Interpolation and sampling sequences for mixed-norm spaces

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Abstract

This paper extends the known characterization of interpolation and sampling sequences for Bergman spaces to the mixed-norm spaces. The Bergman spaces have conformal invariance properties not shared by the mixed-norm spaces. As a result, different techniques of proof were required.

1 Introduction

1.1 The mixed-norm spaces

For a function $f$ analytic in the unit disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$, the integral means are defined by

$$M_p(r, f) = \left[ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right]^{1/p}, \quad 0 < p < \infty,$$

and

$$M_\infty(r, f) = \max_\theta |f(re^{i\theta})|.

For $0 < p < \infty$, $0 < q < \infty$, the mixed-norm space $A(p, q)$ is the set of functions $f$ analytic in $\mathbb{D}$ with

$$\|f\|_{L(p,q)} = \left[ \int_0^1 M_p(r, f)^q 2r \, dr \right]^{1/q}$$

and

$$= \left[ \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{q/p} 2r \, dr \right]^{1/q} < \infty.$$

If $f \in A(p, q)$, we write $\|f\|_{A(p,q)}$ for $\|f\|_{L(p,q)}$.

*Sections 1–3 of this paper are taken from the Ph.D. dissertation of the first author under the direction of the second author. Section 4 on sampling was completed later.
If \( p = q \), \( A(p, q) \) is the Bergman space \( A^p \):

\[
A^p = \left\{ f \text{ analytic in } \mathbb{D} : \| f \|_{A^p} = \left( \int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{1/p} < \infty \right\}.
\]

For \( 0 < p, q < \infty \), \( A(p, q) \) are invariant complete metric spaces with the metric

\[
d(f, g) = \| f - g \|_{A(p, q)}, \text{ where } s = \min(p, q, 1).
\]

If \( 1 \leq p \) and \( 1 \leq q \), \( \| \cdot \|_{A(p, q)} \) is a norm and \( (A(p, q), \| \cdot \|) \) becomes a Banach space.

We are also interested in two other spaces which are related to the mixed-norm space. The first one is the growth space \( A^{-n} \) \( (n > 0) \), which is the set of functions \( f \) analytic in \( \mathbb{D} \) with

\[
\| f \|_{-n} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^n |f(z)| < \infty.
\]

The second one is the weighted Bergman space \( A^p_\alpha \) \( (0 < p < \infty, \alpha > -1) \), which consists of functions \( f \) analytic in \( \mathbb{D} \) with

\[
\| f \|_{p, \alpha} = \left\{ \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) \right\}^{1/p} < \infty.
\]

Note that \( A^p = A^p_0 \). Also note that some authors use that convention that \( A^p_\alpha \) is equal to what we would call \( A^p_{\alpha_{p-1}} \).

### 1.2 Definitions of interpolation sequences

Let \( A \) be a space of functions on \( \Omega \), \( X \) a sequence space and \( \Gamma \equiv (z_m) \subset \Omega \) a sequence that has no limit points in \( \Omega \). Denote by \( R_\Gamma \) the mapping \( f \mapsto (f(z_m)) \). We say \( \Gamma \) is an interpolation sequence for \( (A, X) \) if \( R_\Gamma(A) = X \). In other words, \( \Gamma \) is an interpolation sequence for \( (A, X) \) if \( (f(z_m)) \in X \) for all \( f \in A \) and for every sequence \( (a_m) \in X \), there is a function \( f \in A \) such that \( f(z_m) = a_m \) for every \( m \).

If \( A = A(p, q) \), we let the sequence space \( X \) to be \( L^{p,q} \) defined as folows:

Let \( \beta = 1/L \) for some integer \( L \geq 2 \) and set \( r_j = 1 - j\beta^j, j = 0, 1, 2, \ldots \). Let us divide the unit disc \( \mathbb{D} \) into annuli \( A_j = \{ z \in \mathbb{C} : r_j < |z| < r_{j+1} \} \). Also divide each annulus \( A_j \) by means of equally spaced radii into \( 2L^2 \) equal ‘polar rectangles’ \( Q_{j,k} = \{ z = r\theta : r_j \leq r < r_{j+1}, (k-1)\beta^j \pi \leq \theta < k\beta^j \pi \} \).

We now arrange \( \Gamma \) such that

\[
|z_1| \leq |z_2| \leq \ldots < 1.
\]

For each annulus \( A_j \), let \( L_j \) be the number of points of \( \Gamma \) in \( A_j \) (necessarily finite). Number the points of \( \Gamma \) by \( z_{j,k} \) such that \( z_{j,k} \in A_j, k = 1, 2, \ldots, L_j \) and \( |z_{j,k}| < |z_{j',k'}| \) if \( j < j' \). Specifically, \( (z_{j,k}) \) is a doubly indexed sequence defined by

\[
z_{j,k} = z_m \text{ if } m = k + \sum_{i=0}^{j} L_i. \tag{7}
\]
Unless specified otherwise, every doubly indexed sequence from now on is numbered according to (7).

Now let \((a_m)\) be a sequence in \(\mathbb{C}\) and let \((a_{j,k})\) be its doubly indexed sequence defined in the same manner, i.e. \(a_{j,k} = a_m\) if \(m = k + \sum_{i=0}^{j} L_i\). Let us define a sequence space \(l^{p,q}(\Gamma)\) to consist of all sequence \((a_m)\) with

\[
\| (a_m) \|_{l^{p,q}(\Gamma)} = \left\{ \sum_{j=0}^{\infty} (1 - r_j)^{1+q/p} \left( \sum_{k=1}^{L_j} |a_{j,k}|^p \right)^{q/p} \right\}^{1/q} < \infty. \tag{8}
\]

The sequence space \(l^{p,q}\) is now used for interpolation for \(A(p,q)\). We say \(\Gamma\) is an interpolation sequence for \(A(p,q)\) if \(R_\Gamma(A(p,q)) = l^{p,q}\).

We also require the definitions of interpolation sequences for \(A^{-n}\) and \(A^p_\alpha\). The corresponding sequence spaces are \(l^{\infty}_{-n}(\Gamma)\) and \(l^p_\alpha(\Gamma)\) defined by

\[
l^{\infty}_{-n}(\Gamma) = \left\{ (a_m) \subset \mathbb{C} : \| (a_m) \|_{-n,\Gamma} = \sup_{m} (1 - |z_m|^2)^n |a_m| < \infty \right\}, \tag{9}
\]

\[
l^p_\alpha(\Gamma) = \left\{ (a_m) \subset \mathbb{C} : \| (a_m) \|_{p,\alpha,\Gamma} = \sum_{m} |a_m|^p (1 - |z_m|^2)^{\alpha+2} < \infty \right\}. \tag{10}
\]

Thus we say \(\Gamma\) is an interpolation sequence for \(A^{-n}\) if \(R_\Gamma(A^{-n}) = l^{\infty}_{-n}(\Gamma)\) and \(\Gamma\) is an interpolation sequence for \(A^p_\alpha\) if \(R_\Gamma(A^p_\alpha) = l^p_\alpha(\Gamma)\).

### 1.3 Uniformly Discrete Sequences

Recall that for \(0 < p, q < \infty\), \(A(p,q)\) and \(l^{p,q}\) are invariant complete metric spaces with the metric \(d(x, y) = \| x - y \|_s\), \(s = \min(p, q, 1)\). Let \(\Gamma = (z_m)\) be an interpolation sequence for \(A(p,q)\). A simple verification shows that the mapping \(R_\Gamma : f \to (f(z_m))\) has closed graph and hence is bounded from \(A(p,q)\) into \(l^{p,q}\). Since \(\Gamma\) is an interpolation sequence, \(R_\Gamma\) is also onto. The open mapping theorem implies that there exists the smallest constant \(M(\Gamma)\) such that for every \((a_m) \in l^{p,q}\), there is an \(f \in A(p,q)\) satisfying \(f(z_m) = a_m\) for all \(m\) and \(\|f\|_{A(p,q)} \leq M \| (a_m) \|_{l^{p,q}}\). \(M\) is called the interpolation constant of \(\Gamma\) for \(A(p,q)\).

Interpolation sequences are not too dense anywhere. In fact, a necessary condition for interpolation is being uniformly discrete. A sequence \(\Gamma \equiv (z_m) \subset \mathbb{D}\) is said to be uniformly discrete with separation constant \(\delta\) if

\[
\delta := \inf \{ \rho(z_n, z_m) : n \neq m \} > 0 \tag{11}
\]

where \(\rho(z, w)\) is the pseudohyperbolic distance between two points in \(\mathbb{D}\) given by

\[
\rho(z, w) = \frac{|z - w|}{|1 - \overline{w}z|}. \tag{12}
\]

We also denote the pseudohyperbolic disk of radius \(r\) centered at \(z\) by

\[
E(z, r) = \{ \zeta \in \mathbb{D} : \rho(z, \zeta) < r \}. \tag{13}
\]

The following results on uniformly discrete sequences were proved in [3, Chapter 2]:
Proposition 1. Suppose that $\Gamma = (z_m)$ is a uniformly discrete sequence with $\rho(z_n, z_m) \geq \delta > 0$ for $m \neq n$. Let $n(\Gamma, z, r)$ denote the number of points in $\Gamma \cap E(z, r)$. Then

1. $\sum_{m=1}^{\infty} (1 - |z_m|^2)^2 \leq \frac{4}{\delta^2}$.

2. $n(\Gamma, z, r) \leq \left(\frac{2}{\delta} + 1\right)^2 \frac{1}{1 - r^2}$ for every point $z \in \mathbb{D}$ and $0 < r < 1$. In particular, $n(\Gamma, z, r) = O\left(\frac{1}{1 - r}\right)$.

1.4 Seip’s theorems and their extensions

In [14], Seip characterizes sets of sampling and interpolation for $A^{-n}$ via certain densities which are equivalent to the Korenblum’s densities in [7]. In order to state the results, we require several definitions.

For $0 < s < 1$, let $n(\Gamma, \zeta, s)$ be the number of points of $\Gamma$ contained in $E(\zeta, s)$. Define, for $\Gamma$ uniformly discrete,

$$F(\Gamma, \zeta, r) = \frac{\int_0^r n(\Gamma, \zeta, s) \, ds}{2 \int_0^r a(E(0, s)) \, ds}$$

where

$$a(\Omega) = \int_{\Omega} \frac{1}{(1 - |z|^2)^2} \, dA(z)$$

is the hyperbolic measure of a measurable subset $\Omega$ of $\mathbb{D}$.

The lower and upper uniform densities are defined, respectively, to be

$$D^{-}(\Gamma) = \liminf_{r \to 1} \inf_{\zeta \in \mathbb{D}} F(\Gamma, \zeta, r)$$

and

$$D^{+}(\Gamma) = \limsup_{r \to 1} \sup_{\zeta \in \mathbb{D}} F(\Gamma, \zeta, r).$$

The densities can be reformulated as

$$D^{-}(\Gamma) = \liminf_{r \to 1} \inf_{\zeta \in \mathbb{D}} \frac{\sum_{1/2 < \rho(\zeta, z_n) < r} \log \frac{1}{\rho(\zeta, z_n)} \log \frac{1}{1 - r}}{\log \frac{1}{1 - r}},$$

$$D^{+}(\Gamma) = \limsup_{r \to 1} \sup_{\zeta \in \mathbb{D}} \frac{\sum_{1/2 < \rho(\zeta, z_n) < r} \log \frac{1}{\rho(\zeta, z_n)} \log \frac{1}{1 - r}}{\log \frac{1}{1 - r}}.$$
The following theorems were proved in [14]. (The definition of sampling sequence occurs in section [4].)

