Craig Interpolation and Access Interpolation with Clausal First-Order Tableaux

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Abstract We develop foundations for computing Craig interpolants and similar intermediates of two given formulas with first-order theorem provers that construct clausal tableaux. Provers that can be understood in this way include efficient machine-oriented systems based on calculi of two families: goal-oriented like model elimination and the connection method, and bottom-up like the hyper tableau calculus. The presented method for Craig-Lyndon interpolation involves a lifting step where terms are replaced by quantified variables, similar as known for resolution-based interpolation, but applied to a differently characterized ground formula and proven correct more abstractly on the basis of Herbrand’s theorem, independently of a particular calculus. Access interpolation is a recent form of interpolation for database query reformulation that applies to first-order formulas with relativized quantifiers and constrains the quantification patterns of predicate occurrences. It has been previously investigated in the framework of Smullyan’s non-clausal tableaux. Here, in essence, we simulate these with the more machine-oriented clausal tableaux through structural constraints that can be ensured either directly by bottom-up tableau construction methods or, for closed clausal tableaux constructed with arbitrary calculi, by postprocessing with restructuring transformations.

Keywords Craig interpolation · first-order theorem proving · clausal tableaux · connection method · hyper tableaux · interpolant lifting · query reformulation · relativized quantifiers

1 Introduction

By Craig’s interpolation theorem [19], for two first-order formulas $F$ and $G$ such that $F$ entails $G$ there exists a third first-order formula $H$ that is entailed by $F$, entails $G$ and is such that all predicate and function symbols occurring in it occur in both $F$ and $G$. Such a Craig interpolant $H$ can be constructed from given formulas $F$ and $G$, for example by a calculus that allows to extract $H$ from a proof that $F$ entails $G$, or, equivalently, that the implication $F \rightarrow G$ is valid. Automated construction of interpolants has many applications, in the area of computational logic most notably in symbolic model checking, initiated with [43], and in query reformulation [42,48,14,59,17,8,30,10,9,60]. The foundation for the
latter application field is the observation that a reformulated query can be viewed as a *definiens* of a given query where only symbols from a given set, the target language of the reformulation, occur in the definiens. The existence of such definientia, that is, definability \[58\], or *determinacy* as it is called in the database context, can be expressed as validity and their synthesis as interpolant construction. For example, a *definiens* $H$ of a unary predicate $p$ within a first-order formula $F$ can be characterized by the following conditions:

1. $F$ entails $\forall x (p(x) \leftrightarrow H)$.
2. $p$ does not occur in $H$.

The variable $x$ is allowed there to occur free in $H$. We further assume that $x$ does not occur free in $F$ and let $F'$ denote $F$ with $p$ replaced by a fresh symbol $p'$. Now the characterization of *definiens* by the two conditions given above can be equivalently expressed as

$$H \text{ is a Craig interpolant of the two formulas } F \land p(x) \text{ and } \neg(F' \land \neg p'(x)).$$

A *definiens* $H$ exists if and only if it is valid that the first formula implies the second one.

The construction of Craig interpolants of given first-order formulas has been elegantly specified in the framework of tableaux by Smullyan \[56,24\]. Although this has been taken as foundation for applications of interpolation in query reformulation \[59,10\], it has been hardly used as a basis for the practical computation of first-order interpolants with automated reasoning systems, where the focus so far has been on interpolant extraction from specially constrained resolution proofs (see \[13,35\] for recent overviews and discussions).

Here we approach the computation of interpolants from another paradigm of automated reasoning, the construction of a *clausal tableau*. Expectations are that, on the one hand, the elegance of Smullyan’s interpolation method for non-clausal tableaux can be utilized and, on the other hand, the foundation for efficient practical implementations is laid. Various efficient theorem proving methods can be viewed as operating by constructing a clausal tableau \[37\] (or *clause tableau* \[31\]). They can be roughly divided into two major families: First, methods that are goal-sensitive, typically proceeding with the tableau construction “top-down”, by “backward chaining”, starting with clauses from the theorem in contrast to the axioms. Aside of clausal tableaux in the literal sense, techniques to specify and investigate such methods include model elimination \[40\], the connection method \[11\], and the Prolog technology theorem prover \[57\]. One of the leading first-order proving systems of the 1990s, SETHEO \[26\], followed that approach. The *leanCoP* system \[50\] along with its recent derivations \[32,33\] as well as the *CM* component of *PIE* \[21,64\] are implementations in active duty today. The second major family of methods constructs clausal tableaux “bottom-up”, in a “forward-chaining” manner, by starting with positive axioms and deriving positive consequences. With the focus of their suitability to construct model representations, these methods have been called *bottom-up model generation (BUMG)* methods \[5\]. They include, for example, SATCHMO \[16\] and the hyper tableau calculus \[4\], with implementations such as Hyper. formerly called *E-KRHyper* \[52,62,7\]. Hyper tableau methods are also used in high-performance description logic reasoners \[46\]. It appears that the chase method from the database field, which recently got attention anew in knowledge representation (see, e.g., \[27\]), can also be understood as such a bottom-up tableau construction. Methods of the instance-based approach to theorem proving (see \[6\] for an overview) should in general be applicable to construct a clausal tableau after proving, from the instances involved in the proof, although the proof construction itself might not proceed by tableau construction. For a systematic overview of different variants of tableaux structures and methods, including clausal tableaux with respect to both considered major paradigms see \[31\].

An essential distinction of clausal tableau methods from resolution-based methods is that at the tableau construction only instances of *input clauses* are created and incorporated.
Clauses are not broken apart and joined as in a resolution step. Nevertheless, clausal tableau methods might be complemented by preprocessors that perform such operations. An essential distinction from non-clausal tableau methods is that with the clausal form only a particularly simple formula structuring has to be considered, in essence sets of clauses. Through preprocessing with conversion to prenex form and Skolemization, the handling of quantifications amounts for clausal tableau methods just to the handling of free variables.

The tableau-based method for Craig interpolation presented here proceeds in two stages, with some similarity to resolution-based methods discussed in [29,3,13,35] that compute in a first stage a so-called relational, weak or provisional interpolant which satisfies the vocabulary restriction on interpolants with respect to predicate symbols but not necessarily with respect to function and constant symbols. The result of the first stage is in the second stage lifted to an actual interpolant of the original input formulas by replacing terms with variables and prepending a specific quantifier prefix. In our tableau-based method the two stages are separated at a different place, more directly related to Herbrand’s theorem, without need of an additional notion such as relational interpolant. In the first stage an actual Craig interpolant of a finite unsatisfiable subset of the Herbrand expansion of the Skolemized and clausified input formulas is constructed. The involved ground clauses can be obtained as instances of clauses of the closed tableau computed by a first-order prover for a set of first-order clauses. With respect to interpolation, the closed clausal tableau can be considered just as given, abstracting from the method by which it has been constructed. This leads to a lean formalism for interpolation and justifies the practical implementation of Craig interpolation with arbitrary high-performance first-order theorem provers that construct clausal tableaux, without need to modify inference rules or other prover internals.

There are many known ways to strengthen Craig’s interpolation theorem by ensuring that for given formulas \( F \) and \( G \) that satisfy certain syntactic restrictions there exists an interpolant \( H \) that also satisfies certain syntactic restrictions. For example, that predicates occur in \( H \) only with polarities with which they occur in both \( F \) and \( G \). (A predicate occurs with positive (negative) polarity in a formula if it occurs there in the scope of an even (odd) number of negation operators.) The respective strengthened interpolation theorem has been explicated by Lyndon [41], hence we call Craig interpolants that meet this restriction Craig-Lyndon interpolants. Access interpolation [10] is a variant of Craig-Lyndon interpolation that applies to formulas in which quantifiers only occur relativized by atoms, as for example in

\[
\forall x \ (r(x) \rightarrow \exists y \exists z \ (s(x, y, z) \land true)).
\]  

With each occurrence of a relativizing atom a binding pattern or access pattern is associated, which comprises the predicate, the polarity of the occurrence and the argument positions of those variables that are not quantified by the associated quantifier. For example, in (i) we have for the occurrence of \( r(x) \) the predicate \( r \) in negative polarity with the empty set of argument positions and for the occurrence of \( s(x, y, z) \) the predicate \( s \) in positive polarity and the set \{1\} of argument positions, because \( x \) at the first argument position in the occurrence of \( s(x, y, z) \) is not quantified by \( \exists y \exists z \). Positions specified in the set are also called input positions, while the quantified positions are output positions, corresponding to their role in a naive formula evaluation. Access interpolation strengthens Craig-Lyndon interpolation by requiring that also the binding patterns occurring in the interpolant formula are subsumed by binding patterns occurring in a specific way in the input formulas.

In [10] it has been shown that many tasks in database query reformulation can be expressed in terms of access interpolation, applied to construct definientia of queries that are in a certain vocabulary and involve only certain binding patterns which makes them evaluable
in a certain sense. A variant of Craig-Lyndon interpolation by Otto [51] has been suggested in [48] as a technique to take relativization into account. In [10] access interpolation is presented as a generalization of Otto’s interpolation and constructively proven on the basis of Smullyan’s tableau method following the presentation in [24].

Access interpolants involve only relativized quantification, which seems incompatible with a global quantifier prefix as computed by the lifting technique sketched above for Craig interpolation, at least if predicates used as relativizers are permitted to have empty extensions. Hence, the method for access interpolation presented here extracts the interpolant from a tableau in a single stage, where a form of lifting that only applies to subformulas corresponding to scopes of relativized quantifiers is incorporated. In essence, Smullyan’s techniques for non-clausal tableau are simulated with the more machine-oriented clausal tableaux and variable handling through Skolemization. Correspondence to Smullyan’s tableaux is achieved by a structure preserving normal form and certain structural requirements on the clausal tableaux. These are already met by hyper tableaux. In the general case they can be ensured with restructuring transformations, applied in a postprocessing step to closed clausal tableaux obtained from provers.

The contributions of this work can be summarized as follows:

1. Foundations to perform Craig interpolation and related forms of interpolation for first-order logic with clausal tableau methods are developed. They provide:
   (a) A basis for implementing interpolation with efficient machine-oriented theorem provers for first-order logic that can be understood as constructing clausal tableaux. With methods and systems of two main families, goal-oriented “top-down” and forward-chaining “bottom-up”, there is a wide range of potential applications.
   (b) A relatively simple framework to prove constructively the existence of interpolants with further syntactic properties, beyond the restriction on symbols required by Craig interpolants. The involved constructions are, moreover, suited for realization by practical systems. In the paper such constructions are shown for Craig-Lyndon interpolation, interpolation from a Horn formula, and, with access interpolation, for a form of quantifier relativization.

2. Interpolant lifting, which is in principle known from resolution-based approaches since the mid-nineties, is placed at a new and apparently more natural position within the overall task of first-order interpolation, where it is independent of a particular calculus. A detailed correctness proof that resides on a small technical basis is presented.

3. For access interpolation, a key technique for query reformulation, the first practically implementable methods are described.

4. Conversions between closed clausal tableaux are developed that transform arbitrarily structured inputs to clausal tableaux with a restricted structure that in essence simulates non-clausal tableaux or tableaux that are constrained in specific ways, as, for example, computed by hyper tableau methods. They justify the application of practical methods that construct unrestricted clausal tableaux, such as, for example, goal-oriented “top-

\[ \exists x \ r(x) \models \exists x \ (r(x) \land (F \lor G)) \leftrightarrow \exists x \ (r(x) \land (F \lor G)) \]

\[ \exists x \ r(x) \models (\forall x \ (r(x) \to F) \land G) \leftrightarrow \exists x \ (r(x) \to (F \land G)) \]
down" first-order theorem proving methods, to tasks like access interpolation which require a certain tableau structuring.

Proofs are given for all theorem, lemma and proposition statements that do not pertain to the considered logics in general. Proofs which involve intricacies or subtleties are given in detail.

The rest of this paper is structured in two main parts: Sections 2–8 are concerned with Craig-Lyndon interpolation and Sections 9–13 with access interpolation. After notation and basic terminology have been specified in Sect. 2, precise accounts of clausal tableau and related notions are given in Sect. 3. In Sect. 4 the extraction of ground interpolants from closed clausal ground tableaux is specified and proven correct. The generalization of this method to first-order formulas, which involves preprocessing by Skolemization and postprocessing of ground interpolants by lifting is specified and proven correct in Sect. 5, and in Sect. 6 compared with related approaches from the literature. In Sect. 7 constraints on clausal tableaux are specified that characterize positive hyper tableaux, which are typically computed by “bottom-up” methods. On this basis a construction of Craig-Lyndon interpolants that inherit the Horn property from the first interpolation input is shown. Section 8 concludes the part on Craig-Lyndon interpolation with a discussion of possible refinements of our method and issues for further research. We then turn to access interpolation. In Sect. 9 a brief overview on our approach is given, underlying notions from the literature are recapitulated, and a structure-preserving clausal normalization of the relativized input formulas is described. The extraction of an access interpolant from a closed clausal ground tableau that is for such clauses and meets certain structural constraints is then specified and proven correct in Sect. 10. These structural constraints are met by positive hyper tableaux. For the general case they can be ensured with tableau transformations, specified in Sect. 11 and illustrated with examples in Sect. 12. Section 13 concludes the part on access interpolation with a discussion of possible refinements of our method, issues for further research and related work. Section 14 concludes the paper with an abstract view on its main contributions.

A work-in-progress poster of this research at an earlier stage was presented at the TABLEAUX 2017 conference.

2 Notation and Basic Terminology

We basically consider first-order logic without equality. Atoms are of the form \( p(t_1, \ldots, t_n) \), where \( p \) is a predicate symbol (briefly predicate) with associated arity \( n \geq 0 \) and \( t_1, \ldots, t_n \) are terms formed from function symbols (briefly functions) with associated arity \( \geq 0 \) and individual variables (briefly variables). Function symbols with arity 0 are also called individual constants (briefly constants).

Unless specially noted, a formula is understood as a formula of first-order logic without equality, constructed from atoms, constant operators \( \top, \bot \), the unary operator \( \neg \), binary operators \( \land, \lor \) and quantifiers \( \forall, \exists \) with their usual meaning. Further binary operators \( \rightarrow, \leftrightarrow \) as well as \( n \)-ary versions of \( \land \) and \( \lor \) can be understood as meta-level shorthands. Also quantification upon a set of variables is used as shorthand for successive quantification upon each of its elements. The operators \( \land \) and \( \lor \) bind stronger than \( \rightarrow \) and \( \leftrightarrow \). The scope of \( \neg \), the quantifiers, and the \( n \)-ary connectives is the immediate subformula to the right. Formulas in

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\( ^2 \) This does not preclude to represent equality as a predicate with axioms that express reflexivity, symmetry, transitivity and substitutivity.
which no functions with exception of constants occur are called relational. Formulas in which no predicates with arity larger than zero and no quantifiers occur are called propositional.

A subformula occurrence has in a given formula positive (negative) polarity, or is said to occur positively (negatively) in the formula, if it is in the scope of an even (odd) number of negations. If \( E \) is a term or a formula, then the set of variables that occur free in \( E \) is denoted by \( \text{var}(E) \), the set of functions occurring in \( E \) by \( \text{fun}(E) \), and the set of constants occurring in \( E \) by \( \text{const}(E) \). If \( F \) is a formula, then the set of pairs of predicates occurring in \( F \) coupled with an identifier of the respective polarity of the atom in which they occur is denoted by \( \text{pred}(F) \), the set of pairs of atoms occurring in \( F \) coupled with an identifier of the respective polarity in which they occur as \( \text{lit}(F) \), and the set of terms that occur as argument of a predicate (in contrast to just as argument of a function) as \( \text{arg}(F) \). The notation \( \text{var}(E) \), \( \text{fun}(E) \) and \( \text{const}(E) \) is also used with sets \( E \) of terms or formulas, where it stands for the union of values of the respective function applied to each member of \( E \). A formula without free variables is called a sentence. A term or quantifier-free formula in which no free variable occurs is called ground. A ground formula is thus a special case of a sentence. Symbols not present in the formulas and other items under discussion are called fresh.

A literal is an atom or a negated atom. If \( A \) is an atom, then the complement of \( A \) is \( \neg A \). The complement of a literal \( L \) is denoted by \( \overline{L} \). A clause is a (possibly empty) disjunction of literals. A clausal formula is a (possibly empty) conjunction of clauses, called the clauses in the formula.

The notion of substitution used here follows [2]: A substitution is a mapping from variables to terms which is almost everywhere equal to identity. If \( \sigma \) is a substitution, then the domain of \( \sigma \) is the set of variables \( \text{dom}(\sigma) \equiv \{ x \mid x \sigma \neq x \} \), the range of \( \sigma \) is \( \text{rng}(\sigma) \equiv \bigcup_{x \in \text{dom}(\sigma)} \{ x \sigma \} \), and the restriction of \( \sigma \) to a set \( x \) of variables, denoted by \( \sigma|_x \), is the substitution which is equal to the identity everywhere except over \( x \cap \text{dom}(\sigma) \), where it is equal to \( \sigma \). The identity substitution is denoted by Identity. A substitution can be represented as a function by a set of bindings of the variables in its domain, e.g., \( \{ x_1 \mapsto t_1, \ldots, x_n \mapsto t_n \} \). The application of a substitution \( \sigma \) to a term or a formula \( E \) is written as \( E\sigma \). \( E\sigma \) is called an instance of \( E \) and \( E \) is said to subsume \( E\sigma \). Composition of substitutions is written as juxtaposition. Hence, if \( \sigma \) and \( \gamma \) are both substitutions, then \( E\sigma\gamma \) stands for \( (E\sigma)\gamma \).

For injective substitutions we use the following additional notation: If \( \sigma \) is an injective substitution and \( E \) is a term or a formula, then \( E(\sigma^{-1}) \) denotes \( E \) with all occurrences of subterms \( s \) that are in the range of \( \sigma \) and are not a strict subterm of another subterm in the range of \( \sigma \) replaced by the variable that is mapped by \( \sigma \) to \( s \). As an example let \( \sigma = \{ x \mapsto f(a), y \mapsto g(f(a)) \} \). Then

\[
p(h(f(a), g(f(a))))(\sigma^{-1}) = p(h(x, y)).
\]

The principal functor of a term that is not a variable is its outermost function symbol. If \( S \) is a set of function symbols, then a term with a principal functor in \( S \) is also called an \( S \)-term.

We write \( F \models G \) for \( F \) entails \( G \); \( F \models F \) for \( F \) is valid; and \( F \equiv G \) for \( F \) is equivalent to \( G \), that is, \( F \models G \) and \( G \models F \). On occasion we write a sequence of statements with these operators where the right and left, respectively, arguments of subsequent statements are identical in a chained way, such as, for example, \( F \models G \models H \) for \( F \models G \) and \( G \models H \).
3 Clausal First-Order Tableaux

The following definition makes the variant of clausal tableaux that we use as basis for interpolation precise. It is targeted at modeling tableau structures produced by efficient fully automated first-order proving systems based on different calculi.

Definition 1 (Clausal Tableau and Related Notions)
(i) Let $F$ be a clausal formula. A clausal tableau (briefly tableau) for $F$ is a finite ordered tree whose nodes $N$ with exception of the root are labeled with a literal, denoted by $\text{lit}(N)$, such that the following condition is met: For each node $N$ of the tableau the disjunction of the labels of all its children in their left-to-right order, denoted by $\text{clause}(N)$, is an instance of a clause in $F$. A value of $\text{clause}(N)$ for a node $N$ in a tableau is called a clause of the tableau.

(ii) A node $N$ of a tableau is called closed if and only if it has an ancestor $N'$ with $\text{lit}(N') = \text{lit}(N)$. With a closed node $N$, a particular such ancestor $N'$ is associated as target of $N$, written $\text{tgt}(N)$. A tableau is called closed if and only if all of its leaves are closed.

(iii) A tableau is called ground if and only if for all its nodes $N$ it holds that $\text{lit}(N)$ is ground.

The most immediate relationship of clausal tableaux to the semantics of clausal formulas is that the universal closure of a clausal formula is unsatisfiable if and only if there exists a closed clausal tableau for the clausal formula. Knowing that there are sound and complete calculi that operate by constructing a closed clausal tableau for an unsatisfiable clausal formula, and taking into account Herbrand's theorem we can state the following proposition:

Proposition 2 (Unsatisfiability and Computation of Closed Clausal Tableaux) There is an effective method that computes from a clausal formula $F$ a closed clausal tableau for $F$ if and only if $\forall x_1 \ldots \forall x_n F$, where $\{x_1, \ldots, x_n\} = \text{var}(F)$, is unsatisfiable. Moreover, this also holds if terms in the literal labels of tableau nodes are constrained to ground terms formed from functions occurring in $F$ and, in case there is no constant occurring in $F$, an additional fresh constant.

Our objective is here interpolant construction on the basis of clausal tableaux produced by fully automated systems. This has effect on some aspects of our formal notion of clausal tableau: All occurrences of variables in the literal labels of a tableau according to Definition 1.i are free and the scope of these variables spans all literal labels of the whole tableau. In more technical terms, this means that the tableaux are free variable tableaux (see [37, p. 158ff]) with rigid variables (see [31, p. 114]). Tableaux with only clause-local variables can, however, of course be expressed by just using different variables in each tableau clause. Thus, although our notion of tableaux involves rigid variables, this does not in any way imply that interpolant computation based on it applies only to tableaux whose construction by a prover had involved rigid variables.

