A NEW BASIS FOR THE HOMFLYPT SKEIN MODULE OF THE SOLID TORUS

IOANNIS DIAMANTIS AND SOFIA LAMBROPOULOU

ABSTRACT. In this paper we give a new basis, $\Lambda$, for the Homflypt skein module of the solid torus, $S(ST)$, which was predicted by Jozef Przytycki using topological interpretation. The basis $\Lambda$ is different from the basis $\Lambda'$, discovered independently by Hoste–Kidwell [HK] and Turaev [Tu] with the use of diagrammatic methods, and also different from the basis of Morton–Aiston [MA]. For finding the basis $\Lambda$ we use the generalized Hecke algebra of type $B$, $H_{1,n}$, defined by the second author in [La2], which is generated by looping elements and braiding elements and which is isomorphic to the affine Hecke algebra of type $A$. Namely, we start with the well-known basis of $S(ST)$, $\Lambda'$, and an appropriate linear basis $\Sigma_n$ of the algebra $H_{1,n}$. We then convert elements in $\Lambda'$ to linear combinations of elements in the new basic set $\Lambda$. This is done in two steps: First we convert elements in $\Lambda'$ to elements in $\Sigma_n$. Then, using conjugation and the stabilization moves, we convert these elements to linear combinations of elements in $\Lambda$ by managing gaps in the indices of the looping elements and by eliminating braiding tails in the words. Further, we define an ordering relation in $\Lambda'$ and $\Lambda$ and prove that the sets are totally ordered. Finally, using this ordering, we relate the sets $\Lambda'$ and $\Lambda$ via a block diagonal matrix, where each block is an infinite lower triangular matrix with invertible elements in the diagonal and we prove linear independence of the set $\Lambda$. The infinite matrix is then “invertible” and thus, the set $\Lambda$ is a basis for $S(ST)$.

$S(ST)$ plays an important role in the study of Homflypt skein modules of arbitrary c.c.o. 3-manifolds, since every c.c.o. 3-manifold can be obtained by integral surgery along a framed link in $S^3$ with unknotted components. The new basis, $\Lambda$, of $S(ST)$ is appropriate for computing the Homflypt skein module of the lens spaces. The aim of this paper is to provide the basic algebraic tools for computing skein modules of c.c.o. 3-manifolds via algebraic means.

0. INTRODUCTION

Let $M$ be an oriented 3-manifold, $R = \mathbb{Z}[u^{\pm 1}, z^{\pm 1}]$, $\mathcal{L}$ the set of all oriented links in $M$ up to ambient isotopy in $M$ and let $S$ the submodule of $RL$ generated by the skein expressions $u^{-1}L_+ - uL_- - zL_0$, where $L_+$, $L_-$ and $L_0$ are oriented links that have identical diagrams, except in one crossing, where they are as depicted in Figure 1.

For convenience we allow the empty knot, $\emptyset$, and add the relation $u^{-1}\emptyset - u\emptyset = zT_1$, where $T_1$ denotes the trivial knot. Then the Homflypt skein module of $M$ is defined to be:

$$S(M) = S(M; \mathbb{Z}[u^{\pm 1}, z^{\pm 1}], u^{-1}L_+ - uL_- - zL_0) = RL/\mathcal{S}.$$
Unlike the Kauffman bracket skein module, the Homflypt skein module of a 3-manifold, also known as Conway skein module and as third skein module, is very hard to compute (see [P-2] for the case of the product of a surface and the interval).

Let ST denote the solid torus. In [Tu], [HK] the Homflypt skein module of the solid torus has been computed using diagrammatic methods by means of the following theorem:

**Theorem 1 (Turaev, Kidwell–Hoste).** The skein module $\mathcal{S}(ST)$ is a free, infinitely generated $\mathbb{Z}[u^\pm 1, z^\pm 1]$-module isomorphic to the symmetric tensor algebra $SR\hat{\pi}^0$, where $\hat{\pi}^0$ denotes the conjugacy classes of non trivial elements of $\pi_1(ST)$.

A basic element of $\mathcal{S}(ST)$ in the context of [Tu, HK], is illustrated in Figure 2. In the diagrammatic setting of [Tu] and [HK], ST is considered as Annulus × Interval. The Homflypt skein module of ST is particularly important, because any closed, connected, oriented (c.c.o.) 3-manifold can be obtained by surgery along a framed link in $S^3$ with unknotted components.

A different basis of $\mathcal{S}(ST)$, known as Young idempotent basis, is based on the work of Morton and Aiston [MA] and Blanchet [B].

In [La2], $\mathcal{S}(ST)$ has been recovered using algebraic means. More precisely, the generalized Hecke algebra of type B, $H_{1,n}(q)$, is introduced, which is isomorphic to the affine Hecke algebra of type A, $\tilde{H}_n(q)$. Then, a unique Markov trace is constructed on the algebras $H_{1,n}(q)$ leading to an invariant for links in ST, the universal analogue of the Homflypt polynomial for ST. This trace gives distinct values on distinct elements of the $[Tu, HK]$-basis of $\mathcal{S}(ST)$. The link isotopy in ST, which is taken into account in the definition of the skein module and which corresponds to conjugation and the stabilization moves on the braid level, is captured by the the conjugation property and the Markov property of the trace, while the defining relation of the skein module is reflected into the quadratic relation of $H_{1,n}(q)$. In the algebraic language of [La2] the basis of $\mathcal{S}(ST)$, described in Theorem 1, is given in open braid form by the set $\Lambda'$ in Eq. 4. Figure 8 illustrates the basic element of Figure 2 in braid notation. Note that in the setting of [La2] ST is considered as the complement of the unknot (the bold curve in the figure). The looping elements $t'_i \in H_{1,n}(q)$ in the monomials of $\Lambda'$ are all conjugates, so they are consistent with the trace property and they enable the definition of the trace via simple inductive rules.
In this paper we give a new basis $\Lambda$ for $S(\text{ST})$ conjectured by the J. H. Przytycki, using the algebraic methods developed in [La2]. The motivation of this work is the computation of $S(\text{L}(p,q))$ via algebraic means. The new basic set is described in Eq. 1 in open braid form. The looping elements $t_i$ are in the algebras $H_{1,n}(q)$ and they are commuting. For a comparative illustration and for the defining formulas of the $t_i$’s and the $t'_i$’s the reader is referred to Figure 7 and Eq. 3 respectively. Moreover, the $t_i$’s are consistent with the handle sliding move or band move used in the link isotopy in $L(p,q)$, in the sense that a braid band move can be described naturally with the use of the $t_i$’s (see for example [DL] and references therein).

Our main result is the following:

**Theorem 2.** The following set is a $\mathbb{Z}[q^\pm 1, z^\pm 1]$-basis for $S(\text{ST})$:

$$(1) \quad \Lambda = \{t^{k_0} t_1^{k_1} \ldots t_n^{k_n}, k_i \in \mathbb{Z} \setminus \{0\} \forall i, n \in \mathbb{N}\}.$$ 

Our method for proving Theorem 2 is the following:

- We define total orderings in the sets $\Lambda'$ and $\Lambda$ and
- we show that the two ordered sets are related via a lower triangular infinite matrix with invertible elements on the diagonal.

More precisely, two analogous sets, $\Sigma_n$ and $\Sigma'_n$, are given in [La2] as linear bases for the algebra $H_{1,n}(q)$. See Theorem 4 in this paper. The set $\bigcup_n \Sigma_n$ includes $\Lambda$ as a proper subset and the set $\bigcup_n \Sigma'_n$ includes $\Lambda'$ as a proper subset. The sets $\Sigma_n$ come directly from the works of S. Ariki and K. Koike, and M. Brouè and G. Malle on the cyclotomic Hecke algebras of type $B$. See [La2] and references therein. The second set $\bigcup_n \Sigma'_n$ includes $\Lambda'$ as a proper subset. The sets $\Sigma_n$ play an intrinsic role in the proof of Theorem 2. Indeed, when trying to convert a monomial $\lambda'$ from $\Lambda'$ into a linear combination of elements in $\Lambda$ we pass by elements of the sets $\Sigma_n$. This means that in the converted expression of $\lambda'$ we have monomials in the $t_i$’s, with possible gaps in the indices followed by monomials in the braiding generators $g_i$. So, in order to reach expressions in the set $\Lambda$ we need:

- to manage the gaps in the indices of the $t_i$’s and
- to eliminate the braiding ‘tails’.

The paper is organized as follows. In Section 1 we recall the algebraic setting and the results needed from [La2]. In Section 2 we define the orderings in the two sets $\Lambda$ and $\Lambda'$ and we prove that the sets are totally ordered. In Section 3 we prove a series of lemmas for converting elements in $\Lambda'$ to elements in the sets $\Sigma_n$. In Section 4 we convert elements in $\Sigma_n$ to elements in $\Lambda$ using conjugation and the stabilization moves. Finally in Section 5 we prove that the sets $\Lambda'$ and $\Lambda$ are related through a lower triangular infinite matrix mentioned above. A computer program...
converting elements in \( \Lambda \) to elements in \( \Sigma \) has been developed by K. Karvounis and will be soon available on http://www.math.ntua.gr/~sofia.

The algebraic techniques developed here will serve as basis for computing Homflypt skein modules of arbitrary c.c.o. 3-manifolds using the braid approach. The advantage of this approach is that we have an already developed homogeneous theory of braid structures and braid equivalences for links in c.c.o. 3-manifolds ([LR1, LR2, DL]). In fact, these algebraic techniques are used and developed further in [KL] for knots and links in 3-manifolds represented by the 2-unlink.

1. The Algebraic Settings

1.1. Mixed Links in \( S^3 \). We now view \( ST \) as the complement of a solid torus in \( S^3 \). An oriented link \( L \) in \( ST \) can be represented by an oriented mixed link in \( S^3 \), that is a link in \( S^3 \) consisting of the unknotted fixed part \( \hat{I} \) representing the complementary solid torus in \( S^3 \) and the moving part \( L \) that links with \( \hat{I} \).

A mixed link diagram is a diagram \( \hat{I} \cup \tilde{L} \) of \( \hat{I} \cup L \) on the plane of \( \hat{I} \), where this plane is equipped with the top-to-bottom direction of \( I \).

Consider now an isotopy of an oriented link \( L \) in \( ST \). As the link moves in \( ST \), its corresponding mixed link will change in \( S^3 \) by a sequence of moves that keep the oriented \( \hat{I} \) pointwise fixed. This sequence of moves consists in isotopy in the \( S^3 \) and the mixed Reidemeister moves. In terms of diagrams we have the following result for isotopy in \( ST \):

The mixed link equivalence in \( S^3 \) includes the classical Reidemeister moves and the mixed Reidemeister moves, which involve the fixed and the standard part of the mixed link, keeping \( \hat{I} \) pointwise fixed.

1.2. Mixed Braids in \( S^3 \). By the Alexander theorem for knots in solid torus, a mixed link diagram \( \hat{I} \cup \tilde{L} \) of \( \hat{I} \cup L \) may be turned into a mixed braid \( I \cup \beta \) with isotopic closure. This is a braid in \( S^3 \) where, without loss of generality, its first strand represents \( \hat{I} \), the fixed part, and the other strands, \( \beta \), represent the moving part \( L \). The subbraid \( \beta \) shall be called the moving part of \( I \cup \beta \).

The sets of braids related to the \( ST \) form groups, which are in fact the Artin braid groups type B, denoted \( B_{1,n} \), with presentation:

\[
B_{1,n} = \left\langle t, \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_1 t \sigma_1 t = t \sigma_1 t \sigma_1, \\
\sigma_i t = \sigma_i t, \quad i > 1, \\
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq n-2, \\
\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| > 1 \right\rangle,
\]

where the generators \( \sigma_i \) and \( t \) are illustrated in Figure 6.
Isotopy in ST is translated on the level of mixed braids by means of the following theorem.

**Theorem 3** (Theorem 3, [La1]). Let $L_1, L_2$ be two oriented links in ST and let $I \cup \beta_1, I \cup \beta_2$ be two corresponding mixed braids in $S^3$. Then $L_1$ is isotopic to $L_2$ in ST if and only if $I \cup \beta_1$ is equivalent to $I \cup \beta_2$ in $\bigcup_{n=1}^{\infty} B_{1,n}$ by the following moves:

1. **Conjugation**: $\alpha \sim \beta^{-1} \alpha \beta$, if $\alpha, \beta \in B_{1,n}$.
2. **Stabilization moves**: $\alpha \sim \alpha \sigma_n^{-1} \in B_{1,n+1}$, if $\alpha \in B_{1,n}$.

