MODELING EXTREME VALUES BY THE RESIDUAL COEFFICIENT OF VARIATION

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ABSTRACT. The possibilities of the use of the coefficient of variation over a high threshold in tail modelling are discussed. The paper also considers multiple threshold tests for a generalized Pareto distribution, together with a threshold selection algorithm. One of the main contributions is to extend the methodology based on moments to all distributions, even without finite moments. These techniques are applied to Danish fire insurance losses.

1. INTRODUCTION

Fisher & Tippett [7] and Gnedenko [9] show that, under regularity conditions, the limit distribution for the normalized maximum of a sequence of independent and identically distributed (iid) random variable (r.v.) is a member of the generalized extreme value (GEV) distribution with a cumulative distribution function

\[ H_\xi(x) = \exp\{- (1 + \xi x)^{-1/\xi}\}, \quad (1 + \xi x) > 0, \]

where \( \xi \) is called the extreme value index. This family of continuous distributions contains the Fréchet distribution (\( \xi > 0 \)), the Weibull distribution (\( \xi < 0 \)), and the Gumbell distribution (\( \xi = 0 \), as a limit case), see [12].

The Pickands–Balkema–DeHaan Theorem, see [6] and [12], initiated a new way of studying extreme value theory via distributions above a threshold, which use more information than the maximum data grouped into blocks. This Theorem is a very widely applicable result that essentially says that the generalized Pareto distribution (GPD) is the canonical distribution for modelling excess losses over high thresholds. The cumulative distribution function of GPD(\( \xi, \psi \)) is

\[ F(x) = \frac{1 - (1 + \xi x/\psi)^{-1/\xi}}{1 - F(t)}, \]

where \( \psi > 0 \) and \( \xi \) are scale and shape parameters. For \( \xi > 0 \) the range of \( x \) is \( x > 0 \), in this case the GPD is simply the usual Pareto distribution. The limit case \( \xi = 0 \) corresponds to the exponential distribution. For \( \xi < 0 \) the range of \( x \) is \( 0 < x < \psi/|\xi| \) and GPD has bounded support. The shape parameter \( \xi \) in GPD corresponds to the extreme value index in GEV. The GPD has mean \( \psi/(1 - \xi) \) and variance \( \psi^2/[ (1 - \xi)^2 (1 - 2\xi)] \) provided \( \xi < 1/2 \).

Let \( X \) be a continuous non-negative r.v. with distribution function \( F(x) \). For any threshold, \( t > 0 \), the r.v. of the conditional distribution of threshold excesses \( X - t \) given \( X > t \), denoted \( X_t = (X - t \mid X > t) \), is called the residual distribution of \( X \) over \( t \). The cumulative distribution function of \( X_t \), \( F_t(x) \), is given by

\[ 1 - F_t(x) = (1 - F(x + t))/(1 - F(t)). \]

The quantity \( M(t) = E(X_t) \) is called the residual mean and \( V(t) = \text{var}(X_t) \) the residual variance. The residual coefficient of variation (CV) is given by

\[ \text{CV}(t) \equiv \text{CV}(X_t) = \sqrt{V(t)/M(t)}, \]

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like the usual CV, the function $\text{CV}(t)$ is independent of scale, that is, if $X$ is multiplied by a positive constant, $\text{CV}(t)$ is invariant.

The residual distribution of a GPD is again GPD and for any threshold $t > 0$, the shape parameter $\xi$ is invariant, in fact

$$GPD_t(\xi, \psi) = GPD(\xi, \psi + \xi t).$$

Note that the residual CV is independent of the threshold and the scale parameter, since it is given by

$$\text{CV}(t) = c_\xi = \sqrt{1/(1-2\xi)}.$$

Gupta and Kirmani [10] show that the residual CV characterizes the distribution in the univariate as well as the bivariate case, provided there is a finite second moment. In the case of GPD, the residual CV is constant and is a one to one transformation of the extreme value index suggesting its use to estimate this index.

