EXTERIOR SCHWARZSCHILD INITIAL DATA FOR DEGENERATE APPARENT HORIZONS

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Abstract. In this note we show that if $g$ is a smooth Riemannian metric on $\mathbb{S}^2$ such that the first eigenvalue of the operator $L_g := -\Delta_g + K_g$ is zero, then there exists an asymptotically flat initial data set with $(\mathbb{S}^2, g)$ as a corresponding apparent horizon and with ADM mass arbitrarily close to the Hawking mass $\sqrt{\text{area}(\mathbb{S}^2, g)/16\pi}$. In particular, this determines the Bartnik quasilocal mass (introduced by Bartnik \cite{Bartnik1989} in 1989) associated with $(\mathbb{S}^2, g)$ in this setting. We prove this by modifying the construction of Mantoulidis-Schoen \cite{Mantoulidis2018} who proved the same result in the case that the first eigenvalue of $L_g$ is positive.

1. Introduction

Mantoulidis-Schoen \cite{Mantoulidis2018} proved that if $g$ is a smooth Riemannian metric on $\mathbb{S}^2$ such that the first eigenvalue of the operator $L_g := -\Delta_g + K_g$ is positive, there exists an asymptotically flat initial data set $M$ having $(\mathbb{S}^2, g)$ as an apparent horizon, and with ADM mass arbitrarily close to $\sqrt{\text{area}(\mathbb{S}^2, g)/16\pi}$. This implies that the Bartnik quasilocal mass associated with $(\mathbb{S}^2, g)$ is $\sqrt{\text{area}(\mathbb{S}^2, g)/16\pi}$ by means of the Riemannian-Penrose inequality, proved by Huisken and Ilmanen in \cite{Huisken1997} and in full generality by Bray in \cite{Bray2001}.

In this paper, we extend the above result to include the degenerate case of when the first eigenvalue of $L_g$ is zero. We thus establish the following Theorem in which $\lambda_1(g)$ denotes the first eigenvalue of $L_g$ and $\mathcal{M}_+$ denotes the space of smooth Riemannian metrics $g$ on $\mathbb{S}^2$ with $\lambda_1(g) > 0$:

Theorem 1.1. Let $g \in \mathcal{M}_+$. For any $m > \sqrt{\text{area}(\mathbb{S}^2, g)/16\pi}$ there exists a smooth metric $G(p, t)$ on the manifold with boundary $M^3 = \mathbb{S}^2 \times [0, \infty)$ such that $G(\cdot, 0) = g$ and:

(i) $(M^3, G)$ has non-negative scalar curvature,
(ii) $\partial M^3$ is minimal in $(M^3, G)$,
(iii) the foliating spheres $\mathbb{S}^2 \times \{t\}$ are strictly mean convex in $(M^3, G)$ for all $t > 0$, and
(iv) for some $T > 2m$, the metric $G(p, t)$ is equal to the standard mass-$m$ Riemannian-Schwarzschild metric

$$g_{S,m}(p, t) = t^2 g_{S}(p) + \left(1 - \frac{2m}{t}\right)^{-1} dt^2$$

for $t > T$ where $g_{S}$ is the standard round metric on $\mathbb{S}^2$.

Remark 1.1. We refer to Lemma 2.2 (ii) for the sense of mean convexity we are using here.

The proof is sketched as follows. The first step is to construct a so-called “collar” which extends the metric $g$ on $\mathbb{S}^2 \times \{0\}$ to a warped product metric $\gamma$ on $\mathbb{S}^2 \times [0, 1]$ satisfying certain properties (see Lemma 2.2). The construction closely follows that in \cite[§1]{Mantoulidis2018}, except for a key difference in our choice of the warping factor which will allow us to begin the construction at the boundary $\partial \mathcal{M}_+$. The details of this are carried out in §2. From this point, the construction in \cite[§2]{Mantoulidis2018} can be invoked, providing a way to join the collar to an exterior Schwarzschild region resulting in a Riemannian metric $G$ on $\mathbb{S}^2 \times [0, \infty)$ satisfying the conclusions of Theorem 1.1. This is presented in §3.

