Microstate Counting via Bethe Ansätze
in the 4d $\mathcal{N} = 1$ Superconformal Index

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Abstract

We study the superconformal index of four-dimensional toric quiver gauge theories using a Bethe-Ansatz approach recently pioneered by Benini and Milan. Relying on a particular set of solutions to the corresponding Bethe Ansatz equations we evaluate the superconformal index in the large $N$ limit, thus avoiding to take any Cardy-like limit. We present explicit results for theories arising as a stack of $N$ D3 branes at the tip of toric Calabi-Yau cones: the conifold theory, the suspended pinch point gauge theory, the first del Pezzo theory and $Y^{p,q}$ quiver gauge theories. For generic quiver gauge theories we find a particular correction to the superconformal index in the Cardy limit that happens to vanish in the case of $\mathcal{N} = 4$ supersymmetric Yang-Mills. We estimate how such correction affects the entropy of the would be dual electrically charged rotating AdS$_5$ black holes.
1 Introduction

The understanding of the quantum microstates responsible for the entropy of black holes has long been one of the central questions in the path to a quantum theory of gravity. In the context of the AdS/CFT correspondence it has recently been shown that the entropy of certain asymptotically AdS$_4$ black holes admits a microscopic explanation in terms of a topologically twisted field theory [1] (see [2, 3] for reviews with extensive lists of references).

More recently the question of microstates for asymptotically AdS$_5$ black holes dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) which was originally tackled in [4] has been revisited providing a microscopic entropy matching using various approaches. A broader interpretation of localization was successfully put forward in [5] while an analysis of the free-field partition function in a particular limit led to the entropy in [6] (see also [7]). Both these groups relied on a particular Cardy-like limit to evaluate the path integral. Another approach, put forward by Benini and Milan in [8], attacked the superconformal index using a Bethe Ansatz approach developed in [9]. One key advantage of this approach is that it does not require taking the Cardy limit and thus opens the door for a more in-depth understanding of the superconformal index. In this brief note we simply generalize the large $N$ results obtained for $\mathcal{N} = 4$ SYM using the Bethe-Ansatz approach to a large class of $\mathcal{N} = 1$ supersymmetric field theories.

Other recent studies demonstrating that the Cardy-like limit of the superconformal index of 4d $\mathcal{N} = 4$ SYM accounts for the entropy function, whose Legendre transform corresponds to the entropy of the holographically dual AdS$_5$ rotating black holes were presented in [10, 11]. Such analysis has by now been extended to generic $\mathcal{N} = 1$ supersymmetric gauge theories [12, 13] including a particular description specialized to arbitrary $\mathcal{N} = 1$ toric quiver gauge theories, observing that the corresponding entropy function can be interpreted in terms of the toric data [14].

In this note we verify that a class of holonomies to the Bethe-Ansatz equations used prominently in [8], namely, those of the form $u_i - u_j = \frac{\tau}{N}(i - j)$ (where $u_i$ are the holonomies of the gauge group) exploited for $\mathcal{N} = 4$ SYM can be generalized to evaluate the superconformal index of generic $\mathcal{N} = 1$ four-dimensional superconformal field theories.

The rest of the note is organized as follows. In section 2 we show that a particular class of holonomies solves the Bethe-Ansatz equation for generic 4d $\mathcal{N} = 1$ gauge theories and proceed to evaluate the superconformal index in the large $N$ limit where we find a correction to the Cardy-like limit. Section 3 works out explicitly the index for a number of superconformal field theories. In section 4 we estimate how this leading $O(N^2)$ correction in the superconformal index modifies the entropy of the would-be dual AdS$_5$ black holes. We conclude in 5.

2 Bethe-Ansatz approach to the superconformal index

In this section we generalize the solutions to the Bethe-Ansatz type equations proposed in [8, 9] to evaluate the superconformal index of $\mathcal{N} = 4$ SYM to generic 4d $\mathcal{N} = 1$ supersymmetric gauge theories. For concreteness we will work in the context of toric quiver gauge theories which are naturally decorated with extra global and baryonic symmetries but the results apply more generally to 4d $\mathcal{N} = 1$ supersymmetric gauge theories.

Consider a generic $\mathcal{N} = 1$ theory with semi-simple gauge group $G$, flavor symmetry $G_F$ and non-anomalous $U(1)_R$ R-symmetry. The matter content of this theory is taken to be $n_a$ chiral multiplets $\Phi_a$ in representations $\mathcal{R}_a$ of $G$, with flavor weights $\omega_a$ in some representation $\mathcal{R}_F$ of $G_F$ and superconformal R-charge $r_a$. Let us start by introducing the following quantities which
are related to global fugacities and holonomies in the Cartan of the gauge group:

\[ p = e^{2\pi i r}, \quad q = e^{2\pi i s}, \quad v_a = e^{2\pi i \xi_a}, \quad z_i = e^{2\pi i u_i} \]  

(2.1)

and the R-charge chemical potential which is fixed by supersymmetry to:

\[ \nu_R = \frac{1}{2} (\tau + \sigma) . \]  

(2.2)

With the above data, the integral representation for the superconformal index can be written as

\[ I(p, q; v) = \frac{(p;p)_{r^k(G)}(q;q)_{r^k(G)}}{|W_G|} \int_{\Gamma_{r^k(G)}} \frac{\prod_{i=1}^{n_i} \prod_{\rho_a \in \mathfrak{R}_a} \Gamma_e \left[ (pq)^{r_k/2} z^{\rho_a} v^{\omega_a} ; p, q \right] \prod_{i=1}^{r^k(G)} dz_i}{|z_i|^2 \omega_i} \]  

(2.3)

The integration variables \( z_i \) parametrize the maximal torus of the gauge group \( G \) and the integration contour is the product of \( r_k(G) \) unit circles. Following standard notation, \( \rho_a \) are the weights of the representation \( \mathfrak{R}_a \), \( \alpha \) parametrizes the roots of \( G \) and \( |W_G| \) is the order of the Weyl group. The notation adopted also denotes \( z^{\rho_a} = \prod_{i=1}^{r^k(G)} z_i^{\rho_a} \) and \( v^{\omega_a} = \prod_{i=1}^{r^k(G)} v_i^{\omega_a} \). The other functions involved in the expression for the superconformal index are the Elliptic Gamma function

\[ \Gamma_e (z; p, q) = \prod_{m,n=0}^{\infty} \frac{1 - p^{m+1} q^{n+1} / z}{1 - p^m q^n z}, \quad |p| < 1, \quad |q| < 1, \]  

