Distillability and partial transposition in bipartite systems

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We study the distillability of a certain class of bipartite density operators which can be obtained via depolarization starting from an arbitrary one. Our results suggest that non-positivity of the partial transpose of a density operator is not a sufficient condition for distillability, when the dimension of both subsystems is higher than two.

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I. INTRODUCTION

Maximally entangled states represent an essential ingredient in most applications of Quantum Information (QI) 1. In particular, in Quantum Communication one can use them for transmitting secret messages between two locally separated parties 2. In practice, however, states are mixed due to the interaction with the environment, and are not usable for those applications, even though they may be entangled. The solution to this problem was presented by Bennett et al., Deutsch et al., and Gisin 3–5, who have given a procedure to “distill” maximally entangled states of two qubits out of a set of pairs in a standard form that is characterized by a single parameter. They analyze some properties of those operators, and show that for any given finite number of copies there are density operators ρ for which one cannot find a subspace of dimension 2 in Alice and Bob’s Hilbert spaces in which ρ still has NPPT. We also present some numerical evidence that indicates that this class of states is independent of the number of copies. All these results suggest that there exist states with a NPPT and which are not distillable.

This work is organized as follows: In Section II we review some of the present knowledge concerning distillability and entanglement, and we introduce the definitions and properties that are needed in order to study the problem of distillability of general density operators. In Section III we concentrate on the case in which Alice and Bob have three–level systems, whereas in Section IV we generalize our results to the d–level system case. In Section V we show the basics of the numerical procedure used to study the distillability of 2 and 3 copies. Finally, we summarize our results.

II. ENTANGLEMENT AND DISTILLABILITY

We consider two parties, Alice and Bob, who share several pairs of particles. Each pair is in a state described by the same density operator ρ. We will assume that Alice’s (Bob’s) particles are dA–level systems (dB); that is, the density operator ρ acts on the Hilbert space CdA ⊗ CdB. We will denote by {1}, {2}, . . . , {dA} an orthonormal basis in CdA and analogously for CdB. We will also use the notation |i, j⟩ ≡ |i⟩A ⊗ |j⟩B.

We will assume that Alice and Bob are able to manipulate their particles by only using local actions (operators and measurements) and classical communication. In this case, we say that the density operator ρ is distillable if they can produce a maximally entangled state

|Φd⟩ = 1 √ d d i=1 |i, i⟩,  

(1)
where \( d = \min(d_A, d_B) \). On the other hand, we say that \( \rho \) is separable if it can be prepared out of a product state (e.g., \([1,1]\)).

In this section we will review some of the results derived by Peres and the Horodecki family concerning distillability and entanglement, and will introduce the definitions and properties that are needed in order to study the problem of distillability of general density operators.

**A. Partial transposition**

As shown by Peres \([10]\) and the Horodeckis \([8,11,12]\), the partial transpose of a density operator plays an important role in establishing its distillability and entanglement properties. In general, given an operator \( X \) acting on \( \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \), we define the partial transpose of \( X \) with respect to the first subsystem in the basis \( \{|1\rangle, |2\rangle, \ldots, |d_A\rangle\} \), \( X^{T_A} \), as follows:

\[
X^{T_A} \equiv \sum_{i,j=1}^{d_A} \sum_{k,l=1}^{d_B} (i, k | X | j, l) | j, k \rangle \langle i, l |.
\]

In the following we will use a property of this operation, namely \( \text{tr}(YX^{T_A}) = \text{tr}(Y^{T_A}X) \).

We say that a self-adjoint operator \( X \) has a non-positive partial transpose (NPPT) if \( X^{T_A} \) is not positive; that is, if there exist some \(|\Psi\rangle\) such that \( \langle \Psi | X^{T_A} | \Psi \rangle < 0 \). The positivity of the operator \( \rho^{T_A} \) gives necessary criteria for separability and non-distillability of a density operator \( \rho \). In particular: (1) If \( \rho \) is separable, then \( \rho^{T_A} \geq 0 \) \([10]\); (2) If \( \rho^{T_A} \geq 0 \) then \( \rho \) is not distillable \([12]\). These two necessary conditions turn out to be sufficient for \( d_A = 2 \) and \( d_B \leq 3 \) \([8]\). However, it has been shown that the first condition is not sufficient for separability for the rest of the cases \( (d_A = 2 \text{ and } d_B > 3, \text{ and } d_A, d_B > 2) \) \([12]\). On the other hand, nothing is known about whether the second condition is also sufficient for non-distillability in these cases.

**B. Distillability**

The problem of distillability of general density operators acting on \( \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \) can be expressed in a simpler form \([13]\). A density operator is distillable iff for certain positive integer \( N \), we can find a state of the form

\[
|\Psi\rangle = a|e_1\rangle_A |f_1\rangle_B + b|e_2\rangle_A |f_2\rangle_B,
\]

such that

\[
\langle \Psi | (\rho^{\otimes N})^{T_A} | \Psi \rangle < 0.
\]

Here, \( \{|e_1\rangle_A, |e_2\rangle_A\} \) are two orthonormal vectors in \( (\mathbb{C}^{d_A})^{\otimes N} \), and \( \{|f_1\rangle_B, |f_2\rangle_B\} \) are two orthonormal vectors in \( (\mathbb{C}^{d_B})^{\otimes N} \). This condition basically means that if Alice and Bob share \( N \) pairs, one just has to find a two-dimensional subspace in the whole Hilbert space of Alice, and another in Bob’s such that the projection of \( \rho^{\otimes N} \) in such subspaces has NPPT. The reason is that if one finds such a subspace, then according to what was exposed in the previous subsection one can distill a maximally entangled state in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \), which can be converted into a maximally entangled state in \( \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \). Conversely, if one can create one of those states then one can also produce one in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \), and therefore this ensures that \( N \) must be fulfilled.

