Mean-Field Backward Doubly Stochastic Differential Equations and Applications

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Abstract

Mean-field backward doubly stochastic differential equations (MF-BDSDEs, for short) are introduced and studied. The existence and uniqueness of solutions for MF-BDSDEs is established. One probabilistic interpretation for the solutions to a class of nonlocal stochastic partial differential equations (SPDEs, for short) is given. A Pontryagin’s type maximum principle is established for optimal control problem of MF-BDSDEs. Finally, one backward linear quadratic problem of mean-field type is discussed to illustrate the direct application of above maximum principle.

Keywords. Mean-field backward doubly stochastic differential equations, nonlocal stochastic partial differential equations, maximum principle.

AMS Mathematics subject classification. 60H05, 60H15, 93E20.

1 Introduction

Backward doubly stochastic differential equation (BDSDE for short) of the form

\[ Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)d\widehat{B}_s - \int_t^T Z_s d\widehat{W}_s, \quad 0 \leq t \leq T, \]

was firstly initiated by Pardoux–Peng [22] to give probabilistic interpretation for the solutions of a class of semilinear stochastic partial differential equations (SPDEs for short). BDSDEs have not only emerged as a natural and convenient tool in the context of SPDEs, see Bally–Matoussi [5], Hu–Ren [14], Pardoux–Peng [22], Ren–Lin–Hu [23], Zhang–Zhao ([26], [27]), but also recently

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gained interest in other fields as well, especially in relations to the stochastic optimal control problems, see Bahlali–Gherbal [4], Han–Peng–Wu [13], Zhang–Shi [25].

**McKean–Vlasov** stochastic differential equation of the form

\[ dX(t) = b(X(t), \mu(t))dt + dW(t), \quad t \in [0, T], \quad X(0) = x, \]  

(1.1)

where

\[ b(X(t), \mu(t)) = \int_\Omega b(X(t, \omega), X(t; \omega'))P(d\omega') = \mathbb{E}[b(\xi, X(t))]|_{\xi = X(t)}, \]

\[ b : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R} \] being a (locally) bounded Borel measurable function and \( \mu(t; \cdot) \) being the probability distribution of the unknown process \( X(t) \), was suggested by Kac [16] and firstly studied by McKean [20]. So far numerous works has been done on McKean-Vlasov type SDEs and applications, see for example, Ahmed [1], Ahmed-Ding [2], Borkar-Kumar [6], Chan [10], Crisan-Xiong [11], Kotelenez [17], Potelenez-Kurtz [18], and so on. It is worthy to point out that (1.1) is a particular case of the following general version,

\[ X(t) = x + \int_0^t b(s, X(s), \mathbb{E}b[s, X(s), \xi|_{\xi = X(s)}])ds \]

\[ + \int_0^t \sigma(s, X(s), \mathbb{E}\phi[\sigma[s, X(s), \xi|_{\xi = X(s)}]]dW_s, \]  

(1.2)

which can be regarded as a natural generalization of classical SDEs. Mathematical mean field approaches play a crucial role in diverse areas, such as physics, chemistry, economics, finance and games theory, see for example Lasry–Lions [19], Dawson [12], Huang-Malhame-Caines [15].

In a recent work of Buckdahn–Djehiche–Li–Peng [8], a notion of mean-field backward stochastic differential equation (MF-BSDE for short) of the form

\[ Y_t = \xi + \int_t^T \mathbb{E}' f(s, \omega, \omega', Y_s(\omega), Y_s(\omega'), Z_s(\omega), Z_s(\omega'))ds - \int_t^T Z_s dW_s, \]

with \( t \in [0, T] \) was introduced to investigate one special mean-field problem in a purely stochastic approach.

In this paper, we would like to introduce mean-field backward doubly stochastic differential equation (MF-BDSDE for short) of the form

\[ Y_t = \xi + \int_t^T \mathbb{E}' \Gamma^f(s, Y_s, Z_s) + \Gamma^g(s, Y_s, Z_s)\mathbb{E}^d W_s, \]

(1.3)

where

\[ \Gamma^l(s, Y_s, Z_s)(\omega) = \int_\Omega \theta^l(s, \omega, \omega', Y_s(\omega), Y_s(\omega'), Z_s(\omega), Z_s(\omega'))P(d\omega'), \quad l = f, g. \]

(1.4)

with \( l = f, g. \) For convenience, we also denote

\[ \mathbb{E}'[\theta^l(s, Y_s, Z_s, Y_{s}'', Z_s'')] = \Gamma^l(s, Y_s, Z_s), \]
when there is no abuse of notation. Following the basic ideas in [22], we firstly discuss the existence and uniqueness of solutions for MF-BDSDE (1.3), which obviously extends the results in both [22] and [9]. It is worthy to point out that MF-BDSDEs not just is a natural generalization of BSDEs and MF-BSDEs from the view of mathematics. Our study on them also is motivated by the problems in the following two aspects.

As is well-known to us, the study on stochastic partial differential equations have increasingly been a popular issue in recent years. As one kind of them, stochastic partial differential equations of McKean-Vlasov type were discussed in [18]. In fact, such equations were obtained as continuum limit from empirical distribution of a large number of SDEs, coupled with mean-field interaction. We also refer the reader to [11] and [17] for more details along this. On the other hand, we would also like to mention the work of Buckdahn–Li–Peng [9] who studied one kind of nonlocal deterministic PDEs. In virtue of the “backward semigroup” method they obtained the existence and uniqueness of viscosity solution for nonlocal PDEs via mean-field BSDEs (1.3) in a Markovian framework and McKean-Vlasov forward equation. Motivated by the above two cases, in this paper we will give some discussions on one kind of nonlocal stochastic partial differential equations. Since our backward equation here is allowed to dependent on $Z^0_{x_0}(\cdot)$, therefore the nonlocal SPDEs here is not a direct generalization of deterministic PDEs in [9] to the stochastic case. Some additional necessary and essential terms are required in our SPDE to meet the general case here, see (4.3) below. On the other hand, comparing with the case in [22] and [9], due to the nonlocal property of SPDEs and the interesting measurability of the corresponding solution $u(t, x)$, one fundamental and important term $u'(t, x)$ is required to meet such general case, see also Remark 4.2 below. Instead of investigating the limit result for such equations, we will study the SPDEs from other aspects. A probabilistic interpretation for the solution to such kind of SPDEs is derived by a connection between them and decoupled forward-backward doubly differential equations of mean-field type, which extends the results in [22] to the mean-field case.

The second motivation stems from the study of optimal control problem and certain stochastic differential games problems. Some related works along this have followed two main venues. On the one hand, optimal control of mean-field (forward) stochastic differential equations was discussed in Andersson–Djehiche [3], Buckdahn–Djehiche–Li [7] and Meyer-Brandis–Oksandal–Zhou [21] where stochastic maximum principle were derived as a necessary condition of the optimal control. On the other hand, optimal control problem for backward doubly stochastic differential system (or forward-backward system) were spread out in [1], [13], [25] where the corresponding linear quadratic problem and nonzero sum stochastic differential games were also investigated. Inspired by above two case, it is natural for us to consider the optimal control problem for backward doubly stochastic system of mean-field type. Since the terms $\Gamma^l$ with $l = f, g$ (see (5.1)) have more general feature than the corresponding one in [3], [7] and [21], we have to introduce $E^\tau$ in our adjoint equation, which is slight different from $E^\sigma$. Note that such kind of skill also appears in [21]. On the other hand, since $\theta^f, \theta^g$ and $l$ are allowed to depend on $u(\cdot, \omega')$, some new terms appear and is expressed by $E^\tau$ in our maximum principle (see (5.9)) which is totally different from all the previous literature. We believe that such new features will lead to other interesting things and we hope to explore and study them carefully in the future.

The paper is organized as follows. In Section 2, we will present some preliminary notations needed in the whole paper. In Section 3, we consider the existence and uniqueness of solution for MF-BDSDE (1.3). In Section 4, we give the probabilistic interpretation for the solutions
to a class of nonlocal SPDEs by means of MF-BDSDE. In Section 5 we discuss one optimal control problem of MF-BDSDE. In Section 4 we investigate one backward doubly stochastic LQ problem of mean-field type to show one direct application of result in Section 5

2 Preliminaries

Throughout this paper, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space on which are defined two mutually independent Brownian motion \(\{W_t\}_{t \geq 0}\) and \(\{B_t\}_{t \geq 0}\), with value respectively in \(\mathbb{R}^d\) and \(\mathbb{R}^f\). We denote by

\[
F_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B, \quad \forall t \in [0, T],
\]

where \(\mathcal{N}\) is the class of \(\mathbb{P}\)-null sets of \(\mathcal{F}\) and

\[
\mathcal{F}_t^W := \sigma \{W_r; 0 \leq r \leq t\} \vee \mathcal{N}, \quad \mathcal{F}_{t,T}^B := \sigma \{B_T - B_r; t \leq r \leq T\} \vee \mathcal{N}.
\]

In this case, the collection \(\{\mathcal{F}_t, t \in [0, T]\}\) is neither increasing nor decreasing, while \(\{\mathcal{F}_t^W; t \in [0, T]\}\) is an increasing filtration and \(\{\mathcal{F}_t^B; t \in [0, T]\}\) is a decreasing filtration.

