ALGEBRAICALLY CONSTRUCTIBLE FUNCTIONS AND SIGNS OF POLYNOMIALS

ADAM PARUSIŃSKI AND ZBIGNIEWSZAFRANIEC

Abstract. Let $W$ be a real algebraic set. We show that the following families of integer-valued functions on $W$ coincide: (i) the functions of the form $w \mapsto \chi(X_w)$, where $X_w$ are the fibres of a regular morphism $f : X \to W$ of real algebraic sets, (ii) the functions of the form $w \mapsto \chi(X_w)$, where $X_w$ are the fibres of a proper regular morphism $f : X \to W$ of real algebraic sets, (iii) the finite sums of signs of polynomials on $W$. Such functions are called algebraically constructible on $W$. Using their characterization in terms of signs of polynomials we present new proofs of their basic functorial properties with respect to the link operator and specialization.

1. Introduction

Let $f : X \to W$ be a regular morphism of real algebraic sets. Consider on $W$ an integer-valued function $\varphi(w) = \chi(X_w)$, which associates to $w \in W$ the Euler characteristic of the fibre $X_w = f^{-1}(w)$. The main purpose of this paper is to study the properties of such $\varphi$.

Firstly, by stratification theory, $\varphi$ is (semialgebraically) constructible, that is there exists a semialgebraic stratification $S$ of $W$ such that $\varphi$ is constant on strata of $S$. Equivalently, we may express this property by saying that $\varphi$ is bounded and $\varphi^{-1}(n)$ is semialgebraic for every integer $n$. However it is well-known that not all semialgebraically constructible functions on $W$ are of the form $\chi(X_w)$ for a regular morphism $f : X \to W$. For instance, if $W$ is irreducible, then $\chi(X_w)$ has to be generically constant modulo 2, see for instance [1, Proposition 2.3.2]. Also in the case of $W$ irreducible, as shown in [15], there exists a real polynomial $g : W \to \mathbb{R}$ such that generically on $W \varphi(w) \equiv \text{sgn} g(w) \pmod{4}$, where by $\text{sgn} g$ we denote the sign of $g$. As we show in Theorem 5.5 below, for any regular morphism $f : X \to W$ of real algebraic sets there exist real polynomials $g_1, \ldots, g_s$ on $W$ such that for every $w \in W$

$$\chi(X_w) = \text{sgn} g_1(w) + \cdots + \text{sgn} g_s(w).$$

In particular, taking $g = g_1 \cdot \text{sgn} g_s$ we recover the result of [15].

Constructible functions of the form $\varphi(w) = \chi(X_w)$, for proper regular morphisms $f : X \to W$, were studied in [24] in a different context. Following [24] we call them algebraically constructible. As shown in [24] the family of algebraically constructible functions is preserved by various natural geometric operations such as, for instance, push-forward, duality, and specialization. In a way they behave similarly to constructible functions on complex algebraic varieties. However, unlike their complex counterparts, they cannot be defined neither in terms of stratifications nor as combinations of characteristic functions.
of real algebraic varieties, cf. [24]. Algebraically constructible functions were used in [24] to study local topological properties of real algebraic sets.

In particular, Theorem 5.5 below can be reformulated as follows. Algebraically constructible functions on a real algebraic set \( W \) coincide with finite sums of signs of real polynomials on \( W \), see Theorem 6.1 below. Using this characterization, in section 6, we give new, alternative proofs of basic properties of algebraically constructible functions, without using the resolution of singularities as in [24].

The main result of the paper, Theorem 5.5, is proven in section 5. In sections 2-4 we develop necessary techniques for the proof and recall basic results the proof is based on. In particular, in section 2 we recall the Eisenbud-Levine Theorem 2.2 and the Khimshiashvili formula 2.3. In section 3 we review some basic facts on the Grauert-Hironaka formal division algorithm Theorem 3.1, which we then use to obtain a parametrized version of the Eisenbud-Levine Theorem, Propositions 3.7 and 3.8, with parameter in a given algebraic set \( w \in W \). In section 4 we study polynomial families of polynomial vector fields \( F_w : \mathbb{R}^n \rightarrow \mathbb{R}^n \) parametrized by \( w \in W \).

The proof of Theorem 5.5 can be sketched briefly as follows. First, by an argument similar to the Khimshiashvili Formula, we show that the Euler characteristic \( \chi(X) \) of a real algebraic set \( X \) can be calculated in terms of the local topological degree at the origin of a polynomial vector field, see Proposition 2.5. Then using the theory developed by the second named author, see e.g. [27, 28], we generalize this observation in two directions. Firstly, we show that for a regular morphism \( f : X \rightarrow W \), the Euler characteristic of the fibers \( \chi(X_w) \) can be expressed in terms of the local topological degree \( \deg_0 G_w \) at the origin of a family \( G_w : \mathbb{R}^n \rightarrow \mathbb{R}^n \) of polynomial vector fields, which depends polynomially on \( w \). Secondly, as shown in Lemma 4.1, we may choose all \( G_w \) in such a way that they have algebraically isolated zero at the origin. Then, by the Eisenbud-Levine Theorem 2.2, each \( \deg_0 G_w \) can be calculated algebraically, that is \( \deg_0 G_w \) equals the signature of an associated symmetric bilinear form \( \Psi_w \). By section 3, we may as well require that \( \Psi_w \) depend "polynomially" on \( w \). More precisely, there exists a symmetric matrix \( T(w) \) (representing \( \Psi_w \)) with entries polynomials in \( w \), such that \( \deg_0 G_w \) equals the signature of \( T(w) \), for all \( w \) in a Zariski open subset of \( W \). (See 3.7 and 3.8 for the details.) Finally by Descartes’ Lemma, we express the signature of \( T(w) \) in terms of signs of polynomials in \( w \), see Lemma 2.1 and the proof of Lemma 4.2.

For the definitions and properties of real algebraic sets and maps we refer the reader to [7]. By a real algebraic set we mean the locus of zeros of a finite set of polynomial functions on \( \mathbb{R}^n \).

2. Preliminaries

Let \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \) be a real polynomial. Let \( \Lambda \) be the set of all pairs \((r,s)\) with \( 0 \leq r < s \leq n \) such that \( a_r \neq 0, a_s \neq 0 \), and \( a_i = 0 \) for \( r < i < s \). Denote \( \Lambda' = \{(r,s) \in \Lambda \mid r + s \text{ is odd}\} \).

**Lemma 2.1.** Assume that all roots of \( f(x) \) are real and \( a_0 \neq 0, a_n \neq 0 \). Let \( p_+ \) (resp. \( p_- \)) denote the number of positive (resp. negative) roots counted with multiplicities. Then

\[
p_+ - p_- = - \sum \text{sgn } a_r a_s, \text{ where } (r,s) \in \Lambda',
\]

\[
p_+ - p_- \equiv n + 1 + (-1)^{n+1} \text{sgn } a_0 a_n \pmod{4}.
\]
Proof. We say that the pair of real numbers \((a,b)\) changes sign if \(ab < 0\). If this is the case then \((1 - \text{sgn } ab)/2 = 1\), if \(ab > 0\) then \((1 - \text{sgn } ab)/2 = 0\).

