1 Introduction

In this paper we study a class of fully nonlinear second-order elliptic equations of the form

\[ F(D^2 u) = 0 \]

defined in a domain of \( \mathbb{R}^n \). Here \( D^2 u \) denotes the Hessian of the function \( u \). We assume that \( F \) is a Lipschitz function defined on an open set \( D \subseteq S^2(\mathbb{R}^n) \) of the space of \( n \times n \) symmetric matrices satisfying the uniform ellipticity condition, i.e. there exists a constant \( C = C(F) \geq 1 \) (called an ellipticity constant) such that

\[ C^{-1}||N|| \leq F(M + N) - F(M) \leq C||N|| \]

for any non-negative definite symmetric matrix \( N \); if \( F \in C^1(D) \) then this condition is equivalent to

\[ \frac{1}{C'} \xi_i \xi_j \leq F_{u_{ij}} \xi_i \xi_j \leq C' \xi_i \xi_j, \forall \xi \in \mathbb{R}^n. \]

Here, \( u_{ij} \) denotes the partial derivative \( \partial^2 u / \partial x_i \partial x_j \). A function \( u \) is called a classical solution of (1) if \( u \in C^2(\Omega) \) and \( u \) satisfies (1). Actually, any classical solution of (1) is a smooth \( (C^{\alpha+3}) \) solution, provided that \( F \) is a smooth \( (C^\alpha) \) function of its arguments.

For a matrix \( S \in S^2(\mathbb{R}^n) \) we denote by \( \lambda(S) = \{ \lambda_1 : \lambda_1 \leq ... \leq \lambda_n \} \in \mathbb{R}^n \) the (ordered) set of eigenvalues of the matrix \( S \). Equation (1) is called a Hessian equation.

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if the function \( F(S) \) depends only on the eigenvalues \( \lambda(S) \) of the matrix \( S \), i.e., if

\[
F(S) = f(\lambda(S)),
\]

for some function \( f \) on \( \mathbb{R}^n \) invariant under permutations of the coordinates.

In other words the equation (1) is called Hessian if it is invariant under the action of the group \( O(n) \) on \( S^2(\mathbb{R}^n) \):

\[
(3) \quad \forall O \in O(n), \quad F(O \cdot S \cdot O) = F(S).
\]

If we assume that the function \( F(S) \) is defined for any symmetric matrix \( S \), i.e., \( D = S^2(\mathbb{R}^n) \) the Hessian invariance relation (3) implies the following:

(a) \( F \) is a smooth (real-analytic) function of its arguments if and only if \( f \) is a smooth (real-analytic) function.

(b) Inequalities (2) are equivalent to the inequalities

\[
\frac{\mu}{C_0} \leq f(\lambda_i + \mu) - f(\lambda_i) \leq C_0 \mu, \quad \forall \mu \geq 0,
\]

\( \forall i = 1, ..., n \), for some positive constant \( C_0 \).

(c) \( F \) is a concave function if and only if \( f \) is concave \([Ba, CNS]\] .

Well known examples of the Hessian equations are Laplace, Monge-Ampère, and Special Lagrangian equations.

We are interested also in Isaacs equations which are uniformly elliptic but in general not Hessian. Bellman and Isaacs equations appear in the theory of controlled diffusion processes. The both are fully nonlinear uniformly elliptic equations of the form (1). The Bellman equation is concave in \( D^2u \in S^2(\mathbb{R}^n) \) variables. However, Isaacs operators are, in general, neither concave nor convex.

In a simple homogeneous form the Isaacs equation can be written as follows:

\[
(4) \quad F(D^2 u) = \sup_b \inf_a L_{ab} u = 0,
\]

where \( L_{ab} \) is a family of linear uniformly elliptic operators with an ellipticity constant \( C > 0 \) which depends on two parameters \( a, b \). Consider the Dirichlet problem

\[
(5) \quad \begin{cases}
F(D^2 u) = 0 & \text{in } \Omega \\
u = \varphi & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary \( \partial \Omega \) and \( \varphi \) is a continuous function on \( \partial \Omega \).

We are interested in the problem of existence and regularity of solutions to Dirichlet problem (5) for Hessian and Isaacs equations. Dirichlet problem (5) has
always a unique viscosity (weak) solution for fully nonlinear elliptic equations (not necessarily Hessian equations). The viscosity solutions satisfy the equation (1) in a weak sense, and the best known interior regularity ([C,CC], cf. [T3]) for them is $C^{1+\epsilon}$ for some $\epsilon > 0$. For more details see [CC,CIL]. Until recently it remained unclear whether non-smooth viscosity solutions exist. In [NV1] we proved the existence of viscosity solutions to the fully nonlinear elliptic equations which are not classical in dimension 12. Moreover, we proved in [NV2], that in 24-dimensional space the optimal interior regularity of viscosity solutions of fully nonlinear elliptic equations is no more than $C^{2-\delta}$. Both papers [NV1,NV2] use the function 

$$w = \frac{\text{Re}(q_1 q_2 q_3)}{|x|^3},$$

where $q_i \in \mathbb{H}$, $i = 1, 2, 3$, are Hamiltonian quaternions, $x \in \mathbb{H}^3 = \mathbb{R}^{12}$ which is a viscosity solution in $\mathbb{R}^{12}$ of a uniformly elliptic equation (1) with a smooth $F$.

The proofs use some remarkable algebraic identities verified by (the spectrum of the Hessian of) the function $w$. One notes also that the example by Harvey-Lawson-Osserman [LO,HL] of a Lipshitz non-analytic solution to the associator (minimal surface) equation strongly resembles our function. Moreover a suitable version of an octonion analogue [NV3] of $w$ is reminds the associative calibration and its modifications remind coassociative and Caley calibrations [HL]. In our opinion these connections deserve a further study.

The main goal of this paper is to show that the same function $w$ is a solution to a Hessian equation. Moreover the following theorem holds

**Theorem 1.1.** For any $\delta$, $0 \leq \delta < 1$ the function

$$w/|x|^\delta$$

is a viscosity solution to a uniformly elliptic Hessian equation (1) in a unit ball $B \subset \mathbb{R}^{12}$.

Theorem 1.1 shows that the second derivatives of viscosity solutions of Hessian equations (1) can blow up in an interior point of the domain and that the optimal interior regularity of the viscosity solutions of Hessian equations is no more than $C^{1+\epsilon}$, thus showing the optimality of the result by Caffarelli-Trudinger [C,CC,T3] on the interior $C^{1,\alpha}$-regularity of viscosity solutions of fully nonlinear equations. Our construction provides a Lipschitz functional $F$ in Theorem 1.1. Using a more complicated argument one can make $F$ smooth; we will return to this question elsewhere. However, if we drop the invariance condition (3) we get

**Corollary 1.1.** For any $\delta$, $0 \leq \delta < 1$ the function

$$w/|x|^\delta$$
is a viscosity solution to a uniformly elliptic (not necessarily Hessian) equation (1) in a unit ball $B \subset \mathbb{R}^{12}$ where $F$ is a $(C^\infty)$ smooth functional.

We show that the same function is a viscosity solution to a uniformly elliptic Isaacs equation:

**Theorem 1.2.**

For any $\delta$, $0 \leq \delta < 1$ the function

$$w/|x|^\delta$$

is a viscosity solution to a uniformly elliptic Isaacs equation (1.4) in a unit ball $B \subset \mathbb{R}^{12}$.