1.3. **The Generalized Iwahori-Hecke Algebra of type B.** It is well known that $B_{1,n}$ is the Artin group of the Coxeter group of type B, which is related to the Hecke algebra of type B, $H_n(q, Q)$ and to the cyclotomic Hecke algebras of type B. In [La2] it has been established that all these algebras form a tower of B-type algebras and are related to the knot theory of ST. The basic one is $H_n(q, Q)$, a presentation of which is obtained from the presentation of the Artin group $B_{1,n}$ by adding the quadratic relations

$$g_i^2 = (q - 1)g_i + q$$

and the relation $t^2 = (Q - 1)t + Q$, where $q, Q \in \mathbb{C}\setminus\{0\}$ are seen as fixed variables. The middle B–type algebras are the cyclotomic Hecke algebras of type B, $H_n(q, d)$, whose presentations are obtained by the quadratic relation (2) and $t^d = (t - u_1)(t - u_2) \ldots (t - u_d)$. The topmost Hecke-like algebra in the tower is the **generalized Iwahori–Hecke algebra of type B**, $H_{1,n}(q)$, which, as observed by T.tom Dieck, is isomorphic to the affine Hecke algebra of type A, $\widetilde{H}_n(q)$ (cf. [La2]). The algebra $H_{1,n}(q)$ has the following presentation:
\[ H_{1,n}(q) = \left\{ t, g_1, \ldots, g_{n-1} \mid g_1tg_1 = tg_1t, \quad i > 1 \\
g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad 1 \leq i \leq n - 2 \\
g_i g_j = g_j g_i, \quad |i - j| > 1 \\
g_i^2 = (q - 1) g_i + q, \quad i = 1, \ldots, n - 1 \right\}. \]

That is:

\[ H_{1,n}(q) = \frac{\mathbb{Z}[q^{\pm 1}] B_{1,n}}{\langle \sigma_i^2 - (q - 1) \sigma_i - q \rangle}. \]

Note that in \( H_{1,n}(q) \) the generator \( t \) satisfies no polynomial relation, making the algebra \( H_{1,n}(q) \) infinite dimensional. Also that in [La2] the algebra \( H_{1,n}(q) \) is denoted as \( H_n(q, \infty) \).

In [Jo] V.F.R. Jones gives the following linear basis for the Iwahori-Hecke algebra of type A, \( H_n(q) \):

\[ S = \{(g_{i_1} g_{i_1 - 1} \cdots g_{i_1 - k_1})(g_{i_2} g_{i_2 - 1} \cdots g_{i_2 - k_2}) \cdots (g_{i_p} g_{i_p - 1} \cdots g_{i_p - k_p})\}, \text{ for } 1 \leq i_1 < \ldots < i_p \leq n - 1. \]

The basis \( S \) yields directly an inductive basis for \( H_n(q) \), which is used in the construction of the Ocneanu trace, leading to the Homflypt or 2-variable Jones polynomial.

In \( H_{1,n}(q) \) we define the elements:

\[ t_i := g_i g_{i-1} \cdots g_1 t g_1 \cdots g_{i-1} g_i \] and \( t_i' := g_i g_{i-1} \cdots g_1 t g_1^{-1} \cdots g_{i-1} g_i^{-1} \), as illustrated in Figure 7.

In [La2] the following result has been proved.

**Theorem 4** (Proposition 1, Theorem 1 [La2]). The following sets form linear bases for \( H_{1,n}(q) \):

(i) \[ \Sigma_n = t_{i_1}^{k_1} t_{i_2}^{k_2} \cdots t_{i_r}^{k_r} \cdot \sigma, \text{ where } 1 \leq i_1 < \ldots < i_r \leq n - 1, \]

(ii) \[ \Sigma_n' = t_{i_1}^{k_1} t_{i_2}^{k_2} \cdots t_{i_r}^{k_r} \cdot \sigma, \text{ where } 1 \leq i_1 < \ldots < i_r \leq n, \]

where \( k_1, \ldots, k_r \in \mathbb{Z} \) and \( \sigma \) a basic element in \( H_n(q) \).

**Remark 1.**

(i) The indices of the \( t_i' \)'s in the set \( \Sigma_n' \) are ordered but are not necessarily consecutive, neither do they need to start from \( t \).

(ii) A more straightforward proof that the sets \( \Sigma_n' \) form bases for \( H_{1,n}(q) \) can be found in [D].

In [La2] the basis \( \Sigma_n' \) is used for constructing a Markov trace on \( \bigcup_{n=1}^{\infty} H_{1,n}(q) \).
Theorem 5 (Theorem 6, [La2]). Given $z, s_k$, with $k \in \mathbb{Z}$ specified elements in $R = \mathbb{Z}[q^{\pm 1}]$, there exists a unique linear Markov trace function
\[ \text{tr} : \bigcup_{n=1}^{\infty} H_{1,n}(q) \to R(z, s_k), k \in \mathbb{Z} \]
determined by the rules:
\begin{align*}
(1) \quad & \text{tr}(ab) = \text{tr}(ba) \quad \text{for} \ a, b \in H_{1,n}(q) \\
(2) \quad & \text{tr}(1) = 1 \quad \text{for all} \ H_{1,n}(q) \\
(3) \quad & \text{tr}(ag_n) = z \text{tr}(a) \quad \text{for} \ a \in H_{1,n}(q) \\
(4) \quad & \text{tr}(at^k_n) = s_k \text{tr}(a) \quad \text{for} \ a \in H_{1,n}(q), \ k \in \mathbb{Z}.
\end{align*}

Note that the use of the looping elements $t'_i$ enable the trace $\text{tr}$ to be defined by just extending the three rules of the Ocneanu trace on the algebras $H_n(q)$ [Jo] by rule (4). Using $\text{tr}$ Lambropoulou constructed a universal Homflypt-type invariant for oriented links in ST. Namely, let $\mathcal{L}$ denote the set of oriented links in ST. Then:

Theorem 6 (Definition 1, [La2]). The function $X : \mathcal{L} \to R(z, s_k)$
\[ X_\alpha = \left( \frac{1 - \lambda q}{\sqrt{\lambda} (1 - q)} \right)^{n-1} \left( \sqrt{\lambda} \right)^e \text{tr} (\pi (\alpha)), \]
where $\alpha \in B_{1,n}$ is a word in the $\sigma_i$'s and $t'_i$'s, $e$ is the exponent sum of the $\sigma_i$'s in $\alpha$, and $\pi$ the canonical map of $B_{1,n}$ in $H_{1,n}(q)$, such that $t \mapsto t$ and $\sigma_i \mapsto g_i$, is an invariant of oriented links in ST.

The invariant $X$ satisfies a skein relation [La1]. Theorems 4, 5 and 6 hold also for the algebras $H_n(q, Q)$ and $H_n(q, d)$, giving rise to all possible Homflypt-type invariants for knots in ST. For the case of the Hecke algebra of type $B$, $H_n(q, Q)$, see also [La1] and [LG].

1.4. The basis of $S(\text{ST})$ in algebraic terms. Let us now see how $S(\text{ST})$ is described in the above algebraic language. We note first that an element $\alpha$ in the basis of $S(\text{ST})$ described in Theorem 1 when ST is considered as Annulus $\times$ Interval, can be illustrated equivalently as a mixed link in $S^3$ when ST is viewed as the complement of a solid torus in $S^3$. So we correspond the element $\alpha$ to the minimal mixed braid representation, which has increasing order of twists around the fixed strand. Figure 8 illustrates an example of this correspondence. Denoting
\[ \Lambda' = \{ t_1^{k_1} t_2^{k_2} \ldots t_n^{k_n}, \ k_i \in \mathbb{Z} \setminus \{0\} \ \forall i, \ n \in \mathbb{N} \}, \]
we have that $\Lambda'$ is a subset of $\bigcup_n H_{1,n}$. In particular $\Lambda'$ is a subset of $\bigcup_n \Sigma'_n$. 

![Figure 8. An element in $\Lambda'$.](image)
Applying the inductive trace rules to a word $w$ in $\bigcup_n \Sigma_n'$ will eventually give rise to linear combinations of monomials in $R(z, s_k)$. In particular, for an element of $\Lambda'$ we have:

$$\text{tr}(t_0 t_1' \ldots t_{n-1}' k_{n-1}) = s_{k_{n-1}} \cdots s_{k_1} s_{k_0}.$$ 

Further, the elements of $\Lambda'$ are in bijective correspondence with increasing $n$-tuples of integers, $(k_0, k_1, \ldots, k_{n-1})$, $n \in \mathbb{N}$, and these are in bijective correspondence with monomials in $s_{k_0}, s_{k_1}, \ldots, s_{k_{n-1}}$.

**Remark 2.** The invariant $X$ recovers the Homflypt skein module of $\text{ST}$ since it gives different values for different elements of $\Lambda'$ by rule 4 of the trace.

## 2. An ordering in the sets $\Lambda$ and $\Lambda'$

In this section we define an ordering relation in the sets $\Lambda$ and $\Lambda'$. Before that, we will need the notion of the index of a word in $\Lambda'$ or in $\Lambda$.

**Definition 1.** The *index* of a word $w$ in $\Lambda'$ or in $\Lambda$, denoted $\text{ind}(w)$, is defined to be the highest index of the $t_i'$'s, resp. of the $t_i$'s, in $w$. Similarly, the *index* of an element in $\Sigma_n'$ or in $\Sigma_n$ is defined in the same way by ignoring possible gaps in the indices of the looping generators and by ignoring the braiding part in $H_n(q)$. Moreover, the index of a monomial in $H_n(q)$ is equal to 0.

For example, $\text{ind}(t_{i_0} t_{i_1}' t_{i_2}' \cdots t_{i_n}' k_n) = \text{ind}(t_{j_0} \cdots t_{j_n}) = n$.

**Definition 2.** We define the following *ordering* in the set $\Lambda'$. Let $w = t_{i_1}' t_{i_2}' \ldots t_{i_{\mu}}' k_{\mu}$ and $\sigma = t_{j_1}' t_{j_2}' \ldots t_{j_{\nu}}'$, where $k_i, \lambda_s \in \mathbb{Z}$, for all $t, s$. Then:

(a) If $\sum_{i=0}^{\mu} k_i < \sum_{i=0}^{\nu} \lambda_i$, then $w < \sigma$.

(b) If $\sum_{i=0}^{\mu} k_i = \sum_{i=0}^{\nu} \lambda_i$, then:

(i) if $\text{ind}(w) < \text{ind}(\sigma)$, then $w < \sigma$,

(ii) if $\text{ind}(w) = \text{ind}(\sigma)$, then:

(\(\alpha\)) if $i_1 = j_1, i_2 = j_2, \ldots, i_{s-1} = j_{s-1}, i_s < j_s$, then $w > \sigma$,

(\(\beta\)) if $i_t = j_t \ \forall t$ and $k_\mu = \lambda_\mu, k_{\mu-1} = \lambda_{\mu-1}, \ldots, k_{i_{t+1}} = \lambda_{i_{t+1}}, |k_i| < |\lambda_i|$, then $w < \sigma$,

(\(\gamma\)) if $i_t = j_t \ \forall t$ and $k_\mu = \lambda_\mu, k_{\mu-1} = \lambda_{\mu-1}, \ldots, k_{i_{t+1}} = \lambda_{i_{t+1}}, |k_i| = |\lambda_i|$ and $k_t > \lambda_t$, then $w < \sigma$,

(\(\delta\)) if $i_t = j_t \ \forall t$ and $k_i = \lambda_i, \forall i$, then $w = \sigma$. 

*Figure 9.* An element of $\Lambda$. 

\[tt^2t_3' = \]
(c) In the general case where \( w = t_{i_1}^{\sigma_1} t_{i_2}^{\sigma_2} \ldots t_{i_n}^{\sigma_n} \cdot \sigma_1 \) and \( \sigma = t_{j_1}^{\lambda_1} t_{j_2}^{\lambda_2} \ldots t_{j_p}^{\lambda_p} \cdot \sigma_2 \), where \( \sigma_1, \sigma_2 \in H_n(q) \), the ordering is defined in the same way by ignoring the braiding parts \( \sigma_1, \sigma_2 \).

The same ordering is defined on the set \( \Lambda \), where the \( t_i \)'s are replaced by the corresponding \( t_i' \)'s. Moreover, the same ordering is defined on the sets \( \Sigma_n \) and \( \Sigma_n' \) by ignoring the braiding parts.

**Proposition 1.** The set \( \Lambda' \) equipped with the ordering given in Definition 2, is totally ordered set.

*Proof.* In order to show that the set \( \Lambda' \) is totally ordered set when equipped with the ordering given in Definition 2, we need to show that the ordering relation is antisymmetric, transitive and total. We only show that the ordering relation is transitive. Antisymmetric property follows similarly. Totality follows from Definition 2 since all possible cases have been considered.

Let \( w = t_{i_1}^{\sigma_1} t_{i_2}^{\sigma_2} \ldots t_{i_n}^{\sigma_n} \), \( \sigma = t_{j_1}^{\lambda_1} t_{j_2}^{\lambda_2} \ldots t_{j_p}^{\lambda_p} \) and \( w < \sigma \) and \( \sigma < v \).

