FEYNMAN-KAC PENALISATIONS OF SYMMETRIC STABLE PROCESSES

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Abstract

In K. Yano, Y. Yano and M. Yor (2009), limit theorems for the one-dimensional symmetric \( \alpha \)-stable process normalized by negative (killing) Feynman-Kac functionals were studied. We consider the same problem and extend their results to positive Feynman-Kac functionals of multi-dimensional symmetric \( \alpha \)-stable processes.

1 Introduction

In [9], [10], B. Roynette, P. Vallois and M. Yor have studied limit theorems for Wiener processes normalized by some weight processes. In [16], K. Yano, Y. Yano and M. Yor studied the limit theorems for the one-dimensional symmetric stable process normalized by non-negative functions of the local times or by negative (killing) Feynman-Kac functionals. They call the limit theorems for Markov processes normalized by Feynman-Kac functionals the Feynman-Kac penalisations. Our aim is to extend their results on Feynman-Kac penalisations to positive Feynman-Kac functionals of multi-dimensional symmetric \( \alpha \)-stable processes.

Let \( M^\alpha = (\Omega, \mathcal{F}, \mathcal{F}_t, P_x, X_t) \) be the symmetric \( \alpha \)-stable process on \( \mathbb{R}^d \) with \( 0 < \alpha \leq 2 \), that is, the Markov process generated by \(- (1/2)(-\Delta)^{\alpha/2} \), and \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) the Dirichlet form of \( M^\alpha \) (see (2.1),(2.2)). Let \( \mu \) be a positive Radon measure in the class \( \mathcal{K}_\infty \) of Green-tight Kato measures (Definition 2.1). We denote by \( A^\mu_t \) the positive continuous additive functional (PCAF in abbreviation) in the Revuz correspondence to \( \mu \): for a positive Borel function \( f \) and \( \gamma \)-excessive function \( g \),

\[
(g \mu, f) = \lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^d} \mathbb{E}_x \left[ \int_0^t f(X_s) dA^\mu_s \right] g(x) dx.
\]

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We define the family \( \{Q^\mu_{x,t}\} \) of normalized probability measures by
\[
Q^\mu_{x,t}[B] = \frac{1}{Z^\mu_{t}(x)} \int_B \exp(A^\mu_t(\omega)) \mathbb{P}_x(d\omega), \quad B \in \mathcal{F}_t,
\]
where \( Z^\mu_{t}(x) = \mathbb{E}_x[\exp(A^\mu_t)] \). Our interest is the limit of \( Q^\mu_{x,t} \) as \( t \to \infty \), mainly in transient cases, \( d > \alpha \). They in [16] treated negative Feynman-Kac functionals in the case of the one-dimensional recurrent stable process, \( \alpha > 1 \). In this case, the decay rate of \( Z^\mu_{t}(x) \) is important, while in our cases the growth order is.

We define \( \lambda(\theta) = \inf \{ \mathcal{E}_\theta(u,u) : \int_{\mathbb{R}^d} u^2 d\mu = 1 \} \), \( 0 \leq \theta < \infty \), (1.2)
where \( \mathcal{E}_\theta(u,u) = \mathcal{E}(u,u) + \theta \int_{\mathbb{R}^d} u^2 dx \). We see from [5, Theorem 6.2.1] and [12, Lemma 3.1] that the time changed process by \( A^\mu_t \) is symmetric with respect to \( \mu \) and \( \lambda(0) \) equals the bottom of the spectrum of the time changed process. We now classify the set \( \mathcal{K}_\infty \) in terms of \( \lambda(0) \):

(i) \( \lambda(0) < 1 \)
In this case, there exist a positive constant \( \theta_0 > 0 \) and a positive continuous function \( h \) in the Dirichlet space \( \mathcal{D}(\mathcal{E}) \) such that
\[
1 = \lambda(\theta_0) = \mathcal{E}_{\theta_0}(h,h).
\]
(Lemma 3.1, Theorem 2.3). We define the multiplicative functional (MF in abbreviation) \( L^h_t \) by
\[
L^h_t = e^{-\theta_0 \int_{\mathbb{R}^d} h(X_t) - h(X_0)} e^{A^\mu_t}.
\]
(1.3)

(ii) \( \lambda(0) = 1 \)
In this case, there exists a positive continuous function \( h \) in the extended Dirichlet space \( \mathcal{D}_e(\mathcal{E}) \) such that
\[
1 = \lambda(0) = \mathcal{E}(h,h).
\]
([14, Theorem 3.4]). Here \( \mathcal{D}_e(\mathcal{E}) \) is the set of measurable functions \( u \) on \( \mathbb{R}^d \) such that \( |u| < \infty \) a.e., and there exists an \( \mathcal{E} \)-Cauchy sequence \( \{u_n\} \) of functions in \( \mathcal{D}(\mathcal{E}) \) such that \( \lim_{n \to \infty} u_n = u \) a.e. We define
\[
L^h_t = \frac{h(X_t)}{h(X_0)} e^{A^\mu_t}.
\]
(1.4)