The question on the minimal dimension $n$ for which there exist nontrivial homogeneous order 2 solutions of (1) remains open. We notice that from the result of Alexandrov [A] it follows that any homogeneous order 2 solution of the equation (1) in $\mathbb{R}^3$ with a real analytic $F$ should be a quadratic polynomial. For a smooth and less regular $F$ similar results in the dimension 3 can be found in [HNY].

However, we are able reduce this dimension by one to 11. Moreover the following theorem holds

**Theorem 1.3.** For any hyperplane $H \subset \mathbb{R}^{12}$ the function $w$ restricted to $H = \mathbb{R}^{11}$ is a viscosity solution to a uniformly elliptic Hessian equation (1) in a unit ball $B \subset \mathbb{R}^{11}$ where $F$ is a Lipschitz functional.

If we drop the invariance condition (3) we get

**Corollary 1.2.** For any hyperplane $H \subset \mathbb{R}^{12}$ the function $w$ restricted to $H = \mathbb{R}^{11}$ is a viscosity solution to a uniformly elliptic (not necessarily Hessian) equation (1) in a unit ball $B \subset \mathbb{R}^{11}$ where $F$ is a $(C^\infty)$ smooth functional.

Note, however that our technique here is not sufficient to get singular (i.e. with unbounded second derivatives) solution in eleven dimensions, see Remark 6.2 below.

Ball $B$ in Theorem 1.1 can not be substituted by the whole space $\mathbb{R}^{12}$. In fact, for any $0 < \alpha < 2$ there are no homogeneous order $\alpha$ solutions to the fully nonlinear elliptic equation (1) defined in $\mathbb{R}^n \setminus \{0\}$, [NY]; the essence of the difference with the local problem is that in the case of homogeneous solution defined in $\mathbb{R}^n \setminus \{0\}$ one deals simultaneously with two singularities of the solution: one at the origin and another at the infinity. In the local problem the structure of singularities of solutions is quite different, even in dimension 2,
the function \( u = |x|^\alpha, \; 0 < \alpha < 1, \; x \in B^o \), where \( B^o \) is a punctured ball in \( \mathbb{R}^n, \; n \geq 2 \), \( B^o = \{ x \in \mathbb{R}^n, 0 < |x| < 1 \} \), is a solution to the uniformly elliptic Hessian equation in \( B^o \) (notice that \( u \) is not a viscosity solution of any elliptic equation on the whole ball \( B \)).

We study also the possible singularity of solutions of Hessian equations defined in a neighborhood of a point. We prove the following general result:

**Theorem 1.4.** Let \( u \) be a viscosity solution of a uniformly elliptic Hessian equation in a punctured ball \( B^o \subset \mathbb{R}^n \). Assume that \( u \in C^0(B^o) \). Then \( u = v + l + o(|x|^{1+\varepsilon}) \), where \( v \) is a monotone function of the radius, \( v(x) = v(|x|) \), \( v \in C^\varepsilon(B^o) \), where \( \varepsilon > 0 \) depends on the ellipticity constant of the equation, and \( l \) is a linear function.

As an immediate consequence of the theorem we have

**Corollary 1.3.** Let \( u \) be a homogeneous order \( \alpha, \; 0 < \alpha < 1 \) solution of a uniformly elliptic Hessian equation in a punctured ball \( B^o \subset \mathbb{R}^n \). Then \( u = c|x|^{\alpha} \).

The rest of the paper is organized as follows: in Section 2 we give a sufficient condition for validity of Theorem 1.1, we verify it in Section 3 for \( \delta = 0 \) and then in Section 4 for any \( 1 > \delta \geq 0 \). Section 5 is devoted to a proof of Theorem 1.2, Section 6 proves Theorem 1.3, and Section 7 contains a proof of Theorem 1.4.

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Since the proof of Theorem 1.1 in Sections 3 and 4 is somewhat involved and utilize computer (MAPLE) computations, we give here an account of its logical structure and its principal points. First of all, the criterion of ellipticity in Section 2 reduces Theorem 1.1 for \( \delta = 0 \) to the uniform hyperbolicity of \( \lambda(\text{Hess}(P)(a)) - \lambda(\text{Hess}(P)(b)) = O \) for a pair \( a \neq b \) of unit vectors and an orthogonal matrix \( O \). A classical result by H. Weyl on the eigenvalues of the difference of two symmetric matrices reduces this to the uniform hyperbolicity of the difference \( \lambda(\text{Hess}(P)(a)) - \lambda(\text{Hess}(P)(b)) \). Recall then [NV1, Section 3] that the characteristic polynomial \( CH(P,a)(T) \) of the Hessian \( \text{Hess}(P)(a) \) of the cubic form \( P \) has for \( a \in S^1 \) the following form:

\[
CH(P,a)(T) = (T^3 - T + 2m(a))(T^3 - T - 2m(a))(T^3 - T + 2P(a))^2,
\]

where \( m(a) \geq |P(a)| \) which permits to conclude that the structure of the (ordered) spectrum is as follows

\[
\mu_1 = \mu_1' \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \mu_2 = \mu_2' \geq -\lambda_3 \geq -\lambda_2 \geq -\lambda_1 \geq \lambda_3 \geq \mu_3 = \mu_3'.
\]
Let 2 Ellipticity of Theorem 1.1.

from that of \((\lambda_m + 2m(a))(T^3 - T - 2m(a)).\) The argument of Section 3 is based on the calculation of the (shifted) characteristic polynomial \(CH(w, a)(T - P(a))\) of the full Hessian \(Hess(w)(a)\) which is possible thanks to an action of the group \(Sp(1) \times Sp(1) \times Sp(1)\) which does not change this polynomial. This action permits to bring the matrix \(Hess(w)(a)\) to a simple block form and gives using a MAPLE calculation an explicit formula for \(CH(w, a)(T - P(a)):\)

\[
CH(w, a)(T - P(a)) = P_6(a, T)(T^3 - T + 2P(a))^2
\]

for a certain explicit polynomial \(P_6(a, T);\) in fact \(P_6(a, T)\) is the (shifted) characteristic polynomial of \(Hess(wb)(a')\) for a 6-dimensional version of \(w\) and an appropriate 6-dimensional unit vector \(a'.\) The crucial point then is that the spectrum in this case is not so different from that of \(Hess(P)(a).\) In fact, one has for this ordered spectrum:

\[
\mu_1 = \mu'_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \lambda_3 \geq \mu_2 = \mu'_2 \geq \lambda'_3 \geq \lambda_4 \geq \lambda'_4 \geq \lambda_5 \geq \mu_3 = \mu'_3
\]

where \(\lambda'_1 \geq \lambda'_2 \geq \lambda'_3 \geq \lambda'_4 \geq \lambda'_5 \geq \lambda'_6\) are the roots of \(P_6(a, T).\) To prove this inequalities one verifies it for specific points \(a\) and then explicitly calculates (using MAPLE) the resultant which (miraculously) vanishes nowhere and thus gives the necessary inequalities. This garanties the exact formula for the equal 6th and 7th eigenvalues which permits to get the necessary uniform hyperbolicity of the difference \(\lambda(Hess(P)(a)) - \lambda(Hess(P)(b)).\)