Since \( w < \sigma \), one of the following holds:

(a) Either \( \sum_{i=1}^{n} k_i < \sum_{i=1}^{n} \lambda_i \) and since \( \lambda < v \), we have that \( \sum_{i=1}^{n} \lambda_i \leq \sum_{i=1}^{p} \mu_i \) and so

\[
\sum_{i=1}^{n} k_i < \sum_{i=1}^{p} \mu_i. \quad \text{Thus } w < v.
\]

(b) Either \( \sum_{i=1}^{n} k_i = \sum_{i=1}^{n} \lambda_i \) and \( \text{ind}(w) = m < n = \text{ind}(\sigma) \). Then, since \( \lambda < v \) we have that either \( \sum_{i=1}^{n} \lambda_i < \sum_{i=1}^{p} \mu_i \) (same as in case (a)) or \( \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{p} \mu_i \) and

\[
\text{ind}(\sigma) \leq p = \text{ind}(v). \quad \text{Thus, } \text{ind}(w) = m < p = \text{ind}(v) \text{ and so we conclude that } w < v.
\]

(c) Either \( \sum_{i=1}^{n} k_i = \sum_{i=1}^{n} \lambda_i \), \( \text{ind}(w) = \text{ind}(\sigma) \) and \( i_1 = j_1, \ldots, i_{s-1} = j_{s-1}, i_s > j_s \). Then, since \( \lambda < v \), we have that either:

- \( \sum_{i=1}^{n} \lambda_i < \sum_{i=1}^{p} \mu_i \), same as in case (a), or
- \( \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{p} \mu_i \) and \( \text{ind}(\sigma) < \text{ind}(v) \), same as in case (b), or
- \( \text{ind}(\sigma) = \text{ind}(v) \) and \( j_1 = \varphi_1, \ldots, j_p > \varphi_p \). Then:

(i) if \( p = s \) we have that \( i_s > j_s > \varphi_s \) and we conclude that \( w < v \).

(ii) if \( p < s \) we have that \( i_p = j_p > \varphi_p \) and thus \( w < v \) and if \( s < p \) we have that

\[
i_s > j_s = \varphi_s \text{ and so } w < v.
\]
(d) Either \( \sum_{i=1}^{m} k_i = \sum_{i=1}^{n} \lambda_i \), \( \text{ind}(w) = \text{ind}(\sigma) \), \( i_n = j_n \ \forall n \) and \( k_n = \lambda_n, \ldots, |k_q| < |\lambda_q| \).

Then, since \( \sigma < v \), we have that either:

- \( \sum_{i=1}^{n} \lambda_i < \sum_{i=1}^{p} \mu_i \), same as in case (a), or
- \( \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{p} \mu_i \) and \( \text{ind}(\sigma) < \text{ind}(v) \), same as in case (b), or
- \( \text{ind}(\sigma) = \text{ind}(v) \) and \( j_1 = \varphi_1, \ldots, j_q > \varphi_q \), same as in case (c), or
- \( j_n = \varphi_n \), for all \( n \) and \( |\mu_p| \geq |\lambda_p| \) for some \( p \).

(1) If \( |\mu_p| > |\lambda_p| \), then:

- (i) If \( p \geq q \) then \( |k_p| = |\lambda_p| < |\mu_p| \) and thus \( w < v \).
- (ii) If \( p < q \) then \( |k_q| < |\lambda_q| = |\mu_q| \) and thus \( w < v \).

(2) If \( |\mu_p| = |\lambda_p| \), then:

- (i) If \( p \geq q \) then \( |k_p| = |\lambda_p| = |\mu_p| \) and \( k_p = \lambda_p > \mu_p \). Thus \( w < v \).
- (ii) If \( p < q \) then \( |k_q| < |\lambda_q| = |\mu_q| \) and thus \( w < v \).

So, we conclude that the ordering relation is transitive.

\[ \square \]

**Definition 3.** We define the subset of level \( k \), \( \Lambda_k \), of \( \Lambda \) to be the set

\[ \Lambda_k := \{ t^{k_{0}t_{1}^{k_{1}} \cdots t_{m}^{k_{m}}} | \sum_{i=0}^{m} k_i = k \} \]

and similarly, the subset of level \( k \) of \( \Lambda' \) to be

\[ \Lambda'_k := \{ t^{k_{0}t_{1}'^{k_{1}} \cdots t_{m}'^{k_{m}}} | \sum_{i=0}^{m} k_i = k \} \].

**Remark 3.** Let \( w \in \Lambda_k \) a monomial containing gaps in the indices and \( u \in \Lambda_k \) a monomial with consecutive indices such that \( \text{ind}(w) = \text{ind}(u) \). Then, it follows from Definition 2 that \( w < u \).

**Proposition 2.** The sets \( \Lambda_k \) are totally ordered and well-ordered for all \( k \).

*Proof.* Since \( \Lambda_k \subseteq \Lambda, \ \forall k \), \( \Lambda_k \) inherits the property of being a totally ordered set from \( \Lambda \). Moreover, \( t^k \) is the minimum element of \( \Lambda_k \) and so \( \Lambda_k \) is a well-ordered set. \( \square \)

We also introduce the notion of homologous words as follows:

**Definition 4.** We shall say that two words \( w' \in \Lambda' \) and \( w \in \Lambda \) are homologous, denoted \( w' \sim w \), if \( w \) is obtained from \( w' \) by turning \( t_{i}' \) into \( t_{i} \) for all \( i \).
With the above notion the proof of Theorem 2 is based on the following idea: Every element $w' \in \Lambda'$ can be expressed as linear combinations of monomials $w_i \in \Lambda$ with coefficients in $\mathbb{C}$, such that:

(i) $\exists j$ such that $w_j := w \sim w'$,
(ii) $w_j < w_i$, for all $i \neq j$,
(iii) the coefficient of $w_j$ is an invertible element in $\mathbb{C}$.

3. From $\Lambda'$ to $\Sigma_n$

In this section we prove a series of lemmas relating elements of the two different basic sets $\Sigma_n$, $\Sigma'_n$ of $H_{1,n}(q)$. In the proofs we underline expressions which are crucial for the next step. Since $\Lambda'$ is a subset of $\Sigma'_n$, all lemmas proved here apply also to $\Lambda'$ and will be used in the context of the bases of $S(ST)$.

3.1. Some useful lemmas in $H_{1,n}(q)$. We will need the following results from [La2]. The first lemma gives some basic relations of the braiding generators.

**Lemma 1** (Lemma 1 [La2]). For $\epsilon \in \{\pm 1\}$ the following hold in $H_{1,n}(q)$:

(i) $g_i^m = (q^{m-1} - q^{m-2} + \ldots + (-1)^{m-1}) g_i + (q^{m-1} - q^{m-2} + \ldots + (-1)^{m-2}q)$

$$g_i^{-m} = (q^{-m} - q^{-1-m} + \ldots + (-1)^{m-1}q^{-1}) g_i + (q^{-m} - q^{-1-m} + \ldots + (-1)^{m-1}q^{-1} + (-1)^m)$$

(ii) $g_i^\epsilon (g_{k-1}^{\pm1} g_{k-1}^{\pm1} \ldots g_j^{\pm1}) = (g_{k-1}^{\pm1} g_{k-1}^{\pm1} \ldots g_j^{\pm1}) g_{i-1}^\epsilon$, for $k > i \geq j$,

$$g_i^\epsilon (g_{j+1}^{\pm1} g_{j+1}^{\pm1} \ldots g_k^{\pm1}) = (g_{j+1}^{\pm1} g_{j+1}^{\pm1} \ldots g_k^{\pm1}) g_{i-1}^\epsilon$$, for $k \geq i > j$,

where the sign of the $\pm 1$ exponent is the same for all generators.

(iii) $g_i g_{i-1} \ldots g_{j+1} g_j g_{j+1} \ldots g_i = g_j g_{j+1} \ldots g_{i-1} g_i g_{i-1} \ldots g_{j+1} g_j$

$$g_i^{-1} g_{i-1}^{-1} \ldots g_{j+1}^{-1} g_{j+1} \ldots g_i = g_j g_{j+1} \ldots g_{i-1} g_i^{-1} \ldots g_{j+1}^{-1} g_j^{-1}$$

(iv) $g_1^\epsilon \ldots g_{n-1}^\epsilon g_n 2^\epsilon \ldots g_i^\epsilon = \sum_{r=0}^{n-i+1} (q^\epsilon - 1)^r q^{r+\epsilon} (g_1^\epsilon \ldots g_{n-r}^\epsilon \ldots g_i^\epsilon)$,

where $\epsilon_r = 1$ if $r \leq n - i$ and $\epsilon_{n-i+1} = 0$. Similarly,

(v) $g_1^\epsilon \ldots g_2^\epsilon g_1^2 \epsilon g_2^\epsilon \ldots g_i^\epsilon = \sum_{r=0}^{i} (q^\epsilon - 1)^r q^{r+\epsilon} (g_1^\epsilon \ldots g_{r+2}^\epsilon g_{r+1}^\epsilon g_{r+2}^\epsilon \ldots g_i^\epsilon)$,

where $\epsilon_r = 1$ if $r \leq i - 1$ and $\epsilon_i = 0$.

The next lemma comprises relations between the braiding generators and the looping generator $t$. 
Lemma 2 (cf. Lemmas 1, 4, 5 [La2]). For $\epsilon \in \{\pm 1\}$, $i, k \in \mathbb{N}$ and $\lambda \in \mathbb{Z}$ the following hold in $H_{1,n}(q)$:

(i) \[ t^\lambda g_1 t g_1 = g_1 t g_1 t^\lambda \]

(ii) \[ t^\epsilon g_1 t^k g_1 t^\epsilon = g_1 t^k g_1 t^\epsilon + (q^\epsilon - 1)t^\epsilon g_1 t^k g_1 t^\epsilon + (1 - q^\epsilon) t^k g_1 t^\epsilon t^k g_1 t^\epsilon \]

\[ t^{-\epsilon} g_1 t^k g_1 t^\epsilon = g_1 t^k g_1 t^{-\epsilon} + (q^\epsilon - 1)t^\epsilon g_1 t^k g_1 t^{-\epsilon} + (1 - q^\epsilon) g_1 t^\epsilon t^k g_1 t^\epsilon (k-1) \]

(iii) \[ t_i^\epsilon g_1 t^k g_1 t^\epsilon = g_1 t^k g_1 t^{-\epsilon} + (q^\epsilon - 1) \sum_{j=1}^i t^\epsilon g_1 t^i g_1 t^{(k+i-j)} + (1 - q^\epsilon) \sum_{j=1}^{i-1} t^\epsilon g_1 t^{(k+j)} g_1 t^i t^\epsilon (i-j) \]

\[ t^{-\epsilon} i g_1 t^k g_1 t^\epsilon = g_1 t^k g_1 t^{-\epsilon} + (q^\epsilon - 1) \sum_{j=1}^i t^\epsilon g_1 t^{(k-j)} g_1 t^{-\epsilon} (i-j) + (1 - q^\epsilon) \sum_{j=1}^i t^{-\epsilon} (i-j) g_1 t^{(k-j)} \]

The next lemma gives the interactions of the braiding generators and the loopings $t_i$'s and $t_i$'s.

Lemma 3 (Lemmas 1 and 2 [La2]). The following relations hold in $H_{1,n}(q)$:

(i) \[ g_i t^k \epsilon = t^k_i \epsilon g_i \text{ for } k > i, k < i - 1 \]

\[ g_i t_i = q^{-1} t_i g_i + (q - 1) t_i \]

\[ g_i t_{i-1} - 1 = q^{-1} t_i g_i + (q - 1) t_{i-1} \]

\[ g_i t_{i-1} = q^{-1} t_{i-1} g_i + (q - 1) t_{i-1} - 1 \]

\[ g_i t_{i-1} = q^{-1} t_{i-1} - 1 g_i + (q - 1) t_{i-1} - 1 = t_{i-1} - 1 g_i - 1 \]

(ii) \[ t^k_n g_n = (q - 1) \sum_{j=0}^{k-1} q^j t^j_{n-1} t^{k-j}_n + q^k g_n t^k_{n-1}, \text{ if } k \in \mathbb{N} \]

\[ t^k_n g_n = (1 - q) \sum_{j=0}^{k-1} q^j t^j_{n-1} t^{k-j}_n + q^k g_n t^k_{n-1}, \text{ if } k \in \mathbb{Z} - \mathbb{N} \]

(iii) \[ t_i^k t_j^\lambda = t_j^\lambda t_i^k \text{ for } i \neq j \text{ and } k, \lambda \in \mathbb{Z} \]

(iv) \[ g_i t^\epsilon_i k = t^\epsilon_i k g_i \text{ for } k > i, k < i - 1 \]

\[ g_i t_i^\epsilon = t_{i-1}^\epsilon g_i + (q - 1) t_i^\epsilon + (1 - q) t_i^\epsilon \]

\[ g_i t_{i-1}^\epsilon = t_i^\epsilon g_i \]

(v) \[ t^k_i = g_i \ldots g_i t^k g^{-1}_i \ldots g_i^{-1} \text{ for } k \in \mathbb{Z}. \]

Using now Lemmas 1, 2 and 3 we prove the following relations, which we will use for converting elements in $\Lambda'$ to elements in $\Sigma_n$. Note that whenever a generator is overlined, this means that the specific generator is omitted from the word.

