DISCRETE APPROXIMATIONS FOR COMPLEX KAC-MOODY GROUPS

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Abstract. We construct a map from the classifying space of a discrete Kac-Moody group over the algebraic closure of the field with \( p \) elements to the classifying space of a complex topological Kac-Moody group and prove that it is a homology equivalence at primes \( q \) different from \( p \). This generalises a classical result of Quillen–Friedlander–Mislin for Lie groups. As an application, we construct unstable Adams operations for general Kac-Moody groups compatible with the Frobenius homomorphism. Our results rely on new integral homology decompositions for certain infinite dimensional unipotent subgroups of discrete Kac-Moody groups.

1. Introduction

Cohomological approximations for Lie groups by related discrete groups were developed by Quillen [43], Milnor [39], Friedlander and Mislin [23]. In this paper, we prove that a complex Kac-Moody group is cohomologically approximated by the corresponding discrete Kac-Moody group over \( \mathbb{F}_p \) at primes \( q \) different from \( p \). One application is the construction of unstable Adams operations for arbitrary complex Kac-Moody groups.

Over the complex numbers, topological Kac-Moody groups are constructed by integrating Kac-Moody Lie algebras [29] that are typically infinite dimensional, but integrate to Lie groups when finite dimensional. Kac-Moody Lie algebras are defined via generators and relations encoded in a generalized Cartan matrix [36]. Kac-Moody groups \( K \) have a finite rank maximal torus that is unique up to conjugation by a Coxeter Weyl group; this Weyl group is finite exactly when \( K \) is Lie. For any minimal split topological \( K \) [30,36], Tits [49] constructed a corresponding discrete Kac-Moody group functor \( K(\cdot) \) from the category of commutative rings with unit to the category of groups such that \( K(K) \) is Lie.

Theorem A. Let \( K \) be a topological complex Kac-Moody group and let \( K(\mathbb{F}_p) \) be the value of the corresponding discrete Kac-Moody group functor. Then, there exists a map \( BK(\mathbb{F}_p) \to BK \) that is an isomorphism on homology with \( \mathbb{F}_q \) coefficients for any \( q \neq p \).

Theorem A is proved by extending Friedlander and Mislin’s map [23, Theorem 1.4] between the classifying spaces of appropriate reductive subgroups of discrete and topological Kac-Moody groups to a map between the full classifying spaces of Kac-Moody groups. This extension uses a new homology decomposition of a

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discrete Kac-Moody group over a field away from the ambient characteristic. We state this result now, but see Theorem 4.1 for a more precise statement.

**Theorem B.** Let $K(\mathbb{F})$ be a Kac-Moody group over a field. Then there is a finite collection of subgroups \( \{G_I(\mathbb{F})\}_{I \in S} \) that are the \( \mathbb{F} \)-points of reductive algebraic groups such that the inclusions \( G_I(\mathbb{F}) \hookrightarrow K(\mathbb{F}) \) induce a homology equivalence

\[
\text{hocolim}_{I \in S} BG_I(\mathbb{F}) \longrightarrow BK(\mathbb{F}),
\]

away from the characteristic of \( \mathbb{F} \).

This gives a natural way to propagate cohomological approximations of complex reductive Lie groups to Kac-Moody groups; see also Remark 4.2.

Theorem B in turn depends on a homological vanishing result for key infinite dimensional unipotent subgroups of discrete Kac-Moody groups over fields. As explained in 2.2–2.3, these subgroups play the same role in the subgroup combinatorics of \( K(\mathbb{F}) \) as the unipotent radicals of parabolic subgroups play in that of an algebraic group.

**Theorem C.** Let \( U_I(\mathbb{F}) \) be the unipotent factor of a parabolic subgroup of a discrete Kac-Moody group over a field. Then, \( H_n(BU_I(\mathbb{F}), L) = 0 \) for all \( n > 0 \) and any field \( L \) of different characteristic.

To prove Theorem C, we obtain new structural understanding of these unipotent subgroups—e.g., we obtain new colimit presentations—based in the geometric group theory of Kac-Moody groups but developed on the level of classifying spaces. Theorem 3.1 decomposes the classifying spaces of key unipotent subgroups as homotopy colimits of finite dimensional unipotent subgroup classifying spaces. These new integral homotopy decompositions are natural with respect to Tits’s construction of \( K(-) \) and apply to arbitrary groups with root group data systems (see \( 2.2 \)). We use a functorial comparison of diagrams indexed by the Weyl group to diagrams indexed by its Davis complex \( \{2,3\} \) (Theorem \( \{3,2\} \)) that may be of interest to Coxeter group theorists. This comparison ultimately reduces our homology decompositions to the contractibility of the Tits building. The structural understanding developed also provides a method to compute the non-trivial cohomology of these unipotent subgroups over a field at its characteristic (see Theorem \( 5.7 \)).

Our main application of Theorem A is the construction of unstable Adams operations. These maps are defined as \( \psi \) that fit into the homotopy commutative diagram

\[
\begin{array}{ccc}
BT & \xrightarrow{B((-)^p)} & BT \\
\downarrow & & \downarrow \\
BK & \xrightarrow{\psi} & BK
\end{array}
\]

where the vertical maps are induced by the inclusion \( T \leq K \) of the standard maximal torus. We are more generally interested in \( q \)-local unstable Adams operations where \( 2 \) is replaced by its functorial localization \( (-)^\wedge \) with respect to \( \mathbb{F}_q \)-homology \( 1.4 \). For connected Lie groups, unstable Adams operations (including \( q \)-local versions) are unique, up to homotopy, whenever they exist \( 28 \). In \( 4 \), unstable Adams operations for simply connected rank 2 Kac-Moody groups where constructed and shown to be unique.

Classically \( 23 \), Theorem A was shown for the group functor \( G(-) \) associated to a complex reductive Lie group \( G \). For such \( G \), the \( p \)th unstable Adams operation on \( BG_q^\wedge \) is homotopic to the self-map \( B(G(\varphi))^\wedge \) for \( \varphi \) the Frobenius homomorphism \( x \mapsto x^p \) on \( \mathbb{F}_p \). We use the functoriality of Tits’s construction to show that \( B(K(\varphi))^\wedge \) is an unstable Adams operation for an arbitrary topological Kac-Moody
BK Adams operations can be assembled into a global unstable Adams operation for any prime \( q \neq p \). The Frobenius map induces a proof of Theorem B. Our map from \( BK \) discrete Kac-Moody groups. This structure implies the vanishing theorem C which this paper begins with a structural understanding of key unipotent subgroups of \( K \).

1.1. Organization of the paper. As described above, the logical progression of this paper begins with a structural understanding of key unipotent subgroups of discrete Kac-moody groups. This structure implies the vanishing theorem C which leads to a proof of Theorem B. Our map from \( BK \) to \( BK \) in Theorem A is constructed by a compatible family of Friedlander–Mislin maps [23, Theorem 1.4] which give the desired homological approximation by Theorem B. The functoriality constructed by a compatible family of Friedlander–Mislin maps [23, Theorem 1.4] is desirable.

From the perspective of homotopical group theory, unstable Adams operations on \( p \)-compact groups yield examples of \( p \)-local finite groups after taking homotopy fixed points [8] (see [24] similar results for \( p \)-local compact groups). To the extent that homotopy Kac-Moody groups may generalize homotopy Lie groups (as proposed e.g., in Grodal’s 2010 ICM address [26]), unstable Adams operations on Kac-Moody groups and their homotopy fixed points are interesting. Through cohomology calculations with classifying spaces of compact Lie subgroups, we provide evidence that, in contrast to the classical setting, \( (BK)^{h\psi_k} \) and \( BK \) rarely agree.

Theorem E. Let \( K \) be an infinite dimensional simply connected complex Kac-Moody group of rank 2 and \( \psi \) be its unique \( p \)-local unstable Adams operation for primes \( q \neq p \) with \( q \) odd. Then, \( H^*(BK^{h\psi_k}; \mathbb{F}_q) = H^*(BK; \mathbb{F}_q) \) if and only if they both vanish.

Our comparison methodology for Theorem E is applicable to general Kac-Moody groups, but there are many technical challenges (see the discussion beginning 5.1) and possible issues with uniqueness. Nevertheless, Theorem E does provide a homology decomposition of \( K \) in terms of finite subgroups.
the main unipotent subgroup of \( K(\mathbb{F}_p) \) at the prime \( p \) in most cases where the Weyl group is a free product.

**Notation and Conventions.** As noted above, \( K(R) \), for a commutative ring \( R \), denotes the value of one of Tits’s explicit discrete, minimal, split Kac-Moody group functors \[49, 3.6\]. We generally use \( K \) for a minimal topological Kac-Moody group \[36, 7.4.14\], but to simply typography, we occasionally abbreviate \( K(R) \) to \( K \) when \( R \) is fixed throughout an argument. We define the rank of \( K \) to be the rank of its maximal torus. A generalized Cartan matrix is a square integral matrix \( A = (a_{ij})_{1 \leq i,j \leq n} \) such that \( a_{ii} = 2 \), \( a_{ij} \leq 0 \) and \( a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \) and we reserve the notation \( A \) for the generalized Cartan matrix used in constructing \( K \). To simplify statements, we use \((-\)\( q \))\(^q\) to denote a Bousfield \( \mathbb{F}_q \)–homology localization functor of spaces \[14, 1.E.4\] rather than the more common Bousfield-Kan \( q \)-completion functor \[6\]. These two functors agree, up to homotopy, for any space \( X \) that is nilpotent or has \( H_1(X, \mathbb{F}_q) = 0 \). Math mode bold is used for categories. For instance, each Coxeter group \( W \) has an associated poset \( W \). Our results are stated in category of topological spaces with the usual weak equivalences \( \text{Top} \), but \( \text{Spaces} \)—which can be taken to be topological spaces, \( CW \)-complexes, or simplicial sets—is used in technical results when we wish emphasize that there is little dependence on \( \text{Top} \).

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2. **Combinatorial tool kit**

Like Lie groups, Kac-Moody groups have (typically infinite) Weyl groups that underlie combinatorial structures on subgroups including \( BN \)–pairs and root group data systems. In \[2.1 2.3\] the aspects of these structures we use are assembled. We also collect tools for manipulating homotopy colimits \[2.4 2.6\] which relate to colimit presentations of groups via Seifert-van Kampen theory \[2.7\]. With a homotopy theoretic framework in place, \[2.8\] quickly reviews two known homotopy decompositions associated to Coxeter groups and \( BN \)–pairs and how they apply to Kac-moody groups.

2.1. **Coxeter groups.** Because of their role in \( BN \)–pairs and RGD systems, the structure of Coxeter groups will be important to the arguments here. For our purposes, a Coxeter group will be a finitely generated group with a presentation

\[
\langle s_1, s_2, \ldots, s_n | s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle
\]

for \( 1 \leq i, j \leq n \) and fixed \( 2 \leq m_{ij} = m_{ji} \leq \infty \) with \( m_{ij} = \infty \) specifying a vacuous relation. For example, the presentation the Weyl group of a Kac-Moody group is
determined by its generalized Cartan matrix $A = (a_{ij})_{1 \leq i, j \leq n}$, i.e.

$$m_{ij} = \begin{cases} 2 & a_{ji}a_{ij} = 0 \\ 3 & a_{ji}a_{ij} = 1 \\ 4 & a_{ji}a_{ij} = 2 \\ 6 & a_{ji}a_{ij} = 3 \\ \infty & a_{ji}a_{ij} \geq 4 \end{cases}$$

for all $i \neq j$ [36, p. 25]. The solution to the word problem for Coxeter groups gives us a detailed picture of $W$.

**Theorem 2.1** (Word Problem [1]). Let $W$ be a Coxeter group. For any words $\vec{v}$ and $\vec{w}$ in letters $s_1, s_2, \ldots, s_n$.

- A word $\vec{w}$ is reduced in $W$ if and only if it cannot be shortened by a finite sequence of $s_is_i$ deletions and replacing of the length $m_{ij} < \infty$ alternating words $s_is_j \ldots$ by $s_js_i \ldots$ or vice versa.

- Reduced words $\vec{v}$ and $\vec{w}$ are equal in $W$ if and only if $\vec{v}$ transforms into $\vec{w}$ via a finite sequence of length $m_{ij} < \infty$ alternating word by replacements as describe above.

Thus, every element of a Coxeter group has a well-defined reduced word length. This provides a poset structure on $W$, called the weak Bruhat order, defined as

$$v \leq w \iff w = vx \text{ and } l(w) = l(v) + l(x)$$

where $x, v, w \in W$ and $l(y)$ is the reduced word length of $y$. This poset will be important to us because it dictates the intersection pattern of the subgroups $U_w$.

For any $I \subseteq S$ define $W_I := \langle \{s_I\}_{i \in I \subseteq S} \rangle$ and let

$$S := \{I \subseteq S ||W_I|| < \infty\}$$

be the poset ordered by inclusion. Note that $I \in S$ implies that $wW_I$ has a unique longest word $w_I$. We call $I \in S$ and their associated $W_I$ finite type. We emphasize that a Kac-Moody group is of Lie type if and only if its Weyl group given by (4) is finite.

**2.2. BN–pairs.** Here we follow Buildings [1].

**Definition 2.2.** A group $G$ is a $BN$–pair if it has data $(G, B, N, S)$ where $B$ and $N$ are distinguished normal subgroups that together generate $G$. Furthermore, $T := B \cap N \leq N$ and $W := N/T$ is generated by $S$ so that for $s \in S$ and $w \in W$:

- $BN1$: $\pi B\pi^{-1} \subseteq B\pi B \cup B\pi B$

- $BN2$: $\pi Bs^{-1} \notin B$.

Note that the above sets are well-defined as $B \geq T \leq N$. It is common in the literature, and will be common in this paper, to drop the bars and avoid reference to particular representatives of elements of $W$.

For Kac-Moody groups, $T$ is the standard maximal torus of rank $2n - \text{rank}(A)$, $N$ is the normalizer of $T$, $|S| = n$ is the size of the generalized Cartan matrix and $B$ is the standard Borel subgroup defined analogously to the Lie case. In line with 2.1, we will further require that $S$ is a finite set.

Important properties of $BN$–pairs include that $W$ is a Coxeter group and $G$ admits a Bruhat decomposition

$$G = \prod_{w \in W} BwB$$
so that all subgroups of $G$ containing $B$ are of the form $P_I = \bigcap_{w \in W_I} BwB$ where $I \subseteq S$ and $W_I$ is generated by $I$. These subgroups are called standard parabolic subgroups and inherit the structure of a $BN$–pair with data $(P_I, B, N \cap P_I, I)$ and Weyl group $W_I$. For Kac-Moody groups $A_J := (a_{ij})_{i,j \in J}$ of $A$ determines a Weyl group $W_J$ group for $P_J$ via (4). The $BN$–pair axioms then lead (32) to generalized Bruhat decompositions

$$G = \prod_{w \in W_J \setminus W/W_I} P_J w P_I.$$  

2.3. Root group data systems. Much of the work in this paper could be adapted to refined Tits systems (31) but we prefer to work with root group data (RGD) systems. Notably the framework of RGD systems has been used to prove (1,10) the colimit presentation induced by Theorem 3.1 (see Remark 2.13 in (2.7) which was conjectured by Kac and Petersen in (31).

