Shelving and probe efficiency in trapped ion experiments

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Abstract

A generalized probe sequence typical of trapped ion experiments using shelving is studied. Detection efficiency is analyzed for finite shelved state lifetimes and using multi-modal count distributions. Multi-modal distributions are more appropriate for measurements that use a small number of ions than the simple Poisson counting statistics usually considered and have a larger variance that may be significant in determining uncertainties and in making weighted fits. Optimal probe times and the resulting state detection efficiency and sensitivity are determined for arbitrary cooling rates, initial states, and shelved state lifetimes in terms of a probe coherence time $\tau_p$. A universal optimal probe time of $t_p \approx 0.43\tau_p$ is shown to give close to the optimal probe sensitivity for most systems.

Keywords: ion trap, coherent control, quantum jumps, pump/probe

(Some figures may appear in colour only in the online journal)

1. Introduction

Experiments using trapped ions often provide a superior system for making precision measurements in applications such as determining atomic structure parameters [1], developing microwave or optical atomic clocks [2, 3], searching for time variations of fundamental constants [3, 4], or making precision measurements of atomic parity violation [5–7]. The precision possible in such experiments is fundamentally due to the intrinsic properties of a trapped ion system through its relatively clean and isolated environment but is finally determined, as it is for all experiments, by systematics and sensitivity. Systematic issues are commonly analyzed extensively, but sensitivity has not yet been considered in the detail common to many more traditional precision atomic measurements such as parity violation in atomic systems [8, 9]. A careful consideration of the choice of certain experimental parameters can yield important improvements in sensitivity for many kinds of experiments and a corresponding improvement in the precision possible.

Trapped ion experiments typically consist of many trials of some pump, interaction, probe sequence. The pump stage prepares the ion in a particular state. During the interaction stage the ion is either actively driven by a set of applied fields or is just allowed to evolve in isolation, in both cases for the purposes of studying the dynamics of particular interest. For example, the ion may be driven by another laser beam or a radio-frequency field to characterize a resonant coupling to another state or may be allowed to evolve in the absence of all light in order to measure an excited state lifetime. The probe stage detects whether a transition out of the initially prepared state occurred during this interaction period. By measuring the probe signal as a function of drive parameters such as drive rate, time or frequency, or wait time some quantity of interest relating to the dynamics of the interaction can be determined.

The probe is usually implemented using some variation of the electron shelving method [10]. Shelving makes use of a set of cycling transitions, the probe cycle, that includes at least two states, and another state that is not part of this cycle, the shelved state. These cycling transitions and states are usually related to those that are used to Doppler cool the ions (see for example [11]). Ideally the shelved state is long-lived and well-isolated from the probe cycle, well isolated meaning that there is negligible rate or probability for transitions between this state and any part of the probe cycle when the beams that drive the cycle are applied.

A generalized level diagram of this kind of system is shown in figure 1. This kind of level diagram is qualitatively similar to that of most alkali-like ions. The $C_n$ are typically
part of a laser cooling scheme. The shelved state, S, may be a Zeeman sublevel uncoupled by angular momentum selection rules by using a particular polarization, a hyperfine manifold not part of the nominal cooling cycle, or an entirely separate long-lived state such as the $2^3S_{1/2}$ state in hydrogen. For example, in singly-charged odd isotopes of Ytterbium ions, the Doppler cooling cycle comprises the 370 nm transition from the $^2S_0(F = 1)$ ground state (state $C_1$) to the $^2P_0(F = 0)$ state (state $C_2$) with leakage to $^2D_{3/2}(F = 1)$ state ($C = 3$) in figure 1 necessitating a repump laser beam at 935 nm. The shelved state, S, in this example is the $F = 2$ hyperfine level of the D state, $^2D_{3/2}(F = 2)$, but can be the long-lived $^2F_{7/2}$ (not shown in the figure) state in any isotope.

An ion in a state that is part of the probe cycle when that cycle is driven will fluoresce at some rate $r_c$ that can be detected by a photo-multiplier tube (PMT). In addition there will be some background rate $r_b$ due to ambient light or light scattered from the laser beam driving the probe cycle. This background rate is presumed to be independent of the ion’s state, so the PMT signal for an ion in the cycle corresponds to a total rate $r_T = r_c + r_b$. An ion in the shelved state does not fluoresce when the probe cycle is driven, so the PMT signal will correspond to the background rate. The probe cycle then provides a probe of the ion’s state.

