A General Framework to Analyze Stochastic Linear Bandit

Nima Hamidi∗ Mohsen Bayati†

February 13, 2020

Abstract

In this paper we study the well-known stochastic linear bandit problem where a decision-maker sequentially chooses among a set of given actions in $\mathbb{R}^d$, observes their noisy reward, and aims to maximize her cumulative expected reward over a horizon of length $T$. We introduce a general family of algorithms for the problem and prove that they are rate optimal. We also show that several well-known algorithms for the problem such as optimism in the face of uncertainty linear bandit (OFUL) and Thompson sampling (TS) are special cases of our family of algorithms. Therefore, we obtain a unified proof of rate optimality for both of these algorithms. Our results include both adversarial action sets (when actions are potentially selected by an adversary) and stochastic action sets (when actions are independently drawn from an unknown distribution). In terms of regret, our results apply to both Bayesian and worst-case regret settings. Our new unified analysis technique also yields a number of new results and solves two open problems known in the literature. Most notably, (1) we show that TS can incur a linear worst-case regret, unless it uses inflated (by a factor of $\sqrt{d}$) posterior variances at each step. This shows that the best known worst-case regret bound for TS, that is given by [AG13, AL+17] and is worse (by a factor of $\sqrt{d}$) than the best known Bayesian regret bound given by [RVR14] for TS, is tight. This settles an open problem stated in [RRK+18]. (2) Our proof also shows that TS can incur a linear Bayesian regret if it does not use the correct prior or noise distribution. (3) Under a generalized gap assumption and a margin condition, as in [GZ13], we obtain a poly-logarithmic (in $T$) regret bound for OFUL and TS in the stochastic setting. The result for OFUL resolves an open problem from [DHK08].

1 Introduction

In a bandit problem, a decision-maker sequentially chooses actions from given action sets and receives rewards corresponding to the selected actions. The goal of the decision-maker, also known as the policy, is to maximize the (cumulatively obtained) reward by utilizing the history of the previous observations. This paper considers a variant of this problem, called stochastic linear bandit, in which all actions are elements of $\mathbb{R}^d$ for some integer $d > 0$ and the expected values of the rewards depend on the actions through a linear function. We also let the action sets change over time. The classical multi-armed bandit (MAB) and the $k$-armed contextual multi-armed bandit are special cases of this problem.

Since its introduction by [AL99], the linear bandit problem has attracted a great deal of attention. Several algorithms based on the idea of upper confidence bound (UCB), due to [LR85], have been proposed and analysed (notable examples are [Auc03, DHK08, RT10, AYPS11]). The best known regret bound for these algorithms is $O(d\sqrt{T}\log^{3/2}T)$ which matches the existing lower bounds up to logarithmic factors [DHK08, RT10, Sha15, Lat15, LS19]. The best known algorithm in this family is the optimism in the face of uncertainty linear bandit (OFUL) algorithm by [AYPS11].

A different line of research examines the performance of Thompson sampling (TS), a Bayesian heuristic due to [Tho33] that employs the posterior distribution of the reward function to balance exploration and

∗Department of Statistics, Stanford University, hamidi@stanford.edu
†Graduate School of Business, Stanford University, bayati@stanford.edu
exploitation. TS is also known as posterior sampling. [RVR14, RVR16] and [DVR18] proved an $\mathcal{O}(d\sqrt{T \log T})$ upper bound for the \textit{Bayesian} regret of TS, thereby indicating its near-optimality. The best thus-far known worst-case regret bound for TS, however, is given by [AG13, AL+17] which is worse than the previous bounds by a factor of $\sqrt{d}$. As it is stated in Section 8.2.1 of [RRK+18], it is an open question whether this extra factor can be eliminated by a more careful analysis.

In addition, when there is a gap $\delta$ between the expected rewards of the top two actions, OFUL and TS are shown to have a regret with a poly($\log T$)/$\delta$ dependence in $T$ instead of $\sqrt{T \log T}$. According to [DHK08], it remains an open problem whether this result extends to the cases when $\delta$ can be 0. We defer to [LS19], and references therein, for a more thorough discussion.

On the other hand, in a subclass of the linear bandit problem known as linear $k$-armed contextual bandit, [GZ13] considered a more general version of the gap assumption, a certain type of margin condition for the action set, and proposed a novel extension of the $\epsilon$-greedy algorithm. Their OLS Bandit algorithm explicitly allocates a fraction of rounds to each arm, and uses these \textit{forced} samples to discard obviously sub-optimal arms. They show that this filtering approach can lead to a near-optimal regret bound grows logarithmically in $T$. [BB20] adapted this idea to the setting with very large $d$, and [HBG19] extended that further to when both the number of arms $k$ and $d$ are large. In this paper, we demonstrate that a major generalization of this idea can be used to obtain a unifying technique for analyzing all of the above algorithms, that not only recovers known results in the literature, but also yields a number of new results, and notably, solves the aforementioned two open problems. To be explicit, the main contributions of this paper are:

1. We propose a general family of algorithms, called Two-phase Bandit, for the stochastic linear bandit problem and prove that they are rate optimal. We also show that TS, OFUL, and OLS Bandit are special cases of this family. Therefore, we obtain a universal proof of rate optimality for all of these algorithms, in both Bayesian or worst-case regret settings.

2. We consider the same generalized gap assumption as in [GZ13], that $\delta > 0$ with positive probability, and obtain a poly-logarithmic (in $T$) gap-dependent regret bound for all of the above algorithms, when the action sets are independently drawn from an unknown distribution. To the best of our knowledge, this result is new for OFUL and TS.

3. We show that TS can incur a linear worst-case regret, unless it inflates, by a factor of $\sqrt{d}$, variance of the posterior at each step. Therefore, the best known worst-case regret bound for TS, given by [AG13, AL+17], is tight. This resolves the aforementioned open problem in [RRK+18].

4. Our proof also shows that TS is vulnerable (can incur a linear Bayesian and worst-case regret) if it uses an incorrect prior distribution for the unknown parameter vector of the linear reward function, or when it uses an incorrect noise distribution.

5. As a byproduct of our analyses in 3-4, we obtain a set of conditions under which (a) TS is rate optimal and (b) we can shrink the confidence sets of OFUL by a factor $\sqrt{d}$, without impacting its regret.

\textbf{Organization.} We start by introducing the notation and main assumptions in §2. In §3 we introduce the Two-phase Bandit algorithm and prove that it is rate optimal. In §4 we introduce the ROFUL algorithm, a special case of the Two-phase Bandit algorithm, and in §5 we show that OFUL and TS are special cases of the ROFUL algorithm. Finally, in §6, we prove that TS can incur linear regret in the worst-case or when it does not have correct information on the prior or the noise distribution.

\section{Setting and notations}

For any positive integer $n$, we denote $\{1, 2, \cdots, n\}$ by $[n]$. Letting $\Sigma$ be a positive semi-definite matrix, by $\|X\|_\Sigma$ we mean $\sqrt{X^T \Sigma X}$ for any vector of suitable size. By a \textit{grouped linear bandit} (GLB) problem, we mean a tuple $(P_{\Theta^*}, (\mathbf{V}_t)_{t=1}^T, (\mathbf{x}_t)_{t=1}^T, (\mathbf{Z}_t)_{t=1}^T, (\mathbf{e}_t)_{t=1}^T)$ where:

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1. $\mathcal{P}_\Theta$ is the prior distribution of a parameter vector $\Theta^*$ on $\mathbb{R}^{kd}$.

2. $(V_i)_{i=1}^k$ consists of $k$ orthogonal $d$-dimensional subspaces of $\mathbb{R}^{kd}$. By abuse of notation, we write $V_i$ to also denote the projection matrix from $\mathbb{R}^{kd}$ onto $V_i$.

3. $(X_i)_{i=1}^T$ are random compact subsets of $\bigcup_i V_i$.

4. $(Z_i)_{i=1}^T$ are random objects passed to (randomized) policies to function.

5. $(\varepsilon_i)_{i=1}^T$ is a sequence of independent mean-zero $\sigma^2$-sub-Gaussian random variables.

The main difference between our model and the common linear bandit formulation (for example, as defined in [AYPS11]) is the introduction of $V_i$'s. Loosely speaking, we can consider each $V_i$ as a copy of $\mathbb{R}^d$; thus, $\mathbb{R}^{kd} = \mathbb{R}^d \oplus \mathbb{R}^d \oplus \cdots \oplus \mathbb{R}^d$. In this case, our assumption on the action sets demands each action to have non-zeros entries in only one of the copies of $\mathbb{R}^d$. Our problem can be regarded as a $kd$-dimensional instance of the ordinary (un-grouped) linear bandit; however, we will see in the next sections that this additional structure lets us improve the regret bound by a factor of $k$ in the gap-dependent setting and a factor of $\sqrt{k}$ in the gap-independent setting. Three interesting special cases of this model are:

1. When $d = 1$ and $X_i = \{e_1, e_2, \cdots, e_k\}$ for all $t$, our problem reduces to a simple multi-armed bandit problem.

2. When $k, d > 1$, and each action set $X_i$ contains exactly $k$ copies of a vector $X_i \in \mathbb{R}^d$ (one in each $V_i$), the problem is called $k$-armed contextual bandit.

3. When $k = 1$, we get the ordinary stochastic linear bandit problem.

The optimal and selected actions at time $t$ are denoted by $X^*_t \in X_i$ and $\tilde{X}_t \in X_i$ respectively. The corresponding reward $r_t$ is then revealed to the policy where $r_t := \langle \Theta^*, \tilde{X}_t \rangle + \varepsilon_t$. We denote the history of observations up to time $t$ by $H_t$. More precisely, we define

$$H_0 := \emptyset \quad \text{and} \quad H_t := H_{t-1} \cup \left\{ (X_i, \tilde{X}_t, Z_t, r_t) \right\}.$$  

In this model, a policy $\pi$ is formally defined as a deterministic function that maps $(X_t, H_{t-1}, Z_t)$ to an element of $X_i$. We emphasize that our definition of policies includes randomized policies, as well. Indeed, the random objects $Z_t$'s are the source of randomness used by a policy.