Briefly, a RGD system for a group $G$ is a given by a tuple $(G, \{U_\alpha\}_{\alpha \in \Phi}, T)$ for $\Phi$ associated to a Coxeter system $(W, S)$. The elements of the set $\Phi$ are called roots and $U_\alpha$ are nontrivial subgroups of $G$ known as root subgroups. The RGD subgroups generate $G$, i.e. $G = \langle T, \{U_\alpha\}_{\alpha \in \Phi} \rangle$. In the case of Tits’s Kac-Moody group functor, $U_\alpha$ are isomorphic to the base ring as a group under addition and $T$ is the standard maximal torus isomorphic to a finite direct product of the multiplicative group of units of the base ring. As the complete definition is somewhat involved and the arguments here can be followed with the Kac-Moody example in mind, we refer the reader to (11) for the standard definition of a RGD system and further details but note that (10) provides an alternative formulation.

The RGD structure regulates the conjugation action of $W$ on $U_\alpha$. In particular, $\Phi$ has a $W$–action and is the union of the orbits of the simple roots, $\alpha_i$, corresponding to the elements, $s_i$, of $S$. For $n \in W \ U_{\alpha n} = n U_\alpha n^{-1}$.

In the Kac-Moody case, $\Phi = W\{\alpha_1\}_{1 \leq i \leq n} \bigcup \{1 \leq i \leq n\} \mathbb{Z} \alpha_i$ with the $W$–action given by

$$s_j \left( \sum_{i=1}^{k} n_i \alpha_i \right) = \sum_{1 \leq i \neq j \leq k} n_i \alpha_i - (n_j + \sum_{1 \leq i \neq j \leq k} n_i a_{ij}) \alpha_j.$$  

where $a_{ij}$ are entries in the generalized Cartan matrix.

The set of roots $\Phi$ is divided into positive and negative roots so that $\Phi = \Phi^+ \setminus \Phi^-$. For Kac-Moody groups, $\Phi^-$ and $\Phi^+$ corresponds to the subsets of $\Phi$ with all negative or all positive coefficients, respectively. Define

$$U^\pm = \langle U_\alpha; \alpha \in \Phi^\pm \rangle.$$  

The group $G$ carries the structure of a $BN$–pair for either choice of $B = T U^\pm$ and $N$ the normalizer of $T$. There are also well-defined

$$U_w = \langle U_{\alpha_{i_1}}, U_{s_{i_1} \alpha_{i_2}}, \ldots U_{s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}} \rangle$$  

where $\Theta_w := \{\alpha_1, \alpha_2, \ldots, ws_{i_k} \alpha_1\} = \Phi^+ \cap w \Phi^-$ is well-defined for any reduced expression for $w = s_{i_1} \cdots s_{i_k}$. Moreover, the multiplication map

$$U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{s_{i_k} \alpha_1} \xrightarrow{m} U_w$$  

is an isomorphism of sets for any choice of reduced expression for $w = s_{i_1} \cdots s_{i_k}$. For any index set $I \subseteq W$

$$\bigcap_{I} U_w = U_{\inf W \{ I \}}$$  

where the greatest lower bound is with respect to the weak Bruhat order (5).
For a group with RGD system, there is a symmetry between the positive and negative roots, as in the Kac-Moody case. In particular, there is another RGD system for \(G, \{U_\alpha\}_{\alpha \in \Phi} \), with the positive and negative root groups interchanged. This induces a twin BN\(^{-}\)pair structure on \(G\) which guarantees a Birkhoff decomposition

\[
G = \prod_{w \in W} B^+ w B^-
\]

which when combined with the Bruhat decomposition for \(P_I\) \((13)\) and the fact that \(B = U^+ T\) easily leads to a generalized Birkhoff decomposition

\[
G = \prod_{w_{J} \in W / W_{J}} U^+_I w P^-_I
\]

where \(U^+_I w P^-_I = \prod_{v \in w_{J}} U^+_I w B^- \) (see [20] for details). Using the standard RGD axioms, this symmetry between positive and negative is somewhat subtle. Known proofs employ covering space theory \([1, 10]\).

The loc. cit. covering space arguments imply for \(n \in W\)

\[
U^+_I \cap nU^+_I n^{-1} = U^+_I \cap wB^+ w^{-1} = U^-_w
\]

where \(U^+_w := U_w\) and \(U^-_w = \langle -\alpha_1, U^+_{-s_1 \alpha_2}, \dotsc U^+_{-s_{k_1-1} \alpha_{k_1}} \rangle\). More generally (see Lemma 2.3, for all \(J \in S\) \([10]\)

\[
U^+_I \cap w P^-_J w^{-1} = U^-_{w_{J}}
\]

where \(w_{J}\) is the longest word in \(w_{J} = \{\}

The groups \(U_w\) also provide improved Bruhat decompositions because

\[
B^\pm \cap U^\pm w B^\pm = U^\pm_w B^\pm
\]

where expression on the left hand side factors uniquely for a fixed choice of \(w \in n T \in W\), cf. [1, Lemma 8.52], i.e.

\[
G = \prod_{w \in W} U^+_w w B^+ = \prod_{w \in W} U^-_w w B^-
\]

so that all \(g \in G\) factor uniquely as \(g = u w b\) where \(u \in U^+_w\), \(b \in B^\pm\) and \(w\) is an element in any fixed choice of coset representatives. Parabolic subgroups inherit similar expressions.

As previously mentioned, all groups with RGD systems have the structure of a twin BN\(^{-}\)pair. This structure can be used to define Levi component subgroups

\[
G_I = P^-_I \cap P^+_I.
\]

The \(G_I\) inherit a root group data structure \((G_I, \{U_\alpha\}_{\alpha \in \Phi_I}, T)\) where \(\Phi_I := \{\alpha \in \Phi| U_\alpha \cap G_I \neq \{\}\}\).

When \(I\) has finite type \([10]\), this leads to the semi-direct product decomposition

\[
P^\pm_I \cong G_I \ltimes U^\pm_I
\]

known as the Levi decomposition with

\[
G_I = \langle T, U^+_I, U^-_I \rangle
\]

for \(U^+_I = U^+_I \cap G_I\) called Levi component subgroups. Most of our principal applications will only require \([15]\) for finite type I. For Kac-Moody groups, \([15]\) holds for all I and the submatrix \(A_I := (a_{ij})_{i,j \in I}\) of \(A\) determines a Weyl group \(W_I\) group for \(G_I\) via \([11]\). Moreover, \(G_I\) is a reductive group exactly when \(|W_I| < \infty\) \([35]\). See \([10]\) for further discussion of when this decomposition is known to exist for general RGD systems.
We close the subsection with a simple but important lemma used in the proof of Theorem 3.1.

**Lemma 2.3.** For \( P^{-}_J \) of finite type

\[ U_{wP^{-}_J} = U_{w_J} \]

where \( U_{wP^{-}_J} \) is the stabilizer of \( \{wP^{-}_J\} \) under the left multiplication action of \( U^+ \) on \( G/P \), \( w_J \) is the longest word in \( \omega W_J \), and \( U_w \) is defined in (10).

**Proof.** Note that \( wP^{-}_J = wP^{-}_J \) and \( u \in U^+ \) if and only if \( u \in wP^{-}_J \cap U^+ \).

Recall (10) the (improved) Bruhat decomposition for \( P^{-}_J \)

\[ P^{-}_J = \bigcap_{\nu \in W_J} U^+_\nu \nu B^- \]

We compute

\[ wP^{-}_J w^{-1} = w_J P^{-}_J w_J^{-1} = \bigcup_{\nu \in W_J} w_J U^+_{\nu^{-1}} \nu B^- w_J^{-1} \]

(21)

\[ = \bigcup_{\nu \in W_J} (w_J U^+_{\nu^{-1}})w_J wB^- w_J^{-1}. \]

As \( (w_J U^+_{\nu^{-1}}) \subseteq U^+ \) for all \( \nu \in W_J \) we have

\[ w_J U^+_{\nu^{-1}} wB^- w_J^{-1} \cap U^+ \cong w_J wB^- w_J^{-1} \cap U^+. \]

Each \( w_J wB^- w_J^{-1} \cap U^+ \) isomorphic as a set to

(22)

\[ w_J wB^- \cap U^+ w_J \subseteq U^+ w_J wB^- \cap U^+ w_J B^- \]

via right multiplication by \( w_J \). Together (21) and (22) imply

(23)

\[ w_J wB^- w_J^{-1} \cap U^+ \neq \emptyset \Rightarrow u = e. \]

Combining (21, 22) we have

\[ U_{wP^{-}_J} = wP^{-}_J w^{-1} \cap U^+ = \bigcup_{\nu \in W_J} (w_J U^+_{\nu^{-1}} wB^- w_J^{-1} \cap U^+) \]

\[ = w_J U^+ eB^- w_J^{-1} \cap U^+ \]

\[ = w_J B^- w_J^{-1} \cap U^+ = U_{w_J B^-}. \]

Now, \( (w_J B^- w_J^{-1} \cap U^+) = U_{w_J B^-} \) is known to be \( U_{w_J} \) (10) via covering arguments that appear independently in (11) and (10).

\[ \square \]

2.4. **Pulling back homotopy colimits.** If \( F : J \rightarrow I \) is a fixed functor, then we say \( F \) pulls back homotopy colimits if for any diagram of spaces \( D : I \rightarrow \text{Spaces} \) the natural map

\[ \text{hocolim}_J DF \xrightarrow{\text{hocolim}_F} \text{hocolim}_I D \]

is a weak homotopy equivalence. We will use Theorem 2.6 (see Section 2.5) to pullback homotopy colimits over appropriate functors by assembling pullbacks over subcategories. For such a functor and any \( i \in I \) define the category \( i \downarrow F \) as having objects

\[ \text{Objects}(i \downarrow F) = \{(i \rightarrow i', j') | F(j') = i', i \rightarrow i' \in \text{Hom}_I \} \]

and morphisms

\[ \text{Hom}_{i \downarrow F}((i \xrightarrow{f_1} i_1', j_1'), (i \xrightarrow{f_2} i_2', j_2')) \]

\[ = \{(i_1' \xrightarrow{f} i_2', j_1' \xrightarrow{h} j_2') | f = F(h) \in \text{Hom}_I, h \in \text{Hom}_J \}. \]
i.e. morphisms are pairs of vertical arrow that fit into the following diagram

![Diagram](image)

When $F$ is the identity on $I$, we use the notation $i \downarrow I$ and this definition reduces to that of an under category.

**Theorem 2.4** (Pullback Criterion [17, 27]). Let $D : I \to \text{Spaces}$ be a diagram of spaces and $F : J \to I$ be a fixed functor, then the following are equivalent:

- for all $D$ the canonical $\text{hocolim}_J DF \xrightarrow{\sim} \text{hocolim}_I D$ is a weak equivalence,
- for all $i \in I$ the geometric realization of $i \downarrow F$ is weakly contractible.

In the applications of this paper, $J$ and $I$ will be posets so that $i \downarrow F = F^{-1}(i \downarrow I)$.

### 2.5. Homotopy colimits over subdiagrams

At various times in this paper, we will wish to represent homotopy colimit presentations of a space in terms of homotopy colimits over subdiagrams of the main diagram. This subsection will present tools to do this.

An explicit model for the homotopy colimit of a small diagram $D : C \to \text{Top}$ of (topological) spaces $\text{hocolim}_C D$ may be given as the union of

$$X_n = \bigsqcup_{c_1 \leq \cdots \leq c_k \in C} F(c) \times \Delta^k$$

where $\Delta^k$ is the $k$–simplex and the relations correspond to the composition in $C$ [6]. We call this the standard model of a homotopy colimit. Define the geometric realization or classifying space of a category as $|C| \simeq \text{hocolim}_C \{\ast\}$ for $\{\ast\}$ the one point space.

We will also define a cover of a category $C$ by subcategories $\{C_i\}_{i \in I}$.

**Definition 2.5.** A cover of a small category $C$ is a collection of subcategories $\{C_i\}_{i \in I}$ such that, taking standard models (24), $\{C_i\}_{i \in I}$ covers $|C|$ (2.5).

We may also choose to enrich a cover by giving $I$ the structure of a (commutative) diagram of inclusions of small categories. Such a cover is given as a functor $U : I \to \text{SubCat}(C)$ from a poset into the category of subcategories of $C$. If there are no repetitions of $C_i$’s, then $I$ may be canonically given the poset structure induced by inclusion of subcategories.

We will be interested in the taking homotopy colimits over subdiagrams

$$D_i = D|_{C_i} : C_i \to \text{Top}$$

and one of our main computational tools is provided by the following theorem.

**Theorem 2.6.** Let $D : C \to \text{Top}$ be a diagram of topological spaces and $U : I \to \text{SubCat}(C)$ be a cover $\{C_i\}_{i \in I}$ of $C$ such that

- $I$ has all greatest lower bounds and
- $U(\inf_A (C_a)) = \cap \cup U(C_a)$

for all index sets $A$. Then, $\text{hocolim}_I (\text{hocolim}_C D_i)$ and $\text{hocolim}_C D$ are canonically, weakly equivalent.

By Thomason’s Theorem [17, Theorem 1.2], Theorem 2.6 may be reduced to the following.
Proposition 2.7. Let $D : C \to \text{Spaces}$ be a diagram of spaces (e.g., topological spaces, CW–complexes, or simplicial sets) and $U : I \to \text{SubCat}(C)$ be a cover \{C_i\}_{i \in I} of $C$ such that

- $I$ has all greatest lower bounds and
- $U(\text{inf}_A\{C_a\}) = \cap_a U(C_a)$

for all index sets $A$. Then, the canonical projection of the Grothendieck construction \cite{U} of $U$ onto $C$ pulls back homotopy colimits.

Proof. Recall that the Grothendieck construction of $U$, which we will denote as $I \times U$, is given by

\begin{equation}
\text{Objects}(I \times U) = \{(i,c) : i \in \text{Objects}(I), c \in \text{Objects}(C_i)\},
\end{equation}

\begin{equation}
\text{Hom}_{I \times U}((i,c),(i',c')) = \{(f,g) : f \in \text{Hom}_I(i,i'), g \in \text{Hom}_{C_i}(c,c')\}
\end{equation}

with composition given by pushing forward along $U$, i.e., $(f_2, g_2) \circ (f_1, g_1) = (f_2 f_1, g_2 g_1)$. The projection $P : I \times U \to C$ maps $(i,c)$ to $c$. By Theorem \cite{2.4}, it is sufficient to show that $[c \downarrow P] \simeq \{\ast\}$. Let $i(c)$ be the greatest lower bound for all categories in the cover containing $c$, then $c \downarrow P$ has initial object $(c \downarrow i(c),c)$. \hfill \Box

Dugger and Isaksen \cite{16} present a model, up to weak equivalence, of a general topological space $X$ in terms of an arbitrary open cover $U = \{U_i\}_{i \in I}$ and finite intersections of these $U_i$. This allows a version of Theorem \cite{2.6} to verified depending only on finite greatest lower bounds, but this version will not be needed here and is less natural in the sense that it relies on the standard weak equivalences generating the model category structure.

Now, Theorem \cite{2.6} allows the second criterion of Theorem \cite{2.4} to be verified locally.

