Riemannian Levenberg-Marquardt Method with Global and Local Convergence Properties

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Abstract We extend the Levenberg-Marquardt method on Euclidean spaces to Riemannian manifolds. Although a Riemannian Levenberg–Marquardt (RLM) method was proposed by Peeters in 1993, to the best of our knowledge, there has been no analysis of theoretical guarantees for global and local convergence properties. As with the Euclidean LM method, how to update a specific parameter known as the “damping parameter” has significant effects on its performances. We propose a trust-region-like approach for determining the parameter. We evaluate the worst-case iteration complexity to reach an $\epsilon$-stationary point, and also prove that it has desirable local convergence properties under the local error-bound condition. Finally, we demonstrate the efficiency of our proposed algorithm by numerical experiments.

Keywords Riemannian manifolds · Riemannian optimization · Least squares problem · Levenberg-Marquardt method

1 Introduction

Optimization problems over Riemannian manifolds, manifolds equipped with smoothly varying positive definite symmetric metrics at every point, have been

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studied intensively. On Riemannian manifolds, we can construct counterparts of a variety of basic concepts in Euclidean optimization such as gradient and Hessian. With these extended concepts, classical unconstrained optimization methods on Euclidean spaces such as the steepest descent method and the Newton method have been generalized to Riemannian manifolds [2]. For example, there exist Riemannian quasi-Newton methods [22, 24, 25, 37], Riemannian conjugate gradient methods [2, 37–39], Riemannian trust region methods [1–3, 7, 9, 22, 23, 29], and so forth. Moreover, in the last few years, studies on constrained optimization methods on Riemannian manifolds have been advanced remarkably, too. For instance, Riemannian augmented Lagrangian methods [30, 45], Riemannian SQP methods [34, 40], and Riemannian interior point methods [27] have been proposed together with rigorous convergence analysis.

In this paper, we consider the nonlinear least square problems over an $n$-dimensional connected Riemannian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle)$, i.e.,

$$\min_{x \in \mathcal{M}} f(x) := \frac{1}{2} \|F(x)\|^2 = \frac{1}{2} \sum_{i=1}^{m} F_i(x)^2,$$  \hspace{1cm} (1.1)

where $F_i: \mathcal{M} \to \mathbb{R}$ ($i = 1, \ldots, m$) are continuously differentiable functions and $F: \mathcal{M} \to \mathbb{R}^m$ is defined as $F = (F_1, F_2, \ldots, F_m)^T$. The problem (1.1) in Euclidean spaces, namely, when $\mathcal{M} = \mathbb{R}^n$, has many classic, but important applications ranging from inverse problems [41], regressions [47], to systems of nonlinear equations [48]. In addition, some recent applications such as the CP decomposition of tensors [12], the low-rank matrix completion [43], the Fréchet mean [19], or the geodesic regression [18, 33] are formulated as (1.1) with some more general Riemannian manifold $\mathcal{M}$.

In Euclidean spaces, optimization methods for nonlinear least-square problems have been studied extensively. Especially, the Gauss-Newton (GN) method and the Levenberg-Marquardt (LM) method are the most popular methods specialized in solving this type of problem. Their common strength is that the fast local convergence can be attained without computing the Hessian of $F$ which is often costly when $n$ is large. Indeed, under some assumptions, both the methods admit local quadratic convergence for the zero-residual case which implies $\min_{x \in \mathbb{R}^n} \|F(x)\|^2 = 0$. The GN method [6] generates a search direction in each iteration by solving the linear equation equivalent to the natural approximation problem $\min_{d \in \mathbb{R}^n} \|F(x) + J(x)d\|^2$, where $J(x)$ denotes the Jacobian of $F$. However, it is often pointed out that the GN method fails to work upon confronting the ill-conditioned linear equations derived from the rank deficiency of $J(x)$. On the other hand, the LM method was developed to cope with this matter [28, 31] by solving the regularized linear equations equivalent to $\min_{d \in \mathbb{R}^n} \|F(x) + J(x)d\|^2 + \rho^2 \|d\|^2$, where $\rho$ is a positive parameter called damping parameter. Various versions of LM methods have been proposed so far [4, 5, 32, 42, 46] and their theoretical and practical performances vary mainly depending on a manner of updating the damping parameter.
In summary, the Euclidean LM method enjoys several nice properties. It admits a local quadratic convergence for the zero-residual case under a local error-bound condition, which is weaker than the nonsingularity condition of the Jacobian $J$ at a solution [46]. For the nonzero-residual case, the local linear convergence of the LM method was also shown under the local error-bound condition by [26]. Moreover, the global convergence complexity was studied, e.g. [3,42].

Some researchers worked with GN and LM methods on Riemannian manifolds for solving (1.1). For example, the basic Riemannian GN (RGN) method is described in the textbook, [2, Section 8.4.1]. In [12], it was customized for solving a certain tensor-decomposition problem. Although the local convergence properties were established in [11] under some assumptions on the Jacobian of $F$, it is not equipped with any global convergence property. The first Riemannian LM (RLM) method was considered in [36, but any theoretical results were not presented therein. Moreover, although [2, Section 8.4.1] pointed out that a combination of the trust region method and the RGN method can be regarded as the RLM method, no specific algorithm is presented there. In this paper, we propose the first RLM method equipped with both global and local convergence guarantees by developing a specific trust-region-like manner of tuning the damping parameter.

1.1 Our contribution

Our contribution is summarized as follows:

1. Development of the RLM method: Even though a Riemannian version of the LM method was considered in [36], it only states how to find an LM-like search direction independently from local coordinates. We characterize the search direction as the tangent vector minimizing subproblem of (1.1) and thus, it is intrinsically independent of the choice of local coordinates. Our RLM method is different from the one in [36] especially in the update manner for the damping parameter.

2. Theoretical guarantees for the RLM method: Our method has theoretical convergence guarantees: global iteration complexity and local convergence rates.

- Our method is globally convergent and the worst-case iteration complexity to reach an $\epsilon$-stationary point is $O\left(\log \left(\epsilon^{-1}\right)\epsilon^{-3}\right)$ under standard assumptions such as $L$-smoothness, explained later in Section 3.

- The local convergence analysis evaluates the algorithm’s behavior around a stationary point $x^*$, and the convergence rate differs depending on whether the residual $f(x^*)$ is zero or not. The former case is often called the zero-residual case, while the latter is the nonzero-residual case. We extend the local error bound condition, which is a standard assumption for Euclidean LM methods, to Riemannian manifolds. Under this condition, we prove that the proposed RLM has the quadratic
local convergence for zero-residual cases and the linear one for nonzero-residual cases.

Unlike Euclidean setting, the local convergence analysis is complicated because the search direction \( s_k \) does not generally satisfy \( \| s_k \|_{x_k} = \text{dist}(x_k, x_{k+1}) \) except in special circumstances.\(^1\) This prevents us from applying the standard approach for the local convergence analysis of LM methods in Euclidean spaces to our RLM method. However, we settle this issue by introducing an inequality on the Riemannian distance and the norm of tangent vectors obtained by the inverse retraction. Finally, let us make comparison with two Riemannian methods which are related to the RLM method, the adaptively quadratically regularized Newton (ARN) method \([21]\) and the Riemannian trust region (RTR) method \([1,3,7,9,22,23,29]\):

- The ARN method solves a sequence of quadratic subproblems with proximal regularization and selects the regularization parameter adaptively. This ARN is quite similar to the RLM in that the regularization technique is employed. However, in the article \([21]\), the global complexity is not derived, and the assumptions for the local quadratic convergence of the ARN, which are set in \([21]\), include the nonsingularity of the Jacobian; the regularity assumption is stronger than ours, i.e., the local error bound. It may be worth mentioning that the ARN \([21]\) is limited to manifolds embedded to Euclidean spaces, and the regularization is induced from the squared Euclidean 2-norm. In contrast, our RLM is not the case, and the regularization is described with the norm induced from the Riemannian metric. As a result, search directions of the RLM are determined independently from local coordinates.

- The RTR solves a sequence of quadratic subproblems subject to the so-called trust region on the tangent space of the Riemannian manifold. The trust region radius is tuned so that the quadratic subproblem is a good approximation to the original problem. As observed from the Karush-Kuhn-Tucker conditions for the quadratic subproblem, the trust-region scheme has an effect similar to the regularization technique. However, as well as the ARN method, the regularity assumption that hessians are nondegenerate at a solution is set in \([1,3,23,29]\) for the local convergence. In particular, an attractive point of the RLM is that quadratic convergence is achieved without using Hessians. As regards complexity of the global convergence, the analysis \([9]\) of the RTR method is similar to ours in the present paper. Nevertheless, it is difficult to translate the lemmas and propositions established in \([9]\) as those for our RLM.

In the numerical experiments, we will make numerical comparison with the RTR and ARN methods, and show that the RLM performs very well.

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\(^1\) More specifically, the search direction \( s_k \) does not satisfy \( \| s_k \|_{x_k} = \text{dist}(x_k, x_{k+1}) \) unless \( s_k = \log_{x_k}(x_{k+1}) \) holds. Here, \( \text{dist} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0} \) denotes the Riemannian distance and \( \log_{x_k} : \mathcal{M} \to T_{x_k} \mathcal{M} \) denotes the logarithmic map at \( x_k \).
1.2 Organization of the paper

The rest of this paper is organized as follows: In Section 2, we propose a new RLM method, and in Section 3, we show that the RLM method has a global convergence property and further evaluate the worst-case iteration complexity to reach an $\epsilon$-stationary point. In Section 4, we analyze the local behavior of the proposed RLM method. In Section 5, we conduct some numerical experiments of the RLM method, and in Section 6, we conclude this paper with some remarks.

1.3 Notations and terminologies

For a given Riemannian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle)$, $T_x\mathcal{M}$ denotes the tangent space to $\mathcal{M}$ at $x$ and $T\mathcal{M}$ is the tangent bundle of $\mathcal{M}$, i.e., $T\mathcal{M} := \bigsqcup_{x \in \mathcal{M}} T_x\mathcal{M}$. Let $0_x$ be the zero vector of $T_x\mathcal{M}$ as a vector space. We denote by $\langle \cdot, \cdot \rangle_x$ the inner product induced by the Riemannian metric at the point $x \in \mathcal{M}$ and $\| \cdot \|_x$ is the norm induced by the inner product, i.e., $\|v\|_x := \sqrt{\langle v, v \rangle_x}$ for $v \in T_x\mathcal{M}$. For the sake of brevity, the subscript $x$ is dropped from $\| \cdot \|_x$ when it is clear from the context.

Given manifolds $\mathcal{M}$ and $\mathcal{N}$ together with a smooth map $g : \mathcal{M} \to \mathcal{N}$, $dg$ denotes the differential of $g$, namely, for all $p \in \mathcal{M}$, $dg(p)$ is the linear map from $T_p\mathcal{M}$ to $T_{g(p)}\mathcal{N}$ such that $dg(p)[v] = \frac{d(g \circ c)}{dt} \bigg|_{t=0}$ holds for all $v \in T_p\mathcal{M}$ where $c : (-\epsilon, \epsilon) \to \mathcal{M}$ ($\epsilon > 0$) is any smooth curve satisfying $c(0) = p$ and $\frac{dc(t)}{dt} \bigg|_{t=0} = v$.

We next introduce a retraction, playing an important role for Riemannian optimization algorithms to determine an iteration point on the manifold along a given tangential direction.

**Definition 1** A retraction $R$ is a smooth map from $T\mathcal{M}$ to $\mathcal{M}$ with the following properties. Let $R_x$ denote the restriction of $R$ to $x \in \mathcal{M}$.

1. $R_x(0_x) = x$ for all $x \in \mathcal{M}$.

2. For all $x \in \mathcal{M}$, the differential map $dR_x(0_x) : T_x\mathcal{M} \to T_x\mathcal{M}$ is the identity map on $T_x\mathcal{M}$.

Next, we define some notations and terminologies concerning the function $F$ in (1.1). We refer to the following linear map as the Jacobian matrix of $F$ at $x \in \mathcal{M}$:

$$J(x) : T_x\mathcal{M} \to \mathbb{R}^m$$

$$s \mapsto \begin{pmatrix} \langle \text{grad} F_1(x), s \rangle_x \\ \vdots \\ \langle \text{grad} F_m(x), s \rangle_x \end{pmatrix}.$$
where $\nabla F_i(x)$ is the Riemannian gradient of $F_i$ at $x$ for $1 \leq i \leq m$. We denote the adjoint operator of $J(x)$ by $J(x)^* : \mathbb{R}^m \to T_x M$, i.e.,

$$
\langle J(x)^* u, v \rangle_x = \langle u, J(x)v \rangle
$$

for all $(u, v) \in \mathbb{R}^m \times T_x M$, where the inner product in the right-hand side is the canonical inner product of $\mathbb{R}^m$. Finally, we define the norm $\|J(x)\|$ of the Jacobian matrix $J(x)$ for $x \in M$ by means of the operator norm, namely,

$$
\|J(x)\| := \max_{v \in T_x M \setminus \{0\}} \frac{\|J(x)v\|}{\|v\|_x},
$$

where the norm in the numerator of the right-hand side represents the Euclidean norm. Clearly, $\|J(x)\|$ is equal to the square root of the maximum eigenvalue of $J(x)^* J(x) : T_x M \to T_x M$.

2 Proposed Riemannian Levenberg-Marquardt method

In this section, we describe how the search direction of the RLM is determined in each iteration and present the specific pseudo-code for the RLM. In addition, we will characterize the search direction which plays important roles in theoretical analysis. Our formulation of the RLM can be considered as a natural generalization of the LM in Euclidean spaces. Let $\{x_k\} \subseteq M$ denote a sequence generated by the proposed RLM, and let $J_k := J(x_k)$.

The pseudo-code of our proposed method is Algorithm 1. This trust-region-like updating scheme of the damping parameter is proposed by [5] in the Euclidean setting. Below, we show the details of the algorithm.

**Algorithm 1 RLM method**

**Input:** $x_0 \in M, \eta \in (0, 1), \mu_{\text{min}} > 0, \beta > 1, \text{flag}^{\text{nz}} \in \{\text{true, false}\}$

**Output:** stationary point of (1.1)

1: $\mu_0 \leftarrow \mu_{\text{min}}, \bar{\mu} \leftarrow \mu_0$
2: $k \leftarrow 0$
3: while not convergence do
4: compute $F(x_k), J_k$
5: compute $s_k$ by solving (2.5) with $\lambda_k = \mu_k \|F(x_k)\|^2$
6: compute $\rho_k := \frac{1}{2(\theta(x_k) - \theta(s_k))}$
7: if $\rho_k \geq \eta$ then
8: $x_{k+1} \leftarrow R_{x_k}(s_k), \bar{\mu} \leftarrow \mu_k$
9: if flag$^{\text{nz}}$ then
10: $\mu_{k+1} \leftarrow \bar{\mu}$
11: else
12: $\mu_{k+1} \leftarrow \max(\mu_{\text{min}}, \frac{\rho_k}{\bar{\mu}})$
13: end if
14: else
15: $x_{k+1} \leftarrow x_k, \mu_{k+1} \leftarrow \beta \mu_k$
16: end if
17: $k \leftarrow k + 1$
18: end while
19: return $x_k$
2.1 Subproblem for problem (1.1)

Given $\lambda_k > 0$, define $\theta^k : T_{x_k} \mathcal{M} \to \mathbb{R}$ as

$$
\theta^k(s) := \|F(x_k) + J_k s\|^2 + \lambda_k \|s\|^2
$$

and denote the optimal solution by $s_k$.