Another aspect concerns the definition of closed for nodes and for tableaux: A tableau is closed if all of its leaves are closed, which does, however, not exclude that also an inner node of a closed tableau might be closed. For the construction of a closed tableau in theorem proving it is pointless to attach children to an already closed node. In our context, however, operations such as instantiating literal labels and certain tableau transformations might introduce inner closed nodes. To let the results of such operations be tableaux again, we thus have to permit closed inner nodes. A tableau simplification to eliminate these is shown in Sect. 11.
4 Ground Interpolant Extraction from Clausal Tableaux

As shown by Craig [19], for first-order sentences $F$ and $G$ such that $F \models G$, an “intermediate” sentence $H$ such that $F \models H \models G$ can be constructed, whose predicates and functions are occurring in both $F$ and $G$. That this also holds if in addition the polarities of predicate occurrences in $H$ are constrained to polarities in which they occur in both $F$ and $G$ is attributed to Lyndon [41], such that formulas $H$ are sometimes called Lyndon interpolants in analogy to Craig interpolants. We call them here Craig-Lyndon interpolants:

Definition 3 (Craig-Lyndon Interpolant) Let $F, G$ be sentences such that $F \models G$. A Craig-Lyndon interpolant of $F$ and $G$ is a sentence $H$ such that

1. $F \models H \models G$.
2. $\text{pred}(H) \subseteq \text{pred}(F) \cap \text{pred}(G)$.
3. $\text{fun}(H) \subseteq \text{fun}(F) \cap \text{fun}(G)$.

The notion of Craig-Lyndon interpolant is specified here for sentences in contrast to formulas $F$, $G$ and $H$. This is without loss of generality because free variables in $F$, $G$ and $H$ would, with respect to interpolation, be handled exactly like constants.

Smullyan [56] specifies in his framework of non-clausal tableaux an elegant technique to extract a Craig-Lyndon interpolant from a tableau that represents a proof of $F \models G$, which is also presented in Fitting’s book [24]. The handling of propositional connectives in this method can be straightforwardly transferred to clausal tableaux. Quantifiers, however, have to be processed differently to match their treatment in clausal tableaux by conversion to prenex form and Skolemization. The overall interpolant extraction from a closed clausal tableau then proceeds in two stages, analogously as described for resolution-based methods in [29, 3, 13, 35]. In the first stage a “rough interpolant” is constructed which needs postprocessing by replacing terms with variables and prepending a quantifier prefix on these variables to yield an actual interpolant. This second stage will be specified in Sect. 5 and discussed further in Sect. 6. As we will see now, on the basis of clausal tableaux the first stage can be specified and verified with proofs by a straightforward adaption of Smullyan’s method in an almost trivially simple way.

Our interpolant construction is based on a variant of clausal tableaux where nodes have an additional side label that is shared by siblings and indicates whether the tableau clause is an instance of an input clause derived from the formula of the left side or the formula on the right side of the entailment underlying the interpolation:

Definition 4 (Two-Sided Clausal Tableau and Related Notions)

(i) Let $F_L, F_R$ be clausal formulas. A two-sided clausal tableau for $F_L$ and $F_R$ (or briefly tableau for the two formulas) is a clausal tableau for $F_L \land F_R$ whose nodes $N$ with exception of the root are labeled additionally with a side $\text{side}(N) \in \{L, R\}$, such that the following conditions are met:

1. If $N$ and $N'$ are siblings, then $\text{side}(N) = \text{side}(N')$.
2. If $N'$ is a child of $N$, then $\text{clause}(N)$ is an instance of a clause in $F_{\text{side}(N')}$. The side of a clause $\text{clause}(N)$ in a tableau is the value of the side label of the children of $N$.

(ii) For $S \in \{L, R\}$ and nodes $N$ of a two-sided clausal tableau define

$$\text{branch}_S(N) \triangleq \bigwedge_{N' \in B \text{ and } \text{side}(N') = S} \text{ll}(N')$$

where $B$ is the union of $\{N\}$ with the set of the ancestors of $N$. 


The following definition specifies an adaption of the handling of propositional connectives in [56, Chap. XV] and [24, Chap. 8.12] to construct interpolants from non-clausal tableaux. Differently from these works, the specification is here not in terms of tableau manipulation rules that deconstruct the tableau bottom-up, but inductively, as a function that maps a node to a formula.

**Definition 5 (Interpolant Extraction from a Clausal Ground Tableau)** Let \( N \) be a node of a closed two-sided clausal ground tableau. The value of \( \text{ipol}(N) \) is a ground formula, defined inductively as follows:

i. If \( N \) is a leaf, then the value of \( \text{ipol}(N) \) is determined by the values of \( \text{side}(N) \) and \( \text{side}(	ext{tgt}(N)) \) as specified in the following table:

| \( \text{side}(N) \) | \( \text{side}(	ext{tgt}(N)) \) | \( \text{ipol}(N) \) |
|---------------------|---------------------|---------------------|
| L                   | L                   | \bot               |
| L                   | R                   | \text{lit}(N)      |
| R                   | L                   | \text{lit}(N)      |
| R                   | R                   | \top               |

ii. If \( N \) is an inner node with children \( N_1, \ldots, N_n \) where \( n \geq 1 \), then the value of \( \text{ipol}(N) \) is composed from the values of \( \text{ipol} \) for the children, disjunctively or conjunctively, depending on the side label of the children (which is the same for all of them), as specified in the following table:

| \( \text{side}(N_i) \) | \( \text{ipol}(N) \) |
|-----------------------|---------------------|
| L                     | \bigvee_{i=1}^n \text{ipol}(N_i) |
| R                     | \bigwedge_{i=1}^n \text{ipol}(N_i) |

The following lemma associates semantic and syntactic properties with the formula obtained as value of applying \( \text{ipol} \) to the root of a closed ground tableau. These properties imply the conditions required from a Craig-Lyndon interpolant (Definition 3).

**Lemma 6 (Correctness of Interpolant Extraction from Clausal Ground Tableaux)** Let \( F_L, F_R \) be clausal ground formulas and let \( T \) be a closed two-sided clausal ground tableau for \( F_L \) and \( F_R \). If \( N \) is the root of \( T \), then

1. \( F_L \models \text{ipol}(N) \models \neg F_R. \)
2. \( \text{lit}(\text{ipol}(N)) \subseteq \text{lit}(F_L) \cap \text{lit}(\neg F_R). \)

**Proof** We show the following property of \( \text{ipol} \) that invariantly holds for all nodes of the tableau, including the root, which immediately implies the proposition: For all nodes \( N \) of \( T \) it holds that

(a) \( F_L \land \text{branch}_L(N) \models \text{ipol}(N) \models \neg F_R \lor \neg \text{branch}_R(N). \)
(b) \( \text{lit}(\text{ipol}(N)) \subseteq \text{lit}(F_L \land \text{branch}_L(N)) \cap \text{lit}(\neg F_R \lor \neg \text{branch}_R(N)). \)

This is proven by induction on the tableau structure, proceeding from leaves upwards. We prove the base case, where \( N \) is a leaf, by showing (a) and (b) for all possible values of \( \text{side}(N) \):

- Case \( \text{side}(N) = L \):
  - Case \( \text{side}(	ext{tgt}(N)) = L \): Immediate since then \( \text{branch}_L(N) \models \bot \models \text{ipol}(N). \)
- Case \( \text{side}(\text{tgt}(N)) = R \): Then \( \text{ipol}(N) = \text{lit}(N) \). Properties (a) and (b) follow because \( \text{lit}(N) \) is a conjunct of \( \text{branch}L(N) \) and \( \text{lit}(N) \) is a conjunct of \( \text{branch}R(N) \).

- Case \( \text{side}(N) = R \):
  - Case \( \text{side}(\text{tgt}(N)) = L \): Then \( \text{ipol}(N) = \text{lit}(N) \). Properties (a) and (b) follow because \( \text{lit}(N) \) is a conjunct of \( \text{branch}L(N) \) and \( \text{lit}(N) \) is a conjunct of \( \text{branch}R(N) \).
  - Case \( \text{side}(\text{tgt}(N)) = R \): Immediate since then \( \text{ipol}(N) = T \models \neg \text{branch}R(N) \).

To show the induction step, assume that \( N \) is an inner node with children \( N_1, \ldots, N_n \). Consider the case where the side of the children is \( L \). The induction step for the case where the side of the children is \( R \) can be shown analogously. By the induction hypothesis we can assume that for all \( i \in \{1, \ldots, n\} \) it holds that

\[
F_L \land \text{branch}L(N_i) \models \text{ipol}(N_i) \models \neg F_R \lor \neg \text{branch}R(N_i),
\]

which, since \( \text{side}(N_i) = L \), is equivalent to

\[
F_L \land \text{branch}L(N) \land \text{lit}(N_i) \models \text{ipol}(N_i) \models \neg F_R \lor \neg \text{branch}R(N).
\]

Since \( \text{ipol}(N) = \bigvee_{i=1}^n \text{ipol}(N_i) \) it follows that

\[
F_L \land \text{branch}L(N) \land \bigvee_{i=1}^n \text{lit}(N_i) \models \text{ipol}(N) \models \neg F_R \lor \neg \text{branch}R(N).
\]

Because \( \bigvee_{i=1}^n \text{lit}(N_i) = \text{clause}(N) \) is an instance of a clause in \( F_L \) and thus entailed by \( F_L \) the semantic requirement (a) of the induction conclusion follows:

\[
F_L \land \text{branch}L(N) \models \text{ipol}(N) \models \neg F_R \lor \neg \text{branch}R(N).
\]

The syntactic requirement (b) follows from the induction assumption and because in general for all nodes \( N \) of a two-sided clausal ground tableau for clausal ground formulas \( F_L \) and \( F_R \) it holds that all literals in \( \text{branch}L(N) \) occur in some clause of \( F_L \) and all literals in \( \text{branch}R(N) \) occur in some clause of \( F_R \).

Lemma 6 immediately yields a construction method for Craig-Lyndon interpolants of propositional and, more general, ground formulas, or, in other words, quantifier-free first-order formulas. We call the method \( \text{CTI} \), suggesting \textit{Clausal Tableau Interpolation}. In Sect. 5 below it will be generalized to first-order sentences in full.

**Procedure 7 (The CTI Method for Craig-Lyndon Interpolation on Ground Formulas)**

\textbf{Input}: Ground formulas \( F \) and \( G \) such that \( F \models G \).

\textbf{Method}: Convert \( F \) and \( \neg G \) to equivalent clausal ground formulas and compute a closed two-sided clausal ground tableau for them. Let \( N \) be the root of the tableau and compute the value of \( \text{ipol}(N) \).

\textbf{Output}: Return the value of \( \text{ipol}(N) \). The output is a ground formula that is a Craig-Lyndon interpolant of the input formulas.

The procedure is correct: The existence of a closed two-sided clausal tableau as required follows from Proposition 2, that the result is ground and is a Craig-Lyndon interpolant of \( F \) and \( G \) follows from Lemma 6 and Definition 3.
5 First-Order Interpolant Extraction from Clausal Tableaux

Procedure 7 provides a method to compute Craig-Lyndon interpolants of ground formulas. We now generalize it to first-order sentences with arbitrary quantifications. The starting point is a ground interpolant obtained from a closed clausal ground tableau according to Lemma 6. The tableau is now for two clausal formulas that have been obtained from first-order sentences by Skolemization, conversion to clausal form and instantiation. By a postprocessing lifting operation, the ground interpolant is converted to an interpolant of the two original first-order input sentences. Terms with function symbols that do not occur in both of them are there replaced by variables and a suitable quantifier prefix upon these variables is prepended. The postprocessing is easy to implement, it effects at most a linear increase of the formula size and its computational effort amounts to sorting the replaced terms according to their size. Similar lifting techniques have been shown for resolution-based methods in [29] and [3, Lemma 8.2.2]. We discuss the relationship to these in Sect. 6.

Before we specify the first-order interpolation procedure and prove its correctness we note that to capture the semantics of Skolemization and to eliminate function symbols that occur only in one the two interpolation inputs we use second-order quantification upon functions and predicates in intermediate formulas, that is, formulas used in the procedure specification and within the correctness proof. In particular, we apply the following properties:

**Proposition 8 (Second-Order Skolemization)** Let $F$ be a formula. Assume that $x_1, \ldots, x_n, y$ are variables that do not occur bound in $F$ and that $f$ is an $n$-ary function symbol that does not occur at all in $F$. Then

$$\forall x_1 \ldots \forall x_n \exists y F \equiv \exists f \forall x_1 \ldots \forall x_n F\{y \mapsto f(x_1, \ldots, x_n)\}.$$  

**Proposition 9 (Inessential Quantifications in Entailments)** Let $F, G$ be formulas and let $x, y$ be sets of predicate and function symbols such that $x \cap (\text{pred}(G) \cup \text{fun}(G)) = y \cap (\text{pred}(F) \cup \text{fun}(F)) = \emptyset$. Then

$$\exists x F \models \forall y G \text{ if and only if } F \models G.$$  

Proposition 9 includes the special case of quantification upon nullary functions, that is, constants, which is actually first-order quantification upon them in the role of variables. On the right side of the equivalence stated by the proposition, where they occur free, they can be viewed as constants or as free variables. Notice that $\text{pred}(G)$ and $\text{pred}(F)$ in the preconditions take polarity into account. That is, if a predicate $p$ occurs in $F$ only with, say, positive polarity and in $G$ only with negative polarity, then, by Proposition 9 it holds that $\exists p F \models \forall p G$ holds if and only if $F \models G$, although $p$ occurs in $F$ as well as in $G$.

We are now ready to specify the CTI method in full, which generalizes Procedure 7 by allowing first-order sentences with arbitrary quantifications as inputs:

**Procedure 10 (The CTI Method for Craig-Lyndon Interpolation)**

**INPUT:** First-order sentences $F$ and $G$ such that $F \models G$.

**METHOD:**

- Classify $F$ and $\neg G$ to obtain equivalent sentences $\exists f \forall u F_c$ and $\exists g \forall v G_c$, respectively, where $f_c$ and $g_c$ are the introduced Skolem functions and $F_c$ and $G_c$ are clausal formulas whose variables are $u_c$ and $v_c$, respectively. Assume w.l.o.g. that $f_c$ and $g_c$ are disjoint. Let $k$ be a fresh constant. Construct a closed two-sided clausal ground tableau for $F_c$ and $G_c$ in which all literal labels are instantiated with terms formed from $k$ and functions that occur in $F_c$ or in $G_c$. Let $F_0$ be the conjunction of the clauses of the tableau with side $L$
and let \( G_b \) be the conjunction of the clauses of the tableau with side \( R \). Let \( H_b \) be \( \text{ipol}(N) \), where \( N \) is the root of the tableau. Define:

\[
\begin{align*}
  f & \overset{\text{def}}{=} \text{fun}(F_c) \setminus \text{fun}(G_b), \\
  g & \overset{\text{def}}{=} (\text{fun}(G_b^\perp) \setminus \text{fun}(F_c)) \cup \{ k \}.
\end{align*}
\]

(Alternatively, it is also possible to place \( k \) into \( f \) instead of \( g \). Further possibilities are discussed in Sect. 8.1 below.) Let \( u \) and \( v \) be fresh sequences of variables and let \( \mu \) be an injective substitution with domain \( u \cup v \) such that

\[
\begin{align*}
  \text{rng}(\mu|_u) &= \{ t \mid t \text{ is a } g\text{-term occurring in } H_b \text{ in a position other than as strict subterm of another } f\text{-term or } g\text{-term} \}, \\
  \text{rng}(\mu|_v) &= \{ t \mid t \text{ is an } f\text{-term occurring in } H_b \text{ in a position other than as strict subterm of another } f\text{-term or } g\text{-term} \}.
\end{align*}
\]

Construct \( H_q \) as

\[
H_q \overset{\text{def}}{=} H_b \langle \mu^{-1} \rangle.
\]

Construct the quantifier prefix \( Q_1 z_1 \ldots Q_n z_n \) as follows: Let \( \{ z_1, \ldots, z_n \} \) be the members of \( u \cup v \) that occur in \( H_q \) ordered such that for \( i, j \in \{1, \ldots, n\} \) it holds that if \( z_i \mu \) is a strict subterm of \( z_j \mu \), then \( i < j \) and, for \( i \in \{1, \ldots, n\} \), let \( Q_i \overset{\text{def}}{=} \forall \) if \( z_i \in u \) and let \( Q_i \overset{\text{def}}{=} \exists \) if \( z_i \in v \).

**Output**: Return \( Q_1 z_1 \ldots Q_n z_n H_q \).

The output is a Craig-Lyndon interpolant of the input sentences.

Procedure 10 indeed generalizes Procedure 7: For ground inputs both procedures proceed identically. Correctness of the procedure is stated with the following theorem, which will be proven in detail. The proof is followed by Example 12, which illustrates items mentioned in the proof for a pair of concrete input sentences.

**Theorem 11 (Correctness of the CTI Method)** If \( F \) and \( G \) are first-order sentences such that \( F \models G \), then Procedure 10 applied to \( F \) and \( G \) outputs a Craig-Lyndon interpolant of \( F \) and \( G \).

**Proof** Let symbols have the denotation according to the procedure specification. In addition we will specify further clausal formulas, sets of variables, and substitutions, that relate to the items in the procedure specification and are overviewed in the following two graphs:

**Fig. 1**: Clausal formulas and substitutions used to prove interpolant lifting.
Variables allowed in the respective formulas are shown there in parentheses. Formulas \( F_b \) and \( G_c^- \) are ground. Sets of variables denoted by different symbols (including differences in the subscript) in the figure are disjoint. The superset symbol \( \supseteq \) indicates that all clauses of the formula on the right are clauses of the formula on the left. Arrows \( \rightarrow \) represent the \textit{instance of} relationship, where the formula at the arrow tip under the substitution shown as arrow label is the formula at the arrow origin. Substitutions that are injections are marked with an asterisk (*). The shown substitutions have the following domains:

\[
\begin{align*}
\text{dom}(\sigma) & = u_1 \cup v_1, \\
\text{dom}(\rho) & = u_0 \cup v_1 \cup v_e \cup u_1, \\
\text{dom}(\lambda) & = x \cup y, \\
\text{dom}(\eta) & = u_0 \cup v_e.
\end{align*}
\]

The following additional syntactic constraints are imposed on the involved formulas:

- Members of \( g \) do not occur in \( F, F_e, F_c, F_b, G^-_c, G^-_e \).
- Members of \( f \) do not occur in \( G^-_1, G^-_2, G^-_3, F, F_q \).

We proceed to show the construction of the involved items, stepping out from those mentioned in the procedure description. Sentences \( F \) and \( G \) are given as input. The conversion to \( \exists f \forall u_e F_c \) and \( \exists g \forall v_e G^-_d \) can be obtained by usual first-order normal form transformation. Skolemization can then be understood as equivalence preserving rewriting with Proposition 8. It has to be applied here independently to \( F \) and to \( \neg G \), which is possible since these sentences do not share quantified variables. The required disjointness conditions on sets of variables and Skolem functions can be achieved easily by renaming bound variables. The sets of functions \( f, g \) can then be constructed from \( F_e \) and \( G^-_c \). The following semantic relationships hold:

\[
(1) \quad F \equiv \exists f \forall u_e F_c \models \exists f \forall u_e F_c \models \forall g \exists v_e \neg G^-_e \models \forall g_e \exists v_e \neg G^-_e \equiv G.
\]

Given that \( \forall u_e \forall v_e (F_c \land G^-_c) \) is unsatisfiable, which follows from (1), the existence of a closed two-sided ground tableau as specified in the procedure description follows from “completeness” of clausal ground tableau construction as implied by Proposition 2 (or, in essence, by Herbrand’s theorem). Formulas \( F_b, G^-_b \) and \( H_b \) as specified then exist, since they can be extracted from the tableau. Formulas \( F_d \) and \( G^-_d \) contain clauses of \( F_c \) and \( G^-_c \), respectively, such that each clause with side \( L \) of the tableau is an instance of a clause in \( F_d \) and each clause with side \( R \) is an instance of a clause in \( G^-_d \). The following semantic relationships hold:

\[
(2) \quad \forall u_e F_c \models \forall u_e F_d \models F_b \models \neg G^-_b \models \exists v_e \neg G^-_d \models \exists v_e \neg G^-_e.
\]

We define \( u, v \) and \( \mu \), which are specified in the procedure description, on the basis of larger sets of variables and a substitution with increased domain that are needed for the further internal proceeding of the proof: Let \( x \) and \( y \) be fresh sequences of variables and let \( \lambda \) be an injective substitution with domain \( x \cup y \) such that

\[
\begin{align*}
\text{rng}(\lambda^x) & = \{ t \mid t \text{ is a } g\text{-term occurring in } F_b \text{ or in } G^-_b \}, \\
\text{rng}(\lambda^y) & = \{ t \mid t \text{ is an } f\text{-term occurring in } F_b \text{ or in } G^-_b \}.
\end{align*}
\]

Define \( u \) as the subset of all members \( x \) of \( x \) such that \( x : \lambda \) meets the conditions on the range of \( \mu \) stated in the procedure description, and, analogously, define \( v \) as the subset of all
members \( y \) of \( y \) such that \( y \lambda \) meets the conditions on the range of \( \mu \) stated in the procedure description. Define
\[
\mu \equiv \lambda \vert_{u \cup v}.
\]

The construction of the remaining items specified in the procedure description, that is, the formula \( H_0 \) and a quantifier prefix \( Q_1 z_1 \ldots Q_n z_n \), is straightforward.

