

\textbf{SLE martingales and the Virasoro algebra}

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\textbf{Abstract}

We present an explicit relation between representations of the Virasoro algebra and polynomial martingales in stochastic Loewner evolutions (SLE). We show that the Virasoro algebra is the spectrum generating algebra of SLE martingales. This is based on a new representation of the Virasoro algebra, inspired by the Borel-Weil construction, acting on functions depending on coordinates parametrizing conformal maps.

Fractal critical clusters are the cornerstones of criticality, especially in two dimensions, see e.g. refs.\textsuperscript{1,2,3}. Stochastic Loewner evolutions\textsuperscript{4,5,6} are random processes adapted to a probabilistic description of such fractals. The aim of this Letter is to elaborate on the connection between stochastic Loewner evolutions (SLE) and conformal field theories (CFT) developed in ref.\textsuperscript{7}. We shall construct new representations of the Virasoro algebra which allow us to show explicitly that the Virasoro algebra is the generating algebra of (polynomial) martingales for the SLE processes. Physically, martingales are observables conserved in mean. They are essential ingredients for estimating probability of events. Another approach for connecting SLE to representations of the Virasoro algebra has been described in \textsuperscript{11}.

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Basic definitions of the stochastic Loewner evolutions and of their martingales are recalled in the two first sections. The new representations of the Virasoro algebra we shall construct are described in sections 3 and 4. They are based on a generalization of the Borel-Weil construction, which we apply to the Virasoro algebra. They lead to expressions of the Virasoro generators as first order differential operators acting on (polynomial) functions depending on an infinite set of coordinates parametrizing (germs of) conformal maps. Although motivated by SLE considerations, this is a result independent of SLE which may find other applications in CFT, string theories or connected subjects. The applications to SLE are presented in sections 5 and 6. In particular, we show that all polynomial SLE martingales are in the Virasoro orbit obtained by acting with these Virasoro generators on the constant function. Because it deals with polynomial martingales and well defined Virasoro generators, this construction gives a precise algebraic meaning to the statement \cite{7} that CFT gives all SLE martingales. It is more algebraic but less geometric.

1- SLE basics. Stochastic Loewner evolutions are growth processes defined via conformal maps which are solutions of Loewner’s equation:

\[
\partial_t X_t(z) = \frac{2}{X_t(z) - \xi_t}, \quad X_{t=0}(z) = z
\]

with \( \xi_t \) real. The map \( X_t(z) \) is the uniformizing map for a simply connected domain \( H_t \) of the upper half plane \( H, \Im z > 0 \). The map \( X_t(z) \) is normalized by \( X_t(z) = z + 2t/z + \cdots \) at infinity. For fixed \( z \), it is well-defined up to the time \( \tau_z \leq +\infty \) for which \( X_{\tau_z}(z) = \xi_{\tau_z} \). The sets \( K_t = \{ z \in H : \tau_z \leq t \} \) form an increasing sequence, \( K_{t'} \subset K_t \) for \( t' < t \), and are called the hulls. The domain \( H_t \) is \( H \setminus K_t \). The SLE processes are defined \cite{4} by choosing \( \xi_t = \sqrt{\kappa} B_t \) with \( B_t \) a normalized Brownian motion and \( \kappa \) a real positive parameter so that \( E[\xi_t \xi_s] = \kappa \min(t, s) \). In particular, \( \xi_t \) are white-noise variables: \( E[\xi_t \xi_s] = \kappa \delta(t-s) \). Here and in the following, \( E[\cdots] \) denotes expectation value. It will be convenient to introduce the function \( Y_t(z) \equiv X_t(z) - \xi_t \) whose Itô derivative is:

\[
dY_t(z) = \frac{2}{Y_t(z)} dt - d\xi_t \tag{1}
\]