Castillo et al. [2] suggest a new tool to identify the tail of a distribution based on the residual CV, henceforth called CV-plot, as an alternative to the mean excess plot (ME-plot) that is a commonly used diagnostic tool in risk analysis to justify fitting a GPD, see [8], [6] and [5]. Given a sample $\{x_k\}$ of size $n$ of positive numbers, we denote the ordered sample $\{x(k)\}$, so that $x(1) \leq x(2) \leq \cdots \leq x(n)$. The CV-plot is the CV of the residual samples, that is, the function, $cv(t)$ of the CV of the threshold excesses $(x_j - t)$ for the exceedances $\{x_j : x_j > t\}$, over the order statistics, $t = x(k)$, given by

$$k \to cv(x(k)) = \frac{\text{sd}\{x_j - x(k) \mid x_j > x(k)\}}{\text{mean}\{x_j - x(k) \mid x_j > x(k)\}},$$

where, $k$ ($1 \leq k \leq n$) is the size of the sub-sample removed. This tool has been applied to financial and environmental datasets, see [3].

The CV-plot has some advantages over ME-plot: first, it does not depend on the scale parameter; second, detecting constant functions is easier than linear functions, since linear functions are defined by two parameters and the constants by only one. The uncertainty is essentially reduced from three to one single parameter.

A unconscientious use of some measures of variation can lead to wrong conclusion, see [1]. A serious problem with the residual coefficient of variation is the fact that the proposed method only works when the extreme value index is smaller than 0.25. To fix this, some transformations that relate light-heavy tails are introduced in Section 2.

Section 3 extends some results of Castillo et al. [2] from the exponential distribution to all GPD when the extreme value index is below 0.25. Moreover, multiple threshold tests together with a threshold selection algorithm, designed in a way that avoids subjectivity, are also achieved. In Section 4 the approach developed in the previous sections is illustrated using the Danish fire insurance dataset, a highly heavy-tailed, infinite-variance model.

2. Transformations of heavy-light tails

The transformations introduced to this section make it possible to estimate the extreme value index using methods based on moments in situations where moments are not finite.

A distribution function $F$ is said to be in the maximum domain of attraction of $H_\xi$, written $F \in D(H_\xi)$, if under appropriate normalization the block maxima of a iid sequence of r.v. with distribution $F$ converge to $H_\xi$. For a r.v. $X$ with distribution function $F$ is also written $X \in D(H_\xi)$. A positive function $L$ on $(0, \infty)$
slowly varies at $\infty$ if

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1, \quad t > 0.$$  

Regularly varying functions can be represented by power functions multiplied by slowly varying functions, i.e. $h(x) \in \text{RV}_p$ if and only if $h(x) = x^p L(x)$.

Gnedenko proved, see [12, Theorems 7.8 and 7.10], that the maximum domain of attraction of a Fréchet distribution, with shape parameter $\xi > 0$, is characterized in terms of the tail function, $F(x) = 1 - F(x)$, by

$$F \in D(H_{\xi}) \iff F \in \text{RV}_{-1/\xi} \quad (\xi > 0).$$

Similarly the maximum domain of attraction of a Weibull distribution, with shape parameter $\xi < 0$, is characterized by

$$F \in D(H_{\xi}) \iff F(\xi x_+) - 1/\xi \in \text{RV}_{1/\xi} \quad (\xi < 0),$$

where $x_+ = \sup \{ x : F(x) < 1 \}$.

The following result of practical importance is embedded in the previous characterizations, and which to our knowledge has not been used.

**Corollary 1.** Let $X$ be a r.v. with cumulative distribution function $F$.

1. If $X \in D(H_{\xi})$ with $\xi > 0$, then $X^* = -1/X \in D(H_{-\xi})$.
2. If $X \in D(H_{\xi})$ with $\xi < 0$, then $X^* = x_+ - 1/X \in D(H_{-\xi})$, where $x_+ = \sup \{ x : F(x) < 1 \}$.

**Proof.** (1) The cumulative distribution function of $X^*$ is $F^*(x) = F(-1/x)$ and $x_+ = \sup \{ x : F^*(x) < 1 \} = 0$. By assumption $F(x) = x^{-1/\xi} L(x)$, with $L$ slowly varying at $\infty$, hence $F^*(x_+) = F(x_+) = x^{1/(1-\xi)} L(x)$ and $X^* \in D(H_{-\xi})$.

