Charge-Density-Wave and Superconductor Competition in Stripe Phases of High Temperature Superconductors

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(Dated: July 15, 2010)

We discuss the problem of competition between a superconducting (SC) ordered state with a charge density wave (CDW) state in stripe phases of high $T_c$ superconductors. We consider an effective model for each stripe motivated by studies of spin-gapped electronic ladder systems. We analyze the problem of dimensional crossover arising from inter-stripe SC and CDW couplings using non-Abelian bosonization and renormalization group (RG) arguments to derive an effective $O(4)$-symmetric nonlinear $\sigma$-model in $D = 2 + 1$ for the case of when both inter-stripe couplings are of equal magnitude as well as equally RG relevant. By studying the effects of various symmetry lowering perturbations, we determine the structure of the phase diagram and show that, in general, it has a broad regime in which both orders coexist. The quantum and thermal critical behavior is discussed in detail, and the phase coexistence region is found to end at associated $T = 0$ as well as $T > 0$ tetracritical points. The possible role of hedgehog topological excitations of the theory is considered and argued to be RG irrelevant at the spatially anisotropic higher dimensional low-energy fixed point theory. Our results are also relevant to the case of competing Néel and valence bond solid (VBS) orders in quantum magnets on 2D isotropic square as well as rectangular lattices interacting via nearest neighbor Heisenberg exchange interactions.

PACS numbers: 74.81.-g, 74.40.Kb, 74.62.-c

I. INTRODUCTION

Over the past few years substantial evidence, both experimental and theoretical, has accumulated showing that the pseudogap regime of high $T_c$ superconductors (HTSC) is related to existence of charge and/or spin ordered states, stripe phases that break translation symmetry in the conducting layers, as well as uniform anisotropic nematic states (for reviews see see[1,4] and references therein). In addition, homogeneous phases with “hidden order” involving some degree of time reversal symmetry breaking have also been proposed for the pseudogap phase[5-6]. An important question in this field is to determine if these ordered phases either compete with or are part of the mechanism of high $T_c$.

In principle, superconducting (SC) and charge/spin stripe orders can coexist or compete. For instance, in a well developed SC (gapped) coupled to a weak charge density wave (CDW), the CDW has little effect. Even deep in a $d$-wave SC, weak stripe order (a unidirectional CDW) does not generally affect the low energy physics of the SC quasiparticles as its wave vector generally does not span the nodal points[7-8]. Yet, in a weak coupling BCS approach to both orders, they usually (but not always) compete with each other, and the enhancement of one type of order normally suppresses the other, and one has “competing orders” (see e.g. Ref.[9] and references therein). However, recent work on stripe phases has provided a different perspective on this question by showing that in a strongly correlated system inhomogeneous phases not only are unavoidable[10-11] but may also be part of the mechanism of SC and, moreover, that there exists an optimal degree of inhomogeneity at which the $T_c$ is largest[11,12].

In this work we consider the problem of the interplay between SC and stripe phases of HTSC and formulate an effective field theory of these competing/coexisting orders. We will use as a starting point a strong coupling picture of a stripe phase in which this 2D ordered state is represented as a quasi-1D system, an array of doped Hubbard-type ladders each in a Luther-Emery liquid, a phase with a finite, and typically large, spin gap (denoted by $\Delta[11,13]$. There is extensive numerical evidence for this assumption based, primarily, on density matrix renormalization group (DMRG) calculations of Hubbard and $t - J$ models on up to seven leg ladders[14-15]. There is also supporting analytic evidence for spin gap phases in doped ladders[16-17], although the relation between the parameters of the effective (bosonized) field theory thus derived and microscopic models can only be reliably determined from numerics. In the low-energy regime these systems exhibit spin-charge separation. Furthermore, for a broad range of doping, $0 \leq x \lesssim 0.3$, ladders with only repulsive interactions have a spin gap but do not have a charge gap (provided $x > 0$) (for a review see[18]). The states on the ladders represent the local high energy physics from which the low energy 2D states arise. (These models can also be used to describe ladder materials[19]. Thus, the resulting SC state is not the result of pairing in a system with preexisting quasiparticles. Instead, the quasiparticles of the SC state are emergent low energy excitations.

Based on this strong coupling picture of the stripe phase we use bosonization methods to determine the structure (and symmetries) of the effective low energy physics at scales well below the spin gap and construct
an effective field theory for the resulting 2D state, a problem that has not been addressed before. The power of this approach resides on the fact that allows the treatment of the competing effects of the tendency of the system become a 2D superconductor and a 2D charge ordered phase (a CDW). It turns out that, for a range of parameters, the system naturally has an approximate (enlarged) symmetry, that includes both the SC and stripe orders, that has far reaching consequences on the structure of the phase diagram. In particular, the approximate symmetry allows us to establish the existence of an intermediate phase in the 2D system in which CDW and SC orders coexist rather than being separated by a first order transition. This general question was discussed earlier on using inter-ladder renormalization group (RG) methods \[19,20\] and inter-chain mean field theory (ICMFT) \[21\], suggesting (in both cases) that the transition (at \(T = 0\)) is first order and that there is an associated bicritical point. In this paper we will show that, instead, there is a phase in which SC and CDW coexist and that there is a tetracritical point (both at \(T = 0\) and at \(T > 0\)).

