The Value of the Cosmological Constant in a Unified Field Theory with Enlarged Transformation Group

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Abstract. The geometrical structure of a real four-dimensional space-time has been extended via the Conservation group with basic field variable being the orthonormal tetrad. Field equations were obtained from a variational principle which is invariant under the conservation group. Recently, symmetric solutions of the field equations have been developed. In this note, the free-field solution is investigated in terms of the value of the scalar curvature. The resulting asymptotic value is approximately the negative of the currently accepted value of $\Lambda$, i.e. $R \approx -10^{-120}$. This may add further support to the conclusion that the theory developed by Pandres unifies the fields.

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1. Introduction

Let $X^4$ be a 4-dimensional space with orthonormal tetrad $h^i_\mu$. Then a metric $g_{\mu\nu}$ may be defined on $X^4$ by $g_{\mu\nu} = \eta_{ij} h^i_\mu h^j_\nu$ where $\eta_{ij} = \text{diag}\{-1,1,1,1\}$. Let $\tilde{V}^\alpha$ be a vector density of weight +1. A conservation law of the form $\tilde{V}^\alpha;\alpha = 0$ is invariant under all transformations satisfying

$$x^\nu (x^\mu,_{\nu,\mu} - x^\mu,_{\nu,\nu}) = 0 \quad .$$

This condition (1) defines the conservation group, a group of coordinate transformations that include the group of diffeomorphisms as a proper subgroup [1]. The equation for the propagation of light is also covariant under the conservation group [2].

Although we may view the space $X^4$ as a Riemannian manifold, the space is more general than a manifold [1-5]. Suppose that $x^\mu$ are coordinates on a Riemannian manifold $M$. Then for functions $f$ with continuous second derivatives, the commutator of partial derivatives is zero, i.e., $[\partial_\mu, \partial_\nu]f = 0$. If $x^\alpha_{\mu}$ is non-diffeomorphic but satisfies (1) we see that $[\partial_\mu, \partial_\nu]f$ is nonzero. Thus the geometry associated with the group of conservative transformations is more general than a manifold, although manifolds are included. If one retains the manifold view, then one must regard $x^\mu$ as anholonomic coordinates when $x^\alpha_{\mu}$ is non-diffeomorphic (see [5]). We stipulate that we may begin the setup of our theory as if it were a manifold, defining $h^i_\alpha$ as a function of $x^\mu$. On equal footing there are many coordinate systems related to $x^\mu$ via conservative transformations, however we, as observers, have a natural preference for the manifold view [6]. Thus among all possible coordinates the holonomic coordinates are preferred.

The geometrical content is determined by the vector $C_\alpha \equiv h^i_\nu (h^i_\alpha,_{\nu,\mu} - h^i_\nu,_{\nu,\alpha}) = \gamma^i_{\mu\nu}$, where the Ricci rotation coefficient is given by $\gamma^i_{\mu\nu} = h^i_{\mu,\nu}$ [1-3]. Pandres calls this the curvature vector and shows that $C_\alpha$ is invariant under transformations from $x^\mu$ to $x^\nu$ if and only if the transformation is conservative and thus satisfies (1). A suitable scalar Lagrangian for the free field [6] is given by

$$L_f = \frac{1}{16\pi} \int C^\alpha C_\alpha h \, d^4x \quad (2)$$

where $h = \sqrt{-g}$ is the determinant of the tetrad.

Green has extended the field variables [4] to include the tetrad $h^I_\mu$ and 4 internal vectors $\Lambda^i_I$ and an additional internal space $x^I$. However, that extension is not needed in the present paper.

Using the Ricci rotation coefficients, one finds that

$$C^\alpha C_\alpha = R + \gamma^i_{\alpha\beta} \gamma^\alpha_{i\beta} - 2C^\alpha_{i;\alpha} \quad ,$$

where $R$ is the usual Ricci scalar curvature. (Note: our sign conventions are the same as Misner, Thorne and Wheeler [7].) Thus, when the physical space is interpreted as a manifold, the Lagrangian density of the free field contains terms corresponding to non-gravitational interactions [3,5].
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Setting the variations of $\mathcal{L}_f$ with respect to $h_i^\mu$ equal to zero yields the field equations

$$C_{\mu;\nu} - C_{\mu}^{\gamma} C^{\alpha}_{\mu} - g_{\mu\nu} C^{\alpha}_{,\alpha} - \frac{1}{2} g_{\mu\nu} C^{\alpha} C_{\alpha} = 0 \quad .$$

(4)