Thus, in practice, one can analyze for each number of copies \( N = 1, 2, \ldots \) whether the condition (4) is fulfilled. In order to facilitate this task, we will use the following definitions: If for a given \( N \) condition (4) is fulfilled we will say that \( \rho \) is \( N \)-distillable. On the other hand, if for a certain \( N \) there does not exist any \(|\Psi\rangle\) satisfying Eq. (4), then we will say that \( \rho \) is \( N \)-undistillable. Thus, \( \rho \) is distillable iff there exists an \( N \) for which it is \( N \)-distillable. Conversely, \( \rho \) is non-distillable iff it is \( N \)-undistillable \( \forall N \).

**C. Distillability in \( \mathbb{C}^2 \otimes \mathbb{C}^d \)**

With the properties and definitions given above, one can very easily prove that when \( d_A = 2 \) and \( d_B \geq 2 \) and if \( \rho \) has a NPPT then \( \rho \) is 1-distillable. The reason is that there exists some \(|\Psi\rangle\) such that \( \langle \Psi | \rho^{T_A} | \Psi \rangle < 0 \). On the other hand, since \( d_A = 2 \) then the Schmidt decomposition of \(|\Psi\rangle\) has at most two terms, and therefore can be written in the form (4). Thus, in this case non-positive partial transpose of \( \rho \) is a necessary and sufficient condition for distillability.

**D. Depolarization in \( \mathbb{C}^d \otimes \mathbb{C}^d \)**

In this subsection we will introduce some superoperators which will be useful to study if a given density operator is \( N \)-distillable. We will also show that given a density operator one can reduce it to a standard form which is characterized by a single parameter \([13]\), and that preserves the distillability properties of the original state.

Let us first define some useful projector operators. Given two quantum systems with corresponding Hilbert spaces \( \mathbb{C}^d \), we denote by \( \Pi_d \) the permutation operator, and by

\[
A_d = (1 - \Pi_d)/2, \quad S_d = 1 - A_d = (1 + \Pi_d)/2, \quad \text{tr}(S_d) = d(d + 1)/2 \text{ and } \text{tr}(A_d) = d(d - 1)/2.
\]

We also define the projector operators
\[ P_d = |\phi_d\rangle\langle \phi_d|, \quad Q_d = \mathbf{1} - P_d, \] (6)

where \(|\phi_d\rangle\) is the maximally entangled state defined in [1]. One can easily check that

\[ P_d^{T_A} = \frac{1}{d}(1 - 2A_d), \quad A_d^{T_A} = \frac{1}{2}(1 - dP_d). \] (7)

We define the depolarization superoperator \(\mathcal{D}\), acting on any given operator \(X\), as follows:

\[ \mathcal{D}(X) = A_d \frac{\text{tr}(A_d X)}{\text{tr}(A_d)} + S_d \frac{\text{tr}(S_d X)}{\text{tr}(S_d)}. \] (8)

This superoperator is a projector, is self-adjoint (on the Hilbert–Schmidt space of operators acting on \(\mathcal{C}^d \otimes \mathcal{C}^d\)), and preserves the trace. In the Appendix A we show that we can write

\[ \mathcal{D}(X) = \int d\mu_U(U \otimes U)X(U \otimes U)\dagger \] (9)

where the integral is extended to all unitary operators acting on \(\mathcal{C}^d\) and \(\int d\mu_U = 1 [\mu_U\) represents the standard invariant Haar measure on the group \(SU(d)\)]. We will later on use the partial transpose of \(\mathcal{D}(X)\), and to this aim we define the following superoperator

\[ \mathcal{E}(X) \equiv [\mathcal{D}(X^{T_A})]^{T_A} = \int d\mu_U(U^* \otimes U)X(U^* \otimes U)^\dagger \] (10a)

\[ = P_d \text{tr}(P_d X) + Q_d \frac{\text{tr}(Q_d X)}{\text{tr}(Q_d)}. \] (10b)

where \(U^*\) denotes complex conjugation in the basis in which the partial transposition is defined. This superoperator is also a projector, self-adjoint, and preserves the trace. Note that for any unitary operator \(V\) acting on \(\mathcal{C}^d\) we have

\[ (V \otimes V)\mathcal{D}(X)(V \otimes V)^\dagger = \mathcal{D}(X), \] (11a)

\[ (V^* \otimes V)\mathcal{E}(X)(V^* \otimes V)^\dagger = \mathcal{E}(X). \] (11b)