With high cooling rates, and well isolated shelved states with sufficiently long lifetimes, state detection using shelving can be done with almost perfect efficiency for almost any combination of probe parameters. The results of many trials can be interpreted directly as a transition probability. Under less ideal circumstances the probe signal requires additional information to interpret, and in such cases even limited sensitivity for state detection may require parameters like probe time to be chosen from a particular, narrow range of values. The sensitivity can be maximized by using specific optimal values of these parameters. The improvements in sensitivity when using optimal values may be only a few $\times 10\%$, or a factor of 2–10 and more depending on the starting point. Determining these optimal values is straightforward, and for precision experiments that are primarily limited by statistics it is usually well worth the effort. In the remainder of this article we investigate the sensitivity of ideal and non-ideal systems and derive certain optimal parameters for these systems.

2. Threshold detection for single ions

To determine if an ion is in its shelved state the probe cycling transitions are driven for some probe time $t_p$. The resulting PMT counts collected, $n$, will depend on the initial state of the ion. If the cycling and background rates are constant during the course of the probe, the average number of photons collected will be $n_T = n_T(t_p)$ for an ion that is not shelved and $n_S = n_S(t_p)$ for a ion that is shelved. If $n$ is large enough, uncertainties from Poisson counting statistics will be small, the result of a single probe will give $n$ very close to either $n_T$ or $n_S$, and the initial state of the ion can be determined with negligible uncertainty. An ion was very likely in the shelved state if the probe results in $n < n_0$, for some threshold $n_0$, and likely not shelved if $n > n_0$. With good counting statistics $n_0$ can easily be chosen as almost any value between $n_T$ and $n_S$ and the particular value used is not important. As counting statistics become poor interpretation of the probe count $n$ becomes less straightforward so that measurement precision will be affected by the choice of the threshold, and an optimal value can be determined.

In practice, the complete probe stage may begin with a set of transitions that drive the ion to the shelved state from a particular state involved in the interaction stage. The probability that the ion was in the state of interest at the beginning of the probe stage then corresponds directly to the probability that an ion is shelved after this probe preparation step. In practice the procedure that conditionally shelves the ion may not be perfectly efficient. Generally the probabilities will be linearly related, but the details depend on the particular measurement and vary considerably. Here only determining the resulting shelving probability, $s$, will be considered.

By repeating the probe step many times for an identically prepared ion, $s$ is given directly by the fraction of trials, $N_s$, that resulted in $n < n_0$ giving $s = N_s/N$ where $N$ is the total number of trials. The uncertainty in determining $s$, $\sigma_s$, is then
Figure 2. Scaled probe count probability distribution for shelved and unshelved ions as a function of $n/n_T$ with $n_T = 5$, 15, 50 and $n_b = nr_T/10$.

In some systems a short shelved-state lifetime or losses from the cycling states during the probe mean that the cycling and background rates are not constant and this can limit the time during which the probe is effectively able to distinguish the initial state. This will be considered in detail below, but for now note that if the cycling rate is too low, the resulting $n$ may be too small to allow the results of a single probe to be used to infer the initial state of the ion.

Figure 2 illustrates this by comparing the probability to collect some number of photons per probe time for both shelved and unshelved ions for various $n_T$. It is assumed that the cycling and background rates are constant and that the counts are given by a Poisson distribution

$$p_b(n_i) = \frac{n_i^x e^{-n_i}}{n_i!}$$

with $n_i = n_T$ for $i = b$ or $i = T$ which correspond to a shelved or unshelved ion, respectively. To easily compare the various cases the amplitudes are scaled by $\sqrt{n_i}$, as the height of the peaks roughly decrease by this factor, and $p_b$ is plotted as a function of $n/n_T$ with $n_b/n_T$ fixed so that the peaks always appear in the same position.

For large $n_T$ the distributions are clearly distinguished and a wide range of thresholds could be used as a cutoff to reliably define the difference between shelved and unshelved ions in a single probe trial. As $n_T$ reduces the distributions get wider and the range of possible threshold values is reduced. For particularly small $n_T$ the distributions overlap considerably and there is no threshold that can be used to unambiguously determine the ion’s state. Coarsely, the width of each distribution is given by $\sqrt{n_i}$ so the total region occupied by the non-negligible parts of the distributions is roughly $(\sqrt{n_T} + \sqrt{n_b})/2$. When this becomes comparable to the distance between the peaks, $n_T - n_b$, the distributions start to overlap. A threshold can then be defined only for $\beta = (\sqrt{n_T} + \sqrt{n_b})/2(n_T - n_b) < 1$ which requires sufficiently large $n_T$ and relatively small $n_b$.

