CONTINUITY WITH RESPECT TO FRACTIONAL ORDER OF THE TIME FRACTIONAL DIFFUSION-WAVE EQUATION

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Abstract. This paper studies a time-fractional diffusion-wave equation with a linear source function. First, some stability results on parameters of the Mittag-Leffler functions are established. Then, we focus on studying the continuity of the solution of both the initial problem and the inverse initial value problems corresponding to the fractional-order in our main results. One of the difficulties encountered comes from estimating all constants independently of the fractional orders. Finally, we present some numerical results to confirm the effectiveness of our methods.

1. Introduction. Fractional calculus has many applications in mechanic, physics and engineering science. For example, a fractional diffusion equation is a generalization of a classical diffusion equation which models anomalous diffusive phenomena. Many ideas and methods have been developed to deal with fractional partial differential equations. We refer the reader to [9, 1, 8, 36, 3, 4, 37, 38, 12, 26, 13, 6, 14, 15, 16, 29] and the references therein.

Let $\Omega$ be an open and bounded domain in $\mathbb{R}^N$ with the boundary $\partial \Omega$. Given a function $G$, we seek $u$ such that

\[
\begin{aligned}
\partial_t^\alpha u(x,t) + Au(x,t) &= G(x,t), & (x,t) &\in \Omega \times (0,T), \\
u(x,t) &= 0, & x &\in \partial \Omega, \\
u_t(x,0) &= 0, & x &\in \Omega,
\end{aligned}
\]

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where $T$ is a positive number. Here, the operator $A$ is a symmetric and uniformly elliptic operator on $\Omega$ defined by

$$A\varphi(x) := -\sum_{k=1}^{N} \frac{\partial}{\partial x_k} \left( \sum_{l=1}^{N} a_{kl}(x) \frac{\partial}{\partial x_l} \varphi(x) \right) + a_0(x)\varphi(x), \quad x \in \overline{\Omega},$$

where $a_{kl} \in C^1(\overline{\Omega})$, $a_{kl} = a_{lk}$, $1 \leq k, l \leq N$, and $a_0 \in C(\overline{\Omega}; [0, +\infty))$. Suppose that there exists $\hat{a} > 0$ such that, for $x \in \Omega$, $z = (z_1, z_2, ..., z_N) \in \mathbb{R}^N$,

$$\sum_{1 \leq k, l \leq N} z_k a_{kl}(x) z_l \geq \hat{a} |z|^2,$$

see, e.g., [30]. The fractional derivative in time $\partial_t^\alpha$ for $1 < \alpha < 2$ is understood as the left–sided Caputo fractional derivative of order $\alpha$ with respect to $t$ and defined by

$$\partial_t^\alpha v(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \partial_s^2 v(s)(t-s)^{1-\alpha} ds,$$

where $\Gamma$ is the Gamma function. For $\alpha = 2$, we consider the usual time derivative $\partial_t^2$. The first equation of (1) is called a fractional wave equation and can be used to describe evolution processes between diffusion and wave propagation [23, 24, 25]. In [23] the author shows that the fractional wave equation governs the propagation of mechanical diffusion-waves in viscoelastic media. The physical background for a time-space fractional diffusion-wave equation can be found in [5].

We are interested in the two following problems for time fractional wave equations (1). The first one is to identify $u$ satisfying equations (1) and the initial condition

$$u(x,0) = h(x).$$

(2)

It is called an initial value problem (IVP). The second one is to determine $u$ satisfying equations (1) and the final condition

$$u(x,T) = f(x),$$

(3)

which is called an inverse initial value problem (inverse IVP) or a final value problem (FVP). The initial value problem (1)-(2) has been extensively considereblack in the literature see, for example, [11, 17, 22, 27, 30, 21, 2, 10]. The inverse initial value problem (1),(3) and its applications have been studied in [32]. To the best of our knowledge, there are only a few papers on the inverse initial value problem for the time fractional diffusion-wave equation; see [34, 35, 32, 39, 18, 33]. In practice, many problems on time-space fractional equations depend on fractional parameters, i.e., fractional orders. However, these fractional parameters are not known a priori in the modeling process. Hence the continuity of solutions on these parameters is very important for modeling purposes. Furthermore, numerical computations are not allowed if this continuity does not hold.

Motivated from [7, 31], this paper studies the continuity of the solution of both the initial problem (1)-(2) and the inverse initial value problems (1),(3) with respect to the fractional order $\alpha$. Namely, this work focuses on the question

$$Does \; u_{\alpha_n} \rightarrow u_\alpha \; in \; an \; appropriate \; sense \; as \; n \rightarrow \infty?$$

To answer this question, one of the difficulties encounterblack is estimating all constants independently on the fractional orders.

This paper is organized as follows. Section 2 provides some basic definitions and preliminaries. In Section 3, we present some stability results on parameters of the Mittag-Leffler functions which help to establish our main results in the next
sections. The continuity of the solution of the initial problem (1)-(2) and the inverse initial value problems (1),(3) with respect to the fractional-order \( \alpha \) will be shown in Sections 4, 5 respectively. Finally, we also present some numerical results to confirm the effectiveness of our method.

2. Preliminaries.

2.1. The Mittag-Leffler function. Let us recall some basic properties of the Mittag-Leffler function \( E_{\alpha,\gamma}(z) \) with \( \alpha > 0, \gamma \in \mathbb{R} \) which is defined by

\[
E_{\alpha,\gamma}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}
\]

for \( z \in \mathbb{C} \). The asymptotic behavior and derivatives of this function are presented in the following lemmas. One can see the first one in [28] (Theorem 1.6, p. 35), the second one in [28] (subsubsection 1.2.3, p.21-22), and the last one can be directly implied from [34] (Lemma 2.5).

**Lemma 2.1.** Let \( 1 < \alpha < 2 \). Then

\[
0 \leq E_{\alpha,\alpha}(-t) \leq \frac{M_\alpha}{1+t},
\]

for all \( t \geq 0 \), where \( M_\alpha \) is a positive constant depending only on \( \alpha \).

**Lemma 2.2.** Suppose \( 1 < \alpha < 2 \) and \( \lambda > 0 \). Then

\[
\partial_t E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad \partial_t \left( t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \right) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha),
\]

for all \( t > 0 \).

**Lemma 2.3.** Let \( 1 < \alpha < 2 \). If \( T \) is large enough, then

\[
E_{\alpha,1}(-\lambda_j T^\alpha) \neq 0,
\]

for all \( j \in \mathbb{N}, j \geq 1 \), and there exist positive constants \( m_\alpha, M_\alpha \) such that

\[
\frac{m_\alpha}{1+\lambda_j T^\alpha} \leq \left| E_{\alpha,1}(-\lambda_j T^\alpha) \right| \leq \frac{M_\alpha}{1+\lambda_j T^\alpha}.
\]

2.2. Some Sobolev spaces. We assume that the operator \( A \) acts on \( L^2(\Omega) \) with domain \( D(A) = H^2(\Omega) \cap H^1_0(\Omega) \). Then the spectrum of \( A \) is a sequence of positive real numbers \( \{\lambda_j\}_{j=1,2,...} \) satisfying

\[
0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_j \leq ...
\]

and \( \lim_{j \to \infty} \lambda_j = \infty \). Let us denote by \( \{\phi_j\}_{j=1,2,...} \subset H^2(\Omega) \cap H^1_0(\Omega) \) the set of all orthonormal eigenfunctions of \( A \), i.e., \( A \phi_j = \lambda_j \phi_j \), and \( \phi_j = 0 \) on \( \partial \Omega \), for all \( j \in \mathbb{N}, j \geq 1 \). Then, the sequence \( \{\phi_k\}_{k=1,2,...} \) forms an orthonormal basis of \( L^2(\Omega) \), see e.g. [20]. For a given number \( \gamma \geq 0 \), the Hilbert space

\[
\mathbb{H}^\gamma(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^{2\gamma} |(v,\phi_j)|^2 < \infty \right\}
\]

is endowed with the norm

\[
\|v\|_{\mathbb{H}^\gamma(\Omega)}^2 = \sum_{j=1}^{\infty} \lambda_j^{2\gamma} |(v,\phi_j)|^2.
\]
If \( \gamma = 0 \), we have \( \mathbb{H}^0(\Omega) = L^2(\Omega) \). Let us denote by \( \mathcal{E}((0, T]; \mathbb{H}^\gamma(\Omega)) \) the space of all continuous functions which map \((0, T]\) into \( \mathbb{H}^\gamma(\Omega) \). For a given number \( 0 < \eta < 1 \), we define by \( \mathcal{E}^\eta((0, T]; \mathbb{H}^\gamma(\Omega)) \) the subspace of \( \mathcal{E}((0, T]; \mathbb{H}^\gamma(\Omega)) \) such that

\[
\sup_{0 < t \leq T} t^\eta \|w(t)\|_{\mathbb{H}^\gamma(\Omega)} < \infty, \quad \text{for all } w \in \mathcal{E}((0, T]; \mathbb{H}^\gamma(\Omega)),
\]

which is endowed with the norm, see \([7]\),

\[
\|w\|_{\mathcal{E}^\eta((0, T]; \mathbb{H}^\gamma(\Omega))} := \sup_{0 < t \leq T} t^\eta \|w(t)\|_{\mathbb{H}^\gamma(\Omega)}.
\]

### 3. Stability on parameters of the Mittag-Leffler functions

From Lemma 2.3, \([7]\), and Lemmas 2.1, 2.3, we obtain the following lemma.

