Brownian Motion for the School-Going Child

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I. Introduction

Let us do a “thought experiment”. What is a thought experiment? It is an experiment carried out in thought only. It may or may not be feasible in practice, but by imagining it one hopes to learn something useful. There is a German word for it which is commonly used: \textit{Gedankenexperiment}. It was perhaps A. Einstein who popularized this word by his many gedankenexperiments in the theory of relativity and in quantum mechanics.

Coming back to our thought experiment: Imagine a dark, cloudy, moonless night, and suppose there is a power failure in the entire city. You are sitting in your fourth floor apartment thinking and worrying about your physics test tomorrow. Suddenly you hear a commotion downstairs. You somehow manage to find your torch and rush to the window. Now suppose your torch acts funny: it turns on only for a moment, every 15 seconds. Initially, i.e., at time $t = 0$ seconds, you see a man standing in the large open space in front of your building. Before you make out what is happening, your torch is off. Next time it lights up, i.e., at $t = 15$ sec, you see him at a slightly different location. At $t = 30$ sec, he is somewhere else and has changed his direction too. At $t = 45$ sec, he has again changed his location and direction. You have no idea what is going on. But, you continue observing him for some more time. When the lights come back, you mark his positions on a piece of paper (see Fig. 1). At $t = 0$, he is at point A, at $t = 15$, he is at B, at $t = 30$, he is at C, and so on. Connect point A to B, B to C, C to D, and so on, by straight lines. (Go ahead, grab a pencil and do it.) What do you see? A zigzag path.
What do you think was going on? Did you say “a drunken man wandering around the square”? Right. That was easy. One does not need an Einstein’s IQ to figure that out. In physicists’ language, the above is an example of a random walk in two dimensions: two dimensions because the open area in front of your building has a length and a breadth. (Strictly speaking, a walk can be said to be random if the direction of each step is completely independent of the preceding step. For simplicity, the steps may be taken to be of equal length.) Before you read further, close your eyes and imagine a random walk in 1 dimension and then a random walk in 3 dimensions.

Figure 1

![Random walk diagram]

A
B
C
D
E
F
G
H
I
Random walk in one dimension:

Here is an experiment you can do yourselves. You will need a plain paper, a ruler, a pencil, a one-rupee coin, a small stone and a lot of patience. Draw a number line with markings at $-10, -9, ..., 0, ..., 9, 10$. Place the stone at 0. Toss the coin. The rule is, if it is heads (H), the stone is moved one place to the right and if it is tails (T), it is moved one place to the left. For example, if you get H, H, T, ..., the stone moves from 0 to 1, 1 to 2, 2 to 1, ... . Toss the coin 10 times. Note the final position of the stone. Call it $x_1$. Obviously, $-10 \leq x_1 \leq 10$.

Replace the stone at 0 and repeat the experiment. Again note the final position of the stone. Call it $x_2$. Obviously, $x_2$ may or may not be equal to $x_1$.

If you were to repeat this experiment a very large number of times, say 1000 times, and then take the average ($\bar{x}$) of $x_1, x_2, x_3, ..., x_{1000}$, what result do you think you will get? Since each toss of the coin is equally likely to result in a H or a T, $x_1, x_2, x_3, ..., x_{1000}$ will be distributed symmetrically around the origin 0. Hence $\bar{x}$ is most likely to be zero.

Interestingly, however, the average ($\overline{x^2}$) of $x_1^2, x_2^2, x_3^2, ..., x_{1000}^2$, will not be zero, since these are all non-negative numbers. In fact, $\overline{x^2}$ turns out to be equal to the number of times you toss the coin in each experiment, which is also equal to the number of steps ($N$) in the random walk. (This is 10 in our experiment.) Thus $\overline{x^2} = N$ or $(\overline{x^2})^{1/2} = N^{1/2}$. Since the left-hand-side is the square root of the mean (= average) of the squares, it is called the rms displacement and is denoted by $x_{\text{rms}}$. Thus $x_{\text{rms}} = N^{1/2}$.