We solve the following problem as the subproblem of (1.1) at $x_k$

$$
\text{minimize}_{s \in T_{x_k} \mathcal{M}} \theta^k(s),
$$

and denote the optimal solution by $s_k$, namely

$$
s_k := \arg \min_{s \in T_{x_k} \mathcal{M}} \theta^k(s).
$$

This problem is strongly convex on $T_{x_k} \mathcal{M}$, and thus has a unique optimum. We employ the solution $s_k$ as the search direction at $x_k$. Through the stationary condition of (2.3), $s_k$ is characterized as a solution of a certain linear equation as in the following proposition. This relationship is in fact non-trivial because of $\| \cdot \|_{x_k}$ in the function $\theta^k$.

**Proposition 1** The tangent vector $s_k$ solves problem (2.3) if and only if it satisfies

$$
\langle J_k^* J_k + \lambda_k I_k \rangle s_k = - J_k^* F(x_k)
$$

$$
= - \nabla F(x_k),
$$

where $I_k$ denotes the identity map on $T_{x_k} \mathcal{M}$. In particular, the equation (2.5) has a unique solution.

**Proof** Since the latter assertion is trivial as the linear operator $J_k^* J_k + \lambda_k I_k$ is positive definite, namely, $\langle v, (J_k^* J_k + \lambda_k I_k) v \rangle_{x_k} > 0$ for all $v \in T_{x_k} \mathcal{M} \setminus \{0_{x_k}\}$, we prove only the former one.

Let $(U; x^1, \ldots, x^n)$ be an arbitrary coordinate neighborhood containing $x_k$. Let $G$ be the matrix representation of the Riemannian metric at $x_k$ under this local coordinate, i.e., $G = (g_{ij})_{1 \leq i, j \leq n}$ where $g_{ij} := \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_{x_k}$. Let $s$ be an arbitrary element of $T_{x_k} \mathcal{M}$ and $\tilde{s}$ be its local coordinate expression. In a similar way, we define $\tilde{v}$ as the local coordinate expression of $J_k^* F(x_k)$ and $A$ as the matrix representation of $J_k^* J_k + \lambda_k I_k$ under this local coordinate. Then, we have

$$
\theta^k(s) = \|F(x_k)\|^2 + 2 \langle F(x_k), J_k s \rangle + \langle J_k s, J_k s \rangle_{x_k}
$$

$$
= \|F(x_k)\|^2 + 2 \langle J_k^* F(x_k), s \rangle_{x_k} + \langle (J_k^* J_k + \lambda_k I_k) s, s \rangle_{x_k}
$$

$$
= \|F(x_k)\|^2 + 2 \tilde{v}^T G \tilde{s} + \langle A \tilde{s}, \tilde{s} \rangle
$$

$$
= \|F(x_k)\|^2 + 2 \tilde{v}^T G \tilde{s} + \tilde{s}^T G A \tilde{s} \quad (=: \tilde{\theta}^k(\tilde{s})).
$$

(2.6)
We show that $GA$ is symmetric and positive-definite. Denoting the local coordinate expression of $J_k$ as $\tilde{J}_k$, we can express that of $J^*_k$ as $G^{-1}\tilde{J}_k^T\tilde{J}_k + \lambda_k I$, and hence $A = G^{-1}\tilde{J}_k^T\tilde{J}_k + \lambda_k I$ holds where $I$ denotes the $n$-dimensional identity matrix. Consequently,

$$GA = G \left( G^{-1}\tilde{J}_k^T\tilde{J}_k + \lambda_k I \right) = \tilde{J}_k^T\tilde{J}_k + \lambda_k G,$$

$$A^T G = \left( \tilde{J}_k^T\tilde{J}_k G^{-1} + \lambda_k I \right) G = \tilde{J}_k^T\tilde{J}_k + \lambda_k G,$$

and thus, $GA = (GA)^T$ is shown and the positive-definiteness of $GA$ immediately follows from (2.7). Consequently, the optimality condition of minimizing $\tilde{\theta}_k(\tilde{s})$ is equivalent to the stationary condition

$$\frac{\partial}{\partial \tilde{s}} \tilde{\theta}_k(\tilde{s}) = 0,$$

written as $G (A\tilde{s} + \tilde{v}) = 0$. Furthermore, by the positive definiteness of $G$, this is equivalent to

$$A\tilde{s} = -\tilde{v}.$$

Considering the coordinate-independent form of (2.8), we obtain $(J^*_kJ_k + \lambda_k I_k)s = -J^*_kF(x_k)$. Therefore, we have reached the desired conclusion. □

2.2 How to update the damping parameter

The damping parameter $\lambda_k$ in (2.1) controls the step length and needs to be chosen in a manner reflecting how trustworthy the subproblem is: when the subproblem (2.3) is close to the original problem (1.1), $\lambda_k$ is set relatively small and otherwise, it is set relatively large. In a similar manner to the trust region method [48], we evaluate the quality of the solution $s_k$ of (2.3) in terms of $\rho_k$ defined by

$$\rho_k := \frac{f(x_k) - f(R_{x_k}(s_k))}{\frac{1}{2} \left( \tilde{\theta}^k(0) - \tilde{\theta}^k(s_k) \right)}.$$

Let $\eta \in (0, 1)$ be a prefixed constant. If $\rho_k \geq \eta$ holds, we judge the subproblem is trustworthy, and then update $x_{k+1}$ as $x_{k+1} = R_{x_k}(s_k)$. In this case, we refer to the $k$-th iteration as a successful iteration. Otherwise, it is called unsuccessful. We reject $R_{x_k}(s_k)$ and set $x_{k+1} = x_k$. After setting $\lambda_k$ larger, we solve the subproblem again and check whether $\rho_k \geq \eta$ or not.

We explain how to tune $\lambda_k$. First, we set

$$\lambda_k = \mu_k \|F(x_k)\|^2,$$

which is a standard choice so as to achieve locally fast convergence in the recent Euclidean LM methods [5, 14–17]. The positive parameter $\mu_k$ is updated as shown in Algorithm 1. To ensure global convergence in Section 3 and local convergence for the zero-residual case in Section 4.3, we set $\mu_{k+1} = \max \left( \mu_{\min}, \frac{\mu_k}{2} \right)$, while we set $\mu_{k+1} = \bar{\mu}$ to establish local convergence for the nonzero-residual case in Section 4.4. The parameter flag$_{nz}$ in Algorithm 1 specifies which updating manner for $\mu_k$ is applied and is supposed to be set false except for the nonzero-residual case in Section 4.4.
Remak 1 Even when we update \( \{ \mu_k \} \) by \( \mu_{k+1} = \mu \), we can still ensure the global convergence property. Meanwhile, the iteration complexity analysis in Section 3.2 however, can depend on flag \( nz \) in Algorithm 1. In this paper, we only discuss the iteration complexity of Algorithm 1 with flag \( nz = \text{false} \).

3 Analysis on global convergence and iteration complexity

In this section, we set flag \( nz = \text{false} \) in Algorithm 1. We prove that Algorithm 1 has a global convergence property and then, analyze its iteration complexity.

We begin with giving some assumptions and lemmas.

We define
\[
\rho_k := \min \left\{ \theta_k, \frac{1}{\lambda_k} \right\},
\]
where \( \eta \) is the constant in Algorithm 1. For the sake of convenience in proofs, we express \( S \) as
\[
S = \{ k(0), k(1), \ldots \}.
\]

Lemma 1 The k-th search direction \( s_k \) satisfies 
\[
\theta^k(0) - \theta^k(s_k) \geq \frac{\|\nabla f(x_k)\|^2}{\|J_k\|^2 + \lambda_k}.
\]
Proof Define
\[
s_k^\circ := -\frac{\|\nabla f(x_k)\|^2}{\langle \nabla f(x_k), (J_k^* J_k + \lambda_k I_k) \nabla f(x_k) \rangle} \nabla f(x_k).
\] (3.1)

First, we have
\[
\theta^k(0) - \theta^k(s_k^\circ)
= -2\langle \nabla f(x_k), s_k^\circ \rangle - \|J_k s_k^\circ\|^2 - \lambda_k \|s_k^\circ\|^2
= \frac{2\|\nabla f(x_k)\|^4}{\langle \nabla f(x_k), (J_k^* J_k + \lambda_k I_k) \nabla f(x_k) \rangle} \left( \|J_k s_k^\circ\|^2 + \lambda_k \|s_k^\circ\|^2 \right),
\] (3.2)

where (3.1) is used in the last equality. Moreover, we obtain
\[
\|J_k s_k^\circ\|^2 + \lambda_k \|s_k^\circ\|^2
= \frac{\|\nabla f(x_k)\|^4}{\langle \nabla f(x_k), (J_k^* J_k + \lambda_k I_k) \nabla f(x_k) \rangle} \langle \nabla f(x_k), (J_k^* J_k + \lambda_k I_k) \nabla f(x_k) \rangle
= \frac{\|\nabla f(x_k)\|^4}{\langle \nabla f(x_k), (J_k^* J_k + \lambda_k I_k) \nabla f(x_k) \rangle} \langle \nabla f(x_k), (J_k^* J_k + \lambda_k I_k) \nabla f(x_k) \rangle
= \frac{\|\nabla f(x_k)\|^4}{\langle \nabla f(x_k), (J_k^* J_k + \lambda_k I_k) \nabla f(x_k) \rangle},
\]

where (3.1) is applied in the first equality. Then, from (3.2) we obtain
\[
\theta^k(0) - \theta^k(s_k^\circ) \geq \frac{\|\nabla f(x_k)\|^2}{\|J_k\|^2 + \lambda_k}.
\]
From the above inequality and the fact that \( s_k = \arg \min_{s \in T} \theta^k(s) \), it follows that
\[
\theta^k(0) - \theta^k(s_k) \geq \frac{\|\nabla f(x_k)\|^2}{\|J_k\|^2 + \lambda_k}.
\]
(3.3)

The following lemma will be used not only in this section but also in Section 4.

**Lemma 2** The solution \( s_k \) of (2.5) satisfies the following:
\[
\|s_k\| \leq \frac{\|\nabla f(x_k)\|}{\lambda_k},
\]
(3.4)
\[
-\langle \nabla f(x_k), s_k \rangle \geq \frac{\|\nabla f(x_k)\|^2}{\|J_k\|^2 + \lambda_k}.
\]
(3.5)

**Proof** First, we prove (3.4). By (2.5), \( \|\nabla f(x_k)\|^2 \) satisfies
\[
\|\nabla f(x_k)\|^2 = \langle (J_k^*J_k + \lambda_k I_k)s_k, (J_k^*J_k + \lambda_k I_k)s_k \rangle
\]
\[
= \|(J_k^*J_k)s_k\|^2 + 2\lambda_k\|J_k s_k\|^2 + \lambda_k^2\|s_k\|^2
\]
\[
\geq \lambda_k\|s_k\|^2,
\]
which leads to (3.4).

Next, we show (3.5). Since \( s_k \) can be written as \( s_k = -(J_k^*J_k + \lambda_k I_k)^{-1}\nabla f(x_k) \) from (2.5), we obtain
\[
-\langle \nabla f(x_k), s_k \rangle = \langle \nabla f(x_k), (J_k^*J_k + \lambda_k I_k)^{-1}\nabla f(x_k) \rangle
\]
\[
\geq \frac{\|\nabla f(x_k)\|^2}{\|J_k\|^2 + \lambda_k},
\]
which proves (3.5). \( \square \)

**Assumption 1** The Jacobian matrix \( J : T \rightarrow \mathbb{R}^m \) of \( F \) and its adjoint \( J^* \) are bounded on \( \mathcal{L}(x_0) := \{x \in T \mid f(x) \leq f(x_0)\} \), i.e., there exists \( M > 0 \) such that \( \max\{\|J(x)\|, \|J(x)^*\|\} \leq M \) holds for all \( x \in \mathcal{L}(x_0) \).

3.1 Global convergence

Now, the global convergence theorem of the RLM is presented below. Before moving to the main theorem, we show the following lemma.

**Lemma 3** Under Assumption 1, if \( \liminf_{j \rightarrow \infty} \|F(x_{k(j)})\| > 0 \), then \( \limsup_{j \rightarrow \infty} \mu_{k(j)}\|s_{k(j)}\| < \infty \) holds.
Proof Using (3.4) and \( \mu_k = \frac{\lambda_k}{\|F(x_k)\|} \), we have \( \mu_k \|s_k\| \leq \frac{\|\nabla f(x_k)\|}{\|F(x_k)\|} \). Moreover, we have \( \|\nabla f(x_k)\| = \|J_k^* F(x_k)\| \leq \|J_k^* \| \|F(x_k)\| \leq M \|F(x_k)\| \) from Assumption \( [1] \). Combining these relationships, we obtain \\
\[
\lim_{j \to \infty} M \|F(x_k)\| < \infty.
\]

**Theorem 1** Under Assumption \( [1] \), \( \lim \inf_{k \to \infty} \|\nabla f(x_k)\| = 0. \)

**Proof** Note that \( s_k = 0 \) if and only if \( \nabla f(x_k) = 0 \). Moreover, notice that, for \( k(j), k(j + 1) \in S \), we have
\[
x_\ell = x_{k(j+1)} \text{ for all } \ell \text{ such that } k(j) + 1 \leq \ell \leq k(j+1). \tag{3.6}
\]

We prove the assertion by considering two cases: (i) \( |S| = \infty \) and (ii) \( |S| < \infty \).

**Proof for the case (i):** We consider the case where \( |S| = \infty \). For arbitrary \( j \in \{0, 1, 2, \ldots\} \), it holds that
\[
f(x_{k(j+1)}) - f(x_{k(j)}) = f(x_{k(j)+1}) - f(x_{k(j)}) \quad \text{(by (3.6))}
\]
\[
\leq -\frac{\eta}{2} (\theta^{k(j)}(0) - \theta^{k(j)}(s_{k(j)})) \quad \text{(by (2.9) and } k(j) \in S)\tag{3.7}
\]
\[
\leq -\frac{\eta}{2} \|\nabla f(x_{k(j)})\|^2 \quad \text{(by (3.3))}
\]
\[
\leq -\frac{\eta}{2} M^2 + \mu_k \|F(x_{k(j)})\|^2 \quad \text{(by Assumption [1]).} \tag{3.7}
\]

The above implies that \( \{f(x_k)\}_{k \in S} \) is monotonically decreasing. Furthermore, this property and Algorithm \( [1] \) lead to the fact that \( \{f(x_k)\} \) is monotonically non-increasing. Moreover, by the definition of \( f \), \( f(x_{k(j)}) \geq 0 \) holds for all \( j \in \{0, 1, 2, \ldots\} \). Therefore,
\[
f(x_{k(j+1)}) - f(x_{k(j)}) \to 0 \quad (j \to \infty) \tag{3.8}
\]
holds. In what follows, we will show that
\[
\lim_{j \to \infty} \|\nabla f(x_{k(j)})\| = 0. \tag{3.9}
\]

To this end, we further divide the current case (i) into two cases: (i-a) \( \{\mu_k\} \) is bounded and (i-b) \( \{\mu_k\} \) is unbounded.

