Perturbative S-matrix in discretized light cone quantization of two dimensional $\phi^4$ theory

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Abstract

We study the S-matrix of two-dimensional $\lambda\phi^4$ theory in Discretized Light Cone Quantization and show how the correct continuum limit is reached for various processes in lowest order perturbation theory.

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I. INTRODUCTION

S-matrix elements have been studied in the continuum formulation of light front quantization since its inception [1]. Chang and Yan [2] showed the formal equivalence of S matrix elements in light front quantization and the more familiar instant form of quantization. Some doubts were nevertheless raised [3] regarding the formulation of a consistent scattering theory in the light front formulation. Detailed calculations of phase shifts were carried out [4] in $\phi^3$ theory with emphasis on issues concerning rotational invariance (also see Ref. [5]).

Discretized Light Cone Quantization (DLCQ) [6–8] is a method proposed for the non-perturbative solution of quantum field theories [11]. Most of the applications of the method following the work of Refs. [9,10] have been to bound state spectra. Only very recently have studies focused on the application of DLCQ to the calculation of scattering observables [12].

A calculation of one loop scattering amplitude in two dimensional $\phi^4$ theory was carried out in Ref. [13] for the $s$-channel process below production threshold to illustrate how the correct continuum limit was approached for this process in DLCQ. In a recent work [14] problems associated with compactification near and on the light front have been investigated in detail in the context of perturbative scalar field theory. This work was motivated by the result of Ref. [15] that certain divergences arise in the one loop scattering amplitude in scalar field theory at finite box length as one tried to approach the light front in a formalism of compactification near the light front. By means of detailed calculations in both continuum and discrete versions in three different approaches: (1) quantization on a space-like surface close to a light front; (2) infinite momentum frame calculations; and (3) quantization on the
light front, Ref. [14] concluded that in DLCQ, contributions from "zero mode (ZM) induced" interaction terms decouple in the continuum limit and covariant results are reproduced.

However, the claim of Ref. [14] regarding the continuum limit of DLCQ for processes with $p^+ = 0$ exchange has been challenged in a very recent work [16]. Authors of Ref. [16] agree with the conclusion of Ref. [14] that contributions from ZM induced interaction terms in DLCQ decouple in the continuum limit but they claim that DLCQ yields vanishing forward scattering amplitude in the continuum limit whereas the correct result is finite. In view of the persistent confusion on the subject, it is worthwhile to provide details of our simple, straightforward and unambiguous calculation and reconfirm our original claim. We show how the careful treatment of the process of taking the continuum limit in DLCQ yields the correct result. We also provide detailed numerical results.

The plan of this work is as follows. In Sec. II we present light front perturbation theory calculation of one loop scattering amplitude in $\lambda^4\phi^4$ theory in the continuum formulation and discretized formulation and numerical results. Sec. III contains discussion, summary, and conclusions.

II. LIGHT FRONT PERTURBATION THEORY CALCULATION OF ONE LOOP SCATTERING IN $\lambda^4\phi^4$ THEORY

A. Continuum formulation

1. $t$-channel scattering

Let us first review the forward scattering limit in the continuum formulation. For simplicity we will consider two dimensional theory since extra dimensions do not add or subtract to the essential features and conclusions of the calculation.

Consider the scattering amplitude at one loop level in $\phi^4$ theory. $p_1, p_2$ are the initial momenta and $p_3, p_4$ are the final momenta. Let us denote $s = (p_1 + p_2)^2$ and $t = (p_1 - p_3)^2$. In the light front perturbation theory, we have to consider two cases separately.

1) $p_1^+ > p_3^+$.

The scattering amplitude (Fig. 1a) is

$$M_{fi} = \frac{1}{2} \frac{\lambda^2}{4\pi} \theta(p_1^+ - p_3^+) \int_0^{p_1^+ - p_3^+} dq_1^+ \frac{1}{q_1^+} \frac{1}{p_1^+ - p_3^+ - q_1^+} \frac{1}{p_1^+ + p_2^+ - p_3^+ - p_2^+ - q_1^+ - (p_1 - p_3 - q_1)^-}$$

$$= \frac{1}{2} \frac{\lambda^2}{4\pi m^2 (p_1^+ + p_3^+)} \frac{\theta(p_1^+ - p_3^+)}{p_1^+ - p_3^+} \int_0^{p_1^+ - p_3^+} dq_1^+ \left[ \frac{1}{q_1^+ - p_1^+} - \frac{1}{q_1^+ + p_3^-} \right].$$

(2.1)

2) $p_1^+ < p_3^+$.

