Convergence of Laplacian spectra from random samples

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July 2, 2015

Abstract

Eigenvectors and eigenvalues of discrete graph Laplacians are often used for manifold learning and nonlinear dimensionality reduction. It was previously proved by Belkin and Niyogi [3] that the eigenvectors and eigenvalues of the graph Laplacian converge to the eigenfunctions and eigenvalues of the Laplace-Beltrami operator of the manifold in the limit of infinitely many data points sampled independently from the uniform distribution over the manifold. Recently, we introduced Point Integral method (PIM) [8, 15] to solve elliptic equations and corresponding eigenvalue problem on point clouds. We have established a unified framework to approximate the elliptic differential operators on point clouds. In this paper, we prove that the eigenvectors and eigenvalues obtained by PIM converge in the limit of infinitely many random samples independently from a distribution (not necessarily to be uniform distribution). Moreover, one estimate of the rate of the convergence is also given.

1 Introduction

The Laplace-Beltrami operator (LBO) is a fundamental object associated to Riemannian manifolds, which encodes all intrinsic geometry of the manifolds and has many desirable properties. It is also related to diffusion and heat equation on the manifold, and is connected to a large body of classical mathematics (see, e.g., [12]). In recent years, the Laplace-Beltrami operator has attracted much attention in many applied fields, including machine learning, data analysis, computer graphics and computer vision, and geometric modeling and processing. For instance, the eigensystem of the Laplace-Beltrami operator has been used for representing data in machine learning and data analysis for dimensionality reduction [2, 6], and for representing shapes in computer graphics and computer vision for the analysis of images and 3D models [11, 9].

In general, the underlying Riemannian manifold is unknown and often given by a set of sample points. Thus, in order to exploit the nice properties of the Laplace-Beltrami operator, it is necessary to derive In this paper, we assume that the data points, \( X = \{ x_1, \cdots, x_n \} \), are sampled independently over the manifold \( M \) from a probability distribution \( p(x) \). On the sample points, we consider following discrete eigenvalue problem.

\[
\frac{1}{t} \sum_{j=1}^{n} R \left( \frac{\| x_i - x_j \|^2}{4t} \right) (u_i - u_j) = \lambda \sum_{j=1}^{n} \bar{R} \left( \frac{\| x_i - x_j \|^2}{4t} \right) u_j,
\]  

where \( R : \mathbb{R}^+ \to \mathbb{R}^+ \) is a kernel function, \( \bar{R}(r) = \int_{r}^{+\infty} R(s)ds \).

This eigenvalue problem is closely related with the eigenvalue problem of normalized graph Laplacian. The graph Laplacian is a discrete object associated to a graph, which reveals many properties of the graph as

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†This research was supported by NSFC Grant 11201257 and 11371220.
does the Laplace-Beltrami operator to the manifold \([5]\). In the presence of no boundary and the sample points are uniformly distributed, Belkin and Niyogi \([3]\) showed that the spectra of the normalized graph Laplacian converges to the spectra of \(\Delta_M\). When there is a boundary, it was observed in \([7, 4]\) that the integral Laplace operator \(L_t\) is dominated by the first order derivative and thus fails to be true Laplacian near the boundary. Recently, Singer and Wu \([10]\) showed the spectral convergence in the presence of the Neumann boundary. In the previous approaches, the convergence analysis is based on the connection between the graph Laplacian and the heat operator. The analysis in this paper is very different from the previous ones. We consider this problem from the point of view of solving the Poisson equation on submanifolds, which opens up many tools in the numerical analysis for studying the graph Laplacian.

The purpose of this paper is to study the behavior of discrete eigenvalue problem \((1.1)\) at the limit of \(n \to \infty\) and \(t \to 0\). The main contribution of this paper is that our study reveals that when \(n \to \infty\) and \(t \to 0\), the spectral of \((1.1)\) converge to the spectra of following eigenvalue problem.

\[
\left\{ \begin{array}{l}
\frac{1}{p^2(x)} \text{div} \left( p^2(x) \nabla u(x) \right) = \lambda u(x), \quad x \in \mathcal{M}, \\
\frac{\partial u}{\partial n}(x) = 0, \quad x \in \partial \mathcal{M}.
\end{array} \right. \tag{1.2}
\]

where \(n\) is the outer normal vector of \(\mathcal{M}\).

To analyze the convergence, we introduce an intermediate integral equation.

\[
\frac{1}{t} \int_{\mathcal{M}} R \left( \frac{||x-y||^2}{4t} \right) (u(x) - u(y))p(y)dy = \int_{\mathcal{M}} \tilde{R} \left( \frac{||x-y||^2}{4t} \right) f(y)p(y)dy, \quad x \in \mathcal{M}. \tag{1.3}
\]

Similar integral equation also can be found in previous works. However, the rest of the analysis in this paper is very different as the previous ones. Before presenting the main results, we need to define three solution operators \(T, T_t\) and \(T_{t,n}\).

### 1.1 Solution operators

The solution operators are defined as following.

- **\(T : L^2(\mathcal{M}) \to H^2(\mathcal{M})\)** is the solution operator of the problem \((1.4)\), i.e., for any \(f \in L^2(\mathcal{M})\), \(T(f)\) with \(\int_{\mathcal{M}} T(f) = 0\) is the solution of the following problem:

\[
\left\{ \begin{array}{l}
-\frac{1}{p^2(x)} \text{div} \left( p^2(x) \nabla u(x) \right) = f(x), \quad x \in \mathcal{M}, \\
\frac{\partial u}{\partial n}(x) = 0, \quad x \in \partial \mathcal{M}.
\end{array} \right. \tag{1.4}
\]

where \(n\) is the outer normal vector of \(\mathcal{M}\).

- **\(T_t : L^2(\mathcal{M}) \to L^2(\mathcal{M})\)** is the solution operator of following integral equation \((1.5)\), i.e. \(u = T_t(f)\) with \(\int_{\mathcal{M}} u(x)p(x)dx = 0\) solves the following integral equation

\[
\frac{1}{t} \int_{\mathcal{M}} R_t(x,y) (u(x) - u(y))p(y)dy = \int_{\mathcal{M}} \tilde{R}_t(x,y)f(y)p(y)dy. \tag{1.5}
\]

where

\[
R_t(x,y) = \frac{1}{(4\pi t)^{k/2}} R \left( \frac{||x-y||^2}{4t} \right), \quad \tilde{R}_t(x,y) = \frac{1}{(4\pi t)^{k/2}} \tilde{R} \left( \frac{||x-y||^2}{4t} \right),
\]

- **\(T_{t,n} : C(\mathcal{M}) \to C(\mathcal{M})\)** is defined as follows.

\[
T_{t,n}(f)(x) = \frac{1}{n \ w_{t,n}(x)} \sum_{j=1}^{n} R_t(x,x_j)u_j + \frac{t}{n \ w_{t,n}(x)} \sum_{j=1}^{n} \tilde{R}_t(x,x_j)f(x_j). \tag{1.6}
\]
Let Proposition 1.1. From (1.10) and (1.11), we can see that in some sense, solution operators, and analysis, the operator $T$ using these definitions, we have that

$$\frac{1}{nt} \sum_{j=1}^{n} R_t(x_i, x_j)(u_i - u_j) = \frac{1}{n} \sum_{j=1}^{n} \bar{R}_t(x_i, x_j)f(x_j)$$  

(1.7)

To simplify the notations, we also introduce two operators. For any $f \in L^2(M)$,

$$L_t f(x) = \frac{1}{t} \int_{M} R_t(x, y)(f(x) - f(y))p(y)dy.$$  

(1.8)

and for any $f \in C(M)$,

$$L_{t,n} f(x) = \frac{1}{nt} \sum_{j=1}^{n} R_t(x, x_j)(f(x) - f(x_j)).$$  

(1.9)

Using these definitions, we have that

$$L_t(T_t f)(x) = \int_{M} \bar{R}_t(x, y)f(y)p(y)dy$$  

(1.10)

and

$$L_t(T_{t,n} f)(x) = \frac{1}{n} \sum_{j=1}^{n} \bar{R}_t(x_i, x_j)f(x_j).$$  

(1.11)

From (1.10) and (1.11), we can see that in some sense, solution operators, $T_t, T_{t,n}$, are inverse operators of $L_{t,n}, L_t$. So, it is natural to imagine that their spectra are equivalent.

Proposition 1.1. Let $\theta(u)$ denote the restriction of function $u$ to the sample points $X = (x_1, \cdots, x_n)^t$, i.e., $\theta(u) = (u(x_1), \cdots, u(x_n))^t$.

1. If a function $u$ is an eigenfunction of $T_{t,n}$ with eigenvalue $\lambda$, then the vector $\theta(u)$ is an eigenvector of the eigenproblem (1.11) with eigenvalue $1/\lambda$.

2. If a vector $u$ is an eigenvector of the eigenproblem (1.11) with the eigenvalue $\lambda$, then $I_{\lambda}(u)$ is an eigenfunction of $T_{t,n}$ with eigenvalue $1/\lambda$, where

$$I_{\lambda}(u)(x) = \frac{\lambda t \sum_{j=1}^{n} \bar{R}_t(x, x_j)u_j + \sum_{j=1}^{n} R_t(x, x_j)u_j}{\sum_{j=1}^{n} R_t(x, x_j)}.$$  

3. All eigenvalues of $T, T_t, T_{t,n}$ are real numbers. All generalized eigenvectors of $T, T_t, T_{t,n}$ are eigenvectors.

This proposition can be proved by following the same line as that in [14].

Using this proposition, we only need to analyze the relation among the spectra of $T$ and $T_{t,n}$. In the analysis, the operator $T_t$ plays very important role which bridge $T$ and $T_{t,n}$. The main advantage of using these solution operators instead of $L_t$ and $L_{t,n}$ is that they are compact operators which is proved in following proposition.

Proposition 1.2. For any $t > 0, n > 0$, $T, T_t$ are compact operators on $H^1(M)$ into $H^1(M)$; $T_t, T_{t,n}$ are compact operators on $C^1(M)$ into $C^1(M)$.  

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Proof: First, it is well known that $T$ is compact operator. $T_{t,h}$ is actually finite dimensional operator, so it is also compact. To show the compactness of $T_t$, we need the following formula,

$$T_t u = \frac{1}{w_t(x)} \int_M R_t(x,y) u(y) dy + \frac{t}{w_t(x)} \int_M \hat{R}_t(x,y) u(y) dy, \quad \forall u \in H^1(M).$$

Using the assumption that $R \in C^2$, direct calculation would gives that that $T_t u \in C^2$. This would imply the compactness of $T_t$ both in $H^1$ and $C^1$.

It is well known that compact operator has many good properties. Many powerful theorems in the spectral theory of compact operators can be used which makes our analysis concise and clear.

### 1.2 Main result

The main result in this paper is stated with the help of the Riesz spectral projection. Let $X$ be a complex Banach space and $L : X \to X$ be a compact linear operator. The resolvent set $\rho(L)$ is given by the complex numbers $z \in \mathbb{C}$ such that $z - L$ is bijective. The spectrum of $L$ is $\sigma(L) = \mathbb{C} \setminus \rho(L)$. It is well known that $\sigma(L)$ is a countable set with no limit points other than zero. All non-zero values in $\sigma(L)$ are eigenvalues. If $\lambda$ is a nonzero eigenvalue of $L$, the ascent multiplicity $\alpha$ of $\lambda - L$ is the smallest integer such that $\ker(\lambda - L)^\alpha = \ker(\lambda - L)^{\alpha+1}$.

