FINITE-DIMENSIONAL MODULES OF THE RACAH ALGEBRA AND
THE ADDITIVE DAHA OF TYPE \((C_1', C_1)\)

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Abstract. Assume that \(\mathbb{F}\) is an algebraically closed field with characteristic zero. The Racah algebra \(\mathbb{R}\) is a unital associative \(\mathbb{F}\)-algebra defined by generators and relations. The generators are \(A, B, C, D\) and the relations state that
\[
[A, B] = [B, C] = [C, A] = 2D
\]
and each of
\[
[A, D] + AC - BA, \quad [B, D] + BA - CB, \quad [C, D] + CB - AC
\]
is central in \(\mathbb{R}\). The universal additive DAHA (double affine Hecke algebra) \(\mathcal{H}\) of type \((C_1', C_1)\) is a unital associative \(\mathbb{F}\)-algebra generated by \(t_0, t_1, t_0', t_1'\) and the relations state that
\[
t_0 + t_1 + t_0' + t_1' = -1
\]
and each of \(t_0^2, t_1^2, t_0'^2, t_1'^2\) is central in \(\mathcal{H}\). Each \(\mathcal{H}\)-module is an \(\mathbb{R}\)-module by pulling back via the injection \(\mathbb{R} \to \mathcal{H}\) given by
\[
A \mapsto \frac{(t_1' + t_0')(t_1' + t_0' + 2)}{4}, \quad B \mapsto \frac{(t_1' + t_0')(t_1 + t_1' + 2)}{4}, \quad C \mapsto \frac{(t_0' + t_1)(t_0' + t_1 + 2)}{4}.
\]
We classify the lattices of \(\mathbb{R}\)-submodules of finite-dimensional irreducible \(\mathcal{H}\)-modules. As a consequence, for any finite-dimensional irreducible \(\mathcal{H}\)-module \(V\), the \(\mathbb{R}\)-module \(V\) is completely reducible if and only if \(t_0\) is diagonalizable on \(V\).

Keywords: additive DAHA, Racah algebra, lattices.

1. Introduction

Throughout this paper, we adopt the following conventions. Assume that \(\mathbb{F}\) is an algebraically closed field with characteristic zero. The bracket \([,\,]\) stands for the commutator and the curly bracket \(\{,\,\}\) stands for the anticommutator.

The Racah algebra \(\mathbb{R}\) is a unital associative \(\mathbb{F}\)-algebra with a presentation given by generators \(A, B, C, D\) and the relations state that
\[
[A, B] = [B, C] = [C, A] = 2D
\]
and each of
\[
[A, D] + AC - BA, \quad [B, D] + BA - CB, \quad [C, D] + CB - AC
\]
is central in \(\mathbb{R}\). The algebra \(\mathbb{R}\) was first appeared in the study of the quantum mechanical coupling of three angular momenta \([28]\) and realized by the intermediate Casimir operators.

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of \(\mathfrak{su}(1,1)^{\otimes 3}\) and \(\mathfrak{su}(2)^{\otimes 3}\) \cite{11,16}. The Racah algebra was also explored in a broad range of areas including orthogonal polynomials, distance regular graphs, superintegrable models and Leonard triples \cite{6,8,12,14,17,20,23,27,29,31,32}.

In \cite{21} W. Groenevelt introduced an additive analogue of DAHA (double affine Hecke algebra) of type \((C_1^\vee, C_1)\) and used it to study generalized Fourier transforms. The additive DAHA of type \((C_1^\vee, C_1)\) also showed up in the context of Bannai–Ito polynomials \cite{15}. Given four parameters \(k_0, k_1, k_0^\vee, k_1^\vee \in \mathbb{F}\) the algebra has a presentation \cite[Proposition 2.12]{21} given by generators \(t_0, t_1, t_0^\vee, t_1^\vee\) and relations

\[
t_0 + t_1 + t_0^\vee + t_1^\vee = -1, \\
t_0^2 = k_0, \\
t_1^2 = k_1, \\
t_0^\vee = k_0^\vee, \\
t_1^\vee = k_1^\vee.
\]

In this paper we consider its central extension, denoted by \(\mathfrak{H}\), obtained from the above presentation by reinterpreting the four parameters \(k_0, k_1, k_0^\vee, k_1^\vee\) as central elements.

According to the results from \cite[Section 2]{14} and \cite[Proposition 2]{15}, there exists a unique \(\mathbb{F}\)-algebra homomorphism \(\zeta: \mathcal{R} \to \mathfrak{H}\) that sends

\[
A \mapsto \frac{(t_1^\vee + t_0^\vee)(t_1^\vee + t_0^\vee + 2)}{4},
\]

\[
B \mapsto \frac{(t_1 + t_1^\vee)(t_1 + t_1^\vee + 2)}{4},
\]

\[
C \mapsto \frac{(t_0^\vee + t_1)(t_0^\vee + t_1 + 2)}{4}.
\]

Thus each \(\mathfrak{H}\)-module is an \(\mathcal{R}\)-module by pulling back via \(\zeta\). Note that \(\zeta\) is shown to be injective \cite{23} and the classifications of finite-dimensional irreducible \(\mathcal{R}\)-modules and \(\mathfrak{H}\)-modules are given in \cite{5} and \cite{22}, respectively. The purpose of this paper is to classify the lattices of \(\mathcal{R}\)-submodules of finite-dimensional irreducible \(\mathfrak{H}\)-modules.

The paper is organized as follows. In \S 2 we give some preliminaries on \(\mathcal{R}\) and \(\mathfrak{H}\), as well as review the homomorphism from \(\mathcal{R}\) into \(\mathfrak{H}\). In \S 3 we lay the groundwork for the finite-dimensional irreducible \(\mathcal{R}\)-modules and \(\mathfrak{H}\)-modules. In \S 4 we classify the lattices of \(\mathcal{R}\)-submodules of finite-dimensional irreducible \(\mathfrak{H}\)-modules. In \S 5 we end the paper with a summary of the classification and its consequences.

2. The Racah algebra and the universal additive DAHA of type \((C_1^\vee, C_1)\)

**Definition 2.1** \cite{2,14,16,28}. The Racah algebra \(\mathcal{R}\) is a unital associative \(\mathbb{F}\)-algebra defined by generators and relations in the following way. The generators are \(A, B, C, D\) and the relations state that

\[
[A, B] = [B, C] = [C, A] = 2D
\]

and each of

\[
[A, D] + AC - BA, \\
[B, D] + BA - CB, \\
[C, D] + CB - AC
\]

commutes with \(A, B, C, D\).

Let

\[
\delta = A + B + C.
\]

**Lemma 2.2.** (i) The Racah algebra \(\mathcal{R}\) is generated by \(A, B, C\).

(ii) The Racah algebra \(\mathcal{R}\) is generated by \(A, B, \delta\).
(iii) The element $\delta$ is central in $\mathcal{R}$.

**Proof.** (i): Immediate from (1).

(ii): Since $C = \delta - A - B$ and by (i) the statement (ii) follows.

(iii): By (1) the element $\delta$ commutes with each of $A, B, C$. Hence (iii) follows by (i). □

**Definition 2.3 ([15, 21]).** The universal additive DAHA (double affine Hecke algebra) $\mathcal{H}$ of type $(C^\vee_1, C_1)$ is a unital associative $\mathbb{F}$-algebra defined by generators and relations. The generators are $t_0, t_1, t_0^\vee, t_1^\vee$ and the relations state that

$$t_0 + t_1 + t_0^\vee + t_1^\vee = -1$$

and each of $t_0^2, t_1^2, t_0^\vee 2, t_1^\vee 2$ commutes with $t_0, t_1, t_0^\vee, t_1^\vee$.

Recall from [1, 3, 4, 9, 10, 13, 30] that the Bannai–Ito algebra $\mathfrak{B}_P$ is a unital associative $\mathbb{F}$-algebra generated by $X, Y, Z$ and the relations assert that each of

$$\{X, Y\} - Z, \quad \{Y, Z\} - X, \quad \{Z, X\} - Y$$

is central in $\mathfrak{B}_P$. By [15, Proposition 2] there exists an $\mathbb{F}$-algebra isomorphism $\mathcal{H} \rightarrow \mathfrak{B}_P$ that sends

$$
t_0 \mapsto \frac{X + Y + Z}{2} - \frac{1}{4},
$$

$$
t_1 \mapsto \frac{X - Y - Z}{2} - \frac{1}{4},
$$

$$
t_0^\vee \mapsto \frac{Y - Z - X}{2} - \frac{1}{4},
$$

$$
t_1^\vee \mapsto \frac{Z - X - Y}{2} - \frac{1}{4}.
$$

**Theorem 2.4 ([14, 23]).** There exists a unique $\mathbb{F}$-algebra homomorphism $\zeta : \mathcal{R} \rightarrow \mathcal{H}$ that sends

$$A \mapsto \frac{(t_1^\vee + t_0^\vee)(t_1^\vee + t_0^\vee + 2)}{4},$$

$$B \mapsto \frac{(t_1 + t_1^\vee)(t_1 + t_1^\vee + 2)}{4},$$

$$C \mapsto \frac{(t_0^\vee + t_1)(t_0^\vee + t_1 + 2)}{4},$$

$$\delta \mapsto \frac{t_0^2 + t_1^2 + t_0^\vee 2 + t_1^\vee 2}{4} - \frac{t_0}{2} - \frac{3}{4}.
$$

By Theorem 2.4 each $\mathcal{H}$-module is an $\mathcal{R}$-module by pulling back via $\zeta$.

### 3. Finite-dimensional irreducible $\mathcal{R}$-modules and $\mathcal{H}$-modules

In §3.1 we recall some results on the finite-dimensional irreducible $\mathcal{R}$-modules from [5]. In §3.2 and §3.3 we rephrase some results on the finite-dimensional irreducible $\mathfrak{B}_P$-modules from [22] in terms of the $\mathcal{H}$-modules.
3.1. Finite-dimensional irreducible $\mathbb{R}$-modules.

**Proposition 3.1** ([5]). For any scalars $a, b, c \in \mathbb{F}$ and any integer $d \geq 0$, there exists a $(d + 1)$-dimensional $\mathbb{R}$-module $R_d(a, b, c)$ satisfying the following conditions (i), (ii):

(i) There exists an $\mathbb{F}$-basis for $R_d(a, b, c)$ with respect to which the matrices representing $A$ and $B$ are

\[
\begin{pmatrix}
\theta_0 & \theta_1 & 0 \\
1 & \theta_2 & \\
& \ddots & \ddots \\
0 & \cdots & 1 & \theta_d
\end{pmatrix}, \quad \begin{pmatrix}
\varphi_0^* & \varphi_1^* & 0 \\
\theta_1^* & \varphi_2^* & \\
& \ddots & \ddots \\
0 & \cdots & \theta_d^*
\end{pmatrix},
\]

respectively, where

\[
\theta_i = (a + \frac{d}{2} - i)(a + \frac{d}{2} - i + 1) \quad (0 \leq i \leq d),
\]

\[
\theta_i^* = (b + \frac{d}{2} - i)(b + \frac{d}{2} - i + 1) \quad (0 \leq i \leq d),
\]

\[
\varphi_i = i(i - d - 1)(a + b + c + \frac{d}{2} - i + 2)(a + b - c + \frac{d}{2} - i + 1) \quad (1 \leq i \leq d).
\]

(ii) The element $\delta$ acts on $R_d(a, b, c)$ as scalar multiplication by

\[
d(\frac{d}{2} + 1) + a(a + 1) + b(b + 1) + c(c + 1).
\]

**Proposition 3.2** ([5]). For any scalars $a, b, c \in \mathbb{F}$ and any integer $d \geq 0$, the $\mathbb{R}$-module $R_d(a, b, c)$ is irreducible if and only if

\[
a + b + c + 1, -a + b + c, a - b + c, a + b - c \not\in \left\{ \frac{d}{2} - i \bigg| i = 1, 2, \ldots, d \right\}.
\]

**Theorem 3.3** ([3]). Let $d \geq 0$ denote an integer. If $V$ is a $(d + 1)$-dimensional irreducible $\mathbb{R}$-module then there exist $a, b, c \in \mathbb{F}$ such that $R_d(a, b, c)$ is isomorphic to $V$.

