ON THE VIRTUALLY-CYCLIC DIMENSION OF MAPPING CLASS GROUPS OF PUNCTURED SPHERES

J. ARAMAYONA, D. JUAN-PINEDA, AND A. TRUJILLO-NEGRETE

Abstract. We calculate the virtually-cyclic dimension of the mapping class group of a sphere with at most six punctures. As an immediate consequence, we obtain the virtually-cyclic dimension of the mapping class group of the twice-holed torus and of the closed genus-two surface.

For spheres with an arbitrary number of punctures, we give a new upper bound for the virtually-cyclic dimension of their mapping class group, improving the recent bound of Degrijse-Petrosyan [5].

1. Introduction

Given a discrete group $G$, a family $\mathcal{F}$ of subgroups of $G$ is a set of subgroups of $G$ which is closed under conjugation and taking subgroups. Of particular interest here are the families $\mathcal{FLN}_G$ and $\mathcal{VC}_G$ which consist, respectively, of finite and virtually cyclic subgroups of $G$.

A model for the classifying space $E_{\mathcal{F}}G$ of the family $\mathcal{F}$ is a $G$-CW-complex $X$, such that the fixed point set of $H \in \mathcal{F}$ is contractible, and is empty whenever $H \notin \mathcal{F}$. Using standard terminology, we denote $E_{\mathcal{FLN}}G$ by $\underline{E}G$, and $E_{\mathcal{VC}}G$ by $\mathcal{V}G$. The study of models for these families finds a large part of its motivation in the Baum-Connes and Farrell-Jones conjectures, respectively.

Although a model for the space $E_{\mathcal{F}}G$ always exists, it need not be finite dimensional. The smallest possible dimension of a model of $E_{\mathcal{F}}G$ is called the geometric dimension of $G$ for the family $\mathcal{F}$, and is usually denoted $\text{gd}_{\mathcal{F}}G$. Again using standard terminology, we will write $\text{gd}G = \text{gd}_{\mathcal{FLN}}G$ and $\underline{\text{gd}}G = \text{gd}_{\mathcal{VC}}G$, and refer to them as the proper geometric dimension and the virtually-cyclic dimension of $G$, respectively. For many families of groups these two numbers are related by the inequality

$$\underline{\text{gd}}G \leq \text{gd}G + 1;$$

although it is known to not be true in general [6] Example 6.5. Classes of groups for which it does hold include: CAT(0) groups [16], hyperbolic groups [13], standard braid groups [8], and groups satisfying a certain property (Max) [19] Theorem 5.8], which roughly states that every infinite virtually-cyclic subgroup is contained in a unique maximal such subgroup (see Section 5).
In this note we investigate the relation between \( \text{gd} G \) and \( \text{gd} \overline{G} \) for the mapping class group \( \text{Mod}(S) \) of a connected, orientable surface \( S \), mainly in the case when \( S \) has genus zero. We stress that mapping class groups do not fall in any of the categories above; however, they contain finite-index subgroups with property (Max) \([14, \text{Prop. 5.1}]\); compare with Lemma 5.5 below. For these subgroups the inequality (1) holds, although this does not say anything about whether this is the case for the whole group.

We will denote by \( S^n_{g,b} \) the connected orientable surface of genus \( g \), with \( b \) boundary components and \( n \) punctures. If \( b = 0 \), we will omit \( b \) from the notation. Our main result is as follows:

**Theorem 1.1.** Let \( n \in \{5, 6\} \). Then \( \text{gd} \overline{\text{Mod}(S^n_0)} = \text{gd} \text{Mod}(S^n_0) + 1 \).

We remark that \( \text{Mod}(S^n_0) \) is finite for \( n \leq 3 \), and virtually-free if \( n = 4 \) \([7]\). As an immediate corollary of Theorem 1.1 we will obtain:

**Corollary 1.2.** If \( S \in \{S^2_1, S^2_0\} \), then \( \text{gd} \overline{\text{Mod}(S)} = \text{gd} \text{Mod}(S) + 1 \).

At this point, we remark that the proper geometric dimension of \( \text{Mod}(S^n_g) \) is known \([1]\) to coincide with its virtual cohomological dimension, which in turn was computed by Harer \([10]\) and is an explicit linear function of \( g \) and \( n \) (in the particular case when \( g = 0 \), it is equal to \( n - 3 \)).

As a further corollary of Theorem 1.1 we calculate the exact value of the virtually-cyclic dimension of the spherical braid group \( B_n \) on \( n \) strands, for \( n \in \{5, 6\} \). Indeed, using the classical fact that \( B_n \) is a finite extension of \( \text{Mod}(S^n_g) \), we will obtain:

**Corollary 1.3.** If \( n \in \{5, 6\} \), then \( \text{gd}(B_n) = \text{gd}(B_n) + 1 \).

This latter result should be compared with a recent theorem of Flores and González-Meneses \([8]\), which proves the analogous statement for braid groups of the disk, with an arbitrary number of strands.

In order to prove Theorem 1.1 we will use a result of Lück-Weiermann \([19]\) (stated as Theorem 3.1 below) which relates the virtually-cyclic dimension of a group \( G \) to the proper dimension of certain subgroups associated to infinite-order elements of \( G \). We then use the Nielsen-Thurston classification of mapping classes and a case-by-case analysis, to bound the dimension of such subgroups.

**Remark 1.4.** The argument used in the proof of Theorem 1.1 will not generalize to spheres with an arbitrary number of punctures; see Remark 4.2 below for more details. In spite of this, the interested reader can check that an immediate adaptation of the proof of Theorem 1.1 for \( n = 6 \) gives a direct proof of Corollary 1.2 as well as of the analogous statement for \( \text{Mod}(S^3_1) \). In particular, we obtain that inequality (1) is in fact an equality for all surfaces \( S^n_g \) for which \( 3g - 3 + n \leq 3 \).
For a general number of punctures, a recent result of Degrijse-Petrosyan [5] gives a bound for \(\text{gd} \text{Mod}(S^n_g)\) which is linear in \(g\) and \(n\); see Proposition 5.1 below. In the particular case when \(g = 0\), this bound takes the form:

\[
(2) \quad \text{gd} \text{Mod}(S^n_0) \leq 3n - 8 = 3 \cdot \text{gd} \text{Mod}(S^n_0) + 1
\]

Using the aforementioned result of Lück-Weiermann [19] with a theorem of Cameron-Solomon-Turull [4], we will prove the following slightly improved bound:

**Theorem 1.5.** Suppose \(n \geq 4\). Let \(b_n\) be the number of ones in the binary expression of \(n\). Then

\[
\text{gd} \text{Mod}(S^n_0) \leq n - 4 + \left\lfloor \frac{3n - 1}{2} \right\rfloor - b_n,
\]

where \(\lfloor \cdot \rfloor\) denotes integer part.

**Remark 1.6.** Observe that \(3n - 8 \geq n - 4 + \left\lfloor \frac{3n - 1}{2} \right\rfloor - b_n\) for all \(n \geq 4\), and that the inequality is strict for \(n \geq 5\).

**Pure mapping class groups of spheres.** As we will observe in Lemma 5.5, pure mapping class groups of spheres have property (Max), and hence inequality (1) is satisfied. Moreover, we will remark in Proposition 5.4 that in this case we get an equality, in fact.

**Surfaces with boundary.** It follows from the definition that \(\text{Mod}(S)\) is torsion-free whenever \(S\) has boundary. Combining this with a number of results by various authors, quickly yields that (1) holds for surfaces with boundary. The argument is essentially contained in the paper by Flores and González-Meneses [8]; we offer a short account in the Appendix.

**Acknowledgements.** J. A. was partially supported by grants RYC-2013-13008 and MTM2015-67781. D. Juan-Pineda and A. Trujillo-Negrete were partially supported by CONCAYT FORDECYT 265667. We would like to thank Yago Antolín, John Guaschi and Conchita Martínez for conversations.

2. **Mapping class groups and braid groups**

In this section we give some preliminaries on mapping class groups, and their relation with braid groups. We refer the reader to [7] for a thorough discussion on these and related topics.