The scattering amplitude (Fig. 1b) is

$$M_{fi} = \frac{1}{2} \frac{\lambda^2}{4\pi} \theta(p_3^+ - p_1^+) \int_0^{p_3^+ - p_1^+} dq_1^+ \frac{1}{q_1^+} \frac{1}{p_3^+ - p_1^+ - q_1^+} \frac{1}{p_3^+ + p_3^- - p_1^- - q_1^-}$$

$$= \frac{1}{2} \frac{\lambda^2}{4\pi m^2 (p_3^+ + p_1^+)} \frac{\theta(p_3^+ - p_1^+)}{p_3^+ - p_1^+} \int_0^{p_3^+ - p_1^+} dq_1^+ \left[ \frac{1}{q_1^+ - p_1^-} - \frac{1}{q_1^+ + p_3^-} \right].$$

(2.2)
\[
\frac{1}{p_i + p_2 - p_1 - p_2 - q_1 - (p_3 - p_1 - q_1)^-}
= \frac{1}{2} \frac{\lambda^2}{4 \pi m^2} \frac{p_i^+ p_3^+ \theta(p_i^+ - p_1^+)}{p_3 - p_1} \int_{p_3}^{p_i} dq_1^+ \left[ \frac{1}{q_1^+ - p_3^+} - \frac{1}{q_1^+ + p_1^+} \right].
\] (2.2)

We have used overall energy conservation \(p_i + p_2 = p_3 + p_4\) and hence \(p_2 - p_4 = p_3 - p_1\).

We are interested in the forward scattering amplitude, i.e., in \(|p_i^+ - p_3^+| \to 0\) limit. In this limit \(q_1^+\) is very small compared to both \(p_i^+\) and \(p_3^+\) and it is legitimate to expand the integrands. We get,

\[
\begin{align*}
\frac{1}{q_1^+ - p_1^+} - \frac{1}{q_1^+ + p_3^+} & \approx -\frac{p_1^+ + p_3^+}{q_1^+ + p_1^+}, \\
\frac{1}{q_1^+ - p_3^+} - \frac{1}{q_1^+ + p_1^+} & \approx -\frac{p_1^+ + p_3^+}{q_1^+ + p_3^+}.
\end{align*}
\] (2.3)

Thus, in the forward scattering limit, we get,

\[
M_{fi} = -\frac{1}{2} \frac{\lambda^2}{4 \pi m^2}.
\] (2.4)

Alternatively, we can write the scattering amplitude as

\[
M_{fi} = \frac{1}{2} \frac{\lambda^2}{4 \pi} \int_0^1 dy \frac{1}{y(1-y)t - m^2 + i\epsilon}
\] (2.5)

and calculate it explicitly:

\[
M_{fi}(t) = -\frac{1}{2} \frac{\lambda^2}{4 \pi} \frac{1}{\sqrt{\frac{1}{4} - m^2 t}} \log \left( \frac{2\sqrt{\frac{1}{4} - m^2 t} - 1}{2\sqrt{\frac{1}{4} - m^2 t} + 1} \right).
\] (2.6)

In the forward scattering limit, one again finds the result (2.4).

2. s-channel scattering

For the s-channel scattering we have

\[
T_{fi} = \frac{\lambda^2}{8 \pi} \int_0^{p_i^+ + p_2^+} dq^+ \frac{1}{q^+(p_i^+ + p_2^+ - q^+)} \frac{1}{p_i^+ + p_2^- - m^2 + i\epsilon} - \frac{1}{q^+ - m^2 - p_i^+ + p_2^- - q^+ + i\epsilon}
\] (2.7)

\[
= \frac{\lambda^2}{8 \pi} \int_0^{p_i^+ + p_2^+} dq^+ \frac{1}{q^+(p_i^+ + p_2^+)(p_i^+ + p_2^-) - (q^+)^2(p_i^+ + p_2^-) - m^2(p_i^+ + p_2^-) + i\epsilon}
\]

\[
= \frac{\lambda^2}{8 \pi} \int_0^1 dy \frac{1}{y(1-y)s - m^2 + i\epsilon}.
\] (2.7)