The SLE equation \cite{1} may be turned into a hierarchy of differential equations for the coefficients of the expansion of \( Y_t(z) \) at infinity. Writing \( Y_t(z) \equiv \sum_{n \geq 0} a_n z^{1-n} \) with \( a_0 = 1, a_1 = -\xi_t \), and defining polynomials \( p_j \) in
the variables $a_i$ by $p_1 = 0$, $p_2 = 1$ and $p_j = -\sum_{i=1}^{j-2} a_i p_{j-i}$ for $j \geq 3$, so that $Y_t^{-1}(z) = \sum_{n=1}^{\infty} p_n z^{1-n}$; the Loewner equation (1) becomes:

$$\dot{a}_j = 2 p_j (a_1, \ldots), \quad j \geq 2. \quad (2)$$

Since $a_1(t) = -\xi_t$ is a Brownian motion, with continuous trajectories, eqs. (2) form a set of stochastic differential equations for the $a_j(t)$’s, $j \geq 2$, and the solutions are continuously differentiable functions of $t$ which vanish at $t = 0$ due to the initial condition $Y_0(z) = z$. Thus, the Ito differential of any (polynomial, say) function $Q(a_1(t), a_2(t), \cdots)$ is

$$\left[ -d\xi_t \frac{\partial}{\partial a_1} + dt \left( \frac{\kappa}{2} \frac{\partial^2}{\partial a_1^2} + 2 \sum_{j \geq 2} p_j \frac{\partial}{\partial a_j} \right) \right] Q.$$ 

In particular,

$$\frac{d}{dt} E[Q(a_1(t), a_2(t), \cdots)] = E[(\hat{A} \cdot Q)(a_1(t), a_2(t), \cdots)]$$

where the operator $\hat{A}$ is the coefficient of $dt$ in the previous formula:

$$\hat{A} = \frac{\kappa}{2} \frac{\partial^2}{\partial a_1^2} + 2 \sum_{j \geq 2} p_j \frac{\partial}{\partial a_j}. \quad (3)$$

If one assigns degree $i$ to $a_i$, $\frac{\partial}{\partial a_i}$ is of degree $-i$ and $p_j$ is homogeneous of degree $j - 2$, so that $\hat{A}$ is of degree $-2$.

In the following, we shall treat the functions $a_i(t)$ as independant algebraic indeterminates $a_i$, as already suggested in previous notations. This requires some justification. We need to show that if $Q(a_1, a_2, \cdots)$ is a nonzero abstract polynomial, the function $Q(a_1(t), a_2(t), \cdots)$ cannot vanish for every realization of $\xi_t$ and every $t$. Indeed, assume the contrary and take a counterexample $Q(a_1, a_2, \cdots)$ of minimal degree. As $Q(a_1(t), a_2(t), \cdots) \equiv 0$, the Ito differential of $Q$ vanishes identically as well:

$$\left[ -d\xi_t \frac{\partial}{\partial a_1} + dt \hat{A} \right] Q(a_1(t), a_2(t), \cdots) \equiv 0.$$

Multiplication by $d\xi_t$ yields

$$-dt \frac{\partial Q}{\partial a_1}(a_1(t), a_2(t), \cdots) \equiv 0$$

which can be plugged back into the original equation. So $\hat{A}Q(a_1(t), a_2(t), \cdots) \equiv 0$ and $\frac{\partial Q}{\partial a_1}(a_1(t), a_2(t), \cdots) \equiv 0$. Because $\hat{A}$ and $\frac{\partial}{\partial a_1}$ decrease the degree and $Q$ is a counterexample of minimal degree, $\hat{A}Q$ and $\frac{\partial Q}{\partial a_1}$ vanish as abstract polynomials. So the question whether the functions $a_i(t)$ are algebraically independant is reduced to the purely algebraic question whether the system of linear algebraic equations $\frac{\partial Q}{\partial a_1} = \hat{A}Q = 0$ has only the constant polynomials as solutions. This will be proved at the end of section 4.
2- SLE martingales. The set $\mathcal{F}$ of polynomial functions in the $a_j$ forms a graded vector space $\mathcal{F} \equiv \oplus_{n \geq 0} \mathcal{F}_n$, with elements of $\mathcal{F}_n$ homogeneous polynomials of degree $n$. The operator $\hat{A}$ maps $\mathcal{F}_{n+2}$ into $\mathcal{F}_n$.

Polynomial martingales are, by definition, polynomials in the $a_j$ annihilated by $\hat{A}$. Their set $\mathcal{M} = \ker \hat{A}$ is graded: $\mathcal{M} \equiv \oplus_{n \geq 0} \mathcal{M}_n$ with $\mathcal{M}_n \subset \mathcal{F}_n$.