**Lemma 3.1.** Let \( 1 < \nu_0 < \eta_0 < 2 \), and \( z \geq 0 \). Then, there exist two positive constants \( \omega_-(\nu_0, \eta_0) \), \( \omega_+(\nu_0, \eta_0) \) which only depend on \( \nu_0 \) and \( \eta_0 \) such that

\[
\frac{\omega_-(\nu_0, \eta_0)}{1 + z} \leq |E_{\gamma,1}(-z)| \leq \frac{\omega_+(\nu_0, \eta_0)}{1 + z} \quad \text{if } z \text{ is large enough,}
\]

and

\[
|E_{\gamma,\gamma}(-z)| \leq \frac{\omega_+(\nu_0, \eta_0)}{1 + z} \quad \text{for any } \nu_0 < \gamma < \eta_0.
\]

In the next lemma, we give a useful estimate for the difference of two power functions according to the distance \( |a - b| \) with any \( \epsilon > 0 \). Then, by using some special techniques, we can estimate the differences \(|E_{\alpha,1}(-\lambda \eta t^\alpha) - E_{\alpha',1}(-\lambda_j t^\alpha')|\) and \(|t^{\alpha-1}E_{\alpha,\alpha'(-\lambda \eta t^\alpha) - t^{\alpha'-1}E_{\alpha',\alpha'(-\lambda_j t^\alpha')}|\) in Lemmas 3.3 and 3.4 respectively.

**Lemma 3.2.** Assume that \( 0 \leq a_0 \leq a \leq b \leq b_0 \) and \( 0 < z \leq z_0 \). For any \( \epsilon > 0 \), there always exists \( \overline{c}_\epsilon > 0 \) such that

\[
|z^a - z^b| \leq \max \left( Z_0^{\epsilon(2+2\epsilon)}b, 1 \right) \overline{c}_\epsilon (b - a)^\epsilon z^{a - \epsilon}.
\]

**Proof.** We consider two cases as follows.

**Case 1.** \( 0 < z \leq 1 \). By a simple computation, we have

\[
|z^a - z^b| = z^a(1 - z^{b-a}) = z^a \left[ 1 - \exp \left( -(b-a) \log \left( \frac{1}{z} \right) \right) \right].
\]

We note that, for any \( \epsilon > 0 \), there always exists \( \overline{c}_\epsilon > 0 \) such that \( 1 - e^{-y} \leq \overline{c}_\epsilon y^\epsilon \) for all \( y > 0 \). Therefore, we obtain

\[
1 - \exp \left( -(b-a) \log \left( \frac{1}{z} \right) \right) \leq \overline{c}_\epsilon (b-a)^\epsilon \log^\epsilon \left( \frac{1}{z} \right) \leq \overline{c}_\epsilon (b-a)^\epsilon z^{-\epsilon}.
\]

This implies that

\[
|z^a - z^b| \leq \overline{c}_\epsilon (b-a)^\epsilon z^{a - \epsilon}.
\]

**Case 2.** \( z \geq 1 \). We see that

\[
|z^a - z^b| = z^b(1 - z^{a-b}) \leq z^b \left( 1 - e^{-(b-a) \log(z)} \right).
\]

Using the inequality \( 1 - e^{-y} \leq \overline{c}_\epsilon y^\epsilon \) for any \( \epsilon > 0 \) and \( y > 0 \), we obtain

\[
1 - e^{-(b-a) \log(z)} \leq \overline{c}_\epsilon (b-a)^\epsilon \log^\epsilon(z) \leq \overline{c}_\epsilon (b-a)^\epsilon z^\epsilon.
\]

Therefore,

\[
|z^a - z^b| \leq \overline{c}_\epsilon (b-a)^\epsilon z^{b+\epsilon} \leq Z_0^{\epsilon(2+2\epsilon)} \overline{c}_\epsilon (b-a)^\epsilon z^{a - \epsilon}.
\]

Combining the above cases completes the proof. \( \square \)
Lemma 3.3. Let $1 < \nu_0 < \alpha < \alpha' < \eta_0 < 2$ and $\epsilon > 0$. Then there exists a positive constant $D_2(\nu_0, \eta_0, \epsilon, \beta, T)$ such that
\[
|E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha',1}(-\lambda_j t^{\alpha'})| \\
\leq D_2(\nu_0, \eta_0, \epsilon, \beta, T)\lambda_j^{\beta-1} t^{-\alpha(1-\beta)-\epsilon} \left[|\alpha' - \alpha| + |\alpha' - \alpha|\right],
\]
for any $0 \leq \beta \leq 1$ and $0 < t \leq T$.

Proof. First, it is easy to see that
\[
I = E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha',1}(-\lambda_j t^{\alpha'})
\]
\[
\leq |E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha',1}(-\lambda_j t^{\alpha'})| + |E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha',1}(-\lambda_j t^{\alpha'})|
\]
\[
= I_1 + I_2.
\]

Estimating $I_1$: From Part a of Lemma 2.3 in [7], there exists a positive constant $C(\nu_0, \eta_0)$ depending only on $\nu_0, \eta_0$ such that
\[
\left|\frac{\partial}{\partial \zeta} E_{\zeta,1}(-\lambda_j t^\alpha) + \frac{\partial}{\partial \zeta} E_{\zeta,1}(-\lambda_j t^{\alpha'})\right| \leq C(\nu_0, \eta_0) \frac{1}{(1 + \lambda_j t^{\alpha'})^{1-\beta}}
\]
\[
\leq C(\nu_0, \eta_0) \lambda_j^{\beta-1} t^{-\alpha'(1-\beta)},
\]
for all $1 < \zeta < 2$ and $t > 0$. The latter estimates together with the Fundamental Theorem of Calculus gives
\[
I_1 = \left|E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha',1}(-\lambda_j t^{\alpha'})\right| = \int_{\alpha'}^{\alpha} \left|\frac{\partial}{\partial \zeta} E_{\zeta,1}(-\lambda_j t^\alpha) d\zeta\right|
\]
\[
\leq C(\nu_0, \eta_0) t^{-\alpha'(1-\beta)} \lambda_j^{\beta-1} (\alpha' - \alpha).
\]

Estimating $I_2$: Lemma 2.2 implies that $\partial_\zeta E_{\alpha,1}(-\lambda_j t^\alpha) = -\lambda_j t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha)$, which combined with the Fundamental Theorem of Calculus yields
\[
E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha,1}(-\lambda_j t^{\alpha'}) = \int_{t}^{t_{-\alpha'}} -\lambda_j t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha) dr.
\]
Using Lemma 3.1, we obtain
\[
\lambda_j t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha) \leq \omega_+(\nu_0, \eta_0) \lambda_j^{\beta-1} t^{\alpha(\beta-1)-1} \frac{(\lambda_j t^\alpha)^{2-\beta}}{1 + (\lambda_j t^\alpha)^2}
\]
\[
\leq \omega_+(\nu_0, \eta_0) \lambda_j^{\beta-1} t^{\alpha(\beta-1)-1},
\]
where we have used that $\frac{(\lambda_j t^\alpha)^{2-\beta}}{1 + (\lambda_j t^\alpha)^2} \leq 1$. Hence, we deduce that
\[
I_2 = \left|E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha,1}(-\lambda_j t^{\alpha'})\right| \leq \omega_+(\nu_0, \eta_0) \lambda_j^{\beta-1} t^{\alpha(\beta-1)-1} \int_{\alpha_{-\alpha'}}^{t} t^{\alpha(\beta-1)-1} dr
\]
\[
= \omega_+(\nu_0, \eta_0) \lambda_j^{\beta-1} t^{-\alpha(1-\beta)} t^{-\alpha'(1-\beta)} \left[\frac{t^{\alpha(1-\beta)} - t^{\alpha'(1-\beta)}}{\alpha(1-\beta)}\right].
\]
Next, we will find a bound for $J_1$. If $t \geq 1$, then applying Lemma 3.2 yields

$$J_1 \leq \left| t^{\alpha(1-\beta)} - t^{\alpha(1-\beta)} \right| \leq \frac{T^{\eta_0(1-\beta)+2\epsilon}}{\nu_0(1-\beta)^{1-\epsilon}} \bar{C}_\epsilon(\alpha' - \alpha)'t^{\alpha(1-\beta) - \epsilon}$$

$$\leq \frac{T^{2\eta_0(1-\beta)+2\epsilon}}{\nu_0(1-\beta)^{1-\epsilon}} \bar{C}_\epsilon(\alpha' - \alpha)' \cdot (1-\beta) \leq (1-\beta)^{1-\epsilon} \bar{C}_\epsilon(\alpha' - \alpha)'.$$