This work is is organized as follows. In Section II we present a summary of the stripe model and use it to determine the structure of the effective field theory. The effective field theory is derived in Section III. The quantum and and thermal (classical) critical behaviors are discussed in Section IV. We close with a summary of results and extensions in Section V.

II. ANALYSIS OF THE STRIPE MODEL

We consider an array of weakly coupled spin-gapped 1D systems with a gapless charge sector ("1CS0") \[22\]. We will follow the construction of Refs. \[11,13\] on the bosonized form

\[
H_{\text{intra}} = \sum_{i=1}^{N} \frac{\nu_i}{2} \int dx \left\{ K_c (\partial_x \theta_{c,i})^2 + \frac{1}{K_c} (\partial_x \phi_{c,i})^2 \right\},
\]

(2.1)

where \(i\) is the stripe label, \(K_c\) is the charge Luttinger parameter of each ladder; \(\phi_i\) is the phase field of the CDW along each ladder and \(\theta_i\) are the SC phase fields of each ladder. We will ignore other gapped degrees of freedom of the ladder as we only interested in the physics below the spin gap \(\Delta_s\).

The fields \(\phi_i\) and \(\theta_i\) are dual to each other, and satisfy the equal-time commutation relations \([\phi_i(x), \partial_x \phi_{j}(x')] = i \delta_{ij} \delta(x-x')\). The SC and CDW order parameters on each ladder in a spin gap phase are given by \(O_{SC}(x) \sim \exp(i \sqrt{2\pi} \theta(x))\) and \(O_{CDW}(x) \sim \exp(i \sqrt{2\pi} \phi(x))\), respectively.

DMRG studies of hole-doped ladder systems \[\text{23}\] showed that \(K_c\) is a function of doping \(x\) with \(K_c \to 2\) as \(x \to 0\) to \(K_c \to 1/2\) as \(x \to x_c \simeq 0.3\). In the presence of a spin gap the single particle hopping is irrelevant. The leading relevant inter-stripe operators are the Josephson and CDW interactions\[11,12,19,20\]

\[
H_{\text{inter}} = J_{SC} \sum_{(i,j)} \int dx \cos \left( \sqrt{2\pi} (\theta_{c,i} - \theta_{c,j}) \right) + J_{CDW} \sum_{(i,j)} \int dx \cos \left( \sqrt{2\pi} (\phi_{c,i} - \phi_{c,j}) \right)
\]

(2.2)

Notice that we have used the seemingly wrong sign for the Josephson coupling (as it induces a \(\pi\) phase shift between neighboring ladders). However this is the natural sign for the coupling between the CDW order parameters on neighboring ladders since they want to couple with a \(\pi\) phase shift so as to lower the effects of the repulsive interaction. We will refer to this as an "antiferromagnetic" sign. However, since the two phase fields \(\phi_j\) and \(\theta_j\) are conjugate to each other (as required by their commutation relations), it is possible to carry out a unitary transformation on the SC phase fields \(\theta_j\) on every other ladder (induced by the action of unitary operators involving only the CDW phase fields of the same ladder) and map the problem to one in which both interactions have a "ferromagnetic" sign, and both phases want to lock-in without a \(\pi\) phase shift. Thus, in this simple state in which there is a spin gap on every ladder, the state associated with the sign alternation of a phase-density-wave \[\text{24,25}\] is equivalent to one without the phase alternation. In contrast if both interactions had a "ferromagnetic" sign from the outset (i.e. if the phases wanted to lock without a \(\pi\) phase shift) it is no longer possible to map the system to one in which both are "antiferromagnetic". We will see below that this makes an important difference to the form of the effective field theory.

In the spin gap regime, \(0 \leq x \leq x_C \approx 0.3\), the scaling dimensions of the CDW and Josephson couplings of Eq. (2.2) are \(\Delta_{SC} = 1/K_c\) and \(\Delta_{CDW} = K_c\) respectively. Hence, both couplings are relevant since their scaling dimensions satisfy \(\Delta_{SC, CDW} \leq 2\), with SC being more relevant than CDW for \(x \lesssim 0.1\) (where \(K_c = 1\)) and conversely for \(0.1 \lesssim x \lesssim 0.3\) (where \(\Delta_s \to 0\)). Then, a perturbative RG analysis of these two inter-stripe interactions, valid for \(J_{SC}\) and \(J_{CDW}\) sufficiently small, shows that for \(0 < x \lesssim 0.1\) the system flows to a 2D (anisotropic) SC (with subdominant CDW correlations) whereas for \(0.1 \lesssim x \leq 0.3\) it flows to an (anisotropic) crystal (a bidirectional CDW, with subdominant SC correlations) suggesting a direct first order transition at \(x \sim 0.1\).\[11,13\] We will only consider nearest-neighbor inter-stripe interactions, as couplings for further away stripes become small quickly (although they are just as relevant.)