**Theorem 2.** A sequence $\Gamma$ of distinct points in $\mathbb{D}$ is a set of sampling for $A^{-n}$ if and only if it contains a uniformly discrete subsequence $\Gamma'$ for which $D^-(\Gamma') > n$.

**Theorem 3.** A sequence $\Gamma$ of distinct points in $\mathbb{D}$ is a set of interpolation for $A^{-n}$ if and only if $\Gamma$ is uniformly discrete and $D^+(\Gamma) < n$.

Theorem 3 was extended to the Bergman space $A^p (0 < p < \infty)$ and the weighted Bergman space $A^p_\alpha (0 < p < \infty, \alpha > -1)$. The proofs can be found in [3], [12] and [6].

**Theorem 4.** A sequence $\Gamma$ of distinct points in $\mathbb{D}$ is a set of sampling for $A^\alpha$ if and only if it contains a uniformly discrete subsequence $\Gamma'$ for which $D^{-}(\Gamma') > (1 + \alpha)/p$.

**Theorem 5.** A sequence $\Gamma$ of distinct points in $\mathbb{D}$ is a set of interpolation for $A^\alpha$ if and only if $\Gamma$ is uniformly discrete and $D^+(\Gamma) < (1 + \alpha)/p$.

2 Some properties of interpolation sequences for mixed-norm spaces

2.1 Basic properties of mixed norm spaces

Recall that the mixed norm space $A(p, q)$ consists of analytic functions $f$ in the unit disk with

$$
\|f\|_{A(p, q)} = \left[ \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{q/p} 2r dr \right]^{1/q} < \infty.
$$

It is easy to check that $A(p, q)$ is an invariant metric space with the metric

$$
d(f, g) := \|f - g\|_{A(p, q)}^s
$$

where $s = \min(p, q, 1)$. If $p, q > 1$, $(A(p, q), \|\cdot\|_{A(p, q)})$ is a normed linear space.

The following results can be found in [5] and [1]:

**Proposition 6.**

1. Let $0 < p, q < \infty$ and $f \in A(p, q)$, then for all $z \in \mathbb{D}$:

$$
|f(z)| \leq C\|f\|_{A(p, q)} (1 - |z|)^{-\left(1/p + 1/q\right)}
$$

for some constant $C$ independent of $f$.

2. If $d$ is the metric defined in (19), then $(A(p, q), d)$ is an invariant, complete metric space and $(A(p, q), \|\cdot\|_{A(p, q)})$ is a Banach space if $p, q \geq 1$.

3. If $p, q > 1$ and $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, then there is a continuous linear isomorphism of $A(p', q')$ onto the dual space of $A(p, q)$.
2.2 Discrete versions of mixed norms

Let \( f \) be analytic in \( \mathbb{D} \). Then \( |f|^p \) is subharmonic for any \( p > 0 \). This implies that the integral means

\[
M_p(f, r) = \left( \frac{1}{2\pi} \int_\pi \pi |f(re^{i\theta})|^p d\theta \right)^{1/p}
\]

are increasing function of \( r \). Let us adopt the notations \( A_n, r_n, \beta, L, L_n, Q_{n,k} \) used in Section 1.2. The norm \( \|f\|^q_{A(p,q)} \) can then be replaced with a summation

\[
\|f\|^q_{A(p,q)} = \sum_{n=1}^\infty \int_{r_n}^{r_{n+1}} M_p(f, r)^q 2r \, dr.
\]

If \( r_{n-1} < r < r_n < r' < r_{n+1} \), then

\[
M_p(f, r)^p \leq M_p(f, r')^p.
\]

Integrate this inequality with respect to \( 2r \, dr' \) from \( r_n \) to \( r_{n+1} \) to get

\[
(r_{n+1}^2 - r_n^2)M_p(f, r)^p \leq \int_{r_n}^{r_{n+1}} M_p(f, r')^q 2r' \, dr' \leq \frac{1}{\pi} \int_{A_{n+1}} |f(z)|^p \, dA(z).
\]

This gives

\[
M_p(f, r)^p \leq \frac{1}{|A_{n+1}|} \int_{A_{n+1}} |f(z)|^p \, dA(z)
\]

where absolute bars indicate the area of a set. We can now raise both sides to the power \( q/p \) and integrate with respect to \( 2r \, dr \) from \( r_{n-1} \) to \( r_n \):

\[
\int_{r_{n-1}}^{r_n} M_p(f, r)^q 2r \, dr \leq (r_{n+1}^2 - r_n^2) \left( \frac{1}{|A_{n+1}|} \int_{A_{n+1}} |f(z)|^p \, dA(z) \right)^{q/p}
\]

\[
\leq 2\beta^{n-1} \left( \frac{1}{|A_{n+1}|} \int_{A_{n+1}} |f(z)|^p \, dA(z) \right)^{q/p}.
\]

A similar argument but integrating first in \( r \) and then \( r' \) gives

\[
\int_{r_n}^{r_{n+1}} M_p(f, r)^q 2r \, dr \geq (r_{n+1}^2 - r_n^2) \left( \frac{1}{|A_n|} \int_{A_n} |f(z)|^p \, dA(z) \right)^{q/p}
\]

\[
\geq \beta^n (1 - \beta) \left( \frac{1}{|A_n|} \int_{A_n} |f(z)|^p \, dA(z) \right)^{q/p}.
\]

Summing both these inequalities gives us (for some constant \( C \) depending only on \( \beta \))

\[
\sum_{n=2}^\infty \frac{1 - r_n}{C} \left( \frac{1}{|A_n|} \int_{A_n} |f(z)|^p \, dA(z) \right)^{q/p} \leq \|f\|^q_{A(p,q)}
\]

\[
\leq \sum_{n=1}^\infty C(1 - r_n) \left( \frac{1}{|A_n|} \int_{A_n} |f(z)|^p \, dA(z) \right)^{q/p}.
\]
Since $M_p(f, r)$ is increasing, the integral of $|f|^p$ over $A_1$ is less than a constant times the integral over $A_2$ and so we can include $n = 1$ in the sum on the left, provided we increase the constant $C$. Thus we obtain the following new norm

$$
\|f\|_{A(p, q)} = \sum_{n=1}^{\infty} (1 - r_n) \left( \frac{1}{|A_n|} \int_{A_n} |f(z)|^p \, dA(z) \right)^{q/p}.
$$

which is equivalent to the usual norm $\|f\|_{A(p, q)}$ (that is $\|f\| \leq C \|f\|$ and $\|f\| \leq C \|f\|$ for some constant $C$ depending only on $\beta$).

The integral over the annulus $A_n$ can be written as a sum (on $k$) of integrals over $Q_{n,k}$:

$$
\|f\|_{A(p, q)}^2 = \sum_{n=1}^{\infty} (1 - r_n) \left( \frac{1}{|A_n|} \sum_{k=1}^{L_n} \int_{Q_{n,k}} |f(z)|^p \, dA(z) \right)^{q/p}.
$$

Or, since $|A_n| = 2L^n |Q_{n,k}|$:

$$
\|f\|_{A(p, q)}^2 = \sum_{n=1}^{\infty} (1 - r_n) \left( \frac{L_n}{2} \sum_{k=1}^{L_n} \frac{1}{|Q_{n,k}|} \int_{Q_{n,k}} |f(z)|^p \, dA(z) \right)^{q/p}.
$$

Or, since $L^n = \beta^n = 1 - r_n$:

$$
\|f\|_{A(p, q)}^2 = \sum_{n=1}^{\infty} (1 - r_n) \left( \frac{1 - r_n}{2} \sum_{k=1}^{L_n} \frac{1}{|Q_{n,k}|} \int_{Q_{n,k}} |f(z)|^p \, dA(z) \right)^{q/p}.
$$

We will need to a few properties of the set $Q_{n,k}$, especially in connection with the pseudo-hyperbolic metric $\rho$, defined by

$$
\rho(z, w) \equiv \frac{|z - w|}{|1 - \bar{w}z|} \quad \text{(for } z, w \in \mathbb{D})
$$

and the associated hyperbolic disks defined by

$$
E(z, R) \equiv \{ w : \rho(z, w) < R \}
$$

$$
\bar{E}(z, R) \equiv \{ w : \rho(z, w) \leq R \}
$$

**Theorem 7.** There exist constants $0 < r < R < 1$ depending only on $\beta$ such that

1. For every pair $(n, k)$, if $z \in Q_{n,k}$ then $Q_{n,k} \subset E(z, R)$

2. For every pair $(n, k)$, there exists $z \in Q_{n,k}$ with $E(z, r) \subset Q_{n,k}$.

Let us denote by $r_\beta$ the largest possible $r$ and $R_\beta$ the smallest possible $R$ from this theorem. Let $\tilde{z}_{n,k}$ denote the $z$ from statement 2 for $r = r_\beta$ and let $z_{n,k}$ denote the Euclidean center of $E_{n,k} \equiv E(\tilde{z}_{n,k}, r_\beta)$. 7
A fairly easy estimate of the areas involved show that there are constants $C_1$ and $C_2$ depending only on $\beta$ such that for all pairs $(n, k)$

$$|E_{n,k}| \leq |Q_{n,k}| \leq C_1|E_{n,k}|$$

and for any $z \in Q_{n,k}$

$$|Q_{n,k}| \leq |E(z, R_\beta)| \leq C_2|Q_{n,k}|.$$ 

Then we have the estimate, for every analytic function $f$:

$$|f(z_{n,k})|^p \leq \frac{1}{|E_{n,k}|} \int_{E_{n,k}} |f|^p \, dA \leq \frac{C_1}{|Q_{n,k}|} \int_{Q_{n,k}} |f|^p \, dA.$$ 

Note also that for any $z \in A_n$ we have

$$\frac{1}{C} \leq \frac{1 - |z|^2}{1 - r_n} \leq C$$

for some constant $C$ depending only on $\beta$. And so for any $f$ in $A(p, q)$,

$$\left( \sum_{n=1}^{\infty} (1 - r_n) \left( \sum_{k=1}^{L_n} (1 - |z_{n,k}|^2) |f(z_{n,k})|^p \right)^{q/p} \right)^{1/q} \leq C \|f\|_{A(p, q)} \leq C \|f\|_{A(p, q)}$$

for some constant $C = C(\beta)$. This show that evaluation at the points of the sequence $(z_{n,k})$ is a bounded map from $A(p, q)$ into $L^{p,q}((z_{n,k}))$. This result is also true for a uniformly discrete sequence $\Gamma = (z_m) \equiv (z_{n,k})$ ($(z_{n,k})$ is defined as in Section \textit{1.2}). Proposition \textit{1} implies that there is an upper bound $M$ on the number of points of $\Gamma$ in any $Q_{n,k}$. So, if $\Gamma$ is uniformly discrete then it is the union of at most $M$ sequences each of which has at most one point in any $Q_{n,k}$.

**Theorem 8.** If $\Gamma = \{z_m \in \mathbb{D}\} = \{z_{n,k} \in \mathbb{D} : n \geq 1, 1 \leq k \leq L_n\}$ is uniformly discrete then the operator $R_\Gamma$ taking $f$ to $(f(z_{n,k}))$ is bounded from $A(p, q)$ to $L^{p,q}(\Gamma)$.