Given a closed smooth curve $\Gamma \subset \rho(L)$ which encloses the eigenvalue $\lambda$ and no other elements of $\sigma(L)$, the Riesz spectral projection associated with $\lambda$ is defined by

$$E(\lambda, L) = \frac{1}{2\pi i} \int_\Gamma (z - L)^{-1} dz,$$

where $i = \sqrt{-1}$ is the unit imaginary. The definition does not depend on the chosen of $\Gamma$. It is well known that $E(\lambda, L) : X \to X$ has following properties:

1. $E(\lambda, L) \circ E(\lambda, L) = E(\lambda, L)$, $L \circ E(\lambda, L) = E(\lambda, L) \circ L$, $E(\lambda, L) \circ E(\mu, L) = 0$, if $\lambda \neq \mu$.
2. $E(\lambda, L)X = \ker(\lambda - L)^\alpha$, where $\alpha$ is the ascent multiplicity of $\lambda - L$.
3. If $\Gamma \subset \rho(L)$ encloses more eigenvalues $\lambda_1, \cdots, \lambda_m$, then

$$E(\lambda_1, \cdots, \lambda_m, L)X = \oplus_{i=1}^m \ker(\lambda_i - L)^{\alpha_i}$$

where $\alpha_i$ is the ascent multiplicity of $\lambda_i - L$.

The properties (2) and (3) are of fundamental importance for the study of eigenvector approximation.

To prove the convergence, we need some assumptions on the manifold $\mathcal{M}$, probability distribution $p(x)$ and the kernel function $R$ which are summarized as following:

**Assumption 1.**

- **Assumptions on the manifold:** $\mathcal{M}$ is $k$-dimensional compact and $C^\infty$ smooth manifold isometrically embedded in a Euclidean space $\mathbb{R}^d$.

- **Assumptions on the sample points:** $X = \{x_1, \cdots, x_n\}$ are sampled independently over the manifold $\mathcal{M}$ distribution $p(x) \in C^1(\mathcal{M})$ and $\min_{x \in \mathcal{M}} p(x) > 0$, $\max_{x \in \mathcal{M}} p(x) < \infty$.

- **Assumptions on the kernel function $R(r)$:**
  
  (a) $R \in C^2(\mathbb{R}^+)$;
(b) \( R(r) \geq 0 \) and \( R(r) = 0 \) for \( \forall r > 1 \);
(c) \( \exists \delta_0 > 0 \) so that \( R(r) \geq \delta_0 \) for \( 0 \leq r \leq \frac{1}{2} \).

Now, we are ready to state the main theorem. Since \( T \) and \( T_{t,n} \) are both compact operators, their eigenvalues can be sorted as

\[
0 < \cdots \leq \lambda_i \leq \cdots \leq \lambda_2 \leq \lambda_1, \\
0 < \cdots \leq \lambda_{i,n} \leq \cdots \leq \lambda_{2,n} \leq \lambda_{1,n},
\]

where the same eigenvalue is repeated according to its multiplicity.

For corresponding eigenvalues and eigenfunctions, we have the following theorem.

**Theorem 1.1.** Under the assumptions in Assumption 1 let \( \lambda_i \) be the \( i \)th largest eigenvalue of \( T \) (same eigenvalue is repeated according to its multiplicity) with multiplicity \( \alpha_i \) and \( \phi_i^k, k = 1, \cdots, \alpha_i \) be the linear independent eigenfunctions corresponding to \( \lambda_i \). Let \( \lambda_{i,n} \) be the \( i \)th largest eigenvalue of \( T_{t,n} \). With probability at least \( 1 - 1/n \), there exists a constant \( C_1 > 0, C_2 > 0 \) depend on \( \mathcal{M} \), kernel function \( R \), distribution \( p \) and spectra of \( T \), such that

\[
|\lambda_{i,n} - \lambda_i| \leq C_1 \left( t^{1/2} + \frac{\log n + |\log t| + 1}{t^{k+3} \sqrt{n}} \right),
\]

and

\[
\|\phi_i^k - E(\sigma_{i,n}^T, T_{t,n})\phi_i^k\|_{H^1(\mathcal{M})} \leq C_2 \left( t^{1/2} + \frac{\log n + |\log t| + 1}{t^{k+2} \sqrt{n}} \right),
\]

as long as \( n \) large enough. Here \( \sigma_{i,n}^T = \{\lambda_{j,n} \in \sigma(T_{t,n}) : j \in I_i\} \) and \( I_i = \{j \in \mathbb{N} : \lambda_j = \lambda_i\} \).

This theorem will be proved in Section 2 and 3. Some conclusions are made in Section 4.

## 2 Proof of the main theorem (Theorem 1.1)

The proof of Theorem 1.1 mainly consists of three parts. The first part is to relate the difference of the eigenvalues and eigenfunctions with the difference of operators \( T - T_i \) and \( T_i - T_{t,n} \) (Theorem 2.4). This is achieved by using one theorem in the perturbation of compact operators.

To apply the theorem obtained in the first part, we need to estimate the difference of operators \( T - T_i \) and \( T_i - T_{t,n} \) in \( H^1 \) and \( C^1 \) norm respectively. This is also the most difficult part. Comparing with the pointwise convergence which was proved in previous works, convergence in norm is much stronger and much more difficult to prove. Fortunately, under some mild assumption which are listed in Assumption 1, we could prove that \( T_i \to T \) in \( H^1 \) norm as \( t \to 0 \) (Theorem 2.5) and \( T_{t,n} \to T_i \) in \( C^1 \) norm as \( n \to \infty \) (Theorem 2.6).

To get the rate of the convergence, in the last part of the analysis, we use the theory of the Glivenko-Cantelli class in statistical learning to estimate the error in the Monte-Carlo integration. The key point in this part is to estimate the covering number of the function classes defined as following.

Here, we list some notations which will be used in the proof. Some of them have been defined in previous sections. We also list them here for the convenience of readers.

- \( k \): dimension of the underlying manifold; \( d \): dimension of the ambient Euclidean space;
- \( C \): positive constant independent on \( t \) and sample points \( X_n \). We abuse the notation to denote all the constants independent on \( t \) and sample points \( X_n \) by \( C \). It may be different in different places.
- \( C_t = \frac{1}{(4\pi t)^{d/2}} \) is the normalize constant of kernel function \( R \).
Let $(X, \| \cdot \|_X)$ be an arbitrary Banach space. Let $S$ and $T$ be compact linear operators on $X$ into $X$. Let $z \in \rho(T)$. Assume

$$
\| (T - S)S \|_X \leq \frac{|z|}{\|(z - T)^{-1}\|_X}.
$$

Then $z \in \rho(S)$ and $(z - S)^{-1}$ has the bound

$$
\| (z - S)^{-1} \|_X \leq \frac{1 + \|S\|_X\|(z - T)^{-1}\|_X}{|z| - \|(z - T)^{-1}\|_X\|(T - S)S\|_X}.
$$

Theorem 2.2. (II) Let $(X, \| \cdot \|_X)$ be an arbitrary Banach space. Let $S$ and $T$ be compact linear operators on $X$ into $X$. Let $z_0 \in \mathbb{C}$, $z_0 \neq 0$ and let $\epsilon > 0$ be less than $|z_0|$, denote the circumference $|z - z_0| = \epsilon$ by $\Gamma$ and assume $\Gamma \subset \rho(T)$. Denote the interior of $\Gamma$ by $U$. Let $\sigma_T = U \cap \sigma(T) \neq \emptyset$. $\sigma_S = U \cap \sigma(S)$. Let $E(\sigma_S, S)$ and $E(\sigma_T, T)$ be the corresponding spectral projections of $S$ for $\sigma_S$ and $T$ for $\sigma_T$, i.e.

$$
E(\sigma_S, S) = \frac{1}{2\pi i} \int_{\Gamma} (z - S)^{-1} dz, \quad E(\sigma_T, T) = \frac{1}{2\pi i} \int_{\Gamma} (z - T)^{-1} dz.
$$
Assume

\[ \|(T - S)x\| \leq \frac{|z|}{\min_{z \in \Gamma} \|(z - T)^{-1}\|} \tag{2.4} \]

Then, we have

(1). Dimension \( E(\sigma_S, S)X = E(\sigma_T, T)X \), thereby \( \sigma_S \) is nonempty and of the same multiplicity as \( \sigma_T \).

(2). For every \( x \in X \),

\[ \|E(\sigma_T, T)x - E(\sigma_S, S)x\| \leq \frac{M\epsilon}{c_0} \left( \| (T - S)x \| + \| x \| \|(T - S)S\| \right). \]

where \( M = \max_{z \in \Gamma} \|(z - T)^{-1}\| \), \( c_0 = \min_{z \in \Gamma} |z| \).

**Lemma 2.1.** \( (L_4) \) Let \( T \) be the solution operator of the Neumann problem \( (1.4) \) and \( z \in \rho(T) \), then

\[ \|(z - T)^{-1}\|_{H^1(M)} \leq \frac{1}{\max_{n \in \mathbb{N}} |z - \lambda_n|}, \]

where \( \{\lambda_n\}_{n \in \mathbb{N}} \) is the set of eigenvalues of \( T \).

**Lemma 2.2.** \( (L_4) \) Let \( T_t \) be the solution operator of the integral equation \( (1.5) \). For any \( z \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{N}} B(\lambda_n, r_0) \) with \( r_0 > \|T - T_t\|_{H^1} \), then

\[ \|(z - T_t)^{-1}\|_{C^1} \leq \max \left\{ \frac{2|M|}{|z|^{t(k+2)/4}} \left( \min_{n \in \mathbb{N}} |z - \lambda_n| - \|T - T_t\|_{H^1} \right)^{-1}, \frac{2}{|z|} \right\}. \]

**Theorem 2.3.** \( (L_4) \) Let \( T_t \) be the solution operator of the integral equation \( (1.5) \) and \( \lambda_n \) be eigenvalues of \( T \), then

\[ \sigma(T_t) \subset \bigcup_{n \in \mathbb{N}} B(\lambda_n, 2\|T - T_t\|_{H^1(M)}) \]

The main result in this subsection is stated as following in which the difference of the eigenvalues and eigenfunctions are related with the difference of the solutions operators.

**Theorem 2.4.** Let \( \lambda_m \) be the \( m \)th largest eigenvalue of \( T \) with multiplicity \( \alpha_m \) and \( \phi_m^k, k = 1, \ldots, \alpha_m \) be the eigenfunctions corresponding to \( \lambda_m \). Let \( \lambda_m^{1, n} \) be the \( m \)th largest eigenvalue of \( T_{t,n} \). Let \( \gamma_m = \min_{j \leq m} |\lambda_j - \lambda_{j+1}| \) and

\[ \|(T_{t,n} - T_t)T_{t,n}\|_{C^1} \leq \min \left\{ t, \frac{\gamma_m^{k+3/2}}{24}, \frac{(|\lambda_m| - \gamma_m/3)^{2k+2/4} \gamma_m}{12}, \frac{(|\lambda_m| - \gamma_m/3)^2}{2} \right\}, \]

\[ \|T - T_t\|_{H^1(M)} \leq \gamma_m/12, \quad \|(T - T_t)T_t\|_{H^1(M)} \leq \left( |\lambda_m| - \gamma_m/3 \right) \gamma_m/3 \]

Then there exists a constant \( C_1, C_2 \) depend on \( M \), the kernel function \( R \), \( \gamma_m \) and \( \lambda_m \), such that

\[ |\lambda_m^{1, n} - \lambda_m| \leq \frac{2}{tk^{3/2}} \|(T_{t,n} - T_t)T_{t,n}\|_{C^1} + \|T - T_t\|_{H^1(M)} \]

and

\[ \|\phi_{m}^{k} - E(\sigma_{m}^{1, n}, T_{t,n})\phi_{m}^{k}\|_{H^1(M)} \leq C \left( \|T - T_t\|_{H^1} + \|(T - T_t)T_t\|_{H^1} \right) + \frac{C}{tk^{3/2}} \left( \|(T_{t,n} - T_t)\phi_{m}^{k}\|_{C^1} + \|(T_{t,n} - T_t)T_{t,n}\|_{C^1} \right) \]

Here \( \sigma_{m}^{1, n} = \{ \lambda_j^{1, n} \in \sigma(T_{t,n}) : j \in I_m \} \) and \( I_m = \{ j \in \mathbb{N} : \lambda_j = \lambda_m \} \).
Proof. Let $r_1 = \frac{2}{\sqrt{k/4+3/2}} \frac{\|T_{t,n} - T_t T_{t,n}\|_{C^1} + \|T - T_t\|_{H^1(M)}}{\|T - T_t\|_{H^1(M)}}$, $A = \mathbb{C} \setminus \bigcup_{n \in \mathbb{N}} B(\lambda_n, r_1) \bigcup B(0, t^{1/2})$. For any $z \in A$, using Lemma 2.3 we have

$$\|z - T_t\|_{C^1}^{-1} \leq \frac{2|M|}{|z|^{k+1/2}} \left( \min_{n \in \mathbb{N}} |z - \lambda_n| - \|T - T_t\|_{H^1} \right)^{-1} \leq \frac{2|M|}{t^{1/2}} \left( r_1 - \|T - T_t\|_{H^1} \right)^{-1}$$



or

$$\|z - T_t\|_{C^1}^{-1} \leq \frac{2}{|z|} \leq \frac{2}{t^{1/2}} \leq \frac{\sqrt{t}}{\|T_{t,n} - T_t T_{t,n}\|_{C^1}} \leq \frac{\|z\|}{\|T_{t,n} - T_t T_{t,n}\|_{C^1}}.$$ 

Here, we use the condition that $\|T_{t,n} - T_t T_{t,n}\|_{C^1} \leq t/2$.

