String instantons, fluxes and moduli stabilization

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Abstract

We analyze a class of dual pairs of heterotic and type I models based on freely-acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds in four dimensions. Using the adiabatic argument, it is possible to calculate non-perturbative contributions to the gauge coupling threshold corrections on the type I side by exploiting perturbative calculations on the heterotic side, without the drawbacks due to twisted moduli. The instanton effects can then be combined with closed-string fluxes to stabilize most of the moduli fields of the internal manifold, and also the dilaton, in a racetrack realization of the type I model.
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1 Introduction and Conclusions

In recent years new ways to compute non-perturbative effects in string theory were developed, based on Euclidean $p$-branes ($Ep$-branes) wrapping various cycles of the internal manifold of string compactifications [1, 2, 3, 4, 5, 6, 7]. Some of the instanton effects have an interpretation in terms of gauge theory instantons, whereas others are stringy instanton effects whose gauge theory counterpart is still under investigation (For recent reviews on instanton effects in field and string theory, see e.g. [8]). Whereas the former effects are responsible for the generation of non-perturbative superpotentials via gauge theory strong IR dynamics [9] and of moduli potentials satisfying various gauge invariance constraints [10], the latter could be responsible for generating Majorana neutrino masses or the $\mu$-term in MSSM [4, 5], as well as for inducing other interesting effects at low energy [7].

The purpose of the present paper is to present a class of examples based on freely-acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models, that adds two new ingredients to the discussion, trying to go deeper into the non-perturbative effects analysis. The first new ingredient is the heterotic-type I duality [11], which exchanges perturbative and non-perturbative regimes. As is well known [12], it is possible to construct freely-acting dual pairs with $\mathcal{N} = 1$ supersymmetry in four dimensions which preserve the S-duality structure. As we show explicitly here, the dual pairs can have a rich non-perturbative dynamics exhibiting both types of effects mentioned above. The heterotic-type I duality allows, for example, to obtain the exact $E1$ instantonic summations on the type I side for the non-perturbative corrections to the gauge couplings using the computation of perturbative threshold corrections on the heterotic side. Second, non-perturbative effects also play a potentially important role in addressing the moduli field stabilization issue. Closed string fluxes were invoked in recent years in the framework of type IIB and type IIA string compactifications, following the initial proposal of [14].

\footnote{See [13] for earlier work on instanton effects and heterotic-type I duality.}
to try to stabilize all moduli fields, including the dilaton. The combination of closed string fluxes and freely-acting orbifold actions has the obvious advantage of avoiding to deal with twisted-sector moduli fields, absent in our construction. We show that, besides the Ramond-Ramond (RR) three-form fluxes, also metric fluxes can be turned on in our freely-acting type I models, requiring new quantization conditions and the twisting of the cohomology of the internal manifold. The low-energy effective description is equivalent to the original one, with the addition of a non-trivial superpotential. Moreover, our string constructions allow naturally racetrack models with dilaton stabilization \[15\]. We show how they can be combined with closed string fluxes and stringy instanton effects in order to stabilize most of the moduli fields of the internal manifold.

The plan of the paper is as follows. In Section 2 we discuss the geometric framework of the freely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. In Section 3 we display the explicit type I descendants obtained by quotienting the orbifold with the geometric world-sheet parity operator. Besides some variations of the simplest class with orthogonal gauge groups, we also construct the corresponding heterotic duals in Section 4. In Section 5, we report the calculation of the threshold corrections to the gauge couplings both for the heterotic and for the type I models. The details of the calculations are reported in the Appendices. In particular, we verify that the moduli dependence of the non-perturbative corrections on the type I side is in agreement with the conjectured form \[16\]. In Section 6 we analyze the instanton contributions in the type I framework, that are combined with closed string fluxes in Section 7 in order to attain the stabilization of most of the moduli of the compactification manifold. In particular, in Section 7 we describe an example in which the dilaton can be also stabilized, due to a natural racetrack realization of the type I model in combination with closed metric and RR three-form fluxes.

## 2 The freely-acting orbifold and its moduli

From the point of view of the target space, we take a $T^6$ torus $(y^i = y^i + 1)$ with vielbein vectors $e^i = e^i_\mu dy^\mu$ and metric given by

$$ds^2 = \sum_i e^i_\mu e^{ij}_\mu .$$

(2.1)
This has to be $SL(6,\mathbb{Z})$ invariant. Therefore, performing a general rotation of the lattice vectors one may write a basis as follows\footnote{We use the notation $y^i$, $i = 1 \ldots 6$, to denote the internal compact dimensions and $x^i$, $i = 0 \ldots 3$, for the non-compact space-time dimensions.}

\begin{align}
e^6 &= R^6 dy^6, \tag{2.2} \\
e^5 &= R^5(dy^5 + a^5_6 dy^6), \tag{2.3} \\
e^4 &= R^4(dy^4 + a^4_5 dy^5 + a^4_6 dy^6), \tag{2.4} \\
e^3 &= R^3(dy^3 + a^3_4 dy^4 + a^3_5 dy^5 + a^3_6 dy^6), \tag{2.5} \\
e^2 &= R^2(dy^2 + a^2_3 dy^3 + a^2_4 dy^4 + a^2_5 dy^5 + a^2_6 dy^6), \tag{2.6} \\
e^1 &= R^1(dy^1 + a^1_2 dy^2 + a^1_3 dy^3 + a^1_4 dy^4 + a^1_5 dy^5 + a^1_6 dy^6). \tag{2.7}
\end{align}

Modding by the orbifold action will break the $SL(6,\mathbb{Z})$ symmetry to a smaller subgroup. We define the generators $\{g, f, h\}$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ freely-acting orbifold as,

\begin{align}(y^1, y^2, y^3, y^4, y^5, y^6) \overset{g}{\rightarrow} (y^1 + 1/2, y^2, -y^3, -y^4, -y^5 + 1/2, -y^6), \tag{2.8} \\
(y^1, y^2, y^3, y^4, y^5, y^6) \overset{f}{\rightarrow} (-y^1 + 1/2, -y^2, y^3 + 1/2, y^4, -y^5, -y^6), \tag{2.9} \\
(y^1, y^2, y^3, y^4, y^5, y^6) \overset{h}{\rightarrow} (-y^1, -y^2, -y^3 + 1/2, -y^4, y^5 + 1/2, y^6). \tag{2.10}
\end{align}

Notice that these orbifold operations have no fixed points due to the shifts, hence they act freely (see e.g. \cite{17}). Moreover, for objects localized in the internal space, as will be the case for the $E1_i$ instantons to be discussed in Section 6, orbifold operations will generate inevitably instanton images. This has non-trivial consequences on the instanton spectra, as we shall see later on.

In order for the lattice vectors (2.2) - (2.7) to transform covariantly with respect to the orbifold action, it is required that

\begin{align}a^4_5 = a^4_6 = a^3_5 = a^3_6 = a^2_3 = a^2_5 = a^2_6 = a^1_3 = a^1_4 = a^1_5 = a^1_6 = 0. \tag{2.11}\end{align}

A basis of holomorphic vectors can thus be introduced in the form

\begin{align}z^1 &= e^1 + ie^2 = R_1(dy^1 + iU_1 dy^2), \tag{2.12} \\
z^2 &= e^3 + ie^4 = R_3(dy^3 + iU_2 dy^4), \tag{2.13} \\
z^3 &= e^5 + ie^6 = R_5(dy^5 + iU_3 dy^6). \tag{2.14}
\end{align}
where we have defined

\[ U_1 = \frac{R^2}{R^1} - i a_1^2, \quad U_2 = \frac{R^4}{R^3} - i a_4^3, \quad U_3 = \frac{R^6}{R^5} - i a_6^5. \]  

(2.15)

Hence, the moduli space of the untwisted sector matches precisely the one of an ordinary \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), given by the three complex structure moduli, \( U_i \), together with the three Kähler moduli, \( T_i \), which result from the expansion of the complexified Kähler 2-form in a cohomology basis of even 2-forms,

\[ J_c = e^{-\phi} J + i C_2 = T_1 dy^1 \wedge dy^2 + T_2 dy^3 \wedge dy^4 + T_3 dy^5 \wedge dy^6. \]  

(2.16)

Making use of \( (2.12) - (2.14) \), the real parts of the Kähler moduli can be seen to be

\[ \text{Re} \ T_1 = e^{-\phi} R_1 R_2, \quad \text{Re} \ T_2 = e^{-\phi} R_3 R_4, \quad \text{Re} \ T_3 = e^{-\phi} R_5 R_6. \]  

(2.17)

The effective theory contains also, as usual, the universal axion-dilaton modulus

\[ S = e^{-\phi} \prod_{i=1}^{6} R_i + i \ c, \]  

(2.18)

where \( c \) is the universal axion. On the other hand, since there are no fixed points in the orbifold action, we expect the twisted sector to be trivial. We shall see in next section, from the exchange of massless modes in the vacuum amplitudes, that this is indeed the case.

The internal space of the orbifold is therefore completely smooth and can be interpreted as a Calabi-Yau space with Hodge numbers \((h_{11}, h_{21}) = (3, 3)\). The corresponding type IIB string theory on this orbifold space has the standard left-right worldsheet involution \( \Omega_P \) as a symmetry, which we use, following [18, 19], in order to construct type I freely-acting orbifolds.

### 3 Type I models : vacuum energy and spectra

#### 3.1 Type I with orthogonal gauge groups

We briefly summarize here some of the results of [18]. Following the original notation, the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold generators of eqs. \( (2.8) - (2.10) \) can be written as

\[ g = (P_1, -1, -P_3), \quad f = (-P_1, P_2, -1), \quad h = (-1, -P_2, P_3). \]  

(3.1)
where $P_i$ represents the momentum shift along the real direction $y^{2i-1}$ of the i-th torus. We consider the type I models obtained by gauging the type IIB string with $\Omega_P$, the standard worldsheet orientifold involution. The spectrum can be read from the one-loop amplitudes [20]. In particular, the torus partition function is

$$T = \int \frac{d^2\tau}{\tau_2} \frac{1}{|\eta|^4} \left[ \tau_{oo} + \tau_{og} + \tau_{oh} + \tau_{of} |^2 \Lambda_1 \Lambda_2 \Lambda_3 + | \tau_{oo} + \tau_{og} - \tau_{oh} - \tau_{of} |^2 (-1)^{m_3} \Lambda_1 \right] \frac{4\eta^2}{\vartheta^2} \right|_{1}^{2} + \left| \tau_{oo} - \tau_{og} + \tau_{oh} - \tau_{of} |^2 (-1)^{m_3} \Lambda_3 \right] \frac{4\eta^2}{\vartheta^2} \right|_{1}^{2} + \left| \tau_{go} + \tau_{gg} + \tau_{gh} + \tau_{gf} |^2 \Lambda_1^{n_1+\frac{1}{2}} \right] \frac{4\eta^2}{\vartheta^2} \right|_{1}^{2} + \left| \tau_{ho} + \tau_{hg} + \tau_{hh} + \tau_{hf} |^2 \Lambda_3^{n_3+\frac{1}{2}} \right] \frac{4\eta^2}{\vartheta^2} \right|_{1}^{2} + \left| \tau_{fo} + \tau_{fg} + \tau_{fh} + \tau_{ff} |^2 \Lambda_2^{n_2+\frac{1}{2}} \right] \frac{4\eta^2}{\vartheta^2} \right|_{1}^{2} + \left| \tau_{fo} - \tau_{fg} - \tau_{fh} - \tau_{ff} |^2 (-1)^{m_2} \Lambda_2^{n_2+\frac{1}{2}} \right] \frac{4\eta^2}{\vartheta^2} \right|_{1}^{2}, \quad \text{(3.2)}$$

while the Klein bottle, annulus and Möbius strip amplitudes read in the direct (loop) channel respectively as

$$K = \int_0^\infty \frac{dt}{t^3 \eta^2} \frac{1}{8} (\tau_{oo} + \tau_{og} + \tau_{oh} + \tau_{of}) \left\{ P_1 P_2 P_3 + (-1)^{m_3} P_1 W_2 W_3 + W_1 (-1)^{m_2} P_2 W_3 + W_1 W_2 (-1)^{m_3} P_3 \right\}, \quad \text{(3.3)}$$

$$A = \int_0^\infty \frac{dt}{t^3 \eta^2} \frac{1}{8} \left\{ I_N^2 (\tau_{oo} + \tau_{og} + \tau_{oh} + \tau_{of}) P_1 P_2 P_3 + g_N^2 (\tau_{oo} + \tau_{og} - \tau_{oh} - \tau_{of})(-1)^{m_1} P_1 \frac{4\eta^2}{\vartheta^2} + \text{(3.4)} \right\}$$

$$M = - \int_0^\infty \frac{dt}{t^3 \eta^2} \frac{I_N}{8} \left\{ \frac{4\eta^2}{\vartheta^2} \right\} \left\{ (\tilde{\tau}_{oo} + \tilde{\tau}_{og} + \tilde{\tau}_{oh} + \tilde{\tau}_{of}) P_1 P_2 P_3 + (\tilde{\tau}_{oo} + \tilde{\tau}_{og} - \tilde{\tau}_{oh} - \tilde{\tau}_{of})(-1)^{m_1} P_1 \frac{4\eta^2}{\vartheta^2} + \right\} \left\{ (\tilde{\tau}_{oo} - \tilde{\tau}_{og} + \tilde{\tau}_{oh} - \tilde{\tau}_{of})(-1)^{m_3} P_3 \frac{4\eta^2}{\vartheta^2} + (\tilde{\tau}_{oo} - \tilde{\tau}_{og} - \tilde{\tau}_{oh} + \tilde{\tau}_{of})(-1)^{m_2} P_2 \frac{4\eta^2}{\vartheta^2} \right\}. \quad \text{(3.5)}$$

\footnote{There is an overall normalization that is explicitly written in Appendix A. For other conventions concerning orientifolds, see e.g. the reviews [21].}
Some comments on the notation are to be made. In the torus amplitude, $F$ is the fundamental domain and the $\Lambda_i$ are the lattice sums for the three compact tori, whereas the shorthand notation $(-1)^m \Lambda_i^m \chi^m$ indicates a sum with the insertion of $(-1)^m$ along the momentum in $y^{2i-1}$, with the corresponding winding number shifted by $1/2$. $P_i$ and $W_i$ in (3.3) - (3.5) are respectively the momentum and winding sums for the three two-dimensional tori. More concretely, using for the geometric moduli the conventions of the previous section, one has:

$$P_i \equiv \sum_{m,m'} \exp \left[ -\frac{\pi t}{(\text{Re } T_i)(\text{Re } U_i)} |m' - iU_i m|^2 \right], \quad (3.6)$$

$$(-1)^m P_i \equiv \sum_{m,m'} (-1)^m \exp \left[ -\frac{\pi t}{(\text{Re } T_i)(\text{Re } U_i)} |m' - iU_i m|^2 \right] . \quad (3.7)$$

Moreover, in (3.5) hatted modular functions define a correct basis under the P transformation extracting a suitable overall phase [20]. Indeed, the moduli of the double-covering tori are $\tau = (it/2 + 1/2)$ for the Möbius-strip amplitude, $\tau = 2it$ for the Klein-bottle amplitude and $\tau = it/2$ for the annulus amplitude. In Appendix B we give the definition of the characters used in eqs. (3.3) - (3.5) in terms of $[SO(2)]^4$ characters.