In Section 4 we generalize this argument to any \(\delta \in [0, 1[.\) In this situation we need the uniform hyperbolicity of \(Hess(P)(a) - K' O \cdot Hess(P)(b) \cdot O\) for a pair \(a \neq b\) of unit vectors, any orthogonal matrix \(O\) and any positive constant \(K,\) which follows from that of \(\lambda(Hess(P)(a)) - K\lambda(Hess(P)(b)).\) We begin with the uniform hyperbolicity of the difference \((\mu_1(a), \mu_2(a), \mu_3(a)) - K(\mu_1(b), \mu_2(b), \mu_3(b))\) which is rather elementary since there are simple trigonometric formulas for \(\mu_i.\) Unfortunately, the position of \(\mu_2\) in the ordered spectrum of \(Hess(P)(a)\) is not fixed anymore, which follows from an explicit calculation of \(CH(wb, a)(T - (1+\delta)P(a))\) together with some resultant calculations similar (but more involved) to those in Section 3. However, the position of the double value \(\mu_2 = \mu'_2\) varies from \((5, 6)\) to \((7, 8)\) and an argument using the oddness of \(w_3\) permits to deduce the uniform hyperbolicity of \(\lambda(Hess(P)(a)) - K\lambda(Hess(P)(b))\) from that of \((\mu_1(a), \mu_2(a), \mu_3(a)) - K(\mu_1(b), \mu_2(b), \mu_3(b))\) which finishes the proof of Theorem 1.1.

\section{Ellipticity}

Let \(w\) be a homogeneous function of order \(2 - \delta,\) \(0 \leq \delta < 1,\) defined on a unit ball \(B = B_1 \subset \mathbb{R}^n\) and smooth in \(B \setminus \{0\}.\) Then the Hessian of \(w\) is homogeneous
of order \((-\delta)\). Define the map

\[ \Lambda : B \rightarrow \lambda(D^2 w) \in \mathbb{R}^n. \]

Let \( K \subset \mathbb{R}^n \) be an open convex cone, such that

\[ \{ x \in \mathbb{R}^n : x_i \geq 0, \ i = 1, ..., n \} \subset K. \]

Set

\[ L := \mathbb{R}^n \setminus (K \cup -K). \]

We say that a set \( E \subset \mathbb{R}^n \) satisfy \( K \)-cone condition if \((a - b) \in L\) for any \( a, b \in E \).

Let \( S_n \) be the group of permutations of \( \{1, ..., n\} \). For any \( \sigma \in S_n \), we denote by \( T_\sigma \) the linear transformation of \( \mathbb{R}^n \) given by \( x_i \mapsto x_{\sigma(i)}, \ i = 1, ..., n \).

**Lemma 2.1.** Assume that \( M := \bigcup_{\sigma \in S_n} T_\sigma \Lambda(B) \subset \mathbb{R}^n \)

satisfies the \( K \)-cone condition. If \( \delta > 0 \) we assume additionally that \( w \) changes sign in \( B \). Then \( w \) is a viscosity solution in \( B \) of a uniformly elliptic Hessian equation (1).

**Proof.** Let us choose in the space \( \mathbb{R}^n \) an orthogonal coordinate system \( z_1, ..., z_{n-1}, s \), such that \( s = x_1 + ... + x_n \). Let \( \pi : \mathbb{R}^n \rightarrow Z \) be the orthogonal projection of \( \mathbb{R}^n \) onto the \( z \)-space. Let \( K^* \) denote the adjoint cone of \( K \), that is, \( K^* = \{ b \in \mathbb{R}^n : b \cdot c \geq 0 \text{ for all } c \in K \} \). Notice that \( a \in L \) implies \( a \cdot b = 0 \) for some \( b \in K^* \). We represent the boundary of the cone \( K \) as the graph of a Lipschitz function \( s = e(z) \), with \( e(0) = 0 \), function \( e \) is smooth outside the origin:

\[ e(z) = \inf \{ c : (z + cs) \in K \}. \]

Set \( m = \pi(M) \). We prove that \( M \) is a graph of a Lipschitz function on \( m \),

\[ M = \{ z \in m : s = g(z) \}. \]

Let \( a, \hat{a} \in M, a = (z, s), \hat{a} = (\hat{z}, \hat{s}) \). Since \( a - \hat{a} \in L \), we have

\[ -e(z - \hat{z}) \leq \hat{s} - s \leq e(z - \hat{z}). \]

Since \( e(0) = 0, g(z) := s \) is single-valued. Also

\[ |g(z) - g(\hat{z})| = |s - \hat{s}| \leq |e(z - \hat{z})| \leq C|z - \hat{z}|. \]
The function $g$ has an extension $\tilde{g}$ from the set $m$ to $\mathbb{R}^{n-1}$ such that $\tilde{g}$ is a Lipschitz function and the graph of $\tilde{g}$ satisfies the $K$-cone condition. One can define such extension $\tilde{g}$ simply by the formula

$$\tilde{g}(z) := \inf_{w \in m} \{ g(w) + e(z - w) \} .$$

To show that this formula works let $(z, \tilde{g}(z)), (\hat{z}, \tilde{g}(\hat{z}))$ lie in the graph $\tilde{g}$. We must show

$$-e(z - \hat{z}) \leq \tilde{g}(z) - \tilde{g}(\hat{z}) \leq e(z - \hat{z}).$$

Now

$$\tilde{g}(\hat{z}) = g(w) + e(\hat{z} - w)$$

for some $w \in m$. Thus

$$\tilde{g}(z) - \tilde{g}(\hat{z}) \leq g(w) + e(z - w) - (g(w) + e(\hat{z} - w)) \leq e(z - \hat{z}),$$

since $e(a + b) \leq e(a) + e(b)$, as $e(\cdot)$ is convex, homogenous. Similarly

$$\tilde{g}(z) - \tilde{g}(\hat{z}) \geq -e(z - \hat{z}).$$

Let us set

$$f' := s - \tilde{g}(z).$$

Since the level surface of the function $f'$ satisfies $K$-cone condition it follows that $\nabla f \in K^*$ a.e. where $K^*$ is the adjoint cone to $K$. Moreover the function $w$ satisfies the equation

$$f'(\lambda(D^2 w)) = 0.$$

on $B \setminus \{0\}$. Set

$$f = \sum_{\sigma \in S_n} f'(\sigma(x)).$$

Then $f$ is a Lipschitz function invariant under the action of the group $S_n$ and satisfies the equation

$$f(\lambda(D^2 w)) = 0.$$

on $B \setminus \{0\}$. We show now that $w$ is a viscosity solution of (1) on the whole ball $B$.

Assume first that $\delta = 0$. Let $p(x), x \in B$ be a quadratic form such that $p \leq w$ on $B$. We choose any quadratic form $p'(x)$ such that $p \leq p' \leq w$ and there is a point $x' \neq 0$ at which $p'(x') = w(x')$. Then it follows that $F(p) \leq F(p') \leq 0$. Consequently for any quadratic form $p(x)$ from the inequality $p \leq w$ ($p \geq w$) it follows that $F(p) \leq 0$ ($F(p) \geq 0$). This implies that $w$ is a viscosity solution of (1) in $B$ (see Proposition 2.4 in [CC]).

If $0 < \delta < 1$ then for any smooth function $p$ in $B$ the function $w - p$ changes sign in any neighborhood of 0. Hence, by the same proposition in [CC], it follows that $w$ is a viscosity solution of (1) in $B$. 

