velocities allows the introduction of special sets of relative coordinates, the Jacobi normal relative coordinates that diagonalize the quadratic form and correspond to different patterns of clustering of the centers of mass of the particles. Each set of Jacobi normal relative coordinates organizes the N particles into a hierarchy of clusters, in which each cluster of two or more particles has a mass given by an eigenvalue (reduced masses) of the quadratic form; Jacobi normal coordinates join the centers of mass of pairs of clusters.

However, the non-Abelian nature of the rotation symmetry group whose associated Noether constants of motion (the conserved total angular momentum) are not in involution, prevents the possibility of a global separation of absolute rotations from the relative motions, so that there is no global definition of absolute vibrations. Consequently, an isolated deformable body can undergo rotations by changing its own shape (see the examples of the falling cat and of the diver). It was just to deal with these problems that the theory of the orientation-shape SO(3) principal bundle approach[19] has been developed. Its essential content is that any static (i.e. velocity-independent) definition of body frame for a deformable body must be interpreted as a gauge fixing in the context of a SO(3) gauge theory. Both the laboratory and the body frame angular velocities, as well as the orientational variables of the static body frame, become thereby unobservable gauge variables. This approach is associated with a set of point canonical transformations, which allow to define the body frame components of relative motions in a velocity-independent way.

Since in many physical applications (e.g. nuclear physics, rotating stars,...) angular velocities are viewed as measurable quantities, one would like to have an alternative formulation complying with this requirement and possibly generalizable to special relativity.
This program has been first accomplished in a previous paper [21] and then relativistically extended (Ref.[2]). Let us summarize, therefore, the main points of our formulation.

First of all, for \( N \geq 3 \), we have constructed (see Ref.[21]) a class of non-point canonical transformations which allow to build the already quoted canonical spin bases: they are connected to the patterns of the possible clusterings of the spins associated with relative motions. The definition of these spin bases is independent of Jacobi normal relative coordinates, just as the patterns of spin clustering are independent of the patterns of center-of-mass Jacobi clustering. We have found two basic frames associated to each spin basis: the spin frame and the dynamical body frame. Their construction is guaranteed by the fact that in the relative phase space, besides the natural existence of a Hamiltonian symmetry left action of SO(3) [22, 23], it is possible to define as many Hamiltonian non-symmetry right actions of SO(3) [24] as the possible patterns of spin clustering. While for \( N=3 \) the unique canonical spin basis coincides with a special class of global cross sections of the trivial orientation-shape SO(3) principal bundle, for \( N \geq 4 \) the existing spin bases and dynamical body frames turn out to be unrelated to the local cross sections of the static non-trivial orientation-shape SO(3) principal bundle, and evolve in a dynamical way dictated by the equations of motion. In this new formulation both the orientation variables and the angular velocities become, by construction, measurable quantities in each canonical spin basis.

For each \( N \), every allowed spin basis provides a physically well-defined separation between rotational and vibrational degrees of freedom. The non-Abelian nature of the rotational symmetry implies that there is no unique separation of absolute rotations and relative motions. The unique body frame of rigid bodies is replaced here by a discrete number of evolving dynamical body frames and of spin canonical bases, both of which are grounded in patterns of spin couplings, direct analog of the coupling of quantum angular momenta.

As anticipated at the outset, we want to complete our study of relativistic kinematics for the \( N \)-body system by first evaluating the rest-frame Dixon multipoles [25, 26] and then by analyzing the role of Dixon's multipoles for open subsystems. The basic technical tool will be the standard definition of the energy momentum tensor of the \( N \) positive-energy free particles on the Wigner hyperplane. It will be seen, however, that in order to get a sensible extension of this definition to open subsystems, a physically significant convention is required. On the whole, it turns out that the Wigner hyperplane is the natural framework for reorganizing a lot of kinematics connected with multipoles. Only in this way, moreover, a concept like the barycentric tensor of inertia can be introduced in special relativity, specifically by means of the quadrupole moments.

A first application of the formalism is done for an isolated system of \( N \) positive-energy particles with mutual action-at-a-distance interactions. Then the formalism is applied to the case of an open \( n < N \) particle subsystem of an isolated system consisting of \( N \) charged positive-energy particles (with Grassmann-valued electric charges to regularize the Coulomb self-energies) plus the electro-magnetic field [5]. In the rest frame of the isolated system a suitable definition of the energy-momentum tensor of the open subsystem allows to define its effective mass, 3-momentum and angular momentum.

Then we evaluate the rest-frame Dixon multipoles of the energy-momentum tensor of the open subsystem with respect to various centroids describing possible collective centers of motion. Unlike the case of isolated systems, each centroid generates a different world-line and there are many candidates for the effective center of motion and the effective intrinsic spin. Two centroids (namely the center of energy and Tulczyjew centroid) appear to be
preferable due to their properties. The case \( n = 2 \) is studied explicitly. It is also shown that the pole-dipole description of the 2-particle cluster can be replaced by a description of the cluster as an extended system (its effective spin frame can be evaluated). This can be done, however, at the price of introducing an explicit dependence on the action of the external electro-magnetic field upon the cluster. By comparing the effective parameters of an open cluster of \( n_1 + n_2 \) particles to the effective parameters of the two clusters with \( n_1 \) and \( n_2 \) particles, it turns out in particular that only the effective center of energy appears to be a viable center of motion for studying the interactions of open subsystems.

A review of the rest-frame instant form of dynamics for \( N \) scalar free positive-energy particles and some new original results on the canonical transformation to the internal center of mass and relative variables are given in Section II.

In Section III we evaluate the energy momentum tensor in the Wigner hyper-planes.

Dixon’s multipoles are introduced in Section IV. A special study of monopole, dipole and quadrupole moments is given and the multipolar expansion is defined.

After the extension of the previous results to isolated systems with mutual action-at-a-distance interactions, we study in Section V the behavior of open subsystems of isolated systems, the centroids which are good candidates for the description the collective center of motion, and discuss the determination of the effective parameters (mass, spin, momentum, variables relative to the center of motion) for the open subsystem.

Some comments about standing problems are given in the Conclusions.

The non-relativistic \( N \)-particle multipolar expansion is given in Appendix A, while in Appendix B contains a review of symmetric trace-free (STF) tensors. The Gartenhaus-Schwartz transformation is summarized in Appendix C. Finally, other properties of Dixon’s multipoles are reported in Appendix D.
II. REVIEW OF THE REST-FRAME INSTANT FORM.

We briefly review the treatment of N free scalar positive-energy particles in the framework of parametrized Minkowski theory (see Appendices A and B and Section II of Ref.[2]). Each particle is described by a configuration 3-vector \( \vec{\eta}_i(\tau) \). The particle world-line is \( x_i^\mu(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau)) \), where \( z^\mu(\tau, \vec{\sigma}) \) are the embedding configuration variables describing the space-like hyper-surface \( \Sigma_\tau \).

The foliation is defined by an embedding \( R \times \Sigma \to M^4, (\tau, \vec{\sigma}) \mapsto z^\mu(\tau, \vec{\sigma}) \), with \( \Sigma \) an abstract 3-surface diffeomorphic to \( R^3 \). \( \Sigma_\tau \) is the Cauchy surface of equal time. The metric induced on \( \Sigma_\tau \) is \( g_{AB} = \zeta^\mu \eta_{\mu
u} z_B^\nu \), a functional of \( z^\mu \), and the embedding coordinates \( z^\mu(\tau, \vec{\sigma}) \) are considered as independent fields. We use the notation \( \sigma^A = (\tau, \vec{\sigma}) \) of Refs.[3, 5]. The \( z^\mu_A(\sigma) = \partial z^\mu(\sigma)/\partial \sigma^A \) are flat cotetrad fields on Minkowski space-time with the tangent to \( \Sigma_\tau \).

The dual tetrad fields are \( \zeta^\mu_A(\sigma) = \frac{\partial \sigma^A(z)}{\partial z^\mu} \), \( \zeta^A(\tau, \vec{\sigma}) \zeta^\mu_B(\tau, \vec{\sigma}) = \delta^A_B \). While in Ref.[2] we used the metric convention \( \eta_{\mu\nu} = \epsilon(+-++) \) with \( \epsilon = \pm \), in this paper we shall use \( \epsilon = 1 \) like in Ref.[3].

If we put \( \sqrt{g(\tau, \vec{\sigma})} = \sqrt{-\text{det} g_{AB}(\tau, \vec{\sigma})} \) and \( \sqrt{\gamma(\tau, \vec{\sigma})} = \sqrt{-\text{det} g_{\tau\bar{\tau}}(\tau, \vec{\sigma})} \), we have
\[
\zeta^\mu(\tau, \vec{\sigma}) = \left( \sqrt{\frac{2}{g_{\tau\bar{\tau}}} l^\mu + g_{\tau\bar{\tau}} \gamma_{\tau\bar{s}} z^\mu_{\bar{s}}} \right)(\tau, \vec{\sigma}) \quad (\gamma_{\tau\bar{s}} g_{\bar{s}\bar{t}} = \delta_\tau^\sigma),
\]
(\( \text{normal to } \Sigma_\tau \); \( l^2(\tau, \vec{\sigma}) = 1 \); \( l_\mu(\tau, \vec{\sigma}) z^\mu_{\tau}(\tau, \vec{\sigma}) = 0 \)) and
\[
d^4z = z^\mu_{\tau}(\tau, \vec{\sigma}) d\tau d^3\Sigma_\mu = d\tau [z^\mu_{\tau}(\tau, \vec{\sigma}) l_\mu(\tau, \vec{\sigma})] \sqrt{\gamma(\tau, \vec{\sigma})} d^3\sigma = \sqrt{g(\tau, \vec{\sigma})} d\tau d^3\sigma.
\]

The system is described by the action[3, 4, 5]
\[
S = \int d\tau d^3\sigma \mathcal{L}(\tau, \vec{\sigma}) = \int d\tau L(\tau),
\]
\[
\mathcal{L}(\tau, \vec{\sigma}) = -\sum_{i=1}^N \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) m_i \sqrt{g_{\tau\tau}(\tau, \vec{\sigma}) + 2g_{\tau\bar{\tau}}(\tau, \vec{\sigma}) \dot{\eta}_i^\tau(\tau) + 2g_{\bar{s}\bar{t}}(\tau, \vec{\sigma}) \dot{\eta}_i^\tau(\tau) \dot{\eta}_i^\tau(\tau)},
\]
\[
L(\tau) = -\sum_{i=1}^N m_i \sqrt{g_{\tau\tau}(\tau, \vec{\eta}_i(\tau)) + 2g_{\tau\bar{\tau}}(\tau, \vec{\eta}_i(\tau)) \dot{\eta}_i^\tau(\tau) + 2g_{\bar{s}\bar{t}}(\tau, \vec{\eta}_i(\tau)) \dot{\eta}_i^\tau(\tau) \dot{\eta}_i^\tau(\tau)}. \quad (2.1)
\]

The action is invariant under separate \( \tau - \vec{\sigma} \)-reparametrizations.

The canonical momenta are
\[ \rho_{\mu}(\tau, \vec{\sigma}) = -\frac{\partial L(\tau, \vec{\sigma})}{\partial z^\mu_{\tau}(\tau, \vec{\sigma})} = \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{n}_i(\tau))m_i \]

\[ z_{\tau\mu}(\tau, \vec{\sigma}) + z_{x\mu}(\tau, \vec{\sigma})\eta^s_i(\tau) = \sqrt{g_{\tau\tau}(\tau, \vec{\sigma}) + 2g_{\tau\tau}(\tau, \vec{\sigma})\eta^s_i(\tau) + g_{x\tau}(\tau, \vec{\sigma})\eta^s_i(\tau)\eta^s_i(\tau)} = \\
\left[ (\rho_{\mu}^\nu) l_{\mu} + (\rho_{\nu} z^\nu_{\tau}) \gamma^{x\tau} z_{\eta^s} \right](\tau, \vec{\sigma}), \]

\[ \kappa_{\nu\mu}(\tau) = -\frac{\partial L(\tau)}{\partial \eta^s_i(\tau)} = \\
= m_i \frac{g_{\tau\tau}(\tau, \vec{n}_i(\tau)) + g_{x\tau}(\tau, \vec{n}_i(\tau))\eta^s_i(\tau)}{\sqrt{g_{\tau\tau}(\tau, \vec{n}_i(\tau)) + 2g_{\tau\tau}(\tau, \vec{n}_i(\tau))\eta^s_i(\tau) + g_{x\tau}(\tau, \vec{n}_i(\tau))\eta^s_i(\tau)\eta^s_i(\tau)}, \]

\[ \{z^\mu(\tau, \vec{\sigma}), \rho_{\nu}(\tau, \vec{\sigma})\} = -\eta^s_i \delta^3(\vec{\sigma} - \vec{\sigma}'), \]

\[ \{\eta^s_i(\tau), \kappa_{\xi\xi}(\tau)\} = -\delta_{ij} \delta^s_i, \quad (2.2) \]

The canonical Hamiltonian \( H_c \) is zero, but there are the primary first class constraints

\[ H_\mu(\tau, \vec{\sigma}) = \rho_{\mu}(\tau, \vec{\sigma}) - l_{\mu}(\tau, \vec{\sigma}) \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{n}_i(\tau))\sqrt{m_i^2 - \gamma^{x\tau}(\tau, \vec{\sigma})\kappa_{\nu\mu}(\tau)\kappa_{\xi\xi}(\tau)} - \\
- z_{x\mu}(\tau, \vec{\sigma})\gamma^{x\tau}(\tau, \vec{\sigma}) \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{n}_i(\tau))\kappa_{\xi\xi} \approx 0, \quad (2.3) \]

so that the Dirac Hamiltonian is \( H_D = \int d^3\sigma \lambda^\mu(\tau, \vec{\sigma})H_\mu(\tau, \vec{\sigma}) \), where \( \lambda^\mu(\tau, \vec{\sigma}) \) are Dirac multipliers.

The conserved Poincaré generators are (the suffix “s” denotes the hypersurface \( \Sigma_{\tau} \))

\[ p^\mu = \int d^3\sigma \rho^\mu(\tau, \vec{\sigma}), \]

\[ J_{\mu\nu} = \int d^3\sigma [z^\mu(\tau, \vec{\sigma})\rho^\nu(\tau, \vec{\sigma}) - z^\nu(\tau, \vec{\sigma})\rho^\mu(\tau, \vec{\sigma})]. \quad (2.4) \]

After the restriction to space-like hyper-planes, \( z^\mu(\tau, \vec{\sigma}) = x^\mu(\tau) + b_{x}^\mu(\tau)\vec{\sigma} \), the Dirac Hamiltonian is reduced to \( \hat{H}_D = \lambda_\mu(\tau)\hat{H}^\mu(\tau) + \lambda_{\mu\nu}(\tau)\hat{H}^{\mu\nu}(\tau) \) (only ten Dirac multipliers survive) while the remaining ten constraints are given by
\[ \tilde{H}^\mu(\tau) = \int d^3 \sigma H^\mu(\tau, \vec{\sigma}) = p_s^\mu - l^\mu \sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i^2(\tau)} + b_i^\mu(\tau) \sum_{i=1}^N \kappa_{i\nu}(\tau) \approx 0, \]

\[ \tilde{H}^{\mu\nu}(\tau) = b_r^\mu(\tau) \int d^3 \sigma \sigma^r \hat{H}^\nu(\tau, \vec{\sigma}) - b_r^\nu(\tau) \int d^3 \sigma \sigma^r \hat{H}^\mu(\tau, \vec{\sigma}) = \]

\[ = S_s^{\mu\nu}(\tau) - [b_r^\mu(\tau)b_r^\nu(\tau) - b_r^\nu(\tau)b_r^\mu(\tau)] \sum_{i=1}^N \eta_{i\mu}(\tau) \sqrt{m_i^2 + \vec{\kappa}_i^2(\tau)} - \]

\[ - [b_r^\mu(\tau)b_s^\nu(\tau) - b_r^\nu(\tau)b_s^\mu(\tau)] \sum_{i=1}^N \eta_{i\mu}(\tau) \kappa_{i\nu}(\tau) \approx 0. \tag{2.5} \]

Here \( S_s^{\mu\nu} \) is the spin part of the Lorentz generators

\[ J_s^{\mu\nu} = x_s^\mu p_s^\nu - x_s^\nu p_s^\mu + S_s^{\mu\nu}, \]

\[ S_s^{\mu\nu} = b_r^\mu(\tau) \int d^3 \sigma \sigma^r \hat{\rho}^\nu(\tau, \vec{\sigma}) - b_r^\nu(\tau) \int d^3 \sigma \sigma^r \hat{\rho}^\mu(\tau, \vec{\sigma}). \tag{2.6} \]

The condition \( \dot{l}^\mu = 0 \) that the unit normal \( l^\mu = \epsilon_{\alpha\beta\gamma} b_{1\beta}(\tau) b_{2\gamma}(\tau) b_{3\alpha}(\tau), \ l^2 = 1, \) to the hyper-planes be \( \tau \)-independent, implies three gauge fixing constraints \( \hat{H}^{\mu\nu}(\tau) \approx 0. \) These imply in turn that the \( b_r^\mu(\tau) \)'s depend only on three Euler angles describing a rotating spatial frame. Therefore foliations with parallel hyper-planes have only 7 independent first class constraints.

When \( p_s^2 > 0, \) the restriction to Wigner hyper-planes is done by imposing \( b_A^\mu(\tau) = (b_r^\mu = l^\mu; b_r^\mu) \approx b_A^\mu = L^\mu_{\nu=A}(p_s, \vec{\psi}_s), \) where \( L^\mu_{\nu}(p_s, \vec{\psi}_s) \) is the standard Wigner boost for time-like Poincare’ orbits. The indices \( \hat{r} = r \) are Wigner spin 1 indices and we have \( b_r^\mu = L^\mu_{\nu}(p_s, \vec{\psi}_s) = u^\nu(p_s) = p_s^\mu/\sqrt{p_s^2}, \ b_r^\mu = \epsilon_{\nu r}^\mu(u(p_s)) = L^\nu_{\nu r}(p_s, \vec{\psi}_s). \)

On the Wigner hyperplane \[28], \( z^\mu(\tau, \vec{\sigma}) = x_s^\mu(\tau) + \epsilon_{\nu r}^\mu(u(p_s)) \sigma^r, \) we have the following constraints and Dirac Hamiltonian\[3, 5\]
\[ \tilde{H}^\mu(\tau) = p_\mu^s - u^\mu(p_s) \sum_{i=1}^N \sqrt{m_i^2 + \tilde{\kappa}_i^2} + \epsilon^\mu_i(u(p_s)) \sum_{i=1}^N \kappa_{ir} = \]

\[ = u^\mu(p_s) \left[ \epsilon_s - \sum_{i=1}^N \sqrt{m_i^2 + \tilde{\kappa}_i^2} \right] + \epsilon^\mu_r(u(p_s)) \sum_{i=1}^N \kappa_{ir} \approx 0, \]

or

\[ \epsilon_s - M_{\text{sys}} \approx 0, \quad M_{\text{sys}} = \sum_{i=1}^N \sqrt{m_i^2 + \tilde{\kappa}_i^2}, \]

\[ \tilde{p}_{\text{sys}} = \tilde{\kappa}_+ = \sum_{i=1}^N \tilde{\kappa}_i \approx 0, \]

\[ H_D = \lambda^\mu(\tau) \tilde{H}^\mu(\tau) = \lambda(\tau)[\epsilon_s - M_{\text{sys}}] - \tilde{\lambda}(\tau) \sum_{i=1}^N \tilde{\kappa}_i, \]

\[ \lambda(\tau) \approx -\dot{x}_{s\mu}(\tau) u^\mu(p_s), \quad \lambda_r(\tau) \approx -\dot{x}_{s\mu}(\tau) \epsilon^\mu_r(u(p_s)), \]

\[ \dot{x}_s^\mu(\tau) = -\lambda(\tau) u^\mu(p_s), \]

\[ \dot{x}_s^\mu(\tau) \overset{\circ}{=} \{ x_s^\mu(\tau), H_D \} = \lambda(\tau) \{ x_s^\mu(\tau), \tilde{H}^\mu(\tau) \} \approx \]

\[ \approx -\lambda^\mu(\tau) = -\lambda(\tau) u^\mu(p_s) + \epsilon^\mu_r(u(p_s)) \lambda_r(\tau), \]

\[ \dot{x}_s^2(\tau) = \lambda^2(\tau) - \tilde{\lambda}^2(\tau) > 0, \quad \dot{x}_s \cdot u(p_s) = -\lambda(\tau), \]

\[ U_s^\mu(\tau) = \frac{\dot{x}_s^\mu(\tau)}{\sqrt{\dot{x}_s^2(\tau)}} = \frac{-\lambda(\tau) u^\mu(p_s) + \lambda_r(\tau) \epsilon^\mu_r(u(p_s))}{\sqrt{\lambda^2(\tau) - \tilde{\lambda}^2(\tau)}}, \]

\[ \Rightarrow \quad x_s^\mu(\tau) = x_s^\mu(0) - u^\mu(p_s) \int_0^\tau d\tau_1 \lambda(\tau_1) + \epsilon^\mu_r(u(p_s)) \int_0^\tau d\tau_1 \lambda_r(\tau_1). \quad (2.7) \]

While the Dirac multiplier \( \lambda(\tau) \) is determined by the gauge fixing \( T_s - \tau \approx 0 \), the 3 Dirac’s multipliers \( \bar{\lambda}(\tau) \) describe the classical zitterbewegung of the centroid \( x_s^\mu(\tau) = z^\mu(\tau, 0) \) which is the origin of the 3-coordinates on the Wigner hyperplane. Each gauge-fixing \( \bar{\chi}(\tau) \approx 0 \) to the 3 first class constraints \( \tilde{\kappa}_+ \approx 0 \) (defining the internal rest-frame 3-center of mass \( \bar{\sigma}_{cm} \)) gives a different determination of the multipliers \( \bar{\lambda}(\tau) \) [29]. Therefore each gauge-fixing identifies a different world-line for the covariant non-canonical centroid \( x_s^{(X)}(\tau) \). Of course, inside the Wigner hyperplane, three degrees of freedom of the isolated system [30] become gauge variables. The natural gauge fixing for eliminating the first class constraints \( \tilde{\kappa}_+ \approx 0 \) is \( \bar{\chi}(\tau) = \bar{\sigma}_{com} = \bar{q}_+ \approx 0 \) [vanishing of the location of the internal canonical 3-center of mass, see after Eq.(2.16)]. We have that \( \bar{q}_+ \approx 0 \) implies \( \lambda_+(\tau) = 0 \): in this way the internal 3-center
of mass is located in a unique centroid $x_s^{(\bar{\bar{q}}_s)}(\tau) = z^\mu(\tau, \vec{\sigma} = 0)$ \[ \hat{x}^{(\bar{\bar{q}}_s)}_s = \hat{u}^\mu(p_s) \].

Note that the constant $x_s^\mu(0)$ \[ and, therefore, also \hat{x}_s^\mu(0) \] is arbitrary, reflecting the arbitrariness in the absolute location of the origin of the internal coordinates on each hyperplane in Minkowski space-time. The centroid $x_s^\mu(\tau)$ corresponds to the unique special relativistic center-of-mass-type world-line for isolated systems of Refs. [31, 32], which unifies previous proposals of Synge, Moller and Pryce.

The only remaining canonical variables describing the Wigner hyperplane in the final Dirac brackets are a non-covariant canonical coordinate $\hat{x}_s^\mu(\tau)$ \[ [35] and $p_s^\mu$. The point with coordinates $\hat{x}_s^\mu(\tau)$ is the decoupled canonical external 4-center of mass of the isolated system, which can be interpreted as a decoupled observer with his parametrized clock (point particle clock). Its velocity $\hat{x}_s^\mu(\tau)$ is parallel to $p_s^\mu$, so that it has no classical zitterbewegung.

The relation between $x_s^\mu(\tau)$ and $\hat{x}_s^\mu(\tau)$ (\bar{\sigma} is its 3-location on the Wigner hyperplane) is \[ [2, 3] \]

\[
\hat{x}_s^\mu(\tau) = (\bar{x}_s^0, \vec{\bar{x}}_s(\tau)) = z^\mu(\tau, \vec{\sigma}) = x_s^\mu(\tau) - \frac{1}{\epsilon_s(p_s^0 + \epsilon_s)} \left[ p_s^{\alpha\nu} S_s^{\nu\mu} + \epsilon_s(S_s^{\alpha\mu} - S_s^{\alpha\nu} p_s^{\nu\mu}/\epsilon_s^2) \right], \] (2.8)

After the separation of the relativistic canonical non-covariant external 4-center of mass $\hat{x}_s^\mu(\tau)$, the $N$ particles are described on the Wigner hyperplane by the $6N$ Wigner spin-1 3-vectors $\vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)$ restricted by the rest-frame condition $\vec{k}_+ = \sum_{i=1}^N \vec{\kappa}_i \approx 0$.

The various spin tensors and vectors are \[ [3] \]

\[
J_s^{\mu\nu} = x_s^{\mu}p_s^{\nu} - x_s^{\nu}p_s^{\mu} + S_s^{\mu\nu} = \tilde{x}_s^{\mu}p_s^{\nu} - \tilde{x}_s^{\nu}p_s^{\mu} + \tilde{S}_s^{\mu\nu},
\]

\[
S_s^{\mu\nu} = [u^\mu(p_s)e^\nu(u(p_s)) - u^\nu(p_s)e^\mu(u(p_s))\tilde{S}_s^{\tau\rho} + e^\mu(u(p_s))e^\nu(u(p_s))\tilde{S}_s^{\tau\rho}] \equiv \left\{ e^\mu(u(p_s))u^\nu(p_s) - e^\nu(u(p_s))u^\mu(p_s) \right\} \sum_{i=1}^N \eta_i^r \sqrt{m_i^2c^2 + \vec{k}_i^2} + \left\{ e^\mu(u(p_s))e^\nu(u(p_s))e^\rho(u(p_s)) \right\} \sum_{i=1}^N \eta_i^r \vec{k}_i^s,
\]

\[
\tilde{S}_s^{AB} = e^A_{\mu}(u(p_s))e^B_{\nu}(u(p_s))S_s^{\mu\nu},
\]

\[
\tilde{S}_s^{rs} \equiv \sum_{i=1}^N (\eta_i^r \kappa_i^s - \eta_i^s \kappa_i^r), \quad \tilde{S}_s^{\tau\rho} \equiv - \sum_{i=1}^N \eta_i^r \sqrt{m_i^2c^2 + \vec{k}_i^2},
\]

\[
\tilde{S}_s^{\mu\nu} = \frac{1}{\sqrt{ep_s^2(p_s^0 + \sqrt{ep_s^2})}} \left[ p_s^\beta S_s^{\beta\mu} p_s^\nu - S_s^{\beta\nu} p_s^\mu \right] + \sqrt{p_s^2}(S_s^{\alpha\nu} p_s^\mu - S_s^{\alpha\mu} p_s^\nu),
\]

\[
\tilde{S}_s^{ij} = \delta^{ij} \tilde{S}_s^{rs}, \quad \tilde{S}_s^{oi} = - \frac{\delta^{ij} \tilde{S}_s^{rs} p_s^i}{p_s^0 + \sqrt{ep_s^2}},
\]

\[
\vec{\tilde{S}} \equiv \vec{S} = \sum_{i=1}^N \eta_i \times \vec{\kappa}_i \approx \sum_{i=1}^N \eta_i \times \vec{\kappa}_i - \vec{\eta}_+ \times \vec{\kappa}_+ = \sum_{a=1}^{N-1} \vec{\rho}_a \times \vec{\pi}_a. \quad (2.9)
\]
Note that while $L^{\mu\nu} = x^\mu p^\nu_s - x^\nu p^\mu_s$ and $S^{\mu\nu}$ are not constants of the motion due to the classical zitterbewung (when $\lambda(\tau) \neq 0$), both $\tilde{L}^{\mu\nu} = \tilde{x}^\mu p^\nu_s - \tilde{x}^\nu p^\mu_s$ and $\tilde{S}^{\mu\nu}$ are conserved.

