Cosmological perturbation spectra from SL(4,R)-invariant effective actions

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We investigate four-dimensional cosmological vacuum solutions derived from an effective action invariant under global SL(n,R) transformations. We find the general solutions for linear axion field perturbations about homogeneous dilaton-moduli-vacuum solutions for an SL(4,R)-invariant action and find the spectrum of super-horizon perturbations resulting from vacuum fluctuations in a pre-big-bang scenario. We show that for SL(n,R)-invariant actions with \( n \geq 4 \) there exists a regime of parameter space of non-zero measure where all the axion field spectra have positive spectral tilt, as required if light axion fields are to provide a seed for anisotropies in the microwave background and large-scale structure in the universe.

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I. INTRODUCTION

The pre-big-bang scenario proposed by Gasperini and Veneziano is an alternative model for the very early evolution of our Universe which assumes that its initial state was a low-energy, weakly coupled, vacuum state. Such a regime is well described by the low-energy string effective action which admits two separate branches, labelled (+) and (−), for vacuum solutions of the scale factor in four-dimensional Friedmann-Robertson-Walker (FRW) cosmologies. The (+) branch corresponds to a weakly coupled dilaton in a cold, flat universe in the \( t \rightarrow -\infty \) limit. For \( t \rightarrow 0 \) we have pole driven super-inflation propelled by the dilaton kinetic energy term, with a positive Hubble parameter, \( H \) and a singularity in the future. The (−) branch corresponds to a large spatially flat universe with positive \( H \), but decelerating and can be smoothly joined to a conventional radiation dominated universe at late times. The (+) and (−) branches are related to each other by a string symmetry called scale factor duality, but there is still no compelling dynamical model of the “graceful exit” from the (+) to the (−) branch.

In the absence of a complete theoretical understanding one may still hope to find observational evidence, such as the spectrum of primordial fluctuations that could be generated during the dilaton-driven pre-big-bang phase, in order to test the scenario against more conventional inflation models. Metric perturbations are produced on super-horizon scales during the pre big bang phase have a steep “blue” spectrum, strongly tilted towards small scales. This offers the interesting possibility that there might be a detectable background of relic gravitons on Laser Interferometric Gravitational wave Observatory (LIGO) scales and a related population of primordial black holes. However these metric perturbations are far from the almost scale-invariant (Harrison-Zel’dovich) spectrum of adiabatic density perturbations naturally produced by conventional slow-roll inflation models and leave effectively no metric perturbations on astrophysical scales during the pre-big-bang era.

Instead it has been proposed that a cosmic background of massless axion fluctuations could generate the observed anisotropies in the cosmic microwave background temperature at large angular scales and provide a seed for large-scale structure formation. A pre-big-bang era can produce almost scale invariant spectra of fluctuations in axion fields \( \delta a^2 \propto k^{\Delta n} \) with \( \Delta n \approx 0 \), where \( k \) is the comoving wavenumber. These are isocurvature perturbations to first-order, but assuming the axion field remains effectively massless in the subsequent post big bang era, these fluctuations give rise to a spectrum of density perturbations at horizon crossing \( \Delta n \approx 0.1 \), may be consistent with \( \delta \rho / \rho \sim 10^{-5} \) on astrophysical scales \( k \sim 10^{30} k_s \) for a present-day string coupling \( g_s^2 = e^\phi \sim 10^{-2} \).

The low energy string effective action compactified down to four-dimensions includes a dilaton and axion field, related by an SL(2,R) symmetry. In the absence of all the other moduli fields are fixed during the pre-big-bang phase then it is found that for an SL(2,R) action, with one axion field, the spectral index is fixed to be \( \Delta n = -2\sqrt{2} + 3 = -0.46 \). The addition of a single moduli field gives a spectral index for the axion in the range \(-0.46 \leq \Delta n \leq 3 \) which allows scale-invariant or blue spectra.
The many moduli fields present in any low-energy effective action will have specific symmetry properties inherited from the higher dimensional theory and the details of compactification. For instance, the inclusion of Ramond-Ramond (RR) fields presents in the type II string theories increases the number of degrees of freedom in the four-dimensional effective theory and in Ref. [7] it was shown that the RR 1-form and 3-form field strengths, with a single modulus field determining the size of the 6-torus, combine with the Neveu-Schwarz-Neveu-Schwarz (NS-NS) dilaton and axion to parameterise an SL(3,R)-invariant non-linear sigma model. The symmetries of this action can place constraints on the allowed spectral indices. For an SL(3,R) action [8], with two moduli but three axion fields, the range for each spectral index was the same as for the single axion case \((-0.46 \leq \Delta n_i \leq 3, \ i = 1, 2, 3)\), but there was no point at which all the spectra had \(\Delta n_i > 0\). This poses a threat to the pre-big-bang scenario as all the perturbation spectra have the same normalisation at the string scale \((\delta \rho/\rho \sim e^\phi)\) and if one axion field always has a red spectrum \((\Delta n_i < 0)\) then there would be unacceptably large density fluctuations on large scales.