2.1. **Mapping class groups.** Let \(S\) be a (possibly disconnected) orientable surface with empty boundary and negative Euler characteristic, so that it supports a complete hyperbolic metric of finite area. Sometimes it will be convenient to regard (some of) the punctures of \(S\) as marked points, and we will switch between the two points of view without further mention. As mentioned above, we will write \(S^n_g\) to denote the connected surface of genus \(g\) with \(n\) marked points.
The mapping class group \( \text{Mod}(S) \) is the group of isotopy classes of self-homeomorphisms of \( S \); elements of \( \text{Mod}(S) \) are called mapping classes. The pure mapping class group \( \text{PMod}(S) \) is the subgroup of \( \text{Mod}(S) \) whose elements send every marked point to itself; observe that \( \text{PMod}(S) \) has finite index in \( \text{Mod}(S) \).

Since we will deal mainly with surfaces of genus zero, from now on we will restrict our attention to the case of \( S = S^n_0 \), with \( n \geq 3 \).

2.1.1. Curves and multicurves. By a curve on \( S^n_0 \) we mean the (free) isotopy class of a simple closed curve that does not bound a disk with at most one marked point. A multicurve is then a set of curves that pairwise disjoint, i.e. they may be realized in a disjoint manner on \( S^n_0 \). An easy counting argument shows that a maximal multicurve on \( S^n_0 \) has \( n - 3 \) elements.

2.1.2. Nielsen-Thurston classification. We say that \( f \in \text{Mod}(S^n_0) \) is reducible if there exists a multicurve \( \sigma \subset S^n_0 \) such that \( f(\sigma) = \sigma \); otherwise, we say that \( f \) is irreducible. A notable example of a reducible element is the Dehn twist \( T_\alpha \) about the curve \( \alpha \); see [7] for definitions and properties of Dehn twists. Finally, we note that finite-order elements of \( \text{Mod}(S^n_0) \) may be reducible or irreducible.

The celebrated Nielsen-Thurston classification of mapping classes asserts that an irreducible element of infinite order has a representative which is a pseudo-Anosov homeomorphism; see [7, Ch. 5] for details. For this reason, irreducible mapping classes of infinite order are normally referred to as pseudo-Anosov mapping classes.

2.1.3. Canonical reduction system. Note that, in general, a reducible mapping class may fix more than one multicurve. For this reason, we define the canonical reduction system of a mapping class as the intersection of all the multicurves that it fixes. For instance, the canonical reduction system of the Dehn twist \( T_\alpha \) is equal to \( \alpha \).

2.1.4. The cutting homomorphism. Let \( \sigma \) be a multicurve on \( S^n_0 \), and consider \( (\text{Mod}(S^n_0))_\sigma = \{ g \in \text{Mod}(S^n_0) | g(\sigma) = \sigma \} \). Denote by \( S^n_0 - \sigma \) the (disconnected) surface which results from removing from \( S^n_0 \) a closed regular neighbourhood of each element of \( \sigma \). Write \( S^n_0 - \sigma = Y_1 \sqcup \ldots \sqcup Y_k \), observing that each \( Y_j \) is a sphere with marked points. There is an obvious surjective homomorphism

\[
(\text{Mod}(S^n_0))_\sigma \to \text{Mod}(\sqcup_{i} Y_i, \sigma),
\]

called the cutting homomorphism associated to \( \sigma \). Here, \( \text{Mod}(\sqcup_{i} Y_i, \sigma) \) denotes the subgroup of \( \text{Mod}(\sqcup_{i} Y_i) \) whose elements preserve the set of punctures of \( \sqcup_{i} Y_i \) that correspond to elements \( \sigma \). The cutting homomorphism fits in a short exact sequence

\[
1 \to T_\sigma \to (\text{Mod}(S^n_0))_\sigma \to \text{Mod}(\sqcup_{i} Y_i, \sigma) \to 1
\]
where $T_\sigma$ is the free abelian group generated by the Dehn twists along the elements of $\sigma$.

Armed with these definitions, we can give a canonical form for elements of $\text{PMod}(S^0_n)$. More concretely, let $f \in \text{PMod}(S^0_n)$, and write $\sigma$ for its canonical reduction system, so that $f \in (\text{Mod}(S^0_n))_\sigma$. Again, let $S^0_n - \sigma = Y_1 \sqcup \cdots \sqcup Y_k$. Since $f$ is pure, it follows $f(\alpha) = \alpha$ for every $\alpha \in \sigma$; also, $f(Y_i) = Y_i$ for every $i$. From this discussion, and using the Nielsen-Thurston classification, we have deduced:

**Lemma 2.1.** With the notation above, the image of $f \in \text{PMod}(S^0_n)$ under the cutting homomorphism (3) belongs to $\text{PMod}(Y_1) \times \cdots \times \text{PMod}(Y_k)$. Moreover, the projection of this image onto each factor is either the identity or pseudo-Anosov.

**2.1.5. Normalizers.** We will use the following well-known result about normalizers of pseudo-Anosov elements:

**Lemma 2.2.** Let $f \in \text{Mod}(S^0_n)$ be a pseudo-Anosov. Then its normalizer $N_{\text{Mod}(S^0_n)}(f)$ is virtually cyclic.

It is also possible to describe the normalizer of a multitwist. Indeed, observe that, for any $f \in \text{Mod}(S^0_n)$, we have $fT\sigma f^{-1} = T(f(\sigma))$. In particular, we obtain:

**Lemma 2.3.** For any multicurve $\sigma$, $N_{\text{Mod}(S^0_n)}(T_\sigma) = \text{Mod}(S^0_n)_{\sigma}$.

**2.2. Braid groups.** Given $n \geq 0$, we denote by $F_n$ the configuration space of $n$ distinct points on a sphere. Note that the symmetric group $\Sigma_n$ acts on $F_n$ by permutation the coordinates; the quotient space $J_n = F_n / \Sigma_n$ may then be regarded as the configuration space of $n$ unordered points on the sphere. Birman [2, Prop 1.1] proved that the natural projection $F_n \to J_n$ is a regular $(n!)$-fold covering map.

We define the $n$-strand spherical braid group as $B_n = \pi_1(J_n)$, and its pure subgroup as $P_n = \pi_1(F_n) < \pi_1(J_n)$.

As mentioned in the introduction, braid groups are strongly related to mapping class groups of spheres. More concretely, for $n \geq 3$ there is a short exact sequence (see, for instance, [7, Section 9.4.2]):

\[
1 \to \mathbb{Z}_2 \to B_n \to \text{Mod}(S^0_n) \to 1
\]

where $\mathbb{Z}_2$ is generated by the full twist braid, $\Delta_n$, of $B_n$ and it generates the center of $B_n$. In turn, for pure braid groups we have:

\[
1 \to \mathbb{Z}_2 \to P_n \to \text{PMod}(S^0_n) \to 1
\]

3. General results on geometric dimension

In this section we introduce the main ingredient in our proofs, namely the result of Lück-Weiermann [19] stated as Theorem 3.1 below.
3.1. The main tool. Let $G$ be a group, and $C_G^\infty$ the family of infinite, virtually-cyclic subgroups of $G$. After [16] and [19], we define an equivalence relation $\sim$ on $C_G^\infty$ by:

\begin{equation}
C \sim D \iff |C \cap D| = \infty.
\end{equation}

Let $[C_G^\infty]$ denote the set of equivalence classes and by $[C]$ the equivalence class of $C \in C_G^\infty$. The normalizer of $[C]$ is defined as:

\begin{equation}
N_G[C] := \{ g \in G \mid |gCg^{-1} \cap C| = \infty \};
\end{equation}

in other words, it is the commensurator of $C$ in $G$. We define the following family of subgroups of $N_G[C]$:

\begin{equation}
\mathcal{G}_G[C] = \{ H \in \mathcal{VC}_{N_G[C]} \mid |H \cap C| < \infty \} \cup \mathcal{FIN}_{N_G[C]}.
\end{equation}

After all these definitions, we are ready to give Lück-Weiermann’s bound from [19]:

**Theorem 3.1.** [19, Thm. 2.3] Let $C_G^\infty$ and $\sim$ be as above. Let $\mathcal{I}$ be a complete system of representatives, $[H]$, of the $G$-orbits in $[C_G^\infty]$ under the $G$-action coming from conjugation. Suppose there exists $d \in \mathbb{N}$ with the following properties:

\begin{enumerate}
  \item $g d_G \leq d$,
  \item $g d_{N_G[H]} \leq d - 1$,
  \item $g d_{\mathcal{G}_G[H]} \leq d$,
\end{enumerate}

for each $[H] \in \mathcal{I}$. Then $g d_G \leq d$.