The low degree martingales are:

- $\mathcal{M}_1 : a_1$
- $\mathcal{M}_2 : 2a_1^2 - \kappa a_2$
- $\mathcal{M}_3 : 2a_1^3 - 3\kappa a_1 a_2, a_3 + a_1 a_2$

Let $\text{char} \mathcal{M} = \sum_{n \geq 0} q^n \dim \mathcal{M}_n$ with $\mathcal{M}_n = \ker \hat{A}|_{\mathcal{F}_n}$. Crucial to the sequel is the character formula

$$\text{char} \mathcal{M} = \frac{1 - q^2}{\prod_{j \geq 0} (1 - q^j)} \quad (4)$$

This may be compared to $\text{char} \mathcal{F} = \prod_{j \geq 0} (1 - q^j)^{-1}$, in particular $\dim \mathcal{M}_n < \dim \mathcal{F}_n$ for $n \geq 2$.

We shall give a proof of eq. (4) using the general machinery in section 5. A direct argument can be organized as follows. We want to show that the sequence $\mathcal{M}_{n+2} \xrightarrow{\hat{A}} \mathcal{M}_n \to 0$ is exact, i.e. that $\hat{A}(\mathcal{M}_{n+2}) = \mathcal{M}_n$ for $n \geq 0$. Decompose $\hat{A} = \hat{A}' + \hat{A}''$ where $\hat{A}' \equiv 2 \frac{\partial}{\partial a_2}$. It is clear that $\hat{A}'(\mathcal{M}_{n+2}) = \mathcal{M}_n$ for $n \geq 0$ because if $Q(a_1, a_2, \cdots)$ is any polynomial in $\mathcal{M}_n$, we can set $\hat{I} Q(a_1, a_2, \cdots) \equiv \frac{1}{2} \int_0^{a_2} da \, Q(a_1, a, a_3 \cdots)$, which satisfies clearly $\hat{A}' \hat{I} Q = Q$. Now we do perturbation theory. Starting from $Q \neq 0$, we define two sequences $q_0, n \geq 0$ and $r_n, n \geq 1$ by $q_0 \equiv Q$, $r_1 \equiv \hat{I} q_0$, $q_1 \equiv q_0 - \hat{A} r_1 = -\hat{A}'' \hat{I} q_0$, $r_2 \equiv \hat{I} q_1$, $q_2 \equiv q_1 - \hat{A} r_2 = -\hat{A}'' \hat{I} q_1$, $\cdots$. The key point if that these sequences stop. Indeed, if $q_k$ is non zero, then its total degree is that of $q_{k-1}$ but its degree in $a_2$ is at least one more than that of $q_{k-1}$ : $\hat{I}$ increases the degree in the variable $a_2$ by one unit and $\hat{A}''$ contains no derivative with respect to $a_2$. So $q_k$ and then $r_{k+1}$ have to vanish for large enough $k$. Hence one can sum the definition $\hat{A} r_{k+1} = q_k - q_{k+1}$ over $k$, leading to a telescopic cancellation $\hat{A} \sum_k r_k = q_0$, showing that $\hat{A}$ is onto.

3- Group theoretical background. Let us recall a few basic facts concerning the Borel-Weil construction in group theory. Consider for instance a simply connected compact Lie group $G$. The group acts on itself by left or right multiplication. This induces a representation of the Lie algebra $\text{Lie} G$
on functions on $G$ by left or right invariant vector fields:

$$(X \cdot \nabla^l)f(g) = \frac{d}{du}f(ge^{ux})|_{u=0},$$

$$(X \cdot \nabla^r)f(g) = \frac{d}{du}f(e^{ux}g)|_{u=0}$$

for any $X \in \text{Lie}G$. They form a representation of $\text{Lie}G$ since $[X \cdot \nabla^l, Y \cdot \nabla^l] = [X, Y] \cdot \nabla^l$ and $[X \cdot \nabla^r, Y \cdot \nabla^r] = -[X, Y] \cdot \nabla^r$.

Let us choose a Cartan subgroup $H$, and let $N_{\pm}$ be the associated nilpotent subgroups and $B_{\pm} = HN_{\pm}$ be the corresponding Borel subgroups. At least in a neighbourhood of the identity, elements $g$ of $G$ may be factorized according to the Gaussian decomposition as $g = n_+ h n_-$ with $h \in H$ and $n_\pm \in N_{\pm}$. We set $g_+ = n_+ h$, $g_0 = h$ and $g_- = n_-$, the components of $g$ in $B_+$, in $H$ and in $N_-$, respectively. For elements $X \in \text{Lie}G$, we shall denote by $X_+, X_0$ and $X_-$ their components in $\text{Lie}B_+$, in $\text{Lie}H$ and in $\text{Lie}N_-$, respectively.