Proposition 2.8. Let $F : J \to I$ be a map of posets such that $I$ has all greatest lower bounds. If $F$ restricted to the pullback of all closed intervals in $I$ pull back homotopy colimits, then $F$ pull back homotopy colimits.

Proof. A closed interval $[i_1, i_2]$ is defined to be the full subcategory of $I$ with object set $\{i : i_1 \to i \to i_2 \in I\}$. The category $i \downarrow I$ is covered by $[[i,i']]_{i_1 \to i_2}$ and for any set of objects $A$ the intersection $\cap_{i' \in A} [i_i,i'] = [i,\text{inf}_A i]$. Since covers and intersections pullback over functors, $F^{-1}[[i,i']]_{i_1 \to i_2}$ covers $J$ and is closed under intersection. By hypothesis, $[F^{-1}[[i,i']]_{i_1 \to i_2}] \simeq \{\ast\}$. Theorem \cite{2.3} implies $[F^{-1}F^{-1}(i \downarrow I)]$ and $[i \downarrow I]$ have the same weak homotopy type. The result follows as $i \downarrow I$ has an initial object, namely the identity on $i$. \hfill \Box

2.6. Transport categories. Transport categories are a specialization of the Grothendieck construction \cite{47} for a small diagram of small categories.

Definition 2.9. Given a (small) diagram of sets $X : C \to \text{Sets}$, the transport category of $X$, denoted $\text{Tr}(X)$ has

\begin{equation}
\text{Objects}(\text{Tr}(X)) = \{(c,x) : c \in C, x \in X(c)\}
\end{equation}

\begin{equation}
\text{Hom}_{\text{Tr}(X)}((c,x),(c',y)) = \{f \in \text{Hom}_C(c,c') : X(f)(x) = y\}
\end{equation}

with composition induced by $C$.

For any Coxeter group $W$ with $S$ the set of subsets of generators that generate finite groups \cite{6}, our fundamental example of a transport category is $\text{Tr}(X)$ for $X : S \to \text{Sets}$ defined via $I \mapsto W/W_I$. We call this category $W_S$. Explicitly, $W_S$ is a poset whose elements are finite type cosets and there exists a (unique) morphism from $w_{W_I}$ to $v_{W_J}$ precisely when $w_{W_I} \leq v_{W_J}$. Its geometric realization $|W_S|$ is the Davis complex \cite{12}. 
We also wish to record a proposition which is a specialization and enrichment of Thomason’s Theorem [47, Theorem 1.2]. We indicate an explicit proof.

**Proposition 2.10.** For any diagram of spaces over a transport category \( D : \text{Tr}(X) \to \text{Spaces} \), there is a diagram of spaces \( D' : C \to \text{Spaces} \) over the underlying category such that \( \text{hocolim}_{\text{Tr}(X)} D \) and \( \text{hocolim}_C D' \) are canonically weakly equivalent.

**Proof.** Let \( D' : C \to \text{Spaces} \) be defined by \( D'(c) := \text{hocolim}_{X(c)} D(c, x_c) \). We have a natural weak equivalence

\[
\text{hocolim}_{\text{Tr}(X)} D' \sim \text{hocolim}_C (\text{hocolim}_{X(c)} D|_{X(c)}) = \text{hocolim}_C D'
\]

which can be observed by inspecting the standard models. More explicitly, if one takes standard models the map \( (26) \) is specified by the universal property of the homotopy colimit and the maps

\[
D(c, x_c) \xrightarrow{D} D(c', y_{c'}) \longrightarrow \coprod_{z_{c'} \in X(c')} D(c', z_{c'})
\]

where the second map is the obvious inclusion. \( \square \)

Note that Proposition 2.10 gives a canonical equivalence between \( \text{hocolim}_C X \) and \( \text{[Tr}(X)] \).

2.7. Coset geometry and colimits of groups. Here we will see how homotopy colimit calculations in terms of subdiagrams apply to honest colimits.

**Theorem 2.11.** (Seifert-van Kampen [13]). Let \( D : I \to \text{Spaces} \) be a diagram of pointed connected spaces such that \( I \) has an initial object. Then, there is a natural isomorphism \( \pi_1 \text{hocolim}_ID \cong \text{colim}_I \pi_1 D \).

When \( D : I \to \text{Groups} \) is diagram of inclusions of subgroups of \( H \), we refer to the standard model of the homotopy fibre of \( \text{hocolim}_IH/D(I) \) as the **coset geometry** of the colimit. We call the transport category (see [20]) for the functor \( i \mapsto H/D(i) \) the **poset form of the coset geometry** since its geometric realization is canonically equivalent to the coset geometry. This represents a mild generalization of a notion of coset geometry employed by others, such as Tits [48] and Caprace and Remy [10]. For instance, our notion permits diagrams whose image in the lattice of subgroups is not full and our poset is directed in the opposite direction.

We can always add the trivial group to any diagram of inclusions of groups in an initial position without affecting the colimit of that diagram. Thus, verifying any presentation of a group as a colimit of subgroups reduces to homotopy knowledge of the coset geometry. An example of this technique gives the following additional corollary.

**Corollary 2.12.** Let \( D : I \to \text{Groups} \) be a diagram of the subgroups of \( H \) with each map an inclusion of subgroups such that \( I \) is simply connected. The canonical \( \text{colim}_I D \sim \to H \) is an isomorphism if and only if its coset geometry is simply connected.

**Proof.** We may artificially add an initial object, \( \bullet \), to \( I \) and call this new category \( I \cup \bullet \). We can extend \( D \) to \( I \cup \bullet \) by sending \( \bullet \) to the trivial group. Up to homotopy, the space \( Y := \text{hocolim}_{I \cup \bullet} BD \) is obtained from \( X := \text{hocolim}_I BD \) by coning off the subspace \( Z := \text{hocolim}_{I \cup \bullet}\{\ast\} \), the homotopy colimit base points. Using Theorem 2.11, the fundamental group \( \pi_1(\text{hocolim}_{I \cup \bullet} BD) \cong \text{colim}_I D \). Since \( \pi_1(Z) = 0 \), \( \text{colim} \) agrees with \( \pi_1(\text{hocolim}_I BD) \). By the long exact sequence of a fibration, the coset geometry of \( D \) is simply connected if and only if the canonical \( \text{colim}_I D \to H \) is an isomorphism. \( \square \)
Compare [11, Lemma 1.3.1] and [25, §3]. For example, as observed in [10], if $W$ is a Coxeter group with $S$ the set of subsets of generators of cardinality at most 2 that generate finite groups, then the colimit presentation $\operatorname{colim}_{S} W_{I} \cong W$, which is immediate from the presentation of $W$ [3], implies that the associated coset geometry is simply connected.

Remark 2.13. When the conditions for Corollary 2.12 are satisfied, a homotopy decomposition of a classifying space in terms of the classifying spaces of subgroups induces a colimit presentation. Though not stated explicitly, the homotopy decompositions of Theorem 3.1, Lemma 3.6, and Corollary 3.7 as well as the known homotopy decompositions (26–27) all induce colimit presentations. Of course, a homotopy decomposition is strictly stronger than a colimit presentation. The former is characterized by having a contractible coset geometry whereas latter only requires a simply connected coset geometry. For instance, $\operatorname{colim}_{S} W_{I} \cong W$ is induced by a homotopy decomposition of $BW$ only if $S = S$.

2.8. Known homotopy decompositions. For any Coxeter group $W$ with $S$ the poset of subsets of generators that generate finite groups

$$\operatorname{hocolim}_{I \in S} BW_{I} \cong BW$$

is a homotopy equivalence including, trivially, the cases where $W$ is finite. For non-trivial decompositions, the associated coset geometry is Davis’s version of the Coxeter complex, $[W_{S}]$, which is known to be contractible [12][11]. Theorem 3.2 below provides a combinatorial proof. Likewise, for any discrete $BN$–pair the canonical map induced by inclusion of subgroups

$$\operatorname{hocolim}_{I \in S} BP_{I} \cong BG$$

is a homotopy equivalence including, trivially, the cases where $W$ is finite. Here non-trivial decompositions have Davis’s version of the Tits building as coset geometries. For completeness, we also note that (26) and (27) can be deduced from the contractibility of the usual Coxeter complex (resp. Tits building) inductively because this provides a method to verify (27) for complex topological Kac-Moody groups. More specifically, for any topological $G$, a $BN$–pair with closed parabolic subgroups and infinite Weyl group $W$ the canonical map $\operatorname{hocolim}_{I \in R} BP_{I} \rightarrow BG$ for $R$ the set of all proper subsets of the generating set of $W$ has contractible coset geometry [14]. Thus, this map is a weak homotopy equivalence. Because parabolic subgroups inherit the structure of a $BN$–pair and inductively (on $|I|$) have homotopy decompositions (27) in terms of finite type parabolics, Theorem 2.6 may be applied to obtain (27) for $BG$ whenever all parabolic subgroups are closed. This proof of (27) for topological $BN$–pairs is implicit in [7, 33–35]. Note that (26) and (27) are sharp in the sense that any proper subposet of $S$ will not suffice to give a homotopy decomposition.

Recall that each parabolic subgroup of a Kac-Moody group has a semi-direct product decomposition $P_{I} \cong G_{I} \ltimes U_{I}$ [13] with these $G_{I}$ called Levi component subgroups. For $K$ a topological Kac-Moody group (with its topology induced by the complex numbers) the subgroups $U_{I}$ for $I$ finite type are known to be contractible [30]. This fact with [13] and (27) implies that the inclusion of the Levi component subgroups induces a homotopy decomposition for topological Kac-Moody group classifying spaces

$$\operatorname{hocolim}_{I \in S} BG_{I} \cong \operatorname{hocolim}_{I \in S} BP_{I} \cong BK$$

where $G_{I}$ are reductive, Lie, Levi component subgroups, cf. [33][35]. This decomposition has been one of the fundamental tools for the study $BK$ using homotopy theory [2][3][7][54].
Though discrete $U_I(R)$ over a commutative ring $R$ are in general quite far from being contractible (see 5.3), they are expressible in terms of honestly unipotent subgroups, i.e. each subgroup can be embedded into the subgroup of $GL_n(R)$ of upper triangular matrices with unit diagonal for some $n$. The next section will make this precise.

3. Homotopy Decompositions

The positive unipotent subgroup of a discrete Kac-Moody group $U^+ = U_∅$ is expressible as a colimit of finite dimensional unipotent algebraic subgroups $U_v$ as conjectured by Kac and Petersen [31]. This colimit presentation is induced (see Remark 2.13) by an integral homotopy decomposition and in fact all $BU_I(R)$ are expressible in terms of unipotent subgroup classifying spaces. The proof relies only on the combinatorics of the RGD system (see 2.3) associated to a Kac-Moody and applies to many variants of Kac-Moody groups (cf. [1,10] for discussion) as well other groups with RGD systems.

**Theorem 3.1.** Let $U^+ = U_∅$ be the positive unipotent subgroup of a discrete Kac-Moody group whose Weyl group is viewed as a poset $W$ with respect to the weak Bruhat order [3]. Then, the canonical map induced by inclusion of subgroups

$$\text{holim}_W BU_w \cong U^+$$

as conjectured by Kac and Petersen [31]. This colimit presentation is induced (see Remark 2.13) by an integral homotopy decomposition and in fact all $BU_I(R)$ are expressible in terms of unipotent subgroup classifying spaces. The proof relies only on the combinatorics of the RGD system (see 2.3) associated to a Kac-Moody and applies to many variants of Kac-Moody groups (cf. [1,10] for discussion) as well other groups with RGD systems.

**Theorem 3.2.** For any Coxeter group $W$, the longest element functor $L : W_S \rightarrow W$, $vW_I \rightarrow v_J$ [6], pullbacks homotopy colimits so that the natural map

$$\text{holim}_{W_S} DL \rightarrow \text{holim}_W D$$

is a weak homotopy equivalence for any diagram of spaces $D : W \rightarrow \text{Spaces}$.

This immediately implies that the Davis complex is contractible since the poset $W$ has the identity of $W$ as its initial object. When $W \cong (\mathbb{Z}/2\mathbb{Z})^n$, Theorem 3.2 lets us replace a diagram over an infinite depth tree $(W)$ with a canonical diagram over depth one tree $(W_S)$ obtained via barycentric subdivision. Theorem 3.2 allows this procedure to be extended to posets indexed by more general Coxeter groups.

3.1. The proof of Theorem 3.2

Our proof of Theorem 3.2 will inductively pullback homotopy colimits over closed intervals and apply Proposition 2.8. For instance, the following corollary—which will be used in 5.3—is immediate from our proof by Proposition 2.8.

**Corollary 3.3.** With the definitions of Theorem 3.2, let $X \subseteq W$ be a full subposet covered by a collection of intervals $[v, w]$, then the longest element map $L|_{L^{-1}X}$ pulls back homotopy colimits.
We may also extend Theorem 3.2 to situations where $W$ is the fundamental
domain of a group action on a poset. For our purposes here, we will take $W$ to
be Weyl group of a group with RGD system. We define $V \cdot W$ for any $V \leq U^+$
to be the $V$–orbit under left multiplication of $W$ realized as $\{U_w\}_{w \in W}$
within the poset of cosets of subgroups of $U^+$. We also define $V \cdot W_S$ so that $L$
extends to a $V$–equivariant functor $V \cdot L : V \cdot W_S \to V \cdot W$, i.e. $wW_I$
corresponds to $U_{wi}$.

**Corollary 3.4.** Let $W$ be realized as the poset of subgroups $\{U_w\}_{w \in W}$
ordered by inclusion. Then $V \cdot L : V \cdot W_S \to V \cdot W$ pulls back homotopy colimits
for any $V \leq U^+$.

**Proof.** Observe that $V \cdot W$ is the transport category for $X : W \to \text{Sets}$
defined by $w \mapsto V/(V \cap U_w)$ and $V \cdot W_S$ is that transport category
for $X_S : W_S \to \text{Sets}$ defined by $wW_I \mapsto V/(V \cap U_{wi})$. As in the proof of Proposition 2.10 any fixed
functor $V \cdot D : V \cdot W \to \text{Spaces}$ is associated to $D : W \to \text{Spaces}$
defined by $D(w) := \coprod_{(v \in U_w)} D(v(V \cap U_w))$. If $V \cdot D$ is pulled back along $V \cdot L$, then we obtain
$(V \cdot D)(V \cdot L) : V \cdot W_S \to \text{Spaces}$ which is associated to $DL : W_S \to \text{Spaces}$ by
the same procedure. Thus, we have a diagram

\[
\begin{array}{ccc}
\text{hocolim}_{V \cdot W_S} (V \cdot D)(V \cdot L) & \sim & \text{hocolim}_{W} DL \\
V \cdot L & \downarrow & L \\
\text{hocolim}_{V \cdot W} V \cdot D & \sim & \text{hocolim}_{W} D
\end{array}
\]

that commutes up to homotopy and the horizontal maps are weak equivalences
by Proposition 2.10. Now, by Theorem 3.2, the right vertical map is a weak equivalence
and $V \cdot L$ pulls back homotopy colimits.

For instance, with a bit of reflection, we see that the functor $U^+ \cdot L : U^+ \cdot W_S \to
U^+ \cdot W$ directly relates the poset forms of the coset geometries associated to
the homotopy decompositions of Lemma 3.4 (appearing below) and Theorem 3.1 in the
$I = \emptyset$ case. We also note a useful observation.