Under certain circumstances, Theorem 3.1 becomes a lot easier to work with. In this direction, say that a group $G$ has property (C) (for “conjugation”) if, whenever $f, g \in G$ are elements of infinite order with $gf^m g^{-1} = f^k$, we have that $|m| = |k|$. If $G$ has property (C), then [16, Lem. 4.2] yields that for any $C \in C_G^\infty$,

\[ N_G(C) \subseteq N_G(2!C) \subseteq N_G(3!C) \subseteq \cdots, \]

where $k!C = \{ h^k \mid h \in C \}$ and $N_G[C] = \cup_{k \geq 1} N_G(k!C)$.

Finally, say that $G$ has the property of uniqueness of roots if for any $f, g \in G$ such that $f^n = g^n$ implies that $f = g$. We have:

**Proposition 3.2.** Suppose $G$ satisfies property (C) and has a finite index normal subgroup $H$ with the property of uniqueness of roots. If for any $C \in C_H^\infty$ we have

\begin{enumerate}
  \item $g d_G \leq d$,
  \item $g d_{N_G[H]} \leq d - 1$,
  \item $g d_{W_G[H]} \leq d$,
\end{enumerate}

where $W_G[H] = N_G[H]/C$. Then $g d_G \leq d$. 
Proof. We will use Theorem 3.1. Since $H$ is a normal subgroup of finite index with the property of uniqueness of roots, we have $N_G(D) = N_G(tD)$, for any $D \in \mathcal{C}_H^\infty$ and any $t \in \mathbb{Z}\setminus\{0\}$.

Let $C \in \mathcal{C}_H^\infty$. Combining this with [10, Lem. 4.2], we have that $N_G[C] = N_G[D]$ for some $k \in \mathbb{Z}\setminus\{0\}$ and $kD \in \mathcal{C}_H^\infty$. Thus we may assume that $C \in \mathcal{C}_H^\infty$ and $N_G[C] = N_G(C)$. Further, a model for $E_{\mathcal{G}}(C)$ is a model for $E_{\mathcal{G}}[C]$ with the action given from the projection $p: N_G(C) \to W_G(C)$. Applying Theorem 3.1 we conclude the Proposition. □

Remark 3.3. Let $G$ and $H$ be as in Proposition 3.2. Suppose that $H$ satisfies property (Max). Note that if $D \in \mathcal{C}_H^\infty$ and $D_{\text{max}} \in \mathcal{C}_H^\infty$ is the maximal cyclic subgroup containing $D$, then $N_G(D) = N_G(D_{\text{max}})$; this follows from the property of uniqueness of roots and because $H$ is a normal subgroup of finite index in $G$. Therefore, in Proposition 3.2 we may assume that $C \in \mathcal{C}_H^\infty$ is maximal in $C_H$.

3.2. On proper geometric dimension. In the light of Proposition 3.2 in order to estimate the virtually-cyclic dimension, one needs to be able estimate proper geometric dimension. With this motivation, we now present a number of known results about proper geometric dimension.

First, an immediate consequence of the definition of proper geometric dimension is that, for any two groups $G_1, G_2$, one has

\[(9) \quad \text{gd}(G_1 \times G_2) \leq \text{gd}G_1 + \text{gd}G_2.\]

Another observation is that if $H$ is a subgroup of a group $G$, then

\[(10) \quad \text{gd}H \leq \text{gd}G.\]

Next, a result of Karrass-Pietrowski-Solitar [15] implies that the Bass-Serre tree of a virtually-free group $G$ is a model for $\mathcal{E}G$. In other words, we have:

Lemma 3.4. Let $G$ be a virtually-free group. Then $\text{gd}G \leq 1$, with equality if and only if $G$ is infinite.

The next theorem, due to Lück [18], gives a relation between the geometric dimension of a group and that of finite-index subgroups:

Theorem 3.5. [18, Thm. 2.4] If $H \subseteq G$ is a subgroup of finite index $n$, then $\text{gd}G \leq \text{gd}H \cdot n$.

We will also need to be able to bound the proper geometric dimension of certain extensions of groups. In this direction, we will use the next result, which is a consequence of [17, Thm. 5.16]:

Theorem 3.6. Let $1 \to H \to G \to K \to 1$ be an exact sequence of groups. Suppose that $H$ has the property that for any group $\bar{H}$ which contains $H$ as subgroup of finite index, $\text{gd}\bar{H} \leq n$. If $\text{gd}K \leq k$, then $\text{gd}G \leq n + k$.
Finally, we will make use the following well-known result [9, Prop. 2.6] in order to prove Corollaries 1.2 and 1.3:

**Lemma 3.7.** Suppose $g_d g \geq 3$. Let $1 \to F \to G \to H \to 1$ be a short exact sequence of groups, where $F$ is finite. Then $g_d g = g_f H$.

### 4. Proof of Theorem 1.1

In this section we prove Theorem 1.1 using Proposition 3.2 to $\operatorname{Mod}(S^n_0)$, with $n \leq 6$. In order to do so, we first remark that the second and third named authors showed that $\operatorname{Mod}(S^n_0)$ has property (C) [14], and that [3, Theorem 6.1] implies that $\operatorname{PMod}(S^n_0)$ has unique roots; we recall that $\operatorname{PMod}(S^n_0)$ has finite index in $\operatorname{Mod}(S^n_0)$. Next, we will use the following special case of the main result of [1], combined with Harer’s calculation [10] of the virtual cohomological dimension of the mapping class group:

**Theorem 4.1.** For every $n$, $g_d \operatorname{Mod}(S^n_0) = n - 3$.

In the light of this result, inequality (10) implies that

$$g_d N_{\operatorname{Mod}(S^n_0)}(f) \leq n - 3,$$

for any $f \in \operatorname{Mod}(S^n_0)$. We will show in Lemma 5.5 that $\operatorname{PMod}(S^n_0)$ has the property (Max). Therefore, by Remark 3.3 the proof of Theorem 1.1 boils down to proving that

$$g_d W_{\operatorname{Mod}(S^n_0)}(f) \leq n - 3,$$

for every infinite-order element $f \in \operatorname{PMod}(S^n_0)$ such that $\langle f \rangle$ is maximal. We will do so using a case-by-case analysis depending on the Nielsen-Thurston type of such a mapping class. We have separated the proof in the cases $n = 5$ and $n = 6$, since the combinatorial possibilities are different in these two cases.

**Remark 4.2.** As hinted in Remark 1.4, our methods will not carry over to an arbitrary number of punctures. In a nutshell, the reason lies in the difference between the mapping class group of a disconnected surface, and the product of mapping class groups of the components. While the latter is a finite-index subgroup of the former, the index grows with the topology of the surface. For $n \in \{5, 6\}$, however, this index is amenable to our computations.

On the other hand, we stress that essentially the same analysis as in the case $n = 6$ will give a direct proof of Corollary 1.2 as well as the analogous statement for $S^3$.

#### 4.1. The case of the five-punctured sphere

As indicated above, we need to prove

$$g_d W_{\operatorname{Mod}(S^n_0)}(f) \leq 3$$

for every infinite-order element $f \in \operatorname{PMod}(S^n_0)$ such that $\langle f \rangle$ is maximal. There are two cases to consider:
**Case 1: \( f \) is pseudo-Anosov.** Here, Lemma 2.2 implies that \( N_{\text{Mod}(S_0^5)}(f) \) is virtually cyclic, in which case \( \text{gd} N_{\text{Mod}(S_0^5)}(f) = 1 \) and \( \text{gd} W_{\text{Mod}(S_0^5)}(f) = 0 \).

**Case 2: \( f \) is reducible.** Let \( \sigma \) be its canonical reduction system. We distinguish the following further cases, depending on whether \( \sigma \) has one or two elements.

