Abstract. We examine the relationship between the notion of Frobenius splitting and ordinariness for varieties. We show the following: a) the de Rham-Witt cohomology groups $H^i(X, W(\mathcal{O}_X))$ of a smooth projective Frobenius split variety are finitely generated over $W(k)$. b) we provide counterexamples to a question of V. B. Mehta that Frobenius split varieties are ordinary or even Hodge-Witt. c) a Kummer $K3$ surface associated to an Abelian surface is $F$-split (ordinary) if and only if the associated Abelian surface is $F$-split (ordinary). d) for a $K3$-surface defined over a number field, there is a set of primes of density one in some finite extension of the base field, over which the surface acquires ordinary reduction.

1. Introduction

Let $k$ be a perfect field of characteristic $p > 0$. An abelian variety $A$ over $k$ is said to be ordinary if the $p$-rank of $A$ is the maximum possible, namely equal to the dimension of $A$. The notion of ordinarity was extended by Mazur [19], to a smooth projective variety $X$ over $k$, using notions from crystalline cohomology. A more general definition was given by Bloch-Kato [4] and Illusie-Raynaud [14] using coherent cohomology. With this definition it is easier to see that ordinariness is an open condition. Ordinary varieties tend to have special properties, for example the existence of canonical Serre-Tate liftings for ordinary abelian varieties to characteristic 0, and the comparison theorems between crystalline cohomology and $p$-adic étale cohomology can be more easily established for such varieties. In brief, ordinary varieties play a key role in the study of varieties in characteristic $p > 0$.

One of the motivating questions in this paper is to study the relationship between the concept of Frobenius split varieties introduced by Mehta-Ramanathan [20], and ordinary varieties. Unlike ordinariness, the definition of Frobenius splitting can be extended to singular varieties, and this has proved to be of use in studying the cohomology and singularities of Schubert varieties. It was shown by Mehta and Srinivas [21], that smooth, projective varieties with trivial cotangent bundle, in particular abelian varieties, are ordinary if and only if they are Frobenius split. It can be seen from the theory of Cartier operator, that smooth, projective $F$-split surfaces are ordinary (see Theorem 2.4.1). Moreover any smooth, projective, ordinary variety with trivial canonical bundle is $F$-split (see Theorem 2.4.2).
We show (see Theorem 5.1.2) that if an abelian surface $A$ is $F$-split (hence ordinary), then the associated Kummer $K3$ surface $X$ is also $F$-split (and hence ordinary). We recall that the surface $X$ is obtained by blowing up the singularities of the singular surface $\tilde{A}$, obtained by identifying the points $x$ and $-x$ in $A$. Although this result can be proved by $l$-adic methods when the base field is finite, in this paper we prove this for a perfect field $k$, by relating ordinarity to Frobenius splitting for such varieties. In the case of abelian and $K3$-surfaces, it is possible to compare the notion of ordinarity with that of Frobenius splitting, and this allows us to handle the passage to the singular variety $\tilde{A}$.

Ordinary varieties are Hodge-Witt, in that the de Rham-Witt cohomology groups $H^i(X, W^j\Omega_X^j)$ are finitely generated over $W$. A natural question that arises is whether Frobenius split varieties are Hodge-Witt. We show (see Theorem 3.2.1) that for any smooth projective $F$-split variety over an algebraically closed field the cohomology groups $H^i(X, W(O_X))$ are of finite type as $W$-modules. Using the work of Illusie-Raynaud, we also see that the first differential $d_1^{i,0}$ is zero for all $i \geq 0$. During the course of writing of this paper, the first author refined these methods to control the nature of crystalline torsion for $F$-split varieties (see [15]) and has also shown that any smooth, projective and Frobenius split threefold is Hodge-Witt.

The foregoing results raise the possibility that Frobenius split varieties should be Hodge-Witt or even ordinary and indeed the question of whether or not Frobenius split varieties are ordinary was raised by V. B. Mehta. After the first draft of this paper and [15] were written and circulated, we found however that this general expectation, which had been further strengthened by low dimensional results like Theorem 2.4.1 for surfaces and [15] (for threefolds), turns out to be false in higher dimensions. In [15] it was shown that any Frobenius split, smooth projective threefold is Hodge-Witt. It turns out that this is best possible. We found examples of Frobenius split varieties which are not ordinary (of dimension at least three) and are not even Hodge-Witt (dimension at least four).

We also give an example of a smooth fibration $X \to Y$ of smooth, projective varieties $X$ and $Y$, where the base and fibers are ordinary, but the total space $X$ is not ordinary. This is in contrast to the fact that if $X$ is the projective bundle associated to a vector bundle on $Y$, then $X$ is ordinary if and only if $Y$ is ordinary. A variant of our method also gives an example of a variety defined over a number field, whose reduction modulo all but a finite set of primes is Hodge-Witt (and Frobenius split), but which has non-ordinary reduction for infinitely many primes.

One of the other motivating questions of this paper, is a conjecture of Serre (Conjecture 6.0.1) formulated originally in the context of abelian varieties. Let $K$ be a number field, and let $X$ be a smooth, projective variety defined over $K$. Then the conjecture is that there should be a positive density of primes of $K$, at which $X$ acquires ordinary reduction. We show that if $X$ is either an abelian variety or a $K3$ surface defined over $K$, (Theorem 6.6.2)
then there is a finite extension $L/K$ of number fields, such that the set of primes of $L$ at which $X$ has ordinary reduction in the case of $K3$ surfaces, or has $p$-rank at least two if $X$ is an abelian variety, is of density one. We note here that a proof of the result for a class of $K3$ surfaces was also given by Tankeev (see [31]) under somewhat restrictive hypotheses. Our proof mirrors closely the proof given by Ogus for abelian surfaces. The results of this section are essentially independent of the contents of the rest of the paper.

In the final section we study the relationship between ordinariness and the torsion in the de Rham-Witt cohomology of varieties and discuss some questions and conjectures.

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2. Preliminaries

2.1. Ordinary varieties. Let $X$ be a smooth projective variety over a perfect field $k$ of positive characteristic. Following Bloch-Kato [4] and Illusie-Raynaud [14], we say that $X$ is ordinary if $H^i(X, B^j_X) = 0$ for all $i \geq 0$, $j > 0$, where

$$B^j_X = \text{image} \left( d : \Omega^{j-1}_X \to \Omega^j_x \right).$$

If $X$ is an abelian variety, then it is known that this definition coincides with the usual definition [4]. By [14, Proposition 1.2], ordinarity is an open condition in the following sense: if $X \to S$ is a smooth, proper family of varieties parameterized by $S$, then the set of points $s$ in $S$, such that the fiber $X_s$ is ordinary is a Zariski open subset of $S$. Although the following proposition is well known, we present it here as an illustration of the power of this fact.

Proposition 2.1.1. Let $A_{g,1,n}$ denote the moduli space of principally polarized abelian varieties $A$ of dimension $g$, equipped with a level structure of level $n \geq 5$, and $(n,p) = 1$. Then the set of ordinary points is open and dense in the moduli of principally polarized abelian varieties of dimension $g$.