(2.4)

and the q-Pochhamer symbol

\[ (z; q)_\infty = \prod_{n=0}^{\infty} (1 - z q^n), \quad |q| < 1. \]  

(2.5)

An interesting result of [9] and [8] is to rewrite the above superconformal index in terms of solutions to certain Bethe-Ansatz like system of equations taking the generic form of

\[ Q_i \left( u; \xi, \nu_R, \omega \right) = 1 \quad \forall \quad i = 1, \ldots, r^k(G) \]  

(2.6)

where \( \omega \) is such that \( r\tau = s\sigma \) with \( r \) and \( s \) coprime integer numbers (in practice we will evaluate the equations for \( r = s \)). Furthermore, the “Bethe-Ansatz operator” is defined as:

\[ Q_i \left( u; \xi, \nu_R, \omega \right) = \prod_{a=1}^{n_i} \prod_{\rho_a \in \mathfrak{R}_a} P \left( \rho_a \left( u \right), \omega_a \left( \xi \right) + r_a \nu_R; \omega \right)^{\rho_a^i}, \]  

(2.7)

where

\[ P \left( u; \omega \right) = e^{-\pi i \omega^2 + \pi i u} \theta_0 \left( u ; \omega \right). \]  

(2.8)

Thus,

\[ P \left( \rho_a \left( u \right) + \omega_a \left( \xi \right) + r_a \nu_R; \omega \right) = \frac{e^{-\pi i \xi^2 / (\rho_a (u) + \omega_a (\xi) + r_a \nu_R)^2 + \xi (\rho_a (u) + \omega_a (\xi) + r_a \nu_R)} \theta_0 \left( \rho_a (u) + \omega_a (\xi) + r_a \nu_R; \omega \right)}{\theta_0 \left( \rho_a (u) + \omega_a (\xi) + r_a \nu_R; \omega \right)}, \]  

(2.9)

where:

\[ \theta_0 \left( u; \omega \right) = \left( e^{2\pi i u}, e^{2\pi i \omega} \right)_\infty \left( e^{2\pi i (\omega - u)}, e^{2\pi i \omega} \right)_\infty. \]  

(2.10)
Now we would like to evaluate the Bethe-Ansatz equations for the case of a toric quiver gauge theory. Toric quiver gauge theories describe the low energy dynamics of a stack of $N$ D3 branes probing the tip of a toric Calabi-Yau singularity; there is by now a vast literature detailing how to construct a supersymmetric field theory given toric data (see, for example [15, 16]). Consider a toric quiver gauge theory whose gauge group $G$ has $n_v$ simple factors (in all the $\mathcal{N} = 1$ quiver gauge theories we will deal with, the number of simple factors coincides with the number of vector multiplets). We focus, for concreteness, on the case in which all the gauge group factors are $SU(N_a)$, $a$ goes from 1 to $n_v$, with $N_a = N \forall a$, the same numerical value for all nodes. In these theories the weight vectors $\rho$ are such that for any bi-fundamental field $\Phi_{ab}$ (notice that in the more generic notation used in [9], the index $a$ of $\Phi_a$ would now split into $ab$):

$$\rho_{ab}^{(a,b)}(u) \equiv u_{ij}^{ab} \equiv u_i^a - u_j^b.$$  \hspace{1cm} (2.11)

Let us now evaluate the operator $P(u; \omega)$ for a generic $\Phi_{ab}$ (when $\Phi_{ab}$ transforms in the adjoint representation of $G$ then in this notation $a = b$):

$$Q_{ia}(u; \xi, \tau, \sigma, \omega) = \prod_{(a,b)} \prod_{j_b} P(u_{ia} - u_{jb} + \sum_{l=1}^{d-1} q_{(a,b)}^{l} \Delta_l) \rho_{ij}^{(a,b)},$$  \hspace{1cm} (2.12)

where $(a, b)$ run over all the fields $\Phi_{ab}$ for a fixed $a$. The $d - 1$ fugacities correspond to the flavor symmetries appearing in the generic toric quiver gauge theories that we will study, $d$ is the number of external points of the toric diagram that are related to the quivers defining the theory [14]. If we denote $(a, b) \equiv (a, b)|_{\rho_{ij}^{(a,b)} > 0}$, which implies:

$$Q_{ia}(u; \xi, \tau, \sigma, \omega) = \prod_{(a,b)} \prod_{j_b} \prod_{\rho_{ij}^{(a,b)}} \left[ \frac{P(u_{ia} - u_{jb} + \sum_{l=1}^{d-1} q_{(a,b)}^{l} \Delta_l)}{P(u_{jb} - u_{ia} + \sum_{l=1}^{d-1} q_{(b,a)}^{l} \Delta_l)} \right]^{\rho_{ij}^{(a,b)}} \hspace{1cm} (2.13)$$

$$= \prod_{(a,b)} \prod_{j_b} \prod_{\rho_{ij}^{(a,b)}} \left[ e^{-2\pi i (-u_{ia} + u_{jb})} \theta_0 \left( u_{ia} - u_{jb} + \sum_{l=1}^{d-1} q_{(a,b)}^{l} \Delta_l; \omega \right) \frac{\theta_0 \left( u_{ia} - u_{jb} + \sum_{l=1}^{d-1} q_{(b,a)}^{l} \Delta_l; \omega \right)}{\theta_0 \left( -u_{ia} + u_{jb} + \sum_{l=1}^{d-1} q_{(b,a)}^{l} \Delta_l; \omega \right)} \right]^{\rho_{ij}^{(a,b)}} \hspace{1cm} (2.14)$$