2. The tail function of $X^*$ is now $\overline{F^*(x)} = \overline{F(x_+) - 1/x} = x^{1/\xi} L(x)$. Hence, $\overline{F(x)} \in \text{RV}_{1/\xi}$ and $X^* \in D(H_{-\xi})$. 

**Corollary 1** provides an asymptotic method and is related to an exact result in the GEV model: $X$ has Fréchet distribution if and only if $-1/X$ has Weibull distribution with the same extreme value index, but with the sign changed. However, the corresponding result is not true in GPD, as we discuss below.

For a r.v. $X$, the Pickands–Balkema–DeHaan Theorem shows that $X \in D(H_{\xi})$ if and only if the limiting behavior of the residual distribution of $X$ over $t$, $X_t$, is like a GPD with the same parameter $\xi$, see [12, Theorem 7.20]. Hence, Corollary 1 can be used in applied methods of threshold exceedances.

**Corollary 2.** Let $X$ be a r.v. such that the limiting behavior of the residual distribution of $X$ over a threshold is GPD with parameter $\xi$, then the limiting behaviour of the residual distribution of $X^* = -1/X$ over a threshold is GPD with parameter $-\xi$.

**Corollary 2** enables use of methods to determine the extreme value index for light tails in heavy tailed distributions and vice versa. For instance ME-plot and CV-plot can be used to determine the extreme value index in really heavy tailed distributions, see Example 3 below. These asymptotic results can be improved on GPD for practical applications.

The GPD($\xi, \psi$) distributions are standardized so that all their observations take positive values. The supports of the distributions are $(0, \sigma)$, where $\sigma = \infty$ for $\xi \geq 0$ and $\sigma = \psi/|\xi|$ for $\xi < 0$. The GPD distributions can be expanded to include a location parameter by $Y = X + \mu$. The behavior of $X$ near $\sigma$ is the same as that of $Y$ near $\sigma + \mu$. The transformation $X^* = -1/X$ is also associated with the origin at zero, but can be generalized to $Y = -1/(X + c)$, provided $c \geq 0$, or $c \leq -\sigma$, 

\[ \text{if and only if the limiting behavior of the residual distribution of } X \text{ has Fréchet distribution if and only if } h(x) \in \text{RV}_p \text{ and only if } h(x) = x^p L(x). \]
in order for the transformations to remain monotonic increasing on \((0, \sigma)\). The following result examines these transformations on GPD.

**Theorem 3.** Let \(X\) be a r.v. with GPD\((\xi, \psi)\) distribution in \((0, \sigma)\) and \(c \geq 0\) or \(c \leq -\sigma\), then \(Y = -1/(X + c)\) has distribution GPD with location parameter if and only if \(c = \psi/\xi\). Then \(Z = Y + 1/c\) has GPD\((-\xi, \xi^2/\psi)\) distribution.

**Proof.** From (1) the distribution function of \(X\) is
\[
F(x) = \begin{cases} 
1 - \left(1 - \frac{\xi^+}{\psi} \left(\frac{cy + 1}{y}\right)\right)^{-1/\xi} & \text{if } 1/c < y < -1/(\sigma + c), \\
1 - \left(\frac{\psi y}{y(\psi - \xi c) - \xi}\right)^{1/\xi} & \text{if } -1/(\sigma + c) \leq y < 1/c, \\
0 & \text{otherwise},
\end{cases}
\]
where \(-1/c < y < -1/(\sigma + c)\). The denominator of the right term of (7) is a constant if and only if \(c = \psi/\xi\). In this case the distribution function of \(Z\) is
\[
F_Z(z) = F_Y(y(z)) = 1 - (1 - \psi z/\xi)^{1/\xi} = 1 - (1 - \xi z/(\xi^2/\psi))^{1/\xi},
\]
where \(0 < z < \sigma\), \(\sigma_z = \xi/\psi\) for \(\xi > 0\) and \(\sigma_z = \infty\) for \(\xi < 0\). Hence, \(Z\) has GPD\((-\xi, \xi^2/\psi)\) distribution and \(Y\) has GPD distribution with location parameter. \(\square\)

