The effective potential approach in the limit model of an uninsulated vacuum plane diode

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Abstract. Computational aspects of a vacuum plane diode simulation in a magnetic field are considered. The efficient potential approach is proposed. Results of computational experiments are given.

1. Introduction

The paper is an extension of [1–3]. It considers simulation problems of a vacuum plane diode in a magnetic field [4] and focuses on the numerical aspect of modeling an algebraically equivalent problem within the effective potential function of the diode. An equivalent variant of the problem for the efficient potential of the diode in the form of an ordinary first-order differential equation is given in this paper. The authors propose to proceed to the calculation of the model in a complex form (in the mode of the isolating diode) to overcoming the deficiencies of the original model.

2. Mathematical model based on the effective potential function

The mathematical problem of modeling a vacuum plane diode is described by a system of ordinary differential equations [4]

\[
\begin{align*}
\frac{d^2 \phi}{dx^2}(x) &= j_x \frac{1+\phi(x)}{\sqrt{(1+\phi(x))^2-1-a^2(x)}}, \\
\frac{d^2 a}{dx^2}(x) &= j_x \frac{a(x)}{\sqrt{(1+\phi(x))^2-1-a^2(x)}},
\end{align*}
\]

Equation (1) describes the electric and magnetic fields inside the diode, and its solution must meet the initial and boundary conditions characterizing the natural physics of the process describing the mode of uninsulated diode. So, on the segment \( x = [0, 1] \) we have:

- the initial conditions
  \[
  \phi(0) = 0, \quad \dot{\phi}(0) = 0, \quad a(0) = 0, \quad \dot{a}(0) = \beta;
  \]
the boundary conditions
\[ \varphi(1) = \varphi_L, \ a(1) = a_L. \] (3)

\( \beta \) is a parameter in the initial conditions (2) which characterizes the magnetic field near the cathode. It should be specified that the parameters \( j_x, \beta \) of the system (1)– (3) are determined by the boundary conditions \( \varphi_L \) and \( a_L \).

Since the analytic solution of the problem (1)–(3) is currently unknown, it makes sense to proceed to construction of a simplified equivalent system, in which \( \beta \) is not a parameter in the initial conditions. To do this, define the expression at the root in (1) as a function of \( \theta(x) \):

\[ \theta(x) = (1 + \varphi(x))^2 - 1 - a^2(x). \] (4)

Call this function an effective diode potential (EP).

It is easy to make sure that electrons cannot fly out of the cathode unless there is a non-negative potential attached to the diode. So everywhere we assume \( \dot{\theta}(x) \geq 0 \). The limit is when \( \dot{\theta}(x) = 0 \) is described by the Child-Langmuir law [4]. By analogy with the original problem statement, put \( \theta_L \) equal to the effective potential value (EP) on the anode:

\[ \theta_L = (1 + \varphi_L)^2 - 1 - a_L^2. \]

At \( \theta_L < 0 \) the electrons cannot reach the anode of \( x = 1 \), reflected by the magnetic field back to the cathode. Our diode in this state is called a magnetic insulating diode.

It should be noted that EP’s non-negativity was the assumption in this model. In fact, \( \theta \) may be zero at some points of the diode. This leads to closed particle paths (particles are in a “trap”).

Calculate the first and second derivatives of (4):

\[ \ddot{\theta}(x) = 2[(1 + \varphi(x)) \ddot{\varphi}(x) - a(x)\ddot{a}(x)]; \] (5)

\[ \dddot{\theta}(x) = 2[(1 + \varphi(x)) \dddot{\varphi}(x) - a(x)\dddot{a}(x) + \\
(\dot{\varphi}(x))^2 - (\dot{a}(x))^2]. \] (6)

By substituting (1) in (6) and by making a number of simplifications, we get the problem for the efficient potential of the diode in the form of the following second-order differential equation:

\[ (\ddot{\theta}(x))^2 = 8j_x \theta^3(x) - 4\beta^2 \theta(x) + 8j_x \sqrt{\theta(x)}. \] (7)

The corresponding initial and boundary conditions are calculated directly from the (5):

\[ \theta(0) = 0, \ \dot{\theta}(0) = 0, \ \theta(1) = \theta_L. \] (8)

The resulting second-order differential equation (7) can be converted to an ordinary first-order differential equation. To do this, multiply both parts of the equation by \( \ddot{\theta}(x) \) and combine them with \( \theta(x) \) taking into account the initial conditions of (8) come to the equation:

\[ \frac{1}{2} \frac{d}{dx} (\dot{\theta}(x))^2 = 6j_x \dot{\theta}(x) \sqrt{\theta(x)} + 2j_x \frac{ \dot{\varphi}(x) }{ \sqrt{\theta(x)} } - 2\beta^2 \ddot{\theta}(x), \]

\[ \frac{d}{dx} (\dot{\theta}(x))^2 = \frac{12}{3} j_x \frac{ d }{ dx } \theta^3(x) + 8j_x \frac{ d }{ dx } \theta^2(x) - 4\beta^2 \frac{ d }{ dx } \theta(x), \]

\[ (\dot{\theta}(x))^2 = 8j_x \theta^2(x) + 8j_x \theta^2(x) - 4\beta^2 \theta(x). \]
It is obvious that the inclusion of square-root expressions under the differential does not give an algebraic equivalence. Accordingly, solutions to this equation may be only a part of possible solutions to the problem in the original equation. Thus, the diode model problem for the efficient potential function is represented as two different differential equations with the initial conditions (8):

