Degree of entanglement for two qubits

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In this paper, we present a measure to quantify the degree of entanglement for two qubits in a pure state.

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I. INTRODUCTION

Quantum entanglement is the most surprising nonclassical property of composite quantum systems [1]. As it is well-known, a qubit (or a spin-1/2 particle) is described by the 2 × 2 density matrix \( \rho(n) = (1 + \vec{\sigma} \cdot \vec{n})/2 \), where \( \vec{1} \) is the unit matrix, \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) the Pauli matrices vector, and \( \vec{n} \) the Bloch vector. \( |\vec{n}| = 1 \) corresponds to a pure state, otherwise a mixed state. Whereas, an entangled pairs of two qubits is completely described by the following 4 × 4 density matrix:

\[
\rho_{AB} = \frac{1}{4}(\vec{1} \otimes \vec{1} + \vec{\sigma}^A \cdot \vec{u} \otimes \vec{1} + \vec{1} \otimes \vec{\sigma}^B \cdot \vec{v} + \sum_{i,j=1}^3 \beta_{ij} \sigma_i^A \otimes \sigma_j^B),
\]

from which one could obtain two reduced density matrices

\[
\rho_A = \text{tr}_B(\rho_{AB}) = \frac{1}{2}(1 + \vec{\sigma}^A \cdot \vec{u}),
\]

\[
\rho_B = \text{tr}_A(\rho_{AB}) = \frac{1}{2}(1 + \vec{\sigma}^B \cdot \vec{v}),
\]

for the two qubits A and B, where \( \vec{u} \) and \( \vec{v} \) are Bloch vectors for particles A and B, respectively; \( \beta_{ij} \) are some real numbers.

It has been shown that entangled pairs are a more powerful resource than separable, i.e., disentangled, pairs in a number of applications, such as quantum cryptography [2], dense coding [3], teleportation [4] and investigations of quantum channels [5], communication protocols and computation [6] [7]. The superior potentiality of entangled states has raised a natural question: “How much are two particles entangled?”, since pairs with a high degree of entanglement should be a better resource than less entangled ones. Many measures of entanglement proposed in the past have relied on either the Schmidt decomposition [8] or decomposition in a magic basis [9]. In an interesting paper, Abouraddy et al. devised a new measure of entanglement for pure bipartite states of two qubits, based on a decomposition of the state vector as a superposition of a maximally entangled state vector and an orthogonal factorizable one [10]. Although there are many such decompositions, the weights of the two superposed states are remarkably unique. The square of the weight of the maximally entangled state vector (i.e., \( P_E = p^2 \)) is then defined as the degree of

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entanglement for two qubits, such a measure is consistent with three measures of entanglement previously set forth: maximal violation of Bell’s inequality [11], concurrence [9] and two-particle visibility [12].

The purpose of this paper is to propose a new approach to the problem of defining the degree of entanglement for two qubits in a pure state. In Sec. II, a new measure is formulated to quantify the degree of entanglement. Some examples are given in Sec. III. Conclusion and discussion are made in the last section.

II. FORMALISM

Theorem: If $\rho_{AB}$ is a pure state, then its degree of entanglement $P_E$ is equal to

$$P_E = (-\det(\hat{\alpha}))^{1/4} \quad (3)$$

where the matrix $\hat{\alpha}$ is

$$\hat{\alpha} = \begin{pmatrix}
1 & v_1 & v_2 & v_3 \\
u_1 & \beta_{11} & \beta_{12} & \beta_{13} \\
u_2 & \beta_{21} & \beta_{22} & \beta_{23} \\
u_3 & \beta_{31} & \beta_{32} & \beta_{33}
\end{pmatrix}. \quad (4)$$

Proof: $\rho_{AB}$ is a pure state implies that $\rho_{AB}^2 = \rho_{AB}$, from which one obtains the following constraints among $u_i$, $v_i$ and $\beta_{ij}$ ($i, j = 1, 2, 3$):

$$u_i = \beta_{i1}v_1 + \beta_{i2}v_2 + \beta_{i3}v_3, \quad (5)$$

$$v_i = \beta_{i1}u_1 + \beta_{i2}u_2 + \beta_{i3}u_3, \quad (6)$$

$$\sum_{ij} \beta_{ij}^2 = 3 - |u|^2 - |v|^2, \quad (7)$$

$$\beta_{ij} = u_iv_j - (-1)^{i+j}M_{ij}, \quad (8)$$

where $M_{ij}$ is the algebraic complement of the matrix element $\beta_{ij}$ for the following $\hat{\beta}$ matrix:

$$\hat{\beta} = \begin{pmatrix}
\beta_{11} & \beta_{12} & \beta_{13} \\
\beta_{21} & \beta_{22} & \beta_{23} \\
\beta_{31} & \beta_{32} & \beta_{33}
\end{pmatrix}. \quad (9)$$

