On optimizing low SNR wireless networks using network coding

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On optimizing low SNR wireless networks using network coding

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Abstract—The rate optimization for wireless networks with low SNR is investigated. While the capacity in the limit of disappearing SNR per degree of freedom is known to be linear in the case of fading and non-fading channels, we study the problem of operating in low SNR wireless network with given node locations that use network coding over flows. The model we develop for low SNR physically degraded broadcast channel and multiple access channel respectively operates in a non-trivial feasible rate region. We show that the problem reduces to the optimization of total network power which can be casted as standard linear multi-commodity min-cost flow program with no inherent combinatorially difficult structure when network coding is used with non integer constraints (which is a reasonable assumption). This is essentially due to the linearity of the capacity with respect to vanishing SNR which helps avoid the effect of interference in a low SNR physically degraded broadcast channel and multiple access environment respectively. We propose a fully decentralized Primal-Dual Subgradient Algorithm for achieving optimal rates on each subgraph (i.e. hyperarc) of the network to support the set of traffic demands (multicast/unicast connections).

Index Terms - Low SNR degraded broadcast channel, network coding, rate optimization, Primal-Dual Subgradient Method.

I. INTRODUCTION

Wideband fading channels have been studied since the early 1960’s. Kennedy showed that for the Rayleigh fading channel at the infinite bandwidth limit, the capacity is similar to the capacity of the infinite bandwidth AWGN channel with the same average received power [1, 2]. The robustness of this result in the case of with or without channel state information helps us to model generally the low SNR wideband wireless networks. Using it as our underlying information-theoretic model to approximate the capacity over a link, we model the general traffic (network demands) for this network and show that the linearity of vanishing SNR in SNR per degree of freedom makes for the fundamental reason for simplicity in our model. Hence, we claim, it is possible to do networking over such model with simplistic and essentially linear approach.

In the context of wideband multipath relay channel it is shown in [3] that the min-cut could be achieved using a non-coherent peaky frequency binning scheme. In our model the rate tuples belong to a non-trivial feasible region which is made of the convex hull of all tuples, i.e. capacity achieving in the limit of vanishing SNR, for subsequently defined low SNR physically degraded broadcast and multiple access channel.

The traffic model we use is quite general. It is divided into two classes: unicast and multicast (broadcast is considered as a special case of multicast). Where each pair of source and receiver group in the network form a session for a particular class of traffic. But the problem of successfully establishing multicast connections in wireline or wireless networks has been long thought to be NP-Complete using arbitrary directed and undirected network models. With the advent of network coding (ref., [4], [5], [6]), the breaking of the fluid model for data networks i.e. by performing coding over incoming packets, has been able to intrinsically circumvent the combinatorial hardness of the multicast flow problem. Later, it was shown that minimum cost setting up of multicast connections boils down to optimizing subgraph over coded packet networks [7].

In our problem, since we consider a low SNR wireless network with physically degraded Gaussian broadcast channel (where the number of hyperarcs is equal to $n$ for $n$ receiver nodes, instead of $2^n$ hyperarcs) to optimize the rates over each hyperarc (subgraph) to meet the network traffic demands (which we later show can be cast as a minimum cost multicommodity flow problem for optimizing power over each hyperarc). Also, we consider intra-session network coding for establishing traffic demand sessions.

This paper is organized as follows. Section II is composed of general problem formulation, where we define and develop our underlying information theoretic set-up. Section III consists of a proposal with decentralized solution. We present our results in section IV that support our theory. Finally, we mention concluding remarks in section IV.

II. SET-UP AND PROBLEM FORMULATION

In this section we introduce a general low SNR channel model (Gaussian) and extend it to physically degraded Gaussian broadcast channel, then addressing the interference issue in multiple access. This approach becomes our underlying model and we further develop it to a simple networking model.

A. Low SNR physically degraded Gaussian broadcast channel

Consider a general wideband fading channel where the input waveform is $x$ and the output waveform is $y$, the fading coefficient matrix is given by $h$ and $n$ is the additive white noise. The channel is given by:

$$y = \sqrt{SNR} \cdot hx + n.$$  

(1)
rate for the worst receiver can be approximated as
allocated for the better receiver, as the contribution to the total
the power limited low SNR regime, the effect of the power
the second receiver and then cancels it out from the received
low SNR physically degraded Gaussian broadcast channel, the
noise experienced by the worse receiver is negligible (ref. Fig.
(a) with receivers $d_1$ and $d_2$ (corresponding to better
and worse respectively).

The capacity of the channel, for both Gaussian channels
and fading channels increase sublinearly with the increase in
signal to noise ratio (SNR) but in the low SNR regime the
capacity in the limit is linear in SNR for fading and non-fading
channels:

$$C(SNR) = SNR + o(SNR)(nats/s/Hz).$$

Clearly at low SNR limit, the signal-to-noise ratio per degree of
freedom (SNR) approaches unity in the limit [2, 8, 9].