As a consequence of Descartes’ lemma (see [25], Theorem 6, p.232, or [1], Exercise 1.1.13 (4), p.16), \(p_+\) equals the number of sign changes in the sequence of non-zero coefficients of \(f(x)\), that is,

\[
p_+ = \sum (1 - \text{sgn } a_r a_s)/2, \text{ where } (r,s) \in \Lambda.
\]

According to the same fact, \(p_-\) equals the number of sign changes in the sequence of non-zero coefficients of \(f(-x)\), i.e.

\[
p_- = \sum (1 - (-1)^{r+s} \text{sgn } a_r a_s)/2, \text{ where } (r,s) \in \Lambda.
\]

Hence

\[
p_+ - p_- = - \sum \text{sgn } a_r a_s, \text{ where } (r,s) \in \Lambda'.
\]

The sign of the product of all roots, that is \((-1)^{p_-}\), equals \((-1)^n \text{sgn } a_0 a_n\). Thus \(2p_- \equiv 3 + (-1)^{p_-} = 3 + (-1)^n \text{sgn } a_0 a_n \pmod{4}\). Finally, since \(p_+ + p_- = n\), we conclude that

\[
p_+ - p_- = n - 2p_- \equiv n - 3 - (-1)^n \text{sgn } a_0 a_n \equiv n + 1 + (-1)^{n+1} \text{sgn } a_0 a_n \pmod{4}.
\]

Let \(F : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)\) be a germ of a continuous mapping with isolated zero at \(0\). Then we denote by \(\deg_0 F\) the local topological degree of \(F\) at the origin. Suppose, in addition, that \(F = (f_1, \ldots, f_m)\) is a real analytic germ. Let \(\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]\) denote the ring of formal power series and let \(I\) denote the ideal in \(\mathbb{R}[x]\) generated by \(f_1, \ldots, f_m\). Then \(Q = \mathbb{R}[x]/I\) is an \(\mathbb{R}\)-algebra. If \(\dim_{\mathbb{R}} Q < \infty\), then \(0\) is isolated in \(F^{-1}(0)\) and in this case we say that \(F\) has an algebraically isolated zero at \(0\). Let \(J\) denote the residue class in \(Q\) of the Jacobian determinant

\[
\frac{\partial (f_1, \ldots, f_m)}{\partial (x_1, \ldots, x_m)}.
\]

The next theorem is due to Eisenbud and Levine [10], see also [4, 8, 22] for a proof.

**Theorem 2.2** (Eisenbud&Levine Theorem). Assume that \(\dim_{\mathbb{R}} Q < \infty\). Then

(i) \(J \neq 0\) in \(Q\),

(ii) for any \(\mathbb{R}\)-linear form \(\varphi : Q \to \mathbb{R}\) such that \(\varphi(J) > 0\), the corresponding symmetric bilinear form \(\Phi : Q \times Q \to \mathbb{R}\), \(\Phi(f,g) = \varphi(fg)\), is non-degenerate and signature \(\Phi = \deg_0 F\). \(\Box\)

The next formula was proved by Khimshiashvili [22], for other proofs see [8, 11, 30].

**Theorem 2.3** (Khimshiashvili Formula). Let \(g : (\mathbb{R}^m, 0) \to (\mathbb{R}, 0)\) be a real analytic germ with isolated critical point at \(0\). Let \(S_\epsilon\) denote the sphere of a small radius \(\epsilon\) centered at the origin and let \(A_\epsilon = S_\epsilon \cap \{g \leq 0\}\). (Note that all \(A_\epsilon\) are homeomorphic for \(\epsilon > 0\) small enough.) Then the gradient \(\nabla g : \mathbb{R}^m \to \mathbb{R}^m\) of \(g\) has an isolated zero at \(0\) and

\[
\chi(A_\epsilon) = 1 - \deg_0(\nabla g).
\]

**Lemma 2.4.** Let \(g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}\) be a polynomial vanishing at \(0\) and such that if \(g(x,t) \leq 0\) then either \((x,t) = 0\) or \(t > 0\). Let \(S_\epsilon \subset \mathbb{R}^n \times \mathbb{R}\) (resp. \(B_\epsilon\)) denote the sphere (resp. the open ball) of radius \(\epsilon\) centered at the origin and let \(A_\epsilon = S_\epsilon \cap \{g \leq 0\}\). Let \(P_\eta = \mathbb{R}^n \times \{\eta\}\)
and \( M_{\epsilon, \eta} = P_\eta \cap \{ g \leq 0 \} \cap B_\epsilon \). Then, for \( 0 < \eta \ll \epsilon \ll 1 \), \( A_\epsilon \) and \( M_{\epsilon, \eta} \) have the same homotopy type. In particular,

\[
\chi(A_\epsilon) = \chi(M_{\epsilon, \eta}).
\]

**Proof.** Consider on \( N = \{(x, t) \mid g(x, t) \leq 0 \} \), the functions \( \omega_1(x, t) = \|x\|^2 + t^2 \) and \( \omega_2(x, t) = t \). Both \( \omega_1 \) and \( \omega_2 \) are non-negative on \( N \) and \( \omega_1^{-1}(0) \cap N = \omega_2^{-1}(0) \cap N = \{0\} \). Let \( N^\eta_2 = \{(x, t) \in N \mid 0 < \omega_1(x, t) \leq \eta \} \). By the topological triviality of semi-algebraic mappings, see for instance [10, Theorem 9.3.1] or [12, 20], there is \( \delta > 0 \) such that \( \omega_1 : N^\delta_1 = (0, \delta], i = 1, 2, \) are topologically trivial fibrations. For \( 0 < \eta \leq \delta \) let \( M^\eta_2 \) denote the union of connected components of \( N^\eta_2 \) containing \( 0 \) in their closures. Then \( \omega_2 : M^\delta_2 \to (0, \delta] \) is also topologically trivial.

Hence there exist constants \( 0 < \alpha < \beta < \gamma < \delta \) such that \( M^\alpha_2 \subset N^\beta_1 \subset M^\gamma_2 \subset N^\delta_1 \). By the topological triviality, the inclusions \( M^\alpha_2 \subset M^\beta_2 \) and \( N^\beta_1 \subset N^\delta_1 \) are homotopy equivalencies and hence so are \( M^\alpha_2 \subset N^\beta_1 \) and \( N^\beta_1 \subset M^\gamma_2 \).

