High-accuracy polynomial solutions
of the classical Stefan problem

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Abstract. High-accuracy polynomial solutions of the Stefan problem for a semi-infinite medium with Dirichlet and Neumann boundary conditions and boundary conditions of general form are presented. The initial temperature of the medium was assumed to be equal to its phase-transition temperature. With the use of the integral method of boundary characteristics, based on the multiple integration of the heat-conduction equation, sequences of identical equalities with different boundary conditions were obtained. On the basis of these equalities, polynomial solutions of different degrees were constructed. High efficiency of the approach proposed was demonstrated by different examples. The polynomial solutions of the second and third degrees surpass in approximation accuracy the analogous known solutions. The accuracy of the calculations of the interphase boundary with the use of the fourth- and fifth-degree polynomials is higher by several orders of magnitude than that of numerical methods.

1. Introduction
The solution of the Stefan problem includes the calculation of the temperature (concentration) profile in a body with determination of the law of movement of the interphase boundary in it [1]. In the overwhelming majority of cases, Stefan problems are solved by numerical methods [2, 3]. Problems with phase transitions, with rare exception [1, 4–6], have no exact solutions. At present, approximate analytical methods are used widely for solving such problems [7–13]. We consider the classical Stefan problem in the following mathematical formulation. It is necessary to find the position of the interphase boundary \( s(t) \) and the temperature function \( T(x,t) \), satisfying the equation

\[
\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0
\]

with one of the boundary conditions

(i): \( T = h(t) \);
(ii): \( -\frac{\partial T}{\partial x} = q(t) \);
(iii): \( a \frac{\partial T}{\partial x} + \beta T = \gamma(t), \quad x = 0, \quad t > 0 \)

the conditions at the moving interphase boundary

\[
T(s,t) = 0, \quad \frac{\partial T}{\partial x} \bigg|_{x=s} = \frac{1}{\text{Ste}} \frac{ds}{dt}, \quad t > 0
\]
and the initial condition \( s(0) = 0 \). Here, \( \text{Ste} \) is the Stefan number [1, 14].

2. Identical equalities and their sequences

By analogy with [11, 12] we construct sequences of identical equalities.

2.1. Dirichlet boundary condition

Using the operators \( D_t \equiv \partial / \partial t \) and \( L \equiv \partial^2 / \partial x^2 \), we write equation (1) in the form

\[
D_t T = LT. \tag{4}
\]

Let us introduce the integral operators

\[
L^2_n = \int_{0}^{s} (\cdot) dx^{(2)}, \quad L^1_n = \int_{0}^{s} (\cdot) dx^{(1)}.
\]

Applying the operators \( L^2_n \) and \( L^1_n \) to equation (4), we obtain the following sequence of identical equalities [13]:

\[
\left\{ L_n T + \frac{1}{\text{Ste}} \sum_{k=1}^{n} \int_{0}^{t} \int_{0}^{s} \frac{S^{2k}}{(2k)!} d t d (\cdot)^{(n-k)} \right\} = H_n, \quad \forall n \in \mathbb{Z}_+. \tag{5}
\]

2.2. Neumann boundary condition

Let us integrate equation (4) over the region \( x \in [0, s] \):

\[
\int_{0}^{s} \frac{\partial T}{\partial t} \, dx = \frac{d}{dr} \left[ T \right]_{0}^{s} - T(s, t)s' = \int_{0}^{s} \frac{\partial^2 T}{\partial x^2} \, dx = \left. \frac{\partial T(s, t)}{\partial x} - \frac{\partial T(0, t)}{\partial x} \right|_{0}^{s}.
\]

In view of the boundary conditions (2)-(ii), (3), from equation (6) we obtain the relation

\[
\frac{d}{dr} \left[ T \right]_{0}^{s} = q - \frac{s'}{\text{Ste}}. \tag{7}
\]

Integration of (7) with conditions (3) gives the zero-order equality

\[
\hat{L}_{1} T + \frac{s}{\text{Ste}} \equiv \int_{0}^{t} q \, dt = Q_1, \tag{8}
\]

where \( \hat{L}_{1} \equiv \int_{0}^{1} (\cdot) \, dx \) is an operator. Integration of equation (4) over the region \( x \in [x, s] \) gives

\[
D_t \left( \hat{L}_{1} T \right) + \beta s' = T_s. \tag{9}
\]