What is the meaning of the statement $\bar{x} = 0$, but $x_{\text{rms}} = N^{1/2}$? It means, in a random walk, the object is as likely to be found on one side of the starting point as on the other, making $\bar{x}$ vanish. But at the same time, as the number of steps increases, the object is likely to be found farther and farther from the starting point.

Equivalently, imagine 1000 drunkards standing at the origins of 1000 parallel lines, and then starting simultaneously their random walks along these lines. If you observe them after a while, there will be nearly as many of them to the right of the centres as there are to the left. Moreover, the longer you observe them, the farther they are likely to drift from the centre.

Conclusions: (a) $\bar{x} = 0$. (b) $x_{\text{rms}} = N^{1/2}$ if each step is of unit length. (c) $x_{\text{rms}} = N^{1/2}l$ if each step is of length $l$.

Let us perform another thought experiment. Suppose you are sitting in a big stadium, watching a game of football or hockey, being played between two equally good teams. As in the previous thought experiment, you mark on a piece of paper the position of the ball every 15 seconds, and then connect these positions in sequence. What do you see? Again a zigzag path. The ball is moving almost like the drunken man. Would you say the ball is drunk? Of course, not. The ball is moving that way because it is being hit repeatedly by
the players in two competing teams. This is another example of an (almost) random motion in two dimensions.

Want to impress someone? Remember this: Random processes are also called stochastic processes. Chance or probability plays an essential role in these processes.

What you learnt above is the ABC of the branch of physics, called Statistical Mechanics.

II. History

*He is happiest who hath power to gather wisdom from a flower — Mary Howitt (1799 - 1888).*

Now I want to describe a real (not a gedanken) experiment. Robert Brown was a British botanist. In 1827, he observed through a microscope pollen grains of some flowering plants. To his surprise, he noticed that tiny particles suspended within the fluid of pollen grains were moving in a haphazard fashion.\(^1\) If you were Robert Brown, how would you understand this observation? (Remember, science in 1827 was not as advanced as it is today. Many things written in your science textbook were not known then.) Would you suspect that the pollen grain is alive? Or would you get excited at the thought that you have discovered the very essence of life or a latent life force within every pollen? Or perhaps this is just another property of organic matter? What other experiments would you perform to test your suspicions?

Brown repeated his experiment with other fine particles including the dust of igneous rocks, which is as inorganic as could be. He found that any fine particle suspended in water executes a similar random motion. This phenomenon is now called Brownian Motion. Figure 2 shows the result of an actual experiment: the positions of the particle were recorded at intervals of 30 seconds. (From J. Perrin, *Atoms*, D. Van Nostrand Co., Inc., 1923.) Similar observations were made for tiny particles suspended in gases.

Scientists in the 19th century were puzzled by this mysterious phenomenon. They attempted to understand it with the help of ideas such as convection currents, evaporation, interaction with incident light, electrical forces, etc. But they had no satisfactory explanation for it. With your knowledge of modern science, can you provide a rudimentary explanation? Obviously, the suspended particle is not moving on its own unlike the drunkard in our first gedankenexperiment. Why then is it moving? And why in an erratic way (see Fig. 2)? Think, before you read further.

Want a hint? Recall our second gedankenexperiment.
III. Basic Understanding

If you have not already guessed, here is the rational explanation for the mysterious jerky movement of tiny particles suspended in fluids, which made Mr. Brown famous:

- The size — the radius or diameter — of the suspended particle is roughly of the order of a few microns (1 micron = 10^{-6} m). The size of an atom is of the order of 10^{-10} m. The size of a water molecule (H₂O) is somewhat larger. Thus the suspended particle is a monster, about 10000 times bigger compared to a water molecule. Also note that a spoonful of water contains about 10^{23} water molecules. (The atomic or molecular theory of matter which says that matter consists of atoms and molecules, is well-established today. It was not so in 1827!)

- You also know that molecules of water (or molecules in any sample of a liquid or gas) are not at rest. They are perpetually moving in different directions, some faster than others. As they move, they keep colliding with each other, which can possibly change their speeds and directions of motion.

