Reduction techniques of singular equivalences

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Abstract

It is shown that a singular equivalence induced by tensoring with a suitable complex of bimodules defines a singular equivalence of Morita type with level, in the sense of Wang. This result is applied to homological ideals and idempotents to produce new reduction techniques for testing the properties of syzygy-finite and injectives generation of finite dimensional algebras over a field.

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1 Introduction

Throughout $k$ is a fixed field and all algebras are finite dimensional associative $k$-algebras with identity, and all modules are finitely generated left modules unless stated otherwise. The singularity category $D_{sg}(A)$ of an algebra $A$ is defined as the Verdier quotient of the bounded derived category of finitely generated modules over $A$ by the full subcategory of perfect complexes [6], and two algebras are called singularly equivalent if their singularity categories are equivalent as triangulated categories. In particular, derived equivalent algebras are singularly equivalent, but the converse is not true in general. For this reason, many scholars devote to extend the properties which are preserved under derived equivalences to singular equivalences [13, 14, 34, 36, 40]. In this respect, a special class of singular equivalences induced by bimodules is crucially important. This was first studied by Chen
and Sun [12] under the name of singular equivalence of Morita type, and was
generalized to singular equivalence of Morita type with level in [35]. This
equivalence captures rich structural information, and plays a central role in
the study of homological properties and singular equivalences [13, 34, 36, 40].
Therefore, it is of great interest to construct singular equivalences of Morita
type with level, and to find out which properties are invariant under these
 equivalences. The purpose of this paper is to complement and extend some
results in this literature.

In [11], Chen-Liu-Wang gave a sufficient condition on when a tensor func-
tor with a bimodule defines a singular equivalence Morita type with level,
and in [16], Dalezios proved that for certain Gorenstein algebras, a singular
 equivalence induced from tensoring with a complex of bimodules always in-
duces a singular equivalence of Morita type with level. Our first theorem is
a complex version of Chen-Liu-Wang’s work, and it generalizes the result of
Dalezios to arbitrary algebra (not limited to Gorenstein algebra).

**Theorem I.** (Theorem 3.1) Let $A$ and $B$ be finite-dimensional $k$-algebras
such that $A/\text{rad}(A)$ and $B/\text{rad}(B)$ are separable over $k$. Consider a complex
$X$ of finitely generated $A$-$B$-bimodules which is perfect over $A$ and $B$. Assume
that $\text{RHom}_{A}(X, A)$ is a perfect complex of left $B$-module, and that $X \otimes_{B} ^{L} - : \text{D}_{sg}(B) \to \text{D}_{sg}(A)$ is an equivalence. Then there is an $A$-$B$-bimodule $M$
and a $B$-$A$-bimodule $N$ such that $(M, N)$ defines a singular equivalence of Morita
type with level.

Theorem I can be applied to homological ideals and idempotents to pro-
duce singular equivalences of Morita type with level. Let $A$ be an algebra
and let $J \subseteq A$ be a two-sided ideal. Following [27], $J$ is a homological ideal
if the canonical map $A \to A/J$ is a homological epimorphism, that is, the
naturally induced functor $\text{D}^{b}(\text{mod} A/J) \to \text{D}^{b}(\text{mod} A)$ is fully faithful. In
[9], Chen proved that if $J$ is a homological ideal which has finite projective
dimension as an $A$-$A$-bimodule, then there is a singular equivalence between
$A$ and $A/J$. In this paper we show that this equivalence is a singular equiv-
alence of Morita type with level (cf. Theorem 3.6).

Let $e \in A$ be an idempotent. Then the functor $eA \otimes_{A} - : \text{mod} A \to \text{mod} eAe$ induces a singular equivalence between $A$ and $eAe$ if and only if
$\text{pd}_{A}(\frac{A/eAe}{\text{rad}(A/eAe)}) < \infty$ and $\text{pd}_{eAe}eAe < \infty$, see [3, 28]. Similarly, $Ae \otimes_{eAe} - : \text{mod} eAe \to \text{mod} A$ induces a singular equivalence if and only if $\text{id}_{A}(\frac{A/eAe}{\text{rad}(A/eAe)}) < \infty$ and $\text{pd}Ae_{eAe} < \infty$, see [34]. Applying Theorem I, we show that $A$ and
eAe are singularly equivalent of Morita type with level in these two cases,
see Theorem 4.1 and Theorem 4.2.

Next, we turn to the question that which properties can be preserved under singular equivalence of Morita type with level. It is known that the finitistic dimension conjecture is invariant under this equivalence [35]. In this paper, we focus on the properties of syzygy-finite, Igusa-Todorov, injectives generation and projectives cogeneration, all of which are closely related to finitistic dimension conjecture [31, 37, 41]. We show that these properties are also invariant under singular equivalence of Morita type with level (cf. Proposition 5.1 and Proposition 5.2). As applications, we obtain the following reduction techniques for testing these properties for finite dimensional algebras over a field.

**Corollary I.** (Corollary 5.3) Let $A$ be a finite-dimensional algebra over a field and let $J \subseteq A$ be a homological ideal which has finite projective dimension as an $A$-$A$-bimodule. Then $A$ has the property of syzygy-finite (resp. Igusa-Todorov, injectives generation, projectives cogeneration) if and only if so does $A/J$.

**Corollary II.** (Corollary 5.4) Let $A$ be a finite-dimensional algebra over a field $k$ with separable semisimple quotient, and let $e \in A$ be an idempotent such that $Ae \otimes_{eA} eA$ is bounded in cohomology. If $\text{pd}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty$ or $\text{id}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty$, then $A$ has the property of syzygy-finite (resp. Igusa-Todorov, injectives generation, projectives cogeneration) if and only if so does $eAe$.

Corollary II may be compared with a recent result by Cummings [15]. For a ring $A$, Cummings proved that if $Ae \otimes_{eA} eA$ is bounded in cohomology, then (i) if $\text{id}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty$ and injectives generate for $eAe$, then injectives generate for $A$; (ii) if $\text{pd}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty$ and projectives generate for $eAe$, then projectives generate for $A$. Therefore, if we only consider finite-dimensional $k$-algebras with separable semisimple quotients (for instance, it is the case when $k$ is algebraically closed), then the property of injectives generation (resp. projectives cogeneration) between $A$ and $eAe$ can be displayed more completely.

In Corollary II, the transition from $A$ to $eAe$ is called vertex removal in some literature [18, 20]. If we restrict our discussion to quiver algebras, then we get a practical method for testing the properties of syzygy-finite, Igusa-Todorov, injectives generation and projectives cogeneration — just removing the vertices where no relations start or no relations end (cf. Corollary 5.5).
The paper is organized as follows. In section 2, we will recall some relevant definitions and conventions. In section 3 we prove Theorem I, and we show that a certain homological ideal induces a singular equivalence of Morita type with level. In section 4 we construct singular equivalences of Morita type with level by idempotents. In section 5, we investigate the invariance of syzygy-finite and injectives generation under singular equivalence of Morita type with level, and we prove Corollary I and Corollary II. In particular, we give two examples to illustrate how our reduction techniques can be used.