By the above, the total spaces of fibrations \( \omega_1 : N^\beta_1 \to (0, \delta], \omega_2 : M^\gamma_2 \to (0, \delta] \) are homotopy equivalent to their fibers. Consequently the fibers of both fibrations are homotopy equivalent. Now, to complete the proof, it is enough to observe that these fibers are of the form \( A_\epsilon \) and \( M_{\epsilon, \eta} \), where \( 0 < \eta \ll \epsilon \ll 1 \). \( \Box \)

**Proposition 2.5.** Let \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) be a non-negative homogeneous polynomial of degree \( 2d \) such that \( f(x, 0) = \|x\|^{2d} \). Let \( X = \{x \in \mathbb{R}^n \mid f(x, 1) = 0 \} \) and define \( g(x, t) = f(x, t) - t^{2d+1} \). Then \( g \) has an isolated critical point at the origin and

\[
\chi(X) = 1 - \deg_0(\nabla g).
\]

**Proof.** Let

\[
\Sigma = \left\{ (x, t) \mid \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0 \right\},
\]

\( P_\eta = \mathbb{R}^n \times \{ \eta \} \), and \( \Sigma_\eta = \Sigma \cap P_\eta \). Let \( f_\eta \) (resp. \( g_\eta \)) denote the restriction of \( f \) (resp. \( g \)) to \( P_\eta \). Then \( \Sigma_\eta \) is the set of critical points of both \( f_\eta \) and \( g_\eta \). We have \( \Sigma_0 = \{0\} \). Since the set of critical values of any polynomial is finite, so is each \( f_\eta(\Sigma_\eta) \). Moreover, since \( f \) is non-negative homogeneous of degree \( 2d \) and \( \Sigma \) is a homogeneous set, there is \( D > 0 \) such that any \( y \in f_\eta(\Sigma_\eta) \), if non-zero, satisfies \( y > D \| \eta \|^{2d} \).

If \( \eta < 0 \), then \( g_\eta > 0 \) and \( 0 \in \mathbb{R} \) is a regular value of \( g_\eta \). Clearly \( g_0 \) has a single critical point at the origin.

Consider \( 0 < \eta \ll 1 \). Let \( x \in \Sigma_\eta \). If \( f_\eta(x) > 0 \) then

\[
g_\eta(x) = f_\eta(x) - \eta^{2d+1} > D\eta^{2d} - \eta^{2d+1} > 0.
\]

If \( f_\eta(x) = 0 \) then \( g_\eta(x) < 0 \). Thus \( 0 \in \mathbb{R} \) is a regular value for \( g_\eta \). Hence there is an open neighbourhood \( U \subset \mathbb{R}^n \times \mathbb{R} \) of the origin such that \( 0 \in \mathbb{R} \) is a regular value of \( g \) on \( U \setminus \{0\} \), i.e. \( g \) has an isolated critical point at the origin.

For \( \eta \) fixed, \( \lim f_\eta(x) = +\infty \) as \( \|x\| \to +\infty \). Denote \( N_\eta = \{ x \mid f_\eta(x) = 0 \} \) and \( M_\eta = \{ x \mid g_\eta(x) \leq 0 \} = \{ x \mid f_\eta(x) \leq \eta^{2d+1} \} \). If \( \eta < 0 \), then \( M_\eta \) is empty and \( M_0 = \{0\} \). If \( \eta > 0 \), then \( N_\eta \subset M_\eta \). As we have shown above, for \( 0 < \eta \ll 1 \) both \( f_\eta \) and \( g_\eta \) have no critical points in \( M_\eta - N_\eta \). Hence \( N_\eta \) is a deformation retract of \( M_\eta \) and, in particular, \( \chi(N_\eta) = \chi(M_\eta) \).

Suppose \( 0 < \eta \ll \epsilon \). Then, since \( f_0 = g_0 = \|x\|^{2d} \), \( M_\eta \subset B_\epsilon \), that is \( M_\eta = M_{\epsilon, \eta} \) in the notation of Lemma 2.4. Moreover, let \( A_\epsilon = S_\epsilon \cap \{ g \leq 0 \} \). By Lemma 2.4, \( \chi(A_\epsilon) = \chi(M_\eta) \),
and hence, by the above
\[ \chi(A_\epsilon) = \chi(M_\eta) = \chi(N_\eta). \]
Finally, by the Khimshiashvili formula 3.3,
\[ \chi(A_\epsilon) = 1 - \deg_0(\nabla g), \]
and the lemma follows since \( \chi(X) = \chi(N_1) = \chi(N_\eta) \), for \( \eta > 0 \). \( \square \)

3. The formal division algorithm

In the first part of this section we review some basic facts on the Grauert-Hironaka formal division algorithm for formal power series with polynomial coefficients. In exposition and notation we follow closely [10]. Then we apply the Grauert-Hironaka algorithm to derive a parametrized version of the Eisenbud-Levine Theorem 2.2, with parameter in a given algebraic set \( W \).

Let \( A \) be an integral domain. Let \( A[[y]] = A[[y_1, \ldots, y_n]] \) denote the ring of formal power series in \( n \) variables with coefficients in \( A \).

If \( \beta = (\beta^1, \ldots, \beta^n) \in \mathbb{N}^n \), put \( |\beta| = \beta^1 + \cdots + \beta^n \). We order the \((n+1)\)-tuples \((\beta^1, \ldots, \beta^n, |\beta|)\) lexicographically from the right. This induces a total ordering of \( \mathbb{N}^n \).

Let \( f \in A[[y]] \). Write \( f = \sum_{\beta \in \mathbb{N}^n} f_\beta y^\beta \), where \( f_\beta \in A \) and \( y^\beta \) denotes \( y_1^{\beta_1} \cdots y_n^{\beta_n} \). Let \( \text{supp}(f) = \{ \beta \in \mathbb{N}^n \mid f_\beta \neq 0 \} \) and let \( \nu(f) \) denote the smallest element of \( \text{supp}(f) \). Let \( \text{in}(f) \) denote \( f_{\nu(f)} y^{\nu(f)} \).

Let \( I \) be an ideal in \( A[[y]] \). We define the diagram of initial exponents \( \mathcal{N}(I) \) as \( \{ \nu(f) \mid f \in I \} \). Clearly, \( \mathcal{N}(I) + \mathbb{N}^n = \mathcal{N}(I) \). There is a smallest finite subset \( V(I) \) of \( \mathcal{N}(I) \) such that \( \mathcal{N}(I) = V(I) + \mathbb{N}^n \). We call the elements of \( V(I) \) the vertices of \( \mathcal{N}(I) \).

Let \( \beta_1, \ldots, \beta_t \in V(I) \) be the vertices of \( \mathcal{N}(I) \) and choose \( g^1, \ldots, g^t \in I \) so that \( \beta_i = \nu(g^i), i = 1, \ldots, t \). The \( \beta_1, \ldots, \beta_t \) induce the following decomposition of \( \mathbb{N}^n \): Set \( \Delta_0 = \emptyset \) and then define \( \Delta_i = (\beta_i + \mathbb{N}^n) \setminus \Delta_0 \cup \cdots \cup \Delta_{i-1}, i = 1, \ldots, t \). Put \( \Delta = \mathbb{N}^n \setminus \Delta_0 \cup \cdots \cup \Delta_t = \mathbb{N}^n \setminus \mathcal{N}(I) \).

