The Poincare lemma for codifferential, anticoexact forms, and applications to physics

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Abstract

The linear homotopy theory for codifferential operator on Riemannian manifolds is developed in analogy to the theory for exterior derivative. A new class of anticoexact forms that exist locally in a star-shaped region is defined. Their application to physics, including vacuum Dirac-Kähler equation, coupled Maxwell-Kalb-Ramond system of equations occurring in a bosonic string theory and its reduction to the Dirac equation, is presented.

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1 Introduction

Well known formulation of the Poincaré lemma states [19–25] that

$$H^k(\mathbb{R}^n) = H^k(\text{point}) = \begin{cases} \mathbb{R}, & (k = 0) \\ 0, & (k > 0) \end{cases},$$

for $n > 0$. This means that in $\mathbb{R}^n$ each differential form which is exact, i.e., it is in the kernel of exterior derivative operator $d$, is also closed, i.e., is also in the image of $d$. This can be also extended to a star-shaped open region of a smooth
manifold \( M \), i.e., open set \( U \) of \( M \) that is diffeomorphic to the open ball in \( \mathbb{R}^n \), where \( n = \dim(M) \).

However, in practical calculations, especially in physics, there is need for solving \( d\alpha = 0 \) for some differential form \( \alpha \), i.e., finding the exact formula for a differential form \( \beta \) such that \( d\beta = \alpha \). This can be done locally in a star-shaped region \( U \) using homotopy operator \( [10, 11, 25, 19, 18, 7] \), i.e.,

\[
H\omega := \int_0^1 i_K \omega F(t,x) t^{k-1} dt,
\]  

(2)

where a \( k \)-form \( \omega \in \Lambda^k(U) \), \( k = \deg(\omega) \),

\[
K := (x - x_0)^i \partial_i,
\]  

(3)

and \( F(t,x) = x_0 + t(x - x_0), x_0 \in U \), is a linear homotopy between the constant map \( s_{x_0} : x \to x_0 \) and the identity map \( I : x \to x \). The form \( \omega \) under the integral is evaluated at the point \( F(t,x) \). Here \( i_K = K \downarrow \) is the insertion antiderivative.

The homotopy operator \( H \) has many interesting properties \( [11, 10] \), e.g.,

\[
H^2 = 0, \quad HdH = H, \quad dHd = d, \quad i_K \circ H = 0, \quad H \circ i_K = 0,
\]  

(4) (5)

useful in calculations.

This operator fulfils Homotopy Invariance Formula \( [10, 11, 25, 19, 18] \)

\[
dH + Hd = I - s^*_{x_0},
\]  

(6)

where \( s^*_{x_0} \) is the pullback along the constant map \( s_{x_0}(x) = x_0 \), and \( I \) is the identity map.

We focusing on open star-shaped regions \( U \) of a smooth manifold with or without boundary \( M \). Then the kernel of \( d \) defines the closed (that on \( U \) are also exact) vector space \( E(U) = \{ \omega \in \Lambda(U) | d\omega = 0 \} \) that is subspace of \( \Lambda(U) \). Similarly, the kernel of \( H \) on \( U \) defines a module over \( \Lambda^0(U) \) of antiexact forms \( \mathcal{A} = \{ \omega \in \Lambda(U) | H\omega = 0 \} \), which was described in \( [11, 10] \). It was also proved \( [11, 10] \) that

\[
\Lambda^k(U) = \mathcal{E}^k(U) \oplus \mathcal{A}^k(U),
\]  

(8)

for \( 0 \leq k \leq n \).

On Riemannian manifolds with non-degenerate metric tensor \( g \) one can define the Hodge star operator \( [23, 11] \), \( \ast : \Lambda^r \to \Lambda^{n-r} \), that fulfils

\[
\ast \ast \omega = (-1)^r(n-r) \omega = (-1)^r(n-r) \text{sig}(g)\omega, \quad \omega \in \Lambda^r(U),
\]  

(9)
with the inverse
\[ \star^{-1} = (-1)^{r(n-r)} \text{sig}(g) \star = \text{sig}(g) \eta^{n-1} \star = \text{sig}(g) \star \eta^{n-1}, \]  
(10)

where \( \eta \) is an involutive automorphism: \( \eta \omega = (-1)^p \omega \) for \( \omega \in \Lambda^p \), and where \( \text{sig}(g) = \frac{\det(g)}{\det(g)} \) is the signature of the metric \( g \). For clarity of presentation we will focus on Riemannian case (\( \text{sig}(g) = 1 \)) only and the other signatures, e.g. Lorentzian one, can be analysed similarly.

Then the codifferential is defined as
\[ \delta = \star^{-1} d \star \eta. \]  
(11)

Then the Poincaré lemma for codifferential is a trivial extension of the original lemma. We provide it with the proof since it exists in mathematical jargon, however to our knowledge, without written evidence:

**Theorem 1. (The Poincaré lemma for codifferential)**

For a star-shaped region \( U \), if \( \delta \omega = 0 \) for \( \omega \in \Lambda^k(U) \), then there exists \( \alpha \in \Lambda^{k+1}(U) \) for \( k < n = \dim(U) \), such that \( \omega = \delta \alpha \).