2 Definitions and conventions

Let \( C \) be a triangulated category which has all (set-indexed) products and coproducts. An object \( X \) of \( C \) is \textit{compact} if the functor \( \text{Hom}_C(X, -) \) preserves coproducts. For a set \( S \) of objects of \( C \), we denote by \( \text{tria} S \) the smallest triangulated subcategory of \( C \) containing \( S \), and by \( \text{thick} S \) (resp. \( \text{Tria} S \), \( \text{Coloc} S \)) the smallest triangulated subcategory of \( C \) containing \( S \) and closed under taking direct summands (resp. coproducts, products). \( S \) is called a set of \textit{compact generators} of \( C \) if all objects in \( S \) are compact and \( C = \text{Tria} S \). In this paper, all functors between triangulated categories are assumed to be triangle functors.

\textbf{Definition 2.1.} \((3)\) Let \( \mathcal{T}_1, \mathcal{T} \) and \( \mathcal{T}_2 \) be triangulated categories. A \textit{recollement} of \( \mathcal{T} \) relative to \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) is given by

\[
\begin{array}{c}
\mathcal{T}_1 & \xrightarrow{i_*} & \mathcal{T} & \xleftarrow{i^*} & \mathcal{T}_2 \\
\xrightarrow{i^*} & \xrightarrow{i_*} & \xleftarrow{i} & \xleftarrow{j^*} & \xrightarrow{j} \\
\end{array}
\]

such that

\begin{itemize}
  \item[(R1)] \((i^*, i_*),(i_*, i^*),(j_!, j^*)\) and \((j^*, j_*)\) are adjoint pairs;
  \item[(R2)] \(i_*\), \(j_!\) and \(j_*\) are full embeddings;
  \item[(R3)] \(j^*i_* = 0\) (and thus also \(i_!j_* = 0\) and \(i^*j_! = 0\));
  \item[(R4)] for each \( X \in \mathcal{T} \), there are triangles
    \[
    j_!j^*X \rightarrow X \rightarrow i_*i^*X \rightarrow \\
i_!i^*X \rightarrow X \rightarrow j_*j^*X \rightarrow
    \]
\end{itemize}

where the arrows to and from \( X \) are the counits and the units of the adjoint pairs respectively.
Definition 2.2. ([31, 30]) Let $\mathcal{T}_1$, $\mathcal{T}$ and $\mathcal{T}_2$ be triangulated categories, and $n$ a positive integer. An $n$-recollement of $\mathcal{T}$ relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ is given by $n + 2$ layers of triangle functors

$$\mathcal{T}_1 \xrightarrow{\gamma} \mathcal{T} \xrightarrow{\delta} \mathcal{T}_2$$

such that every consecutive three layers form a recollement.

Let $A$ be a finite dimensional associative algebra over a field $k$. Denote by rad($A$) the Jacobson radical of $A$. The semisimple quotient $A/\text{rad}(A)$ is called separable if $A/\text{rad}(A)$ remains semisimple under any extension of scalars to a field $K$ containing $k$. In particular, $A/\text{rad}(A)$ is separable if $k$ is an algebraically closed field.

Denote by Mod$_A$ the category of left $A$-modules, and by mod$_A$, proj$_A$ and inj$_A$ the full subcategories consisting of all finitely generated modules, finitely generated projective modules and finitely generated injective modules, respectively. We denote by mod$_A$ (resp. mod$_A$) the projective (resp. injective) stable category of mod$_A$ modulo morphisms factoring through projective (resp. injective) modules.

Let $\mathcal{X}$ be a subcategory of Mod$_A$. A (chain) complex $X$ over $\mathcal{X}$ is a set $\{X_i \in \mathcal{X}, i \in \mathbb{Z}\}$ equipped with a set of homomorphisms $\{d_X^i : X_i \to X_{i-1}, i \in \mathbb{Z} | d_X^i d_X^{i+1} = 0\}$. We usually write $X = \{X_i, d_X^i\}$. A chain map $f$ between complexes, say from $\{X_i, d_X^i\}$ to $\{Y_i, d_Y^i\}$ is a set of maps $f = \{f_i : X_i \to Y_i\}$ such that $f_{i-1} d_X^i = d_Y^i f_i$. A complex $X = \{X_i, d_X^i\}$ is right (resp. left) bounded if $X_i = 0$ for all but finitely many negative (resp. positive) integers $i$. A complex $X$ is bounded if it is both left and right bounded, equivalently, $X_i = 0$ for all but finitely many $i$. We denote by $[1]$ the left shift functor on complexes.

Let $\mathcal{D}(\text{Mod}_A)$ (resp. $\mathcal{D}^b(\text{mod}_A)$) be the derived category (resp. bounded derived category) of complexes over Mod$_A$ (resp. mod$_A$). Let $K^b(\text{proj}_A)$ (resp. $K^b(\text{inj}_A)$) be the bounded homotopy category of complexes over proj$_A$ (resp. inj$_A$). Up to isomorphism, the objects in $K^b(\text{proj}_A)$ are precisely all the compact objects in $\mathcal{D}(\text{Mod}_A)$. For convenience, we do not distinguish $K^b(\text{proj}_A)$ from the perfect derived category $\mathcal{D}_{\text{per}}(A)$ of $A$, i.e., the full triangulated subcategory of $\mathcal{D}(\text{Mod}_A)$ consisting of all compact objects, which will not cause any confusion. Moreover, we also do not distinguish $K^b(\text{inj}_A)$ (resp. $\mathcal{D}^b(\text{mod}_A)$) from their essential images under the canonical full em-
beddings into $\mathcal{D} \text{(Mod } A\text{)}. Usually, we just write $\mathcal{D}A$ (resp. $\mathcal{D}^b(A)$) instead of $\mathcal{D} \text{(Mod } A\text{)}$ (resp. $\mathcal{D}^b(\text{mod } A\text{)})$.

Let $A$ and $B$ be finite dimensional algebras over a field $k$ and $F : \mathcal{D}A \to \mathcal{D}B$ be a triangle functor. We say that $F$ restricts to $K^b(\text{proj})$ (resp. $\mathcal{D}^b(\text{mod})$, $K^b(\text{inj})$) if $F$ sends $K^b(\text{proj } A)$ (resp. $\mathcal{D}^b(\text{mod } A)$, $K^b(\text{inj } A)$) to $K^b(\text{proj } B)$ (resp. $\mathcal{D}^b(\text{mod } B)$, $K^b(\text{inj } B)$).

Following [6, 26], the singularity category of $A$ is defined to be the Verdier quotient $D_{sg}(A) = \mathcal{D}^b(\text{mod } A)/K^b(\text{proj } A)$. Let $A^e = A \otimes_k A^{\text{op}}$ be the enveloping algebra of $A$. We identify $A$-$A$-bimodules with left $A^e$-modules. Denote by $\Omega^A_{\text{sg}}(\cdot)$ the syzygy functor on the stable category $\text{mod} A^e$ of $A$-$A$-bimodules. The following terminology is due to Wang [35].

**Definition 2.3.** Let $A M_B$ and $B N_A$ be an $A$-$B$-bimodule and a $B$-$A$-bimodule, respectively, and let $n \geq 0$. We say $(M, N)$ defines a singular equivalence of Morita type with level $n$, provided that the following conditions are satisfied:

1. The four one-sided modules $A M$, $M_B$, $B N$ and $N_A$ are all finitely generated projective.
2. There are isomorphisms $M \otimes_B N \cong \Omega^n_{A^e}(A)$ and $N \otimes_A M \cong \Omega^n_{B^e}(B)$ in $\text{mod } A^e$ and $\text{mod } B^e$, respectively.