Eqs. (5) and (6) can be recast as $\hat{\beta}v = u$, $\hat{\beta}^Tu = v$, where $T$ represents transpose and $u = (u_1, u_2, u_3)^T$. An interesting result, i.e., $|u| = |v|$, will be obtained immediately from Eqs. (5) and (6) for the pure state $\rho_{AB}$ [13]. From Eq. (8) we have

$$\beta_{11}^2 + \beta_{12}^2 + \beta_{13}^2 = \beta_{11}u_1v_1 + \beta_{12}u_1v_2 + \beta_{13}u_1v_3$$

$$- [(-1)^{i-1}\beta_{11}M_{11} + (-1)^{i+2}\beta_{12}M_{12} + (-1)^{i+3}\beta_{13}M_{13}]. \quad (10)$$

Due to $\det(\hat{\beta}) = \beta_{11}M_{11} - \beta_{12}M_{12} + \beta_{13}M_{13}$ and Eq. (10), one obtains

$$\beta_{11}^2 + \beta_{12}^2 + \beta_{13}^2 - u_1^2 = -\det(\hat{\beta}). \quad (11)$$

Similarly,
\[
\beta_{21}^2 + \beta_{22}^2 + \beta_{23}^2 - u_2^2 = -\det \hat{\beta},
\]
\[
\beta_{31}^2 + \beta_{32}^2 + \beta_{33}^2 - u_3^2 = -\det \hat{\beta}. 
\tag{12}
\]

After combining Eqs. (7), (11), (12), and taking \(|u| = |v|\) into account, one easily obtains \(-\det \hat{\beta} = 1 - |u|^2\). Consequently, we have
\[
(-\det \hat{\alpha})^{1/4} = \left(\frac{1}{2}\right)^{1/4} = \sqrt{1 - |u|^2}. \tag{13}
\]

One can know from Ref. [10] that \(P_E = 2\kappa_1\kappa_2\), where \(\kappa_1\) and \(\kappa_2\) are the two coefficients in the Schmidt decomposition \(|\Psi\rangle = \kappa_1|x_1, y_1\rangle + \kappa_2|x_2, y_2\rangle\), \(\rho_{AB} = |\Psi\rangle\langle\Psi|\), where \(|x_1\rangle, |x_2\rangle\) and \(|y_1\rangle, |y_2\rangle\) are orthogonal bases for the Hilbert spaces of particles \(A\) and \(B\), respectively. It is easy to prove that \(\kappa_1 = \sqrt{(1 + |u|)/2}, \kappa_2 = \sqrt{(1 - |u|)/2}\), which are square-roots of the two eigenvalues of the reduced matrix \(\rho_A\) or \(\rho_B\). Therefore we have \(P_E = (-\det \hat{\alpha})^{1/4}\). This ends the proof.

III. EXAMPLES

Example 1. For the state \(|\Psi\rangle = (|00\rangle + |01\rangle + |11\rangle)/\sqrt{3}\), one obtains the density matrix
\[
\rho_{AB} = |\Psi\rangle\langle\Psi| = \frac{1}{3} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},
\]
with the Bloch vectors \(u = (2/3, 0, 1/3)^T\), \(v = (2/3, 0, -1/3)^T\), and the alpha matrix
\[
\hat{\alpha} = \frac{1}{3} \begin{pmatrix} 3 & 2 & 0 & -1 \\ 2 & 2 & 0 & -2 \\ 0 & 0 & -2 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}. \tag{14}
\]
One can have \(P_E = 2/3\), which is consistent with the result in Ref. [11].

Example 2. For the state \(|\Psi\rangle = (|00\rangle + 2(|01\rangle + |11\rangle))/3\), the density matrix is
\[
\rho_{AB} = |\Psi\rangle\langle\Psi| = \frac{1}{9} \begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 4 \end{pmatrix},
\]
with \(u = (8/9, 0, 1/9)^T\), \(v = (4/9, 0, -7/9)^T\), and the alpha matrix
\[
\hat{\alpha} = \frac{1}{9} \begin{pmatrix} 9 & 4 & 0 & -7 \\ 8 & 4 & 0 & -8 \\ 0 & 0 & -4 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}. \tag{15}
\]
Hence the degree of entanglement is \(P_E = 4/9\).

Example 3. For the maximally entangled state \(|\Psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}\), one obtains the density matrix
\[
\rho_{AB} = |\Psi\rangle\langle\Psi| = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.
\]
with the Bloch vectors \( \mathbf{u} = \mathbf{v} = (0, 0, 0)^T \), and the alpha matrix

\[
\hat{\alpha} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(16)

Thus \( P_E = 1 \) reaches the highest value.