**Subcase 2(a): \( \sigma \) has exactly one element.** Write \( \sigma = \{ \alpha \} \), observing that \( S_0^5 \setminus \alpha = S_0^3 \sqcup S_0^4 \). Let \( \rho \) be the cutting homomorphism (3) associated to \( \sigma \). Suppose first that \( \rho(f) \) is trivial, so that \( f \in \langle T_\alpha \rangle \). By Lemma 2.3, \( N_{\text{Mod}(S_0^5)}(T_\alpha) = \text{Mod}(S_0^5) \alpha \), and thus we have:

\[
\begin{align*}
1 &\longrightarrow \langle T_\alpha \rangle \longrightarrow N_{\text{Mod}(S_0^5)}(f) \longrightarrow \text{Mod}(S_0^3, q_1) \times \text{Mod}(S_0^4, q_2) \longrightarrow 1
\end{align*}
\]

where the punctures \( q_1, q_2 \) are those that appear when the surface is cut along \( \alpha \) (see Figure 1). Therefore

\[
W_{\text{Mod}(S_0^5)}(f) \simeq \text{Mod}(S_0^3, q_1) \times \text{Mod}(S_0^4, q_2)
\]

Since \( \text{Mod}(S_0^3) \) is finite and \( \text{Mod}(S_0^4) \) is virtually free, the combination of Lemma 3.4 with equations (9) and (10) implies that \( \text{gd} W_{\text{Mod}(S_0^5)}(f) = 1 \), as desired.

Suppose now that \( \rho(f) \) is not trivial, so that the restriction of \( f \) to \( \text{Mod}(S_0^4, q_2) \) (using the notation above) is a pseudo-Anosov, which we denote \( f_2 \). In this case, we have:

\[
\begin{align*}
1 &\longrightarrow \langle T_\alpha \rangle \longrightarrow N_{\text{Mod}(S_0^5)}(f) \longrightarrow \text{Mod}(S_0^3, q_1) \times N_{\text{Mod}(S_0^4, q_2)}(f_2) \longrightarrow 1
\end{align*}
\]

By Lemma 2.2, \( N_{\text{Mod}(S_0^4, q_2)}(f_2) \) is virtually cyclic, hence taking quotients we obtain:

\[
\begin{align*}
1 &\rightarrow \mathbb{Z} \rightarrow W_{\text{Mod}(S_0^5)}(f) \rightarrow F \rightarrow 1,
\end{align*}
\]

where \( F \) is a finite group. In other words, \( W_{\text{Mod}(S_0^5)}(f) \) is virtually cyclic, and thus \( \text{gd} W_{\text{Mod}(S_0^5)}(f) = 1 \) by Lemma 3.4.

**Subcase 2(b): \( \sigma \) has two elements.** Write \( \sigma = \{ \alpha, \beta \} \), and \( S \setminus \sigma = Y_1 \sqcup Y_2 \sqcup Y_3 \). Note that \( Y_j \) is homeomorphic to \( S_0^3 \), for \( j = 1, 2, 3 \).
Since \( f \) is pure, it follows that \( f \in \langle T_\alpha, T_\beta \rangle \). Therefore, the normalizer of \( f \) in \( \text{Mod}(S^6_0) \) coincides with \( \text{Mod}(S^5_0)_\sigma \), by Lemma 2.3. The cutting homomorphism (3) reads

\[
1 \longrightarrow \langle T_\alpha, T_\beta \rangle \longrightarrow \text{Mod}(S^5_0)_\sigma \longrightarrow \text{Mod}(Y_1 \sqcup Y_2 \sqcup Y_3, \sigma) \longrightarrow 1.
\]

Since \( \text{Mod}(Y_1 \sqcup Y_2 \sqcup Y_3) \) is a finite group, we obtain

\[
1 \longrightarrow \mathbb{Z} \longrightarrow W_{\text{Mod}(S^5_0)}(f) \longrightarrow F',
\]

where \( F' \) is a finite group. Therefore, \( \text{gd} W_{\text{Mod}(S^5_0)}(f) = 1 \), again by Lemma 3.4. This finishes the proof of Theorem 1.1 in the case \( n = 5 \).

4.2. The case of the six-punctured sphere. We now prove Theorem 1.1 in the case \( n = 6 \). Again, it suffices to prove that

\[
\text{gd} W_{\text{Mod}(S^5_0)}(f) \leq 4
\]

for every \( f \in \text{PMod}(S^6_0) \) of infinite order such that \( \langle f \rangle \) is maximal. Let \( f \) be such an element. As in the case \( n = 5 \), if \( f \) is pseudo-Anosov, then \( \text{gd} N_{\text{Mod}(S^5_0)}(f) = 1 \) and \( \text{gd} W_{\text{Mod}(S^5_0)}(f) = 0 \). Therefore, from now on we assume that \( f \) is reducible. Write \( \sigma \) for the canonical reduction system of \( f \), noting that \( 1 \leq |\sigma| \leq 3 \).

**Case 1:** \( |\sigma| = 1 \). We write \( \sigma = \{\alpha_1\} \), and distinguish the following subcases:

**Subcase 1(i):** \( \alpha_1 \) bounds a disk with exactly two punctures. In this case, \( S^6_0 \setminus \alpha_1 = S^3_0 \sqcup S^5_0 \), and the cutting homomorphism (3) associated to \( \sigma \) reads:

\[
1 \longrightarrow \langle T_{\alpha_1} \rangle \longrightarrow \text{Mod}(S^6_0)_{\sigma} \longrightarrow \text{Mod}(S^3_0 \sqcup S^5_0, \sigma) \longrightarrow 1
\]

Since the two components of \( S^6_0 \setminus \alpha_1 \) are not homeomorphic (or by Lemma 2.1) we deduce that \( \rho_{\sigma}(\text{Mod}(S^6_0)_{\sigma}) = \text{Mod}(S^3_0, q_1) \times \text{Mod}(S^5_0, q_2) \), where \( q_1 \) and \( q_2 \) are the punctures that appear when cutting \( S^6_0 \) along \( \alpha_1 \). Restricting this sequence to \( N_{\text{Mod}(S^6_0)}(f) \), and observing that \( \text{Mod}(S^3_0, q_1) \cong \mathbb{Z}_2 \), we obtain:

\[
1 \longrightarrow \langle T_{\alpha_1} \rangle \longrightarrow N_{\text{Mod}(S^6_0)}(f) \longrightarrow \mathbb{Z}_2 \times \text{Mod}(S^5_0, q_2) \longrightarrow 1
\]

Suppose first that \( f \) has no pseudo-Anosov components; in other words, the projection of \( f \) under the cutting homomorphism is trivial. In this case, \( f \) is central in \( N_{\text{Mod}(S^6_0)}(f) \), and from (17) we obtain

\[
W_{\text{Mod}(S^6_0)}(f) \cong \mathbb{Z}_2 \times \text{Mod}(S^5_0, q_2).
\]

Note that any model for \( E\mathbb{G}(\text{Mod}(S^6_0)) \) is also a model for \( E\mathbb{G}(\mathbb{Z} \times \text{Mod}(S^5_0)) \) also; thus \( \text{gd} W_{\text{Mod}(S^6_0)}(f) \leq \text{gd} \text{Mod}(S^5_0) \leq 2 \).

Thus, we may assume that the restriction of \( f \) to the \( S^5_0 \)-component of \( S^6_0 - \alpha_1 \) is pseudo-Anosov. In this case, Lemma 2.2 and equation (16) yield:
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(18) \[ 1 \to \mathbb{Z} \to N_{\text{Mod}(S^6_0)}(f) \to \mathbb{Z}_2 \times V \to 1, \]
where \( V \subseteq N_{\text{Mod}(S^6_0)}(f^2) \) is infinite and virtually cyclic. Thus, taking quotients:

(19) \[ 1 \to \mathbb{Z} \to W_{\text{Mod}(S^6_0)}(f) \to \mathbb{Z}_2 \times F \to 1, \]
and hence \( \text{gd} W_{\text{Mod}(S^6_0)}(f) \leq 1 \), as desired.