We have introduced \(s = (p_i^+ + p_2^+)(p_i^- + p_2^-)\), \(y = \frac{q^+}{p_i^+ + p_2^+}\). An explicit evaluation leads to

\[
\text{Re } T_{fi} = -\frac{\lambda^2}{4 \pi} \frac{1}{s \sqrt{1 - 4m^2/s}} \ln \frac{1 - y_+}{y_+}
\] (2.8)

where \(y_+ = \frac{1}{2} \left[ 1 + \sqrt{1 - 4(m^2/s)} \right] \).
B. Discretized formulation

1. Periodic Boundary Condition

In order to calculate the one-loop scattering amplitude in DLCQ perturbation theory for the $\lambda/(4!)^{-1}\phi^4$ (1+1) model with periodic boundary conditions, we need to derive the light front Hamiltonian with $O(\lambda^2)$ effective interactions. However, since it was already shown \cite{14} that contributions from ZM induced effective interactions decouple in the continuum limit, we shall ignore these contributions from the very beginning. The mode expansion for the normal mode field $\phi_n(x^-)$ is

$$\phi_n(x^-) = \frac{1}{\sqrt{2L}} \sum_{k_n^+ > 0} \frac{1}{\sqrt{k_n^+}} [a_n e^{-ikx} + a_n^* e^{ikx}]. \quad (2.9)$$

Here we have used the notation $kx \equiv \frac{1}{2} k^+_n x^-$ and $k^+_n = \frac{2\pi}{L} n, n = 1, 2, \ldots \infty$.

The scattering amplitude can be calculated by the old fashioned perturbation theory formula

$$T_{fi} = \sum_j \frac{\langle p' | H_I | j \rangle \langle j | H_I | p \rangle}{p^- - p_j}, \quad (2.10)$$

where $H_I$ denotes the interacting Hamiltonian. Using the formula (2.10) with $|p\rangle \rightarrow |p^+_1, p^+_2\rangle$, $|p'\rangle \rightarrow |p^+_1, p^+_2\rangle$ and with four-particle intermediate states, one finds the following expression for the second-order normal mode scattering amplitude

$$T_{fi} = \frac{\delta_{p^+_1+p^+_2+p^+_1+1} \theta(p^+_3-p^-_1)}{(2L)^2 \sqrt{p^+_1 p^+_2 p^+_3}} \frac{1}{q^-_1 (p^+_3-p^-_1-q^-_1)} \frac{1}{p^-_1 - q^-_1 - (p^-_3 - p^-_1 - q^-_1)}, \quad (2.11)$$

plus another term with $1 \leftrightarrow 3$. The above equation must be treated with care. Due to the presence of the $\theta$-function, $p^+_1$ may approach $p^+_3$ to an arbitrary precision but not to the exact value. In DLCQ, we have,

$$t = (p^+_1 - p^+_3)(p^-_1 - p^-_3) = -m^2 \frac{(p^+_1 - p^+_3)^2}{p^+_1 p^+_3} = -m^2 \frac{(n_1 - n_3)^2}{n_1 n_3}, \quad (2.12)$$

independent of $L$. For convenience, we set $m^2 = 1.0$ and without loss of generality take $p^+_1 > p^+_3$. The scattering amplitude (we have taken out the irrelevant factor $\frac{\lambda^2}{8\pi}$) is

$$M(t) = \frac{n_1 n_3}{n_1 + n_3} \frac{1}{n_1 - n_3} \sum_{n=1}^{n_1-n_3} \left[ \frac{1}{n - n_1} - \frac{1}{n + n_3} \right]. \quad (2.13)$$

2. Anti Periodic Boundary Condition

With anti periodic boundary condition, the mode expansion for the field is
\[
\phi(x^-) = \frac{1}{\sqrt{2\pi}} \sum_{n=1,2,\ldots} \frac{1}{\sqrt{n}} \left[a_m e^{-\frac{i}{2} \sqrt{m} x^-} + a_m^\dagger e^{-\frac{i}{2} \sqrt{m} x^-}\right].
\] (2.14)

The scattering amplitude (we have taken out the irrelevant factor \(\frac{1}{8\pi}\)) in the \(t\)-channel is

\[
M(t) = 2n_1n_3 \frac{1}{n_1 + n_3 - n_1 - n_3} \sum_{n=1}^{n_1-n_3-1} \left[\frac{1}{n-n_1} - \frac{1}{n+n_3}\right].
\] (2.15)

In the discretized version, the \(s\)-channel scattering amplitude is given by

\[
M(s) = 2 \sum_{n=1}^{2n_{max}} \frac{1}{(2n-1)(\frac{1}{2n_1-1}) + \frac{1}{2n_2-1})[2n_{max} - (2n - 1)] - 2n_{max} + i\epsilon}
\] (2.16)

where \(2n_{max} = (2n_1 - 1) + (2n_2 - 1)\) and \(s = [(2n_1 - 1) + (2n_2 - 1)]\left[\frac{1}{2n_1-1} + \frac{1}{2n_2-1}\right].

