ZETA-FUNCTIONS OF ROOT SYSTEMS AND POINCARÉ POLYNOMIALS OF WEYL GROUPS

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Abstract. We consider a certain linear combination $S(s,y;I;\Delta)$ of zeta-functions of root systems, where $\Delta$ is a root system of rank $r$ and $I \subset \{1, 2, \ldots, r\}$. Showing two different expressions of $S(s,y;I;\Delta)$, we find that a certain signed sum of zeta-functions of root systems is equal to a sum involving Bernoulli functions of root systems. This identity gives a non-trivial functional relation among zeta-functions of root systems, if the signed sum does not identically vanish. This is a generalization of the authors’ previous result proved in [7], in the case when $I = \emptyset$. We present several explicit examples of such functional relations. A criterion of the non-vanishing of the signed sum, in terms of Poincaré polynomials of associated Weyl groups, is given. Moreover we prove a certain converse theorem, which implies that the generating function for the case $I = \emptyset$ essentially knows all information on generating functions for general $I$.

1. Introduction

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0$ the set of non-negative integers, $\mathbb{Z}$ the set of rational integers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. For any set $S$, denote by $|S|$ the cardinality of $S$.

Let $V$ be an $r$-dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. The dual space $V^*$ is identified with $V$ via this inner product. Let $\Delta$ be a finite reduced root system in $V$ and $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ its fundamental system. Let $\Delta^+$ and $\Delta^-$ be the sets of all positive roots and negative roots, respectively: $\Delta = \Delta^+ \coprod \Delta^-$. We denote by $\alpha^\vee$ the coroot associated with a root $\alpha$. Let $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ be the set of fundamental weights defined by $\langle \alpha^\vee_i, \lambda_j \rangle = \delta_{ij}$ (Kronecker’s delta). Let $Q^\vee$ be the coroot lattice, $P$ the weight lattice, $P_+$ the set of integral dominant weights, and $P_{++}$ the set of integral strongly dominant weights, respectively, defined by

$$Q^\vee = \bigoplus_{i=1}^r \mathbb{Z}\alpha^\vee_i, \quad P = \bigoplus_{i=1}^r \mathbb{Z}\lambda_i, \quad P_+ = \bigoplus_{i=1}^r \mathbb{N}_0\lambda_i, \quad P_{++} = \bigoplus_{i=1}^r \mathbb{N}\lambda_i.$$
Let \( y \in V \), and \( s = (s_\alpha)_{\alpha \in \Delta_+} \in \mathbb{C}^{\lvert \Delta_+ \rvert} \). The zeta-function of the root system \( \Delta \) is defined by

\[
\zeta_r(s, y; \Delta) = \sum_{\lambda \in P_+} e^{2\pi i \sqrt{-1} \langle y, \lambda \rangle} \prod_{\alpha \in \Delta_+} \frac{1}{(\alpha^\vee, \lambda)^{s_\alpha}}.
\]

This function was introduced and has been studied by the authors in [5] [6] [7] [8] [9] [11] [15] and [19].

Let \( g \) be a complex semisimple Lie algebra. If \( \Delta = \Delta(g) \) is the root system associated with \( g \), and \( s_\alpha = s \) for all \( \alpha \in \Delta_+ \), then \( \zeta_r(s, s, \ldots, s, 0; \Delta) \) is essentially equal to the Witten zeta-function of \( g \) studied by Witten [27] and Zagier [28], up to a certain simple factor (see [6, (1.7)]). It is well known that simple Lie algebras are classified into seven types; we denote them by \( X_r \), where \( X = A, B, C, D, E, F, G \) and \( r \) denotes its rank. When \( g \) is of type \( X_r \), we frequently write its root system as \( \Delta(X_r) \), and its zeta-function as \( \zeta_r(s, y; X_r) \). In particular, \( \zeta_1(s, 0; A_1) = \zeta(s) \), the classical Riemann zeta-function, and

\[
\zeta_2((s_1, s_2, s_3), 0; A_2) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} m_1^{-s_1} m_2^{-s_2} (m_1 + m_2)^{-s_3},
\]

which is sometimes called the (Mordell-)Tornheim double sum [24].

On the other hand, we can see that the Euler-Zagier \( r \)-ple zeta-function

\[
\zeta_{EZ,r}(s_1, \ldots, s_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} m_1^{-s_1} (m_1 + m_2)^{-s_2} \cdots (m_1 + \cdots + m_r)^{-s_r}
\]

may be regarded as a special case of zeta-functions of root systems of type \( A_r \) (see [10]), or of type \( C_r \) (see [13]).

Therefore we can say that the notion of zeta-functions of root systems gives a unification of two important class of multiple zeta-functions, of Witten and of Euler and Zagier.

A lot of relations among special values (at integer points) of Euler-Zagier multiple zeta-functions are known. Are there any functional relations which interpolate those relations? This question was raised, around 2000, by the second-named author (cf. [18]). Needless to say, harmonic product formulas such as

\[
\zeta(s_1)\zeta(s_2) = \zeta_{EZ,2}(s_1, s_2) + \zeta_{EZ,2}(s_2, s_1) + \zeta(s_1 + s_2)
\]

are valid not only at integer points, but also at any other complex values of \( s_1 \) and \( s_2 \), so these give an answer. But what else? So far, no other functional relations has been discovered among Euler-Zagier multiple zeta-functions (except for a kind of functional equation for the case \( r = 2 \) discovered by the second-named author [17]), and in fact, a kind of negative answer was obtained recently by Ikeda and Matsuoka [4].
However, if we extend the range of the search, we may find such functional relations. The first example is a functional relation between \( \zeta(s) \) and \( \zeta_2((s_1, s_2, s_3), 0; A_2) \) discovered by the third-named author [26], which interpolates certain value relations among \( \zeta(k) \) and \( \zeta_{EZ, 2}(k_1, k_2) \ (k, k_1, k_2 \in \mathbb{N}) \). After this discovery, various other functional relations among zeta-functions of root systems have been reported (the aforementioned papers of the authors, Nakamura [20] [21], Zhou et al. [29], Onodera [22], and Ikeda and Matsuoka [3]).

On the other hand, the structural background of the existence of those functional relations has been studied in [11], [7]. The present paper is a continuation of these two papers. The mainactor of the present paper is the “Weyl-group-symmetric” linear combination \( S(s, y; I; \Delta) \) of zeta-functions of root systems defined by (2.1) below. This \( S(s, y; I; \Delta) \) has two different expressions: (i) It is a signed sum of zeta-functions of root systems, and on the other hand, (ii) it can be expressed in terms of certain generalization of Bernoulli functions \( P(k, y, \lambda; I; \Delta) \); the exact form of these facts will be stated in Section 2 ((2.2) and Theorem 2.3).

Combining these two expressions, (if they do not identically vanish) we can obtain certain functional relations among zeta-functions of root systems. We will state several explicit forms of such functional relations. However, since the expression (i) is a signed sum, there is the possibility that it vanishes identically. Whether it vanishes or not can be seen by observing the associated Poincaré polynomials. This is another main theme of the present paper.

## 2. Fundamental Formulas

Let \( I \subset \{1, 2, \ldots, r\} \), and \( \Psi_I = \{\alpha_i \mid i \in I\} \subset \Psi \). Let \( V_I \) be the subspace of \( V \) spanned by \( \Psi_I \). Then \( \Delta_I = \Delta \cap V_I \) is the root system in \( V_I \) whose fundamental system is \( \Psi_I \). For \( \Delta_I \), we denote the corresponding coroot lattice, weight lattice etc. by \( Q_I^\vee = \bigoplus_{\alpha_i \in I} \mathbb{Z} \alpha_i^\vee \), \( P_I = \bigoplus_{\lambda_i \in I} \mathbb{Z} \lambda_i \) etc. Let \( \iota : Q_I^\vee \to Q^\vee \) be the natural embedding, and \( \iota^*: P \to P_I \) the projection induced from \( \iota \); that is, for \( \lambda \in P \), \( \iota^*(\lambda) \) is defined as a unique element of \( P_I \) satisfying \( \langle \iota(q), \lambda \rangle = \langle q, \iota^*(\lambda) \rangle \) for all \( q \in Q_I^\vee \).

Let \( \text{Aut}(\Delta) \) be the subgroup of \( \text{GL}(V) \), consisting of all automorphisms which stabilizes \( \Delta \). Let \( \sigma_\alpha \in \text{Aut}(\Delta) \) be the reflection with respect to \( \alpha \), and denote by \( W = W(\Delta) \) the Weyl group of \( \Delta \), namely the group generated by \( \{\sigma_i \mid 1 \leq i \leq r\} \), where \( \sigma_i = \sigma_{\alpha_i} \). This is a normal subgroup of \( \text{Aut}(\Delta) \). For \( w \in W \), we put \( \Delta_w = \Delta_+ \cap w^{-1}\Delta_- \). Let \( W_I \) be the subgroup of \( W \).
generated by all the reflections associated with the elements in \( \Psi_I \), and \( W^I = \{ w \in W \mid \Delta_+^I \subset w\Delta_+^I \} \).

The fundamental Weyl chamber is defined by

\[
C = \{ v \in V \mid \langle \alpha^\vee_i, v \rangle \geq 0 \text{ for } 1 \leq i \leq r \}.
\]

Then \( W \) acts on the set of Weyl chambers \( \{ wC \mid w \in W \} \) simply transitively. For any subset \( A \subset \Delta \), let \( H_{A^\vee} \) be the set of all \( v \in V \) which satisfies \( \langle \alpha^\vee_i, v \rangle = 0 \) for some \( \alpha \in A \). In particular, \( H_{\Delta^\vee} \) is the set of all walls of Weyl chambers.

Now define

\[
(2.1) \quad S(s, y; I; \Delta) = \sum_{\lambda \in \pi^{-1}(P_{+I}) \setminus H_{\Delta^\vee}} e^{2\pi i \langle (y, \lambda) \rangle} \prod_{\alpha \in \Delta^+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}}.
\]

This sum was first introduced in \[11\] (110). (Note that \[11\] was published later than \[7\], but was written earlier, already in 2007). In \[11\] Theorems 5 and 6], we showed

\[
(2.2) \quad S(s, y; I; \Delta) = \sum_{w \in W^I} \left( \prod_{\alpha \in \Delta_{w-1}} (-1)^{-s_\alpha} \right) \zeta_r(w^{-1}s, w^{-1}y; \Delta).
\]

In the above statement, the action of \( W \) to \( s \) is defined by \( (ws)_\alpha = s_{w^{-1}\alpha} \) for \( w \in W \) with the convention that, if \( \alpha \in \Delta_- \), then we understand that \( s_\alpha = s_{-\alpha} \).

We also proved in \[11\] Theorems 5 and 6] a certain multiple integral expression of \( S(s, y; I; \Delta) \). When \( I = \emptyset \), we further observed that, in some special cases, those integrals may be regarded as generalizations of classical (Seki-)Bernoulli functions, which we denoted by \( P(k, y; \Delta) \) (which is actually the special case \( \lambda = 0, I = \emptyset \) of \( P(k, y, \lambda; I; \Delta) \) defined later in \( (2.7) \)).

We gave multiple integral expressions of generating functions of \( P(k, y; \Delta) \) (see \[11\] Theorem 7]), but it requires extremely huge task if we want to calculate that expression more explicitly. This situation was improved in \[7\], in which more accessible expressions were given (see \[7\] Theorem 4.1]).

However, these are the results in the case \( I = \emptyset \). In order to develop the full theory of functional relations, it is necessary to obtain the analogous results for general \( I \). This is the first aim of the present paper. To state the results, we need some more notations.

Let \( \Delta^* = \Delta_+ \setminus \Delta_{I+} \) and \( d = |I^c| \). We may find \( V_I = \{ \gamma_1, \ldots, \gamma_d \} \subset \Delta^* \) such that \( V = V_I \cup \Psi_I \) becomes a basis of \( V \). Let \( \gamma'_I = \gamma'(\Delta^*) \) be the set of all such bases. In particular, \( \gamma'_I = \gamma'_0 \) be the set of all linearly independent subsets \( V = \{ \beta_1, \ldots, \beta_s \} \subset \Delta_+^I \).

For \( V \in \gamma'_I \), the lattice \( L(V^\vee) = \bigoplus_{\beta \in V} \mathbb{Z}\beta^\vee \) is a sublattice of \( Q^\vee \). Let \( \{ \mu^V_\gamma \}_{\gamma \in V} \) be the dual basis of \( \bigoplus_{\gamma \in V} \mathbb{Z}\mu^V_\gamma \), namely \( \langle \gamma^\vee, \mu^V_\gamma \rangle = \delta_{kl} \).
\[ \langle \alpha_i^\vee, \mu_{\alpha_j}^\vee \rangle = \delta_{ij}, \text{ and } \langle \gamma_k^\vee, \mu_{\alpha_i}^\vee \rangle = \langle \alpha_i^\vee, \mu_{\gamma_k}^\vee \rangle = 0. \]

Let \( p_{V_I}^\vee \) be the projection defined by
\[
(2.3) \quad p_{V_I}^\vee (v) = v - \sum_{\gamma \in I} \mu_{\gamma}^\vee \langle \gamma^\vee, v \rangle = \sum_{\alpha \in \Phi_I} \mu_{\alpha}^\vee \langle \alpha^\vee, v \rangle,
\]
for \( v \in V. \) (The second equality can be easily seen by expressing \( v = \sum_k a_k \mu_{\gamma_k}^\vee + \sum_i b_i \mu_{\alpha_i}^\vee. \)) It is to be noted that the projection \( p_{V_I}^\vee \) depends only on \( V_I \) in the following sense.

**Lemma 2.1.** For any linearly independent subset \( \Phi_I = \{\beta_1, \ldots, \beta_{|I|}\} \subset \Delta_I^+ \) and \( U = V_I \cup \Phi_I, \) we have
\[
(2.4) \quad p_{V_I}^\vee (v) = v - \sum_{\gamma \in I} \mu_{\gamma}^\vee \langle \gamma^\vee, v \rangle = \sum_{\beta \in \Phi_I} \mu_{\beta} \langle \beta^\vee, v \rangle.
\]

**Proof.** Put \( u = \sum_{\beta \in \Phi_I} \mu_{\beta}^\vee \langle \beta^\vee, v \rangle \) and we show \( p_{V_I}^\vee (v) = u. \) It is enough to check that \( \langle \gamma^\vee, p_{V_I}^\vee (v) \rangle = \langle \gamma^\vee, u \rangle \) for all \( \gamma \in U. \) For \( \gamma \in V_I, \) we have
\[
\langle \gamma^\vee, p_{V_I}^\vee (v) \rangle = \langle \gamma^\vee, v - \sum_{\gamma_i \in I} \mu_{\gamma_i}^\vee \langle \gamma_i^\vee, v \rangle \rangle = \langle \gamma^\vee, v \rangle - \langle \gamma^\vee, v \rangle = 0,
\]
\[
\langle \gamma^\vee, u \rangle = \langle \gamma^\vee, v - \sum_{\gamma_i \in I} \mu_{\gamma_i}^\vee \langle \gamma_i^\vee, v \rangle \rangle = \langle \gamma^\vee, v \rangle - \langle \gamma^\vee, v \rangle = 0,
\]
because \( V_I \subset V, U. \) For \( \beta \in \Phi_I, \) we have
\[
\langle \beta^\vee, u \rangle = \sum_{\beta \in \Phi_I} \langle \beta^\vee, \mu_{\beta_i}^\vee \langle \beta_i^\vee, v \rangle \rangle = \sum_{\beta \in \Phi_I} \langle \beta^\vee, v \rangle \langle \beta^\vee, \mu_{\beta_i}^\vee \rangle = \langle \beta^\vee, v \rangle,
\]
while by writing \( \beta^\vee = \sum_{\alpha \in \Psi_I} a_{\alpha} \alpha^\vee, \) we have
\[
\langle \beta^\vee, p_{V_I}^\vee (v) \rangle = \sum_{\alpha \in \Psi_I} \langle \beta^\vee, \mu_{\alpha_i}^\vee \rangle \langle \alpha_i^\vee, v \rangle = \sum_{\alpha \in \Psi_I} a_{\alpha} \langle \alpha^\vee, v \rangle = \langle \sum_{\alpha \in \Psi_I} a_{\alpha} \alpha^\vee, v \rangle = \langle \beta^\vee, v \rangle.
\]

\[ \square \]

Next we introduce a generalization of the notion of “fractional part” of real numbers. Let \( \mathcal{Y} \) be the set of linearly independent subsets \( V = \{\beta_1, \ldots, \beta_r\} \subset \Delta_+. \) Also, let \( \mathcal{B} \) be the set of all linearly independent subsets \( R = \{\beta_1, \ldots, \beta_{r-1}\} \subset \Delta, \) and let \( \delta_{R^\vee} = \bigoplus_{i=1}^{r-1} \mathbb{R}_{\beta_i^\vee} \) be the hyperplane passing through \( R^\vee \cup \{0\}. \) We fix a non-zero vector
\[
\phi \in V \setminus \bigcup_{R \in \mathcal{B}} \delta_{R^\vee}.
\]
Then \( \langle \phi, \mu_R^\vee \rangle \neq 0 \) for all \( V \in \mathcal{Y} \) and \( \beta \in V. \) For \( y \in V, V \in \mathcal{Y} \) and \( \beta \in V, \) we define
\[
(2.5) \quad \{y\}_{V, \beta} = \left\{ \begin{array}{ll}
\{\langle y, \mu_{\beta}^\vee \rangle\}, & \langle \phi, \mu_R^\vee \rangle > 0, \\
1 - \{\langle y, \mu_{\beta}^\vee \rangle\}, & \langle \phi, \mu_R^\vee \rangle < 0,
\end{array} \right.
\]
where \{·\} on the right-hand sides denotes the usual fractional part of real numbers.