3.2. Even-dimensional irreducible $\mathbb{F}$-modules.

**Proposition 3.4** ([22]). For any scalars $a, b, c \in \mathbb{F}$ and any odd integer $d \geq 1$, there exists a $(d + 1)$-dimensional $\mathbb{F}$-module $E_d(a, b, c)$ that has an $\mathbb{F}$-basis $\{v_i\}_{i=0}^d$ such that

\[
t_0v_i = \begin{cases} 
  i(d - i + 1)v_{i-1} - \frac{d - 2i + 1}{2}v_i & \text{for } i = 2, 4, \ldots, d - 1, \\
  \frac{d - 2i - 1}{2}v_i + v_{i+1} & \text{for } i = 1, 3, \ldots, d - 2,
\end{cases}
\]

\[
t_0v_0 = -\frac{d + 1}{2}v_0, \quad t_0v_d = -\frac{d + 1}{2}v_d,
\]

\[
t_1v_i = \begin{cases} 
  i(i - d - 1)v_{i-1} + av_i + v_{i+1} & \text{for } i = 2, 4, \ldots, d - 1, \\
  -av_i & \text{for } i = 1, 3, \ldots, d,
\end{cases}
\]

\[
t_1v_0 = av_0 + v_1,
\]

\[
t_1v_d = av_d + v_{d+1},
\]

\[
t_0^tv_i = \begin{cases} 
  bv_i & \text{for } i = 0, 2, \ldots, d - 1, \\
  -\sigma v_i & \text{for } i = 1, 3, \ldots, d - 2,
\end{cases}
\]

\[
t_0^tv_d = -\sigma v_d,
\]

\[
t_0^{t'}v_i = \begin{cases} 
  bv_i & \text{for } i = 0, 2, \ldots, d - 1, \\
  -\sigma v_i & \text{for } i = 1, 3, \ldots, d - 2.
\end{cases}
\]
Odd-dimensional irreducible $H$-modules

3.3. For any scalars $a, b, c \in \mathbb{F}$ and any odd integer $d \geq 1$, the elements $t_0^2, t_1^2, t_0^\vee, t_1^\vee$ act on $E_d(a, b, c)$ as scalar multiplication by $(d+1)^2, a^2, b^2, c^2$ respectively.

Proof. Apply Proposition 3.3 to evaluate the actions of $t_0^2, t_1^2, t_0^\vee, t_1^\vee$ on $E_d(a, b, c)$. \hfill \Box

**Proposition 3.6** (22). For any scalars $a, b, c \in \mathbb{F}$ and any odd integer $d \geq 1$, the $\mathcal{H}$-module $E_d(a, b, c)$ is irreducible if and only if

$$a + b + c, -a + b + c, a - b + c, a + b - c \notin \left\{ \frac{d-1}{2} - i \right\} \text{ for } i = 0, 2, \ldots, d-1.$$

Observe that there exists a unique $\{\pm 1\}^2$-action on $\mathcal{H}$ such that each $(\varepsilon, \varepsilon') \in \{\pm 1\}^2$ acts on $\mathcal{H}$ as an $\mathbb{F}$-algebra automorphism in the following way:

| $u$   | $t_0$ | $t_1$ | $t_0^\vee$ | $t_1^\vee$ |
|-------|-------|-------|-------------|-------------|
| $u^{(1,1)}$ | $t_0$ | $t_1$ | $t_0^\vee$ | $t_1^\vee$ |
| $u^{(1,-1)}$ | $t_1$ | $t_0$ | $t_1^\vee$ | $t_0^\vee$ |
| $u^{(-1,1)}$ | $t_1^\vee$ | $t_0^\vee$ | $t_0$ | $t_1$ |
| $u^{(-1,-1)}$ | $t_1$ | $t_0$ | $t_1$ | $t_0$ |

**Table 1.** The $\{\pm 1\}^2$-action on $\mathcal{H}$

For any $(\varepsilon, \varepsilon') \in \{\pm 1\}^2$, we define

$$E_d(a, b, c)^{(\varepsilon, \varepsilon')}$$

to be the $\mathcal{H}$-module obtained by pulling back $E_d(a, b, c)$ via $(\varepsilon, \varepsilon')$. Note that the $\mathcal{H}$-modules $E_d(a, b, c)$ and $E_d(a, b, c)^{(1,1)}$ are identical.

**Theorem 3.7** (22). Let $d \geq 1$ denote an odd integer. If $V$ is a $(d+1)$-dimensional irreducible $\mathcal{H}$-module then there exist $a, b, c \in \mathbb{F}$ and $(\varepsilon, \varepsilon') \in \{\pm 1\}^2$ such that $E_d(a, b, c)^{(\varepsilon, \varepsilon')}$ is isomorphic to $V$.

3.3. Odd-dimensional irreducible $\mathcal{H}$-modules.

**Proposition 3.8** (22). For any scalars $a, b, c \in \mathbb{F}$ and any even integer $d \geq 0$, there exists a $(d+1)$-dimensional $\mathcal{H}$-module $O_d(a, b, c)$ that has an $\mathbb{F}$-basis $\{v_i\}_{i=0}^d$ such that

$$t_0v_i = \begin{cases} 
-i(\sigma + i)v_{i-1} + \frac{\sigma + 2i}{2}v_i & \text{for } i = 2, 4, \ldots, d, \\
\frac{\sigma + 2i}{2}v_i + v_{i+1} & \text{for } i = 1, 3, \ldots, d-1, \\
\frac{\sigma}{2}v_0 & \text{for } i = 0.
\end{cases}$$

\[ (9) \]

where

$$\sigma = a + b + c - \frac{d+1}{2}, \quad \tau = a + b - c - \frac{d+1}{2}.$$
Theorem 3.11

Lemma 3.9. For any scalars \(a, b, c \in \mathbb{F}\) and any even integer \(d \geq 0\), the elements \(t_0^2, t_1^2, t_0^\vee_2, t_1^\vee_2\) act on \(O_d(a, b, c)\) as scalar multiplication by

\[
\begin{align*}
\left(\frac{a + b + c - d + 1}{2}\right)^2, & \quad \left(\frac{a - b - c - d + 1}{2}\right)^2, \\
\left(\frac{c - a - b - d + 1}{2}\right)^2, & \quad \left(\frac{b - a - c - d + 1}{2}\right)^2,
\end{align*}
\]

respectively.

Proof. Apply Proposition 3.8 to evaluate the actions of \(t_0^2, t_1^2, t_0^\vee_2, t_1^\vee_2\) on \(O_d(a, b, c)\).

Proposition 3.10 \((\text{[22]})\). For any scalars \(a, b, c \in \mathbb{F}\) and any even integer \(d \geq 0\), the \(\mathfrak{H}\)-module \(O_d(a, b, c)\) is irreducible if and only if

\[
a + b + c, a - b - c, -a + b - c, -a - b + c \notin \{d + 1 - \frac{i}{2} \mid i = 2, 4, \ldots, d\}.
\]

Theorem 3.11 \((\text{[22]})\). Let \(d \geq 0\) denote an even integer. If \(V\) is a \((d + 1)\)-dimensional irreducible \(\mathfrak{H}\)-module then there exist \(a, b, c \in \mathbb{F}\) such that \(O_d(a, b, c)\) is isomorphic to \(V\).

4. THE CLASSIFICATION OF LATTICES OF \(\mathbb{R}\)-SUBMODULES OF FINITE-DIMENSIONAL IRREDUCIBLE \(\mathfrak{H}\)-MODULES

In \((\text{[41.1]}\) we investigate the role of \(t_0\) in the \(\mathbb{R}\)-submodules of an \(\mathfrak{H}\)-module. According to Theorems 3.7 and 3.11 it is enough to contemplate the lattices of \(\mathbb{R}\)-submodules of the irreducible \(\mathfrak{H}\)-modules \(E_d(a, b, c)^{(c,e)}\) and \(O_d(a, b, c)\). In \((\text{[41.2]}\) \text{[41.6]} we individually classify those lattices.
4.1. **The eigenspaces of** $t_0$ **as** $\mathbb{R}$-modules.

**Lemma 4.1.** The following equations hold in $\mathfrak{H}$:

\[
\{t_0 + t_1, [t_1, t_0]\} = 0, \\
\{t_0 + t_0^\vee, [t_0^\vee, t_0]\} = 0, \\
\{t_0 + t_1^\vee, [t_1^\vee, t_0]\} = 0.
\]

**Proof.** A direct calculation yields that

\[
\{t_0 + t_1, [t_1, t_0]\} = t_1^2 t_0 + t_1 t_0^2 - t_0 t_1 - t_0 t_1^2.
\]

Since $t_0^2$ and $t_1^2$ are central in $\mathfrak{H}$ by Definition 2.3, the right-hand side of (10) is zero. By similar arguments the other two equations follow.

By [23, Theorem 6.4] the $\mathbb{F}$-algebra homomorphism $\zeta$ given in Theorem 2.4 is injective. Thus the Racah algebra $\mathbb{R}$ can be considered as a subalgebra of $\mathfrak{H}$.

**Lemma 4.2.** The element $t_0$ is in the centralizer of $\mathbb{R}$ in $\mathfrak{H}$.

**Proof.** By Lemma 2.2(i) it suffices to show that $t_0$ commutes with each of $A, B, C$. Any elements $x, y, z$ in a ring satisfy

\[
[x y, z] = x [y, z] + [x, z] y.
\]

Applying (11) with $(x, y, z) = (t_0^\vee + t_1^\vee, t_0^\vee + t_1^\vee + 2, t_0)$, the right-hand side of the resulting equation is

\[
(t_0^\vee + t_1^\vee)(t_0^\vee + t_1^\vee + 2, t_0) + [t_0^\vee + t_1^\vee, t_0](t_0^\vee + t_1^\vee + 2)
\]

and the left-hand side is $4[A, t_0]$ by Theorem 2.4. Using (2) yields that (12) is equal to $\{t_0 + t_1, [t_1, t_0]\}$. Combined with Lemma 4.1, we have $[A, t_0] = 0$. By similar arguments, each of $[B, t_0]$ and $[C, t_0]$ is zero. The lemma follows.

Given any $\mathfrak{H}$-module $V$ and any $\theta \in \mathbb{F}$ we let

\[
V(\theta) = \{v \in V \mid t_0 v = \theta v\}.
\]

**Proposition 4.3.** If $V$ is an $\mathfrak{H}$-module then $V(\theta)$ is an $\mathbb{R}$-submodule of $V$ for any $\theta \in \mathbb{F}$.

**Proof.** For any $\theta \in \mathbb{F}$ it follows from Lemma 4.1 that $V(\theta)$ is $x$-invariant for all $x \in \mathbb{R}$.

**Proposition 4.4.** Let $V$ denote a finite-dimensional irreducible $\mathfrak{H}$-module. For any irreducible $\mathbb{R}$-submodule $W$ of $V$, there exists a scalar $\theta \in \mathbb{F}$ such that $W \subseteq V(\theta)$.

**Proof.** Recall from Lemma 2.2(iii) that $\delta$ is central in $\mathbb{R}$. Recall from Definition 2.3 that each of $t_0^2, t_1^2, t_0^\vee, t_1^\vee$ is central in $\mathfrak{H}$. It follows from Schur’s lemma that the action of $\delta$ on $W$ and the actions of $t_0^2, t_1^2, t_0^\vee, t_1^\vee$ on $V$ are scalar multiplication. By Theorem 2.4 the element $t_0$ is an $\mathbb{F}$-linear combination of $1, \delta, t_0^2, t_1^2, t_0^\vee, t_1^\vee$. Hence $t_0$ acts on $W$ as scalar multiplication. The proposition follows.
4.2. The lattice of $\mathcal{R}$-submodules of $E_d(a,b,c)$. Throughout §4.2–§4.5 we adopt the notation from §3.2 and let
\[ \rho_i = c^2 - \left( a + b - \frac{d+1}{2} + i \right)^2 \quad \text{for } i = 1,3,\ldots,d. \]

**Lemma 4.5.** The matrix representing $t_0$ with respect to the $\mathbb{F}$-basis

\[ v_0, \quad v_d, \quad v_i - iv_{i-1} \quad \text{for } i = 2,4,\ldots,d-1, \quad v_i \quad \text{for } i = 1,3,\ldots,d-2 \]

for $E_d(a,b,c)$ is

\[
\begin{pmatrix}
-\frac{d+1}{2}I_2 & 0 & 0 \\
0 & -\frac{d+1}{2}I_{d-1} & \frac{1}{2}I_{d-1} \\
0 & 0 & \frac{d+1}{2}I_{d-1}
\end{pmatrix}.
\]

**Proof.** Applying (3) and (4) it is routine to verify the lemma.

**Lemma 4.6.**
(i) If $d = 1$ then $t_0$ is diagonalizable on $E_d(a,b,c)$ with exactly one eigenvalue $-\frac{d+1}{2}$.

(ii) If $d \geq 3$ then $t_0$ is diagonalizable on $E_d(a,b,c)$ with exactly two eigenvalues $\pm \frac{d+1}{2}$.

**Proof.** Immediate from Lemma 4.5

It follows from Proposition 4.3 that $E_d(a,b,c)(-\frac{d+1}{2})$ is an $\mathcal{R}$-submodule of $E_d(a,b,c)$. We now go into the $\mathcal{R}$-modules $E_d(a,b,c)(-\frac{d+1}{2})$ and $E_d(a,b,c)/E_d(a,b,c)(-\frac{d+1}{2})$.