One may define two actions of $G$ on $N_-$ by:

$$l_x(g) \equiv (gxg^{-1})_+^{-1} gx = (gxg^{-1})_- g$$

$$r_x(g) \equiv x^{-1} g (g^{-1}xg)_+ = g (g^{-1}xg)_-^{-1}$$

for $g \in N_-$ and $x \in g$. They act on $N_-$ since $l_x(g) \in N_-$ and $r_x(g) \in N_-$ for $g \in N_-$. They form anti-representations of $G$:

$$l_y(l_x(g)) = l_{xy}(g), \quad r_y(r_x(g)) = r_{xy}(g)$$

because $(gxg^{-1})_+ = (gxg^{-1})_+ (l_x(g)r_x(g)^{-1})_+$. The Borel-Weil construction consists in defining an action of the group $G$ on sections of line bundles over the quotient space $B_+ \backslash G^C$. Sections of $B_+ \backslash G^C$ may be viewed as functions $S^l(g)$ on $G^C$ such that

$$S^l(n_+ g) = S^l(g), \quad n_+ \in N_+$$

$$S^l(hg) = \chi(h) S^l(g), \quad h \in H$$

with $\chi$ a $C$-valued $H$-character such that $\chi(h_1 h_2) = \chi(h_1) \chi(h_2)$. Such a character is specified by a weight $\omega \in (\text{Lie}H)^\ast$ via $\chi(e^X) = \exp - (\omega, X)$ for $X \in \text{Lie}H$.

The group $G$ acts on such sections by right multiplication: $(L_x \cdot S^l)(g) \equiv S^l(gx)$ for $x \in G$. This action defines a representation of $G$: $L_y \cdot (L_x \cdot S^l)(g) = S^l(\gamma^{-1})$, where $

\footnote{For $G = SU(N)$, $H$ is made of diagonal complex matrices and $N_\pm$ of lower (upper) triangular matrices with 1 along the diagonal.}
\((L_x \cdot S^l)(g)\). Since \(B_+ \backslash G^C\) may locally be identified with \(N_-\), we may choose a gauge in which \(g \in N_-\) and view \(S^l\) as functions on \(N_-\) with specific transformation properties. The action of \(G\) then reads:
\[
(L_x \cdot S^l)(g) = \chi(((gxg^{-1})_0) S^l(l_x(g))), \quad x \in G, \ g \in N_-
\]
Infinitesimally, this action may be presented as first order differential operator:
\[
(D^l_X \cdot S^l)(g) = \frac{d}{du}(L_{e^uX} \cdot S^l)(g)|_{u=0}
\]
\[
= \left((X \cdot \nabla^l) - ((gxg^{-1})_+ \cdot \nabla^r)\right)S^l(g) - (\omega, (gxg^{-1})_0)S^l(g)
\]
with \(\nabla^l\) and \(\nabla^r\) the left and right invariant vector fields. By construction \([D^l_X, D^l_Y] = D^l_{[X,Y]}\). Note that \(D^l_X\) coincides with \(X \cdot \nabla^l\) for \(X \in \text{Lie}N_-\).

A similar construction applies to the right quotient \(G^C/B_+\) and its sections \(S^r(g)\), defined similarly as in eq.(5) but with right instead of left multiplications. The group \(G\) acts on sections \(S^r\) by left multiplications:
\[
(R_x \cdot S^r)(g) = S^r(g^{-1}x). \quad x \in G, \ g \in N_-
\]
Infinitesimally:
\[
(D^r_X \cdot S^r)(g) = \left(((x^{-1}X g)_{+} \cdot \nabla^r) - (X \cdot \nabla^r)\right)S^r(g) - (\omega, (x^{-1}X g)_{0})S^r(g)
\]
By construction \([D^r_X, D^r_Y] = D^r_{[X,Y]}\).

These left and right actions are linked by the relation \(r_x(g)l_x(g^{-1}) = 1\). Therefore, if \(D^l_X\) are represented as first order differential operators in a specific set of coordinates parametrizing \(g \in N_-\), then \(D^r_X\) will be represented by the same differential operators but in the coordinates parametrizing the inverse element \(g^{-1}\).

