Deformation Expression for Elements of Algebras (VIII)
– SU(2)-vacuum and the regular representation space–

Hideki Omori∗ Yoshiaki Maeda† Naoya Miyazaki‡
Tokyo University of Science Keio University Keio University

Akira Yoshioka §
Tokyo University of Science

Keywords: SU(2)-vacuum, Weyl algebra, Weyl’s equation, Dirac’s equation, Pseudo-vacuums, Minkowski space.
Mathematics Subject Classification(2000): Primary 53D55, Secondary 53D17, 53D10

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∗Department of Mathematics, Faculty of Sciences and Technology, Tokyo University of Science, 2641, Noda, Chiba, 278-8510, Japan, email: omori@ma.noda.tus.ac.jp
†Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1, Hiyoshi, Yokohama, 223-8522, Japan, email: maeda@math.keio.ac.jp
‡Department of Mathematics, Faculty of Economics, Keio University, 4-1-1, Hiyoshi, Yokohama, 223-8521, Japan, email: miyazaki@hc.cc.keio.ac.jp
§Department of Mathematics, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Tokyo, 102-8601, Japan, email: yoshioka@trs.kagu.tus.ac.jp
1 Introduction

Rotations in the 3-dimensional space is expressed in terms of vector analysis as a linear combination of rotations w.r.t. each axis

\[ R_t = (\tilde{e}_2 \times \tilde{e}_3)p_1 + (\tilde{e}_3 \times \tilde{e}_1)p_2 + (\tilde{e}_2 \times \tilde{e}_3)p_3, \quad p_1^2 + p_2^2 + p_3^2 = \mu^2. \]

Consider the problem “Give the equation of the conceptional rotations in \( \mathbb{R}^3 \) without using the parameter expressing individual rotations”, just as the conceptional motion of constant velocity along straight lines (Galilley motions) is expressed as \( \frac{df}{dt} = 0 \). The best answer may by given as follows:

Let \( 1, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \) be the natural basis of quaternion field \( (\mathbb{Q}, \ast) \). The answer is the quaternion valued one component equation

\[(1.1) \quad \partial_t \phi(t, x_1, x_2, x_3) = \frac{1}{2} [\tilde{e}_1 \partial_{x_1} + \tilde{e}_2 \partial_{x_2} + \tilde{e}_3 \partial_{x_3}, \phi(t, x_1, x_2, x_3)] \]

where \([ , ]\) is the commutator bracket in the quaternion field. \( (1.1) \) splits into the 3-component equation by setting \( \phi(t, x) = \phi_1(t, x)\tilde{e}_1 + \phi_2(t, x)\tilde{e}_2 + \phi_3(t, x)\tilde{e}_3 \)

\[
\partial_t \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 & -\partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & -\partial_{x_1} \\ -\partial_{x_2} & \partial_{x_1} & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}
\]

Behind this equation, we see the partial differential equation

\[ i\partial_t \phi(t, x_1, x_2, x_3) = (\tilde{e}_1 \partial_{x_1} + \tilde{e}_2 \partial_{x_2} + \tilde{e}_3 \partial_{x_3}) \ast \phi(t, x_1, x_2, x_3). \]

This is called the **Weyl equation**.

Setting \( \phi(t, x) = \phi_0(t, x) + \phi_1(t, x)\tilde{e}_1 + \phi_2(t, x)\tilde{e}_2 + \phi_3(t, x)\tilde{e}_3 \), we have

\[
\begin{align*}
    i\partial_t i\phi_0(t, x) &= i\partial_{x_1} \phi_1 + i\partial_{x_2} \phi_2 + i\partial_{x_3} \phi_3 \\
    i\partial_t \phi_1(t, x) &= \partial_{x_2} \phi_3 - \partial_{x_3} \phi_2 + \partial_{x_1} \phi_0 \\
    i\partial_t \phi_2(t, x) &= \partial_{x_3} \phi_1 - \partial_{x_1} \phi_3 + \partial_{x_2} \phi_0 \\
    i\partial_t \phi_3(t, x) &= \partial_{x_1} \phi_2 - \partial_{x_2} \phi_1 + \partial_{x_3} \phi_0
\end{align*}
\]
This is viewed as the equation of the conceptional 3-dim rotations taking the time parameter in mind. Replacing \( t' = -it \), the original equation changes into:

\[
\partial_{t'} \begin{bmatrix} 0 -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} \\ \partial_{x_1} & 0 & -\partial_{x_2} \\ -\partial_{x_2} & \partial_{x_3} & 0 \\ -\partial_{x_2} & \partial_{x_1} & 0 \end{bmatrix} \begin{bmatrix} i\phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix},
\]

these are \( O(1,3) \)-invariant w.r.t. 4-components \((\phi_0, \phi_1, \phi_2, \phi_3)\).

At a first glance, there is no parameter indicating individual rotations. However, Fourier-transform \( \hat{\phi}(t,p) = \int_{\mathbb{R}^3} e^{i\sum_{j=1}^{3} x_j \phi(t,\mathbf{x})} d\mathbf{x} \) changes the original equation into

\[
\partial_t \hat{\phi}(t,p) = (\hat{\epsilon}_1 p_1 + \hat{\epsilon}_2 p_2 + \hat{\epsilon}_3 p_3) \ast \hat{\phi}(t,p),
\]

and individual rotations are involved in \( p_1, p_2, p_3 \).

Namely, via Fourier transformation space-variables are transformed into momentum-variables. As a result individual rotations appear instead of Lorentz covariance.

The square of the Weyl equation is the (mass-less) Klein-Gordon equation:

\[
\partial_t^2 \phi = (\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2) \phi.
\]

Weyl equation is viewed as the “field equation” of relativistic mass-less particles. The massive Klein-Gordon equation is

\[
\partial_t^2 \phi = (\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2 + m^2) \phi.
\]

The reason why \( m \) is the mass term is based on Einstein’s relation \( E^2 = p^2 + m^2 c^2 \).

On the other hand in the theory of Dirac, the equation of massive \((m>0)\) relativistic particles is given by

\[
(1.2) \quad i\partial_{\xi_0} \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} \mu I_2 & D \\ D & -\mu I_2 \end{bmatrix} \begin{bmatrix} \Phi \\ \Psi \end{bmatrix}, \quad D = \frac{1}{\hbar}(\rho_1 \partial_{\xi_1} + \rho_2 \partial_{\xi_2} + \rho_3 \partial_{\xi_3})
\]

where \( \rho_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \) with Pauli matrices \( \sigma_i \) and \( \mu \) is called the mass-term. As this is considered in the Clifford algebra of 4 generators, this is viewed as an equation for “Fermions”.

However, in [11] we found Clifford algebras are contained in the transcendentally extended Weyl algebra. In contrast, it is known that the Clifford algebra of infinite generators contains Weyl algebra.

On the other hand in physics, equations written in Weyl algebra (not transcendentally extended) is often called “Bosonic equations”. Note that the Bose-Fermi symmetry is widely believed in physics. Thus, it is mathematically interesting to write the “Bosonic equation” corresponding to the (massive) Dirac equation.

In this note, we propose an equation which is covariant (not invariant) under Lorentz transformations. The fundamental solution will be constructed in the form of product integrals. But unfortunately, it is still difficult to negotiate with causality conditions in the theory of relativity, as we do not have definite idea of treating the time parameter.
Now, we have to note that the term “vacuums” used often in this note may not be the same that physicists uses. A $G$-vacuum is only a group $G$-invariant idempotent element in some extended Weyl algebra. It is still difficult to obtain $SU(2)$-vacuum. This is because we have to restrict the expression parameters to a very special class which is very easy to be broken. Much work seem to be done in simulating the $SU(2)$-vacuum, with the aim of learning more about the vacuum structure of QCD. But our $SU(2)$-vacuum may have nothing to do with these.

Readers will find many obstructions in mathematical detail, but we think these suggest many interesting subjects. In particular, throughout this series we are concerned with parameters of expressing elements of algebras. In a very beginning these parameters are thought inessential for physics. However, we found there are many phenomena that suggest such parameters must have some significance in physics. This note is only a trial to understand mathematically the significance of expression parameters.

2 Weyl algebra with hermitian structures and $*$-exponentials

First of all, we define the Weyl algebra $(W_2,*)$ as the associative algebra of 2 generators $u, v$ with the commutation relation $[u,v]=−i\hbar$. $W_2$ extends transcendentally under suitable topology so that the exponential functions of quadratic forms are treated.

An hermitian structure is defined by setting $u^*=u, v^*=v, i^*=-i, [u,v]^*=[v,u]$ on generators $u, v$ as an involutive anti-automorphism. Here $u, v$ are treated as hermite elements. Hence $2uv=uv+vu, u^2+v^2, u^2-v^2$ are hermite elements.

There is another hermite structure defined by $u^*=iv, v^*=iu, i^*=-i$. Hence $u^2, iv, i(u^2+v^2), (u^2-v^2)$ are hermite elements.

Although it is involutive only on even elements, if we define an anti-automorphism by $u^*=v, v^*=-u, i^*=-i$, then $iu=iv, i(u^2+v^2), u^2+v^2$ are hermite elements.

2.1 Product formulas and linear change of generators

To make a transcendental extension, we use a concrete product formula by setting $(u, v)=(u_1, u_2)$

$$f_K g = fe^{\frac{i\hbar}{2} \sum_{ij} \Lambda_{ij} \partial_j g}, \quad \Lambda=K+J, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

where $K=(K_{ij})$ is any $2 \times 2$ complex symmetric matrix. The algebraic structure does not depend on $K$, but the expression of elements depend on $K$. $K$ is called an expression parameter. For instance, $u^2, u*\nu$ are defined in $W_2$, but $u^2_K=u^2+2i\hbar K_{11}, u*v=uv+2i\hbar(K_{12}-1)$. We denote these by

$$u^2_K, \quad u*\nu_K=uv+2i\hbar(K_{12}-1).$$

Such a manner to express elements is redundant while we treat $*$-polynomials or $*$-exponential functions of linear forms. However, expression parameters play important roles in Jacobi’s $\theta$-functions (cf.[10]) and these play essential roles when $*$-exponential functions of quadratic forms are treated. Indeed it is a main target of this series of notes to understand the significance of expression parameters.
If \( K=0 \), (2.1) is called the **Moyal product formula**. By restricting \((u,v)\) in \( \mathbb{R}^2 \), this formula may be given by the integral form

\[
(2.2) \quad f \ast_0 g(u,v) = \text{os-} \int_{\mathbb{R}^2} f(u+\frac{\hbar}{2} s, v+\frac{\hbar}{2} t) g(u+s', v+t') e^{i(ts'-st')} ds dt ds' dt'.
\]

If \( K=K_0= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) then (2.1) is called the product formula of pseudo-differential operators (ΨDO-product formula in short). Its integral form is

\[
(2.3) \quad f \ast_{K_0} g(u,v) = \text{os-} \int_{\mathbb{R}^2} f(u, v+\hbar s) g(u+t, v) e^{-ist} ds dt.
\]

These integral formulas are convenient to treat “lower order terms”, while (2.1) is used to compute “positive order terms”. In (2.1), \((u,v)\) are viewed as complex variables.

Next, we consider the effect of a linear change of generators

\[
u_i' = \sum u_k S^k_i, \quad S \in \text{SL}(2,\mathbb{C}), \quad (u'=uS).
\]

By the help that

\[
\partial u_i = \sum S^k_i \partial u'_k,
\]

the product formula is rewritten by using new generators as

\[
(2.4) \quad f \ast_K g = \exp \left( i\hbar \sum_{i,j} (K'_{ij} - K_{ij}) \partial u_i \partial u_j \right) f \ast_{K'} \left( \exp \left( i\hbar \sum_{i,j} (S^k_i S^l_j - S^l_j S^k_i) \partial u_i \partial u_j \right) g \right).
\]

As \( SJS = J \), the algebraic structure of \((\mathbb{C}[u]; \ast_K)\) does not change. Thus the notation \( \ast_K \) is better to be replaced by \( \ast_{K'} \) where \( K'=SKS \).

For every \( K, K' \), the intertwiner is defined by

\[
I_{K'}^K(f) = \exp \left( \frac{i\hbar}{4} \sum_{i,j} (K'_{ij} - K_{ij}) \partial u_i \partial u_j \right) f \ast_{K'} \left( \exp \left( i\hbar \sum_{i,j} (S^k_i S^l_j - S^l_j S^k_i) \partial u_i \partial u_j \right) \right).
\]

This gives an isomorphism \( I_{K'}^K : (\mathbb{C}[u]; \ast_{K+J}) \rightarrow (\mathbb{C}[u]; \ast_{K'+J}) \). Namely, the following identity holds for any \( f, g \in \mathbb{C}[u] \):

\[
(2.5) \quad I_{K'}^K(f \ast_K g) = I_{K'}^K(f) \ast_{K'} I_{K'}^K(g).
\]

Intertwiners do not change the algebraic structure \( \ast \), but do change the expression of elements by the ordinary commutative structure.

Thus, a symplectic change of generators is recovered by the intertwiner \( I_{K}^{KS} \). Change of generators are viewed often as coordinate transformations, but note here that \( I_{K}^{KS} \) is something like the “square root” of symplectic coordinate transformations, and these behaves as 2-to-2 mappings on the space of \( \ast \)-exponential functions of quadratic forms. (Cf. [10], [11].)
2.2 Star-exponential functions of quadratic forms and several properties

The $K$-ordered expression of the $*$-exponential function $e_{\frac{1}{2\pi}}^{\frac{1}{gKg}}$ for $g\in SL(2, \mathbb{C})$ is given by

\[
(2.6) \quad \mathcal{E}_{K}^{(u, v)} :_{K} = \frac{1}{\sqrt{\det(\cos tI - (\sin t)gKg)}} e^{\frac{1}{2\pi} (u, v)}.
\]

where $u = (u, v)$, $[u, v] = -i\hbar$, and $K$ is any complex symmetric matrix. Note that the phase part and the inside of $\sqrt{}$ is $\pi$-periodic (not $2\pi$-periodic). It is easy to rewrite (2.6) as

\[
\mathcal{E}_{K}^{(u, v)} :_{K} = \frac{1}{\sqrt{\det(\cos tI - (\sin t)gKg)}} e^{\frac{1}{2\pi} (u, v)}.
\]

Now setting $t=\pi$ in (2.6), we have $\mathcal{E}_{K}^{(u, v)} :_{K} = 1$. For a while we assume $\det K = 1$. To manage the sign of $\sqrt{}$, we first fix the value at $t = 0$ as $\mathcal{E}_{0}^{(u, v)} :_{K} = 1 = 1$.

Now let $\mu, \nu$ be the eigenvalues of $gKg$. Then, $\mu \nu = \det gKg = 1$ and $\mu + \nu = \text{tr}(gKg)$

\[
\sqrt{\det(\cos tI - (\sin t)gKg)} = \sqrt{(\cos t - \mu \sin t)(\cos t - \nu \sin t)}
\]

\[
= \frac{1}{2} \sqrt{(1 + i\mu)(1 + i\nu)} e^{-it} \sqrt{e^{2it} - \frac{i - (-\mu)}{i + (-\mu)}} \sqrt{e^{2it} - \frac{i - (-\nu)}{i + (-\nu)}}.
\]

Note that $\left| \frac{i - (-\mu)}{i + (-\mu)} \right|$ is the ratio of the distance of $-\mu$ from $i$ and $-i$. Hence if $(-\mu)$ is in the upper half-plane, then $\left| \frac{i - (-\mu)}{i + (-\mu)} \right| < 1$ and therefore $\sqrt{e^{2it} - \frac{i - (-\mu)}{i + (-\mu)}}$ changes the sign when $t$ moves from 0 to $\pi$. (Note that the inside of $\sqrt{}$ is $\pi$-periodic.) Denote for simplicity

\[
R_{\mu} = \frac{i - (-\mu)}{i + (-\mu)}, \quad R_{\nu} = \frac{i - (-\nu)}{i + (-\nu)}.
\]

Furthermore considering $re^{2it} - R_{\mu}$ for some $0 < r < 1$, we see singular points appears $\pi$-periodically on the line in lower half plane parallel to the real axis.

As $\det(gKg) = \det K = \mu \nu = 1$, if $\mu, \nu \in \mathbb{R}$, then $\mathcal{E}_{K}^{(u, v)} :_{K}$ is not singular on $t \in \mathbb{R}$ and only one of $\sqrt{e^{2it} - \frac{i - (-\mu)}{i + (-\mu)}}$, $\sqrt{e^{2it} - \frac{i - (-\nu)}{i + (-\nu)}}$ changes the sign when $t$ moves from 0 to $\pi$. As $e^{-it}$ changes sign when $t$ moves from 0 to $\pi$, $\sqrt{\det(\cos tI - (\sin t)gKg)}$ does not change sign. It follows $\mathcal{E}_{K}^{(u, v)} :_{K} = 1$.

**Lemma 2.1** If $\text{tr}(gKg)$ is not a real number, or if $\text{tr}(gKg)$ is a real number with $(\text{tr}(gKg))^2 < 4$, then $\mu, \nu$ are not real number and $\sqrt{\det(\cos tI - (\sin t)gKg)}$ does not change sign.

Setting $t = \pi/2$ in (2.6), we have

\[
\mathcal{E}_{K}^{(u, v)} :_{K} = \frac{1}{\sqrt{\det K}} e^{-\frac{1}{2\pi} (u, v)}.
\]
which looks independent of \( g \in SL(2\mathbb{C}) \). But recall here that the formula of changing generators give
\[
\varepsilon_{K}^{t} = \varepsilon_{K}^{t}(ug,ug).
\]
Change of expression parameters sometimes give a change of generators. The above observation relating to Lemma 2.1 shows this must depend on \( g \) discontinuously.

Denoting \( \varepsilon_{00}^{t} = \varepsilon_{00}^{t}(ug,ug) \), we call this the polar element. The exponential law gives in general
\[
\varepsilon_{00}^{2} = \pm 1
\]
depending on \( K \).

