RICCI CURVATURE OF CONFORMAL DEFORMATION ON COMPACT 2-MANIFOLDS

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Abstract. In this paper, we consider Ricci curvature of conformal deformation on compact 2-manifolds. And we prove that, by the conformal deformation, the resulting manifold is an Einstein manifold.

1. Introduction. A Riemannian manifold \((M, g)\) with a metric \(g\) is an Einstein manifold provided Ricci curvature \(\text{Ric} = cg\) for some constant \(c\). It is well known that if \(M\) is connected, \(n = \text{dim} \ M \geq 3\), and \(\text{Ric} = k(p)g\), then \(M\) is Einstein([1, p.7],[3],[6]).

In general, if \(M\) is connected and 2-dimensional, then \(\text{Ric} = k(p)g\) for some function \(k(p)\). In this case, we have the following question:

Question A. Is it possible that \((M, \tilde{g})\) is an Einstein manifold \((\text{Ric}_{\tilde{g}} = C\tilde{g}\), where \(C\) is a constant\) by conformal deformation when \(\tilde{g} = e^{2f}g\) for some function \(f\)?

Now if a given metric \(g\) on \(M\), where \(\text{dim} \ M = 2\), has Ricci curvature such that \(\text{Ric}_g = k(p)g\) for some function \(k(p)\), and we seek \(K(p)\) as the Ricci curvature of the metric \(\tilde{g} = e^{2f}g\) pointwise conformal to \(g\) such that \(\text{Ric}_g = K(p)\tilde{g} = K(p)e^{2f}g\), then \(f\) must satisfy

\[
\triangle f - k(p) + K(p)e^{2f} = 0, \quad (1.1)
\]

where \(\triangle\) is the Laplacian in the metric \(g\). Several authors have studied the solutions of equation (1.1) (cf. [2],[4],[5], etc.).

In this paper, to solve Question A, when \(K(p) \equiv C\) for some constant, instead of equation (1.1), we consider the solvability of the following equation

\[
\triangle f - k(p) + Ce^{2f} = 0. \quad (1.2)
\]

In particular, using the change of variables, instead of equation (1.1), Kazdan and Warner consider the following form, which has a similar form with (1.2),

\[
\triangle u - c + he^u = 0, \quad (1.3)
\]

where \(c\) is a constant, and \(h\) is some prescribed function, with neither \(c\) nor \(h\) depending on the geometry of \((M, g)\) ([5]). Kazdan and Warner discussed the
solutions of equation (1.3) according to the value of $c$ ([4], [5]). See the following Remark 1.1.

**Remark 1.1.** In fact, in [5], Kazdan and Warner had shown the following results:

i) Case $c < 0$. If equation (1.3) is solvable, then $\tilde{h} < 0$, where $\tilde{h} = \frac{1}{\text{Vol}(M)} \int_M h dM$. And there exists a critical strictly negative constant $-\infty \leq c_-(h) < 0$ such that equation (1.3) is solvable if $c_- (h) < c < 0$, but not solvable if $c < c_- (h)$.

ii) Case $c = 0$. Then, excluding the trivial case $h \equiv 0$, a solution of equation (1.3) exists if and only if both $\tilde{h} < 0$ and $h$ changes signs.

iii) Case $c > 0$. Then there is a constant $0 < c_+(h) \leq \infty$, possibly depending on $M$, such that a solution exists if $h$ is positive somewhere and if $0 < c < c_+(h)$. They had shown that, in the case $c > 0$, there exists some obstructions of the solvability of (1.3). They proved that, in the case $c = 2$ on the sphere $S^2$, equation (1.3) has no solutions for any function $h$ such that $\nabla h \cdot \nabla F_0$ has a fixed sign for some spherical harmonic $F_0$ of degree 1, in particular, for all functions $h$ of the form $F_0 + \text{constant}$. And they proved also that, in the case $c > 2$ on $S^2$, if $h = F_0$ is a spherical harmonic of degree 1, then (1.3) has no solutions.