Let \( \text{in}(g^i) = g_{\beta_i}^i y^\beta_i \). Then \( g_{\beta_i}^i \neq 0 \). Let \( A_0 \) denote the field of fractions of \( A \). We denote by \( S \) the multiplicative subset of \( A \) generated by the \( g_{\beta_i}^i \) and by \( S^{-1}A \) the corresponding localization of \( A \); i.e. the subring of \( A_0 \) comprising the quotients with denominators in \( S \). Then \( S^{-1}A[[y]] \subset A_0[[y]] \).

**Theorem 3.1** (Grauert, Hironaka, [3, 4, 11, 19]). For every \( f \in S^{-1}A[[y]] \) there exist unique \( g_i \in S^{-1}A[[y]], i = 1, \ldots, t, \) and \( r \in S^{-1}A[[y]] \) such that \( \beta_i + \text{supp}(g_i) \subset \Delta_i, \) \( \text{supp}(r) \subset \Delta, \) and
\[ f = \sum_{i=1}^{t} g_i g^i + r. \] \( \square \)

**Corollary 3.2.** \( \nu(f) \leq \nu(r) \). In particular, if \( \Delta \) is finite and \( \beta < \nu(f) \) for all \( \beta \in \Delta, \) then \( r = 0 \) and \( f \) belongs to the ideal in \( S^{-1}A[[y]] \) generated by \( g^1, \ldots, g^t. \) \( \square \)

Let \( S^{-1}I[[y]] \) denote the ideal in \( S^{-1}A[[y]] \) generated by \( I \). Then \( S^{-1}A[[y]]/S^{-1}I[[y]] \) is finitely generated if and only if \( \Delta \) is finite. If this is the case then \( S^{-1}A[[y]]/S^{-1}I[[y]] \) is a free \( S^{-1}A \) module and we take the monomials \( y^\beta, \beta \in \Delta, \) as a basis.
Let $W \subset \mathbb{R}^n$ be an irreducible real algebraic set and let $A$ denote the ring of real polynomial functions on $W$. Each $w \in W$ defines an evaluation homomorphism $h \mapsto h(w)$ of $A$ onto $\mathbb{R}$. For $f = \sum_{\beta} f_{\beta} y^\beta \in A[[y]]$ we write $f(x; y) = \sum_{\beta} f_{\beta}(x) y^\beta$, and $f(w; y) = \sum_{\beta} f_{\beta}(w) y^\beta$ when the coefficients are evaluated at $x = w$.

Let $f^1, \ldots, f^s \in A[[y]]$ and let $I$ denote the ideal in $A[[y]]$ generated by $f^1, \ldots, f^s$. Let $N = N(I) = \{ \nu(g) | g \in I \}$ denote the diagram of initial exponents (here $A = A$). Given $w \in W$. We denote by $I_w$ the ideal in $R[[y]]$ generated by $f^1(w; y), \ldots, f^s(w; y)$ and by $N_w = N(I_w)$ the diagram of initial exponents of $I_w$ (so here $A = R$).

The next theorem was proved by Bierstone and Milman [3].

**Theorem 3.3.** Assume that $W$ is irreducible (so that $A$ is an integral domain). Let $\beta_1, \ldots, \beta_t$ denote the vertices of $N$ and choose $g^i \in I$ such that $\nu(g^i) = \beta_i$. Let

$$\Sigma = \bigcup_{i=1}^t \{ w \in W | g^i_{\beta_i}(w) = 0 \}.$$ 

Then $\Sigma$ is a proper algebraic subset of $W$, $N_w = N$ for all $w \in W - \Sigma$, $\nu(g^i) = \beta_i = \nu(g^i(w; \cdot))$ for every vertex $\beta_i \in N$ and $w \in W \setminus \Sigma$. $\square$

**Corollary 3.4.** Suppose that $\Delta_w = N^n \setminus N_w$ is finite for each $w \in W \setminus \Sigma$. Then $\Delta = N^n \setminus N$ is also finite and $\Delta = \Delta_w$ for all $w \in W \setminus \Sigma$. $\square$

Suppose that $\Delta$ is finite and let $\beta$ denote the largest element in $\Delta$. Let $j = y^\beta$. Then for $w \in W - \Sigma$, the residue class of $j$ in $Q_w = R[[y]]/I_w$ is nonzero.

**Definition.** Let $\varphi_w : Q_w \longrightarrow R$ be the linear form given by $\varphi_w(j) = 1$ and $\varphi_w(y^\beta) = 0$ for $\beta \in \Delta - \{ \beta \}$.

Let $\Phi_w : Q_w \times Q_w \longrightarrow R$ be the corresponding symmetric bilinear form, $\Phi_w(f, g) = \varphi_w(fg)$. Let $M_w$ denote the matrix of $\Phi_w$ in the basis $y^\beta$, $\beta \in \Delta$. Let, as before, $S$ denote the multiplicative subset of $A$ generated by $g^i_{\beta_i}$.

**Lemma 3.5.** There is a symmetric matrix $M$ with entries in $S^{-1}A$ such that $M_w = M(w)$ for $w \in W \setminus \Sigma$. $\square$

From now on we suppose that $F = (f_1, \ldots, f_n) : W \times R^n \longrightarrow R^n$ is a polynomial mapping with $F(w; 0) = 0$ for every $w \in W$. Denote

$$J = \frac{\partial(f_1, \ldots, f_n)}{\partial(y_1, \ldots, y_n)}$$

and $J_w = J(w; \cdot)$.

Let $I$ be the ideal in $A[[y]]$ generated by $f_1, \ldots, f_n$ and $I_w \subset R[[y]]$ that generated by $f_1(w; \cdot), \ldots, f_n(w; \cdot)$. We assume that $\dim R Q_w < \infty$ for every $w \in W$. Hence, $\Delta$ and all $\Delta_w$ are finite.

**Lemma 3.6.** If $w \in W \setminus \Sigma$ then there is $0 \neq \lambda_w \in R$ such that $J_w = \lambda_w J$ in $Q_w$.

**Proof.** By the Eisenbud-Levine Theorem [2], $J_w \neq 0$ in $Q_w$. By Theorem 3.1, $J_w = \sum_{\beta \in \Delta} \lambda_{\beta} y^\beta$ in $Q_w$. Suppose, contrary to our claim, that $\lambda_{\beta'} \neq 0$ for a $\beta' < \beta$. Then, define a linear form $\psi : Q_w \longrightarrow R$ by the formula $\psi(f) = f_{\beta'} \lambda_{\beta'}$, where $f = \sum_{\beta \in \Delta} f_{\beta} y^\beta \in Q_w$.

We show that the corresponding symmetric bilinear form $\Psi(f, g) = \psi(fg)$ is degenerate. For any $f \in Q_w$ we have $\nu(fj) = \nu(f) + \nu(j) \geq \nu(j) = \beta$. Therefore, by Corollary [2], $\psi(fj) = 0$ for any $f \in Q_w$ and hence $\Psi(f, g)$ is degenerate. On the other hand $\psi(J_w) = \lambda_{\beta'}^2 > 0$, and hence the existence of $\psi$ contradicts Theorem 2.2.