Consider the case (i-a). Let \( \bar{\mu} := \sup_k \mu_k < \infty \). By (3.7), we have
\[
f(x_{k(j+1)}) - f(x_{k(j)}) \leq -\frac{\eta}{2} M^2 + \bar{\mu} \|F(x_{k(j)})\|^2
\]
\[
\leq -\frac{\eta}{2} M^2 + \bar{\mu} \|F(x_0)\|^2,
\]
where the second inequality follows from $\|F(x_{k(j)})\| \leq \|F(x_0)\|$ by the monotonically non-increasing property of $\{f(x_k)\}$. Combining this with (3.8) yields $\lim_{j \to \infty} \|\nabla f(x_{k(j)})\| = 0$. Thus, (3.9) is established in the case (i-a).

We next consider the case (i-b) where $\{\mu_k\}$ is unbounded. From the construction of Algorithm 1 along with the assumptions that $|S| = \infty$ and $\{\mu_k\}$ is unbounded, it follows that $\{\mu_{k(j)}\}$ is unbounded. Noting $\|\nabla f(x_{k(j)})\|^2 \geq \lambda_{k(j)}^2 \|s_{k(j)}\|^2$ from Lemma 2 and (2.5), we obtain

$$J_{j}(\|F(x_{k(j)})\|^4 \|F(x_{k(j)})\| \|s_{k(j)}\|^2 \leq -\frac{\eta M^2 + \mu_{k(j)}\|F(x_{k(j)})\|}{2} \|s_{k(j)}\|^2, \quad (3.10)$$

where the second inequality follows from $\|F(x_{k(j)})\| \leq \|F(x_0)\|$. From (3.10) and (3.8), it follows that

$$\lim_{j \to \infty} \frac{\mu_{k(j)}^2 \|F(x_{k(j)})\|^4}{M^2 + \mu_{k(j)}\|F(x_0)\|^2} \|s_{k(j)}\|^2 = 0. \quad (3.11)$$

Since $\{\mu_{k(j)}\}$ is unbounded, there exists a subsequence such that $\frac{\mu_{k(j)}^2 \|F(x_{k(j)})\|^4}{M^2 + \mu_{k(j)}\|F(x_0)\|^2}$ diverges as $j \to \infty$. This fact along with (3.11) results in

$$\lim_{j \to \infty} \|F(x_{k(j)})\|^2 \|s_{k(j)}\| = 0, \quad (3.12)$$

from which we will derive (3.9) below. Notice that

$$0 = \liminf_{j \to \infty} \|F(x_{k(j)})\|^2 \|s_{k(j)}\| \geq \left( \liminf_{j \to \infty} \|F(x_{k(j)})\|^2 \right) \left( \liminf_{j \to \infty} \|s_{k(j)}\| \right).$$

Suppose $\liminf_{j \to \infty} \|F(x_{k(j)})\|^2 > 0$. We then obtain $\liminf_{j \to \infty} \|s_{k(j)}\| = 0$. Then, there exists some $J \subseteq \{0, 1, 2, \ldots\}$ such that $\lim_{j \to \infty} \|s_{k(j)}\| = 0$. Using this and (2.5), we obtain

$$\lim_{j \to \infty} \|\nabla f(x_{k(j)})\| = \lim_{j \to \infty} \|\left( J^*_k \cdot J_{k(j)} + \mu_{k(j)} \|F(x_{k(j)})\| \|I_{k(j)}\| \right) \|s_{k(j)}\| \leq \lim_{j \to \infty} \|\left( J^*_k \cdot J_{k(j)} + \mu_{k(j)} \|F(x_{k(j)})\| \|I_{k(j)}\| \right) \|s_{k(j)}\| \leq \lim_{j \to \infty} \left( M^2 \|F(x_0)\|^2 \mu_{k(j)} \right) \|s_{k(j)}\| = 0,$$

where $I_{k(j)}$ denotes the identity mapping on $T_{x_{k(j)}} \mathcal{M}$ and the last equality follows from $\lim_{j \to \infty} \|s_{k(j)}\| = 0$ and Lemma 3. Next, suppose $\liminf_{j \to \infty} \|F(x_{k(j)})\|^2 = 0$. Since $\nabla f(x_{k(j)}) = J^*_k \cdot F(x_{k(j)})$ and $\|J^*_k \cdot F(x_{k(j)})\|$ is bounded by Assumption 4, (3.9) is ensured. Consequently, (3.9) is established in the case (i-b).
Now, combining the cases (i)-a and (i)-b, we gain (3.9) in the whole case (i). Lastly, by noting (3.6) again,
\[ \lim \inf_{k \to \infty} \| \nabla f(x_k) \| \leq \lim \inf_{j \to \infty} \| \nabla f(x_{(j)}) \| = 0 \]
is ensured.

**Proof for the case (ii):** In turn, we consider the case (ii) where \(|S| < \infty\). For each \( \mu > 0 \), define
\[ \theta^k_{\mu}(s) := \| F(x_k) + J_k s \|^2 + \mu \| F(x_k) \|^2 \| s \|^2, \]
\[ s_k(\mu) := \arg \min_{s \in T_{x_k} M} \theta^k_{\mu}(s). \]
Then, by replacing \( \lambda_k \) with \( \mu \| F(x_k) \|^2 \) in Lemma 2, we have
\begin{align*}
\| s_k(\mu) \| & \leq \frac{\| \nabla f(x_k) \|}{\mu \| F(x_k) \|^2}, \quad (3.13) \\
- \langle \nabla f(x_k), s_k(\mu) \rangle & \geq \frac{\| \nabla f(x_k) \|^2}{\| J_k \|^2 + \mu \| F(x_k) \|^2}. \quad (3.14)
\end{align*}
To derive a contradiction, we suppose \( \| \nabla f(x_k) \| \neq 0 \). Since \( \bar{k} := \max_{k \in S} k < \infty \) by assumption, all iterations after the \( \bar{k} \)-th iteration are unsuccessful, implying that
\[ f(x_k) - f(R_{x_k}(s_k(\mu))) < \frac{\eta}{2} (\theta^k_{\mu}(0) - \theta^k_{\mu}(s_k(\mu))) \quad \text{for all} \quad \mu \geq \mu_{\bar{k}}. \]
As it holds that
\begin{align*}
\theta^k_{\mu}(0) - \theta^k_{\mu}(s_k(\mu)) & = -2 \langle \nabla f(x_k), s_k(\mu) \rangle - \| J_k s_k(\mu) \|^2 - \mu \| F(x_k) \|^2 \| s_k(\mu) \|^2 \\
& \leq -2 \langle \nabla f(x_k), s_k(\mu) \rangle, \quad (3.15)
\end{align*}
where the equality holds in a way analogous to (2.2), we obtain
\[ f(x_k) - f(R_{x_k}(s_k(\mu))) < -\eta \langle \nabla f(x_k), s_k(\mu) \rangle \quad (3.16) \]
for all \( \mu \geq \mu_{\bar{k}} \). By (3.13) with \( k = \bar{k} \),
\[ \lim_{k \to \infty} \| s_k(\mu) \| = 0. \quad (3.17) \]
From the \( C^1 \) property of \( f \), Taylor’s expansion yields that
\[ f(R_{x_k}(s_k(\mu))) = f(x_k) + \langle \nabla f(x_k), s_k(\mu) \rangle + e_\mu \quad (3.18) \]
with \( e_\mu = o(\| s_k(\mu) \|) \). By combining (3.18) with (3.16), we find that
\[ -e_\mu < (1 - \eta) \langle \nabla f(x_k), s_k(\mu) \rangle \quad (3.19) \]
holds. Since $\eta \in (0, 1)$, the right-hand side is negative and thus $e_{\mu} > 0$ follows, from which we have
\[
\frac{e_{\mu}}{\|s_k(\mu)\|} > (1 - \eta) \frac{\|\nabla f(x_k)\|^2}{\left\| J_k \right\|^2 + \mu \|F(x_k)\|^2} \frac{1}{\|s_k(\mu)\|} \quad \text{(by (3.14) and (3.19))}
\]
\[
\geq (1 - \eta) \frac{\|\nabla f(x_k)\|^2}{\left\| J_k \right\|^2 + \mu \|F(x_k)\|^2} \frac{2\mu \|F(x_k)\|^2}{\|s_k(\mu)\|} \quad \text{(by (3.13))},
\]
which is equivalent to
\[
\|\nabla f(x_k)\| < \frac{1}{1 - \eta} \left( \frac{\|J_k\|^2 + \mu \|F(x_k)\|^2}{\mu \|F(x_k)\|^2} \right) e_{\mu} \frac{\|s_k(\mu)\|}{\|s_k(\mu)\|}.
\]
By driving $\mu \to \infty$ in the above, the right-hand side converges to 0 since (3.17) and $e_{\mu} = o(\|s_k(\mu)\|)$ hold. This contradicts the assumption $\nabla f(x_k) \neq 0$. As a result, we conclude that $\|\nabla f(x_k)\| = 0$. In this case (ii), any iteration points never vary after the $k$-th iteration and hence $\lim_{k \to \infty} \|\nabla f(x_k)\| = 0$ holds.

The whole proof is complete. \hfill \Box

In view of the proof of Theorem 1 for the cases (i)-a and (ii), we have the following corollary:

**Corollary 1** If \{\$\mu_k\$\} is bounded and Assumption 7 holds, then
\[
\lim_{k \to \infty} \|\nabla f(x_k)\| = 0.
\]

As we will show later, if $\nabla f$ is Lipschitz continuous and (1.1) is nonzero-residual, then, \{\$\mu_k\$\} is ensured to be bounded.

### 3.2 Iteration complexity

Next, we analyze the iteration complexity of RLM. For this purpose, we require the Lipschitz continuity of $\nabla f$ as in the Euclidean setting.

**Assumption 2** $\nabla f$ is $L$-Lipschitz continuous on $\mathcal{L}(x_0)$ where $\mathcal{L}(x_0)$ is defined as in Assumption 7.

Under this assumption, the following useful lemma holds, where the second-order retraction defined below plays an important role.

**Definition 2** A retraction $R$ is a second-order retraction if and only if for all $(x, v) \in TM$, the smooth curve $c(t)$ defined as $c(t) := R_x(tv)$ has zero acceleration at $t = 0$, i.e., $c''(0) = 0$.

For instance, a map known as exponential map is a second-order retraction.
**Lemma 4** Suppose that the retraction $R$ is second-order and Assumption 2 holds. Then, for any $(x, s)$ such that $(x, R_x(s)) \in \mathcal{L}(x_0) \times \mathcal{L}(x_0)$, we have
\[
f(R_x(s)) \leq f(x) + \langle \nabla f(x), s \rangle + \frac{L}{2} \|s\|^2.
\]

**Proof** The proof follows from [8, Exercise 10.56] easily. □

To use Lemma 4, we restrict retractions to second-order ones throughout the analysis of iteration complexity.

For any positive number $\epsilon$, we define some notations as below:
- $j_\epsilon := \min\{j \mid \|\nabla f(x_j)\| < \epsilon \text{ or } f(x_j) < \epsilon\}$,
- $\mathcal{S}_\epsilon := \{0, 1, \ldots, j_\epsilon - 1\} \cap \mathcal{S}$,
- $\mathcal{U}_\epsilon := \{0, 1, \ldots, j_\epsilon - 1\} \setminus \mathcal{S}_\epsilon$.

Our objective is to evaluate the worst-case iteration number which is needed to reach a point $x$ such that $\|\nabla f(x)\| < \epsilon$ or $f(x) < \epsilon$ hold. In other words, we wish to evaluate $j_\epsilon (= |\mathcal{S}_\epsilon| + |\mathcal{U}_\epsilon|)$ in the worst case. The analysis will be conducted by tracing the following three steps one by one:

1. We give a sufficient condition for the $k$-th iteration to be successful.
2. We derive an upper bound of $|\mathcal{S}_\epsilon|$.
3. We evaluate the maximum number of unsuccessful iterations occurring consecutively, and then give an upper bound of $|\mathcal{U}_\epsilon|$.

Hereinafter, we denote
\[
\kappa := \frac{L + \sqrt{L^2 + 2(1 - \eta)LM^2}}{2(1 - \eta)}.
\]

For the above step (1), we introduce the following lemma.

**Lemma 5** Under Assumptions 1 and 2 if $\lambda_k \geq \kappa$, then the $k$-th iteration is successful.

**Proof** We have
\[
f(x_k) - f(R_{x_k}(s_k)) - \frac{\eta}{2} (\theta^k(0) - \theta^k(s_k))
\]
\[
\geq -\langle \nabla f(x_k), s_k \rangle - \frac{L}{2} \|s_k\|^2 - \frac{\eta}{2} (\theta^k(0) - \theta^k(s_k)) \quad \text{(by (3.20) in Lemma 4)}
\]
\[
\geq -\langle \nabla f(x_k), s_k \rangle - \frac{L}{2} \|s_k\|^2 + \eta \langle \nabla f(x_k), s_k \rangle \quad \text{(in a manner similar to (3.15))}
\]
\[
\geq (1 - \eta) \langle \nabla f(x_k), -s_k \rangle - \frac{L}{2} \|s_k\|^2
\]
\[
\geq (1 - \eta) \frac{\|\nabla f(x_k)\|^2}{\|J_k\|^2 + \lambda_k} - \frac{L}{2\lambda_k} \|\nabla f(x_k)\|^2 \quad \text{(by (3.4) and (3.5))}
\]
\[
\geq (1 - \eta) \frac{\|\nabla f(x_k)\|^2}{M^2 + \lambda_k} - \frac{L}{2\lambda_k^2} \|\nabla f(x_k)\|^2 \quad \text{(by Assumption 1)}
\]
\[
= \left(1 - \eta \frac{L}{M^2 + \lambda_k} - \frac{L}{2\lambda_k^2}\right) \|\nabla f(x_k)\|^2.
\]
Therefore, if \( \frac{1-\eta}{M^2 + \lambda_k} - \frac{f}{2\lambda_k} \geq 0 \) holds, then the \( k \)-th iteration is successful. Lastly, since \( \frac{1-\eta}{M^2 + \lambda_k} - \frac{f}{2\lambda_k} \geq 0 \) is equivalent to \( (1-\eta)\lambda_k^2 - \frac{f}{2}\lambda_k - \frac{LM^2}{2} \geq 0 \), we conclude that if \( \lambda_k \geq \frac{\xi + \sqrt{\xi^2 + 2(1-\eta)LM^2}}{2(1-\eta)} = \kappa \), then the \( k \)-th iteration is successful.

Recall that the parameter \( \beta > 1 \) is set in Algorithm 1. Using Lemma 5, we can show that \( \{\mu_k\}_{0 \leq k \leq j} \) is bounded by

\[
\mu_{\max}(\epsilon) := \frac{\beta \kappa}{2\epsilon},
\]

namely, it holds that

\[
\max_{0 \leq k \leq j_r} \mu_k \leq \mu_{\max}(\epsilon). \tag{3.22}
\]

Indeed, \( \lambda_k(=\mu_k\|F(x_k)\|^2) \) is bounded by \( \beta \kappa \) because of Lemma 5 and thus,

\[
\mu_k \leq \frac{\beta \kappa}{\|F(x_k)\|^2} \leq \frac{\beta \kappa}{2\epsilon} = \mu_{\max}(\epsilon),
\]

where the second inequality follows from \( f(x_k) = \frac{1}{2}\|F(x_k)\|^2 \geq \epsilon \) for all \( k = 0, \ldots, j_r - 1 \).

