Rational points on certain homogeneous varieties

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Abstract
Let $L$ be a simply-connected simple connected algebraic group over a number field $F$, and $H$ be a semisimple absolutely maximal connected $F$-subgroup of $L$. Let $\Delta(H)$ be the image of $H$ diagonally embedded in $L^n$. Under a cohomological condition, we prove an asymptotic formula for the number of rational points of bounded height on projective equivariant compactifications of $\Delta(H)\backslash L^n$ with respect to a balanced line bundle.

Keywords
Rational points · Height · Counting

Mathematics Subject Classification
14G05 · 11G50

1 Introduction
Let $X$ be a smooth projective variety over a number field $F$. $X$ is called Fano if the anticanonical divisor $-K_X$ is ample. Given an ample line bundle $\mathcal{L}$ on $X$, and an adelic metrization $\mathcal{L}$ of $\mathcal{L}$, we can define an associated height function

$$H_\mathcal{L} : X(F) \to \mathbb{R}_{\geq 0}$$

on the set of $F$-rational functions (see [1, Section 2]). Take a suitable Zariski open $X^\circ \subset X$, Manin’s conjecture [2] predicts the asymptotic growth of the number of rational points with height at most $T$ in $X^\circ(F)$, as $T \to \infty$. Consider the counting function

$$N(X^\circ(F), \mathcal{L}, T) = \# \{ x \in X^\circ(F) : H_\mathcal{L}(x) \leq T \},$$

it is conjectured that
\[ N(X^\circ(F), \mathcal{L}, T) \sim c(\mathcal{L}) T^{a(\mathcal{L})} (\log T)^{b(\mathcal{L}) - 1}, \]

where \(c(\mathcal{L}) > 0\), and \(a(\mathcal{L}), b(\mathcal{L})\) are geometric constants attached to \(X\) and \(\mathcal{L}\), which we define below.

Let \(\mathcal{L}\) be an ample line bundle on \(X\), and let \(\Lambda_{\text{eff}}(X)\) denote the pseudo-effective cone in the real Néron–Severi group \(\text{NS}(X, \mathbb{R})\). We define

\[
\begin{align*}
    a(\mathcal{L}) &= \inf \{ t \in \mathbb{Q} : t[\mathcal{L}] + [K_X] \in \Lambda_{\text{eff}}(X) \}, \\
    b(\mathcal{L}) &= \text{the maximal codimension of the face containing } a(\mathcal{L})[\mathcal{L}] + [K_X].
\end{align*}
\]  

(1.1)

The property of these two constants was systematically studied in [6], where the notion of \textit{balanced} line bundle was defined.

In the equivariant setting, let \(G\) be an algebraic group over \(F\), and \(H\) be an \(F\)-subgroup of \(G\). Take \(X^\circ = H \backslash G\), and let \(X\) be a smooth \(G\)-equivariant compactification of \(X^\circ\). Let \(\mathcal{L}\) be an ample line bundle on \(X\). Several cases have been studied in recent years using ergodic-theoretical methods. Gorodnik, Maucourant and Oh [3] studied the case where \(G = H \times H\) using adelic mixing, and this case was studied by Shalika, Takloo-Bighash and Tschinkel [8] independently using deep results from the theory of automorphic forms. Later Gorodnik and Oh [4] proved Manin’s conjecture for \(G\) a connected semisimple \(F\)-group, and \(H\) a semisimple maximal connected \(F\)-subgroup of \(G\), under certain cohomological condition. Gorodnik, Takloo-Bighash and Tschinkel [5] settled the case where \(H\) is a simple connected \(F\)-group diagonally embedded into \(G = H^n\), and \(\mathcal{L} = -K_X\) is the anticanonical bundle.

**Definition 1.1** Let \(X\) be an equivariant compactification of \(X^\circ = H \backslash G\) and \(H' \subset G\) any closed proper subgroup containing the diagonal, i.e. \(H \subset H'\). Let \(X' \subseteq X\) be the induced equivariant compactification of \(H \backslash H'\). A line bundle \(\mathcal{L}\) on \(X\) is called \textit{balanced with respect to} \(H'\) if

\[
(a(\mathcal{L}|_{X'}), b(\mathcal{L}|_{X'})) < (a(\mathcal{L}), b(\mathcal{L})),
\]

in the lexicographic ordering. It is called \textit{balanced} if this property holds for every such \(H' \subset G\).