The performance measure for evaluating the policies is the standard cumulative Bayesian regret defined as

$$\text{Regret}(T, \pi, \mathcal{P}_\Theta) := \sum_{i=1}^T \mathbb{E} \left[ \sup_{X \in X_t} \langle \Theta^*, X \rangle - \langle \Theta^*, \tilde{X}_t \rangle \right].$$

The expectation is taken with respect to the entire randomness in our model, including the prior distribution. Although we describe our setting in a Bayesian fashion, our results cover fixed $\Theta^*$ setting by defining the prior distribution to be the one with a point mass at $\Theta^*$.

### 2.1 Action sets

In the next sections, we derive our regret bounds for various types of action sets, thereby dedicating this subsection to the definitions and notations we will use for the action sets. We start off by defining the extremal points of an action set.

**Definition 1** (Extremal points). For an action set $X$, define its extremal points $E(X)$ to be all $X \in X$ for which there are no $\{Y_1, \cdots, Y_n\} \subseteq X \setminus \{X\}$ and $\{c_1, \cdots, c_n\} \subset [0, 1]$ satisfying

$$X = \sum_{i=1}^n c_i Y_i \quad \text{and} \quad \sum_{i=1}^n c_i = 1.$$
The importance of this definition is that all the algorithms studied in this paper only choose extremal points in action sets. This observation implies that the rewards attained by any of these algorithms, and an action set \( \mathcal{X} \), belong to the reward profile of \( \mathcal{X} \) defined by
\[
\Pi_{\mathcal{X}} := \{ \langle X, \Theta^* \rangle : X \in \mathcal{E}_X \}.
\]

The maximum attainable reward and gap of an action set \( \mathcal{X} \) for the parameter vector \( \Theta^* \in \mathbb{R}^{kd} \) are defined respectively as
\[
M_{\mathcal{X}} := \sup(\Pi_{\mathcal{X}}) \quad \text{and} \quad \Delta_{\mathcal{X}} := M_{\mathcal{X}} - \sup(\Pi_{\mathcal{X}} \setminus \{M_{\mathcal{X}}\}).
\]

For any \( z \geq 0 \), write
\[
\mathcal{X}^z := \{ X \in \mathcal{X} : \langle \Theta^*, X \rangle \geq M_{\mathcal{X}} - z \}.
\]

In the above notations, for the sake of simplicity, we may use subscript \( t \) to refer to \( \mathcal{X}_t \). For instance, by \( M_t \) we mean \( M_{\mathcal{X}_t} \). We now define a gapped problem as follows:

**Definition 2** (Gapped problem). We call a GLB problem gapped if for some \( \delta, q > 0 \) the following inequality holds:
\[
P(\Delta_t \geq \delta) \geq q \quad \text{for all } t \in [T]. \tag{1}
\]

Moreover, for a fixed gap level \( \delta \), we let \( G_t \) be the indicator of the event \( \{\Delta_t \geq \delta\} \).

**Remark 1.** Note that all problems are gapped for all \( \delta > 0 \) and \( q = 0 \). This observation will help us obtain gap-independent bounds.

**Remark 2.** Our notion of gap is more general than the well-known gap assumption in the literature (e.g., as in [AYPS11]) which always holds which means \( q \) would be 1.

We conclude this section by defining near-optimal space followed by diversity condition and margin condition. The two conditions will enable us to enhance a \( T\delta \) term in the our regret bound, that will appear in Eq. (6), to an expression that grows sub-linearly in terms of \( T \).

**Definition 3** (Near-optimal space). Let \( l \leq k \) be the smallest number such that there exists \( I \subseteq [k] \) with \( |I| = l \) and
\[
P \left( \mathcal{X}_t^I \subseteq \bigoplus_{j \in I} V_j \right) = 1 \quad \text{for all } t \in [T],
\]

Let us denote \( W := \bigoplus_{j \in I} V_j \), and as before, we also treat \( W \) as the projection of \( \mathbb{R}^{kd} \) onto the subspace \( W \).

**Remark 3.** The main purpose of this notion is to handle sub-optimal arms in the special case of \( k \)-armed contextual bandit. One might harmlessly assume that \( W = \mathbb{R}^{kd} \) (or equal to the identity function if viewed as an operator) and follow the rest of the paper.

**Definition 4** (Diversity condition). We say that a GLB problem satisfies the diversity condition with parameter \( \gamma_{\text{max}} \) if \( \mathcal{X}_t \) is independent of \( H_{t-1} \) and
\[
\lambda_{\text{min}} \left( W^T \mathbb{E} \left[ X_t^* X_t^{*\top} \cdot G_t \right] W \right) \geq \gamma_{\text{max}} \quad \text{for all } t \in [T]. \tag{2}
\]

**Definition 5** (Margin condition). In a GLB problem, the margin condition holds if
\[
P(\Delta_t \leq z) \leq c_0 e^{z^\alpha} \quad \text{for all } t \in [T] \text{ and } 0 \leq z \leq \delta. \tag{3}
\]

where \( \alpha, c_0 > 0 \) are two constants.
Algorithm 1 Two-phase Bandit

**Input:** Forced-sampling rule $F$, blurry selector $B$, and vivid selector $V$.

1. for $t = 1, 2, \ldots$ do
2. Observe $X_t$,
3. if $F(X_t | H_{t-1}) \neq \text{Null}$ then
4. $\tilde{X}_t \leftarrow F(X_t | H_{t-1})$
5. else
6. $\tilde{X}_t \leftarrow B(X_t | H_{t-1})$
7. $\tilde{X}_t \leftarrow V(\tilde{X}_t | H_{t-1})$
8. end if
9. end for

### 3 Two-phase bandit algorithm

In this section, we describe the Two-phase Bandit algorithm, an extension of the OLS Bandit algorithm, that was introduced by [GZ13] for the special case of $k$-armed contextual bandit setting, to our more general grouped linear bandit problem. Two-phase Bandit algorithm (presented in Algorithm 1) has two separate phases to deal with the exploration-exploitation trade-off. At each time $t$, a forced-sampling rule $F$ determines which arm to pull (i.e., when $F$ is an element of $X_t$) or it refuses to pick one which implies that the best arm should be chosen by exploiting the information gathered thus far (i.e., when $F = \text{Null}$). The forced-sampling rule is allowed to depend on the history $H_{t-1}$ as well as the current action set $X_t$. The notation $F(X_t | H_{t-1})$ expresses this dependence explicitly.

In the exploitation phase, two selectors are used to decide which action to choose. The blurry selector $B(X_t, H_{t-1})$ first selects a candidate set $\tilde{X}_t \subseteq X_t$ which is later passed to the vivid selector $V(\tilde{X}_t, H_{t-1})$ to pick a single action from. The idea behind this architecture is that the blurry selector eliminates all the actions that are suboptimal with a constant margin with high probability. The name “blurry” indicates the low accuracy of this selector. The vivid selector, on the other hand, should become more accurate as $t$ grows with high probability. The diversity condition is the assumption that plays a crucial role in proving this type of result. We now state and briefly discuss the assumptions that need to be met such that our results are valid.

**Assumption 1** (Boundedness). There exist constants $x, \theta > 0$ such that $\|\Theta^*\|_2 \leq \theta$ and $\|X\|_2 \leq x$ for all $X \in X_t$ and for all $t \in [T]$ almost surely.

The next assumption concerns how much regret the forced-sampling rule incurs to assure that the blurry selector works properly.

**Assumption 2** (Forced-sampling cost). Let $F_t$ be the indicator function for the event $\{F(X_t | H_{t-1}) \neq \text{Null}\}$. Then there exists some $c_1 \geq 0$ such that

$$E \left[ \sum_{t \in [T]} \left( M_t - \langle \Theta^*, \tilde{X}_t \rangle \right) \cdot F_t \right] \leq c_1. \quad (4)$$

We are now ready to formalize what it means for the blurry selector to work properly.

**Assumption 3** (Blurry selector bound). Let $\tilde{X}_t = B(X_t | H_{t-1})$. There exists some $c_2 \geq 0$ such that for all $t \in [T]$, we have that

$$P(\tilde{X}_t \not\subseteq X_t) \leq c_2^2 \frac{t^2}{T}. \quad (5)$$

The above assumptions are sufficient to prove a bound which scales with $T$ as $T\delta(1 - q)$. Furthermore, we will discuss in §4, that $\delta$ can be tuned such that the regret grows as $O(\sqrt{T})$. Furthermore, we can obtain
even sharper regret bounds, under additional assumptions that will be stated below. For example, the vivid selector should satisfy certain properties. In order to highlight the main idea, we restrict our attention to a rather concrete scenario in which these properties hold and defer the most general cases to a longer version of the paper. More specifically, we assume that the vivid selector is a \textit{greedy} selector defined as follows.

**Definition 6 (Greedy selector).** Let $\tilde{\Theta}_t : H_{t-1} \rightarrow \mathbb{R}^{kd}$ be an estimator for $\Theta^*$. By the greedy selector with respect to $\tilde{\Theta}_t$, we mean a selector $V(\tilde{X}_t | H_{t-1})$ given by

$$V(\tilde{X}_t | H_{t-1}) \in \arg \max_{X \in \tilde{X}_t} \langle \tilde{\Theta}_{t-1}, X \rangle.$$ 

**Definition 7 (Reasonableness).** Let $\lambda, \rho > 0$ be fixed. For an estimator $\tilde{\Theta}_t$, define

$$\Sigma_t := \left( \lambda I + \sum_{j=1}^t \tilde{X}_j \tilde{X}_j^\top \right)^{-1}$$

and

$$C_t := \left\{ \Theta \in \mathbb{R}^{kd} : \max_{i \in [k]} \| V_i(\Theta - \tilde{\Theta}_t) \|_{\Sigma_t^{-1}} \leq \rho \right\}.$$ 

The estimator is called \textit{reasonable} if

$$P(\Theta^* \notin C_t \text{ and } F_t = 0) \leq \frac{c_3}{T^3},$$

for some $c_3 \geq 0$.

**Assumption 4.** The vivid selector $V(\tilde{X}_t, H_{t-1})$ is a greedy selector for a reasonable estimator $\tilde{\Theta}_t^V$ for all $t \in [T]$.