**Corollary 4.** Let \(\xi > 0, \psi > 0\) and \(c = \psi/\xi\), then a r.v. \(X\) has GPD\((\xi, \psi)\) distribution if and only if \(Z = X/(c(X + c))\) has GPD\((\xi_z, \psi_z)\) distribution with \(\xi_z = -\xi, \psi_z = \xi^2/\psi\) and the support \((0, \xi/\psi)\).

**Proof.** In the direct sense, this is proved by the Theorem 3 because \(c > 0\) and \(Z = X/(c(X + c)) = -1/(X + c) + 1/c\).

The converse is also a consequence of Theorem 3 because the inverse of the above transformation is
\[
X = c^2 Z/(1 - c) = Z/(c_2(Z + c_2)) = -1/(Z + c_2) + 1/c_2
\]
where \(c_2 = -1/c = -\xi/\psi\). The support of \(Z\) is \((0, \psi_z/\xi_z) = (0, \xi/\psi)\) and \(Z + c_2 < 0\) (equivalently \(c_2 \leq -\xi/\psi\)), then \(X\) is a monotonous increasing function of \(Z\) and Theorem 3 proves the result. \(\square\)

In practical applications of the previous results, a first estimate of the shape and the scale parameters is required in order to define the transformation to a lighter tail, after which the residual empirical CV plot is constructed.

### 3. Multiple threshold test

Some results of Castillo et al. [2] on the residual CV extend directly from the exponential distribution to all GPD, provided there is a finite fourth moment. Therefore, the proof of the following theorem is omitted. The asymptotic distribution of the residual CV as a random process indexed by the threshold provides pointwise error limits for CV-plot in (6) and a multiple thresholds test for GPD that really does reduce the multiple testing problem. The multiple thresholds test provides a clear sense of significance levels and p-values.

**Theorem 5.** Let \(X\) be a GPD\((\xi, \psi)\) distributed r.v., with \(\xi < 1/4\). Then \(\sqrt{n}(c_{n}(t) - c_{2})\), where \(c_{n}(t)\) and \(c_{2}\) were respectively defined in (6) and (5), converges to a Gaussian process with zero mean and covariance function given by
\[
\rho_0(s, t) = \exp((s \wedge t)/\psi),
\]
for \(\xi = 0\), and
\[
\rho_\xi(s, t) = \left((\psi + \xi s)/\psi\right)^{1/\xi}(1 + \xi)^2(6\xi^4 t^2 + 12\psi^3 t + 8\xi^3 st - 9\xi^3 t^2 + 6\psi^2 \xi^2 + 8\psi^2 s - 10\psi^2 t - 2\xi^2 s + 3\xi^2 t^2 - \psi^2 \xi - 2\psi \xi s + 4\psi \xi t + \psi^2) /
\]
\[
(1 - 3\xi)(1 - 2\xi)(1 - 4\xi)(\psi + \xi s)^3)
\]
for \( \xi \neq 0 \) and \( s \leq t \).

Pointwise error limits of the CV-plot under GPD follow from the next result.

**Corollary 6.** Given a sample \( \{X_j\} \) of a GPD(\( \xi, \psi \)) distribution (\( \xi < 0.25 \)) and a threshold \( t \), the asymptotic distribution of the residual CV is

\[
D_{I}(\xi) = \sqrt{n(t)}(cv(t) - c_{\xi}) \xrightarrow{d} N(0, \sigma_{\xi}^2).
\]

where \( c_{\xi} \) is in \( [0,1] \), \( n(t) = \sum_{j=1}^{n} 1_{\{X_j > t\}} \) and

\[
\sigma_{\xi}^2 = \frac{(1 - \xi^2(6\xi^2 - \xi + 1))}{(1 - 2\xi^2)(1 - 3\xi)(1 - 4\xi)}.
\]