**Proof.** By the remarks preceding the theorem, we can suppose that there is at most one point of $\Gamma$ in each $Q_{n,k}$. If $z_{n,k} \in \Gamma \cap Q_{n,k}$, let it be the Euclidean center of a disk $E_{n,k}$ with hyperbolic radius $r_\beta$. $E_{n,k}$ needs not be contained in $Q_{n,k}$, but if it intersects the boundary of $Q_{n,k}$ then it is contained in the union of those $Q_{n',k'}$ that are adjacent to the part of the boundary intersected. This union contains at most one additional $Q_{n,k'}$ in the same annulus $A_n$ as $Q_{n,k}$ and at most $L + 1$ additional $Q_{n+1,k'}$ in the next annulus $A_{n+1}$ (or at most 2 additional $Q_{n-1,k'}$ in the previous annulus $A_{n-1}$). Thus

$$|f(z_{n,k})|^p \leq \frac{1}{|E_{n,k}|} \int_{E_{n,k}} |f|^p \, dA \leq \sum_{Q_{n',k'} \cap D_{n,k} \neq \emptyset} \frac{1}{|Q_{n',k'}|} \int_{Q_{n',k'}} |f|^p \, dA$$

It is now easy to see from 22 that $\|f\|_{L^{p,q}(\Gamma)}$ is less than a finite multiple of $\|f\|_{A(p, q)}$. \qed
2.3 Necessity of separation

One might wonder why $l^{p,q}$ is chosen to be the target space for interpolation. It turns out that relatively mild assumptions on $A$ and $X$ force an interpolation sequence $\Gamma$ to be uniformly discrete. Moreover, Theorem 8 shows that if $\Gamma \equiv (z_k)$ is uniformly discrete then $R_\Gamma(A(p,q)) \subset l^{p,q}$ with $R_\Gamma(f) = (f(z_k))$. This suggests we choose $l^{p,q}$ for the interpolation problem. We now turn to the proofs of these assertions.

Let $\Gamma = \{z_k : k = 1, 2, 3, \ldots\} \subset \mathbb{D}$. Let us define $M_a(z) = (a - z)/(1 - \bar{a}z)$. Let $e^{(k)}$ denote the sequence having a 1 in position $k$ and 0 elsewhere. Let $P_k$ be the operator of projection onto the $k$th component: if $w = (w_j)$ then $P_k(w) = w_k e^{(k)}$.

**Theorem 9.** Let $A$ be a Banach space of analytic functions on $\mathbb{D}$ and $\Gamma$ a sequence of distinct points in $\mathbb{D}$. Let $X$ be a Banach space of sequences, the sequences being indexed the same as $\Gamma$. Let $R_\Gamma$ be the operator that takes functions to sequences via $R_\Gamma(f)_k = f(z_k)$.

Assume the following:

1. For every $k$, the sequence $e^{(k)}$ belongs to $X$.
2. For every $k$, $P_k$ takes $X$ continuously into $X$ and $\sup_k \|P_k\| < \infty$.
3. There is a constant $C_A$ such that for any $k$, if $f \in A$ and $f(z_k) = 0$, then $f/M_{z_k} \in A$ and $\|f/M_{z_k}\|_A \leq C_A \|f\|_A$.
4. Convergence in $A$ implies pointwise convergence on $\Gamma$.

If the operator $R_\Gamma$ satisfies $R_\Gamma(A) = X$, then $\Gamma$ is uniformly discrete.

**Proof.** Condition 2 and 4 imply that $R_\Gamma$ has closed graph and so is bounded. Since it is also onto we can apply the open mapping principle to obtain an interpolation constant $K$: every sequence $w \in X$ is $R_\Gamma f$ for some $f \in A$ with $\|f\|_A \leq K \|w\|_X$.

Let $z_k, z_n \in \Gamma$, $k \neq n$. The assumptions imply that the sequence $w = M_{z_n}(z_k)e^{(k)}$ belongs to $X$. Let $f \in A$ with $\|f\|_A \leq K \|w\|_X$. Since $f$ vanishes at $z_n$ we have $g = f/M_{z_n} \in A$. Note that $g(z_k) = 1$, and so $P_k(R_\Gamma(g)) = e^{(k)}$. Then we have the following inequalities:

$$\|e^{(k)}\| \leq C_1 \|R_\Gamma(g)\| \leq C_2 \|g\| \leq C_3 \|f\| \leq C_4 \|w\| = C_4 |M_{z_n}(z_k)| \|e^{(k)}\|,$$

where $C_1 = \sup_k \|P_k\|$, $C_2 = C_1 \|R_\Gamma\|$, $C_3 = C_A C_2$, and $C_4 = K C_3$. We immediately obtain $\rho(z_k, z_n) = |M_{z_n}(z_k)| \geq 1/C_4$. □

Certainly, $l^{p,q}(\Gamma)$ satisfies the requirements. It is not immediately obvious that $A(p,q)$ satisfies condition 3. We address that in the next result.

**Corollary 10.** Interpolation sequences for $A(p,q)$ are uniformly discrete.
Proof. We will show that $A(p, q)$ satisfies condition 3 of the previous theorem. Let $f$ vanish at a fixed $z_k$. Consider the disk $D = D(z_k, 1/2) = \{ z : |M_{z_k}(z)| < 1/2 \}$ and let $g = f/M_{z_k}$. On the complement of this disk it is clear that $|g| < 2|f|$. It will be enough to show that $\|g\|_{A(p, q)} \leq \|g\chi_{D'}\|_{L^p(p, q)}$ for any analytic function $g$, where the constant $C$ does not depend on $z_k$. For this it suffices to show that

$$\|g\chi_D\| \leq \|g\chi_{D'}\| .$$

The proof of this is essentially a sort of maximum principle. Its proof would be a great deal easier if we were able to use a conformal map to turn integral centered around $z_k$ to integrals centered around 0.

We consider the equivalent norm $\|\cdot\|$ on $A(p, q)$ obtained in Section 2.2. Let $D'$ be the slightly larger disk with pseudohyperbolic radius $\frac{1/2 + 1/2}{1 + (1/2)(1/2)} = 4/5$. If we choose the parameter $\beta$ appropriately, we can arrange for $D'$ to overlap at most two annuli $A_n$. For simplicity, let us temporarily assume that $D'$ is contained in exactly one annulus $A_n$. We did not extend the new norm to all of $L^{p, q}$, but it is clear that we can apply it to any measurable function, though it needs not always be finite. In the case $g\chi_D$ we get a single nonzero term in the infinite sum:

$$\|g\chi_D\|_{A(p, q)}^q = (1 - r_n) \left( \frac{1}{|A_n|} \int_D |g|^p \, dA \right)^{q/p} .$$

It is relatively straightforward to show that $\int_D |g|^p \, dA \leq C \int_{D' \setminus D} |g|^p \, dA$ (conformally map to the disks of radius 1/2 and 4/5 centered at 0, use polar coordinates and the fact that the integral on circles increases as the radius increases). This gives us

$$\|g\chi_D\|_{A(p, q)}^q \leq C(1 - r_n) \left( \frac{1}{|A_n|} \int_{D' \setminus D} |g|^p \, dA \right)^{q/p} .$$

If $D'$ overlaps two annuli $A_{n-1}$ and $A_n$, then the second line can be replaced by a sum of two terms.

So we established that $A(p, q)$ satisfies the conditions of Theorem 9. Now if $\Gamma$ is an interpolation sequence for $A(p, q)$, i.e. $R_{\Gamma}(A(p, q)) = L^{p, q}(\Gamma)$, then $\Gamma$ is uniformly discrete thanks to Theorem 9. 

2.4 Stability under perturbation

A property of interpolation sequences is their stability under (hyperbolically) small perturbations. We start with the following lemma:
Lemma 11. Let $\Gamma = (z_m)$ and $\Gamma' = (z'_m)$ be two sequences in $\mathbb{D}$ with no limits in $\mathbb{D}$ such that $\rho(z_m, z'_m) < \delta$ for all $m$. If $\delta$ is sufficiently small; then for every sequence $(a_m)$ in $l^{p,q}(\Gamma)$, $(a_m)$ also belongs in $l^{p,q}(\Gamma')$.

Proof. First, for $a,b,c > 0$ and $\alpha > 0$, a simple application of Holder’s and Minkowski’s inequalities gives us

$$ (a + b + c)^\alpha \leq C_\alpha (a^\alpha + b^\alpha + c^\alpha) $$

(25)

where $C_\alpha = \max\{1, 3^{\alpha-1}\}$.

Second, since $z'_m \in E(z_m, \delta)$, $|z_m - z'_m| < 2R$ where

$$ R = \frac{\delta(1 - |z_m|^2)}{1 - \delta^2|z_m|^2} $$

is the Euclidean radius of the hyperbolic disk $E(z_m, \delta)$. This gives us

$$ |z_m - z'_m| < \frac{2\delta(1 - |z_m|^2)}{1 - \delta^2|z_m|^2} < \frac{4\delta}{1 - \delta^2}(1 - |z_m|). $$

(26)

Thus for $\delta < 1/20$:

$$ |z_m - z'_m| < \frac{4\delta}{1 - \delta^2}(1 - |z_m|) < \frac{1}{4}(1 - |z_m|). $$

Now let $(z_{j,k})$ be the doubly indexed sequence of $(z_m)$ defined as in (7). Since $r_j < |z_{j,k}| < r_{j+1}$ and $1 - r_j = \beta^j$, we have

$$ |z_m - z'_m| < \frac{1}{4}(1 - r_j) < \frac{\beta(r_j - r_{j-1})}{4(1 - \beta)}. $$

It follows that $z'_m \in A_{j-1} \cup A_j \cup A_{j+1}$. This and inequality (25) imply

$$ \|a_m\|_{l^{p,q}(\Gamma')}^q \leq 3\beta^{-(1+q/p)}C_{q/p}\|a_m\|_{l^{p,q}(\Gamma)}^q < \infty. $$

(27)

Hence, $(a_m)$ is also a sequence in $l^{p,q}(\Gamma')$. □

The following theorem shows that if $\Gamma$ is an interpolation sequence for $A(p,q)$ then a small perturbation on $\Gamma$ still results in an interpolation sequence for $A(p,q)$. The proof is taken after Lemma 1.9 in [6] and Theorem 5.1 in [8], save a few minor changes to work for mixed-norm spaces.

Theorem 12. For $0 < p,q < \infty$, let $\Gamma = (u_m)$ be an interpolation sequence for $A(p,q)$ and $(u'_m)$ be another sequence in $\mathbb{D}$. There exists $\delta > 0$ such that if

$$ \rho(u_m, u'_m) < \delta \quad \text{for all } m $$

then $\Gamma' = (u'_m)$ is also an interpolation sequence for $A(p,q)$. 11
Proof. Let \((v_m) \in l^{p,q}(\Gamma')\). By Lemma \[1\] \((v_m) \in l^{p,q}(\Gamma)\) for \(\delta\) small enough. Denote \(u = (u_m), u' = (u'_m)\) and \(v^0 = v\). Since \(u\) is an interpolation sequence of \(A(p,q)\), there exists \(f_0 \in A(p,q)\) such that \(f_0(u) = v^0\) (i.e., \(f_0(u_m) = v_m\) for all \(m\)). Suppose \(v^1 \in l^{p,q}(\Gamma)\), take now \(v^1 := v^0 - f_0(u')\), \(f_1 \in A(p,q)\) with \(f_1(u) = v^1\), and define \(v^2 = v^1 - f_1(u')\). An iteration of this construction provides functions \(f_m \in A(p,q)\) with \(f_0(u') + f_1(u') + \cdots + f_m(u') + f_m(u) = v^0 = v\). If we can prove that there exists \(0 < \gamma < 1\) such that \(\|v_{m+1}\| \leq \gamma \|v_m\|\) for \(m = 0, 1, 2, \ldots\) then

\[
\|f_m\| \leq M \|v^m\| \leq M \gamma^m \|v^0\|,
\]

where \(M\) is the interpolation constant of \((u_m)\). The interpolation problem for \((u'_m)\) is then solved by the function \(f = \sum_m f_m\). To this end, we use a general estimate for analytic function which can be found in \[6\] or \[8\]:

\[
|f(z) - f(w)|^p \leq C \rho^p(z, w) \int_{E(w,r)} |f(\zeta)|^p (1 - |\zeta|^2)^{-2} dA(z)
\]

where \(C = C(p, r) > 0\) and \(r \geq 2\rho(z, w)\). Provided that \(\delta\) is chosen small enough so that the hyperbolic disks \(E(z_m, r)\) are pairwise disjoint, we have:

\[
\|v^1\|_{l^{p,q}(\Gamma)}^q = \sum_j (1 - r_j)^{1+q/p} \left( \sum_k |f_0(u_{j,k}) - f_0(u'_{j,k})|^p \right)^{q/p} \\
\leq \sum_j (1 - r_j)^{1+q/p} \left( \sum_k C_1 \delta^p \int_{E(u_{j,k}, r)} |f_0(\zeta)|^p (1 - |\zeta|^2)^{-2} dA(\zeta) \right)^{q/p} \\
\leq C_1^{q/p} \delta^q \sum_j (1 - r_j)^{1+q/p} \left( \sum_k \int_{E(u_{j,k}, r)} \frac{1}{|E(u_{j,k}, r)|} |f_0(\zeta)|^p dA(\zeta) \right)^{q/p} \\
\leq C_2 \delta^q \sum_j (1 - r_j)^{1+q/p} \left( \sum_k \int_{E(u_{j,k}, r)} \frac{1}{|E(u_{j,k}, r)|} |f_0(\zeta)|^p dA(\zeta) \right)^{q/p}
\]