Subcase 1(ii): Each component of \( S^6_0 \setminus \alpha_1 \) contains three punctures. In this case, \( S^6_0 \setminus \alpha_1 = S^4_0 \sqcup S^4_0 \), and thus

\[ \text{Mod}(S^6_0 \sqcup S^4_0) \xrightarrow{\psi} (\text{Mod}(S^6_0) \times \text{Mod}(S^4_0)) \times \mathbb{Z}_2, \]
where \( \mathbb{Z}_2 \) is generated by a mapping class that interchanges the two components of \( S^6_0 \setminus \alpha_1 \). Furthermore, the image of \( \text{Mod}(S^6_0)_{\sigma} \) under the cutting homomorphism (3) is equal to

\[ \rho_{\sigma}(\text{Mod}(S^6_0)_{\sigma}) \psi_{\rho_{\sigma}}(\text{Mod}(S^4_0, q_1) \times \text{Mod}(S^4_0, q_2)) \times \mathbb{Z}_2, \]
where \( q_1 \) and \( q_2 \) are again the new punctures of \( S^6_0 \setminus \alpha_1 \). Let \( \text{Mod}(S^6_0)_{\sigma}^* \subseteq \text{Mod}(S^6_0)_{\sigma} \) be the subgroup whose elements do not permute the components of \( S^6_0 \setminus \alpha_1 \), and let \( \rho_{\sigma}^* := \rho_{\sigma}|_{\text{Mod}(S^6_0)_{\sigma}^*} \). We have the following diagram:

\[
\begin{array}{ccccccccc}
1 & \to & \langle T_{\alpha_1} \rangle & \to & \text{Mod}(S^6_0)_{\sigma}^* & \xrightarrow{\psi_{\rho_{\sigma}}^*} & \text{Mod}(S^4_0, q_1) \times \text{Mod}(S^4_0, q_2) & \to & 1 \\
\downarrow & & \downarrow Id & & \downarrow \text{inclusion} & & \downarrow & & \downarrow \\
1 & \to & \langle T_{\alpha_1} \rangle & \to & \text{Mod}(S^6_0)_{\sigma} & \xrightarrow{\psi_{\rho_{\sigma}}} & (\text{Mod}(S^4_0, q_1) \times \text{Mod}(S^4_0, q_2)) \times \mathbb{Z}_2 & \to & 1 \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & & & \mathbb{Z}_2 & \to & & & 1 \\
\end{array}
\]

With the above diagram in mind, we distinguish the following two cases, depending on the image of \( f \) under the cutting homomorphism associated to \( \sigma \):

(a) Suppose first that \( f \) has no pseudo-Anosov components. In this case, \( N_{\text{Mod}(S^6_0)}(f) = \text{Mod}(S^6_0)_{\sigma} \). Since \( \text{Mod}(S^4_0) \) is virtually free, equation (9) and Lemma 3.4 imply that

\[ \text{gd} (\text{Mod}(S^4_0, q_1) \times \text{Mod}(S^4_0, q_2)) \leq 2, \]
which in turn yields

\[ \text{gd} (\text{Mod}(S^4, q_1) \times \text{Mod}(S^4, q_2)) \cong \mathbb{Z}_2 \leq 4, \]

by Theorem 3.5. Finally, using Theorems 3.6 and Lemma 3.4, we obtain \( \text{gd} N_{\text{Mod}(S^6)}(f) \leq 5 \) and \( \text{gd} W_{\text{Mod}(S^6)}(f) \leq 4 \), as desired.

(b) Now suppose that \( f \) has at least one pseudo-Anosov component; equivalently, assume that \( \psi_{\rho_\sigma}(f) = (f_1, f_2, \text{Id}_{\mathbb{Z}_2}) \) is not trivial. Again, there are two cases to consider.

Suppose first that there exists \( (g_1, g_2, \gamma) \) in the image \( \psi_{\rho_\sigma}(N_{\text{Mod}(S^6)}(f)) \) with \( \gamma \neq \text{Id}_{\mathbb{Z}_2} \). In particular, \( f_1 \) is conjugate to \( f_2^{\pm 1} \), and hence both \( f_1 \) and \( f_2 \) are pseudo-Anosov. Let

\[ N_{\text{Mod}(S^6)}(f)^* = N_{\text{Mod}(S^6)}(f) \cap \text{Mod}(S^6)^*. \]

By restricting the diagram (20) we have

\[ \begin{array}{ccccccc}
1 & \longrightarrow & \langle T_{\alpha_1} \rangle & \longrightarrow & N_{\text{Mod}(S^6)}(f)^* & \psi_{\rho_\sigma}^* & V_1 \times V_2 & \longrightarrow & 1 \\
& & Id & \downarrow & inclusion & \downarrow & & \\
1 & \longrightarrow & \langle T_{\alpha_1} \rangle & \longrightarrow & N_{\text{Mod}(S^6)}(f) & \psi_{\rho_\sigma} & (V_1 \times V_2) \times \mathbb{Z}_2 & \longrightarrow & 1 \\
& & & & & & \text{Z}_2 & \downarrow & \uparrow \\
& & & & & & & & 1
\end{array} \]

where \( V_1 \times V_2 \subseteq N_{\text{Mod}(S^6, q_1)}(f_1) \times N_{\text{Mod}(S^6, q_2)}(f_2) \), which is a product of virtually cyclic subgroups by Lemma 2.2. Taking quotients in (21) we obtain:

\[ \begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & W_{\text{Mod}(S^6)}(f) & \longrightarrow & V_3 & \longrightarrow & 1 \\
\end{array} \]

where \( V_3 \) is a virtually cyclic subgroup. In particular, this implies that

\[ \text{gd} W_{\text{Mod}(S^6)}(f) \leq 2, \]

using Theorem 3.6. This finishes the proof of the case in consideration.

Next, suppose that for any element \( g \in N_{\text{Mod}(S^6)}(f) \), \( \psi_{\rho_\sigma}(g) = (g_1, g_2, \text{Id}_{\mathbb{Z}_2}) \), and thus \( N_{\text{Mod}(S^6)}(f) = N_{\text{Mod}(S^6)}(f)^* \). Hence \( N_{\text{Mod}(S^6)}(f) \) and \( W_{\text{Mod}(S^6)}(f) \)
We conclude that $g_d$.

From the exact sequences (28) and (29), plus (9) and Theorem 3.6, we conclude that $g_d$

in the case when $f_1$ is pseudo-Anosov and $f_2$ is the identity, or into

$$1 \rightarrow \mathbb{Z} \rightarrow N_{\text{Mod}(S_0^6)}(f) \xrightarrow{\psi_{\rho \sigma}} V_1 \times \text{Mod}(S_0^4, q_2) \xrightarrow{} 1$$

$$1 \rightarrow \mathbb{Z} \rightarrow W_{\text{Mod}(S_0^6)}(f) \rightarrow F \times \text{Mod}(S_0^4, q_2) \rightarrow 1,$$

when or both $f_1$ and $f_2$ are pseudo-Anosov; here, $V_1$, $V_2$ and $V_3$ are virtually cyclic subgroups and $F$ is finite. Proceeding as above, in both cases (24) and (25) we conclude that $\text{gd}(W_{\text{Mod}(S_0^6)}(f) \leq 2$. This finishes the discussion of Case 1.

**Case 2.** $|\sigma| = 2$. Write $\sigma = \{\alpha_1, \alpha_2\}$. Again, there are some cases to consider, depending on the topological type of $\alpha_1$ and $\alpha_2$.

Subcase 2(i): $\alpha_i$ bounds a disc with exactly two punctures, for $i = 1, 2$. Observe that $S_0^6 \setminus (\alpha_1 \cup \alpha_2) = S_0^4 \cup S_0^3 \cup S_0^3$. The cutting homomorphism (3) yields the exact sequence

$$1 \rightarrow \langle T_{\alpha_1}, T_{\alpha_2} \rangle \rightarrow \text{Mod}(S_0^6, \sigma) \rightarrow \text{Mod}(S_0^4 \cup S_0^3 \cup S_0^3, \sigma) \rightarrow 1,$$

noting that $(T_{\alpha_1}, T_{\alpha_2}) \simeq \mathbb{Z}^2$. Observe that

$$\text{Mod}(S_0^4 \cup S_0^3 \cup S_0^3, \sigma) \xrightarrow{\phi} \text{Mod}(S_0^4, q_1, q_2) \times (\text{Mod}(S_0^3, q_3) \times \text{Mod}(S_0^3, q_4)) \times \mathbb{Z}_2$$

Again, there are different cases depending on the image of $f$ under the cutting homomorphism. In this direction, suppose first that $f = T_{\alpha_1}^{k_1} T_{\alpha_2}^{k_2}$ with $\gcd(k_1, k_2) = 1$. In this case $N_{\text{Mod}(S_0^6)}(f) = \text{Mod}(S_0^6, \sigma)$, and we have:

$$1 \rightarrow \mathbb{Z}^2 \rightarrow N_{\text{Mod}(S_0^6)}(f) \xrightarrow{\phi_{\rho \sigma}} \text{Mod}(S_0^4, q_1, q_2) \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_2 \rightarrow 1$$

$$1 \rightarrow \mathbb{Z} \rightarrow W_{\text{Mod}(S_0^6)}(f) \rightarrow \text{Mod}(S_0^4, q_1, q_2) \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_2 \rightarrow 1$$

From the exact sequences (28) and (29), plus (9) and Theorem 3.6, we conclude that $\text{gd}(W_{\text{Mod}(S_0^6)}(f) \leq 2$, as desired.

