FUNCTIONAL INEQUALITIES FOR THE BICKLEY FUNCTION

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ABSTRACT. In this paper our aim is to deduce some complete monotonicity properties and functional inequalities for the Bickley function. The key tools in our proofs are the classical integral inequalities, like Chebyshev, Hölder-Rogers, Cauchy-Schwarz, Carlson and Grüss inequalities, as well as the monotone form of l’Hospital’s rule. Moreover, we prove the complete monotonicity of a determinant function of which entries involve the Bickley function.

1. Introduction

The Bessel function fractional integral

\[ K_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} K_0(t) \, dt, \]

where \( K_0 \) is the modified Bessel function of the second kind of zero order, was first introduced for \( \alpha \in \{1, 2, \ldots\} \) by Bickley [10] in connection with the solution of heat convection problems. This function appears also in neutron transport calculations, and is frequently used in nuclear reactor computer codes. An alternative representation of the Bickley function, which will be used frequently in the sequel, is the following

\[ (1.1) \quad K_\alpha(x) = \int_0^\infty e^{-x \cosh t} (\cosh t)^{-\alpha} \, dt, \]

where \( \alpha \) is an arbitrary real number and \( x > 0 \). For properties of the Bickley function, including asymptotic expansions and generalizations we refer to [1, 2, 11, 14], [18, Chapter 8], [21], [26, p. 259] and to the references therein.

In this paper, by using the classical integral inequalities, like Chebyshev, Hölder-Rogers, Cauchy-Schwarz, Carlson and Grüss, and the monotone form of l’Hospital’s rule we present some complete monotonicity properties and functional inequalities for the Bickley function. Moreover, we prove the complete monotonicity of a determinant function of which entries involve the Bickley function. For similar functional inequalities involving other special functions we refer for example to the papers [7, 8] and to the references therein.

Before we present the main results of this paper we recall some definitions, which will be used in the sequel. A function \( f : (0, \infty) \to \mathbb{R} \) is said to be completely monotonic if \( f \) has derivatives of all orders and satisfies

\[ (-1)^m f^{(m)}(x) \geq 0 \]

for all \( x > 0 \) and \( m \in \{0, 1, 2, \ldots\} \).

The exponentially convex functions form a sub-class of convex functions introduced by Bernstein in [9] (see also [4]). A function \( g : \mathcal{I} \mapsto \mathbb{R} \) which is continuously differentiable on \( \mathcal{I} \subseteq \mathbb{R} \) if it is continuous and satisfies

\[ H_\xi(f) = \sum_{j,k=1}^n \xi_j \xi_k f(x_j + x_k) \geq 0, \]

for all \( n \in \{1, 2, \ldots\} \) and all \( \xi_j \in \mathbb{R}, j \in \{1, 2, \ldots, n\} \) such that \( x_j + x_k \in \mathcal{I} \) for \( j, k \in \{0, 1, \ldots, n\} \).

A function \( h : (0, \infty) \to (0, \infty) \) is said to be logarithmically convex, or simply log-convex, if its natural logarithm \( \ln h \) is convex, that is, for all \( x, y > 0 \) and \( \lambda \in [0, 1] \) we have

\[ h(\lambda x + (1-\lambda)y) \leq [h(x)]^\lambda [h(y)]^{1-\lambda}. \]

A similar characterization of log-concave functions also holds. We also note that every completely monotonic function is log-convex, see [28, p. 167]. The same conclusion holds true for the exponentially convex...
functions on $(0, \infty)$, that is, if $h : (0, \infty) \to (0, \infty)$ is exponentially convex, then it is log-convex. See [1 Corollary 2] for more details.

By definition, a function $q : (0, \infty) \to (0, \infty)$ is said to be geometrically (or multiplicatively) convex if it is convex with respect to the geometric mean, that is, for all $x, y > 0$ and all $\lambda \in [0, 1]$ the inequality

$$q(x^\lambda y^{1-\lambda}) \leq [q(x)]^\lambda[q(y)]^{1-\lambda}$$

holds. The function $q$ is called geometrically concave if the above inequality is reversed. Observe that, actually the geometrical convexity of a function $q$ means that the function $\ln q$ is a convex function of $\ln x$ in the usual sense. We also note that the differentiable function $h$ is log-convex (log-concave) if and only if $x \mapsto h'(x)/h(x)$ is increasing (decreasing), while the differentiable function $q$ is geometrically convex (concave) if and only if the function $x \mapsto xq'(x)/q(x)$ is increasing (decreasing). See for example [5] for more details.

Finally, let us recall the concept of relative convexity. This concept has been considered by Hardy et al. [15 p. 75]: if $\varphi, \psi : [a, b] \to \mathbb{R}$ are two continuous functions and $\psi$ is strictly monotone, then we say that $\varphi$ is convex (concave) with respect to $\psi$ if $\varphi \circ \psi^{-1}$ is convex (concave) in the usual sense on the interval $\psi([a, b])$. The usual convexity of a function $\varphi$ in this manner means actually that the function $\varphi$ is convex with respect to the identity function, the log-convexity of $\varphi$ is exactly the fact that the function $\ln \varphi$ is convex with respect to the identity function, while the geometrical convexity of $\varphi$ means that $\ln \varphi$ is convex with respect to logarithm function. See [25] for more details. It is also known (see [12]) that the increasing function $\varphi$ with respect to an increasing function $\psi$ if and only if the function $\varphi'/\psi'$ is increasing, or if and only if the inequality $\psi''(x)/\psi'(x) \leq \varphi''(x)/\varphi'(x)$ is valid for all $x \in (a, b)$.

2. Bickley function: Monotonicity patterns and functional inequalities

Our first main result is the following theorem.