**Remark 2** When Assumptions 1 and 2 hold and the global optimal value of (1.1) is positive, then \( \{\mu_k\} \) is bounded by \( \frac{\beta \kappa}{\sum_{j \in S_k} f(x_j)} \).

As the next step (2), we give an upper-bound of \( |S_k| \) specifically in the following lemma.

**Lemma 6** Under Assumptions 1 and 2, \( |S_k| \leq 2f(x_0)\frac{M^2 + \mu_{\max}(\epsilon)}{\eta}\|F(x_0)\|^2 - 2 \) holds.

**Proof** First, for all \( j \in \{0, 1, \ldots, |S_k| - 1\} \), the inequality (3.17) is obtained in a similar manner to Theorem 1. Then, \( \mu_k\|F(x_k)\|^2 \leq \mu_{\max}(\epsilon)\|F(x_0)\|^2 \) holds and thus, we have

\[
f(x_{k(j)}) - f(x_{k(j+1)}) \geq \frac{\eta}{2(M^2 + \mu_{\max}(\epsilon))\|F(x_0)\|^2}\sum_{j=0}^{\|S_k\| - 1}\|\text{grad}f(x_{k(j)})\|^2.
\]

By summing up the above inequality from \( j = 0 \) to \( |S_k| - 1 \) and noting \( f(x_{k(\|S_k\|)}) \geq 0 \), we obtain

\[
f(x_0) = f(x_{k(0)}) \geq \frac{\eta}{2(M^2 + \mu_{\max}(\epsilon))\|F(x_0)\|^2}\sum_{j=0}^{\|S_k\| - 1}\|\text{grad}f(x_{k(j)})\|^2 + f(x_{k(\|S_k\|)})
\]

\[
\geq \frac{\eta}{2(M^2 + \mu_{\max}(\epsilon))\|F(x_0)\|^2}\sum_{j=0}^{\|S_k\| - 1}\|\text{grad}f(x_{k(j)})\|^2.
\]

\[
\geq \frac{\eta \epsilon^2}{2(M^2 + \mu_{\max}(\epsilon))\|F(x_0)\|^2}\|S_k\|.
\]
where the last inequality follows from the assumption \( \| \nabla f(x_{k(j)}) \| \geq \epsilon \) for \( j = 0, 1, \ldots, |S_\epsilon| - 1 \). Consequently, we ensure
\[
|S_\epsilon| \leq 2f(x_0) \frac{M^2 + \mu_{\max}(\epsilon)\|F(x_0)\|^2}{\eta} \epsilon^{-2}.
\]

As the final step (3), we prove the following lemma.

**Lemma 7** Suppose that Assumptions 1 and 2 hold. Then, \( |U_\epsilon| \leq c_{\max}(\epsilon)|S_\epsilon| \) holds where \( c_{\max}(\epsilon) := \lceil \log_\beta \left( \frac{\kappa}{2\mu_{\min}} \epsilon^{-1} \right) \rceil \). Here, \( \mu_{\min} \) is the constant prefixed in Algorithm 1 and \( \lceil \cdot \rceil \) is the ceiling function.

**Proof** Recall \( \beta > 1 \). Since \( \|F(x_k)\|^2 \geq 2\epsilon \) holds for an arbitrarily chosen \( k < j_\epsilon \), we have
\[
\beta^{c_{\max}(\epsilon)} \mu_{\min} \|F(x_k)\|^2 \geq \beta^{\log_\beta \left( \frac{\kappa}{2\mu_{\min}} \epsilon^{-1} \right)} \mu_{\min} \|F(x_k)\|^2 = \kappa \frac{\|F(x_k)\|^2}{2\epsilon} \geq \kappa.
\]
Hence, if the \( k \)-th iteration is right after consecutive \( c_{\max}(\epsilon) \) unsuccessful iterations, the assumptions of Lemma 5 are fulfilled because \( \lambda_k \geq \beta^{c_{\max}(\epsilon)} \mu_{\min} \).

Therefore, the \( k \)-th iteration is successful. This implies that the number of consecutive unsuccessful iterations is at most \( c_{\max}(\epsilon) \).

Now we can upper-bound \( |U_\epsilon| \) by \( c_{\max}(\epsilon)|S_\epsilon| \), because there occur alternately at most \( c_{\max}(\epsilon) \) unsuccessful iterations and one successful iteration until the number of iterations reaches \( j_\epsilon \).

Finally, we obtain the following result about the iteration complexity of Algorithm 1.

**Theorem 2** Under Assumptions 1 and 2, the iteration complexity of Algorithm 1 to find a solution satisfying \( \| \nabla f(x) \| < \epsilon \) or \( f(x) < \epsilon \) is bounded by \( O \left( \log \left( \epsilon^{-1} \right) \epsilon^{-3} \right) \).

**Proof** The number \( j_\epsilon \) is bounded from above as follows:
\[
\begin{align*}
j_\epsilon &= |S_\epsilon| + |U_\epsilon| \\
&\leq (1 + c_{\max}(\epsilon))|S_\epsilon| \\
&\leq (1 + c_{\max}(\epsilon)) \left( 2f(x_0) \frac{M^2 + \mu_{\max}(\epsilon)\|F(x_0)\|^2}{\eta} \epsilon^{-2} \right). \quad (3.23)
\end{align*}
\]

From (3.23) and the definitions of \( \mu_{\max}(\epsilon) \) and \( c_{\max}(\epsilon) \), the assertion follows. □
4 Analysis on local convergence

In this section, we show the local convergence properties of Algorithm 1 by dividing it into (1) zero-residual and (2) nonzero-residual cases. First, we study the local convergence behavior of the algorithm around a zero-residual stationary point, namely, \( x^* \) such that \( f(x^*) = 0 \), implying that \( x^* \) is a stationary point since it is optimal. Second, we analyze the behavior around a solution \( x^* \) which is a stationary point but \( f(x^*) \neq 0 \).

4.1 Notations for local convergence analysis

We first introduce additional notations. Let \( X^* \) denote the set of stationary points with the same residual \( f^* := f(x^*) \) as \( x^* \), i.e.,
\[
X^* := \{ x \in M \mid \text{grad} f(x) = 0, \ f(x) = f^* \}.
\]
Given \( x \in M \), we define the distance between \( x \) and \( X^* \) as
\[
\text{Dist}(x, X^*) := \min_{\hat{x} \in X^*} \text{dist}(x, \hat{x}).
\]
Moreover, we write \( \bar{x} \) to denote a point which is the closest to \( x \) in \( X^* \), that is,
\[
\bar{x} \in \arg \min_{\hat{x} \in X^*} \text{dist}(x, \hat{x}).
\]
Hereinafter, we often use \( \overline{\mathcal{B}} \) defined by setting \( x := x_k \) above. Let \( B(x, b) \subset M \) be the ball with radius \( b \) centered at \( x \), i.e., \( B(x, b) := \{ y \in M \mid \text{dist}(x, y) \leq b \} \). Note that the Jacobian matrix \( J \) is ensured to be bounded over \( B(x^*, b) \) without any specific assumptions. This is due to the \( C^1 \) property of \( F \) and the compactness of \( B(x^*, b) \). Let \( K \) denote the upper bound of the operator norm of \( J \). Namely,
\[
\| J(x) \| \leq K
\]
holds for all \( x \in B(x^*, b) \).

4.2 Basic assumptions and lemmas

In this subsection, we give common assumptions and lemmas, which are used throughout the analysis for zero- and nonzero-residual cases.

From the inverse function theorem, there exists an open set \( U \) of \( T_x \mathcal{M} \) containing \( 0_x \) such that \( R_x : U \to R_x(U) (\subseteq \mathcal{M}) \) is a diffeomorphism. Let \( R^{-1}_x : R_x(U) \to U \) be the inverse function.

**Assumption 3** The stationary point \( x^* \in X^* \) satisfies the following conditions:
(a) There exist $b \in (0, \infty)$ and $c_1 \in (0, \infty)$ such that $\|J(y)R_y^{-1}(x) - (F(x) - F(y))\| \leq c_1\|R_y^{-1}(x)\|^2$ holds for all $x, y \in B(x^*, b)$.

(b) There exists $c_2 > 0$ such that $c_2\|R_x^{-1}(\bar{x})\| \leq \|F(x) - F(\bar{x})\|$ holds for all $x \in B(x^*, b)$. In particular, in the zero-residual case, i.e., $F(\bar{x}) = 0$, the inequality is reduced to $c_2\|R_x^{-1}(\bar{x})\| \leq \|F(x)\|$.

(c) $\{\mu_k\}$ is upper-bounded by some positive constant, say $\mu_{\text{max}}$.

The first and second assumptions are often made in the local convergence analysis for the Euclidean LM method. Indeed, they correspond to \cite{46} Assumption 2.1 (a) and (b), respectively. In particular, the second one is often referred to as the local error-bound condition in many articles regarding the Euclidean LM following \cite{46}. This condition is weaker than the injectiveness of the Jacobian matrix supposed in the local analysis for the RGN \cite{11}.

By taking the constant $b$ in the above assumption to be sufficiently small, the following relation (4.2) is ensured under the above assumptions.

\[ B(x^*, b) \subset R(B^{T_x^*M}(\text{Inj}^R(x^*))), \tag{4.2} \]

where $\text{Inj}^R(x^*)$ denotes the injectivity radius of $R$ at $x^*$ defined formally as follows:

**Definition 3** The injectivity radius of $R$ at $x \in M$, denoted by $\text{Inj}^R(x)$, is the supremum over radii $r > 0$ such that $R(x)$ is a diffeomorphism on the open ball

\[ B^{T_x^*M}(r) := \{v \in T_xM \mid \|v\|_x < r\}. \]

Hereinafter, we additionally suppose (4.2) holds.

Since any bounded and closed set is compact in a complete Riemannian manifold, so is $B(x^*, b)$. Hence, there exist the minimum and maximum eigenvalues of the matrix of Riemannian metric $\langle \cdot, \cdot \rangle$ on $B(x^*, b)$, denoted by $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$, respectively. Define the following constant $c$ in terms of $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$:

\[ c := \sqrt{\frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}} \geq 1. \]

**Lemma 8** The following holds:

\[ c^{-1}\text{dist}(x, y) \leq \|R_x^{-1}(y)\|_x \leq c\text{dist}(x, y) \text{ for all } x, y \in B(x^*, b). \tag{4.3} \]

**Proof** Note that $R_x(U)$ is an open neighborhood of $x$ and $U$ can be identified with an open set of $\mathbb{R}^d$. Therefore, for $x \in M$, $(R_x(U), R_x^{-1})$ forms a coordinate neighborhood around $x$. Hence, we have

\[ R_x^{-1}(y) = R_x^{-1}(y) \tag{4.4} \]

for $y \in B(x^*, b)$. 

Choose $v \in T_x \mathcal{M}$ with $x \in B(x^*, b)$ arbitrarily. By the definition of $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$, we have

$$\lambda_{\text{min}} \| \hat{v} \|^2 \leq \langle v, v \rangle_x \leq \lambda_{\text{max}} \| \hat{v} \|^2,$$

where the norm in the left and right sides stands for the Euclidean norm. This yields

$$\frac{\| v \|_x}{\sqrt{\lambda_{\text{max}}}} \leq \| \hat{v} \| \leq \frac{\| v \|_x}{\sqrt{\lambda_{\text{min}}}}.$$ 

Substituting $v = R^{-1}_x(y)$ into the above and noting (4.4), we obtain

$$\| R^{-1}_x(y) \|_x \leq \| R^{-1}_x(y) \| \leq \| R^{-1}_x(y) \|_x \sqrt{\lambda_{\text{min}}}. \tag{4.5}$$

Given the formulation of coordinate neighborhood, the following holds:

$$\| R^{-1}_x(y) \| = \| R^{-1}_x(x) - R^{-1}_x(y) \| = \| \hat{x} - \hat{y} \|. \tag{4.6}$$

Using the inequalities $\sqrt{\lambda_{\text{min}}} \| \hat{x} - \hat{y} \| \leq \text{dist}(x, y) \leq \sqrt{\lambda_{\text{max}}} \| \hat{x} - \hat{y} \|$ in [20] for the current local coordinate system, we obtain

$$\sqrt{\lambda_{\text{min}}} \| R^{-1}_x(y) \| \leq \text{dist}(x, y) \leq \sqrt{\lambda_{\text{max}}} \| R^{-1}_x(y) \| \tag{4.7},$$

where (4.6) is used in the place of $\| \hat{x} - \hat{y} \|$. Noting that

$$\begin{cases} \sqrt{\lambda_{\text{min}}} \| R^{-1}_x(y) \|_x \leq \lambda_{\text{min}} \| R^{-1}_x(y) \| \\ \sqrt{\lambda_{\text{max}}} \| R^{-1}_x(y) \|_x \leq \lambda_{\text{max}} \| R^{-1}_x(y) \|_x \end{cases}$$

holds by (4.5), we find that (4.7) leads to

$$\sqrt{\lambda_{\text{min}}} \| R^{-1}_x(y) \|_x \leq \text{dist}(x, y) \leq \sqrt{\lambda_{\text{max}}} \| R^{-1}_x(y) \|_x,$$

equivalently,

$$\sqrt{\lambda_{\text{min}}} \text{dist}(x, y) \leq \| R^{-1}_x(y) \|_x \leq \sqrt{\lambda_{\text{max}}} \text{dist}(x, y).$$

□

**Lemma 9** Suppose that Assumption [3(a)] holds. Then, there exists some $L > 0$ such that $\| F(x) - F(y) \| \leq L \| R^{-1}_x(y) \|$ holds for all $x, y \in B(x^*, b)$.
Proof For \(x, y \in B(x^*, b)\), we have
\[
\|F(x) - F(y)\| \leq \|(F(x) - F(y)) - J(y)R_y^{-1}(x)\| + \|J(y)R_y^{-1}(x)\|
\]
\[
\leq c_1\|R_y^{-1}(x)\|^2 + \sum_{i=1}^{m} \|\text{grad} F_i(y)\|_y^2 \|R_y^{-1}(x)\|
\]
\[
= \left( c_1\|R_y^{-1}(x)\| + \sum_{i=1}^{m} \|\text{grad} F_i(y)\|_y^2 \right) \|R_y^{-1}(x)\|
\]
\[
\leq 2c_1 cb + \sum_{i=1}^{m} \|\text{grad} F_i(y)\|_y^2 \|R_y^{-1}(x)\|
\]
where Assumption 3 (a) is applied in the second inequality and the final one follows from \(\|R_y^{-1}(x)\| \leq c \text{ dist}(x, y) \leq 2cb\).

Here, \(\text{grad} F_i\) \((1 \leq i \leq m)\) are continuous because \(F\) is a \(C^1\) function. Moreover, the norm defined by the Riemannian metric is continuous. Consequently, \(\sum_{i=1}^{m} \|\text{grad} F_i(y)\|_y^2\) is a continuous function on the compact set \(B(x^*, b)\) and hence it attains the maximum value on \(B(x^*, b)\).

