CLOSED IDEALS AND LIE IDEALS OF $C_0(X) \otimes^\text{min} A$

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Abstract. In this article, we prove that a closed ideal of $C_0(X) \otimes^\text{min} A$ is a finite sum of product ideals, where $X$ is a locally compact Hausdorff space and $A$ is a $C^*$-algebra with only finitely many closed ideals. As an application, we characterize the closed Lie ideals of $C_0(X) \otimes^\text{min} B(H)$, $H$ being a separable Hilbert space. Further, for a $C^*$-algebra $A$, the primitive ideals of $C_0(X) \otimes^\text{min} A$ are characterized. We also prove that for a simple unital $C^*$-algebra $A$ and a compact Hausdorff space $X$, $A$ has Dixmier property if and only if $C(X) \otimes^\text{min} A$ has the same.

1. Introduction

Let $A \otimes^\alpha B$ denote the completion of the algebraic tensor product of two $C^*$-algebras $A$ and $B$ under an algebra cross norm $\|\cdot\|_\alpha$, then one may ask the following natural question: Can the closed ideals of $A \otimes^\alpha B$ be identified in terms of the closed ideals of $A$ and $B$? In 1978, Wassermann [22] established an astonishing result, which he termed as a pathology, that not every closed ideal of $B(H) \otimes^\text{min} B(H)$ is a finite sum of product ideals, $\otimes^\text{min}$ being the minimal $C^*$-tensor norm (for definition, see [19]).

On the other hand, closed ideals of the Banach algebras $A \otimes^h B$, $A \otimes^\gamma B$ and $A \otimes \delta$ are directly related to the closed ideals of $A$ and $B$, where $\otimes^h$, $\otimes^\gamma$ and $\otimes$ are the Haagerup tensor product, Banach space projective tensor product and operator space projective tensor product, respectively.

It is interesting to note that every closed ideal of these spaces is a finite sum of product ideals if either $A$ or $B$ possesses finitely many closed ideals (see, for instance, [1, 7, 11]). In the present article we deal with their counterpart for the minimal $C^*$-tensor norm assuming one of the $C^*$-algebras to be commutative. In particular, we prove that every closed ideal of $C_0(X) \otimes^\text{min} A$ is a finite sum of product ideals, where $X$ is a locally compact Hausdorff space and $A$ is a $C^*$-algebra with finitely many closed ideals. In order to prove this we establish an appropriate (surjective) correspondence between a class of closed subsets of $X$ and the closed ideals of $C_0(X,A)$, for any $C^*$-algebra $A$.

Interestingly, this correspondence turns out to be bijective if $A$ has finitely many closed ideals. This correspondence further enables us to give a nice characterization of primitive ideals of $C_0(X) \otimes^\alpha A$, namely, every primitive ideal of $C_0(X) \otimes^\alpha A$ is precisely of the form $C_0(X) \otimes^\alpha I + J \otimes^\alpha A$ for some primitive ideals $I$ of $A$ and $J$ of $C_0(X)$, where $\otimes^\alpha$ is either $\otimes^\text{min}$ or $\otimes^h$ or $\otimes$. Note that if $A \otimes^\text{min} B$ has property (F) of Tomiyama and $I \subseteq A$ and $J \subseteq B$ are primitive ideals then $I \otimes^\text{min} B + A \otimes^\text{min} J$ is a primitive ideal of $A \otimes^\text{min} B$ ([3, Theorem 5]). Similar results for Haagerup tensor norm and operator space projective norm are also known (see, [11, Theorem 5.13], [11, Theorem 7]).

If $B$ is a Banach algebra, it naturally imbibes a Lie algebra structure with the Lie bracket given by $[a,b] = ab - ba$ for every $a, b \in B$. A closed subspace $L$ of $B$ is said to be a Lie ideal if $[B,L] \subseteq L$ where $[B,L] = \text{span}\{[b,l] : b \in B, l \in L\}$. The closed Lie ideals for $C^*$-algebras are extensively studied, one may refer to the expository article [16] for details. Recently some research has been done to identify the closed Lie ideals for the various tensor products of $C^*$-algebras. In [16, Section 5], [9, Section 4]) the closed Lie ideals of $C_0(X) \otimes^\text{min} A$ have been characterized in terms of closed subspaces of $X$, $X$ being a locally compact Hausdorff space and $A$ being a simple $C^*$-algebra with at most one tracial state. However, if $A$ is not simple nothing is known about the closed Lie ideals of such spaces. In Section 3, we characterize the closed Lie ideals of $C_0(X,A)$ in terms of some closed subspaces of $X$. As an application, we first prove that a closed subspace $L$ of $C_0(X) \otimes^\text{min} B(H)$ is a Lie ideal if and only if there exist two closed subsets $S_1 \subseteq S_2$ of $X$ and a closed subspace $K$
of $C_0(X) \otimes \mathbb{C}1$ such that $L = \overline{J(S_1) \otimes K(H) + J(S_2) \otimes B(H) + K}$, where $H$ is a separable Hilbert space and for $F \subseteq X$, $J(F) := \{ f \in C_0(X) : f(F) \subseteq \{0\} \}$. Finally, as another application we prove that for a compact Hausdorff space $X$, $C(X) \otimes_{\min} A$ has Dixmier property if and only if $A$ has the same, $A$ being simple unital $C^*$-algebra.

2. Closed ideals of $C_0(X) \otimes_{\min} A$

Let $X$ be a locally compact Hausdorff space and $A$ be any $C^*$-algebra. It is a well known fact that there is a bijective correspondence between the closed subsets of $X$ and the closed ideals of $C_0(X)$ given by $F \leftrightarrow J(F)$. However, if we move from complex valued functions to the vector valued functions, such a correspondence is not known. Although, in the literature, it is established that every closed ideal of $C_0(X,A)$ is of the form $\{ f \in C_0(X,A) : f(x) \in I_x, \forall x \in X \}$ where for every $x \in X$, $I_x$ is a closed ideal of $A$ [17 V.26.2.1], but this description fails to be fruitful while moving from $C_0(X,A)$ to $C_0(X) \otimes_{\min} A$ in order to determine the closed ideals.

We first generalize the former notion to the continuous vector valued functions by establishing a correspondence between the closed subsets of $X$ and closed ideals of $C_0(X,A)$. This correspondence will further enable us to characterize closed ideals of $C_0(X) \otimes_{\min} A$ in terms of closed ideals of $A$ and subsets of $X$, when $A$ has finitely many closed ideals. This is due to the fact that there exists an isometric $^*$-isomorphism $\hat{\varphi} : C_0(X) \otimes_{\min} A \rightarrow C_0(X,A)$, which takes $f \otimes a$ to $a f$ for every $f \in C_0(X)$ and $a \in A$, where $(a f)(x) = f(x) a$ (see, [19 Theorem 4.14 (iii)], [14 Proposition 1.5.6]).

Let us first fix some notations for further use. For any $t \in \mathbb{N}$, the set $\{1, 2, 3, \ldots, t\}$ be denoted by $N_t$. The spaces $C_b(X,A)$ and $C_c(X,A)$, as usual, denote the $C^*$-algebras of all bounded continuous functions and compactly supported continuous functions, respectively, from $X$ to $A$ endowed with sup norm. For a non-unital $C^*$-algebra $A$, $\hat{A}$ will denote its unitization. For a locally compact Hausdorff space $X$ and any function $g \in C_0(X)$, we define $\hat{g} \in C_0(X,\hat{A})$ (resp., $\hat{g} \in C_0(X,A)$) by $\hat{g}(x) = g(x) 1$, where 1 is the unit of $\hat{A}$ (resp., of $A$) if $A$ is non unital (resp., if $A$ is unital).