**Remark 2.4.** If $(M, N)$ defines a singular equivalence of Morita type with level $n$, then the functor $M \otimes_B -$ induces a singular equivalence between $A$ and $B$, that is, $M \otimes_B -$ : $D_{sg}(B) \to D_{sg}(A)$ is a triangle equivalence. This equivalence preserves many homological properties and homological conjectures, such as Hochschild homology [35], Fg condition [34], Keller’s conjecture [10] and the finitistic dimension conjecture [35].

Recall that an algebra $A$ is syzygy-finite provided that there is an integer $s$ such that the class of all $n$-th syzygies, where $n > s$, is representation finite, or equivalently, the number of nonisomorphic indecomposable modules in the class is finite. Such class of algebras include algebras of finite global dimension, algebras of finite representation type, monomial algebras and serial algebras.

From [37], an algebra $A$ is called Igusa-Todorov if there are a fixed $A$-module $V$ and an integer $n$ such that every $n$-th syzygy module $M$ fits into an exact sequence $0 \to V_1 \to V_0 \to M \to 0$, where $V_1$, $V_0$ are some direct summands of finite direct sums of $V$. Examples of such algebras include syzygy-finite algebras, algebras with representation dimension not more than three and algebras with infinite-layer length not more than three [22]. It is known that Igusa-Todorov algebras satisfy the finitistic dimension conjecture,
and the invariance of syzygy-finite and Igusa-Todorov under recollements and derived equivalences is discussed in [38, 39].

Let $A$ be an algebra and $D := \text{Hom}_k(-, k)$ be the standard duality. If $\text{Tria}DA = DA$ then we say that \textit{injectives generate} for $A$, and dually, if $\text{Coloc}A = DA$ then we say \textit{projectives cogenerate} for $A$. These concepts were proposed by Keller [23] as they are well-connected with some homological conjectures. In particular, if injectives generate for an algebra $A$, then $A$ satisfies the Nunke condition, the Generalised Nakayama conjecture and the finitistic dimension conjecture [23, 31]. Moreover, if projectives cogenerate for $A$, then its opposite algebra $A^{\text{op}}$ satisfies the finitistic dimension conjecture [31].

Nowadays, there is no known example of a finite dimensional algebra over a field for which injectives do not generate, and the property of injectives generation has been verified for commutative algebras, Gorenstein algebras and monomial algebras [31, Theorem 8.1]. On the other hand, the properties of injectives generation and projectives cogeneration are shown invariant under recollements and derived equivalences of algebras [15, 31].

3 Singular equivalences induced by complexes

In this section, we will investigate when a tensor functor giving by a bi-module complex induces a singular equivalence of Morita type with level. Let us make some notations. We denote by $\Omega_A(-)$ (resp. $\Omega_{A-B}(-)$) the syzygy functor on the stable category of $A$-modules (resp. $A$-$B$-bimodules), and $\Omega_{D^b(A)}(-)$ the syzygy functor on derived category, up to some direct summands of projective modules. We point that $\Omega_A(M) = \Omega_{D^b(A)}(M)$ for any $M \in \text{mod}_A$, and we refer to [2, 38] for more details on syzygies of complexes.

**Theorem 3.1.** Suppose that both $A/\text{rad}(A)$ and $B/\text{rad}(B)$ are separable over $k$. Consider a complex $X$ of finitely generated $A$-$B$-bimodules which is perfect over $A$ and $B$. Assume that $\text{RHom}_A(X, A)$ is a perfect complex of left $B$-module, and that $X \otimes^L_B : D_{\text{sg}}(B) \to D_{\text{sg}}(A)$ is an equivalence. Then there is an $A$-$B$-bimodule $M$ and a $B$-$A$-bimodule $N$ such that $(M, N)$ defines a singular equivalence of Morita type with level.

**Proof.** Set $Y = \text{RHom}_A(X, A)$. Since $AX$ is compact, we have an isomorphism of functors

$$\text{RHom}_A(X, -) \cong Y \otimes^L_A - : \mathcal{D}(A) \to \mathcal{D}(B).$$
Hence, there is an adjoint pair

$$
\mathcal{D}(B) \xrightarrow{X \otimes_B^L -} \mathcal{D}(A)
$$

with unit \( \eta \) and counit \( \epsilon \). Since \( B Y \in K^b(\text{proj}B) \), these adjoint functors restrict to one at the level of singularity categories (ref. \text{[26, Lemma 1.2]}). By assumption, \( X \otimes_B^L - : D_{sg}(B) \to D_{sg}(A) \) is an equivalence, and it follows from \text{[16] Theorem 3.6} that there are two isomorphisms \( B \cong Y \otimes_A^L X \) in \( D_{sg}(B) \) and \( A \cong X \otimes_B^L Y \) in \( D_{sg}(A) \) (for this we need the assumption on separability). Therefore, the mapping cones of \( \eta_B : B \to Y \otimes_A^L X \) and \( \epsilon_A : X \otimes_B^L Y \to A \) are perfect complexes of bimodules, and by \text{[38, Proposition 3.8]}, there exists some \( l \in \mathbb{Z} \) such that for any \( i \geq l \), there are two isomorphisms \( \Omega_{D^b(B')}^r(B) \cong \Omega_{D^b(B')}^r(Y \otimes_A^L X) \) and \( \Omega_{D^b(A')}^r(A) \cong \Omega_{D^b(A')}^r(X \otimes_B^L Y) \), up to some direct summands of projective bimodules (see \text{[38, Proposition 3.5]}).

From \text{[16] Proposition 4.4], \( X \) is isomorphic in \( \mathcal{D}(A \otimes_k B^{op}) \) to a complex

$$
0 \to U \to P_n \to \cdots \to P_m \to 0,
$$

where all \( P_i \) are finitely generated projective \( A \)-\( B \)-bimodules and \( U \) is finitely generated projective as a left \( A \)-module and as a right \( B \)-module. Similarly, \( Y \) is isomorphic in \( \mathcal{D}(B \otimes_k A^{op}) \) to a complex

$$
0 \to V \to Q_{n'} \to \cdots \to Q_{m'} \to 0,
$$

where all \( Q_i \) are finitely generated projective \( B \)-\( A \)-bimodules and \( V \) is finitely generated projective as a left \( B \)-module and as a right \( A \)-module. Therefore, \( Y \otimes_A^L X \) is quasi-isomorphic to the tensor product complex:

$$
0 \to V \otimes_A U \to Z_{n+n'+1} \to \cdots \to Z_{m+m'} \to 0,
$$

where all \( Z_i \) are projective over \( B^e \). Hence, for any \( i \geq n + n' + 2 \), we have

$$
\Omega_{D^b(B')}^r(Y \otimes_A^L X) \cong \Omega_{D^b(B')}^r(U \otimes_B V) \cong U \otimes_B \Omega_{D^b(A')}^r(U \otimes_B V),
$$