It is worth to analyze the effects of the freely-acting operation on the geometry of the models. In general, $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifolds contain $O9$-planes and three sets of $O5_r$-planes defined as the fixed tori of the operations $\Omega_P \circ g$, $\Omega_P \circ f$, $\Omega_P \circ h$, each wrapping one of the three internal tori $T_i$. In our freely-acting orbifold case, the overall $O5_r$-plane charges are zero and the $O5_r$-planes couple only to massive (odd-windings) states. A geometric picture of this fact can be obtained T-dualizing the two directions the $O5_r$ planes wrap, so that they become $O3_r$-planes. In this way, the freely acting operation replaces the $O3_{i,-}$ planes by ($O3_{i,-}$-$O3_{i,-}$) pairs, separated by half the lattice spacing in the coordinate affected by the free action. Since there are no global background charges from $O5_r$-planes, the model contains only background $D9$ branes. Finally, the Chan-Paton $D9$ charges are defined as,

$$I_N = n_o + n_g + n_h + n_f , \quad g_N = n_o + n_g - n_h - n_f ,$$

$$h_N = n_o - n_g + n_h - n_f , \quad f_N = n_o - n_g - n_h + n_f , \quad (3.8)$$

In what follows we set the string tension $\alpha' = 1/2$. 

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with \( I_N = 32 \) fixed by the tadpole cancellation condition. The massless spectrum has \( \mathcal{N} = 1 \) supersymmetry. The gauge group is \( SO(n_o) \otimes SO(n_g) \otimes SO(n_h) \otimes SO(n_f) \), with chiral multiplets in the bifundamental representations

\[
(n_o, n_g, 1, 1) + (n_o, 1, n_f, 1) + (n_o, 1, 1, n_h) + \\
+ (1, n_g, n_f, 1) + (1, n_g, 1, n_h) + (1, 1, n_f, n_h) .
\] (3.9)

The existence of four different Chan-Paton charges can be traced to the various consistent actions of the orbifold group on the Chan-Paton space or, alternatively, to the number of independent sectors of the chiral Conformal Field Theory. It can be useful for the reader to make a connection with the alternative notation of [23]. The original Chan-Paton charges can be grouped into a \( 32 \times 32 \) matrix \( \lambda \). In this Chan-Paton matrix space, the three orbifold operations \( g, f \) and \( h \) act via matrices \( \gamma_g, \gamma_f, \gamma_h \) which, correspondingly to (3.8), are given by

\[
\gamma_g = (I_{n_o}, I_{n_g}, -I_n, -I_n), \\
\gamma_f = (I_{n_o}, -I_{n_g}, I_n, -I_n), \\
\gamma_h = (I_{n_o}, -I_{n_g}, -I_n, I_n),
\] (3.10)

where \( I_{n_o} \) denote the identity matrix in the \( n_o \times n_o \) block diagonal Chan-Paton matrix, and the same for the other multiplicities \( n_i \). For \( n_g = n_h = n_f = 0 \) one recovers a pure \( SO(32) \) SYM with no extra multiplets, a theory where gaugino condensation is expected to arise. Finally, let us notice that even if perturbatively \( n_o, n_g, n_f, n_h \) can be arbitrary positive integers subject only to the tadpole condition \( n_o + n_g + n_f + n_h = 32 \), non-perturbative consistency asks all of them to be even integers.

### 3.2 Type I racetrack model

In a variation of the previous \( SO(32) \) model, we may add a discrete deformation along one of the unshifted directions, similar to a Wilson line \( A_2 = (e^{2\pi i a}) \) along \( y^2 \), with \( a = (0_p, 1/2_{32-p}) \) and breaking \( SO(32) \rightarrow SO(p) \otimes SO(32-p) \). The annulus and Möbius
amplitudes, (3.4) and (3.5), get correspondingly modified to the following expressions:

\[
A = \int_0^\infty \frac{dt}{t^3 \eta^4} \sum_{m, m'} \exp \left[ -\frac{\pi t}{(\text{Re } T_1)(\text{Re } U_1)} |m' - iU_1 m + 1/2|^2 \right].
\]

As mentioned, \( I_N = p + q = 32 \),

\[
P_{m_1'} P_{m_1} = P_1,
\]

and

\[
P_{m_1'} \frac{1}{2} P_{m_1} = \sum_{m, m'} \exp \left[ -\frac{\pi t}{(\text{Re } T_1)(\text{Re } U_1)} |m' - iU_1 m + 1/2|^2 \right].
\]

Hence, the resulting \( SO(p) \otimes SO(32 - p) \) gauge group is accompanied by a pure \( \mathcal{N} = 1 \) SYM theory on both factors, leading to a racetrack scenario with two gaugino condensates. Indeed, in the four-dimensional effective supergravity Lagrangian, the tree-level gauge kinetic functions on the two stacks of \( D9 \) branes are equal,

\[
f_{SO(p)} = f_{SO(q)} = S,
\]

where \( S \) is the universal dilaton-axion chiral multiplet. Gaugino condensation on both stacks then generates the non-perturbative superpotential

\[
W_{np} = A_p^{(k)} e^{-a_p S} + A_q^{(l)} e^{-a_q S},
\]

where \( A_p^{(k)} = (p - 2) \exp(2\pi ik/(p - 2)) \) and \( A_q^{(l)} = (q - 2) \exp(2\pi il/(q - 2)) \), with \( k = 1 \ldots p - 2 \) and \( l = 1 \ldots q - 2 \), provide the requested different phases of the SYM vacua [24]. Moreover, \( a_p = 2/(p - 2) \) \( (a_q = 2/(q - 2)) \) is related to the one-loop beta function of the \( SO(p) \) \( (SO(q)) \) SYM gauge factor. In addition to the massless states, the model contains massive states, in particular a massive vector multiplet in the \((p, q)\) bifundamental
representation, with a lowest mass of the order of the compactification scale $M_c \sim 1/R$. Since the four-dimensional effective theory is valid anyway below $M_c$, these states are heavy and their effects on the low-energy physics can be encoded in threshold effects which we shall compute later on.

An interesting question is the geometrical interpretation of the present model. The natural interpretation is in terms of a Wilson line breaking of the $SO(p) \otimes SO(32-p)$ model. The absence of scalars describing positions of the branes corresponding to each $SO$ factor indicates that the corresponding branes are fractional and, as such, cannot move outside the fixed points. However, by giving vev’s to the scalars in bifundamentals, one converts fractional branes into regular branes. The resulting gauge group is $SO(2P) \otimes SO(16 - P)$, where the first factor comes from the branes sitting at the fixed point, while the second factor describe brane pairs in the bulk having scalars in the symmetric representation corresponding to their positions. Moving the bulk branes to another fixed-point, one gets, as usual, an enhancement of the gauge group to $SO(2P) \otimes SO(32 - 2P)$.

3.3 Type I with unitary groups

It is interesting to analyze the non-perturbative dynamics of the gauge theory on the D9 branes in the case of an orbifold action on the Chan-Paton space that produces unitary gauge groups. This can be done in a very simple way by choosing a different Chan-Paton assignment compared to (3.8). Consider the same cylinder amplitude (3.4) equipped with the following parametrization of the Chan-Paton charges:

$$I_N = n + \bar{n} + m + \bar{m} \quad , \quad g_N = n + \bar{n} - m - \bar{m} \quad ,$$

$$f_N = i(n - \bar{n} + m - \bar{m}) \quad , \quad h_N = i(n - \bar{n} - m + \bar{m}) . \quad (3.17)$$

The Möbius amplitude has to be changed for consistency into

$$M = - \int_0^\infty \frac{dt}{t^3 \eta^4} \frac{I_N}{8} \left\{ (\hat{r}_{oo} + \hat{r}_{og} + \hat{r}_{oh} + \hat{r}_{of}) P_1 P_2 P_3 + (\hat{r}_{oo} + \hat{r}_{og} - \hat{r}_{oh} - \hat{r}_{of})(-1)^{m_1} P_1 \frac{4\eta^2}{\partial^2} 
$$

$$- (\hat{r}_{oo} - \hat{r}_{og} + \hat{r}_{oh} - \hat{r}_{of})(-1)^{m_3} P_1 \frac{4\eta^2}{\partial^2} - (\hat{r}_{oo} - \hat{r}_{og} - \hat{r}_{oh} + \hat{r}_{of})(-1)^{m_2} P_2 \frac{4\eta^2}{\partial^2} \right\} , \quad (3.18)$$

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5E.D. is grateful to C. Angelantonj and M. Bianchi for illuminating discussions on this and the other string models presented in the present paper.
where the changes of sign in the $D_9$-$O_5$ propagations, needed to enforce the unitary projection, are interpreted as discrete Wilson lines on the $D_9$ branes in the last two torii [20]. The massless open string amplitudes,

\[
A_0 + M_0 = (n\bar{n} + m\bar{m})\tau_{oo} + \left[ \frac{n(n-1)}{2} + \frac{\bar{n}(\bar{n}-1)}{2} + \frac{m(m-1)}{2} + \frac{\bar{m}(\bar{m}-1)}{2} \right] \tau_{og} \\
+ (n\bar{m} + \bar{n}m)\tau_{of} + (nm + \bar{n}\bar{m})\tau_{oh},
\]

(3.19)

exhibit the spectrum of an $\mathcal{N} = 1$ supersymmetric $U(n) \otimes U(m)$ theory, with $n + m = 16$ due to the $(D_9/O_9)$ RR tadpole cancellation condition. Matter fields fall into massless chiral multiplets in the representations

\[
\left( \frac{n(n-1)}{2} + \frac{\bar{n}(\bar{n}-1)}{2}, 1 \right) + \left( 1, \frac{m(m-1)}{2} + \frac{\bar{m}(\bar{m}-1)}{2} \right) + \\
+ (n, \bar{m}) + (\bar{n}, m) + (n, m) + (\bar{n}, \bar{m}).
\]

(3.20)

Notice that the choice $m = 0$ with a gauge group $U(16)$, in contrast to the $SO(32)$ case, is not pure SYM, since it contains massless chiral multiplets in the $(120 + \overline{120})$ representation.

The gauge theory on $D_9$ branes is not really supersymmetric QCD with flavors in the fundamental and antifundamental representation, whose non-perturbative dynamics is known with great accuracy [9]. One way to get a more interesting example is the following. Moving $p$ $D_9$ branes out of the total 16 to a different orientifold fixed point not affected by the shift, one gets a gauge group $U(n) \otimes U(m) \otimes U(p)$, with $n + m + p = 16$. Strings stretched between the $p$ $D_9$ branes and the remaining $n + m$ are massive, and therefore they disappear from the effective low-energy gauge theory, whereas the $U(n) \otimes U(m)$ gauge sector has the massless spectrum displayed in (3.20). Choosing $n = 3$ and $m = 1$, a gauge group $SU(3) \otimes U(1)^2$ results, together with a factor $U(12)$ decoupled from it. Using the fact that the antisymmetric representation of $SU(3)$ coincides with the antifundamental $\overline{3}$, one ends up with a SQCD theory with gauge group $SU(3)$ and $N_f = 3$ flavors of quarks-antiquarks. This is the regime $N_c = N_f = N$ described in [25], where the composite mesons $M = QQ$ and baryons (antibaryons) $B = Q_1 \cdots Q_n$ ($\bar{B} = \bar{Q}_1 \cdots \bar{Q}_n$) have a quantum-deformed moduli space such that

\[
\det M - B\bar{B} = \Lambda^{2N},
\]

(3.21)
where \( \Lambda^{2N} = \exp(-8\pi^2/g^2) \) is the dynamical scale of the \( SU(3) \) gauge theory. As a consequence, the deformation in (3.21) originates only from the one-instanton contribution.

4 Heterotic dual models

4.1 Heterotic \( \text{SO}(32) \) model

Due to the freely-acting nature of the type I orbifold, according to the adiabatic argument \[12\] the S-duality between the type I and the \( \text{SO}(32) \) heterotic string is expected to be preserved. In this section we explicitly construct the heterotic S-dual of the \( \text{SO}(32) \) type I model\[6\]. The natural guess is to use the same freely-acting orbifold generators with a trivial action on the internal gauge degrees of freedom, consistently with the fact that in its type I dual the action on the Chan-Paton factors is trivial as well. There is however one subtlety, already encountered in similar situations and explained in other examples in \[12\]. Modular invariance forces us to change the geometric freely-orbifold actions (2.8)-(2.10) into a non-geometric one. Let us consider for simplicity one circle of radius \( R \) and one of the geometric shift in (2.8)-(2.10)

\[
X \rightarrow X + \pi R. 
\] (4.1)

Our claim is that its S-dual on the heterotic side is the non-geometric action\[7\]

\[
X_L \rightarrow X_L + \frac{\pi R}{2} + \frac{\pi \alpha'}{2R}, \quad X_R \rightarrow X_R + \frac{\pi R}{2} - \frac{\pi \alpha'}{2R}. 
\] (4.2)

In order to prove this claim, we use the fermionic formulation of the sixteen dimensional heterotic gauge lattice, with 16 complex fermions. Guided by the type I dual model, we take a trivial orbifold action on the 16 gauge fermions. The adiabatic argument of \[12\] allows identification of the orbifold action only in the large radius limit, where the shift (4.2) is indistinguishable from (4.1). In the twisted sector of the theory, the masses of the lattice states \((m, n)\) are shifted according to \[6\]

\[
(m, n) \rightarrow (m + s_1, n + s'_1), 
\] (4.3)

\[6\]We are grateful to M. Bianchi and E. Kiritsis for helpful discussions and comments on this point.

\[7\]As shown recently \[26\], such asymmetric shifts in type I models are consistent only if they act in an even number of coordinates.
where \((s_1, s'_1) = (1/2, 0)\) for (4.1) and \((s_1, s'_1) = (1/2, 1/2)\) for (4.2). The Virasoro generators of the left and right CFT’s are

\[
L_0 = N + 2 \times \left(-\frac{1}{12} - \frac{1}{24}\right) + 2 \times \left(\frac{1}{24} + \frac{1}{12}\right),
\]

\[
\bar{L}_0 = \bar{N} + 10 \times \left(-\frac{1}{12}\right) + 2 \times \frac{1}{24},
\]

(4.4)

where \(N (\bar{N})\) contains the oscillator contributions whereas the other terms are the zero-point energy in the NS sector from the spacetime and the gauge coordinates. Level-matching in the twisted sector is then

\[
L_0 - \bar{L}_0 = N - \bar{N} + \frac{3}{4} = -(m + s_1)(n + s'_1) \pmod{1}.
\]

(4.5)