For an indexing set $\Delta$, let $S = \{ S_i \}_{i \in \Delta}$ and $T = \{ T_i \}_{i \in \Delta}$ be collections of subsets of some sets $Y$ and $Z$. We define $S$ to be compatible with $T$ if whenever for some subset $\gamma$ of $\Delta$, $\cap_{j \in \gamma} T_j = T_i$ for some $i \in \Delta$, then $\cap_{j \in \gamma} S_j = S_i$. For a locally compact Hausdorff space $X$, $\alpha \in \Delta$ and $T$ as above, define a collection $T^\alpha := \{ S = \{ S_i \}_{i \in \Delta} : S_i$ is a closed subset of $X$ for every $i \in \Delta, S$ is compatible with $T \text{ and } S_\alpha = X \}$. 

**Theorem 2.1.** Let $X$ be a locally compact Hausdorff space, $A$ be a $C^*$-algebra and $\mathcal{I} = \{ I_i \}_{i \in \Delta}$ be the collection of all closed ideals of $A$ with $I_\beta = A$. Then there exists a surjection $\theta$ from $T^\Delta$ into $\mathcal{K}$, the set of all closed ideals of $C_0(X,A)$.

**Proof.** Define $\theta : T^\Delta \rightarrow \mathcal{K}$ by $\theta(S) = J(S)$ where $J(S) = \{ f \in C_0(X,A) : f(S_i) \subseteq I_i, \forall i \in \Delta \}$. Clearly $\theta$ is well defined. To see that $\theta$ is onto, consider $J \in \mathcal{K}$. For each $i \in \Delta$, set $S_i = \cap_{j \in \gamma} I_j$ for some subset $\gamma$ of $\Delta$, then

$$S_i = \cap_{j \in \gamma} I_j \cap_{j \in \gamma} I_j^{-1} \cap_{j \in \gamma} I_j^{-1} \cap_{j \in \gamma} I_j^{-1} \cap_{j \in \gamma} I_j^{-1} \cap_{j \in \gamma} I_j^{-1} \cap_{j \in \gamma} S_j,$$

so that $S = \{ S_i \}_{i \in \Delta}$ is compatible with $\mathcal{I}$. Since, $S_\beta = \cap_{j \in \gamma} I_j^{-1} \cap_{j \in \gamma} I_j^{-1} \cap_{j \in \gamma} I_j^{-1} \cap_{j \in \gamma} I_j^{-1} \cap_{j \in \gamma} I_j^{-1} \cap_{j \in \gamma} I_j^{-1} \cap_{j \in \gamma} S_j = \cap_{j \in \gamma} I_j$, we obtain $S \in T^\Delta$. Clearly $J \subseteq J(S)$, so it is sufficient to prove that $J$ is dense in $J(S)$. For this, let $f \in J(S)$ be non zero and $\epsilon > 0$ be arbitrary.

We first claim that for any $x \in X$, there exists an element $h_x \in J$ such that $\| h_x(x) - f(x) \| \leq \frac{\epsilon}{2}$. Indeed, if $\gamma' = \{ i \in \Delta : x \in S_i \}$, then $x \in \cap_{i \in \gamma'} S_i$ and $f(x) \in \cap_{i \in \gamma'} I_i = I_j$ for some $j \in \Delta$. Let $I_j$ denote the closed ideal of $A$ generated by the set $\{ g(x) : g \in \hat{J} \}$. Then $x \in S_r$, which implies that $f(x) \in I_r$. Hence $\| \sum_{i=1}^n a_i g_i(x) b_i - f(x) \| \leq \frac{\epsilon}{2}$ for some $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$ in $A$ (resp., in $\hat{A}$) if $A$ is unital (resp., if $A$ is non unital). From [12 Corollary 4.2.10], we know that $J$ is a closed ideal of $C_0(X,\hat{A})$, thus we have a function $h_x = \sum_{i=1}^n a_i g_i b_i$ in $J$ which satisfies the required condition.

Denote by $\hat{X} = X \cup \{ \infty \}$, the one point compactification of $X$. For each $x \in X$, let $\hat{h}_x$ and $\hat{f}$ be the continuous extensions of $h_x$ and $f$ to $\hat{X}$ which take $\infty$ to 0, and let $\hat{h}_x 0$ be the continuous
Let \( A \) be a \( C^* \)-algebra and \( I = \{ I_1, I_2, \ldots, I_n = A \} \), \( n \in \mathbb{N} \), be the set of all closed ideals of \( A \). Then the map \( \theta \) from \( T_\mathcal{F}^\circ \) to the set of all closed ideals of \( C_0(X, \mathcal{I}) \) is a bijection. In particular, every closed ideal of \( C_0(X, \mathcal{I}) \) is precisely of the form \( \{ f \in C_0(X, \mathcal{I}) : f(x) = 0, \forall x \in F \} \), for some closed subset \( F \) of \( X \).

**Theorem 2.2.** Let \( X \) be a locally compact Hausdorff space, \( A \) be a \( C^* \)-algebra and \( \mathcal{I} = \{ I_1, I_2, \ldots, I_n = A \} \), \( n \in \mathbb{N} \), be the set of all closed ideals of \( A \). Then the map \( \theta \) from \( T_\mathcal{F}^\circ \) to the set of all closed ideals of \( C_0(X, \mathcal{I}) \) is a bijection. In particular, every closed ideal of \( C_0(X, \mathcal{I}) \) is precisely of the form \( \{ f \in C_0(X, \mathcal{I}) : f(x) = 0, \forall x \in F \} \), for some closed subset \( F \) of \( X \).

**Proof.** Let \( S = \{ S_i \}_{i \in \mathbb{N}_n} \) and \( S' = \{ S'_i \}_{i \in \mathbb{N}_n} \) be two distinct elements of \( T_\mathcal{F}^\circ \) so that \( S_i \neq S'_i \) for some \( i \in \mathbb{N}_n \setminus \{ n \} \). Without loss of generality, we may assume that there exists \( x \in S_i \setminus S'_i \).

Define a non-empty subset \( \gamma = \{ j \in \mathbb{N}_n : x \notin S'_j \} \) of \( \mathbb{N}_n \). By Urysohn’s Lemma [18; Theorem 2.12], as \( \mathcal{V}_x = \{ \cup_{i \in \gamma} S'_i \}^\circ \) is an open set containing \( x \), there exists \( g \in C_c(X) \) such that \( g(x) = 1 \), \( g(X) \subseteq [0, 1] \) and \( \text{supp}(g) \subseteq V_x \). It is easy to see that \( (\cap_{j \in \gamma} I_j) \setminus I_i \neq \emptyset \), because if \( \cap_{j \in \gamma} I_j \subseteq I_i \) then \( \cap_{j \in \gamma} S'_j \subseteq S'_i \), which is not true as \( x \in \cap_{j \in \gamma} S'_j \) but \( x \notin S'_i \). Let \( a \in (\cap_{j \in \gamma} I_j) \setminus I_i \). Consider the function \( h = a \cdot g \in C_0(X, \mathcal{I}) \). Observe that \( h \notin J(S_i) \), since \( x \in S_i \) but \( h(x) = a'(x) \hat{g}(x) = a \notin I_i \). However, we assert that \( h \in J(S') \) which proves \( J(S) \neq J(S') \). For \( k \in \gamma \), \( y \in S'_k \) implies \( y \in V_x^\circ \), so that \( h(y) = a \hat{g}(y) = 0 \in I_k \). Also, if \( k \in \gamma^c \), then \( h(y) = a \hat{g}(y) \in (\cap_{j \in \gamma^c} I_j) \subseteq I_k \) for every \( y \in S'_k \).