Thus $\lambda_{\beta'} = 0$ for every $\beta' \in \Delta \setminus \{ \beta \}$ and we take $\lambda_w = \lambda_{\beta}$. $\square$
Proposition 3.7. The forms $\psi_x$ and $\Psi_w$ defined above satisfy

(i) $\psi_x(J_w) > 0$,
(ii) $\Psi_w$ is non-degenerate,
(iii) the entries and the determinant of the matrix of $\Psi_w$ in basis $y^\beta$, $\beta \in \Delta$, belong to $S^{-1}A$.

Proof. $\psi_x(J_w) = \lambda_w \varphi_w(J_w) = \lambda_w^2 \varphi_w(j) = \lambda_w^2 > 0$, so the statement follows from the Eisenbud-Levine Theorem 2.2 and Lemma 3.5. $\square$

Clearly multiplication by a positive scalar does not change the signature of a symmetric matrix. So if we multiply the matrix of $\Psi_w$ by the product of squares of the denominators of its entries we get

Proposition 3.8. Assume that $W$ is irreducible. Then there are a symmetric matrix $T$ with entries polynomials in $w \in W$ and a proper algebraic subset $\Sigma \subset W$ such that for every $w \in W \setminus \Sigma$

(i) $T(w)$ is non-degenerate,
(ii) signature $\Psi_w = \text{signature } T(w)$. $\square$

4. Families of vector fields

Lemma 4.1. Let $F : W \times \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial mapping. For any $w \in W$ let $F_w = F(w; \cdot) : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that for all $w \in W$, $0 \in \mathbb{R}^n$ is isolated in $F_w^{-1}(0)$. (Hence $\deg_G F_w$ is always well-defined.) Then there is a polynomial mapping $G : W \times \mathbb{R}^n \to \mathbb{R}^n$ such that for every $w \in W$

(i) $G_w : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ has an algebraically isolated zero at $0$,
(ii) $\deg G_w = \deg G_w$.

Proof. By the parametrized version of the Lojasiewicz Inequality of [17], there is $\alpha > 0$ such that

$$\|F_w(y)\| \geq C\|y\|^\alpha$$

for every $w \in W$ and $\|y\| < \delta$, where $C = C(w) > 0$ and $\delta = \delta(w) > 0$ depend on $w$.

Choose an integer $k \gg 0$. Define $G(w; y) = F(w; y) + (y_1^k, \ldots, y_n^k)$. Let $G_{C,w} : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ denote the complexification of $G_w$. Then, for every $w \in W$, $G_{C,w}(0)$ is a bounded complex algebraic set and hence finite. So $0$ is isolated in $G_{C,w}(0)$ and hence $G_w$ has an algebraically isolated zero at $0$.

We may assume that $k > \alpha$. So if $w \in W$ and $y$ is close enough to the origin then

$$\|tG_w(y) + (1 - t)F_w(y)\| = \|F_w(y) + t(y_1^k, \ldots, y_n^k)\| \geq C\|y\|^\alpha - t\|(y_1^k, \ldots, y_n^k)\| \geq \frac{C}{2}\|y\|^\alpha$$
for $0 \leq t \leq 1$. Hence $\deg_0 F_w = \deg_0 G_w$ as required. □

**Lemma 4.2.** Under the assumptions of Lemma 4.4, if moreover $W$ is irreducible, then there exist a proper algebraic subset $\Sigma \subset W$, an integer $\mu$, and polynomials $q_1, \ldots, q_1, q_n$ nowhere vanishing in $W \setminus \Sigma$ such that for every $w \in W \setminus \Sigma$

(i) $\deg_0 F_w = \text{sgn } q_1(w) + \cdots + \text{sgn } q_1(w),$
(ii) $\deg_0 F_w \equiv \mu + 1 \pmod{2},$
(iii) $\deg_0 F_w \equiv \mu + \text{sgn } q(w) \pmod{4}.$

**Proof.** Let $F = (f_1, \ldots, f_n)$, let $I_w$ denote the ideal in $R[[y]]$ generated by $f_1(w; \cdot), \ldots, f_n(w; \cdot)$ and let $Q_w = R[[y]]/I_w$. By Lemma 4.1 we may assume that each $F_w$ has an algebraically isolated zero at 0. Let

$$J = \frac{\partial(f_1, \ldots, f_n)}{\partial(y_1, \ldots, y_n)}$$

and let $J_w$ denote the residue class of $J(w; \cdot)$ in $Q_w$.

Let $\psi_w : Q_w \to R$ be the linear form defined in section 3. By Proposition 3.7, $\psi_w$ satisfies the assumptions of the Eisenbud-Levine Theorem 2.2. Hence the corresponding symmetric bilinear form $\Psi_w$ is non-degenerate and $\deg_0 F_w = \text{signature } \Psi_w$. In particular, by Proposition 3.8, there is a symmetric matrix $T$ with polynomial entries and a proper algebraic set $\Sigma' \subset W$ such that $T(w)$ is non-degenerate and $\deg_0 F_w = \text{signature } T(w)$ for every $w \in W \setminus \Sigma'$.

Let $P_w(\lambda) = a_N \lambda^N + a_{N-1}(w) \lambda^{N-1} + \cdots + a_0(w)$, $a_N \equiv (-1)^N$, denote the characteristic polynomial of $T(w)$. Clearly its coefficients are polynomials in $w$ and $a_0(w)$ does not vanish in $W \setminus \Sigma'$. If $w \in W \setminus \Sigma'$ then all roots of $P_w$ are real and non-zero. Let $p_+(w)$ (resp. $p_-(w)$) denote the number of positive (resp. negative) roots. Then

$$\text{signature } T(w) = p_+(w) - p_-(w),$$

and, by Lemma 2.3, it is easy to see that there are proper algebraic $\Sigma \subset W$, polynomials $q_1, \ldots, q_1, q_n$ nowhere vanishing on $W \setminus \Sigma$, and an integer $\mu$ such that