where the last isomorphism follows from the fact that \( U \) is projective as a left \( A \)-module and as a right \( B \)-module. Similarly, we obtain that

$$
\Omega_{D^b(A')}^r(X \otimes_B^L Y) \cong \Omega_{D^b(A')}^r(U \otimes_B V) \cong U \otimes_B \Omega_{D^b(A')}^r(U \otimes_B V),
$$

for any \( i \geq n + n' + 2 \). Taking \( r = \text{sup}\{l, n + n' + 2\} \), we have isomorphisms

$$
\Omega_{D^b(B')}^r(B) \cong \Omega_{D^b(B')}^r(Y \otimes_A^L X) \cong \Omega_{D^b(A')}^r(U \otimes_B V) \cong U \otimes_B \Omega_{D^b(A')}^r(U \otimes_B V)
$$
and
\[ \Omega^r_{A^e}(A) \cong \Omega^r_{\mathcal{D}^b(A^e)}(X \otimes_B Y) \cong U \otimes B \Omega^r_{B-A} \otimes B = \Omega^r_{B-A} \otimes B \otimes X \otimes L_B Y \cong U \otimes B \Omega^r_{B-A} \ominus n' = 2 \otimes B - A(V) , \]
up to some projective direct summands.

Since \( V \) is finitely generated projective as a left \( B \)-module and as a right \( A \)-module, \( \Omega^r_{B-A} \ominus n' = 2 \otimes B - A(V) \) is also finitely generated projective as an one-side module. Above all, we conclude that \( (U, \Omega^r_{B-A} \ominus n' = 2 \otimes B - A(V)) \) defines a singular equivalence of Morita type with level \( r \).

**Proposition 3.2.** Let \( A, B \) and \( C \) be finite dimensional \( k \)-algebras such that either \( A/\text{rad}(A) \) or \( B/\text{rad}(B) \) is separable over \( k \). If \( \text{gl.dim} B < \infty \) and \( \mathcal{D}A \) admits a 2-recollement relative to \( \mathcal{D}B \) and \( \mathcal{D}C \), then \( A \) and \( C \) are singularly equivalent of Morita type with level.

**Proof.** By [30, Proposition 1], there is a standard 2-recollement
\[ \mathcal{D}B \begin{array}{c} i^* \end{array} \mathcal{D}A \begin{array}{c} i_! \end{array} \mathcal{D}C . \]
Then, it follows from [1, Lemma 2.9] that \( i^! \) restricts to \( \mathcal{D}^b(\text{mod}) \), and then \( i^!(A) \in \mathcal{D}^b(\text{modB}) \cong \text{K}^b(\text{projB}) \) since \( \text{gl.dim} B < \infty \). Therefore, this 2-recollement can be extended one step downwards, see [1, Proposition 3.2].

So, there are four bimodule complexes \( C X_A, A Y_C, A U_B \) and \( B V_A \) (which are perfect complexes of one-side modules) such that \( i_*= U \otimes_B \cdot, i^*= V \otimes_A \cdot, j^*= X \otimes_A \cdot \) and \( j_* = Y \otimes_C \cdot \), where \( Y = R\text{Hom}_C(X, C) \). On the other hand, it follows from [29, Proposition 3] that \( j^* \) and \( j_* \) induce a mutually inverse equivalence between \( D_{sg}(A) \) and \( D_{sg}(C) \).

Since \( j_* \) is fully faithful, there is an isomorphism \( C \cong X \otimes_A Y \) in \( D_C \). Moreover, the canonical map
\[ X \otimes_A Y = X \otimes_A \text{RHom}_C(X, C) \rightarrow C, x \otimes f \mapsto f(x) \]
is a morphism of \( C\)-\( C \)-bimodules. Therefore, we get \( C \cong X \otimes_A Y \) in \( D(C^e) \) and thus \( C \cong X \otimes_A Y \) in \( D_{sg}(C^e) \). Now we claim \( A \cong Y \otimes_C X \) in \( D_{sg}(A^e) \) and then we are done by the proof of Theorem 3.1.

Let \( \eta : 1_{\mathcal{D}(A)} \rightarrow Y \otimes_C X \otimes_A \cdot \) be the unit of the adjoint pair
\[ \mathcal{D}(A) \begin{array}{c} X \otimes_A \cdot \end{array} Y \otimes_C \mathcal{D}(C) \begin{array}{c} \cdot \end{array} . \]
Since these functors induce a singular equivalence, we have that \( \eta_A \otimes_A Z \) is an isomorphism in \( D_{sg}(A) \), for any \( Z \in D^b(A) \). If \( A/\text{rad}(A) \) is separable over \( k \), then it follows from [16, Lemma 3.5] that the mapping cone of \( \eta_A \) is a perfect complex of \( A \)-\( A \)-bimodules, and thus \( A \cong Y \otimes_C X \) in \( D_{sg}(A^e) \).

If \( B/\text{rad}(B) \) is separable over \( k \), then the condition \( \text{gl.dim } B < \infty \) implies that \( B \in K^b(\text{proj}B^e) \), see [32, Lemma 7.2]. Hence \( U \otimes_B^L V \cong U \otimes_B^L B \otimes_B^L V \in K^b(\text{proj}A^e) \), because the functors \( - \otimes_B^L V : D(B^e) \to D(B \otimes_k A^op) \) and \( U \otimes_B^L - : D(B \otimes_k A^{op}) \to D(A^e) \) restrict to \( K^b(\text{proj}) \). On the other hand, it follows from [21, Theorem 1] that there is a recollement

\[
\begin{array}{c}
D(B \otimes_k A^{op}) \ar{r}{U \otimes^L_B} & D(A^e) \ar{r}{X \otimes^L_A} & D(C \otimes_k A^{op}) \ar{l}{V \otimes^L_B}
\end{array}
\]

Consequently, we have a triangle \( U \otimes_B V \to A \to Y \otimes_C X \to \) in \( D(A^e) \). Using the fact \( U \otimes_B V \in K^b(\text{proj}A^e) \), we obtain that \( A \cong Y \otimes_C X \) in \( D_{sg}(A^e) \).

**Corollary 3.3.** Let \( A, B \) and \( C \) be finite dimensional \( k \)-algebras such that \( B \) has finite projective dimension as a \( B \)-\( B \)-bimodule. Assume \( \mathcal{DA} \) admit a 2-recollement relative to \( \mathcal{DB} \) and \( \mathcal{DC} \). Then \( A \) and \( C \) are singularly equivalent of Morita type with level.

**Proof.** From [7, Chapter IX, Propositions 7.6], we have that \( \text{gl.dim } B < \infty \). Now note that \( B \in K^b(\text{proj}B^e) \), the statement can be proved in the same way as Proposition 3.2. \( \square \)

**Corollary 3.4.** (Compare [35, Section 3]) Let \( A = \begin{pmatrix} B & 0 \\ C & M_B \\ C & B \end{pmatrix} \), where \( B, C \) are finite dimensional \( k \)-algebras and \( M \) a finitely generated \( C \)-\( B \)-bimodules. Then following statements hold:

1. If \( B \) has finite projective dimension as a \( B \)-\( B \)-bimodule, then \( A \) and \( C \) are singularly equivalent of Morita type with level;
2. If \( C \) has finite projective dimension as a \( C \)-\( C \)-bimodule, then \( A \) and \( B \) are singularly equivalent of Morita type with level.