Suppose now that $\psi_{\rho \sigma}(f) = (f_1, Id) \in \text{Mod}(S_0^4, q_1, q_2) \times F$, with $f_1$ is pseudo-Anosov. Then

$$1 \rightarrow \mathbb{Z}^2 \rightarrow N_{\text{Mod}(S_0^6)}(f) \xrightarrow{\phi_{\psi_{\rho \sigma}}} V \times F \rightarrow 1$$
and

\[ 1 \to \mathbb{Z}^2 \to W_{\text{Mod}(S_0^6)}(f) \xrightarrow{\psi \rho \sigma} F \times F \to 1 \]

where \( F, F' \) are finite groups and \( V \) is virtually cyclic. By Theorem 3.6 we conclude

\[ \text{gcd} N_{\text{Mod}(S_0^6)}(f) \leq 3 \quad \text{and} \quad \text{gcd} N_{\text{Mod}(S_0^6)}(f) \leq 2, \]

as desired.

**Subcase 2(ii):** \( \alpha_1 \) bounds a disc with exactly two punctures, and \( \alpha_2 \) bounds a disc with three punctures. In this case, the cutting homomorphism (3) again gives:

\[ 1 \to \langle \tau \alpha_1, \tau \alpha_2 \rangle \to \text{Mod}(S_0^6, \sigma) \to \text{Mod}(S_3^0 \sqcup S_3^0 \sqcup S_3^0, \sigma) \to 1. \]

However, in this case we have:

\[ \psi \rho \sigma (\text{Mod}(S_0^6, \sigma)) = \text{Mod}(S_0^6, q_1) \times \text{Mod}(S_0^6, q_2, q_3) \times \text{Mod}(S_0^6, q_4) \]

\[ \cong \mathbb{Z}_2 \times \text{Mod}(S_0^6, q_4). \]

Again, we distinguish two cases depending on the image of \( f \) under the cutting homomorphism. First, assume that \( f = T_{\alpha_1}^k T_{\alpha_2}^k \) with \( \text{gcd}(k_1, k_2) = 1 \). Then \( N_{\text{Mod}(S_0^6)}(f) = \text{Mod}(S_0^6, \sigma) \), and therefore we have the sequences

\[ 1 \to \mathbb{Z}^2 \to N_{\text{Mod}(S_0^6)}(f) \xrightarrow{\nu \rho \sigma} \mathbb{Z}_2 \times \text{Mod}(S_0^6, q_4) \to 1 \]

and

\[ 1 \to \mathbb{Z} \to W_{\text{Mod}(S_0^6)}(f) \xrightarrow{\nu \rho \sigma} \mathbb{Z}_2 \times \text{Mod}(S_0^6, q_4) \to 1. \]

From these sequences, we conclude that

\[ \text{gcd} N_{\text{Mod}(S_0^6)}(f) \leq 3 \quad \text{and} \quad \text{gcd} W_{\text{Mod}(S_0^6)}(f) \leq 2, \]

as desired.

Suppose now that \( \nu \rho \sigma f = (\text{Id}_{\mathbb{Z}_2}, f_1) \), where \( f_1 \) is pseudo-Anosov. From Lemma 2.2 we have the sequences

\[ 1 \to \mathbb{Z}^2 \to N_{\text{Mod}(S_0^6)}(f) \xrightarrow{\nu \rho \sigma} V \to 1 \]

and

\[ 1 \to \mathbb{Z} \to W_{\text{Mod}(S_0^6)}(f) \xrightarrow{\nu \rho \sigma} F \to 1, \]

where \( V' \) is a virtually cyclic subgroup and \( F \) is finite. Therefore \( \text{gcd} W_{\text{Mod}(S_0^6)}(f) \leq 2 \) again. This finishes the discussion of Case 2.

**Case 3.** \( |\sigma| = 3 \). Write \( \sigma = \{\alpha_1, \alpha_2, \alpha_3\} \), observing that \( S_0^6 \setminus (\alpha_1 \cup \alpha_2 \cup \alpha_3) \) is the disjoint union of four copies of \( S_3^3 \). Thus the cutting homomorphism (3) gives

\[ 1 \to \langle T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3} \rangle \to \text{Mod}(S_0^6, \sigma) \xrightarrow{\rho \sigma} \text{Mod}(S_0^3 \sqcup S_0^3 \sqcup S_0^3 \sqcup S_0^3, \sigma) \to 1, \]
observing that \( \langle T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3} \rangle \simeq \mathbb{Z}^3 \). Note that \( \text{Mod}(S_0^3 \sqcup S_0^3 \sqcup S_0^3) \) is a finite subgroup, and that \( f \) is in the kernel of \( \rho_{\sigma} \); moreover, \( f = T_{\alpha_1}^{k_1} T_{\alpha_2}^{k_2} T_{\alpha_3}^{k_3} \) with \( \gcd(k_1, k_2, k_3) = 1 \). Therefore we have the sequences

\[
1 \to \mathbb{Z}^3 \to N_{\text{Mod}(S_0^3)}(f) \to F \to 1
\]

and

\[
1 \to \mathbb{Z}^2 \to W_{\text{Mod}(S_0^3)}(f) \to F \to 1.
\]

In particular, \( \text{gd} W_{\text{Mod}(S_0^3)}(f) \leq 2 \), as desired. This finishes the discussion of Case 3, and also the proof of Theorem 1.1.

4.3. Proof of Corollaries 1.2 and 1.3. We now explain how to prove Corollaries 1.2 and 1.3. First, the latter follows immediately from the combination of Theorem 1.1, equation (4), and Lemma 3.7.

Now, Corollary 1.2 follows along equal lines, recalling that there are short exact sequences

\[
1 \to \mathbb{Z}_2 \to \text{Mod}(S_1^2) \to \text{Mod}(S_0^5) \to 1
\]

and

\[
1 \to \mathbb{Z}_2 \to \text{Mod}(S_0^5) \to \text{Mod}(S_6^0) \to 1;
\]

in both cases, the \( \mathbb{Z}_2 \) is generated by a hyperelliptic involution, see [7].

5. A GENERAL BOUND

In this section we prove Theorem 1.5. Before doing so, we remark that Degrijse-Petrosyan [5] have recently given the following bound for the virtually cyclic dimension of \( \text{Mod}(S_g^n) \):

**Theorem 5.1** ([5]). Let \( g, n \geq 0 \) with \( 3g - 3 + n \geq 1 \). Then

\[
\text{gd} \text{Mod}(S_g^n) \leq 9g + 3n - 8.
\]

The above result is stated in [5] for closed surfaces only; however the argument remains valid in full generality. For completeness we include a sketch here, which uses known facts about the geometry of the Weil-Petersson metric on Teichmüller space. We refer the reader to [21] for a thorough discussion on these and many other topics.

**Proof of Theorem 5.1**. Denote by \( T_{g,n} \) the Teichmüller space of \( S_g^n \), which is homeomorphic to \( \mathbb{R}^{6g+2n-6} \). Endow \( T_{g,n} \) with its Weil-Petersson metric, on which \( \text{Mod}(S_g^n) \) acts by semisimple isometries. The metric completion \( \overline{T}_{g,n} \) of \( T_{g,n} \) is a complete separable CAT(0) space, and the action of \( \text{Mod}(S_g^n) \) on \( T_{g,n} \) extends to a semisimple isometric action on \( \overline{T}_{g,n} \). Moreover, the stabiliser of a point is a virtually abelian group of rank \( \leq 3g + n - 3 \). At this point, [5] Corollary 3(iii)] implies that

\[
\text{gd} \text{Mod}(S_g^n) = (6g + 2n - 6) + (3g + n - 3) + 1 = 9g + 3n - 8,
\]

as desired. \( \Box \)
We now proceed to prove Theorem 1.5. Again, the main tool will be Proposition 3.2, this time combined with a result of Martínez-Pérez [20]. Before stating the latter, we need the following definition. Let $G$ be a group, and $F \in \mathcal{F}\mathcal{L}\mathcal{N}_G$ a finite subgroup. The length $l(F)$ of $F$ is defined as the largest natural number $k$ for which there is a chain $1 = F_0 < F_1 < \cdots < F_k = F$. The length of $G$ is

$$l(G) = \sup\{l(F) \mid F \in \mathcal{F}\mathcal{L}\mathcal{N}_G\}.$$ 

**Theorem 5.2.** [20] Thm. 3.10, Lem. 3.9] Suppose that $3 \leq \text{gd}(G) < \infty$. If $l(G)$ is finite, then

$$\text{gd}(G) \leq \text{vcd}(G) + l(G),$$

where $\text{vcd}(\cdot)$ denotes virtual cohomological dimension.