**Proposition 3.5.** Define $w(I)$ as the longest $v \in W_I$ such that $v \leq w$ in $W$.
The functor $L_I : W \to W_I$ given by $w \mapsto w(I)$ pulls back homotopy colimits.
Moreover, $L_{L^{-1}[v,w]}$ pulls back homotopy colimits for all intervals and $V \cdot L_I$ pulls
back homotopy colimits for all $V \leq U^+$ as in Corollary 3.4.

**Proof.** Note that $w \downarrow L_I = L_I^{-1}(w(I) \downarrow W_I) = w(I) \downarrow W$ and $w(I) \downarrow W$ has
initial object $w(I)$. For closed intervals, $L_I^{-1}([v,w])$ still has initial object $w(I)$.
The functor $V \cdot L_I$ is defined as in Corollary 3.4 with respect to the base $W_I$, i.e.
$W_I$ is realized as the poset of subgroups $\{U_w\}_{w \in W_I}$. With this definition, $V \cdot L_I$
pulls back homotopy colimits for all $V \leq U^+$ as in the proof Corollary 3.4. \qed

**Proof of Theorem 3.2.** By Proposition 2.3 it is sufficient to show that the geometric
realization of the poset

\[ X^w_v := L^{-1}[v,w]. \]

is contractible for all $v \leq w \in W$. Define $I[v,w]$ to be the maximal chain length
in $[v,w]$. Let us proceed by induction on this length. If $v = w$, then $wW_{I_v}$ is the
(unique) terminal object of $X^w_v = w$ and $|X^w_v| \simeq \{\ast\}$.

Fix $(v,w)$ with $v < w$. We will cover $X^w_v$ as a category and apply Theorem 2.6
to show that $|X^w_v| \simeq \{\ast\}$, inductively. Let us start to define the elements of this
cover precisely.

For any $w \in W$, there is a unique greatest $I_w \in S$ such that $w$ is longest in
$wW_{I_w}$. By Theorem 2.4 $I_w$ is precisely the set of all $i$ such that $s_i$ is the right most
letter of some reduced word expression of $w$. Define $Y_w$ to be the full subcategory of the poset $W_S$ with
\[ \text{Objects}(Y_w) = \{ vW_J \in W_S | vW_J \leq wW_{I_w} \}. \]

Now, $X^v_w$ is a full subcategory of $W_S$ and
\[
\text{Objects}(X^v_w) \subseteq \bigcup_{v \leq x \leq w} \text{Objects}(Y_x)
\]
where $v \leq w$ refers to the weak Bruhat order on $W$ [5]. Because all chains in $W_S$ are contained within some $Y_x$, $X^v_w$ is covered by $\{Y_x \cap X^v_w \}_{v \leq x \leq w}$, as a category (see Definition 2.5).

We now see that $X^v_w$ is covered as a category by $X^v_{wI}$ for all $i \in I_w$ and $\{X^v_w \cap Y_w\}$ since
\[
\bigcup_{i \in I_w} \text{Objects}(X^v_{wI}) = \bigcup_{v \leq x \leq w} \text{Objects}(Y_x \cap X^v_{wI})
\]
where $v \leq w$ refers to the weak Bruhat order on $W$. Define $y$ to be the shortest element of $wW_{I_w}$. Any (non-empty) element of this category contains the singleton coset $\{z := \sup\{v, y\}\}$ which exists and is less than or equal to $w$ since $v < w$ and $y < w$. Thus, we can close this cover under intersection without introducing empty categories.

If $m \leq |I_w| + 1$ is the number of elements of this cover, we may define $U : \Delta_{m+1} \to \text{Cat}$ via $\{i_1, \ldots, i_k\} \mapsto U_{i_1} \cap \ldots \cap U_{i_k}$ for some enumeration of this cover. Here $\Delta_{m+1}$ is the category of inclusions of facets in the standard $m + 1$-simplex. By Theorem 2.4, $\text{hocolim}[U]$ is weakly homotopy equivalent to $|X^v_w|$. As $|\Delta_m| \simeq \{\ast\}$, it is enough to show that each $|U_{i_1} \cap \ldots \cap U_{i_k}| \simeq \{\ast\}$ by induction on $l[v, w]$. In fact, we will show that each element of this cover is isomorphic to some $X^v_w$ with $l[v, w] < l[v, w]$. Observe that $|X^v_w| \simeq \{\ast\}$.

**Case 1**: $\bigcap_{i \in I_w} X^v_{wI}$. Any non-empty intersection of $X^v_{wI}$ for $i \in I_w$ is equal to some $X^v_x$ with $x < w$ because any intersection of intervals $[v, w]$ will be some interval $[v, x]$ and intersections pullback along functors. Inductively, $|X^v_x| \simeq \{\ast\}$.

**Case 2**: $\bigcap_{i \in I_w} X^v_{wI} \cap Y_w$. Any intersection of $Y_w$ and at least one non-empty $X^v_{wI}$ with $i \in I_w$ will be equal to $X^v_{wI} \cap Y_w$ for some $x < w$. Multiplication by $y$ induces an isomorphism of posets
\[
X^v_{wI} \xrightarrow{\cong} Y_w
\]

for $e_{I_w}$, the longest word in $W_{I_w}$. Now, $X^v_x \cap Y_w = X^v_z \cap Y_w$ for $z = \sup\{v, y\}$ and $X^v_z \cap Y_w$ is in bijection with $X^v_{y^{-1}z}$ under (32). Observe that $l[y^{-1}z, y^{-1}x] = l[z, x] \leq l[v, x] < l[v, w]$ as $v \leq z \leq x < w$. By induction,
\[ |X^v_z \cap Y_w = X^v_z \cap Y_w \cong X^v_{y^{-1}z} \simeq \{\ast\}. \]

**Case 3**: $X^v_w \cap Y_w$. In this case, there is terminal object, namely $wW_{I_w}$, and $|X^v_w \cap Y_w| \simeq \{\ast\}$. This completes the proof. \hfill \square

### 3.2. New homotopy decompositions for groups with RGD systems

Theorem 3.1 will follow from Lemma 3.6 and Lemma 3.8 (below) by pulling back appropriate homotopy colimits.
Lemma 3.6. Let $U^+ = U_\emptyset$ be the positive unipotent subgroup of a group with RGD system and $W_S$ be the poset underlying the Davis complex (see [2.0]). Then the canonical map

$$\text{hocolim}_{W_S} BU \xrightarrow{\sim} BU^+$$

induces a homotopy equivalence where $w_J$ is the longest word in $wW_J$ under the weak Bruhat order [23]. More generally, if the standard parabolic subgroup $P_I$ has a Levi decomposition then

$$\text{hocolim}_{U^+_I \cdot W_S} BU(\hat{u}, w) \xrightarrow{\sim} BU^+_I$$

where $U^+_I \cdot W_S$ is the poset defined in Corollary 3.4 and each $U(\hat{u}, w)$ is isomorphic to a subgroup of some $U_v$. 

Proof. Let us first show the $I = \emptyset$ case. We calculate

$$BU^+ \simeq EU^+ \times_{U^+} \{\ast\} \simeq EU^+ \times_{U^+} \text{hocolim}_G G/P^-_J$$

where the fourth equivalence requires the generalized Birkhoff decomposition [13] and the fifth uses the isomorphism of posets $wP^-_J \mapsto wW_J$ and Proposition 2.10 for $X : S \to \text{Sets}$ defined via $I \mapsto W/W_I$. Alternatively, it is not overly difficult to check directly that the map $u wP^-_J \mapsto (wW_J, u \text{Stab}_{U^+_I} \{wP^-_J\})$ is an isomorphism of the poset forms of the coset geometries associated to $\text{hocolim}_{W_S} B(\text{Stab}_{U^+_I} \{wP^-_J\})$ and $\text{hocolim}_{W_S} B(\text{Stab}_{U^+_I} \{wP^-_J\})$.

Thus, $U^+_I \simeq U^+_I \times U^+_I \text{hocolim}_G G/P^-_J$

where

$$G = \prod_{w \in W/W_J} U^+ wP^-_J = \prod_{w \in W/W_J} U^+_I \hat{u} wP^-_J$$

by the generalized Birkhoff decomposition [13]. Thus,

$$BU^+_I \simeq EU^+ \times_{U^+_I} \text{hocolim}_G G/P^-_J$$

as in [12]. Here the map $u \hat{u} wP^-_J \mapsto (wW_J, u \text{Stab}_{U^+_I} \{u \hat{u} wP^-_J\})$ is an isomorphism of the poset forms of the coset geometries associated to $\text{hocolim}_{U^+_I \cdot W_S} B(\text{Stab}_{U^+_I} \{u \hat{u} wP^-_J\})$ and $\text{hocolim}_{W_S} B(\text{Stab}_{U^+_I} \{u \hat{u} wP^-_J\})$, respectively, for $u \in U^+_I$ and $\hat{u} \in \hat{U}^+_I$.

Let us characterize $\text{Stab}_{U^+_I} \{u \hat{u} wP^-_J\} = U^+_I \cap \hat{u} wP^-_J w^{-1} \hat{u}^{-1}$. We see

$$\hat{u}^{-1} U^+_I \hat{u} \cap wP^-_J w^{-1} = \hat{u}^{-1} U^+_I \hat{u} \cap U^+ \cap wP^-_J w^{-1}$$

$$= \hat{u}^{-1} U^+_I \hat{u} \cap wP^-_J$$

This $\text{Stab}_{U^+_I} \{u \hat{u} wP^-_J\} = \hat{u} U wP^-_J \hat{u}^{-1} \cap U^+_I \cap wP^-_J w^{-1}$. As $J$ has finite type, each $\hat{u} U wP^-_J \hat{u}^{-1} = \hat{u} U wP^-_J \hat{u}^{-1} \simeq U w, \hat{u}^{-1} \simeq U w$ by Lemma 2.3. This completes the proof. 

Recall that $G_I$ [17] has a root group data structure with Weyl group $W_I$. The positive unipotent subgroup of this root group data structure is $\hat{U}^+_I$. Applying Theorem 3.2 to (33) gives the following.
Corollary 3.7. Under the assumptions of Lemma 3.6, the canonical map induced by inclusions of subgroups
\[
\text{hocolim}_{w \in \mathcal{W}I} BU_w \xrightarrow{\sim} B\hat{U}_I^+
\]
is a homotopy equivalence where \( \mathcal{W}I \) is a poset under the weak Bruhat order (5).

We are now ready to prove our final lemma for Theorem 3.1.

Lemma 3.8. The posets \( \hat{U}_I^+ \cdot W_S \) and \( \hat{U}_I^+ \cdot W \), defined in Corollary 3.4, have contractible geometric realizations.

Proof. Recall the definition of \( w(I) \) as the longest \( v \in \mathcal{W}I \) such that \( v \leq w \) in \( \mathcal{W} \). Note that (12) implies \( U_w \cap \hat{U}_I^+ = \bigcap_{v \in \mathcal{W}I} U_v \cap U_w = U_w(I) \). Thus, we have a commutative diagram of fibrations over \( BU_I^+ \) induced by inclusions of subgroups

with the vertical maps induced by the indicated functors of index categories as defined in 3.1. In particular, the vertical maps are weak equivalences by Theorem 3.2, Corollary 3.4 and Proposition 3.5. The bottom fibration is simply the homotopy decomposition (33) of Lemma 3.6 for the (not necessarily compact) Levi factor \( G_I \) which carries a RGD structure with positive unipotent subgroup \( \hat{U}_I^+ \). Thus, we have \( |\hat{U}_I^+ \cdot W_S| \simeq |\hat{U}_I^+ \cdot W| \simeq |\hat{U}_I^+ \cdot W| \simeq \{\ast\} \) which completes the proof. □

Proof of Theorem 3.1. Note that for discrete Kac-Moody groups the subgroups \( U_w \) are unipotent and Levi decompositions always exist (see Section 2.3). Lemma 3.6 implies that all the statements of Theorem 3.1 follow from Theorem 3.2 and Corollary 3.4 except the claim that \( |\hat{U}_I^+ \cdot W| \) is contractible. This final claim is shown in Lemma 3.8 □

4. Main Results

We now can express the classifying space of the unipotent factor of any parabolic subgroup of a discrete Kac-Moody group \( BU_I^+(R) \) in terms of a countable collection of classifying spaces of unipotent subgroups. In particular, when all these unipotent subgroups have vanishing homology, our decompositions imply that the homology of \( BU_I^+(R) \) must vanish. This provides sufficient input to express \( BK(R) \) as in Theorem B. For simplicity, we state our results in this section for Kac-Moody groups over fields.
4.1. Vanishing and simplification away from $p$. Our main application of the results of \[2\] is Theorem \[C\]

Proof of Theorem \[A\]  Fix fields $\mathbb{F}$ and $\mathbb{L}$ of different characteristics. First note that $BU_w(\mathbb{F})$ is a $\mathbb{L}$–homology point since each $U_w(\mathbb{F})$ has a normal series of length $l(w)$, i.e. the reduced word length of $w$, with quotient groups isomorphic to $(\mathbb{F},+)$, of course, $H_n(\mathbb{F},\mathbb{L}) = 0$ for all $n > 0$, cf. \[13\] Theorem 6.4, p. 123. For example, if $\mathbb{F} = \mathbb{F}_p$ each $U_w(\mathbb{F}_p)$ is a $p$–group with $p^{\ell(w)}$ elements; compare \[11\].

By the homology spectral sequence for homotopy colimits \[6\], $H_*(BU^+(\mathbb{F}),\mathbb{L})$ is the homology of the poset $W_S$ underlying the homotopy decomposition \[33\]. Since the Davis complex $|W_S|$ is contractible, $H_n(BU^+(\mathbb{F}),\mathbb{L}) = 0$ for all $n > 0$, completing the proof in the case of $I = \emptyset$. Theorem \[3,1\] implies that same proof will work for all $I$ since all $U_{(\bar{u},w)}(\mathbb{F})$ are unipotent and $|\hat{U}_I^J(\mathbb{F}) \cdot W_S| \simeq \{\ast\}$. □

An independent proof of the rank two case of Theorem \[C\] appears in \[5\]. Notice that the proof of Lemma \[3,5\] brings together the full combinatorial tool kit to show that $|\hat{U}_I^J(\mathbb{F}) \cdot W_S|$ has vanishing homology. Theorem \[A\] will follow from an analog of the decomposition of a topological Kac-Moody group \[28\] for Kac-Moody groups over a field of characteristic $p$. We now state a precise version of Theorem \[B\]

Theorem 4.1. Let $K(\mathbb{F})$ be a Kac-Moody group over a field of characteristic $c$ with standard parabolic subgroups $P_I(\mathbb{F})$ and $G_I(\mathbb{F})$ the reductive, $\mathfrak{L}$ Levi component subgroups for all $I \in S$. The canonical map induced by inclusions of subgroups

\[(36) \quad \text{hocolim}_{I \in S}BG_I(\mathbb{F}) \xrightarrow{\sim} BK(\mathbb{F})\]

is a $q$–equivalence for any prime $q \neq c$ and a rational equivalence for $c > 0$.

