On the derived category of the Hilbert scheme of points on an Enriques surface

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Abstract We use semi-orthogonal decompositions to construct autoequivalences of Hilbert schemes of points on Enriques surfaces and of Calabi–Yau varieties which cover them. While doing this, we show that the derived category of a surface whose irregularity and geometric genus vanish embeds into the derived category of its Hilbert scheme of points.

Keywords Derived categories · Semi-orthogonal decompositions · Fourier–Mukai functors · Hilbert schemes of points on surfaces

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1 Introduction

The bounded derived category of coherent sheaves on a smooth projective complex variety $Z$, denoted by $\mathcal{D}^b(Z)$, is now widely recognised as an important invariant which can be used to study the geometry of $Z$. It is therefore quite natural to consider the group of autoequivalences $\operatorname{Aut}(\mathcal{D}^b(Z))$. This group always contains the subgroup $\operatorname{Aut}^{sl}(\mathcal{D}^b(Z))$ of standard autoequivalences, namely those generated by automorphisms of $Z$, the shift functor and tensor products with line bundles. Note that all these equivalences send coherent sheaves to (shifts of) coherent sheaves. A classical
result by Bondal and Orlov, see [8], states that \( \text{Aut}^\text{st}(\mathcal{D}^b(Z)) \simeq \text{Aut}(\mathcal{D}^b(Z)) \) if \( \omega_Z \) is either ample or anti-ample. Therefore, we expect the most interesting behaviour if the canonical bundle is trivial. For instance, if \( \omega_Z \simeq \mathcal{O}_Z \) and \( H^i(Z, \omega_Z) = 0 \) for all \( 0 < i < \dim(Z) \), then \( \text{Aut}(\mathcal{D}^b(Z)) \) contains spherical twists, which are more interesting than the ones mentioned above since, in general, they do not preserve the abelian category of coherent sheaves, see [29].

It is usually quite difficult to construct new autoequivalences of a given variety \( Z \). However, if \( A \) is some triangulated category and an exact functor \( \Phi: A \to \mathcal{D}^b(Z) \) is a so-called spherical or \( \mathbb{P}^n \)-functor, see Sect. 2.5, we do get an autoequivalence of \( \mathcal{D}^b(Z) \). For example, let \( X \) be a K3 surface and \( \mathbb{X}^{[n]} \) its Hilbert scheme of \( n \) points. It was shown in [1] that the Fourier–Mukai functor \( \mathcal{D}^b(\mathbb{X}) \to \mathcal{D}^b(\mathbb{X}^{[n]}) \) with the ideal sheaf of the universal family as kernel is a \( \mathbb{P}^{n-1} \)-functor whose associated autoequivalence is new. Interestingly, for an abelian surface \( A \), the corresponding functor is not a \( \mathbb{P}^{n-1} \)-functor, but pulling everything to the generalised Kummer variety does give one, see [22]. Another example was given in [18] where it was shown that, given any surface \( S \), there is a \( \mathbb{P}^{n-1} \)-functor \( \mathcal{D}^b(S) \to \mathcal{D}^b(S^{[n]}) \) which is defined using equivariant methods.

In this paper, we will construct new examples of spherical functors using Enriques surfaces and Hilbert schemes of points on them. Let \( X \) be an Enriques surface and \( X^{[n]} \) the Hilbert scheme of \( n \) points on \( X \). The canonical cover of \( X^{[n]} \) is a Calabi–Yau variety (see [23] or [24]) and will be denoted by \( \text{CY}_n \). Write \( \pi: \text{CY}_n \to X^{[n]} \) for the quotient map. Consider the Fourier–Mukai functor \( F: \mathcal{D}^b(X) \to \mathcal{D}^b(X^{[n]}) \) induced by the ideal sheaf of the universal family.

**Theorem 1.1** The functor

\[
\tilde{F} = \pi^* F : \mathcal{D}^b(X) \to \mathcal{D}^b(\text{CY}_n)
\]

is split spherical for all \( n \geq 2 \) and the associated twist \( \tilde{T} \) is equivariant, so descends to an autoequivalence of \( X^{[n]} \). The autoequivalence \( \tilde{T} \) of \( \mathcal{D}^b(\text{CY}_n) \) is not standard and not a twist around a spherical object.

Under some conditions, we can also compare our twist \( \tilde{T} \) to the autoequivalences constructed in [26], see Proposition 3.18.

Once we establish Theorem 1.2 below, the first part of the theorem is an incarnation of the following general principle, see Theorem 3.4 and Remark 3.11:

If \( Y \) is a smooth projective variety whose canonical bundle is of order 2, \( \tilde{Y} \) its canonical cover with quotient map \( \pi: \tilde{Y} \to Y \), and \( A \) an admissible subcategory of \( \mathcal{D}^b(Y) \) with embedding functor \( i: A \to \mathcal{D}^b(Y) \), then \( \pi^* i \) is a split spherical functor and the associated twist is equivariant.

The following result might be of independent interest.

**Theorem 1.2** If \( S \) is any surface with \( p_g = q = 0 \), then the FM transform \( F: \mathcal{D}^b(S) \to \mathcal{D}^b(S^{[n]}) \) whose kernel is the ideal sheaf of the universal family, is fully faithful, hence \( \mathcal{D}^b(S) \) is an admissible subcategory of \( \mathcal{D}^b(S^{[n]}) \).

Since there are many semi-orthogonal decompositions of \( \mathcal{D}^b(X^{[n]}) \), we have many, potentially non-standard, twists associated with them. Note that in general it seems
difficult to describe the complement of the image of $F$ explicitly but see Remark 3.14 and Sect. 5.3 for statements related to this question.

The paper is organised as follows. In Sect. 2, we present some background information, before proving our main results in Sect. 3. In Sect. 4, we give a general construction of exceptional sequences on the Hilbert scheme $S^{[n]}$ out of exceptional sequences on a surface $S$. This construction has been independently considered by Evgeny Shinder. In particular, we have the following result which is probably well known to experts.

**Proposition 1.3** If a surface $S$ has a full exceptional collection, then so too does $S^{[n]}$.

In the last section, we describe what we call truncated ideal functors which provide us with a further example of a fully faithful functor $D^b(X) \rightarrow D^b(X^{[n]})$ for $X$ an Enriques surface, and, in some cases, $\mathbb{P}^n$-functors on smooth Deligne–Mumford stacks. The last section also gives some background on the proof of Proposition 3.1, the main ingredient in the proof of Theorem 1.2.

**Conventions.** We will work over the complex numbers, and all functors are assumed to be derived. Furthermore, all varieties are assumed to be smooth and projective unless stated otherwise. We will write $\mathcal{H}^i(E)$ for the $i$th cohomology object of a complex $E \in D^b(Z)$ and $\mathbb{H}^*(E)$ for the complex $\bigoplus_i \mathbb{H}^i(Z, E)[-i]$. If $F$ is a functor, its right adjoint will be denoted by $F^R$ and its left adjoint by $F^L$.

## 2 Preliminaries

### 2.1 Hilbert schemes of surfaces with $p_g = q = 0$

Let $S$ be a surface with $p_g = q = 0$ and consider $S^{[n]}$, the Hilbert scheme of $n$ points on $S$. Then, we have $H^k(S^{[n]}, \mathcal{O}_{S^{[n]}}) = 0$ for all $k > 0$, compare [24]. Indeed, by the Künneth formula, $H^*(S^n, \mathcal{O}_{S^n}) = H^*(S, \mathcal{O}_S)^{\otimes n}$ is concentrated in degree zero. As a consequence, the structure sheaf of the $n$th symmetric product has no higher cohomology, and the same then also holds for $S^{[n]}$, because the symmetric product has rational singularities.

For example, we can consider an *Enriques surface*, which is a smooth projective surface $X$ with $p_g = q = 0$ such that the canonical bundle $\omega_X$ is of order 2.

### 2.2 Canonical covers

Let $Y$ be a variety with torsion canonical bundle of (minimal) order $k$. The *canonical cover* $\widetilde{Y}$ of $Y$ is the unique (up to isomorphism) variety with trivial canonical bundle and an étale morphism $\pi: \widetilde{Y} \rightarrow Y$ of degree $k$ such that $\pi_* \mathcal{O}_{\widetilde{Y}} = \bigoplus_{j=0}^{k-1} \omega_{\widetilde{Y}}$. In this case, there is a free action of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ on $\widetilde{Y}$ such that $\pi$ is the quotient morphism.

As an example, the canonical cover of an Enriques surface $X$ is a K3 surface $\widetilde{X}$ and $X$ is the quotient of $\widetilde{X}$ by the action of a fixed-point free involution.

Furthermore, the canonical bundle of $X^{[n]}$ has order 2, and the associated canonical cover, denoted here by $\text{CY}_n$, is a Calabi–Yau variety, see [23, Prop. 1.6] or [24, Thm. 3.1].
2.3 Fourier–Mukai transforms and kernels

Recall that given an object $E$ in $D^b(Z \times Z')$, where $Z$ and $Z'$ are smooth and projective, we get an exact functor $D^b(Z) \to D^b(Z')$, $\alpha \mapsto p_* Z' (E \otimes p^* Z' \alpha)$. Such a functor, denoted by $\text{FM}_E$, is called a Fourier–Mukai transform (or FM transform), and $E$ is its kernel. See [16] for a thorough introduction to FM transforms. For example, $\text{FM}/\Delta(Z)^* L(\alpha) = \alpha \otimes L$, where $\Delta: Z \to Z \times Z$ is the diagonal map and $L \in \text{Pic}(Z)$. In particular, $\text{FM} O/\Delta$ is the identity functor.

**Convention.** We will write $M_L$ for the functor $\text{FM}/\Delta(Z)^* L$.

Let $S$ be any smooth projective surface, $Z_n \subset S \times S$ be the universal family and consider its structure sequence $0 \to I_{Z_n} \to O_{S \times S} \to O_{Z_n} \to 0$. We can use the objects from the above sequence as kernels to get a triangle $F \to F' \to F''$ of functors $D^b(S) \to D^b(S[n])$. Since all these functors are FM transforms, they have left and right adjoints, see [16, Prop. 5.9].