8
3 Non-classical solution

This section is devoted to a proof of Theorem 1.1 in the case of $\delta = 0$ i.e. for a non-classical, but not singular, solution.

We define the cubic form $P$ which is used to construct our non-classical and singular solutions. Let $X = (r, s, t) \in \mathbb{R}^{12}$ be a variable vector with $r, s,$ and $t \in \mathbb{R}^4$.

For any $t = (t_0, t_1, t_2, t_3) \in \mathbb{R}^4$ we denote by $qt = t_0 \cdot i + t_1 \cdot j + t_2 \cdot k \in \mathbb{H}$ (Hamilton quaternions).

Define the cubic form $P(X) = P(r, s, t)$ as follows

$$P(r, s, t) = \text{Re}(qr \cdot qs \cdot qt) = r_0s_0t_0 - r_0s_1t_1 - r_0s_2t_2 - r_0s_3t_3$$

$$-r_1s_0t_1 - r_1s_1t_0 - r_1s_2t_3 + r_1s_3t_2 + r_2s_0t_2 - r_2s_1t_3 - r_2s_2t_0 - r_2s_3t_1$$

$$-r_3s_0t_3 - r_3s_1t_2 + r_3s_2t_1 - r_3s_3t_0;$$

and denote

$$w(X) = \frac{P(X)}{|X|}.$$

Note that by definition one has $|P(X)| \leq \frac{|X|^3}{3\sqrt{3}}$, since

$$|P(r, s, t)| \leq |r| \cdot |s| \cdot |t| \leq \left(\frac{r^2 + s^2 + t^2}{3}\right)^{3/2}.$$

In particular for $X \in S_{11}$ one has $|P(X)| = |w(X)| \leq \frac{1}{3\sqrt{3}}$. For $a \in \mathbb{R}^{12} - \{0\}$ we denote by $H(a)$ the Hessian $D^2w(a)$.

**Proposition 3.1.** Let $a \neq b \in S_{11}$ and let $O \in O(12)$ be an orthogonal matrix s.t. $H(a, b, O) := H(a) - tO \cdot H(b) \cdot O \neq 0$. Denote $\Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_{12}$ the eigenvalues of the matrix $H(a, b, O)$. Then

$$\frac{1}{26} \leq \frac{\Lambda_1}{-\Lambda_{12}} \leq 26.$$

We need the following property of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of real symmetric matrices of order $n$:

**Property 3.1.** Let $A, B$ be two real symmetric matrices with the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\lambda'_1 \geq \lambda'_2 \geq \ldots \geq \lambda'_n$ respectively. Then for the eigenvalues $\Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_n$ of the matrix $A + B$ we have

$$\Lambda_i \geq \lambda_i + \lambda'_j, \quad \Lambda_n \leq \lambda_i + \lambda'_j$$

whenever $i + j = n$.

This is a classical result by Hermann Weyl [We], cf. [Fu], p. 211.
Let $A, B$ be two real symmetric matrices with the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\lambda'_1 \geq \lambda'_2 \geq \ldots \geq \lambda'_n$ respectively. Then for the eigenvalues $\Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_n$ of the matrix $A - B$ we have

$$\Lambda_1 \geq \max_{i=1,\ldots,n} (\lambda_i - \lambda'_i), \quad \Lambda_n \leq \min_{i=1,\ldots,n} (\lambda_i - \lambda'_i).$$

**Main Lemma 3.1.** Let $A := H(a), B := \cdot O \cdot H(b) \cdot O$.

(i) If $P(a) - P(b) \geq 0$ then $\text{Tr}(B - A) = 15(P(a) - P(b)) \leq 15\Lambda_1$;

(ii) If $P(a) - P(b) \leq 0$ then $\text{Tr}(B - A) = 15(P(a) - P(b)) \geq 15\Lambda_{12}$.

**Proof of Proposition 3.1.** We consider only the case $\text{Tr}(A - B) = 15(P(b) - P(a)) \geq 0$, the proof in the other case being symmetric. Since $\text{Tr}(A - B) = \Lambda_1 + \Lambda_2 + \ldots + \Lambda_{12} \geq 0$ one gets $11\Lambda_1 \geq -\Lambda_{12}$. On the other hand,

$$-15\Lambda_{12} \geq \text{Tr}(A - B) = \Lambda_1 + \Lambda_2 + \ldots + \Lambda_{12}$$

implies

$$-26\Lambda_{12} \geq -15\Lambda_1 - \Lambda_2 - \Lambda_3 - \ldots - \Lambda_{12} \geq \Lambda_1$$

which finishes the proof.

To prove Main Lemma we need two lemmas which constitute our principal technical tool. We postpone their proof until the end of the section.

**Lemma 3.2.** Let $a = (r, s, t) \in S_1^{11}$; define

$$W = W(a) = P(a), \quad m = m(a) = |s|, \quad n = n(a) = |t|.$$

Then the characteristic polynomial of the matrix $A := H(a)$ is given by

$$P_A(T) = P_1(T)^2 \cdot P_2(T)$$

where

$$P_1(T) = T^3 + 3WT^2 + 3W^2T - T + W + W^3,$$

$$P_2(T) = T^6 + 9WT^5 + (21W^2 + 3L - 2)T^4 + 2W(7W^2 + 3L - 4)T^3 +
(1 - 6W^2 - 9W^4 - 3L + 9M)T^2 - (15W^4 + 6W^2L - 4W^2 - 6L + 1)WT
-5W^6 - 3LW^4 + 4W^4 - 3(3M + L)W^2 + W^2 - M$$

with $L := L(m, n) = m^2 + n^2 - n^2m^2 - n^4 - m^4 \in \left[M, \frac{1}{3}\right]$, $M := M(m, n) = m^2n^2(1 - n^2 - m^2) \in \left[W^2, \frac{1}{27}\right]$. 

Lemma 3.3. Let \( a = (r, s, t) \in S_{11}^{11} \), \( A = H(a) \). Let \( \mu_1 \geq \mu_2 \geq \mu_3 \) be the roots of \( P_1(T) \), \( \nu_1 \geq \nu_2 \geq \ldots \geq \nu_6 \) be the roots of \( P_2(T) \). Then

\[
\mu_1 \geq \nu_1 \geq \nu_2 \geq \nu_3 \geq \mu_2 \geq \nu_4 \geq \nu_5 \geq \nu_6 \geq \mu_3.
\]

Corollary 3.1. Let \( a = (r, s, t) \in S_{11}^{11} \). Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{12} \) be the eigenvalues of \( A = H(a) \). Then

\[
\lambda_6 = \lambda_7 = \frac{2}{\sqrt{3}} \cos \left( \arccos(3\sqrt{3}P(a)) + \pi \right) - P(a).
\]

Proof of Corollary. By Lemmas 3.1 and 3.2 \( \lambda_6 = \lambda_7 = \mu_2 \). One easily verifies that \( Q_1(X) := P_1(X-W) = X^3 - X + 2W \). If we set \( X = 2 \cos(\beta) / \sqrt{3} \), \( 3 \sqrt{3}W = \cos(\alpha) \) we get \( \cos(3\beta) = \cos(\alpha) \) which implies

\[
\mu_1 = \frac{2}{\sqrt{3}} \cos \left( \frac{\arccos(3\sqrt{3}W) - \pi}{3} \right) - W, \quad \mu_2 = \frac{2}{\sqrt{3}} \cos \left( \frac{\arccos(3\sqrt{3}W) + \pi}{3} \right) - W,
\]

\[
\mu_3 = \frac{2}{\sqrt{3}} \cos \left( \frac{\arccos(3\sqrt{3}W) + 3\pi}{3} \right) - W.
\]