**Theorem 1.** The following assertions are true:

- a. The function $x \mapsto K_{i\alpha}(x)$ is completely monotonic on $(0, \infty)$ for all $\alpha \in \mathbb{R}$.
- b. The function $\alpha \mapsto K_{i\alpha}(x)$ is completely monotonic on $\mathbb{R}$ for all $x > 0$.
- c. The function $\alpha \mapsto K_{i\alpha}(x)$ is log-convex on $\mathbb{R}$ for all $x > 0$.
- d. The function $x \mapsto K_{i\alpha}(x)$ is log-convex on $(0, \infty)$ for all $\alpha \in \mathbb{R}$.
- e. The function $x \mapsto K_{i\alpha}(x)$ is geometrically concave on $(0, \infty)$ for all $\alpha \in \{-1, 0, 1, \ldots \}$. Consequently, for all $\alpha \in \{-1, 0, 1, \ldots \}$ and $x, y > 0$ we have

\begin{equation}
K_{i\alpha}(\sqrt{xy}) \geq \sqrt{K_{i\alpha}(x)K_{i\alpha}(y)} \geq K_{i\alpha}\left(\frac{x+y}{2}\right),
\end{equation}


\begin{equation}
1 \leq \frac{K_{i\alpha}(x)K_{i\alpha-2}(x)}{[K_{i\alpha-1}(x)]^2} \leq 1 + \frac{K_{i\alpha}(x)}{xK_{i\alpha-1}(x)}.
\end{equation}

Moreover, the right-hand side of (2.1) and the left-hand side of (2.2) are valid for all real $\alpha$.

- f. The inequality

\begin{equation}
K_{i-\beta}(x)K_{i+\beta}(x) \leq K_{i0}(x)K_{i\alpha}(x)
\end{equation}

is valid for all $x > 0$ and $\alpha + \beta \leq 0 \leq \beta$ or $\alpha + \beta \geq 0 \geq \beta$. If $\alpha \geq 0$, $\alpha + \beta \geq 0$ or $\beta \leq 0$, $\alpha + \beta \leq 0$, then (2.3) is reversed. In particular, when $\beta = -1$ and $\alpha$ is changed to $\alpha - 1$ the inequality (2.3) becomes

\begin{equation}
\frac{K_{i\alpha'}(x)}{K_{i\alpha}(x)} = \frac{K_{i\alpha-2}(x)}{K_{i\alpha-1}(x)} \geq \frac{K_{i0}(x)}{K_{i1}(x)} = \frac{K_{i\alpha}(x)}{K_{i2}(x)}
\end{equation}

where $\alpha \geq 2$, i.e. $x \mapsto -K_{i\alpha}(x)$ is convex with respect to $x \mapsto -K_{i2}(x)$ for $\alpha \geq 2$.

- g. If $\alpha + \beta \geq 0$, $\beta \leq 0$ and $x > 0$, then the following inequality is valid

\begin{equation}
|K_{i0}(x)K_{i\alpha}(x) - K_{i-\beta}(x)K_{i\alpha+\beta}(x)| \leq \frac{[K_{i0}(x)]^2}{4}.
\end{equation}

- h. The inequality

\begin{equation}
\frac{2\Gamma\left(\frac{\alpha+\beta}{4}\right)}{\sqrt{\pi} \Gamma\left(\frac{\beta}{4}\right)} K_{i\alpha}(x)K_{i\alpha}(y) \leq K_{i\alpha}(x+y) \leq K_{i\alpha}(x) + K_{i\alpha}(y) \leq K_{i\alpha}(x+y) + \frac{\sqrt{\pi} \Gamma\left(\frac{\beta}{4}\right)}{2\Gamma\left(\frac{\alpha+\beta}{4}\right)}.
\end{equation}
Moreover, if we let \( r, s \geq 1 \), then for all \( \alpha, x, y > 0 \),
\begin{equation}
    r \text{Ki}_\alpha(x) + s \text{Ki}_\alpha(y) \leq \text{Ki}_\alpha(rx + sy) + (r + s - 1)\frac{\sqrt{\pi} \Gamma \left( \frac{q}{2} \right)}{2\Gamma \left( \frac{a + 1}{2} \right)}.
\end{equation}

i. The inequality
\begin{equation}
    \frac{1}{\text{Ki}_0(x)} \text{Ki}_\alpha(x) \text{Ki}_\beta(x) \leq \text{Ki}_{\alpha + \beta}(x) \leq \text{Ki}_\alpha(x) + \text{Ki}_\beta(x) \leq \text{Ki}_0(x) + \text{Ki}_{\alpha + \beta}(x)
\end{equation}
holds for all \( \alpha, \beta > 0 \) and \( x > 0 \).

j. The function \( x \mapsto \text{Ki}_\alpha(x) \) is exponentially convex on \((0, \infty)\) for all \( \alpha > 0 \).

k. The function \( \alpha \mapsto \text{Ki}_\alpha(x) \) is exponentially convex on \( \mathbb{R} \) for all \( x > 0 \).

l. For all \( \alpha, \beta \in \mathbb{R} \) and \( x > 0 \),
\begin{equation}
    \text{Ki}_{\alpha + \beta}(x) + \text{Ki}_{\alpha - \beta}(x) \geq 2 \text{Ki}_\alpha(x).
\end{equation}

m. The inequality
\begin{equation}
    \text{Ki}_{\alpha + \mu}(x) \text{Ki}_{\alpha - \mu}(x) + \text{Ki}_{\alpha - \nu}(x) \text{Ki}_{\alpha + \nu}(x) \geq 2 |\text{Ki}_\alpha(x)|^2
\end{equation}
holds for all \( \alpha, \nu, \mu \in \mathbb{R} \) and \( x > 0 \).

n. The function \((\alpha, x) \mapsto \text{Ki}_\alpha(x)\) is log-convex for all \( x > 0 \) and \( \alpha \in \mathbb{R} \). In particular,
\begin{equation}
    |\text{Ki}_\alpha(x)|^2 \leq \text{Ki}_{\alpha(1+\mu)}((1+\nu)x) \text{Ki}_{\alpha(1-\mu)}((1-\nu)x)
\end{equation}
is valid for all \( \alpha, \nu, \mu \in \mathbb{R} \) and \( x > 0 \).