Lemma 4. The following relations hold in $H_{1,n}(q)$ for $k \in \mathbb{N}$:

(i) \[ g_m + 1 t^k_{m+1} = q^{-(k-1)} t^k_{m+1} g_{m+1}^{-1} + \sum_{j=1}^{k-1} q^{-(k-1-j)} (q^{-1} - 1) t^j m t^j_{m+1} \]

(ii) \[ g_{m-1} t^{-k}_{m+1} = q^{(k-1)} t^{-k}_{m+1} g_{m+1}^{-1} + \sum_{j=1}^{k-1} q^{(k-1-j)} (q - 1) t^{-j}_{m+1} \]

\[ t^j m t^j_{m+1} \]
Proof. We prove relations (i) by induction on \( k \). Relations (ii) follow similarly. For \( k = 1 \) we have that \( g_{m+1} t_m^k = t_{m+1} g_m^{-1} \), which holds from Lemma 3 (i). Suppose that the relation holds for \( k - 1 \). Then, for \( k \) we have:

\[
g_{m+1} t_m^k = g_{m+1} t_m^{k-1} t_m \overset{\text{ind.}}{=} q^{-(k-2)} t_m^{k-1} g_{m+1}^{-1} t_m + \sum_{j=1}^{k-2} q^{-(k-2-j)} (q^{1} - 1) t_m^j t_m^{k-1-j} = \]

\[
= q^{-(k-1)} g_{m+1} t_m + q^{-(k-2)} (q^{1} - 1) t_m^{k-1} + \sum_{j=1}^{k-2} q^{-(k-2-j)} (q^{1} - 1) t_m^j t_m^{k-1-j} = \]

\[
= q^{-(k-1)} t_m^{k-1} g_{m+1}^{-1} + \sum_{j=1}^{k} q^{-(k-1-j)} (q^{1} - 1) t_m^j t_m^{k-j}. \]

\( \square \)

Lemma 5. In \( H_{1,n}(q) \) the following relations hold:

(i) For the expression \( A = (g_r g_{r-1} \ldots g_{r-s}) \cdot t_k \) the following hold for the different values of \( k \in \mathbb{N} \):

1. \( A = t_k (g_r \ldots g_{r-s}) \) for \( k > r \) or \( k < r - s - 1 \)

2. \( A = t_r (g_r^{-1} \ldots g_{r-s}^{-1}) \) for \( k = r - s - 1 \)

3. \( A = qt_{r-1} (g_r \ldots g_{r-s}) + (q - 1) t_r (g_{r-1} \ldots g_{r-s}) \) for \( k = r \)

4. \( A = qt_{r-s-1} (g_r \ldots g_{r-s}) + (q - 1) t_r (g_r^{-1} \ldots g_{r-s+1}) \) for \( k = r - s \)

5. \( A = t_{m-1} (g_r \ldots g_{r-s}) + (q - 1) t_r (g_r^{-1} \ldots g_{m+s+1}) (g_{m-1} \ldots g_{r-s}) \) for \( k = m \in \{r - s + 1, \ldots, r - 1\} \).

(ii) For the expression \( A = (g_r g_{r-1} \ldots g_{r-s}) \cdot t_k^{-1} \) the following hold for the different values of \( k \in \mathbb{N} \):

1. \( A = t_k^{-1} (g_r \ldots g_{r-s}) \) for \( k > r \) or \( k < r - s - 1 \)

2. \( A = t_{r-s-1}^{-1} (g_r \ldots g_{r-s+1} g_{r-s}^{-1}) \) for \( k = r - s \)

3. \( A = t_{m-1}^{-1} (g_r g_{r-1} \ldots g_{m+s+1} g_{m-s} g_{m-1} \ldots g_{r-s}) \) for \( k = m \in \{r - s + 1, \ldots, r\} \)

4. \( A = q^{s+1} t_{r-1}^{-1} (g_r \ldots g_{r-s}) + (q - 1) \sum_{j=1}^{s+1} q^{s-j+1} t_{r-1}^{-1} (g_r \ldots g_{r-j+2} g_{r-j} \ldots g_{r-s}) \) for \( k = r - s - 1 \).

Proof. We only prove relations (ii) for \( k = r - s - 1 \) by induction on \( s \) (case 4). All other relations follow from Lemma 3 (i).

For \( s = 1 \) we have:

\[
gr_r g_{r-1} t_{r-2}^{-1} = gr_r [qt_{r-1}^{-1} g_{r-1} + (q - 1) t_{r-2}^{-1}] = q g_r t_{r-1}^{-1} g_{r-1} + (q - 1) g_r t_{r-2}^{-1} \]

\[
= qt_{r-1}^{-1} g_r + (q - 1) t_{r-1}^{-1} g_{r-1} + (q - 1) t_{r-2}^{-1} g_r \]

\[
= q^2 t_{r-1}^{-1} (g_r g_{r-1}) + (q - 1) [qt_{r-1}^{-1} g_{r-1} + q^0 t_{r-2}^{-1}]. \]
and so the relation holds for \( s = 1 \). Suppose that the relation holds for \( s = n \). We will show that it holds for \( s = n + 1 \). Indeed we have:

\[
(g_r \cdots g_r^{-1}) t_{r-2}^{-1} = (g_r \cdots g_r^{-1})(g_r^{-1}t_{r-2}^{-1}) = (g_r \cdots g_r^{-1})[qt_{r-1}^{-1}g_r^{-1} + (q-1)t_{r-2}^{-1}] = \\
q(g_r \cdots g_r^{-1} t_{r-1}) g_r^{-1} + (q-1)(g_r \cdots g_r^{-1})t_{r-2}^{-1} ind. step = q^{n+1} t_{r-1}^{-1}(g_r \cdots g_r^{-1}) + \\
+ (q-1) \sum_{j=1}^{n+1} q^{n-j+1} (g_r \cdots g_r^{-j+2} g_r^{-j} \cdots g_r^{-1}) + (q-1)t_{r-2}^{-1}(g_r \cdots g_r^{-1}) = \\
q^{n+1} t_{r-1}^{-1}(g_r \cdots g_r^{-1}) + (q-1) \sum_{j=1}^{n+2} q^{n+1-j+1} t_{r-j}^{-1}(g_r \cdots g_r^{-j+2} g_r^{-j} \cdots g_r^{-1}).
\]

Before proceeding with the next lemma we introduce the notion of length of \( w \in H_n(q) \). For convenience we set \( \delta_{k,r} := g_kg_{k-1} \cdots g_r \cdot g_r \) for \( k > r \) and by convention we set \( \delta_{k,k} := g_k \).

**Definition 5.** We define the length of \( \delta_{k,r} \in H_n(q) \) as \( l(\delta_{k,r}) := k - r + 1 \) and since every element of the Iwahori-Hecke algebra of type \( A \) can be written as \( \prod_{i=1}^{n-1} \delta_{k_i,r_i} \), so that \( k_j < k_{j+1} \forall j \), we define the length of an element \( w \in H_n(q) \) as

\[
l(w) := \sum_{i=1}^{n-1} l_i(\delta_{k_i,r_i}) = \sum_{i=1}^{n-1} k_i - r_i + 1.
\]

Note that \( l(g_k) = l(\delta_{k,k}) = k - k + 1 = 1 \).

**Lemma 6.** For \( k > r \) the following relations hold in \( H_{1,n}(q) \):

\[
t_k \delta_{k,r} = \sum_{i=0}^{k-r} q^i(q-1) \delta_{k,k-i,r} t_{k-i} + q^i(\delta_{k,r}) \delta_{k,r} t_{r-1},
\]

where \( \delta_{k,k-i,r} := g_kg_{k-1} \cdots g_{k-i+1}g_{k-i} \cdots g_r := g_k \cdots g_{k-i} \cdots g_r \).

**Proof.** We prove relations by induction on \( k \). For \( k = 1 \) we have that \( t_1g_1 = (q-1)t_1 + qg_1t \), which holds. Suppose that the relation holds for \( (k-1) \), then for \( k \) we have:

\[
t_k \delta_{k,r} = t_k g_k \cdot \delta_{k-1,r} = (q-1)t_k \delta_{k-1,r} + qg_k t_{k-1} \delta_{k-1,r} = \\
= (q-1) \delta_{k-1,r} t_k + qg_k \sum_{i=0}^{k-r} q^i(q-1) \delta_{k-1,k-1-i,r} t_{k-1-i} + q^i(q-1) \delta_{k-1,r} t_{r-1} = \\
= \sum_{i=0}^{k-r} q^i(q-1) \delta_{k,k-1-i,r} t_{k-1-i} + q^i(\delta_{k,r}) \delta_{k,r} t_{r-1}.
\]

**Lemma 7.** In \( H_{1,n}(q) \) the following relations hold:

(i) For the expression \( A = (g_r g_{r+1} \cdots g_{r+s}) \cdot t_k \) the following hold for the different values of \( k \in \mathbb{N} \):

1. \( A = t_k (g_r \cdots g_{r+s}) \) for \( k \geq r + s + 1 \) or \( k < r - 1 \)
2. \( A = t_{k+1} (g_r \cdots g_k g_{k+1}^{-1} g_{k+2} \cdots g_{r+s}) \) for \( r - 1 \leq k < r + s \)
3. \( A = (q-1) \sum_{i=r}^{r+s} q^{r+s-i} t_i (g_r \cdots g_i \cdots g_{r+s}) + q^{s+1} t_{r-1} (g_r \cdots g_{r+s}) \) for \( k = r + s \)
(ii) For the expression

$$A = (g_r g_{r+1} \ldots g_{r+s}) \cdot t^{-1}_k$$

the following hold for the different values of $k \in \mathbb{N}$:

1. $A = t^{-1}_k (g_r g_{r+1} \ldots g_{r+s})$ for $k \geq r + s + 1$ or $k < r - 1$

2. $A = q t^{-1}_{k+1} (g_r \ldots g_{r+s}) + (q - 1) t^{-1}_{r-1} (g_{r-1} \ldots g_{k+2} \ldots g_{r+s})$ for $r - 1 \leq k < r + s$

3. $A = t^{-1}_{r-1} (g_{r-1} \ldots g_{r+s})$ for $k = r + s$

**Proof.** We prove relation (i) for $r + s = k$ by induction on $k$ (case 3). All other relations follow from Lemmas 1 and 3.

For $k = 1$ we have: $g_1 t_1 = g^2 t g_1 = q t g_1 + (q - 1) t_1$. Suppose that the relation holds for $k = n$. Then, for $k = n + 1$ we have:

$$g_r \ldots g_{n+1} t_{n+1} = q (g_r \ldots g_n t_n) g_{n+1} + (q - 1) (g_r \ldots g_n) t_{n+1} \overset{ind.step}{=}$$

$$= q [(q - 1) \sum_{i=r}^{n} q^{i-r} t_i (g_r \ldots g_i \ldots g_{n+1}) + q^{n-r+1} t_{r-1} (g_r \ldots g_n)] g_{n+1} + (q - 1) t_{n+1} (g_r \ldots g_n) =$$

$$= (q - 1) \sum_{i=r}^{n} q^{n+1-i} t_i (g_r \ldots g_i \ldots g_{n+1}) + (q - 1) t_{n+1} (g_r \ldots g_n) + q^{n+1-r+1} t_{r-1} (g_r \ldots g_n g_{n+1}) =$$

$$= (q - 1) \sum_{i=r}^{n+1} q^{n+1-i} t_i (g_r \ldots g_i \ldots g_{n+1}) + q^{n+1-r+1} t_{r-1} (g_r \ldots g_{n+1}).$$

\[\square\]

**Lemma 8.** The following relations hold in $H_{1,n}(q)$ for $k \in \mathbb{N}$:

(i) $(g_1 \ldots g_{i-1} g_{i+1}^2 g_{i-1} \ldots g_1) \cdot t = (q - 1) \sum_{k=1}^{i} q^{i-k} t_k (g_1 \ldots g_k \cdot g_{k-1} g_{k+1}^{-1} \ldots g_1^{-1}) + q^2 t$

(ii) $(g_1^{-1} \ldots g_{i-1}^{-1} g_{i+1}^{-1} \ldots g_1^{-1}) \cdot t^{-1} = (q - 1) \sum_{k=1}^{i} q^{-i-k} t_k^{-1} (g_1^{-1} \ldots g_k^{-1} g_{k+1} \ldots g_1) + q^{-i} t^{-1}$

(iii) $(g_k^{-1} \ldots g_2^{-1} g_1^{-1} g_2^{-1} \ldots g_k^{-1}) \cdot k = (q - 1) \sum_{i=1}^{k-1} q^{-k} t_i (g_k^{-1} \ldots g_{i+2} g_{i+1} g_{i+2} \ldots g_k) + q^{-k} t_k$