The form [3] shows that the superoperators \(\mathcal{D}\) and \(\mathcal{E}\) can be implemented by means of local operations. In particular, it shows that any density operator \(\rho\) can be transformed, using local operations, to the form

\[ \mathcal{D}(\rho) = \rho_a = \frac{1}{N(\alpha)}(S_d + \alpha A_d) \] (12)

where \(\alpha\) is such that \(\text{tr}(A_d \rho_a) = \text{tr}(A_d \rho)\) and \(N(\alpha) = \text{tr}(S_d) + \alpha \text{tr}(A_d)\) is a normalization constant. That is, one can depolarize any density operator to the one parameter family [3] while keeping the weight in the antisymmetric subspace. We will be more interested in the partial transpose of \(\rho_a\), which is given by

\[ [\mathcal{D}(\rho)]^{T_A} = \mathcal{E}(\rho^{T_A}) = \frac{1}{M(\beta)}(Q_d - \beta P_d), \] (13)

where \(M(\beta) = \text{tr}(Q_d) - \beta\) is a normalization constant, and the relationship between \(\alpha\) and \(\beta\) is \(\beta = [(\alpha - 1)(d - 1)/d + \alpha - 1] / (\alpha + 1) < d - 1\). Note that since

\[ \rho_a = \mathcal{D}(\rho_a), \quad \rho_a^{T_A} = \mathcal{E}(\rho_a^{T_A}), \] (14)

we have that

\[ (U \otimes U)\rho_a(U \otimes U)^\dagger = \rho_a, \] (15a)

\[ (U^* \otimes U)\rho_a^{T_A}(U^* \otimes U)^\dagger = \rho_a^{T_A}. \] (15b)

for any unitary operator \(U\).

Using the properties derived above, one can easily check that

\[ \rho_a\text{ is separable } \iff \rho_a^{T_A} \geq 0 \iff \beta \leq 0. \] (16)

The last equivalence follows directly from (13). For the first one we have: \((\iff)\) See subsection [1A]; \((\iff)\) We have that for \(\beta = 0\) [i.e. \(\alpha = \alpha_0 \equiv (d + 1)/(d - 1) = \text{tr}(S_d)/\text{tr}(A_d)\)]

\[ \rho_a^{T_A} \propto Q_d \propto \mathcal{E}((0, 1)(0, 1)), \] (17)

which is obviously positive and separable, in which case the same holds for \(\rho_a\). For \(\beta < 0\) (\(\alpha < \alpha_0\) we can always obtain \(\rho_a\) by adding the identity operator (which is separable) to \(\rho_{aa}\) (this is due to the fact that \(\text{tr}(S_d) > \text{tr}(A_d)\)).

Thus, for \(\beta > 0\) (\(\alpha > \alpha_0\) \(\rho_a\) is non-separable. One can easily check that this condition is equivalent to

\[ \text{tr}(A_d \rho_a) > 1/2. \] (18)

This last form allows us to show that for any given density operator \(\rho\) with NPPT, one can always transform it using local actions to the form [3] such that it still has NPPT [3]. Let us show that. Suppose that for a given \(|\Psi\rangle\), \(|\Psi|\rho A |\Psi\rangle < 0\). We can write \(|\Psi\rangle = \sum_{i=1}^{n} c_i |u_i, v_i\rangle\) where \(|u_i\rangle\rangle = 1 \text{ and } |v_i\rangle\rangle = 1\) form an orthonormal basis. The operator \(\rho\) can be transformed by local operations to \(\rho_a \propto (A_i^\dagger \otimes B_i^\dagger) \rho (A_i \otimes B_i)\), with \(\text{tr}(\rho_a^{T_A} P_n) < 0\) (by simply taking \(A = \sum_{i=1}^{n} |u_i^*\rangle\langle i|/c_i + \sum_{i=n+1}^{d} |u_i^*\rangle\langle i| \text{ and } B = \sum_{i=n+1}^{d} |v_i^*\rangle\langle i|\)). Using that \(0 > \text{tr}(\rho_a^{T_A} P_n) = \text{tr}(\rho_a P_n^{T_A})\) and [3] we immediately obtain that \(\text{tr}(A_d \rho_a) > \text{tr}(A_n \rho_a) \geq 1/2\). Since \(\mathcal{D}\) conserves this quantity, we obtain that \(\mathcal{D}(\rho_a)\) has a negative partial transpose.

As pointed out by the Horodecki [8], the problem of distillability can be reduced to the study of density operators of the form [3]. If we find that all those operators with a NPPT are distillable, we will have shown that NPPT is a necessary and sufficient condition for distillability. On the contrary, if we find that there exist an operator of the form [3] which has a NPPT, but is not distillable, we will have shown that such a condition is not sufficient. In the following Sections we will show that
there are density operators of the standard form with a NPPT which are not \(N\)-distillable for certain values of \(N\). As we have seen in subsection II we can study that by checking whether there exist vectors of the form (1) fulfilling condition (1).

III. \(N\)-DISTILLABILITY IN \(G^3 \otimes G^3\)

We consider the case \(d_A = d_B = 3\) and a density operator of the form

\[
\rho_\alpha = \frac{1}{N(\alpha)}(S + \alpha A),
\]

where we have omitted the superindices \(d = 3\), and \(N(\alpha) = 6 + 3\alpha\). According to the discussion in Subsection II we just have to consider \(\alpha \geq \alpha_0 = 2\), since otherwise \(\rho_\alpha\) is separable. We also have for the partial transpose

\[
\rho_{\beta}^{T_A} = \frac{1}{M(\beta)}(Q - \beta P)
\]

where \(\beta = (2\alpha - 4)/(\alpha + 1)\), with \(2 \geq \beta \geq 0\) and \(M(\beta) = 8 - \beta\).