**Proof.** If \( \delta \omega = 0 \), then also \( \star d \star \omega = 0 \), and therefore \( d \star \omega = 0 \). By Poincaré lemma for \( d \), we have that there exists \( \beta \in \Lambda^{n-k-1}(U) \) that \( d \beta = \star \omega \), so \( \omega = \star^{-1} d \beta \).

Since the Hodge star is an isomorphism, so there exists \( \alpha \in \Lambda^{k+1}(U) \) such that \( \beta = \star \circ \eta(\alpha) \). Then \( \delta \alpha = \omega \), as required.

This paper aims to build a theory analogous to antiexact forms, which is centered at codifferential \( \delta \)-anticoexact forms, and then apply it to various equations and systems of equations containing \( d \) and \( \delta \) operators. Anti(co)exact forms are defined only locally on a star-shaped open region of a manifold; however, local problems are essential to physics applications. Therefore, we also provide some examples of physics equations that can be solved locally by the methods presented here.

The paper is organized as follows: In the next section, we develop the theory of anticoexact forms that allows us to decompose arbitrary differential form into coexact and anticoexact part. Then we relate this decomposition with exact-antiexact decomposition of [10, 11]. Next, the connection with de Rham theory and Clifford algebras will be presented. Finally, application of (anti)(co)exact decomposition to the vacuum Dirac(-Kähler) equations [1], Maxwell equations of classical electrodynamics and with the Kalb-Ramond equations of bosonic string theory [17, 27] will be presented.

## 2 Anticoexact forms

This section defines an analog of the theory for antiexact forms, which we call anticoexact forms. The presentation will be along with Chapter 5 of [10] with marking differences between antiexact and defined below anticoexact forms.

We start from the homotopy operator for \( \delta \):

3
Definition 1. We define the cohomotopy operator for δ for a star-shaped region $U$ as
$$h : \Lambda(U) \to \Lambda(U), \quad h = \eta \star^{-1} H \star. \quad (12)$$

In particular,
$$h_r : \Lambda^r(U) \to \Lambda^{r-1}(U), \quad h_r = (-1)^{r+1} \star^{-1} H \star, \quad r > 0. \quad (13)$$

Fig. 1 presents interplay between all the operators.

Since $(H\omega)|_{x=x_0} = 0$, so $(h\omega)|_{x=x_0} = 0$.

Such a definition makes the Homotopy Invariance Formula for δ and h similar to (6), namely,

Proposition 1.
$$\delta h + h\delta = I - S_{x_0}, \quad (14)$$

where $S_{x_0} = \star^{-1}s_{x_0}\star$. The operator $S_{x_0}$ is nonzero for $\Lambda^n(U)$ and it evaluates top forms at $x = x_0$, i.e., for $\omega \in \Lambda^n(U)$, $S_{x_0}\omega = \omega|_{x=x_0}$.

Proof. Using the Homotopy Invariance Formula restricted to $\Lambda^r$, we have
$$\star^{-1}(dH + Hd)\star = ((-1)^{r+1} \star^{-1} d\star)((-1)^{r+1} \star^{-1} H\star) + ((-1)^r \star^{-1} H\star)((-1)^{r} \star^{-1} d\star) = \delta h + h\delta = I - S_{x_0}, \quad (15)$$
since $\eta|_{\Lambda^r} = (-1)^r I$.

As a simple extension of the properties of $d$ and $H$, we have

Proposition 2.
$$h^2 = 0, \quad \delta h\delta = \delta, \quad h\delta h = h. \quad (16)$$

Proof. Since $H^2 = 0$ so, by (12), $h^2 = 0$. For the second property, we have
$$\delta h\delta = \star^{-1}d\star \eta \star^{-1} h \star \star^{-1}d\star \eta = \star^{-1}d\star \eta = \delta, \quad \text{since} \quad \eta^2 = 1 \quad \text{and} \quad dHd = d.$$ 
Similarly, using $HdH = H$, we get the third property.

Define now the coexact (that in a star-shaped $U$ is also coclosed) vector space
$$C := \{\omega \in \Lambda(U)|\delta\omega = 0\}. \quad (17)$$

Note that $C^n = \mathbb{R} \star 1$. Since $A^0$ consists of constant functions, coexact top forms are dual antiexact ones: $\star A^0 = C^n$.

We have that
Proposition 3. The operator $\delta h$ is the projector $\delta h : \Lambda \rightarrow C$.