In the quest of proving the main result regarding the characterization of closed ideals of \( C_0(X) \otimes_{min} A \), we require few more ingredients.

**Lemma 2.3.** Let \( X \) be a locally compact Hausdorff space and \( A \) be a \( C^* \)-algebra. Then for a closed subspace \( C \) of \( A \) and a closed ideal \( J(Y) \) of \( C_0(X) \), \( Y \subseteq X \) being closed, there is an isometric isomorphism of Banach spaces

\[
J(Y) \otimes C^\min \cong \{ f \in C_0(X, A) : f(Y) = \{ 0 \}, f(X) \subseteq C \}.
\]

**Proof.** Denote by \( J \) the closed subspace \( \{ f \in C_0(X, A) : f(Y) = \{ 0 \}, f(X) \subseteq C \} \) of \( C_0(X, A) \), and \( I = J(Y) \). Let \( \varphi \) denote the restriction of \( \hat{\varphi} \) to \( \overline{T \otimes C^\min} \), where \( \hat{\varphi} : C_0(X) \otimes_{min} A \to C_0(X, A) \)
is the isometric $*$-isomorphism as discussed earlier. Then for $\sum_{j=1}^{n} f_j \otimes c_j \in I \otimes C$, we have $\varphi(\sum_{j=1}^{n} f_j \otimes c_j)(Y) = \{0\}$, so that $\varphi(I \otimes C) \subseteq J$. Since $\varphi$ is an isometry, it is sufficient to prove that $\varphi(I \otimes C)$ is dense in $J$.

Let $g \in J$ and $\epsilon > 0$ be arbitrary. Since $J$ is also a closed subspace of $C_0(X, C)$ and $C_0(X, C)$ is dense in $C_0(X, C)$, there exists a function $h \in C_0(X, C)$ such that $\|g - h\| < \epsilon/2$. Let $K := \text{supp}(h)$, $B_r(b) := \{c \in C : \|c - b\| < r\}$ and $B^r_r(b) := B_r(b) \setminus \{0\}$, where $b \in C$ and $r > 0$. Since $\|h(y)\| = \|g(y) - h(y)\| < \epsilon/2$ for every $y \in Y$, the collection $\{h^{-1}(B^r_r(h(x)) \setminus \overline{h(Y)}) : x \in K \setminus Y\} \cup h^{-1}(B_r(0))$ forms an open cover of the compact set $K$. Fix a finite subcover, say, $h^{-1}(B_r(0)) \cup \{h^{-1}(B^r_r(h(x_i)) \setminus \overline{h(Y)}) : 1 \leq i \leq n\}$. Since $K$ is a compact subset of a locally compact Hausdorff space $X$, there exists a partition of unity subordinate to this finite subcover, i.e. there exist functions $f_0, f_1, \ldots, f_n$ in $C_c(X)$ such that $0 \leq f_i \leq 1$ for all $0 \leq i \leq n$, supp$(f_0) \subseteq U_0 := h^{-1}(B_r(0))$, supp$(f_i) \subseteq U_i := h^{-1}(B^r_r(h(x_i)) \setminus \overline{h(Y)})$ for all $1 \leq i \leq n$ and $\sum_{i=0}^{n} f_i(x) = 1$ for $x \in K$ (see [13, Theorem 2.13]).

Let $V = (\sum_{i=0}^{n} f_i)^{-1}(0, 3/2)$. Then $V \cap (\cup_{i=0}^{n} U_i)$ is an open set containing $K$. Pick $\hat{f} \in C_c(X)$ such that $\hat{f}$ is 1 on $K$, supp$(\hat{f}) \subseteq V \cap (\cup_{i=1}^{n} U_i)$ and $0 \leq \hat{f} \leq 1$. Then for $\hat{f} = \sum_{i=0}^{n} f_i(x)$, supp$(\hat{f}) \subseteq V \cap U_i$ because supp$(f_i) \subseteq U_i$ and supp$(\hat{f}) \subseteq V$. Now for $x \in K$, we have

$$\hat{f}(x) = \sum_{i=0}^{n} f_i(x)$$

Also notice that $0 \leq \sum_{i=0}^{n} \hat{f}_i \leq 3/2$ because for $x \in V \cap (\cup_{i=0}^{n} U_i)$, $\sum_{i=0}^{n} \hat{f}_i(x) = \sum_{i=0}^{n} f_i(x) \leq \sum_{i=0}^{n} f_i(x) = 3/2$ and for $x \in (V \cap (\cup_{i=0}^{n} U_i))^c$, we have $\sum_{i=0}^{n} f_i(x) = 0$.

Now for $1 \leq i \leq n$, the open set $U_i$, and thus $V \cap U_i$ is disjoint from $Y$ so that $\sum_{i=1}^{n} \hat{f}_i \otimes h(x_i) \in I \otimes C$. Fix $x_0 \in K^c$, then for each $x \in X$

$$\|h(x) - \sum_{i=1}^{n} \hat{f}_i(x)h(x_i)\| = \|h(x)\| \sum_{i=0}^{n} \hat{f}_i(x) - \sum_{i=0}^{n} \hat{f}_i(x)h(x_i)\|$$

$$\leq \sum_{i=0}^{n} \|h(x) - h(x_i)\| \hat{f}_i(x)$$

$$= \sum_{i : x \in U_i \cap V} \|h(x) - h(x_i)\| \hat{f}_i(x)$$

$$< \epsilon.$$  

Hence we obtain $\|g - \varphi(\sum_{i=1}^{n} \hat{f}_i \otimes h(x_i))\| < \frac{\epsilon}{2}$, proving that $\varphi(I \otimes C)$ is dense in $J$. \hfill \Box

As a consequence of the above result, we have an interesting observation which identifies certain closed ideals of $C_0(X, A)$ with some closed ideals of $C_0(X) \otimes_{\min} A$. Below, we will use the notation $J(x)$ for $J(\{x\})$, $x \in X$, for convenience. Note that, in a $C^*$-algebra, finite sum of closed ideals is again a closed ideal.

**Corollary 2.4.** Let $X$ be a locally compact Hausdorff space and $I$ be a closed ideal of a $C^*$-algebra $A$. Then for any $x \in X$

$$C_0(X) \otimes_{\min} I + J(x) \otimes_{\min} A = \{f \in C_0(X, A) : f(x) \in I\}$$

**Proof.** Let $J_1 = C_0(X) \otimes_{\min} I + J(x) \otimes_{\min} A$ and $J_2 = \{f \in C_0(X, A) : f(x) \in I\}$, then $J_1$ and $J_2$ are closed ideals of $C_0(X) \otimes_{\min} A$ and $C_0(X, A)$, respectively. From Lemma 2.3, it is clear that $J_1 \subseteq J_2$. For the other containment, consider $f \in J_2$. Let $\hat{f} \in C_c(X)$ be such that $\hat{f}$ takes values in $[0, 1]$ and $\hat{f}(x) = 1$. For every $y \in X$, define $g(y) := f(x)f(y)$, then by Lemma 2.3, $g \in C_0(X) \otimes_{\min} I$. Now for $h = f - g$, $h(x) = f(x) - f(x)f(x) = f(x) - f(x) = 0$, which shows that $h \in J(x) \otimes_{\min} A$ and hence $f = h + g \in J_1$. \hfill \Box
We would now extend the above result from the singleton \( \{ x \} \) to any closed subset of \( X \) and the proof requires little more efforts.