We begin with the following Lemma:

**Lemma 5.3.** Suppose $n \geq 4$. Let $f \in \text{PMOD}(S^n_0)$ and suppose that $(f)$ is maximal in $C^\infty_{\text{PMOD}(S^n_0)}$. Then

$$\text{gd}_W(\text{PMOD}(S^n_0))(f) \leq n - 4.$$ 

**Proof.** Suppose that $f$ has canonical reduction system $\sigma = (\alpha_1, \ldots, \alpha_k)$, the restriction $f|_{Y_j}$ is pseudo-Anosov for $j \in \{1, \ldots, r\}$ and $f|_{Y_j}$ is the identity for $i \in \{r+1, \ldots, k+1\}$. Note that for $j \in \{1, \ldots, r\}$, $Y_j$ is a sphere with at least four punctures. Thus from the definition of the cutting homomorphism [3] and the comment after it, we have

$$(30) \quad 1 \to \langle T_\sigma \rangle \to N_{\text{PMOD}(S^n_0)}(f) \xrightarrow{\rho_{\sigma}} U \times \prod_{i=r+1}^{k+1} \text{PMOD}(Y_i) \to 1$$

where $U$ is a finite-index subgroup of $\prod_{i=1}^{r-1}N_{\text{PMOD}(Y_i)}(f|_{Y_i}) \simeq \mathbb{Z}^r$. If $f \in \langle T_\sigma \rangle$, then $r = 0$ and

$$(31) \quad 1 \longrightarrow \langle T_\sigma \rangle/(f) \longrightarrow W_{\text{PMOD}(S^n_0)}(f) \xrightarrow{\tilde{\rho}_{\sigma}} \prod_{i=1}^{k+1} \text{PMOD}(Y_i) \longrightarrow 1$$

where $\langle T_\sigma \rangle/(f) \simeq \mathbb{Z}^{k-1}$. By Theorem 3.6, equation (9), and [11] Corollary 10.5 we have

$$\text{gd}_W(\text{PMOD}(S^n_0))(f) \leq k - 1 + \sum_{i=1}^{k+1} \text{gd}(\text{PMOD}(Y_i))$$

$$= k - 1 + \sum_{i=1}^{k+1} \text{vcd}(\text{PMOD}(Y_i))$$

$$= k - 1 + n - k - 3$$

$$= n - 4.$$

If, on the other hand, $f \notin \langle T_\sigma \rangle$, then $r \geq 1$. Let $\bar{f} = (f|_{Y_1}, \ldots, f|_{Y_r})$, then

$$(32) \quad 1 \to \langle T_\sigma \rangle \to W_{\text{PMOD}(S^n_0)}(f) \xrightarrow{\rho_{\sigma}} U/(\bar{f}) \times \prod_{i=r+1}^{k+1} \text{PMOD}(Y_i) \to 1$$
Note that $U/\langle \bar{f} \rangle$ is virtually $\mathbb{Z}^{r-1}$, and thus $\text{gd}(U/\langle \bar{f} \rangle) = r$. Again, applying Theorem [3.6] equation (9), and [11] Corollary 10.5 to (32) we have

$$
\text{gd} W_{\text{PMod}(S^n_0)}(f) \leq k + 1 + r - 1 + \sum_{i=r+1}^{k+1} \text{vcd} \text{PMod}(Y_i)
$$

$$
\leq k + r - 1 + \sum_{i=1}^{k+1} \text{vcd} \text{PMod}(Y_i) - r
$$

$$
= n - 4.
$$

□

We are finally in a position for proving Theorem 1.5:

**Proof of Theorem 1.5.** We will use Proposition 3.2. Let $\langle f \rangle$ maximal in $C_\infty^{\text{PMod}(S^n_0)}$, and note that $\text{gd} N_{\text{Mod}(S^n_0)}(f) \leq \text{gd} \text{Mod}(S^n_0) = n - 3$, by [1]. We now give a bound for $\text{gd} W_{\text{Mod}(S^n_0)}(f)$. We have the following exact sequence

$$
1 \longrightarrow N_{\text{PMod}(S^n_0)}(f) \longrightarrow N_{\text{Mod}(S^n_0)}(f) \longrightarrow A \longrightarrow 1
$$

where $A \subseteq \Sigma_n$. Since $\langle f \rangle \subset \text{PMod}(S^n_0)$ we have

$$
1 \longrightarrow W_{\text{PMod}(S^n_0)}(f) \longrightarrow W_{\text{Mod}(S^n_0)}(f) \longrightarrow A \longrightarrow 1
$$

We remark that $W_{\text{PMod}(S^n_0)}(f)$ is torsion-free since $\text{PMod}(S^n_0)$ is, and $\langle f \rangle$ is maximal in $C_\infty^{\text{PMod}(S^n_0)}$. Then $l(W_{\text{Mod}(S^n_0)}(f)) \leq l(A) \leq l(\Sigma_n)$. By a result of Cameron-Solomon-Turull [4, Theorem 1],

$$
l(\Sigma_n) = \left[ 3n - 1 \right] - b_n.
$$

At this point, Theorem 5.2 and Lemma 5.3 together imply

$$
\text{gd} W_{\text{Mod}(S^n_0)}(f) \leq \text{vcd} W_{\text{PMod}(S^n_0)}(f) + l(A)
$$

$$
= \text{vcd} W_{\text{PMod}(S^n_0)}(f) + l(A)
$$

$$
\leq n - 4 + \left[ \frac{3n-1}{2} \right] - b_n,
$$

where the equality holds since $\text{PMod}(S^n_0)$ has finite index in $\text{Mod}(S^n_0)$. □

### 5.1. Pure subgroups

As mentioned in the introduction, if one considers the pure mapping class group instead of the full mapping class group, then the situation is a lot easier. Indeed, after Harer’s calculation of the virtual cohomological dimension of the (pure) mapping class group, we get:

**Proposition 5.4.** Let $n \geq 4$. Then

$$
\text{gd} \text{PMod}(S^n_0) = \text{gd}(\text{PMod}(S^n_0)) + 1 = n - 2.
$$

The main ingredient of the proof is the following result. After [19], we say that a group $G$ satisfies (Max) if every subgroup $H \in \mathcal{VC}_G^\infty$ is contained in a unique $H_{\text{max}} \in \mathcal{VC}_G^\infty$ which is maximal in $\mathcal{VC}_G^\infty$. 

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Lemma 5.5. Let \( n \geq 4 \). Then the pure mapping class group \( \text{PMOD}(S^n_{0}) \) satisfies property (Max).