Proof. In this proof all groups mentioned are over a fixed $\mathbb{F}$ that is suppressed in the notation. Consider the fibration sequence

\[(37) \quad BU_I \longrightarrow BP_I \longrightarrow BG_I\]

arising from the semidirect product decomposition $P_I \cong G_I \ltimes U_I$ \[18\]. By Theorem \[C\] $BU_I(\mathbb{F})$ is a $\mathbb{F}_q$–homology point (or $\mathbb{Q}$–homology point for $q = 0$) for all $I \in S$. The Serre spectral sequence for the fibration \[37\] shows $B(P_I \xrightarrow{\pi} G_I)$ induces an isomorphism on $\mathbb{F}_q$–homology. The inclusion $G_I \xrightarrow{i} P_I$ is a section of $\pi$ and must induce the inverse of $H_*(B(P_I \xrightarrow{\pi} G_I),\mathbb{F}_q)$ on homology by naturality. Recalling \[27\] the natural maps induced by inclusion of subgroups

\[
\text{hocolim}_{I \in S}BG_I \xrightarrow{\sim} \text{hocolim}_{I \in S}BP_I \xrightarrow{\sim} BK
\]

compose to yield the desired $q$–equivalence. □

Proof of Theorem \[A\]  In the case at hand, Theorem 4.1 implies

\[(38) \quad \text{hocolim}_{I \in S}BG_I(\overline{\mathbb{F}}_p) \xrightarrow{\sim} BK(\overline{\mathbb{F}}_p)\]

where $\overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$. Referring back to the construction Friedlander and Mislin used to produce Theorem \[A\] for reductive $G$ \[23\] Theorem 1.4], the maps $BG_I(\overline{\mathbb{F}}_p) \rightarrow BG_I$ are induced by the zig-zag of groups

\[(39) \quad G_I(\overline{\mathbb{F}}_p) \longleftarrow G_I(\text{Witt}(\mathbb{F}_p)) \longrightarrow G_I(\mathbb{C}) \longrightarrow G_I(\mathbb{R})\]

where Witt$(\mathbb{F}_p)$ → $\mathbb{C}$ is a fixed choice of embedding of the Witt vectors of $\mathbb{F}_p$ into $\mathbb{C}$ and $G_I(R)$ denotes the discrete group over $R$ of the same type as the (topological) complex reductive Lie group $G_I$. Moreover, the maps of \[39\] are natural with
respect to the maps of group functors $G_I(-) \leftrightarrow G_J(-)$. Taking classifying spaces, we have compatible maps

$$BG_I(\mathbb{F}_p) \leftrightarrow BG_I(\text{Witt}(\mathbb{F}_p)) \rightarrow BG_I(\mathbb{C}) \rightarrow BG_I$$

which are all $q$–equivalences. Localizing at a prime $q$ distinct from $p$, we obtain compatible $BG_I(\mathbb{F}_p)^\wedge_q \xrightarrow{\sim} BG_I^\wedge_q$. Compatible $BG_I(\mathbb{F}_p) \rightarrow BG_I$ are then produced via an arithmetic fibre square \[ \text{[45]} \] and induce a $q$–equivalence

\[ (40) \quad \text{hocolim}_{I \in S} BG_I(\mathbb{F}_p)^\wedge_q \xrightarrow{\sim} \text{hocolim}_{I \in S} BG_I. \]

Thus, equations \[ (27),(29), (39) \] and \[ (40) \] induce

$$BK(\mathbb{F}_p) \xrightarrow{\sim} \text{hocolim}_{I \in S} BP_I(\mathbb{F}_p) \xrightarrow{\sim} \text{hocolim}_{I \in S} BG_I(\mathbb{F}_p) \xrightarrow{\sim} \text{hocolim}_{I \in S} BG_I \xrightarrow{\sim} BK.$$

Choosing a fixed homotopy inverse for the arrow pointing to the left, we obtain the desired map. \[ \square \]

Remark 4.2. Theorem [A] for reductive $G$ is one instance of the Friedlander–Milnor conjecture \[ [22] \] in which $BG(\mathbb{F}_p)$ approximates $BG$ homologically. More generally, natural discrete approximations of $BG$ by $BG(\mathbb{F})$ can be extended to discrete approximations of Kac-Moody groups via Theorem \[ [4,4] \] as in the proof of Theorem \[ [4] \]. For example, Morel’s \[ [10] \] confirmation of the Friedlander–Milnor conjecture for specific $G$ of small rank extends to homological approximations of $BK$ by $BK(\mathbb{F})$ for any separably closed field $\mathbb{F}$ whenever $K$ has a cofinal subposet $C$ of $S$ with the property that $I \in C$ implies $G_I = H \times T$ for some $T = (\mathbb{C}, \times)^k$ and $H \in \{ SL_3, SL_4, SO_5, G_2 \}$.

4.2. Unstable Adams operations for Kac-Moody groups. For $q \neq p$, we will construct a local unstable Adams operation $\psi^k : BK^\wedge_q \rightarrow BK^\wedge_q$ compatible with the Frobenius map. When $W$ has no element of order $p$, we can assemble the local Adams maps via the arithmetic fibre square to obtain a global unstable Adams operation $\psi^k : BK \rightarrow BK$.

Proof of Theorem [D]. Let us first construct $\psi^k : BK^\wedge_q \rightarrow BK^\wedge_q$ for $q \neq p$ by noting that, localizing of the map \[ (40) \], we have homotopy equivalences

$$\text{hocolim}_{I \in S} BG_I(\mathbb{F}_p)^\wedge_q \xrightarrow{\sim} \text{hocolim}_{I \in S} BG_I^\wedge_q \xrightarrow{\sim} BK^\wedge_q.$$ 

As in the proof of Theorem \[ [A] \] we have compatible $BG_I(\mathbb{F}_p)^\wedge_q \xrightarrow{\sim} BG_I^\wedge_q$ induced by the zig-zag of groups

$$G_I(\mathbb{F}_p) \leftarrow G_I(\text{Witt}(\mathbb{F}_p)) \rightarrow G_I(\mathbb{C}) \rightarrow G_I$$

where Witt($\mathbb{F}_p$) $\rightarrow \mathbb{C}$ is a fixed embedding of the Witt vectors of $\mathbb{F}_p$ into $\mathbb{C}$ and $G_I(R)$ denotes the discrete algebraic group over $R$ of the same type as the (topological) complex reductive Lie group $G_I$. From the naturality of the group functors, we have explicit topological models so that

$$BG_I(\mathbb{F}_p) \xrightarrow{BG_I(\psi^k)^\wedge_q} BG_I(\mathbb{F}_p)$$

commutes for all $I \subset J$ in $S$. Localizing, we have a map of diagrams and $\psi^k := (\phi^k)^\wedge_q$ extends to $\text{hocolim}_{I \in S} BG_I(\mathbb{F}_p)^\wedge_q \simeq BK(\mathbb{F}_p)^\wedge_q \simeq BK^\wedge_q$. 

\[ \square \]
When $W$ has no elements of order $p$, we will work one prime at a time and assemble the maps using an arithmetic fibre square. We have already constructed $\psi^k : BK_q^\wedge \to BK_q^\wedge$ for $q \neq p$. At $p$, we have a homotopy equivalence

$$BN_p^\wedge \simto BK_p^\wedge$$

where $N$ is the normalizer of the maximal torus and this map is induced by the inclusion of groups [33]. Kumar [36, 6.1.8] presents $N$ as being generated by $T$ and $\{s_1, \ldots, s_n\}$ so that under the projection $\pi : N \to W$, $s_i$ maps to a standard generator of $W$. Thus, we can attempt to define $\theta : N \to N$ in terms of these generators and need only check that Kumar’s relations are satisfied to obtain a group homeomorphism. To get an unstable Adams operation we choose $t \mapsto t^{v_k}$ and $s_i \mapsto s_i$. Note that here $p$ is odd and this is needed to verify Kumar’s relations.

Because $BK$ is a simply connected $CW$--complex [34], it is given as a homotopy pullback

$$\begin{array}{c}
\begin{array}{ccc}
BK & \simeq & BK_2^\wedge \prod^{(-)^\wedge_q} \Pi BK_q^\wedge \\
\downarrow & & \downarrow \\
BK_q^\wedge & \simeq & \prod^{(-)^\wedge_q} \Pi (BK_q^\wedge)_q \\
\end{array}
\end{array}$$

known as the arithmetic fibre square where $(-)^\wedge_q$ and $(-)^\wedge_N$ denote localization with respect to $\mathbb{Z}$ and $\mathbb{Q}$ homology, respectively [15]. Now, by (28) we have homotopy equivalence

(42)  
$$BK_2^\wedge \simeq (\hocolim_{I \in S} BG_I^\wedge)_Q \simeq (\hocolim_{I \in S} \prod K(2m_i, Q))^\wedge_I$$

where the $m_i$ vary for different $BG_I$ and $K(n, Q)$ denotes the $n^{th}$ Eilenberg-MacLane space [18]. The homomorphism $t \mapsto t^{v_k}$ of $S^1$ induces compatible self maps of the $K(2m_i, Q) \simeq B^{2m_i-1}(S^1)_q$ appearing in (42). We may now define the desired map with (11).

**Remark 4.3.** In general, twisted Adams operations beyond those constructed here are expected. See [4] for rank two examples. For instance, when ever pullback up to homotopy, among maps that restrict to the self-map $\theta$ in rank two examples. Generally, we do not expect homotopy limits and colimits to commute. Our calculations in the next section that show that they rarely do in rank two examples.

**Question 4.4.** Are the unstable Adams operations constructed in Theorem [28] unique, up to homotopy, among maps that restrict to the self-map $B(t \mapsto t^{v_k})$ of $BT$?

Now, that we have $\psi^k : BK_q^\wedge \to BK_q^\wedge$ let us note that when comparing $(BK_q^\wedge)^{\psi^k}$ and $BK(\mathbb{F}_{p^k})^\wedge_q$ there is a natural map

(43)  
$$BK(\mathbb{F}_{p^k})^\wedge_q \simeq (\hocolim_{I \in S} BG_I(\mathbb{F}_{p^k})^\wedge_q)^\wedge \to (BK_q^\wedge)^{\psi^k}$$

arising from the diagram $D : \mathbb{Z} \times \mathbb{S} \to \text{Spaces}$ via $(\cdot, W_I) \mapsto BG_I^\wedge$ on objects and $(n, W_I \leftrightarrow W_J) \mapsto (\psi^k)^n B(G_I \leftrightarrow G_J) = B(G_I \leftrightarrow G_J)(\psi^k)^n$ on morphisms. In particular, (43) is the localization of the canonical

(44)  
$$\hocolim_{\mathbb{Z}} (BG_I(\mathbb{F}_{p^k})^\wedge_q) \to \hocolim_{\mathbb{Z}} (\hocolim_{\mathbb{S}} BG_I(\mathbb{F}_{p^k})^\wedge_q).$$

Generally, we do not expect homotopy limits and colimits to commute. Our calculations in the next section that show that they rarely do in rank two examples.

**Question 4.5.** What is the structure of the homotopy fibre of (44)?
5. Cohomology Calculations

Theorem 4.1 allows us to study $BK(\mathbb{F}_p)$ in terms of finite reductive algebraic group classifying spaces, $BG_I(\mathbb{F}_p)$ for $I \in S$, after localizing at a prime $q$ distinct from $p$. Furthermore, 21 reduces the study of $BG_I(\mathbb{F}_p)$ to understanding homotopy fixed points $(BG_I(\mathbb{F}_p))^h\psi^I$ under stable Adams operations, $\psi^I$. Thus, in principle, only the cohomology of compact Lie group classifying spaces is needed as input data to compute $H^*(BK(\mathbb{F}_p), \mathbb{F}_q)$.

In this section, we begin such calculations. Recent work by Kishimoto and Kono 22 facilitates the determination of cohomology between $(BG_I(\mathbb{F}_p))^h\psi^I$ as $I$ varies. We then compare the results with $H^*((BK(\mathbb{F}_p))^h\psi^I, \mathbb{F}_q)$ for rank two, infinite dimensional Kac-Moody groups. We close with explicit calculations of $H^*(BU^+(\mathbb{F}_p), \mathbb{F}_q)$ in specific cases where $W \cong (\mathbb{Z}/2\mathbb{Z})^n$.

5.1. Restriction to Rank 2. To compare $(BK(\mathbb{F}_p))^h\psi^I$ and $BK(\mathbb{F}_p)$, we will partially compute their $\mathbb{F}_q$-cohomology rings. These computations will become more tractable by restricting to simply connected rank 2 Kac-Moody groups and $q$ odd. Notably $H^*(BK, \mathbb{F}_q)$ and its $\psi^I$-action can be determined explicitly. In the rank 2, non-Lie, case $BK(\mathbb{F}_p)^h\psi^I \simeq \text{hocolim}_{I \in S}(BG_I(\mathbb{F}_p))^h\psi^I$ is simply a homotopy pushout. To compute cohomology, we use the Mayer-Vietoris sequence. We will also restrict to $q$ odd, so that for $I \in S$

$$H^*(BG_I, \mathbb{F}_q) \cong H^*(BT, \mathbb{F}_q)^{W_I}$$

with the restriction map from $H^*(BG_I, \mathbb{F}_q)$ to $H^*(BT, \mathbb{F}_q)$ inducing this isomorphism 18. The determination of $H^*((BK(\mathbb{F}_p))^h\psi^I, \mathbb{F}_q)$ for $I \in S$ will occur in 5.2 this subsection will investigate $H^*((BK(\mathbb{F}_p))^h\psi^I, \mathbb{F}_q)$.

In the simply connected rank 2 case, we have 33

$$H^*(BK, \mathbb{F}_q) = \mathbb{F}_q[x_4, x_{2l}] \otimes A(x_{2l+1})$$

as a ring where $\Lambda$ denotes an exterior algebra and $l := l([a, b], q) \geq 2$ is a positive integer depending on $q$ and the generalized Cartan matrix for $K$, i.e. some non-singular $2 \times 2$ matrix given by

$$A = \begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix}$$

for $a$ and $b$ positive integers such that $ab \geq 4$. An explicit description of $l$ is given in 33. Notice non-singularity implies $ab > 4$ and each $A$ is associated with one simply connected $K$. Work in 3 will determine the map $\psi^I$ induces on $\mathbb{F}_q$-cohomology.

**Proposition 5.1.** For $K$ a rank 2, infinite dimensional complex Kac-Moody group and $q$ odd, $\psi^I$ acts on $H^*(BK, \mathbb{F}_q)$ via $(x_4, x_{2l}, x_{2l+1}) \mapsto (p^{2k}x_4, p^{2k}x_{2l}, p^{2k}x_{2l+1})$.