We now consider further items introduced with Fig. 1. The clauses of \( F_0 \) are ground instances of clauses of \( F_0 \). Hence, there must exist a clausal formula \( F_0 \) that subsumes both formulas \( F_0 \) and \( F_b \). Specifically, the formula \( F_0 \) can be understood as conjunction of "copies" of clauses of \( F_0 \), that is, clauses of \( F_0 \) with variables renamed to fresh symbols. The set of variables \( u_e \) consists of all variables occurring in these copies. Analogous considerations hold for \( G_0^e \). We can then supplement the semantic relationships in (2) to
\[
(3) \quad \forall u_e F_e \models \forall u_e F_b \equiv \forall u_e F_0 \models \neg G_0^e \models \exists v_e \neg G_0^e \equiv \exists v_e \neg G_0^e \models \exists v_e \neg G_0^e.
\]

Define \( F_\bar{0} \equiv F_\bar{0}(\lambda^{-1}) \) and \( G_\bar{0}^e \equiv G_\bar{0}^e(\lambda^{-1}) \), in analogy to the specification of \( H_0 \) in the procedure description. The formula \( F_1 \) subsumes both \( F_b \) and \( F_0 \). Together with the substitution \( \sigma \) it can be characterized as follows: Let \( \sigma \) be an injective substitution such that
\[
\text{rng}(\sigma \vert_{u_e}) = \{ t \mid t \text{ is an } f \text{-term occurring in } F_b \},
\]
and define \( F_1 \equiv F_b(\sigma^{-1}) \). Intuitively, \( F_1 \) can be understood as obtained from \( F_b \) by replacing each term whose principal function symbol does not occur in \( G \) (which includes the Skolem functions \( f_e \)) and which is not a proper subterm of another such term with a dedicated variable from \( v_1 \). Analogous considerations apply to \( G_1^e \). We complete the characterization of \( \sigma \) with
\[
\text{rng}(\sigma \vert_{v_1}) = \{ t \mid t \text{ is a } g \text{-term occurring in } G_0^e \}
\]
and define \( G_1^e \equiv G_0^e(\sigma^{-1}) \). It still needs to be shown that \( F_0 \) is an instance of \( F_1 \) and that \( G_0^e \) is an instance of \( G_1^e \) obtained by applying the substitution \( \rho \). Define
\[
\rho \equiv \{ x \mapsto x \sigma \eta(\lambda^{-1}) \mid x \in u_e \cup v_1 \cup v_e \cup u_1 \}.
\]

That \( F_1 \rho = F_1 \rho \) can then be shown as follows: Since the range of \( \lambda \) only includes \( f \)-terms and \( g \)-terms, whereas in \( F_1 \) members of \( f \) and \( g \) do not occur at all it holds that \( F_1 \sigma \eta(\lambda^{-1}) = F_1 \{ x \mapsto x \sigma \eta(\lambda^{-1}) \mid x \in u_e \cup v_1 \} \). Since members of \( v_e \cup u_1 \) do not occur in \( F_1 \) it follows that \( F_1 \sigma \eta(\lambda^{-1}) = F_1 \{ x \mapsto x \sigma \eta(\lambda^{-1}) \mid x \in u_e \cup v_1 \} \). As Fig. 1 makes evident, \( F_1 \sigma \eta(\lambda^{-1}) = F_0 \). By definition \( F_0 = F_0(\lambda^{-1}) \). Hence \( F_1 \rho = F_0 \sigma \eta(\lambda^{-1}) = F_0(\lambda^{-1}) = F_0 \). With analogous considerations it follows that \( G_1^e \rho = G_0^e \).

We are now done with showing the construction of the items introduced in the procedure description and in Fig. 1. It remains to show on this basis that the constructed output formula \( Q_1 z_1 \ldots Q_n z_n H_0 \) is indeed a Craig-Lyndon interpolant. Let \( Q \) be a shorthand for \( Q_1 z_1 \ldots Q_n z_n H_0 \). From the construction of \( Q H_0 \) by replacing in a ground formula ground terms with variables that are bound by a prepended quantifier prefix it follows that \( Q H_0 \) is a sentence. The further syntactic properties of a Craig-Lyndon interpolant, as specified with items (2) and (3) of Definition 3, are:

\[
(4) \quad \text{pred}(Q H_0) \subseteq \text{pred}(F) \cap \text{pred}(G).
\]

\[
(5) \quad \text{fun}(Q H_0) \subseteq \text{fun}(F) \cap \text{fun}(G).
\]

They can be shown as follows: Recall that \( H_0 \) is a Craig-Lyndon interpolant of \( F_0 \) and \( \neg G_0^e \). Hence \( \text{pred}(H_0) \subseteq \text{pred}(F_0) \cap \text{pred}(\neg G_0^e) \). Statement (4) then follows since \( \text{pred}(Q H_0) \subseteq \text{pred}(F) \cap \text{pred}(\neg G_0^e) \).
pred(H_q) = pred(H_b), pred(F_b) ⊆ pred(F_3) ⊆ pred(F) and pred(¬G^{-}_q) ⊆ pred(¬G^{-}_q) ⊆ pred(G). All members of fun(H_b) that are not in fun(F) ∩ fun(G) are in f ∪ g. Statement (5) then follows since H_q is defined as H_b(μ^{-1}). which implies fun(QH_q) = fun(H_q) ⊆ fun(H_q) and, with the specification of μ, that there are no occurrences of members of f ∪ g in H_q.

It remains to prove that QH_q has the semantic characteristics of a Craig-Lyndon interpolant as specified with item (1.) of Definition 3. Let \{w_1, . . . , w_m\} be x \cup y ordered such that for i, j \in \{1, . . . , m\} it holds that if w_iA is a strict subterm of w_jA then i < j and the ordering of \{z_1, . . . , z_n\} is extended, that is, if w_a = z_c, w_b = z_d and c < d, then a < b. For i \in \{1, . . . , m\} let R_i = \{w_i \mid x \in x\} and let R_i = \{w_i \mid y \in y\}. Let R = R_1w_1 . . . R_mw_m. Since F_b \models H_b \models ¬G^{-}_b, F_q = F_b(\lambda^{-1}), G^{-}_q = G^{-}_q(\lambda^{-1}), and H_q = G^{-}_q(\mu^{-1}) = G^{-}_q(\lambda^{-1}) it follows that F_q \models H_q \models ¬G^{-}_q. Hence RF_q = RH_q \models R¬G^{-}_q. Since the quantifier prefix Q consists of exactly those quantifications in R that are upon variables occurring in H_q, in the same order as in R, it holds that RH_q \models QH_q. Thus

\( 6 \) RF_q \models QH_q \models R¬G^{-}_q.

The semantic property of a Craig-Lyndon interpolant that we are going to prove is F \models QH_q \models G. Given (6), this follows from F \models RF_q and R¬G^{-}_q \models G. We will now show the first of these entailments, F \models RF_q. For this we need a further substitution, ϕ, which is not displayed in Fig. 1. Its key properties are stated as (9) and (10) below. They will later serve to justify the base cases of an induction. Their proof depends on a further property of ϕ:

\( 7 \) σϕ = ϕ.

Equality (7) can be shown as follows: For members x of dom(ϕ)\dom(σ) = u_x ∪ v_x it is evident that xσϕ = xϕ. It remains to consider further members of dom(σ) \dom(ρ) = dom(σ) = x ∈ u_x ∪ v_x ∪ u_1, that is, members of v_1 ∪ u_1. Let x be a member of this set. Observe that var(xσ) ⊆ u_x ∪ v_x. Since dom(σ) \cap (u_x ∪ v_x) = \emptyset it holds that xσϕ = xϕ. Hence, by the definition of ϕ it holds that xσϕ = xσϕ(λ^{-1}) = xσϕ(λ^{-1}) = xϕ, which concludes the proof of (7). We now define the substitution ϕ with dom(ϕ) = x \cup y by

ϕ ≜ \{ x \mapsto xλ(λ^{-1}) \mid x \in x\} \cup \{ y \mapsto yλ(λ^{-1}) \mid y \in y\}.

It then holds that

\( 8 \) ∀u_x F_0 = ∀u_x F_0σσϕ|_y = ∀x F_0σϕ|_y = ∀x F_0ϕ|_y.

That ∀u_x F_0σσϕ|_y = ∀x F_0ϕ|_y follows since F_0σϕ|_y is an instance of F_0σ and the free variables in both formulas are universally quantified on both sides of the entailment. That ∀x F_0σϕ|_y = ∀x F_0ϕ|_y follows from (7). The remaining identities in (8) are immediate from the relationships displayed in Fig. 1. From (8), (3) and (1) it follows that

\( 9 \) F \models ∃ϕ∀x F_0ϕ|_y.

Analogously, it can be shown that

\( 10 \) ∀g∃y ¬G^{-}_qϕ|_x \models G.

In preparation of an inductive argument, we define fragments of R, x and ϕ for all i \in \{0, . . . , m\}:

\[ R_i \equiv R_1w_1 . . . R_iw_i, \]

\[ x_i \equiv x \cap \{w_{i+1}, . . . , w_m\}, \]

\[ ϕ_i \equiv ϕ|_{x \cap \{w_{i+1}, . . . , w_m\}}. \]
Observe that then

(11) \( R_0 = \epsilon, \, x_0 = x, \, \phi_0 = \phi_y \).

(12) \( R_m = R, \, x_m = \{ \}. \, \phi_m = \text{Identity}. \)

We now show by induction that for all \( i \in \{ 0, \ldots, m \} \) it holds that

\[
\text{(IP)} \quad F \models \exists f R_i \forall x_i, F_q \phi_i.
\]

If \( i = m \), then, by (12) the entailment (IP) equals \( F \models \exists f R_m \). Since \( \text{fun}(F_q) \cap \emptyset = \emptyset \) this is equivalent to \( F \models R F_q \), the statement to prove. In the base case \( i = 0 \) of the induction, the entailment (IP) is by (11) identical with \( F \models \exists f \forall x F_q \phi_y \), which we have already shown as (9). To show the induction step we assume as induction hypothesis that (IP) holds for some \( i \in \{ 0, \ldots, m - 1 \} \). The variable \( w_{i+1} \) must be either in \( x \) or in \( y \). In the case \( w_{i+1} \in x \) it holds that \( x_i = \{ w_{i+1} \} \cup x_{i+1} \) and that \( \phi_{i+1} = \phi_i \). Thus

\[
(13) \quad \exists f R_i \forall x_i, F_q \phi_i \equiv \exists f R_i \forall w_{i+1} \forall x_{i+1} F_q \phi_{i+1} \equiv \exists f R_{i+1} \forall x_{i+1} F_q \phi_{i+1}.
\]

In the case \( w_{i+1} \in y \) it holds that \( x_{i+1} = x_i \) and \( \phi_i = \{ w_{i+1} \mapsto w_{i+1} \} \phi_{i+1} \). Moreover, it holds that

\[
(14) \quad w_{i+1} \not\in \text{var}(\text{rng}(\phi_{i+1})).
\]

(15) \( x_{i+1} \cap \text{var}(w_{i+1}) = \emptyset \).

Statement (14) follows since \( w_{i+1} \in y \) and \( \text{var}(\text{rng}(\phi_{i+1})) \subseteq \text{var}(\text{rng}(\phi_y)) \subseteq x \). Statement (15) can be shown as follows: Assume that (15) does not hold. Then there must be a number \( j \) such that \( w_j \in x_{i+1} \cap \text{var}(w_{i+1}) \). By the definition of \( w_j \) it follows that \( j > i + 1 \). It also follows that \( w_j \lambda \) is a strict subterm of \( w_{i+1} \phi \lambda \), and thus, since \( \phi \lambda = \lambda \), which is not hard to verify, also of \( w_{i+1} \lambda \). From the specification of the ordering of \( \{ w_1, \ldots, w_m \} \) it follows that \( j < i + 1 \), which contradicts with the condition \( j > i + 1 \) just derived. Hence (15) must hold. We now can state the following relationships, where the entailment step is justified by (14) and (15):

\[
\begin{align*}
(16) & \quad R_i \forall x_i, F_q \phi_i \\
& \equiv R_i \forall x_{i+1}, F_q \{ w_{i+1} \mapsto w_{i+1} \} \phi_{i+1} \\
& \models R_i \exists w_{i+1} \forall x_{i+1}, F_q \phi_{i+1} \\
& \equiv R_{i+1} \forall x_{i+1}, F_q \phi_{i+1}.
\end{align*}
\]

Given the induction hypothesis \( F \models \exists f R_i \forall x_i, F_q \phi_i \), the induction conclusion

\( F \models \exists f R_{i+1} \forall x_{i+1}, F_q \phi_{i+1} \)

follows in the case \( w_{i+1} \in x \) from (13) and in the case \( w_{i+1} \in y \) from (16). Hence we have established

\[
(17) \quad F \models R F_q.
\]

Analogously it can be shown that

\[
(18) \quad R \vdash G_q \models G.
\]

Combining (4), (5), (6), (17) and (18) and recalling that \( Q \) was defined as shorthand for \( Q_1 z_1 \ldots Q_n z_n \) we can finish the proof of Theorem 11 by concluding that the output of the CTI procedure is indeed a Craig-Lyndon interpolant:

\[
(19) \quad Q_1 z_1 \ldots Q_n z_n H_q \text{ is a Craig-Lyndon interpolant of } F \text{ and } G. \quad \Box
\]
The following example illustrates the proof of Theorem 11:

**Example 12 (Items in the Proof of Theorem 11)** Consider computation of a Craig-Lyndon interpolant by the CTI method for the sentences:

\[ F = \forall x_1 \forall x_2 \, p(x_1, h(f_1(x_1)), x_2)). \]

\[ G = \exists x_1 \exists x_2 \, (p(h(g_2(x_1)), x_2, g_1) \land p(g_1, x_1, h(g_2(x_1)))). \]

The common symbols of both sentences are the predicate \( p \), in positive polarity, and the function \( h \). Alternatively, the non-common functions \( f_1, g_1, g_2 \) might be viewed as Skolem functions for original sentences

\[ F' = \forall x_1 \exists y \forall x_2 \, p(x_1, h(y), x_2)). \]

\[ G' = \forall x_1 \exists y_1 \forall x_2 \exists y_2 \, (p(h(x_2), y_2, x_1) \land p(x_1, y_1, h(x_2))). \]

Under this view, however, \( F' \) and \( G' \) themselves both qualify as Craig-Lyndon interpolants of \( F' \) and \( G' \). Nevertheless, the proceeding in the example can also be understood as computing a further interpolant of \( F' \) and \( G' \) which actually is strictly weaker than \( F' \) and strictly stronger than \( G' \).

We return back to the original view of \( f_1, g_1, g_2 \) as functions occurring in just one of the interpolation inputs \( F \) and \( G \) and show the respective values of the items mentioned the description of Procedure 10 and in the proof of its correctness, Theorem 11. Converting \( F \) and \( \neg G \) to clausal form yields the following formulas, variables and sets of distinguished functions:

\[ F_c = p(u_2^c, h(f_1(u_2^c)), v_2^c, g_2^c). \]

\[ G_c = \neg p(h(g_2(v_2)), v_2^c, g_2^c) \lor \neg p(g_1, v_2^c, h(g_2(v_2))). \]

\[ u_c = \{u_2^c, u_2^c\}. \]

\[ v_c = \{v_2^c, v_2^c\}. \]

\[ f = \{f_1\}. \]

\[ g = \{g_1, g_2, k\}. \]

Formulas \( F_b \) and \( G_b^c \) are clausal ground formulas obtained from instantiating clauses of \( F_c \) and \( G_c^c \). Actually it is easy to verify syntactically that \( F_b \equiv \neg G_b^c \), hence \( F_b \land G_b^c \) is unsatisfiable, implying that a two-sided ground tableau for \( F_b \) and \( G_b^c \) can be constructed. From that tableau we can extract \( H_b \), a Craig-Lyndon interpolant of \( F_b \) and \( \neg G_b^c \). Since \( F_b \), \( G_b^c \) and \( H_b \) are built up from the same two ground atoms, we introduce shorthands \( A, B \) for these to facilitate readability:

\[ A = p(h(g_2(h(f_1(g_1)))), h(f_1(h(g_2(h(f_1(g_1))))))), g_1). \]

\[ B = p(g_1, h(f_1(g_1))), h(g_2(h(f_1(g_1))))). \]

\[ F_b = H_b = A \land B. \]

\[ G_b^c = \neg A \lor \neg B. \]

Formulas \( F_q, G_q^c, H_q \) can be viewed as obtained from \( F_b, G_b^c, H_b \) by replacing \( f \)-terms and \( g \)-terms in occurrences that are not as subterm of another such term with dedicated variables, that is, different terms are replaced by different variables and identical terms with the same variable:

\[ F_q = H_q = p(h(x_1), h(y_1), x_2) \land p(x_2, h(y_2), h(x_1)). \]

\[ G_q^c = \neg p(h(x_1), h(y_1), x_2) \lor \neg p(x_2, h(y_2), h(x_1)). \]

\[ x = \{x_1, x_2\}. \]

\[ y = \{y_1, y_2\}. \]

\[ A = \{x_1 \mapsto g_2(h(f_1(g_1))), x_2 \mapsto g_1, y_1 \mapsto f_1(h(g_2(h(f_1(g_1))))), y_2 \mapsto f_1(g_1)\}. \]
Recall that in our example the \( \phi \) and is used in the proof to justify the semantic property of the lifted interpolant. In our case

\[
\text{The base case } F_0 \text{ provides that clause in two "copies":}
\]

\[
F_0 = \neg p(u_i^o, h(f_1(u_i^o)), u_i^o)), \wedge \neg p(u_i^o, h(f_1(u_i^o)), u_i^o)).
\]

\[
G_0^- = \neg p(h(g_2(v_i^o)), v_i^o, \sigma) \vee \neg p(g_1, v_i^o, h(g_2(v_i^o))).
\]

\[
u_i = \{u_i^o, u_i^e, u_i^e, u_i^e\}.
\]

\[
\eta = \{u_i^o \mapsto h(g_1(h(f_1(g_1)))), u_i^o \mapsto g_1, u_i^o \mapsto h(g_2(h(f_1(g_1)))),
\]

\[
u_i \mapsto h(f_1(h(g_1(g_1)))), v_i^o \mapsto h(f_1(h(g_1(g_1)))), v_i^o \mapsto h(f_1(h(g_1(g_1))))\}.
\]

A clausal formula \( F_1 \) that subsumes both \( F_0 \) and \( F_0^- \) along with the respective substitutions can be specified as:

\[
F_1 = \neg p(u_i^o, h(v_i^o), u_i^o)) \wedge
\]

\[
\neg p(u_i^o, h(v_i^o), u_i^o)).
\]

\[
G_1^- = \neg p(h(u_i^o), v_i^o, u_i^o) \vee \neg p(u_i^e, v_i^e, h(u_i^e));
\]

\[
r_i = \{v_i^e, v_i^e\}.
\]

\[
u = \{v_i^e \mapsto f_1(u_i^o), v_i^e \mapsto f_1(u_i^o), u_i^e \mapsto g_2(v_i^o), u_i^e \mapsto g_1\}.
\]

\[
\rho = \{u_i^o \mapsto h(x_1), v_i^o \mapsto y_1, u_i^o \mapsto x_2, v_i^o \mapsto y_2, u_i^o \mapsto h(x_1),
\]

\[
u_i \mapsto x_1, v_i^o \mapsto h(y_1), u_i^e \mapsto x_2, v_i^e \mapsto h(y_2)\}.
\]

The proof involves an induction where it is shown that for all \( i \in \{0, \ldots, m\} \) it holds that

\[
F \models \exists f (\forall x_i F_i \phi_i).
\]

The base case \( i = 0 \) is equal to: \( F \models \exists f (\forall x_i F_i \phi_i) \). The case \( i = m \) is equal to \( F \models R F_0 \)

and is used in the proof to justify the semantic property of the lifted interpolant. In our case

\[
\text{m is 4. The substitution } \phi \text{ is determined by } \lambda \text{ as follows:}
\]

\[
\phi = \{x_1 \mapsto g_2(h(y_2)), x_2 \mapsto g_1, y_1 \mapsto f_1(h(x_1)), y_2 \mapsto f_1(x_1)\}
\]

Recall that in our example the ordered set \( \{w_1, w_2, w_3, w_4\} \) is \( \{x_2, y_2, x_1, y_1\} \). The substitutions \( \phi_i \) used in the induction property are then:

\[
\phi_1 = \phi_0 = \phi_1 = \{y_2 \mapsto f_1(x_1), y_1 \mapsto f_1(h(x_1))\}.
\]

\[
\phi_2 = \phi_3 = \{y_1 \mapsto f_1(h(x_1))\}.
\]

\[
\phi_4 = \text{Identity}.
\]
We finish the example with showing the induction property for \( i \in \{0, \ldots, m = 4 \} \), where changes in the matrix compared to the previous step are highlighted by underlining:

\[
\begin{array}{ccc}
\begin{array}{c}
0 \quad F \models \exists f \\
1 \quad F \models \forall x_2 \\
2 \quad F \models \exists y_2 \\
3 \quad F \models \forall x_2 \exists y_2 \\
4 \quad F \models \forall x_2 \exists y_2 \exists y_1
\end{array}
\quad &
\begin{array}{c}
R_i \\
x_i \quad F_0 \phi_i
\end{array}
\quad &
\begin{array}{c}
\forall x_2 x_1 (p(h(x_1), h(f_1(h(x_1)))) \land p(x_2, h(h(x_1)))) \\
\forall x_1 (p(h(x_1), h(f_1(h(x_1)))) \land p(x_2, h(f_1(h(x_1)))) \\
\forall x_1 (p(h(x_1), h(f_1(h(x_1)))) \land p(x_2, h(y_2), h(x_1))) \\
\forall x_2 y_2 x_1 (p(h(x_1), h(f_1(h(x_1)))) \land p(x_2, h(y_2), h(x_1))) \\
\forall x_2 y_2 x_1 y_1 (p(h(x_1), h(y_1), x_2) \land p(x_2, h(y_2), h(x_1)))
\end{array}
\end{array}
\]

We conclude this section with a proposition that shows some properties of Craig-Lyndon interpolants constructed with the CTI procedure that go beyond the requirements of a Craig-Lyndon interpolant (Definition 3), are useful in certain applications, such as Theorem 17 below and easily follow from the specification of the CTI procedure:

**Proposition 13 (Properties of Interpolants Constructed with CTI)** Let \( QF \) and \( RG \) be first-order sentences such that \( Q \) and \( R \) are quantifier prefixes, \( F \) and \( G \) are quantifier-free formulas and it holds that \( QF \models RG \). Then, by the CTI method a first-order sentence \( SH \) can be constructed such that \( S \) is a quantifier prefix, \( H \) is a quantifier-free formula, \( SH \) is a Craig-Lyndon interpolant of \( QF \) and \( RG \), and it holds that:

1. If there is an existential quantification in \( S \), then there is an existential quantification in \( Q \) or there is a member of \( \text{fun}(F) \) that is not in \( \text{fun}(G) \).
2. If there is a universal quantification in \( S \), then there is a universal quantification in \( R \) or there is a member of \( \text{fun}(G) \) that is not in \( \text{fun}(F) \).
3. If \( F_c \) and \( G^c_\sigma \) are clausal formulas obtained from classifying \( QF \) and \( \lnot RG \), respectively, and \( N \) is the root of a closed two-sided ground tableau for \( F_c \) and \( G^c_\sigma \), then \( H \sigma = \text{ipol}(N) \) for some substitution \( \sigma \) whose domain is the set of the variables quantified in \( S \).