Let \((\Omega^2, \mathcal{F}^2, \mathbb{P}^2) = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, \mathbb{P} \otimes \mathbb{P})\) be the completion of the product probability space of the above \((\Omega, \mathcal{F}, \mathbb{P})\) with itself, where we define \(\mathcal{F}_t^2 = \mathcal{F}_t \otimes \mathcal{F}_t\) with \(t \in [0, T]\) and \(\mathcal{F}_t \otimes \mathcal{F}_t\) being the completion of \(\mathcal{F}_t \times \mathcal{F}_t\). It is worthy of noting that any random variable \(\xi = \xi(\omega)\) defined on \(\Omega\) can be extended naturally to \(\Omega^2\) as \(\xi'(\omega, \omega') = \xi(\omega)\) with \((\omega, \omega') \in \Omega^2\). For \(H = \mathbb{R}^n\), etc., let \(L^1(\Omega^2, \mathcal{F}^2, \mathbb{P}^2; H)\) be the set of random variable \(\xi : \Omega^2 \to H\) which is \(\mathcal{F}^2\)-measurable such that \(\mathbb{E}^2|\xi| \equiv \int_{\Omega^2} |\xi(\omega, \omega')|^p \mathbb{P}(d\omega') \mathbb{P}(d\omega) < \infty\). For any \(\eta \in L^1(\Omega^2, \mathcal{F}^2, \mathbb{P}^2; H)\), we denote

\[
\mathbb{E}^p \eta(\omega, \cdot) = \int_{\Omega} \eta(\omega, \omega') \mathbb{P}(d\omega'), \quad \mathbb{E}^* \eta(\cdot, \omega) = \int_{\Omega} \eta(\omega', \omega) \mathbb{P}(d\omega').
\]

Particularly, for example, if \(\eta_1(\omega, \omega') = \eta(\omega')\) and \(\eta_2(\omega, \omega') = \eta_2(\omega)\), then

\[
\mathbb{E}^p \eta_1 = \int_{\Omega} \eta_1(\omega') \mathbb{P}(d\omega') = \mathbb{E}^p \eta_1, \quad \mathbb{E}^* \eta_2 = \int_{\Omega} \eta_2(\omega) \mathbb{P}(d\omega) = \mathbb{E}^p \eta_2.
\]

Hence, in what follows, \(\mathbb{E}^p\) and \(\mathbb{E}^*\) will be used when we need to distinguish \(\omega'\) from \(\omega\), which is the case of both \(\omega\) and \(\omega'\) appearing at the same time. On the one hand, the well definition of \(\mathbb{E}^p\) above gives the precise meaning of \(\Gamma^f\) and \(\Gamma^g\) in (4.1). On the other hand, it also indicates that, for example, the operator \(\Gamma^f\) is nonlocal in the sense that the value \(\Gamma^f(s, \omega, Y(s, \omega), Z(s, \omega))\) of \(\Gamma^f(s, Y(s), Z(s))\) at \(\omega\) depends on the whole set

\[
\{(Y(s, \omega'), Z(s, \omega')) \mid \omega' \in \Omega\},
\]

not just on \((Y(s, \omega), Z(s, \omega))\).
At last, we would like to introduce some spaces of functions required in the sequel.

\[
S^2([0, T]; \mathbb{R}^n) = \{ \varphi : [0, T] \times \Omega \to \mathbb{R}^n, \left| \varphi_t \right| \text{ is } \mathcal{F}_t \text{-measurable process such that } \mathbb{E} \left( \sup_{0 \leq t \leq T} |\varphi_t|^2 \right) < \infty \},
\]

\[
M^2(0, T; \mathbb{R}^n) = \{ \varphi_t : [0, T] \times \Omega \to \mathbb{R}^n, \varphi_t \text{ is } \mathcal{F}_t \text{-measurable process such that } \mathbb{E} \int_0^T |\varphi_t|^2 dt < \infty \},
\]

\[
L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n) = \{ \xi : [0, T] \times \Omega \to \mathbb{R}^n, \xi \text{ is } \mathcal{F}_T \text{-measurable random variable such that } \mathbb{E}|\xi|^2 < \infty \}.
\]

3 The unique solvability of MF-BDSDEs

In this section, we will discuss the existence and uniqueness of adapted solution for MF-BDSDE (1.3) which is rewritten below (for convenience):

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \Gamma^f(s, Y_s, Z_s)) ds + \int_t^T g(s, Y_s, Z_s, \Gamma^g(s, Y_s, Z_s)) d\tilde{B}_s - \int_t^T Z_s d\tilde{W}_s,
\]

with \(l = f, g\). Before it, we make the following assumptions.

(H1) (i) \(\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)\). \(f : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{k_1} \to \mathbb{R}^n\) is measurable and for all \((t, y, z, \gamma) \in [0, T] \times \mathbb{R}^{n+n \times d+k_1}\), \((t, \omega) \mapsto f(t, \omega, y, z, \gamma)\) is \(\mathcal{F}_t\)-measurable. \(g : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{k_2} \to \mathbb{R}^{m \times d}\) is measurable and for all \((t, y, z, \gamma) \in [0, T] \times \mathbb{R}^{n+n \times d+k_2}\), \((t, \omega) \mapsto g(t, \omega, y, z, \gamma)\) is \(\mathcal{F}_t\)-measurable. The map \(\theta^i : [0, T] \times \Omega^2 \times \mathbb{R}^{2n+2n \times 2d} \to \mathbb{R}^m\) is measurable and for all \((t, y, z, y', z') \in [0, T] \times \mathbb{R}^{2n+2n \times 2d}\), the map \((t, \omega, \omega') \mapsto \theta^i(t, \omega, \omega', y, z, y', z')\) is \(\mathcal{F}_t^2\)-measurable on \([0, T]\), \(l = f, g\).

(ii) \(f\) and \(g\) satisfy uniformly Lipschitz condition to \((y, z, \gamma)\), that is, there exist positive constants \(L_i K_i\) and \(\alpha_j\) with \(i = y, z, y', z', \gamma, j = 1, 2, 3, 4\), such that

\[
|f(t, y_1, z_1, \gamma_1) - f(t, y_2, z_2, \gamma_2)| \leq L_y|y_1 - y_2| + L_z|z_1 - z_2| + L_\gamma|\gamma_1 - \gamma_2|,
\]

\[
|g(t, y_1, z_1, \gamma_1) - g(t, y_2, z_2, \gamma_2)|^2 \leq K_y^2|y_1 - y_2|^2 + K_z^2|z_1 - z_2|^2 + K_\gamma^2|\gamma_1 - \gamma_2|^2,
\]

\[
|\theta^i(t, \omega, \omega', y_1, z_1, y'_1, z'_1) - \theta^i(t, \omega, \omega', y_2, z_2, y'_2, z'_2)| \leq L_y|y_1 - y_2| + L_z|z_1 - z_2| + L_{y'}|y'_1 - y'_2| + L_{z'}|z'_1 - z'_2|,
\]

\[
|\theta^g(t, \omega, \omega', y_1, z_1, y'_1, z'_1) - \theta^g(t, \omega, \omega', y_2, z_2, y'_2, z'_2)|^2 \leq K_y^2|y_1 - y_2|^2 + K_z^2|z_1 - z_2|^2 + K_{y'}^2|y'_1 - y'_2|^2 + K_{z'}^2|z'_1 - z'_2|^2,
\]

\[\forall (t, \omega, \omega') \in [0, T] \times \Omega^2, (y_i, z_i, y'_i, z'_i) \in \mathbb{R}^{4n+2d}, i = 1, 2,\]
and
\[ E \int_0^T |E'\theta_0'(t, \omega, \omega')|^2 dt < \infty, \quad E \int_0^T |l_0(s, \omega)|^2 ds < \infty, \quad i = f, g, \]
where \( \theta_0'(t, \omega, \omega') = \theta_0'(t, \omega, \omega', 0, 0, 0, 0) \), \( l_0(s, \omega) = l_0(s, \omega, 0, 0, 0) \). Here we assume that \( \alpha_1 + \alpha_2 \alpha_3 + \alpha_2 \alpha_4 < 1 \).

**Remark 3.1** Under (H1), since we have
\[
|\theta^i(t, \omega, \omega', Y(t, \omega), Z(t, \omega), y, z)| \\
\leq L(1 + |Y(t, \omega)| + |Z(t, \omega)| + |y| + |z|), \quad l = f, g,
\]
consequently,
\[
|\Gamma^i(t, Y(t), Z(t))| \leq L(1 + |Y(t)| + |Z(t)| + E|Y(t)| + E|Z(t)|), \quad l = f, g.
\]
Likewise, for any \((Y_1(\cdot), Z_1(\cdot)), (Y_2(\cdot), Z_2(\cdot)) \in S^2([0, T]; \mathbb{R}^n) \times M^2([0, T]; \mathbb{R}^{n \times d})\),
\[
|\Gamma^f(t, Y_1(t), Z_1(t)) - \Gamma^f(t, Y_2(t), Z_2(t))| \\
\leq L_y|Y_1(t) - Y_2(t)| + L_z|Z_1(t) - Z_2(t)| \\
+ L_y E|Y_1(t) - Y_2(t)| + L_z E|Z_1(t) - Z_2(t)|, \\
|\Gamma^g(t, Y_1(t), Z_1(t)) - \Gamma^g(t, Y_2(t), Z_2(t))| \\
\leq K_y^2|Y_1(t) - Y_2(t)|^2 + K_z^2 E|Y_1(t) - Y_2(t)|^2 \\
+ \alpha_3 |Z_1(t) - Z_2(t)|^2 + \alpha_4 E|Z_1(t) - Z_2(t)|^2.
\]
The above two estimates will play an interesting role in the next theorem.

**Theorem 3.1** Suppose (H1) holds. Then MF-BDSDE (3.1) admits a unique adapted solution \((Y, Z) \in S^2([0, T]; \mathbb{R}^n) \times M^2(0, T; \mathbb{R}^{n \times d})\).