Here, \( \hat{L}_{1} \equiv \int_{0}^{1} (\cdot) \, dx \) is an additional integral operator and \( T_s \equiv \partial T / \partial x \). Let us apply the operator \( L_{s} \) to equation (9). For its left side we write

\[
L_{s} \left( D_t \left( \hat{L}_{1} T \right) \right) + L_{s} \left( s' / \text{Ste} \right) = D_t \left( L_{s} \hat{L}_{1} T \right) + s' \left( x - s \right)^2 / (2 \text{Ste}). \tag{10}
\]

For the right side of (9) we have

\[
L_{s} T_s = \int_{s}^{1} dx \int_{0}^{s} \hat{L}_{1} T, \quad dx = \int_{s}^{1} (T - T(s, t)) \, dx = \int_{s}^{1} T \, dx. \tag{11}
\]
Using (9)–(11), we arrive at the equation

\[ D_s \left( \mathcal{L}_s^2 T \right) + s' (x - s)^2/(2 \text{Ste}) = \frac{\dd}{ds} T \]  
(12)

Equation (12) at the point \( x = 0 \) has the form

\[ D_s \left( \mathcal{L}_s^2 T \right) + s' s^2/(2 \text{Ste}) = \mathcal{L}_s^2 T \]  
(13)

or, in view of equality (8), the form

\[ D_s \left( \mathcal{L}_s^2 T \right) + s(s' + 2)/2 \text{Ste} = Q. \]  
(14)

Integration of equation (14) gives the first-order identical equality

\[ \mathcal{L}_s^2 \left( \mathcal{L}_s^2 T \right) + \text{Ste}^{-1} \left( s^3/6 + \int_0^s s \, ds \right) \equiv Q. \]  
(15)

Identical equalities of higher orders are obtained in the same way, and we arrive at the sequence

\[ \mathcal{L}_s^2 \left( \mathcal{L}_s^2 T \right) + \text{Ste}^{-1} \left( s^{n+3}/6 + \int_0^s s^n \, ds \right) \equiv Q_n, \quad \forall n \in \mathbb{Z}_+. \]  
(16)

2.3. Robin boundary condition

On multiple integration of the boundary condition (2)-(iii), we obtain

\[ \alpha \int_0^1 \int_0^1 \partial T(0,t) \frac{\partial x}{\partial y} \, dx \, dt + \beta \int_0^1 \int_0^1 T(0,t) \, dx \, dt = \gamma \int_0^1 \int_0^1 \gamma \, dx \, dt = Y_n, \quad n \in \mathbb{Z}_+. \]  
(17)

Performing the formal changes \(-\partial T(0,t) / \partial x \to q, T(0,t) \to h\) in (17), we arrive at the relation

\[ \beta \int_0^1 \int_0^1 h \, dx \, dt - \alpha \int_0^1 \int_0^1 q \, dx \, dt = \gamma \int_0^1 \int_0^1 \gamma \, dx \, dt = Y_n, \quad n \in \mathbb{Z}_+. \]  
(18)

On the basis of the identical equalities (5) and (16), from (2.27) we obtain the sequence

\[ \mathcal{L}_s^2 \left( \mathcal{L}_s^2 T - \alpha \mathcal{L}_s^2 T \right) + \frac{1}{\text{Ste}} \sum_{n=1}^{N} \int_0^1 \int_0^1 \left( \beta \frac{s^{2k}}{(2k)!} - \alpha \frac{s^{2k-1}}{(2k-1)!} \right) \, dx \, dt \equiv Y_n, \quad \forall n \in \mathbb{Z}_+. \]  
(19)

3. Polynomial solutions of the Stefan problem

We represent the solutions of the Stefan problems (1)–(3) in the form of the polynomial

\[ T = (1-x/s) \sum_{n=0}^{N-1} a_n(t) (x/s)^n \]  
providing the fulfillment of the boundary condition (2). In this case, the problem is reduced to the determination of the coefficients \( a_n(t) \) and the function \( s(t) \).