(a) $\deg_0 F_w = p_+(w) - p_-(w) = \text{sgn } q_1(w) + \cdots + \text{sgn } q_1(w),$
(b) $\deg_0 F_w \equiv \mu + \text{sgn } q(w) \pmod{4}$

for every $w \in W \setminus \Sigma$ which completes the proof. □

Let $P$ be any non-negative polynomial with $P^{-1}(0) \cap W = \Sigma$. Then

$$\sum \text{sgn } P(w) q_i(w) = \sum \text{sgn } q_i(w)$$
on $W \setminus \Sigma$ and

$$\sum \text{sgn } P(w) q_i(w) = 0$$
on $\Sigma$. Similarly, let $p_1, \ldots, p_r$ be another set of polynomials. Then

$$\sum \text{sgn } p_j(w) + \sum \text{sgn } (-P(w)p_j(w)) = 0$$
on $W \setminus \Sigma$ and

$$\sum \text{sgn } p_j(w) + \sum \text{sgn } (-P(w)p_j(w)) = \sum \text{sgn } p_j(w)$$
on $\Sigma$. Hence, by induction on $\dim W$ we get
Theorem 4.3. Let \( W \) be a real algebraic set and let \( F : W \times \mathbb{R}^n \to \mathbb{R}^n \) be a polynomial mapping such that \( 0 \) is isolated in \( F^{-1}(0) \) for all \( w \in W \). Then there are polynomials \( g_1, \ldots, g_s \) such that for every \( w \in W \)
\[
\deg_0 F_w = \text{sgn } g_1(w) + \cdots + \text{sgn } g_s(w). \quad \square
\]

5. Families of algebraic sets

Let \( X \subseteq W \times \mathbb{R}^n \) be a real algebraic set such that \( W \times \{0\} \subseteq X \). There is a non-negative polynomial \( f : W \times \mathbb{R}^n \to \mathbb{R} \) such that \( X = f^{-1}(0) \). Denote \( f_w(y) = f(w;y) \). Then \( 0 \) is contained in the set of critical points of each \( f_w \). By the parametrized version of the Lojasiewicz Inequality of \([17]\), there is \( \alpha > 0 \) such that for every \( w \in W \) there are positive \( C = C(w) \) and \( \delta = \delta(w) \) such that
\[
f_w(y) \geq C\|y\|^\alpha,
\]
for all critical points \( y \) of \( f_w \) with \( \|y\| < \delta \) and \( f_w(y) \neq 0 \).

Let \( k \) be an integer such that \( 2k > \alpha \). Define
\[
g(w; y) = f(w; y) - \|y\|^{2k}
\]
and let
\[
G = \left( \frac{\partial g}{\partial y_1}, \ldots, \frac{\partial g}{\partial y_n} \right) : W \times \mathbb{R}^n \to \mathbb{R}^n.
\]
Clearly, \( G \) is a polynomial family of vector fields such that \( G_w(0) = 0 \).

For every \( w \in W \) let \( L(w) = \{y \in S_r^{n-1} \mid (w;y) \in X\} \), where \( r > 0 \) is small. It is well-known that \( L(w) \) is well-defined up to a homeomorphism. Then \( \chi(L(w)) = 1 - \deg_0 G_w \). Indeed, this can be proven by an argument similar to that of proof of Lemma 2.3, if we replace \( t^{2d+1} \) by \( \|y\|^{2k} \), \( P_n \) by the sphere \( S_r \), and \( \Sigma_n \) by the set of critical points of \( f \) restricted to \( S_r \), see \([27]\) for the details. Therefore, Theorem 4.3 implies

Theorem 5.1. For all \( w \in W \), \( \mathbb{R}^n \ni 0 \) is isolated in \( G_w^{-1}(0) \) and \( \chi(L(w)) = 1 - \deg_0 G_w \).

In particular, there are polynomials \( g_1, \ldots, g_s \) such that for every \( w \in W \)
\[
\chi(L(w)) = \text{sgn } g_1(w) + \cdots + \text{sgn } g_s(w). \quad \square
\]

Similarly, let \( S(w) = \{y \in S_R^{n-1} \mid (w;y) \in X\} \), where \( R > 0 \) is very large. \( S(w) \) is well-defined up to a homeomorphism.

Corollary 5.2. There is a polynomial family of vector fields \( H_w : \mathbb{R}^n \to \mathbb{R}^n \) such that \( \mathbb{R}^n \ni 0 \) is isolated in \( H_w^{-1}(0) \) for all \( w \in W \) and \( \chi(S(w)) = 1 - \deg_0 H_w \).

Proof. Let \( d \) denote the degree of \( f \), where as above, \( f \) is a non-negative polynomial defining \( X \). Then, there is a non-negative polynomial \( h : W \times \mathbb{R}^n \to \mathbb{R} \) such that \( h(w;y) = \|y\|^{2d} f(w; y/\|y\|^2) \) for \( y \neq 0 \). Clearly \( h(w; 0) \equiv 0 \) and \( S(w) \) is homeomorphic to \( L(w) = \{y \in S_r^{n-1} \mid (w;y) \in h^{-1}(0)\} \), where \( r > 0 \) is small. So the corollary follows from Theorem 5.1. \( \square \)

It is well-known (see, for instance, \([4, 8, 10]\)) that the single point Aleksandrov compactification of a real algebraic set is homeomorphic to a real algebraic set. We shall recall briefly the proof.

Suppose \( X = \{y \in \mathbb{R}^n \mid f_1(y) = \cdots = f_p(y) = 0\} \), where \( f_1, \ldots, f_p : \mathbb{R}^n \to \mathbb{R} \) are polynomials of degree \( \leq p - 1 \). Set \( h(y, y_{n+1}) = y_{n+1}^2 (f_1^2(y) + \cdots + f_p^2(y)) + (y_{n+1} - 1)^2 \), so that \( h^{-1}(0) \) is homeomorphic to \( X \) and \( h \) is a non-negative polynomial of degree \( \leq 2p \).
Put $y' = (y, y_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ and $H(y') = \|y'\|^4 b(y'\|y'\|)^2$. Then, it is easy to see that $H$ extends to a non-negative polynomial on $\mathbb{R}^n \times \mathbb{R}$ such that $H(0,0) = 0$ and $H(y') = \|y'\|^4 b + \text{monomials of lower degree}$. Clearly $\tilde{X} = H^{-1}(0)$ is the single point compactification of $X$ (If $X$ is compact then $\tilde{X} = X \sqcup \{\text{point}\}$). Note that $t^4 p H(y'/t)$ extends to a non-negative homogeneous polynomial $f(y',t)$ on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ of degree $4p$ such that $f(y',0) = \|y'\|^{4p}$ and $\tilde{X}$ is homeomorphic to $\{y' \mid f(y',1) = 0\}$. Proceeding exactly in the same way we may prove the following parametrized version of the above compactification method.

**Lemma 5.3.** Let $X \subset W \times \mathbb{R}^n$ be a real algebraic set. Then there is a non-negative polynomial $f : W \times \mathbb{R}^{n+1} \times \mathbb{R} \rightarrow \mathbb{R}$ such that for every $w \in W$

(i) $f_w(y',t) = f(w; y', t) : \mathbb{R}^{n+1} \times \mathbb{R} \rightarrow \mathbb{R}$ is a non-negative homogeneous polynomial of degree $4p$,

(ii) $f_w(y',0) = \|y'\|^{4p}$,

(iii) $\tilde{X}_w = \{y' \in \mathbb{R}^{n+1} \mid f_w(y',1) = 0\}$ is homeomorphic to the single point compactification of $X_w = \{y \in \mathbb{R}^n \mid (w;y) \in X\}$.