The canonical variables $\tilde{x}^\mu_s, p^\mu_s$ for the external 4-center of mass, can be replaced by the canonical pairs \[36\]

\[
T_s = \frac{p_s \cdot \tilde{x}_s}{\epsilon_s} = \frac{p_s \cdot x_s}{\epsilon_s}, \quad \epsilon_s = \pm \sqrt{p^2_s},
\]

\[
\tilde{z}_s = \epsilon_s (\tilde{x}_s - \frac{\tilde{p}_s}{p_s} \tilde{x}^0_s), \quad \tilde{k}_s = \frac{\tilde{p}_s}{\epsilon_s},
\] (2.10)

which make explicit the interpretation of it as a point particle clock. The inverse transformation is

\[
\tilde{x}^0_s = \sqrt{1 + \tilde{k}_s^2} (T_s + \frac{\tilde{k}_s \cdot \tilde{z}_s}{\epsilon_s}), \quad \tilde{x}_s = \frac{\tilde{z}_s}{\epsilon_s} + (T_s + \frac{\tilde{k}_s \cdot \tilde{z}_s}{\epsilon_s}) \tilde{k}_s, \quad p_s^0 = \epsilon_s \sqrt{1 + \tilde{k}_s^2},
\]

\[
\tilde{p}_s = \epsilon_s \tilde{k}_s.
\] (2.11)

This non-point canonical transformation can be summarized as $[\epsilon_s - M_{sys} \approx 0, \kappa_+ = \sum_{i=1}^N \kappa_i \approx 0]$

\[
\begin{array}{c|c|c|c|c}
\tilde{x}^\mu_s & \tilde{p}^\mu_s & \tilde{y}_i & T_s & \kappa_s \\
\hline
\tilde{p}^\mu_s & \tilde{y}_i & T_s & \kappa_s & \kappa_i
\end{array}
\] . (2.12)

After the addition of the gauge-fixing $T_s - \tau \approx 0$ \[37\], the invariant mass $M_{sys}$ of the system, which is also the internal energy of the isolated system, replaces the non-relativistic Hamiltonian $H_{rel}$ for the relative degrees of freedom: this reminds of the frozen Hamilton-Jacobi theory, in which the time evolution can be re-introduced by using the energy generator of the Poincaré group as Hamiltonian \[38\].

After the gauge fixings $T_s - \tau \approx 0$ [implying $\lambda(\tau) = -1$], the final Hamiltonian, the embedding of the Wigner hyperplane into Minkowski space-time and the Hamilton equations become
\[ H_D = M_{sys} - \Lambda(\tau) \cdot \vec{k}_+ , \]

\[ z^\mu(\tau, \sigma) = x^\mu_s(\tau) + \epsilon^\mu_s(u(p_s))\sigma^r = \]

\[ = x^\mu_s(0) + u^\mu(p_s)\tau + \epsilon^\mu_s(u(p_s))\left( \sigma^r + \int_0^\tau d\tau_1 \lambda_r(\tau_1) \right), \]

with

\[ \dot{x}^\mu_s(\tau) = u^\mu(p_s) + \epsilon^\mu_s(u(p_s))\lambda_r(\tau), \]

\[ \tilde{\eta}_i(\tau) = \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2(\tau)}} - \Lambda(\tau), \Rightarrow \]

\[ \vec{k}_i(\tau) = m_i \frac{\dot{\eta}_i(\tau) + \Lambda(\tau)}{\sqrt{1 - [\dot{\eta}_i(\tau) + \Lambda(\tau)]^2}}, \]

\[ \dot{\vec{k}}_i(\tau) = 0. \quad (2.13) \]

The particles’ world-lines in Minkowski space-time and the associated momenta are

\[ x^\mu_s(\tau) = z^\mu(\tau, \tilde{\eta}_i(\tau)) = x^\mu_s(\tau) + \epsilon^\mu_s(u(p_s))\eta^r_i(\tau), \]

\[ p_i^\mu(\tau) = \sqrt{m_i^2 + \vec{k}_i^2(\tau)}u^\mu(p_s) + \epsilon^\mu_s(u(p_s))\kappa_i^r(\tau) \Rightarrow p_i^2 = m_i^2. \quad (2.14) \]

The external rest-frame instant form realization of the Poincaré generators [40] with non-fixed invariants \( p_s^2 = \epsilon_s^2 \approx M_{sys}^2, -\vec{p}_s^2 \approx -M_{sys}^2 \vec{S}_s \), is obtained from Eq.(2.9):

\[ p_s^\mu, \quad J_s^{\mu\nu} = \vec{x}_s^\mu p_s^\nu - \bar{x}_s^\nu p_s^\mu + \vec{S}_s^{\mu\nu}, \]

\[ p_s^0 = \sqrt{\epsilon_s^2 + \vec{p}_s^2} = \epsilon_s \sqrt{1 + \vec{k}_s^2} \approx \sqrt{M_{sys}^2 + \vec{p}_s^2} = M_{sys} \sqrt{1 + \vec{k}_s^2}, \]

\[ \vec{p}_s = \epsilon_s \vec{k}_s \approx M_{sys} \vec{k}_s, \]

\[ J_s^{ij} = \vec{x}_s^i p^j_s - \bar{x}_s^j p^i_s + \delta^{ir} \delta^{js} \sum_{s=1}^{N} (\eta_i^s \kappa^r_j - \eta_j^s \kappa^r_i) = z_s^i k_s^j - z_s^j k_s^i + \delta^{ir} \delta^{js} \epsilon^{rsu} \vec{S}_s^u, \]

\[ K_s^i = J_s^{0i} = \vec{x}_s^0 p^i_s - \bar{x}_s^i \sqrt{\epsilon_s^2 + \vec{p}_s^2} - \frac{1}{\epsilon_s + \sqrt{\epsilon_s^2 + \vec{p}_s^2}} \delta^{ir} \delta^{js} \sum_{s=1}^{N} (\eta_i^s \kappa^r_j - \eta_j^s \kappa^r_i) = \]

\[ = -\sqrt{1 + \vec{k}_s^2} \bar{x}_s^i \frac{\delta^{ir} k_s^e \epsilon^{esu} \vec{S}_u}{1 + \sqrt{1 + \vec{k}_s^2}} \approx \bar{x}_s^i p_s^0 - \bar{x}_s^i \sqrt{M_{sys}^2 + \vec{p}_s^2} - \frac{\delta^{ir} p_s^e \epsilon^{esu} \vec{S}_u}{M_{sys} + \sqrt{M_{sys}^2 + \vec{p}_s^2}} \quad (2.15) \]
On the other hand the \textit{internal} realization of the Poincaré algebra is built inside the Wigner hyperplane by using the expression of $S_{s}^{AB}$ given by Eq.(2.9) \cite{41}

$$M_{\text{sys}} = H_{M} = \sum_{i=1}^{N} \sqrt{m_{i}^{2} + \vec{\kappa}_{i}^{2}},$$

$$\vec{\kappa}_{+} = \sum_{i=1}^{N} \vec{\kappa}_{i} (\approx 0),$$

$$\vec{J} = \sum_{i=1}^{N} \vec{\eta}_{i} \times \vec{\kappa}_{i}, \quad J^{r} = \vec{S}^{r} = \frac{1}{2} \epsilon^{rav} \vec{S}^{av} \equiv \vec{S}_{s}^{r},$$

$$\vec{K} = - \sum_{i=1}^{N} \sqrt{m_{i}^{2} + \kappa_{i}^{2}} \vec{\eta}_{i} = - M_{\text{sys}} \vec{R}_{+}, \quad K^{r} = J^{or} = \vec{S}_{s}^{rr},$$

$$\Pi = M_{\text{sys}}^{2} - \vec{\kappa}_{+}^{2} \approx M_{\text{sys}}^{2} > 0,$$

$$W^{2} = - \epsilon (M_{\text{sys}}^{2} - \vec{\kappa}_{+}^{2}) \vec{S}_{s}^{2} \approx - \epsilon M_{\text{sys}}^{2} \vec{S}_{s}^{2}. \quad (2.16)$$

The constraints $\epsilon_{s} - M_{\text{sys}} \approx 0, \vec{\kappa}_{+} \approx 0$ have the following meaning:

i) the constraint $\epsilon_{s} - M_{\text{sys}} \approx 0$ is the bridge connecting the \textit{external} and \textit{internal} realizations \cite{42};

ii) the constraints $\vec{\kappa}_{+} \approx 0$, together with $\vec{K} \approx 0$ (i.e. $\vec{R}_{+} \approx \vec{q}_{+} \approx \vec{y}_{+} \approx 0$) \cite{43}, imply an unfaithful \textit{internal} realization, in which the only non-zero generators are the conserved energy and spin of an isolated system.

The determination of $\vec{q}_{+}$ for the N particle system has been carried out by the group theoretical methods of Ref.[44] in Section III of Ref.[2]. Given a realization of the ten Poincaré generators on the phase space, one can build three 3-position variables in terms of them only. For N free scalar relativistic particles on the Wigner hyperplane with $\vec{p}_{\text{sys}} = \vec{\kappa}_{+} \approx 0$ within the \textit{internal} realization (2.16) they are:

i) a canonical \textit{internal} 3-center of mass (or 3-center of spin) $\vec{q}_{+}$;

ii) a non-canonical \textit{internal} Møller 3-center of energy $\vec{R}_{+}$;

iii) a non-canonical \textit{internal} Fokker-Pryce 3-center of inertia $\vec{y}_{+}$.

It can be shown\cite{2} that, due to $\vec{\kappa}_{+} \approx 0$, they all coincide: $\vec{q}_{+} \approx \vec{R}_{+} \approx \vec{y}_{+}$.

Therefore the gauge fixings $\vec{\chi}(\tau) = \vec{q}_{+} \approx \vec{R}_{+} \approx \vec{y}_{+} \approx 0$ imply $\vec{\lambda}(\tau) \approx 0$ and force the three \textit{internal} collective variables to coincide with the origin of the coordinates, which now becomes

$$x_{s}^{(Q_{+})\mu}(T_{s}) = x_{s}^{\mu}(0) + u^{\mu}(p_{s})T_{s}. \quad (2.17)$$

As shown in Section IV, the addition of the gauge fixings $\vec{\chi}(\tau) = \vec{q}_{+} \approx \vec{R}_{+} \approx \vec{y}_{+} \approx 0$ also implies that the Dixon center of mass of an extended object\cite{27} and the Pirani\cite{45} and Tulczyjew\cite{34, 46, 47} centroids \cite{48} all simultaneously coincide with the centroid $x_{s}^{(\vec{q}_{+})\mu}(\tau)$. 

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The *external* realization (2.15) allows to build the analogous *external* 3-variables $\vec{q}_s$, $\vec{R}_s$, $\vec{Y}_s$. Eq.(4.4) of Ref.[2] shows the construction of the associated *external* 4-variables $\vec{\tau}_s^\mu$, $Y_s^\mu$, $\vec{R}_s^\mu$ and their locations $\vec{\tau}$, $\vec{\sigma}_Y$, $\vec{\sigma}_R$ on the Wigner hyperplane. It appears that the *external* Fokker-Pryce non-canonical covariant 4-center of inertia $Y_s^\mu$ coincides with the centroid (2.17).

In Ref.[2] there is the definition of the following sequence of canonical transformations

$$
\begin{align*}
\vec{\eta}_i &\rightarrow \vec{\eta} + \vec{\rho}_a \\
\vec{\kappa}_i &\rightarrow \vec{\kappa} + \vec{\pi}_a
\end{align*}
$$

leading to the canonical separation of the internal 3-center of mass ($\vec{q}_+^i$, $\vec{\kappa}_+$) from the internal relative variables $\vec{\rho}_a$, $\vec{\pi}_a$. Since the rest-frame condition $\vec{\kappa}_+ \approx 0$ implies [2] $\vec{\rho}_a \approx \vec{\rho}$, $\vec{\pi}_a \approx \vec{\pi}$, in the gauge $\vec{q}_+ \approx 0$ and in terms of the associated Dirac brackets we get an internal reduced phase space whose canonical basis is $\vec{\rho}_a \equiv \vec{\rho}$, $\vec{\pi}_a \equiv \vec{\pi}$, $a = 1, .., N - 1$.

The intermediate linear point canonical transformation in (2.18) is [actually this is a family of canonical transformations, since the $\gamma_j$’s are any set of numbers satisfying $\sum_i \gamma_i = 0$, $\sum_a \gamma_i \gamma_j = \delta_{ij} - \frac{1}{N}$, $\sum_{\gamma} \gamma_i \gamma_j = \delta_{ab}$]

$$
\begin{align*}
\vec{\eta}_i & = \vec{\eta} + \frac{1}{\sqrt{N}} \sum_a \gamma_i \vec{\rho}_a, \\
\vec{\kappa}_i & = \frac{1}{N} \vec{\kappa} + \sqrt{N} \sum_a \gamma_i \vec{\pi}_a, \\
\vec{\eta} & = \frac{1}{N} \sum_i \vec{\eta}_i, \quad \vec{\rho}_a = \sqrt{N} \sum_i \gamma_i \vec{\tilde{\eta}}, \\
\vec{\kappa} & = \sum_i \vec{\kappa}_i, \quad \vec{\pi}_a = \frac{1}{\sqrt{N}} \sum_i \gamma_i \vec{\kappa}_i.
\end{align*}
$$

The second canonical transformation has been defined in Section V of Ref.[2] by using a singular Gartenhaus-Schwartz transformation (see Appendix C), but it was not written explicitly for $\vec{\kappa}_+ \neq 0$. By using Eqs. (C14) and (C15) we get the following results:

1) For $N = 2$ ($\gamma_{11} = - \gamma_{12} = \frac{1}{\sqrt{2}}$) we have
\[\vec{\eta}_1 = \vec{\eta}_+ + \frac{1}{2} \vec{\rho}, \quad \vec{\kappa}_1 = \frac{1}{2} \vec{\kappa}_+ + \vec{\pi},\]
\[\vec{\eta}_2 = \vec{\eta}_+ - \frac{1}{2} \vec{\rho}, \quad \vec{\kappa}_2 = \frac{1}{2} \vec{\kappa}_- - \vec{\pi},\]
\[\vec{\eta}_+ = \frac{1}{2} (\vec{\eta}_1 + \vec{\eta}_2), \quad \vec{\kappa}_+ = \vec{\kappa}_1 + \vec{\kappa}_2,\]
\[\vec{\rho} = \vec{\eta}_1 - \vec{\eta}_2, \quad \vec{\pi} = \frac{1}{2} (\vec{\kappa}_1 - \vec{\kappa}_2),\]
\[\vec{J} = \vec{\eta}_1 \times \vec{\kappa}_1 + \vec{\eta}_2 \times \vec{\kappa}_2 = \vec{\eta}_+ \times \vec{\kappa}_+ + \vec{S} = \vec{q}_+ \times \vec{\kappa}_+ + \vec{S}_q, \quad \vec{S} = \vec{\rho} \times \vec{\pi}, \quad \vec{S}_q = \vec{\rho}_q \times \vec{\pi}_q,\]
\[\vec{R}_+ = \frac{\sqrt{m_1^2 + \vec{\kappa}_1^2} \vec{\eta}_1 + \sqrt{m_2^2 + \vec{\kappa}_2^2} \vec{\eta}_2}{\sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 + \vec{\kappa}_2^2}} = \vec{\eta}_+ + \frac{1}{2} \frac{\sqrt{m_1^2 + \vec{\kappa}_1^2} - \sqrt{m_2^2 + \vec{\kappa}_2^2}}{\sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 + \vec{\kappa}_2^2}} \vec{\rho},\]
\[\vec{q}_+ = \vec{R}_+ + \frac{\vec{S}_q \times \vec{\kappa}_+}{(\sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 + \vec{\kappa}_2^2}) (\sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 + \vec{\kappa}_2^2}) + \sqrt{(\sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 + \vec{\kappa}_2^2})^2 - \vec{\kappa}_+^2}}.\]

Then after some straightforward algebra we get (note that in Ref. [2] we used the notation \(M_{\text{free}} = M_{\text{sys}}\) and \(M_{\text{free}}^2 = \Pi\))
\[
\begin{align*}
\sqrt{m_i^2 + \kappa_i^2} &= \frac{1}{2} \sqrt{M_{(\text{free})}^2 + \kappa_i^2} \left(1 + (-)^{i+1} \frac{m_i^2 - m_{i+1}^2}{M_{(\text{free})}^2}\right) + (-)^{i+1} \frac{\pi_i \cdot \kappa_i}{M_{(\text{free})}}, \\
M_{(\text{free})} &= \sqrt{m_1^2 + \kappa_1^2 + \sqrt{m_2^2 + \kappa_2^2}} = \sqrt{M_{(\text{free})}^2 + \kappa_2^2} \approx M_{(\text{free})} \overset{\text{def}}{=} \sqrt{m_1^2 + \pi_2^2 + \sqrt{m_2^2 + \pi_2^2}}, \\
\sqrt{m_1^2 + \pi_1^2} - \sqrt{m_2^2 + \pi_2^2} &= \frac{2 \pi_1 \cdot \pi_2}{M_{(\text{free})}} + \frac{m_1^2 - m_2^2}{M_{(\text{free})}^2} \sqrt{M_{(\text{free})}^2 + \kappa_2^2} \overset{\text{def}}{=} E, \\
\pi &= \pi_q + \frac{\kappa_i}{M_{(\text{free})} \sqrt{M_{(\text{free})}^2 + \kappa_i^2} \left[ \pi_i \cdot \kappa_i \left(1 - \left(\sqrt{M_{(\text{free})}^2 + \kappa_i^2} - M_{(\text{free})}\right) \frac{M_{(\text{free})}}{\kappa_i^2}\right)\right]} + \\
+ &\left(m_i^2 - m_{i+1}^2\right) \frac{M_{(\text{free})}^2 + \kappa_i^2}{M_{(\text{free})}^2 + \kappa_i^2} \overset{\text{def}}{=} \pi_q + F \kappa_i, \\
\rho &= \rho_q - \frac{A}{B} \frac{\kappa_i}{M_{(\text{free})} \sqrt{M_{(\text{free})}^2 + \kappa_i^2} \left[ \pi_i \cdot \kappa_i \left(1 - \left(\sqrt{M_{(\text{free})}^2 + \kappa_i^2} - M_{(\text{free})}\right) \frac{M_{(\text{free})}}{\kappa_i^2}\right)\right]} \overset{\text{def}}{=} \rho_q + C \pi_q, \\
A &= \frac{\sqrt{m_1^2 + \pi_1^2}}{\sqrt{m_2^2 + \pi_2^2}} + \frac{\sqrt{m_2^2 + \pi_2^2}}{\sqrt{m_1^2 + \pi_1^2}}, \\
B &= 1 + \frac{A \pi_i \cdot \kappa_i}{M_{(\text{free})} \sqrt{M_{(\text{free})}^2 + \kappa_i^2}}, \\
\bar{\pi}_i &= \pi_i - \frac{\kappa_i}{M_{(\text{free})} \sqrt{M_{(\text{free})}^2 + \kappa_i^2} \left[ \pi_i \cdot \kappa_i \left(1 - \left(\sqrt{M_{(\text{free})}^2 + \kappa_i^2} - M_{(\text{free})}\right) \frac{M_{(\text{free})}}{\kappa_i^2}\right)\right]} - \frac{\kappa_i \cdot \bar{\pi}_i}{\kappa_i^2} \left[M_{(\text{free})} - \sqrt{M_{(\text{free})}^2 - \kappa_i^2}\right] \overset{\text{def}}{=} \bar{\pi}_i - D \kappa_i, \\
\bar{\rho}_i &= \rho_i + \frac{A \kappa_i \cdot \bar{\rho}_i}{M_{(\text{free})} \sqrt{M_{(\text{free})}^2 - \kappa_i^2}} \bar{\pi}_i, \\
\bar{S}_q &= \bar{S} - D \bar{\rho} \times \bar{\kappa}_i, \\
\bar{q}_i &= \bar{R}_i + G \bar{S}_q \times \bar{\kappa}_i, \\
G &= \frac{1}{M_{(\text{free})} \left(M_{(\text{free})} + \sqrt{M_{(\text{free})}^2 - \kappa_i^2}\right)} = \frac{1}{\sqrt{M_{(\text{free})}^2 + \kappa_i^2} \left(\sqrt{M_{(\text{free})}^2 + \kappa_i^2} + M_{(\text{free})}\right)}, \\
\bar{\eta}_i &= \bar{q}_i - \frac{E}{2 \sqrt{M_{(\text{free})}^2 + \kappa_i^2}} \left[ \bar{\rho}_i + C \bar{\pi}_i\right] - G \bar{S}_q \times \bar{\kappa}_i, \\
\bar{\eta}_i &= \bar{q}_i + \frac{1}{2} \left(\left(-\right)^{i+1} - \frac{E}{\sqrt{M_{(\text{free})}^2 + \kappa_i^2}}\right) \left(\bar{\rho}_i + C \bar{\pi}_i\right) - G \bar{S}_q \times \bar{\kappa}_i, \\
\bar{\kappa}_i &= \left(\frac{1}{2} + \left(-\right)^{i+1} F\right) \bar{\kappa}_i + \left(-\right)^{i+1} \bar{\pi}_i, \\
\end{align*}
\]
(2.21)
2) For \( N > 2 \) the results concerning the coordinates (and also \( q_+ \)) are much more involved due to the complexity of Eqs. (C15) so that we give only the following results for the momenta

\[
M_{\text{(free)}} = \sum_i \sqrt{m_i^2 + \kappa_i^2} = \sqrt{M_{\text{(free)}^2}^2 + \kappa_+^2} \approx M_{\text{(free)}} = \sum_i \sqrt{m_i^2 + N \sum_{ab} \gamma_{ai} \gamma_{bi} \bar{\pi}_{qa} \cdot \bar{\pi}_{qb},}
\]

\[
\sqrt{m_i^2 + \kappa_i^2} = \frac{1}{M_{\text{(free)}}} \left( \sqrt{N} \sum_a \gamma_{ai} \bar{\kappa}_+ \cdot \bar{\pi}_{qa} + \sqrt{m_i^2 + N \sum_{ab} \gamma_{ai} \gamma_{bi} \bar{\pi}_{qa} \cdot \bar{\pi}_{qb} \sqrt{M_{\text{(free)}^2}^2 + \kappa_+^2}} \right),
\]

\[
\bar{\kappa}_i = \frac{\bar{\kappa}_+}{N} + \sqrt{N} \sum_a \gamma_{ai} \left[ \bar{\pi}_{qa} + \frac{M_{\text{(free)}}}{M_{\text{(free)}}^2 - 1} \frac{\bar{\kappa}_+ \cdot \bar{\pi}_{qa}}{\kappa_+^2} \bar{\kappa}_+ + \frac{\bar{\kappa}_+}{M_{\text{(free)}} \sqrt{N}} \sum_i \gamma_{ai} \sqrt{m_i^2 + N \sum_{ab} \gamma_{ai} \gamma_{bi} \bar{\pi}_{qa} \cdot \bar{\pi}_{qb}} \right].
\]

(2.22)
III. THE ENERGY-MOMENTUM TENSOR ON THE WIGNER HYPERPLANE.

A. The Euler-Lagrange Equations and the Energy-Momentum Tensor of Parametrized Minkowski Theories.

The Euler-Lagrange equations associated with the Lagrangian (2.1) are (the symbol \(\overset{\circ}{\cdot}\) means evaluated on the solutions of the equations of motion)

\[
(\frac{\partial L}{\partial x^\mu} - \frac{\partial}{\partial \tau} \frac{\partial L}{\partial \dot{x}_A^\mu})(\tau, \vec{\sigma}) = \eta_{\mu \nu} \partial_A [\sqrt{g} T^{AB} z^\nu_B](\tau, \vec{\sigma}) \overset{\circ}{=},
\]

\[
\frac{\partial L}{\partial \dot{\eta}_i^A} - \frac{\partial}{\partial \tau} \frac{\partial L}{\partial \ddot{\eta}_i^A} = -\left[ \frac{1}{2} \frac{\delta S}{\sqrt{g} \delta g_{AB}} \right]_{\vec{\sigma} = \vec{\eta}_i}(\tau, \vec{\sigma}) = -\sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i) \left( \frac{m_i \dot{\eta}_i^A(\tau) \dot{\eta}_i^B(\tau)}{\sqrt{g_{rr} + 2 g_{ru} \dot{\eta}_i^u + g_{uu} \dot{\eta}_i^u \dot{\eta}_i^u}} \right),
\]

where we have introduced the energy-momentum tensor [here \(\dot{\eta}_i^A(\tau) = (1; \dot{\eta}_i(\tau))\)]

\[
T^{AB}(\tau, \vec{\sigma}) = -\left[ \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{AB}} \right](\tau, \vec{\sigma}) = -\sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i) \left( \frac{m_i \dot{\eta}_i^A(\tau) \dot{\eta}_i^B(\tau)}{\sqrt{g_{rr} + 2 g_{ru} \dot{\eta}_i^u + g_{uu} \dot{\eta}_i^u \dot{\eta}_i^u}} \right).
\]

Because of the delta functions, the Euler-Lagrange equations for the fields \(z^\mu(\tau, \vec{\sigma})\) are trivial \((0 \overset{\circ}{=} 0)\) everywhere except at the positions of the particles. They may be rewritten in a form valid for every isolated system

\[
\partial_A T^{AB} z^\mu_B \overset{\circ}{=} -\frac{1}{\sqrt{g}} \partial_A [\sqrt{g} z^\mu_B] T^{AB}.
\]

When \(\partial_A [\sqrt{g} z^\mu_B] = 0\), as it happens on the Wigner hyper-planes in the gauge \(\vec{q}_+ \approx 0\) and \(T_s - \tau \approx 0\), we get the conservation of the energy-momentum tensor \(T^{AB}\), i.e. \(\partial_A T^{AB} \overset{\circ}{=} 0\). Otherwise, there is a compensation coming from the dynamics of the surface.