**Lemma 4.7.** $E_d(a,b,c)(-\frac{d+1}{2})$ is of dimension $\frac{d+3}{2}$ with the $\mathbb{F}$-basis

\[ v_0, \quad v_d, \quad v_i - iv_{i-1} \quad \text{for } i = 2,4,\ldots,d-1. \]

**Proof.** It is straightforward to verify the lemma by using Lemma 4.5

**Lemma 4.8.** The actions of $A$ and $B$ on the $\mathcal{R}$-module $E_d(a,b,c)$ are as follows:

\[
Av_i = \begin{cases} 
\theta_i v_i - \frac{1}{2}v_{i+1} + \frac{1}{4}v_{i+2} & \text{for } i = 0,2,\ldots,d-3, \\
\theta_i v_i + \frac{1}{2}v_{i+2} & \text{for } i = 1,3,\ldots,d-2,
\end{cases}
\]

$Av_{d-1} = \theta_{d-1}v_{d-1} - \frac{1}{2}v_d, \quad Av_d = \theta_d v_d$,

\[
Bv_i = \begin{cases} 
\theta^*_i v_i + \frac{i(d-i+1)}{4} \rho_{i-1}v_{i-2} & \text{for } i = 2,4,\ldots,d-1, \\
\theta^*_i v_i - \frac{\rho_i}{2}v_{i-1} + \frac{(i-1)(d-i+2)}{4} \rho_i v_{i-2} & \text{for } i = 3,5,\ldots,d,
\end{cases}
\]

$Bv_0 = \theta^*_0 v_0, \quad Bv_1 = \theta^*_1 v_1 - \frac{\rho_1}{2}v_0$,

where

\[
\theta_i = \left( \frac{a}{2} - \frac{d-1}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \left( \frac{a}{2} - \frac{d+3}{4} + \left\lceil \frac{i}{2} \right\rceil \right), \quad (0 \leq i \leq d),
\]

\[
\theta^*_i = \left( \frac{b}{2} - \frac{d-1}{4} + \left\lceil \frac{i}{2} \right\rceil \right) \left( \frac{b}{2} - \frac{d+3}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right), \quad (0 \leq i \leq d).
\]
Proof. Apply Theorem 2.4 and Proposition 3.4 to evaluate the actions of $A$ and $B$ on $E_d(a, b, c)$. □

Lemma 4.9. The matrices representing $A$ and $B$ with respect to the $\mathbb{F}$-basis

\begin{equation}
\begin{aligned}
(v_0, \quad \frac{1}{2^i}(v_i - iv_{i-1}) \quad \text{for } i = 2, 4, \ldots, d - 1, \quad \frac{(d + 1)}{2d + 1}v_d
\end{aligned}
\end{equation}

for the $\mathbb{R}$-module $E_d(a, b, c)(-\frac{d+1}{2})$ are

\[
\begin{pmatrix}
\theta_0 & 0 \\
1 & \theta_1 \\
1 & \theta_2 \\
\vdots & \ddots \\
0 & \cdots & 1 & \theta_{\frac{d+1}{2}}
\end{pmatrix},
\begin{pmatrix}
\varphi_0^* & 0 \\
\varphi_1^* & \varphi_2^* \\
\varphi_2^* & \ddots \\
0 & \cdots & \varphi_{\frac{d+1}{2}}^*
\end{pmatrix}
\]

respectively, where

\[
\begin{align*}
\theta_i &= \frac{(2a - d + 4i - 3)(2a - d + 4i + 1)}{16} \quad (0 \leq i \leq \frac{d+1}{2}), \\
\theta_i^* &= \frac{(2b - d + 4i - 3)(2b - d + 4i + 1)}{16} \quad (0 \leq i \leq \frac{d+1}{2}), \\
\varphi_i &= \frac{i(2i - d - 3)(2a + 2b + 2c - d + 4i - 3)(2a + 2b - 2c - d + 4i - 3)}{32} \quad (1 \leq i \leq \frac{d+1}{2}).
\end{align*}
\]

The element $\delta$ acts on the $\mathbb{R}$-module $E_d(a, b, c)(-\frac{d+1}{2})$ as scalar multiplication by

\begin{equation}
\begin{aligned}
\frac{(d+1)(d+5)}{16} + \frac{(a-1)(a+1)}{4} + \frac{(b-1)(b+1)}{4} + \frac{(c-1)(c+1)}{4}.
\end{aligned}
\end{equation}

Proof. By Lemma 4.7 the vectors (14) are an $\mathbb{F}$-basis for $E_d(a, b, c)(-\frac{d+1}{2})$. Applying Lemma 4.8 a direct calculation yields the matrices representing $A$ and $B$ with respect to (14). By Theorem 2.4 and Lemma 3.5 the element $\delta$ acts on $E_d(a, b, c)(-\frac{d+1}{2})$ as scalar multiplication by (15). The lemma follows. □

Proposition 4.10. The $\mathbb{R}$-module $E_d(a, b, c)(-\frac{d+1}{2})$ is isomorphic to

\[
R_{\frac{d+1}{2}} \left(\frac{-a + 1}{2}, \frac{-b + 1}{2}, \frac{-c + 1}{2}\right).
\]

Moreover the $\mathbb{R}$-module $E_d(a, b, c)(-\frac{d+1}{2})$ is irreducible provided that the $\mathcal{H}$-module $E_d(a, b, c)$ is irreducible.

Proof. Set $(a', b', c', d') = (\frac{-a + 1}{2}, \frac{-b + 1}{2}, \frac{-c + 1}{2}, \frac{d+1}{2})$. Comparing Proposition 3.1 with Lemma 4.9 it follows that the $\mathbb{R}$-module $E_d(a, b, c)(-\frac{d+1}{2})$ is isomorphic to $R_{\frac{d'}{2}}(a', b', c')$. Suppose that the $\mathcal{H}$-module $E_d(a, b, c)$ is irreducible. Using Proposition 3.6 yields that

\[
a' + b' + c' + 1, -a' + b' + c', a' - b' + c', a' + b' - c' \not\subset \left\{\frac{d'}{2} - i \middle| i = 1, 2, \ldots, d'\right\}.
\]

By Proposition 3.2 the $\mathbb{R}$-module $R_{\frac{d'}{2}}(a', b', c')$ is irreducible. The proposition follows. □
Lemma 4.11. Suppose that \( d \geq 3 \). Then the matrices representing \( A \) and \( B \) with respect to the \( \mathbb{F} \)-basis

\[
\frac{1}{2i-1}v_i + E_d(a, b, c)(-\frac{d+1}{2}) \quad \text{for } i = 1, 3, \ldots, d - 2
\]

for the \( \mathcal{R} \)-module \( E_d(a, b, c)/E_d(a, b, c)(-\frac{d+1}{2}) \) are

\[
\begin{pmatrix}
\theta_0 & 0 \\
1 & \theta_1 \\
1 & \theta_2 \\
\vdots & \ddots \\
0 & 1 & \theta_{\frac{d+1}{2}}
\end{pmatrix}, \quad \begin{pmatrix}
\theta_0 & \varphi_1 & 0 \\
\theta_1 & \varphi_2 & \ddots \\
0 & \ddots & \ddots \\

\end{pmatrix}
\]

respectively, where

\[
\theta_i = \frac{(2a - d + 4i + 5)(2a - d + 4i + 1)}{16} \quad \text{for } 0 \leq i \leq \frac{d-3}{2},
\]

\[
\varphi_i = \frac{i(2i - d + 1)(2a + 2b + 2c - d + 4i + 1)(2a + 2b - 2c - d + 4i + 1)}{32} \quad \text{for } 1 \leq i \leq \frac{d-3}{2}.
\]

The element \( \delta \) acts on the \( \mathcal{R} \)-module \( E_d(a, b, c)/E_d(a, b, c)(-\frac{d+1}{2}) \) as scalar multiplication by

\[
\frac{(d-3)(d+1)}{16} + \frac{(a-1)(a+1)}{4} + \frac{(b-1)(b+1)}{4} + \frac{(c-1)(c+1)}{4}.
\]

Proof. By Lemma 4.7 the cosets (16) are an \( \mathbb{F} \)-basis for \( E_d(a, b, c)/E_d(a, b, c)(-\frac{d+1}{2}) \). Applying Lemma 4.8 a direct calculation yields the matrices representing \( A \) and \( B \) with respect to (16). By Lemma 4.5 the element \( t_0 \) acts on \( E_d(a, b, c)/E_d(a, b, c)(-\frac{d+1}{2}) \) as scalar multiplication by \( \frac{d+1}{2} \). Combined with Theorem 2.4 and Lemma 3.5 it follows that \( \delta \) acts on \( E_d(a, b, c)/E_d(a, b, c)(-\frac{d+1}{2}) \) as scalar multiplication by (17). The lemma follows. \( \square \)

Proposition 4.12. Suppose that \( d \geq 3 \). Then the \( \mathcal{R} \)-module \( E_d(a, b, c)/E_d(a, b, c)(-\frac{d+1}{2}) \) is isomorphic to

\[
R_{\frac{d+1}{2}} \left( -\frac{a+1}{2}, -\frac{b+1}{2}, -\frac{c+1}{2} \right).
\]

Moreover the \( \mathcal{R} \)-module \( E_d(a, b, c)/E_d(a, b, c)(-\frac{d+1}{2}) \) is irreducible provided that the \( \mathcal{S} \)-module \( E_d(a, b, c) \) is irreducible.

Proof. Set \( (a', b', c', d') = (-\frac{a+1}{2}, -\frac{b+1}{2}, -\frac{c+1}{2}, -\frac{d-3}{2}) \). Comparing Proposition 3.1 with Lemma 4.11 the quotient \( \mathcal{R} \)-module \( E_d(a, b, c)/E_d(a, b, c)(-\frac{d+1}{2}) \) is isomorphic to \( R_{d'}(a', b', c') \). Suppose that the \( \mathcal{S} \)-module \( E_d(a, b, c) \) is irreducible. Using Proposition 3.6 yields that

\[
a' + b' + c' + 1, -a' + b' + c', a' - b' + c', a' + b' - c' \not\in \left\{ \frac{d'}{2} - i \left| i = 0, 1, \ldots, d' + 1 \right. \right\}.
\]

By Proposition 3.2 the \( \mathcal{R} \)-module \( R_{d'}(a', b', c') \) is irreducible. The proposition follows. \( \square \)

Theorem 4.13. Assume that the \( \mathcal{S} \)-module \( E_d(a, b, c) \) is irreducible. Then the following hold:
(i) If \( d = 1 \) then the \( \mathbb{R} \)-module \( E_d(a, b, c) \) is irreducible.

(ii) If \( d \geq 3 \) then

\[
\begin{array}{c}
E_d(a, b, c) \\
\downarrow \\
E_d(a, b, c)(-\frac{d+1}{2}) \\
\downarrow \\
E_d(a, b, c)(\frac{d+1}{2}) \\
\downarrow \\
\{0\}
\end{array}
\]

is the lattice of \( \mathbb{R} \)-submodules of \( E_d(a, b, c) \).

Proof. (i): Suppose that \( d = 1 \). Then \( E_d(a, b, c) = E_d(a, b, c)(-\frac{d+1}{2}) \) by Lemma 4.6(i). It follows from Proposition 4.10 that the \( \mathbb{R} \)-module \( E_d(a, b, c) \) is irreducible. The statement (i) follows.

(ii): Suppose that \( d \geq 3 \). Combining Propositions 4.10 and 4.12 yields that

\[
\{0\} \subset E_d(a, b, c)(-\frac{d+1}{2}) \subset E_d(a, b, c)
\]

is a composition series for the \( \mathbb{R} \)-module \( E_d(a, b, c) \). By Proposition 4.13 and Lemma 4.6(ii), \( E_d(a, b, c)(\frac{d+1}{2}) \) is a nonzero \( \mathbb{R} \)-submodule of \( E_d(a, b, c) \). By Jordan–Hölder theorem the sequence

\[
\{0\} \subset E_d(a, b, c)(\frac{d+1}{2}) \subset E_d(a, b, c)
\]

is a composition series for the \( \mathbb{R} \)-module \( E_d(a, b, c) \). It follows from Proposition 4.4 that there is no other irreducible \( \mathbb{R} \)-submodule of \( E_d(a, b, c) \). Hence (18) and (19) are the unique two composition series for the \( \mathbb{R} \)-module \( E_d(a, b, c) \). The statement (ii) follows.

4.3. The lattice of \( \mathbb{R} \)-submodules of \( E_d(a, b, c)\)\(^{(1, -1)}\).

Lemma 4.14. The matrix representing \( t_0 \) with respect to the \( \mathbb{F} \)-basis

\[
v_1, \quad v_{i+1} - i(d - i + 1)v_{i-1} \quad \text{for } i = 2, 4, \ldots, d - 1, \quad v_i \quad \text{for } i = 0, 2, \ldots, d - 1
\]

for \( E_d(a, b, c)\)\(^{(1, -1)}\) is

\[
\begin{pmatrix}
-aI_{\frac{d+1}{2}} & I_{\frac{d+1}{2}} \\
0 & aI_{\frac{d+1}{2}}
\end{pmatrix}.
\]

Proof. By Table 1 the action of \( t_0 \) on \( E_d(a, b, c)\)\(^{(1, -1)}\) corresponds to the action of \( t_1 \) on \( E_d(a, b, c) \). By (5) and (6) it is routine to verify the lemma.