In this article we confirm the following asymptotic formula of Manin type.

**Theorem 1.2** Let \(L\) be a simply-connected absolutely-simple connected algebraic group over a number field \(F\), and \(H\) be a semisimple absolutely maximal connected \(F\)-subgroup of \(L\). Let \(G = L^n\), and \(\Delta(H)\) be the image of the diagonal embedding of \(H\) into \(G\). Let \(X\) be a \(G\)-equivariant compactification of \(X^\circ = \Delta(H) \backslash G\). Let \(\mathcal{L}\) be a balanced line bundle on \(X\), with a smooth adelic metrization \(\mathcal{L}\). Suppose that for any completion \(F_v\) of \(F\), the map of Galois cohomology \(H^1(F_v, H) \to H^1(F_v, L)\) is injective. Then

\[ N(X^\circ(F), \mathcal{L}, T) \sim c_L \cdot T^{a(\mathcal{L})} (\log T)^{b(\mathcal{L}) - 1}, \]

(1.2)

as \(T \to \infty\), for some \(c_L > 0\).
Our proof is based heavily on [5], and also combines techniques from [1, 4].

**Example 1.3** Let $L = \text{SL}_{2m}$ and $H = \text{Sp}_{2m}$, where $m \geq 2$ is an integer. Let $X$ be an equivariant compactification of $\Delta(H) \backslash L^n$, and we take $L$ to be its anticanonical bundle $-K_X$. By [6, Theorem 1.3] and Proposition 2.1 below, $-K_X$ is balanced. Moreover, since $H^1(F_v, H)$ is trivial, we know that $H^1(F_v, H) \to H^1(F_v, L)$ is injective. Hence the conditions of Theorem 1.2 are satisfied, and the asymptotic formula (1.2) holds for this case.

### 2 Intermediate subgroups

Let $F$ be an algebraically closed field of characteristic 0. Let $L$ be a simply-connected simple connected algebraic group defined over $F$, and $H$ be a semisimple maximal connected $F$-subgroup of $L$. $G = L^n$ is the $n$-fold direct product of $L$.

Let $\Delta_1(H)$ denote the diagonal embedding of $H$ into $G$. In this section we classify all the subgroups of $G$ which contain $\Delta_1(H)$. For any group $N$ and integer $r$, let $\Delta_r(N)$ denote the image of the diagonal embedding of $N$ into $N^r$.

**Proposition 2.1** (cf. [5, Proposition 4.1]) Suppose $M$ is a connected algebraic group such that $\Delta_1(H) \subset M \subset G$. Then there exist positive integers $n_1, \ldots, n_k; m_1, \ldots, m_l$ such that $n_1 + \cdots + n_k + m_1 + \cdots + m_l = n$, and that up to permutation of indices, $M$ is the image of the morphism

$$\prod_{i=1}^k \Delta_{n_i}(H) \prod_{j=1}^l \Delta_{m_j}(L) \to L^n$$

$$(h_1, \ldots, h_s, g_1, \ldots, g_t) \mapsto (h_1, \ldots, h_s, \rho_1(g_1), \ldots, \rho_t(g_t)),$$

where $\rho_i : L \to L$ is an automorphism of $L$ fixing each element in $H$.

**Proof** We prove by induction on $n$. The case $n = 1$ follows from the maximality of $H$. Suppose the proposition holds for $n - 1$. For $G = L^n$, let $p_1 : M \to L^{n-1}$ denote the projection onto the first $n - 1$ entries, and $p_2 : M \to L$ the projection onto the last entry. After possibly applying a permutation, we can assume $p_2$ is onto, i.e. $p_2(M) = L$. Otherwise $M$ is contained in $H^n$ and the conclusion holds by [5, Proposition 4.1].