Using these notions, we now define Bernoulli functions of the root system \(\Delta\) associated with \(I\) and their generating functions.

**Definition 2.2.** For \(t = (t_\alpha)_{\alpha \in \Delta^*} \in \mathbb{C}^{|\Delta^*|}\) and \(\lambda \in P_I\), let

\[
F(t, y, \lambda; I; \Delta) = \sum_{V \in V_I} \left( \prod_{\gamma \in \Delta^* \setminus V} t_\gamma \right)
\left( \sum_{\beta \in V} t_\beta \langle \gamma^\vee, \mu_{\beta} \rangle - 2\pi \sqrt{-1} \langle \gamma^\vee, p_{V_I}^I(\lambda) \rangle \right)
\frac{1}{|Q^V/L(V^\vee)|} \sum_{q \in Q^V/L(V^\vee)} \exp(2\pi \sqrt{-1} \langle y + q, p_{V_I}^I(\lambda) \rangle)
\prod_{\beta \in V} t_\beta \exp(t_\beta \{y + q\}_{V, \beta}) e^{t_\beta - 1},
\]

and define **Bernoulli functions** \(P(k, y, \lambda; I; \Delta)\) of the root system \(\Delta\) associated with \(I\) by the expansion

\[
F(t, y, \lambda; I; \Delta) = \sum_{k \in \mathbb{N}^{|\Delta^*|}} P(k, y, \lambda; I; \Delta) \prod_{\alpha \in \Delta^*} \frac{k_\alpha}{k_\alpha!}.
\]

The fundamental formula in our theory is the following theorem.

**Theorem 2.3.** Let \(s_\alpha = k_\alpha \in \mathbb{N}\) for \(\alpha \in \Delta^*\) and \(s_\alpha \in \mathbb{C}\) for \(\alpha \in \Delta_{I+}\). We assume

(1) If \(\alpha\) belongs to an irreducible component of type \(A_1\), then the corresponding \(k_\alpha \geq 2\).

Then we have

\[
S(s, y; I; \Delta) = (-1)^{|\Delta^*|} \left( \prod_{\alpha \in \Delta^*} \frac{(2\pi \sqrt{-1})^{k_\alpha}}{k_\alpha!} \right) \sum_{\lambda \in \Lambda_{I+}^+} \left( \prod_{\alpha \in \Delta_{I+}} \frac{1}{(\alpha^\vee, \lambda)^{s_\alpha}} \right) P(k, y, \lambda; I; \Delta).
\]

In the case \(I = \emptyset\), clearly \(\Psi_I = \emptyset, V_I = V, V^\perp_I = V, \Delta^* = \Delta_{I+}, P_I = \{0\}\), and hence the only possible \(\lambda\) is \(\lambda = 0\). Write \(t = t_\emptyset = (t_\alpha)_{\alpha \in \Delta_{I+}}\). Then we see that

\[
F(t, y; \Delta) = F(t, y; 0; \emptyset; \Delta)
\]

\[
= \sum_{V \in \mathcal{V}} \left( \prod_{\gamma \in \Delta_{I+} \setminus V} t_\gamma \right)
\left( \sum_{\beta \in V} t_\beta \langle \gamma^\vee, \mu_{\beta} \rangle - 2\pi \sqrt{-1} \langle \gamma^\vee, p_{V_I}(\lambda) \rangle \right)
\frac{1}{|Q^V/L(V^\vee)|} \sum_{q \in Q^V/L(V^\vee)} \prod_{\gamma \in V} t_\gamma \exp(t_\gamma \{y + q\}_{V, \gamma}) e^{t_\gamma - 1},
\]
which is exactly equal to \( [7, \text{(4.3)}] \). Therefore Theorem 2.3 is a generalization of \( [7, \text{Theorem 4.1}] \). We note that, when \( \Delta = \Delta(A_1) \), it follows that

\[
F(t, y; A_1) = \frac{te^{t\{y\}}}{e^t - 1}
\]

(see \( [7, \text{(3.14)}] \)), the usual generating function of classical Bernoulli functions \( B_k(\cdot) \), that is

\[
\frac{te^{t\{y\}}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(\{y\}) \frac{t^k}{k!}.
\]

Therefore we may regard that \( P(k, y, \lambda; I; \Delta) \) is a root-theoretic generalization of classical Bernoulli functions.

The fundamental philosophy of the present paper is to combine two expressions of \( S(s, y; I; \Delta) \), that is \( (2.2) \) and \( (2.8) \). It will produce the relation of the form

\[
\tag{2.12}
(A \text{ signed sum of zeta-functions of root systems}) = (A \text{ sum involving Bernoulli functions of root systems}).
\]

If the left-hand side of \( (2.12) \) does not vanish identically, then this relation gives a non-trivial functional relation among zeta-functions of root systems. In Sections 3 and 4 we will show explicit computations on the right-hand side of \( (2.12) \), and in Section 7 we state several examples. On the other hand, concerning the left-hand side, it is important to know when it does not vanish. A criterion of the non-vanishing in terms of Poincaré polynomials will be given in Sections 5 and 6.

The proof of Theorems 2.3 itself will be given in Section 8. It is to be stressed that, though Theorem 2.3 is a generalization of the previous result in \( [17] \) where the case \( I = \emptyset \) was treated, there are much more technical difficulties when we consider the case of general \( I \). To overcome those difficulties, the authors had to write a separate paper \( [14] \), whose results will be used essentially in the course of the proof of Theorem 2.3.

Another main result in the present paper is the following converse theorem. The generating function \( F(t_I, y, \lambda; I; \Delta) \) for general \( I \) can be deduced, in the following sense, from the case \( I = \emptyset \). That is, the generating function for \( I = \emptyset \) knows ‘everything’. This theorem will be proved in the last section.

**Theorem 2.4.** Let \( I \subset \{1, \ldots, r\} \). For \( \lambda \in P_I^{++} \), we have

\[
F(t_I, y, \lambda; I; \Delta) = \text{Res}_{\alpha = \sqrt{-1}(\alpha; I, \lambda)} \left( \prod_{\alpha \in \Delta_I^+} \frac{1}{t_\alpha} \right) F(t, y; \Delta),
\]

(2.13)
where \( t_I = (t_\alpha)_{\alpha \in \Delta^*} \) and the meaning of Res is as explained in Remark 2.5 below.

**Remark 2.5.** The function \( \widetilde{F}(t, y; \Delta) = \left( \prod_{\alpha \in \Delta I^+} t_\alpha \right) F(t, y; \Delta) \) is in \(|\Delta^*|\) complex variables. We give an order of the elements of the set \( \Delta I^+ \) by numbering as \( \alpha_1, \ldots, \alpha_n \), where \( n = |\Delta I^+| \). The definition of Res in the above theorem is given by the following iterated procedure:

\[
\text{Res}_{t_\alpha = 2\pi \sqrt{-1} \langle \alpha^\vee, \lambda \rangle} \left( \prod_{\alpha \in \Delta I^+} t_\alpha \right) F(t, y; \Delta) = \text{Res}_{t_{\alpha_1} = 2\pi \sqrt{-1} \langle \alpha_{\alpha_1}^\vee, \lambda \rangle} \ldots \text{Res}_{t_{\alpha_n} = 2\pi \sqrt{-1} \langle \alpha_{\alpha_n}^\vee, \lambda \rangle} \widetilde{F}(t, y; \Delta),
\]

where on the right-hand side, each Res means the usual residue of a one-variable function. In the proof of the theorem we will see that the value (2.14) does not depend on the choice of the order of iteration. This is to be regarded as a part of the statement of Theorem 2.4, because this does not hold in general for iterated residues. The point \( 2\pi \sqrt{-1} \langle \alpha^\vee, \lambda \rangle \) is on some singular hyperplane of \( \widetilde{F}(t, y; \Delta) \), but the limit process \( t_\alpha \to 2\pi \sqrt{-1} \langle \alpha^\vee, \lambda \rangle \) of calculating the residue is along some ‘generic’ path in the sense that, when the limit process on \( t_{\alpha_j} \) is carried out, the remaining variables \( (t_{\alpha_{j+1}}, \ldots, t_{\alpha_n}) \) are located outside any singular hyperplanes.

### 3. Explicit functional relations \((|I| = (r - 1) \text{ case})\)

In this and the next section we evaluate the generating functions more explicitly, and deduce explicit functional relations. We only discuss the cases \( |I| = r - 1 \) and \( |I| = 1 \), because the other cases are much more complicated. We use the notation \( \delta_\Box \), where some condition is inserted in \( \Box \), defined as \( \delta_\Box = 1 \) if the condition is satisfied, and \( = 0 \) otherwise.

We first introduce the transposes \( p_{V_I^+}^* \) of the projections \( p_{V_I^+} \) (defined by (2.3)) by

\[
\langle u, p_{V_I^+}^* (v) \rangle = \langle u, v \rangle - \sum_{\beta \in V} \langle u, \mu_{V^\beta}^\vee \rangle \langle \beta^\vee, v \rangle = \langle u - \sum_{\beta \in V} \langle u, \mu_{V^\beta}^\vee \rangle \beta^\vee, v \rangle = \langle p_{V_I^+}^* (u), v \rangle
\]

for \( v \in V \), that is

\[
p_{V_I^+}^* (u) = u - \sum_{\beta \in V} \langle u, \mu_{V^\beta}^\vee \rangle \beta^\vee.
\]
Let $I \subset \{1, \ldots, r\}$. In this section we consider the situation $|I^c| = 1$, and put $I^c = \{k\}$. Then $\Psi_I = \{\alpha_i\}_{i \in I}$, and we see that

$$\Delta^* = \{\alpha \in \Delta^* | a_k = \langle \alpha, \lambda_k \rangle \neq 0\}.$$

Since $|V_I| = 1$ in the present case, we have

$$\{\beta \in V = \{\beta \} \cup \Psi_I \}_{\beta \in \Delta^*}.$$

For $V = \{\beta \} \cup \Psi_I \in \mathcal{V}_I$ and $\gamma \in \Delta^* \setminus \{\beta\}$, from (3.2) we have

$$p^*_V(\gamma) = \gamma - \langle \gamma, \mu_\beta \rangle \beta.$$

We put $b_i = b_i(\beta) = \langle \beta, \lambda_i \rangle$ $(1 \leq i \leq r)$ so that

$$\beta = \sum_{i=1}^r b_i \alpha_i.$$

Then we find

$$\nu_V(\lambda) = \lambda - \frac{\lambda_k b_k}{b_k} \langle \beta, \lambda \rangle.$$

Write $y = y_1 \alpha_1 + \cdots + y_r \alpha_r$ and $\lambda = \sum_{i=1}^r m_i \lambda_i \in P_I$. Then

$$p^*_V(\lambda) = \lambda - \frac{\lambda_k}{b_k} \langle \beta, \lambda \rangle.$$

Note that $Q^*/L(V) = \{a_k \alpha_k \}_{0 \leq a_k < b_k}$ and $|Q^*/L(V)| = b_k$. For $q = a_k \alpha_k \in Q^*$, from (3.7) we obtain

$$\langle y + q, p^*_V(\lambda) \rangle = \sum_{i=1}^r m_i \left( y_i - \frac{b_i}{b_k} (y_k + a_k) \right).$$

We fix $\phi$ such that $\langle \phi, \lambda_k \rangle > 0$. Then for $q = a_k \alpha_k \in Q^*$,

$$\{y + q \}_{\gamma, \beta} = \left\{ \frac{y_i + a_k}{y_k} \right\} = \left\{ \frac{y_k + a_k}{b_k} \right\}.$$

Substituting the above results into (2.8), we obtain the following form of the generating function (under the identification $y = (y_i)_{1 \leq i \leq r}$ and $\lambda = (m_i)_{1 \leq i (\neq k) \leq r}$):

$$F((t_\beta)_{\beta \in \Delta^*}, (y_i)_{1 \leq i \leq r}, (m_i)_{1 \leq i (\neq k) \leq r}; I; \Delta)$$
where \( b_i = b_i(\beta) = \langle \beta^\vee, \lambda_i \rangle \).

For \( \gamma^\vee \in \Delta^* \), we see that

\[
(3.11) \quad p_{V_j}^\vee(\gamma^\vee) = \gamma^\vee - \frac{\langle \gamma^\vee, \lambda_k \rangle g^\vee}{b_k} \in \frac{1}{b_k}Q_{I}^y,
\]

which may not be proportional to a coroot in \( \Delta_1 \). Therefore, observing \( (3.10) \) with \( (3.11) \) we see that \( S(s, y; I; \Delta) \) may not be necessarily written in terms of zeta-functions of root systems of lower ranks. However, in the classical root systems of type \( A, B, C \), special choices of \( I \) give rise to recursive structures among zeta-functions of root systems. To show this, in the following we will give explicit forms of \( (3.11) \). Moreover we will present associated functional relations in the \( A_r \) cases.

3.1. \( A_r \) Case. We realize \( \Delta^*_y(A_r) = \{ e_i - e_j \mid 1 \leq i < j \leq r + 1 \} \), where \( e_j \) is the \( j \)th unit vector in \( \mathbb{R}^{r+1} \). Then the variable \( s \) can be parametrized as \( s = (s_{ij})_{1 \leq i < j \leq r} \), and the zeta-function of type \( A_r \) reads as

\[
(3.12) \quad \zeta_r((s_{ij})_{1 \leq i < j \leq r}, (y_k)_{1 \leq i \leq r}; A_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_{r+1}=1}^{\infty} \prod_{1 \leq i < j \leq r+1} \exp(2\pi \sqrt{-1} \sum_{i=1}^{m_j} y_i) \prod_{1 \leq i < j \leq r+1} (m_i + \cdots + m_{j-1})^{s_{ij}}
\]

Choose \( I = \{2, \ldots, r\} \) and \( I^c = \{1\} \) as in the following diagram.

\[
(3.13) \quad \begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \cdots & \iota_j & \cdots & \alpha_r
\end{array}
\]

Then

\[
(3.14) \quad \Psi^\vee_I = \{ \alpha_2^\vee = e_2 - e_3, \ldots, \alpha_r^\vee = e_r - e_{r+1} \}
\]

and \( \Delta^* = \{ e_1 - e_j \mid 2 \leq j \leq r + 1 \} \). Hence

\[
\gamma^\vee_I = \{ V^\vee = \{ e_1 - e_j \} \cup \Psi^\vee_I \}_{2 \leq j \leq r+1}.
\]

Temporarily we fix one

\[
\beta^\vee = e_1 - e_j = \alpha_1^\vee + \cdots + \alpha_{j-1}^\vee
\]

(\( 2 \leq j \leq r + 1 \)). Then we see that

\[
(3.15) \quad b_i = \langle \beta^\vee, \lambda_i \rangle = \begin{cases} 1 & (1 \leq i < j), \\ 0 & (j \leq i \leq r), \end{cases}
\]
and in particular \( p^*_f(\gamma) = 1 \). The following form of the generating function:

\[
\langle \gamma \rangle = e_1 - e_i - (e_1 - e_i, \lambda_1) (e_1 - e_j)
\]

\[
= e_1 - e_i - (\alpha_i^+ \cdots + \alpha_{i-1}^+, \lambda_1) (e_1 - e_j)
\]

\[
= e_1 - e_i - (e_1 - e_j)
\]

\[
e_j - e_i = \begin{cases} 
- (\alpha_i^+ \cdots + \alpha_{j-1}^-) & (2 \leq i \leq j - 1), \\
\alpha_i^+ \cdots + \alpha_{i-1}^- & (j + 1 \leq i \leq r + 1), 
\end{cases}
\]

we find that \( \langle p^*_f(\gamma) \rangle, \lambda = -m_{ij} \) for \( \lambda = m_2 \beta_2 + \cdots + m_r \beta_r \), where

\[
m_{ij} = \begin{cases} 
m_i + \cdots + m_{i-1} & (2 \leq i \leq j - 1), \\
-(m_i + \cdots + m_{i-1}) & (j + 1 \leq i \leq r + 1).
\end{cases}
\]

By putting \( t_{e_i - e_j} = t_i \) for \( 2 \leq i \leq r + 1 \), we can deduce from (3.10) (with \( k = 1 \)) the following form of the generating function:

\[
F(t_1, t_2, \ldots, t_r, \lambda) = \sum_{j=0}^{r+1} \prod_{i=1}^{j-1} \frac{t_i}{t_i - t_j + 2\pi \sqrt{-1} m_{ij}} 
\times \exp \left( 2\pi \sqrt{-1} \left( \sum_{i=2}^{j} m_i(y_i - y_1) + \sum_{i=j}^{r} m_i y_i \right) \right) t_j \exp(t_j(y_1)).
\]

Next we calculate the Taylor expansion of the right-hand side of the above formula. First we note that

\[
\frac{t_i}{t_i - t_j + 1/x_{ij}} = \frac{t_i x_{ij}}{1 - (t_j - t_i) x_{ij}} = \sum_{n=0}^{\infty} x_{ij}^{n+1} t_i (t_j - t_i)^n
\]

\[
= \sum_{n=0}^{\infty} x_{ij}^{n+1} \sum_{k,l=0}^{\infty} (-1)^k \binom{n}{k} \frac{1}{l!} \frac{k!}{l!} t_i^{k+1}
\]

\[
= \sum_{k,l=0}^{\infty} (-1)^{k-1} \binom{k+l-1}{l} \frac{1}{l!} \frac{k!}{l!} x_{ij}^{k+l} t_i^l t_j^k
\]

(\text{with a certain indeterminate } x_{ij}). \text{ By use of this result and (2.11) we have}

\[
\left( \prod_{2 \leq i \leq r+1} \frac{t_i}{t_i - t_j + 1/x_{ij}} \right) t_j \exp(t_j(y_1)) \frac{t_j \exp(t_j(y_1))}{e^{t_j} - 1}
\]
We compute (2.2) and (2.8) in the case of type 1.