For the use in the later section, we give the formula of \(*\)-exponential function of degenerate quadratic from
\[
(2.7)
\]
which is decreasing in \( t \) of the order \( \sqrt{|t|}^{-1/2} \).

### 2.3 Notes from linear algebra

Now setting \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{C}) \), we have
\[
\langle ug,ug \rangle = (u,v) \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = (a^2+b^2)u^2 + (c^2+d^2)v^2 + 2(ac+bd)uv.
\]

The space
\[
(2.8)
\]
is the space \( D_{-1} = \{XY-Z^2=1\} \) in \( \mathbb{C}^3 \) (The space of all quadratic forms of discriminant \(-1\)). Replacing \( X=x+y \), \( Y=x-y \), \( Z=z \), we have \( x^2-y^2-z^2=1 \). \( D_{-1} \) may be understood as the complexified 2-sphere \( x^2+(iy)^2+(iz)^2=1 \).

To get the real sphere, we set as follows together with \( \alpha^2 + \beta^2 + \gamma^2 = 1 \)
\[
(2.9)
\]
We denote this set by \( \tilde{S}^2 \). This is the space of all symmetric elements in \( SU(2) \). It is easy to see that \( \tilde{S}^2 = SU(2)/SO(2) \). We denote \( S' = \{g; g'g \in \tilde{S}^2\} \), e.g.
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \sqrt{1-\gamma^2} e^{i\theta/2} \cosh \xi & \sqrt{1-\gamma^2} i e^{i\theta/2} \sinh \xi \\ \sqrt{1-\gamma^2} i e^{-i\theta/2} \sinh \eta & \sqrt{1-\gamma^2} e^{-i\theta/2} \cosh \eta \end{bmatrix},
\]
where \( \xi, \eta, \theta \in \mathbb{R} \), If \( \gamma = \pm 1 \), we set
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.
\]
Lemma 2.2 The eigenvalue of $T \in SU(2)$ is real if and only if $T = \pm I$. If $K=I$, and $y \in S^2$, then except the case $y \in SU(2)$, $\sqrt{\det \cos T-(\sin t)yKg}$ does not change sign.

Proof The first one is wellknown. Next one is the case $\text{tr}(yKg)$ is real and $2\alpha$ in (2.9). As $\alpha^2+\beta^2+\gamma^2=1$, $\text{tr}(yKg)^2 \leq 4$, and the equality takes place only by $\alpha=1$, $\beta=\gamma=0$.

Suppose $g'y$ be a complex matrix $g'y=\left[\begin{array}{cc} -i\gamma & i\gamma \\ i\gamma & -i\gamma \end{array}\right]$, $\xi^2+\eta^2+\zeta^2=1$, $\xi, \eta, \zeta \in \mathbb{C}$ and suppose $\left[\begin{array}{c} \alpha+i\beta \\ \alpha-i\beta \end{array}\right]$ be in $SU(2)/SO(2)$. We take $K=R\left[\begin{array}{cc} \alpha+i\beta & i\gamma \\ i\gamma & -\alpha-i\beta \end{array}\right]$ as an expression parameter. The multiplier $R=\exp \theta$ will be used for singular points to avoid real line.

Then, a direct computation gives by setting $\xi=(\xi, \eta, \zeta), \alpha=(\alpha, \beta, \gamma)$ that

$$\text{tr} \left[ \begin{array}{cc} \xi -i\eta & -i\zeta \\ -i\zeta & \xi +i\eta \end{array} \right] R \left[\begin{array}{cc} \alpha+i\beta & i\gamma \\ i\gamma & -\alpha-i\beta \end{array}\right] \right] = 2R\langle \alpha, \xi \rangle,$$

$$\det \left( \cos tI-(\sin t)R \left[\begin{array}{cc} \xi -i\eta & -i\zeta \\ -i\zeta & \xi +i\eta \end{array} \right] \right] = (\cos t-R\langle \alpha, \xi \rangle \sin t)^2 + \sin^2 tR^2 \langle \alpha \times \xi, \alpha \times \xi \rangle$$

where $\langle \alpha \times \xi, \alpha \times \xi \rangle=\langle \xi-\eta \alpha \rangle^2+\langle \eta-\xi \alpha \rangle^2+\langle \xi-\zeta \alpha \rangle^2$. Split $\xi=x+ix'$ by the real and the imaginary parts. As $\langle \xi, \xi \rangle=1$ by the condition $\det g'y=1$, we see $\langle x, x' \rangle=0, \|x\|^2-\|x'\|^2=1$ and

$$\langle \alpha, \xi \rangle = \langle \alpha, x \rangle + i\langle \alpha, x' \rangle, \quad \alpha \times \xi = \alpha \times x + i\alpha \times x'.$$

By setting $t=\frac{\pi}{2}$, the determinant gives $R^2 \langle \alpha, \xi \rangle^2+R^2 \langle \alpha \times \xi, \alpha \times \xi \rangle=R^2$.

It follows

$$\langle \alpha, x \rangle \langle \alpha, x' \rangle + \langle \alpha \times x, \alpha \times x' \rangle = 0, \quad \langle \alpha, x \rangle^2 - \langle \alpha, x' \rangle^2 + \langle \alpha \times x, \alpha \times x \rangle - \langle \alpha \times x', \alpha \times x' \rangle = 1$$

Hence if $x'=0$, all terms are nonnegative and $(\cos t-\langle \alpha, x \rangle \sin t)^2 + \sin^2 t(\alpha \times x, \alpha \times x)$ has a double multiple root only at

$$(\cos t-\sin t(\alpha, x), x) = 0, \quad x = \pm \alpha, \quad t = \frac{\pi}{4}, \frac{3\pi}{4}.$$ 

But these are in fact the same point as $\cos t-\sin t(\alpha, x)$ is $\pi$-periodic. As this does not depend on $R$, it is impossible to move the singular point away from the real line by selecting suitable $R$.

If $x \neq \pm \alpha$ and $x'=0$, then

$$\sqrt{(\cos t-\langle \alpha, x \rangle \sin t)^2 + \sin^2 t(\alpha \times x)^2}$$

does not change sign when $t$ moves from 0 to $\pi$. (Note this is $\pi$-periodic.)

If $x' \neq 0$, then choosing $\alpha=x$, we see $\langle \alpha, x' \rangle=0$ and

$$\langle \alpha, x \rangle^2 = 1 + \|\alpha \times x'\|^2.$$

It follows $(\cos t-\langle \alpha, x \rangle \sin t)^2 + \sin^2 t(\alpha \times x)^2 = 0$ has two different simple real roots.
Thus, taking \( K = r \begin{bmatrix} \alpha + i\beta & i\gamma \\ i\gamma & \alpha - i\beta \end{bmatrix} \), \( r > 1 \), or \( r < 1 \), as an expression parameter, we see there is 
\[
\begin{bmatrix} \alpha + i\beta & i\gamma \\ i\gamma & \alpha - i\beta \end{bmatrix} \in SU(2)
\] such that
\[
\sqrt{\det \left( \cos tI - (\sin t) \begin{bmatrix} \xi - i\eta & -i\zeta \\ -i\zeta & \xi + i\eta \end{bmatrix} r \begin{bmatrix} \alpha + i\beta & i\gamma \\ i\gamma & \alpha - i\beta \end{bmatrix} \right)}
\]
changes sign when \( t \) moves from 0 to \( \pi \). That is, this is alternating \( \pi \)-periodic. Furthermore, taking a suitable \( e^{i\theta} \) one can make \( \mu, \nu \) pure imaginary numbers. Hence,
\[
\sqrt{\det \left( \cos tI - (\sin t) e^{i\theta} \begin{bmatrix} \xi - i\eta & -i\zeta \\ -i\zeta & \xi + i\eta \end{bmatrix} r \begin{bmatrix} \alpha + i\beta & i\gamma \\ i\gamma & \alpha - i\beta \end{bmatrix} \right)}
\]
does not change sign when \( t \) moves from 0 to \( \pi \). That is, this is \( \pi \)-periodic.

**Remark 1.** By switching \( g_t \) and \( K \), we regard 
\[
\begin{bmatrix} \xi - i\eta & -i\zeta \\ -i\zeta & \xi + i\eta \end{bmatrix}
\] as an expression parameter and we regard 
\[
\begin{bmatrix} \alpha + i\beta & i\gamma \\ i\gamma & \alpha - i\beta \end{bmatrix} \]
as \( g_t^2 \). Fix 
\[
\begin{bmatrix} \xi - i\eta & -i\zeta \\ -i\zeta & \xi + i\eta \end{bmatrix}
\]
so that \( x' \neq 0 \). Then \( \langle x, x \rangle - \langle x', x' \rangle = 1 \) gives \( \langle x, x \rangle > 1 \) and there is \( \alpha \) such that \( \langle \alpha, \xi \rangle = \langle \alpha, x \rangle > 1 \). This implies if we multiply \( r \neq 0 \) to the expression parameter 
\[
K = r \begin{bmatrix} \xi - i\eta & -i\zeta \\ -i\zeta & \xi + i\eta \end{bmatrix}
\]
then there is \( g \in S' \) such that \( \sqrt{\det(\cos tI - (\sin t)^2 gKg)} \) changes sign. Note that expression parameters are not necessarily \( \det K = 1 \).

Now suppose \( g'g \) satisfies \( \langle x', \alpha \rangle = 0 \). This is the case where \( \text{tr.}(g'Kg) \) is a real number, but in what follows we restrict \( g \in S' \) i.e. \( g'g \in \tilde{S}^2 \), and investigate where eigenvalues of \( g'Kg \) are real numbers. In such a restricted case, we see that \( \xi = x \) and \( \langle x, x \rangle = 1 \). Thus,
\[
\text{tr.}(\begin{bmatrix} \xi - i\eta & -i\zeta \\ -i\zeta & \xi + i\eta \end{bmatrix}) = 2 \langle \xi, \alpha \rangle \leq 2
\]
The equality holds only if \( \xi = \pm \alpha \) and \( g'gK = \pm I \).

Thus we have

**Theorem 2.1** For an arbitrarily fixed \( K \) in \( \tilde{S}^2 \), \( \sqrt{\det(\cos tI - (\sin t)^2 gKg)} \) does not change sign for any \( g \in S' \) except the case \( g'gK = \pm I \).

This theorem shows that excluding only one point \( g_K \) in \( S' \) such that \( g'g = \pm K^{-1} \), the \(*\)-exponential function 
\[
\left( \frac{1}{2\pi} \log(u, u) \right)_K \]
is \( 2\pi \)-periodic and has no singular point on \( \mathbb{R} \).

At a first glance it looks we have two exceptional points. In fact these are the same point, because the same singular point is expressed by opposite parameter \( -t \) on the interval \( (0, 2\pi) \).

**Note** Proposition 3.2 in the previous note [10] is not correct where the singular point \( g_K \) is not cared. However we can apply the result in [11].

Thus, we have
Proposition 2.1 For every \( g \in S' \), \( g \neq g_K \): \( \varepsilon_{\frac{1}{2}\pi\langle u_g,u_g \rangle}_{\cdot K} \) is 2\( \pi \)-periodic, and the singular points distributed 2\( \pi \)-periodically on two lines sitting in both upper and lower half planes. The real line is between these. At the point \( g = g_K \), \( \varepsilon_{\frac{1}{2}\pi\langle u_g,u_g \rangle}_{\cdot K} \) has a not branching singular point at \( t = \pi/2 \), but alternating 2\( \pi \)-periodic.

Definition 2.1 An expression parameter \( K \) is called a nice expression parameter if all \( g \in S' \) except only one \( \varepsilon_{\frac{1}{2}\pi\langle u_g,u_g \rangle}_{\cdot K} \) is 2\( \pi \)-periodic, and the two lines of singular points are sitting in both upper and lower half planes.

Proposition 2.1 shows that any \( K \in \tilde{S}^2 \) is a nice expression parameter, but the above Remark 1 shows that any other expression parameters are not nice expression parameters. Note that \( \tilde{S}^2 \) is only 2-dimensional, while complex symmetric matrices of determinant 1 is 4-dimensional. This is the reason why we said \( SU(2) \)-vacuum is easy to be broken.

Recall now \( \tilde{S}^2 = SU(2)/SO(2) \). Hence \( SU(2) \) may be viewed as the collection of one parameter subgroups \( S_{\alpha} \) parameterized by \( \alpha \in \tilde{S}^2 \). (This is not the Hopf fibration by itself.)

For any expression parameter \( K \) chosen in \( \tilde{S}^2 \), the set
\[
\{ \varepsilon_{\frac{1}{2}\pi\langle u_g,u_g \rangle}_{\cdot K}, \quad t \in [0, \pi], \quad g \in S' \}
\]
covers the group \( SU(2) \) except only one 1-parameter subgroup \( S_{g_K} = \varepsilon_{\frac{1}{2}\pi\langle u_g,u_g \rangle}_{\cdot K}; t \in [0, 2\pi] \). Denote by \( \infty \) the singular point \( \varepsilon_{\frac{1}{2}\pi\langle u_g,u_g \rangle}_{\cdot K}, \quad g_K \neq g_K \in \tilde{S}^2 \).

We identify \( \tilde{S}^2 \setminus \{ \infty \} \) with the complex plane \( \mathbb{C} \) by stereographic projection. \( SU(2) \backslash S_{g_K} \) is identified \( S^1 \times \mathbb{C} \), which is naturally embedded in \( \mathbb{C}^2 \). By these observation, we see

Proposition 2.2 In any \( K \)-ordered expression such that \( K \in \tilde{S}^2 \), \( \varepsilon_{\frac{1}{2}\pi\langle u_g,u_g \rangle}_{\cdot K} \) is holomorphic w.r.t. \( g \in \tilde{S}^2 \setminus \{ \infty \} \).

Note that \( \gamma = (\cos t, \sin t(x, y, z)) \) corresponds to the element
\[
\gamma_x = \varepsilon_{\frac{1}{2}\pi\left(\frac{1}{2}(x^2+y^2)+ixy\left(\frac{1}{2}(x^2-y^2)+2iyz\right)\right)} \in SU(2)
\]

On the other hand, by viewing \( \tilde{S}^2 \) as the Riemann sphere the group \( SL(2, \mathbb{C})/\{ \pm 1 \} \) acts transitively on \( \tilde{S}^2 \) as Möbius transformations. This fact supports also Proposition 2.2.

2.3.1 Quaternion group

Fix arbitrarily \( K \in \tilde{S}^2 \) such that
\[
\varepsilon_{\frac{1}{2}\pi\langle u_g,u_g \rangle}_{\cdot K}, \quad \varepsilon_{\frac{1}{2}\pi\langle u^2+v^2 \rangle}_{\cdot K}, \quad \varepsilon_{\frac{1}{2}\pi\langle u^2-v^2 \rangle}_{\cdot K}
\]
are welldefined. Then, at \( t = \pi \) these give the same element \( \varepsilon_{00} \), called the polar element. We have seen in [12] that square roots of polar element \( \varepsilon_{00} \):
\[
e_1 = i\varepsilon_{\frac{1}{2}\pi\langle u^2+v^2 \rangle}, \quad e_2 = i\varepsilon_{\frac{1}{2}\pi\langle u^2+v^2 \rangle}, \quad e_3 = i\varepsilon_{\frac{1}{2}\pi\langle u^2-v^2 \rangle}
\]
form something like a double cover of the quaternion group. That is, in a nice expression parameter
\( K \), it holds
\[
e_i * e_j = \varepsilon_{00} * e_j * e_i.
\]
Note that \( \frac{1}{2}(1 \pm \varepsilon_{00}) \) are idempotent elements and
\[
1 = \frac{1}{2}(1 - \varepsilon_{00}) + \frac{1}{2}(1 + \varepsilon_{00}), \quad \frac{1}{2}(1 - \varepsilon_{00}) * \frac{1}{2}(1 + \varepsilon_{00}) = 0.
\]
Hence using the projection \( \pi = \frac{1}{2}(1 - \varepsilon_{00}) \), where \( \pi(\varepsilon_{00}) \) is treated as \(-1\), we have the equation same to (1.1)
\[
\partial_t \pi \phi(t, x) = \frac{1}{2} \left[ \pi e_1 \partial_x + \pi e_2 \partial_y + \pi e_3 \partial_z, \pi \phi(t, x) \right].
\]
On the other hand, quadratic forms on the phase part
\[
le_1 = \frac{1}{i\hbar} u \cdot v, \quad le_2 = \frac{1}{2\hbar} (u^2 + v^2), \quad le_3 = \frac{1}{2i\hbar} (u^2 - v^2)
\]
satisfies in any ordered expression the same commutation relations as \( \pi(e_1), \pi(e_2), \pi(e_3) \). That is
\[
[le_1, le_2] = 2ile_3, \quad [le_2, le_3] = 2ile_1, \quad [le_3, le_1] = 2ile_2.
\]
Thus, the equation
\[
i\partial_t \phi_t = \frac{1}{2\hbar} \left[ (le_1) \partial_{\xi_1} + (le_2) \partial_{\xi_2} + (le_3) \partial_{\xi_3}, \phi_t \right]
\]
express the conceptional 3-dimensional rotations mentioned in the introduction. Now under the thought of the “second quantization” we have to treat the equation behind this:
\[
i\partial_t \phi_t = \frac{1}{\hbar} \left( (le_1) \partial_{\xi_1} + (le_2) \partial_{\xi_2} + (le_3) \partial_{\xi_3} \right) * \phi_t.
\]
This may be viewed as the counter part of Weyl equation. Taking the Fourier transform, this is rewritten as
\[
(2.11) \quad \partial_t \phi_t = \frac{1}{i\hbar} \left( (le_1) \alpha + (le_2) \beta + (le_3) \gamma \right) * \phi_t.
\]
In the next section it will be shown this is covariant under Lorentz transformations.