Proof. Since the level $n \geq 5$, it is known that we obtain a fine moduli space over $\mathbb{Z} \left[ \frac{1}{n} \right]$. Further since the points of the moduli space over $\mathbb{C}$, are uniformized by the Siegel upper half plane, $A_{g,1,n}$ is an irreducible smooth variety over $\mathbb{Z} \left[ \frac{1}{n} \right]$. Since $p$ is coprime to $n$, the moduli problem specializes, and we see that $A_{g,1,n} \otimes \mathbb{Z} \text{Spec}(\mathbb{F}_p)$ is irreducible. By [13, Proposition 1.2], the ordinary locus is open. To show it is dense it suffices to prove it is not empty. But this is easily done by choosing an ordinary elliptic curve
(and there is always one in every characteristic) together with its principal polarization. Then we can take our ordinary abelian variety with principal polarization to be the product of this ordinary elliptic curve and we are done.

2.2. Cartier Operator. Let $X$ be a smooth proper variety over a perfect field of characteristic $p > 0$, and let $F_X$ (or $F$) denote the absolute Frobenius of $X$. We recall a few basic facts about Cartier operators from [11]. The first fact we need is that we have a fundamental exact sequence of locally free sheaves

$$0 \to B^i_X \to Z^i_X \xrightarrow{C} \Omega^i_X \to 0,$$

where $Z^i_X$ is the sheaf of closed $i$-forms, where $C$ is the Cartier operator. The existence of this sequence is the fundamental theorem of Cartier (see [11]).

Since the Cartier operator is also the trace map in Grothendieck duality theory for the finite flat map $F$, we have a perfect pairing

$$F^*_X(\Omega^j_X) \otimes F^*_X(\Omega^{n-j}_X) \to \Omega^n_X$$

where $n = \dim(X)$, and the pairing is given by $(\omega_1, \omega_2) \mapsto C(\omega_1 \wedge \omega_2)$. This pairing is perfect and $O_X$-bilinear (see [21]).

In particular, on applying $\text{Hom}(\cdot, \Omega^n_X)$ to the exact sequence

$$0 \to B^n_X \to Z^n_X \xrightarrow{C} \Omega^n_X \to 0$$

we get

$$0 \to \mathcal{O}_X \xrightarrow{\text{Frob}} \mathcal{O}_X \to B^1_X \to 0$$

2.3. $F$-split varieties. In this section we recall a few basic facts about ordinary and Frobenius split ($F$-split) varieties. In this section $X$ is a normal, projective variety over a perfect field $k$. Let $F : X \to X$ be the Frobenius morphism of $X$.

Recall that $X$ is $F$-split if the canonical exact sequence of sheaves

$$0 \to \mathcal{O}_X \xrightarrow{\text{Frob}} \mathcal{O}_X \to B^1_X \to 0$$

splits. Note that when $X$ is smooth this is an exact sequence of locally free $\mathcal{O}_X$-modules.

Frobenius splitting was introduced by Mehta and Ramanathan in [20] and a number of remarkable properties were also investigated in that paper. It is known for instance that an abelian variety is $F$-split if and only if it is ordinary in the usual sense (see [21]).

In analogy with ordinary varieties, we now consider the openness of the $F$-split condition. Let $f : X \to S$ be a smooth projective morphism. Let $X'$ denote the fiber product $X \times_{(S,F_S)} S$. Then one has a morphism

$$F_{X/S} : X \to X'$$

which is called the relative Frobenius morphism (see [11]). The restriction of $F_{X/S}$ to the fibers of $f$ induce the Frobenius morphism on the fibers.

The following proposition is the relative version of Proposition 9 of [20].
Proposition 2.3.1. Let \( f : X \to S \) be a smooth projective morphism of schemes in characteristic \( p \). Let \( F_{X/S} : X \to X' \) be the relative Frobenius morphism. Then for a point \( s \in S \), the fiber \( X_s = f^{-1}(s) \) is F-split if and only if the natural map

\[
R^n f_*(\Omega^d_{X/S}) \otimes k(s) \to R^n f_*(F_{X/S}^*\Omega^d_{X/S}) \otimes k(s)
\]

is injective.

We note that the map

\[
F_S^*(R^n f_*(\Omega^d_{X/S})) \to R^n f_*(F_{X/S}^*(\Omega^d_{X/S}))
\]

is \( O_S \)-linear. From this and Proposition 2.3.1 we get:

Proposition 2.3.2. With the notations of Proposition 2.3.1, there exists a Zariski open subset \( U \subset S \) such that all the fibers of \( f \) over points of \( U \) are F-split.

2.4. Ordinary and F-split varieties. In this subsection we record a key lemma which we need and also record our proof that Frobenius split surfaces are ordinary. Our main tool here is the duality induced by the Cartier operator (see [21]).

We have the following lemma:

Lemma 2.4.1. Let \( X \) be any smooth projective variety over a perfect field. Assume \( X \) is F-split. Then for all \( i \geq 0 \),

\[
H^i(X, B^1_X) = 0.
\]

Proof. As \( X \) is F-split, it follows that \( F_* (\mathcal{O}_X) = \mathcal{O}_X \oplus B^1_X \). Hence

\[
H^i(X, F_* (\mathcal{O}_X)) = H^i(\mathcal{O}_X) \oplus H^i(X, B^1_X).
\]

But by the Leray spectral sequence applied to the Frobenius morphism and the projection formula we see that

\[
H^i(X, F_* (\mathcal{O}_X)) \simeq H^i(X, \mathcal{O}_X)
\]

and hence we see that

\[
\dim H^i(\mathcal{O}_X) = \dim H^i(\mathcal{O}_X) + \dim H^i(B^1_X)
\]

and so the lemma is proved. \( \square \)

Theorem 2.4.1. Let \( X \) be any smooth projective, F-split surface over a perfect field. Then \( X \) is ordinary.

Proof. By Lemma 2.4.1 we know that for any F-split variety \( X \),

\[
H^i(X, B^1_X) = 0
\]

for all \( i \). So when \( X \) is a surface we need to check that the same vanishing is also valid for \( B^2_X \). But this is immediate from Serre duality and the
following fact: Cartier operator induces a perfect pairing (we write it under assumption that $X$ is a surface)

$$F_*(\mathcal{O}_X) \otimes F_*(\Omega^2_X) \to \Omega^2_X,$$

given by $(f, w) \mapsto C(fw)$ and this induces a perfect pairing $B^1_X \otimes B^2_X \to \Omega^2_X = \omega_X$ (see [21]).

**Theorem 2.4.2.** Let $X$ be any smooth projective, ordinary variety over a perfect field $k$. If the canonical bundle of $X$ is trivial, then $X$ is Frobenius split.

**Proof.** The obstruction to the splitting of the sequence

$$0 \to \mathcal{O}_X \to F_*(\mathcal{O}_X) \to B^1_X \to 0,$$

is an element of $\text{Ext}^1(B^1_X, \mathcal{O}_X) \simeq H^1(X, (B^1_X)^*)$. The duality pairing induced by the Cartier operator implies that

$$(B^1_X)^* \simeq B^n_X \otimes \omega_X,$$

where $\omega_X$ denotes the canonical bundle. Since we have assumed that $\omega_X$ is trivial, it follows that

$$\text{Ext}^1(B^1_X, \mathcal{O}_X) \simeq H^1(X, (B^1_X)^*) \simeq H^1(X, B^n_X) = 0$$

where the vanishing follows from the ordinarity assumption. Hence $X$ is Frobenius split.