Inter-chain mean field theory (ICMFT) \[11,13,26-27\] also found a direct SC-CDW first order transition \[21\]. ICMFT yields a good qualitative picture of the dimensional crossover and, indeed, it has the correct asymptotic scaling as \(J_{SC}/J_{CDW} \to 0\) (see, e.g. Refs. \[11,13,28\] pre-
dicting the critical temperatures \( T_{SC} \propto \mathcal{J}_{SC}^{K_c} \), and \( T_{CDW} \propto \mathcal{J}_{CDW}^{1/(2-K_c)} \), respectively. Within ICMFT, close enough to the critical temperatures \( T_{SC} \) and \( T_{CDW} \), the order parameter is small and follows the critical behaviors \( \Delta_{SC}(T) \propto \Delta_{SC}(0) \left( 1 - \frac{T}{T_{SC}} \right)^{1/4-K_c} \), and \( \Delta_{CDW}(T) \propto \Delta_{CDW}(0) \left( 1 - \frac{T}{T_{CDW}} \right)^{1/4-K_c} \). These results are consistent with those of Ref.\[13\] for the case of \( K_c = 1/2 \) (weakly coupled Luther-Emery stripes), which predicts a 2D SC phase with BCS-type scaling. The problem of the competition of SC and CDW ordering was examined in considerable detail in Ref.\[21\] at the level of ICMFT, supplemented by a random phase approximation (RPA) analysis of fluctuation effects, and concluded that, at this level of approximation, there is a direct first order transition from SC to CDW order at \( T = 0 \).

However, the problem of whether in the 2D system there is a direct SC-CDW first order transition (and an associated bicritical point) or a phase in which both order parameters coexist (controlled by a tetracritical point) cannot be addressed at the level of ICMFT or within the quasi-1D regime, and requires an RG analysis of the 2D problem. In answering this question, we take an alternate route, and derive an effective field theory of the 2D system, and study it as a problem in (quantum and thermal) critical phenomena.

In the regime where the SC/CDW competition is strongest, i.e. near the self-dual point (SDP) at \( K_c = 1 \), both the SC and CDW interactions have the same scaling dimension \( \Delta_{SC} = \Delta_{CDW} = 1 \), and hence must be treated on an equal footing. Under duality, \( \theta_i \leftrightarrow \phi_i \) and \( K_c \leftrightarrow K_c^{-1} \), and at \( K_c = 1 \) the Hamiltonian \( H = H_{\text{intra}} + H_{\text{inter}} \) of the 2D system is exactly self-dual (if the inter-stripe couplings are equal). Moreover, at \( K_c = 1 \) the system has a (dynamical) \( SU(2) \) symmetry, a feature reproduced by ICMFT\[24\]. The chiral fields on each ladder \( \phi_{R/L} = (\theta \pm \phi)/2 \) can be used to define three chiral current operators

\[
J_{R,L}^\pm = \partial_x \phi_{R,L}, \\
J_{R,L}^x = \exp \left( \pm i 2 \sqrt{2} \pi \phi_{R,L} \right)
\]

each with scaling dimension 1, which generate an \( SU(2) \) Kac-Moody algebra independently for the right and left moving currents\[25\]. As shown in Refs.\[19,20\], sufficiently strong and long-ranged inter-stripe current-current interactions can turn the inter-stripe SC and CDW interactions RG irrelevant and lead to the low-energy fixed point of the sliding TLL (or smectic liquid). We will focus here, however, with the opposite case, i.e., that of dominant SC and CDW inter-stripe interactions.

The WZW model is a non-linear sigma model (NLSM) in \( 1+1 \) dimensions whose degree of freedom is a field \( g(x) \) that takes values on a (compact) Lie group, \( SU(2) \) in this case. The action of the WZW models for each stripe is\[32\]

\[
S_{WZW}^k[g] = \frac{k}{16 \pi} \int d^2 x \text{tr} \left( \partial^\mu g^\dagger \partial_\mu g \right) - \frac{k}{24 \pi} \int_B d^2 x e^{\alpha \beta \gamma} \text{tr} \left( g^\dagger \partial_\alpha g g^\dagger \partial_\beta g g^\dagger \partial_\gamma g \right)
\]

where \( k \) is the level of the Kac-Moody algebra; here we are interested in the case \( SU(2) \) and \( k = 1 \). The second term in Eq.\[33\] is the WZW term, where \( B \) denotes a
3D solid sphere whose boundary $S^2$ is 1 + 1 dimensional space-time. The $SU(2)$-field $g$ and $(\phi, \theta)$ fields are related by

$$
g_{\sigma \sigma'} \sim \begin{pmatrix} e^{-i\sqrt{2}\pi \phi} & -e^{i\sqrt{2}\pi \theta} \\
 e^{-i\sqrt{2}\pi \theta} & e^{i\sqrt{2}\pi \phi} \end{pmatrix}
$$

