Weak* closures and derived sets for convex sets in dual Banach spaces

Mikhail I. Ostrovskii

December 14, 2021

Abstract: The paper is devoted to the convex-set counterpart of the theory of weak* derived sets initiated by Banach and Mazurkiewicz for subspaces. The main result is the following: For every nonreflexive Banach space $X$ and every countable successor ordinal $\alpha$, there exists a convex subset $A$ in $X^*$ such that $\alpha$ is the least ordinal for which the weak* derived set of order $\alpha$ coincides with the weak* closure of $A$. This result extends the previously known results on weak* derived sets by Ostrovskii (2011) and Silber (2021).

Keywords: weak* closure, weak* derived set, weak* sequential closure

MSC 2020 classification: primary 46B10, secondary 46B20

1 Introduction

Let $X$ be a Banach space. For a subset $A$ of the dual Banach space $X^*$, we denote the weak* closure of $A$ by $\overline{A}^*$. The weak* derived set of $A$ is defined as

$$A^{(1)} = \bigcup_{n=1}^{\infty} A \cap nB_{X^*},$$

where $B_{X^*}$ is the unit ball of $X^*$. That is, $A^{(1)}$ is the set of all limits of weak* convergent bounded nets in $A$. If $X$ is separable, $A^{(1)}$ coincides with the set of all limits of weak* convergent sequences from $A$, called the weak* sequential closure. The strong closure of a set $A$ in a Banach space is denoted by $\overline{A}$. We set $A^{(0)} := A$.

It was noticed in the early days of Banach space theory by Mazurkiewicz [20] that $A^{(1)}$ does not have to coincide with $\overline{A}^*$ even for a subspace $A$, and $(A^{(1)})^{(1)}$ can be different from $A^{(1)}$. In this connection, it is natural to introduce derived sets for all ordinal numbers as: (1) If $A^{(\alpha)}$ has been already defined, then $A^{(\alpha+1)} := (A^{(\alpha)})^{(1)}$; (2) If $\alpha$ is a limit ordinal and $A^{(\beta)}$ has been already defined for all $\beta < \alpha$, then

$$A^{(\alpha)} := \bigcup_{\beta < \alpha} A^{(\beta)}.$$ (1)

The study of weak* derived sets was initiated by Banach and Mazurkiewicz (see [20]). Its early developments are discussed at length in the Appendix to the classical monograph
by Banach [3]. Later, this study was continued by many authors and found significant applications. Since the well-known survey [31] in the fields of Banach space theory initiated by [3] does not mention developments stemmed from Banach’s “Appendix” [3 Annexe], it looks beneficial to present here a short historical account.

Banach and Mazurkiewicz were primarily interested in the case of a separable Banach space $X$. Banach asked whether the weak* sequential closure of a subspace does not have to be weak* sequentially closed, and Mazurkiewicz [20] gave an affirmative answer to this question. This result was the reason for Banach to introduce weak* sequential closures of all transfinite orders.

In [3 Annexe] Banach proved that weak* sequential closures of finite orders do not have to be weak* sequentially closed. Furthermore, Banach stated that in his paper, which was going to appear in Studia Math., volume 4, he proved a similar result for $X = c_0$ and an arbitrary countable ordinal. Nevertheless, the cited paper has never been published. A possible explanation of this situation can be that Banach found a mistake in his proof when it was late to delete the statement and the reference in [3]. It is regrettable that the story was left uncommented in the reprint of [3] in [4] and the survey [31] because the editors of [4] might have known the actual story.

In the late 20s and early 30s, Banach and his school focused on the sequential approach to weak* topology and did not use the notion of weak* topology. The subject of what is now called General Topology already existed, see [2], but was not yet well known. Using General Topology significant part of the theory was made more elegant (see an account in [10]). However, the sequential approach developed in [3 Annexe] has its advantages and has led to significant applications. An early application of weak* sequential closures to the study of sets of uniqueness for Fourier series was discovered by Piatetski-Shapiro [33], and further developed in [18] and [19].

As for further development of theoretical aspects of weak* sequential closures, it is worth pointing out that Banach’s claim mentioned above (see [3 Annexe, §1]) was proved in 1968 by McGehee [21], using results by Piatetski-Shapiro [33]. At the same time, Sarason [39, 40] proved similar results for some other spaces.

Davis and Johnson [6] developed an essential tool for investigation of non-quasi-reflexive Banach spaces (that is, spaces for which the canonical image of $X$ in $X^{**}$ has infinite codimension). This tool was used by Godun [13] to prove that for any finite ordinal the dual of any non-quasi-reflexive Banach space contains a subspace whose weak* sequential closures of finite orders form a strictly increasing sequence (this result was rediscovered later by Moscatelli [23]). Godun [12] also made an attempt to prove similar results for infinite countable ordinals, but his argument in [12] contains gaps.

The result which completes the investigation for separable Banach spaces was proved in [26] for general non-quasi-reflexive separable Banach spaces. Namely, it was proved that, for every separable non-quasi-reflexive Banach space $X$ and every countable ordinal $\alpha$, the space $X^*$ contains a linear subspace $\Gamma$ for which $\Gamma^{(\alpha)} \neq \Gamma^{(\alpha+1)} = X^*$. This result completes the investigation for separable Banach spaces for the following reasons: (1) It is easy to see that if $X$ a separable quasi-reflexive Banach space and $\Gamma$ is a subspace of $X^*$, then $\Gamma^{(1)} = \Gamma$. (2) It is known (a proof is sketched in [3 page 213]) that for a separable
Banach space $X$ and any convex subset $A \subset X^*$, there is a countable ordinal $\alpha$ such that

$$A^{(\alpha)} = \overline{A}^*.$$  

This result of [26] is presented in [14]. Regrettably, the historical information on weak* sequential closures in [14] is inaccurate.

Meanwhile, the theory of weak* sequential closures found applications in many different fields:

- The structure theory of Fréchet spaces ([5, 8, 22, 24, 27]);
- The Borel and Baire classification of linear operators, including applications to the theory of ill-posed problems ([37, 32, 35, 36]);
- The operator theory ([41]);
- The theory of universal Markushevich bases ([34, 14]).

The survey [28] contains a more detailed historical account on weak* sequential closures, which was up-to-date in 2000.

It is worth mentioning that the proof of nonexistence of universal Markushevich bases in [34] uses the existence of subspaces satisfying $\Gamma^{(\alpha)} \neq \Gamma^{(\alpha+1)} = X^*$ in the same way as Szlenk [11] uses the existence of reflexive spaces with an arbitrary large Szlenk index in his proof of nonexistence of universal reflexive Banach spaces.

Recently, sequential closures and derived sets became objects of interest in some other areas, such as:

- Extension problems for holomorphic functions on dual Banach spaces ([11]);
- Valuations ([1]);
- The mathematical economics ([7]);
- The duality operators/spaces ([38]).