For $\beta \geq 1$ probing an unshelved ion may yield a count less than any chosen threshold $n_b$, in a sense a false negative, with non-negligible probability

$$P_{sc} = P(n > n_b | s = 0) = \sum_{n=n_0}^{n-1} p_b(n_T) = \frac{\Gamma(n_0, n_T)}{\Gamma(n_0)}$$

with $\Gamma(n_0)$ and $\Gamma(n_0, n_T)$ the complete and incomplete gamma functions, respectively [12]. There is a corresponding false positive probability to measure $n > n_0$ for a shelved ion

$$P_{ss} = P(n > n_0 | s = 1) = \frac{\Gamma(n_0, n_b)}{\Gamma(n_0)}$$

as well as the correctly interpreted complements $P_{sc} = P(n > n_0 | s = 0) = 1 - P_{sc}$ and $P_{ss} = P(n < n_0 | s = 1) = 1 - P_{ss}$.

The probability of ‘detecting’ an ion to be shelved, $P_s \equiv P(n < n_0) = N_s/N$ will be given by the probability that the ion is actually shelved, $s$, times the probability it is detected as shelved, $P_{ss}$, plus the probability that it is not shelved, $1 - s$, times the probability it is nevertheless detected as shelved, $P_{sc}$.

$$P_s = sP_{ss} + (1-s)P_{sc} = P_{ss} + (P_{ss} - P_{sc}).$$

The corresponding $P_s = (1 - s)P_{sc} + sP_{ss} = 1 - P_s$ is no longer given exactly by $s$ except in the limit of $P_{ss}, P_{sc} \rightarrow 0$. For all other values $P_{ss}$ is linearly related to $s$. If $s$ must be determined directly from $s = (P_s - P_{sc})/(P_{ss} - P_{sc})$, then the $P_s$ must be known as well and may need to be determined experimentally.

Binomial statistics now give the uncertainty in determining $P_s$ from $N$ trials

$$\sigma_p^2 = \frac{P_s(1-P_s)}{N}.$$ 

Simple error propagation using $\sigma_p$ and $\partial P_s/\partial s = P_{ss} - P_{sc}$ gives an estimate for the uncertainty in determining $s$ using this threshold method,

$$\sigma_s^{2_{s_{ss}}} \approx \left( \frac{\partial P_s}{\partial s} \right)_s^{-2} \sigma_p^2 = \frac{P_s(1-P_s)}{N(p_{ss} - P_{sc})^2}. \quad (2)$$

The ratio, $\eta$, of $\sigma_s$ for the binomial distribution, equation (1), to $\sigma_{s_{ss}}$ for this non-ideal case,

$$\eta = \frac{\sqrt{s(1-s)}}{P_s(1-P_s)} (P_{ss} - P_{sc}).$$

can be viewed as the efficiency for detecting the state of the ion [14].

Obviously, $\eta$ depends on the threshold chosen, $n_b$, and some choices of $n_b$ yield better sensitivity of $P_s$ to $s$ than others. This optimal threshold value depends on details of the system including cycling and background rates, probe time (through $n_T$ and $n_b$), and the expected shelved state probability, $s$, so finding a general result is not trivial. However, it is easily determined numerically for any particular set of
parameters. Figure 3 shows the optimal threshold, determined as the value of \( n_0 \) that maximizes the detection efficiency, \( \eta \), and the resulting maximal \( \eta \) as a function of \( s \) for \( n_T = \{5, 15, 50\} \) and \( n_b = n_T/10 \). The \( n_T = 50 \) detection efficiency is effectively 1 independent of \( s \). Dashed curves show detection efficiency for a fixed threshold \( n_0 = 0.5n_T \) for the same range of \( n_T \).

![Figure 3](image)

**Figure 3.** Optimal threshold, \( n_{(optimal)}^T \), (lower solid curves) and resulting detection efficiency (upper solid curves) as a function of \( s \) for \( n_T = \{5, 15, 50\} \) and \( n_b = n_T/10 \). The \( n_T = 50 \) detection efficiency is effectively 1 independent of \( s \). Dashed curves show detection efficiency for a fixed threshold \( n_0 = 0.5n_T \) for the same range of \( n_T \).

3. Probe count distribution and statistics

For systems with poor probe counting statistics a threshold detection scheme may not provide sufficient information to confidently determine the state of a single ion in one trial or how many ions were shelved if using more than one ion. Instead consider using all the information in the full distribution of probe counts when determining the shelving probability \( s \).