The (combined) inter-stripe terms in the Hamiltonian density now take the form

$$
\mathcal{H}_{\text{inter}} = -\mathcal{J}_+ \sum_j \text{tr}(g_j^\dagger(x)g_{j+1}(x))
- \mathcal{J}_- \sum_j \text{tr}(g_j^\dagger(x)\sigma z g_{j+1}(x)\sigma z)
$$

where $\sigma z$ is the diagonal Pauli matrix, and we have defined the couplings $\mathcal{J}_\pm = \mathcal{J}_\text{SC} \pm \mathcal{J}_\text{CDW}$. Notice that for $\mathcal{J}_- = 0$ the 2 + 1-dimensional system still enjoys an $SU(2) \times SU(2)$ symmetry, which is broken down to $U(1) \times U(1)$ if $\mathcal{J}_- \neq 0$. In addition, away from $K_c = 1$, the intra-stripe Hamiltonian has additional terms of the form $\text{tr}(\partial_x g^* \sigma \partial^\mu g_x)$ that also break the symmetry down to $U(1) \times U(1)$.

Depending on the signs of $\mathcal{J}_\pm$, the inter-stripe couplings favor ordered phases in 2 + 1 dimensions in which the $SU(2)$-valued matrix field $g$ is either uniform across the system or changes sign (i.e. be staggered by an element of the $\mathbb{Z}_2$ center of the group $SU(2)$) from one stripe to the next. In a (relatively) recent paper, Senthil and Fisher discussed the behavior of an array of antiferromagnetic quantum Heisenberg model in the quasi-1D regime and proposed a description of that system which is similar in spirit to the one we use here for the stripe state in the spin gap regime. One important difference is that in the Senthil-Fisher the $SU(2)$ group is associated with the spins which, in the 1D limit, have quasi-long range antiferromagnetic commensurate order with wave vector $\pi$. As a result, in Ref. [34] the interchain coupling has the form $\text{tr}(g_{j+1}(x)g_j^\dagger(x)) + \text{h.c.}$ that breaks the symmetry to a single global $SU(2)$ symmetry. This coupling favors a staggered order in which $g$ alternates with $g^\dagger$ between neighboring chains. In the problem we discuss here this form of coupling is not allowed by gauge invariance (in the case of the SC order) and by translation invariance (in the case of CDW order). As we will see below this leads to some differences in the structure of the effective field theory.

The derivation of the effective field theory in 2 + 1 dimensions proceeds in two stages, e.g. along the lines discussed early on by Affleck and Halperin. In the quasi-1D limit the leading relevant operators are the inter-stripe couplings shown in Eq. (3.5). In the 1D limit at the $SU(2)$ invariant system (with $K_c = 1$), in the absence of the inter-stripe interactions, the low energy physics is controlled by the WZW fixed point, an infrared stable fixed point at a finite value of the NLSM coupling constant. At the WZW fixed point the inter-stripe operators have scaling dimension 1, strongly destabilize the fixed point of the WZW model, and favor 2 + 1-dimensional ordered states at which the field $g$ picks up a non-vanishing expectation value. In the presence of the “internal anisotropy” terms that break $SU(2) \times SU(2) \rightarrow U(1) \times U(1)$ the resulting ordered phases correspond to 2D SC, 2D CDW and possibly coexistence phases (which is the main question we address here).

After this initial step of quasi-1D renormalization, the system becomes coarse-grained and flows to a full 2 + 1-dimensional theory with the proper symmetries, with an effective field theory of the form of a relativistic-like 2 + 1-dimensional NLSM. The effective theory is generally spatially anisotropic and also has terms “internal” anisotropy terms. However, the infrared unstable (quantum critical) fixed point of the 2 + 1-dimensional NLSM occurs at a finite value of the NLSM effective coupling constant and, hence, cannot be accessed perturbatively from the quasi-1D regime (at least not in a controllable way). Thus, derivations of the effective theory based on naive gradient expansions of the quasi-1D Hamiltonian must be regarded as being only suggestive (at best) of the structure of the effective field theory near the quantum critical point. Nevertheless it is possible to use the powerful constraints of locality and symmetry to write down the structure of the effective field theory. This is the approach we use here. With one significant caveat concerning the role of topology (that we will discuss below), the spatial anisotropies discussed above lead to redundant operators whose effects can be taken into account by a suitable rescaling of the spatial coordinates and time. In contrast, the internal anisotropy terms play a key role. In what follows we will work in imaginary time.

For reasons of clarity we find it useful to represent the $SU(2)$-valued matrix field $g$ in terms of a four-component vector $N^a$ ($a = 0, 1, 2, 3$) that takes values on the three-sphere $S^3$, i.e. $N^a = \frac{1}{2} \text{tr}(g^a \phi)$, where we have used the basis of Hermitian $2 \times 2$ matrices, $\sigma^0 = I$ and the three Pauli matrices for $a = 1, 2, 3$. The four-component field $N^a$ satisfies the constraint $N^2 = 1$, and as such takes values on $S^3$.