Let us now introduce a Lagrange multiplier $\lambda_a$ that accounts for the constraint ensuring the condition $\sum_i u_i^a = 0$ [8], with its help, equation (2.13) can be written as:

$$Q_{ia}(u; \xi, \tau, \sigma, \omega) = e^{2\pi i \left( \sum_i \lambda_a - \sum_j u_i^a \right)} \prod_{(a,b)} \prod_{j_b} \theta_0 \left( -u_{ij}^{ab} + \sum_{l=1}^{d-1} q_{(a,b)}^{l} \Delta_l; \omega \right) \frac{\theta_0 \left( u_{i}^{ab} + \sum_{l=1}^{d-1} q_{(b,a)}^{l} \Delta_l; \omega \right)}{\theta_0 \left( -u_{i}^{ab} + \sum_{l=1}^{d-1} q_{(b,a)}^{l} \Delta_l; \omega \right)},$$  \hspace{1cm} (2.14)

where we have denoted $u_{ia} - u_{jb} \equiv u_{ij}^{ab}$. Restricting ourselves to the case with $\tau = \sigma$, we would like to propose a set of $u_{ij}^{ab}$ that makes (2.14) equal to 1, thus solving the Bethe-Ansatz equation (2.6). It is natural to make an attempt with a direct generalization of the type of solution encountered in [8], namely: $u_{ij}^{ab} = i(\xi_i - \xi_j)$ and $\lambda_a = (N - 1)/2$. These solutions appeared first in [17] while evaluating the topologically twisted of 4d $\mathcal{N} = 1$ theories on $T^2 \times S^2$ in the high temperature limit; it was later shown in [18] that such configuration provides an exact solution on the Bethe-Ansatz equations.
Consider one generic factor entering in (2.14) for a fixed value of $b$:

\[
\prod_{j_{b}} \frac{\theta_{0}(u_{ij}^{ab} + \Delta_{ab}; \omega)}{\theta_{0}(-u_{ij}^{ab} + \Delta_{ab}; \omega)} = \frac{\prod_{j=0}^{n_{a}-1} \theta_{0} \left( \frac{\tau}{N} k + \Delta_{ab} \right) \times \prod_{k=1}^{1} \theta_{0} \left( \frac{\tau}{N} k + \Delta_{ab} \right)}{\prod_{j=0}^{N-1} \theta_{0} \left( \frac{\tau}{N} k + \Delta_{ab} \right) \times \prod_{k=1}^{1} \theta_{0} \left( \frac{\tau}{N} k + \Delta_{ab} \right)}
\]

with \( \Delta_{ab} = \sum_{l=1}^{d-1} q_{(a,b)}^{l} \Delta_{l} \).

In (2.15) we have used the following properties of the \( \theta_{0} \) function:

\[
\theta_{0}(u + n + m \tau; \tau) = -e^{2\pi iu - \pi im \tau(m-1)} \theta_{0}(u; \tau)
\]

\[
\theta_{0}(u; \tau) = \theta_{0}(\tau - u; \tau) = -e^{2\pi iu} \theta_{0}(-u; \tau).
\]

Inserting (2.15) back into (2.14) leads to multiplying all the results obtained in (2.15) for all \( n_{a} \) values of \( b \) connected with \( a \) via some field \( \Phi_{ab} \):

\[
Q_{i_{a}}(u; \xi, \tau) = e^{2\pi i \left( n_{a} \frac{i_{1}}{N} - \sum_{j_{b}} \frac{i_{j_{b}}}{i_{b}}(i_{a} - j_{b}) \right)} (-1)^{n_{a}(N-1)} \left( e^{2\pi i \sum_{b} \Delta_{ab}} \right)^{N-2i_{a}+1} \left( e^{2\pi i n_{a} \tau} \right)^{i_{a} - \frac{N_{a} - 1}{2}}
\]

\[
\sum_{b=1}^{n_{a}} \Delta_{ab} = 0
\]

\[
\downarrow
\]

\[
Q_{i_{a}}(u; \xi, \tau) = e^{2\pi i \left( n_{a} \frac{i_{1}}{N} - \sum_{j_{b}} \frac{i_{j_{b}}}{i_{b}}(i_{a} - j_{b}) \right)} (-1)^{n_{a}(N-1)} \left( e^{2\pi i n_{a} \tau} \right)^{i_{a} - \frac{N_{a} - 1}{2}}
\]

\[
= e^{2\pi i \left( n_{a} \frac{i_{1}}{N} - \sum_{j_{b}} \frac{i_{j_{b}}}{i_{b}}(N_{a} - i_{1} - \frac{N_{a} - 1}{2}) \right)} (-1)^{n_{a}(N-1)} \left( e^{2\pi i n_{a} \tau} \right)^{i_{a} - \frac{N_{a} - 1}{2}} = 1.
\]

### 2.1 Evaluation of the index

The formula for the index reads:

\[
\mathcal{I}(p, q; v) = \kappa_{G} \sum_{\hat{u} \in \mathbb{B}_{ABE}} \mathcal{Z}_{\text{tot}}(\hat{u}; \xi, \nu_{R}, r \omega, s \omega) H(\hat{u}; \xi, \nu_{R}, \omega)^{-1}
\]

\[
\kappa_{G} = \frac{(p; p)_{\infty}^{r k(G)} (q; q)_{\infty}^{r k(G)}}{|W_{G}|}
\]

\[
\mathcal{Z}_{\text{tot}}(u; \xi, \nu_{R}, r \omega, s \omega) = \sum_{(m_{a})=1}^{r} \mathcal{Z}(u - m \omega; \xi, \nu_{R}, r \omega, s \omega)
\]

\[
\mathcal{Z}(u; \xi, \nu_{R}, r \omega, s \omega) = \frac{\prod_{\Phi_{ab}} \Gamma_{e} \left( u_{i_{a}} - u_{j_{b}} + \sum_{l=1}^{d-1} q_{(a,b)}^{l} \Delta_{l}; \tau, \sigma \right)}{\prod_{\alpha \in \Delta} \Gamma_{e}(\alpha(u); \tau, \sigma)}
\]

\[
H(u; \xi, \nu_{R}, \omega) = \det \left[ \frac{1}{2\pi i} \frac{\partial Q_{i_{a}}(u; \xi, \nu_{R}, \omega)}{\partial u_{j_{b}}} \right]_{i_{a}j_{b}}.
\]