Lemma 4.15. (i) If \( a = 0 \) then \( t_0 \) is not diagonalizable on \( E_d(a, b, c)\)\(^{(1, -1)}\) with exactly one eigenvalue 0.

(ii) If \( a \neq 0 \) then \( t_0 \) is diagonalizable on \( E_d(a, b, c)\)\(^{(1, -1)}\) with exactly two eigenvalues \( \pm a \).

Proof. Immediate from Lemma 4.14.

Lemma 4.16. \( E_d(a, b, c)\)\(^{(1, -1)}\)(\(-a\)) is of dimension \( \frac{d+1}{2} \) with the \( \mathbb{F} \)-basis

\[
v_i \quad \text{for } i = 1, 3, \ldots, d.
\]

Proof. Immediate from Lemma 4.14.
Lemma 4.17. The actions of $A$ and $B$ on the $\mathfrak{N}$-module $E_d(a, b, c)^{(1, -1)}$ are as follows:

$$Av_i = \begin{cases} 
\theta_i v_i - \frac{1}{2} v_{i+1} + \frac{1}{4} v_{i+2} & \text{for } i = 0, 2, \ldots, d - 3, \\
\theta_i v_i + \frac{1}{4} v_{i+2} & \text{for } i = 1, 3, \ldots, d - 2,
\end{cases}$$

$$Av_{d-1} = \theta_{d-1} v_{d-1} - \frac{1}{2} v_d, \quad Av_d = \theta_d v_d,$$

$$Bv_i = \begin{cases} 
\theta_i^* v_i + \frac{i (d - i + 1)}{4} v_{i-1} + \frac{i (d - i + 1)}{4} \rho_{i-1} v_{i-2} & \text{for } i = 2, 4, \ldots, d - 1, \\
\theta_i^* v_i + \frac{(i - 1)(d - i + 2)}{4} \rho_{i-2} v_{i-2} & \text{for } i = 3, 5, \ldots, d,
\end{cases}$$

where

$$\theta_i = \left( \frac{a}{2} - \frac{d - 1}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \left( \frac{a}{2} - \frac{d + 3}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \quad (0 \leq i \leq d),$$

$$\theta_i^* = \left( \frac{b}{2} - \frac{d - 3}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \left( \frac{b}{2} - \frac{d + 1}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \quad (0 \leq i \leq d).$$

Proof. By Theorem 2.4 and Table 1 the actions of $A$ and $B$ on $E_d(a, b, c)^{(1, -1)}$ correspond to the actions of

$$\frac{(t_0^\gamma + t_1^\gamma)(t_0^\gamma + t_1^\gamma + 2)}{4}, \quad \frac{(t_0 + t_0^\gamma)(t_0 + t_0^\gamma + 2)}{4}$$
on $E_d(a, b, c)$, respectively. Applying Proposition 3.4 it is routine to verify the lemma. $\square$

Lemma 4.18. The matrices representing $A$ and $B$ with respect to the $\mathbb{F}$-basis

$$\frac{1}{2^{i-1}} v_i \quad \text{for } i = 1, 3, \ldots, d$$

for the $\mathfrak{N}$-module $E_d(a, b, c)^{(1, -1)}(-a)$ are

$$\begin{pmatrix}
\theta_0 & 1 & \theta_1 & 0 \\
1 & \theta_2 & \ddots & \ddots \\
& 0 & \ddots & \ddots \\
0 & \cdots & 1 & \theta_{d-1}
\end{pmatrix}, \quad \begin{pmatrix}
\theta_0^* & \varphi_1 & 0 \\
\varphi_1 & \theta_1^* & \varphi_2 \\
& \ddots & \ddots \\
0 & \ddots & \varphi_{d-1} & \theta_{d-1}^*
\end{pmatrix},$$

respectively, where

$$\theta_i = \frac{(2a - d + 4i + 1)(2a - d + 4i + 5)}{16} \quad (0 \leq i \leq \frac{d-1}{2}),$$

$$\theta_i^* = \frac{(2b - d + 4i - 1)(2b - d + 4i + 3)}{16} \quad (0 \leq i \leq \frac{d-1}{2}),$$

$$\varphi_i = \frac{i(2i - d - 1)(2a + 2b + 2c - d + 4i + 1)(2a + 2b - 2c - d + 4i + 1)}{32} \quad (1 \leq i \leq \frac{d-1}{2}).$$

The element $\delta$ acts on the $\mathfrak{N}$-module $E_d(a, b, c)^{(1, -1)}(-a)$ as scalar multiplication by

$$\frac{(d - 1)(d + 3)}{16} + \frac{a(a + 2)}{4} + \frac{(b - 1)(b + 1)}{4} + \frac{(c - 1)(c + 1)}{4},$$

for $i = 1, 3, \ldots, d$. The elements $\theta_i, \theta_i^*, \varphi_i, \varphi_{i+1}$ are as follows:
Proposition 4.19. The \( \mathcal{R} \)-module \( E_d(a, b, c)^{(1, -1)}(-a) \) is isomorphic to

\[
R_{d+1} \left( -\frac{a}{2} - 1, -\frac{b+1}{2}, -\frac{c+1}{2} \right).
\]

Moreover the \( \mathcal{R} \)-module \( E_d(a, b, c)^{(1, -1)}(-a) \) is irreducible if the \( \mathcal{S} \)-module \( E_d(a, b, c)^{(1, -1)} \) is irreducible.

Proof. Set \( (a', b', c', d') = (-\frac{a}{2} - 1, -\frac{b+1}{2}, -\frac{c+1}{2}, \frac{d+1}{2}) \). Comparing Proposition 3.1 with Lemma 4.18 it follows that the \( \mathcal{R} \)-module \( E_d(a, b, c)^{(1, -1)}(-a) \) is isomorphic to \( R_d(a', b', c') \). Suppose that the \( \mathcal{S} \)-module \( E_d(a, b, c)^{(1, -1)} \) is irreducible. Using Proposition 3.6 yields that

\[
a' + b' + c' + 1, a' - b' + c', a' + b' + c' \not\subset \left\{ \frac{d' - i}{2} \mid i = 1, 2, \ldots, d' + 1 \right\}
\]

and

\[
-a' + b' + c' \not\subset \left\{ \frac{d' - i}{2} \mid i = 0, 1, \ldots, d' \right\}.
\]

By Proposition 3.2 the \( \mathcal{R} \)-module \( R_d(a', b', c') \) is irreducible. The proposition follows. \( \square \)

Lemma 4.20. The matrices representing \( A \) and \( B \) with respect to the \( \mathbb{F} \)-basis

\[
\frac{1}{2^i} v_i + E_d(a, b, c)^{(1, -1)}(-a) \quad \text{for } i = 0, 2, \ldots, d - 1
\]

for the \( \mathcal{R} \)-module \( E_d(a, b, c)^{(1, -1)}/E_d(a, b, c)^{(1, -1)}(-a) \) are

\[
\begin{pmatrix}
\theta_0 & \varphi_{1} & 0 \\
1 & \theta_1 & 0 \\
& \ddots & \ddots \\
0 & 1 & \theta_{d+1}/2
\end{pmatrix},
\]

\[
\begin{pmatrix}
\theta_0^* & \varphi_{1} & 0 \\
\theta_1^* & \varphi_{2} & 0 \\
& \ddots & \ddots \\
0 & \theta_{d+1}^*/2 & \varphi_{d+1}/2
\end{pmatrix}
\]

respectively, where

\[
\theta_i = \frac{(2a - d + 4i - 3)(2a - d + 4i + 1)}{16} \quad (0 \leq i \leq \frac{d-1}{2}),
\]

\[
\theta_i^* = \frac{(2b - d + 4i - 1)(2b - d + 4i + 3)}{16} \quad (0 \leq i \leq \frac{d-1}{2}),
\]

\[
\varphi_i = \frac{i(2i - d - 1)(2a + 2b + 2c - d + 4i - 3)(2a + 2b - 2c - d + 4i - 3)}{32} \quad (1 \leq i \leq \frac{d-1}{2}).
\]

The element \( \delta \) acts on the \( \mathcal{R} \)-module \( E_d(a, b, c)^{(1, -1)}/E_d(a, b, c)^{(1, -1)}(-a) \) as scalar multiplication by

\[
\frac{(d - 1)(d + 3)}{16} + \frac{a(a - 2)}{4} + \frac{(b - 1)(b + 1)}{4} + \frac{(c - 1)(c + 1)}{4}.
\]
Proof. By Lemma 4.16 the cosets (22) are an $F$-basis for $E_d(a, b, c)^{(1,-1)}/E_d(a, b, c)^{(1,-1)}(-a)$. Applying Lemma 4.17 a direct calculation yields the matrices representing $A$ and $B$ with respect to (22). By Lemma 4.14 the element $t_0$ acts on $E_d(a, b, c)^{(1,-1)}/E_d(a, b, c)^{(1,-1)}(-a)$ as scalar multiplication by $a$. Combined with Theorem 2.4 and Lemma 3.5 the element $\delta$ acts on $E_d(a, b, c)^{(1,-1)}/E_d(a, b, c)^{(1,-1)}(-a)$ as scalar multiplication by (23). The lemma follows. □

Proposition 4.21. The $R$-module $E_d(a, b, c)^{(1,-1)}/E_d(a, b, c)^{(1,-1)}(-a)$ is isomorphic to

$$R_{d+1}^d \left( \frac{-a}{2}, \frac{b+1}{2}, \frac{-c+1}{2} \right).$$

Moreover the $R$-module $E_d(a, b, c)^{(1,-1)}/E_d(a, b, c)^{(1,-1)}(-a)$ is irreducible provided that the $H$-module $E_d(a, b, c)^{(1,-1)}$ is irreducible. Using Proposition 3.6 yields that $$a' + b' + c' + 1', a' - b' + c', a' + b' - c' \not\in \left\{ \frac{d'}{2} - i \bigg| i = 0, 1, \ldots, d' \right\}$$ and $$-a' + b' + c' \not\in \left\{ \frac{d'}{2} - i \bigg| i = 1, 2, \ldots, d' + 1 \right\}.$$

By Proposition 3.2 the $R$-module $R_d^d(a', b', c')$ is irreducible. The proposition follows. □

Theorem 4.22. Assume that the $H$-module $E_d(a, b, c)^{(1,-1)}$ is irreducible. Then the following hold:

(i) If $a = 0$ then

$$E_d(a, b, c)^{(1,-1)}$$

$$\left| \begin{array}{c}
E_d(a, b, c)^{(1,-1)}(0) \\
\{0\}
\end{array} \right.$$  

is the lattice of $R$-submodules of $E_d(a, b, c)^{(1,-1)}$.

(ii) If $a \neq 0$ then

$$E_d(a, b, c)^{(1,-1)}$$

$$\left| \begin{array}{c}
E_d(a, b, c)^{(1,-1)}(-a) \\
\{0\}
\end{array} \right|$$

$$\left| \begin{array}{c}
E_d(a, b, c)^{(1,-1)}(a) \\
\{0\}
\end{array} \right.$$  

is the lattice of $R$-submodules of $E_d(a, b, c)^{(1,-1)}$.  

Proof. (i): Suppose that \( a = 0 \). Combining Propositions 4.19 and 4.21 yields that
\[
(24) \quad \{0\} \subset E_d(a, b, c)^{(1, -1)}(0) \subset E_d(a, b, c)^{(1, -1)}
\]
is a composition series for the \( \mathbb{R} \)-module \( E_d(a, b, c)^{(1, -1)} \). By Proposition 4.3 and Lemma 4.15 (i) every irreducible \( \mathbb{R} \)-submodule of \( E_d(a, b, c)^{(1, -1)} \) is contained in \( E_d(a, b, c)^{(1, -1)}(0) \). Hence (24) is the unique composition series for the \( \mathbb{R} \)-module \( E_d(a, b, c)^{(1, -1)} \). Therefore (i) follows.