Garcia-Kalenda-Maestre [11] initiated the theory of weak* derived sets for convex sets. This theory was developed by Ostrovskii [29], who proved that, for an arbitrary non-reflexive Banach space $X$ (not necessarily non-quasi-reflexive), there a convex set $A \subset X$ for which $A^{(1)} \neq A^{(2)}$. This result showed that the theory for convex sets is different from the theory for subspaces - for convex sets quasi-reflexivity does not imply that $A^{(1)} = \overline{A}^*$. Silber [42] developed this result further by proving that, for an arbitrary non-reflexive space $X$ and an arbitrary $n \in \mathbb{N}$, the space $X^*$ contains a convex subset $A$ for which $n$ is the least ordinal satisfying $A^{(n)} = \overline{A}^*$ and a convex subset $D$ for which $\omega + 1$ is the least ordinal satisfying $D^{(\alpha)} = \overline{D}^*$. Our goal is to develop this theory further by proving the following result.

**Theorem 1.1.** Let $X$ be a non-reflexive Banach space and $\alpha$ be a countable ordinal. Then there exists a convex subset $A \subset X^*$ such that $\alpha$ is the least ordinal for which $A^{(\alpha)} \neq A^{(\alpha+1)} = \overline{A}^*$.

This theorem is proved in Section 5.
2 Trees and Forests

Our construction of sets is quite different from the one in [29, 42]. For our construction of sets whose existence is stated in Theorem 1.1, we need families of both finite and infinite forests - that is, graphs with no cycles, and trees - that is, connected forests. It has to be pointed out that although our terminology and constructions are related to the ones in [9, p. 161], [16, Section 2], [17], [18], [25, Section 2], and [45], they are different.

To achieve the goal of this work, we need a family of graphs labelled by countable ordinals $\alpha$, including all finite ordinals. The graph structure of these graphs is induced by a partial order on their vertex sets. To be specific, they are constructed using the definition below.

**Definition 2.1** (Graph structure on partially ordered sets). Given be a nonempty partially ordered set $V$ such that the set $\{y : y \leq x\}$ is finite and linearly ordered for each $x \in V$. The graph $F_V$ is defined in the following way. The vertex set of $F_V$ is taken to be $V$. Two vertices $x, y \in F_V$ are adjacent if and only if they are comparable and the set of vertices between them is empty. For a vertex $x \in F_V$, its up-degree is defined as the cardinality of the set of vertices $y$ adjacent to $x$ and satisfying $x < y$. Each such $y$ is called an up-neighbor of $x$, which is written as $y \succ x$. Likewise, the down-degree of a vertex $x \in F_V$ is defined as the cardinality of the set of vertices $y$ adjacent to $x$ and satisfying $y < x$. Each such $y$ is called a down-neighbor of $x$, and we write $y \prec x$.

If the context specifies a set $V$, we omit the lower index of $F_V$ and denote it by $F$ whenever it does not lead to misunderstanding.

It is clear that the obtained graph $F$ is a forest and that the down-degree of a vertex in $F$ can be either 0 or 1, while the up-degree can be any cardinal. Vertices with down-degree 0 are called initial. Vertices with up-degree 0 are called terminal.

For each ordinal $\alpha < \omega_1$ ($\omega_1$ is our notation for the least uncountable ordinal), we construct a forest $F_\alpha$ of the type described above. The forest $F_\alpha$ will be called a forest of order $\alpha$. Our construction of $F_\alpha$ is inductive.

Throughout the paper, all ordinals (except $\omega_1$) are assumed to be countable, that is, satisfying $< \omega_1$.

The forest $F_0$ of order 0 contains one vertex, no edges. Suppose that we have already defined the forest $F_\alpha$ of order $\alpha$. To get the forest $F_{\alpha+1}$ of order $\alpha + 1$ in the case where $\alpha$ is a successor ordinal, we introduce $\aleph_0$ (notation for the cardinality of $\mathbb{N}$) disjoint copies of $F_\alpha$, by this we mean that any two elements of different copies are incomparable in the partial order, and each copy inherits the partial order of $F_\alpha$. After that, we add to the vertex set one more vertex as a vertex which is $< \alpha$ each of the vertices of each copy of $F_\alpha$ and define $F_{\alpha+1}$ as the forest obtained using Definition 2.1 for the resulting partially ordered set. Observe that the added vertex is adjacent to the initial vertex of each copy of $F_\alpha$, and has no other adjacent vertices.

If $\beta < \omega_1$ is a limit ordinal, we consider a strictly increasing sequence $\{\beta_n\}_{n=1}^\infty$ of successor ordinals converging to $\beta$ and define $F_\beta$ as a disjoint union of $F_{\beta_n}$ over all $n \in \mathbb{N}$. This means that we use Definition 2.1 for the partial order on the union of $F_{\beta_n}$'s in
which each \( F_{\beta_n} \) keeps its partial order, and vertices of different \( F_{\beta_n} \)'s are not comparable. Therefore, in this case \( F_\beta \) is disconnected with \( \aleph_0 \) connected components.

Finally, to define \( F_{\alpha+1} \) in the case where \( \alpha \) is a limit ordinal, we add to the graph \( F_\alpha \) one more vertex which is \(<\) each of the vertices of \( F_\alpha \) and use Definition 2.1. The definition of \( F_\alpha \) implies that this new vertex has up-degree \( \aleph_0 \).

Observe that up-degrees of vertices in the forests \( \{F_\gamma\} \) can be only 0 or \( \aleph_0 \). This statement about up-degrees follows by observing that each new - that is, not a copy of an introduced before - vertex which is used in the construction of \( F_\alpha \) for \( \alpha \geq 1 \) has up-degree \( \aleph_0 \). Only vertices obtained from the vertex forming \( F_0 \) by repeated copying have up-degree 0.

Note that \( F_n \) for a finite ordinal \( n \) is what is called the (infinitely) countably branching tree of depth \( n \).

For a forest \( F \) of the described above type, we define the derived forest \( F^1 \) as a subgraph of \( F \) from which we delete all infinite sets of terminal vertices having the same down-neighbor. If we have already defined the derived forest \( F^\beta \) of order \( \beta \), we define the derived forest of order \( \beta+1 \) as \( (F^\beta)^1 \). If \( \beta \) is a limit ordinal and we have already defined derived forests for smaller ordinals, we set \( F^\beta = \cap_{\gamma<\beta} F^\gamma \).

Note that the derived forest \( (F_\alpha)^\gamma \) (we write \( F_\alpha^\gamma \) since it does not create any confusion) of order \( \gamma \) is not necessarily of the form \( F_\beta \) for some ordinal \( \beta \). For example, \( F_{\omega_0}^{\omega_0} \) (\( \omega_0 \) is the least infinite ordinal) is a collection of isolated vertices of cardinality \( \aleph_0 \).