Proof. a. & b. It is known [22, Theorem 4] that if the kernel \( K(x, t) \) is completely monotonic in \( x \) for all \( t > 0 \) and \( f \) is a nonnegative locally integrable function such that the integral
\begin{equation}
    \int_a^b \frac{\partial^n}{\partial x^n} K(x, t) f(t) dt
\end{equation}
converges uniformly for all \( n \in \{0, 1, 2, \ldots\} \) and \( 0 < a < b < \infty \) in a neighborhood of any point \( x > 0 \), then the function
\begin{equation}
    x \mapsto \int_a^b K(x, t) f(t) dt
\end{equation}
is completely monotonic on \((0, \infty)\). Now, since the function \( x \mapsto e^{-x \cosh t} \) is completely monotonic on \((0, \infty)\) for all \( t > 0 \), the above result implies that indeed the function \( x \mapsto \text{Ki}_\alpha(x) \) is completely monotonic on \((0, \infty)\) for all \( \alpha \in \mathbb{R} \). Similarly, since the function \( \alpha \mapsto (\cosh t)^{-\alpha} \) is completely monotonic on \( \mathbb{R} \) for all \( t > 0 \), by using [22, Theorem 4] again we obtain that the function \( \alpha \mapsto \text{Ki}_\alpha(x) \) is completely monotonic on \( \mathbb{R} \) for all \( x > 0 \). It should be mentioned here that
\begin{equation}
    \text{Ki}_\alpha'(x) = - \text{Ki}_{\alpha-1}(x)
\end{equation}
and by induction we have
\begin{equation}
    (-1)^m \text{Ki}_\alpha^{(m)}(x) = \text{Ki}_{\alpha-m}(x) > 0
\end{equation}
for all \( x > 0 \), \( \alpha \in \mathbb{R} \) and \( m \in \{0, 1, 2, \ldots\} \), which provides an alternative proof for part a. Similarly,
\begin{equation}
    (-1)^m \frac{\partial^m \text{Ki}_\alpha(x)}{\partial \alpha^m} = \int_0^\infty e^{-x \cosh t} (\cosh t)^{-\alpha} [\log(\cosh t)]^m dt > 0
\end{equation}
for all \( x > 0 \), \( \alpha \in \mathbb{R} \) and \( m \in \{0, 1, 2, \ldots\} \), which provides an alternative proof for part b.

c. & d. These results follow from parts a & b, since every completely monotonic function is log-convex (see [28, p. 167]). However, we give here an alternative proof by using the classical Hölder-Rogers inequality for integrals [23, p. 54],
\begin{equation}
    \int_a^b |f(t)g(t)| dt \leq \left[ \int_a^b |f(t)|^p dt \right]^{1/p} \left[ \int_a^b |g(t)|^q dt \right]^{1/q},
\end{equation}
where \( p > 1 \), \( 1/p + 1/q = 1 \), \( f \) and \( g \) are real functions defined on \([a, b]\) and \(|f|^p, |g|^q\) are integrable functions on \([a, b]\). Using (2.13) we obtain that
\begin{equation}
    \text{Ki}_{\lambda \alpha + (1-\lambda)\beta}(x) = \int_0^\infty e^{-x \cosh t} (\cosh t)^{-(\lambda \alpha + (1-\lambda)\beta)} dt
\end{equation}
is valid for all \( \alpha, \beta > 0 \).
\[
\alpha \text{ holds for all } K_i, \text{ i.e. the function } \alpha \mapsto K_i(x) \text{ is log-convex on } \mathbb{R}. \text{ Similarly, by using (2.13) we get }
\]
\[
K_i(\mu x + (1 - \mu)y) = \int_0^\infty e^{-\mu x + (1 - \mu)y} \cosh t \cosh t = 0 \quad (\text{3.15})
\]
\[
= \left[ K_i(\alpha) \right]^\lambda \left[ K_i(\beta) \right]^{1 - \lambda}
\]
holds for all \( \lambda \in [0, 1] \), \( \alpha, \beta \in \mathbb{R} \) and \( x > 0 \), i.e. the function \( \alpha \mapsto K_i(x) \) is log-convex on \( \mathbb{R} \).

Alternatively, to prove part d we may use part c of this theorem. More precisely, since the function \( \alpha \mapsto K_i(x) \) is log-convex, the following Turán type inequality holds for all \( \alpha_1, \alpha_2 \in \mathbb{R} \) and \( x > 0 \)
\[
(2.14) \quad \left[ K_{\alpha_1 + \alpha_2}(x) \right]^2 \leq K_{\alpha_1}(x) K_{\alpha_2}(x).
\]

Now, if we choose \( \alpha_1 = \alpha - 2 \) and \( \alpha_2 = \alpha \) and apply (2.12), then we obtain
\[
(2.15) \quad \frac{K^\prime_i(x)}{K_i(x)} = \frac{K_{i-2}(x) K_i(x) - [K_{i-1}(x)]^2}{[K_i(x)]^2} \geq 0,
\]
i.e. the function \( x \mapsto K^\prime_i(x)/K_i(x) \) is increasing on \((0, \infty)\) for all \( \alpha \in \mathbb{R} \).

To prove the asserted result, first we verify the following statement: For each real \( \alpha \) if the function \( K_{i-1} \) is geometrically concave on \((0, \infty)\), then the function \( K_i \) is also geometrically concave on \((0, \infty)\). Since \( K_{i-1} \) is geometrically concave it follows that the function
\[
x \mapsto 1 + \frac{xK_{i-1}(x)}{K_{i-1}(x)} = \frac{K_{i-1}(x) - xK_{i-2}(x)}{K_{i-1}(x)} = -\frac{[xK_{i-1}(x)]^\prime}{K_i(x)}
\]
is decreasing on \((0, \infty)\) and by the monotone form of l'Hospital’s rule [3, Lemma 2.2] the function
\[
x \mapsto -\frac{xK_{i-1}(x)}{K_{i}(x)} + \frac{xK^\prime_{i}(x)}{K_i(x)}
\]
is also decreasing on \((0, \infty)\), that is, the function \( K_{i} \) is geometrically concave on \((0, \infty)\). Here we used tacitly that \( xK_{i-1} \) and \( K_{i} \) tend to zero as \( x \to \infty \). Now, because \( K_0 = K_1 \) and \( K_{i-1} = -K_{i-1} = K_1 \) and according to [3, Theorem 2] the function \( K_{i} \) is geometrically concave on \((0, \infty)\) for all \( \alpha \in \mathbb{R} \), we obtain that \( K_{i-1}, K_0, K_1, K_2, \ldots \) are geometrically concave on \((0, \infty)\).