**Proof.** For any \((y, z) \in S^2(0, T; \mathbb{R}^n) \times M^2(0, T; \mathbb{R}^{n \times d})\), we consider the following MF-BDSDE
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \mathbb{E}'[\theta^i(s, Y_s, Z_s, y_s', z_s')]) ds \\
\quad + \int_t^T g(s, Y_s, Z_s, \mathbb{E}'[\theta^i(s, Y_s, Z_s, y_s', z_s')]) dB_s - \int_t^T Z_s d\mathbb{W}^s.
\]
According to Theorem 1.1 in [22], there exists a unique pair of solution \((Y, Z) \in S^2([0, T]; \mathbb{R}^n) \times M^2(0, T; \mathbb{R}^{n \times d})\). Hence, if we define \(\Theta(y, z) = (Y, Z)\), then \(\Theta\) maps from \(S^2([0, T]; \mathbb{R}^n) \times M^2(0, T; \mathbb{R}^{n \times d})\) to itself. We now show that \(\Theta\) is contractive. To this end, take any \((y^i, z^i) \in S^2([0, T]; \mathbb{R}^n) \times M^2(0, T; \mathbb{R}^{n \times d}), (i = 1, 2)\), and let
\[
(Y^i, Z^i) = \Theta(y^i, z^i).
\]
We denote by \((\hat{Y}, \hat{Z}) = (Y^1 - Y^2, Z^1 - Z^2)\) and \((\hat{y}, \hat{z}) = (y^1 - y^2, z^1 - z^2)\). Using Itô formula to \(e^{\beta t}|\hat{Y}_t|^2\) we have

\[
\E e^{\beta t}|\hat{Y}_t|^2 + \E \int_t^T e^{\beta s}|\hat{Z}_s|^2 \, ds + \E \int_t^T \beta e^{\beta s}|\hat{Y}_s|^2 \, ds
= \quad 2\E \int_t^T e^{\beta s}\hat{Y}_s\hat{f}(s) \, ds + \E \int_t^T e^{\beta s}\hat{g}(s) \, ds,
\]

where

\[
\hat{f}(s) = f(s, Y^1_s, Z^1_s, \E'[\theta f(s, Y^1_s, Z^1_s, y^1_s, z^1_s)]) - f(s, Y^2_s, Z^2_s, \E'[\theta f(s, Y^2_s, Z^2_s, y^2_s, z^2_s)]),
\]

\[
\hat{g}(s) = g(s, Y^1_s, Z^1_s, \E'[\theta g(s, Y^1_s, Z^1_s, y^1_s, z^1_s)]) - g(s, Y^2_s, Z^2_s, \E'[\theta g(s, Y^2_s, Z^2_s, y^2_s, z^2_s)]).
\]

Hence from (H1), Remark 3.1 above and the inequality \(ab \leq \frac{1}{\delta}a^2 + \delta b^2\), we have

\[
\E e^{\beta t}|\hat{Y}_t|^2 + \E \int_t^T e^{\beta s}|\hat{Z}_s|^2 \, ds + \E \int_t^T \beta e^{\beta s}|\hat{Y}_s|^2 \, ds
\leq \quad 2\E \int_t^T e^{\beta s}[(1 + L_\gamma)|\hat{Y}_s|(L_y|\hat{Y}_s| + L_z|\hat{Z}_s|) + L_\gamma|\hat{Y}_s|(L_{y'}|\hat{y}_s| + L_{z'}|\hat{z}_s|)] \, ds
+ \E \int_t^T e^{\beta s}[(1 + \alpha_2)|\hat{Y}_s|^2 + (\alpha_1 + \alpha_2\alpha_3)|\hat{Z}_s|^2 + \alpha_2K_y|\hat{y}_s|^2 + \alpha_2\alpha_4|\hat{z}_s|^2] \, ds
\leq \quad M_1\E \int_t^T e^{\beta s}|\hat{Y}_s|^2 \, ds + M_2\E \int_t^T e^{\beta s}|\hat{Z}_s|^2 \, ds
+ M_3\E \int_t^T e^{\beta s}|\hat{y}_s|^2 \, ds + M_4\E \int_t^T e^{\beta s}|\hat{z}_s|^2 \, ds,
\]

where

\[
M_1 = K_y^2(1 + \alpha_2) + (1 + L_\gamma)(L_y + L_z \cdot C + \frac{1}{2}L_\gamma^2 + L_\gamma L_{z'} \cdot C),
\]

\[
M_2 = \frac{(1 + L_\gamma)L_z}{C} + (\alpha_1 + \alpha_2\alpha_3),
\]

\[
M_3 = \frac{1}{2}L_{y'}^2 + \alpha_2K_y; \quad M_4 = \frac{L_\gamma L_{z'}}{C} + \alpha_2\alpha_4.
\]

After some simple calculations, it is easy to see that

\[
\E \int_t^T \beta - \frac{M_1}{1 - M_2} e^{\beta s}|\hat{Y}_s|^2 \, ds + \E \int_t^T e^{\beta s}|\hat{Z}_s|^2 \, ds
\leq \quad \frac{M_4}{1 - M_2} \left[ \E \int_t^T M_3|\hat{y}_s|^2 \, ds + \E \int_t^T e^{\beta s}|\hat{z}_s|^2 \, ds \right].
\]

By the assumption imposed on \(\alpha_i\), \(\Theta\) is a contraction on \(M^2(0, T; \mathbb{R}^{n+n\times d})\), thus there is a unique fixed point \((Y, Z) \in M^2(0, T; \mathbb{R}^{n+n\times d})\) which is the solution of (3.1). Moreover, it is easy to check that \(Y(\cdot) \in S^2(0, T; \mathbb{R}^n)\). The proof is complete. \(\square\)
Remark 3.2 Note that our result here can fully cover the corresponding results in [9] and [22]. In fact, if $K_y = \alpha_1 = \alpha_2 = 0$ or $L_\gamma = \alpha_2 = 0$, our result degenerates respectively to the case in [9] and [22].

Next we introduce a type of forward doubly stochastic differential equation with mean field type as follows,

$$
P_t = \eta + \int_0^t f(s, P_s, Q_s, \Gamma^f(s, P_s, Q_s))ds + \int_0^t g(s, P_s, Q_s, \Gamma^g(s, P_s, Q_s))d\tilde{W}_s - \int_0^t Q_s d\tilde{B}_s,
$$

(3.2)

where $\eta$ is $\mathcal{F}_0$-measurable. Note that such kind of equations appear as adjoint equation in the optimal control problem below. Here we will transform (3.2) into the similar form of (3.1). If we define

$$
\tilde{B}_t = B_T - B_{T-t}, \quad \tilde{W}_t = W_T - W_{T-t}, \quad \tilde{\mathcal{F}}_t = \mathcal{F}_{T-t} \vee \tilde{\mathcal{F}}_t,
$$

$$
\tilde{P}_t = P_{T-t}, \quad \tilde{Q}_t = Q_{T-t}, \quad t \in [0, T],
$$

then $\tilde{\mathcal{F}}_t$ is $\mathcal{F}_{T-t}$-measurable, $\tilde{P}_t, \tilde{Q}_t$ are $\tilde{\mathcal{F}}_t$-measurable, and

$$
\tilde{P}_t = \eta + \int_t^T f(s, \tilde{P}_s, \tilde{Q}_s, \Gamma^f(s, \tilde{P}_s, \tilde{Q}_s))ds + \int_t^T g(s, \tilde{P}_s, \tilde{Q}_s, \Gamma^g(s, \tilde{P}_s, \tilde{Q}_s))d\tilde{W}_s - \int_t^T \tilde{Q}_s d\tilde{B}_s.
$$

(3.3)

Note that (3.3) have the similar form as (3.1), then by Theorem 3.1 there exists a unique pair of $(\tilde{P}, \tilde{Q})$ solving (3.3), and we have

Theorem 3.2 Suppose (H1) holds. Then (3.2) admits a unique adapted solution $(P, Q) \in S^2([0, T]; \mathbb{R}^n) \times M^2(0, T; \mathbb{R}^{n \times d})$.

4 Probabilistic interpretation for a class of nonlocal SPDEs

The connection between BDSDEs and systems of second-order quasilinear SPDEs was firstly observed by Pardoux and Peng [22], where the probabilistic interpretation for second-order SPDEs of parabolic types was derived. Thereafter, this subject has attracted a lot of research, such as [5], [14], [23], [26], [27]. This section can be regarded as a continuation of such a theme. In other words, we will exploit the above theory of MF-BDSDE in order to provide a probabilistic formula for the solution of a class of nonlocal SPDE.