**Proof** Follows from the specification of Procedure 10 and its correctness, Theorem 11.

The first two items of Proposition 13 concern quantifiers in the interpolant in a coarse way, just with respect to their kind, existential or universal, without taking dependencies on their order into account. The third item states in essence that whenever for first-order inputs there is a ground interpolant of the respective clausalizations whose formula has a certain structure, then there is a first-order interpolant of the original inputs whose matrix has the same structure.

**6 Interpolant Lifting: Related Work**

The interpolant lifting of Procedure 10 by replacing terms in a ground interpolant \( H_\sigma \) with fresh variables \( z_1, \ldots, z_n \) and prepending a quantifier prefix \( Q_1z_1 \ldots Q_nz_n \) whose ordering respects the subterm relationship among the replaced terms has been already shown in essence by Huang [29]. Although this interpolant lifting can be expressed as a simple formula conversion, independently of any particular calculus, its correctness seems not trivial to prove and subtle issues arise. For example, as observed in [35], there is an error in [29] that concerns equality handling. Another example is a version of interpolant lifting developed in [13] where only constants are replaced by variables but which, as indicated in [13], does not generalize to compound terms in a way that is compatible with other techniques.
shown there. It seems that so far two proofs for interpolant lifting with respect to compound terms can be found in the literature: The proof of [29, Theorem 15] and the proof of [3, Lemma 8.2.2], seemingly obtained independently. Interpolant lifting is called abstraction in [3]. Further discussions and references can be found in [13,35,9]. Our use of lifting and our correctness proof differs from the related methods and proofs described in the literature [29, 3,13,35] in two important respects:

1. We apply lifting to a ground formula that actually is a Craig-Lyndon interpolant of two intermediate ground formulas that relate in a certain way to the input sentences. In contrast, lifting is applied in [29] to a so-called relational interpolant of the original input sentences, which is specified like a Craig interpolant, except that constraints on the functions need not to be satisfied. Similarly less constrained variants of a Craig interpolant of the original input sentences are used as basis for lifting in [3] (weak interpolant) and in [13] (provisional interpolant).

2. Our proof of the correctness of interpolant lifting is independent of a particular calculus. The correctness proofs in [29] and [3] are both based on modifying proofs as data structures, resolution proofs in the case of [29] and natural deduction proofs in the case of [3]. We assume more abstractly just Herbrand’s theorem, expressed in the form that for an unsatisfiable clausal first-order formula a closed ground tableau can be constructed, where terms are formed from input functions, Skolem functions and, if needed, an additional constant. The tableau enters our method as “given”, where the actual way in which it had been constructed is irrelevant. Provers would typically operate on non-ground clauses and hand over a closed non-ground tableau which is instantiated to a ground tableau only just before extraction of the ground interpolant. For practical implementation, this approach has the advantage that any system which computes a clausal tableau for an unsatisfiable first-order formula can be applied unaltered to the computation of first-order interpolants.

We now look into the details of some interesting aspects of Huang’s result in [29] in comparison to ours. There are similarities in the involved formulas or resolution derivations, respectively, used internally in both proofs: Huang’s proof uses a conversion of the given resolution deduction to what he calls binary tree deduction, where each clause is used at most once. Analogously, in our formulas $F_e$ and $G_e$ of the proof of Theorem 11 each variable is instantiated to a ground term by the substitution $\eta$. In Huang’s proof, the binary tree deduction is converted further to what he calls a propositional deduction, which correspond to our ground formulas $F_\eta = F_e \eta$ and $G_\eta = G_e \eta$.

In [29] equality handling with paramodulation is explicitly taken into account, which, however, leads to the mentioned error in Huang’s lifting theorem [35]. The proof of [3, Lemma 8.2.2] applies just to formulas without equality. A possibility to integrate equality handling into our method is described in Sect. 8.3.

A minor difference between our and Huang’s lifting technique is that Huang orders variables in the quantifier prefix by the length of the associated terms, more constrained than the strict subterm relationship used here.

In contrast to Huang’s method for constructing relational interpolants, the method of [3] to construct weak interpolants involves certain cases where quantified variables are introduced. In Huang’s relational interpolants free variables are allowed, upon which extra quantifiers will be added after lifting. As indicated in [29, p. 188], this can be done in an arbitrary way: the extra quantifiers can be existential or universal, at any position in the prefix. In our formalization, the base formulas used for lifting have to be ground. The effects
described by Huang are subsumed by the alternate possibilities to instantiate non-ground tableaux delivered by provers as discussed in Sect. 8.1.

The input formulas in Huang’s interpolation method are clausal formulas. In the symbolism of the proof of our Theorem 11, his method computes interpolants of \( \forall u_c F_c \) and \( \exists v_c \neg G_c \). The handling of arbitrary first-order formulas by Skolemization incorporated in our proof needs to be wrapped around Huang’s core theorem, which is, however not difficult: Staying in the symbolism of the proof of our theorem, the set \( f \cup g \) includes the involved Skolem functions. An interpolant of \( \forall u_c F_c \) and \( \exists v_c \neg G_c \) in which – after lifting – no members of \( f \cup g \) occur is also an interpolant of \( F \) and \( G \), the original formulas before Skolemization.

7 Positive Hyper Tableaux and Interpolation from a Horn Sentence

So far, our interpolant construction based on clausal tableaux applies to arbitrarily structured closed clausal ground tableaux. To obtain interpolants that, in dependence of syntactic properties of the input formulas, have specific syntactic properties beyond those required from Craig-Lyndon interpolants, it is useful to consider clausal tableaux with structural restrictions. Two basic restrictions are specified with the following definition:

Definition 14 (Tableau Properties: Regular, Leaf-Only) Define the following properties of clausal tableaux:

(i) Regular: No node has an ancestor with the same literal label.
(ii) Leaf-only for a set \( S \) of pairwise non-complementary literals: Members of \( S \) do not occur as literal labels of inner nodes.

Regularity is a well-known standard notion to avoid redundancies in tableaux, see, e.g., [37,31]. Any closed clausal tableaux for some clausal formula can be converted with a tableau simplification to a regular closed clausal tableau for the same formula ([38], see also Sect. 11). The leaf-only property can be applied to model constraints on clausal tableaux that are constructed by “bottom-up” methods, as shown with Definition 15 below. In Sect. 10 we will apply it together with a further tableau restriction to essentially simulate non-clausal tableaux with clausal tableaux. Any closed clausal tableau can be transformed to a closed clausal tableau for the same formula that is leaf-only for a given set of pairwise non-complementary literals, although the required transformations are potentially expensive (see Sect. 11).

In the introduction we mentioned the important family of methods that can be understood as constructing a clausal tableaux “bottom-up”, in a “forward-chaining” manner, by starting with positive axioms and deriving positive consequences, with the hyper tableaux calculus [4] as a representative. The following definition, expressed in terms of properties from Definition 14, renders structural constraints that are typically observed by tableaux constructed with these methods:

Definition 15 (Positive Hyper Tableau) A clausal tableau that is regular and leaf-only for the set of all negative literals occurring as labels the tableau is called a positive hyper tableau.

In a closed positive hyper tableau the leaves are exactly the nodes with negative literal label. The term positive hyper tableau is from [31], where methods that construct such tableaux are investigated as specializations in a general setting of clausal tableau methods with selection functions. Availability of complete methods ensures that for any unsatisfiable clausal formula a closed positive hyper tableau can be constructed. These construction methods typically
observe further constraints that are not modeled in Definition 15 since they are not relevant for extracting interpolants from a given closed tableau. This includes in particular that variable scopes are only clause-local and that nodes labeled with a negative literal are always closed ("weakly connected"), also during construction when the overall tableau is not yet closed.

To demonstrate how the clausal tableau framework for first-order Craig-Lyndon interpolation can be applied to derive further properties of constructed interpolants in dependency of properties of the input formulas we now show that, if the first interpolation argument is a Horn sentence, then for arbitrary sentences as second arguments an interpolant that is a Horn sentence can be constructed. We first specify some syntactically characterized formula classes:

**Definition 16 (Formula Classes: Universal, Existential, Positive, Negative, Horn)**

(i) A sentence is called **universal** (**existential**) if it is a first-order sentence of the form $QF$, where $Q$ is an individual quantifier prefix with only universal (existential) quantifications and $F$ is quantifier-free.

(ii) A formula is called **positive** (**negative**) if and only if all occurrences of atoms in the formula have positive (negative) polarity.

(iii) A sentence is called **Horn** if and only if it is a first-order sentence of the form $QF$ where $Q$ is a quantifier prefix and $F$ is a quantifier-free clausal formula with at most one positive literal in each of its clauses.

Now the claimed property of interpolants where the first argument is Horn can be made precise as follows:

**Theorem 17 (Interpolation from a Horn Sentence)** Let $F$, $G$ be first-order sentences such that $F$ is Horn and $F \models G$. Then a Craig-Lyndon interpolant $H$ of $F$ and $G$ can be constructed such that

1. $H$ is a Horn sentence.
2. If $F$ and $G$ are universal sentences and $\text{fun}(F) \subseteq \text{fun}(G)$, then $H$ is a universal sentence.
3. If $F$ and $G$ are existential sentences and $\text{fun}(G) \subseteq \text{fun}(F)$, then $H$ is an existential sentence.

**Proof** Let is essentially a Horn ground formula stand for is a Horn ground formula or can be converted to an equivalent Horn ground formula by distributing disjunction upon conjunction. Existence of a closed two-sided positive hyper ground tableau for any clausification results of $F$ and $\neg G$ follows from the completeness of proving methods that construct positive hyper tableaux. Since $F$ is Horn, it can be clausified such that the respective clausal formula is Horn. The theorem then follows from Proposition 13 since, as we will show, if $N$ is the root of a closed two-sided positive hyper ground tableau for clausal formulas $F_L$ and $F_R$ where $F_L$ is Horn, then the formula $\text{ipol}(N)$ is a essentially a Horn ground formula. We prove the latter claim by showing with induction on the tableau structure the following more general statement:

**IP** For all nodes $N$ of a closed two-sided clausal positive hyper ground tableau for clausal formulas $F_L$ and $F_R$ where $F_L$ is Horn the formula $\text{ipol}(N)$ is essentially a Horn ground formula.

For the base case where $N$ is a leaf it is immediate from Definition 5 that $\text{ipol}(N)$ is a ground literal or a truth value constant and thus obviously a Horn ground formula. To show the induction step, let $N$ be an inner node with children $N_1, \ldots, N_n$ where $n \geq 1$. As induction
hypothesis assume that for all $i \in \{1, \ldots, n\}$ it holds that $\text{ipol}(N_i)$ is essentially a Horn ground formula. We prove the induction step by showing that then also $\text{ipol}(N)$ is essentially a Horn ground formula:

- Case $\text{side}(N_1) = L$: Observe that since the tableau is leaf-only for all negative literals it holds in this case for all $i \in \{1, \ldots, n\}$ such that $\text{lit}(N_i)$ is negative that either $\text{ipol}(N) = \bot$ or $\text{ipol}(N)$ is a negative ground literal.
  - Case $\text{clause}(N)$ is negative: Then $\text{ipol}(N) = \bigvee_{i=1}^{n} \text{ipol}(N_i)$ is a disjunction of negative ground literals, hence a Horn ground formula.
  - Case $\text{clause}(N)$ is not negative: Since $\text{FL}$ is Horn, $\text{clause}(N)$ has exactly one child whose literal label is positive. Let $N_j$ with $j \in \{1, \ldots, n\}$ be that child. By the induction hypothesis $\text{ipol}(N_j)$ is essentially a Horn ground formula. Since, as observed above, for all $i \in \{1, \ldots, n\} \setminus \{j\}$ the formula $\text{ipol}(N_i)$ is $\bot$ or a negative ground literal it follows that $\text{ipol}(N) = \bigvee_{i=1}^{n} \text{ipol}(N_i)$ is essentially a Horn ground formula.

- Case $\text{side}(N_1) = R$: From the induction hypothesis it follows that $\text{ipol}(N) = \bigwedge_{i=1}^{n} \text{ipol}(N_i)$ is essentially a Horn ground formula.

As we have seen with Theorem 17, the framework for interpolation based on clausal tableaux allows to prove the existence of interpolants that meet certain constraints quite easily and, moreover, in a constructive way that can be realized directly by practical automated reasoning systems. An apparently weaker property has been shown in [45, § 4] with techniques from model theory: For two universal Horn formulas there exists a universal Horn formula that is like a Craig interpolant, except that function symbols occurring in it are not constrained.

8 Craig-Lyndon Interpolation: Refinements and Issues

8.1 Choices in Grounding and Side Assignment

Procedure 10 for the construction of first-order interpolants leaves at several stages alternate choices that have effect on the formula returned as interpolant. We discuss some of these here, although a thorough investigation of ways to integrate the exploration and evaluation of these into interpolant construction seems a nontrivial topic on its own.

The first choice concerns the instantiation of variables in the tableaux returned by provers. Typically, provers instantiate variables just as much “as needed” by the calculus to compute a closed tableau. To match with our interpolant lifting technique, variables in the literal labels of such non-ground tableaux have to be instantiated by ground terms. There are different possibilities to do so, all yielding a closed tableau for the input clauses, but leading to different interpolants: A variable can be instantiated by a term whose functions all occur in both interpolation inputs. The term will then occur in the interpolant. Alternatively, the variable can be instantiated by a term with a principal functor that has been introduced at Skolemization, either of the first or of the second input formula, or that occurs just in one of the input formulas. In the procedure description the fresh constant $k$ that is handled like a Skolem constant in the second input formula has been introduced to have such a term available in any case. By interpolant lifting the term will then be replaced by a variable whose kind, existential or universal, depends on the principal functor of the term, and whose quantifier position in the prefix is constrained by subterms. Of course, also a combination of these two ways is possible, that is, instantiating with a term whose principal functor occurs in both input formulas but which has subterms with a functor that does not occur in one of the input formulas.
Aside of these alternate possibilities that concern the instantiation of each variable individually, there are also choices to instantiate different variables by the same term or by different terms: Arbitrary subsets of the free variables of the literal labels of the tableau can be instantiated with the same ground term, leading in the interpolant to fewer quantified variables but to more variable sharing. In the description of Procedure 10 the fresh constant \( k \) is uniformly used to instantiate all variables.

The second possibility for choice concerns the assignment of the side \( L \) and \( R \) to tableau clauses. Existing systems for tableau construction typically would require changes to their internal data structures to maintain such side information, which is undesirable. However, assuming that sides are associated with the given input clauses, such systems can be actually used unaltered to construct a two-sided clausal tableau: Sides can then be assigned to the clauses of the returned tableau "in retrospect", by matching against the input clauses. In some cases there are choices: A tableau clause can be an instance of some input clause with side \( L \) as well as of some input clause with side \( R \), or a clause can be present in copies for each side. In these cases it is possible to assign either side to the tableau clause, where both assignments may lead to different interpolants.

8.2 Goal-Sensitivity

Model elimination and the connection method are goal sensitive: They construct a clausal tableau by starting with a clause from a designated subset of the input clauses, the “goal clauses”. Without loss of completeness the set of negative clauses can, for example, be taken as goal clauses, or, if a theorem is to be proven from a consistent set of axioms, the clauses representing the (negated) theorem. It remains to be investigated what choices of goal clauses are particularly useful for the computation of interpolants.

8.3 Equality Handling

So far we considered only first-order logic without equality. Nevertheless, our method to compute interpolants can be used together with the well-known encoding of equality as a binary predicate with axioms that express its reflexivity, symmetry and transitivity as well as axioms that express substitutivity of predicates and functions. If the input formulas of interpolant computation involve equality, these axioms have to be added. The clauses of substitutivity axioms for predicate or function symbols that occur only in the first (second) input formula then receive side \( L \) (\( R \)). The side of clauses of axioms for reflexivity, symmetry and transitivity can be assigned arbitrarily, including the possibility to have two copies of the clauses, one for each side.

For relational formulas, more can be said about the polarity in which equality may occur in the interpolant in cases where it occurs only in the first (second) of input formula: Then the clauses of axioms for reflexivity, symmetry and transitivity can be assigned to the corresponding side to ensure that in the computed interpolant equality only occurs positively (negatively). This follows from the “Lyndon property”, the condition that predicates occur in the interpolant only in polarities in which they occur in both input formulas, since in substitutivity clauses for predicates, which are then the only clauses with equality literals whose side is \( R \) (\( L \)), equality only occurs negatively. Stronger possible constraints on interpolants with respect to equality are stated in an interpolation theorem due to Oberschelp and Fujiwara (see [47]).
The example from [35] to demonstrate the mentioned error in [29] in presence of equality is finding an interpolant of \( r(a) \neq r(b) \) and \( a \neq b \): Huang’s proof would imply \( \exists x \ x \neq r(b) \models a \neq b \), which does not hold in general. With our suggested encoding we would obtain \( a \neq b \) as ground interpolant of the ground formulas \( r(a) \neq r(b) \wedge (a \neq b \lor r(a) = r(b)) \) and \( a \neq b \), and, because lifting has no effect, also correctly as interpolant of the original inputs \( r(a) \neq r(b) \) and \( a \neq b \).

8.4 Preprocessing for Interpolation

Sophisticated preprocessing is a crucial component of automated reasoning systems with high performance. While formula simplifications such as removal of subsumed clauses and removal of tautological clauses preserve equivalence, others only preserve unsatisfiability. For example, purity simplification, that is, removal of clauses that contain a literal with a predicate that occurs only in a single polarity in the formula. Many simplifications of the latter kind actually preserve not just unsatisfiability, but, moreover, equivalence with respect to a set of predicates, or, more precisely, a second-order equivalence

\[
\exists p_1 \ldots \exists p_n \ F \equiv \exists p_1 \ldots \exists p_n \ \text{simplify}(F),
\]

where \( \text{simplify}(F) \) stands for the result of the simplification operation applied to \( F \). One might say that the semantics of the predicates not in \( \{p_1, \ldots, p_n\} \) is preserved by the simplification.

For the computation of Craig-Lyndon interpolants it is possible to preprocess the first as well as the negated second input formula independently from each other in ways such that the semantics of the predicates occurring in both formulas is preserved in this sense. Preprocessors that support simplification operations that can be parameterized with a set of predicates whose semantics has to be preserved (see [64, Section 2.5] for a discussion) can be applied for that purpose.

For clausal tableau methods some of these simplifications are particularly relevant as they complement tableau construction with techniques which break apart and join clauses and may thus introduce some of the benefits of resolution. Techniques for propositional logic that preserve equivalence (ii) for certain sets of predicates include variable elimination by resolution and blocked clause elimination. For first-order generalizations of these, the handling of equality seems the most difficult issue. Predicate elimination can in general introduce equality also for inputs without equality. In a semantic framework where the Herbrand universe is taken as domain this can be avoided to some degree, as shown in [63] with a variant of the SCAN algorithm [25] for predicate elimination. Blocked clause elimination in first-order logic [34] comes in two variants, for formulas without and with equality, respectively.

8.5 Issues Related to Definer Predicates

Another way to utilize equivalence (ii) is by introducing fresh “definer” predicates for example by structure-preserving normal forms such as the Tseitin transformation and first-order generalizations of it [55,61,23,53]. Actually, in our approach to compute access interpolants this will play an important role. If disjoint sets of definer predicates are used for the first and for the second interpolation input, then, by the definition of Craig-Lyndon interpolant, definer predicates do not occur in the interpolant. In certain situations, which need further investigation, it might be useful to relax this constraint. For example, if in preprocessing two definers whose associated subformulas are equal should be identified,
even if one was introduced for the first and the other for the second interpolation input. Another example would be allowing definers occurring in the interpolant in cases where this permits a condensed representation of a formula whose equivalent without the definers would be much larger but straightforward to obtain.

8.6 Implementation – Current State

The PIE system [64] includes an implementation of the described approach to Craig-Lyndon interpolation. Currently the goal-sensitive first-order prover CM included with PIE is supported as underlying theorem prover. Support for using also Hyper [52,62,7] in that role has been implemented in part. The clausal tableaux used for interpolant extraction are represented as Prolog terms, providing a potential interface also to further provers. Configurable preprocessing which respects preservation of predicate semantics as required for interpolation is included. Symmetric interpolation [20, Lemma 2] (the name is due to [44]) with consideration of predicate polarity is implemented as iterated interpolation with two inputs. Other implementations of interpolation will be discussed in Sect. 13.4 in the context of query reformulation.