2.4 Equivalences of canonical covers

The relation between autoequivalences of a variety $Y$ with torsion canonical bundle and those of the canonical cover $\tilde{Y}$ was studied in [10]. We recall some facts in the special case where the order of $\omega_Y$ is 2. We will write $\tau$ for the fixed-point free involution of $\tilde{Y}$ such that $\tilde{Y}/\langle \tau \rangle \simeq Y$ and $\pi$ for the quotient map.

An autoequivalence $\phi$ of $D^b(\tilde{Y})$ is **equivariant** if $\tau_* \phi \simeq \phi \tau_*$. By [10, Sect. 4], an equivariant functor $\tilde{\phi}$ descends to a functor $\phi \in \text{Aut}(D^b(Y))$ with functor isomorphisms $\pi_* \phi \simeq \phi \pi_*$ and $\pi^* \phi \simeq \phi \pi^*$; moreover, the two descents $\phi, \phi'$ of $\tilde{\phi}$ differ by tensoring with $\omega_Y$.

In the other direction, it is also shown in [10, Sect. 4] that every autoequivalence of $\text{Aut}(D^b(Y))$ has an equivariant lift. Two lifts differ by the action of $\tau$ in $\text{Aut}(D^b(\tilde{Y}))$.

2.5 Spherical functors

Now consider two triangulated categories $\mathcal{A}$ and $\mathcal{B}$ and any exact functor $F: \mathcal{A} \to \mathcal{B}$ with left and right adjoints $F_L, F_R: \mathcal{B} \to \mathcal{A}$. Define the **twist** $T = T_F$ to be the cone on the counit $\epsilon: F F_R \to \text{id}_B$ of the adjunction and the **cotwist** $C$ to be the cone on the unit $\eta: \text{id}_A \to F_R F$.

**Remark 2.1** Of course, one needs to make sure that the above cones actually exist. If one works with Fourier–Mukai transforms, this is not a problem, because the maps between the functors come from the underlying kernels and everything works out, even for (reasonable) schemes which are not necessarily smooth and projective, see [3]. More generally, everything works out if one uses an appropriate notion of a spherical DG-functor, see [4].
So, as we will see, in the cases of interest to us, we have the triangles
\[ FF R \rightarrow \text{id}_B \rightarrow T \] and \[ \text{id}_A \rightarrow F^R F \rightarrow C. \] Following [4], we call \( F \) spherical if \( C \) is an equivalence and \( F^R \simeq CF^L \). If \( \mathcal{A} \) and \( \mathcal{B} \) admit Serre functors \( S_{\mathcal{A}} \) and \( S_{\mathcal{B}} \), the last condition is equivalent to \( S_{\mathcal{B}} FC \simeq FS_{\mathcal{A}} \). If \( F \) is a spherical functor, then \( T \) is an equivalence. If the triangle \( \text{id}_A \rightarrow F^R F \rightarrow C \) splits, we call \( F \) split spherical.

For an example of a (split) spherical functor consider a \( d \)-dimensional variety \( Z \) and a spherical object \( E \in D^b(Z) \), that is, \( E \otimes \omega_Z \simeq E \) and \( \text{Hom}^*(E, E) \simeq \mathbb{C} \oplus \mathbb{C}[-d] \).

The functor
\[ F = - \otimes E : D^b(\text{Spec}(\mathbb{C})) \rightarrow D^b(Z) \]
is then spherical, and the associated autoequivalence of \( D^b(Z) \) is the spherical twist from the introduction, denoted by \( \text{ST}_E \) and called Seidel–Thomas twist in the following.

### 2.6 \( \mathbb{P}^n \)-functors

Following [1, Def. 3.1], a \( \mathbb{P}^n \)-functor is a functor \( F : \mathcal{A} \rightarrow \mathcal{B} \) of triangulated categories such that

1. There is an autoequivalence \( H_F = H \) of \( \mathcal{A} \) such that
\[ F^R F \simeq \text{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^n. \]

2. The map
\[ HH^R F \rightarrow F^R F \rightarrow F^R F \]
with \( \epsilon \) being the counit of the adjunction, is, when written in the components
\[ H \oplus H^2 \oplus \cdots \oplus H^n \oplus H^{n+1} \rightarrow \text{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^n, \]
of the form
\[ \begin{pmatrix} * & * & \cdots & * & * \\ 1 & * & \cdots & * & * \\ 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \end{pmatrix}, \quad (2.1) \]

3. \( F^R \simeq H^n F^L \). If \( \mathcal{A} \) and \( \mathcal{B} \) have Serre functors, this condition is equivalent to \( S_{\mathcal{B}} F H^n \simeq FS_{\mathcal{A}} \).

If \( F \) is a \( \mathbb{P}^n \)-functor, then there is also an associated autoequivalence of \( \mathcal{B} \), denoted by \( P_F = P \). A \( \mathbb{P}^1 \)-functor is precisely a split spherical functor, and for the associated
equivalences, we have \( T^2 \simeq P \). If \( \mathcal{X} \) is a K3 surface, the functor \( F = \text{FM}_{I,_{\mathcal{X}}} \) defined in Sect. 2.3 is a \( \mathbb{P}^{n-1} \)-functor, see [1].

2.7 Semi-orthogonal decompositions

References for the following facts are, for example, [6] and [7].

Let \( \mathcal{T} \) be a triangulated category. A semi-orthogonal decomposition of \( \mathcal{T} \) is a sequence of strictly full triangulated subcategories \( \mathcal{A}_1, \ldots, \mathcal{A}_m \) such that (a) if \( \mathcal{A}_i \in \mathcal{A}_i \) and \( A_j \in \mathcal{A}_j \), then \( \text{Hom}(A_i, A_j[l]) = 0 \) for \( i > j \) and all \( l \), and (b) the \( \mathcal{A}_i \) generate \( \mathcal{T} \), that is, the smallest triangulated subcategory of \( \mathcal{T} \) containing all the \( \mathcal{A}_i \) is already \( \mathcal{T} \).

We write \( \mathcal{T} = (\mathcal{A}_1, \ldots, \mathcal{A}_m) \). If \( m = 2 \), these conditions boil down to the existence of a functorial exact triangle \( A_2 \longrightarrow T \longrightarrow A_1 \) for any object \( T \in \mathcal{T} \).

A subcategory \( \mathcal{A} \) of \( \mathcal{T} \) is right admissible if the embedding functor \( i \) has a right adjoint \( i^R \), left admissible if \( i \) has a left adjoint \( i^L \), and admissible if it is left and right admissible. Note that if \( \mathcal{T} \) admits a Serre functor, then the existence of one of the adjoints implies the existence of the other.

Given any subcategory \( \mathcal{A} \), the category \( \mathcal{A}^\perp \) consists of objects \( b \) such that \( \text{Hom}(a, b[k]) = 0 \) for all \( a \in \mathcal{A} \) and all \( k \in \mathbb{Z} \). If \( \mathcal{A} \) is right admissible, then \( \mathcal{T} = (\mathcal{A}^\perp, \mathcal{A}) \) is a semi-orthogonal decomposition. Similarly, \( \mathcal{T} = (\mathcal{A}, \mathcal{A}^\perp) \) is a semi-orthogonal decomposition if \( \mathcal{A} \) is left admissible, where \( \mathcal{A}^\perp \) is defined in the obvious way.

Examples typically arise from exceptional objects. Recall that an object \( E \in \mathcal{D}^b(Z) \) (or any \( \mathbb{C} \)-linear triangulated category) is called exceptional if \( \text{Hom}(E, E) = \mathbb{C} \) and \( \text{Hom}(E, E[k]) = 0 \) for all \( k \neq 0 \). The smallest triangulated subcategory containing \( E \) is then equivalent to \( \mathcal{D}^b(\text{Spec}(\mathbb{C})) \), and this category, by abuse of notation again denoted by \( E \), is admissible, leading to a semi-orthogonal decomposition \( \mathcal{D}^b(Z) = (E^\perp, E) \). We call a sequence of exceptional objects \( E_1, \ldots, E_n \) an exceptional collection if \( \mathcal{D}^b(Z) = \langle (E_1, \ldots, E_n)^\perp, E_1, \ldots, E_n \rangle \), where \( (E_1, \ldots, E_n)^\perp \) is the category of objects \( F \) which satisfy \( \text{Hom}(E_i, F[k]) = 0 \) for all \( i, k \). The collection is called full if \( (E_1, \ldots, E_n)^\perp = 0 \).

Note that any fully faithful FM transform \( i : \mathcal{A} = \mathcal{D}^b(Z') \longrightarrow \mathcal{D}^b(Z) \) gives a semi-orthogonal decomposition \( \mathcal{D}^b(Z) = \langle i(\mathcal{A})^\perp, i(\mathcal{A}) \rangle \).

We will need the following well-known and easy fact.

**Lemma 2.2** If \( \mathcal{T} \) has a Serre functor \( S_\mathcal{T} \) and \( \mathcal{A} \) is an admissible subcategory, then \( \mathcal{A} \) has a Serre functor \( S_\mathcal{A} \simeq i^R S_\mathcal{T} i \).

**Proof** Given \( a, a' \in \mathcal{A} \), we compute

\[
\text{Hom}_\mathcal{A}(a, a') \simeq \text{Hom}_\mathcal{T}(i(a), i(a')) \simeq \text{Hom}_\mathcal{T}(i(a'), S_\mathcal{T} i(a))^{\vee} \\
\simeq \text{Hom}_\mathcal{A}(a', i^R S_\mathcal{T} i(a))^{\vee}.
\]

\( \square \)

**Remark 2.3** Assume \( \mathcal{A} \) is an admissible subcategory of \( \mathcal{D}^b(Z) \). The embedding functor \( i \) lifts to DG-enhancements (see, for example, [21] for this notion). It can be checked
that the adjoints \( i^R \) and \( i^L \) also lift to so-called DG quasi-functors, which follows, for example, from [21, Lem. 4.4 and Prop. 4.10]. So the composition of \( i \) with any FM transform will lift to the DG-level; hence, we are in a position to use the results of [4] and all required cones of natural transformations will exist.