In this paper, for the given Ricci curvature $k(p)$, we prove the solvability of equation (1.2), using the variational method, for some constant $C$. The aspect of the solvability of equation (1.2) is different from that of equation (1.3). In equation (1.3), they consider the solvability of (1.3) for $h$ according to $c$, but we consider the solvability of (1.2) for some constant $C$ according to $k$.

Let $M$ be a compact connected 2-dimensional manifold, which is not necessarily orientable and possesses a given Riemannian structure $g$. We denote the volume element of this metric by $dV$, the gradient by $\nabla$, and the associated Laplacian by $\Delta$ (we use the sign convention which gives $\Delta f = 1$ for the standard metric on $\mathbb{R}^2$). The mean value of a function $f$ on $M$ is written $\overline{f}$, that is,

$$\overline{f} = \frac{1}{\text{vol}(M)} \int_M f dV.$$

We let $H_{s,p}(M)$ denote the Sobolev space of functions on $M$ whose derivatives through order $s$ are in $L_p(M)$. The norm on $H_{s,p}(M)$ will be denoted by $\| \cdot \|_{s,p}$. In the special case $s = 0$, $H_{s,p}(M)$ is just $L_p(M)$, and we denote the norm by $\| \cdot \|_p$.

If $\dim M = 2$ and $u \in C^\infty(M)$ with $\overline{u} = 0$, then for any $p \geq 1$ there is a constant $c_1$ independent of $p$ and $u$ such that

$$\| u \|_p \leq c_1 \ p^{\frac{1}{2}} \ \| \nabla u \|_2. \quad (1.4)$$

The point here is the sharp control of the dependence of the right side on $p$ ([5, p.21-22]).

Another immediate consequence of the Poincaré inequality (1.4) is that there is a constant $c_2$ such that for any $u \in C^\infty(M)$ with $\overline{u} = 0$, one has

$$\| u \|_{1,2} \leq c_2 \ \| \nabla u \|_2. \quad (1.5)$$

**Proposition 1.2.** Assume $\dim M = 2$. If $u_j \in H_{1,2}(M)$ and $u_j \to u$ weakly in $H_{1,2}(M)$, then $e^{u_j} \to e^u$ strongly in $L_2(M)$.

**Proof.** See [5, p.23].
2. Preminaries. In this section, we consider the Ricci curvature of conformal deformation on a 2-dimensional manifold.

**Theorem 2.1.** If $(M, g)$ is connected and 2-dimensional, then there exists $k(p)$ such that

$$R_{ij} = k(p)g_{ij}, \ i, j = 1, 2,$$

where $R_{ij}$ is a Ricci tensor and $g_{ij}$ is a metric tensor.

**Proof.** Using the normal coordinate system, let

$$ds^2 = f(u, v)du^2 + h(u, v)dv^2.$$

Since

$$\Gamma^k_{ij} = \frac{1}{2} \sum_m g^{km} (\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m})$$

for all $i, j, k = 1, 2$, then by boring computation,

$$\Gamma^1_{11} = \frac{1}{2} f_u, \ \Gamma^1_{12} = \frac{1}{2} f_v, \ \Gamma^1_{21} = \frac{1}{2} f_v, \ \Gamma^1_{22} = \frac{1}{2} (h_u),$$

$$\Gamma^2_{11} = \frac{1}{2} (-f_v), \ \Gamma^2_{12} = \frac{1}{2} h_u, \ \Gamma^2_{21} = \frac{1}{2} h_u, \ \Gamma^2_{22} = \frac{1}{2} h_v.$$

And since

$$R^i_{jkl} = \frac{\partial}{\partial x^j} \Gamma^i_{kl} - \frac{\partial}{\partial x^k} \Gamma^i_{jl} + \sum_m \Gamma^i_{jm} \Gamma^m_{kl} - \sum_m \Gamma^i_{km} \Gamma^m_{jl},$$

for all $i, j, k, l = 1, 2$, the boring computation leads that

$$R^1_{112} = \frac{f_{uv} - f_{vu}}{2f}, \ R^1_{121} = \frac{f_{vu} - f_{uw}}{2f},$$