9 Access Interpolation with Clausal Tableaux: Overview and Basic Notions

Access interpolation [10] is a recently introduced form of interpolation for applications in query reformulation where the two input formulas as well as the computed interpolants are in a fragment of first-order logic, first-order logic with relativized quantifiers. This fragment allows to associate a binding pattern, or access pattern, with each atom occurrence: a representation of its polarity, of its predicate, and of a division of argument positions into input and output positions. The technical framework for access interpolation has been developed in [10] on the basis of Smullyan’s non-clausal tableaux [56,24], which follow the formula structure, proceeding from the overall input into subformulas, which allows to integrate relativized quantifiers whose scope is a subformula in an elegant way. This correspondence to the formula structure is as such not available in clausal tableaux, obtained after clausification, Skolemization and with techniques targeted at automated processing that follow inner connection structures instead of the formula structure. The basic approach adopted here is to “simulate” aspects of Smullyan’s tableaux by clausal tableaux as much as needed for the extraction of interpolants that are in the target fragment with relativized quantifiers. This is achieved by first converting the input formulas into a structure preserving normal form. Then there are two ways to proceed, which we will both consider: The first is to compute a closed clausal tableau that is constrained in a particular way such that an access interpolant can be extracted. The second is to compute an arbitrary closed clausal tableau and convert it such that it meets the constraints required to extract an access interpolant. With the following Definitions 18–20 we recapitulate precise notions underlying access interpolation, adapted from [10]:

**Definition 18 (RQFO Formula)**

(i) The formulas of first-order logic with relativized quantifiers, briefly RQFO formulas, are the relational formulas that are generated by the grammar

\[ F ::= \top | \bot | F \land F | F \lor F | \forall v (\neg R \lor F) | \exists v (R \land F), \]

where in the last two grammar rules \( v \) matches a (possibly empty) set of variables and \( R \) matches a relational atom in which all members of \( v \) occur.
(ii) If $F$ is an RQFO formula, then $\neg F$ denotes the RQFO formula obtained from rewriting $\neg F$ exhaustively with equivalences that propagate negation inwards, that is: $\neg \top \equiv \bot$; $\neg \bot \equiv \top$; $\neg (F \land G) \equiv \neg F \lor \neg G$; $\neg (F \lor G) \equiv \neg F \land \neg G$; $\neg \forall v (\neg A \lor F) \equiv \exists v (A \land \neg F)$; $\neg \exists v (A \land F) \equiv \forall v (\neg A \lor \neg F)$.

Definition 19 (Binding Pattern and Related Notions)

(i) A binding pattern is a triple $\langle \text{Sign}, \text{Predicate}, \text{InputPositions} \rangle$, where $\text{Sign} \in \{+,−\}$, $\text{Predicate}$ is a predicate and $\text{InputPositions}$ is a set of numbers larger than or equal to 1 and smaller than or equal to the arity of $\text{Predicate}$. A binding pattern with sign $+$ ($-$) is called existential (universal).

(ii) A binding pattern $\langle S, P, O \rangle$ is covered by a binding pattern $\langle S', P', O' \rangle$ if and only if $S = S'$, $P = P'$ and $O' \subseteq O$. A set $B$ of binding patterns is covered by a set of binding patterns $B'$ if and only if each member of $B$ is covered by some member of $B'$.

(iii) The binding patterns $\text{bp}(F)$ of an RQFO formula $F$ is a set of binding patterns defined inductively as follows:

- $\text{bp}(\top) \equiv \text{bp}(\bot) \equiv \{\}$.
- $\text{bp}(G \land H) \equiv \text{bp}(G \lor H) \equiv \text{bp}(G) \cup \text{bp}(H)$.
- $\text{bp}(\forall v (\neg r(t_1, \ldots, t_n) \lor G)) \equiv \{\langle r, \{i \mid t_i \notin v\}\rangle \} \cup \text{bp}(G)$.
- $\text{bp}(\exists v (r(t_1, \ldots, t_n) \land G)) \equiv \{\langle +, r, \{i \mid t_i \notin v\}\rangle \} \cup \text{bp}(G)$.

For example, if $F = \forall x (\neg r(x) \lor \exists y \exists z (s(x, y, z) \land \top))$, then $\text{bp}(F) = \{\langle -, r, \{\rangle, \langle +, s, \{1\}\rangle\}$.

Definition 20 (Access Interpolant) Let $F, G$ be RQFO sentences such that $F \models G$. An access interpolant of $F$ and $G$ is an RQFO sentence $H$ such that

1. $F \models H \models G$.
2. $\text{pred}(H) \subseteq \text{pred}(F) \cap \text{pred}(G)$.
3. Every existential binding pattern of $H$ is covered by an existential binding pattern of $G$.
   Every universal binding pattern of $H$ is covered by a universal binding pattern of $F$.
4. $\text{const}(H) \subseteq \text{const}(F) \cap \text{const}(G)$.

Our approach to compute access interpolants with clausal tableau resides on a structure preserving, also called definitional, normal form \cite{55,61,23,53} for clausifying the two input RQFO formulas. Auxiliary “definer” predicates for subformulas are introduced there. By using disjoint sets of definer predicates for the conversion of each of the two input formulas it is ensured that definer predicates do not occur in interpolants. The normalization yields only clauses of certain specific forms. To specify the subformula definers we use the following common notions of subformula position and subformula at a position, specialized to RQFO formulas by considering the relativizer literals not as subformulas on their own but as belonging to the associated quantifications:

Definition 21 (Position within an RQFO Formula)

(i) A position of a subformula occurrence within an RQFO formula is a finite sequence of integers.

(ii) The positions $\text{pos}(F)$ of an RQFO formula is a set of positions defined inductively as follows: If $F$ is $\top$ or $\bot$, then $\text{pos}(T) \equiv \{\}$, if $F$ is of the form $F_1 \land F_2$ or $F_1 \lor F_2$, then

\footnote{3 Compared to \cite[Thm. 3.12]{10} in this definition of access interpolant from the condition (3.) the explicit requirements that the predicate of an existential (universal) binding pattern of $H$ occurs positively in $F$ (negatively in $G$) have been dropped because these are already implied by condition (2.).}
We assume a total order on the set of all variables, called the letpredicates as specified in Definition 22. Then Let

Proposition 23 (Semantic Properties of the Definitional Form of an RQFO Formula)

formula and its definitional form as specified in Definition 22 is captured by a second-order alences are known as

Definition 22 (Definitional Form of an RQFO Formula)
yield conjunctions of first-order formulas of certain shapes.
The following definition specifies structure preserving conversions of RQFO formulas that

Proposition 24 (Semantic Property of Interpolants for RQFO Formulas in Definitional Form)

interpolant

We assume a total order on the set of all variables, called the standard order of variables. The following definition specifies structure preserving conversions of RQFO formulas that yield conjunctions of first-order formulas of certain shapes.

Definition 22 (Definitional Form of an RQFO Formula) Let F be an RQFO formula.
(i) For all positions \( p \in \text{pos}(F) \) let \( x_p \) denote the sequence of the members of \( \text{var}(F|_p) \) ordered according to the standard order of variables and let \( D_p \) denote the atom \( d_p(x_p) \), where \( d_p \) is a fresh predicate, also called a definer predicate. For all positions \( p \in \text{pos}(F) \) define the sentence \( \text{def}_p(F) \) depending on the form of \( F|_p \) as shown in the following table:

\[
\begin{array}{ccc}
F|_p & \text{def}_p(F) \\
\top & D_p \rightarrow \top \\
\bot & D_p \rightarrow \bot \\
G \land H & \forall x_p(D_p \rightarrow (D_{p1} \land D_{p2})) \\
G \lor H & \forall x_p(D_p \rightarrow (D_{p1} \lor D_{p2})) \\
\exists x(\neg R \lor G) & \forall x_p(D_p \rightarrow \exists x (R \lor D_{p1})) \\
\exists x (R \land G) & \forall x_p(D_p \rightarrow \exists x (R \land D_{p1}))
\end{array}
\]

(ii) Define the following formula:

\[
\text{DEF}(F) \equiv d_k(x_e) \land \land_{p \in \text{pos}(F)} \text{def}_p(F).
\]

Structural normal forms that are like Definition 22.i based on of implications instead of equivalences are known as Plaited-Greenbaum form [53]. The semantic relationship between a formula and its definitional form as specified in Definition 22 is captured by a second-order equivalence, which is easy to verify with Ackermann’s lemma [1,22]:

Proposition 23 (Semantic Properties of the Definitional Form of an RQFO Formula)

Let F be an RQFO formula, let \( \{p_0, \ldots, p_n\} = \text{pos}(F) \) and let \( d_{p_1}, \ldots, d_{p_n} \) be definer predicates as specified in Definition 22. Then

\[
F \equiv \exists d_{p_1} \ldots \exists d_{p_n} \text{DEF}(F).
\]

Proposition 23 allows to express the semantic requirement (1.) of the definition of access interpolant (Definition 20) in terms of the normalized formulas:

Proposition 24 (Semantic Property of Interpolants for RQFO Formulas in Definitional Form) Let F, G be RQFO formulas. Let \( \{p_1, \ldots, p_m\} = \text{pos}(F) \), let \( \{q_1, \ldots, q_n\} = \text{pos}(G) \), and let predicates \( d_{p_1}, \ldots, d_{p_m} \) and \( e_{q_1}, \ldots, e_{q_n} \) be the definer predicates introduced with forming \( \text{DEF}(F) \) and \( \text{DEF}(\neg G) \), respectively, according to Definition 22. Let H be a formula such that \( \text{pred}(H) \cap \{d_{p_1}, \ldots, d_{p_m}, e_{q_1}, \ldots, e_{q_n}\} = 0 \). Then the following statements are equivalent:

1. \( F \models H \models G \).
2. \( \exists d_{p_1} \ldots \exists d_{p_m} \text{DEF}(F) \models H \models \neg \exists e_{q_1} \ldots \exists e_{q_n} \text{DEF}(\neg G) \).
3. \( \text{DEF}(F) \models H \models \neg \text{DEF}(\neg G) \).
As basis for computing an access interpolant we thus can take a closed two-sided clausal tableau for a clausal form of $\text{DEF}(F)$ as $F_i$ and a clausal form of $\text{DEF}(\neg G)$ as $F_R$. The following lemma specifies the clause forms obtained and introduces symbolic notation to refer to particular literals, variables and Skolem functions occurring in them:

Lemma 25 (Definitional Clausification of an RQFO Formula) Let $F$ be an RQFO formula. For all $p \in \text{pos}(F)$ let $d_p$ denote the definer predicate for $p$ introduced at forming $\text{DEF}(F)$. Let $x_p$ denote the sequence of the members of $\text{var}(F_p)$ ordered according to the standard order of variables, and let $D_p$ denote the atom $d_p(x_p)$. For all $p \in \text{pos}(F)$ where $F_p$ is of the form $\forall x_1 \ldots \forall x_n (\neg R \lor F')$ or $\exists x_1 \ldots \exists x_n (R \land F')$ let $R_p$ denote $R$ and let $v_p$ denote $\{x_1, \ldots, x_n\}$. For all $p \in \text{pos}(F)$ where $F_p$ is of the form $\exists x_1 \ldots \exists x_n (R \land F')$ let $f_{(p, 1)}, \ldots, f_{(p, n)}$ be fresh functions and let $\sigma_p$ be the substitution $\{x_1 \mapsto f_{(p, 1)}(x_p), \ldots, x_n \mapsto f_{(p, n)}(x_p)\}$. Then $\text{DEF}(F)$ is equivalent to the existential quantification upon Skolem functions of the universal closure of a clausal formula, where the Skolem functions are the introduced $f_{(p, i)}$ and the clauses are of the following forms, satisfying restrictions on arguments of atoms and free variables as indicated:

| No. | Clause Form | Restrictions |
|-----|-------------|--------------|
| 1   | $D_x$       | If $F$ is a sentence, then $\text{arg}(D_x) = \emptyset$ |
| 2   | $\neg D_p$  | $\text{arg}(D_p) \subseteq \text{arg}(\neg D_p)$ |
| 3   | $\neg D_p \lor D_{p1}$ | $\text{arg}(D_{p1}) \subseteq \text{arg}(\neg D_p)$ |
| 4   | $\neg D_p \lor D_{p2}$ | $\text{arg}(D_{p2}) \subseteq \text{arg}(\neg D_p)$ |
| 5   | $\neg D_p \lor D_{p1} \lor D_{p2}$ | $\text{arg}(D_{p1} \lor D_{p2}) \subseteq \text{arg}(\neg D_p)$ |
| 6   | $\neg D_p \lor \neg R_p \lor D_{p1}$ | $\text{arg}(D_{p1}) \subseteq \text{arg}(\neg D_p \lor \neg R_p)$, $\text{arg}(\neg R_p \lor D_{p1}) \subseteq \text{arg}(\neg D_p) \cup v_p$ |
| 7   | $\neg D_p \lor R_p \sigma_p$ | $\text{var}(R_p \sigma_p) \subseteq \text{arg}(\neg D_p)$ |
| 8   | $\neg D_p \lor D_{p1} \sigma_p$ | $\text{var}(D_{p1} \sigma_p) \subseteq \text{arg}(\neg D_p)$ |

Proof The required clausal form would be obtained by common CNF transformation methods, provided Skolemization is applied individually to each implication of the form $\forall x_p (D_p \rightarrow \exists x (R \land D_{p1}))$. □

The order within blocks 1–8 of Lemma 25 corresponds to the order in which clauses would be obtained by a straightforward CNF translator applied on the definitional implications in the order displayed in Definition 22.i.

The applied variant of Skolemization is inner Skolemization [49]. This follows because the universal quantifications upon $x_p$ that precedes the quantification upon the Skolemized variables $v$ is exactly upon the free variables of the argument formula of the quantification upon $v$, that is, $R \land d_{p1}(x_{p1})$. Considering that the arguments of $x_p$ are exactly the free variables of $F_p$ the applied Skolemization also corresponds to inner Skolemization with respect to the original formula $F$ before translation to definitional form.

10 Access Interpolant Extraction from Clausal Tableaux

To permit extraction of access interpolants, clausal tableaux have to satisfy certain restrictions that are specified with Definition 27 below. Aside of the regular, closed and leaf-only properties, which have already been specified, a further property is now needed:
**Definition 26 (Contiguous)** A clausal tableau is called **contiguous** for an unordered pair of literals if and only if whenever both members of the pair occur as literal labels of two nodes on the same branch, one of the nodes is the parent of the other.

The **contiguous** property is used to represent relativized quantification by handling conjuncts in the scope of an existential quantifier simultaneously, specifically the atom that relativizes the quantified variables and a second atom with a definer predicate that represents the argument of the relativized quantification. For this application the **contiguous** property can be ensured by a tableau simplification, Procedure 45 shown in Sect. 11. We have now specified all prerequisites to define the constraint package on clausal tableaux for access interpolation and call tableaux that satisfy it **ACI-tableaux**, suggesting **Access Interpolation**

**Definition 27 (ACI-Tableau)** Let $F$, $G$ be RQFO sentences. An ACI-tableau for $F$ and $G$ is a closed two-sided clausal ground tableau for two clausal formulas obtained from $F$ and $G$ by clausifying $\text{DEF}(F)$ and $\text{DEF}(\neg G)$ as specified in Lemma 25 that is regular, leaf-only for the set of all negative literals that occur as literal labels in it, and contiguous for all pairs of ground literals that occur as literal labels in it and have, referring to the notation of Lemma 25, the form $\{R_p\sigma_p\mu, D_p1\sigma_p\mu\}$ for some position $p$ in $F$ or in $G$ and some ground substitution $\mu$.

Note that an ACI-tableau is a special case of a closed positive hyper tableau (Definition 15). The specification of the extraction of an access interpolant from a clausal tableau involves a form of lifting that differs from the lifting described for Craig-Lyndon interpolants with Procedure 10. For access interpolation lifting can not be performed globally on a ground interpolant but on subformulas that correspond to the scopes of relativized quantifiers. To specify this form of lifting we need further auxiliary concepts that concern those occurrences of ground terms in a formula that are as argument of an atom, in contrast to embedded in another term. Symbolic notation for referring to these occurrences as well as for systematically replacing these occurrences with variables is provided. Preconditions are made precise under which an entailment relationship between formulas still holds after such a replacement by variables.

**Definition 28 (Set of Ground Arguments of Atoms)** If $F$ is a formula, then $garg(F)$ denotes the set of ground terms in $\text{arg}(F)$.

For example, if $x, y$ are variables and $a, b$ are constants, then

$$garg(\forall x p(a, g(a), b, x, f(y, b))) = \{a, g(a), g(b)\}.$$  

For relational formulas $F$ it holds that $garg(F) = \text{const}(F) = \text{fun}(F)$. Based on $garg(F)$, we define for injective substitution $\sigma$ the following restricted variant of $F\langle\sigma^{-1}\rangle$:

**Definition 29 (Inverse Substitution of Ground Arguments of Atoms)** If $F$ is a formula and $\sigma$ is an injective ground substitution such that $\text{rng}(\sigma) \subseteq garg(F)$, then let $F\langle\sigma^{-1}\rangle$ denote $F$ with all occurrences of members $t$ of $\text{rng}(\sigma)$ that are as argument of an atom replaced with the variable mapped by $\sigma$ to $t$.

While in $F\langle\sigma^{-1}\rangle$ occurrences of terms that are not strict subterms of some other member of $\text{rng}(\sigma)$ are replaced, in $F\langle\sigma^{-1}\rangle$ only occurrences that are arguments of atoms are replaced. The following proposition relates these two forms of “inverse substitution”:
Proposition 30 (Inverse Substitution of Arguments of Atoms and of Terms) Let \( F \) be a formula in which all non-ground terms are variables, let \( \sigma \) be an injective ground substitution such that \( \text{rng}(\sigma) \subseteq \text{garg}(F) \) and no member of \( \text{dom}(\sigma) \) occurs in \( F \), and let \( \gamma \) be an injective substitution such that \( \text{rng}(\gamma) = \text{garg}(F) \), no member of \( \text{dom}(\gamma) \) occurs in \( F \) and \( \gamma_{\text{dom}(\sigma)} = \sigma \).

Then

\[
F \langle \langle \sigma^{-1} \rangle \rangle = F \langle \langle \gamma^{-1} \rangle \rangle_{\text{dom}(\gamma)}|\text{dom}(\sigma).
\]

The following proposition states a variant of Proposition 9 where occurrences of possibly complex ground terms that themselves are not subterms of other terms are replaced by quantified variables. We will apply it later to justify lifting from ground terms introduced through Skolemization to quantified variables.

Proposition 31 (Inessential Quantifications in Entailments for Terms) Let \( F, G \) be formulas in which no non-ground terms with the exception of variables occur. Let \( \sigma \) be a ground substitution such that \( \text{rng}(\sigma) \subseteq \text{garg}(F) \), \( \text{rng}(\sigma) \cap \text{garg}(G) = \emptyset \) and no member of \( \text{dom}(\sigma) \) occurs in \( F \) or in \( G \). Let \( x \) stand for \( \text{dom}(\sigma) \). Then

\[
\exists x \ F \langle \langle \sigma^{-1} \rangle \rangle \models G \text{ if and only if } F \models G.
\]

We are now equipped with the prerequisites to specify the extraction of an access interpolant from an ACI-tableau, that is, a constructive mapping from an ACI-tableau for two RQFO sentences \( F \) and \( G \) such that \( F \models G \) to an access interpolant of them. The correctness of the mapping is then stated and proven as Theorem 33.

Definition 32 (Access Interpolant Extraction from an ACI-Tableau) Let \( F, G \) be RQFO sentences such that \( F \models G \) and let \( T \) be an ACI-tableau for \( F \) and \( G \). For all inner nodes \( N \) of \( T \) define \( \text{acc-ipol}(N) \) inductively as follows, where \( N_1, \ldots, N_k \) with \( k \leq 1 \) are the children of \( N \), and clause forms are understood as specified in Lemma 25:

i. Case \text{clause}(N) \text{ is an instance of form } 1: \ acc-ipol(N) \equiv \text{acc-ipol}(N_1).

ii. Case \text{clause}(N) \text{ is an instance of one of forms } 2–5 \text{ or } 7–8:

a. Case \text{side}(N_i) = \text{L}:

\[
\text{acc-ipol}(N) \equiv \bigwedge_{i=2}^{k} \text{acc-ipol}(N_i).
\]

b. Case \text{side}(N_i) = \text{R}:

\[
\text{acc-ipol}(N) \equiv \bigwedge_{i=2}^{k} \text{acc-ipol}(N_i).
\]

iii. Case \text{clause}(N) \text{ is an instance } \neg D \lor \neg R \lor D' \text{ of form } 6: \text{ Since the tableau is closed and regular there is a unique ancestor } \text{tgt}(N_2) \text{ of } N_2 \text{ with } \text{lit}(\text{tgt}(N_2)) = \text{lit}(N_2).

a. Case \text{side}(\text{tgt}(N_2)) = \text{side}(N_1): \ acc-ipol(N) \equiv \text{acc-ipol}(N_3).

b. Case \text{side}(N_1) = \text{L} \text{ and } \text{side}(\text{tgt}(N_2)) = \text{R}: \text{ Let }

\[
\{t_1, \ldots, t_n\} \equiv \text{garg}(\neg R) \setminus \text{garg}(\text{DEF}(F) \land \text{branch}_L(N)),
\]

let \( v_1, \ldots, v_n \) be fresh variables, let \( \theta \equiv \{v_1 \mapsto t_1, \ldots, v_n \mapsto t_n\} \), and define

\[
\text{acc-ipol}(N) \equiv \forall v_1 \ldots \forall v_n (\neg R \lor \text{acc-ipol}(N_3))\langle \theta^{-1} \rangle.
\]

---

4 Definition 32 is slightly different from a straightforward transfer of the corresponding specification in terms of tableau rules in [10]. In case (iii.b) of Definition 32 the range \( \{t_1, \ldots, t_n\} \) of \( \theta \) is specified as a subset of \( \text{garg}(\neg R) \), whereas according to [10, Figure 2.8] one would expect \( \{t_1, \ldots, t_n\} = \text{garg}(\neg R \lor \text{acc-ipol}(N_3)) \setminus \text{garg}(\text{DEF}(F) \land \text{branch}_L(N)) \). The inclusion of \( \text{acc-ipol}(N_3) \) on the left side of the \( \setminus \) operator would, however, actually be redundant. Analogous considerations hold for the case (iii.c).
c. Case side(N₁) = R and side(tgt(N₂)) = L: Let

\{t₁, \ldots, tₙ\} = \text{garg}(R) \setminus \text{garg}(\text{DEF}(\neg G) \land \text{branch}_{₁}(N))

let \(v₁, \ldots, vₙ\) be fresh variables, let \(θ = \{v₁ \mapsto t₁, \ldots, vₙ \mapsto tₙ\}\), and define

\(\text{acc-ipol}(N) \equiv \exists v₁ \ldots \exists vₙ (R \land \text{acc-ipol}(N₃))(θ^{-1})\).