(ii): Similar to the proof of Theorem 4.13 (ii). \( \square \)

4.4. The lattice of \( \mathbb{R} \)-submodules of \( E_d(a, b, c)^{(-1,1)} \).

Lemma 4.23. Assume that the \( \mathfrak{H} \)-module \( E_d(a, b, c)^{(-1,1)} \) is irreducible. Then
\[
(25) \quad \rho_i v_{i-2} - v_i \quad \text{for } i = 2, 4, \ldots, d - 1, \quad \rho_d v_{d-1}, \quad v_i \quad \text{for } i = 1, 3, \ldots, d
\]
form an \( \mathbb{F} \)-basis for \( E_d(a, b, c)^{(-1,1)} \). The matrix representing \( t_0 \) with respect to the \( \mathbb{F} \)-basis (25) for \( E_d(a, b, c)^{(-1,1)} \) is
\[
\begin{pmatrix}
bI_{d+1} & I_{d+1} \\
0 & -bI_{d+1}
\end{pmatrix}.
\]

Proof. It follows from Proposition 3.6 that \( \rho_i \neq 0 \) for all \( i = 1, 3, \ldots, d \). Hence (25) is an \( \mathbb{F} \)-basis for \( E_d(a, b, c)^{(-1,1)} \). By Table 11 the action of \( t_0 \) on \( E_d(a, b, c)^{(-1,1)} \) corresponds to the action of \( t_0^* \) on \( E_d(a, b, c) \). Using (7) and (8) it is routine to verify the lemma. \( \square \)

Lemma 4.24. Assume that the \( \mathfrak{H} \)-module \( E_d(a, b, c)^{(-1,1)} \) is irreducible. Then the following hold:

(i) If \( b = 0 \) then \( t_0 \) is not diagonalizable on \( E_d(a, b, c)^{(-1,1)} \) with exactly one eigenvalue 0.

(ii) If \( b \neq 0 \) then \( t_0 \) is diagonalizable on \( E_d(a, b, c)^{(-1,1)} \) with exactly two eigenvalues \( \pm b \).

Proof. Immediate from Lemma 4.23. \( \square \)

Lemma 4.25. If the \( \mathfrak{H} \)-module \( E_d(a, b, c)^{(-1,1)} \) is irreducible then \( E_d(a, b, c)^{(-1,1)}(b) \) is of dimension \( \frac{d-1}{2} \) with the \( \mathbb{F} \)-basis
\[
v_i \quad \text{for } i = 0, 2, \ldots, d - 1.
\]

Proof. Immediate from Lemma 4.23. \( \square \)

Lemma 4.26. The actions of \( A \) and \( B \) on the \( \mathfrak{H} \)-module \( E_d(a, b, c)^{(-1,1)} \) are as follows:
\[
Av_i = \begin{cases}
\theta_i v_i + \frac{1}{4} v_{i+2} & \text{for } i = 0, 2, \ldots, d - 3, \\
\theta_i v_i + \frac{1}{2} v_{i+1} + \frac{1}{4} v_{i+2} & \text{for } i = 1, 3, \ldots, d - 2,
\end{cases}
Av_{d-1} = \theta_{d-1} v_{d-1}, \quad Av_d = \theta_d v_d,
\]
\[
Bv_i = \begin{cases}
\theta_i^* v_i + \frac{i(d - i + 1)}{4} \rho_{i-1} v_{i-2} & \text{for } i = 2, 4, \ldots, d - 1, \\
\theta_i^* v_i - \rho_i v_{i-1} + \frac{(i-1)(d - i + 2)}{4} \rho_{i-1} v_{i-2} & \text{for } i = 3, 5, \ldots, d,
\end{cases}
Bv_0 = \theta_0^* v_0, \quad Bv_1 = \theta_1^* v_1 - \frac{\rho_1}{2} v_0.
\]
Lemma 4.26. A straightforward calculation yields the matrices representing the actions of \( A \) and \( B \) on \( E_d(a, b, c)^{(1, -1)} \) corresponding to the elements \( \delta \) and \( \varphi \) respectively, where

\[
\theta_i = \left( \frac{a}{2} - \frac{d - 3}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \left( \frac{a}{2} - \frac{d + 1}{4} + \left\lceil \frac{i}{2} \right\rceil \right) (0 \leq i \leq d),
\]

\[
\theta_i^* = \left( \frac{b}{2} - \frac{d - 1}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \left( \frac{b}{2} - \frac{d + 3}{4} + \left\lceil \frac{i}{2} \right\rceil \right) (0 \leq i \leq d).
\]

Proof. By Theorem 2.4 and Table 1, the actions of \( A \) and \( B \) on \( E_d(a, b, c)^{(1, -1)} \) correspond to the elements \( \delta \) and \( \varphi \) respectively, where \( \delta \) acts on \( E_d(a, b, c) \) and \( \varphi \) acts as scalar multiplication by \( \frac{1}{2} \). Applying Proposition 3.4, it is routine to verify the lemma.

Lemma 4.27. Assume that the \( \mathcal{F} \)-module \( E_d(a, b, c)^{(-1,1)} \) is irreducible. Then the matrices representing \( A \) and \( B \) with respect to the \( \mathbb{F} \)-basis

\[
(26) \quad \frac{1}{2^i} v_i \quad \text{for } i = 0, 2, \ldots, d - 1
\]

for the \( \mathbb{R} \)-module \( E_d(a, b, c)^{(-1,1)}(b) \) are

\[
\begin{pmatrix}
\theta_0 & 1 & \theta_2 & \cdots & 0 \\
1 & \theta_1 & \theta_3 & \cdots & \\
0 & 1 & \theta_{d-1} & \cdots & \theta_{d-1}
\end{pmatrix},
\]

\[
\begin{pmatrix}
\theta_0^* & \varphi_1 & \theta_2^* & \cdots & \\
\varphi_1 & \theta_1^* & \varphi_2 & \cdots & \\
\theta_2^* & \varphi_2 & \theta_3^* & \cdots & \\
\vdots & \ddots & \ddots & \ddots & \\
0 & \cdots & \cdots & \varphi_{d-1} & \theta_{d-1}^*
\end{pmatrix}
\]

respectively, where

\[
\theta_i = \frac{(2a - d + 4i - 1)(2a - d + 4i + 3)}{16} (0 \leq i \leq \frac{d-1}{2}),
\]

\[
\theta_i^* = \frac{(2b - d + 4i - 3)(2b - d + 4i + 1)}{16} (0 \leq i \leq \frac{d-1}{2}),
\]

\[
\varphi_i = \frac{i(2i - d - 1)(2a + 2b + 2c + 4i - d - 3)(2a + 2b - 2c + 4i - d - 3)}{32} (1 \leq i \leq \frac{d-1}{2}).
\]

The element \( \delta \) acts on the \( \mathbb{R} \)-module \( E_d(a, b, c)^{(-1,1)}(b) \) as scalar multiplication by

\[
(27) \quad \frac{(d-1)(d+3)}{16} + \frac{(a-1)(a+1)}{4} + \frac{b(b-2)}{4} + \frac{(c-1)(c+1)}{4}.
\]

Proof. By Lemma 4.25, the vectors (26) are a \( \mathbb{F} \)-basis for \( E_d(a, b, c)^{(-1,1)}(b) \). Applying Lemma 4.26, a straightforward calculation yields the matrices representing \( A \) and \( B \) with respect to (26). By Theorem 2.4 and Lemma 3.5, the element \( \delta \) acts on \( E_d(a, b, c)^{(-1,1)}(b) \) as scalar multiplication by (27). The lemma follows.

Proposition 4.28. Assume that the \( \mathcal{F} \)-module \( E_d(a, b, c)^{(-1,1)} \) is irreducible. The \( \mathbb{R} \)-module \( E_d(a, b, c)^{(-1,1)}(b) \) is isomorphic to

\[
R_{\frac{d-1}{2}} \left[ \frac{-a + 1}{2}, \frac{-b + c + 1}{2} \right].
\]

Moreover the \( \mathbb{R} \)-module \( E_d(a, b, c)^{(-1,1)}(b) \) is irreducible.
Proof. Set \((a', b', c', d') = (-\frac{a+1}{2}, -\frac{b}{2}, -\frac{c+1}{2}, \frac{d+1}{2})\). Comparing Proposition 3.1 with Lemma 4.27 yields that the \(\mathcal{R}\)-module \(E_d(a, b, c)^{(-1,1)}(b)\) is isomorphic to \(R_d'(a', b', c')\). It follows from Proposition 3.6 that
\[
a' + b' + c' + 1, -a' + b' + c', a' + b' - c' \notin \left\{ \frac{d'}{2} - i \right\}_{i=0,1,\ldots,d'}
\]
and
\[
a' - b' + c' \notin \left\{ \frac{d'}{2} - i \right\}_{i=1,2,\ldots,d'+1}.
\]
By Proposition 3.2 the \(\mathcal{R}\)-module \(R_d'(a', b', c')\) is irreducible. The proposition follows. \(\square\)

Lemma 4.29. Assume that the \(\mathfrak{H}\)-module \(E_d(a, b, c)^{(-1,1)}\) is irreducible. Then the matrices representing \(A\) and \(B\) with respect to the \(\mathcal{F}\)-basis
\[(28) \quad \frac{1}{2^{i-1}}v_i + E_d(a, b, c)^{(-1,1)}(b) \quad \text{for} \ i = 1, 3, \ldots, d
\]
for the \(\mathcal{R}\)-module \(E_d(a, b, c)^{(-1,1)}/E_d(a, b, c)^{(-1,1)}(b)\) are
\[
\begin{pmatrix}
\theta_0 \\ 1 \\
\theta_1 \\ 1 \\
\vdots \\ \theta_{d-1}/2 \\ 1 \\
\end{pmatrix}, \quad \begin{pmatrix}
\theta_0^* \\ \phi_1 \\ \theta_1^* \\ \phi_2 \\ \vdots \\ \theta_{d-1}^*/2 \\ \phi_d-1/2 \\
\end{pmatrix}
\]
respectively, where
\[
\theta_i = \frac{(2a - d + 4i - 1)(2a - d + 4i + 3)}{16} \quad (0 \leq i \leq \frac{d-1}{2}),
\]
\[
\theta_i^* = \frac{(2b - d + 4i + 1)(2b - d + 4i + 5)}{16} \quad (0 \leq i \leq \frac{d-1}{2}),
\]
\[
\phi_i = \frac{i(2i-d-1)(2a + 2b + 2c + 4i - d + 1)(2a + 2b - 2c + 4i - d + 1)}{32} \quad (1 \leq i \leq \frac{d-1}{2}).
\]
The element \(\delta\) acts on the \(\mathcal{R}\)-module \(E_d(a, b, c)^{(-1,1)}/E_d(a, b, c)^{(-1,1)}(b)\) as scalar multiplication by
\[(29) \quad \frac{(d-1)(d+3)}{16} + \frac{(a-1)(a+1)}{4} + \frac{b(b+2)}{4} + \frac{(c-1)(c+1)}{4}.
\]
Proof. By Lemma 4.25 the cosets (28) are an \(\mathcal{F}\)-basis for \(E_d(a, b, c)^{(-1,1)}/E_d(a, b, c)^{(-1,1)}(b)\). Applying Lemma 4.26 we obtain the matrices representing \(A\) and \(B\) with respect to (28). By Lemma 4.23 the element \(v_0\) acts on \(E_d(a, b, c)^{(-1,1)}/E_d(a, b, c)^{(-1,1)}(b)\) as scalar multiplication by \(-b\). Combined with Theorem 2.4 and Lemma 3.5 the element \(\delta\) acts on \(E_d(a, b, c)^{(-1,1)}/E_d(a, b, c)^{(-1,1)}(b)\) as scalar multiplication by (29). The lemma follows. \(\square\)

Proposition 4.30. Assume that the \(\mathfrak{H}\)-module \(E_d(a, b, c)^{(-1,1)}\) is irreducible. The \(\mathcal{R}\)-module \(E_d(a, b, c)^{(-1,1)}/E_d(a, b, c)^{(-1,1)}(b)\) is isomorphic to
\[
R_{d-1}^+ \left( -\frac{a+1}{2}, -\frac{b}{2} - 1, -\frac{c+1}{2} \right).
\]
Moreover the \(\mathcal{R}\)-module \(E_d(a, b, c)^{(-1,1)}/E_d(a, b, c)^{(-1,1)}(b)\) is irreducible.
Proof. Let \((a', b', c', d') = (-\frac{a+1}{2}, -\frac{b}{2} - 1, -\frac{c+1}{2}, \frac{d+1}{2})\). Comparing Proposition 3.1 with Lemma 4.29 yields that the quotient \( \mathcal{R} \)-module \( \mathcal{E}_{d'}(a, b, c)(-1,1) / \mathcal{E}_d(a, b, c)(-1,1)(b) \) is isomorphic to \( \mathcal{E}_d(a', b', c') \). It follows from Proposition 3.6 that

\[
\begin{align*}
    a' + b' + c' + 1, &\quad -a' + b' + c', \\
    a' + b' - c' &\not\in \left\{ \frac{d'}{2} - i \mid i = 1, 2, \ldots, d' + 1 \right\}
\end{align*}
\]

and

\[
    a' - b' + c' \not\in \left\{ \frac{d'}{2} - i \mid i = 0, 1, \ldots, d' \right\}.
\]

By Proposition 3.2 the \( \mathcal{R} \)-module \( \mathcal{R}_{d'}(a', b', c') \) is irreducible. The proposition follows. \(\Box\)

**Theorem 4.31.** Assume that the \( \mathcal{S}_3 \)-module \( \mathcal{E}_d(a, b, c)(-1,1) \) is irreducible. Then the following hold:

(i) If \( b = 0 \) then

\[
\begin{array}{c}
\mathcal{E}_d(a, b, c)(-1,1) \\
\mathcal{E}_d(a, b, c)(-1,1)(0) \\
\{0\}
\end{array}
\]

is the lattice of \( \mathcal{R} \)-submodules of \( \mathcal{E}_d(a, b, c)(-1,1) \).