3.1. Dirichlet boundary condition

Let us define the temperature profile by a quadratic parabola \((N = 2)\). From the first-order equality of sequence (5) we determine the coefficient \( a_1(t) \) and, as a result, obtain
For determining the function \( s(t) \) we will use the second-order identical equality of sequence (5), from which we obtain the constitutive equation

\[
\left( \frac{30}{\text{Ste}} + 4h \right)s^2 - \left( \frac{h'}{6} \right)s^4 + \left( \left( \frac{1}{\text{Ste}} - \frac{h}{3} \right)s^2 + 4H_1 \right) (s^2)' = 60H_1. \tag{21}
\]

Let \( h(t) = 1 \). The exact solution for the semi-space being considered has the form [1, 2]

\[
T = 1 - \text{erf} \left( \frac{x}{(2\sqrt{t})} \right) / \text{erf} (\alpha), \quad s(t) = 2\alpha \sqrt{t}, \tag{22}
\]

where \( \text{erf} \) is the error function and the constant \( \alpha \) is determined from the transcendental equation

\[
\sqrt{\pi} \alpha \exp(\alpha^2) \text{erf}(\alpha) = \text{Ste}. \tag{23}
\]

Substitution of the function \( s(t) \) in the form \( s(t) = 2\alpha \sqrt{t} \) into (21) gives the quadratic equation

\[
4\beta^2 (\text{Ste} - 3) - 6\beta (15 + 4\text{Ste}) + 45\text{Ste} = 0, \tag{24}
\]

where \( \beta = \alpha^2 \). From (24) we determine the real positive root

\[
\alpha = \frac{1}{2} \sqrt{\frac{15 + 4\text{Ste}}{\text{Ste} - 3} - 1}. \tag{25}
\]

We will estimate the accuracy of formula (25) by the \( \varepsilon_\alpha = \left| \alpha - \alpha_{\text{ex}} \right| / \alpha_{\text{ex}} \times 100\% \), where \( \alpha_{\text{ex}} \) is determined from equation (23). At \( \text{Ste} = 1 \) formula (25) gives \( \alpha = 0.619975 \) to which corresponds \( \varepsilon_\alpha = 0.014\% \). The calculation by the CIM method [8] gives \( \alpha = 0.618497 \) with an error \( \varepsilon_\alpha = 0.252\% \).

### 3.2. Neumann boundary condition

At \( N = 4 \) we will determine the polynomial coefficients of the temperature profile at condition (2)-(ii) with the use of the first three equality of (16) and, as a result, obtain the system of equations

\[
\begin{bmatrix}
1 & -1 & 0 & 0 & \alpha_0 \\
\gamma_2 & \gamma_6 & \gamma_{12} & \gamma_{20} & a_1 \\
\gamma_{24} & \gamma_{50} & \gamma_{60} & \gamma_{64} & a_2 \\
\gamma_{30} & \gamma_{108} & \gamma_{344} & \gamma_{1728} & a_3
\end{bmatrix}
= \begin{bmatrix}
sq \\
Q_1 / s - 1 / \text{Ste} \\
Q_2 / s^3 - 1 / (1/6 + S_1^3 / s^3) \\
Q_3 / s^3 - 1 / (1/120 + (S_1^3 + S_2^3) / s^3)
\end{bmatrix}. \tag{26}
\]

From here we obtain the temperature function

\[
T = \left( 1 - \frac{x}{s} \right) \left[ \frac{5s}{64} \left( 1 - \frac{59x}{5s} + \frac{151x^2}{5s^2} - 21x^3/s^3 \right) + \frac{13}{8\text{Ste}} \left( 1 + \frac{x}{s} - \frac{597x}{13s^3} + \frac{315x^3}{13s^3} \right) + \frac{75}{16s} \left( 1 + \frac{x}{s} - \frac{13x^2}{s^2} \right) \right] \\
+ \left( \frac{63x^3}{5s^3} \right) Q_1 + \frac{525}{8s^3} \left( S_1^3 / \text{Ste} - Q_2 \right) \left( 1 + \frac{x}{s} - \frac{121x^2}{5s^2} + \frac{27x^3}{s^3} \right) + \frac{945}{16s} \left( Q_3 - S_1^3 / \text{Ste} \right) \left( 1 + \frac{x}{s} - \frac{29x^2}{3s} + \frac{35x^3}{s^3} \right),
\]