A. 1-distillability

We look for a vector of the form (11) such that (1) is fulfilled. Choosing \(U\) such that \(U|e_{1,2}\rangle = |1,2\rangle\) and using the property (15b) we see that we can restrict ourselves to the subspace spanned by \(\{1,2\}\) in Alice’s Hilbert space. Defining by \(1_2\) the projection operator into this subspace and by \(1_3\) the identity operator in \(G^3\), we obtain after projecting \(\rho_{\beta}^{T_A}\) onto this subspace

\[
1_2^A \rho_{\beta}^{T_A} 1_2^B \propto 1_2^A \otimes 1_3^B \left[\frac{2(1+\beta)}{3} P_2\right]
\]

which is positive if \(\beta \leq 1/2\). Thus, we obtain that \(\rho_\alpha\) is 1-distillable iff \(\beta > 1/2\) (or, equivalently, \(\alpha > 3\)).

B. 2-distillability

Let us consider now two pairs, in a state \(\rho_\alpha\). We will show that for \(\beta \leq 1/4\) the state \(\rho_\alpha\) is 2-distillable. For any state \(|\Psi\rangle\) of the form (12) we have

\[
\langle\Psi|Q^{(1)} \otimes (Q^{(2)} - P^{(2)}/2)|\Psi\rangle = \text{tr}_2[\text{tr}_1(\langle\Psi|Q^{(1)}\rangle(Q^{(2)} - P^{(2)/2})]\]

where the superscripts 1 and 2 refer to the first and second pair, respectively. Using the fact that \(Q\) is separable, and therefore that it can be written as \(Q = \sum_ic_i|a_i,b_i\rangle\langle a_i,b_i|\) with \(c_i > 0\), we have

\[
\text{tr}_1(|\Psi\rangle\langle\Psi|Q^{(1)}) = \sum_i c_i|\Psi_i\rangle\langle\Psi_i|
\]

where \(|\Psi_i\rangle = \langle a_i,b_i|\Psi\rangle\) is a state acting on the second pair which itself has the form (11). Thus, according to the results of the previous subsection we have that \(\langle\Psi_i|((Q^{(2)} - P^{(2)}/2)|\Psi_i\rangle \geq 0\) and therefore \(\langle\Psi|Q^{(1)} \otimes (Q^{(2)} - P^{(2)}/2)|\Psi\rangle \geq 0\). In the same way we have that \(\langle\Psi|Q^{(1)} - P^{(1)}/2 \otimes Q^{(2)}|\Psi\rangle \geq 0\), i.e.

\[
0 \leq \langle\Psi|Q^{(1)} - P^{(1)}/4 \otimes (Q^{(2)} - P^{(2)}/4) - P^{(1)} \otimes P^{(2)}/16|\Psi\rangle.
\]

Using the fact that \(P^{(1)} \otimes P^{(2)} \geq 0\) we obtain the desired result. Note that our results do not imply that for \(1/4 \leq \beta < 1/2\) \(\rho_\alpha\) is 2-distillable. In fact, as shown in the next section, numerical calculations indicate that it is 2-distillable.

C. \(N\)-distillability

We consider now \(N\) pairs, in a state \(\rho_\alpha\). We will show that for \(\beta \leq 4^{-N}\) the state \(\rho_\alpha\) is \(N\)-distillable. For any state \(|\Psi\rangle\) of the form (12) one can check the following relations: (i) \(\langle\Psi|P^{(k)}|\Psi\rangle \leq \frac{1}{N}\), (ii) \(\langle\Psi|Q^{(N-k)} \otimes P^{(k)}|\Psi\rangle \leq \frac{1}{N}\) and (iii) \(\langle\Psi|Q^{(N)}|\Psi\rangle \geq 1\). To show (i) one uses \(P^{(k)} = P_d\) and the property (15b), from which follows that the projection into the subspace spanned by \(\{1,2\}\) gives the maximum value for \(\langle\Psi|P^{(k)}|\Psi\rangle\). From (i) we immediately obtain (ii) by using that \(\langle\Psi|Q^{(N-k)} \otimes P^{(k)}|\Psi\rangle \leq \langle\Psi|P^{(k)} |\Psi\rangle\) for all positive operators \(X\). Relation (iii) can be obtained by using (i) and the separability of \(Q\) in a similar way as in the previous Section. Combining (ii) and (iii) we find

\[
\langle\Psi|a_k Q^{(N)} - \sum_{\text{perm}} Q^{\otimes N-k} \otimes P^{(k)}|\Psi\rangle \geq 0,
\]

with \(a_k = 2(N-k)/3\) and the sum runs over all possible permutations of the pairs. By summing (22) for all odd \(k\) and using that \(\sum_{k \text{ odd}} a_k = 4^N - 2^N \equiv \beta_1^N\), one finds for \(\beta_N \leq \beta_1^N\)

\[
0 \leq \langle\Psi|Q^{(N)} - \beta_1^N \sum_{k \text{ odd}} \sum_{\text{perm}} Q^{\otimes N-k} \otimes P^{(k)}|\Psi\rangle
\]

\[
\leq \langle\Psi|Q^{(N)} - \sum_{k \text{ odd}} \beta_1^N \sum_{\text{perm}} Q^{\otimes N-k} \otimes P^{(k)}|\Psi\rangle
\]

\[
\leq \langle\Psi|(Q - \beta_1^N P)^{\otimes N}|\Psi\rangle.
\]

We used that \(\sum_{k \text{ odd}} \beta_1^N \leq \beta_1^N\) (for \(\beta_N \leq \beta_1^N \leq 1\)) to obtain the first inequality (line 2), while we added all positive terms (even \(k\)) in the second step (line 3). This is already the desired bound, i.e. for \(0 \leq \beta \leq 4^{-N} \leq \beta_1^N \equiv \frac{1}{N}\), the state \(\rho_\alpha\) is \(N\)-distillable. Again, this does not mean that \(\rho_\alpha\) is \(N\)-distillable for \(4^{-N} \leq \beta \leq 1/2\).