In terms of
\[
L := 2c_1 cb + \max_{z \in B(x^*, b)} \sum_{i=1}^{m} \|\text{grad} g_i(z)\|_z^2 > 0,
\]
the following inequality is established:
\[
\|F(x) - F(y)\| \leq L \|R_y^{-1}(x)\| \text{ for all } x, y \in B(x^*, b).
\]

\[\square\]

4.3 Quadratic convergence for zero-residual cases

In this subsection, we consider the zero-residual case, namely, \(f^* := f(x^*) = 0\) for the stationary point \(x^*\). We suppose that flag\(^nz\) = false in Algorithm 1.

Lemma 10 Suppose \(x_k \in B(x^*, \frac{b}{2})\) with some \(k\). Under Assumption 3, we have
\[
\|s_k\| \leq c_3 \|R_x^{-1}(x_k)\|, \quad \text{ (4.8)}
\]
\[
\|J_k s_k + F(x_k)\| \leq c_4 \|R_x^{-1}(x_k)\|^2, \quad \text{ (4.9)}
\]
where
\[
c_3 := \sqrt{\frac{c_1^2 + c_2^2 \mu_{\text{min}}}{c_2 \mu_{\text{min}}}}, \quad c_4 := \sqrt{c_1^2 + L^2 \mu_{\text{max}}^2}.
\]
Proof It follows that $\theta^k(s_k) \leq \theta^k(R_{x_k}^{-1}(\overline{x}))$ from (2.4). Moreover, $\lambda_k \|s_k\|^2 \leq \theta^k(s_k)$ holds from (2.1). Therefore, we have

$$\|s_k\|^2 \leq \frac{1}{\lambda_k} \theta^k(R_{x_k}^{-1}(\overline{x})) = \frac{1}{\lambda_k} \left( \|F(x_k) + J_k R_{x_k}^{-1}(\overline{x})\|^2 + \lambda_k \|R_{x_k}^{-1}(\overline{x})\|^2 \right).$$

(4.10)

Next, we can ensure $\overline{x} \in B(x^*, b)$ under $x_k \in B(x^*, \frac{b}{2})$ as follows:

$$\text{dist}(\overline{x}, x^*) \leq \|F(x_k) - F(x^*)\| + \|x_k - x^*\| \leq 2 \|x_k - x^*\| \leq b.$$

Thus, applying Assumption 3 and noting $F(\overline{x}) = 0$ from $\overline{x} \in X^*$, we obtain

$$\|F(x_k) + J_k R_{x_k}^{-1}(\overline{x})\|^2 = \|J_k R_{x_k}^{-1}(\overline{x}) - (F(\overline{x}) + F(x_k))\|^2 \leq c_2^2 \|R_{x_k}^{-1}(\overline{x})\|^4,$$

(4.11)

and moreover, by $c_2 \|R_{x_k}^{-1}(\overline{x})\| \leq \|F(x_k)\|$ together with $\lambda_k = \mu_k \|F(x_k)\|^2$ and $\mu_k \geq \mu_{\min}$, we gain

$$c_2^2 \|R_{x_k}^{-1}(\overline{x})\|^4 \leq \frac{\lambda_k}{\mu_{\min}}.$$

Combining these relations with (1.10) yields

$$\|s_k\|^2 \leq \left( \frac{c_2^2 \|R_{x_k}^{-1}(\overline{x})\|^2}{\lambda_k} + 1 \right) \|R_{x_k}^{-1}(\overline{x})\|^2 \leq \frac{c_2^2 \|R_{x_k}^{-1}(\overline{x})\|^2}{\lambda_k c_2^2 \mu_{\min}} + 1 \|R_{x_k}^{-1}(\overline{x})\|^2 \leq c_2^2 \|R_{x_k}^{-1}(\overline{x})\|^2.$$

Therefore, $\|s_k\| \leq c_3 \|R_{x_k}^{-1}(\overline{x})\|$ has been proved.

Next, we show that $\|J_k s_k + F(x_k)\| \leq c_4 \|R_{x_k}^{-1}(\overline{x})\|^2$.

$$\|J_k s_k + F(x_k)\|^2 \leq \theta^k(s_k) \text{ (by (2.1))} \leq \theta^k(R_{x_k}^{-1}(\overline{x})) \text{ (by (2.4))} \leq c_2^2 \|R_{x_k}^{-1}(\overline{x})\|^4 + \lambda_k \|R_{x_k}^{-1}(\overline{x})\|^2,$$

(4.12)

where the last inequality follows from (2.1) and (4.11). Furthermore, from Lemma 9, it follows that $\sqrt{\lambda_k} = \sqrt{\mu_k \|F(x_k)\|} = \sqrt{\mu_k \|F(\overline{x}) - F(x_k)\|} \leq L \sqrt{\mu_k \|R_{x_k}^{-1}(\overline{x})\|}$. Hence, by Assumption 3(c),

$$\lambda_k \leq L^2 \mu_k \|R_{x_k}^{-1}(\overline{x})\|^2 \leq L^2 \mu_{\max} \|R_{x_k}^{-1}(\overline{x})\|^2 \leq \mu_{\min} \|R_{x_k}^{-1}(\overline{x})\|^2 \leq L^2 \mu_{\max} \|R_{x_k}^{-1}(\overline{x})\|^2$$

(4.13)

Applying (4.13) to (4.12), we conclude $\|J_k s_k + F(x_k)\|^2 \leq (c_2^2 + L^2 \|R_{x_k}^{-1}(\overline{x})\|)^4 \leq (c_2^2 + L^2 \|R_{x_k}^{-1}(\overline{x})\|)^4$, which is equivalent to

$$\|J_k s_k + F(x_k)\| \leq c_4 \|R_{x_k}^{-1}(\overline{x})\|^2.$$

The proof is complete. □
Let
\[ c^* := -cc_2c_4 + \sqrt{c^2c_2^2c_4^2 + 4c^2\mu_{\max}^2c_2^2c_4^2L^2} \]
\[ \frac{2c^2\mu_{\max}^2c_2^2L^2}{2c^2\mu_{\max}^2c_2^2L^2}. \tag{4.14} \]
Note that \( c^* \) is a strictly positive constant.

**Lemma 11** Suppose that Assumption 3 holds. Moreover, assume that \( b \) satisfies \( b \leq \frac{2cc_2}{c_4} \). Then, there exists some \( r^* > 0 \) such that if \( \{x_k\} \) satisfies \( x_k \in B(x^*, \min\{\frac{b}{2}, r^*\}) \) and \( \text{dist}(x_k, x^*) < c^* \) for all \( k \in \{0, 1, 2, \ldots\} \), then all iterations are successful.

**Proof** Let \( x_k \in B(x^*, \frac{b}{2}) \). From the definition (2.9), we have
\[
1 - \rho_k = \frac{1}{2} \theta^k(0) - f(x_k) + f(R_k(s_k)) - \frac{1}{2} \theta^k(s_k) \\
= \frac{f(R_k(s_k)) - \frac{1}{2} \theta^k(s_k)}{\frac{1}{2} \theta^k(0) - \theta^k(s_k)},
\tag{4.15}
\]
where the second equality follows from \( \frac{1}{2} \theta^k(0) = f(x_k) \). Before evaluating the denominator in (4.15), we first show that \( \|F(x_k)\| - \|F(x_k) + J_k s_k\| \) is ensured to be positive under the assumption that \( b \leq \frac{2cc_2}{c_4} \). This can be verified as follows:
\[
\|F(x_k)\| - \|F(x_k) + J_k s_k\| \\
\geq cc_2\|R_{x_k}(\overline{\tau})\| - \|F(x_k) + J_k s_k\| \quad \text{(by Assumption 3(b))} \\
\geq \|R_{x_k}(\overline{\tau})\|/2 - cc_4\|R_{x_k}(\overline{\tau})\|/2 \quad \text{(by (4.9) in Lemma 10)} \\
\geq \|R_{x_k}(\overline{\tau})\|/2 - cc_4\|R_{x_k}(\overline{\tau})\|/2 \quad \text{(by Lemma 8)} \\
\geq \|R_{x_k}(\overline{\tau})\|/2 \quad \text{(by the definition of \( \overline{\tau} \))} \\
\geq \|R_{x_k}(\overline{\tau})\|/2 \quad \text{(by the assumption } x_k \in B(x^*, \min\{\frac{b}{2}, r^*\})) \\
\geq 0. \quad \text{(by the assumption } b \leq \frac{2cc_2}{c_4}) \tag{4.16}
\]

Then, by Lemma 10, the denominator in (4.15) is evaluated as
\[
\begin{aligned}
&\frac{1}{2} \theta^k(0) - \theta^k(s_k) \\
&= \frac{1}{2} \|F(x_k)\|^2 - \frac{1}{2} \|F(x_k) + J_k s_k\|^2 - \frac{\lambda_k}{2} \|s_k\|^2 \\
&= \frac{1}{2} \left( \|F(x_k)\| + \|F(x_k) + J_k s_k\| \right) \left( \|F(x_k)\| - \|F(x_k) + J_k s_k\| \right) - \frac{\mu_k}{2} \|F(x_k)\|^2 \|s_k\|^2 \\
&\geq \frac{1}{2} \left( \|F(x_k)\| \right) \left( \|F(x_k)\| - \|F(x_k) + J_k s_k\| \right) - \frac{\mu_k}{2} \|F(x_k)\|^2 \|s_k\|^2 \text{(by (4.16))} \\
&\geq \frac{1}{2} cc_2 \|R_{x_k}(\overline{\tau})\| \left( \|F(x_k)\| - \|F(x_k) + J_k s_k\| \right) - \frac{\mu_k}{2} \|F(x_k)\|^2 \|s_k\|^2 \text{(by Assumption 3(b),(c), Lemma 9)} \\
&\geq \frac{1}{2} cc_2 \|R_{x_k}(\overline{\tau})\|^2 \left( cc_2 \|R_{x_k}(\overline{\tau})\| \right) - \frac{\mu_k}{2} cc_4 \|R_{x_k}(\overline{\tau})\|^2 \text{(by Assumption 3(b), Lemma 10)} \tag{4.17}
\end{aligned}
\]
Note that
\[ \|F(x_k)\| \leq \|J_{k-1}s_{k-1} - (F(x_k) - F(x_{k-1}))\| + \|J_{k-1}s_{k-1} + F(x_{k-1})\| \]
\[ \leq c_1\|s_{k-1}\|^2 + c_4\|R_{x_{k-1}}^{-1}(J_{k-1})\|^2 \]
\[ \leq (c_1c_3^2 + c_4)\|R_{x_{k-1}}^{-1}(J_{k-1})\|^2. \] (by Assumption 3(a) and 4.9)

The absolute value of the numerator in (4.15) is bounded as
\[
\left| f(R_{x_k}(s_k)) - \theta_k(s_k) \right| \leq \frac{1}{2} \left( \|F(R_{x_k}(s_k))\| + \|F(x_k) + J_k s_k\| \right) \left( \|F(R_{x_k}(s_k))\| - \|F(x_k) + J_k s_k\| \right) - \mu_k\|F(x_k)\|^2\|s_k\|^2
\leq \frac{1}{2} \left( \|F(R_{x_k}(s_k))\| + \|F(x_k)\| + \|J_k s_k\| \right) \|J_k s_k - (F(R_{x_k}(s_k)) - F(x_k))\| + \frac{1}{2}\mu_k\|F(x_k)\|^2\|s_k\|^2
\leq \frac{1}{2} \left( \|F(R_{x_k}(s_k))\| + L\|R_{x_k}^{-1}(\overline{xT})\| + c_1\|R_{x_k}^{-1}(\overline{xT})\| \right) \|J_k s_k - (F(R_{x_k}(s_k)) - F(x_k))\| + \frac{\mu_{\text{max}}^2}{2}L^2c_3^2\|R_{x_k}^{-1}(\overline{xT})\|^4
\leq \frac{c_1c_3^2(L + c_3K)}{2}\|R_{x_k}^{-1}(\overline{xT})\|^3 + \frac{c_3^2}{2} (c_1c_3^2 + c_4) + \mu_{\text{max}}^2L^2\|R_{x_k}^{-1}(\overline{xT})\|^4.
\]

Using (4.17) and (4.19) for (4.15), we have
\[ |1 - \rho_k| \leq \frac{c_1c_3^2(L + c_3K)}{2}\|R_{x_k}^{-1}(\overline{xT})\|^3 + \frac{c_3^2}{2} (c_1c_3^2 + c_4) + \mu_{\text{max}}^2L^2\|R_{x_k}^{-1}(\overline{xT})\|^2
\leq \frac{c_1c_3^2(L + c_3K)\text{Dist}(x_k, X^*) + c_3^2c_3^2 (c_1c_3^2 + c_4) + \mu_{\text{max}}^2L^2c_3^2\text{Dist}(x_k, X^*)^2}{c_2 (c_2 - cc_4\text{Dist}(x_k, X^*)) - \mu_{\text{max}}^2c_3^2L^2\text{Dist}(x_k, X^*)^2}
\leq \frac{c_1c_3^2(L + c_3K)\text{dist}(x_k, x^*) + c_3^2c_3^2 (c_1c_3^2 + c_4) + \mu_{\text{max}}^2L^2c_3^2\text{dist}(x_k, x^*)^2}{c_2 (c_2 - cc_4\text{dist}(x_k, x^*)) - \mu_{\text{max}}^2c_3^2L^2\text{dist}(x_k, x^*)^2},
\]
where the second inequality follows from \(\|R_{x_k}^{-1}(\overline{xT})\| \leq c\text{Dist}(x_k, X^*)\) by Lemma 8
and the last one follows from \(\text{Dist}(x_k, X^*) \leq \text{dist}(x_k, x^*)\) by their definitions.
Note that the denominator of (4.20) is positive because of the assumption \( \text{dist}(x_k, x^*) < c^* \), where \( c^* \) is defined by (4.14). From (4.20), it follows that 
\[ |1 - \rho_k| \to 0 \] as \( \text{dist}(x_k, x^*) \to 0 \), which implies that there exists some \( r^* > 0 \) such that if \( x_k \in B(x^*, \min \{ \frac{b}{2}, r^* \}) \), then \( \rho_k \geq \eta \) holds for the given \( \eta \in (0, 1) \).

Therefore, if \( \{ x_k \} \) satisfies \( x_k \in B(x^*, \min \{ \frac{b}{2}, r^* \}) \) and \( \text{dist}(x_k, x^*) < c^* \) for all \( k \in \{0, 1, 2, \ldots \} \), then all iterations are successful. The proof is complete. \( \square \)

Hereinafter, we assume \( b \leq \frac{2c_2}{c_4}, \frac{b}{2} \leq r^* \), and \( \frac{b}{2} < c^* \). This condition is fulfilled by re-taking a sufficiently small \( b \) if necessary.

**Lemma 12** Suppose that Assumption [3] holds. If \( x_k, x_{k-1} \in B(x^*, \frac{b}{2}) \) hold with some \( k \geq 1 \), then \( \text{Dist}(x_k, X^*) \leq c_5 \text{Dist}(x_{k-1}, X^*)^2 \) holds, where \( c_5 := \frac{c^3(c_1c_3^2 + c_4)}{c_2} \).