On the other hand, although its square does not become Klein-Gordon equation without taking 2×2-matrix representation, we regard
\[
(2.12) \quad \partial^2_{t} \phi_t = \frac{1}{\hbar^2} \left( \left( (le_1)p_1 + (le_2)p_2 + (le_3)p_3 \right)^2 + m^2 \right) * \phi_t
\]
as the counter part of Klein-Gordon equation, where \( p_1, p_2, p_3 \) are regarded as parameters. In the later section, it will be shown that one can take its square root as a differential equation of infinite components of order one without using Clifford algebra.
3 Action of Lorentz group to the space of quadratic forms

By taking $X, Y$ in $(2, \mathbb{R})$ independent pure imaginary numbers,

$$\begin{bmatrix} i\beta & \rho+iy \\\n-\rho+iy & i\beta' \end{bmatrix}, \quad \beta, \beta', \gamma, \rho \in \mathbb{R}$$

is the space $\mathfrak{sh}(2)$ of all skew-hermitian matrices. Note that

$$i\mathfrak{h}(2) = \mathfrak{sh}(2), \quad \mathfrak{sl}(2, \mathbb{C}) = \mathfrak{h}_0(2) + \mathfrak{su}(2)$$

where $\mathfrak{h}_0(2)$ is the space of all traceless hermitian matrices.

For $x_0, x_1, x_2, x_3 \in \mathbb{R}$, the space of all hermite matrices

$$\mathfrak{h}(2) = \left\{ \begin{bmatrix} x_0+x_3 & x_1+ix_2 \\ x_1-ix_2 & x_0-x_3 \end{bmatrix}; x_i \in \mathbb{R} \right\} \quad \text{with, } \det X = x_0^2-(x_1^2+x_2^2+x_3^2)$$

is the Minkowski space with Minkowski-metrix $\det X$, on which $\text{SL}(2, \mathbb{C})$ acts as $X \rightarrow AXA^*$, $A \in \text{SL}(2, \mathbb{C})$. Strictly speaking, the Minkowski space is the 4-dimensional affine space with metric tensor of $(1,3)$-type. The expression above is one of coordinate expressions.

Note that Lorentz group $SO(1,3)$ is isomorphic to $\text{SL}(2, \mathbb{C})/\{\pm 1\}$, and also that $\mathfrak{sl}(2, \mathbb{C})J = Q(2)$ space of quadratic forms. Hence we see

$$\mathfrak{su}_1(2)J = SU(2)/SO(2) = S^2, \quad \mathfrak{su}(2)J = \mathbb{R}S^2 = \mathbb{R}^3,$$

where $\mathfrak{su}_1(2)$ is the space of all skew-hermitian matrices with determinant 1. Although $iI$ is not traceless, consider the space

$$\mathfrak{sh}(2)J = (\mathbb{R}iI \oplus \mathfrak{su}(2))J$$

and we view this as

$$(u, v)(\mathbb{R}iI \oplus \mathfrak{su}(2))J \begin{bmatrix} u \\ v \end{bmatrix}$$

where $(u, v)iJ \begin{bmatrix} u \\ v \end{bmatrix}$ is $i(-u^*v+v^*u) = -\hbar$. By this way the term of non-vanishing trace may be changed into a term $\rho I$, and hence Lorentz group acts on the space $(\mathbb{R}iI \oplus \mathfrak{su}(2))J$ by $X \rightarrow AX\bar{A}^{-1} = AXA^*J$.

Now, note that Lorentz group $SO(1, 3)$ is generated by $SU(2)$ and $\text{diag}(\lambda, \lambda^{-1}), \lambda \neq 0$. Note that

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a & \lambda^2b \\ \lambda^{-2}b & c \end{bmatrix}.$$  

Using these, one can write down the Lorentz action concretely.

Note first that $i\mathfrak{h}(2) = \mathbb{R}iI \oplus \mathfrak{su}(2)$. We have

$$A(\mathbb{R}iI \oplus \mathfrak{su}(2))J\bar{A^{-1}} = \mathbb{R}iAA^*J \oplus Asu(2)A^*J.$$  

If $A \in SU(2)$ then $A(\mathbb{R}iI \oplus \mathfrak{su}(2))J\bar{A^{-1}} = \mathbb{R}iI \oplus \mathfrak{su}(2))J$. 

If \( A = \text{diag}(\lambda, \lambda^{-1}) \), then decomposing

\[
\begin{align*}
iAA^*J &= \frac{1}{2} \text{tr}.AA^*iJ + (iAA^*J - \frac{1}{2} \text{tr}.AA^*iJ), \\
\text{Asu}(2)A^* &= \frac{1}{2} \text{tr}.(\text{Asu}(2))A^*J + \text{Asu}(2))A^* - \frac{1}{2} \text{tr}.(\text{Asu}(2))A^*
\end{align*}
\]
and involving \( (iAA^*J - \frac{1}{2} \text{tr}.AA^*iJ) \) to the traceless term and involving \( \text{diag}(\text{Asu}(2))A^* - \frac{1}{2} \text{tr}.(\text{Asu}(2))A^* \) term to the constant term by using the relation \( u*v - v*u = -i\hbar \), these are rewritten in the same form. As Lorentz group is generated by \( SU(2), \text{diag}(a, a^{-1}), a > 0 \), this gives an action of Lorentz group.

### 3.1 Lorentz covariance of the counterpart of Weyl’s equation

Now joining a new variable \( \rho \) corresponding to the diagonal part, we set

\[
h(t, \alpha, \beta, \gamma, \rho) = e^{\frac{i}{\hbar}(u_g, u_g) + \rho} = e^{\frac{i}{\hbar}(u_g, u_g) + \rho}.
\]

This satisfies the equation

\[
\partial_t h(t, \alpha, \beta, \gamma, \rho) = \frac{1}{i\hbar}(u_g, u_g) + \rho \star h(t, \alpha, \beta, \gamma, \rho) = \frac{1}{i\hbar}(\rho + \alpha(\lambda e_2 + \beta(\lambda e_3) + \gamma(\lambda e_1)) \star h(t, \alpha, \beta, \gamma, \rho).
\]

This means that \( \hat{h}(t, \alpha, \beta, \gamma, \rho) = \text{Ad}(h(t, \alpha, \beta, \gamma, \rho))(h(0, \alpha, \beta, \gamma, \rho)) \) is a solution of the equation

\[
\partial_t \hat{h}(t, \alpha, \beta, \gamma, \rho) = \frac{1}{i\hbar}(\rho + \alpha(\lambda e_2 + \beta(\lambda e_3) + \gamma(\lambda e_1)) \star \hat{h}(t, \alpha, \beta, \gamma, \rho).
\]

Note that if we regard \( t \rho \) as an independent variable, then as \( \partial_{t \rho} e^{\frac{i}{\hbar}t \rho} = \frac{1}{i\hbar} e^{\frac{i}{\hbar}t \rho} \), the equation (3.1) is written as

\[
i(\partial_t - \partial_{t \rho})h(t, \alpha, \beta, \gamma, \rho) = \frac{1}{i\hbar}(\alpha(\lambda e_2 + \beta(\lambda e_3) + \gamma(\lambda e_1)) \star h(t, \alpha, \beta, \gamma, \rho).
\]

In precise, viewing \((t, \rho)\) as coordinate variables we take a singular coordinate change \((\xi_0, z) = (t, t \rho)\). (i.e. \( t = \xi_0, \rho = z / \xi_0. \) Then

\[
\begin{align*}
\partial_{\xi_0} &= \partial_t - \frac{\rho}{\xi_0} \partial_{\rho} = \partial_t - \partial_{t \rho} \\
\partial_z &= \rho \partial_{\rho}
\end{align*}
\]

Constant term which appear by an adjoint transformation is involved in the (universal) time parameter \( \xi_0 \). Hence we have

**Proposition 3.1** The counterpart of the Weyl equation

\[
\partial_{\xi_0} \phi = \frac{1}{i\hbar}(\lambda e_1 + \lambda e_2 + \lambda e_3) \gamma \star \phi.
\]

is covariant under \( SL(2, \mathbb{C})/\{\pm 1\} \).

**Remark 2** In the later section, we have to use singular coordinate transformations such as

\[
\partial_{t b} e^{\frac{i}{\hbar}(t b(u, v, a))} = \frac{1}{i\hbar} e^{\frac{i}{\hbar}(t b(u, v, a))}
\]

for every fixed \((u, v) \in \mathbb{R}^2\) to obtain the Lorentz covariance of the counterpart of the massive Dirac’s equation. The problem is that this procedure changes the Minkowski structure.
### 3.2 Joining massive terms

As Weyl equation may be viewed as a 2-component equation, the massive Dirac equation is written as a 4-component equation. The reason is one has to use the identity

\[
\begin{bmatrix}
\mu & D
\end{bmatrix} = \begin{bmatrix}
D^2 + \mu^2 & 0 \\
0 & D^2 + \mu^2
\end{bmatrix}
\]

where \( D \) may be an arbitrary operator. Hence, if 2-component equations are permitted, then one can put a mass term to (3.2). Namely setting \( Q = \frac{1}{\hbar} (le_1 \alpha + (le_2) \beta + (le_3) \gamma) \), we have an equation

\[
\partial_\xi_0 \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} \mu Q \\ -\mu \end{bmatrix} \begin{bmatrix} \Phi \\ \Psi \end{bmatrix},
\]

which gives

\[
\partial^2_\xi_0 \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} Q^2 + \mu^2 \\ 0 \\ 0 \\ Q^2 + \mu^2 \end{bmatrix} \begin{bmatrix} \Phi \\ \Psi \end{bmatrix}.
\]

This may be regarded as the equation corresponding to (2.12). However when we treat algebra valued equations, this equation is written formally as one component equation (cf. the next section).

#### 3.2.1 Making matrix algebra \( M(2) \)

Recall the polar element \( \varepsilon_{00} \) is defined by setting \( \pi/2 \) in (2.6). As a nice expression parameter is used here we see:\n
\[
\varepsilon_{00}^2 = 1, \quad \frac{1}{2} (1 \pm \varepsilon_{00}) = \text{idempotent elements such that}
\]

\[
1 = \frac{1}{2} (1 + \varepsilon_{00}) + \frac{1}{2} (1 - \varepsilon_{00}), \quad \frac{1}{2} (1 + \varepsilon_{00}) \ast \frac{1}{2} (1 - \varepsilon_{00}) = 0,
\]

the bumping identity used often in previous notes (cf. [11] for instance) gives

**Proposition 3.2** In generic ordered expression, \( \varepsilon_{00} \) anti-commutes with generators \( u, v \). Hence \( \varepsilon_{00} \) commutes with every even element.

In what follows, we use half-inverses (cf. [13]) defined as follows:

\[
u^* = v \ast (u \ast v)^{-1}, \quad (u \ast v)^{-1} = - \frac{1}{i\hbar} \int_0^\infty e^{s \ast u \ast v} ds.
\]

These have the properties

\[
u \ast u^* = 1, \quad u^* \ast u = 1 - \overline{\nu}_{00}, \quad \overline{\nu}_{00} \ast \overline{\nu}_{00} = \overline{\nu}_{00}, \quad u \ast \overline{\nu}_{00} = 0 = \overline{\nu}_{00} \ast u^*,
\]

where \( \overline{\nu}_{00} = \lim_{s \to \infty} \frac{1}{2} e^{s \ast u \ast v} u \). We see that \( (u^*)^k \ast \overline{\nu}_{00} \ast u^\ell \) are \( (k, \ell) \)-matrix element.

Moreover as \( (u \ast v)^{-1} \) is an even element, we see

\[
\varepsilon_{00} \ast u = - u \ast \varepsilon_{00}, \quad \varepsilon_{00} \ast u^* = - u^* \ast \varepsilon_{00}.
\]

We now define

\[
\phi = \frac{1}{2} (1 + \varepsilon_{00}) u = (u \frac{1}{2} (1 - \varepsilon_{00})), \quad \psi = \frac{1}{2} (1 - \varepsilon_{00}) u^* = u^* \frac{1}{2} (1 + \varepsilon_{00}).
\]
Next formulas are easy to see

\[
\phi^2 = 0 = \psi^2, \quad \phi*\psi + \psi*\phi = 1.
\]

As \( \phi, \psi \) commute with every even elements, one can rewrite (3.4) as a one component equation

\[
i\partial_t f_t = ((\phi*\psi*\phi)\mu + (\phi + \psi)\frac{1}{i\hbar}(le_1)e + (le_2)\beta + (le_3)\gamma)\hat{f}_t.
\]

The solution is obtained by computing

\[
e^{it((\phi*\psi*\phi)\mu + (\phi + \psi)\hat{Q})}.
\]

Taking Fourier transform, we have

\[
\partial_t \hat{f}_t = ((\phi*\psi*\phi)\mu = (\phi + \psi)\hat{Q})e^{\frac{i}{\hbar}(mt + \mu \tau)\hat{Q}}d\tau.
\]

This may be viewed as the counter part of the Dirac equation.

### 3.2.2 Symmetry by Fourier transform

Note that (1.2) may be written as

\[
\begin{bmatrix}
i\partial_t - \mu & 0 \\
0 & i\partial_t + \mu
\end{bmatrix}
\begin{bmatrix}
\Phi_t(\alpha, \beta, \gamma) \\
\Psi_t(\alpha, \beta, \gamma)
\end{bmatrix}
= \begin{bmatrix}
0 & Q \\
Q & 0
\end{bmatrix}
\begin{bmatrix}
\Phi_t(\alpha, \beta, \gamma) \\
\Psi_t(\alpha, \beta, \gamma)
\end{bmatrix}.
\]

Dirac equation is given by taking its Fourier transform w.r.t. \((\alpha, \beta, \gamma)\). Now, consider the Fourier transform w.r.t. \((t, \mu)\). Take the standard Fourier transform \(f(t, \mu) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{f}(m, \tau)e^{-\frac{i}{\hbar}(mt + \mu \tau)}dmd\tau\). Then we have

\[
i\partial_t f(t, \mu) = \int_{\mathbb{R}^2} (-m)\hat{f}(m, \tau)e^{-\frac{i}{\hbar}(mt + \mu \tau)}dmd\tau, \quad \mu f(t, \mu) = \int_{\mathbb{R}^2} i\partial_t \hat{f}(m, \tau)e^{-\frac{i}{\hbar}(mt + \mu \tau)}dmd\tau.
\]

Hence the above equation becomes

\[
i\partial_t \begin{bmatrix}
\phi(\tau, m, \alpha, \beta, \gamma) \\
\psi(\tau, m, \alpha, \beta, \gamma)
\end{bmatrix} = \begin{bmatrix}
m & -Q \\
Q & -m
\end{bmatrix}
\begin{bmatrix}
\phi(\tau, m, \alpha, \beta, \gamma) \\
\psi(\tau, m, \alpha, \beta, \gamma)
\end{bmatrix}.
\]

This is acceptable, recalling that the time and the energy are canonical conjugate variables in general mechanics.

### 4 The square root of the counterpart of Klein-Gordon equation

Recall the counter part of the Klein-Gordon equation (2.12) is written in the form

\[
-\partial_t^2 \phi_t = \frac{1}{\hbar^2}(Q^2 \alpha(u, v) + m^2)\phi_t, \quad Q_\alpha(u, v) = \langle ug, ug \rangle, \quad g'g = \alpha \in S^2.
\]

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As \( Q_\alpha(u, v) \) is given by \( g'y \in \tilde{S}^2 \), this is viewed as a mapping from \( \tilde{S}^2 \) into the Weyl algebra \( W_2 \).

We want to consider the square root of the equation

\[
i\partial_t \phi_t = \pm \frac{1}{\hbar} \sqrt{(Q_\alpha(u, v))^2 + m^2*\phi_t}, \quad \sqrt{Q_\alpha^2 + m^2} = \pm \sqrt{1 + m^2Q_\alpha^{-2}}*Q_\alpha.
\]

We want to change this into an infinite component differential equation of order one written in the form

\[
i\partial_t \Phi_t(u, v) = M(u, v)*(le_1 \partial_{\xi_1} + le_2 \partial_{\xi_2} + le_3 \partial_{\xi_3})*\Phi_t(u, v).
\]

The point of this trick is that we regard \( \sqrt{1 + m^2Q_\alpha^{-2}}*k \) as a function of \( \alpha \) on the space \( \tilde{S}^2 \) and we use expansions of spherical functions.

To this aim, we recall first the formula (2.10) in a nice expression parameter \( K \).

\[
\varepsilon^{\frac{i}{\hbar}(u_g, u_g)}_{+} = \sqrt{\frac{1}{\det(\cos tI - (\sin t)i\hbar gKg)}} e^{\frac{i}{\hbar}(u_g, u_g)}
\]

and note

\[
g \frac{\sin t}{\cos tI - \sin t} \frac{\hbar g}{i\hbar Kg} g = \frac{1}{\cos t - (\sin t)\hbar Kg} \hbar g 1_{\hbar} g.
\]

Recall in a nice expression parameter \( K \) singular points w.r.t. \( t \) appear in very restricted way:

1. If \( t \) is not real, then singular points are double branched simple singularity distributed \( \pi \)-periodically. This will be used in the next section for the case \( t \) is the pure imaginary number.
2. If \( t \) is real, such a singular point appears only at \( t=\pi/2 \). Hence, this case does not appear if \( t \) is restricted in the pure imaginary number.

### 4.1 Computation of \( \sqrt{1 + m^2Q_\alpha^{-2}} \)

First we note the integral

\[
\int_{-\infty}^{0} e^{i\frac{1}{\hbar}(u_g, u_g)}_{+} dt
\]

in generic \( K \)-ordered expression converges to give a \( * \)-inverse \( \left( \frac{1}{\hbar}(u_g, u_g) \right)_{+}^{-1} \). If there is a singular point on \( (-\infty, 0] \), then make a detour by small half circle to avoid the singularity. As singular points are branching singularity in generic ordered expression, the secondary residue does not appear (cf. [13]) and the integral does not depend on the detour.