3.1. Single ion Bi-modal distribution

Take \( P_n \) to be the probability of collecting \( n \) photons during a single probe of an ion that has probability \( s \) to be in the shelved state. For background, cycling, and total rates \( r_b, r_c, \) and \( r_T = n_b + r_c \), respectively, a particular probe time \( t_p \) gives \( n_i = n r_p \) counts. \( P_n \) will be the probability that the ion is shelved, \( s \), times the probability of a shelved ion giving \( n \) counts, plus the probability that the ion is not shelved, \( 1 - s \), times the probability of an unshelved ion giving \( n \) counts. If the counts for each pure state probe are given by a Poisson distribution, \( P_n \) will be

\[
P_n = sP_{n_i}(n_b) + (1 - s)P_{n_i}(n_T),
\]

where \( P_{n_i}(n) \) is the usual Poisson distribution. Familiar properties of the Poisson distribution are reviewed in many texts [13]. Figure 4 shows an example of \( P_n \) as a function of \( n \), using continuous values of \( n \) for clarity, for various \( s \) in the case of relatively poor counting statistics. Essentially, the \( s = 1 \) and \( s = 0 \) curves of this figure are the equivalents of the unshelved and shelved curves in figure 2, respectively, but with poorer counting statistics. Again, the lack of a clear distinction between shelved and unshelved ions is apparent.

![Figure 4](image)

**Figure 4.** Probe count distribution as a function of \( n \) for \( s = \{0, 0.2, 0.4, 0.6, 0.8, 1.0\} \) with \( n_T = 5, n_b = 1 \).

In this case the quantity of interest is the number of counts collected during a probe cycle. The mean of this quantity in this distribution is given by

\[
\langle n \rangle_p = \sum_n n P_n = s \langle n \rangle_{p(n_b)} + (1 - s) \langle n \rangle_{p(n_T)}
\]

\[
= sn_b + (1 - s)n_T
\]

\[
= n_T - sn_c,
\]

and the variance can be shown to be [14]

\[
\sigma_{(n)}^2 = \langle n \rangle_p + s(1 - s)n_c^2.
\]

As might be expected \( \langle n \rangle \) is linearly related to \( s \), allowing \( \langle n \rangle \) to be easily used to measure the dependence of \( s \) on drive parameters. Though as with the threshold probe, if the actual value of \( s \) is needed, two other parameters must be known, in this case \( n_T \) and \( n_c \).
The previous result for $\langle n \rangle_P$ may be substituted into this result, but in this form it is easy to see that $\sigma_{\langle n \rangle}^2$ reduces to $\langle n \rangle$ as in the Poisson distribution only for the trivial cases of $s = 0, 1$. In general there is an extra contribution as large as $(n_s/2)^2$ which dominates as $n_s$ increases. Figure 5 shows $\sigma_{\langle n \rangle}$ as a function of $s$ for various $n_s$. The deviation from Poisson statistics becomes significant for large $n_s$. This extra contribution has not been considered in previous experiments that use shelving in trapped ions and is usually missed if the variance is not determined from the data but instead assumed to be given by $\langle n \rangle$. This will result in underestimating the errors of derived quantities and will result in choosing the wrong optimal values for the probe time as considered below.

For $N$ trials the probe count probability distribution can be shown to be given by [14]

$$P_n^{(N)} = \sum_{m=0}^{N} \binom{N}{m} s^m (1-s)^{N-m} p_b (m n_b + (N-m)n_T)$$

$$= \sum_{m=0}^{N} B_n^N (s)p_b (N n_T - mn_c),$$

where $B_n^N (s)$ is the binomial distribution. The mean and variance of this distribution are given by [14]

$$\langle n \rangle_{P^N} = N \langle n \rangle_P = N (n_T - sn_c)$$

$$\sigma_{\langle n \rangle_{P^N}}^2 = N \sigma_{\langle n \rangle_P}^2 = N \langle n \rangle_P + N s (1-s) n_c^2$$

which are just $N$ times the $N = 1$ results as should be expected for $N$ independent trials. Here $\sigma_{\langle n \rangle_P}$ is the uncertainty in the determination of $\langle n \rangle$ after $N$ trials.

