The $R$–matrix of the symplecto–orthogonal quantum superalgebra $U_q(\mathfrak{spo}(2n|2m))$

in the vector representation

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Abstract

The $R$–matrix of the symplecto–orthogonal quantum superalgebra $U_q(\mathfrak{spo}(2n|2m))$
in the vector representation is calculated, and its basic properties are derived.
1 Introduction

The present work is the first of two papers devoted to the construction of the symplecto–orthogonal quantum supergroup $\text{SPO}_q(2n|2m)$ and of some of its comodule superalgebras. In this work, I am going to calculate the $R$–matrix of the symplecto–orthogonal quantum superalgebra $U_q(\mathfrak{spo}(2n|2m))$ in the vector representation. Once this has been done, we can use the techniques of Ref. [1] (generalized to the super case) to define the corresponding quantum supergroup $\text{SPO}_q(2n|2m)$ and to introduce its basic comodule superalgebras. This will be carried out in the subsequent paper [2].

As the reader will immediately notice, my approach is slightly different from what he/she presumably might expect. Hence a few words of explanation are in order. The starting point, and one of the main goals of the present investigation, was to construct a deformed Weyl superalgebra (i.e., a deformed oscillator algebra) $W_q(n|m)$, describing $n$ deformed bosons and $m$ deformed fermions, and covariant under deformed orthosymplectic transformations (I am grateful to V. Rittenberg for insisting that this problem should be solved). Classically, the bosonic/fermionic commutation relations are invariant under symplectic/orthogonal transformations. Since, in supersymmetry, bosons/fermions are regarded to be even/odd, the natural supersymmetric generalization of the above is that a combined system consisting of $n$ bosons and $m$ fermions is invariant under the action of the symplecto–orthogonal Lie superalgebra $\mathfrak{spo}(2n|2m)$, rather than under the action of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m|2n)$. From a practical point of view, this distinction is not really important. It is well–known that the Lie superalgebras $\mathfrak{osp}(2m|2n)$ and $\mathfrak{spo}(2n|2m)$ are naturally isomorphic: Basically, the transition from $\mathfrak{osp}(2m|2n)$ to $\mathfrak{spo}(2n|2m)$ amounts to a shift of the gradation of the vector representation. Nevertheless, I prefer to work from the outset with the natural gradations, and to avoid any shift of gradations.

The second point where I am going to depart from the more familiar formulation is more serious. Since Kac’s basic papers on Lie superalgebras [4], [5] it has become customary to split the family of Lie superalgebras $\mathfrak{osp}(2m|2n)$ into two subfamilies, the $C$–type algebras, which are those with $m = 1$, and the $D$–type algebras, which are those with $m \geq 2$. Accordingly, the so–called distinguished basis of the root system is chosen differently for these two subfamilies.

Needless to say, there are good reasons for considering the $C$– and $D$–type Lie superalgebras separately. In the standard terminology, the former are of type I, while the latter are of type II. This has serious consequences for the general representation theory of these algebras. On the other hand, one must not forget that the root systems of all of the $\mathfrak{osp}(2m|2n)$ algebras have some bases which resemble those of the $C$–type Lie algebra, and others which are similar to those of the $D$–type Lie algebras (so that I would prefer to say that these algebras are of $CD$–type). In particular, for each of the $\mathfrak{osp}(2m|2n)$ algebras, the root system has a basis, which
is of $C$–type and contains only one odd simple root (see Section 2). This is the basis I am going to choose (but, of course, for the $\mathfrak{spo}(2n|2m)$ algebras).

Since the quantum superalgebra associated to a basic classical Lie superalgebra depends on the choice of the basis of the root system, any such choice has non–trivial consequences. The advantage of my choice is that it allows of a simultaneous treatment of all cases, resulting in a unified construction of the corresponding quantum supergroups $\text{SPO}_q(2n|2m)$ and of the deformed Weyl superalgebras $W_q(n|m)$. The reader might wonder whether the differences between the $C$–type and $D$–type Lie superalgebras will not show up at some stage of our investigations. But since in the following we only have to consider the vector module $V$ of $U_q(\mathfrak{spo}(2n|2m))$ and its tensorial square $V \otimes V$, such is not the case.

In principle, the $R$–matrix in question could be calculated by specializing the formula for the universal $R$–matrix given in Ref. [6] (see also Ref. [7]), or by using the results of Ref. [8] (I am grateful to M. Jimbo and M. Okado for drawing my attention to the latter reference). However, I prefer to proceed differently and to determine the corresponding braid generator $\hat{R}$ by investigating the module structure of $V \otimes V$. This procedure has the advantage of yielding the spectral decomposition of $\hat{R}$ as well, moreover, at several places it can serve to check the general theory.

The present work is set up as follows. In Section 2 we introduce the Lie superalgebra $\mathfrak{spo}(2n|2m)$ and fix some notation. In particular, we specify the basis of the root system that we are going to use, and we introduce the corresponding Chevalley–Serre generators of the algebra. Using these data, we define in Section 3 the quantum superalgebra $U_q(\mathfrak{spo}(2n|2m))$ in the sense of Drinfeld [9] and Jimbo [10] (generalized to the super case). Basically, we follow Ref. [6], but some details are different. In Section 4 we introduce the vector module $V$ of $U_q(\mathfrak{spo}(2n|2m))$. This is almost trivial, since (in the usual sloppy terminology) this module is undeformed. We also show that, as in the undeformed case, there exists on $V$ a $U_q(\mathfrak{spo}(2n|2m))$–invariant bilinear form, which is unique up to scalar multiples.

In Section 5 we investigate the structure of the $U_q(\mathfrak{spo}(2n|2m))$–module $V \otimes V$, in particular, we determine its module endomorphisms. This section is central to the present work. Using the results obtained therein, we can calculate the $R$–matrix $R$ (equivalently, the braid generator $\hat{R}$) of $U_q(\mathfrak{spo}(2n|2m))$ in the vector representation. This will be carried out in Section 6. In Section 7 we collect some of the basic properties of $R$ and $\hat{R}$. Section 8 contains a comparison of our results with known special cases. A brief discussion in Section 9 closes the main body of the paper. There are two appendices: In Appendix A we comment on invariant bilinear forms, in Appendix B we introduce what we have called the partial (super)transposition.

We close this introduction by explaining some of our conventions. The base field will be the field $\mathbb{C}$ of complex numbers (in the appendices, we allow for an arbitrary field $\mathbb{K}$ of characteristic zero). If $A$ is an algebra, and if $V$ is an arbitrary (left) $A$–module, the representative of an element $a \in A$ under the corresponding representation will be denoted by $a_V$, and the image of an element $x \in V$ under the
module action of $a$ will be written in the form $a \cdot V(x) = a \cdot x$. The multiplication in a Lie superalgebra will be denoted by pointed brackets $\langle \ , \rangle$. All algebraic notions and constructions are to be understood in the super sense, i.e., they are assumed to be consistent with the $\mathbb{Z}_2$–gradations and to include the appropriate sign factors.

2 Notation and a few comments on the Lie superalgebra $\mathfrak{spo}(2n|2m)$

Essentially, we use the same type of notation as in Ref. [11] (see also Ref. [12]), but adapted to the present setting.

We choose two integers $m, n \geq 1$ and set

$$r = m + n.$$ 

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$–graded vector space such that

$$\dim V_0 = 2n, \quad \dim V_1 = 2m,$$

let $b$ be a non–degenerate, even, super–skew–symmetric, bilinear form on $V$, and let $\mathfrak{spo}(b)$ be the Lie superalgebra consisting of all vector space endomorphisms of $V$ that leave the form $b$ invariant. Then $\mathfrak{spo}(b)$ is isomorphic to $\mathfrak{spo}(2n|2m)$.

According to Ref. [12], the Lie superalgebra $\mathfrak{spo}(b)$ can be described as follows (note that in the cited reference we have written $\mathfrak{osp}(b)$ instead of $\mathfrak{spo}(b)$). Let $\mathfrak{gl}(V_0 \oplus V_1)$ be the general linear Lie superalgebra of the $\mathbb{Z}_2$–graded vector space $V$, and let

$$\theta : V \otimes V \longrightarrow \mathfrak{gl}(V_0 \oplus V_1)$$

be the linear map defined by

$$\theta(x \otimes y)z = b(y, z)x + (-1)^{\xi\eta}b(x, z)y,$$

for all $x \in V_\xi$, $y \in V_\eta$, $z \in V$, with $\xi, \eta \in \mathbb{Z}_2$. Then the kernel of $\theta$ is equal to the subspace of all super–skew–symmetric tensors in $V \otimes V$, its image is equal to the subalgebra $\mathfrak{spo}(b)$ of $\mathfrak{gl}(V_0 \oplus V_1)$, and $\theta$ is an $\mathfrak{spo}(b)$–module homomorphism. In particular, $\theta$ induces an $\mathfrak{spo}(b)$–module isomorphism of the submodule of all super–symmetric tensors in $V \otimes V$ onto the adjoint $\mathfrak{spo}(b)$–module.

Let us make all this more explicit by introducing a suitable basis of $V$. In order to do that we need some more notation. Define the index sets

$$I = \{-r, -r + 1, \ldots, -2, -1, 1, 2, \ldots, r - 1, r\}$$

$$I_\theta = \{-n, -n + 1, \ldots, -2, -1, 1, 2, \ldots, n - 1, n\}$$

$$I_\tau = \{-r, -r + 1, \ldots, -n - 2, -n - 1, n + 1, n + 2, \ldots, r - 1, r\}$$
and also 
\[ J = \{-r, -r + 1, \ldots, -2, -1\} \]
\[ J_\alpha = \{-n, -n + 1, \ldots, -2, -1\} = J \cap I_\alpha \]
\[ J_T = \{-r, -r + 1, \ldots, -n - 2, -n - 1\} = J \cap I_T. \]

Moreover, define the elements \( \eta_i \in \mathbb{Z}_2; i \in I \), by
\[ \eta_i = \alpha \quad \text{if} \quad i \in I_\alpha, \alpha \in \mathbb{Z}_2, \]
and the sign factors 
\[ \sigma_i = (-1)^{h_i}, \quad \sigma_{ij} = (-1)^{h_i h_j} \quad \text{for all} \quad i, j \in I \]
\[ \tau_j = 1 \quad \text{and} \quad \tau_{-j} = -\sigma_j \quad \text{for all} \quad j \in J. \]

Note that 
\[ \tau_i \tau_{-i} = -\sigma_i \quad \text{for all} \quad i \in I \]
(note also that the mapping \( \pi : I \to I \) used in Ref. [11] is given by \( \pi(i) = -i \) for all \( i \in I \)).

Then there exists a homogeneous basis \((e_i)_{i \in I}\) of \( V \) such that \( e_i \) is homogeneous of degree \( \eta_i \), for all \( i \in I \), and such that 
\[ b(e_i, e_j) = \tau_j \delta_{i,-j} \quad \text{for all} \quad i, j \in I. \]

We shall also use the notation 
\[ C_{ij} = b(e_i, e_j), \quad i, j \in I. \quad (2.1) \]

If \( C \) is the \( I \times I \)-matrix with elements \( C_{ij} \), and if \( G \) is the \( I \times I \)-matrix defined by 
\[ G_{ij} = \sigma_i \delta_{ij} \quad \text{for all} \quad i, j \in I, \quad (2.2) \]
we have
\[ C^2 = -G. \quad (2.3) \]

Besides the basis \((e_i)_{i \in I}\) of \( V \), we also use the basis \((f_i)_{i \in I}\), which is dual to \((e_i)\) with respect to \( b \) and is defined by 
\[ b(f_j, e_i) = \delta_{ij} \quad \text{for all} \quad i, j \in I. \]

Obviously, \( f_i \) is homogeneous of degree \( -\eta_i \). Explicitly, we have 
\[ f_i = \sum_{j \in I} (C^{-1})_{ij} e_j = \tau_i e_{-i} \quad \text{for all} \quad i \in I. \]

Using the two bases \((e_i)\) and \((f_i)\) of \( V \), we define the following elements of \( \mathfrak{so}(b) \):
\[ X_{ij} = \theta(e_i \otimes f_j) \quad \text{for all} \quad i, j \in I. \]
Let \((E_{ij})_{i,j \in I}\) be the basis of \(\mathfrak{gl}(V_0 \oplus V_1)\) that canonically corresponds to the basis \((e_i)_{i \in I}\) of \(V\), i.e.,
\[
E_{ij}(e_k) = \delta_{jk}e_i \quad \text{for all } i, j, k \in I .
\]
Then we obtain
\[
X_{ij} = E_{ij} + \sigma_{ij} \sum_{k,\ell} C_{ik}(C^{-1})_{j\ell}E_{\ell k} = E_{ij} + \sigma_{ij} \tau_{-i} \tau_j E_{-j,-i} \quad \text{for all } i, j \in I .
\]
In particular, we have
\[
X_{ii} = E_{ii} - E_{-i,-i} \quad \text{for all } i \in I .
\]
According to the properties of the map \(\theta\), the elements \(X_{ij}\) generate the vector space \(\mathfrak{sp}(b)\), moreover, the super–symmetry of \(\theta\) implies that
\[
\tau_{-j}X_{i,-j} = \sigma_{ij} \tau_{-i} X_{j,-i} \quad \text{for all } i, j \in I .
\]
Thus we have
\[
X_{i,-i} = 0 \quad \text{for all } i \in I .
\]
Let \(\mathfrak{h}\) be the subspace of \(\mathfrak{sp}(b)\) that is spanned by the elements \(X_{ii}, i \in I\). Obviously, \(\mathfrak{h}\) consists of those elements of \(\mathfrak{sp}(b)\) whose matrices with respect to the basis \((e_i)\) are diagonal, and the \(X_{jj}\) with \(j \in J\) form a basis of \(\mathfrak{h}\).

Define, for every \(i \in I\), the linear form \(\varepsilon_i\) on \(\mathfrak{h}\) by
\[
H(e_i) = \varepsilon_i(H)e_i \quad \text{for all } H \in \mathfrak{h} .
\]
Then it is easy to see that
\[
\varepsilon_{-i} = -\varepsilon_i \quad \text{for all } i \in I ,
\]
and that
\[
\varepsilon_i(X_{jj}) = \delta_{ij} \quad \text{for all } i, j \in J .
\]
Thus \((\varepsilon_j)_{j \in J}\) is the basis of \(\mathfrak{h}^*\) that is dual to the basis \((X_{jj})_{j \in J}\) of \(\mathfrak{h}\).