**Corollary 2.5.** Let \( Y \) be a closed subset of a locally compact Hausdorff space \( X \). For any closed ideal \( I \) of a \( C^* \)-algebra \( A \), we have

\[
C_0(X) \otimes \min I + J(Y) \otimes \min A = \{ f \in C_0(X, A) : f(Y) \subseteq I \}
\]

**Proof.** Let \( I = \{ I_i \}_{i \in \Delta} \) be the set of all closed ideals of \( A \) with \( I_\beta = A \), for some \( \beta \in \Delta \), and set \( I = I_\beta \). If \( t = \beta \) then the result is trivial. Otherwise, let \( J_1 = C_0(X) \otimes \min I + J(Y) \otimes \min A \) and \( J_2 = \{ f \in C_0(X, A) : f(Y) \subseteq I_i \} \). Then, by Theorem 2.4 there exist elements \( S = \{ S_i \}_{i \in \Delta} \) and \( S' = \{ S'_i \}_{i \in \Delta} \) in \( T^\Delta_2 \) such that \( J_1 = J(S) \) and \( J_2 = J(S') \). It is sufficient to prove that \( S = S' \).

We first mention a common trick used in the proof. For any \( x \in X \) and a closed subset \( F \) of \( X \) with \( x \not\in F \), Urysohn’s Lemma implies that there exists \( f \in C_c(X) \) such that \( f(x) = 1 \) and \( f(F) = 0 \). Then for any fixed \( a \in A \) and any \( y \in X \), there exists a function \( g(y) := f(y)a \) in \( C_0(X, A) \) such that \( g(x) = a \) and \( g \) vanishes on \( F \).

We now claim that \( S_t = \cap_{f \in A} f^{-1}(I_t) = Y \). For \( f \in J_1 \), \( f = f_1 + f_2 \) for some \( f_1 \in C_0(X) \otimes \min I_t \) and \( f_2 \in J(Y) \otimes \min A \). Thus, for any \( y \in Y \), \( f(y) = f_1(y) + f_2(y) \subseteq I_t \) as \( f_2(y) = 0 \) by Lemma 2.3, so that \( Y \subseteq S_t \).

For the reverse containment assume that \( Y \nsubseteq S_t \). Pick \( a \in A \setminus I_t \) and \( x \in S_t \setminus Y \), then there exists a function in \( C_0(X, A) \) which vanishes on \( Y \) and maps \( x \) to \( a \) which is a contradiction to the definition of \( S_t \). On the similar lines, using the fact that \( S'_t = \cap_{g \in J_t} g^{-1}(I_t) \), one can easily deduce that \( S_t = Y = S'_t \).

Now fix \( i \in \Delta \) with \( i \neq \beta, i \neq t \). Note that \( J_2 = \cap_{i \in \Delta} \{ f \in C_0(X, A) : f(S_i') \subseteq I_i \} \), so that \( G_t := \{ f \in C_0(X, A) : f(S_i') \subseteq I_i \} \subseteq \{ f \in C_0(X, A) : f(S_i') \subseteq I_i \} = G_t \) (say), for every \( i \in \Delta \).

Case(i): \( I_t \subseteq I_i \), then \( S'_i = \emptyset = S_i \). For \( y \in S'_i \subseteq S_t \) and \( a \in I_t \setminus I_i \), there is a function in \( C_0(X, A) \) which takes \( y \) to \( a \). Then such a function is in \( G_t \) but not in \( G_i \). Also, if there exists an \( x \in S_t \subseteq S_i = Y \), then \( a'g \) as defined above will be a function in \( C_0(X) \otimes \min I_t \subset I_i \) which takes an element \( x \) of \( S_t \) to \( a \) which does not belong to \( I_i \), which is a contradiction to the definition of \( S_i \).

Case(ii): \( I_t \supseteq I_i \), then \( S_i = S'_t = S'_i \). To see this, if \( S'_t \) is properly contained in \( S'_i \), then for \( x \in S'_t \setminus S'_i \) and \( a \not\in I_i \), there exists a function in \( C_0(X, A) \) which takes \( x \) to \( a \) and \( S'_i \) to 0. This function belongs to \( G_i \) but does not belong to \( G_i \), which is a contradiction. Similarly, if \( S_i \) is properly contained in \( S_t \), then for \( x \in S_t \setminus S_i \) and \( a \not\in I_i \), there is a function in \( J(Y) \otimes \min A \subset J_i \) which takes \( x \) outside \( I_i \) which contradicts the definition of \( S_i \).

Case(iii): \( I_i \) is neither a subset nor a superset of \( I_t \), then we claim that \( S_i = S'_i = \emptyset \). If \( I_i \cap I \neq \emptyset \), then \( I \subseteq I_i \) so that by Case(i), \( S_i \cap S_i = \emptyset = S'_i = S'_i \cap S'_I \). Now, for \( x \in S'_i \), \( x \not\in S_i' \), as argued in Case (ii), we obtain that \( G_i \) is not contained in \( G_i \), which is a contradiction, thus \( S_i' = \emptyset \). Similarly, for \( x \in S_t \), \( x \) is not a member of \( Y = S_i \). So for any \( a \in I_i \setminus I_i \), applying the technique mentioned in the beginning, we get a function \( g \) in \( J(Y) \otimes \min A \subset J_i \) such that \( g(x) \not\in I_i \), a contradiction.

This proves that \( S = S' \), and hence \( J_1 = J_2 \).

We are now ready to prove the main result of this section. Note that a product ideal is a closed ideal of the form \( I \otimes \min J \), where \( I \) and \( J \) are closed ideals of \( A \) and \( B \), respectively.

**Theorem 2.6.** Let \( X \) be a locally compact Hausdorff space and \( A \) be a \( C^* \)-algebra with finitely many closed ideals, say, \( I_1, I_2, \ldots, I_n \) with \( I_1 = \{0\} \) and \( I_n = A \). Then for any closed ideal \( K \) of \( C_0(X) \otimes \min A \), there exists \( S = \{ S_i \}_{i \in \mathbb{N}_n} \in T_2^n \) where \( I = \{ I_i \}_{i \in \mathbb{N}_n} \), such that

\[
K = \sum_{j=2}^{n} J(\cup_{k \in \gamma_j} S_k) \otimes \min I_j,
\]

where \( \gamma_j = \{ i \in \mathbb{N}_n : I_j \not\subset I_i \} \), for every \( j \in \{2, 3, \ldots, n\} \).

In particular, every closed ideal of \( C_0(X) \otimes \min A \) is a finite sum of product ideals.