Applying this to the right-hand side of (3.18), we obtain the following conclusion.

**Theorem 3.1.** In the case $\Delta = \Delta(A_r)$ and $I = \{2, \ldots, r\}$, we have

$$F((t_i)_{2 \leq i \leq r+1}, (y_j)_{1 \leq j \leq r}, (m_i)_{2 \leq i \leq r}; A_r)$$

$$= \sum_{k_1, \ldots, k_{r+1} \geq 0} \frac{\prod_{l=2}^{r+1} B_l((y_1)_{l \neq j})}{t_j!} \prod_{2 \leq i \leq r+1} \frac{1}{t_i - t_j + 2\pi\sqrt{-1}m_{ij}}$$

$$\times \exp\left(2\pi\sqrt{-1}\left(\sum_{i=2}^{j-1} m_i(y_i - y_1) + \sum_{i=j}^{r} m_iy_i\right)\right) \frac{t_j \exp(t_j(y_1))}{e^{t_j} - 1}$$

$$= \sum_{k_2, \ldots, k_{r+1} \geq 0} \frac{\prod_{l=2}^{r+1} B_l((y_1)_{l \neq j})}{t_j!} \prod_{2 \leq i \leq r+1} \frac{1}{t_i - t_j + 2\pi\sqrt{-1}m_{ij}}$$

$$\times \exp\left(2\pi\sqrt{-1}\left(\sum_{i=2}^{j-1} m_i(y_i - y_1) + \sum_{i=j}^{r} m_iy_i\right)\right) \frac{t_j \exp(t_j(y_1))}{e^{t_j} - 1}$$

where

$$P((k_i)_{2 \leq i \leq r+1}, (y_i)_{1 \leq i \leq r}, (m_i)_{2 \leq i \leq r}; A_r)$$

$$= k_2! \cdots k_{r+1}! \sum_{j=2}^{r+1} \left(\prod_{l=2}^{r+1} \delta_{k_l \neq 0}\right) \exp\left(2\pi\sqrt{-1}\left(\sum_{i=2}^{j-1} m_i(y_i - y_1) + \sum_{i=j}^{r} m_iy_i\right)\right)$$

$$\times \left(\prod_{l=2}^{r+1} \frac{B_l((y_1)_{l \neq j})}{t_j!} \prod_{2 \leq i \leq r+1} \frac{1}{t_i - t_j + 2\pi\sqrt{-1}m_{ij}}\right)^{k_i}.$$
where \( \sigma_j = (j+1) \). In fact, we can easily check that

\[
w = \sigma_1 \sigma_2 \cdots \sigma_j = (1 \cdots j + 1)
\]

for \( 0 \leq j \leq r \) satisfies \( w^{-1} \Delta^\vee_{\mathcal{I}} \subset \Delta^\vee_{\mathcal{I}} \), and those elements exhausts \( W^I \) because \( |W^I| = |W|/|W_I| = (r+1)!/r! = r+1 \) (see (3.21) below).

Next, for \( 2 \leq i \leq r \) and \( 1 \leq j \leq r \), we have

\[
(3.23) \quad \sigma_j \cdots \sigma_1 \alpha_i^\vee = e_{j+1} - e_1 = -\alpha_1^\vee - \cdots - \alpha_j^\vee,
\]

\[
(3.24) \quad \sigma_j \cdots \sigma_1 \alpha_i^\vee = \begin{cases} 
\alpha_{i-1}^\vee & (i \leq j), \\
\alpha_{i-1}^\vee + \alpha_i^\vee & (i = j + 1), \\
\alpha_i^\vee & (i \geq j + 2).
\end{cases}
\]

Therefore, for \( w = \sigma_1 \cdots \sigma_j \in W^I \), we have

\[
w^{-1} y = \sigma_j \cdots \sigma_1 (y_1 \alpha_1^\vee + \cdots + y_r \alpha_r^\vee) = -y_1 (\alpha_1^\vee + \cdots + \alpha_j^\vee) + \sum_{i=2}^{j} y_i \alpha_{i-1}^\vee + y_{j+1} (\alpha_j^\vee + \alpha_{j+1}^\vee) + \sum_{i=j+2}^{r} y_i \alpha_i^\vee = \sum_{i=1}^{j} (y_{i+1} - y_1) \alpha_i^\vee + \sum_{i=j+1}^{r} y_i \alpha_i^\vee.
\]

Lastly for \( w = \sigma_1 \cdots \sigma_j \in W^I \), we can see that

\[
(3.25) \quad \Delta_{w^{-1}} = \{ \alpha_1^\vee + \cdots + \alpha_j^\vee = e_1 - e_{i+1} | 1 \leq i \leq j \}.
\]

Therefore from (2.22) we obtain

\[
(3.26) \quad S(s, y; \{2, \ldots, r\}; A_r) = \sum_{j=0}^{r-1} \prod_{i=1}^{j} (-1)^{k_{1,i+1}}
\times \zeta_r ((s_{p,j})_{1 \leq p < q \leq r+1}, (y_2 - y_1, \ldots, y_{j+1} - y_1, y_{j+1}, \ldots, y_r); A_r),
\]

where \( p_j = (1 \cdots j + 1)p \) and \( q_j = (1 \cdots j + 1)q \).

On the other hand, (2.23) in the present case reads as

\[
(3.27) \quad S(s, y; \{2, \ldots, r\}; A_r) = (-1)^r \left( \prod_{j=2}^{r+1} \frac{(2\pi \sqrt{-1})^{k_{1,j}}}{k_{1,j}!} \right)
\times \sum_{m_2, \ldots, m_r = 1}^{\infty} \left( \prod_{2 \leq p < q \leq r+1} \frac{1}{(e_p - e_q, m_2 \lambda_2 + \cdots + m_r \lambda_r)^{s_{p,q}}} \right)
\times P(k, y, (m_i)_{2 \leq i \leq r}; \{2, \ldots, r\}; A_r)
\]

with \( k = (k_{1,j})_{2 \leq j \leq r+1} \). Substituting (3.21) (with \( k_i = k_{1,i} \)) into the right-hand side, we obtain

\[
(3.27) \quad S(s, y; \{2, \ldots, r\}; A_r)
\]
Theorem 3.2. For \((s_{pq})_{1 \leq p < q \leq r + 1}\) with \(s_{1j} = k_{1j}\) \((2 \leq j \leq r + 1)\), we have

\[
\sum_{j=0}^{r} \left( \prod_{i=1}^{j} (-1)^{k_{1i}+1} \right) \times \zeta_{r}((s_{pq})_{1 \leq p < q \leq r + 1}, (y_2 - y_1, \ldots, y_j+1 - y_1, y_j+1, \ldots, y_r); A_r)
\]

\[= - \sum_{j=2}^{r+1} \sum_{l_2, \ldots, l_{r-1} \geq 0 \atop l_2 + \cdots + l_{r-1} = k_{1j}} (-1)^{k_{12} + \cdots + k_{1j-1} + l_j + 1 + \cdots + l_{r-1}} (2\pi \sqrt{-1})^l_j
\]

\[
\times \frac{B_{l_j}(\{y_1\})}{l_j!} \left( \prod_{2 \leq i \neq j} \left( \frac{k_{1i} + l_i - 1}{l_i} \right) \right)
\]

\[
\times \zeta_{r-1}((s_{pq} + \delta_p < j \delta_q = j (k_{1p} + l_p) + \delta_p = j \delta_q > j (k_{1q} + l_q))_{2 \leq p < q \leq r+1}, (y_2 - y_1, \ldots, y_j - y_1, y_j, \ldots, y_r); A_{r-1}),
\]

where in the last equality we have used

\[
(3.28) \quad \sum_{2 \leq i \leq r+1, i \neq j} (k_{1i} + l_i) = -l_j + \sum_{2 \leq i \leq r+1} k_{1i}.
\]

Comparing \((3.28)\) and \((3.27)\), we now arrive at the following explicit form of functional relations.

**Theorem 3.2.** For \((s_{ij})_{1 \leq i < j \leq r+1}\) with \(s_{1j} = k_{1j}\) \((2 \leq j \leq r + 1)\), we have

\[
(3.29) \quad \sum_{j=0}^{r} \left( \prod_{i=1}^{j} (-1)^{k_{1i}+1} \right) \times \zeta_{r}((s_{pq})_{1 \leq p < q \leq r + 1}, (y_2 - y_1, \ldots, y_j+1 - y_1, y_j+1, \ldots, y_r); A_r)
\]

\[= - \sum_{j=2}^{r+1} \sum_{l_2, \ldots, l_{r-1} \geq 0 \atop l_2 + \cdots + l_{r-1} = k_{1j}} (-1)^{k_{12} + \cdots + k_{1j-1} + l_j + 1 + \cdots + l_{r-1}} (2\pi \sqrt{-1})^l_j
\]

\[
\times \frac{B_{l_j}(\{y_1\})}{l_j!} \left( \prod_{2 \leq i \neq j} \left( \frac{k_{1i} + l_i - 1}{l_i} \right) \right)
\]

\[
\times \zeta_{r-1}((s_{pq} + \delta_p < j \delta_q = j (k_{1p} + l_p) + \delta_p = j \delta_q > j (k_{1q} + l_q))_{2 \leq p < q \leq r+1}, (y_2 - y_1, \ldots, y_j - y_1, y_j, \ldots, y_r); A_{r-1}),
\]

where \(p_j = (12 \cdots j+1)p\) and \(q_j = (12 \cdots j+1)q\).
The cases \( r = 2, 3 \) of this theorem will be written down as Examples 7.1, 7.2 in Section 7.

3.2. \( C_r \) Case. We realize
\[
\Delta^\vee_+ = \{ e_i \pm e_j \mid 1 \leq i < j \leq r \} \cup \{ e_j \mid 1 \leq j \leq r \}.
\]
Then \( \Psi^\vee = \{ \alpha_1^\vee = e_1 - e_2, \ldots, \alpha_{r-1}^\vee = e_{r-1} - e_r, \alpha_r^\vee = e_r \} \), and the fundamental weights are given by \( \lambda_i = e_1 + \cdots + e_i \) for \( 1 \leq i \leq r \). Choose \( I = \{2, \ldots, r\} \) and \( I^c = \{1\} \) as in the following diagram.

\[
(3.30)
\]

Then
\[
(3.31)
\]
and \( \Delta^*\vee = \{ e_1 \pm e_j \mid 2 \leq j \leq r \} \cup \{ e_1 \} \). Hence
\[
(3.32)
\]
For \( \beta^\vee = e_1 - e_j \) \((2 \leq j \leq r)\), we see that
\[
(3.33)
\]
and in particular \( \mu^\vee_\beta = \lambda_1 = e_1 \). Therefore for \( \gamma^\vee = e_1 - e_i \in \Delta^{*\vee} \) \((i \neq j)\) and \( \gamma^\vee = e_1 + e_i \in \Delta^{*\vee} \),
\[
(3.34)
\]
and
\[
(3.35)
\]
Next for \( \beta^\vee = e_1 + e_j \) \((2 \leq j \leq r)\), we see that
\[
(3.36)
\]
and in particular \( \mu^\mathbf{V}_{\beta} = \lambda_1 = e_1 \). Therefore for \( \gamma^\mathbf{V} = e_1 + e_i \in \Delta^{*\mathbf{V}} \) and \( \gamma^\mathbf{V} = e_1 - e_i \in \Delta^{*\mathbf{V}} \),

\[
p^\mathbf{V}_{ij}(\gamma^\mathbf{V}) = e_1 \pm e_i - (e_1 \pm e_i, \lambda_1)(e_1 + e_j)
\]

\[
= e_1 \pm e_i - (e_1 + e_j)
\]

\[
= \begin{cases} 
- e_j \pm e_i \in \Delta_i^\mathbf{V} & (i \neq j), \\
-2e_j \in 2\Delta_i^\mathbf{V} & (i = j),
\end{cases}
\]

and

\[
p^\mathbf{V}_{ij}^\ast(e_1) = e_1 - (e_1, \lambda_1)(e_1 + e_j)
\]

\[
= e_1 - (e_1 + e_j)
\]

\[
= -e_j \in \Delta_i^\mathbf{V}.
\]

Lastly for \( \beta^\mathbf{V} = e_1 \), we see that

\[
b_l = \langle \beta^\mathbf{V}, \lambda_l \rangle = 1 \quad (1 \leq l \leq r),
\]

and in particular \( \mu^\mathbf{V}_{\beta} = \lambda_1 = e_1 \). Therefore for \( \gamma^\mathbf{V} = e_1 \pm e_i \in \Delta^{*\mathbf{V}} \),

\[
p^\mathbf{V}_{ij}^\ast(\gamma^\mathbf{V}) = e_1 \pm e_i - (e_1 \pm e_i, \lambda_1)e_1
\]

\[
= e_1 \pm e_i - e_1
\]

\[
= \pm e_i \in \Delta_i^\mathbf{V}.
\]

Now calculate \( \langle p^\mathbf{V}_{ij}^\ast(\gamma^\mathbf{V}), \lambda \rangle \). Consider the case when \( \beta^\mathbf{V} = e_1 - e_j \) \( (2 \leq j \leq r) \). For \( \gamma^\mathbf{V} = e_1 - e_i \) \( (i \neq j) \), we have

\[
\langle p^\mathbf{V}_{ij}^\ast(\gamma^\mathbf{V}), \lambda \rangle = \langle e_j - e_i, m_2 \lambda_2 + \cdots + m_r \lambda_r \rangle = -m_{ij},
\]

where the second equality is obtained by using \( \lambda_i = e_1 + \cdots + e_i \), or by using (the second member of) the facts

\[
e_i = \sum_{i \leq k \leq r} \alpha_k^\mathbf{V} \quad (1 \leq i \leq r),
\]

\[
e_i - e_j = \sum_{i \leq k \leq j} \alpha_k^\mathbf{V} \quad (1 \leq i < j \leq r),
\]

\[
e_i + e_j = \sum_{i \leq k < j} \alpha_k^\mathbf{V} + 2 \sum_{j \leq k \leq r} \alpha_k^\mathbf{V} \quad (1 \leq i < j \leq r).
\]

Using again \( \lambda_i = e_1 + \cdots + e_i \) or \( (3.41) \), for \( \gamma^\mathbf{V} = e_1 + e_i \) we have

\[
\langle p^\mathbf{V}_{ij}^\ast(\gamma^\mathbf{V}), \lambda \rangle = \langle e_j + e_i, m_2 \lambda_2 + \cdots + m_r \lambda_r \rangle = m_{ijr},
\]

where

\[
m_{ijr} = \begin{cases} 
2(m_j + \cdots + m_r) & (2 \leq i < j), \\
2(m_j + \cdots + m_r) & (i = j), \\
m_j + \cdots + m_{i-1} + 2(m_i + \cdots + m_r) & (j < i \leq r).
\end{cases}
\]

For \( \gamma^\mathbf{V} = e_1 \),

\[
\langle p^\mathbf{V}_{ij}^\ast(\gamma^\mathbf{V}), \lambda \rangle = \langle e_j, m_2 \lambda_2 + \cdots + m_r \lambda_r \rangle = m_j + \cdots + m_r = m_{j,r+1}.
\]

Similarly we evaluate \( \langle p^\mathbf{V}_{ij}^\ast(\gamma^\mathbf{V}), \lambda \rangle \) for other \( \beta^\mathbf{V} \).