2.8 Group actions and derived categories

Let \( G \) be a finite group acting on a smooth projective variety \( Z \). The *equivariant derived category*, denoted by \( \text{D}^b_G(Z) \), is defined as \( \text{D}^b(\text{Coh}^G(Z)) \), see, for example, [25] for details. Recall that for every subgroup \( H \subset G \), the restriction functor \( \text{Res}: \text{D}^b_G(Z) \to \text{D}^b_H(Z) \) has the inflation functor \( \text{Inf}: \text{D}^b_H(Z) \to \text{D}^b_G(Z) \) as a left and right adjoint (see e.g. [25, Sect. 1.4]). It is given for \( A \in \text{D}^b(Z) \) by

\[
\text{Inf}(A) = \bigoplus_{[g] \in H/G} g^*A
\]  

with the linearisation given by permutation of the summands.

If \( G \) acts trivially on \( Z \), there is also the functor \( \text{triv}: \text{D}^b(Z) \to \text{D}^b_G(Z) \) which equips an object with the trivial \( G \)-linearisation. Its left and right adjoint is the functor \( (-)^G: \text{D}^b_G(Z) \to \text{D}^b(Z) \) of invariants.

**Convention.** When working with Fourier–Mukai transforms, we will frequently identify the functor with its kernel.

3 Proofs of the main results

3.1 Surfaces with \( p_g = q = 0 \)

Recall the FM transforms from Sect. 2.3. To compute \( F^R F \) in the examples known so far, one usually works out the various compositions such as, for example, \( F''^R F' \). In our case, \( F^R F \) has a rather simple shape.

**Proposition 3.1** If \( S \) is a surface with \( p_g = q = 0 \), then the composition \( F^R F \) is isomorphic to the identity.

**Proof.** Firstly, we note that, using results in Section 6 of [22] and the isomorphism \( H^*(S^{[n]}, \mathcal{O}_{S^{[n]}}) \simeq \mathbb{C} \), the following holds:

\[
\begin{align*}
F''^R F' &\simeq \mathcal{O}_{S \times S}, \\
F'^R F' &\simeq (\mathcal{O}_S \boxtimes \omega_S)[2], \\
F''^R F'' &\simeq \mathcal{O}_\Delta \oplus \mathcal{O}_{S \times S}, \\
F'^R F'' &\simeq (\mathcal{O}_S \boxtimes \omega_S)[2].
\end{align*}
\]

Next, \( F'^R F \simeq 0 \) and \( F''^R F \simeq \mathcal{O}_\Delta[-1] \). Indeed, the map \( F' \to F'' \) induces an isomorphism \( F'^R F' \to F'^R F'' \) and the component \( \mathcal{O}_{S \times S} \to \mathcal{O}_{S \times S} \) of the induced
map $F'' F' \rightarrow F'' F''$ is an isomorphism too, see [22, Sect. 6] or Section 5. Hence, the first assertion follows from the triangle $F'^{R} F \rightarrow F'^{R} F' \rightarrow F'^{R} F''$.

We then consider the triangle $F'' F \rightarrow F'' F' \rightarrow F'' F''$ and check that the cokernel of the map $F'' F' \rightarrow F'' F''$ is isomorphic to $O_\Delta$. Indeed, if

$$\varphi = (\varphi_1, \varphi_2): A \rightarrow A \oplus B$$

is a map in an abelian category such that the first component is an isomorphism (so $\varphi$ is an injection), the cokernel has to be isomorphic to $B$. To see this, just note that the map $A \oplus B \rightarrow B$ defined by $(-\varphi_2 \varphi^{-1}, \text{id}_B)$ satisfies the universal property of the cokernel.

To finally prove the claim, use the triangle $F'' F \rightarrow F'^{R} F \rightarrow F^{R} F$ and what was proved above. \qed

**Proof of Theorem 1.2** Since $F^{R} F \simeq \text{id}$, $F$ is fully faithful. On the other hand, $F$ has adjoints, so $F(D^b(S))$ is an admissible subcategory of $D^b(S^{[n]})$. \qed

**Remark 3.2** The above shows that for any surface with $p_g = q = 0$, the functor $F$ is quite far from being a spherical or a $\mathbb{P}^n$-functor.

**Remark 3.3** There exist surfaces $S$ of general type with $p_g = q = 0$ such that $D^b(S)$ contains an admissible subcategory whose Hochschild homology is trivial and whose Grothendieck group is finite or torsion, see, for example, [5]. Therefore, by Theorem 1.2, $D^b(S^{[n]})$ also contains such a (quasi-)phantom category.

### 3.2 Application to Enriques surfaces

**Theorem 3.4** Let $X$ be an Enriques surface, $\text{CY}_n$ the canonical cover of $X^{[n]}$, $\pi : \text{CY}_n \rightarrow X^{[n]}$ the covering map and $\tau : \text{CY}_n \rightarrow \text{CY}_n$ the deck transformation. If $i : A \rightarrow D^b(X^{[n]})$ is the embedding of an admissible subcategory $A$ of $D^b(X^{[n]})$, then $\pi^* i$ is a split spherical functor whose induced twist $\widehat{T}_A := T_{\pi^* i}$ is equivariant and thus descends to an autoequivalence of $D^b(X^{[n]})$ for all $n \geq 2$.

**Proof** First, the functor $\pi^* : D^b(X^{[n]}) \rightarrow D^b(\text{CY}_n)$ is split spherical with cotwist $C = (-) \otimes \omega_{X^{[n]}}$ and twist $T_{\pi^*} = \pi^*[1]$. Indeed, this follows from the identities $\pi_* \pi^* \simeq (-) \otimes (\mathcal{O}_{X^{[n]}} \oplus \omega_{X^{[n]}})$, $\pi^* \omega_{X^{[n]}} \simeq \omega_{\text{CY}_n}$, and $\pi^* \pi_* \simeq \text{id} \oplus \tau^*$.

Next, $\pi^* i$ is also split spherical as follows from the first step together with Lemma 2.2; compare [1, Prop. 1.1].

Finally, to see that $\widehat{T}_A$ is equivariant, we note that $\pi^* i i^R \pi_* \tau_s \simeq \pi^* i i^R \pi_* \simeq \tau_s \pi^* i i^R \pi_* \tau_s$ so $\tau_s \widehat{T}_A \simeq \widehat{T}_A \tau_s$, since both are a cone of $\pi^* i i^R \pi_* \rightarrow \tau_s$.

**Example 3.5** If $A \simeq D^b(\text{Spec}(\mathbb{C}))$ is the category generated by an exceptional object $E$, then the twist $\widehat{T}_A$ associated to $\pi^* i$ is the Seidel–Thomas twist $\text{ST}_A$, where $A \simeq \pi^* i(E)$ is a spherical object by [29, Prop. 3.13].

**Example 3.6** Setting $i = F : D^b(X) \rightarrow D^b(X^{[n]})$ as the FM transform along the universal ideal sheaf gives the first part of Theorem 1.1.
Remark 3.7 The functor $\hat{F}: D^b(\tilde{X}) \to D^b(X^{[n]})$ defined as $F\pi_{\tilde{x}*}$ fails to be spherical in an interesting way. Note that $\pi_{\tilde{x}}^{\dagger} = \pi_{\tilde{x}*}$ in this case, so $R\hat{F} \simeq \pi_{\tilde{x}*}\pi_{\tilde{x}*} \simeq \id \oplus \tau_{\tilde{x}}$ and $C \simeq \tau_{\tilde{x}}$ is an autoequivalence of $D^b(\tilde{X})$. But the condition $\hat{F}S_{\tilde{x}} \simeq S_{X^{[n]}}\hat{F}C$ is not satisfied. Indeed, $\hat{F}S_{\tilde{x}}(\alpha) \simeq \hat{F}(\alpha)[2]$, whereas $S_{X^{[n]}}\hat{F}C(\alpha) \simeq \hat{F}(\alpha) \otimes \omega_{X^{[n]}}[2n]$, so these objects are not isomorphic for $\alpha \in D^b(\tilde{X})$ since their non-vanishing cohomologies lie in different degrees.

One could call $\hat{F}$ a sphere-like functor in analogy with the sphere-like objects of [14].

Furthermore, it is easy to check that the functor $\overline{F}: D^b(\tilde{X}) \to D^b(CY_n)$ defined as $\pi^*F\pi_{\tilde{x}*}$ is not spherical as well.

Remark 3.8 Let $B = \perp A$ so that $D^b(Y) = \langle A, B \rangle$ is a semi-orthogonal decomposition. Then, by [2, Thm. 11] we have $\overline{T}_A\overline{T}_B \simeq \tau^*[1]$.

Remark 3.9 One of the two descents of $\overline{T}_A$ is

\[ T_A := \text{cone} \left( i^R i^* M_{\omega_{\tilde{x}^{[n]}}} \rho_{\tilde{x}*} \overline{\epsilon} \right) \]

where $\epsilon$ is the counit of the adjunction. To see this, first note that the twist $\overline{T}_A$ along the spherical functor $\pi^*i$ is given by

\[ \overline{T}_A := \text{cone} \left( \pi^*i^R \pi_* \overline{\epsilon} \right) \]

Since $\pi_*\pi^* \simeq \id \oplus M_{\omega_{\tilde{x}^{[n]}}}$ and $\pi^*M_{\omega_{\tilde{x}^{[n]}}} \simeq \pi^*$, we have

\[(\pi^*i^R \pi_*) \pi^* \simeq \pi^*i^R \oplus \pi^*i^R M_{\omega_{\tilde{x}^{[n]}}} \simeq \pi^* (i^R \oplus M_{\omega_{\tilde{x}^{[n]}}} i^R M_{\omega_{\tilde{x}^{[n]}}})\]

from which $\overline{T}_A \pi^* \simeq \pi^*T_A$ follows. Using $M_{\omega_{\tilde{x}^{[n]}}} \pi_* \simeq \pi_*$ one can similarly show that $\pi_*\overline{T}_A \simeq T_A\pi_*$.