**Proof.** By Theorem 2.2 there exists \( S = \{ S_i \}_{i \in \mathbb{N}_n} \in T_2^n \) such that \( K = J(S) = \{ f \in C_0(X, A) : f(S_i) \subseteq I_i, i \in \mathbb{N}_n \} \). Set \( K' = \sum_{j=2}^{n} J(\cup_{k \in \gamma_j} S_k) \otimes \min I_j \), then by Lemma 2.3, \( K' \) can be considered as...
a closed ideal of $C_0(X, A)$. By virtue of Theorem 2.8, it is sufficient to prove that $S_i = \cap_{f \in K} f^{-1}(I_i)$ for every $i \in \mathbb{N}_n$.

It is clear that $S_n = X = \cap_{f \in K} f^{-1}(A)$. Fix $i \in \mathbb{N}_{n-1}$ and consider any $x \in S_i$. For $f \in K'$, $f = f_2 + f_3 + \cdots + f_n$, where $f_r \in J(\cup_{k \in \gamma_r} S_k) \otimes^{\text{min}} I_r$ for every $r \in \{2, 3, \ldots, n\}$. Then for any such $r$, either $i \in \gamma_r$ or $i \in \gamma_r^c$. If $i \in \gamma_r$, then $f_r(x) = 0 \in I_i$. If $i \in \gamma_r^c$ then $f_r(x) \in I_r \subseteq I_i$. These two conclusions together imply that $S_i \subseteq \cap_{f \in K} f^{-1}(I_i)$.

Next, pick $x \notin S_i$ and define $\alpha_i = \{j \in \mathbb{N}_n : I_j \not\subseteq I_i\}$. Note that $\alpha_i$ is non empty as $n \in \alpha_i$. It is sufficient to prove the existence of a function $f \in K'$ such that $f(x) \notin I_i$. We shall actually prove that such a function exists in the subset $\sum_{r \in \alpha_i} J(\cup_{k \in \gamma_r} S_k) \otimes^{\text{min}} I_r$ of $K'$. It is further enough to prove that there exists an $r \in \alpha_i$ such that $x \notin \cup_{k \in \gamma_r} S_k$, so that the required function $f$ exists in $J(\cup_{k \in \gamma_r} S_k) \otimes^{\text{min}} I_r$. In fact, by Urysohn’s Lemma there exists a function $g \in C_r(X)$ such that $0 \leq g \leq 1$, $g(x) = 1$ and $g(\cup_{k \in \gamma_r} S_k) = \{0\}$. Then by Lemma 2.3 for $a \in I_r \setminus I_i$ (since $I_r \not\subseteq I_i$), the function $g' \hat{g}$ serves the purpose. We claim that $\cap_{r \in \alpha_i}(\cup_{k \in \gamma_r} S_k) = S_i$, which will ensure the existence of such an $r$.

When $r \in \alpha_i$, we have $i \in \gamma_r$ and hence $S_i \subseteq \cap_{r \in \alpha_i}(\cup_{k \in \gamma_r} S_k)$. We now prove the reverse inclusion. Set $\alpha_i = \{r_1, r_2, \ldots, r_q\}$ and for each $r_j \in \alpha_i$, let there be $p_{r_j}$ number of elements in $\gamma_{r_j}$, say $\gamma_{r_j} = \{r_{1, r_j}, \ldots, r_{p_{r_j}, r_j}\}$. So

$$\bigcap_{r \in \alpha_i}(\cup_{k \in \gamma_r} S_k) = \bigcup_{1 \leq r_j \leq p_{r_j}} (S_{r_{1, r_j}} \cap S_{r_{2, r_j}} \cap \cdots \cap S_{r_{p_{r_j}, r_j}}).$$

We have obtained that $\cap_{r \in \alpha_i}(\cup_{k \in \gamma_r} S_k)$ is a union of $\Pi_{r_j=1}^{p_{r_j}} r_{r_j}$ objects, each of which is an intersection of $q$ objects which looks like $S_{r_{1, r_j}} \cap S_{r_{2, r_j}} \cap \cdots \cap S_{r_{p_{r_j}, r_j}}$. Pick an ideal $I_{r_{1, r_j}} \cap I_{r_{2, r_j}} \cap \cdots \cap I_{r_{p_{r_j}, r_j}}$. Then there exists an $m \in \mathbb{N}_n$ such that $I_m = I_{r_{1, r_j}} \cap I_{r_{2, r_j}} \cap \cdots \cap I_{r_{p_{r_j}, r_j}}$, and hence $S_m = S_{r_{1, r_j}} \cap S_{r_{2, r_j}} \cap \cdots \cap S_{r_{p_{r_j}, r_j}}$. If $S_m \subseteq S_i$, we are done. Otherwise, $S_m \not\subseteq S_i$ will imply $I_m \not\subseteq I_i$ and hence $m \in \alpha_i$. Thus $m = r_j$ for some $j \in \{1, 2, \ldots, k\}$. Then $r_{1, r_j} \in \gamma_{r_j}$, which implies $(I_m = I_{r_j}) \not\subseteq I_{r_{1, r_j}}$, which is a contradiction to the fact that $I_m = I_{r_{1, r_j}} \cap I_{r_{2, r_j}} \cap \cdots \cap I_{r_{p_{r_j}, r_j}}$.

As an important consequence of the above result, we now describe the precise form of closed ideals of $C_0(X) \otimes^{\text{min}} B(H)$, in terms of product ideals. Note that for a Hilbert space $H$, the set of all closed ideals forms a chain (see, [15 Corollary 6.2]). In the following, $w_0$ denotes the cardinality of the set of all natural numbers and for every $i \in \mathbb{N}$, let $w_i = 2^{w_0-1}$ so that $w_1$ is continuum.

**Corollary 2.7.** Let $X$ be a locally compact Hausdorff space and $H$ be a Hilbert space with $w_n$ ($n \in \mathbb{N}$) as its Hilbert dimension. Then the closed ideals of $C_0(X) \otimes^{\text{min}} B(H)$ are of the form $\sum_{j=1}^{n+2} J(S_{j-1}) \otimes^{\text{min}} I_j$, where $I_j$’s are closed ideals of $B(H)$ and $S_j$’s are some closed subsets of $X$.

**Proof.** Let $\{0\} = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{n+2} = B(H)$ be the chain of closed ideals of $B(H)$ and $J$ be a closed ideal of $C_0(X) \otimes^{\text{min}} B(H)$. By Theorem 2.1 there exists $n + 3$ closed subsets $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{n+2} = X$ such that $J = J(S)$ where $S = \{S_j\}_{j=0}^{n+2}$. As in Theorem 2.2 for $j \in \mathbb{N}_{n+2}, \cup_{k \in \gamma_j} S_k = S_{j-1}$ and hence $J = \sum_{j=1}^{n+2} J(S_{j-1}) \otimes^{\text{min}} I_j$.

We now move on to characterize the primitive ideals of these spaces. Recall that a **primitive ideal** of a $C^*$-algebra is the kernel of some irreducible $*$-representation. Also, a closed ideal $K$ of a $C^*$-algebra $A$ is called **prime** if whenever two closed ideals $I$ and $J$ in $A$ satisfy $IJ \subseteq K$, then either $I \subseteq K$ or $J \subseteq K$. Since $C_0(X) \otimes^{\text{min}} A$ has property (F) of Tomiyama, a closed ideal $K$ of the $C^*$-algebra $C_0(X) \otimes^{\text{min}} A$ is prime if and only if there exist an element $x \in X$ and a closed prime ideal $I \subseteq A$ such that $K = C_0(X) \otimes^{\text{min}} I + J(x) \otimes^{\text{min}} A$ (see [14 Proposition 5.1]). We prove a similar characterization for the primitive ideals. Note that every maximal ideal of a $C^*$-algebra is primitive, and every prime ideal of $C_0(X)$ is maximal.