Now consider $N := p_2(\ker p_1)$. Since $M$ contains $\Delta(H)$, it follows that $N$ is normalized by $H$. If $N$ is contained in $H$, then $N$ is a normal subgroup of $H$. If $N$ is not contained in $H$, then $H$ is a proper subgroup of $NH$, but $H$ is maximal, hence $NH = L$. Since $L$ is simple and $N$ is normal in $L$, we conclude that $N = L$. Now we discuss all the possible cases.

**Case 1.** $N = L$. In this case, $M = p_1(M) \times L$. By inductive hypothesis we know that $p_1(M)$ is of the form as in the proposition. Hence $M = p_1(M) \times L$ also satisfies the conclusion.
Case 2. \( N \) is an infinite normal subgroup of \( H \). Let \( N_L(N) \) denote the normalizer of \( N \) in \( L \), then it is a proper subgroup of \( L \) containing \( H \). Since \( p_2(M) = L \), we can take \( g \in p_2(M) \) such that \( g \notin N_L(N) \). Then there exists \( a \in L^{\pi-1} \) such that \( (a, g) \in M \). It follows that \( (a^{-1}, Ng^{-1}) \subset M \), and this implies \( gNg^{-1} \subset N \). Therefore \( g \in N_L(N) \), contradicting the choice of \( g \).

Case 3. \( N \) is finite. Then \( p_1 \) is an isogeny. By inductive hypothesis we may assume that \( p_1(M) = L^r H^s \). Since \( p_2(M) = L \), we get a surjection \( \overline{p_2} : L^r H^s \to L/N \), whose kernel is denoted by \( K \). Since \( L \) and \( H \) are both semisimple, by Lemma 2.2 below, the neutral component \( K^0 \) of \( K \) is of the form \( L^{r-1} H^s \).

\[
\begin{array}{ccc}
p_1(M) & \sim & L^r H^s \\
\downarrow & & \downarrow \overline{p_2} \\
p_1(M)/K_0 & \mathrel{\mathrel{\mathrel{\mathrel{\sim}}}} & L \\
\downarrow \phi & & \downarrow \pi \\
p_1(M)/K & \xrightarrow{\psi} & L/N
\end{array}
\]

Since \( \psi \) is an automorphism of \( L/N \), by the following Theorem 2.3, \( \psi \) induces an isomorphism \( \phi : p_1(M) \to L \). Further lifting to \( p_1(M) \), we can say that \( \overline{p_2} \) induces \( \overline{\phi} : p_1(M) \to L \), whose mapping graph \( M_0 \) is contained in \( M \). Since \( M \) is connected, we know that \( M_0 = M \). In other words, \( p_1 \) is surjective. Therefore \( \overline{p_2} \) induces an automorphism \( \rho : L \to L \). Since \( \Delta(H) \) is contained in \( M \), we know that \( \rho \) fixes each element in \( H \).

Lemma 2.2 Suppose \( G_i \)'s are simple algebraic groups, and \( H \) is a connected normal subgroup of \( \prod_{i=1}^{n} G_i \). Then there exist \( 1 \leq i_1 < \cdots < i_k \leq n \) such that \( H = \prod_{j=1}^{k} G_{i_j} \).

Proof We prove by induction on \( n \). Without loss of generality, we may assume that the projection \( p_2(H) \) of \( H \) onto the last summand is nontrivial. Since \( G_n \) is simple, and \( p_2(H) \) is connected normal in \( G_n \), we know that \( p_2(H) = G_n \). Suppose \( h = (h_1, h_2) \) where \( h_1 \in \prod_{i=1}^{n-1} G_i \). Then for any \( g \in G_n \) we have \( (h_1, gh_2 g^{-1}) \in H \), as \( H \) is normal. This implies that \( gh_2 g^{-1} h_2^{-1} \in p_2(\ker p_1) \). Hence \( p_2(\ker p_1) \) contains \([G_n, G_n], \) the commutator group of \( G_n \). Since \( G_n \) is simple, we know that \( p_2(\ker p_1) = G_n \). Therefore \( H = p_1(H) \times G_n \). And by inductive hypothesis \( p_1(H) \) is already a direct product.