To obtain a description of derived forests, we start with the following construction. If \( \{K_i\}_{i=1}^k \) are forests constructed from partially ordered sets as in Definition 2.1, by \( \bigcup_{i=1}^k K_i \) we denote the disjoint union of \( \{K_i\}_{i=1}^k \), which can also be described as a graph constructed using Definition 2.1 for the partial order on the union of \( \{K_i\}_{i=1}^k \) which on sets \( K_i \) coincides with their original partial order, and any two elements of different \( K_i \) are incomparable. This notation is also used in the case where \( k = \infty \). We also shall use the notation

\[
\left( \bigcup_{i=1}^k K_i \right) \bigcup \left( \bigcup_{i=k+1}^\infty K_i \right),
\]

which is self-explanatory at this point.

The following two lemmas contain results on derived forests needed for our construction.

**Lemma 2.2.** (1) Let \( n \) and \( k \), \( 1 \leq k \leq n \), be finite ordinals, then \( (F_n)^k = F_{n-k} \).

(2) Let \( \alpha \) be any ordinal. Then \( (F_{\alpha+1})^\alpha = F_1 \) and \( (F_{\alpha+1})^{\alpha+1} = F_0 \).

(3) Let \( \alpha \) be a limit ordinal and \( \{\beta_n\}_{n=1}^\infty \) be an increasing sequence of successor ordinals converging to \( \alpha \), used to define \( F_\alpha \). Then, for \( \beta < \alpha \),

\[
(F_\alpha)^\beta = \left( \bigcup_{n, \beta \geq \beta_n} F_0 \right) \bigcup \left( \bigcup_{n, \beta < \beta_n} (F_{\beta_n})^\beta \right).
\]
and

\[(F_\alpha)^\alpha = \biguplus_{i=1}^\infty F_0.\]

**Proof.** (1) The case of finite ordinals follows easily from the definition of \(F_n\).

It is easy also to see from the definition of \(F_\alpha\) for a successor ordinal that the conditions

\[(F_{\alpha+1})^\alpha = F_1 \text{ and } (F_{\alpha+1})^{\alpha+1} = F_0\]

imply \((F_{\alpha+2})^{\alpha+1} = F_1\) and \((F_{\alpha+2})^{\alpha+2} = F_0\).

Before proving (2) for limit ordinals (provided we proved it for all smaller ordinals), we need to prove the equality (2).

To get equality (2) we use the obvious \(F_0^\gamma = F_0\) for every \(\gamma\), its generalization

\[\left( \bigcup_{\alpha, \beta \geq \beta_n} F_0 \right)^\gamma = \left( \bigcup_{\alpha, \beta \geq \beta_n} F_0 \right),\]

and the definition of \(F_\alpha\). The equality (2) implies that \(F_{\alpha+1}^\alpha = F_1\) and \(F_{\alpha+1}^{\alpha+1} = F_0\) for a limit ordinal \(\alpha\), provided the statement (2) is known for all ordinals \(\beta < \alpha\).

This implies all formulas stated in Lemma 2.2.

**Lemma 2.3.** If a vertex in \((F_\alpha)^\beta\) has infinitely many terminal up-neighbors, then all of its up-neighbors are terminal vertices.

**Proof.** We prove this using induction on \(\alpha\). For finite \(\alpha\), the statement immediately follows from the description of \(F_\alpha^\beta\) in Lemma 2.2 (1).

If the statement is true for all ordinals which are strictly less that a limit ordinal \(\alpha\), then it is true for \(F_\alpha\). In fact, since \(F_\alpha\) consists of components which are \(F_\beta\) for \(\beta < \alpha\), and derived forests of disconnected forests are taken componentwise, the conclusion follows.

Now we assume that the statement is true for \(F_\alpha\), and derive it for \(F_{\alpha+1}\). There are two cases:

(a) \(\alpha\) is a limit ordinal,

(b) \(\alpha\) is a successor ordinal,

In both cases the induction hypothesis implies that the condition holds for all vertices of all derived forest except, possibly the initial vertex.

**Case (a):** For the initial vertex it is also satisfied because Lemma 2.2 (3) implies that for \(\beta < \alpha\), the initial vertex has finitely many terminal up-neighbors in the derived forest \(F_{\alpha+1}^\beta\). On the other hand, all of the up-neighbors of the initial vertex are terminal in the derived forest \((F_{\alpha+1})^\alpha\).

**Case (b):** In this case, each of the derived forests consists of \(\aleph_0\) copies of the same tree attached to the initial vertex. The conclusion follows.

\[\square\]
3 Reduction to separable case

Our main goal is to prove Theorem 1.1. The main part of our proof of Theorem 1.1 is its proof for a separable Banach space with a basis satisfying special condition. In this section we show that this special case implies the general case of Theorem 1.1.

We need the following notation. Let $Z$ be a closed subspace in a Banach space $X$ and $E : Z \to X$ be the natural isometric embedding. Then $E^* : X^* \to Z^*$ is a quotient mapping which maps each functional in $X^*$ onto its restriction to $Z$. Let $A$ be a subset of $Z^*$. It is clear that $D = (E^*)^{-1}(A)$ is the set of all extensions of all functionals in $A$ to the space $X$.

**Lemma 3.1** (cf. [26, 29]). For any ordinal $\alpha$ the equality

$$D^{(\alpha)} = (E^*)^{-1}(A^{(\alpha)})$$

holds, where the derived set $D^{(\alpha)}$ is taken in $X^*$ and the derived set $A^{(\alpha)}$ – in $Z^*$.

**Proof.** The inclusion $D^{(\alpha)} \subset (E^*)^{-1}(A^{(\alpha)})$ follows from the weak* continuity of the operator $E^*$ using transfinite induction.

To prove the inverse inclusion by transfinite induction, it suffices to show that for every bounded net $\{f_\nu\} \subset Z^*$ with $\lim f_\nu = f$ and every $g \in (E^*)^{-1}(\{f\})$ there exist $g_\nu \in (E^*)^{-1}(\{f_\nu\})$ such that some subnet of $\{g_\nu\}$ is bounded and weak* convergent to $g$. Let $h_\nu$ be such that $h_\nu \in (E^*)^{-1}(\{f_\nu\})$ and $\|h_\nu\| = \|f_\nu\|$ (Hahn-Banach extensions). Then $\{h_\nu\}_\nu$ is a bounded net in $X^*$. Hence it has a weak* convergent subnet, let $h$ be its limit. Then $g - h \in (E^*)^{-1}(\{0\})$, therefore $g_\nu = h_\nu + g - h$ is a desired net. \hfill \blacksquare

**Reduction of Theorem 1.1 to its special case.** We are going to use the following result proved in [30, 43]: If a Banach space $X$ is non-reflexive, then it contains a bounded basic sequence $\{z_i\}_{i=1}^\infty$ such that $\|z_i\| \geq 1$ for every $i \in \mathbb{N}$, but

$$\sup_{1 \leq k < \infty} \left\| \sum_{i=1}^k z_i \right\| = C < \infty. \quad (4)$$

Let $Z$ be the closed linear span of the sequence $\{z_i\}_{i=1}^\infty$ and $\{z_i^*\}_{i=1}^\infty \subset Z^*$ be the biorthogonal functionals of $\{z_i\}_{i=1}^\infty$. Let $z^{**}$ be a weak*-cluster point of the sequence $\{(\sum_{i=1}^k z_i)^\infty_{k=1}\}$ in $Z^{**}$.