This is possible only for \((s_1, s'_1) = (1/2, 1/2)\) which therefore fixes (4.2) to be the correct choice. The S-dual of the type I freely-acting \(SO(32)\) is then defined by the modular invariant torus amplitude

\[
T = \int_X d^2x \frac{1}{2\pi \eta^2 \bar{\eta}^2} \left[(\tau_{oo} + \tau_{og} + \tau_{oh} + \tau_{of})\Lambda_1 \Lambda_2 \Lambda_3 + \right.
\]

\[
(\tau_{oo} - \tau_{oh} - \tau_{of})(-1)^{m_1+n_1} \Lambda_1 \left. \left[\frac{4\eta^2}{\partial \bar{\eta}^2}\right]^2 + (\tau_{oo} - \tau_{og} + \tau_{oh} - \tau_{of})(-1)^{m_3+n_3} \Lambda_3 \right. \left[\frac{4\eta^2}{\partial \bar{\eta}^2}\right]^2 + \right.
\]

\[
(\tau_{oo} - \tau_{og} - \tau_{oh} + \tau_{of})(-1)^{m_2+n_2} \Lambda_2 \left. \left[\frac{4\eta^2}{\partial \bar{\eta}^2}\right]^2 + (\tau_{go} + \tau_{gg} + \tau_{gh} + \tau_{gf})\Lambda_1^{m_1+n_1} \left[\frac{4\eta^2}{\partial \bar{\eta}^2}\right]^2 + \right. \left. \right.
\]

\[
(\tau_{ho} + \tau_{hg} + \tau_{hh} + \tau_{hf})\Lambda_3^{m_3+n_3} \left[\frac{4\eta^2}{\partial \bar{\eta}^2}\right]^2 + (\tau_{fo} + \tau_{fg} + \tau_{fh} + \tau_{ff})\Lambda_2^{m_2+n_2} \left[\frac{4\eta^2}{\partial \bar{\eta}^2}\right]^2 \right. + \left. \right.
\]

\[
- (\tau_{go} + \tau_{gg} - \tau_{gh} - \tau_{gf})(-1)^{m_1+n_1} \Lambda_1^{m_1+n_1} \left[\frac{4\eta^2}{\partial \bar{\eta}^2}\right]^2 - \right. \left. \right.
\]

\[
- (\tau_{ho} - \tau_{hg} + \tau_{hh} - \tau_{hf})(-1)^{m_3+n_3} \Lambda_3^{m_3+n_3} \left[\frac{4\eta^2}{\partial \bar{\eta}^2}\right]^2 - \right. \left. \right.
\]

\[
- (\tau_{fo} - \tau_{fg} + \tau_{fh} + \tau_{ff})(-1)^{m_2+n_2} \Lambda_2^{m_2+n_2} \left[\frac{4\eta^2}{\partial \bar{\eta}^2}\right]^2 \right] \times \left(\Omega_{32} + \Sigma_{32}\right). \quad (4.6)
\]

Indeed, the massless spectrum matches perfectly with its type I counterpart. Compared to its type I S-dual cousin, the heterotic model has the same spectrum for the Kaluza-Klein modes, whereas it has a different spectrum for the winding modes. This is precisely what is expected from S-duality [[1]], which maps KK states into KK states, whereas it maps perturbative winding states into non-perturbative states in the S-dual theory.
4.2 Dual heterotic models with orthogonal gauge groups

In the fermionic formulation, the dual of the type I $SO(n_0) \otimes SO(n_g) \otimes SO(n_f) \otimes SO(n_h)$,

$n_0 + n_g + n_f + n_h = 32$ can be constructed by splitting the 16 complex fermions of the
gauge lattice into $n_0/2 + n_g/2 + n_f/2 + n_h/2$ groups. We then embed the orbifold action
into the gauge lattice as shown in Table 1.

| orb. actions | $SO(n_0)$ | $SO(n_g)$ | $SO(n_f)$ | $SO(n_h)$ |
|--------------|-----------|-----------|-----------|-----------|
| $g$          | +         | +         | -         | -         |
| $f$          | +         | -         | +         | -         |
| $h$          | +         | -         | -         | +         |

Table 1: Orbifold actions in the gauge degrees of freedom in the fermionic formulation.

Level matching in this case can be readily worked out with the result, in the $g$, $f$ and
$h$ twisted sectors respectively

$$L_0 - \bar{L}_0 = N - \tilde{N} - \frac{5}{4} + \frac{n_o + n_g}{16} = -(m_1 + s_1)(n_1 + s'_1) \pmod{1},$$

$$L_0 - \bar{L}_0 = N - \tilde{N} - \frac{5}{4} + \frac{n_o + n_f}{16} = -(m_2 + s_2)(n_2 + s'_2) \pmod{1},$$

$$L_0 - \bar{L}_0 = N - \tilde{N} - \frac{5}{4} + \frac{n_o + n_h}{16} = -(m_3 + s_3)(n_3 + s'_3) \pmod{1}. \quad (4.7)$$

The various possibilities are then as follows

- $n_o + n_g = 8 \pmod{8} \rightarrow s_1 = s'_1 = 1/2$ ,
- $n_o + n_g = 4 \pmod{8} \rightarrow s_1 = 1/2 , s'_1 = 0$ ,

and similarly for the other pairs $n_o + n_f, n_o + n_h$. It is interesting to notice the restrictions
on the rank of the gauge group. While the restriction on the even $SO(2n)$ gauge factors
was expected from the beginning, the above conditions are actually stronger.

Let us take a closer look to the particular case of the gauge group $SO(p) \otimes SO(q)$ with
$p + q = 32$, in order to better understand this point. The corresponding setting is $n_o = p, n_g = q$ and $n_f = n_h = 0$. Level matching in the $f$ and $h$ twisted sectors reads

$$L_0 - \bar{L}_0 = N - \tilde{N} - \frac{5}{4} + \frac{p}{16} = -(m + s_1)(n + s_2) \pmod{1}, \quad (4.8)$$
which leads to the following options:

- $p = 8 \pmod{8} \rightarrow s_1 = s_2 = 1/2$
- $p = 4 \pmod{8} \rightarrow s_1 = 1/2, s_2 = 0$

Surprisingly, we do not find solutions for $p = 2 \pmod{2}$. We can only speculate that, perhaps, a more subtle orbifold actions on the gauge lattice and/or the introduction of discrete Wilson lines could help in finding the $p = 2$ models, which the dual type I models suggest that have to exist.

For the first case, $p = 8, 16, 24$, it is convenient, in the fermionic formulation of the gauge degrees of freedom, to define the following characters

$$
\chi_0 = O_p O_q + C_p C_q \quad , \quad \chi_v = V_p V_q + S_p S_q \quad , \\
\chi_s = O_p C_q + C_p O_q \quad , \quad \chi_c = V_p S_q + S_p V_q .
$$

The complete partition function of the heterotic model is then

$$
T = \int \frac{d^2 \tau}{\tau^2 \eta^4 \tau^2} \frac{1}{4} \left\{ \left[ (\tau_{oo} + \tau_{og} + \tau_{oh} + \tau_{of}) \Lambda_1 \Lambda_2 \Lambda_3 + \\
+ (\tau_{oo} + \tau_{og} - \tau_{oh} - \tau_{of})(-1)^{m_1+n_1} \Lambda_1 \left| \frac{4 \eta^2}{\theta_{2}^2} \right|^2 + (\tau_{go} + \tau_{gg} + \tau_{gh} + \tau_{gf}) \Lambda_1^{m_1+n_1} \left| \frac{4 \eta^2}{\theta_{3}^2} \right|^2 + \\
+ (\tau_{go} + \tau_{gg} - \tau_{gh} - \tau_{gf})(-1)^{m_1+n_1} \Lambda_1^{m_1+n_1} \left| \frac{4 \eta^2}{\theta_{4}^2} \right|^2 \right] (\chi_o + \chi_v) + \\
\left[ (\tau_{oo} - \tau_{og} + \tau_{oh} - \tau_{of})(-1)^{m_3+n_3} \Lambda_3 + (\tau_{oo} - \tau_{og} - \tau_{oh} + \tau_{of})(-1)^{m_2+n_2} \Lambda_2 \right] \left| \frac{4 \eta^2}{\theta_{2}^2} \right|^2 (\chi_o - \chi_v) + \\
+ \left[ (\tau_{ho} + \tau_{hg} + \tau_{hh} + \tau_{hf}) \Lambda_3^{m_3+n_3} + (\tau_{fo} + \tau_{fg} + \tau_{fh} + \tau_{ff}) \Lambda_2^{m_2+n_2} \right] \left| \frac{4 \eta^2}{\theta_{3}^2} \right|^2 (\chi_s + \chi_c) \\
- (-1)^q/8 \left[ (\tau_{ho} - \tau_{hg} + \tau_{hh} - \tau_{hf})(-1)^{m_3+n_3} \Lambda_3^{m_3+n_3} + \\
+ (\tau_{fo} - \tau_{fg} - \tau_{fh} + \tau_{ff})(-1)^{m_2+n_2} \Lambda_2^{m_2+n_2} \right] \left| \frac{4 \eta^2}{\theta_{4}^2} \right|^2 (\chi_s - \chi_c) \right\} .
$$

As for the $SO(32)$ model, the whole KK spectrum precisely match the corresponding one on the type I S-dual side, whereas the massive winding states and the massive twisted spectra are, as expected, quite different.
On the other hand, for the second case $p = 4, 12, 20$, the correct characters are

\[
\chi_o = O_p O_q + C_p C_q \quad , \quad \chi_v = V_p V_q + S_p S_q
\]

\[
\chi_s = V_p C_q + S_p O_q \quad , \quad \chi_c = O_p S_q + C_p V_q
\]

The complete partition function is now

\[
T = \int \frac{d^2 \tau}{\tau_{\Lambda}^2 \eta^2} \frac{1}{4} \left\{ \left( \tau_{oo} + \tau_{og} + \tau_{oh} + \tau_{of} \right) \Lambda_1 \Lambda_2 \Lambda_3 + \\
+ \left( \tau_{oo} + \tau_{og} - \tau_{oh} - \tau_{of} \right) (-1)^{m_1 + n_1} \Lambda_1 \frac{4 \eta^2}{\partial^2} \right\}^2 + (\tau_{go} + \tau_{gg} + \tau_{gh} + \tau_{gf}) \Lambda_1^{m_1 + \frac{1}{2}, n_1 + \frac{1}{2}} \frac{4 \eta^2}{\partial^2} + \\
\left( \tau_{go} + \tau_{gg} - \tau_{gh} - \tau_{gf} \right) (-1)^{m_1 + n_1} \Lambda_1^{m_1 + \frac{1}{2}, n_1 + \frac{1}{2}} \frac{4 \eta^2}{\partial^2} \right\} \left( \chi_o + \chi_v \right) + \\
\left( \tau_{oo} - \tau_{og} + \tau_{oh} - \tau_{of} \right) (-1)^{m_3} \Lambda_3 + (\tau_{oo} - \tau_{og} - \tau_{oh} + \tau_{of}) (-1)^{m_2} \Lambda_2 \frac{4 \eta^2}{\partial^2} \right\} \left( \chi_o - \chi_v \right) + \\
+ \left( \tau_{ho} + \tau_{hg} + \tau_{hh} + \tau_{hf} \right) \Lambda_3^{m_3, n_3 + \frac{1}{2}} + (\tau_{fo} + \tau_{fg} + \tau_{fh} + \tau_{ff}) \Lambda_2^{m_2, n_2 + \frac{1}{2}} \frac{4 \eta^2}{\partial^2} \right\} \left( \chi_s + \chi_c \right) + \\
- (-1)^{(p+4)/8} \left( \tau_{ho} - \tau_{hg} + \tau_{hh} - \tau_{hf} \right) (-1)^{m_3} \Lambda_3^{m_3, n_3 + \frac{1}{2}} + \\
(\tau_{fo} - \tau_{fg} - \tau_{fh} + \tau_{ff}) (-1)^{m_2} \Lambda_2^{m_2, n_2 + \frac{1}{2}} \frac{4 \eta^2}{\partial^2} \right\} \left( \chi_s - \chi_c \right) \right) .
\]

It should be noticed that while the KK spectra are actually the same for the two cases $p = 4$ and $p = 8 \, (\text{mod} \, 8)$, they are very different in the massive winding sector, in perfect agreement with the modular invariance constraints (4.7).

We expect that appropriate orbifold action in the sixteen dimensional gauge lattice will also produce the S-dual of the type I racetrack and of the unitary gauge group cases, discussed in the previous sections. The required action, however, cannot correspond to a standard Wilson line in the adjoint of the gauge group, but rather to a non-diagonal action in the Cartan basis, like the ones considered in [27].

## 5 Threshold corrections to the gauge couplings

In this section we perform the one-loop calculation of the threshold corrections to the gauge couplings of some of the models described in the previous sections. The effective field theory quantities can be then easily extracted from the one-loop computation.
The threshold correction \( \Lambda_2 \) is generically written as

\[
\left. \frac{4\pi^2}{g^2} \right|_{\text{1-loop}} = \left. \frac{4\pi^2}{g^2} \right|_{\text{tree}} + \Lambda_{2,a},
\] (5.1)

with

\[
\Lambda_{2,a} = \int_{\mathcal{F}} \frac{d^2 \tau}{4\tau_2} B_a(\tau)
\] (5.2)

for the heterotic string, and

\[
\Lambda_{2,a} = \int_0^\infty \frac{dt}{4t} B_a(t)
\] (5.3)

for the type I string. In these expressions, \( B_a \) flows in the infrared to

\[
b_a = -3T_a(G) + \sum_r T_a(r),
\] (5.4)

the one-loop beta function for the gauge group factor \( G_a \), with \( r \) running over the gauge group representations with Dynkin index \( T_a(r) \). From the one-loop expression of the gauge coupling it is possible to extract \[30\] the holomorphic gauge couplings \( f_a(M_i) \), where \( M_i \) denote here collectively the moduli chiral (super)fields, using the relation \[31\]

\[
\frac{4\pi^2}{g_a^2(\mu^2)} = \text{Re} f_a + \frac{b_a}{4} \log \frac{M_P^2}{\mu^2} + \frac{c_a}{4} K + \frac{T_a(G)}{2} \ln g_a^{-2}(\mu^2) - \sum_r \frac{T_a(r)}{2} \ln \det Z_r(\mu)^2,
\] (5.5)

where \( K \) is the Kahler potential, \( Z_r \) is the wave-function normalization matrix for the matter fields and \( c_a = \sum_r T_a(r) - T_a(G) \). With this definition, the holomorphic non-perturbative scale \( \Lambda_a \) of an asymptotically-free gauge theory \( (b_a < 0) \) is given by

\[
\Lambda_a = M_P e^{-\frac{2\pi}{|b_a|}}.
\] (5.6)

### 5.1 Type I \( \text{SO}(n_o) \otimes \text{SO}(n_g) \otimes \text{SO}(n_f) \otimes \text{SO}(n_h) \) model

For the computation of threshold corrections to the gauge couplings in the freely-acting type I model with orthogonal gauge groups, we make use of the background field method \[28, 29, 30\]. Therefore, we introduce a magnetic field along two of the spatial non-compact directions, say \( F_{23} = BQ \). In the weak field limit, the one-loop vacuum energy can be expanded in powers of \( B \), providing

\[
\Lambda(B) = \Lambda_0 + \frac{1}{2} \left( \frac{B}{2\pi} \right)^2 \Lambda_2 + \ldots .
\] (5.7)
For supersymmetric vacua $\Lambda_0 = 0$, and the quadratic term accounts exactly for the threshold corrections in eq. (5.1).