**Example 4.** For the disentangled pure state

\[
\rho_{AB} = \frac{1}{2}(1 + \sigma^A \cdot \mathbf{u}) \otimes \frac{1}{2}(1 + \sigma^B \cdot \mathbf{v}),
\]

where \( |\mathbf{u}| = |\mathbf{v}| = 1 \), we have the alpha matrix as

\[
\hat{\alpha} = \begin{pmatrix}
1 & v_1 & v_2 & v_3 \\
u_1 & u_1 v_1 & u_1 v_2 & u_1 v_3 \\
u_2 & u_2 v_1 & u_2 v_2 & u_2 v_3 \\
u_3 & u_3 v_1 & u_3 v_2 & u_3 v_3
\end{pmatrix}.
\]

(17)

Obviously \( P_E = 0 \) indicates that \( \rho_{AB} \) is disentangled.

**IV. CONCLUSION AND DISCUSSION**

In conclusion, we have presented a measure to quantify the degree of entanglement for two qubits in a pure state. We would like to make some discussion in the following:

(i) The similar idea developed in this paper could be generalized to quantify the degree of entanglement for two quNits (i.e., \( N \)-state quantum systems, \( N = 2 \) and \( N = 3 \) correspond to a qubit and a qutrit, respectively) \[14\] \[15\] in a pure state. For instance, the density matrix for two entangled qutrits could be written as

\[
\rho_{AB} = \frac{1}{9}(1 \otimes 1 + \sqrt{3} \lambda^A \cdot \mathbf{u} \otimes 1 + \sqrt{3} \lambda^B \cdot \mathbf{v} + \frac{3}{2} \sum_{i,j=1}^{8} \beta_{ij} \lambda^A_i \otimes \lambda^B_j),
\]

(18)

where \( \lambda_i \) \( (i = 1, 2, ..., 8) \) are the eight hermitian generators of \( SU(3) \) (namely the usual Gellmann matrices). For the state of two entangled qutrits

\[
|\Psi\rangle = \frac{1}{3}(|00\rangle + |11\rangle + |22\rangle),
\]

(19)

its corresponding density matrix is \[13\]

\[
\rho_{AB} = \frac{1}{9}(1 \otimes 1 + \frac{3}{2} \sum_{i,j=1}^{8} \beta_{ij} \lambda^A_i \otimes \lambda^B_j),
\]

(20)

with the non-zero coefficients \( \beta_{11} = \beta_{22} = \beta_{33} = \beta_{44} = \beta_{55} = \beta_{66} = 1, \beta_{22} = \beta_{55} = \beta_{77} = -1 \). The elements \( \beta_{ij}, 1, \mathbf{u} \) and \( \mathbf{v} \) form a \( 9 \times 9 \) matrix \( \hat{\alpha} \), it is easy to show that \( P_E = (-\det \hat{\alpha})^{1/4} = 1 \), which indicates that the state \(|\Psi\rangle\) in Eq. (19) is just a maximally entangled state.

(ii) After making the parametrization \( \mathbf{u} = \hat{\mathbf{u}} \tanh \phi_u \), where \( \hat{\mathbf{u}} = \mathbf{u}/|\mathbf{u}| \), the density matrix of a qubit \( \rho(\mathbf{u}) = (1 + \hat{\sigma} \cdot \mathbf{u})/2 \) can be connected to the Lorentz boost matrix \( L(\mathbf{u}) = \exp(\phi_u \hat{\sigma} \cdot \hat{\mathbf{u}}/2) = 1 \cosh(\phi_u/2) + \hat{\sigma} \cdot \mathbf{u} \sinh(\phi_u/2) \) as \[16\]

\[
\rho(\mathbf{u}) = \frac{L(\mathbf{u})}{2 \cosh \phi_u}, \quad \phi_u = \phi_u/2.
\]

(21)

Obviously, \( \rho(\mathbf{u}) \) and \( L(\mathbf{u}) \) are in one-to-one correspondence. For the former, the physical meaning of the vector \( \mathbf{u} \) is the Bloch vector in quantum mechanics, while for the latter the relativistic velocity. Due
to the rapidity $\varphi$, i.e., the hyperbolic angle, special relativity can be formulated in terms of hyperbolic geometry. As a result, some physical quantities have been found to have geometric significance, such as the Thomas rotation angle corresponds to the defect of a hyperbolic triangle [17] [18]. After viewing the Bloch vector $u$ as an analogous relativistic velocity, the Bures fidelity $F(\rho_1, \rho_2) = \left| \text{tr} \left( \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \right) \right|^2$ was found to have a geometric interpretation in the framework of hyperbolic geometry [10]. Similarly, with the aid of the parametrization $u = \hat{u} \tanh \varphi u$, it is not difficult to find that the entanglement degree $P_E = \sqrt{1 - |u|^2} = 1 / \cosh \varphi u$ for two qubits in a pure state is the reciprocal of the Lorentz factor in the hyperbolic geometry. The extension of our approach to the mixed states of two entangled qubits will be discussed elsewhere.

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