In order to study whether this remains a problem in larger symmetry groups with more moduli and axion degrees of freedom we will study the spectrum of cosmological perturbations generated in fields which parameterise an SL(4,R) non-linear sigma model in the low-energy effective action. The presence of a global SL(n,R) symmetry is a completely general consequence of dimensional reduction from \(D + n\) to \(D\) dimensions [9,10]. We will investigate the perturbation spectra generated in axion fields at late times or large scales from vacuum fluctuations at early times or small scales in a pre-big-bang era, and discuss whether these might be compatible with an almost scale invariant spectrum of small primordial density perturbations. We will show that perturbation spectra with \(\Delta n > 0\) for all fields are indeed possible in models whose scalar fields parameterise an SL(n,R) group where \(n \geq 4\).

II. SL(N,R) INVARIANT ACTIONS

We begin with a discussion of the representation of the most familiar SL(2,R)-invariant effective action is string theory, and how we can extend this to provide a representation of larger SL(n,R)-invariant effective actions.

The NS-NS sector of string theory contains the dilaton, \(\phi\), graviton, \(G_{AB}\), and 2-form potential, \(B_{AB}\), and is common to both heterotic and type II string theories. The low energy effective action is

$$S = \frac{1}{16\pi \alpha'^4} \int d^{10}x \sqrt{|G|} e^{-\phi}\left(\mathcal{R}_{10} + (\nabla \phi)^2 - \frac{1}{12} \hat{H}^2\right),$$

(2)

where \(\alpha'\) is the inverse string tension and \(\hat{H} = dB\) is a 3-form field strength. Considering the simplest Kaluza-Klein compactification on a static six-torus, the ten-dimensional line element has the form

$$ds_{10}^2 = e^\phi g_{\mu\nu} dx^\mu dx^\nu + \delta_{ab} dy^a dy^b,$$

where \(g_{\mu\nu}\) is the four-dimensional metric in the Einstein frame and \(\delta_{ab}\) is the Kronecker delta-function. In four-dimensions the 3-form field strength is dual to a one-form written as the gradient of a pseudo-scalar axion field, \(\phi = e^\phi \nabla \sigma\), and the four-dimensional effective action is then [7]

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} e^{2\phi} (\nabla \sigma)^2\right],$$

(3)

where \(\kappa^2 \equiv 8\pi G\). Solutions to the equations of motion from this action respect the invariance of this action under an arbitrary global SL(2,R) transformation [20]

$$\lambda \rightarrow \alpha\lambda + \beta \gamma \lambda + \delta,$$

(4)

where \(\lambda = \sigma + ie^{-\phi}\) and the real parameters \(\alpha, \beta, \gamma\) and \(\delta\) obey the constraint \(\alpha \delta - \beta \gamma = 1\). In order to extend the effective action to larger symmetry groups it is convenient to re-write the action Eq. (3) in terms of the symmetric matrix

$$M = \begin{pmatrix} e^\phi & \sigma e^\phi \\ \sigma e^\phi & e^{-\phi} + \sigma^2 e^\phi \end{pmatrix},$$

(5)

which parameterises the SL(2,R)/U(1) maximal coset of SL(2,R). The SL(2,R) transformation in Eq. (4) is given by \(M \rightarrow \Theta M \Theta^T\) where

$$\Theta = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}.$$  

Any member of SL(2,R) obeys the relation \(M^T J M = J\) where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$  

and hence, using the expression \(\Theta^T J \Theta = J\) we can show that the line element

$$dS^2 = \frac{1}{2} Tr(JdMDJdM) = d\phi^2 + e^{2\phi} d\sigma^2$$

(6)

is invariant under the global SL(2,R) transformation \(M \rightarrow \Theta M \Theta^T\). Thus the effective action Eq. (3) written in the form

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R + \frac{1}{4} \text{Tr} (\nabla M \nabla M^{-1})\right]$$

(7)

is manifestly invariant under global SL(2,R) transformations. More generally the action for scalar fields parameterising an SL(n,R)/SO(n) non-linear sigma model can be written as
where $U_n$ is a symmetric SL($n,R$) matrix. This action is invariant under the global transformation $U_n \rightarrow \Theta U_n \Theta^T$ where $\Theta$ is a member of SL($n,R$).