4- Differential representations of the Virasoro algebra. We now apply the previous construction to the Virasoro algebra with generators \(L_n\), \(n\) integers, and relations:
\[
[L_n, L_m] = (n-m)L_{n+m} + \frac{\hat{c}}{12}n(n^2-1)\delta_{n+m,0}
\]
with \(\hat{c}\) central. By convention, \(\text{Lie}N_-\) is generated by the \(L_n\) with \(n < 0\), \(\text{Lie}H\) by \(L_0\) and \(\hat{c}\), and \(\text{Lie}N_+\) by the \(L_n\) with \(n > 0\).
We have to select a set of coordinates in $N_-$, at least in a neighbourhood of the identity. This is provided by looking at the representation in which the Virasoro generators are represented by $\ell_n = -z^{n+1}\partial_z$. The $N_-$ orbits of a point $z$ in the complex plane define (germs of) complex maps $w(z)$ with a simple pole at infinity:

$$w(z) \equiv g z g^{-1} = \sum_{n \geq 0} a_n z^{1-n}, \quad g \in N_-$$

(10)

with $a_0 = 1$. The $a_n$ form a set of coordinates parametrizing elements of $N_-$. We shall also need the inverse map $z(w)$:

$$z(w) = \sum_{n \geq 0} b_n w^{1-n}$$

(11)

The $b_n$ are polynomials in the $a_j$ of degree $n$ and $b_0 = 1$.

Let us first deal with sections of the left quotient $B_+ \backslash G_C$. To define them we have to specify the $H$–character, or equivalently the weight $\omega \in (\text{Lie } H)^*$. It is specified by two numbers $\delta$ and $c$ such that $(\omega, L_0) = \delta$ and $(\omega, \hat{c}) = c$.

The action of the Virasoro generators on functions of the $a_j$ is then defined by the formula (7). One may view $w(z)$ as functions of the $a_j$ and use it as generating functions. We then have\footnote{Here, we use the convention that for a Laurent series $h(z) = \sum_j h_j z^j$, $(h(z))_0 = h_0$ and $(h(z))_+ = \sum_{j \geq 1} h_j z^j$.}:

**Proposition.**

i) The action of the Virasoro algebra on $w(z) = \sum_{n \geq 0} a_n z^{1-n}$ specified by eq.(7) reads:

$$D_n^l \cdot w(z) = -w(z)^{n+1} + \left( w^{n+1} \frac{dz}{dw} + \frac{dw}{dz} \right) + \left( \delta (gL_n g^{-1})(L_0) + c (gL_n g^{-1})(\hat{c}) \right) w(z)$$

(12)

with

$$(gL_n g^{-1})(L_0) = \left( z^{-1} w^{n+1} \frac{dz}{dw} \right)_0, \quad (gL_n g^{-1})(\hat{c}) = -\frac{1}{12} \left( zw^{n+1} S_z(w) \frac{dz}{dw} \right)_0$$

where $S_z(w) = \frac{w'''}{w'} - \frac{3}{2} \left( \frac{w''}{w'} \right)^2$ is the Schwarzian derivative of $w(z)$ with respect to $z$.

ii) The first order differential operators $D_n^l$ in the $a_j$ satisfy the Virasoro algebra

$$[D_n^l, D_m^l] = (n - m)D_{n+m}^l - \frac{c}{12} n(n^2 - 1)\delta_{n+m,0}$$
and are such that \( D_l^n \cdot 1 = 0 \) for \( n < 0 \) and \( D_l^n \cdot 1 = -\delta \).

**Proof.** The three terms in eq. (12) correspond to the three terms in eq. (7). They may be computed one by one using the following relations, which we just quote without proofs. First one has:

\[
(L_n, \nabla^l) w(z) = g[\ell_n, z] g^{-1} = -w^{n+1}
\]
\[
(L_n, \nabla^r) w(z) = [\ell_n, g z g^{-1}] = -z^{n+1}(\frac{dw}{dz})
\]

Then, \( gL_n g^{-1} \) is evaluated using the transformation properties [8] of the stress tensor \( T(z) = \sum_n L_n z^{-n-2} \):

\[
gT(w) g^{-1} dw^2 = T(z(w)) dz^2 - \frac{c}{12} S_z(w) dz^2
\]