We now insert those data into \( (3.41) \). Putting \( t_{e_1 \pm e_i} = t_{\pm i} \) for \( 2 \leq i \leq r \) and \( t_{2e_1} = t_{1} \), we obtain the following form of the generating function:
Theorem 3.3. In the case $\Delta = \Delta(C_r)$ and $I = \{2, \ldots, r\}$ we have

\[(3.43)\]

\[
F(t_1, (t_{\pm i})_{2 \leq i \leq r}, (y_j)_{1 \leq j \leq r}, (m_i)_{2 \leq i \leq r}; \{2, \ldots, r\}; C_r)
\]

\[
= \sum_{j=2}^r \prod_{2 \leq i \leq r, \ i \neq j} \frac{t_{i-j}}{t_{i-j} - 2\pi \sqrt{-1}m_{ij}} \prod_{2 \leq i \leq r} \frac{t_{i-j}}{t_{i-j} - 2\pi \sqrt{-1}m_{ij}}
\]

\[
\times \exp\left(2\pi \sqrt{-1} \left(\sum_{i=2}^{j-1} m_i(y_i - y_1) + \sum_{i=j}^r m_i(y_i - 2y_1)\right)\right) \frac{t_{i-j}}{t_{i-j} - 1}
\]

\[
+ \sum_{j=2}^r \prod_{2 \leq i \leq r, \ i \neq j} \frac{t_{i-j}}{t_{i-j} - 2\pi \sqrt{-1}m_{ij}} \prod_{2 \leq i \leq r} \frac{t_{i-j}}{t_{i-j} - 2\pi \sqrt{-1}m_{ij}}
\]

\[
\times \exp\left(2\pi \sqrt{-1} \left(\sum_{i=2}^{j-1} m_i(y_i - y_1) + \sum_{i=j}^r m_i(y_i - 2y_1)\right)\right) \frac{t_{i-j}}{t_{i-j} - 1}
\]

From this theorem, it is possible to deduce the explicit forms of Bernoulli functions and functional relations for the root system of type $C_r$. The case $r = 3$ will be discussed more explicitly in Section 7.

4. Explicit functional relations ($|I| = 1$ case)

In this section we consider the case $I = \{i\}$ and $y = 0$. In this case the result can be stated in terms of the Lerch zeta-function

\[(4.1)\]

\[
\phi(s, u) = \sum_{n=1}^{\infty} \frac{e^{2\pi \sqrt{-1}un}}{n^s}.
\]

Theorem 4.1. Let $s_\alpha = k_\alpha \in \mathbb{Z}_{\geq 2}$ for $\alpha \in \Delta_+ \setminus \{\alpha_i\}$ and $s_{\alpha_i} \in \mathbb{C}$. Let $|k| = \sum_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} k_\alpha$. Let $X_i = \{\nu = \{(q, p)\} \mid V \in \mathcal{V}, q \in Q^V / L(V^\vee)\} \subset \mathbb{Q}$. Then we have

\[(4.2)\]

\[
\sum_{w \in \mathbb{W}} \left( \prod_{\alpha \in \Delta_{w-1}} (-1)^{-k_\alpha} \right) \zeta_r(w^{-1}s, 0; \Delta)
\]
where $b_{kj} \in \mathbb{Q}$ is given by

\begin{equation}
\sum_{k \in \mathbb{N}^3} \sum_{\nu \in X_i} b_{kj} \sum_{j=0}^{\nu} \frac{t_{\nu j}}{\nu!} \phi(s_{\alpha_i} + j, \nu),
\end{equation}

where $b_{kj} \in \mathbb{Q}$ is given by

\begin{align}
&= (-1)^{\Delta_+ - 1} \left( \prod_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} \frac{(2\pi \sqrt{-1})^{k_\alpha}}{k_\alpha!} \right) \sum_{\nu \in X_i} \sum_{j=0}^{\nu} \frac{b_{kj}}{(2\pi \sqrt{-1})^j} \phi(s_{\alpha_i} + j, \nu),
\end{align}

where $b_{kj} \in \mathbb{Q}$ is given by

\begin{align}
&= \sum_{k \in \mathbb{N}^3} \sum_{\nu \in X_i} b_{kj} \sum_{j=0}^{\nu} \frac{t_{\nu j}}{\nu!} \phi(s_{\alpha_i} + j, \nu),
\end{align}

where $x$ and $y$ are indeterminates.

**Proof.** Since $I = \{i\}$, we have $\Delta_{I+} = \{\alpha_i\}$ and $\Delta^* = \Delta_+ \setminus \{\alpha_i\}$. Note that for $\lambda = n\lambda_i \in P_{I+}$, we have $p_{\lambda, V_i}(\lambda) = n\mu_{\alpha_i}$ by (2.3), and hence for $u \in V_i$

\begin{equation}
\langle u, p_{\lambda, V_i}(\lambda) \rangle = n\langle u, \mu_{\alpha_i} \rangle.
\end{equation}

Therefore

\begin{equation}
F(t, 0, \lambda; I; \Delta) = \sum_{V \in \mathbb{N}^3} \prod_{\gamma \in \Delta_+ \setminus \Delta^*} t_{\gamma} \prod_{\beta \in \Delta^*} t_{\beta} \exp(t_{\beta}(\gamma^\vee, \mu_{\alpha_i})) \prod_{\beta \in \Delta^*} t_{\beta} \exp(t_{\beta}(\gamma^\vee, \mu_{\alpha_i}))(2\pi \sqrt{-1n} \langle \gamma^\vee, \mu_{\alpha_i} \rangle)
\end{equation}

Consider the Taylor expansion of the right-hand side with respect to $t_i$. Putting $x = (2\pi \sqrt{-1n})^{-1}$ and $y = \exp(2\pi \sqrt{-1n})$, the right-hand side of (4.5) is equal to the right-hand side of (4.3). We write the Taylor expansion as the left-hand side of (4.3) with rational coefficients $b_{kj}$; here we remark that in (4.3), the highest degree of $x$ is at most $|k|$, because when $\langle \gamma^\vee, \mu_{\alpha_i} \rangle \neq 0$

\begin{equation}
\frac{t_{\gamma} - \sum_{\beta \in \Delta^*} t_{\beta} \langle \gamma^\vee, \mu_{\alpha_i} \rangle / x}{xt_{\gamma} - \sum_{\beta \in \Delta^*} t_{\beta} \langle \gamma^\vee, \mu_{\alpha_i} \rangle / x}
\end{equation}

and hence $x$ in the expansion appears necessarily together with $t_a$.

From (4.3) we have

\begin{equation}
P(k, 0, \lambda; I; \Delta) = \sum_{\nu \in X_i} \sum_{j=0}^{\nu} \frac{b_{kj}}{(2\pi \sqrt{-1n})^j} \exp(2\pi \sqrt{-1n} \nu).
\end{equation}
Therefore from Theorem 2.3 we obtain
\[(4.8)\]
\[S(s, 0; I; \Delta) = (-1)^{\Delta_+ - 1} \left( \prod_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} \frac{(2\pi \sqrt{-1})^{k_{\alpha}}}{k_{\alpha}!} \right) \sum_{n=1}^{\infty} \frac{1}{n^{s_{\alpha}}} P(k, 0, \lambda; I; \Delta) = (-1)^{\Delta_+ - 1} \left( \prod_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} \frac{(2\pi \sqrt{-1})^{k_{\alpha}}}{k_{\alpha}!} \right) \sum_{\nu X, j=0}^{\nu} \frac{b_{k\nu j}}{(2\pi \sqrt{-1})^{j}} \phi(s_{\alpha} + j, \nu).\]

Combining this and (2.2) we obtain the assertion of the theorem. \( \square \)

5. Poincaré polynomials

Now we turn our attention to the left-hand side of (2.12). We begin with the relation
\[(5.1)\]
\[W_I W^I = W,\]
which follows by the definitions of \(W_I, W^I\) or [11, Lemma 1]. Therefore \(w \in W\) is uniquely written as \(w = xy\) with \(x \in W_I, y \in W^I\). We note
\[(5.2)\]
\[\Delta_{w^{-1}} = x\Delta y^{-1} \sqcup \Delta x^{-1}.\]

This is well-known, but we supply a proof. Let \(x = \sigma_{\nu_1} \cdots \sigma_{\nu_j}, y = \sigma_{\nu_{j+1}} \cdots \sigma_{\nu_k}\) be reduced expressions by simple reflections. Then \(w = \sigma_{\nu_1} \cdots \sigma_{\nu_k}\) is also a reduced expression because of the length identity \(l(w) = l(x) + l(y)\) ( [2, Section 1.10, Proposition (c)]). Let \(\beta_{ik} = \sigma_{\nu_i} \sigma_{\nu_{i+1}} \cdots \sigma_{\nu_{k-1}} (\nu_i)\) (1 \(\leq i \leq k - 1\) and \(\beta_{kk} = \alpha_k\). Then \(\Delta_w = \{\beta_{1k}, \ldots, \beta_{kk}\}\) ( [2, Section 1.7, p.14]). Similarly we have \(\Delta_x = \{\beta_{ij}, \ldots, \beta_{jj}\}\) and \(\Delta_y = \{\beta_{j+1,k}, \ldots, \beta_{kk}\}\). Therefore \(\Delta_w = (\nu_i - 1)\Delta_x \sqcup \Delta_y\). Applying this argument to \(w^{-1} = y^{-1}x^{-1}\), we obtain (5.2).

Let \(u = (u_\alpha)_{\alpha \in \Delta_+}\) be a vector of indeterminates. For \(X \subset \text{Aut}(\Delta)\), we define a generalization of Poincaré polynomials due to Macdonald [16] as
\[(5.3)\]
\[X(u) = \sum_{w \in X} \prod_{\alpha \in \Delta_{w^{-1}}} u_\alpha.\]

When \(X = W^I\) and \(u = ((-1)^{s_\alpha})_{\alpha \in \Delta_+}\), the corresponding Poincaré polynomial is
\[(5.4)\]
\[W^I(u) = \sum_{w \in W^I} \prod_{\alpha \in \Delta_{w^{-1}}} (-1)^{s_\alpha},\]
which is the sum of coefficients of (2.2). If this is not zero, then the left-hand side of (2.12) does not vanish identically.

**Lemma 5.1** (cf. [2, Section 1.11, p.21]). Assume \(u_\alpha = u_\beta\) if \(\|\alpha\| = \|\beta\|\). Then
\[(5.5)\]
\[W_I(u)W^I(u) = W(u).\]
Proof. The assumption for $u_\alpha$ implies $u_\alpha = u_{x-1, \alpha}$. Therefore, using (5.1) and (5.2) we have
\begin{equation}
W(u) = \sum_{w \in W} \prod_{\alpha \in \Delta_{y-1}} u_\alpha = \sum_{x \in W_I, y \in W'} \left( \prod_{\alpha \in x \Delta_{y-1}} u_\alpha \right) \left( \prod_{\alpha \in \Delta_{y-1}} u_\alpha \right) = W_I(u) W'(u).
\end{equation}

Assume $u_\alpha = u$ for all $\alpha \in \Delta_+$, that is, $u = (u, \ldots, u)$, which we write $(u)$ for brevity. By Lemma 5.1 we have
\begin{equation}
W'(((u))) = \frac{W((u))}{W_I((u))}.
\end{equation}
In this case the numerator and the denominator of the right-hand side are classical Poincaré polynomials, which have the following product expression (due to Chevalley):
\begin{align}
W((u)) &= \prod_{i=1}^r \frac{u^{d_i} - 1}{u - 1} = \prod_{i=1}^r (1 + \cdots + u^{e_i}), \\
W_I((u)) &= \prod_{i \in I} \frac{u^{d'_i} - 1}{u - 1} = \prod_{i \in I} (1 + \cdots + u^{e'_i}),
\end{align}
where $d_i$ and $d'_i$ (resp. $e_i$ and $e'_i$) are degrees (resp. exponents) of the Weyl groups $W$ and $W_I$. In fact, the first equalities are [2, Section 3.15, Theorem], and then the second equalities are immediate in view of [2, Section 3.19, Theorem]. The following is the table of degrees ([2, Section 3.7]). Here we note that $W(B_r) = W(C_r)$ and hence the their degrees agree.

| Type | $\{d_1, \ldots, d_r\} = \{e_1 + 1, \ldots, e_r + 1\}$ |
|------|----------------------------------|
| $A_r$ | $2, 3, \ldots, r + 1$ |
| $B_r$ | $2, 4, \ldots, 2r$ |
| $C_r$ | $2, 4, \ldots, 2r$ |
| $D_r$ | $2, 4, \ldots, 2r - 2, r$ |
| $E_6$ | $2, 5, 6, 8, 9, 12$ |
| $E_7$ | $2, 6, 8, 10, 12, 14, 18$ |
| $E_8$ | $2, 8, 12, 14, 18, 20, 24, 30$ |
| $F_4$ | $2, 6, 8, 12$ |
| $G_2$ | $2, 6$ |

Assume that $\Delta$ is a non-simply laced root system and $\Delta = \Delta_1 \cup \Delta_2$, where each $\Delta_i$ consists of all roots of the same length. Then each $\Delta_i$ is a root system. Let $\Delta_{1+} = \Delta_+ \cap \Delta_1$ and $\Delta_{1-} = \Delta_- \cap \Delta_1$. Let $J$ be the set of indices determined by $\Psi_2 = \Psi \cap \Delta_2 = \{\alpha_i\}_{i \in J}$. 
Proposition 5.2. $W(\Delta)((u),(1)) = |W_J| W(\Delta_1)((u))$, where the left-hand side means that $u_\alpha = u$ for $\alpha \in \Delta_1$ and $u_\alpha = 1$ for $\alpha \in \Delta_2$.

Proof. We show $W(\Delta_1) \cap W_J = \{\text{id}\}$. Let $w \in W(\Delta_1) \cap W_J$. Then $w = \sigma_{i_1} \cdots \sigma_{i_l}$ with $i_k \in J$. Since $\sigma_{i_k} \Delta_+ = (\Delta_+ \setminus \{\alpha_{i_k}\}) \cup \{-\alpha_{i_k}\}$, we have $\sigma_{i_k} \Delta_- = (\Delta_- \setminus \{-\alpha_{i_k}\}) \cup \{\alpha_{i_k}\} = \Delta_-$ and hence $\sigma_{i_k} \Delta_1^- = \Delta_1^-$ (because $\pm \alpha_{i_k} \notin \Delta_1$). This implies that $\Delta_1^+ \cap w^{-1} \Delta_1^- = \emptyset$. Therefore the length of $w$ in $W(\Delta_1)$ is 0 and hence $w = \text{id}$.

We define $f : W(\Delta_1) \to W_J$ by $f(w) = w^J$ using the decomposition $w = w^J w_J$ for $w \in W(\Delta_1)$. We show that $f$ is bijective. For $w, \tilde{w} \in W(\Delta_1)$, assume $f(w) = f(\tilde{w})$. Then $w w_J^{-1} = \tilde{w} \tilde{w}_J^{-1}$ and $w^{-1} \tilde{w} = w_J^{-1} \tilde{w}_J \in W(\Delta_1) \cap W_J = \{\text{id}\}$ by the previous paragraph, which implies $w = \tilde{w}$ and the injectivity of $f$. Fix $x \in W_J$. Let $x = \sigma_{i_1} \cdots \sigma_{i_m}$ be a reduced expression. We decompose $(i_1, \ldots, i_m)$ into two subsequences as $(j_1, \ldots, j_q)$ and $(k_1, \ldots, k_{m-q})$ such that $j_l \in J$ and $k_l \notin J$. Let $y = \sigma_{j_1} \cdots \sigma_{j_q} \in W_J$ and consider $w = xy$. Then by use of $\sigma_\alpha \sigma_\beta = \sigma_\beta \sigma_\sigma_\alpha$, we carry each $\sigma_{j_i}$ forward in $y$ until it cancels the same element at the original position in $x$. Hence we see that $w$ can be written as $\sigma_{\beta_1} \cdots \sigma_{\beta_{m-q}}$ with $\beta_p = \sigma_{j_1} \cdots \sigma_{j_h} \alpha_{k_p} \in \Delta_1$ for some $h$. Thus we have $w = xy \in W(\Delta_1)$. By the uniqueness of the decomposition, we see that $f$ is surjective and hence bijective.

