Non-vanishing of Artin-twisted $L$-functions of Elliptic Curves

Thomas Ward

Abstract: Let $E$ be an elliptic curve and $\rho$ an Artin representation, both defined over $\mathbb{Q}$. Let $p$ be a prime at which $E$ has good reduction. We prove that there exists an infinite set of Dirichlet characters $\chi$, ramified only at $p$, such that the Artin-twisted $L$-values $L(E, \rho \otimes \chi, \beta)$ are non-zero when $\beta$ lies in a specified region in the critical strip (assuming the conjectural continuations and functional equations for these $L$-functions). The new contribution of our paper is that we may choose our characters to be ramified only at one prime, which may divide the conductor of $\rho$.

1 Introduction

An Artin representation of $\text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q})$ is a continuous finite-dimensional complex representation which factors through a finite extension $K/\mathbb{Q}$. If $E$ is an elliptic curve over $\mathbb{Q}$, there exists an $L$-series $L(E, \rho, s)$ associated to $E$ and $\rho$, which appears naturally as a factor of the $L$-function of $E$ over $K$.

The function $L(E, \rho, s)$ is defined by an Euler product which converges only on the region $\text{Re}(s) > 3/2$, but conjecturally it may be analytically continued to the entire complex plane and will satisfy a functional equation (we give further details in §3). Assuming this continuation, we are interested in the vanishing properties of $L(E, \rho \otimes \chi, \beta)$ for Dirichlet twists $\chi$, where $\beta$ lies in the critical strip.

Following the work of Langlands, there conjecturally exists a tempered cuspidal automorphic representation $\pi$ of $\text{GL}_n(\mathbb{A}_F)$ (where $n = 2 \dim(\rho)$) such that $L(E, \rho, s) = L(\pi, s - 1/2)$. The existence of such a $\pi$ immediately implies the claimed functional equation and analytic continuation. There are many results on the non-vanishing of automorphic $L$-functions in the literature: in particular, in [10] Rohrlich shows the existence of infinitely many ray class characters $\chi$ such that $L(\pi \otimes \chi, \beta) \neq 0$ for any $\beta \in \mathbb{C}$, where $\pi$ is an irreducible cuspidal representation of $\text{GL}_n(\mathbb{A}_F)$ with $n = 1$ or 2, for any number field $F$. In [1] Barthel and Ramakrishnan extend this result to all $n$ for $\text{Re}(\beta) \notin [1/n, 1 - 1/n]$, or $\text{Re}(\beta) \notin [2/(n + 1), 1 - 2/(n + 1)]$ when $\pi$ is tempered. In [7] Luo extends this further to the region $\text{Re}(\beta) \notin [2/n, 1 - 2/n]$ when the base field is $\mathbb{Q}$.

These non-vanishing theorems are initially proved for twists which are unramified at the primes dividing the conductor of $\pi$. It is simple to relax this assumption: one can choose a character $\chi_0$ with no restriction on its ramification,
and then apply the non-vanishing theorem to the automorphic representation \( \pi \otimes \chi_0 \) to obtain an infinite set of twists \( \chi \) such that \( L(\pi \otimes \chi \otimes \chi_0, \beta) \neq 0 \). In this way we get a set of characters \( \chi \otimes \chi_0 \) which satisfy the desired non-vanishing property, but which can be arbitrarily ramified at the primes dividing the conductor \( N_\pi \). However, this method cannot produce a set of twists ramified at only one prime, if that prime divides \( N_\pi \). The purpose of this paper is to prove a non-vanishing result for such a set of twists, for the case of the Artin-twisted \( L \)-function \( L(E, \rho, s) \) over \( \mathbb{Q} \). The precise statement of our main theorem is as follows.