### 3. De Rham-Witt cohomology of $F$-split varieties

#### 3.1. De Rham-Witt cohomology.** The standard reference for de Rham-Witt cohomology is [11]. Throughout this section, the following notations will be in force. Let $k$ be an algebraically closed field of characteristic $p > 0$, and $X$ a smooth, projective variety over $k$. Let $W = W(k)$ be the ring of Witt vectors of $k$. Let $K = W[1/p]$ be the quotient field of $W$. Note that as $k$ is perfect, $W$ is a Noetherian local ring with a discrete valuation and with residue field $k$. For any $n \geq 1$, let $W_n = W(k)/p^n$. $W$ comes equipped with a lift $\sigma : W \to W$, of the Frobenius morphism of $k$, which will be called the Frobenius of $W$. We define a non-commutative ring $R^0 = W[\sigma, V, F]$, where $F, V$ are two indeterminate subject to the relations $FV = VF = p$ and $Fa = \sigma(a)F$ and $aV = V\sigma(a)$. The ring $R^0$ is called the Dieudonne ring of $k$. The notation is borrowed from [11].

Let $\{W_n\Omega^i_X\}_{n \geq 1}$ be the de Rham-Witt pro-complex constructed in [11]. It is standard that for each $n \geq 1, i, j \geq 0$, $H^i(X, W_n\Omega^j_X)$ are of finite type over $W_n$. We define

$$H^i(X, W\Omega^j_X) = \lim_{\leftarrow} H^i(X, W_n\Omega^j_X),$$

which are $W$-modules of finite type up to torsion. These cohomology groups are called Hodge-Witt cohomology groups of $X$. 


Definition 3.1.1. $X$ is Hodge-Witt if for $i, j \geq 0$, the Hodge-Witt cohomology groups $H^i(X, W\Omega^j_X)$ are finite type $W$-modules.

The properties of the de Rham-Witt pro-complex are reflected in these cohomology modules and in particular we note that for each $i, j$, the Hodge-Witt groups $H^i(X, W\Omega^j_X)$ are left modules over $R^0$. The complex $W\Omega^*_{\mathcal{X}}$ defined in a natural manner from the de Rham-Witt pro-complex computes the crystalline cohomology of $X$ and in particular there is a spectral sequence

$$E_1^{i,j} = H^i(X, W\Omega^j_X) \Rightarrow H^*_{\text{cris}}(X/W)$$

This spectral sequence induces a filtration on the crystalline cohomology of $X$ which is called the slope filtration and the spectral sequence above is called the slope spectral sequence (see [11]). It is standard (see [11] and [14]) that the slope spectral sequence degenerates at $E_1$ modulo torsion (i.e. the differentials are zero on tensoring with $K$) and at $E_2$ up to finite length (i.e. all the differential have images which are of finite length over $W$).

In dealing with the slope spectral sequence it is more convenient to work with a bigger ring than $R^0$. This ring was introduced in [14]. Let $R = R^0 \oplus R^1$ be a graded $W$-algebra which is generated in degree 0 by variables $F, V$ with the properties listed earlier (so $R^0$ is the the Dieudonne ring of $k$) and $R^1$ is a bimodule over $R^0$ generated in degree 1 by $d$ with the properties $d^2 = 0$ and $FdV = d$, and $da = ad$ for any $a \in W$. The algebra $R$ is called the Raynaud-Dieudonne ring of $k$ (see [14]). The complex $(E_1^{*,i}, d_1)$ is a graded module over $R$ and is in fact a coherent, left $R$-module (in a suitable sense, see [14]).

3.2. A finiteness result. For a general variety $X$, the de Rham-Witt cohomology groups are not of finite type over $W$, and the structure of these groups reflects the arithmetical properties of $X$. For instance, in [4], [14] it is shown that for ordinary varieties $H^i(X, W\Omega^j)$ are of finite type over $W$. The following theorem lends more evidence towards the general expectation that Frobenius split varieties should be ordinary.

Theorem 3.2.1. Let $X/k$ be any smooth, projective, $F$-split variety over an algebraically closed field $k$ of characteristic $p > 0$. Then for each $i \geq 0$, $H^i(X, W(O_X))$ is a finite type $W(k)$-module.

Remark 3.2.1. When the formal Brauer group $\hat{Br}(X)$ of $X$ (associated by Artin and Mazur; see [4]) is representable, $H^i(X, W(O_X))$ is the Cartier module of this formal group. When this module is free of finite type over $W$, the formal Brauer group is a $p$-divisible group of height equal to the dimension $\dim_K H^i(X, W(O_X)) \otimes W K$.

Corollary 3.2.1. Let $X$ be as in Theorem 3.2.1. Then for all $i \geq 0$, the differential

$$d^{i,0}_1: H^i(X, W(O_X)) \to H^{i+1}(X, W\Omega^1_X)$$
is zero.

Proof. This is immediate from Theorem 3.2.1 and [14].

Remark 3.2.2. Let \( X \) be any smooth projective variety. In [14] it is has been shown that for all \( j \), \( H^1(X, W^j \Omega^1_X) \) are finite type \( W \)-modules.

Remark 3.2.3. By combining the theory of dominoes with Theorem 3.2.1 and the foregoing remark, it can be shown that Frobenius split, smooth, projective threefolds are Hodge-Witt [15].

Before we give the proof of the above theorem, we need a few preparatory lemmas. Our proofs use the theory of higher Cartier operators as outlined in [11]. To set up conformity with notations from previous sections we recall Illusie’s notations and our definitions

\begin{equation}
B_1 \Omega^1_X = B_1^1 = \text{image}(d : \mathcal{O}_X = \Omega^0_X \to \Omega^1_X)
\end{equation}

and

\begin{equation}
Z_1 \Omega^1_X = Z^1 = \ker(d : \Omega^1_X \to \Omega^2_X)
\end{equation}

The higher Cartier sheaves, \( Z_n \Omega^i_X, B_n \Omega^i_X \) are defined inductively in [11] (see page 519 of [11]). The formation of these sheaves is compatible with arbitrary base change (see [11], page 519).

Lemma 3.2.1. Let \( X \) be any smooth, projective, \( F \)-split variety over \( k \). Then for all \( n \geq 0 \) and for all \( i \geq 0 \) we have \( H^i(X, B_n \Omega^1_X) = 0 \).

Proof. The case \( n = 1 \) is Lemma 2.4.1. We prove the result by induction on \( n \). We recall that we have an exact sequence (the arrow on the extreme right is the Cartier operator):

\[ 0 \to B_1 \Omega^1_X \to B_{n+1} \Omega^1_X \xrightarrow{C^{-1}} B_n \Omega^1_X \to 0. \]

(see [11], page 519). This is essentially the definition of \( B_{n+1} \) using \( B_n \). The result now follows trivially from the above exact sequence and the result for \( n = 1 \).