Given arbitrary $x_0 \in \mathbb{R}^m$, $(t, x) \in [0, T] \times \mathbb{R}^m$ being the initial condition, let us consider the following forward SDE in $\mathbb{R}^m$,

$$
X_{s}^{t,x} = x + \int_t^s \Gamma^b(r, X_r^{t,x})dr + \int_t^s \Gamma^a(r, X_r^{t,x})d\tilde{W}_r, \quad s \geq t,
$$

(4.1)
It is known that SDE (4.1) has a unique solution if coefficients satisfy linear growth and the obtain naturally the wellposedness of (4.2) by means of Theorem 3.1 above. and the backward equation

\[
Y_t^{t,x} = \mathbb{E}'[h(X_T^{t,x},(X_T^{0,x_0}))] + \int_s^T \Gamma^{f}_1(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr - \int_s^T Z_r^{t,x}d\tilde{W}_r
\]

\[
+ \int_s^T \Gamma^{g}_1(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})d\tilde{B}_r, \quad s \geq t, \tag{4.2}
\]

where

\[
\Gamma^{k}(r,X_r^{t,x}) = \int_{\Omega} h(r,X_r^{t,x}(\omega),X_r^{0,x_0}(\omega'))d\mathbb{P}(d\omega') =: \mathbb{E}'[h(r,X_r^{t,x},(X_r^{0,x_0}))],
\]

\[
\Gamma^{f}_1(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}) = \int_{\Omega} \theta^{f}(r,X_r^{t,x}(\omega),Y_r^{t,x}(\omega),Z_r^{t,x}(\omega),X_r^{0,x_0}(\omega'),Y_r^{0,x_0}(\omega'),Z_r^{0,x_0}(\omega'))d\mathbb{P}(d\omega')
\]

\[
=: \mathbb{E}'[\theta^{f}(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x},(X_r^{0,x_0}),(Y_r^{0,x_0}),(Z_r^{0,x_0}))],
\]

with \(k = b, \sigma, l = f, g, \text{ and} \)

\[
b : [0,T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m, \quad \sigma : [0,T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m \times d},
\]

\[
\theta^{f} : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n,
\]

\[
\theta^{g} : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{n \times l},
\]

\[
h : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n.
\]

It is known that SDE (4.1) has a unique solution if coefficients satisfy linear growth and the Lipschitz condition, see [3]. Similarly under suitable assumptions of \(h, \theta^{f} \) and \(\theta^{g}\), we can also obtain naturally the wellposedness of (4.2) by means of Theorem 3.1 above.

Suppose \(\theta^{f}(t,x,x',\cdots) \) and \(\theta^{g}(t,x,x',\cdots)\) satisfy the conditions in Theorem 3.1 uniformly for \(t, x \) and \(x'\), and \(\mathbb{E}|h(X_T^{t,x},(X_T^{0,x_0}))^2| < \infty\), it follows from Theorem 3.1 that (4.2) admits a unique solution \((Y_t,Z_t) \in S^2([0,T]; \mathbb{R}^n) \times M^2(0,T; \mathbb{R}^{n \times d})\).

We now relate MF-BDSDE (4.2) to the following nonlinear SPDE:

\[
du(t,x) = -\left(\mathcal{L}u(t,x) + \mathbb{E}'\theta^{f}(t,x,u(t,x),\nabla u(t,x) \cdot \mathbb{E}'[\sigma(t,x,(X_t^{0,x_0}))])
\]

\[
(X_t^{0,x_0}), u'(t,X_t^{0,x_0}), \nabla u'(t,a) \cdot \mathbb{E}'[\sigma(t,a,(X_t^{0,x_0}))] \bigg|_{a=X_t^{0,x_0}} \right) dt \]

\[
- \mathbb{E}'\theta^{g}(t,x,u(t,x),\nabla u(t,x) \cdot \mathbb{E}'[\sigma(t,x,(X_t^{0,x_0}))],(X_t^{0,x_0}))^T 
\]

\[
u'(t,X_t^{0,x_0}), \nabla u'(t,a) \cdot \mathbb{E}'[\sigma(t,a,(X_t^{0,x_0}))] \bigg|_{a=X_t^{0,x_0}} \right) d\tilde{B}_t,
\]

\[
u(T,x) = \mathbb{E}'[h(x,(X_T^{0,x_0}))],
\]

where \(u : \mathbb{R} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^n, \quad \mathcal{L}u = (Lu_1, \cdots, Lu_n)^T, \) with

\[
Lu_i(t,x) = \frac{1}{2} tr(\mathbb{E}'[\sigma(t,x,(X_t^{0,x_0}))][\mathbb{E}'[\sigma(t,x,(X_t^{0,x_0}))]^T D^2u(t,x))
\]

\[+ \nabla u(t,x) \cdot \mathbb{E}'[b(t,x,(X_t^{0,x_0}))],
\]
and
\[ u'(t, (X_t^{0,x_0})') = u(t, \omega', X_t^{0,x_0}(\omega')), \quad t \in [0, T]. \] (4.4)

We can assert that

**Theorem 4.1** Suppose that \( b, \sigma, f \) and \( g \) satisfy suitable linear growth and Lipschitz condition, \( h_{xx}(x, (X_T^{0,x_0})) \) exists and \( \mathbb{E}[e^{|h(X_T^{0,x_0})|}]^2 < \infty \). Suppose SPDE (4.3) has a solution \( u(t, x) \in C^{1,2}(\Omega \times [0, T] \times \mathbb{R}^m; \mathbb{R}^n) \). Then, for any given \((t, x)\), \( u(t, x) \) has the following interpretation

\[ u(t, x) = Y_t^{t, x}, \] (4.5)

where \( Y_t^{t, x} \) is determined by (4.1) and (4.2). Moreover, the solution \( u(t, x) \) of (4.3) is unique too.

**Proof** Applying Itô’s formula to \( u(t, X_t) \), we obtain

\[
u'(t, X_t^{t, x}) - u(t, x) = \int_t^T \partial_r u(r, X_r^{t, x}) + \mathcal{L}u(r, X_r^{t, x}) dr + \int_t^T \nabla u(r, X_r^{t, x}) \cdot \mathbb{E}[\sigma(r, X_r^{t, x}, (X_r^{0,x_0}'))] d\overrightarrow{W_r}.
\]

Because \( u(t, x) \) satisfies SPDE (4.3), it holds that

\[
u(t, X_t^{t, x}) - u(t, x) = - \int_t^T \mathbb{E}[\theta^f(s, X_s^{t, x}, u(s, X_s^{t, x}), \nabla u(s, X_s^{t, x}) \cdot \mathbb{E}[\sigma(s, X_s^{t, x}, (X_s^{0,x_0}'))], (X_s^{0,x_0}'), u'(s, (X_s^{0,x_0}'))], [\nabla u'(s, a) \cdot \mathbb{E}[\sigma(s, a, (X_s^{0,x_0}'))]|_{a=X_s^{0,x_0}}] ds
\]

\[
- \int_t^T \mathbb{E}[\theta^g(s, X_s^{t, x}, u(s, X_s^{t, x}), \nabla u(s, X_s^{t, x}) \cdot \mathbb{E}[\sigma(s, X_s^{t, x}, (X_s^{0,x_0}'))], (X_s^{0,x_0}'), u'(s, (X_s^{0,x_0}'))], [\nabla u'(s, a) \cdot \mathbb{E}[\sigma(s, a, (X_s^{0,x_0}'))]|_{a=X_s^{0,x_0}}] d\overrightarrow{B_s}
\]

\[
+ \int_t^T \nabla u(s, X_s^{t, x}) \cdot \mathbb{E}[\sigma(s, X_s^{t, x}, (X_s^{0,x_0}'))] d\overrightarrow{W_s}.
\]

By the uniqueness of solution for (4.1) and (4.2), it is easy to check that \((u(s, X_s^{t, x}), \nabla u(s, X_s^{t, x}) \cdot \mathbb{E}[\sigma(s, X_s^{t, x}, (X_s^{0,x_0}'))])\) with \( s \in [0, T] \) is a solution of (4.2). Hence it follows that

\[ u(t, x) = Y_t^{t, x}, \quad t \in [0, T], x \in \mathbb{R}^m. \]

Under the above condition, the solution \( Y_t^{t, x}(\cdot) \) is unique, then the solution \( u(t, x) \) of SPDE (4.3) is also unique. \( \square \)

**Remark 4.1** (4.5) can be regarded a stochastic Feynman-Kac formula for SPDE (4.3), which is a useful tool in the study of the property for SPDE. For example, from Theorem 4.1 we know that the solution of SPDE (4.3) must be unique if it exists.
Remark 4.2 Note that the introduction of function $u'$ in (1.4) coincides with the general setting in our discussion. Actually, on the one hand, comparing with the SPDEs in [22], the term $u'(t, (X^0_t, x_0))$ is necessary since our SPDE here is nonlocal. On the other hand, comparing with the case in [9], here we replace $u(t, (X^0_t, x_0))$ frequently used there with a slight new term $u'(t, (X^0_t, x_0))$ because of the special measurability of $u(t, x)$. As we know, since $Y^0_t, x_0(s, \omega) = u(s, \omega, X^0_t, x_0(s, \omega))$, thus $Y^0_t, x_0(s, \omega') = u(s, \omega', X^0_t, x_0(s, \omega')) = u'((X^0_t, x_0))^\prime)$. Obviously, if $\theta^g$ equals to zero, i.e., (1.3) becomes a deterministic PDE, hence $u'(t, (X^0_t, x_0))$ will degenerates into $u(t, (X^0_t, x_0))$ which is the case in [9].

Remark 4.3 In our framework, (4.2) is allowed to depend on $Z^{0, x_0}(\cdot)$, hence some more necessary terms representing the nonlocal property of (1.3) are needed. This is totally different from the discussion in [9].

5 An optimal control problem for MF-BDSDEs

In this section, we would like to consider one optimal control problem for MF-BDSDEs. As a necessary condition, we will derive one maximum principle. For the reason of simplicity, we assume $m = n = d = l = k_1 = k_2 = 1$.