In the presence of $F_{23}$, the oscillator modes along the non-compact complex plane $x^2 + ix^3$ get shifted by an amount $\epsilon$ such that

$$\pi \epsilon = \arctan(\pi q_L B) + \arctan(\pi q_R B) \simeq \pi (q_L + q_R) B + O(B^3) ,$$

(5.8)

where $q_L$ and $q_R$ are the eigenvalues of the gauge group generator $Q$, acting on the Chan-Paton states localized at the two endpoints of the open strings. In the vacuum energy, the contribution of the non-compact bosons and fermions gets replaced by

$$\vartheta_\alpha(0|\tau) \frac{\vartheta_\alpha(\tau|\tau)}{\eta^2(\tau)} \rightarrow 2\pi \epsilon \tau \frac{\vartheta_\alpha(\tau|\tau)}{\vartheta_1(\tau|\tau)} \quad \text{for } \alpha = 2, 3, 4$$

(5.9)

in the annulus and Möbius amplitudes. In addition, the momentum operator along the non-compact dimensions becomes,

$$p^\mu p_\mu \rightarrow -(p_0)^2 + (p_1)^2 + (2n + 1)\epsilon + 2\epsilon \Sigma_{23} ,$$

(5.10)

where $\Sigma_{23}$ is the spin operator in the (23) direction, while $n$ is an integer that labels the Landau levels. The supertrace operator becomes now

$$\text{STr} \rightarrow \left( \sum_{\text{bos}} - \sum_{\text{ferm}} \right) \frac{(q_L + q_R) B}{2\pi} \int \frac{d^2 p}{(2\pi)^2} ,$$

(5.11)

where $(q_L + q_R) B/2\pi$ is the density of the Landau levels and the integral is performed only over the momenta in the non-compact directions $x^0$ and $x^1$.

The details of the computation can be found in Appendix C.1. Collecting the results obtained there, and assuming $Q$ to be in a $U(1)$ inside $SO(n_o), SO(n_g), SO(n_f)$ or $SO(n_h)$, the moduli dependent threshold corrections for the respective gauge couplings can be written as follows,

$$\Lambda_{2,o} = -\frac{1}{4} \text{Tr}(Q^2) \left[ (2 - g_N) \left( \pi \text{Re} U_1 + \log[(\text{Re} U_1)(\text{Re} T_1)] \mu^2 \left| \frac{\vartheta_4}{\eta^4}(2iU_1) \right|^{-2} \right) + 
+(2 - f_N) \left( \pi \text{Re} U_2 + \log[(\text{Re} U_2)(\text{Re} T_2)] \mu^2 \left| \frac{\vartheta_4}{\eta^4}(2iU_2) \right|^{-2} \right) + 
+(2 - h_N) \left( \pi \text{Re} U_3 + \log[(\text{Re} U_3)(\text{Re} T_3)] \mu^2 \left| \frac{\vartheta_4}{\eta^4}(2iU_3) \right|^{-2} \right) \right] ,$$

(5.12)
\[ \Lambda_{2,g} = -\frac{1}{4} \text{Tr}(Q^2) \left[ (2 - g_N) \left( \pi \text{Re} \, U_1 + \log[(\text{Re} \, U_1)(\text{Re} \, T_1)] \mu^2 \left\{ \frac{4}{\eta^2} |(2iU_1)|^2 \right\} \right) + 
+ (2 + f_N) \left( \pi \text{Re} \, U_2 + \log[(\text{Re} \, U_2)(\text{Re} \, T_2)] \mu^2 \left\{ \frac{4}{\eta^2} |(2iU_2)|^2 \right\} \right) + 
+ (2 + h_N) \left( \pi \text{Re} \, U_3 + \log[(\text{Re} \, U_3)(\text{Re} \, T_3)] \mu^2 \left\{ \frac{4}{\eta^2} |(2iU_3)|^2 \right\} \right) \right], \quad (5.13) \]

\[ \Lambda_{2,f} = -\frac{1}{4} \text{Tr}(Q^2) \left[ (2 + g_N) \left( \pi \text{Re} \, U_1 + \log[(\text{Re} \, U_1)(\text{Re} \, T_1)] \mu^2 \left\{ \frac{4}{\eta^2} |(2iU_1)|^2 \right\} \right) + 
+ (2 - f_N) \left( \pi \text{Re} \, U_2 + \log[(\text{Re} \, U_2)(\text{Re} \, T_2)] \mu^2 \left\{ \frac{4}{\eta^2} |(2iU_2)|^2 \right\} \right) + 
+ (2 + h_N) \left( \pi \text{Re} \, U_3 + \log[(\text{Re} \, U_3)(\text{Re} \, T_3)] \mu^2 \left\{ \frac{4}{\eta^2} |(2iU_3)|^2 \right\} \right) \right], \quad (5.14) \]

\[ \Lambda_{2,h} = -\frac{1}{4} \text{Tr}(Q^2) \left[ (2 + g_N) \left( \pi \text{Re} \, U_1 + \log[(\text{Re} \, U_1)(\text{Re} \, T_1)] \mu^2 \left\{ \frac{4}{\eta^2} |(2iU_1)|^2 \right\} \right) + 
+ (2 + f_N) \left( \pi \text{Re} \, U_2 + \log[(\text{Re} \, U_2)(\text{Re} \, T_2)] \mu^2 \left\{ \frac{4}{\eta^2} |(2iU_2)|^2 \right\} \right) + 
+ (2 - h_N) \left( \pi \text{Re} \, U_3 + \log[(\text{Re} \, U_3)(\text{Re} \, T_3)] \mu^2 \left\{ \frac{4}{\eta^2} |(2iU_3)|^2 \right\} \right) \right]. \quad (5.15) \]

The \( \beta \)-function coefficients can also be extracted in the form

\[ b_o = -\left[ 3(n_o - 2) - (n_f + n_g + n_h) \right], \]
\[ b_g = -\left[ 3(n_g - 2) - (n_f + n_o + n_h) \right], \]
\[ b_f = -\left[ 3(n_f - 2) - (n_o + n_g + n_h) \right], \]
\[ b_h = -\left[ 3(n_h - 2) - (n_f + n_g + n_o) \right], \quad (5.16) \]

and, using the definition \[(5.5)\], the holomorphic one-loop gauge kinetic functions are then

\[ f_o = S + \frac{1}{2} \left[ (2 - g_N) \log \frac{\vartheta_4}{e^{\pi \eta_1/2 \eta_3}} |(2iU_1)| + (2 - f_N) \log \frac{\vartheta_4}{e^{\pi \eta_2/2 \eta_3}} |(2iU_2)| + 
+ (2 - h_N) \log \frac{\vartheta_4}{e^{\pi \eta_3/2 \eta_3}} |(2iU_3)| \right], \]

\[ f_g = S + \frac{1}{2} \left[ (2 - g_N) \log \frac{\vartheta_4}{e^{\pi \eta_1/2 \eta_3}} |(2iU_1)| + (2 + f_N) \log \frac{\vartheta_4}{e^{\pi \eta_2/2 \eta_3}} |(2iU_2)| + 
+ (2 + h_N) \log \frac{\vartheta_4}{e^{\pi \eta_3/2 \eta_3}} |(2iU_3)| \right]. \]
\[ f_f = S + \frac{1}{2} \left[ (2 + g_N) \log \frac{\partial_4}{\eta^3} e^{\pi U_1/2 \eta^3} (2iU_1) + (2 - f_N) \log \frac{\partial_4}{\eta^3} e^{\pi U_2/2 \eta^3} (2iU_2) + \
+ (2 + h_N) \log \frac{\partial_4}{\eta^3} e^{\pi U_3/2 \eta^3} (2iU_3) \right], \]

\[ f_h = S + \frac{1}{2} \left[ (2 + g_N) \log \frac{\partial_4}{\eta^3} e^{\pi U_1/2 \eta^3} (2iU_1) + (2 + f_N) \log \frac{\partial_4}{\eta^3} e^{\pi U_2/2 \eta^3} (2iU_2) + \
+ (2 - h_N) \log \frac{\partial_4}{\eta^3} e^{\pi U_3/2 \eta^3} (2iU_3) \right]. \] (5.17)

It is very important to stress the linear dependence of the above threshold corrections on the \((\pi \text{Re } U_i)\) factors. Indeed, the presence of such terms in a loop contribution may seem surprising. However, expanding the factor \(\partial_4 \eta^{-3}\), it can be realized that this term exactly cancels the contributions coming from the factor \(q^{1/24}\) contained in the \(\eta\)-function. Thus, the total dependence on the moduli of the threshold corrections turns out to be exclusively of logarithmic form. This phenomenon can be physically understood making the observation that, beyond the Kaluza-Klein scale, \(\mathcal{N} = 4\) supersymmetry is effectively recovered. Therefore, in the large volume limit only logarithmic corrections in the moduli should be present. The price one has to pay is that modular invariance in the target space is lost, as evident from the above expressions. The breaking of modular invariance in the target space by the shift \(\mathbb{Z}_2 \times \mathbb{Z}_2\) orbifold is very different from what happens in the ordinary \(\mathbb{Z}_2 \times \mathbb{Z}_2\) case where, beyond the Kaluza-Klein scale, the effective supersymmetry for each sector is still \(\mathcal{N} = 2\). The threshold corrections in that case turn out to be proportional to \((\text{Re } U) \log |\eta(iU)|^4\). Therefore, they preserve modular invariance, but have a non-logarithmic dependence on the moduli, due to the term \(q^{1/24}\) inside the \(\eta\)-function.

### 5.2 Type I racetrack model

The details of the calculation can be found again in Appendix C. Using the background field method, the moduli dependent part of the gauge coupling threshold corrections is
given by
\[ \Lambda_{2,p} = -\frac{1}{4} \text{Tr}(Q^2) \left[ (2 - p) \sum_{j=1}^{3} \left( \pi \text{Re} U_j + \log[(\text{Re} U_j)(\text{Re} T_j)\mu^2 \left| \frac{\partial_4}{\eta^3}(2iU_j) \right|^{-2}] - q \left( \log \left| \frac{\partial_4}{\eta^3}(2iU_1) \right|^2 - \log \left| \frac{\partial_4}{\eta^3}(4iU_1) \right|^2 + \pi \text{Re} U_1 \right) \right) \right] , \tag{5.18} \]

together with a similar expression for the \( SO(q) \) factor, with the obvious replacements. The corresponding \( \beta \)-function coefficients of the \( SO(p) \) and \( SO(q) \) gauge group factors are
\[ b_p = -3 \ (p - 2) \ , \quad b_q = -3 \ (q - 2) \ , \tag{5.19} \]
and the one-loop holomorphic gauge functions read
\[ f_p = S + \frac{2 - p}{2} \sum_{i=1}^{3} \log \left| \frac{\partial_4}{e^{\pi U_i/2\eta^3}}(2iU_i) \right| - q \left[ \log \frac{\partial_4}{e^{\pi U_1/2\eta^3}}(2iU_1) - \log \frac{\partial_4}{e^{\pi U_1/2\eta^3}}(4iU_1) \right] , \]
\[ f_q = S + \frac{2 - q}{2} \sum_{i=1}^{3} \log \left| \frac{\partial_4}{e^{\pi U_i/2\eta^3}}(2iU_i) \right| - p \left[ \log \frac{\partial_4}{e^{\pi U_1/2\eta^3}}(2iU_1) - \log \frac{\partial_4}{e^{\pi U_1/2\eta^3}}(4iU_1) \right] . \]
The non-perturbative superpotential can be written, in analogy with (3.16),
\[ W_{np} = A_p(U_i) \ e^{-a_p S} + A_q(U_i) \ e^{-a_q S} , \tag{5.20} \]
where
\[ a_p = \frac{2}{p - 2} \ , \quad A_p = \left[ \prod_{i=1}^{3} e^{-\pi U_i/2\eta^3} \left| \frac{\partial_4}{\eta^3}(2iU_i) \right| \left[ e^{\pi U_1/2\eta^3} \frac{\partial_4}{\eta^3}(2iU_1) \right] \right]^{\frac{q}{p-2}} , \]
\[ a_q = \frac{2}{q - 2} \ , \quad A_q = \left[ \prod_{i=1}^{3} e^{-\pi U_i/2\eta^3} \left| \frac{\partial_4}{\eta^3}(2iU_i) \right| \left[ e^{\pi U_1/2\eta^3} \frac{\partial_4}{\eta^3}(2iU_1) \right] \right]^{\frac{p}{q-2}} . \tag{5.21} \]

### 5.3 Heterotic \( SO(32) \) model

For the heterotic string, several procedures are available in literature to extract the threshold corrections \[32, 33, 34\]. The general expression for the threshold corrections to the gauge couplings, valid in the \( \overline{\text{DR}} \) renormalization scheme, is given by
\[ \Lambda_{a,a} = \int_\mathcal{F} d^2 \tau \frac{i}{\tau_2} \frac{1}{4\pi |\ell|^2} \sum_{\alpha, \beta = 0, 1/2} \partial_\tau \left( \frac{\partial \left[ \alpha \right]}{\eta} \right) \left( Q^2_a - \frac{1}{4\pi \tau_2} \right) C \left[ \alpha \right] \ , \tag{5.22} \]
where $Q_a$ is the charge operator of the gauge group $G_a$, and $C^a_\beta$ is the internal six-dimensional partition function, which, for the particular case of the $SO(32)$ model, can be read from (4.6). As noticed in [33], only the $\mathcal{N} = 2$ sectors of the theory contribute to the moduli dependent part of this expression.

Again, the details of the computation are relegated to Appendix C.3. The expression for the gauge threshold corrections of the heterotic $SO(32)$ model is

$$\Lambda_2 = -\frac{1}{96} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{i=1}^{3} \left[ (-1)^{m_i+n_i} \hat{Z}_i \bar{v}_3 v_4^2 - \hat{Z}_i^{m_i+\frac{1}{2},n_i+\frac{1}{2}} \bar{v}_3 v_4^2 - (-1)^{m_i+n_i} \hat{Z}_i^{m_i+\frac{1}{2},n_i+\frac{1}{2}} \right] \frac{E_4(\bar{E}_2 E_4 - E_6)}{\bar{\eta}^{24}} ,$$

(5.23)

where $E_{2n}$ are the Eisenstein series (given explicitly in the Appendix D), and the three toroidal lattice sums, $\hat{Z}_i \equiv |\eta|^4 \Lambda_i$, read

$$\hat{Z}_i \begin{bmatrix} h \\ g \end{bmatrix} = \frac{\text{Re} \ T_i}{\tau_2} \sum_{n_1,\ell_1,n_2,\ell_2} (-1)^{h n_1+g \ell_1} \exp \left[ 2\pi i T_i \text{det}(A) - \frac{\pi (\text{Re} \ T_i)}{\tau_2 (\text{Re} \ U_i)} \begin{pmatrix} 1 & i U_i \end{pmatrix} A \begin{pmatrix} \tau \\ 1 \end{pmatrix}^2 \right] ,$$

(5.24)

with

$$A = \begin{pmatrix} n_1 + \frac{h}{2} & \ell_1 + \frac{h}{2} \\ n_2 & \ell_2 \end{pmatrix}$$

(5.25)

and

$$(-1)^{m_i+n_i} \hat{Z}_i = \hat{Z}_i \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \quad \hat{Z}_i^{m_i+\frac{1}{2},n_i+\frac{1}{2}} = \hat{Z}_i \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \quad ( -1 )^{m_i+n_i} \hat{Z}_i^{m_i+\frac{1}{2},n_i+\frac{1}{2}} = \hat{Z}_i \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

Notice that $\begin{bmatrix} h \\ g \end{bmatrix}$ labels the three $\mathcal{N} = 2$ sectors associated to the $i$-th 2-torus, $i = 1, 2, 3$. Although the full expression (5.23) is worldsheet modular invariant, each of these $\mathcal{N} = 2$ sectors is not worldsheet modular invariant by itself, contrary to what happens in orbifolds with a trivial action on the winding modes.