**Proof.** Because $H^*(BG_I, \mathbb{F}_q)$ is concentrated in even degrees for $I \in S$, the Mayer-Vietoris sequence associated to the homotopy pushout presentation of $BK$ 25 reduces to an exact sequence

$$0 \to H^{2s}(BK, \mathbb{F}_q) \to H^*(BG_I, \mathbb{F}_q) \oplus H^*(BG_2, \mathbb{F}_q) \to H^*(BT, \mathbb{F}_q) \to 0$$

where $H^{2s}(BK, \mathbb{F}_q) = H^*(BG_I, \mathbb{F}_q) \cap H^*(BG_2, \mathbb{F}_q) = \mathbb{F}_q[x_4, x_{2l}]$ and $H^{2s+1}(BK, \mathbb{F}_q) = (x_{2l+1})H^*(BK, \mathbb{F}_q)$ for $x_{2l+1}$ the image of a homogeneous degree $2l$ class under the connecting homomorphism 3. The $k^{th}$ unstable Adams operation $\psi^I$ acts on $H^*(BT, \mathbb{F}_q)$ via multiplication by $p^k$ on generators, and commutes with the restriction 15 $H^*(BG_I, \mathbb{F}_q) \to H^*(BT, \mathbb{F}_q)$ 10.
Moore spectral sequence associated to $H_2$. However, there is sufficient information at
$\eta$ and $\zeta$, which implies
\begin{align*}
\text{Table 1. The } & F_q\text{-algebra structure for } E_2^{*,*} \text{ converging to } \\
H^*((BK_q^\wedge)^{h\psi^k}, F_q) & \text{ with bidegrees } |x_i| = (-1, i + 1), |\gamma_n(x_i)| = \\
& (-n, (i + 1)n), \text{ and } |s_1| = (0, i).
\end{align*}

| $p_i^\ast \equiv 1$ (mod $q$) | $p_i^\ast \not\equiv 1$ (mod $q$) |
|---|---|
| $p^{\neq} = 1$ (mod $q$) | $p^{\neq} \neq 1$ (mod $q$) |
| $\Lambda(x_3, x_{2l-1}) \otimes \Gamma(x_{2l}) \otimes$ | $\Lambda(x_{2l-1}) \otimes \Gamma(x_{2l}) \otimes$
| $\Lambda(s_{2l+1}) \otimes \Lambda(s_{2l+1})$ | $\Lambda(s_{2l+1})$
| $p^{\neq} \not\equiv 1$ (mod $q$) | $\Lambda(x_3) \otimes F_q[s_{2l}] \otimes \Lambda(s_{2l+1})$

We are presently unable to fully compute $H^*((BK_q^\wedge)^{h\psi^k}, F_q)$ for all rank 2 Kac-Moody groups. However, there is sufficient information at $E_2^{*,*}$ in the Eilenberg-Moore spectral sequence associated to $H^*((BK_q^\wedge)^{h\psi^k}, F_q)$ to determine that in most rank 2 cases $H^*(BK(F_q^{p_k}), F_q) \neq H^*((BK_q^\wedge)^{h\psi^k}, F_q)$, in contrast to the Lie case.

**Theorem 5.2.** Consider the Eilenberg-Moore spectral sequence (EMSS) for the homotopy pullback of the diagram

\[
\begin{array}{c}
BK_q^\wedge \\
\Delta \\
BK_q^\wedge \times \psi^k \circ \Delta \\
\rightarrow (BK \times BK_q^\wedge)
\end{array}
\]

converging to $H^*((BK_q^\wedge)^{h\psi^k}, F_q)$ where $\psi^k$ is the $k$th unstable Adams operation.

For $q$ odd, $E_2^{*,*}$, as a $F_q$-algebra, is given in Table 1.

**Proof.** The $E_2^{*,*}$–page for the cohomological EMSS for (19) is given by

\[
\begin{align*}
\text{Tor}^{*,*}_{H^*(BK(F_q^{p_k}), H^*(BK(F_q), H^*(BK_q^\wedge)))} \\
\cong \text{Tor}^{*,*}_{F_q[x_3, x_{2l}, x_{2l+1}, x_{2l+2}]}(F_q[s_{2l}], \Lambda(s_{2l+1}), F_q[t_{2l}], \Lambda(t_{2l+1}))
\end{align*}
\]

Here left copy of $H^*(BK_q^\wedge)$ a $H^*(BK(F_q), H^*(BK_q^\wedge))$–module via $1 \times \psi^k \circ \Delta$ which is given by the ring homomorphism $(x_{2l}, y_{2l+1}) \mapsto (p^{\neq} s_{2l}, p^{\neq} s_{2l+1})$ and $x_i \mapsto t_i$ on generators by Proposition 5.1. Of course, the $H^*(BK_q^\wedge)$–module structure on the right $H^*(BK_q^\wedge)$ is given by $y_i, x_i \mapsto t_i$.

To simplify $E_2^{*,*}$ we will employ a change of ring isomorphism.

**Proposition 5.3** (Change of Rings [28], p. 280). Let $A$, $B$, and $C$ be $k$–algebras over a field $k$. Then,

\[
\text{Tor}^n_A(M, N) = \text{Tor}^n_C(M \otimes_A N, L) = 0
\]

for all $n > 0$ implies

\[
\text{Tor}^n_A(M \otimes_B N \otimes_C L) \cong \text{Tor}^n_B(M \otimes_C N, L)
\]

where $M$, $N$ and $L$ have the appropriate module structures so that (73) is well defined.

Let us use this change of ring isomorphisms with

\[
\begin{align*}
(A, B, C) &= (F_q[x_4, x_{2l}] \otimes \Lambda(x_{2l+1}), F_q[y_4 - x_4, y_{2l} - x_{2l}] \otimes \Lambda(y_{2l+1} - x_{2l+1}), F_q), \\
(M, N, L) &= (F_q[s_4, s_{2l}] \otimes \Lambda(s_{2l+1}), F_q[t_4, t_{2l}] \otimes \Lambda(t_{2l+1}, F_q)).
\end{align*}
\]

This gives

\[
E_2^{*,*} \cong \text{Tor}^{*,*}_{F_q[x_4, x_{2l}] \otimes \Lambda(x_{2l+1})}(F_q[s_4, s_{2l}] \otimes \Lambda(s_{2l+1}), F_q)
\]
Table 2. $E_2^{*,*}$ converging to $H^*((BK_q^\wedge)^{h\psi^k}, \mathbb{F}_q)$ in terms of Tor$^{*,*}$ where $z_i = y_i - x_i$ for $y_i$ and $x_i$.

| $(mod \ q)$       | $E_2^{*,*}$                                                                 |
|-------------------|-----------------------------------------------------------------------------|
| $p^j + 1$ and $p^k = 1$ | $\text{Tor}_{E_q^*[z_4, z_{2l}] \otimes \Lambda (z_{2l+1})}^* (\mathbb{F}_q [s_4, s_{2l}] \otimes \Lambda (s_{2l+1}), \mathbb{F}_q)$ |
| $p^j \neq 1$ and $p^k = 1$ | $\text{Tor}_{E_q^*[z_4, z_{2l}] \otimes \Lambda (z_{2l+1})}^* (\mathbb{F}_q [s_4] \otimes \Lambda (s_{2l+1}), \mathbb{F}_q)$ |
| $p^j = 1$ and $p^k \neq 1$ | $\text{Tor}_{E_q^*[z_4]}^* (\mathbb{F}_q [s_4], \mathbb{F}_q)$ |
| $p^j \neq 1$ and $p^k \neq 1$ | $\text{Tor}_{E_q^*[z_4]}^* (\mathbb{F}_q, \mathbb{F}_q)$ |

where $z_i = y_i - x_i$ acts via $(z_4, z_{2l}, z_{2l-1}) \mapsto ((p^{2k} - 1)s_4, (p^j - 1)s_{2l}, (p^j - 1)s_{2l+1})$. We may employ Proposition 5.3 further if $p^k - 1$ or $p^j - 1$ is nonzero modulo $q$. In this way, we obtain Table 2.

For example, the $p^j = 1(\mod \ q)$ and $p^k \neq 1(\mod \ q)$ case is obtained via the choices $(A, B, C, M, N, L) = (\mathbb{F}_q, \mathbb{F}_q[z_{2l}] \otimes \Lambda (z_{2l+1}), \mathbb{F}_q[z_4], \mathbb{F}_q[s_{2l}] \otimes \Lambda (s_{2l+1}), \mathbb{F}_q[s_4], \mathbb{F}_q)$ and the other entries are obtained similarly. In Table 2, all the modules are trivial over their respective $\mathbb{F}_q$–algebras.

For $q$ odd, all the $\mathbb{F}_q$–algebras in Table 2 are finitely generated free graded commutative. Let $\mathbb{F}_q(X)$ denote a free graded commutative algebra on the graded set $X$.

We follow [38], p. 258-260 to compute $E_2^{*,*}$. If one resolves the trivial $\mathbb{F}_q(X)$–module $\mathbb{F}_q$ using the Koszul complex $\Omega(X)$, then

$$T_{\text{Tor}}^* (L, \mathbb{F}_q) = H(\Omega(X) \otimes L, d_L) \tag{54}$$

where $L$ is any $\mathbb{F}_q(X)$–module. In particular, $\Omega(X)$ is $\Lambda(s^{-1}X^{\text{even}}) \otimes \Gamma(s^{-1}X^{\text{odd}})$ where $\Gamma$ denotes the divided power algebra. Here $s^{-1}X^{\text{even}}$ is a set of odd degree generators obtained from the even degree elements of $X$ by decreasing the their degree by one and $s^{-1}X^{\text{odd}}$ is defined analogously. In all the cases at hand, $L$ is a trivial $\mathbb{F}_q(X)$–module which implies $d_L$ is zero. Thus, $E_2^{*,*}$ is given by $\Omega(X) \otimes L$ (see Table 1). As the EMSS is a spectral sequence of $\mathbb{F}_q$–algebras [38], we have computed $E_2^{*,*}$ as an algebra.

Note that from Table 1 the EMSS computing $H^*((BK_q^\wedge)^{h\psi^k}, \mathbb{F}_q)$ collapses at $E_2^{*,*}$ if $p^j \neq 1(\mod \ q)$ for degree reasons. However, if $p^j = 1(\mod \ q)$, $d_r(\gamma_r(x_{2l}))$ is potentially nonzero.

5.2. Levi component calculations and a proof of Theorem 5. Note that for odd $q$ and $I = \{i\}$

$$H^*(BG_l, \mathbb{F}_q) \cong H^*(BT, \mathbb{F}_q)^* = \mathbb{F}_q[p_2(x, y), p_4(x, y)] = \mathbb{F}_q[s_2, s_4]$$

with $r_i$ generating $W_i$ and $p_j(x, y)$ homogenous polynomials of degree $j$ in the degree 2 generators of $H^*(BT, \mathbb{F}_q)$, i.e. $p_2(x, y)$ is linear in $x$ and $y$. The action of $\psi^k$ is given by a simple scalar multiplication on generators and we can compute $H^*((BG_l^\wedge)^{h\psi^k}, \mathbb{F}_q)$ for $I \in S = \{\emptyset, \{1\}, \{2\}\}$.

**Theorem 5.4.** The Eilenberg-Moore spectral sequence (EMSS) for the homotopy pullback computes $H^*((BG_l^\wedge)^{h\psi^k}, \mathbb{F}_q)$ where $\psi^k$ is the $k$th unstable Adams operation. For $q$ odd, we may compute $H^*((BG_l^\wedge)^{h\psi^k}, \mathbb{F}_q)$ for $I \in S = \{\emptyset, \{1\}, \{2\}\}$.
Table 3. \( H^*((BG_{1,q})^{\psi k},\mathbb{F}_q) \cong H^*((BG_{2,q})^{\psi k},\mathbb{F}_q) \) for rank 2 Kac-Moody groups of infinite type and \( q \) odd.

| \( p^k \) | \( H^*((BG_{1,q})^{\psi k},\mathbb{F}_q) \cong H^*((BG_{2,q})^{\psi k},\mathbb{F}_q) \) |
|---|---|
| \( 1 \pmod{q} \) | \( \Lambda(z_1,z_2) \otimes \mathbb{F}_q[s_2,s_4] \) |
| \( -1 \pmod{q} \) | \( \Lambda(z_3) \otimes \mathbb{F}_q[s_4] \) |
| \( \not\equiv \pm 1 \pmod{q} \) | \( \mathbb{F}_q \) |

Table 4. \( E_2^{*,*} \) collapsing to \( H^*((BG_{1,q})^{\psi k},\mathbb{F}_q) \cong H^*((BG_{2,q})^{\psi k},\mathbb{F}_q) \) for \( z_i = x_i - x_i' \), the difference of the generators for two copies of \( H^*((BG_{1,q})^{\psi k},\mathbb{F}_q) \cong H^*((BG_{2,q})^{\psi k},\mathbb{F}_q) \).

| \( p^k \) | \( E_2^{*,*} \) |
|---|---|
| \( 1 \pmod{q} \) | \( Tor_{\mathbb{F}_q}[z_2,z_4](\mathbb{F}_q[s_2,s_4],\mathbb{F}_q) \) |
| \( -1 \pmod{q} \) | \( Tor_{\mathbb{F}_q}[z_3](\mathbb{F}_q[s_4],\mathbb{F}_q) \) |
| \( \not\equiv \pm 1 \pmod{q} \) | \( Tor_{\mathbb{F}_q}^{*,*}(\mathbb{F}_q,\mathbb{F}_q) \) |

and, up to isomorphism, it is given in Table 3 and

\[
H^*((BT_{1,q})^{\psi k},\mathbb{F}_q) = \Lambda(z_1,z_2') \otimes \mathbb{F}_q[s_2,s_4'] \iff p^k = 1(\text{mod } q) \quad \text{ and } \quad \mathbb{F}_q \iff p^k \neq 1(\text{mod } q).
\]

(55)

**Proof.** Let us first compute \( H^*((BG_{1,q})^{\psi k},\mathbb{F}_q) \cong H^*((BG_{2,q})^{\psi k},\mathbb{F}_q) \). If we employ the techniques in the proof of Theorem 5.2, we obtain Table 4 in which all \( \mathbb{F}_q(X) \)-modules are trivial. Koszul resolutions give Table 3, but here the spectral sequence collapses at \( E_2^{*,*} \) for degree reasons. Equation (55) is obtained similarly.

**Proof of Theorem 5.** Let us set notation for the Mayer-Vietoris sequence for computing \( H^*(\hocolim(BG_{1,q})^{\psi k},\mathbb{F}_q) \)

\[
\cdots \to H^*(\hocolim(BG_{1,q})^{\psi k},\mathbb{F}_q) \to H^*(BG_{1,q})^{\psi k},\mathbb{F}_q) \oplus H^*(BG_{2,q})^{\psi k},\mathbb{F}_q) \to \Delta \to H^*(BT_{1,q})^{\psi k},\mathbb{F}_q) \to \cdots.
\]

(56)

We will proceed by cases.

**Case 1:** \( p^k \neq 1(\text{mod } q) \). Here the vanishing of \( H^*((BT_{1,q})^{\psi k},\mathbb{F}_q) \) gives the second and third rows of Table 5 by (55). If \( p^k \neq \pm 1(\text{mod } q) \), then the Theorem clearly holds. If instead \( p^k = -1(\text{mod } q) \), then comparing Tables 1 and 5 in degrees 3 and 5 gives \( H^*(BK_{1,q})^{\psi k},\mathbb{F}_q) \neq H^*(\hocolim_{\mathbf{I} \in \mathbf{S}}(BG_{I,q})^{\psi k},\mathbb{F}_q) \).