We will need the following observations about these vectors:

(a) $|z^{**}(x)| \leq C\|x\|$ for every $x \in Z^*$. This is an immediate consequence of $\|z^{**}\| \leq C$ which follows from (4).

(b) If $x$ is a linear combination of $\{z_i^*\}_{i=1}^\infty$ with nonnegative coefficients, then $z^{**}(x) \geq c\|x\|$ for some $c > 0$.

In fact, let $x = \sum_{i=1}^k a_i z_i^*$. Then $z^{**}(x) = \sum_{i=1}^k a_i z_i$. On the other hand,

$$\|x\| = \left\| \sum_{i=1}^k a_i z_i^* \right\| \leq \sup_i \|z_i^*\| \sum_{i=1}^k a_i.$$
Since \( \{ z_i \} \) is a basic sequence satisfying \( \| z_i \| \geq 1 \), then \( \sup_i \| z_i^* \| \) is finite, and the conclusion follows.

Note that analysis of the proof of \([30, 43]\) leads to reasonably small absolute bounds for \( \sup_i \| z_i^* \| \) and \( C \) from above, but we do not need such bounds.

Let us show that Lemma 3.1 implies that to prove Theorem 1.1 it suffices to find a convex subset \( A \subset Z^* \) such that \( \overline{A} = A^{(a+1)} \neq A^{(a)} \). In fact, if we construct such \( A \), we let \( D = (E^*)^{-1}(A) \). We have, by Lemma 3.1 \( D^{(a)} = (E^*)^{-1}(A^{(a)}) \) and \( D^{(a+1)} = (E^*)^{-1}(A^{(a+1)}) \). Since each functional has a continuous extension, \( A^{(a+1)} \neq A^{(a)} \) implies \( D^{(a+1)} \neq D^{(a)} \).

To show that \( \overline{A} = A^{(a+1)} \) implies \( \overline{D} = D^{(a+1)} \) we observe that \( \overline{A} = A^{(a+1)} \) implies that \( A^{(a+1)} = A^{(a+2)} \). By Lemma 3.1 the last equality implies \( D^{(a+1)} = D^{(a+2)} \). By the Krein-Smulian theorem \([10, p. 429]\), the condition \( D^{(a+1)} = D^{(a+2)} \) implies \( \overline{D} = D^{(a+1)} \).

For this reason from now on our goal is to prove Theorem 1.1 for \( X = Z \).

### 4 Construction of suitable convex sets

We introduce an injective map of \( F_\alpha \) into \( \mathbb{N} \) and identify each vertex of \( F_\alpha \) with its image in \( \mathbb{N} \). We may and shall assume that if \( x < y \) in \( F_\alpha \), then the images \( \bar{x} \) and \( \bar{y} \) of \( x, y \) in \( \mathbb{N} \) satisfy \( \bar{x} < \bar{y} \). In fact, to establish such identification we reserve for each component of \( F_\alpha \) an infinite subsequence in \( \mathbb{N} \) (we reserve the whole \( \mathbb{N} \) if \( F_\alpha \) is connected). Then we assign to the initial vertex of each of the components of \( F_\alpha \) the least number of the corresponding subsequence and delete the initial vertices from \( F_\alpha \). Unless an initial vertex in a component \( K \) of \( F_\alpha \) was a terminal vertex, deletion of it will split \( K \) into infinitely many (incomparable with respect to the partial order) components. We reserve for each of these components a subsequence of the sequence reserved for \( K \), and continue in an obvious way.

For each terminal vertex \( v \equiv n_k \in \mathbb{N} \) (the symbol \( \equiv \) means that we identify the vertex \( v \) and number \( n_k \) of \( F_\alpha \)) we consider the path joining \( v \) with the initial vertex of the component of \( F_\alpha \) containing \( v \). Let the path be \( n_k, \ldots, n_1 \). In the path we list vertices only and observe that (by the result of the previous paragraph) \( n_k, \ldots, n_1 \) is a decreasing sequence in \( \mathbb{N} \).

Introduce for a terminal vertex \( v \) of \( F_\alpha \), corresponding to \( \{ n_k, \ldots, n_1 \} \), a vector in \( Z^* \) given by

\[
 z^*(v) = \sum_{i=1}^{k} n_{i-1} z_{n_i}^*, \tag{5}
\]

where we set \( n_0 = n_1 \).

Let

\[
 X = X_\alpha = \{ z^*(v) : v \text{ is a terminal vertex in } F_\alpha \}, \tag{6}
\]

and let \( A = A_\alpha = \text{conv}(X) \).

Our next goal is to analyze the structure of the weak* derived sets of \( A \).
In this connection, for each countable ordinal \( \beta \), we define a shortening \( X^\beta \) of the set \( X \) as

\[
X^\beta = \left\{ \sum_{n_i \in F_\alpha^\beta} n_i z_i^* : \sum_{i=1}^k n_i z_i^* \in X \right\},
\]

where \( n_i \in F_\alpha^\beta \) means that the vertex of \( F_\alpha \) corresponding to \( n_i \) belongs to the (defined in Section 2) derived forest \( (F_\alpha)^\beta \) of order \( \beta \). We let \( X^0 = X \).

Remark 4.1. Observe that each vector \( y \) in any of \( X^\beta \) including \( X^0 \) is supported on the vertex set of a finite path in \( F_\alpha \) whose vertex set is linearly ordered (recall that the vertex set of \( F_\alpha \) is partially ordered). We denote the largest vertex in this path by \( v(y) \).

Corollary 4.2 (Of Lemma 2.2). The difference \( X^{\beta+1} \setminus \left( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \right) \) is nonempty for every \( \beta < \alpha \).

In fact, Lemma 2.2 implies that \( X^{\beta+1} \) contains some “shortened” vectors which are not in \( X^\beta \).

Corollary 4.3 (Of Lemma 2.3). It is impossible for the support of a vector \( y \in X^{\beta+1} \) to contain the support of any of the vectors \( z \in \left( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \right) \).