On the Wigner hyperplane, where we have

\[
x_i^\mu(\tau) = z_i^\mu(\tau, \vec{\eta}_i(\tau)) = x_i^\mu(\tau) + \epsilon_i^\mu(u(p_s)) \eta_i^\nu(\tau),
\]

\[
\dot{x}_i^\mu(\tau) = \frac{\partial x_i^\mu}{\partial \dot{\eta}_i^A}(\tau, \vec{\eta}_i(\tau)) + \dot{x}_i^\mu(\tau, \vec{\eta}_i(\tau)) \dot{\eta}_i^A(\tau) = \dot{x}_i^\mu(\tau) + \epsilon_i^\mu(u(p_s)) \dot{\eta}_i^A(\tau),
\]

\[
\ddot{x}_i^2(\tau) = g_{rr}(\tau, \vec{\eta}_i(\tau)) + 2 g_{ru}(\tau, \vec{\eta}_i(\tau)) \dot{\eta}_i^r(\tau) + g_{su}(\tau, \vec{\eta}_i(\tau)) \dot{\eta}_i^r(\tau) \dot{\eta}_i^u(\tau) = \dot{x}_i^2(\tau) + 2 \epsilon_i^r(u(p_s)) \dot{\eta}_i^r(\tau) - \dot{\eta}_i^2(\tau),
\]

\[
p_i^\mu(\tau) = \sqrt{m_i^2 - \gamma^{rs}(\tau, \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau) l^\mu(\tau, \vec{\eta}_i(\tau)) - \kappa_{ir}(\tau) \gamma^{rs}(\tau, \vec{\eta}_i(\tau)) z_i^\mu(\tau, \vec{\eta}_i(\tau)) =}
\]

\[
= \sqrt{m_i^2 + \kappa_i^2(\tau) u^\mu(p_s) + \epsilon_i^r(u(p_s)) \kappa_i^r(\tau) \Rightarrow p_i^2 = m_i^2},
\]

\[
p_s^\mu = \int d^3 \sigma \rho_\mu(\tau, \vec{\sigma}) \approx \sum_{i=1}^{N} p_i^\mu(\tau),
\]

(3.4)
the energy-momentum tensor \( T^{AB}(\tau, \vec{\sigma}) \) takes the form

\[
T^{\tau\tau}(\tau, \vec{\sigma}) = -\sum_{i=1}^{N} \frac{\delta^{3}(\vec{\sigma} - \vec{\eta}_{i}(\tau)) m_{i}}{\sqrt{\dot{x}_{a}^{2}(\tau) + 2\dot{x}_{s\mu}(\tau)\epsilon_{\mu}^{\nu}(u(p_{s})) - \dot{\eta}_{i}^{2}(\tau)}},
\]

\[
T^{\tau\sigma}(\tau, \vec{\sigma}) = -\sum_{i=1}^{N} \frac{\delta^{3}(\vec{\sigma} - \vec{\eta}_{i}(\tau)) m_{i}\dot{\eta}_{i}^{\nu}(\tau)}{\sqrt{\dot{x}_{a}^{2}(\tau) + 2\dot{x}_{s\mu}(\tau)\epsilon_{\mu}^{\nu}(u(p_{s})) - \dot{\eta}_{i}^{2}(\tau)}},
\]

\[
T^{\nu\sigma}(\tau, \vec{\sigma}) = -\sum_{i=1}^{N} \frac{\delta^{3}(\vec{\sigma} - \vec{\eta}_{i}(\tau)) m_{i}\dot{\eta}_{i}^{\nu}(\tau)\dot{\eta}_{i}^{\sigma}(\tau)}{\sqrt{\dot{x}_{a}^{2}(\tau) + 2\dot{x}_{s\mu}(\tau)\epsilon_{\mu}^{\nu}(u(p_{s})) - \dot{\eta}_{i}^{2}(\tau)}},
\]

(3.5)

B. The Energy-Momentum Tensor in the Standard Lorentz-Covariant Theory.

With the position \( x^{\mu} = z^{\mu}(\tau, \vec{\sigma}) \), the same form is obtained from the energy momentum tensor of the standard manifestly Lorentz covariant theory with Lagrangian \( S_{S} = \int dt L_{S}(\tau) = -\sum_{i=1}^{N} m_{i} \int d\tau \sqrt{\dot{x}_{i}^{2}(\tau)}, \) restricted to positive energies[51]

\[
T^{\mu\nu}(z(\tau, \vec{\sigma})) = -\left( \frac{2}{\sqrt{g}} \frac{\delta S_{S}}{\delta g_{\mu\nu}} \right)_{|z=x(\tau, \vec{\sigma})} = \sum_{i=1}^{N} m_{i} \int d\tau \frac{\dot{x}_{i}^{\mu}(\tau_{1})\dot{x}_{i}^{\nu}(\tau_{1})}{\sqrt{\dot{x}_{i}^{2}(\tau_{1})}} \delta^{4}(x_{i}(\tau_{1}) - z(\tau, \vec{\sigma})) = \epsilon_{A}^{\mu}(u(p_{s}))\epsilon_{B}^{\nu}(u(p_{s}))T^{AB}(\tau, \vec{\sigma}).
\]

(3.6)

1) On arbitrary space-like hyper-surfaces we have
\[ T^{\mu\nu}(z(\tau, \bar{\sigma})) = \sum_{i=1}^{N} m_i \int d\tau_{1} \frac{\dot{x}_{i}^{\mu}(\tau_{1})\dot{x}_{i}^{\nu}(\tau_{1})}{\sqrt{x_{i}^{2}(\tau_{1})}} \delta^{4}(x_{i}(\tau_{1}) - z(\tau, \bar{\sigma})) = \]

\[ = \sum_{i=1}^{N} m_i \int \frac{d\tau_{1}}{\sqrt{x_{i}^{2}(\tau_{1})}} \delta^{4}(z(\tau_{1}, \bar{\eta}(\tau_{1})) - z(\tau, \bar{\sigma})) \]

\[ [z_{\nu}^{\mu}(\tau_{1}, \bar{\eta}(\tau_{1})) + z_{\nu}^{\mu}(\tau_{1}, \bar{\eta}(\tau_{1}))\delta^{\mu}(\tau_{1})] = \]

\[ = \sum_{i=1}^{N} m_i \int \frac{d\tau_{1}}{\sqrt{x_{i}^{2}(\tau_{1})}} \delta^{4}(z(\tau_{1}, \bar{\eta}(\tau_{1})) - z(\tau, \bar{\sigma})) \]

\[ = \sum_{i=1}^{N} \frac{m_i}{\sqrt{g}(\tau)} \delta^{3}(\bar{\sigma} - \bar{\eta}(\tau)) (\det |z_{A}^{\mu}(\tau, \bar{\sigma})|)^{-1} \]

\[ [z_{\nu}^{\mu}(\tau, \bar{\eta}(\tau)) z_{\nu}^{\mu}(\tau, \bar{\eta}(\tau)) + (z_{\nu}^{\mu}(\tau, \bar{\eta}(\tau))z_{\nu}^{\mu}(\tau, \bar{\eta}(\tau)) + \]

\[ + z_{\nu}^{\mu}(\tau, \bar{\eta}(\tau))z_{\nu}^{\mu}(\tau, \bar{\eta}(\tau))] = \]

\[ = \sum_{i=1}^{N} \frac{m_i}{\sqrt{g}(\tau)} \delta^{3}(\bar{\sigma} - \bar{\eta}(\tau)) \]

\[ [z_{\nu}^{\mu}(\tau, \bar{\eta}(\tau)) z_{\nu}^{\mu}(\tau, \bar{\eta}(\tau)) + (z_{\nu}^{\mu}(\tau, \bar{\eta}(\tau))z_{\nu}^{\mu}(\tau, \bar{\eta}(\tau)) + \]

\[ + z_{\nu}^{\mu}(\tau, \bar{\eta}(\tau))z_{\nu}^{\mu}(\tau, \bar{\eta}(\tau))] = \]

\[ \text{since } \det |z_{A}^{\mu}| = \sqrt{g} = \sqrt{\gamma(g_{\tau\tau} - g_{\tau\tau}g_{\tau\tau}g_{\tau\tau})}, \gamma = |\det g_{\tau\tau}|. \]

2) On arbitrary space-like hyper-planes, where it holds[3]
\[ z^\mu(\tau, \bar{\sigma}) = x^\mu_s(\tau) + b^\mu_s(\tau)\sigma^\nu, \quad x^\mu_s(\tau) = x^\mu(\tau) + b^\mu_s(\tau)\eta^s_\nu(\tau), \]
\[ z^\mu_r(\tau, \bar{\sigma}) = b^\mu_r(\tau), \quad z^\mu_r(\tau, \bar{\sigma}) = \dot{x}^\mu_s(\tau) + \dot{b}^\mu_s(\tau)\sigma^\nu = l^\mu/\sqrt{g^{rr}} - g_{rr}z^\mu_r, \]
\[ g_{rr} = [\dot{x}^\mu_s(\tau) + \dot{b}^\mu_s(\sigma^\nu)]^2, \quad g_{rr} = b_{r\mu}[\dot{x}^\mu_s + \dot{b}^\mu_s\sigma^\nu], \]
\[ g_{rs} = -\delta_{rs}, \quad \gamma^{rs} = -\delta^{rs}, \quad \gamma = 1, \]
\[ g = g_{rr} + \sum_r g_{rr}^2, \]
\[ g^{rr} = 1/[l_{\mu}(\dot{x}^\mu_s + \dot{b}^\mu_s\sigma^\nu)]^2, \quad g^{rr}g_{rr} = b_{r\mu}(\dot{x}^\mu_s + \dot{b}^\mu_s(\sigma^\nu))/[l_{\mu}(\dot{x}^\mu_s + \dot{b}^\mu_s(\sigma^\nu))]^2 \]
\[ (3.8) \]
\[ \text{we have} \]
\[ T^{\mu\nu}[x^2_s(\tau) + b^2_s(\tau)\sigma^\nu] = \sum_{i=1}^{N} \frac{m_i\delta^3(\bar{\sigma} - \bar{\eta}_i(\tau))}{\sqrt{g(\tau, \bar{\sigma})}\sqrt{g_{rr}(\tau, \bar{\sigma}) + 2g_{rr}(\tau, \bar{\sigma})\bar{\eta}_i(\tau) - \bar{\eta}_i^2(\tau)}} \]
\[ \left[ (\dot{x}^\mu_s(\tau) + \dot{b}^\mu_s(\sigma^\nu))\eta^s_\nu(\tau) + \dot{b}^\nu_s(\tau)\eta^s_\nu(\tau) + (\dot{x}^\mu_s(\tau) + \dot{b}^\mu_s(\sigma^\nu))\eta^s_\nu(\tau) + \right] \bar{\eta}^r_i(\tau) + \]
\[ b^\nu_r(\tau)(\dot{b}^\nu_s(\tau)\eta^s_\nu(\tau)\bar{\eta}^r_i(\tau)) . \quad (3.9) \]

3) Finally, on Wigner hyper-planes, where it holds[3]

\[ z^\mu(\tau, \bar{\sigma}) = x^\mu_s(\tau) + e^\mu_s(u(p_s))\sigma^\nu, \quad x^\mu_s(\tau) = x^\mu(\tau) + e^\mu_s(u(p_s))\eta^s_\nu(\tau), \]
\[ z^\mu_r = e^\mu_r(u(p_s), \quad l^\mu = u^\mu(p_s), \quad x^\mu_r = \dot{x}^\mu_s(\tau), \]
\[ g = [\dot{x}_s(\tau)\cdot u(p_s)]^2, \quad g_{rr} = \dot{x}_s^2, \quad g_{rr} = \dot{x}_s^2, \quad g_{rs} = -\delta_{rs}, \]
\[ g^{rr} = 1/[\dot{x}_s(\tau)u^\mu(p_s)]^2, \quad g^{rr} = \dot{x}_s^2u^\mu(p_s)/[\dot{x}_s(\tau)u^\mu(p_s)]^2, \]
\[ g^{rs} = -\delta^{rs} + \dot{x}_s^2u^\mu(p_s)\dot{x}_s^2u^\mu(p_s)/[\dot{x}_s(\tau)u^\mu(p_s)]^2, \]
\[ ds^2 = \dot{x}_s(\tau)d\tau^2 + 2\dot{x}_s(\tau)\cdot u(p_s)d\tau d\sigma^r - d\sigma^2, \quad (3.10) \]
\[ \text{we have} \]
Since the volume element on the Wigner hyperplane is $u^\mu(p_s)d^3\sigma$, we obtain the following total 4-momentum and total mass of the N free particle system (Eqs.(2.7) are used)

$$L_{\mu} = \int d^3\sigma T^\mu_{\nu}[x^\beta_s(\tau) + e^\beta_u(u(p_s))\sigma^u]u_{\nu}(p_s) =$$

$$= \sum_{i=1}^{N} \frac{m_i}{\sqrt{\lambda^2(\tau) - [\eta_i(\tau) + \vec{\Lambda}(\tau)]^2}} \left[ -\lambda(\tau)u^\mu(p_s) + [\eta_i^r(\tau) + \vec{\Lambda}^r(\tau)]e^\mu_u(u(p_s)) \right]$$

$$= |_{\lambda(\tau) = -1} \sum_{i=1}^{N} \left[ \sqrt{m_i^2c^2 + \vec{\kappa}_i^2(\tau)u^\mu(p_s) + \kappa_i^r(\tau)e^\mu_u(u(p_s))} \right] = \sum_{i=1}^{N} p^\mu_i(\tau) = p^\mu_s,$$

$$M_{sys} = P^\mu_T u_\mu(p_s) = -\lambda(\tau) \sum_{i=1}^{N} \frac{m_i}{\sqrt{\lambda^2(\tau) - [\eta_i(\tau) + \vec{\Lambda}(\tau)]^2}}$$

$$= |_{\lambda(\tau) = -1} \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{\kappa}_i^2(\tau)},$$

which turn out to be in the correct form only if $\lambda(\tau) = -1$. This shows that the agreement with parametrized Minkowski theories on arbitrary space-like hyper-surfaces is obtained only on Wigner hyper-planes in the gauge $T_s - \tau \approx 0$, which indeed implies $\lambda(\tau) = -1$. 

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C. The Phase-Space Version of the Standard Energy-Momentum Tensor.

The same result may be obtained by first rewriting the standard energy-momentum
tensor in phase space and then imposing the restriction $p_i^\mu(\tau) =$

\[
\sqrt{m_i^2 - \gamma_i^s(\tau, \tilde{\eta}_i(\tau))\kappa_{ir}(\tau)\kappa_{is}(\tau)}u^\mu(\tau, \tilde{\eta}_i(\tau)) \rightarrow \sqrt{m_i^2 + \kappa_i^2(\tau)u^\mu(p_s) + \kappa_i^2\epsilon_r^\mu(u(p_s))} \quad [52] \quad [the \ last \ equality \ refers \ to \ the \ Wigner \ hyper-plane, \ see \ Eq.(2.14)]. \ We \ have:
\]

\[
T_{\mu\nu}(z(\tau, \bar{\sigma})) = \sum_{i=1}^{N} \frac{1}{m_i} \int d\tau_i \sqrt{\dot{x}_i^2(\tau_i)p_i^\mu(\tau_i)p_i^\nu(\tau_i)} \delta^4(x_i(\tau_i) - z(\tau, \bar{\sigma})) = \\
= \sum_{i=1}^{N} \frac{\sqrt{\dot{x}_i^2(\tau)}}{m_i \sqrt{\dot{x}_i \cdot u(p_s)}} p_i^\mu(\tau)p_i^\nu(\tau) \delta^3(\bar{\sigma} - \tilde{\eta}_i(\tau)),
\]

\[\downarrow \quad on \ Wigner'\ hyper \ - \ planes\]

\[
T_{\mu\nu}[x_s^\beta(\tau) + \epsilon_r^\mu(u(p_s))\sigma^\nu] = \sum_{i=1}^{N} \sqrt{\dot{x}_i^2 + 2\dot{x}_s \beta \epsilon_r^\beta(u(p_s))\tilde{\eta}_i^\mu - \tilde{\eta}_i^{\mu^2}(\tau)}p_i^\mu(\tau)p_i^\nu(\tau) \delta^3(\bar{\sigma} - \tilde{\eta}_i(\tau)) = \\
= \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \tilde{\eta}_i(\tau)) \sqrt{\dot{x}_i^2 + 2\dot{x}_s \beta \epsilon_r^\beta(u(p_s))\tilde{\eta}_i^\mu - \tilde{\eta}_i^{\mu^2}(\tau)}
\]

\[
\left( [m_i^2 + \kappa_i^2(\tau)]u^\mu(p_s)u^\nu(p_s) + \\
k_i^r(\tau) \sqrt{m_i^2 + \kappa_i^2(\tau)} \left( u^\mu(p_s)\epsilon_r^\mu(u(p_s)) + u^\nu(p_s)\epsilon_r^\nu(u(p_s)) \right) + \\
\kappa_i^r(\tau)\kappa_i^s(\tau)\epsilon_r^\mu(u(p_s))\epsilon_s^\nu(u(p_s)) \right) = \\
= \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \tilde{\eta}_i(\tau)) \sqrt{\lambda^2(\tau) - [\tilde{\eta}_i(\tau) + \tilde{\lambda}(\tau)]^2} \sqrt{-\lambda(\tau)}
\]

\[
\frac{\sqrt{m_i^2 + \kappa_i^2(\tau)}}{m_i} \left[ \sqrt{m_i^2 + \kappa_i^2(\tau)}u^\mu(p_s)u^\nu(p_s) + \\
k_i^r(\tau) \left( u^\mu(p_s)\epsilon_r^\mu(u(p_s)) + u^\nu(p_s)\epsilon_r^\nu(u(p_s)) \right) + \\
\kappa_i^r(\tau)\kappa_i^s(\tau)\epsilon_r^\mu(u(p_s))\epsilon_s^\nu(u(p_s)) \right].
\] (3.13)

The total 4-momentum and total mass are then
\[ P_T^\mu = \int d^3 \sigma T^{\mu \nu}[x^\beta_s(\tau) + \epsilon^\beta_u(u(p_s))\sigma^u]u_v(p_s) = \]
\[ = \sum_{i=1}^N \frac{\dot{x}_s^2 + 2\dot{x}_s\dot{\epsilon}^\beta_u(u(p_s))\dot{\eta}_i^\mu - \dot{\eta}_i^2}{\dot{x}_s \cdot u(p_s)} \sqrt{m_i^2 + \kappa_i^2(\tau)} \]
\[ \left[ \sqrt{m_i^2 + \kappa_i^2(\tau)}u^\mu(p_s) + \kappa_i^2(\tau)e^\mu_r(u(p_s)) \right] = \]
\[ = \sum_{i=1}^N \frac{\sqrt{\lambda^2(\tau) - [\dot{\eta}_i(\tau) + \bar{\lambda}(\tau)]^2} \sqrt{m_i^2 + \kappa_i^2(\tau)}}{\sqrt{-\lambda(\tau)} m_i} p_i^\mu(\tau), \]
\[ \sum_{i=1}^N \frac{\sqrt{\lambda^2(\tau) - [\dot{\eta}_i(\tau) + \bar{\lambda}(\tau)]^2} \sqrt{m_i^2 + \kappa_i^2(\tau)}}{\sqrt{-\lambda(\tau)} m_i} \sqrt{m_i^2 + \kappa_i^2(\tau)}. \quad (3.14) \]

Since in this case we have \( m_i/\sqrt{1 - [\dot{\eta}_i(\tau) + \bar{\lambda}(\tau)]^2} \equiv \sqrt{m_i^2 c^2 + \kappa_i^2(\tau)} \), the equations above show that the total 4-momentum evaluated from the energy-momentum tensor of the standard theory restricted to positive energy particles is consistent \cite{53} with the description on the Wigner hyper-planes with its gauge freedom \( \lambda(\tau), \bar{\lambda}(\tau) \), provided one works with Dirac brackets of the gauge \( T_s \equiv \tau \), where one has \( \lambda(\tau) = -1 \) and

\[ \dot{x}_s^\mu(\tau) = u^\mu(p_s) + \epsilon^\mu_r(u(p_s))\lambda_r(\tau), \]
\[ x_s^\mu(\tau) = x_s^\mu(0) + \tau u^\mu(p_s) + \epsilon^\mu_r(u(p_s)) \int_0^\tau d\tau_1 \lambda_r(\tau_1), \quad (3.15) \]

Therefore, for every \( \bar{\lambda}(\tau) \), we have
\[ T^{\mu\nu}[x^\beta_s(T_s) + \epsilon^\beta_u(u(p_s))\sigma^u] = \epsilon^\mu_A(u(p_s))\epsilon^\nu_B(u(p_s))T^{AB}(T_s, \bar{\sigma}) = \]
\[ = \sum_{i=1}^{N} \delta^3(\bar{\bar{\sigma}} - \bar{\eta}_i(T_s)) \left[ \sqrt{m_i^2 + \kappa_i^2(T_s)}u^\mu(p_s)u^\nu(p_s) + \kappa_i^\tau(T_s)u^\mu(p_s)\epsilon^\nu_r(u(p_s)) + \frac{\kappa_i^\tau(T_s)\epsilon^\mu_r(u(p_s))\epsilon^\nu_s(u(p_s))}{\sqrt{m_i^2 + \kappa_i^2(T_s)}} \right], \]
\[ T^{\tau\tau}(T_s, \bar{\sigma}) = \sum_{i=1}^{N} \delta^3(\bar{\bar{\sigma}} - \bar{\eta}_i(T_s)) \sqrt{m_i^2 + \kappa_i^2(T_s)}, \]
\[ T^{\tau\bar{\tau}}(T_s, \bar{\sigma}) = \sum_{i=1}^{N} \delta^3(\bar{\bar{\sigma}} - \bar{\eta}_i(T_s))\kappa_i^\tau(T_s) , \]
\[ T^{\tau s}(T_s, \bar{\sigma}) = \sum_{i=1}^{N} \delta^3(\bar{\bar{\sigma}} - \bar{\eta}_i(T_s)) \frac{\kappa_i^\tau(T_s)\epsilon^\mu_r(u(p_s))}{\sqrt{m_i^2 + \kappa_i^2(T_s)}}. \]
\[ P^\mu_T = p^\mu_s = Mu^\mu(p_s) + \epsilon^\mu_r(u(p_s))\kappa_r^+ \approx Mu^\mu(p_s), \]
\[ M = \sum_{i=1}^{N} \sqrt{m_i^2c^2 + \kappa_i^2(T_s)}, \]
\[ T^{\mu\nu}[x^\beta_s(T_s) + \epsilon^\beta_u(u(p_s))\sigma^u] u_\nu(p_s) = \epsilon^\mu_A(u(p_s))T^{A\bar{\tau}}(T_s, \bar{\sigma}) = \]
\[ = \sum_{i=1}^{N} \delta^3(\bar{\bar{\sigma}} - \bar{\eta}_i(T_s)) \left[ \sqrt{m_i^2c^2 + \kappa_i^2(T_s)}u^\mu(p_s) + \kappa_i^\tau(T_s)\epsilon^\mu_r(u(p_s)) \right], \]
\[ T^\mu_\mu[x^\beta_s(T_s) + \epsilon^\beta_u(u(p_s))\sigma^u] = T^{A\bar{\tau}}(T_s, \bar{\sigma}) = \]
\[ = \sum_{i=1}^{N} \delta^3(\bar{\bar{\sigma}} - \bar{\eta}_i(T_s)) \frac{m_i^2}{\sqrt{m_i^2 + \kappa_i^2(T_s)}}, \quad (3.16) \]
IV. DIXON’S MULTIPOLES FOR FREE PARTICLES ON THE WIGNER HYPERPLANE.

In this Section we shall define the special relativistic Dixon multipoles on the Wigner hyperplane with \( T_s - \tau \equiv 0 \) for the N-body problem [54] [see Eqs.(2.13) with \( x_\mu^s(\tau) = x_\mu^o + u^\mu(p_s) \int_\tau^o d\tau_1 \lambda_r(\tau_1) = x_\mu^{(\vec{q}+)}(\tau) + \int_\tau^o d\tau_1 \lambda_r(\tau_1) \)]. By comparison, a list of the non-relativistic multipoles for N free particles is given in Appendix A.

Consider an arbitrary time-like world-line \( w_\mu(\tau) = z_\mu(\tau, \vec{\eta}(\tau)) = x_\mu^s(\tau) + \epsilon_\mu^s(u(p_s)) \eta^r(\tau) = x_\mu^{(\vec{q}+)}(\tau) + \epsilon_\mu^s(u(p_s)) \tilde{\eta}^r(\tau) \) [\( \tilde{\eta}^r(\tau) = \eta^r(\tau) + \int_\tau^o d\tau_1 \lambda_r(\tau_1) \)] and evaluate the Dixon multipoles [25] [61] on the Wigner hyper-planes in the natural gauge with respect to the given world-line. A generic point will be parametrized by

\[
\begin{align*}
z^\mu(\tau, \vec{\sigma}) &= x^\mu(\tau) + \epsilon^\mu(\tau, \vec{\sigma}) = \\
&= x_\mu^{(\vec{q}+)}(\tau) + \epsilon_\mu^s(u(p_s)) \left[ \sigma^r + \int_\tau^o d\tau_1 \lambda_r(\tau_1) \right] = \\
&= w^\mu(\tau) + \epsilon_\mu^s(u(p_s)) \left[ \sigma^r - \tilde{\eta}^r(\tau) \right] \overset{\text{def}}{=} w^\mu(\tau) + \delta z^\mu(\tau, \vec{\sigma}),
\end{align*}
\]

so that \( \delta z^\mu(\tau, \vec{\sigma}) w^\mu(p_s) = 0 \).

While for \( \tilde{\eta}(\tau) = 0 \) [\( \tilde{\eta}(\tau) = \int_\tau^o d\tau_1 \lambda_r(\tau_1) \)] we get the multipoles relative to the centroid \( x^\mu(\tau) \), for \( \eta(\tau) = 0 \) we get those relative to the centroid \( x^{(\vec{q}+)}(\tau) \). In the gauge \( \vec{R}_+ \approx \vec{q}_+ \approx \vec{y}_+ \approx 0 \), where \( \vec{L}(\tau) = 0 \), it follows that \( \tilde{\eta}(\tau) = \tilde{\eta}(\tau) = 0 \) identifies the barycentric multipoles with respect to the centroid \( x^{(\vec{q}+)}(\tau) \), which now carries the internal 3-center of mass.

A. Dixon’s Multipoles.

Lorentz covariant Dixon’s multipoles and their Wigner covariant counterparts on the Wigner hyper-planes are then defined as
\[ t_{\mu_1 \cdots \mu_n \nu}^{T \mu_1 \cdots \mu_n \nu}(T_s, \vec{\eta}) = t_{T}^{(\mu_1 \cdots \mu_n)(\mu \nu)}(T_s, \vec{\eta}) = \]
\[ = \epsilon_{\tau_1}^{\mu_1}(u(p_s)) \cdots \epsilon_{\tau_n}^{\mu_n}(u(p_s)) \epsilon_{A}^{\nu}(u(p_s)) \epsilon_{\nu}^{\mu}(u(p_s)) q_{T}^{\tau_1 \cdots \tau_n AB}(T_s, \vec{\eta}) = \]
\[ = \int d^3 \sigma \delta z^{\mu_1}(T_s, \vec{\sigma}) \cdots \delta z^{\mu_n}(T_s, \vec{\sigma}) T^{\mu\nu}[x_{s}(\vec{q}_{s})^{\nu}(T_s) + \epsilon_{u}^{\nu}(u(p_s)) \sigma^{u}] = \]
\[ = \epsilon_{A}(u(p_s)) \epsilon_{\nu}^{\mu}(u(p_s)) \int d^3 \sigma \delta z^{\mu_1}(T_s, \vec{\sigma}) \cdots \delta z^{\mu_n}(T_s, \vec{\sigma}) T^{AB}(T_s, \vec{\sigma}) = \]
\[ = \epsilon_{\tau_1}^{\mu_1}(u(p_s)) \cdots \epsilon_{\tau_n}^{\mu_n}(u(p_s)) \]
\[ \left[ u^{\mu}(p_s) u^{\nu}(p_s) \sum_{i=1}^{N} [\eta^{\gamma_1}_i(T_s) - \eta^{\gamma_1}(T_s)] [\eta^{\gamma_n}_i(T_s) - \eta^{\gamma_n}(T_s)] \sqrt{m_{\tau}^2 + \kappa_{\tau}^2(T_s)} + \right. \]
\[ + \left. \epsilon_{\nu}^{\mu}(u(p_s)) \epsilon_{\nu}^{\nu}(u(p_s)) \sum_{i=1}^{N} [\eta^{\gamma_1}_i(T_s) - \eta^{\gamma_1}(T_s)] [\eta^{\gamma_n}_i(T_s) - \eta^{\gamma_n}(T_s)] \kappa_{\tau}^n(T_s) \kappa_{\tau}^n(T_s) \right] \]
\[ q_{T}^{\tau_1 \cdots \tau_n AB}(T_s, \vec{\eta}) = \int d^3 \sigma [\sigma^{\gamma_1} - \eta^{\gamma_1}(T_s)] [\cdots [\sigma^{\gamma_n} - \eta^{\gamma_n}(T_s)] T^{AB}(T_s, \vec{\sigma}) = \]
\[ = \delta^{A}_{\gamma} \delta^{B}_{\tau} \sum_{i=1}^{N} [\eta^{\gamma_1}_i(T_s) - \eta^{\gamma_1}(T_s)] [\eta^{\gamma_n}_i(T_s) - \eta^{\gamma_n}(T_s)] \sqrt{m_{\tau}^2 + \kappa_{\tau}^2(T_s)} + \]
\[ + \delta^{A}_{\gamma} \delta^{B}_{\tau} \sum_{i=1}^{N} [\eta^{\gamma_1}_i(T_s) - \eta^{\gamma_1}(T_s)] [\eta^{\gamma_n}_i(T_s) - \eta^{\gamma_n}(T_s)] \kappa_{\tau}^n(T_s) \kappa_{\tau}^n(T_s) + \]
\[ + (\delta^{A}_{\gamma} \delta^{B}_{\tau} + \delta^{A}_{\tau} \delta^{B}_{\tau}) \sum_{i=1}^{N} [\eta^{\gamma_1}_i(T_s) - \eta^{\gamma_1}(T_s)] [\eta^{\gamma_n}_i(T_s) - \eta^{\gamma_n}(T_s)] \kappa_{\tau}^n(T_s), \]
\[ u_{\mu_1}(p_s) \quad t_{T}^{\mu_1 \cdots \mu_n \nu}(T_s, \vec{\eta}) = 0, \]
\[ t_{T}^{\mu_1 \cdots \mu_n \mu}(T_s, \vec{\eta}) \overset{\text{def}}{=} \epsilon_{\tau_1}^{\mu_1}(u(p_s)) \cdots \epsilon_{\tau_n}^{\mu_n}(u(p_s)) q_{T}^{\tau_1 \cdots \tau_n A}(T_s, \vec{\eta}) = \]
\[ = \int d^3 \sigma \delta z^{\mu_1}(\tau, \vec{\sigma}) \cdots \delta z^{\mu_n}(\tau, \vec{\sigma}) T^{\mu\nu}[x_{s}(\vec{q}_{s})^{\nu}(T_s) + \epsilon_{u}^{\nu}(u(p_s)) \sigma^{u}] = \]
\[ = \epsilon_{\tau_1}^{\mu_1}(u(p_s)) \cdots \epsilon_{\tau_n}^{\mu_n}(u(p_s)) \]
\[ \sum_{i=1}^{N} [\eta^{\gamma_1}_i(T_s) - \eta^{\gamma_1}(T_s)] [\eta^{\gamma_n}_i(T_s) - \eta^{\gamma_n}(T_s)] \frac{m_{\tau}^2}{\sqrt{m_{\tau}^2 + \kappa_{\tau}^2(T_s)}} = \]
\[ \bar{t}_{T}^{\mu_1 \cdots \mu_n}(T_s, \vec{\eta}) = t_{T}^{\mu_1 \cdots \mu_n}(T_s, \vec{\eta}) u_{\nu}(p_s) = \]
\[ = \epsilon_{\tau_1}^{\mu_1}(u(p_s)) \cdots \epsilon_{\tau_n}^{\mu_n}(u(p_s)) q_{T}^{\tau_1 \cdots \tau_n \tau \tau}(T_s, \vec{\eta}) = \]
\[ = \epsilon_{\tau_1}^{\mu_1}(u(p_s)) \cdots \epsilon_{\tau_n}^{\mu_n}(u(p_s)) \]
Related multipoles are

\[ p_T^{\mu_1 \ldots \mu_n}(T_s, \vec{\eta}) = t_T^{\mu_1 \ldots \mu_n \nu}(T_s, \vec{\eta}) u_\nu(p_s) = \]

\[ = \epsilon^{\mu_1}(u(p_s)) \ldots \epsilon^{\mu_n}(u(p_s)) \epsilon^\nu_A(u(p_s)) q_T^{r_1 \ldots r_n \nu}(T_s, \vec{\eta}) = \]

\[ = \epsilon^{\mu_1}(u(p_s)) \ldots \epsilon^{\mu_n}(u(p_s)) \]

\[ \sum_{j=1}^N [\eta_i^{r_1}(T_s) - \eta_i^{r_1}(T_s)] \ldots [\eta_i^{r_n}(T_s) - \eta_i^{r_n}(T_s)] \]

\[ \left[ \sqrt{m^2_i + \tilde{k}_i^2(\tau)} u^\mu(p_s) + \tilde{k}_i^\nu(T_s) \epsilon^\nu(u(p_s)) \right], \]

\[ u_{\mu_1}(p_s) p_T^{\mu_1 \ldots \mu_n}(T_s, \vec{\eta}) = 0, \]

\[ p_T^{\mu_1 \ldots \mu_n}(T_s, \vec{\eta}) u_\mu(p_s) = \tilde{p}_T^{\mu_1 \ldots \mu_n}(T_s, \vec{\eta}), \]

\[ n = 0 \Rightarrow p_T^\mu(T_s, \vec{\eta}) = \epsilon^\mu_A(u(p_s)) q_T^{A\tau}(T_s) = P_T^\mu \approx p_s^\mu. \quad (4.3) \]

The inverse formulas, giving the multipolar expansion, are

\[ T^{\mu\nu}[\omega^\beta(T_s) + \delta z^\beta(T_s, \vec{\sigma})] = T^{\mu\nu}[x_s^\nu(x_s^\nu)] + \epsilon^\beta(u(p_s)) \sigma^r = \]

\[ = \epsilon^\nu_A(u(p_s)) \epsilon^\nu_B(u(p_s)) T^{AB}(T_s, \vec{\sigma}) = \]

\[ = \epsilon^\nu_A(u(p_s)) \epsilon^\nu_B(u(p_s)) \sum_{n=0}^{\infty} (-1)^n q_T^{r_1 \ldots r_n AB}(T_s, \vec{\eta}) \]

\[ \left[ \frac{\partial^n}{\partial \sigma_{r_1} \ldots \partial \sigma_{r_n}} \delta^3(\vec{\sigma} - \vec{\eta}(T_s)) \right] = \]

\[ = \sum_{n=0}^{\infty} (-1)^n \frac{t_T^{r_1 \ldots \mu_n \nu}(T_s, \vec{\eta})}{n!} \epsilon_{r_1 \mu_1}(u(p_s)) \ldots \epsilon_{r_n \mu_n}(u(p_s)) \]

\[ \left[ \frac{\partial^n}{\partial \sigma_{r_1} \ldots \partial \sigma_{r_n}} \delta^3(\vec{\sigma} - \vec{\eta}(T_s)) \right]. \quad (4.4) \]

Note however that, as pointed out by Dixon [25], the distributional equation (4.4) is valid only if analytic test functions are used, defined on the support of the energy-momentum tensor.