where \( S_n^k(t) = \int_0^t \int_0^t s^{2k-1} / (2k-1)! dt^{(n-k)} \). Substitution of this temperature profile into the third-order equality (16) gives the equation
\[ s^8 q' - 8 s^6 (10 - s s') q + 240 s^4 (21 - 2 s s') Q_0 + 20160 s^2 \left( \frac{S_1}{\text{Ste}} - Q_0 \right)(12 - s s') - 120960 \left( \frac{S_1 + S_2}{\text{Ste}} - Q_0 \right)(55 - 4 s s') - \frac{448 s^3}{\text{Ste}}(45 + s s') = 0. \]  
(27)

The solution obtained will be analyzed with the use of the test problem [17]: \{ \( -T'_x(0,t) = e^t \text{Ste} = 1 \), \( T = e^{-t} - 1, \quad s(t) = t \} \). Substitution of the function \( s(t) \) in the form \( s(t) = at + bt^m \) into (27) gives \( s(t) = t + t^9/1814400 + O(t^{10}) \). It is seen (figure 1) that at times as large as \( t \approx 3 \) the approximate and exact profiles are practically completely coincident. The calculated error norm \( \|e_r(0.5)\|_1 = 6.27 \cdot 10^{-6} \) is substantially smaller as compared to that of the numerical solutions [15, 16].

Using the function \( s(t) = t + t^9/1814400 \), we expand the temperature function \( T(0,t) \) into a Taylor series and apply the Pade approximation to it. In the long run we obtain

\[ T(0,t) \approx \frac{t + 0.49210t^2 + 0.02328t^3 + 0.008123t^4}{1 - 0.007905t - 0.139435t^2 + 0.037506t^3 - 0.003511t^4}. \]  
(28)

Calculation of the function \( T(0,t) \) (figure 2) gives the error \( e_0 = (T(0,t) - T^*(0,t))/T^*(0,t) \) equal to \( e_0 = 1.4 \cdot 10^{-5} \% \) and \( e_0 = 2.6 \cdot 10^{-6} \% \) at \( t = 0.5 \) and \( t = 1 \) respectively. In comparison with the numerical schemes [16, 17], these values of \( e_0(0.5) \) are smaller by two orders of magnitude. Note that formula (28) gives a negligibly small error \( t = 0.5 \) \( e_0 = 1.06 \cdot 10^{-7} \% \).

**Figure 1.** Temperatures given by the exact (full lines) and approximate \((N = 4)\) (dotted lines) solutions at \( t = 1 \) (1), 1.5 (2), 2 (3), 2.5 (4), 3 (5).

**Figure 2.** Time change in the surface temperature given by the exact (full line) and the approximate \((N = 4)\) (dotted line) solutions.

3.3. Robin boundary condition

The coefficients of the temperature profile \((N = 3)\) will be determined at the boundary condition (2)-(ii) with the use of the first two identical equalities of sequence (19):

\[ \int_0^1 (1 + x) T \, dx + \frac{s}{2 \text{Ste}} (2 + s) = Y_1, \quad \int_0^1 T \frac{x^2}{2} \left( 1 + \frac{x}{3} \right) \, dx + \frac{1}{2 \text{Ste}} \left( \int_0^1 (2s + x^2) \, dx + \frac{s^2}{12} (4 + s) \right) = Y_2. \]  
(29)

A constitutive equation for the function \( s(t) \) can be obtained from the Stefan boundary condition (3), the third-order identical equality of sequence (19), the heat-balance integral (7), or the first-order
integral relation (16). We dwell on one of the above-listed variants, namely, the third-order identical equality differentiated with respect to the time $t$ of sequence (16):

$$
\frac{d}{dr} \int_0^s x^4 \left(1 + \frac{x}{5}\right) dx + \frac{1}{\text{Ste}} \left( \int_0^s \left(s + \frac{s^2}{2}\right) dr + \frac{s^3}{6} + \frac{s^4}{120} + \frac{s^5}{24}(1 + s') \right) = Y_2.
$$