An identity for the Einstein tensor is (using Misner, Thorne and Wheeler sign conventions [7])

$$G_{\mu\nu} = C_{\mu;\nu} - C^{\gamma}_{\mu} C_{\nu}^{\alpha} - g_{\mu\nu} C^{\alpha}_{,\alpha} - \frac{1}{2} g_{\mu\nu} C^{\alpha} C_{\alpha} + \gamma^{\alpha}_{\mu\nu} + \gamma^{\alpha}_{\sigma\nu} \gamma^{\sigma}_{\mu\alpha} + \frac{1}{2} g_{\mu\nu} \gamma^{\alpha\beta\gamma} \gamma^{\alpha}_{\sigma\beta}$$

This expression is not manifestly symmetric in $\mu$ and $\nu$, but the left-hand side is symmetric in its lower indices and hence the right-hand side must be as well. Thus we use a symmetrized expression to ensure this. Define for general $K_{\mu\nu}$, the symmetrized tensor by

$$K_{\mu\nu} = \frac{1}{2}(K_{\mu\nu} + K_{\nu\mu})$$

Using (4) we see that the field equations may be also expressed in the form

$$G_{\mu\nu} = \gamma^{\alpha}_{(\mu\nu),\alpha} + \gamma^{\alpha}_{\sigma(\nu} \gamma^{\sigma}_{\mu)\alpha} + \frac{1}{2} g_{\mu\nu} \gamma^{\alpha\beta\gamma} \gamma^{\alpha}_{\sigma\beta} \equiv 8\pi(T_f)_{\mu\nu}$$

(5)

with free field stress energy tensor $T_f$. The terms of $T_f$ suggest that, when interpreted in Riemannian geometry [7-9], this new geometry produces a stress energy tensor with additional terms that could be the stress energy tensor for dark matter or dark energy [5].

2. Spherically symmetric solutions and the asymptotic value of the scalar curvature.

Let $\Phi(r)$ be an arbitrary function of radial coordinate $r$. Define a tetrad in spherical coordinates [6] by

$$h_i^\mu = \begin{bmatrix} e^{\Phi(r)} & 0 & 0 & 0 \\ 0 & (1 + \frac{1}{2} r \Phi') \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ 0 & (1 + \frac{1}{2} r \Phi') \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ 0 & (1 + \frac{1}{2} r \Phi') \cos \theta & -r \sin \theta & 0 \end{bmatrix}$$

(6)

where the upper index refers to the row and the prime indicates differentiation of $\Phi(r)$ with respect to $r$. We find that $C_\mu = 0$ for this tetrad and hence the free-field field equations (4) are satisfied. The metric

$$ds^2 = -e^{2\Phi(r)} dt^2 + (1 + \frac{1}{2} r \Phi')^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

(7)

differs from the Schwarzschild metric [7-9]. From (6) we find that the Einstein tensor is diagonal: $G_{\mu\nu} = 8\pi(T_f)_{\mu\nu}$. The non-zero components are (with $\Phi$ representing $\Phi(r)$)

$$G_{tt} = 8\pi(T_f)_{tt} = \frac{e^{2\Phi} \left( \frac{1}{8}(r \Phi')^3 + \frac{3}{4}(r \Phi')^2 + 2r \Phi' + r^2 \Phi'' \right)}{r^2 \left( 1 + \frac{1}{2} r \Phi' \right)^3} \quad ,$$

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\[ G_{rr} = 8\pi (T_t)_{rr} = \frac{r\Phi' - \frac{1}{4}(r\Phi')^2}{r^2} \]  

(9)

and

\[ \frac{G_{\theta\theta}}{r^2} = \frac{8\pi T_{\theta\theta}}{r^2} = \frac{\frac{1}{2}(r\Phi')^3 + (r\Phi')^2 + \frac{1}{2}r\Phi' + \frac{1}{2}r^2\Phi''}{r^2(1 + \frac{1}{2}r\Phi')^3} \]  

(10)

\[ \frac{G_{\phi\phi}}{r^2 \sin^2 \theta} = \frac{8\pi T_{\phi\phi}}{r^2 \sin^2 \theta} = \frac{\frac{1}{2}(r\Phi')^3 + (r\Phi')^2 + \frac{1}{2}r\Phi' + \frac{1}{2}r^2\Phi''}{r^2(1 + \frac{1}{2}r\Phi')^3}. \]

From (8) we find [6] that

\[ e^{-2\Phi(r)}G_{tt} = \frac{2}{r^2} \cdot \frac{d}{dr} \left( \frac{r}{2} - \frac{2w(r)}{r} \right) \equiv \frac{2}{r^2}w'(r) \equiv 8\pi \rho_f, \]  