2. Extending to $\mathbb{S}^2 \times [0, 1]$: A Collar Extension of $g$

We begin with the following.

Lemma 2.1. For any $g \in \mathcal{M}_+$, there exists a smooth path of metrics $t \mapsto g(t) \in \mathcal{M}_+$, $t \in [0, 1]$ with the following properties:

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Proposition 1.1 shows that decreasing function \( \lambda \) Conditions (i)-(iv) were proved in [4, Lemma 1.2] and we only sketch the proof here, then highlight Remark 2.1.

It is then shown in [4, Lemma 1.2] that the family where \( \phi \) prescribe (1), we only need to prove that inequality (1) holds for 0 \( t \) and some \( \rho > 0 \) let us estimate \( g \) such that \( u \) is a first eigenfunction of \( u \). Now fix some \( t \) \( \in \mathbb{R} \) such that \( u_t := u(t, \cdot) \) is a smooth path in \( M_+ \) from \( g \) to the round metric \( g_s \). Let \( h(t) = e^{2\zeta(t)+2\alpha(t)}g_s \) where \( a(t) \) is smooth, \( a(0) = 0 \) and \( a'(t) = -\zeta'(t) \int g_2 w e^{2\zeta(t)}g_s \).

It is then shown in [4, Lemma 1.2] that the family \( g(t) = \phi^*_t h(t) \) satisfies conditions (i)-(iv) in the Lemma where \( \phi_t \) is the integral flow along the vector field \( X_t \) satisfying \( \text{div}_h X_t = -2(\zeta(t) w + a'(t)) \).

We now complete the proof of the Lemma by showing that part (v) holds when \( g \in \partial M_+ \) provided we prescribe \( \zeta'(0) < 0 \). Note that \( \lambda_1(g(t)) = \lambda_1(h(t)) = e^{-2\alpha(t)} \lambda_1(e^{2\zeta(t)}w g_s) \). Thus, it is sufficient to prove

\[
\lambda_1(e^{2\zeta(t)}w g_s) \geq \beta t
\]

for all \( t \) and some \( \beta > 0 \). Moreover, since \( \rho(t) := e^{2\zeta(t)}w g_s \in M_+ \) for all \( t \in [c, 1] \) for any \( c > 0 \) (see Remark 2.1), we only need to prove that inequality (1) holds for \( 0 < t \ll 1 \) and some \( \beta > 0 \).

We prove inequality (1) as follows. Since \( \zeta : [0, 1] \to [0, 1] \) is smooth, we can write \( \zeta(t) = 1 + \zeta'(0)t + O(t^2) \).

As explained in [4, appendix], there is a smooth positive function \( u : [0, 1] \times S^2 \to \mathbb{R}_{>0} \) such that \( u_t := u(t, \cdot) \) is a first eigenfunction of \( \rho(t) \) with unit \( L^2 \) norm (with respect to the area form \( dA_{\rho(t)} \)).

Now fix some \( t \in [0, 1] \). For any \( s \in [0, 1] \), the formula for \( K_{\rho(s)} \) gives

\[
\int |\nabla^{(s)} u|_{\rho(s)}^2 + K_{\rho(s)} u_s^2 \, dA_{\rho(s)} = \int \left| \nabla^s u_{\rho(s)}^2 + (1 - \zeta(s) \Delta_s w) u_s^2 \right| \, dA_s
\]

which we view as a linear function \( L_t(\zeta(s)) \) of the variable \( \zeta(s) \) and can thus be written as

\[
L_t(\zeta(s)) = \zeta(s) L_t(\zeta(0)) + (1 - \zeta(s)) L_t(\zeta(1))
\]

for all \( s \in [0, 1] \) where we have used the above properties of \( \zeta \). In particular, evaluating the above line at \( s = t \) lets us estimate

\[
\lambda_1(\rho(t)) = \int |\nabla^{(t)} u|_{\rho(t)}^2 + K_{\rho(t)} u_t^2 \, dA_{\rho(t)}
\]

\[
= \zeta(t) \int |\nabla^{(0)} u|_{\rho(0)}^2 + K_{\rho(0)} u_t^2 \, dA_{\rho(0)} + (1 - \zeta(t)) \int |\nabla^{(t)} u|_{\rho(1)}^2 + K_{\rho(1)} u_t^2 \, dA_{\rho(1)}
\]