**Theorem 1.1.** Let \( p \) be a prime at which \( E \) has good reduction. Suppose that the \( L \)-function \( L(E, \rho \otimes \chi, s) \) satisfies the conjectural analytic continuation and functional equation for all Dirichlet twists \( \chi \). Then, for any \( \beta \) in the critical strip \( \{ s \in \mathbb{C} : 1/2 \leq \Re(s) \leq 3/2 \} \) satisfying

\[
\Re(\beta) \notin \left[ \frac{1}{2} + \frac{2}{2 \dim \rho + 1}, 3 - \frac{2}{2 \dim \rho + 1} \right]
\]

if \( \dim \rho \geq 2 \), and \( \Re(\beta) \neq 1 \) if \( \dim \rho = 1 \), we have \( L(E, \rho \otimes \chi, \beta) \neq 0 \) for infinitely many Dirichlet characters \( \chi \) which are ramified only at \( p \).

We must point out that our theorem is weaker than that of Barthel and Ramakrishnan from [1]: it holds for a more restricted class of tempered cuspidal automorphic representations and only for base field \( \mathbb{Q} \). Further, it holds on a smaller region than Luo’s result from [7]. Our new contribution is to allow the twists \( \chi \) to be ramified at only a single prime, which may divide the conductor of \( \rho \). To prove Theorem 1.1 we follow a standard technique of using the approximate functional equation and averaging over twists. In particular we follow the method of Luo from [7] (previously used by Iwaniec in [5]).

We conclude the introduction with the following remark: suppose \( \rho \) is 2-dimensional, irreducible and odd. Under these assumptions, except for certain icosahedral examples of \( \rho \), it is known that \( \rho \) is equivalent to the representation given by a weight 1 newform (this is proved by Buzzard, Dickinson, Shepherd-Barron and Taylor in [3]). As we also know that \( E \) is modular by the work of Wiles et al in [12] and [2] we may write \( L(E, \rho, s) \) as a Rankin convolution of modular forms. For such an \( L \)-function, the functional equation and continuation are known (in fact, the existence of the automorphic representation \( \pi \) of \( \text{GL}_4 \) has been proved by Ramakrishnan in [9]) so Theorem 1.1 holds unconditionally in this case.

### 2 Conductors of twisted Artin representations

In our main theorem, we impose no restrictions on the ramification of \( \rho \) at the prime \( p \). For this reason we first prove a result on the Artin conductors of the twists \( \rho \otimes \chi \); we begin with a preparatory lemma on ramification groups.

**Lemma 2.1.** Let \( p \) be a prime. Let \( L/\mathbb{Q}_p \) be a finite Galois extension of local fields, and suppose

\[
\phi : \text{Gal}(L/\mathbb{Q}_p) \to A
\]
is a homomorphism of groups. Let \( K_n = \mathbb{Q}_p(\mu_p^n) \) and regard \( \rho \) as a homomorphism \( \text{Gal}(LK_n/\mathbb{Q}_p) \to A \) by extension through the quotient map. Then there exists a fixed integer \( i_\phi \) such that

\[
G_i(LK_n/\mathbb{Q}_p) \subseteq \ker \phi
\]

for all \( i \geq i_\phi \), for all sufficiently large \( n \). Here \( G_i(LK_n/\mathbb{Q}_p) \) denotes the \( i \)-th ramification group of the extension \( LK_n/\mathbb{Q}_p \).

**Proof.** Let \( H = \text{Gal}(LK_n/L) \), so that \( H \subseteq \ker \phi \). Quoting [8, Chapter II, §10] we have the formula

\[
G_i(LK_n/\mathbb{Q}_p) H H = G_t(L/Q_p)
\]

where \( t = \eta_{LK_n/L}(i) \). Here, \( \eta_{LK_n/L} \) is the function which defines the upper numbering for the ramification groups of \( LK_n/L \) (see [8]). Let \( t' \) be the some integer such that \( G_{t'}(L/Q_p) = \{1\} \). We will show that there exists a fixed \( i_\phi \) such that \( \eta_{LK_n/L}(i) \geq t' \) when \( i \geq i_\phi \), for all sufficiently large \( n \). This will establish the claim, as the formula above implies \( G_i(LK_n/\mathbb{Q}_p) \subseteq H \subseteq \ker \phi \) for such \( i \).