In the first paragraph of the proof we have seen that $w_J \Delta_1^- = \Delta_1^-$ for $w_J \in W_J$. Noting this fact and the bijectivity of $f$, we obtain

$$W(\Delta_1)((u)) = \sum_{w \in W(\Delta_1) \cap \Delta_1^-} \prod_{\alpha \in \Delta_1^-} u = \sum_{w \in W(\Delta_1) \cap \Delta_1^+ \cap w^J \Delta_1^-} \prod_{\alpha \in \Delta_1^-} u$$

(5.10)

Furthermore we have

$$W(\Delta)((u),(1)) = \sum_{w \in W(\Delta) \cap \Delta_1^- \cap \Delta_1} \left( \prod_{\alpha \in \Delta_1} u \right) \left( \prod_{\alpha \in \Delta_1} 1 \right) = \sum_{w \in W(\Delta) \cap \Delta_1^- \cap \Delta_2} \prod_{\alpha \in \Delta_1} u$$

(5.11)

Combining (5.10) and (5.11), we obtain the result. \qed
We denote \( W(X_r) = W(\Delta) \) for a root system of type \( X_r \). For a root system of type \( X_r = B_r, C_r, F_4, G_2 \), let \( \Delta_1 = \Delta_L(X_r) \) be the set of all long roots, \( \Delta_2 = \Delta_S(X_r) \) that of all short roots.

**Corollary 5.3.**

\[
W(B_r)((u), (1)) = W(C_r)((1), (u)) = |W(A_1)|W(D_r)((u)) = 2W(D_r)((u)), \\
W(C_r)((u), (1)) = W(B_r)((1), (u)) = |W(A_{r-1})|W(A_1')((u)) = r!(W(A_1)((u)))^r, \\
W(F_4)((u), (1)) = W(F_4)((1), (u)) = |W(A_2)|W(D_4)((u)) = 6W(D_4)((u)), \\
W(G_2)((u), (1)) = W(G_2)((1), (u)) = |W(A_1)|W(A_2)((u)) = 2W(A_2)((u)).
\]

**Proof.** These are simple consequences of Proposition \( \text{5.2} \). We just notice that

\( \Delta_L(B_r) \asymp \Delta_S(C_r) \) and \( \Delta_S(B_r) \asymp \Delta_L(C_r) \),

\( \Delta_L(B_r) \asymp \Delta(D_r), \Delta_L(C_r) \asymp \Delta(A_1'), \Delta_L(F_4) \asymp \Delta(D_4), \Delta_L(G_2) \asymp \Delta(A_2) \), and \( \Psi_2(B_r) \asymp \Psi(A_1), \Psi_2(C_r) \asymp \Psi(A_{r-1}), \Psi_2(F_4) \asymp \Psi(A_2), \Psi_2(G_2) \asymp \Psi(A_1) \) (which can be seen from the list of roots of each system; see [Planche], [23, Section 2.14]),

\( \Delta(L) \) for a root system of type \( X \)

6. The non-vanishing of the sum of coefficients

Now we carry out the case study when \( \text{[5.3]} \) vanishes, or does not vanish. Assume \( s_\alpha = s_{w\alpha} \) for all \( w \in W^r \) and \( \alpha \in \Delta \). Assume \( s_\alpha = k_\alpha \in \mathbb{N} \) for all \( \alpha \in \Delta \setminus \Delta_1 \), and let \( u_\alpha = (-1)^{-k_\alpha} \) for those \( \alpha \).

**Lemma 6.1.** Let \( y \in W^r \). Then \( \Delta_{y^{-1}} \cap \Delta_{I^+} = \emptyset \).

**Proof.** Assume \( \alpha \in \Delta_{y^{-1}} \cap \Delta_{I^+} \). Then \( s_\alpha \in W_I \). Let \( w = s_\alpha y \); this expression gives the unique decomposition of \( w \). Then \( \Delta_{w^{-1}} = s_\alpha \Delta_{y^{-1}} \sqcup \Delta_{s_\alpha} \) by \( \text{[5.2]} \). By the definition \( \Delta_{w^{-1}} \subset \Delta_+ \), while \( -\alpha = s_\alpha \alpha \in s_\alpha \Delta_{y^{-1}} \), which leads to the contradiction. \( \square \)

From the above lemma, we see that the product on the right-hand side of

\[
W^r(u) = \sum_{w \in W^r} \prod_{\alpha \in \Delta_{w^{-1}}} u_\alpha
\]

consists of only \( \alpha \in \Delta_+ \setminus \Delta_{I^+} \).

Now consider several cases. Assume \( I \subset \{1, \ldots, r\} \).

6.1. **Case 1.** This is the trivial case. If all \( k_\alpha \) are even, then \( u_\alpha = 1 \) for all \( \alpha \in \Delta \), and we have \( W^r((1)) = |W^r| > 0 \).
6.2. **Case 2.** If all \( k_\alpha \) are odd, then we have to evaluate (5.7) at \( u = (-1) \).

Let \( K \) and \( K_I \) be the sets of indices of even degrees (i.e., odd exponents) of the Weyl groups \( W \) and \( W_I \) respectively, given as

\[
(6.2) \quad K = \{ i \mid 1 \leq i \leq r, d_i \in 2\mathbb{Z} \} = \{ i \mid 1 \leq i \leq r, e_i \in 2\mathbb{Z} + 1 \},
\]

\[
(6.3) \quad K_I = \{ i \mid i \in I, d_i \in 2\mathbb{Z} \} = \{ i \mid i \in I, e_i \in 2\mathbb{Z} + 1 \}.
\]

From (5.8) and (5.9) we see that

\[ W((-1)) = 0 \] if \( K \neq \emptyset \), and \( W_I((-1)) = 0 \) if \( K_I \neq \emptyset \). Comparing the orders of zeros of the both sides of (5.5), we find that

\[
(6.4) \quad |K| \geq |K_I|,
\]

hence \( W_I((-1)) \) does not vanish if and only if

\[
(6.5) \quad |K| = |K_I|.
\]

If it holds, then by (5.8), (5.9) and l'Hôpital's rule

\[
W_I((-1)) = \lim_{u \to -1} W_I(u) = \lim_{u \to -1} \frac{\prod_{i \in K} (1 + \cdots + u^{e_i}) \prod_{i \in K_I} (u^{d_i} - 1)}{\prod_{i \in K_I} (1 + \cdots + u^{e_i}) \prod_{i \in K} (u^{d_i} - 1)}
\]

\[
= \frac{\prod_{i \in K} d_i}{\prod_{i \in K_I} d_i} \in \mathbb{N}.
\]

The above argument especially implies the following

**Claim 6.2.** If all degrees of a root system \( X_r \) are even, then \( W_I((-1)) \) vanishes for any choice of \( I \subset \{1, \ldots, r\} \).

This is obvious from (6.5), because for any \( I \subset \{1, \ldots, r\} \), we have \( |K| > |K_I| \). From this claim and the table in Section 5 we find that \( W_I((-1)) \) always vanishes for \( B_r, C_r, D_{2k}, E_7, E_8, F_4, \) and \( G_2 \). Therefore the only root systems for which (6.6) can be applied are \( A_r, D_{2k+1}, \) and \( E_6 \).

**Theorem 6.3.** \( W_I((-1)) \) does not vanish only in one of the following cases.
\[ \Delta \]

where \([r_1 + 1]/2 + \cdots + [r_n + 1]/2 = (r + 1)/2\).

**Proof.** Assume that \(\Delta\) is of type \(E_6\). Then \(\Delta_I\) must be a subroot system such that there are 4 even degrees in it, which implies that \(\Delta_I\) is either of type \(D_4\) or \(D_5\).

Assume that \(\Delta\) is of type \(D_{2k+1}\). Then \(\Delta_I\) must be a subroot system such that there are \(2k\) even degrees in it, which implies that \(\Delta_I\) is of type \(D_{2k}\).

Assume that \(\Delta\) is of type \(A_r\). By noting that there are \([r_1 + 1]/2\) even degrees and they are \(2, 4, \ldots, 2[r_1 + 1]/2\), and that their product is

\[ (6.7) \quad 2 \cdot 4 \cdot \cdots (2[r_1 + 1]/2) = 2^{(r_1 + 1)/2}[(r_1 + 1)/2]!, \]

we obtain the result. \(\square\)

For example, see the following diagrams for the pairs \((A_{2k}, A_{2k-1})\) and \((A_4, A_1 \times A_2)\), where the set of enclosed nodes is \(\Psi_I\).

\[ (6.8) \]

\[ (6.9) \]

Here are some examples of the case when \(\Delta\) is of type \(A\).

| Type of \(\Delta\) | Type of \(\Delta_I\) | \(W^I((-1))\) |
|-------------------|---------------------|----------------|
| \(A_r\)          | \(A_{r_1} \times \cdots \times A_{r_n}\) | \([r + 1]/2\)! \| \((r + 1)/2\)! \cdots [r_n + 1]/2\)! \n| \(D_{2k+1}\)     | \(D_{2k}\)          | \(2 \cdot 4 \cdots (4k - 2) \cdot 4k\) = 2 |
| \(E_6\)       | \(D_5\)             | \(2 \cdot 6 \cdot 8 \cdot 12 = 3\) |
| \(E_6\)       | \(D_4\)             | \(2 \cdot 4 \cdot 6 \cdot 8 = 3\) |

6.3. **Case 3.** Assume \(k_\alpha\) are odd for \(\alpha \in \Delta_1\) and \(k_\beta\) are even for \(\beta \in \Delta_2\). Let \(u = ((u), (1))\). Then we have

\[ (6.10) \quad W^I(u) = \frac{W((u), (1))}{W^I((u), (1))} = \frac{|W_{I\cap J}|W(\Delta_1)((u))}{|W_{I\cap J}|W(\Delta_1 \cap \Delta_I)((u))} \]
by Proposition 5.2. By using an argument similar to Case 2, we can calculate $W^I((-1), (1))$ as follows. Let $K_1$ and $K_1^I$ be the sets of indices of even degrees of the Weyl groups $W(\Delta_1)$ and $W(\Delta_1 \cap \Delta_I)$ respectively, given as

$K_1 = \{ i \mid \alpha_i \in \Delta_1, d_i \in 2\mathbb{Z} \}$,

$K_1^I = \{ i \mid \alpha_i \in \Delta_1 \cap \Delta_I, d_i^I \in 2\mathbb{Z} \}$.

Then we see that $W^I((-1), (1))$ does not vanish if and only if

$|K_1| = |K_1^I|$.

If it holds, then

$W^I((-1), (1)) = \lim_{u \to -1} W^I((u), (1)) = |W_J| \lim_{u \to -1} \frac{\prod_{i \in K_1^I} (1 + \cdots + u^{d_i}) \prod_{i \in K_1}(u^{d_i} - 1)}{\prod_{i \in K_1^I}(1 + \cdots + u^{d_i^I}) \prod_{i \in K_1}(u^{d_i} - 1)}$

$= \frac{|W_J| \prod_{i \in K_1^I} 1 \prod_{i \in K_1} d_i}{|W_I \cap J| \prod_{i \in K_1^I} 1 \prod_{i \in K_1} d_i} \in \mathbb{N}$,

where the complement of sets of indices means that in $\{ i \mid \alpha_i \in \Delta_1 \}$.

**Theorem 6.4.** $W^I((-1), (1))$ does not vanish only in one of the following cases.

| Type of $\Delta$ | Type of $\Delta_I$ | $\Delta_1$ | $W^I((-1), (1))$ |
|------------------|-------------------|------------|-----------------|
| $B_{2k+1}$      | $B_{2k}$         | $\Delta_L$ | $\frac{2 \cdot 4 \cdots (4k - 2) \cdot 4k}{2 \cdot 4 \cdots (4k - 2) \cdot 2k} = 2$ |
| $C_{2k+1}$      | $C_{2k}$         | $\Delta_S$ | $\frac{2 \cdot 4 \cdots (4k - 2) \cdot 4k}{2 \cdot 4 \cdots (4k - 2) \cdot 2k} = 2$ |
| $G_2$           | $A_1$ (long)     | $\Delta_L$ | $\frac{2 \cdot 4}{1 \cdot 2} = 2$ |
| $G_2$           | $A_1$ (short)    | $\Delta_S$ | $\frac{2 \cdot 4}{1 \cdot 2} = 2$ |

**Proof.** The root systems which have roots of two (long and short) lengths are of type $B_r, C_r, F_4, G_2$.

When $\Delta$ is of type $F_4$, by Corollary 5.3, $\Delta_1$ is of type $D_4$. Since the degrees of $\Delta(D_4)$ are all even, we see that $W^I((-1), (1))$ vanishes for any choice of $I$ in $\Delta(F_4)$.

Assume that $\Delta$ is of type $B_r$. Then by Corollary 5.3, $\Delta_1$ must be $\Delta_L$ and $r$ must be odd because otherwise all the degrees are even. When $\Delta_1 = \Delta_L$,
(r – 1) degrees are even. Hence \( \Delta_I \) must be of type \( B_{r-1} \) whose degrees are all even, which is the unique choice such that \(|K_1| = |K_{1I}|\).

In the case of \( C_r \), a similar argument works and we obtain the result.

Assume that \( \Delta \) is of type \( G_2 \). Then \( \Delta_1 \) is either \( \Delta_L \) or \( \Delta_S \). However \( \Delta_I \) must be determined uniquely so that \(|K_1| = |K_{1I}|\), that is, \( \Delta_I \subset \Delta_1 \).

\[ \square \]

For example, see the following diagram for the pair \((B_{2k+1}, B_{2k})\), where the set of enclosed nodes is \( \Psi_I \).

(6.15)

7. Examples

In this section, we give several explicit examples of the functional relations among zeta-functions of root systems. For the case \( y = (0) \), we write \( \zeta_r(s; \Delta) \) instead of \( \zeta_r(s, y; \Delta) \) for short. First we consider the \( A_r \)-type.

Example 7.1. Set \( r = 2 \), \( s_{23} \in \mathbb{R}_{>1} \) and \((y_1, y_2) = (0, 0)\) in (3.29). By considering its real part, we obtain

\[
\zeta_2(k_{12}, k_{13}, s_{23}; A_2) + (-1)^{k_{12}} \zeta_2(k_{12}, s_{23}, k_{13}; A_2)
\]

\[ + (-1)^{k_{12} + k_{13}} \zeta_2(s_{23}, k_{12}, k_{13}; A_2) \]

\[ = 2 \sum_{j_2=0}^{[k_{12}/2]} (-1)^{j_2} \left( k_{12} + k_{13} - 1 - 2j_2 \over k_{13} - 1 \right) \zeta(2j_2) \zeta(k_{12} + k_{13} + s_{23} - 2j_2) \]

\[ + 2 \sum_{j_3=0}^{[k_{13}/2]} (-1)^{k_{13}} \left( k_{12} + k_{13} - 1 - 2j_3 \over k_{12} - 1 \right) \zeta(2j_3) \zeta(k_{12} + k_{13} + s_{23} - 2j_3), \]

where we use the well-known formula

\[ \zeta(2k) = -\frac{B_{2k}(0)(2\pi \sqrt{-1})^{2k}}{2(2k)!} \quad (k \in \mathbb{Z}_{\geq 0}). \]

Dividing the both sides by \((-1)^{k_{12}}\), we recover the known result in [8, Theorem 3.1], which is equivalent to [26, Theorem 4.5] (see also [20]). Here we note that

\[ \zeta_2(s_{12}, s_{13}, s_{23}; A_2) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{1}{m_1^{s_{12}}(m_1 + m_2)^{s_{13}}m_2^{s_{23}}}, \]
where the order of indices is slightly different from that in \[8\].