**Theorem 2.8.** Let $X$ be a locally compact Hausdorff space and $A$ be a $C^*$-algebra. A closed ideal $K$ of $C_0(X) \otimes^{\text{min}} A$ is primitive if and only if it is of the form $K = C_0(X) \otimes^{\text{min}} I + J(x) \otimes^{\text{min}} A$ for some primitive ideal $I$ of $A$ and for some $x \in X$. 
proof. Since \( J(x) \) is primitive ideal being a maximal ideal, the if part follows from \([5 \text{ Theorem } 5]\). For the converse, let \( K \) be a primitive ideal of \( C_0(X) \otimes_{\min} A \). Since every primitive ideal of a \( C^* \)-algebra is prime, there exist an element \( x \in X \) and a closed prime ideal \( I \subseteq A \) such that \( K = J(x) \otimes_{\min} A + C_0(X) \otimes_{\min} I \). We prove that \( I \) is the kernel of some irreducible *-representation. Let \( \varphi_1 : C_0(X) \to C_0(X)/J(x) \) and \( \varphi_2 : A \to A/I \) be the natural quotient maps. By \([20 \text{ Theorem } 5]\), we have \((C_0(X) \otimes_{\min} A)/K \cong (C_0(X)/J(x)) \otimes_{\min} (A/I) \cong C \otimes_{\min} (A/I) \cong A/I \). Let \( \psi \) be the isometric *-isomorphism from \( A/I \) to \((C_0(X) \otimes_{\min} A)/K\); \((\psi', H)\) be an irreducible *-representation of \( C_0(X) \otimes_{\min} A \) whose kernel is \( K \) and \( \psi'' \) \( C_0(X) \otimes_{\min} A \) be the isomorphism induced from \((\psi', H)\). Then \( \psi'' \circ \psi \circ \varphi_2 \) is an irreducible *-representation of \( A \) whose kernel is \( I \).

\[ \text{Remark 2.9. Corollary 2.4 and Theorem 2.5 together imply that a closed ideal } J \text{ in } C_0(X,A) \text{ is primitive if and only if there exist a primitive ideal } I \text{ of } A \text{ and an element } x \in X \text{ such that } J = \{ f \in C_0(X,A) : f(x) \in I \}. \]

Let \( \otimes \) denote the operator space projective or the Haagerup tensor norm. It is known that for \( C^* \)-algebras \( A \) and \( B \), every primitive ideal of \( A \otimes \beta B \) is of the form \( A \otimes \beta J + \beta I \otimes B \) for some prime ideals \( I \subseteq A \) and \( J \subseteq B \), and that if \( I \subseteq A \) and \( J \subseteq B \) are primitive ideals, then \( A \otimes \beta J + I \otimes B \) is a primitive ideal of \( A \otimes \beta B \) (see, \([1 \text{ Theorem 5.13}], [11 \text{ Theorem 7}]\) ). However, if \( A \) and \( B \) are separable, then a closed ideal \( K \) of \( A \otimes \beta B \) is primitive if and only if there exist primitive ideals \( I \subseteq A \) and \( J \subseteq B \) such that \( K = A \otimes \beta J + I \otimes \beta B \). We extend this result partially to the non-separable case. In the following result \( \text{Prim}(A) \) represents the space of primitive ideals of \( A \).

\[ \text{Corollary 2.10. Let } X \text{ be a locally compact Hausdorff space and } A \text{ be a } C^* \text{-algebra. A closed ideal } K \text{ of } C_0(X) \otimes \beta A \text{ is primitive if and only if it is of the form } K = C_0(X) \otimes \beta I + J(x) \otimes \beta A \text{ for some } \text{primitive ideal } I \text{ of } A \text{ and for some } x \in X. \]

\[ \text{Proof. We just need to prove the ‘only if’ part. Since } C_0(X) \otimes_{\min} A \text{ has property } (F) \text{ of Tomiyama and } C_0(X) \text{ is nuclear, by }[2 \text{ Corollary 4.4}], \text{ the map } \Phi : \text{Prim}(C_0(X) \otimes_{\min} A) \to \text{Prim}(C_0(X) \otimes h A) \text{ given by } \Phi(L) = L \cap (C_0(X) \otimes h A) \text{ is a homeomorphism. Also, by }[2 \text{ Lemma 1.1}], \text{ we have } (C_0(X) \otimes_{\min} I + J(x) \otimes_{\min} A) \cap (C_0(X) \otimes h A) = C_0(X) \otimes h I + J(x) \otimes h A. \text{ On the similar lines of the proof given in }[2 \text{ Lemma 1.1, Corollary 4.4}], \text{ one can easily check that this equality also holds for the operator space projective norm. The result is now an easy consequence of Theorem 2.5} \square \]

3. Closed Lie ideals of \( C_0(X,A) \)

The Lie normalizer of a subspace \( S \) of a Lie algebra \( A \) is defined by \( N(S) := \{ a \in A : [a, A] \subseteq S \} \). It can be easily verified that \( N(I) \) is a closed subalgebra of \( A \) for a closed ideal \( I \subseteq A \). The Lie normalizer plays an important role in determining the Lie ideals of \( A \) (for instance, see \([9, 3]\)). We identify the Lie normalizer of ideals of \( C_0(X,A) \) and use this identification to characterize its closed Lie ideals.

\[ \text{Theorem 3.1. Let } X \text{ be a locally compact Hausdorff space and } A \text{ be a } C^* \text{-algebra with } I = \{ I_i \}_{i \in \Delta} \text{ as the collection of all closed ideals such that } I_i = A. \text{ Then a closed subspace } L \text{ of } C_0(X,A) \text{ is a closed Lie ideal if and only if there is an element } S = \{ S_i \}_{i \in \Delta} \subseteq T_\Delta I \text{ such that } \{ f \in C_0(X,A) : f(S_i) \subseteq [I_i, A], \forall i \in \Delta \} \subseteq L \subseteq \{ f \in C_0(X,A) : f(S_i) \subseteq N(I_i), \forall i \in \Delta \}. \]

\[ \text{Proof. We know that a closed subspace } L \text{ of the } C^* \text{-algebra } C_0(X,A) \text{ is a Lie ideal if and only if there exists a closed ideal } J \subseteq C_0(X,A) \text{ such that } [J, C_0(X,A)] \subseteq L \subseteq N(J) \text{ (}[4 \text{ Proposition 5.25, Theorem 5.27}]\). By Theorem 2.4, } J = J(S) \text{ for some } S = \{ S_i \}_{i \in \Delta}. \text{ Since for any fixed } a \in A
and \( x \in X \), there is an element in \( C_0(X, A) \) which takes \( x \) to \( a \), we have

\[
N(J) = \{ f \in C_0(X, A) : [f, g] \in J(S), \forall g \in C_0(X, A) \} = \{ f \in C_0(X, A) : [f, g](x) \in I_i, \forall x \in S_i, g \in C_0(X, A), \forall i \in \Delta \} = \{ f \in C_0(X, A) : f(x) \in N(I_i), \forall x \in S_i, i \in \Delta \} = \{ f \in C_0(X, A) : f(S_i) \subseteq N(I_i), \forall i \in \Delta \}.
\]

We know from [1] Proposition 5.25 that \([I, B] = I \cap [B, B]\) for closed ideal \( I \) of a \( C^* \)-algebra \( B \). This fact, along with Lemma 2.3 gives

\[
[J, C_0(X, A)] = [C_0(X, A), C_0(X, A)] \cap J = [C_0(X) \otimes_{\min} A, C_0(X) \otimes_{\min} A] \cap J = C_0(X) \otimes [A, A] \cap J = C_0(X) \cap [A, A] \cap J = \{ f \in C_0(X, A) : f(N(I_i)) \subseteq [A, A] \cap J, \forall i \in \Delta \} = \{ f \in C_0(X, A) : f(S_i) \subseteq [I_i, A], \forall i \in \Delta \}.
\]

Hence the result. \( \square \)

Recall that a \( C^* \)-algebra \( A \) is said to have the \emph{centre-quotient property} if \( Z(A/I) = (Z(A) + I)/I \) for every closed ideal \( I \) of \( A \), where \( Z(A) \) is the centre of \( A \) [3].