Our last assumption demands the selected actions to be diverse in the near-optimal space. This condition is not a mere property of a model or a policy; it is a characteristic of a policy in combination with a model.

**Assumption 5 (Linear expansion).** We say that linear expansion holds if

$$P \left( \| W^\top \Sigma_t W \|_{\text{op}} \geq \frac{c_5^2}{t} \right) \leq \frac{c_4}{t^2}$$

for some constants $c_5, c_4 > 0$ and all $t \in [T]$.

Now, with all these assumptions, we are ready to state our main regret bound that results in both gap-independent and gap-dependent settings. The result is also applicable to the cases that the action selected by an adversary.

**Theorem 1.** If Assumptions 1-3 hold, the cumulative regret of Algorithm 1 (denoted by policy $\pi^{TP}$) satisfies the following inequality:

$$\text{Regret}(T, \pi^{TP}, P_{\Theta^*}) \leq c_1 + 4x\theta c_2 + T\delta(1 - q). \quad (6)$$

Furthermore, under Assumptions 1-5 and the margin condition (3), we have

$$\text{Regret}(T, \pi^{TP}, P_{\Theta^*}) \leq c_1 + 4x\theta c_2 + 2\delta(c_3 + c_4) + c_0(2x\rho c_5)^{\alpha+1} \left( 1 + \int_1^T t^{-\frac{\alpha+1}{2}} dt \right). \quad (7)$$

**Remark 4.** Theorem 1 provides a gap-independent bound, Eq. (6), as well as a gap-dependent bound, Eq. (7). For the special case of OLS Bandit, when $\alpha = 1$, the latter produces an $O(\log^2 T)$ bound on the regret. Note that we can follow the same peeling argument as in [GZ13, BB20] and obtain an $O(\log T)$ for OLS Bandit. However, this peeling argument may not apply for more general algorithms that we will be analyzing in next sections.
**Proof of Theorem 1.** We split the regret of the algorithm into the following three cases. We will then bound each term separately.

(a) Forced-sampling phase $F_t = 1$,

(b) When $F_t = 0$ and $B_t^B = 1$, where $B_t^B$ is the indicator function for $\bar{X}_t \not\subseteq X_t^\delta$,

(c) When $F_t = 0$ and $B_t^B = 0$.

By $R_T^{(a)}$, $R_T^{(b)}$ and $R_T^{(c)}$, we denote the regrets of the above items up to time $T$. Clearly, we have that

$$\text{Regret}(T, \pi^{TP}, P_{\Theta^*}) = R_T^{(a)} + R_T^{(b)} + R_T^{(c)}.$$ 

Assumption 2 gives us an upper bound for $R_T^{(a)}$

$$R_T^{(a)} \leq c_1.$$ 

Next, notice that the maximum regret that can occur in each round is bounded above by

$$\sup_{X, X' \in X_t} |\langle X - X', \Theta^* \rangle| \leq 2\|\Theta^*\|_2 \cdot \sup_{X \in X_t} \|X\|_2 \leq 2\theta.$$ 

It follows from the definition of $B_t^B$ and the inequality (5) in Assumption 3 that the number of times that $B_t = 1$ and $F_t = 0$ is controlled by

$$E\left[\sum_{t=1}^T (1 - F_t) B_t^B\right] = \sum_{t=1}^T P(B_t^B = 1 \text{ and } F_t = 0) \leq \sum_{t=1}^T \frac{c_2}{t^2} \leq 2c_2.$$ 

It thus entails that

$$R_T^{(b)} \leq 2\theta E\left[\sum_{t=1}^T (1 - F_t) B_t^B\right] \leq 4\theta c_2.$$ 

From the definition of $B_t^B$, we can infer that whenever $F_t = 0$ and $B_t = 0$, the regret at time $t$ can not exceed $\delta$, and in the case that the action set is gapped (at this level $\delta$), the regret is equal to zero. Using this observation, we get that

$$R_T^{(c)} = E\left[\sum_{t=1}^T (M_t - r_t) \cdot (1 - B_t^B)(1 - F_t)\right] = E\left[\sum_{t=1}^T (M_t - r_t) \cdot (1 - B_t^B)(1 - F_t)(1 - G_t)\right] \leq \delta \sum_{t=1}^T E((1 - B_t^B)(1 - F_t)(1 - G_t)) \leq \delta \sum_{t=1}^T E(1 - G_t).$$
As in the proof of (6), the regret would be zero if 
\[ \Delta \]
which completes the proof of (6).

In order to prove (7), we use Assumptions 4-5 and the margin condition to bound \( R_T^{(c)} \) in a different way. Define
\[
E_t^Y := \begin{cases} 
1 & \text{if } \Theta^* \not\in C_t \text{ or } \|W^T \Sigma_t W\|_{op} \geq \frac{c^2}{T}, \\
0 & \text{otherwise.}
\end{cases}
\]

The key idea in bounding \( R_T^{(c)} \) is that under \( E_t^B = 0, \tilde{X}_t \subseteq X_t^c \); thereby, we get, with probability 1,
\[ \tilde{X}_t \subseteq W. \]

Hence, we have that
\[
R_T^{(c)} = E \left[ \sum_{t=1}^{T} \left( M_t - \langle \Theta^*, \tilde{X}_t \rangle \right) \cdot (1 - B_t^B)(1 - F_t) \right]
= E \left[ \sum_{t=1}^{T} \langle \Theta^*, X_t^* - \tilde{X}_t \rangle \cdot (1 - B_t^B)(1 - F_t) \right]
= E \left[ \sum_{t=1}^{T} \langle \Theta^*, X_t^* - \tilde{X}_t \rangle \cdot (1 - B_t^B)(1 - F_t) \cdot \left( E_t^Y + (1 - E_t^Y) \right) \right]
\leq E \left[ \sum_{t=1}^{T} \langle \Theta^*, X_t^* - \tilde{X}_t \rangle \cdot (1 - B_t^B)(1 - E_t^Y) \right] + E \left[ \sum_{t=1}^{T} \delta \cdot (1 - F_t)E_t^Y \right]
\leq E \left[ \sum_{t=1}^{T} \langle \Theta^*, X_t^* - \tilde{X}_t \rangle \cdot (1 - B_t^B)(1 - E_t^Y) \right] + \delta \sum_{t=1}^{T} E_t^Y = 0, F_t = 0)
\leq E \left[ \sum_{t=1}^{T} \langle \Theta^*, X_t^* - \tilde{X}_t \rangle \cdot (1 - B_t^B)(1 - E_t^Y) \right] + 2\delta(c_3 + c_4).
\]

Note that, whenever \( E_t^B = 0 \) and \( E_t^Y = 0 \), we have
\[
\langle X_t^* - \tilde{X}_t, \Theta^* \rangle = \langle X_t^*, \Theta^* - \tilde{\Theta}_t^Y \rangle + \langle X_t^* - \tilde{X}_t, \tilde{\Theta}_t^Y \rangle + \langle \tilde{X}_t, \tilde{\Theta}_t^Y - \Theta^* \rangle
\leq \langle X_t^*, \Theta^* - \tilde{\Theta}_t^Y \rangle + \langle \tilde{X}_t, \tilde{\Theta}_t^Y - \Theta^* \rangle
\leq \left( \|X_t^*\|_2 + \|\tilde{X}_t\|_2 \right) \cdot \|\Theta^* - \tilde{\Theta}_t^Y\|_2
\leq 2\sigma \cdot \|\Theta^* - \tilde{\Theta}_t^Y\|_2
\leq 2\sigma c_5 \sqrt{\frac{t}{T}} \cdot \|\Theta^* - \tilde{\Theta}_t^Y\|_{\Sigma^{-1}}
\leq 2\sigma c_5 \sqrt{\frac{T}{t}}
\]

As in the proof of (6), the regret would be zero if \( \Delta_t \) is larger than the above. This, in turn, implies that
\[
R_T^{(c)} \leq \sum_{t=1}^{T} 2\sigma c_5 \sqrt{\frac{T}{t}} \cdot P \left( \Delta_t \leq 2\sigma c_5 \sqrt{\frac{T}{t}} \right) + 2\delta(c_3 + c_4)
\]
Algorithm 2 ROFUL

Input: Estimator \( \tilde{\Theta} \).
1: for \( t = 1, 2, \ldots \) do
2: Observe \( X_t \),
3: \( \tilde{X}_t \leftarrow \max_{X \in \mathcal{X}_t} \langle X, \tilde{\Theta}_{t-1} \rangle \)
4: end for

\[
\leq \sum_{t=1}^{T} 2x_\rho c_5 \sqrt{\frac{1}{t}} \cdot c_0 \left( 2x_\rho c_5 \sqrt{\frac{1}{t}} \right)^\alpha + 2\delta (c_3 + c_4)
\]
\[
= c_0 (2x_\rho c_5)^{\alpha+1} \sum_{t=1}^{T} t^{-\frac{\alpha+1}{2}} + 2\delta (c_3 + c_4)
\]
\[
\leq c_0 (2x_\rho c_5)^{\alpha+1} \left( 1 + \int_1^{T} t^{-\frac{\alpha+1}{2}} dt \right) + 2\delta (c_3 + c_4),
\]
which is the desired result.

4 Randomized OFUL

In this section, we present an extension of the OFUL algorithm of [AYPS11], and prove that under mild conditions it enjoys the same regret bound as the original OFUL. We call this extension Randomized OFUL (or ROFUL) and present its pseudo-code version in Algorithm 2. It receives an arbitrary estimator \( \tilde{\Theta} \), and at each round, it makes greedy decision using this estimator. We require this estimator to be reasonable (Definition 7) and optimistic (Definition 8), and in Theorem 2, we use these assumptions to provide our regret bounds for this algorithm. We now define optimism. Recall that we defined

\[
M_t = \sup_{X \in \mathcal{X}_t} \langle X, \Theta^* \rangle.
\]

Similarly, we write

\[
\tilde{M}_t := \sup_{X \in \mathcal{X}_t} \langle X, \tilde{\Theta}_{t-1} \rangle.
\]

Definition 8 (Optimism). We say that the estimator \( \tilde{\Theta}_t \) is optimistic if for some \( p \in (0, 1] \) we have

\[
P \left( \tilde{M}_t \geq M_t - \frac{\delta}{4} \mid \mathcal{X}_t, \mathcal{H}_{t-1}, \mathbb{T}_t = 1 \right) \geq p
\]  \( (8) \)

with probability at least \( 1 - \frac{c_6}{T} \) where \( c_6 \geq 0 \) is a fixed constant and \( \mathbb{T}_t \) is the indicator function for the typical event \( \tilde{\Theta}_{t-1} \in \mathcal{C}_{t-1} \).