In the large volume limit, $\text{Re} \ T_i \gg 1$, the winding modes decouple and only Kaluza-Klein modes with small $q$ contribute to the integral. In that case, the threshold correction receives contributions only from $A$ matrices with zero determinant in the sector $\begin{bmatrix} h \\ g \end{bmatrix} =$
\( (1,0), \) in such a way that \((5.23)\) becomes
\[
\Lambda_2 \mid_{\text{Re } T_i \gg 1} \simeq \frac{b}{3} \left[ -\pi \text{Re } U_i - \log[(\text{Re } U_i)(\text{Re } T_i)] \left| \frac{\eta^3}{\theta^4} (2iU_i) \right|^2 \mu^2 \right], \tag{5.26}
\]
matching exactly the threshold corrections for the dual type I \( \text{SO}(32) \) model.

For arbitrary \( T_i \), however, the winding modes do not decouple from the low energy physics and corrections due to worldsheet instantons appear:
\[
\Lambda_2 \simeq \Lambda_2 \mid_{\text{Re } T_i \gg 1}(U_i) + \Lambda_{\text{inst.}}(U_i, T_i). \tag{5.27}
\]
They correspond to \( E1 \) instanton contributions in the dual type I \( \text{SO}(32) \) model, and therefore are absent in \((5.17)\).

For example, consider the \( q \to 0 \) contributions to \( \Lambda_{\text{inst.}} \) of winding modes in the sector \((h, g) = (1, 0)\). These result in
\[
\Lambda_{\text{inst.}}(1,0) \bigg|_{q \to 0} \simeq -\frac{2b}{3} \sum_{n=1}^{\infty} (-1)^n \log \prod_{i=1}^{3} \left( 1 - e^{-2\pi n T_i} \right) + \text{c.c.} \tag{5.28}
\]
Since the axionic part of \( T_i \) in type I corresponds to components of the RR 2-form, \( C_2 \), it is natural to expect that these contributions come from \( E1 \) instantons wrapping \( n \) times the (1,1)-cycle associated to \( T_i \). Notice that the dependence on \( T_i \) perfectly agrees with general arguments in \cite{16} for the mirror type IIA picture.

The corresponding holomorphic gauge kinetic function reads
\[
f = S - 15 \sum_{i=1}^{3} \left[ \log \frac{\theta^4}{e^{\pi U_i/2} \eta^3} (2iU_i) - 2 \sum_{n=1}^{\infty} (-1)^n \log(1 - e^{-2\pi n T_i}) \right] + \ldots, \tag{5.29}
\]
where the dots denote further contributions from \( \Lambda_{\text{inst.}} \). Hence, the non-perturbative superpotential generated by gaugino condensation receives an extra dependence in the Kähler moduli,
\[
W_{np} = A(U_i, T_i) \, e^{-aS}, \tag{5.30}
\]
with
\[
a = \frac{1}{15}, \quad A = \prod_{i=1}^{3} \left[ e^{-\pi U_i/2} \frac{\theta^4}{\eta^3} (2iU_i) \prod_{n=1}^{\infty} \left( \frac{1 - e^{-4\pi (n+1/2) T_i}}{1 - e^{-4\pi n T_i}} \right)^2 \right] \times \ldots. \tag{5.31}
\]
\(^8\)We have neglected an extra term coming from the non-holomorphic regularization of \( \hat{E}_2 \), which in the dual type I side would presumably correspond to contact contributions in two-loop open string diagrams.
Unfortunately, a complete analytic evaluation of the non-perturbative corrections in (5.23) is subtle, as worldsheet modular invariance mix orbits within different \( N = 2 \) sectors and the unfolding techniques of [33, 13] cannot be applied straightforwardly to this case.

6 Euclidean brane instantons in the type I freely-acting \( SO(32) \) model

The model has two types of BPS brane instantons, denoted as \( E5 \) and \( E1 \). The \( E5 \) branes are interpreted as gauge instantons within the four dimensional gauge theory on the compactified \( D9 \) branes and map, in the heterotic dual, to non-perturbative euclidean NS5 corrections. The \( E1_i \) type I instantons wrapping the internal torus \( T^i \), instead, are stringy instantons from the gauge theory perspective and are responsible, in the heterotic dual, for the perturbative world-sheet instantons effects, that we have computed in section 5.

The configurations of the various \( Op \) planes and \( (D/E)p \) branes in the models are pictorially provided in table 2.

6.1 \( E5 \) instantons

A convenient way to describe the \( E5 \) instantons is to write the partition functions coming from the cylinder amplitudes (for \( E5-E5 \) and \( E5-D9 \) strings) and the Möbius amplitudes (for \( E5-O9 \) and \( E5-O5_i \)). In order to extract the spectrum, it is useful to express the result using the subgroup of \( SO(10) \) involved in a covariant description, namely \( SO(4) \times SO(2)^3 \) in our present case. Considering \( p \) coincident \( E5 \) instantons, one gets

\[
A_{E5-E5} = \frac{p^2}{16} \int_0^\infty \frac{dt}{t \eta^2} \sum_{\alpha, \beta} c_{\alpha \beta} \frac{\vartheta[\alpha]}{\eta} \left( P_1 P_2 P_3 \frac{\vartheta[\alpha]^3}{\eta^9} + (-1)^m P_1 \frac{\vartheta[\alpha] \vartheta[\beta] \vartheta[\beta+1/2] \vartheta[\beta-1/2]}{4 \eta^5} \right)
\]

\[
+ (-1)^m \frac{P_2}{4 \eta^5} \frac{\vartheta[\beta-1/2] \vartheta[\beta] \vartheta[\beta+1/2]}{\eta^5} \frac{4 \eta^2}{\vartheta_2^2} + (-1)^m \frac{P_3}{4 \eta^5} \frac{\vartheta[\beta+1/2] \vartheta[\beta-1/2]}{\eta^5} \frac{4 \eta^2}{\vartheta_2^2} \right) \tag{6.1}
\]

Notice that generically there will be also massless modes stretching between both kind of instantons, \( E5 \) and \( E1_i \). From the gauge theory perspective, these modes are presumably responsible of the \( E1 \) instanton corrections to the Veneziano-Yankielowicz superpotential, discussed at the end of section 5.
Table 2: Op-planes and D9/Ep branes present in the type I models. A - denotes a coordinate parallel to the Op-plane/Dp-brane, while a • represent an orthogonal coordinate.

\[
M_{E5-O9} = -\frac{p}{16} \int_0^\infty \frac{dt}{t} \frac{4\eta_2^2}{\vartheta_2^2} \sum_{\alpha\beta} \frac{\partial [\alpha}_{\beta+1/2] [\alpha}_{\beta-1/2]}{\eta^2} \vartheta [\alpha}_{\beta+1/2] \vartheta [\alpha}_{\beta-1/2] \{P_1 P_2 P_3 \vartheta [\alpha]}_{\beta+1/2} + \\
\left[ (-1)^{m_1} P_1 \frac{\vartheta [\alpha]}{\eta^3} \vartheta [\alpha}_{\beta+1/2] - (-1)^{m_2} P_2 \frac{\vartheta [\alpha]}{\eta^3} \vartheta [\alpha}_{\beta-1/2] + (-1)^{m_3} P_3 \frac{\vartheta [\alpha]}{\eta^3} \vartheta [\alpha}_{\beta+1/2] \right] \times \frac{1}{\vartheta_2^2}, \quad (6.2)
\]

where \( c_{\alpha\beta} \) are the usual GSO projection coefficients. In terms of covariant \( SO(4) \times SO(2)^3 \) characters, the massless instanton zero-modes content results

\[
A^{(0)}_{E5-E5} + M^{(0)}_{E5-O9} = \frac{p(p+1)}{2} (V_4 O_2 O_2 O_2 - C_4 C_2 C_2 C_2) - \frac{p(p-1)}{2} S_4 S_2 S_2 S_2. \quad (6.3)
\]

From a four-dimensional perspective, \( V_4 O_2 O_2 O_2 \) describe vector zero-modes, \( a_\mu \), while \( C_4 C_2 C_2 C_2 \) is a spinor \( M^{\alpha-\cdots-\cdots} \), where \( \alpha \) denotes an \( SO(4) \) spinor index of positive chirality, whereas \( (- - - -) \) denote the \( SO(2)^3 \) internal chiralities. Analogously, \( S_4 S_2 S_2 S_2 \) are fermionic zero modes \( \lambda^{\dot{\alpha}-\cdots-\cdots} \). Notice that in the one-instanton \( p = 1 \) sector, \( \lambda \) is projected out by the orientifold projection.

The charged instanton spectrum is obtained from strings stretched between the \( E5 \)
instanton and the $D9$ background branes. The corresponding cylinder amplitude is

$$A_{E5-D9} = \frac{Np}{8} \int_0^\infty \frac{dt}{t} \frac{\eta^2}{\vartheta^4} \sum_{\alpha\beta} \frac{\vartheta^{[\alpha+1/2]}_\beta}{\vartheta^{[\alpha]}_\beta} \{P_1P_2P_3 \frac{\vartheta^{[\alpha]}_\beta}{\eta^9} +$$

$$+ (-1)^{m_1}P_1 \frac{\vartheta^{[\alpha]}_\beta}{\eta^3} \frac{\vartheta^{[\beta+1/2]}_\alpha}{\vartheta^{[\beta-1/2]}_\alpha} \frac{\vartheta^{[\alpha]}_\beta}{\vartheta^{[\alpha]}_\beta} \} + (-1)^{m_2}P_2 \frac{\vartheta^{[\beta-1/2]}_\alpha}{\eta^3} \frac{\vartheta^{[\beta]}_\alpha}{\vartheta^{[\beta]}_\alpha} \frac{\vartheta^{[\alpha]}_\beta}{\vartheta^{[\alpha]}_\beta}$$

$$+ (-1)^{m_3}P_3 \frac{\vartheta^{[\beta+1/2]}_\alpha}{\eta^3} \frac{\vartheta^{[\beta-1/2]}_\alpha}{\vartheta^{[\beta-1/2]}_\alpha} \frac{\vartheta^{[\alpha]}_\beta}{\vartheta^{[\alpha]}_\beta} \times \frac{1}{2} \frac{4\eta^2}{\vartheta^2} \} . \quad (6.4)$$

The massless states are described by the contributions

$$A^{(0)}_{E5-D9} = Np (S_4O_2O_2O_2 - O_4C_2C_2C_2) \ . \quad (6.5)$$

In particular, the state $S_4O_2O_2O_2$, coming from the NS sector, has a spinorial $SO(4)$ index $\omega_\alpha$, whereas $O_4C_2C_2C_2$, coming from the R sector, is an $SO(4)$ scalar with a spinorial $SO(6)$ index or, which is the same, a fundamental $SU(4)$ index $\mu^A$.

### 6.2 $E1$ instantons

The case of the $E1$ instantons is more subtle. Indeed, they wrap one internal torus while they are orthogonal to the two remaining ones, thus feeling the nontrivial effects of the freely-acting operations. The explicit discussion can be limited to the case of the $E1_1$ instantons, the other two cases $E1_{2,3}$ being obviously completely similar. It is useful to separately discuss the two distinct possibilities:

i) the $E1_1$ instantons sit at one of the fixed points (tori) of the $g$ orbifold generator in the $y^1 \ldots y^6$ directions.

ii) the $E1_1$ instantons are located off the fixed points (tori) of the $g$ orbifold generator in the $y^1 \ldots y^6$ directions.

It is worth to stress that, strictly speaking, the freely acting $g$ has no fixed tori, due, of course, to the shift along $T^1$. However, since the instanton $E1_1$ wraps $T^1$, while it is localized in the $(T^2, T^3)$ directions, it is convenient to analyze the orbifold action in the space perpendicular to the instanton world-volume.

In the following, we discuss the first configurations with the instantons on the fixed tori, which are the relevant ones for matching the dual heterotic threshold corrections.
Since the freely-acting operations \((f, h)\) identify points in the internal space perpendicular to the instanton world-volume, they enforce the presence of doublets of \(E_1\) instantons, in complete analogy with similar phenomena happening in the case of background D5 branes in [18, 19]. Indeed, the \(g\)-operation is the only one acting in a nontrivial way on the instantons. The doublet nature of the \(E_1\) instantons can be explicitly figured out in the following geometric way. Let the location of the \(E_1\) instanton be fixed at a point of the \((y^3, y^4, y^5, y^6)\) space, which is left invariant by the \(g\)-operation. For instance, \(|E_1\rangle = |0, 0, \pi R_5/2, 0\rangle\). Then, the \(f\) and \(h\) operations both map the point \(|E_1\rangle\) into \(|E_1'\rangle = |\pi R_3, 0, 3\pi R_5/2, 0\rangle\), so that an orbifold invariant instanton state is provided by the combination ("doublet")

\[
\frac{1}{\sqrt{2}} \left[ |0, 0, \pi R_5/2, 0\rangle + |\pi R_3, 0, 3\pi R_5/2, 0\rangle \right].
\] (6.6)

The corresponding open strings can be stretched between fixed points and/or images, and can be described by the following amplitudes

\[
A_{E_1-E_1} = \frac{q^2}{32} \int_0^\infty \frac{dt}{t \eta^2} \sum_{\alpha\beta} c_{\alpha\beta} \frac{\varphi^{[\alpha]}_{\beta}}{\eta} \left\{ P_1 (W_2 W_3 + W_2^{n+1/2} W_3^{n+1/2}) \frac{\varphi^{[\alpha]}_{\beta}}{\eta^9} \right\},
\] (6.7)

\[
M_{E_1-O9} = -\frac{q}{16} \int_0^\infty \frac{dt}{t} 4\eta^2 \sum_{\alpha\beta} \frac{\varphi^{[\alpha]}_{\beta}}{\eta^2} \left\{ P_1 \frac{\varphi^{[\alpha]}_{\beta}}{\eta^9} \left\{ (-1)^{m_1} P_1 W_2 W_3 \frac{\varphi^{[\alpha]}_{\beta}}{\eta^9} \right\} \right\}.
\] (6.8)

Since only the \(Z_2\) \(g\)-operation acts non-trivially on the characters, it is convenient in this case to use covariant \(SO(4) \times SO(2) \times SO(4)\) characters in order to describe the massless instanton zero-modes. Due to the doublet nature of the instantons, particle interpretation asks for a rescaling of the "charge" \(q = 2Q\), meaning that the tension of the elementary instanton is twice the tension of the standard D1-brane. The result is

\[
A^{(0)}_{E_1-E_1} + M^{(0)}_{E_1-O9} = \frac{Q(Q+1)}{2} (V_4 O_2 O_4 - C_4 C_2 S_4) + \frac{Q(Q-1)}{2} (O_4 V_2 O_4 - S_4 S_2 S_4).
\] (6.9)

These zero-modes describe the positions \(x^\mu\) of the \(E_1\) instantons in spacetime, scalars \(y^i\) along the torus wrapped by the instanton and fermions \(\Theta^{\alpha, -\alpha}\), \(\Theta^{\alpha, +\alpha}\). The charged \(E_1\)-\(D9\) instanton spectrum is obtained from strings stretched between the \(E_1\) instantons and
the $D9$ background branes. The corresponding cylinder amplitude is

$$A_{E1-D9} = \frac{Nq}{8} \int_0^\infty \frac{dt}{t} \eta_4^2 \sum_{\alpha\beta} c_{\alpha\beta} \frac{\partial^{[\alpha+1/2]}_{\beta}}{\eta^2} \frac{\partial^{[\alpha+1/2]}_{\beta+1/2}}{\eta^2} \frac{\partial^{[\alpha+1/2]}_{\beta+1/2}}{\eta^2} \frac{\eta^2}{\eta_4^2}$$

$$+ (-1)^{m_1} P_1 \frac{\partial^{[\alpha]}_{\beta} \partial^{[\alpha+1/2]}_{\beta+1/2} \partial^{[\alpha+1/2]}_{\beta+1/2}}{\eta^2}$$

(6.10)

The surviving massless states are now described by

$$A^{(0)}_{E1-D9} = NQ (-O_4S_2O_4)$$

(6.11)

and correspond to the surviving “would be” world-sheet current algebra fermionic modes in the “heterotic string” interpretation (with $Q = 1$ and $N = 32$ [38 [11]]).