We now see that the study of automorphism group plays an import role here. Hence let us recall the following result.

Theorem 2.3 ([7, Theorem 2.8]) For any simply connected semisimple group \( G \), the automorphism group \( \text{Aut } G \) is the semidirect product of \( \text{Int } G \simeq \overline{G} \) by \( \text{Sym}(R) \), where \( R \) is the Dynkin diagram. If \( G \) is an arbitrary semisimple group and \( \widetilde{G} \xrightarrow{\pi} G \) is a universal covering, then \( \text{Aut } \widetilde{G} \) is isomorphic to the subgroup of \( \text{Aut } G \) fixing \( \ker \pi \), the fundamental group.

Here we also give a corollary of this theorem, which will be used in the next section.
Corollary 2.4 Let $G$ be a simply-connected semisimple group and $H$ be a maximal connected subgroup of $G$. Then the set

$$\{ \rho \in \text{Aut } G : \rho(h) = h \text{ for all } h \in H \}$$

is finite.

**Proof** By Theorem 2.3, it suffices to consider inner automorphisms only, as $\text{Sym } R$ is already finite. Suppose $\rho(g) = g_0g_0^{-1}$ for some $g_0 \in G$, then $g_0 \in Z_H(G)$. Since $H$ is maximal and $G$ is semisimple, we know that $Z_H(G)$ is finite. Hence we only have finitely many choices of such automorphisms. $\square$

3 Equidistribution of translated measures

Let $F$ be a number field. Let $L$ be a connected semisimple group defined over $F$, and $H$ a semisimple maximal connected subgroup of $L$. Notice that in this section we drop the assumption that $L$ is simply-connected. Let $\pi: \widetilde{L} \to L$ be the universal cover of $L$ and $W$ a compact subgroup of $L(\mathbb{A})$ such that $W \cap L(\mathbb{A}_f)$ is open in $L(\mathbb{A}_f)$. Define

$$L_W := L(F) \pi(\widetilde{L}(\mathbb{A}))W$$

and

$$Y_W := L(F) \backslash L_W.$$

Let $C_c(Y_W)^W$ denote the space of compactly supported $W$-invariant continuous functions on $Y_W$. Let $G = L^n$ be the $n$-fold direct product of $L$, and $V = W^n$ is an open subgroup of $G$. Set

$$G_V := G(F) \pi(\widetilde{G}(\mathbb{A}))V = L_W \times \cdots \times L_W.$$

**Theorem 3.1** Let $L$ be a connected simple group defined over $F$. Let $H$ be a simple maximal connected subgroup of $L$. Suppose

$$\{(b^{(k)}_1, \ldots, b^{(k)}_n)\} \subset L_W \times \cdots \times L_W$$

is a sequence such that

(1) For any $i \neq j$,

$$\lim_{k \to \infty} (b^{(k)}_i)^{-1}b^{(k)}_j = \infty.$$

(2) For any $i$, $b^{(k)}_i \to \infty$ modulo $H(\mathbb{A})$, as $k \to \infty$.

Then for all $f_1, \ldots, f_n \in C_c(Y_W)^W$, we have

$$\lim_{k \to \infty} \int_{Y_W} f_1(yb^{(k)}_1) \cdots f_n(yb^{(k)}_n) \, d\nu(y) = \int_{Y_W} f_1 \, d\mu \cdots \int_{Y_W} f_n \, d\mu.$$
where $\nu$ is the invariant probability measure supported on $H(F) \setminus (H(\tilde{A}) \cap L_W)$ considered as a measure on $Y_{W}$ via pushing forward by the natural injection, and $\mu$ is the probability Haar measure on $Y_{W}$.