In the Appendix C we present a better bound for \(\beta\).
IV. N–DISTILLABILITY IN $\mathcal{G}^D \otimes \mathcal{G}^D$

We consider the case $d_A = d_B = d$ and a density operator of the form \( \rho = \sum_i c_i |i\rangle \langle i| \) with the partial transposition given by \( \rho^T \).

Similar techniques as in the $d = 3$ case can be used to obtain bounds also for arbitrary $d$. One finds for example that $\rho_\alpha$ is 1–distillable iff $\beta > \frac{d}{2} - 1$. We also obtain that $\rho_\alpha$ is 2–undistillable for $\beta < \frac{d^2}{4} - 1$ and $N$–undistillable for $\beta < \min(\tilde{\beta}_N, \beta_{1/N}^N)$ with $\tilde{\beta}_N = \sum_{k=1}^N \frac{1}{N^k}$. Note that the minimum is required here, since differently to $d=3$ case - one can have that $\tilde{\beta}_N > 1$. In this case one has to choose $\beta > \beta_{1/N}^N$ which implies $\max_k \tilde{\beta}_N^k \leq \beta_{1/N}^N$ to ensure that the first inequality in (26) remains valid.

Furthermore, there is an interesting relation between the states $\rho_\alpha$ for different $d$. Imagine we would like to convert a single copy of a state $\rho_\alpha$ into $\mathcal{G}^k \otimes \mathcal{G}^k$, to a state in $\mathcal{G}^k \otimes \mathcal{G}^k (k < d)$ in an optimal way, i.e. to obtain a new $\alpha$ which is as large as possible. We show here that whenever we convert a state $\rho_\alpha$ to some lower dimension, there will always be some states which loose the negativity of their partial transposition. In order to prove this, we consider vectors $|\Psi_k\rangle$ with $k$ Schmidt coefficients and show that $\langle \Psi_k | \rho_\alpha^T | \Psi_k \rangle < 0$, while $\langle \Psi_k | \rho_\alpha^T | \Psi_k \rangle > 0 \forall |\Psi_k\rangle$. Due to the property (15), one can restrict oneself to the subspace spanned by $\{|1\ldots k\rangle\}$. Let us denote the identity operator in this subspace by $I_k$. One finds after projecting $\rho_\alpha^T$ onto $I_k$ in $A$ and $B$

$$I_k^A \otimes I_k^B \rho_\alpha^T I_k^A \otimes I_k^B \propto I_k^A \otimes I_k^B \left( k(1 + \beta) \right) \left( -1 \right) \left( \frac{1}{d} \right) P_k$$

which is positive iff $\beta < \frac{d}{4} - 1$, while $\rho_\alpha^T$ before the projection was positive iff $\beta \leq 0$. Thus all states with $0 < \beta < \frac{d}{4} - 1$ lose the negativity of their partial transposition after the optimal projection onto a $k$-dimensional subspace. The new $\beta_k$ can be calculated from $\beta_d$ of the initial state by $\beta_k = \frac{k}{d}(\beta_d + 1) - 1$.

Finally, let us consider $N$ copies of $\rho_\alpha$ of dimension $d$, which can be viewed as a state in $\mathcal{G}^{N \otimes d^N} \otimes \mathcal{G}^{N \otimes d^N}$. With $\text{tr}(A \rho_\alpha) = \lambda_d$, one finds that the state $D(\rho_\alpha^N) \equiv \tilde{\rho}_\alpha$ - the state in the high dimensional Hilbert space after depolarization - has $\lambda_d = \frac{1}{N} \text{tr}(A \rho_\alpha^N) = \frac{1 - (1 - 2\lambda_d)^N}{2}$.

One checks that for $\lambda_d > 1/2$ (i.e. $\rho_\alpha$ is inseparable) we have $\lambda_d, \lambda_d < \lambda_d \forall N$, which simply means that the weight in the antisymmetric subspace decreases when going to more copies. Note that using this notation, we have that the state $\rho_\alpha$ is separable for $\lambda \leq 1/2$, while it is 1-distillable for $\lambda > \frac{3(d-1)}{2(d-1)}$, which tends to $\frac{3}{2}$ for $d \to \infty$.