Now, we focus on the inequalities (2.11) and (2.22). Inequality (2.11) follows by definition. The left-hand side of (2.22) is a particular case of the Turán type inequality (2.14), while the right-hand side of (2.22) follows from the geometric concavity. More precisely, since \( K_{i} \) is geometrically concave, it follows that
\[
\frac{xK^\prime_i(x)}{K_{i}(x)} = \frac{\left[ K_{i-1}(x) \right]^2}{[K_i(x)]^2} \leq 0
\]
for all \( \alpha \in \{-1, 0, 1, \ldots\} \) and \( x > 0 \).

We recall the Chebyshev integral inequality [23, p. 40]: If \( f, g : [a, b] \to \mathbb{R} \) are integrable functions, both increasing or both decreasing, and \( p : [a, b] \to \mathbb{R} \) is a positive integrable function, then
\[
(2.16) \quad \int_a^b p(t)f(t)dt \int_a^b p(t)g(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)f(t)g(t)dt.
\]
Note that if one of the functions \( f \) or \( g \) is decreasing and the other is increasing, then (2.16) is reversed. We shall use this inequality. For this we write \( K_i(x) \) as
\[
K_i(x) = \int_0^\infty e^{-x \cosh t \cosh t} \cosh t \cosh t dt
\]
and let \( p(t) = e^{-x \cosh t} \), \( f(t) = (\cosh t)^\beta \) and \( g(t) = (\cosh t)^{-(\alpha + \beta)} \). The function \( f \) is increasing (decreasing) on \((0, \infty)\) if and only if \( \beta \geq 0 \) (\( \beta \leq 0 \)), while \( g \) is increasing (decreasing) on \((0, \infty)\) if and only if \( \alpha + \beta \leq 0 \) (\( \alpha + \beta \geq 0 \)). Observe that

\[
\int_0^\infty p(t)dt = \int_0^\infty e^{-x \cosh t}dt = K_0(x),
\]

\[
\int_0^\infty p(t)f(t)dt = \int_0^\infty e^{-x \cosh t}(\cosh t)^\beta dt = K_{-\beta}(x)
\]

and

\[
\int_0^\infty p(t)g(t)dt = \int_0^\infty e^{-x \cosh t}(\cosh t)^{-(\alpha + \beta)} dt = K_{\alpha + \beta}(x).
\]

Thus, appealing to Chebyshev integral inequality (2.16), the proof of the inequality (2.3) is complete.

Finally, if we consider the functions \( \varphi, \psi : (0, \infty) \to \mathbb{R} \), defined by \( \varphi(x) = -K_i(x) \) and \( \psi(x) = -K_i(x) \), then by using the inequality (2.21) we obtain that

\[
\frac{\varphi''(x)}{\varphi(x)} = \frac{K_i''(x)}{K_i(x)} \geq \frac{K_i''(x)}{K_i'(x)} \frac{\psi''(x)}{\psi'(x)}
\]

for all \( x > 0 \) and \( \alpha \geq 2 \). In other words, the function \( \varphi \) is convex with respect to \( \psi \) on \((0, \infty)\) for \( \alpha \geq 2 \).

Let us consider the following interpolation of the Grüss inequality [12]: If the integrable functions \( f, g : [a, b] \to \mathbb{R} \) satisfies the inequalities \( m_1 \leq f(x) \leq M_1 \) and \( m_2 \leq g(x) \leq M_2 \) for all \( x \in [a, b] \) and \( p : [a, b] \to [0, \infty) \) is integrable such that \( \int_a^b p(t)dt > 0 \), then

\[
\left| \int_a^b p(t)dt \int_a^b p(t)f(t)g(t)dt - \int_a^b p(t)f(t)dt \int_a^b p(t)g(t)dt \right| \leq \frac{1}{4}(M_2 - m_2)(M_1 - m_1) \left( \int_a^b p(t)dt \right)^2.
\]

We use this inequality for the functions \( f, g \) and \( p \) as in the proof of part f. Observe that when \( \beta \leq 0 \) and \( \alpha + \beta \geq 0 \), then we have \( 0 < f(t) < 1 \) and \( 0 < g(t) < 1 \) for all \( t > 0 \).

Owing to Kimberling [17] it is known that if the function \( f \), defined on \((0, \infty)\), is continuous and completely monotonic and maps \((0, \infty)\) into \((0, 1)\), then \( f \) is super-additive, that is for all \( x, y > 0 \) we have

\[
\log f(x + y) \geq \log f(x) + \log f(y) \quad \text{or} \quad f(x + y) \geq f(x)f(y).
\]

In view of part a the Bickley function \( K_i \) is completely monotonic and so is \( x \mapsto K_i(x)/K_i(0) \), which maps \((0, \infty)\) into \((0, 1)\). Similarly, the function \( \alpha \mapsto K_i(x)/K_i(0) \) is completely monotonic on \((0, \infty)\), according to part b of this theorem, and maps \((0, \infty)\) into \((0, 1)\). Consequently, applying Kimberling’s result, the proof of the left-hand side of the inequalities (2.6) and (2.8) is complete. Here we used that [26], p. 259

\[
(2.17)
\]

\[
K_i(0) = \sqrt{\pi} \frac{\Gamma\left(\frac{3}{2}\right)}{2 \Gamma\left(\frac{\alpha + 1}{2}\right)}
\]

for all \( \alpha > 0 \).

For the proof of the second inequalities in (2.6) and (2.8) recall the well-known fact that if for a function \( g : (0, \infty) \to (0, \infty) \) we have that \( x \mapsto g(x)/x \) is decreasing, then we have that \( g \) is sub-additive, that is, for all \( x, y > 0 \) one has

\[
g(x + y) \leq g(x) + g(y).
\]

Now, both of functions \( x \mapsto K_i(x)/x \) and \( \alpha \mapsto K_i(x)/\alpha \) are decreasing on \((0, \infty)\), and hence the functions \( x \mapsto K_i(x) \) and \( \alpha \mapsto K_i(x) \) are sub-additive.