The second configuration, where the $E1_1$ instantons are off the fixed points (tori) of the $g$ orbifold generator in $y^1 \ldots y^6$, for instance $|E1_1\rangle = |0,0,0,0\rangle$, can be worked out as well. In this case a quartet structure of instantons is present, in a situation again similar to the ones described in [18 [19]). Indeed, $g$ produces the image $g : |0,0,0,0\rangle \rightarrow |0,0,\pi R_5,0\rangle$, while $f$ and $h$ produce two other images $f : |0,0,0,0\rangle \rightarrow |\pi R_3,0,0,0\rangle$, $h : |0,0,0,0\rangle \rightarrow |\pi R_3,0,\pi R_5,0\rangle$. In conclusion, the orbifold-invariant linear superposition of the instanton images is now the combination

$$\frac{1}{2} \left[ |0,0,0,0\rangle + |0,0,\pi R_5,0\rangle + |\pi R_3,0,0,0\rangle + |\pi R_3,0,\pi R_5,0\rangle \right] .$$

(6.12)

For a given number of “bulk” $E1$ instantons, they have twice the number of neutral (uncharged) fermionic zero modes as compared to their “fractional” instantons cousins [6.9], whose minimal number of uncharged zero modes is four. On the other hand, their tension is twice bigger. If $n$ “fractional” $E1$ instanton doublets wrap the torus $T^i$, one expects a contribution proportional to $e^{-4\pi n T_i}$, whereas if they wrap half of the internal torus, consistently with the shift identification, the contributions should be proportional to $e^{-4\pi (n+1/2) T_i}$.

These considerations are perfectly in agreement with the $\mathcal{N} = 2$ nature of the threshold corrections appearing in the heterotic computation [5.23], [5.29] and [5.31]. On the other hand, the quartet structure of the “bulk” instantons is probably incompatible with them. It should be also noticed that the absence of $\mathcal{N} = 1$ sectors contributing to the threshold corrections (moduli-independent threshold corrections) on the heterotic side reflects the fact that only the $f$ and $h$ action create instanton images.
A similar analysis to the one carried out in this section can be performed for the more general type I $SO(n_o) \otimes SO(n_g) \otimes SO(n_f) \otimes SO(n_h)$ model presented in section 3.1. However, we do not find any remarkable difference in nature between different choices of $n_o, n_g, n_f$ and $n_h$, contrary to what the heterotic dual model seems to suggest. It would be interesting to clarify this issue and to understand why type I models differing only in the Chan-Paton charges lead to so different models in the heterotic dual side.

7 Fluxes and moduli stabilization

7.1 $\mathbb{Z}_2 \times \mathbb{Z}_2$ freely-acting orbifolds of twisted tori

Background fluxes for the RR and NS-NS fields have been shown to be relevant for lifting some of the flat directions of the closed string moduli space. From the four dimensional effective field theory perspective, the lift can be properly understood in terms of a non-trivial superpotential encoding the topological properties of the background. Many models based on ordinary abelian orientifolds of string theory have appeared in the literature (for recent reviews and references see for instance [37]). Here we would like to extend this construction to the case of orientifolds with a free action. The motivation is two-fold. First, in these models the twisted sector modes are massive, as has been previously shown. The same happens for the open string moduli transforming in the adjoint. Second, we have enough control over the non-perturbative regime, so that this model provides us with a laboratory on which to explicitly test the combined effect of fluxes and non-perturbative effects.

For the particular type I (heterotic) orbifolds considered here, the orientifold projection kills a possible constant $H_3$ ($F_3$) background, so that the only possibilities left, apart from non-geometric deformations, are RR (NSNS) 3-form fluxes and metric fluxes [38, 39, 40]. The latter correspond to twists of the cohomology of the internal manifold $\mathcal{M}$,

$$d\omega_i = M_i^j \alpha_j + N_i^j \beta^i ,$$

(7.1)

where $\omega_i$ is a basis of harmonic 2-forms in $\mathcal{M}$, and $(\alpha_i, \beta^i)$ a symplectic basis of harmonic 3-forms. The resulting manifold $\tilde{\mathcal{M}}$ is in general no longer Calabi-Yau, but rather it
possesses $SU(3)$-structure. Duality arguments show, however, that the light modes of the compactification in $\hat{M}$ can be suitably described in terms of a compactification in $\hat{M}$, together with a non-trivial superpotential $W_{\text{twist}}$ accounting for the different moduli spaces.

Here we want to take a further step in the models of the previous sections and to consider geometries which go beyond the toroidal one by adding metric fluxes to the original torus. In terms of the global 1-forms of the torus, the cohomology twist reads,

$$de^i = \frac{1}{2} f_{jk}^i e^j \wedge e^k,$$

(7.2)

the resulting manifold being a group manifold $\hat{M} = G/\Gamma$ with structure constants $f_{jk}^i$ and $\Gamma$ a discrete subgroup of $G$. Modding (7.2) by the orbifold action (2.8)-(2.10) will in general put restrictions on the structure constants $f_{jk}^i$ and the lattice $\Gamma$. More concretely, the surviving structure constants are

$$
\begin{aligned}
\begin{pmatrix}
f_{35}^2 \\
f_{51}^4 \\
f_{14}^6
\end{pmatrix}
&= 
\begin{pmatrix}
h_1 \\
h_2 \\
h_3
\end{pmatrix},
\quad
\begin{pmatrix}
f_{35}^1 & f_{52}^4 & f_{23}^6
\end{pmatrix}
&= 
\begin{pmatrix}
b_{11} & b_{12} & b_{13}
\end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\begin{pmatrix}
f_{46}^1 \\
f_{62}^3 \\
f_{24}^5
\end{pmatrix}
&= 
\begin{pmatrix}
h_1 \\
h_2 \\
h_3
\end{pmatrix},
\quad
\begin{pmatrix}
f_{46}^2 & f_{61}^3 & f_{14}^5
\end{pmatrix}
&= 
\begin{pmatrix}
b_{11} & b_{12} & b_{13}
\end{pmatrix},
\end{aligned}
$$

as in an ordinary $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. The Jacobi identity of the algebra $G$ requires in addition $f_{ijk}^m f_{mij}^n = 0$, [22, 38]. The set of metric fluxes transforms trivially under S-duality, so one can build heterotic-type I dual pairs by simply exchanging $F_3 \leftrightarrow H_3$.

The low energy physics of the $G/[\Omega_P \times \Gamma \times (\tilde{\mathbb{Z}}_2 \times \tilde{\mathbb{Z}}_2)]$ compactification can be then suitably described in terms of a $T^6/[\Omega_P \times (\mathbb{Z}_2 \times \mathbb{Z}_2)]$ compactification, with $\mathbb{Z}_2 \times \mathbb{Z}_2$ being the freely-acting orbifold action described in section 2, together with a superpotential [42],

$$W_{\text{twist}} = \sum_{i=1}^{3} T_i [-i\bar{h}_i + \sum_{j=1}^{3} \bar{b}_{ji} U_j + ib_{11} U_2 U_3 + b_{21} U_2 U_3 + ib_{31} U_1 U_2 - h_i U_1 U_2 U_3].$$

(7.3)
Notice that the freely-acting $\tilde{\mathbb{Z}}_2 \times \tilde{\mathbb{Z}}_2$ orbifold of the full ten dimensional picture will in general differ from the freely-acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the effective description. For illustration, consider the following simple example given by,

$$de^1 = b_{11}e^3 \wedge e^5, \quad de^2 = de^3 = de^4 = de^5 = de^6 = 0.$$  \hspace{1cm} (7.4)

We may integrate these equations as,

$$e^1 = dy^1 + b_{11}y^3dy^5, \quad e^i = dy^i \quad \text{for } i \neq 1,$$  \hspace{1cm} (7.5)

so that $G$ is a fibration of $y^5$ over $y^1$. The lattice $\Gamma$ is then suitably chosen as,

$$\Gamma : \begin{cases} y^3 \to y^3 + 1, & y^1 \to y^1 - b_{11}y^5 \\ y' \to y' + 1 & \text{for } i \neq 3 \end{cases},$$  \hspace{1cm} (7.6)

with $b_{11} \in \mathbb{Z}$ so that the vielbein vectors remain invariant under $\Gamma$ transformations. Acting now with the orbifold generators $\{\tilde{g}, \tilde{f}, \tilde{h}\}$ defined as,

$$(y^1, y^2, y^3, y^4, y^5, y^6) \xrightarrow{\tilde{g}} (y^1 + 1/2, y^2, -y^3, -y^4, -y^5 + 1/2, -y^6),$$  \hspace{1cm} (7.7)

$$(y^1, y^2, y^3, y^4, y^5, y^6) \xrightarrow{\tilde{f}} (-y^1 + 1/2 + b_{11}y^5/2, -y^2, y^3 + 1/2, y^4, -y^5, -y^6),$$

$$(y^1, y^2, y^3, y^4, y^5, y^6) \xrightarrow{\tilde{h}} (-y^1 - b_{11}y^5/2, -y^2, -y^3 + 1/2, -y^4, y^5 + 1/2, y^6).$$

The generators $\{\tilde{g}, \tilde{f}, \tilde{h}\}$ still define a $\mathbb{Z}_2 \times \mathbb{Z}_2$ discrete group. Indeed, requiring the quantization condition $b_{11} \in 2\mathbb{Z}$, one can prove that $\tilde{g}^2 = \tilde{h}^2 = \tilde{f}^2 = 1$ and $\tilde{g}\tilde{f} = \tilde{f}\tilde{g} = \tilde{h}$, $\tilde{g}\tilde{h} = \tilde{h}\tilde{g} = \tilde{f}$, $\tilde{h}\tilde{f} = \tilde{f}\tilde{h} = \tilde{g}$, up to discrete transformations of the lattice $\Gamma$. Hence, the light modes of the $SU(3)$-structure orientifold defined by the group manifold (7.5), together with the lattice (7.6) and the orbifold generators (7.7), can be consistently described by a $T^6$ compactification with an orbifold action given by eqs. (2.8) and a superpotential term,

$$W_{\text{twist}} = ib_{11}T_1U_2U_3.$$  \hspace{1cm} (7.8)
7.2 Moduli stabilization in a $S^3 \times T^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbifold

To illustrate the interplay between non-perturbative effects and metric fluxes we consider in this section the following one-parameter family of twists,

$$\begin{align*}
    de^1 &= \alpha e^4 \wedge e^6, \\
    de^2 &= \alpha e^4 \wedge e^6, \\
    de^3 &= \alpha e^6 \wedge e^2, \\
    de^4 &= \alpha e^6 \wedge e^2, \\
    de^5 &= \alpha e^2 \wedge e^4, \\
    de^6 &= \alpha e^2 \wedge e^4.
\end{align*}$$

The particular solution to these equations

$$\begin{align*}
    e^1 &= dy^1 + e^2, \\
    e^2 &= \sin(\alpha y^6)dy^4 + \cos(\alpha y^6) \cos(\alpha y^4)dy^2, \\
    e^3 &= dy^3 + e^4, \\
    e^4 &= -\cos(\alpha y^6)dy^4 + \sin(\alpha y^6) \cos(\alpha y^4)dy^2, \\
    e^5 &= dy^5 + e^6, \\
    e^6 &= dy^6 + \sin(\alpha y^4)dy^2,
\end{align*}$$

is corresponding to a product of a 3-sphere and a 3-torus. Consistency requires $\alpha$ to be multiple of $2\pi$. On the other hand, in this particular case the orbifold action remains unaffected by the fluxes and is still given by (2.8)-(2.10).

We will also add a possible RR 3-form flux along the 3-sphere,

$$F_3 = m e^2 \wedge e^4 \wedge e^6. \quad (7.9)$$

One may easily check that this flux, together with the above twists, does not give rise to tadpole contributions.

The model can be effectively described by a $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ compactification with Kähler potential and superpotential,

$$\begin{align*}
    K &= -\log(S + S^*) - \sum_{i=1}^3 \log(U_i + U_i^*) - \sum_{i=1}^3 \log(T_i + T_i^*), \\
    W &= m + \alpha \sum_{j=1}^3 T_j (-i + U_j) + W_{np}(S, T_1, T_2, T_3, U_1, U_2, U_3), \quad (7.10)
\end{align*}$$

where we have introduced a generic non-perturbative superpotential possibly depending on all moduli, as shown in the previous sections\textsuperscript{10}.

\textsuperscript{10}Perturbative corrections to the Kahler potential could also play a role in the moduli stabilization. We restrict here to the tree-level form of the Kahler potential, for the possible effect of $\alpha'$ or quantum corrections to it, see e.g. \[43\].
For $\text{Re} \, T_i \gg 1$ and $\text{Re} \, U_i \gg 1$, the dependence of the non-perturbative superpotential on the Kähler and complex structure moduli can be neglected, $\partial_{U_i} W_{np} \simeq \partial_{T_i} W_{np} \simeq 0$, and the above superpotential has a perturbative vacuum given by

$$\text{Im} \, U_i \simeq 1, \quad \text{Re} \, W_{np} + m \simeq \alpha (\text{Re} \, T_i)(\text{Re} \, U_i),$$

$$\text{Im} \, T_i \simeq 0, \quad \text{Im} \, W_{np} \simeq 0, \quad DSW = 0,$$

(7.12)

with $DSW = \partial S W - (S + S^*)^{-1} W$, as usual. Then, for $W_{np}$ the racetrack superpotential (5.20), one may stabilize $S$ at a reasonably not too big coupling.