Since \(\theta\) is an \(\mathfrak{sp}(b)\)–module homomorphism, it follows that
\[
\langle H, X_{ij} \rangle = (\varepsilon_i - \varepsilon_j)(H)X_{ij}
\]
for all \(H \in \mathfrak{h}\) and all \(i, j \in I\) (recall that the multiplication in a Lie superalgebra is denoted by pointed brackets). We conclude that \(\mathfrak{h}\) is a Cartan subalgebra of \(\mathfrak{sp}(b)\), that
\[
\Delta = \{\varepsilon_i - \varepsilon_j | i, j \in I; j \neq i \text{ and } j < -i, \text{ or } j = -i \in I_0\}
\]
is the root system of \(\mathfrak{sp}(b)\) with respect to \(\mathfrak{h}\), and that \(X_{ij}\) is a (non–zero) root vector corresponding to the root \(\varepsilon_i - \varepsilon_j\) (with \(i, j\) as specified on the right hand side of Eqn. (2.4)). The root \(\varepsilon_i - \varepsilon_j\) is even/odd depending on whether \(\sigma_i \sigma_j = \pm 1\).
In order to introduce an adequate bilinear form on \( h^* \), we recall that the invariant bilinear form
\[ (X,Y) \mapsto \frac{1}{2} \text{Str}(XY) \]
on \( \mathfrak{spo}(b) \) is non–degenerate and super–symmetric; consequently, its restriction to \( h \) is likewise. Let \((\mathbf{\cdot} | \mathbf{\cdot})\) denote the bilinear form on \( h^* \) that is inverse to this restriction. By definition, we have
\[ (\lambda | \mu) = \frac{1}{2} \text{Str}(H_\lambda H_\mu) \]
for all \( \lambda, \mu \in h^* \), where, for example, the element \( H_\lambda \in h \) is uniquely determined through the equation
\[ \lambda(H) = \frac{1}{2} \text{Str}(H_\lambda H) \text{ for all } H \in h. \quad (2.5) \]
It is easy to check that
\[ H_\lambda = \sum_{j \in J} \sigma_j \lambda(X_{jj})X_{jj} \text{ for all } \lambda \in h^*, \]
and that
\[ (\varepsilon_i | \varepsilon_j) = \sigma_i \delta_{ij} \text{ for all } i, j \in J. \]

Let us now specify the basis of the root system \( \Delta \) that we are going to use in the following. It is equal to \((\alpha_j)_{j \in J}\), where the simple roots \( \alpha_j \) are defined by
\[ \alpha_j = \begin{cases} 
\varepsilon_j - \varepsilon_{j+1} & \text{for } -r \leq j \leq -2 \\
2\varepsilon_{-1} & \text{for } j = -1 .
\end{cases} \]
Note that \( \alpha_{-n-1} \) is the sole odd simple root. The corresponding Chevalley–Serre generators of \( \mathfrak{spo}(b) \) are denoted by \( E^v_j, F^v_j, H^v_j; \ j \in J \), and are introduced as follows. First of all, we choose
\[
E^v_j = \begin{cases}
X_{j+1,j} = E_{j+1,j} - \sigma_j \sigma_{j+1} E_{-j-1,-j} & \text{for } -r \leq j \leq -2 \\
\frac{1}{2} X_{-1,1} = E_{-1,1} & \text{for } j = -1 
\end{cases}
\]
\[
F^v_j = \begin{cases}
X_{j+1,j} = E_{j+1,j} - \sigma_{j+1} \sigma_{j+1,j} E_{-j-1,-j} & \text{for } -r \leq j \leq -2 \\
\frac{1}{2} X_{1,-1} = E_{1,-1} & \text{for } j = -1 
\end{cases}
\]

**Remark 2.1.** It is easy to check that
\[ \sigma_{j,j+1} = \sigma_{j+1} \text{ for } -r \leq j \leq -2 , \quad (2.6) \]
or, equivalently, that
\[ \sigma_{j,j+1} = \sigma_j \text{ for } 1 \leq j \leq r - 1. \quad (2.7) \]
This implies that in the equation for \( E^v_j, -r \leq j \leq -2 \), the factor \( \sigma_{j,j+1} \) might be replaced by \( \sigma_{j+1} \), and in the equation for \( F^v_j, -r \leq j \leq -2 \), the \( \sigma \)-factors might be dropped. I prefer to keep the \( \sigma \)-factors as they stand: They have an
immediate meaning in terms of the sign rules of supersymmetry, and by modifying the equations for the $E^v_j$ and $F^v_j$ as mentioned above, we might well end up in the unpleasant situation where we would have to check (possibly implicitly) the equations (2.6) or (2.7) again and again.

Using the elements $E^v_j$ and $F^v_j$, we define the generators $H^v_j \in \mathfrak{h}$ as usual by

$$H^v_j = \langle E^v_j, F^v_j \rangle \quad \text{for all } j \in J.$$  \hspace{1cm} (2.8)

More explicitly, we find

$$H^v_j = \begin{cases} X_{jj} - \sigma_j \sigma_{j+1} X_{j+1,j+1} & \text{for } -r \leq j \leq -2 \\ X_{-1,-1} & \text{for } j = -1. \end{cases}$$

Then the generators $E^v_j$, $F^v_j$, $H^v_j$ satisfy the following familiar relations, which hold for all $i, j \in J$:

$$\langle H^v_i, H^v_j \rangle = 0 \quad \text{(2.9)}$$

$$\langle E^v_i, F^v_j \rangle = \delta_{ij} H^v_j \quad \text{(2.10)}$$

$$\langle H^v_i, E^v_j \rangle = a_{ij} E^v_j, \quad \langle H^v_i, F^v_j \rangle = -a_{ij} F^v_j. \quad \text{(2.11)}$$

Here, $A = (a_{ij})_{i,j \in J}$ is the Cartan matrix, whose elements are defined by

$$a_{ij} = \alpha_i(H^v_j) \quad \text{for all } i, j \in J.$$

The Cartan matrix is tridiagonal. For $n \geq 2$, it takes the following form:

$$A = \begin{pmatrix} 2 & -1 & \\
-1 & 2 & -1 \\
\end{pmatrix}_{-r \leq j \leq -2}$$

$$A = \begin{pmatrix} -1 & 2 & -1 \\
-1 & 0 & 1 \\
-1 & 2 & -1 \\
\end{pmatrix}_{-r \leq j \leq -2}$$

where the zero on the diagonal has the row and column number $-n-1$. For $n = 1$, the zero is in position $(-2, -2)$, and the lower right corner of $A$ is equal to

$$\begin{pmatrix} 2 & -1 & 0 \\
-1 & 0 & 2 \\
0 & -1 & 2 \end{pmatrix}.$$

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Finally, for \( m = n = 1 \) the Cartan matrix is given by
\[
A = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}.
\]
Thus the Dynkin diagram of \( \mathfrak{spo}(b) \simeq \mathfrak{spo}(2n | 2m) \) with respect to our basis of the root system takes the form

![Dynkin diagram of the Lie superalgebra \( \mathfrak{spo}(2n | 2m) \)]

Remark 2.2. It may be helpful to comment on the rules according to which the generators \( H^v_i \) (and hence \( E^v_i \) and \( F^v_i \)) have been chosen. If the simple root \( \alpha_i \) is even, we choose \( H^v_i \) such that \( \alpha_i(H^v_i) = 2 \). For odd simple roots, the situation is more complicated. In the present case, the sole odd simple root \( \alpha_i, i = -n - 1 \), is such that \( (\alpha_i | \alpha_i) = 0 \). Then it follows that \( a_{ii} = 0 \), and the element \( H^v_i \) is usually chosen such that (for this index \( i \))
\[
a_{ij} \in \mathbb{Z} \quad \text{for all } j \in J,
\]
and such that these \( a_{ij} \) don’t have a common divisor. This fixes the \( a_{ij} \) up to a common sign factor, which (according to Kac) is chosen such that \( a_{i,i+1} > 0 \) (assuming that \( i + 1 \in J \) and that \( a_{i,i+1} \neq 0 \)). These conventions are introduced simply for convenience, and they are of little (if any) importance. Note that, for \( m = n = 1 \), we haven’t followed these conventions: The first row of the Cartan matrix could be divided by 2, and for \( \mathfrak{sl}(2 | 1) \simeq \mathfrak{spo}(2 | 2) \) this is usually done. Our choice is motivated by the wish for a unified treatment of all cases.

The relations (2.9) – (2.11) given above are not sufficient to characterize the Lie superalgebra \( \mathfrak{spo}(2n | 2m) \) completely, there are certain Serre-type and supplementary relations which must also be satisfied. We don’t give these relations here, but only mention that they can be read off from the relations (3.5) – (3.13) by setting \( q = 1 \).

### 3 Definition of the quantum superalgebra \( U_q(\mathfrak{spo}(2n | 2m)) \)

The notation introduced in the preceding section will now be used to define the quantum superalgebra \( U_q(\mathfrak{spo}(2n | 2m)) \). Basically, we are going to follow Ref. [5], however, there will be differences in detail.
Define the diagonal \( J \times J \)-matrix \( D \) by
\[
D = (d_i \delta_{ij})_{i,j \in J} = \text{diag}(-1, -1, \ldots, -1, 1, 1, \ldots, 1, 2)
\]
It is chosen such that
\[
(DA)_{ij} = (\alpha_i | \alpha_j) \quad \text{for all } i, j \in J.
\]
In particular, the matrix \( DA \) is symmetric.

Let \( q \in \mathbb{C} \) be a non-zero complex number, and assume that \( q \) is not a root of unity. We use the abbreviation
\[
q_i = q^{d_i}.
\]
Then we have
\[
q_i^{a_{ij}} = q^{(\alpha_i | \alpha_j)} \quad \text{for all } i, j \in J.
\]

Now the quantum superalgebra \( U_q(\mathfrak{spo}(2n|2m)) \) is defined to be the universal associative superalgebra (with unit element) with generators \( K_i, K_i^{-1}, E_i, F_i; i \in J \), and certain relations to be specified below. The \( \mathbb{Z}_2 \)-gradation is fixed by requiring that \( E_{-n-1} \) and \( F_{-n-1} \) be odd, while all the other generators are even. (Needless to say, one has to check that the relations are compatible with this requirement.) The relations are the following, they are assumed to hold for all \( i, j \in J \):
\[
K_i K_i^{-1} = K_i^{-1} K_i = 1 \quad (3.1)
\]
\[
K_i K_j = K_j K_i \quad (3.2)
\]
\[
K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j \quad (3.3)
\]
\[
\langle E_i, F_j \rangle = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}. \quad (3.4)
\]
In addition, the generators \( E_i \) satisfy certain Serre–type and supplementary relations among themselves, as do the generators \( F_i \). We only write the relations for the \( E_i \), those for the \( F_i \) are obtained from these by simply replacing \( E \) by \( F \).

In the subsequent relations, it is always assumed that \( i, j \in J \). Suppose first that the root \( \alpha_i \) is even, i.e., that \( i \neq -n - 1 \). Then we have
\[
\langle E_i, E_j \rangle = 0 \quad \text{for } a_{ij} = 0 \quad (3.5)
\]
\[
E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{for } i \leq -3, |i - j| = 1. \quad (3.6)
\]
If \( \alpha_{-2} \) is even, i.e., if \( n \geq 2 \), we have (as in the case of symplectic Lie algebras)
\[
E_{-2}^2 E_{-3} - (q + q^{-1}) E_{-2} E_{-3} E_{-2} + E_{-3} E_{-2}^2 = 0 \quad (3.7)
\]
\[
E_{-2}^3 E_{-1} - (q^2 + 1 + q^{-2}) E_{-2} E_{-1} E_{-2} + (q^2 + 1 + q^{-2}) E_{-2} E_{-1} E_{-2}^2 - E_{-1} E_{-2}^3 = 0. \quad (3.8)
\]
In all cases, the generators $E_{-2}$ and $E_{-1}$ satisfy

$$E_{-1}^2 E_{-2} - (q^2 + q^{-2}) E_{-1} E_{-2} E_{-1} + E_{-2} E_{-1}^2 = 0. \quad (3.9)$$

Next we recall that $\alpha_{-n-1}$ is the sole odd simple root, and that this root is isotropic. Correspondingly, we have

$$\langle E_{-n-1}, E_j \rangle = 0 \quad \text{for} \quad a_{-n-1,j} = 0, \quad (3.10)$$
in particular,

$$E_{-n-1}^2 = 0. \quad (3.11)$$

Finally, there are the following supplementary relations. If $m, n \geq 2$, we have

$$\langle E_{-n-1}, \langle E_n, \langle E_{-n-1}, E_{-n-2} \rangle_q \rangle_q^{-1} \rangle_q \rangle_q = 0, \quad (3.12)$$

and for $n = 1, m \geq 3$ we have

$$\langle E_{-2}, \langle E_{-3}, \langle E_{-2}, \langle E_{-1}, \langle E_{-3}, E_{-4} \rangle_q \rangle_q^{-2} \rangle_q \rangle_q^{-1} \rangle_q \rangle_q = 0. \quad (3.13)$$

The last two relations are expressed in terms of so–called $q$–supercommutators. We recall the definition: If $A$ is any associative superalgebra, if $p$ is any non–zero complex number, and if $X \in A_\xi$ and $Y \in A_\eta$, with $\xi, \eta \in \mathbb{Z}_2$, the $p$–supercommutator of $X$ and $Y$ is defined by

$$\langle X, Y \rangle_p = XY - p(-1)^{\xi \eta} YX.$$

Obviously, we have

$$\langle X, Y \rangle_1 = \langle X, Y \rangle.$$

As shown below, the Serre–type relations can also be expressed in terms of $q$–supercommutators.

The superalgebra $U_q(\mathfrak{sp}(2n|2m))$ is converted into a Hopf superalgebra by means of structure maps, which are fixed by the following equations:

- **coproduct**
  \[
  \Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1} \\
  \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \\
  \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i
  \]

- **counit**
  \[
  \varepsilon(K_i^{\pm 1}) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0
  \]

- **antipode**
  \[
  S(K_i^{\pm 1}) = K_i^{\pm 1}, \quad S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i.
  \]

In the subsequent series of remarks, we collect some elementary properties of the Hopf superalgebra $U_q(\mathfrak{sp}(2n|2m))$. 

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Remark 3.1. Let $Q$ be the root lattice of $\mathfrak{spo}(2n|2m)$, i.e.,

$$Q = \sum_{i \in J} \mathbb{Z} \alpha_i .$$

Then the algebra $U_q(\mathfrak{spo}(2n|2m))$ admits a unique $Q$–gradation such that, for all $i \in J$, the element $E_i$ is homogeneous of degree $\alpha_i$, $F_i$ is homogeneous of degree $-\alpha_i$, and $K_i$ is homogeneous of degree 0. In view of a more general definition to be given later, the $Q$–degree of a $Q$–homogeneous element is called its weight. If an element $X \in U_q(\mathfrak{spo}(2n|2m))$ has the weight $\lambda \in Q$, it satisfies

$$K_j X K_j^{-1} = q^{(\alpha_j|\lambda)} X \quad \text{for all } j \in J .$$

Conversely, if an element $X \in U_q(\mathfrak{spo}(2n|2m))$ satisfies this condition, it is $Q$–homogeneous of weight $\lambda$ (since $q$ is not a root of unity). Note that the structure maps $\Delta$, $\varepsilon$, and $S$ are $Q$–homogeneous of degree zero.