Now, we consider the proof of the last inequalities in (2.6) and (2.8). In view of parts a and b the functions \( x \mapsto K_i'(x) \) and \( \alpha \mapsto \partial \frac{\partial}{\partial \alpha} K_i(x)/\partial \alpha \) are increasing on \((0, \infty)\). Hence by using the monotone form of l’Hospital’s rule [24, Lemma 2.2], the functions \( x \mapsto (K_i(x) - K_i(0))/x \) and \( \alpha \mapsto (K_i(x) - K_i(0))/\alpha \) are increasing too on \((0, \infty)\), which implies that the functions \( x \mapsto K_i(x) - K_i(0) \) as well as \( \alpha \mapsto K_i(x) - K_i(0) \) are super-additive on \((0, \infty)\). Note that the last inequalities in (2.6) and (2.8) can proved also by using Petrović’s result [23], p. 22: if \( f : [0, \infty) \to \mathbb{R} \) is convex, then for all \( x, y > 0 \) we have

\[
f(x) + f(y) \leq f(x + y) + f(0).
\]
Finally, let us consider Vasić’s extension of Petrović inequality [27] which reads: for a function $f$ convex on $[0, a]$, $x_j \in [0, a]$, $p_j \geq 1$, $j \in \{1, 2, \ldots, n\}$; $\sum_{j=1}^{n} p_j x_j \in [0, a]$ there holds

$$\sum_{j=1}^{n} p_j f(x_j) \leq f \left( \sum_{j=1}^{n} p_j x_j \right) + \left( \sum_{j=1}^{n} p_j - 1 \right) f(0).$$

Specifying $f = K_{\alpha}$; $n = 2$, $p_1 = r$, $p_2 = s$; $x_1 = x$, $x_2 = y$ and by the above exposed Vasić’s result we deduce (2.7). Let us point out that $r = s = 1$ in (2.7) gives the right-hand side inequality in (2.8).

j. By the definition of exponential convexity and by using the left-hand side of (2.6) we conclude

$$H_\xi(K_{\alpha}) = \sum_{j,k=1}^{n} \xi_j \xi_k K_{\alpha}(x_j + x_k) \geq 2\Gamma \left( \frac{\alpha + 1}{2} \right) \sum_{j,k=1}^{n} \xi_j \xi_k K_{\alpha}(x_j) K_{\alpha}(x_k) = \frac{2\Gamma \left( \frac{\alpha + 1}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{\alpha}{2} \right)} \left[ \sum_{j=1}^{n} \xi_j K_{\alpha}(x_j) \right]^2 > 0,$$

where $n \in \{1, 2, \ldots\}$ and $\xi_j \in \mathbb{R}$, $x_j > 0$ for $j \in \{0, 1, \ldots, n\}$. Thus, $K_{\alpha}$ is exponentially convex on $(0, \infty)$ for $\alpha > 0$. Now, because the exponential convexity implies log-convexity, we proved part d for $\alpha > 0$ as well.

k. Similarly, by using the left-hand side of (2.5) we conclude

$$H_\eta(K_{\alpha}) = \sum_{j,k=1}^{n} \eta_j \eta_k K_{\alpha+\alpha_k}(x) \geq \frac{1}{K_{\alpha}(x)} \sum_{j,k=1}^{n} \eta_j \eta_k K_{\alpha+j}(x) K_{\alpha+k}(x) = \frac{1}{K_{\alpha}(x)} \left[ \sum_{j=1}^{n} \eta_j K_{\alpha+j}(x) \right]^2 > 0,$$

where $n \in \{1, 2, \ldots\}$ and $\eta_j, \alpha_j \in \mathbb{R}$ for each $j \in \{0, 1, \ldots, n\}$. Consequently, $\alpha \mapsto K_{\alpha}(x)$ is exponentially convex on $\mathbb{R}$ for $x > 0$. Since the exponential convexity implies log-convexity, we proved part c as well.

l. & m. Employing the inequality $x + 1/x \geq 2$ we conclude (2.9). Indeed, making use of the integral form (1.1) of $K_{\alpha}(x)$ we have

$$K_{\alpha+\beta}(x) + K_{\alpha-\beta}(x) = \int_{0}^{\infty} e^{-x \cosh t} \left( \cosh t \right)^{\alpha} \left[ \left( \cosh t \right)^{\beta} + \left( \cosh t \right)^{-\beta} \right] dt \geq 2 \int_{0}^{\infty} e^{-x \cosh t} \left( \cosh t \right)^{-\alpha} dt = 2 K_{\alpha}(x).$$

Repeating this procedure to the left-hand side expression in (2.10) we get

$$K_{\alpha+\nu}(x) \cdot K_{\alpha-\mu}(x) + K_{\alpha-\nu}(x) \cdot K_{\alpha+\mu}(x) = \int_{0}^{\infty} \int_{0}^{\infty} e^{-x \cosh t - \cosh s} \left( \cosh t \cosh s \right)^{-\alpha} \times \left[ \left( \cosh t \right)^{\nu} \left( \cosh s \right)^{-\mu} + \left( \cosh t \right)^{-\nu} \left( \cosh s \right)^{\mu} \right] dt ds \geq 2 \int_{0}^{\infty} \int_{0}^{\infty} e^{-x \cosh t - \cosh s} \left( \cosh t \cosh s \right)^{-\alpha} dt ds = 2 \left[ K_{\alpha}(x) \right]^2,$$

which finishes the proof of l. It should be mentioned here that inequality (2.9) is actually a consequence of part b or c. More precisely, since $\alpha \mapsto K_{\alpha}(x)$ is convex on $\mathbb{R}$ for all $x > 0$, we have

$$K_{\alpha+(1-\lambda)\nu}(x) \leq \lambda K_{\alpha}(x) + (1 - \lambda) K_{\nu}(x)$$

for all $\nu, \mu \in \mathbb{R}$ and $x > 0$. Now, choosing $\nu = \alpha + \beta$, $\mu = \alpha - \beta$ and $\lambda = 1/2$, we get (2.9).