We consider low SNR physically degraded Gaussian broad-
channel, let’s look at the standard model of a single sender
and 2 receivers with noise variances $N_1$ and $N_2$ respectively
(ref. Fig 1(a)). The capacity region is given by:

$$r_1 < C\left(\frac{\lambda_1 P}{N_1}\right), r_2 < C\left(\frac{\lambda_2 P}{N_1 + N_2}\right).$$

where $C(x) = W(ln(1 + x))$, $\lambda_1 + \lambda_2 = 1$ and $P$ is the
total power (ref. Fig. 1(b)). As the channel in consideration
is a degraded broadcast channel, the high-resolution receiver
(in this case $r_1$) always get enough information to decode for
the second receiver and then cancels it out from the received
signal to decode information for itself.

The rate region defined in (3), when looked under the
low SNR lens comes across as a rather simpler picture. For
the power limited low SNR regime, the effect of the power
allocated for the better receiver, as the contribution to the total
noise experienced by the worse receiver is negligible (ref. Fig.
1(b), for the rate region for low SNR in the limit). So, for the
low SNR physically degraded Gaussian broadcast channel, the
rate for the worst receiver can be approximated as

$$r_2 \approx C\left(\frac{\lambda_2 P}{N_2}\right).$$

Generalizing the same idea for the case of a given source $i$ with
power $P_i$ and $n$ receiver nodes, where the receiver set $J = \{1, \ldots, n\}$ can be broken into $n$ subsets as $J^k = \{1, 2, \ldots, k\}$ for
$k \in [1, n]$. The rate region defined for each hyperarc $i, J^k$
in the low SNR limit is given as

$$r_{i, J^k} \approx C\left(\frac{(\lambda_k) P_i}{\parallel L_i - L_k \parallel^{\alpha/2} N_2}\right), \forall k \in [1, n].$$

where, $\sum_{k=1}^{n} \lambda_k \leq 1$, which when combined appropriately gives
the rate region of the set $J^k$. The equation (6) comes from the
fact that the SNR is linear in the limit of disappearing SNR per
degree of freedom, where $L_i$ for all $i \in [1, n]$ is the location of
the node and $\alpha$ is the path loss exponent. We formalize the
above mentioned concepts and motivate our next definition.

Let $\lambda_k P_i = P_{i, j^k}, \forall k$.

**Definition 1:** For a given sender $i$ with total power $P_i$ and
a receiver set $J = [1, K]$ in low SNR physically degraded
Gaussian broadcast channel, the set $J$ can be decomposed into
$K$ hyperarcs where each hyperarc is defined as the connection
from the sender $i$ to the receiver set $J^k = [1, k]$, where $k \in [1, K]$, with individual receiver rates $r_{ij^k}, \forall l \in J^k$ equal
to an associated common rate ($r_{ij^k}$) of this hyperarc. The rate
for each receiver $l \in k$ in the hyperarc $J^k$ is defined as
$$r_{ij^k} = r_{i, j^k} = \frac{P_{i, j^k}}{\parallel L_i - L_j \parallel^{\alpha/2} N_2},$$
where, $\sum_{k=1}^{n} P_{i, j^k} \leq P_i, \forall k \in J$ and
the set $J^k$ ranges from best to worst receiver (ref. Fig 1
(c)).

**B. Interference issues in multiple access at low SNR.**

Now, let’s consider the case of multiple access where more
than one node tries to access the channel at the given instance.
Let there be $U$ nodes in the system at an instance, and $u \in U$ of
them are trying to access the channel at this instance, if node $i \in u$ intends to communicate with node $j \in U$ among
others in $u$, the signal to interference and noise ratio (SINR,
denoted as $\mu_{ij}$) experienced at node $j$ is given by:

$$\mu_{ij} = \frac{\frac{P_i}{\parallel L_i - L_j \parallel^{\alpha/2} N_0}}{2N_0 + \sum_{v \in u, v \neq i} \frac{P_v}{\parallel L_i - L_j \parallel^{\alpha/2} N_0}}.$$ (7)

Note that, since every node in $u$ is interested only in a common
receiver, we allocate the whole power of the node over this
single hyperarc, so $k = 1$ and $P_{i, j^1} = P_i$ for every transmitter.
But as we are operating in the low SNR regime, the intuition
suggests that the effect of the interference should be negligible.
We straightforwardly include it in our assumption, thus we
define the rate (denoted with $R$) experienced at the receiver $j$ as:

$$R_{ij} = W(ln\left(1 + \frac{\frac{P_i}{\parallel L_i - L_j \parallel^{\alpha/2} N_0}}{2N_0 + \sum_{v \in u, v \neq i} \frac{P_v}{\parallel L_i - L_j \parallel^{\alpha/2} N_0}}\right))$$ (8)

$$\approx W(ln\left(1 + \frac{P_i}{\parallel L_i - L_j \parallel^{\alpha/2} N_0}\right))$$ (9)

$$\approx W\left(\frac{P_i}{\parallel L_i - L_j \parallel^{\alpha/2} N_0}\right).$$ (10)