Proceeding as before the variance in the determination of the shelving probability in this method is given by

$$\sigma_{\sigma_{\langle n \rangle_P}}^2 \approx \left( \frac{\partial \langle n \rangle_P}{\partial s} \right)^2 \sigma_{\langle n \rangle_P}^2 = \left( \frac{\sigma_{\langle n \rangle_P}}{N n_c} \right)^2.$$

3.2. Relative sensitivity and stability

We now compare the uncertainty in $s$ using either the mean of the probe counts or a threshold for a single ion. These quantities are $\sigma_{\langle n \rangle_P}$ and $\sigma_{\langle n \rangle_{P^N}}$, respectively, and are given in equation (4) and equation (2), respectively. Figure 6 shows the ratio $\sigma_{\langle n \rangle_P} / \sigma_{\langle n \rangle_{P^N}}$ as a function of $n_s$ with various $n_T$ for $s = 1/2$. The threshold, $n_T$, is taken to be $(n_T + n_b)/2$ which is close to the optimal value for most values of $n_T$.

For very large $n_s$, there is no advantage in one method over the other for any $n_T$, as $\sigma_{\langle n \rangle_P} / \sigma_{\langle n \rangle_{P^N}} \rightarrow 1$. As $n_T$ decreases, the threshold method clearly yields a higher precision for $n_T$ above a particular value with an eventual transition to the opposite case as $n_T$ decreases further. The value of $n_T$ at which this transition occurs depends upon the absolute value of $n_T$. For very small $n_T$, as shown in section 2, it is not possible to unambiguously assign a threshold value, and one is better off using the mean, $\langle n \rangle_P$, as is clearly indicated in the figure.

Note also that as the background increases, the threshold method is superior only for larger $n_T$.

These results for the variance favor using a threshold probe and $s_{n_0}$ in systems with good counting statistics. Threshold detection is also initially more robust to variations of the count rates. For relatively large $n_T$, the peaks of the probe count distribution are well separated. Qualitatively, modest variations of the $n_T$ will not result in the peaks moving far enough that they begin to overlap and a large range of threshold values, $n_0$, will result in reliable determination of the ion state.

More formally the $P_n$ resemble step functions as a function of $n_T$ and are very insensitive to changes in $n_T$. Then $P_n$, which is given by

$$P_n = P_{n_0} + s (P_{n+1} - P_{n_0}),$$

is stable, that is it doesn’t change significantly as $n_T$ is varied. More rigorously, $\partial P_n / \partial n_T$ is small. As a result $\delta n_0 / \partial n_T$ is small when determined from $P_n$, and

$$s_{n_0} = (P_{n+1} - P_{n_0})/(P_{n+1} - P_n)$$

is also stable. Conversely

$$\langle n \rangle_P = n_T - s_{n_0}$$

varies directly as the $n_T$ vary so that for $\sigma_{\langle n \rangle_P} = (n_T - \langle n \rangle_P)/n_T$, $\sigma_{\langle n \rangle_P}$ must increase due to $\sigma_{\langle n \rangle_P}$ and $\partial n_T$. As $n_T$ decreases

![Figure 6. Relative precision, $\sigma_{\langle n \rangle_P} / \sigma_{\langle n \rangle_{P^N}}$ of mean probe compared to threshold probe as a function of $n_s$ for the case of $s = 1/2$ with $n_b = 0.002, 0.01, 0.05, 0.2, 1$ and 5.](image)

![Figure 7. Relative stability of mean, $\partial \langle n \rangle_P / \partial n_T$, versus threshold probe, $\partial \langle n \rangle_{P^N} / \partial n_T$ (solid), as a function of $n_T$.](image)
the peaks of the probe count distribution start to overlap, and
the $P_0$ will start to become sensitive to variations of $n_e.$ At that
point $s_{n_0}$ will also start to include contributions from these
variations. As $n_1$ decrease further $s_{n_0}$ actually becomes
more sensitive to variations than $s_{n_0}.$ Figure 7 shows $\delta s_{n_0}/\delta n_1$ and
$\delta s_{n_0}/\delta n_2$ as a function of $n_T$ for $n_b = 1.$ $s_{n_0}$ becomes
more robust to variations of $n_T$ than $s_{n_0}$ around the same point
that it also becomes more precise.

4. Finite probe time

The development thus far has assumed that the shelved state
is perfectly isolated from the states in the cycling transition
during the probe and that it is effectively infinitely long-lived.
This latter condition is satisfied if $t_p \ll 1/\Gamma_{\text{shelved}}.$ Here $t_p$ is
the probe time and $1/\Gamma_{\text{shelved}}$ is the shelved state lifetime. In
practical experiments this condition cannot always be satis-
fied. In addition, the ion may be driven out of or back into the
shelved state by off-resonant couplings driven by the probe
laser beams. Together these couplings can effectively make a
weak pump.