(iii) \( \lambda(0) > 1 \)
In this case, the measure \( \mu \) is gaugeable, that is,
\[
\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ e^{\lambda_\mu \alpha} \right] < \infty
\]
([15, Theorem 3.1]). We put \( h(x) = \mathbb{E}_x \left[ e^{\lambda_\mu \alpha} \right] \) and define
\[
L^h_t = \frac{h(X_t)}{h(X_0)} e^{A^\mu_t}.
\]
(1.5)
The cases (i), (ii), and (iii) are corresponding to the supercriticality, criticality, and subcriticality of the operator, \(-(1/2)(-\Delta)^{\alpha}/2 + \mu\), respectively (\cite{15}). We will see that \(L^h_t\) is a martingale \(\mathbb{P}^h\) for each case, i.e., \(\mathbb{E}_x[L^h_t] = 1\). Let \(M^h = (\Omega, \mathcal{F}_t, X_t)\) be the transformed process of \(M^\alpha\) by \(L^h_t\):

\[
\mathbb{P}^h_x(B) = \int_B L^h_t(\omega)\mathbb{P}_x(d\omega), \quad B \in \mathcal{F}_t.
\]

We then see from [3. Theorem 2.6] and Proposition 3.8 below that if \(\lambda(0) \leq 1\), then \(M^h\) is an \(h^2dx\)-symmetric Harris recurrent Markov process.

To state the main result of this paper, we need to introduce a subclass \(\mathcal{N}^S_\infty\) of \(\mathcal{N}_\infty\); a measure \(\mu \in \mathcal{N}_\infty\) is said to be in \(\mathcal{N}^S_\infty\) if

\[
\sup_{x \in \mathbb{R}^d} |x|^{d-\alpha} \int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^{d-\alpha}} < \infty. \tag{1.6}
\]

This class is relevant to the notion of special PCAF’s which was introduced by J. Neveu (\cite{6}); we will show in Lemma 4.4 that if a measure \(\mu\) belongs to \(\mathcal{N}^S_\infty\), then \(\int_0^t \int (1/h(X_s))dA^\mu_s\) is a special PCAF of \(M^h\). This fact is crucial for the proof of the main theorem below. In fact, a key to the proof lies in the application of the Chacon-Ornstein type ergodic theorem for special PCAF’s of Harris recurrent Markov processes (\cite{2. Theorem 3.18}).

We then have the next main theorem.

**Theorem 1.1.** (i) If \(\lambda(0) \neq 1\), then

\[
\mathcal{Q}^{\mu}_{x,t} \overset{t \to \infty}{\longrightarrow} \mathbb{P}^h_x \quad \text{along } (\mathcal{F}_t),
\]

that is, for any \(s \geq 0\) and any bounded \(\mathcal{F}_s\)-measurable function \(Z,

\[
\lim_{t \to \infty} \frac{\mathbb{E}_x[Z \exp(A^\mu_t)]}{\mathbb{E}_x[\exp(A^\mu_t)]} = \mathbb{P}^h_x[Z].
\]

(ii) If \(\lambda(0) = 1\) and \(\mu \in \mathcal{N}^S_\infty\), then (1.7) holds.

Throughout this paper, \(B(R)\) is an open ball with radius \(R\) centered at the origin. We use \(c, C, \ldots, \text{etc}\) as positive constants which may be different at different occurrences.

## 2 Preliminaries

Let \(M^\alpha = (\Omega, \mathcal{F}_t, \theta_t, \mathbb{P}_x, X_t)\) be the symmetric \(\alpha\)-stable process on \(\mathbb{R}^d\) with \(0 < \alpha \leq 2\). Here \(\mathcal{F}_t\) is the minimal (augmented) admissible filtration and \(\theta_t, \ t \geq 0\), is the shift operators satisfying \(X_s(\theta_t) = X_{s+t}\), identically for \(s, t \geq 0\). When \(\alpha = 2\), \(M^\alpha\) is the Brownian motion. Let \(p(t,x,y)\) be the transition density function of \(M^\alpha\) and \(G_\beta(x,y), \beta \geq 0\), be its \(\beta\)-Green function,

\[
G_\beta(x,y) = \int_0^\infty e^{-\beta t} p(t,x,y) dt.
\]

For a positive measure \(\mu\), the \(\beta\)-potential of \(\mu\) is defined by

\[
G_\beta \mu(x) = \int_{\mathbb{R}^d} G_\beta(x,y) \mu(dy).
\]
Let $P_t$ be the semigroup of $M^\alpha$,

$$P_tf(x) = \int_{\mathbb{R}^d} p(t, x, y)f(y)dy = \mathbb{E}_x[f(X_t)].$$

Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the Dirichlet form generated by $M^\alpha$: for $0 < \alpha < 2$

\[\mathcal{E}(u, v) = \frac{1}{2} \mathcal{A}(d, \alpha) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dxdy,\]

\[\mathcal{D}(\mathcal{E}) = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dxdy < \infty \right\},\]

where $\Delta = \{(x, x) : x \in \mathbb{R}^d\}$ and

$$\mathcal{A}(d, \alpha) = \frac{\alpha 2^{d-1} \Gamma\left(\frac{\alpha + d}{2}\right)}{\pi^{d/2} \Gamma\left(1 - \frac{\alpha}{2}\right)}$$