In the case of nice expression parameter, as in the remark mentioned in the last part of the previous section, singularities appeared in \( (-\infty, 0] \) are double branched simple singularity. Hence the integral \( \int_{-\infty}^{0} e^{i\frac{1}{\hbar}(u_g, u_g)}_{+} dt \) is welldefined for every \( g \in S' \) to give a smooth function of \( \alpha = g^\dagger g \in \tilde{S}^2 \).

By using the exponential law with \( e^{iz} \), we see if \( |\text{Re} z| < 1/2 \), then the integral gives an inverse \( (z + Q_\alpha)_{++}^{-1} = \int_{-\infty}^{0} e^{i\frac{1}{\hbar}(z + Q_\alpha)} dt \) and its derivative \( \frac{d^k}{dz^k}(z + Q(u, v))_{++}^{-1} \) for every \( k \):

\[
\frac{d^k}{dz^k} \frac{1}{(u_g, u_g)_{++} + z \rho} = (-1)^k \frac{k! \rho^k}{(k^\dagger \rho^k)_{+} + z \rho} \frac{1}{\hbar} \int_{-\infty}^{0} (t^k \rho^k)_{+} e^{i\frac{1}{\hbar}(u_g, u_g)_{+} + z \rho} dt
\]
Hence roughly speaking $\sqrt{1+m^2Q(u,v)a^2}$ may be viewed as
\[(4.3)\]
$$\sqrt{1+m^2Q(u,v)a^2}=1+\frac{1}{2}m^2Q(u,v)a^2-\frac{1}{4}m^4Q(u,v)a^2+\ldots.$$ 

In precise, we take the Taylor expansion for $|z|<1$, $\sqrt{1+z}=\sum c_kz^k$. We see then
\[
\sqrt{1+m^2Q(u,v)a^2}=1-\frac{1}{\hbar}\int_{-\infty}^{0}\sum_{k\geq1}\frac{m^{2k}k^{2k-1}}{(2k-1)!}e^{i\frac{t}{\hbar}(u_\alpha u_\beta)}dt.
\]

As $\sum_{k\geq1}c_km^{2k}k^{2k-1}$ is an entire function, we get $\sqrt{1+m^2Qa^2}$ as a smooth function of $\alpha$ and $(u,v)$ where $(u,v)$ is regarded as coordinate functions of $\mathbb{R}^2$. Denote this by $a(\alpha;u,v)$. By this concrete form $a(\alpha;u,v)$ belongs to the symbol class $\tilde{\Sigma}^0(\mathbb{R}^2)$ defined in the next section.

For every fixed $u,v$ we take the expansion by spherical functions by setting $\alpha=(\theta,\phi)$
\[
a(\alpha;u,v)=\sum_{n=0}^{\infty}(A_{n,0}(u,v)P_n(\cos\phi)+\sum_{m=1}^{n}(A_{n,m}(u,v)\cos m\phi+B_{n,m}(u,v)\sin m\phi)P_n^m(\cos\theta).
\]
Now, setting
\[
\Phi(\theta,\phi)=\sum_{n=0}^{\infty}(x_{n,0}(u,v)P_n(\cos\phi)+\sum_{m=1}^{n}(x_{n,m}(u,v)\cos m\phi+y_{n,m}(u,v)\sin m\phi)P_n^m(\cos\theta),
\]
we have an equation of infinite components
\[(4.4)\]
$$\partial_\theta\Phi(\theta,\phi)=\frac{1}{i\hbar}a(\alpha;u,v)\star_{K}(le_1\alpha+le_2\beta+le_3\gamma)\star_{K}\Phi(\theta,\phi).$$

As $le_i$ are quadratic form of $u,v$, the first $\star_{K}$-products are welldefined.

By Fourier transform this is changed into a differential equation of first order:
\[(4.5)\]
$$\partial_\theta\hat{\Phi}(\theta,\phi)=\frac{1}{\hbar}a(\alpha;u,v)\star_{K}(le_1\partial_{\xi_1}+le_2\partial_{\xi_2}+le_3\partial_{\xi_3})\star_{K}\hat{\Phi}(\theta,\phi).$$

As it will be shown in the next section, this is an equation written in the algebra $(\tilde{\Sigma}^0(\mathbb{R}^2),\star_{K})$. This algebra is the non-positive part of the extended Weyl algebra. Note that $(\Sigma^0(\mathbb{R}^2)\star_{K},\star_{K})$ plays the same role as the Clifford algebra in Dirac’s equation.

The fundamental solution of $(4.4)$ is given by
\[
\Phi(\theta,\phi)=e^{i\hbar(a(\alpha;u,v)\star_{K}(le_1\alpha+le_2\beta+le_3\gamma))}.
\]

As $a(\alpha;u,v)\star_{K}(le_1\alpha+le_2\beta+le_3\gamma)$ is an element of $le_1\alpha+le_2\beta+le_3\gamma+b(u,v,\alpha),b(u,v,\alpha)\in\tilde{\Sigma}_{ev}^0(\mathbb{R}^2)$, this is obtained as a one parameter subgroup of Fourier integral operators.

**Remark 3** $U(1)$-gauge principle may be applied to treat $(4.5)$ under the effect of electro-magnetic fields. That is to make a $U(1)$-connection and replace $\partial_{\xi_i}$ by $\nabla_{\xi_i}$. Another word, electro-magnetic fields are written by the terms involving differentials by the space-time coordinates. Thus, a quantization of electro-magnetic fields under the thought of deformation quantization is to write differentials by the space-time coordinates as adjoint operators of Weyl algebras. Such a procedure will be suggested in the later section $\S5.4$. 

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4.1.1 Weyl symbol class $\hat{\Sigma}^0(\mathbb{R}^2)$ and the product formula

In this section, $u, v$ are regarded as the coordinate function of $\mathbb{R}^2$ and let $\rho=\sqrt{1+u^2+v^2}$. $\hat{\Sigma}^0(\mathbb{R}^2)$ is a class of $C^\infty$ functions $f$ on $\mathbb{R}^2$ having asymptotic expansions

$$f \sim f_0(\theta) + \rho^{-1} f_{-1}(\theta) + \cdots + \rho^{-k} f_{-k}(\theta) + \cdots$$

where $f_j(\theta)$ is a smooth function on $S^1$.

Depending on the expression parameter $K$ we will make $*$-functions for every $f \in \hat{\Sigma}^0(\mathbb{R}^2)$ in what follows.

First of all, keeping the Fourier transform of 1 in mind, we define $*$-delta functions of full-variables $\delta_u^{(R^2)}(u-x)$ as in [16] by

$$\langle \delta_u^{(R^2)}(u-x) : \kappa \rangle = \int_{\mathbb{R}^2} e^{\frac{i}{\hbar}(\xi \cdot u - \xi \cdot x)} : \kappa \rangle d\xi d\eta$$

We note here that the exponential law gives

$$e^{\frac{i}{\hbar}(\xi \cdot u - \xi \cdot x)} = e^{\frac{i}{\hbar}(\xi \cdot u)} e^{-\frac{i}{\hbar}(\xi \cdot x)}, \quad \text{and} \quad \int_{\mathbb{R}^2} e^{\frac{i}{\hbar}(\xi \cdot u - \xi \cdot x)} d\xi = \delta^{(R^2)}(\xi) e^{\frac{i}{\hbar}(\xi \cdot u)}.$$

It follows

$$\int_{\mathbb{R}^2} \delta_u^{(R^2)}(u-x) d\xi = 1. \quad (4.6)$$

Let $f(x)$ be a tempered distribution on $\mathbb{R}^2$ and let $\tilde{f}^{(R^2)}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-\frac{i}{\hbar}(\xi \cdot x)} d\xi$ be the inverse Fourier transform. Noting the wellknown reciprocity formula $f(x) = \int_{\mathbb{R}^2} \tilde{f}^{(R^2)}(\xi) e^{\frac{i}{\hbar}(\xi \cdot x)} d\xi$, we define $*$-functions corresponding to $f(x)$ as

$$\langle f_u^{(R^2)}(u-x) : \kappa \rangle = \int_{\mathbb{R}^2} \langle f^{(R^2)}(x) : \kappa \rangle e^{\frac{i}{\hbar}(\xi \cdot u - \xi \cdot x)} d\xi d\eta = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x') e^{-\frac{i}{\hbar}(\xi \cdot x')} : \kappa \rangle d\xi' d\xi.$$

We denote by $\hat{\Sigma}^0(\mathbb{R}^2)_K$ the totality of obtained $*$-functions by this way.

The Weyl ordered ($K=0$) expression of $\delta_u^{(R^2)}(u-x)$ is given by $\delta_u^{(R^2)}(u-x) : 0 = \delta^{(R^2)}(u-x)$. Thus we see

$$\langle f_u^{(R^2)}(u) : 0 \rangle = \int f(x) \delta^{(R^2)}(u-x) d\xi = f(u). \quad (4.8)$$

**Proposition 4.1** The inverse of the correspondence $f(x) \rightarrow f_u^{(R^2)}(u)$ is given by its Weyl ordered expression and replacement of $u$ by $x$.

Hence the $*$-product formula of these $*$-functions is given by the Moyal product formula given in the integral form [22]. Namely

$$\langle f_u^{(R^2)}(u)* g_u^{(R^2)}(u) : \kappa \rangle = (f* g)^{(R^2)}(u).$$

Hence we have
Proposition 4.2 \((\tilde{\Sigma}^0(\mathbb{R}^2)_{*K}; *_K)\) becomes an associative algebra.

Now, by using the formula of Laplace transform, we define

\[
\rho^-_s = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} e^{-\frac{1}{4}t(u^2+v^2+1)} dt, \quad \rho_s = (u^2+v^2+1)^{s} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} e^{-\frac{1}{4}t(u^2+v^2+1)} dt.
\]

In fact, \((\tilde{\Sigma}^0(\mathbb{R}^2)_{*K}; *_K)\) is a \(\rho^-_s\)-regulated algebra defined in \([15]\). Roughly speaking this is filtered by \(\rho^-_s\) satisfying

\[
[\rho^-_s, \tilde{\Sigma}^0(\mathbb{R}^2)_{*K}] \subset \rho^-_s \tilde{\Sigma}^0(\mathbb{R}^2)_{*K} * \rho^-_s
\]

and

\[
[\tilde{\Sigma}^0(\mathbb{R}^2)_{*K}, \tilde{\Sigma}^0(\mathbb{R}^2)_{*K}] \in \rho^-_s \tilde{\Sigma}^0(\mathbb{R}^2)_{*K}, \quad [\rho_s, \tilde{\Sigma}^0(\mathbb{R}^2)_{*K}] \subset \tilde{\Sigma}^0(\mathbb{R}^2)_{*K}.
\]

The factor space is given by

\[
\tilde{\Sigma}^0(\mathbb{R}^2)_{*K}/\rho^-_s \tilde{\Sigma}^0(\mathbb{R}^2)_{*K} \simeq C^\infty(S^1).
\]

As for Clifford algebras it is not necessary to consider transcendental extension because these are finite dimensional. On the contrary, Weyl algebras are infinite dimensional and elements are represented by “unbounded operators”. \(\tilde{\Sigma}^0(\mathbb{R}^2)_{*K}\) is the part of “bounded operators” in the extended Weyl algebra. This \(\rho^-_s\)-regulated algebra will be used also in the next section.

In fact, as \(\text{ad}(\rho^2_s)\) leaves the space of quadratic forms invariant, it is sometimes convenient to restrict our system to a smaller class by using only “even elements”. Now, let \(\tilde{\Sigma}^0_{ev}(\mathbb{R}^2)\) is the class of \(C^\infty\) functions \(f\) on \(\mathbb{R}^2\) having asymptotic expansions

\[
f \sim f_0(\theta) + \rho^{-2}f_{-2}(\theta) + \cdots + \rho^{-2k}f_{-2k}(\theta) + \cdots
\]

where \(f_j(\theta)\) is a smooth function on \(S^1\). As \([\rho^2_s, \tilde{\Sigma}^0_{ev}(\mathbb{R}^2)_{*K}] \subset \tilde{\Sigma}^0_{ev}(\mathbb{R}^2)_{*K} \tilde{\Sigma}^0_{ev}(\mathbb{R}^2)\) is in fact a \(\rho^2_s\)-regulated algebra filtered by \(\rho^2_s \tilde{\Sigma}^0_{ev}(\mathbb{R}^2)_{*K}\) such that

\[
[\rho^2_s \tilde{\Sigma}^0_{ev}(\mathbb{R}^2)_{*K}, \rho^{-2k} \tilde{\Sigma}^0_{ev}(\mathbb{R}^2)_{*K}] \subset \rho^{-2(k+\ell+1)} \tilde{\Sigma}^0_{ev}(\mathbb{R}^2)_{*K}.
\]

As for \((4.4)\), \(a(\alpha; u, v) *_K (le_1\alpha + le_2\beta + le_3\gamma)\) is written in the form

\[
(le_1\alpha + le_2\beta + le_3\gamma) + \rho^{-2} \tilde{\Sigma}^0_{ev}(\mathbb{R}^2)
\]

without the term of order 0. Hence the fundamental solution is obtained by the \(*\)-exponential function given in the form of Fourier integral operators: Recalling \((4.3)\) we set

\[
a(\alpha; u, v) *_K (le_1\alpha + le_2\beta + le_3\gamma) = \langle ug, ug \rangle + b(\alpha; u, v), \quad g \in S', \quad b(\alpha; u, v) \in \rho^{-2} \tilde{\Sigma}^0_{ev}(\mathbb{R}^2).
\]

and write

\[
\frac{\partial}{\partial t}(ug, ug) + b(\alpha; u, v) = :t^\frac{1}{4}(ug, ug) *_K f(t, u, v).
\]

Then the equation we have to solve is

\[
\partial_t f(t, u, v) \equiv \text{Ad}(e_{\frac{1}{4}}(ug, ug)) (b(\alpha; u, v)) *_K f(t, u, v).
\]
As \( \text{Ad}(e^{-\frac{1}{\hbar}(\langle uuug,uuug \rangle)_{\alpha}})(b(\alpha; u, v)) \) is a smooth curve in \( \rho^{-2} \ast \tilde{\Sigma}^0_{ev}(\mathbb{R}^2) \) the existence of the solution written in the form \( e^{t \hbar \langle uuug,uuug \rangle_{\alpha}} \), \( c(t, u, v) \in \rho^{-2} \ast \tilde{\Sigma}^0_{ev}(\mathbb{R}^2) \), is ensured by the product integrals, although the concrete form is hard to write down. A precise treatment of these product integrals will be seen in the future note.

If we apply \( \text{Ad}(e^{-\frac{1}{\hbar}(\langle uuug',uuug' \rangle)_{\alpha}}) \), \( g' \in SL(2, \mathbb{C}) \), to both sides of (4.4), then a constant term (the term of order 0) appears in the phase. This term will be involved in the time parameter by using a singular coordinate transformation as in Remark 2.

5 SU(2)-vacuum

Note first that the proof of existence of \( \Omega^* \) in [16] is not correct, as the property of the singular point was not cared. First, we give a correction.

Proposition 5.1 Under a nice expression parameter \( K \), the integral \( \frac{1}{2\pi} \int_{0}^{2\pi} e^{\frac{\pi}{\hbar}(\langle uuug,uuug \rangle_{\alpha})} \cdot_{K} d\alpha \) which gives a pseudo-vacuum does not depend on \( \alpha'=g \cdot g' \).

Proof For a fixed \( K \), one parameter subgroup \( e^{\frac{\pi}{\hbar}(\langle uuug,uuug \rangle_{\alpha})} \) is well-defined except one \( g_{K} \). Non-vanishing of this integral is proved in [16].

Now, let \( g(s), s \in [0, 1] \) be a curve avoiding \( g_{K} \). We see

\[
e^{\frac{\pi}{\hbar}(\langle uuug,uuug \rangle_{\alpha}(s))}=1, \quad e^{\frac{\pi}{\hbar}(\langle uuug,uuug \rangle_{\alpha}(s))} = \varepsilon_{00}.
\]

Hence by minding Proposition 2.2 on the domain \((t, s) \in [0, \pi] \times [0, 1] \), Cauchy’s integration theorem shows that the integral does not depend on \( s \). As a result, we have no need to care about the singular point. \( \Box \)

Hence we have the nontrivial existence of the integral

\[
\Omega_{s} = \int_{(t, g) \in SU(2)} :e^{\frac{\pi}{\hbar}(\langle uuug,uuug \rangle_{\alpha})} :_{K} d\mu = \frac{1}{8\pi^2} \int_{S^2} \int_{0}^{2\pi} :e^{\frac{\pi}{\hbar}(\langle uuug,uuug \rangle_{\alpha})} :_{K} d\alpha d\alpha' \quad \text{K}
\]

where \( d\mu \) is the invariant volume form with total volume 1. Let us call this the SU(2)-vacuum.

It is clear that \( e^{\frac{\pi}{\hbar}(\langle uuug,uuug \rangle_{\alpha})} \ast_{K} \Omega_{s} = \Omega_{s} \), \( (\mathfrak{su}(2), J) \ast \Omega_{s} = \{0\} \). By viewing as quadratic forms, we see

\[
(\mathfrak{sl}(2, \mathbb{C}), J) \ast \Omega_{s} = \mathfrak{h}_{0}(2). J \ast \Omega_{s}.
\]

The next one is trivial

Proposition 5.2 \( \mathfrak{h}_{0}(2) = i\mathfrak{su}(2) \) is a Lie algebra over \( \mathbb{R} \) under the bracket product \([X, Y] = i[X, iY] \). Hence \( \mathfrak{h}(2) = \mathbb{R} \oplus \mathfrak{h}_{0}(2) \) is a Lie algebra under this new bracket product. This is viewed as the Lie algebra of U(2).