Dominant contributions to the index in the large \( N \) limit will come from terms analogous to the dominating the expression obtained in [3] for the \( \mathcal{N} = 4 \) SYM theory. This implies that in
order to investigate the large $N$ limit of Eq. (2.18), we only need to consider the following term:

$$
\Gamma_e(u_{ij}^a + \Delta_{ab}; \tau, \tau) = \frac{e^{-\pi i Q(u_{ij}^a + \Delta_{ab}; \tau, \tau)}}{\theta_0(\frac{u_{ij}^a + \Delta_{ab}}{\tau}; -1)} \prod_{k=0}^{\infty} \psi\left(\frac{k+1+u_{ij}^a - \Delta_{ab}}{\tau}\right) \prod_{k=0}^{\infty} \psi\left(\frac{k-u_{ij}^a - \Delta_{ab}}{\tau}\right)
$$

(2.19)

$$
Q(u; \tau, \sigma) = \frac{u^3}{3\tau^2} - \frac{\tau + \sigma - 1}{2\tau^2} u^2 + \left(\frac{\tau + \sigma}{2}\right)^2 + \tau\sigma - 3\left(\tau + \sigma\right) + 1 (\tau + \sigma - 1) (\tau + \sigma - \tau\sigma)
$$

$$
Q(u + \Delta; \tau, \tau) = \frac{u^3}{3\tau^2} + u^2 \left(\frac{\Delta}{\tau^2} - \frac{2\tau - 1}{2\tau^2}\right) + u \left(\frac{1 - 6\tau + 5\tau^2}{6\tau^2} + \frac{\Delta^2}{\tau^2} - \frac{2\tau - 1}{\tau}\right) - \frac{\Delta^2}{2\tau^2} (2\tau - 1) + \frac{\Delta}{6\tau^2} (5\tau^2 - 6\tau + 1) + \frac{1}{12\tau^2} (2\tau - 1) (2\tau - \tau^2) + \frac{\Delta^3}{\tau^2}.
$$

Now recalling that solutions of the form $u_{ij}^a = \frac{\tau}{N}(i_a - j_b)$ satisfy the Bethe-Ansatz equations, we proceed to analyze the large $N$ limit. Thus, the leading contribution in $N$ to $\log I$ takes the form:

$$
\log I = \frac{i\pi N^2}{3\tau^2} \sum_{\Phi_{ab}} (\Delta_{ab}) \left(\Delta_{ab} - \tau + \frac{1}{2}\right)(\Delta_{ab} + \tau + 1) - \frac{i\pi N^2}{3\tau^2} \sum_{v} \tau\left(\tau - \frac{1}{2}\right)(\tau - 1) - \frac{i\pi N^2}{3\tau^2} \sum_{\Phi_{ab}} \left[\Delta_{ab}\right] \left(\Delta_{ab} + \frac{1}{2}\right)(\Delta_{ab} + 1) + 3\tau \left[\Delta_{ab}\right] - 3\tau \left[\Delta_{ab}\right]^2 + 3\tau \left[\Delta_{ab}\right] - 3\tau \left[\Delta_{ab}\right] + \frac{i\pi N^2}{3\tau^2} (n_\chi - n_v) \tau\left(\tau - \frac{1}{2}\right)(\tau - 1),
$$

where $[\Delta_{ab}]$ is defined such that $[\Delta]_\tau = \Delta$ mod 1 [8], the sum $\sum_v$ is carried over the $n_v$ vector multiplets and $n_\chi$ is the number of chiral fields. The conservation of $U(1)$ charges implies $\sum_{\Phi_{ab}} [\Delta_{ab}]_\tau = 0$, which allows us to eliminate every linear term in $[\Delta_{ab}]_\tau$ appearing in Eq. (2.21), therefore we can write:

$$
\log I = \frac{i\pi N^2}{3\tau^2} \sum_{\Phi_{ab}} \left[\Delta_{ab}\right] \left(\Delta_{ab} + \frac{1}{2}\right)(\Delta_{ab} + 1) - 3\tau \left[\Delta_{ab}\right]^2 + \frac{i\pi N^2}{3\tau^2} (n_\chi - n_v) \tau\left(\tau - \frac{1}{2}\right)(\tau - 1).
$$

(2.21)

Let us now analyse the properties of the function we have obtained. Recalling that:

$$
[\Delta + 1]_\tau = [\Delta]_\tau,
$$

$$
[-\Delta]_\tau = -[\Delta]_\tau - 1
$$

$$
[\Delta + \tau]_\tau = [\Delta]_\tau + \tau,
$$

Then, it is possible to write:

$$
K(\Delta, \tau) \equiv \left([\Delta]_\tau + \frac{1}{2}\right)([\Delta]_\tau + 1) - 3\tau \left[\Delta\right]^2 = \frac{1}{2} (2\Delta^3 - 3\Delta|\Delta + \Delta - 6\tau|\Delta) \tau (\tau - \frac{1}{2}) (\tau - 1),
$$

(2.23)

which holds when $|\Delta| < 1$.

Equation (2.21) is very similar to the one obtained in [11] when analyzed n the Cardy-like limit of the index, however, there is an extra contribution of the form $\frac{i\pi N^2}{3\tau^2} (n_\chi - n_v) \tau\left(\tau - \frac{1}{2}\right)(\tau - 1)$ which is still of order $O(N^2)$ but sub-leading when $\tau \to 0$. Notice that at this point there is
no dependence on the holonomies of the gauge groups since we have already evaluated in the solutions of the Bethe-Ansatz equations.

Finally we have:

$$\log I = -\frac{i\pi N^2}{3\tau^2} \sum_{ab} K(\Delta_{ab}, \tau) + \frac{i\pi N^2}{3\tau^2} (n_\chi - n_v) \tau \left( \tau - \frac{1}{2} \right) (\tau - 1).$$

(2.24)

Defining $\Delta_d$ such that: $\sum_{l=1}^d \Delta_l - 2\tau = -1$ [8], it can be shown that $\log I$ can be written as:

$$\log I = -\frac{i\pi N^2}{6\tau^2} C_{IJK} \Delta_I \Delta_J \Delta_K + \frac{i\pi N^2}{3\tau^2} (n_\chi - n_v) \tau \left( \tau - \frac{1}{2} \right) (\tau - 1)$$

(2.25)

The coefficients $C_{IJK}$ in (2.25) correspond, as pointed out in [14], to the Chern-Simons couplings of the holographic dual gravitational description as elucidated in [19]. It is worth noting that the new contribution to the index cancels in the prototypical example of $\mathcal{N} = 4$ SYM since when written in $\mathcal{N} = 1$ language, $n_\chi = n_v = 3$. In that case, as demonstrated in [8], the large $N$ and Cardy limits coincide. The correction we highlight in equation (2.25) becomes important for theories with $n_\chi \neq n_v$ which are generic in the space of toric quiver gauge theories. In the following section we proceed to evaluate the superconformal index for various models, some of them recently discussed in a similar context in [14], and compare our results with (2.25). We will, in one very relevant example, explore how the new contribution affects the would-be macroscopic entropy of the dual black holes.