(ii) If \( b \neq 0 \) then

\[
\begin{array}{ccc}
\mathcal{E}_d(a, b, c)(-1,1) & \mathcal{E}_d(a, b, c)(-1,1)(-b) & \mathcal{E}_d(a, b, c)(-1,1)(b) \\
\{0\} & \{0\}
\end{array}
\]

is the lattice of \( \mathcal{R} \)-submodules of \( \mathcal{E}_d(a, b, c)(-1,1) \).

Proof. Using the above lemmas and propositions, the result follows by an argument similar to the proof of Theorem 4.22. \(\Box\)

4.5. **The lattice of \( \mathcal{R} \)-submodules of \( \mathcal{E}_d(a, b, c)(-1,-1) \).**

**Lemma 4.32.** The matrix representing \( t_0 \) with respect to the \( \mathcal{F} \)-basis

\[
v_i + (\tau + i)v_{i-1} \quad \text{for} \quad i = 1, 3, \ldots, d, \quad v_i \quad \text{for} \quad i = 0, 2, \ldots, d - 1
\]

for \( \mathcal{E}_d(a, b, c)(-1,-1) \) is

\[
\begin{pmatrix}
    cI_{d+1} & -I_{d+1} \\
    0 & -cI_{d+1}
\end{pmatrix}.
\]

Proof. By Table 4 the action of \( t_0 \) on \( \mathcal{E}_d(a, b, c)(-1,-1) \) corresponds to the action of \( t_1^\vee \) on \( \mathcal{E}_d(a, b, c) \). Applying (9) it is routine to verify the lemma. \(\Box\)
Lemma 4.33. (i) If \( c = 0 \) then \( t_0 \) is not diagonalizable on \( E_d(a,b,c)^{(-1,-1)} \) with exactly one eigenvalue 0.
(ii) If \( c \neq 0 \) then \( t_0 \) is diagonalizable on \( E_d(a,b,c)^{(-1,-1)} \) with exactly two eigenvalues \( \pm c \).

Proof. Immediate from Lemma 4.32 \( \square \)

Lemma 4.34. \( E_d(a,b,c)^{(-1,-1)}(c) \) is of dimension \( \frac{d+1}{2} \) with the \( \mathbb{F} \)-basis
\[ v_i + (\tau + i)v_{i-1} \quad \text{for} \quad i = 1,3, \ldots, d. \]

Proof. Immediate from Lemma 4.32 \( \square \)

Lemma 4.35. The actions of \( A \) and \( B \) on the \( \mathfrak{F} \)-module \( E_d(a,b,c)^{(-1,-1)} \) are as follows:
\[
Av_i = \begin{cases} 
\theta_i v_i + \frac{1}{4} v_{i+2} & \text{for } i = 0,2, \ldots, d-3, \\
\theta_i v_i + \frac{1}{2} v_{i+1} + \frac{1}{4} v_{i+2} & \text{for } i = 1,3, \ldots, d-2,
\end{cases}
\]
\[
Av_{d-1} = \theta_d v_{d-1}, \quad Av_d = \theta_d v_d,
\]
\[
Bv_i = \begin{cases} 
\theta_i^* v_i + \frac{i(d-i+1)}{4} v_{i-1} + \frac{i(d-i+1)}{4} \rho_{i-1} v_{i-2} & \text{for } i = 2,4, \ldots, d-1, \\
\theta_i^* v_i + \frac{(d-i)(d+i+2)}{4} \rho_i v_{i-2} & \text{for } i = 3,5, \ldots, d,
\end{cases}
\]
\[
Bv_0 = \theta_0^* v_0, \quad Bv_1 = \theta_1^* v_1,
\]
where
\[
\theta_i = \left( \frac{a}{2} - \frac{d-3}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \left( \frac{a}{2} - \frac{d+1}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \quad (0 \leq i \leq d),
\]
\[
\theta_i^* = \left( \frac{b}{2} - \frac{d-3}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \left( \frac{b}{2} - \frac{d+1}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \quad (0 \leq i \leq d).
\]

Proof. By Theorem 2.4 and Table 1 the actions of \( A \) and \( B \) on \( E_d(a,b,c)^{(-1,-1)} \) correspond to the actions of
\[
\frac{(t_0 + t_1)(t_0 + t_1 + 2)}{4}, \quad \frac{(t_0 + t_0^\vee)(t_0 + t_0^\vee + 2)}{4}
\]
on \( E_d(a,b,c) \), respectively. Using Proposition 3.4 it is routine to verify the lemma. \( \square \)

Lemma 4.36. The matrices representing \( A \) and \( B \) with respect to the \( \mathbb{F} \)-basis
\[
\frac{1}{2^{i-1}}(v_i + (\tau + i)v_{i-1}) \quad \text{for} \quad i = 1,3, \ldots, d
\]
for the \( \mathfrak{R} \)-module \( E_d(a,b,c)^{(-1,-1)}(c) \) are
\[
\begin{pmatrix}
\theta_0 & 0 \\
1 & \theta_1 \\
& & \ddots \\
0 & & & & \theta_{d-1}
\end{pmatrix}, \quad \begin{pmatrix}
\theta_0^* & \varphi_1 & 0 \\
\theta_1^* & \varphi_2 & \ddots \\
& & \ddots & \varphi_{d-2} \\
0 & & & & \theta_{d-1}^*
\end{pmatrix}
\]
respectively, where
\[
\theta_i = \frac{(2a - d + 4i - 1)(2a - d + 4i + 3)}{16} \quad (0 \leq i \leq \frac{d-1}{2}),
\]
\[
\theta_i^* = \frac{(2b - d + 4i - 1)(2b - d + 4i + 3)}{16} \quad (0 \leq i \leq \frac{d-1}{2}),
\]
\[
\varphi_i = \frac{i(2i - d - 1)(2a + 2b + 2c - d + 4i - 3)(2a + 2b - 2c - d + 4i + 1)}{32} \quad (1 \leq i \leq \frac{d-1}{2}).
\]
The element \( \delta \) acts on the \( \mathcal{R} \)-module \( E_d(a, b, c)^{(-1,-1)}(c) \) as scalar multiplication by
\[
\frac{(d - 1)(d + 3)}{16} + \frac{(a - 1)(a + 1)}{4} + \frac{(b - 1)(b + 1)}{4} + \frac{c(c - 2)}{4}.
\]

**Proof.** By Lemma 4.34 the vectors (30) are an \( \mathbb{F} \)-basis for \( E_d(a, b, c)^{(-1,-1)}(c) \). Applying Lemma 4.35 a straightforward calculation yields the matrices representing \( A \) and \( B \) with respect to (30). Using Theorem 2.4 and Lemma 3.5 yields that \( \delta \) acts on \( E_d(a, b, c)^{(-1,-1)}(c) \) as scalar multiplication by (31). The lemma follows. \( \square \)

**Proposition 4.37.** The \( \mathcal{R} \)-module \( E_d(a, b, c)^{(-1,-1)}(c) \) is isomorphic to
\[
R_{d+1} \left( - \frac{a + 1}{2}, - \frac{b + 1}{2}, - \frac{c}{2} \right).
\]
Moreover the \( \mathcal{R} \)-module \( E_d(a, b, c)^{(-1,-1)}(c) \) is irreducible if the \( \mathcal{S} \)-module \( E_d(a, b, c)^{(-1,-1)} \) is irreducible.

**Proof.** Set \((a', b', c', d') = (-\frac{a+1}{2}, -\frac{b+1}{2}, -\frac{c}{2}, \frac{d-1}{2})\). Comparing Proposition 3.1 with Lemma 4.36 it follows that the \( \mathcal{R} \)-module \( E_d(a, b, c)^{(-1,-1)}(c) \) is isomorphic to \( R_{d'}(a', b', c') \). Suppose that the \( \mathcal{S} \)-module \( E_d(a, b, c)^{(-1,-1)} \) is irreducible. Using Proposition 3.6 yields that
\[
a' + b' + c' + 1, -a' + b' + c', a' - b' + c' \not\in \left\{ \frac{d' - i}{2} \right\}_{i=0,1,\ldots,d'}
\]
and
\[
a' + b' - c' \not\in \left\{ \frac{d' - i}{2} \right\}_{i=1,2,\ldots,d'+1}.
\]
By Proposition 3.2 the \( \mathcal{R} \)-module \( R_{d'}(a', b', c') \) is irreducible. The proposition follows. \( \square \)

**Lemma 4.38.** The matrices representing \( A \) and \( B \) with respect to the \( \mathbb{F} \)-basis
\[
\frac{1}{2i} v_i + E_d(a, b, c)^{(-1,-1)}(c) \quad \text{for } i = 0, 2, \ldots, d - 1
\]
for the \( \mathcal{R} \)-module \( E_d(a, b, c)^{(-1,-1)}(c) / E_d(a, b, c)^{(-1,-1)}(c) \) are
\[
\begin{pmatrix}
\theta_0 & \theta_1 & 0 \\
1 & \theta_1 & \theta_2 \\
0 & \theta_{d+1} & \theta_d \\
\end{pmatrix},
\begin{pmatrix}
\theta_0^* & \varphi_1 \\
\theta_1^* & \varphi_2 \\
\theta_{d+1}^* & \varphi_{d+1} \\
\end{pmatrix}.
\]
respectively, where

\[
\begin{align*}
\theta_i &= \frac{(2a - d + 4i - 1)(2a - d + 4i + 3)}{16} \quad (0 \leq i \leq \frac{d-1}{2}), \\
\theta_i^* &= \frac{(2b - d + 4i - 1)(2b - d + 4i + 3)}{16} \quad (0 \leq i \leq \frac{d-1}{2}), \\
\varphi_i &= \frac{i(2i - d - 1)(2a + 2b + 2c - d + 4i + 1)(2a + 2b - 2c - d + 4i - 3)}{32} \quad (1 \leq i \leq \frac{d-1}{2}).
\end{align*}
\]

The element \( \delta \) acts on the \( \mathcal{R} \)-module \( E_d(a, b, c)^{(-1,-1)} / E_d(a, b, c)^{(-1,-1)}(c) \) as scalar multiplication by

\[
\frac{(d-1)(d+3)}{16} + \frac{(a-1)(a+1)}{4} + \frac{(b-1)(b+1)}{4} + \frac{c(c+2)}{4}.
\]

**Proof.** By Lemma 4.34 the cosets \( \{\bar{a}\} \) are an \( \mathbb{F} \)-basis for \( E_d(a, b, c)^{(-1,-1)} / E_d(a, b, c)^{(-1,-1)}(c) \). Applying Lemma 4.35 a direct calculation yields the matrices representing \( A \) and \( B \) with respect to \( \{\bar{a}\} \). By Lemma 4.32 the element \( t_{\theta} \) acts on \( E_d(a, b, c)^{(-1,-1)} / E_d(a, b, c)^{(-1,-1)}(c) \) as scalar multiplication by \(-c\). Combined with Theorem 2.41 and Lemma 3.5 the element \( \delta \) acts on \( E_d(a, b, c)^{(-1,-1)} / E_d(a, b, c)^{(-1,-1)}(c) \) as scalar multiplication by \( \varphi_{d-1} \). The lemma follows. \( \square \)

**Proposition 4.39.** The \( \mathcal{R} \)-module \( E_d(a, b, c)^{(-1,-1)} / E_d(a, b, c)^{(-1,-1)}(c) \) is isomorphic to

\[
R_{\mathcal{A}^{(2)}} \left( -\frac{a+1}{2}, -\frac{b+1}{2}, \frac{c}{2} - 1 \right).
\]

Moreover the \( \mathcal{R} \)-module \( E_d(a, b, c)^{(-1,-1)} / E_d(a, b, c)^{(-1,-1)}(c) \) is irreducible provided that the \( \mathcal{S} \)-module \( E_d(a, b, c)^{(-1,-1)} \) is irreducible.