Remark 4. Note that the complementary subspace of \( \mathfrak{su}(2) \) is not unique. Furthermore, as \( \mathfrak{su}(2) \ast \Omega_{s} = 0 \) we have to restrict the coefficients to \( \mathbb{R} \).
5.1 What are remained under the vacuum $\ast \Omega_s$

The universal enveloping algebra of $(\mathfrak{h}(2); [\ , \ ]_\mathfrak{h})$ is an infinite dimensional noncommutative algebra over $\mathbb{R}$ generated by Pauli-matrices $\sigma_1, \sigma_2, \sigma_3$ and $I$ with only the commutation relations

$$\sigma_1 \ast \sigma_2 - \sigma_2 \ast \sigma_1 = 2\sigma_3, \quad \sigma_2 \ast \sigma_3 - \sigma_3 \ast \sigma_2 = 2\sigma_1, \quad \sigma_3 \ast \sigma_1 - \sigma_1 \ast \sigma_3 = 2\sigma_2, \quad I \ast \sigma_i = \sigma_i \ast I$$

We denote this by $Env(\mathfrak{h}(2); [\ , \ ]_\mathfrak{h})$. Note that Pauli-matrices are the bases of $2 \times 2$-hermite matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$  

However, we do not have $I^2=I$, $\sigma_i^2=I$ e.t.c. Furthermore, we do not set $-\sigma_i=(i\sigma_i)^2$ e.t.c.. We have only the commutation relations.

Note also that

$$(\mathfrak{sl}(2; \mathbb{C}; [\ , \ ]_{\mathfrak{sl}}) \cong (Q(u,v); [\ , \ ]_s), \quad Q(u,v)=\{\text{quadratic forms}\}$$

Hence a linear base of $\mathfrak{h}_0(2)J$ is

$$he_1 = \frac{1}{i\hbar}u, \quad he_2 = \frac{1}{2\hbar}(u^2+v^2), \quad he_3 = \frac{1}{2i\hbar}(u^2-v^2).$$

They satisfy in any expression the commutation relations

$$[[he_1, he_2],_s] = 2he_3, \quad [[he_2, he_3],_s] = 2he_1, \quad [[he_3, he_1],_s] = 2he_2.$$  

The Casimir element in the enveloping algebra vanishes: $\zeta = he_1^2 + he_2^2 + he_3^2 = 0$.

Now for every $g \in SL(2, \mathbb{C})$, Lie algebra isomorphism $Ad(g) : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}(2, \mathbb{C})$ maps $\mathfrak{h}(2)$ to other space $\mathfrak{h}'(2)$. By the observation in §8 one can make a projection $\mathfrak{h}'(2) \to \mathfrak{h}(2)$.

**Proposition 5.3** $SL(2, \mathbb{C})/\{\pm 1\}$ acts on $(\mathfrak{h}(2); [\ , \ ]_\mathfrak{h})$ as Lie algebra isomorphisms. Hence the action extends on its universal enveloping algebra.

**Proof** It is enough to prove the first one. For every $X,Y \in \mathfrak{h}_0(2)$, we set for every $g \in SL(2, \mathbb{C})$ $Ad(g)X=X'+c(X)I, Ad(g)Y=Y'+c(Y)I$. Then, $[[X,Y],_s] \in \mathfrak{h}_0(2)$ and

$$Ad(g)[[X,Y],_s] = [[Ad(g)X, Ad(g)Y],_s] = [[X',Y'],_s].$$

It follows $Ad(g)$ is a Lie homomorphism.

**Remark 5.** It is difficult to join linear terms to the Lie algebra $\mathfrak{h}_0(2)$ so that the enveloping algebra $Env(\mathfrak{g})$ of the real Lie algebra $(\mathfrak{g}, [\ , \ ]_\mathfrak{g}) = \{a+be^{i\xi}u+ce^{i\eta}v+(\mathfrak{h}_0(2)J); a, b, c \in \mathbb{R}\}$ over $\mathbb{R}$ do not crash out by the multiplication $\ast \Omega$ from the r.h.s..

Here we give a proof under the assumption that the constant term of the enveloping algebra forms a field over $\mathbb{R}$. If $e^{i\xi+n} = \pm e^{\frac{i\pi}{p}}$, then $[e^{i\xi}u, e^{i\eta}v] = \mp he^{\frac{i\pi}{p}}$. If $p=2^{m+1}$, then $(\mp he^{\frac{i\pi}{p}})^{2^m} = \pm i\hbar 2^m$, and $\pm i\mathfrak{h}_0(2) = \pm \mathfrak{su}(2)$. In general, under the assumption, the constant elements $e^{i(\xi+n)}$ is in the enveloping algebra. Multiplying its inverse to $e^{i\xi}u, e^{i\eta}v$, we see that $e^{i(\xi-n)}u, e^{i(\eta-\xi)}v$ are in $Env(\mathfrak{g})$. Thus,

$$[[e^{i(\xi-n)}u], e^{i(\eta-\xi)}v] = -i[u^2, v^2] = 4\hbar^2 i.$$  

Hence $Env(\mathfrak{g}) \ni \mathfrak{h}_0(2) = \mathfrak{su}(2)$.  

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5.1.1 Lie subalgebras contained in a suitably extended universal enveloping algebra

Although \( \mathfrak{su}(2)*\Omega_s = 0 \), the complementary subspace is not unique. What we have to consider is the universal enveloping algebra of \( \mathfrak{sl}(2, \mathbb{C})/\mathfrak{su}(2) \). Although \( h \in \mathfrak{h}(2) \), a certain nontrivial element may remain in

\[
\begin{align*}
\mathfrak{h} &= \mathfrak{h}(2), \\
\mathfrak{su}(2) &= \mathfrak{su}(2)
\end{align*}
\]

Now, note that

\[
2i\hbar (h \in \mathfrak{h}(2)) = (u + iv)^2, \quad 2i\hbar (h \in \mathfrak{h}(2)) = (u - iv)^2.
\]

In this section, we make \( \mathfrak{h} \in \mathfrak{h}(2) \), \( \mathfrak{su}(2) \in \mathfrak{su}(2) \) a suitably extended \( \mathfrak{Env}(\mathfrak{h}(2); [X, Y]) \) of \( \mathfrak{Env}(\mathfrak{h}(2); [X, Y]) \). To this end, we use the formula of Laplace transform

\[
\frac{\sqrt{\pi}}{\sqrt{p}} = \int_0^\infty \frac{1}{\sqrt{t}} e^{-pt} dt
\]

First we recall \([2.7]\) that

\[
: e^{\frac{i}{\hbar}(u \pm iv)^2} :_K = \frac{1}{\sqrt{1 - \tau t}} e^{\frac{i}{\hbar}(u \pm iv)^2}, \quad \tau = K_{11} - K_{22} + 2iK_{12}
\]

In a case of nice expression parameter, we see \( \tau = 2(i \beta + \gamma) \). But the argument below can be applied in the case \( \tau \neq 0 \) in general. Now we see the integral

\[
\int_0^\infty \frac{1}{\sqrt{t}} e^{-\frac{i}{\hbar}(u \pm iv)^2} :_K dt = \int_0^\infty \frac{1}{\sqrt{t}} \frac{1}{\sqrt{1 + \tau t}} e^{-\frac{i}{\hbar}(u \pm iv)^2} dt
\]

converges. Define elements

\[
: L(u + iv) :_K = : L(u + iv) :_K = \frac{1}{i\hbar} (u + iv)_s^* \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} e^{\frac{i}{\hbar}(u \pm iv)^2} dt :_K
\]

\[
: L(u - iv) :_K = : L(u - iv) :_K = \frac{1}{i\hbar} (u - iv)_s^* \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} e^{\frac{i}{\hbar}(u \pm iv)^2} dt :_K
\]

These satisfy

\[
( L(u + iv) - (u + iv) ) * ( L(u + iv) + (u + iv) ) = 0 = ( L(u - iv) - (u - iv) ) * ( L(u - iv) + (u - iv) ).
\]

These are elements of \( \rho_s \Sigma^0(\mathbb{R}^2)_s \), which may be written as

\[
: L(u + iv) :_K = u + iv + \phi_s, \quad : L(u - iv) :_K = u - iv + \psi_s, \quad \phi_s, \psi_s \in \Sigma^0(\mathbb{R}^2)_s
\]

Some care will be required, for \( \phi_s, \psi_s \) are 0-divisors, but by suitable replacements (cf. [8], § XIII.5), one can reduce the remainder terms arbitrarily “small” \( \phi_s, \psi_s \in \rho_s^N \Sigma^0(\mathbb{R}^2)_s \), but these do not vanish in general. This is because \( L(u + iv) \) commutes with the polar element \( \epsilon_{00} \) while \( u + iv \) anti-commutes with \( \epsilon_{00} \). These remain as terms of “smoothing operators”.

Hence, we have
Proposition 5.4 Suitably extended $\widetilde{\text{Env}}(\mathfrak{h}(2))$ contains for instance a Lie algebra over $\mathbb{R}$:

$$\tilde{\mathfrak{g}} = \left\{ x_0 + x_1 u + x_2 i v + \sigma \frac{1}{2\hbar} (u^2 - v^2) \right\} + \text{some elements in } \rho_{*N_0}^* (\mathbb{R}^2)_{*K}.$$

Note that if we use all quadratic forms $Q(u, v)$, then $\text{Ad}(e^{\frac{i}{\hbar} Q(u, v)})$ generates the group $SL(2, \mathbb{C})$.

By Proposition 5.3, we can apply $\text{Ad}(e^{\frac{i}{\hbar} Q(u, v)})$ to the above $\widetilde{\text{Env}}(\mathfrak{h}(2))$ and obtain a family of Lie algebras over $\mathbb{R}$

$$\left\{ \text{Ad}(e^{\frac{i}{\hbar} Q(u, v)}) \tilde{\mathfrak{g}} ; \ Q(u, v) = \text{quadratic forms} \right\}$$

parameterized by $SL(2, \mathbb{C})/\Gamma$ where $\Gamma$ is the stabilizer of $\mathbb{C} \otimes \tilde{\mathfrak{g}}$.

Although they are not equal to $(\mathfrak{h}(2), [\ , \ ])$ as Lie algebras, their enveloping algebras equal to $\widetilde{\text{Env}}(\mathfrak{h}(2))$. Thus, one can make the complexifications of these Lie algebras and one may use their enveloping algebra as if it were the regular representation space of $\Omega_\star$.

Several comments about the stabilizer will be worthwhile. In a strict sense, $\Gamma$ is the identity. However what we are concerned is the “space-time” as the Minkowski space, or a certain class of specific coordinate expressions. Here, we concern only on Minkowski space with a light-cone frame. In the next section, we see that the Lie algebra $\mathfrak{g}$ gives a light-cone frame of the Minkowski space. Hence, we see the following:

**Proposition 5.5** The light-cone frame of the Minkowski space given by the Lie algebra $\tilde{\mathfrak{g}}$ is determined by the highest term $\frac{1}{2\hbar} (u^2 - v^2)$. Hence the stabilizer $\Gamma$ is $e^{\frac{z}{\hbar} (u^2 - v^2)}$, $z \in \mathbb{C}$ and the factor space $SL(2, \mathbb{C})/\Gamma$ is a 4-dimensional space.

### 5.2 Noncommutative Minkowski spaces

In this section we treat the Lie algebra 

$$\mathfrak{g} = \left\{ \left\{ x_0 + x_1 u + x_2 i v + \sigma \frac{1}{2\hbar} (u^2 - v^2) \right\}, \ [\ , \ ]_*, \ x_i, \sigma \in \mathbb{R} \right\}.$$

This is linearly isomorphic over $\mathbb{R}$ to the linear space

$$\mathfrak{h}(2) = \left\{ \frac{1}{2} \begin{bmatrix} x_0 + \sigma & x_1 + i x_2 \\ x_1 - i x_2 & x_0 - \sigma \end{bmatrix}, \ [\ , \ ]_* \right\}$$

by the natural correspondence. Namely, Proposition 5.3 shows there are a family of $\star$-Lie algebras over $\mathbb{R}$ that give same universal enveloping algebra as $(\mathfrak{h}(2), [\ , \ ])_*$. Note the latter is isomorphic to the Lie algebra of $U(2)$.

In this section, we show the following

**Theorem 5.1** The Lie group structure with the Lie algebra $\mathfrak{g}$ is constructed on the space $\mathfrak{g}$ itself and this group has an adjoint invariant Lorentz metric.
The group structure is given as follows: First we take the central extension of $\mathbb{R}^2$ by the skew-symmetric form $J$ on $\mathbb{R}^2$. We denote this by $\mathbb{R}_J \times \mathbb{R}^2$. The group $G_0$ we want to make is the semi-direct product

$$(\mathbb{R}_J \times \mathbb{R}^2) \rtimes \text{Ad} \mathbb{R}$$

by the adjoint action of $e^{\frac{1}{\hbar^2} (u^2-v^2)}$.

To be precise, recall first the $\ast_K$-product of $\ast_K$-exponential functions is given by

$$(5.1)\quad e^{\frac{1}{\hbar^2} (\xi,u)} \ast e^{\frac{1}{\hbar^2} (\eta,u)} : e^{\frac{1}{\hbar^2} (\xi,u)} : K_e^{\frac{1}{\hbar^2} (\eta,u)}. \quad \xi, \eta \in \mathbb{C}^3, \quad \Lambda = K + J, \quad u = (u, v).$$

Using this formula consider next the semi-direct product group $\mathbb{C} \times e^{\frac{1}{\hbar^2} (\xi,u)} \ast e^{\frac{1}{\hbar^2} (u^2-v^2)}$. To fix the group structure, we note first that

$$\begin{aligned}
\text{ad}(\frac{1}{2i\hbar} (u^2-v^2)) \begin{bmatrix} u \\ v \end{bmatrix} &= - \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \\
\text{Ad}(e^{\frac{1}{\hbar^2} (u^2-v^2)})(\begin{bmatrix} u \\ v \end{bmatrix}) &= \Theta(\sigma) \begin{bmatrix} u \\ v \end{bmatrix}, \\
\Theta(\sigma) &= \begin{bmatrix} \cos \sigma & -i \sin \sigma \\ i \sin \sigma & \cos \sigma \end{bmatrix}.
\end{aligned}$$

The group structure is given by

$$(ae^{\frac{1}{\hbar^2} (\xi,u)} \ast e^{\frac{1}{\hbar^2} (u^2-v^2)}) \ast (be^{\frac{1}{\hbar^2} (\eta,u)} \ast e^{\frac{1}{\hbar^2} (u^2-v^2)}) = ab \ e^{\frac{1}{\hbar^2} (\xi, \eta)} \ast e^{\frac{1}{\hbar^2} ((\sigma + \tau) (u^2-v^2))}.$$  

(5.1) is used to compute the first $\ast$-product of r.h.s.. Denote this group by $G_J$.

Using this in particular case, we see

$$e^{\frac{1}{\hbar^2} (\tau (x_1 u + x_2 iv))} \ast e^{\frac{1}{\hbar^2} (u^2-v^2)} \tau, x_1, x_2, \sigma \in \mathbb{R},$$

forms a group $G_0$. We denote by $\hat{G}_0$ its universal covering group. It will be shown below that $\hat{G}_0$ is given by defining the group structure on the tangent space $\mathfrak{g}_0 = \mathbb{R}^4$ at the identity

Indeed, $e^{\frac{1}{\hbar^2} (\tau (x_1 u + x_2 iv))}$ is a central extension of the group $e^{\frac{1}{\hbar^2} (x_1 u + x_2 iv)}$ by the 2-cocycle $e^{\frac{1}{\hbar^2} (x_1 u + x_2 iv)}.$ The universal covering group is the central extension of $\mathbb{R}^2$ by the skew-symmetric form $J$ on $\mathbb{R}^2$.

The tangent space at the identity 1 forms a Lie algebra

$$\mathfrak{g}_0 = \left\{ \frac{1}{\hbar} \tau + \frac{1}{i\hbar} (x_1 u + x_2 iv) + \frac{i}{2\hbar} (u^2-v^2) \right\}$$

with bracket products

$$\left[ \frac{1}{\hbar} (x_1 u + x_2 iv), \frac{1}{\hbar} (y_1 u + y_2 iv) \right] = \frac{1}{\hbar} (x_2 y_1 - x_1 y_2), \quad \left[ \frac{1}{i\hbar} (x_1 u + x_2 iv), \frac{1}{2\hbar} (u^2-v^2) \right] = - \frac{1}{i\hbar} (x_2 u - x_1 iv).$$

We define a Lorentz metric on $\mathfrak{g}_0 = \mathbb{R} \oplus E^2 \oplus \mathbb{R}$ by the bilinear form $Q$ as follows:

$$(5.2)\quad Q = ((\dot{\tau}, \dot{x}_1, \dot{x}_2, \dot{\sigma}), (\dot{\tau}', \dot{y}_1, \dot{y}_2, \dot{\sigma}')) = \frac{1}{2} (\dot{\tau} \dot{\sigma}' + \dot{\sigma} \dot{\tau'}) - \langle \dot{x}, \dot{y} \rangle,$$

where $\langle \dot{x}, \dot{y} \rangle$ is the Euclidean inner product on $\mathbb{R}^2$.