C. Numerical Results

1. Periodic Boundary Condition

Let us evaluate the scattering amplitude given in Eq. (2.13) in DLCQ. Note that the minimum allowed value for \(n_1, n_3\) is 1. Thus we start from \(n_1 = 2\). In this case \(n_3 = 1\) and DLCQ gives the answer -1 for the scattering amplitude for \(t = -1/2\) which is obviously wrong. It is easy to check that for each \(n_1\), since the maximum \(n_3\) is \(n_1 - 1\), the corresponding minimum \(t\) is \(-\frac{1}{n_1(n_1-1)}\) and for this particular \(t\) DLCQ always gives the answer -1 for the scattering amplitude which is wrong for finite \(n_1\) but is correct for \(n_1 \to \infty\). The next maximum value of \(n_3\) is \(n_1 - 2\) and we denote the corresponding \(t\) by \(\tilde{t} = -\frac{4}{n_1(n_1-2)}\). In table I we present the behavior of \(M(\tilde{t})\) with \(n_1\) as \(\tilde{t} \to 0\). It is clear from Table I that DLCQ produces the correct answer which is -1 in our units, for the limit of forward scattering. Again, the limit may be approached to an arbitrary numerical precision.

For a given \(n_1\), we increase \(n_3\) by steps of 2 and study the behavior of \(M(t)\) as a function of \(t\) for small values of \(t\). The result is plotted in Fig. 2. Recall that for \(n_1 = 2, n_3 = 1, t = -1/2\) and \(M(t) = -1\). For \(n_1 = 4, n_3 = 2, t = -1/2\) and \(M(t) = -0.94\) which is close to the continuum limit (-0.92). Thus, for very small \(n_1\), with periodic boundary condition, the convergence is from below. We can see that results for very small \(n_1\) are affected by discretization but reliable results emerge already for \(n_1=10\). This is further confirmed by Fig. 3 where we present the results for \(n_1=10, 20\) and 30 and also present the continuum result given in Eq. (2.6) for comparison. In Fig. 4 we present the result for \(n_1 = 2000\) and the continuum result. It is evident that DLCQ reproduces the continuum answer for the entire range of \(t\) including the forward scattering limit \(t = 0\).

2. Anti Periodic Boundary Condition

We evaluate the scattering amplitude given in Eq. (2.15) for anti periodic boundary condition in DLCQ. For the minimum value of \(n_1 = 3, n_3 = 1, t = -4/3, M(t) = -3/4\) which is away from the continuum limit. For \(n_1 = 9, n_3 = 3, t = -4/3, M(t) = -0.81\)
which is closer to the continuum limit (−0.82). Thus for very small \( n_1 \), with anti periodic boundary condition, the convergence is from above. We can see that results for very small \( n_1 \) are affected by discretization but reliable results emerge already for \( n_1 = 9 \). The behavior of \( M(t) \) as a function of \( t \) for small values of \( n_1 \) is plotted in Fig. 5. In Fig. 6 we present the result for \( n_1 = 2001 \) and the continuum result. It is evident that DLCQ reproduces the continuum answer for the entire range of \( t \) including the forward scattering limit \( t = 0 \) also for anti periodic boundary condition.

3. \( s \)-channel scattering

We choose antiperiodic boundary condition and evaluate the real part of the \( s \)-channel scattering amplitude given in Eq. (2.19). For \( s = 10 \) and 4.2, we start from small \( n_1 \) and solve for \( n_2 \) and calculate the real part of the scattering amplitude. The real part of the amplitude converges rapidly in DLCQ to the continuum result with increasing \( n_1, n_2 \) at fixed \( s \) (which represents the continuum limit for scattering problems in DLCQ). The results for the real part of the amplitude are presented in Tables II and III where the approach to continuum limit is shown to be quicker for values of \( s \) away from the threshold value (4.0).

III. DISCUSSION, SUMMARY AND CONCLUSIONS

The question whether DLCQ can produce the correct continuum limit is nontrivial in 3+1 dimensions due to divergences, renormalization etc.. Two dimensional scalar field theory allows us to unambiguously answer this question.