$$R^1_{212} = -\frac{f_u f_v}{4f^2} + \frac{f_{uv}}{2f} - \frac{f_u h_u}{2f} + \frac{h_{uu} - h_u h_u}{4h} - \frac{f_v}{4h},$$

$$R^2_{221} = \frac{f_{h_u}}{4h^2} + \frac{h_{uu}}{2h} - \frac{f_{uv} + f_v}{4h} + \frac{f_{h_v}}{4h},$$

$$R^1_{12} = -\frac{h_{uu} h_u}{2h} + \frac{h_{uu}}{2h} - \frac{h_{vv}}{4h^2} + h_{uv} - \frac{f_{h_u}}{4h} - \frac{f_{h_v}}{4h},$$

$$R^2_{21} = \frac{h_{uu} - h_{vv}}{2h}, \ R^2_{221} = \frac{h_{vv} - h_{uu}}{2h},$$

$$R^l_{i j k} = 0,$$ otherwise.

By definition of Ricci tensor, $R_{ij} = \sum_k R^k_{i j k}$ for $i, j = 1, 2$. Hence

$$R_{11} = \frac{f_{h_v}}{4h^2} + h_{uu} h_u - \frac{f_{uv}}{2h} + \frac{h_{uu}}{2h} + \frac{f_{h_u}}{4h} + \frac{f_{h_v}}{4h},$$

$$R_{12} = R_{21} = 0,$$

$$R_{22} = \frac{f_{h_u}}{4f^2} + \frac{f_{h_v}}{4f^2} - \frac{h_u h_u}{2h} + \frac{h_{uu}}{2h} + \frac{f_{h_u}}{4h} + \frac{f_{h_v}}{4h}.$$

If we let

$$k(u, v) = \frac{f(h_u h_u + f_v h_v) + h(f_u h_u + f_v f_v) - 2fh(f_{uv} + h_{vv})}{4f^2 h^2},$$
then
\[ R_{11} = k(u,v)g_{11}, \quad R_{22} = k(u,v)g_{22}, \]
so \( R_{ij} = k(u,v)g_{ij} \) for all \( i, j = 1, 2 \), which is our desired one.

In fact, the above \( k(u,v) \) is a Gaussian curvature on \((M,g)\).

**Example 2.2.** For compact 2-manifolds, we have the metric, for example, as follows:
\[ ds^2 = du^2 + (u^2 + 1)dv^2. \]
Then
\[ \Gamma_{22}^1 = -u, \quad \Gamma_{12}^2 = \frac{u}{u^2 + 1}, \quad \Gamma_{21}^2 = \frac{u}{u^2 + 1}, \quad \Gamma_{ij}^k = 0, \text{ otherwise}, \]
and
\[ R_{212}^l = \frac{1}{(u^2 + 1)^2}, \quad R_{221}^l = -\frac{1}{u^2 + 1}, \quad R_{112}^2 = -\frac{1}{(u^2 + 1)^2}, \quad R_{121}^2 = \frac{1}{(u^2 + 1)^2}, \quad R_{ijk}^l = 0, \text{ otherwise}. \]
Since the Ricci tensors are \( R_{ij} = \sum_k R_{ikj}^k \) for \( i, j = 1, 2 \), then
\[ R_{11} = -\frac{1}{(u^2 + 1)^2}, \quad R_{12} = R_{21} = 0, \quad R_{22} = -\frac{1}{u^2 + 1}. \]
Hence
\[ R_{11} = k(u,v)g_{11}, \quad R_{22} = k(u,v)g_{22}, \]
where \( k(u,v) = -\frac{1}{(u^2 + 1)^2} \).