We can build a symmetric SL(3,R) matrix $U$, from the SL(2,R) matrix $M$ in the following way [21,17]:

$$U = \left( \begin{array}{ccc} e^\nu M & e^\nu M \sigma \\ e^\nu \sigma^T M & e^{-2\nu} + e^\nu \sigma^T M \sigma \end{array} \right),$$

where $\nu$ is a modulus field and the two additional degrees of freedom are the components of the $2 \times 1$ vector

$$\sigma = \left( \frac{\sigma_2}{\sigma_3} \right).$$

The SL(3,R)-invariant trace of the $3 \times 3$ matrix $\nabla U \nabla U^{-1}$ which appears in the effective action is

$$\text{Tr} \left( \nabla U \nabla U^{-1} \right) = \text{Tr} \left( \nabla M \nabla M^{-1} \right) - 6 (\nabla \nu)^2 - 2 \text{Tr} \left( e^{3\nu} \nabla \sigma^T M \nabla \sigma \right).$$

More generally, the same method can be used to construct an SL($n+1,R$) matrix $U_{n+1}$ from an SL($n,R$) matrix $U_n$, where

$$U_{n+1} = \left( \begin{array}{ccc} e^{v_{n+1}} U_n & e^{v_{n+1}} U_n \sigma_n \\ e^{v_{n+1}} \sigma_n^T U_n & e^{-n v_{n+1}} + e^{v_{n+1}} \sigma_n^T U_n \sigma_n \end{array} \right).$$

In addition to the fields in the SL($n,R$) matrix $U_n$, the SL($n+1,R$) matrix $U_{n+1}$ includes an additional modulus field $v_{n+1}$ and $n$ additional axion fields contained in the $n \times 1$ vector $\sigma_n$. This is sufficient to define our representation of the SL($n,R$)/SO($n$) coset starting from $U_1 = 1$. Thus the SL($n,R$) matrix $U_n$ contains $n-1$ moduli and $n(n-1)/2$ axions in total.

The SL($n+1,R$)-invariant trace of the $(n+1) \times (n+1)$ matrix $\nabla U_{n+1} \nabla U_{n+1}^{-1}$ which appears in the effective action [see Eq. (8)] can be calculated iteratively as

$$\text{Tr} \left( \nabla U_{n+1} \nabla U_{n+1}^{-1} \right) = \text{Tr} \left( \nabla U_n \nabla U_n^{-1} \right) + (-n^2 - n) (\nabla v_{n+1})^2 - 2 \text{Tr} \left( e^{(n+1) v_{n+1}} \nabla \sigma_n^T U_n \nabla \sigma_n \right).$$

III. SL(4,R) DILATON-MODULI-VACUUM COSMOLOGIES

We first investigate the homogeneous dilaton-moduli vacuum solutions where we set $\sigma_i =$constant. The form of the dilaton-moduli solutions are invariant under a constant shift of the axion fields, so without loss of generality we set $\sigma_i = 0$. This amounts to setting to zero the off-diagonal terms in the SL($n,R$) matrix. Thus we have the vacuum SL(2,R) matrix (putting $\phi = v_2$)

$$U_2^{(0)} = \left( \begin{array}{cc} e^{v_2} & 0 \\ 0 & e^{-v_2} \end{array} \right)$$

and the SL(3,R) matrix given in Eq. (9) becomes

$$U_3^{(0)} = \left( \begin{array}{ccc} e^{v_3+v_2} & 0 & 0 \\ 0 & e^{-v_2} & 0 \\ 0 & 0 & e^{-2v_3} \end{array} \right)$$

while, from Eq. (12), an SL($n+1,R$) matrix is given in terms of an SL($n,R$) matrix as

$$U_{n+1}^{(0)} = \left( \begin{array}{ccc} e^{v_{n+1}} U_n^{(0)} & 0 \\ 0 & e^{-n v_{n+1}} \end{array} \right).$$

The trace of $\nabla U_n^{(0)} \nabla U_n^{(0)-1}$ for a vacuum SL($n,R$) matrix, using Eq. (13), is

$$\text{Tr} \left( \nabla U_n^{(0)} \nabla U_n^{(0)-1} \right) = - \sum_i n(i - 1) (\nabla v_i)^2.$$

Substituting this into the SL($n,R$)-invariant action, Eq. (8), we can now write down the effective action for vacuum SL(4,R)

$$S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ R - 3(\nabla v_4)^2 - \frac{3}{2} (\nabla v_3)^2 - \frac{1}{2} (\nabla v_2)^2 \right].$$