**Example 7.2.** Set \( r = 3 \), \((s_{23}, s_{24}, s_{34}) \in (\mathbb{R}_{>1})^3 \) and \((y_1, y_2, y_3) = (0, 0, 0)\) in \((3.29)\). By considering its real part, we obtain

\[
\zeta_3(k_{12}, k_{13}, k_{14}, s_{23}, s_{24}, s_{34}; A_3) \\
+ (-1)^{k_{12}} \zeta_3(k_{12}, s_{23}, s_{24}, k_{13}, k_{14}, s_{34}; A_3) \\
+ (-1)^{k_{12}+k_{13}} \zeta_3(s_{23}, k_{12}, s_{24}, k_{13}, s_{34}, k_{14}; A_3) \\
+ (-1)^{k_{12}+k_{13}+k_{14}} \zeta_3(s_{23}, s_{24}, k_{12}, s_{34}, k_{13}, k_{14}; A_3)
\]

\[= 2 \sum_{j_2=0}^{[k_{12}/2]} \sum_{l_3,l_4 \geq 0} (-1)^{k_{12}} \left( k_{13} + l_3 - 1 \right) \left( k_{14} + l_4 - 1 \right) \]

\[\times \zeta(2j_2) \zeta_2(s_{23} + k_{13} + l_3, s_{24} + k_{14} + l_4, s_{34}; A_2)\]

\[= 2 \sum_{j_3=0}^{[k_{13}/2]} \sum_{l_2,l_4 \geq 0} (-1)^{k_{12}+l_4} \left( k_{12} + l_2 - 1 \right) \left( k_{14} + l_4 - 1 \right) \]

\[\times \zeta(2j_3) \zeta_2(s_{23} + k_{12} + l_2, s_{24}, s_{34} + k_{14} + l_4; A_2)\]

\[= 2 \sum_{j_4=0}^{[k_{14}/2]} \sum_{l_2,l_3,l_4 \geq 0} (-1)^{k_{12}+k_{13}} \left( k_{12} + l_2 - 1 \right) \left( k_{13} + l_3 - 1 \right) \]

\[\times \zeta(2j_4) \zeta_2(s_{23} + k_{12} + l_2, s_{24} + k_{13} + l_3; A_2)\]

This formula is apparently different from the previous result in \([5]\) Theorem 4.4. For example, we obtain from \((7.2)\) with \((k_{ij}) = (2)\) and \((s_{ij}) = (2)\) that

\[4\zeta_3(2, 2, 2, 2, 2, 2; A_3) = 2\zeta(2) \left\{ 2\zeta_2(4, 4, 2; A_2) + \zeta_2(4, 2, 4; A_2) \right\} \]

\[+ 6\zeta_2(6, 4, 2; A_2) + 6\zeta_2(6, 2, 4; A_2) - 8\zeta_2(5, 5, 2; A_2) \]

\[+ 4\zeta_2(5, 2, 5; A_2) - 6\zeta_2(4, 6, 2; A_2).\]

On the other hand, we already obtained from \([5]\) Eq. (4.28)] that

\[4\zeta_3(2, 2, 2, 2, 2, 2; A_3) = 8\zeta(2) \left\{ \zeta_2(4, 4, 2; A_2) + \zeta_2(3, 5, 2; A_2) \right\} \]

\[+ 12\zeta_2(6, 4, 2; A_2) - 12\zeta_2(5, 5, 2; A_2) - 6\zeta_2(4, 6, 2; A_2).\]

These two right-hand sides are apparently different. However we can check that both of the right-hand sides are equal to \(887\pi^{12}/3831077250\) by the method of partial fraction decompositions (see \([5]\) Example 4.5).

Next we consider the \(C_r\)-type.

**Example 7.3.** We define zeta-functions of root systems of \(C_2\)-type and \(C_3\)-type by

\[
\zeta_2(s_1, s_2, s_3, s_4; C_2) = \sum_{m,n \geq 1} \frac{1}{m^{s_1} n^{s_2} (m + n)^{s_3} (m + 2n)^{s_4}}.
\]
\[ \zeta_3(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9; C_3) \]
\[ = \sum_{m_1, m_2, m_3 \geq 1} \frac{1}{m_1^s m_2^s m_3^s (m_1 + m_2)^{s_4} (m_2 + m_3)^{s_5} (m_1 + 2m_2)^{s_6}} \times \frac{1}{(m_1 + m_2 + m_3)^{s_7} (m_1 + m_2 + 2m_3)^{s_8} (m_1 + 2m_2 + 2m_3)^{s_9}}. \]

These were already studied in \([6, 8]\). Here we give explicit forms of functional relations among them as follows.

Let \( r = 3, \Delta = \Delta(C_3), I = \{2, 3\} \) and \((y_1, y_2, y_3) = (0, 0, 0)\) in \([8, 23]\). Then we have

\[
(7.3) \quad F((t_1, t_{\pm 2}, t_{\pm 3}), 0, (m_2, m_3); \{2, 3\}; C_3)
\]
\[
= \frac{t_{-3}}{t_{-3} - t_{-2} - 2\pi \sqrt{-1}(2m_2 + m_3)} \times \frac{t_{+2}}{t_{+2} - 2\pi \sqrt{-1}(2m_2 + m_3)} \times \frac{t_{+3}}{t_{+3} - t_{+2} - 2\pi \sqrt{-1}(2m_2 + m_3)} \times \frac{t_{1}}{t_{1} - t_{-2} - 2\pi \sqrt{-1}(2m_2 + m_3) e^{t_{-2}} - 1}
\]
\[
+ \frac{t_{-3}}{t_{-3} + t_{-2} + 2\pi \sqrt{-1}(2m_2 + m_3)} \times \frac{t_{+3}}{t_{+3} - t_{+2} + 2\pi \sqrt{-1}(2m_2 + m_3)} \times \frac{t_{1}}{t_{1} - t_{-2} + 2\pi \sqrt{-1}(2m_2 + m_3) e^{t_{-2}} - 1}
\]
\[
+ \frac{t_{-3}}{t_{-3} - t_{-2} + 2\pi \sqrt{-1}(2m_2 + m_3)} \times \frac{t_{+3}}{t_{+3} - t_{+2} + 2\pi \sqrt{-1}(2m_2 + m_3)} \times \frac{t_{1}}{t_{1} - t_{-2} + 2\pi \sqrt{-1}(2m_2 + m_3) e^{t_{-2}} - 1}
\]
\[
+ \frac{t_{-3}}{t_{-3} + t_{-2} + 2\pi \sqrt{-1}(2m_2 + m_3)} \times \frac{t_{+3}}{t_{+3} - t_{+2} + 2\pi \sqrt{-1}(2m_2 + m_3)} \times \frac{t_{1}}{t_{1} - t_{-2} + 2\pi \sqrt{-1}(2m_2 + m_3) e^{t_{-2}} - 1}
\]

Hence we can compute \( P(k, y, \lambda; I; C_3) \) and give some functional relations from Theorem \([2, 3]\). For example, we obtain

\[
(7.4) \quad P((2, 1, 1, 1, 1), 0, (m_2, m_3); \{2, 3\}; C_3) =
\]
\[
\frac{1}{32\pi^6 m_3^2 (m_2 + 2m_3)^3} + \frac{1}{32\pi^6 m_2^2 (m_2 + m_3)^4 (m_2 + 2m_3)}
- \frac{1}{32\pi^6 m_1^3 (m_2 + m_3)^2} - \frac{1}{64\pi^6 m_2 m_3^2 (m_2 + 2m_3)}
- \frac{1}{32\pi^6 m_2 m_3^2 (m_2 + 2m_3)^2} \frac{1}{32\pi^6 m_3^2 (m_2 + m_3)^2} + \frac{1}{96\pi^4 m_3^2 (m_2 + m_3)^2}
+ \frac{1}{64\pi^6 m_2 (m_2 + m_3)^3 (m_2 + 2m_3)} + \frac{32\pi^6 m_2 (m_2 + m_3)^3 (m_2 + 2m_3)^2}.\]

Hence we have

\[\zeta_3(1, s, t, 1, u, v, 2, 1, 1; C_3) - \zeta_3(1, 1, t, 1, u, v, 1; C_3)\]
\[+ \zeta_3(s, 1, 2, 1, t, 1, u, v; C_3) + \zeta_3(s, 1, 2, 1, t, 1, u, v; C_3)\]
\[- \zeta_3(1, 1, t, 1, s, t, 1, u, v, 1; C_3) + \zeta_3(s, 1, t, 1, u, v, 2, 1, 1; C_3)\]
\[= (-1)^{\frac{5}{2}} \frac{(2\pi \sqrt{1-\frac{1}{4}})}{2!!3!!4!!} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} \frac{P((2, 1, 1, 1, 1, 0, (m_2, m_3); \{2, 3\}; C_3)}{m_2 m_3^2 (m_2 + m_3)^{m_2 + m_3}}\]
\[= \zeta_2(s + 2, t + 3, u + v + 1; C_2) + \zeta_2(s + 2, t, u + 3, v + 1; C_2)\]
\[- \zeta_2(s, t + 4, u + 2, v; C_2) - \frac{5}{2} \zeta_2(s + 1, t + 4, v + 1; C_2)\]
\[- \zeta_2(s + 1, t + 3, u, v + 2; C_2) + \zeta_2(s, t + 2, u + 4, v; C_2)\]
\[+ \frac{\pi^2}{3} \zeta_2(s, t + 2, u + 2, v; C_2) + \frac{5}{2} \zeta_2(s + 1, t, u + 4, v + 1; C_2)\]
\[- \zeta_2(s + 1, t + 3, v + 2; C_2).\]

Setting \((s, u, v) = (1, 2, 1)\), we obtain

\[\zeta_3(1, 1, 2, 1, 1, 1, 2, 1, 1; C_3)\]
\[= \zeta_2(3, t + 3, 2, 2; C_2) + \zeta_2(3, t, 5, 2; C_2) - \zeta_2(1, t + 4, 4, 1; C_2)\]
\[- \frac{5}{2} \zeta_2(2, t + 4, 2, 2; C_2) - \zeta_2(2, t + 3, 2, 3; C_2)\]
\[- \zeta_2(1, t + 2, 6, 1; C_2) + \frac{\pi^2}{3} \zeta_2(1, t + 2, 4, 1; C_2)\]
\[- \frac{5}{2} \zeta_2(2, t, 6, 2; C_2) + \zeta_2(2, t, 5, 3; C_2).\]

In particular when \(t = 2\), this is an example of the assertion in Section 6.3 corresponding to \(\Delta = \Delta(C_3), \Delta_I = \Delta(C_2)\) with \(I = \{1\}\) and \(W^I((-1), (1)) = 2\), because the left-hand does not vanish and its coefficient is 2.

Moreover we here recall the known fact that

\[\zeta_2(a, b, c, d; C_2) \in \mathbb{Q} \left[\pi^2, \{\zeta(2j + 1)\}_{j \in \mathbb{N}}\right]\]

for \(a, b, c, d \in \mathbb{N}\) with \(2 \nmid (a + b + c + d)\), which was given by the third-named author (see [25]). If we set \(t = 2k - 1 (k \in \mathbb{N})\) in (7.6), we obtain from (7.7) that

\[\zeta_3(1, 1, 2, 1, 2k - 1, 1, 2, 1, 1; C_3) \in \mathbb{Q} \left[\pi^2, \{\zeta(2j + 1)\}_{j \in \mathbb{N}}\right]\]
for $k \in \mathbb{N}$. Setting $t = 1, 3$, we have

$$2\zeta_3(1, 1, 2, 1, 1, 2, 1, 1; C_3) = \frac{3}{20}\zeta(7)\pi^4 - \frac{233}{16}\zeta(9)\pi^2 + \frac{4135}{32}\zeta(11),$$

$$2\zeta_3(1, 1, 2, 1, 3, 1, 2, 1; C_3) = -\frac{7}{15}\zeta(9)\pi^4 + \frac{681}{16}\zeta(11)\pi^2 - \frac{5995}{16}\zeta(13).$$

More generally, we can prove that

$$\zeta_3(2q - 1, 2q - 1, 2r, 2p - 1, 2k - 1, 2q - 1, 2r, 2p - 1, 2q - 1; C_3) \in \mathbb{Q}\left[\frac{\pi^2}{2}, \{\zeta(2j + 1)\}_{j \in \mathbb{N}}\right]$$

for $p, q, r, k \in \mathbb{N}$. For example, we can give

$$2\zeta_3(1, 1, 2, 3, 1, 1, 2, 3, 1; C_3)$$

$$= \frac{1}{10080}\zeta(7)\pi^8 + \frac{7}{180}\zeta(9)\pi^6 + \frac{13217}{576}\zeta(11)\pi^4$$

$$- \frac{4132493}{1536}\zeta(13)\pi^2 + \frac{24864015}{1024}\zeta(15),$$

$$2\zeta_3(1, 1, 2, 3, 5, 1, 2, 3, 1; C_3)$$

$$= \frac{43}{90720}\zeta(11)\pi^8 + \frac{859}{2520}\zeta(13)\pi^6 + \frac{140051}{576}\zeta(15)\pi^4$$

$$- \frac{42288073}{1536}\zeta(17)\pi^2 + \frac{253652169}{1024}\zeta(19).$$

Finally we consider the $G_2$-type.

**Example 7.4.** We define the zeta-function of root system of $G_2$-type by

$$\zeta_2(s_1, s_2, s_3, s_4, s_5, s_6; G_2)$$

$$= \sum_{m, n \geq 1} \frac{1}{m^{s_1}n^{s_2}(m + n)^{s_3}(m + 2n)^{s_4}(m + 3n)^{s_5}(2m + 3n)^{s_6}}.$$  

This was already studied in [9,15]. In fact, it follows from [15, Theorem 6.1] that for $p, q, r, u, v \in \mathbb{N}$,

$$\zeta_2(p, s, q, r, u, v; G_2) + (-1)^p\zeta_2(p, q, s, r, v, u; G_2)$$

$$+ (-1)^{p+q}\zeta_2(v, q, r, s, p, u; G_2) + (-1)^{p+q+v}\zeta_2(v, r, q, s, u, p; G_2)$$

$$+ (-1)^{p+q+r+u}\zeta_2(u, r, s, q, p, v; G_2)$$

$$+ (-1)^{p+q+r+u+v}\zeta_2(u, s, r, q, p, v; G_2)$$

(7.10)

can be expressed in terms of the Riemann zeta-function. In particular, setting $(p, q, r, s, u, v) = (2a, 2b - 1, 2b - 1, 2b - 1, 2a, 2a)$ for $a, b \in \mathbb{N}$, we can compute (7.10) as

$$2\zeta_2(2a, 2b - 1, 2b - 1, 2b - 1, 2a, 2a; G_2).$$

This is an example of the assertion in Section 6.3 corresponding to $\Delta = \Delta(G_2)$, $\Delta_I = \Delta(A_1)$ with $I = \{2\}$ and $W^I((-1), (1)) = 2$. For example, we can obtain from [15, Theorem 6.1] that

$$2\zeta_2(2, 1, 1, 1, 2, 2; G_2) = -\frac{187}{972}\zeta(7)\pi^2 + \frac{11149}{5832}\zeta(9).$$
Denote by $d$ in \cite{14}, but in the present paper it is written as $k \in \vec{Y}$. In this section we prove Theorem 2.3. First we review a result proved in \cite{14}.

Under the assumption (#) in the statement of Theorem 2.3, we see that $\Lambda \cap \{\cdot\}$, where

$$
\begin{align*}
2\zeta_2(4, 1, 1, 1, 4, 4; G_2) &= -\frac{15337}{4723920}\zeta(11)\pi^4 - \frac{157303}{2834352}\zeta(13)\pi^2 \\
+ \frac{14696765}{17006112}\zeta(15),
\end{align*}
$$

$$
2\zeta_2(2, 3, 3, 3, 2, 2; G_2) &= -\frac{16171}{3888}\zeta(13)\pi^2 + \frac{957697}{2328}\zeta(15).
$$

8. Proof of Theorem 2.3

Now we proceed to prove fundamental results stated in Section 2. In this section we prove Theorem 2.3. First we review a result proved in \cite{14}.

Denote the usual inner product on $\mathbb{R}^d$ by $\langle \cdot, \cdot \rangle$. For each $f \in \mathbb{R}^d$, denote by $(\vec{f})$ the Z-span of $\vec{f}$. For $Y \subset \mathbb{R}^d \times \mathbb{C}$, put $\vec{Y} = \{\vec{f} \mid f = (\vec{f}, \hat{f}) \in Y\}$.

Let $\Lambda$ be a finite subset of $(\mathbb{Z}^d \setminus \{0\}) \times \mathbb{C}$ such that the rank of $\langle \vec{\Lambda} \rangle$ is $d$. Denote by $\tilde{\Lambda}$ the set of all $f \in \Lambda$ such that the rank of $\langle \vec{\Lambda} \setminus \{\vec{f}\} \rangle$ is less than $d$. For each $f \in \Lambda$ we associate a number $k_f \in \mathbb{N}$, and put $\mathbf{k} = (k_f)_{f \in \Lambda}$. For $k \in \mathbb{N}$, define $\Lambda_k = \{f \in \Lambda \mid k_f = k\}$. Then $\Lambda = \bigcup_{k \geq 1} \Lambda_k$.