By definition, we have

\[
\eta_{LK_n/L}(i) = \frac{1}{g_0}(g_1 + g_2 + \cdots + g_i)
\]

where we write \( g_i = |G_i(LK_n/L)| \). Enlarging \( L \) if necessary, assume that \( \mathbb{Q}_p(\mu_p^{\infty}) \cap L = K_m \). Suppose we have \( p^s - 1 \geq i \geq p^{s-1} \), where \( n \geq s \geq m \). One checks that

\[
G_i(LK_n/L) \supseteq \text{Gal}(LK_n/LK_s)
\]

so for such \( i \) we have \( g_i \geq p^{n-s} \). We also observe that \( g_i = p^{n-m} \) for \( p^{m-1} \geq i \geq 0 \). Therefore we have

\[
\eta_{LK_n/L}(p^s - 1) \geq \frac{1}{g_0} \sum_{i=1}^{p^{m-1}-1} g_0 + \frac{1}{g_0} \sum_{t=m}^s p^{n-s}(p^s - p^{s-1})
\]

\[
= p^{m-1}(1 + (s-m)(p-1)) - 1.
\]

We may choose \( s \) to make this quantity as large as we like, independently of \( n \) (provided \( n \) is sufficiently large), which completes the proof.

Now let \( \rho \) be an Artin representation of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). The global Artin conductor associated to \( \rho \) may be given as a product

\[
N_\rho = \prod_{p \text{ prime}} p^{n_p(\rho)}
\]

where the local conductors \( n_p(\rho) \) are almost all zero (their precise definition is given below).

**Lemma 2.2.** We have

\[
n_p(\rho \otimes \chi) = n_p(\chi) \dim \rho
\]

when \( n_p(\chi) \) is sufficiently large.
Proof. Fix a character \( \chi \) of conductor \( p^n \); we know that \( \chi \) factors through \( \text{Gal}(K_n/\mathbb{Q}) \) where \( K_n = \mathbb{Q}(\mu_{p^n}) \). As \( \rho \) is an Artin representation, \( \rho \) factors through \( \text{Gal}(L/\mathbb{Q}) \) for some finite extension \( L/\mathbb{Q} \). We may regard \( \rho \otimes \chi \) as a representation of \( \text{Gal}(LK_n/\mathbb{Q}) \).

Let \( G \) be the decomposition group of \( LK_n/\mathbb{Q} \) at \( p \), and let \( G_i = G_i((LK_n)_{p^n}/\mathbb{Q}_p) \) be the \( i \)-th ramification group, where \( p_n \) denotes a prime of \( LK_n \) above \( p \). By definition of the Artin conductor (see [8, Chapter VII]) we have

\[
n_p(\rho \otimes \chi) = \sum_{i=0}^{\infty} \frac{g_i}{g_0} \left( \dim \rho - \frac{1}{g_i} \sum_{s \in G_i} \text{Tr} \rho(s) \chi(\rho) \right)
\]

where \( g_i = |G_i| \). We observe that

\[
\frac{1}{g_i} \sum_{s \in G_i} \chi(s) \text{Tr}(\rho)
\]

is the inner product of the characters \( \rho \) and \( \chi \). This equals the number of copies of \( \chi \) in \( \rho \), as representations restricted to \( G_i \).

Therefore if \( i < i_\rho \), the sum above must be zero as \( \rho \) restricted to \( G_{i_\rho} \) is a sum of copies of the trivial representation, and \( \chi \) is non-trivial. Also, if \( i \geq i_\rho \) then \( \text{Tr} \rho(s) = \dim \rho \) for any \( s \in G_i \) by the definition of \( i_\rho \). Putting this into the formula above we deduce that \( n_p(\rho \otimes \chi) = n_p(\chi) \dim \rho \).