- Now you can very well imagine the fate of the particle unfortunate enough to be placed in the mad crowd of water molecules. The poor fellow is getting hit, at any instant, from all sides, by millions of water molecules. The precise number of water molecules which hit the particle at any instant, the exact points where they hit it, their speeds and directions — all keep changing from time to time. (It is practically impossible and also unnecessary to have this information.) This results in a net force which keeps fluctuating in time, i.e., its magnitude and direction keep changing from one instant to another. The particle keeps getting kicks in the direction of the instantaneous net force. The end result is that its position
keeps changing randomly as in Fig. 2.

IV. A Quiz

Ready for a quiz? Here are a few easy questions:

(1) Imagine the game of football played by invisible players. (Nothing except the ball is visible.)

(2) See Fig. 2. If the positions of the particle were recorded every 60 seconds, instead of every 30 seconds, how will the pattern look like?

(3) How will the Brownian Motion be affected if (a) water is cooled, or (b) instead of water a more viscous liquid is taken, or (c) the experiment is done with a bigger particle?

V. Einstein’s Contribution

You now have a qualitative understanding of the Brownian Motion. But that is usually not enough. Scientists like to develop a quantitative understanding of a phenomenon. This allows them to make precise numerical predictions which can be tested in the laboratory. For example, one would like to know how far (on an average) will the particle drift from its initial position in say 10 minutes? How will its motion be affected if the water is cooled by say 5°C, or if the viscosity of the liquid is increased by 10%, or if the particle size is exactly doubled?

In 1905, Einstein published a detailed mathematical theory of the Brownian Motion, which allowed one to answer these and many other interesting questions. How did he do it? I will only give you a flavour of what is involved.
What is an ensemble?:

Recall that chance or probability plays an important role in random processes. Hence, in the Introduction, when we discussed the random walk (see the box), you were asked to do the experiment 1000 times and then average the results. If you do it only a few times, \( \bar{x} \) may not vanish and \( \bar{x^2} \) may not equal 10. If you are too lazy to do the experiment 1000 times, there is a way out: Get hold of 1000 friends of yours, ask each of them to prepare a similar experimental set-up, and let each of them do the experiment only once. If you then take the average of the results obtained by them, you will find \( \bar{x} \approx 0 \) and \( \bar{x^2} \approx 10 \).

Similarly, if you observe the Brownian Motion of a particle only a few times, based on these observations, you would not be able to make quantitative statements about its average behaviour. You need to repeat the experiment a large number of times and take the average of all the results. Alternately, you could prepare a large assembly of identical particles, observe each of them once under identical experimental conditions, and then take the average.

Physicists use the word *ensemble* to describe such an imaginary assembly of a very large number of similarly prepared physical systems. The average taken over all the members of the ensemble is called an ensemble average. In the following, when we talk about an average behaviour of a Brownian particle, we mean an ensemble average.
Let us ask ourselves a few simple questions about the average behaviour of a tiny particle suspended in a liquid. Taking the initial location of the particle as the origin, imagine drawing x, y and z axes in the liquid. Let \((x, y, z)\) denote the coordinates of the particle.

\[ x: \] Where will the particle be after some time? In other words, what will be the values of \(\bar{x}, \bar{y}\) and \(\bar{z}\) after some time? (Remember the overhead lines denote ensemble averages.) Since this is a case of a random walk in 3 dimensions, \(\bar{x} = \bar{y} = \bar{z} = 0\).

\[ \bar{v}_x: \] What will be the average velocity of the particle parallel to the \(x\) axis? When we talk of the velocity of a particle, we have \textit{two} things in our mind: its speed (fast or slow) and its direction of motion. Since the particle is as likely to move in the positive-\(x\) direction as in the negative-\(x\) direction, \(v_x\) is as likely to be positive as negative. Hence \(\bar{v}_x = 0\). Similarly, \(\bar{v}_y = \bar{v}_z = 0\).