Theorem 2. For any reasonable (Definition 7) and optimistic (Definition 8) estimator \( \tilde{\Theta} \) the corresponding ROFUL algorithm admits the following regret bound:

\[
\text{Regret}(T, \pi_{\text{ROFUL}}, \mathcal{P}_{\Theta^*}) \leq \frac{128 \rho^2 k d}{\delta p} \log \left( \lambda + \frac{T x^2}{d} \right) + T \delta (1 - q) + 4x_\theta (c_3 + c_6). \quad (9)
\]

Furthermore, if the diversity condition (2) and the margin condition (3) also hold, we have the following bound:

\[
\text{Regret}(T, \pi_{\text{ROFUL}}, \mathcal{P}_{\Theta^*}) \leq \frac{128 \rho^2 k d x_\theta}{\delta^2 p} \log(\lambda + \frac{T x^2}{d}) + \frac{4(x_\theta)^2 (c_3 + c_6)}{\delta} + c_0 \left( \frac{30x_\rho}{\sqrt{\gamma_{\min}}} \right)^{\alpha+1} \left( 1 + \int_1^{T} t^{-\frac{\alpha+1}{2}} dt \right). \quad (10)
\]
Proof. For simplicity, we introduce some new notations to keep the expressions in this proof shorter; thereby, increasing the readability. Define:

1. Well-posed action set indicator:
   \[ W_t := \mathbb{I} \left( \mathbb{P} \left( \bar{M}_t \geq M_t - \frac{\delta}{4} \bigg| \mathcal{H}_t \right) \geq p \right) \]

2. Upper and lower confidence bounds:
   \[ U_t(X) := \sup_{\Theta \in \mathcal{C}_{t-1}} \langle X, \Theta \rangle, \]
   \[ L_t(X) := \inf_{\Theta \in \mathcal{C}_{t-1}} \langle X, \Theta \rangle. \]

3. Acceptance threshold:
   \[ A_t := \sup_{X \in \mathcal{X}_t} L_t(X). \]

Our strategy is to first represent ROFUL as an instance of Two-phase Bandit, and then, verify the assumptions of Theorem 1. In order to do so, let \( F_t \) be given by
\[
F(X_t | \mathcal{H}_{t-1}) := \begin{cases} 
\tilde{X}_t & \text{if } T_t = 1, \ W_t = 1, \text{ and } \langle \Theta^*, X_t^* - \tilde{X}_t \rangle \geq \delta, \\
\text{Null} & \text{otherwise}.
\end{cases}
\]

The blurry selector is also defined as
\[
B(X_t | \mathcal{H}_{t-1}) := X_t^\delta \cup \{ \tilde{X}_t \}.
\]

We also set the vivid selector \( B(X_t | \mathcal{H}_{t-1}) \) to be the greedy selector with respect to the estimator \( \tilde{\Theta}_t \). It is straight-forward to verify that Two-phase Bandit algorithm with \((F, B, V)\) as defined in the above is equivalent to Algorithm 2. We thus need to show that the assumptions of Theorem 1 hold. We begin with computing the forced-sampling cost \( c_1 \) in Assumption 2.

\[
\mathbb{E} \left[ \langle \Theta^*, X_t^* - \tilde{X}_t \rangle \cdot F_t \right] \leq \frac{1}{\delta} \mathbb{E} \left[ (\Theta^*, X_t^* - \tilde{X}_t)^2 \cdot F_t \right] \leq \frac{1}{\delta} \mathbb{E} \left[ (2 \langle \Theta^*, X_t^* - \tilde{X}_t \rangle^2 - \delta^2) \cdot F_t \right].
\]

(11)

It follows from the definition of \( T_t \) that \( T_t = 1 \) implies
\[
\langle \Theta^*, X_t^* - \tilde{X}_t \rangle = M_t - \langle \Theta^*, \tilde{X}_t \rangle \leq M_t - L_t(\tilde{X}_t).
\]

This gives us
\[
\mathbb{E} \left[ (\Theta^*, X_t^* - \tilde{X}_t)^2 \cdot F_t \right] \leq \mathbb{E} \left[ (M_t - L_t(\tilde{X}_t))^2 \cdot F_t \right] \leq \mathbb{E} \left[ 2 (M_t - A_t)^2 + (A_t - L_t(\tilde{X}_t))^2 \right] \cdot F_t,
\]

which in combination with (11) leads to
\[
\mathbb{E} \left[ \langle \Theta^*, X_t^* - \tilde{X}_t \rangle \cdot F_t \right] \leq \frac{1}{\delta} \mathbb{E} \left[ 4 (M_t - A_t)^2 + 4 (A_t - L_t(\tilde{X}_t))^2 - \delta^2 \right] \cdot F_t.
\]

(12)

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Next, letting $U_t := I \left( M_t - A_t \geq \frac{\delta}{2} \right)$, we get

$$
\left( M_t - A_t \right)^2 \leq \frac{\delta^2}{4} + 4 \left( M_t - \frac{\delta}{4} - A_t \right)^2 \cdot U_t,
$$

which in turn yields

$$
E \left[ \left( M_t - A_t \right)^2 \cdot F_t \right] \leq E \left[ \frac{\delta^2}{4} \cdot F_t \right] + E \left[ 4 \left( M_t - \frac{\delta}{4} - A_t \right)^2 \cdot U_t T_t W_t \right]
$$

$$
= E \left[ \frac{\delta^2}{4} \cdot F_t \right] + E \left[ 4 \left( M_t - \frac{\delta}{4} - A_t \right)^2 \cdot U_t T_t W_t \left| X_t, H_{t-1}, T_t \right. \right].
$$

We deduce from optimism (8) that

$$
0 \leq E \left\{ \left( \frac{1}{p} \left( M_t - A_t \right)^2 - \left( M_t - \frac{\delta}{4} - A_t \right)^2 \right) \cdot U_t W_t T_t \left| X_t, H_{t-1}, T_t \right. \right\},
$$

almost surely. Hence, we have

$$
E \left[ \left( M_t - A_t \right)^2 \cdot F_t \right] \leq E \left[ \frac{\delta^2}{4} \cdot F_t \right] + E \left[ 4 \left( M_t - \frac{\delta}{4} - A_t \right)^2 \cdot U_t T_t W_t \left| X_t, H_{t-1}, T_t \right. \right]
$$

$$
\leq E \left[ \frac{\delta^2}{4} \cdot F_t \right] + E \left[ 4 \left( M_t - A_t \right)^2 \cdot U_t \right]
$$

$$
\leq E \left[ \frac{\delta^2}{4} \cdot F_t + 4 \left( U_t (\hat{X}_t) - A_t \right)^2 \cdot T_t \right]
$$

$$
\leq E \left[ \frac{\delta^2}{4} \cdot F_t + 4 \left( U_t (\hat{X}_t) - A_t \right)^2 \right].
$$

Substituting the above inequality into (12), we obtain

$$
E \left[ \langle \Theta^*, X^*_t - \tilde{X}_t \rangle \cdot F_t \right] \leq \frac{1}{\delta} E \left[ \frac{16}{p} \left( U_t (\hat{X}_t) - A_t \right)^2 + 4 \left( A_t - L_t (\hat{X}_t) \right)^2 \cdot F_t \right]
$$

$$
\leq \frac{16}{\delta p} E \left[ \left( U_t (\hat{X}_t) - A_t \right)^2 \right]
$$

$$
= \frac{64}{\delta p} E \left[ \left( U_t (\hat{X}_t) - M_t \right)^2 \right]
$$

$$
= \frac{64}{\delta p} E \left[ \sup_{\Theta \in C_{t-1}} \langle \tilde{X}_t, \Theta - \tilde{\Theta}_t \rangle \right].
$$

Recall we assumed that for each $X \in X_t$, there exists $j \in [k]$ such that $X \in V_j$. Now, let $j_t \in [k]$ be such that $\tilde{X}_t \in V_{j_t}$. We have that

$$
E \left[ \langle \Theta^*, X^*_t - \tilde{X}_t \rangle \cdot F_t \right] \leq \frac{64}{\delta p} E \left[ \sup_{\Theta \in C_{t-1}} \langle V_{j_t}, \tilde{X}_t, V_{j_t}, (\Theta - \tilde{\Theta}_t) \rangle \right]
$$

$$
\leq \frac{64}{\delta p} E \left[ \left\| V_{j_t}, \tilde{X}_t \right\|_{\Sigma_t} \cdot \sup_{\Theta \in C_{t-1}} \left\| V_{j_t}, (\Theta - \tilde{\Theta}_t) \right\|_{\Sigma_t} \right]
$$

$$
= \frac{64 \rho^2}{\delta p} E \left[ \left\| V_{j_t}, \tilde{X}_t \right\|_{\Sigma_t} \right].
$$
Finally, we apply Lemma 10 and Lemma 11 in [AYPS11] for each \( i \in [k] \) separately:

\[
\sum_{t=1}^{T} \| V_{jt} \tilde{X}_t \|_{\Sigma_t}^2 \cdot I(j_t = i) \leq 2 \log \left( \frac{\det V_i \Sigma_{T-1} V_i}{\det V_i \Sigma_{0}^{-1} V_i} \right)
\]

\[
\leq 2 \log \left( \lambda + \frac{T x^2}{d} \right).
\]