The same argument as in the proof of Theorem \[8\] shows that

\[
\int_{E(u_{j,k}, r)} \frac{1}{|E(u_{j,k}, r)|} |f_0(\zeta)|^p dA(\zeta) \leq C_3(r, \beta) \sum_{Q_{j',k'} \cap E_j \neq \emptyset} \frac{1}{|Q_{j',k'}|} \int_{Q_{j',k'}} |f|^p dA
\]

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where the number of \( Q_{j,k'} \cap E_{j,k} \neq \emptyset \) is at most \( L + 4 \). Hence,

\[
\|v^1\|^q \leq C_4 \delta^q \sum_j (1 - r_j)^{1+q/p} \left( \sum_k \frac{1}{|Q_{j,k}|} \int_{Q_{j,k}} |f|^p dA \right)^{q/p} \\
\leq C_5 \delta^q \|f_0\|^q \\
\leq C_5 \delta^p M^q \|v^0\|^q ,
\]

where \( C_5 \) depends only on \( p, q, \delta \) and \( \beta \). Let \( \gamma^q = C_5 \delta^q M^q \) and choose \( \delta \) small enough so that \( \gamma < 1 \). The same estimate shows that \( \|v_{m+1}\| \leq \gamma \|v_m\| \) for \( m = 0, 1, 2, \ldots \). This completes the proof.

We see that interpolation sequences are stable under small perturbations. On the other hand, a small perturbation can increase the upper uniform density of a sequence. The following lemma was proved in [2]:

**Lemma 13.** Let \( \Gamma = (z_n) \) be a uniformly discrete sequence in \( \mathbb{D} \) with \( \rho(z_m, z_n) > \beta \) for \( m \neq n \) and \( 0 < \delta < \beta < 1/2 \). If \( D^+(\Gamma) < \infty \), then there is a sequence \( \Gamma' = (z'_n) \subset \mathbb{D} \) with \( \rho(z_n, z'_n) \leq \delta \) for all \( n \), such that

\[
D^+(\Gamma') \geq (1 + \delta) D^+(\Gamma)
\]

**Remark 1.** Suppose that \( D^+(\Gamma) \leq \gamma \) for every interpolation sequence \( \Gamma \) for \( A(p,q) \). Then by Theorem 12 and Lemma 13, there exists an interpolation sequence \( \Gamma' \) for \( A(p,q) \) such that \( D^+(\Gamma') > D^+(\Gamma) \). Since \( D^+(\Gamma') \leq \gamma \), we must have \( D^+(\Gamma) < \gamma \). Thus in order to prove that \( D^+(\Gamma) < \gamma \) for every interpolation sequence \( \Gamma \) for \( A(p,q) \), we only need to show that \( D^+(\Gamma) \leq \gamma \) for all \( \Gamma \).

### 3 Interpolation sequences for mixed-norm spaces

We propose to prove

**Theorem 14.** A sequence \( \Gamma \) of distinct points in \( \mathbb{D} \) is a set of interpolation for \( A(p,q) \) if and only if \( \Gamma \) is uniformly discrete and \( D^+(\Gamma) < 1/q \).

The proof is split into several cases and requires the following inequalities

**Lemma 15.**

1. For \( 0 < p \leq 1 \),

\[
\left( \sum_{n=1}^{\infty} |b_n| \right)^p \leq \sum_{n=1}^{\infty} |b_n|^p \quad (29)
\]

2. For \( M > 1 \), there exists a constant \( C = C(M) \) such that for all \( 0 < \rho < 1 \)

\[
\frac{1/C}{(1 - \rho)^{M-1}} \leq \int_{-\pi}^{\pi} \frac{1}{|1 - \rho e^{i\theta}|^M} d\theta \leq \frac{C}{(1 - \rho)^{M-1}} \quad (30)
\]
3. For $-1 < a < B - 1$, there exists a constant $C = C(a, B)$ such that for all $0 < \rho < 1$

$$\frac{1}{C} \leq \frac{1}{(1 - \rho)^{B-a-1}} \leq \int_0^1 \frac{(1 - r)^a}{(1 - r\rho)^B} \, dr \leq \frac{C}{(1 - \rho)^{B-a-1}}$$  \hfill (31)

4. For $-1 < a < M - 2$, there exists a constant $C = C(a, M)$ such that for all $w \in \mathbb{D}$

$$\frac{1}{C} \leq \frac{1}{(1 - |w|)^{M-a-2}} \leq \int_\mathbb{D} \frac{(1 - |z|^2)^a}{|1 - \overline{w}z|^M} \, dA(z) \leq \frac{C}{(1 - |w|)^{M-a-2}}$$  \hfill (32)

5. Let $\Gamma = (z_k)$ be a uniformly discrete sequence in $\mathbb{D}$ with separation constant $\delta = \delta(\Gamma)$. Then for $1 < t < s$, there is a constant $C = C(t, s, \delta) > 0$ such that

$$\sum_{k=1}^\infty \frac{(1 - |z_k|^2)^t}{|1 - \overline{z}z_k|^s} \leq C(1 - |z|^2)^{t-s}$$  \hfill (33)

for all $z \in \mathbb{D}$.

Proof. The techniques for the proof can be found in [Lemma 1-3, Chapter 6, [3]]. \hfill \Box

3.1 Sufficiency part of Theorem 14

Suppose $\Gamma = (z_{mk})$ is uniformly discrete and $D^+(\Gamma) < 1/q$. Then $\Gamma$ is an interpolation sequence for $A^{-n}$ where $n = 1/q - \epsilon$ for some $\epsilon > 0$. Hence there exists a sequence $(g_{mk}) \in A^{-n}$ such that $g_{mk}(z_{mk}) = (1 - |z_{mk}|^2)^{-n}$, $g_{mk}(z_{m'k'}) = 0$ for all $(m', k') \neq (m, k)$, $\|g_{mk}\| \leq M(\Gamma)$ for all $m$ and $k$; where $M(\Gamma)$ is the interpolation constant of $\Gamma$ for $A^{-n}$.

The interpolation problem is then solved by the formula

$$f(z) = \sum_m \sum_k a_{mk}g_{mk}(z) \frac{(1 - |z_{mk}|^2)^{n+s}}{(1 - \overline{z_{mk}}z)^s}$$  \hfill (34)

where $s$ is sufficiently large.

It is clear that if the series converges point-wise to $f$ then $f(z_{mk}) = a_{mk}$ for all $(m, k)$. For each $0 < R < 1$, we will show that the series converges uniformly on the disk $\{z: |z| \leq R\}$. First, we see that

$$\frac{|g_{mk}(z)|}{(1 - \overline{z_{mk}}z)^s} \leq (1 - |z|^2)^{-n} \|g_{mk}(z)\|_{A^{-n}} (1 - R)^{-s} \leq (1 - R^2)^{-n}(1 - R)^{-s}M = C(R, M(\Gamma)).$$

Thus, it suffices to show that

$$\sum_m \sum_k |a_{mk}|(1 - |z_{mk}|^2)^{n+s} < \infty.$$  \hfill (35)
If $p \leq 1$, then
\[
\sum_k |a_{mk}| \leq \left( \sum_k |a_{mk}|^p \right)^{1/p}
\]  
(36)
and if $p > 1$ then, since the number of points $z_{mk}$ in the annulus $A_m$ is $O(1/(1 - r_m))$,
\[
\sum_k |a_{mk}| \leq \left( \frac{C}{(1 - r_m)} \right)^{1/p'} \left( \sum_k |a_{mk}|^p \right)^{1/p}
\]  
(37)
where $C$ depends only on $R$ and the separation constant of $\Gamma$. We also have $1 - |z_{mk}| \approx 1 - r_m$. In either case the series in (35) is bounded by
\[
C \sum_m (1 - r_m)^{n + s - 1} \left( \sum_k |a_{mk}|^p \right)^{1/p}.
\]  
(38)
If $q \leq 1$ this sum is bounded by
\[
C \left( \sum_m (1 - r_m)^{(n + s - 1)q} \left( \sum_k |a_{mk}|^p \right)^{q/p} \right)^{1/q}.
\]  
(39)
otherwise Hölder’s inequality gives a bound of
\[
C \left( \sum_m (1 - r_m)^{(n + s - 1)} \left( \sum_k |a_{mk}|^p \right)^{q/p} \right)^{1/q} \left( \sum_m (1 - r_m)^{(n + s - 1)} \right)^{1/q'}.
\]  
(40)
Now note that $\sum_m (1 - r_m^2)$ is finite. Consequently, a straightforward estimation shows that, for $s$ sufficiently large,
\[
\sum_m (1 - r_m)^{n + s - 1} \sum_k |a_{mk}| \leq C \|a_{mk}\|_{p,q}^q < \infty
\]  
(41)
where $C$ depends only on $R, p, q, M(\Gamma)$ and the separation constant of $\Gamma$. It follows that the series in (34) converges uniformly to an analytic function $f$ on each compact set of $\mathbb{D}$. To prove that $f$ belongs in $A(p, q)$, we only need to verify that $\|f\|_{A(p,q)} < \infty$. The proof is split into four cases. The only property of the $g_{mk}$ that we will use is that for all $m$ and $k$, $|g_{mk}(z)| \leq M(\Gamma)(1 - r^2)^{-n}$. Thus, it suffices to show that
\[
h(z) := \sum_m \sum_k |a_{mk}|(1 - |z|^2)^{-n} \frac{(1 - |z_{mk}|^2)^{n + s}}{|1 - z_{mk}z|^s}
\]  
(42)
belongs to $L(p, q)$.
Case 1: \( q/p \leq 1, p > 1 \).
Let \( x_1, x_2, y_1, y_2 \) be real numbers (to be specified later) satisfying \( x_1 + y_1 = n + s, x_2 + y_2 = s \). By Holder’s inequality, we have

\[
\|h\|_{p,q}^q = \left( \int_0^1 \left( \int_0^{2\pi} \left| \sum_{m} \sum_{k} \left| a_{mk} \right| (1 - r^2)^{-n} \left( \frac{1 - |z_{mk}|^2}{|1 - z_{mk}z|^s} \right)^p \right| \right)^{q/p} d\theta \right)^{q/p} 2r \ dr \\
\leq \int_0^1 (1 - r^2)^{-nq} \left( \int_0^{2\pi} \left[ \sum_{m} \sum_{k} \left| a_{mk} \right| \left( \frac{1 - |z_{mk}|^2}{|1 - z_{mk}z|^2} \right)^{x_1p} \right] \times \left( \sum_{m} \sum_{k} \left( \frac{1 - |z_{mk}|^2}{|1 - z_{mk}z|^2} \right)^{y_1p'} \right)^{p'/p} d\theta \right)^{q/p} 2r \ dr \\
= C \int_0^1 (1 - r^2)^{-q(n-y_1+y_2)} \left[ \sum_{m} \sum_{k} |a_{mk}|^p \left( \frac{1 - |z_{mk}|^2}{|1 - z_{mk}z|^{x_2p-1}} \right) \int_0^{2\pi} \frac{1}{|1 - z_{mk}z|^{x_2p}} d\theta \right]^{q/p} 2r \ dr.
\]