Let $\mathbf{y} \in \mathbb{R}^d$, and define

$$
S(\mathbf{k}, \mathbf{y}; \Lambda) = \lim_{N \to \infty} \sum_{u = (u_1, \ldots, u_d) \in \mathbb{Z}^d \atop |u_j| \leq N \,(1 \leq j \leq d) \atop f(u) \neq 0 \,(f \in \Lambda)} \frac{1}{f(\mathbf{u})^{k_f}}.
$$

Under the assumption (#) in the statement of Theorem 2.3, we see that $\tilde{\Lambda} \cap \Lambda_1 = \emptyset$. Therefore by \cite{14} Theorem 2.2, the right-hand side of (8.1) converges (and defines $S(\mathbf{k}, \mathbf{y}; \Lambda)$) for any $\mathbf{y} \in \mathbb{R}^d$.

Let $\mathcal{B}(\Lambda)$ be the collection of all subsets $B = \{f_1, \ldots, f_d\} \subset \Lambda$ such that $\vec{B}$ forms a basis of $\mathbb{R}^d$. We write the dual basis of $\vec{B} = \{\vec{f}_1, \ldots, \vec{f}_d\}$ as $\vec{B}^* = \{\vec{f}_{d-1}^*, \ldots, \vec{f}_1^*\}$. For any $\mathbf{y} \in \mathbb{R}^d$ and any $\mathbf{t} = (t_f)_{f \in \Lambda} \in \mathbb{C}^{\Lambda}$, define

$$
F(\mathbf{t}, \mathbf{y}; \Lambda) = \sum_{B \in \mathcal{B}(\Lambda)} \left( \prod_{g \in \Lambda \setminus B} t_g \right) \left( \prod_{g \in B} t_g - 2\pi \sqrt{-1} \hat{g} - \sum_{f \in B} (t_f - 2\pi \sqrt{-1} \hat{f})(\hat{g}, \vec{f}^B') \right) \frac{t_g}{\exp(t_g - 2\pi \sqrt{-1} \hat{f}) - 1}.
$$

where $\{\cdot\}_{B,f}$ is the multi-dimensional fractional part in the sense of \cite{14} Section 2] (using the standard inner product on $\mathbb{R}^d$, which is written as $\langle \cdot, \cdot \rangle$ in \cite{14}, but in the present paper it is written as $\langle \cdot, \cdot \rangle'$), and define $C(\mathbf{k}, \mathbf{y}; \Lambda)$
by the expansion

\[ F(t, y'; \Lambda) = \sum_{k \in \mathbb{N}^{\left| \Lambda \right|}} C(k, y'; \Lambda) \prod_{f \in \Lambda} \frac{t_{f}^{k_{f}}}{k_{f}!}. \]

Then [14, Theorem 2.5] asserts

\[ S(k, y'; \Lambda) = \left( \prod_{f \in \Lambda} \frac{-2 \pi \sqrt{-1}}{k_{f}^{2}} \right) C(k, y'; \Lambda) \]

for any \( k = (k_{f})_{f \in \Lambda} \in \mathbb{N}^{\left| \Lambda \right|} \) and any \( y' \in \mathbb{R}^{d} \). This is the key formula for the proof of Theorem 2.3.

**Proof of Theorem 2.3.** For \( \lambda \in P_{I^{+}} \) and \( \mu \in P_{I^{c}} \), the relation \( \lambda + \mu \notin H_{\Delta_{+}} = H_{\Delta_{+}^{y}} \cup H_{\Delta_{-}^{\nu}} \) is equivalent to

\[ (8.5) \quad \lambda \notin H_{\Delta_{+}^{y}}, \quad \text{and} \quad \lambda + \mu \notin H_{\Delta_{-}^{\nu}}. \]

Hence for \( s \in \mathbb{C}^{\left| \Delta_{+} \right|} \) and \( y \in V \) we have

\[ S(s, y; I; \Delta) = \sum_{\lambda \in P_{I^{+}}} \sum_{\mu \in P_{I^{c}}} \left( \prod_{\alpha \in \Delta_{+}} \frac{1}{(\alpha^{\vee}, \lambda + \mu)^{s_{\alpha}}} \right) e^{2 \pi \sqrt{-1} T(y, \lambda)} \left( e^{2 \pi \sqrt{-1} T(y, \mu)} \prod_{\alpha \in \Delta_{-}^{\nu}} \frac{1}{(\alpha^{\vee}, \lambda + \mu)^{k_{\alpha}}} \right) \]

\[ = \sum_{\lambda \in P_{I^{+}}} \left( \prod_{\alpha \in \Delta_{+}} \frac{1}{(\alpha^{\vee}, \lambda)^{s_{\alpha}}} \right) e^{2 \pi \sqrt{-1} T(y, \lambda)} S_{\lambda}(k, y; I; \Delta), \]

where

\[ S_{\lambda}(k, y; I; \Delta) = \sum_{\mu \in P_{I^{c}}} \left( e^{2 \pi \sqrt{-1} T(y, \mu)} \prod_{\alpha \in \Delta_{-}^{\nu}} \frac{1}{(\alpha^{\vee}, \lambda + \mu)^{k_{\alpha}}} \right). \]

We apply the aforementioned result of [14] to find the generating function of \( S_{\lambda}(k, y; I; \Delta) \). For this purpose, we need to identify each symbol appearing here and in [14]. We identify \( \mathbb{R}^{d} \simeq \bigoplus_{i \in I^{c}} \mathbb{R} \lambda_{i}^{y} \) and through this identification, we define the projection \( \pi_{I^{c}} : V \rightarrow \mathbb{R}^{d} \) by

\[ (8.8) \quad \pi_{I^{c}}(v) = (\langle v, \lambda_{i} \rangle)_{i \in I^{c}} \quad (v \in V). \]

For \( \alpha \in \Delta_{+} \), we set the affine linear form

\[ f_{\alpha}(\mu) = \langle \alpha^{\vee}, \mu \rangle + \langle \alpha^{\vee}, \lambda \rangle \]

for \( \mu \in \bigoplus_{i \in I^{c}} \mathbb{R} \lambda_{i} \).

First we note that \( f_{\alpha} \) is naturally regarded as an affine linear form on \( \mathbb{R}^{d} \) via \( \mathbb{R}^{d} \simeq \bigoplus_{i \in I^{c}} \mathbb{R} \lambda_{i} \). In fact, for \( \alpha^{\vee} = \sum_{i \in I^{c}} a_{i} \alpha_{i}^{y} \) and \( \mu = \sum_{i \in I^{c}} b_{i} \lambda_{i} \), we
have
\[ \langle \alpha', \mu \rangle = \sum_{i \in I^e} a_i b_i = \langle (a_i)_{i \in I^e}, (b_i)_{i \in I^e} \rangle' = \langle ((\alpha', \lambda_i))_{i \in I^e}, (b_i)_{i \in I^e} \rangle' \]
(8.10)
\[ = \langle \pi_{I^e}(\alpha'), (b_i)_{i \in I^e} \rangle'. \]
Thus we may choose
\[ \Lambda = \{ f_\alpha = (\tilde{f}_\alpha, \check{f}_\alpha) \mid \alpha \in \Delta^* \} \subset (\mathbb{Z}^d \setminus \{0\}) \times \mathbb{Z}, \]
where \( \tilde{f}_\alpha = \pi_{I^e}(\alpha') \) and \( \check{f}_\alpha = \langle \alpha', \lambda \rangle. \)

Next, let \( C \subset \Delta^* \) with \( |C| = d \) and \( B = \{ f_\alpha \mid \alpha \in C \} \subset \Lambda. \) Secondly we show that \( B \in \mathcal{B}(\Lambda) \) if and only if \( V = C \cup \Psi_I \in \mathcal{Y}_I. \) Since the matrix of the pairings \( \langle \beta', \lambda_i \rangle \) for \( \beta \in C \cup \Psi_I \) and \( 1 \leq i \leq r \) is written as the form
\[ \left( \begin{array}{cc} \langle (\beta', \lambda_i) \rangle_{\beta \in C, i \in I^e} & \langle (\beta', \lambda_i) \rangle_{\beta \in \Psi_I, i \in I^e} \\ \langle (\beta', \lambda_i) \rangle_{\beta \in \Psi_I, i \in I^e} & \langle (\beta', \lambda_i) \rangle_{\beta \in \Psi_I \cup I^e} \end{array} \right) = \left( \begin{array}{cc} A & \ast \\ 0 & \text{id}_{|I|} \end{array} \right), \]
(8.12)
it is sufficient to show that that \( A \) is regular if and only if \( B \in \mathcal{B}(\Lambda) \). The definition of \( \mathcal{B}(\Lambda) \) implies that \( B \in \mathcal{B}(\Lambda) \) if and only if
\[ A' = ((\tilde{f}_\alpha, e_i)')_{\alpha \in C, i \in I^e} \]
is a regular matrix, where \( \{ e_i \}_{i \in I^e} \) is the standard basis of \( \mathbb{R}^d \). Here by (8.10), we have
\[ \langle \tilde{f}_\alpha, e_i \rangle' = \langle \pi_{I^e}(\alpha'), e_i \rangle' = \langle \alpha', \lambda_i \rangle \]
and so \( A' \) is rewritten as
\[ A' = ((\alpha', \lambda_i))_{\alpha \in C, i \in I^e} = A. \]
Thus we showed the desired claim. When these equivalent conditions hold, we write \( C = V_I \), which agrees with the notation in Section 2. In this case we have
\[ Q' / L(V') = \left( \bigoplus_{i=1}^r \mathbb{Z} \alpha_i' \right) / \left( \bigoplus_{\beta \in V_I} \mathbb{Z} \beta' \oplus \bigoplus_{i \in I} \mathbb{Z} \alpha_i' \right) \]
(8.16)
\[ \simeq \pi_{I^e} \left( \bigoplus_{i \in I^e} \mathbb{Z} \alpha_i' \right) / \pi_{I^e} \left( \bigoplus_{\beta \in V_I} \mathbb{Z} \beta' \right), \]
and hence \( \mathbb{Z}^d / \langle \tilde{B} \rangle \simeq Q' / L(V'). \)

Thirdly we show that \( (\tilde{f}_\gamma, \check{f}_\alpha)^B = \langle \gamma', \mu'_{\alpha} \rangle \) for \( \gamma \in \Delta^* \) and \( \alpha \in V_I \) (and hence \( B = \{ f_\alpha \mid \alpha \in V_I \} \in \mathcal{B}(\Lambda) \) and \( V = V_I \cup \Psi_I \)). It is sufficient to show that for any \( v \in V, \)
\[ \langle \pi_{I^e}(v), \check{f}_\alpha^B \rangle' = \langle v, \mu'_{\alpha} \rangle \]
holds. Write
\[ v = \sum_{\beta \in \Psi_I} b_{\beta} \beta' + \sum_{\beta \in V_I} c_{\beta} \beta'. \]
Then
\[ \pi_{I^*}(v) = \sum_{\beta \in \mathcal{V}_I} c_{\beta} \pi_I(\beta^\ast). \]

Since \( (\pi_I(\beta^\ast), \tilde{f}_{B}^\ast) = (\tilde{f}_{\beta}, \tilde{f}_{A}^\ast) = \delta_{\beta A} \) for \( \beta \in \mathcal{V}_I \), we have
\[ \langle \pi_I(v), \tilde{f}_{A}^\ast \rangle = \sum_{\beta \in \mathcal{V}_I} c_{\beta} \langle \pi_I(\beta^\ast), \tilde{f}_{A}^\ast \rangle = c_{\alpha} = \langle v, \mu_{\alpha} \rangle. \]

Therefore \( \langle \tilde{f}_{\gamma}, \tilde{f}_{A}^\ast \rangle = \langle \pi_I(\gamma^\ast), \tilde{f}_{A}^\ast \rangle = \langle \gamma^\ast, \mu_{\alpha} \rangle \).

From the above arguments, we see that (8.2) under the choice (8.11) now reads as
\[ F(t, y'; \Lambda) \]
\[ = \sum_{\mathcal{V} \in \mathcal{V}_I} \left( \prod_{\gamma \in \Delta^* \setminus \mathcal{V}_I} t_{\gamma} - 2\pi \sqrt{-1} \langle \gamma^\ast, \lambda \rangle - \sum_{\beta \in \mathcal{V}_I} (t_{\beta} - 2\pi \sqrt{-1} \langle \beta^\ast, \lambda \rangle) \langle \gamma^\ast, \mu_{\beta}^\ast \rangle \right) \]
\[ \times \frac{1}{|Q^\ast / L(\mathcal{V}^\ast)|} \sum_{q \in Q^\ast / L(\mathcal{V}^\ast)} \prod_{\beta \in \mathcal{V}_I} t_{\beta} \exp((t_{\beta} - 2\pi \sqrt{-1} \langle \beta^\ast, \lambda \rangle) \langle y' + q \rangle) \]
\[ \exp(t_{\beta} - 2\pi \sqrt{-1} \langle \beta^\ast, \lambda \rangle) - 1. \]

In fact, there are the following one-to-one correspondences:
\begin{align*}
B \in \mathcal{B}(\Lambda) & \leftrightarrow \mathcal{V} \in \mathcal{V}_I, \\
g \in \Lambda \setminus B & \leftrightarrow \gamma \in \Delta^* \setminus \mathcal{V}_I, \\
f \in B & \leftrightarrow \beta \in \mathcal{V}_I, \\
w \in \mathbb{Z}^d / (\mathcal{B}) & \leftrightarrow q \in Q^\ast / L(\mathcal{V}^\ast),
\end{align*}

hence \( \langle \tilde{g}, \tilde{f}_{A}^\ast \rangle \) corresponds to \( \langle \tilde{f}_{\gamma}, \tilde{f}_{A}^\ast \rangle \), the latter being equal to \( \langle \gamma^\ast, \mu_{\beta}^\ast \rangle \) as we have already seen.

From (8.11) we see that
\[ \langle y, \mu \rangle = \langle \pi_I(\gamma), (b_i)_{i \in I^*} \rangle \]
for any \( y \in V \). Therefore comparing (8.1) (with \( y' = \pi_I(y) \)) and (8.7) we find that
\[ S_{\lambda}(k, y; I; \Delta) = S(k, \pi_I(y); \Lambda), \]
and also \( \{ \pi_I(y) + q \}_{B, \beta} = \{ y + q \}_{\mathcal{V}, \beta} \). Moreover
\[ t_{\gamma} - 2\pi \sqrt{-1} \langle \gamma^\ast, \lambda \rangle - \sum_{\beta \in \mathcal{V}_I} (t_{\beta} - 2\pi \sqrt{-1} \langle \beta^\ast, \lambda \rangle) \langle \gamma^\ast, \mu_{\beta}^\ast \rangle \]
\[ = t_{\gamma} - \sum_{\beta \in \mathcal{V}_I} t_{\beta} \langle \gamma^\ast, \mu_{\beta}^\ast \rangle - 2\pi \sqrt{-1} \left( \langle \gamma^\ast, \lambda \rangle - \sum_{\beta \in \mathcal{V}_I} \langle \beta^\ast, \lambda \rangle \langle \gamma^\ast, \mu_{\beta}^\ast \rangle \right) \]
\[ = t_{\gamma} - \sum_{\beta \in \mathcal{V}_I} t_{\beta} \langle \gamma^\ast, \mu_{\beta}^\ast \rangle - 2\pi \sqrt{-1} \langle \gamma^\ast, p_{\mathcal{V}_I^\ast} (\lambda) \rangle \]
by (2.3), and
\[
\exp\left(-2\pi \sqrt{-1} \sum_{\beta \in \mathcal{V}_t} \langle \beta^\vee, \lambda \rangle \{y + q\} v_\beta \right) = \exp\left(-2\pi \sqrt{-1} \sum_{\beta \in \mathcal{V}_t} \langle \beta^\vee, \lambda \rangle \{y + q, \mu_\beta^\vee \} \right) = \exp(-2\pi \sqrt{-1}(y + q, \lambda)) \exp\left(2\pi \sqrt{-1}(y + q, p_{\mathcal{V}_t}(\lambda)) \right) = \exp(-2\pi \sqrt{-1}(y, \lambda)) \exp\left(2\pi \sqrt{-1}(y + q, p_{\mathcal{V}_t}(\lambda)) \right)
\]
by (2.3) again and the facts \(\langle \beta^\vee, \lambda \rangle, \langle q, \lambda \rangle \in \mathbb{Z}\). Collecting these facts, we obtain
\[
F(t, \pi_I(y); \lambda) = \exp(-2\pi \sqrt{-1}(y, \lambda)) F(t, y, \lambda; I; \Delta)
\]
and hence
\[
C(k, \pi_I(y); \lambda) = \exp(-2\pi \sqrt{-1}(y, \lambda)) P(k, y, \lambda; I; \Delta).
\]
Therefore by (8.24) and (8.25) we have
\[
S_\lambda(k, y; I; \Delta) = \left(\prod_{\alpha \in \Delta^+} \frac{-2\pi \sqrt{-1} \langle \alpha_k \rangle}{k!} \right) \times \exp(-2\pi \sqrt{-1}(y, \lambda)) P(k, y, \lambda; I; \Delta).
\]
Substituting this into (8.6), we arrive at the assertion of the theorem. \(\square\)

9. PROOF OF THEOREM 2.4

In this final section we prove Theorem 2.4. In what follows, let \([A]\) denote the \(\mathbb{R}\)-span of \(A\). Let \(n = |\Delta_{I^+}|\). Fix an order \(\Delta_{I^+} = \{\beta_1, \beta_2, \ldots, \beta_n\}\) and put \(A_j = \{\beta_1, \ldots, \beta_j\}\) for \(0 \leq j \leq n\). Note that
\[
(9.1) \quad \emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = \Delta_{I^+}.
\]
Fix \(V \in \mathcal{V}\) and \(q \in Q^\vee / L(V^\vee)\). Consider the following condition for \(j\):
\[
(9.2) \quad \beta_k \in V \cup [A_{k-1}] \text{ for all } k \leq j.
\]
Under this condition it follows that, in the sequence of nondecreasing linear spaces
\[
(9.3) \quad \{0\} = [A_0] \subset [A_1] \subset \cdots \subset [A_n] = [\Delta_{I^+}]
\]
if \([A_{k-1}] \not\subset [A_k]\), then \(\beta_k \notin [A_{k-1}]\) and hence \(\beta_k \in V\).