\[
= \zeta(t) \int |\nabla^g u|^2 + K_g u_t^2 \, dA_g + (1 - [1 + \zeta'(0)t + O(t^2)]) \int |\nabla^g u|^2 + K_g u_t^2 \, dA_g
\]

\[
\geq \zeta(t) \lambda_1(g) \int u_t^2 \, dA_g + [-\zeta'(0)t + O(t^2)] \lambda_1(g_s) \int u_t^2 \, dA_g.
\]

\[
\geq |\zeta(0)| \lambda_1(g_s) \inf_x u_t(x)^2 + O(t^2).
\]

In particular, recalling that \( u : [0, 1] \times S^2 \to \mathbb{R}_{>0} \) is smooth and positive we may conclude from the last line above that for \( t \) sufficiently small we have \( \inf_x u_t(x)^2 \geq \inf_x u_0(x)^2 / 2 > 0 \) and thus

\[
\lambda_1(g_t) \geq [-\pi \zeta'(0) \lambda_1(g_s) \inf_x u_0(x)] t =: \beta t.
\]

Recall that \( \zeta'(0) < 0 \) and \( \lambda_1(g_s) = 1 \) since \( g_s \) is round. This completes the proof of the Lemma.
Now we fix some \( g \in \partial \mathcal{M}_+ \) and consider the path \( t \mapsto g(t) \) constructed above. Fix some smooth positive function \( u : [0, 1] \times S^2 \to \mathbb{R}_{>0} \) such that \( u(t, \cdot) \) is a first eigenfunction for \( L_{g(t)} \) with unit \( L^2 \) norm with respect to the area form \( dA_{g(t)} \) (see [1] appendix).

**Lemma 2.2.** There exists \( 0 < \epsilon_0 \ll 1 \) and \( A_0 \gg 1 \) depending on \( g(0) \) such that for all \( 0 < \epsilon \leq \epsilon_0 \), \( A \geq A_0 \), the topological cylinder \( S^2 \times (0, 1] =: \Sigma \) endowed with the metric

\[
\gamma = (1 + \epsilon t^2)g(t) + \Phi(t)^2u(t, \cdot)^2dt^2
\]

has the following properties:

- (i) \( \gamma \) has positive scalar curvature,
- (ii) the foliating spheres \( S^2 \times \{t\} \) are mean convex for all \( t > 0 \) in the sense: \( H_t := -\operatorname{tr}_\gamma \rho > 0 \) for all \( t > 0 \) where \( \rho \) is the scalar second fundamental form of \( S^2 \times \{t\} \) in \( (\Sigma, \gamma) \) relative to the outward normal direction \( \partial_t \),
- (iii) \( H_t \to 0 \) as \( t \to 0 \).

Here \( \Phi(t) : (0, 1] \to \mathbb{R}_{>0} \) is defined as

\[
\Phi(t) = \begin{cases} \frac{4}{\sqrt{\epsilon}} & : t \in (0, 1/4] \\ \varphi(t) & : t \in (1/4, 1/2] \\ 2A - 1 & : t \in (1/2, 1) \end{cases}
\]

where \( \varphi \) is a smooth, decreasing, convex function chosen so that \( \Phi \in C^\infty((0, 1]) \).

**Remark 2.2.** When \( g \in \mathcal{M}_+ \) it was proved in [1] Lemma 1.3 that the metric \( \gamma = (1 + \epsilon t^2)g(t) + A^2u(t, \cdot)^2dt^2 \) satisfies the same conclusions (i)-(iii) for a sufficiently large constant \( A \). Their proof relies on the fact that \( \lambda_1(g) > 0 \), and thus does not extend to the case \( g \in \partial \mathcal{M}_+ \). We get around this by replacing this constant \( A \) with our choice of \( \Phi(t) \) above, and using the fact that \( \lambda_1(g(t)) \geq \epsilon t \).