Both above two inequalities implies that

$$\|T_{t,n} - T_t T_{t,n}\|_{C^1} \leq \frac{|z|}{\|z - T_t\|_{C^1}}.$$ 

Then using Lemma 2.3, we have $z \in \rho(T_{t,n})$.

Since $z$ is arbitrary in $A$, we get $A \subset \rho(T_{t,n})$. This means that

$$\sigma(T_{t,n}) = \mathbb{C} \setminus \rho(T_{t,n}) \subset \mathbb{C} \setminus A = \bigcup_{n \in \mathbb{N}} B(\lambda_n, r_1) \bigcup B(0, t^{1/2}). \quad (2.5)$$

Moreover, using Theorem 2.3 and the definition of $r_1$, we have

$$\sigma(T_t) \subset \bigcup_{n \in \mathbb{N}} B(\lambda_n, 2r_1). \quad (2.6)$$

For any fixed eigenvalue $\lambda_m \in \sigma(T)$, let $\gamma_m = \min_{j \leq m} |\lambda_j - \lambda_{j+1}|$. Using the structure of $\sigma(T)$, we know that $\gamma_m > 0$. Since

$$2 \frac{2}{t^{k/4+3/2}} \|T_{t,n} - T_t T_{t,n}\|_{C^1} \leq \frac{\gamma_m/12}{\|T - T_t\|_{H^1(M)}} \leq \frac{\gamma_m/12}{\|T_{t,n} - T_t T_{t,n}\|_{C^1}},$$

we know that $r_1 < \gamma_m/6$.

Let $\Gamma_j = \{z \in \mathbb{C} : |z - \lambda_j| = \gamma_j/3\}$, $U_j$ be the area enclosed by $\Gamma_j$. Let

$$\sigma_{t,j} = \sigma(T_t) \bigcap U_j, \quad \sigma_{t,n,j} = \sigma(T_{t,n}) \bigcap U_j.$$ 

Using the definition of $\Gamma_j$, we know for any $j \leq m$, $\Gamma_j \subset \rho(T), \rho(T_t)$ and $\rho(T_{t,n})$.

In order to apply Theorem 2.3 we need to verify the conditions

$$\|T - T_t\|_{H^1} \leq \min_{z \in \Gamma_j} \frac{|z|}{\|z - T_t\|_{H^1}}, \quad (2.7)$$ 

$$\|T_{t,n} - T_t T_{t,n}\|_{C^1} \leq \min_{z \in \Gamma_j} \frac{|z|}{\|z - T_t\|_{C^1}}. \quad (2.8)$$

Using Lemma 2.3 and the choice of $\Gamma_j$, we have

$$\min_{z \in \Gamma_m} \frac{|z|}{\|z - T_t\|_{H^1}} \geq \min_{z \in \Gamma_m} \frac{|z|}{\max_{z \in \Gamma_m} \|z - T_t\|_{H^1}} \geq (|\lambda_m| - \gamma_m/3) \min_{z \in \Gamma_m, n \in \mathbb{N}} |z - \lambda_m| = (|\lambda_m| - \gamma_m/3) \gamma_m/3.$$
Then, using the assumption that \( \|(T - T_i)T_i\|_{H^1(M)} \leq (|\lambda_m| - \gamma_m/3)\gamma_m/3 \), condition (2.7) is true.

Using Lemma 2.2 we have
\[
\min_{z \in \Gamma_m} \frac{\|z\|}{\|z - T_i\|^{-1}_C} \geq \frac{\min_{z \in \Gamma_m} |z|}{\max_{z \in \Gamma_m} \|z - T_i\|^{-1}_C} \geq \frac{(|\lambda_m| - \gamma_m/3)^2 t^{(k+2)/4}}{2} \left( \min_{z \in \Gamma_m, n \in \mathbb{N}} |z - \lambda_m| - \|T - T_i\|_{H^1} \right) \geq \frac{(|\lambda_m| - \gamma_m/3)^2 t^{(k+2)/4} \gamma_m}{12},
\]
(2.9)

or
\[
\min_{z \in \Gamma_m} \frac{\|z\|}{\|z - T_i\|^{-1}_C} \geq \frac{\min_{z \in \Gamma_m} |z|}{\max_{z \in \Gamma_m} \|z - T_i\|^{-1}_C} \geq \frac{(|\lambda_m| - \gamma_m/3)^2}{2}. \quad (2.10)
\]

To get the last inequality of (2.9), we use the assumption that \( \|T - T_i\|_{H^1} \leq \gamma/6 \) and \( \min_{z \in \Gamma_m, n \in \mathbb{N}} |z - \lambda_m| = \gamma_m/3 \).

Using the assumption that \( \|(T - T_i)T_i\|_{C^1(M)} \leq \min \left\{ \frac{(|\lambda_m| - \gamma_m/3)^2 t^{(k+2)/4} \gamma_m}{12}, \frac{(|\lambda_m| - \gamma_m/3)^2}{2} \right\} \), condition (2.8) is satisfied.

Then using Theorem 2.2 we have
\[
\text{dim}(E(\lambda_m, T_i)) = \text{dim}(E(\sigma_{t,m}, T_i)) = \text{dim}(E(\sigma_{t,n,m}, T_i, T_n)). \quad (2.11)
\]

Using (2.5), above equality would imply that
\[
|\lambda_m| - \gamma_m \leq r_1 = \frac{2}{t^{k+3/2}} \|(T_i - T_i)T_i\|_{C^1} + \|T - T_i\|_{H^1(M)}. \quad (2.12)
\]

The convergence of eigenspace is also given by Theorem 2.2. For any \( x \in E(\lambda_m, T_i), \|x\|_{C^1} = 1 \),
\[
\|x - E(\sigma_{t,m}, T_i)x\|_{H^1} \leq \frac{\max_{z \in \Gamma_m} \|z\|}{\min_{z \in \Gamma_m} |z|} \frac{\|z - T_i\|_{H^1}}{\gamma_m} \frac{\|z - T_i\|_{H^1}}{3} \frac{\|z - T_i\|_{H^1}}{\gamma_m} \|x\|_{H^1}.
\]

Using Lemma 2.1 we know that
\[
\max_{z \in \Gamma_m} \|z - T_i\|_{H^1} \leq \max_{j \in \mathbb{N}} \frac{1}{|z - \lambda_j|} \leq \frac{3}{2\gamma_m},
\]
and \( \min_{z \in \Gamma_m} |z| = |\lambda_m| - \gamma_m/3 \). This implies that from Theorem 2.5
\[
\|x - E(\sigma_{t,m}, T_i)x\|_{H^1} \leq C(\|(T - T_i)x\|_{H^1} + \|(T - T_i)T_i\|_{H^1} \|x\|_{H^1}). \quad (2.13)
\]

Regarding the convergence from \( T_{i,n} \) to \( T_i \), using Theorem 2.2 again, we have
\[
\|E(\sigma_{t,m}, T_i)x - E(\sigma_{t,n,m}, T_{i,n})x\|_{C^1} \leq \frac{\gamma_m}{3 \min_{z \in \Gamma_m} |z|} \left( \|(T_i - T_{i,n})x\|_{C^1} + \|(T_i - T_{i,n})T_{i,n}\|_{C^1} \right). \quad (2.14)
\]

Using Lemma 2.2 we know that
\[
\max_{z \in \Gamma_m} \|z - T_i\|_{C^1} \leq \max_{z \in \Gamma_m} \left\{ \frac{2}{|z| t^{(k+2)/4}} \left( \min_{j \in \mathbb{N}} |z - \lambda_j| - \|T - T_i\|_{H^1} \right)^{-1}, \frac{2}{|z|} \right\} \leq \max \left\{ \frac{12}{\gamma_m (|\lambda_m| - \gamma_m/3)^{t^{(k+2)/4}}}, \frac{2}{|\lambda_m| - \gamma_m/3} \right\}. \quad (2.15)
\]

To get the last inequality, we use that \( \|T - T_i\|_{H^1} \leq \gamma_m/6 \) and \( |z - \lambda_m| = \gamma_m/3, |z| \geq |\lambda_m - \gamma_m/3| \) for \( z \in \Gamma_m \).

Then the proof is completed by combining (2.12), (2.13), (2.14) and (2.15).
2.2 Convergence of solution operators

To apply Theorem 2.3, we need to estimate the difference of the solution operators. More precisely, we need to estimate $\|T - T_t\|_{H^1}$ and $\|T_t - T_{t,n}\|_{C^1}$ as $t \to 0$ and $n \to \infty$. These results are summarized in Theorem 2.5 and Theorem 2.6 respectively.

**Theorem 2.5.** Under the assumptions in Assumption 1, there exists a constant $C > 0$ only depends on $\mathcal{M}$ and the kernel function $R$, such that

$$\|T - T_t\|_{H^1} \leq Ct^{1/2}, \quad \|T_t\|_{H^1} \leq C.$$  

The proof of this theorem can be found in [13].

The other theorem is about $\|T_t - T_{t,n}\|_{C^1}$.

**Theorem 2.6.** Under the assumptions in Assumption 1 and

$$C_t \sup_{f \in \mathcal{K}_{t,n} \cup \mathcal{K}_t} |p(f) - p_n(f)| \leq w_{\text{min}}/2, \quad (2.16)$$

$$C_t \sup_{f \in \mathcal{K}_{t,n} \cup \mathcal{K}_t} |p(f) - p_n(f)| \leq \frac{\delta^2}{2 \max\{w_{\text{max}} + w_{\text{min}}/2, 2/w_{\text{min}}\}}, \quad (2.17)$$

where $\delta = \frac{w_{\text{min}}}{4w_{\text{max}} + 3w_{\text{min}}}, t' = t/18$. There exists a constant $C$ only depends on $\mathcal{M}$ and kernel function $R$, such that

$$\|(T_{t,n} - T_t)T_{t,n}\|_{C^1} \leq \frac{Ch_0}{t^{3k/4 - 3/2}}, \quad \|(T_{t,n} - T_t)f\|_{C^1} \leq \frac{Ch(f)}{t^{3k/4 - 3/2}}.$$  

where

$$h_0 = \sup_{g \in \mathcal{K}_{t,n} \cup \mathcal{K}_t} |p_n(g) - p(g)| + t \sup_{g \in \mathcal{D}_{t,n} \cup \mathcal{D}_t} \sup_{\mathcal{K}_{t,n} \cup \mathcal{K}_t} |p_n(g) - p(g)| \quad (2.18)$$

$$h(f) = \sup_{g \in \mathcal{K}_{t,n} \cup \mathcal{K}_t} |p_n(g) - p(g)| + t \sup_{g \in \mathcal{D}_{t,n} \cup \mathcal{D}_t} \sup_{\mathcal{K}_{t,n} \cup \mathcal{K}_t} |p_n(g) - p(g)| \quad (2.19)$$

The proof of this theorem will be deferred to Section 3.

2.3 Entropy bound

In this subsection, we will verify the assumption (2.16), (2.17) in Theorem 2.6 and estimate $h_0$ and $h(f)$ defined in (2.18) and (2.19) to get the convergence number. The method we use is to estimate the covering number of function classes defined in previous subsection. First we introduce the definition of covering number.