Proof of Main Lemma 3.1. Let \( W = P(a) \), \( W' = P(b) \) and \( W - W' \geq 0 \). By Property 3.1.

\[
\Lambda_1 \geq \lambda_6(A) - \lambda_6(B) = \frac{2}{\sqrt{3}} \left( \cos \left( \frac{\arccos(3\sqrt{3}W) + \pi}{3} \right) - \cos \left( \frac{\arccos(3\sqrt{3}W') + \pi}{3} \right) \right) - W + W'.
\]

Since \( \cos \left( \frac{\arccos(3\sqrt{3}W) + \pi}{3} \right) \geq \sqrt{3}|W| \) and \( \cos \left( \frac{\arccos(3\sqrt{3}W') + \pi}{3} \right) \geq \sqrt{3}|W'| \) we get the conclusion. The case \( P(a) - P(b) \leq 0 \) is symmetric.

Proof of Lemma 3.2. Note that the function \( w \) is invariant under the action of the group \( \text{Sp}_1 \times \text{Sp}_1 \times \text{Sp}_1 \) by conjugation on each factor, i.e.

\[
(g_1, g_2, g_3) : (r, s, t) \mapsto (g_1rg_1^{-1}, g_2sg_2^{-1}, g_3tg_3^{-1})
\]

for \( g_1, g_2, g_3 \in \text{Sp}_1 = \{ q \in H : \ |q| = 1 \} \), and hence the spectrum \( \text{Sp}(H(a)) \) is invariant under this action as well.

Applying this action one can suppose that \( r_2 = r_3 = s_2 = s_3 = t_2 = t_3 = 0 \), i.e. that \( (r, s, t) \in C^3 \subset H^3 \). In this case the matrix \( A = H(a) \) becomes a block matrix

\[
A = \begin{pmatrix} A_6 & 0 \\ 0 & M_6 \end{pmatrix}
\]
where \( A_6 = D^2 w_6(a') \) is the Hessian of the function

\[
  w_6(a') = \frac{P_6(a')}{|a'|} = \text{Re}(cr \cdot cs \cdot ct) = \frac{r_0 s_0 t_0 - r_0 s_1 t_1 - r_1 s_0 t_1 - r_1 s_1 t_0}{\sqrt{r_0^2 + s_0^2 + t_0^2 + r_1^2 + s_1^2 + t_1^2}},
\]

\( a' = (cr, cs, ct) = (r_0 + r_1 i, s_0 + s_1 i, t_0 + t_1 i) \in \mathbb{C}^3 \), and \( M_6 \) is the following matrix:

\[
  M_6 = \begin{pmatrix}
  -W & 0 & -t_0 & -t_1 & -s_0 & s_1 \\
  0 & -W & t_1 & -t_0 & -s_1 & -s_0 \\
  -t_0 & t_1 & -W & 0 & -r_0 & -r_1 \\
  -t_1 & -t_0 & 0 & -W & r_1 & -r_0 \\
  -s_0 & -s_1 & -r_0 & r_1 & -W & 0 \\
  s_1 & -s_0 & -r_1 & -r_0 & 0 & -W 
  \end{pmatrix}.
\]

A direct calculation shows that the characteristic polynomial of

\[
  N_6 = M_6 + W \cdot J_6 = \begin{pmatrix}
  0 & 0 & -t_0 & -t_1 & -s_0 & s_1 \\
  0 & 0 & t_1 & -t_0 & -s_1 & -s_0 \\
  -t_0 & t_1 & 0 & 0 & -r_0 & -r_1 \\
  -t_1 & -t_0 & 0 & 0 & r_1 & -r_0 \\
  -s_0 & -s_1 & -r_0 & r_1 & 0 & 0 \\
  s_1 & -s_0 & -r_1 & -r_0 & 0 & 0 
  \end{pmatrix},
\]

is given by

\[
  P_{N_6}(X) = (X^3 - X + 2W)^2
\]

(one uses that \(|a|^2 = |a'|^2 = |r|^2 + |s|^2 + |t|^2 = 1\) which gives the formula for the first factor. To calculate the characteristic polynomial of \( A_6 \) one notes an action of the group

\[
  T^2 = S^1 \times S^1 = \{(u_1, u_2, u_3) \in \mathbb{C}^3 : u_1 = u_2 = u_3 = 1, u_1 u_2 u_3 = 1\}
\]
on \( \mathbb{C}^3 \) respecting \( w_6 \):

\[
  (u_1, u_2, u_3) : (r, s, t) \mapsto (u_1 r, u_2 s, u_3 t).
\]

This action permits to suppose that \( s_1 = t_1 = 0 \), \( s', t' \in \mathbb{R}^+ \) and thus \( s' = s_0 = m, t' = t_0 = n \), \( W = P(r, s, t) = r_0 mn \). Applying MAPLE one gets the characteristic polynomial \( P_2(T) \).

One notes also that in this case a direct calculation gives for \( A_6 = (N_{ij}) \):

\[
  N_{11} = (3r_0^2 - 3)W, \ N_{12} = (3W r_0 - mn t_0) r_1, \ N_{13} = n(1 - r_0^2 - m^2) + 3W r_0 m, \ N_{14} = r_0 n r_1,
\]

\[
  N_{15} = m(1 - r_0^2 - n^2) + 3r_0 n W, \ N_{16} = r_0 m r_1, \ N_{21} = (3W r_0 - mn) r_1, \ N_{22} = 3W(r_1^2 - 1),
\]

\[
  N_{23} = (3W s_0 - mn) r_1, \ N_{24} = n(r_1^2 - 1), \ N_{25} = (3W r_0 m) r_1, \ N_{26} = m(r_1^2 - 1),
\]

\[
  N_{31} = (1 - r_0^2 - m^2) n + 3r_0 m W, \ N_{32} = (3mW - r_0 n) r_1, \ N_{33} = (3m^2 - 3) W, \ N_{34} = mn r_1,
\]

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\[ N_{35} = (1-m^2-n^2)r_0 + 3mr_0W, \quad N_{36} = (m^2-1)r_1, \quad N_{41} = r_0nr_1, \quad N_{42} = (r_1^2-1)n, \quad N_{43} = mn, \quad N_{44} = -W, \quad N_{45} = (n^2-1)r_1, \quad N_{46} = -r_0, \quad N_{51} = (1-r_0^2-n^2)m + 3r_0nW, \quad N_{52} = (3nW - mn)r_1, \quad N_{53} = (1-m^2-n^2)r_0 + 3mnW, \quad N_{54} = (n^2-1)r_1, \quad N_{55} = (3n^2-3)W, \quad N_{56} = mn, \quad N_{61} = mn, \quad N_{62} = (r_1^2-1)m, \quad N_{63} = (m^2-1)r_1, \quad N_{64} = -r_0, \quad N_{65} = mn, \quad N_{66} = -W \]

which permits a human (albeit very tedious) calculation of the polynomial.