We will assume that the external four dimensional spacetime is a spatially-flat FRW metric, with the line element

$$ds^2 = a^2(\eta) \left( -d\eta^2 + dx^2 + dy^2 + dz^2 \right)$$

and scale factor, $a(\eta)$, where $\eta$ is the conformal time. The effective action in Eq. (18) can then be written for homogeneous dilaton-moduli fields as

$$S = \frac{1}{2\kappa^2} \int d^3 x \int d\eta \left[ 6 a''^2 - 3 a'' v_4'^2 - \frac{3}{2} a^2 v_3'^2 - \frac{1}{2} a^2 v_2'^2 \right].$$

where a prime denotes differentiation with respect to $\eta$. We use the Euler-Lagrange equations derived from this action to calculate the equations of motion related to this action. The resulting evolution equations for the dilaton-moduli fields in a FRW metric are

$$12 a'' - 6 a v_4' = -6 v_4'^2 - 3 v_3'^2 - v_2'^2,$$

$$v_4'' + 2 a' v_4' = 0,$$

$$v_3'' + 2 a' v_3' = 0,$$

$$v_2'' + \frac{a'}{a} v_2' = 0.$$
subject to the constraint equation (from general relativistic invariance under time-reparameterisation)

$$\left( \frac{a^\prime}{a} \right)^2 = \frac{1}{12} \left( 6v_4^2 + 3v_3^2 + v_2^2 \right).$$  \(25\)

Substituting Eq. \(23\) into Eq. \(21\) yields a second-order equation for \(a(\eta)\) which can be integrated twice to obtain

$$a = a_\ast |\eta|^3,$$  \(26\)

where \(a_\ast\) is one integration constant, and we have used our freedom to choose \(a = 0\) as the origin for the time coordinate \(\eta\) in order to eliminate the other constant of integration. Equation \(26\) is the standard solution in the Einstein frame for a spatially flat FRW cosmology with free scalar fields. In terms of the proper time \(v \equiv \int a \, d\eta\) we have \(a \propto |\eta|^{1/3}\), which describes a non-accelerating expanding universe for \(t > 0\), or an accelerating contraction for \(t < 0\). It is this phase of accelerated contraction that is the basis of the pre-big-bang scenario \([1]\).

Equations \(22\) to \(24\) can now be integrated using the solution for \(a(\eta)\) in Eq. \(23\) to give

$$v_i^\prime = \frac{C_i}{a^2},$$  \(27\)

where \(C_i\) are constants of integration, \(i = 1 \ldots 3\). This allows the constraint equation \(23\) to be rewritten in the form

$$\frac{C_1^2}{3} + C_2^2 + 2C_3^2 = 1.$$  \(28\)

The constants of integration \(C_1/\sqrt{3}, C_2\) and \(\sqrt{2}C_3\) can be interpreted as points on the surface of a sphere. It is therefore convenient to move to spherical coordinates

$$C_1/\sqrt{3} = \cos \xi_1,$$

$$C_2 = \sin \xi_1 \cos \xi_2,$$

$$\sqrt{2}C_3 = \sin \xi_1 \sin \xi_2,$$  \(29\)

where the constraint is automatically satisfied and the new constants of integration are \(0 \leq \xi_1 \leq \pi\) and \(0 \leq \xi_2 < 2\pi\).

Using Eq. \(26\) in Eqs. \(27\) gives monotonic power law solutions for \(v_4, v_3\) and \(v_2\):

$$e^{v_4} = e^{v_4^\ast} |\eta|^{\sin \xi_1 \sin \xi_2 / \sqrt{2}},$$

$$e^{v_3} = e^{v_3^\ast} |\eta|^{\sin \xi_1 \cos \xi_2},$$

$$e^{v_2} = e^{v_2^\ast} |\eta|^{\sqrt{2} \cos \xi_1},$$  \(30\)

where \(v_4^\ast, v_3^\ast, v_2^\ast\) are constants of integration. Dilaton-moduli-vacuum solutions related by SL(3,R) transformations \([18]\) are recovered by setting \(\xi_2 = 0\), while for \(\xi_2 = 0\) and \(\xi_1 = 0\) or \(\pi\) we recover dilaton-vacuum cosmological solutions related by SL(2,R) transformations \([22]\).