Remark 3.10 As in the case of spherical or $\Proj^n$-twists (see [18, Lem. 2.3]), we have for any $\Psi \in \text{Aut}(D^b(X^{[n]}))$ the relation $\Psi T_A \simeq T_{\Psi(A)} \Psi$. For this, one uses the cone description of $T_A$ given by Eq. (3.1) and the fact that the embedding functor of $\Psi(A)$ is $\Psi i$.

Remark 3.11 The reader will note that the proof of Theorem 3.4 shows more generally that the functor $\pi^*i$ is split spherical with equivariant twist for any canonical cover $\pi: \tilde{Y} \to Y$ of degree 2 and any fully faithful admissible embedding $i: A \to D^b(Y)$. More generally, $\pi^*$ is a $\Proj^{n-1}$-functor if $\pi: \tilde{Y} \to Y$ is a canonical cover of order $n \geq 2$. But for $n \geq 3$ the composition $\pi^*i$ is in general not a $\Proj^{n-1}$-functor for a fully faithful admissible embedding $i: A \to D^b(Y)$. The reason is that Lemma 2.2 does not generalise to powers of the Serre functor, that is, in general, $S_{\tilde{A}}^k \not\simeq i^R S_{\tilde{Y}}^k i$ for $k \geq 2$. 
3.3 Comparison to known autoequivalences

Let $X$ be an Enriques surface and $F : \mathcal{D}^b(X) \longrightarrow \mathcal{D}^b(X[n])$ the FM transform induced by the ideal sheaf of the universal family. We denote the twist along the spherical functor $\widetilde{F} = \pi^*F$ by $\widetilde{T} \in \text{Aut}(\mathcal{D}^b(\text{CY}_n))$ and its descent as described in Remark 3.9 by $T \in \text{Aut}(\mathcal{D}^b(X[n]))$. The second descent is given by $\widetilde{M}_{\omega X} T$. Now we want to compare these new autoequivalences to the known ones. Of course, on any variety, there are the standard autoequivalences. On $\text{CY}_n$ there are also Seidel-Thomas twists and the equivalences constructed in [26], while on $X[n]$ there is the $\mathbb{P}^{n-1}$-functor constructed in [18]. We will need the following statement.

**Lemma 3.12** Let $S$ be any surface and let $k(\xi) \in \mathcal{D}^b(S[n])$ be the skyscraper sheaf of a point $\xi \in S[b]$. Then, $F \mathcal{F}^R(k(\xi))$ has rank $\chi - 2n$ where $\chi := \chi(\omega_S) = \chi(\mathcal{O}_S)$.

**Proof** The kernel of $F \mathcal{F}^R$ is given by $\mathcal{I}_{\mathbb{Z}_n} \otimes p^*_n \omega_S[2]$. Since $\mathcal{I}_{\mathbb{Z}_n}$ is flat over $X[n]$, we get $F \mathcal{F}^R(k(\xi)) = \mathcal{I}_{\xi} \otimes \omega_S[2]$ where $\xi$ on the left-hand side denotes a point of $S[n]$ and $\xi$ on the right-hand side denotes the subscheme of $S$ represented by this point. Note that for any object $E \in \mathcal{D}^b(M)$ on a smooth variety $M$ and $x \in M$, we have $\text{rk } E = \chi(E, k(x))$. Thus,

$$\text{rk } F \mathcal{F}^R(\alpha) = \chi(F \mathcal{F}^R(\alpha), \alpha) = \chi(F \mathcal{F}^R(\alpha), F \mathcal{F}^R(\alpha)) = \chi(\mathcal{I}_{\xi}, \mathcal{I}_{\xi}) = \chi - n - n + 0$$

where the last equality uses the equation $\mathcal{I}_{\xi} = \mathcal{O}_S - \mathcal{O}_{\xi}$ in the Grothendieck group $K_0(S)$.

**Proposition 3.13** The autoequivalence $T$ is not contained in the subgroup of $\text{Aut}(\mathcal{D}^b(X[n]))$ generated by the standard autoequivalences $\text{Aut}^s(\mathcal{D}^b(X[n]))$ and the equivalence $P$ arising from the $\mathbb{P}^{n-1}$-functor constructed in [18]. Similarly, $\widetilde{T} \notin (\text{Aut}^s(\mathcal{D}^b(\text{CY}_n)), \widetilde{P})$, where $\widetilde{P}$ is a lift of $P$.

**Proof** The first assertion follows from the second, since standard autoequivalences lift to standard autoequivalences.

The twist $P$ is rank preserving since all the objects in the image of the corresponding $\mathbb{P}^{n-1}$-functor are supported on a proper subset of $X[n]$, see [18, Rem. 4.7]. Thus, the lift $\widetilde{P}$ is rank preserving too. The same holds for every standard autoequivalence (up to the sign -1 occurring for odd shifts). But by Lemma 3.12, we have for $\tilde{\xi} \in \text{CY}_n$ with $\tilde{\xi} := \pi(\xi) \in S[n]$ the equalities

$$2 \cdot \text{rk}(\tilde{T}(k(\tilde{\xi}))) = \text{rk}(T(k(\xi))) = 4n - 2 \neq 0.$$  

Before proceeding further with the comparisons, we need to compute the values of $\widetilde{T}$ and $T$ on certain spanning classes (for details on spanning classes, see [16, Sect. 1.3]). Following the proof of Theorem 3.4, we see that the cotwist of $\widetilde{F}$ is $C = S_X[-2n] = M_{\omega X}[2 - 2n]$. By [1, Sect. 1.4], $\tilde{T}$ is given on the spanning class $\text{im}(\tilde{F}) \cup \pi(\tilde{F})$ by

$$\tilde{T} \tilde{F} \simeq \tilde{F}C[1] \simeq \tilde{F}M_{\omega X}[3 - 2n], \quad \tilde{T}(\beta) = \beta \quad \text{for } \beta \in \text{im}(\tilde{F})^\perp.$$  

(3.2)
Note that \( \text{im}(\tilde{F})^\perp = \ker(\tilde{F}^R) = \{ \beta \in \mathbb{D}^b(CY_n) \mid \tilde{F}^R(\beta) = 0 \} \).

Using the cone description (3.1) of \( T \) we also get

\[
T \mathcal{F} \simeq M_{\omega_X[n]} F M_{\omega_X} [3 - 2n], \quad T M_{\omega_X[n]} F \simeq F M_{\omega_X} [3 - 2n] \tag{3.3}
\]

by Lemma 2.2. Also, \( T \) acts as the identity on \( \langle \mathcal{A}, \mathcal{A} \otimes \omega_X[n] \rangle^\perp = \ker(\tilde{F}^R) \cap \ker(F^L) \) where \( \mathcal{A} = \text{im}(F) \). So we have a description of the restriction of \( T \) to the spanning class \( \mathcal{C} := \mathcal{A} \cup \mathcal{A} \otimes \omega_X[n] \cup \langle \mathcal{A}, \mathcal{A} \otimes \omega_X[n] \rangle^\perp \). To see that \( \mathcal{C} \) is indeed a spanning class, note that if \( \text{Hom}(\beta, \gamma) = 0 \) for all \( \gamma \in \mathcal{C} \), then, in particular, \( \text{Hom}(\beta, \alpha) = 0 = \text{Hom}(\beta, \alpha \otimes \omega_X[n]) \) for all \( \alpha \in \mathcal{A} \). By Serre duality, \( \beta \in \langle \mathcal{A}, \mathcal{A} \otimes \omega_X[n] \rangle^\perp \subset \mathcal{C} \), hence \( \beta \simeq 0 \). Similarly, one proves that \( \text{Hom}(\gamma, \beta) = 0 \) implies that \( \beta \simeq 0 \).

**Remark 3.14** We will see in Section 5.3 that there exist non-trivial objects in \( \ker(\tilde{F}^R) \cap \ker(F^L) \). By applying \( \pi^* \), we also get non-trivial objects in \( \ker(\tilde{F}^R) \cap \ker(F^L) \). So there are \( 0 \neq \beta \in \mathbb{D}^b(X[n]) \) and \( 0 \neq \beta \in \mathbb{D}^b(CY_n) \) such that \( T(\beta) = \beta \) and \( T(\tilde{\beta}) = \beta \).

**Lemma 3.15** Let \( E \in \mathbb{D}^b(CY_n) \) be a spherical object and \( ST_E \) the induced Seidel–Thomas twist. Let \( 0 \neq \alpha \in \mathbb{D}^b(CY_n) \) with \( ST_E(\alpha) = \alpha[\ell] \). Then \( \ell = 0 \) or \( \ell = 1 - 2n \).

**Proof** We have \( ST_E(E) = E[1 - 2n] \) and \( ST_E(\beta) = \beta \) for all \( \beta \in E^\perp \). The assertion follows by [1, Prop. 1.2] together with the fact that \( E \cup E^\perp \) is a spanning class of \( \mathbb{D}^b(CY_n) \). See also Lemma 3.17 below for a similar statement with a similar proof.

**Proposition 3.16** The twist \( \tilde{T} \) does not equal a shift of a Seidel–Thomas twist.

**Proof** By (3.2), we have \( \tilde{T}(\tilde{F}(\alpha)) = \tilde{F}(\alpha)[3 - 2n] \) for \( \alpha \in \mathbb{D}^b(X) \) with \( \omega_X \otimes \alpha \simeq \alpha \), e.g. \( \alpha = k(x) \) a skyscraper sheaf. By Remark 3.14, there is also a \( 0 \neq \beta \in \mathbb{D}^b(CY_n) \) such that \( \tilde{T}(\beta) = \beta \). Thus, the assumption that \( \tilde{T}[m] = ST_E \) for some \( m \in \mathbb{Z} \) contradicts the previous lemma.