Recall that \( S_w(z) dw^2 = -S_z(w)dz^2 \). This gives:

\[
gL_n g^{-1} = \sum_{s \leq n} L_s \left( \frac{w^{n+1}}{z^{s+1}} \frac{dz}{dw} \right)_0 - \frac{c}{12} \left( z w^{n+1} S_z(w) \frac{dz}{dw} \right)_0
\]

In particular, \( (gL_1 g^{-1})|_{L_0} = 2a_1 \), \( (gL_2 g^{-1})|_{L_0} = 3a_1^2 + 4a_2 \) and \( (gL_2 g^{-1})|_{\hat{c}} = a_2/2 \). \( \blacksquare \)

The differential operators \( D_l^n \) may be written explicitly. They are of the form \( D_l^n = \mathcal{D}_l^n(a) - \delta d_\delta^{(n)} - c d_c^{(n)} \) with \( \mathcal{D}_l^n(a) = \sum_j q_j^{(n)} \partial_{a_j} \), and \( d_\delta^{(n)}, d_c^{(n)} \) and \( q_j^{(n)} \) homogeneous polynomials. The first few are:

\[
D_{-2}^l = \mathcal{D}_2^l(a)
\]
\[
D_{-1}^l = \mathcal{D}_1^l(a)
\]
\[
D_0^l = \mathcal{D}_0^l(a) - \delta
\]
\[
D_1^l = \mathcal{D}_1^l(a) - 2\delta a_1
\]
\[
D_2^l = \mathcal{D}_2^l(a) - (3a_1^2 + 4a_2)\delta - \frac{c}{2} a_2
\]

with \( \mathcal{D}_n^l(a) \) vector fields given by:

\[
\mathcal{D}_{-2}^l(a) = -\sum_{j \geq 2} p_j(a) \frac{\partial}{\partial a_j}
\]
\[
\mathcal{D}_{-1}^l(a) = -\frac{\partial}{\partial a_1}
\]
\[
\mathcal{D}_0^l(a) = -\sum_{j \geq 1} j a_j \frac{\partial}{\partial a_j}
\]

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The above sums include the terms with \(a_0 = 1\). These five operators generate the whole Virasoro algebra. They act on polynomial functions of the \(a_j\).

Note that it was imperative to consider sections of \(B_+ \backslash G^C\) associated to non trivial \(H\)–characters in order to get representations of the Virasoro algebra with non vanishing central charges. To make contact with usual highest weight representations, one may define generators \(\tilde{L}_n\) which satisfy the Virasoro algebra with central charge \(c\). They are such that \(\tilde{L}_n \cdot 1 = \delta\) for \(n > 0\) and \(\tilde{L}_0 \cdot 1 = \delta\), so that \(1\) is a highest weight vector for the \(\tilde{L}_n\).

As explained above, one may define another action of the Virasoro algebra using the right quotient \(G^C/B_+\). Its generators \(D^r_n\) are defined via eq.(9).

According to the last remark of the previous section, one goes from \(D^l_n\) to \(D^r_n\) by exchanging the role played by \(g\) and its inverse, so that \(D^r_n\) coincides with \(D^l_n\) but with the variables \(a_j\) replaced by those parametrizing the inverse map:

\[
D^r_n = D^l_n(a_j \rightarrow b_j) \tag{14}
\]

The two representations \(D^r_n\) and \(D^l_n\) do not commute.

Let us remark that although the space of (polynomial) functions in the \(a_j\) may be identified with a Fock space, these representations are not the usual free field representations used in conformal field theory [9]. As far as we know, these representations were not previously described in the literature. They are similar in spirit to the representations of affine Kac-Moody algebras studied in [10].