n. By using (1.1) and the Hölder-Rogers inequality (2.13) we obtain

$$K_{\lambda \alpha+(1-\lambda)\beta}(\lambda x + (1 - \lambda)y) = \int_{0}^{\infty} e^{-(\lambda x + (1-\lambda)y) \cosh t} \left( \cosh t \right)^{-(\lambda \alpha + (1-\lambda)\beta)} dt.$$
and \( y \) by choosing in the above inequality the generalized Cauchy-Schwarz inequality [19, 20]

\[
(2.20) \quad K_i \alpha 
\]

where holds for all \( \lambda \in [0,1], \alpha, \beta \in \mathbb{R} \) and \( x, y > 0 \), i.e. the function \((\alpha, x) \mapsto K_{i\alpha}(x)\) is log-convex. Now, by choosing in the above inequality \( \lambda = 1/2 \), and changing \( \alpha \) to \((1 + \mu)\alpha\), for \( (1 + \mu)\alpha \), \( x \) to \((1 + \nu)x\) and \( y \) to \((1 - \nu)x\), we obtain the inequality (2.11). We note that inequality (2.11) also follows from the generalized Cauchy-Schwarz inequality [19, 20]

\[
\left[ \int_a^b f(t)g(t)dt \right]^2 \leq \int_a^b [f(t)]^{1+\nu} [g(t)]^{1+\mu} dt \int_a^b [f(t)]^{-\nu} [g(t)]^{-\mu} dt,
\]

where \( \nu, \mu \in \mathbb{R} \) and \( f, g : [a, b] \to \mathbb{R} \) are integrable functions such that the above integrals exist.

The next theorem contains some other functional inequalities for the Bickley function.

**Theorem 2.**

**a.** For all \( \alpha > 1/4 \) and \( x > 0 \) the following inequality holds

\[
(2.18) \quad K_{i\alpha}(x) \leq \frac{\sqrt{\pi} e^{-x} \Gamma(\alpha - \frac{1}{4})}{2\Gamma(\alpha) x^{1/4}}.
\]

**b.** The inequality

\[
(2.19) \quad \frac{1}{2} \left[ K_{i\alpha+1}(x) - K_{i\alpha-1}(x) \right] < \frac{\partial K_{i\alpha}(x)}{\partial \alpha} < \frac{1}{2} \left[ K_{i\alpha+2}(x) - K_{i\alpha}(x) \right]
\]

is valid for all \( \alpha \in \mathbb{R} \) and \( x > 0 \).

**c.** The inequality

\[
(2.20) \quad [K_{i\alpha}(x)]^4 \leq \frac{\pi^2}{2} K_{2\alpha}(2x) K_{2\alpha-2}(2x)
\]

holds for all \( \alpha \in \mathbb{R} \) and \( x > 0 \).

**d.** If \( \alpha > 0 \) and \( x > 0 \), then

\[
(2.21) \quad K_{i\alpha}(x) \leq \frac{\sqrt{\pi} \alpha^\alpha \Gamma(\alpha)}{2(e\alpha)\alpha \Gamma(\alpha + \frac{1}{4})}.
\]

**e.** Let \( p, q \) be conjugated Hölder exponents, \( 1/p + 1/q = 1 \), \( \min\{p, q\} > 1 \). Then for all \( \alpha > 0 \) and \( x > 0 \) the inequality

\[
(2.22) \quad [K_{i\alpha}(0) - x K_{i\alpha-1}(0)]_+ \leq K_{i\alpha}(x) \leq [K_0(xp)]^{1/p} [K_{i\alpha+q}(0)]^{1/q}
\]

holds. Here \([A]_+ = \max\{A, 0\}\).

**Proof.**

**a.** Let us recall the familiar formula for the gamma function [20, p. 139]

\[
\frac{\Gamma(\alpha)}{z^\alpha} = \int_0^\infty e^{-xy} y^{\alpha-1} dy,
\]

where \( z, \alpha > 0 \). Putting \( z = \cosh t \) for \( t \in (0, \infty) \) we get

\[
(cosh t)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y \cosh t} y^{\alpha-1} dy.
\]

This together with the integral form (1.1) of \( K_{i\alpha}(x) \) yields the double integral representation

\[
K_{i\alpha}(x) = \int_0^\infty e^{-x \cosh t} (cosh t)^{-\alpha} dt = \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^\infty e^{-(x+y) \cosh t} y^{\alpha-1} dt dy.
\]

Since the integrand is positive and \( \cosh t \geq 1 + t^2/2 \) for all \( t \in \mathbb{R} \), it follows that

\[
K_{i\alpha}(x) \leq \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^\infty e^{-(x+y)(1+t^2/2)} y^{\alpha-1} dt dy = \frac{e^{-x}}{\Gamma(\alpha)} \int_0^\infty e^{-y^{\alpha-1}} \left( \int_0^\infty e^{-\frac{x+y+t^2}{2}} dt \right) dy = R.
\]
The integration order exchange and the variable substitution $t \sqrt{(x + y)^2} \mapsto s$ lead to
\[
R = \frac{\sqrt{2}}{\Gamma(\alpha)} \int_0^\infty e^{-s} y^{\alpha-1} \left( \int_0^\infty e^{-s^2} ds \right) dy = \frac{\sqrt{\pi}}{2 \Gamma(\alpha)} \int_0^\infty e^{-y^{\alpha-1}} dy.
\]
Applying the arithmetic mean - geometric mean inequality to the denominator of the integrand, we get
\[
R \leq \frac{\sqrt{\pi e^{-x}}}{2 \Gamma(\alpha)} \int_0^\infty e^{-y^{\alpha-1}/4} dy = \frac{\sqrt{\pi e^{-x} \Gamma(\alpha - \frac{1}{2})}}{2 \Gamma(\alpha) x^{1/4}},
\]
which makes sense for all $\alpha > \frac{1}{4}$. This completes the proof of (2.13).