Recall that if \( Z, Z' \) are smooth projective varieties and \( a < b \) two integers, an exact functor \( G : \mathbb{D}^b(Z) \rightarrow \mathbb{D}^b(Z') \) is said to have *cohomological amplitude* \( [a, b] \) if for every complex \( E \in \mathbb{D}^b(Z) \) whose cohomology is concentrated in degrees between \( p \) and \( q \), the cohomology of \( G(E) \) is concentrated in degrees between \( p - a \) and \( q + b \). We will need the following result.

**Lemma 3.17** Let \( 0 \neq \alpha \in \mathbb{D}^b(X[n]) \) with \( T(\alpha) = \alpha[\ell] \) or \( T(\alpha) = \alpha \otimes \omega_X[n][\ell] \). Then \( \ell = 0 \) or \( \ell = 3 - 2n \). Similarly, if \( \gamma \in \mathbb{D}^b(CY_n) \) with \( \tilde{T}(\gamma) = \gamma[\ell] \) then \( \ell = 0 \) or \( \ell = 3 - 2n \).

**Proof** Let \( \alpha \in \mathbb{D}^b(X[n]) \) with \( T(\alpha) = \alpha[\ell], \ell \notin \{0, 3 - 2n\} \). To simplify notation, write \( \Phi = M_{\omega_X[n]}, \Psi = M_{\omega_X} \) and \( j = 3 - 2n \). By Eq. (3.3), we have \( TF \simeq \Phi F \Psi[j] \).

Hence, for any \( \beta \in \mathbb{D}^b(X) \), we have

\[
\text{Hom}(\alpha, F(\beta)[k]) \simeq \text{Hom}(T^m(\alpha), T^m F(\beta)[k]) \\
\simeq \text{Hom}(\alpha[m\ell], \Phi^m F \Psi^m(\beta)[mj + k]) \\
\simeq \text{Hom}(\Psi^{-m} F^L \Phi^{-m}(\alpha), \beta[(j - \ell)m + k])
\]
for every \( k \in \mathbb{Z} \). This vanishes for \( m \gg 0 \) since \( \Phi = M_{\omega_X[n]} \) and \( \Psi = M_{\omega_X} \) have cohomological amplitude \([0, 0]\) and \( F^L \) has finite cohomological amplitude by \([20, \text{Prop. 2.5}]\). Therefore, \( \alpha \in \perp_A \) where \( A = \im F \). Similarly, we get \( \alpha \in \perp (A \otimes \omega_X[\eta]) \).

The object \( \alpha \) is also orthogonal to \( B := \langle A, A \otimes \omega_X[\eta] \rangle^\perp = A^\perp \cap \perp_A \) on which \( T \) acts trivially. Indeed, for \( \beta \in B \) and \( k \in \mathbb{Z} \), we have

\[
\hom(\alpha, \beta[k]) = \hom(T^m(\alpha), T^m(\beta))[k] = \hom(\alpha[m\ell], \beta[k])
\]

which vanishes for \( m \gg 0 \).

Therefore, \( \alpha \) is orthogonal to the spanning class \( A \cup A \otimes \omega_X[\eta] \cup \langle A, A \otimes \omega_X[\eta] \rangle^\perp \), hence is zero. The proof in the case that \( T(\alpha) = \alpha \otimes \omega_X[\ell] \) is similar and the statement about \( \tilde{T} \) and \( \gamma \) follows by applying \( \pi_n \).

Now we want to compare our autoequivalence to those described in \([26]\). While recalling the construction of the latter, we also introduce some more general facts which will be useful in the next section.

Let \( Z \) be a smooth projective variety and \( n \geq 2 \). We consider the cartesian power \( Z^n \) equipped with the natural \( \mathfrak{S}_n \)-action given by permuting the factors.

For \( E \in \mathcal{D}^b(Z) \) an exceptional object, the box product \( E \boxtimes_n \) is again exceptional, since by the Küneth formula

\[
\text{Ext}^*_{\mathcal{D}^b_{\mathfrak{S}_n}(Z^n)}(E \boxtimes_n, E \boxtimes_n) \simeq \text{Ext}^*_X(E \boxtimes_n, E \boxtimes_n)_{\mathfrak{S}_n} \simeq S^n \text{Ext}^*_X(E, E) \simeq \mathbb{C}[0].
\]

More generally, for \( \rho \) an irreducible representation of \( \mathfrak{S}_n \), the object \( E \boxtimes_n \otimes \rho \) is exceptional and the objects obtained this way are pairwise orthogonal, i.e. \( \text{Ext}^*(E \boxtimes_n \otimes \rho, E \boxtimes_n \otimes \rho') = 0 \) for \( \rho \neq \rho' \).

The case of biggest interest is if \( Z \) is a surface. Then, there is the Bridgeland–King–Reid–Haiman equivalence (see \([9]\) and \([13]\))

\[
\Phi : \mathcal{D}^b(Z[n]) \xrightarrow{\sim} \mathcal{D}^b_{\mathfrak{S}_n}(Z^n).
\]

In particular, when \( Z = X \) is an Enriques surface, we get for every exceptional object \( E \in \mathcal{D}^b(X) \) further induced autoequivalences given by the Seidel–Thomas twists \( \tilde{T}_\rho := \text{ST}_{\pi^*\Phi^{-1}(E \boxtimes \rho)} \in \text{Aut}(\mathcal{D}^b(CY_n)) \) and their descents \( T_\rho \in \text{Aut}(\mathcal{D}^b(X[n])) \). We assume from now on that there exists an object \( 0 \neq F \in \langle E, E \otimes \omega_X \rangle^\perp \). This is equivalent to \( \tilde{E}^\perp \) being non-trivial, where \( \tilde{E} := \pi^*E \) is the corresponding spherical object on the K3 cover \( \tilde{X} \) of \( X \) (for example, if \( E \) is a line bundle, we can take \( \tilde{E} \otimes I^\perp_{x,y} I_y \in \tilde{E}^\perp \) for two distinct points \( x, y \)). In this case, \( F \boxtimes n \) is orthogonal to every \( E \boxtimes_n \otimes \rho \) and \( \omega_X \otimes E \boxtimes_n \otimes \rho \). Thus, \( \tilde{T}_\rho(\alpha) = \alpha \) for \( \alpha = \pi^*\Phi^{-1}(F \boxtimes m) \). Since also \( \tilde{T}_\rho(\beta) = \beta[1-2n] \) for \( \beta = \pi^*\Phi^{-1}(E \boxtimes n \otimes \rho) \), we see that \( \tilde{T}_\rho \) and thus also its descent \( T_\rho \) are non-standard.

By construction, the object \( \tilde{E} \) is invariant under \( \tau_X \), hence the associated Seidel–Thomas twist descends to an equivalence \( \Phi_E \) of \( \mathcal{D}^b(X) \). In turn, this gives an equivalence \( \Phi_E = FM_{\mathfrak{P} \boxtimes n} \in \text{Aut}(\mathcal{D}^b(X[n])) \), where \( \mathfrak{P} \in \mathcal{D}^b(X \times X) \) denotes the Fourier–
Mukai kernel of $\Phi_E$, see [25]. It is possible to lift this equivalence to $\widetilde{\Phi_E} \in \text{Aut}(\text{CY}_n)$, see [26] for details.

Recall that the isomorphism classes of irreducible representations of $\mathfrak{S}_n$ are in bijection to the set $P(n)$ of partitions of $n$. For an exceptional object $E \in D^b(X)$, we set

$$G_E := \langle [1], \Phi_{\overline{E}} : T_\rho : \rho \in P(n) \rangle \subset \text{Aut}\left(D^b(X^{[n]})\right).$$

By Remark 3.10, the latter shows that $\widetilde{\Phi_E}$ commutes with all the $T_\rho$. Note that for $k < n$ and $\rho \neq \rho'$, we have

$$\text{Ext}^n\left(E^{\overline{E}} \otimes \rho, E^k \cdot F^{n-k}\right) = 0 = \text{Ext}^n\left(E^{\overline{E}} \otimes \rho, E^{\overline{E}} \otimes \rho'\right).$$

Similarly to (3.3) we get by the cone description (3.1) of $T_\rho$

$$T_\rho : E^k \cdot F^{n-k} \mapsto E^k \cdot F^{n-k} \text{ for } k < n,$$

$$T_\rho : E^{\overline{E}} \otimes \rho' \mapsto \begin{cases} E^{\overline{E}} \otimes \rho' & \text{if } \rho \neq \rho' \\ \omega_{X^n} \otimes E^{\overline{E}} \otimes \rho'[-(2n-1)] & \text{if } \rho = \rho' \end{cases}$$

which, in particular, shows that the $T_\rho$ pairwise commute.

Now consider $\Psi = (\Phi_{\overline{E}})^a \circ \prod_\rho T^{b_\rho}_\rho$ $[c] \in G_E$ and assume that $\Psi \simeq \text{id}$. We have

$$\Psi(E^{\overline{E}}) = F^{\overline{E}}[c]$$

and thus $c = 0$. It follows that $\Psi(E^1 \cdot F^{n-1}) = E^1 \cdot F^{n-1}[-a]$ and thus $a = 0$. Finally, $\Psi(E^{\overline{E}} \otimes \rho) = \omega_{X^n}^{b_\rho} \otimes E^{\overline{E}} \otimes \rho[-b_\rho(2n-1)]$ shows that $b_\rho = 0$ for all $\rho$. The assertion that $T \notin G_E$ follows in a similar way using Lemma 3.17.

The identities (3.4) and (3.5) lift to identities in $D^b(\text{CY}_n)$. Therefore, one can analogously show that $\widetilde{G_E} \simeq \mathbb{Z}^{p(n)+2}$ and $\widetilde{T} \notin G_E$. □
Remark 3.19 Let \( a \) be the sign representation, i.e. the one-dimensional representation on which \( S_n \) acts by multiplication by \( \text{sgn} \). By Remark 3.10, we have \( T_a \simeq M_a \circ T \mathbb{C} \circ M_a \) where \( \mathbb{C} \) denotes the trivial representation and \( M_a \in \text{Aut}(D^b_{\mathfrak{S}_n}(X^n)) \) is the involution \((-) \otimes a\). Note that for higher dimensional irreducible representations \( \rho \), there is no such relation since \( M_\rho \) is not an equivalence.