If $0 < t \leq 1$, then by the same techniques as in Lemma 3.2 one has

$$J_1 = t^{-\alpha'(1-\beta)} \left| 1 - t^{(\alpha'-\alpha)(1-\beta)} \right| \leq \frac{T^{-\alpha'(1-\beta)}}{\nu_0(1-\beta)^{1-\epsilon}} \bar{C}_\epsilon(1-\beta) \leq (1-\beta)^{1-\epsilon} \bar{C}_\epsilon(\alpha' - \alpha)'t^{-\epsilon}.$$  

The above observations yield

$$J_2 \leq D_1(\nu_0, \eta_0, \epsilon, \beta, T) \lambda_j^{\beta-1} t^{-\eta_0(1-\beta) - \epsilon} (\alpha' - \alpha)' \epsilon,$$

where

$$D_1(\nu_0, \eta_0, \epsilon, \beta, T) = \frac{\omega_+(\nu_0, \eta_0)}{\nu_0(1-\beta)^{1-\epsilon}} \max \left( T^{3\eta_0(1-\beta) + 3\epsilon}, 1 \right) \bar{C}_\epsilon.$$  

From the above steps, we deduce that

$$\left| E_{\alpha,1}(-\lambda_j t^{\alpha}) - E_{\alpha',1}(-\lambda_j t^{\alpha'}) \right| \leq D_2(\nu_0, \eta_0, \epsilon, \beta, T) \lambda_j^{\beta-1} t^{-\eta_0(1-\beta) - \epsilon} \left[ (\alpha' - \alpha)' + (\alpha' - \alpha) \right],$$

where

$$D_2(\nu_0, \eta_0, \epsilon, \beta, T) = D_1(\nu_0, \eta_0, \epsilon, \beta) + C(\nu_0, \eta_0) \max \left( T^{(\eta_0 - \eta_0)(1-\beta) + \epsilon}, 1 \right).$$

This completes the proof.

**Lemma 3.4.** Let $\alpha, \alpha'$ be as defined in Lemma 3.3. For any $0 \leq \beta \leq 1$ and $\epsilon > 0$, there exists a positive constant $D_0(\nu_0, \eta_0, \epsilon, \beta, T)$ such that

$$\left| t^{\alpha-1} E_{\alpha,\alpha'}(\alpha' t^{\alpha}) - t^{\alpha'-1} E_{\alpha',\alpha'}(-\lambda_j t^{\alpha'}) \right| \leq D_0(\nu_0, \eta_0, \epsilon, \beta, T) \lambda_j^{\beta-1} t^{\nu_0(1-\beta) - \epsilon} \left[ (\alpha' - \alpha)' + (\alpha' - \alpha) \right].$$

**Proof.** First, we see that

$$\left| t^{\alpha-1} E_{\alpha,\alpha'}(-\lambda_j t^{\alpha}) - t^{\alpha'-1} E_{\alpha',\alpha'}(-\lambda_j t^{\alpha'}) \right| \leq J_2 + J_3 + J_4,$$

where we let

$$J_2 = \left| t^{\alpha-1} E_{\alpha,\alpha'}(-\lambda_j t^{\alpha}) - t^{\alpha'-1} E_{\alpha,\alpha'}(-\lambda_j t^{\alpha}) \right|,$$

$$J_3 = \left| t^{\alpha'-1} E_{\alpha,\alpha'}(-\lambda_j t^{\alpha}) - t^{\alpha'-1} E_{\alpha',\alpha'}(-\lambda_j t^{\alpha'}) \right|,$$

$$J_4 = \left| t^{\alpha'-1} E_{\alpha',\alpha'}(-\lambda_j t^{\alpha}) - t^{\alpha'-1} E_{\alpha',\alpha'}(-\lambda_j t^{\alpha'}) \right|.$$

We divide the proof into three steps as follows.

**Estimating $J_2$:** From Lemma 3.2, we get

$$\left| t^{\alpha-1} - t^{\alpha'-1} \right| \leq \max \left( T^{\eta_0+2\epsilon}, 1 \right) \bar{C}_\epsilon t^{\alpha-1-\epsilon} (\alpha' - \alpha)' \cdot (1-\beta)^{1-\epsilon} \bar{C}_\epsilon(\alpha' - \alpha)'t^{-\epsilon}. (12)$$
Moreover, the following inequality holds:

\[ |E_{\alpha, \alpha}(-\lambda_j r^{\alpha'})| \leq \frac{\omega^+(\nu_0, \eta_0)}{1 + \lambda_j t^{\alpha'}} \leq \frac{\omega^+(\nu_0, \eta_0)}{(1 + \lambda_j t^{\alpha'})^{1-\beta}} \leq \omega^+\left(\nu_0, \eta_0\right)t^{-\alpha(1-\beta)}\lambda_j^{\beta-1}. \quad (13) \]

By taking (12) and (30) together, we obtain

\[ J_2 \leq \max\left(T^{\nu_0+2\epsilon}, 1\right)\omega^+\left(\nu_0, \eta_0\right)C\epsilon t^{\alpha\beta-1-\epsilon}\lambda_j^{\beta-1}(\alpha' - \alpha)^\epsilon \]

\[ \leq \mathcal{D}_3(\nu_0, \eta_0, \epsilon, \beta, T)t^{\nu_0\beta-1-\epsilon}\lambda_j^{\beta-1}(\alpha' - \alpha)^\epsilon, \quad (14) \]

where

\[ \mathcal{D}_3(\nu_0, \eta_0, \epsilon, \beta, T) = \max\left(T^{\nu_0+2\epsilon}, 1\right)\omega^+\left(\nu_0, \eta_0\right)C\epsilon. \]

**Estimating \( J_3 \):** Using the same argument as in (5) we deduce that

\[ J_3 = t^{\alpha'-1} \left| \int_t^{\alpha'} \partial_x E_{\alpha, \alpha}(-\lambda_j r^{\alpha'}) \right| \leq t^{\alpha'-1} \frac{\omega^+\left(\nu_0, \eta_0\right)}{(1 + \lambda_j r^{\alpha'})^{1-\beta}}(\alpha' - \alpha) \]

\[ \leq t^{\alpha'-1} \frac{\omega^+\left(\nu_0, \eta_0\right)}{1 + \lambda_j r^{\alpha'}}(\alpha' - \alpha) \leq t^{\alpha'-\alpha+\epsilon}\epsilon t^{\nu_0\beta-1-\epsilon}\lambda_j^{\beta-1}C(\nu_0, \eta_0)(\alpha' - \alpha) \]

\[ \leq \mathcal{D}_4(\nu_0, \eta_0, \epsilon, \beta, T)t^{\nu_0\beta-1-\epsilon}\lambda_j^{\beta-1}(\alpha' - \alpha), \quad (15) \]

where

\[ \mathcal{D}_4(\nu_0, \eta_0, \epsilon, \beta, T) = \max\left(T^{\nu_0+\epsilon}, 1\right)\max\left(T^{(\nu_0-\nu_0)\beta}, 1\right)\omega^+\left(\nu_0, \eta_0\right)C\epsilon. \]

**Estimating \( J_4 \):** Using Lemma 2.2 and applying the product rule of differentiation we see that

\[ \partial_x E_{\alpha', \alpha}(-\lambda_j r^{\alpha'}) = \partial_r \left(r^{1-\alpha'}(r^{\alpha'}-1)E_{\alpha', \alpha}(-\lambda_j r^{\alpha'})\right) \]

\[ = r^{-\epsilon}\left[ (1 - \alpha')E_{\alpha', \alpha}(-\lambda_j r^{\alpha'}) + E_{\alpha', \alpha-1}(-\lambda_j r^{\alpha'}) \right]. \]

Therefore, by Part (a) of Lemma 2.3 in [7], we can find a constant \( \mathcal{C}(\nu_0, \eta_0) > 0 \) such that

\[ \left| \partial_x E_{\alpha', \alpha}(-\lambda_j r^{\alpha'}) \right| \leq \frac{\mathcal{C}(\nu_0, \eta_0)}{r(1 + \lambda_j r^{\alpha'})}. \]