It is worthwhile to contrast the calculations of mass spectra and scattering amplitudes in DLCQ. For the bound state spectra one is solving for the invariant mass for various values of \( K \) and \( K \to \infty \) gives the continuum limit. Calculations of scattering amplitudes present a different situation. For \( s \)-channel scattering we fixed \( s \), picked an \( n_1 \) and solved for \( n_2 \) and calculated the amplitude for these values of the external discretized momenta. Then we increased \( n_1 \) and solved for \( n_2 \) for the same value of \( s \). By going to larger values of \( n_1 \) we showed how the continuum limit is approached in DLCQ. For \( t \)-channel scattering we fixed \( n_1 \), and for allowed values of \( n_3 \) such that \( p_1^+ > p_3^+ \) we calculated the scattering amplitude as a function of \( t \). For increasing values of \( n_1 \) we showed how continuum limit was reached.

We have provided details of the straightforward calculations in the continuum and DLCQ versions of light front perturbation theory for the one loop scattering diagram in two dimensional scalar field theory. We have shown that the continuum limit of DLCQ produces the correct covariant limit for processes with \( p^+ = 0 \) exchange in the \( t \)-channel. It is also important to demonstrate that DLCQ can produce the absorptive part of the scattering amplitude above the particle production threshold, a subject for future research.

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\begin{table}
\begin{tabular}{|c|c|c|}
\hline
$n_1$ & $\tilde{t}$ & $M(\tilde{t})$ \\
\hline
6 & .166667 \times 10^0 & -.980000 \\
8 & .833333 \times 10^{-1} & -.989796 \\
10 & .500000 \times 10^{-1} & -.993827 \\
20 & .111111 \times 10^{-1} & -.998615 \\
30 & .476190 \times 10^{-2} & -.999405 \\
100 & .408163 \times 10^{-3} & -.999949 \\
500 & .160643 \times 10^{-4} & -.999998 \\
1000 & .400802 \times 10^{-5} & -.999999 \\
\hline
\end{tabular}
\end{table}

Table I. $M(\tilde{t})$ versus $\tilde{t}$ in DLCQ. For the definition of $\tilde{t}$, see the text. The correct answer is $-1$ in our choice of units.
Table II. Real part of $\frac{1}{8\pi} M(s)$ in DLCQ. $\epsilon = .0001$. The continuum answer is .0211985.

| $n_1$ | $n_2$ | $s$  | $\text{Re} \frac{1}{8\pi} M(s)$ |
|-------|-------|------|---------------------------------|
| 4     | 28    | 9.984| .0207590                        |
| 12    | 91    | 9.996| .0210703                        |
| 20    | 154   | 9.999| .0211225                        |
| 44    | 343   | 10.001| .0211631                       |
| 83    | 650   | 10.000| .0211807                       |
| 162   | 1272  | 10.000| .0211891                       |
| 327   | 2571  | 10.000| .0211941                       |
| 650   | 5114  | 10.000| .0211962                       |

Table III. Real part of $\frac{1}{8\pi} M(s)$ in DLCQ. $\epsilon = .0001$. The continuum answer is .03851.

| $n_1$ | $n_2$ | $s$  | $\text{Re} \frac{1}{8\pi} M(s)$ |
|-------|-------|------|---------------------------------|
| 4     | 6     | 4.208| -.00426                         |
| 13    | 20    | 4.201| .02612                          |
| 29    | 45    | 4.202| .03310                          |
| 58    | 90    | 4.199| .03580                          |
| 115   | 179   | 4.200| .03716                          |
| 230   | 358   | 4.200| .03784                          |
| 461   | 718   | 4.200| .03818                          |
| 923   | 1438  | 4.200| .03834                          |
| 1846  | 2876  | 4.200| .03843                          |
Figures

Fig. 1. $\phi^4$ scattering diagrams in old fashioned perturbation theory

Fig. 2. $M(t)$ versus $t$ in DLCQ for $n_1=2, 4, 6, 8, \text{and } 10$ plotted for small $t$. 
Fig. 3. $M(t)$ versus $\log(-t)$ in DLCQ for $n_1=10, 20$ and $30$ compared with the continuum result.

Fig. 4. $M(t)$ versus $\log(-t)$ for $n_1 = 2000$ compared with the continuum result.
Fig. 5. $M(t)$ versus $t$ in DLCQ for $n_1=3$, 5, 7, and 9 plotted for small $t$ (anti-periodic boundary condition).

Fig. 6. $M(t)$ versus $\log(-t)$ for $n_1 = 2001$ compared with the continuum result (anti-periodic boundary condition).