3 The superconformal index of various SCFT’s

We will test our result (2.21) in various cases in each of which we follow the prescription of charge assignment used in [14]. We will restrict ourselves to the regime of fugacities $\Delta_i$ of the $d - 1\ U(1)$ global symmetries such that:

$$0 \leq \Delta_i \leq \frac{1}{2} \quad \forall i, \quad 0 \leq \sum_{i=1}^{d-1} \Delta_i \leq 1 \quad (3.1)$$

which in our case will be useful to evaluate the function $K(\Delta, \tau)$ using equation (2.23). This regime also coincides with the one in which the existence of a universal saddle point in which all the holonomies vanish according to the analysis carried in [14] can be ensured.

3.1 The conifold theory

We would like to study the index in the large $N$ limit and thus investigate it beyond the Cardy-like limit. We will explicitly evaluate the correction determined in equation (2.25). To do so we start with one of the simplest examples of toric quiver gauge theories – the conifold theory [20] whose quiver diagram is given below. We take the ranks of all the gauge groups equal ($N_1 = N_2 = N$) and the sub-index in $N_i$ helps describe the representations of the matter fields:
The superpotential is

\[ W \propto \epsilon_{ij} \epsilon_{kl} \text{Tr} \left[ A^i B^k A^j B^l \right] \]  

(3.2)

The global charges of the conformal field theory are: a \( U(1)_R \) factor, two \( SU(2) \) factors and finally there is a \( U(1)_B \) baryonic symmetry. A fascinating fact about this theory is that it admits a gravity dual in terms of strings in \( \text{AdS}_5 \times T^{1,1} \). The isometries of \( T^{1,1} \) realize the mesonic symmetries of the field theory in terms of the isometries of \( \mathbb{C}P^1 \times \mathbb{C}P^1 \); the \( U(1)_B \) baryonic symmetry is associated to the unique non-trivial three-cycle of the geometry. It is worth pointing out that the rotating electrically charged black holes dual to the superconformal index have not yet been constructed on the supergravity side, and that remains an outstanding problem.

We use the basis for the charges suggested by the toric diagram discussed in [14] and we summarize them in the following table:

| Field | \( U(1)_R \) | \( U(1)_1 \) | \( U(1)_2 \) | \( U(1)_3 \) |
|-------|-------------|-------------|-------------|-------------|
| \( A_1 \) | 1/2 | 1 | 0 | 0 |
| \( A_2 \) | 1/2 | 0 | 0 | 1 |
| \( B_1 \) | 1/2 | 0 | 1 | 0 |
| \( B_2 \) | 1/2 | -1 | -1 | -1 |

With this information we are ready to evaluate equation (2.21):

\[
\log \mathcal{I} = -\frac{i \pi N^2}{3 \tau^2} \left[ K (\Delta_1, \tau) + K (\Delta_2, \tau) + K (\Delta_3, \tau) + K (-\Delta_1 - \Delta_2 - \Delta_3, \tau) \right] + \\
+ \frac{i \pi N^2}{3 \tau^2} 2 \tau \left( \tau - \frac{1}{2} \right) (\tau - 1) \\
= -\frac{i \pi N^2}{\tau^2} \left[ -\Delta_1^2 (\Delta_2 + \Delta_3) - \Delta_2 \Delta_3 (1 - 2 \tau + \Delta_2 + \Delta_3) - \Delta_1 (\Delta_2 + \Delta_3) (1 - 2 \tau + \Delta_2 + \Delta_3) \right] + \\
+ \frac{i \pi N^2}{3 \tau^2} 2 \tau \left( \tau - \frac{1}{2} \right) (\tau - 1),
\]

(3.3)

where we have used \( n_\chi = 4 \) and \( n_v = 2 \). After imposing the condition \( \sum_{I=1}^4 \Delta_I - 2 \tau = -1 \) yields :

\[
\log \mathcal{I} = -\frac{i \pi N^2}{\tau^2} [\Delta_2 \Delta_3 \Delta_4 + \Delta_1 \Delta_3 \Delta_4 + \Delta_1 \Delta_2 \Delta_3 + \Delta_1 \Delta_2 \Delta_4] + \\
+ \frac{i \pi N^2}{3 \tau^2} 2 \tau \left( \tau - \frac{1}{2} \right) (\tau - 1).
\]

(3.4)

We see that \( \log \mathcal{I} \) presents the behavior proposed in (2.25) with the explicit \( O(N^2) \) correction to the index.

### 3.2 The Suspended Pinch Point

The suspended pinch point (SPP) gauge theory corresponds to the near horizon limit of a stack of \( N \) D3 branes probing the tip of the conical singularity , \( x^2y = wz \). The SPP gauge theory is described by the following quiver
where we have used:

\[ \log \mathcal{I} = -i \pi N^2 \left[ K(\Delta_1 + \Delta_2, \tau) + K(\Delta_4, \tau) + K(-\Delta_1 - \Delta_2 - \Delta_4, \tau) + K(\Delta_2, \tau) + K(\Delta_1, \tau) + \right. \]

\[ \left. K(-\Delta_1 - \Delta_2 - \Delta_3, \tau) + K(\Delta_4, \tau) \right] + \frac{i \pi N^2}{3 \tau^2} 4\tau \left( \tau - \frac{1}{2} \right) (\tau - 1) \]  

Each field \( X_{ij} \) transforms in the \( N \) representation of the index \( i \)-th node and in the \( \overline{N} \) of the \( j \)-th node. The field \( \phi \) transforms in the adjoint representation of the corresponding gauge group.