Let $(Y, d)$ be a metric space and set $F \subset Y$. For every $\epsilon > 0$, denote by $N(\epsilon, F, d)$ the minimal number of open balls (with respect to the metric $d$) needed to cover $F$. That is, the minimal cardinality of the set \{\{y_1, \ldots, y_m\} \subset Y with the property that every $f \in F$ has is some $y_i$ such that $d(f, y_i) < \epsilon$. The set \{\{y_1, \ldots, y_m\} is called an $\epsilon$-cover of $F$. The logarithm of the covering numbers is called the entropy of the set. For every sample \{x_1, \ldots, x_n\} let $\mu_n$ be the empirical measure supported on that sample. For $1 \leq p < \infty$ and a function $f$, put $\|f\|_{L_p(\mu_n)} = \left(\frac{1}{n} \sum_{i=1}^{n} |f(x_i)|^p\right)^{1/p}$ and set $\|f\|_{\infty} = \max_{1 \leq i \leq n} |f(x_i)|$. Let $N(\epsilon, F, L_p(\mu_n))$ be the covering numbers of $F$ at scale $\epsilon$ with respect to the $L_p(\mu_n)$ norm.

We will use following theorem which is well known in empirical process theory.
Theorem 2.7. (Theorem 2.3 in [10]) Let $F$ be a class of functions from $\mathcal{M}$ to $[-1,1]$ and set $\mu$ to be a probability measure on $\mathcal{M}$. Let $(x_i)_{i=1}^{\infty}$ be independent random variables distributed according to $\mu$. For every $\epsilon > 0$ and any $n \geq 8/\epsilon^2$,

$$\mathbb{P}\left(\sup_{f \in F} \frac{1}{n} \sum_{i=1}^{n} f(x_i) - \int_{\mathcal{M}} f(x) \mu(x) dx > \epsilon\right) \leq 8\mathbb{E}_\mu[N(\epsilon/8, F, L_1(\mu_n))] \exp(-n\epsilon^2/128) \quad (2.20)$$

Notice that $L_1(\mu_n) \leq L_\infty(\mu_n) \leq L_\infty$ where $\|f\|_{L_\infty} = \max_{x \in \mathcal{M}} |f(x)|$. Then we get following corollary.

Corollary 2.1. Let $F$ be a class of functions from $\mathcal{M}$ to $[-1,1]$ and set $\mu$ to be a probability measure on $\mathcal{M}$. Let $(x_i)_{i=1}^{\infty}$ be independent random variables distributed according to $\mu$. For every $\epsilon > 0$ and any $n \geq 8/\epsilon^2$,

$$\mathbb{P}\left(\sup_{f \in F} \frac{1}{n} \sum_{i=1}^{n} f(x_i) - \int_{\mathcal{M}} f(x) \mu(x) dx > \epsilon\right) \leq 8N(\epsilon/8, F, L_\infty) \exp(-n\epsilon^2/128) \quad (2.21)$$

where $N(\epsilon, F, L_\infty)$ be the covering numbers of $F$ at scale $\epsilon$ with respect to the $L_\infty$ norm.

Corollary 2.2. Let $F$ be a class of functions from $\mathcal{M}$ to $[-1,1]$. Let $(x_i)_{i=1}^{\infty}$ be independent random variables distributed according to $p$, where $p$ is the probability distribution in Assumption 1. Then with probability at least $1 - \delta$,

$$\sup_{f \in F} |p(f) - p_n(f)| \leq \sqrt{\frac{128}{n} \left( \ln N\left(\sqrt{\frac{2}{n}}, F, L_\infty\right) + \ln \frac{8}{\delta} \right)},$$

where

$$p(f) = \int_{\mathcal{M}} f(x) p(x) dx, \quad p_n(f) = \frac{1}{n} \sum_{i=1}^{n} f(x_i). \quad (2.22)$$

Proof. Using Corollary 2.1 with probability at least $1 - \delta$,

$$\sup_{f \in F} |p(f) - p_n(f)| \leq \epsilon_\delta,$$

where $\epsilon_\delta$ is determined by

$$\epsilon_\delta = \sqrt{\frac{128}{n} \left( \ln N(\epsilon_\delta/8, F, L_\infty) + \ln \frac{8}{\delta} \right)}.$$

Obviously,

$$\epsilon_\delta \geq \sqrt{\frac{128}{n}} = 8\sqrt{\frac{2}{n}}$$

which gives that

$$N(\epsilon_\delta/8, F, L_\infty) \leq N(\sqrt{\frac{2}{n}}, F, L_\infty)$$

Then, we have

$$\epsilon_\delta \leq \sqrt{\frac{128}{n} \left( \ln N(\sqrt{\frac{2}{n}}, F, L_\infty) + \ln \frac{8}{\delta} \right)}$$

which proves the corollary. \qed
Above corollary provides a tool to estimate the integral error on random samples. The key point is to obtain the estimates of the covering number.

Let us start from the function class $\mathcal{R}_t$. The functions in $\mathcal{R}_t$ are bounded uniformly, and the bound only depends on the kernel function $R$. To apply above corollary, we need to normalize $\mathcal{R}_t$ to make it lie in $[-1, 1]$. Here we also use $\mathcal{R}_t$ to denote the normalized function class and absorb the bound of $\mathcal{R}_t$ into the generic constant $C$. We do same normalize procedure for all function classes defined in Section 2.

Since the kernel $R \in C^2(\mathcal{M})$ and $\mathcal{M} \subset C^\infty$, we have for any $x, y \in \mathcal{M}$

$$|R \left( \frac{\|x - y\|^2}{4t} \right) - R \left( \frac{\|z - y\|^2}{4t} \right)| \leq \frac{C}{\sqrt{t}} \|x - z\|.$$  

This gives an easy bound of $N(\epsilon, \mathcal{R}_t, L_\infty)$,

$$N(\epsilon, \mathcal{R}_t, L_\infty) \leq \left( \frac{C}{\epsilon \sqrt{t}} \right)^k$$  

(2.23)

Using Corollary 2.2 with probability at least $1 - 1/(2n)$,

$$\sup_{f \in \mathcal{R}_t \cup \mathcal{R}_t \cup \mathcal{R}_{st}} |p(f) - p_n(f)| \leq \frac{C}{\sqrt{n}} (\ln n - \ln t + 1)^{1/2}$$  

(2.24)

Then, we have

**Corollary 2.3.** With probability at least $1 - 1/(2n)$,

$$\sup_{f \in \mathcal{R}_t \cup \mathcal{R}_t \cup \mathcal{R}_{st}} |p(f) - p_n(f)| \leq \frac{w_{\text{min}}}{2}$$

as long as $n$ is large enough such that the right hand side of (2.24) is less than $w_{\text{min}}/2$.

To get the covering number $N(\epsilon, \mathcal{K}_{t,n}, L_\infty)$, we need the assumption that $\sup_{f \in \mathcal{K}_t} |p(f) - p_n(f)| \leq \frac{w_{\text{min}}}{2}$. Then we have

$$|\frac{1}{w_{t,n}(y)} \left[ R \left( \frac{\|x - y\|^2}{4t} \right) - R \left( \frac{\|z - y\|^2}{4t} \right) \right]| \leq \frac{2}{w_{\text{min}}} |R \left( \frac{\|x - y\|^2}{4t} \right) - R \left( \frac{\|z - y\|^2}{4t} \right)| \leq \frac{C}{\sqrt{t}} \|x - y\|$$  

The first inequality comes from the fact that $\min_{x \in \mathcal{M}} w_{t,n}(x) \geq w_{\text{min}}/2$ which is guaranteed by the assumption that $\sup_{f \in \mathcal{K}_t} |p(f) - p_n(f)| \leq \frac{w_{\text{min}}}{2}$. Then we have

$$N(\epsilon, \mathcal{K}_{t,n}, L_\infty) \leq \left( \frac{C}{\epsilon \sqrt{t}} \right)^k$$  

(2.25)

Similarly, we can get

$$N(\epsilon, \mathcal{K}_{t,n} \cdot \mathcal{K}_{t,n}, L_\infty) \leq \left( \frac{C}{\epsilon \sqrt{t}} \right)^{2k}$$  

(2.26)

Using Corollary 2.2 if $\sup_{f \in \mathcal{K}_t} |p(f) - p_n(f)| \leq \frac{w_{\text{min}}}{2}$, then

$$\sup_{f \in \mathcal{K}_{t,n} \cup \mathcal{K}_{t,n} \cup \mathcal{K}_{t,n}} |p(f) - p_n(f)| \leq C \sqrt{\frac{k}{n}} (\ln n - \ln t + 1)^{1/2}$$  

(2.27)

with probability at least $1 - 1/(2n)$. From Corollary 2.3, we know that the assumption $\sup_{f \in \mathcal{K}_t} |p(f) - p_n(f)| \leq \frac{w_{\text{min}}}{2}$ holds with probability at least $1 - 1/(2n)$. By integrating these results together, we obtain
Corollary 2.4. With probability at least \(1 - 1/n\),
\[
\sup_{f \in K_{t,n} \cup K_{t,n}' \cup K_{t,n}} |p(f) - p_n(f)| \leq \frac{\delta^2}{2 \max\{w_{\max} + w_{\min}/2, 2/w_{\min}\}}
\]
as long as \(n\) is large enough. Here \(\delta = \frac{w_{\min}}{4w_{\max} + 3w_{\min}}\).

Using similar techniques, we can get the estimate of \(h_0\) and \(h(f)\) in (2.18) and (2.19). Together with Theorem 2.5, we get

Theorem 2.8. Let \(\phi\) be an eigenfunction of \(T\). With probability at least \(1 - 1/n\),
\[
\| (T - T_{t,n}) T_{t,n} \|_{C^1} \leq \frac{C}{t^{3k/4 + 1/2} \sqrt{n}} (\ln n - \ln t + 1)^{1/2},
\]
\[
\| (T - T_{t,n}) \phi \|_{C^1} \leq \frac{C_\phi}{t^{3k/4 + 1/2} \sqrt{n}} (\ln n - \ln t + 1)^{1/2}
\]
as long as \(n\) is large enough. Here \(C_\phi\) is a constant depends on \(M\), kernel function \(R\), distribution \(p\) and eigenfunction \(\phi\).

3 Proof of Theorem 2.6

To prove Theorem 2.6 we need following two theorems.

Theorem 3.1. Under the assumption in Assumption 4 and assume (2.16), (2.17) hold. There exist constants \(C > 0\) only depends on \(M\) and kernel function \(R\), so that for any \(u = (u_1, \cdots, u_n)^t \in \mathbb{R}^d\) with \(\sum_{i=1}^n u_i = 0\),
\[
\frac{1}{n^2t} \sum_{i,j=1}^n R_t(x_i, x_j)(u_i - u_j)^2 \geq \frac{C}{n} \sum_{i=1}^n u_i^2.
\]
(3.1)

The proof of this theorem can be found in Appendix.

Theorem 3.2. Suppose \(u = (u_1, \cdots, u_n)^t \) with \(\sum_i u_i = 0\) solves the problem (1.7) and \(f \in C(M)\). Then there exists a constant \(C > 0\) only depends on \(M\) and kernel function \(R\), such that
\[
\left( \frac{1}{n} \sum_{i=1}^n u_i^2 \right)^{1/2} \leq C \left( \frac{1}{n} \sum_{i=1}^n f(x_i)^2 \right)^{1/2} \leq C \|f\|_\infty,
\]
as long as (2.16), (2.17) are satisfied.