([5, Example 1.4.1]); for $\alpha = 2$

$$\mathcal{E}(u, v) = \frac{1}{2} D(u, v), \quad \mathcal{D}(\mathcal{E}) = H^1(\mathbb{R}^d),$$

where $D$ denotes the classical Dirichlet integral and $H^1(\mathbb{R}^d)$ is the Sobolev space of order 1 ([5, Example 4.4.1]). Let $\mathcal{D}(\mathcal{E})$ denote the extended Dirichlet space ([5, p.35]). If $\alpha < d$, that is, the process $M^\alpha$ is transient, then $\mathcal{D}(\mathcal{E})$ is a Hilbert space with inner product $\mathcal{E}$ ([5, Theorem 1.5.3]).

**Definition 2.1.** (I) A positive Radon measure $\mu$ on $\mathbb{R}^d$ is said to be in the **Kato class** ($\mu \in \mathcal{K}$ in notation), if

$$\lim_{\beta \to \infty} \sup_{x \in \mathbb{R}^d} G_\beta \mu(x) = 0. \quad (2.3)$$

(II) A measure $\mu$ is said to be $\beta$-**Green-tight** ($\mu \in \mathcal{K}_\infty(\beta)$ in notation), if $\mu$ is in $\mathcal{K}$ and satisfies

$$\lim_{\beta \to \infty} \sup_{R \to \infty} \int_{|y| > R} G_\beta(x, y)\mu(dy) = 0. \quad (2.4)$$

We see from the resolvent equation that for $\beta > 0$

$$\mathcal{K}_\infty(\beta) = \mathcal{K}_\infty(1).$$

When $d > \alpha$, that is, $M^\alpha$ is transient, we write $\mathcal{K}_\infty$ for $\mathcal{K}_\infty(0)$. For $\mu \in \mathcal{K}$, define a symmetric bilinear form $\mathcal{E}^\mu$ by

$$\mathcal{E}^\mu(u, u) = \mathcal{E}(u, u) - \int_{\mathbb{R}^d} \bar{u}^2 d\mu, \quad u \in \mathcal{D}(\mathcal{E}), \quad (2.5)$$

where $\bar{u}$ is a quasi-continuous version of $u$ ([5, Theorem 2.1.3]). In the sequel, we always assume that every function $u \in \mathcal{D}(\mathcal{E})$ is represented by its quasi continuous version. Since $\mu \in \mathcal{K}$ charges no set of zero capacity by [11, Theorem 3.3], the form $\mathcal{E}^\mu$ is well defined. We see from
Theorem 4.1] that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}))$ becomes a lower semi-bounded closed symmetric form. Denote by $\mathcal{H}^\mu$ the self-adjoint operator generated by $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E})): \mathcal{E}^\mu(u, v) = (\mathcal{H}^\mu u, v)$. Let $P_t^\mu$ be the $L^2$-semigroup generated by $\mathcal{H}^\mu$: $P_t^\mu = \exp(-t\mathcal{H}^\mu)$. We see from [1 Theorem 6.3(iv)] that $P_t^\mu$ admits a symmetric integral kernel $p^\mu(t, x, y)$ which is jointly continuous function on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

For $\mu \in \mathcal{K}$, let $A^\mu_t$ be a PCAF which is in the Revuz correspondence to $\mu$ (Cf. [5 p.188]). By the Feynman-Kac formula, the semigroup $P_t^\mu$ is written as

$$P_t^\mu f(x) = \mathbb{E}_x[\exp(A^\mu_t f(X_t))].$$

Theorem 2.2 ([11]). Let $\mu \in \mathcal{K}$. Then

$$\int_{\mathbb{R}^d} u^2(x) \mu(dx) \leq \|G_\beta \mu\|_\infty \mathcal{E}_\beta(u, u), \quad u \in \mathcal{D}(\mathcal{E}),$$

(2.7)

where $\mathcal{E}_\beta(u, u) = \mathcal{E}(u, u) + \beta \int_{\mathbb{R}^d} u^2 dx$.

Theorem 2.3. ([14 Theorem 10], [13 Theorem 2.7]) If $\mu \in \mathcal{K}_\infty(1)$, then the embedding of $\mathcal{D}(\mathcal{E})$ into $L^2(\mu)$ is compact. If $d > \alpha$ and $\mu \in \mathcal{K}_\alpha$, then the embedding of $\mathcal{D}_\alpha(\mathcal{E})$ into $L^2(\mu)$ is compact.

3 Construction of ground states

For $d \leq \alpha$ (resp. $d > \alpha$), let $\mu$ be a non-trivial measure in $\mathcal{K}_\infty(1)$ (resp. $\mathcal{K}_\alpha$). Define

$$\lambda(\theta) = \inf \left\{ \mathcal{E}_\theta(u, u) : \int_{\mathbb{R}^d} u^2 d\mu = 1 \right\}, \quad \theta \geq 0. \quad (3.1)$$

Lemma 3.1. The function $\lambda(\theta)$ is increasing and concave. Moreover, it satisfies $\lim_{\theta \rightarrow \infty} \lambda(\theta) = \infty$.