The charge assignment for the \( U(1)_R \) and the extra \( U(1)_i \) global symmetries can be taken as:

| Field | \( U(1)_R \) | \( U(1)_1 \) | \( U(1)_2 \) | \( U(1)_3 \) | \( U(1)_4 \) |
|-------|---------------|---------------|---------------|---------------|---------------|
| \( \phi \) | \( 4/5 \) | 1 | 1 | 0 | 0 |
| \( X_{12} \) | \( 2/5 \) | 0 | 0 | 0 | 1 |
| \( X_{21} \) | \( 4/5 \) | -1 | -1 | 0 | -1 |
| \( X_{23} \) | \( 2/5 \) | 0 | 1 | 0 | 0 |
| \( X_{32} \) | \( 2/5 \) | 1 | 0 | 0 | 0 |
| \( X_{31} \) | \( 4/5 \) | -1 | -1 | -1 | 0 |
| \( X_{13} \) | \( 2/5 \) | 0 | 0 | 1 | 0 |

All the ranks are taken to be the same with \( N_1 = N_2 = N_3 = N \) and the sub-indices are meant to help understand the representation properties of the matter fields. The superpotential is

\[ W = \text{Tr} \left[ X_{21}X_{12}X_{23}X_{32} - X_{21}X_{32}X_{31}X_{13} + X_{13}X_{31}\phi - X_{12}X_{21}\phi \right]. \]

This result is in agreement with equation (2.25) which is what is expected from toric geometry and reinforces the validity of the analysis of [14] which was limited to the Cardy-Like limit but contains a correction.
3.3 The dP\textsubscript{1} theory

We consider now the theory arising from a stack of $N$ D3 branes at the tip of the complex Calabi-Yau cone whose base is the first del Pezzo surface. The quiver associated to this theory is:

![Quiver Diagram]

where $N_1 = N_2 = N_3 = N_4 = N$ and the superpotential is given by:

$$W = \epsilon_{\alpha\beta} \text{Tr} \left[ X_{34}^{(\alpha)} X_{41}^{(\beta)} X_{13} + X_{34}^{(\alpha)} X_{23}^{(\beta)} X_{42} + X_{12} X_{34}^{(\alpha)} X_{41}^{(\beta)} \right].$$ \hspace{1cm} (3.8)

The charge assignment specified by the toric data is given by:

| Field   | $U(1)_R$ | $U(1)_1$ | $U(1)_2$ | $U(1)_3$ |
|---------|----------|----------|----------|----------|
| $X_{12}$ | 1/2      | 0        | 0        | 1        |
| $X_{23}^{(1)}$ | 1/2      | 0        | 1        | 0        |
| $X_{23}^{(2)}$ | 1/2      | -1       | -1       | -1       |
| $X_{34}^{(1)}$ | 1        | 0        | 1        | 1        |
| $X_{34}^{(2)}$ | 1        | -1       | -1       | 0        |
| $X_{34}^{(3)}$ | 1/2      | 1        | 0        | 0        |
| $X_{41}^{(1)}$ | 1/2      | 0        | 1        | 0        |
| $X_{41}^{(2)}$ | 1/2      | -1       | -1       | -1       |
| $X_{13}$ | 1/2      | 1        | 0        | 0        |
| $X_{41}$ | 1/2      | 1        | 0        | 0        |

Let us now evaluate the leading in $N$ part of the superconformal index according to (2.21):

$$\log \mathcal{I} = -\frac{i\pi N^2}{3\tau^2} \left[ 2K(\Delta_1, \tau) + K(\Delta_3, \tau) + 2K(\Delta_2, \tau) + 2K(-\Delta_1 - \Delta_2 - \Delta_3, \tau) + \right. \hspace{1cm} (3.9)$$

$$+ \ K(\Delta_2 + \Delta_3, \tau) + K(-\Delta_1 - \Delta_2, \tau) \right] +$$

$$+ \frac{i\pi N^2}{\tau^2} 2\tau \left( \tau - \frac{1}{2} \right) (\tau - 1)$$

$$= -\frac{i\pi N^2}{\tau^2} \left[ -(2\Delta_1 + \Delta_2) \Delta_3^2 - 3\Delta_1 \Delta_2 (\Delta_1 + \Delta_2 + 1) - \right.$$ \hspace{1cm}

$$- \left( \Delta_2^2 + 4\Delta_1 \Delta_2 + \Delta_2 + 2\Delta_1 (\Delta_1 + 1) \right) \Delta_3 + \tau(6\Delta_1 \Delta_2 + 2(2\Delta_1 + \Delta_2)\Delta_3) + \frac{i\pi N^2}{\tau^2} 2\tau \left( \tau - \frac{1}{2} \right) (\tau - 1). \right.$$  

We notice that $n_\chi = 10$ and $n_v = 4$. Introducing now $\Delta_4$ via the constraint $\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - 2\tau = -1$ we obtain:

$$\log \mathcal{I} = -\frac{i\pi N^2}{\tau^2} \left[ 2\Delta_1 \Delta_2 \Delta_3 + 3\Delta_1 \Delta_2 \Delta_4 + 2\Delta_1 \Delta_3 \Delta_4 + 2\Delta_2 \Delta_3 \Delta_4 \right] +$$

$$+ \frac{i\pi N^2}{\tau^2} 2\tau \left( \tau - \frac{1}{2} \right) (\tau - 1).$$ \hspace{1cm} (3.10)
3.4 \( Y^{p,q} \) quiver gauge theories

The \( Y^{p,q} \) model corresponds to quiver gauge theories with 2\( p \) gauge groups and a chiral field content of bifundamental fields. The charge assignment and the corresponding multiplicity of the fields are shown below:

| Multiplicity | \( U(1)_1 \) | \( U(1)_2 \) | \( U(1)_3 \) | \( U(1)_R \) |
|--------------|-------------|-------------|-------------|-------------|
| \( p+q \)    | 1           | 0           | 0           | \( 1/2 \)   |
| \( p \)      | 0           | 1           | 0           | \( 1/2 \)   |
| \( p-q \)    | 0           | 0           | 1           | \( 1/2 \)   |
| \( q \)      | -1          | -1          | -1          | 1           |
| \( q \)      | -1          | 0           | 1           | 1           |

where we have used the prescription of [14]. Now we evaluate the leading, order \( \mathcal{O}(N^2) \), part of the superconformal index (2.21):

\[
\log \mathcal{I} = \frac{-i\pi N^2}{3\tau^2} [(p+q)K(\Delta_1, \tau) + pK(\Delta_2, \tau) + (p-q)K(\Delta_3, \tau) + pK(-\Delta_1 - \Delta_2 - \Delta_3, \tau) + qK(\Delta_2 + \Delta_3, \tau) + qK(-\Delta_1 - \Delta_2, \tau)] + \frac{i\pi N^2}{3\tau^2} 2(p+q)\tau \left( \tau - \frac{1}{2} \right) (\tau - 1) \quad \text{(3.11)}
\]