**Case 2:** \( p^k = 1(\text{mod } q) \). In this case, \( H^*(BG_{1,q})^{\psi k},\mathbb{F}_q) \) is isomorphic to the \( \mathbb{F}_q \)-cohomology of free loops on \( BG_{I} \) all for \( I \in \mathbf{S} \). Furthermore, we know \( \Delta \) from the Mayer-Vietoris sequence (55) almost completely by loc. cit. A careful count of dimensions will show \( H^*(BG_{1,q})^{\psi k},\mathbb{F}_q) \neq H^*(BK_{1,q})^{\psi k},\mathbb{F}_q) \).

Let \( | \cdot | \) denote the rank of a vector space for this purpose.

On \( E_2^{*,*} \) of the EMSS for \( H^*(BG_{1,q})^{\psi k},\mathbb{F}_q) \) total degree 4l has rank 6 for \( l \) odd and rank 8 for \( l \) even. Because this is a spectral sequence of algebras, \( d_r \) of all these generators vanish since they are products of permanent cocycles for
Examples of 5.3. \( BT \) satisfies the Leibnitz rule. In the terminology of [32], \( \Lambda(z_3) \) computes preliminary calculations for the group cohomology of \( U \). We perform no such calculations here but note that [5, Theorem 8.3] computes \( \Lambda(z_3) \otimes \Lambda(z_3') \otimes F_q[s'_4] \).

Degree reasons noting that \( q \neq 2 \) implies \( \gamma_2(x_{2l}) = \frac{\gamma_2(x_{2l})}{2} \). Indeed, they are all permanent cocycles as they are not the target of any differential for degree reasons and we recover \( H^4((BKq)^{h\psi^k}, F_q) \).

Considering [35] and Table 3 we see that

\[
|H^4((BGq)^{h\psi^k}, F_q)| = 2 \cdot |(BGq)^{h\psi^k}, F_q| = 4l
\]

for \( i \in \{1, 2\} \). It follows that

\[
|H^4((\text{hocolim}_{1 \in S} (BGq)^{h\psi^k}, F_q)| = |\ker(\Delta_{-1})| + |\ker(\Delta_{1})|
\]

by elementary homological methods. By [3][32], there are isomorphisms of graded abelian groups

\[
\ker(\Delta) \cong H^4((BGq)^{h\psi^k}, F_q) \cong F_q[s_4, s_2] \otimes \Lambda(x_3, x_{2l-1})
\]

where \( W \) is the infinite dihedral Weyl group of \( K \) which acts on \( H^4((BGq)^{h\psi^k}, F_q) \cong \Lambda(x_3, x_{2l-1}) \otimes F_q[s_2, s'_2] \). Here we extend the standard action on \( H^4(BT, F_q) \cong F_q[s_2, s'_2] \) via the identification \( ds_i = x_i \), so that \( d \) commutes with the action and satisfies the Leibnitz rule. In the terminology of [32], \((BGq)^{h\psi^k}\) is a twisted loop space with associated derivation \( d \) and \( d \) commutes with restriction. Considering \( F_q[s_4, s_2] \otimes \Lambda(x_3, x_{2l-1}) \) in degree \( 4l \) and \( 4l-1 \) gives that \( H^4((\text{hocolim}_{1 \in S} (BGq)^{h\psi^k}, F_q) \) has rank 5 for \( l \) odd and rank 6 for \( l \) even. Hence \( p^k = 1(\mod q) \) implies \( H^4((BKq)^{h\psi^k}, F_q) \) and \( H^4((\text{hocolim}_{1 \in S} (BGq)^{h\psi^k}, F_q) \) are distinct (and nontrivial) for odd \( q \).

Table 5 summarizes our deductions about the structure of

\[
H^4((\text{hocolim}_{1 \in S} (BGq)^{h\psi^k}, F_q) \cong H^4(BK, F_q)
\]

within the proof of Theorem 5.3. The methods of [3], where [33] is used to compute \( H^4(BK, F_q) \), and a good understanding of the derivation associated to a twisted loop space should in principle allow further computation in the \( p^k = 1(\mod q) \) case. We perform no such calculations here but note that [5] Theorem 8.3 computes \( H^4(BK(F_{pq}), F_q) \) in most of these interesting cases.

Table 5. \( H^4(BK(F_{pq}), F_q) \) for rank 2 Kac-Moody groups of infinite type.

| Degree | Group Cohomology |
|--------|-----------------|
| \( p^k = 1(\mod q) \) | \( H^4(BK(F_{pq}), F_q) \) |
| \( p^k = -1(\mod q) \) | \( \Lambda(z_3) \otimes F_q[s_4] \otimes \Lambda(z_3') \otimes F_q[s'_4] \) |
| \( p^k \neq \pm 1(\mod q) \) | \( F_q \) |

5.3 Examples of \( H^4(BU^+(F_{pq}), F_q) \). In this subsection, we perform some preliminary calculations for the group cohomology of \( U^+(F_{pq}) \) at \( p \) using our new homotopy decomposition [31] and briefly discuss considerations in the general case. In our examples, we will see that \( H^4(BU^+(F_{pq}), F_q) \) is infinitely generated as a ring. Along the way, we give explicit descriptions of the groups \( U^+(R) \) underlying our cohomology calculations. Unless otherwise stated, when we refer to Kac-Moody groups in this subsection we mean some image of Tits's [49] explicit (discrete, minimal, split) Kac-Moody group functor \( K(-) \): \text{Rings} \rightarrow \text{Groups} \) from the category of commutative rings with unit to the category of groups. Likewise, \( U^+(R) \) will be Tits's subgroup generated by the positive (real) root groups [49]. When \( R \) is clear from context, we will simply write \( K \) or \( U^+ \).
Our simplest (and most explicit) calculations will be for \( U_2 := U_2^+ (R) \leq K_2(R) \) the positive unipotent subgroup of the infinite dimensional Kac-Moody group \( K_2(R) \) with generalized Cartan matrix
\[
\begin{bmatrix}
2 & -a \\
-b & 2
\end{bmatrix}
\]
where \( a, b \geq 2 \). In this case the relevant diagram of unipotent subgroups of \( U_2^+ (R) \) indexed over \( W \) is
\[
\begin{array}{c}
U_{sts} \leftarrow U_{st} \leftarrow U_s \leftarrow e \leftarrow U_t \leftarrow U_{st} \leftarrow U_{sts} \\
\end{array}
\]
as here \( W \) is the infinite dihedral group. When \( a = b = 2 \) and \( R \) is a field, we have an affine Kac-Moody group \( K_2(R) \cong GL_2(R[t, t^{-1}]) \) and \( U_2^+ \) is known explicitly (cf. [48]). We will identify \( U_w(R) \) directly from the presentation of Tits [49] and recover \( U_w(R) \) from the colimit presentation induced by \([50]\). In fact, for any Kac-Moody group with generalized Cartan matrix \( A = (a_{ij})_{1 \leq i,j \leq k} \) such that \(|a_{ij}| \geq 2 \) for all \( 1 \leq i, j \leq k \) the finite dimensional unipotent subgroups \( U_w(R) \) have a very simple form.

**Lemma 5.5.** Let \( K(R) \) be the discrete Kac-Moody group over \( R \) with generalized Cartan matrix \( A = (a_{ij})_{1 \leq i,j \leq k} \) such that \(|a_{ij}| \geq 2 \) for all \( 1 \leq i, j \leq k \). Then, for \( U_w(R) \leq K(R) \) defined by \([17]\) we have
\[
U_w(R) \cong (R, +)^{\leq l(w)}
\]
such that the image of \( U_w(R) \to U_w(R) \) for \( l(w) + 1 = l(ws) \) is the first \( l(w) \) factors.

**Proof.** Recall that \( w^{-1} \Phi^+ \cap \Phi^- = \Theta_w := \{\alpha_{i_1}, \alpha_{i_2}, \ldots, ws_{i_1} \alpha_{i_1}\} \) and that the multiplication map
\[
U_{\alpha_{i_1}} \times U_{\alpha_{i_2}} \times \cdots \times U_{ws_{i_1} \alpha_{i_1}} \to U_w
\]
is an isomorphism of sets \([11]\). It is immediate from the presentation of Tits [49, 3.6] that for \( \alpha, \beta \in \Phi^+, \{N_\alpha + \beta\} \cap \Phi^+ = \{\alpha, \beta\} \) implies \( U_{\alpha} \times U_{\beta} \). Moreover, the W of action on \( \Phi \) guarantees that \( \{N_\alpha + \beta\} \cap \Phi^+ \subset \Theta_w \) whenever \( \alpha, \beta \in \Theta_w \). Direct computation in the hypothesized cases of the action of \( W \) on the simple roots \([5]\) shows that each of the positive coefficients of \( ws_{i_1} \alpha_{i_1} \) expressed in terms of the simple roots is maximal and one coefficient is strictly largest within \( \Theta_w \). In particular, if \( \alpha \in \Theta_w \) is distinct from \( ws_{i_1} \alpha_{i_1} \) then \( U_\alpha \) and \( U_{ws_{i_1} \alpha_{i_1}} \) commute because the coefficients demand that if \( m_1 \alpha + m_2 ws_{i_1} \alpha_{i_1} \in \Theta_w \), then \( (m_1, m_2) \) is \((0, 1)\) or \((m_1, 0)\). However, \( m_1 \alpha \in \Phi^+ \implies m_1 = 1 \) \([50]\).

Since \( u, v \in U_w \) implies
\[
wv = (u_1 u_2 \cdots u_{l-1}) u_l (v_1 v_2 \cdots v_{l-1}) v_l = (u_1 u_2 \cdots u_{l-1}) (v_1 v_2 \cdots v_{l-1}) u_l v_l
\]
for \( u_j, v_j \in U_{s_{i_1} \cdots s_{i_{l-1}} \alpha_{i_j}} \), we have \( U_w \cong U_{ws_{i_1} \times U_{i_1}} \) for \( \gamma = ws_{i_1} \alpha_{i_1} \). The lemma follows by induction on \( l(w) \) and the fact that all root subgroups are isomorphic to \((R, +)\).

Returning to \( U_2(R) \) and considering the right telescope of \([55]\), we obtain a projective limit of graded abelian groups upon taking group cohomology
\[
\begin{array}{c}
H^*(e, F_p) \leftarrow H^*(U_s, F_p) \leftarrow H^*(U_{st}, F_p) \leftarrow H^*(U_{sts}, F_p) \ldots
\end{array}
\]
which is a sequence of surjections by the Kunneth theorem. In particular,
\[
\lim_{\leftarrow} H^*(U_w(R), F_p) = 0
\]
and the cohomology spectral sequence of the mapping telescope of classifying spaces \([38]\) collapses at \( E_2^{*,*} \).
There is some subtly in defining the positive unipotent subgroup of a Kac-Moody group over an arbitrary ring (cf. Remark 3.10 (d) in [10]) so we will state the rest of our results in this section only over fields.

**Theorem 5.6.** Let \( U_2 := U_2^+(\mathbb{L}) \subseteq K_2^+(\mathbb{L}) \) be the positive unipotent subgroup of the Kac-Moody group \( K_2^+(\mathbb{L}) \) over a field \( \mathbb{L} \) with generalized Cartan matrix given by \( \Gamma \). Then we may compute the group cohomology of \( U_2^+ \) as

\[
H^*(BU_2^+(\mathbb{L}), \mathbb{F}) = \bigotimes_{\mathbb{Z} > 0} H^*(U_2^+(\mathbb{L}), \mathbb{F}) \bigotimes_{\mathbb{Z} > 0} H^*(\mathbb{L}, \mathbb{F})
\]

where \( \mathbb{L} \) is considered as an abelian group and \( \mathbb{F} \) is a field.

Proof. As \( \lim_{\mathbb{W}^0} H^*(U_w(\mathbb{L}), \mathbb{F}) = 0 \) in this case [39], the spectral sequence of the homotopy colimit of classifying spaces collapses at \( E_2^{s,s} \) and computes \( H^*(BU_2^+(\mathbb{L}), \mathbb{F}) \) as the projective limit of graded abelian groups \( \lim_{\mathbb{W}^0} H^*(U_w(\mathbb{L}), \mathbb{F}) \). Working one telescope at a time, we see the limit of graded abelian groups \( \lim_{\mathbb{Z} > 0} H^*(U_w^{(\mathbb{L})^n}, \mathbb{F}) \) by Lemma 5.5. This limit is computed in each gradation so that, unlike the limit of underlying abelian groups, only finite tensor words are possible in each gradation. The left telescope is computed in the same way. As the double telescope is the one point union of the left and right telescopes, \( H^*(BU_2^+(\mathbb{L}), \mathbb{F}) \) is simply the direct sum of their \( \mathbb{F} \)-cohomologies. \( \square \)

For the case of interest, \( H^*(BU^+(\mathbb{F}_p^\infty), \mathbb{F}_p) \), cohomology classes are represented by finite tensor words in the generators of \( H^*(\mathbb{F}_p^\infty, \mathbb{F}_p) \). For instance, for \( k = 1 \) we have

\[
H^*((\mathbb{Z}/p\mathbb{Z})^n, \mathbb{F}_p) = \left\{ \begin{array}{ll} \mathbb{F}_2[x_1, \ldots, x_n] & p = 2 \\ \Lambda[x_1, \ldots, x_n] \otimes \mathbb{F}_p[y_1, \ldots, y_n] & p \text{ odd} \end{array} \right.
\]

where \( |x_i| = 1 \) and \( |y_i| = 2 \). Applying the double telescope construction

\[
H^*(BU^+((\mathbb{Z}/2\mathbb{Z})^n), \mathbb{F}_2) = \lim_{\mathbb{F}_2[x_1, \ldots, x_n]} H^* \bigotimes_{\mathbb{F}_2[x_1, \ldots, x_n]} \mathbb{F}_2[x_1, \ldots, x_n] \oplus \mathbb{F}_2[\overline{x_1}, \ldots, \overline{x_n}].
\]

For \( p \) odd

\[
H^*(BU^+((\mathbb{Z}/p\mathbb{Z})^n), \mathbb{F}_p) = \lim_{\mathbb{F}_p[x_1, \ldots, x_n]} H^* \bigotimes_{\mathbb{F}_p[x_1, \ldots, x_n]} \mathbb{F}_p[x_1, \ldots, x_n] \oplus \mathbb{F}_p[\overline{x_1}, \ldots, \overline{x_n}, \ldots] \oplus \Lambda[^{\overline{x_1}} x_1, \ldots, ^{\overline{x_n}} x_n, \ldots] \oplus \mathbb{F}_p[y_1, \ldots, y_n, \ldots].
\]

Lemma 5.5 lets us determine the group structure of more general \( U_+^+(R) \) which have an interesting presentation. Briefly, when Lemma 5.5 applies we have

\[
U_+^+(R) = \text{colim}_{w \in W} R^{l(w)}
\]

where all maps are inclusions \( R^{l(v)} \rightarrow R^{l(w)} \) with image the first \( l(v) \) factors. This leads to the presentation

\[
U_+^+(R) = \langle U_\alpha u_\alpha u_\beta w_\alpha^{-1} u_\beta^{-1} = 1, \text{ if } \exists w \text{ with } w\{\alpha, \beta\} \subset \Phi^+ \rangle
\]

where \( w \in W, u_\gamma \) is an arbitrary element of \( U_\gamma \) and \( \gamma \in \Phi^+ \). Pictorially, there is a generating \( U_\alpha \subseteq (R, +) \) for each node in Hasse diagram of the poset \( W \), i.e. the directed graph with vertices and edges \( (W; \{(w, ws) | l(w) + 1 = l(ws)\}) \) (see, e.g., Figure 3.2). Generating groups \( U_\alpha \) and \( U_\beta \) commute exactly when there is a (unique) path between them in this directed graph. If there is no path between the corresponding nodes, \( U_\alpha \ast U_\beta \) embeds into \( U_+^+ \).
Figure 1. A picture of the graph underlying the poset $W = \langle s_1, s_2, s_3 \mid s_1^2 = 1 \rangle$ and the weak Bruhat order with elements of length at most 3 labeled.