(iv) $(g_k^{-1} \ldots g_2^{-1} g_1^{-1} g_2^{-1} \ldots g_k^{-1}) \cdot t^{-1} = t^{-1} q^{-k} (q - 1) g_{k}^{-1} \ldots g_1^{-1} \ldots g_k^{-1} +$

$$+ \sum_{i=0}^{k-1} t_i^{-1} q^{-k+i} (q - 1) g_{k+1}^{-1} \ldots g_1^{-1} +$$

$$+ t_{k}^{-1} \left[ \sum_{i=0}^{k} q^{-i} (q - 1) q_{k+2}^{-1} \ldots g_2^{-1} g_{k+1}^{-1} \ldots g_{k}^{-1} + q^{-1} \right].$$

**Proof.** We prove relations (i) by induction on $i$. All other relations follow similarly. For $i = 1$ we have: $g_1^2 t = g_1 g_1 t g_1 g_1^{-1} = g_1 t_1 g_1^{-1} = (q - 1) t_1 g_1^{-1} + q t$. Suppose that the relation holds for $i = n$. Then, for $i = n + 1$ we have:

$$(g_1 \ldots g_n g_{n+1}^2 g_n \ldots g_1) \cdot t = (q - 1) (g_1 \ldots g_{n+1} g_n \ldots g_1) \cdot t + q (g_1 \ldots g_{n-1} g_{n+1}^2 g_{n-1} \ldots g_1) \cdot t =$$

$$= (q - 1) g_1 \ldots g_n t_{n+1} g_{n+1}^{-1} \ldots g_1^{-1} + q \sum_{k=1}^{n} q^{-k} (q - 1) t_k (g_1 \ldots g_{k-1} g_{k+1}^{-1} \ldots g_1^{-1}) + q^n t.$$
= (q - 1)t_{n+1} (g_1 \ldots g_n g_{n+1}^{-1} \ldots g_i^{-1}) + \sum_{k=1}^n q^{n+1-k}(q - 1)t_k (g_1 \ldots g_k g_{k+1}^{-1} \ldots g_i^{-1}) + q^{n+1}t =
= \sum_{k=1}^{n+1} q^{n+1-k}(q - 1)t_k (g_1 \ldots g_k g_{k+1}^{-1} \ldots g_i^{-1}) + q^{n+1}t.

\[\square\]

3.2. Converting elements in $\Lambda'$ to elements in $\Sigma_n$. We are now in the position to prove a set of relations converting monomials of $t_i'$s to expressions containing the $t_i$'s. In [D] we provide lemmas converting monomials of $t_i'$s to monomials of $t_i$'s in the context of giving a simple proof that the sets $\Sigma_n'$ form bases of $H_{1,n}(q)$.

**Lemma 9.** The following relations hold in $H_{1,n}(q)$ for $k \in \mathbb{N}$:

(i) $t_1'^{-k} = q^k t_1^{-k} + \sum_{j=1}^k q^{k-j}(q - 1)t^{-j} t_1^{-j-k} g_1^{-1},$

(ii) $t_1'^k = q^{-k} t_1^k + \sum_{j=1}^k q^{-k-j}(q - 1)t^{-j} t_1^{-k-1-j} g_1^{-1}.$

**Proof.** We prove relations (i) by induction on $k$. Relations (ii) follow similarly. For $k = 1$ we have: $t_1'^{-1} = g_1 t^{-1} g_1^{-1} = q g_1^{-1} t^{-1} g_1^{-1} + (q - 1) t^{-1} g_1^{-1} = q t_1^{-1} + (q - 1) t^{-1} g_1^{-1}.$

Suppose that the relation holds for $k - 1$. Then, for $k$ we have:

$t_1'^{-k} = t_1'^{-1} t_1'^{-k-1} \text{ ind. step } q^{k-1} t_1'^{-1} t_1'^{-k-1} + \sum_{j=1}^{k-1} q^{k-1-j}(q - 1)t^{-j} t_1'^{-1-j} g_1^{-1} t_1'^{-1} =
= q^k t_1'^{-k} + q^{k-1-j} t_1^{-1} g_1^{-1} + \sum_{j=1}^{k-1} q^{-j-k}(q - 1)t^{-j} t_1^{-1-j} g_1^{-1} t_1'^{-1} =
= q^k t_1'^{-k} + \sum_{j=1}^k q^{-k-j}(q - 1)t^{-j} t_1'^{-j-k} g_1^{-1}.

\[\square\]

**Lemma 10.** The following relations hold in $H_{1,n}(q)$ for $k \in \mathbb{N}$:

$t_1'^{-k} = q^k t_1^{-k} + (q - 1) \sum_{i=0}^{k-1} q^i t_1^{-1} (g_k g_{k-1} \ldots g_{i+2} g_{i+1}^{-1} \ldots g_{k-1} g_1^{-1}).$

**Proof.** We prove the relations by induction on $k$. For $k = 1$ we have:

$t_1'^{-1} = g_1 t^{-1} g_1^{-1} = q g_1^{-1} t^{-1} g_1^{-1} + (q - 1) t^{-1} g_1^{-1} = q t_1^{-1} + (q - 1) t^{-1} g_1^{-1}.$

Suppose that the relations hold for $k = n$. Then, for $k = n + 1$ we have that:

$t_1'^{-n+1} = g_{n+1} t_1'^{-1} g_{n+1}^{-1} \text{ ind. step } q^n t_1'^{-1} (g_n \ldots g_{n+1} g_{n+1}^{-1} \ldots g_1^{-1}) g_{n+1}^{-1} =
= q^n g_{n+1} t_1'^{-1} g_{n+1}^{-1} + (q - 1) \sum_{i=0}^{n-1} q^i t_1'^{-1} (g_n \ldots g_{n+1} g_{n+1}^{-1} \ldots g_1^{-1}) g_{n+1}^{-1} =
= q^n [q t_1'^{-1} g_{n+1} + (q - 1) t_1'^{-1}] g_{n+1}^{-1} + (q - 1) \sum_{i=0}^{n-1} q^i t_1'^{-1} (g_n \ldots g_{n+1} g_{n+1}^{-1} \ldots g_1^{-1}) g_{n+1}^{-1} =
= q^{n+1} t_1'^{-1} + q^n (q - 1) t_1'^{-1} g_{n+1}^{-1} + (q - 1) \sum_{i=0}^{n-1} q^i t_1'^{-1} (g_n \ldots g_{n+1} g_{n+1}^{-1} \ldots g_1^{-1}) g_{n+1}^{-1} =
= q^{n+1} t_1'^{-1} + (q - 1) \sum_{i=0}^{n} q^i t_1'^{-1} (g_n \ldots g_{n+1} g_{n+1}^{-1} \ldots g_1^{-1}).

\[\square\]
Lemma 11. The following relations hold in $H_{1,n}(q)$ for $k \in \mathbb{Z}\backslash\{0\}$:

$$t^k_m = q^{-mk}t^k_m + \sum_i f_i(q)t^k_i w_i + \sum_i g_i(q)t^{\lambda_0\lambda_1^1} \cdots t^{\lambda_m u_i},$$

where $w_i, u_i \in H_{m+1}(q)$, $\forall i$, $\sum_{i=0}^{m} \lambda_i = k$ and $\lambda_i \geq 0$, if $k_m > 0$ and $\lambda_i \leq 0$, if $k_m < 0$.

Proof. We prove relations by induction on $m$. The case $m = 1$ is Lemma 9. Suppose now that the relations hold for $m - 1$. Then, for $m$ we have:

$$t^k_m = g_m t^k_m - g_m^{-1} \frac{\text{ind. step}}{q^{-(m-1)}k} g_{m-1}t^k_m g_m^{-1} + \sum_i f_i(q)g_m t^k_m u_i g_m^{-1} + \sum_i g_i(q)t^{\lambda_0\lambda_1^1} \cdots t^{\lambda_{m-2} u_i g_m^{-1}} = q^{-(m-1)}k q^{-(k-1)}t^k_m g_m^{-2} + \sum_{j=1}^{k-1} q^{-(k-1)-j}(q^{-1})t^j_{m-1}t^{k-j} g_m^{-1} + \sum + \sum = q^{-mk}t^k_m + \sum_i f_i(q)t^k_i w_i + \sum_i g_i(q)t^{\lambda_0\lambda_1^1} \cdots t^{\lambda_m u_i}. \square$$

Using now Lemma 11 we have that every element $u \in \Lambda'$ can be expressed to linear combinations of elements $v_i \in \Sigma_n$, where $\exists j : v_j \sim u$. More precisely:

Theorem 7. The following relations hold in $H_{1,n}(q)$ for $k \in \mathbb{Z}$:

$$t^{k_0} t^{k_1} \cdots t^{k_m} = q^{-\sum_{i=1}^{m} nk_i} t^{k_0} t^{k_1} \cdots t^{k_m} + \sum_i f_i(q) t^{k_0} t^{k_1} \cdots t^{k_m} t^i w_i + \sum_j g_j(q) t_j u_j,$$

where $w_i, u_j \in H_{m+1}(q), \forall i, \tau_j \in \Lambda$, such that $\tau_j < t^{k_0} t^{k_1} \cdots t^{k_m}, \forall i$.

Proof. We prove relations by induction on $m$. Let $k_1 \in \mathbb{N}$, then for $m = 1$ we have:

$$t^{k_0} t^{k_1} = q^{-k_1} t^{k_0} t^{k_1} + \sum_{j=1}^{k_1} q^{-(k_1-j)(q^{-1})-1} t^{k_0} t^{k_1} t^{1-i-j} g^{-1} = q^{-k_1} t^{k_0} t^{k_1} + q^{-(k_1-1)} t^{k_0} t^{k_1} \cdot g^{-1} + \sum_{j=2}^{k_1} q^{-(k_1-j)}(q^{-1})-1 t^{k_0} t^{k_1} t^{1-i-j} g^{-1}. \ (L.9)$$

On the right hand side we obtain a term which is the holomorphic word of $t^{k_0} t^{k_1}$ with scalar $q^{-k_1} \in \mathbb{C}$, the homologous word again followed by $g^{-2} \in H_2(q)$ and with scalar $q^{-(k_1-1)}(q^{-1}-1) \in \mathbb{C}$ and the terms $t^{k_0} t^{k_1-j} t^{1-i-j}$, which are of less order than the holomorphic word $t^{k_0} t^{k_1}$, since $k_1 > k_1 + 1 - j$, for all $j \in \{2, 3, \ldots k_1\}$. So the statement holds for $m = 1$ and $k_1 \in \mathbb{N}$. The case $m = 1$ and $k_1 \in \mathbb{Z}\backslash\mathbb{N}$ is similar.

Suppose now that the relations hold for $m - 1$. Then, for $m$ we have:

$$t^{k_0} t^{k_1} \cdots t^m = \sum_{i=1}^{m-1} n_k \cdot t^{k_0} t^{k_1} \cdots t^{k_{m-1}} t^i w_i + \sum_i f_i(q) t^{k_0} t^{k_1} \cdots t^{k_{m-1}} w_i \cdot t^i w_i t^m \cdot t^{k_m} w_i + \sum_j g_j(q) t_j u_j \cdot t^m \cdot t^{k_m} w_i. \ (L.9)$$

Now, since $w_i, u_i \in H_{m+1}(q), \forall i$ we have that $w_i t^m \cdot t^{k_m} w_i$ and $u_i t^m \cdot t^{k_m} w_i, \forall i$. Applying now Lemma 11 to $t^m \cdot t^{k_m}$ we obtain the requested relation. \ (L.9)

Example 1. We convert the monomial $t^{-1} t^{3} t^{-2} \in \Lambda'$ to linear combination of elements in $\Sigma_n$. We have that:
4.1. Managing the gaps. Before proceeding with the proof of Theorem 2 we need to discuss the following situation. According to Lemma 9, for a word \( w \) we have that:

\[
t'_2 = q^2 t'_2 + q(q-1)t^{-1} g_2^{-1} g_1^{-1} + (q-1) t_1^{-1} g_1^{-1},
\]

\[
t_2'^{-1} = q^2 t_2'^{-1} + q(q-1)t^{-1} g_2^{-1} g_1^{-1} + q(q-1)t_1^{-1} g_2^{-1} + (q-1)^2 t^{-1} g_1^{-1} g_2^{-1},
\]

and so:

\[
t^{-1} t'_1 t_2'^{-1} = t^{-1} t_1'^{-1} t_2'^{-1} \cdot (1 + q^2(q-1)g_1^{-1}) + t^{-2} t_1'^{-2} (q^{-1}(q-1)g_2^{-1} g_1^{-1} g_2^{-1}) +
\]

\[+ t^{-1} t_1 \cdot (q^{-1}(q-1)g_2^{-1} + (q-1)(q-1)g_2^{-1} g_1^{-1} + (q-1)(q-1)g_1^{-1} g_2^{-1}) +
\]

\[+ 1 \cdot (-q^{-1}g_2^{-1} g_1^{-1} + t_1 t_2'^{-1} (q^2(q-1)g_1^{-1}).
\]

We obtain the homologous word \( w = t^{-1} t_1'^{-1} t_2'^{-1} \), the homologous word again followed by the braiding generator \( g_1^{-1} \) and all other terms are of less order than \( w \) since, either they contain gaps in the indices such as the term \( t_1 t_2'^{-1} \), or their index is less than \( \text{ind}(w) \) (the terms \( t^{-1} t_1 \), \( t^{-2} t_1'^{-1} \).