3 The approximate functional equation

Let \( E \) be an elliptic curve and \( \rho \) an Artin representation, both defined over \( \mathbb{Q} \). The \( L \)-series of \( E \) twisted by \( \rho \) is defined by setting

\[
L(E, \rho, s) = \prod_{q \text{ prime}} P_q(E, \rho, q^{-s})^{-1}
\]

where the local polynomial at the prime \( q \) is given by

\[
P_q(E, \rho, T) = \det \left( 1 - \text{Frob}_q^{-1} \cdot T \big| (E_1^1) \otimes_{\mathbb{Q}_l} V_{\rho,l} \right).
\]

where \( l \) is a prime different from \( q \). Here, \( E_1^1 \) is the dual of the \( l \)-adic Tate module of \( E \), and \( V_{\rho,l} \) is the representation obtained from \( \rho \) by extending scalars to \( \mathbb{Q}_l \). This Euler product converges only on the right half-plane \( \text{Re}(s) > 3/2 \) but conjecturally it may be continued to a holomorphic function on the whole complex plane. Let us put

\[
L_\infty(s) = \left( 2(2\pi)^{-s} \Gamma(s) \right)^{\dim \rho}
\]

and define the completed \( L \)-function

\[
\hat{L}(E, \rho, s) := L_\infty(s) L(E, \rho, s).
\]

We also write \( N(E, \rho) \) for the conductor associated to the twist of \( E \) by \( \rho \). Then, we have the following conjecture (see [11] or [4, §2] for example):
Conjecture 3.1. The completed $L$-function $\hat{L}(E, \rho, s)$ has an analytic continuation to the whole complex plane, and satisfies the functional equation

$$\hat{L}(E, \rho, s) = w(E, \rho) N(E, \rho)^{1-s} \hat{L}(E, \rho^*, 2-s)$$

where $\rho^*$ is the contragredient representation to $\rho$, and the root number $w(E, \rho)$ is a complex number of absolute value 1.

Suppose $\rho$ is an Artin representation for which Conjecture 3.1 is satisfied. Let

$$L(E, \rho, s) = \sum_{n \geq 1} c_n n^{-s}$$

be the Dirichlet series expression which is valid for $\text{Re}(s) > 3/2$. For $\beta \in \mathbb{C}$ and positive $u \in \mathbb{R}$ we define the function

$$F_{\beta}(u) := \frac{1}{2\pi i} \int_{\text{Re}(s)=2} L_{\infty}(s+\beta) u^{-s} \frac{ds}{s}$$

Now we assume that $1 \leq \text{Re}(\beta) \leq 3/2$. Using Cauchy’s theorem and the functional equation, we obtain the following formula (the approximate functional equation):

$$\hat{L}(E, \rho, \beta) = \sum_{n \geq 1} c_n n^{-\beta} F_{\beta}(ny) + w N^{1-\beta} \sum_{n \geq 1} c_n^* n^{\beta-2} F_{2-\beta}(\frac{n}{Ny})$$

for $y > 0$, where $N = N(E, \rho)$ and $w = w(E, \rho)$. We also note that the function $F_{\beta}(u)$ has the following properties:

$$u^k F_{\beta}(u) \to 0 \quad \text{as} \quad u \to \infty$$

for any $k \geq 0$, and

$$F_{\beta}(u) \to L_{\infty}(\beta) \quad \text{as} \quad u \to 0.$$

We can check these by shifting the line of integration to $\text{Re}(s) = u^{k+\theta}$ and $\text{Re}(s) = u^{-\theta}$ respectively, for $\theta$ small and positive. Further, one can show that

$$F_{\beta}(u) \ll 1 + u^{\theta}$$

as $u$ approaches zero, for any small $\theta > 0$. These statements also hold for $F_{2-\beta}(u)$.