**Proof.** (1) By [11, Example 3.4], \( \mathcal{DA} \) admits a 2-recollement relative to \( \mathcal{DC} \) and \( \mathcal{DB} \). Moreover, \( B \in K^b(\text{proj}B^e) \) implies that \( \text{gl.dim } B < \infty \), and then \( \text{pd}_M B < \infty \). Therefore, this 2-recollement can be extended one step upwards, that is, \( \mathcal{DA} \) admits a 2-recollement relative to \( \mathcal{DB} \) and \( \mathcal{DC} \). Now the statement follows from Corollary 3.3.

(2) It follows from Corollary 3.3. \( \square \)
Recall that an algebra $A$ is said to \textit{satisfy the Fg condition} if the Hochschild cohomology ring $HH^*(A)$ is a Noetherian ring, and the Yoneda algebra $\text{Ext}^*_A(A/\text{rad} A, A/\text{rad} A)$ is a finitely generated $HH^*(A)$-module. The following corollary generalizes \cite{29} Theorem 5 on the assumption of separability.

\textbf{Corollary 3.5.} (Compare \cite{29} Theorem 5) Let $A$, $B$ and $C$ be finite dimensional $k$-algebras such that either $A/\text{rad}(A)$ or $B/\text{rad}(B)$ is separable over $k$. If $(DB, DA, DC, i^*, i_*, i^!, j_!, j^*, j^*)$ is a recollement and $j^*$ is an eventually homological isomorphism, then $A$ satisfies the Fg condition if and only if so does $C$.

\textit{Proof.} Since $j^*$ is an eventually homological isomorphism, it follows from \cite{29} Theorem 1 that $\text{gl.dim}B < \infty$, and the functor $i_*$ restricts to both $K^b(\text{proj})$ and $K^b(\text{inj})$. By \cite{1} Proposition 3.2, this recollement can be extended one downwards, and by Proposition 3.2, $A$ and $C$ are singularly equivalent of Morita type with level. Now assume either $A$ or $C$ satisfies Fg. Then, it follows from \cite{17} Theorem 1.5 (a)] that either $A$ or $C$ is Gorenstein, and by \cite{29} Theorem 3, both $A$ and $C$ are Gorenstein. Now we finish our proof by \cite{34} Theorem 7.4).

In \cite{9}, the author proved that a certain homological epimorphism between two algebras induces a triangle equivalence between their singularity categories. Now we will show that this equivalence is a singular equivalence of Morita type with level.

\textbf{Theorem 3.6.} (Compare \cite{9} Theorem)] Let $A$ be a finite dimensional $k$-algebra and let $J \subseteq A$ be a homological ideal which has finite projective dimension as an $A$-$A$-bimodule. Then $A$ and $A/J$ are singularly equivalent of Morita type with level.

\textit{Proof.} consider the adjoint pair $D(A) \xrightarrow{\eta} D(A/J)$ with the unit $\eta$ and the counit $\epsilon$. Since $J$, as an $A$-$A$-bimodule, has finite projective dimension, so it has finite projective dimension both as a left and right $A$-module. Therefore, $\text{pd}_A(A/J) < \infty$ and $\text{pd}(A/J)_A < \infty$, and these adjoint functors restrict to one at the level of singularity categories. By \cite{19} Theorem 4.4 (1), the counit $\epsilon_{A/J} : A/J \otimes^L_A A/J \to A/J$ is an isomorphism in $D((A/J)^c)$ and thus $A/J \cong A/J \otimes^L_A A/J$ in $D_{sg}((A/J)^c)$. Moreover, the unit map $\eta_A : A \to A/J$ is an isomorphism in $D_{sg}(A^c)$ since $\text{pd}_{A^c}(J) < \infty$. As a consequence, we obtain that $A$ and $A/J$ are singularly equivalent of Morita type with level by the same way as we did in Theorem 3.1.

\hfill $\square$
4 Singular equivalences induced by idempotents

Let \( A \) be a finite-dimensional algebra over a field \( k \), and let \( e \in A \) be an idempotent. From \([28]\), the functor \( eA \otimes_A - : \text{mod} A \rightarrow \text{mod} eA \) induces a singular equivalence between \( A \) and \( eA \) if and only if \( \text{pd}_{eA} eA < \infty \), see also \([3]\) Theorem 2.1. Similarly, \( Ae \otimes_{eA} - : \text{mod} eA \rightarrow \text{mod} A \) induces a singular equivalence if and only if \( \text{id}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty \) and \( \text{pd}_A Ae < \infty \), see \([33]\) Theorem II. In this section, we will show that \( A \) and \( eA \) are singularly equivalent of Morita type with level in these two cases.

**Theorem 4.1.** Let \( A \) be a finite-dimensional algebra over a field \( k \) such that \( A/\text{rad}(A) \) is separable over \( k \) and let \( e \in A \) be an idempotent. If \( \text{pd}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty \) and \( \text{pd}_{eA} eA < \infty \), then \( A \) and \( eA \) are singularly equivalent of Morita type with level.

**Proof.** Consider the recollement

\[
\begin{array}{ccc}
\text{DB} & \xrightarrow{i} & \text{DA} \\
\xleftarrow{i^*} & & \xrightarrow{j^* = eA \otimes_A A} \text{D}(eA),
\end{array}
\]

where \( B \) is a dg algebra. Since \( \text{pd}_{eA} eA < \infty \), it follows from \([1]\) Lemma 2.5 that \( j^* \) restricts to \( K^b(\text{proj}) \), and then \( j_* \) restricts to \( D^b(\text{mod}) \) by \([1]\) Lemma 2.7. Moreover, it follows from \([1]\) Lemma 2.9 (e) that \( j^* \) restricts to \( D^b(\text{mod}) \). Now we claim that \( j_* \) restricts to \( K^b(\text{proj}) \), and then \( j^* \) and \( j_* \) induce an adjoint pair between the corresponding singularity categories.

Since \( j^* \) and \( j_* \) restrict to \( D^b(\text{mod}) \), we infer that \( j_* j^* A \in D^b(\text{mod} A) \), and then \( i_* i^* A \in D^b(\text{mod} A) \) by the triangle

\[
i_* i^* A \rightarrow A \rightarrow j_* j^* A \rightarrow .
\]

As \( i_* i^* A \in \text{Ker} j^* \), it follows that all homologies of \( i_* i^* A \) are in \( \text{mod} A/AeA \) (ref. \([24]\) Proposition 2.17)). Thus \( i_* i^* A \in \text{tria}(\text{mod} A/AeA) = \text{tria}(\frac{A/AeA}{\text{rad}(A/AeA)}) \), and then the assumption \( \text{pd}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty \) implies that \( i_* i^* A \in K^b(\text{proj} A) \). Applying the triangle given above, we conclude that \( j_* j^* A \in K^b(\text{proj} A) \).