4. From \( \Sigma_n \) to \( \Lambda \)

4.1. Managing the gaps. Before proceeding with the proof of Theorem 2 we need to discuss the following situation. According to Lemma 9, for a word \( w' = t^k t_1'^{-\lambda} \in \Lambda' \), where \( k, \lambda \in \mathbb{N} \) and \( k < \lambda \) we have that:

\[
w' = t^k t_1'^{-\lambda} = t^{k-1} t_1^{-\lambda+1} \alpha_1 + t^{k-2} t_1^{-\lambda+2} \alpha_2 + \ldots + t^{k-(k-1)} t_1^{-\lambda+k-1} \alpha_{k-1} +
\]

\[+ t^0 t_1^{-\lambda+k} \alpha_k + t^{-1} t_1^{-\lambda+k+1} \alpha_{k+1} + \ldots + t^{-\lambda+k} \alpha_{\lambda},
\]

where \( \alpha_i \in H_n(q), \forall i \). We observe that in this particular case, in the right hand side there are terms which do not belong to the set \( \Lambda \). These are the terms of the form \( t_1^m \). So these elements cannot be compared with the highest order term \( w \sim w' \). The point now is that a term \( t_1^m \) is an element of the basis \( \Sigma_n \) on the Hecke algebra level, but, when we are working in \( S(ST) \), such an element must be considered up to conjugation by any braiding generator and up to stabilization moves. Topologically, conjugation corresponds to closing the braiding part of a mixed braid. Conjugating \( t_1 \) by \( g_1^{-1} \) we obtain \( t g_1^2 \) (view Figure 10) and similarly conjugating \( t_1^m \) by \( g_1^{-1} \) we obtain \( t g_1^2 t_1^2 \ldots g_1^2 \). Then, applying Lemma 3 we obtain the expression \( \sum_{k=1}^{m-1} k t_1^{m-k} v_k \), where \( v_k \in H_n(q) \), for all \( k \), that is, we obtain new elements in the \( \bigcup_n H_n(q) \)-module \( \Lambda \).

We shall next treat this situation in general. For the expressions that we obtain after appropriate conjugations we shall use the notation \( \Xi \). We will call gaps in monomials of the \( t_i \)’s, gaps occurring in the indices and size of the gap \( t_i'^k t_j'^k \) the number \( s_{i,j} = j - i \in \mathbb{N} \).
Lemma 12. For $k_0, k_1 \ldots k_i \in \mathbb{Z}$, $\epsilon = 1$ or $\epsilon = -1$ and $s_{i,j} > 1$ the following relation holds in $H_{1,n}(q)$:

$$p_0^{k_0}t_1^{k_1} \ldots t_{i-1}^{k_{i-1}}t_i^\epsilon \cdot t_j^\epsilon \cong p_0^{k_0}t_1^{k_1} \ldots t_{i-1}^{k_{i-1}}t_i^\epsilon \cdot t_{i+1}^\epsilon \left( g_{i+2}^\epsilon \ldots g_j^\epsilon g_j^\epsilon 2^\epsilon g_j^\epsilon 1 \ldots g_i^\epsilon + \right).$$

Proof. We have that $t_j^\epsilon = (g_j^\epsilon \ldots g_i^\epsilon + 2) \cdot t_{i+1}^\epsilon (g_{i+2}^\epsilon \ldots g_j^\epsilon)$ and so:

$$p_0^{k_0}t_1^{k_1} \ldots t_{i-1}^{k_{i-1}}t_i^\epsilon \cdot t_j^\epsilon = p_0^{k_0}t_1^{k_1} \ldots t_{i-1}^{k_{i-1}}t_i^\epsilon \left( g_j^{\epsilon+2} \ldots g_j^{\epsilon} \right) t_{i+1}^\epsilon \left( g_{i+2}^\epsilon \ldots g_j^\epsilon \right)$$

$$= (g_j^\epsilon \ldots g_i^\epsilon + 2) \cdot t_1^{k_1} \ldots t_{i-1}^{k_{i-1}}t_i^\epsilon t_{i+1}^\epsilon \left( g_{i+2}^\epsilon \ldots g_j^\epsilon \right) \cong$$

$$\cong t_0^{k_0} \ldots t_{i-1}^{k_{i-1}}t_i^\epsilon \left( g_j^\epsilon \ldots g_i^\epsilon \right).$$

□

In order to pass to a general way for managing gaps in monomials of $t_i$’s we first deal with gaps of size one. For this we have the following.

Lemma 13. For $k \in \mathbb{N}$, $\epsilon = 1$ or $\epsilon = -1$ and $\alpha \in H_{1,n}(q)$ the following relations hold:

$$t_i^{\epsilon k} \cdot \alpha \cong \sum_{u=1}^{k-1} q^{(\epsilon u-1)}(q^\epsilon - 1)t_{i-1}^{\epsilon u}t_i^{\epsilon (k-u)}(\alpha g_i^\epsilon) + q^{(k-1)}t_{i-1}^{\epsilon k}(g_i^\epsilon \alpha g_i^\epsilon).$$

Proof. We prove the relations by induction on $k$. For $k = 1$ we have $t_i^\epsilon \cdot \alpha \cong g_0^\epsilon t_i^\epsilon \cdot \alpha \cong t_i^\epsilon g_i^\epsilon \cdot \alpha$. Suppose that the assumption holds for $k-1 > 1$. Then for $k$ we have:

$$t_i^{\epsilon k} \cdot \alpha \cong t_i^{\epsilon (k-1)}(t_i^\epsilon \cdot \alpha) \quad \text{(ind. step)}$$

$$= \sum_{u=1}^{k-2} q^{(\epsilon u-1)}(q^\epsilon - 1)t_{i-1}^{\epsilon u}t_i^{\epsilon (k-u)}(\alpha g_i^\epsilon) + q^{(k-2)}t_{i-1}^{\epsilon (k-1)}(g_i^\epsilon \beta g_i^\epsilon) \quad (\beta = t_i^\epsilon \cdot \alpha)$$

We now introduce the following notation.

Notation 1. We set $\tau_{i,i+m}^{k_i \ldots k_{i+m}} := p_i^{k_i} \ldots t_{i+m}^{k_{i+m}}$, where $m \in \mathbb{N}$ and $k_j \neq 0$ for all $j$ and

$$\delta_{i,j} := \begin{cases} g_i g_{i+1} \ldots g_{j-1} g_j & \text{if } i < j \\ g_i g_{i+1} \ldots g_j & \text{if } i > j \end{cases}, \quad \delta_{i,k,j} := \begin{cases} g_i g_{i+1} \ldots g_{j-1} g_{k+1} \ldots g_j & \text{if } i < j \\ g_i g_{i+1} \ldots g_k + g_{j+1} & \text{if } i > j \end{cases}$$

We also set $w_{i,j}$ an element in $H_{j+1}(q)$ where the minimum index in $w$ is $i$.

Using now the notation introduced above, we apply Lemma 13 $s_{i,j}$-times to 1-gap monomials of the form $\tau_{0,i}^{k_i} \cdot t_j^{k_j}$ and we obtain monomials with no gaps in the indices, followed by words in $H_n(q)$. 
Example 2. For \( s_{i,j} > 1 \) and \( \alpha \in \mathcal{H}_n(q) \) we have:

\[
\begin{align*}
(i) \quad & \tau_{0,i}^{k_i} \cdot t_j \cdot \alpha \cong \tau_{0,i}^{k_i} \cdot t_{i+1} \cdot \delta_{i+2,j} \alpha \delta_{j,i+2} \\
(ii) \quad & \tau_{0,i}^{k_i} \cdot t_j^2 \cdot \alpha \cong \tau_{0,i}^{k_i} \cdot t_{i+1}^2 \cdot \delta_{i+2,j} \alpha \delta_{j,i+2} + \tau_{0,i}^{k_i} \cdot t_{i+1} t_{i+2} \alpha \beta, \text{ where} \\
& \beta = \left[ (q-1) \sum_{s=i+2} q^{j-s} \delta_{i+3,s} \delta_{i+2,s} \delta_{s+1,j} \alpha \delta_{j,i+2} \delta_{s,i+3} \right] \\
(iii) \quad & \tau_{0,i}^{k_i} \cdot t_j^3 \cdot \alpha \cong \left[ q^{-(i+2)+1} \right] \tau_{0,i}^{k_i} \cdot t_{i+1} \cdot \delta_{i+2,j} \alpha \delta_{j,i+2} + \tau_{0,i}^{k_i} \cdot t_{i+1} t_{i+2} \alpha \beta + \\
& + \tau_{0,i}^{k_i} \cdot t_{i+1} t_{i+2} \gamma + \tau_{0,i}^{k_i} \cdot t_{i+1} t_{i+2} t_{i+3} \mu, \text{ where} \\
& \gamma = q^{-(i+3)+1} (q-1) \delta_{i+3,j} \delta_{i+2,s} \delta_{s+1,j} \alpha \delta_{j,i+2} \delta_{s,i+3}, \text{ and} \\
& \mu = \sum_{s=i+2} q^{j-r-s} (q-1) \delta_{i+4,r} \delta_{i+2,s} \delta_{s+1,r} \delta_{r+1,j} \alpha \delta_{j,i+2} \delta_{s,i+3} \delta_{r,i+4} \\
& + \sum_{s=i+2} q^{j-r-s} (q-1) \delta_{i+4,r} \delta_{i+3,s} \delta_{s+1,j} \delta_{s+2,i} \delta_{s+1,j} \alpha \delta_{j,i+2} \delta_{s,i+3} \\
\end{align*}
\]

Applying Lemma 13 to the one gap word \( \tau_{0,i}^{k_{0,i}} \cdot t_j^{k_j} \), where \( k_j \in \mathbb{Z}\setminus\{0\} \) and \( \alpha \in \mathcal{H}_n(q) \) we obtain:

\[
\tau_{0,i}^{k_{0,i}} \cdot t_j^{k_j} \alpha \cong \left\{ \begin{array}{l}
\sum_{\lambda_{k_{i,j}}} \tau_{0,i}^{k_{0,i}} t_{i+1}^{\lambda_{i+1}} \ldots t_{i+k_j} \alpha' \quad \text{if} \quad k_j < s_{i,j} \\
\sum_{\lambda_{k_{i,j}}} \tau_{0,i}^{k_{0,i}} t_{i+1}^{\lambda_{i+1}} \ldots t_j^{\lambda_j} \beta' \quad \text{if} \quad k_j \geq s_{i,j}
\end{array} \right.
\]

where \( \alpha', \beta' \in \mathcal{H}_n(q) \), \( \sum_{\mu=i+1}^{i+k_j} \lambda_{\mu} = k_j, \lambda_{\mu} \geq 0, \forall \mu \) and if \( \lambda_u = 0 \), then \( \lambda_v = 0, \forall v \geq u \).

More precisely:

Lemma 14. For the 1-gap word \( A = \tau_{0,i}^{k_{0,i}} \cdot t_j^{k_j} \cdot \alpha \), where \( \alpha \in \mathcal{H}_n(q) \) we have:

\[
\begin{align*}
(i) \quad & \text{If} \ |k_j| < s_{i,j}, \text{ then: } A \cong \left( q^{k_j-1} \right)^{j-(i+1)} \tau_{0,i}^{k_{0,i}} \cdot t_{i+1}^{k_j} \delta_{i+2,j} \alpha \delta_{j,i+2} + \\
& + \sum \sum_{k_{i+1,i+k_j} = k_j} f(q,z) \tau_{0,i}^{k_{0,i}} \cdot t_{i+1,i+k_j} \beta \alpha \beta'. \\
(ii) \quad & \text{If} \ |k_j| \geq s_{i,j}, \text{ then: } A \cong \left( q^{k_j-1} \right)^{j-(i+1)} \tau_{0,i}^{k_{0,i}} \cdot t_{i+1}^{k_j} \delta_{i+2,j} \alpha \delta_{j,i+2} + \\
& + \sum \sum_{k_{i+1,i+k_j} = k_j} f(q) \tau_{0,i}^{k_{0,i}} \cdot t_{i+1,i+k_j} \beta \alpha \beta'.
\end{align*}
\]

where \( \beta \) and \( \beta' \) are of the form \( w_{i+1,j} \in \mathcal{H}_{i+1,j}(q) \) and \( \sum_{k_{i+1,k_j} = k_j} \) such that \( |k_{i+1}| < |k_j| \) and if \( k_{\mu} = 0 \), then \( k_s = 0, \text{ for all } s > \mu \).

Proof. We prove the relations by induction on \( k_j \). Let \( 0 < k_j < j-i \).