IV. AXION PERTURBATIONS

It is known \([23]\) that inhomogeneous linear perturbations in the dilaton and other moduli fields about the homogeneous FRW solutions given by Eq. \(23\) have the general solution \(\delta v_i = Z_0(|\eta|)\), where \(Z_0\) is any linear combination of Bessel functions of order zero, independent of the various integration constants that appear in the solutions in Eq. \(30\). In the pre-big-bang scenario this solution inevitably leads to a cosmological spectrum of vacuum fluctuations steeply tilted towards small scales, with essentially no perturbations on super-horizon scales \([14]\). If the pre-big bang scenario is to produce any observable perturbations on large scales it must be through vacuum fluctuations in the axion fields \([14]\) which are non-minimally coupled to the dilaton-moduli fields in the four-dimensional Einstein frame and hence sensitive to the integration constants that parameterise their evolution.

Hence we will investigate inhomogeneous axion field perturbations about the dilaton-moduli vacuum solutions. We will calculate the field equations for these linear perturbations including axion fields by constructing the effective action to second-order in the perturbations \([14]\).

This is constructed iteratively using Eq. \(17\) as

$$\text{Tr} \left( \nabla U_{n+1} \nabla U_{-1} \right) = \text{Tr} \left( \nabla U_{n} \nabla U_{-1} \right)$$

$$+ (-n^2 - n) \left( \nabla v_{n+1} \right)^2$$

$$- 2 \text{Tr} \left( e^{(n+1)\nu_0} \nabla \sigma^T U_n^{(0)} \nabla \sigma_n \right),$$  \(31\)

where we can use the vacuum solution, \(U_n^{(0)}\), in the last term of the trace equation, in order to calculate the action to second-order in the axion fields.

Expressions can be constructed from the initial SL(2,R) matrix Eq. \(8\) and we can therefore write the trace of \(\nabla U_3 \nabla U_3^{-1}\) for SL(3,R) to second order in the axion perturbations as

$$\text{Tr} \left( \nabla U_3 \nabla U_3^{-1} \right) = \text{Tr} \left( \nabla U_2 \nabla U_2^{-1} \right) - 6 \left( \nabla v_2 \right)^2$$

$$- 2 \text{Tr} \left( e^{3v_2} \nabla \sigma^T U_2^{(0)} \nabla \sigma \right).$$  \(32\)

Rewriting Eq. \(8\) as

$$U_2 = \left( e^{v_2} \sigma_1 e^{v_2} - e^{-v_2} + \sigma_1^2 e^{2v_2} \right)$$  \(33\)

and then adding on the second order axion term with the vacuum case given in Eq. \(17\) in the above expression we obtain \([18]\).

*Because \(\sigma_1^0 = 0\) in the background solution there is no metric back-reaction to lowest-order and the perturbations are automatically gauge-invariant \([4]\).
\[ \text{Tr} \left( \nabla U_3 \nabla U_3^{-1} \right) = -6(\nabla v_3)^2 - 2(\nabla v_2)^2 - 2e^{2v_2}(\nabla \sigma_1)^2 - 2e^{3v_3+v_2}(\nabla \sigma_2)^2 - 2e^{3v_1-v_3}(\nabla \sigma_3)^2, \quad (34) \]

while for SL(4,R) we have

\[ \text{Tr} \left( \nabla U_4 \nabla U_4^{-1} \right) = -12(\nabla v_4)^2 - 6(\nabla v_3)^2 - 2(\nabla v_2)^2 - 2e^{2v_2}(\nabla \sigma_1)^2 - 2e^{3v_3+v_2}(\nabla \sigma_2)^2 - 2e^{3v_1-v_3}(\nabla \sigma_3)^2 - 2e^{4v_4+v_3+v_2}(\nabla \sigma_4)^2 - 2e^{4v_4-v_3-v_2}(\nabla \sigma_3)^2 - 2e^{4v_4-2v_3}(\nabla \sigma_6)^2. \quad (35) \]

In the Einstein frame the action becomes

\[ S = \frac{1}{2\kappa^2} \int d^3x \int d\eta \left[ -6a''^2 - 3a'v_4^2 - \frac{3}{2} a'' v_3^2 - \frac{1}{2} a'' v_2^2 - \frac{1}{2} a'' e^{2v_2} \sigma_1^2 - \frac{1}{2} a'' e^{3v_3+v_2} \sigma_2^2 - \frac{1}{2} a'' e^{3v_1-v_3} \sigma_3^2 - \frac{1}{2} a'' e^{4v_4+v_3+v_2} \sigma_4^2 - \frac{1}{2} a'' e^{4v_4-v_3-v_2} \sigma_5^2 - \frac{1}{2} a'' e^{4v_4-2v_3} \sigma_6^2 \right]. \quad (36) \]