Remark 3.2. The antipode $S$ is bijective. To prove this, we show that $S^2$ is bijective. Indeed, it is easy to check that, for all $i \in J$,

$$S^2(K_i) = K_i \quad , \quad S^2(E_i) = q^{-\langle \alpha_i|\alpha_i \rangle} E_i \quad , \quad S^2(F_i) = q^{\langle \alpha_i|\alpha_i \rangle} F_i . \quad (3.14)$$

Since $S^2$ is an algebra endomorphism of $U_q(\mathfrak{spo}(2n|2m))$, and since a suitable set of monomials in the generators $K_i^{\pm 1}$, $E_i$, and $F_i$ forms a basis of $U_q(\mathfrak{spo}(2n|2m))$, this implies our claim.

Actually, $S^2$ is an inner automorphism of the algebra $U_q(\mathfrak{spo}(2n|2m))$. Let $2\rho$ denote the sum of the even positive roots minus the sum of the odd positive roots of $\mathfrak{spo}(2n|2m)$. Explicitly, we have

$$2\rho = -2 \sum_{i=-r}^{-n-1} (i + 2n + 1)\varepsilon_i - 2 \sum_{i=-n}^{-1} i\varepsilon_i ,$$

and it is easy to check that

$$(2\rho|\alpha_i) = (\alpha_i|\alpha_i) \quad \text{for all } i \in J .$$

Given an arbitrary linear combination of the simple roots $\alpha_i$ with coefficients $r_i \in \mathbb{Z}$,

$$\lambda = \sum_{i \in J} r_i \alpha_i \in Q ,$$

we define

$$K_\lambda = \prod_{i \in J} K_i^{r_i} .$$

In particular, we have

$$K_{\alpha_i} = K_i \quad \text{for all } i \in J .$$
Then the Eqns. (3.14) immediately imply that
\[ S^2(X) = K_{-2\rho} X K_{-2\rho}^{-1} \text{ for all } X \in U_q(\mathfrak{sp}(2n|2m)). \]

**Remark 3.3.** Obviously, there is a certain symmetry between the \(E\) and \(F\) generators of \(U_q(\mathfrak{sp}(2n|2m))\). To make this more explicit, we note that there is a unique algebra endomorphism \(\varphi: U_q(\mathfrak{sp}(2n|2m)) \to U_q(\mathfrak{sp}(2n|2m))\), such that for all \(i \in J\)
\[ \varphi(E_i) = F_i, \quad \varphi(F_i) = (-1)^{\gamma_i} E_i, \quad \varphi(K_i^{\pm 1}) = K_i^{\mp 1}, \]
where \(\gamma_i \in \mathbb{Z}_2\) is the degree of \(E_i\). This endomorphism is homogeneous of \(\mathbb{Z}_2\)-degree zero, and we have \(\varphi^4 = \text{id}\). Consequently, \(\varphi\) is an automorphism of the (associative) superalgebra \(U_q(\mathfrak{sp}(2n|2m))\).

Now let \(U_q(\mathfrak{sp}(2n|2m))^{\text{cop}}\) be the bi–superalgebra which, regarded as a \(\mathbb{Z}_2\)–graded algebra, coincides with \(U_q(\mathfrak{sp}(2n|2m))\), but whose coalgebra structure is opposite (in the super sense) to that of \(U_q(\mathfrak{sp}(2n|2m))\). Then it is easy to check that
\[ \varphi: U_q(\mathfrak{sp}(2n|2m)) \to U_q(\mathfrak{sp}(2n|2m))^{\text{cop}} \quad (3.15) \]
is a homomorphism of bi–superalgebras. Since \(\varphi\) is bijective, this implies that \(U_q(\mathfrak{sp}(2n|2m))^{\text{cop}}\) is a Hopf superalgebra, and that \(\varphi\) is a Hopf superalgebra isomorphism. As is well-known, it follows (once again) that \(S\) is bijective, and that \(S^{-1}\) is the antipode of \(U_q(\mathfrak{sp}(2n|2m))^{\text{cop}}\).

The Serre–type and the supplementary relations can be written in various ways. Before we do that, we remind the reader of the definition of the adjoint representation of a Hopf superalgebra \(H\). This is a (graded) representation of the superalgebra \(H\) on the graded vector space \(H\), it is denoted by \(\text{ad}\), and is defined as follows. Let \(X\) be an arbitrary element of \(H\), and set
\[ \Delta(X) = \sum_a X^1_a \otimes X^2_a, \]
with homogeneous elements \(X^1_a, X^2_a \in H\), of degree \(\xi^1_a\) and \(\xi^2_a\), respectively. Then \(\text{ad}X\) (the representative of \(X\)) is given by
\[ (\text{ad}X)(Y) = \sum_a (-1)^{\xi_a^2 \eta} X^1_a Y S(X^2_a), \]
for all homogeneous elements \(Y \in H_\eta\), where \(\eta \in \mathbb{Z}_2\). We note that \(\text{ad}X\) is a generalized derivation in the sense that, if \(Y'\) is another element of \(H\),
\[ (\text{ad}X)(YY') = \sum_a (-1)^{\xi_a^2 \eta}(\text{ad}X^1_a)(Y)(\text{ad}X^2_a)(Y'). \]
Now suppose that $S$ bijective. This implies that $H^{\text{cop}}$ (see the analogous definition of $U_q(\mathfrak{spo}(2n|2m))^{\text{cop}}$ given above) is a Hopf superalgebra with antipode $S^{-1}$. Let $\overline{\text{ad}}$ be the adjoint representation of $H^{\text{cop}}$. Then $\overline{\text{ad}}$ is a graded representation of $H$ in $H$, it is given by

$$
(\overline{\text{ad}} X)(Y) = \sum_a (-1)^{\xi_i^1(\xi_i^2+\eta)} X_a^2 Y S^{-1}(X_a^1),
$$

and it satisfies

$$
(\overline{\text{ad}} X)(Y Y') = \sum_a (-1)^{\xi_i^1(\xi_i^2+\eta)} (\overline{\text{ad}} X_a^2)(Y)(\overline{\text{ad}} X_a^1)(Y').
$$

Let us now choose $H = U_q(\mathfrak{spo}(2n|2m))$. Then the isomorphism $\varphi$ given in (3.15) shows that $\overline{\text{ad}}(\varphi(X)) = \varphi \circ (\text{ad} X) \circ \varphi^{-1}$ for all $X \in U_q(\mathfrak{spo}(2n|2m))$. (3.16)

Moreover, for every element $X \in U_q(\mathfrak{spo}(2n|2m))$ of weight $\lambda$ we have

$$
(\text{ad} E_i)(X) = \langle E_i, X \rangle q^{(\alpha_i|\lambda)}
$$

(3.17)

$$
(\text{ad} F_i)(X) = \langle F_i, X \rangle q^{-1(\alpha_i|\lambda)}.
$$

(3.18)

We note that in the proof of these equations we only have to use the first resp. second of the relations (3.3) but none of the other defining relations.

Now Eqn. (3.17) implies that the left hand side of Eqn. (3.5) is equal to

$$
\langle E_i, E_j \rangle = (\text{ad} E_i)(E_j),
$$

the left hand side of Eqn. (3.6) is equal to

$$
\langle E_i, \langle E_i, E_j \rangle q^{\pm 1} \rangle q^{\pm 1} = (\text{ad} E_i)^2(E_j),
$$

the left hand side of Eqn. (3.7) is equal to

$$
\langle E_{-2}, \langle E_{-2}, E_{-3} \rangle q^{\pm 1} \rangle q^{\pm 1} = (\text{ad} E_{-2})^2(E_{-3}),
$$

the left hand side of Eqn. (3.8) is equal to

$$
\langle E_{-2}, \langle E_{-2}, \langle E_{-2}, E_{-1} \rangle q^{\pm 1} \rangle q^{\pm 1} \rangle q^{\pm 2} = (\text{ad} E_{-2})^3(E_{-1}),
$$

the left hand side of Eqn. (3.9) is equal to

$$
\langle E_{-1}, \langle E_{-1}, E_{-2} \rangle q^{\pm 2} \rangle q^{\pm 2} = (\text{ad} E_{-1})^2(E_{-2}),
$$

the left hand side of Eqn. (3.10) is equal to

$$
\langle E_{-n-1}, E_j \rangle = (\text{ad} E_{-n-1})(E_j),
$$
the left hand side of Eqn. (3.12) is equal to
\[ \langle E_{-n-1}, \langle E_{-n}, \langle E_{-n-1}, E_{-n-2} \rangle_q \rangle_{q^{-1}} \rangle = (\text{ad} E_{-n-1})(\text{ad} E_{-n})(\text{ad} E_{-n-1})(E_{-n-2}), \quad (3.19) \]
and the left hand side of Eqn. (3.13) is equal to
\[ \langle E_{-2}, \langle E_{-3}, \langle E_{-2}, \langle E_{-1}, \langle E_{-2}, \langle E_{-3}, E_{-4} \rangle_q \rangle_{q^{-1}} \rangle_{q^{-2}} \rangle_q \rangle_{q^{-1}} \rangle = (\text{ad} E_{-2})(\text{ad} E_{-3})(\text{ad} E_{-2})(\text{ad} E_{-1})(\text{ad} E_{-2})(\text{ad} E_{-3})(E_{-4}). \quad (3.20) \]

**Remark 3.4.** Using Eqn. (3.11) and the fact that \( E_{-n-2} \) and \( E_{-n} \) commute, it is easy to see that the left hand side of Eqn. (3.19) is invariant under the substitution \( q \to q^{-1} \). Somewhat unexpectedly, it seems that this is not the case for the left hand side of Eqn. (3.20), even if one assumes that all the relations for the \( E \)-generators except Eqn. (3.13) are satisfied.

**Remark 3.5.** Taking the defining relations for granted except (3.13), one can show that the expressions in Eqn. (3.20) are annihilated by all \( \text{ad} F_i \). Since, quite generally, we have
\[ (\text{ad} F_i)(X) = \langle F_i, X \rangle K_i \quad \text{for all } i \in J \text{ and all } X \in U_q(\mathfrak{spo}(2n|2m)), \]
the same is true when acting with \( \langle F_i, \cdot \rangle \). This shows that by “acting” on the relation (3.13) with the generators \( F_i \), we cannot derive new relations for the \( E \)-generators.

Up to now we have only discussed the Serre–type and supplementary relations for the \( E \)-generators. Of course, similar comments can be made for the \( F \)-generators as well, but with \( \text{ad} \) replaced by \( \overline{\text{ad}} \). In fact, all we have to do is to apply the isomorphism \( \varphi \) given in (3.15) and to recall Eqn. (3.16).

We close this section by a remark on the weights of a \( U_q(\mathfrak{spo}(2n|2m)) \)-module \( W \). In the present work, all \( U_q(\mathfrak{spo}(2n|2m)) \)-modules will be weight modules, in the sense that the representatives \( (K_j)_W, j \in J \), are simultaneously diagonalizable, and such that, for any common eigenvector \( x \) of these operators, we have
\[ K_j \cdot x = q^{(\alpha_j|\lambda)} x \quad \text{for all } j \in J, \]
with a linear form
\[ \lambda \in \sum_{i \in J} \mathbb{Z} \varepsilon_i. \]
Since \( q \) is not a root of unity, the linear form \( \lambda \) is uniquely fixed by these conditions and is called the **weight** of \( x \). This definition generalizes the definition of the weight of an element of \( U_q(\mathfrak{spo}(2n|2m)) \): In that case the representation considered is the adjoint representation \( \text{ad} \) or its modified version \( \overline{\text{ad}} \).
4 The vector module $V$ of $U_q(\mathfrak{spo}(2n|2m))$

Let us now discuss the vector module $V$ of $U_q(\mathfrak{spo}(2n|2m))$. The definition of $V$ is easy, since (in the usual sloppy terminology) the vector module of $U_q(\mathfrak{spo}(2n|2m))$ is undeformed. More precisely, let $V$ be the graded vector space introduced in Section 2, and let $E_i^v$, $F_i^v$, and $H_i^v$; $i \in J$, be the linear operators on $V$ defined there. Define the linear operators $K_i^v$, $i \in J$, by

$$K_i^v = q_i H_i^v$$

for all $i \in J$.

Since the operators $H_i^v$ are diagonalizable, with eigenvalues $0, \pm 1$, the operators $K_i^v$ are well-defined and, obviously, invertible. It is easy to see that the operators $E_i^v$, $F_i^v$, and $(K_i^v)^{\pm 1}$ satisfy the defining relations of the generators $E_i$, $F_i$, and $K_i^{\pm 1}; i \in J$. Hence there exists a unique graded representation $\pi$ of the algebra $U_q(\mathfrak{spo}(2n|2m))$ in $V$ such that

$$\pi(E_i) = E_i^v, \quad \pi(F_i) = F_i^v, \quad \pi(K_i^{\pm 1}) = (K_i^v)^{\pm 1} \quad \text{for all } i \in J.$$

The graded vector space $V$, endowed with this representation, will be called the vector module of $U_q(\mathfrak{spo}(2n|2m))$.

Remark 4.1. The reader might suspect that checking the seventh order relation (3.13) might be quite tedious. Actually, this is not the case. Let $\text{Lgr}(V)$ be the superalgebra of all linear operators in $V$. It is well-known that $\text{Lgr}(V)$ is an $\mathfrak{spo}(b)$–module in a canonical way, and its weights (with respect to the Cartan subalgebra $\mathfrak{h}$) are the linear forms $\varepsilon_i - \varepsilon_j; i, j \in I$. Since $\mathfrak{spo}(b)$ acts on $\text{Lgr}(V)$ by superderivations, any product with one factor $E_{-1}^v$, three factors $E_{-2}^v$, two factors $E_{-3}^v$, and one factor $E_{-4}^v$ has the weight $\varepsilon_{-4} + \varepsilon_{-3} + \varepsilon_{-2} - \varepsilon_{-1}$. Since this is not a weight of $\text{Lgr}(V)$, every such product is equal to zero, and this implies the relation to be proved.