For \( k_j = 1 \) we have \( A \cong \left( q^{1-1} \right)^{j-(i+1)} \tau_{0,i}^{k_{0,i}} \cdot t_{i+1} \delta_{i+2,j} \alpha \delta_{j,i+2} \) (Lemma 12). Suppose that the relation holds for \( k_j - 1 > 1 \). Then for \( k_j \) we have:
\[
A = \frac{k_{0,i}}{\tau_{0,i}} \cdot t_j^{k_j-1} \cdot (t_j \alpha) \quad \overset{\text{ind.step}}{=} \left[ q^{k_j-2} \right]^{j-(i+1)} \frac{k_{0,i}}{\tau_{0,i}} \cdot t_j^{k_j-1} \cdot \frac{1}{\delta_{i+2,j}} \cdot t_j \alpha \cdot \delta_{j,i+2} + \\
+ \sum_{k_{i+1}+k_j-1} f(q, z) \frac{k_{0,i}}{\tau_{0,i}} \cdot t_{i+1,i+1} \beta t_j \beta'.
\]

We now consider \(B\) and \(C\) separately and apply Lemma 4 to both expressions:

\(B \overset{(L. 4)}{=} \)

\[
= \left[ q^{k_j-2} \right]^{j-(i+1)} \frac{k_{0,i}}{\tau_{0,i}} \cdot t_j^{k_j-1} \left[ (q-1) \sum_{k+i+2} q^{j-k} t_k \delta_{i+2,k-1} \delta_{k+1,j} + q^{j-(i+2)+1} t_{i+1} \delta_{i+2,j} \right] \alpha \delta_{j,i+2} + \\
= \left[ q^{k_j-2} \right]^{j-(i+1)} (q-1) \frac{k_{0,i}}{\tau_{0,i}} t_{i+1} \cdot \sum_{k+i+2} q^{j-k} t_k \delta_{i+2,k-1} \delta_{k+1,j} \alpha \delta_{j,i+2} + \\
+ \left[ q^{k_j-2} \right]^{j-(i+1)} \frac{k_{0,i}}{\tau_{0,i}} \cdot t_j^{k_j} \delta_{i+2,j} \alpha \delta_{j,i+2}.
\]

We now do conjugation on the \((j-(i+3))-\)one-gap words that occur and since \(t_k \cdot \beta \overset{\text{ind.step}}{=} t_{i+2} \cdot \delta_{i+3,k} \beta \delta_{k,i+3}\) we obtain:

\(B \overset{\text{ind.step}}{=} \)

\[
= \left[ q^{k_j-2} \right]^{j-(i+1)} \frac{k_{0,i}}{\tau_{0,i}} \cdot t_{i+1} \alpha \delta_{j,i+2} + \\
+ \frac{k_{0,i}}{\tau_{0,i}} t_{i+1} + \sum_{k+i+2} f(q, z) q^{j-k} t_k \delta_{i+2,k-1} \delta_{k+1,j} \alpha \delta_{j,i+2} \delta_{k,i+3} = \\
= \left[ q^{k_j-2} \right]^{j-(i+1)} \frac{k_{0,i}}{\tau_{0,i}} \cdot t_{i+1} \delta_{i+2,j} \alpha \delta_{j,i+2} + \frac{k_{0,i}}{\tau_{0,i}} t_{i+1} + \frac{k_{0,i}}{\tau_{0,i}} t_{i+1} \beta_{i},
\]

where \(\beta_1 \in H_{j+1}(q)\).

Moreover, \(C = \sum_{k_j} f(q) \frac{k_{0,i}}{\tau_{0,i}} \cdot t_{i+1,i+k_j-1} \beta t_j \beta'\) and since \(\beta = w_{i+k_j-1,j}\), we have that:

\(\beta \cdot t_j \overset{\text{(L. 4)}}{=} \sum_{s=i+k_j-1} t_s \cdot \gamma_s\), where \(\gamma_s \in H_{j+1}(q)\) and so: \(C \overset{\text{ind.step}}{=} \sum_{u_s} f(q) \frac{k_{0,i}}{\tau_{0,i}} \cdot \left( t_{i+1,i+k_j} \cdot \beta_{2} \right)\),

where \(\beta_2 \in H_{j+1}(q)\).

This concludes the proof. \(\square\)

We now pass to the general case of one-gap words.

**Proposition 3.** For the 1-gap word \(B = \frac{k_{0,i}}{\tau_{0,i}} \cdot \frac{k_{j,i+m}}{\tau_{j,i+m}} \cdot \alpha\), where \(\alpha \in H_n(q)\) we have:

\[
B \overset{\text{ind.step}}{=} \prod_{s=0}^{m} (q^{k_j+s-1})^{j-(i+1)} \frac{k_{0,i}}{\tau_{0,i}} \cdot \frac{k_{j,i+m}}{\tau_{i+1,i+m}} + \\
+ \prod_{s=0}^{m} (\delta_{i+m+2-s,j+s}) \cdot \alpha \cdot \prod_{s=0}^{m} (\delta_{j,s,i+m+2-s}) + \\
+ \sum_{u_s} f(q) \frac{k_{0,i}}{\tau_{0,i}} \cdot \left( t_{i+1,i+m} \right) \cdot \alpha'
\]

where \(\alpha' \in H_n(q)\), \(\sum u_{1,m} = k_j\) such that \(u_1 < k_j\) and if \(u_{i} = 0\), then \(u_{s} = 0, \forall s > \mu\).
Theorem 8. For the \( \phi \)-gap word \( C = \tau_{k_{0,i}} \cdot \tau_{i+1+u+s+\mu+\alpha} \cdot \tau_{i+s+1+u+s+\mu+\alpha} \cdot \tau_{i+s+1+u+s+\mu+\alpha} \cdot \ldots \tau_{i+s+1+u+s+\mu+\alpha} \cdot \alpha \), where \( k_i \in \mathbb{Z} \setminus \{0\} \) for all \( i, \alpha \in H_n(q) \), \( s_j, \mu_j \in \mathbb{N} \), such that \( s_1 > 1 \) and \( s_j > s_{j-1} + \mu_{j-1} \) for all \( j \) we have:

\[
C \equiv \prod_{j=1}^{\phi} \left( q^{k_{1+s}-1} \right)^{s_j-j-\sum_{\mu_p=1}^{j-1} \mu_p} \cdot \tau_{i+1+u+s+\mu+\alpha} \cdot \tau_{i+s+1+u+s+\mu+\alpha} \cdot \cdots \tau_{i+s+1+u+s+\mu+\alpha} \cdot \alpha \cdot \left( \prod_{p=1}^{\phi} \alpha_p' \right)
\]

\[
+ \sum_v f_v(q) \tau_{k_{0,v}} \cdot w_v, \quad \text{where}
\]

(i) \( \alpha_j = \prod_{j=0}^{\phi} \delta_{i+j+1+\sum_{\mu_p=1}^{j-1} \mu_p-\lambda_j, \lambda_j, \lambda_j, \lambda_j, j = \{1, 2, \ldots, \phi\} \),

(ii) \( \alpha_j' = \prod_{j=0}^{\phi} \delta_{i+j+1+\sum_{\mu_p=1}^{j-1} \mu_p-\lambda_j, \lambda_j, \lambda_j, \lambda_j, j = \{1, 2, \ldots, \phi\} \),

(iii) \( \tau_{0,i+1+u+s+\mu+\alpha} = \tau_{k_{0,i}} \cdot \prod_{j=1}^{\phi} \tau_{i+j+1+\sum_{\mu_p=1}^{j-1} \mu_p, i+j+1+\sum_{\mu_p=1}^{j} \mu_p} \),

(iv) \( \tau_{k_{0,v}} < \tau_{0,i+1+u+s+\mu+\alpha} \), for all \( v \),

(v) \( w_v \) of the form \( w_{i+2, i+s+\mu} \in H_{i+s+\mu+1}(q) \), for all \( v \),

(vi) the scalars \( f_v(q) \) are expressions of \( q \in \mathbb{C} \) for all \( v \).

Proof. We prove the relations by induction on the number of gaps. For the 1-gap word \( \tau_{k_{0,i}} \cdot \tau_{i+1+u+s+\mu+\alpha} \), where \( \alpha \in H_n(q) \), we have:

\[
A \equiv \left[ \prod_{\lambda=0}^{\phi} \left( q^{k_{1+s}-1} \right)^{s_j-j-\sum_{\mu_p=1}^{j-1} \mu_p} \cdot \tau_{k_{0,i}} \cdot \tau_{i+1+1+u+s+\mu+\alpha} \cdot \prod_{\lambda=0}^{\phi} \delta_{i+2+\mu-\lambda, i+s+\mu-\lambda, \lambda} \cdot \cdots \cdots \right]
\]

\[
+ \sum_v f_v(q) \cdot \tau_{k_{0,v}} \cdot w_v, \quad \text{which holds from Proposition 3.}
\]

Suppose that the relation holds for \( (\phi-1) \)-gap words. Then for a \( \phi \)-gap word we have:

\[
\prod_{j=1}^{\phi} \left( q^{k_{1+s}-1} \right)^{s_j-j-\sum_{\mu_p=1}^{j-1} \mu_p} \cdot \tau_{0,i+1+u+s+\mu+\alpha} \cdot \tau_{i+1+1+u+s+\mu+\alpha} \cdot \cdots \cdots \cdot \tau_{i+1+1+u+s+\mu+\alpha} \cdot \alpha \cdot \left( \prod_{p=1}^{\phi} \alpha_p' \right)
\]

\[
\sum_v f_v(q) \cdot \tau_{k_{0,v}} \cdot w_v \cdot \tau_{i+1+1+u+s+\mu+\alpha} \cdot \cdots \cdots \cdot \tau_{i+1+1+u+s+\mu+\alpha} \cdot \alpha \cdot \left( \prod_{p=1}^{\phi} \alpha_p' \right)
\]

\[
\sum_v f_v(q) \cdot \tau_{k_{0,v}} \cdot w_v \cdot \tau_{i+1+1+u+s+\mu+\alpha} \cdot \cdots \cdots \cdot \tau_{i+1+1+u+s+\mu+\alpha} \cdot \alpha \cdot \left( \prod_{p=1}^{\phi} \alpha_p' \right)
\]

\[
\sum_v f_v(q) \cdot \tau_{k_{0,v}} \cdot w_v \cdot \tau_{i+1+1+u+s+\mu+\alpha} \cdot \cdots \cdots \cdot \tau_{i+1+1+u+s+\mu+\alpha} \cdot \alpha \cdot \left( \prod_{p=1}^{\phi} \alpha_p' \right)
\]
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the highest index to be \((-1)\) when the exponent of the corresponding loop generator is in \(\mathbb{N}\) and \((+1)\) when the exponent of the corresponding loop generator is in \(\mathbb{Z}\backslash \mathbb{N}\). We then apply Lemma 3 and 4 in order to interact \(t_{n}^{\pm k_{n}}\) with \(g_{n}^{\pm 1}\) and obtain words of the following form:

1. \(\tau_{0,p}^{\lambda_{0,p}} \cdot v\), where \(\tau_{0,p}^{\lambda_{0,p}} < \tau_{0,n}^{k_{n}}\) and \(v \in H_{n+1}(q)\) of any length, or
2. \(\tau_{0,q}^{k_{0,q}} \cdot u\), where \(\tau_{0,q}^{k_{0,q}} < \tau_{0,n}^{k_{n}}\) and \(u \in H_{n}(q)\) such that \(l(u) < l(w)\).

In the first case we obtain monomials of \(t_{i}s\) of less order than the initial monomial, followed by a word in \(H_{n+1}(q)\) of any length. After at most \((k_{n} + 1)\)-interactions of \(t_{n}\) with \(g_{n}\), the exponent of \(t_{n}\) will become zero and so by applying a stabilization move we obtain monomials of \(t_{i}s\) of less index, and thus of less order (Definition 2), followed by words of the braiding generators of any length. We then apply stabilization moves and repeat the same procedure until the braiding ‘tails’ are eliminated.

**Theorem 9.** Applying conjugation and stabilization moves on a word in the \(\bigcup_{\infty} H_{n}(q)\)-module, \(\Lambda\) we have that:

\[
\tau_{0,m}^{k_{0,m}} \cdot w_{n} \simeq f(q,z) \cdot \sum_{j} f_{j}(q,z) \cdot \tau_{0,u_{j}}^{v_{0,u_{j}}},
\]

such that \(\sum v_{0,u_{j}} = \sum k_{0,m} \) and \(\tau_{0,u_{j}}^{v_{0,u_{j}}} < \tau_{0,m}^{k_{0,m}}\), for all \(j\).

The logic for the induction hypothesis is explained above. We shall now proceed with the proof of the theorem.

**Proof.** We prove the statement by double induction on the length of \(w_{n} \in H_{n}(q)\) and on the order of \(\tau_{0,m}^{k_{0,m}} \in \Lambda\), where order of \(\tau_{0,m}^{k_{0,m}}\) denotes the position of \(\tau_{0,m}^{k_{0,m}}\) in \(\Lambda\) with respect to total-ordering.