We can now use this to derive the field equations for inhomogeneous axion perturbations evolving in the homogeneous vacuum background solutions

\[
\begin{align*}
\delta \sigma_1'' + \left( 2 \frac{a'}{a} + 2v_2 \right) \delta \sigma_1' + k^2 \delta \sigma_1 &= 0, \quad (37) \\
\delta \sigma_2'' + \left( 2 \frac{a'}{a} + 3v_3 + v_2 \right) \delta \sigma_2' + k^2 \delta \sigma_2 &= 0, \quad (38) \\
\delta \sigma_3'' + \left( 2 \frac{a'}{a} + 3v_3 - v_2 \right) \delta \sigma_3' + k^2 \delta \sigma_3 &= 0, \quad (39) \\
\delta \sigma_4'' + \left( 2 \frac{a'}{a} + 4v_4 + v_3 + v_2 \right) \delta \sigma_4' + k^2 \delta \sigma_4 &= 0, \quad (40) \\
\delta \sigma_5'' + \left( 2 \frac{a'}{a} + 4v_4 + v_3 - v_2 \right) \delta \sigma_5' + k^2 \delta \sigma_5 &= 0, \quad (41) \\
\delta \sigma_6'' + \left( 2 \frac{a'}{a} + 4v_4 - 2v_3 \right) \delta \sigma_6' + k^2 \delta \sigma_6 &= 0, \quad (42)
\end{align*}
\]

where \( k \) is the comoving wavenumber. Note that because the axion field is zero in the vacuum background solution, their back-reaction upon the moduli fields and the four-dimensional spacetime metric vanishes to first-order and the perturbations are independent of the spacetime gauge \[ \Box. \]

The field equations (37) to (42) can be written in the standard form for a free scalar field evolving in a FRW metric

\[ \delta \sigma_i'' + \frac{a''}{a_i} \delta \sigma_i' + k^2 \delta \sigma_i = 0 \quad (43) \]

with \( i = 1 \ldots 6 \), where we introduce the conformally rescaled scale factor, \( \bar{a}_i \), in the corresponding “axion frame” \[ \Box. \]

Substituting the solutions from Eqs. (26) and (30) into Eqs. (43), we obtain

\[ \bar{a}_i \propto \eta^{p_i}, \quad (50) \]

where

\[ p_1 = \sqrt{3} \cos \xi_1, \]
\[ p_2 = \sqrt{3} \left( \frac{\sqrt{3}}{2} \sin \xi_1 \cos \xi_2 + \frac{1}{2} \cos \xi_1 \right), \]
\[ p_3 = \sqrt{3} \left( \frac{\sqrt{3}}{2} \sin \xi_2 - \frac{1}{2} \cos \xi_1 \right), \]
\[ p_4 = \sqrt{3} \left( \frac{\sqrt{3}}{2} \sin \xi_1 \left( \frac{2\sqrt{3}}{3} \sin \xi_2 + \frac{1}{3} \cos \xi_2 \right) + \frac{1}{2} \cos \xi_1 \right), \]
\[ p_5 = \sqrt{3} \left( \frac{\sqrt{3}}{2} \sin \xi_2 + \frac{2\sqrt{3}}{3} \sin \xi_2 + \frac{1}{3} \cos \xi_2 \right) - \frac{1}{2} \cos \xi_1 \right), \]
\[ p_6 = \sqrt{3} \sin \xi_1 \left( \frac{\sqrt{3}}{3} \sin \xi_2 - \frac{1}{3} \cos \xi_2 \right). \quad (51) \]

Equation (43) has the standard form for perturbations of a free scalar field evolving in an FRW cosmology with scale factor \( \bar{a}_i \). Thus we can define the canonically normalised variables \[ \Box. \]

\[ u_i = \frac{1}{\sqrt{2} \kappa} \bar{a}_i \delta \sigma_i, \quad (52) \]

which enables us to rearrange Eq. (43) in the form of a simple harmonic oscillator with time-dependent mass

\[ u_i'' + \left( k^2 + \frac{\left( -\bar{a}_i'' \right)}{\bar{a}_i} \right) u_i = 0. \quad (53) \]

For a power-law expansion given by Eq. (50) this corresponds to Bessel’s equation

\[ u_i'' + \left( k^2 + \frac{(1/4 - p_i^2)}{|\eta|^2} \right) u_i = 0. \quad (54) \]

Equation (54) has a standard solution

\[ u_i = |\eta|^{1/2} Z_{p_i} (|\eta|), \quad (55) \]

where \( Z_{p_i} \) is any linear combination of Bessel (or Hankel) functions and for each axion field the order of the Bessel function is \( p_i \) given in Eq. (50).
V. PRE-BIG-BANG PERTURBATION SPECTRA