b. To prove (2.19) observe that by (1.1) we have
\[
\frac{\partial K(x)}{\partial t} = - \int_0^\infty e^{-x \cosh t} \log(cosh t)(\cosh t)^{-\alpha} dt.
\]
On the other hand, it is known [24, Theorem 3.3] that for all $t \in \mathbb{R}$ the inequality
\[
2 \left( \tanh \frac{t}{2} \right)^2 < \log(cosh t) < \frac{\sinh t \tanh t}{2}
\]
is valid. Applying this inequality together with
\[
\tanh \frac{t}{2} = \frac{\sinh t}{\cosh t + 1} \geq \frac{\tanh t}{2},
\]
where $t \in \mathbb{R}$, we obtain that for all $\alpha \in \mathbb{R}$ and $x > 0$
\[
\frac{\partial K(x)}{\partial \alpha} > - \frac{1}{2} \int_0^\infty e^{-x \cosh t} (\sinh^2 t)(\cosh t)^{-(\alpha+1)} dt
= \frac{1}{2} \int_0^\infty e^{-x \cosh t} (1 - \cosh^2 t)(\cosh t)^{-(\alpha+1)} dt
= \frac{1}{2} \left( \int_0^\infty e^{-x \cosh t} (\cosh t)^{-(\alpha+1)} dt - \int_0^\infty e^{-x \cosh t} (\cosh t)^{-(\alpha+1)} dt \right)
= \frac{1}{2} [K_{\alpha+1}(x) - K_{\alpha-1}(x)]
\]
and
\[
\frac{\partial K(x)}{\partial \alpha} < -2 \int_0^\infty e^{-x \cosh t} \left( \tanh \frac{t}{2} \right)^2 (\cosh t)^{-(\alpha+1)} dt
\]
\[
\leq \frac{1}{2} \int_0^\infty e^{-x \cosh t} (\tanh^2 t)(\cosh t)^{-(\alpha+1)} dt
= \frac{1}{2} \int_0^\infty e^{-x \cosh t} (1 - \cosh^2 t)(\cosh t)^{-(\alpha+2)} dt
= \frac{1}{2} \left( \int_0^\infty e^{-x \cosh t} (\cosh t)^{-(\alpha+2)} dt - \int_0^\infty e^{-x \cosh t} (\cosh t)^{-(\alpha+1)} dt \right)
= \frac{1}{2} [K_{\alpha+2}(x) - K_{\alpha}(x)].
\]
c. We shall apply Carlson’s inequality [23, p. 370] which states that if for the function $f : [0, \infty) \rightarrow (0, \infty)$ the functions $x \mapsto |f(x)|^2$ and $x \mapsto |xf(x)|^2$ are integrable on $[0, \infty)$, then
\[
\left[ \int_0^\infty f(t) dt \right]^4 \leq \pi^2 \left[ \int_0^\infty |f(t)|^2 dt \right] \left[ \int_0^\infty |tf(t)|^2 dt \right].
\]
Thus we obtain that for all $\alpha \in \mathbb{R}$ and $x > 0$
\[
\left[ \int_0^\infty e^{-x \cosh t} (\cosh t)^{-(\alpha+1)} dt \right]^4 \leq \pi^2 \left[ \int_0^\infty e^{-2x \cosh t} (\cosh t)^{-(2\alpha)} dt \right] \left[ \int_0^\infty t^2 e^{-2x \cosh t} (\cosh t)^{-(2\alpha)} dt \right].
\]
Applying again the inequality $\cosh t \geq 1 + t^2/2 \geq \sqrt{2}t$, we get $\cosh^2 t \geq 2t^2$, for all $t \geq 0$. Consequently,
\[
\int_0^\infty t^2 e^{-2x \cosh t} (\cosh t)^{-(2\alpha+2)} dt \leq \frac{1}{2} \int_0^\infty e^{-2x \cosh t} (\cosh t)^{-(2\alpha+2)} dt.
\]
Now, using the representation (1.21), we complete the proof of (2.20).
d. Applying the inequality \[ p. 266] e^{-y} \leq (a/e)^{y-a}, \] where \( y > 0 \) and \( a > 0 \), for \( y = x \cosh t \) and \( a = \alpha \), we obtain
\[
K_{i_0}(x) = \int_0^\infty e^{-x \cosh t} (\cosh t)^{-\alpha} \, dt \leq (a/e)^{\alpha} \int_0^\infty (\cosh t)^{-2\alpha} \, dt = (a/e)^{\alpha} K_{2\alpha}(0).
\]
Thus, in view of (2.11), the inequality (2.41) follows.

e. Applying the estimate \( e^{-\alpha} \geq 1 - a \) with \( a = x \cosh t \) in (1.1) we obtain
\[
K_{i_0}(x) \geq \int_0^\infty (1 - x \cosh t)(\cosh t)^{-\alpha} \, dt = K_{i_0}(0) - x K_{i_0-1}(0).
\]
This proves the left-hand side of (2.22). Now, let \( p, q \) with \( \min\{p, q\} > 1 \) be conjugated H"older exponents. Then by the H"older–Rogers inequality (2.13) we conclude
\[
K_{i_0}(x) = \int_0^\infty e^{-x \cosh t} (\cosh t)^{-\alpha} \, dt
\leq \left[ \int_0^\infty e^{-xp \cosh t} \, dt \right]^{1/p} \left[ \int_0^\infty (\cosh t)^{-aq} \, dt \right]^{1/q}
= [K_{i_0}(xp)]^{1/p} \cdot [K_{i_0}(0)]^{1/q} = [K_{0}(x)]^{1/p} \left[ \frac{\sqrt{\pi} \Gamma \left( \frac{aq}{2} \right)}{2 \Gamma \left( \frac{aq+1}{2} \right)} \right]^{1/q}.
\]
On the other hand, since \( \cosh t \geq 1 + t^2/2 \),
\[
K_{0}(xp) \leq \int_0^\infty e^{-xp(1+t^2/2)} \, dt = \frac{\sqrt{\pi} e^{-xp}}{2xp},
\]
which completes the proof of Theorem 2. \( \square \)

Finally, observe that the Turán type inequality (2.15) can be deduced also from the representation
\[
K_{i_0-2}(x) K_{i_0}(x) - [K_{i_0-1}(x)]^2 = \frac{1}{2} \int_0^\infty \int_0^\infty e^{-x(\cosh t + \cosh s)} (\cosh t \cosh s)^{-\alpha} (\cosh t - \cosh s)^2 \, dt \, ds.
\]
Moreover, the above integral representation yields the following complete monotonicity result: the function
\[
x \mapsto \begin{vmatrix} K_{i_0}(x) & K_{i_0-1}(x) \\ K_{i_0-1}(x) & K_{i_0-2}(x) \end{vmatrix}
\]
is not only positive, but also completely monotonic on \((0, \infty)\) for all \( \alpha \in \mathbb{R} \). The next result generalizes this property of the Bickley function concerning Turán determinants.