Let $(e_i)_{i \in I}$ be the basis of $V$ used in Section 2. Then we have

$$K_j \cdot e_i = q_i^{(a_i | e_i)} e_i \quad \text{for all } j \in J \text{ and all } i \in I.$$

Stated differently, $e_i$ is a weight vector with weight $\varepsilon_i$, just as in the undeformed case.

Our next goal is to show that there exists a unique (up to scalar multiples) $U_q(\mathfrak{spo}(2n|2m))$–invariant bilinear form on $V$. (For a few comments on invariant bilinear forms, see Appendix A.) Let $b$ be a bilinear form on $V$, and let $\tilde{b}$ be the linear form on $V \otimes V$ canonically corresponding to $b$. Then $b$ is $U_q(\mathfrak{spo}(2n|2m))$–invariant if and only if

$$\tilde{b}(X \cdot (x \otimes y)) = \varepsilon(X) \tilde{b}(x \otimes y)$$

for all $X \in U_q(\mathfrak{spo}(2n|2m))$ and all $x, y \in V$ (see Eqn. (A.2)). The condition that

$$\tilde{b}(K_j \cdot (e_i \otimes e_k)) = \tilde{b}(e_i \otimes e_k) \quad \text{for all } j \in J \text{ and all } i, k \in I$$

for all $X \in U_q(\mathfrak{spo}(2n|2m))$ and all $x, y \in V$ (see Eqn. (A.2)). The condition that
is satisfied if and only if
\[ \tilde{b}(e_i \otimes e_k) = 0 \quad \text{for all } i, k \in I \text{ with } i + k \neq 0 . \] (4.1)

In particular, this implies that \( b \) must be homogeneous of degree zero.

Taking Eqn. (4.1) for granted, the conditions
\[ \tilde{b}(E_j \cdot (e_i \otimes e_k)) = 0 \quad \text{for all } j \in J \text{ and all } i, k \in I \]
and
\[ \tilde{b}(F_j \cdot (e_i \otimes e_k)) = 0 \quad \text{for all } j \in J \text{ and all } i, k \in I \]
both yield the same system of linear equations for the elements \( b(e_i, e_k) \). This system has a unique (up to scalar multiples) solution. Choosing a suitable normalization, the invariant bilinear form \( b^q \) we are looking for is given by
\[ b^q(e_i, e_k) = C_{ik}^q \quad \text{for all } i, k \in I , \]
where
\[ C_{i, k}^q = C_{i, -i}^q \delta_{i, -k} \quad \text{for all } i, k \in I , \]
and where the coefficients \( C_{i, -i}^q \) are given by
\[
C_{i, -i}^q = \begin{cases} 
-q^i & \text{for } -1 \geq i \geq -n \\
q^{-i-2n-2} & \text{for } -n - 1 \geq i \geq -r \\
q^i & \text{for } 1 \leq i \leq n \\
q^{2n-i} & \text{for } n + 1 \leq i \leq r . 
\end{cases}
\]

Obviously, the matrix \( C^q = (C_{ij}^q)_{i,j \in I} \) is invertible, i.e., the bilinear form \( b^q \) is non-degenerate. We note that \( C^q=1 = C \) (see Eqn. (2.1)), moreover, we have
\[
C_{i, -i}^q C_{-i, i}^q = \begin{cases} 
-1 & \text{for } i \in I_0 \\
q^{-2} & \text{for } i \in I_T . 
\end{cases}
\] (4.2)

Thus the matrix \((C^q)^2\) is not equal to \(-G\) (recall the Eqns. (2.2), (2.3)).

5 The structure of the module \( V \otimes V \)

We now are ready to tackle a crucial intermediate problem, namely, to determine the structure of the tensorial square of the vector module \( V \) of \( U_q(\mathfrak{spo}(2n|2m)) \). In the undeformed case, this structure is known. It turns out that in the deformed case, the structure is completely analogous. In particular, for \( n = m \), the module \( V \otimes V \) is not completely reducible. (Actually, if adequately interpreted, the investigations of the present section apply also to the case \( q = 1 \).)
To begin with, we stress that the \( U_q(\mathfrak{sp}(2n|2m)) \)-module \( V \otimes V \) has the same weights (with the same multiplicities) as in the undeformed case: For all \( i, j \in I \), the tensor \( e_i \otimes e_j \) has the weight \( \varepsilon_i + \varepsilon_j \).

As expected, \( V \otimes V \) contains a unique (up to scalar multiples) \( U_q(\mathfrak{sp}(2n|2m)) \)-invariant element, i.e., a non-zero element \( a \) such that\footnote{Note that \( \tilde{b}^q(a) = 0 \) if \( n = m \). This is a first indication that there will be problems in the case \( n = m \).}

\[
X \cdot a = \varepsilon(X) a \quad \text{for all} \ X \in U_q(\mathfrak{sp}(2n|2m)) .
\]

The invariance of \( a \) under the action of the generators \( K_j ; j \in J \), is equivalent to the fact that \( a \) has the weight zero, i.e., that \( a \) is a linear combination of the following form

\[
a = \sum_{i \in I} c_i e_i \otimes e_{-i} ,
\]

with some coefficients \( c_i , i \in I \).

For an element \( a \) of this form, the conditions

\[
E_j \cdot a = 0 \quad \text{for all} \ j \in J
\]

and

\[
F_j \cdot a = 0 \quad \text{for all} \ j \in J
\]

both yield the same system of linear equations for the coefficients \( c_i \). This system has a unique (up to scalar multiples) solution. Choosing a suitable normalization, the element \( a \) is given by

\[
a = \sum_{i,k \in I} ((C^q)^{-1})_{ik} e_i \otimes e_k ,
\]

where \( C^q \) is the matrix found in Section 4. Of course, this result might have been anticipated. More explicitly, we have

\[
a = \sum_{i=1}^n (q^{-i} e_{-i} \otimes e_i - q^i e_i \otimes e_{-i}) + q \sum_{i=n+1}^r (q^{i-2n-1} e_{-i} \otimes e_i + q^{-i+2n+1} e_i \otimes e_{-i}) .
\]

It is useful to calculate \( \tilde{b}^q(a) \), where \( \tilde{b}^q \) is the linear form on \( V \otimes V \) defined in Section 4. Setting\footnote{Note that \( \tilde{b}^q(a) = 0 \) if \( n = m \). This is a first indication that there will be problems in the case \( n = m \).}

\[
d = n - m ,
\]

we obtain

\[
\tilde{b}^q(a) = \frac{1}{q^d - 1} (q^{-2d} - 1 - q^2 (q^{2d} - 1)) = -\frac{q^d - q^{-d}}{q - q^{-1}} (q^{d+1} + q^{-d-1}) .
\]
The rest of the present section will now be devoted to prove the following statements.

a) The \(U_q(\mathfrak{sp}(2n|2m))\)–module \(V \otimes V\) is the direct sum of two submodules \((V \otimes V)_s\) and \((V \otimes V)_a\):

\[
V \otimes V = (V \otimes V)_s \oplus (V \otimes V)_a,
\]

where in the undeformed case \((V \otimes V)_s\) corresponds to the subspace of super–symmetric and \((V \otimes V)_a\) to the subspace of super–skew–symmetric tensors in \(V \otimes V\).

b) The submodule \((V \otimes V)_s\) is irreducible.

c) The submodule \((V \otimes V)_a\) contains a submodule \((V \otimes V)_0\) of codimension one, and \((V \otimes V)_s \oplus (V \otimes V)_0\) is the kernel of the linear form \(\tilde{b}^a\) found in Section 4.

d) If \(n \neq m\), the \(U_q(\mathfrak{sp}(2n|2m))\)–module \((V \otimes V)_0\) is irreducible, and \((V \otimes V)_a\) is the direct sum of the submodules \((V \otimes V)_0^a\) and \(\mathbb{C} a\):

\[
(V \otimes V)_a = (V \otimes V)_0^a \oplus \mathbb{C} a\quad \text{if } n \neq m.
\]

e) If \(n = m\), we have

\[
a \in (V \otimes V)_0^a\quad \text{if } n = m,
\]

and \((V \otimes V)_0^a\) does not have a module complement in \((V \otimes V)_a\).

f) If \(n = m \geq 2\), the \(U_q(\mathfrak{sp}(2n|2m))\)–module \((V \otimes V)_0^a/\mathbb{C} a\) is irreducible.

g) For \(n = m = 1\), there exist two submodules \(V_4\) and \(\overline{V}_4\) of \((V \otimes V)_0^a\) such that

\[
V_4 + \overline{V}_4 = (V \otimes V)_0^a,\quad V_4 \cap \overline{V}_4 = \mathbb{C} a,
\]

and the modules \(V_4/\mathbb{C} a\) and \(\overline{V}_4/\mathbb{C} a\) are irreducible.

h) Let \(P_s\) be the projector of \(V \otimes V\) onto \((V \otimes V)_s\) with kernel \((V \otimes V)_a\), and let

\[
K : V \otimes V \rightarrow V \otimes V
\]

be the linear map defined by

\[
K(u) = \tilde{b}^a(u)a\quad \text{for all } u \in V \otimes V.
\]

Then \(\text{id}_{V \otimes V}, P_s\), and \(K\) form a basis of the space of all \(U_q(\mathfrak{sp}(2n|2m))\)–module endomorphisms of \(V \otimes V\).

In the proof of these claims, we shall obtain more detailed information on the submodules mentioned above. In particular, we shall construct bases of the vector spaces \((V \otimes V)_s\), \((V \otimes V)_0^a\), and \((V \otimes V)_a\).

### 5.1 The module \((V \otimes V)_s\)

As already mentioned above, in the undeformed case the module \((V \otimes V)_s\) corresponds to the subspace of all super–symmetric tensors in \(V \otimes V\). This subspace is
an irreducible $\mathfrak{spo}(b)$–submodule of $V \otimes V$ and has the highest weight $\varepsilon_{-r} + \varepsilon_{-r+1}$. If there exists a corresponding primitive vector in the $U_q(\mathfrak{spo}(2n|2m))$–module $V \otimes V$, it must be a linear combination of $e_{-r} \otimes e_{-r+1}$ and $e_{-r+1} \otimes e_{-r}$. Indeed, there is a unique (up to scalar multiples) linear combination of these tensors that is annihilated by all $E_j$, $j \in J$. Choosing a suitable normalization, it is equal to

$$s_{-r,-r+1} = e_{-r} \otimes e_{-r+1} + \sigma_{-r,-r+1} q^{-1} e_{-r+1} \otimes e_{-r}.$$  

By definition, $(V \otimes V)_s$ is the submodule of $V \otimes V$ generated by $s_{-r,-r+1}$.

Let us define the following elements of $V \otimes V$:

$$s_{i,j} = e_i \otimes e_j + \sigma_{i,j} q^{-1} e_j \otimes e_i \quad \text{for } i,j \in I; \ i < j \text{ but } i \neq -j$$
$$s_{i,i} = e_i \otimes e_i \quad \text{for } i \in I_0,$$  

(5.4)

furthermore,

$$s_1 = e_{-1} \otimes e_1 + q^{-2} e_1 \otimes e_{-1};$$  

(5.5)

and for $2 \leq j \leq r$

$$s_j = q^{-\sigma_{j-1}} e_{-j+1} \otimes e_{j-1} + \sigma_{j-1} q^{-1} e_{j-1} \otimes e_{-j+1} - \sigma_{j-1} \sigma_{j} e_{-j} \otimes e_{j} - \sigma_{j-1} q^{-1} q^{-\sigma_{j}} e_{j} \otimes e_{-j}. $$  

(5.6)

It turns out that the tensors (5.4), (5.5), and (5.6) form a basis of $(V \otimes V)_s$.

First of all, one shows that the tensors (5.4), (5.5), and (5.6) can be obtained by iterated action of the $F$–generators on $s_{-r,-r+1}$. Next one proves that the vector space spanned by these tensors is a $U_q(\mathfrak{spo}(2n|2m))$–submodule of $V \otimes V$. Since these tensors obviously are linearly independent, this implies our claim.

Next we have to show that the $U_q(\mathfrak{spo}(2n|2m))$–module $(V \otimes V)_s$ is irreducible. This can be done as follows. First, we prove the following statement:

If $x$ is a non–zero element of $(V \otimes V)_s$, there exists a monomial $P$ in the $F$–generators such that $P \cdot x$ is a non–zero scalar multiple of $s_{r-1,r}$.

A moment’s thought shows that this is a consequence of the following fact:

If $x$ is a (non–zero) weight vector of $(V \otimes V)_s$ whose weight is different from $\varepsilon_{r-1} + \varepsilon_r$ (i.e., if $x$ is not a scalar multiple of $s_{r-1,r}$), there exists an index $j \in J$ such that $F_j \cdot x \neq 0$.

Since $s_{-r,-r+1}$ is a cyclic vector of the $U_q(\mathfrak{spo}(2n|2m))$–module $(V \otimes V)_s$, the irreducibility of this module will follow if we can show that there exists a monomial in the $E$–generators that maps $s_{r-1,r}$ onto a non–zero scalar multiple of $s_{-r,-r+1}$. Similar as above, this is a consequence of the following fact:

If $x$ is a (non–zero) weight vector of $(V \otimes V)_s$ whose weight is different from $\varepsilon_{-r} + \varepsilon_{-r+1}$ (i.e., if $x$ is not a scalar multiple $s_{-r,-r+1}$), there exists an index $j \in J$ such that $E_j \cdot x \neq 0$. 

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The proof of the foregoing statements amounts to easy but lengthy calculations. Let us mention that one needs the intermediate result that
\[ a \notin (V \otimes V)_s . \]
Summarizing, we have proved that the statement b) above is correct.

Obviously, we have
\[ \tilde{b}^q(s_{-r}, r+1) = 0 . \]
Since the \( U_q(s\mathfrak{po}(2n|2m)) \)–module \( (V \otimes V)_a \) is irreducible, we conclude that it is contained in the kernel of \( \tilde{b}^q \). This proves part of statement c).

### 5.2 The module \( (V \otimes V)_a^0 \)

 Basically, our treatment of the \( U_q(s\mathfrak{po}(2n|2m)) \)–module \( (V \otimes V)_a^0 \) follows similar lines to that of \( (V \otimes V)_s \), however, in the cases \( n = m \) there are several complications.

In the undeformed case, the module \( (V \otimes V)_a^0 \) corresponds to the subspace of all super–skew–symmetric tensors in \( V \otimes V \) with a vanishing “symplectic trace” (i.e., which belong to the kernel of \( \tilde{b} \)). As an \( s\mathfrak{po}(b) \)–submodule of \( V \otimes V \), it is generated by the tensors \( e_{-r} \otimes e_{-r} \) and \( e_r \otimes e_r \), and for \( (n,m) \neq (1,1) \), each of these tensors alone is already sufficient.