The approximation (9) comes from the fact that the contri-
bution of other signals being transmitted from other sources
in the system with low SNR channel to the interference is
negligible and the approximation (10) comes from the linearity
of SINR in the limit of disappearing SNR per degree of freedom
(ref. Fig 2(a) and 2(c)). In Fig. 2(b), we can see that the SINR
curve approaches the capacity curve in the limit, corroborating

![Diagram](image_url)

Fig. 1. (a): Two receiver physically degraded Gaussian broadcast channel with $Z_1 \sim N(0, N_1)$ and $Z_2 \sim N(0, N_2 - N_1)$. (b): Rate region for the channel in (a), dotted line denotes the flatness of the rate region due to the limit of vanishing SNR with $C_1$ and $C_2$ as max rates for each receiver respectively. (c): Decomposition into hyperarcs $\{(s, (d_1)), (s, (d_1, d_2))\}$ with their common rates for the case in (a) with receivers $d_1$ and $d_2$ (corresponding to better and worse respectively).
∥L − Lki∥α/2N0 = γiJkiPki.

Then, the minimum cost optimization problem for the low SNR network can be formulated as:

\[
\min \sum_{(i,J) ki} P_{ki} x_{ki}
\]

subject to:

\[
y_{ki}(m) \geq \max_{t m} (x_{ki}^m (m)), \forall (i, J) ki \in A, \forall m
\]

\[
z_{ki} \leq \gamma_{ki} P_{ki}, \forall (i, J) ki \in A
\]

\[
K_i \sum_{k_i = 1} P_{ki} \leq P, \forall i \in N.
\]

where \( P_i \) is given \( \forall \) \( i \), \( x_{ki}^m (m) \in F_{ki}^m (m), \) and \( F_{ki}^m (m) \) a bounded polyhedron made of flow conservation constraints:

\[
\sum_{(i,J) ki} x_{ki}^m (m) = \sum_{(i,J) ki} x_{ki}^m (m) = s_i (m),
\]

\[\forall i \in N, \forall t m, \forall m\]

\[
x_{ki}^m (m) = \sum_{(i,J) ki} x_{ki}^m (m),
\]

\[\forall (i, J) ki \in A, \forall t m, \forall m\]

As opposed to standard multicommodity flow problem in which flows are simply added over a link, the constraint (12) in fact catches the essence of network coding by taking only the maximum among all the flows of a session (note that we only consider intra-session network coding). Since \( F_{ki}^m (m) \) is the polyhedron formed by the law of flow conservation, constraint (17) translates the flow conservation laws from the underlying directed graph \( A' \) to the hypergraph \( A \) (the wireless network) by adding the flows on all hyperarcs between node \( i \) and \( J_{ki} \) i.e. flow in \( (i, J_{ki}) \) \( \in A' \) is the sum of all the flows on the hyperarcs \( (i, J_{ki}), \forall J_{ki} \geq J_j \).
As we can see, the above mentioned problem is a convex optimization problem. The only nonlinear constraint is (12), and could be readily replaced by the set of linear inequality constraints \( y_{ij}(m) \geq \left(x_{ij}(m)\right)^T A_{ij}(m) \forall t_m \in [1, T_m] \). The modified problem results in a standard linear multicommodity flow problem with linear objective and linear constraint set.

\[
\text{minimize} \sum_{(i,j,k) \in A} P_{kj}^i \quad \text{(B)}
\]

subject to:

\[
y_{ij}(m) \geq \left(x_{ij}(m)\right)^T A_{ij}(m), \forall t_m, \forall m, \forall (i,j,k) \in A \tag{19}
\]

\[
z_{ij} \leq \sum_{m=1}^{M} y_{ij}(m), \forall (i,j,k) \in A \tag{20}
\]

\[
z_{ij} \leq \gamma_{ij} \cdot P_{kj}^i, \forall (i,j,k) \in A \tag{21}
\]

\[
\sum_{k_i=1}^{K_i} P_{kj}^i \leq P_t, \forall i \in N. \tag{22}
\]

where \( x_{ij}(m) \in F_{ij}(m) \), and \( F_{ij}(m) \) is a bounded polyhedron made of flow conservation constraints. Note that we optimize the power over each hyperarc, to determine the optimal rates for each hyperarc that satisfies the network demands, we simply need to multiply the optimal power with \( \gamma_{ij} \). We will prefer to solve the problem by proposing a decentralized algorithm for generally understood and appreciated reasons.