(11)

where \( w(r) \equiv \frac{r}{2} - \frac{2w(r)}{r}. \) Hence

\[ \Phi'(r) = \frac{2}{r} \left[ (1 - \frac{2w(r)}{r})^{-\frac{1}{2}} - 1 \right]. \]  

(12)

Thus

\[ g_{rr} = (1 + \frac{1}{2}r\Phi')^2 = \left( 1 - \frac{2w(r)}{r} \right)^{-1}, \]  

(13)

and

\[ g_{tt} = -e^{2\Phi(r)}, \text{ where } \Phi(r) = \int \frac{2}{r} \left[ (1 - \frac{2w(r)}{r})^{-\frac{1}{2}} - 1 \right] dr \]  

(14)

(this defines \( \Phi(r) \) up to a constant). The function \( w(r) \) (as shown below) is related to the mass inside a ball of radius \( r \) for the free field and \( \rho_f \) represents the density of the free field in the manifold interpretation.

Let \( p_R \) represent the radial pressure of the free field. Then we find [7] that the radial pressure of the free field is given by

\[ 8\pi p_R = \frac{G_{rr}}{(1 + \frac{1}{2}r\Phi')^2} = \frac{r\Phi' - \frac{1}{4}(r\Phi')^2}{r^2(1 + \frac{1}{2}r\Phi')^2} \]  

(15)

and from (12) we find that

\[ 8\pi p_R = \frac{4r\sqrt{1 - \frac{2w(r)}{r}} - 4r + 6w(r)}{r^3}. \]  

(16)

Let the tangential pressure [7] of the free field be denoted by \( p_T \). We also find that

\[ 8\pi p_T = \frac{G_{\theta\theta}}{r^2 \sin^2 \theta} = \frac{G_{\phi\phi}}{r^2 \sin^2 \theta} \]  

and thus,

\[ 8\pi p_T = \frac{\frac{1}{2}(r\Phi')^3 + (r\Phi')^2 + \frac{1}{2}r\Phi' + \frac{1}{2}r^2\Phi''}{r^2(1 + \frac{1}{2}r\Phi')^3}. \]  

(17)

Using (12), the tangential pressure may be expressed in terms of \( w(r) \) and \( r \) by

\[ 8\pi p_T = \frac{8r - 9w(r) - 8r\sqrt{1 - \frac{2w(r)}{r}} + rw'(r)}{r^3}. \]  

(18)

Note that \( p_R \neq p_T \).
We now take the trace of the Einstein tensor and using (11), (16) and (18). Note that $G_{\mu}^\nu = 8\pi \cdot \text{diag}(-\rho_f, p_f, p_T, p_T)$. Hence, when we take the trace, the $w'(r)$ terms cancel and the remaining terms are easily combined to yield

$$G_{\mu}^\mu = \frac{12}{r^2} \left( 1 - \sqrt{1 - \frac{2w(r)}{r}} - \frac{w(r)}{r} \right)$$

which can be factored to

$$G_{\mu}^\mu = \frac{6}{r^2} \left( \sqrt{1 - \frac{2w(r)}{r}} - 1 \right)^2$$

Assuming that $\frac{w(r)}{r} < 1$, we find that $\sqrt{1 - \frac{2w(r)}{r}} - 1 = \frac{w(r)}{r}(1 + \frac{w(r)}{2r} + \ldots)$. Hence

$$G_{\mu}^\mu = \frac{6[w(r)]^2}{r^4} \left( 1 + \frac{w(r)}{2r} + \ldots \right)^2$$

Let $M$ be the total mass that is within a radius of $R_0$ of the center of our coordinate system. This could apply to a star, a galaxy or the entire universe. Suppose that for $r > R_0$, no ordinary matter exists. In order to agree with the weak-field solution as $r \to \infty$, we require that $\lim_{r \to \infty} w(r) = \frac{1}{2}M$. From (14), this requirement implies that $\Phi(r) = \int \frac{2}{r} \left[ (1 - \frac{M}{r})^{\frac{1}{2}} - 1 \right] dr \approx \int \frac{M}{r^2} dr$ for $r >> R_0$. Thus from (14) we see that $g_{tt} \approx -(1 - \frac{2M}{r})$ which agrees with the weak-field solution of standard general relativity.