Note that the characteristic polynomial \( Q_2(X) = P_2(X - W) \) of \( A_6 + W \cdot I_6 \) equals

\[
Q_2(X) = X^6 + 3WX^5 - (9W^2 - 3L + 2)X^4 - 6WLX^3 + (6W^2 - 3L + 9M + 1)X^2 - 3(6M - 4L - 1)WX + 3W^2 - 12 LW^2 - M.
\]

In fact, one can directly apply the MAPLE directive

\[
P2 := \text{sort(sortify(charpoly(hessian(w_6, v)), S))}, S;
\]

for the coordinate vector \( v \), but in this case the calculation takes about a minute, 100 MB of space (and the result need many dozens lines to be written), while the same directive applied to the case with two zero coordinates gives the result in less than a second.

**Remark 3.1.** Since \(|a|^2 = |a'|^2 = |r|^2 + |s|^2 + |t|^2 = r_0^2 + r_1^2 + m^2 + n^2 = 1\) one gets \( r_0^2 + m^2 + n^2 \leq 1 \) and an application

\[
\Phi : S_1^{11} \rightarrow \mathbb{B}_3^{1^+, \star}, \quad a = (r, s, t) \mapsto \Phi(a) := (r_0, m, n) = (W_{mn}, m, n)
\]

where \( \mathbb{B}_3^{1^+, \star} = \mathbb{B}_3^1 \cap \{m \geq 0, n \geq 0\} \).

**Proof of Lemma 3.3.** Let \( \mu_i'' = \mu_i + W, \ \nu_j'' = \nu_j + W \) for \( i = 1, 2, 3, \ j = 1, \ldots, 6; \) be the roots of \( Q_1(X) \) and \( Q_2(X) \), respectively. We have to show that

\[
\mu_1' \geq \nu_1' \geq \nu_2' \geq \mu_2' \geq \nu_4' \geq \nu_5' \geq \nu_6' \geq \mu_3'.
\]

One notes that \( \mu_j''(W) = \mu_j'(-W), \ \nu_j''(W) = \nu_j'(-W) \). Therefore we can suppose w.r.g. that \( W \geq 0 \). For \( n = 0 \) we have \( W = mn_0 = 0 \) and

\[
Q_2(X) = X^6 - 2X^4 + 3mX^4 - 3mX^4 + X^2 - 3mX^2 + 3mX^2 =
\]

\[
X^2(X - 1)(X + 1)(X^2 - 3m^2 - 1 + 3m^2)
\]

\[
X^3 - X - 2W = X^3 - X = X(X - 1)(X + 1).
\]

Thus \( \mu_1' = \nu_1' = 1, \ \nu_2' = \sqrt{1 - 3m^2 + 3m^2} \in (0, 1], \ \nu_4' = \nu_5' = \mu_2' = 0, \ \nu_6' = -\sqrt{1 - 3m^2 + 3m^2} \in [-1, 0), \ \mu_3' = -1 \), and the inequalities take place.
Symmetrically this is true for $m = 0$ as well. We can suppose thus that $m \neq 0, n \neq 0$.

We suppose then that $r_0^2 + m^2 + n^2 \neq 1$; without loss one supposes also $(m, n, r_0) \in B_1^3 \cap \mathbb{R}_+^3$. We begin with a particular choice: $m = n = r_0 = 1/2$, $W = 1/8$. For that choice easy brute force calculations show that $\mu_1' \in [0.83, 0.84], \mu_2' \in [0.26, 0.27], \mu_3' \in [-1.11, -1.1], \nu_2' \in [0.54, 0.55], \nu_4' \in [0.42, 0.43], \nu_4' \in [-0.39, -0.38], \nu_5' \in [-0.71, 0.7], \nu_6' \in [-0.96, 0.95]$ and the inequalities hold. Then we consider the resultant $R = R(m, n, r_0)$ of the polynomials $Q_2(X)$ and $X^3 - X + 2W$; a brute force (MAPLE) calculations give

$$R = 16(-n^2m^2 + W^2 + n^2m^4 + n^4m^2)(27W^2 + 4)(1 - 27W^2) =$$

$$16n^2m^2(-1 + r_0^2 + m^2 + n^2)^3(27W^2 + 4)(1 - 27W^2) < 0$$

since the condition $W^2 = 1/27$ implies $r_0^2 + m^2 + n^2 = 1$. For any $(m, n, r_0) \in B_1^3 \cap \mathbb{R}_+^3$ there is a line segment joining it to the triple $(1/2, 1/2, 1/2)$, the set $B_1^3 \cap \mathbb{R}_+^3$ being convex. The value of $R(m, n, r_0)$ on the whole segment is strictly negative and thus the order of the roots at $(m, n, r_0)$ is the same as at $(1/2, 1/2, 1/2)$ which finishes the proof of the inequalities for $r_0^2 + m^2 + n^2 \neq 1$.

Let finally $m^2 + n^2 + r_0^2 = 1$. Then easy brute force calculations show that

$$Q_2(X) = (X^3 - X + 2W)(X^3 + 3WX^2 - 9W^2X - X + 3LX + W - 6WL).$$

Thus by continuity we get $\lambda_1 = \lambda_2 = \lambda_3 = \mu_1, \lambda_6 = \lambda_7 = \mu_2, \lambda_{10} = \lambda_{11} = \lambda_{12} = \mu_3$ which is sufficient to conclude.

Remark 3.2. We use extensively MAPLE calculations in Sections 3 and 4. These calculations concern algebraic identities, do not use any approximation and are thus completely rigorous. Besides, all of them need only few seconds on a modest laptop.

Proposition 3.1 and Lemma 2.1 give a proof of Theorem 1.1 in the case of $\delta = 0$. Indeed, we set $K$ to be the dual cone $K := K^*_\lambda$ where

$$K_\lambda = \{ (\lambda_1, ..., \lambda_n) \in [C/\lambda, C\lambda] : \text{ for some } C > 0 \}$$

with $n = 12, \lambda = 26$. Then Proposition 3.1 gives the $K$–cone condition in Lemma 2.1 on $T_{\sigma_0}A(B)$ for $\sigma_0 = \text{id} \in S_{12}$ which implies the same condition on the whole $M = \bigcup_{\sigma \in S_{12}} T_\sigma A(B)$ as well.

Remark 3.3. The ellipticity constant $C$ of thus obtained functional $F$ verifies $C \leq 4 \cdot 26^2 \sqrt{12} < 10^5$ (cf. [NV1, Lemma 2.2]).
4 Singular solutions

In this section we prove Theorem 1.1 for any $\delta \in [0, 1)$. For this it is sufficient to show by Lemma 2.1 that the ellipticity condition (the $K$–cone condition) valid for the function $w$ remains to hold for the function $w_\delta(X) := w(X)|X|^{-\delta}$.