Proof. For any form $\omega$, the form $\delta h \omega$ is coexact, so $\delta h : \Lambda \rightarrow C$. It is idempotent since $\delta(h\delta h) = \delta h$. Finally, for $\omega \in C$ we have from the Poincaré lemmat that there exists $\alpha$ such that $\delta\alpha = \omega$, and $h\omega = h\delta\alpha$. Then $\delta h \omega = \delta h^2 \alpha = \delta\alpha = \omega$. Therefore on $C$ the operator $\delta h$ is the identity.

We can therefore define the coexact part of the form by the projection

$$\omega_c := \delta h \omega.$$ (18)

By treating Homotopy Invariance Formula (14) as the partition of unity, we can define

Definition 2. (Anticoexact part of a form)
We define anticoexact part of a form $\omega \in \Lambda^k(U), k < n$, on a star-shaped region $U$ as

$$\omega_{ac} := h\delta \omega = \omega - \delta h \omega.$$ (19)

For $k = n$ the anticoexact part is

$$\omega_{ac} := h\delta \omega = \omega - \omega|_{x=x_0}.$$ (20)

Note that the anticoexact part of the form is of the type $\omega_{ac} = h\alpha$ and so these parts are in the kernel of the operator $h$ by its nilpotency.

We can define the anticoexact vector space as

$$\mathcal{Y} := \{\omega \in \Lambda(U) | \omega = h\delta \omega\},$$ (21)

that is the vector space of spanned by all anticoexact parts. Note that $\mathcal{Y}^0 = 0$.

Anticoexact space can be alternatively defined by the vector $K$. To this end we have to use the following lemma

Lemma 1. (Equation (1.4.7) of [1])

$$i_\alpha \star \phi = \star(\phi \wedge \alpha),$$ (22)

for $\alpha \in \Lambda^1$, $\phi$ an arbitrary form, and where $\sharp$ is a musical isomorphism such that $g(\alpha^2, X) = \alpha(X)$ for an arbitrary vector field $X$.

Using this Lemma, we have

Proposition 4.

$$(K^\flat \wedge) \circ h = 0, \quad h \circ K^\flat \wedge = 0.$$ (23)

Proof. Since we have (3), i.e., $i_K \circ H = 0$, therefore $i_K \star^* \circ H = 0$. Using (22), we have $(K^\flat \wedge) \circ \star^{-1} H = 0$. From the definition (12) of $h$ and the fact that $\star$ is an isomorphism, we have the result.

For the second identity, using (3), i.e., $H \circ i_K = 0$, we get $H \circ i_K \star = 0$, so using (22) we get $H \star \circ K^\flat \wedge \circ \eta = 0$. Therefore, $h \circ K^\flat \wedge = 0$, as required.
Using this we can characterize the vector space of anticoexact forms in an alternative way,

**Proposition 5.**

On a star-shaped region $U$,

$$\mathcal{Y} = \{ \omega \in \Lambda(U) | K^{\flat} \land \omega = 0, \quad \omega|_{x=x_0} = 0 \}. \quad (24)$$

**Proof.** If $\omega \in \mathcal{Y}$, i.e., $\omega = h\delta \omega$ then from (23) we get $K^{\flat} \land \omega = 0$ and $\omega|_{x=x_0} = 0$.

In the opposite direction, let $K^{\flat} \land \alpha = 0$ and take $\omega = \alpha + \delta \beta$. Since $K|_{F(t,x)} = tK_{x}$ and $(iK \ast \alpha)|_{F(t,x)} = t \ast K^{\flat}|_{x} \land \eta \alpha|_{F(t,x)} = 0$, so $h\alpha = 0$. Then $h\omega = h\delta \beta$, so $\omega_c = \delta h \omega = \delta h \delta \beta = \delta \beta$. Therefore the remaining part $\omega - \omega_c = \omega_{ac} = \alpha$. Since $\alpha$ is the anticoexact part of $\omega$ so $\alpha = h\delta \omega$ and therefore $\omega|_{x=x_0} = 0$.

Contrary to $C$ being a vector space, we have

**Proposition 6.**

$\mathcal{Y}$ is a $C^\infty$-module.

**Proof.** The conditions (24) defining $\mathcal{Y}$: $K^{\flat} \land \omega = 0$ and $\omega|_{x=x_0} = 0$ is preserved under wedge multiplication of two elements from $\mathcal{Y}$ and under $C^\infty$ multiplication.