**Remark 2.3.** The presumed singularity at \( t = 0 \) is superficial in the sense that changing to the new coordinate \( s = \sqrt{t} \) on \( S^2 \times (0, 1/4] \) gives

\[
\gamma = (1 + \epsilon s^2)g(s^2) + 4A^2u(s^2, \cdot)^2ds^2
\]

which extends smoothly to \( S^2 \times [0, 1/4] \). Moreover, since the mean curvature is coordinate invariant and continuous along the foliating spheres \( S^2 \times \{s\} \), we obtain that the boundary sphere \( \{s = 0\} \) is minimal in \( S^2 \times [0, 1] \) relative to the extension by part (ii) of the Lemma. In fact, Lemma 2.2 could have been stated and proved for this simpler parametrization as well, but we chose to use the parameter \( t \) in the proof for ease of reference to [1] and in §3.

**Proof of Lemma 2.2** The proof of the Lemma is the same as the proof of [1] Lemma 1.3 except for part (i) where we deviate in our choice of the component \( \Phi(t) \). We provide the full details of all three parts of the proof for the readers convenience.

Write \( h(t) = (1 + \epsilon t^2)g(t) \), \( v(t, x) = \Phi(t)u(t, x) \), and \( \mu(t) = (1 + \epsilon t^2)^{-1}\lambda(t) \) to simplify our notation to \( \gamma = h(t) + v(t, x)^2dt^2 \) and \( v(t, \cdot) \) now being an eigenfunction of \( -\Delta_{h(t)} + K_{h(t)} \) with eigenvalue \( \mu(t) \).

Now fix any \( t \in (0, 1] \). The mean curvature of the sphere \( S^2 \times \{t\} \) as a submanifold of \( (\Sigma, \gamma) \) is \( H_t = \frac{1}{2}\operatorname{tr}_{h(t)}(\rho) \) where \( \rho = \langle N, II \rangle \gamma \). Here \( N = \frac{1}{v(t, \cdot)} \frac{\partial}{\partial t} \) is the (outward) unit normal and \( II \) is the second fundamental form. To calculate this, let \( E_i, E_j \) be a local coordinate frame field on \( S^2 \times \{t\} \), which extend naturally to coordinate vector fields in the product space \( \Sigma \). Then

\[
\rho_{ij} = \langle N, II(E_i, E_j) \rangle \gamma = \gamma_{ab}N^a((\nabla_{E_i}E_j)^{a})^{b} = \gamma_{tt}N^4((\nabla_{E_i}E_j)^{a})^{t} = \gamma_{tt}N^4((\nabla_{E_i}E_j)^{a})^{t} = \frac{1}{v(t, \cdot)} \frac{\gamma_{tt}}{2v(t, \cdot)} h_{ij; tt}.
\]

Therefore

\[
H_t = \frac{1}{2}\operatorname{tr}_{h(t)}(\rho) = -\frac{1}{4v(t, \cdot)} \operatorname{tr}_h(h) = -\frac{1}{4v(t, \cdot)} \operatorname{tr}_h(h) = -\frac{1}{v(t, \cdot)} \epsilon t(1 + \epsilon t^2)^{-1}
\]

where in the last equality we have used that \( \hat{h} = 2\epsilon t g + (1 + \epsilon t^2)\hat{g} \) and the fact that \( \operatorname{tr}_g \hat{g} \equiv 0 \) by Lemma 2.1 (iii). In particular, the spheres \( S^2 \times \{t\} \) are strictly mean-convex relative to the outward normal direction since \( u(t, \cdot) > 0 \) while \( H_t \to 0 \) as \( t \to 0^+ \). This establishes (ii) and (iii).
To prove (i), we calculate
\[ \dot{h} = 2\epsilon g + 4\epsilon t \dot{g} + (1 + \epsilon t^2) \ddot{g} \quad \text{and} \quad \text{tr}_h \dot{h} = 4\epsilon (1 + \epsilon t^2)^{-1} + \text{tr}_g \ddot{g}. \]