In view of inequality (30), we obtain

\[
\|h\|_{p,q}^q \leq C \int_0^1 (1 - r^2)^{-q(n-y_1+y_2)} \left[ \sum_{m} \sum_{k} |a_{mk}|^p \left( \frac{1 - |z_{mk}|^2}{|1 - z_{mk}z|^{x_2p-1}} \right) \right]^{q/p} 2r \ dr.
\]

From inequalities (29) and (23), we have

\[
\|h\|_{p,q}^q \leq C \int_0^1 (1 - r^2)^{-q(n-y_1+y_2)} \left[ \sum_{m} \sum_{k} |a_{mk}|^p \left( \frac{1 - |z_{mk}|^2}{|1 - z_{mk}z|^{x_2p-1}} \right) \right]^{q/p} 2r \ dr.
\]

Finally, inequality (31) implies

\[
\|h\|_{p,q}^q \leq C \sum_{m} (1 - r_m^2)^{x_1q} (1 - r_m^2)^{-q(n-y_1+y_2)x_2-1/p+1} \left( \sum_k |a_{mk}|^p \right)^{q/p} \\
= C \sum_{m} (1 - r_m^2)^{1+q/p} \left( \sum_k |a_{mk}|^p \right)^{q/p} < \infty.
\]
To apply the inequalities above, $x_1, x_2, y_1, y_2$ must be chosen such that
\[
\begin{aligned}
1 < y_1 p' < y_2 p', \\
1 < x_2 p,
\end{aligned}
\]

and
\[-1 < -q(n - y_1 + y_2) < q x_2 - q/p - 1\]

The following works for $\epsilon$ so small such that $n + \epsilon < 1/q$:
\[
\begin{aligned}
y_1 = 1/p' + \epsilon, & \quad y_2 = 1/p' + 2\epsilon \\
x_1 = n + s - y_1, & \quad x_2 = s - y_2 \\
s > 1 - \epsilon
\end{aligned}
\]

We conclude that $f \in A(p, q)$.

**Case 2:** $q/p \leq 1, p \leq 1$.

From inequality (29), we obtain
\[
\|h\|^q_{p,q} = \int_0^1 \left\{ \int_0^{2\pi} \left[ \sum_m \sum_k |a_{mk}|(1 - r^2)^{-n} \frac{(1 - |z_{mk}|^2)^{2(n+s)}}{|1 - \overline{z_{mk}} x|^s} \right]^p dr \right\}^{q/p} 2r dr
\]

\[
\leq \int_0^1 \left[ \int_0^{2\pi} \sum_m \sum_k |a_{mk}|^p (1 - r^2)^{-np} \frac{(1 - |z_{mk}|^2)^{(n+s)p}}{|1 - \overline{z_{mk}} x|^p} dr \right]^{q/p} 2r dr
\]

\[
= \int_0^1 \left[ \sum_m \sum_k |a_{mk}|^p (1 - r^2)^{-np} (1 - |z_{mk}|^2)^{(n+s)p} \int_0^{2\pi} \frac{1}{|1 - \overline{z_{mk}} x|^p} d\theta \right]^{q/p} 2r dr
\]

In light of inequalities (30) and (23), this implies that
\[
\|h\|^q_{p,q} \leq C \int_0^1 \left[ \sum_m \sum_k |a_{mk}|^p (1 - r^2)^{-np} (1 - |z_{mk}|^2)^{(n+s)p} \frac{1}{(1 - |z_{mk}|r)^{sp-1}} \right]^{q/p} 2r dr
\]

\[
\leq C \int_0^1 \left[ \sum_m (1 - r^2)^{-np} (1 - r_m^2)^{(n+s)p} \sum_k |a_{mk}|^p \right]^{q/p} 2r dr,
\]

provided $s > 1/p$. By inequality (29),
\[
\|h\|^q_{p,q} \leq C \int_0^1 \sum_m (1 - r^2)^{-nq} \frac{(1 - r_m^2)^{(n+s)q}}{(1 - r_m^2)^{sq-q/p}} \left( \sum_k |a_{mk}|^p \right)^{q/p} 2r dr
\]

\[
= C \sum_m (1 - r_m^2)^{(n+s)q} \int_0^1 \frac{(1 - r^2)^{-nq}}{(1 - r_m^2)^{sq-q/p}} 2r dr \left( \sum_k |a_{mk}|^p \right)^{q/p}
\]
By inequality (31), if \( s > 1/p + 1/q - n \), we have

\[
\|h\|^q_{p,q} \leq C \sum_{m} (1 - r_m^2)^{(n+s)q}(1 - r_m^2)^{-nq-sq+p+1} \left( \sum_{k} |a_{mk}|^p \right)^{q/p} 
= C \sum_{m} (1 - r_m^2)^{1+q/p} \left( \sum_{k} |a_{mk}|^p \right)^{q/p} < \infty.
\]

**Case 3:** \( q/p > 1, \ p > 1 \).

Let \( x_1, y_1, x_2, y_2 \) be real numbers such that \( x_1 + y_1 = n + s, x_2 + y_2 = s \) and apply Holder’s inequality to get

\[
\|h\|^q_{p,q} = \int_{0}^{2\pi} \left\{ \int_{0}^{2\pi} \left[ \sum_{m} \sum_{k} |a_{mk}|(1 - r_m^2)^{-n} \left( \frac{1 - |z_{mk}|^{2}}{1 - \bar{z}_{mk}z} \right)^{p} \right] \, d\theta \right\} \, 2r \, dr
\]

\[
\leq \int_{0}^{1} (1 - r^2)^{-nq} \left\{ \int_{0}^{2\pi} \left[ \sum_{m} \sum_{k} |a_{mk}|p \left( \frac{1 - |z_{mk}|^{2}}{1 - \bar{z}_{mk}z} \right)^{p} \right] \, d\theta \right\} \, 2r \, dr
\]

If \( p'y_1 < p'y_2 \), we can apply inequality (33) to get

\[
\|h\|^q_{p,q} \leq C \int_{0}^{1} (1 - r^2)^{-nq} \left\{ \int_{0}^{2\pi} \left[ \sum_{m} \sum_{k} |a_{mk}|p \left( \frac{1 - |z_{mk}|^{2}}{1 - \bar{z}_{mk}z} \right)^{p} \right] (1 - r^2)^{(y_1 - y_2)p} \, d\theta \right\} \, 2r \, dr
\]

\[
= C \int_{0}^{1} (1 - r^2)^{(y_1 - y_2 - n)q} \left[ \sum_{m} \sum_{k} |a_{mk}|p \left( 1 - |z_{mk}|^{2} \right)^{px_1} \right] \int_{0}^{2\pi} \frac{1}{1 - \bar{z}_{mk}z} \, d\theta \right\} \, 2r \, dr
\]

If \( px_2 > 1 \), inequality (30) gives us

\[
\|h\|^q_{p,q} \leq C \int_{0}^{1} (1 - r^2)^{(y_1 - y_2 - n)q} \left[ \sum_{m} \sum_{k} |a_{mk}|p \left( 1 - |z_{mk}|^{2} \right)^{px_1} \frac{1}{(1 - |z_{mk}|^{2} r)^{px_2 - 1}} \right] \, 2r \, dr
\]

Let \( x_3, x_4, y_3, y_4 \) be real numbers such that \( x_3 + y_3 = x_1, x_4 + y_4 = x_2 - 1/p. \) By Holder’s inequality,

\[
\|h\|^q_{p,q} \leq C \int_{0}^{1} (1 - r^2)^{(y_1 - y_2 - n)q} \left[ \sum_{m} \frac{(1 - r_m^{2})^{qx_3}}{(1 - r_m)^{qx_4}} \left( \sum_{k} |a_{mk}|^p \right)^{q/p} \right] \times
\]

\[
\times \left[ \sum_{m} \frac{(1 - r_m^{2})^{p(y_3/q)/p}}{(1 - r_m)^{p(y_4/q)/p}} \right] \, 2r \, dr
\]
If \( y_3 < y_4 \), inequality (33) gives us
\[
\|h\|_{p,q}^q \leq C \int_0^1 (1 - r^2)^{(y_1 - y_2 - n)q} \left[ \sum_m \frac{(1 - r_m^2)^{qx_m}}{(1 - r/r_m)^{qx_{m+1}}} \right]^{q/p} \times (1 - r^2)^{(y_3 - y_4)2r} \, dr 
\]
\[
= C \sum_m \left( \sum_k |a_{mk}|^p \right)^{q/p} (1 - r_m^2)^{q(x_3)(1 - r_m^2)} \int_0^1 \frac{(1 - r^2)^{(y_1 - y_2 + y_3 - y_4 - n)}}{(1 - r/r_m)^{qx_{m+1}}} \, 2r \, dr 
\]
If \(-1 < q(y_1 - y_2 + y_3 - y_4 - n) < qx_4 - 1\) then we can apply inequality (31) to get
\[
\|h\|_{p,q}^q \leq C \sum_m \left( \sum_k |a_{mk}|^p \right)^{q/p} (1 - r_m^2)^{q(x_3)} (1 - r_m^2)^{(y_1 - y_2 + y_3 - y_4 - x_4 - n) + 1} 
\]
\[
= C \sum_m (1 - r_m^2)^{1+q/p} \left( \sum_k |a_{mk}|^p \right)^{q/p} < \infty. 
\]
It is easy to see that there exist \( x_1, x_2, \ldots \) satisfying the conditions for the inequalities. Thus \( f \in A(p, q) \).