It is not hard to see that the Lorentz metric $Q$ is adjoint invariant. Thus, this extends to an invariant bilinear metric on $\hat{G}_0$ by left-translations.
6 Pseudo-vacuum representations

We start with giving some comments about various vacuums. For every $K$, the $K$-ordered expression of $e_{s}^{(s+it)\frac{1}{2m}(ug, ug)}$ has a remarkable periodicity property in $t$. There is an interval $[a, b]$, called the exchanging interval, such that

$$e_{s}^{(s+it)\frac{1}{2m}(ug, ug)}:_{K} = \begin{cases} 
2\pi \text{-periodic} & a < s < b \\
\text{alternating} 2\pi \text{-periodic} & b < s \\
2\pi \text{-periodic} & s < b 
\end{cases}$$

In Weyl ordered expression, $K=0$, the exchanging interval of $e_{s}^{(s+it)\frac{1}{2m}(ug, ug)}$ is $a=b=\frac{\pi}{2}$, and in fact $e_{s}^{(s+it)\frac{1}{2m}(ug, ug)}:_{0}$ is singular at $t=\frac{\pi}{2}$. In the normal ordered expression, the exchanging interval of $e_{s}^{(s+it)1bu^{v}}$ looks to be $a=-\infty, b=\infty$ as this is an entire element, but it should be regarded that there is no exchanging interval because $e_{s}^{(s+it)1bu^{v}}:_{K}$ is alternating 2\pi-periodic. (Cf. \textbf{[11]})

We see $a < b$ in generic $K$. Pseudo-vacuums are defined whenever $a < 0 < b$ by

$$:\varpi_{s}(0):_{K} = \frac{1}{2\pi} \int_{0}^{2\pi} e_{s}^{(s+it)\frac{1}{2m}(ug, ug)}:_{K} dt.$$ 

as an idempotent element. This does not depend on $s$ whenever $a < s < b$ by Cauchy’s integration theorem. By the periodicity mentioned above, we have

$$\frac{1}{4\pi} \int_{-2\pi}^{2\pi} e_{s}^{(s+it)\frac{1}{2m}(ug, ug)}:_{K} = \begin{cases} 
0 & s < b \\
:\varpi_{s}(0):_{K} & a < s < b \\
0 & b < s 
\end{cases}$$

In what follows of this section we fix a nice expression parameter $K$. For every $g \in S'$ we see that $e_{s}^{(\frac{1}{2m}(ug, ug))}$ is $\pi$-periodic and singular points are distributed $\pi$-periodically along two lines sitting upper and lower half-plane. In what follows, we treat the case $(ug, ug)=2uv$ as a representative of general case by setting $g=\frac{1}{2} \begin{bmatrix} 1 & i \\
 i & 1 \end{bmatrix}$. All others are obtained by taking adjoint transformations given by $e_{s}^{(\frac{1}{2m}(ug, ug))}$, $g \in S'$, which gives the adjoint action of $SU(2)$.

Furthermore, argument below can be applied by taking adjoint transformations $\text{Ad}(e_{s}^{(\frac{1}{2m}(ug, ug))})$ by using any $g \in SL(2, \mathbb{C})$. Noting that $(ug, ug)$, covers all quadratic forms with discriminant $-1$, $2uv$ is a representative of all quadratic forms with discriminant $-1$. Note that these are Lorentz transformations bigger than the above.

In \textbf{[10]}, we see the integral

$$:\varpi_{s}(0):_{K} = \frac{1}{2\pi} \int_{0}^{2\pi} e_{s}^{(\frac{1}{2m}(uv))} dt$$

gives an idempotent element, and we called this the pseudo-vacuum.

Different from other vacuums used in \textbf{[11]}, \textbf{[12]} e.t.c. the regular representation space does not form an algebra by the $*$-product, but

$$(uv)*:\varpi_{s}(0):=0=:\varpi_{s}(0)*(uv), \quad \mathbb{C}[u, v]*:\varpi_{s}(0):=(\mathbb{C}[u]+\mathbb{C}[v])*:\varpi_{s}(0).$$
Regular representations on this space \((C[u] + C[v]) * \varpi_s(0)\) is already discussed in [13] by using Laurent expansions. As a result, we have bilateral matrix elements \(\{D_{k,l} : k, l \in \mathbb{Z}\}\) such that \(\sum_{k \in \mathbb{Z}} D_{k,k}\) converges to 1. (See comments given in this note §7.1.)

In what follows, we give another treatment by using half-inverses (cf. [13]) defined in the previous section §3.2.1.

\[
u = u*(u*v)^{-1}, \quad (u*v)^{-1} = -\frac{1}{i\hbar} \int_0^\infty e^{\frac{i}{\hbar} u*v} ds
\]

where \(\varpi_{00} = \lim_{s \to \infty} \frac{1}{i\hbar} e^{\frac{i}{\hbar} u*v}\). We see that \((u^*)^k * \varpi_{00} * v^\ell\) are \((k, \ell)\)-matrix element.

**Note** The algebra above is isomorphic to the algebra of “calculus” by setting

\[
u = \int_0^x dx, \quad u = \frac{d}{dx}, \quad \varpi_{00} = \delta_0, \quad \text{where} \quad \delta_0(f(x)) = f(0).
\]

However note that the double integral

\[
\int_0^\infty \int_0^{2\pi} e^{\frac{i}{\hbar} u*v} e^{i\frac{t}{\hbar} u*v} ds dt
\]

does not converge suffered by a singular point in the domain. In spite this, the repeated integrals give certain values. Note that

\[
e^{\frac{i}{\hbar} u*v} e^{i\frac{t}{\hbar} u*v} = e^{(s+it)\frac{1}{2\hbar} u*v} e^{-\frac{1}{2}s} = e^{(s+it)\frac{1}{2\hbar} u*v} e^{\frac{1}{2}t}
\]

Hence

\[
\int_0^\infty \frac{1}{4\pi} \left( \int_{-\pi}^{2\pi} e^{(s+it)\frac{1}{2\hbar} u*v}. :\varpi_s(0):_K \quad a < s < b \right) ds = \int_0^b e^{-\frac{1}{\hbar} s} ds = \frac{1}{1/2} (1 - e^{-\frac{1}{\hbar} b}) :\varpi_s(0):_K
\]

It follows :\(u^* * \varpi_s(0):_K = 2(1 - e^{-\frac{1}{\hbar} b}) v * \varpi_s(0)\). Next, we compute \((u*v)^{-1} * v^l * \varpi_s(0)\) as follows:

\[
\int_0^\infty \frac{1}{4\pi} \int_{-\pi}^{2\pi} e^{s\frac{1}{2\hbar} u*v} e^{it\frac{1}{2\hbar} u*v}. :\varpi_s(0):_K ds = \int_0^b e^{-s\frac{1}{2\hbar} b} ds = \frac{1}{1/2} (1 - e^{-\frac{1}{\hbar} b}) :v^l * \varpi_s(0):_K
\]

Hence we see

\[
(u^*)^n * \varpi_s(0):_K = \frac{i^n}{\hbar^n (1/2)_n} \left( \prod_{l=1}^{n-1} (1 - e^{-b(l+\frac{1}{2})}) \right) :v^n * \varpi_s(0):_K
\]

Note that the associativity is broken:

\[
:\varpi_s(0):_K = :u^n * (u^*)^n * \varpi_s(0):_K \neq :u^n * (u^* * \varpi_s(0)):_K = :u^n * \frac{i^n}{\hbar^n (1/2)_n} \left( \prod_{l=1}^{n-1} (1 - e^{-b(l+\frac{1}{2})}) \right) :v^n * \varpi_s(0):_K
\]

\[
= \prod_{l=1}^{n-1} (1 - e^{-b(l+\frac{1}{2})}) :\varpi_s(0):_K
\]
It is remarkable that the result is very sensitive on expression parameters. It follows: \( u \ast \varpi_\ast(0) \colon_\mathcal{K} = 2(1-e^{-\frac{b}{2}}) \). Next, we compute \( (u \ast v)^{-1} \ast u^l \ast \varpi_\ast(0) \) as follows:

\[
\int_0^\infty \frac{1}{4\pi} \int_{-2\pi}^{2\pi} e^{i\lambda_\ast(u \ast v)} e^{i\lambda_\ast(u \ast v)-(l+\frac{1}{2})} e^{i\lambda_\ast(l+\frac{1}{2})} d\lambda d\lambda' =: \int_0^b e^{-s(l+\frac{1}{2})} ds \varpi_\ast(0) :_\mathcal{K} = \frac{1}{l+\frac{1}{2}}(1-e^{-b(l+\frac{1}{2}))} :_\mathcal{K} \varpi_\ast(0) :_\mathcal{K}.
\]

Hence we see

\[
(6.3) \quad : (u^*)^n \ast \varpi_\ast(0) :_\mathcal{K} = \frac{i^n}{h^n(1/2)^n} (\prod_{l=1}^{n-1} (1-e^{-b(l+\frac{1}{2}))}) : (v^n \ast \varpi_\ast(0) :_\mathcal{K}.
\]

Note that the associativity is broken:

\[
\varpi_\ast(0) :_\mathcal{K} = (u^n \ast (u^*)^n) \ast \varpi_\ast(0) :_\mathcal{K} \neq u^n \ast (u^* \ast \varpi_\ast(0)) :_\mathcal{K} = u^n \ast \prod_{l=1}^{n-1} (1-e^{-b(l+\frac{1}{2}))} v^n \ast \varpi_\ast(0) :_\mathcal{K} = \prod_{l=1}^{n-1} (1-e^{-b(l+\frac{1}{2}))} : \varpi_\ast(0) :_\mathcal{K}
\]

It is remarkable that the result is very sensitive on expression parameters. The equality holds only in the case \( b = \infty \). At a first glance, this looks the case of normal ordered expression, but the exchanging interval and the pseudo-vacuum \( \varpi_\ast(0) \) do not appear in the normal ordered expression.

Anyway, losing associativity does not suffer to the argument. The regular representation space \( (\mathbb{C}[u] + \mathbb{C}[v]) \ast \varpi_\ast(0) \) is linearly isomorphic to the space \( \mathbb{C}\{u^*, u\} \ast \varpi_\ast(0) \), where \( \mathbb{C}\{u^*, u\} \) is viewed simply as a linear space

\[
\cdots + a_n(u^*)^n + \cdots + a_1 u^* + a_0 + b_1 u + \cdots + b_m u^m + \cdots \quad \text{(finite sum)}.
\]

### 6.1 Generated algebra

On the other hand, let \( \mathcal{A} \) be the algebra generated by \( u, u^* \):

\[
\mathcal{A} = \mathbb{C}\{u^*, u\} \odot \mathcal{M}, \quad \mathcal{M} = \{(u^*)^k * \varpi_{00}; k, \ell \in \mathbb{N}\}.
\]

Beside the regular representation space mentioned above, the algebra \( \mathcal{A} \) is represented on the linear space spanned by \( \{(u^*)^k * \varpi_{00}; k \in \mathbb{N}\} \) as infinite matrices

\[
\sum_{|k-l| = \text{finite}} A_{k,l} (u^*)^k * \varpi_{00} * u^l
\]

e.g.

\[
(6.4) \quad u^l = \sum_{k \geq 0} (u^*)^k * \varpi_{00} * u^{k+l}, \quad (u^*)^\ell = \sum_{k \geq 0} (u^*)^k * \varpi_{00} * u^k.
\]

As the factor algebra \( \mathcal{A}/\mathcal{M} \) is naturally isomorphic to the algebra \( \mathbb{C}[z, z^{-1}] \) of all Laurent polynomials,

\[
0 \to \mathcal{M} \to \mathcal{A} \to \mathbb{C}[z^{-1}, z] \to 0.
\]
\( \mathcal{A} \) may be regarded as a nontrivial extension of \( \mathbb{C}[z^{-1}, z] \) by \( \mathcal{M} \) such that

\[
z \ast z^{-1} = 1, \quad z^{-1} \ast z = 1 - \varpi_{00}, \quad z^{-2} \ast z = z^{-1} - u \cdot \varpi_{00}, \quad z^{-2} \ast z^2 = 1 - \varpi_{00} - u \cdot \varpi_{00} \ast u, \quad \text{e.t.c.}
\]

As \( \mathbb{C}[z^{-1}, z] \) is abelian, we see \([\mathcal{M}, \mathcal{M}] \subset [\mathcal{A}, \mathcal{A}] \subset \mathcal{M} \). As \( \mathfrak{sl}(n, \mathbb{C}), n > 1 \), are simple Lie algebras, we see \( \mathcal{M}/[\mathcal{M}, \mathcal{M}] \cong \mathbb{C} \). All elements of \([\mathcal{M}, \mathcal{M}]\) are traceless, but some elements of \([\mathcal{A}, \mathcal{A}]\) have non-vanishing trace. Hence, \( \mathcal{M} = [\mathcal{A}, \mathcal{A}], [\mathcal{M}, \mathcal{M}] \) is a Lie ideal of \( \mathcal{A} \). Thus we have a Lie algebra extension

\[
0 \to \mathbb{C} \to \mathcal{A}/[\mathcal{M}, \mathcal{M}] \to \mathbb{C}[z^{-1}, z] \to 0,
\]

where \( \mathbb{C}[z^{-1}, z] \) is viewed as abelian Lie algebra. The projection \( \mathcal{M} \to \mathcal{M}/[\mathcal{M}, \mathcal{M}] \) is given by taking the trace. Consider the complementary subspace \( \mathcal{A}_0 \) of \( \mathbb{C} \) in \( \mathcal{A}/[\mathcal{M}, \mathcal{M}] \), but we denote these as

\[
\cdots + a_n \hat{u}_{-n} + \cdots + a_1 \hat{u}_{-1} + a_0 + b_1 \hat{u}_1 + \cdots + b_m \hat{u}_m + \cdots \quad (\text{finite sum}).
\]

\( \mathcal{A}_0 \) generates a meta abelian Lie algebra \( \mathbb{C} \oplus \mathcal{A}_0 \) under the bracket product

\[
[[X, Y]] = \text{Tr}[X, Y], \quad [[\text{Tr}[X, Y], Z]] = 0, \quad [[\hat{u}_m, \hat{u}_n]] = m \delta_{m+n,0}
\]

**Proposition 6.1** The regular representation space \( \mathbb{C}\{u^*, u\} \) generates a Lie algebra \( \mathbb{C} \oplus \mathcal{A}_0 \) such that \( [[\hat{u}_m, \hat{u}_n]] = m \delta_{m+n,0} \).

\[
0 \to \mathbb{C} \to \mathbb{C} \oplus \mathcal{A}_0 \to \mathbb{C}[z^{-1}, z] \to 0.
\]

Hence \( \mathbb{C} \oplus \mathcal{A}_0 \) is a central extension of the (abelian) Lie algebra \( \mathbb{C}[z^{-1}, z] \).

Its enveloping algebra \( \tilde{\mathcal{A}}_0 \) is isomorphic to the Weyl algebra with infinitely many generators. This is called often the free Boson algebra. Elements can be expressed univalently as linear combinations of

\[
u^{\alpha_1, \beta_1} = \hat{u}_m^{\alpha_m} \cdots \hat{u}_1^{\alpha_1} \hat{u}_0^{\beta_1} \cdots \hat{u}_n^{\beta_n}, \quad (\text{called normal ordering})
\]

by using the commutation relations given in Proposition 6.1.

There is a natural homomorphism \( \pi \) of \( \mathcal{A}_0 \) onto \( \mathbb{C}[z^{-1}, z] \) defined by

\[
\pi(u^{\alpha_1, \beta_1}) = z^{\beta_1 + \cdots + \beta_1 - \alpha_1 - \cdots - \alpha_m}.
\]

such that \( \text{Ker} \pi \) is the ideal \( \mathcal{I} \) generated by \( [[\tilde{\mathcal{A}}_0, \tilde{\mathcal{A}}_0]] \) i.e.

\[
0 \to \mathcal{I} \to \mathcal{A}_0 \to \mathbb{C}[z^{-1}, z] \to 0.
\]

The Fock representation of \( \mathcal{A}_0 \) is given by setting the vacuum \( |0\rangle \) by \( \hat{u}_n |0\rangle = 0 \) for \( n \geq 0 \). As this is equivalent with \( u^n \cdot \varpi_{00} = 0 \), the representation is equivalent with (6.4).

Once we have a vacuum, the regular representation space is defined as the remainder space by the operation to the vacuum. Evolution equations represented in the regular representation space is refered often as a Schrödinger equations. Now as in the procedure of the second quantization, we make a representation of the regular representation space. But this is simply to make an embedding into the matrices (6.4).
As any automorphism $\psi : \tilde{A}_0 \to \tilde{A}_0$ leaves $\mathcal{I}$ invariant, $\psi$ must yield an isomorphism of $\mathbb{C}[z, z^{-1}]$, that is, a holomorphic mapping of $\mathbb{C} \setminus \{0\}$ onto itself leaving $\mathbb{C}[z^{-1}, z]$ invariant. This is not a Möbius transformation. Similarly, any derivation $D : \tilde{A}_0 \to \tilde{A}_0$ satisfies

$$D[|X, Y|] = [DX, Y] + [X, DY]$$

and hence $D \mathcal{I} \subset \mathcal{I}$.

Hence $D$ yields a derivation/complex vector field $\tilde{D}$ on $\mathbb{C}[z^{-1}, z]$.

On the other hand, for every quadratic form $Q(u) \in A_0$ (finite summation), $\text{ad}(Q(u)) = [Q(u), \ ]$ is a derivation of $\tilde{A}_0$. However, as $||Q(u), f||=0$ for almost all $f \in \tilde{A}_0$, and for every $z^n$ there is an element $f$ in its representative such that $||Q(u), f||=0$. Hence, this yields the trivial derivation 0 on $\mathbb{C}[z^{-1}, z]$.

Similarly, by using the $\ast$-exponential function $e^{Q(u)}$, $\text{Ad}(e^{Q(u)})$ is an automorphism of $\tilde{A}_0$, but this yields the identity on $\mathbb{C}[z^{-1}, z]$ by the same reason.

Note that $z^{n+1} \partial_z$ is a derivation of $\mathbb{C}[z^{-1}, z]$. Its integral curve $\psi_t(z)$ starting at $z$ is given by the differential equation

$$\frac{d}{dt} \psi_t(z) = (\psi_t(z))^{n+1}, \quad \psi_0(z) = z.$$

But the solution is

$$\psi_t(z) = \frac{z}{\sqrt{1-ntz^n}}, \quad (n \geq 1), \quad =ze^t, \quad (n=0), \quad =\sqrt[|n|]{\sqrt{|n|}-nt}, \quad (n \leq -1).$$

These are not in the group $\text{Aut}(\mathbb{C} \setminus \{0\})$, except the case $n = 0$. Exponential functions have multivalued nature in general just like $\ast$-exponential functions of quadratic forms. It is natural to expect that the Lie algebra of all derivations on $\tilde{A}_0$ generate a blurred covering group of $\text{Aut}(\mathbb{C} \setminus \{0\}) = \{e^z \to ce^{\pm z}; c > 0\}$.