Proof. In fact, if the support of \( z \) is contained in the support of \( y \), then, on one hand, the vertex \( v(z) \) has infinitely many terminal up-neighbors in some \( X^\gamma, \gamma < \beta \). On the other hand, it means that not all of up-neighbors of \( v(z) \) are in that \( X^\gamma \), because otherwise in could not happen that \( y \in X^{\beta+1} \). We get a contradiction with Lemma 2.3. \( \square \)

5 Proof of Theorem 1.1

The main steps in our proof of Theorem 1.1 are the following:

(A) For every \( \beta \leq \alpha \)

\[
\text{conv} \left( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \right) \subset A^{(\beta)}.
\] (7)

(B) For every \( \beta \leq \alpha \)

\[
A^{(\beta)} \subset \text{conv} \left( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \right).
\] (8)

(C) If \( \beta < \alpha \), then \( X^{\beta+1} \setminus \text{conv} \left( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \right) \neq \emptyset. \)
(D) The weak* sequential closure of \( \text{conv} \left( \bigcup_{0 \leq \gamma \leq \alpha} X^\gamma \right) \) coincides with \( \text{conv} \left( \bigcup_{0 \leq \gamma \leq \alpha} X^\gamma \right) \).

Therefore \( \text{conv} \left( \bigcup_{0 \leq \gamma \leq \alpha} X^\gamma \right) \) is weak* closed.

(E) The inclusion in (8) becomes an equality if \( \alpha \) is a successor ordinal and \( \beta = \alpha \).

5.1 Proof of item (A)

Since convexity is preserved under weak* sequential closures, to prove (A) by induction it suffices to show that \( X^\beta \subset A^{(\beta)} \) for every \( \beta \leq \alpha \).

The inclusion \( X^1 \subset A^{(1)} \) can be derived from the definitions as follows. The definitions imply that \( y \in X^1 \setminus X^0 \) if and only if there is an infinite sequence of terminal vertices \( \{v_n\} \subset F_\alpha \) having the same down-neighbor \( u \), such that \( y = z^*(u) \) (we use for non-terminal vertices the same notation as in (5)). Let \( x_n = z^*(v_n) \). Then, as it is easy to see, \( x_n \in X \) and \( y = w^* - \lim_{n \to \infty} x_n \).

In a similar way, if we know that \( X^\beta \subset A^{(\beta)} \) we derive \( X^{\beta+1} \subset A^{(\beta+1)} \) from the fact that each element of \( X^{\beta+1} \setminus X^\beta \) is a limit of a weak* convergent sequence of elements of \( X^\beta \).

On limit ordinals. The definition \( (F_\beta)^\gamma = \bigcap_{\gamma < \beta} (F_\alpha)^\gamma \) for the derived forest of order \( \beta \) with a limit ordinal \( \beta \) implies that

\[
X^\beta \subset \bigcup_{\gamma < \beta} X^\gamma
\]

for a limit ordinal \( \beta \). Combining inclusion (9) with the definition of the weak* derived set \( A^{(\beta)} \) for a limit ordinal \( \beta \) (see (11)), we get that the validity of inclusion (7) for all ordinals \( \tau < \beta \) implies its validity for a limit ordinal \( \beta \).

5.2 Proof of item (B)

We prove (8) by induction. Of course, we have the inclusion for \( \beta = 0 \).

The next step is to suppose that we have

\[
A^{(\beta)} \subset \text{conv} \left( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \right),
\]

and to use this inclusion to derive

\[
A^{(\beta+1)} \subset \text{conv} \left( \bigcup_{0 \leq \gamma \leq \beta+1} X^\gamma \right).
\]

To achieve this it is clearly enough to show that

\[
\left( \text{conv} \left( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \right) \right)^{(1)} \subset \text{conv} \left( \bigcup_{0 \leq \gamma \leq \beta+1} X^\gamma \right).
\]
Proof of the step $\beta \rightarrow \beta + 1$ will complete the proof of (8), because for a limit ordinal $\beta$ the inclusion (8) follows immediately from the definition of $A^{(\beta)}$ for a limit ordinal $\beta$, provided (8) has been already proved for all $\tau < \beta$.

So we prove the step $\beta \rightarrow \beta + 1$.

Since the set $\text{conv} \left( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \right)$ is a subset of the dual of a separable Banach space, any element of its weak* derived set is a weak* limit of a bounded sequence of the form

$$\left\{ \sum_{x \in W} a_{x,i}x \right\}_{i=1}^{\infty}, \text{ where } a_{x,i} \geq 0, \sum_{x \in W} a_{x,i} = 1, \quad (10)$$

where $W = \bigcup_{0 \leq \gamma \leq \beta} X^\gamma$ and the set $\{a_{x,i}\}_{x \in W}$ is finitely nonzero for any $i \in \mathbb{N}$.

For each $x \in W$ we consider the vertex $v(x)$ in $F_\alpha$ (see the definition in Remark 4.1). It can happen that for some $x \in W$ the vertex $v(x)$ is an initial vertex. We denote the set of all such $x \in W$ by $I$.

For $x \in (W \setminus I)$ denote by $v(y)$ the down-neighbor of $v(x)$ in $F_\alpha$ and denote by $y = y(x)$ the vector in $Z^*$ obtained if we replace by 0 the component of $x$ corresponding to $v(x)$, so that $v(y)$ agrees with the definition in Remark 4.1.

We group the summands of $\sum_x a_{x,i}x$ for $x \in (W \setminus I)$ according to vectors $y = y(x)$ defined in the previous paragraph. We denote the set of all such vectors $y$ obtained for different $x \in (W \setminus I)$ by $D$. We can write

$$\sum_{x \in W} a_{x,i}x = \sum_{x \in I} a_{x,i}x + \sum_{y \in D} \sum_{\{x: v(x) > v(y)\}} a_{x,i}x,$$

where $v(x) \succ v(y)$ means that $v(y)$ is a down-neighbor of $v(x)$.

We may assume without loss of generality that $\lim_{i \to \infty} \sum_{\{x: v(x) > v(y)\}} a_{x,i}$ exists for every $y \in D$ and denote this limit by $s_y$. We may also assume that $\lim_{i \to \infty} a_{x,i}$ exists for every $x \in W$ and denote this limit by $p_x$.

Lemma 5.1. If the sequence $\left\{ \sum_{x \in W} a_{x,i}x \right\}_{i=1}^{\infty}$ is bounded, then $\sum_{x \in I} p_x + \sum_{y \in D} s_y = 1$.