Proof. It follows from the definition of $\lambda(\theta)$ that it is increasing. For $\theta_1, \theta_2 \geq 0, 0 \leq t \leq 1$

$$\lambda(t \theta_1 + (1-t) \theta_2) = \inf \left\{ \mathcal{E}_{t \theta_1 + (1-t) \theta_2}(u, u) : \int_{\mathbb{R}^d} u^2 d\mu = 1 \right\}$$

$$\geq t \inf \left\{ \mathcal{E}_{\theta_1}(u, u) : \int_{\mathbb{R}^d} u^2 d\mu = 1 \right\} + (1-t) \inf \left\{ \mathcal{E}_{\theta_2}(u, u) : \int_{\mathbb{R}^d} u^2 d\mu = 1 \right\}$$

$$= t \lambda(\theta_1) + (1-t) \lambda(\theta_2).$$

We see from Theorem 2.2 that for $u \in \mathcal{D}(\mathcal{E})$ with $\int_{\mathbb{R}^d} u^2 d\mu = 1$, $\mathcal{E}_\theta(u, u) \geq 1/\|G_\theta \mu\|_\infty$. Hence we have

$$\lambda(\theta) \geq \frac{1}{\|G_\theta \mu\|_\infty}. \quad (3.2)$$

By the definition of the Kato class, the right hand side of (3.2) tends to infinity as $\theta \rightarrow \infty$. \hfill \Box

Lemma 3.2. If $d \leq \alpha$, then $\lambda(0) = 0$.

Proof. Note that for $u \in \mathcal{D}(\mathcal{E})$

$$\lambda(0) \int_{\mathbb{R}^d} u^2 d\mu \leq \mathcal{E}(u, u).$$

Since $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is recurrent, there exists a sequence $\{u_n\} \subset \mathcal{D}(\mathcal{E})$ such that $u_n \uparrow 1$ q.e. and $\mathcal{E}(u_n, u_n) \rightarrow 0$ ([5 Theorem 1.6.3, Theorem 2.1.7]). Hence if $\lambda(0) > 0$, then $\mu = 0$, which is contradictory. \hfill \Box
We see from Theorem 2.3 and Lemma 3.2 that if $d \leq \alpha$, then there exist $\theta_0 > 0$ and $h \in \mathcal{D}(\mathcal{F})$ such that
\[
\lambda(\theta_0) = \inf \left\{ \mathcal{E}_{\theta_0}(h, h) : \int_{\mathbb{R}^d} h^2 d\mu = 1 \right\} = 1.
\]
We can assume that $h$ is a strictly positive continuous function (e.g. Section 4 in [14]).

Let $M_t^{[h]}$ be the martingale part of the Fukushima decomposition ([5] Theorem 5.2.2):
\[
h(X_t) - h(X_0) = M_t^{[h]} + N_t^{[h]}.
\]
(3.3)

Define a martingale by
\[
M_t = \int_0^t \frac{1}{h(X_s)} dM_s^h
\]
and denote by $L_t^h$ the unique solution of the Doléans-Dade equation:
\[
Z_t = 1 + \int_0^t Z_s dM_s.
\]
(3.4)

Then we see from the Doléans-Dade formula that $L_t^h$ is expressed by
\[
L_t^h = \exp \left( M_t - \frac{1}{2} \langle M^c \rangle_t \right) \prod_{0 \leq s \leq t} \left( 1 + \Delta M_s \right) \exp(-\Delta M_s) \exp \left( \frac{h(X_s)}{h(X_{s-})} \right) \exp \left( 1 - \frac{h(X_s)}{h(X_{s-})} \right).
\]
Here $M_t^c$ is the continuous part of $M_t$ and $\Delta M_s = M_s - M_{s-}$. By Itô's formula applied to the semi-martingale $h(X_t)$ with the function $\log x$, we see that $L_t^h$ has the following expression:
\[
L_t^h = e^{-\theta_0 t} \frac{h(X_t)}{h(X_0)} \exp(A_{\mu}^t).
\]
(3.5)

Let $d > \alpha$ and suppose that $\theta_0 = 0$, that is,
\[
\lambda(0) = \inf \left\{ \mathcal{E}(u, u) : \int_{\mathbb{R}^d} u^2 d\mu = 1 \right\} = 1.
\]

We then see from [14] Theorem 3.4 that there exists a function $h \in \mathcal{D}(\mathcal{F})$ such that $\mathcal{E}(h, h) = 1$.