**Proof** First of all, note that the \((k-1)\)-th iteration is successful by Lemma 11 and \( x_{k-1} \in B(x^*, \frac{b}{2}) \). It follows that

\[
\frac{c_2}{c} \text{Dist}(x_k, X^*) \\
\leq c_2 \| R_{x_k}^{-1}(x_k) \| \\
\leq \| F(x_k) \| \\
\leq (c_1c_3^2 + c_4) \| R_{x_{k-1}}^{-1}(x_{k-1}) \|^2 \\
\leq c^2(c_1c_3^2 + c_4) \text{Dist}(x_{k-1}, X^*)^2.
\]

Therefore, we conclude

\[
\text{Dist}(x_k, X^*) \leq c_5 \text{Dist}(x_{k-1}, X^*)^2.
\]

\( \square \)

In order to prove Lemma 15, we will show \( \text{dist}(x_k, x_{k+1}) \leq c \| s_k \| \) for each \( k \). This inequality trivially holds true in the Euclidean case. For the verification of the inequality in the present manifold setting, we need the following lemma concerning \( \text{Inj}^R \) that is defined in Definition 3:

**Lemma 13** There exists some \( b^* > 0 \) such that if \( 0 < b \leq b^* \), then

\[
\inf_{x \in B(x^*, \frac{b}{2})} \text{Inj}^R(x) \geq \frac{bc_3}{2}
\]

holds.

**Proof** First, we show that there exists some \( b^* > 0 \) such that

\[
\inf_{x \in B(x^*, \frac{b}{2})} \text{Inj}^R(x) \geq \frac{b^*c_3}{2}
\]
holds. To derive a contradiction, suppose that
\[ \inf_{x \in B(x^*, \frac{b_n}{2})} \text{Inj}^R(x) < \frac{b_n c c_3}{2} \tag{4.22} \]
holds for all \( b > 0 \). Let \( \{b_n\} \) be a monotonically decreasing sequence satisfying \( b_n \downarrow 0 \). By the assumption,
\[ \inf_{x \in B(x^*, \frac{b_n}{2})} \text{Inj}^R(x) < \frac{b_n c c_3}{2} \tag{4.23} \]
holds for all \( n \in \{0, 1, 2, \ldots\} \). By [8, Corollary 10.24], \( \text{Inj}^R : \mathcal{M} \to (0, \infty) \) is continuous. Combining this with the compactness of \( B(x^*, \frac{b_n}{2}) \), we have
\[ \inf_{x \in B(x^*, \frac{b_n}{2})} \text{Inj}^R(x) = \min_{x \in B(x^*, \frac{b_n}{2})} \text{Inj}^R(x) > 0. \tag{4.24} \]
for all \( n \in \{0, 1, 2, \ldots\} \). Since the left-hand side of (4.23) is monotonically non-decreasing with respect to \( n \), we have
\[ 0 < \inf_{x \in B(x^*, \frac{b_n}{2})} \text{Inj}^R(x) \quad \text{(by (4.24))} \]
\[ \leq \inf_{x \in B(x^*, \frac{b_n}{2})} \text{Inj}^R(x) \]
\[ < \frac{b_n c c_3}{2} \tag{4.25} \]
Taking \( n \to \infty \) in (4.25) leads to a contradiction as desired. Therefore, there exists some \( b^* > 0 \) such that
\[ \inf_{x \in B(x^*, \frac{b^*}{2})} \text{Inj}^R(x) \geq \frac{b^* c c_3}{2} \tag{4.26} \]
holds. Moreover, for all \( 0 < b \leq b^* \), we have
\[ \inf_{x \in B(x^*, \frac{b}{2})} \text{Inj}^R(x) \geq \inf_{x \in B(x^*, \frac{b^*}{2})} \text{Inj}^R(x) \geq \frac{b c c_3}{2} \geq \frac{b c c_3}{2}, \]
which is the desired assertion. \( \square \)

Hereinafter, we assume that \( b \) satisfies \( b \leq b^* \) where \( b^* \) is the constant in Lemma 13. Using (4.21), we can prove that \( \text{dist}(x_k, x_{k+1}) \leq c\|s_k\| \) holds for each \( k \).

**Lemma 14** Under \( x_k \in B(x^*, \frac{b}{2}) \), \( \text{dist}(x_k, x_{k+1}) \leq c\|s_k\| \) holds.
Proof Since it holds that
\[ \|s_k\|_{x_k} \leq c_3 \|R_{x_k}^{-1}(x_k)\| \quad \text{(by (4.8))} \]
\[ \leq c_3 \|R_{x_k}(x_k)\| \quad \text{(by Lemma 8)} \]
\[ \leq c_3 \text{dist}(x_k, x^*) \]
\[ \leq \frac{b c_3}{2} \quad \text{(by } x_k \in B(x^*, \frac{b}{2}) \text{)} \]
\[ \leq \inf_{x \in B(x^*, \frac{b}{2})} \text{Inj}^R(x) \quad \text{(by (4.21) in Lemma 13)} \]
\[ \leq \text{Inj}^R(x_k) \quad \text{(by } x_k \in B(x^*, \frac{b}{2}) \text{)}, \]
we find that \( x_{k+1} = R_{x_k}(s_k) \in R(\text{Inj}^R(x_k)) \) and thus by (4.3), \( \text{dist}(x_k, x_{k+1}) \leq c\|s_k\| \) holds. The proof is complete.

Lemma 15 Suppose that Assumption 3 holds and let \( r := \min \left\{ \frac{b}{2 + 4c^2c_3}, \frac{1}{2c_3} \right\} \).

If \( x_0 \in B(x^*, r) \) and every iteration is successful, then \( x_k \in B(x^*, \frac{b}{2}) \) holds for all \( k \geq 0 \).

Proof When \( k = 0 \), \( x_0 \in B(x^*, \frac{b}{2}) \) clearly holds since \( r \leq \frac{b}{2} \) by definition. In what follows, we show the assertion for \( k \geq 1 \) by induction. We first show \( x_1 \in B(x^*, \frac{b}{2}) \). Note that
\[
\text{dist}(x_1, x^*) \\
\leq \text{dist}(x_0, x^*) + \text{dist}(x_0, x_1) \\
\leq \text{dist}(x_0, x^*) + c\|s_0\| \quad \text{(by Lemma 8 and } s_0 = R_{x_0}^{-1}(x_0).) \]
\[ \leq \text{dist}(x_0, x^*) + c^2c_3 \text{Dist}(x_0, x^*) \quad \text{(by (4.8) and Lemma 8)} \]
\[ \leq (1 + c^2c_3) \text{dist}(x_0, x^*) \quad \text{(by } \text{Dist}(x_0, x^*) \leq \text{dist}(x_0, x^*)) \quad (4.27) \]
which together with
\[
(1 + c^2c_3) \text{dist}(x_0, x^*) \leq (1 + 2c^2c_3)r \leq \frac{1 + 2c^2c_3b}{2 + 4c^2c_3} = \frac{b}{2}
\]
implies \( x_1 \in B(x^*, \frac{b}{2}) \).

Next, we prove that \( x_{k+1} \in B(x^*, \frac{b}{2}) \) for each \( k \geq 1 \) by supposing that \( x_l \in B(x^*, \frac{b}{2}) \) \( (l = 0, 1, \ldots, k) \) holds with some \( k \geq 1 \). By this assumption and Lemma 12, we have
\[
\text{Dist}(x_l, x^*) \leq c_3 \text{Dist}(x_{l-1}, x^*)^2 \leq \cdots \leq c_3^2 \text{Dist}(x_0, x^*)^2l. \quad (4.28)
\]
Moreover, as $c_5 \leq \frac{1}{2r}$ by the choice of $r$ and $x_0 \in B(x^*, r)$,

$$c_5^{l-1} \operatorname{Dist}(x_0, X^*)^{2^l} \leq \left( \frac{1}{2} \right)^{2^l-1} \left( \frac{r^{2^l}}{2^{2^l-1}} \right) = r \left( \frac{1}{2} \right)^{2^l-1},$$

which along with (4.28) implies

$$\operatorname{Dist}(x_l, X^*) \leq r \left( \frac{1}{2} \right)^{2^l-1} \quad (4.29)$$

for each $l = 0, 1, 2, \ldots, k$. Let us upper-bound $\operatorname{dist}(x_{k+1}, x^*)$ by the following two-steps: First, applying the triangle inequality and Lemma 14 to $\operatorname{dist}(x_{k+1}, x^*)$ successively, we have

$$\operatorname{dist}(x_{k+1}, x^*) \leq \operatorname{dist}(x_k, x^*) + \operatorname{dist}(x_k, x_{k+1}) \leq \operatorname{dist}(x_k, x^*) + c\|s_k\|_{x_k} \leq \operatorname{dist}(x_1, x^*) + c \sum_{l=1}^{k} \|s_l\|_{x_l}. \quad (4.30)$$

Second, (4.30) is further bounded as follows:

$$\begin{align*}
(4.30) & \leq (1 + c^2 c_3) r + c \sum_{l=1}^{k} \|s_l\|_{x_l} \\
& \leq (1 + c^2 c_3) r + c c_3 \sum_{l=1}^{k} R_{x_l}^{-1}(\mathcal{F}) \\
& \leq (1 + c^2 c_3) r + c^2 c_3 \sum_{l=1}^{k} \operatorname{Dist}(x_l, X^*) \\
& \leq (1 + c^2 c_3) r + c^2 c_3 r \sum_{l=1}^{k} \left( \frac{1}{2} \right)^{2^l-1} ,
\end{align*}$$

where the first inequality follows from (4.27) and the second one does from Lemma 10 with $x = x_l$ and $x_l \in B(x^*, \frac{b}{2})$. Moreover, the third and last ones are implied by Lemma 8 with $x = x_l \in B(x^*, \frac{b}{2})$ and (4.29), respectively.

Consequently, we have

$$\operatorname{dist}(x_{k+1}, x^*) \leq (1 + c^2 c_3) r + c^2 c_3 r \sum_{l=1}^{k} \left( \frac{1}{2} \right)^{2^l-1}. \quad (4.31)$$

Finally, by $2^l - 1 \geq l$ for $l \geq 1$,

$$\sum_{l=1}^{k} \left( \frac{1}{2} \right)^{2^l-1} \leq \frac{1}{2} + \left( \frac{1}{2} \right)^2 + \cdots + \left( \frac{1}{2} \right)^k \leq 1 - \left( \frac{1}{2} \right)^k \leq 1.$$
Hence, from this fact and (4.31), \( \text{dist}(x_{k+1}, x^*) \leq (1 + 2c^2c_3)r \leq \frac{b}{2} \) holds and thus \( x_{k+1} \in B(x^*, \frac{b}{2}) \). The proof is completed. \( \square \)

**Theorem 3** Suppose that Assumptions 3 holds and let \( r > 0 \) be the same as in Lemma 15. Moreover, assume \( x_0 \in B(x^*, r) \). Then, \( \{\text{Dist}(x_k, X^*)\} \) converges to 0 quadratically and furthermore, \( \{x_k\} \) converges to some \( \hat{x} \in B(x^*, \frac{b}{2}) \).

**Proof** The first assertion can be verified as follows. By the assumptions and Lemma 15, \( \{x_k\} \subseteq B(x^*, b) \) holds. Thus, we can repeatedly apply Lemma 12 and consequently, we conclude that \( \{x_k\} \) quadratically converges to 0. Next, we show the second claim. Since \( (\mathcal{M}, \langle \cdot, \cdot \rangle) \) is a complete Riemannian manifold and thus it is a complete metric space with respect to \( \text{dist}(\cdot, \cdot) \), it suffices to show that \( \{x_k\} \) is a Cauchy sequence.

For arbitrary \( m > n \), we have

\[
\text{dist}(x_m, x_n) \leq \sum_{l=n}^{m-1} \text{dist}(x_l, x_{l+1}) \\
\leq c \sum_{l=n}^{m-1} \|s_l\|_{x_l} \quad \text{(by \( \{x_k\} \subseteq B\left(x^*, \frac{b}{2}\right) \) and Lemma 8)} \\
\leq cc_3 \sum_{l=n}^{m-1} \|R^{-1}_x(x_l)\| \quad \text{(by (4.8))} \\
\leq c^2c_3 \sum_{l=n}^{m-1} \text{Dist}(x_l, X^*) \quad \text{(by Lemma 8)} \\
\leq c^2c_3r \sum_{l=n}^{m-1} \left(\frac{1}{2}\right)^{2^l-1} \quad \text{(by (4.29))}
\]

Using \( \sum_{l=n}^{m-1} \left(\frac{1}{2}\right)^{2^l-1} \leq \sum_{l=n}^{\infty} \left(\frac{1}{2}\right)^{2^l-1} = \frac{1}{4} \left(\frac{1}{2}\right)^{2n-3} \), we obtain

\[
\text{dist}(x_m, x_n) \leq c^2c_3r \left(\frac{1}{2}\right)^{2n-3}.
\]

This inequality indicates that \( \{x_k\} \) is a Cauchy sequence and consequently, the second claim has been proved. The proof is completed. \( \square \)

From Theorem 3 the RLM has a local quadratic convergence property for zero-residual cases. Next, we study the local behavior of our RLM when (4.1) is nonzero-residual.

### 4.4 Linear convergence for nonzero-residual cases

Recall the definitions of \( f^* \) and \( X^* \) in the beginning of subsection 4.1. In this subsection, we consider the nonzero-residual case, namely, \( f^* > 0 \).
Besides Assumption 3, we suppose that flag_{nx} = true in Algorithm 1 and further make the following assumptions on x^* ∈ X^*.

**Assumption 4** \(\nabla f\) is \(L_0\)-Lipschitz continuous on \(B(x^*, \frac{\beta}{2})\).

It is shown in Corollary 10.45 of \([8]\) that this assumption is satisfied when \(f\) is twice continuously differentiable on \(B(x^*, \frac{\beta}{2})\).

We remark that Assumption 3 (c) can be removed if the problem is globally nonzero-residual, namely, the global optimal value is larger than 0. Indeed, as discussed in Remark 2, by combining \(\|J(x)\| \leq K\) of \((4.1)\) with Assumption 4 and using an algorithmic parameter \(\beta > 1\) and \(\kappa\) defined by \((3.21)\) with \(L = L_0\) and \(M = K\), we derive an upper bound of \(\{\mu_k\}\) by

\[
\mu_{nx}^\ast := \frac{\beta K}{2 \min_{x \in M} f(x)}.
\]

We begin by introducing a lemma similar to Lemma 10.

**Lemma 16** Suppose \(x_k \in B(x^*, \frac{\beta}{2})\) with some \(k\). Under Assumptions 3 and 4

\[
\|s_k\| \leq \hat{c}_3 \|R_{x_k}^{-1}(\pi_k)\|, \tag{4.32}
\]

\[
\|J_k s_k + F(x_k)\| - \sqrt{2f^*} \leq \hat{c}_4 \|R_{x_k}^{-1}(\pi_k)\|^2 \tag{4.33}
\]

hold where \(f^* = f(\pi_k), \hat{c}_3 := \frac{c L_0}{2 \min f}, \) and \(\hat{c}_4 := \frac{\mu_{nx}^\ast b L}{2} + \sqrt{\mu_{nx}^\ast L + c_1 + \sqrt{2f^* \mu_{nx}^\ast}}\).