The model can be viewed in the S-dual heterotic side as an asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of some Freedman-Gibbons electrovac solution [44, 45]. In particular, the full string ground state includes a SU(2) Wess-Zumino-Witten model describing the radial stabilization of the 3-sphere by $m$ units of $H_3$ flux, provided by $F_3 \rightarrow H_3$ in (7.9). In terms of the radii $R_i$, $i = 1 \ldots 6$, equations (7.12) lead to

$$(R_2)^2 = (R_4)^2 = (R_6)^2 \simeq \frac{\text{Re} \, W_{np} + m}{\alpha},$$

(7.13)

whereas the radii of the 3-torus, $R_1, R_3, R_5$, remain as flat directions. Having $\text{Re} \, T_i \gg 1$ and $\text{Re} \, U_i \gg 1$ then requires the volume of the 3-sphere to be much bigger than the volume of the 3-torus, i.e. $m/\alpha \gg 1$.

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11We thank E. Kiritsis for pointing out to us this connection.
A Normalization of string amplitudes

For sake of brevity, throughout the paper we ignored the overall factors coming from integrating over the noncompact momenta. For arbitrary string tension $\alpha'$, the complete string amplitudes $T, K, A, M$ are related to the ones used in the main text by

$$ T = \frac{1}{(4\pi^2 \alpha')^2} T \quad , \quad K = \frac{1}{(8\pi^2 \alpha')^2} K , $$

$$ A = \frac{1}{(8\pi^2 \alpha')^2} A \quad , \quad M = \frac{1}{(8\pi^2 \alpha')^2} M . \quad (A.1) $$

B Characters for $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds

In the light-cone RNS formalism, the vacuum amplitudes involve the following characters

$$ \tau_{oo} = V_2 I_2 I_2 I_2 + I_2 V_2 V_2 V_2 - S_2 S_2 S_2 S_2 - C_2 C_2 C_2 C_2 , $$
$$ \tau_{og} = I_2 V_2 I_2 I_2 + V_2 I_2 V_2 V_2 - C_2 C_2 S_2 S_2 - S_2 S_2 C_2 C_2 , $$
$$ \tau_{oh} = I_2 I_2 V_2 I_2 + V_2 V_2 I_2 V_2 - C_2 S_2 S_2 S_2 - S_2 C_2 C_2 S_2 , $$
$$ \tau_{of} = I_2 I_2 V_2 I_2 + V_2 I_2 V_2 V_2 - C_2 S_2 C_2 S_2 - S_2 C_2 S_2 C_2 , $$
$$ \tau_{go} = V_2 I_2 S_2 C_2 + I_2 V_2 C_2 S_2 - S_2 S_2 V_2 I_2 - C_2 C_2 I_2 V_2 , $$
$$ \tau_{gg} = I_2 V_2 S_2 C_2 + V_2 I_2 C_2 S_2 - S_2 S_2 I_2 V_2 - C_2 C_2 V_2 I_2 , $$
$$ \tau_{gh} = I_2 I_2 S_2 S_2 + V_2 V_2 C_2 C_2 - C_2 S_2 V_2 V_2 - S_2 C_2 I_2 I_2 , $$
$$ \tau_{gf} = I_2 I_2 C_2 C_2 + V_2 V_2 S_2 S_2 - S_2 C_2 V_2 V_2 - C_2 S_2 I_2 I_2 , $$
$$ \tau_{ho} = V_2 S_2 C_2 I_2 + I_2 C_2 S_2 V_2 - C_2 I_2 V_2 C_2 - S_2 V_2 I_2 S_2 , $$
$$ \tau_{hg} = I_2 C_2 C_2 I_2 + V_2 S_2 S_2 V_2 - C_2 I_2 I_2 S_2 - S_2 V_2 C_2 C_2 , $$
$$ \tau_{hh} = I_2 S_2 C_2 V_2 + V_2 C_2 S_2 I_2 - S_2 I_2 V_2 S_2 - C_2 V_2 I_2 C_2 , $$
$$ \tau_{hf} = I_2 S_2 S_2 I_2 + V_2 C_2 C_2 V_2 - C_2 V_2 V_2 S_2 - S_2 I_2 C_2 I_2 , $$
$$ \tau_{fo} = V_2 S_2 I_2 C_2 + I_2 C_2 V_2 S_2 - S_2 V_2 S_2 I_2 - C_2 I_2 C_2 V_2 , $$
$$ \tau_{fg} = I_2 C_2 I_2 C_2 + V_2 S_2 V_2 S_2 - C_2 I_2 S_2 I_2 - S_2 V_2 C_2 V_2 , $$
$$ \tau_{fh} = I_2 S_2 I_2 S_2 + V_2 C_2 V_2 C_2 - C_2 V_2 S_2 V_2 - S_2 I_2 C_2 I_2 , $$
$$ \tau_{ff} = I_2 S_2 V_2 C_2 + V_2 C_2 I_2 S_2 - C_2 V_2 C_2 I_2 - S_2 I_2 S_2 V_2 \quad , \quad (B.1) $$
where each term is a tensor product of the characters of the vector representation ($V_2$), the scalar representation ($I_2$), the spinor representation ($S_2$) and the conjugate-spinor representation ($C_2$) of the four $SO(2)$ factors that enter the light-cone restriction of the ten-dimensional Lorentz algebra.

## C Details on the threshold correction computations

### C.1 Threshold corrections in the type I $SO(n_o) \otimes SO(n_g) \otimes SO(n_f) \otimes SO(n_h)$ models

In order to implement the background field method, it is convenient to express the orbifold characters in terms of the corresponding $\vartheta$-functions:

\[
\tau_{oo} + \tau_{og} + \tau_{oh} + \tau_{of} = \frac{1}{2\eta^4}(\vartheta^4_3 - \vartheta^4_4 - \vartheta^4_2 - \vartheta^4_1), \quad (C.1)
\]

\[
\tau_{oo} + \tau_{og} - \tau_{oh} - \tau_{of} = \frac{1}{2\eta^4}(\vartheta^2_2 \vartheta^2_1 + \vartheta^2_2 \vartheta^2_2 - \vartheta^2_3 \vartheta^2_3 + \vartheta^2_3 \vartheta^2_4), \quad (C.2)
\]

\[
\tau_{oo} - \tau_{og} + \tau_{oh} - \tau_{of} = \frac{1}{2\eta^4}(\vartheta_1 \vartheta_2 \vartheta_3 \vartheta_1 + \vartheta_1 \vartheta_2 \vartheta_3 \vartheta_2 + \vartheta_3 \vartheta_4 \vartheta_3 - \vartheta_4 \vartheta_3 \vartheta_4), \quad (C.3)
\]

\[
\tau_{oo} - \tau_{og} - \tau_{oh} + \tau_{of} = \frac{1}{2\eta^4}(\vartheta_2 \vartheta_2 \vartheta_1 \vartheta_2 + \vartheta_1 \vartheta_2 \vartheta_2 \vartheta_2 - \vartheta_4 \vartheta_3 \vartheta_4 \vartheta_3 + \vartheta_3 \vartheta_4 \vartheta_3 \vartheta_4). \quad (C.4)
\]

Making use of the expansion (valid for even spin structure $\alpha$))

\[
\frac{\vartheta_{\alpha}(\epsilon \tau | \tau)}{\vartheta_1(\epsilon \tau | \tau)} = \frac{1}{2\pi \epsilon \tau} \frac{\vartheta_{\alpha}}{\eta^4} + \frac{\epsilon \tau}{4\pi \eta^2} \vartheta''_{\alpha} + \ldots , \quad (C.5)
\]

and the modular identities (D.2) and (D.3) in Appendix D, the expansions of the characters in terms of the (small) magnetic field or, equivalently, in terms of the $\epsilon$ of eq. (5.8), are

\[
(\tau_{oo} + \tau_{og} + \tau_{oh} + \tau_{of})(\epsilon \tau, \tau) \simeq -\frac{i\epsilon \tau}{8\pi \eta^4}(\vartheta''_{3} \vartheta^2_{3} - \vartheta''_{4} \vartheta^2_{4} - \vartheta''_{2} \vartheta^2_{2}) = 0, \quad (C.6)
\]

\[
(\tau_{oo} + \tau_{og} - \tau_{oh} - \tau_{of})(\epsilon \tau, \tau) = (\tau_{oo} - \tau_{og} + \tau_{oh} - \tau_{of})(\epsilon \tau, \tau) = (\tau_{oo} - \tau_{og} + \tau_{oh} - \tau_{of})(\epsilon \tau, \tau) \simeq -\frac{i\epsilon \tau}{8\pi \eta^4}(-\vartheta''_{4} \vartheta^2_{4} + \vartheta''_{3} \vartheta^2_{3} \vartheta^2_{4}) = \frac{i\pi \epsilon}{2} \tau \eta^2 \vartheta^2_{2}. \quad (C.6)
\]
The one-loop threshold corrections on any of the gauge group factors can therefore be written in the form

\[ \Lambda_2 = 16\pi^2 \int_0^\infty \frac{dt}{t} \left\{ [2 \text{Tr}(Q^2) - \text{Tr}(\gamma_g \text{Tr}(\gamma_g Q^2)))(-1)^{m_1}P_1 + 
\right. \\
\left. [2 \text{Tr}(Q^2) - \text{Tr}(\gamma_f \text{Tr}(\gamma_f Q^2)))(-1)^{m_2}P_2 + [2 \text{Tr}(Q^2) - \text{Tr}(\gamma_h \text{Tr}(\gamma_h Q^2)))(-1)^{m_3}P_3 \right\} , \quad (C.7) \]

where the action induced by the orbifold on the CP matrices, defined in (3.10), has been used. The last step is to compute the momentum sums \((-1)^m P\). To this end, it is useful to reexpress (3.7) as

\[ \Gamma \equiv \int_0^\infty \frac{dt}{t} (-1)^{mP} = \int_0^\infty \frac{dt}{t} \exp \left\{ -\frac{\pi(\text{Re} T)}{4t(\text{Re} U)} \right\} \sum_{m,m'} \exp \left\{ -\pi(m - b)^T A(m - b) \right\} , \quad (C.8) \]

with

\[ m - b = \left( \begin{array}{c} m - \frac{i(\text{Re} T)}{2t(\text{Re} U)} \\ m' + \frac{i(\text{Re} T)(\text{Im} U)}{2t(\text{Re} U)} \end{array} \right) , \quad A = \frac{t}{(\text{Re} T)(\text{Re} U)} \left( \begin{array}{cc} |U|^2 & \text{Im} U \\ \text{Im} U & 1 \end{array} \right) . \quad (C.9) \]

Making use of the Poisson summation formula (D.11) and redefining \(t \to 1/\ell\) in order to move to the transverse channel picture, one gets

\[ \Gamma = (\text{Re} T) \int_0^\infty d\ell \sum_{n_1,n_2} \exp \left\{ -\frac{\pi\ell(\text{Re} T)}{\text{Re} U} [(n_1 + \frac{1}{2} - n_2 \text{Im} U)^2 + (n_2 \text{Re} U)^2] \right\} . \quad (C.10) \]

As expected, the integral contains infrared (IR) divergences as \(\ell \to 0\), corresponding to loops of massless modes. It can be regularized introducing an IR regulator \(\mu\) via a factor \(F_\mu = (1 - e^{-1/\mu^2})\). Performing the integral in \(\ell\) the result is

\[ \Gamma = \lim_{\mu^2 \to 0} \left[ \frac{\text{Re} U}{\pi} \sum_{n_1,n_2} \left( \frac{1}{(n_1 + \frac{1}{2} - n_2 \text{Im} U)^2 + (n_2 \text{Re} U)^2} - \frac{1}{(n_1 + \frac{1}{2} - n_2 \text{Im} U)^2 + (n_2 \text{Re} U)^2 + \frac{\text{Re} U}{\pi\mu^2(\text{Re} T)}} \right) \right] . \quad (C.11) \]

Finally, using the Dixon, Kaplunovsky and Louis (DKL) formula [33] to evaluate the sum over \(n_1\), the expression become

\[ \Gamma = -\sum_{n_2 > 0} \left[ \frac{1}{n_2} \left( \frac{q^{n_2} - 1}{q^{n_2} + 1} + \frac{\tilde{q}^{n_2} - 1}{\tilde{q}^{n_2} + 1} \right) + \frac{2}{\sqrt{n_2^2 + ((1/\pi(\text{Re} U)(\text{Re} T)\mu^2)}} \right] , \quad (C.12) \]
with \( q \equiv \exp[-2\pi U] \) and where we have taken \( \mu^2 \ll 1 \) (in string units). A Taylor expansion (using eq. (D.19)) produces

\[
\Gamma = \sum_{n_2 > 0} \left( \frac{2}{n_2} - \frac{2}{\sqrt{n_2^2 + (1/\pi \text{Re}(U) \text{Re}(T) \mu^2)}} \right) + 2 \sum_{n_2, m_2 > 0} \frac{(-1)^m}{n_2} q^{mn_2} + 2 \sum_{n_2, m_2 > 0} \frac{(-1)^m}{n_2} q^{mn_2} 
\]

\[
= \sum_{n_2 > 0} \left( \frac{2}{n_2} - \frac{2}{\sqrt{n_2^2 + (1/\pi \text{Re}(U) \text{Re}(T) \mu^2)}} \right) - 2 \sum_{m_2 > 0} \log(1-q^{2m}) + 2 \sum_{m_2 > 0} \log(1-q^{2m-1}) + \text{c.c.} \tag{C.13}
\]

Taking the \( \mu^2 \to 0 \) limit and at the same time subtracting the finite and the cut-off dependent parts, in terms of the modular functions (D.17) and (D.16) one gets

\[
\int_0^\infty \frac{dt}{t} (-1)^m F_\mu P = \log \left| \frac{\eta_4}{\eta_3} (2iU) \right|^2 - \pi \text{Re} U - \log[(\text{Re} U)(\text{Re} T) \mu^2]. \tag{C.14}
\]