Given these symmetry requirements, the only allowed effective action of the effective field theory in 2 + 1 dimensions is that of an $O(4)$ NLSM. Ignoring for the moment spatial anisotropies and internal anisotropy terms, the effective Lagrangian is

$$
\mathcal{L}_0[N] = \frac{1}{2g_0}(\partial_\mu N)^2
$$

where $\mu = 0, 1, 2$ label time and the two spatial coordinates respectively. The bare value of the effective coupling constant $g_0$ of the NLSM in 2 + 1 dimensions has units of (length)$^{-1}$. Its value is essentially given by the geometric mean of the (suitably dimensionalized) coupling constant of the WZW model, $1/(4\pi a_0)$ (where $a_0$ is the stripe spacing) and the inter-stripe coupling $\mathcal{J}_\pm$. This value of $g_0$ will be substantially modified by renor-
The competition/coexistence of SC and CDW orders can be now be discussed by considering the effects of operators that break the large $SU(2) \times SU(2)$ symmetry down to $U(1) \times U(1)$. The symmetry lowering arises from (i) a finite asymmetry in the bare values of the SC and CDW couplings, $J_{SC/CDW} \neq 0$, and (ii) a departure from the SDP as $K_c - 1 \neq 0$. Up to spatially anisotropic gradient terms that we will omit for now, we obtain the following effective Lagrangian

$$L_{\text{eff}}[N] = \frac{1}{2g_0} (\partial_\mu N)^2 - w \partial_\mu N_a O_{ab} \partial_\mu N_b - h N_a O_{ab} N_b$$

where $a, b = 0, 1, 2, 3$. In Eq. (3.7) the second term with coupling constant $w \sim K_c - 1$ parametrizes the departure from the SDP, the third term with coupling constant $h \sim J_{CDW}$ describes the unequal couplings between CDW and SC order parameters, and $O$ is a fixed matrix that breaks $SU(2) \times SU(2) \to U(1) \times U(1)$.

There is still one more operator that can be part of the effective action that needs to be considered: the topological charge $Q_{\text{top}}$ defined by

$$Q_{\text{top}}[N] = \frac{1}{12\pi^2} \int d^4 x \epsilon_{\mu\nu\lambda\sigma} N_\mu \partial_\nu N_\lambda \partial_\sigma N_\nu$$

(3.8)

$Q_{\text{top}}[N]$ is an integer, a topological invariant that classifies the non-trivial maps of the 2 + 1-dimensional Euclidean space-time (compactified to $S^3$) to the target manifold of the $O(4)$ NLSM which is also isomorphic to $S^3$. In the language of homotopy theory these maps are classified by $\Pi_3(S^3) \simeq \mathbb{Z}$. Since the system at hand is time-reversal invariant, the only allowed topological term in the effective (Euclidean) action must have the form $S_{\text{top}} = \pi p Q_{\text{top}}[N]$, where $p \in \mathbb{Z}$. Since $Q_{\text{top}}[N]$ is also an integer, such a topological term has an effect only for $p$ odd.

IV. CLASSICAL AND QUANTUM CRITICAL BEHAVIOR

A. $T = 0$

We begin the discussion of quantum criticality by considering the role of the topological term. Terms of this type play a crucial role in 1D spin-1/2 quantum Heisenberg antiferromagnets, also described by an $SU(2)_1$ WZW model. Extensions to higher dimensional antiferromagnets have also been proposed before and examined in some detail, but never in the context of stripe phases in superconducting systems. Particularly relevant here is the work of Senthil and Fisher, who discussed this same effective field theory in the context of the quantum phase transition from a 2D Neél antiferromagnetic state to a four-fold degenerate valence bond solid (VBS) state on a square lattice, which has been conjectured to be controlled by a deconfined quantum critical (DQC) point. The topological excitations that carry a non-trivial winding number $Q[N]$ are monopole ("hedgehog") configurations in 3D Euclidean space-time, the instantons of this theory. Contrary to their 2D cousins, the Euclidean action of instantons of 3D NLSMs is linearly divergent and hence are suppressed throughout the ordered phase. Nevertheless, within the DQC scenario they are still argued to play a key role both at the quantum critical point and in the quantum disordered phase (which becomes a topological phase). However, the analysis of Ref. [34] shows that the 3D DQC fixed point is unstable to the effects of spatially anisotropic perturbations, such as the ones we have in this theory, and becomes inaccessible. The resulting effective field theory of our system is then the anisotropic NLSM of Eq. (3.7) but without the topological term which becomes an irrelevant operator at the accessible fixed point.