- the first variant
  \[ \ddot{\theta}(x) = 2 \left[ j_x \frac{3\theta(x) + 1}{\sqrt{\theta(x)}} - \beta^2 \right] ; \]  
  \[ (9) \]

- the second variant
  \[ \dot{\theta}(x) = \pm \sqrt{8j_x \theta^3(x) + 8j_x \theta_1^2(x) - 4\beta^2 \theta(x)} . \]
  \[ (10) \]

As can be seen from these equations, the first form (9) of the model defines a singular boundary value problem, as there is a gap at the starting point \( \theta(x) = 0 \). The resulting model in the second form (10) allows to overcome this limitation. Also, both forms of differential equations, due to the presence of root in both cases, define phase-level solutions in \( \mathbb{R}^+ \) domain.

3. Computational experiment

To determine trajectories of the equations (9)-(10) in \( \mathbb{R}^+ \) domain, we used the LSODA adaptive integration algorithm developed by Petzold [5]. This algorithm automatically chooses between the nonstiff predictor-corrector Adams method and the stiff Gear method [6,7] with the backward differentiation formula. In the first stages of a solution process that is not usually stiff, LSODA uses more efficient Adams techniques. If the presence of stiffness is detected, an automatic transition to a stiff Gear method is carried out. It may also be noted that in Adams method errors made at any step have no tendency to exponential growth.

To apply the formulated integration method for the equation (9) make a substitution of \( \theta(x) = x_1 \) and \( \dot{\theta}(x) = x_2 \). As a result, we have

\[ \begin{cases} \dot{x}_1 = x_2 , \\ \dot{x}_2 = 2 \left( j_x \frac{3x_1 + 1}{\sqrt{x_1}} - \beta^2 \right) . \end{cases} \]  
\[ (11) \]

To overcome the problem of singularity at zero with the numerical solution of the boundary value problem in the first form, it is proposed to select the initial value \( \theta_L(0) \) non-zero but sufficiently small, linking the small parameter to the integration step.

The most interesting is the modelling of the diode in the limit (boundary) mode, which occurs when the magnetic field is substantially increased, regulated by the parameter \( \beta \) in (1), relative to the current density of \( j_x \) flowing through the diode. Based on earlier calculations [3] a range of 50,000 points \( \theta \in [0; 1] \) was chosen to calculate the phase trajectories with a value of \( j_x = 0.7 \) and a range of parameter change \( \beta \in [1.7; 2.9] \) with a step of 0.1. The results of numerical integration in computing (11) are presented in figure 1. Hear initial conditions \( \theta_L(0) = 1e - 8 \).

Figure 1 shows that in the “boundary” region for large values \( \beta \) relatively small \( j_x \) we observe a distorted “center” in the phase plane \( (\theta, \dot{\theta}) \). It explains a heterogeneity of the solution of the original boundary problem. At the same time, within the region the phase dynamics character is defined as an unstable node and therefore the solution in this area is unique (figure 2).

To estimate the influence of non-zero condition \( \theta_L(0) = 1e - 8 \) by the values of \( \theta \) points obtained during model calculation for (9), recalculation of the phase trajectory using the (10) was made (figure 3). As a result, differences in the calculation were estimated between two forms of the EP model in the form of an absolute deviation of the values of \( \Delta \dot{\theta}(x) \) with the same values of the points \( \theta \) for the both modes of the model.

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Figure 1. Border EP model.

Figure 2. EP differential model diode within the domain.

Figure 3. Refined border EP model.
Figure 4 illustrates an absolute deviation of values $\Delta \dot{\theta}(x)$ for models (9) and (10). Small initial conditions of $\theta(0)$ generate substantial errors (exceeding by several orders the non-zero small initial condition) for $\dot{\theta} \approx 0$. However, in the area of non-zero values $\dot{\theta} \neq 0$ two forms of the model are numerically equivalent. However, both models have a clear limitation on the phase-level solution in the $\mathbb{R}^+$ domain.

As shown above, the phase trajectories of EP model may be determined not only by numerical differentiation by the model equation in the first form (9), but also by direct analysis in analytical form in the second form of the (10) using a first-order differential equation.

4. Conclusion
For both variants of the EP model transition from the space $\mathbb{R}$ to $\mathbb{R}^+$ domain has been a significant limitation. In the authors’ opinion the use of complex numbers will allow to analyze the process in general. This is a possible direction of further activities.

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