V. NUMERICAL PROCEDURES

In general, we are interested in showing there exists a $|\Psi\rangle$ of the form $|\Psi\rangle = \sum_i c_i |i\rangle \langle i|$ for which condition (4) is fulfilled, i.e.

$$\lambda \equiv \langle \Psi | R | \Psi \rangle < 0, \quad (28)$$

where we have defined $R = (Q - \beta P)^\otimes N$ with $0 < \beta \leq 1/2$. In order to check that, we can minimize $\lambda$ with respect to $|e_{1,2}\rangle$, $|f_{1,2}\rangle$ and $a$ while keeping the normalization and orthogonality relations. One can readily check that the minimization implies

$$(e_{1} | R | e_{1}) = \lambda_0 a | f_1 \rangle| f_1 \rangle,$$

$$(e_{2} | R | e_{2}) = \lambda_0 b | f_2 \rangle| f_2 \rangle,$$

$$(f_1 | R | e_{1}) = \lambda_0 a | e_1 \rangle| e_1 \rangle,$$

$$(f_2 | R | e_{2}) = \lambda_0 b | e_2 \rangle| e_2 \rangle.$$
VI. CONCLUSIONS

We have shown that in order to study the distillability properties of bipartite $d$-level systems, it is sufficient to consider only the one-parameter class of states $\rho_\alpha$ (12). By investigating the distillability properties of this family of states, we found strong indications that this family provides examples for non-distillable states with non-positive partial transposition. In particular, we found that for any given number of copies $N$ there exist $N$-undistillable states which have NPPT. Guided by the results of the numerical investigations, we conjecture that for $d = 3$ the states $\rho_\alpha$ are non-distillable for $\beta \leq 1/2$, while they have NPPT for $\beta > 0$ (see also Fig. I).

Note added: After completing this work we became aware of the results of D. P. DiVincenzo et al. [21], in which they also found evidences for the existence of non-distillable states with NPPT.

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APPENDIX A: INTEGRAL REPRESENTATION OF $\mathcal{D}(X)$

We show here that the superoperator $\mathcal{D}$ as defined in (8) can also be written in the form (1). We restrict ourselves here to operators $X$ which are density operators $\rho$ for convenience, but exactly the same line of arguments holds for arbitrary self adjoint-operators $X$. As shown in Appendix B, we have that the depolarization superoperator $\mathcal{D}$ can be implemented by a finite sequence of bi-local operations (13) of the form $U \otimes U$. Furthermore we have that the projector onto the antisymmetric subspace is invariant under unitary operations of the form $U \otimes U$, i.e. $U \otimes U A_d U^\dagger \otimes U^\dagger = A_d$, which can be easily seen by using that $A_d = (1 - \Pi_d)/2$. From this property automatically follows that also $\rho_\alpha$ (12) is invariant under unitary operations of the form $U \otimes U$, since $\rho_\alpha \propto (1 + \alpha A_d)$. It is now straightforward to show (1) by using for any $V$ with $U'V = U$. Taking $p_k$ such that $\sum_k p_k = 1$ we can write

$$\int d\mu_U(U \otimes U)\rho(U \otimes U)^\dagger =$$

$$\sum_k p_k \int d\mu_{U'}(U' \otimes U')(U_k \otimes U_k)\rho(U_k \otimes U_k)^\dagger(U' \otimes U')^\dagger =$$

$$\int d\mu_U(U' \otimes U')\rho_\alpha(U' \otimes U')^\dagger = \rho_\alpha = \mathcal{D}(\rho),$$

(36)

where we used (13) in the first equality, while the second equality follows from (37) and we finally used the invariance of $\rho_\alpha$ under operations of the form $U \otimes U$. This already shows that (1) is fulfilled, i.e. $\mathcal{D}(\rho) = \int d\mu_U(U \otimes U)\rho(U \otimes U)^\dagger$.

APPENDIX B: DEPOLARIZATION

We are going to show now that an arbitrary state $\rho$ can be depolarized to the standard form (12) by a finite sequence of bi-local (random) operations without changing the weight in the antisymmetric subspace $\mathcal{H}_a(d)$, i.e we show that there exist unitary operators $U_k$ and probabilities $p_k$ such that

$$\sum_k p_k U_k \otimes U_k \rho U_k^\dagger \otimes U_k^\dagger = \mathcal{D}(\rho) = \rho_\alpha$$

(37)

with $\text{tr}(A_d \rho) = \text{tr}(A_d \rho_\alpha)$. We first introduce a basis of $\mathcal{C}^d \otimes \mathcal{C}^d$:

$$|\varphi_{ij}^+\rangle = \frac{1}{\sqrt{2}}(|i\rangle_A |j\rangle_B \pm |j\rangle_A |i\rangle_B)$$

$$|\chi_k\rangle = |k\rangle_A |k\rangle_B.$$  

(38)

with $i < j$ and $(i, j, k) \in \{1, ..., d\}$. Note that $|\varphi_{ij}^+\rangle$ provides a basis of the antisymmetric subspace $\mathcal{H}_a(d)$, while $\{|\varphi_{ij}^+\rangle, |\chi_k\rangle\}$ is a basis of the symmetric subspace $\mathcal{H}_s(d)$. The projectors into the symmetric/antisymmetric subspace can thus be written as

$$A_d = \sum_{i, j = 1(i<j)}^d |\varphi_{ij}^+\rangle \langle \varphi_{ij}^+|$$

$$S_d = \sum_{i, j = 1(i<j)}^d |\varphi_{ij}^+\rangle \langle \varphi_{ij}^+| + \sum_{k=1}^d |\chi_k\rangle \langle \chi_k|.$$  

(39)

Let us write $\rho$ in the basis (38). In order to prove the statement (17), we devide the depolarization procedure into three steps:

- we show that one can make $\rho$ diagonal in the basis (38) without changing the diagonal elements.
- we prove that the antisymmetric subspace $\mathcal{H}_a(d)$ can be mixed up, i.e. one can equalize the coefficients of $|\varphi_{ij}^+\rangle, |\varphi_{ij}^-\rangle$ without changing the weight in $\mathcal{H}_a(d)$.
• finally we show that also the symmetric subspace can be completely mixed up without changing the weight in $\mathcal{H}_a(d)$.