**Proof** We first prove \((4.33)\). It follows that \(\theta^k(s_k) \leq \theta^k(R_{x_k}^{-1}(\pi_k))\) from \((2.4)\). Moreover, \((2.1)\) immediately yields \(\|J_k s_k + F(x_k)\|^2 \leq \theta^k(s_k)\). Therefore, we have

\[
\|J_k s_k + F(x_k)\|^2 \leq \theta^k(s_k)
\]

\[
\leq \theta^k(R_{x_k}^{-1}(\pi_k))
\]

\[
= \|F(x_k) + J_k R_{x_k}^{-1}(\pi_k)\|^2 + \lambda_k \|R_{x_k}^{-1}(\pi_k)\|^2. \tag{4.34}
\]

By the same argument as in the proof for Lemma 10 we have \(\pi_k \in B(x^*, b)\). Hence, from Assumption \(3\) (i), we obtain

\[
\|F(x_k) + J_k R_{x_k}^{-1}(\pi_k)\|^2 \leq (\|J_k R_{x_k}^{-1}(\pi_k) - (F(\pi_k) - F(x_k))\| + \|F(\pi_k)\|)^2
\]

\[
\leq (c_1 \|R_{x_k}^{-1}(\pi_k)\|^2 + \sqrt{2f^*})^2. \tag{4.35}
\]

By combining \((4.35)\) with \((4.34)\), we get

\[
\|J_k s_k + F(x_k)\|^2 \leq (c_1 \|R_{x_k}^{-1}(\pi_k)\|^2 + \sqrt{2f^*})^2 + \lambda_k \|R_{x_k}^{-1}(\pi_k)\|^2. \tag{4.36}
\]

Furthermore, we have \(\sqrt{\frac{\lambda_k}{\mu_k}} = \|F(x_k)\| \leq \|F(x_k) - F(\pi_k)\| + \|F(\pi_k)\| \leq L \|R_{x_k}^{-1}(\pi_k)\| + \sqrt{2f^*}\) from Lemma 3. Consequently, with \(\mu_k \leq \mu_{nx}^\ast\), \(\lambda_k\) can be bounded from the above as follows:

\[
\lambda_k \leq \mu_{nx}^\ast \left\{ L^2 \|R_{x_k}^{-1}(\pi_k)\|^2 + 2\sqrt{2f^*}L \|R_{x_k}^{-1}(\pi_k)\| + 2f^* \right\}. \tag{4.37}
\]
Lemma 17 Suppose that Assumptions 3 and 4 hold. Moreover, if
\[ x \text{ and } \mu \]
Hereinafter we set \( \mu \). We need this property for establishing the local convergence as shown below.
\[ \square \]
Therefore, we have
\[ \|J_k s_k + F(x_k)\|^2 \]
Moreover, since \( \|R_{x_k}^{-1}(\pi_k)\| \leq c \text{ dist}(x_k, \pi_k) \leq c \text{ dist}(x_k, x^*) \leq \frac{b}{2} \) holds by Lemma 8 and the assumption \( x_k \in B(x^*, \frac{b}{2}) \), we have
\[ 2\sqrt{2f^* \mu_{\text{max}}^n L} ||R_{x_k}^{-1}(\pi_k)||^3 + (2\sqrt{2f^* c_1 + 2f^* \mu_{\text{max}}^n}) ||R_{x_k}^{-1}(\pi_k)||^2 + 2f^* \]
\[ \leq \frac{2f^* \mu_{\text{max}}^n bc L + 2\sqrt{2f^* c_1 + 2f^* \mu_{\text{max}}^n}) ||R_{x_k}^{-1}(\pi_k)||^2 + 2f^* \]
\[ \leq \left( \frac{\mu_{\text{max}}^n bc L}{2} + \sqrt{\mu_{\text{max}}^n^2 L + c_1 + \frac{2f^* \mu_{\text{max}}^n}{2}} \right) ||R_{x_k}^{-1}(\pi_k)||^2 + 2f^* \]
\[ = \left( \frac{c_4 ||R_{x_k}^{-1}(\pi_k)||^2 + 2f^* \right)^2. \]
Therefore, we have
\[ ||J_k s_k + F(x_k)|| - \sqrt{2f^*} \leq \hat{c}_4 \|R_{x_k}^{-1}(\pi_k)\|^2. \]
In turn, we show \( ||s_k|| \leq \hat{c}_3 ||R_{x_k}^{-1}(\pi_k)||. \)
\[ \|s_k\| \]
\[ \leq \frac{1}{\lambda_k} \|\text{grad} f(x_k)\| \]
(\text{by } (3.4))
\[ \leq \frac{L_0}{2\mu_{\text{min}} f^*} \text{Dist}(x_k, X^*) \]
(\text{by Assumption 4 and } \lambda_k = \mu_k ||F(x_k)||^2 \geq 2\mu_{\text{min}} f^*)
\[ \leq \frac{c L_0}{2\mu_{\text{min}} f^*} ||R_{x_k}^{-1}(\pi_k)|| = \hat{c}_3 ||R_{x_k}^{-1}(\pi_k)||. \]
(\text{by Lemma 8})
Hence, the proof is completed. \( \square \)

When we fix \( \mu_k \) as \( \mu_{\text{max}}^n \) for all \( k \in \{0, 1, 2, \ldots \} \), every iteration is successful. We need this property for establishing the local convergence as shown below. Hereinafter we set \( \mu_k = \mu_{\text{max}}^n \). Note that this is consistent with Algorithm 4 since \( \mu_k = \mu_{\text{max}}^n \) holds for all \( k \in \{0, 1, 2, \ldots \} \) by setting \( \mu_{\text{min}} = \mu_{\text{max}}^n \).

Lemma 17 Suppose that Assumptions 3 and 5 hold. Moreover, if \( f^* < \frac{c_2^2}{8c_1^2 \mu_{\text{min}}} \), and \( x_k, x_{k-1} \in B(x^*, \frac{b}{2}) \) hold, then it follows that
\[ \text{Dist}(x_k, X^*) \leq c_5 \text{Dist}(x_{k-1}, X^*), \]
where
\[ c_5 := \frac{c_4 (c_1 c_2^2 + \hat{c}_4) \left( 4c_1^2 c_2^2 + 2\sqrt{2f^*} \right)}{c_2^2 - 2\sqrt{2f^*} c_4 c_1}. \]
Then, from Assumption 3(a) and Lemma 8, we have

\[ c_2 \text{Dist}(x_k, X^*) \]
\[ \leq c_2 c_2 \| R_{x_k}^{-1}(\bar{x}_k) \| \quad \text{(by Lemma 8)} \]
\[ \leq c \| F(x_k) - F(\bar{x}_k) \| \quad \text{(by Assumption 3(b))}. \]  

Moreover, the following inequality holds:

\[ \| F(x_k) \| \]
\[ \leq \| J_{k-1}s_{k-1} - (F(x_k) - F(x_{k-1})) \| + \| J_{k-1}s_{k-1} + F(x_{k-1}) \| \]
\[ \leq c_1 \| s_{k-1} \|^2 + \hat{c}_4 \| R_{x_{k-1}}^{-1}(\bar{x}_{k-1}) \|^2 + \sqrt{2f^*} \quad \text{(by Assumption 3(a) and (4.33))} \]
\[ \leq (c_1 \hat{c}_3^2 + \hat{c}_4) \| R_{x_{k-1}}^{-1}(\bar{x}_{k-1}) \|^2 + \sqrt{2f^*} \quad \text{(by (4.32))} \]
\[ \leq c^2 (c_1 \hat{c}_3^2 + \hat{c}_4) \text{Dist}(x_{k-1}, X^*)^2 + \sqrt{2f^*} \quad \text{(by Lemma 8)} \]
\[ = A + \sqrt{2f^*}, \quad (4.41) \]

where \( A := c^2 (c_1 \hat{c}_3^2 + \hat{c}_4) \text{Dist}(x_{k-1}, X^*)^2 \).

Next, we evaluate \( \| F(x_k) - F(\bar{x}_k) \| \) by using (4.41). Then, we find

\[ \| F(x_k) - F(\bar{x}_k) \|^2 = \| F(x_k) \|^2 - 2F(\bar{x}_k)^T F(x_k) + 2f^* \]
\[ \leq f^2 + 2\sqrt{2f^*}A + 4f^* - 2F(\bar{x}_k)^T F(x_k). \]  

Now, we analyze \( 4f^* - 2F(\bar{x}_k)^T F(x_k) \) in (4.42). Let

\[ r_{x_k} := -J(\bar{x}_k)R_{\bar{x}_k}^{-1}(x_k) + (F(x_k) - F(\bar{x}_k)) \]

so as to satisfy

\[ F(x_k) = F(\bar{x}_k) + J(\bar{x}_k)R_{\bar{x}_k}^{-1}(x_k) + r_{x_k}. \]  

Then, from Assumption 3(a) and Lemma 8, we have

\[ \| r_{x_k} \| \leq c_1 \| R_{\bar{x}_k}^{-1}(x_k) \|^2 \leq c^2 c_1 \text{Dist}(x_k, X^*)^2, \]

which yields

\[ 4f^* - 2F(\bar{x}_k)^T F(x_k) \]
\[ = 2F(\bar{x}_k)^T F(x_k) - 2F(\bar{x}_k)^T F(x_k) \]
\[ = -2F(\bar{x}_k)^T (F(x_k) - F(\bar{x}_k)) \]
\[ = -2F(\bar{x}_k)^T J(\bar{x}_k)R_{\bar{x}_k}^{-1}(x_k) - 2F(\bar{x}_k)^T r_{x_k} \quad \text{(by (4.43))} \]
\[ = -2(J(\bar{x}_k))^* F(\bar{x}_k), R_{\bar{x}_k}^{-1}(x_k))_{\bar{x}_k} - 2F(\bar{x}_k)^T r_{x_k} \quad \text{(by (1.2))} \]
\[ = -2(\text{grad} f(\bar{x}_k), R_{\bar{x}_k}^{-1}(x_k))_{\bar{x}_k} - 2F(\bar{x}_k)^T r_{x_k} \]
\[ \leq 2\| F(\bar{x}_k) \| \| r_{x_k} \| \]
\[ \leq 2\sqrt{2f^*} c^2 c_1 \text{Dist}(x_k, X^*)^2. \]  

(4.45)
Lemma 18
Suppose that Assumptions 3, 4, and 5. The proof is completed.

Therefore, we obtain

\[ \| F(x_k) - F(x_*^k) \| \leq A^2 + 2\sqrt{2f^r}c_1 \text{ Dist}(x_k, X^*)^2. \quad (4.46) \]

Using (4.46) in the right-hand side of (4.40), we find

\[ c_2^2 \text{ Dist}(x_k, X^*)^2 \leq c^2 \left( A^2 + 2\sqrt{2f^r}c_1 \text{ Dist}(x_k, X^*)^2 \right), \]

implying

\[ (c_2^2 - 2\sqrt{2f^rc_1}) \text{ Dist}(x_k, X^*)^2 \leq c^2 (A + 2\sqrt{2f^r})A. \quad (4.47) \]

Using the definition of \( A \) together with \( \text{ Dist}(x_{k-1}, X^*) \leq \frac{b_1}{2}, \) (4.47) leads to

\[ (c_2^2 - 2\sqrt{2f^rc_1}) \text{ Dist}(x_k, X^*)^2 \leq c^4 (c_1 c_3^2 + \hat{c}_4) \left( \frac{b^2 c_2^2 c_3^2 + \hat{c}_4}{4} + 2\sqrt{2f^r} \right) \text{ Dist}(x_{k-1}, X^*)^2. \]

Since \( f^* < \frac{c_4^4}{2\sqrt{2f^r}} \) holds by the assumption, we have \( c_2^2 - 2\sqrt{2f^rc_1} > 0. \)

Therefore, we obtain

\[ \text{ Dist}(x_k, X^*) \leq c_0 \text{ Dist}(x_{k-1}, X^*). \]

The proof is completed. \( \square \)

Note that \( c, c_1 \) and \( c_2 \) are constants independent of \( f^* \) and hence the condition \( f^* < \frac{c_4^4}{2\sqrt{2f^r}} \) in Lemma 17 makes sense. Considering characteristics of these constants, \( f^* \) tends to be required to be relatively small so as to satisfy this condition. Moreover, even though \( c_5 < 1 \) is required to establish the linear convergence of Algorithm 1, we are not sure about how reasonable this condition is since \( c_5 \) depends on constants appearing in Assumption 3. Hereinafter, We assume, however, that \( c_5 < 1 \) is satisfied in addition to \( f^* < \frac{c_4^4}{2\sqrt{2f^r}} \) for our analysis.

Lemma 18
Suppose that Assumptions 3, 4, and \( c_5 < 1 \) hold and define

\[ r := \frac{b}{2} \left( 1 + \frac{c^2 c_5}{c_3} \right)^{-1}. \]

If \( x_0 \in B(x^*, r) \), then \( x_k \in B(x^*, \frac{b}{2}) \) holds for all \( k \in \{0,1,2,... \} \).

Proof
When \( k = 0 \), \( x_0 \in B(x^*, \frac{b}{2}) \) holds clearly since \( r \leq \frac{b}{4} \) by the definition.

For \( k \geq 1 \), we prove \( x_k \in B(x^*, \frac{b}{2}) \) by induction. First, we consider \( k = 1 \). Noting

\[ \text{ dist}(x_1, x^*) \]

\[ \leq \text{ dist}(x_0, x^*) + \text{ dist}(x_0, x_1) \]

\[ \leq r + c \| x_0 \| \quad \text{ (by } x_0 \in B(x^*, r), \text{ Lemma 14 \text{ and Lemma 8})} \]

\[ \leq r + c^2 \hat{c}_3 \text{ Dist}(x_0, X^*) \quad \text{ (by 4.32 \text{ and Lemma 8})} \]

\[ \leq r + c^2 \hat{c}_3 \text{ dist}(x_0, x^*) \leq (1 + c^2 \hat{c}_3) r. \quad (4.48) \]
and \((1+c^2c_2)r = \frac{1}{2}(1+c^2c_2)(1+c^2c_3)^{-1} \leq \frac{b}{2}\) hold, we find that \(x_1 \in B(x^*, \frac{b}{2})\).

In what follows, supposing \(x_l \in B(x^*, \frac{b}{2})\) \((l = 0, 1, \ldots, k)\) holds for some \(k \geq 1\), we show \(x_{k+1} \in B(x^*, \frac{b}{2})\). By these assumptions together with Lemma 17, we have

\[
\text{Dist}(x_l, X^*) \leq c_5 \text{Dist}(x_{l-1}, X^*) \leq \cdots \leq c_5^k \text{ Dist}(x_0, X^*) \leq r c_5^l \tag{4.49}
\]

for all \(1 \leq l \leq k\), which yields

\[
\begin{align*}
dist(x_{k+1}, x^*) & \leq dist(x_k, x^*) + dist(x_k, x_{k+1}) \\
& \leq dist(x_k, x^*) + c\|s_k\|_{x_k} \quad \text{(by Lemma 8)} \\
& \leq dist(x_{k-1}, x^*) + c\|s_{k-1}\|_{x_{k-1}} + c\|s_k\|_{x_k} \\
& \quad \vdots \\
& \leq dist(x_1, x^*) + c \sum_{l=1}^{k} \|s_l\|_{x_l} \\
& \leq (1+c^2c_3)r + c^2c_3 \sum_{l=1}^{k} \text{Dist}(x_l, X^*) \quad \text{(by (4.48), (4.32), and Lemma 8)} \\
& \leq (1+c^2c_3)r + c^2c_3r \sum_{l=1}^{k} c_5^l, \quad \text{(by (4.49)}
\end{align*}
\]

which together with \(\sum_{l=1}^{k} c_5^l \leq \sum_{l=1}^{\infty} c_5^l = \frac{c_5}{1-c_5}\implies\)

\[
\text{dist}(x_{k+1}, x^*) \leq \left(1+\frac{c^2c_3}{1-c_5}\right)r = \frac{b}{2},
\]

thus we conclude \(x_{k+1} \in B(x^*, \frac{b}{2})\). Hence, we obtain the desired assertion.