We can also assume that $h$ is a strictly positive continuous function and satisfies
\[
\frac{c}{|x|^{d-\alpha}} \leq h(x) \leq \frac{C}{|x|^{d-\alpha}}, \quad |x| > 1
\]
(3.6)

(see (4.19) in [14]). We define the MF $L_t^h$ by
\[
L_t^h = \frac{h(X_t)}{h(X_0)} \exp(A_{\mu}^t).
\]
(3.7)

We denote by $M_t^h = (\Omega, \mathbb{P}_x^h, X_t)$ the transformed process of $M^\theta$ by $L_t^h$,
\[
\mathbb{P}_x^h(d\omega) = L_t^h(\omega) \cdot \mathbb{P}_x(d\omega).
\]
Proposition 3.3. The transformed process $M^h = (P^h_x, X_t)$ is Harris recurrent, that is, for a non-negative function $f$ with $m(\{x : f(x) > 0\}) > 0$,
\[ \int_0^\infty f(X_t) dt = \infty \text{ P}_x\text{-a.s.}, \] (3.8)
where $m$ is the Lebesgue measure.

Proof. Set $A = \{x : f(x) > 0\}$. Since $M^h$ is an $h^2 dx$-symmetric recurrent Markov process,
\[ \mathbb{P}_x[S_A \circ \theta_n < \infty, \forall n \geq 0] = 1 \text{ for q.e. } x \in \mathbb{R}^d \] (3.9)
by [5, Theorem 4.6]. Moreover, since the Markov process $M^h$ has the transition density function
\[ e^{-\theta_0 t} \cdot \frac{p^h(t, x, y)}{h(x)h(y)} \]
with respect to $h^2 dx$, (3.9) holds for all $x \in \mathbb{R}^d$ by [5, Problem 4.6.3]. Using the strong Feller property and the proof of [3, Chapter X, Proposition (3.11)], we see from (3.9) that $M^h$ is Harris recurrent.

We see from [14, Theorem 4.15]: If $\theta_0 > 0$, then $h \in L^2(\mathbb{R}^d)$ and $M^h$ is positive recurrent. If $\theta_0 = 0$ and $\alpha < d \leq 2\alpha$, then $h \not\in L^2(\mathbb{R}^d)$ $M^h$ is null recurrent. If $\theta_0 = 0$ and $d \geq 2\alpha$, then $h \in L^2(\mathbb{R}^d)$ $M^h$ is positive recurrent.

4 Penalization problems

In this section, we prove Theorem 1.1.

(1°) Recurrent case ($d \leq \alpha$)

Theorem 4.1. Assume that $d \leq \alpha$. Then there exist $\theta_0 > 0$ and $h \in \mathcal{D}(\mathcal{F})$ such that $\lambda(\theta_0) = 1$ and $\mathcal{E}_{\theta_0}(h, 1) = 1$. Moreover, for each $x \in \mathbb{R}^d$
\[ e^{-\theta_0 t} \mathbb{E}_x \left[ e^{\mathcal{L}_t x} \right] \rightarrow h(x) \int_{\mathbb{R}^d} h(x) dx \text{ as } t \rightarrow \infty. \] (4.1)

Proof. The first assertion follows from Theorem 2.3 and Lemma 3.2. Note that
\[ e^{-\theta_0 t} \mathbb{E}_x \left[ e^{\mathcal{L}_t x} \right] = h(x) \mathbb{E}_x \left[ \frac{1}{h(X_t)} \right] \]
Then by [13, Corollary 4.7] the right hand side converges to $h(x) \int_{\mathbb{R}^d} h(x) dx$. 

Theorem 4.1 implies 1.7. Indeed,
\[
\begin{align*}
E_x \left( \exp(A^\mu_t) | \mathcal{F}_t \right) &= \frac{e^{-\theta_0 t} \mathbb{E}_x \left( \exp(A^\mu_t) | \mathcal{F}_t \right)}{\mathbb{E}_x \left( \exp(A^\mu_t) \right)} \\
&= \frac{e^{-\theta_0 t} \mathbb{E}_x \left( \exp(A^\mu_t) \right) \mathbb{E}_x \left( \exp(A^\mu_t) \right) h(X_t) \int_{\mathbb{R}^d} h(x) dx}{\mathbb{E}_x \left( \exp(A^\mu_t) \right) h(X_t) \int_{\mathbb{R}^d} h(x) dx} = \mathbb{L}^h \text{ as } t \rightarrow \infty.
\end{align*}
\]
We showed in [3, Theorem 2.6 (b)] that the transformed process $M^h$ is recurrent. We see from this fact that $L^h_t$ is martingale, $\mathbb{E}(L^h_t) = 1$. Therefore Scheff’s lemma leads us to Theorem 1.1 (i) (e.g. [9]).

(2°) Transient case ($d > \alpha$)

If $\lambda(0) < 1$, there exist $\theta_0 > 0$ and $h \in \mathcal{D}(E)$ such that $\lambda(\theta_0) = 1$ and $\mathcal{E}_{\theta_0}(h, h) = 1$. Then we can show the equation (4.1) in the same way as above. If $\lambda(0) > 1$, then $A^\mu_t$ is gaugeable (see Theorem 4.1 below), that is,

$$
\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ e^{A^\mu_t} \right] < \infty,
$$

and thus

$$
\lim_{t \to \infty} \mathbb{E}_x \left[ e^{A^\mu_t} \right] = \mathbb{E}_x \left[ e^{A^\mu_\infty} \right].
$$

Hence for any $s \geq 0$ and any $\mathcal{F}_s$-measurable bounded function $Z$

$$
\frac{\mathbb{E}_x \left[ Ze^{A^\mu_t} \right]}{\mathbb{E}_x \left[ e^{A^\mu_t} \right]} = \frac{\mathbb{E}_x \left[ Ze^{A^\mu_t} \mathbb{E}_x \left[ e^{A^\mu_t-s} \right] \right]}{\mathbb{E}_x \left[ e^{A^\mu_t} \right]} \rightarrow \frac{1}{h(x)} \mathbb{E}_x \left[ Ze^{A^\mu_t} h(X_s) \right] = \mathbb{E}_x^h \left[ Z \right]
$$

as $t \to \infty$.