\[ \bar{v}_x^2: \] What will be the value of \(\bar{v}_x^2\)? This ensemble average will not be zero since \(v_x^2\) is either positive or zero — never negative. I already said that molecules of water are not at rest. They are perpetually moving in different directions, some faster than others. Now, heat is a form of energy. When we heat water, we give energy to it. As a result, the water molecules start moving faster. Their average kinetic energy rises. On the other hand, when we heat water, its temperature also rises. Thus temperature is a measure of the average kinetic energy of the water molecules. It turns out that when the suspended particle is in thermal equilibrium with the water, its average kinetic energy is proportional to the temperature:

\[
\frac{1}{2}mv_x^2 = \frac{1}{2}kT,
\]

where \(m\) is the mass of the suspended particle, \(k\) is a constant and \(T\) is the absolute temperature of the water. Hence \(\bar{v}_x^2 = kT/m\). This implies that heavier particles will have smaller \(\bar{v}_x^2\). Similar statements can be made about the motion in \(y\) and \(z\) directions.

\[ x^2: \] How far from the origin will the particle be after some time? Equivalently, what will be the value of \(x^2\)? Before I answer this question, note the important complication in the present problem. Now, not only the direction but also the size of each step is a variable and is completely independent of the preceding step. (Why? Remember the fluctuating force mentioned in section III.) Using the ideas from Statistical Mechanics, Einstein derived the following result:

\[
\overline{x^2} = \frac{kT}{3\pi \eta a} t,
\]

where \(\eta\) is the viscosity of the liquid, \(a\) is the radius of the suspended particle (assumed to be spherical) and \(t\) is the elapsed time. Thus the mean square displacement \(\overline{x^2}\) increases \textit{linearly} with time (i.e., the power of \(t\) in the above equation is unity).

Looking at the last equation, can you now answer the question no. (3) in the Quiz above? Please find out yourselves, before you read the answers given here:
(a) The Brownian Motion is less vigorous in cold water than in hot water. (b) The Brownian Motion will be damped if water is replaced by a more viscous liquid. (c) A bigger particle will drift less than a smaller particle — we do not notice the Brownian Motion of fish, people or boats. Do we?

Using Einstein’s result, one can also answer quantitatively the more specific questions listed at the beginning of section V.

VI. Importance

- In 1908, the French physicist Jean-Baptiste Perrin verified Einstein’s result experimentally. He measured $x^2$ as a function of time $t$. Knowing the temperature $T$ and viscosity $\eta$ of the water, and radius $a$ of the particle, he could obtain the value of the constant $k$. Using this, he obtained a reasonably good value for Avogadro’s number (no. of molecules in a mole of a substance).

- Einstein’s explanation of the Brownian Motion and its subsequent experimental verification by Perrin were historically important because they provided a convincing evidence for the molecular theory of matter. In other words, they showed that atoms and molecules are real physical objects. Skeptics who doubted their existence were silenced.

- Fluctuating force on a Brownian particle is but one example of a fluctuating physical quantity even when the system is in equilibrium. Another example is a fluctuating current in some electric circuits. Einstein’s work on the Brownian Motion laid the foundations of the study of fluctuation phenomena as a branch of statistical mechanics.

We have reached the end of our story of the Brownian Motion. You must have realized how a lowly pollen grain can tell us so much about the constitution of matter. Note that nothing of this would have been possible without the inquisitive mind of the scientist. The following quotation comes to my mind:

*There is something fascinating about science. One gets such wholesale returns of conjecture out of such a trifling investment of fact* — Mark Twain (1835 - 1910).
Footnotes:

1 Sometimes it is wrongly stated that Brown observed irregular motion of the pollen grains themselves. Secondly, he was not the first to notice this phenomenon. But he was the first to stress its ubiquitousness and to rule out its explanations based on the so-called life force. As a result of his work, this subject was removed from the realm of biology into the realm of physics.

2 However good a theory may appear, if its predictions do not agree with experimental data, it is discarded.

3 Here we assumed that successive time intervals are very small compared with the observation time but still large enough that the motion of the suspended particle in any time interval can be considered to be completely independent of its motion in the preceding time interval.

4 Einstein showed that it is practically impossible to measure $\bar{v}_x^2$ and suggested that experimentalists should rather measure $\bar{x}^2$.

5 Perrin was honoured with the Nobel Prize for Physics in 1926, for this work.