Antiexact forms can be written as $iK \alpha$ for some $\alpha$. Likewise, we have

**Proposition 7.** If $\omega \in \mathcal{Y}(U)$ then there exists $\alpha \in \Lambda^{r-1}(U)$ such that

$$\omega = K^{\flat} \land \alpha. \quad (25)$$

**Proof.** Since $h\delta$ is the projector onto $\mathcal{Y}$, so for $\omega$ there is $\beta$ such that $\omega = h\beta$. Since $h$ is linear, so we can focus on a simple form which has local expression $\beta = f(x)dx^I$ for some multiindex $I$. Then

$$h\beta = \eta^{-1} H \ast \beta = \eta^{-1} iK \ast dx^I \int_0^1 dt f(F(t,x))t^{I|-1} = K^{\flat} \land \alpha, \quad (26)$$

where

$$\alpha = \left( \int_0^1 dt f(F(t,x))t^{I|-1} \right) dx^I. \quad (27)$$

The final point of this section is the following

**Theorem 2.** For a star-shaped $U$ there is the direct sum decomposition

$$\Lambda^k(U) = C^k(U) \oplus \mathcal{Y}^k(U). \quad (28)$$

**Proof.** For $0 < k < n$ from the Homotopy Invariance Formula (14), we have $h\delta + \delta h = I$. Moreover we know that both summands are projection operators. Therefore there is the unique decomoposition $\omega = \omega_{ac} + \omega_c$ with $\omega_{ac} = h\delta \omega$ and $\omega_c = \delta h \omega$.

For $k = 0$ we get $h\delta = I$ (i.e., $\Lambda^0(U) = \mathcal{Y}^0(U)$) since then $\delta \omega = 0$.

For $k = n$ we have $h\omega = 0$, so $h\delta = I - S_{x_0}$. Therefore the decomoposition is $\omega_{ac} = h\delta \omega = (I - S_{x_0})\omega$ and $\omega_c = S_{x_0}\omega$. We have $C^n(U) = \mathbb{R} \ast 1$.

Finally, if $\omega \in \mathcal{Y} \cap C$ then $\omega_{ac} = 0 = \omega_c$ and so $\omega = 0$. 

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Figure 2: On a locally star-shaped region $U$ of a manifold $M$, forms can be decomposed into for components $\mathcal{E}$ - exact, $\mathcal{A}$ - antiexact, $\mathcal{C}$ - coexact, and $\mathcal{Y}$ - anticoexact.

3 (Anti)coexact vs. (anti)exact forms

The theory developed in the previous section can be related to the theory of antiexact and exact forms, namely,

**Proposition 8.** In a star-shaped region

- $\mathcal{E}^k = \star \mathcal{C}^{n-k}$.
- $\mathcal{A}^k = \star \mathcal{Y}^{n-k}$

**Proof.** If $\omega \in \mathcal{E}$, then $d\omega = 0$. Therefore, $\delta \star \omega = 0$, and so $\star \omega \in \mathcal{C}$. In a result $\star \mathcal{E} \subset \mathcal{C}$. Similarly one can prove the opposite inclusion. Since $\star$ operator is an isomorphism we get the first point.

For the second point, we note that from (22) we can relate in a unique way $\mathcal{A}$ with $\mathcal{Y}$.

Due to this proposition, the theory of exact and antiexact forms is dual, with respect to the $\star$ isomorphism, to the theory of coexact and anticoexact forms developed above. However, all the work done above is not futile. Extraction of the notion of (anti)coexact forms and operator $h$ is useful in applications, as we will see below.

Finally, we have

**Theorem 3.** On a star-shaped open region $U$ of a manifold $M$, there is the unique direct sum decomposition presented in Fig. 2. We will call elements of $\mathcal{H}(U) := \mathcal{E}(U) \cap \mathcal{C}(U) = \{ \omega \in \Lambda(U) | d\omega = 0 = \delta \omega \}$ almost harmonic forms. By analogy, we will call elements of $\bar{\mathcal{H}} := \mathcal{A} \cap \mathcal{Y}$ almost antiharmonic forms.

The almost harmonic forms are in general not harmonic since on $U$ and on $M$, not necessarily closed and coclosed forms are harmonic - they are not in the kernel of the Laplace-Beltrami operator $\triangle = \delta d + d\delta$. We will discuss this issue concerning de Rham theory in the next section.
Proof. The decomposition results from the uniqueness of the direct sum decompositions $\Lambda(U) = \mathcal{E}(U) \oplus \mathcal{A}(U)$ and $\Lambda(U) = \mathcal{C}(U) \oplus \mathcal{Y}(U)$.

For example if $\omega \in C$, that is $\delta \omega = 0$, then there is the direct sum decomposition $\omega = H d \omega + d H \omega$ into exact and antieexact parts. Since it is a direct sum decomposition, so $\delta H d \omega = 0$ and $\delta d H \omega = 0$.

One also have to check if the projection operators $H d$, $d H$, $\delta h$ and $\delta h$ respect action of $d$, $\delta$, $H$ and $h$. It is straightforward, and we provide it in a case when $\omega \in C$, i.e., $\delta h \omega = \omega$. Then

$$d \omega = d \delta h \omega = d *^{-1} d * \eta \eta *^{-1} H \star \omega = d *^{-1} d H \star \omega = d \omega,$$

(29)

since if $\omega \in C$, then $* \omega \in E$, so $d H \omega = \omega$. Similarly,

$$H \omega = H \delta h \omega = H *^{-1} d * \eta \eta *^{-1} H \star \omega = H \omega,$$

(30)

by the same argument.