In the remainder of this section, we consider the case when $\lambda(0) = 1$. It is known that a measure $\mu \in \mathcal{H}_\infty$ is Green-bounded,

$$
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^{d-\alpha}} < \infty. \quad (4.2)
$$

To consider the penalisation problem for $\mu$ with $\lambda(0) = 1$, we need to impose a condition on $\mu$.

**Definition 4.2.** (I) A measure $\mu \in \mathcal{H}$ is said to be special if

$$
\sup_{x \in \mathbb{R}^d} (|x|^{d-\alpha} \int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^{d-\alpha}}) < \infty. \quad (4.3)
$$

We denote by $\mathcal{H}^S_\infty$ the set of special measures.

(II) A PCAF $A_t$ is said to be special with respect to $M^h$, if for any positive Borel function $g$ with $\int_{\mathbb{R}^d} g \, dx > 0$

$$
\sup_{x \in \mathbb{R}^d} \mathbb{E}_x^h \left[ \int_0^\infty \exp \left( -\int_0^t g(X_s) \, ds \right) \, dA_t \right] < \infty.
$$

A Kato measure with compact support belongs to $\mathcal{H}^S_\infty$. The set $\mathcal{H}^S_\infty$ is contained in $\mathcal{H}_\infty$,

$$
\mathcal{H}^S_\infty \subset \mathcal{H}_\infty. \quad (4.4)
$$

Indeed, since for any $R > 0$

$$
M(\mu) := \sup_{x \in \mathbb{R}^d} \left( |x|^{d-\alpha} \int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^{d-\alpha}} \right) \geq R^{d-\alpha} \sup_{x \in B(R)} \int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^{d-\alpha}},
$$
we have
\[ \sup_{x \in \mathbb{R}^d} \int_{B_R(y)} \frac{d\mu(y)}{|x-y|^{d-\alpha}} = \sup_{x \in \mathbb{B}(R,y)} \int_{\mathbb{B}(R,y)} \frac{d\mu(y)}{|x-y|^{d-\alpha}} \leq \frac{M(\mu)}{R^{d-\alpha}} \to 0, \ R \to \infty. \]

**Lemma 4.3.** Let \( B_t \) be a PCAF. Then
\[ \mathbb{E}_x \left[ \int_0^\infty e^{(\alpha_s - \alpha_B) t} dA^\mu_s \right] = h(x) \mathbb{E}_x^h \left[ \int_0^\infty e^{-B_t} \frac{dA^\mu_t}{h(X_t)} \right]. \]

**Proof.** We have
\[
\begin{align*}
\mathbb{E}_x \left[ \int_0^s e^{-B_t} \frac{dA^\mu_t}{h(X_t)} \right] &= \mathbb{E}_x \left[ e^{\alpha_s h(X_s)} \int_0^s e^{-B_t} \frac{dA^\mu_t}{h(X_t)} \right] \\
&= \mathbb{E}_x \left[ \int_0^s e^{\alpha_s h(X_s)} e^{-B_t} \frac{dA^\mu_t}{h(X_t)} \right].
\end{align*}
\]

Put \( Y_t = e^{\alpha_s h(X_s)} e^{-B_t} / h(X_t) \). Then since \( Y_t \) is a right continuous process, its optional projection is equal to \( \mathbb{E}_x[Y_t|\mathcal{F}_t] \) (e.g. [15] Theorem 7.10). Hence the right hand side equals
\[ \mathbb{E}_x \left[ \int_0^\infty \mathbb{E}_x \left[ Y_t|\mathcal{F}_t \right] dA^\mu_t \right] = \mathbb{E}_x \left[ \int_0^\infty e^{\alpha_s h(X_s)} e^{-B_t} \frac{1}{h(X_t)} \mathbb{E}_{X_s} \left[ e^{\alpha_s h(X_s)} \right] dA^\mu_t \right]. \]

Since \( \mathbb{E}_{X_s} \left[ e^{\alpha_s h(X_s)} \right] = h(X_s) \), the right hand side equals
\[ \mathbb{E}_x \left[ \int_0^\infty e^{\alpha_s - \alpha_B} dA^\mu_t \right]. \]

Hence the proof is completed by letting \( s \to \infty \). \( \square \)

The next theorem was proved in [15].