In the present deformed case, it is easy to see that \( e_{-r} \otimes e_{-r} \) is annihilated by the \( E \)–generators, and that \( e_r \otimes e_r \) is annihilated by the \( F \)–generators. Accordingly, we define \( (V \otimes V)_a^0 \) to be the \( U_q(s\mathfrak{po}(2n|2m)) \)–submodule of \( V \otimes V \) generated by \( e_{-r} \otimes e_{-r} \) and \( e_r \otimes e_r \).

Let us define the following elements of \( V \otimes V \):

\[
\begin{align*}
  a_{i,j} &= e_i \otimes e_j - \sigma_{i,j} q e_j \otimes e_i \quad \text{for } i, j \in I; \ i < j \text{ but } i \neq -j \\
  a_{i,i} &= e_i \otimes e_i \quad \text{for } i \in I_1 ,
\end{align*}
\]

and for \( 2 \leq j \leq r \)

\[
a_j = q^{-\sigma_j} e_{-j+1} \otimes e_{j-1} - \sigma_{j-1} q e_{j-1} \otimes e_{-j+1} \\
- \sigma_{j-1} \sigma_j e_{-j} \otimes e_{j} + \sigma_{j-1} q q^{-\sigma_j} e_j \otimes e_{-j} .
\]

(I hope there is no risk to confound the tensors \( a_{i,j} \) with the elements of the Cartan matrix.) It turns out that the tensors \( (5.7) \) and \( (5.8) \) form a basis of \( (V \otimes V)_a^0 \).

First one proves that the vector space \( U \) spanned by the tensors \( (5.7) \) and \( (5.8) \) is a \( U_q(s\mathfrak{po}(2n|2m)) \)–submodule of \( V \otimes V \). Since these tensors obviously are linearly independent, they form a basis of \( U \).

Next one shows that \( a_{-r,-r} \) generates this module, \textit{provided} that \( r \geq 3 \) (i.e., provided that \( (n,m) \neq (1,1) \)). In the case \( n = m = 1 \), we have

\[ a_2 = a \quad \text{if } m = n = 1 , \]
and $a_{-2,-2}$ generates the $U_q(\mathfrak{spo}(2|2))$–submodule
\[ V_4 = \mathbb{C} a_{-2,-2} \oplus \mathbb{C} a_{-2,-1} \oplus \mathbb{C} a_{-2,1} \oplus \mathbb{C} a , \]
while $a_{2,2}$ generates the $U_q(\mathfrak{spo}(2|2))$–submodule
\[ \overline{V}_4 = \mathbb{C} a_{2,2} \oplus \mathbb{C} a_{1,2} \oplus \mathbb{C} a_{-1,2} \oplus \mathbb{C} a . \]
Obviously, we have
\[ V_4 + \overline{V}_4 = U , \quad V_4 \cap \overline{V}_4 = \mathbb{C} a , \]
and it is easy to see that the $U_q(\mathfrak{spo}(2|2))$–modules $V_4/\mathbb{C} a$ and $\overline{V}_4/\mathbb{C} a$ are irreducible. This proves statement g), and we also have shown that in all cases
\[ U = (V \otimes V)_a^0 . \]

Let us next show that the sum of the subspaces $(V \otimes V)_s$ and $(V \otimes V)_a^0$ of $V \otimes V$ is direct. Obviously, it is sufficient to prove the analogous statement for the corresponding weight spaces. For non–zero weights, this is trivial. To prove the claim for the weight zero, it is sufficient to show that the tensors $s_i$, $1 \leq i \leq r$, and $a_j$, $2 \leq j \leq r$, are linearly independent. To show this, we observe that
\[ s_j = u_j + q^{-1} v_j \quad , \quad a_j = u_j - q v_j \quad \text{for} \quad 2 \leq j \leq r , \tag{5.9} \]
where the tensors $u_j$ and $v_j$; $2 \leq j \leq r$, are defined by
\[
\begin{align*}
  u_j &= q^{-\sigma_j} e_{-j} e_{j+1} - \sigma_j e_{j} e_j \\
  v_j &= \sigma_j e_{j+1} e_{j-1} - \sigma_{j-1} q^{-\sigma_j j} e_{j} e_{-j} .
\end{align*}
\]
Consequently, we have to prove that the tensors $s_1$ and $u_j$, $v_j$, $2 \leq j \leq r$, are linearly independent. This follows from the obvious fact that the $2r$ tensors $e_{-1} \otimes e_1$, $s_1$, $u_j$, $v_j$ span the same subspace of $V \otimes V$ as the $2r$ tensors $e_i \otimes e_{-i}$, $i \in I$, namely, the weight space of $V \otimes V$ corresponding to the weight zero.

The proof above shows that the codimension of $(V \otimes V)_s \oplus (V \otimes V)_a^0$ in $V \otimes V$ is equal to one. Obviously, $\tilde{b}^q$ vanishes on the tensors $e_{-r} \otimes e_r$ and $e_r \otimes e_{-r}$, hence also on the submodule $(V \otimes V)_a^0$ generated by them. As noted earlier, $\tilde{b}^q$ also vanishes on $(V \otimes V)_s$. Since $\tilde{b}^q$ is a non–zero linear form on $V \otimes V$, it follows that $(V \otimes V)_s \oplus (V \otimes V)_a^0$ is the kernel of $\tilde{b}^q$. This proves the last claim of statement c).

Using Eqn. (5.2), our last result implies that
\[ a \in (V \otimes V)_s \oplus (V \otimes V)_a^0 \quad \text{if and only if} \quad n = m . \tag{5.10} \]

Since the $U_q(\mathfrak{spo}(2n|2m))$–module $(V \otimes V)_s$ is irreducible and not one–dimensional, it follows that
\[ a \in (V \otimes V)_a^0 \quad \text{for} \quad n = m . \tag{5.11} \]
Indeed, it can be shown that

\[ a = \sum_{j=2}^{n+1} [j-1]a_j - \sum_{j=n+2}^{2n} [2n+1-j]a_j \quad \text{for } n = m, \quad (5.12) \]

where, for all integers \( s \), the \( q \)-number \([s]\) is defined by

\[ [s] = [s]_q = \frac{q^s - q^{-s}}{q - q^{-1}}. \]

For \( n = 1 \), Eqn. (5.12) is just the equation \( a = a_2 \) mentioned earlier.

Using Eqn. (5.10) and the fact that \((V \otimes V)_s \oplus (V \otimes V)_a^0\) is equal to the kernel of \( \tilde{b}^0 \), it follows that, for \( n = m \), this submodule does not have a module complement in \( V \otimes V \). In fact, any such complement would have to be a trivial one-dimensional submodule of \( V \otimes V \), and hence would be spanned by an invariant element of \( V \otimes V \). But the invariant elements of \( V \otimes V \) are the scalar multiples of \( a \). Combined with Eqn. (5.11), this yields statement e).

Finally, to answer questions of irreducibility, we prove the following technical results.

Suppose that \( r \geq 3 \), and that \( x \in (V \otimes V)_a^0 \) is a (non-zero) weight vector which is neither proportional to \( a_{r,r} \), nor to \( a \). Then there exists an index \( j \in J \) such that \( F_j \cdot x \notin \mathbb{C}a \) (in particular, we have \( F_j \cdot x \neq 0 \)).

Suppose that \( r \geq 3 \), and that \( x \in (V \otimes V)_a^0 \) is a (non-zero) weight vector which is neither proportional to \( a_{-r,-r} \), nor to \( a \). Then there exists an index \( j \in J \) such that \( E_j \cdot x \notin \mathbb{C}a \) (in particular, we have \( E_j \cdot x \neq 0 \)).

As in the case of \((V \otimes V)_s\), these results follow from easy but lengthy calculations. Once they are established, it is easy to draw the following conclusions.

If \( n \neq m \), the \( U_q(\text{spo}(2n|2m)) \)-module \((V \otimes V)_a^0\) is irreducible.

If \( n = m \geq 2 \), the \( U_q(\text{spo}(2n|2m)) \)-module \((V \otimes V)_a^0/\mathbb{C}a\) is irreducible.

These results prove statement f) and the first claim of statement d).

### 5.3 The module \((V \otimes V)_a\)

Our next task is to construct the submodule \((V \otimes V)_a\) of \( V \otimes V \). In the case \( n \neq m \), this is easy. Recalling Eqn. (5.10) and the fact that \((V \otimes V)_s \oplus (V \otimes V)_a^0\) has the codimension one in \( V \otimes V \), it follows that

\[ V \otimes V = (V \otimes V)_s \oplus (V \otimes V)_a^0 \oplus \mathbb{C}a \quad \text{if } n \neq m. \quad (5.13) \]

As we know, the submodules on the right hand side of this equation are irreducible, moreover, they are obviously non-isomorphic. This implies that Eqn. (5.13) is the
unique decomposition of the $U_q(\mathfrak{spo}(2n|2m))$–module $V \otimes V$ into irreducible submodules. Setting

$$(V \otimes V)_a = (V \otimes V)_a^0 \oplus \mathbb{C} a,$$

we have proved the statements a)–d) in the case $n \neq m$.

Unfortunately, this type of reasoning is not possible in the case $n = m$. Since we want to obtain a unified treatment of the problem, we start all over again and present an approach which is applicable in all cases.

To begin with, we note that every module complement of $(V \otimes V)_s$ in $V \otimes V$ must take the form $(V \otimes V)_s^0 \oplus \mathbb{C} g$, where $g$ is a weight vector of $V \otimes V$ of weight zero. Indeed, since the tensors $e_{-r} \otimes e_{-r}$ and $e_r \otimes e_r$ do not belong to $(V \otimes V)_s$, and since the corresponding weights have multiplicity one, these two tensors and hence the submodule generated by them must be contained in every module complement. Obviously, a subspace of this type is a module complement of $(V \otimes V)_s$ if and only if $g \notin (V \otimes V)_s^0 \oplus (V \otimes V)_a^0$ and if $E_j \cdot g$ and $F_j \cdot g$ belong to $(V \otimes V)_a^0$, for all $j \in J$.

Since $g$ is of zero weight, it takes the form

$$g = \sum_{i \in I} g_i e_i \otimes e_{-i},$$

with some coefficients $g_i \in \mathbb{C}$. If $E_j \cdot g$ is non–zero, it is a weight vector with non–zero weight and hence belongs to $(V \otimes V)_s \oplus (V \otimes V)_a^0$. Consequently, $E_j \cdot g$ lies in $(V \otimes V)_s^0$ if and only if its component in $(V \otimes V)_s$ is equal to zero. It follows that we have $E_j \cdot g \in (V \otimes V)_a^0$ if and only if the coefficients $g_i$ satisfy the following system of linear equations:

$$g_1 + q^2 g_{-1} = 0$$
$$q g_{j+1} - q g_j = \sigma_j g_{-j} - \sigma_{j+1} q^{j+1} g_{-j-1} \quad \text{for } -r \leq j \leq -2.\quad (5.16)$$

The condition that $F_j \cdot g \in (V \otimes V)_a^0$ for all $j \in J$ is equivalent to the same system of equations.

The general solution of this system can easily be described: We can choose $g_{-1}, g_{-2}, \ldots, g_{-r}$ arbitrarily, and then the coefficients $g_1, g_2, \ldots, g_r$ are uniquely fixed.

Let $X_a$ be the subspace of $V \otimes V$ consisting of all tensors of the form $(5.14)$ such that the coefficients $g_i$ satisfy the system $(5.15), (5.16)$. According to the foregoing result, this subspace is $r$–dimensional. Obviously, $X_a$ contains the $(r - 1)$–dimensional weight space of $(V \otimes V)_a^0$ corresponding to the weight zero. On the other hand, $X_a$ does not contain any non–zero elements of $(V \otimes V)_s$ (indeed, any such element would be invariant and hence proportional to $a$). It follows that

$$(V \otimes V)_a = (V \otimes V)_a^0 + X_a$$

is a module complement of $(V \otimes V)_s$ in $V \otimes V$.

By this latter property, $(V \otimes V)_a$ is uniquely fixed. In fact, let $(V \otimes V)'_a$ be an arbitrary module complement of $(V \otimes V)_s$ in $V \otimes V$. As noted at the beginning of
this discussion, \( (V \otimes V)'_a \) contains the submodule \( (V \otimes V)_0' \). Let \( X'_a \) be the weight space of \( (V \otimes V)'_a \) corresponding to the weight zero. Of course, \( X'_a \) is \( r \)-dimensional. Consider an arbitrary element \( g \in X'_a \). For every \( j \in J \), the elements \( E_j \cdot g \) and \( F_j \cdot g \) lie in \( (V \otimes V)'_a \) and have a non-zero weight, which implies that they are elements of \( (V \otimes V)_0' \). By the definition of \( X_a \), this proves that \( g \in X_a \). Thus we have shown that \( X'_a \subset X_a \), and for reasons of dimension, this implies that \( X'_a = X_a \). It follows that \( X_a \subset (V \otimes V)'_a \), hence that

\[
(V \otimes V)_a \subset (V \otimes V)'_a ,
\]

and finally

\[
(V \otimes V)_a = (V \otimes V)'_a ,
\]

as claimed.

We proceed by choosing, in a unified way, an element \( t \in X_a \) that does not belong to \( (V \otimes V)_0' \). Let \( t \) be the tensor of the form \( (5.14) \) whose coefficients \( g^t_i \) are fixed by the requirement that

\[
g^t_{-1} = 1 \quad \text{for } -r \leq j \leq -2 .
\]

Recall that the coefficients \( g^t_i \) with \( 1 \leq i \leq r \) can then be calculated by means of the system \( (5.13), (5.16) \). We obtain

\[
t = e_{-1} \otimes e_1 - q^2 e_1 \otimes e_{-1} + (q - q^{-1}) \sum_{i=2}^{n+m} ((C^a)^{-1})_{i,-i} e_i \otimes e_{-i} . \tag{5.17}
\]

It is easy to check that

\[
\tilde{b}^q(t) = -q^{-1} (q^{2d+2} + 1)
\]

(where \( d = n - m \); see Eqn. \( (5.1) \)). Thus \( t \) does not belong to the kernel of \( \tilde{b}^q \) (since \( q \) is not a root of unity).

According to the results of the present subsection, the tensor \( t \) and the tensors \( (5.7) \) and \( (5.8) \) form a basis of the vector space \( (V \otimes V)_a \).

Remark 5.1. We note that some other, simpler looking choices for the tensor \( t \) are possible. For example, the tensor \( e_{-r} \otimes e_r + e_r \otimes e_{-r} \) is a candidate. However, it is not at all obvious that such a choice would simplify the subsequent calculations.

The reader will easily convince himself/herself that, at this stage, we have proved the statements a)–g).