On various manifolds, some of the parts in Fig. 2 may not be permitted due to topological reasons. We will provide such an example in the next section.

4 Relation to de Rham theory

The de Rham theory has the simplest form in the compact Riemannian manifolds without boundary [12, 8]. For such a manifold $M$, when defining the inner product [12, 21, 8] on $\Lambda(M)$:

$$(\alpha, \beta) := \int_M \alpha \wedge \star \beta,$$

(31)

we have that the adjoint of $d$ is $d^\dagger = \delta$. The adjoint relation is also valid [8] when the forms have compact support in a non-compact $M$.

The simplest setup for relating the Poincaré lemma with de Rham theory is to consider the compact manifold $M$ without boundary and some open star-shaped region $U \subset M$. Then on $M$ we have the Hodge decomposition theorem [12, 8], that is, for every form $\omega$ there is the unique decomposition

$$\omega = \omega_d + \omega_\delta + \omega_h,$$

(32)

where $\omega_d \in \mathcal{E}(M)$ is the closed part, $\omega_\delta \in \mathcal{C}(M)$ is the coclosed part and $\omega_h \in \{\alpha | d \alpha = 0 = \delta \alpha \Leftrightarrow \Delta \alpha = 0\}$ is the harmonic part, where $\Delta = d \delta + \delta d$ is the Laplace-Beltrami operator. Therefore, we have the decomposition presented in Fig. 3. In this situation, we have

Corollary 1. For a compact manifold $M$ without boundary, when restricting differential forms to an open star-shaped region $U \subset M$ we have

- In Fig. 3, $1 + 3 = \mathcal{E}$, $2 = \mathcal{A}$;
Figure 3: Visualization of the Hodge decomposition: 1 + 3 - closed forms, 2 + 3 - coclosed forms, 3 - harmonic forms that are both closed and coclosed.

- In Fig. 3, $2 + 3 = \mathcal{C}$, $1 = \mathcal{Y}$;

We therefore have that $\mathcal{H}(U) = \mathcal{A}(U) \cap \mathcal{Y}(U) = 0$, that is there is no almost antiharmonic forms.

Another suitable common setup for the Poincaré lemma and de Rham theory is the open (non-compact) star-shaped (sub)manifold $U$ and the forms with compact supports on them [8]. Then the Laplace-Beltrami operator $\triangle = d\delta + \delta d$ under the product behaves as

$$\langle \alpha, \triangle \alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle \delta \alpha, \delta \alpha \rangle \geq 0.$$ (33)

Therefore, $\alpha$ is harmonic ($\triangle \alpha = 0$) iff $d\alpha = 0 = \delta \alpha$. However, in such a space there is no nontrivial harmonic forms of compact support [8]. Therefore the intersection of $\mathcal{C}$ and $\mathcal{E}$ is empty. Using the decomposition of $\Lambda(U)$ we have

**Corollary 2.** For an open star-shaped $U$ and forms with compact support there is

$$\mathcal{E}(U) = \mathcal{Y}(U), \quad \mathcal{C}(U) = \mathcal{A}(U).$$ (34)

## 5 Relation to Clifford algebras

For a Riemannian manifold $(M, g)$ the Clifford bundle is isomorphic to $\Lambda(TM)$ pointwise by defining the Clifford multiplication of a vector $v \in T_xM$ by $\psi \in \Lambda(T_xM)$ as

$$v\psi := v \wedge \psi + iv\psi,$$ (35)

see e.g., [13, 2, 3, 1, 6, 22].

Then for the unique metric-compatible torsion-free connection $\nabla$ we can define for an orthonormal co-frame $\{e^a\}_{a=1}^n$

$$d := e^a \wedge \nabla e_a, \quad \delta := -i e^a \nabla e_a.$$ (36)

In these terms, the Dirac(-Kähler) operator $\Pi$ on a Clifford bundle is defined as

$$D := e^a \nabla e_a = d - \delta.$$ (37)

When $M$ is parallelizable, i.e., there is a frame $\{e_a\}_{a=1}^n$ such that $\nabla X e_a = 0$ for all $X \in \Gamma(TM)$ then from this basis one can construct global idempotents
and use them to project Clifford algebra to minimal left ideals obtaining spinor subbundle of a Clifford bundle \([1, 13]\). However, the existence of this subbundle is more restrictive (\(M\) must be flat) than the existence of the spinor bundle.

Note that for the form \(\omega\) Clifford multiplied by \(K\) is

\[
K\omega = K^0 \wedge \omega + i_K \omega,
\]

which is the decomposition of \(\omega\) into anticoexact and antiexact parts, see Proposition \([7]\).