Lemma 3.2.2. Let \( X \) be a smooth, projective, \( F \)-split variety over an algebraically closed field \( k \) of characteristic \( p > 0 \). Then for all \( n \geq 0 \) and for all \( i \geq 0 \), we have \( H^i(X, Z_n \Omega^1_X) \simeq H^i(X, \Omega^1_X) \). In particular we have \( \forall i > 0, \forall n \geq 1 \),

\[ \dim H^i(X, Z_n \Omega^1_X) = \dim H^i(X, \Omega^1_X). \]

Proof. We recall the exact sequence (page 531, 2.5.1.2 of [11]):

\[ 0 \to B_n \Omega^1_X \to Z_n \Omega^1_X \to \Omega^1_X \to 0. \]

Then proof follows from the vanishing of cohomology of \( B_n \Omega^1_X \).
Proof. [of Theorem 3.2.1] By ([11], page 613, Proposition 2.16), it suffices to prove that for all \( j \geq 0 \) and for all \( n \geq 0 \), \( H^j(X, Z_n \mathcal{O}_X) \) has bounded dimension. But \( Z_1 \mathcal{O}_X = \ker(d : \mathcal{O}_X \to \Omega^1_X) \) and by definition \( Z_n \mathcal{O}_X \to Z_{n-1} \mathcal{O}_X \) is an isomorphism for all \( n \geq 2 \) (the arrow in this isomorphism is the Cartier operator, ([11], see 2.5.1.2, page 531). Thus the required cohomology has dimension independent of \( n \). Next we need to check \( H^j(X, B_n \Omega^1_X) \) has bounded dimension for all \( n \). But by Lemma 3.2.1 this group is zero! Thus we can apply Proposition 2.16 of [11] to deduce that \( H^i(X, W(\mathcal{O}_X)) \) is a finite type \( W(k) \)-module.

4. Examples

4.1. Two Questions. One of the main motivations for this paper, are the following questions raised by V. B. Mehta.

Question 4.1.1. Is any smooth projective, Frobenius split variety over a perfect field of characteristic \( p \) of Hodge-Witt type?

The above question is a weaker variant of the following.

Question 4.1.2. Is any smooth projective, Frobenius split variety Bloch-Kato ordinary?

We know by Theorem 2.4.1 that a smooth, projective Frobenius split surface is ordinary. In [13], it is shown that any Frobenius split smooth, projective three fold is Hodge-Witt. Further it is known that for abelian varieties, the notions of Frobenius splitting and ordinarity coincide [21]. However in contrast to the expectation created by these results, we show in this section that the first question is false in dimensions greater than 3, and the second question is false in dimensions bigger than two.

Our examples also give examples of varieties which are Hodge-Witt, but are not ordinary. These examples also provide examples of smooth, projective varieties \( f : X \to Y \), such that both \( Y \) and the the (smooth) fibers of \( f \) are ordinary, but \( X \) is not ordinary.

4.2. Let \( X \) be a smooth, projective variety with canonical bundle \( \omega \). We recall that by Cartier duality there is functorial isomorphism [20, Proposition 5], [22],

\[ F_s \omega^{1-p} \simeq \text{Hom}_{\mathcal{O}_X}(F_s \mathcal{O}_X, \mathcal{O}_X). \]

By means of this, we obtain a natural identification

\[ H^0(X, \omega^{1-p}_X) \simeq \text{Hom}_{\mathcal{O}_X}(F_s \mathcal{O}_X, \mathcal{O}_X). \]

Definition 4.2.1. A section \( \sigma \in H^0(X, \omega^{1-p}_X) \), such that under the above isomorphism, \( \sigma \) provides a splitting of \( X \), will be called as a splitting section.

The key result we need is a criteria on the relative embedding of a smooth subvariety in a smooth, Frobenius split variety, such that the blow up along the subvariety remains Frobenius split. We recall now some of the concepts.
and results regarding compatible Frobenius splitting of subvarieties. Let $X$ be a Frobenius split variety, and let
\[ \sigma : F_*(\mathcal{O}_X) \to \mathcal{O}_X \]
be a splitting of the Frobenius morphism. Suppose $Y$ is a subvariety of $X$, defined by a sheaf of ideals $\mathcal{I}_Y \subset \mathcal{O}_X$. In this case we have a notion of $Y$ being compatibly Frobenius split in $X$ as follows:

**Definition 4.2.2.** $Y$ is said to be compatibly split by $\sigma$ in $X$, if
\[ \sigma(F_*(\mathcal{I}_Y)) \subset \mathcal{I}_Y. \]

Let $X$ be a nonsingular variety and $Y$ a nonsingular subvariety of codimension $d \geq 2$. Denote by $B_Y(X)$ the blow up of $X$ along $Y$, and by $E$ the exceptional divisor. The following result follows quite easily from \[18, Proposition 2.1\].

**Proposition 4.2.1.** Let $s \in H^0(X, \omega_{X}^{-1})$. Suppose that $s^{p^{-1}}$ is a splitting section of $X$, and that it vanishes to order $(d-1)$ or $d$ generically along $Y$. Then $s^{p^{-1}}$ extends to a splitting of $B_Y(X)$. Moreover if $s$ vanishes to order $d$ generically along $Y$, then $E$ is compatibly split in $B_Y(X)$.

On the other hand, we have the following criterion for a blow-up to be ordinary or Hodge-Witt \[13, 9\]:

**Proposition 4.2.2.** $B_Y(X)$ is ordinary (or Hodge-Witt) if and only if both $X$ and $Y$ are ordinary (resp. Hodge-Witt).

The proof of this proposition follows from the decomposition of $W$-modules, compatible with the action of the Frobenius \[9, IV 1.1.9\],
\[ H^i(X, W\Omega^j_X) \oplus \bigoplus_{0 \leq i < d} H^{i-1}(Y, W\Omega^{j-1}_Y) \to H^i(B_Y(X), W\Omega^j_{B_Y(X)}), \]
and the fact that a smooth, proper variety $Z$ is ordinary if and only if
\[ F : H^j(Z, W\Omega^i_Z) \to H^j(Z, W\Omega^i_Z) \]
is an isomorphism for all $i$ and $j$.

Combining the above propositions, we obtain the following theorem:

**Theorem 4.2.3.** Let $X$ be a smooth, projective Frobenius split variety, with a splitting section $s^{p^{-1}}$ as above. Suppose that $s$ vanishes to order precisely $(d-1)$ generically along a smooth subvariety $Y$ of codimension $d$ in $X$. Further assume that $Y$ is not ordinary (or not Hodge-Witt). Then $B_Y(X)$ is Frobenius split but not ordinary (resp. not Hodge-Witt).

4.3. We can now give the examples of Frobenius split varieties which are not ordinary or Hodge-Witt.

**Example 4.3.1.** Let $E$ be a supersingular elliptic curve in $\mathbb{P}^3$. $E$ is contained in the zero locus of non-degenerate quadric $q$. Let $t_1$ and $t_2$ be linear polynomials such that the ideals generated by choosing any combination of $q$, $t_1$, $t_2$ define complete intersection subvarieties in $\mathbb{P}^3$. Then it can be
checked using \[20, \text{Proposition 7}\] that the section \(s = qt_1t_2 \in \mathcal{O}(4)\) gives rise to a splitting of \(\mathbb{P}^3\), vanishing to order 1 along \(E\). Hence the blow up of \(\mathbb{P}^3\) is Frobenius split (and is Hodge-Witt) but is not ordinary.