We will write the solutions of the Bessel equation in terms of Hankel functions so that a general solution of Eq. (54) is given by

\[ u_i = |k\eta|^{1/2} \left[ u_+ H_{|p_i|}^{(1)} (|k\eta|) + u_- H_{|p_i|}^{(2)} (|k\eta|) \right]. \tag{56} \]

For wavelengths much smaller than the horizon scale \(|k\eta| \gg 1\) the equation of motion Eq. (53) reduces to that for a free-scalar field \(u_i\) in flat Minkowski spacetime with a well-defined vacuum state. Allowing only positive frequency modes in a flat-spacetime vacuum state requires that

\[ u_i \to e^{-ik\eta} \sqrt{2k}. \tag{57} \]

The classic horizon problem of the standard hot big bang is that all modes start outside the horizon at the big bang \((k\eta \to 0)\) and so there is no reason to expect modes to start in this vacuum state.

However in the pre-big-bang scenario all modes start within the horizon in the infinite past as \(\eta \to -\infty\). As \(\eta \to 0\) on the (+) branch modes leave the horizon, giving a well-defined spectrum of super-horizon perturbations in all fields, even though there is no inflation in the conventional sense \((\dot{a} > 0)\) in the Einstein frame.

Allowing only positive frequency modes in a flat-spacetime vacuum state at early times \((as \, k\eta \to -\infty)\) i.e. large \(-\eta\) yields

\[ u_+ = \frac{\sqrt{\pi}}{2\sqrt{k}} e^{i(2|p_i|+1)\pi/4}, \quad u_- = 0. \tag{58} \]

Therefore using Eq. (52) we can write

\[ \delta \sigma_i = k\sqrt{\frac{\pi}{2k}} e^{i(2|p_i|+1)\pi/4} \frac{\sqrt{2k}}{a_i} H_{|p_i|}^{(1)} (-k\eta). \tag{59} \]

At late times on super-horizon scales \((|k\eta| \ll 1)\) we have

\[ \delta \sigma_i = \pm ik \sqrt{\frac{\pi}{k}} e^{i(2|p_i|+1)\pi/4} \left( -\frac{\Gamma(|p_i|)}{\pi a_i} \right) \left( \frac{2}{-\eta|p_i|}\right) |p_i|^{-1/2} \] \tag{60}

The power spectrum for these axion perturbations is denoted by

\[ P_{\delta \sigma_i} \equiv \frac{k^3}{2\pi^2} |\delta \sigma_i|^2 \] \tag{61}

which represents the dispersion \(\langle \delta \sigma_i^2 \rangle\) due to fluctuations on comoving scales \(\sim k^{-1}\) \cite{10}. It can be calculated from Eq. (60) to be

\[ P_{\delta \sigma_i} = 2k^2 \left( \frac{C(|p_i|)}{2\pi} \right)^2 \frac{k^2}{a_i^2} (-k\eta)^{1-2|p_i|}. \tag{62} \]

where the coefficient term

\[ C(|p_i|) = 2^{2|p_i|/3} 2^{3/2} \Gamma(3/2). \tag{63} \]

approaches unity for \(|p_i| = 3/2|p_i|\)

The spectral tilt of the perturbation spectra is defined by

\[ \Delta n_i = \frac{d \ln P_{\delta \sigma_i}}{d \ln k}. \tag{64} \]

It follows from Eq. (62) that the spectral tilt for each of the axion fields is constant and takes the values

\[ \Delta n_i = 3 - 2|p_i|, \tag{65} \]

where the \(p_i\) are given in Eq. (61) in terms of the two integration constants \(\xi_1\) and \(\xi_2\) which parameterise the dilaton-moduli vacuum background solutions.

From Eqs. (63) and (64) it is then possible to find the spectral tilts as functions of the integration constants \(\xi_1\) and \(\xi_2\). (See Figs. 1 and 2.) The maximum absolute value for any \(p_i\) is \(3/\sqrt{3}\) and thus the minimum value of the spectral tilt for any axion field is \(\Delta n = -2\sqrt{3} + 3 = -0.46\) as found previously with SL(2,R) \cite{14} or SL(3,R) \cite{18} symmetry groups. For any axion field the allowed range for the spectral tilt is

\[ -2\sqrt{3} + 3 \leq \Delta n_i \leq 3. \tag{66} \]