Therefore, we have

\[
c_1 = \sum_{i=1}^{T} E \left[ \langle \Theta^*, X_t^* - \tilde{X}_t \rangle \cdot F_t \right]
\]

\[
\leq \frac{64 \rho^2}{\delta p} \sum_{i=1}^{T} \| V_{jt} \tilde{X}_t \|_{\Sigma_t}^2
\]

\[
\leq \frac{64 \rho^2}{\delta p} \sum_{i=1}^{k} \sum_{t=1}^{T} \| V_{jt} \tilde{X}_t \|_{\Sigma_t}^2 \cdot I(j_t = i)
\]

\[
\leq \frac{64 \rho^2}{\delta p} \cdot \sum_{i=1}^{k} 2d \log \left( \lambda + \frac{T x^2}{d} \right)
\]

\[
= \frac{128 \rho^2 kd}{\delta p} \log \left( \lambda + \frac{T x^2}{d} \right).
\]

This completes the computation of the forced-sampling cost. We now need to compute \( c_2 \) in Assumption 3. In other words, we ought to find \( c_2 \) such that

\[
P \left( \tilde{X}_t \not\subseteq X_t^s \text{ and } F_t = 0 \right) \leq \frac{c_2}{t^2}.
\]

(13)

The definition of \( F(X_t | H_{t-1}) \) implies that \( F_t = 0 \) if \( T_t = 0, W_t = 0, \) or \( \langle \Theta^*, X_t^* - \tilde{X}_t \rangle < \delta \). Using the fact that \( \tilde{X}_t = B(X_t | H_{t-1}) = X_t^s \cup \{ \tilde{X}_t \} \), we realize that

\[
P \left( \tilde{X}_t \not\subseteq X_t^s \text{ and } F_t = 0 \right) = P \left( T_t = 0 \text{ or } W_t = 0 \right)
\]

\[
\leq P \left( T_t = 0 \right) + P \left( W_t = 0 \right)
\]

\[
\leq c_3 + c_6 s.
\]

From this, we conclude that \( c_2 = c_3 + c_6 \) satisfies (13). Therefore, the regret bound (9) is a direct consequence of (6) in Theorem 1.

Next, we turn to proving (10). The main step involves demonstrating that the linear expansion assumption holds for ROFUL. We first show that there exists some \( t_0 \geq 1 \) such that for all \( t \geq t_0 \), we have

\[
P \left( \| W^T \Sigma_t W \|_{op} \geq \frac{c_3}{t} \right) \leq \frac{c_4}{t^2},
\]

(14)

where \( c_4 := \) and \( c_5 := \). For any \( t \in [T] \), let \( O_t \) be a Bernoulli random variable such that

\[
P(O_t = 1 | X_t, H_{t-1}, T_t) = p, \quad \text{a.s.}
\]

and

\[
W_t = 1 \text{ and } O_t = 1 \implies \tilde{M}_t \geq M_t - \frac{\delta}{4}.
\]
The existence of these random variables is a direct consequence of the optimism assumption. The equality (14) implies that $O_t$ is independent of $(X_t, H_{t-1})$.

Next, fixing $t \geq t_0$, we have
\[
\sum_{i=\frac{t}{2}}^{t} \bar{X}_i \bar{X}_i^\top \cdot (1 - F_t)G_iO_iW_i \preceq \sum_{i=\frac{t}{2}}^{t} \bar{X}_i \bar{X}_i^\top \cdot (1 - F_t)G_i \preceq \Sigma_t^{-1}.
\]
It follows from the definition of $F_t$ and $G_t$ that
\[
\bar{X}_i \bar{X}_i^\top \cdot (1 - F_t)G_i = X_i^* X_i^* \top \cdot (1 - F_t)G_i
\]
for all $i \in [T]$. Therefore, we get that
\[
\sum_{i=\frac{t}{2}}^{t} X_i^* X_i^* \top \cdot (1 - F_t)G_iO_iW_i \preceq \Sigma_t^{-1}.
\]
and thus, letting $\Gamma_i := \bar{W}^\top X_i^* X_i^* \top \bar{W} \cdot G_iO_i$, we have
\[
\lambda_{\min} (\bar{W}^\top \Sigma_t^{-1} \bar{W}) \geq \lambda_{\min} \left( \sum_{i=\frac{t}{2}}^{t} \Gamma_i \cdot (1 - F_t)W_i \right)
\geq \lambda_{\min} \left( \sum_{i=\frac{t}{2}}^{t} \Gamma_i \cdot W_i \right) - \lambda_{\max} \left( \sum_{i=\frac{t}{2}}^{t} \Gamma_i \cdot F_tW_i \right)
\geq \lambda_{\min} \left( \sum_{i=\frac{t}{2}}^{t} \Gamma_i \cdot W_i \right) - \lambda_{\max} \left( \sum_{i=\frac{t}{2}}^{t} \Gamma_i \cdot F_t \right)
\geq \lambda_{\min} \left( \sum_{i=\frac{t}{2}}^{t} \Gamma_i \cdot W_i \right) - \lambda_{\max} \left( \sum_{i=\frac{t}{2}}^{t} \Gamma_i \cdot W_i \right).
\]
We now bound each term separately. We first note that
\[
\mathbb{P} \left( \sum_{i=\frac{t}{2}}^{t} \Gamma_i \cdot W_i \neq \sum_{i=\frac{t}{2}}^{t} \Gamma_i \right) \leq \mathbb{P} \left( \exists i \geq \frac{t}{2} : W_i = 0 \right) \leq \frac{12c_0}{t^2}.
\]
The above inequality implies that we can safely ignore $W_i$ terms. This is helpful in our proof as $\Gamma_i$’s are independent. Next, we need to prove that the smallest singular value of $\sum_{i=\frac{t}{2}}^{t} \Gamma_i$ grows linearly with high probability. This is a direct consequence of tail bound inequalities for random matrices. Finally, a simpler version of our analysis for the forced-sampling cost gives us a deterministic bound for $\sum_{i=\frac{t}{2}}^{t} F_tG_iO_iW_i$ which completes the proof of this Theorem.

5 Examples

5.1 Bayesian regret for Thompson sampling

In this section, we prove that Thompson sampling (TS) is an instance of the ROFUL algorithm and therefore, applying Theorem 2, we recover the Bayesian regret bound of [RVR14] for TS.

Let $\mathcal{P}_{\Theta^*}$ be the prior distribution on $\mathbb{R}^{kd}$ such that
\[
\mathbb{P} \left( \max_{i \in [k]} \|V_i \Theta^*\|_2 \leq \theta \right) = 1.
\]
and assume that
\[ P(\forall t \in [T], \forall X \in \mathcal{X}_t : \|X\|_2 \leq x) = 1. \]

Furthermore, let \( (\varepsilon_t)_{t=1}^T \) be \( \tau^2 \)-sub-Gaussian. Then, Thompson sampling using the corresponding update rule is an instance of ROFUL by taking \( \Theta_t \) to be the sampled vector at time \( t \). We now verify the assumptions of Theorem 2 to derive regret bounds for this policy. We first show that this estimator is reasonable with
\[
\lambda := x^2, \quad \rho := 2r \sqrt{d \log (2kt^2 (1 + t))} + 2\sqrt{\lambda}, \quad \text{and} \quad c_3 := 1.
\]

To see this, define
\[
\tilde{\Theta}_t := \left( \lambda I + \sum_{t=1}^T \bar{X}_t \bar{X}_t^\top \right)^{-1} \cdot \left( \sum_{t=1}^T \bar{X}_t r_t \right).
\] (15)

Theorem 2 in [AYPS11] yields
\[
P\left( \max_{i \in [k]} \left\| V_i (\tilde{\Theta}_t - \Theta^*) \right\|_{\Sigma_t^{-1}} \leq \frac{\rho}{2} \left| \Theta^* \right| \right) \leq \sum_{i=1}^k P\left( \left\| V_i (\tilde{\Theta}_t - \Theta^*) \right\|_{\Sigma_t^{-1}} \leq \frac{\rho}{2} \right|
\]
\[
\leq \sum_{i=1}^k \frac{1}{2kt^2}
\]
\[
= \frac{1}{2t^2}
\]
for all \( \Theta^* \) with \( \max_i \left\| V_i \Theta^* \right\| \leq \theta \). From this we can deduce that
\[
P\left( \max_{i \in [k]} \left\| V_i (\tilde{\Theta}_t - \Theta^*) \right\|_{\Sigma_t^{-1}} \leq \frac{\rho}{2} \right) \leq \frac{1}{2t^2}. \tag{16}
\]

Next, we prove a similar tail bound for \( \tilde{\Theta}_t \). In order to do so, we notice that \( \tilde{\Theta}_t \) and \( \Theta^* \) are identically distributed when conditioned on \( \mathcal{H}_{t-1} \). Hence, we have
\[
P\left( \max_{i \in [k]} \left\| V_i (\tilde{\Theta}_t - \tilde{\Theta}_t) \right\|_{\Sigma_t^{-1}} \leq \frac{\rho}{2} \right)
\]
\[
= \mathbb{E} \left[ I \left\{ \max_{i \in [k]} \left\| V_i (\tilde{\Theta}_t - \tilde{\Theta}_t) \right\|_{\Sigma_t^{-1}} \leq \frac{\rho}{2} \right\} \right]
\]
\[
= \mathbb{E} \left[ \mathbb{E} \left[ I \left\{ \max_{i \in [k]} \left\| V_i (\tilde{\Theta}_t - \tilde{\Theta}_t) \right\|_{\Sigma_t^{-1}} \leq \frac{\rho}{2} \right\} \left| \mathcal{H}_{t-1} \right. \right] \right]
\]
\[
= \mathbb{E} \left[ \mathbb{E} \left[ I \left\{ \max_{i \in [k]} \left\| V_i (\tilde{\Theta}_t - \Theta^*) \right\|_{\Sigma_t^{-1}} \leq \frac{\rho}{2} \right\} \left| \mathcal{H}_{t-1} \right. \right] \right]
\]
\[
= \mathbb{E} \left[ \max_{i \in [k]} \left\| V_i (\tilde{\Theta}_t - \Theta^*) \right\|_{\Sigma_t^{-1}} \leq \frac{\rho}{2} \right]
\]
\[
= \frac{1}{2t^2}.
\]

Application of the triangle inequality together with the union bound gives
\[
P(\Theta^* \not\in C_t) = P\left( \max_{i \in [k]} \left\| V_i (\tilde{\Theta}_t - \Theta^*) \right\|_{\Sigma_t^{-1}} \leq \frac{\rho}{2} \right) \leq \frac{1}{t^2}.
\]

Our next goal is to show that \( \tilde{\Theta}_t \) is optimistic with \( p := \frac{1}{2} \). Indeed, we show the slightly stronger inequality
\[
P\left( \tilde{M}_t \geq M_t \left| \mathcal{X}_t, \mathcal{H}_{t-1}, T_t = 1 \right. \right) \geq \frac{1}{2} \quad \text{a.s.}
\]
Algorithm 3 \textbf{OFUL}

\textbf{Input:} Estimator $\hat{\Theta}$.