**Case 4:** \( q/p > 1, p \leq 1 \).

By inequalities (29) and (30), we have
\[
\|h\|_{p,q}^q \leq C \int_0^{2\pi} \left( \int_0^1 \sum_m \sum_k |a_{mk}| (1 - r^2)^{-n} \left( \frac{1 - |z_{mk}|^2}{1 - \bar{z}_{mk}z} \right)^{(n+s)q/p} \right) \, d\theta \, 2r \, dr 
\]
\[
\leq C \int_0^{2\pi} \left( \int_0^1 \sum_m \sum_k |a_{mk}|^p (1 - r^2)^{-np} \left( \frac{1 - |z_{mk}|^2}{1 - \bar{z}_{mk}z} \right)^{(n+s)p} \, d\theta \right)^{q/p} \, 2r \, dr 
\]
\[
= C \int_0^1 \left( \sum_m \sum_k |a_{mk}|^p (1 - r^2)^{-np} \left( \frac{1 - |z_{mk}|^2}{1 - \bar{z}_{mk}z} \right)^{(n+s)p} \right)^{q/p} \, 2r \, dr 
\]
Let \( x_1, y_1, x_2, y_2 \) be real numbers such that \( x_1 + y_1 = n + s, x_2 + y_2 = s - 1/p \) and apply Holder’s inequality to get
\[
\|h\|_{p,q}^q \leq C \int_0^1 (1 - r^2)^{-nq} \left[ \sum_m \left( \sum_k |a_{mk}|^p \right)^{q/p} \right] \times \left[ \sum_m \frac{(1 - r_m^2)^{qx_1}}{(1 - r/r_m)^{qx_{m+1}}} \right]^{(q/p)/(q/p)'} \times \left[ \sum_m \frac{(1 - r_m^2)^{pq_1(q/p)'}}{(1 - r/r_m)^{pq_2(q/p)'}} \right]^{(q/p)/(q/p)'} \, 2r \, dr 
\]
Inequalities (33) and (31) then give us
\[ \|h\|_{p,q}^q \leq C \int_0^1 (1 - r^2)^{-nq} \left[ \sum_m \left( \sum_k |a_m|^p \right)^{q/p} \frac{(1 - r_m^2)^{q_{x_1}}}{(1 - rr_m)^{q_{x_2}}} \right] (1 - r^2)(y_1 - y_2)^{q_2} r \, dr \]
\[ = C \sum_m (1 - r_m^2)^{q_{x_1}} \int_0^1 \frac{(1 - r^2)^{q(y_1 - y_2 - n)}}{(1 - rr_m)^{q_{x_2}}} \, dr \left( \sum_k |a_m|^p \right)^{q/p} \]
\[ \leq C \sum_m (1 - r_m^2)^{q_{x_1}} (1 - r_m^2)^{q(y_1 - y_2 - x_2 - n) + 1} \left( \sum_k |a_m|^p \right)^{q/p} \]
\[ = C \sum_m (1 - r_m^2)^{1+q/p} \left( \sum_k |a_m|^p \right)^{q/p} < \infty \]

The last step is choosing \( x_1, x_2, y_1, y_2 \) for applying the inequalities. Having verified all the cases, we conclude that \( f \in A(p, q) \).

This completes the proof of the sufficiency part of Theorem 14.

3.2 Necessity part of Theorem 14

Suppose \( \Gamma \) is an interpolation sequence for \( A(p, q) \). Corollary 10 shows that \( \Gamma \) is uniformly discrete. To prove \( D^+(\Gamma) \leq 1/q \), we will show that \( \Gamma \) is an interpolation sequence for either \( A^q \) or \( A^p_{\alpha} \) for all \( \alpha \) with \( (1 + \alpha)/p > 1/q \). In the latter case, the known results for the Bergman spaces will imply that the density is less than or equal to 1/q and then strict equality follows thanks to the stability of interpolation sequences under small perturbations (see Remark 1).

The interpolation problem is solved by the formula
\[ f(z) = \sum_j a_j g_j(z) \frac{(1 - |z_j|^2)^{n+s}}{(1 - zz_j)^s} \quad (43) \]
provided the sum converges for the space in question. Here \( n = 1/p + 1/q \), and \( g_j \) are functions in \( A(p, q) \) satisfying \( g_j(z_j) = (1 - |z_j|^2)^{-n} \), \( g_j(z_{j'}) = 0 \) for all \( j' \neq j \), and \( \|g_j\|_{A(p, q)} \leq M \) independent of \( j \). (\( M \) is the interpolation constant of \( A(p, q) \).)

First, we show that for \( s \) sufficiently large, the series above converges uniformly on each compact set of the unit disk to an analytic function \( f \). This is done similarly to the sufficiency case. The proof now is split into four cases. We verify that \( \|f\| \) is bounded in \( A^q \) in the first two cases and that it is bounded in \( A^p_{\alpha} \) (when \( (1 + \alpha)/p > 1/q \)) in the last two. Since the case \( p = q \) is known for Bergman spaces, we do not need to include it.

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**Case 1:** $q/p < 1$, $q \leq 1$.

By inequality (29),

$$
\|f\|^q_{A^q} = \int_0^{1} \int_0^{2\pi} \left| \sum_j a_j g_j(z) \frac{(1 - |z_j|^2)^{(n+s)q}}{(1 - z \bar{z}_j)^s} \right|^q d\theta \, 2r \, dr
$$

$$
\leq \int_0^{1} \int_0^{2\pi} \sum_j |a_j|^q |g_j(z)|^q \frac{(1 - |z_j|^2)^{(n+s)q}}{|1 - z \bar{z}_j|^{sq}} d\theta \, 2r \, dr
$$

$$
= \int_0^{1} \sum_j |a_j|^q (1 - |z_j|^2)^{(n+s)q} \left[ \int_0^{2\pi} \frac{|g_j(z)|^q}{|1 - z \bar{z}_j|^{sq}} d\theta \right] 2r \, dr
$$

Apply Holder’s inequality, we have

$$
\|f\|^q_{A^q} \leq \int_0^{1} \sum_j |a_j|^q (1 - |z_j|^2)^{(n+s)q} \left[ \int_0^{2\pi} |g_j(z)|^p d\theta \right]^{q/p} \times
$$

$$
\times \left[ \int_0^{2\pi} \frac{1}{|1 - z \bar{z}_j|^{sq(p/q')}} d\theta \right]^{1/(p/q')} \, 2r \, dr
$$

If $s$ is sufficiently large, inequality (31) then gives us

$$
\|f\|^q_{A^q} \leq \int_0^{1} \sum_j |a_j|^q (1 - |z_j|^2)^{(n+s)q} \left[ \int_0^{2\pi} \frac{1}{(1 - r|z_j|)^{q(s+1/p-1/q)}} d\theta \right] 2r \, dr
$$

$$
= \sum_j |a_j|^q (1 - |z_j|^2)^{(n+s)q} \int_0^{1} \frac{1}{(1 - r|z_j|)^{q(s+1/p-1/q)}} \left[ \int_0^{2\pi} |g_j(z)|^p d\theta \right]^{q/p} 2r \, dr
$$

$$
\leq C \sum_j |a_j|^q (1 - |z_j|^2)^{(n+s)q} \left[ \int_0^{1} \frac{1}{(1 - r|z_j|)^{q(s+1/p-1/q)}} \left[ \int_0^{2\pi} |g_j(z)|^p d\theta \right]^{q/p} \right] 2r \, dr
$$

$$
\leq C \sum_j |a_j|^q (1 - |z_j|^2)^2 \|g_j\|_{A(p,q)}^q 2r \, dr
$$

$$
\leq C \sum_j (1 - |z_j|^2)^2 |a_j|^q < \infty.
$$

**Case 2:** $q/p < 1$, $q > 1$.

Let $x_1, x_2, y_1, y_2$ be real numbers satisfying $x_1 + y_1 = n + s, x_2 + y_2 = s$. Then by Holder’s
inequality and inequality \([33]\), we have

\[
\|f\|_{A^q}^q = \int_0^1 \int_0^{2\pi} \left| \sum_j a_j g_j(z) \frac{(1 - |z_j|^2)^{n+s}}{(1 - z\bar{z}_j)^s} \right|^q \, d\theta \, 2r \, dr
\]

\[
\leq C \int_0^1 \int_0^{2\pi} \left[ \sum_j |a_j|^q |g_j(z)|^q \frac{(1 - |z_j|^2)^{q'q_1}}{|1 - z\bar{z}_j|^{q'q_2}} \right] \left[ \sum_j \frac{(1 - |z_j|^2)^{q'q_1}}{|1 - z\bar{z}_j|^{q'q_2}} \right]^{q'/q} \, d\theta \, 2r \, dr
\]

\[
\leq C \int_0^1 \int_0^{2\pi} \left[ \sum_j |a_j|^q |g_j(z)|^q \frac{(1 - |z_j|^2)^{q'q_1}}{|1 - z\bar{z}_j|^{q'q_2}} \right] (1 - r^2)^{q(y_1 - y_2)} \, d\theta \, 2r \, dr
\]

\[
= C \int_0^1 \sum_j |a_j|^q (1 - r^2)^{q(y_1 - y_2)} (1 - |z_j|^2)^{q'q_1} \left[ \int_0^{2\pi} |g_j(z)|^q \, d\theta \right] \, dr
\]

Holder’s inequality again gives us

\[
\|f\|_{A^q}^q \leq C \int_0^1 \sum_j |a_j|^q (1 - r^2)^{q(y_1 - y_2)} (1 - |z_j|^2)^{q'q_1} \left( \int_0^{2\pi} |g_j(z)|^p \, d\theta \right)^{q/p} \times \left( \int_0^{2\pi} \frac{1}{|1 - z\bar{z}_j|^{q'q_2(p'/q')}} \, d\theta \right)^{1/(p'/q')} \, 2r \, dr.
\]

By inequality \([30]\), we have

\[
\|f\|_{A^q}^q \leq C \int_0^1 \sum_j |a_j|^q (1 - |z_j|^2)^{q'q_1} (1 - r^2)^{q(y_1 - y_2)} \frac{(1 - r^2)^{q(y_1 - y_2)}}{(1 - r|z_j|)^{q(x_2 + 1/p - 1/q)}} \times \frac{1}{(1 - |z_j|^2)^{q'q_2} (1 - r^2)^{q(y_1 - y_2)}} \leq C(1 - |z_j|^2)^2.
\]

Since \(x_1 - x_2 + y_1 - y_2 = n\), if we choose \(x_1 = 2/q\) then \(y_1 - y_2 = x_2 + 1/p - 1/q\) and thus,

\[
(1 - |z_j|^2)^{q'q_1} (1 - r^2)^{q(y_1 - y_2)} \leq C(1 - |z_j|^2)^2 (1 - r^2)^{q(y_1 - y_2)} \leq C(1 - |z_j|^2)^2.
\]

It follows that

\[
\|f\|_{A^q}^q \leq C \sup_j \left\{ \|g_j\|_{p,q}^q \right\} \sum_j (1 - |z_j|^2)^2 |a_j|^q < \infty.
\]

**Case 3:** \(q/p > 1, \ p \leq 1\).

Given \(\alpha\) with \((1 + \alpha)/p > 1/q\),

\[
\|f\|_{A^p}^p = \int_D |f(z)|^p (1 - |z|^2)^\alpha \, dA(z)
\]

\[
= \int_0^1 \int_0^{2\pi} \left| \sum_k a_k g_k(z) \frac{(1 - |z_k|^2)^{n+s}}{(1 - z\bar{z}_k)^s} \right|^p (1 - r^2)^\alpha \, d\theta \, 2r \, dr
\]
By inequality (29), we have
\[
\|f\|_{A^p_\alpha}^p \leq \int_0^1 \int_0^{2\pi} \sum_k |a_k|^p |g_k(z)|^p \left(1 - |z_k|^2\right)^{(n+s)p} \left(1 - r^2\right)^\alpha \, d\theta \, 2r \, dr
\]
\[
= \sum_k |a_k|^p \left(1 - |z_k|^2\right)^{(n+s)p} \int_0^1 \left(1 - r^2\right)^\alpha \left[\int_0^{2\pi} \frac{|g_k(z)|^p}{|1 - z\bar{z}_k|^p} \, d\theta\right] \, 2r \, dr
\]
\[
\leq \sum_k |a_k|^p \left(1 - |z_k|^2\right)^{(n+s)p} \int_0^1 \left(1 - r^2\right)^\alpha \left[\int_0^{2\pi} |g_k(z)|^p \, d\theta\right] \, 2r \, dr
\]