4 Exceptional sequences on \( X^{[n]} \)

Let \( Z \) be a smooth projective variety and \( n \geq 2 \). In this section, we will construct exceptional sequences in the equivariant derived category \( D^b_{\mathfrak{S}_n}(Z^n) \) out of exceptional sequences in \( D^b(Z) \).

**Proposition 4.1** ([27, Cor. 1]). Let \( Z \) and \( Z' \) be smooth projective varieties with full exceptional sequences \( E_1, \ldots, E_k \) and \( F_1, \ldots, F_\ell \), respectively. Then

\[
E_1 \boxtimes F_1, E_1 \boxtimes F_2, \ldots, E_1 \boxtimes F_\ell, E_2 \boxtimes F_1, \ldots, E_k \boxtimes F_\ell
\]

is a full exceptional sequence of \( D^b(Z \times Z') \).

Let now \( E_1, \ldots, E_k \) be an exceptional sequence on \( Z \). We consider for every multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in [1, k]^n := \{1, \ldots, k\}^n \) the object

\[
E(\alpha) := E_{\alpha_1} \boxtimes \cdots \boxtimes E_{\alpha_n} \in D^b(Z^n).
\]

**Remark 4.2** Let \( \alpha, \beta \in [1, k]^n \). By the Künneth formula

\[
\text{Ext}^*(E(\alpha), E(\beta)) = \text{Ext}^*(E_{\alpha_1}, E_{\beta_1}) \otimes \cdots \otimes \text{Ext}^*(E_{\alpha_n}, E_{\beta_n}). \tag{4.1}
\]

Therefore, if the original sequence is full then, by Proposition 4.1, these objects form a full exceptional sequence of \( D^b(Z^n) \) when considering them with the ordering given by the lexicographical order \( <_{\text{lex}} \) on \([1, k]^n\). Thus, we have \( \text{Ext}^*(E(\alpha), E(\beta)) = 0 \), whenever \( \alpha_i > \beta_i \) for some \( i \in [1, n] \).

**Theorem 4.3** ([11]). Let \( G \) be a finite group acting on a variety \( M \). Consider an exceptional sequence of \( D^b(M) \) of the form

\[
E_1^{(1)}, \ldots, E_{k_1}^{(1)}, E_1^{(2)}, \ldots, E_{k_2}^{(2)}, \ldots, E_1^{(\ell)}, \ldots, E_{k_\ell}^{(\ell)}
\]

such that \( G \) acts transitively on every block \( E_i^{(i)} \), \( i = 1, \ldots, \ell \), i.e. for every \( i \in [\ell] \) and every pair \( a, b \in [1, k_i] \) there is a \( g \in G \) such that \( g^* E_a^{(i)} \simeq E_b^{(i)} \) (and conversely for every \( g \in G \) and every \( a \in [1, k_i] \) there is an element \( b \in [1, k_i] \) such that \( g^* E_a^{(i)} \simeq E_b^{(i)} \)). Let \( H_i := \text{Stab}_G(E_1^{(i)}) \) and assume that \( E_1^{(i)} \) carries an \( H_i \)-linearisation, i.e. there exists an \( E^{(i)} \in D^b_{H_i}(M) \) such that \( \text{Res}(E^{(i)}) = E_1^{(i)} \). Then
\[
\text{Inf}^G_{H_1}(\mathcal{E}^{(1)} \otimes V_1^{(1)}), \ldots, \text{Inf}^G_{H_1}(\mathcal{E}^{(1)} \otimes V_{m_1}^{(1)}), \ldots, \\
\text{Inf}^G_{H_{\ell}}(\mathcal{E}^{(\ell)} \otimes V_1^{(\ell)}), \ldots, \text{Inf}^G_{H_{\ell}}(\mathcal{E}^{(\ell)} \otimes V_{m_{\ell}}^{(\ell)})
\]

is an exceptional sequence of \(D^b_G(M)\) with \(V_1^{(i)}, \ldots, V_{m_i}^{(i)}\) being all the irreducible representations of \(H_i\). The induced exceptional sequence of \(D^b_G(M)\) is full if and only if the original exceptional sequence of \(D^b(M)\) is full.

**Proof** In [11], the theorem is only stated in the case of full exceptional sequences. But one can easily infer from the proof that non-full exceptional sequences also induce non-full exceptional sequences. \(\square\)

In order to apply Theorem 4.3, we have to reorder the sequence consisting of the \(E(\alpha)\) as follows. For a multi-index \(\alpha \in [1, k]^n\), we denote the unique non-decreasing representative of its \(\mathfrak{S}_n\)-orbit by \(\text{nd}(\alpha)\). Then, we define a total order \(\prec\) of \([1, k]^n\) by

\[
\alpha \prec \beta : \iff \begin{cases} 
\text{nd}(\alpha) \leq \text{nd}(\beta) \text{ or } \\
\text{nd}(\alpha) = \text{nd}(\beta) \text{ and } \alpha \leq \beta
\end{cases}
\]

Now the group \(\mathfrak{S}_n\) acts transitively on the blocks consisting of all \(E(\alpha)\) with fixed \(\text{nd}(\alpha)\) because of \(\sigma^*E(\alpha) \simeq E(\sigma^{-1} \cdot \alpha)\). Furthermore, every \(E(\alpha)\) has a canonical \(\text{Stab}(\alpha)\)-linearisation given by permutation of the factors in the box product. It remains to show that \((E(\alpha))_\alpha\) with the ordering given by \(\prec\) is still an exceptional sequence. This follows by Remark 4.2 and the last item of the following lemma.

**Lemma 4.4** Let \(\alpha, \beta \in [1, k]^n\).

1. Let \(\text{nd}(\alpha) = \text{nd}(\beta)\) but \(\alpha \neq \beta\). Then, there exists an \(i \in [n]\) such that \(\alpha_i < \beta_i\).
2. Let \(\sigma \in \mathfrak{S}_n\). Then, there exists an \(i \in [1, n]\) such that \(\alpha_i < \beta_i\) if and only if there exists a \(j \in [1, n]\) such that \((\sigma \cdot \alpha)_j < (\sigma \cdot \beta)_j\).
3. If \(\text{nd}(\alpha) \leq \text{nd}(\beta)\), then there exists an \(i \in [1, n]\) with \(\alpha_i < \beta_i\).
4. Let \(\alpha \prec \beta\). Then, there exists an \(i \in [1, n]\) such that \(\alpha_i < \beta_i\).

**Proof** If \(\text{nd}(\alpha) = \text{nd}(\beta)\), we have \(\alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n\). This shows (1). By setting \(j = \sigma(i)\) we obtain (2). In order to show (3) we may now assume using (2) that \(\alpha = \text{nd}(\alpha)\). Let \(\sigma \in \mathfrak{S}_n\) be such that \(\beta = \sigma \cdot \text{nd}(\beta)\) and let \(m := \min\{\ell \in [1, n] \mid \text{nd}(\alpha)_\ell \neq \text{nd}(\beta)_\ell\}\). Then \(\text{nd}(\alpha)_m < \text{nd}(\beta)_m\). If \(\sigma(m) \leq m\), we have \(\alpha_{\sigma(m)} = \text{nd}(\alpha)_{\sigma(m)} \leq \text{nd}(\alpha)_m < \text{nd}(\beta)_m = \beta_{\sigma(m)}\). If \(\sigma(m) > m\), there exists \(\ell > m\) such that \(\sigma(\ell) \leq m\). This yields

\[
\alpha_{\sigma(\ell)} = \text{nd}(\alpha)_{\sigma(\ell)} \leq \text{nd}(\alpha)_m < \text{nd}(\beta)_m \leq \text{nd}(\beta)_\ell = \beta_{\sigma(\ell)}.
\]

Finally, (4) follows from (1) and (3). \(\square\)

We summarise all of the above in the following

**Proposition 4.5** If \(\alpha \in [1, k]^n\) is a non-decreasing multi-index and \(V_i^{(\alpha)}\) is an irreducible representation of \(H_\alpha = \text{Stab}(\alpha)\), then the collection of objects \(\mathcal{E}(\alpha, V_i^{(\alpha)}) := \)
Inf${\mathcal{E}}_{\alpha}(E(\alpha) \otimes V^\alpha_i)$ forms an exceptional sequence of $\text{D}^b_{\mathfrak{S}}(Z^n)$. The induced sequence is full if and only if the original sequence on $\text{D}^b(Z)$ is full.

**Remark 4.6** An exceptional sequence is called **strong** if all the higher extension groups between its members vanish. Using Eq. (4.1), one can show that $(E(\alpha, V^\alpha_i))_{\alpha, i}$ is strong if and only if $(E^\ell)_{\ell}$ is strong. Thus, in the case that the full exceptional sequence $E_1, \ldots, E_k$ of $\text{D}^b(Z)$ is strong, there is an equivalence of triangulated categories $\text{D}^b_{\mathfrak{S}}(Z^n) \simeq \text{D}^b(\text{Mod} - \text{End}_{\mathfrak{S}}(\mathcal{M}))$ where $\mathcal{M} := \bigoplus_{\alpha, i} E(\alpha, V^\alpha_i)$, see [6].

**Remark 4.7** Using [12], one can also construct semi-orthogonal decompositions of $\text{D}^b_{\mathfrak{S}}(Z^n)$ out of general semi-orthogonal decompositions of $\text{D}^b(Z)$ in a similar way. Let us describe the special case that $\text{D}^b(Z) = \langle A, E \rangle$ with $E \in \text{D}^b(Z)$ an exceptional object. Then, there is an induced semi-orthogonal decomposition of $\text{D}^b_{\mathfrak{S}}(Z^n)$ with components

$$B(k, \rho) := \left(\text{Inf}_{\mathfrak{S}}(E^k \otimes \rho) \otimes B) \mid \text{Res}(B) \simeq A_{k+1} \boxtimes \cdots \boxtimes A_n, A_i \in \mathcal{A}\right)$$

for $k = 0, \ldots, n$ and $\rho$ an irreducible representation of $\mathfrak{S}_k$. Note that $B(n, \rho)$ is spanned by the exceptional object $E^\otimes n \otimes \rho$ and $B(n - 1, \rho) \simeq A$.