Proof. Since \((u_1, \cdots, u_n)\) satisfies that
\[
\frac{1}{n} \sum_{j=1}^n R_t(x_i, x_j)(u_i - u_j) = \frac{1}{n} \sum_{j=1}^n \tilde{R}_t(x_i, x_j)f(x_j)
\]
using Theorem 3.1 we have

\[
\frac{C}{n} \sum_{i=1}^{n} u_i^2 \leq \frac{1}{n^2} \sum_{i,j=1}^{n} R_t(x_i, x_j)(u_i - u_j)^2 = \frac{2}{n^2} \sum_{i,j=1}^{n} R_t(x_i, x_j)(u_i - u_j) u_i
\]

\[
= \frac{2}{n^2} \sum_{i,j=1}^{n} R_t(x_i, x_j)f(x_j) u_i
\]

\[
\leq \left( \frac{1}{n^2} \sum_{i,j=1}^{n} R_t(x_i, x_j)f^2(x_j) \right)^{1/2} \left( \frac{1}{n^2} \sum_{i,j=1}^{n} R_t(x_i, x_j)u_i^2 \right)^{1/2}
\]

\[
\leq C \left( \frac{1}{n} \sum_{j=1}^{n} f^2(x_j) \right)^{1/2} \left( \frac{1}{n} \sum_{i=1}^{n} u_i^2 \right)^{1/2}
\]

\[
\leq C\|f\|_{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} u_i^2 \right)^{1/2}
\]

\[
\square
\]

**Theorem 3.3.** (\cite{1} \cite{3}) Under the assumptions in Assumption 4, assume \(u(x)\) solves the following equation

\[-Lu = r,\]  (3.2)

where

\[L_t u = \frac{C_t}{t} \int_{\mathcal{M}} R \left( \frac{|x - y|^2}{4t} \right) (u(x) - u(y))p(y)dy.\]  (3.3)

Then, there exist constants \(C > 0, T_0 > 0\) independent on \(t\), such that

\[
\|u\|_{L^2(\mathcal{M})} \leq C\|r\|_{L^2(\mathcal{M})}.\]  (3.4)

as long as \(t \leq T_0\).

The proof of above theorem can be found in \cite{15}.

**Theorem 3.4.** Under the assumptions in Assumption 4. Let \(f \in C(\mathcal{M})\) in both problems, then there exists constants \(C > 0\), so that

\[
\|(T_{t_n} - T_t)T_{t,n} f\|_{L^2(\mathcal{M})} \leq \frac{C}{t^{k/2+1}} \|f\|_{\infty} \left( \sup_{g \in \mathcal{K} \cup \mathcal{K}_t} |p_n(g) - p(g)| + t \sup_{g \in \mathcal{K}_{t,n} \cup \mathcal{K}_t} |p_n(g) - p(g)| \right)
\]

\[
\|(T_{t_n} - T_t) f\|_{L^2(\mathcal{M})} \leq \frac{C}{t^{k/2+1}} \|f\|_{\infty} \left( \sup_{g \in \mathcal{K} \cup \mathcal{K}_t} |p_n(g) - p(g)| + t \sup_{g \in \mathcal{K}_{t,n} \cup f, \mathcal{K}_t} |p_n(g) - p(g)| \right),
\]

as long as \(t\) small enough and (2.14), (2.17) are satisfied.

**Proof.** of Theorem 3.4

First, denote

\[
u_{t,n}(x) = T_{t,n} f = \frac{1}{n} \frac{1}{w_{t,n}(x)} \left( \sum_{j=1}^{n} R_t(x, x_j) u_j - t \sum_{j=1}^{n} R_t(x, x_j) f_j \right)
\]  (3.5)
where \( \mathbf{u} = (u_1, \ldots, u_n) \) with \( \sum_{i=1}^{n} u_i = 0 \) solves the problem (1.7), \( f_j = f(x_j) \) and \( w_{t,n}(x) = \frac{1}{n} \sum_{j=1}^{n} R_t(x, x_j) \). And denote

\[
 v_{t,n}(x) = T_{t,n}u_{t,n} = \frac{1}{n \ w_{t,n}(x)} \left( \sum_{j=1}^{n} R_t(x, x_j)v_j - t \sum_{j=1}^{n} \tilde{R}_t(x, x_j)u_j \right) \quad (3.6)
\]

where \( \mathbf{v} = (v_1, \ldots, v_n) \) with \( \sum_{i=1}^{n} v_i = 0 \) solves

\[
 - \frac{1}{nt} \sum_{j=1}^{n} R_t(x_i, x_j)(v_i - v_j) = \frac{1}{n} \sum_{j=1}^{n} \tilde{R}_t(x_i, x_j)u_j. \quad (3.7)
\]

It follows from Theorem 3.1 that there exists a constant \( C > 0 \) independent on \( t \) and \( n \) such that

\[
 \left( \frac{1}{n} \sum_{i=1}^{n} u_i^2 \right)^{1/2} \leq C \|f\|_{\infty}, \quad \left( \frac{1}{n} \sum_{i=1}^{n} v_i^2 \right)^{1/2} \leq C \left( \frac{1}{n} \sum_{i=1}^{n} u_i^2 \right)^{1/2} \leq C \|f\|_{\infty} \quad (3.8)
\]

The idea to prove the theorem is using Theorem 3.3. Then we need to estimate \( \|L_t(T_{t,n} - T_t)T_{t,n}f\|_{2} \) and \( \|L_t(T_{t,n} - T_t)f\|_{2} \) for any \( f \in C(M) \).

For any \( f \in C(M) \),

\[
 L_t(T_{t,n} - T_t)T_{t,n}f = (L_tT_{t,n}T_{t,n}f - L_tT_{t,n}T_{t,n}f) + (L_tT_{t,n}T_{t,n}f - L_tT_{t,n}f) = (L_tT_{t,n}v_{t,n} + (L_tT_{t,n}u_{t,n} - L_tT_{t,n}u_{t,n}) \quad (3.9)
\]

Next, we estimate two terms of right hand side of (3.9) separately. For convenience, we split \( v_{t,n} = a_{t,n} + b_{t,n} \) and

\[
 a_{t,n}(x) = \frac{1}{n \ w_{t,n}(x)} \sum_{j=1}^{n} R_t(x, x_j)v_j, \quad (3.10)
\]

\[
 b_{t,n}(x) = -\frac{t}{n \ w_{t,n}(x)} \sum_{j=1}^{n} \tilde{R}_t(x, x_j)u_j. \quad (3.11)
\]

For \( \|L_t b_{t,n} - L_t b_{t,n}\|_{2} \), we have

\[
 \left| \left( L_t b_{t,n} - L_t b_{t,n} \right)(x) \right| = \frac{1}{t} \left| \int_{M} R_t(x, y)(b_{t,n}(x) - b_{t,n}(y))p(y)dy - \frac{1}{n} \sum_{j=1}^{n} R_t(x, x_j)(b_{t,n}(x) - b_{t,n}(x_j)) \right|
\]

\[
 \leq \frac{1}{t} \ |b_{t,n}(x)| \left| \int_{M} R_t(x, y)p(y)dy - \frac{1}{n} \sum_{j=1}^{n} R_t(x, x_j) \right|
\]

\[
 + \frac{1}{t} \left| \int_{M} R_t(x, y)b_{t,n}(y)p(y)dy - \frac{1}{n} \sum_{j=1}^{n} R_t(x, x_j)b_{t,n}(x_j) \right| \quad (3.12)
\]

The first term of (3.12) can be bounded as following,

\[
 \left\| b_{t,n}(x) \left( \int_{M} R_t(x, y)p(y)dy - \frac{1}{n} \sum_{j=1}^{n} R_t(x, x_j) \right) \right\|_{L^2} \leq C_t \|b_{t,n}\|_{L^2} \sup_{g \in \mathcal{R}_t} |p_n(g) - p(g)| \quad (3.13)
\]
Let \( \|b_{t,n}\|_2^2 = \frac{t^2}{n^2} \int_\mathcal{M} \left( \frac{1}{w_{t,n}(x)} \sum_{j=1}^n \tilde{R}_t(x, x_j) u_j \right)^2 p(x) dx \)

\[
\leq \frac{C t^2}{n} \int_\mathcal{M} \left( \frac{1}{n} \sum_{j=1}^n \tilde{R}_t(x, x_j) \right) \left( \sum_{j=1}^n \tilde{R}_t(x, x_j) u_j^2 \right) p(x) dx \\
\leq \frac{C t^2}{n} \sum_{j=1}^n \left( u_j^2 \int_\mathcal{M} \tilde{R}_t(x, x_j) p(x) dx \right) \\
\leq \frac{C t^2}{n} \sum_{j=1}^n u_j^2 \leq C t^2 \|f\|_\infty, \tag{3.14}
\]

where the second inequality comes from (3.8).

For the second term of (3.12),

\[
\left| \int_\mathcal{M} \tilde{R}_t(x, y) b_{t,n}(y) p(y) dy - \frac{1}{n} \sum_{j=1}^n \tilde{R}_t(x, x_j) b_{t,n}(x_j) \right| \\
= \frac{t}{n} \left| \int_\mathcal{M} \frac{\tilde{R}_t(x, y)}{w_{t,n}(y)} \left( \sum_{x_k \in P} \tilde{R}_t(y, x_k) u_k \right) p(y) dy - \frac{1}{n} \sum_{j=1}^n \frac{\tilde{R}_t(x, x_j)}{w_{t,n}(x_j)} \sum_{x_k \in P} \tilde{R}_t(x_j, x_k) u_k \right| \\
\leq \frac{t}{n} \sum_{k=1}^n |u_k| \left| \int_\mathcal{M} \frac{\tilde{R}_t(x, y)}{w_{t,n}(y)} \tilde{R}_t(y, x_k) p(y) dy - \frac{1}{n} \sum_{j=1}^n \frac{\tilde{R}_t(x, x_j)}{w_{t,n}(x_j)} \tilde{R}_t(x_j, x_k) \right| \tag{3.15}
\]

Let

\[
A = C t \int_\mathcal{M} \frac{1}{w_{t,n}(y)} \tilde{R} \left( \frac{|x - y|^2}{4t} \right) \tilde{R} \left( \frac{|x_i - y|^2}{4t} \right) p(y) dy \\
- \frac{C_t}{n} \sum_{j=1}^n \frac{1}{w_{t,n}(x_j)} \tilde{R} \left( \frac{|x - x_j|^2}{4t} \right) \tilde{R} \left( \frac{|x_i - x_j|^2}{4t} \right). \tag{3.16}
\]

We have

\[
|A| < C_t \sup_{g \in \mathcal{K}_{t,n}} |p_n(g) - p(g)| \tag{3.17}
\]

for some constant \( C \) independent of \( t \). In addition, notice that only when \( |x - x_i|^2 \leq 16t \) is \( A \neq 0 \), which implies

\[
|A| \leq \frac{1}{\delta_0} |A| R \left( \frac{|x - x_i|^2}{32t} \right). \tag{3.18}
\]

Using these properties of \( A \), we obtain

\[
\left| \int_\mathcal{M} \tilde{R}_t(x, y) b_{t,n}(y) p(y) dy - \frac{1}{n} \sum_{j=1}^n \tilde{R}_t(x, x_j) b_{t,n}(x_j) \right| \\
\leq \frac{C t}{n} |A| \|f\|_\infty \sum_{k=1}^n |u_k| R \left( \frac{|x_k|^2}{32t} \right) \\
\leq \frac{C t}{n} \sum_{k=1}^n C t |u_k| R \left( \frac{|x_k|^2}{32t} \right) C_t \sup_{g \in \mathcal{K}_{t,n}} |p_n(g) - p(g)| \tag{3.19}
\]
And consequently, 

$$\left\| \int_{\mathcal{M}} R_t(x, y)b_{t,n}(y)p(y)dy - \frac{1}{n} \sum_{j=1}^{n} R_t(x,x_j)b_{t,n}(x_j) \right\|_2 \leq C_t \left( \int_{\mathcal{M}} \left( \frac{1}{n} \sum_{k=1}^{n} C_t |u_k| R_t(\frac{|x-x_k|^2}{32t}) \right)^2 p(x)dx \right)^{1/2} \leq C_t \sup_{g \in \mathcal{K}_{t,n,\mathcal{R}_t}} |p_n(g) - p(g)|$$

Now we have complete upper bound of \(\|L_t b_{t,n} - L_{t,n} b_{t,n}\|_{L^2} \) using (3.12), (3.13) and (3.20) and \(C_t = \frac{1}{(4\pi t)^{k/2}}\).