**Example 4.3.2.** A natural question that arises in the study of the geometry of ordinary varieties, is whether a variety is ordinary, if it is fibered over an ordinary variety, such that the smooth fibres are ordinary. The above example also provides an example of a variety fibered over \(\mathbb{P}^1\), such that the (smooth) fibers are ordinary but the variety itself is not ordinary. The elliptic curve is defined as the complete intersection of two non-degenerate quadrics which generates a pencil of quadrics. The strict transform of these quadrics in the blow-up gives a fibration of \(B_E(\mathbb{P}^3)\) over \(\mathbb{P}^1\) by Frobenius split varieties. However Frobenius split (smooth) surfaces are ordinary, and this gives us the desired example.

**Example 4.3.3.** The above example can be generalized. Let \(Y\) be a smooth hypersurface in \(\mathbb{P}^{n+1} \subset \mathbb{P}^{n+2}\), for example a Fermat hypersurface of degree \(m\). Choose a system of coordinates \(x_0, \ldots, x_{n+1}\) on \(\mathbb{P}^{n+1}\), where \(\mathbb{P}^{n+1}\) is given by \(x_0 = 0\). Then \(s^{p-1} = (x_0 \cdots x_{n+2})^{p-1}\) is a splitting section vanishing precisely to order \(p - 1\) generically along \(Y\). The blow-up of \(\mathbb{P}^{n+2}\) along \(Y\) is then Frobenius split. Recall that results of \[32\] give explicit conditions on \((n, p, m)\) under which the Fermat hypersurface \(Y\) is not Hodge-Witt. For instance assume that \(p \not\equiv 1 \mod m\), \(p\) does not divide \(m\) and \(n \geq 5, m \geq 5\). Then this hypersurface is not Hodge-Witt and so the blowup is not Hodge-Witt. But the blowup is Frobenius split but neither Hodge-Witt nor ordinary. The results of \[32\] can also be used to give examples in dimensions four and five as well.

**Example 4.3.4.** Let \(E/\mathbb{Q}\) be an elliptic curve. We embed \(E\) in \(\mathbb{P}^3\) by using the embedding given by the linear system \(4(\infty)\). The blowup of \(\mathbb{P}^3\) along \(E\) has \(F\)-split and Hodge-Witt reduction at all but finite number of primes. In fact, by the blowup formula for Hodge-Witt cohomology, blowup of \(\mathbb{P}^3\) along any smooth projective embedded curve is Hodge-Witt. By \[8\] we know that the reduction of \(E\) is supersingular at infinitely many primes, and if \(E\) has CM, then it has supersingular reduction at a set of primes of density \(1/2\). Hence there are infinitely many primes where the blowup has \(F\)-split (and Hodge-Witt) but non-ordinary reduction.

**Example 4.3.5.** The above examples might lead one to raise the question whether Fano \(F\)-split varieties are ordinary or Hodge-Witt. But even this turns out to be false. See Example \[7.1.1\].

5. **Kummer surfaces**

In this section we explore further the relationship between Frobenius split and ordinary varieties, especially in the context of Kummer surfaces.
5.1. Kummer Surfaces over perfect fields. Let $A$ be an abelian surface over a perfect field $k$ of odd, positive characteristic $p$. Denote by $\iota : A \to A$, the involution $x \mapsto -x$ on the abelian surface. Let $\hat{A}$ denote the quotient variety of $A$ with respect to this involution. It is known that $\hat{A}$ is Gorenstein having only quotient singularities. Denote by $\omega_{\hat{A}}$ the dualizing sheaf. It is known that $\omega_{\hat{A}} \cong \mathcal{O}_{\hat{A}}$, the structure sheaf on $\hat{A}$.

There exists a smooth, projective variety $X$, which is a blow up of $\hat{A}$, at the sixteen singular points of $\hat{A}$. $X$ is a $K3$-surface, in that it is simply connected and $H^2(X, \mathcal{O}_X)$ is one dimensional. $X$ is the Kummer $K3$-surface associated to the abelian surface $A$.

**Theorem 5.1.1.** With notation as in the above theorem, $A$ is Frobenius split if and only if $X$ is Frobenius split.

Since the canonical bundles of $A$ and $X$ are trivial, we see by Theorems 2.4.1 and 2.4.2, that the above theorem is equivalent to proving the following:

**Theorem 5.1.2.** Let $A$ be an abelian surface over a perfect field $k$ of characteristic $p > 0$. Let $X$ be the associated Kummer $K3$ surface. Then $X$ is ordinary if and only if $A$ is ordinary.

**Proof.** We will need the formalism of Section 4.2 for Gorenstein varieties. We recall this from [22].

Let $Z$ be a scheme over $k$. In order to split $Z$, we need a map $F_* \mathcal{O}_Z \to \mathcal{O}_Z$ such that the composite with the $\mathcal{O}_Z \to F_* \mathcal{O}_Z \to \mathcal{O}_Z$ is the identity.

Suppose now that $Z$ is a reduced equidimensional Gorenstein $k$-scheme. By applying duality for the Frobenius morphism, we obtain a canonical isomorphism of sheaves on $Z$, as in [22, Lemma 1],

$$F_* \omega_Z^{1-p} \cong \text{Hom}_{\mathcal{O}_Z}(F_* \mathcal{O}_Z, \mathcal{O}_Z),$$

(5.1.1)

where $\omega_Z$ denotes the dualizing sheaf of $Z$. In particular, Frobenius splittings of $Z$ are induced by sections of $H^0(Z, \omega_Z^{1-p})$.

Let $Z$ denote any one of the varieties $A$, $\hat{A}$, $X$. By definition $A$ and $X$ are smooth, and it is known that $\hat{A}$ is Gorenstein. Further the dualizing sheaf of $Z$ is the structure sheaf $\mathcal{O}_Z$ in each of the above cases. Moreover the $\mathbb{Z}/2\mathbb{Z}$ action is trivial on $H^0(A, \mathcal{O}_A)$, we have a natural isomorphism

$$H^0(A, \mathcal{O}_A) \cong H^0(\hat{A}, \mathcal{O}_{\hat{A}}) \cong H^0(X, \mathcal{O}_X).$$

Let $s$ denote a section in any one of the above cohomology groups, and we continue to denote by $s$, its image in the other cohomology groups. By the isomorphism (5.1.1), $s$ gives rise to a morphism $F_* \mathcal{O}_Z \to \mathcal{O}_Z$. To check that $s$ gives a splitting section, that the composite $\mathcal{O}_Z \to F_* \mathcal{O}_Z \to \mathcal{O}_Z$ is the identity, it is enough to check at a point $P$ on $Z$, since $Z$ is projective and any global map $\mathcal{O}_Z \to \mathcal{O}_Z$ is a constant. By the local nature of duality, the morphism (5.1.1) is an isomorphism of sheaves, and it is enough to check the splitting condition in the formal neighborhood of a smooth point $P$ on $Z$. 
We now choose $P$ to be a non 2-torsion point on $A$. We continue to denote by $P$, the image of $P$ in $\tilde{A}$ and $X$. We then have an isomorphism of the formal completions, 

$$\hat{O}_{A,P} \cong \hat{O}_{\tilde{A},P} \cong \hat{O}_{X,P}$$

compatible with the isomorphism 5.1.1. Hence a section $s$ gives a splitting section for $A$ if and only if it gives a splitting section for $\tilde{A}$, or equivalently for $X$. Hence $A$ is $F$-split is equivalent to $X$ being $F$-split, and this is equivalent to $\tilde{A}$ being $F$-split.