Hence, the Fundamental Theorem of Calculus gives

\[ J_4 = t^{\alpha'-1} \left| \int_t^{\alpha'} \partial_x E_{\alpha', \alpha}(-\lambda_j r^{\alpha'}) \right| \leq t^{\alpha'-1} \int_t^{\alpha'} \frac{\mathcal{C}(\nu_0, \eta_0)}{r(1 + \lambda_j r^{\alpha'})^{1-\beta}} dr \]

\[ \leq \mathcal{C}(\nu_0, \eta_0)t^{\alpha'-1}\lambda_j^{\beta-1} \int_t^{\alpha'} r^{-\alpha'(1-\beta)-1} dr = \mathcal{C}(\nu_0, \eta_0)t^{\alpha'-1}\lambda_j^{\beta-1}J_1 \]

\[ \leq \mathcal{C}(\nu_0, \eta_0)t^{\alpha'-1}\lambda_j^{\beta-1} \left( \mathcal{D}_1(\nu_0, \eta_0, \epsilon, \beta, T)\omega^+\left(\nu_0, \eta_0\right)t^{-\alpha'(1-\beta)-\epsilon}(\alpha' - \alpha)^\epsilon \right) \]

\[ \leq \mathcal{D}_5(\nu_0, \eta_0, \epsilon, \beta, T)t^{\nu_0\beta-1-\epsilon}\lambda_j^{\beta-1}(\alpha' - \alpha)^\epsilon, \quad (16) \]

where we have used the estimates (7), (8) for \( J_1 \). Here the latter constant is given by

\[ \mathcal{D}_5(\nu_0, \eta_0, \epsilon, \beta, T) = \mathcal{C}(\nu_0, \eta_0)\frac{\mathcal{D}_1(\nu_0, \eta_0, \epsilon, \beta, T)}{\omega^+\left(\nu_0, \eta_0\right)} \max\left(T^{(\nu_0-\nu_0)\beta}, 1\right). \]
By combining (11), (14), (15), and (16), we deduce that
\[
\left| t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha) - t^{\alpha'-1} E_{\alpha',\alpha'}(-\lambda_j t^{\alpha'}) \right|
\leq D_0(\nu_0, \eta_0, \epsilon, \beta, T) \lambda_j^{\beta-1} t^{\nu_0 \beta - 1 - \epsilon} \left( (\alpha' - \alpha) \epsilon + (\alpha' - \alpha) \right),
\] (17)
where
\[
D_0(\nu_0, \eta_0, \epsilon, \beta, T) = D_3(\nu_0, \eta_0, \epsilon, \beta, T) + D_4(\nu_0, \eta_0, \epsilon, \beta, T) + D_5(\nu_0, \eta_0, \epsilon, \beta, T).
\]

This completes the proof. \(\square\)

4. **Continuity with respect to fractional order of the IVP.** In this section, we present the continuous dependence of the solution of Problem (1)-(2) on the input data (the fractional order \(\alpha\), and the initial condition \(h\)).

**Theorem 4.1.** Given a number \(\nu \geq 0\) and let \(h \in H^\nu(\Omega)\), \(G \in L^\infty(0, T; H^\nu(\Omega))\). Assume that \(1 < \nu_0 < \alpha < \alpha' < \eta_0 < 2\). Let \(u_\alpha\) and \(u_{\alpha'}\) be the solutions of Problem (1)-(2) with respect to the fractional orders \(\alpha\) and \(\alpha'\). If the numbers \(\beta, \epsilon\) satisfy \(1 - 1/\eta_0 < \beta < 1\), and \(0 < \epsilon < \min\left(\nu_0 \beta; 1 - \eta_0 (1 - \beta)\right)\), then
\[
\left\| u_\alpha - u_{\alpha'} \right\|_{C([0,T]; H^\nu(\Omega))} \leq C_1(\nu_0, \eta_0, \epsilon, \beta, T) \left( \|h\|_{H^\nu(\Omega)} + \|G\|_{L^\infty(0, T; H^\nu(\Omega))} \right) \left( (\alpha' - \alpha) \epsilon + (\alpha' - \alpha) \right),
\] (18)
where \(C_1(\nu_0, \eta_0, \epsilon, \beta, T)\) is a positive constant.

**Proof.** The proof will be based on the results given in Section 3 which helps to estimate some differences of Mittag-Leffler functions. From [30], the solution of Problem (1)-(2) is given by
\[
u_\alpha(x, t) = \sum_{j=1}^{\infty} E_{\alpha,1}(-\lambda_j t^\alpha) \langle h(x), \varphi_j \rangle \varphi_j(x)
+ \sum_{j=1}^{\infty} \left( \int_0^t (t-r)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha) \langle G(x,r), \varphi_j \rangle dr \right) \varphi_j,
\]
\[
u_{\alpha'}(x, t) = \sum_{j=1}^{\infty} E_{\alpha',1}(-\lambda_j t^{\alpha'}) \langle h(x), \varphi_j \rangle \varphi_j(x)
+ \sum_{j=1}^{\infty} \left( \int_0^t (t-r)^{\alpha'-1} E_{\alpha',\alpha'}(-\lambda_j (t-r)^{\alpha'}) \langle G(x,r), \varphi_j \rangle dr \right) \varphi_j.
\]
Therefore, we have
\[
u_\alpha(x, t) - \nu_{\alpha'}(x, t) = \mathcal{F}^{(1)}_{\alpha,\alpha'}(t) + \mathcal{F}^{(2)}_{\alpha,\alpha'}(t),
\] (19)
where
\[
\mathcal{F}^{(1)}_{\alpha,\alpha'}(t, x) = \sum_{j=1}^{\infty} \left[ E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha',1}(-\lambda_j t^{\alpha'}) \right] \langle h(x), \varphi_j \rangle \varphi_j(x),
\]
\[
\mathcal{F}^{(2)}_{\alpha,\alpha'}(t, x) = \sum_{j=1}^{\infty} \left( \int_0^t \mathcal{H}_1(\alpha, \alpha', \lambda_j, t-r) \langle G(x,r), \varphi_j \rangle dr \right) \varphi_j(x),
\]
\[
\mathcal{H}_1(\alpha, \alpha', \lambda_j, t-r) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha) - t^{\alpha'-1} E_{\alpha',\alpha'}(-\lambda_j t^{\alpha'}).\]
(20)
By applying Lemma 3.3, the term $\mathcal{F}^{(1)}_{\alpha, \alpha'}$ can be bounded as follows:

$$
\left\| \mathcal{F}^{(1)}_{\alpha, \alpha'} (\cdot, t) \right\|_{\mathbb{H}^\nu(\Omega)} \leq \sum_{j=1}^{\infty} \lambda_j^{2\nu} \left[ E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha',1}(-\lambda_j t^{\alpha'}) \right] \| h(x), \varphi_j \|_2^2 
$$

$$
\leq D_2(\nu_0, \eta_0, \epsilon, \beta, T) \lambda_1^{\beta - 1} t^{-\eta_0(1-\beta) - \epsilon} \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right] \sum_{j=1}^{\infty} \lambda_j^{2\nu} \| h(x), \varphi_j \|_2^2 
$$

$$
= D_2(\nu_0, \eta_0, \epsilon, \beta, T) \lambda_1^{\beta - 1} t^{-\eta_0(1-\beta) - \epsilon} \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right] \| h \|_{\mathbb{H}^\nu(\Omega)}. \quad (21)
$$

In order to bound the term $\left\| \mathcal{F}^{(1)}_{\alpha, \alpha'} (\cdot, t) \right\|_{\mathbb{H}^\nu(\Omega)}$, we will use Lemma 3.4. Indeed, by applying this lemma and using some direct computations, one can see that

$$
\left\| \mathcal{F}^{(2)}_{\alpha, \alpha'} (\cdot, t) \right\|_{\mathbb{H}^\nu(\Omega)} \leq \int_0^t \left\| \sum_{j=1}^{\infty} \mathcal{H}_1(\alpha, \alpha', \lambda_j, t - r) G_j(r) \varphi_j \right\|_{\mathbb{H}^\nu(\Omega)} \, dr 
$$

$$
\leq \int_0^t \left[ \sum_{j=1}^{\infty} \lambda_j^{2\nu} \left[ \mathcal{H}(\alpha, \alpha', \lambda_j, t - r) \right] \right] |G_j(r)|^2 \, dr 
$$

$$
\leq D_3(\nu_0, \eta_0, \epsilon, \beta) \lambda_1^{\beta - 1} \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right] \int_0^t (t - r)^{\nu_0(1-\beta) - \epsilon} \| G(r) \|_{\mathbb{H}^\nu(\Omega)} \, dr. 
$$

Assumption $0 < \epsilon < \nu_0 \beta$ yields that the number $\nu_0 \beta - 1 - \epsilon$ is strictly greater than $-1$. Hence, we now observe from assumption $G \in \mathcal{L}^{\infty}(0, T; \mathbb{H}^\nu(\Omega))$ that

$$
\int_0^t (t - r)^{\nu_0 \beta - 1 - \epsilon} \| G(r) \|_{\mathbb{H}^\nu(\Omega)} \, dr \leq \| G \|_{\mathcal{L}^{\infty}(0, T; \mathbb{H}^\nu(\Omega))} \left( \int_0^t (t - r)^{\nu_0 \beta - \epsilon} \, dr \right) 
$$

$$
\leq \| G \|_{\mathcal{L}^{\infty}(0, T; \mathbb{H}^\nu(\Omega))} \frac{t^{\nu_0 \beta - \epsilon}}{\nu_0 \beta - \epsilon}. 
$$

This estimate implies that

$$
\left\| \mathcal{F}^{(2)}_{\alpha, \alpha'} (\cdot, t) \right\|_{\mathbb{H}^\nu(\Omega)} 
$$

$$
\leq D_3(\nu_0, \eta_0, \epsilon, \beta, T) \| G \|_{\mathcal{L}^{\infty}(0, T; \mathbb{H}^\nu(\Omega))} t^{\nu_0 \beta - \epsilon} \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right], \quad (22)
$$

where we denote by $D_3(\nu_0, \eta_0, \epsilon, \beta, T) = D_3(\nu_0, \eta_0, \epsilon, \beta) \lambda_1^{\beta - 1}(\nu_0 \beta - \epsilon)^{-1}$.