$$\|L_t b_{t,n} - L_{t,n} b_{t,n}\|_{L^2(\mathcal{M})} \leq C_{k/2} \|f\|_{\infty} \left( \sup_{g \in \mathcal{R}_{t,n,\mathcal{K}_t,\mathcal{R}_t}} |p_n(g) - p(g)| \right).$$

Mimicking the derivation of (3.21), we have

$$\|L_t a_{t,n} - L_{t,n} a_{t,n}\|_{L^2(\mathcal{M})} \leq C_{k/2+1} \|f\|_{\infty} \left( \sup_{g \in \mathcal{R}_{t,n,\mathcal{K}_t,\mathcal{R}_t}} |p_n(g) - p(g)| \right)$$

And consequently,

$$\|L_t v_{t,n} - L_{t,n} v_{t,n}\|_{L^2(\mathcal{M})} \leq \|L_t a_{t,n} - L_{t,n} a_{t,n}\|_{L^2(\mathcal{M})} + \|L_t b_{t,n} - L_{t,n} b_{t,n}\|_{L^2(\mathcal{M})} \leq C_{k/2+1} \|f\|_{\infty} \left( \sup_{g \in \mathcal{R}_{t,n,\mathcal{K}_t,\mathcal{R}_t}} |p_n(g) - p(g)| \right).$$

The second term of (3.21) can be bounded as following,

$$\begin{align*}
L_t(T_t u_{t,n}) - L_{t,n}(T_{t,n} u_{t,n}) & \leq \int_{\mathcal{M}} \tilde{R}_t(x, y) u_{t,n}(y)p(y)dy - \frac{1}{n} \sum_{j=1}^{n} \tilde{R}_t(x,x_j) u_j \\
& \leq \frac{1}{n^2} \sum_{j=1}^{n} \frac{\tilde{R}_t(x,x_j)}{w_{t,n}(x_j)} \left( \sum_{k=1}^{n} R_t(x_j,x_k) u_k - t \sum_{k=1}^{n} \tilde{R}_t(x_j,x_k) f_k \right) \\
& \quad - \frac{1}{n} \int_{\mathcal{M}} \tilde{R}_t(x,y) \left( \sum_{k=1}^{n} R_t(y,x_k) u_k - t \sum_{k=1}^{n} \tilde{R}_t(y,x_k) f_k \right) p(y)dy \\
& = \frac{1}{n} \sum_{k=1}^{n} u_k \left( \frac{1}{n} \sum_{j=1}^{n} \frac{\tilde{R}_t(x,x_j)}{w_{t,n}(x_j)} R_t(x_j,x_k) - \int_{\mathcal{M}} \frac{\tilde{R}_t(x,y)}{w_{t,n}(y)} R_t(y,x_k) p(y)dy \right) \\
& \quad - \frac{t}{n} \sum_{k=1}^{n} f_k \left( \frac{1}{n} \sum_{j=1}^{n} \frac{\tilde{R}_t(x,x_j)}{w_{t,n}(x_j)} \tilde{R}_t(x_j,x_k) - \int_{\mathcal{M}} \frac{\tilde{R}_t(x,y)}{w_{t,n}(y)} \tilde{R}_t(y,x_k) p(y)dy \right).\end{align*}$$
Using the similar derivation from (3.15) to (3.21), we get
\[
\|L_{t}(T_{t}u_{t,n}) - L_{t,n}(T_{t,n}u_{t,n})\|_{L^{2}} \leq C \left( \frac{1}{n} \sum_{j=1}^{n} u_{j}^{2} \right)^{1/2} C_{t} \sup_{g \in K_{t,n}, K_{t}} |p_{n}(g) - p(g)| + C_{t} \|f\|_{\infty} C_{t} \sup_{g \in K_{t,n}, K_{t}} |p_{n}(g) - p(g)|
\]
\[
\leq \frac{C}{t^{k/2}} \|f\|_{\infty} \left( \sup_{g \in K_{t,n}, K_{t}} |p_{n}(g) - p(g)| + t \sup_{g \in K_{t,n}, K_{t}} |p_{n}(g) - p(g)| \right). \tag{3.25}
\]

The complete estimate follows from Equation (3.23) and (3.24).
\[
\|L_{t}(T_{t,n} - T_{t})T_{t,n}f\|_{L^{2}(\mathcal{M})} \leq \frac{C}{t^{k/2+1}} \|f\|_{\infty} \left( \sup_{g \in K_{t,n}, K_{t}} |p_{n}(g) - p(g)| + t \sup_{g \in K_{t,n}, K_{t}} |p_{n}(g) - p(g)| \right). \tag{3.26}
\]

Similarly, we can also get
\[
\|L_{t}(T_{t,n} - T_{t})f\|_{L^{2}(\mathcal{M})} \leq \frac{C}{t^{k/2+1}} \|f\|_{\infty} \left( \sup_{g \in K_{t,n}, K_{t}} |p_{n}(g) - p(g)| + t \sup_{g \in K_{t,n}, K_{t}} |p_{n}(g) - p(g)| \right). \tag{3.27}
\]

The theorem is proved by using Theorem 3.3 and above two estimates (3.26), (3.27).

**Theorem 3.5.** Under the assumption in Assumption 7 and assume (2.16), (2.17) hold. Then, there exist constants $C > 0$ only depends on $\mathcal{M}$ and kernel function $R$, such that for any $f \in C(\mathcal{M})$,
\[
\|T_{t,n}f\|_{\infty} \leq C t^{-k/4} \|f\|_{\infty}, \quad \|T_{t,n}f\|_{L^{2}} \leq C \|f\|_{\infty}.
\]

**Proof.** From the definition of $T_{t,n}$, we have for any $f \in C(\mathcal{M})$
\[
T_{t,n}f = \frac{C_{t}}{nw_{t,n}(x)} \sum_{i=1}^{n} R \left( \frac{|x - x_i|}{4t} \right) u_i + \frac{t C_{t}}{nw_{t,n}(x)} \sum_{i=1}^{n} R \left( \frac{|x - x_i|}{4t} \right) f(x_i)
\]
where $(u_1, \cdots, u_n)$ satisfies the equation
\[
\frac{C_{t}}{nt} \sum_{j=1}^{n} R \left( \frac{|x_i - x_j|}{4t} \right) (u_i - u_j) = \frac{C_{t}}{n} \sum_{j=1}^{n} R \left( \frac{|x_i - x_j|}{4t} \right) f(x_j).
\]

Using Theorem 3.1, it is easy to get that
\[
\left( \frac{1}{n} \sum_{i=1}^{n} u_{j}^{2} \right)^{1/2} \leq C \|f\|_{\infty}
\]
where $C > 0$ is a constant only depends on $\mathcal{M}$ and kernel function $R$. 

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Using Theorem 3.2, we have
\[ u_t \text{ and} \]
respectively
\[ \text{Denote} \]
\[ \text{Proof. of Theorem 2.6} \]
the regularity of the kernel function. The details are given as following.

For any \( f \in C^1(M) \), let \( u_{t,n} = T_{t,n}f \) and \( v_i = T_{i,n}u_{t,n}(x_i), \ i = 1, \cdots, n \). Using the definition of \( T_t \) and \( T_{i,n}, T_t u_{t,n} \) and \( T_{i,n} u_{t,n} \) have following representations
\[
T_{t,n}u_{t,n} = \frac{1}{w_t(x)} \int_M R_t(x,y)T_{t,n}u_{t,n}(y)p(y)dy + \frac{t}{w_t(x)} \int_M \bar{R}(x,y)u_{t,n}(y)p(y)dy,
\]
\[
T_{i,n}u_{t,n} = \frac{1}{n w_{t,n}(x)} \sum_{i=1}^n R_t(x,x_i)v_i + \frac{t}{n w_{t,n}(x)} \sum_{i=1}^n \bar{R}(x,x_i)u_i. \tag{3.28}
\]
where \( u_i = u_{t,n}(x_i), \ i = 1, \cdots, n \). We know that \( (u_1, \cdots, u_n) \) and \( (v_1, \cdots, v_n) \) satisfy following equations respectively
\[
\frac{1}{nt} \sum_{j=1}^n R_t(x_i,x_j)(u_i - u_j) = \frac{1}{n} \sum_{i=1}^n R_t(x_i,x_j)f(x_j),
\]
\[
\frac{1}{nt} \sum_{j=1}^n R_t(x_i,x_j)(v_i - v_j) = \frac{1}{n} \sum_{i=1}^n R_t(x_i,x_j)u_j.
\]
Using Theorem 3.2 we have
\[
\left( \frac{1}{n} \sum_{i=1}^n u_i^2 \right)^{1/2} \leq C \|f\|_\infty, \quad \left( \frac{1}{n} \sum_{i=1}^n v_i^2 \right)^{1/2} \leq C \left( \frac{1}{n} \sum_{i=1}^n u_i^2 \right)^{1/2} \leq C \|f\|_\infty. \tag{3.29}
\]

Denote
\[
T_{i,n}^1u_{t,n} = \frac{1}{w_{t,n}(x)} \int_M R_t(x,y)T_{i,n}u_{t,n}(y)p(y)dy + \frac{t}{w_{t,n}(x)} \int_M \bar{R}(x,y)u_{t,n}(y)p(y)dy,
\]
\[
T_{i,n}^2u_{t,n} = \frac{1}{w_{t,n}(x)} \int_M R_t(x,y)T_{i,n}u_{t,n}(y)p(y)dy + \frac{t}{w_{t,n}(x)} \int_M \bar{R}(x,y)u_{t,n}(y)p(y)dy.
\]

\[
|T_{i,n}f| \leq \left( \frac{C_t}{nw_{t,n}(x)} \right) \sum_{i=1}^n R_t \left( \frac{|x - x_i|^2}{4t} \right) u_i^2 + t \|f\|_\infty \leq \left( \frac{C_t}{nw_{t,n}(x)} \right) \sum_{i=1}^n R_t \left( \frac{|x - x_i|^2}{4t} \right) t \|f\|_\infty \leq C \left( \frac{1}{n} \sum_{i=1}^n u_i^2 + t \|f\|_\infty \right)^{1/2} \leq C \|f\|_\infty. \tag{3.30}
\]
Secondly, using Theorem 3.4 we have

\[
|T_t u_{t,n} - T_t^1 u_{t,n}| \leq \frac{1}{w_{t,n}(x)} - \frac{1}{w_1(x)} \left| \int_M R_t(x,y) T_t u_{t,n}(y)p(y)dy \right| + t \left| \int_M R(x,y) u_{t,n}(y)p(y)dy \right|
\]

\[
\leq 2C_1 \sup_{g \in \mathcal{D}} \|p_n(g) - p(g)\| \left( \int_M R_t(x,y) T_t u_{t,n}(y)p(y)dy \right) + t \left| \int_M R(x,y) u_{t,n}(y)p(y)dy \right|
\]

\[
\leq C \frac{t^{3k/4}}{n} \left( ||u_{t,n}||_{L^2} + t\|u_{t,n}\|_{L^2} \right) \sup_{g \in \mathcal{D}} \|p_n(g) - p(g)\|
\]

\[
\leq C \frac{t^{3k/4}}{n} \|f\|_{\infty} \sup_{g \in \mathcal{D}} \|p_n(g) - p(g)\|.
\]

Similarly, we have

\[
|\nabla (T_t u_{t,n} - T_t^1 u_{t,n})| \leq \frac{C}{t^{(3k+2)/4}} \|f\|_{\infty} \sup_{g \in \mathcal{D}} \|p_n(g) - p(g)\|
\]

which proves that

\[
\|T_t u_{t,n} - T_t^1 u_{t,n}\|_{C^1} \leq \frac{C}{t^{(3k+2)/4}} \|f\|_{\infty} \sup_{g \in \mathcal{D}} \|p_n(g) - p(g)\|. \quad (3.30)
\]