1: \textbf{for} $t = 1, 2, \ldots$ \textbf{do}
2: \hspace{1em} Observe $X_t$,
3: \hspace{1em} $X_t \leftarrow \arg \max_{X \in \mathcal{X}_t} \sup_{\Theta \in S_t} \langle X, \Theta \rangle$
4: \textbf{end for}

The key observation here is that, conditional on $(\mathcal{X}_t, H_{t-1}, T_t)$, the random object $(\tilde{\Theta}_t, \Theta^*)$ is identically distributed as $(\Theta^*, \hat{\Theta}_t)$. This exchangeability implies

\[ \mathbb{P}(\tilde{M}_t \geq M_t \mid \mathcal{X}_t, H_{t-1}, T_t = 1) = \mathbb{P}(M_t \geq \tilde{M}_t \mid \mathcal{X}_t, H_{t-1}, T_t = 1) \quad \text{a.s.} \]

which leads to the desired result.

\subsection{5.2 Worst-case regret for \textbf{OFUL}}

Algorithm 3 displays \textbf{OFUL} as introduced in [AYPS11]. The confidence set (or search set) $S_t$ is defined as

\[ S_t := \left\{ \Theta \in \mathbb{R}^{kd} : \max_{i \in [k]} \left\| V_i (\Theta - \hat{\Theta}_{t-1}) \right\|_{\Sigma_{t-1}} \leq \frac{\rho}{2} \right\}. \quad (17) \]

The main difference between Algorithm 3 and the one in [AYPS11] is in the radius of this confidence set. Note that the radius of the confidence sets depends on $k$ logarithmically. We now show that this version of \textbf{OFUL} is an instance of ROFUL. For $\tilde{\Theta}_t$ given by

\[ \tilde{\Theta}_t := \arg \max_{\Theta \in S_t} \max_{X \in \mathcal{X}_t} \langle X, \Theta \rangle, \]

it is evident that \textbf{ROFUL} is identical to Algorithm 3. It thus suffices to verify reasonability and optimism of this estimator. It follows from the definition of $\tilde{\Theta}_t$ and (16) that

\[ \mathbb{P}(\Theta^* \not\in C_t) \leq \frac{1}{2t^2}. \]

This completes the proof of reasonability. For optimism, observe that whenever $T_t = 1$, then

\[ \tilde{M}_t = \langle X_t, \hat{\Theta}_t \rangle \geq \sup_{\Theta \in S_t} \langle X^*_t, \Theta \rangle = \langle X^*_t, \tilde{\Theta}_{t-1} \rangle + \sup_{\Theta \in S_t} \langle X^*_t, \Theta - \tilde{\Theta}_{t-1} \rangle \]

\[ = \langle X^*_t, \Theta^* \rangle + \langle X^*_t, \tilde{\Theta}_{t-1} - \Theta^* \rangle + \sup_{\Theta \in S_t} \langle X^*_t, \Theta - \tilde{\Theta}_{t-1} \rangle. \]

Next, it follows from Cauchy-Schwartz that

\[ \langle X^*_t, \tilde{\Theta}_{t-1} - \Theta^* \rangle + \sup_{\Theta \in S_t} \langle X^*_t, \Theta - \tilde{\Theta}_{t-1} \rangle = \langle X^*_t, \tilde{\Theta}_{t-1} - \Theta^* \rangle + \frac{\rho}{2} \left\| X^*_t \right\|_{\Sigma_t} \geq 0. \quad (18) \]

We call this inequality \textit{compensation inequality}. Therefore, we have

\[ \tilde{M}_t \geq \langle X^*_t, \Theta^* \rangle = M_t. \]

This implies that

\[ \mathbb{P}(\tilde{M}_t \geq M_t \mid \mathcal{X}_t, H_{t-1}, T_t = 1) = 1, \]

almost surely. Therefore, we can apply Theorem 2 which means bounds (9) and (10) hold for \textbf{OFUL}, and we recover the result of [AYPS11].
Algorithm 4 Linear Thompson sampling with inflated posterior

1: Initialize $\Sigma_1 \leftarrow I$ and $\hat{\Theta}_1 \leftarrow 0$
2: for $t = 1, 2, \ldots$ do
3: Observe $X_t$
4: Sample $\hat{\Theta}_t \sim \mathcal{N}(\hat{\Theta}_t, \beta^2 \Sigma_t)$
5: $\tilde{X}_t \leftarrow \arg\max_{X \in X_t} \langle X, \hat{\Theta}_t \rangle$
6: Observe reward $r_t$
7: $\Sigma_{t+1}^{-1} = \Sigma_t^{-1} + \tilde{X}_t \tilde{X}_t^\top$
8: $\hat{\Theta}_{t+1} \leftarrow \Sigma_{t+1}^{-1} \hat{\Theta}_t + \tilde{X}_t r_t$
9: end for

5.3 Worst-case regret for TS

In this section, we show how our framework can be applied to replicate the results of [AG13, AL+17]. Algorithm 4 presents TS with the inflated posterior where $\beta$ is the inflation parameter. We show that for any value of $\beta$, the worst-case regret of Algorithm 4 depends on $T$ as $\sqrt{T}$. However, for small values of $\beta$, the regret can grow exponentially in terms of $d$. As before, we start by showing that $\hat{\Theta}_t$ is reasonable. We need to use a tail-bound inequality for chi-squared distribution. Let $A$ be distributed as chi-squared with $d$ degrees of freedom. Then, for any $a > 0$, we have

$$P(A \geq da) \leq 2d\Phi(-\sqrt{a}) \leq 2d \exp\left(-\frac{a}{2}\right).$$

Next, setting $a := 2\log(4kdT^2)$, for any $i \in [k]$, we have

$$P\left(\|V_i(\tilde{\Theta}_t - \hat{\Theta}_t)\|_{\Sigma_t^{-1}}^2 \leq \beta^2 da\right) \leq 2d \exp\left(-\frac{a}{2}\right) \leq \frac{1}{2kt^2}.$$

It then follows from the union bound that

$$P\left(\max_{i \in [k]} \|V_i(\tilde{\Theta}_t - \hat{\Theta}_t)\|_{\Sigma_t^{-1}} \leq \beta \sqrt{2d\log(4kdT^2)}\right) \leq \frac{1}{2t^2}.$$

This inequality in combination with our previous bound (16) imply that $\tilde{\Theta}_t$ is reasonable with radius $\frac{\sqrt{2}}{2} + \beta \sqrt{2d\log(4kdT^2)}$. We next need to prove optimism. We use $T_t'$ we denote the indicator function of $\max_{i \in [k]} \|V_i(\tilde{\Theta}_t - \Theta^*)\|_2 \leq \frac{\sqrt{2}}{2}$. Clearly, $T_t'$ is $\mathcal{H}_{t-1}$-measurable. Whenever $T_t' = 1$, we have

$$\tilde{M}_t - M_t = \langle \tilde{X}_t, \hat{\Theta}_t \rangle - \langle X^*_t, \Theta^* \rangle \geq \langle X^*_t - X^*_t, \hat{\Theta}_t - \Theta^* \rangle \geq \langle X^*_t, \hat{\Theta}_t - \hat{\Theta}_t \rangle - \frac{\rho}{2}\|X^*_t\|_{\Sigma_t}.$$ 

It also follows from the definition of $\hat{\Theta}_t$ that $\langle X^*_t, \hat{\Theta}_t - \hat{\Theta}_t \rangle \sim \mathcal{N}(0, \beta^2 \|X^*_t\|_{\Sigma_t}^2)$. Hence, we get

$$P\left(\tilde{M}_t - M_t \geq 0, T_t' = 1 \mid X_t, \mathcal{H}_{t-1}\right) \geq \Phi\left(-\frac{\rho}{2\beta}\right).$$

Therefore, if one sets $\beta := \rho$, this becomes independent of the parameters in the problem. However, recall that $\rho \in \mathcal{O}(\sqrt{d})$. This thus implies that if one sets $\beta = 1$ (as in the uninflated version), $\Phi\left(-\frac{\rho}{2\beta}\right) \in \exp(\mathcal{O}(d))$ which matches our lower bound in the next section.
5.4 Improved worst-case bound for TS and OFUL

As shown in Theorem 2, reasonableness and optimism are sufficient conditions for an estimator to provide sub-linear regret bounds. TS and OFUL employ mechanisms for modifying the penalized least squares solution \( \hat{\Theta}_t \) to ensure optimism while preserving reasonableness. We now show that these mechanisms can be enhanced to reduce the exploration under some conditions. In order to do so, we introduce thinness coefficient at time \( t \) defined as

\[
\eta_t := \max_{i \in [k]} \frac{\lambda_{\max}(V_i^T \Sigma_t V_i)}{\lambda_{\min}(V_i^T \Sigma_t V_i)}.
\]

It is straightforward to see that \( 1 \leq \eta_t \), and we will show that smaller \( \eta_t \) can lead to reduced exploration.

We next make an assumption on the optimal arm \( X^*_t \).

**Assumption 6.** Assume that for any \( Y \in \mathbb{R}^{kd} \) with \( \max_{i \in [k]} \| V_i Y \|_2 = 1 \), we have

\[
P(\langle X^*_t, Y \rangle \geq \psi \| X^*_t \|_2) \leq \frac{1}{t^3},
\]

for some fixed \( \psi \in [0, 1] \).