In order to understand the structure of the Dirac operator and its relation to the Poincaré lemma, we have to split it into grading of the base of Fig. 1. To this end, introduce the base of grading \(\Lambda = \Lambda^0 \oplus \ldots \oplus \Lambda^n\), where a form \(\omega\) is written as the vector

\[
\omega = \begin{bmatrix}
\omega^0 \\
\vdots \\
\omega^n
\end{bmatrix},
\]

where \(\omega^i \in \Lambda^i\). In this base the exterior derivative \(d\) has the simpler form

\[
d = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
d_0 & 0 & \ldots & 0 & 0 \\
0 & d_1 & \ldots & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \ldots & d_{n-1} & 0
\end{bmatrix}.
\]

Since \(d_k d_{k-1} = 0\) therefore \(d^2 = 0\). The operator is nilpotent due to combination of the matrix multiplication and its (operator) elements. Likewise, we have

\[
\delta = \begin{bmatrix}
0 & \delta_1 & 0 & \ldots & 0 \\
0 & 0 & \delta_2 & \ldots & 0 \\
0 & 0 & \ldots & \ddots & 0 \\
0 & 0 & \ldots & 0 & \delta_n \\
0 & 0 & \ldots & 0 & 0
\end{bmatrix},
\]

where \(\delta^2 = 0\) by \(\delta_{k-1} \delta_k = 0\). For a star-shaped region \(U\) we can define analogously the homotopy operators from Fig. 1.

\[
H = \begin{bmatrix}
0 & H_1 & 0 & \ldots & 0 \\
0 & 0 & H_2 & \ldots & 0 \\
0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & \ldots & 0 & H_n \\
0 & 0 & \ldots & 0 & 0
\end{bmatrix}, \quad h = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
h_0 & 0 & \ldots & 0 & 0 \\
0 & h_1 & \ldots & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \ldots & h_{n-1} & 0
\end{bmatrix}.
\]
Then the Dirac operator is

\[ D = d - \delta = \begin{bmatrix} 0 & -\delta_1 & 0 & \ldots & 0 \\ d_0 & 0 & -\delta_1 & \ldots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ddots & 0 & -\delta_n \\ 0 & \ldots & \ldots & d_{n-1} & 0 \end{bmatrix}, \]  

(43)

with the Laplace-Beltrami operator

\[ D^2 = (d - \delta)^2 = -\begin{bmatrix} \delta_1 d_0 & 0 & \ldots & 0 & 0 \\ 0 & d_0 \delta_1 + \delta_2 d_1 & 0 & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & d_{n-2} \delta_n - 1 + \delta_n d_{n-1} & 0 \\ 0 & \ldots & \ldots & 0 & d_{n-1} \delta_n \end{bmatrix}. \]  

(44)

One then see that the Dirac operator mix different grades and involves two neighbour grades.

Similarly, on \( U \) we can define the anti-Dirac operator by \( \Gamma := h - H \) and the anti-Laplace-Beltrami operator \( \Gamma^2 = (h - H)^2 = -(hH + Hh) \).

In the next section, we examine local solutions of the Dirac and other equations of physics using the machinery of decomposition of arbitrary form into (anti)(co)exact forms.

6 Application to equations of physics

In this section, the application of the above theory to some equations of physics will be presented. All considerations will be made in a star-shaped region \( U \) of a manifold \( M \) since then it is possible to use the machinery presented above. The restriction to local star-shaped open regions of some general manifold is not general, however, for many applications in physics usually sufficient. The existence of global solutions that projects down to the local ones is usually connected with topological conditions on the manifold and requires proper ’sheafication’ procedure [24, 12]. Therefore we will restrict ourselves to the local considerations only.

6.1 Local solutions of the Dirac equation

First, we start from the obvious vacuum solutions, i.e., the solutions for the vacuum Dirac equation \( \mathcal{D} \psi = 0 \) in a star-shaped region \( U \) of a manifold is an arbitrary almost harmonic form \( \psi \in \mathcal{H}(U) \). Such forms are also the solution of the Laplace equation \( D^2 \psi = 0 \). On a compact manifold, \( \psi \) is also a harmonic form.

However, (almost) harmonic forms can be seen as a ‘gauge modes’ that allows to shift other solutions, since they nullify both terms \( d \) and \( \delta \) independently. More
Figure 4: The procedure of solution for the vacuum Dirac equation.

complicated solutions are such that engage three neighbour spaces Λ\(_{k-1}\), Λ\(_k\) and Λ\(_{k+1}\) with 0 < k < n. Take two forms \(\alpha \in \Lambda^{k-1}\) and \(\beta \in \Lambda^{k+1}\) and set \(\psi = \alpha + \beta\). Then the vacuum Dirac equation, under splitting into grades, gives the system

\[
\begin{align*}
\delta \alpha &= 0 \\
\delta \beta &= 0 \\
d\alpha - \delta \beta &= 0 \\
d\beta &= 0
\end{align*}
\]

If one take \(\alpha \in \mathcal{E}^{k-1}\) then also \(\alpha \in \mathcal{C}^{k-1}\), and then \(\beta \in \mathcal{E}^{k+1} \cap \mathcal{C}^{k+1}\). Such case can be considered as a gauge for \(\Lambda^{k-1}\) and \(\Lambda^{k+1}\) forms.