Having determined the form of the effective field theory we can proceed to study its quantum and classical critical behaviors. The analysis of the phase diagram can now be determined using well established methods of (classical and quantum) critical phenomena and, in particular, of multritical phenomena. At this new, more conventional, fixed point spatial anisotropies become mild redundant operators and their presence no longer affect the critical behavior. We will see now that the effective action of Eq. (3.7) will allow us to find a solution to the problem of SC/CDW competition vs. coexistence, both at $T = 0$ and at $T > 0$. A summary of the results is presented in the phase diagrams shown in Fig. 1 and Fig. 2.

We thus have a 2 + 1-dimensional NLSM with symmetry breaking fields. By power counting we see that the third term in Eq. (3.7) is the most relevant perturbation. From this point of view the problem of the CDW/SC competition is conceptually similar to other problem in which there is a partial breaking of the internal symmetry, e.g. the spin flop transition in magnets. This point of view will give us the solution of the problem. The critical behavior obeyed by this system must be approached either by means of a) a $2 + \epsilon$ expansion of the NLSM (here 2 means $1 + 1$ Euclidean space-time dimensions), or b) a $4 - \epsilon$ expansion of the associated Landau-Ginzburg-Wilson (LGW) theory, usually known as $\phi^4$ field theory (again, here 4 means $3 + 1$ Euclidean space-time dimensions). Although we already have the problem expressed as a NLSM we will have to use the $4 - \epsilon$ expansion approach. The reason is that the $4 - \epsilon$ expansion of the $\phi^4$ has a property known as Borel-summability which allows for an accurate determination of its critical exponents directly in $D = 3$ Euclidean space-time dimensions (see, e.g. Ref. [31]). In contrast, the conceptually simpler $2 + \epsilon$ approach has poor convergence properties and has never been successfully used to compute exponents in $D = 3$ even for the simplest NLSM.

For this reason we will replace the NLSM effective
action for the field \( N \) with \( O(4) \) symmetry by a theory with the same symmetries but in which the constraint of the NLSM is replaced by a suitable potential. Let us denote by \( N_{\phi} \) the upper two (CDW) components and \( N_{\theta} \) the lower two (SC) components of the four-component scalar field \( N \), respectively. In this form we are describing the breaking of \( O(4) \rightarrow O(2) \times O(2) \). This is a particular case of the breaking of \( O(n) \) down to \( O(n_1) \times O(n_2) \), with \( n = n_1 + n_2 \), that has been studied in detail in the literature. It was originally studied to one-loop order in the \( 4 - \epsilon \) expansion by Kosterlitz et al., and was more recently reexamined by Calabrese et al. who used a five-loop \( 4 - \epsilon \) expansion with Borel resummation and were able to determine the critical behavior in \( D = 3 \) with high precision.

The resulting LGW Lagrangian (or “free energy density”) has the form

\[
\mathcal{L}[\tilde{N}_\phi, \tilde{N}_\theta] = \frac{1}{2} \left( \partial_\mu \tilde{N}_\phi \right)^2 + \frac{1}{2} \left( \partial_\mu \tilde{N}_\theta \right)^2 + \frac{r_\phi}{2} \tilde{N}_\phi^2 + \frac{r_\theta}{2} \tilde{N}_\theta^2 + \frac{u_\phi}{4!} \left( \tilde{N}_\phi^2 \right)^2 + \frac{u_\theta}{4!} \left( \tilde{N}_\theta^2 \right)^2 + \frac{w_\phi}{4!} \tilde{N}_\phi^2 \tilde{N}_\theta^2
\]

(4.1)

As usual, \( r_\phi \) and \( r_\theta \) measure the departure from the (classical or mean field) critical point; \( r = (r_\theta + r_\phi)/2 \) qualitatively plays the role of the coupling constant \( g_0 \) of the NLSM, \( \tilde{h} = (r_\theta - r_\phi)/2 \) of the symmetry breaking field, and \( u_\phi, u_\theta \) and \( w \) are four coupling constants that parameterize the potential. For \( u = u_\phi = u_\theta = w \) the quartic terms have \( O(n) \) symmetry. Notice also that in this form the spatial anisotropies can always be absorbed by a suitable rescalings of the coordinates and fields.

Below 4 dimensions, the free field (Gaussian) fixed point is always unstable. Kosterlitz et al. showed that theories of these type can describe either a bicritical point (the endpoint of a line of direct first order transitions between the two phases with order parameters \( \tilde{N}_\phi \) and \( \tilde{N}_\theta \)) or a tetracritical point (the endpoint of a region of phase coexistence of the two order parameters). They also showed that when the bicritical scenario holds, the fixed point associated with the critical endpoint has maximal symmetry, \( O(n) \). In the tetracritical scenario, the \( O(n) \) fixed point is unstable and two possibilities arise for the endpoint: a) either it is a decoupled fixed point (DFP) at which the \( O(n_1) \times O(n_2) \) theory is effectively decoupled (\( w \to 0 \)), or b) its is a biconical fixed point (BFP) at which the \( O(n_1) \times O(n_2) \) has a non-trivial coupling \( w^* \).