Let us consider the conformal metric \( \tilde{g} = e^{2f}g \) with \( f \in C^\infty(M) \). By Definition 1.19 in [1], if \( \tilde{\Gamma}_{ij}^l \) and \( \Gamma_{ij}^l \) denote the Christoffel symbols relating to \( \tilde{g} \) and \( g \), respectively, for \( i, j, l = 1, 2 \), then
\[ \tilde{\Gamma}_{ij}^l - \Gamma_{ij}^l = \sum_k (g_{kj} \frac{\partial f}{\partial i} + g_{ki} \frac{\partial f}{\partial j} - g_{ij} \frac{\partial f}{\partial k}) \tilde{g}^{kl} = (\delta_i^l \frac{\partial f}{\partial j} + \delta^l_j \frac{\partial f}{\partial i} - g_{ij} \nabla^l f). \]
And the Ricci tensors are
\[ \tilde{R}_{ij} = \sum_k \tilde{R}_{ikj}^k = R_{ij} - (n - 2)\nabla_i f + (n - 2)\nabla_j f \nabla_i f - (\nabla_v f + (n - 2)\nabla_v f \nabla_v f)g_{ij}. \]
for \( i, j = 1, 2 \). If \( \tilde{R}_{ij} = K(p)\tilde{g}_{ij} = K(p)e^{2f}g_{ij} \) and \( R_{ij} = k(p)g_{ij} \) for \( i, j = 1, 2 \) on a 2-dimensional manifold \( M \), then
\[ K(p)e^{2f} = k(p) - \triangle f. \]
In other words,
\[ \triangle f - k(p) + K(p)e^{2f} = 0. \quad (2.1) \]
For Question A, if \( K(p) \equiv C(\text{constant}) \), then equation (2.1) is changed into the following equation:
\[ \triangle f - k(p) + Ce^{2f} = 0. \quad (2.2) \]
Instead of equation (2.1), we will prove the solvability of equation (2.2), using the variational method.
Theorem 2.3. Let $(M, g)$ be a 2-dimensional compact and connected manifold. If there exists a solution of equation (2.2), then

$$C \int_M e^{2f} \, dM = \int_M k(p) \, dM = 2\pi \chi(M),$$

where $\chi(M)$ is an Euler characteristic of $M$.

Proof. Since $k(p)$ is a Gaussian curvature on $(M, g)$, Gauss-Bonnet formula implies that our theorem holds trivially. \qed

Necessarily, if there exists a solution of equation (2.2), then from Theorem 2.3 we can see that the sign of $C$ is the same with that of $\chi(M)$.

Theorem 2.4. If there exists a solution of equation (2.2) for a constant $C(\neq 0)$, then we also have a solution of equation (2.2) for some constant $C'(CC' > 0)$.

Proof. If $f$ is a solution of equation (2.2) for $C$ and $C' = e^{2b}$ for some constant $b$,

$$\Delta(f + b) - k(p) + C'e^{2f + 2b} = \Delta f - k(p) + Ce^{2f} = 0,$$

hence $f + b$ is a solution of equation (2.2) for $C'$. \qed

Lemma 2.5. If $M$ is a 2-dimensional compact manifold without boundary. Then there exists $u$ such that $\Delta u = k(p)$ if and only if $K = 0$.

We know that the solution $u$ in Lemma 2.5 is unique up to a constant. If $k(p) \in C^{r+\alpha}(M)$, $(r \geq 0$ an integer or $r = +\infty$, $0 < \alpha < 1)$, then $u \in C^{r+2+\alpha}(M)$ (cf. 1, Theorem 4.7]).

It is trivial that if $k(p) \equiv C_1(\text{constant})$ and $C_1C > 0$, then equation (2.2) has a constant solution. Hence, from now on, we assume that $k(p) \neq \text{constant}$. Then $f(\equiv \text{constant})$ is not a solution of equation (2.2).

Theorem 2.6. If $\varphi = 0(\neq 0)$ for $\varphi \in H_{1.2}(M)$, then $\int_M e^\varphi \, dM > 0$.

Proof. If $\varphi \neq 0$ and $\varphi = 0$, we put $v^+(x) = \max\{\varphi(x), 0\}$ and $v^-(x) = \max\{-\varphi(x), 0\}$. Trivially $v(x) = v^+(x) - v^-(x)$ and $|v(x)| = v^+(x) + v^-(x)$. And $M^+_v = \{x \in M | v^+(x) > 0\}$, $M^-_v = \{x \in M | v^-(x) \geq 0\}$. Then for each $v$, $M = M^+_v \cup M^-_v$

$$\int_M e^\varphi \, dM = \int_{M^+_v} e^\varphi \, dM + \int_{M^-_v} e^\varphi \, dM$$