Holder’s inequality then gives us
\[
\|f\|_{A^p_\alpha}^p \leq \sum_k |a_k|^p \left(1 - |z_k|^2\right)^{(n+s)p} \left[\int_0^1 \left(\int_0^{2\pi} |g_k(z)|^p \, d\theta\right)^{q/p} \, 2r \, dr\right]^{p/q} \times
\]
\[
\times \left[\int_0^1 \frac{\left(1 - r^2\right)^\alpha (q/p)'}{(1 - r|z_k|)^{sp(q/p)'}} \, 2r \, dr\right]^{1/(q/p)'}
\]

From inequality (31), if \( s \) is sufficiently large, we get
\[
\|f\|_{A^p_\alpha}^p \leq C \sum_k |a_k|^p \left(1 - |z_k|^2\right)^{(n+s)p} \left\|g_k\right\|^p_{A(p,q)} \left(1 - |z_k|^2\right)^{sp+1/(q/p)'}
\]
\[
\leq C \sum_k \left(1 - |z_k|^2\right)^{2+\alpha} |a_k|^p < \infty.
\]

**Case 4:** \( q/p > 1, p > 1 \).

The norm of \( f \) (if it exists) in the weighted Bergman space \( A^p_\alpha \) is
\[
\|f\|_{A^p_\alpha}^p = \int_D |f(z)|^p (1 - |z|^2)^\alpha \, dA(z)
\]
\[
= \int_0^1 \left(1 - r^2\right)^\alpha \int_0^{2\pi} \left|\sum_k a_k g_k(z) \frac{(1 - |z_k|^2)^{n+s}}{(1 - z\bar{z}_k)^s}\right|^p \, d\theta \, 2r \, dr
\]

Let \( x_1, y_1, x_2, y_2 \) be real numbers satisfying \( x_1 + y_1 = n + s, x_2 + y_2 = s \). Then by Holder’s inequality, we have
\[
\|f\|_{A^p_\alpha}^p \leq \int_0^1 \left(1 - r^2\right)^\alpha \int_0^{2\pi} \left[\sum_k |a_k|^p |g_k(z)|^p \frac{(1 - |z_k|^2)^{px_1}}{|1 - z\bar{z}_k|^{px_2}}\right] \times
\]
\[
\times \left[\sum_k \left(\frac{(1 - |z_k|^2)^{py_1}}{|1 - z\bar{z}_k|^{py_2}}\right)^{p/p'} \, d\theta \, 2r \, dr\right]
\]
In light of inequality (33), this implies that

\[
\|f\|_{A^p}^p \leq C \int_0^{2\pi} \left[ \sum_k |a_k|^p |g_k(z)|^p \left( \frac{1 - |z_k|^2}{1 - z\bar{z}_k} \right)^{\frac{p(1 - r^2)\rho(y_1 - y_2 + \alpha/p)}{1 - r^2\rho(y_1 - y_2 + \alpha/p)}} \right] dr \, d\theta
\]

\[
= C \sum_k |a_k|^p (1 - |z_k|^2)^{px_1} \int_0^{2\pi} \left( \frac{|g_k(z)|^p}{1 - z\bar{z}_k} \right)^{\frac{p(1 - r^2)\rho(y_1 - y_2 + \alpha/p)}{1 - r|z_k|^2}} dr \, d\theta
\]

\[
\leq C \sum_k |a_k|^p (1 - |z_k|^2)^{px_1} \left[ \int_0^{2\pi} \left( \int_0^1 |g_k(z)|^p d\theta \right)^{\frac{q/p}{2r}} dr \right] \times \left[ \int_0^1 \left( \frac{(1 - r^2)^{\rho(y_1 - y_2 + \alpha/p)(q/p)'}}{(1 - r|z_k|^2)^{px_2(q/p)'}} \right)^{\frac{1/(q/p)'}{2r}} dr \right]
\]

Inequality (31) then gives us

\[
\|f\|_{A^p}^p \leq C \sum_k |a_k|^p \|g_k\|_{A(p,q)}^p (1 - |z_k|^2)^{px_1 + \rho(y_1 - y_2 + \alpha/p) + 1/(q/p)'}
\]

\[
\leq C \sum_k (1 - |z_k|^2)^{2+\alpha} |a_k|^p < \infty.
\]

Finally, we choose \(x_1, x_2, y_2, y_2\) such that they satisfy the conditions of the inequalities above.

This completes the proof of the necessity part of Theorem 14.

4 Sampling sequences for mixed-norm spaces

Interpolation problems often go together with sampling problems. For a function space \((A, \|\cdot\|_A)\) on \(\Omega\), a sequence of distinct points \(\Gamma = (z_m) \subset \Omega\) and a sequence space \((X, \|\cdot\|_X)\), \(\Gamma\) is said to be a sampling sequence for \((A, X)\) if there exist positive constants \(K_1, K_2\) such that for all \(f \in A\)

\[
K_1 \|f\|_A \leq \|f(z_m)\|_X \leq K_2 \|f\|_A
\]  

(44)

The sampling problem for mixed-norm spaces is to characterize sampling sequences for the pair \((A(p,q), \|\cdot\|^{p,q}(\Gamma))\). Our method is based on the paper [10]. There it is shown that a sequence \(\Gamma\) is sampling for \(A^q\) if and only if there exists \(r < q\) such that every limit set under a sequence of Möbius transformations of \(\Gamma\) is a set of uniqueness for \(A^r\). Thus, the density condition that is known to be characteristic of sampling in \(A^q\) must be equivalent to this limit condition. We will show the following theorem.
Theorem 16. The following are equivalent

1. $\Gamma$ is sampling for $A(p, q)$.

2. $\Gamma$ is sampling for $A^q$.

3. $\Gamma$ is sampling for $A^{p\alpha}$ where $(1 + \alpha)/p = 1/q$.

4. $\Gamma$ contains a uniformly discrete subsequence $\Gamma'$ with $D^{-}(\Gamma') > 1/q$.

The equivalence of the last three conditions is known work of K. Seip [14] and A. Schuster [13]. We will prove that the first condition is equivalent to the second when $q < p$ and to the third when $p < q$.

But to start, we need a mixed norm version of [10, Lemma 3.4].

Lemma 17. Let $f \in A(p, q)$ and let $r < \min(p, q)$ and $\alpha > r/q - 1$. For any $z \in \mathbb{D}$ we have

$$|f(z)|^r \leq \frac{\alpha + 1}{\pi} \int_{\mathbb{D}} |f(w)|^r \frac{(1 - |z|^2)^{2+\alpha}(1 - |w|^2)^\alpha}{|1 - \overline{w}z|^{4+2\alpha}} dA(w).$$  \hfill (45)

Moreover, there is a constant $C$ depending only on $p$, $q$ and $\alpha$ such that if $B_\epsilon = B_\epsilon(f)$ is the set of points where

$$|f(z)|^r \leq \epsilon \int_{\mathbb{D}} |f(w)|^r \frac{(1 - |z|^2)^{2+\alpha}(1 - |w|^2)^\alpha}{|1 - \overline{w}z|^{4+2\alpha}} dA(w).$$  \hfill (46)

(we think of $B_\epsilon$ as the set of ‘bad’ points) then

$$\|f\chi_{B_\epsilon}\|_{L(p,q)} \leq C\epsilon \|f\|_{L(p,q)}.$$  \hfill (47)

Therefore, $\epsilon > 0$ may be chosen independent of $f$ so that if $G_\epsilon = \mathbb{D}\setminus B_\epsilon$ is the set of ‘good’ points for $f$ then

$$\|f\chi_{G_\epsilon}\|_{L(p,q)} \geq (1/2) \|f\|_{L(p,q)}.$$  \hfill (48)

Note that the inequality $\alpha > r/q - 1$ will allow us to take $\alpha \leq p/q - 1$ when $p \leq q$ and $\alpha < 0$ when $q < p$.

Proof. The proof is exactly as in [10], except we need to show that the integral operator $T$ with kernel

$$K_\alpha(z, w) = \frac{(1 - |z|^2)^{2+\alpha}(1 - |w|^2)^\alpha}{|1 - \overline{w}z|^{4+2\alpha}} = \frac{(1 - r^2)^{2+\alpha}(1 - \rho^2)^\alpha}{|1 - r\rho e^{i(\theta - t)}|^{4+2\alpha}}$$

(where $z = re^{i\theta}$ and $w = \rho e^{it}$) is bounded from $L(p/r, q/r)$ to itself.

This is essentially well-known, but we include a proof for convenience. To simplify the notation, let us show that this operator is bounded on $L(p, q)$, when $p > 1$ and $q > 1$ and $\alpha > 1/q - 1$. Then the proof will apply to $L(p/r, q/r)$ with $\alpha > r/q - 1$.

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Given a function $g(w) = g(\rho e^i) \in L(p, q)$,

$$\|Tg\|_{L^p(\rho d\theta)} \leq \int_0^1 \|K_\alpha * g\|_{L^p(\rho d\theta)} 2\rho \, d\rho \leq \int_0^1 \|K_\alpha\|_{L^1(dt)} \|g\|_{L^p(dt)} \, d\rho$$

in which the convolution is taken in the angle variables. We use part 2 of Lemma 15 (with $M = 4 + 2\alpha$, which is greater than 1 because $\alpha > -1$) to obtain

$$\|K_\alpha\|_{L^1(dt)} \leq C \frac{(1 - r^2)^{2+\alpha}(1 - \rho^2)^\alpha}{|1 - r\rho|^{3+2\alpha}}$$

Since $\|g\|_{L^p(dt)}$ belongs to $L^q(2r \, dr)$, it now suffices to show that the function in the inequality above defines a bounded integral operator on $L^q(2r \, dr)$. This is a consequence of the Schur method (Lemma 18 below) and the inequalities

$$\int_0^1 \frac{(1 - r^2)^{2+\alpha}(1 - \rho^2)^\alpha - \epsilon q'}{|1 - r\rho|^{3+2\alpha}} 2\rho \, d\rho \leq C(1 - r^2)^{-\epsilon q'}$$

and

$$\int_0^1 \frac{(1 - r^2)^{2+\alpha - \epsilon q}(1 - \rho^2)^\alpha}{|1 - r\rho|^{3+2\alpha}} 2r \, dr \leq C(1 - \rho^2)^{-\epsilon q}$$

These both follow from part 3 of Lemma 15 if $\epsilon$ is chosen so that $2 + 2\alpha > \alpha - \epsilon q' > -1$ and $2 + 2\alpha + \alpha - \epsilon q > -1$. If we solve for $\epsilon$ and see that we must have simultaneously

$$\frac{-2 - \alpha}{q'} < \epsilon < \frac{1 + \alpha}{q'} \quad \text{and} \quad \frac{-\alpha}{q} < \epsilon < \frac{3 + \alpha}{q}$$

Such an $\epsilon$ exists if we have

$$\max \left( \frac{-2 - \alpha}{q'}, \frac{-\alpha}{q} \right) < \min \left( \frac{1 + \alpha}{q'}, \frac{3 + \alpha}{q} \right)$$

Of the four inequalities that this leads to, two are equivalent to $\alpha > -3/2$ and the others are respectively equivalent to $\alpha > -2 - 1/q$ and $\alpha > 1/q - 1$. All of these follow from the last, and that was one of the assumptions. \(\square\)

The Schur method for establishing boundedness of integral operators on $L^p$ spaces can be found in [4]. Here we use the following form:

**Lemma 18.** Let $k(x, y)$ be a non-negative measurable kernel on the product space $(X \times Y, \mu \otimes \nu)$ where $\mu$ and $\nu$ are $\sigma$-finite measures. Let $q > 1$ and assume there exist real constants $C_1$ and $C_2$ and positive functions $h_1(x)$ and $h_2(y)$ such that

$$\int_X k(x, y)h_1(x) \, d\mu(x) \leq C_1 h_2(y)$$

and

$$\int_Y k(x, y)h_2(y) \, d\mu(y) \leq C_2 h_1(x)$$

Then the integral operator $K$ defined by $Kf(y) = \int k(x, y)f(x) \, d\mu(x)$ is bounded from $L^q(X, \mu)$ to $L^q(Y, \nu)$ and $\|K\| \leq C_1^{1/q'} C_2^{1/q}$.
The equations (49) establish the hypotheses of Lemma 18 for the functions \( h(z) = k(z) = (1 - \rho^2)^{-\epsilon} \), thus finishing the proof of Lemma 17.