Secondly, using Theorem 3.4 we have

\[
\|T_t^1 u_{t,n} - T_t^2 u_{t,n}\| = \left| \frac{1}{w_{t,n}(x)} \int_M R_t(x,y) (T_t u_{t,n}(y) - T_{t,n} u_{t,n}(y)) p(y)dy \right|
\]

\[
\leq C t^{-k/4} \|T_t u_{t,n} - T_{t,n} u_{t,n}\|_{L^2}
\]

\[
= C t^{-k/4} \|T_t^1 - T_t^2\|_{L^2}
\]

\[
\leq C \frac{t^{3k/4+1}}{n} \|f\|_{\infty} \left( \sup_{g \in \mathcal{D}} |p_n(g) - p(g)| + t \sup_{g \in \mathcal{D}} |p_n(g) - p(g)| + t^2 \sup_{g \in \mathcal{D}} |p_n(g) - p(g)| \right).
\]

and

\[
|\nabla (T_t^1 u_{t,n} - T_t^2 u_{t,n})| = \left| \nabla \left( \frac{1}{w_{t,n}(x)} \int_M R_t(x,y) (T_t u_{t,n}(y) - T_{t,n} u_{t,n}(y)) p(y)dy \right) \right|
\]

\[
\leq C t^{-k/4+1/2} \|T_t u_{t,n} - T_{t,n} u_{t,n}\|_{L^2}
\]

\[
= C t^{-k/4+1/2} \|T_t^1 - T_t^2\|_{L^2}
\]

\[
\leq C \frac{t^{3k/4+3/2}}{n} \|f\|_{\infty} \left( \sup_{g \in \mathcal{D}} |p_n(g) - p(g)| + t \sup_{g \in \mathcal{D}} |p_n(g) - p(g)| + t^2 \sup_{g \in \mathcal{D}} |p_n(g) - p(g)| \right).
\]

This implies that

\[
\|T_t^1 u_{t,n} - T_t^2 u_{t,n}\|_{C^1} \leq \frac{C}{t^{k/4+3/2}} \|f\|_{\infty} \left( \sup_{g \in \mathcal{D}} |p_n(g) - p(g)| + t \sup_{g \in \mathcal{D}} |p_n(g) - p(g)| + t^2 \sup_{g \in \mathcal{D}} |p_n(g) - p(g)| \right). \quad (3.31)
\]
Now, we turn to estimate $T_{t,n}u_{t,n} - T_t^2u_{t,n}$. Using (3.28), we have

$$T_{t,n}u_{t,n} - T_t^2u_{t,n} = \frac{1}{w_{t,n}(x)} \left( \frac{1}{n} \sum_{i=1}^{n} R_t(x, x_i) v_i - \int_{\mathcal{M}} R_t(x, y) T_{t,n} u_{t,n}(y) p(y) dy \right)$$

$$+ \frac{t}{w_{t,n}(x)} \left( \frac{1}{n} \sum_{i=1}^{n} \bar{R}(x, x_i) u_i - \int_{\mathcal{M}} \bar{R}(x, y) u_{t,n}(y) p(y) dy \right).$$

Using (3.28) again, the first term becomes

$$\left| \frac{1}{n} \sum_{i=1}^{n} R_t(x, x_i) v_i - \int_{\mathcal{M}} R_t(x, y) T_{t,n} u_{t,n}(y) p(y) dy \right|$$

$$\leq \left| \frac{1}{n} \sum_{i=1}^{n} R_t(x, x_i) \left( \frac{1}{nw_{t,n}(x_i)} \sum_{j=1}^{n} R_t(x_i, x_j) v_j + \frac{t}{nw_{t,n}(x_i)} \sum_{j=1}^{n} \bar{R}_t(x_i - x_j) u_j \right) \right|$$

$$- \int_{\mathcal{M}} R_t(x, y) \left( \frac{1}{nw_{t,n}(y)} \sum_{j=1}^{n} R_t(y, x_j) v_j + \frac{t}{nw_{t,n}(y)} \sum_{j=1}^{n} \bar{R}_t(y - x_j) u_j \right) p(y) dy \right|$$

$$\leq \frac{1}{n} \sum_{j=1}^{n} v_j \left( \frac{1}{n} \sum_{i=1}^{n} \frac{R_t(x_i, x_j)}{w_{t,n}(x_i)} R_t(x_i, x_j) - \int_{\mathcal{M}} \frac{R_t(x_i, x_j)}{w_{t,n}(y)} R_t(y, x_j) p(y) dy \right)$$

$$+ \frac{t}{n} \sum_{j=1}^{n} u_j \left( \frac{1}{n} \sum_{i=1}^{n} \frac{R_t(x_i, x_j)}{w_{t,n}(x_i)} \bar{R}_t(x_i - x_j) - \int_{\mathcal{M}} \frac{R_t(x_i, x_j)}{w_{t,n}(y)} \bar{R}_t(y - x_j) p(y) dy \right)$$

Using the similar derivation from (3.15) to (3.21), we can get

$$\left| \frac{1}{n} \sum_{i=1}^{n} R_t(x, x_i) v_i - \int_{\mathcal{M}} R_t(x, y) T_{t,n} u_{t,n}(y) p(y) dy \right|$$

$$\leq C \frac{\|f\|_\infty}{t^{k/4}} \left( \frac{1}{n} \sum_{j=1}^{n} v_j^2 \right)^{1/2} C_t \sup_{g \in \mathcal{K}_{t,n}} \|p_n(g) - p(g)\| + C \frac{\|f\|_\infty}{t^{k/4}} \left( \frac{1}{n} \sum_{j=1}^{n} u_j^2 \right)^{1/2} C_t \sup_{g \in \mathcal{K}_{t,n}} \|p_n(g) - p(g)\|$$

$$\leq C \frac{\|f\|_\infty}{t^{k/4}} \left( \sup_{g \in \mathcal{K}_{t,n}} \|p_n(g) - p(g)\| + t \sup_{g \in \mathcal{K}_{t,n}} \|p_n(g) - p(g)\| \right)$$

The second term can be bounded similarly,

$$\left| \frac{1}{n} \sum_{i=1}^{n} \bar{R}(x, x_i) u_i - \int_{\mathcal{M}} \bar{R}(x, y) u_{t,n}(y) p(y) dy \right|$$

$$\leq C \frac{\|f\|_\infty}{t^{k/4}} \left( \frac{1}{n} \sum_{j=1}^{n} u_j^2 \right)^{1/2} C_t \sup_{g \in \mathcal{K}_{t,n}} \|p_n(g) - p(g)\| + C \frac{\|f\|_\infty}{t^{k/4}} \left( \frac{1}{n} \sum_{j=1}^{n} f_j^2 \right)^{1/2} C_t \sup_{g \in \mathcal{K}_{t,n}} \|p_n(g) - p(g)\|$$

$$\leq C \frac{\|f\|_\infty}{t^{k/4}} \left( \sup_{g \in \mathcal{K}_{t,n}} \|p_n(g) - p(g)\| + t \sup_{g \in \mathcal{K}_{t,n}} \|p_n(g) - p(g)\| \right)$$

Now, we have

$$|T_{t,n}u_{t,n} - T_t^2u_{t,n}| \leq C \frac{\|f\|_\infty}{t^{k/4}} \left( \sup_{g \in \mathcal{K}_{t,n}} \|p_n(g) - p(g)\| + t \sup_{g \in \mathcal{K}_{t,n}} \|p_n(g) - p(g)\| + t^2 \sup_{g \in \mathcal{K}_{t,n}} \|p_n(g) - p(g)\| \right)$$
Using the similar method, we can get
\[
|\nabla (T_{t,n}u_{t,n} - T_{t,n}^2 u_{t,n})| \leq \frac{C}{t^{3k/4+1/2}} \|f\|_\infty \left( \sup_{g \in K_{t,n}} |p_n(g) - p(g)| + t \sup_{g \in K_{t,n}} |p_n(g) - p(g)| + t^2 \sup_{g \in K_{t,n}} |p_n(g) - p(g)| \right)
\]

The estimate of \(\| (T_t - T_{t,n}) T_{t,n} \|_{C^1} \) in Theorem 2.6 is proved.

Similarly, we can obtain the estimate of \(\| (T_t - T_{t,n}) f \|_{C^1} \) for any \(f \in C(M)\) which complete the proof.

\[\square\]

4 Conclusions

In this paper, we proved that the spectra of the normalized graph laplacian \([1,1]\) will converge to the spectral of a weighted Laplace-Beltrami operator with Neumann boundary condition \([1,2]\) as \(t \to 0\) and the number of sample points goes to infinity. The samples points are assumed to be drawn on a smooth manifold according to some probability distribution \(p\). Moreover, we also give an estimate of the convergence rate. Up to our knowledge, this is the first result about the spectra convergence rate of graph laplacian. However, the estimate of the convergence rate in this paper is far from optimal. There are mainly two places in the analysis which can be improved in the future. The first one is the estimate of the integral equation \([1,3]\). Now, we only get \(L^2\) estimate, however, in the spectra convergence analysis, we need \(C^1\) estimate. In this paper, the regularity is lifted by using the regularity of the kernel function. The trade off is that a large number \(t^{-k/4}\) emerge which reduce the rate of convergence. The other place is the estimate of the covering number. The estimate of the covering number is very rough in this paper. More delicate method would give better estimate which could help to improve the estimate of the convergence rate.

Appendix A: Proof of Theorem 3.1

Proposition A.1. \([1,3]\) Assume both \(M\) and \(\partial M\) are \(C^2\) smooth. There are constants \(w_{\min} > 0, w_{\max} < +\infty\) and \(T_0 > 0\) depending only on the geometry of \(M\), so that
\[
w_{\min} \leq w_t(x) = \int_M R_t(x, y) dy \leq w_{\max}
\]
as long as \(t < T_0\).

We have the following lemma about the function \(w_{t,n}\).

Lemma A.1. Under the assumptions in Assumption \([2]\) if \(C_t \sup_{f \in \mathcal{R}_t} |p(f) - p_n(f)| \leq w_{\min}/2,\)
\[
w_{\min}/2 \leq w_{t,n}(x) \leq w_{\max} + w_{\min}/2.
\]

This lemma is a direct consequence of Proposition A.1 and the fact that
\[
|w_{t,n}(x) - C_t \int_M R_t \left( \frac{|x - y|^2}{4t} \right) p(y) dy| \leq C_t \sup_{f \in \mathcal{R}_t} |p(f) - p_n(f)|.
\]

Lemma A.2. \([1,3]\) For any function \(u \in L^2(M)\), there exists a constant \(C > 0\) only depends on \(M\), such that
\[
\int_M \int_M R_t(x, y) (u(x) - u(y))^2 p(x) p(y) dx dy \geq C \int_M |u(x) - \bar{u}|^2 p(x) dx,
\]
(A.1)
where
\[ \bar{u} = \int_{\mathcal{M}} u(x)p(x)\,dx. \]

Now, we can prove Theorem 3.1.

**Proof. of Theorem 3.1**

First, we introduce a smooth function \( u \) that approximates \( \mathbf{u} \) at the samples \( X_n \).

\[ u(x) = \frac{C_t}{n w_{\nu \cdot n}(x)} \sum_{i=1}^{n} R \left( \frac{|x - x_i|^2}{4t'} \right) u_i, \quad \text{(A.2)} \]

where \( w_{\nu \cdot n}(x) = \frac{C_t}{n} \sum_{i=1}^{n} R \left( \frac{|x - x_i|^2}{4t'} \right) \) and \( t' = t/18 \).