Here \( n_\chi = 4p + 2q \) and \( n_\nu = 2p \). Finally we eliminate \( \tau \) from (3.11) using \( \Sigma_{I=1}^4 \Delta_I - 2\tau = -1 \), which successfully reproduce the structure of (2.25):

\[
\log \mathcal{I} = \frac{-i\pi N^2}{3\tau^2} [p\Delta_1\Delta_2\Delta_3 + (p+q)\Delta_1\Delta_2\Delta_4 + p\Delta_1\Delta_3\Delta_4 + (p-q)\Delta_2\Delta_3\Delta_4] + \frac{i\pi N^2}{3\tau^2} 2(p+q)\tau \left( \tau - \frac{1}{2} \right) (\tau - 1). \quad \text{(3.12)}
\]

4 Corrections to the dual black hole entropy

Let us now investigate how the extra term in (2.25) modifies the entropy obtained by taking the Legendre transform of \( \log \mathcal{I} \). Our starting point it to organize the computation as to maximally take advantage of the scaling properties (with respect to \( \Delta_I \) and \( \tau \)) of the superconformal index in the Cardy limit. Thus we write:

\[
\log \mathcal{I} = S_E + S_\tau \quad \text{(4.1)}
\]

\[
S_E = \frac{i\pi N^2}{6\tau^2} C_{IJK} \Delta_I \Delta_J \Delta_K
\]

\[
S_\tau = \frac{i\pi N^2}{3\tau^2} (n_\chi - n_\nu) \tau \left( \tau - \frac{1}{2} \right) (\tau - 1)
\]
Since $S_\tau$ is independent of $\Delta_I$ we have:

\[
\begin{align*}
\frac{\partial \log I}{\partial \Delta_I} &= \frac{\partial S_E}{\partial \Delta_I} \\
\frac{\partial \log I}{\partial \tau} &= \frac{\partial S_E}{\partial \tau} + \frac{\partial S_\tau}{\partial \tau} 
\end{align*}
\]

(4.2)

The Legendre transform leading to the entropy can be written as:

\[
S(Q, \Lambda) = \log I + 2\pi i \left( \sum_{I=1}^{d} \Delta_I Q_I - 2\tau J \right) + 2\pi i \Lambda \left( \sum_{I=1}^{d} \Delta_I - 2\tau + 1 \right),
\]

(4.3)

where $\Lambda$ is a Lagrange multiplier imposing the constraint. The extremization condition implies:

\[
\begin{align*}
\frac{\partial S}{\partial \Delta_I} &= 0, \Rightarrow \frac{\partial \log I}{\partial \Delta_I} = -2\pi i (Q_I + \Lambda) \\
\frac{\partial S}{\partial \tau} &= 0 \Rightarrow \frac{\partial S_E}{\partial \tau} = -4\pi i (\bar{J} - \Lambda)
\end{align*}
\]

(4.4)

\[
\bar{J} \equiv J + \frac{1}{4\pi i} \frac{\partial S_\tau}{\partial \tau}.
\]

The homogeneity of $S_E$ leads to the important relation:

\[
S_E = \sum_{I=1}^{d} \Delta_I \frac{\partial S_E}{\partial \Delta_I} + \tau \frac{\partial S_E}{\partial \tau}.
\]

(4.5)

Following [14], we insert (4.5) in (4.4) and evaluating on the extremization solutions we find:

\[
S(Q,J) = 2\pi i \Lambda (Q,J) + S_\tau - 4\pi i \tau \left( \bar{J} - J \right)
\]

(4.6)

\[
= 2\pi i \Lambda (Q,J) + S_\tau - \tau \frac{\partial S_\tau}{\partial \tau}
\]

\[
= 2\pi i \Lambda (Q,J) + \frac{i\pi N^2}{3} (n_\chi - n_\nu) \left( \frac{1}{\tau(J)} - \frac{3}{2} \right).
\]

The properties of $S_E$ allow us to reconstruct $S_E^2$ from suitable combinations of products of its derivatives with respect to $\Delta_I$ which generically leads to a cubic equation to determine $L(Q,J)$. To make the analysis somehow concrete we now focus on a particular model.

### 4.1 Entropy corrections to the $Y^{p,p}$ theory

Let us track the implications of our correction in the particular case of gauge theories obtained as a stack of D3 branes placed at the cone over $Y^{p,p}$, equivalently, a stack of $N$ D3-branes placed at the tip of $\mathbb{C}^3/\mathbb{Z}_{2p}$, which leads to an orbifold of $\mathcal{N} = 4$ SYM. The gravity dual is expected to be simply related to string theory on $\text{AdS}_5 \times Y^{p,p}$ and one can expect to be able to read off the entropy on the gravity side readily following the extremization prescription put forward in [21]. Our goal in this section is to estimate the correction to the entropy in the Cardy limit – $S_{|\text{Cardy}}$ – due to the correction we have computed for the superconformal index.

If we specialize equation (3.12) to $(p = q)$ case we obtain:

\[
\log I = -\frac{i\pi p N^2}{3\tau^2} \left[ \Delta_1 \Delta_2 \Delta_3 + 2\Delta_1 \Delta_2 \Delta_4 + \Delta_1 \Delta_3 \Delta_4 \right] + \frac{i\pi N^2}{3\tau^2} 2\pi \left( \tau - \frac{1}{2} \right) (\tau - 1).
\]

(4.7)
This particular model – the $\mathcal{N}=p$ quiver gauge theory – has been recently analyzed in the Cardy limit in \cite{14}, where the second term in equation (4.7) was not considered. A further simplification (seding the baryonic charge $Q_2 \to 0$ and further renaming $Q_1 \to Q_2$) was previously presented in \cite{13}. In the notation of \cite{13} which we follow in this note the above simplifications lead to an entropy of the form

$$S|_{\text{Cardy}} = 2\pi \sqrt{Q_1 Q_3 + Q_1 Q_4 + Q_3 Q_4 - \frac{p N^2}{2} 2J}. \quad (4.8)$$

This is quite similar to the $\mathcal{N}=4$ SYM expression for the entropy and will guide our intuition for corrections.