The quantities \( q_T^{r_1 \ldots r_n RT}(T_s, \vec{\eta}) \), \( q_T^{r_1 \ldots r_n RT}(T_s, \vec{\eta}) \), \( q_T^{r_1 \ldots r_n RT}(T_s, \vec{\eta}) \), \( q_T^{r_1 \ldots r_n uu}(T_s, \vec{\eta}) \) are the \textit{mass density}, \textit{momentum density} and \textit{stress tensor multipoles} with respect to the world-line \( u^\mu(T_s) \) (barycentric for \( \vec{\eta} = \vec{\eta} = 0 \)).
B. Monopoles.

The monopoles correspond to \( n = 0 \) [64] and have the following expression [65] (see Appendix C for the definition of \( \rightarrow_{\alpha \rightarrow \infty} \))

\[
q_{T}^{AB}(T_s, \vec{\eta}) = \delta_{\tau}^{A} \delta_{\tau}^{B} M + \delta_{u}^{A} \delta_{u}^{B} \sum_{i=1}^{N} \frac{\kappa_{u}^{\mu} \kappa_{i}^{\nu}}{\sqrt{m_{i}^{2} + \kappa_{i}^{2}}} + (\delta_{\tau}^{A} \delta_{\tau}^{B} + \delta_{u}^{A} \delta_{u}^{B}) \kappa_{u}^{\nu} \approx \\
\rightarrow_{\alpha \rightarrow \infty} \delta_{\tau}^{A} \delta_{\tau}^{B} \sum_{i=1}^{N} \sqrt{m_{i}^{2} + N \sum_{de} \gamma_{di} \gamma_{ei} \vec{\pi}_{qd} \cdot \vec{\pi}_{qe}} + \\
+ \delta_{u}^{A} \delta_{u}^{B} \sum_{i=1}^{N} \frac{\sum_{1..N-1}^{N} \gamma_{ai} \gamma_{bi} \vec{\pi}_{qa} \cdot \vec{\pi}_{qb}}{\sqrt{m_{i}^{2} + N \sum_{de} \gamma_{di} \gamma_{ei} \vec{\pi}_{qd} \cdot \vec{\pi}_{qe}}}.
\]

\[
q_{T}^{\tau\tau}(T_s, \vec{\eta}) \rightarrow_{c \rightarrow \infty} \sum_{i=1}^{N} m_{i} c^{2} + \frac{1}{2} \sum_{ab} \sum_{i=1}^{N} \frac{N \gamma_{ai} \gamma_{bi} \vec{\pi}_{qa} \cdot \vec{\pi}_{qb}}{m_{i}} + O(1/c) = \\
= \sum_{i=1}^{N} m_{i} c^{2} + H_{rel,nr} + O(1/c),
\]

\[
q_{T}^{r\tau}(T_s, \vec{\eta}) = \kappa_{+}^{\nu} \approx 0, \quad \text{rest - frame condition (also at the non - relativistic level)},
\]

\[
q_{T}^{w}(T_s, \vec{\eta}) \rightarrow_{c \rightarrow \infty} \sum_{ab} \sum_{i=1}^{N} \frac{N \gamma_{ai} \gamma_{bi} \vec{\pi}_{qa}^{\nu} \vec{\pi}_{qb}^{\nu} + O(1/c)}{m_{i}} = \\
= \sum_{ab} k_{ab}^{-1} \vec{\pi}_{qa}^{\nu} \vec{\pi}_{qb}^{\nu} + O(1/c) = \sum_{ab} k_{ab} \hat{\rho}_{a}^{\nu} \hat{\rho}_{b}^{\nu} + O(1/c),
\]

\[
q_{T}^{A}(T_s, \vec{\eta}) = \mu_{\mu}^{A}(T_s, \vec{\eta}) = \sum_{i=1}^{N} \frac{m_{i}^{2}}{\sqrt{m_{i}^{2} + \kappa_{i}^{2}}} \\
\rightarrow_{\alpha \rightarrow \infty} \sum_{i=1}^{N} \sqrt{m_{i}^{2} + N \sum_{de} \gamma_{di} \gamma_{ei} \vec{\pi}_{qd} \cdot \vec{\pi}_{qe}} \\
\rightarrow_{c \rightarrow \infty} \sum_{i=1}^{N} m_{i} c^{2} - \frac{1}{2} \sum_{ab} \sum_{i=1}^{N} \frac{N \gamma_{ai} \gamma_{bi} \vec{\pi}_{qa} \cdot \vec{\pi}_{qb}}{m_{i}} + O(1/c) = \\
= \sum_{i=1}^{N} m_{i} c^{2} - H_{rel,nr} + O(1/c).
\]

where we have exploited Eqs. (5.10), (5.11) of Ref.[2] to obtain the expression in terms of the internal relative variables.

Therefore, independently of the choice of the world-line \( w^{\mu}(\tau) \), in the rest-frame instant form the mass monopole \( q_{T}^{\tau\tau} \) is the invariant mass \( M = \sum_{i=1}^{N} \frac{m_{i}^{2}}{\sqrt{m_{i}^{2} + \kappa_{i}^{2}}} \), while the momentum monopole \( q_{T}^{r\tau} \) vanishes and \( q_{T}^{w} \) is the stress tensor monopole.
C. Dipoles.

The mass, momentum and stress tensor dipoles correspond to \( n = 1 \) [66]

\[
q_{TB}^{AB}(T_s, \eta) = \delta^A_r \delta^B_r M[R_+^r(T_s) - \eta^r(T_s)] + \delta^A_u \delta^B_v \left[ \sum_{i=1}^{N} \frac{\eta^i_r \kappa^u_i \kappa^v_i}{\sqrt{m^2_i + \kappa^2_i}}(T_s) - \eta^r(T_s)q_{uv}^{uv}(T_s, \eta) \right] +
\]

\[
+ \left( \delta^A_u \delta^B_u + \delta^A_u \delta^B_r \right) \left[ \sum_{i=1}^{N} [\eta^i_r \kappa^u_i](T_s) - \eta^r(T_s)\kappa^u_+ \right],
\]

\[
q_{TA}^{RA}(T_s, \eta) = \varepsilon^r_{\mu_1}(u(p_s)) t^{\mu_1 \mu}(T_s, \eta) = \sum_{i=1}^{N} \frac{[\eta^i_r - \eta^r]m^2_i}{\sqrt{m^2_i + \kappa^2_i}}(T_s),
\]

(4.6)

The vanishing of the mass dipole \( q_{TT}^{TT} \) implies \( \tilde{\eta}(\tau) = \tilde{\eta}(\tau) - \int_{0}^{\tau} dt_1 \tilde{\lambda}(t_1) = \tilde{R}_+ + \tilde{\eta}(\tau) \) and identifies the world-line \( w^\mu(\tau) = x^{\bar{q}_+}(\tau) + \varepsilon^r_{\mu_1}(u(p_s)) [R_+^r + \int_{0}^{\tau} dt_1 \lambda_+(t_1)] \). In the gauge \( \tilde{R}_+ \approx \bar{q}_+ \approx \bar{y}_+ \approx 0 \), where \( \tilde{\lambda}(\tau) = 0 \), this is the world-line \( w^\mu(\tau) = x^{\bar{q}_+}(\tau) \) of the centroid associated with the internal Møller 3-center of energy and, as a consequence of the rest frame condition, also with the rest-frame internal 3-center of mass \( \bar{q}_+ \). Therefore we have the implications following from the vanishing of the barycentric (i.e. \( \tilde{\lambda}(\tau) = 0 \)) mass dipole

\[
q_{TT}^{TT}(T_s, \eta) = \varepsilon^r_{\mu_1}(u(p_s)) \bar{t}_{TT}^{\mu_1}(T_s, \eta) = M \left[ R_+^r(T_s) - \eta^r(T_s) \right] = 0, \quad \text{and} \quad \tilde{\lambda}(\tau) = 0,
\]

\[
\Rightarrow \tilde{\eta}(T_s) = \tilde{\eta}(T_s) = \tilde{R}_+ \approx \bar{q}_+ \approx \bar{y}_+.
\]

(4.7)

In the gauge \( \tilde{R}_+ \approx \bar{q}_+ \approx \bar{y}_+ \approx 0 \), Eq.(4.7) with \( \tilde{\eta} = \tilde{\eta} = 0 \) implies the vanishing of the time derivative of the barycentric mass dipole: this identifies the center-of-mass momentum-velocity relation (or constitutive equation) for the system

\[
\frac{dq_{TT}^{TT}(T_s, \eta)}{dT_s} \cong \kappa^r_+ - M \dot{R}_+^r = 0.
\]

(4.8)

The expression of the barycentric dipoles in terms of the internal relative variables, when \( \tilde{\eta} = \tilde{\eta} = \tilde{R}_+ \approx \bar{q}_+ \approx 0 \) and \( \kappa_+ \approx 0 \), is obtained by using the results of Appendix C.
\[ q_T^{rr}(T_s, \vec{R}_+) = 0, \]

\[ q_T^{ru}(T_s, \vec{R}_+) = \sum_{i=1}^{N} \eta_i^r \kappa_i^u - R^r_+ \kappa_+^u = \sum_{a=1}^{N-1} \rho_a^r \pi_a^u + (\eta_+^r - R^r_+) \kappa_+^u \]

\[ \rightarrow_{a \rightarrow \infty} \sum_{a=1}^{N-1} \rho_a^r \pi_a^u \]

\[ \rightarrow_{c \rightarrow \infty} \sum_{a=1}^{N-1} \rho_a^r \pi_a^u = \sum_{ab} k_{ab} \rho_a^r \rho_b^u, \]

\[ q_T^{ru}(T_s, \vec{R}_+) = \sum_{i=1}^{N} \eta_i^r \kappa_i^u - R^r_+ \sum_{i=1}^{N} \kappa_i^u \kappa_i^u = \]

\[ = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sum_{a=1}^{N-1} \gamma_{ia} \rho_a^r \kappa_i^u \kappa_i^v \kappa_i^v \kappa_i^v \frac{m_j^2 + N \sum_{de} \gamma_{de} \gamma_{de} \pi_{d} \pi_{d}}{m_i^2 + N \sum_{de} \gamma_{de} \gamma_{de} \pi_{d} \pi_{d}} \times \]

\[ \sum_{k=1}^{N} \frac{\sum_{bc} \gamma_{bc} \pi_{b} \pi_{c}}{m_k^2 + N \sum_{de} \gamma_{de} \pi_{d} \pi_{d}} \rho_{qa} \]

\[ \rightarrow_{c \rightarrow \infty} \sum_{ij} \frac{\gamma_{ai} - \gamma_{aj}}{\sqrt{N}} \rho_a^r \frac{m_j N}{m_i m} \sum_{bc} \gamma_{bc} \pi_{b} \pi_{c} + O(1/c) = \]

\[ = \frac{1}{\sqrt{N}} \sum_{abc} \left[ N \sum_{i=1}^{N} \gamma_{ai} \gamma_{ai} \gamma_{ci} \gamma_{ci} \frac{m_j^2 \gamma_{aj}}{m_i} - \sum_{j=1}^{N} \frac{m_j \gamma_{aj}}{m} \right] \rho_a^r \pi_{qb} \pi_{qc} + O(1/c), \]

\[ q_T^{ru}(T_s, \vec{R}_+) = \sum_{i=1}^{N} (\eta_i^r - R^r_+) \frac{m_i^2}{H_i} = \]

\[ \rightarrow_{a \rightarrow \infty} \sum_{a=1}^{N-1} \left( \frac{1}{\sqrt{N}} \sum_{ij} (\gamma_{ai} - \gamma_{aj}) \frac{m_i^2 + N \sum_{de} \gamma_{de} \gamma_{de} \pi_{d} \pi_{d}}{m_j^2 + N \sum_{de} \gamma_{de} \gamma_{de} \pi_{d} \pi_{d}} \times \right. \]

\[ \left. \sum_{k=1}^{N} \sqrt{m_k^2 + N \sum_{de} \gamma_{de} \pi_{d} \pi_{d}} \rho_a^r \right) \]

\[ \rightarrow_{c \rightarrow \infty} \sum_{ij} \sqrt{N} (\gamma_{ai} - \gamma_{aj}) \rho_a^r \frac{m_j N}{m_i m} \sum_{bc} \gamma_{bc} \pi_{b} \pi_{c} + O(1/c) = \]

\[ = \sqrt{N} \sum_{abc} \left[ N \sum_{i=1}^{N} \gamma_{ai} \gamma_{ai} \gamma_{ci} \gamma_{ci} \frac{m_j^2 \gamma_{aj}}{m_i} - \sum_{j=1}^{N} \frac{m_j \gamma_{aj}}{m} \right] \rho_a^r \pi_{qb} \pi_{qc} + O(1/c) \tag{4.9} \]

The antisymmetric part of the related dipole \( p_T^{1+1}(T_s, \vec{n}) \) identifies the spin tensor. Indeed, the spin dipole [67] is
\[ S_{\mu\nu}^{\mu}(T_s)[\vec{\eta}] = 2p_{\mu}^{[\mu]}(T_s, \vec{\eta}) = 2\epsilon_{\tau}[u(p_s))] e_{\lambda}^\rho(u(p_s)) q_{TR}^{\lambda\rho}(T_s, \vec{\eta}) = M [R_s^\tau(T_s) - \eta^\tau(T_s)] [\epsilon_{\tau}^\rho(u(p_s))u^\nu(p_s) - \epsilon_{\rho}^\nu(u(p_s))u^\mu(p_s)] + \sum_{i=1}^N [\eta_i^\tau(T_s) - \eta^\tau(T_s)] [\epsilon_{\rho}^\nu(u(p_s))] \epsilon_{\lambda}^\sigma(u(p_s)) - \epsilon_{\rho}^\sigma(u(p_s))\epsilon_{\lambda}^\nu(u(p_s))]\]

\[ m_{\mu}^\nu(T_s, \vec{\eta}) = u_\mu(p_s)S_{\mu\nu}^{\nu}(T_s)[\vec{\eta}] = -\epsilon_{\tau}^\nu(u(p_s))[\bar{S}^{\tau\tau} - M\eta^\tau(T_s)] = -\epsilon_{\tau}^\nu(u(p_s))M[R_s^\tau(T_s) - \eta^\tau(T_s)] = -\epsilon_{\tau}^\nu(u(p_s))q_{TR}^{\tau\tau}(T_s, \vec{\eta}), \]

\[ \Rightarrow u_\mu(p_s)S_{\mu\nu}^{\nu}(T_s)[\vec{\eta}] = 0, \quad \Rightarrow \vec{\eta} = \vec{R}_+, \]

\[ \downarrow \quad \text{barycentric spin for } \vec{\eta} = \vec{\eta} = 0, \text{ see Eq}(2.9), \]

\[ S_{\mu\nu}^{\nu}(T_s)[\vec{\eta}] = 0 = S_{\mu}^{\nu} \circ \]

\[ \circ \sum_{i=1}^N \frac{m_i \eta_i^\tau(T_s)}{1 - \bar{\eta}_i^2(T_s)} [\epsilon_{\tau}^\rho(u(p_s))u^\nu(p_s) - \epsilon_{\rho}^\nu(u(p_s))u^\mu(p_s)] + \]

\[ + \sum_{i=1}^N \frac{m_i \eta_i^\tau(T_s) \bar{\eta}_i^\tau(T_s)}{1 - \bar{\eta}_i^2(T_s)} [\epsilon_{\tau}^\rho(u(p_s))] \epsilon_{\lambda}^\sigma(u(p_s)) - \epsilon_{\rho}^\sigma(u(p_s))\epsilon_{\lambda}^\nu(u(p_s))] = \]

\[ \circ \sum_{i=1}^N \eta_i^\tau(T_s) \sqrt{m_i^2 + \bar{\eta}_i^2} [\epsilon_{\tau}^\rho(u(p_s))u^\nu(p_s) - \epsilon_{\rho}^\nu(u(p_s))u^\mu(p_s)] + \]

\[ + \epsilon_{\tau}^{\rho\nu} S_{\mu}^{\nu} \epsilon_{\tau}^\rho(u(p_s))e_{\lambda}^\nu(u(p_s)). \]

(4.10)

This explains why \( m_{\mu}^\nu(T_s, \vec{\eta}) \) is also called the mass dipole moment.

We find, therefore, that in the gauge \( \vec{R}_+ \approx \vec{q}_+ \approx \vec{y}_+ \approx 0 \) with \( P_{\mu}^\nu = M u^\mu(p_s) = M \tilde{x}_s^{(\vec{q}_+)^\mu}(T_s) \) the Møller and barycentric centroid \( x_s^{(\vec{q}_+)^\mu}(T_s) \) is simultaneously the Tulczyjew centroid \( 34, 46, 47, 71 \) (defined by \( S_{\mu\nu}^\nu P_{\nu} = 0 \)) and also the Pirani centroid \( 45, 72 \) (defined by \( S_{\mu\nu}^{\nu\nu}(\vec{q}_+) = 0 \)). In general, lacking a relation between 4-momentum and 4-velocity, they are different centroids \( 73 \).

Note that non-covariant centroids could also be connected with the non-covariant external center of mass \( \tilde{x}_s^{\mu} \) and the non-covariant external Møller center of energy.

D. Quadrupoles and the Barycentric Tensor of Inertia.

The quadrupoles correspond to \( n = 2 \) \( 75 \)
\[
q_{T_t}^{r_2^{AB}}(T_s, \vec{\eta}) = \delta^A_\tau \delta^B_\tau \sum_{i=1}^{N} \left[ \eta_{r_i}^{r_1}(T_s) - \eta_{r_i}^{r_1}(T_s) \right] \left[ \eta_{r_i}^{r_2}(T_s) - \eta_{r_i}^{r_2}(T_s) \right] \sqrt{m_i^2 + \kappa_i^2(T_s)} + \\
+ \delta^A_u \delta^B_v \sum_{i=1}^{N} \left[ \eta_{r_i}^{r_1}(T_s) - \eta_{r_i}^{r_1}(T_s) \right] \left[ \eta_{r_i}^{r_2}(T_s) - \eta_{r_i}^{r_2}(T_s) \right] \frac{\kappa_i^u \kappa_i^v}{\sqrt{m_i^2 + \kappa_i^2(T_s)}} + \\
+ (\delta^A_\tau \delta^B_u + \delta^A_u \delta^B_\tau) \sum_{i=1}^{N} \left[ \eta_{r_i}^{r_1}(T_s) - \eta_{r_i}^{r_1}(T_s) \right] \left[ \eta_{r_i}^{r_2}(T_s) - \eta_{r_i}^{r_2}(T_s) \right] \kappa_i^u(T_s),
\]

and when the mass dipole vanishes, i.e. \( \vec{\eta} = \vec{R}_+ = \sum_i \vec{\eta}_i \sqrt{m_i^2 + \kappa_i^2}/M \), we get

\[
q_{T_t}^{r_1^{r_2}}(T_s, \vec{R}_+) = \sum_{i=1}^{N} \left( \eta_{r_i}^{r_1} - R_+^{r_1} \right) \left( \eta_{r_i}^{r_2} - R_+^{r_2} \right) \sqrt{m_i^2 + \kappa_i^2(T_s)},
\]

\[
q_{T_t}^{r_1^{r_2}u}(T_s, \vec{R}_+) = \sum_{i=1}^{N} \left( \eta_{r_i}^{r_1} - R_+^{r_1} \right) \left( \eta_{r_i}^{r_2} - R_+^{r_2} \right) \kappa_i^u,
\]

\[
q_{T_t}^{r_1^{r_2}v}(T_s, \vec{R}_+) = \sum_{i=1}^{N} \left( \eta_{r_i}^{r_1} - R_+^{r_1} \right) \left( \eta_{r_i}^{r_2} - R_+^{r_2} \right) \frac{\kappa_i^u \kappa_i^v}{\sqrt{m_i^2 + \kappa_i^2(T_s)}}
\]

\[
= \frac{1}{N} \sum_{ijk}^{1..N} \sum_{ab}^{1..N-1} (\gamma_{ai} - \gamma_{aj}).
\] (4.11)

Following the non-relativistic pattern, Dixon starts from the \textit{mass quadrupole}

\[
q_{T_t}^{r_1^{r_2}r_2}(T_s, \vec{R}_+) = \sum_{i=1}^{N} \left[ \eta_{r_i}^{r_1} \eta_{r_i}^{r_2} \sqrt{m_i^2 + \kappa_i^2(T_s)} \right] - M R_+^{r_1} R_+^{r_2},
\] (4.12)

and defines the following \textit{barycentric tensor of inertia}
\[ I_{dixon}^{r_1 r_2}(T_s) = \delta^{r_1 r_2} \sum_u q^{u u r r}_T(T_s, \vec{R}+) - q^{r_1 r_2 r r}_T(T_s, \vec{R}+) = \]

\[ = \sum_{i=1}^N \left[ (\delta^{r_1 r_2}(\vec{\eta}_i - \vec{R}^+)^2 - (\eta_i^{r_1} - R_i^{r_1})(\eta_i^{r_2} - R_i^{r_2})) \sqrt{m_i^2 + \vec{\kappa}_i^2} \right](T_s) \]

\[ \to_{\alpha \to \infty} \sum_{a b 1..N} \left( \frac{1}{N} \sum_{i j k} (\gamma_{ai} - \gamma_{aj})(\gamma_{bi} - \gamma_{bk}) \right) \]

\[ \frac{\sqrt{m_i^2 + N \sum_{de} \gamma_{i d} \vec{\pi}_{qd} \cdot \vec{\pi}_{qe}}}{(\sum_{h=1}^N \sqrt{m_h^2 + N \sum_{de} \gamma_{h d} \vec{\pi}_{qd} \cdot \vec{\pi}_{qe})^2}} \]

\[ \sqrt{m_j^2 + N \sum_{de} \gamma_{j d} \vec{\pi}_{qd} \cdot \vec{\pi}_{qe}} \sqrt{m_k^2 + N \sum_{de} \gamma_{k d} \vec{\pi}_{qd} \cdot \vec{\pi}_{qe}} \]

\[ \left[ \vec{\rho}_{qa} \cdot \vec{\rho}_{qe} \delta^{r_1 r_2} - \rho_{qa} \rho_{qe} \right] \]

\[ \to_{c \to \infty} \sum_{a b 1..N} \sum_{i j k} m_i m_j m_k \left( \gamma_{ai} - \gamma_{aj} \right) \left( \gamma_{bi} - \gamma_{bk} \right) \vec{\rho}_{qa} \cdot \vec{\rho}_{qe} \delta^{r_1 r_2} - \rho_{qa} \rho_{qe} \right] \times \]

\[ \times \left[ 1 + \frac{1}{c} \left( N \sum_{cd}^{1..N-1} \frac{\gamma_{cd} \vec{\pi}_{q_c} \cdot \vec{\pi}_{q_d}}{2m_i^2} + N \sum_{cd}^{1..N-1} \frac{\gamma_{cd} \vec{\pi}_{q_c} \cdot \vec{\pi}_{q_d}}{2m_j^2} + \right. \right. \]

\[ + \left. \left. N \sum_{cd}^{1..N-1} \frac{\gamma_{cd} \vec{\pi}_{q_c} \cdot \vec{\pi}_{q_d}}{2m_k^2} - \frac{1}{m} \sum_{h=1}^N N \sum_{cd}^{1..N-1} \frac{\gamma_{cd} \vec{\pi}_{q_c} \cdot \vec{\pi}_{q_d}}{m_h} \right) + O(1/c^2) \right] = \]

\[ = \sum_{a b} k_{ab} \left[ \vec{\rho}_{qa} \cdot \vec{\rho}_{qe} \delta^{r_1 r_2} - \rho_{qa} \rho_{qe} \right] + O(1/c) = \]

\[ = I^{r_1 r_2}[\vec{q}_{nr}] + O(1/c). \quad (4.13) \]

Note that in the non-relativistic limit we recover the tensor of inertia of Eqs.(A11).