**Lemma 3.2.** A \( C^* \)-algebra \( A \) has centre-quotient property if and only if \( N(I) = I + Z(A) \) for every closed ideal \( I \) in \( A \).

**Proof.** Let \( I \) be a closed ideal of \( A \) and \( \pi : A \to A/I \) be the natural quotient map. Then \( N(I) = \pi^{-1}(Z(A/I)) \). So that \( Z(A/I) = (I + Z(A))/I \) if and only if \( N(I) = \pi^{-1}((I + Z(A))/I) = I + Z(A) \).

\( \square \)

A unital \( C^* \)-algebra \( A \) is called \emph{weakly central} if the continuous surjection \( \psi : \text{Max}(A) \to \text{Max}(Z(A)) \) given by \( \psi(I) = I \cap Z(A) \) is an injection, where \( \text{Max}(B) \) denotes the space of all maximal ideals of a \( C^* \)-algebra \( B \) endowed with hull-kernel topology. It is well known that a unital \( C^* \)-algebra with unique maximal ideal must have one dimensional centre [3] Lemma 2.1. Since weak centrality and centre-quotient property are equivalent in unital \( C^* \)-algebras (see, [2] Theorem 1 and 2), presence of unique maximal ideal of a unital \( C^* \)-algebra \( A \) implies that \( A \) has centre-quotient property. In [8] Lemma 4.6 it was observed that for a simple unital \( C^* \)-algebra \( A \) and a closed ideal \( I \subseteq A \), \( N(I) = I + C_0(X, C1) \). In the following we generalize this result to a unital \( C^* \)-algebra with unique maximal ideal. It can also be observed that centre-quotient property passes to \( C_0(X, A) \) even though \( C_0(X, A) \) does not have a unique maximal ideal.

**Theorem 3.3.** Let \( X \) be a locally compact Hausdorff space and \( A \) be a unital \( C^* \)-algebra with a unique maximal ideal. Then \( N(J) = J + C_0(X, C1) \), for any closed ideal \( J \) of \( C_0(X, A) \).

**Proof.** Note that \( Z(C_0(X, A)) = C_0(X, C1) \) [8] Corollary 1, and hence \( J + C_0(X, C1) \subseteq N(J) \). Let \( \mathcal{I} = \{ I_i \}_{i \in \Delta} \) be the collection of all closed ideals of \( A \) with \( A = I_\beta \) and let \( I_\beta \) be the unique maximal ideal of \( A, \beta, \beta' \in \Delta \). Then, Theorem [21] there exists an element \( S = \{ S_i \}_{i \in \Delta} \in \mathcal{T}_\beta \) such that \( J = J(S) \). Since \( A \) has centre-quotient property, by Lemma 3.2, \( N(I_i) = I_i + C1 \) for every \( i \in \Delta \). Let \( f \in N(J) = \{ g \in C_0(X, A) : g(S_i) \subseteq I_i + C1, \forall i \in \Delta \} \) as noted in Theorem 3.1. Since \( I_\beta \) is the unique maximal ideal of \( A \) and \( S \in \mathcal{T}_\beta \), we have \( S_{\alpha} \subseteq S_{\beta'} \) for every \( \alpha \in \Delta \setminus \{ \beta \} \).

On \( S_{\beta'} \), write \( f = g + h \) which satisfy \( h(S_{\beta'}) \subseteq C1 \) and \( g(S_i) \subseteq I_i \) for every \( i \in \Delta \setminus \{ \beta \} \). This is possible as no proper ideal of \( A \) can intersect \( C1 \). Since \( I_{\beta'} \cap C1 = \{ 0 \} \), by Hahn-Banach Theorem, there exists \( T \in A^* \) such that \( ||T|| = 1 \), \( T(I_{\beta'}) = \{ 0 \} \) and \( T(\lambda I) = \lambda \) for \( \lambda \in \mathbb{C} \). Then \( TF = Th \)
on $S^\beta$. Also $f$ vanishes at infinity and $\|T\| = 1$, so we obtain that $Tf$ vanishes at infinity because $\|Tf\| \leq f\|f\|$. Since $\|Th\| = \|h\|$, $Th$ and hence $h$ is a continuous function vanishing at infinity. So $g = f - h$ is continuous on $S^\beta$ and is vanishing at infinity. By [8, Theorem 4.5], there exists an $h' \in C_0(X)$ such that $h''_{S^\beta} = h$. For $x \in S^\beta$, define $g'(x) = f(x) - h'(x)$. Then $f = g' + h'$ with $g' \in J(S)$ and $h' \in C_0(X, \mathbb{C})1$ and we are done.  

With this identification of Lie normalizer, we can now characterize the Lie ideals of a class of $C^*$-algebras. Recall that a bounded linear functional $f$ on a $C^*$-algebra $B$ is said to be a tracial state if $f$ is positive of norm $1$ and $f([a, b]) = 0$ for every $a, b \in B$.

**Corollary 3.4.** Let $X$ be a locally compact Hausdorff space and $A$ be a unital $C^*$-algebra with unique maximal ideal and no tracial state. Then a closed subspace $L$ of $C_0(X, A)$ is a Lie ideal if and only if it is of the form $J + K$ for some closed ideal $J$ of $C_0(X, A)$ and a closed subspace $K$ of $C_0(X, \mathbb{C}1)$.

**Proof.** Let $I = \{I_i\}_{i \in \Delta}$ be the collection of all closed ideals of $A$. Because $A$ has no tracial state, from [8, Lemma 2.4], $C_0(X, A)$ has no tracial state. Thus, by [4, Proposition 5.25], $[J, A] = J$ for every closed ideal $J$ of $C_0(X, A)$. From [4, Theorem 5.27] and Theorem 3.3 it can be concluded that a closed subspace $L$ of $C_0(X, A)$ is a Lie ideal if and only if there exists a closed ideal $J$ of $C_0(X, A)$ such that $J \subseteq L \subseteq J + C_0(X, \mathbb{C}1)$. Hence $L$ must be of the form $J + K$ for some closed subspace $K$ of $C_0(X, \mathbb{C}1)$.

As a consequence, we can now characterize all closed ideals of $C_0(X) \otimes \min B(H)$.