For \(l(w) = 0\), that is for \(w = e\) we have that \(\tau_{0,m}^{k_{0,m}} \simeq \tau_{0,m}^{k_{0,m}}\) and there’s nothing to show. Moreover, the minimal element in the set \(\Lambda\) is \(t^{k}\) and for any word \(w \in H_{n}(q)\) we have that \(t^{k} \cdot w \simeq f(q,z) \cdot t^{k}\), by the quadratic relation and stabilization moves.

Suppose that the relation holds for all \(\tau_{0,p}^{\lambda_{0,p}} \cdot w'\), where \(\tau_{0,p}^{\lambda_{0,p}} \leq \tau_{0,m}^{k_{0,m}}\) and \(l(w') = l\) and for all \(\tau_{0,q}^{\lambda_{0,q}} \cdot w\), where \(\tau_{0,q}^{\lambda_{0,q}} < \tau_{0,m}^{k_{0,m}}\) and \(l(w) = l + 1\). We will show that it holds for \(\tau_{0,m}^{k_{0,m}} \cdot w\). Let the exponent of \(t_{r}, k_{r} \in \mathbb{N}\) and let \(w \in H_{r+1}(q)\). Then, \(w\) can be written as \(w' \cdot g_{r}^{\pm 1} \cdot \delta_{r-1,d}\), where \(w' \in H_{r}(q)\) and \(d < r\). We have that:
In this example we demonstrate how to eliminate the braiding 'tail' in a word in \( \Sigma_n \) and so

\[
S_{r, r-1}^k \cdot w' \cdot g_r^{-1} \delta_r^{-1, d} =
\]

\[
= \frac{\tau_{0, r-1}^k \tau_{r+1, m}}{\tau_{0, r-1}^k \tau_{r+1, m}} \cdot w' \cdot g_r \cdot \delta_r^{-1, d} \quad \text{L.6}
\]

\[
= \frac{\tau_{0, r-1}^k \tau_{r+1, m}}{\tau_{0, r-1}^k \tau_{r+1, m}} \cdot w' \cdot g_r
\]

\[
\cdot \left( \sum_{j=0}^{r-1-d} q^j (q-1) \delta_{r-1, r-1-j, d} t_{r-1-j} + q^{(r_1-r-1, d)} \delta_{r-1, d-1} \right)
\]

\[
\cong \sum_{j=0}^{r-1-d} q^j (q-1) \tau_{0, r-1}^k \tau_{r+1, m} \cdot t_{r-1-j} \cdot w' \cdot g_r \delta_{r-1, r-1-j, d} +
\]

\[
q^{(r_1-r-1, d)} \tau_{0, r-1}^k t_{r+1, m} \cdot t_{d-1} \cdot w.
\]

We have that \( \left( \tau_{0, r-1}^k \tau_{r+1, m} \cdot t_{r-1-j} \right) \left( \tau_{0, r-1}^k \tau_{r+1, m} \cdot t_{d-1} \right) \left( \tau_{0, r-1}^k \tau_{r+1, m} \cdot t_{r-1-j} \right) \) < \( \left( \tau_{0, m}^k \right) \), for all \( j \in \{1, 2, \ldots r-1-d\} \) and \( l \left( w' \cdot g_r \delta_{r-1, r-1-j, d} \right) = l \) and \( \left( \tau_{0, r-1}^k \tau_{r+1, m} \cdot t_{d-1} \right) \left( \tau_{0, r-1}^k \tau_{r+1, m} \cdot t_{d-1} \right) \) < \( \left( \tau_{0, m}^k \right) \). So, by the induction hypothesis, the relation holds.

\[\Box\]

**Example 4.** In this example we demonstrate how to eliminate the braiding 'tail' in a word in \( \Sigma_n \).

\[
t^{-1} t_2^{-1} g_1^{-1} = t^{-1} t_1 t_2^{-1} t_1 g_1^{-1} = t^{-1} t_1 t_2^{-1} g_1 \cong t_1 t_2^{-1} g_1 = t_2^{-1} t_1 g_1 =
\]

\[
= (q-1) t_1 t_2^{-1} + q t_2^{-1} g_1 \cong (q-1) t t_2^{-1} g_1 + q t t_2^{-1} g_1
\]

\[
= (q-1) t t_1^{-1} g_1^{-1} g_2^{-1} + q t t_1^{-1} g_1^{-1} g_2^{-1}.
\]

We have that:

\[
g_2^{-1} g_1 g_2^{-1} = q^{-2} g_1 g_2 g_1 + q^{-1}(q-1) g_2 g_1 + q^{-1}(q-1) g_1 g_2 + (q^{-1} - 1)^2 g_1,
\]

\[
g_2^{-1} g_1 g_2^{-1} = q^{-2}(q-1) g_1 g_2 g_1 - (q^{-1} - 1)^2 g_2 g_1 - (q^{-1} - 1)^2 g_1 g_2 + (q-1)(q^{-1} - 1)^2 g_1
\]

\[
+ q(q^{-1} - 1) g_2^{-1} + 1,
\]

and so

\[
(q-1) t t_1^{-1} g_1^{-1} g_2^{-1} \cong ((q-1) + q^{-1}(q-1)^3) \cdot t t_1^{-1} - q^{-3}(q-1)^3 z^2 \cdot 1 +
\]

\[
+ 3q^{-3}(q-1)^4 z^2 \cdot 1 - q^{-1}(q-1)^2 z^2 \cdot 1 - q^{-3}(q-1)^5 \cdot 1,
\]

\[
q t t_1^{-1} g_1 g_2^{-1} \cong z \cdot t t_1^{-1} + q^{-1}(q-1) z^2 \cdot 1 + 2(q^{-1} - 1) z^2 \cdot 1 + q(q^{-1} - 1)^3 \cdot 1.
\]

### 5. The basis \( \Lambda \) of \( S(ST) \)

In this section we shall show that the set \( \Lambda \) is a basis for \( S(ST) \), given that \( \Lambda' \) is a basis of \( S(ST) \). This is done in two steps:
• We first relate the two sets Λ and Λ′ via an infinite lower triangular matrix with invertible elements in the diagonal. Since Λ′ is a basis for $S(\text{ST})$, the set Λ spans $S(\text{ST})$.
• Then, we prove that the set Λ is linear independent and so we conclude that Λ forms a basis for $S(\text{ST})$.

5.1. The infinite matrix. With the orderings given in Definition 2 we shall show that the infinite matrix converting elements of the basis Λ′ to elements of the set Λ is a block diagonal matrix, where each block is an infinite lower triangular matrix with invertible elements in the diagonal. Note that applying conjugation and stabilization moves on an element of some Λ_k followed by a braiding part won’t alter the sum of the exponents of the loop generators and thus, the resulted terms will belong to the set of the same level Λ_k. Fixing the level $k$ of a subset of Λ′, the proof of Theorem 2 is equivalent to proving the following claims:

1. A monomial $w' \in \Lambda_k' \subseteq \Lambda'$ can be expressed as linear combinations of elements of $\Lambda_k \subseteq \Lambda$, $v_i$, followed by monomials in $H_n(q)$, with scalars in $\mathbb{C}$ such that $\exists \ j : v_j = w \sim w'$.
2. Applying conjugation and stabilization moves on all $v_i$’s results in obtaining elements in $\Lambda_k$, $u_i$’s, such that $u_i < v_i$ for all $i$.
3. The coefficient of $w$ is an invertible element in $\mathbb{C}$.
4. $\Lambda_k \ni w < u \in \Lambda_{k+1}$.

Indeed we have the following: Let $w' \in S_k' \subseteq \Lambda'$. Then, by Theorem 7 the monomial $w'$ is expressed to linear combinations of elements of $\Sigma_n$, where the only term that isn’t followed by a braiding part is the homologous monomial $w \in \Lambda$. Other terms in the linear combinations involve lower order terms than $w$ (with possible gaps in the indices) followed by a braiding part and words of the form $w \cdot \beta$, where $\beta \in H_n(q)$. Then, by Theorem 8 elements of $\Sigma_n$ are expressed to linear combinations of elements of the $H_n(q)$-module $\Lambda$ (regularizing elements with gaps) and obtaining words which are of less order than the initial word $w$. In Theorem 9 all elements who are followed by a braiding part are expressed as linear combinations of elements of $\Lambda$ with coefficients in $\mathbb{C}$. It is essential to mention that when applying Theorem 9 to a word of the form $w \cdot \beta$ one obtains elements in $\Lambda$ that are less ordered that $w$. Thus, we obtain a lower triangular matrix with entries in the diagonal of the form $q^{-A}$ (see Theorem 7), which are invertible elements in $\mathbb{C}$. The fourth claim follows directly from Definition 2.

If we denote as $[\Lambda_k]$ the block matrix converting elements in $\Lambda_k'$ to elements in $\Lambda_k$ for some $k$, then the change of basis matrix will be of the form:

$$
S = \begin{bmatrix}
\ddots & 0 & 0 & 0 & 0 & 0 \\
[\Lambda_{k-2}] & 0 & 0 & 0 & 0 & 0 \\
0 & [\Lambda_{k-1}] & 0 & 0 & 0 & 0 \\
0 & 0 & [\Lambda_k] & 0 & 0 & 0 \\
0 & 0 & 0 & [\Lambda_{k+1}] & 0 & 0 \\
0 & 0 & 0 & 0 & [\Lambda_{k+2}] & \ddots
\end{bmatrix}
$$

The infinite block diagonal matrix

5.2. Linear independence of Λ. Consider an arbitrary subset of Λ with finite many elements $\tau_1, \tau_2, \ldots, \tau_k$. Without loss of generality we consider $\tau_1 < \tau_2 < \ldots < \tau_k$ according to Definition 2. We convert now each element $\tau_i \in \Lambda$ to linear combination of elements in $\Lambda'$ according to the infinite matrix. We have that
\[ \tau_i \simeq A_i \tau'_i + \sum_j A_j \tau'_j, \]

where \( \tau'_i \sim \tau_i, \ A_i \in \mathbb{C} \setminus \{0\}, \ \tau'_j < \tau'_i \) and \( A_j \in \mathbb{C}, \forall j. \)

So, we have that:

\[ \tau_1 \simeq A_1 \tau'_1 + \sum_j A_{1j} \tau'_{1j} \]
\[ \tau_2 \simeq A_2 \tau'_2 + \sum_j A_{2j} \tau'_{2j} \]
\[ \vdots \]
\[ \tau_{k-1} \simeq A_{k-1} \tau'_{k-1} + \sum_j A_{(k-1)j} \tau'_{(k-1)j} \]
\[ \tau_k \simeq A_k \tau'_k + \sum_j A_{kj} \tau'_{kj} \]

Note that each \( \tau'_i \) can occur as an element in the sum \( \sum_j A_{pj} \tau'_{pj} \) for \( p > i \). We consider now the equation \( \sum_{i=1}^{k} \lambda_i \cdot \tau_i = 0 \), \( \lambda_i \in \mathbb{C}, \forall i \) and we show that this holds only when \( \lambda_i = 0, \forall i. \)

Indeed, we have:

\[ \sum_{i=1}^{k} \lambda_i \cdot \tau_i = 0 \iff \lambda_k A_k \tau'_k + \sum_{i=1}^{k} \sum_j \lambda_i A_{ij} \tau'_{ij} = 0, \]

where \( \tau'_k > \tau'_{ij}, \forall i, j. \) So we conclude that \( \lambda_k = 0. \) Using the same argument we have that:

\[ \sum_{i=1}^{k} \lambda_i \cdot \tau_i = 0 \iff \sum_{i=1}^{k-1} \lambda_i \cdot \tau_i = 0 \iff \lambda_{k-1} A_{k-1} \tau'_{k-1} + \sum_{i=1}^{k-1} \sum_j \lambda_i A_{ij} \tau'_{ij} = 0, \]

where \( \tau'_{k-1} > \tau'_{ij}, \forall i, j. \) So, \( \lambda_{k-1} = 0. \) Retrospectively we get:

\[ \sum_{i=1}^{k} \lambda_i \cdot \tau_i = 0 \iff \lambda_i = 0, \forall i, \]

and so an arbitrary finite subset of \( \Lambda \) is linear independent. Thus, the set \( \Lambda \) is linear independent and it forms a basis for \( S(ST) \).

The proof of our main theorem is now concluded. QED

6. Conclusions

In this paper we gave a new basis \( \Lambda \) for \( S(ST) \), different from the Turaev-Hoste-Kidwell basis and the Morton-Aiston basis. This basis has been conjectured by J.H. Przytycki. The new basis is appropriate for describing the handle sliding moves, whilst the old basis \( \Lambda' \) is consistent with the trace rules [La2]. In a sequel paper we shall use the bases \( \Lambda' \) and \( \Lambda \) of \( S(ST) \) and the change of basis matrix in order to compute the Homflypt skein module of the lens spaces \( L(p, 1) \).
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Department of Mathematics, National Technical University of Athens, Zografou Campus, GR-15780 Athens, Greece.

E-mail address: diamantis@math.ntua.gr

Department of Mathematics, National Technical University of Athens, Zografou campus, GR-15780 Athens, Greece.

E-mail address: sofia@math.ntua.gr

URL: http://www.math.ntua.gr/~sofia