4 Proof of Theorem 1.1

We now fix a prime $p$ at which $E$ has good reduction. We fix the Artin representation $\rho$ and let $\chi$ be a primitive Dirichlet character modulo $p^a$. Taking $a = n_p(\chi)$ to be sufficiently large, by Lemma 2.2 we know that $n_p(\rho \otimes \chi) = a \dim \rho$. As we assumed $E$ has good reduction at $p$, we have

$$N(E, \rho \otimes \chi) = N p^{2a \dim \rho},$$

where $N$ is the prime-to-$p$ part of the conductor, which will remain fixed.
We consider the twisted $L$-series

$$L(E, \rho \otimes \chi, s) = \sum_{n \geq 1} \chi(n) c_n n^{-s}.$$  

In order to apply the approximate functional equation, we assume that it has the analytic continuation and functional equation specified in Conjecture 3.1 for every Dirichlet character $\chi$ of conductor $p^a$ (which is hypothesised in Theorem 1.1).

By the functional equation, it suffices to consider the values of $L(E, \rho \otimes \chi, s)$ in the right half of the critical strip. Let $\beta \in \mathbb{C}$ satisfy $1 \leq \text{Re}(\beta) \leq 3/2$; we apply equation (1) to write

$$\hat{L}(E, \rho \otimes \chi, \beta) = \sum_{n \geq 1} \chi(n) c_n n^{-\beta} F_{\beta} \left( \frac{ny}{Np^{2ad}} \right)$$

for $y > 0$; here we have re-normalised the variable $y$ via $y \mapsto yN^{-1}p^{-2ad}$. We observe that the $\chi$-twist does not change the Gamma-factor $L(\infty, s)$ so it does not affect the function $F$.

We now average over primitive characters modulo $p^a$: we define

$$A(n) := \sum_{\chi \mod p^a}^* \chi(n) F_{\beta} \left( \frac{ny}{Np^{2ad}} \right)$$

and

$$B(n) := \sum_{\chi \mod p^a}^* w(E, \rho, \chi) \chi(n) p^{2ad(1-\beta)}$$

where the symbol $\sum^*$ denotes the sum over primitive characters only. Then we have

$$\sum_{\chi \mod p^a}^* \hat{L}(E, \rho \otimes \chi, \beta) = \sum_{n \geq 1} A(n) c_n n^{-\beta}$$

$$+ N^{1-\beta} \sum_{n \geq 1} B(n) c_n^* n^{\beta-2} F_{2-\beta} \left( \frac{n}{y} \right).$$

We will proceed to show that this sum tends to infinity with $a$ when

$$\frac{3}{2} - \frac{2}{2d+1} < \text{Re}(\beta) \leq \frac{3}{2}$$

which will establish Theorem 1.1. We write the sum above in the form

$$\sum_{\chi \mod p^a}^* \hat{L}(E, \rho \otimes \chi, \beta) = A(1) + \Sigma_1 + \Sigma_2$$

where we have put

$$\Sigma_1 = \sum_{n \geq 2} A(n) c_n n^{-\beta} \quad \text{and} \quad \Sigma_2 = N^{1-\beta} \sum_{n \geq 1} B(n) c_n^* n^{\beta-2} F_{2-\beta} \left( \frac{n}{y} \right).$$

We will show that the parameter $y$ may be specified so that $|A(1)| \gg |\Sigma_1|$ and $|A(1)| \gg |\Sigma_2|$ as $a \to \infty$ which will prove the result.
5 Estimating the sums

We introduce a parameter \( x \) such that \( xy = p^{2da} \) and assume both \( x \) and \( y \) are fixed positive powers of \( p^a \) (we will discuss how they can be specified in \( \S 6 \)). We have

\[
A(1) = \sum_{\chi \mod p^a} F_{\beta} \left( \frac{y}{Np^{2ad}} \right).
\]