VI. DISCUSSION

The amplitude of the axion power spectra at the end of the pre-big bang phase when the comoving horizon scale is given by \(\eta_s = -1/k_s\) can be given, from Eqs. (62) and (65), as

\[ P_{\delta \sigma_i} |_{s} = 2k^2 \left( \frac{C(|p_i|)}{2\pi} \right)^2 \left( \frac{H_i}{2\pi} \right)^2 \left( \frac{k}{K_s} \right)^{\Delta n_i}, \tag{67} \]

\[ FIG. 1. \quad \text{The spectral tilts, } \Delta n_i, \quad \text{for all six axion fields as a function of } \xi_1 \text{ when } \xi_2 = 0. \]

\[ 
\begin{align*}
\text{FIG. 2. The maximum absolute value of the spectral tilt, } \Delta n_i, \text{ for all six axion fields as a function of } \xi_2 \text{ when } \xi_1 = 0. 
\end{align*}
\]
maximum $\Delta$ function of $\xi$ scale re-enters the cosmological horizon contribute a perturbation to the density when a given matter dominated era is given by

$$\rho_{\text{total}} = \frac{3}{\kappa^2} H^2 \kappa^2.$$ 

The Hubble rate in the axion frame can also be related to $H$, the expansion rate in the Einstein frame, via

$$H_i = \frac{2(p_i + 1/2)}{\Omega_i}.$$ 

Assuming that modes remain frozen-in on large scales ($|k\eta| \ll 1$) during the uncertain transition from pre to post big bang phase, then the energy density contributed by massless axion perturbations in the Einstein frame can be estimated as

$$\rho_i \approx \Omega_i C(|p_i|) \left(\frac{H_s}{2\pi}\right)^2 \left(\frac{k}{k_s}\right)^\Delta n_i.$$ 

Although massless axion fields never come to dominate the total energy density in the universe, they do contribute a perturbation to the density when a given scale re-enters the cosmological horizon

$$\frac{\delta \rho}{\rho} = \frac{\rho_i}{\rho_{\text{total}}} = \frac{C^2(|p_i|)}{3} \left(\frac{\kappa H_s}{2\pi}\right)^2 \left(\frac{k}{k_s}\right)^\Delta n_i.$$ 

where the expression for $C(|p_i|)$ is given in Eq. (53) and the Hubble rate in the axion frame is given by

$$H_i^2 = \frac{(p_i + 1/2)^2}{\bar{a}^2 \eta^2}.$$ 

The region of parameter space for which $\Delta n_i > 0$ is smaller than the region of parameter space for which $\Delta n_i < 0$. We find $\Delta n_i$ agrees with $\xi_1 = \pi/4$ or $3\pi/4$ and $\tan \xi_2 = -\sqrt{2}$ (shown in Fig. 3 where we obtain $\Delta n_i = 3 + \sqrt{6} = 0.55$). Table 1 shows the previously calculated maximum values for $\Delta n_i$ for different (n,R) groups. The maximum allowed spectral tilt gets progressively larger (bluer) as the symmetry group gets larger. Since the larger groups always contain the previous groups we see that it is indeed possible for all the axion fields to have blue spectral indices in SL(n,R) nonlinear sigma models where $n \geq 4$. On the other hand requiring all the spectral indices to be positive represents a restriction on the allowed initial conditions.

We have focused our discussion on the cosmological perturbations induced by an effectively massless axion field proposed in Ref. [11], in which case the primordial axion perturbation spectra can be directly related to seed density perturbations. If the axions become non-relativistic during the radiation-dominated era then the overall amplitude of perturbations is increased by a factor $(m/H_{\text{eq}})^{1/2}$, where the Hubble rate at matter-radiation equality is $H_{\text{eq}} \approx 10^{-27}$ eV. If any of the primordial axion spectra have a negative spectral tilt, $\Delta n_i < 0$ (as occurs in a large regime of parameter space shown in Fig. 3) then the only way to suppress the large-scale density perturbation appears to be for the axion to acquire a periodic potential where large field fluctuations may have a small effect on the energy density [27]. In such a
scenario, the massive axion could be a novel form of dark matter and lead to a very different model for large-scale structure formation [28].

Finally, we note that the presence of non-trivial background axion fields in the general solutions of the full SL(2,R) and SL(3,R) symmetric axion-dilaton-moduli cosmologies restricts the allowed asymptotic vacuum state [22,23,24]. The effect of the background axion fields upon the perturbation spectra has only been calculated for a single axion field where the perturbation spectra remain invariant under SL(2,R) transformations of the background solutions [14,31]. The cosmological perturbation spectra in axion-dilaton-moduli solutions with more degrees of freedom remain to be investigated.

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