One half of the sampling requirement for \( \Gamma = \{ z_n \} \) is an upper estimate:

\[
\| (f(z_n)) \|_{p,q} \leq \| f \|_{L(p,q)}, \quad f \in A(p,q).
\]

A necessary and sufficient condition for this is that the number of points of the sequence in \( Q_{j,k} \) is bounded above independent of \( j \) and \( k \). We will say such a sequence has bounded density and also call it a Carleson sequence.

If \( \mu \) is sum of unit masses at each point of \( \Gamma \), then \( \mu(Q_{j,k}) \leq C \) with \( C \) independent of \( j \) and \( k \). We will use \( C \) for the set of measures satisfying this inequality for some constant \( C \) depending on the measure.

If \( \phi \) is a Möbius transformation of the disk, write \( \mu_{\phi} \) for the measure defined by \( \mu_{\phi}(E) = \mu(\phi^{-1}(E)) \). We say a sequence of measures \( \mu_n \) converges weakly to \( \mu \) if \( \int h \, d\mu_n \to \int h \, d\mu \) for all continuous \( h \) with compact support in \( \mathbb{D} \). Let \( W_\mu \) denote the set of all weak limits of measures of the form \( \mu_{\phi_n} \) for sequences \( \{ \phi_n \} \) of Möbius transformations of \( \mathbb{D} \).

The main result of [10] is that a measure \( \mu \) is sampling for the weighted Bergman space \( A^p_\alpha \) if and only there exists \( r < p \) such that the support of every measure in \( W_\mu \) is a set of uniqueness for \( A^r_\alpha \). When applied to a sum of point masses on a sequence \( \Gamma \), this condition must be equivalent to \( D^{-}(\Gamma') > (1 + \alpha)/p \) for some uniformly discrete subsequence \( \Gamma' \) of \( \Gamma \). In particular, if the lower uniform density of such a \( \Gamma' \) is greater than \( 1/q \), then \( \Gamma \) is a sampling sequence for both \( A^q \) and \( A^p_{p/q-1} \).

We also need the following lemma from [10] (Lemma 3.7).

**Lemma 19.** Let \( 0 < r < \infty \) and \( \alpha > 0 \). Let \( \epsilon > 0 \) be given, and define

\[
\mathcal{U}_\epsilon = \left\{ f \in A^r_\alpha : \| f \|_{A^\alpha_\epsilon} \leq 1 \text{ and } |f(0)| > \epsilon \right\}.
\]

Assume that \( \mu \in C \) is such that the support of every measure in \( W_\mu \) is a set of uniqueness for \( A^r_\alpha \). Then there is \( \delta > 0 \) such that \( \int |f|^r (1 - |z|^2)^{\alpha+2} \, d\mu_\phi > \delta \) for all \( \phi \in \mathcal{M} \) and all \( f \in \mathcal{U}_\epsilon \).

When \( \mu \) is the sum of unit point masses on \( \Gamma = \{ z_n \} \) then this lemma says that if 0 is one of the ‘good’ points from Lemma 17 that is

\[
|f(0)|^r > \epsilon \| f \|_{L^r_{\alpha}}
\]

then there exists a \( \delta > 0 \) such that for all Möbius transformations \( \varphi \) we have

\[
\sum_{n=1}^{\infty} |f(\varphi(z_n))|^r (1 - |\varphi(z_n)|^2)^{\alpha+2} > \delta \| f \|_{L^r_{\alpha}}^r > \delta |f(0)|^r
\]

We can now prove the sufficiency of the density condition

**Theorem 20.** If there exists a uniformly discrete subsequence \( \Gamma' \) of \( \Gamma \) such that \( D^{-}(\Gamma') > 1/q \) then \( \Gamma \) is a sampling sequence for \( A^{p,q} \).
Proof. The case $p = q$ is the known result for $A^q$ and so we divide the proof into two cases, $p < q$ and $p > q$. We start with the first. Throughout the proof, let $\alpha = p/q - 1$ so that $\alpha > -1$ and $(1 + \alpha)/p = 1/q$. Also let $\mu = \sum_{z_n \in \Gamma} \delta_{z_n}$, the sum of point masses for points in $\Gamma$.

Thus $\Gamma$ is a sampling sequence for $A^p_q$ and so the measure $\mu$ satisfies the conditions of \cite{10}. In particular, Lemma 19 holds for this $\mu$. We have seen that this implies if $0$ belongs to the good set $G_\epsilon$ from Lemma 17

$$|f(0)|^r \leq C \int_{\mathbb{D}} |f(z)|^r (1 - |z|^2)^2 + \alpha \, d\mu \phi$$

for some $r < p$ and for all M"obius transformations $\phi$. As in \cite{10} we can consider compositions of $f$ with M"obius transformations and conclude that if $\zeta \in G_\epsilon$

$$|f(\zeta)|^r \leq C \int |f(z)|^r \left| \phi'_\zeta(z) \right|^{2+\alpha} (1 - |z|^2)^{2+\alpha} \, d\mu(z)$$

Expanding the expression $|\phi'_\zeta(z)|$, this gives

$$|f(\zeta)|^r \leq C \int |f(z)|^r \frac{(1 - |\zeta|^2)^{2+\alpha}(1 - |z|^2)^{2+\alpha}}{|1 - \zeta z|^{4+2\alpha}} \, d\mu(z)$$

Let us use $\mathcal{K}$ to represent the operator of integration against

$$\mathcal{K}(\zeta, z) = \frac{(1 - |\zeta|^2)^{2+\alpha}(1 - |z|^2)^{2+\alpha}}{|1 - \zeta z|^{4+2\alpha}} \, d\mu(z)$$

We claim that $\mathcal{K}$ maps $L^{p/r,q/r}(\Gamma)$ boundedly into $L^{p,q}$, viewing a sequence in $L^{p/r,q/r}(\Gamma)$ as a function $f(z)$ on the measure space $(\Gamma, \mu)$. Then we will have

$$\|f\|_{A^p_q} = \|f^r\|_{L^{p/r,q/r}}$$
$$\leq 2 \|\|f^r\|_{L^{p/r,q/r}} \chi_{G_\epsilon} \|_{L^{p/r,q/r}}$$
$$\leq 2 C \mathcal{K}(\|f(z_m)^r\|)$$
$$\leq 2 C \|\mathcal{K}\| \|\|f(z_m)^r\|\|_{L^{p/r,q/r}}$$
$$= 2 C \|\mathcal{K}\| \|\|f(z_m)^r\|\|_{L^{p,q}}$$

Thus we will have obtained the sampling inequality and completed the proof of sufficiency.

The proof that $\mathcal{K}$ is bounded is almost identical to the proof of the boundedness of a similarly defined operator in Lemma 17. The difference between the two operators is that the latter involves integration against $dA(z)$ whereas the former replaces this with the measure $(1 - |z|^2)^2 \, d\mu(z)$. However, because the kernel satisfies $K(\zeta, z) < CK(\zeta, w)$ for pairs $z, w \in Q_{j,k}$, with $C$ independent of $\zeta, (j, k)$, Integration with respect to $(1 - |z|^2)^2 \, d\mu(z)$ can be estimated in terms of integration with respect to $dA(z)$ because $(1 - |z|^2)^2 \, d\mu(z)$ is a Carleson measure. Details are omitted.

The case $q < p$ proceeds similarly and is even simpler because we can take $\alpha = 0$. What is important is the the exponent $r$ must be chosen to make both $p/r$ and $q/r$ greater than 1.
Turning to the necessity we have the following.

**Theorem 21.** If $\Gamma$ is a sampling sequence for $A^{p,q}$ then it contains a uniformly discrete subsequence $\Gamma'$ satisfying $D^-(\Gamma') > 1/q$.

I will simply sketch the proof. We need the following variant of Theorem 12:

**Lemma 22.** For $0 < p, q < \infty$, let $\Gamma = (u_m)$ be a sampling sequence for $A(p,q)$ and $(u'_m)$ be another sequence in $\mathbb{D}$. There exists $\delta > 0$ such that if

$$\rho(u_m, u'_m) < \delta$$

for all $m$ then $\Gamma' = (u'_m)$ is also a sampling sequence for $A(p,q)$.

The proof is omitted. It is very much the same as that for Theorem 12. A crucial point of that proof is that the sequence $\Gamma$ is uniformly discrete. That is stronger than necessary for this lemma: all that is needed is that the sequence be Carleson.

As a first step toward the proof of necessity, we need the following mixed-norm version of Theorem 3.9 of [10].

**Lemma 23.** Let $Z$ be a zero sequence for $A^{p,q}$ and assume that $Z$ is a Carleson sequence. Let $0 < \gamma < 1$ and suppose there exists another set $Z'$ and a one-to-one correspondence

$$\sigma: Z \to Z'$$

such that $1 - |\sigma(a)|^2 = \gamma(1 - |a|^2)$ and $\rho(\sigma(a), \sigma(a))$ is bounded on $Z$. Then $Z'$ is a zero set for $A^{p,\gamma, q, \gamma}$.

The proof of this is almost identical to that of Theorem 3.9 of [10]. Instead of Theorem 3.8 of [10], taken from [9], we can make use of this similar characterization of zero sets of the mixed-norm space, also from [9]:

**Lemma 24.** $Z$ is a zero sequence for $A^{p,q}$ if and only if there exists a harmonic function $h$ in $\mathbb{D}$ such that the function $\exp\left[k_Z(\zeta) - h(z)\right]$ belongs to $L^{p,q}$.

See either [10] or [9] for the definition of $k_Z$.

The next thing we need is the appropriate relationship between mixed norm spaces and related Bergman spaces.

**Lemma 25.** If $p < q$ let $\alpha = p/q - 1$. Then $A^p_\alpha \subset A^{p,q}$ and for all positive $\lambda < 1$, $A^{p,q} \subset A^p_\lambda$.

If $q < p$ then $A^{p,q} \subset A^q$ and for all positive $\lambda < 1$, $A^q \subset A^{\lambda p, \lambda q}$.

If $p < q$ we can deduce from this the following: $\Gamma$ is a set of uniqueness for $A^r_\alpha$, for some $r < p$, if and only if it is a set of uniqueness for $A^{\lambda p, \lambda q}$, for some $\lambda < 1$.

If $q < p$ we can deduce: $\Gamma$ is a set of uniqueness for $A^r$, for some $r < q$, if and only if it is a set of uniqueness for $A^{\lambda p, \lambda q}$, for some $\lambda < 1$.

The main result of [10] then implies that the density criterion is equivalent to the following condition: there exist $\lambda < 1$ such that the support of every measure in $W_\mu$ is a set of uniqueness for $A^{\lambda p, \lambda q}$. So our proof of necessity requires the following:
Theorem 26. Suppose for every $\lambda < 1$ there exists a measure in $W_\mu$ whose support is a zero set for $A^{\lambda p, \lambda q}$. Then $\Gamma$ is not a sampling sequence for $A^{p,q}$.

Finally, the proof of this theorem is essentially identical to that of the corresponding result in [10]: Theorem 5.1, (a) $\Rightarrow$ (b).

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