Then, we have
\[
\int_{\mathcal{M}} \int_{\mathcal{M}} R_\nu(x, y) (u(x) - u(y))^2 p(x)p(y)\,dx\,dy \\
= \int_{\mathcal{M}} \int_{\mathcal{M}} R_\nu(x, y) \left( \frac{1}{n w_{\nu \cdot n}(x)} \sum_{i=1}^{n} R_\nu(x, x_i) u_i - \frac{1}{n w_{\nu \cdot n}(y)} \sum_{j=1}^{n} R_\nu(x, y) u_j \right)^2 p(x)p(y)\,dx\,dy \\
= \int_{\mathcal{M}} \int_{\mathcal{M}} R_\nu(x, y) \left( \frac{1}{n^2 w_{\nu \cdot n}(x) w_{\nu \cdot n}(y)} \sum_{i,j=1}^{n} R_\nu(x, x_i) R_\nu(x, y) u_i u_j \right)^2 p(x)p(y)\,dx\,dy \\
\leq \int_{\mathcal{M}} \int_{\mathcal{M}} \sum_{i,j=1}^{n} \left( \int_{\mathcal{M}} \int_{\mathcal{M}} R_\nu(x, x_i) R_\nu(x, y) R_\nu(x, y) p(x)p(y)\,dx\,dy \right) (u_i - u_j)^2. 
\]

Denote
\[ A = \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{1}{w_{\nu \cdot n}(x) w_{\nu \cdot n}(y)} R_\nu(x, x_i) R_\nu(x, y) R_\nu(x, y) p(x)p(y)\,dx\,dy \]
and then notice only when \( |x_i - x_j|^2 \leq 36t' \) is \( A \neq 0 \). For \( |x_i - x_j|^2 \leq 36t' \), we have
\[
A \leq \int_{\mathcal{M}} \int_{\mathcal{M}} R_\nu(x, x_i) R_\nu(x, y) R_\nu(x, y) R_\nu(x, y) R \left( \frac{|x_i - x_j|^2}{72t'} \right)^{-1} R \left( \frac{|x_i - x_j|^2}{72t'} \right) p(x)p(y)\,dx\,dy \\
\leq \frac{CC_t}{\delta_0} \int_{\mathcal{M}} \int_{\mathcal{M}} R_\nu(x, x_i) R_\nu(x, y) R \left( \frac{|x_i - x_j|^2}{72t'} \right) p(x)p(y)\,dx\,dy \\
\leq CC_t \int_{\mathcal{M}} \int_{\mathcal{M}} R_\nu(x, x_i) R_\nu(x, y) R \left( \frac{|x_i - x_j|^2}{72t'} \right) p(x)p(y)\,dx\,dy \\
\leq CC_t R \left( \frac{|x_i - x_j|^2}{4t} \right). 
\]

Combining Equation (A.3), (A.4) and Lemma A.2, we obtain
\[
\frac{CC_t}{n^2t} \sum_{i,j=1}^{n} R \left( \frac{|x_i - x_j|^2}{4t} \right) (u_i - u_j)^2 \geq \int_{\mathcal{M}} (u(x) - \bar{u})^2 p(x)\,dx 
\]
(4.5)

We now lower bound the RHS of the above equation using \( \frac{1}{n} \sum_{j=1}^{n} u_j^2 \).

\[
|\bar{u}| = \left| \int_{\mathcal{M}} u(x)p(x)\,dx \right| = \left| \frac{1}{n} \sum_{j=1}^{n} u_j \int_{\mathcal{M}} \frac{C_t}{w_{\nu \cdot n}(x)} R \left( \frac{|x - x_j|^2}{4t'} \right) p(x)\,dx \right|. 
\]
Notice that
\[
\left| \int_\mathcal{M} \frac{C_t}{w_{t',n}(x)} \left( \frac{|x - x_j|^2}{4t'} \right) p(x) \, dx - \frac{1}{n} \sum_{i=1}^n \frac{C_t}{w_{t',n}(x_i)} \left( \frac{|x_i - x_j|^2}{4t'} \right) \right| \leq C_t \sup_{f \in \mathcal{K}_{t',n}} |p(f) - p_n(f)|.
\]
Thus we have
\[
\begin{align*}
|\bar{u}| & \leq \left| \frac{1}{n^2} \sum_{i,j=1}^n \frac{C_t}{w_{t',n}(x_i)} R \left( \frac{|x_i - x_j|^2}{4t'} \right) u_j \right| + \left( \frac{1}{n} \sum_{j=1}^n |u_j| \right) \sup_{f \in \mathcal{K}_{t',n}} |p(f) - p_n(f)| \\
& \leq \left| \frac{1}{n} \sum_{i=1}^n u(x_i) \right| + \left( \frac{1}{n} \sum_{j=1}^n |u_j| \right) \sup_{f \in \mathcal{K}_{t',n}} |p(f) - p_n(f)| \\
& \leq \left| \frac{1}{n^2} \sum_{i,j=1}^n \frac{C_t}{w_{t',n}(x_i)} R \left( \frac{|x_i - x_j|^2}{4t'} \right) (u_j - u_i) \right| + \left( \frac{1}{n} \sum_{j=1}^n u_j^2 \right)^{1/2} \sup_{f \in \mathcal{K}_{t',n}} |p(f) - p_n(f)| \\
& \leq \frac{2}{w_{\min}} \left( \frac{C_t}{n^2} \sum_{i,j=1}^n \sum_{j=1}^n R \left( \frac{|x_i - x_j|^2}{4t'} \right) (u_i - u_j)^2 \right)^{1/2} + \left( \frac{1}{n} \sum_{j=1}^n u_j^2 \right)^{1/2} \sup_{f \in \mathcal{K}_{t',n}} |p(f) - p_n(f)|, \tag{A.7}
\end{align*}
\]
Denote
\[
A = \int_\mathcal{M} \frac{C_t}{w_{t',n}(x)} R \left( \frac{|x - x_i|^2}{4t'} \right) R \left( \frac{|x - x_i|^2}{4t'} \right) p(x) \, dx - \frac{1}{n} \sum_{j=1}^n \frac{C_t}{w_{t',n}(x_j)} R \left( \frac{|x_j - x_i|^2}{4t'} \right) R \left( \frac{|x_j - x_i|^2}{4t'} \right)
\]
and then \( |A| \leq C_t \sup_{f \in \mathcal{K}_{t',n}} |p(f) - p_n(f)| \). At the same time, notice that only when \( |x_i - x_i|^2 < 16t' \) is \( A \neq 0 \). Thus we have
\[
|A| \leq \frac{1}{\delta_0} |A| R\left(\frac{|x_i - x_i|^2}{72t'}\right).
\]
Then
\[
\begin{align*}
\left| \int_\mathcal{M} u^2(x) \, dx - \frac{1}{n} \sum_{j=1}^n u^2(x_j) \right| \\
& \leq \frac{1}{n^2} \sum_{i,j=1}^n |C_t u_i u_j| |A| \\
& \leq \frac{C_t}{n^2} \sup_{f \in \mathcal{K}_{t',n}} |p(f) - p_n(f)| \sum_{i,j=1}^n \left| C_t R\left(\frac{|x_i - x_j|^2}{72t'}\right) u_i u_j \right| \\
& \leq \frac{C_t}{n^2} \sup_{f \in \mathcal{K}_{t',n}} |p(f) - p_n(f)| \sum_{i,j=1}^n \left| C_t R\left(\frac{|x_i - x_j|^2}{72t'}\right) u_i^2 \right| \\
& \leq (w_{\max} + w_{\min}/2)C_t \sup_{f \in \mathcal{K}_{t',n}} |p(f) - p_n(f)| \left( \frac{1}{n} \sum_{i=1}^n u_i^2 \right). \tag{A.8}
\end{align*}
\]
In the last inequality, we use the condition that \( C_t \sup_{f \in \mathcal{K}_{t',n}} |p(f) - p_n(f)| \leq w_{\min}/2 \).
Now combining Equation (A.5), (A.7) and (A.8), we have for small $t$

\[
\frac{1}{n} \sum_{i=1}^{n} u^{2}(x_{i}) = \int_{\mathcal{M}} u^{2}(x) p(x) dx + (w_{\text{max}} + w_{\text{min}}/2)C_{t} \sup_{f \in K_{t'}, K_{t'}} |p(f) - p_{n}(f)| \left( \frac{1}{n} \sum_{i=1}^{n} u_{i}^{2} \right)
\]

\[
\leq 2 \int_{\mathcal{M}} (u(x) - \bar{u})^{2} p(x) dx + 2\bar{u}^{2} + (w_{\text{max}} + w_{\text{min}}/2)C_{t} \sup_{f \in K_{t'}, K_{t'}} |p(f) - p_{n}(f)| \left( \frac{1}{n} \sum_{i=1}^{n} u_{i}^{2} \right)
\]

\[
\leq \frac{CC_{t}}{n^{2}t} \sum_{i,j=1}^{n} R \left( \frac{|x_{i} - x_{j}|^{2}}{4t} \right) (u_{i} - u_{j})^{2}
\]

\[
+ \max\{w_{\text{max}} + w_{\text{min}}/2, 2/w_{\text{min}}\}C_{t} \sup_{f \in K_{t'}, K_{t'}} |p(f) - p_{n}(f)| \left( \frac{1}{n} \sum_{i=1}^{n} u_{i}^{2} \right).
\]

Let $\delta = \frac{w_{\text{min}}}{4w_{\text{max}} + 3w_{\text{min}}}$. If $\frac{1}{n} \sum_{i=1}^{n} u^{2}(x_{i}) \geq \frac{\delta^{2}}{n} \sum_{i=1}^{n} u_{i}^{2}$, and

\[
\max\{w_{\text{max}} + w_{\text{min}}/2, 2/w_{\text{min}}\}C_{t} \sup_{f \in K_{t'}, K_{t'}} |p(f) - p_{n}(f)| \leq \frac{\delta^{2}}{2}
\]

then we have completed the proof. Otherwise, we have

\[
\frac{1}{n} \sum_{i=1}^{n} (u_{i} - u(x_{i}))^{2} = \frac{1}{n} \sum_{i=1}^{n} u_{i}^{2} + \frac{1}{n} \sum_{i=1}^{n} u(x_{i})^{2} - \frac{2}{n} \sum_{i=1}^{n} u_{i} u(x_{i}) \geq \frac{(1 - \delta)\delta}{n} \sum_{i=1}^{n} u_{i}^{2}.
\]

(A.9)

This enables us to prove the theorem in the case of $\frac{1}{n} \sum_{i=1}^{n} u^{2}(x_{i}) < \frac{\delta^{2}}{n} \sum_{i=1}^{n} u_{i}^{2}$ as follows.

\[
\frac{C_{t}}{n^{2}} \sum_{i,j=1}^{n} R \left( \frac{|x_{i} - x_{j}|^{2}}{4t} \right) (u_{i} - u_{j})^{2}
\]

\[
= \frac{2C_{t}}{n^{2}} \sum_{i,j=1}^{n} R \left( \frac{|x_{i} - x_{j}|^{2}}{4t} \right) u_{i}(u_{i} - u_{j})
\]

\[
= \frac{2}{n} \sum_{i=1}^{n} u_{i}(u_{i} - u(x_{i})) w_{t'}, n(x_{i})
\]

\[
= \frac{2}{n} \sum_{i=1}^{n} (u_{i} - u(x_{i}))^{2} w_{t'}, n(x_{i}) + \frac{2}{n} \sum_{i=1}^{n} u(x_{i})(u_{i} - u(x_{i})) w_{t'}, n(x_{i})
\]

\[
\geq \frac{2}{n} \sum_{i=1}^{n} (u_{i} - u(x_{i}))^{2} w_{t'}, n(x_{i}) - 2 \left( \frac{1}{n} \sum_{i=1}^{n} u^{2}(x_{i}) w_{t'}, n(x_{i}) \right)^{1/2} \left( \frac{1}{n} \sum_{i=1}^{n} (u_{i} - u(x_{i}))^{2} w_{t'}, n(x_{i}) \right)^{1/2}
\]

\[
\geq \frac{w_{\text{min}}}{n} \sum_{i=1}^{n} (u_{i} - u(x_{i}))^{2} - 2(w_{\text{max}} + w_{\text{min}}/2) \left( \frac{1}{n} \sum_{i=1}^{n} u^{2}(x_{i}) \right)^{1/2} \left( \frac{1}{n} \sum_{i=1}^{n} (u_{i} - u(x_{i}))^{2} \right)^{1/2}
\]

\[
\geq (w_{\text{min}}(1 - \delta) - 2(w_{\text{max}} + w_{\text{min}}/2)\delta) \left( \frac{1}{n} \sum_{i=1}^{n} u_{i}^{2} \right)^{1/2} \left( \frac{1}{n} \sum_{i=1}^{n} (u_{i} - u(x_{i}))^{2} \right)^{1/2}
\]

\[
\geq w_{\text{min}}(1 - \delta) \left( \frac{1}{n} \sum_{i=1}^{n} u_{i}^{2} \right).
\]

(A.10)

\[ \square \]

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