Now note that $\text{ad}(Q(u))$ by a quadratic forms of infinite summation gives also a derivation of $\tilde{A}_0$. For instance, let $L_0 = -\frac{1}{2} \sum_{k \in \mathbb{Z}} \hat{u}_{-k} \hat{u}_k$ where $X \cdot Y = \frac{i}{2}(X \cdot Y + Y \cdot X)$. Then $[L_0, \hat{u}_m] = m \hat{u}_m$. Hence $L_0 : \tilde{A}_0 \to \tilde{A}_0$ is a derivation corresponding to $z \partial_z$. Similarly, for every $n \in \mathbb{Z}$, let

$$L_n = -\frac{1}{2} \sum_{k \in \mathbb{Z}} \hat{u}_{n-k} \hat{u}_k,$$

Then, we see

$$|[L_n, \hat{u}_m]| = m \hat{u}_{m+n}, \quad|[L_{n}, \hat{u}_m]| = m \hat{u}_{m-n}, \quad \text{etc.}$$

Hence $L_n, n \in \mathbb{Z}$, are derivations on $\tilde{A}_0$ corresponding naturally to $z^{n+1} \partial_z$. It is easy to see that

$$|[L_k, [L_{k+\ell}, \hat{u}_m]]| = [(k-\ell)L_{k+\ell}, \hat{u}_m].$$

Hence $\{\text{ad}(L_n); n \in \mathbb{Z}\}$ is a representation of the Lie algebra $\mathbb{C}[z, z^{-1}] \partial_z$ of all Laurent polynomial vector fields. However, note that

$$L_n \cdot L_{-n} - L_{-n} \cdot L_n = \frac{1}{4} \sum_{k, \ell} (\hat{u}_{-k} \circ \hat{u}_{k+n}) \cdot (\hat{u}_{-(\ell+n)} \circ \hat{u}_\ell) - \frac{1}{4} \sum_{k, \ell} (\hat{u}_{-(\ell+n)} \circ \hat{u}_\ell) \cdot (\hat{u}_{-k} \circ \hat{u}_{k+n})$$

$$= 2nL_0 - \frac{1}{2} \sum_{k=1}^{n-1} (n-k) \hat{u}_{-k} \cdot \hat{u}_k = 2nL_0 - \frac{1}{2} \sum_{k=1}^{n-1} k(n - k).$$

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Thus,

\[(6.6) \quad [L_n, L_{-n}] = 2nL_0 + \frac{1}{12}(n-1)n(n+1).\]

It is easy to see that if \(n+m\neq 0\), then \([L_n, L_m] = (n-m)L_{n+m}\).

This implies that \(L_n, n \in \mathbb{Z}\), generates a Lie algebra which is nontrivial central extension of \(\mathbb{C}[z, z^{-1}]\partial_z\). \(\mathbb{C}[z, z^{-1}]\partial_z\) is called the Witt Lie algebra in the conformal field theory. Denoting \(\mathbb{C}[z, z^{-1}]\partial_z\) by \(g\) for simplicity, any central extension of \(g\) is caused by a Chevalley 2-cocycle \(\omega\), i.e. skew-symmetric bilinear form \(\omega : g \times g \to \mathbb{C}\) such that \(d\omega = 0\), that is,

\[\sum_{cyclic} \omega(X, [Y, Z]) = 0, \quad [X, Y] = [X, Y] + \omega(X, Y).\]

The Chevalley 2-cohomology group of the Witt Lie algebra is known to be 1 dimensional. The standard one is known as the Virasoro Lie algebra given by

\[(6.7) \quad [L_n, L_m] = (n-m)L_{n+m} + c \frac{n(n^2-1)}{12} \delta_{n+m, 0}.\]

Denote this Lie algebra by \(Vir(c)\). It will be mentioned a group theoretical treatment of this algebra from a viewpoint of infinite dimensional Lie groups will be discussed in the next note.

Note here that there is no obstruction to restrict our system to the real coefficients.

### 6.2 Restriction to the real coefficients

To make the situation clearer, we take the topological completion \(C^\infty(S^1)\) (all smooth \(\mathbb{C}\)-valued functions on \(S^1\)) of \(\mathbb{C}[z, z^{-1}]\partial_z\) by regarding these as polynomial functions on the unit circle \(S^1\) and considering Fourier series. \(C^\infty(S^1)\) is the space of the Fourier series

\[\{ \sum a_n e^{in\theta}; \sum (1+n^2)^k |a_n|^2 < \infty, \forall k \in \mathbb{N} \}\]

It is a little hard task to fix the topological completions \(\overline{\mathcal{A}}\) and \(\overline{\mathcal{M}}\) to obtain the exact sequence

\[0 \to \overline{\mathcal{M}} \to \overline{\mathcal{A}} \to C^\infty(S^1) \to 0.\]

We have to leave the strict treatment to future notes, but it is natural to think there is a central extension of the (abelian) Lie algebra

\[0 \to \mathbb{C} \to \overline{\mathcal{A}}_0 \to C^\infty(S^1) \to 0.\]

\(\overline{\mathcal{A}}_0\) is spanned by \(\{u_n; n \in \mathbb{Z}\}\), where \(\hat{u}_n\) correspond to \(e^{in\theta}\). We denote by \(\tilde{\mathcal{A}}_0\) the enveloping algebra of \(\overline{\mathcal{A}}_0\).

Let \(\Gamma(TS^1)\) be the Lie algebra over \(\mathbb{R}\) of the smooth diffeomorphism group \(D(S^1)\) i.e. the space of \(C^\infty\) vector fields on \(S^1\). \(\Gamma(TS^1)\) is linearly isomorphic to \(C^\infty_{\mathbb{R}}(S^1)\). The Lie bracket of \(\Gamma(TS^1)\) is expressed in \(C^\infty_{\mathbb{R}}(S^1)\) as \([f, g] = fg' - gf'\). We now restrict our attention to the derivations of \(\tilde{\mathcal{A}}_0\).
which induce $C^\infty$-vector fields on $S^1$. One may use the same derivations as in (6.5) by taking the real part. But it is known in [3] that a nontrivial 2-cocycle $\alpha$ is given by

$$\alpha(f,h) = \int_{S^1} (f'h'' - f''h')dt \quad \text{(essentially same to (6.7))}$$

and any other 2-cocycles are cohomologous to $a\alpha$ for some constant $a$. Thus, there is an $\mathbb{R}$ central extension $\Gamma(T_{S^1}) \ltimes_{\alpha} \mathbb{R}$ of $\Gamma(T_{S^1})$.

Now, we want to lift this cocycle to the Hochschild cocycle on the group $\mathcal{D}(S^1)$. Let $\mathcal{D}_0(S^1)$ be the identity component of $\mathcal{D}(S^1)$. Recall that the homotopy type of $\mathcal{D}_0(S^1)$ is $\mathbb{S}^1$. Therefore the de Rham cohomology group $H^2(\tilde{\mathcal{D}}_0(S^1))$ of the universal covering group $\tilde{\mathcal{D}}_0(S^1)$ of $\mathcal{D}_0(S^1)$ vanishes.

By a technique similar to Koszur construction, we can construct a Hochschild 2-cocycle $\omega$ from $\alpha$ (cf. [8] §X.6). Thus, there exists a regular Fréchet Lie group $\tilde{G}$ with the Lie algebra $\mathbb{R} \ltimes_{\alpha} \Gamma(T_{S^1})$. $\tilde{G}$ is the $\mathbb{R}$ central extension of $\tilde{\mathcal{D}}_0(S^1)$. Thus, the multivalued nature appears when we consider on the space $S^1$.

Future problems

Recall here that $\frac{1}{m}u\cdot v$ was a representative of all quadratic forms $\frac{1}{m}(ug,ug)$, $g \in SL(2,\mathbb{C})$ that is a representative of all quadratic forms of discriminant $-1$. Hence, we have a family of extensions:

$$0 \to \mathcal{L}_\gamma \to \tilde{A}_\gamma \to C^\infty_\gamma(S^1) \to 0$$

parameterized by $\gamma \in \mathcal{L} = SL(2,\mathbb{C})/\Gamma$.

As $SL(2,\mathbb{C})$ acts on $\mathcal{L}$, we have bridges between two exact sequences:

$$0 \to \mathcal{L}_\gamma \to \tilde{A}_\gamma \to C^\infty_\gamma(S^1) \to 0$$

On the other hand, infinitesimal adjoint action $\text{ad}(\frac{1}{\sqrt{m}}Q(u,v)) : \tilde{A}_\gamma \to \tilde{A}_\gamma$ is a derivation for every $\gamma \in \mathcal{L}$ and this yields a vector field on the space $\mathcal{L} \times S^1$.

Hence, what we really have to consider is a central extension of the Lie algebra of vector fields on $\mathcal{L} \times S^1$. This will be treated in forthcoming note.

7 Spontaneous splitting of polar element

$SU(2)$-vacuum is defined only by using a nice expression parameter, which is by no means generic. By Remark 1 in §2.3 we see that for a generic $K \in D_{-1}$, there is $g$ such that $:e^{\pi \frac{1}{m}(ug,ug)}:_{K} = -1$.

Now, we repeat the result in [12] for the case $m = 1$.

**Proposition 7.1** Suppose there is $g \in Sp(1,\mathbb{C}) = SL(2,\mathbb{C})$ such that $:e^{[0,\pi]\frac{1}{m}(ug,ug)}:_{K} = -1$. Then, there must exist $h \in Sp(1,\mathbb{C})$ such that $:e^{[0,\pi]\frac{1}{m}(uh,uh)}:_{K} = 1$, and $\hat{h} \in Sp(1,\mathbb{C})$ such that the path $:e^{[0,\pi]\frac{1}{m}(u\hat{h},u\hat{h})}:_{K}$ must hit a singular points.
However, as singular points are double branched in generic ordered expression, there must be a one parameter group going through near the singular point that cuts a slit and goes into another sheet and hence 1 at $t=\pi$. Hence we have

**Theorem 7.1** In generic ordered expression, $S'$ splits into three open sets $S_0 \cup S_+ \cup S_-$ such that $S_0 \cup S_+ \cup S_- = S'$ such that

$$
\begin{align*}
&g \in S_0 \quad e^{\frac{\pi}{4}i(u,v)} = 1 : \quad \text{Real axis is between two lines of singular points} \\
&g \in S_+ \quad e^{\frac{\pi}{4}i(u,v)} = -1 : \quad \text{Singular points are in upper half-plane} \\
&g \in S_- \quad e^{\frac{\pi}{4}i(u,v)} = -1 : \quad \text{Singular points are in lower half-plane}
\end{align*}
$$

All of these are on one-parameter subgroups. To make the argument simpler, we take one parameter groups

$$
e^{\frac{t}{2\pi}(u^2+v^2)}, \quad e^{\frac{t}{2\pi}u, v}, \quad e^{\frac{t}{2\pi}(u^2-v^2)}
$$

as representatives of $S_0, S_+, S_-$ respectively. We denote their square roots by $\varepsilon_0, \varepsilon^*, \varepsilon'$ respectively. These are the polar element under a nice expression parameters. We have also the square roots of these such that

$$
e^{\frac{i}{2\pi}(u^2+v^2)}, \quad e^{\frac{t}{2\pi}u, v}, \quad e^{\frac{i}{2\pi}(u^2-v^2)}
$$

where we have $e_1^4 = 1, e_2^4 = e_3^4 = -1$.

### 7.1 Regular representation on each sector

In spite of these difficulties, every one parameter subgroup makes vacuum or pseudo-vacuum and their own regular representation spaces. We discuss these in the case of each representative

$$
e^{\frac{t}{2\pi}(u^2+v^2)}, \quad e^{\frac{t}{2\pi}u, v}, \quad e^{\frac{t}{2\pi}(u^2-v^2)}
$$

(1) $e^{\frac{t}{2\pi}(u^2+v^2)}$: By setting $u' = \sqrt{i}(u+iv), \quad v' = \sqrt{i}(u-iv)$ we have $u^2 + v^2 = u'^2v'$. This is the case discussed in §5 where we have a pseudo-vacuum $\Psi_*(0)$. If the expression parameter is restricted to nice expression parameters, we have to treat only the case (1).

Thus, in what follows we treat $u', v'$ as $u, v$.

Now, by the observation in [13], if we use the convention

$$
\zeta^k = \begin{cases} u^k, & k \geq 0 \\ v^{|k|}, & k < 0 \end{cases}, \quad \zeta^\ell = \begin{cases} u^\ell, & \ell \geq 0 \\ v^{|\ell|}, & \ell < 0 \end{cases}
$$

then we have

$$
D_{k,\ell}(K) = \frac{1}{\sqrt{\left(\frac{i}{2}\right)_k \left(\frac{i}{2}\right)_\ell e(i\hbar)|k|+|\ell|}} \zeta^k \Psi_*(0) :_k \ast \zeta^\ell , \quad :\Psi_*(0) :_k \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{t}{2\pi}(u^2+v^2)} dt :_k dt
$$

are matrix elements for every $k, \ell \in \mathbb{Z}$. 

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7.1.1 Relation to matrix algebra \( \mathcal{A} \)

To relate these to \( \mathcal{A} \), we have to know more about the property of \( e_s^{(s+it)\frac{1}{\pi}uv} \), \( e_s^{(s+it)\frac{1}{\pi}vu} \). First for \( e_s^{(s+it)\frac{1}{\pi}vu} \) in generic ordered expression, there is an interval \([a, b]\) called the \textit{exchanging interval} (cf.\([14]\)) such that

\[
\begin{align*}
\begin{cases}
  s < a & : e_s^{(s+it)\frac{1}{\pi}vu} :_K \text{ alternating } 2\pi\text{-periodic} \\
  a < s < b & : e_s^{(s+it)\frac{1}{\pi}vu} :_K \quad 2\pi\text{-periodic} \\
  s < b & : e_s^{(s+it)\frac{1}{\pi}vu} :_K \text{ alternating } 2\pi\text{-periodic}
\end{cases}
\end{align*}
\]

In a nice expression parameter \( K \), the exchanging interval has the property \( a < 0 < b \). Note that pseudo-vacuums requires only the property \( a < 0 < b \).

Furthermore, we see (cf.\([13], [14]\)) that \( \pi_{\ast}(0) \) is given also by

\[
\pi_{\ast}(0) = \frac{1}{2\pi} \int_{0}^{2\pi} e_s^{(s+it)\frac{1}{\pi}vu} dt, \quad a < \forall s < b
\]

Furthermore, we see easily

\[
\pi_{00} = \frac{1}{2\pi} \int_{0}^{2\pi} e_s^{(s+it)\frac{1}{\pi}vu} dt, \quad b < \forall s.
\]

Now, noting \( \frac{1}{2\pi} \pi_{vu} = \frac{1}{2\pi} \pi_{vu} + \frac{1}{2} \) we compute as follows:

\[
(2\pi)^2 \pi_{00} \ast u' \ast \pi_{\ast}(0) = \int \int e_s^{(s+it)\frac{1}{\pi}(v^2 + l^2)} e_s^{(s+it')\frac{1}{\pi}(v^2 + l^2)} dt dt' = \int \int e_s^{(s+it')\frac{1}{\pi}(v^2 + l^2)} e_s^{(s+it')\frac{1}{\pi}(v^2 + l^2)} dt dt'.
\]

Now for a fixed \( l \), we choose \( s \) and \( \sigma \) so that

\[
a < s + \sigma < b, \quad b < s, \quad l\sigma + \frac{1}{2}s = 0.
\]

Then, we have

\[
(2\pi)^2 \pi_{00} \ast u' \ast \pi_{\ast}(0) = \frac{4}{2l-1} \pi_{\ast}(0) \ast u'.
\]

Hence we have

\[
(u^\ast)^k \ast \pi_{00} \ast u' \ast \pi_{\ast}(0) = \frac{1}{(2l-1)^{\frac{1}{2}}} (u^\ast)^k \ast \pi_{\ast}(0) \ast u', \quad k, l \in \mathbb{N}.
\]

Hence by applying \( \pi_{\ast}(0) \) from r.h.s. the algebra \( \mathcal{A} \) in \( \S 33 \) is translated to uni-lateral matrices of \( D_{k,l} \).

It is interesting that this procedure mentioned above gives a method to change bilateral matrix-elements to uni-lateral matrices.

(2) \( e_s^{(s+it)\frac{1}{\pi}uv} \): This gives the most typical vacuum representations. The vacuum \( \pi_{00} \) is given by

\[
\pi_{00} = \frac{1}{2\pi} \int_{0}^{2\pi} e_s^{(s+it)\frac{1}{\pi}vu} dt, \quad \forall s < a.
\]

In generic ordered expressions, \( E_{p,q} = \frac{1}{\sqrt{p(q + 1)q^2}} u^p \ast \pi_{00} \ast v^q \) is the \((p, q)\)-matrix element, that is \( E_{p,q} \ast E_{r,s} = \delta_{q,p} E_{p,s} \). The \( K \)-expression \( :E_{p,q} :_K \) of \( E_{p,q} \) will be denoted by \( E_{p,q}(K) \). Note that
\( E_{0,0}(K) = \varpi_{00} \). As the singular points are in the upper half-plane, i.e. \( 0 < a < b \) where \([a, b]\) is the exchanging interval, the radius of convergence of \( e^{\log \frac{1}{\sqrt{2}} (u+v)} \) is \( > 1 \). Hence we have

\[
\sum E_{n,n} = 1.
\]

(3) \( e^{i t \frac{1}{\sqrt{2}} u^2 - v^2} \): We set \( u' = \frac{1}{\sqrt{2}} (u+v), v' = \frac{1}{\sqrt{2}} (u-v) \), and treat \( (u', v') \) as \( (u, v) \). A similar calculation, we see that \( \overline{E}_{p,q} = \frac{\sqrt{-p+q}}{\sqrt{p!q!}} v^p \overline{u}^q \) is the \((p, q)\)-matrix element in generic ordered expressions. The \( K \)-expression of \( \overline{E}_{p,q} \) will be denoted by \( \overline{E}_{p,q}(K) \). Note that \( \overline{E}_{0,0}(K) = \varpi_{00} \).