**Remark 4.8** If $B \in \text{D}^b(Z)$ is a tilting object, so is $\bigoplus_{\rho \in P(n)} (B^\otimes n \otimes \rho) \in \text{D}^b_{\mathfrak{S}}(Z^n)$.

**Remark 4.9** Any Enriques surface has a completely orthogonal exceptional sequence $(E_i)_i$ of length 10 consisting of line bundles, see [30]. The induced exceptional sequence in $\text{D}^b_{\mathfrak{S}}(X^n) \simeq \text{D}^b(X^{[n]})$ is again completely orthogonal, and the same holds for the corresponding sequence of spherical objects on CY$_n$. By [18, Cor. 2.4], it follows that the associated spherical twists give an embedding $\mathbb{Z}^{[10, n]} \rightarrow \text{Aut}(\text{D}^b(CY_n))$ where $\ell(10, n)$ denotes the length of the induced sequence. By arguments similar to those in the proof of Proposition 3.18, we also get an embedding $\mathbb{Z}^{[10, n]} \rightarrow \text{Aut}(\text{D}^b(X^{[n]}))$.

### 5 The truncated universal ideal functor

The arguments in this section follow those of [22, Sect. 6]. Recall that for any surface $Z$ and any $n \geq 2$, there is the Bridgeland–King–Reid–Haiman equivalence

$$\Phi : \text{D}^b(Z^{[n]}) \xrightarrow{\sim} \text{D}^b_{\mathfrak{S}}(Z^n).$$

Note that $Z^{[n]}$ is smooth under our assumptions, while this is not true anymore when $\dim Z \geq 3$ and $n \geq 3$. The key new observation is that the functor $\hat{F} := \Phi F : \text{D}^b(Z) \rightarrow \text{D}^b_{\mathfrak{S}}(Z^n)$ for $Z = S$ a surface can be truncated to a functor $G$ in such a way that $G^R \hat{G} \simeq \hat{F}^R \hat{F}$ and that this functor generalises in a nicer way to varieties of arbitrary dimension than $\hat{F}$ does.
If $Z = S$ is a surface, then $\hat{F}'' = \Phi \text{FM}_{O_{Z^n}} = \text{FM}_{C^\bullet}$, where $C^\bullet$ is the complex concentrated in degrees $0, \ldots , n - 1$ given by

$$0 \rightarrow \bigoplus_{i=0}^{n} O_{D_1} \rightarrow \bigoplus_{|I|=2} O_{D_I} \otimes a_I \rightarrow \bigoplus_{|I|=3} O_{D_I} \otimes a_I \rightarrow \cdots \rightarrow O_{D_{[n]}} \otimes a_{[n]} \rightarrow 0;$$

see [28]. For $I \subset \{1, \ldots , n\}$, the reduced subvariety $D_I \subset S \times S^n$ is given by $D_I = \{(y, x_1, \ldots , x_n) \mid y = x_i \forall i \in I\}$ and $a_I$ is the sign representation of $G_I$. Furthermore, $\hat{F}' = \Phi \text{FM}_{O_{S \times S^n}} \simeq \text{FM}_{O_{S \times S^n}}$ and the induced map $\hat{F}' \rightarrow \hat{F}''$ is given by the morphism of kernels $O_{S \times S^n} \rightarrow O^0 = \oplus_i O_{D_i}$ whose components are given by restriction of sections. We set $G'' := \text{FM}_{O^0}$. As explained in [22, Sect. 6], the main steps in the computation of the formulas of [28] and [17] can be translated into the following statement

$$\hat{F}'' \simeq \hat{F}' \hat{F}'' \simeq \hat{F}' \hat{F}'' \simeq G'' \hat{F}', \quad \hat{F}'' \simeq G'' \hat{F}.'$$

(5.1)

**Definition 5.1** Let $Z$ be a smooth projective variety of arbitrary dimension $d$ and $n \geq 2$. The **truncated universal ideal functor** $G = \text{FM}_G : D^b(Z) \longrightarrow D^b_{\mathbb{C}^n}(Z^n)$ is the Fourier–Mukai transform whose kernel is the complex

$$G := G^\bullet := (0 \rightarrow O_{Z \times Z^n} \rightarrow \oplus_{i=1}^{n} O_{D_i} \rightarrow 0) \in D^b_{\mathbb{C}^n}(Z \times Z^n).$$

Thus, there is the triangle of FM transforms $G ightarrow G' ightarrow G''$ with

$$G' := \text{FM}_{G'} = \hat{F}' = H^*(Z, -) \otimes O_{Z^n}, \quad G'' := \text{FM}_{G''} = \text{Inf}_{\mathbb{C}^n} p^n_{*} \text{triv}.$$

For $E \in D^b(Z)$, we have $G''(E) = \oplus_{i=1}^{n} p_{i*} E$ (see also Sect. 2.8 for details about the inflation functor $\text{Inf}$ and its right adjoint $\text{Res}$). The right adjoints are

$$G'^R = \text{FM}_{G'^R} = H^*(Z^n, -) \otimes_{\mathbb{C}^n} \otimes \omega_Z[d], \quad G'^R = O_{Z^n} \otimes \omega_Z[d],$$

$$G''R = \text{FM}_{G''R} = [-] \otimes p_{n*} \circ \text{Res}_{\mathbb{C}^n}^{\mathbb{C}^n-1}, \quad G''R = \oplus_{i=0}^{n} O_{D_i}.$$

In the surface case Eq. (5.1) gives the following

**Lemma 5.2** If $Z = S$ is a surface, $\hat{F}^R \hat{F} \simeq G^R G$.

See [19, Sec. 5.5] for a further investigation of the relation between the functors $F$ and $G$.

We compute the compositions of the kernels:

$$G'^R G' = (O_Z \otimes \omega_Z) \otimes S^n H^*(O_Z)[d],$$

$$G'^R G'' = (O_Z \otimes \omega_Z) \otimes S^{n-1} H^*(O_Z)[d],$$

$$G''R G' = (O_Z \otimes O_Z) \otimes S^{n-1} H^*(O_Z),$$

$$G''R G'' = O_\Delta \otimes S^{n-1} H^*(O_Z) \oplus (O_Z \otimes O_Z) \otimes S^{n-2} H^*(O_Z).$$
The induced map $G' \rightarrow G''$ under these isomorphisms is given by evaluation as follows. Let $(e_i)_i$ be a basis of $H^*(O_Z) = \text{Hom}(O_Z, O_Z[*])$. Then, the component

$$(O_Z \boxtimes \omega Z) \cdot e_1 \cdots e_n [d] \rightarrow (O_Z \boxtimes \omega Z) \cdot e_1 \cdots \hat{e}_k \cdots e_n [d]$$

is $e_k \boxtimes \text{id}[d]$. The component

$$(O_Z \boxtimes O_Z) \otimes S^{n-1} H^*(O_Z) \rightarrow (O_Z \boxtimes O_Z) \otimes S^{n-2} H^*(O_Z)$$

of $G'' \rightarrow G''$ is given in the same way and the component

$$(O_Z \boxtimes O_Z) \otimes S^{n-1} H^*(O_Z) \rightarrow O_\Delta \otimes S^{n-1} H^*(O_Z)$$

is the restriction map. The map $G'' \rightarrow G''$ is given as follows. Let $(\theta_i)_i$ be the basis of $H^*(O_Z) \vee$ dual to $(e_i)_i$. Furthermore, let $\theta_i$ correspond to the morphism $\tilde{\theta}_i \in \text{Hom}(O_Z, \omega Z[d - *]) \simeq \text{Hom}(O_Z, O_Z[*]) \vee$ under Serre duality. Then, the component

$$(O_Z \boxtimes O_Z) \cdot e_1 \cdots \hat{e}_k \cdots e_n \rightarrow (O_Z \boxtimes \omega Z) \cdot e_1 \cdots e_n [d]$$

is $\tilde{\theta}_k$.

In the following, we will use the commutative diagram

$$
\begin{array}{c}
G'' \rightarrow G'' \rightarrow G'' \\
\downarrow \quad \downarrow \quad \downarrow \\
G' \rightarrow G' \rightarrow G'
\end{array}
\quad (5.2)
$$

with exact triangles as columns and rows in order to deduce formulae for $G' = FM_{G''}$. 

5.1 The case of an even dimensional Calabi–Yau variety

If $Z$ is a Calabi–Yau variety of even dimension $d$, then $H^*(O_Z) = \mathbb{C}[0] \oplus \mathbb{C}[-d]$ and $S^k H^*(O_Z) = \mathbb{C}[0] \oplus \mathbb{C}[-d] \oplus \cdots \oplus \mathbb{C}[-dk]$. Let $u \in H^d(O_Z)$ be the basis vector whose dual $\theta \in \text{Hom}(O_Z, O_Z[d]) \vee$ corresponds to $\text{id} \in \text{Hom}(O_Z, O_Z)$ under Serre duality. We denote the induced degree $d\ell$ basis vector of $S^k H^*(O_Z)$ by $u^\ell$.

**Lemma 5.3** For a Calabi–Yau variety $Z$ of even dimension $d$, we have

$$G' \simeq \text{id} \oplus [-d] \oplus \cdots \oplus [-d(n - 1)].$$
Proof. By the description of the last subsection, the components $\mathcal{O}_{Z^2} \cdot u^\ell \longrightarrow \mathcal{O}_{Z^2} \cdot u^\ell$ of $\mathcal{G}''R \mathcal{G}' \longrightarrow \mathcal{G}''R \mathcal{G}''$ and $\mathcal{G}'R \mathcal{G} \longrightarrow \mathcal{G}'R \mathcal{G}''$ equal the identity. By [15, Lem. 5], the top of the diagram (5.2) is isomorphic to

\[
\begin{array}{c}
\mathcal{G}''R \mathcal{G} \longrightarrow \mathcal{O}_{Z^2}[-d(n-1)] \longrightarrow \mathcal{O}_\Delta([0] \oplus [-d] \oplus \cdots \oplus [-d(n-1)]) \\
\mathcal{G}'R \mathcal{G} \longrightarrow \mathcal{O}_{Z^2}[-d(n-1)] \longrightarrow 0
\end{array}
\]

where the middle vertical map is the component $\mathcal{O}_{Z^2} \cdot u^{n-1} \longrightarrow \mathcal{O}_{Z^2} u^n[d]$ of the morphism $\mathcal{G}''R \mathcal{G}' \longrightarrow \mathcal{G}'R \mathcal{G}'$. By the above description, it is the identity. Now the claim follows by the octahedral axiom. Alternatively, chase through long exact sequences of cohomology as in [1, Sect. 2.4].