$$= \int_{M^+_v} e^\varphi_+ \, dM - \int_{M^-_v} e^{-\varphi^-} \, dM$$

$$> \int_{M^+_v} v^+ \, dM - \int_{M^-_v} v^- \, dM = \int_M v^+ \, dM - \int_M v^- \, dM$$

$$= \int_M (v^+ - v^-) \, dM = 0.$$

Jessen inequality ([7, p.62]) implies that for all $\varphi \in H_{1.2}(M)$,

$$\frac{1}{\text{vol}(M)} \int_M e^\varphi \, dM \geq \frac{1}{\text{vol}(M)} \int_M \varphi \, dM.$$

Hence, if $\varphi = 0$, then $\int_M e^\varphi \, dM \geq \text{vol}(M)$. 

Corollary 2.7. If \( \varpi = 0 (v \neq 0) \) for \( v \in H_{1,2}(M) \), then \( \int_M e^{cv} \, dM > \text{vol}(M) \) for all \( c \neq 0 \) and \( \int_M e^{cv} \, dM \to \infty \) as \( c \to \infty \).

Proof. Put \( f(c) = \int_M e^{cv} \, dM \) for some \( v \in H_{1,2}(M) (v \neq 0 \text{ and } \varpi = 0) \). Then \( f'(c) = \int_M v e^{cv} \, dM \). Theorem 2.6 implies that if \( c > 0 \), then \( f'(c) > 0 \) and if \( c < 0 \), then \( f'(c) < 0 \). Therefore \( f(c) \) has the minimum at \( c = 0 \), which means that \( \int_M e^{cv} \, dM > \text{vol}(M) \) for all \( c \neq 0 \). And since \( f'(c) > 0 \) and \( f''(c) > 0 \) for \( c > 0 \), \( f(c) \to \infty \) as \( c \to \infty \). \( \square \)

We consider the following functional \( J \) on \( B_{\lambda_0} = \{ v \in H_{1,2}(M) \mid \varpi = 0, v \neq 0, \int_M e^{2v} \, dM = \lambda_0 \} \) for some \( \lambda_0 > 1 \),

\[
J(v) = \frac{\int_M |\nabla v|^2 \, dM + 2 \int_M k(p)v \, dM}{\int_M e^{2v} \, dM}.
\]

Theorem 2.8. Let \( \{v_i\} \) be a minimizing sequence in \( B_{\lambda_0} \) such that \( J(v_i) \to C \) for some constant \( C \) depending on some \( \lambda_0 > 1 \). If \( v_i \to v_0 \) in \( B_{\lambda_0} \) and \( J(v_0) = C \), then equation (2.2) has a solution \( v_0 \).

Proof. Let \( v_0 \) satisfy

\[
J(v_0) = \frac{\int_M |\nabla v_0|^2 \, dM + 2 \int_M k(p)v_0 \, dM}{\int_M e^{2v_0} \, dM} = C.
\]

For all \( \psi \in H_{1,2}(M) \),

\[
\left. \frac{dJ(v_0 + t\psi)}{dt} \right|_{t=0} = \frac{d}{dt} \left[ \frac{\int_M (|\nabla v_0 + t\nabla \psi|^2 + 2k(p)(v_0 + t\psi)) \, dM}{\int_M e^{2v_0 + 2t\psi} \, dM} \right] \bigg|_{t=0}
\]

\[
= \frac{1}{(\int_M e^{2v_0 + 2t\psi} \, dM)^2} \left\{ \int_M 2\nabla v_0 \nabla \psi \, dM + 2 \int_M k(p)\psi \, dM + 2t \int_M |\nabla v_0|^2 \, dM \right\} - \left\{ \int_M |\nabla v_0 + t\nabla \psi|^2 \, dM \right\} \bigg|_{t=0}
\]

\[
= \frac{1}{(\int_M e^{2v_0} \, dM)^2} \left[ \left\{ \int_M \nabla v_0 \nabla \psi \, dM + 2 \int_M k(p)\psi \, dM \right\} \int_M e^{2v_0} \, dM \right.
\]

\[
- \left\{ \int_M |\nabla v_0|^2 \, dM + 2 \int_M k(p)v_0 \, dM \right\} \left\{ \int_M e^{2v_0} \, dM \right\} \bigg|_{t=0} = 0.
\]