We also assume that $\frac{M}{r}$ is near zero for $r >> R_0$. In this case (21) yields the following result:

$$G_{\mu}^\mu = \frac{3M^2}{2r^4} \left( 1 + \frac{M}{4r} + \ldots \right)^2$$

and using $G_{\mu}^\mu = -R$ we see that the scalar curvature is given by

$$R = -\frac{3M^2}{2r^4} \left( 1 + \frac{M}{4r} + \ldots \right)^2$$

Since $C_{\mu} = 0$ for this tetrad field, (3) yields $0 = R + \gamma^{\alpha\beta\nu} \gamma_{\alpha\nu\beta}$. Thus

$$\gamma^{\alpha\beta\nu} \gamma_{\alpha\nu\beta} = \frac{3M^2}{2r^4} \left( 1 + \frac{M}{4r} + \ldots \right)^2$$

If we assume that our spherically symmetric coordinate system is centered at the center of the universe, then an approximate value of radius of the entire universe in Planck units is $R_u \approx 10^{60}$. An approximate value for the mass of the universe in Planck units is $M_u \approx 10^{60}$ as well. The resulting value of the scalar curvature computed in Planck units with $r = R_u$ and $M = M_u$ is

$$R \approx -\frac{3M^2}{2r^4} \approx -10^{-120}$$

We remark that this value is close to the value of the cosmological constant [10] and occurs in a natural way in our theory. Let us investigate the situation further. In standard general relativity, in regions $\Omega$ free of sources and fields, we have the curvature tensor is identically zero and hence at the extremum,

$$\mathcal{L}_{f(\text{ext})} = \frac{1}{16\pi} \int_{\Omega} R h d^4x = 0$$
If we introduce a cosmological constant, \( \Lambda \), into the standard theory the free-field Lagrangian \([7,10]\) is

\[
\mathcal{L}_f = \frac{1}{16\pi} \int (R - 2\Lambda) \, h \, d^4x
\]

resulting in field equations \( G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \). Suppose we assume that \( \Lambda > 0 \) and view the introduction of \( \Lambda \) as a perturbation of general relativity. Our first guess would be that the value of \( \mathcal{L}_f \) would become negative on \( \Omega \). However the trace of the Einstein tensor yields \( R = 4\Lambda \) and hence \( R \approx 4 \times 10^{-120} \). At the extremum, on a region \( \Omega \) which contains no ordinary mass-energy, with \( t \) coordinate values between 0 and \( \Delta t \),

\[
\mathcal{L}_{f(\text{ext})} = \frac{1}{16\pi} \int_\Omega 2\Lambda \, h \, d^4x = \frac{\Lambda \cdot vol(\Omega)\Delta t}{8\pi}
\]

We see that this yields an arbitrary positive value and that our perturbation actually increases the value of \( \mathcal{L}_f \) on \( \Omega \). Also, there is no particular reason to choose \( \Lambda \approx 10^{-120} \). No such dilemma occurs with the theory based on the conservation group. From (2) and (3) with the \( C_{\alpha\beta} \) term dropped (since a pure divergence contributes nothing to the field equations), we have

\[
\mathcal{L}_{f(\text{ext})} = \frac{1}{16\pi} \int_\Omega 0 \cdot h \, d^4x = 0 \quad .
\] (25)

Another consideration is that the theory based on the conservation group, basically says that although the mass-energy of an object may be localized, the curvature effects are not localized. That implies that every massive object in the universe contributes curvature to the entire (reachable) universe (limited by the speed of light). This may yield an explanation of Mach’s principal.

These considerations and calculations give support to the conclusion that the theory based on the conservation group is the correct model for gravitation.

3. Example Solutions.

3.1. Example with compact density.

For this example \([6]\) we take \( w(r) = \frac{1}{2}M \) for \( r > R \), the radius of the closed system of mass \( M \). Thus from (21) we have \( \frac{1}{2}M = \frac{r}{2} - \frac{r}{2(1 + \frac{r}{2\sqrt{r}})^2} \) and hence

\[
\Phi(r) = \int \left[ \frac{2}{r\sqrt{1 - \frac{M}{r}}} - \frac{2}{r} \right] \, dr
\] (26)

which can be easily integrated to find \( \Phi(r) = 4 \ln\left(1 + \sqrt{1 - \frac{M}{r}}\right) + \frac{1}{2} \ln C \) for some arbitrary \( C > 0 \). The arbitrary constant \( C \) is determined by the usual weak field approximation \([8]\) which is \( g_{tt} \approx -1 + \frac{2M}{r} \). This implies that \( C = \frac{1}{256} \). Hence

\[
g_{tt} = -\frac{1}{256} \left(1 + \sqrt{1 - \frac{M}{r}}\right)^8
\] (27)
We thus obtain the following line element:

\[ ds^2 = -\frac{1}{256} \left( 1 + \sqrt{1 - \frac{M}{r}} \right)^8 dt^2 + \left( 1 - \frac{M}{r} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \]  

(28)