Let us now return to the treatment explain at the beginning of this section. It can be shown that $S_E$ in our case (the first line in equation (4.7)) satisfies:

$$0 = \left( \frac{\partial S_E}{\partial \Delta_1} \right) \left[ 2 \left( \frac{2 \partial S_E}{\partial \Delta_3} + \frac{\partial S_E}{\partial \Delta_4} \right) \left( \frac{\partial S_E}{\partial \Delta_2} \right) - \left( \frac{\partial S_E}{\partial \Delta_3} \right)^2 - \left( \frac{2 \partial S_E}{\partial \Delta_4} \right)^2 \right] + 4pN^2 \left( \frac{\partial S_E}{\partial \tau} \right)^2. \quad (4.9)$$

Using equation (4.1), we can obtain a cubic equation for $\Lambda$ in the same spirit that \cite{13}, however we are able to keep track of the modification produced in the entropy by the presence of $S_\tau$ in \cite{22,25}, hence, we have:

$$0 = (Q_1 + \Lambda)[2 \left( 2(Q_3 + \Lambda) + (\Lambda + Q_4) \right) (\Lambda + Q_2) - (\Lambda + Q_2)^2 - (2(\Lambda + Q_3) - (\Lambda + Q_4))^2] + 4pN^2(\Lambda - \bar{J})^2. \quad (4.10)$$

The solution of (4.10) when plugged into equation (4.6) leads to an entropy of the form:

$$S(Q, \Lambda) = 2\pi \sqrt{(Q_1 + Q_3)(Q_2 + Q_4) + \frac{Q_4 Q_2}{2} - \frac{Q_3^2}{4} - \frac{Q_4^2}{4} - 2N^2 p \bar{J}}$$

$$+ \frac{i\pi N^2}{3} (n_{\chi} - n_{\nu}) \left( \frac{1}{\tau} - \frac{3}{2} \right)$$

$$= 2\pi \sqrt{(Q_1 + Q_3)(Q_2 + Q_4) + \frac{Q_4 Q_2}{2} - \frac{Q_3^2}{4} - \frac{Q_4^2}{4} - 2N^2 p \left[ J + \frac{N^2}{12} (n_{\chi} - n_{\nu}) \left( 1 - \frac{1}{2\tau^2(J)} \right) \right]}$$

$$+ \frac{i\pi N^2}{3} (n_{\chi} - n_{\nu}) \left( \frac{1}{\tau(J)} - \frac{3}{2} \right)$$

In the above expression we have left $(n_{\chi} - n_{\nu})$ explicitly in the corrections to highlight its effect. The angular velocity $\tau(J)$ appears only formally, it should be substituted by the extremization procedure, we have indicated such operation as $\tau(J)$. The most dramatic effect is a shift in the angular momentum.

## 5 Conclusions

In this brief note we have explored the superconformal index following the Bethe Ansatz approach introduced by Benini and Milan \cite{19}. We have shown that a class of solutions can be extended to solve the Bethe Ansatz equation for a large class of 4d $\mathcal{N}=1$ supersymmetric gauge theories. The Bethe Ansatz approach has the advantage that it does not require to take the Cardy limit and therefore provides a more complete large $N$ expression. Indeed, for generic toric quiver gauge theories we determined that there is a contribution of the order $\mathcal{O}(N^2)$
which is sub-leading in the Cardy limit. We hope that more work along this direction might eventually allow to understand the growth of states in the index in a more systematic fashion. For example, by exploiting the Bethe Ansatz approach to the topologically twisted index a systematic study of $1/N$ corrections for the ABJM index was performed in [22]; a similar study for a Chern-Simons matter theory dual to massive IIA black holes was reported in [23]. Such understanding of $1/N$ corrections will naturally translate into interesting aspects in the dual quantum gravity side for AdS$_5$ black holes. For example, the statistical entropy of certain magnetically charged AdS$_5$ black holes has recently been given a microscopic explanation in terms of the topologically twisted index [1] (see [2, 3] for a reviews with comprehensive lists of references). The investigation of sub-leading (logarithmic in $N$) corrections such as those performed recently [24, 25] have helped clarify the nature of the degrees of freedom on the gravitational side of the duality. One would hope for similar developments in the context of AdS$_5$ black holes.

There are many other interesting open problems. At the technical level, it would be interesting to generalize the Bethe Ansatz approach to arbitrary fugacities such that a general expression depending on both angular momenta can be achieved. There is little doubt that such generalization will yield the expected results but it will clarify the inner workings of the evaluation of the superconformal index. In this manuscript we have completely avoided the subtle discussion concerning the space of solutions of the Bethe Ansatz equations, we limited ourselves to just one class and showed that it yields a contribution sufficient to extract the dual black hole entropy and its potential corrections. It would be very illuminating to have a better understanding of all the solutions and how one should weight their contributions to the index.

Finally, it is an important open problem to construct explicitly the black holes dual to the field theories discussed in this manuscript. Our computation, as well as those in a number of recent publications [12, 13, 14], shows that it is relatively easy to find the superconformal index in a large class of supersymmetric four-dimensional field theories some of which have known supergravity dual. Moreover, using the entropy formula one can evaluate the entropy and realize that it corresponds to that of large black holes in AdS$_5$. However, the explicit black hole construction on the gravity side is still in its infancy, not much is known beyond the AdS$_5$ black holes dual to $\mathcal{N} = 4$ SYM (and some of its orbifolds). It remains an outstanding challenge for the supergravity community to explicitly construct rotating electrically charged black holes which could be understood as dual of available field theory results. One particular example that comes to mind among the class discussed in this note would be the black holes in asymptotically $\text{AdS}_5 \times T^{1,1}$ and, more generally, $\text{AdS}_5 \times Y^{p,q}$.

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