**III. DECENTRALIZED ALGORITHM**

For developing a decentralized solution for problem (B) we need to understand the structure of the primal problem first and then to be able to solve each of its subproblems. The advantage of this approach is that at each iteration we get a pair of points \((p_d, \lambda, \mu, \nu, \xi)\), there exist a vector \( \{x_{ij}(m)\} \) for which the inequality can be strict.

Let us represent the set of primal vectors as \( p = \{x, y, z, P\} \in S_1 \) where \( S_1 \) is the feasible set for the primal problem, and similarly we can do it for the dual problem, \( d = \{\lambda, \nu, \mu, \xi, \zeta\} \in S_2 \). As we can see that the primal and dual optimal are equal (thanks to strong duality), we can express our problem in the standard saddle point form \( \max \min \phi(p, d) = \min \max \phi(p, d) \), where function \( \phi(p, d) \) is the Lagrangian dual problem of the problem (B). This implies that for (C), we get the hyperarc separable saddle-point form

\[
\max_{d \in S_2} q_{ij}(d) = \min_{p \in S_1} \max_{d \in S_2} \phi(p, d). \tag{25}
\]

Now we are in the position where we can solve the problem, separable in hyperarchs using any saddle-point optimization method for non-smooth functions. For our problem setup, we propose a Primal-Dual Subgradient Algorithm by Nesterov for nonsmooth optimization [13]. Nesterov’s method generates a subgradient scheme intelligently based on Dual-Averaging method which beats the lower case complexity bound for any black-box subgradient scheme. The algorithm works in both primal and dual spaces, generating a sequence of feasible points, and ultimately squeezing the duality gap to zero by finally approaching the optimal solution. A positive consequence of the Primal-Dual approach is that at each iteration we get a pair of points \((p, d)\) which are primal and dual feasible, hence, we get the primal feasible solution with essentially no extra effort. As opposed to many subgradient type methods where there needs to be a method for primal recovery, specially for large and ill-posed problems.

**A. Primal-Dual Subgradient Algorithm.**

Since the dual function is hyperarc separable, we can optimize the power over each hyperarc separately and add each of the optimal solutions to construct the optimal solution of the dual problem (C), ultimately achieving the primal optimal solution for problem (B). The algorithm is as follows:

1) Initialization: Set \( s_0 = 0 \in Q \). Choose \( \theta > 0 \).

2) Iteration \((k \geq 0)\):

\[
\text{maximize} \sum_{(i,j,k) \in A} q_{ij}(d) = \min_{p \in S_1} \max_{d \in S_2} \phi(p, d).
\]
where \((g_k, g_d)\) is the set of primal and dual subgradients and \(\sigma_k, s_k\) and \(S_k\) are aggregated sequence of points.

IV. SIMULATIONS

We now show the results of our simulations that support the claims of the algorithm presented. We solved the dual problem in a decentralized way by solving it for every hyperarc separately and then adding up the respective solutions to construct the dual optimal solution of the problem \((C)\), which when optimal is the primal optimal solution for problem \((A)\) in our case.

The setup consists of uniformly placed nodes on a chosen area of \(a \times a m^2\), with given node locations. We start our simulations with smaller networks of only 4 nodes on a \(10 \times 10 m^2\) area with the area size increasing as the number of nodes in the network increase to keep the node density/area in a controlled range. Each node has a single hyperarc and it can communicate with all the nodes in the network, this is just a simple generalization of our case where a node can communicate with only a subset of total nodes in the network. For each network we randomly choose a set of \(m\) multicast sessions and \(T_m\) set of receivers for each session respectively with the required rate demand associated with each session that need to be established, but making sure the the traffic demands are \(\leq\) the respective min-cut for each session to make the problem feasible.

In Figure 3, we compare the optimal solution approximations of the Primal-Dual Subgradient Method for problem \((C)\) with the standard infeasible path following method for problem \((B)\). It can be seen that the our proposed algorithm gives close approximations of the primal solution of the problem \((B)\). Note that the path following method is directly applied to the primal problem and the Primal-Dual subgradient method is applied to the dual problem, to compute the dual solution of the problem \((C)\), which will be give us the close approximation to the primal solution of problem \((A)\).

V. CONCLUSION

We develop an efficient optimization model that provides an achievable rate region. And we do this by showing that rate optimization for the Low SNR physically degraded broadcast wireless network can be formulated as a standard linear multicommodity flow problem for optimizing power over each hyperarc using network coding. Our model is relieved from interference related issues, this is due to the fact that the capacity of the low SNR wideband channel is essentially linear in SNR per degree of freedom for vanishing SNR in the limit, which relieves the system from interference and related issues. Our model operates in the non-trivial feasible rate region that achieves capacity in the limit of disappearing SNR with appropriate encoding scheme.

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