We end this section by showing that a polynomial \(Q\) such that \(\frac{\partial Q}{\partial a_1} = 0\) and \(\hat{A} \cdot Q = 0\) is constant, thereby completing the proof, started below eq.(3), that the functions \(a_n(t)\) are algebraically independent. The key observation is that \(D^l_{-n}\) has the form \(D^l_{-n} = -\frac{\partial}{\partial a_n} - \sum_{m \geq n+1} p_{n,m} \frac{\partial}{\partial a_m}\), where \(p_{n,m}\) is a polynomial, \(p_{1,m} = 0, p_{2,m} = p_m, \cdots\). This results from the recursive definition of \(D^l_{-n}\) and the fact that, as a polynomial in \(a_1\), \(p_n = -(-a_1)^{n-2} + \) terms of lower degree. By hypothesis, the polynomial \(Q\) is annihilated by \(\frac{\partial}{\partial a_1} = -D^l_{-1}\) and \(\hat{A} = \frac{\partial}{\partial a_1} D^l_{-1} - 2D^l_{-2}\). Hence it is annihilated by all \(D^l_{-n}\)’s. The polynomial \(Q\) depends effectively on only a finite number of variables : there is a minimal
n such that $\frac{\partial Q}{\partial a_m} = 0$, $m > n$. If $n > 0$, $D_l^n Q = 0$ implies that $\frac{\partial Q}{\partial a_n} = 0$ contradicting minimality. So $n = 0$ and $Q$ is constant, as was to be proved.

5- Martingale generating algebra. Let us now make contact between the stochastic Loewner equation and the representations of the Virasoro algebra we just define.

Comparing the evolution operator $\hat{A}$, eq.(3), and the operator $D_l^n$, eqs.(13), it is clear that one has the following identification:

$$\hat{A} = \frac{\kappa}{2} (D_{l-1}^l)^2 - 2D_{l-2}$$

In other words, the stochastic evolution $(\hat{A})$ is associated with the action the Virasoro algebra on sections of the left quotient $B_+ \backslash G^C$.

On the other hand, the martingale generating algebra is not constructed using the representation $D_l^n$, since it does not commute with $\hat{A}$, but using the representation $D_r^n$ based on the right quotient $G^C/B_+$. Indeed, we have:

**Proposition.** For $c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}$ and $\delta = \frac{6 - \kappa}{2\kappa}$, one has the commutation relations:

$$[\hat{A}, D_r^n] = \hat{q}_n(a_1, \cdots) \hat{A}$$

with $\hat{q}_n(a_1, \cdots)$ homogeneous polynomials in the $a_j$ of degree $n$.

In particular, the generators $D_r^n$ act on $\ker \hat{A}$, so that if $f \in \ker \hat{A}$ then $D_r^n f \in \ker \hat{A}$.

**Proof.** This follows by construction. For $n < 0$, $D_r^n$ coincide with the right invariant vector fields which commute with the left invariant vector fields and thus with $\hat{A}$. For $n \geq 0$, eq.(15) may be checked directly using the following relation

$$[D_l^Y, D_r^X] \cdot f(g) = \left( [Y, (g^{-1}Xg)_+] - [Y, (g^{-1}Xg)]_+ \right) \cdot \nabla f(g)$$

valid for $Y \in \text{Lie}N_-$ and $X \in \text{Lie}G$. For instance, applying this formula for $X = L_1$ leads to:

$$[\hat{A}, D_l^1] \cdot f = (6 - \kappa(2\delta + 1))(L_{-1} \cdot \nabla f) \cdot f - 4b_1 (\hat{A} \cdot f)$$

The first term in the right hand side vanishes for $2\kappa \delta = 6 - \kappa$. Similarly, $X = L_2$ gives:

$$[\hat{A}, D_l^2] \cdot f = 3(6 - \kappa - 2\kappa \delta)b_1 (L_{-1} \cdot \nabla f) \cdot f$$

$$+((8 - 3\kappa)\delta + c)f - 2(3b_1^2 + 4b_2)(\hat{A} \cdot f)$$
The first two terms in the r.h.s. vanish for $2\kappa \delta = 6 - \kappa$ and \( c = (3\kappa - 8)\delta \). Note that \( \hat{q}_1 = -4b_1 \) and \( \hat{q}_2 = -2(3b_1^2 + 4b_2) \). The higher degree polynomials \( \hat{q}_n \) are recursively determined by \((n - m)\hat{q}_{n+m} = [D_{\kappa}^r, \hat{q}_m] - [D_{\kappa}^m, \hat{q}_n] \).

Let us first prove eq. (4) using this proposition. Since \( \hat{A} \) has degree \( -2 \), proving formula (4) amounts to show that \( \hat{A} : F_{n+2} \to F_n \) is surjective. We do it by recursion using the fact that \( D_{\kappa}^r - j \), \( j \geq 1 \), commute with \( \hat{A} \).