(30)

Assuming that $\alpha = 1$, $\beta = -1$, $\text{Ste} = 1$, and $\gamma(t) = 2e - 1$ in (2)-(iii), we have the known test problem [16–18]. Expanding the function $e'$ into a Taylor series and representing the function $s(t)$ in the form $s(t) = at + bt^n$, from equation (30) we find $s(t) = t - t^2 / 11025 + O(t^3)$. The mean square error given by the approach proposed at $t = 1$ comprises $\|E_t\| = 8.545 \times 10^{-4}$.

We also obtained analytical solutions in the form of the fourth- and fifth-order polynomials. At $N = 5$ the polynomial coefficients of the temperature profile were determined at the Robin boundary condition (2)-(iii) and the Stefan condition (3) with the use of three identical equalities of sequence (19). Equation (7) was used as a constitutive equation. In this case, the error norm $(t = 1)$ was $\|E_t\| = 2.747 \times 10^{-6}$. The indicated problem was solved numerically in the work [17] where the errors at $t = 0.9$ comprised $\|E_t\| = 3.951 \times 10^{-5} (n = 11, \Delta t = 0.10)$ and $\|E_t\| = 1.212 \times 10^{-6} (n = 51, \Delta t = 0.01)$.

As is seen (table 1), the new approach proposed for solving the Stefan problem allows one to calculate the temperature profile with an accuracy close to that of numerical schemes, and these solutions substantially surpass the numerical solutions in the accuracy of calculating the interphase boundary.

4. Conclusion

On the basis of integral equations, high-accuracy solutions of the classical Stefan problem have been obtained. The approach proposed can be used to advantage in different applied problems.

| Parameters $(t=1)$ | Accurate values | New scheme $(N=5)$ | Absolute error $(E_i / E_f)$ | Relative error $(\varepsilon_i / \varepsilon_f)$ |
|-------------------|-----------------|--------------------|----------------------------|------------------------------------------|
| $s$               | 1               | 1.00000056362      | 5.63621$\times 10^{-8}$    | 5.63621$\times 10^{-6}$                  |
| $T(0,t)$          | 1.718281828459  | 1.71828176243      | 3.47784$\times 10^{-7}$    | 2.02402$\times 10^{-5}$                  |

References

[1] Hill J M 1989 One-Dimensional Stefan Problems: An Introduction (London: Chapman & Hall)
[2] Caldwell J 2009 Therm. Sci. vol 13 pp 61–72
[3] Mitchell S L and Vynnycky M 2009 Appl. Math. Comput. vol 215 pp 1609–21
[4] Lunardini V J 1991 Heat Transfer with Freezing and Thawing (London: Elsevier)
[5] Witula R 2010 Archives of Foundry Engineering vol 10 pp 83–8
[6] Tao L N 1979 Quart. J. Appl. Math. Phys. vol 30 pp 416–26
[7] Mitchell S L and Myers T G 2012 Int. J. Diff. Eq. vol 2012 pp 22
[8] Myers T G 2010 Int. J. Heat Mass Transf. vol 53 pp 1119–27
[9] Mitchell S L and Myers T G 2010 SIAM Rev. vol 52 pp 57–86
[10] Sadoun N 2006 Appl. Math. Model. vol 30 pp 531–44
[11] Kor V A 2016 Heat Transf. Res. vol 47 pp 1035–55
[12] Kor V A 2016 Heat Transf. Res. vol 47 pp 927–44
[13] Kor V A 2017 J. Eng. Phys. Thermophys. vol 89 pp 1289–314
[14] Mosally F A, Wood A S and Al-Fhaid A 2002 Appl. Math. Comput. vol 130 pp 87–100
[15] Kutluay S, Bahadir A R and Ozdes A 1997 J. Comput. Appl. Math. vol 81 pp 134–144
[16] Kutluay S and Elsen A 2004 Appl. Math. Comput. vol 150 pp 59–67
[17] Whue-Teong Ang 2008 Num. Meth. Part. Diff. Eq. vol 24 pp 939–49