On the other hand, Thorne's definition of barycentric tensor of inertia[76] is
I_{thorne}^{r_1 r_2} (T_s) = \delta^{r_1 r_2} \sum_u q_T^{iu} A(T_s, \vec{R}_+^u) - q_T^{r_1 r_2} A(T_s, \vec{R}_+) =

= \sum_{i=1}^N m_i^2 (\delta^{r_1 r_2} (\vec{\eta}_i - \vec{R}_+)^2 - (\eta_i^{r_1} - R_+^{r_1})(\eta_i^{r_2} - R_+^{r_2})) (T_s)

\rightarrow_{\alpha \rightarrow \infty} \sum_{ab} \sum_{ijk} \left( \frac{c}{N} \sum_{i=1}^N (\gamma_{ai} - \gamma_{aj})(\gamma_{bi} - \gamma_{bk}) \right) 

\frac{m_i^2 \sqrt{m_j^2 + N \sum_{de} \gamma_{de} \vec{\eta}_{qd} \cdot \vec{\eta}_{qe} \sqrt{m_k^2 + N \sum_{de} \gamma_{de} \vec{\eta}_{qd} \cdot \vec{\eta}_{qe} \sum_{h=1}^N \sqrt{m_h^2 + N \sum_{de} \gamma_{de} \vec{\eta}_{qd} \cdot \vec{\eta}_{qe}^2}}}{[\vec{\rho}_{qa} \cdot \vec{\rho}_{qb} \delta^{r_1 r_2} - \vec{\rho}_{qa} \vec{\rho}_{qb}^2]}

\rightarrow_{c \rightarrow \infty} \sum_{ab} \sum_{ijk} \frac{m_j m_k}{N m^2} \left( \frac{1}{c} \sum_{i=1}^N \frac{N \sum_{de} \gamma_{ci} \gamma_{di} \vec{\eta}_{qc} \cdot \vec{\eta}_{qd}}{2 m_i} + \frac{N \sum_{i=1}^N \gamma_{c} \gamma_{d} \vec{\eta}_{qc} \cdot \vec{\eta}_{qd}}{2 m_j} + \frac{N \sum_{i=1}^N \gamma_{c} \gamma_{d} \vec{\eta}_{qc} \cdot \vec{\eta}_{qd}}{2 m_k} \right) =

\sum_{ab} k_{ab} [\vec{\rho}_{qa} \cdot \vec{\rho}_{qb} \delta^{r_1 r_2} - \vec{\rho}_{qa} \vec{\rho}_{qb}^2] + O(1/c) =

= I^{r_1 r_2} (\vec{q}_{nr}) + O(1/c). \quad (4.14)

In this case too we recover the tensor of inertia of Eq.(A11).

Note that the Dixon and Thorne barycentric tensors of inertia differ at the post-Newtonian level

\[ I_{\text{dixon}}^{r_1 r_2} (T_s) - I_{\text{thorne}}^{r_1 r_2} (T_s) = \frac{1}{c} \sum_{ab} \sum_{ijk} \frac{m_j m_k}{N m^2} \left( \gamma_{ai} - \gamma_{aj} \right) \left( \gamma_{bi} - \gamma_{bk} \right) \]

\[ \left[ \vec{\rho}_{qa} \cdot \vec{\rho}_{qb} \delta^{r_1 r_2} - \vec{\rho}_{qa} \vec{\rho}_{qb}^2 \right] \frac{N \sum_{i=1}^N \gamma_{c} \gamma_{d} \vec{\eta}_{qc} \cdot \vec{\eta}_{qd}}{m_i} + O(1/c^2). \]

E. The Multipolar Expansion.

By further using the types of Dixon’s multipoles analyzed in Appendix D as well as the consequences of Hamilton equations for an isolated system (equivalent to \( \partial_\mu T^{\mu\nu} = 0 \)), it turns out that the multipolar expansion (4.4) can be rearranged [see Eqs.(D11)] in the following form
\[ T^{\mu\nu}[\varepsilon^\beta_{(s)}(\beta)(T_s) + \varepsilon^{\beta}(u(p_s))\sigma^\tau] = T^{\mu\nu}[\varepsilon^\beta(u(p_s))\sigma^\tau - \eta^\tau(T_s)] = \]

\[ = u^{(\mu}(p_s)\varepsilon^\nu_{(A)}(u(p_s))[\delta^A_{\tau} M + \delta_{\mu
u}^A] \delta^\beta(\sigma - \eta(T_s)) + \]

\[ + \frac{1}{2} S^\mu_{(s)}(T_s)[\eta] u^{(\nu}(p_s)\varepsilon^\nu_{(\rho)}(u(p_s))\frac{\partial}{\partial \sigma^\tau} \delta^\beta(\sigma - \eta(T_s)) + \]

\[ + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} I^{\mu_1...\mu_n}_{(s)}(T_s, \eta) \varepsilon_{(\mu_1}^{\tau_1} (u(p_s))..\varepsilon_{\mu_n}^{\tau_n} (u(p_s)) \frac{\partial^n}{\partial \tau_1..\partial \tau_n} \delta^\beta(\sigma - \eta(T_s)), \quad (4.15) \]

where for \( n \geq 2 \) and \( \eta = 0 \) \( I^{\mu_1...\mu_n}_{(s)}(T_s) = \frac{4(n-1)}{n+1} I^{\mu_1...\mu_n}_{(s)}(T_s) \), with \( J^{\mu_1...\mu_n\rho\sigma}\(T_s\) \) being the Dixon \( 2^{2+n}\)-pole inertial moment tensors given in Eqs.(D10). With this form of the multipolar expansion, the quadrupole term \( (n = 2) \) has the form [see Eq.(D11)]

\[ \frac{1}{2} \left[ \frac{5}{3} u^{(\mu}(p_s) u^{\nu}(p_s) q^{(r_1 r_2 \tau\tau}_{(s)}(T_s, \eta) + \frac{1}{2} [u^{(\mu}(p_s) \varepsilon^{\nu}_{(u)}(u(p_s)) + u^{\nu}(p_s) \varepsilon^{(\mu}_{(u)}(u(p_s))] q^{(r_1 r_2 \tau\tau}_{(s)}(T_s, \eta) + \]

\[ + \varepsilon_{u_1}^{(\mu}(u(p_s)) \varepsilon_{u_2}^{\nu}(u(p_s)) \left[ q^{(r_1 r_2 u_1 u_2}_{(s)}(T_s, \eta) - \frac{3}{2} \left( q^{(r_1 r_2 u_1}_{(s)}(T_s, \eta) + q^{(r_1 r_2 u_2}_{(s)}(T_s, \eta) \right) + \]

\[ + q^{(r_1 r_2 u_1 u_2}_{(s)}(T_s, \eta) \right) . \]

Note that, as said in Appendix D, Eq.(4.15) holds only if the multipoles are evaluated with respect to world-lines \( w^{(\mu}(\tau) = z^{(\mu}(\tau, \eta(\tau)) \) with \( \eta(\tau) = \eta = \text{const.}, \) namely with respect to one of the integral lines of the vector field \( z^{(\mu}(\tau, \eta) \partial_\mu \).

For an isolated system described by the multipoles appearing in Eq.(4.15) [this is not true for those in Eq.(4.4)] the equations \( \partial_{(\mu} T^{\nu\rho}_{(s)} = 0 \) [see Eqs.(D4) and (D7)] imply no more than the following Papapetrou-Dixon-Souriau equations of motion [27, 68, 77, 78] for the total momentum \( P_{(s)}^{(\mu}_{(s)}(T_s) = \varepsilon_{(A}(u(p_s)) q^{(A}_{(s)}(T_s) \approx p_{(s)}^{(\mu}\) and the spin tensor \( S_{(s)}^{\mu\nu}(T_s)[\eta = 0] \)

\[ \frac{dP_{(s)}^{(\mu}_{(s)}(T_s)}{dT_s} \stackrel{\circ}{=} 0, \]

\[ \frac{dS_{(s)}^{\mu\nu}(T_s)[\eta = 0]}{dT_s} \stackrel{\circ}{=} 2 P_{(s)}^{(\mu}_{(s)}(T_s) u^{(\nu}(p_s) = 2k_{(s)}^{(s)} \varepsilon^{(\mu}_{(s)}(u(p_s)) u^{(\nu}(p_s) \approx 0, \]

\[ \text{or} \quad \frac{dM}{dT_s} = 0, \quad \frac{d(k_{(s)}^{(s)}}{dT_s} = 0, \quad \frac{dS_{(s)}^{\mu\nu}}{dT_s} = 0. \quad (4.16) \]
V. DIXON’S MULTIPOLES AND RELEVANT CENTROIDS FOR CLOSED AND OPEN SYSTEMS OF INTERACTING RELATIVISTIC PARTICLES.

In this Section we present new applications of the multipolar expansion to interacting systems of particles and fields. We first deal with the case of an isolated system of positive-energy relativistic particles with mutual action-at-a-distance interaction (see Section VIII of Ref.[2] and Section VI of Ref.[5]); then we deal with the case of an open particle subsystem of an isolated system consisting of \( N \) charged positive-energy relativistic particles (with Grassmann-valued electric charges to regularize the Coulomb self-energies) plus the electro-magnetic field [5].

A. An isolated System of Positive-Energy Particles with Action-at-a-Distance Interactions.

As said in Section VIII of Ref.[2], in the rest-frame instant form the most general expression of the internal energy for an isolated system of \( N \) positive-energy particles with mutual action-at-a-distance interactions is

\[
M = \sum_i \sqrt{m_i^2 + U_i + (\vec{\kappa}_i - \vec{V}_i)^2 + V},
\]

(5.1)

where all the potentials \( U_i, \vec{V}_i, V \) are functions of \( \vec{\kappa}_i \cdot \vec{\kappa}_j, |\vec{\eta}_i - \vec{\eta}_j|, \vec{\kappa}_k \cdot (\vec{\eta}_i - \vec{\eta}_j) \). On the other hand, as shown at the end of Section II, in the free case we have

\[
M_{(\text{free})} = \sum_i \sqrt{m_i^2 + \vec{\kappa}_i^2} = \sqrt{M_{(\text{free})}^2 + \vec{\kappa}_i^2} \approx \sqrt{M^2 + \vec{\kappa}_i^2} \approx M_{(\text{free})} = \sum_i \sqrt{m_i^2 + N \sum_{ab} \gamma_{ai} \gamma_{bi} \pi_{qa} \cdot \pi_{qb}}.
\]

(5.2)

Since the 3-centers \( \vec{R}_+ \) and \( \vec{q}_+ \) became interaction dependent, in the interacting case we do not know the final canonical basis \( \vec{q}_+, \vec{\kappa}_+, \vec{\rho}_{qa}, \vec{\pi}_{qa} \) explicitly. For an isolated system, however, we have

\[
M = \sqrt{M^2 + \vec{\kappa}_i^2} \approx M \quad \text{with} \quad M \quad \text{independent of} \quad \vec{q}_+ \quad \{M, \vec{\kappa}_+\} = 0 \quad \text{in the internal Poincare’ algebra}. \]

This suggests that also in the interacting case the same result should hold true. Indeed, by its definition, the Gartenhaus-Schwartz transformation gives \( \vec{\rho}_{qa} \approx \vec{\rho}_a, \vec{\pi}_{qa} \approx \vec{\pi}_a \) also in presence of interactions, so that we get

\[
M|_{\vec{\kappa}_+ = 0} = \left( \sum_i \sqrt{m_i^2 + U_i + (\vec{\kappa}_i - \vec{V}_i)^2 + V} \right)|_{\vec{\kappa}_+ = 0} = \sqrt{M^2 + \vec{\kappa}_i^2}|_{\vec{\kappa}_+ = 0} = \sqrt{M^2 + \vec{\kappa}_i^2} = \sum_i \sqrt{m_i^2 + \vec{U}_i + (\vec{\kappa}_i - \vec{V}_i)^2 + \vec{V}},
\]

(5.3)

where the potentials \( \vec{U}_i, \vec{V}_i, \vec{V} \) are now functions of \( \vec{\pi}_{qa} \cdot \vec{\pi}_{qb}, \vec{\pi}_{qa} \cdot \vec{\rho}_{qb}, \vec{\rho}_{qa} \cdot \vec{\rho}_{qb} \).
A relevant example of this type of isolated system has been studied in Ref.[5] starting from the isolated system of \( N \) charged positive-energy particles (with Grassmann-valued electric charges \( Q_i = \theta^* \theta, Q_i^2 = 0, Q_i Q_j = Q_j Q_i \neq 0 \) for \( i \neq j \)) plus the electro-magnetic field. After a Shanmugadhasan canonical transformation, this system can be expressed only in terms of transverse Dirac observables corresponding to a radiation gauge for the electro-magnetic field. The expression of the energy-momentum tensor in this gauge will be shown in the next Subsection. In the semi-classical approximation of Ref.[5], the electro-magnetic degrees of freedom are re-expressed in terms of the particle variables by means of the Lienard-Wiechert solution in the framework of the rest-frame instant form. In this way it has been possible to derive the exact semi-classical relativistic form of the action-at-a-distance Darwin potential in the reduced phase space of the particles. Note that this form is independent of the choice of the Green function in the Lienard-Wiechert solution. In Ref.[5] the associated energy-momentum tensor for the case \( N = 2 \) [Eqs.(6.48)] is also given. The internal energy is

\[
M = \sqrt{\mathcal{M}^2 + \mathcal{K}_+^2} \approx \mathcal{M} = \sum_{i=1}^{2} \sqrt{m_i^2 + \pi_i^2 + \frac{Q_i Q_j}{4 \pi \rho} [1 + \tilde{V}(\pi_i^2, \pi_i \cdot \rho)]}
\]

where \( \tilde{V} \) is given in Eqs.(6.34), (6.35) [in Eqs. (6.36), (6.37) for \( m_1 = m_2 \)]. The internal boost \( \mathcal{K}_+ \) [Eq.(6.46)] allows the determination of the 3-center of energy \( \vec{R}_+ = -\frac{\mathcal{K}_+}{\mathcal{M}} \approx \vec{q}_+ \approx \vec{y}_+ \) in the present interacting case.

The knowledge of the energy-momentum tensor \( T^{AB}(\tau, \sigma) \) and of \( \vec{R}_+ \approx \vec{q}_+ \approx \vec{y}_+ \) allows to apply our formalism to find the barycentric multipoles of this interacting case. It turns out that, in the gauge \( \vec{R}_+ \approx \vec{q}_+ \approx \vec{y}_+ \approx 0 \), all the formal properties studied in the previous Section (like the coincidence of all the relevant centroids) are reproduced in presence of mutual action-at-a-distance interactions.

**B. Open Subsystem of the Isolated System of N Positive-Energy Particles with Grassmann-valued Electric Charge plus the Electromagnetic Field.**

Let us now consider an open sub-system of the isolated system of \( N \) charged positive-energy particles plus the electro-magnetic field in the radiation gauge. The energy-momentum tensor and the Hamilton equations on the Wigner hyper-plane of the isolated system are, respectively, \( [\mathcal{K}_+ = \sum_i \mathcal{K}_i; \overset{\circ}{=} \text{means evaluated on the equations of motion; to avoid degenerations we assume that all the masses } m_i \text{ are different} \]
\[T^{rr}(\tau, \bar{\sigma}) = \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau))\sqrt{m_i^2 + [\bar{\kappa}_i(\tau) - Q_i\bar{A}_\perp(\tau, \bar{\eta}_i(\tau))]^2} + \]
\[+ \frac{1}{2} \left( \bar{\pi}_\perp + \sum_{i=1}^{N} Q_i \overline{\partial} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \right)^2 + \bar{B}^2](\tau, \bar{\sigma}) = \]
\[= \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau))\sqrt{m_i^2 + [\bar{\kappa}_i(\tau) - Q_i\bar{A}_\perp(\tau, \bar{\eta}_i(\tau))]^2} + \]
\[+ \sum_{i=1}^{N} Q_i \bar{\pi}_\perp(\tau, \bar{\sigma}) \times \overline{\partial} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) + \frac{1}{2} [\bar{\pi}_\perp^2 + \bar{B}^2](\tau, \bar{\sigma}) + \]
\[+ \frac{1}{2} \sum_{i,k,i \neq k}^{N} Q_i Q_k \bar{\partial} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \cdot \overline{\partial} \delta^3(\bar{\sigma} - \bar{\eta}_k(\tau)), \]
\[T^{rs}(\tau, \bar{\sigma}) = \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) [\kappa_i^r(\tau) - Q_iA_{\perp}^r(\tau, \bar{\eta}_i(\tau))] + \]
\[+ \left[ \left( \bar{\pi}_\perp + \sum_{i=1}^{N} Q_i \overline{\partial} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \right) \times \bar{B} \right](\tau, \bar{\sigma}), \]
\[T^{rs}(\tau, \bar{\sigma}) = \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \frac{[\kappa_i^r(\tau) - Q_iA_{\perp}^r(\tau, \bar{\eta}_i(\tau))] [\kappa_i^s(\tau) - Q_iA_{\perp}^s(\tau, \bar{\eta}_i(\tau))]}{\sqrt{m_i^2 + [\bar{\kappa}_i(\tau) - Q_i\bar{A}_\perp(\tau, \bar{\eta}_i(\tau))]^2} - \]
\[= \frac{1}{2} \delta^{rs} \left[ \left( \bar{\pi}_\perp + \sum_{i=1}^{N} Q_i \overline{\partial} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \right)^2 + \bar{B}^2 \right] - \]
\[= \left[ \left( \bar{\pi}_\perp + \sum_{i=1}^{N} Q_i \overline{\partial} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \right)^r \left( \bar{\pi}_\perp + \sum_{i=1}^{N} Q_i \overline{\partial} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \right)^s + \right. \]
\left. + B^r B^s \right](\tau, \bar{\sigma}). \] (5.4)
Let us note that in this reduced phase space there are only either particle-field interactions or action-at-a-distance 2-body interactions.

The particle world-lines are \( x_i^\mu(\tau) = x_i^\mu + u^\mu(p_s) \tau + \epsilon_i^\mu(u(p_s)) \eta_i(\tau) \), while their 4-momenta are \( p_i^\mu(\tau) = \sqrt{m_i^2 + [\vec{\kappa}_i - Q_i \vec{A}_\perp(\tau, \vec{\eta}_i(\tau))]^2} + e_i^\mu(u(p_s)) [\kappa_i^\mu - Q_i A_i^\mu(\tau, \vec{\eta}_i(\tau))] \).

The generators of the internal Poincaré group are

\[
\mathcal{P}_{\text{int}}^\tau = M = \sum_{i=1}^N \sqrt{m_i^2 + (\vec{\kappa}_i(\tau) - Q_i \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)))^2} + \frac{1}{2} \sum_{i \neq j} Q_i Q_j \frac{4 \pi}{4 \pi} \left| \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \right| + \int d^3 \sigma \frac{1}{2} \left[ 2 \vec{\pi}_\perp + 2 \vec{B} \right](\tau, \vec{\sigma}),
\]

\[
\vec{P}_{\text{int}} = \vec{\kappa}_+(\tau) + \int d^3 \sigma [\vec{\pi}_\perp \times \vec{B}](\tau, \vec{\sigma}) \approx 0,
\]

\[
\mathcal{J}_{\text{int}}^\tau = \sum_{i=1}^N (\vec{\kappa}_i(\tau) \times \vec{\kappa}_i(\tau))^\tau + \int d^3 \sigma (\vec{\sigma} \times [\vec{\pi}_\perp \times \vec{B}]^\tau)(\tau, \vec{\sigma}),
\]

\[
\mathcal{K}_{\text{int}}^\tau = -\sum_{i=1}^N \vec{\kappa}_i(\tau) \sqrt{m_i^2 + [\vec{\kappa}_i(\tau) - Q_i \vec{A}_\perp(\tau, \vec{\eta}_i(\tau))]^2} + \frac{1}{2} \left[ Q_i \sum_{j=1}^N \sum_{j \neq i} Q_j \int d^3 \sigma \sigma^\tau \vec{c}(\vec{\sigma} - \vec{\eta}_i(\tau)) \cdot \vec{c}(\vec{\sigma} - \vec{\eta}_j(\tau)) + Q_i \int d^3 \sigma \sigma^\tau [\vec{\pi}_\perp + \vec{B}](\tau, \vec{\sigma}) \right] - \frac{1}{2} \int d^3 \sigma \sigma^\tau \left[ \vec{\pi}_\perp + \vec{B} \right](\tau, \vec{\sigma}),
\]
with \( c(\vec{n}_i - \vec{n}_j) = 1/(4\pi |\vec{n}_i - \vec{n}_j|) \) \[ |\Delta c(\vec{\sigma}) = \delta^3(\vec{\sigma}), \Delta = -\partial^2, \vec{c}(\vec{\sigma}) = \partial c(\vec{\sigma}) = \vec{\sigma}/(4\pi |\vec{\sigma}|^3) \].

Note that \( P'_{(\text{int})} = q^{\tau \tau} \) and \( P_{(\text{int})} = q^{\tau \tau} \) are the mass and momentum monopoles, respectively.

For the sake of simplicity let us consider the sub-system formed by the two particles of mass \( m_1 \) and \( m_2 \). Our considerations may be extended to any cluster of particles both in this case and in the case discussed in the previous Subsection. This sub-system is open: besides their mutual interaction the two particles have Coulomb interaction with the other \( N - 2 \) particles and feel the transverse electric and magnetic fields.

By using the multipoles we will select a set of effective parameters (mass, 3-center of motion, 3-momentum, spin) describing the two-particle cluster as a global entity subject to external forces in the global rest-frame instant form. This was the original motivation of the multipolar expansion in general relativity: actually replacing an extended object (an open system due to the presence of the gravitational field) with a set of multipoles concentrated on one center of motion. In the rest-frame instant form it is possible to show that there is no preferred centroid for an open system, namely different centers of motion may be selected according to different conventions unlike the case of isolated systems where, in the rest frame \( \kappa_+ \approx 0 \), all these conventions identify the same centroid. We will see, however, that there is a choice which seems preferable due to its properties.

Given the energy-momentum tensor \( T^{AB}(\tau, \vec{\sigma}) \) (5.4) of the isolated system, it would seem natural to define the energy-momentum tensor \( T^{AB}_{c(n)}(\tau, \vec{\sigma}) \) of an open sub-system composed by a cluster of \( n \leq N \) particles as the sum of all the terms in Eq.(5.4) containing a dependence on the variables \( \vec{n}_i, \vec{\kappa}_i \), of the particles of the cluster. Besides kinetic terms, this tensor would contain internal mutual interactions as well as external interactions of the cluster particles with the environment composed by the other \( N - n \) particles and by the transverse electro-magnetic field. There is an ambiguity, however: why attributing just to the cluster all the external interactions with the other \( N - n \) particles (no such ambiguity exists for the interaction with the electro-magnetic field)? Since we have 2-body interactions, it seems more reasonable to attribute only half of these external interactions to the cluster and consider the other half as a property of the remaining \( N - n \) particles. Let us remark that according to the first choice, if we consider two clusters composed by two non-overlapping sets of \( n_1 \) and \( n_2 \) particles respectively, we would get \( T^{AB}_{c(n_1+n_2)} \neq T^{AB}_{c(n_1)} + T^{AB}_{c(n_2)} \), since the mutual Coulomb interactions between the two clusters are present in both \( T^{AB}_{c(n_1)} \) and \( T^{AB}_{c(n_2)} \). Instead according to the second choice we would get \( T^{AB}_{c(n_1+n_2)} = T^{AB}_{c(n_1)} + T^{AB}_{c(n_2)} \). Since this property is important for studying the mutual relative motion of two clusters in actual cases, we will adopt the convention that the energy-momentum tensor of a \( n \) particle cluster contains only half of the external interaction with the other \( N - n \) particles.

Let us remark that, in the case of \( k \)-body forces, this convention should be replaced by the following rule: i) for each particle \( m_i \) of the cluster and each \( k \)-body term in the energy-momentum tensor involving this particle, we write \( k = h_i + (k - h_i) \), where \( h_i \) is the number of particles of the cluster participating to this particular \( k \)-body interaction; ii) then only the fraction \( h_i/k \) of this particular \( k \)-body interaction term containing \( m_i \) has to be attributed to the cluster.

Let us consider the cluster composed by the two particles with mass \( m_1 \) and \( m_2 \). The knowledge of \( T^{AB}_{c(1)} \) on the Wigner hyper-plane of the global rest-frame in-
stant form allows to find the following 10 non conserved charges [due to $Q_i^2 = 0$ we have
\[ \sqrt{m_i^2 + [\vec{r}_i - Q_i \vec{A}_i(\tau, \vec{\eta}_i)]^2} = \sqrt{m_i^2 + \vec{r}_i^2} - Q_i \frac{\vec{r}_i \cdot \vec{A}_i(\tau, \vec{\eta}_i)}{\sqrt{m_i^2 + \vec{r}_i^2}} \]

\[ M_c = \int d^3\sigma T^{rr}_c(\tau, \vec{\sigma}) = \sum_{i=1}^2 \sqrt{m_i^2 + [\vec{r}_i - Q_i \vec{A}_i(\tau, \vec{\eta}_i)]^2} + \frac{Q_1 Q_2}{4\pi |\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau)|^2} + \frac{1}{2} \sum_{i=1}^2 \sum_{k \neq 1, 2} Q_i Q_k 4\pi |\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)|^2 = \]

\[ = M_c(\text{int}) + M_c(\text{ext}), \]

\[ M_c(\text{int}) = \sum_{i=1}^2 \sqrt{m_i^2 + \vec{r}_i^2} - \frac{Q_1 Q_2}{4\pi |\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau)|^2}, \]

\[ \vec{P}_c = \{ \int d^3\sigma T^{rr}_c(\tau, \vec{\sigma}) \} = \vec{r}_1(\tau) + \vec{r}_2(\tau), \]

\[ \vec{J}_c = \{ \epsilon^{rru} \int d^3\sigma [\sigma^u T^{ur}_c - \sigma^u T^{ur}_c](\tau, \vec{\sigma}) \} = \]

\[ = \vec{\eta}_1(\tau) \times \vec{r}_1(\tau) + \vec{\eta}_2(\tau) \times \vec{r}_2(\tau), \]

\[ \vec{K}_c = -\int d^3\sigma \vec{\sigma} T^{rr}_c(\tau, \vec{\sigma}) = \]

\[ = -\sum_{i=1}^2 \vec{\eta}_i(\tau) \sqrt{m_i^2 + [\vec{r}_i - Q_i \vec{A}_i(\tau, \vec{\eta}_i)]^2} - \]

\[ - \sum_{i=1}^2 Q_i \int d^3\sigma \vec{\pi}_i(\tau, \vec{\sigma}) c(\vec{\sigma} - \vec{\eta}_i(\tau)) - \]

\[ - Q_1 Q_2 \int d^3\sigma \vec{\sigma} c(\vec{\sigma} - \vec{\eta}_1(\tau)) \cdot \vec{c}(\vec{\sigma} - \vec{\eta}_2(\tau)) - \]

\[ - \frac{1}{2} \sum_{i=1}^2 \sum_{k \neq 1, 2} Q_i Q_k \int d^3\sigma \vec{\sigma} c(\vec{\sigma} - \vec{\eta}_i(\tau)) \cdot \vec{c}(\vec{\sigma} - \vec{\eta}_k(\tau)) = \]

\[ = \vec{K}_c(\text{int}) + \vec{K}_c(\text{ext}), \]

\[ \vec{K}_c(\text{int}) = -\sum_{i=1}^2 \vec{\eta}_i(\tau) \sqrt{m_i^2 + \vec{r}_i^2} - Q_1 Q_2 \int d^3\sigma \vec{\sigma} c(\vec{\sigma} - \vec{\eta}_1(\tau)) \cdot \vec{c}(\vec{\sigma} - \vec{\eta}_2(\tau)) (5.7) \]

which do not satisfy the algebra of an internal Poincare’ group due to the openness of the system. Since we work in an instant form of dynamics only the cluster internal energy and boosts depend on the (internal and external) interactions. Again $M_c = q_c^{rr}$ and $P_c = q_c^{r}$ are the mass and momentum monopoles of the cluster.