Combining the equation (19) and the estimates (21) and (22), there exists a positive constant $C_1(\nu_0, \eta_0, \epsilon, \beta, T)$ such that

$$
\left\| u(\cdot, t) - u(\cdot, t) \right\|_{\mathbb{H}^\nu(\Omega)} \leq \left\| \mathcal{F}^{(1)}_{\alpha, \alpha'} (\cdot, t) \right\|_{\mathbb{H}^\nu(\Omega)} + \left\| \mathcal{F}^{(2)}_{\alpha, \alpha'} (\cdot, t) \right\|_{\mathbb{H}^\nu(\Omega)} 
$$

$$
\leq C_1(\nu_0, \eta_0, \epsilon, \beta, T) \| h \|_{\mathbb{H}^\nu(\Omega)} t^{-\eta_0(1-\beta) - \epsilon} \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right] 
$$

$$
+ C_1(\nu_0, \eta_0, \epsilon, \beta, T) \| G \|_{\mathcal{L}^{\infty}(0, T; \mathbb{H}^\nu(\Omega))} t^{\nu_0 \beta - \epsilon} \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right]. 
$$

Therefore, we can find a positive constant $C_1(\nu_0, \eta_0, \epsilon, \beta, T)$ satisfying

$$
t^{\nu_0(1-\beta)+\epsilon} \left\| u(\cdot, t) - u(\cdot, t) \right\|_{\mathbb{H}^\nu(\Omega)} 
$$

$$
\leq C_1(\nu_0, \eta_0, \epsilon, \beta, T) \left( \| h \|_{\mathbb{H}^\nu(\Omega)} + \| G \|_{\mathcal{L}^{\infty}(0, T; \mathbb{H}^\nu(\Omega))} \right) \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right],
$$

which directly implies inequality (18).
Remark 1. If we set $\nu = 0$ and $G = 0$ in Theorem 4.1, then
\[
\sup_{0 < t \leq T} t^{\nu_0(1-\beta) + \epsilon} \|u_{\alpha}(:, t) - u_{\alpha^*}(:, t)\|_{L^2(\Omega)} \leq 2C_1(\nu_0, \eta_0, \epsilon, \beta, T)\|h\|_{L^2(\Omega)}(\alpha' - \alpha)\beta.
\]  
(23)

where we note that $(\alpha' - \alpha) \leq (\alpha' - \alpha)^\beta$ since $0 < \alpha' - \alpha < 1$ and $0 < \epsilon < 1$. Hence, we can estimate the output error $u_{\alpha}(:, t) - u_{\alpha^*}(:, t)$ on $L^2(\Omega)$ with the assumption $h \in L^2(\Omega)$. This comes from the estimate (21). Namely, it was based on Lemma 3.3 corresponding to the estimate
\[
\left| E_{\alpha, 1}(\lambda_{j, \alpha}^1 t^\alpha) - E_{\alpha', 1}(\lambda_{j, \alpha'}^1 t^{\alpha'}) \right| \leq D_2(\nu_0, \eta_0, \epsilon, \beta, T)\lambda_{j, \alpha}^{\beta - 1} t^{\nu_0(1-\beta) - \epsilon} (\alpha' - \alpha)\beta,
\]  
(24)

for any $0 \leq \beta \leq 1$, where the power of $\lambda_{j, \alpha}$ on the right hand side is $\beta - 1 \in [-1; 0]$. This can be compared with some existing methods as follows. In [7] (see Theorem 4.3), estimates for the output error are based on the estimate
\[
\left| E_{\alpha, 1}(\lambda_{j, \alpha}^1 t^\alpha) - E_{\alpha', 1}(\lambda_{j, \alpha'}^1 t^{\alpha'}) \right| \leq C\lambda_{j, \alpha}^{\beta + 1} |\alpha - \alpha'|,
\]  
(25)

where $C$ does not depend on $\alpha, \alpha' \in [\alpha_0, \alpha_1] \subset (0, 1)$, and $\gamma, \gamma' \in [\gamma_0, \gamma_1] \subset (0, 1)$. If we try to establish the same estimate as in (25) for $\alpha, \alpha' \in [\alpha_0, \alpha_1] \subset (1, 2)$, and apply it instead of (24), then we require $h \in H^{2\gamma_1}(\Omega)$ to estimate the output error on $L^2(\Omega)$. In [19] (see Lemma 5), the authors gave the following estimate by using solutions of two ordinary differential equations
\[
\left| E_{\alpha, 1}(\lambda_{j, \alpha}^1 t^\alpha) - E_{\alpha', 1}(\lambda_{j, \alpha'}^1 t^{\alpha'}) \right| \leq K_0(\alpha, \beta, T, \lambda_1)\lambda_{j, \alpha}^{\beta - 1} t^{-\max(\alpha, \alpha')} |\alpha - \alpha'|,
\]  
(26)

for $0 < \beta < 1$ and $\alpha, \alpha' \in (0, 1)$. Since the constant $K_0(\alpha, \beta, T, \lambda_1)$ is not known exactly, it is not suitable to establish the same estimate as in (26) for $\alpha, \alpha' \in (1, 2)$ and apply it to our problem.

Similarly, in the case $h = 0$ and $G \neq 0$, one can compare our method with the existing method in [7].

Remark 2. We now present an example to simulate the theory. Let us consider the IVP of finding $u = u(x, t)$, $(x, t) \in (0, \pi) \times (0, 1)$, such that
\[
\begin{align*}
\partial_t^\alpha u(x, t) - \Delta u(x, t) &= \sqrt{2/\pi} t^{\alpha - 1} \sin(2x), \quad (x, t) \in \Omega \times (0, 1), \\
u(x, t) &= 0, \quad (x, t) \in (0, \pi) \times (0, 1), \\
u_t(x, 0) &= 0, \quad x \in \Omega, \\
u(x, 0) &= \sqrt{2/\pi} \sin(x), \quad x \in \Omega.
\end{align*}
\]  
(27)

We consider the negative Laplace operator $-\Delta$ associated with the Dirichlet boundary condition on $H^2_0(0, \pi) \cap H^2(0, \pi)$. Then, it has the eigenvalues $\tilde{b}_j = j^2$, $j \geq 1$, and corresponding eigenfunctions $\varphi_j(x) = \sqrt{2/\pi} \sin(jx)$, $j \geq 1$. The analytic solution of this problem is given by
\[
u(x, t) = \sqrt{\frac{2}{\pi}} \left[ E_{\alpha, 1}(-t^\alpha) \sin(x) + \left( \int_0^t (tr - r^2)^{\alpha - 1} E_{\alpha, \alpha}(-4(t - r)^\alpha)dr \right) \sin(2x) \right].
\]

Let us recall the composite Simpson’s rule. Suppose that the interval $[a, b]$ is split up into $n$ sub-intervals, with $n$ being an even number. Then, the composite
Simpson’s rule is given by
\[
\int_a^b \phi(z) \, dz = \frac{b-a}{3} \left[ \phi(z_0) + 2 \sum_{j=1}^{n/2-1} \phi(z_{2j}) + 4 \sum_{j=1}^{n/2} \phi(z_{2j-1}) + \phi(z_n) \right],
\]
where \( z_j = a + jh \) for \( j = 0, \ldots, n \) with \( h = \frac{b-a}{n} \), and in particular, \( z_0 = a \) and \( z_n = b \). In the following simulation results, we will use the finite difference method to discretize the time and spatial variables as follows
\[
x_p = p \Delta x, \quad 0 \leq p \leq N_x, \quad t_q = q \Delta t, \quad 0 \leq q \leq N_t, \quad \Delta x = \frac{\pi}{N_x}, \quad \Delta t = \frac{1}{N_t},
\]
where \( N_x \geq 1 \), \( N_t \geq 1 \) are two given integer numbers.