Therefore

\[
\int_M \nabla v_0 \nabla \psi \, dM + \int_M k(p)\psi \, dM - C \int_M e^{2v_0} \, dM = 0,
\]

for all \( \psi \in H_{1,2}(M) \). Hence

\[
\Delta v_0 - k(p) + Ce^{2v_0} = 0.
\]

\( \square \)

Lemma 2.9. If \( \varpi = 0 \) for \( v \in H_{1,2}(M) \), then

\[
\frac{\int_M |v| \, dM}{\int_M e^v \, dM} \leq 2.
\]
Proof. If \( v \equiv 0 \), then it is trivial. If \( v \neq 0 \) and \( \overline{v} = 0 \), we put \( v^+(x) = \max\{v(x), 0\} \) and \( v^-(x) = \max\{-v(x), 0\} \). Trivially \( v(x) = v^+(x) - v^-(x) \) and \( |v(x)| = v^+(x) + v^-(x) \). Since \( \int_M v^+(x) \, dM = \int_M v^-(x) \, dM \), we have \( \int_M |v(x)| \, dM = 2 \int_M v^+(x) \, dM \).

And \( M^+ = \{ x \in M | v^+(x) > 0 \} \), \( M^- = \{ x \in M | v^-(x) \geq 0 \} \). Then for each \( v \), \( M = M^+ \cup M^- \) and \( \int_M |v(x)| \, dM = 2 \int_M v^+(x) \, dM = 2 \int_{M^+} v^+(x) \, dM \). Since \( \int_M e^v \, dM = \int_{M^+} e^{v^+} \, dM + \int_{M^-} e^{-v^-} \, dM \), we have
\[
\frac{\int_M |v(x)| \, dM}{\int_M e^v \, dM} \leq \frac{2 \int_{M^+} v^+(x) \, dM}{\int_{M^+} e^{v^+} \, dM} \leq 2.
\]

\( \square \)

**Theorem 2.10.** On \( B = \{ v \in H_{1,2}(M) \ : \ \overline{v} = 0, v \neq 0 \} \), the functional \( J(v) \) is lower bounded.

**Proof.** Since \( M \) is compact, \( \max_{p \in M} |k(p)| \leq N_0 \) for some positive constant \( N_0 \). If \( v \in B \), then
\[
J(v) \geq \frac{\int_M |\nabla v|^2 \, dM - N_0 \int_M 2|v| \, dM}{\int_M e^{2v} \, dM}.
\]

Since \( \frac{\chi v}{\int_M e^{2v} \, dM} \leq 2 \) by Lemma 2.9, \( J(v) \geq -2N_0 \) for all \( v \in B \). This means that \( J(v) \) is lower bounded on \( B \).

We consider the following functional \( J \) on \( B_{\lambda_0} = \{ v \in H_{1,2}(M) | \int_M e^v \, dM = \lambda_0, \overline{v} = 0, v \neq 0 \} \) for some \( \lambda_0 > 1 \),
\[
J(v) = \frac{\int_M |\nabla v|^2 dM + 2 \int_M k(p)v \, dM}{\int_M e^{2v} \, dM} = \frac{\int_M |\nabla v|^2 dM + 2 \int_M k(p)v \, dM}{\lambda_0}.
\]

**Theorem 2.11.** Let \( C = \inf_{v \in B_{\lambda_0}} J(v) \) for some \( \lambda_0 > 1 \), where \( C \) is a constant depending on \( \lambda_0 \). Then there exists a solution of equation (2.2) and \( \chi(M) \) and \( C \) are the same signs, where \( \chi(M) \) is an Euler characteristic of \( M \).