Proof. The asymptotic variance is given by \( \sigma_{\xi}^2 = \rho_{\xi}(0,0) \), where the covariance function is in Theorem 5. The Theorem can be applied to the threshold excesses \( \{X_j - t \mid X > t\} \), replacing \( n \) with \( n(t) \) and \( cv(0) \) with \( cv(t) \). From (1) the threshold excesses are again GPD with the same parameter \( \xi \) and the CV does not depend on \( \psi \).

\[\square\]

From the last result the asymptotic confidence intervals of the CV-plot for exponential distribution are obtained with \( c_0 = 1 \) and \( \sigma_0^2 = 1 \) and for uniform distribution with \( c_{-1} = 1/\sqrt{3} \) and \( \sigma_{-1}^2 = 8/45 \).

### 3.1. Simple null hypothesis

Corollary 6 makes it possible to test whether the empirical CV of a sample, or of threshold excesses, fit the CV of a GPD with fixed values \( \xi \) and \( t \). However, from [10], in order to have a consistent test in GPD, \( CV(t) = c_{\xi} \) must be checked for all threshold \( t \). From Theorem 5, a multiple threshold test for a number \( m \) of thresholds as large as necessary for practical applications can also be constructed using the building blocks \( D_{I}(\xi)/\sigma_{\xi}^2 \), regardless of the scale parameter, with asymptotic distribution \( \chi_1^2 \) under the null hypothesis of GPD (\( \xi < 0.25 \)).

The choice of thresholds could be arbitrary, but the multiple thresholds test, \( T(\xi) \), is designed to avoid subjectivity as much as possible, to the limit of the number of thresholds \( m \). If the thresholds are selected as empirical quantiles or order statistics, then \( T(\xi) \) is invariant when the sample is multiplied by a positive number while maintaining the set of probabilities, since CV is invariant. This first condition ensures that the test results do not depend on units used for the observations.

Given a sample \( \{x_j\} \) of size \( n \) of non-negative numbers, \( Q_n(p) \) denotes the inverse of the empirical distribution function,

\[
Q_n(p) = \inf[x : F_n(x) \geq p].
\]

From a set of probabilities \( \{0 = p_0 < p_1 < \cdots < p_m\} \) let \( \{0 = q_0 < q_1 < \cdots < q_m\} \) be the corresponding empirical quantiles of the sample, \( q_k = Q_n(p_k) \), then a multiple thresholds statistic can be constructed as

\[
T(\xi) = \sum_{k=0}^{m} D_{q_k}^2.
\]

The asymptotic expectation is \( (m + 1)\sigma_{\xi}^2 \), hence \( T(\xi)/(m + 1) \) is an estimator of the asymptotic variance \( \sigma_{\xi}^2 \); when \( \xi \) is known or estimated. Note that the distribution of \( T(\xi) \) is independent of the scale parameter \( \psi \). \( T(\xi) \) makes it possible to test the null hypothesis that the sample comes from a distribution with the residual CV corresponding to previous quantiles all equal to \( c_{\xi} \).

\( H_0 : CV(q_k) = c_{\xi}, \quad k = 0, 1, \ldots, m. \)
Hence, if $H_0$ is accepted and $m$ is large enough, say 20 or 50, it will be reasonable to assume that the sample comes from a distribution $\text{GPD}(\xi, \psi)$. The previous test $T(\xi)$ is a global test in the sense that some $D_{\hat{q}_{1/2}}^2$ may be significant and others not but with one test alone the equality of all CV for all quantiles is checked.

A second desirable condition is to select the set of probabilities that determine the statistic $T(\xi)$ so that the corresponding thresholds are approximately equally spaced. This can be achieved for the exponential distribution by taking $0 < p < 1$, $p_k = 1 - p^k$, $(k = 0, \ldots, m)$ and $q_k$ the corresponding quantiles, since for a random variable, $X$, with exponential distribution $\text{Pr}\{X > (\mu \log(1/p)) k\} = p^k$, where $\mu$ is the expected value. Then the condition holds for $\xi = 0$ and is fairly approximate for $\xi$. Selecting the probabilities this way, $q_k = Q_n(p_k) \approx x_{(n - np^k)}$, $n(q_k) \approx np^k$ and $T(\xi)$ becomes