We note that \( g_{tt} \approx -1 + \frac{2M}{r} - \frac{5M^2}{4r^2} \) to second order in \( \frac{M}{r} \). Using (11), (16) and (18), the Einstein field equations for the external solution are

\[ G_{tt} = 8\pi T_{tt} = 0 \]
\[ G_{rr} = 8\pi T_{rr} = \frac{M \left( 3\sqrt{1 - \frac{M}{r}} - 1 \right)}{r^3 \left( 1 + \sqrt{1 - \frac{M}{r}} \right)} \]  

(29)

\[ G_{\theta\theta} = \frac{G_{\phi\phi}}{r^2 \sin^2 \theta} = \frac{8\pi T_{\theta\theta}}{r^2 \sin^2 \theta} = \frac{8\pi T_{\phi\phi}}{r^2 \sin^2 \phi} = -\frac{M \left( 9\sqrt{1 - \frac{M}{r}} - 7 \right)}{2r^2 \left( 1 + \sqrt{1 - \frac{M}{r}} \right)} \]

which are also imply

\[ 8\pi \rho = 0 \]
\[ 8\pi p_R = \frac{M \left( 3\sqrt{1 - \frac{M}{r}} - 1 \right)}{r^3 \left( 1 + \sqrt{1 - \frac{M}{r}} \right)} \]  

(30)

\[ 8\pi p_T = -\frac{M \left( 9\sqrt{1 - \frac{M}{r}} - 7 \right)}{2r^3 \left( 1 + \sqrt{1 - \frac{M}{r}} \right)} . \]

Thus from (21)

\[ R = -\frac{6M \left( 1 - \sqrt{1 - \frac{M}{r}} \right)}{r^3 \left( 1 + \sqrt{1 - \frac{M}{r}} \right)} . \]  

(31)

Asymptotically for \( r >> M \), we have

\[ 8\pi p_R \approx \frac{M}{r^3} \left( 1 - \frac{M}{2r} \right) \]
\[ 8\pi p_T \approx -\frac{M}{2r^3} \left( 1 - \frac{2M}{r} \right) \]  

(32)

and hence \( R \approx -\frac{3M^2}{2r^3} \). Note the negative tangential pressure.

### 3.2. Examples with non-compact densities.

One particularly simple model is given by

\[ w(r) = \frac{M}{2} - \frac{M^2}{8r} . \]  

(33)

With this choice, \( 1 - \frac{2w(r)}{r} = 1 - \frac{M}{r} + \frac{M^2}{4r^2} = (1 - \frac{M}{2r})^2 \). Thus, using (11), (13-14), (16) and (18) we have

\[ ds^2 = -\left( 1 - \frac{M}{2r} \right)^4 dt^2 + \left( 1 - \frac{M}{2r} \right)^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 , \]

(34)
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and

\[ 8\pi \rho = \frac{M^2}{4r^4} \]
\[ 8\pi \rho_r = \frac{M}{r^3} \left( 1 - \frac{3M}{4r} \right) \]  
\[ 8\pi \rho_T = -\frac{M}{2r^2} \left( 1 - \frac{5M}{2r} \right) \]  

and hence

\[ R = -\frac{3M^2}{2r^4} \]  

These equations are exact.

A second non-compact example is given by

\[ w(r) = \frac{M(1 - \frac{7M}{4r})}{2(1 - \frac{3M}{2r})^2} \]  

With this choice, we find from (11-18) that the metric is

\[ ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 + \frac{(1 - \frac{3M}{2r})^2}{(1 - \frac{3M}{2r})^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]  

whose \( g_{tt} \) term matches with the exterior Schwarzschild metric. Furthermore

\[ 8\pi \rho = -\frac{5M^2(1 - \frac{19M}{10r})}{4r^4(1 - \frac{3M}{2r})^3} \]
\[ 8\pi \rho_r = \frac{M(1 - \frac{9M}{4r})}{r^3(1 - \frac{3M}{2r})^2} \]  
\[ 8\pi \rho_T = -\frac{M(1 - \frac{M}{r})(1 - \frac{3M}{r})}{2r^2(1 - \frac{3M}{2r})^3} \]  

and

\[ R = -\frac{3M^2}{2r^4 \left( 1 - \frac{3M}{2r} \right)^2} \]  

Note the negative density term and the negative tangential pressure.

4. Conclusion.

Asymptotic values of the scalar curvature have been developed from the theory based on the conservation group. The computation of the scalar curvature yields \( R \approx -\Lambda \approx -10^{-120} \). This may lead to an explanation of the value of the cosmological constant. This result supports the conclusion that the theory based on the conservative transformation group provides a theoretical basis for a unifying the fields of nature as well as explaining dark matter.
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