### C.2 Threshold corrections in the type I racetrack models

The procedure for the racetrack models is completely analogous to the one in the previous section. Plugging (C.6) into (3.11) and (3.12) one gets

\[
\Lambda_{2,p} = 16\pi^2 \text{Tr}(Q^2) \int_0^\infty \frac{dt}{t} \left[ (2-p) P_1 - q(P_{m'+1} P_{m_1}) (-1)^{m_1} + (2-p) P_2 (-1)^{m_2} + (2-p) P_3 (-1)^{m_3} \right], \tag{C.15}
\]

where the \( Q \) generator has been taken in the \( SO(p) \) factor. In this case there is a new lattice summation to compute, namely

\[
\Gamma' = \int_0^\infty \frac{dt}{t} (-1)^m P_{m'+1} P_m = \int_0^\infty \frac{dt}{t} \sum_{m, m'} (-1)^{m'} \exp \left[ -\frac{\pi t}{(\text{Re} T)(\text{Re} U)} |m' + \frac{1}{2} - i U m|^2 \right] = \int_0^\infty \frac{dt}{t} \exp \left[ -\frac{\pi (\text{Re} T)}{4t(\text{Re} U)} \right] \sum_{m, m'} \exp[-\pi (m - b)^T A (m - b)]\tag{C.16}
\]

where now

\[
m - b = \left( m - \frac{i(\text{Re} T)}{2t(\text{Re} U)} \right) \left( m' + \frac{i(\text{Re} T)(\text{Im} U)}{2t(\text{Re} U)} + \frac{1}{2} \right), \quad A = \frac{t}{(\text{Re} T)(\text{Re} U)} \begin{pmatrix} |U|^2 & \text{Im} U \\ \text{Im} U & 1 \end{pmatrix}. \tag{C.17}
\]

\[\text{The finite term can be actually reabsorbed into the value of the gauge coupling at the compactification scale.}\]
Thus, the integration in the transverse channel gives

\[ \Gamma' = \frac{\text{Re} \, U}{\pi} \sum_{n_1,n_2} \frac{(-1)^{n_2}}{(n_1 + \frac{1}{2} - n_2 \text{Im} \, U)^2 + (n_2 \text{Re} \, U)^2}. \]  

(C.18)

Using again the (DKL) formula, after some algebra, the \( \Gamma' \) can be written

\[ \Gamma' = \sum_{n_2 > 0} \frac{1}{n_2} \left( q^{n_2 - 1} - \frac{q^{2n_2} - 1}{q^{2n_2} + 1} \right) + \text{c.c.}, \]  

(C.19)

with \( q = \exp[-2\pi U] \). It should be noticed that in this case there is no need of an IR regulator for this sum. In terms of modular functions the integral becomes

\[ \int_0^\infty \frac{dt}{t} (-1)^m P_{m'+\frac{1}{2}} P_m = \log \left| \frac{\partial_4}{\eta^3} (4iU) \right|^2 - \log \left| \frac{\partial_4}{\eta^3} (2iU) \right|^2 - \pi \text{Re} \, U \]  

(C.20)

and the moduli dependent part of the gauge coupling threshold corrections is

\[ \Lambda_{2,p} = -16\pi^2 \text{Tr}(Q^2) \left[ (2 - p) \sum_{j=1}^{3} \left( \pi \text{Re} \, U_j + \log[(\text{Re} \, U_j)(\text{Re} \, T_j)\mu^2] - \log \left| \frac{\partial_4}{\eta^3} (2iU_j) \right|^2 \right) \right. \]

\[ + q \left( \log \left| \frac{\partial_4}{\eta^3} (2iU_1) \right|^2 - \log \left| \frac{\partial_4}{\eta^3} (4iU_1) \right|^2 + \pi \text{Re} \, U_1 \right], \]  

(C.21)

with a \( \beta \)-function coefficient,

\[ b_p = -3(p - 2), \]  

(C.22)

that can be easily extracted from the previous expression.

### C.3 Threshold corrections in the heterotic models

We consider separately the contributions from left- and right-mover oscillators in (5.22).

The left-mover contributions read

\[ \Lambda_{\text{left}} = \frac{1}{8\eta^2} \sum_{i=1}^{3} \left[ \left( \partial_\tau \left( \frac{\partial_3}{\eta} \right) \partial_3 \partial_4^2 - \partial_\tau \left( \frac{\partial_4}{\eta} \right) \partial_4 \partial_3^2 \right) (-1)^{m_i + \frac{1}{2}} \Lambda_i \left| \frac{4\eta^2}{\partial_4^2} \right|^2 \right. \]

\[ + \left( \partial_\tau \left( \frac{\partial_3}{\eta} \right) \partial_3 \partial_2^2 - \partial_\tau \left( \frac{\partial_2}{\eta} \right) \partial_4 \partial_3^2 \right) \Lambda_i^{m_i + \frac{1}{2}, m_i + \frac{1}{2}} \left| \frac{4\eta^2}{\partial_3^2} \right|^2 \right. \]

\[ - \left( \partial_\tau \left( \frac{\partial_4}{\eta} \right) \partial_2 \partial_4^2 - \partial_\tau \left( \frac{\partial_4}{\eta} \right) \partial_4 \partial_3^2 \right) (-1)^{m_i + \frac{1}{2}} \Lambda_i^{m_i + \frac{1}{2}, m_i + \frac{1}{2}} \left| \frac{4\eta^2}{\partial_3^2} \right|^2 \]  

(C.23)
Making use of the identities \((D.4) - (D.8)\), we get after some small algebra
\[
\Lambda_{\text{left}} = \frac{\pi i}{2\eta^6} \left[ (-1)^{m_i+n_i} \hat{Z}_i \overline{\vartheta}_3 \overline{\vartheta}_4^2 - \hat{Z}_i^{m_i+n_i+\frac{1}{2}} \overline{\vartheta}_2 \overline{\vartheta}_3^2 \right] \left( -1 \right)^{m_i+n_i} \hat{Z}_i^{m_i+n_i+\frac{1}{2}} \overline{\vartheta}_2 \overline{\vartheta}_4^2 \right], \quad (C.24)
\]
where the toroidal lattice sums \(\hat{Z}_i \equiv |\eta|^4 A_i\) are provided by \((5.24) - (5.25)\), after Poisson resummation in \(m_1\) and \(m_2\).

Regarding the contributions from the right-mover fermionic oscillators, we get
\[
\Lambda_{\text{right}} = \left( Q_{SO(32)}^2 - \frac{1}{4\pi \tau_2} \right) \frac{1}{2} \sum_{a,b} \frac{\overline{\vartheta}_{[a]}^{[a]} 16}{\eta^{16}} - \frac{1}{8\pi^2} \sum_{a,b} \frac{\overline{\vartheta}_{[a]}^{[a]} \overline{\vartheta}_{[a]}^{[a]} 15}{\eta^{16}} - \frac{1}{8\pi \tau_2} \sum_{a,b} \frac{\overline{\vartheta}_{[a]}^{[a]} 16}{\eta^{16}}. \quad (C.25)
\]
Making use of relations \((D.5) - (D.12)\), these terms can be rearranged in the very compact expression
\[
\Lambda_{\text{right}} = \frac{E_4(\overline{E}_4 \overline{E}_2 - \overline{E}_6)}{12\eta^{24}}, \quad (C.26)
\]
corresponding to the modular covariant derivative of \(\overline{E}_8\).

Putting all together we then arrive to the final expression for the gauge kinetic threshold corrections to the SO(32) heterotic model,
\[
\Lambda_2 = \frac{i}{4\pi} \int_F \frac{d^2 \tau}{\tau_2} \Lambda_{\text{left}} \Lambda_{\text{right}} = -\frac{1}{96} \int_F \frac{d^2 \tau}{\tau_2} \sum_{i=1}^3 \left[ (-1)^{m_i+n_i} \hat{Z}_i \overline{\vartheta}_3 \overline{\vartheta}_4^2 - \hat{Z}_i^{m_i+n_i+\frac{1}{2}} \overline{\vartheta}_2 \overline{\vartheta}_3^2 \right] - \left( -1 \right)^{m_i+n_i} \hat{Z}_i^{m_i+n_i+\frac{1}{2}} \overline{\vartheta}_2 \overline{\vartheta}_4^2 \right] \frac{E_4(\overline{E}_2 \overline{E}_4 - \overline{E}_6)}{\eta^{24}}. \quad (C.27)
\]
In the limit of large volume, \(\text{Re } T_i \gg 1\), or equivalently \(q \rightarrow 0\) and \(n_i = 0\), only degenerate orbits consisting of \(A\) matrices \((5.25)\) with zero determinant in the sector \((h, g) = (1, 0)\) contribute to the toroidal lattice sums. Following \([33]\), then we can pick an element \(A_0\) in each orbit and to integrate its contribution over the image under \(V\) of the fundamental domain, for all \(V \in SL(2)\) yielding \(A_0 V \neq A_0\). The representatives can be chosen to be,
\[
A_0 = \begin{pmatrix} 0 & j + \frac{1}{2} \\ 0 & p \end{pmatrix}, \quad (C.28)
\]
enforcing the identification
\[
(j, p) \sim (-j - 1, -p). \quad (C.29)
\]
With this representation, $A_0 V' = A_0 V''$ if and only if
\[
V' = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} V''.
\]
Therefore, the contributions are integrated over \( \{ \tau_2 > 0, |\tau_1| < \frac{1}{2} \} \), and the double covering is taking into account by summing over all \( p \) and \( j \),
\[
I_d = (\text{Re } T) \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_0^\infty \frac{d\tau_2}{\tau_2^2} \sum_{j,p} \exp \left( -\frac{\pi \text{Re } T}{\tau_2 \text{Re } U} |j + \frac{1}{2} + iU_i p|^2 \right).
\] (C.30)
This is exactly the same expression as (C.10), so the contributions of the degenerate orbits perfectly match the perturbative type I threshold corrections,
\[
I_d = \log \left| \frac{\varphi_4}{\eta^3} (2iU_i) \right|^2 - \pi \text{Re } U_i - \log([\text{Re } U_i](\text{Re } T_i)) .
\] (C.31)
Analogously, in the limit \( q \to 0 \) but \( n_i \neq 0 \) also the non-degenerate orbits in the sector \( (h,g) = (1,0) \) contribute. The representative in this class can be chosen to have the form
\[
A_0 = \begin{pmatrix} k & j + \frac{1}{2} \\ 0 & p \end{pmatrix},
\] (C.32)
with \( k > j \geq 0, p \neq 0 \). For these, \( V' \neq V'' \) implies \( A_0 V' \neq A_0 V'' \), and therefore these contributions must be integrated over the double cover of the upper half plane \( (\tau_2 > 0) \),
\[
I_{nd} = 2(\text{Re } T) \sum_{0 \leq j < k, p \neq 0} e^{2\pi T k p} \int_{-\infty}^\infty d\tau_1 \int_0^\infty \frac{d\tau_2}{\tau_2^2} (-1)^k \exp \left[ -\frac{\pi \text{Re } T}{\tau_2 \text{Re } U} |k\tau + j + \frac{1}{2} + ipU|^2 \right].
\] (C.33)
Evaluating the gaussian integral over \( \tau_1 \) and summing on \( j \), one gets
\[
I_{nd} = 2 \sum_{0 < k, p \neq 0} e^{2\pi T k p} \int_0^\infty d\tau_2 \sqrt{\frac{(\text{Re } U)(\text{Re } T)}{\tau_2^2}} (-1)^k \exp \left[ -\frac{\pi \text{Re } T}{\tau_2 \text{Re } U} (k\tau_2 + p\text{Re } U)^2 \right],
\] (C.34)
and the contribution of this sector becomes
\[
I_{nd} = \log \left| \frac{\varphi_4}{\eta^3} (2iT) \right|^2 - \pi \text{Re } T .
\] (C.35)
It corresponds to \( E1 \) instanton corrections in the type I \( SO(32) \) dual model. Indeed, expanding the \( \eta \)-function in (C.35), \( I_{nd} \) can be expressed as
\[
I_{nd} = -2 \sum_{n=1}^\infty (-1)^n \log (1 - e^{-2\pi nT}) + \text{c.c.},
\] (C.36)
which should correspond to a sum over the contributions of $E1$-instantons wrapping $n$ times the (1,1)-cycle associated to $T$, a fact that would be very interesting to verify explicitly. Notice that the dependence on $T$ perfectly agrees with the general arguments in [16] for the mirror type IIA picture.

D Some useful formulae

- Poisson summation formula:

\[
\sum \text{Exp} \left[ -\pi (m - b)^T A (m - b) \right] = \frac{1}{\sqrt{\det A}} \sum \text{Exp} \left[ -\pi n^T A^{-1} n + 2i\pi b^T n \right]. \tag{D.1}
\]

- Modular identities:

\[
\begin{align*}
\vartheta''_3 \vartheta^3_3 - \vartheta''_4 \vartheta^3_4 - \vartheta''_2 \vartheta^3_2 &= 0, \\
\vartheta''_3 \vartheta_3^2 - \vartheta''_4 \vartheta_4^2 &= -4\pi^2 \eta^6 \vartheta^3_2, \\
\vartheta_2 \vartheta_3 \vartheta_4 &= 2\eta^3, \\
\vartheta''_2 &= 4\pi i \partial_\tau \vartheta_2 = -\frac{\pi^2}{3} \vartheta_2 (E_2 + \vartheta^4_3 + \vartheta^4_4), \\
\vartheta''_3 &= 4\pi i \partial_\tau \vartheta_3 = -\frac{\pi^2}{3} \vartheta_3 (E_2 + \vartheta^4_2 - \vartheta^4_4), \\
\vartheta''_4 &= 4\pi i \partial_\tau \vartheta_4 = -\frac{\pi^2}{3} \vartheta_4 (E_2 - \vartheta^4_2 - \vartheta^4_3),
\end{align*}
\]

- Eisenstein series:

\[
\begin{align*}
E_2 &= \hat{E}_2 + \frac{3}{\pi \tau_2} = \frac{12}{i\pi} \partial_\tau \log \eta = 1 - 24q - \ldots, \\
E_4 &= \frac{1}{2} (\vartheta^8_2 + \vartheta^8_3 + \vartheta^8_4) = 1 + 240q - \ldots, \\
E_6 &= \frac{1}{2} (\vartheta^4_2 + \vartheta^4_3)(\vartheta^4_3 + \vartheta^4_4)(\vartheta^4_4 - \vartheta^4_2) = 1 - 540q - \ldots, \\
E_8 &= E_4^2 = \frac{1}{2} (\vartheta^{16}_2 + \vartheta^{16}_3 + \vartheta^{16}_3) = 1 + 480q + \ldots, \\
E_{10} &= E_4 E_6 = -\frac{1}{2} [\vartheta^{16}_2 (\vartheta^4_3 + \vartheta^4_4) + \vartheta^{16}_3 (\vartheta^4_2 - \vartheta^4_4) - \vartheta^{16}_4 (\vartheta^4_2 + \vartheta^4_3)].
\end{align*}
\]
- Series expansions:

\[
\log(1 - Q) = - \sum_{n=1}^{\infty} \frac{Q^n}{n}, \quad (D.13)
\]

\[
\log(1 + Q) = - \sum_{n=1}^{\infty} (-1)^n \frac{Q^n}{n}, \quad (D.14)
\]

\[
\log \vartheta_2 = \log 2^{1/8} + \sum_{n=1}^{\infty} \log(1 - q^n) + 2 \sum_{n=1}^{\infty} \log(1 + q^n), \quad (D.15)
\]

\[
\log \vartheta_4 = \sum_{n=1}^{\infty} \log(1 - q^n) + 2 \sum_{n=1}^{\infty} \log(1 - q^{n-\frac{1}{2}}), \quad (D.16)
\]

\[
\log \eta = \log q^{1/24} + \sum_{n=1}^{\infty} \log(1 - q^n), \quad (D.17)
\]

\[
\frac{1 + Q}{1 - Q} = 1 + \sum_{m=1}^{\infty} Q^m, \quad (D.18)
\]

\[
\frac{Q - 1}{Q + 1} = -1 - 2 \sum_{m=1}^{\infty} (-1)^m Q^m. \quad (D.19)
\]

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