$$T_m(\xi) = n \sum_{k=0}^{m} p^k (cv(q_k) - c_\xi)^2.$$  

In applications, given the number of single tests that will be included in the multivariate test, $m$, we choose the value of $p$, which determines the distance between the quantiles, such that $n p^m \approx ns$, where $ns$ is the sample size such that irrelevant information comes from smaller sub-samples. Hence, given $m$, $p = (ns/n)^{1/m}$ is suggested. In this paper $ns \approx 8$ is used in numerical algorithms. Note that this way $T_m(\xi)$ depends only on $\xi$ and $m$ and the researcher chose only the number of thresholds used in the analysis, essentially eliminating subjectivity. These multiple thresholds tests generalize those developed by Castillo et al. [2] for $\xi = 0$ and $p = 1/2$.

The asymptotic distribution of $T_m(\xi)$ is easily calculated from Theorem 5 following the steps suggested by Castillo et al. [2], whenever $\xi < 0.25$. However, taking into account the different values of the extreme value index and the diverse small sample sizes, it is easier in practice to calculate the $p$-value for $T_m(\xi)$ using simulation methods, which are especially simple in this case. Assuming GPD for simulations, only the sample size, the number of thresholds, $m$, and $\xi$ are needed. Since the distribution does not depend on scale, parameter $\psi = 1$ will be used.

3.2. Composite null hypothesis. In most cases the parameter $\xi$ is unknown and its estimate should be incorporated in the statistic $T_m(\xi)$ (see the R code below). The method for estimating $\xi$ leads to slight variations in the statistic, but it leads to essentially equivalent inference whenever we use the same estimation method in simulations to obtain the $p$-value. The null hypothesis is now that the sample comes from a distribution in which all $(m + 1)$ residual CV are equal.

$$H_0 : \text{CV}(q_k) = \cdots = \text{CV}(q_m), \quad k = 0, 1, \ldots, m.$$  

The alternative hypothesis is that the residual CV are equal from a threshold $q_r$ $(0 < r \leq m)$ to the threshold $q_m$.

The most recommended estimation method is maximum likelihood estimation (MLE), although in GPD it is only asymptotically efficient provided $-0.5 < \xi$, see [5]. For this distribution, the CV is a one-to-one transformation of $\xi$, see [5], and the empirical CV of the residual sample, CV($t$), provides an alternative method of estimation. It is asymptotically normal whenever $\xi < 0.25$, see Corollary 6. The multiple thresholds tests [10] suggest estimating $\xi$ as the value such that $c_\xi$ achieves the minimum $T_m(\xi)$, namely

$$c_\xi = \frac{\sum_{k=0}^{m} p^k \text{cv}(q_k)}{\sum_{k=0}^{m} p^k} = (1 - p) \sum_{k=0}^{m} p^k \text{cv}(q_k)/(1 - p^{m+1}).$$
From Corollary 6 the estimator is also asymptotically normal. The main advantage of this method is that under the alternative hypothesis it is a better estimator than CV or MLE, since the sample is only GPD over a threshold $q_r$. Since the main interest is in samples that are not GPD, but in the tail, and results are often used in small samples with $\xi < 0$, the estimation method (11) is included in the statistic $T_m = T_m(\tilde{\xi})$. The following R code for $T_m$ is used in the algorithms, see [13].

```r
#Statistic $T_m$ of a sample given the number of thresholds $m$. 
Tm<-function(m,sample){
  sam<-sample-min(sample);
  n<-length(sam);ns<-8;
  p<-round(exp(log(ns/n)/m),digits=2);
  Ws<-Ps<-Qs<-Cs<-numeric(m+1);
  for(k in 1:(m+1)){Ws[k]<-p^(k-1)};
  Ps<-1-Ws;Qs<-as.vector(quantile(sam,Ps));
  for(k in 1:(m+1)){Cs[k]<-sd(sam[sam>=Qs[k]]-Qs[k])/mean(sam[sam>=Qs[k]]-Qs[k])};
  cx<-(1-p)*sum(Ws*Cs)/(1-p^(m+1));xi<-(cx^2-1)/(2*cx^2);
  tm<-n*sum(Ws*(Cs-cx)^2);list(CV=cx,Tm=tm,Xi=xi)
}
```