In Table 1 and Figures 1-3, for \( t \in \{0.1; 0.5; 0.9\}, \alpha \in \{1.5\}, \) and \( \alpha^* \in \{1.500; 1.505; 1.510; 1.515; 1.520\} \), we present the solutions and the output error between \( u_\alpha(.,t) \) and \( u_{\alpha^*}(.,t) \). Observing from the table and figures, we can conclude that the smaller input error between \( \alpha \) and \( \alpha^* \) generates the smaller output error. In addition, we also present a 3D graph of the solution \( u \) for different values of \( \alpha \) in \( \{1.1; 1.2; 1.3; 1.4; 1.5; 1.6; 1.7; 1.8; 1.9\} \), see Figure (4).

### Table 1. The output errors for \( t \in \{0.1; 0.5; 0.9\}; \ x \in (0, \pi) \)

| \( \{\alpha, \alpha^*\} \) | \( N_x = 40, N_t = 40 \) |
|-----------------------------|-----------------------------|
| \( \{1.500, 1.505\} \)    | 0.005 0.000932294832753 0.020776926030918 0.043799703445885 |
| \( \{1.500, 1.510\} \)    | 0.010 0.001846284758162 0.041394223517507 0.087636643120345 |
| \( \{1.500, 1.515\} \)    | 0.015 0.002742293511105 0.061850625744690 0.131503482942371 |
| \( \{1.500, 1.520\} \)    | 0.020 0.003620640443306 0.082144942461500 0.1753929962381 |

5. **Continuity with respect to fractional order of the inverse IVP.** In this section, we present the continuous dependence of the solution of Problem (1), (3) on the input data (the fractional order \( \alpha \), and the final condition \( f \)).

**Theorem 5.1.** Given \( \nu \geq 0 \), and let \( f \in H^{\nu+\beta} (\Omega), \ G \in L^\infty (0, T; H^{\nu+\beta} (\Omega)). \) Let \( u_\alpha \) and \( u_{\alpha^*} \) be the solutions of Problem (1), (3) with respect to the fractional orders \( \alpha \) and \( \alpha^* \). Assume that the numbers \( \beta, \epsilon \) satisfies that \( 1 - 1/\eta_0 < \beta < 1 \), and \( 0 < \epsilon < \min (\nu_0 \beta; 1 - \eta_0 (1 - \beta/2)) \). If assumption (4) holds, then

\[
\| u_\alpha - u_{\alpha^*} \|_{Q^{\nu_0(1-\beta/2)+\epsilon} (0,T;H^{\nu+\beta} (\Omega))} \leq C_2 (\nu_0, \eta_0, \epsilon, \beta, T) \left[ (\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha)^{\nu_0} \right] \left( \| f \|_{L^\infty (0,T;H^{\nu} (\Omega))} + \| G \|_{L^\infty (0,T;H^{\nu+\beta} (\Omega))} \right),
\]

where \( C_2 (\nu_0, \eta_0, \epsilon, \beta, T) \) is a positive constant.

**Proof.** In order to prove the desiblack result, we will apply the results given in Section 3, and then we make some suitable choices of solution spaces. We first refer
Figure 1. A comparison between $u_\alpha$, $u_\alpha^*$ for $\alpha \in \{1.500\}$, $\alpha^* \in \{1.505; 1.510; 1.515; 1.520\}$ at $t = 0.1$, $x \in (0, \pi)$. Here $N_x = N_t = 40$.

the reader to the formula (20), page 5, in [32] to see the precise formulations of $u_\alpha$ and $u_{\alpha^*}$ which gives

$$ u_\alpha(t) - u_{\alpha^*}(t) = \mathcal{G}^{(1)}_{\alpha,\alpha^*}(t) + \mathcal{G}^{(2)}_{\alpha,\alpha^*}(t) + \mathcal{G}^{(3)}_{\alpha,\alpha^*}(t) + \mathcal{G}^{(4)}_{\alpha,\alpha^*}(t), $$

where we let

$$ \mathcal{G}^{(1)}_{\alpha,\alpha^*}(t) = \sum_{j=1}^{\infty} \mathcal{H}_2(\alpha, \alpha', \lambda_j, t) f_j \varphi_j, \quad \mathcal{G}^{(2)}_{\alpha,\alpha^*}(t) = \sum_{j=1}^{\infty} \left( \int_{0}^{t} \mathcal{H}_1(\alpha, \alpha', \lambda_j, t-r) G_j(r) dr \right) \varphi_j, $$

$$ \mathcal{G}^{(3)}_{\alpha,\alpha^*}(t) = -\sum_{j=1}^{\infty} \left( \int_{0}^{T} (T-r)^{\alpha-1} \mathcal{H}_2(\alpha, \alpha', \lambda_j, t) E_{\alpha, \alpha}(-\lambda_j(T-r)^\alpha) G_j(r) dr \right) \varphi_j, $$
CONTINUITY WITH RESPECT TO FRACTIONAL ORDER

(a) The solutions $u_\alpha$ and $u_{\alpha^*}$

Figure 2. A comparison between $u_\alpha$, $u_{\alpha^*}$ for $\alpha \in \{1.500\}$, $\alpha^* \in \{1.505, 1.510, 1.515, 1.520\}$ at $t = 0.5$, $x \in (0, \pi)$. Here $N_x = N_t = 40$.

\[
\Phi^{(4)}_{\alpha, \alpha'}(t) = -\sum_{j=1}^{\infty} \left( \int_0^T \frac{E_{\alpha', 1}(-\lambda_j t^{\alpha'})}{E_{\alpha', 1}(-\lambda_j T^{\alpha'})} \mathcal{H}_1(\alpha, \alpha', \lambda_j, T-r) G_j(r) dr \right) \varphi_j,
\]

and

\[
\mathcal{H}_2(\alpha, \alpha', \lambda, t) = \frac{E_{\alpha, 1}(-\lambda_j t^{\alpha})}{E_{\alpha, 1}(-\lambda_j T^{\alpha})} - \frac{E_{\alpha', 1}(-\lambda_j t^{\alpha'})}{E_{\alpha', 1}(-\lambda_j T^{\alpha'})}.
\]

Note that notation $\mathcal{H}_1$ was given by (20).

**Step 1. Estimating $\|\Phi^{(1)}_{\alpha, \alpha'}(t)\|_{L^p(\Omega)}$.** By using the triangle inequality, we observe that

\[
\left| \mathcal{H}_2(\alpha, \alpha', \lambda, t) \right| = \left| \frac{E_{\alpha, 1}(-\lambda_j t^{\alpha})}{E_{\alpha, 1}(-\lambda_j T^{\alpha})} - \frac{E_{\alpha', 1}(-\lambda_j t^{\alpha'})}{E_{\alpha', 1}(-\lambda_j T^{\alpha'})} \right|
\]

\[
\quad = \left| \frac{E_{\alpha, 1}(-\lambda_j t^{\alpha}) E_{\alpha', 1}(-\lambda_j T^{\alpha'}) - E_{\alpha', 1}(-\lambda_j t^{\alpha'}) E_{\alpha, 1}(-\lambda_j T^{\alpha})}{E_{\alpha, 1}(-\lambda_j T^{\alpha}) E_{\alpha', 1}(-\lambda_j T^{\alpha'})} \right|
\]
To bound the term \( J_5 \), we will use Lemma 3.3. Indeed, using this lemma gives
\[
\left| E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha',1}(-\lambda_j t^{\alpha'}) \right|
\leq D_2(\nu_0, \eta_0, \epsilon, \beta, T) \lambda_j^{\beta-1} t^{-\eta_0(1-\beta)-\epsilon} \left[ (\alpha' - \alpha) \epsilon + (\alpha' - \alpha) \right].
\] (30)

This with Lemma 2.3 gives
\[
J_5 = \frac{\left| E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha',1}(-\lambda_j t^{\alpha'}) \right|}{\left| E_{\alpha,1}(-\lambda_j T^\alpha) \right|}.
\]
CONTINUITY WITH RESPECT TO FRACTIONAL ORDER

\[ u_\alpha \text{ for } (x,t) \in (0,\pi) \times \{0.5\} \]

\[ \text{Figure 4. The solution } u_\alpha \text{ for } \alpha \in \{1.1; 1.2; 1.3; 1.4; 1.5; 1.6; 1.7; 1.8; 1.9\}, N_x = N_t = 40. \]

\[ \leq \frac{(1 + \lambda_j T^\alpha)}{\omega_{-}(\nu_0, \eta_0)} D_2(\nu_0, \eta_0, \epsilon, \beta, T) \lambda_j^\beta t^{-\eta_0(1-\beta)-\epsilon} \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right] \]

\[ \leq D_7(\nu_0, \eta_0, \epsilon, \beta, T) \lambda_j^\beta t^{-\eta_0(1-\beta)-\epsilon} \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right], \quad (31) \]

where

\[ D_7(\nu_0, \eta_0, \epsilon, \beta, T) = \left( \max \left( T^{\nu_0}, 1 \right) + \lambda_1^{-1} \right) \frac{D_2(\nu_0, \eta_0, \epsilon, \beta, T)}{\omega_{-}(\nu_0, \eta_0)}. \]