Another needed quantity is the momentum dipole
\[ p_{ru}^c = \int d^3\sigma \, \sigma^r T_{ru}^c(\tau, \bar{\sigma}) = \]
\[ = \sum_{i=1}^{2} \eta_i^r(\tau) \kappa_i^u(\tau) - \sum_{i=1}^{2} Q_i \int d^3\sigma \, c(\bar{\sigma} - \bar{\eta}_i(\tau)) [\partial^r A^s_\perp + \partial^s A^r_\perp](\tau, \bar{\sigma}), \]
\[ p_{ru}^c + p_{ur}^c = \sum_{i=1}^{2} [\eta_i^r(\tau) \kappa_i^u(\tau) + \eta_i^u(\tau) \kappa_i^r(\tau)] - \]
\[ - 2 \sum_{i=1}^{2} Q_i \int d^3\sigma \, c(\bar{\sigma} - \bar{\eta}_i(\tau)) [\partial^r A^s_\perp + \partial^s A^r_\perp](\tau, \bar{\sigma}), \]
\[ p_{ru}^c - p_{ur}^c = \epsilon^{ruv} J^v_c. \]

(5.8)

The time variation of the 10 charges (5.7) can be evaluated by using the equations of motion (5.5)
\[
\frac{dM_c}{d\tau} = \sum_{i=1}^{2} Q_i \left( \frac{\vec{k}_i(\tau) \cdot \vec{\pi}_i(\tau, \vec{\eta}_i(\tau))}{\sqrt{m_i^2 + \vec{k}_i^2}} \right) + \frac{1}{2} \sum_{k \neq 1, 2} Q_k \left[ \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} + \frac{\vec{k}_k(\tau)}{\sqrt{m_k^2 + \vec{k}_k^2}} \right] \cdot \frac{\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)}{4\pi |\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)|^3},
\]

\[
\frac{d\mathcal{P}^r_c}{d\tau} = \sum_{i=1}^{2} Q_i \left( \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} \cdot \partial A^r_i(\tau, \vec{\eta}_i(\tau)) \right) + \sum_{k \neq 1, 2} Q_k \frac{\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)}{4\pi |\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)|^3},
\]

\[
\frac{d\mathcal{J}_c}{d\tau} = \sum_{i=1}^{2} Q_i \left( \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} \times \vec{A}_i(\tau, \vec{\eta}_i(\tau)) + \vec{\eta}_i(\tau) \times \left[ \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} \cdot \frac{\partial}{\partial \vec{\eta}_i} \vec{A}_i(\tau, \vec{\eta}_i(\tau)) \right] - \frac{\vec{\eta}_i(\tau) \times \vec{\eta}_k(\tau)}{4\pi |\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)|^3},
\]

\[
\frac{d\mathcal{K}_c^r}{d\tau} = -\frac{\mathcal{P}^r_c}{c} - \sum_{i=1}^{2} Q_i \sum_{k \neq i} \int d^3\sigma \ c(\vec{\sigma} - \vec{\eta}_i(\tau)) \left( \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} \right) \cdot \vec{\sigma} \left[ \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} \right] \cdot \vec{\sigma} \left[ \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} \right] - \sum_{k \neq i} Q_k \int d^3\sigma \ c(\vec{\sigma} - \vec{\eta}_1(\tau)) \left( \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} \right) \cdot \vec{\sigma} \left[ \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} \right] + \vec{\sigma} \cdot \left[ \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} \right] - \frac{1}{2} \sum_{i=1}^{2} Q_i \sum_{k \neq 1, 2} Q_k \int d^3\sigma \ c(\vec{\sigma} - \vec{\eta}_1(\tau)) \left( \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} \right) \cdot \vec{\sigma} \left[ \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} \right] + \vec{\sigma} \cdot \left[ \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2}} \right].
\]

Let us remark that, if we have two clusters of \(n_1\) and \(n_2\) particles respectively, our definition of cluster energy-momentum tensor implies

\[
M_{c(n_1+n_2)} = M_{c(n_1)} + M_{c(n_2)},
\]
\[
\mathcal{P}^r_{c(n_1+n_2)} = \mathcal{P}^r_{c(n_1)} + \mathcal{P}^r_{c(n_2)},
\]
\[
\mathcal{J}_{c(n_1+n_2)} = \mathcal{J}_{c(n_1)} + \mathcal{J}_{c(n_2)},
\]
\[
\mathcal{K}^r_{c(n_1+n_2)} = \mathcal{K}^r_{c(n_1)} + \mathcal{K}^r_{c(n_2)}.
\]
The main problem is the determination of an effective center of motion $\zeta_c^e(\tau)$ with world-line $u^e_c(\tau) = x^e_\mu + u^\mu(p_s) \tau + \epsilon^e_\mu(u(p_s)) \zeta^e_c(\tau)$ in the gauge $T_s \equiv \tau$, $\bar{q}_+ = \bar{R}_+ = \bar{y}_+ \equiv 0$ of the isolated system. The unit 4-velocity of this center of motion is $u^e_c(\tau) = \dot{u}^e_c(\tau)/\sqrt{1 - \dot{\zeta}^2_c(\tau)}$ with $\dot{\zeta}^e_c(\tau) = u^\mu(p_s) + \epsilon^e_\mu(u(p_s)) \dot{\zeta}^e_c(\tau)$. By using $\delta z^e(\tau, \bar{\sigma}) = \epsilon^e_\mu(u(p_s)) (\sigma^\tau - \zeta^\tau(\tau))$ we can define the multipoles of the cluster with respect to the world-line $w^\mu_c(\tau)$

$$ q^{r1-rnAB}_c(\tau) = \int d^3\sigma [\sigma^{r1} - \zeta^{r1}_c(\tau)] [\sigma^{rn} - \zeta^{rn}_c(\tau)] T^{AB}_c(\tau, \bar{\sigma}). \quad (5.11) $$

The mass and momentum monopoles and the mass, momentum and spin dipoles are respectively

$$ q^{r\tau}_c = M_c, \quad q^{\tau\tau}_c = P^\tau_c, $$

$$ q^{r\tau\tau}_c = -K_c^\tau - M_c \zeta^\tau_c(\tau) = M_c (R^\tau_c(\tau) - \zeta^\tau_c(\tau)), $$

$$ q^{\tau\tau\tau}_c = p^{\tau\tau}_c(\tau) - \zeta^{\tau\tau}_c(\tau) P^\tau_c, $$

$$ S^{\mu\nu}_c = [\epsilon^\mu_\tau(u(p_s)) u^\nu(p_s) - \epsilon^\nu_\tau(u(p_s)) u^\mu(p_s)] q^{r\tau\tau}_c + \epsilon^\mu_\tau(u(p_s)) \epsilon^\nu_\tau(u(p_s)) (q^{\tau\tau\tau}_c - q^{\tau\tau\tau}_c) = $$

$$ = [\epsilon^\mu_\tau(u(p_s)) u^\nu(p_s) - \epsilon^\nu_\tau(u(p_s)) u^\mu(p_s)] M_c (R^\tau_c - \zeta^\tau_c) + $$

$$ + \epsilon^\mu_\tau(u(p_s)) \epsilon^\nu_\tau(u(p_s)) \left[\epsilon^{\tau\mu} J^{\tau\nu} - (\zeta^\nu_c P^\nu_c - \zeta^\nu_c \bar{P}^\nu_c)\right]. $$

$$ \Rightarrow m^{\mu}_{c(p_s)} = -S^{\mu\nu}_c u^\nu(p_s) = -\epsilon^\mu_\tau(u(p_s)) q^{r\tau\tau}_c. \quad (5.12) $$

Let us now consider the following possible definitions of effective centers of motion (many other possibilities exist)

1) **Center of energy as center of motion**, $\bar{\zeta}_{c(E)}(\tau) = \bar{R}_c(\tau)$, where $\bar{R}_c(\tau)$ is a 3-center of energy for the cluster built by means of the standard definition

$$ \bar{R}_c = -\frac{\bar{K}_c}{M_c}. \quad (5.13) $$

It is determined by the requirement that either the mass dipole vanishes, $q^{r\tau\tau}_c = 0$ or the mass dipole moment with respect to $u^\mu(p_s)$ vanishes, $m^{\mu}_{c(p_s)} = 0$.

The center of energy seems to be the only center of motion enjoying the simple composition rule

$$ \bar{R}_{c(n_1+n_2)} = \frac{M_{c(n_1)} \bar{R}_{c(n_1)} + M_{c(n_2)} \bar{R}_{c(n_2)}}{M_{c(n_1+n_2)}}. \quad (5.14) $$

The constitutive relation between $\bar{P}_c$ and $\bar{R}_c(\tau)$, see Eq.(4.8), is
\[ 0 = \frac{dq_{c\tau\tau}}{d\tau} = -\dot{K}_c^\tau - \dot{M}_c R_c^\tau - M_c \ddot{R}_c, \]

\[ \Downarrow \]

\[ \ddot{\mathcal{P}}_c = M_c \ddot{R}_c + \dot{M}_c \ddot{R}_c - \sum_{i=1}^{2} Q_i \tilde{\eta}_i(\tau) \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2(\tau)}} \left[ \vec{\pi}_\perp(\tau, \tilde{\eta}_i(\tau)) + \sum_{k \neq i} Q_k \vec{c}(\tilde{\eta}_k(\tau) - \tilde{\eta}_k(\tau)) \right] + \]

\[ + \sum_{i=1}^{2} Q_i \left[ \sum_{k \neq i} Q_k \int d^3\sigma \vec{c}(\vec{\sigma} - \vec{\eta}_1(\tau)) \left( \frac{\vec{k}_k(\tau)}{\sqrt{m_k^2 + \vec{k}_k^2(\tau)}} \cdot \vec{\sigma} \right) \vec{c}(\vec{\sigma} - \vec{\eta}_1(\tau)) - \right. \]
\[ \left. \left[ \frac{\vec{k}_2(\tau)}{\sqrt{m_2^2 + \vec{k}_2^2(\tau)}} \cdot \vec{\sigma} \right] \vec{c}(\vec{\sigma} - \vec{\eta}_2(\tau)) \right] - \]
\[ - \frac{1}{2} \sum_{i=1}^{2} Q_i \sum_{k \neq 1,2} Q_k \int d^3\sigma \vec{c}(\vec{\sigma} - \vec{\eta}_1(\tau)) \left( \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2(\tau)}} \cdot \vec{\sigma} \right) \vec{c}(\vec{\sigma} - \vec{\eta}_1(\tau)) + \]
\[ + \vec{c}(\vec{\sigma} - \vec{\eta}_1(\tau)) \cdot \left[ \frac{\vec{k}_i(\tau)}{\sqrt{m_i^2 + \vec{k}_i^2(\tau)}} \cdot \vec{\sigma} \right] \vec{c}(\vec{\sigma} - \vec{\eta}_1(\tau)) \right]. \tag{5.15} \]

From Eq.(4.10) the associated cluster spin tensor is

\[ S_{c\mu} = \epsilon_{c\mu}^e(u(p_s)) \epsilon_{c\mu}^e(u(p_s)) [q_{c\tau\tau}^\mu - q_{c\tau\tau}^\nu] = \epsilon_{c\mu}^e(u(p_s)) \epsilon_{c\mu}^e(u(p_s)) \epsilon^{\tau\nu} \left[ J_c^\tau - (\ddot{R}_c \times \ddot{\mathcal{P}}_c)^\nu \right]. \tag{5.16} \]

2) Pirani centroid \( \tilde{\zeta}(\mu) \) as center of motion. It is determined by the requirement that the mass dipole moment with respect to 4-velocity \( \dot{w}_c^\mu(\tau) \) vanishes (it involves the anti-symmetric part of \( p_c^{\mu\nu} \))

\[ m_{c(\dot{w}_c)}^\mu = -S_{c\tau\tau} \dot{w}_c = 0, \quad \Rightarrow \dot{\tilde{\zeta}}_c(\mu) = \tilde{\zeta}_c(\mu) \cdot \ddot{R}_c, \]

\[ \Downarrow \]

\[ \tilde{\zeta}_c(\mu)(\tau) = \frac{1}{M_c - \mathcal{P}_c \cdot \ddot{\mathcal{P}}_c(\tau)} \left[ M_c \ddot{R}_c - \ddot{R}_c \cdot \ddot{\mathcal{P}}_c - \ddot{\mathcal{P}}_c - \ddot{\mathcal{P}}_c(\tau) \times \mathcal{J}_c \right]. \tag{5.17} \]
Therefore this centroid is implicitly defined as the solution of these three coupled first order ordinary differential equations.

3) Tulczyjew centroid $\vec{\zeta}(\tau)$ as center of motion. If we define the cluster 4-momentum $P^\mu = M_c u^\mu(p_s) + \mathcal{P}_c^s e_s^\mu(u(p_s))$ [$P^2_c = M^2_c - \vec{P}^2_c = M^2_c$], its definition is the requirement that the mass dipole moment with respect to $P^\mu_c$ vanishes (it involves the anti-symmetric part of $p^\mu ur_c$)

$$m^\mu c(\vec{P}_c) = -S^\mu cP_c = 0, \quad \Rightarrow \vec{P}_c \cdot \vec{\zeta}(\vec{P}_c) = \vec{P}_c \cdot \vec{R}_c,$$

$$\vec{\zeta}(\vec{P}_c) = \frac{1}{M^2_c - \vec{P}^2_c} [M^2_c \vec{R}_c - \vec{P}_c \cdot \vec{R}_c \vec{P}_c - \vec{P}_c \times \vec{J}_c]. \quad (5.18)$$

Let us show that this centroid satisfies the free particle relation as constitutive relation

$$\vec{P}_c = M_c \vec{\zeta}(\vec{P}_c),$$

$$\downarrow$$

$$P^\mu_c = M_c [u^\mu(p_s) + \mathcal{P}_c^n e^n_s(u(p_s))],$$

$$q^{\tau \tau} = \frac{M_c}{M^2_c - \vec{P}^2_c} [\vec{P}^2 \vec{R}_c + \vec{P}_c \cdot \vec{R}_c \vec{P}_c + \vec{P}_c \times \vec{J}_c],$$

$$S^{cuv} = [e^u(p_s)) u^v(p_s) - e^s_r(u(p_s)) u^\mu(p_s)] q^{\tau \tau} +$$

$$+e^\mu_r(u(p_s)) e^\nu_s(u(p_s)) \epsilon^{r uv} [\vec{J}_c^v - (\vec{\zeta}(\vec{P}_c) \times \vec{P}_c)^v]. \quad (5.19)$$

If we use Eq.(5.17) to find a Pirani centroid such that $\vec{\zeta}_c = \vec{P}_c/M_c$, it turns out that the condition (5.17) becomes Eq.(5.18) and this implies Eq.(5.19).

The equations of motion

$$M_c(\tau) \dot{\vec{\zeta}_c}(\tau) = \dot{\vec{P}_c}(\tau) - \dot{M}_c(\tau) \vec{\zeta}_c(\tau), \quad (5.20)$$

contain both internal and external forces. Notwithstanding the nice properties (5.19) and (5.20) of the Tulczyjew centroid, this effective center of motion suffers the drawback of not satisfying a simple composition property. The relation among the Tulczyjew centroids of clusters with $n_1$, $n_2$ and $n_1 + n_2$ particles respectively is much more complicated of the composition (5.14) of the centers of energy.

All the previous centroids coincide for an isolated system in the rest-frame instant form with $\vec{P}_c = \vec{R}_+ \approx 0$ in the gauge $\vec{q}_+ \approx \vec{R}_+ \approx \vec{g}_+ \approx 0$.  

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4) The Corinaldesi-Papapetrou centroid with respect to a time-like observer with 4-velocity $v^\mu(\tau)$, $\zeta_{c(CP)}^{(\nu)}(\tau)$ as center of motion.

$$m_{c(\nu)}^\mu = -S_{c(\nu)}^{\mu\nu} v_\nu = 0. \quad (5.21)$$

Clearly these centroids are unrelated to the previous ones being dependent on the choice of an arbitrary observer.

5) The Pryce center of spin or classical canonical Newton-Wigner centroid $\zeta_{c(NW)}$. It defined as the solution of the differential equations implied by the requirement \{$\zeta_{c(NW)}^r$, $\zeta_{c(NW)}^s$\} = 0, \{$\zeta_{c(NW)}^r$, $P_c^s$\} = $\delta^{rs}$. Let us remark that, being in an instant form of dynamics, we have \{$P_c^r$, $P_c^s$\} = 0 also for an open system.

The two effective centers of motion which look more useful for applications seems to be the center of energy $\zeta_{c(E)}(\tau)$ and Tulczyjew’s centroid $\zeta_{c(T)}(\tau)$, with $\zeta_{c(E)}(\tau)$ preferred for the study of the mutual motion of clusters due to Eq.(5.14).

Therefore, in the spirit of the multipolar expansion, our two-body cluster may be described by an effective non-conserved internal energy (or mass) $M_c(\tau)$, by the world-line $w_c^\mu(\tau) = x^\mu + u^\mu (\rho_s) \tau + c^\mu (u(\rho_s)) \zeta_{c(E or T)}^r(\tau)$ associated with the effective center of motion $\zeta_{c(E or T)}(\tau)$ and by the effective 3-momentum $\vec{P}_c(\tau)$, with $\zeta_{c(E or T)}(\tau)$ and $\vec{P}_c(\tau)$ forming a non-canonical basis for the collective variables of the cluster. A non-canonical effective spin for the cluster in the 1) and 3) cases is defined by

\[\begin{align*}
\text{a) case of the center of energy,} & \\
\zeta_{c(E)}(\tau) & = \vec{J}_c(\tau) - \vec{R}_c(\tau) \times \vec{P}_c(\tau), \\
\frac{d\zeta_{c(E)}(\tau)}{d\tau} & = \frac{d\vec{J}_c(\tau)}{d\tau} - \frac{d\vec{R}_c(\tau)}{d\tau} \times \vec{P}_c(\tau) - \vec{R}_c(\tau) \times \frac{d\vec{P}_c(\tau)}{d\tau},
\end{align*}\]

\[\begin{align*}
\text{b) case of the Tulczyjew centroid,} & \\
\zeta_{c(T)}(\tau) & = \vec{J}_c(\tau) - \zeta_{c(T)}(\tau) \times \vec{P}_c(\tau) = \\
& = \frac{M_c^2(\tau) \zeta_{c(E)}(\tau) - \vec{P}_c(\tau) \cdot \vec{J}_c(\tau) \vec{P}_c(\tau)}{M_c^2(\tau) - \vec{P}_c^2(\tau)}, \\
\frac{d\zeta_{c(T)}(\tau)}{d\tau} & = \frac{d\vec{J}_c(\tau)}{d\tau} - \zeta_{c(T)}(\tau) \times \frac{d\vec{P}_c(\tau)}{d\tau}.
\end{align*}\] (5.22)

Since our cluster contains only two particles, this pole–dipole description concentrated on the world-line $w_c^\mu(\tau)$ is equivalent to the original description in terms of the canonical variables $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ (all the higher multipoles are not independent quantities in this case).
Let us see whether it is possible to replace the description of the two body system as an effective pole-dipole system with a description as an effective extended two-body system by introducing two non-canonical relative variables \( \vec{\rho}_{c(E\text{ or } T)}(\tau), \vec{\pi}_{c(E\text{ or } T)}(\tau) \) with the following definitions

\[
\begin{align*}
\vec{\eta}_1 & \equiv \vec{\zeta}_{c(E\text{ or } T)} + \frac{1}{2} \vec{\rho}_{c(E\text{ or } T)}; & \vec{\zeta}_{c(E\text{ or } T)} & = \frac{1}{2} (\vec{\eta}_1 + \vec{\eta}_2), \\
\vec{\eta}_2 & \equiv \vec{\zeta}_{c(E\text{ or } T)} - \frac{1}{2} \vec{\rho}_{c(E\text{ or } T)}; & \vec{\rho}_{c(E\text{ or } T)} & = \vec{\eta}_1 - \vec{\eta}_2, \\
\vec{\kappa}_1 & \equiv \frac{1}{2} \vec{\rho}_c + \vec{\pi}_{c(E\text{ or } T)}; & \vec{\rho}_c & = \vec{\kappa}_1 + \vec{\kappa}_2, \\
\vec{\kappa}_2 & \equiv \frac{1}{2} \vec{\rho}_c - \vec{\pi}_{c(E\text{ or } T)}; & \vec{\pi}_{c(E\text{ or } T)} & = \frac{1}{2} (\vec{\kappa}_1 - \vec{\kappa}_2), \\
\vec{\mathcal{J}}_c & = \vec{\eta}_1 \times \vec{\kappa}_1 + \vec{\eta}_2 \times \vec{\kappa}_2 = \vec{\zeta}_{c(E\text{ or } T)} \times \vec{\rho}_c + \vec{\rho}_{c(E\text{ or } T)} \times \vec{\pi}_{c(E\text{ or } T)}, \\
\Rightarrow \quad \mathcal{S}_{c(E\text{ or } T)} & = \vec{\rho}_{c(E\text{ or } T)} \times \vec{\pi}_{c(E\text{ or } T)}. \quad (5.23)
\end{align*}
\]

Even if suggested by a canonical transformation, it is not a canonical transformation and it only exists because we are working in an instant form of dynamics in which both \( \vec{\rho}_c \) and \( \vec{\mathcal{J}}_c \) do not depend on the interactions.

Note that we know everything about this new basis except for the unit vector \( \vec{\rho}_{c(E\text{ or } T)}/|\vec{\rho}_{c(E\text{ or } T)}| \) and the momentum \( \vec{\pi}_{c(E\text{ or } T)} \). The relevant lacking information can be extracted from the symmetrized momentum dipole \( p_c^{ru} + p_c^{ur} \), which is a known effective quantity due to Eq.(5.9) having the following expression in terms of the variables (5.23)

\[
\begin{align*}
p_c^{ru} + p_c^{ur} & + 2 \sum_{i=1}^{2} Q_i \int d^3\sigma \ c(\sigma - \vec{\eta}_1(\tau)) \left[ \partial^r A_\perp^i + \partial^s A_\perp^r \right]|(\tau, \vec{\sigma}) = \\
& = \sum_{i=1}^{2} (\eta_i^r \kappa_i^u + \eta_i^u \kappa_i^r) = \zeta_{c(E\text{ or } T)}^r \mathcal{P}_c^u + \zeta_{c(E\text{ or } T)}^u \mathcal{P}_c^r + \\
& + \rho_{c(E\text{ or } T)}^r \pi_{c(E\text{ or } T)}^u + \rho_{c(E\text{ or } T)}^u \pi_{c(E\text{ or } T)}^r. \quad (5.24)
\end{align*}
\]

A strategy for getting this information is to construct a spin frame, which, following Ref.[2] for the \( N = 2 \) case, is defined by \( \mathcal{S}_{c(E\text{ or } T)} = \hat{\mathcal{S}}_{c(E\text{ or } T)}/|\hat{\mathcal{S}}_{c(E\text{ or } T)}|, \mathcal{R}_{c(E\text{ or } T)} = \mathcal{R}_{c(E\text{ or } T)} \times \mathcal{S}_{c(E\text{ or } T)}, \mathcal{V}_{c(E\text{ or } T)} = \mathcal{V}_{c(E\text{ or } T)}^2 \mathcal{R}_{c(E\text{ or } T)}^2 = 1 \). Then we get the following decomposition ((\( \mathcal{S}_{c(E\text{ or } T)} = |\hat{\mathcal{S}}_{c(E\text{ or } T)}| \))

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\[ \tilde{\rho}_{c(E\text{or} \ E)} = \rho_{c(E\text{or} \ E)} \hat{R}_{c(E\text{or} \ E)}, \]  
\[ \tilde{\pi}_{c(E\text{or} \ E)} = \tilde{\pi}_{c(E\text{or} \ E)} \hat{R}_{c(E\text{or} \ E)} - \frac{S_{c(E\text{or} \ E)}}{\rho_{c(E\text{or} \ E)}} \dot{V}_{c(E\text{or} \ E)}, \]  
\[ \tilde{\pi}_{c(E\text{or} \ E)} = \tilde{\pi}_{c(E\text{or} \ E)} \frac{\tilde{\rho}_{c(E\text{or} \ E)}}{\rho_{c(E\text{or} \ E)}}, \]

(5.25)

where \( \rho_{c(E\text{or} \ E)} \) is just the relative variable appearing in the Coulomb potential. Eqs.(5.25) show that only the three variables \( \tilde{\pi}_{c(E\text{or} \ E)} \) and \( \hat{R}_{c(E\text{or} \ E)} = \frac{\tilde{\rho}_{c(E\text{or} \ E)}}{\rho_{c(E\text{or} \ E)}} \) are still unknown. Then from Eqs.(5.23) and (5.24) we get

\[ \rho_{c(E\text{or} \ E)}^r \pi_{c(E\text{or} \ E)}^r + \rho_{c(E\text{or} \ E)}^u \pi_{c(E\text{or} \ E)}^u = 2 \rho_{c(E\text{or} \ E)} \tilde{\pi}_{c(E\text{or} \ E)} \hat{R}_{c(E\text{or} \ E)}^r \hat{R}_{c(E\text{or} \ E)}^u - \frac{S_{c(E\text{or} \ E)}}{\rho_{c(E\text{or} \ E)}} \left( \hat{R}_{c(E\text{or} \ E)}^r \dot{V}_{c(E\text{or} \ E)} + \hat{R}_{c(E\text{or} \ E)}^u \dot{V}_{c(E\text{or} \ E)} \right) = \rho_{c(E\text{or} \ E)}^r + \rho_{c(E\text{or} \ E)}^u - \left( \zeta_{c(E\text{or} \ E)} \mathcal{P}_{c} + \mathcal{S}_{c(E\text{or} \ E)} \mathcal{P}_{c} \right) + \\
+ 2 \sum_{i=1}^{2} Q_{i} \int d^{3} \sigma A_{u}(\tau, \tilde{\eta}) [\partial^{r} A_{u}^{r} + \partial^{u} A_{u}^{r}] (\tau, \tilde{\sigma}) = \frac{F_{c(E\text{or} \ E)}^{ru}}{2 \rho_{c(E\text{or} \ E)}}, \]

(5.26)

But these are three independent equations for \( \tilde{\pi}_{c(E\text{or} \ E)} \) and for the two degrees of freedom in the unit vector \( \hat{R}_{c(E\text{or} \ E)} \) in terms of the known quantities \( F_{c(E\text{or} \ E)}^{ru} \), \( S_{c(E\text{or} \ E)} \), \( \rho_{c(E\text{or} \ E)} = |\tilde{\eta}_{1} - \tilde{\eta}_{2}| \). For instance we get \( \tilde{\pi}_{c(E\text{or} \ E)} = (\sum_{r} F_{c(E\text{or} \ E)}^{ru}) / 2 \rho_{c(E\text{or} \ E)} \): due to the transversality of the vector potential \( \tilde{\pi}_{c(E\text{or} \ E)} \) does not depend on it. In conclusion the external electromagnetic potential \( \tilde{A}_{u} \) enters only in the determination of the axis \( \hat{R}_{c(E\text{or} \ E)} \) of the spin frame.

This completes the construction of the effective relative variables and of the effective spin frame using the extra input of the 3-momentum dipole. In this way we get a description of the two-body cluster as an effective two-body system instead of a pole-dipole system. However, the weak point of this description of the open system as an extended object is that, whatever definition of effective center of motion one uses, the symmetrized momentum dipole \( p_{c}^{ru} + p_{c}^{ur} \) does not depend only on the cluster properties but also on the external electromagnetic transverse vector potential at the particle positions, as shown by Eq.(5.8). As a consequence the spin frame, or equivalently the 3 Euler angles associated with the internal spin, depends upon the external fields.

If we accept this drawback, it is reasonable that, by taking into account higher multipoles, it is possible to give a description of a cluster of \( n \geq 3 \) particles in terms of as many effective \( n \)-body systems as effective dynamical body frames following the scheme of Ref.[2].

This would open the possibility to have effective descriptions of two clusters of \( n_{1} \) and \( n_{2} \) particles, respectively, and to compare it with the effective description of the cluster composed by the same \( n_{1} + n_{2} \) particles to find the relation between the three centers of
motion of the \((n_1 + n_2)\), \(n_1\) and \(n_2\) clusters and the relative motion of the two \(n_1\) and \(n_2\) clusters. To this end the use of the center of energy as center of motion seems unavoidable due to the simple composition law (5.14). Whatever choice we adopt, however, it turns out that the relative motion of the two clusters depends on the external fields besides the effective parameters of the clusters.

These techniques can be extended to relativistic perfect fluids, if described in the rest-frame instant form as done in Refs. [1]. Moreover, they are needed for the determination of the post-Minkowskian approximation to the quadrupole formula for the emission of gravitational waves (re-summation of the post-Newtonian approximations) in the background-independent Hamiltonian linearization of tetrad gravity [79] plus a perfect fluid [80].
VI. CONCLUSIONS.

A relativistic description of open systems like binary stars embedded in the gravitational field would be an important achievement nowadays in view of the construction of templates for the gravitational radiation. Even by approximating such description by means of a multipolar expansion in a way suitable for doing actual calculations, either analytical or numerical, a big amount of kinematical technical preliminaries is needed anyway. With this in view, we had in mind to develop methods which could be useful in general relativity with relativistic perfect fluids as matter, where single or binary stars could be described by open fluid subsystems of the isolated system formed by the gravitational field plus the fluid in the rest-frame instant form of either metric or tetrad gravity [79, 80].

To pursue our program, in the present paper we have first of all completed the study of the relativistic kinematics of the system of $N$ free scalar positive-energy particles in the rest-frame instant form of dynamics on Wigner hyper-planes, initiated in Ref.[2].

Then, we have evaluated the energy momentum tensor of the system on the Wigner hyperplane and then determined Dixon’s multipoles for the N-body problem with respect to the internal 3-center of mass located at the origin of the Wigner hyperplane [81]. For an isolated system most of the existing definitions of a collective centroid identify a unique world-line, associated with the internal canonical 3-center of mass. In the rest-frame instant form these multipoles are Cartesian (Wigner-covariant) Euclidean tensors. While the study of the monopole and dipole moments in the rest frame gives information on the mass, the spin and the internal center of mass, the quadrupole moment provides the only (though not unique) way of introducing the concept of barycentric tensor of inertia for extended systems in special relativity.