If the total loss rate from the shelved state due to its
natural lifetime and non-ideal couplings can be characterized
by a single simple $\Gamma_{\text{loss}}$ and the parasitic pump rate by $\Gamma_{\text{pump}},$
a familiar first order rate equation gives the probability for the
ion to be shelved as a function of time to be

$$s(t) = s_0 e^{-t/\tau_p} + s_\infty \left(1 - e^{-t/\tau_p}\right)$$

with $s_0$ the initial probability to be shelved at the beginning
of the probe, $s_\infty = \Gamma_{\text{pump}}/(\Gamma_{\text{pump}} + \Gamma_{\text{loss}})$ and
$1/\tau_p = \Gamma_{\text{pump}} + \Gamma_{\text{loss}}.$

The conventional interpretation of shelving transitions as
quantum jumps assumes that an ion is either shelved, having
100% probability to be in the shelved state, or unshelved with
zero probability of being in the shelved state. In the former
case, the ion does not fluoresce and the count rate will be
given by $r_\text{u}$ while in the latter case the count rate is given by
$r_T.$ As before, the count distribution will therefore be given by
the sum of two Poisson distributions, corresponding to these
two rates, weighted by the probability that the ion is either
shelved or unshelved. However, in this case $s$ is no longer
constant, and the weight is instead the time average of $s$:

$$s = \frac{1}{t_p} \int_0^{t_p} ds(t)$$

$$= (s_0 - s_\infty) \left(1 + s_\infty/s_0\right) + s_\infty$$

$$\gamma(\alpha) \equiv \frac{1 - e^{-\alpha}}{\alpha}.$$  

Note that the results scale simply with $t_p$ through $\alpha = t_p/\tau_p,$
and that $\gamma \rightarrow 1$ for $\alpha \rightarrow 0,$ and $\gamma \rightarrow 0$ for $\alpha \rightarrow \infty.$

An ion with time average probability $s$ to be in
the shelved state is then understood to be shelved for $s$ fraction
of the time. This gives the same bimodal distribution as found
in section 3.1 with the same rates but with $s \rightarrow \bar{s}$:

$$P_n = s p_n(n_b) + (1 - s) p_n(n_T).$$

In this case the interpretation of the distribution is slightly
different as previously a single probe would follow a single
Poisson distribution corresponding to one of two different
rates. The composite bimodal distribution results from many
trials where the rate for each trial is chosen to be one or the
other of those rates with probability $s,$ and the possible results
are summed. Now, since an ion can decay during a probe, a
single trial will have contributions from distributions corre-
sponding to both rates weighted by $\bar{s}$ so that this composition
happens during a single trial.

If this picture of the shelving dynamics is not correct then
the observed distribution will be different than that deter-
mined here. For example, if $s = 0.5$ instead gives a constant
scattering rate of $n_f = 0.5 n_c$ then, in general, as $s$ evolves the
distribution would be the time integral of a Poisson dis-
tribution with mean $(n) = n_f - s(t) n_c,$ which does not gen-
erally reduce to any particular simple distribution. Such
a distinction may provide a means of determining the degree to
which a probe functions as an ‘observation’ to collapse the
ion to either a shelved or unshelved state. For present
purposes the usual quantum jump understanding of the shelving
dynamics will be assumed and the corresponding simple
modifications to the count distribution will be used.

Since the distribution is formally the same as that found
in section 3.1, the expectation value and variance will be
given by the same replacement. In particular

$$\langle n \rangle \rho = n_f - \bar{s} n_c$$

$$\sigma_n^2 = \langle n \rangle + s(1-s) n_c^2$$

which can then be rewritten in terms of $s_0$ and $s_{\infty}$ as

$$\langle n \rangle = \bar{n}_f - s_0 n_c,$$

$$n_f = n_f - s_{\infty} n_c (1 - \gamma),$$

$$\bar{n}_c = \gamma n_c.$$  

The $n_c$ are not themselves averages but derived param-
eters, effective renormalized counts as a consequence of $s(t)$
varying over the probe time, and the notation indicates their
correspondence to $s.$ For $\alpha = t_p/\tau_p \ll 1,$ $\gamma \rightarrow 1,$ so that