**Theorem 4.1.** (15) Suppose \( d > \alpha \). For \( \mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty^\prime \) let \( A^\mu_t = A^\mu_t^+ - A^\mu_t^- \). Then the following conditions are equivalent:

(i) \( \sup_{x \in \mathbb{R}^d} \mathbb{E}_x[e^{\alpha x}] < \infty \).

(ii) There exists the Green function \( G^\mu(x,y) < \infty (x \neq y) \) of the operator \( -\frac{1}{2}(-\Delta)^{\alpha/2} + \mu \) such that
\[ \mathbb{E}_x \left[ \int_0^\infty e^{\alpha x} f(X_t) dt \right] = \int_{\mathbb{R}^d} G^\mu(x,y)f(y)dy. \]

(iii) \( \inf \left\{ \mathbb{E}(u,u) + \int_{\mathbb{R}^d} u^2 d\mu^- : \int_{\mathbb{R}^d} u^2 d\mu^+ = 1 \right\} > 1. \)

We see from (4.19) in [14] that if one of the statements in Theorem 4.1 holds, then \( G^\mu(x,y) \) satisfies
\[ G(x,y) \leq G^\mu(x,y) \leq CG(x,y). \] (4.5)
Lemma 4.4. If $\mu \in \mathcal{K}_\infty^S$, then $\int_0^t \frac{dA^\mu_t}{h(X_t)}$ is special with respect to $\mathbf{M}^h$.

Proof. We may assume that $g$ is a bounded positive Borel function with compact support. Note that by Lemma 4.3

$$E_x \left[ e^{\int_0^t g(X_s)ds} \right] = \frac{1}{h(x)} \int_0^t \exp \left( \frac{A^\mu_t - \int_0^t g(X_s)ds}{h(X_t)} \right) \frac{dA^\mu_t}{h(X_t)}$$

If the measure $\mu$ satisfies $\lambda(0) = 1$, then $\mu - g \cdot dx \in \mathcal{K}_\infty^ \mathcal{K}_\infty$ satisfies Theorem 4.1 (iii), and $G^{\mu - {g \cdot dx}}(x,y)$ is equivalent with $G(x,y)$ by (4.5). Therefore the equation (3.6) implies that (4.3) is equivalent to that sup$_{x \in \mathbb{R}^d} \left\{ (1/h(x))G^{\mu - {g \cdot dx}}(x) \right\} < \infty$.

We note that by Lemma 4.3

$$E_x \left[ e^{\int_0^t g(X_s)ds} \right] = 1 + \int_0^t e^{\int_0^s g(X_u)du} \frac{dA^\mu_s}{h(X_s)}.$$

Thus for a finite positive measure $\nu$,

$$E_\nu \left[ e^{\int_0^t g(X_s)ds} \right] = \nu(\mathbb{R}^d) + \langle \nu, h \rangle E^h \left[ \int_0^t \frac{dA^\mu_s}{h(X_s)} \right].$$

(4.6)

where $\nu^h = h \cdot \nu / \langle \nu, h \rangle$. For a positive smooth function $k$ with compact support, put

$$\psi(t) = E_x \left[ \int_0^t k(X_s)ds \right].$$

Then $\lim_{t \to \infty} \psi(t) = \infty$ by the Harris recurrence of $\mathbf{M}^h$. Moreover,

$$\lim_{t \to \infty} \frac{\psi(t+s)}{\psi(t)} = 1.$$ (4.7)

Indeed,

$$\psi(t+s) = E_x \left[ \int_0^t k(X_u)du \right] + E_x \left[ \int_0^t k(X_u)du \right] \leq \psi(t) + \|k\|_\infty s,$$

and

$$1 \leq \frac{\psi(t+s)}{\psi(t)} \leq 1 + \frac{\|k\|_\infty s}{\psi(t)}.$$ (4.7)

We see from [4, Lemma 4.4] that the Revuz measure of $A^\mu_t$ is $h^2 \mu$ as a PCAF of $\mathbf{M}^h$. Since by (4.6)

$$\frac{1}{\psi(t)} E_\nu \left[ e^{\psi(t)} \right] = \frac{\nu(\mathbb{R}^d)}{\psi(t)} + \langle \nu, h \rangle E^h \left[ \int_0^t \frac{dA^\mu_s}{h(X_s)} \right] \frac{1}{\psi(t)} \int_0^t k(X_u)du$$
and \( \int_0^t (1/h(X_s))dA^h_s \) and \( \int_0^t k(X_s)ds \) are special with respect to \( M^h \), we see from Chacon-Ornstein type ergodic theorem in [2, Theorem 3.18] that

\[
\frac{1}{\psi(t)} \mathbb{E}_v \left[ e^{\psi^h_t} \right] \to (v, h) \cdot \frac{\langle \mu, h \rangle}{\int_{\mathbb{R}^d} kh^2 dx} \quad (4.8)
\]
as \( t \to \infty \). Note that \( \langle \mu, h \rangle < \infty \) by (3.6) and (4.2).