**Proof.** Since \( k(p) \) is smooth on \( M \), Theorem 2.10 implies that \( J \) is bounded on \( B_{\lambda_0} \). Hence there exists a minimizing sequence \( \{ v_n \} \) in \( B_{\lambda_0} \) such that \( J(v_1) \to C \). Because \( B_{\lambda_0} \) is not empty, there is some \( v_1 \in B_{\lambda_0} \). Let \( b = J(v_1) \). We may assume that \( b > 0 \) and \( J(v_n) \leq b \) for all \( n \).

For \( v_n \in B_{\lambda_0} \),
\[
J(v_n) = \frac{\int_M |\nabla v_n|^2 \, dM + 2 \int_M k(p)v_n \, dM}{\lambda_0} \geq \frac{\int_M |\nabla v_n|^2 \, dM}{\lambda_0} - 2.
\]

Hence \( \int_M |\nabla v_n|^2 \, dM \leq (b + 2)\lambda_0 \). It follows from equation (1.4) and equation (1.5) that \( ||v_n||^2_{2,2} \leq \text{constant} \) for all \( n \). Because the unit ball in any Hilbert space is weakly compact, we conclude that there is some \( v_0 \in H_{1,2}(M) \) such that a subsequence of \( \{ v_n \} \), which we relabel \( v_n \), converges weakly to \( v_0 \). This implies that \( \int_M v_0 \, dM = 0 \) and \( \int_M k(p)v_0 \, dM \to \int_M k(p)v_0 \, dM \).

Since, by Proposition 1.2, \( e^{v_n} \) converges to \( e^{v_0} \) in \( L_2(M) \), we obtain \( \int_M e^{v_0} \, dM = \lambda_0 \). Therefore \( v_0 \in B_{\lambda_0} \). Hence \( J(v_0) \geq C \).

To conclude that \( v_0 \) minimizes \( J \) for all \( v \in B_{\lambda_0} \), we use the general result that whenever \( v_n \) converges to \( v_0 \) weakly in a Hilbert space, then \( ||\nabla v_n||_2 \leq \lim \inf ||\nabla v_0||_2 \). Thus \( J(v_0) \leq J(v_n) \) for all \( n \) and \( J(v_0) \leq C \). Therefore \( v_0 \) minimizes \( J \) in \( B \). By Theorem 2.8, there exists a solution of equation (2.2). The fact that \( \chi(M) \) and \( C \) are the same signs follows from the integration of equation (2.2). \( \square \)
3. Manifolds with $\chi(M) \leq 0$. We will be considering the operator

$$L(u) = \triangle u - a(p)u,$$

where $a(p) \geq \text{const} > 0$. The following lemma for equation (3.1) is well known.

**Lemma 3.1.** $L : H_{1,2}(M) \rightarrow L_2(M)$ is a bijective operator.

**Proof.** See p. 24 in [5].

In the following theorem, we will solve Question A for compact 2- manifolds with $\chi(M) < 0$, i.e., for $q$-handle($q > 2$) torus, $\chi(M) = 2 - 2q$ or Dyck’s surface, $\chi(M) = -1$, etc.

**Theorem 3.2.** If $\chi(M) < 0$, then there exists a nonconstant solution of equation (2.2) for some constant $C < 0$. Hence, by conformal deformation, the resulting manifold is an Einstein manifold.

**Proof.** Since $\chi(M) < 0$, Gauss-Bonnet formula implies that $\overline{K} < 0$. By Lemma 3.1, there exists a unique solution $u_0$ such that

$$\triangle u_0 - bu_0 = 2k(p) - 2\overline{K}$$

for some constant $b > 0$. We assume that $k \neq \text{constant}$, so $u_0$ is not constant. And we know that $\overline{\omega} = 0$. Thus there exists $\lambda_0$ such that $\int_M e^{2u_0} \, dM = \lambda_0$. If we set $B_{\lambda_0} = \{ u \in H_{1,2}(M) \mid \overline{\omega} = 0, u \neq 0, \int_M e^{2u} \, dM = \lambda_0 \}$, then $u_0 \in B_{\lambda_0}$, which means $\partial B_{\lambda_0}$ is not empty. Multiplying both sides of equation (3.2) by $u_0$ and integrating,

$$\int_M |\nabla u_0|^2 \, dM + \int_M 2k(p)u_0 \, dM = -b \int_M |u_0|^2 \, dM < 0,$$

which implies that $C = \inf_{u \in B_{\lambda_0}} J(u) < 0$. Hence by Theorem 2.11 there exists a solution $u$ such that

$$\triangle u - k(p) + Ce^{2u} = 0,$$

where $u \in B_{\lambda_0}$ means that the solution $u$ is not constant.