For any \( X \in D(eA) \), we have \( X \cong j^* j_* X \subseteq j^* (\text{Tria} A) \subseteq \text{Tria} j^* A \), that is, \( j^* A \) is a compact generator of \( D(eA) \). So, it follows that \( \text{thick} j^* A = \text{thick} (eA) \), and thus \( \text{thick} j_* j^* A = \text{thick} j_* (eA) \) for \( j_* \) is a full embedding.
cf. [25, Lemma 2.2]. Now note that \( j_* j^* A \in K^b(\text{proj} A) \), and so we get \( j_*(eAe) \in K^b(\text{proj} A) \), that is, the functor \( j_* \) restricts to \( K^b(\text{proj}) \).

Next, we claim that \( (j^*, j_*) \) induces a mutually inverse equivalence between the corresponding singularity categories, and then we obtain that \( A \) and \( eAe \) are singularly equivalent of Morita type with level by the same way as we did in Theorem 3.1. For this, take \( X \in D^b(\text{mod} A) \) and consider the triangle

\[
i_* i^! X \to X \to j_* j^* X \to .
\]

By a similar argument as above, we can prove that \( i_* i^! X \in K^b(\text{proj} A) \). Therefore, \( X \cong j_* j^* X \) in \( D_{sg}(A) \), and for any \( Y \in D^b(\text{mod} eAe) \) the isomorphism \( Y \cong j^* j_* Y \) in \( D_{sg}(eAe) \) is clear. Hence, \( (j^*, j_*) \) induces a mutually inverse equivalence between the corresponding singularity categories.

**Theorem 4.2.** Let \( A \) be a finite-dimensional algebra over a field \( k \) such that \( A/\text{rad}(A) \) is separable over \( k \) and let \( e \in A \) be an idempotent. If \( \text{id}_A(\frac{A/eA}{\text{rad}(A/eA)}) < \infty \) and \( \text{pd}_{eAe} eAe < \infty \), then \( A \) and \( eAe \) are singularly equivalent of Morita type with level.

**Proof.** Consider the recollement

\[
\begin{array}{c}
\leftarrow i^* \leftarrow j_! = A \otimes^{L}_{A/\text{rad}(A/eA)} \rightarrow j^* \rightarrow \end{array}
\]

\[
D_B \leftarrow i_! \leftarrow D_A \leftarrow j^! \leftarrow D(eAe) \leftarrow j_* \leftarrow
\]

where \( B \) is a dg algebra. Since \( \text{pd}_{eAe} eAe < \infty \), it follows from [1, Lemma 2.8] that \( j_! \) has a left adjoint \( j^a = R\text{Hom}_{eAe}(eAe, eAe) \otimes^{L}_{A} \). By [1, Lemma 2.2], \( i^* \) also has a left adjoint \( i_{th} \). Therefore, we have a 2-recollement

\[
\begin{array}{c}
\leftarrow j^a \leftarrow \leftarrow j_{th} \leftarrow \leftarrow \end{array}
\]

\[
D(eAe) \leftarrow j^a \leftarrow D_A \leftarrow i^* \leftarrow D_B \leftarrow
\]

It follows from [1, Lemma 2.9 (e)] that \( j^a \) restricts to \( K^b(\text{proj}) \), and \( j_! \) restricts to \( D^b(\text{mod}) \) and \( K^b(\text{proj}) \). Now we claim that \( j^a \) restricts to \( D^b(\text{mod}) \), and then \( (j^a, j_!) \) induces an adjoint pair between the corresponding singularity categories.

Since \( j^* \) and \( j_! \) restrict to \( D^b(\text{mod}) \), we infer that \( j_! j^* DA \in D^b(\text{mod} A) \), and then \( i_* i^* DA \in D^b(\text{mod} A) \) by the triangle

\[
j_! j^* DA \to DA \to i_* i^* DA \to .
\]
As $i_*i^*DA \in \text{Ker} j^*$, it follows that all homologies of $i_*i^*DA$ are in $\text{mod}A/AeA$. Therefore, $i_*i^*DA \in \text{tria}(\text{mod}A/AeA) = \text{tria}(\frac{A/AeA}{\text{rad}(A/AeA)})$, and the assumption $\text{id}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty$ implies that $i_*i^*DA \in K^b(\text{inj}A)$. Applying the triangle given above, we conclude that $j_!j^*DA \in K^b(\text{inj}A)$. Therefore,

$$j_!D(eAe) \cong j_!j^*j_*D(eAe) \subseteq j_!j^*(\text{thick}DA) \subseteq \text{thick}j_!j^*DA \subseteq K^b(\text{inj}A).$$

Here, the first inclusion follows from the fact that $j_*$ restrict to $K^b(\text{inj}A)$, see [30, Lemma 1]. As a result, $j_!$ restricts to $K^b(\text{inj})$ and then $j^\theta$ restricts to $D^b(\text{mod})$ by [30, Lemma 1].

Next, we claim that $(j^\theta, j_!)$ induces a mutually inverse equivalences between the corresponding singularity categories, and then we obtain that $A$ and $eAe$ are singularly equivalent of Morita type with level by the same way as we did in Theorem 3.1. For this, take $X \in D^b(\text{mod}A)$ and consider the triangle

$$i_!i_*X \to X \to j_!j^\theta X \to .$$

Now we will use [1, Lemma 2.4 (c)] to show that $i_!i_*X \subseteq K^b(\text{proj}A)$. For any $Y \in D^b(\text{mod}A)$ and $n \in \mathbb{Z}$, we have isomorphisms

$$\text{Hom}_{DA}(i_!i_*X, Y[n]) \cong \text{Hom}_{DB}(i^*X, i^*Y[n]) \cong \text{Hom}_{DA}(X, i_*i^*Y[n]).$$

By a similar argument as above, we can prove that $i_!i_*Y \subseteq K^b(\text{inj}A)$, and then $\text{Hom}_{DA}(X, i_*i^*Y[n]) = 0$ for all but finitely many $n$. As a result, $\text{Hom}_{DA}(i_!i_*X, Y[n]) = 0$ for all but finitely many $n$, and then $i_!i_*X \subseteq K^b(\text{proj}A)$ by [1, Lemma 2.4 (c)]. Applying the triangle given above, we infer that $X \cong j_!j^\theta X$ in $D_{sg}(A)$. For any $Z \in D^b(\text{mod}eAe)$, the isomorphism $Z \cong j^\theta j_!Z$ in $D_{sg}(eAe)$ is clear. Therefore, $(j^\theta, j_!)$ induces a mutually inverse equivalences between the corresponding singularity categories.

Next, we will show that the condition $\text{pd}_{eAe}eA < \infty$ (resp. $\text{pd}_{A/AeA} < \infty$) in Theorem 3.1 (resp. Theorem 4.2) can be removed if $\text{pd}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) \leq 1$ (resp. $\text{id}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) \leq 1$).

**Proposition 4.3.** Let $A$ be a finite-dimensional algebra over a field $k$ such that $A/\text{rad}(A)$ is separable over $k$ and $e \in A$ be an idempotent. If $\text{pd}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) \leq 1$ or $\text{id}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) \leq 1$, then $A$ and $eAe$ are singularly equivalent of Morita type with level.