In particular, by Proposition 2.3 we get

**Proposition 5.4.** Let $X \subset W \times \mathbb{R}^n$ be a real algebraic set. Then there is a polynomial family of vector fields $F_w : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for every $w \in W$

(i) $F_w(0) = 0$,

(ii) $0$ is isolated in $F_w^{-1}(0)$,

(iii) $\chi(X_w) = 1 - \deg F_w$. □

Let $S(w) = X_w \cap S_{R}^{n-1}$, where $R > 0$ is sufficiently large. Then it is easy to check that

$$\chi(X_w) = \chi(\tilde{X}_w) + \chi(S(w)) - 1.$$  

By 5.4, 5.2, 4.3, and since $\text{sgn } a + \text{sgn } b \equiv \text{sgn } (ab) + 1 \pmod{4}$, provided $a \neq 0$ and $b \neq 0$, we get

**Theorem 5.5.** Let $X \subset W \times \mathbb{R}^n$ be a real algebraic set. Then there are polynomials $g_1, \ldots, g_s$ on $W$ such that

$$\chi(X_w) = \text{sgn } g_1(w) + \cdots + \text{sgn } g_s(w).$$

In particular, if $W$ is irreducible, then there are a proper algebraic subset $\Sigma \subset W$, an integer $\mu$, and a polynomial $g$ nowhere vanishing in $W - \Sigma$ such that for every $w \in W - \Sigma$

$$\chi(X_w) \equiv \mu + \text{sgn } g(w) \pmod{4},$$

In particular $\chi(X_w) \equiv \mu + 1 \pmod{2}$. □

6. Algebraically Constructible Functions

Let $W$ be a real algebraic set. An integer-valued function $\varphi : W \rightarrow \mathbb{Z}$ is called *(semialgebraically)* constructible if it admits a presentation as a finite sum

$$\varphi = \sum m_i 1_{W_i},$$

where for each $i$, $W_i$ is a semialgebraic subset of $W$, $1_{W_i}$ is the characteristic function of $W_i$, and $m_i$ is an integer. Constructible functions, well-known in complex domain, were studied in real algebraic set-up by Viro [29], and in sub-analytic set-up by Kashiwara and
If the support of constructible function $\varphi$ is compact, then we may choose all $W_i$ in (1) compact. Then, cf. [29, 26, 24], the Euler integral of $\varphi$ is defined as

$$\int \varphi = \sum m_i \chi(W_i).$$

It follows from the additivity of Euler characteristic that the Euler integral is well-defined and does not depend on the presentation (1) of $\varphi$, provided all $W_i$ are compact. Let $f: W \to Y$ be a (continuous) semialgebraic map of real algebraic sets, $\varphi$ a constructible function on $W$ and suppose that $f: W \to Y$ restricted to the support of $\varphi$ is proper. Then the direct image $f_* \varphi$ is given by the formula

$$f_* \varphi(y) = \int_{f^{-1}(y)} \varphi,$$

where by $\int_{f^{-1}(y)} \varphi$ we understand the Euler integral of $\varphi$ restricted to $f^{-1}(y)$. It follows from the existence of a stratification of $f$ that $f_* \varphi$ is a constructible function on $Y$.

Another more restrictive class of constructible functions, was introduced in [24] in order to study local topological properties of real algebraic sets. An integer-valued function $\varphi: W \to \mathbb{Z}$ is called algebraically constructible if there exists a finite collection of algebraic sets $Z_i$, regular proper morphisms $f_i: Z_i \to W$, and integers $m_i$, such that

$$\varphi = \sum m_i f_i^* 1_{Z_i}.$$

It is obvious that every algebraically constructible function is semialgebraically constructible but the converse is false for $\dim W > 0$. For instance, a constructible function on $\mathbb{R}$ is algebraically constructible if and only if it is generically constant mod 2. The reader may consult [24] for other examples. As a consequence of section 5 we obtain the following simple description of algebraically constructible functions.

**Theorem 6.1.** Let $W$ be a real algebraic set. Then $\varphi: W \to \mathbb{Z}$ is algebraically constructible if and only if there exist polynomial functions $g_1, \ldots, g_s$ on $W$ such that

$$\varphi(w) = \text{sgn} g_1(w) + \cdots + \text{sgn} g_s(w).$$

**Proof.** It is easy to see that the sign of a polynomial function $g$ on $W$ defines an algebraically constructible function. Indeed, let $\bar{W} = \{(w, t) \in W \times \mathbb{R} \mid g(w) = t^2\}$ and let $\pi: \bar{W} \to W$ denote the standard projection. Then $\text{sgn} f = \pi_* 1_{\bar{W}} - 1_W$ is algebraically constructible.

The opposite implication follows from Theorem 5.3. $\Box$

**Corollary 6.2.**

(i) Let $F: W \times \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial mapping satisfying the assumptions of 5.3. Then $w \to \deg_0 F_w$ is an algebraically constructible function on $W$.

(ii) Let $X_w$ be an algebraic family of affine real algebraic sets parametrized by $w \in W$ as in 5.3. Then $w \to \chi(X_w)$ is an algebraically constructible function on $W$. $\Box$

The next corollary is virtually equivalent to the main result of [15].

**Corollary 6.3.** Let $\varphi$ be an algebraically constructible function on an irreducible real algebraic set $W$. Then there exist a proper real algebraic subset $\Sigma \subset W$, an integer $\mu$, and a polynomial $g$ on $W$, such that $g$ does not vanish on $W \setminus \Sigma$ and

$$\varphi(w) \equiv \mu + \text{sgn} g(w) \pmod 4$$

for $w \in W - \Sigma$. In particular, for such $w$, $\varphi(w) \equiv \mu + 1 \pmod 2$. 
Proof. Let $g_1, \ldots, g_s$ be polynomials given by [3,1]. We may suppose that all of them are not identically equal to zero. Since $\text{sgn} a + \text{sgn} b \equiv \text{sgn} (ab) + 1 \pmod{4}$, for $a$ and $b$ non-zero, the polynomial $g = g_1 \cdots g_s$ satisfies the statement. This ends the proof. □

Let $\varphi$ be a constructible function on $W$. Following [24] we define the link of $\varphi$ as the constructible function on $W$ given by

$$\Lambda \varphi(w) = \int_{S(w,\varepsilon)} \varphi,$$

where $\varepsilon > 0$ is sufficiently small, and $S(w,\varepsilon)$ denotes the $\varepsilon$-sphere centered at $w$. It is easy to see that $\Lambda \varphi$ is well defined and independent of the embedding of $W$ in $\mathbb{R}^n$. Then the duality operator $D$ on constructible functions, introduced by Kashiwara and Schapira in [21, 26], satisfies

$$D \varphi = \varphi - \Lambda \varphi.$$ 

As shown in [24] the following general statement generalizes various previously known restrictions on local topological properties of real algebraic sets. In particular it implies Akbulut and King's numerical conditions of [1] and the conditions modulo 4, 8, and 16 of Coste and Kurdyka [13, 14] generalized in [23].