Therefore, we can restrict ourselves to \(\alpha \in \mathcal{A}^{k-1}\) and \(\beta \in \mathcal{Y}^{k+1}\). From the first equation we must have \(\alpha \in \mathcal{A}^{k-1} \cap \mathcal{C}^{k-1}\), and from the last one \(\beta \in \mathcal{Y}^{k+1} \cap \mathcal{E}^{k+1}\).

Then, \(d\alpha \in \mathcal{C}^{k}\) and \(\delta \beta \in \mathcal{E}^{k}\) and the third equation implies \(d\alpha = \delta \beta \in \mathcal{H}^{k} = \mathcal{E}^{k} \cap \mathcal{C}^{k}\). This procedure of generating solutions for the vacuum Dirac equation is presented in Fig. 4

A similar procedure can be applied to the solutions of the vacuum anti-Dirac equation \((D\psi = 0)\). The gauge modes are almost antiharmonic forms \(\psi \in \bar{\mathcal{H}}\). The non-gauge solutions are of the form \(\psi = \alpha + \beta\), where \(\alpha \in \mathcal{E}^{k-1} \cap \mathcal{Y}^{k-1} \subset \Lambda^{k-1}\) and \(\beta \in \mathcal{C}^{k+1} \cap \mathcal{A}^{k+1} \subset \Lambda^{k+1}\) with the condition \(h\alpha = H\beta \in \bar{\mathcal{H}} \subset \Lambda^{k}\).

There are no almost antiharmonic forms on a compact manifold, and therefore the anti-Laplace-Beltrami operator does not exist since it would have the empty domain.

## 6.2 Maxwell equations

As a preparation for describing the Kalb-Ramond equations in the next subsection, we provide application of the above theory to the solutions of the Maxwell system on Minkowski space \(M\).

\[
dF = 0, \quad \delta F = j,
\]

where \(F \in \Lambda^2\) and the external current is \(j \in \Lambda^1\). This current is conserved since \(\delta j = 0\).

The typical approach on a star-shaped region \(U\) is to take a potential \(A \in \Lambda^1 \subset \Lambda^1\) such that \(dA = F\), since \(F \in \mathcal{E}^2\). Then the gauge transform \(A \rightarrow A + \chi\), where \(\chi \in E^1\), that is \(\chi = df\) for some \(f \in \Lambda^0\), does not change \(F\). The second equation
is $\delta dA = (\Delta - d\delta) A = j$. Since $A \in \mathcal{A}^1$, so we can decompose $\mathcal{A}^1 = \mathcal{C}^1 \oplus \mathcal{Y}^1$. Using $\delta j = 0$ we can remove anticoexact part by fixing Lorentz gauge: $\delta A = 0$, and we obtain the wave equation $\Delta A = j$ that can be solved by standard propagator methods. Then there is still a gauge freedom $A \to A + \phi$, where $\Delta \phi = 0$.

However, simpler approach is to use the above developed theory. First, use $dF = 0$, i.e., $F \in \mathcal{E}^2$, to select, as before, $A \in \mathcal{A}^1$ such that $dA = F$. Then the second equation is $\delta dA = j$. Since the current is conserved, so $j \in \mathcal{C}^1$, and therefore, $j = \delta h j$. We have, $\delta(dA - hj) = 0$, or $\delta(F - hj) = 0$. So the solution is $F = \delta \alpha + hj$, \hfill (47)

where $\alpha \in \mathcal{Y}^3$ and is unique up to an element of $\mathcal{C}^3$. From this solution one sees that $j$ can be changed by an element of $\mathcal{Y}^1$ without affecting $F$. The additional constraint is $dF = 0 = d\delta \alpha + dhj$, i.e.,

$$\delta \alpha + hj \in \mathcal{E}^2.$$ \hfill (48)

Note that, in this approach the existence of a specific $A$ was not needed - only the fact that $F \in \mathcal{E}^2$ is sufficient. In this approach a co-potential $\alpha$ is more important and it has also gauge freedom. Moreover, the current also can be modified by $\mathcal{Y}^1$ without affecting $F$. We can also recover $A$. Since $\delta \alpha + hj \in \mathcal{E}^2$, so $dA = F = dH(\delta \alpha + hj)$, and therefore,

$$A = df + H(\delta \alpha + hj),$$ \hfill (49)

where $f \in \Lambda^0$.

This approach is simpler than that presented in [10] (Chapter 9), since we have the complete theory of (anti)exact and (anti)coexact forms at our disposal.

The full picture is well visible when we rewrite the system (46) using the Dirac operator [11]

$$DF = -j.$$ \hfill (50)

Then we can split $F = \psi + \gamma$, where $\psi = \alpha + \beta$ such that $D\psi = 0$ is the solution of the vacuum Dirac equation with $\alpha \in \mathcal{C}^1$, $\beta \in \mathcal{E}^3$, and $\gamma \in \mathcal{E}^2$ is the solution of the nonhomogenous Dirac equation $D\gamma = -j$, that is $d\gamma = 0$ and $\delta \gamma = j$.