**Theorem 5.4** If $Z$ is a Calabi–Yau variety of even dimension, then the truncated universal ideal functor $G : \mathcal{D}^b(Z) \longrightarrow \mathcal{D}^b_{\Sigma n}(Z^n)$ is a $\mathbb{P}^{n-1}$-functor.

Proof. By the previous lemma, condition (1) of the definition of a $\mathbb{P}^{n-1}$-functor is satisfied.

The proof that condition (2) is satisfied is analogous to the proof in the case of the non-truncated functor when $X$ is a K3 surface. Basically, one has to go through [1, Sect. 2.5] and replace $F$ by $G$, $q^*$ by $p_n^* \circ \text{triv} : \mathcal{D}^b(Z) \longrightarrow \mathcal{D}^b_{\Sigma n-1}(Z^n)$ and its right adjoint $q_*$ by $(-)_{\Sigma n-1} \circ p_{n*} \circ \text{triv}$, $\text{Inf} : \mathcal{D}^b_{\Sigma n-1}(Z^n) \longrightarrow \mathcal{D}^b_{\Sigma n}(Z^n)$ and its right adjoint $g_!$ by $\text{Res}$, $H^*(\mathcal{O}_{\Sigma n}) \simeq H^*(\mathcal{O}_{Z^n})_{\Sigma n}$, 2 by $d$, and the symplectic form $\sigma$, which occurs in [1, Sect. 2.5], by a generator of $H^d(\mathcal{O}_Z)$.

Very roughly, the idea is the following: $G^R \mathcal{G}$ is identified with the direct summand $(p_n^* \circ \text{triv})^R (p_n^* \circ \text{triv}) \simeq \text{id} \otimes H^*(Z^{n-1}, \mathcal{O}_{Z^{n-1}})_{\Sigma n-1}$ of $\mathcal{G}''R \mathcal{G}''$ and the monad structure of $(p_n^* \circ \text{triv})^R (p_n^* \circ \text{triv})$ is given by the cup product on $H^*(Z^{n-1}, \mathcal{O}_{Z^{n-1}})_{\Sigma n-1}$ which has the right form.

Condition (3) is easy to check since all occurring autoequivalences are simply shifts.

**Remark 5.5** If $Z$ is an odd dimensional Calabi–Yau variety, then $G$ fails to be spherical in an interesting way (compare Remark 3.7). Indeed, in this case, $H^*(\mathcal{O}_Z) = \mathbb{C}[0] \oplus \mathbb{C}[-d]$ and $S^k H^*(\mathcal{O}_Z) = \mathbb{C}[0] \oplus \mathbb{C}[-d]$ for $k \geq 1$. The reason for the vanishing of the higher degrees is that the symmetric product is taken in the graded sense.

We can then check that for $n \geq 3$, we have $G^R \mathcal{G} \simeq \mathcal{O}_\Delta([0] \oplus [-d])$ by using that the component $\mathcal{O}_{Z^2} \otimes S^{n-1} H^*(\mathcal{O}_Z) \longrightarrow \mathcal{O}_{Z^2} \otimes S^{n-2} H^*(\mathcal{O}_Z)$ of the map $\mathcal{G}''R \mathcal{G}' \longrightarrow \mathcal{G}''R \mathcal{G}''$ as well as the whole $\mathcal{G}'R \mathcal{G}' \longrightarrow \mathcal{G}'R \mathcal{G}''$ are isomorphisms.

The above means that the cotwist of $G$ is an equivalence, but for dimension reasons, the second axiom of a spherical functor cannot hold for $G$.

5.2 The case $H^*(\mathcal{O}_Z) = \mathbb{C}[0]$

In this case, also $S^k H^*(\mathcal{O}_Z) = \mathbb{C}[0]$ for $k \geq 1$. 
Lemma 5.2, the proof of Theorem 1.2.

4.7. Thus, we get a description of \( \text{im} \ L \) bundles

5.3 The orthogonal complement of \( \text{im} \ G \)

Let, as in the previous subsection, \( Z \) be a smooth projective variety such that the structure sheaf \( \mathcal{O}_Z \) is exceptional. We set \( A := \mathcal{O}_Z^{\perp} \) which means that we have the semi-orthogonal decomposition \( \mathcal{D}^b(Z) = \langle A, \mathcal{O}_Z \rangle \). For \( A \in \mathcal{A} \) we have \( G'(A) = 0 \) and thus \( G(A) = \text{Inf}(\mathcal{O}_{Z^n} \otimes A) \). Let \( \rho_n \) be the standard representation of \( \mathcal{S}_n \) which is given by the short exact sequence

\[
0 \rightarrow \mathbb{C} \rightarrow \mathbb{C}^n \rightarrow \rho_n \rightarrow 0
\]

where \( \iota: \mathbb{C} \rightarrow \mathbb{C}^n \) denotes the diagonal embedding of the trivial representation into the permutation representation. We have \( G'(\mathcal{O}_Z) = \mathcal{O}_{Z^n} \) and \( G''(\mathcal{O}_Z) = \mathcal{O}_{Z^n} \otimes \mathbb{C}^n \) from which \( G(\mathcal{O}_Z) = \mathcal{O}_{Z^n} \otimes \rho_n \) follows. We see that \( \text{im}(G) \) equals the component \( \langle B(n - 1, \mathbb{C}), \mathcal{O}_{Z^n} \otimes \rho_n \rangle \) of the semi-orthogonal decomposition described in Remark 4.7. Thus, we get a description of \( \text{im}(G) \) and \( \perp \text{im}(G) \). For example, in the case \( n = 2 \), we have \( \text{im}(G) \) with

\[
\mathcal{B}(0, \mathbb{C}) = \langle \text{Inf}(A_1 \otimes A_2) \mid A_1, A_2 \in \mathcal{A} \rangle.
\]

Let now \( Z = X \) be an Enriques surface. By tensoring the exceptional sequence of Remark 4.9 by \( E_{10}^{-1} \), we get the completely orthogonal exceptional sequence of line bundles \( \mathcal{L}_1, \ldots, \mathcal{L}_9, \mathcal{O}_X \) in \( \mathcal{D}^b(X) \). Using the above description of the image of \( G \), one can check that \( \mathcal{L}_{2n}^{\perp} \in \text{im}(G) \cap \text{im}(G) = \ker(G^R) \cap \ker(G^L) \) for \( j = 1, \ldots, 9 \).

The objects \( \mathcal{L}_{2n}^{\perp} \) are in fact also contained in \( \ker(G^R) \cap \ker(F^L) \) which proves Remark 3.14. Indeed, for \( i = 2, \ldots, n \) the FM transform \( \text{FM}_{\mathcal{C}^i} \) is the composition

\[
\mathcal{D}^b(X) \xrightarrow{M_{a_i} \text{triv}} \mathcal{D}^b_{\mathcal{S}_i \times \mathcal{S}_{n-i}}(X) \xrightarrow{p^*_X} \mathcal{D}^b_{\mathcal{S}_i \times \mathcal{S}_{n-i}}(X \times X^{n-i}) \xrightarrow{(\delta_i \times \text{id})_a} \mathcal{D}^b_{\mathcal{S}_i \times \mathcal{S}_{n-i}}(X^i \times X^{n-i}) \xrightarrow{\text{Inf}} \mathcal{D}^b_{\mathcal{S}_i \times \mathcal{S}_{n-i}}(X^i \times X^{n-i})
\]

where \( a_i \) is the sign representation of \( \mathcal{S}_i \) and \( \delta_i: X \rightarrow X^i \) is the diagonal embedding.

The left adjoint \( \text{FM}_{\mathcal{C}^i}^L \) is
\[ D^b(X) \xrightarrow{(-)^{\otimes i} \times (-)_{\operatorname{op}}^i M_{\alpha i}} D^b_{\mathcal E_i \times \mathcal E_{n-i}}(X) \xrightarrow{p_{X_!}} D^b_{\mathcal E_i \times \mathcal E_{n-i}}(X \times X^{n-i}) \]

(5.3)

Now, let \( \mathcal L \) be one of the \( \mathcal L_j \). Then \( (\delta_i \times \operatorname{id})^* \mathcal L = \mathcal L^{\otimes i} \boxtimes \mathcal L^{\otimes i} \). The \( \mathcal E_i \)-action on \( \mathcal L^{\otimes i} \) is given by permuting the tensor factors. Since \( \mathcal L \) is a line bundle, this action is trivial. Thus, after tensoring by \( M_{\alpha i} \) the \( \mathcal E_i \)-action vanishes, hence \( \mathcal F M^{C_{i-1}}_i(\mathcal L^{\otimes i}) = 0 \). Since we already know that \( G^L(\mathcal L^{\otimes i}) = 0 \) we get \( \mathcal L^{\otimes i} \in \ker(F^L) \). The functor \( \mathcal F M^{R}_{C_{i-1}} \) is given by the composition (5.3) with \( (\delta_i \times \operatorname{id})^* \) replaced by \( (\delta_i \times \operatorname{id})^! \) and \( p_{X_!} \) replaced by \( p_{X*} \). Thus, also \( \mathcal F M^{R}_{C_{i-1}}(\mathcal L^{\otimes i}) = 0 \) and hence \( \mathcal L^{\otimes i} \in \ker(F^R) \).

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