These three steps together ensure that $\rho$ can be depolarized to the standard form $\rho_a$.

A. Diagonalizing $\rho$

By mixing we understand in the following that a certain operation $U$ is (randomly) performed with probability $p$ by Alice and Bob, while with probability $1-p$ no operation is performed. The resulting density operator after this mixing–operation reads

$$\rho_{\text{new}} = p(U \otimes U \rho U^\dagger \otimes U^\dagger) + (1-p)\rho.$$  

We define the operation $U_l$ as follows:

$$U_l |k\rangle = e^{i\pi \delta_{kl}} |k\rangle,$$  

i.e. the state $|l\rangle$ picks up a minus sign while all others remain unchanged. Let us perform a sequence of $d$ mixing operations, using $p = \frac{1}{2}$ and $U_l$ with $l = \{1,\ldots,d\}$ respectively. One can easily check that all diagonal elements remain unchanged, while all off–diagonal elements of the form $|\varphi_{ij}^+\rangle\langle \chi_k|$ and $|\varphi_{ij}^-\rangle\langle \chi_{j'}|$ for $(i \neq i', j \neq j')$ are eliminated.

Let us define now the operation $\tilde{U}_l$, which introduces a phase $i$ for the state $|l\rangle$ while it leaves all other states unchanged. Performing again the same sequence of $d$ mixing operations as described above, but with $U_l$ replaced by $\tilde{U}_l$, one can check that all off–diagonal elements of the form $|\chi_k\rangle\langle \chi_{j'}|$ are eliminated.

We finally define the operation $U_{kl}$, which simply swaps the states $|k\rangle$ and $|l\rangle$ while leaving all other states untouched. Performing now a sequence of mixing operations using all possible combinations $k < l \in \{1,\ldots,d\}$ for $U_{kl}$ and $P = 1/2$, one gets rid of the remaining off–diagonal elements of the form $|\varphi_{ij}^-\rangle\langle \varphi_{ij}^-|$ without changing the diagonal ones. Thus $\rho$ can be made diagonal in the basis $|\chi_k\rangle$ by a sequence of bi–local operations.

B. Mixing of $\mathcal{H}_{a}(d)$

Let $\mathcal{H}_{io}$ be the subspace spanned by $\{|\varphi_{00}^-\rangle\}$ for fixed $i_0$. In a first step we will show that one can depolarize all subspaces $\mathcal{H}_{io}$ independently, while in the second step we are going to prove that these subspaces can be mixed with each other.

To depolarize $\mathcal{H}_1$, one just has to keep the state $|1\rangle$ and performs (randomly) one of the cyclic permutations of the states $\{|2\rangle,\ldots,|d\rangle\}$ each with probability $p = \frac{1}{d-1}$. Similarly, one depolarizes $\mathcal{H}_2$ by keeping the states $\{|1\rangle,|2\rangle\}$ and performing with probability $p = \frac{1}{d-2}$ one of the cyclic permutations of the states $\{|3\rangle,\ldots,|d\rangle\}$. Since $\mathcal{H}_1$ is already depolarized, it is not affected by this operation. One can continue in the same way until one has depolarized $\mathcal{H}_{d-1}$. So the antisymmetric part of the density operator has now the form

$$A_d \rho A_d = \sum_{i=1}^{d-1} a_i \sum_{j=j+1}^d |\varphi_{ij}^-\rangle\langle \varphi_{ij}^-|.$$  

The second step starts by mixing of $\mathcal{H}_{d-1}$ with $\mathcal{H}_{d-2}$, i.e. equalizing the coefficients $a_{d-1}$ and $a_{d-2}$. To achieve this, both Alice and Bob swap the states $|d-1\rangle$ and $|d-2\rangle$ with probability $p_2 = \frac{2}{d}$, or both apply the identity operator with probability $1-p_2$. If one now depolarizes $\mathcal{H}_{d-1}$ and $\mathcal{H}_{d-2}$ independently as described in step 1, one finds that $|\varphi_{(d-1)j}\rangle\langle \varphi_{(d-1)j}|$ and $|\varphi_{(d-2)j}\rangle\langle \varphi_{(d-2)j}|$ have all the same weight now, i.e. the coefficients are equal. Thus the subspace $\mathcal{H}_{d-1}$ is completely mixed up with $\mathcal{H}_{d-2}$. One now continues by mixing the subspaces $\mathcal{H}_{d-3}$ with $\mathcal{H}_{d-2}$ and $\mathcal{H}_{d-1}$ and so on, until one reaches $\mathcal{H}_1$.