Figure 8. Effective probe rate as a function of probe time through
$t_p = \alpha \tau_p$ for (top trace to bottom trace)
$s_0 = (0, 0.2, 0.4, 0.6, 0.8, 1.0),$ $s_\infty = 0.3,$ $n_f=1,$ $n_T=10.$
\[ \tilde{s} \rightarrow \tilde{s}_0, \text{ a constant value as before, and } n_i \rightarrow n_i \text{ thereby recovering the previous results. For the complementary case of } \alpha \gg 1, \gamma \rightarrow 0 \text{ so that } (n) \rightarrow n_T = s_p n_i, \text{ independent of } s, \text{ and no information about the ion's state at the end of the interaction stage can be obtained in this limit. Hence, } \tau_p \text{ can be regarded as a probe coherence time. When the probe time exceeds the probe coherence time sensitivity is reduced.} \]

\[ \text{functions as another measure of detection efficiency in this case relative to the ideal shelved state system.} \]

Figure 8 shows the average probe count and the effective rate \( \langle n \rangle/\tau_p \) as a function of \( \alpha = \tau_T/\tau_p \) using \( n_i = n_T \) for various \( \tilde{s} \). For \( \tau_p \gg \tau_p \) the rate becomes insensitive to \( s \) as just suggested. The difference between initial states is only detected during the beginning of the probe time when a probe of an unshelved ion yields some fixed number of extra counts compared to a shelved ion. With longer probe times this difference becomes negligible compared to the total counts and may become less than the variations due to counting statistics.

Though \( (n) \) appears most sensitive to \( s \) for low \( \tau_p \), \( \sigma_s \) diverges as \( \tau_p \rightarrow 0 \) and counting statistics become poor, so that an optimal probe time that minimizes \( \sigma_s \) must be some finite, intermediate value. The resulting optimal probe time can be determined directly from the explicit form of \( \sigma_s \).

5. Sensitivity and optimal probe

Returning to the expression for the variance in the shelving probability after \( N \) trials, equation (4), but taking into account the finite nature of the probe as developed above, we have

\[ \sigma_s = \frac{\sigma}{} \]

\[ = \frac{1}{\gamma} \sqrt{\frac{1}{N}} \left[ \tilde{s} \left( 1 - \tilde{s} \right) \right] \]

where we have written the number of counts in terms of the count rates, \( r_i (i = c, T) \) and the probe time \( \tau_p \). Note that if \( r_c \tau_p \gg 1 \), then

\[ \sigma_s \approx \frac{1}{\gamma} \sqrt{\frac{1}{N}} \tilde{s} \left( 1 - \tilde{s} \right) \]

since \( n_T/r_c \sim O(1) \) for this case and \( \tilde{s} \sim O(1) \). If \( r_c \) is sufficiently high and \( \tau_p \) sufficiently long, then we can choose \( \tau_p \) to satisfy both \( r_c \tau_p \gg 1 \) and \( \alpha = \frac{\tau_T}{\tau_p} \ll 1 \). Then \( \gamma \approx 1 \) and

\[ \sigma_s \approx \sqrt{\frac{1}{N}} \tilde{s} \left( 1 - \tilde{s} \right) \]

As one would expect for good counting statistics, this result is the same as that for binomial statistics for effectively perfect threshold detection efficiency. For example, if \( \alpha \lesssim 0.2 \) then \( 1/\gamma \lesssim 1.1 \), and \( \sigma_s \) is within 10% of the perfect detection limit provided \( r_c \tau_p = r_c \alpha \tau_p \gg 1 \) or \( \tau_p \gg \tilde{s}/r_c \).

\[ \text{Figure 9. } \alpha \text{-dependent terms of the sensitivity versus probe time for the limiting case of poor counting statistics that give Poisson count distributions.} \]

For the opposite case, \( r_c \tau_p \ll 1 \), the expression for the uncertainty in terms of count rates can be written

\[ \sigma_s \approx \frac{\sqrt{\tilde{s} \tau_p \left( 1 - \tilde{s} \right) \gamma}}{r_c \tau_p \gamma N} \]

\[ \approx \frac{1}{\gamma} \sqrt{\frac{\tilde{s} \tau_p \left( 1 - \tilde{s} \right) \gamma}{N}} \]

The first factor, \( 1/\gamma \sqrt{\tilde{s}} = \sqrt{\tilde{s}}/(1 - e^{-a}) \) contains all the probe time dependence of \( \sigma_s \). Figure 9 shows this quantity as a function of \( \alpha \). Note that it takes on the minimum value of 1.57 for \( \alpha_{\text{opt}} = 1.25 \) giving \( \sigma_s \lesssim 1.57 \sqrt{\tilde{s} \tau_p \left( 1 - \tilde{s} \right)} \) and that \( \sigma_s \) is fairly flat around its minimum. Further from that minimum it increases significantly for \( \alpha < \alpha_{\text{opt}} \) but less dramatically for \( \alpha > \alpha_{\text{opt}} \). So there is a large penalty for using too short a probe time, but less for using a longer probe time. Generally, taking \( \tau_p \) to be within a factor of 2 of its optimal value will give \( \sigma_s \) to within about 10% of its minimum value. Note that for this situation \( \sigma_s \approx 1/\sqrt{\tilde{s} \tau_p} \), improves quickly and simply as the probe coherence time, \( \tau_p \), increases so that improving \( \tau_p \) in a particular system, if possible, or using a different system allowing for longer \( \tau_p \) may be worthwhile.