5.4 The module endomorphisms of \( V \otimes V \)

In the present subsection we are going to prove statement h), i.e., that the linear operators \( \text{id}_{V \otimes V}, P_s, \) and \( K \) form a basis of the space of all endomorphisms of
the $U_q(\mathfrak{spo}(2n|2m))$–module $V \otimes V$ (recall that $P_s$ is the projector of $V \otimes V$ onto $(V \otimes V)_s$ with kernel $(V \otimes V)_a$, and that the map $K$ has been defined in Eqn. (5.3)). Obviously, the maps $\text{id}_{V \otimes V}$ and $P_s$ are module endomorphisms, and since $\bar{b}^q$ and $a$ are invariant, the same is true of $K$.

Now let $Q$ be a module endomorphism of $V \otimes V$, i.e., an even linear map of $V \otimes V$ into itself that commutes with the action of $U_q(\mathfrak{spo}(2n|2m))$. Since every weight vector of $V \otimes V$ of weight $\varepsilon_{-r} + \varepsilon_{-r+1}$, that is annihilated by all $E_j$, $j \in J$, is proportional to $s_{-r,-r+1}$, there exists a constant $c_s$ such that

$$Q(s_{-r,-r+1}) = c_s s_{-r,-r+1}.$$ 

Similarly, since the multiplicity of the weight $2\varepsilon_{-r}$ is equal to one, we have

$$Q(a_{-r,-r}) = c_a a_{-r,-r},$$

with some constant $c_a$. It follows that

$$Q(x) = c_s x \quad \text{for all } x \in (V \otimes V)_s$$

$$Q(y) = c_a y \quad \text{for all } y \in (V \otimes V)_a^0.$$ 

(5.18)

Indeed, since the tensor $s_{-r,-r+1}$ generates the submodule $(V \otimes V)_s$, the first of these equations follows immediately. Similarly, for $r \geq 3$ the tensor $a_{-r,-r}$ generates the submodule $(V \otimes V)_a^0$, which implies the second equation in this case.

In the case $r = 2$, i.e., for $m = n = 1$, we argue as follows. Quite generally, we also have

$$Q(a_{r,r}) = \tau_a a_{r,r},$$

with some constant $\tau_a$. Then, for $m = n = 1$, we conclude as above that

$$Q(y) = c_a y \quad \text{for all } y \in V_4$$

$$Q(\overline{y}) = \tau_a \overline{y} \quad \text{for all } \overline{y} \in \overline{V}_4.$$ 

Since $a$ lies in $V_4$ and in $\overline{V}_4$, this implies that

$$c_a = \tau_a,$$

and since

$$V_4 + \overline{V}_4 = (V \otimes V)_a^0,$$

it follows that the second of the Eqns. (5.18) holds for $n = m = 1$ as well.

Now let

$$P_a = \text{id}_{V \otimes V} - P_s$$

be the projector of $V \otimes V$ onto $(V \otimes V)_a$ with kernel $(V \otimes V)_s$. Then the equations derived above show that

$$Q' = Q - c_s P_s - c_a P_a$$

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is a module endomorphism of $V \otimes V$, that vanishes on
\[ \ker \tilde{b}^g = (V \otimes V)_s \oplus (V \otimes V)_a^0. \]
Consequently, it induces a module homomorphism
\[ (V \otimes V)/\ker \tilde{b}^g \rightarrow V \otimes V. \]
The module on the left hand side is one-dimensional and trivial, and all invariants in $V \otimes V$ are proportional to $a$. This implies that
\[ Q' = c_0 K, \]
with some constant $c_0$, which proves our claim.

We close this subsection by the remark that the maps $\text{id}_{V \otimes V}$, $P_s$, and $K$ commute one with another.

6 Calculation of the $R$–matrix

At last, we are prepared to calculate the $R$–matrix $R$ or, equivalently, the braid generator $\hat{R}$ of $U_q(\mathfrak{spo}(2n|2m))$ in the vector representation. By definition, $R$ is the representative of the universal $R$–matrix $\mathcal{R}$ in the vector representation,
\[ R = \mathcal{R}_{V \otimes V}, \]
and $\hat{R}$ is given by
\[ \hat{R} = PR, \]
where
\[ P : V \otimes V \rightarrow V \otimes V \]
denotes the twist operator (in the super sense), which is given by
\[ P(x \otimes y) = (-1)^{\xi \eta} y \otimes x, \]
for all $x \in V_\xi$, $y \in V_\eta$, with $\xi, \eta \in \mathbb{Z}_2$. To calculate the $R$–matrix (or the braid generator) means to calculate its matrix elements with respect to the basis $(e_i \otimes e_j)_{i,j \in I}$ of $V \otimes V$.

Remark 6.1. Due to the fact that $\mathcal{R}$ is given in terms of a formal power series, the foregoing remarks need to be amended. See below for further details.

In order to perform the calculation we observe that $\hat{R}$ is an endomorphism of the $U_q(\mathfrak{spo}(2n|2m))$–module $V \otimes V$. According to Section 5.4, this implies that $\hat{R}$ is a linear combination of $\text{id}_{V \otimes V}$, $K$, and $P_s$. Since the matrix elements of $\text{id}_{V \otimes V}$ and $K$ are known, our task consists of two pieces, namely, to calculate the projector $P_s$. 

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(or any module endomorphism of $V \otimes V$ “containing” $P_s$ in a non–trivial way), and
to find the aforementioned linear combination. As we are going to see, the second
problem can easily be dealt with once the first problem has been solved.

Obviously, every module endomorphism of $V \otimes V$ maps each of the weight spaces
into itself. Consequently, the first problem splits into a number of subproblems, one
for each of the weight spaces of $V \otimes V$. Since the weight spaces corresponding to
the non–zero weights are at most two–dimensional, the corresponding subproblems
are trivial, and we are left with the subproblem corresponding to the zero weight.
Basically, this latter problem amounts to writing the tensors $e_i \otimes e_{-i}$, $i \in I$, as
linear combinations of the tensors (5.4) – (5.8) and $t$, i.e., we have to invert a certain
$(2r \times 2r)$–matrix, whose elements are rational functions of $q$.

Unfortunately, the corresponding calculations turn out to be rather tedious. Ac-
cordingly, I don’t present the details of this calculation but only mention two points.
First, in the course of the calculations I have taken advantage of the tensors $u_j$
and $v_j$ introduced in Eqn. (5.9) and of the resulting equations

$$P_s(u_j) = q(q + q^{-1})^{-1}s_j, \quad P_s(v_j) = (q + q^{-1})^{-1}s_j.$$

Secondly, I have applied the following simple trick. In Ref. [1], the formulae for $\hat{R}$
are slightly simpler than those for $P_s$. On the other hand, again according to Ref. [1]
(see also Ref. [13]), it is tempting to conjecture that

$$\hat{R} = \hat{R}',$$  \hfill (6.1)

where the endomorphism $\hat{R}'$ of $V \otimes V$ is defined by

$$\hat{R}' = (q + q^{-1})P_s - q^{-1}I \otimes I - (q - q^{-1})(1 + q^{2d+2})^{-1}K.  \hfill (6.2)$$

Here and in the following, $I$ denotes the unit operator of $V$:

$$I = \text{id}_V.$$

Accordingly, I haven’t calculated $P_s$ but rather the operator $\hat{R}'$. The Eqn. (6.1) can
then be proved at a later stage (which solves the second problem mentioned at the
beginning of this section).

A long calculation shows that for $1 \leq j \leq n$

$$\hat{R}'(e_{-j} \otimes e_j) = (q - q^{-1}) \sum_{i=1}^{n} q^{-j-i}e_{-i} \otimes e_i$$

$$+ (q - q^{-1}) \sum_{i=n+1}^{n+m} q^{-2n+i-j}e_{-i} \otimes e_i$$

$$- (q - q^{-1}) \sum_{i=1}^{j-1} q^{i-j}e_i \otimes e_{-i}$$

$$+ (q - q^{-1}) e_{-j} \otimes e_j + q^{-1}e_j \otimes e_{-j}$$

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\[ \hat{R}'(e_j \otimes e_{-j}) = -(q - q^{-1}) \sum_{i=j+1}^{n} q^{j-i} e_{-i} \otimes e_i \]
\[ - (q - q^{-1}) \sum_{i=n+1}^{n+m} q^{-2n+i+j} e_{-i} \otimes e_i \]
\[ + q^{-1} e_{-j} \otimes e_j , \]
and for \( n+1 \leq j \leq n+m \)
\[ \hat{R}'(e_{-j} \otimes e_j) = -(q - q^{-1}) \sum_{i=1}^{n} q^{-2n-i+j-2} e_{-i} \otimes e_i \]
\[ - (q - q^{-1}) \sum_{i=n+1}^{n+m} q^{-4n+i+j-2} e_{-i} \otimes e_i \]
\[ + (q - q^{-1}) \sum_{i=1}^{j-1} q^{j-i} e_i \otimes e_{-i} \]
\[ - (q - q^{-1}) \sum_{i=n+1}^{j-1} q^{j-i} e_i \otimes e_{-i} \]
\[ + (q - q^{-1}) e_{-j} \otimes e_j - q e_j \otimes e_{-j} \]
\[ \hat{R}'(e_j \otimes e_{-j}) = -(q - q^{-1}) \sum_{i=j+1}^{n+m} q^{-j-i} e_{-i} \otimes e_i \]
\[ - q e_{-j} \otimes e_j . \]

It might have been difficult to unify these equations in a concise formula. Fortunately, Ref. [1] suggests that, for all \( i \in I \), we have
\[ \hat{R}'(e_i \otimes e_{-i}) = \sigma_i q^{-\sigma_i} e_{-i} \otimes e_i + (q - q^{-1}) \theta(-i > i) e_i \otimes e_{-i} \]
\[ - (q - q^{-1}) C_{i,-i}^q \sum_{k>i} ((C^q)^{-1})_{-k,k} e_{-k} \otimes e_k , \]
where, for all \( i, j \in I \), the symbol \( \theta(j > i) \) is defined by
\[ \theta(j > i) = \begin{cases} 
1 & \text{if } j > i \\
0 & \text{otherwise}.
\end{cases} \]

It is not difficult to see that this is indeed the case.

The remaining tensors \( \hat{R}'(e_i \otimes e_j) \), with \( i, j \in I \), are easily determined:
If \( i < j \), but \( i \neq -j \), we obtain
\[ \hat{R}'(e_i \otimes e_j) = \sigma_{i,j} e_j \otimes e_i + (q - q^{-1}) e_i \otimes e_j \]
\[ \hat{R}'(e_j \otimes e_i) = \sigma_{j,i} e_i \otimes e_j . \]

On the other hand, we find for all \( i \in I \)
\[ \hat{R}'(e_i \otimes e_i) = \sigma_i q^{\sigma_i} e_i \otimes e_i . \]
Let us next prove Eqn. (6.1). Needless to say, at this point we have to make contact with the theory of the universal $R$–matrix. Fortunately, only very little of that theory is needed. Hence it should be sufficient to recall a few simple facts. For more details, we refer the reader to Ref. [6].

Usually, the theory is formulated in the framework of formal power series in one indeterminate $h$. In particular, the complex parameter $q$ is replaced by

$$q = e^h,$$

and the corresponding quantum superalgebra $U_h$ is a topological Hopf superalgebra over the ring $\mathbb{C}[[h]]$ of formal power series in $h$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{spo}(2n|2m)$ is regarded as a subspace of $U_h$, and in this sense, the elements $K_i$ are given by

$$K_i = \exp(hH_{\alpha_i}) \ , \quad H_{\alpha_i} = d_i H_i ,$$

where the elements $H_\lambda$ have been defined in Eqn. (2.5), and the elements $H_i = H_i^\prime$ in Eqn. (2.8).

In this setting, instead of $V$ we have to consider the $\mathbb{C}[[h]]$–module $V[[h]]$ that is obtained from $V$ by an extension of the domain of scalars from $\mathbb{C}$ to $\mathbb{C}[[h]]$:

$$V[[h]] = V \otimes_{\mathbb{C}} \mathbb{C}[[h]] .$$

It is well–known that $V[[h]]$ is a graded $U_h$–module in a natural way, and it is this module which in the present setting takes the role of the vector module. More explicitly, the elements $e_i \otimes 1 \in V[[h]]$, $i \in I$, form a basis of the $\mathbb{C}[[h]]$–module $V[[h]]$, and it is customary to identify $e_i \otimes 1$ with $e_i$. With this convention, the action of the generators $E_j$, $F_j$, $K_j$; $j \in J$, on the basis elements is given by the same formulae as in Section 4 (of course, with a different meaning of $q$).

The tensor product (over $\mathbb{C}[[h]]$) of $V[[h]]$ with itself is also a graded $U_h$–module. On the other hand, it is known that

$$V[[h]] \otimes_{\mathbb{C}[[h]]} V[[h]] \simeq (V \otimes V) \otimes \mathbb{C}[[h]]$$

(as graded $\mathbb{C}[[h]]$–modules), and the elements $e_i \otimes e_j \otimes 1$; $i, j \in I$, form a basis of this $\mathbb{C}[[h]]$–module. Once again, we identify $e_i \otimes e_j \otimes 1$ with $e_i \otimes e_j$, and then the action of the generators $E_j$, $F_j$, $K_j$ is given by the same formulae as in Section 5.

Using these conventions, the arguments of Section 5 can be adopted almost verbatim. Of course, we have to keep in mind that $\mathbb{C}[[h]]$ is not a field but only a ring. Correspondingly, at various instances we have to observe that a scalar is not only different from zero but even invertible. Moreover, the usual concept of an irreducible module is not useful here: If $W$ is a graded $U_h$–module, then $hW$ is a graded submodule of $W$ and, in general, different from $W$.

In particular, the $U_h$–module endomorphisms of $V[[h]] \otimes_{\mathbb{C}[[h]]} V[[h]]$ are linear
combinations of the identity map, $K$, and $\hat{R}'$ (interpreted as $\mathbb{C}[[\hbar]]$–linear maps of $(V \otimes V) \otimes \mathbb{C}[[\hbar]]$ into itself). Now we can prove Eqn. (6.1) (in the present setting), but let us first complete our survey. Eqn. (6.1) shows that $\hat{R}$ depends on $\hbar$ only through $e^{\hbar}$ (a result which immediately follows by inspection of the formula for the universal $R$–matrix). In fact, its matrix elements are Laurent polynomials in $e^{\hbar}$. Substituting for $e^{\hbar}$ the complex number $q$ we started with, we obtain the braid generator that we want to calculate.