By exploiting the canonical spin bases of Refs.[2, 21], after the elimination of the internal 3-center of mass ($\vec{q}_+ = \vec{r}_+ = 0$), the Cartesian multipoles $q_{T}^{r_1 \ldots r_n AB}$ can be expressed in terms of 6 orientational variables (the spin vector and the three Euler angles identifying the dynamical body frame) and of $6N - 6$ (rotational scalar) shape variables, i.e. in terms of the canonical pairs of a canonical spin basis.

Having completed the discussion of the isolated system of $N$ positive energy free scalar particles the previous formalism has been applied to an isolated system of $N$ positive-energy particles with mutual action-at-a-distance interactions. Here again we find a unique world-line describing the collective motion of the system.

On the other hand, in the case of an open $n < N$ particle subsystem of an isolated system consisting of $N$ charged positive-energy particles plus the electro-magnetic field a more complex description appears. In the rest frame of the isolated system a suitable definition of the energy-momentum tensor of the open subsystem allows to define its effective mass, 3-momentum and angular momentum. However, unlike the case of isolated systems, each centroid putatively describing the collective centers of motion, gives rise to a different world-line. Starting from the evaluation of the rest-frame Dixon multipoles of the energy-momentum tensor of the open subsystem with respect to various centroids we are given therefore many candidates for an effective center of motion and for an effective intrinsic spin. Two centroids (the center of energy and Tulczyjew centroid) seems to be preferred because of their specific properties. In the case $n = 2$ it is possible to replace the pole-dipole description of the 2-particle cluster with a description of the cluster as an extended system (whose effective spin frame can be evaluated) at the price of introducing an explicit
dependence on the action of the external electro-magnetic field upon the cluster.

Finally, by comparing the effective parameters of an open cluster of $n_1 + n_2$ particles with the effective parameters of the two clusters with $n_1$ and $n_2$ particles, it is shown that only the effective center of energy can in fact play the role of a useful center of motion.

The kinematical concepts we have defined for closed and open N-body systems are enough for the treatment of relativistic continua like relativistic fluids. In [1] a preliminary extension to closed relativistic fluids is given.
APPENDIX A: NON-RELATIVISTIC MULTIPOLAR EXPANSIONS FOR N FREE PARTICLES.

In the review paper of Ref.[27] it can be found a study of the Newtonian multipolar expansions for a continuum isentropic distribution of matter characterized by a mass density $\rho(t, \vec{\sigma})$, a velocity field $U^r(t, \vec{\sigma})$, and a stress tensor $\sigma^{rs}(t, \vec{\sigma})$, with $\rho(t, \vec{\sigma}) \vec{U}(t, \vec{\sigma})$ the momentum density. In case the system is isolated, the only dynamical equations are the mass conservation and the continuum equations of motion, respectively

\[
\frac{\partial \rho(t, \vec{\sigma})}{\partial t} - \frac{\partial \rho(t, \vec{\sigma})}{\partial \sigma^r} U^r(t, \vec{\sigma}) = 0, \quad \frac{\partial \rho(t, \vec{\sigma})}{\partial t} U^r(t, \vec{\sigma}) - \frac{\partial [\rho U^r U^s - \sigma^{rs}]}{\partial \sigma^s}(t, \vec{\sigma}) = 0. \tag{A1}
\]

We can adapt this description to an isolated system of N particles in the following way. The mass density

\[
\rho(t, \vec{\sigma}) = \sum_{i=1}^N m_i \delta^3(\vec{\sigma} - \vec{\eta}_i(t)), \tag{A2}
\]

satisfies

\[
\frac{\partial \rho(t, \vec{\sigma})}{\partial t} = - \sum_{i=1}^N m_i \dot{\vec{\eta}}_i(t) \cdot \vec{\sigma} \delta^3(\vec{\sigma} - \vec{\eta}_i(t)) \overset{\text{def}}{=} \frac{\partial}{\partial \sigma^r} [\rho U^r](t, \vec{\sigma}), \tag{A3}
\]

while the momentum density [82] is

\[
\rho(t, \vec{\sigma}) U^r(t, \vec{\sigma}) = \sum_{i=1}^N m_i \dot{\vec{\eta}}_i(t) \delta^3(\vec{\sigma} - \vec{\eta}_i(t)), \tag{A4}
\]

The associated constant of motion is the total mass $m = \sum_{i=1}^N$.

If we define a function $\zeta(\vec{\sigma}, \vec{\eta}_i)$ concentrated in the N points $\vec{\eta}_i$, $i=1,...,N$, such that $\zeta(\vec{\sigma}, \vec{\eta}_i) = 0$ for $\vec{\sigma} \neq \vec{\eta}_i$ and $\zeta(\vec{\eta}_i, \vec{\eta}_j) = \delta_{ij}$ [83], the velocity field associated to N particles becomes

\[
\vec{U}(t, \vec{\sigma}) = \sum_{i=1}^N \dot{\vec{\eta}}_i(t) \zeta(\vec{\sigma}, \vec{\eta}_i(t)). \tag{A5}
\]

The continuum equations of motion are replaced by

\[
\frac{\partial}{\partial t} [\rho(t, \vec{\sigma}) U^r(t, \vec{\sigma})] \overset{\text{def}}{=} \frac{\partial}{\partial \sigma^r} \sum_{i=1}^N m_i \dot{\vec{\eta}}_i(t) \dot{\vec{\eta}}_i(t) \delta^3(\vec{\sigma} - \vec{\eta}_i(t)) + \sum_{i=1}^N m_i \ddot{\vec{\eta}}_i(t) = \frac{\partial [\rho U^r U^s - \sigma^{rs}]}{\partial \sigma^s}(t, \vec{\sigma}). \tag{A6}
\]
For a system of free particles we have $\ddot{\eta}(t) = 0$ so that $\sigma^{rs}(t, \vec{\sigma}) = 0$. If there are inter-particle interactions, they will determine the effective stress tensor.

Let us consider an arbitrary point $\vec{\eta}(t)$. The multipole moments of the mass density $\rho$ and momentum density $\rho \vec{U}$ and of the stress-like density $\rho U^r U^s$ with respect to the point $\vec{\eta}(t)$ are defined by setting ($N \geq 0$)

$$m^{r_1 \ldots r_N}[\vec{\eta}(t)] = \int d^3\sigma [\sigma^{r_1} - \eta^{r_1}(t)] \ldots [\sigma^{r_N} - \eta^{r_N}(t)] \rho(\tau, \vec{\sigma}) =$$

$$= \sum_{i=1}^{N} m_i [\eta^{r_1}_i(\tau) - \eta^{r_1}(t)] \ldots [\eta^{r_N}_i(t) - \eta^{r_N}(t)],$$

$$n = 0 \quad m[\vec{\eta}(t)] = m = \sum_{i=1}^{N} m_i,$$

$$p^{r_1 \ldots r_N}[\vec{\eta}(t)] = \int d^3\sigma [\sigma^{r_1} - \eta^{r_1}(t)] \ldots [\sigma^{r_N} - \eta^{r_N}(t)] \rho(t, \vec{\sigma}) U^r(t, \vec{\sigma}) =$$

$$= \sum_{i=1}^{N} m_i \dot{\eta}^{r_1}_i(t)[\eta^{r_1}_i(t) - \eta^{r_1}(t)] \ldots [\eta^{r_N}_i(t) - \eta^{r_N}(t)],$$

$$n = 0 \quad p^r[\vec{\eta}(t)] = \sum_{i=1}^{N} m_i \dot{\eta}^r_i(t) = \sum_{i=1}^{N} \kappa^r_i = \kappa^r_+ \approx 0,$$

$$p^{r_1 \ldots r_N s}[\vec{\eta}(t)] = \int d^3\sigma [\sigma^{r_1} - \eta^{r_1}(t)] \ldots [\sigma^{r_N} - \eta^{r_N}(t)] \rho(t, \vec{\sigma}) U^r(t, \vec{\sigma}) U^s(t, \vec{\sigma}) =$$

$$= \sum_{i=1}^{N} m_i \dot{\eta}^{r_1}_i(t) \dot{\eta}^s_i(t)[\eta^{r_1}_i(t) - \eta^{r_1}(t)] \ldots [\eta^{r_N}_i(t) - \eta^{r_N}(t)]. \quad (A7)$$

The mass monopole is the conserved mass, while the momentum monopole is the total 3-momentum, vanishing in the rest frame.

If the mass dipole vanishes, the point $\vec{\eta}(t)$ is the center of mass:

$$m^r[\vec{\eta}(t)] = \sum_{i=1}^{N} m_i [\eta^r_i(t) - \eta^r(t)] = 0 \Rightarrow \vec{\eta}(t) = \vec{q}_{nr}. \quad (A8)$$

The time derivative of the mass dipole is

$$\frac{dm^r[\vec{\eta}(t)]}{dt} = p^r[\vec{\eta}(t)] - m \dot{\eta}^r(t) = \kappa^r_+ - m \dot{\eta}^r(t). \quad (A9)$$

When $\vec{\eta}(t) = \vec{q}_{nr}$, from the vanishing of this time derivative we get the momentum-velocity relation for the center of mass

$$p^r[\vec{q}_{nr}] = \kappa^r_+ = m \dot{q}^r_+ \quad [\approx 0 \text{ in the rest frame}]. \quad (A10)$$

The mass quadrupole is
so that the barycentric mass quadrupole and tensor of inertia are, respectively

\[ m^{rs}[\vec{q}_{nr}] = \sum_{i=1}^{N} m_i \eta_i^r(t) \eta_i^s(t) - m q_{nr}^r q_{nr}^s, \]

\[ I^{rs}[\vec{q}_{nr}] = \delta^{rs} \sum_{a} m^u [\vec{q}_{nr}] - m^{rs}[\vec{q}_{nr}] = \]

\[ = \sum_{i=1}^{1...N-1} k_{ab} (\vec{\rho}_a \cdot \vec{\rho}_b \delta^{rs} - \rho_a^r \rho_b^s), \]

\[ \Rightarrow m^{rs}[\vec{q}_{nr}] = \delta^{rs} \sum_{a,b=1}^{N-1} k_{ab} (\vec{\rho}_a \cdot \vec{\rho}_b - I^{rs}[\vec{q}_{nr}]). \] (A12)

The antisymmetric part of the barycentric momentum dipole gives rise to the spin vector in the following way

\[ p^{rs}[\vec{q}_{nr}] = \sum_{i=1}^{N} m_i \eta_i^r(t) \dot{\eta}_i^s(t) - q_{nr}^r q_{nr}^s, \]

\[ S^u = \frac{1}{2} \epsilon^{urs} p^{rs}[\vec{q}_{nr}] = \sum_{a=1}^{N-1} (\vec{\rho}_a \times \vec{\pi}_{qa})^u. \] (A13)

The multipolar expansions of the mass and momentum densities around the point \( \vec{\eta}(t) \) are

\[ \rho(t, \vec{\sigma}) = \sum_{n=0}^{\infty} \frac{m^{r_1...r_n}[\vec{\eta}^n]}{n!} \frac{\partial^n}{\partial \sigma^{r_1}...\partial \sigma^{r_n}} \delta^3(\vec{\sigma} - \vec{\eta}(t)), \]

\[ \rho(t, \vec{\sigma}) U^r(t, \vec{\sigma}) = \sum_{n=0}^{\infty} \frac{p^{r_1...r_n}[\vec{\eta}^n]}{n!} \frac{\partial^n}{\partial \sigma^{r_1}...\partial \sigma^{r_n}} \delta^3(\vec{\sigma} - \vec{\eta}(t)). \] (A14)

Finally, for the barycentric multipolar expansions we have
\[ \rho(t, \vec{\sigma}) = m \delta^3(\vec{\sigma} - \vec{q}_{nr}) - \frac{1}{2}(\dot{I}^{rs}[\vec{q}_{nr}]) - \frac{1}{2} \delta^{rs} \sum_u I^{uy}[\vec{q}_{nr}] \frac{\partial^2}{\partial \sigma^r \partial \sigma^s} \delta^3(\vec{\sigma} - \vec{q}_{nr}) + \]
\[ + \sum_{n=3}^{\infty} m^{r_1 \ldots r_n} \frac{\partial^n}{\partial \sigma^{r_1} \ldots \partial \sigma^{r_n}} \delta^3(\vec{\sigma} - \vec{q}_{nr}), \]
\[ \rho(t, \vec{\sigma}) U^r(t, \vec{\sigma}) = \kappa_+^r \delta^3(\vec{\sigma} - \vec{q}_{nr}) + \left[ \frac{1}{2} \epsilon^{rsu} S^u + p^{(sr)}[\vec{q}_{nr}] \right] \frac{\partial}{\partial \sigma^s} \delta^3(\vec{\sigma} - \vec{q}_{nr}) + \]
\[ + \sum_{n=2}^{\infty} p^{r_1 \ldots r_n} \frac{\partial^n}{\partial \sigma^{r_1} \ldots \partial \sigma^{r_n}} \delta^3(\vec{\sigma} - \vec{q}_{nr}). \] (A15)
**APPENDIX B: SYMMETRIC TRACE-FREE TENSORS.**

In the applications to gravitational radiation, irreducible symmetric trace-free Cartesian tensors (STF tensors) \[76, 84, 85, 86\] are needed instead of Cartesian tensors. While a Cartesian multipole tensor of rank \(l\) (like the rest-frame Dixon multipoles) on \(R^3\) has \(3^l\) components, \(\frac{1}{2}(l+1)(l+2)\) of which are in general independent, a spherical multipole moment of order \(l\) has only \(2l + 1\) independent components. Even if spherical multipole moments are preferred in calculations of molecular interactions, spherical harmonics have various disadvantages in numerical calculations: for analytical and numerical calculations Cartesian moments are often more convenient (see for instance Ref.\[87\] for the case of the electrostatic potential). It is therefore preferable using the irreducible Cartesian STF tensors\[88, 89\] (having \(2l + 1\) independent components if of rank \(l\)), which are obtained by using Cartesian spherical (or solid) harmonic tensors in place of spherical harmonics.

Given an Euclidean tensor \(A_{k_1 \ldots k_l}\) on \(R^3\), one defines the completely symmetrized tensor

\[
S_{k_1 \ldots k_I} \equiv \frac{1}{I!} \sum_{\pi} A_{k_{\pi(1)} \ldots k_{\pi(I)}}.
\]

Then, the associated STF tensor is obtained by removing all traces \(([I/2] = \text{largest integer} \leq I/2)\)

\[
A^{(STF)}_{k_1 \ldots k_j} = \sum_{n=0}^{[I/2]} a_n \delta(k_1 k_2 \ldots \delta k_{2n+1} k_{2n} S_{k_{2n+1} \ldots k_I})_{i_1 \ldots j_n j_n},
\]

\[
a_n \equiv (-1)^n \frac{l!(2l - 2n - 1)!!}{(l - 2n)!(2l - 1)!!(2n)!!}.
\]

For instance \((T_{abc})^{STF} \equiv T_{(abc)} - \frac{1}{5} \left[ \delta_{ab} T_{(iec)} + \delta_{ac} T_{(ibc)} + \delta_{bc} T_{(aai)} \right].\)
APPENDIX C: THE GARTENAHUS-SCHWARTZ TRANSFORMATION.

In Ref. [2] we defined canonical internal relative variables with respect to the internal 3-center of mass \( \vec{q}_+ \) by exploiting a Gartenhaus-Schwartz canonical transformation. The canonical generator of this transformation is

\[
G = \vec{q}_+ \cdot \vec{\kappa}_+,
\]

so that the finite transformation, depending on a parameter \( \alpha \), on a generic function \( F \) on the phase space is

\[
F(\alpha) = F + \int_0^\alpha d\alpha \{ F(\alpha), G(\alpha) \}.
\]

In particular we have

\[
\lim_{\alpha \to \infty} \vec{\kappa}_+(\alpha) = 0, \quad \lim_{\alpha \to \infty} \vec{q}_+(\alpha) = \infty.
\]

As said in Section II, if we define the canonical transformation (2.19), then the quantities

\[
\vec{\pi}_{qa} = \lim_{\alpha \to \infty} \vec{\pi}_a(\alpha), \quad \vec{\rho}_{qa} = \lim_{\alpha \to \infty} \vec{\rho}_a(\alpha),
\]

are well defined and the transformation

\[
\vec{\eta}_i, \vec{\kappa}_i \rightarrow \vec{\kappa}_+, \vec{q}_+, \vec{\rho}_{qa}, \vec{\pi}_{qa},
\]

is a canonical transformation

\[
\{ q^r_+, \kappa^s_+ \} = \delta^{rs}, \quad \{ \rho^r_{qa}, \pi^s_{qb} \} = \delta^{rs} \delta_{ab},
\]

as said in Eq. (2.18).

The quantities \( \vec{\rho}_{qa}, \vec{\pi}_{qa} \) are the searched internal relative variables: they describe the system after the gauge fixing \( \vec{q}_+ \approx 0, \vec{\kappa}_+ \approx 0 \). We have also

\[
\vec{\kappa}_+ \approx 0 \Rightarrow \vec{\rho}_{qa} \approx \vec{\rho}_a, \quad \vec{\pi}_{qa} \approx \vec{\pi}_a.
\]

Thanks to these results, we can calculate a function \( F \) independent of \( \vec{q}_+ \) on the phase space, under the constraint \( \vec{\kappa}_+ \approx 0 \), by simply performing the limit

\[
F \bigg|_{\vec{\kappa}_+ \approx 0} (\vec{\rho}_{qa}, \vec{\pi}_{qa}) = \lim_{\alpha \to \infty} F(\alpha).
\]

This method is applied in Section IV for calculating the multipoles after the gauge fixing \( \vec{q}_+ \approx 0, \vec{\kappa}_+ \approx 0 \). These multipoles depend on \( \vec{\kappa}_i \), so that (see Ref. [2])

\[
\lim_{\alpha \to \infty} \vec{\kappa}_i(\alpha) = \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\pi}_{qa},
\]

and on \( (\vec{\eta}_i - \vec{R}_+) \), so that
\[ (\vec{\eta}_i - \bar{R}_+) = \sum_j (\vec{\eta}_i - \vec{\eta}_j) \frac{\sqrt{m_j^2 + \bar{\kappa}_j^2}}{\sum_k \sqrt{m_k^2 + \bar{\kappa}_k^2}} = \]
\[ = \sum_j \sum_a \sqrt{N} (\gamma_{ai} - \gamma_{aj}) \tilde{\rho}_a \frac{\sqrt{m_j^2 + \bar{\kappa}_j^2}}{\sum_k \sqrt{m_k^2 + \bar{\kappa}_k^2}}. \quad (C10) \]

Then using the (C9) and (C4) we have

\[
\lim_{\alpha \to \infty} (\vec{\eta}_{\alpha} - \bar{R}_+) = \sum_j \sum_a \sqrt{N} (\gamma_{ai} - \gamma_{aj}) \tilde{\rho}_a \frac{\sqrt{m_j^2 + N \sum_{ab} \bar{\pi}_{qa} \cdot \bar{\pi}_{qb} \gamma_{aj} \gamma_{bj}}}{\sum_k \sqrt{m_k^2 + N \sum_{ab} \bar{\pi}_{qa} \cdot \bar{\pi}_{qb} \gamma_{ak} \gamma_{bk}}} \quad (C11)
\]

The following notation is used to denote the limits (C8)

\[
F \to_{\alpha \to \infty} F |_{\bar{\kappa}_+ \approx 0} (\tilde{\rho}_a, \bar{\pi}_{qa}). \quad (C12)
\]

For example Eqs. (C9) and (C11) become

\[
\bar{R}_i \to_{\alpha \to \infty} \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \bar{\pi}_{qa},
\]
\[
(\vec{\eta}_i - \bar{R}_+) \to_{\alpha \to \infty} \sum_j \sum_a \sqrt{N} (\gamma_{ai} - \gamma_{aj}) \tilde{\rho}_a \frac{\sqrt{m_j^2 + N \sum_{ab} \bar{\pi}_{qa} \cdot \bar{\pi}_{qb} \gamma_{aj} \gamma_{bj}}}{\sum_k \sqrt{m_k^2 + N \sum_{ab} \bar{\pi}_{qa} \cdot \bar{\pi}_{qb} \gamma_{ak} \gamma_{bk}}} \quad (C13)
\]

The closed form (2.20) - (2.22) of the canonical transformation (C5) was not given in Ref.[2], but it can derived from the following two equations of that paper [its Eq.(5.13) and (5.24)]
\[ \bar{\pi}_a(\alpha) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_{ai} \bar{K}_i(\alpha), \]

\[ \bar{\pi}_{qa} \stackrel{def}{=} \bar{\pi}_a(\infty) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \bar{K}_i(\infty) = \]

\[ = \bar{\pi}_a + \frac{\bar{n}_+}{\mathcal{M}} \left[ (M_{sys} - \mathcal{M}) \bar{n}_+ \cdot \bar{\pi}_a - |\bar{\kappa}_+| H_a \right] = \]

\[ = \bar{\pi}_a - \frac{\bar{\kappa}_+}{\sqrt{M_{sys}^2 - \bar{\kappa}_+^2}} \left[ H_a - \frac{M_{sys} - \sqrt{M_{sys}^2 - \bar{\kappa}_+^2}}{\bar{\kappa}_+} \bar{\kappa}_+ \cdot \bar{\pi}_a \right] \approx \bar{\pi}_a, \]

\[ H_a = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_{ai} H_i = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_{ai} \sqrt{m_i^2 + \bar{\kappa}_+^2}, \]

\[ \bar{K}_i(\infty) = \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \bar{\pi}_{qa}, \]

\[ H_{(rel)i} = H_i(\infty) = \sqrt{m_i^2 + N \sum_{ab}^{1..N-1} \gamma_{ai} \gamma_{bij} \bar{\pi}_{qa} \cdot \bar{\pi}_{qb},} \]

\[ M_{sys} = \sum_{i=1}^{N} H_i = \sqrt{\mathcal{M}^2 + \bar{\kappa}_+^2} \approx H_{(rel)} = H_M(\infty) = \mathcal{M} = \]

\[ = \sum_{i=1}^{N} H_i(\infty) = \sum_{i=1}^{N} \sqrt{m_i^2 + N \sum_{ab}^{1..N-1} \gamma_{ai} \gamma_{bij} \bar{\pi}_{qa} \cdot \bar{\pi}_{qb}.} \quad (C14) \]

\[ \bar{\rho}_{qa} \stackrel{def}{=} \bar{\rho}_a(\infty) = \bar{\rho}_a - \]

\[ - \sum_{i,j=1}^{N} \sum_{b=1}^{N-1} \gamma_{aj}(\gamma_{bi} - \gamma_{bj}) \frac{H_i}{M_{sys}} \left[ \bar{\kappa}_+ \bar{K}_j(\infty) \frac{H_j(\infty)}{\sqrt{11}} + \frac{M_{sys}}{\sqrt{11} - 1} \bar{n}_+ \cdot \bar{\rho}_b \right] = \]

\[ = \bar{\rho}_a - \sum_{i,j=1}^{N} \sum_{b=1}^{N-1} \gamma_{aj}(\gamma_{bi} - \gamma_{bj}) \frac{H_i}{M_{sys}} \frac{\bar{K}_j(\infty)}{H_j(\infty) \sqrt{11}} \bar{\kappa}_+ \cdot \bar{\rho}_b \approx \bar{\rho}_a. \quad (C15) \]
APPENDIX D: MORE ON DIXON’S MULTIPOLES.

In this Appendix we shall consider multipoles with respect to the origin, i.e. with \( \vec{\eta} = 0 \) [we use the notation \( t^{\mu_1 \cdots \mu_n \nu}(T_s) = \int_T^{T_s} t^{\mu_1 \cdots \mu_n \nu}(T_s,0) \) and we shall give some of their properties following Ref.[25]]. The proofs of these results are identical to those given in Ref.[25] after the following kinematical modifications of Eqs.(3.6) and (3.7) of that paper:

i) let \( W \) be the world-tube containing the compact support of the isolated system and \( w^{\mu}(\tau) = z^{\mu}(\tau, \vec{\eta}(\tau)) \) a time-like world-line inside it, with tangent vector \( \dot{w}^{\mu}(\tau) = z^{\mu}(\tau, \vec{\eta}(\tau)) + z^{\mu}(\tau, \vec{\eta}(\tau)) \), used to evaluate the multipoles (see Section IV);

ii) the Wigner hyper-planes \( \Sigma_{W,\tau} \) of the rest-frame instant form are in general not orthogonal to the world-line (differently from the hyper-surfaces \( \Sigma(s) \) of Ref.[25]);

iii) Eq.(3.6) is replaced with

\[
\int d\tau \int_{\Sigma_{\tau}} d^3z \frac{1}{\sqrt{g(\tau, \vec{\sigma})}} f(z(\tau, \vec{\sigma})) = \int d\tau \int_{\Sigma_{\tau}} d^3z \frac{1}{\sqrt{g(\tau, \vec{\sigma})}} \dot{f}(z(\tau, \vec{\sigma})) \]

(see before Eq.(2.1) for the notations);

iv) since

\[
l_{\mu}(\tau, \vec{\sigma}) = [l_\rho z^{\rho}_{\mu}]_{\mu}(\tau, \vec{\sigma}) = [\sqrt{2} z^{\mu}_{\mu}]_{\mu}(\tau, \vec{\sigma}),
\]

we get

\[
\int_{\Sigma_{\tau}} d^3z \frac{1}{\sqrt{g(\tau, \vec{\sigma})}} f(z(\tau, \vec{\sigma})) = \int_{\Sigma_{\tau}} d^3\sigma \frac{\sqrt{\gamma(\tau, \vec{\sigma})}}{\sqrt{g(\tau, \vec{\sigma})}} l_{\mu}(\tau, \vec{\sigma}) f(z(\tau, \vec{\sigma})) = \int_{\Sigma_{\tau}} d^3\sigma \frac{\sqrt{\gamma(\tau, \vec{\sigma})}}{\sqrt{g(\tau, \vec{\sigma})}} z_{\mu}(\tau, \vec{\sigma}) f(z(\tau, \vec{\sigma}))
\]

so that we have

\[
\frac{d}{d\tau} \int_{\Sigma_{\tau}} d^3z \frac{1}{\sqrt{g(\tau, \vec{\sigma})}} f(z(\tau, \vec{\sigma})) = \int_{\Sigma_{\tau}} d^3\sigma \frac{\sqrt{\gamma(\tau, \vec{\sigma})}}{\sqrt{g(\tau, \vec{\sigma})}} z_{\mu}(\tau, \vec{\sigma}) \frac{\partial f(z(\tau, \vec{\sigma}))}{\partial z_{\mu}}
\]

v) as a consequence the results quoted in this Appendix hold if the multipoles are taken with respect to world-lines \( w^{\mu}(\tau) = z^{\mu}(\tau, \vec{\eta}) \) such that \( \vec{\eta}(\tau) = \vec{\eta} = const. \) (they are the integral lines of the vector field \( z^{\mu}(\tau, \vec{\sigma}) \partial_{\mu} \)), so that \( \dot{w}^{\mu}(\tau) = z^{\mu}(\tau, \vec{\eta}) \).

As shown in Ref.[25], if a field has a compact support \( W \) on the Wigner hyper-planes \( \Sigma_{W,\tau} \) and if \( f(x) \) is a \( C^\infty \) complex-valued scalar function on Minkowski space-time with compact support \[90\], we have

\[
<T^{\mu \nu}, f> = \int d^4x T^{\mu \nu}(x) f(x) = \int dT_s \int d^3\sigma f(x_s + \delta x_s) T^{\mu \nu}[x_s(T_s) + \delta x_s(\vec{\sigma})][\phi] = \int dT_s \int d^3\sigma \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{-ik \cdot [x_s(T_s) + \delta x_s(\vec{\sigma})]} T^{\mu \nu}[x_s(T_s) + \delta x_s(\vec{\sigma})][\phi] = \int dT_s \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{-ik \cdot x_s(T_s)} \int d^3\sigma T^{\mu \nu}[x_s(T_s) + \delta x_s(\vec{\sigma})][\phi] \]

\[
\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} [k_{\mu} \epsilon_u(\mu(p_s)) \sigma^u]^n = \int dT_s \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{-ik \cdot x_s(T_s)} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} k_{\mu_1} \cdots k_{\mu_n} T_{T, \mu_1 \cdots \mu_n \nu}(T_s),
\]

(D1)
If, in particular, \( f(x) \) is analytic on \( W \), we get

\[
< T^{\mu \nu}, f > = \int dT_s \sum_{n=0}^{\infty} \frac{1}{n!} t^{\mu_1 \ldots \mu_n \mu \nu}(T_s) \frac{\partial^n f(x)}{\partial x^{\mu_1} \ldots \partial x^{\mu_n}} |_{x=x_s(T_s)},
\]

\[
T^{\mu \nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^{\mu_1} \ldots \partial x^{\mu_n}} \int dT_s \delta^4(x - x_s(T_s)) t^{\mu_1 \ldots \mu_n \mu \nu}(T_s). \tag{D2}
\]

For a \( N \) particle system this equation may be rewritten as Eq.(4.4).