Let \(t_j = (t_{\gamma})_{\gamma \in \Delta_{A_j} \setminus A_j}\) (hence \(t_0 = t\)), \(t_j \cdot V = (t_{\beta})_{\beta \in V \setminus [A_j]}\), and
\[
S_{j, \gamma}(t_{\gamma}, t_j \cdot V) = t_{\gamma} - \sum_{\beta \in V \setminus [A_j]} t_{\beta} (\gamma^\vee, \mu_\beta^\vee) - 2\pi \sqrt{-1} \sum_{\beta \in V \setminus [A_j]} \langle \beta^\vee, \lambda \rangle \langle \gamma^\vee, \mu_\beta^\vee \rangle.
\]
For \( j \) satisfying (9.2), define
\( f_j(t_j; V, q) \)
\[
\left( \prod_{\gamma \in \Delta_+ \setminus (V \cup [A_j])} t_\gamma \right) \left( \prod_{\gamma \in (\Delta_+ \cap [A_j]) \setminus V} \frac{t_\gamma}{t_\gamma - 2\pi \sqrt{-1} \langle \gamma, \lambda \rangle} \right) \times \exp \left( 2\pi \sqrt{-1} \sum_{\beta \in V \cap [A_j]} \langle \beta^\vee, \lambda \rangle \langle y + q, \mu_\beta \rangle \right) \times \left( \prod_{\gamma \in V \cap [A_j]} \frac{t_\gamma \exp(t_\gamma \langle y + q, \nu \rangle)}{e^{t_\gamma} - 1} \right)
\]
\( = f_{j1} \times f_{j2} \times f_{j3} \times f_{j4}, \)
say. If \( j \) does not satisfy (9.2), we let \( f_j(t_j; V, q) = 0 \). Then
\( F(t, y; \Delta) = \sum_{V \in \mathcal{Y}} \frac{1}{|Q^/ / L(V^+)\rangle} \sum_{q \in Q^/ / L(V^+)} f_0(t; V, q). \)

**Lemma 9.1.** We have
\( f_j(t_j; V, q) = \text{Res}_{\tau = 2\pi \sqrt{-1} \langle \beta_j^\vee, \lambda \rangle} \frac{f_{j-1}(t_{j-1}; V, q)}{t_{\beta_j}}, \)
where the limit procedure of calculating the residue is along some ‘generic’ path in the sense of Remark 2.7.

**Proof.** We begin with two preparatory statements in relation to (9.2). First we show that under the condition (9.2),
\( \beta_j \in V \setminus [A_{j-1}] \) or \( \beta_j \in [A_{j-1}] \setminus V \)
holds. Under the condition (9.2) we have \( \beta_j \in V \cup [A_{j-1}] \). Assume \( \beta_j \in V \cap [A_{j-1}] \). Then there exists \( k \leq j - 1 \) such that \( \beta_j \in [A_k] \setminus [A_{k-1}] \).
Since \( [A_k] \setminus [A_{k-1}] \neq \emptyset \), we have \( \beta_k \in V \) by the argument just below (9.3).
Then \( \beta_j, \beta_k \in (V \cap [A_k]) \setminus (V \cap [A_{k-1}]) \) and \( |V \cap [A_k]| - |V \cap [A_{k-1}]| \geq 2 \). Since the elements of \( V \) are linearly independent, this contradicts to \( \dim[A_k] - \dim[A_{k-1}] \leq 1 \). Therefore (9.7) holds.

Next we show that under the condition (9.2),
\( [A_j] = [V \cap [A_j]] \)
holds. Since \( [A_j] \supseteq [V \cap [A_j]] \) is trivial, we show the opposite inclusion. If \( [A_{k-1}] \subseteq [A_k] \), then \( \beta_k \in V \cap [A_k] \) (again by the argument just below (9.3)).
Hence \( \dim[A_j] = |V \cap [A_j]|, \) which implies (9.8).

Now we start to prove (9.6).

**Case 1.** The case when \( j \) satisfies (9.2). Then by (9.7) either \( \beta_j \in V \setminus [A_{j-1}] \) or \( \beta_j \in [A_{j-1}] \setminus V \) holds.

**Case 1-1.** Assume \( \beta_j \in V \setminus [A_{j-1}] \). Then \( [A_{j-1}] \subseteq [A_j] \) and we see that, in the expression (9.4) of \( f_{j-1}(t_{j-1}; V, q) \), \( \beta_j \) appears only in \( f_{j-1,1} \) and \( f_{j-1,4} \). When \( t_{\beta_j} = 2\pi \sqrt{-1} \langle \beta_j^\vee, \lambda \rangle \), the singularity appears only on the
factor corresponding to $\gamma = \beta_j$ in $f_{j-1,4}$, whose contribution to the residue (9.6) is

$$
(9.9) \quad \text{Res}_{t_{\beta_j}=2\pi\sqrt{-1}(\beta_j^\vee,\lambda)} \frac{1}{t_{\beta_j}} e^{t_{\beta_j}} - 1 = \exp(2\pi\sqrt{-1}(\beta_j^\vee,\lambda)\{y + q\}v_{\beta_j}) = \exp(2\pi\sqrt{-1}(\beta_j^\vee,\lambda)\{y + q, \mu_{\beta_j}^V\})
$$

because $\langle \beta_j^\vee, \lambda \rangle \in \mathbb{Z}$, which becomes a member of $f_{j\beta}$. Therefore the contribution of $f_{j-1,3}f_{j-1,4}$ to the residue (9.6) is

$$
(9.10) \quad = \exp\left(2\pi\sqrt{-1} \sum_{\beta \in \mathbf{V}\cap [A_{j-1}]} \langle \beta^\vee, \lambda \rangle \{y + q, \mu_{\beta}^V\}\right)
$$

$$
\times \exp(2\pi\sqrt{-1}\langle \beta_j^\vee, \lambda \rangle \{y + q, \mu_{\beta_j}^V\}) \prod_{\gamma \in \mathbf{V}\setminus([A_{j-1}]\cup\{\beta_j\})} t_\gamma \exp(t_\gamma \{y + q\}v_\gamma) \prod_{\gamma \in \mathbf{V}\setminus[A_j]} \frac{t_\gamma \exp(t_\gamma \{y + q\}v_\gamma)}{e^{t_\gamma} - 1}
$$

$$
= f_{j\beta}f_{j4}.
$$

Next consider the denominator of $f_{j-1,1}$ for $\gamma \in (\Delta_+ \cap [A_j]) \setminus (\mathbf{V} \cup [A_{j-1}])$ at $t_{\beta_j} = 2\pi\sqrt{-1}(\beta_j^\vee, \lambda)$. It is

$$
(9.11) \quad S_{j-1,\gamma}(t_\gamma, t_{j-1}, v) \Bigg|_{t_{\beta_j}=2\pi\sqrt{-1}(\beta_j^\vee,\lambda)} = t_\gamma - \sum_{\beta \in \mathbf{V}\cap [A_{j-1}]} t_\beta \langle \gamma^\vee, \mu_{\beta}^V \rangle \Bigg|_{t_{\beta_j}=2\pi\sqrt{-1}(\beta_j^\vee,\lambda)} - 2\pi\sqrt{-1} \sum_{\beta \in \mathbf{V}\cap [A_{j-1}]} \langle \beta^\vee, \lambda \rangle \langle \gamma^\vee, \mu_{\beta}^V \rangle
$$

$$
= t_\gamma - \sum_{\beta \in \mathbf{V}\cap [A_j]} t_\beta \langle \gamma^\vee, \mu_{\beta}^V \rangle - 2\pi\sqrt{-1} \sum_{\beta \in \mathbf{V}\cap [A_j]} \langle \beta^\vee, \lambda \rangle \langle \gamma^\vee, \mu_{\beta}^V \rangle,
$$

which is further

$$
(9.12) \quad = t_\gamma - 2\pi\sqrt{-1} \langle \gamma^\vee, \lambda \rangle
$$

because by (9.8), $\gamma^\vee$ is a linear combination of $\mathbf{V}^\vee \cap [A_j]$, so $\langle \gamma^\vee, \mu_{\beta}^V \rangle = 0$ for $\beta \in \mathbf{V} \setminus [A_j]$ and by writing $\gamma^\vee = \sum_{\beta \in \mathbf{V}\cap [A_j]} a_{\beta}^V \beta^\vee$, we have

$$
\sum_{\beta \in \mathbf{V}\cap [A_j]} \langle \beta^\vee, \lambda \rangle \langle \gamma^\vee, \mu_{\beta}^V \rangle = \langle \gamma^\vee, \lambda \rangle.
$$
Thus each corresponding member of $f_{j-1, 1}$ becomes a member of $f_{j2}$. Therefore
\begin{equation}
(9.13)
\begin{align*}
 f_{j-1, 1} f_{j-1, 2} &= 2 \sqrt{-1} \beta_j' \lambda \\
 &= \prod_{\gamma \in (\Delta_+ \setminus [A_j]) \setminus (V \cup [A_{j-1}])} \frac{t_{\gamma}}{S_{j-1, \gamma}(t_{\gamma}, t_{j-1}, V)} \times \prod_{\gamma \in A_{j-1}} \frac{t_{\gamma}}{t_{\gamma} - 2 \sqrt{-1} \langle \gamma', \lambda \rangle},
\end{align*}
\end{equation}
where
\begin{equation}
A_{j-1} = ((\Delta_+ \cap [A_j]) \setminus (V \cup [A_{j-1}]) \cup ((\Delta_+ \cap [A_{j-1}]) \setminus (V \cup A_{j-1})).
\end{equation}

We note that
\begin{equation}
(9.14)
A_{j-1} = (\Delta_+ \cap [A_j]) \setminus (V \cup A_{j-1})
\end{equation}
holds. In fact, it is obvious that $A_{j-1}$ is included in the right-hand side, while an element of the right-hand side does not belong to $\Delta_+ \cap [A_{j-1}]$, it does not belong to $V \cup [A_{j-1}]$. Therefore (9.14) holds, and noting $V \cup A_{j-1} = V \cup A_j$, we further find that $A_{j-1} = (\Delta_+ \cap [A_j]) \setminus (V \cup A_j)$. As for the first product on the right-hand side of (9.13), we see that $(\Delta_+ \setminus [A_j]) \setminus (V \cup A_{j-1}) = \Delta_+ \setminus (V \cup [A_j])$ and $S_{j-1, \gamma}(t_{\gamma}, t_{j-1}, V) = S_{j, \gamma}(t_{\gamma}, t_j, V)$ (because $\langle \gamma', \mu_{\beta_j} \rangle = 0$ for $\gamma \in \Delta_+ \setminus [A_j]$). Therefore (9.13) can be read as
\begin{equation}
(9.15)
\begin{align*}
 f_{j-1, 1} f_{j-1, 2} &= 2 \sqrt{-1} \beta_j' \lambda \\
 &= f_{j1} f_{j2}.
\end{align*}
\end{equation}

Combining (9.10) and (9.15) we arrive at (9.6) in the present case.

Case 1-2. Assume $\beta_j \in [A_{j-1}] \setminus V$. Then $[A_{j-1}] = [A_j]$ and we see that $\beta_j$ appears only in $f_{j-1, 2}$ in the expression (9.4) of $f_{j-1}(t_{j-1}; V, q)$. Furthermore
\begin{equation}
(9.16)
\begin{align*}
 \text{Res}_{t_{\beta_j} = 2 \sqrt{-1} \beta_j' \lambda} \frac{1}{t_{\beta_j} - 2 \sqrt{-1} \beta_j' \lambda} = 1
\end{align*}
\end{equation}
and hence (9.6) holds.

Case 2. The case when $j$ does not satisfy (9.2). There exists $k \leq j$ such that $\beta_k \notin V \cup [A_{k-1}]$. Take the minimum $k$ among them. If $k \leq j - 1$, then $f_j(t_j; V, q) = f_{j-1}(t_{j-1}; V, q) = 0$ and (9.6) holds. Assume $k = j$. Then $f_{j-1}(t_{j-1}; V, q)$ is given by (9.14) and $f_j(t_j; V, q) = 0$. In this case $\beta_j$ appears only in $f_{j-1, 1}$ in the expression (9.4) of $f_{j-1}(t_{j-1}; V, q)$. If $\langle \beta_j', \mu_{\beta_j} \rangle = 0$ for all $\beta \in V \setminus [A_{j-1}]$, then $\beta_j \in [V \cap [A_{j-1}] \subset [A_{j-1}]$, which contradicts to $\beta_j \notin V \cup [A_{j-1}]$. Hence at least one of the coefficients of $t_{\beta}$ for $\beta \in V \setminus [A_{j-1}]$ does not vanish in
\begin{equation}
(9.17)
\begin{align*}
 S_{j-1, \gamma}(t_{\gamma}, t_{j-1}, V) &= 2 \sqrt{-1} \beta_j' \lambda \\
 &\bigg|_{t_{\beta_j} = 2 \sqrt{-1} \beta_j' \lambda}
\end{align*}
\end{equation}
Proof of Theorem 2.4.

(9.18)

which implies that the residue is 0 (by the calculation along some 'generic' path, in the sense of Remark 2.5) and so (9.6) holds.

Thus we showed that (9.6) holds in any case. □

By the repeated use of Lemma 9.1 we obtain

(9.18)

Proof of Theorem 2.4. Let \( \Phi_I = \{ \beta_k \mid \beta_k \notin [A_{k-1}] \} \subset \Delta_{I+} \). Then \( \Phi_I \) is a linearly independent set with \( |\Phi_I| = |I| \).

Assume \( \Phi_I \subset V \). Let \( V_J = V \setminus \Phi_I \). Since \( [A_n] = [\Delta_{I+}] \), we see that \( V \setminus [A_n] = V_I \),

\[
\Delta_+ \setminus (V \cup [A_n]) = \Delta_+ \setminus (V_I \cup [\Delta_{I+}]) = \Delta^* \setminus V_I,
\]

\[
(\Delta_+ \cap [A_n]) \setminus (V \cup A_n) = \Delta_{I+} \setminus (V \cup \Delta_{I+}) = \emptyset.
\]

Further we have

\[
V \cap [A_n] = V \setminus V_I = \Phi_I.
\]

Therefore (9.3) for \( j = n \) implies

(9.19)

If \( \Phi_I \not\subset V \), then for \( \beta_k \in \Phi_I \setminus V \), we have \( \beta_k \notin V \cup [A_{k-1}] \) and hence \( f_n(t_n; V, q) = 0 \).

Applying Lemma 2.1 to the right-hand side of (2.6), and noting that there is a one-to-one correspondence between

(9.20)

we obtain

(9.21)

by using \( t_n = t_I \). Combining this with (9.18), and using (9.5), we finally obtain the formula (2.13) stated in Theorem 2.4.
So far we have discussed under the fixed choice of the order $\beta_1, \ldots, \beta_n$ of the elements of $\Delta_{I^+}$. The right-hand side of (9.18) apparently depends on this choice. However the left-hand side of (2.13), defined by (2.6), does not depend on the choice of the order. So is the right-hand side of (2.13). This implies the claim mentioned in Remark 2.5, and hence the proof of Theorem 2.4 is thus complete.

□

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