Before we can proceed to the calculation proper, we remind the reader of the general form of the universal $R$–matrix. Using the fact that the vector spaces $(\mathfrak{h} \otimes \mathfrak{h})^*$ and $\mathfrak{h}^* \otimes \mathfrak{h}^*$ are canonically isomorphic, it is obvious that there exists a unique tensor $B \in \mathfrak{h} \otimes \mathfrak{h}$ such that

$$(\lambda \otimes \mu)(B) = (\lambda|\mu)$$

for all $\lambda, \mu \in \mathfrak{h}^*$. In terms of this tensor, we have

$$R = e^{\hbar B} (1 \otimes 1 + \ldots) ,$$

where the dots stand for an infinite sum of terms of the form $X \otimes X'$, in which $X$ and $X'$ are weight vectors of the quantum superalgebra with non-zero (and opposite) weights (see Refs. [6], [7]).

Now we are ready to prove Eqn. (6.1). According to the preceding discussion, $\hat{R}$ is a $\mathbb{C}[[\hbar]]$–linear combination of $\hat{R}'$, $\mathbb{I} \otimes \mathbb{I}$, and $K$ (regarded as $\mathbb{C}[[\hbar]]$–linear maps of $(V \otimes V) \otimes \mathbb{C}[[\hbar]]$ into itself). Equivalently, this means that

$$\mathcal{R}_{(V \otimes V) \otimes \mathbb{C}[[\hbar]]} = aP \hat{R}' + bP + cPK ,$$

with some coefficients $a, b, c \in \mathbb{C}[[\hbar]]$. In order to determine these coefficients, we apply Eqn. (6.4) to the tensors $e_i \otimes e_j$ and keep only the diagonal terms, i.e., the terms proportional to $e_i \otimes e_j$. On the left hand side, the terms indicated by the dots in Eqn. (6.3) do not contribute, and we are left with

$$(e^{\hbar B})(e_i \otimes e_j) = q^{(e_i|e_j)} e_i \otimes e_j$$

$$= \begin{cases} 
  e_i \otimes e_j & \text{if } i \neq j, -j \\
  q^{\sigma_i} e_i \otimes e_i & \text{if } j = i \\
  q^{-\sigma_i} e_i \otimes e_{-i} & \text{if } j = -i 
\end{cases} .$$

Using the formulae for $\hat{R}'(e_i \otimes e_j)$ obtained above, the analogous terms on the right hand side can easily be calculated. Comparing both sides, we obtain the following equations:

For $i \neq j, -j$

$$1 = a ,$$

for $i = j$

$$q^{\sigma_i} = a q^{\sigma_i} + b \sigma_i ,$$

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for $i = -j$

$$q^{-\sigma_i} = a q^{-\sigma_i} + c \sigma_i .$$

The unique solution of these equations is

$$a = 1 , \quad b = c = 0 .$$

This implies that

$$\mathcal{R}_{(V \otimes V) \otimes \mathbb{C}[h]} = P \hat{R} ,$$

which proves our claim.

**Remark 6.1.** It is well-known that if $\mathcal{R}$ is a universal $R$-matrix for a Hopf superalgebra $H$, then so is $\mathcal{R}_{21}^{-1}$ (we are using the standard notation). Moreover, if $V$ is any graded $H$-module, and if $\hat{R}$ is the braid generator in $V \otimes V$ with respect to $\mathcal{R}$, then $\hat{R}^{-1}$ is the braid generator in $V \otimes V$ with respect to $\mathcal{R}_{21}^{-1}$. Thus we can apply the preceding discussion to $\mathcal{R}_{21}^{-1}$ and $\hat{R}^{-1}$. First of all, we conclude that

$$P \hat{R}^{-1} = (\mathcal{R}_{21}^{-1}(V \otimes V) \otimes \mathbb{C}[h]) = a' P \hat{R} + b' P + c' P K ,$$

with some coefficients $a', b', c' \in \mathbb{C}[h]$. Since the tensor $B$ obviously is symmetric, we conclude from Eqn. (6.3) that

$$\mathcal{R}_{21}^{-1} = (1 \otimes 1 + \ldots) e^{-hB} ,$$

where the dots stand for terms similar to those in Eqn. (6.3). Proceeding as above, we can show that

$$a' = 1 , \quad c' = -b' = q - q^{-1} ,$$

which implies that

$$\hat{R}^{-1} = \hat{R}' - (q - q^{-1}) \mathbb{1} \otimes \mathbb{1} + (q - q^{-1}) K$$

$$= (q + q^{-1}) P - q \mathbb{1} \otimes \mathbb{1} + (q - q^{-1})(1 + q^{-2d-2})^{-1} K .$$

It is easy to show directly that the operator on the right hand side really is the inverse of $\hat{R} = \hat{R}'$, which is a first check that our calculations are correct.

Summarizing part of the results of the present section, we have shown that

$$\hat{R} = \sum_{i} \sigma_i q^{\sigma_i} E_{i,i} \otimes E_{i,i} + \sum_{i} \sigma_i q^{-\sigma_i} E_{-i,i} \otimes E_{i,-i}$$

$$+ \sum_{i \neq j, -j} \sigma_{i,j} E_{j,i} \otimes E_{i,j}$$

$$+ (q - q^{-1}) \sum_{i < j} E_{i,i} \otimes E_{j,j}$$

$$- (q - q^{-1}) \sum_{i < j} ((C^q)^{-1})_{-j,i} C_{i,-i} C_{i,j} E_{-j,i} \otimes E_{j,-i} ,$$

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or equivalently, that

\[ R = \sum_i q^{a_i}E_{i,i} \otimes E_{i,i} + \sum_i q^{-a_i}E_{i,i} \otimes E_{-i,-i} \]

\[ + \sum_{i \neq j, -j} E_{i,i} \otimes E_{j,j} \]

\[ + (q - q^{-1}) \sum_{i<j} \sigma_{i,j} E_{j,i} \otimes E_{i,j} \]

\[ - (q - q^{-1}) \sum_{i<j} \sigma_j ((C_q^{-1})_{j,j} C^{-q}_{q,i} E_{j,i} \otimes E_{-j,-i}). \]

In these equations, the indices \( i \) and \( j \) run through the index set \( I \), subject to conditions as specified. The equations hold in both settings, the one in terms of formal power series, and the one where \( q \) is a complex number. It should also be noted that \( E_{i,j} \otimes E_{k,\ell} \) denotes the normal (non–super) tensor product of linear mappings (the graded tensor product of two linear maps \( f \) and \( g \) is denoted by \( f \otimes g \)).

7 Properties of \( R \) and \( \hat{R} \)

In the present section, we want to collect some of the basic relations satisfied by \( R \) and \( \hat{R} \). First of all, we recall the following equations, which have been derived in the preceding section:

\[ \hat{R} = (q + q^{-1}) P_s - q^{-1} \mathbb{I} \otimes \mathbb{I} - (q - q^{-1})(1 + q^{2d+2})^{-1} K \] (7.1)

\[ \hat{R}^{-1} = (q + q^{-1}) P_s - q \mathbb{I} \otimes \mathbb{I} + (q - q^{-1})(1 + q^{-2d-2})^{-1} K. \] (7.2)

Using the results of Section 5 (in particular, Eqn. (5.2)), the first of these equations implies that

\[ \hat{R}(x) = qx \quad \text{for all } x \in (V \otimes V)_s \]

\[ \hat{R}(y) = -q^{-1}y \quad \text{for all } y \in (V \otimes V)^0_a \]

\[ \hat{R}(a) = -q^{-2d-1}a. \]

Moreover, the linear map induced by \( \hat{R} \) in the one–dimensional \( U_q(\mathfrak{spo}(2n|2m)) \)–module \( (V \otimes V)_a/(V \otimes V)^0_a \) is equal to the multiplication by \( -q^{-2d-1} \). For \( n \neq m \), this follows from the last equation.

Next we recall that the \( U_q(\mathfrak{spo}(2n|2m)) \)–module endomorphisms of \( V \otimes V \) commute one with another. In particular, we have

\[ P_s K = KP_s = 0 \]

\[ \hat{R} K = K \hat{R} = -q^{-2d-1} K. \]
On the other hand, the Eqns. (7.1) and (7.2) imply that
\[ \hat{R} - \hat{R}^{-1} = (q - q^{-1})(I \otimes I - K) . \]
Obviously, this equation is equivalent with
\[ \hat{R}^2 - (q - q^{-1})\hat{R} - I \otimes I = -(q - q^{-1})\hat{R}K , \]
and hence also with
\[ (\hat{R} - qI \otimes I)(\hat{R} + q^{-1}I \otimes I) = (q - q^{-1})q^{-2d-1}K . \] (7.3)
Since the image of the operator \( K \) is contained in \( \mathbb{C}a \), it follows that
\[ (\hat{R} - qI \otimes I)(\hat{R} + q^{-1}I \otimes I)(\hat{R} + q^{-2d-1}I \otimes I) = 0 . \] (7.4)
It is easy to see that the polynomial \((X - q)(X + q^{-1})(X + q^{-2d-1})\) involved in Eqn. (7.4) is the minimal polynomial of the operator \( \hat{R} \).

The preceding equations can be used to write the spectral projectors of \( \hat{R} \) as polynomials in \( \hat{R} \) (to the extent in which these projectors exist). For example, we find
\[ P_s = \frac{(\hat{R} + q^{-1})(\hat{R} + q^{-2d-1})}{(q - q^{-1})(q + q^{-2d-1})} . \]
Moreover, we stress that according to Eqn. (7.3), the operator \( K \) can be written as a polynomial in \( \hat{R} \). This fact (which is not true in the undeformed case \( q = 1 \)) will turn out to be crucial in the construction of the quantum supergroup \( S^P_q(2n|2m) \).

Let us now derive two relations which are related to the fact that on \( V \) there exists an invariant bilinear form, namely, the form \( b^q \) found in Section 4. We use the results of Appendix A and argue as in Ref. [14], for the original setting in which \( q \) is a complex number. The reader who is not satisfied by this sloppy procedure may either reformulate everything in terms of formal power series, or else regard the final result Eqn. (7.5) as a conjecture which has to be (and has been) checked independently.

Let
\[ f_\ell : V \longrightarrow V^{*\text{gr}} , \quad f_r : V \longrightarrow V^{*\text{gr}} \]
be the linear maps associated to \( b^q \) (see Appendix A). Like \( b^q \) they are homogeneous of degree zero.

We write the universal \( R \)–matrix of \( U_q(\mathfrak{spo}(2n|2m)) \) in the form
\[ \mathcal{R} = \sum_s R^1_s \otimes R^2_s , \]
where \( R^1_s, R^2_s \in U_q(\mathfrak{spo}(2n|2m)) \). It is well–known that
\[ \mathcal{R}^{-1} = (S \otimes \text{id})(\mathcal{R}) . \]
This implies that
\[ R^{-1} = \mathcal{R}^{-1}_{V \otimes V} = (S \otimes \text{id})(\mathcal{R})_{V \otimes V} = \sum_s S(R^1_s)_V \overline{\otimes} (R^2_s)_V, \]
where \( \overline{\otimes} \) denotes the tensor product of linear maps in the graded sense. Using Eqn. (A.5), we conclude that
\[ R^{-1} = \sum_s f^{-1}_r \circ (R^1_s)_V \otimes f_r \circ (R^2_s)_V \]
\[ = (f^{-1}_r \otimes I) \circ \left( \sum_s (R^1_s)_V \otimes (R^2_s)_V \right) \circ (f_r \otimes I) \]
\[ = (f^{-1}_r \otimes I) \circ R^{st1} \circ (f_r \otimes I), \]
where \( st1 \) denotes the super–transposition of the first tensorial factor (for more details, see Appendix B).

Similarly, we can start from the equation
\[ \mathcal{R}^{-1} = (\text{id} \otimes S^{-1})(\mathcal{R}) \]
and derive that
\[ R^{-1} = \mathcal{R}^{-1}_{V \otimes V} = \sum_s (R^1_s)_V \overline{\otimes} S^{-1}(R^2_s)_V. \]
According to Eqn. (A.3), this implies that
\[ R^{-1} = \sum_s (R^1_s)_V \overline{\otimes} f^{-1}_r \circ (R^2_s)_V \]
\[ = (I \otimes f^{-1}_r) \circ \left( \sum_s (R^1_s)_V \otimes (R^2_s)_V \right) \circ (I \otimes f_r) \]
\[ = (I \otimes f^{-1}_r) \circ R^{st2} \circ (I \otimes f_r), \]
where \( st2 \) denotes the super–transposition of the second tensorial factor. Summarizing, we have shown that
\[ R^{-1} = (f^{-1}_r \otimes I) \circ R^{st1} \circ (f_r \otimes I) = (I \otimes f^{-1}_r) \circ R^{st2} \circ (I \otimes f_r). \] (7.5)
Note that these equations imply that \( R^{st1} \) and \( R^{st2} \) are invertible.

The equations (7.5) can be checked directly. To do that we need the matrices of the linear maps \( f_r \) and \( f_l \). If \( (e'_i)_{i \in I} \) is the basis of \( V^{*gr} \) dual to \( (e_i)_{i \in I} \), we find for all \( j \in I \)
\[ f_l(e_j) = \sum_{i \in I} C^q_{j,i} e'_i, \quad f_r(e_j) = \sum_{i \in I} \sigma_{j,i} C^q_{i,j} e'_i. \]
We also need a formula for \( R^{-1} \). Using the equation
\[ R^{-1} = \hat{R}^{-1} P = P R P - (q - q^{-1})(P - K P), \]
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we derive that
\[
R^{-1} = \sum_i q^{-\sigma_i} E_{i,i} \otimes E_{i,i} + \sum_i q^{\sigma_i} E_{i,i} \otimes E_{-i,-i}
\]
\[
+ \sum_{i \neq j, -j} E_{i,i} \otimes E_{j,j}
\]
\[
- (q - q^{-1}) \sum_{i < j} \sigma_{i,j} E_{j,i} \otimes E_{i,j}
\]
\[
+ (q - q^{-1}) \sum_{i < j} \sigma_i ((Cq)^{-1})_{j,-j} C^q_{-i,i} E_{j,i} \otimes E_{-j,-i}
\].

Recalling the formulae for the partial super–transpose given in Appendix B, it is now not difficult to show that the equations (7.5) are indeed satisfied.

A closer look at the formula for \(R^{-1}\) reveals that, somewhat unexpectedly, \(R^{-1}q\) is not equal to \(Rq\) (we are using the obvious notation). This fact is closely related to Eqn. (4.2).

In the purely symplectic case considered in Ref. [1] it is known that \(\hat{R}t^1t^2\) is equal to \(\hat{R}\) (where \(t^1\) and \(t^2\) denote the usual transposition of the first resp. second tensorial factor). For reasons similar to those above, I have not been able to derive an analogous equation in the present setting.