Proof. In fact, suppose $\sum_{x \in I} p_x + \sum_{y \in D} s_y = \omega < 1$. For every pair of finite subsets $G \subset I$ and $F \subset D$, and every $\varepsilon > 0$, there is $j \in \mathbb{N}$ such that

$$\sum_{x \in G} (a_{x,i} - p_x) + \sum_{y \in F} \left( \left( \sum_{\{x: v(x) > v(y)\}} a_{x,i} \right) - s_y \right) < \varepsilon \text{ for } i \geq j.$$

Therefore

$$\sum_{x \in (I \setminus G)} a_{x,i} + \sum_{y \in (D \setminus F)} \left( \sum_{\{x: v(x) > v(y)\}} a_{x,i} \right) > 1 - (\omega + \varepsilon) \text{ for } i \geq j.$$

For any $M \in \mathbb{N}$, we can pick $F$ in such a way that for all $y \in (D \setminus F)$ the natural number corresponding to $v(y)$ in the identification described in Section 4 is at least $M$. (For this and the next statement we need to recall that $n_0 = n_1$ in (5).)
Similarly, we can pick $G$ in such a way that for all $x \in (I \setminus G)$, the natural number corresponding to $x$ (recall that $n_0 = n_1$, see the line after (5)) is at least $M$. Then, by (a) in Section 3,

$$
\left\| \sum_{x \in W} a_{x,i}x \right\| \geq \frac{1}{C} z^{**} \left( \sum_{x \in W} a_{x,i}x \right)
\geq \frac{1}{C} z^{**} \left( \sum_{x \in (I \setminus G)} a_{x,i}x + \sum_{y \in (D \setminus F)} \left( \sum_{\{x: v(x) > v(y)\}} a_{x,i}x \right) \right)
\geq \frac{M(1 - (\omega + \varepsilon))}{C}.
$$

Since this can be done for every $M \in \mathbb{N}$ and every $\varepsilon > 0$, we conclude that $\{\sum_{x \in W} a_{x,i}x\}_{i=1}^\infty$ is unbounded. This contradiction proves the lemma.

**Lemma 5.2.** If $\sum_{x \in I} p_x + \sum_{y \in D} s_y = 1$, then the vectors $\left\{ \sum_{x: v(x) > v(y)} a_{x,i} \right\}_{y \in D}$ converge to the vector $\{s_y\}_{y \in D}$ strongly in $\ell_1(D)$ and the vectors $\{a_{x,i}\}_{x \in I}$ converge to $\{p_x\}_{x \in I}$ in $\ell_1(I)$.

Lemma 5.2 is an immediate consequence of the fact that a sequence of normalized vectors $\{v_i\}$ in $\ell_1$ which converges pointwise to a normalized vector $v$, converges to $v$ strongly.

**Lemma 5.3.** The series $\sum_{x \in W} p_x x$ and $\sum_{x \in I} p_x x$ are strongly convergent.

**Proof.** In fact, otherwise by items (a) and (b) in Section 3 the series $\sum_{x \in W} p_x z^{**}(x)$ diverges to infinity. This divergence implies that $\sum_{x \in W} a_{x,i} z^{**}(x) \leq C \left\| \sum_{x \in W} a_{x,i}x \right\|$ tend to infinity as $i \to \infty$ contradicting the boundedness of the sequence $\{\sum_{x \in W} a_{x,i}x\}_{i=1}^\infty$.

**Corollary 5.4.** If the sequence $\{\sum_{x \in W} a_{x,i}x\}_{i=1}^\infty$ is bounded in $Z^*$, then the sequence $\{\sum_{x \in I} a_{x,i}x\}_{i=1}^\infty$ weak* converges to the vector $\sum_{x \in I} p_x x$.

**Proof.** In fact, the mentioned sequence is uniformly bounded and convergent coordinate-wise (this is very easy to see because $x \in I$ are vectors with one nonzero coordinate each). By the well-known simple fact, they converge in the weak* topology.

**Lemma 5.5.** The series $\sum_{y \in D} s_y y$, and thus $\sum_{y \in D} \left( s_y - \sum_{v(x) > v(y)} p_x \right) y$, is strongly convergent.

The first statement implies the second statement by virtue of an easy inequality $s_y \geq \sum_{x: v(x) > v(y)} p_x$. 

Proof. Assume the contrary. Since the vectors $y$ are nonnegative and $s_y \geq 0$, the contrary, by (a) and (b) on page 7 implies that $\sum y s_y z** (y)$ diverges to $\infty$. On the other hand, for each $y \in D$ and sufficiently large $i = i(y)$ we have

$$z** \left( \sum_{x : v(x) \succ v(y)} a_{x,i} x \right) \geq \frac{1}{2} s_y z** (y).$$

Since the sets $\{ x : v(x) \succ v(y) \}$ with different $y$ are disjoint, we conclude that sums $\sum_{x \in W} a_{x,i} x$ cannot be uniformly bounded.

Lemma 5.6. The sequence of vectors

$$\sum_{x \in W} a_{x,i} x = \sum_{x \in I} a_{x,i} x + \sum_{y \in D} \sum_{\{ x : v(x) \succ v(y) \}} a_{x,i} x, \ i \in \mathbb{N}, \quad (11)$$

converges to the vector

$$\sum_{x \in I} p_x x + \sum_{y \in D} \left( \sum_{\{ x : v(x) \succ v(y) \}} p_x x + \left( s_y - \sum_{\{ x : v(x) \succ v(y) \}} p_x \right) y \right), \quad (12)$$

in the weak* topology.

Proof. We know that vectors (11) are uniformly bounded. Because of this it is enough to prove that the vectors (11) converge to the vector (12) componentwise.

Let us consider their $m$th components. Assume that $m$ is in the image of $F_\alpha$ and that the path to $m$ from the initial vertex of the component containing $m$ is $n_1, \ldots, n_k = m$. There are two slightly different cases: $k = 1$ and $k \geq 2$. We consider the case $k \geq 2$. The change which should be made if $k = 1$ is: replace $n_{k-1}$ by $n_1$ in the formulas below and add the corresponding term for $x \in I$.

In the case $k \geq 2$ the $m$th component of the vector (12) is

$$p_z n_{k-1} + \sum_{y \in D} s_y n_{k-1},$$

where in the first term, $z$ is such that $m = v(z)$. On the other hand, the $m$th components of the vectors (11) are

$$a_{z,i} n_{k-1} + \sum_{y \in D} \sum_{m \in \text{supp} y} a_{x,i} n_{k-1},$$

where in the first term $z$ is such that $m = v(z)$.

We have the convergence of $\sum_{m \in \text{supp} y} \sum_{x : v(x) \succ v(y)} a_{x,i} n_{k-1}$ to $\sum_{m \in \text{supp} y} s_y n_{k-1}$ by Lemma 5.2.

To complete the proof of item (B), it suffices:
(1) To recall that (see Lemma 5.1)

\[ \sum_{x \in I} p_x + \sum_{y \in D} \left( \sum_{\{x : v(x) > v(y)\}} p_x \right) + \sum_{y \in D} \left( s_y - \sum_{\{x : v(x) > v(y)\}} p_x \right) = 1. \]

(2) To derive from the previous item and Lemmas 5.3 and 5.5 that the vector in (12) is an infinite convergent convex combination of \( x \in W \) and \( y \in D \).