On the other hand, for non-analytic functions \( f(x) \) we have

\[
< T^{\mu \nu}, f > = \int dT_s \sum_{n=0}^{N} \frac{1}{n!} t^{\mu_1 \ldots \mu_n \mu \nu}(T_s) \frac{\partial^n f(x)}{\partial x^{\mu_1} \ldots \partial x^{\mu_n}} |_{x=x_s(T_s)} + \int dT_s \int \frac{d^4k}{(2\pi)^4} \tilde{f}^k(k) e^{-i k \cdot x_s(T_s)} \sum_{n=N+1}^{\infty} \frac{(-i)^n}{n!} k_{\mu_1} \ldots k_{\mu_n} t^{\mu_1 \ldots \mu_n \mu \nu}(T_s), \tag{D3}
\]

and, as shown in Ref.[25], from the knowledge of the moments \( t^{\mu_1 \ldots \mu_n \mu}(T_s) \) for all \( n > N \), we can get \( T^{\mu \nu}(x) \) and, therefore, all the moments with \( n \leq N \).

In the case of \( N \) free particles and more in general for isolated systems, the Hamilton equations [92] for the multipoles (4.3), with \( p_T^{\mu_1 \ldots \mu_n \mu}(T_s) \overset{def}{=} p_T^{\mu_1 \ldots \mu_n \mu}(T_s, \vec{0}) \), imply

\[
\frac{dp_T^{\mu}(T_s)}{dT_s} \overset{\circ}{=} 0, \quad \text{for } n = 0,
\]

\[
\frac{dp_T^{\mu_1 \ldots \mu_n \mu}(T_s)}{dT_s} \overset{\circ}{=} -nu^{\mu_1}(p_s)p_T^{\mu_2 \ldots \mu_n \mu}(T_s) + nt^{(\mu_1 \ldots \mu_n) \mu}(T_s), \quad n \geq 1. \tag{D4}
\]

If we define for \( n \geq 1 \)
\[ b^{\mu_1 \ldots \mu_n \mu}(T_s) \overset{\text{def}}{=} p_T^{(\mu_1 \ldots \mu_n \mu)}(T_s) =\]
\[ = \epsilon_{r_1}^{\mu_1}(u(p_s)) \ldots \epsilon_{r_n}^{\mu_n}(u(p_s)) \epsilon_{A}^{\mu}(u(p_s)) \ q_{T}^{r_1 \ldots r_n A \tau}(T_s), \]

\[ c^{\mu_1 \ldots \mu_n \mu}(T_s) \overset{\text{def}}{=} c_T^{(\mu_1 \ldots \mu_n \mu)}(T_s) = p_T^{\mu_1 \ldots \mu_n \mu}(T_s) - p_T^{(\mu_1 \ldots \mu_n \mu)}(T_s) =\]
\[ = [\epsilon_{r_1}^{\mu_1}(u(p_s)) \ldots \epsilon_{r_n}^{\mu_n} \epsilon_{A}^{\mu}(u(p_s)) - \]
\[ - \epsilon_{r_1}^{(\mu_1}(u(p_s)) \ldots \epsilon_{r_n}^{\mu_n}(u(p_s)) \epsilon_{A}^{\mu}(u(p_s))] q_{T}^{r_1 \ldots r_n A \tau}(T_s), \]

\[ c_T^{(\mu_1 \ldots \mu_n \mu)}(T_s) = 0, \quad S_{\mu}^{\mu} = 2p_T^{[\mu]} = 2\epsilon_{\mu}^{\tau}, \]

\[ \epsilon_{r_1}^{\mu_1}(u(p_s)) \ldots \epsilon_{r_n}^{\mu_n}(u(p_s)) b_T^{\mu_1 \ldots \mu_n \mu}(T_s) =\]
\[ = \frac{1}{n + 1} u^{\mu}(p_s) q_{T}^{r_1 \ldots r_n \tau \tau}(T_s) + \epsilon_{r_1}^{\mu}(u(p_s)) q_{T}^{r_1 \ldots r_n \tau \tau}(T_s),\]

\[ \epsilon_{r_1}^{\mu_1}(u(p_s)) \ldots \epsilon_{r_n}^{\mu_n}(u(p_s)) c_T^{\mu_1 \ldots \mu_n \mu}(T_s) =\]
\[ = \frac{n}{n + 1} u^{\mu}(p_s) q_{T}^{r_1 \ldots r_n \tau \tau}(T_s) +\]
\[ + \epsilon_{r_1}^{\mu}(u(p_s)) [q_{T}^{r_1 \ldots r_n \tau \tau}(T_s) - q_{T}^{(r_1 \ldots r_n) \tau \tau}(T_s)], \quad (D5)\]

and for \( n \geq 2 \)

\[ d_T^{\mu_1 \ldots \mu_n \mu}(T_s) = d_{T}^{(\mu_1 \ldots \mu_n \mu)}(T_s) \overset{\text{def}}{=} d_{T}^{\mu_1 \ldots \mu_n \mu}(T_s) -\]
\[ = \frac{n + 1}{n} [d_{T}^{(\mu_1 \ldots \mu_n \mu)}(T_s) + d_{T}^{(\mu_1 \ldots \mu_n \mu)}(T_s)] +\]
\[ + \frac{n + 2}{n} \epsilon_{T_{\mu_1 \ldots \mu_n \mu}}(T_s) =\]
\[ = \left[ \epsilon_{r_1}^{\mu_1} \ldots \epsilon_{r_n}^{\mu_n} \epsilon_{A}^{\mu} \epsilon_{B}^{\nu} - \frac{n + 1}{n} \left( \epsilon_{r_1}^{(\mu_1} \ldots \epsilon_{r_n}^{\mu_n} \epsilon_{A}^{\mu} \epsilon_{B}^{\nu} +\right.\]
\[ + \frac{n + 2}{n} \epsilon_{r_1}^{(\mu_1} \ldots \epsilon_{r_n}^{\mu_n} \epsilon_{B}^{\nu} \epsilon_{A}^{\mu} \right] (u(p_s)) d_{T}^{r_1 \ldots r_n A B \tau}(T_s),\]

\[ d_{T}^{(\mu_1 \ldots \mu_n \mu)}(T_s) = 0, \]

\[ \epsilon_{r_1}^{\mu_1}(u(p_s)) \ldots \epsilon_{r_n}^{\mu_n}(u(p_s)) d_T^{\mu_1 \ldots \mu_n \mu}(T_s) =\]
\[ = \frac{n - 1}{n + 1} u^{\mu}(p_s) u^{\nu}(p_s) q_{T}^{r_1 \ldots r_n \tau \tau}(T_s) +\]
\[ + \frac{1}{n} [u^{\mu}(p_s) \epsilon_{r_1}^{\mu}(u(p_s)) + u^{\nu}(p_s) \epsilon_{r_1}^{\nu}(u(p_s))]\]
\[ + \frac{1}{n} [(n - 1) q_{T}^{r_1 \ldots r_n \tau \tau}(T_s) + q_{T}^{(r_1 \ldots r_n) \tau \tau}(T_s)] +\]
\[ + \epsilon_{s_1}^{\mu}(u(p_s)) \epsilon_{s_2}^{\nu}(u(p_s)) [q_{T}^{r_1 \ldots r_n s_1 s_2}(T_s) -\]
\[ - \frac{n + 1}{n} (q_{T}^{(r_1 \ldots r_n s_1) s_2}(T_s) + q_{T}^{(r_1 \ldots r_n s_2) s_1}(T_s)) +\]
\[ + q_{T}^{(r_1 \ldots r_n s_1 s_2)}(T_s)], \quad (D6)\]
then Eqs.(D4) may be rewritten in the form

1) \( n = 1 \)

\[
\begin{align*}
t^{\mu\nu}(T_s) &= t^{(\mu\nu)}_T(T_s) = p_T^{\mu}(T_s)u^\nu(p_s) + \frac{1}{2} \frac{d}{dT_s}(S_T^{\mu\nu}(T_s) + 2b_T^{\mu\nu}(T_s)), \\
&\downarrow \\
t^{\mu\nu}(T_s) &\overset{\circ}{=} p_T^{(\mu\nu)}(T_s)u^\nu(p_s) + \frac{d}{dT_s}b_T^{\mu\nu}(T_s) + Mu^{\mu}(p_s)u^\nu(p_s) + \\
&+ \kappa^e_+ [u^{(\nu}(p_s)e^\nu_r)](u(p_s)) + e^{(\mu}(u(p_s))u^{\nu)}(p_s)] + \\
&\kappa^e_+ e^{(\mu}(u(p_s))u^{\nu)}(p_s) \sum_{i=1}^{N} \frac{k_i^e}{\sqrt{m_i^2 + k_i^e}} \\
\frac{d}{dT_s}S_T^{\mu\nu}(T_s) &\overset{\circ}{=} 2p_T^{(\mu\nu)}(T_s)u^\nu(p_s) = 2\kappa^e_+ e^{(\mu}(u(p_s))u^{\nu)}(p_s) \approx 0,
\end{align*}
\]

2) \( n = 2 \) [identity \( t^{\rho\mu\nu}_T = t^{(\rho\mu\nu)}_T + t^{(\rho\mu)}_T + t^{(\rho\mu\nu)}_T \)]

\[
\begin{align*}
t^{(\rho\mu\nu)}_T(T_s) &\overset{\circ}{=} 2u^{(\rho}(p_s)b_T^{(\nu\mu)}(T_s) + u^{(\rho}(p_s)S_T^{(\nu\mu)}(T_s) + \frac{d}{dT_s}(b_T^{\rho\mu\nu}(T_s) + \rho_T^{\rho\mu\nu}(T_s)), \\
&\downarrow \\
t^{\rho\mu\nu}(T_s) &\overset{\circ}{=} u^{(\rho}(p_s)b_T^{\mu\nu}(T_s) + S_T^{\rho\mu\nu}(T_s)u^{\nu)}(p_s) + \frac{d}{dT_s} \left( \frac{1}{2} b_T^{\rho\mu\nu}(T_s) - \rho_T^{\rho\mu\nu}(T_s) \right),
\end{align*}
\]

3) \( n \geq 3 \)

\[
\begin{align*}
t^{\mu_1...\mu_n\nu}(T_s) &\overset{\circ}{=} d_T^{\mu_1...\mu_n\nu}(T_s) + u^{(\mu_1}(p_s)b_T^{\mu_2...\mu_n)\nu}(T_s) + 2u^{(\mu_1}(p_s)\rho_T^{\mu_2...\mu_n)\nu}(T_s) + \\
&= 2 \frac{1}{n} c_T^{\mu_1...\mu_n\nu}(T_s)u^{\nu)}(p_s) + \frac{d}{dT_s} \left[ \frac{1}{n+1} b_T^{\mu_1...\mu_n\nu}(T_s) + \frac{2}{n} \rho_T^{\mu_1...\mu_n\nu}(T_s) \right]. \quad \text{(D7)}
\end{align*}
\]

This allows to rewrite \(<T^{\mu\nu},f>\) in the form [25]

\[
\begin{align*}
<T^{\mu\nu},f> &= \int dT_s \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k)e^{-i k \cdot x_s(T_s)} \left[ u^{(\mu}(p_s)p_T^{\nu)}(T_s) - ik_\rho S_T^{\rho(\mu}(T_s)u^{\nu)}(p_s) + \\
&+ \sum_{n=2}^{\infty} \frac{(-i)^n}{n!} k_{\rho_1}...k_{\rho_n} I_T^{\rho_1...\rho_n\nu}(T_s) \right], \quad \text{(D8)}
\end{align*}
\]

with
\[ I_{T}^{\mu_1 \ldots \mu_n \nu \nu}(T_s) = I_{T}^{(\mu_1 \ldots \mu_n)\nu}(T_s) \overset{\text{def}}{=} d_{T}^{\mu_1 \ldots \mu_n \nu \nu}(T_s) - \]
\[ \frac{2}{n-1} u^{(\mu_1}(p_s) c_{T}^{\mu_2 \ldots \mu_n)\nu}(T_s) + \]
\[ + \frac{2}{n} c_{T}^{\mu_1 \ldots \mu_n}(\nu(T_s)) = \]
\[ = \left[ \epsilon_{r_1}^{\mu_1} \ldots \epsilon_{r_n}^{\mu_n} e_{A}^{\nu} - \frac{n+1}{n} \left( \epsilon_{r_1}^{(\mu_1} \ldots \epsilon_{r_n}^{\mu_n)} e_{A}^{\nu} \right) + \]
\[ + \epsilon_{r_1}^{(\mu_1} \ldots \epsilon_{r_n}^{\nu)} e_{A}^{\nu} \right] u(p_s) \]
\[ q_{T}^{r_1 \ldots r_n AB}(T_s) - \]
\[ - \left[ \frac{2}{n-1} \left( \epsilon_{r_1}^{(\mu_2} \ldots \epsilon_{r_n}^{\mu_n)} - \epsilon_{r_1}^{(\mu_2} \ldots \epsilon_{r_n}^{\mu_n)} \right) \right] \]
\[ - \frac{2}{n} \left[ \epsilon_{r_1}^{\mu_1} \ldots \epsilon_{r_n}^{\nu}(\nu u(p_s)) \right] u(p_s) \]
\[ q_{T}^{r_1 \ldots r_n A \nu}(T_s), \]
\[ I_{T}^{\mu_1 \ldots \mu_n \nu}(T_s) = 0, \]
\[ \epsilon_{\mu_1 r_1}(u(p_s)) \ldots \epsilon_{\mu_n r_n}(u(p_s)) I_{T}^{\mu_1 \ldots \mu_n \nu \nu}(T_s) = \]
\[ \frac{n+3}{n+1} u^{\mu}(p_s) u^{\nu}(p_s) q_{T}^{r_1 \ldots r_n \tau \tau}(T_s) + \]
\[ + \frac{1}{n} [u^{\mu}(p_s) q_{T}^{\nu}(u(p_s)) + u^{\nu}(p_s) q_{T}^{\mu}(u(p_s))] q_{T}^{r_1 \ldots r_n \tau \tau}(T_s) + \]
\[ + \epsilon_{s_1}^{\mu}(u(p_s)) \epsilon_{s_2}^{\nu}(u(p_s)) [q_{T}^{r_1 \ldots r_n s_1 s_2}(T_s) - \]
\[ - \frac{n+1}{n} (q_{T}^{r_1 \ldots r_n s_1 s_2}(T_s) + q_{T}^{r_1 \ldots r_n s_2 s_1}(T_s)) + \]
\[ + q_{T}^{(r_1 \ldots r_n s_1 s_2)}(T_s). \]

For a \( N \) particle system, Eq.(D8) implies Eq.(4.15).

Finally, a set of multipoles equivalent to the \( I_{T}^{\mu_1 \ldots \mu_n \nu \nu} \) is [93]:

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for \( n \geq 0 \)

\[
J_{\mu_1 \ldots \mu_n}^{\mu \nu \rho \sigma} (T_s) = J_{\mu_1 \ldots \mu_n}^{\mu} [\mu \nu] [\rho \sigma] (T_s) \equiv I_{\mu_1 \ldots \mu_n}^{\mu} [\mu \nu] [\rho \sigma] (T_s) =
\]

\[
= \frac{1}{n + 1} + \left[ u_{\mu}(p_s) p_{\nu}^{\mu_1 \ldots \mu_n} (T_s) + u_{\rho}^{\sigma} (p_s) p_{\mu_1 \ldots \mu_n}^{\sigma} (T_s) \right] =
\]

\[
\epsilon_{\mu_1}^{\mu} (u(p_s)) \ldots \epsilon_{\mu_n}^{\mu} (u(p_s)) J_{\mu_1 \ldots \mu_n}^{\mu_1 \ldots \mu_n} (T_s) = \frac{(n + 1)(3n + 5)}{4(n - 1)} I_{\mu_1 \ldots \mu_n}^{\mu_1 \ldots \mu_n} (T_s), \quad \text{for} \ n \geq 2,
\]

\[
(q_{1 \ldots n}^{\mu} T_{1 \ldots n}^{\nu} + q_{2 \ldots n}^{\mu} T_{2 \ldots n}^{\nu} + q_{3 \ldots n}^{\mu} T_{3 \ldots n}^{\nu}). \quad \quad (D^{10})
\]

The \( J_{T}^{\mu_1 \ldots \mu_n}^{\mu_1 \ldots \mu_n} \) are the Dixon \( 2^{n+2} \)-pole inertial moment tensors of the extended system: they (or equivalently the \( I_{T}^{\mu_1 \ldots \mu_n}^{\mu_1 \ldots \mu_n} \)) determine the energy-momentum tensor together with the monopole \( P_{\mu}^{\mu} \) and the spin dipole \( S_{\mu}^{\mu} \).

As shown in Section 5 of Ref.[25], the equations of motion \( \partial_{\mu} T_{\mu \nu} = 0 \) do not imply equations of motion for the multipoles \( I_{T}^{\mu_1 \ldots \mu_n}^{\mu_1 \ldots \mu_n} (n \geq 2) \) or \( J_{T}^{\mu_1 \ldots \mu_n}^{\mu_1 \ldots \mu_n} (n \geq 0) \), but only Eqs.(4.16) for the multipoles \( P_{\mu}^{\mu} \) and \( S_{\mu}^{\mu} \) [94]. Instead the multipoles \( I_{T}^{\mu_1 \ldots \mu_n}^{\mu_1 \ldots \mu_n} \) and \( p_{T}^{\mu_1 \ldots \mu_n}^{\mu_1 \ldots \mu_n} \) have non trivial equations of motion.

When all the multipoles \( J_{T}^{\mu_1 \ldots \mu_n}^{\mu_1 \ldots \mu_n} \) are zero (or negligible) one speaks of a pole-dipole system.

On the Wigner hyperplane, the content of these \( 2^{n+2} \)-pole inertial moment tensors is replaced by the Euclidean Cartesian tensors \( q_{T}^{1 \ldots n} T_{1 \ldots n}^{1 \ldots n}, q_{T}^{2 \ldots n} T_{2 \ldots n}^{1 \ldots n}, q_{T}^{3 \ldots n} T_{3 \ldots n}^{1 \ldots n} \). As shown in Appendix
B, we can decompose these Cartesian tensors in their irreducible STF (symmetric trace-free)
parts (the STF tensors).

Thus the multipolar expansion (4.4) may be rewritten as

\[
T^{\mu\nu}[x^i(\bar{q}_r^i)\beta(T_s) + \epsilon^i_r(u(p_s))\sigma^r] = T^{\mu\nu}[w^\beta(T_s) + \epsilon^\beta(u(p_s)) (\sigma^r - \eta^r(T_s))] =
\]

\[
= u^\mu(p_s)\epsilon^i\rho(u(p_s))[\delta^A\gamma + \delta^A u^\kappa]\delta^3(\bar{\sigma} - \bar{\eta}(T_s)) +
\]

\[
+ \frac{1}{2} S_T^\mu(\bar{\eta}) u^\nu(p_s) \epsilon^{\rho}(u(p_s)) \frac{\partial}{\partial\sigma^r} \delta^3(\bar{\sigma} - \bar{\eta}(T_s)) +
\]

\[
+ \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \left[ \frac{n+3}{n+1} u^\mu(p_s) u^\nu(p_s) q^{\tau_1...\tau_n\eta}(T_s, \bar{\eta}) +
\right.
\]

\[
+ \frac{1}{n} [u^\mu(p_s) \epsilon^\nu_r(u(p_s)) + u^\nu(p_s) \epsilon^\mu_r(u(p_s))] q^{\tau_1...\tau_n\eta}(T_s, \bar{\eta}) +
\]

\[
+ \epsilon^\mu_{s_1}(u(p_s)) \epsilon^\nu_{s_2}(u(p_s)) [q^{\tau_1...\tau_n\eta_{s_1} s_2}(T_s, \bar{\eta}) -
\]

\[
- \frac{n+1}{n} (q^{\tau_1...\tau_n s_1} s_2(T_s, \bar{\eta}) + q^{\tau_1...\tau_n s_2} s_1(T_s, \bar{\eta})) + q^{\tau_1...\tau_n s_1 s_2}(T_s, \bar{\eta})]
\]

\[
\left[ \frac{\partial^\mu}{\partial\sigma^{\tau_1}...\partial\sigma^{\tau_n}} \delta^3(\bar{\sigma} - \bar{\eta}(T_s)),
\right]
\]

leading to Eq.(4.15).

For open systems, subsystems of an isolated system like in Section V, we have
\[
\partial\nu T^{\mu\nu} = F^\mu \neq 0,
\]
with \(F^\mu\) an external force. As shown in Ref.[25] for the case in which
\(F^\mu = -F^{\mu\nu} J_\nu (\partial_\mu J^\mu = 0)\) is the Lorentz force, in this case the multipolar expansion (4.15)
is still valid, while the equations of motion (4.16) become (\(P^\mu\) and \(T_s\) are the conserved
4-momentum and the rest-frame time of the global isolated system)

\[
\frac{dP^\mu_c(T_s)}{dT_s} \overset{\sigma}{=} \int d^3\sigma F^\mu(T_s, \bar{\sigma}),
\]

\[
\frac{dS^\mu(T_s)|\bar{\eta} = 0}{dT_s} \overset{\sigma}{=} 2 P^\mu_c(T_s) \epsilon^\nu(P) - \int d^3\sigma \epsilon^\nu(\epsilon^\mu(u(P)) F^\nu(T_s, \bar{\sigma}) - \epsilon^\nu(u(P)) F^\mu(T_s, \bar{\eta})),
\]

leading to Eq.(4.15).
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[8] The Rest-Frame Instant Form of Dynamics and Dirac’s Observables, talk given at the Int.Workshop Physical Variables in Gauge Theories, Dubna 1999.
[9] Such hyper-surfaces are the leaves of a foliation of Minkowski space-time, associated with one among its 3+1 splittings, and are equivalent to a congruence of time-like accelerated observers.
[10] P. A. M. Dirac, Can.J.Math. 2, 129 (1950); Lectures on Quantum Mechanics, Belfer Graduate School of Science, Monographs Series (Yeshiva University, New York, N.Y., 1964).
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[15] A. D. Fokker, Relativiteitstheorie (P. Noordhoff, Groningen, 1929), p.171.
[16] G. N. Fleming, Phys.Rev. 137B, 188 (1965); 139B, 963 (1965).
[17] For instance \( \vec{q}_t \approx 0 \) implies that the internal 3-center of mass is put in the centroid \( x_s^\mu(\tau) = z^\mu(\tau, \vec{\sigma} = 0) \), origin of the 3-coordinates on the Wigner hyper-planes.
[18] S. Gartenhaus and C. Schwartz, Phys.Rev. 108, 842 (1957).
[19] R. G. Littlejohn and M. Reinsch, Rev.Mod.Phys. 69, 213 (1997).
[20] See Ref.[2] for a review of this approach used in molecular physics for the definition and study of the vibrations of molecules.
[21] D. Alba, L. Lusanna and M. Pauri, J.Math.Phys. 43, 373 (2002) (hep-th/0011014).
[22] We adhere to the definitions used in Ref.[19]; in the mathematical literature our left action is a right action.
[23] Their generators are the center-of-mass angular momentum Noether constants of motion.
Their generators are not constants of motion.

W.G.Dixon, J.Math.Phys. 8, 1591 (1967).

See Ref.[27] for the definition of Dixon’s multipoles in general relativity.

W.G.Dixon, Proc.Roy.Soc.London A314, 499 (1970) and A319, 509 (1970); Gen.Rel.Grav. 4, 199 (1973); Phil. Trans.Roy.Soc.London A277, 59 (1974); Extended Bodies in General Relativity: Their Description and Motion, in Isolated Systems in General Relativity, ed.J.Ehlers (North Holland, Amsterdam, 1979).

On it we use the notation $A = (\tau, r)$. The 3-vectors $\vec{B} = \{b^r\}$ on the Wigner hyper-planes are Wigner spin-1 3-vectors. $\epsilon^r_\tau(u(p_s)) = L^\mu_\tau(p_s, \hat{p}_s)$ and $\epsilon^\mu_\tau(u(p_s)) = u^\mu(p_s) = L^\mu_0(p_s, \hat{p}_s)$ are the columns of the standard Wigner boost for time-like Poincaré orbits. See Appendix B of Ref.[2].

Obviously each choice $\vec{\chi}(\tau)$ leads to a different set of conjugate canonical relative variables.

As already said, they describe an internal center-of-mass 3-variable $\vec{\sigma}_{\text{com}}$ defined inside the Wigner hyperplane and conjugate to $\vec{\kappa}_+$; when the $\vec{\sigma}_{\text{com}}$ are canonical variables they are denoted $\vec{q}_+^\tau$.

W.Beiglböck, Commun.Math.Phys. 5, 106 (1967).

See Refs.[33, 34] for the definition of this concept in general relativity. By using the interpretation of Ref.[33], also the special relativistic limit of the general relativistic Dixon centroid of Ref.[27] gives the centroid $x^\mu_s(\tau)$: it coincides with the special relativistic Dixon centroid of Ref.[26] defined by using the conserved energy momentum tensor, as we shall see in Section IV.

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It describes a point living on the Wigner hyper-planes and has the covariance of the little group $O(3)$ of time-like Poincaré orbits, like the Newton-Wigner position operator.

G.Longhi and L.Lusanna, Phys.Rev. D34, 3707 (1986).

It implies $\lambda(\tau) = -1$ and identifies the time parameter $\tau$ with the Lorentz scalar time of the center of mass in the rest frame, $T_s = p_s \cdot \vec{x}_s/M_{\text{sys}}$; $M_{\text{sys}}$ generates the evolution in this time.

See Refs.[39] for a different derivation of this result.

J.M.Pons and L.Shepley, Class.Quant.Grav. 12, 1771 (1995) (gr-qc/9508052). J.M.Pons, D.C.Salisbury and L.Shepley, Phys.Rev. D55, 658 (1997) (gr-qc/9612037).

As in every instant form of dynamics, there are four independent Hamiltonians $p^\alpha_s$ and $J^{\alpha i}_s$, functions of the invariant mass $M_{\text{sys}}$; we give also the expression in the basis $T_s, \epsilon_s, \vec{z}_s, \vec{k}_s$.

This internal Poincaré algebra realization should not be confused with the previous external one based on $S^{\mu\nu}_s$; $\Pi$ and $W^2$ are the two non-fixed invariants of this realization.

The external spin coincides with the internal angular momentum due to Eqs.(A11) of Ref.[2].

As we shall see in the next Section, $\vec{K} \approx 0$ is implied by the natural gauge fixing $\vec{q}_+ \approx 0$.

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W.G.Dixon, Nuovo Cimento 34, 317 (1964).

See Ref. [49] for the application of these methods to find the center of mass of a configuration of the Klein-Gordon field after the preliminary work of Ref.[50] on the center of phase for a real Klein-Gordon field.

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In this way we are sure to have imposed the restriction to positive energy particles and to have excluded the other $2^N - 1$ branches of the total mass spectrum. We have also excluded the other 2 branches of the total mass spectrum.

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[84] So that the origin is also the Fokker-Pryce center of inertia and Pirani and Tulczyjew centroids of the system.

[85] This can be taken as the definitory equation for the velocity field, even if strictly speaking we do not need it in what follows.

[86] It is a limiting concept deriving from the characteristic function of a manifold.

[87] R. Sachs, Proc. Roy. Soc. London A264, 309 (1961), especially Appendix B.

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[93] So that its Fourier transform \( \hat{f}(k) = \int d^4x f(x) e^{ik \cdot x} \) is a slowly increasing entire analytic function on Minkowski space-time \((|x^a + iy^a| q_a ...(x^3 + iy^3) q_3 f(x^\mu + iy^\mu)| < C_{q_0...q_3} e^{a_0|y^0| + ... + a_3|y^3|}, a_\mu > 0, q_\mu \) positive integers for every \( \mu \) and \( C_{q_0...q_3} > 0 \), whose inverse is \( f(x) = \int \frac{d^4k}{(2\pi)^4} \hat{f}(k)e^{-ik \cdot x} \).

[94] See this paper for related results of Mathisson and Tulczyjew.[46, 55, 62, 63].

[95] In Ref.[25] this is a consequence of \( \partial_\mu T^{\mu \nu} = 0 \).

[96] \( A^{[\mu \nu \rho \sigma]} \stackrel{def}{=} \frac{1}{4} (A^{\mu \nu \rho \sigma} - A^{\nu \rho \mu \sigma} - A^{\mu \sigma \nu \rho} + A^{\nu \sigma \mu \rho}). \)

[97] The so called Papapetrou-Dixon-Souriau equations given in Eq.(4.16).