Remark 5.1.1. Over finite fields, it is possible to give a different proof using $l$-adic methods. For a smooth, projective surface $X$ with trivial canonical bundle, the condition for being Frobenius split is that the Frobenius $F : H^2(X, O_X) \to H^2(X, O_X)$ is an isomorphism. When the surfaces are defined over finite fields, it follows from the Katz congruence formula for the zeta function [16], applied to a $K3$-surface, that there is precisely one eigenvalue of the crystalline Frobenius acting on $H^2_{\text{crys}}(X/W)$ which is a $p$-adic unit. From the shape of the Hodge polygon and duality in the case of abelian and $K3$-surfaces, we then conclude that this is equivalent to $K3$-surface being ordinary. By comparing $H^2(A, O_A)$ and $H^2(X, O_X)$, we obtain a different proof over finite fields, that the Kummer $K3$-surface is ordinary if the abelian surface is ordinary. However these $l$-adic methods do not seem to generalize to an arbitrary perfect base field.

Remark 5.1.2. In the course of the proof of the Tate conjecture for ordinary $K3$-surfaces $X$ over a finite field [26], it is shown that the Kuga-Satake abelian variety $K(X)$ associated to $X$ is ordinary, provided $X$ is ordinary. If $X$ is a Kummer $K3$-surface associated to an abelian surface $A$, then it is known that $K(X)$ is isogeneous to a sum of copies of $A$. It follows that if $X$ is ordinary, then $A$ is ordinary.

Remark 5.1.3. Combining the above theorem with a theorem of Ogus [27, page 372], it follows that for a Kummer $K3$-surface defined over a number field, there is a finite extension over which the variety acquires ordinary reduction at a set of primes of density one. In the next section, we will show that Ogus’ proof extends to prove this result for any $K3$-surface defined over a number field.

6. PRIMES OF ORDINARY REDUCTION FOR $K3$ SURFACES

The following more general question, which is one of the motivating questions for this paper, is the following conjecture which is well-known and was raised initially for abelian varieties by Serre:

Conjecture 6.0.1. Let $X/K$ be a smooth projective variety over a number field. Then there is a positive density of primes $v$ of $K$ for which $X$ has good ordinary reduction at $v$. 

Let $K$ be a number field, and let $X$ denote either an abelian variety or a $K3$ surface defined over $K$. Our aim in this section is to show that there is a finite extension $L/K$ of number fields, such that the set of primes of $L$ at which $X$ has ordinary reduction in the case of $K3$ surfaces, or has $p$-rank at least two if $X$ is an abelian variety of dimension at least two, is of density one. Our proof closely follows the method of Ogus for abelian surfaces (see [27, page 372]).

We note here that a proof of the result for a class of $K3$ surfaces was also given by Tankeev (see [31]) under some what restrictive hypothesis. The question of primes of ordinary reduction for abelian variet ies has also been treated recently by R. Noot (see [24]), R. Pink (see [28]) and more recently A. Vasiu (see [33]) has studied the question for a wider class of varieties. The approach adopted by these authors is through the study of Mumford-Tate groups.

Let $O_K$ be the ring of integers of $K$; for a finite place $v$ of $K$ lying above a rational prime $p$, let $O_v$ be the completion of $O_K$ with respect to $v$ and let $k_v$ be the residue field at $v$ of cardinality $q_v = p^{e_v}$. Assume that $v$ is a place of good reduction for $X$ as above and write $X_v$ for the reduction of $X$ at $v$. We recall here the following facts:

6.1. (Weil, Deligne, Ogus) [27]: The Frobenius endomorphism $F_v$ is a semi-simple endomorphism of the $l$-adic cohomology groups $H^i_l := H^i_{et}(X \otimes \overline{K}, \mathbb{Q}_l)$ for a prime $l \neq p$. The $l$-adic characteristic polynomial $P_{i,v}(t) = \det(1 - tF_v | H^i_{et}(X \otimes \overline{K}, \mathbb{Q}_l))$ is an integral polynomial and is independent of $l$. Let

$$a_v = \text{Tr}(F_v | H^2_{et}(X \otimes \overline{K}, \mathbb{Q}_l))$$

denote the trace of the $l$-adic Frobenius acting on the second étale cohomology group. $a_v$ is a rational integer.

6.2. (Deligne-Weil estimates) [6]: It follows from Weil estimates proved by Weil for abelian varieties and by Deligne in general that

$$|a_v| \leq dp$$

where $d = \dim H^2$ is a constant independent of the place $v$.

6.3. (Katz-Messing theorem) [17]: Let $\phi_v$ denote the crystalline Frobenius on $H^i_{cris}(X/W(k_v))$. $\phi_v^{cr}$ is linear over $W(k_v)$, and the characteristic polynomials of the crystalline Frobenius and the $l$-adic Frobenius are equal:

$$P_{i,v}(t) = \det(1 - t\phi_v^{cr} | H^i_{cris}(X/W(k_v)) \otimes K_v).$$

6.4. (Semi-simplicity of the crystalline Frobenius) [27]: If $X$ is a $K3$-surface, then it is known by [27, Theorem 7.5] that the crystalline Frobenius $\phi_v^{cr}$ is semi-simple. We remark that although this result is not essential in the proof of the theorem, it simplifies the proof.
6.5. (Mazur’s theorem) [19], [7], [3]: There are two parts to the theorem of Mazur that we require. After inverting finitely many primes \( v \in S \) in \( K \), we can assume that \( X \) has good reduction outside \( S \). Using Proposition 6.6.1 (see below) we can assume that \( H^i(X, \Omega^j_X) \) and \( H^i(X/W(k_v)) \) are torsion-free outside a finite set of primes of \( K \). As \( X \) is defined over characteristic zero, the Hodge to de Rham spectral sequence degenerates at \( E_1 \) stage. Thus all the hypothesis of Mazur’s theorem are satisfied. The two parts of Mazur’s theorem that we require are the following:

6.5.1. (Mazur’s proof of Katz’s conjecture): Let \( L_v \) be a finite extension of field of fractions of \( W(k_v) \), over which the polynomial \( P_{i,v}(t) \) splits into linear factors. Let \( w \) denote a valuation on \( L_v \), such that \( w(p) = 1 \). The Newton polygon of the polynomial \( P_{i,v}(t) \) lies above the Hodge polygon in degree \( i \), defined by the Hodge numbers \( h^{j,i} \) of degree \( i \). Moreover they have the same endpoints.

6.5.2. (Divisibility property): The crystalline Frobenius \( \phi_v \) is divisible by \( p^i \) when restricted to \( F^iH^d_{dR}(X/W(k_v)) := H(X/W(k_v), \Omega^d_{X}) \).

6.6. (Crystalline torsion): We will also need the following proposition which is certainly well-known but as we use it in the sequel, we record it here for convenience.

**Proposition 6.6.1.** Let \( X/K \) be a smooth projective variety. Then for all but finitely many nonarchimedean places \( v \), the crystalline cohomology \( H^i_{\text{cris}}(X_v/W(k_v)) \) is torsion free for all \( i \).