**Proof** Set

$$W^{(k)} = \bigcap_{i=1}^{n} b_i^{(k)} W(b_i^{(k)})^{-1}.$$  

From the proof of [4, Corollary 4.14], we know that $H(F) \pi(\tilde{H}(A))(W^{(k)} \cap H(A_f))$ is a normal subgroup of $H(\tilde{A}) \cap L_W$ with finite index, for any $k$. Hence there exists a finite subset $\Delta^{(k)} \subseteq H(\tilde{A}) \cap L_W$ such that

$$H(\tilde{A}) \cap L_W = \bigcup_{x \in \Delta^{(k)}} H(F) \pi(\tilde{H}(A)) x (W^{(k)} \cap H(A_f)),$$

where the union is a disjoint union. We note that by [4, Corollary 4.10], $H(F) \pi(\tilde{H}(A))$ is normal in $H(\tilde{A}) \cap L_W$. Observe that the function

$$y \mapsto f_1(yb_1^{(k)}) \cdots f_n(yb_n^{(k)})$$

is right invariant under $W^{(k)}$. Therefore

$$\int_{Y_{W}} f_1(yb_1^{(k)}) \cdots f_n(yb_n^{(k)}) \, d\nu(y)$$

$$= \sum_{x \in \Delta^{(k)}} \int_{W^{(k)}} \int_{x_0 \pi(\tilde{H}(A))x} f_1(ub_1^{(k)}) \cdots f_n(ub_n^{(k)}) \, d\mu_{x}^{(k)}(u) \, dw$$  

$$= \sum_{x \in \Delta^{(k)}} \frac{1}{\# \Delta^{(k)}} \int_{x_0 \pi(\tilde{H}(A))x} f_1(ub_1^{(k)}) \cdots f_n(ub_n^{(k)}) \, d\mu_{x}^{(k)}(u),$$  

where $\mu_{x}^{(k)}$ is the invariant probability measure supported on $x_0 \pi(\tilde{H}(A))x$, and $dw$ is the normalized invariant measure on $W^{(k)}$.  

Now for any $k$, choose $x^{(k)} \in \Delta^{(k)}$, and set

$$c^{(k)} = (x^{(k)}b_1^{(k)}, \ldots, x^{(k)}b_n^{(k)}).$$

Since $(c_i^{(k)})^{-1} c_j^{(k)} = (b_i^{(k)})^{-1} b_j^{(k)}$ and $x^{(k)} \in H(\tilde{A})$, we can see that the sequence $\{c^{(k)}\}$ still satisfies both conditions in the theorem. According to the definition we can rewrite the last integral in (3.1) as

$$\sum_{x \in \Delta^{(k)}} \frac{1}{\# \Delta^{(k)}} \int_{x_0 \pi(\tilde{H}(A))x} f_1(ub_1^{(k)}) \cdots f_n(ub_n^{(k)}) \, d\mu_{x}^{(k)}(u),$$
\[
\int_{x_0\pi(H(\mathbb{A}))} f_1(u b_1^{(k)}) \cdots f_n(u b_n^{(k)}) \, d\mu_x(u) = \int_{G(F) \backslash G_V} f_1 \otimes \cdots \otimes f_n \, d(c^{(k)} \cdot \lambda_H).
\]

(3.2)

Now it remains to determine the limit points of \(c^{(k)} \cdot \lambda_H\), where \(\lambda_H\) is the invariant probability measure on \(\pi(\Delta(H)(\mathbb{A}))\).

Since \(H\) is maximal in \(L\), the centralizer \(Z_L(H)\) of \(H\) in \(L\) is anisotropic. Hence the centralizer of \(\Delta(H)\) in \(G = L^n\), which equals \(Z_L(H) \times \cdots \times Z_L(H)\), is also anisotropic over \(F\). Therefore, by [4, Theorem 1.7(1)] we know that \(\{c^{(k)} \cdot \lambda_H\}\) is relatively compact. Suppose \(\mu\) is a limit point. By [4, Theorem 1.7(2)] we get a connected \(F\)-subgroup \(M\) of \(G\) and a sequence \(\delta^{(k)} \in G(F)\) such that

\[
\Delta(H) \subset (\delta^{(k)})^{-1} M \delta^{(k)} \subset G.
\]

Hence

\[
M = \delta^{(k)} N_k \delta^{(k)}^{-1},
\]

(3.3)

where \(N_k\) is an intermediate subgroup as described in Proposition 2.1. Now it suffices to show that \(M = G\), and the theorem will follow by a similar argument to [5, Theorem 5.1].