Then by a known formula for the scalar curvature of the warped product metric \( \gamma \), we have
\[
R_\gamma = 2K_h - 2v^{-1} \Delta_h v + v^{-2} \left[ -\text{tr}_h \dot{h} - \frac{1}{4}(\text{tr}_h \dot{h})^2 + \frac{\partial v}{v} \text{tr}_h \dot{h} + \frac{3}{4} |h|^2_h \right] 
= 2\mu + v^{-2} \left[ -\text{tr}_h \dot{h} - \frac{1}{4}(\text{tr}_h \dot{h})^2 + \frac{\partial v}{v} \text{tr}_h \dot{h} + \frac{3}{4} |h|^2_h \right] 
\geq 2\mu + v^{-2} \left[ -\text{tr}_h \dot{h} + \frac{\partial v}{v} \text{tr}_h \dot{h} \right] 
= 2(1 + \epsilon t^2)^{-1} \lambda + \Phi^{-2}u^{-2} \left[ -4\epsilon (1 + \epsilon t^2)^{-1} - \text{tr}_g \ddot{g} + 4\epsilon t (1 + \epsilon t^2)^{-1} \frac{\Phi \partial_u \Phi + u \partial_\Phi \Phi}{\Phi} \right] 
= \Phi^{-2}(1 + \epsilon t^2)^{-1}u^{-2} \left[ 2\Phi^2 \lambda u^2 - 4\epsilon - (1 + \epsilon t^2) \text{tr}_g \ddot{g} + 4\epsilon t \frac{\partial u}{u} + 4\epsilon t \frac{\partial \Phi}{\Phi} \right] 
=: \Phi^{-2}(1 + \epsilon t^2)^{-1}u^{-2} [I + II + III + IV + V].
\]

By the smoothness of the family \( g(t) \), the definition of \( \Phi \) and the fact that \( \inf_{t,x} u^2 > 0 \) (see \[4\] appendix) it follows that
\[
|II| + |III| + |IV| + |V| \leq C_1
\]
for some constant \( C_1 \) depending only on \( \epsilon_0 \) and \( g(0) \). On the other hand, combining Lemma 2.2 part (v), the definition of \( \Phi \), and again that \( \inf_{t,x} u^2 > 0 \) yields
\[
I \geq C_2 A_0
\]
for some positive constant \( C_2 \) depending only on \( g(0) \).

From \[2\] we obtain the estimate
\[
R_\gamma \geq \Phi(t)^{-2}(1 + \epsilon t^2)^{-1}u^{-2}[C_2 A_0 - C_1]
\]
from which part (i) of the Lemma follows readily.

This completes the proof of the Lemma. \( \square \)

3. Extending to \( S^2 \times [0, \infty) \): Joining collar to an exterior Schwarzschild region

We can now complete the proof of Theorem 1.1. Let \( g \in \mathcal{M}_+ \) Consider the Riemannian “collar” \( (S^2 \times (0,1], \gamma(p,t)) \) constructed in Lemma 2.2 where
\[
\gamma = (1 + \epsilon t^2)g(t) + \Phi(t)^2u(t,\cdot)^2 dt^2.
\]

Recall that \( \epsilon > 0 \) can be chosen arbitrarily small, \( g(1) = g_* \) while \( \Phi(t) \) and \( u(p,t) \) are both constant functions for \( t \in [1/2, 1] \). Now for any \( m > \sqrt{\text{area}(S^2,g)/16\pi} \) consider the Riemannian mass-\( m \) Schwarzschild manifold \( (S^2 \times (2m, \infty), g_{S,m}) \) where
\[
g_{S,m} = t^2 g_* + \left( 1 - \frac{2m}{t} \right)^{-1} \text{dt}^2.
\]

Under these exact conditions it was proved in §2 of \[4\] that by choosing \( \epsilon \) sufficiently small, a positive scalar curvature “bridge” can be constructed between an interior region of the collar and an exterior region of the Schwarzschild manifold to ultimately give a metric \( G(p,t) \) on \( S^2 \times [0, \infty) \) which satisfies

(i) \( G \) has non-negative scalar curvature,
(ii) \( G(p,t) = \gamma(p,t) \) for \( t \in (0,1/2] \), and
(iii) For some \( T > 2m \) we have \( G(p,t) = g_{S,m}(p,t) \) for \( t \geq T \).

From Lemma 2.2 and Remark 2.3 we conclude that \( G \) satisfies the conclusions in Theorem 1.1 thus completing its proof.
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