**Proof.** Let \( (a', b', c', d') = (-\frac{a+1}{2}, -\frac{b+1}{2}, \frac{c}{2} - 1, \frac{d-1}{2}) \). Comparing Proposition 3.1 with Lemma 4.38 yields that the quotient \( \mathcal{R} \)-module \( E_d(a, b, c)^{(-1,-1)} / E_d(a, b, c)^{(-1,-1)}(c) \) is isomorphic to \( R_{\mathcal{A}}(a', b', c') \). Suppose that the \( \mathcal{S} \)-module \( E_d(a, b, c)^{(-1,-1)} \) is irreducible. By Proposition 3.6 we have

\[
a' + b' + c' + 1', -a' + b' + c', a' - b' + c' \notin \left\{ \frac{d'}{2} - i \mid i = 1, 2, \ldots, d' + 1 \right\}
\]

and

\[
a' + b' - c' \notin \left\{ \frac{d'}{2} - i \mid i = 0, 1, \ldots, d' \right\}.
\]

Combined with Proposition 3.2 the \( \mathcal{R} \)-module \( R_{\mathcal{A}}(a', b', c') \) is irreducible. The proposition follows. \( \square \)

**Theorem 4.40.** Assume that the \( \mathcal{S} \)-module \( E_d(a, b, c)^{(-1,-1)} \) is irreducible. Then the following hold:

(i) If \( c = 0 \) then
is the lattice of $\mathbb{R}$-submodules of $E_d(a, b, c)^{(-1, -1)}$.

(ii) If $c \neq 0$ then

\[
\begin{array}{ccc}
E_d(a, b, c)^{(-1, -1)} & \rightarrow & E_d(a, b, c)^{(-1, -1)}(-c) \\
\downarrow & & \downarrow \\
\{0\} & \rightarrow & \{0\}
\end{array}
\]

is the lattice of $\mathbb{R}$-submodules of $E_d(a, b, c)^{(-1, -1)}$.

Proof. Using the above lemmas and propositions, the result follows by an argument similar to the proof of Theorem 4.22.

\[\square\]

4.6. The lattice of $\mathbb{R}$-submodules of $O_d(a, b, c)$. Throughout this subsection we adopt the notation of §3.3.

Lemma 4.41. The matrix representing $t_0$ with respect to the $\mathbb{F}$-basis

\[
v_0, \quad v_i - iv_{i-1} \quad \text{for } i = 2, 4, \ldots, d, \quad v_i \quad \text{for } i = 1, 3, \ldots, d - 1
\]

for $O_d(a, b, c)$ is

\[
\begin{pmatrix}
\frac{\sigma^2}{2} & 0 & 0 \\
0 & -\frac{\sigma^2}{2}I_d & I_d \\
0 & 0 & -\frac{\sigma^2}{2}I_d
\end{pmatrix}
\]

Proof. It is straightforward to verify the lemma by using Proposition 3.8.

\[\square\]

Lemma 4.42. (i) If $d = 0$ then $t_0$ is diagonalizable on $O_d(a, b, c)$ with exactly one eigenvalue $\frac{\sigma}{2}$.

(ii) If $d \geq 2$ and $a + b + c = \frac{d+1}{2}$ then $t_0$ is not diagonalizable on $O_d(a, b, c)$ with exactly one eigenvalue 0.

(iii) If $d \geq 2$ and $a + b + c \neq \frac{d+1}{2}$ then $t_0$ is diagonalizable on $O_d(a, b, c)$ with exactly two eigenvalues $\pm \frac{\sigma}{2}$.

Proof. Immediate from Lemma 4.41.

\[\square\]

Lemma 4.43. $O_d(a, b, c)(\frac{\sigma}{2})$ is of dimension $\frac{d}{2} + 1$ with the $\mathbb{F}$-basis

\[
v_0, \quad v_i - iv_{i-1} \quad \text{for } i = 2, 4, \ldots, d.
\]

Proof. Immediate from Lemma 4.41.

\[\square\]
Lemma 4.44. The actions of $A$ and $B$ on the $\mathfrak{N}$-module $O_d(a, b, c)$ are as follows:

$$Av_i = \begin{cases} 
\theta_i v_i - \frac{1}{4} v_{i+1} + \frac{1}{4} v_{i+2} & \text{for } i = 0, 2, \ldots, d - 2, \\
\theta_i v_i + \frac{1}{4} v_{i+2} & \text{for } i = 1, 3, \ldots, d - 3,
\end{cases}$$

$$Av_{d-1} = \theta_{d-1} v_{d-1}, \quad Av_d = \theta_d v_d.$$

$$Bv_i = \begin{cases} 
\theta_i^* v_i + \frac{i(i - d - 2)(\sigma + i)(\tau + i - 1)}{4} v_{i-2} & \text{for } i = 2, 4, \ldots, d, \\
\theta_i^* v_i + \frac{(i - d - 1)(\tau + i)}{2} \left( v_{i-1} + \frac{(i - 1)(\sigma + i - 1)}{2} v_{i-2} \right) & \text{for } i = 3, 5, \ldots, d - 1,
\end{cases}$$

$$Bv_0 = \theta_0^* v_0, \quad Bv_1 = \theta_1^* v_1 - \frac{d(\tau + 1)}{2} v_0.$$ 

where

$$\theta_i = \left( \frac{a}{2} - \frac{d + 3}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \left( \frac{a}{2} - \frac{d - 1}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \quad (0 \leq i \leq d),$$

$$\theta_i^* = \left( \frac{b}{2} - \frac{d + 3}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \left( \frac{b}{2} - \frac{d - 1}{4} + \left\lfloor \frac{i}{2} \right\rfloor \right) \quad (0 \leq i \leq d).$$

Proof. Apply Theorem 2.4 and Proposition 3.8 to evaluate the actions of $A$ and $B$ on $O_d(a, b, c)$. \qed

Lemma 4.45. The matrices representing $A$ and $B$ with respect to the $\mathcal{F}$-basis

$$v_0, \quad \frac{1}{2^i} (v_i - iv_{i-1}) \quad \text{for } i = 2, 4, \ldots, d$$

for the $\mathcal{R}$-module $O_d(a, b, c)(\frac{\mathcal{F}}{}$)

are

$$\begin{pmatrix}
\theta_0 & 0 & 0 & \\
\theta_1 & \theta_2 & 0 & \\
\vdots & \vdots & \ddots & \\
0 & 1 & \theta_4 & 
\end{pmatrix}, \quad \begin{pmatrix}
\theta_0^* & \varphi_1 & 0 & \\
\varphi_1 & \theta_2^* & \varphi_2 & \\
\vdots & \vdots & \ddots & \\
0 & \varphi_3 & \varphi_4 & 
\end{pmatrix},$$

respectively, where

$$\theta_i = \frac{(2a - d + 4i - 3)(2a - d + 4i + 1)}{16} \quad (0 \leq i \leq \frac{d}{2}),$$

$$\theta_i^* = \frac{(2b - d + 4i - 3)(2b - d + 4i + 1)}{16} \quad (0 \leq i \leq \frac{d}{2}),$$

$$\varphi_i = \frac{i(2i - d - 2)(2a + 2b + 2c - d + 4i - 5)(2a + 2b - 2c - d + 4i - 3)}{32} \quad (1 \leq i \leq \frac{d}{2}).$$

The element $\delta$ acts on the $\mathcal{R}$-module $O_d(a, b, c)(\frac{\mathcal{F}}{)}$ as scalar multiplication by

$$\frac{d(d + 4)}{16} + \frac{(2a - 3)(2a + 1)}{16} + \frac{(2b - 3)(2b + 1)}{16} + \frac{(2c - 3)(2c + 1)}{16}. $$
Proof. By Lemma 4.43 the vectors (34) are an R-basis for \( O_d(a, b, c)(\frac{\sigma}{2}) \). Applying Lemma 4.44 a straightforward calculation yields the matrices representing \( A \) and \( B \) with respect to (34). Applying Theorem 2.4 and Lemma 3.9 yields that \( \delta \) acts on \( O_d(a, b, c)(\frac{\sigma}{2}) \) as scalar multiplication by (35). The lemma follows. \( \square \)

**Proposition 4.46.** The R-module \( O_d(a, b, c)(\frac{\sigma}{2}) \) is isomorphic to

\[
R_{\frac{\sigma}{2}} \left( \frac{a}{2} - \frac{1}{4}, \frac{b}{2} - \frac{1}{4}, \frac{c}{2} - \frac{1}{4} \right).
\]

Moreover the R-module \( O_d(a, b, c)(\frac{\sigma}{2}) \) is irreducible provided that \( a + b + c \neq \frac{d+1}{2} \) and the \( H \)-module \( O_d(a, b, c) \) is irreducible.

Proof. Set \( (a', b', c', d') = \left( -\frac{a}{2} - \frac{1}{4}, -\frac{b}{2} - \frac{1}{4}, -\frac{c}{2} - \frac{1}{4}, \frac{d}{2} - \frac{1}{4} \right) \). Comparing Proposition 3.1 with Lemma 4.47 yields that the \( R \)-module \( O_d(a, b, c)(\frac{\sigma}{2}) \) is isomorphic to \( R_{d'}(a', b', c') \). Suppose that \( a + b + c \neq \frac{d+1}{2} \) and the \( H \)-module \( O_d(a, b, c) \) is irreducible. It follows from Proposition 3.10 that

\[
a' + b' + c' + 1 \not\in \left\{ \frac{d'}{2} - i \bigg| i = 0, 1, \ldots, d' - 1 \right\}
\]

and

\[-a' + b' + c', a' - b' + c', a' + b' - c' \not\in \left\{ \frac{d'}{2} - i \bigg| i = 1, 2, \ldots, d' \right\}.
\]

By the assumption \( a + b + c \neq \frac{d+1}{2} \) we have \( a' + b' + c' + 1 \neq \frac{d'}{2} \). By Proposition 3.2 the R-module \( R_{d'}(a', b', c') \) is irreducible. The proposition follows. \( \square \)

**Lemma 4.47.** Assume that \( d \geq 2 \). The matrices representing \( A \) and \( B \) with respect to the R-basis

\[
(36) \quad \frac{1}{2^{i-1}} v_i + O_d(a, b, c)(\frac{\sigma}{2}) \quad \text{for } i = 1, 3, \ldots, d - 1
\]

for the \( R \)-module \( O_d(a, b, c)/O_d(a, b, c)(\frac{\sigma}{2}) \) are

\[
\begin{pmatrix}
\theta_0 & 0 \\
1 & \theta_1 \\
\vdots & \ddots \\
0 & \cdots & \cdots & 1 & \theta_{\frac{d}{2}-1}
\end{pmatrix},
\begin{pmatrix}
\theta_0^* & \varphi_1 & 0 \\
\theta_1^* & \varphi_2 & \ddots \\
\vdots & \ddots & \ddots \\
0 & \cdots & \cdots & \varphi_{\frac{d}{2}-1}
\end{pmatrix}
\]

respectively, where

\[
\theta_i = \frac{(2a - d + 4i + 1)(2a - d + 4i + 5)}{16} \quad (0 \leq i \leq \frac{d}{2} - 1),
\]

\[
\theta_i^* = \frac{(2b - d + 4i + 1)(2b - d + 4i + 5)}{16} \quad (0 \leq i \leq \frac{d}{2} - 1),
\]

\[
\varphi_i = \frac{i(2i - d)(2a + 2b + 2c + d + 4i + 3)(2a + 2b - 2c - d + 4i + 1)}{32} \quad (1 \leq i \leq \frac{d}{2} - 1).
\]

The element \( \delta \) acts on \( O_d(a, b, c)/O_d(a, b, c)(\frac{\sigma}{2}) \) as scalar multiplication by

\[
(37) \quad \frac{d^2 - 13}{16} + \frac{a(a + 1)}{4} + \frac{b(b + 1)}{4} + \frac{c(c + 1)}{4}.
\]
Proof. By Lemma 4.43 the cosets (36) are an $\mathbb{F}$-basis for $O_d(a,b,c)/O_d(a,b,c)(\frac{a}{2})$. Applying Lemma 4.44 a direct calculation yields the matrices representing $A$ and $B$ with respect to (36). By Theorem 2.4 and Lemma 3.9 the element $\delta$ acts on $O_d(a,b,c)/O_d(a,b,c)(\frac{a}{2})$ as scalar multiplication by (37). The lemma follows. \hfill $\Box$

Proposition 4.48. Assume that $d \geq 2$. Then the $\mathbb{R}$-module $O_d(a,b,c)/O_d(a,b,c)(\frac{a}{2})$ is isomorphic to

$$R_{\frac{d}{2} - 1} \left( \begin{array}{cccc} -a & 3 & b & 3 \\ 4 & -2 & 4 & -2 \\ 2 & 2 & 2 & 2 \\ \end{array} \right).$$

Moreover the $\mathbb{R}$-module $O_d(a,b,c)/O_d(a,b,c)(\frac{a}{2})$ is irreducible provided that the $\mathbb{S}$-module $O_d(a,b,c)$ is irreducible.