We prove \(M = G\) by contradiction. Suppose \(M\) is a proper subgroup of \(G\). By Corollary 2.4, the number of intermediate subgroups is finite, and thus by passing to a subsequence we may assume that \(N_k = N\) for all \(k\). Here \(N\) is a proper subgroup of \(G = L^n\).

Case 1. There is no \(\Delta_{n_1}(H)\) part. Since \(N\) is a proper subgroup of \(G\), we can find \(i \neq j\) such that \(\pi_{ij}(N) = \Delta_2(L)\). Set \(\sigma^{(k)} = (\delta^{(1)})^{-1} \delta^{(k)}\), we see from equation (3.3) that

\[
z^{(k)} := (\sigma_i^{(k)})^{-1} \sigma_j^{(k)} \in Z(L)(F).
\]

By [4, Theorem 1.7(2)], there exists \(h^{(n)} \in \pi(\Delta(H)(\mathbb{A}))\) such that \(\delta^{(k)} h^{(n)} c^{(k)}\) converges. In particular, \((z^{(k)})^{-1} c_i^{(k)})^{-1} c_j^{(k)}\) converges. Since \(Z(L)\) is finite, \(\{z^{(k)}\}\) is a compact set. Hence \((c_i^{(k)})^{-1} c_j^{(k)}\) converges, but this contradicts to the fact that pairwise ratios diverge.

Case 2. There is a \(\Delta_{n_1}(H)\) part. We may assume that \(\pi_1(N) = H\). We see from (3.3) that \(\sigma_1^{(k)}\) is in the normalizer \(N_L(H)\) of \(H\). But \(H\) has finite index in \(N_L(H)\), hence \(\{\sigma_1^{(k)}\}\) is bounded modulo \(H\). Again by [4, Theorem 1.7], the sequence \(\sigma_1^{(k)} h_1^{(k)} c_1^{(k)}\) converges. This contradicts to the fact that \(\{c_1^{(k)}\}\) diverges modulo \(H\).

Therefore, \(M = G\), and still by [4, Theorem 1.7] we know that there exists a normal subgroup \(M_0\) of \(M(\mathbb{A}) = G(\mathbb{A})\) containing \(G(F) \pi(G(\mathbb{A}))\) and \(g \in \pi(G(\mathbb{A}))\) such that for any \(f \in \mathcal{C}_c(G(F))\), we have the following:

\[
\int_{G(F) \backslash G_V} f \, d\mu = \int_{G(F) \backslash G_V} f \, d(g \cdot v_{M_0}) = \int_{G(F) \backslash G_V} f \, dv_{M_0} = \int_{G(F) \backslash G_V} f(uv) \, dv \, dv_{M_0} = \int_{G(F) \backslash G_V} f \, dz,
\]

(3.4)
where $d\mu_{M_0}$ is the pushforward of the Haar measure on $x_0M_0$, and $dz$ is the Haar measure on $G(F)\backslash G_V$.

Finally, combining equations (3.1), (3.2), (3.4) we finish the proof of the theorem. $\square$

4 Volume computation

In this section, let $L$ be a simply-connected simple connected algebraic group over a number field $F$, and $H$ be a simple maximal connected $F$-subgroup of $L$. Denote by $G$ the $n$-fold direct product of $L$. We treat $H$ as a subgroup of $G$ via diagonal embedding. Let $X$ be a smooth projective equivariant compactification of $X^0 = H \backslash G$. Let $L$ be an ample line bundle on $X$. By [5, Proposition 2.1], we can write

$$L = \sum_{\alpha \in A} \lambda_\alpha D_\alpha, \quad \lambda_\alpha \in \mathbb{Q}_{>0},$$

and

$$-K_X = \sum_{\alpha \in A} \kappa_\alpha D_\alpha,$$

where all $\kappa_\alpha \geq 1$.

Given a smooth metrization $L$ of $L$, as in [1, Section 2.1] we have a corresponding height function

$$H = H_L : X^0(F) \to \mathbb{R}_{>0}.$$

There exists a compact open subgroup $V$ of $G(\AA_f)$ such that the adelic height function $H$ is invariant under $V$. By possibly replacing $V$ with a smaller compact open subgroup, we may assume that $V = W \times \cdots \times W$ for a compact open subgroup $W$ of $H(\AA_f)$.