Although base cases are not explicitly distinguished in the inductive definition of \(\text{acc-ipol}(N)\), they are covered by the specification in Definition 32: If clause(N) is an instance of form 2 of Lemma 25, then \(k = 1\) and \(\text{acc-ipol}(N) = \bot\) or \(\text{acc-ipol}(N) = \top\), respectively.

Correctness of the access interpolant extraction according to Definition 32 is stated with the following theorem:

**Theorem 33 (Correctness of Access Interpolant Extraction from an ACI-Tableau)** Let \(F, G\) be RQFO sentences such that \(F \models G\) and let \(N\) be the root of an ACI-tableau for \(F\) and \(G\). Then \(\text{acc-ipol}(N)\) is an access interpolant of \(F\) and \(G\).

Before we can prove this theorem, we need some auxiliary concepts and propositions. An ACI-tableau is based on the conjunction of two clausal formulas, each obtained from one of the two input sentences. *Global position specifiers* allow to refer unambiguously to each literal occurrence and further items in this conjunction:

**Definition 34 (Global Position Specifier)** Consider an ACI-tableau for RQFO sentences \(F\) and \(G\). It is a clausal tableau for two clausal formulas obtained by clausifying \(\text{DEF}(F)\) and \(\text{DEF}(\neg G)\). For \(s \in \{L, R\}\) define \(D_{sp}, R_{sp}, x_{sp}, v_{sp}, t_{(p,i)}, σ_{sp}\) to denote \(D_p, R_p, x_p, v_p, t_{(p,i)}, σ_p\), respectively, as specified in Definition 25, in case \(s = L\) referring to \(p \in \text{pos}(F)\) and clauses obtained from \(\text{DEF}(F)\) and in case \(s = R\) referring to \(p \in \text{pos}(\neg G)\) and clauses obtained from \(\text{DEF}(\neg G)\). Position specifiers of the form \(sp\), where \(s \in \{L, R\}\) and \(p\) denotes a position as specified in Definition 21.ii are called *global position specifiers*. The symbol \(s\) is called the *side* of a global position identifier \(sp\).

To mimic the \(δ\)-rule of non-clausal tableaux we specify the notion of *introducer literal* and *introducer node* associated with each “Skolem term”, that is, ground term whose principal functor is a Skolem function:

**Definition 35 (Ground Term Introducers)** Let \(T\) be an ACI-tableau. The *introducer literals* for a ground term \(t_{(p,i)}(x_{sp})μ\) occurring in a literal label of a tableau node are \(R_{sp}σ_{sp}μ\) and \(D_{sp}σ_{sp}μ\). An *introducer node* for a ground term is a node whose literal label is an introducer literal for the term.

The following proposition shows a relationship of occurrences of Skolem terms and their introducers that holds for ACI-tableaux:

**Proposition 36 (Precedence of Ground Term Introducers in ACI-Tableaux)** Let \(N\) be a node of an ACI-tableau and \(t \in \text{garg}(\text{lht}(N))\) where the principal functor of \(t\) is a Skolem function \(t_{(p,i)}\). Then \(N\) is an introducer node for \(t\) or \(N\) has an ancestor that is an introducer node for \(t\).

**Proof** Assume that \(t \in \text{garg}(\text{lht}(N))\) and \(N\) is not an introducer node for \(t\). We show that \(N\) then has an ancestor \(N'\) such that \(t \in \text{garg}(\text{lht}(N'))\). The proposition then follows from finiteness of the tableau branch length. Numbers of clause forms refer to Lemma 25. Let \(\text{parent}(N)\) denote the parent of \(N\). Then:
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i. If \( \text{lit}(N) \) is negative, then there must exist an ancestor \( N' \) of \( N \) with \( \text{lit}(N') = \overline{\text{lit}(N)} \), and thus \( t \in \text{garg}(\overline{\text{lit}(N)}) \), as claimed.

ii. Else, if \( \text{lit}(N) \) is of the form \( D_{tp}\mu \), then \( \text{clause}(\text{parent}(N)) \) must be an instance of a clause of one of the forms 2–6. (Form 1 can be excluded since \( \text{arg}(D_{te}) = \emptyset \), contradicting our assumption \( t \in \text{garg}(\text{lit}(N)) \).) In all cases it can be verified that \( \text{garg}(D_{tp}\mu) \subseteq \text{garg}(E) \), where \( E \) is the disjunction of the negative literals in \( \text{clause}(\text{parent}(N)) \). Hence \( N \) must have a sibling \( N'' \) with a negative literal label and such that \( t \in \text{garg}(\overline{\text{lit}(N'')}) \). The existence of an ancestor \( N' \) of \( N \) as claimed then follows from (i).

iii. Else \( \text{lit}(N) \) must be of the form \( R_{tp}\sigma_p \mu \) or \( D_{tp1}\sigma_p \mu \) and \( \text{clause}(\text{parent}(N)) \) must be an instance of a clause of form 7 or 8. Because \( N \) is not an introducer node for \( t \) it follows that \( t \in \{ x\mu \mid x \in \text{var}(R_{tp}\sigma_p) \} \) or \( t \in \{ x\mu \mid x \in \text{var}(D_{tp1}\sigma_p) \} \), respectively. With \( t \in \text{garg}(\overline{\text{lit}(N)}) \) it follows from the specification of clause forms 7 and 8 that \( t \in \text{garg}(\overline{D_{tp}\mu}) \). Because \( N \) has a sibling whose literal label is \( \overline{\text{lit}(N)} \), the existence of an ancestor \( N' \) of \( N \) as claimed follows from (i).

In the proof of Theorem 33 semantic and syntactic properties of intermediate formulas constructed during access interpolant extraction need to be considered. The notions of RQFO formula and access interpolant as such are not adequate to express the relevant properties of these intermediate formula, but generalizations of them, defined as follows:

**Definition 37 (RQFOT Formula, Weak Access Interpolant)**

(i) Formulas of first-order logic with relativized quantifiers and ground terms, briefly RQFOT formulas, are defined like RQFO formulas (Definition 18.i) with the exception that as arguments of atoms not just variables and constants, but also ground terms with function symbols of arbitrary arity are allowed.

(ii) Let \( F, G \) be RQFO sentences and let \( F', G' \) be quantifier-free first-order formulas such that \( F \land F' \models G \lor G' \). A **weak access interpolant** of the quadruple \( \langle F, F', G, G' \rangle \) is an RQFOT sentence \( H \) such that

1. \( F \land F' \models H \models G \lor G' \).
2. \( \text{pred}(H) \subseteq \text{pred}(F) \cap \text{pred}(G) \).
3. Every existential binding pattern of \( H \) is covered by an existential binding pattern of \( G \).
4. Every universal binding pattern of \( H \) is covered by a universal binding pattern of \( F \).

Access interpolants are special cases of weak access interpolants:

**Proposition 38 (Weak and Standard Access Interpolants)** Let \( F, G \) be RQFO sentences. A formula \( H \) is an **access interpolant** of \( F \) and \( G \) if and only if \( H \) is a weak access interpolant of \( \langle F, T, G, \bot \rangle \).

**Proof** Easy to see from the definitions of weak access interpolant (Definition 37.ii) and access interpolant (Definition 20).

We are now ready to prove the core property that underlies the correctness of the access interpolant extraction from ACI-tableaux:

**Lemma 39 (Core Invariant of Access Interpolant Extraction from ACI-Tableaux)** Let \( F, G \) be RQFO sentences such that \( F \models G \) and let \( T \) be an ACI-tableau for \( F \) and \( G \). For all inner nodes \( N \) of \( T \) the formula \( \text{acc-pol}(N) \) is a weak access interpolant of \( \langle \text{DEF}(F), \text{branch}_h(N), \neg \text{DEF}(\neg G), \neg \text{branch}_h(N) \rangle \).
Proof By induction on the tableau structure. The property to show for all nodes \( N \) of the tableau is:

(IP) If \( N \) is an inner node, then \( \text{acc-\text{ipol}}(N) \) is a weak access interpolant of
\[
(\text{DEF}(F), \text{branch}_L(N), \neg\text{DEF}(\neg G), \neg\text{branch}_R(N)).
\]

In the base case where \( N \) is a leaf it satisfies (IP) trivially. To prove the induction step assume as induction hypothesis that \( N \) is an inner node with children \( N_1, \ldots, N_k \) where \( k \geq 1 \) and that (IP) holds for all children, that is:

(IH) For all \( i \in \{1, \ldots, k\} \) it holds that if \( N_i \) is an inner node, then \( \text{acc-\text{ipol}}(N_i) \) is a weak access interpolant of
\[
(\text{DEF}(F), \text{branch}_L(N_i), \neg\text{DEF}(\neg G), \neg\text{branch}_R(N_i)).
\]

We prove the induction step by showing that (IH) implies that (IP) holds for \( N \), that is, \( \text{acc-\text{ipol}}(N) \) is a weak access interpolant of
\[
(\text{DEF}(F), \text{branch}_L(N), \neg\text{DEF}(\neg G), \neg\text{branch}_R(N)).
\]

We will now prove this for the case where the children of \( N \) have side label \( L \) for all possible forms of \( \text{clause}(N) \) according to Lemma 25. The case where the children have side label \( R \) can be shown analogously. We thus assume that \( N \) is an inner node of \( T \) and that the children of \( N \) have side label \( L \). The following general lemma is then easy to verify:

(LR) For all \( i \in \{1, \ldots, k\} \) it holds that \( \text{branch}_R(N) = \text{branch}_R(N_i) \).

For all clause forms with exception of form 1 the literal \( \text{lit}(N_i) \) must be an instance of a literal of the form \( \neg D_{L,i} \). Since the tableau is closed and leaf-only it follows for ground substitutions \( \mu \) such that \( \neg D_{L,i} \mu = \text{lit}(N_i) \) that:

(LD) \( D_{L,i} \mu \) (which is equal to \( \text{lit}(N_i) \)) occurs as a conjunct in \( \text{branch}_L(N) \).

The formula \( \text{clause}(N) \) must be an instance of a clause of one of the forms listed in Lemma 25. We now consider each possible case in subsections headed with the respective clause forms. For each case we verify that \( \text{acc-\text{ipol}}(N) \) satisfies the characteristics 1–4 of weak access interpolant according to Definition 37.ii. We label the respective subproofs with \( \text{WAI} 1 \) left, \( \text{WAI} 1 \) right, \( \text{WAI} 2 \), \( \text{WAI} 3 \), and \( \text{WAI} 4 \), respectively. Condition \( \text{WAI} 1 \) is there split up into a left component, that is, \( \text{DEF}(F) \land \text{branch}_L(N) \models \text{acc-\text{ipol}}(N) \) and a right component, that is, \( \text{acc-\text{ipol}}(N) \models \neg\text{DEF}(\neg G) \lor \neg\text{branch}_R(N) \), or, equivalently, expressed as contrapositive, \( \text{DEF}(\neg G) \land \text{branch}_R(N) \models \neg\text{acc-\text{ipol}}(N) \). If appropriate, proof steps are shown in tabular symbolic form followed by explanations.

**Clause Form 1.** For this clause form \( \text{acc-\text{ipol}}(N) \) is defined as \( \text{acc-\text{ipol}}(N_i) \). This case can be proven with a simplified variant of the proof for clause forms 2–5 below. The role of \( N_2 \) in that other proof is taken here by \( N_1 \) and properties \( \text{DEF}(F) \models D_{L,i} \) as well as \( \text{garg}(D_{L,i}) = \emptyset \) can be utilized.

**Clause Form 2–5.** For these clause forms \( \text{acc-\text{ipol}}(N) \) is defined as \( \sqrt[k=2]{\text{acc-\text{ipol}}(N_i)} \). Let \( \mu \) be a ground substitution such that \( \text{dom}(\mu) = \text{arg}(\neg D_{L,i}) \) and \( \text{lit}(N_i) = \neg D_{L,i} \mu \).
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WAI 1 left:

1. \( \text{DEF}(F) \models \forall x_{i,p} (D_{x,p} \rightarrow D_{x,p1} \lor \ldots \lor D_{x,p(k-1)}) \).
2. \( \text{DEF}(F) \models D_{x,p} \mu \rightarrow D_{x,p1} \mu \lor \ldots \lor D_{x,p(k-1)} \mu \).
3. \( \text{branch}_{1}(N) \models D_{x,p} \mu \).
4. \( \text{DEF}(F) \land \text{branch}_{1}(N) \models D_{x,p1} \mu \lor \ldots \lor D_{x,p(k-1)} \mu \).
5. \( \text{DEF}(F) \land \text{branch}_{1}(N) \models \text{acc-ipol}(N_{2}) \lor \ldots \lor \text{acc-ipol}(N_{k}) \).
6. \( \text{DEF}(F) \land \text{branch}_{1}(N) \models \text{acc-ipol}(N) \).

Entailment (1) holds since its right side is a conjunct of its left side. Entailment (2) follows from (1) by instantiating universally quantified variables. Entailment (3) follows from (LD). Entailment (4) follows from (3) and (2). Entailment (5) follows from (4) and (IH). Entailment (6) follows from (5) and the definition of \( \text{acc-ipol}(N) \) for the considered clause forms.

WAI 1 right:

1. \( \text{DEF}(\neg G) \land \text{branch}_{2}(N) \models \neg \text{acc-ipol}(N_{2}) \land \ldots \land \neg \text{acc-ipol}(N_{k}) \).
2. \( \text{DEF}(\neg G) \land \text{branch}_{2}(N) \models \neg \text{acc-ipol}(N) \).

Entailment (1) follows from (IH) and (LR). Entailment (2) follows from (1) and the definition of \( \text{acc-ipol}(N) \) for the considered clause forms.

WAI 2 and 3:

Immediate from (IH) and the definition of \( \text{acc-ipol}(N) \) for the considered clause forms.

WAI 4:

1. \( \text{garg}(D_{x,p} \mu) \subseteq \text{garg}(\neg D_{x,p} \mu) \), for all \( i \in \{1, \ldots, k-1\} \).
2. \( \text{garg}(\text{branch}_{1}(N)) = \text{garg}(\text{branch}_{1}(N_{i})) \), for all \( i \in \{2, \ldots, k\} \).
3. \( \text{garg}(\text{branch}_{2}(N)) = \text{garg}(\text{branch}_{2}(N_{i})) \), for all \( i \in \{2, \ldots, k\} \).
4. \( \text{garg}(\text{acc-ipol}(N)) \subseteq \text{garg}(\text{DEF}(F) \land \text{branch}_{1}(N)) \cap \text{garg}(\neg \text{DEF}(\neg G) \lor \neg \text{branch}_{2}(N)) \).

Subsumption (1) follows from the specification of the considered clause forms. Equality (2) follows from (1), since for \( i \in \{2, \ldots, k\} \) it holds that \( \text{lft}(N_{i}) = D_{x,p(i-1)} \mu \).

Equality (3) follows from (LR). Subsumption (4) follows from (IH), (3) and (2), given that \( \text{garg}(\text{acc-ipol}(N)) = \bigcup_{i=2}^{k} \text{garg}(\text{acc-ipol}(N_{i})) \).

Clause Form 6, Case side(\text{tgt}(N_{3})) = L. In this case \( \text{acc-ipol}(N) \) is defined as \( \text{acc-ipol}(N_{3}) \).

Le \( \mu \) be a ground substitution such that \( \text{dom}(\mu) = \arg(\neg D_{x,p} \cup \forall y_{p}) \) and clause(\( N \)) = \( \neg D_{x,p} \lor \neg R_{x,p} \lor D_{x,p1} \mu \). We note the following lemma, which can be derived similarly as (LD):

(L3) The literal \( R_{x,p} \mu \) (that is, \( \text{lit}(N_{3}) \)) occurs as a conjunct in \( \text{branch}_{1}(N) \).

WAI 1 left:

1. \( \text{DEF}(F) \land \text{branch}_{1}(N) \models \text{acc-ipol}(N_{3}) \).
2. \( \text{DEF}(F) \land \text{branch}_{1}(N) \land D_{x,p} \mu \models \text{acc-ipol}(N_{3}) \).
3. \( \text{DEF}(F) \models \forall x_{i,p} (D_{x,p} \rightarrow \forall y_{p} (\neg R_{x,p} \lor D_{x,p1})) \).
4. \( \text{DEF}(F) \models \neg D_{x,p} \mu \lor \neg R_{x,p} \mu \lor D_{x,p1} \mu \).
5. \( \text{branch}_{1}(N) \models D_{x,p} \mu \).
6. \( \text{branch}_{1}(N) \models R_{x,p} \mu \).
7. \( \text{DEF}(F) \land \text{branch}_{1}(N) \models \text{acc-ipol}(N_{3}) \).
8. \( \text{DEF}(F) \land \text{branch}_{1}(N) \models \text{acc-ipol}(N) \).
Entailment (1) follows from (IH). Entailment (2) is obtained from (1) by expressing \( \text{branch}_h(N_3) \) with its last conjunct made explicit. Entailment (3) holds since its right side is a conjunct of its left side. Entailment (4) follows from (3) by instantiating universally quantified variables. Entailments (5) and (6) follow from (LD) and (L3), respectively. Entailment (7) follows from (4)–(6) and (2). Step (8) follows from (7) since \( \text{acc-ipol}(N) = \text{acc-ipol}(N_5) \).

\textbf{WAI 1 right:}
Immediate from (IH) and (LR) since \( \text{acc-ipol}(N) = \text{acc-ipol}(N_3) \).

\textbf{WAI 2 and 3:}
Immediate from (IH) since \( \text{acc-ipol}(N) = \text{acc-ipol}(N_3) \).

\textbf{WAI 4:}

(1) \( \text{garg}(\neg R_{\bar{p}}) \cup \text{garg}(\neg D_{\bar{p}}) \subseteq \text{garg}(\text{branch}_h(N)) \).

(2) \( \text{garg}(\neg R_{u_{1}}) \cup \text{garg}(\neg D_{u_{1}}) \subseteq \text{garg}(\text{branch}_h(N_3)) \).

(3) \( \text{garg}(\text{branch}_h(N)) = \text{garg}(\text{branch}_h(N_3)) \).

(4) \( \text{garg}(\text{branch}_h(N)) = \text{garg}(\text{branch}_h(N_3)) \).

(5) \( \text{garg}(\text{acc-ipol}(N)) \subseteq \text{garg}(\text{DEF}(F) \land \text{branch}_h(N)) \land \text{garg}(\neg \text{DEF}(\neg G) \lor \neg \text{branch}_h(N)) \).

Subsumption (1) follows from (LD) and (L3). Subsumption (2) follows from the definition of clause form 6. Equality (3) follows from (2) and (1) since \( \text{branch}_h(N_3) = \text{branch}_h(N) \land D_{u_{1}}. \) Equality (4) follows from (LR). Subsumption (5) follows from (IH), (4) and (3), since \( \text{acc-ipol}(N) = \text{acc-ipol}(N_3) \).

\textit{Clause Form 6, Case side(tgt(N_2)) = R.} Let \( \mu \) be a ground substitution such that \( \text{dom}(\mu) = \text{arg}(\neg D_{u_{1}}) \cup \forall \theta \text{ and clause}(N) = (\neg D_{u_{1}} \lor \neg R_{u_{1}} \lor D_{u_{1}}) \mu. \) Formula \( \text{acc-ipol}(N) \) is for this case then defined as

\[ \forall v_1 \ldots \forall v_n (\neg R_{u_{1}} \lor \text{acc-ipol}(N_3))^{\langle \theta^{-1} \rangle}, \]

where \( \{v_1, \ldots, v_n\} = \text{garg}(\neg R_{u_{1}}) \cup \text{garg}(\text{DEF}(F) \land \text{branch}_h(N)) \), \( v_1, \ldots, v_n \) are fresh variables, and \( \theta = \{v_1 \mapsto t_1, \ldots, v_n \mapsto t_n\} \).

\textbf{WAI 1 left:}

(1) \( \text{DEF}(F) \land \text{branch}_h(N_3) \models \text{acc-ipol}(N_5) \).

(2) \( \text{DEF}(F) \land \text{branch}_h(N) \land D_{u_{1}} \mu \models \text{acc-ipol}(N_5) \).

(3) \( \text{DEF}(F) \models \forall \theta (D_{u_{1}} \rightarrow \forall \theta (\neg R_{u_{1}} \lor D_{u_{1}})) \).

(4) \( \text{DEF}(F) \models \neg D_{u_{1}} \mu \lor \neg R_{u_{1}} \lor D_{u_{1}} \mu \).

(5) \( \text{branch}_h(N) \models D_{u_{1}} \mu \).

(6) \( \text{DEF}(F) \land \text{branch}_h(N) \models \neg R_{u_{1}} \lor \text{acc-ipol}(N_3) \).

(7) \( \text{DEF}(F) \land \text{branch}_h(N) \models \forall v_1 \ldots \forall v_n (\neg R_{u_{1}} \lor \text{acc-ipol}(N_3))^{\langle \theta^{-1} \rangle} \).

(8) \( \text{DEF}(F) \land \text{branch}_h(N) \models \text{acc-ipol}(N) \).