Given a convex subset $U \subset \mathbb{R}^k$, for any admissible control $v \in \mathcal{U}_{ad}$, where

$$
\mathcal{U}_{ad} = \left\{ v : [0, T] \times \Omega \to U | v \text{is } \mathcal{F}_t \text{-measurable, } \mathbb{E} \int_0^T |v_t|^2 dt < +\infty \right\},
$$

and $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$, we consider the following MF-BDSDE:

$$
Y^v_t = \xi + \int_t^T \Gamma^f(s, Y^v_s, Z^v_s, v_s)ds + \int_t^T \Gamma^g(s, Y^v_s, Z^v_s, v_s)d\mathbb{B}_s - \int_t^T Z^v_s d\mathbb{W}_s,
$$

(5.1)

where $i = f, g$,

$$
\Gamma^i(s, Y^v_s, Z^v_s, v_s) = \int_{\Omega} \theta^i(s, \omega, \omega', Y^v_s(\omega), Z^v_s(\omega), v_s(\omega), Y^v_s(\omega'), Z^v_s(\omega'), v_s(\omega')) \mathbb{P}(d\omega'),
$$

and

$$
\theta^f : \Omega^2 \times [0, T] \times \mathbb{R} \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R},
$$

$$
\theta^g : \Omega^2 \times [0, T] \times \mathbb{R} \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}.
$$

The control problem is to find an admissible control to minimize over $\mathcal{U}_{ad}$ the cost function of

$$
J(v(\cdot)) = \mathbb{E} \int_0^T \Gamma^I(s, Y^v_s, Z^v_s, v_s)ds + \mathbb{E}[\mathbb{E}'h(Y^v_0(\omega), Y^v_0(\omega'))],
$$

(5.2)

where

$$
\Gamma^I(s, Y^v_s, Z^v_s, v_s) = \int_{\Omega} l(s, \omega, \omega', Y^v_s(\omega), Z^v_s(\omega), v_s(\omega), Y^v_s(\omega'), Z^v_s(\omega'), v_s(\omega')) \mathbb{P}(d\omega'),
$$

and

$$
\mathbb{E}' : \Omega \times \mathbb{R} \to \mathbb{R}.
$$
and \( h : \Omega^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \),

\[
l : \Omega^2 \times [0, T] \times \mathbb{R} \times U \times \mathbb{R} \times U \to \mathbb{R}.
\]

The following is our main assumptions on the above mappings in this section:

(H2)  
(i) \( \theta^f, \theta^g, l \) and \( h \) are continuous and continuously differentiable with respect to \( y, y', z, z', v, v' \), and the derivatives of \( l \) and \( h \) are allowed to be linear growth.

(ii) \( \theta^f \) and \( \theta^g \) satisfy uniformly Lipschitz condition to \( (y, z, y', z', v, v') \), that is, there exist positive constants \( L_i \) and \( \alpha_j \) with \( i = y, z, y', z', v, v', j = 3, 4 \), such that

\[
|\theta^f(t, \omega, \omega', y_1, z_1, y'_1, z'_1, v_1, v'_1) - \theta^f(t, \omega, \omega', y_2, z_2, y'_2, z'_2, v_2, v'_2)| \\
\leq L_y|y_1 - y_2| + L_z|z_1 - z_2| + L_{y'}|y'_1 - y'_2| \\
+ L_{z'}|z'_1 - z'_2| + L_v|v_1 - v_2| + L_{v'}|v'_1 - v'_2|,
\]

\[
|\theta^g(t, \omega, \omega', y_1, z_1, y'_1, z'_1, v_1, v'_1) - \theta^g(t, \omega, \omega', y_2, z_2, y'_2, z'_2, v_2, v'_2)|^2 \\
\leq K_y^2|y_1 - y_2|^2 + K_{y'}^2|y'_1 - y'_2|^2 + K_v^2|v_1 - v_2|^2 \\
+ K_{v'}^2|v'_1 - v'_2|^2 + \alpha_3|z_1 - z_2|^2 + \alpha_4|z'_1 - z'_2|^2,
\]

\( \forall (t, \omega, \omega') \in [0, T] \times \Omega^2 \), \((y_i, z_i, y'_i, z'_i, v_i, v'_i) \in \mathbb{R}^6, i = 1, 2, \)

and

\[
\mathbb{E} \int_0^T |\mathbb{E}^t \theta^l_0(t, \omega, \omega')|^2 ds < \infty, \quad l = f, g,
\]

where \( \theta^l_0(t, \omega, \omega') = \theta^l_0(t, \omega, \omega', 0, 0, 0, 0, 0) \), and \( \alpha_3 + \alpha_4 < 1 \).

Under the above hypotheses, for every \( v(\cdot) \in \mathcal{U}_{ad} \), by Theorem 3.1, (5.1) admits a unique strong solution \((Y^v, Z^v) \in S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}) \), and the cost functional \( J \) is well-defined.

Suppose that \( \bar{u}(\cdot) \) is an optimal control and \((\bar{Y}(\cdot), \bar{Z}(\cdot)) \) is the corresponding optimal trajectory. Let \( v(\cdot) \) be such that \( \bar{u}(\cdot) + v(\cdot) \in \mathcal{U}_{ad} \). Since \( \mathcal{U}_{ad} \) is convex, then for any \( 0 \leq \varepsilon \leq 1 \), \( w^\varepsilon(\cdot) = \bar{u}(\cdot) + \varepsilon v(\cdot) \) is also in \( \mathcal{U}_{ad} \). From Theorem 3.1 we know that state equation (5.1) has a unique solution, denoted by \((Y^\varepsilon(\cdot), Z^\varepsilon(\cdot)) \) corresponding to \( w^\varepsilon \). Before the main result we require to prove some basic results.

**Lemma 5.1** Under assumption (H2), for any \( t \in [0, T] \), we have

\[
E|Y^\varepsilon_t - \hat{Y}_t|^2 \leq C\varepsilon^2, \quad E\int_t^T |Z^\varepsilon_s - \hat{Z}_s|^2 ds \leq C\varepsilon^2.
\]

**Proof.** Notice that \( Y^\varepsilon_t - \hat{Y}_t \) satisfies the following MF-BDSDE:

\[
Y^\varepsilon_t - \hat{Y}_t = \int_t^T \left[ \Gamma^f(s, Y^\varepsilon_s, Z^\varepsilon_s, u^\varepsilon_s) - \Gamma^f(s, \hat{Y}_s, \hat{Z}_s, \hat{u}_s) \right] ds \\
+ \int_t^T \left[ \Gamma^g(s, Y^\varepsilon_s, Z^\varepsilon_s, u^\varepsilon_s) - \Gamma^g(s, \hat{Y}_s, \hat{Z}_s, \hat{u}_s) \right] d\tilde{W}_s \\
- \int_t^T (Z^\varepsilon_s - \hat{Z}_s) d\tilde{W}_s.
\]
Applying Itô’s formula to $|Y_t^\varepsilon - \bar{Y}_t|^2$, we have
\[
\begin{align*}
\mathbb{E} \left( |Y_t^\varepsilon - \bar{Y}_t|^2 + \int_t^T |Z_s^\varepsilon - \bar{Z}_s|^2 ds \right) \\
= 2 \mathbb{E} \int_t^T \left( Y_s^\varepsilon - \bar{Y}_s, \Gamma^f(s, Y_s^\varepsilon, Z_s^\varepsilon, u_s^\varepsilon) \right) ds \\
+ \mathbb{E} \int_t^T \left| \Gamma^g(s, Y_s^\varepsilon, Z_s^\varepsilon, u_s^\varepsilon) - \Gamma^g(s, \bar{Y}_s, \bar{Z}_s, \bar{u}_s) \right|^2 ds.
\end{align*}
\]
From (H2), we have
\[
\mathbb{E}[Y_t^\varepsilon - \bar{Y}_t]^2 + \mathbb{E} \int_t^T |Z_s^\varepsilon - \bar{Z}_s|^2 ds \\
\leq k_1 \mathbb{E} \int_t^T |Y_s^\varepsilon - \bar{Y}_s|^2 ds + k_2 \varepsilon^2 \mathbb{E} \int_t^T |v_s|^2 ds,
\]
where $k_i$ ($i = 1, 2$) are two constants depending on the parameters in (H2). By Gronwall’s inequality the Burkholder-Davis-Gundy inequality, we obtain the results. \hfill \Box