To compute the volume of the height ball via standard Tauberian argument, we need the following result.

**Theorem 4.1** ([5, Theorem 3.3]) Let $G$ be a connected semisimple algebraic group and $H \subset G$ a closed subgroup, defined over a number field $F$, such that the map $H^1(E, H) \to H^1(E, G)$ is injective, for $E$ being either $F$ or a completion of $F$. Let $X$ be a smooth projective equivariant compactification of $X^0 = H \backslash G$ with normal crossing boundary $\cup_{\alpha \in A} D_\alpha$ and

$$H : \mathbb{C}^A \times X^0(\AA) \to \mathbb{C}$$

an adelic height system. Then there exists a function $\Phi$, holomorphic and bounded in vertical strips for $\Re s_\alpha > \kappa_\alpha - \epsilon$, for some $\epsilon > 0$, such that for $s = (s_\alpha)$ in this domain one has

$$\int_{X^0(\AA)} H(s, x)^{-1} dx = \prod_{\alpha \in A} \zeta_F(s_\alpha - \kappa_\alpha + 1) \cdot \Phi(s),$$

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where $\zeta_F$ is the Dedekind zeta function.

Let $B_T$ be the height ball in $X_V = X^0(\mathbb{A})$ defined by

$$B_T = B_{T,L} = \{x \in X_V : H_L(x) < T\}.$$ 

We have the following asymptotic formula for the volume of the height ball.

**Lemma 4.2** ([5, Lemma 6.3]) Let $\mathcal{L}$ be an ample line bundle on $X$, and $\mathcal{L}$ be a smooth metrization of $\mathcal{L}$. Then

$$\text{vol}(B_T) \sim c_L \cdot T^{a_L} (\log T)^{b_L - 1}$$

with $a_L, b_L$ as in (1.1) and $c_L > 0$.

The following lemma is a generalized version of [5, Lemma 6.6].

**Lemma 4.3** Let $H \subset M \subset G$ be semisimple connected algebraic groups defined over $F$, and $\mathcal{L}$ be a balanced line bundle on $H \setminus G$. Let $K$ be a compact subset of $G(\mathbb{A})$ such that $k_1k_2^{-1} \notin M(\mathbb{A})$ for all distinct $k_1, k_2 \in K$. Suppose $H^1(F_v, H) \to H^1(F_v, G)$ is injective for any place $v$ of $F$, then for any smooth adelic metrization of $\mathcal{L}$, we have

$$\lim_{T \to \infty} \frac{\text{vol}(B_T \cap (H \setminus M)(\mathbb{A}) \cdot K)}{\text{vol}(B_T)} = 0.$$ 

**Proof** Since we have injectivity of cohomology, we can apply Theorem 4.1 to $H \setminus M$ and $H \setminus G$. Now we know the poles of the height integral and their orders, and we can apply the standard Tauberian argument to obtain the volume asymptotics (see [1, Appendix A]). The proof of [5, Lemma 6.6] works with no changes needed. \qed

**Corollary 4.4** Let $G = L^n$ and $H$ embeds in $G$ diagonally. Let the volume be given by the Tamagawa measure with respect to a balanced line bundle $\mathcal{L}$.

1. Let $K_1$ be a compact subset of $L(\mathbb{A})$. For any $1 \leq i < j \leq n$, we have

$$\lim_{T \to \infty} \frac{\text{vol}(B_T \cap \{(x_1, \ldots, x_n) \in (H \setminus G)(\mathbb{A}) : (x_i)^{-1}x_j \in K_1\})}{\text{vol}(B_T)} = 0.$$ 

2. Let $K_2$ be a compact subset of $H \setminus L(\mathbb{A})$. Fix $1 \leq i \leq n$, we have

$$\lim_{T \to \infty} \frac{\text{vol}(B_T \cap \{(x_1, \ldots, x_n) \in (H \setminus G)(\mathbb{A}) : x_i \in K_2\})}{\text{vol}(B_T)} = 0.$$ 

**Proof** In (1) we take

$$M = \{(x_1, \ldots, x_n) \in G : x_i = x_j\},$$

and

$$K = \{(x_k) \in G(\mathbb{A}) : x_j \in K_1; x_k = e \text{ for all } k \neq \hat{j}\}.$$
In (2) we take

\[ M = \{(x_1, \ldots, x_n) \in G : x_i \in H\}, \]

and

\[ K = \{(x_k) \in G(\mathbb{A}) : x_i \in K_2; x_k = e \text{ for all } k \neq i\}. \]

Now it remains to apply Lemma 4.3.