3.3. Threshold Selection Algorithms. To select the number of extremes used in applying the peaks over a high threshold method, threshold selection algorithms are developed in this section to estimate the point above which the GPD distribution can be used to estimate the extreme value index for a set of extreme events, $\{x_j\}$, of size $n$. For this purpose the previous statistical tests will be adapted.

Note that in the $T_m$ calculation the number of thresholds $m$ is the only parameter that must be fixed by the researcher. This determines the thresholds (quantiles) where the CV is calculated, $\{q_0 < q_1 < \cdots < q_m\}$, which are fixed throughout the procedure. Then, by simulation of GPD, the associated $p$-value is calculated (running $10^4$ samples). After that, we accept or reject the null hypothesis with the estimated shape parameter using all the thresholds.

If the hypothesis is rejected, the threshold excesses $\{x_j - q_1\}$ are calculated for the sub-sample $\{x_j \geq q_1\}$. The previous steps are repeated, but removing one threshold, to accept or reject the null hypothesis that the sample is from a GPD. At every stage only statistics associated to thresholds $k = r, \ldots, m$, where $0 \leq r \leq m$, are calculated:

\[
T_m^r(\xi) = n \sum_{k=r}^{m} p_k (cv(q_k) - c_\xi)^2.
\]

In summary, the steps of the general algorithm are

1. Given $m$ find $p$ such that $np^n \approx ns$, where $ns$ is the smaller sample size used to calculate CV (here $ns = 8$ is used, but it can be modified).
2. Calculate $\{0 = p_0 < p_1 < \cdots < p_m\}$, where $p_k = 1 - p^k$, and $\{0 = q_0 < q_1 < \cdots < q_m\}$, where $q_k = Q_n(1 - p^k)$, $k = 1, \ldots, m$.
3. Estimate $\tilde{\xi}$ minimizing the value of $T_m(\xi)$ with the specific values in the previous steps.
4. Calculate by simulation of GPD the $p$-value associated to the minimum $T_m(\tilde{\xi})$ and accept or reject the null hypothesis with the estimated shape parameter using all the thresholds (starting with $q_0 = 0$).
5. If the hypothesis is rejected, compute the threshold excesses $\{x_j - q_1\}$ for the sub-sample $\{x_j \geq q_1\}$ and repeat the previous steps with $\{p_1 < \cdots < p_m\}$ and $\{q_1 < \cdots < q_m\}$, to accept or reject the null hypothesis that the sample is from a GPD, but removing a threshold.
(6) Continue the process for the next value in the index of thresholds while the hypothesis is rejected.

Several authors recommend giving a prominent role to the exponential distribution in the model GPD, see [3]. The usual method for doing this is to consider the exponential models as the null hypothesis testing against GPD, see [11]. Alternatively, one can consider the Akaike or Bayesian information criteria for model selection, see [4]. The previous algorithm can be adapted to the case when $\xi = 0$ (or simply known) skipping step-3.

4. Danish fire insurance data

An interesting aspect of this article is the combination of the results of sections [2] and [3] when applying the peaks over threshold technique for tails in any maximum domain of attraction. This approach is illustrated here using a popular dataset.

The Danish fire insurance data are a well-studied set of losses to illustrate the basic ideas of extreme value theory. The dataset consists of 2,156 fire insurance losses over one million Danish kroner from 1980 to 1990 inclusive, see [6, Example 6.2.9], [13] and [12, Example 7.23].

In this example the authors agree to assume iid observations and a heavy tailed model. They also agree to set the threshold at $u = 10$ million Danish kroner, the exceedances over the threshold, denoted $\{x_j\}$, are $n_{10} = 109$.