(3) To observe that \( y \in D \) can be involved in this combination with nonzero coefficient only if \( (s_y - \sum_{\{x : v(x) > v(y)\}} p_x) > 0 \), and this can happen only if \( y \in \bigcup_{0 \leq \gamma \leq \beta+1} X^\gamma \).

### 5.3 Proof of item (C)

If \( \beta < \alpha \), some vertices of \((F_\alpha)^\beta\) are not in \((F_\alpha)^{\beta+1}\) (see Lemma 2.2) and therefore there is an infinite family of terminal vertices of \((F_\alpha)^\beta\) with the common down-neighbor \( u \). Then there is \( y \in X^{\beta+1} \setminus \left( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \right) \) satisfying \( v(y) = u \) (see Section 4).

To complete the proof of (C) it suffices to show that \( y \notin \text{conv} \left( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \right) \).

Note that \( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \) contains vectors of two types:

1. Extensions of \( y \), that is, vectors coinciding with \( y \) on its support, but also having at least one more positive coordinate. According to the definitions in Section 4, the coordinate has to be \( \geq m \), where \( m \) is the image of \( u \) in \( \mathbb{N} \), see the beginning of Section 4. We denote this set of all such vectors in \( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \) by \( E \).

2. Vectors whose \( m \)th coordinate is equal to 0 and some coordinates which are not in the support of \( y \) are positive. We denote the set of all such vectors in \( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \) by \( R \). By Corollary 4.3 all vectors in \( R \) have nonzero coordinates which are not in the support of \( y \).

Clearly,

\[
\text{conv} \left( \bigcup_{0 \leq \gamma \leq \beta} X^\gamma \right) = \text{conv}(E \cup R).
\]

So we need to find a continuous linear functional on \( Z^* \) which separates \( y \) from \( \text{conv}(E \cup R) \). In this connection, we consider the following two continuous linear functionals on \( Z^* \).

The first is the sum of all coordinates of a vector with respects to the basis \( \{z_i^*\}_{i=1}^\infty \), except the coordinates which are nonzero for \( y \). This functional is continuous because it is a linear combination of \( z^{**} \) (introduced in Section 3) and finitely many functionals of the sequence \( \{z_i\}_{i=1}^\infty \) considered as elements of \( Z^{**} \). We denote this functional by \( \tilde{z} \).

The second functional is \( z_m \) (that is, \( m \)th coordinate functional).

We claim that \( z_m - \tilde{z} \) separates \( y \) from \( \text{conv}(E \cup R) \). To see this observe that \( (z_m - \tilde{z})(y) = a > 0 \), where \( a \) is the value of the largest coordinate of \( y \).
On the other hand, \((z_m - \tilde{z})|_R \leq 0\) because \(z_m\) is zero on \(R\) and \(\tilde{z}\) is nonnegative for all vectors in \(R\).

Also \((z_m - \tilde{z})|_E \leq 0\) because for each vector in the extension further coordinates cannot be smaller than the previous ones.

5.4 Proof of item (D)

To prove this statement, we repeat the argument used to prove (B) and observe that we get into the closure of the same set because \(D\), in this case, is a subset of \(W\). By the Krein-Smulian theorem [10, p. 429]), this completes the proof.

5.5 Proof of item (E)

Let \(\alpha = \tau + 1\). By item (A), we have

\[ A^{(\tau)} \supset \text{conv} \left( \bigcup_{1 \leq \gamma \leq \tau} X^\gamma \right).\]

Since the weak* derived set (for any set \(A\)) contains the strong closure of the set, we get

\[ A^{(\tau+1)} \supset \text{conv} \left( \bigcup_{1 \leq \gamma \leq \tau} X^\gamma \right).\]

Denote by \(j\) the initial vertex of the tree \(F_{\tau+1}\), and let \(r \in Z^*\) be the vector whose only nonzero coordinate in \(\{z^*_i\}_{i=1}^{\infty}\) is for \(k \in \mathbb{N}\) which corresponds to \(j\) according to the injective map constructed at the beginning of Section 4. So \(r = kz^*_k\), the definition (5) takes this form in the case where \(j\) is the initial vertex of \(F_{\tau+1}\). By Lemma 2.2 and item [A] we have \(r \in A^{(\tau+1)}\). Our goal is to prove

\[ A^{(\tau+1)} \supset \text{conv} \left( \left( \bigcup_{1 \leq \gamma \leq \tau} X^\gamma \right) \bigcup \{r\} \right).\]

So we consider a strongly convergent sequence

\[ \left\{ \sum_{x \in W} a_{x,i}x + a_{r,i}r \right\}_{i=1}^{\infty}, \]

where \(W = \bigcup_{1 \leq \gamma \leq \tau} X^\gamma, a_{x,i} \geq 0, a_{r,i} \geq 0,\) and \(\sum_{x \in W} a_{x,i} + a_{r,i} = 1\). Since \(0 \leq a_{r,i} \leq 1\), we may assume that the sequence \(\{a_{r,i}r\}_{i=1}^{\infty}\) is convergent. Since \(A^{(\tau+1)}\) is convex, the conclusion follows.

5.6 End of the proof of Theorem 1.1

It is clear that combination of items (A)-(E) proves Theorem 1.1 in all cases except case \(\alpha = 1\). Our proof can be adjusted to cover this case - just consider a forest consisting
of infinitely many disjoint copies of $F_1$. We do not provide the details because the case $\alpha = 1$ is covered by Silber [42].

5.7 Comment

The following question asked in [42] remains unanswered.

Question 5.7 ([42, Section 3, Question 1]). Does there exist a convex subset $A$ in the dual to a separable Banach space for which the least ordinal $\alpha$ satisfying $A^{(\alpha)} = A^*$ is a limit ordinal?

It should be mentioned that it is an easy consequence of the Baire theorem that this cannot happen if we additionally require that $A$ is subspace. To the best of our knowledge, Godun [12] was the first to make this observation (later it was repeatedly rediscovered, see [18], [15]).

Acknowledgement

The author gratefully acknowledges the support by the National Science Foundation grant NSF DMS-1953773.