Let us define recursively \( w_j \in F_{n-j} \) by \((j - 1)w_{j+1} + 1 = D_{\kappa}^r w_j - D_{\kappa}^r w_{j+1} \). By construction they satisfy: i) \( D_{\kappa}^r w_j = \hat{A} w_j \) and ii) \( D_{\kappa}^r w_i - D_{\kappa}^r w_j = (i-j)w_{i+j} \).

Relation ii) is the integrability condition for the existence of \( v \in F_n \) such that \( w_j = D_{\kappa}^r v \). Relation i) then gives \( D_{\kappa}^r u = D_{\kappa}^r \hat{A} v \) for all \( j \geq 1 \). This implies \( u = \hat{A} v \), meaning that \( \hat{A} \) is surjective. Note that this proof is dual to the proof given at the end of section 4 that the functions \( a_n(t) \) are algebraically independant.

Let us now remark that acting successively with \( D_{\kappa}^r \) on the constant function 1 generates \( SLE \) martingales. For instance:

\[
D_{\kappa}^r \cdot 1 = \left( \frac{6 - \kappa}{\kappa} \right) a_1
\]

\[
4D_{\kappa}^r \cdot 1 = -\kappa D_{\kappa}^r \cdot 1 = 3\left( \frac{6 - \kappa}{\kappa} \right) (\kappa a_2 - 2a_1^2)
\]

Recall that the operators \( D_{\kappa}^r \) are obtained from the \( D_{\kappa}^l \), eq. (13), by replacing \( a_j \) by \( b_j \). More generally, the space \( \mathcal{P} \) of polynomials in the \( a_j \) generated by successive actions of the \( D_{\kappa}^r \) on the constant function, that is

\[ \mathcal{P} = \text{vect.} \langle \prod_j D_{n_j}^r \cdot 1 \rangle, \]

is made of martingales and so it is embedded in \( \mathcal{M} \). By construction it carries a representation of the Virasoro algebra. It is well know [9] that its character is

\[ \text{char} \mathcal{P} = \frac{1 - q^2}{\prod_{j \geq 0} (1 - q^j)} \]

So \( \text{char} \mathcal{P} = \text{char} \mathcal{M} \) and \( \mathcal{P} \equiv \mathcal{M} \).

In other words, all polynomial \( SLE \) martingales are generated by successive actions of the Virasoro differential operators associated to the right quotient.

6- Lifted SLE. Let us finally make contact with the group theoretical
formulation of the stochastic Loewner evolution proposed in ref. [7]. There, the SLE was lifted to a Markov process in the nilpotent subgroup $N_-$ of the Virasoro group defined by:

$$dg_t = g_t(-2L_{-2}dt + L_{-1}d\xi_t), \quad g_{t=0} = 1.$$ 

The associated stochastic evolution operator, acting on function of $g_t$, was identified with:

$$A = -2(L_{-2} \cdot \nabla^l) + \frac{\kappa}{2}(L_{-1} \cdot \nabla^l)^2$$

See ref. [7] for details. The random group element $g_t$ is related to the random conformal map $Y_t$ by $Y_t(z) = g_t z g_t^{-1}$ in the representation with $\ell_n = -z^{n+1} \partial_z$. Since $D_n^l$ is simply $(L_n \cdot \nabla^l)$ for $n < 0$, the operators $A$ and $\hat{A}$ clearly coincide.

A generating function of SLE martingales was identified in [7]. It is given by the vector $g_t|\omega_\kappa\rangle$ which takes values in the irreducible (for generic $\kappa$) Virasoro module, called $\mathcal{H}_{1,2}$, with highest weight vector $|\omega_\kappa\rangle$ of central charge $c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$ and conformal weight $\delta = \frac{6-\kappa}{2\kappa}$. In a graded basis of $\mathcal{H}_{1,2}$, the components of $g_t|\omega_\kappa\rangle$ are polynomial SLE martingales by construction. As is well known, $\text{char} \mathcal{H}_{1,2} = \text{char} \mathcal{P}$ and this allows us to identifies $\mathcal{H}_{1,2}$ with $\mathcal{P}$, so that the SLE martingales generated by $g_t|\omega_\kappa\rangle$ coincide with those obtained above by successive actions with the $D_n^l$. This actually follows by construction, since in the Borel-Weil construction, sections $S^r(g)$ of $G\mathcal{C}/B_+$ may be identified with matrix element $S^r(g) = \langle \nu | g | \omega \rangle$ with $|\omega\rangle$ highest weight vector.

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