\(\square\)

**Theorem 4** Suppose that \(f^* < \frac{c_4}{8c_5c_3^2}\), \(c_5 < 1\) and Assumptions \(3, 4\) hold.

Let \(r > 0\) be the same as in Lemma 18. Moreover, assume \(x_0 \in B(x^*, r)\).

Then, \(\{\text{Dist}(x_k, X^*)\}\) converges to 0 linearly and furthermore, \(\{x_k\}\) converges to some point \(\hat{x} \in B(x^*, \frac{b}{2})\).

**Proof** The former claim follows from the assumptions, Lemma 17 and Lemma 18. We next show the latter one. Using the assumption that \((\mathcal{M}, \langle \cdot, \cdot \rangle)\) is a complete Riemannian manifold, it suffices to show that \(\{x_k\}\) is a Cauchy sequence with respect to the Riemannian distance dist(\(\cdot, \cdot\)). For arbitrary \(m > \)
\[ \text{dist}(x_m, x_n) \]
\[ \leq \sum_{l=n}^{m-1} \text{dist}(x_l, x_{l+1}) \]
\[ \leq c \sum_{l=n}^{m-1} \| s_l \|_x \quad \text{(by Lemma 8)} \]
\[ \leq c^2 \hat{c}_3 \sum_{l=n}^{m-1} \text{Dist}(x_l, X^*) \quad \text{(by (4.32) and Lemma 8)} \]
\[ \leq c^2 \hat{c}_3 r \sum_{l=n}^{m-1} c_5^l, \quad \text{(by (4.49))} \]

which together with \( \sum_{l=n}^{m-1} c_5^l \leq \sum_{l=n}^{\infty} c_5^l = \frac{c_5^n}{1-c_5} \) implies
\[ \text{dist}(x_m, x_n) \leq \frac{c^2 \hat{c}_3 r}{1-c_5} c_5^n. \]

Therefore, noting \( 0 < c_5 < 1 \), we ensure that \( \{x_k\} \) is a Cauchy sequence. The proof is complete. \( \square \)

5 Numerical experiments

We apply the proposed RLM to two kinds of problems: CANDECOMP/PARAFAC (CP) decomposition of tensors and low-rank matrix completion. All experiments were conducted on a machine with an Intel Core i5 CPU and 8.0 GB RAM. Regarding implementations, all methods were implemented in Matlab.

5.1 CP decomposition of tensors

Here we apply the RLM method to the CP decomposition of tensors.

5.1.1 Brief introduction to tensor rank approximation problem (TAP)

Let \( S_i \) denote the set of rank one tensors of format \( n_i \times \cdots \times n_d \), \( S^r := S_1 \times \cdots \times S_1 \), and let \( \cdot \|_F \) denote the Frobenius norm. For a given tensor \( A \in \mathbb{R}^{n_1 \times \cdots \times n_d} \) and \( r > 0 \), rank \( r \) CP decomposition of \( A \) is formulated as the following optimization problem with the map \( \Phi(p) : (p_1, \ldots, p_r) \mapsto \sum_{i=1}^r p_i, \) where \( p_i \in S_1, i = 1, \ldots, r. \)

\[ \text{(TAP)} \min_{p \in S^r} \quad f(p) = \frac{1}{2} \| \Phi(p) - A \|_F^2. \]
For solving (TAP), the article [12] proposed the trust-region-based Riemannian Gauss-Newton method “RGN-HR” with a manipulation named “hot-restart” specialized for solving TAP. We utilize the same retraction and geometry as their work and compare performances of our proposal, RGN-HR, and RGN (i.e., RGN without hot-restart), where we used the Matlab code provided by [12] for RGN-HR. Since RGN can frequently encounter ill-conditioned linear equations, as a remedy, we employ the Moore-Penrose pseudo-inverse matrix in solving them.

5.1.2 Experimental setting of TAP

We sampled a tensor \( A \in \mathbb{R}^{13 \times 11 \times 9} \) or \( A \in \mathbb{R}^{50 \times 50 \times 50} \) from “Model 2” in [12] and generated an input tensor \( B \) according to \( B = \frac{A}{\|A\|_F} + 10^{-p} \frac{\mathcal{E}}{\|\mathcal{E}\|_F} \), where \( \mathcal{E} \) is a tensor with the same size as \( A \), whose each element of the tensor is independently and identically distributed random variable from \( \mathcal{N}(0, 1) \). The parameter \( p \) controls the degree of perturbation.

In this experiment, \( p \) is fixed as \( p = 5 \) and the decomposition rank is \( r = 5 \). As the hyperparameters \( \eta, \mu_\min, \) and \( \beta \) in Algorithm 1, we set \( \eta = 0.2, \mu_\min = 0.1, \beta = 5.0 \) in RLM. As the stopping rule, we make each algorithm terminate when any one of the following conditions is satisfied:

\begin{enumerate}
  \item[(c1)] The iteration number exceeds MAX ITER = 1000.
  \item[(c2)] \( f(x_k) \leq 10^{-10} \) holds.
  \item[(c3)] \( \|\nabla f(x_k)\| \leq 10^{-6} \) holds.
\end{enumerate}

5.1.3 Comparison by averaged performances

To compare averaged performances of our RLM, RGN-HR and RGN, we generated 10 tensors \( B_i \) (1 \( \leq \) i \( \leq \) 10) in the above way and we set 50 randomized starting points for each tensor. We show the results of our experiments in Table 1 where their each row represents the following:

\begin{itemize}
  \item success: the number of runs terminated due to fulfilling the stopping rules (c2) or (c3) among 500 runs. The left and right numbers in the parentheses show the number of iterations terminated due to the stopping rule (c2) and the stopping rule (c3), respectively.
  \item fail: the left and right numbers show the number of iterations terminated due to the stopping rule (c1) and due to some numerical error, respectively.
  \item {\( t_{\text{success}} \): the averaged computational time among the successful runs.}
\end{itemize}

As Table 1 shows, RLM outperforms RGN in all items, though it is defeated by RGN-HR, which is specialized for solving TAP without any theoretical guarantees. RGN-HR and RGN contain five instances in which they could
Table 1 Comparison of RLM, RGN, and RGN-HR.

| Bi ∈ R^{13×11×9}, i = 1,⋯, 10 | RLM          | RGN          | RGN-HR       |
|-------------------------------|--------------|--------------|--------------|
| success                       | 493 (375 : 118) | 475 (203 : 272) | 500 (500 : 0) |
| fail                          | 7 : 0        | 25 : 0       | 0 : 0        |
| t_{success} (sec.)            | 1.232        | 1.674        | 6.519 × 10^{-1} |

| Bi ∈ R^{50×50×50}, i = 1,⋯, 10 | RLM          | RGN          | RGN-HR       |
|-------------------------------|--------------|--------------|--------------|
| success                       | 492 (335 : 157) | 481 (242 : 239) | 495 (485 : 10) |
| fail                          | 8 : 0        | 14 : 5       | 0 : 5        |
| t_{success} (sec.)            | 1.088 × 10   | 3.616 × 10   | 3.150        |

Fig. 1 CP decomposition for Bi ∈ R^{13×11×9}

not reach MAX ITER without the stopping rules satisfied. In all such cases, the progress in the computation stalled at the calculation of the retraction which uses the sequentially-truncated higher order singular value decomposition (ST-HOSVD) [13,44]. Thus, we infer they were provoked due to numerically unstable calculations of the ST-HOSVD. While RLM employs the same retraction as RGN-HR and RGN, it did not cause such an instance as long as we experimented. These observations may support the stability of RLM in comparison with RGN-HR and RGN.

Figure 1 shows an example of the change of the objective value of RLM and RGN-HR as the iteration proceeds for Bi ∈ R^{13×11×9}. As this figure indicates, \{f(x_k)\} is monotonically non-increasing as proved in the proof of Theorem 1. Moreover, in this example, the objective value of RLM starts to drastically decrease when it gets relatively small. In most cases, we observed this tendency for RLM.
5.2 Low-rank matrix completion

Next, we apply the RLM method to low-rank matrix completion problems.

5.2.1 Brief introduction to low-rank matrix completion

This problem is to recover a low-rank matrix from a matrix, say $A \in \mathbb{R}^{m \times n}$, whose elements are known only partially in advance. Specifically, letting $\mathbb{R}^{m \times n}_k$ be the set of $m \times n$ matrices with rank $k$, the problem is formulated as follows:

$$\min_{X \in \mathbb{R}^{m \times n}} f(x) := \frac{1}{2} \| P_\Omega(X) - A \|_F^2, \tag{5.1}$$

where $\Omega \subset \{1, \ldots, m\} \times \{1, \ldots, n\}$ denotes the set of indices for which elements of $A$ are known in advance and $P_\Omega : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ is given by

$$P_\Omega(X) = \begin{cases} X_{i,j} & (i,j) \in \Omega \\ 0 & \text{(otherwise)} \end{cases}.$$

It is known that $\mathbb{R}^{m \times n}_k$ has a structure as a $k(m+n-k)$-dimensional smooth manifold embedded into $\mathbb{R}^{m \times n}$ (e.g. [8]). Thus, (5.1) can be regarded as a least-square Riemannian optimization problem.

5.2.2 Experimental setting

We compare the RLM with other four Riemannian methods: Riemannian trust-region (RTR) method with Gauss-Newton approximation for its Hessian approximation, Riemannian gradient descent (RSD) method, Riemannian conjugate gradient (RCG) method provided by Manopt [10], which is a Matlab optimization toolbox on Riemannian manifolds, and adaptive quadratically regularized Newton (ARNT) method proposed by [21]. We refer to them respectively as “manoptRTR”, “manoptRSD”, “manoptRCG”, and “ARNT”.

Given natural numbers $m, n, k$, the oversampling factor $r_s$ (i.e., the ratio of observed elements in $A$) for a low-rank matrix completion is defined as

$$r_s := \frac{|\Omega|}{k(m+n-k)}.$$ 

Once $r_s$ is given, we set $\Omega$ by repeatedly sampling $(i,j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ such that $(i,j) \notin \Omega$ and adding it to $\Omega$ until $|\Omega|$ gets equal to $k(m+n-k)r_s$.

Next, we generate an input matrix $A \in \mathbb{R}^{m \times n}$ in the following manner: First, we sample $A_L \in \mathbb{R}^{m \times k}$ and $A_R \in \mathbb{R}^{n \times k}$ such that their each element independently and identically follows $N(0,1)$ and secondly, define $A$ as $P_\Omega(A_LA_R^T)$.

As the parameters $\eta, \mu_{\min}$, and $\beta$ in Algorithm 1, we set $\eta = 0.2, \mu_{\min} = 0.1, \beta = 5.0$. As the stopping rule, we make each algorithm terminate when any one of the following conditions is satisfied:
(c1) The CPU time exceeds 300 seconds.
(c2) \( \| \nabla f(x_k) \| \leq 10^{-8} \) holds.

In the same manner as in [21], we generate an initial point as follows: With matrices \( A_L \) and \( A_R \) sampled in the same way as in producing the matrix \( A \) above, we first compute \( A_L A_R^T \), from which we run manoptRSD to gain a refined point such that the norm of Riemannian gradient gets less than or equal to \( 10^{-3} \). The last point is used as an initial solution. This procedure is executed for the sake of observing the performance of our algorithm when the residual of (1.1) is sufficiently small.

5.2.3 Comparison by averaged performances

We compare averaged performances of those methods among 10 starting points generated in the way described in Section 5.2.2 in the following two types of setting of \( m,n,k \), and \( r_s \):

(I) \( r_s = 0.9 + 0.01i \) (0 \( \leq \) i \( < \) 10), \( m = n = 30, k = 3 \)

(II) \( r_s = 1.2, m = n = 200 + 200i \) (0 \( \leq \) i \( < \) 5), \( k = \frac{m}{10} \)

The setting (I) aims to examine the performances against different \( r_s \)s with fixed \( (m,n,k) \), while (II) for different \( (m,n,k) \) with fixed \( r_s \). We evaluate the quality of performances in terms of the following criteria:

- **success**: the number of runs terminated due to fulfilling the stopping rules (c2)
- **iter\_success**: the averaged number of iterations among the successful runs
- **t\_success**: the averaged computational time among the successful runs.

Figure 2 shows, from top to bottom, the changes of success, iter\_success (log-scale) and t\_success (log-scale) versus the ratio \( r_s \) in the setting (I). Note that manoptRSD has zero success in all \( r_s \) as the top of Figure 2 shows and thus iter\_success and t\_success of manoptRSD cannot be computed. Due to this issue, manoptRSD does not appear in the second and third plots of Figure 2.

Figure 2 indicates that RLM has superiority over manoptRTR, manoptRSD, and manoptRCG in terms of both computational time and iteration number. Moreover, RLM is more robust against the change of \( r_s \) compared with those methods. In the comparison of RLM and ARNT, RLM still shows superiority over ARNT.

Figure 3 shows the averaged performances in the setting (II). Since RLM and ARNT show very similar performances, some plots of their results overlap in the figure. According to Figure 3, from the perspective of both computational time and iteration number, RLM is as efficient as manoptRTR and ARNT, and is superior to the other methods.

Figure 4 illustrates how each method decreases the objective value against CPU time in an instance of problem with \( m = n = 30, r_s = 0.97 \). While manoptRTR, manoptRSD, and manoptRCG get stuck at some point, RLM accomplishes a considerable reduction.
Fig. 2 Comparison of RLM to existing methods in setting (I). In the middle and bottom figures, manoptRSD and manoptRCG do not appear because of failures for all the instances.

6 Conclusion

We proposed a Riemannian Levenberg-Marquardt (RLM) method for the nonlinear least-squares problem on the Riemannian manifold of the form (1.1). We proved the global and local convergence properties of the algorithm and conducted two types of numerical experiments: the CP decomposition of tensors and the low-rank matrix completion. In both of them, we found the RLM efficiently converges when the residual of (1.1) is sufficiently small.

Possible directions of future work would be to extend the theoretical guarantees to (1.1) where $\mathcal{M}$ is a manifold with boundary. Furthermore, we are interested in the establishment of a theory pertaining to the desirable affine transformation for RLM and its relation with the Riemannian metric.

Compliance with Ethical Standards

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Fig. 3 Comparison of RLM to existing methods in setting (II) (the same legend with Figure 2) the results of RLM are covered with those for manoptRTR and ARNT in the top and middle figures and those for manoptRTR in the bottom one.

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Fig. 4  The objective value versus computation time for $m = n = 30$, $k = 3$, $r_s = 0.97$. 

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