Let us proceed to bound the term \( J_6 \). By noting that \( 0 \leq \beta/2 \leq 1 \) and applying Lemma 3.3 with \( \beta/2 \) instead of \( \beta \), there exists a positive constant \( \overline{D}_2(\nu_0, \eta_0, \epsilon, \beta, T) \) such that

\[ \left| E_{\alpha,1}(\lambda_j T^\alpha) - E_{\alpha',1}(\lambda_j T^{\alpha'}) \right| \]

\[ \leq \overline{D}_2(\nu_0, \eta_0, \epsilon, \beta, T) \lambda_j^\beta t^{-\eta_0(1-\beta)-\epsilon} \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right]. \quad (32) \]
On the other hand, using Lemma 3.1, we have the following:

$$
\begin{align*}
\left| \frac{E_{\alpha',1}(-\lambda_j T^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha) E_{\alpha',1}(-\lambda_j T^\alpha')} \right| &\leq \frac{\omega_+(\nu_0, \eta_0)}{|\omega_-(\nu_0, \eta_0)|^2} \frac{(1 + \lambda_j T^\alpha)(1 + \lambda_j T^\alpha')}{1 + \lambda_j t^\alpha'} \\
&\leq \frac{\omega_+(\nu_0, \eta_0)}{|\omega_-(\nu_0, \eta_0)|^2} (1 + \lambda_j T^\alpha) \left( \frac{1 + \lambda_j T^\alpha'}{1 + \lambda_j t^\alpha'} \right)^{1 - \frac{\alpha}{2}} \left( \frac{1 + \lambda_j T^\alpha'}{1 + \lambda_j t^\alpha'} \right)^{\frac{\beta}{2}}.
\end{align*}
$$

Since $\alpha, \alpha' \in [\nu_0, \eta_0]$, it is easy to see that

$$1 + \lambda_j T^\alpha \leq \left( \lambda_1^{-1} + T^\nu \right) \lambda_j \leq \left( \lambda_1^{-1} + \max(T^{\nu_0}, 1) \right) \lambda_j,$$

and

$$\left( \frac{1 + \lambda_j T^\alpha'}{1 + \lambda_j t^\alpha'} \right)^{1 - \frac{\alpha}{2}} \leq \left( \frac{T^\alpha'}{t^\alpha'} \right)^{1 - \frac{\alpha}{2}} \leq T^\alpha(1 - \frac{\alpha}{2}) T^{-\alpha'(1 - \frac{\beta}{2})} \leq T^{\mu_0(1 - \frac{\alpha}{2})} t^{-\nu_0(1 - \frac{\beta}{2})},$$

and

$$\left( \frac{1 + \lambda_j T^\alpha'}{1 + \lambda_j t^\alpha'} \right)^{\frac{\beta}{2}} \leq \left( 1 + \lambda_j T^\alpha' \right)^{\frac{\beta}{2}} \leq \left( \lambda_1^{-1} + T^\nu \right)^{\frac{\beta}{2}} \lambda_j^\beta \leq \left( \lambda_1^{-1} + \max(T^{\nu_0}, 1) \right)^{\frac{\beta}{2}} \lambda_j^\beta.$$

From the above observations, we find that

$$\left| \frac{E_{\alpha',1}(-\lambda_j T^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha) E_{\alpha',1}(-\lambda_j T^\alpha')} \right| \leq D_8(\nu_0, \eta_0, T) \lambda_j^{1 + \frac{\beta}{2}} t^{-\nu_0(1 - \frac{\beta}{2})}, \quad (33)$$

where

$$D_8(\nu_0, \eta_0, T) = \left( \lambda_1^{-1} + \max(T^{\nu_0}, 1) \right)^{1 + \frac{\beta}{2}} T^{\mu_0(1 - \frac{\beta}{2})}.$$

It immediately follows from (32) and (33) that

$$\mathcal{J}_6 = \left| \frac{E_{\alpha',1}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha) E_{\alpha',1}(-\lambda_j T^\alpha')} \right| E_{\alpha,1}(-\lambda_j T^\alpha) - E_{\alpha',1}(-\lambda_j T^\alpha') \leq D_8(\nu_0, \eta_0, \epsilon, \beta, T) \lambda_j^{\beta} t^{-\nu_0(1 - \frac{\beta}{2}) / \epsilon} \left[ (\alpha' - \alpha) \epsilon + (\alpha' - \alpha) \right], \quad (34)$$

where $D_8(\nu_0, \eta_0, \epsilon, \beta, T) = D_8(\nu_0, \eta_0, T) D_2(\nu_0, \eta_0, \epsilon, \beta, T) T^{-\nu_0(1 - \frac{\beta}{2}) - \epsilon}$. We now observe that the quantities $t^{-\nu_0(1 - \beta)/\epsilon}$, $t^{-\nu_0(1 - \beta/2)}$ are less than or equal to the product of $t^{-\nu_0(1 - \beta/2) - \epsilon}$ and $\max(T^{\nu_0/2}, 1, T^\epsilon)$. Henceforth, the following estimate can be obtained by taking (31), (34) together:

$$\mathcal{H}_2(\alpha, \alpha', \lambda_j, t) \leq D_9(\nu_0, \eta_0, \epsilon, \beta, T) t^{-\nu_0(1 - \frac{\beta}{2}) - \epsilon} \lambda_j^{\beta} \left[ (\alpha' - \alpha) \epsilon + (\alpha' - \alpha) \right], \quad (35)$$

where $D_9(\nu_0, \eta_0, \epsilon, \beta, T) = (D_7(\nu_0, \eta_0, \epsilon, \beta, T) + D_8(\nu_0, \eta_0, \epsilon, \beta, T))(1 + \max(T^{\nu_0/2}, 1, T^\epsilon))$. Thus

$$\|g^{(1)}_{\alpha, \alpha'}(t)\|^{2}_{\mathcal{H}^\nu(\Omega)} = \sum_{j=1}^{\infty} \lambda_j^{2\nu} \mathcal{H}_2(\alpha, \alpha', \lambda_j, t)^2 |f_j|^2$$

$$\leq |D_9(\nu_0, \eta_0, \epsilon, \beta, T)| t^{-\nu_0(1 - \frac{\beta}{2}) - 2\epsilon} \left( \sum_{j=1}^{\infty} \lambda_j^{2\nu + 2\beta} |f_j|^2 \right)^2 \left[ (\alpha' - \alpha) \epsilon + (\alpha' - \alpha) \right]^2,$$

which consequently leads to

$$\|g^{(1)}_{\alpha, \alpha'}(t)\|^{2}_{\mathcal{H}^\nu(\Omega)} \leq D_9(\nu_0, \eta_0, \epsilon, \beta, T) t^{-\nu_0(1 - \frac{\beta}{2}) - \epsilon} \left[ (\alpha' - \alpha) \epsilon + (\alpha' - \alpha) \right] \|f\|^2_{\mathcal{H}^{\nu+\beta}(\Omega)}.$$
Step 2. Estimating $\|g_{\alpha,\alpha'}^{(2)}(t)\|_{H^\nu(\Omega)}$. By noting that $\lambda_j^{\beta-1} \leq \lambda_1^{\beta/2-1}\lambda_j^{\beta/2}$ for all $j \geq 1$, and using Lemma 3.4, we derive the estimate

$$\|\mathcal{H}_1(\alpha, \alpha', \lambda_j, t)\| \leq D_6(\nu_0, \eta_0, \epsilon, \beta, T)\lambda_1^{\beta/2-1}t^{\nu_0\beta-1-\epsilon}(\alpha' - \alpha) + (\alpha' - \alpha),$$

where $D_6(\nu_0, \eta_0, \epsilon, \beta, T) = D_6(\nu_0, \eta_0, \epsilon, \beta, T)\lambda_1^{\beta/2-1}$. Therefore, we have

$$\|g_{\alpha,\alpha'}^{(2)}(t)\|_{H^\nu(\Omega)} \
\leq \int_0^T \left( \sum_{j=1}^{\infty} (\mathcal{H}_1(\alpha, \alpha', \lambda_j, t-r)G_j(r))^2 \right)^{\frac{1}{2}} \, dr \\n\leq D_6(\nu_0, \eta_0, \epsilon, \beta, T) \int_0^T (t-r)^{\nu_0\beta-1-\epsilon} \|G(r)\|_{H^{\nu+\frac{\beta}{2}}(\Omega)} \, dr,$$

and with $G \in L^\infty(0, T; H^{\nu+\frac{\beta}{2}}(\Omega))$ we have

$$\|g_{\alpha,\alpha'}^{(2)}(t)\|_{H^\nu(\Omega)} \\n\leq D_{10}(\nu_0, \eta_0, \epsilon, \beta, T) \|G\|_{L^\infty(0, T; H^{\nu+\frac{\beta}{2}}(\Omega))} t^{\nu_0\beta-\epsilon}(\alpha' - \alpha) + (\alpha' - \alpha),$$

where $D_{10}(\nu_0, \eta_0, \epsilon, \beta, T) = D_6(\nu_0, \eta_0, \epsilon, \beta, T)\lambda_1^{\beta-1}(\eta_0\beta - \epsilon)^{-1}$.