**Corollary 3.5.** For a separable Hilbert space $H$ and a locally compact Hausdorff space $X$, a closed subspace $L$ of $C_0(X) \otimes \min B(H)$ is a Lie ideal if and only if there exist two closed subsets $S_1 \subseteq S_2$ of $X$ and a closed subspace $K$ of $C_0(X) \otimes \mathbb{C}1$ such that

$$L = J(S_1) \otimes K(H) + J(S_2) \otimes B(H) + K.$$

**Proof.** The result is an easy consequence of Corollary 3.4 and Corollary 3.4 using the fact that $B(H)$ has no tracial state.

Recall that a unital $C^*$-algebra $A$ is said to have singleton Dixmier property if for any $a \in A$, the closed convex hull of the set $\{u^*au : u$ is a unitary in $A\}$ intersects the centre $Z(A)$ exactly once. If this intersection is non-empty for every $a \in A$, then $A$ is said to have Dixmier property. For $C^*$-algebras $A$ and $B$, it can be very difficult to see when $A \otimes \min B$ has Dixmier property because the unitaries of $A \otimes \min B$ are not known. A $C^*$-algebra $A$ is said to be postliminal if for every irreducible representation $(\psi, H)$ of $A$, $K(H) \subseteq \psi(A)$. Every commutative $C^*$-algebra is postliminal since its non zero irreducible representations are one dimensional.

**Corollary 3.6.** Let $X$ be a compact Hausdorff space and $A$ be a unital postliminal $C^*$-algebra with a unique maximal ideal. Then $C(X) \otimes \min A$ has singleton Dixmier property.

**Proof.** The $C^*$-algebra $C(X) \otimes \min A$ is postliminal. Now, the centre-quotient property and singleton Dixmier property are equivalent in $C(X, A)$ ([3, Theorem 2.12]). Thus Lemma 3.2 and Theorem 3.3 gives the result.

**Theorem 3.7.** Let $A$ be a simple unital $C^*$-algebra and $X$ be a compact Hausdorff space. Then $A$ has Dixmier property if and only if $C(X) \otimes \min A$ has Dixmier property.

**Proof.** We will make use of the fact that a unital $C^*$-algebra $B$ has Dixmier property if and only if $B$ is weakly central, for every maximal ideal $I \subset B$, $B/(I \cap Z(B))B$ has at most one tracial state and if $B/(I \cap Z(B))B$ has a tracial state $\tau$ then $\tau(I/(I \cap Z(B))B) = \{0\}$ [3, Theorem 2.6]. From Lemma 3.2, Theorem 3.3 and [21, Theorem 1 and 2], it can be easily seen that $C(X) \otimes \min A$ is weakly central. Suppose $A$ has Dixmier property. Then $A$ has at most one tracial state being a simple unital $C^*$-algebra having Dixmier property [9]. Let $K$ be a maximal ideal of $C(X) \otimes \min A$. Then there exists an $x \in X$ such that $K = J(x) \otimes \min A$ [6, Theorem 3.1]. Set $B = C(X) \otimes \min A$. 

As noted earlier, $Z(B) = C(X) \otimes_{\min} C1$. So, $K \cap Z(B) = J(x) \otimes_{\min} C1$ from Lemma 23 and $(J(x) \otimes_{\min} C1)(B) = J(x) \otimes_{\min} A$. Thus

$$B/(K \cap Z(B))(B) = B/K \cong (C(X)/J(x)) \otimes_{\min} A \cong C \otimes_{\min} A \cong A,$$

where the second isomorphism follows from [20] Theorem 5. So $B/(K \cap Z(B))(B)$ has at most one tracial state. If it has a tracial state, then

$$K/(K \cap Z(B))(B) = K/K = \{0\}$$

hence the last condition is also satisfied.

Conversely, suppose $C(X) \otimes_{\min} A$ has Dixmier property. Since $A$ has a unique maximal ideal it is clearly weakly central. Rest follows from [3] Theorem 2.6], since every simple quotient of $C(X) \otimes_{\min} A$ is isomorphic to $A$. \hfill $\square$

References

[1] S. D. Allen, A. M. Sinclair and R. R. Smith, The ideal structure of the Haagerup tensor product of C*-algebras, J. Reine Angew. Math. 442 (1993), 111–148.
[2] R. J. Archbold et al., Ideal spaces of the Haagerup tensor product of C*-algebras, Internat. J. Math. 8 (1) (1997), 1–20.
[3] R. J. Archbold, L. Robert and A. Tikuisis, The Dixmier property and tracial states for C*-algebras, J. Funct. Anal. 273 (8) (2017), 2655–2718.
[4] M. Brešar, E. Kissin and V. S. Shulman, Lie ideals: from pure algebra to C*-algebras, J. Reine Angew. Math. 623 (2008), 73–121.
[5] A. Guichardet, Tensor products of C*-algebras Part I. Finite tensor products, Math. Inst. Aarhus Univ. Lecture notes series 12 (1969).
[6] V. P. Gupta and R. Jain, On closed Lie ideals of certain tensor products of C*-algebras, Math. Nachr. 291 (8-9) (2018), 1297–1309.
[7] V.P. Gupta and R. Jain, On Banach space projective tensor product of C*-algebras, Preprint at arxiv: 1801.06705 [math.OA].
[8] V.P. Gupta, R. Jain and B. Talwar, On closed Lie ideals of certain tensor products of C*-algebras II, Preprint at arxiv: 1801.06705v4 [math.OA].
[9] U. Haagerup and L. Zsidó, Sur la propriété de Dixmier pour les C*-algèbres, C. R. Acad. Sci. Paris Sér. I Math. 298 (8) (1984), 173–176.
[10] R.Haydon and S. Wassermann, A commutation result for tensor products of C*-algebras, Bull. London Math. Soc. 5 (1973), 283–287.
[11] R. Jain and A. Kumar, Ideals in operator space projective tensor product of C*-algebras, J. Aust. Math. Soc. 91 (2) (2011), 275–288.
[12] R. V. Kadison and J. R. Ringrose, Fundamentals of the theory of operator algebras, Vol. I, Pure and Applied Mathematics, 100, Academic Press, Inc., New York, 1983.
[13] E. Kaniuth, A course in commutative Banach algebras, Graduate Texts in Mathematics, 246, Springer, New York, 2009.
[14] A. J. Lazar, The space of ideals in the minimal tensor product of C*-algebras, Math. Proc. Cambridge Philos. Soc. 148 (2) (2010), 243–252.
[15] E. Luft, The two-sided closed ideals of the algebra of bounded linear operators of a Hilbert space, Czechoslovak Math. J. 93 (18) (1968), 595–605.
[16] L.W. Marcoux, Projections, commutators and Lie ideals in C*-algebras, Math. Proc. R. Ir. Acad, 110A (1) (2010), 31–55.
[17] M. A. Naimark, Normed rings, Groningen, 1964.
[18] W. Rudin, Real and complex analysis, third edition, McGraw-Hill Book Co., New York, 1987.
[19] M. Takesaki, Theory of operator algebras I, Springer-Verlag, New York, 1979.
[20] J. Tomiyama, Applications of Fubini type theorem to the tensor products of C*-algebras, Tôhoku Math. J. 19 (2) (1967), 213–226.
[21] J. Vesterstrøm, On the homomorphic image of the center of a C*-algebra, Math. Scand. 29 (1971), 134–136.
[22] S. Wassermann, A pathology in the ideal space of $L(H) \otimes L(H)$, Indiana Univ. Math. J. 27 (6) (1978), 1011–1020.

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