**Proof.** We choose a smooth model \( X \to \text{Spec}(O_K) - V(I) \) for some non-zero proper ideal \( I \subset O_K \). The relative de Rham cohomology of the smooth model \( X \) is a finitely generated \( O_K \)-module and has bounded torsion. After inverting a finite set \( S \) of primes, we can assume that \( H^i_{dR}(X, O_S) \) is a torsion-free \( O_S \) module, where \( O_S \) is the ring of \( S \)-integers in \( K \). By the comparison theorem of Berthelot (see [4]), there is a natural isomorphism of the crystalline cohomology of \( X_p \) to that of the de Rham cohomology of the generic fiber of a lifting to \( \mathbb{Z}_p \).

\[
H^i_{\text{cris}}(X_v/W(k_v)) \equiv H^i_{dR}(X, O_S) \otimes W(k_v).
\]

This proves our proposition. Moreover the proof shows that we can assume after inverting some more primes, that the Hodge filtrations \( F^jH^i_{dR}(X, O_S) \) are also locally free over \( O_S \), such that the sub-quotients are also locally free modules. \( \Box \)

We first note the following lemma which is fundamental to the proof.

**Lemma 6.6.1.** With notation as above, assume the following:

a) if \( X \) is a K3-surface, then \( X \) does not have ordinary reduction at \( v \).

b) if \( X \) is an abelian variety, then the \( p \)-rank of the reduction of \( X \) at \( v \), is at most 1. Then \( p|a_v \).
Proof. Let $w$ be a valuation as in 6.5.1 above. If $X$ is an abelian variety defined over the finite field $k_v$, then the $p$-rank of $X$ is precisely the number of eigenvalues of the correct power of the crystalline Frobenius acting on $H^1(X/W(k_v)) \otimes \mathbb{Q}_p$, which are $p$-adic units. Suppose now $\alpha$ is an eigenvalue of the crystalline Frobenius $\phi_v^{c}\nu$ acting on $H^2(X/W(k_v)) \otimes \mathbb{Q}_p$. In case b), the hypothesis implies that $w(\alpha)$ is positive, and hence $w(\alpha)$ is strictly positive. As $\alpha$ is a rational integer, the lemma follows.

When $X$ is a K3-surface, it follows from the shapes of the Newton and Hodge polygons, that ordinarity is equivalent to the fact that precisely one eigenvalue of $\phi_v^{c}\nu$ acting on $H^2(X/W(k_v)) \otimes \mathbb{Q}_p$ is a $p$-adic unit. Hence if $X$ is not ordinary, then for any $\alpha$ as above, we have $w(\alpha)$ is positive. Again since $\alpha$ is a rational integer, the lemma follows. \qed

Theorem 6.6.2. Let $X$ be a K3 surface or an abelian variety of dimension at least two defined over a number field $K$. Then there is a finite extension $L/K$ of number fields, such that

1. if $X$ is a K3-surface, then $X \times_K L$ has ordinary reduction at a set of primes of density one in $L$.
2. if $X$ is an abelian variety, then there is set of primes $O$ of density one in $L$, such that the reduction of $X \times_K L$ at a prime $p \in O$ has $p$-rank at least two.

Proof. Our proof follows closely the method of Serre and Ogus (see [27]). Fix a prime $l$, and let $\rho_l$ denote the corresponding Galois representation on $H^2_l$. The Galois group $G_K$ leaves a lattice $V_l$ fixed, and let $\overline{\rho}_l$ denote the representation of $G_L$ on $V_l \otimes \mathbb{Z}/l\mathbb{Z}$. Let $L$ be a Galois extension of $\mathbb{Q}$, containing $K$ and the $l^{th}$ roots of unity, and such that for $\sigma \in G_L$, $\overline{\rho}_l(\sigma) = 1$. We have

$$a_v \equiv d(mod \ l),$$

where $d = \dim H^2_l$.

Let $v$ be a prime of $L$ of degree 1 over $\mathbb{Q}$, lying over the rational prime $p$. Since $p$ splits completely in $L$, and $L$ contains the $l^{th}$ roots of unity, we have $p \equiv 1(mod \ l)$. Now choose $l > d$. Since $|a_v| \leq dp$ and is a rational integer divisible by $p$ from the above lemma, it follows on taking congruences modulo $l$ that

$$a_v = \pm dp.$$

Now $a_v$ is the sum of $d$ algebraic integers each of which is of absolute value $p$ with respect to any embedding. It follows that all these eigenvalues must be equal, and equals $\pm p$. Hence we have that

$$F_v = \pm pI$$

as an operator on $H^2_l$. By the semi-simplicity of the crystalline Frobenius for abelian varieties and K3-surfaces [27], it follows that $\phi_v = \pm pI$. But this contradicts the divisibility property of the crystalline Frobenius 6.5.2, that the crystalline Frobenius is divisible by $p^2$ on $F^2H^2_{dR}(X \times k_v)$. Hence
v has to be a prime of ordinary reduction, and this completes the proof of our theorem.

Ogus’ method can in fact be axiomatized to give positive density results whenever certain cohomological conditions are satisfied. We present this formulation for the sake of completeness.

**Proposition 6.6.3.** Let $X$ be a smooth projective variety over a number field $K$. Assume the following conditions are satisfied:

1. $\dim H^2(X, \mathcal{O}_X) = 1$,
2. The action of the crystalline Frobenius of the reduction $X_\wp$ of $X$ at a prime $\wp$ is semi-simple for all but finite number of primes $\wp$ of $K$.

Then the Galois representation $H^2_{\text{ét}}(X, \mathbb{Q}_\ell)$ is ordinary at a set of primes of positive density in $K$ and the $F$-crystal $H^2_\text{cris}(X_\wp/W)$ is ordinary for these primes. In other words, the motive $H^2(X)$ has ordinary reduction for a positive density of primes of $K$.

7. **Primes of Hodge-Witt reduction**

### 7.1. Hodge-Witt reduction

Let $X$ be a smooth projective variety over a number field $K$. We fix a model $\mathcal{X} \to \text{Spec}(\mathcal{O}_K)$ which is regular, proper and flat and which is smooth over a suitable non-empty subset of $\text{Spec}(\mathcal{O}_K)$. All our results are independent of the choice of the model. In what follows we will be interested in the smooth fibers of the map $\mathcal{X} \to \text{Spec}(\mathcal{O}_K)$, in other words we will always consider primes of good reduction. Henceforth $p$ will always denote such a prime and the fiber over this prime will be denoted by $X_p$.

Since ordinary varieties are Hodge-Witt, we can formulate a weaker version of Conjecture 6.0.1.

**Conjecture 7.1.1.** Let $X/K$ be a smooth projective variety over a number field then $X$ has Hodge-Witt reduction modulo a set of primes of $K$ of positive density.

For surfaces the geometric genus appears to detect the size of the set of primes which is predicted in Conjecture 7.1.1.

**Theorem 7.1.1.** Let $X$ be a smooth, projective surface with $p_g(X) = 0$, defined over a number field $K$. Then for all but finitely many primes $\wp$, $X$ has Hodge-Witt reduction at $\wp$.