For $a \in \mathbb{R}^{12} - \{0\}$ we denote by $H_\delta(a)$ the Hessian $D^2w_\delta(a)$. The following result is sufficient to prove Theorem 1.1:

**Proposition 4.1.** Let $0 \leq \delta < 1$. Then for any $a \neq b \in \mathbb{R}^{12} - \{0\}$ and any orthogonal matrix $O \in O(12)$ with $H_\delta(a, b, O) := H_\delta(a) - tO \cdot H_\delta(b) \cdot O \neq 0$ the eigenvalues $\Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_{12}$ of $H_\delta(a, b, O)$ verify

$$\frac{1}{C_\delta} = \frac{1 - \delta}{26 + 3\delta - \delta^2} \leq \frac{\Lambda_1}{-\Lambda_{12}} \leq \frac{26 + 3\delta - \delta^2}{1 - \delta} =: C_\delta.$$

**Proof.** We can suppose without loss that $|a| \leq |b|$, moreover, by homogeneity we can suppose that $a \in S_{11}^1$ and thus $|b| \geq 1$. Let $\bar{b} := b/|b| \in S_{11}^1$ then $D^2w_\delta(b) = D^2w_\delta(\bar{b})|b|^{-\delta}$. One needs then the following result for the points $a, \bar{b} \in S_{11}^1$:

**Lemma 4.1.** Let $\delta \in [0, 1)$, $a, \bar{b} \in S_{11}^1$, $W = W(a)$, $\bar{W} = W(\bar{b})$, and let

$$\mu_1(\delta) = \frac{2}{\sqrt{3}} \cos \left( \frac{\arccos(3\sqrt{3}W) + \pi}{3} \right) - W(1 + \delta) \geq$$

$$\mu_2(\delta) = \frac{2}{\sqrt{3}} \cos \left( \frac{\arccos(3\sqrt{3}W) - \pi}{3} \right) - W(1 + \delta) \geq$$

$$\mu_3(\delta) = -\frac{2}{\sqrt{3}} \cos \left( \frac{\arccos(3\sqrt{3}W)}{3} \right) - W(1 + \delta)$$

(resp., $\bar{\mu}_1(\delta) \geq \bar{\mu}_2(\delta) \geq \bar{\mu}_3(\delta)$) be the roots of the polynomial

$$P_{1,\delta}(T, W) := Q_1(T + W + \delta W) =$$

$$T^3 + 3W(1 + \delta)T^2 + (3W^2(1 + \delta)^2 - 1)T + W(1 - \delta) + W^3(1 + \delta)^3$$

(resp. of the polynomial

$$\bar{P}_{1,\delta}(T, \bar{W}) := Q_1(T + \bar{W} + \delta \bar{W}) =$$

$$T^3 + 3\bar{W}(1 + \delta)T^2 + (3\bar{W}^2(1 + \delta)^2 - 1)T + \bar{W}(1 - \delta) + \bar{W}^3(1 + \delta)^3).$$

Then for any $K > 0$ verifying $|K - 1| + |\bar{W} - W| \neq 0$ one has

$$\frac{1 - \delta}{5 + \delta} =: \varepsilon \leq \frac{\mu_+(K)}{-\mu_-(K)} \leq \frac{1}{\varepsilon} = \frac{5 + \delta}{1 - \delta}$$

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where
\[
\begin{align*}
\mu_-(K) &:= \min\{\mu_1(\delta) - K\bar{\mu}_1(\delta), \mu_2(\delta) - K\bar{\mu}_2(\delta), \mu_3(\delta) - K\bar{\mu}_3(\delta)\}, \\
\mu_+(K) &:= \max\{\mu_1(\delta) - K\bar{\mu}_1(\delta), \mu_2(\delta) - K\bar{\mu}_2(\delta), \mu_3(\delta) - K\bar{\mu}_3(\delta)\}.
\end{align*}
\]

Proof of Lemma 4.1. In the proof we will repeatedly use the following elementary fact:

Claim. Let \(l_1 \geq l_2 \geq l_3, l_1 + l_2 + l_3 = t \geq 0, l_3 \leq -ht, \text{ with } h > 0. \) Then \(-l_1/l_3 \in [h/(2h + 1), (2h + 1)/h]\) for \(t > 0, \) \(-l_1/l_3 \in [1/2, 2]\) for \(t = 0.\)

If \(W = \bar{W}, K = 1\) there is nothing to prove. If \(K = 1\) one can suppose that \(W > \bar{W};\) we have

\[
(\mu_1(\delta) - K\bar{\mu}_1(\delta)) + (\mu_2(\delta) - K\bar{\mu}_2(\delta)) + (\mu_3(\delta) - K\bar{\mu}_3(\delta)) = 3(\bar{W} - W)(1 + \delta)
\]
and

\[
\mu_2(\delta) - K\bar{\mu}_2(\delta) = \frac{2}{\sqrt{3}} \left( \cos \left( \frac{\arccos(3\sqrt{3}W) + \pi}{3} \right) - \cos \left( \frac{\arccos(3\sqrt{3}\bar{W}) + \pi}{3} \right) \right) - (W - \bar{W})(1 + \delta) \geq (1 - \delta)(W - \bar{W}).
\]

Therefore, one can take \(\varepsilon = (1 - \delta)/(5 + \delta)\) in this case. We can suppose then \(W > \bar{W}, K \neq 1.\) Using the relations

\[
\mu_1(\delta)(-W) = -\mu_3(\delta)(W), \mu_2(\delta)(-W) = -\mu_2(\delta)(W), \mu_3(\delta)(-W) = -\mu_1(\delta)(W)
\]
we can suppose without loss that \(K < 1.\)

We distinguish then three cases corresponding to different signs of \(W - K\bar{W}.\) If \(W - K\bar{W} = 0\) then one can take \(\varepsilon = 1/2\) since

\[
(\mu_1(\delta) - K\bar{\mu}_1(\delta)) + (\mu_2(\delta) - K\bar{\mu}_2(\delta)) + (\mu_3(\delta) - K\bar{\mu}_3(\delta)) = 0.
\]

Let \(W - K\bar{W} = W - \bar{W} + (1 - K)\bar{W} < 0.\) Then

\[
(\mu_1(\delta) - K\bar{\mu}_1(\delta)) + (\mu_2(\delta) - K\bar{\mu}_2(\delta)) + (\mu_3(\delta) - K\bar{\mu}_3(\delta)) = -3(W - K\bar{W})(1 + \delta) > 0
\]
and

\[
\mu_3(\delta) - K\bar{\mu}_3(\delta) = \mu_3(\delta) - \bar{\mu}_3(\delta) + (1 - K)\bar{\mu}_3(\delta) = \mu'_3(\delta)(W' - \bar{W}) + (1 - K)\bar{\mu}_3(\delta)
\]
for \(W' \in (W, \bar{W}).\) Since

\[
\bar{\mu}_3(\delta) \leq \frac{\delta - 2}{3\sqrt{3}} < \frac{-1}{3\sqrt{3}} \leq -\bar{W}, \mu'_3(\delta)(W') \leq -5/3 - \delta \leq -5/3 < -1
\]
we get

\[
\mu_3(\delta) - K\bar{\mu}_3(\delta) < -(W - \bar{W} + (1 - K)\bar{W}) = -(W - K\bar{W})
\]

and one can take \( \varepsilon = (2 + (3 + 3\delta))^{-1} = 1/(5 + 3\delta) \geq (1 - \delta)/(5 + \delta) \).