We investigate now one particular step in this procedure, namely the mixing of $\mathcal{H}_k$ with $\{\mathcal{H}_{k+1},\mathcal{H}_{k+2},\ldots,\mathcal{H}_{d-1}\}$. Both Alice and Bob swap the states $|k\rangle$ and $|k+1\rangle$ with probability $p_2 = \frac{2}{d-k}$ or both apply the identity operator with probability $1-p_2$. After this one depolarizes $\mathcal{H}_j (j \leq k)$ independently, then mixes $\mathcal{H}_j$ with $\mathcal{H}_{d-2}$ as described above. Next one mixes $\mathcal{H}_{d-3}$ with $\{\mathcal{H}_{d-2},\mathcal{H}_{d-1}\}$ and continues in this way until one has mixed $\mathcal{H}_k$ and $\{\mathcal{H}_{k+1},\mathcal{H}_{k+2},\ldots,\mathcal{H}_{d-1}\}$. It can be checked that after this procedure all weight factors $a_j$ are equal for $(j \leq k)$.

So once one has mixed $\mathcal{H}_1$ with $\{\mathcal{H}_2,\mathcal{H}_3,\ldots,\mathcal{H}_{d-1}\}$, one has achieved that the whole antisymmetric subspace is completely depolarized, i.e. it can be written as $a_d A_d$.

C. Mixing of $\mathcal{H}_a(d)$

Note that the depolarization of the antisymmetric subspace also mixes the subspace spanned by $\{|\varphi_{ij}^+\rangle\}$ in a similar way. Here we show now that one can also depolarize the subspace spanned by $\{|\chi_k\rangle\}$ and finally that also these two subspaces together can be mixed up. To prove the first step, let us define the operation $\tilde{U}_l$ as follows:

$$\tilde{U}_l |k\rangle = |(k+1) \mod d\rangle >.$$  

Performing now $\tilde{U}_l \otimes \tilde{U}_l, l = \{1,\ldots,d\}$ each with probability $p = \frac{1}{d}$ ensures that the subspace spanned by $\{|\chi_k\rangle\}$ is completely depolarized, i.e. that $\rho$ has now the form

$$\rho = a A_d + b \sum_{i,j=1 \text{ (}i<j\text{)}}^d |\varphi_{ij}^+\rangle\langle \varphi_{ij}^+| + c \sum_{k=1}^d |\chi_k\rangle\langle \chi_k|.$$  

(44)

For the second step, we define the unitary operation $T$:
\[ T|j\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} e^{i2\pi(j-1)(k-1)/d}|k\rangle. \tag{45} \]

One can check that if we perform the operation \( T \otimes T \) with probability \( p_T = \frac{d}{d+1} \) and the identity operation with probability \( p_I = 1-p_T \), the diagonal elements of the symmetric subspace will each be identical to \( \frac{d}{d+1} + \frac{2d}{d+1} \). The introduced off-diagonal elements can be eliminated using the procedure explained above. Note that the antisymmetric subspace is not affected by this kind of operations and will thus remain untouched. So finally we managed to show that \( p \) can be converted to the standard form \( [12] \) by a sequence of local operations \( [17] \). The weight in \( \mathcal{H}_\alpha(d) \) was not affected by any of the used mixing operations, which ensures that \( \text{tr}(A_d \rho) = \text{tr}(A_d \rho_\alpha) \).

**APPENDIX C: BETTER BOUND FOR \( \beta \)**

In this Appendix we prove that for any vector \(|\Psi\rangle\) of the form \( [3] \)
\[ \langle \Psi|(Q - \beta P)^{\otimes N}|\Psi\rangle \geq \frac{1}{3^N} \left( 1 - \frac{\beta}{\beta_N} \right), \tag{46} \]
for \( \beta \leq \beta_N \), where
\[ \beta_N = \frac{x_N}{3^{N/3}N^{1/3}}, \tag{47} \]
and \( x_N = x^*(1 - O(1/N)) \) as \( N \to \infty \), where \( x^* = 3(1 - 3^{-1/3})^{1/3} \), and \( O(1/N) \) denotes a quantity of the order of \( 1/N \).

The proof is by induction. For \( N = 1 \), we have \( \beta_1 = 1/2 \). Assuming that the statement holds for \( N - 1 \), we observe that
\[ \langle \Psi|(Q - \beta P)^{\otimes N-1}|\Psi\rangle \geq \frac{1}{3^N} \left( 1 - \frac{\beta}{\beta_{N-1}} \right), \tag{48} \]
and the same holds for all possible permutations with respect to the copies. Adding LHS’s, dividing the sum by \( N \), and rearranging various terms, we find for large \( N \)
\[ \langle \Psi|(Q - \frac{N-1}{N} \beta P)^{\otimes N}|\Psi\rangle \geq \frac{1}{3^N} \left( 1 - \frac{\beta}{\beta_{N-1}} \right) - \frac{N\beta^3}{27} (1 + O(1/N)) - O(\beta^5). \tag{49} \]
For \( \beta \simeq O(\beta_N) \), the last correction term in the above inequality can be safely neglected. It is easy to check that the right hand side has zero at \( \beta = \beta_N \), i.e. that
\[ \langle \Psi|(Q - \beta P)^{\otimes N}|\Psi\rangle \geq \frac{1}{3^N} \left( 1 - \frac{\beta}{\beta_N} \right), \tag{50} \]
for \( \beta \leq \beta_N \); the statement holds thus for every \( N \).

The above result provides a better bound for \( N \)-undistillability: the states \( \rho_\alpha \) such that
\[ \beta \leq \frac{x^*}{3^{N/3}N^{1/3}(1 - O(1/N))} \tag{51} \]
are \( N \)-undistillable.

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