By our choice of \( x \) and \( y \) we have

\[
y Np^{2ad} \to 1 \quad \text{as} \quad a \to \infty.
\]

Recalling that \( F_{\beta}(u) \) tends to the non-zero constant \( L_{\infty}(\beta) \) as \( u \to 0 \) we have

\[
|A(1)| \sim \sum_{\chi \mod p^a} 1
\]

and therefore \( |A(1)| \gg p^a \) (recalling that \( p \) is constant and \( a \) is tending to infinity). Next we consider \( \Sigma_1 \); we have

\[
|A(n)| = \left| F_{\beta} \left( \frac{ny}{Np^{2ad}} \right) \sum_{\chi \mod p^a} \chi(n) \right|.
\]

Suppose first that \( n \leq x^{1+\epsilon} \) for \( \epsilon \) small and positive. Using the estimate \( F_{\beta}(u) \ll 1 + u^\theta \), we obtain

\[
|A(n)| \ll x^\epsilon \sum_{\chi \mod p^a} \chi(n).
\]

By basic properties of character sums we have

\[
\sum_{\chi \mod m} \chi(n) = \sum_{b \parallel (n-1, m)} \varphi(b) \mu \left( \frac{m}{b} \right)
\]

if \( (n, m) = 1 \) (see [6, 3.8]). From this we infer that the character sum factor in \( |A(n)| \) is \( \ll p^a \) if \( n-1 \) is divisible by \( p^{a-1} \), and zero otherwise. The Dirichlet coefficients \( c_n \) are known to satisfy \( |c_n| \leq n^{1/2+\epsilon} \) for any \( \epsilon > 0 \), and putting these facts together we get

\[
\left| \sum_{2 \leq n \leq x^{1+\epsilon}} A(n) c_n n^{-\beta} \right| \ll p^a x^\epsilon \sum_{2 \leq n \leq x^{1+\epsilon}} n^{1/2 - \text{Re}(\beta) + \epsilon}.
\]  

(2)

where \( \sum^\dagger \) denotes the sum over integers congruent to 1 mod \( p^{a-1} \) only. We then observe that

\[
\sum_{2 \leq n \leq x^{1+\epsilon}} n^{-\theta} = \sum_{1 \leq r \leq x^{1+\epsilon} p^{1-a}} (1 + rp^{a-1})^{-\theta}
\]

\[
\ll p^{-a} x^{1+\epsilon} p^{1-a} \sum_{1 \leq r \leq x^{1+\epsilon} p^{1-a}} r^{-1-\epsilon}
\]

\[
\ll p^{-a+\epsilon} x^{1+\epsilon}.
\]
for any constant $\theta > 0$. Putting $\theta = \text{Re}(\beta) - 1/2 - \epsilon$ and combining this with \[2\] we deduce

$$\left| \sum_{2 \leq n \leq x^{1+\epsilon}} A(n) c_n n^{-\beta} \right| \ll x^{3/2 - \text{Re}(\beta) + \epsilon}.$$ 

We must deal with the terms for $n > x^{1+\epsilon}$; however the fact that $u^k F_\beta(u) \to 0$ as $u \to \infty$ for any $k \geq 0$ shows that the contribution of these terms is negligible. We conclude that

$$|\Sigma_1| \ll x^{3/2 - \text{Re}(\beta) + \epsilon}. \quad (3)$$

We now consider the sum

$$\Sigma_2 = N^{1-\beta} \sum_{n \geq 1} B(n) c_n^* n^{\beta-2} F_{2-\beta} \left( \frac{n}{y} \right).$$