Fitting a GPD to $\{x_j\}$ by MLE, the parameter estimates in [12] are $\hat{\xi} = 0.50$ and $\hat{\psi} = 7.0$ with standard errors 0.14 and 1.1, respectively. Thus the fitted model is a very heavy-tailed, infinite-variance model and the method in Section 3 cannot be applied directly. However, they can be used through the results shown in Section 2.

First of all, let us suppose we want to use CV to check whether the above data correspond to a GPD distribution with the estimated extreme value index. Applying Theorem 3 with $c = \hat{\psi}/\hat{\xi} = 14$, let $z_j = -1/(x_j + c) + 1/c$ be, then the set $\{z_j\}$ has light tails and the same extreme value index with the sign changed, provided that the estimated parameters are the true parameters. The CV of $\{z_j\}$ is $cv = 0.697$ which provides a new estimation of $\xi$, solving [5] by $\hat{\xi}_Z = (cv^2 - 1)/(2cv^2) = -0.530$, then, according to Theorem [3], $\hat{\xi} = -\hat{\xi}_Z = 0.53$, not far from 0.50, since the standard error is 0.14. Alternatively, the multiple thresholds statistic $T_m$, from [12], can be used to check $\xi = 0.5$. The corresponding CV under GPD is $c_\xi = 0.707$. Taking $m = 20$, we get $T_m = 4.89$ with a $p$-value 0.421 (by simulation with $10^4$ samples), accepting the null hypothesis.

Now consider the problem of choosing the threshold to estimate the extreme value index. In this example, most researchers use a visual observation of the $ME$-plot on the full Danish dataset. The algorithm in Section 3.3 with the transformations from Section 2 comes to similar solutions automatically and opens up new perspectives.

Figure 1 shows the CV-plots of the full Danish dataset, transformed according to the Corollary 2 plot (a), and Theorem 3 plot (b). The first, corresponding to the transformation $X^* = -1/X$, shows an increasing CV and the second, corresponding to $Z = -1/(X + c) + 1/c$, shows a stabilized CV close to a constant, indicating that the original dataset is close to a GPD, which is also shown by $ME$-plot.

Applying the algorithm of Section 3.3 with $m = 20$ after transformation $X^*$, constant residual CV is rejected in the first 11 steps (each one reduces the sample size by $(1 - p) = 24\%$). Step 12, for the last 106 observations, accepts constant residual CV ($p$-value = 0.269) with estimates $c_\xi = 0.673$ and $\xi = 0.603$. The estimated threshold is approximately the same ($u = 10.2$ instead of 10), while the extreme value index is different but within the confidence interval.

The algorithm in Section 3.3 with $m = 20$ after transformation $Z$ with $c = 0.932/0.611 = 1.524$, rejects constant residual CV in the first three steps. Step 4,
for the last 951 observations, accepts constant residual CV (p-value = 0.167) with estimates $c_\xi = 0.675$ and $\xi = 0.599$. The number of observations is much higher, the extreme value index being very close to that obtained with the transformation $X^*$ and within the confidence interval. The $p$-value remains similar in the following steps up until the 12th, where it jumps up to 0.474. The number of observations is again 106 and the estimation $\xi = 0.548$, nearer to 0.50.

The conclusions from using the new methodology to analyze this dataset are the following. First, the results obtained by previous investigators are validated, in particular GPD can be accepted with parameter $\xi = 0.5$, for the 109 larger observations see [12]. This also shows the consistency of the presented methodology with other common techniques.

Moreover, from examining the extreme value index it is now known that for the 951 larger observations GPD can also be accepted, where the MLE parameter estimate is $\xi = 0.680$, with standard error 0.055 ($\xi = 0.599$ obtained by $T_m$ is within the confidence interval). The estimated extreme value index is now much more accurate because the sample size is much larger. We also note that the tails are heavier than was assumed, which means that higher risks should be considered.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Residual empirical CV for The Danish fire insurance losses under transformation of the data. (a): Dataset, transformed by $-1/X$. (b): Dataset, transformed by $-1/(X + \psi/\xi)$. The dotted lines correspond to the asymptotic confidence intervals (90%) under the estimated parameter, the dashed line is its CV.}
\end{figure}

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