Proof. Set $(a',b',c',d') = (-\frac{a}{2} - \frac{3}{4}, -\frac{b}{2} - \frac{3}{4}, -\frac{c}{2} - \frac{3}{4}, -\frac{d}{2} - 1)$. Comparing Proposition 3.1 with Lemma 4.47 it follows that the $\mathbb{R}$-module $O_d(a,b,c)/O_d(a,b,c)(\frac{a}{2})$ is isomorphic to $R_d(a',b',c')$. Suppose that the $\mathbb{S}$-module $O_d(a,b,c)$ is irreducible. Using Proposition 3.10 yields that

$$a' + b' + c' + 1, -a' + b' + c', a' - b' + c', a' + b' - c' \not\in \left\{ \frac{d'}{2} - i \mid i = 1, 2, \ldots, d' + 1 \right\}.$$ 

By Proposition 3.2 the $\mathbb{R}$-module $R_d(a',b',c')$ is irreducible. The proposition follows. \hfill $\Box$

For the rest of this subsection we let $O_d(a,b,c)(0)'$ denote the $\mathbb{F}$-subspace of $O_d(a,b,c)(0)$ spanned by

$$v_i - iv_{i-1} \quad \text{for all } i = 2, 4, \ldots, d.$$

Lemma 4.49. Assume that $d \geq 2$ and $a + b + c = \frac{d + 1}{2}$. Then $O_d(a,b,c)(0)'$ is an $\mathbb{R}$-module and the actions of $A, B, \delta$ on $O_d(a,b,c)(0)'$ are as follows: The matrices representing $A$ and $B$ with respect to the $\mathbb{F}$-basis

$$(38) \quad \frac{1}{2i - 2}(v_i - iv_{i-1}) \quad \text{for } i = 2, 4, \ldots, d$$

for the $\mathbb{R}$-module $O_d(a,b,c)(0)'$ are

$$\begin{pmatrix} \theta_0 & 0 \\ 1 & \theta_1 \\ \vdots & \vdots \\ 0 & \theta_{\frac{d}{2} - 1} \end{pmatrix}, \quad \begin{pmatrix} \theta_0^* & \varphi_1 \\ \theta_1^* & \varphi_2 \\ \vdots & \vdots \\ \theta_{\frac{d}{2} - 1}^* & \varphi_{\frac{d}{2} - 1} \end{pmatrix}$$

respectively, where

$$\theta_i = \frac{(2a - d + 4i + 1)(2a - d + 4i + 5)}{16} \quad (0 \leq i \leq \frac{d}{2} - 1),$$

$$\theta_i^* = \frac{(2b - d + 4i + 1)(2b - d + 4i + 5)}{16} \quad (0 \leq i \leq \frac{d}{2} - 1),$$

$$\varphi_i = \frac{i(2i - d)(2a + 2b + 2c - d + 4i + 3)(2a + 2b - 2c - d + 4i + 1)}{32} \quad (1 \leq i \leq \frac{d}{2} - 1).$$

The element $\delta$ acts on $O_d(a,b,c)(0)'$ as scalar multiplication by

$$(39) \quad \frac{d^2 + 13}{16} + \frac{a(a + 1)}{4} + \frac{b(b + 1)}{4} + \frac{c(c + 1)}{4}.$$
Proof. It follows from Lemma 4.45 that \( O_d(a, b, c)(0)' \) is invariant under \( A \) and \( \delta \); under the assumption \( a + b + c = \frac{d+1}{2} \) it is also invariant under \( B \). Hence \( O_d(a, b, c)(0)' \) is an \( \mathbb{R} \)-module by Lemma 2.2(ii).

By Lemma 4.45 the matrix representing \( A \) with respect to the \( F \)-basis (38) for \( O_d(a, b, c)(0)' \) is as stated. Under the assumption \( a + b + c = \frac{d+1}{2} \) the matrix representing \( B \) with respect to (38) is as stated and the scalars (35) and (39) are identical. The lemma follows. \( \square \)

**Proposition 4.50.** Assume that \( d \geq 2 \) and \( a + b + c = \frac{d+1}{2} \). Then the \( \mathbb{R} \)-module \( O_d(a, b, c)(0)' \) is isomorphic to

\[
R_{\frac{d}{2}-1} \left( \frac{-a}{2} - \frac{3}{4}, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{4} \right).
\]

Moreover the \( \mathbb{R} \)-module \( O_d(a, b, c)(0)' \) is irreducible provided that the \( \mathcal{H} \)-module \( O_d(a, b, c) \) is irreducible.

**Proof.** Similar to the proof of Proposition 4.48. \( \square \)

**Theorem 4.51.** Assume that the \( \mathcal{H} \)-module \( O_d(a, b, c) \) is irreducible. Then the following hold:

(i) If \( d = 0 \) then the \( \mathbb{R} \)-module \( O_d(a, b, c) \) is irreducible.

(ii) If \( d \geq 2 \) and \( a + b + c = \frac{d+1}{2} \) then

\[
\begin{align*}
O_d(a, b, c) \\
O_d(a, b, c)(0) \\
O_d(a, b, c)(0)' \\
\{0\}
\end{align*}
\]

is the lattice of \( \mathbb{R} \)-submodules of \( O_d(a, b, c) \).

(iii) If \( d \geq 2 \) and \( a + b + c \neq \frac{d+1}{2} \) then

\[
\begin{align*}
O_d(a, b, c) \\
O_d(a, b, c)(-\frac{a}{2}) \\
O_d(a, b, c)(\frac{a}{2}) \\
\{0\}
\end{align*}
\]

is the lattice of \( \mathbb{R} \)-submodules of \( O_d(a, b, c) \).

**Proof.** (i): If \( d = 0 \) then \( O_d(a, b, c) \) is one-dimensional and hence an irreducible \( \mathbb{R} \)-module.

(ii): Suppose that \( d \geq 2 \) and \( a + b + c = \frac{d+1}{2} \). Since the \( \mathbb{R} \)-submodule \( O_d(a, b, c)(0)' \) of \( O_d(a, b, c) \) is of codimension 1, the quotient \( \mathbb{R} \)-module \( O_d(a, b, c)/O_d(a, b, c)(0)' \) is irreducible. Combined with Propositions 4.48 and 4.50 the sequence

\[
\{0\} \subset O_d(a, b, c)(0)' \subset O_d(a, b, c)(0) \subset O_d(a, b, c)
\]

(40)
is a composition series for the \( \mathbb{R} \)-module \( O_d(a, b, c) \).

By Proposition 4.4 and Lemma 4.14(ii), every irreducible \( \mathbb{R} \)-submodule of \( O_d(a, b, c) \) is contained in \( O_d(a, b, c)(0) \). To see (iii), it remains to show that \( O_d(a, b, c)(0)' \) is the unique irreducible \( \mathbb{R} \)-submodule of \( O_d(a, b, c)(0) \). Suppose on the contrary that \( W \) is an irreducible \( \mathbb{R} \)-submodule \( O_d(a, b, c)(0) \) different from \( O_d(a, b, c)(0)' \). By irreducibility, we have \( O_d(a, b, c)(0)' \cap W = \{0\} \). Since \( O_d(a, b, c)(0)' \) is of codimension 1 in \( O_d(a, b, c) \), it follows that \( W \) is of dimension 1 and

\[
O_d(a, b, c)(0) = O_d(a, b, c)(0)' \oplus W.
\]

Applying Jordan–Hölder theorem to (40) the one-dimensional \( \mathbb{R} \)-module \( W \) is isomorphic to \( O_d(a, b, c)(0)' \) when \( d = 2 \) or \( O_d(a, b, c)(0)/O_d(a, b, c)(0)' \).

First we suppose that \( d = 2 \) and the \( \mathbb{R} \)-module \( W \) is isomorphic to \( O_d(a, b, c)(0)' \). By Lemma 4.45 the eigenvalues of \( A \) in \( O_d(a, b, c)(0)' \) are \( \frac{(2a-5)(2a-1)}{16} \) and \( \frac{(2a-1)(2a+3)}{16} \). By Lemma 4.49 the eigenvalue of \( A \) is \( O_d(a, b, c)(0)' \) is \( \frac{(2a-1)(2a+3)}{16} \). Combined with (41) this implies

\[
(2a - 5)(2a - 1) = (2a - 1)(2a + 3),
\]

Solving (42) for \( a \) yields that \( a = \frac{1}{2} \). By considering the eigenvalues of \( B \) in \( O_d(a, b, c)(0) \) and \( O_d(a, b, c)(0)' \), a similar argument implies \( b = \frac{1}{2} \). Moreover \( c = \frac{1}{2} \) by the assumption \( a + b + c = \frac{d+1}{2} \). Then

\[
a - b - c = -a + b - c = -a - b + c = -\frac{1}{2}.
\]

This leads to a contradiction to the irreducibility of the \( \mathfrak{N} \)-module \( O_d(a, b, c) \) by Proposition 3.10.

Next we suppose that \( W \) is isomorphic to \( O_d(a, b, c)(0)/O_d(a, b, c)(0)' \). By Lemma 4.45 the elements \( A \) and \( B \) act on \( O_d(a, b, c)(0)/O_d(a, b, c)(0)' \) as the scalars \( \frac{(2a-d-3)(2a-d+1)}{16} \) and \( \frac{(2b-d-3)(2b-d+1)}{16} \), respectively. By Lemma 4.45 the \( \frac{(2a-d-3)(2a-d+1)}{16} \)-eigenspace of \( A \) in \( O_d(a, b, c)(0) \) is one-dimensional and hence is equal to \( W \). Consequently \( W \) contains a vector \( w \) in which the coefficient of \( \frac{1}{4}(v_d - dv_{d-1}) \) with respect to the \( \mathbb{F} \)-basis (34) for \( O_d(a, b, c)(0) \) is 1. By Lemma 4.45 the coefficient of \( \frac{1}{4}(v_d - dv_{d-1}) \) in \( Bw \) with respect to (34) is \( \frac{(2b-d-3)(2b-d+1)}{16} \). Since \( w \) is a \( \frac{(2b-d-3)(2b-d+1)}{16} \)-eigenvector of \( B \) it follows that

\[
(2b - d - 3)(2b - d + 1) = (2b + d - 3)(2b + d + 1).
\]

Solving (43) for \( b \) yields that \( b = \frac{1}{2} \). Combined with the assumption \( a + b + c = \frac{d+1}{2} \) we have

\[
-a + b - c = \frac{1 - d}{2}.
\]

This leads to a contradiction to the irreducibility of the \( \mathfrak{N} \)-module \( O_d(a, b, c) \) by Proposition 3.10. We have shown that \( O_d(a, b, c)(0)' \) is the unique irreducible \( \mathbb{R} \)-submodule of \( O_d(a, b, c)(0) \). Therefore (ii) follows.

(iii): Using the above lemmas and propositions, the statement (iii) follows by an argument similar to the proof of Theorem 4.13(ii).

\[\square\]

5. The summary

We summarize the results of 4.2–4.6 as follows:
Theorem 5.1. Let $V$ denote a finite-dimensional irreducible $\mathfrak{H}$-module. Given any $\theta \in \mathbb{F}$ let $V(\theta)$ denote the null space of $t_0 - \theta$ in $V$. Then the following hold:

(i) Suppose that $t_0$ is not diagonalizable on $V$. Then 0 is the unique eigenvalue of $t_0$ in $V$. Moreover the following hold:

(a) If the dimension of $V$ is even then the lattice of $\mathbb{R}$-submodules of $V$ is as follows:

\[
\begin{array}{c}
V \\
\mid \\
V(0) \\
\mid \\
\{0\}
\end{array}
\]

(b) If the dimension of $V$ is odd then the lattice of $\mathbb{R}$-submodules of $V$ is as follows:

\[
\begin{array}{c}
V \\
\mid \\
V(0) \\
\mid \\
V(0)' \\
\mid \\
\{0\}
\end{array}
\]

Here $V(0)'$ is the irreducible $\mathbb{R}$-submodule of $V(0)$ that has codimension 1.

(ii) Suppose that $t_0$ is diagonalizable on $V$. Then there are at most two eigenvalues of $t_0$ in $V$. Moreover the following hold:

(a) If $t_0$ has exactly one eigenvalue in $V$ then the $\mathbb{R}$-module $V$ is irreducible of dimension less than or equal to 2.

(b) If $t_0$ has exactly two eigenvalues in $V$ then there exists a nonzero scalar $\theta \in \mathbb{F}$ such that $\pm \theta$ are the eigenvalues of $t_0$ and the lattice of $\mathbb{R}$-submodules of $V$ is as follows:

\[
\begin{array}{c}
V \\
\mid \\
V(-\theta) \\
\mid \\
V(\theta) \\
\mid \\
\{0\}
\end{array}
\]

As byproducts of Theorem 5.1 we have the following corollaries:

Corollary 5.2. Let $V$ denote a finite-dimensional irreducible $\mathfrak{H}$-module. If $\theta$ is an eigenvalue of $t_0$ in $V$ then either $V = V(\theta)$ or the $\mathbb{R}$-module $V/V(\theta)$ is irreducible.

Corollary 5.3. For any finite-dimensional irreducible $\mathfrak{H}$-module $V$, the $\mathbb{R}$-module $V$ is completely reducible if and only if $t_0$ is diagonalizable on $V$. 

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