We recall the definition of \( L_W, Y_W \) and \( G_V \) from Sect. 3. Since \( L \) is simply-connected, we have \( L_W = L(F) \pi(\tilde{L}(\mathbb{A})) W = L(\mathbb{A}), G_V = (L_W)^n = G(\mathbb{A}). \) Denote \( Y = Y_W = L(F) \backslash L(\mathbb{A}) \) and \( Z = Z_V = G(F) \backslash G(\mathbb{A}). \) Let \( \nu \) be the invariant probability measure supported on \( H(F) \backslash H(\mathbb{A}). \) Let \( dx \) denote the Tamagawa measures on \( X^\circ(\mathbb{A}), \) and \( dz \) denote the invariant probability measure on \( G(F) \backslash G(\mathbb{A}). \)

We recall the following result from [5].

**Proposition 4.5** ([5, Corollary 6.8]) For any \( f \in C_c(Z), \)

\[
\lim_{T \to \infty} \frac{1}{\text{vol}(B_T)} \int_{B_T} dx \int_Y f(yx) d\nu(y) = \int_Z f \, dz.
\]

**Proof** We follow the proof of [5, Corollary 6.8]. By the Stone–Weierstrass theorem, it suffices to consider functions of the form \( f = f_1 \otimes \cdots \otimes f_n \) with \( f_i \in C_c(Y), \) and it is shown that we may assume \( f_i \) to be \( W \)-invariant. In this case,

\[
I(x) = \int_Y f(yx) \, d\nu(y) = \int_Y f_1(yx_1) \cdots f_n(yx_n) \, d\nu(y).
\]

Given compact subsets \( K_1 \) of \( L(\mathbb{A}) \) and \( K_2 \) of \( (H \backslash L)(\mathbb{A}), \) we set

\[
B_{T,K_1,K_2} = \left\{ x \in B_T : (x_i)^{-1}x_j \notin K_1, i \neq j; x_i \notin K_2 \right\}.
\]

By Theorem 3.1, for every \( \epsilon > 0, \) there exists \( K_1 \) and \( K_2 \) such that for all \( x = (x_1, \ldots, x_n) \in B_{T,K_1,K_2}, \) we have

\[
|I(x) - \int_Y f_1 \, d\mu \cdots \int_Y f_n \, d\mu| < \epsilon,
\]

and

\[
\int_{B_{T,K_1,K_2}} I(x) \, dx = \text{vol}(B_{T,K_1,K_2}) \int_Y f_1 \, d\mu \cdots \int_Y f_n \, d\mu + O(\epsilon \text{vol}(B_{T,K_1,K_2})). \tag{4.1}
\]
One also has
\[ \int_{B_T \setminus B_T, K_1, K_2} I(x) \, dx = O(\text{vol}(B_T \setminus B_T, K_1, K_2)). \tag{4.2} \]

Since the line bundle \( \mathcal{L} \) is balanced, it follows from Lemma 4.3 that
\[ \frac{\text{vol}(B_T \setminus B_T, K_1, K_2)}{\text{vol}(B_T)} \to 0, \quad T \to \infty. \]

Since \( \epsilon > 0 \) is arbitrary, combining (4.1) and (4.2) we finish the proof of the proposition. \( \square \)

**Proof of Theorem 1.2** By [5, Lemma 3.4], the height balls are well-rounded. Then Theorem 1.2 follows from Proposition 4.5 via the standard unfolding argument. See e.g. [4, Proposition 5.3] and [5, Theorem 6.9].

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