**Proof.** If $\text{pd}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) \leq 1$ or $\text{id}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) \leq 1$, then the restriction functor $\text{mod}A/AeA \to \text{mod}A$ is a homological embedding (ref. [20, Proposition 3.5 (iv), Remark 5.9]). Therefore, there is a recollement of derived
where $i_*$ sends all $A/AeA$-modules to $A$-modules. If $\text{pd}_A(A/AeA) \leq 1$, then $\text{pd}_A(i_*M) \leq 1$ for any $M \in \text{mod}A/AeA$. Therefore, the functor $i_*$ restricts to $K^b(\text{proj})$, and so does $j^*$ (see [1, Lemma 2.5 and Lemma 4.3]. Hence, $j^*A = eA \in K^b(\text{proj}eAe)$ and then $\text{pd}_{eAe}eA < \infty$. Therefore, the desired result follows from Theorem 4.1 and the case of $\text{id}_A(A/AeA) \leq 1$ can be proved dually.

The following example illustrates that our result can be applied to construct singular equivalence of Morita type with level where previous method seems too complicated.

**Example 4.4.** Let $A$ be the triangular matrix algebra \( \begin{pmatrix} k & k \\ 0 & k[x]/\langle x^2 \rangle \end{pmatrix} \), where $k$ is an algebraically closed field. In [34, Example 7.5], the author proved that $A$ and $k[x]/\langle x^2 \rangle$ are singularly equivalent of Morita type with level by constructing two explicit bimodules. Now we will use Theorem 4.1 to give a brief proof. Let $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ be an idempotent. Then it is shown in [28, Example 5.5] that $\text{pd}_A(A/AeA) < \infty$ and $\text{pd}_{eAe}eA < \infty$. Therefore, it follows from Theorem 4.1 that $A$ and $eAe = k[x]/\langle x^2 \rangle$ are singularly equivalent of Morita type with level.

## 5 Syzygy-finite, injectives generation and singular equivalences

In [35, Lemma 4.13], Wang proved that the finiteness of the finitistic dimension is invariant under singular equivalences of Morita type with level. In this section, we will show that the properties of syzygy-finite, Igusa-Todorov, injectives generation and projectives cogeneration are also preserved under singular equivalences of Morita type with level.

**Proposition 5.1.** Let $A$ and $B$ be two finite dimensional $k$-algebras which are singularly equivalent of Morita type with level. Then $A$ is syzygy-finite (resp. Igusa-Todorov) if and only if $B$ is syzygy-finite (resp. Igusa-Todorov).
Proof. Assume that \((A_{MB, B} N_A)\) defines a singular equivalence of Morita type of level \(n\) between \(A\) and \(B\). Consider the functors \(F = N \otimes_A - : \mathcal{D}A \to \mathcal{D}B\) and \(G = M \otimes_B - : \mathcal{D}B \to \mathcal{D}A\). Since \(M\) and \(N\) are projective as one-side modules, it follows that the functors \(F\) and \(G\) restrict to \(\mathcal{D}^b(\text{mod})\) and \(K^b(\text{proj})\). Assume that \(B\) is syzygy-finite (resp. Igusa-Todorov). Then \(F(\text{mod} A)\) is syzygy-finite (resp. Igusa-Todorov) since \(F(\text{mod} A) \subseteq \text{mod} B\), and by \([39, \text{Lemma 3.3}]\), \(GF(\text{mod} A) = \Omega^n(\text{mod} A)\) is also syzygy-finite (resp. Igusa-Todorov). Hence, \(A\) is syzygy-finite (resp. Igusa-Todorov) and the “only if” part can be proved parallelly.

Proposition 5.2. Let \(A\) and \(B\) be two finite dimensional \(k\)-algebras which are singularly equivalent of Morita type with level. Then injectives generate (resp. projectives cogenerate) for \(A\) if and only if injectives generate (resp. projectives cogenerate) for \(B\).

Proof. Assume that \((A_{MB, B} N_A)\) defines a singular equivalence of Morita type of level \(n\) between \(A\) and \(B\). Then the functors \(F = \text{Hom}_A(M, -) : \mathcal{D}A \to \mathcal{D}B\) and \(G = \text{Hom}_B(N, -) : \mathcal{D}B \to \mathcal{D}A\) preserve bounded complexes of injectives and set indexed coproducts. Suppose that injectives generate for \(B\). Then \(\mathcal{D}B = \text{Tri}_BDB\), and thus \(GF(\text{mod} A) \subseteq \text{Tri}_BDB\). On the other hand, \(GF(\text{mod} A) = \text{Hom}_B(N, \text{Hom}_A(M, -)) \cong \text{Hom}_A(M \otimes_B N, A) \cong \text{Hom}_A(\Omega^n_A(A), A)\), which is isomorphic to \(\text{Hom}_A(A, \Omega^n_A(A))\) in \(\text{mod} A\), see \([35, \text{Lemma 4.14}]\). Therefore, we conclude that \(GF(\text{mod} A) \cong \Omega^n_A(A)\), up to some direct summands of injective \(A\)-modules. Since \(GF(\text{mod} A) \subseteq \text{Tri}_A\mathcal{D}A\), we get \(\Omega^n_A(A) \subseteq \text{Tri}_A\mathcal{D}A\) and then \(A \in \text{Tri}_A\mathcal{D}A\). Thus, \(\mathcal{D}A = \text{Tri}_A \subseteq \text{Tri}_A\mathcal{D}A\), and then \(\mathcal{D}A = \text{Tri}_A\mathcal{D}A\), that is, injectives generate for \(A\). The “only if” part can be proved parallelly.

Clearly, both \(N \otimes_A - : \mathcal{D}A \to \mathcal{D}B\) and \(M \otimes_B - : \mathcal{D}B \to \mathcal{D}A\) preserve bounded complexes of projective modules. Further, by \([1, \text{Lemma 2.8}]\), these two functors have left adjoints, and then they preserve set indexed products. Therefore, the statement on “projectives cogenerate” can be proved similarly.

The following corollary follows from Theorem 3.6, Proposition 5.1 and Proposition 5.2.

Corollary 5.3. Let \(A\) be a finite dimensional \(k\)-algebra and let \(J \subseteq A\) be a homological ideal which has finite projective dimension as an \(A\)-\(A\)-bimodule. Then \(A\) has the property of syzygy-finite (resp. Igusa-Todorov, injectives generation, projectives cogeneration) if and only if so does \(A/J\).
In \[15\] Proposition 7.6], the author investigated the invariance of injective
generation (resp. projectives cogeneration) under the operation of vertex
removal. Now we mention that \[15\] Proposition 7.6] can be completed as
follow if we only consider algebras with the conditions of separability.

**Corollary 5.4.** (Compare \[15\] Proposition 7.6]) Let \( A \) be a finite-dimensional
\( k \)-algebra such that \( A/\text{rad}(A) \) is separable over \( k \). Let \( e \in A \) be an idempotent
and assume \( Ae \otimes^L_{eA} eA \) is bounded in cohomology. If \( \text{pd}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty \) or
\( \text{id}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty \), then \( A \) has the property of syzygy-finite (resp. Igusa-
Todorov, injectives generation, projectives cogeneration) if and only if so does \( eA \).