We can then reduce the radius of the confidence set \( S_t \) in (17) and define

\[
S'_t := \left\{ \Theta \in \mathbb{R}^{kd} : \max_{i \in [k]} \| V_i (\Theta - \hat{\Theta}_{t-1}) \|_{\Sigma_{t-1}^{-1}} \leq \frac{\rho \min(1, \psi \eta_t)}{2} \right\}.
\]

(19)

We demonstrate that substituting \( S_t \) in Algorithm (3) with \( S'_t \) enjoys the same regret bound. It is clear from the definition of \( S'_t \) that the corresponding estimator \( \tilde{\Theta}'_t \) as defined in OFUL meets reasonableness condition with the same (and potentially smaller) radius. We thus need to show that optimism also holds in this case. We just need to verify the compensation inequality (18) for the case \( \psi \eta_t \leq 1 \). Note that

\[
\| X^*_t \|_2 \cdot \| \hat{\Theta}_{t-1} - \Theta^* \|_2 \leq \frac{1}{\lambda_{\min}(\Sigma_{t-1})} \| X^*_t \|_{\Sigma_{t-1}} \cdot \| \hat{\Theta}_{t-1} - \Theta^* \|_{\Sigma_{t-1}^{-1}} \\
\leq \frac{\lambda_{\max}(\Sigma_{t-1})}{\lambda_{\min}(\Sigma_{t-1})} \| X^*_t \|_{\Sigma_{t-1}^{-1}} \cdot \| \hat{\Theta}_{t-1} - \Theta^* \|_{\Sigma_{t-1}^{-1}} \\
\leq \frac{\rho \eta_t}{2} \| X^*_t \|_{\Sigma_{t-1}} \cdot \| \hat{\Theta}_{t-1} - \Theta^* \|_{\Sigma_{t-1}^{-1}}.
\]

Hence, the compensation inequality holds with probability at least \( 1 - \frac{1}{t^3} \). We thus have

\[
P(M_t \geq M_t \mid X_t, \mathcal{H}_{t-1}) \leq \frac{1}{t^3}.
\]

A similar approach can be applied to Thompson sampling to get

\[
P(M_t - M_t \geq 0, \mathcal{T}'_t = 1 \mid X_t, \mathcal{H}_{t-1}) \geq \Phi \left( -\frac{\rho \min(1, \psi \eta_t)}{2\beta} \right), \quad \text{w.p. at least } 1 - \frac{1}{t^3}
\]

Therefore, under the same assumptions the inflation rate \( \beta \) can be reduced while preserving the optimism.

6 Vulnerabilities of Thompson sampling: necessity of inflating the variance of posterior

In this section, we demonstrate that Thompson sampling with proper posterior update rule may incur linear regret when the assumptions are *slightly* violated. These examples, in particular, solve an open question
Algorithm 5 Linear Thompson sampling

1: Initialize $\Sigma_1 \leftarrow I$ and $\hat{\Theta}_1 \leftarrow 0$
2: for $t = 1, 2, \cdots$ do
3: Observe $X_t$
4: Sample $\tilde{\Theta}_t \sim N(\hat{\Theta}_t, \Sigma_t)$
5: $\tilde{X}_t \leftarrow \arg \max_{X \in X_t} \langle X, \tilde{\Theta}_t \rangle$
6: Observe reward $r_t$
7: $\Sigma_{t+1}^{-1} \leftarrow \Sigma_t^{-1} + \tilde{X}_t \tilde{X}_t^\top$
8: $\hat{\Theta}_{t+1} \leftarrow \Sigma_t^{-1}\left[\Sigma_t^{-1}\hat{\Theta}_t + \tilde{X}_t r_t\right]$
9: end for

mentioned in [RRK+18, §8.1.2]. More precisely, we show that Thompson sampling’s Bayesian regret (and thereby, worst case regret) can grow linearly up to time $\exp(O(d))$ whenever the prior distribution or the noise distribution mismatches with the one that Thompson sampling uses. It, furthermore, follows from our strategy that one needs the inflation rate of at least $\Omega\left(\frac{d}{\log d}\right)$ to avoid these problems. Our strategy for proving these results involves the following two steps:

1. We first construct small problem instances for which $\tilde{\Theta}_t$ is marginally biased.
2. We then show that by combining independent copies of these biased instances Thompson sampling can get linear Bayes regret.

6.1 Bias introducing action sets

In this section, we construct an example in which $\tilde{\Theta}_t$ is marginally biased provided that either the prior distribution or the noise distribution mismatches the one that Thompson sampling uses. Fix $\sigma^2, \tau^2 \geq 0$ and let $\Theta^* \sim N(0, \sigma^2 I_d)$ be the vector of unobserved parameters. At time $t \in \{1, 2, 3\},$ we reveal the following action sets to the policy:

$$X_t := \begin{cases} \{e_1\} & \text{if } t = 1, \\ \{e_2\} & \text{if } t = 2, \\ \{e_1, e_2\} & \text{if } t = 3. \end{cases}$$

For $t \leq 2,$ Thompson sampling has only one choice $e_t$ and thus $\tilde{X}_t = e_t.$ Assume that $r_t = \Theta^*_t + \epsilon_t$ is revealed to the algorithm where $\epsilon_t \sim N(0, \tau^2).$

At time $t = 3$ for the first time, Thompson sampling has two choices. Let $i$ be such that $\tilde{X}_3 = e_i.$ Then, $r_3 = \Theta^*_i + \epsilon_3$ is given to the algorithm where $\epsilon_3 \sim N(0, 1).$ We now show that $\hat{\Theta}_4$ is marginally biased provided that $\sigma^2 \neq \tau^2,$ i.e.,

$$\mathbb{E}\left[\hat{\Theta}_4\right] \neq 0.$$ 

It follows from the definition of $\hat{\Theta}_3$ that

$$\hat{\Theta}_3 = \frac{1}{2} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \Theta^*_1 + \epsilon_1 \\ \Theta^*_2 + \epsilon_2 \end{bmatrix}.$$ 

Next, at $t = 3,$ the $i$-th entry is updated according to

$$\hat{\Theta}_{4,i} = \frac{r_i + r_3}{3}.$$
\[
\begin{align*}
\hat{\Theta}^*_3 &= \frac{2\Theta^*_i + \varepsilon_i + \varepsilon_3}{3} \\
&= \frac{\Theta^*_i + \varepsilon_i}{2} + \frac{\Theta^*_i - \varepsilon_i}{6} + \frac{\varepsilon_3}{3} \\
&= \hat{\Theta}_{3,i} + \frac{\Theta^*_i - \varepsilon_i}{6} + \frac{\varepsilon_3}{3}.
\end{align*}
\]

Moreover, the other entry remains unchanged, in other words
\[
\hat{\Theta}_{4,3-i} = \hat{\Theta}_{3,3-i}.
\]

Therefore, setting \(V = e_1 + e_2\), we have
\[
\langle \hat{\Theta}_4, V \rangle = \langle \hat{\Theta}_4, V \rangle \\
= \hat{\Theta}_{4,1} + \hat{\Theta}_{4,2} \\
= \hat{\Theta}_{3,1} + \hat{\Theta}_{3,2} + \frac{\Theta^*_i - \varepsilon_i}{6} + \frac{\varepsilon_3}{3}.
\]

(20)

and in particular
\[
\langle E[\hat{\Theta}_4], V \rangle = E \left[ \frac{\Theta^*_i - \varepsilon_i}{6} \right].
\]

(21)

We can now compute this expression in terms of the selection bias coefficient given by
\[
\beta := E[\max\{A, B\}] > 0,
\]
where \(A\) and \(B\) are two independent standard normal random variables. Our main tool in this calculation is Theorem 3 in Appendix A. Recall that
\[
i = \arg\max_{j \in [1, 2]} \bar{\Theta}_{3,j}.
\]

By definition, \(\bar{\Theta}_3 \sim \mathcal{N} \left(0, \left(\frac{\sigma^2 + \tau^2 + 2}{4}\right) I_2 \right)\). Therefore, we have
\[
E[\bar{\Theta}_{3,i}] = \sqrt{\frac{\sigma^2 + \tau^2 + 2}{4}} \cdot \beta.
\]

On the other hand, it follows from the symmetry that
\[
E[\bar{\Theta}_{3,i}] = 2E \left[ \bar{\Theta}_{3,1} \cdot I(i = 1) \right] \\
= 2E \left[ \bar{\Theta}_{3,1} \cdot I(\bar{\Theta}_{3,1} \geq \bar{\Theta}_{3,2}) \right].
\]

Using Theorem 3 for the sequence
\[
X_1 := \frac{\Theta^*_1}{2}, \quad X_2 := \frac{\varepsilon_1}{2}, \quad \text{and} \quad X_3 := \bar{\Theta}_3 - \bar{\Theta}_3 \sim \mathcal{N} \left(0, \frac{1}{2}\right),
\]
we infer that
\[
E \left[ \frac{\Theta^*_1}{2} \cdot I(\bar{\Theta}_{3,1} \geq \bar{\Theta}_{3,2}) \right] = \frac{\sigma^2}{\sigma^2 + \tau^2 + 2} \cdot \frac{\sqrt{\sigma^2 + \tau^2 + 2}}{4} \cdot \beta \\
= \frac{\sigma^2 \beta}{4\sqrt{\sigma^2 + \tau^2 + 2}}.
\]

Consequently, we can write
\[
E[\Theta^*_i] = 2E \left[ \Theta^*_1 \cdot I(\bar{\Theta}_{3,1} \geq \bar{\Theta}_{3,2}) \right]
\]
\[
\begin{align*}
&= 4 \mathbb{E} \left[ \frac{\Theta_1}{2} \cdot I(\Theta_{3,1} \geq \Theta_{3,2}) \right] \\
&= \frac{\sigma^2 \beta}{\sqrt{\sigma^2 + \tau^2 + 2}}. \quad (22)
\end{align*}
\]

Similarly, we can conclude that
\[
\mathbb{E}[\varepsilon_i] = \frac{\tau^2 \beta}{\sqrt{\sigma^2 + \tau^2 + 2}}. \quad (23)
\]

Combining (21) with (22) and (23), we obtain
\[
\langle \mathbb{E}[\hat{\Theta}_4], V \rangle = \frac{(\sigma^2 - \tau^2) \beta}{6 \sqrt{\sigma^2 + \tau^2 + 2}}. \quad (24)
\]