The one-loop analysis of Kosterlitz et al. predicts a tetracritical point with a DFP for some sufficiently large value of \( n > 4 \), and a bicritical point for the spin-flop transition \( O(3) \rightarrow O(2) \times \mathbb{Z}_2 \). However, the one-loop results are unable to resolve the case of interest here, \( O(4) \rightarrow O(2) \times O(2) \). On the other hand, the five-loop results of Calabrese et al. show without ambiguity that in \( D = 3 \) the \( O(4) \rightarrow O(2) \times O(2) \) theory has a tetracritical point and not a bicritical point as the \( O(4) \) invariant fixed point is unstable. However, the five-loop results do not have sufficient accuracy to distinguish between a tetracritical point controlled by a DFP or by a BFP. Nevertheless, regardless of this technical issue, this analysis predicts the existence of a phase in which the SC and CDW order parameters coexist at \( T = 0 \). This is shown in the schematic phase diagram of Fig.1 as the phase labeled by SC+CDW. The relative strength of the SC and CDW order parameters varies continuously across this phase as the magnitude (and sign) of the coupling \( \tilde{h} \) to the symmetry breaking field varies. (The broken line in Fig.1 is not a phase transition and denotes the manifold with higher symmetry.) All the phase boundaries are in the universality class of the 3D classical XY model (i.e. the \( O(2) \) Wilson-Fisher fixed point in \( D = 3 \)). Whether the tetracritical point is governed by a different coupled fixed point (BFP) or a decoupled one (DFP) is not presently established.

We end this subsection by discussing briefly the extension of this analysis to a stripe phase in 3D. For simplicity we will assume that the stripe phase consists of a stack of 2D stripe phases and, hence, that the order parameter theory is the same as in 2D. Much of the analysis above here follows here too. The main difference is that the effective field theory is now in \( D = 4 \) Euclidean space-time dimensions. In this case the \( O(4) \) NLSM does not have a topological term to begin with. In \( D = 4 \) Euclidean space-time dimensions the free-field (Gaussian) fixed point is marginally stable and the classical (mean field) results are correct up to logarithmic corrections to scaling. In this case there is clearly a phase in which SC and CDW orders coexist. The resulting phase diagram in 3D at \( T = 0 \) has the same topology as in Fig.1.
We now turn to the phase diagram at finite temperature, \( T > 0 \). It is not possible to replace the NLSM by the LGW theory in 2D at finite temperature, since in 2D there is a drastic difference in the thermal critical behavior of the system if it is fine-tuned to have the larger \( O(4) \) symmetry from where it has a lower \( O(2) \times O(2) \) symmetry. (Recall that topological terms do not play a role in the thermal critical behavior as they always contain time derivatives of the fields.) (This was already emphasized by Carr and Tsvelik\(^{21}\).) In both cases there is no long range order in 2D as required by the Mermin-Wagner theorem. However, the critical temperature of the \( O(4) \)-invariant classical 2D NLSM is zero and it is in a classically disordered phase at all \( T > 0 \), as shown in the schematic phase diagrams of Fig. 2 (a-c).

Away from the \( O(4) \) symmetric theory, i.e. for \( \tilde{h} \neq 0 \), there are phase transitions at finite \( T > 0 \). Since the symmetry is now reduced to \( O(2) \times O(2) \) these are, generically Kosterlitz-Thouless (KT) transitions. At a fixed value of the NLSM coupling \( g_0 \) and anisotropy \( \tilde{h} \) different sequences of phase transitions take place. Some details of the phase diagram depends on whether the \( T = 0 \) quantum phase transition is described by a DFP or a BFP. In general there are three situations, shown in the phase diagrams of Fig. 2 (a-c).

In Fig. 2 (a) we depict the case in which at \( T = 0 \) the isotropic \( O(4) \) NLSM is quantum disordered, \( g_0 > g_c \), corresponding to line c in the \( T = 0 \) phase diagram of Fig. 1. For some range of anisotropy \( \tilde{h} \) the system remains disordered at all temperatures. At some critical anisotropy \( \tilde{h}(g_0) \) there is a \( T = 0 \) quantum phase transition to a SC or a CDW phase (depending on the sign of \( \tilde{h} \)), which is in the universality of the 3D XY classical model. For \( T > 0 \) this transition becomes a KT transition and the SC (or the CDW phase depending of the case) has power law correlations and not long range order. Here we assumed that either the \( T = 0 \) tetracritical point is a DFP or, in the case of of a BFP, that \( g_0 \) is above the region of the SC+CDW coexistence phase.

In Fig. 2 (b) we depict the case in which at \( T = 0 \) the isotropic \( O(4) \) NLSM is quantum disordered, \( g_0 > g_c \), corresponding to line c in the \( T = 0 \) phase diagram of Fig. 1. For some range of anisotropy \( \tilde{h} \) the system remains disordered at all temperatures. At some critical anisotropy \( \tilde{h}(g_0) \) there is a \( T = 0 \) quantum phase transition to a SC or a CDW phase (depending on the sign of \( \tilde{h} \)), which is in the universality of the 3D XY classical model. For \( T > 0 \) this transition becomes a KT transition and the SC (or the CDW phase depending of the case) has power law correlations and not long range order. Here we assumed that either the \( T = 0 \) tetracritical point is a DFP or, in the case of of a BFP, that \( g_0 \) is above the region of the SC+CDW coexistence phase.