**Proof.** Consider the functors \( i^*, i_*, i^!, j_!, j^* \) and \( j_* \) in the proof of Theorem \[4.1\]
If \( Ae \otimes^L_{eA} eA \in \mathcal{D}^b(A) \) and \( \text{pd}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty \), then \( j_*j^*(A) \in \mathcal{D}^b(A) \) and
thus \( i_*i^*A \in \mathcal{D}^b(A) \) by the canonical triangle \( jj^*A \to A \to i_*i^*A \to \). Since
\( i_*i^*A \in \text{Ker} j^* \), it follows that all cohomologies of \( i_*i^*A \) are in \( \text{mod}A/AeA \).
Therefore, \( i_*i^*A \in \text{tria}(\text{mod}A/AeA) = \text{tria}(\frac{A/AeA}{\text{rad}(A/AeA)}) \) in \( \mathcal{K}^b(\text{proj}A) \). Applying
the triangle given above, we have \( j_*j^*A \in \mathcal{K}^b(\text{proj}A) \) and then \( j^*A = eA \in \mathcal{K}^b(\text{proj}A) \),
see \[1\] Lemma 4.2]. Therefore, \( \text{pd}_eA < \infty \) and by Theorem \[4.1\] \( A \) and \( eA \) are singularly equivalent of Morita type with level.

Now the statement follows from Proposition \[5.1\] and Proposition \[5.2\].

If \( Ae \otimes^L_{eA} eA \in \mathcal{D}^b(A) \) and \( \text{id}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty \), then \( j_*j^*(DA) \in \mathcal{D}^b(A) \).
Therefore, we obtain \( j_*j^*(DA) \in \mathcal{D}^b(A) \) by the isomorphisms

\[
H^n(j_*j^*(DA)) \cong \text{Hom}_{DA}(A, j_*j^*(DA)[n]) \cong \text{Hom}_{DA}(j_*j^*(A), DA[n]),
\]
and then \( i_*i^!(DA) \in \mathcal{D}^b(A) \) by the canonical triangle \( i_*i^!(DA) \to DA \to j_*j^*(DA) \to \). Since \( i_*i^!(DA) \in \text{Ker} j^* \), it follows that all cohomologies of \( i_*i^!(DA) \) are in \( \text{mod}A/AeA \). Therefore, \( i_*i^!(DA) \in \text{tria}(\text{mod}A/AeA) = \text{tria}(\frac{A/AeA}{\text{rad}(A/AeA)}) \) in \( \mathcal{K}^b(\text{inj}A) \). Applying the triangle given above, we get \( j_*j^*(DA) \in \mathcal{K}^b(\text{inj}A) \). Now we claim \( j_! \) restricts to \( \mathcal{D}^b(\text{mod}) \). For this,
take \( X \in \mathcal{D}^b(eA) \) and consider the isomorphisms

\[
DH^n(j!X) \cong H^{-n}(D(j!X)) \cong \text{Hom}_{Dk}(j!X, k[-n]) \cong \text{Hom}_{DA}(j!X, DA[-n]),
\]
where the last isomorphism follows by adjunction. Therefore,

\[
DH^n(j!X) \cong \text{Hom}_{D(eAe)}(X, j^*DA[-n]) \cong \text{Hom}_{DA}(j_*X, j_*j^*DA[-n]). \tag{5.1}
\]

Since \( H^n(j_*X) \cong \text{Hom}_{DA}(j_*X, j_*X[n]) \cong \text{Hom}_{D(eAe)}(j^*A, X[n]) \), we infer that \( H^n(j_*X) = 0 \), for any sufficiently small \( n \). As \( j_*j^*(DA) \in \mathcal{K}^b(\text{inj}A) \), it follows
from the formula \[5.1\] that \(H^n(j_i X) = 0\) for sufficiently small \(n\). Moreover, by [1, Lemma 2.10 (b)], we get \(H^n(j_i X) = 0\) for sufficiently large \(n\). Therefore, \(j_i\) restricts to \(D^b(\text{mod})\) and thus \(\text{pd} A e Ae < \infty\), see [1, Lemma 2.8]. Now the statement follows from Theorem 4.2, Proposition 5.1 and Proposition 5.2.

Applying Proposition 4.3, Proposition 5.1 and Proposition 5.2, we get the following result for path algebras.

**Corollary 5.5.** Let \(A = kQ/I\) be a quotient of a path algebra such that \(A/\text{rad}(A)\) is separable over \(k\). Choose a vertex \(v\) in \(Q\) where no relations start or no relations end, and let \(e\) be the sum of all idempotents corresponding to vertices except \(v\). Then \(A\) has the property of syzygy-finite (resp. Igusa-Todorov, injectives generation, projectives cogeneration) if and only if so does \(eAe\).

**Proof.** Clearly, \(\frac{A/AeA}{\text{rad}(A/AeA)}\) is nothing but the simple module corresponding to the vertex \(v\). Hence, it follows from [5, Corollary, Section 1.1] that \(\text{pd}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) \leq 1\) (resp. \(\text{id}_A(\frac{A/AeA}{\text{rad}(A/AeA)}) \leq 1\)) if and only if no relation starts (resp. ends) in \(v\). Now, the statement follows from Proposition 4.3, Proposition 5.1 and Proposition 5.2.

Next, we will apply our results to produce some examples of algebras which have the properties of syzygy-finite, Igusa-Todorov and injectives generation.

**Example 5.6.** ([9, Example 3.3]) Let \(\Gamma\) be the \(k\)-algebra given by the following quiver

\[
\begin{array}{cccc}
1 & \alpha & x & 2 \\
\beta & \delta & \gamma & \\
\end{array}
\]

with relations \(\{x^2, \delta x, \beta x, x\gamma, x\alpha, \beta\gamma, \delta\alpha, \beta\alpha, \delta\gamma, \alpha\beta - \gamma\delta\}\). We write the concatenation of paths from right to left. The singularity category of \(\Gamma\) is studied in [9, Example 3.3]. Indeed, there is an equivalence \(D_{sg}(\Gamma) \cong D_{sg}(\Gamma/\Gamma e_1 \Gamma)\) induced by a homological ideal. Note that \(\Gamma/\Gamma e_1 \Gamma\) is a monomial algebra, and then it has the properties of syzygy-finite and injectives generation. Applying Corollary 5.3, we obtain that \(\Gamma\) also has the properties of syzygy-finite and injectives generation.
Example 5.7. ([IN Example 4.4]) Let $A$ be the $k$-algebra given by the following quiver

$$
\begin{array}{c}
1 \\
\uparrow \alpha \\
\downarrow \gamma \\
\downarrow \beta \\
3 \\
\downarrow \delta \\
\nearrow \epsilon \\
2
\end{array}
$$

with relations $\{\delta \alpha, \gamma \alpha, \beta \gamma, \gamma \epsilon \beta, \delta \epsilon \delta, \alpha \gamma - \epsilon \delta\}$. Let $e = e_1 + e_2$, and then $eAe$ is the monomial algebra $1 \xrightarrow{\tau_\alpha} \tau_\gamma \xrightarrow{\tau_\epsilon \beta} 2$ with relations $\tau_\gamma \tau_\alpha, \tau_\epsilon \beta \tau_\gamma, \tau_\gamma \tau_\epsilon \beta$. Therefore, the algebra $eAe$ has the properties of syzygy-finite and injectives generation. Clearly, there is no relation starts in 3, and it follows from Corollary 5.5 that $A$ also has the properties of syzygy-finite and injectives generation.

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