**Theorem 3.** If \( \alpha \in \mathbb{R} \) and \( n \in \{1, 2, \ldots\} \), then the function
\[
x \mapsto \det_{K_i}(x) = \begin{vmatrix} K_{i_0}(x) & K_{i_0-1}(x) & \cdots & K_{i_0-n}(x) \\ K_{i_0-1}(x) & K_{i_0-2}(x) & \cdots & K_{i_0-n-1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ K_{i_0-n}(x) & K_{i_0-n-1}(x) & \cdots & K_{i_0-2n}(x) \end{vmatrix}
\]
is completely monotonic on \((0, \infty)\).

**Proof.** Recently, Baricz and Ismail [6, Theorem 5] proved the following result (see also [16, Remark 2.9]): If the sequence of functions \( \{f_n\}_{n \geq 0} \) is of the form
\[
f_n(x) = \int_0^\beta [\phi(t, x)]^n \, d\mu(t, x),
\]
where \( \phi, \mu : [\alpha, \beta] \times \mathbb{R} \to \mathbb{R} \) and \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha < \beta \), then the determinant
\[
\det_{n}(x) = \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_{n}(x) \\ f_1(x) & f_2(x) & \cdots & f_{n+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n}(x) & f_{n+1}(x) & \cdots & f_{2n}(x) \end{vmatrix}
\]
can be rewritten as follows
\[
\det_{n}(x) = \frac{1}{(n+1)!} \int_{[\alpha, \beta]^{n+1}} \prod_{0 \leq j < k \leq n} [\phi(t_j, x) - \phi(t_k, x)]^2 \prod_{j=0}^n d\mu(t_j, x).
\]
Applying this result for the sequence of functions \( \{ \text{Ki}_{n} \}_{n \geq 0} \) we obtain that
\[
\text{Det}_{\text{Ki}_n}(x) = \frac{1}{(n+1)!} \int_{0}^{\infty} \exp \left( -x \sum_{j=0}^{n} \cosh t_j \right) \prod_{0 \leq j < k \leq n} (\cosh t_j - \cosh t_k)^2 \prod_{j=0}^{n} (\cosh t_j)^{-\alpha} \, dt_j,
\]
which shows that indeed the function \( x \mapsto \text{Det}_{\text{Ki}_n}(x) \) is completely monotonic on \((0, \infty)\) for all \( x > 0 \). \( \square \)

**Acknowledgement.** The authors are very grateful to the eagle-eyed referee for his/her several constructive comments and suggestions, which improved the quality of the paper.

**References**

1. D.E. Amos, Uniform asymptotic expansions for exponential integrals \( E_n(x) \) and Bickley functions \( \text{Ki}_n(x) \), ACM Trans. Math. Software 9(4) (1983) 467–479.
2. D.E. Amos, Repeated integrals and derivatives of \( K \) Bessel functions, SIAM J. Math. Anal. 20(1) (1989) 169–175.
3. G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, Inequalities for quasiconformal mappings in space, Pacific J. Math. 160(1) (1993) 1–18.
4. M. Anwar, J. Jukić, J. Pečarić, A. Ur Rehman, Exponential convexity, positive semi-definite matrices and fundamental inequalities, J. Math. Inequal. 4(2) (2010) 171–189.
5. Á. Baricz, Geometrically concave univariate distributions, J. Math. Anal. Appl. 363(1) (2010) 182–196.
6. Á. Baricz, M.E.H. Ismail, Turán type inequalities for Tricomi confluent hypergeometric functions, Constr. Approx. 37(2) (2013) 195–221.
7. Á. Baricz, D. Jankov, T.K. Pogány, Turán type inequalities for Krätzel functions, J. Math. Anal. Appl. 388(2) (2012) 716–724.
8. Á. Baricz, S. Ponnusamy, M. Vuorinen, Functional inequalities for modified Bessel functions, Expo. Math. 29(4) (2011) 399–414.
9. S.N. Bernstein, Sur les fonctions absolument monotones, Acta Math. 52 (1929) 1-66.
10. W.G. Bickley, Some solutions of the problem of forced convection, Philos. Mag. 20 (1935) 322–343.
11. J.M. Blair, C.A. Edwards, J.H. Johnson, Rational Chebyshev approximations for the Bickley functions \( \text{Ki}_n(x) \), Math. Comp. 32(143) (1978) 876–886.
12. G.T. Cargo, Comparable means and generalized convexity, J. Math. Anal. Appl. 12 (1965) 387-392.
13. S.S. Dragomir, Some integral inequalities of Grüss type, Indian J. Pure and Appl. Math. 31(4) (2000) 397–415.
14. L.A.-M. Hanna, S.L. Kalla, On a generalized secant integral, Radiat. Phys. Chem. 59(3) (2000) 281–285.
15. G. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge Mathematical Library, 2nd ed., 1952.
16. M.E.H. Ismail, A. Laforgia, Monotonicity properties of determinants of special functions, Constr. Approx. 26 (2007) 1–9.
17. C.H. Kimberling, A probabilistic interpretation of complete monotonicity, Aequationes Math. 10 (1974) 152–164.
18. Y.L. Luke, Integrals of Bessel functions, McGraw-Hill Book Co., New York, 1962.
19. M. Masjed-Jamei, A functional generalization of the Cauchy-Schwarz inequality and some subclasses, Appl. Math. Lett. 22(9) (2009) 1335–1339.
20. M. Masjed-Jamei, A main inequality for several special functions, Comput. Math. Appl. 60(5) (2010) 1280–1289.
21. M.S. Milgram, On the properties of collision probability integrals in annular geometry, J. Analysis, J. Math. Phys. 18(12) (1977) 2456–2467.
22. K.S. Miller, S.G. Samko, Completely monotonic functions, Integral Transf. Spec. Funct. 12(4) (2001) 389–402.
23. D.S. Mitrinović, Analytic Inequalities, Springer-Verlag, Berlin, 1970.
24. E. Neuman, J. Sándor, Inequalities for hyperbolic functions, Appl. Math. Comput. 218 (2012) 9291–9295.
25. C.P. Niculescu, Convexity according to the geometric mean, Math. Inequal. Appl. 3(2) (2000) 155-167.
26. F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark (Eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.
27. P. M. Vasić, Sur une inégalité de M. Petrović, Mat. Vesnik 5(20) (1968), 473–478.
28. D.V. Widder, The Laplace Transform, Princeton Univ. Press, Princeton, 1941.