As above, it will suffice to bound the terms with $n \leq y^{1+\epsilon}$ as the tail will become constant due to the rapid decay of $F_{2-\beta}(u)$ as $u \to \infty$. By the estimates for $F_{2-\beta}(u)$ in \[4\] we have the bound

$$F_{2-\beta}(u) \ll \frac{1 + u^\epsilon}{1 + u^2}$$

for any small $\epsilon > 0$. Following Luo’s proof from \[7\], we apply Cauchy’s inequality and split up the product to get

$$\left| \sum_{1 \leq n \leq y^{1+\epsilon}} B(n) c_n^* n^{\beta-2} F_{2-\beta} \left( \frac{n}{y} \right) \right| \ll \sum_{1 \leq n \leq y^{1+\epsilon}} |B(n)| n^{\text{Re}(\beta)-2} \left( 1 + \left( \frac{n}{y} \right)^2 \right)^{-1/2} \left( \sum_{n \leq y^{1+\epsilon}} |B(n)|^2 \left( 1 + \left( \frac{n}{y} \right)^2 \right)^{-1} \right)^{1/2} \ll y^{\text{Re}(\beta)-1+\epsilon} \left( \sum_{n \leq y^{1+\epsilon}} |B(n)|^2 \left( 1 + \left( \frac{n}{y} \right)^2 \right)^{-1} \right)^{1/2}. $$

Here we have bounded the first factor in the product above by a method analogous to that used for $\Sigma_1$. Let us define

$$H(u) := \frac{1}{\pi(1 + u^2)}$$

which is the Fourier transform of $e^{-2\pi |u|}$. By the estimate above we have

$$|\Sigma_2| \ll y^{\text{Re}(\beta)-1+\epsilon} \left( \sum_{n \in \mathbb{Z}} |B(n)|^2 H \left( \frac{n}{y} \right) \right)^{1/2}. $$

Here we have increased the range of the sum in order to apply Poisson summation later. Recall that

$$B(n) = \sum_{\chi \mod p^n}^* w_\chi \overline{\chi}(n) p^{2n\delta(1-\beta)},$$
where we have written \( w_\chi \) for the root number \( w(E, \rho \otimes \chi) \). We have

\[
\sum_{n \in \mathbb{Z}} |B(n)|^2 H \left( \frac{n}{y} \right) \leq \sum_{\chi \mod p\alpha} \sum_{\psi \mod p\alpha} p^{4\alpha d(1-\text{Re}(\beta))} \left| w_\chi \prod_{n \in \mathbb{Z}} \chi \psi(n) H \left( \frac{n}{y} \right) \right|.
\]

First we consider the diagonal terms in this sum: those with \( \chi = \psi \). For these terms we get

\[
\sum_{\chi \mod p\alpha} p^{4\alpha d(1-\text{Re}(\beta))} \left| w_\chi \prod_{n \in \mathbb{Z}} H \left( \frac{n}{y} \right) \right| \ll p^{4\alpha d(1-\text{Re}(\beta)) + a} \sum_{n \in \mathbb{Z}} H \left( \frac{n}{y} \right)
\]

recalling that \( |w_\chi| = 1 \). By the Poisson summation formula we have

\[
\sum_{n \in \mathbb{Z}} H \left( \frac{n}{y} \right) = y \sum_{h \in \mathbb{Z}} T(yh)
\]

where \( T(u) = e^{-2\pi |u|} \). All terms in this sum decay exponentially with \( y \), except that for \( h = 0 \). So \( y \sum_{h \in \mathbb{Z}} T(yh) \ll y \) which implies that the diagonal terms are

\[
\ll y p^{4\alpha d(1-\text{Re}(\beta)) + a}.
\]

Now we consider the terms with \( \chi \neq \psi \). The characters \( \chi \psi \) are not primitive in general, therefore we use Poisson summation in the following form:

\[
\sum_{n \in \mathbb{Z}} \chi \psi(n) H \left( \frac{n}{y} \right) = \frac{y}{p^a} \sum_{h \in \mathbb{Z}} \tau_h(\chi \psi) T \left( \frac{yh}{p^a} \right).
\]