Proof. Let \( C \subset D \) be an inclusion of infinite cyclic subgroups. Then the centralizers of \( C \) and \( D \) in \( \text{PMOD}(S^n_{0}) \) are equal, since \( \text{PMOD}(S^n_{0}) \) has unique roots. If \( C \) is generated by a pseudo-Anosov class, then its centralizer is cyclic, and in particular is the unique maximal cyclic subgroup that contains \( C \). If \( C \) is generated by a reducible element \( f \), by Lemma 2.1 and the case of pseudo-Anosov classes, we obtain a unique maximal cyclic subgroup that contains \( C \). \( \square \)

Proof of Proposition 5.4. As mentioned in the introduction, Lück and Weiermann [19, Theorem 5.8] proved that every group with property (Max) satisfies inequality (1). Now, a combination of Harer’s calculation [10] of the virtual cohomological dimension of \( \text{Mod}(S^n_{0}) \) and [11, Corollary 10.5] yields \( \text{gd} \text{PMOD}(S^n_{0}) = n - 3 \). Therefore,

\[
\text{gd} \text{PMOD}(S^n_{0}) \leq n - 2.
\]

Now, \( S^n_{0} \) contains \( n - 2 \) disjoint essential curves \( \alpha_1, \ldots, \alpha_{n-3} \), and the subgroup \( \langle T_{\alpha_1}, \ldots, T_{\alpha_{n-3}} \rangle \) is isomorphic to \( \mathbb{Z}^{n-3} \). By property (10), we conclude \( \text{gd} \text{Mod}(S^n_{0}) \geq \text{gd} \mathbb{Z}^{n-3} = n - 2 \). \( \square \)

6. Appendix. Surfaces with boundary

Finally, we explain how to establish inequality (1) in the case of mapping class groups of surfaces with non-empty boundary. As indicated in the introduction, the arguments appear in the paper of Flores and González-Meneses [8] in the case when the surface has genus zero. For completeness, we give a self-contained argument here.

For \( S \) a surface with non-empty boundary, its mapping class group \( \text{Mod}(S) \) is again defined as the group of isotopy classes of self-homeomorphisms of \( S \), but this time the homeomorphisms and isotopies are required to fix each boundary component pointwise. As a by-product of this definition, \( \text{Mod}(S) \) has no torsion.

The main ingredient will be the following result of Martínez-Pérez [20]; again, \( \text{vcd}(\cdot) \) denotes virtual cohomological dimension:

Theorem 6.1. Let \( G \) be a group such that any finite subgroup is nilpotent. Suppose \( \text{vcd} G < \infty \) and \( \text{gd} G \geq 3 \), then

\[
\text{gd} G \leq \max_{F \in \mathcal{F}_{\mathbb{Z}^G}} \{ \text{vcd} G + \text{rk}(W_G F) \},
\]

where \( \text{rk}(\cdot) \) denotes the biggest rank of a finite elementary abelian subgroup.

Denoting by \( S^n_{g,b} \) the connected, orientable surface of genus \( g \), with \( n \) marked points and \( b \) boundary components. We remark that for \( m \geq 3 \) the congruence subgroups \( \text{Mod}(S^n_{g,b})[m] \) are finite-index subgroups that have property (Max) [14 Prop. 5.11], and property of uniqueness of roots [3].

We will make use of the following lemma:
Lemma 6.2. Let $b \geq 1$. If $g = 0$, suppose $b + n \geq 4$, and if $g \geq 1$, suppose $2g + b + n \geq 3$. Fix $m \geq 3$ and let $C \in C_{\text{Mod}(S^n_{g,b})[m]}^\infty$ maximal, then

\[ \underline{\text{gd}} W_{\text{Mod}(S^n_{g,b})}(C) \leq \underline{\text{gd}} \text{Mod}(S^n_{g,b}) + 1. \] (33)

Proof. We will use Theorem 6.1. First, the hypotheses imply that $\text{Mod}(S^n_{g,b})$ contains $\mathbb{Z}^k$ with $k \geq 3$, and thus $\text{vcd} \text{Mod}(S^n_{g,b}) + 1 \geq 3$. Also, observe that $\underline{\text{gd}} \text{Mod}(S^n_{g,b}) = \text{vcd} \text{Mod}(S^n_{g,b})$ since $\text{Mod}(S^n_{g,b})$ has no torsion.

Let $C \in C_{\text{Mod}(S^n_{g,b})[m]}^\infty$ be maximal. Note that any finite subgroup of $W_{\text{Mod}(S^n_{g,b})}(C)$ is of the form $V/C$ where $V$ is an infinite cyclic subgroup of $N_{\text{Mod}(S^n_{g,b})}(C)$. Again since $\text{Mod}(S^n_{g,b})$ has no torsion, it follows that finite subgroups of $W_{\text{Mod}(S^n_{g,b})}(C)$ are cyclic.

Write, for compactness, $Q = W_{\text{Mod}(S^n_{g,b})}(C)$. Applying Theorem 6.1 we have

\[ \underline{\text{gd}} Q \leq \max_{F \in F_{\text{FL}} Q} \{ \text{vcd} Q + r k \{ W_Q(F) \} \} = \text{vcd} Q + 1. \] (34)

We will give a bound for $\text{vcd} Q$. Consider the short exact sequence

\[ 1 \longrightarrow N_{\text{Mod}(S^n_{g,b})[m]}(C) \longrightarrow N_{\text{Mod}(S^n_{g,b})}(C) \longrightarrow K \longrightarrow 1, \] (35)

where $K$ is a subgroup of the finite group $\text{Aut}(H_1(S^n_{g,b}, \mathbb{Z}_m))$. Passing to the quotient, we have

\[ 1 \longrightarrow W_{\text{Mod}(S^n_{g,b})[m]}(C) \longrightarrow W_{\text{Mod}(S^n_{g,b})}(C) \longrightarrow K' \longrightarrow 1, \] (36)

where $K' \simeq K$. Since $C$ is maximal in $C_{\text{Mod}(S^n_{g,b})[m]}^\infty$, then $W_{\text{Mod}(S^n_{g,b})[m]}(C)$ is torsion-free, and thus

\[ \text{vcd} W_{\text{Mod}(S^n_{g,b})}(C) = \text{vcd} W_{\text{Mod}(S^n_{g,b})[m]}(C) \]
\[ \leq \underline{\text{gd}} W_{\text{Mod}(S^n_{g,b})[m]}(C) \]
\[ \leq \underline{\text{gd}} N_{\text{Mod}(S^n_{g,b})[m]}(C) \]
\[ \leq \underline{\text{gd}} \text{Mod}(S^n_{g,b}) \] (37)
\[ \leq \underline{\text{gd}} \text{Mod}(S^n_{g,b}) \] (38)
\[ \leq \underline{\text{gd}} \text{Mod}(S^n_{g,b}) \] (39)

where the equality holds since $\text{Mod}(S^n_{g,b})[m]$ has finite index in $\text{Mod}(S^n_{g,b})$, inequality (37) is given in the proof of [19, Thm. 5.8], and inequality (38) follows from subgroup inclusion. Thus the result follows.

Finally, we have the desired bound for surfaces with boundary:

**Proposition 6.3.** Let $b \geq 1$. If $g = 0$, suppose $b + n \geq 4$, and if $g \geq 1$, suppose $2g + b + n \geq 3$. Then

\[ \underline{\text{gd}} \text{Mod}(S^n_{g,b}) \leq \underline{\text{gd}} \text{Mod}(S^n_{g,b}) + 1, \]

and equality holds if $g \in \{0, 1\}$.
Proof. For the inequality \( \text{gd} \, \text{Mod}(S_{g,b}^n) \leq \text{gd} \, \text{Mod}(S_{g,b}^n) + 1 \), the proof is again an immediate consequence of Proposition 3.2, Remark 3.3, and Lemma 6.2, with \( d = \text{gd} \, \text{Mod}(S_{g,b}^n) \).

By [10, Thm. 4.1], \( \text{vcd} \, \text{Mod}(S_{g,b}^n) \) and the maximal rank of an abelian subgroup of \( \text{Mod}(S_{g,b}^n) \) are equal if only if \( g \in \{0,1\} \). Let \( \Lambda \) that abelian subgroup of rank \( \text{vcd} \, \text{Mod}(S_{g,b}^n) \), then \( \text{gd} \, \Lambda = \text{vcd} \, \text{Mod}(S_{g,b}^n) + 1 \), therefore \( \text{gd} \, \text{Mod}(S_{g,b}^n) \geq \text{vcd} \, \text{Mod}(S_{g,b}^n) + 1 \). \[ \square \]

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Universidad Autónoma de Madrid & ICMAT
E-mail address: aramayona@gmail.com

Centro de Ciencias Matemáticas., Universidad Nacional Autónoma de México, Campus Morelia, Ap.Postal 61-3 Xangari, Morelia, Michoacán. MÉXICO 58089
E-mail address: daniel@matmor.unam.mx

Centro de Ciencias Matemáticas., Universidad Nacional Autónoma de México, Campus Morelia, Ap.Postal 61-3 Xangari, Morelia, Michoacán. MÉXICO 58089
E-mail address: aletn@matmor.unam.mx