For the general case it is informative to rewrite the full expression for \( \sigma_s \) in the form

\[ \sigma_s = \frac{f_0 (\alpha)}{\gamma} \sqrt{\frac{1}{N}} \tilde{s} \left( 1 - \tilde{s} \right) \]

where

\[ f_0 (\alpha) \equiv \frac{1}{\gamma} \alpha^{-1} \]

\[ a \equiv \frac{\tilde{s} \tau_p - \tilde{s}}{\tilde{s} \tau_p} \]

The function \( f_0 (\alpha) \) is the ratio of the uncertainty in the measurement of \( \tilde{s} \) to that for the binomial statistics case, and the quantity \( a \) is a measure of the quality of the counting statistics. Note that when \( r_c \tau_p \gg 1 \) then \( a/\alpha \approx 0 \), and we recover the binomial statistics case as above. The optimal
proportionality constant for species with fixed $s_0$ one could calculate the optimal probe time for a given, expected $s_0$. One could then recalculate the optimal $a$ if $s_0$ changed during an experiment. However, figure 11 suggests that if one uses a probe time of just under half of the probe coherence time the resulting uncertainty will exceed the optimal value by at most 20% with the deviation being much less than this over a broad range of experimental conditions as parameterized by $a$. Specifically, using fixed $a = 0.43$ yields the lowest deviation of $f_a(a)$ from $f_a(a^0)$ while eliminating the need to measure or estimate many of experiment parameters. Obviously, the probe coherence time must still be determined by measurement or calculation. In some situations this parameter will be set by the lifetime of the shelved state.

6. Conclusion

Shelving provides a highly sensitive probe of atomic state population probabilities. In systems allowing good counting statistics analysis is simple, intuitively understood, and the sensitivity is independent of modest instabilities in probe parameters. For less ideal systems the sensitivity can degrade considerably if the probe parameters are not carefully chosen and stabilized. For a wide range of real systems, uncertainties in determining $s$ with a shelved state probe are minimized by using a probe time on the order of half the probe coherence time and deriving $s$ from the mean of a set of probe counts. Using this probe time gives a sensitivity that is generally within 30% of the exact optimal value.

The importance of this, and the potential improvements, are especially significant in systems with probe coherence times short compared to the directly detected cycling time. This is often the case when trying to isolate close hyperfine sublevels for use as shelved states where off-resonant couplings can result in non-negligible spurious transitions, or when trying to isolate individual Zeeman levels using angular momentum selection rules where impurities in the polarization of probe beams also significantly limit discrimination between states in practice [6]. Even in the optimally probed case the resulting probe count distributions are still multimodal and can have variances significantly different than that expected from Poisson statistics. Neglecting such details can result in underestimation of the uncertainty of fit-derived parameters or inaccurately weighting points in a fit resulting in less precise fits or systematic shifts in the fits.

We have used these results to measure the lifetime of the $^{2}\text{D}_{3/2}(F = 2)$ state in $^{171}\text{Yb}^{+}$ ions with roughly 15 times better statistical precision than previous measurements [14]. This measurement was part of a larger effort to evaluate Ytterbium ions as potential candidates for an atomic parity violation experiment. In the course of this work we achieved roughly an order of magnitude improvement in the precision of our measurement of the hyperfine levels of the $S\,_{1/2}$ ground state largely through improved sampling and probing of the system. Similar improvement in the measurement precision of the Zeeman splitting of these hyperfine levels has brought the system to within an order of magnitude of the sensitivity necessary for an initial parity non-conservation measurement. It should be noted that these measurements were done in an ion trap without any magnetic shielding [15].

Finally we note that the methods outlined in this paper are broadly applicable to any experiment of the pump-probe type and would appear to be of great interest to optical frequency standard development. Of particular note are experiments to drive transitions directly with the modes of an optical frequency comb [16, 17], especially in the case of weak transitions where one would expect to be statistics limited (see for example [18]).
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