In the following we make the convention that
\[
\hat{\alpha} (\cdot) = \alpha (\cdot, \omega, \omega', \bar{Y} (\cdot, \omega), \bar{Z} (\cdot, \omega), \bar{\mu} (\cdot, \omega), \bar{\mu}' (\cdot, \omega')),
\]
where $\omega, \omega' \in \Omega$, $\alpha = \theta^f, \theta^g, \theta^l$. We introduce the variational equation as follows,
\[
\xi_t = \psi_t + \int_t^T F_1(s, \xi_s, \eta_s) ds + \int_t^T G_1(s, \xi_s, \eta_s) d\bar{B}_s - \int_t^T \eta_s d\bar{W}_s, \tag{5.3}
\]
where
\[
\begin{align*}
F_1(s, \xi_s, \eta_s) &= \mathbb{E}'[\hat{\theta}^f_\gamma (s) \xi_s + \hat{\theta}^{f'}_\gamma (s) \xi'_s + \hat{\theta}^f_\gamma (s) \eta_s + \hat{\theta}^{f'}_\gamma (s) \eta'_s], \\
G_1(s, \xi_s, \eta_s) &= \mathbb{E}'[\hat{\theta}^g_\gamma (s) \xi_s + \hat{\theta}^{g'}_\gamma (s) \xi'_s + \hat{\theta}^g_\gamma (s) \eta_s + \hat{\theta}^{g'}_\gamma (s) \eta'_s],
\end{align*}
\]
and
\[
\psi(t) = \int_t^T \mathbb{E}'[\hat{\theta}^f_\gamma v(s) v_s + \hat{\theta}^{f'}_\gamma v(s) v'_s] ds + \int_t^T \mathbb{E}'[\hat{\theta}^g_\gamma v(s) v_s + \hat{\theta}^{g'}_\gamma v(s) v'_s] d\bar{B}_s.
\]
Here we denote, for example,
\[
\begin{align*}
\mathbb{E}'[\hat{\theta}^f_\gamma (s) \xi_s] &= \int_\Omega \hat{\theta}^f_\gamma (s, \omega, \omega') \xi(s, \omega) \mathbb{P}(d\omega') \\
\mathbb{E}'[\hat{\theta}^{f'}_\gamma (s) \xi'_s] &= \int_\Omega \hat{\theta}^{f'}_\gamma (s, \omega, \omega') \xi(s, \omega') \mathbb{P}(d\omega').
\end{align*}
\]
Under (H2), from Theorem 3.1 there exists a unique $(\xi_t, \eta_t) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R})$ satisfying (5.3).
Lemma 5.2 If we denote by
\[ y^\varepsilon_t = \frac{Y^\varepsilon_t - \hat{Y}_t}{\varepsilon} - \xi_t, \quad z^\varepsilon_t = \frac{Z^\varepsilon_t - \hat{Z}_t}{\varepsilon} - \eta_t. \]
Then we have
\[ \lim_{\varepsilon \to 0} \sup_{t \in [0, T]} \mathbb{E}|y^\varepsilon_t|^2 = 0, \quad \lim_{\varepsilon \to 0} \mathbb{E} \int_0^T |z^\varepsilon_t|^2 dt = 0. \] (5.4)

Proof. First we can express \( y^\varepsilon_t \) and \( z^\varepsilon_t \) as
\[
\begin{cases}
-dy^\varepsilon_t = & \mathbb{E}[f^\varepsilon_y(t)y^\varepsilon_t + f^\varepsilon_z(t)z^\varepsilon_t + f^\varepsilon_{y'}(t)y^\varepsilon_t + f^\varepsilon_{z'}(t)z^\varepsilon_t + f^\varepsilon_{z''}(t)z^\varepsilon_t]dt \\
+ & \mathbb{E}[g^\varepsilon_y(t)y^\varepsilon_t + g^\varepsilon_z(t)z^\varepsilon_t + g^\varepsilon_{y'}(t)y^\varepsilon_t + g^\varepsilon_{z'}(t)z^\varepsilon_t + g^\varepsilon_{z''}(t)z^\varepsilon_t]d\hat{W}_t \\
- & z^\varepsilon_t d\hat{W}_t,
\end{cases}
\]
where, for example, we denote \( \delta = f, g, \bar{Y}_{t,\omega} = \bar{Y}_{t,\omega} + \lambda(Y^\varepsilon_{t,\omega} - \bar{Y}_{t,\omega}), \bar{u}_{t,\omega} = \bar{u}_{t,\omega} + \lambda(u^\varepsilon_{t,\omega} - \bar{u}_{t,\omega}) \)
\[ \delta^\varepsilon_y(\cdot) = \int_0^1 \theta^\varepsilon_y(\cdot, \bar{Y}_{t,\omega}, \bar{Z}_{t,\omega}, \bar{u}_{t,\omega}, \bar{Y}_{t,\omega}', \bar{Z}_{t,\omega}', \bar{u}_{t,\omega}')d\lambda, \]
and
\[ \delta^\varepsilon_z(\cdot) = [\delta^\varepsilon_y(\cdot) - \hat{\theta}^\varepsilon_y(\cdot)]\xi_{t,\omega} + [\delta^\varepsilon_z(\cdot) - \hat{\theta}^\varepsilon_z(\cdot)]\eta_{t,\omega} + [\delta^\varepsilon_{y'}(\cdot) - \hat{\theta}^\varepsilon_{y'}(\cdot)]\bar{\xi}_{t,\omega} + [\delta^\varepsilon_{z'}(\cdot) - \hat{\theta}^\varepsilon_{z'}(\cdot)]\bar{\eta}_{t,\omega}. \]

Applying Itô’s formula to \( |y^\varepsilon_t|^2 \) on \([t, T]\), we get
\[
\mathbb{E}|y^\varepsilon_t|^2 + \mathbb{E} \int_t^T |z^\varepsilon_s|^2 ds
= 2\mathbb{E} \int_t^T \left\langle y^\varepsilon_s, \mathbb{E}[f^\varepsilon_y(s)y^\varepsilon_s + f^\varepsilon_z(s)z^\varepsilon_s + f^\varepsilon_{y'}(s)y^\varepsilon_s + f^\varepsilon_{z'}(s)z^\varepsilon_s + f^\varepsilon_{z''}(s)z^\varepsilon_s + f^\varepsilon_{z''}(s)z^\varepsilon_s]ds \right\rangle
+ \mathbb{E} \int_t^T |g^\varepsilon_y(s)y^\varepsilon_s + g^\varepsilon_z(s)z^\varepsilon_s + g^\varepsilon_{y'}(s)y^\varepsilon_s + g^\varepsilon_{z'}(s)z^\varepsilon_s + g^\varepsilon_{z''}(s)z^\varepsilon_s + g^\varepsilon_{z''}(s)z^\varepsilon_s|^2 ds.
\]
From (H2), we have
\[
\mathbb{E}|y^\varepsilon_t|^2 + \mathbb{E} \int_t^T |z^\varepsilon_s|^2 ds \leq k \mathbb{E} \int_t^T |y^\varepsilon_s|^2 ds + C, \]
where \( k \) is some suitable constant, \( C \varepsilon \to 0 \) as \( \varepsilon \to 0 \). By Gronwall’s inequality, we obtain the desired result. \( \square \)

Since \( \hat{u}(\cdot) \) is an optimal control, then
\[ \varepsilon^{-1}[J(u^\varepsilon(\cdot)) - J(\hat{u}(\cdot))] \geq 0. \] (5.5)

From this and Lemma 5.2 we have the following
Lemma 5.3 Let assumption (H2) hold. Then the following variational inequality holds:

$$
\mathbb{E} \int_0^T \mathbb{E}'[\hat{l}_y(s)\xi_s + \hat{l}_z(s)\eta_s + \hat{l}_{y'}(s)\xi'_s + \hat{l}_{z'}(s)\eta'_s + \hat{l}_u(s)v_s + \hat{l}_{v'}(s)v'_s]ds + \mathbb{E}\mathbb{E}'[h_y(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'})\xi_{0,\omega} + h_{y'}(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'})\xi_{0,\omega'}] \geq 0.
$$

(5.6)

Here we denote, for example,

$$
\mathbb{E}'[\hat{l}_{y'}(s)\xi'_s] = \int_{\Omega} \hat{l}_{y'}(s, \omega, \omega')\xi(s, \omega')\mathbb{P}(d\omega').
$$

Proof. From the first result of (5.4), we derive

$$
\mathbb{E}\mathbb{E}'\varepsilon^{-1}[h(Y_{0,\omega}^\varepsilon, Y_{0,\omega'}^\varepsilon) - h(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'})]
= \mathbb{E}\mathbb{E}'\varepsilon^{-1} \int_0^1 h_y(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'}) (Y_{0,\omega}^\varepsilon - \hat{Y}_{0,\omega}) d\lambda
+ \mathbb{E}\mathbb{E}'\varepsilon^{-1} \int_0^1 h_{y'}(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'}) (Y_{0,\omega}^\varepsilon - \hat{Y}_{0,\omega'}) d\lambda
\rightarrow \mathbb{E}\mathbb{E}'[h_y(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'})\xi_{0,\omega} + h_{y'}(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'})\xi_{0,\omega'}], \quad \varepsilon \rightarrow 0,
$$

where for example, $\hat{Y}_{0,\omega} = \hat{Y}_{0,\omega} + \lambda(Y_{0,\omega}^\varepsilon - \hat{Y}_{0,\omega})$. Similarly, we have $\varepsilon \rightarrow 0$,

$$
\varepsilon^{-1} \left\{ \mathbb{E} \int_0^T \mathbb{E}'[\hat{l}^\varepsilon(t) - \hat{l}(t)]dt \right\}
\rightarrow \mathbb{E} \int_0^T \mathbb{E}'[\hat{l}_y(s)\xi_s + \hat{l}_z(s)\eta_s + \hat{l}_{y'}(s)\xi'_s + \hat{l}_{z'}(s)\eta'_s + \hat{l}_u(s)v_s + \hat{l}_{v'}(s)v'_s]ds.
$$

Thus (5.6) follows. 

Now we consider the adjoint equation:

$$
p_t = \mathbb{E}'h_y(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'}) + \mathbb{E}^*h_{y'}(\hat{Y}_{0,\omega'}, \hat{Y}_{0,\omega})
+ \int_0^t F_2(s, \omega, \omega) ds + \int_0^t G_2(s, \omega, \omega) d\hat{W}_s - \int_0^t q_s d\hat{B}_s,
$$

(5.7)

where

$$
F_2(s, \omega, \omega) = \mathbb{E}'[\hat{\theta}_y(s)p_s + \hat{\theta}_y(s)q_s + \hat{l}_y(s)]
+ \mathbb{E}^*[\hat{\theta}_{y'}(s)p_s^* + \hat{\theta}_{y'}(s)q_s^* + \hat{l}_{y'}(s)],
$$

$$
G_2(s, \omega, \omega) = \mathbb{E}'[\hat{\theta}_z(s)p_s + \hat{\theta}_z(s)q_s + \hat{l}_z(s)]
+ \mathbb{E}^*[\hat{\theta}_{z'}(s)p_s^* + \hat{\theta}_{z'}(s)q_s^* + \hat{l}_{z'}(s)].
$$

Here we denote by, for example,

$$
\mathbb{E}'[\hat{l}_{y'}(s)\xi'_s] = \int_{\Omega} \hat{l}_{y'}(s, \omega, \omega')\xi(s, \omega')\mathbb{P}(d\omega'),
$$

$$
\mathbb{E}^*[\hat{\theta}_{y'}(s)p_s^*] = \int_{\Omega} \hat{\theta}_{y'}(s, \omega, \omega) p(s, \omega')\mathbb{P}(d\omega').
$$
The adjoint equation \([5.7]\) is a special form of \([3.2]\) with bounded coefficients. Under (H2), it follows from Theorem 3.2 that \([5.7]\) admits a unique solution \((p_t, q_t)\).