Note that if it would be that $j \in \Lambda^3$ (hypothetical magnetic monople current) and the equations would be $dF = j$, $\delta F = 0$, then the procedure is similar as above with restriction $F \in \mathcal{C}^2$. Since now $dj = 0$, so $j = dHj$ and the first equation is $d(F - Hj) = 0$, which gives $F = d\alpha + Hj$ for $\alpha \in \Lambda^3$ with the additional constraint $\delta(d\alpha + Hj) = 0$, i.e., $d\alpha + Hj \in \mathcal{C}^2$. Since $\delta F = 0$ so by the Poincaré lemma there exists $A \in \Lambda^3$ such that $F = \delta A$. Then the solution for $A$ is $A = \delta \beta + h(d\alpha + Hj)$ for some $\beta \in \Lambda^4$. This is dual to the classical electrodynamics presented above.
6.3 Kalb-Ramond equations

The Kalb-Ramond equations [17, 27] were postulated for describing charged bosonic string and unlike Electrodynamics is the theory of a two-form $F$, they are equations for three-form. The literature on the subject is vast, including both physical variations, e.g., [20, 9] and generalizations of an idea of using $p$-forms, e.g., [4, 15, 16, 26, 14].

The equations have the following form

$$dK = 0, \quad \delta K = J,$$

where $K \in \Lambda^3$ and $J \in \Lambda^2$. This can be further generalized to $p$-form electrodynamics [16], but we restrict ourselves to this simple example, since extension to different cases is straightforward.

From the first equation we have that $K \in \mathcal{E}^3$ and therefore there is a Kalb-Ramond field $B \in \Lambda^2$ such that $dB = K$. $B$ is defined up to $\mathcal{E}^2$, therefore we can chose it as $B \in \mathcal{A}^2$. From the second equation $J \in \mathcal{C}^2$ and therefore $J = \delta h J$ and so $\delta(K - h J) = 0$. We get, analogously to the electrodynamics, that $K = \delta \beta + h J$, where $\beta \in \Lambda^4$ is defined up to gauge $\mathcal{C}^4$. The constraint is $\delta \beta + h J \in \mathcal{E}^3$.

The equations (51) can be written in the Dirac form

$$DK = -J,$$

with the solution $K = \psi + \gamma$, where $D\psi = 0$ and $\delta \gamma = J$, as in the case of Maxwell equations.

Finally, we can couple Kalb-Ramond field with the Maxwell equation. It is possible since $B \in \mathcal{A}^2$ up to closed forms, and $F \in \mathcal{E}^2$. Therefore $F$ is a gauge field for $B$. We define [27]

$$R = B + F,$$

which is possible since there is unique decomposition $\Lambda^2 = \mathcal{E}^2 \oplus \mathcal{A}^2$. Then we have $\delta R = \delta F + \delta B = j$ and therefore $\delta B = 0$ by Maxwell equations [16]. So $B \in \mathcal{A}^2 \cap \mathcal{C}^2$. Taking exterior derivative $dR = dB = K$, and then $\delta K = \delta dB = J$.

Using the Dirac operator, the Kalb-Ramond-Maxwell system can be written in the compact form

$$DR = K - j, \quad DK = -J,$$

and can be analyzed similarly to the Maxwell equations.

6.4 Cohomotopic fermionic harmonic oscillator

In [18] there was presented a homotopy analogy of a fermionic quantum harmonic oscillator and its relation to Bittner’s calculus of abstract derivative and integral [5]. Since operators $\delta, h$ are analogous to $d$ and $H$, therefore, we can define the co-version of this equation. The hamiltonian operator is

$$\bar{H} := h \delta - \delta h,$$
and the anticommutator relation is played by nilpotency of \( \delta, h \) and the Homotopy Invariance Formula \([14]\) for \( \Lambda^r \), where \( 0 < r < n \). Then the eigenvalue problem is

\[
\tilde{H}\omega = \lambda \omega, \quad \omega \in \Lambda^r
\]

where \( \lambda \in \mathbb{R} \) are eigenvalues. As in \([18]\), the eigenvalues are \( \lambda = \pm 1 \) and eigenvectors are coexact and anticoexact forms.

7 Conclusions

The theory of anticoexact forms in analogy to antiexact forms of \([10]\) was developed in full detail. Then the relation to de Rham theory on compact manifolds was described. The most useful is the relation to Clifford algebra that allows us to solve the vacuum Dirac equation using (anti)(co)exact decomposition. Finally, the application of this decomposition in solving Maxwell equations of classical electrodynamics and Kalb-Ramond equations of bosonic string theory was presented. Moreover, (anti)(co)exact decomposition allows to trace all ingredients of the solutions of these and other similar equations.

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