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Finally, in Fig.2 (c) we have the case $g_0 < g_c$, line $\alpha$ in Fig.1. Once again we have $T_c = 0$ at $\tilde{h} = 0$ as we are in 2D. As a function of $\tilde{h}$ (at fixed $T$) one enters first the SC (or CDW) phase (with power law correlations, once again). There is a range of $\tilde{h}$ in which the coexistence phase appears (with KT transitions at both ends) and eventually a KT transition to the high temperature phase.

Let us finally discuss, briefly, the finite temperature behavior of the 3D stripe state. The main change compared to the 2D case is that the $O(4)$ symmetric NLSM has now a finite temperature transition, in the universality class of the 3D $O(4)$ Wilson-Fisher fixed point. Hence, although $T_c$ for the $O(4)$ NLSM is lower than that of the system with a lower $O(2) \times O(2)$ symmetry, it is not suppressed down to $T = 0$ as in 2D. The 3D case has a phase diagram with a more conventionally looking tetracritical point.

V. DISCUSSION

In this paper we have considered the problem of competition and coexistence of SC and CDW orders in stripe models of strongly correlated systems (e.g. weakly coupled ladders in the spin gap regime). We have shown that the natural effective field theory in the $2 + 1$-dimensional regime is a spatially anisotropic $O(4)$ NLSM with additional interactions that break the symmetry down to $U(1) \times U(1)$. We examined the quantum and thermal critical behaviors of this system. In the quantum regime we used the $\phi^4$ version of this theory (i.e. the Landau-Ginzburg-Wilson theory) with $O(4)$ global symmetry explicitly broken down to $U(1) \times U(1)$. Using relatively recent high precision five loop results (improved with Borel-Padé resummation of the $\epsilon$ expansion) by Calabrese et al. we showed that there is a phase in which SC and CDW orders coexist and that this phase transition is controlled by a tetra-critical point (likely of the decoupled type). Our results for the quantum and thermal phase diagrams (and phase transitions) are summarized in the Figures 1, 2, and 3. It is worth noting that our RG phase diagrams are essentially the same irrespective of the sign of the CDW and SC couplings: we can map one problem onto the other by translating the $(\theta_i, \phi_i)$ fields in equation (2.2) by $\pi$ on all odd-numbered stripes. Thus this analysis also describes quantum phase transitions in a spin-gapped version of the pair-density-wave (PDW) state, a phase in which charge and SC orders are intertwined.

We end by noting that our results are also applicable to the problem of a two dimensional array of weakly coupled spin-1/2 Heisenberg antiferromagnetic spin chains. As shown by Senthil and Fisher, a super-spin representation used by SF for the $O(4)$ NLSM describing Néel and VBS order parameters for this problem is identical to that employed here for us for the problem of coupled stripes. The SC and CDW couplings in the stripe problem correspond to couplings between the Néel and VBS order parameters respectively on nearest neighbor spin chains, while departures from $K_c = 1$ in our problem of stripes correspond to departures from the Heisenberg point to the XXZ model. Thus, our results suggest that in the problem of coupled spin-1/2 chains, there is also a phase in which Néel and columnar VBS orders coexist, ending at a tetracritical point. Within the coexistence region, the VBS singlet order must have spin triplet excitations. Further, the spatially isotropic $O(4)$ NLSM theory of Refs. [39,40] for the competition of Néel and VBS orders in the case of the 2D spin-1/2 Heisenberg antiferromagnetic system admits a deconfined quantum critical fixed point at which an anisotropy in the nearest neighbor Néel and VBS couplings is a relevant operator; the UV stable fixed point reached is the anisotropic $O(4)$ NLSM studied here. It is important to note, however, that our $O(4)$ symmetric theory does not have deconfined topological excitations. As we have seen, for $(J_{SC}, J_{CDW}) > 0$, the competing orders are uniform SC and CDW respectively, while for $(J_{SC}, J_{CDW}) < 0$, the orders are staggered SC (period-2 pair density wave state) and CDW respectively. In the equivalent spin problem, these two cases correspond to the competition between antiferromagnetic Néel vs. staggered columnar VBS orders and ferromagnetic vs. uniform columnar VBS orders respectively.

Acknowledgments

We thank S. A. Kivelson and T. Senthil for very stimulating discussions. This work was supported in part by the National Science Foundation, under grant DMR 0758462 at the University of Illinois, and by the Office of Science, U.S. Department of Energy, under Contract DE-FG02-91ER45439 through the Frederick Seitz Materials Research Laboratory of the University of Illinois. One of us (S.L.) also thanks NSF DMR 09-06521 for support.

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