We define the Hamiltonian function \(H : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) as follows:

\[
H(t, y_1, z_1, v_1, y_2, z_2, v_2, p, q) = \theta^f(t, \omega, \omega', y_1, z_1, v_1, y_2, z_2, v_2)p + \theta^g(t, \omega, \omega')y_1, z_1, v_1, y_2, z_2, v_2)q + l(t, \omega, \omega', y_1, z_1, v_1, y_2, z_2, v_2).
\]

From variational inequality \([5.6]\), we can state the stochastic maximum principle of optimal control problem for MF-BDSDEs.

**Theorem 5.1** (Stochastic maximum principle). Let \((\widehat{Y}(\cdot), \widehat{Z}(\cdot), \widehat{u}(\cdot))\) be an optimal triple of the control problem \([5.1, 5.2]\). Then by definition of \(l E\), it follows from Theorem 3.2 that \([5.7]\) admits a unique solution \((p_t, q_t)\).

\[
[H(t, \omega, \omega')] = H(t, \omega, \omega') + [H^w(t, \omega, \omega)] \cdot (v - \widehat{u}_t) \geq 0,
\]

where for convenience we denote by

\[
\begin{align*}
\widehat{H}(t, \omega, \omega') &= H(t, \omega, \omega', \widehat{Y}_t(\omega), \widehat{Z}_t(\omega), \widehat{u}_t(\omega), \widehat{Y}_t(\omega'), \widehat{Z}_t(\omega'), \widehat{u}_t(\omega'), p_t(\omega), q_t(\omega)).
\end{align*}
\]

**Proof.** Applying Itô’s formula to \((\xi_t, p_t)\), we obtain

\[
-E\xi_0 p_0 = E^T \int_0^T E^f[l_y(s)\xi_s + l_z(s)\eta_s + l_{\omega'}(s)\xi_{s}' + l_{\omega''}(s)\eta_{s}']ds
\]

\[
- E^T \int_0^T E^f[\widehat{J}_{\omega'}(s)v_s'p_s + \widehat{\theta}g_{v'}(s)v_s'q_s]ds
\]

\[
- E^T \int_0^T E^f[\widehat{J}_v(s)v_s p_s + \widehat{\theta}g_v(s)v_s q_s]ds.
\]

Then by definition of \(E^*\) and variational inequality \([5.6]\) above, we have

\[
E^T \int_0^T E^f[\widehat{J}_{\omega'}(s)v_s'p_s + \widehat{\theta}g_{v'}(s)v_s'q_s + \widehat{l}_{v'}(s)v_s']ds
\]

\[
+ E^T \int_0^T E^f[\widehat{J}_v(s)v_s p_s + \widehat{\theta}g_v(s)v_s q_s + \widehat{l}_v(s)v_s]ds \geq 0.
\]

From the definition of Hamiltonian function in \([5.8]\),

\[
E^T \int_0^T \left[E^*H_{v'}(t, \widehat{Y}_t(\omega'), \widehat{Z}_t(\omega'), \widehat{u}_t(\omega'), \widehat{Y}_t(\omega), \widehat{Z}_t(\omega), \widehat{u}_t(\omega), p_t(\omega'), q_t(\omega'))
\]

\[
+ E'H_v(t, \widehat{Y}_t(\omega), \widehat{Z}_t(\omega), \widehat{u}_t(\omega), \widehat{Y}_t(\omega'), \widehat{Z}_t(\omega'), \widehat{u}_t(\omega'), p_t(\omega), q_t(\omega))\right] \cdot v_t dt \geq 0.
\]
For \( v \in U, F \) be an arbitrary element of the \( \sigma \)-algebra \( \mathcal{F}_t \), set
\[
\varpi(s) = \begin{cases} 
\hat{u}_s, & s \in [0, t), \\
v, & s \in [t, t + \varepsilon), \omega \in F, \\
\hat{u}_s, & s \in [t, t + \varepsilon), \omega \in \Omega - F, \\
\hat{u}_s, & s \in [t + \varepsilon, T],
\end{cases}
\]
we have \( \varpi(s) \in \mathcal{U}_{ad} \). Since \( v_t \) satisfies \( \hat{u}_t + v_t \in \mathcal{U}_{ad} \), then by taking \( v_t = \varpi_t - \hat{u}_t \), we can rewrite above inequality as
\[
\mathbb{E} 1_F \int_t^{t + \varepsilon} \left[ \mathbb{E}' \hat{H}_v(s, \omega, \omega') + \mathbb{E}^* \hat{H}_{v'}(s, \omega, \omega) \right] \cdot (v - \hat{u}_s) \, ds \geq 0,
\]
where \( \hat{H} \) is defined in (5.10). Differentiating with respect to \( \varepsilon \) at \( \varepsilon = 0 \) gives
\[
\mathbb{E} 1_F \left[ \mathbb{E}' \hat{H}_v(t, \omega, \omega') + \mathbb{E}^* \hat{H}_{v'}(t, \omega, \omega) \right] \cdot (v - \hat{u}_t) \geq 0,
\]
and (5.9) holds naturally. \( \square \)

6 One mean-field backward LQ problem

In this section, we are dedicated to apply the previous maximum principle to one backward doubly stochastic LQ problem of mean field type. In this case, by supposing \( h(Y_0, Y_0, \omega) = \frac{1}{2} Q_0 Y_0^2 + \frac{1}{2} Q_0^2 Y_0^2, \)
\[
f(s, \omega, \omega', Y_{s, \omega, \omega}', Z_{s, \omega, \omega}', v_{s, \omega, \omega}, Y_{s, \omega, \omega}', Z_{s, \omega, \omega}', v_{s, \omega, \omega}) \\
= A_1 s Y_{s, \omega, \omega'} + B_1 s Z_{s, \omega, \omega'} + C_1 s v_{s, \omega, \omega} + A_2 s Y_{s, \omega, \omega'} + B_2 s Z_{s, \omega, \omega'} + C_2 s v_{s, \omega, \omega}, \\
g(s, \omega, \omega', Y_{s, \omega, \omega}', Z_{s, \omega, \omega}', v_{s, \omega, \omega}, Y_{s, \omega, \omega}', Z_{s, \omega, \omega}', v_{s, \omega, \omega}) \\
= D_1 s Y_{s, \omega, \omega'} + E_1 s Z_{s, \omega, \omega'} + F_1 s v_{s, \omega, \omega} + D_2 s Y_{s, \omega, \omega'} + E_2 s Z_{s, \omega, \omega'} + F_2 s v_{s, \omega, \omega}, \\
l(s, \omega, \omega', Y_{s, \omega, \omega}', Z_{s, \omega, \omega}', v_{s, \omega, \omega}, Y_{s, \omega, \omega}', Z_{s, \omega, \omega}', v_{s, \omega, \omega}) \\
= \frac{1}{2} [M_1 s Y_{s, \omega, \omega'}^2 + N_1 s Z_{s, \omega, \omega'}^2 + R_1 s v_{s, \omega, \omega}^2 + M_2 s Y_{s, \omega, \omega'}^2 + N_2 s Z_{s, \omega, \omega'}^2 + R_2 s v_{s, \omega, \omega}^2],
\]
with, for example, \( A_t : [0, T] \times \Omega^2 \to \mathbb{R} \) being bounded, \( (s, \omega, \omega') \mapsto A_t(s, \omega, \omega') \) being \( \mathcal{F}_s^2 \)-measurable (such assumption also hold for the other coefficients), \( M^t, R^t \) being nonnegative, \( R^t \) being positive, we can write the state equation and the cost functional as
\[
Y_{t, \omega} = \xi + \int_t^T \left[ \mathbb{E}' A_1 s Y_{s, \omega, \omega'} + \mathbb{E}' B_1 s Z_{s, \omega, \omega'} + \mathbb{E}' C_1 s v_{s, \omega, \omega} + \mathbb{E}' [A_2 s Y_{s, \omega, \omega'}] \right] ds \\
+ \int_t^T \mathbb{E}'[B_2 s Z_{s, \omega, \omega'} + C_2 s v_{s, \omega, \omega}] ds + \int_t^T \left[ \mathbb{E}' D_1 s Y_{s, \omega, \omega'} + \mathbb{E}' E_1 s Z_{s, \omega, \omega'} \right] d\tilde{B}_{s, \omega} \\
+ \int_t^T \left[ \mathbb{E}' E_1 s v_{s, \omega, \omega} + \mathbb{E}' [D_2 s Y_{s, \omega, \omega'} + E_2 s Z_{s, \omega, \omega'} + F_2 s v_{s, \omega, \omega}] \right] d\tilde{B}_{s, \omega} - \int_t^T Z_{s, \omega, \omega'} d\tilde{W}_{s, \omega, \omega'},
\]
(6.1)
and

\[ J(v(\cdot)) = \frac{1}{2} \mathbb{E} \left( \int_0^T \mathbb{E}'[M_s^1 | Y_{s,\omega}^v|^2 + N_s^1 | Z_{s,\omega}^v|^2 + R_s^1 v_{s,\omega}^2]ds + \mathbb{E}'[Q_0^1 | Y_{0,\omega}^v|^2] \right) \]
\[ + \frac{1}{2} \mathbb{E} \left( \int_0^T \mathbb{E}'[M_s^2 | Y_{s,\omega'}^v|^2 + N_s^2 | Z_{s,\omega'}^v|^2 + R_s^2 | v_{s,\omega'}|^2]ds + \mathbb{E}'[Q_0^2 | Y_{0,\omega'}^v|^2] \right). \]