Step 3. Estimating $\|g_{\alpha,\alpha'}^{(3)}(t)\|_{H^\nu(\Omega)}$. Applying Lemma 3.1 gives

$$|E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)| \leq \frac{\omega_+(\nu_0, \eta_0)}{[1 + \lambda_j(T-r)^\alpha]^\beta} \leq \omega_+(\nu_0, \eta_0)\lambda_1^{-\beta/2}\lambda_j^{-\beta/2}(T-r)^{-\alpha\beta}.$$

Moreover, by using the same techniques as in (35) we have

$$\|g_{\alpha,\alpha'}^{(3)}(t)\|_{H^\nu(\Omega)} \\n\leq \int_0^T (T-r)^{\alpha-1}\sum_{j=1}^{\infty} \|\mathcal{H}_2(\alpha, \alpha', \lambda_j, t)E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r)\|_{H^\nu(\Omega)} \, dr \\n\leq D_9(\nu_0, \eta_0, \epsilon, \beta, T) \int_0^T (T-r)^{\alpha-1} \|\sum_{j=1}^{\infty} E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r)\|_{H^{\nu+\beta}(\Omega)} \, dr \\n\leq D_{11}(\nu_0, \eta_0, \epsilon, \beta, T) \|G\|_{L^\infty(0, T; H^{\nu+\frac{\beta}{2}}(\Omega))} t^{-\eta_0(1-\frac{\beta}{2})-\epsilon}(\alpha' - \alpha) + (\alpha' - \alpha),$$

where we used $G \in L^\infty(0, T; H^{\nu+\frac{\beta}{2}}(\Omega))$, and

$D_{11}(\nu_0, \eta_0, \epsilon, \beta, T) = D_9(\nu_0, \eta_0, \epsilon, \beta, T)\omega_+(\nu_0, \eta_0)\lambda_1^{-\beta/2} \max(T^{\eta_0(1-\beta)}, 1)(\alpha(1-\beta))^{-1}$.
Step 4. Estimating $\|\mathcal{G}_{\alpha,\alpha'}^{(4)}(t)\|_{H^\nu(\Omega)}$. Using (33) and Lemma 3.1, there exists $D_{12}(\nu_0, \eta_0, T) > 0$ such that

$$\frac{E_{\alpha',1}(\lambda_j t^{\alpha'})}{E_{\alpha',1}(\lambda_j T^{\alpha'})} = |E_{\alpha,1}(\lambda_j T^{\alpha'})| \frac{E_{\alpha',1}(\lambda_j t^{\alpha'})}{E_{\alpha,1}(\lambda_j T^{\alpha'})} \leq D_{12}(\nu_0, \eta_0, T) \lambda_j^\frac{2}{\nu_0(1 - \frac{2}{\nu}) - \epsilon}. $$

Hence, by noting that $\lambda_{j-1}^\beta \leq \lambda_1^\beta$, we obtain

$$\|\mathcal{G}_{\alpha,\alpha'}^{(4)}(t)\|_{H^\nu(\Omega)}$$

$$\leq \int_0^T \left| \sum_{j=1}^\infty \frac{E_{\alpha',1}(\lambda_j t^{\alpha'})}{E_{\alpha',1}(\lambda_j T^{\alpha'})} \mathcal{H}_1(\alpha, \alpha', \lambda_j, T - r) G_j(r) \varphi_j \right| \|G(r)\|_{H^\nu} \, dr$$

$$\leq D_{13}(\nu_0, \eta_0, \epsilon, \beta, T) t^{-\nu_0(1 - \frac{2}{\nu}) - \epsilon} \int_0^T \left| \sum_{j=1}^\infty \mathcal{H}_1(\alpha, \alpha', \lambda_j, T - r) G_j(r) \varphi_j \right| \|G(r)\|_{H^{\nu + \frac{2}{\nu}}} \, dr,$$

where $D_{13}(\nu_0, \eta_0, \epsilon, \beta, T) = \frac{D_{12}(\nu_0, \eta_0, T)}{D_0(\nu_0, \eta_0, \epsilon, \beta, T)} \lambda_1^\beta$. Since $\nu_0 \beta - 1 - \epsilon$ is strictly greater than $-1$ as $0 < \epsilon < \nu_0 \beta$, we deduce from assumption $G \in \mathcal{L}_{\infty}(0, T; H^{\nu + \frac{2}{\nu}}(\Omega))$ that

$$\int_0^T (T - r)^{\nu_0 \beta - 1 - \epsilon} \|G(r)\|_{H^{\nu + \frac{2}{\nu}}(\Omega)} \, dr$$

$$\leq \|G\|_{\mathcal{L}_{\infty}(0, T; H^{\nu + \frac{2}{\nu}}(\Omega))} \left( \int_0^T (T - r)^{\nu_0 \beta - 1 - \epsilon} \, dr \right)$$

$$\leq \|G\|_{\mathcal{L}_{\infty}(0, T; H^{\nu + \frac{2}{\nu}}(\Omega))} T^{\nu_0 \beta - \epsilon}.$$

Therefore, one can find a positive constant $D_{14}(\nu_0, \eta_0, \epsilon, \beta, T)$ such that

$$\|\mathcal{G}_{\alpha,\alpha'}^{(4)}(t)\|_{H^\nu(\Omega)}$$

$$\leq D_{14}(\nu_0, \eta_0, \epsilon, \beta, T) \|G\|_{\mathcal{L}_{\infty}(0, T; H^{\nu + \frac{2}{\nu}}(\Omega))} t^{-\nu_0(1 - \frac{2}{\nu}) - \epsilon} \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right].$$

Finally, it follows from the above steps that there exists $C_2(\nu_0, \eta_0, \epsilon, \beta, T) > 0$ with

$$\|u(., t)\|_{H^\nu(\Omega)}$$

$$\leq \|\mathcal{G}_{\alpha,\alpha'}^{(1)}(t)\|_{H^\nu(\Omega)} + \|\mathcal{G}_{\alpha,\alpha'}^{(2)}(t)\|_{H^\nu(\Omega)} + \|\mathcal{G}_{\alpha,\alpha'}^{(3)}(t)\|_{H^\nu(\Omega)} + \|\mathcal{G}_{\alpha,\alpha'}^{(4)}(t)\|_{H^\nu(\Omega)}$$

$$\leq C_2(\nu_0, \eta_0, \epsilon, \beta, T) \|f\|_{H^{\nu + \beta}(\Omega)} t^{-\nu_0(1 - \frac{2}{\nu}) - \epsilon} \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right]$$

$$+ C_2(\nu_0, \eta_0, \epsilon, \beta, T) \|G\|_{\mathcal{L}_{\infty}(0, T; H^{\nu + \frac{2}{\nu}}(\Omega))} t^{\nu_0 \beta - \epsilon} \left[ (\alpha' - \alpha)^\epsilon + (\alpha' - \alpha) \right].$$
CONTINUITY WITH RESPECT TO FRACTIONAL ORDER

Thus, one can find $C_2(\nu_0, \eta_0, \epsilon, \beta, T) > 0$ such that
\[
\| u_{\alpha}(\cdot, t) - u_{\alpha'}(\cdot, t) \|_{H^\nu(\Omega)} \\
\leq C_2(\nu_0, \eta_0, \epsilon, \beta, T) \left( (\alpha' - \alpha)^{\epsilon} + (\alpha' - \alpha) \right) \left( \| f \|_{H^{\nu+\beta}(\Omega)} + \| G \|_{L^\infty(0, T; H^{\nu+\beta}(\Omega))} \right),
\]
which gives (29).

Remark 3. By intuitive observations, one can see that the assumptions on $f$ and $G$ given in Theorem 5.1 are stronger than the assumptions on $h$ and $G$ in Theorem 4.1. Due to the inclusion
\[
C^\eta_0(1-\beta)^{\epsilon}((0, T]; H^{\nu}(\Omega)) \subset C^\eta_0(1-\beta)^{\epsilon}((0, T]; H^{\nu}(\Omega)),
\]
the output error estimate (29) is weaker than (18). Moreover, treating the inverse IVP (1),(3) certainly requires more complexity than the IVP (1)-(2).

6. Conclusions and future works. This work presents some stability results on the parameter $\alpha$ of the Mittag-Leffler functions $E_{\alpha,1}, E_{\alpha,\alpha}$. Then, the continuity of the solutions with respect to the fractional order $\alpha$ of the IVP (1)-(2) and the inverse IVP (1),(3) was obtained under some suitable assumptions on the initial data $h$, the final data $f$, and the source function $G$. Some useful comparisons with the existing methods are also given. We hope to study, in the future, nonlinear problem containing a globally Lipschitz nonlinearity or a critical nonlinearity.

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