In the following theorem, we will solve Question A for compact 2- manifolds with $\chi(M) = 0$, i.e., for torus and Klein bottle.

**Theorem 3.3.** If $\chi(M) = 0$, then there exists a nonconstant solution of equation (2.2) for the constant $C \equiv 0$. Hence, by conformal deformation, the resulting manifold is an Einstein manifold.

**Proof.** If $\chi(M) = 0$, then Gauss-Bonnet theorem implies that $\overline{K} = 0$, so, by Lemma 2.5, there exists $u_0$ such that $\triangle u_0 = k(p)$. Since $k(p) \neq \text{constant}$, $u_0$ is a nonconstant solution and unique up to constants. And $u_0$ satisfies that $\triangle u_0 - k(p) + Ce^{2u_0} = 0$, which is our desired one.

4. Manifolds with $\chi(M) > 0$. In this section, we consider Question A for compact 2-dimensional manifolds with $\chi(M) > 0$, i.e., for the sphere, $\chi(M) = 2$ or the real projective space, $\chi(M) = 1$, etc.

**Lemma 4.1.** If $M$ is a compact 2-dimensional manifold and if $\overline{\omega} = 0$ on $M$, then there exist constants $c_3, c_4$ (not depending on $u$) such that

$$\int_M e^u \, dM \leq c_3 e^{c_4 ||\nabla u||^2}.$$

**Proof.** See equation (3.5) in [5, p.23].
In the following theorem, we will solve Question A for a compact 2-dimensional manifold.

**Theorem 4.2.** If \( \chi(M) > 0 \), then there exists a nonconstant solution of equation (2.2) for some constant \( C > 0 \). Hence, by conformal deformation, the resulting manifold is an Einstein manifold.

**Proof.** Choose a nonconstant function \( u_1 \) such that \( \int_M u_1 \, dM = 0 \). If \( f(c) = \int_M e^{2c} \, dM \), then Corollary 2.7 implies that there exists \( c \) such that \( \int_M e^{2c} \, dM = \lambda_0 \) for large \( \lambda_0 \). If we put \( B_{\lambda_0} = \{ u \in H^{1,2}(M) \mid u \equiv 0, \int_M e^{2u} \, dM = \lambda_0 \} \), then \( B_{\lambda_0} \) is not empty.

By Lemma 4.1, if \( \lambda_0 \) is large, then \((\int_M |\nabla u|^2 \, dM)^{\frac{1}{2}} > (N_1 + 1)c_1\)
for all \( u \in B_{\lambda_0} \), where \( N_1 = \max_{p \in M} |2k(p)| \) and \( c_1 \) is the coefficient in equation (1.4). Hence equation (1.4) implies that for all \( u \in B_{\lambda_0} \)

\[
J(u) = \frac{\int_M |\nabla u|^2 \, dM + \int_M 2k(p)u \, dM}{\lambda_0} \\
\geq \frac{(N_1 + 1)c_1 \int_M |u| \, dM - N_1 \int_M |u| \, dM}{\lambda_0} \geq 0,
\]

where \( N_1 = \max_{p \in M} |2k(p)| \). Therefore \( \inf_{u \in B_{\lambda_0}} J(u) = C \geq 0 \). Hence there exists \( u_0 \in B_{\lambda_0} \) such that \( u_0 \) is the minimizer of the functional \( J \). Here \( C \) is positive. (If \( C = 0 \), then \( \int_M |u_0| \, dM = 0 \), which means \( u_0 = 0 \). It is impossible because \( \lambda_0 \) is large.) Hence by Theorem 2.11 there exists a solution \( u \) such that

\[
\Delta u - k(p) + Ce^{2u} = 0.
\]

Since \( u \in B_{\lambda_0} \), \( u \) is a nonconstant solution. \( \square \)

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