References

[1] S. Alesker, On repeated sequential closures of constructible functions in valuations. Geometric aspects of functional analysis, 1–14, Lecture Notes in Math., 2169, Springer, Cham, 2017.
[2] P. Alexandroff, P. Urysohn, Zur Theorie der topologischen Räume, Math. Annalen, 92 (1924), 258–266.
[3] S. Banach, Théorie des opérations linéaires, Monografje Matematyczne, Warszawa, 1932.
[4] S. Banach, Œuvres. Vol. II. Travaux sur l’analyse fonctionnelle. Edited by C. Bessaga, S. Mazur, W. Orlicz, A. Pełczyński, S. Rolewicz, and W. Żelazko. PWN Éditions Scientifiques de Pologne, Warsaw, 1979.
[5] E. Behrends, S. Dierolf, P. Harmand, On a problem of Bellenot and Dubinsky, Math. Ann., 275 (1986), pp. 337–339.
[6] W. J. Davis, W. B. Johnson, Basic sequences and norming subspaces in non-quasi-reflexive Banach spaces, Israel J. Math., 14 (1973), 353–367.
[7] F. Delbaen, K. Owari, Convex functions on dual Orlicz spaces. Positivity 23 (2019), no. 5, 1051–1064.
[8] S. Dierolf, V. B. Moscatelli, A note on quotjections, Functiones et approximation, 17 (1987), 131–138.
[9] R. Diestel, Graph decompositions. A study in infinite graph theory. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1990.
[10] N. Dunford, J. T. Schwartz, Linear Operators. I. General Theory. With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7, Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1958.
[11] D. Garcia, O. F. K. Kalenda, M. Maestre, Envelopes of open sets and extending holomorphic functions on dual Banach spaces, J. Math. Anal. Appl., 363 (2010) 663–678.
[12] B. V. Godun, Weak∗ derivatives of transfinite order for sets of linear functionals. (Russian) Sibirsk. Mat. Zh. 18 (1977), no. 6, 1289–1295.

[13] B. V. Godun, Weak∗ derivatives of sets of linear functionals. (Russian) Mat. Zametki 23 (1978), no. 4, 607–616.

[14] P. Hajek, V. Montesinos, J. Vanderwerff, V. Zizler, Biorthogonal systems in Banach spaces, Berlin, Springer-Verlag, 2007.

[15] A. J. Humphreys, S. G. Simpson, Separable Banach space theory needs strong set existence axioms, Trans. Amer. Math. Soc. 348 (1996), no. 10, 4231–4255.

[16] R. Judd, E. Odell, Concerning Bourgain’s ℓ1-index of a Banach space. Israel J. Math. 108 (1998), 145–171.

[17] A. S. Kechris, Classical descriptive set theory. Graduate Texts in Mathematics, 156. Springer-Verlag, New York, 1995.

[18] A. S. Kechris, A. Louveau, Descriptive set theory and the structure of sets of uniqueness, Cambridge University Press, 1987.

[19] R. Lyons, A new type of sets of uniqueness, Duke Math. J., 57 (1988), 431–458.

[20] S. Mazurkiewicz, Sur la dérivée faible d’un ensemble de fonctionnelles linéaires, Studia Math., 2 (1930), 68–71.

[21] O. C. McGehee, A proof of a statement of Banach about the weak∗ topology, Michigan Math. J., 15 (1968), 135–140.

[22] G. Metafune, V. B. Moscatelli, Quojections and prequojections, in: Advances in the Theory of Fréchet spaces (ed.: T. Terzioglu), Kluwer Academic Publishers, Dordrecht, pp. 235–254, 1989.

[23] V. B. Moscatelli, On strongly nonnorming subspaces. Note Mat. 7 (1987), no. 2, 311–314 (1988).

[24] V. B. Moscatelli, Strongly nonnorming subspaces and prequojections, Studia Math., 95 (1990), 249–254.

[25] E. Odell, Ordinal indices in Banach spaces. Extracta Math. 19 (2004), no. 1, 93–125.

[26] M. I. Ostrovskii, w∗-derived sets of transfinite order of subspaces of dual Banach spaces, Dokl. Akad. Nauk Ukrain. SSR, 1987, no. 10, 9–12 (in Russian and Ukrainian); An English version of this paper is available at https://arxiv.org/abs/math/9303203

[27] M. I. Ostrovskii, Quojections and their duals, Revista Mat. Univ. Complutense Madrid., 11 (1998), 59–77.

[28] M. I. Ostrovskii, Weak∗ sequential closures in Banach space theory and their applications, in: General Topology in Banach Spaces, ed. by T. Banakh and A. Plichko, New York, Nova Sci. Publishers, 2001, pp. 21–34; Available at https://arxiv.org/abs/math/0203139.

[29] M. I. Ostrovskii, Weak∗ closures and derived sets in dual Banach spaces. Note Mat. 31 (2011), no. 1, 129–138.

[30] A. Pelczyński, A note on the paper of I. Singer “Basic sequences and reflexivity of Banach spaces”, Studia Math., 21 (1961/1962), 371–374.

[31] A. Pelczyński, C. Bessaga, Some aspects of the present theory of Banach spaces. In: S. Banach, Œuvres. Vol. II. Travaux sur l’analyse fonctionnelle. Edited by C. Bessaga, S. Mazur, W. Orlicz, A. Pelczyński, S. Rolewicz and W. Żelazko. PWN Éditions Scientifiques de Pologne, Warsaw, 1979.

[32] Y. I. Petunin, A. N. Plichko, The theory of characteristic of subspaces and its applications (Russian), Vyshcha Shkola, Kiev, 1980.
[33] I. I. Piatetski-Shapiro, An addition to the work “On the problem of uniqueness of expansion of a function in a trigonometric series” (Russian), Moskov. Gos. Univ. Uˇ c. Zap. Mat., 165 (1954), no. 7, 79–97; English translation in: I. Piatetski-Shapiro, Selected works, Edited by J. Cogdell, S. Gindikin, P. Sarnak, American Mathematical Society, Providence, RI, 2000.

[34] A. Plichko, On bounded biorthogonal systems in some function spaces, Studia Math., 84 (1986), 25–37.

[35] A. Plichko, Decomposition of Banach space into a direct sum of separable and reflexive subspaces and Borel maps, Šer dica Math. J., 23 (1997) 335–350.

[36] M. Raja, Borel properties of linear operators, J. Math. Anal. Appl., 290 (2004), 63–75.

[37] J. Saint-Raymond, Espaces a modèle séparable, Ann. Inst. Fourier (Grenoble), 26 (1976), 211–256.

[38] M. de la Salle, A duality operators/Banach spaces, arXiv:2101.07666v2

[39] D. Sarason, A remark on the weak-star topology of $\ell^\infty$, Studia Math., 30 (1968), 355–359.

[40] D. Sarason, On the order of a simply connected domain. Michigan Math. J. 15 (1968), 129–133.

[41] D. Sarason, Weak-star density of polynomials J. Reine Angew. Math. 252 (1972), 1–15.

[42] Z. Silber, Weak$^*$ derived sets of convex sets in duals of non-reflexive spaces, J. Funct. Anal. 281 (2021), no. 12, Paper No. 109259; DOI:10.1016/j.jfa.2021.109259

[43] I. Singer, Basic sequences and reflexivity of Banach spaces, Studia Math., 21 (1961/1962), 351–369.

[44] W. Szlenk, The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces. Studia Math., 30 (1968), 53–61.

[45] W.T. Trotter, Combinatorics and partially ordered sets. Dimension theory. Johns Hopkins Series in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 1992.

Department of Mathematics and Computer Science, St. John’s University, 8000 Utopia Parkway, Queens, NY 11439, USA

E-mail address: ostrovsm@stjohns.edu