This equality implies that \(\hat{\Theta}_4\) is marginally biased whenever \(\sigma^2 \neq \tau^2\). Finally, (20) gives
\[
\|\langle \hat{\Theta}_4, V \rangle\|_\psi^2 = \|\hat{\Theta}_{3,1} + \hat{\Theta}_{3,2} + \frac{\Theta_3^* - \varepsilon_i}{6} + \frac{\varepsilon_3}{3}\|_\psi^2 \\
\leq \|\hat{\Theta}_{3,1}\|_\psi^2 + \|\hat{\Theta}_{3,2}\|_\psi^2 + \frac{1}{6}\|\Theta_3^*\|_\psi^2 + \frac{1}{6}\|\varepsilon_i\|_\psi^2 + \frac{1}{3}\|\varepsilon_3\|_\psi^2 \\
\leq \sqrt{\sigma^2 + \tau^2} + \frac{1}{6}\|\Theta_3^*\|_\psi^2 + \frac{1}{6}\|\varepsilon_i\|_\psi^2 + \frac{1}{3}.
\]

Noting that
\[
\|\Theta_3^*\|_\psi^2 = \|\Theta_i^*\|_\psi^2 \leq \|\Theta_3^*\|_\psi^2 \leq 2\|\Theta_i^*\|_\psi^2 = 2\sigma
\]

and similarly for \(\varepsilon_i\), we get that
\[
\|\langle \hat{\Theta}_4, V \rangle\|_\psi^2 \leq \sqrt{\sigma^2 + \tau^2} + \frac{1}{3}(\sigma + \tau) + \frac{1}{3} \\
\leq 2(\sigma + \tau) + 1.
\]

Therefore, we have
\[
\|\langle \hat{\Theta}_4, V \rangle - \mathbb{E}[\langle \hat{\Theta}_4, V \rangle]\|_\psi^2 \leq \|\langle \hat{\Theta}_4, V \rangle\|_\psi^2 + \|\mathbb{E}[\langle \hat{\Theta}_4, V \rangle]\|_\psi^2 \\
\leq 4(\sigma + \tau) + 2.
\]

This implies that the mgf of \(\langle \hat{\Theta}_4, V \rangle - \mathbb{E}[\langle \hat{\Theta}_4, V \rangle]\) satisfies
\[
\mathbb{E} \left[ \exp \left( s \left( \langle \hat{\Theta}_4, V \rangle - \mathbb{E}[\langle \hat{\Theta}_4, V \rangle] \right) \right) \right] \leq \exp \left( \frac{s^2(4\sigma + 4\tau + 2)^2}{2} \right), \quad \text{for all } s \in \mathbb{R}. \quad (25)
\]

### 6.2 Combining biased settings

In this section, we demonstrate that by combining copies of the example in the previous section Thompson sampling can choose an incorrect action for at least \(\exp(O(d))\) rounds. Let \(d\) be a positive integer and define \(\Theta^* \sim \mathcal{P}_{\Theta^*} = \mathcal{N}(0, \sigma^2 I_{2d})\). In the first \(3d\) rounds, follow the action sets in the previous section for each pairs \((\Theta_{2i-1}^*, \Theta_{2i}^*)\) for \(i \in [d]\). Namely, define
\[
\mathcal{X}_t := \begin{cases} 
\{e_t\} & \text{if } t \leq 2d, \\
\{e_{2(t-2d)-1}, e_{2(t-2d)}\} & \text{if } 2d + 1 \leq t \leq 3d, \\
\{0, X\} & \text{otherwise,}
\end{cases}
\]
where

\[ X := \frac{\text{sgn}(\tau^2 - \sigma^2)}{\sqrt{d}} \cdot \sum_{i=1}^{2d} e_i. \]

It follows from (21) that

\[ \mathbb{E}\left[ \langle \hat{\Theta}_{3d+1}, X \rangle \right] = -C_1 \sqrt{d} \]

where \( C_1 := \frac{\mid \sigma^2 - \tau^2 \mid \cdot \beta}{12 \sqrt{\sigma^2 + \tau^2}} \). Assuming \( \sigma^2 \neq \tau^2 \), we observe \( C_1 > 0 \). Moreover, (25) implies that

\[ \mathbb{E}\left[ \exp \left( s \left( \langle \hat{\Theta}_{3d+1}, X \rangle + C_1 \sqrt{d} \right) \right) \right] \leq \exp \left( \frac{s^2(4\sigma + 4\tau + 2)^2}{2} \right), \quad \text{for all } s \in \mathbb{R}. \]

which means

\[ \left\| \langle \hat{\Theta}_{3d+1}, X \rangle + C_1 \sqrt{d} \right\|_{\psi_2} \leq (4\sigma + 4\tau + 2). \]

Using this inequality in combination with (26), we assert the following concentration inequality

\[ \mathbb{P}\left( \langle \hat{\Theta}_{3d+1}, X \rangle \leq -\frac{C_1 \sqrt{d}}{2} \right) = \mathbb{P}\left( \langle \hat{\Theta}_{3d+1}, X \rangle + C_1 \sqrt{d} \leq -\frac{C_1 \sqrt{d}}{2} \right) \geq 1 - \exp (-C_2 d), \]

where \( C_2 := \frac{C_1^2}{8} \).

Next, note that \( \langle \Theta^*, X \rangle \sim \mathcal{N}(0, 2\sigma^2) \), and thus, we have

\[ \mathbb{P}\left( \langle \Theta^*, X \rangle \geq \sqrt{2}\sigma \right) = 1 - \Phi(1). \]

For sufficiently large values of \( d \), we have \( \exp(-C_2 d) \leq \frac{1}{2}(1 - \Phi(1)) \), and hence

\[ \mathbb{P}\left( \langle \Theta^*, X \rangle \geq \sqrt{2}\sigma \right. \text{ and } \langle \hat{\Theta}_{3d+1}, X \rangle \leq -\frac{C_1 \sqrt{d}}{2} \right) \geq p_0, \]

where \( p_0 := \frac{1}{2}(1 - \Phi(1)) > 0 \). We denote the above event by \( \mathcal{B} \). Under this event, for all \( t > 3d \), the optimal arm is \( X \) and the regret incurred by choosing 0 is at least \( \sqrt{2}\sigma \). Next, for \( t > 3d \), let \( Z_t \) be given by

\[ Z_t := \begin{cases} 1 & \text{if action } X \text{ is never selected up to time } t, \\ 0 & \text{otherwise.} \end{cases} \]

We now have the following lower bound for the regret of Algorithm 5:

\[
\text{Regret}(T, \pi_{TS}^*, \mathcal{P}_\Theta^*) \geq \mathbb{P}(\mathcal{B}) \cdot \mathbb{E}\left[ \sqrt{2}\sigma \cdot \sum_{t=3d+1}^{T} Z_t \mid \mathcal{B} \right] \\
\geq \sqrt{2}\sigma p_0 \cdot \sum_{t=3d+1}^{T} \mathbb{E}[Z_t \mid \mathcal{B}] \\
= \sqrt{2}\sigma p_0 \cdot \sum_{t=3d+1}^{T} \mathbb{P}(Z_t \mid \mathcal{B}).
\]
Define \( q := P(Z_{3d+1} \mid B) \). We get that
\[
P(Z_t \mid B) = P(Z_t \mid B, Z_{t-1}) \cdot P(Z_{t-1} \mid B)
= q \cdot P(Z_{t-1} \mid B)
= q^{t-3d}.
\]
Furthermore, it follows from the definition of \( q \) that
\[
q \geq P\left(\hat{\Theta}_{3d+1} - \hat{\Theta}_{3d+1}, X\right) \leq \frac{C_1 \sqrt{d}}{2}
\]
\[
\geq P\left(\mathcal{N}(0,1) \leq \frac{C_1 \sqrt{d}}{2}\right)
\]
\[
\geq 1 - \exp(-C_2d).
\]
By combining the above, we have that
\[
\text{Regret}(T, \pi_{\text{TS}}, P_{\Theta^*}) \geq \sqrt{2} \sigma p_0 \sum_{t=3d+1}^{T} (1 - \exp(-C_2d))^{t-3d}
\]
\[
= \sqrt{2} \sigma p_0 \max_{t=1}^{T-3d} \left\{ 1 - t \exp(-C_2d), 0 \right\}.
\]
This immediately follows that
\[
\text{Regret}(T, \pi_{\text{TS}}, P_{\Theta^*}) \geq \sqrt{2} \sigma p_0 \frac{e}{e} \left( \min\{T, \exp(C_2d - 1)\} - 3d \right),
\]
which demonstrates that the regret of Thompson sampling grows linearly up to time \( \exp(C_2d - 1) \).

A Auxiliary lemmas

**Theorem 3** (Bias decomposition). Let \((X_i)_{i=1}^n\) be a sequence of independent random variables where \(X_i \sim \mathcal{N}(0, \sigma_i^2)\). By \(Y\) we denote their sum and let \(Z\) be any independent random variable. Then, for any function \(g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\), we have
\[
\mathbb{E}[X_i \cdot g(Y, Z)] = \frac{\sigma_i^2}{\sum_{i=1}^{n} \sigma_i^2} \mathbb{E}[Y \cdot g(Y, Z)].
\]

*Proof.* It is straight-forward to see that \(X_i \mid Y\) follows Gaussian distribution with mean \(\frac{\sigma_i^2}{\sum_{i=1}^{n} \sigma_i^2} \cdot Y\). We thus get
\[
\mathbb{E}[X_i \cdot g(Y, Z)] = \mathbb{E}[\mathbb{E}[X_i \cdot g(Y, Z) \mid Y, Z]]
= \mathbb{E}[\mathbb{E}[X_i \mid Y, Z] \cdot g(Y, Z)]
= \frac{\sigma_i^2}{\sum_{i=1}^{n} \sigma_i^2} \mathbb{E}[Y \cdot g(Y, Z)].
\]

B Acknowledgement

The authors gratefully acknowledge support of the National Science Foundation (CAREER award CMMI: 1554140) and Stanford Graduate School of Business and Stanford Data Science Initiative.
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