Here we write

\[
\tau_h(\chi \psi) = \sum_{r \mod p^a} \chi \psi(r) e^{2\pi i hr/p^a}
\]

for the discrete Fourier transform of the character \( \chi \psi \) which is defined modulo \( p^a \). This may be difficult to estimate when \( \chi \psi \) is not primitive, but for our purposes the trivial bound \( |\tau_h(\chi \psi)| \leq p^a \) will suffice. We also note that \( \tau_0(\chi \psi) = 0 \) as \( \chi \psi \) is non-trivial here, and we obtain

\[
\left| \sum_{n \in \mathbb{Z}} \chi \psi(n) H \left( \frac{n}{y} \right) \right| \leq y \sum_{h \in \mathbb{Z}, h \neq 0} T \left( \frac{yh}{p^a} \right).
\]

Recall that \( y \) is a positive power of \( p^a \). We assume that \( y \) is chosen so that \( y/p^a \to \infty \) (we will show in the next section that this assumption is consistent with other restrictions on the parameter \( y \)) and therefore all terms in the sum over \( h \) decay exponentially with \( p^a \). This removes the contribution from the factor \( y \), so the right hand side of the above inequality is \( O(1) \).

We deduce that the contribution of the terms with \( \chi \neq \psi \) in the sum above is \( \ll p^{4\alpha d(1-\text{Re}(\beta)) + 2a} \). As we have now assumed \( y/p^a \to \infty \) we see that diagonal terms in the sum are dominant, and we get

\[
\sum_{n \in \mathbb{Z}} |B(n)|^2 H \left( \frac{n}{y} \right) \ll p^{4\alpha d(1-\text{Re}(\beta)) + a} y.
\]

Recalling the earlier bound for \( \Sigma_{2} \) obtained from Cauchy’s inequality, we have

\[
|\Sigma_{2}| \ll y^{\text{Re}(\beta)-1/2+\epsilon} p^{2\alpha d(1-\text{Re}(\beta)) + a/2}.
\]
6 Conclusion

Let us write $p^a = P$ for our parameter which is tending to infinity. We will now specify $x$ and $y$: we require them to satisfy $xy = P^{2d}$ and to tend to infinity with $P$, so we may write

$$y = P^{2d\gamma} \quad \text{and} \quad x = P^{2d(1-\gamma)}$$

for $0 < \gamma < 1$. Further, we required $y/P$ to tend to infinity, so we assume $\gamma > 1/2d$. Putting $\text{Re}(\beta) = \sigma$, we may write estimates (3) and (4) as follows:

$$|\Sigma_1| \ll P^{2d(1-\gamma)(3/2-\sigma+\epsilon)}$$

and

$$|\Sigma_2| \ll P^{2d\gamma(\sigma-1/2+\epsilon)+2d(1-\sigma)+1/2}$$

Recall that we have $|A(1)| \gg P$, so $A(1)$ will eventually grow more rapidly than $\Sigma_1$ and $\Sigma_2$ if the following two inequalities hold:

$$2d(1-\gamma)(3/2-\sigma) < 1, \quad (5)$$
$$2d\gamma(\sigma-1/2) + 2d(1-\sigma) < 1/2. \quad (6)$$

Observe that we have neglected $\epsilon$ which may be chosen as small as we like. From (5) and (6) we deduce the bounds

$$1 - \frac{1}{2d(3/2-\sigma)} < \gamma \quad \text{and} \quad \frac{1 + 4d(\sigma - 1)}{4d(\sigma - 1/2)} < \gamma.$$ 

One checks that there exists a value of $\gamma$ satisfying both of these inequalities precisely when $\sigma > (6d - 1)/(4d + 2)$. Further, combining the upper bound on $\gamma$ with the assumption $\gamma > 1/2d$ we obtain $\sigma > 1$; this is a strictly weaker condition, except in the case $d = 1$. Given the functional equation, these are the hypotheses imposed in Theorem 1.1, and this completes the proof.

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