For a bounded \( \mathcal{F}_t \)-measurable function \( Z \), define a positive finite measure \( \nu \) by

\[
\nu(B) = \mathbb{E}_x \left[ Z e^{\psi^h_{t-s}} ; X_s \in B \right], \quad B \in \mathcal{B}(\mathbb{R}^d).
\]

Then by the Markov property,

\[
\mathbb{E}_x \left[ Z e^{\psi^h_t} \right] = \mathbb{E}_x \left[ e^{\psi^h_{t-s}} \right].
\]

Therefore

\[
\lim_{t \to \infty} \frac{\mathbb{E}_x \left[ Z e^{\psi^h_t} \right]}{\mathbb{E}_x \left[ e^{\psi^h_t} \right]} = \lim_{t \to \infty} \frac{\mathbb{E}_x \left[ Z e^{\psi^h_t} / \psi(t) \right]}{\mathbb{E}_x \left[ e^{\psi^h_t} / \psi(t) \right] / \psi(t)} = \lim_{t \to \infty} \frac{(\psi(t-s)/\psi(t)) \mathbb{E}_x \left[ e^{\psi^h_{t-s}} \right] / \psi(t-s)}{\mathbb{E}_x \left[ e^{\psi^h_t} / \psi(t) \right]}.
\]

By (4.7) and (4.8), the right hand side equals

\[
\frac{(v, h)(\mu, h)}{(h(x)(\mu, h))} = \frac{1}{h(x)} \mathbb{E}_x \left[ Z e^{\psi^h_t} h(X_s) \right] = \mathbb{E}_x^h[Z]. \quad (4.9)
\]

**Remark 4.5.** We suppose that \( d > \alpha \) and \( \lambda(0) = 1 \). If \( d > 2\alpha \), then \( h \in L^2(\mathbb{R}^d) \) on account of (3.6). Hence \( M^h \) is an ergodic process with the invariant probability measure \( h^2 dx \), and thus for a smooth function \( k \) with compact support,

\[
\frac{\psi(t)}{t} = \frac{1}{t} \mathbb{E}_x^h \left[ \int_0^t k(X_s)ds \right] \to \int_{\mathbb{R}^d} gh^2 dx.
\]

Hence we see that for \( \mu \in \mathcal{N}^h \infty \)

\[
\lim_{t \to \infty} \frac{1}{t} \mathbb{E}_x \left[ e^{\psi^h_t} \right] = h(x)(\mu, h). \quad (4.10)
\]

**References**

[1] Albeverio, S., Blanchard, P., Ma, Z.M.: Feynman-Kac semigroups in terms of signed smooth measures, in "Random Partial Differential Equations" ed. U. Hornung et al., Birkhäuser, (1991). [MR1185735]

[2] Brancovan, M.: Fonctionnelles additives speciales des processus recurrents au sens de Harris, Z. Wahrsch. Verw. Gebiete. 47, 163-194 (1979). [MR0523168]

[3] Chen, Z.-Q., Fitzsimmons, P.J., Takeda, M., Ying, J., Zhang, T.S: Absolute continuity of symmetric Markov processes, Ann. Probab. 32, 2067-2098 (2004). [MR2073186]
[4] Fitzsimmons, P.J., Absolute continuity of symmetric diffusions, Ann. Probab. 25, 230-258 (1997). MR1428508

[5] Fukushima, M., Oshima, Y., Takeda, M.: Dirichlet Forms and Symmetric Markov Processes, Walter de Gruyter, Berlin (1994). MR1303354

[6] Neveu, J.: Potentiel Markovien recurrent des chaines de Harris. Ann. Inst. Fourier 22, 85-130 (1972). MR0380992

[7] Rogers, L., Williams, D.: Diffusions, Markov Processes, and Martingales, Vol. 2, John Wiley (1987). MR0921238

[8] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, Third edition, Springer-Verlag, Berlin (1999). MR1725357

[9] Roynette, B., Vallois, P., Yor, M.: Some penalisations of the Wiener measure. Jpn. J. Math. 1, 263-290 (2006). MR2261065

[10] Roynette, B., Vallois, P., Yor, M.: Limiting laws associated with Brownian motion perturbed by normalized exponential weights. I. Studia Sci. Math. Hungar. 43, 171-246 (2006). MR2229621

[11] Stollmann, P., Voigt, J.: Perturbation of Dirichlet forms by measures, Potential Analysis 5, 109-138 (1996). MR1378151

[12] Takeda, M.: Exponential decay of lifetimes and a theorem of Kac on total occupation times, Potential Analysis 11, 235-247, (1999). MR1717103

[13] Takeda, M.: Large deviations for additive functionals of symmetric stable processes. J. Theoret. Probab. 21, 336-355 (2008). MR2391248

[14] Takeda, M., Tsuchida, K.: Differentiability of spectral functions for symmetric $\alpha$-stable processes, Trans. Amer. Math. Soc. 359, 4031-4054 (2007). MR2302522

[15] Takeda, M., Uemura, T.: Subcriticality and gaugeability for symmetric $\alpha$-stable processes, Forum Math. 16, 505-517 (2004). MR2044025

[16] Yano, K., Yano, Y., Yor, M.: Penalising symmetric stable Lévy paths, J. Math. Soc. Japan. 61, 757-798 (2009). MR2552915