Entailments (1)–(5) follow in the same way as in the as shown above for clause form 6, case \( \text{side(tgt(N_2)) = L.} \) \textbf{WAI 1 left:} Entailment (6) follows from (5), (4) and (2). Given the specified properties of \( \theta \), entailment (7) follows from (6) by Proposition 31. Step (8) is obtained from (7) by contracting the definition of \( \text{acc-ipol}(N) \) for the considered case.
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WAI 1 right:

(1) \( \text{DEF}(\neg G) \land \text{branch}_{\mu}(N_3) \models \neg \text{acc}-\text{ipol}(N_3). \)
(2) \( \text{branch}_{\mu}(N) \models R_{N,\mu}. \)
(3) \( \text{DEF}(\neg G) \land \text{branch}_{\mu}(N) \models R_{N,\mu} \land \neg \text{acc}-\text{ipol}(N_3). \)
(4) \( \text{DEF}(\neg G) \land \text{branch}_{\mu}(N) \models \exists \nu_1 \ldots \exists \nu_n (R_{N,\mu} \land \neg \text{acc}-\text{ipol}(N_3)) \langle\theta^{-1}\rangle. \)
(5) \( \text{DEF}(\neg G) \land \text{branch}_{\mu}(N) \models \neg \forall \nu_1 \ldots \forall \nu_n (\neg R_{N,\mu} \lor \text{acc}-\text{ipol}(N_3)) \langle\theta^{-1}\rangle. \)
(6) \( \text{DEF}(\neg G) \land \text{branch}_{\mu}(N) \models \neg \text{acc}-\text{ipol}(N). \)

Entailment (1) follows from (IH). Entailment (2) holds since \( \text{lt}(N_2) = \neg R_{N,\mu} \) and \( \text{side}(\text{lt}(N_2)) = R. \) Entailment (3) follows from (2), (LR) and (1). Entailment (4) follows from (3), since the formula on the right side of (4) is entailed by the formula on the right side of (3). The formula on the right side of (5) is equivalent to that on the right side of (4). Entailment (6) is obtained from (5) by contracting the definition of \( \text{acc}-\text{ipol}(N) \) for the considered case.

WAI 2:

The formula \( \text{acc}-\text{ipol}(N) \) contains, compared to \( \text{acc}-\text{ipol}(N_3) \) one additional predicate occurrence, a negative occurrence of the predicate of \( R_{N,\mu}. \) This predicate occurs in an instance of a clause of form 6, hence negatively in \( \text{DEF}(F). \) It also occurs positively in the literal label of \( \text{lt}(N_2), \) where \( \text{side}(\text{lt}(N_2)) = R, \) hence positively in a clause of form 7 obtained from normalizing \( \text{DEF}(\neg G), \) hence negatively in \( \neg \text{DEF}(\neg G). \)

WAI 3:

From (IH) it follows that all existential binding patterns of \( \text{acc}-\text{ipol}(N_3) \) are covered by \( \neg \text{DEF}(\neg G) \) and all universal binding patterns of \( \text{acc}-\text{ipol}(N_3) \) are covered by \( \text{DEF}(F). \) That the binding patterns of \( \text{acc}-\text{ipol}(N), \) defined as

\[ \forall \nu_1 \ldots \forall \nu_n (\neg R_{N,\mu} \lor \text{acc}-\text{ipol}(N_3)) \langle\theta^{-1}\rangle, \]

are also covered in that way by \( \neg \text{DEF}(\neg G) \) and \( \text{DEF}(F) \) then follows if the outermost quantification of \( \text{acc}-\text{ipol}(N) \) is covered by the quantification upon \( \forall v_{L,\mu}, \) in the formula \( \forall \nu_{L,\mu} (D_{L,\mu} \rightarrow \forall \nu_{L,\mu} (\neg R_{L,\mu} \lor D_{L,\mu})), \) which is a conjunct of \( \text{DEF}(F). \) This, in turn, follows if each member of the range of \( \theta \) occurs in \( R_{L,\mu} \) in an argument position that is an “output position” of \( R_{L,\mu}, \) that is, the argument of \( R_{L,\mu} \) at that position is a member of \( v_{L,\mu}. \)

We show the latter statement. Let \( t \) be a member of the range of \( \theta. \) From the definition of \( \theta \) it follows that \( t \in \text{garg}(\neg R_{L,\mu}). \) Assume that \( t \) occurs in \( \neg R_{L,\mu} \) in a “non-output” position, that is, at an argument position of \( R_{L,\mu} \) at which the argument of \( R_{L,\mu} \) is no member of \( v_{L,\mu}. \) From the definition of the considered clause form 6 it follows that then \( t \in \text{garg}(\neg D_{L,\mu}) \) or \( t \in \text{garg}(\neg R_{L,\mu}). \) By (LD) and since \( \text{garg}(\neg R_{L,\mu}) \subseteq \text{garg}(\text{DEF}(F)) \) it follows that \( t \in \text{garg}(\text{branch}_{\mu}(N)) \cup \text{garg}(\text{DEF}(F)). \) From the specification of \( \theta \) it follows that its \( \text{rng}(\theta) \cap (\text{garg}(\text{branch}_{\mu}(N)) \cup \text{garg}(\text{DEF}(F))) = \emptyset. \) Hence \( t \notin \text{rng}(\theta), \) contradicting our initial assumption about \( t. \) Thus \( t \) must occur in \( \text{garg}(\neg R_{L,\mu}) \) at an “output position” of \( R_{L,\mu}. \)

WAI 4:

(1) \( \text{garg}(\text{acc}-\text{ipol}(N_1)) \subseteq \text{garg}(\text{DEF}(F) \land \text{branch}_{\mu}(N_1)). \)
(2) \( \text{garg}(\text{acc}-\text{ipol}(N_1)) \subseteq \text{garg}(\text{DEF}(F) \land \text{branch}_{\mu}(N) \land D_{L,\mu}). \)
(3) \( \text{garg}(D_{L,\mu}) \subseteq \text{garg}(\neg D_{L,\mu}) \cup \text{garg}(\neg R_{L,\mu}) \subseteq \text{garg}(\text{branch}_{\mu}(N)) \cup \text{garg}(\neg R_{L,\mu}). \)
(4) \( \text{garg}(\text{acc}-\text{ipol}(N_1)) \subseteq \text{garg}(\text{DEF}(F) \land \text{branch}_{\mu}(N)) \cup \text{garg}(\neg R_{L,\mu}). \)
Subsumption (1) follows from (IH). Subsumption (2) is obtained from (1) by expressing \( \text{branch}_h(N_1) \) with its last conjunct made explicit. Subsumptions (3) follow from the definition of clause form 6 and (LD). Subsumption (4) follows from (2) and (3). Subsumptions (5) and (6) follow from the definition of acc-ipol(N) in the considered case. Subsumption (7) follows from (6), (5) and (4). Subsumption (8) can be shown as follows: Let \( t \) be a member of acc-ipol(N). From the definition of acc-ipol(N) for the considered case it follows that \( t \in \text{garg}(\text{acc-ipol}(N_1)) \). If \( t \in \text{garg}(\text{acc-ipol}(N_1)) \), then, since \( \text{lit}(N_2) = \text{lit}(\text{acc-ipol}(N_1)) \), hence, by \( \text{LR}, \) \( t \in \text{garg}(\text{acc-ipol}(N_1)) \), which completes the proof of (8). Subsumption (9) follows from (8) and (7).

**Clause Form 7.** For this clause forms acc-ipol(N) is defined as \( \bigvee_{i=1}^{k} \text{acc-ipol}(N_i) = \text{acc-ipol}(N_2) \).

Let \( \mu \) be a ground substitution such that \( \text{dom}(\mu) = \text{arg}(\neg D_{ip}) \) and \( \text{lit}(N_1) = \neg D_{ip} \).

**WAI I left:**

\[
\begin{align*}
(1) & \quad \text{DEF}(F) \land \text{branch}_h(N_2) \models \text{acc-ipol}(N_2).
(2) & \quad \text{DEF}(F) \land \text{branch}_h(N) \land R_{ip}\sigma_{ip}\mu \models \text{acc-ipol}(N_2).
(3) & \quad \gamma \models (\sigma_{ip}\mu) |_{V_{ip}}.
(4) & \quad \text{DEF}(F) \land \text{branch}_h(N'') \land R_{ip}\sigma_{ip}\mu \land D_{ip1}\sigma_{ip}\mu \models \text{acc-ipol}(N_2).
(5) & \quad \text{rng}(\gamma) \land \text{garg}(\text{branch}_h(N'')) = \emptyset.
(6) & \quad \text{rng}(\gamma) \land \text{garg}(\text{branch}_h(N_2)) = \emptyset.
(7) & \quad \text{rng}(\gamma) \land \text{garg}(\text{DEF}(F)) = \emptyset.
(8) & \quad \text{rng}(\gamma) \land \text{garg}(\neg \text{DEF}(F)) = \emptyset.
(9) & \quad \text{rng}(\gamma) \land \text{garg}(\text{acc-ipol}(N_2)) = \emptyset.
(10) & \quad \exists ! \nu_{ip} (\text{DEF}(F) \land \text{branch}_h(N'') \land R_{ip}\sigma_{ip}\mu \land D_{ip1}\sigma_{ip}\mu \langle \gamma^{-1} \rangle) \models \text{acc-ipol}(N_2).
(11) & \quad \text{DEF}(F) \land \text{branch}_h(N'') \land \exists \nu_{ip} (R_{ip}\sigma_{ip}\mu \land D_{ip1}\sigma_{ip}\mu \langle \gamma^{-1} \rangle) \models \text{acc-ipol}(N_2).
(12) & \quad \text{DEF}(F) \land \text{branch}_h(N) \land \exists \nu_{ip} (R_{ip}\sigma_{ip}\mu \land D_{ip1}\sigma_{ip}\mu \langle \gamma^{-1} \rangle) \models \text{acc-ipol}(N_2).
(13) & \quad \text{DEF}(F) \models \exists ! \nu_{ip} (R_{ip}\mu \land D_{ip1}\mu) \models \text{acc-ipol}(N_2).
(14) & \quad \text{DEF}(F) \models \exists ! \nu_{ip} (R_{ip}\mu \land D_{ip1}\mu).
(15) & \quad \text{DEF}(F) \models D_{ip}\mu \rightarrow \exists ! \nu_{ip} (R_{ip}\mu \land D_{ip1}\mu).
(16) & \quad \text{DEF}(F) \land \text{branch}_h(N) \models D_{ip}\mu.
(17) & \quad \text{DEF}(F) \land \text{branch}_h(N) \models \exists \nu_{ip} (R_{ip}\mu \land D_{ip1}\mu).
(18) & \quad \text{DEF}(F) \land \text{branch}_h(N) \models \text{acc-ipol}(N_2).
(19) & \quad \text{DEF}(F) \land \text{branch}_h(N) \models \text{acc-ipol}(N).
\end{align*}
\]

Entailment (1) follows from (IH). Entailment (2) is obtained from (1) by expressing \( \text{branch}_h(N_2) \) with its last conjunct made explicit. The substitution \( \gamma \), defined in (3), is an injection and \( R_{ip}\sigma_{ip}\mu \) as well as \( D_{ip1}\sigma_{ip}\mu \) are the introducer literals for exactly the members of \( \text{rng}(\gamma) \). From the contiguity property of the tableau it follows that if a node with literal label \( D_{ip1}\sigma_{ip}\mu \) is an ancestor of \( N_2 \), then it is the parent of \( N_2 \), that is, \( N \). Hence there exists node \( N'' \) which is the parent of \( N \) or identical with \( N \).
(depending on whether \( \text{lit}(N) = D_{L,p_1}(\sigma_{\gamma_1} \mu) \) such that (4) holds and, by Proposition 36, also (5) and (6) hold. (Entailment (4) holds also if \( \text{lit}(N) \neq D_{L,p_1}(\sigma_{\gamma_1} \mu) \). The conjunct \( D_{L,p_1}(\sigma_{\gamma_1} \mu) \) then just redundantly strengthens the left side.) Equalities (7) and (8) hold since the principal functor of all members of \( \text{rng}(\gamma) \) is a Skolem functor and thus does occur neither in \( \text{DEF}(F) \) nor in \( \text{DEF}(\neg G) \). Equality (9) follows from (8), (6) and (IH). Entailment (10) follows from (9) and (4) by Proposition 31. Entailment (11) follows from (10), (7) and (5). Entailment (12) follows from (11) since \( N'' \) is identical to \( N \) or the parent of \( N \). Entailment (13) follows from (12) since \( R_{L,p_1}(\sigma_{\gamma_0} \mu(\langle \gamma^{-1} \rangle)) = R_{L,p} \mu \) and \( D_{L,p_1}(\sigma_{\gamma_0} \mu(\langle \gamma^{-1} \rangle)) = D_{L,p} \mu(\langle \gamma^{-1} \rangle) \). Entailment (14) holds since its right side is a conjunct of its left side. Entailment (15) follows from (14) by instantiating universally quantified variables. Entailment (16) follows from (LD). Entailment (17) follows from (16) and (14). Entailment (18) follows from (17) and (13). Entailment (19) follows from (18) since \( \text{acc-ipol}(N) = \text{acc-ipol}(N_2) \).

**WAI 1 right:**

Immediate from (IH) and (LR) since \( \text{acc-ipol}(N) = \text{acc-ipol}(N_2) \).

**WAI 2 and 3:**

Immediate from (IH) since \( \text{acc-ipol}(N) = \text{acc-ipol}(N_2) \).

**WAI 4:**

1. \( t \in \text{garg}(\text{acc-ipol}(N_2)) \cap \text{garg}(R_{L,p}(\sigma_{\gamma_1} \mu)). \)
2. \( t \notin \text{rng}(\gamma). \)
3. \( t \in \text{garg}(\neg D_{L,p} \mu) \cup \text{garg}(R_{L,p}). \)
4. \( t \in \text{garg}(\text{branch}_L(N)) \cup \text{garg}(R_{L,p}). \)
5. \( \text{garg}(\text{acc-ipol}(N_2)) \cap \text{garg}(R_{L,p}(\sigma_{\gamma_1} \mu)) \subseteq \text{garg}(\text{branch}_L(N)) \cup \text{garg}(R_{L,p}). \)
6. \( \text{garg}(\text{acc-ipol}(N_2)) \subseteq \text{garg}(\text{DEF}(F) \land \text{branch}_L(N)). \)
7. \( \text{garg}(\text{acc-ipol}(N_2)) \subseteq \text{garg}(\neg \text{DEF}(\neg G) \lor \neg \text{branch}_R(N)). \)
8. \( \text{garg}(\text{acc-ipol}(N)) \subseteq \text{garg}(\text{DEF}(F) \land \text{branch}_L(N)) \cap \text{garg}(\neg \text{DEF}(\neg G) \lor \neg \text{branch}_R(N)). \)

Let \( t \) be a term that satisfies (1). Let substitution \( \gamma \) be defined as in step (3) of the proof of **WAI 1 left** above. Statement (2) then follows from step (1) of that proof. By (2), the literal \( R_{L,p}(\sigma_{\gamma_1} \mu) \) is not an introducer literal for \( t \). Hence (3) follows from the definition of clause form 7. Statement (4) follows from (3) and (LD). Subsumption (5) then follows since (1) implies (4), for all ground terms \( t \). Subsumption (6) follows from (IH) and (5) because \( \text{garg}(R_{L,p}) \subseteq \text{garg}(\text{DEF}(F)) \) and \( \text{branch}_L(N_2) = \text{branch}_L(N) \land R_{L,p}(\sigma_{\gamma_0} \mu). \) Subsumption (7) follows from (IH) and (LR). Since \( \text{acc-ipol}(N) = \text{acc-ipol}(N_2) \), subsumption (8) follows from (7) and (6).

**Clause Form 8.** Can by show in the same way as for clause form 7, with the roles of \( R_{L,p}(\sigma_{\gamma_1} \mu) \) and \( D_{L,p_1}(\sigma_{\gamma_1} \mu) \) switched.

Theorem 33 stated above can now be proven on the basis of Lemma 39:

**Theorem 33 (Correctness of Access Interpolant Extraction from an ACI-Tableau).** Let \( F, G \) be RQFO sentences such that \( \mathcal{F} \models G \) and let \( N \) be the root of an ACI-tableau for \( F \) and \( G \). Then \( \text{acc-ipol}(N) \) is an access interpolant of \( F \) and \( G \).

**Proof** From Lemma 39 and Proposition 38 it follows that \( \text{acc-ipol}(N) \) is an access interpolant of \( \text{DEF}(F) \) and \( \neg \text{DEF}(-G) \). With Proposition 24 if follows that \( \text{acc-ipol}(N) \) is an access interpolant of \( F \) and \( G \).
11 Ensuring the Requirements on ACI-Tableaux

An ACI-tableau has clauses of specific forms according to Lemma 25 and certain structural properties, namely, it is closed, regular, leaf-only for the set of all negative literals occurring as literal labels, and contiguous for certain pairs of literals. A closed positive hyper tableau has all these structural properties, with exception of the contiguity requirement. Hence, a closed positive hyper tableau whose clauses match the forms of Lemma 25 that also satisfies the required contiguity property can be directly used to extract an access interpolant. Actually, contiguity can in this case be ensured with an inexpensive tableau transformation, shown as Procedure 45 below.

Closed clausal tableaux with arbitrary structure can be restructured to meet the structural properties required by ACI-tableaux with a series of tableau conversions that we will now specify. All of them preserve closedness and for all of them the clauses of the converted tableau are clauses of the respective input tableau. With exception of the conversion that ensures the leaf-only property all considered tableau conversions are simplifications, that is, procedures that require typically linear and at most polynomial effort. Termination is for these conversions easy to see. For the potentially expensive leaf-only conversion we state it explicitly as a proposition and provide a proof. Examples that illustrate the conversions will be given in Sect. 12. First we need to specify an additional auxiliary tableau property:

**Definition 40 (Eager)** A clausal tableaux is called eager if and only if no closed node is a descendant of another closed node.

The eager property is typically ensured implicitly by tableau construction calculi, since there it is pointless to attach children to a closed node. In addition, the leaf-only property for the set of all negative literals, which is presupposed for positive hyper tableaux and ACI-tableau, implies eagerness. Operations such as instantiating literal labels and tableau structure transformations as considered here might, however, result in non-eager tableaux, also for eager inputs, such that it is useful to take the this property here explicitly into account.

The following conversions to ensure eagerness and regularity are described as destructive tableau manipulation procedures. The procedure for ensuring regularity is from [38] and is illustrated by Fig. 2. Both conversions can be considered as tableau simplifications.

**Procedure 41 (Removal of Uneagerness)**
**Input:** A clausal tableau.

**Method:** Repeat the following operation until the resulting tableau is eager: Select an inner node $N$ that is closed. Remove the edges originating in $N$.

**Output:** An eager clausal tableau, whose clauses are also clauses of the input tableau. The following properties of the input tableau are preserved: closed, regular, leaf-only.

**Procedure 42 (Removal of Irregularities [38, Section 2.1.3])**
**Input:** A clausal tableau.

**Method:** Repeat the following operation until the resulting tableau is regular: Select a node $N$ in the tableau with an ancestor $N'$ such that \( \text{lit}(N') = \text{lit}(N) \). Remove the edges originating in the parent $N''$ of $N$ and replace them with the edges originating in $N$.

**Output:** A regular clausal tableau whose clauses are also clauses of the input tableau. The following properties of the input tableau are preserved: closed, eager, leaf-only.
Fig. 2: Tableau simplification step for removal of irregularities with Procedure 42 [38]. Node $N$ in the procedure description corresponds to $N_i$ in the figure, where $1 \leq i \leq k$. It is possible that $N'' = N'$. A triangle below a node represents the edges originating in the node (which might be none) together with the descendants of the node and all edges between them.

The following conversion ensures the leaf-only property. It is again specified as a procedure that destructively manipulates a tableau. We use there the notion of a fresh copy of an ordered tree $T$, which is an ordered tree $T'$ with fresh nodes and edges, related to $T$ through a bijection $\epsilon$ such that any node $N$ of $T$ has the same labels (e.g., literal label and side label) as node $\epsilon(N)$ of $T'$ and such that the $i$-th edge originating in node $N$ of $T$ ends in node $\epsilon(M)$ if and only if the $i$-th edge originating in node $\epsilon(N)$ of $T'$ ends in node $\epsilon(M)$. The procedure is illustrated by Fig. 3 and 4. Its termination is then shown with Proposition 44.

Procedure 43 (Leaf-Only Conversion)

**Input:** A closed, eager and regular clausal tableau and a set $S$ of pairwise non-complementary literals that occur as literal labels of nodes of the tableau.

**Method:** Repeat the following operations until the tableau is leaf-only for $S$:

1. Let $N$ be the inner node whose literal label is in $S$ that is first visited by traversing the tableau in pre-order. Let $N'$ be the parent of $N$.
2. Create a fresh copy $U$ of the subtree rooted at $N'$. In $U$ remove the edges that originate in the node corresponding to $N$.
3. Remove the edges originating in $N'$ and replace them with the edges originating in $N$.
4. For each leaf descendant $M$ of $N'$ with $\text{lit}(M) = \text{lit}(N)$: Create a fresh copy $U'$ of $U$. Change the origin of the edges originating in the root of $U'$ to $M$.
5. Ensure eagerness and regularity by simplifying with Procedure 41 and 42.

**Output:** A closed, eager and regular clausal tableau whose clauses are also clauses of the input tableau and which is leaf-only for $S$.