**Proof.** By the results of [23], [14], [11], it suffices to verify that $H^2(X_\wp, W(\mathcal{O}_{X_\wp}))$ is zero for all but finite number of primes $\wp$ of $K$. But the assumption that $p_g(X) = 0$ entails that $H^2(X, \mathcal{O}_X) = H^0(X, K_X) = 0$. Hence by the semicontinuity theorem, for all but finite number of primes $\wp$ of $K$, the reduction $X_\wp$ also has $p_g(X_\wp) = 0$. Then by [11], [23] one sees that $H^2(X_\wp, W(\mathcal{O}_{X_\wp})) = 0$ and so $X_\wp$ is Hodge-Witt at any such prime. □
Corollary 7.1.1. Let $X/K$ be an Enriques surface over a number field $K$. Then $X$ has Hodge-Witt reduction modulo all but finite number of primes of $K$.

Proof. This is immediate from the fact that for an Enriques surface over $K$, $p_g(X) = 0$. \qed

When $X$ is a smooth Fano surface over a number field, one can prove a little more:

**Theorem 7.1.2.** Let $X$ be a smooth, projective Fano surface, defined over a number field $K$. Then for all but finitely many primes $p$, $X$ has ordinary reduction at $p$ and moreover the de Rham-Witt cohomology of $X_p$ is torsion free.

Proof. It follows from the results of [23] and [10], that if $X$ is a smooth, projective and Fano variety $X$ over a number field, for all but finitely many primes $p$, the reduction $X_p$ is $F$-split. By Theorem 2.4.1 we see the reduction modulo all but finitely many primes $p$ of $K$ gives an ordinary surface. Then by Lemma 9.5 of [4] and Proposition 6.6.1, the result follows. \qed

**Example 7.1.1.** Let $K = \mathbb{Q}$ and $X \subset \mathbb{P}^n$ be any Fermat hypersurface of degree $m$ and $n \geq 6$. If $m < n+1$ then this hypersurface is Fano but by [32] this hypersurface does not have Hodge-Witt reduction at primes $p$ satisfying $p \not\equiv 1 \pmod{m}$. This gives examples of Fano varieties which are $(F\text{-split but are})$ not Hodge-Witt or ordinary.

**Remark 7.1.1.** It is clear from Example 7.1.1 that there exist Fano varieties over number fields which do not have Hodge-Witt reduction modulo an infinite set of primes and thus this indicates that in higher dimension $p_g(X)$ is not a good invariant for measuring this behavior. The following question and subsequent examples suggests that the Hodge level may intervene in higher dimensions.

**Question 7.1.1.** Let $X/K$ be a smooth, projective Fano variety over a number field. Assume that $X$ has Hodge level $\leq 1$ in the sense of [3]. Then does $X$ have Hodge-Witt reduction modulo all but a finite number of primes of $K$?

**Remark 7.1.2.** A list of all the smooth complete intersection in $\mathbb{P}^n$ which are of Hodge level $\leq 1$ is given in [29] and one knows from [31] that complete intersections of Hodge level 1 are Hodge-Witt.

**Remark 7.1.3.** The first author has answered the Question 7.1.1 affirmatively (in [15]) for $\dim(X) = 3$ where the Hodge level condition is automatic.

Recall that an abelian variety $A$ over a perfect field is Hodge-Witt if and only if the $p$-rank of $A$ is at least $\dim(A) - 1$ (see [12]). This together with Theorem 6.6.2 gives
Theorem 7.1.3. Let $A/K$ be an abelian threefold over a number field $K$. Then there exists a set of primes of positive density in $K$ such that $A$ has Hodge-Witt reduction at these primes.

7.2. Hodge-Witt torsion. We include here some observations probably well-known to the experts, but we have not found them in print. We assume as in the previous section that $X/K$ is smooth projective variety over a number field and that we have fixed a regular, proper model smooth over some open subscheme of the ring of integers of $K$ and whose generic fiber is $X$.

Before we proceed we record the following:

Proposition 7.2.1. Let $X/K$ be a smooth projective variety over a number field $K$. Then there exists an integer $N$ such that for all primes $\mathfrak{p}$ in $K$ lying over any rational prime $p \geq N$, the following dichotomy holds

1. either for all $i, j \geq 0$, the Hodge-Witt groups are free $W$-modules (of finite type), or
2. there is some pair $i, j$ such that $H^i(X_{\mathfrak{p}}, W\Omega^j_{X_{\mathfrak{p}}})$ has infinite torsion.

Proof. Choose a finite set of primes $S$ of $K$, such that $X$ has a proper, regular model over $\text{Spec}(O_{K,S})$, where $O_{K,S}$ denotes the ring of $S$-integers in $K$. Choose $N$ large enough so that for any prime $\mathfrak{p}$ lying over a rational prime $p > N$, we have $\mathfrak{p} \not\in S$, and all the crystalline cohomology groups of $X_{\mathfrak{p}}$ are torsion-free. We note that this choice of $N$ may depend on the choice of a regular proper model for $X$ over $\text{Spec}(O_{K,S})$. If $\mathfrak{p}$ is such that $H^i(X_{\mathfrak{p}}, W\Omega^j_{X_{\mathfrak{p}}})$ are all finite type, then by the degeneration of the slope spectral sequence at the $E_1$-stage by Bloch-Nygaard (see Theorem 3.7 of [1]), and the fact that the crystalline cohomology groups are torsion free, it follows that the Hodge-Witt groups are free as well. If, on the other hand, some Hodge-Witt group of $X_{\mathfrak{p}}$ is not of finite type over $W$, then we are in the second case.

Question 7.2.1. Let $X/K$ be a smooth projective variety over a number field. When does there exist an infinite set of primes of $K$ such that the Hodge-Witt cohomology groups of the reduction $X_{\mathfrak{p}}$ at $\mathfrak{p}$ are not Hodge-Witt?

We would like to explicate the information encoded in such a set of primes (when it exists).

Proposition 7.2.2. Let $A/K$ be an abelian surface over a number field $K$. Then there exists infinitely many primes $\mathfrak{p}$ such that $H^2(X_{\mathfrak{p}}, W(O_{X_{\mathfrak{p}}}))$ has infinite torsion if and only if there exists infinitely many primes $\mathfrak{p}$ of supersingular reduction for $X$. In particular, let $E$ be an elliptic curve over $\mathbb{Q}$ and let $X = E \times_{\mathbb{Q}} E$. Then for an infinite set of primes of $\mathbb{Q}$, the Hodge-Witt groups $H^i(X_{\mathfrak{p}}, W\Omega^j_{X_{\mathfrak{p}}})$ are not torsion free for $(i, j) \in \{(2, 0), (2, 1)\}$. 

Proof. The first part follows from the results of [11, Section 7.1(a)]. The second part follows from combining the first part with Elkies’s theorem (see [8]), that given an elliptic curve $E$ over $\mathbb{Q}$, there are infinitely many primes $p$ of $\mathbb{Q}$ such that $E$ has supersingular reduction.

Example 7.2.1. Results of [32] on Fermat hypersurfaces together with foregoing discussion indicate similar examples as above. These are the only examples of this phenomena we know so far related to the above question.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARIZONA, 617 N SANTA RITA, P O BOX 210089, TUCSON, AZ 85721, USA. SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, BOMBAY - 400 005, INDIA.

E-mail address: kirti@math.arizona.edu, rajan@math.tifr.res.in