In general the \(*\)-product \( \varpi_{00} \overline{\varpi}_{00} \) can not be defined univalently, as this depend on the manner of calculation.

The next identities are easy to see

\[
\varpi_{00} \overline{\varpi}_{00} = \frac{1}{i \hbar} u \overline{v} = \frac{1}{2} \varpi_{00}, \quad \frac{1}{i \hbar} u \overline{v} \overline{\varpi}_{00} = -\frac{1}{2} \overline{\varpi}_{00}.
\]

Note that in order to keep the associativity

\[(\varpi_{00} \overline{\varpi}_{00} \overline{\varpi}_{00}) \overline{\varpi}_{00} = \varpi_{00} \overline{\varpi}_{00} (\frac{1}{i \hbar} u \overline{v} \overline{\varpi}_{00}),\]

we have to define

\[
\frac{1}{2} \varpi_{00} \overline{\varpi}_{00} = -\frac{1}{2} \overline{\varpi}_{00} \varpi_{00} = 0.
\]

### 7.2 Failure of binary operations

In the previous section we saw that

\[
(\varepsilon_0)^2 = 1, \quad (\varepsilon^*)^2 = -1, \quad (\varepsilon')^2 = -1.
\]

In spite of this, we have \((\varepsilon^*)^{-1} = \varepsilon'\). This is because when \( t \) is replaced by \(-t\), the lines of singular points are switched upside down, and this is proved by Cauchy’s integration theorem. At a first glance it looks to contradict the exponential law.

Indeed, this is the reason why the polar element and \(*\)-exponential functions of quadratic forms are viewed as double valued elements in generic ordered expressions (cf. [11]). Intertwiners are defined only as 2-to-2 mappings on such spaces. However, computations such as \( \sqrt{a} \overline{\sqrt{b}} = \sqrt{ab} \) arrows us to treat these double valued elements safely. As a result, we have some extraordinary “blurred groups” such as a double covering group of \( SL(2, \mathbb{C}) \). (Cf. [12].) An extraordinary phenomenon mentioned above is then a natural conclusion when we treat \(*\)-exponential functions of quadratic forms as strictly single valued elements.

Thus, we have to conclude that these elements such as \( \varepsilon_0, \varepsilon^*, \varepsilon' \) and \( e_1, e_2, e_3 \) cannot be used as members of binary operations. We have to accept that there are many such elements in the
extended Weyl algebra because of double-branched singular points. However, if one treat every element together with a path avoiding singular points from the origin \( t = 0 \) where we assign 1 always, then we obtain the value of \(*\)-exponential functions such as \( \mathcal{K}_{\text{univalent}} \). By this way, one may give some groupoid structure for the space of \(*\)-exponential functions with paths of quadratic forms. But this is too complicated to treat the objects safely, as we have to use two sheets and slits setting between singular points.

### 7.2.1 Path-connecting products by restricting paths

One way to treat these safely is to restrict the paths although every element is given together with a path avoiding singular points from the origin \( t = 0 \).

We consider here in various products of elements 

\[
a(x) = e^{\frac{1}{i\hbar} u^2 + v^2}, \quad b(y) = e^{\frac{1}{i\hbar} u^v v}, \quad c(z) = e^{\frac{1}{i\hbar} u^2 - v^2}
\]

for \( x, y, z \in \mathbb{R} \) such as 

\[
a(x_1)c(z_1)b(y_1)a(x_2)c(z_2)a(x_2) \cdot \cdot \cdot .
\]

To treat these we take the 3-dimensional lattice \( L \) such that 

\[
cL = \{(x, y, z); \text{two of them are integers and the other is a real number}\}
\]

where \( c \) is a scaling unit. We assume there is no singular point of \( a(x) * b(y) * c(z) \) on \( cL \).

The set \( \{(x, y, z); \text{only one of them is an integer, others are real}\} \) will be called the wall. We assume further that the singular set of \( a(x) * b(y) * c(z) \) intersect the wall transversally.

As the singular set forms a subset of complex codimension one, singular set on the wall must be discrete set. It is hard to fix this set, but by observations in [11] we see that if a box in the wall contain a singular point, then travelling around the edge of the box makes the sign change.

Now project \( cL \) to \( \mathbb{R}^2 \) so that each cube projects down as hexagon with 6 triangles inside. We assume for simplicity that every triangle contains at most one singular point.

In the next section, we set a virtual, but somewhat realistic distribution of singular points, and we investigate how path-connecting products are defined.

### 7.3 Virtual experiment for binary operations

Here put several relations to these products, which might violate the associativity. First, we consider \( \pm \)-sign at every vertex. Denote in the figure of the next page, where \( (a, b, c) \) are replaced sometimes by \( (u, d, s) \). Consider paths starting at the (red) origin to some other lattice point along any of three lines directed \( a^\pm 1, b^\pm 1, c^\pm 1 \) at each lattice point. We denote these by 

\[
= c^{-1} * b * a * 1,
\]

\[
= a^{-1} * b * b * a * a * c * 1. \quad \text{(Be careful about the order.)}
\]

Paths are considered by parallel translations when we want to connect two paths. Define the product for instance by 

\[
(b * c^{-1} * a * c * 1) * (c^{-1} * b * a * 1) = b * c^{-1} * a * b * a * 1
\]

where we set \( c * c^{-1} = c^{-1} * c = 1 \). These are called a free path connecting product. Apparently these form a group and \( (a^\pm 1, b^\pm 1, c^\pm 1) \) is the generator. We denote this group by \( \Gamma \). In what follows considering the \( \pm \) sign for each element, we treat \( \mathbb{Z}_2 \otimes \Gamma \).
Relations are set on each small triangle. If a triangle $\alpha \beta \gamma$ contains $\bullet$, or $\circ$ then we set $\alpha \beta \gamma = -1$ and if not we set $\alpha \beta \gamma = 1$. If bullets are distributed at random, then one can not fix the sign for almost all paths. Our main concern is to what extent we can control the $\pm$ sign, if certain periodic conditions are imposed.

\begin{equation}
1 = c^{-1} b^{-1} * a^{-1} * 1 = c * b * a * 1, \quad \text{and} \quad c = b * a = c^{-1}.
\end{equation}

Hence, $a * b = c$. Thus we obtain the Klein's 4 group $K_4$. Replacing $(a, b, c)$ by $(-a, -b, -c)$ makes the changing relation $a * b * c * 1 = 1$ to $a * b * c * 1 = -1$. This is the case where every triangle contains a singular point $\bullet$, but the generated group is isomorphic. The group ring $RK_4$ of $K_4$ over $\mathbb{C}$ is by identifying $\mathbb{R} \oplus \mathbb{R}ia$ with $\mathbb{C}$

\begin{equation}
\mathbb{C} \otimes (\mathbb{R} \oplus \mathbb{R}ia) \oplus \mathbb{C} \otimes (b * (\mathbb{R} \oplus \mathbb{R}ia)) = \mathbb{C} \oplus \mathbb{C} * b, \quad (z + wb) * (z' + w'b) = zz' + w'w' + (zw + wz') * b.
\end{equation}

(1): First of all, the case there is no singular point $\bullet$, nor $\circ$. Set as follows:

$$a^2 * 1 = 1, \quad b^2 * 1 = 1, \quad c^2 * 1 = 1, \quad a * b * c * 1 = 1.$$
Hence, the group of all invertible elements is \( \mathbb{C}_* \times \mathbb{C}_* b \). Its compact part is \( U(1) \times U(1) \). \( RK_4 \) has a natural nondegenerate bilinear form \( \langle x, y \rangle \) as the coefficient of the identity element of \( xy \). This satisfies

\[
\langle x, y \rangle = \langle x \rangle \langle y \rangle, \quad \langle a, a \rangle = \langle b, b \rangle = (a \ast b, a \ast b) = 1.
\]

(2): Next, the case every triangle contains a singular point. We set then

\[
a^2 \ast 1 = -1, \quad b^2 \ast 1 = -1, \quad c^2 \ast 1 = -1, \quad a \ast b \ast c \ast 1 = -1.
\]

Then, \( 1 = -c^{-1} \ast b^{-1} \ast a^{-1} \ast 1 = c \ast b \ast a \ast 1 \), and \( b \ast a \ast 1 = -c \ast 1 = c^{-1} \ast 1 \). Hence \( a \ast b \ast 1 = c = -b \ast a \ast 1 \). Similarly, \( b \ast c \ast 1 = a \ast 1, \quad c \ast a \ast 1 = b \ast 1 \) are obtained easily. Thus we obtain the quaternion group.

Replacing \( (a, b, c) \) by \( (-a, -b, -c) \) makes the changing relation \( a \ast b \ast c \ast 1 = -1 \) to \( a \ast b \ast c \ast 1 = 1 \). The cases (1) and (2) are already appeared in [12].

The group ring over \( \mathbb{R} \) (resp. \( \mathbb{C} \)) of the quaternion group is the quaternion field \( \mathbb{H} \) (resp. \( \mathbb{C} \otimes \mathbb{H} \)). The compact part of all invertible elements is the group \( SU(2) \) (resp. \( U(2) \)).

(3): The cases (1) and (2) are already appeared in [12]. On the other hand, we found in [12] there is a class of expression parameters such that square roots of polar elements have strange properties that

\[
a^4 \ast 1 = -1, \quad b^4 \ast 1 = -1, \quad c^4 \ast 1 = 1, \quad a \ast b \ast c \ast 1 = 1, \quad c \ast a \ast b \ast 1 = -1.
\]

(See also [16].) To treat this case, we now consider the consider the paths shown in Magical Lattice with singular points in the previous page.

These read at each triangle, the product of the three edges makes 1 when there is no singular point (●) inside, and it makes \(-1\) when there is a singular point (●) inside. as a matter of cause, such relations destroys the possibility of binary operations. The the relation requested for the path connected product is \( c \ast b \ast a \ast 1 = -1 \) at the right lower triangle, but there are right lower triangles with no ● inside. Same holds for \( b \ast a \ast c \ast 1 = 1 \) at the left lower triangle.

Such a trouble is caused by the discordance between the periodicity of the tiling by triangles and that of bullets. Any way, it is impossible to use \( a, b, c \) as fundamental elements of binary operations. However, it may be possible to make binary operations between pairs of \( a, b, c \). Note that once binary operations are established by path connecting products on a family of suitably combined elements, then the associativity holds automatically.

First setting \( \varepsilon = a^2, \quad \varepsilon^* = b^2, \quad \varepsilon' = c^2 \), we investigate the system generated by \( \varepsilon, \varepsilon^*, \varepsilon' \). (See inside the large blue box.)
We see \( \varepsilon^2 = -1 \), \( \varepsilon^{*2} = -1 \), \( \varepsilon^2 = 1 \). The number in each triangle is the cardinality of singular point sitting inside, and this pattern repeated periodically. Hence, we cannot obtain a consistent definition of path connecting products of these elements. For instance, counting the number of singular points in the path, we have to write \( \varepsilon \ast \varepsilon^* = \varepsilon \ast \varepsilon^* = -\varepsilon \ast \varepsilon^* \), \( \varepsilon \ast \varepsilon' = \varepsilon \ast \varepsilon' = -\varepsilon \ast \varepsilon' \).

Hence in \( \mathcal{R}_{im} \) expression, these polar elements cannot be a member of binary operation, while in \( \mathcal{R}_{re} \) expressions, the polar element is a single element such that \( \varepsilon^2 = 1 \).

Recall that \( \varepsilon = a^2 \), \( \varepsilon^* = b^2 \), \( \varepsilon' = c^2 \). (Cf. [12])

Note that the number of singular points in every small triangle in Magical lattice is one or zero, and such a difference destroys the binary operations. However, we may use elements of binary operations, just as \( \{a, b, c\} \) for fundamental elements of binary operations, just as \( \{\varepsilon, \varepsilon^*, \varepsilon'\} \). On the other hand, note that every hexagon in the Magical lattice contains three singular points. Fig.1 is a tilings containing the origin at the common edges of three tiles. Now we permit to use only invertible paths sitting on the edges of hexagons on a tiling. Single paths \( a, b, c \) are not involved, for these are not invertible within the admissible paths.

Now we see all closed paths around a hexagon make \(-1\), hence we have relations as follows:

\[
\begin{align*}
-1 & = c^{-1} * a^{-1} * b * c^{-1} * a * 1, \\
-1 & = c^{-1} * a * b^{-1} * c * a^{-1} * b * 1, \\
-1 & = a^{-1} * b * c^{-1} * a * b^{-1} * c * 1.
\end{align*}
\]

It does not depend on the orientation of closed paths, but such relations are forbidden:

\[
\begin{align*}
-1 & = c^{-1} * a^{-1} * b * c^{-1} * a * 1, \\
-1 & = c * b * 1 = a^{-1} * 1, \\
-1 & = a * c * 1 = b^{-1} * 1, \\
-1 & = a * b * 1 = c^{-1} * 1, \\
-1 & = b * c * 1 = a^{-1} * 1, \\
-1 & = c * a * 1 = b^{-1} * 1.
\end{align*}
\]

Consider now compositions of two generators not included in the above: For simplicity and by minding the similarity to the quark model, we denote these as follows:

\[
\begin{align*}
\bar{s}u = c^{-1} * a * 1, \\
\bar{u}d = a^{-1} * b * 1, \\
\bar{d}s = b^{-1} * c * 1, \\
\bar{u}s = a^{-1} * c * 1, \\
\bar{d}u = b^{-1} * a * 1, \\
\bar{s}d = c^{-1} * b * 1.
\end{align*}
\]

The second row are inverses of the first row, and

\[
(\bar{d}s) \ast (\bar{s}u) = \bar{d}u, \quad (\bar{u}s) \ast (\bar{s}d) = \bar{u}d.
\]

As the tiling and singularities are distributed periodically, these form a group by the path connecting product. Here, the origin \( \bullet \) has \( \pm \) sign. We fix \( + \bullet \) is the identity element of the group.
We denote this group by $\Gamma_{uds}$. Note that polar elements $\varepsilon, \varepsilon^*, \varepsilon'$ are not on the admissible paths. By tracing in Magical lattice, it is easy to see

\[(\bar{s}u)^4=-1=(\bar{s}u)^4, \quad (\bar{u}d)^4=1=(\bar{u}d)^4, \quad (\bar{d}s)^4=-1=(\bar{d}s)^4, \quad (\bar{s}u)^* (\bar{u}d)^* (\bar{d}s)^*=1.\]

Moreover as every hexagon contains three singular points, we have another relation

\[(\bar{d}s)^* (\bar{u}d)^* (\bar{s}u)^* = -1,\]

and these show that they anti-commute each other: i.e.

\[(\bar{s}u)^* (\bar{u}d)=-(\bar{u}d)^* (\bar{s}u), \quad (\bar{u}d)^* (\bar{d}s)=-(\bar{d}s)^* (\bar{u}d), \quad (\bar{s}u)^* (\bar{d}s)=-(\bar{d}s)^* (\bar{s}u), \quad \text{e.t.c.}\]

It follows that the squares $\bar{s}u^2, \quad (\bar{u}d)^2, \quad (\bar{d}s)^2$ commute with all other elements. Hence these generate the center $Z_2=\mathbb{Z}_4 \times \mathbb{Z}_2$:

\[(7.3) \quad Z_2=\{\pm 1, \pm (\bar{s}u)^2, \pm (\bar{u}d)^2, \pm (\bar{s}u)^2 (\bar{u}d)^2\} \quad (6 \text{ colored points in Fig.1}).\]

Let $\pi : \Gamma_{uds} \to \Gamma_{uds}/Z_2$ be the projection onto the factor group. We have then

\[(\pi(\bar{s}u))^2=(\pi(\bar{u}d))^2=(\pi(\bar{d}s))^2=1, \quad \pi(\bar{s}u)^* \pi(\bar{u}d)=\pi(\bar{u}d)^* \pi(\bar{s}u), \quad \pi(\bar{u}d)^* \pi(\bar{d}s)=\pi(\bar{d}s)^* \pi(\bar{u}d), \quad \pi(\bar{s}u)^* \pi(\bar{d}s)=\pi(\bar{d}s)^* \pi(\bar{s}u).\]

Hence we have

**Proposition 7.2** $Z_2$ is the center of $\Gamma_{uds}$, and $\pi(\bar{s}u), \pi(\bar{u}d), \pi(\bar{d}s)$ generates the quaternion group. Its group ring over $\mathbb{C}$ is $\mathbb{C} \otimes \mathbb{H}$.

It is convenient to represent these as follows by using Pauli matrices: Set $\alpha=e^{\frac{i\pi}{4}}, \beta=e^{\frac{i\pi}{2}}$ and

\[(7.4) \quad \bar{s}u=\begin{bmatrix} 0 & i\alpha & 0 \\ i\alpha & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \quad \bar{d}s=\begin{bmatrix} i\alpha & 0 & 0 \\ 0 & -i\alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad \bar{u}d=\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.\]

Note that $Z_2$ is not the polar elements. It is interesting that we obtain a kind of double cover of the quaternion group. Recall that $\{1, \varepsilon, \varepsilon^*, \varepsilon'\}$ does not form a group.

**Other 5 isomorphic groups**. Beside $\Gamma_{uds}$, there are 5 groups which is isomorphic to $\Gamma_{uds}$:

\[u\Gamma_{uds}\bar{u}, \quad d\Gamma_{uds}\bar{d}, \quad s\Gamma_{uds}\bar{s}, \quad \bar{u}^2\Gamma_{uds}u^2, \quad \bar{d}^2\Gamma_{uds}d^2.\]

These 6 groups all together seem to be deeply related to the octerions.

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