**Proposition 44 (Termination of Leaf-Only Conversion)** Procedure 43 terminates.

**Proof** We give a measure that strictly decreases in each round of the procedure. Consider a single round of the steps 1.–5. of Procedure 43 with $N$ and $N'$ as determined in step 1. Then:

i. All tableau modifications made in the round are in the subtree rooted at $N'$.

ii. At finishing the round all descendants of $N'$ with the same literal label as $N$ are leaves.

iii. All literal labels of inner nodes that are descendants of $N'$ and are different from $\text{lit}(N)$ at finishing the round are already literal labels of inner nodes that are descendants of $N'$ when entering the round.
We can now specify the measure that strictly decreases in each round of Procedure 43. For a node $N$ define bad-literals($N$) as the set of literal labels that occur in inner (i.e., non-leaf) descendants of $N$ and are members of $S$. From the above items (ii.) and (iii.) it follows that for $N'$ as determined in step 1 of Procedure 43 the cardinality of bad-literals($N'$) is strictly decreased in a round of steps 1.–5. of the procedure. However, a different node might be determined as $N'$ in step 1 of the next round. To specify a globally decreasing measure we define a further auxiliary notion: Let $N_0$ be a node whose ancestors are in root-to-leaf order the nodes $N_1, \ldots, N_{n-1}$. Define path-string($N_0$) as the string $I_1 \ldots I_n \omega$ of numbers, where for $i \in \{1, \ldots, n\}$ the number $I_i$ is the number of right siblings of $N_i$. With item (i.) it then follows that the following string of numbers, determined at step 1 of a round, is strictly reduced from round to round w.r.t. the lexicographical order of strings of numbers:

$$\text{path-string}(N')|\text{bad-literals}(N')|.$$

Regularity ensures that the length of the strings to be considered can not be larger than the finite number of literal labels of nodes of the input tableau plus 3 (a leading 0 for the root, which has no literal label; $\omega$; and $|\text{bad-literals}(N')|$). With the lexicographical order restricted to strings up to that length we have a well-order and the strict reduction ensures termination.

The following conversion ensures contiguity as far as required for ACI-tableaux. It is illustrated by Fig. 5.

**Fig. 3:** Conversion step for ensuring the leaf-only property with Procedure 43. Node $N$ in the procedure description corresponds to $N_i$ in the figure, where $1 \leq i \leq k$. A triangle below a node represents the edges originating in the node (which might be none, except for $T_i$) together with the descendants of the node and all edges between them. Triangle $T_i'$ is obtained from $T_i$ with steps illustrated in Fig. 4.

**Fig. 4:** Conversion step of a leaf $M$ with $\text{lit}(M) = \text{lit}(N)$ in Procedure 43 to obtain $T_i'$ from $T_i$ (see Fig. 3). An asterisk indicates leaves for which it is ensured that they are closed. Node $N$ in the procedure description corresponds to $N_i$ here. Superscripts $C$ indicate that copies of the subtrees referenced in Fig. 3 are used. Node $N'$, depicted also in Fig. 3, is included here just to indicate explicitly that all affected nodes $M$ are descendants of $N'$.
Fig. 5: First steps of a round for establishing contiguity with Procedure 45. Node \( M \) and the third node between \( N \) and \( M \) mentioned in the procedure definition are descendants of \( N \) and not shown explicitly. Nodes \( N_0 \) and \( M_0' \) have the same labels. The depicted conversion in steps 1–4 is followed by the application of Procedure 42 to re-establish regularity.

Procedure 45 (Ensuring Contiguity in Special Cases)

Input: An eager and regular clausal tableau and a set \( S \) of unordered pairs of literals such that for each such pair \( \{L_1, L_2\} \) it holds that:

- \( L_1 \) and \( L_2 \) occur as literal labels of nodes of the tableau.
- There is a literal \( L_0 \) such that all clauses of the tableau in which \( L_1 \) or \( L_2 \) occur as literals are of the form \( L_0 \lor L_1 \) or \( L_0 \lor L_2 \).
- All nodes of the tableau with \( L_0 \) as literal label are leaves.

Method: Repeat the following until the resulting tableau is contiguous for all members of \( S \):

1. Select an inner node \( N \) that has a descendant \( M \) such that \( \{\text{lit}(N), \text{lit}(M)\} \in S \) and there is a third node that is a descendant of \( N \) and an ancestor of \( M \).
2. Create fresh nodes \( M_0' \) and \( M' \) where \( M_0' \) has the same label values (i.e., the literal label and, if applicable, the side label) as the sibling of \( M \), and \( M' \) has the same label values as \( M \).
3. Remove the outgoing edges from \( N \) and attach them to \( M' \).
4. Add \( M_0' \) and \( M' \) as children to \( N \).
5. Apply Procedure 42 to ensure regularity.

Output: An eager and regular clausal tableau whose clauses are also clauses of the input tableau and which is contiguous for all members of the input set \( S \). The following further properties of the input tableau are preserved: closed, leaf-only for a set of literals that does not contain members of the pairs in \( S \).

Termination of Procedure 45 follows since the number of nodes that can be selected in step 1 strictly decreases in each round. Like Procedure 41 and 42, the procedure can be considered as a tableau simplification.

The procedures defined in this section suggest to apply them in the presented order, that is, Procedure 41 (eagerness), Procedure 42 (regularity), Procedure 43 (leaf-only property) and Procedure 45 (contiguity) to the closed clausal tableau obtained by a prover from the structure preserving clausifications of \( \text{DEF}(F) \) and \( \text{DEF}(\neg G) \). The converted tableau is then an ACI-tableau, suitable for extracting the access interpolant according to Definition 32. If the closed clausal tableau obtained by the prover is already a positive hyper tableau, then it is, of course, sufficient ensure contiguity with Procedure 45.
12 Examples for Conversion to ACI-Tableaux

In this section the definitional normalization of RQFO formulas for access interpolation and the conversion of closed clausal tableaux for them to ACI-tableaux is illustrated with examples. We consider computing an access interpolant for the single RQFO sentence

\[ F = G = \forall x (\neg r(x) \lor \exists y (s(x, y) \land \top)) \]

in the role of both interpolation inputs. Of course, the sentence itself is then trivially also an access interpolant. Nevertheless, with this example different structuring possibilities of clausal tableaux as obtained by provers and the effects of the conversions show up.

The definitional normal forms of \( \text{DEF}(F) \) and \( \neg \text{DEF}(G) \), conjoined together, yield the following clausal formula, where \( f \) and \( g \) are Skolem functions, as basis for interpolant computation. The respective clause form according to Lemma 25 is there annotated in the right column. Clauses obtained from \( \text{DEF}(F) \) are shown against grey background.

\[
\begin{align*}
\text{d}_{\text{L}} & \land 1 \\
(\neg \text{d}_{\text{L}} \lor \neg r(x) \lor \text{d}_{\text{L1}}(x)) & \land 5 \\
(\neg \text{d}_{\text{L1}}(x) \lor s(x, f(x))) & \land 7 \\
\text{d}_{\text{R}} & \land 9 \\
(\neg \text{d}_{\text{R}} \lor r(g)) & \land 14 \\
(\neg \text{d}_{\text{R}} \lor \text{d}_{\text{R1}}(g)) & \land 15 \\
(\neg \text{d}_{\text{R1}}(x) \lor \neg s(x, y) \lor \text{d}_{\text{R11}}) & \land 11 \\
\neg \text{d}_{\text{R11}} & 10
\end{align*}
\]

Using global position specifiers (Definition 34), the value of some of the symbolic designators in Lemma 25 is as follows: \( R_{L0} = R_{R0} = r \), \( R_{L1} = R_{R1} = s \), \( \sigma_{L1} = \{ y \mapsto f(x) \} \), \( \sigma_{R0} = \{ x \mapsto g \} \). The Skolem functions \( f_{(L,1)} \) and \( f_{(R,1)} \) are expressed by \( f \) and \( g \), respectively, for readability.

In the examples shown below we will consider closed clausal tableaux for the clausal formula (iv), where the tableau clauses are the following instances of the clauses of formula (iv). Again the respective clause form according to Lemma 25 is annotated in the right column.

\[
\begin{align*}
\text{d}_{\text{L}} & 1 \\
(\neg \text{d}_{\text{L}} \lor \neg r(g) \lor \text{d}_{\text{L1}}(g)) & 5 \\
(\neg \text{d}_{\text{L1}}(g) \lor s(g, f(g))) & 7 \\
\text{d}_{\text{R}} & 9 \\
(\neg \text{d}_{\text{R}} \lor r(g)) & 14 \\
(\neg \text{d}_{\text{R}} \lor \text{d}_{\text{R1}}(g)) & 15 \\
(\neg \text{d}_{\text{R1}}(g) \lor \neg s(g, f(g)) \lor \text{d}_{\text{R11}}) & 11 \\
\neg \text{d}_{\text{R11}} & 10
\end{align*}
\]

To qualify as ACI-tableau the ground tableau then has to be leaf-only for the set

\[
\{ \neg \text{d}_{\text{L}}, \neg \text{d}_{\text{L1}}(g), \neg r(g), \neg \text{d}_{\text{R}}, \neg \text{d}_{\text{R1}}(g), \neg \text{d}_{\text{R11}}, \neg s(g, f(g)) \}
\]
and contiguous for the pair
\[
\{ r(g), d_{R1}(g) \}.
\] (vii)

As noted in Sect. 11, positive hyper tableaux which satisfy a certain contiguity condition are already ACI-tableaux. Such a tableau is typically constructed by “bottom-up” calculi that would start with the positive “root definers” \( d_L \) and \( d_R \) and proceed by “applying” clauses like rules that fire in a forward-chaining manner, that is, extending a branch only with a clause whose negative literals all have complements in the branch. The following tableau gives an example:

**Example 46 (Positive Hyper Tableau)** Figure 6 shows a closed positive hyper tableau for the clausal formula (iv) that is an ACI-tableau for \( F \) and \( G \) and thus allows direct extraction of an access interpolant. Nodes with side label \( L \) are shown with grey background.

![Fig. 6: Example 46 – Positive Hyper Tableau](image)

The remaining examples shown in this section follow start from “connection tableaux”, or, more precisely, tightly connected tableaux (see, e.g., [39]): Each inner node with exception of the root has a child with complementary literal label. Such tableaux are constructed from provers based on model elimination or the connection method, which maintain the tightly connected property throughout tableau construction. Typically they build the tableau “top-down” in a goal-sensitive way by starting in a theorem proving setting with a clause obtained from the theorem in contrast to the axioms. This connectedness property of the tableau returned by provers might get lost by our conversion to ACI-tableaux. Moreover, also the weaker property of path connectedness, that is, among siblings (except for the root and its children) there exists a node that has an ancestor with complementary literal label, is not ensured by the conversions.

**Example 47 (Connection Tableau I)** Figure 7 shows a closed tightly connected clausal tableau for the clausal formula (iv). Nodes with side label \( L \) are shown with grey background. Edges that connect nodes with complementary literal labels are emphasized. The node picked as \( N \) in the next round of Procedure 43 is marked by a surrounding rectangle. Figure 8 shows
the result of applying a round of Procedure 43. Again the node picked as \( N \) in the next round is marked. Further rounds yield the tableaux of Fig. 9 and Fig. 10. The latter is leaf-only for the set (vi) of literals, but not contiguous for the pair (vii). The literals that are chosen as \( N \) and \( M \) in Procedure 45 are displayed in oval markings. The result of applying Steps 1.\,-4. of Procedure 45 is then shown in Fig. 11. The tableau now also is contiguous for the pair (vii), but violates regularity with the nodes marked by a flag. The regularity simplification of Procedure 42 finally yields the tableau in Fig. 12, which is an ACI-tableau and actually identical to the positive hyper tableau in Fig. 6.
Example 48 (Connection Tableau II) Like Example 47, this example starts with a closed tightly connected clausal tableau for the clausal formula (iv) and proceeds in rounds of Procedure 43 (Fig. 13–15), steps 1.-4. of Procedure 45 (Fig. 16) and regularity simplification with Procedure 42 to an ACI-tableau (Fig. 17).
Fig. 15: Example 48 – Stage 3

Fig. 16: Example 48 – Stage 4

Fig. 17: Example 48 – Stage 5
Example 49 (Connection Tableau III) Like Example 47 and 48, this example starts with a closed tightly connected clausal tableau for the clausal formula (iv) and proceed in rounds of Procedure 43 (Fig. 18–22), steps 1.-4. of Procedure 45 (Fig. 23) and regularity simplification with Procedure 42 to an ACI-tableau (Fig. 24).

Fig. 18: Example 49 – Stage 1

Fig. 19: Example 49 – Stage 2

Fig. 20: Example 49 – Stage 3

Fig. 21: Example 49 – Stage 4
13 Access Interpolation: Refinements, Issues and Related Work

13.1 Alternate Tableaux – Alternate Interpolants

In general there are different closed clausal tableaux for a single given unsatisfiable clausal formula. A different interpolant would be extracted from each tableau. For an application such as query reformulation some of these interpolants might be more preferable than others. For example, a query might be preferably reformulated in terms of more specific access patterns such that available parameter instantiations are best utilized when the reformulated query is evaluated.

In [59,30,60] query reformulation is indeed based on computing alternate interpolants – each considered as representing a query plan – and comparing them with a cost function. This approach has been refined in [30,60] with a condensed representation of a set of tableaux in a single structure. An advanced system that interleaves the generation of a pair of such condensed tableaux, one for each of the two interpolation input formulas, with detecting when their combination would be closed is described in [60].

Enumeration of closed clausal tableaux for a given set of clauses is quite natural for goal-sensitive clausal tableau methods such as model elimination and the connection method which operate with backtracking anyways. If such a calculus is not stopped after finishing construction of a closed clausal tableau it backtracks to generate further closed tableaux (CM [64] can for example be configured in that way). However, requirements for theorem proving and for the computation of interpolants as query plans seem to contradict: Theorem provers typically aim to prevent the search for alternate proofs as much as possible without compromising completeness or experimental success, whereas, if interpolants are considered as query plans with associated costs it would make sense to compare even proofs with “trivial” differences, for example, if they correspond to interpolants just distinguished by a different order of conjuncts in a subformula.

Heuristics that can be configured to give priority to tableaux that are preferred with respect to the application seem in general useful. For goal-sensitive provers based on model elimination or on the connection calculus the order in which clauses are picked for inclusion in the tableau is relatively easy to influence. With backtracking and iterative deepening completeness is preserved by ensuring that lower ranked clauses will eventually be considered as long as no closed tableau with more highly ranked clauses has been found. A common heuristics is to rank clauses by their length, shortest first. Application specific orderings can, for example, effect that clauses associated with more specific reformulations are given priority.5

13.2 Preprocessing Issues and Resolution

Preserving the second-order equivalence (ii) as discussed in Sect. 8.4 for preprocessing inputs of Craig-Lyndon interpolation is too weak for access interpolation, as the latter depends on further constraints on the clause form. It seems, however, possible to generalize some of the clause forms of Lemma 25 such that they are closed under certain preprocessing steps that break apart and join clauses. Forms 2–5, for example, could be easily generalized to a single form with one negative and an arbitrary number of positive definer literals whose arguments,

5 The CM prover included with the PIE system [64] supports this with an experimental option. For an example see Section 2.2 in http://cs.christophwernhard.com/pie/downloads/pie/scratch/scratch_views_lit.pdf.
and hence also free variables, all occur in the negative literal. This generalized form is closed under resolution. The forms 2–5 are already handled in a generalized way together in proof of Lemma 39. Form 6 could be generalized by allowing further positive literals. However, unrestricted resolution among such clauses may yield clauses with multiple negative “$R$”-literals (literals that are not definers). Exploring ways to restrict resolution for these clauses is an issue for future research.

13.3 Application to Query Reformulation

The actual specification of axiom schemas used as inputs of interpolation tasks for query reformulation is beyond the scope of this paper. We refer to the literature, in particular to [10,9] and [59]. Nevertheless, certain aspects of the axiom schemas in [10] seem to be closely connected to the precise way in which input formulas are expressed and processed by a theorem prover, suggesting further investigations in the context of our method. This concerns in particular the consideration of access interpolation for “non-Boolean” queries, that is, queries whose results are relations of non-zero arity, discussed in [10, Sec. 3.8] and the axiom schema $\text{AltAcSch}$ from [10, Section 3.6], which is apparently obtained from a more abstract specification $\text{AcSch}$ with the involvement of unfolding predicates, raising the question whether a more condensed representation without the need of unfolding is possible.

The considered approach to query reformulation is not principally limited to relational database queries, but applies to logic-based knowledge representation mechanisms in general. So far, the main application direction is to optimize a given query with respect to a given access schema. The approach can, however, in also be applied conversely to determine from given parameterized queries an access schema that would be required to answer these queries. That inferred access schema can then be used to determine caches and precomputed indexes as basis for evaluating the queries at a later time.

13.4 Implementations of Interpolation for Query Reformulation

In [9] different approaches to implement query reformulation based on Craig-Lyndon interpolation have been experimentally investigated, however only for ground inputs. One approach considered there was an extension of the general first-order prover Vampire that supports interpolant computation [28]. Apparently it does not ensure the polarity constraint on Craig-Lyndon interpolants (the “Lyndon property”), had many timeouts and produced poor result formulas, indicating that the requirements of interpolation in verification, the main objective of the Vampire extension, and in query reformulation are quite different. Better results were obtained with methods for interpolant extraction from resolution proofs, based on algorithms from [29,12,43]. The MathSAT SMT solver [18] and the E first-order prover [54] have been used to compute the underlying resolution proofs (apparently E was the only first-order resolution prover that produced sufficiently detailed proofs). MathSAT was there superior to E. For extraction of the propositional interpolants an optimized variant of the algorithm of [29] introduced in [9] as well as the method of [43] showed best. Also a method based on the chase technique, implemented with DLV [36] as model generator, has been evaluated in [9]. In essence, a uniform interpolant (that is, the result of predicate

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* Aside of Vampire, also the Princess theorem prover (http://www.philipp.ruemmer.org/princess.shtml) [15] supports Craig interpolation, mainly with respect to theories targeted at applications in verification. The PIE system is another first-order prover with support for Craig interpolation (see Sect. 8.6).
elimination) is computed there by removing literals whose predicate is not allowed in the interpolant from a disjunctive normal form. In cases where the expected reformulation is a disjunctive normal form this approach is only slightly worse than the best resolution-based approach. It seems that access interpolation has so far not been implemented. As already mentioned, the approach of [59,30] has been implemented with advanced dedicated techniques [60]. Small examples with a simple form of axiom schemas that are processed by Craig-Lyndon interpolation on the basis of a general first-order clausal tableau prover come with the PIE system [64].

14 Conclusion

We investigated the computation of Craig-Lyndon interpolants and of access interpolants, a recent form of interpolation with applications in query reformulation, by means of clausal tableau methods. Aspects of the elegance of an established interpolant construction based on non-clausal tableaux were combined with the suitability of clausal tableaux for machine processing. The framework of clausal tableaux as a basis in contrast to resolution leads to a natural way to decompose the overall task of computing first-order interpolants into subtasks, which seems useful for theoretical considerations as well as practical implementation. This new modularization concerns three different aspects: the lifting from ground interpolants to quantified formulas, the roles of “local” versus “non-local” techniques, and the interplay of proof search with structural requirements on the proof:

1. Computation of Craig-Lyndon interpolants is performed on the basis of clausal tableaux in two stages, like in most resolution-based interpolation methods. In contrast to these, however, the second stage, the lifting to a quantified first-order formula, is performed here on an actual Craig-Lyndon interpolant of a finite unsatisfiable subset of the Herbrand expansion of the Skolemized and clausified input formulas instead of some formula that is characterized as “almost” a Craig-Lyndon interpolant of the original input formulas. This allows to consider the lifting conversion more abstractly, just based on Herbrand’s theorem, independently from a particular calculus. For practical implementation, it means that any method that computes a closed clausal tableau for an unsatisfiable first-order input formula can be directly applied to interpolant computation, without need to alter its internal workings, benefiting directly from refinements and efficient data structures.

2. Differently from resolution, the construction of a clausal tableau does not involve breaking apart and joining clauses. However, such operations can be performed during preprocessing for a clausal tableau prover. We then get an overall picture of the tableau-based interpolation where first operations that break apart and join clauses, like resolution, are performed only “locally”, that is, on each of the two input formulas individually. These operations must preserve the semantics of the predicates and functions that are allowed in the interpolant, but can eliminate or semantically alter other predicates and functions. After this preprocessing stage is completed, for example, because no further conversion operations are possible or because potential further operations would increase the formula size in an undesired way, the actual tableau construction comes in to handle the “non-local” joint processing of both preprocessed inputs.

7 http://cs.christophwernhard.com/pie/downloads/pie/scratch/scratch_access_demo_01.pdf. http://cs.christophwernhard.com/pie/downloads/pie/scratch/scratch_views_lit.pdf
3. In our approach to the computation of access interpolants the underlying clausal tableaux are required to meet specific structural constraints. “Bottom-up” methods for constructing clausal tableaux, such as the hyper tableau calculus, typically can be configured to compute tableaux that satisfy these constraints. For other calculi the tableau construction is split in two phases: First, a theorem prover computes an arbitrarily structured closed clausal tableau. Second, the structure of the tableau output by the prover is converted such that it satisfies the required restrictions. This conversion is, however, potentially costly since it involves steps that might involve duplication of subtableaux.

These aspects suggest a number of challenging follow-up questions. With respect to aspect (1.): Can the justification of lifting for Craig-Lyndon interpolants also be applied to resolution-based interpolation methods? Can this lifting method, which prepends a single quantifier prefix to the whole formula be reconciled with requirements of relativized quantification as in access interpolation, where quantifier scopes that are limited to subformulas seem essential? With respect to (2.): Does the observation of the roles of “local” versus “non-local” inferences in interpolation based on clausal tableaux indicate some interesting property where clausal tableaux and resolution diverge? Is interpolation a field for which clausal tableau methods are better suited than resolution in some substantial sense? With respect to (3.): Are there interesting implications of the “calculus that preserves a structure” versus “calculus has more freedom followed by potentially costly conversion” approaches? Can it be shown that the first approach implies that the costs of conversion have to be incorporated in essence into the proof search?

The presented material provides foundations to practically implement Craig-Lyndon interpolation and access interpolation on the basis of a variety of machine-oriented theorem-proving methods for first-order logic that can be considered as constructing a closed clausal tableaux. These fall into two families, goal-oriented “top-down” methods such as model elimination and the connection method, and data-oriented “bottom-up” methods such as the hyper tableau calculus. A first implementation of Craig-Lyndon interpolation on the basis of a prover of the first family is already available [64]. Since the presented methods and most discussed refinements incorporate first-order provers that compute clausal tableaux abstractly, without imposing requirements on tableau construction methods, it should only be a short way from the foundations provided with this work to experimental evaluations.

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