**Theorem 6.4.** Let $\varphi$ be an algebraically constructible function on a real algebraic set $W$. Then $\frac{1}{2} \Lambda \varphi$ is integer-valued and algebraically constructible.

The above theorem was proven in [24] using the resolution of singularities. As we show below it is a simple consequence of Theorem 6.1.

**Proof.** We begin the proof by some preparatory observations.

**Lemma 6.5.** $W$ be a real algebraic set and let $\gamma$ be an algebraically constructible function on $W \times \mathbb{R}$. Then

$$\psi_+(w) = \lim_{t \to 0^+} \gamma(w,t), \quad \psi_-(w) = \lim_{t \to 0^+} \gamma(w,-t), \quad \psi(w) = \frac{1}{2}(\psi_+(w) - \psi_-(w))$$

are integer-valued and algebraically constructible on $W_0 = W \times \{0\}$.

**Proof.** We show the lemma for $\psi$. The proofs for $\psi_+$ and $\psi_-$ are similar.

We proceed by induction on $\dim W$. Without loss of generality we may assume that that $W$ is affine and irreducible. We shall show that the statement of lemma holds generically on $W_0$, that is to say there exists a proper algebraic subset $W'_0$ of $W_0$ and an algebraically constructible function $\psi'$ on $W_0$ which equals $\psi$ in the complement of $W'_0$. Then the lemma follows from the inductive assumption since $\dim W'_0 < \dim W_0$.

By Theorem 6.4 we may assume that $\gamma = \text{sgn} g$, where $g(w,t)$ is a polynomial function on $W \times \mathbb{R}$. We may also assume that $g$ does not vanish identically, and then there exists a nonnegative integer $k$ such that

$$g(w,t) = t^k h(w,t),$$

where $h(w,t)$ is a polynomial function on $W \times \mathbb{R}$ not vanishing identically on $W \times \{0\}$. Then, in the complement of $W'_0 = \{w|h(w,0) = 0\}$, either $\psi(w) = \text{sgn} h(w,0)$ for $k$ odd or $\psi'(w) = 0$ for $k$ even satisfies the statement. This ends the proof of lemma. □

Let $\tilde{W} = \{(w,y,t) \in W \times W \times \mathbb{R} \mid \|w-y\|^2 = t\}$ and let $\pi : \tilde{W} \to W \times \mathbb{R}$ be given by $\pi(w,y,t) = (w,t)$. Let $\tilde{\varphi}(w,y,t) = \varphi(y)$. Then $\tilde{\varphi}$ is algebraically constructible and hence $\gamma = \pi_* \tilde{\varphi}$ is an algebraically constructible function on $W \times \mathbb{R}$ and

$$\lim_{t \to 0^+} \gamma(w,t) = \Lambda \varphi(w).$$
Since $\gamma(w,t) = 0$ for $t < 0$
\[ \frac{1}{2} \Delta \varphi(w) = \frac{1}{2} \lim_{t \to 0, e} (\gamma(w,t) - \gamma(w,-t)) \]
is algebraically constructible by Lemma 6.5. This ends the proof of Theorem 6.4. \qed

Suppose that $f : W \to \mathbb{R}$ is regular and let $w \in W_0 = f^{-1}(0)$. Then we define the positive, resp. negative, Milnor fibre of $f$ at $w$ by
\[ F_f^+(w) = B(w, \varepsilon) \cap f^{-1}(\delta) \]
\[ F_f^-(w) = B(w, \varepsilon) \cap f^{-1}(-\delta), \]
where $B(w, \varepsilon)$ is the ball of radius $\varepsilon$ centered at $w$ and $0 < \delta \ll \varepsilon \ll 1$. Let $\varphi$ be an algebraically constructible function on $W$. Following [24] we define the positive (resp. negative) specialization of $\varphi$ with respect to $f$ by
\[ (\Psi_f^+ \varphi)(w) = \int_{F_f^+(w)} \varphi, \quad (\Psi_f^- \varphi)(w) = \int_{F_f^-(w)} \varphi. \]
It is easy to see that both specializations are well-defined and that they are constructible functions supported in $W_0$. Moreover, as shown in [24], they are also algebraically constructible. We present below an alternative proof of this fact.

**Theorem 6.6.** Let $f : W \to \mathbb{R}$ be a regular function on a real algebraic set $W$. Let $\varphi$ be an algebraically constructible function on $W$. Then $\Psi_f^+ \varphi$, $\Psi_f^- \varphi$, and $\frac{1}{2}(\Psi_f^+ \varphi - \Psi_f^- \varphi)$ are integer valued and algebraically constructible.

**Proof.** The proof is similar to that of Theorem 6.4. Since the Milnor fibres are defined not only by equations but also by inequalities we use the following auxiliary construction.

Let $\widehat{W} = \{(w,y,t,r,s) \in W \times W \times \mathbb{R}^3 | |w - y|^2 + t^2 = r, f(y) = s\}$ and let $\pi : \widehat{W} \to W \times \mathbb{R}^2$ be given by $\pi(w,y,t,r,s) = (w,r,s)$. Note that for $w \in W_0$, $0 < s \ll r \ll 1$, $\widetilde{F} = \pi^{-1}(w,r,s)$ is a double cover of the Milnor fibre $F = F_f^+(w)$ branched along its boundary $\partial F = S(w, \sqrt{r}) \cap f^{-1}(s)$. Hence $\chi(\widetilde{F}) = 2\chi(F) - \chi(\partial F)$. Let $\tilde{\varphi}(w,y,t,r,s) = \varphi(y)$. Then
\[ \Psi_f^+ \varphi(w) = \frac{1}{2}(\int_{\widetilde{F}} \tilde{\varphi} + \int_{\partial F} \varphi) = \frac{1}{2} \pi_*(\tilde{\varphi} + \varphi|_{t=0})(w,r,s), \]
for $0 < s \ll r \ll 1$. Clearly an analogous formula holds for $\Psi_f^- \varphi(x)$.

Let $\gamma = \pi_*(\tilde{\varphi} + \varphi|_{t=0})$. Then $\gamma(w,r,s)$ is algebraically constructible and $\gamma(w,r,s) = 0$ for $r < 0$. Hence, by Lemma 6.5, the following functions are algebraically constructible
\[ \Psi_f^+ \varphi = \frac{1}{2} \lim_{r \to 0, s \to 0} \gamma(w,r,\pm s), \]
\[ \frac{1}{2}(\Psi_f^+ - \Psi_f^-) \varphi = \frac{1}{4} \lim_{r \to 0, s \to 0} \lim (\gamma(w,r,s) - \gamma(w,r,-s)), \]
as required. \qed
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Département de Mathématiques, Université d’Angers, 2 bd. Lavoisier, 49045 Angers Cedex, France, and School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia

E-mail address: parus@tonton.univ-angers.fr, parusinski_a@maths.su.oz.au

Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, 80-952 Gdańsk, Poland

E-mail address: szafra@ksinet.univ.gda.pl