For convenience, we write the random coefficients, for example, \( A_i(s, \omega, \omega') \) as \( A_i^s \) in the above and the following part. The Hamiltonian in such setting becomes

\[ H(s, \omega, \omega', y_1, z_1, v_1, y_2, z_2, v_2, p, q) = \]
\[ = [A_1^s y_1 + B_1^s z_1 + C_1^s v_1 + A_2^s y_2 + B_2^s z_2 + C_2^s v_2]p \]
\[ + [D_1^s y_1 + E_1^s z_1 + F_1^s v_1 + D_2^s y_2 + E_2^s z_2 + F_2^s v_2]q \]
\[ + \frac{1}{2} [M_1^s y_1^2 + N_1^s z_1^2 + R_1^s v_1^2 + M_2^s y_2^2 + N_2^s z_2^2 + R_2^s v_2^2]. \]

It follows from Theorem 5.1 that
\[ 0 = \mathbb{E}'[C_1^1 p_s + F_1^1 q_s + R_1^1 \hat{u}_s] + \mathbb{E}^* [C_1^2 p_s + F_2^1 q_s^* + R_2^1 \hat{u}_s], \]
(6.2)

where, for example,
\[ \mathbb{E}^* [C_1^1 p_s] = \int_{\Omega} C_1^1 (s, \omega, \omega') p(s, \omega) \mathbb{P}(d\omega'), \]
\[ \mathbb{E}^* [R_1^2 \hat{u}_s] = \int_{\Omega} R_1^2 (s, \omega^*, \omega) u(s, \omega) \mathbb{P}(d\omega^*), \]
\[ \mathbb{E}^* [C_2^2 p_s] = \int_{\Omega} C_2^2 (s, \omega^*, \omega) p(s, \omega^*) \mathbb{P}(d\omega^*), \]

and
\[ p_t = \mathbb{E}'[Q_1^1 \hat{Y}_{0,\omega}] + \mathbb{E}^* [Q_0^2 \hat{Y}_{0,\omega}] \]
\[ + \int_0^t F_2(s, p_s, q_s)ds + \int_0^t G_2(s, p_s, q_s)d\hat{W}_s - \int_0^t q_s d\hat{B}_s, \]
(6.3)

with
\[ F_2(s, p_s, q_s) = \mathbb{E}'[A_1^1 p_s + D_1^1 q_s + M_1^1] + \mathbb{E}^* [A_1^2 p_s + D_2^1 q_s^* + M_1^2], \]
\[ G_2(s, p_s, q_s) = \mathbb{E}'[B_1^1 p_s + E_1^1 q_s + N_1^1] + \mathbb{E}^* [B_1^2 p_s + E_2^1 q_s^* + N_1^2]. \]

**Theorem 6.1** Suppose there exists \( \hat{u} \) satisfies (6.2), where \((p, q)\) satisfy (6.3), then it must be the unique optimal control of above backward LQ problem.
Proof. First we have

\[ J(v) - J(\widehat{u}) = \frac{1}{2} \mathbb{E} \int_0^T \mathbb{E}'[M_s^1(Y_v^v_{s,\omega} - \widehat{Y}_{s,\omega})^2 + N_s^1(Z_v^v_{s,\omega} - \widehat{Z}_{s,\omega})^2]ds \]

\[ + \frac{1}{2} \mathbb{E} \int_0^T \mathbb{E}'[R_s^1(|v_{s,\omega}|^2 - |\widehat{u}_{s,\omega}|^2) + M_s^2(|Y_v^v_{s,\omega}|^2 - |\widehat{Y}_{s,\omega}|^2)]ds \]

\[ + \frac{1}{2} \mathbb{E} \int_0^T \mathbb{E}'[N_s^2(|Z_v^v_{s,\omega}|^2 - |\widehat{Z}_{s,\omega}|^2) + R_s^2(|v_{s,\omega}|^2 - |\widehat{u}_{s,\omega}|^2)]ds \]

\[ + \frac{1}{2} \mathbb{E} \mathbb{E}'[Q_0^1(|Y_v^v_{0,\omega}|^2 - |\widehat{Y}_{0,\omega}|^2) + Q_0^2(|Y_v^v_{0,\omega}|^2 - |\widehat{Y}_{0,\omega}|^2)] \]

\[ \geq \mathbb{E} \int_0^T \mathbb{E}'[M_s^1(\widehat{Y}_{s,\omega})^2 - (\widehat{Y}_{s,\omega})^2 + N_s^1(\widehat{Z}_{s,\omega})^2 - (\widehat{Z}_{s,\omega})^2]ds \]

\[ + \mathbb{E} \int_0^T \mathbb{E}'[R_s^1(\widehat{u}_{s,\omega} - \widehat{u}_{s,\omega}) + M_s^2(\widehat{Y}_{s,\omega}' - \widehat{Y}_{s,\omega}')ds \]

\[ + \mathbb{E} \int_0^T \mathbb{E}'[N_s^2(\widehat{Z}_{s,\omega}' - \widehat{Z}_{s,\omega}') + R_s^2(\widehat{u}_{s,\omega}' - \widehat{u}_{s,\omega}')ds \]

\[ + \mathbb{E} \mathbb{E}'[Q_0^1(\widehat{Y}_{0,\omega}' - \widehat{Y}_{0,\omega}) + Q_0^2(\widehat{Y}_{0,\omega}' - \widehat{Y}_{0,\omega})]. \]

On the other hand, by using Itô formula to \( p_{s,\omega}(Y_v^v_{s,\omega} - \widehat{Y}_{s,\omega}) \) on \([0, T]\), we have

\[ \mathbb{E} \mathbb{E}'[Q_0^1(\widehat{Y}_{0,\omega}' - \widehat{Y}_{0,\omega}) + Q_0^2(\widehat{Y}_{0,\omega}' - \widehat{Y}_{0,\omega})] \]

\[ = \mathbb{E} \int_0^T \mathbb{E}'[C_s^1 p_{s,\omega} + F_s^1 q_{s,\omega}](v_{s,\omega} - \widehat{u}_{s,\omega})ds \]

\[ - \mathbb{E} \int_0^T \mathbb{E}'[M_s^1(\widehat{Y}_{s,\omega}' - \widehat{Y}_{s,\omega}) + N_s^1(\widehat{Z}_{s,\omega}' - \widehat{Z}_{s,\omega})]ds \]

\[ + \mathbb{E} \int_0^T \mathbb{E}'[C_s^2(v_{s,\omega}' - \widehat{u}_{s,\omega})]p_{s,\omega} + F_s^2(v_{s,\omega}' - \widehat{u}_{s,\omega})q_{s,\omega}]ds \]

\[ - \mathbb{E} \int_0^T \mathbb{E}'[M_s^2(\widehat{Y}_{s,\omega}' - \widehat{Y}_{s,\omega}) + N_s^2(\widehat{Z}_{s,\omega}' - \widehat{Z}_{s,\omega})]ds. \]
Thus we have
\[ J(v) - J(\hat{u}) \geq \mathbb{E} \int_0^T \mathbb{E}'[C_s^1 p_{s,\omega} + F_s^1 q_{s,\omega}](v_{s,\omega} - \hat{u}_{s,\omega})ds \]
\[ + \mathbb{E} \int_0^T \mathbb{E}'[R_s^1 u_{s,\omega}(v_{s,\omega} - \hat{u}_{s,\omega}) + R_s^2 u_{s,\omega'}(v_{s,\omega'} - \hat{u}_{s,\omega'})]ds \]
\[ + \mathbb{E} \int_0^T \mathbb{E}'[C_s^2 (v_{s,\omega'} - \hat{u}_{s,\omega'})p_{s,\omega} + F_s^2 (v_{s,\omega'} - \hat{u}_{s,\omega'})q_{s,\omega}]ds \]
\[ = \mathbb{E} \int_0^T \mathbb{E}'[C_s^1 p_{s,\omega} + F_s^1 q_{s,\omega}](v_{s,\omega} - \hat{u}_{s,\omega})ds \]
\[ + \mathbb{E} \int_0^T \mathbb{E}'[R_s^1 v_{s,\omega}(v_{s,\omega} - \hat{u}_{s,\omega}) + \mathbb{E}'[R_s^2 \hat{u}_s](v_{s,\omega} - \hat{u}_{s,\omega})]ds \]
\[ + \mathbb{E} \int_0^T \mathbb{E}'[C_s^2 p_s + F_s^2 q_s^*](v_{s,\omega} - \hat{u}_{s,\omega})ds. \]

Thus by (6.2) we have \( J(v) - J(\hat{u}) \geq 0 \), which means \( \hat{u} \) is an optimal control. Since the method of proving the result of uniqueness is classical and similar to the case in [13], we omit it here.

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