For example, if $K(R)$ has rank 3 with generalized Cartan matrix such as

$$\begin{bmatrix}
2 & -2 & -3 \\
-2 & 2 & -2 \\
-2 & -4 & 2
\end{bmatrix},$$

then the poset $W$ is generated by the directed valence–3 tree pictured in Figure 1. Two elements commute exactly when they are both contained in some $U_w(R)$.

Starting with the homotopy decomposition \[\text{(61)}\] of Theorem 3.1, it is straightforward to apply Theorem 2.6 and Theorem 2.11 to observe

$$\text{colim}_{w \in W} B(R^l(w)) \simeq \text{colim}_{m \in N}(\text{colim}_{w \in W_m} B(R^l(w)),$$

for $N$ the linearly ordered poset and $W_m$ is the full subposet of $W$ of elements of length at most. Set $U_{W_m} := \text{colim}_{w \in W_m} U_w$. By Corollary 3.3, we have a commutative diagram

$$\begin{array}{c}
\text{hocolim}_{L-1(W_m)} U_{W_m} / R^l(w) \longrightarrow \text{hocolim}_{L-1(W_m)} B(R^l(w)) \\
\downarrow \quad \downarrow L \\
\text{hocolim}_{W_m} U_{W_m} / R^l(w) \longrightarrow \text{hocolim}_{W_m} B(R^l(w)) \longrightarrow BU_{W_m}
\end{array}$$

whose vertical maps are weak equivalences. Since $\text{hocolim}_{L-1(W_m)} U_{W_m} / R^l(w)$ can be modeled by a 1–dimensional CW–complex and is simply connected by Corollary
Thus, we have homotopy equivalences
\[
BU^+(R) \simeq \operatorname{hocolim}_{w \in W} B(R^{i(w)}) \\
\simeq \operatorname{hocolim}_{w} (\operatorname{hocolim}_{m} B(R^{i(w)})) \\
\simeq \operatorname{hocolim}_{m} B(\operatorname{hocolim}_{w} R^{i(w)})
\]
induced by the obvious maps. Cohomology calculations for \(B(\operatorname{colim}_{w} R^{i(w)}) \simeq \operatorname{hocolim}_{m} B(R^{i(w)})\) can be performed by iterated Mayer-Vietoris calculations. For instance, for \(K\) with the generalized Cartan matrix \((g)\) considered above we have
\[
\operatorname{colim}_{m} R^{i(w)} = \left[ R^m \ast_{R^m} R^{m-1} \ast_{R^m} \cdots \ast_{R^m} R^{m_1} \ast_{R^m} R^m \ast_{R^m} \cdots \ast_{R^m} R^{m} \right]^{i(w)}
\]
and each \(R^i\) of the amalgamated products denotes the first \(i\) factors. For example, this implies
\[
H^*(\operatorname{colim}_{w} (\mathbb{Z}/2\mathbb{Z})^{i(w)}, \mathbb{F}_2) = (\mathbb{F}_2[x, x_0, x_1, x_{00}, x_{01}, x_{10}, x_{11}]/I)^{i(w)}
\]
when \(m = 3\) where
\[
I = \langle x_0 x_1, x_0 \{x_{10}, x_{11}\}, x_1 \{x_{00}, x_{01}\}, x_{00} \{x_{10}, x_{11}\}, x_{01} \{x_{10}, x_{11}\}, x_{10} x_{11} \rangle
\]
as nonidentity nodes in the directed graph underlying \(W_3\) (and pictured in Figure 1) contribute generators and the multiplicative structure is determined by whether the corresponding root groups commute or generate a free product. The three direct summands correspond to each of the three main branches of the tree underlying \(W_3\). For instance, on the \(s_2\) branch nodes correspond to generators via \(s_2 \leftrightarrow x, s_1 s_2 \leftrightarrow x_0, s_3 s_2 \leftrightarrow x_1, s_2 s_1 s_2 \leftrightarrow x_{00}, s_3 s_1 s_2 \leftrightarrow x_{01}, s_1 s_3 s_2 \leftrightarrow x_{10}\) and \(s_2 s_3 s_2 \leftrightarrow x_{11}\). The other branches have a similar correspondence. This cohomology may be obtained inductively from the expression of \(\operatorname{colim}_{w} R^{i(w)}\) given in (64).

More generally, this process gives in the following result.

**Theorem 5.7.** Let \(K(L)\) be a discrete Kac-Moody over a field \(L\) with generalized Cartan matrix \(A = (a_{ij})_{1 \leq i, j \leq n}\) such that \(|a_{ij}| \geq 2\) for all \(1 \leq i, j \leq n\) with positive unipotent subgroup \(U^+(L)\). Then the cohomology of \(BU^+(L)\) with coefficients in any field \(F\) is given as
\[
H^*(BU^+(L), F) = \lim_{\leftarrow} H^*(\operatorname{colim}_{w} B(L^{i(w)}), F) = \otimes_{W_{-\{e\}}} H^*(L, F)/I
\]
where the ideal \(I\) is generated by all products of pairs of elements in different copies of \(H^*(L, F)\) corresponding to nonidentity nodes in the directed graph underlying the poset \(W\) that are disconnected.

For the rank 3 example discussed above (61), \(H^*(BU^+(F_2), F_2)\) is the \(\mathbb{F}_2\)-algebra generated by the nodes (except the root) of an infinitely extended Figure 1 with relations \(xy = 0\) for generators \(x\) and \(y\) whose nodes are not connected.

All the examples described here concern Kac-Moody groups with Weyl groups whose Hasse diagrams are trees so that (62) is available to clarify inductive arguments and diagrams over \(W_m\) can be seen as iterated pushouts. Because the cohomology spectral sequence calculation is trivialized for limits satisfying the Mittag–Leffler condition, we pose the following question.

**Question 5.8.** Let \(K(F_{p^k})\) be a discrete minimal Kac-Moody group over \(F_{p^k}\) with unipotent \(p\)-subgroups \(U_{w}(F_{p^k})\) defined by (10). When does the diagram of group
cohomologies $D_K : W \to \text{Groups} \to \text{Graded Abelian Groups}$ given by $w \mapsto U_w(F_p^k) \mapsto H^*(U_w(F_p^k), F_p)$ satisfy the Mittag–Leffler condition?

In the tree shaped cases not covered here, a positive answer to this question would permit cohomology calculations as outlined in this section. Obviously, Lemma 5.5 greatly simplifies our calculations. For example, for an infinite dimensional Kac-Moody group with generalized Cartan matrix

$$
\begin{bmatrix}
2 & -a \\
-1 & 2
\end{bmatrix}
$$

for $a \geq 4$, it is a priori possible that the 3-dimensional unipotent group $U_{sts}$ is a Heisenberg group (see [5] for details on $\Phi^+$ in this case).

References

[1] Peter Abramenko and Kenneth S. Brown, Buildings, Graduate Texts in Mathematics, vol. 248, Springer, New York, 2008. Theory and applications.
[2] Jaume Aguadé, $p$-compact groups as subgroups of maximal rank of Kac-Moody groups, J. Math. Kyoto Univ. 49 (2009), no. 1, 83–112.
[3] Jaume Aguadé, Carles Broto, Nitu Kitchloo, and Laia Saumell, Cohomology of classifying spaces of central quotients of rank two Kac-Moody groups, J. Math. Kyoto Univ. 45 (2005), no. 3, 449–488.
[4] Jaume Aguadé and Albert Ruiz, Maps between classifying spaces of Kac-Moody groups, Adv. Math. 178 (2003), no. 1, 66–98.
[5] Jaume Aguadé, Cohomology of Kac-Moody groups over a finite field, Preprint (2012), pp. 24, available at arXiv:1206.1352v3[math.AT].
[6] A. K. Bousfield and D. M. Kan, Homotopy limits, completions and localizations, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin, 1972.
[7] Carles Broto and Nitu Kitchloo, Classifying spaces of Kac-Moody groups, Math. Z. 240 (2002), no. 3, 621–649.
[8] Carles Broto and Jesper M. Møller, Chevalley $p$-local finite groups, Algebr. Geom. Topol. 7 (2007), 1809–1919.
[9] Kenneth S. Brown, Cohomology of groups, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1982.
[10] Pierre-Emmanuel Caprace and Bertrand Rémy, Groups with a root group datum, Innov. Incidence Geom. 9 (2009), 5–77.
[11] Ruth Charney and Michael W. Davis, The $K(\pi, 1)$-problem for hyperplane complements associated to infinite reflection groups, J. Amer. Math. Soc. 8 (1995), no. 3, 597–627.
[12] Michael W. Davis, The geometry and topology of Coxeter groups, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008.
[13] Emmanuel Dror Farjoun, Fundamental group of homotopy colimits, Adv. Math. 182 (2004), no. 1, 1–27.
[14] Cellulare spaces, null spaces and homotopy localization, Lecture Notes in Mathematics, vol. 1622, Springer-Verlag, Berlin, 1996.
[15] E. Dror, W. G. Dwyer, and D. M. Kan, An arithmetic square for virtually nilpotent spaces, Illinois J. Math. 21 (1977), no. 3, 347–361.
[16] Daniel Dugger and Daniel C. Isaksen, Topological hypercovers and $K^1$-realizations, Math. Z. 240 (2002), no. 4, 667–689.
[17] W. G. Dwyer and D. M. Kan, A classification theorem for diagrams of simplicial sets, Topology 23 (1984), no. 2, 139–155.
[18] W. G. Dwyer and C. W. Willerson, The elementary geometric structure of compact Lie groups, Bull. London Math. Soc. 30 (1998), no. 4, 337–364.
[19] Mikhail Ershov, Golod-Shafarevich groups with property $(T)$ and Kac-Moody groups, Duke Math. J. 145 (2008), no. 2, 309–339.
[20] John D. Foley, Comparing Kac-Moody groups over the complex numbers and fields of positive characteristic, PhD Thesis (2012).
[21] Eric M. Friedlander, Étale homotopy of simplicial schemes, Annals of Mathematics Studies, vol. 104, Princeton University Press, Princeton, N.J., 1982.
[22] The Friedlander-Milnor Conjecture, Annals of Mathematics Studies, vol. 2, 2008.
[23] Eric M. Friedlander and Guido Mislin, Cohomology of classifying spaces of complex Lie groups and related discrete groups, Comment. Math. Helv. 59 (1984), no. 3, 347–361.
[24] Alex González, Unstable Adams operations acting on $p$-local compact groups and fixed points, Algebr. Geom. Topol. 12 (2012), no. 2, 867–908.
[25] Andrés Haefliger, Complexes of groups and orbifolds, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 504–540.
[26] Jesper Grodal, The Classification of $p$-Compact Groups and Homotopical Group Theory, Proceedings of the International Congress of Mathematicians, Vol. 1, Hyderabad, India, 2010.
[27] J. Hollender and R. M. Vogt, Modules of topological spaces, applications to homotopy limits and $E_{\infty}$ structures, Arch. Math. (Basel) 59 (1992), no. 2, 115–129.
[28] Stefan Jackowski, James McClure, and Bob Oliver, Self-homotopy equivalences of classifying spaces of compact connected Lie groups, Fund. Math. 147 (1995), no. 2, 99–126.
[29] Victor G. Kac, Constructing groups associated to infinite-dimensional Lie algebras, Infinite-dimensional groups with applications (Berkeley, Calif., 1984), Math. Sci. Res. Inst. Publ., vol. 4, Springer, New York, 1985, pp. 167–216.
[30] Victor G. Kac and Dale H. Peterson, Regular functions on certain infinite-dimensional groups, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 141–166.
[31] V. G. Kac and D. H. Peterson, Defining relations of certain infinite-dimensional groups, Astérisque Numero Hors Serie (1985), 165–208. The mathematical heritage of Élie Cartan (Lyon, 1984).
[32] Daisuke Kishimoto and Akira Kono, On the cohomology of free and twisted loop spaces, J. Pure Appl. Algebra 214 (2010), no. 5, 646–653.
[33] Nitu Kitchloo, Topology of Kac-Moody groups, MIT Thesis (1998).
[34] , On the topology of Kac-Moody groups, Manuscript (2008).
[35] , Dominant $K$-theory and integrable highest weight representations of Kac-Moody groups, Adv. Math. 221 (2009), no. 4, 1191–1226.
[36] Shrawan Kumar, Kac-Moody groups, their flag varieties and representation theory, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.
[37] Bertram Kostant and Shrawan Kumar, The nil Hecke ring and cohomology of $G/P$ for a Kac-Moody group $G$, Adv. in Math. 62 (1986), no. 3, 187–237.
[38] John McCleary, A user’s guide to spectral sequences, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001.
[39] J. Milnor, On the homology of Lie groups made discrete, Comment. Math. Helv. 58 (1983), no. 1, 72–85.
[40] Fabien Morel, On the Friedlander-Milnor conjecture for groups of small rank, Current developments in mathematics, 2010, Int. Press, Somerville, MA, 2011, pp. 45–93.
[41] Gabor Moussong, Hyperbolic Coxeter Groups, PhD Thesis (1998).
[42] Daniel Quillen, Higher algebraic $K$-theory. I, Algebraic $K$-theory, I: Higher $K$-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.
[43] , On the cohomology and $K$-theory of the general linear groups over a finite field, Ann. of Math. (2) 96 (1972), 552–586.
[44] Stephen A. Mitchell, Quillen’s theorem on buildings and the loops on a symmetric space, Enseign. Math. (2) 34 (1988), no. 1-2, 123–166.
[45] Bertrand Rémy, Kac-Moody groups as discrete groups, Essays in geometric group theory, Ramanujan Math. Soc. Lect. Notes Ser., vol. 9, Ramanujan Math. Soc., Mysore, 2009, pp. 105–124.
[46] Stefan Jackowski, James McClure, and Bob Oliver, Homotopy classification of self-maps of $BG$ via $G$-actions, I, Ann. of Math. (2) 135 (1992), no. 1, 183–226.
[47] R. W. Thomason, Homotopy colimits in the category of small categories, Math. Proc. Cambridge Philos. Soc. 85 (1979), no. 1, 91–109.
[48] Jacques Tits, Ensembles ordonnés, immuables et sommes amalgamées, Bull. Soc. Math. Belg. Sér. A 38 (1986), 367–387 (1987) (French).
[49] , Uniqueness and presentation of Kac-Moody groups over fields, J. Algebra 105 (1987), no. 2, 542–573.
[50] Clarence Wilkerson, Self-maps of classifying spaces, (Sympos., Battelle Seattle Res. Center, Seattle, Wash., 1974), Springer, Berlin, 1974, pp. 150–157. Lecture Notes in Math., Vol. 418.

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