We proceed by recalling that the general theory of quasitriangular Hopf superalgebras implies that \(R\) satisfies the graded Yang–Baxter equation. Equivalently, this means that \(\hat{R}\) satisfies the braid relation
\[
(\hat{R} \otimes \mathbb{I})(\mathbb{I} \otimes \hat{R})(\hat{R} \otimes \mathbb{I}) = (\mathbb{I} \otimes \hat{R})(\hat{R} \otimes \mathbb{I})(\mathbb{I} \otimes \hat{R})
\].

It would be worth–while to check this relation directly, but I haven’t done that.

Finally, I have shown by explicit calculation that \(\hat{R}\) and \(K\) satisfy the following relations:
\[
(\mathbb{I} \otimes K)(\hat{R}^{\pm1} \otimes \mathbb{I})(\mathbb{I} \otimes K) = -q^{\pm(2d+1)}(\mathbb{I} \otimes K)
\]
\[
(K \otimes \mathbb{I})(\mathbb{I} \otimes \hat{R}^{\pm1})(K \otimes \mathbb{I}) = -q^{\pm(2d+1)}(K \otimes \mathbb{I})
\].

Summarizing part of the results of the present section, we conclude that \(\hat{R}\) and \(K\) generate representations of the Birman, Wenzl, Murakami algebras [15], [16] as defined in Ref. [17] (with \(z = -q^{2d+1}\)).

8 Comparison with known special cases

In a few special cases, the \(R\)–matrix calculated in this work has already been known. 1. The case \(m = 0\)

It should be obvious to the reader that our results apply in the case \(m = 0\) as
well, and this case has been settled in Ref. [1]. Actually, I have used this fact throughout the whole investigation: It enabled me to check my calculations and to guess a concise expression for the $R$–matrix. For greater clarity, let us mark the entries of the present work by the subscript “here” and those of Ref. [1] by the subscript “RTF”, moreover, let us indicate the dependence on the parameter $q$ by the superscript $q$. Then we have

$$R^{q}_{\text{here}} = R^{q}_{\text{RTF}} , \quad \hat{R}^{q}_{\text{here}} = \hat{R}^{q}_{\text{RTF}} ,$$

and also

$$C^{q}_{\text{here}} = -C^{q}_{\text{RTF}} \quad , \quad K^{q}_{\text{here}} = K^{q}_{\text{RTF}} .$$

2. The case $n = 0$

This case is more interesting. Once again, this case has been treated in Ref. [1]. On the other hand, the calculations of the present work don’t make sense in this case from the outset, since the root system of the Lie algebra $\mathfrak{o}(2m)$ does not have a basis of the type used here. Nevertheless, we find

$$R^{q}_{\text{here}} = R^{(q^{-1})}_{\text{RTF}} , \quad \hat{R}^{q}_{\text{here}} = -\hat{R}^{(q^{-1})}_{\text{RTF}} ,$$

furthermore, we have

$$C^{q}_{\text{here}} = q^{-1}C^{(q^{-1})}_{\text{RTF}} \quad , \quad K^{q}_{\text{here}} = K^{(q^{-1})}_{\text{RTF}} .$$

The change from $q$ to $q^{-1}$ under the transition from the even to the odd case is a known phenomenon. On the other hand, the sign factor in the formula for $\hat{R}$ is easily understood: In the purely odd case, the supersymmetric twist is equal to minus the normal (non–graded) twist.

3. The case $m = n = 1$

In this case, the universal $R$–matrix and the $R$–matrix in the vector representation have been given in Ref. [18]. However, these authors have worked with a basis of the root system which consists of two odd roots (see Ref. [19]). Actually, I have applied the approach of the present paper also to this case, and (after some obvious adjustments) indeed have obtained the $R$–matrix of Ref. [18].

9 Discussion

We have calculated the $R$–matrix of the symplecto–orthogonal quantum superalgebra $U_{q}(\mathfrak{spo}(2n|2m))$ in the vector representation, and we have derived its most important properties. In a subsequent work [2], we shall use this $R$–matrix to construct the corresponding quantum supergroup $\text{SPO}_{q}(2n|2m)$ and its basic comodule superalgebras.
A special feature of the present work is that we have used a somewhat unusual basis of the root system of $\mathfrak{spo}(2n|2m)$. This was dictated by the wish for a unified treatment of all cases, and by the assumption that the basis of the root system should contain only one odd root. If one drops this last requirement, there is another possibility: For $m \geq 2$, one chooses Kac’s distinguished basis, whereas for $m = 1$ (i.e., for the $C$-type Lie superalgebras) one chooses a basis containing two odd simple roots. In the latter case, the Dynkin diagram looks as follows:

![Dynkin diagram of the Lie superalgebra $\mathfrak{spo}(2n|2)$](image)

For $n = m = 1$, this is just the choice mentioned in the preceding section. It should be interesting to calculate the $R$–matrix also under these assumptions.

In Section 6 we had to use a formulation of the theory in terms of formal power series. This made our arguments somewhat clumsy. Of course, we could exclusively use the language of formal power series. The present formulation has been chosen in view of possible applications.

## Appendix

### A Invariant bilinear forms

In the following, the base field will be an arbitrary field $\mathbb{K}$ of characteristic zero. Let $\Gamma$ be an abelian group, and let $\sigma$ be a commutation factor on $\Gamma$ with values in $\mathbb{K}$. It is well-known that the class of $\Gamma$–graded vector spaces, endowed with the usual tensor product of graded vector spaces and with the twist maps defined by means of $\sigma$, forms a tensor category (see Ref. [12] for details). A (generalized) Hopf algebra $H$ living in this category is called a $\sigma$–Hopf algebra. More explicitly, $H$ is a $\Gamma$–graded associative algebra with a unit element, and it is endowed with a coproduct $\Delta$, a counit $\varepsilon$, and an antipode $S$, which satisfy the obvious axioms (in the category). In particular, this implies that the structure maps $\Delta$, $\varepsilon$, and $S$ are homogeneous of degree zero. In the following, we shall freely use the notation and results of Ref. [12]. (For generalized Hopf algebras living in more general categories, see Ref. [20].)

Let $V$ and $W$ be two graded (left) $H$–modules. Then $V \otimes W$ and $\text{Lgr}(V,W)$ have a canonical structure of a graded $H$–module as well. (Recall that $\text{Lgr}(V,W)$ denotes the space of all linear maps of $V$ into $W$ which can be written as a sum of
homogeneous linear maps of $V$ into $W$.) If $U$ is a third graded $H$–module, there exists a canonical isomorphism of graded $H$–modules,

$$\lambda : \text{Lgr}(V, \text{Lgr}(W, U)) \rightarrow \text{Lgr}(V \otimes W, U), \quad (A.1)$$

which is defined by

$$\lambda(f)(x \otimes y) = (f(x))(y),$$

for all $f \in \text{Lgr}(V, \text{Lgr}(W, U))$, $x \in V$, and $y \in W$.

The next thing to be mentioned is that an element $x$ of a graded $H$–module $V$ is said to be $H$–invariant (or simply invariant) if

$$h \cdot x = \varepsilon(h)x \quad \text{for all } h \in H.$$ (Recall that, quite generally, the dot denotes a module action.) Let $g \in \text{Lgr}(V, W)$ be homogeneous of degree $\gamma$. Then $g$ is invariant if and only if it is $H$–linear in the graded sense, i.e., if and only if

$$g(h \cdot x) = \sigma(\gamma, \eta) h \cdot g(x),$$

for all elements $h \in H_\eta$, $\eta \in \Gamma$, and all $x \in V$.

In the following, we choose $U = \mathbb{K}$, the trivial $H$–module. Then $\text{Lgr}(V \otimes W, \mathbb{K}) = (V \otimes W)^{*gr}$ is the graded dual of $V \otimes W$. It is well–known that, regarded as a graded vector space, this space is canonically isomorphic to $\text{Lgr}_2(V, W; \mathbb{K})$, the space of all bilinear forms on $V \times W$ that can be written as a sum of homogeneous bilinear forms on $V \times W$. The canonical isomorphism is used to transfer the $H$–module structure from $\text{Lgr}(V \otimes W, \mathbb{K})$ to $\text{Lgr}_2(V, W; \mathbb{K})$. For every bilinear form $b \in \text{Lgr}_2(V, W; \mathbb{K})$, the corresponding linear form on $V \otimes W$ will be denoted by $\tilde{b}$.

Now let

$$b : V \times W \rightarrow \mathbb{K}$$

be a bilinear form on $V \times W$ that is homogeneous of degree $\beta$, let

$$\tilde{b} : V \otimes W \rightarrow \mathbb{K}$$

be the associated linear form on $V \otimes W$, and let

$$f_\ell : V \rightarrow W^{*gr}$$

be the linear map canonically corresponding to $\tilde{b}$, i.e.,

$$(f_\ell(x))(y) = b(x, y)$$

for all $x \in V$ and $y \in W$. (Choosing $U = \mathbb{K}$ in Eqn. (A.1), this is to say that $\lambda(f_\ell) = \tilde{b}$.) Then $\tilde{b}$ and $f_\ell$ are homogeneous of degree $\beta$. According to the preceding discussion, the following statements are equivalent:
1) The bilinear form $b$, or equivalently, the linear form $\tilde{b}$, is $H$–invariant, i.e., we have
$$\sigma(\eta, \beta) \tilde{b} \circ S(h)_{V \otimes W} = \varepsilon(h) \tilde{b}$$
for all elements $h \in H_\eta, \eta \in \Gamma$.
2) The linear form $\tilde{b}$ is $H$–linear in the graded sense, i.e., we have
$$\tilde{b}(h \cdot (x \otimes y)) = \sigma(\beta, \eta) \varepsilon(h) \tilde{b}(x \otimes y)$$
for all elements $h \in H_\eta, \eta \in \Gamma$, and all $x \in V, y \in W$. Since $\varepsilon(h) = 0$ if $\eta \neq 0$, this is equivalent with
$$\tilde{b}(h \cdot (x \otimes y)) = \varepsilon(h) \tilde{b}(x \otimes y)$$
(A.2)
for all elements $h \in H, x \in V$, and $y \in W$.
3) The linear map $f_\ell$ is $H$–invariant.
4) The linear map $f_\ell$ is $H$–linear in the graded sense, i.e., we have
$$f_\ell(h \cdot x) = \sigma(\beta, \eta) h \cdot f_\ell(x)$$
for all elements $h \in H_\eta, \eta \in \Gamma$, and all $x \in V$. This is equivalent with
$$f_\ell \circ h_{V \rightarrow} = \sigma(\eta, \xi) b(x, S(h) \cdot y)$$
for all elements $h \in H_\eta, \eta \in \Gamma$, and also with
$$b(h \cdot x, y) = \sigma(\eta, \xi) b(x, S(h) \cdot y)$$
(A.4)
for all elements $h \in H_\eta, \eta \in \Gamma$, all $x \in V_\xi, \xi \in \Gamma$, and all $y \in W$. Recall that $^T$ denotes the $\sigma$–transposition. (In the super case, the super–transposition will be denoted by $^st$.)

To proceed, we note that, apart from $f_\ell$, the bilinear form $b$ yields a second linear map, namely, the map
$$f_r : W \rightarrow V^{*_{gr}}$$
which, for all elements $x \in V_\xi, \xi \in \Gamma$, and $y \in W_\eta, \eta \in \Gamma$, is given by
$$(f_r(y))(x) = \sigma(\eta, \xi) b(x, y).$$
If
$$\nu_V : V \rightarrow (V^{*_{gr}})^{*_{gr}}, \quad \nu_W : W \rightarrow (W^{*_{gr}})^{*_{gr}}$$
are the canonical injections (in the graded sense, see Ref. [12]), we have
$$f_r = f_\ell^T \circ \nu_W, \quad f_\ell = f_r^T \circ \nu_V.$$
Then the condition (A.3) is equivalent with
$$f_r \circ S(h)_W = \sigma(\beta, \eta) (h_{V \rightarrow})^T \circ f_r,$$
(A.5)
for all elements $h \in H_\eta, \eta \in \Gamma$. Indeed, the Eqns. (A.3) and (A.5) can be derived from each other by composing their $\sigma$–transposes with $\nu_W$ and $\nu_V$, respectively.
B Partial transposition

In the following, the base field will be an arbitrary field $\mathbb{K}$ of characteristic zero. Let $\Gamma$ be an abelian group, and let $\sigma$ be a commutation factor on $\Gamma$ with values in $\mathbb{K}$. All gradations considered in this appendix will be $\Gamma$–gradations. We shall freely use the notation and results of Ref. [12].

Let $V$ and $W$ be two finite–dimensional graded vector spaces. Then there exists a unique linear map

$$T_1 : \text{Lgr}(V \otimes W, V \otimes W) \rightarrow \text{Lgr}(V^{*\gr} \otimes W, V^{*\gr} \otimes W),$$

such that

$$(f \overline{\otimes} g)^{T_1} = f^T \overline{\otimes} g,$$

and a unique linear map

$$T_2 : \text{Lgr}(V \otimes W, V \otimes W) \rightarrow \text{Lgr}(V \otimes W^{*\gr}, V \otimes W^{*\gr}),$$

such that

$$(f \overline{\otimes} g)^{T_2} = f \overline{\otimes} g^T,$$

for all $f \in \text{Lgr}(V, V)$ and all $g \in \text{Lgr}(W, W)$. (Recall that $\overline{\otimes}$ denotes the graded tensor product of linear maps, and that $^T$ denotes the $\sigma$–transposition.) We call $T_1$ and $T_2$ the partial $\sigma$–transposition of the first and second tensorial factor, respectively. (In the super case, we shall write $^{\text{st}1}$ and $^{\text{st}2}$.)

It is easy to see that

$$(f \otimes g)^{T_2} = f \otimes g^T,$$

for all $f \in \text{Lgr}(V, V)$ and all $g \in \text{Lgr}(W, W)$. On the other hand, under our present general assumptions, $(f \otimes g)^{T_1}$ is not, in general, equal to $f^T \otimes g$. Using our standard notation for $V$, and a similar notation for $W$, but with all entries overlined, we find instead that

$$(E_{ij} \otimes \overline{E}_{rs})^{T_1} = \sigma(\eta_i + \eta_j, \overline{\eta}_r - \overline{\eta}_s) E_{ij}^T \otimes \overline{E}_{rs}.$$ \(\text{In the super case, which is the case we are mainly interested in, this equation implies that}

$$(f \otimes g)^{T_1} = \sigma(\varphi, \gamma)f^T \otimes g,$$

where $f \in \text{Lgr}(V, V)$ is homogeneous of degree $\varphi$ and $g \in \text{Lgr}(W, W)$ is homogeneous of degree $\gamma$.

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