Let then \( W - K\bar{W} = W - \bar{W} + (1 - K)\bar{W} > 0 \). We get

\[
(\mu_1(\delta) - K\bar{\mu}_1(\delta)) + (\mu_2(\delta) - K\bar{\mu}_2(\delta)) + (\mu_3(\delta) - K\bar{\mu}_3(\delta)) = -3(W - K\bar{W}) < 0.
\]

If \( \bar{W} \geq 0 \) then

\[
\mu_2(\delta) - K\bar{\mu}_2(\delta) = \mu_2(\delta) - \bar{\mu}_2(\delta) + (1 - K)\bar{\mu}_2(\delta) = \mu_2(\delta)(W' - \bar{W}) + (1 - K)\bar{\mu}_2(\delta) \geq (1 - \delta)(W - \bar{W}) + (1 - K)(1 - \delta)\bar{W} \geq (1 - \delta)(W - K\bar{W})
\]

which gives again \( \varepsilon = (1 - \delta)/(5 + \delta) \).

Let \( \bar{W} < 0 \), \( W \geq 0 \). Then

\[
\mu_2(\delta) - K\bar{\mu}_2(\delta) \geq (1 - \delta)W + K(1 - \delta)\bar{W} = (1 - \delta)(W - K\bar{W}).
\]

Let finally \( \bar{W} < 0 \), \( W < 0 \). Then the same inequality holds since the function

\[
f(W) := \mu_2(\delta)(W) / W
\]

is decreasing for \( W \in \left[ \frac{1}{5 + \delta}, 0 \right] \) and \( f(0) = (1 - \delta) \).

This result can be applied to our situation thanks to the following formulas generalizing those of Section 3; the proofs remain essentially the same as for Lemma 3.2 (i.e. brute force MAPLE calculation together with invariance properties of \( w \)). Namely, the matrix \( A_\delta = H_\delta(a) \) becomes a block matrix

\[
A_\delta = \begin{pmatrix}
A_{6,\delta} & 0 \\
0 & M_{6,\delta}
\end{pmatrix}
\]

where \( A_{6,\delta} = D^2w_{6,\delta}(a') \) is the Hessian of the function

\[
w_{6}(a') = P_0(a') / |a'|^{1+\delta} = \frac{r_0s_0l_0 - r_0s_1t_1 - r_1s_0l_1 - r_1s_1t_0}{(r_0^2 + s_0^2 + t_0^2 + r_1^2 + s_1^2 + t_1^2)^{1+\delta/2}}
\]

and \( M_{6,\delta} = N_6 - (1 + \delta)W \cdot I_6 \).

**Lemma 4.2.** Let \( \delta \in [0, 1] \) and let \( a = (r, s, t) \in S_1^1 \); define

\[
W = W(a) = P(a), \ m = m(a) = |s|, \ n = n(a) = |t|.
\]

Then the characteristic polynomial of the matrix \( A_\delta = H_\delta(a) := D^2w_\delta(a) \) is given by

\[
P_{A,\delta}(T) = P_{1,\delta}(T)^2 \cdot P_{2,\delta}(T)
\]

where
\[ P_{1,\delta}(T) = P_{1,\delta}(T, W) := Q_1(T + W + \delta W) = \\
T^3 + 3W(1 + \delta)T^2 + (3W^2(1 + \delta)^2 - 1)T + W(1 - \delta) + W^3(1 + \delta)^3; \]
\[ P_{2,\delta}(T) = P_{2,\delta}(T, W) := T^6 + a_{5,\delta}T^5 + a_{4,\delta}T^4 + a_{3,\delta}T^3 + a_{2,\delta}T^2 + a_{1,\delta}T + a_{0,\delta} \]
where
\[ a_{5,\delta} := W(\delta + 1)(9 - \delta), \]
\[ a_{4,\delta} := W^2(\delta + 1)(21 + 28\delta - 5\delta^2) + L(\delta + 1)(3 - \delta) - 2, \]
\[ a_{3,\delta} := -2W(1 + \delta) \cdot (W^2(\delta + 1)(5\delta^2 - 26\delta - 7) - L(2\delta + 1)(3 - \delta) + 4), \]
\[ a_{2,\delta} := -W^4(10\delta^2 - 53\delta + 9)(\delta + 1)^3 - 2W^2(\delta + 1)(3L\delta^3 - 6L\delta^2 - 9L\delta + 7\delta + 3) \\
+ L\delta^2 - 3M\delta^2 - 2L\delta + 6M\delta - 3L + 9M + 1 \]
\[ a_{1,\delta} := -\delta(\delta + 1)(W^4(5\delta - 3)(\delta - 5)(\delta + 1)^3 - 2(\delta + 1)(-2L\delta^3 + 5L\delta^2 + 4L\delta - 6\delta - 3L + 2)W^2) \\
+ 2(3 - \delta)(-3\delta M + L\delta - L) + 1 - \delta) W \]
\[ a_{0,\delta} = (1 - \delta)(W^6(\delta - 5)(\delta + 1)^5 + W^4(\delta + 1)^3(L\delta^2 - 2L\delta - 3L + 4) \\
- W^2(\delta + 1)(L\delta^2 - 3M\delta^2 + \delta + 6M\delta - 4L\delta - 1 + 3L + 9M) - M(1 - \delta)) \],
with \( L = m^2 + n^2 - n^2m^2 - n^4 - m^4, M = m^2n^2(1 - n^2 - m^2) \) as before.

A MAPLE calculation gives then for the resultant
\[ R_\delta(r_0, m, n) := Res(P_{1,\delta}, P_{2,\delta}) = 16m^4n^4(1 - n^2 - m^2 - r_0^2)^3 \cdot R(W, \delta) \]
where
\[ R(W, \delta) = 27(\delta + 1)^3(3 - \delta)^3W^4 + 9(\delta - 1)^2(\delta - 3)^2(\delta + 1)^2W^2 - (\delta - 1)^2(\delta^2 - 2\delta - 2)^2. \]

Denote by \( W_0(\delta) \in (0, 1/3\sqrt{3}) \) the unique positive root of \( R(W, \delta) \). Recall that the set \( \Phi(S^{11}) \) of possible triples \( \Phi(a) = (r_0, m, n) : r_0 = r_0(a), m = m(a), n = n(a) \) for \( a \in S^{11} \) is a quarter \( B_+ := B_1 \cap \{m \geq 0, n \geq 0\} \) of the closed unit ball \( B = B_1 \subset V \); recall also that \( W(a) = r_0mn \). Let \( B_+ (\delta) \) (resp. \( B_-(\delta), B_0(\delta) \)) be the subset of \((r_0, m, n) \in \Phi(S^{11})\) where \( R_\delta(W) > 0 \) (resp. \( R_\delta(W) < 0, R_\delta(W) = 0 \)). Then
\[ \begin{align*}
B_0(\delta) &= S_+ \cup D_{r_0} \cup D_m \cup D_n \text{ with } D_m = \overline{B}_+ \cap \{m = 0\} \text{ etc., } \\
B_+(\delta) &= B_+ \cap \{r_0mn > W_0(\delta)\}, \quad \overline{B}_+(\delta) = \overline{B}_+ \cap \{r_0mn \geq W_0(\delta)\}, \\
B_-(\delta) &= B_+ \cap \{0 < r_0mn < W_0(\delta)\}, \quad \overline{B}_-(\delta) = \overline{B}_+ \cap \{r_0mn \leq W_0(\delta)\}.
\end{align*} \]

Note that these sets are invariant under the reflection \( Refl : (r_0, m, n) \mapsto (-r_0, m, n) \); \( B_0(\delta) \) and \( B_- (\delta) \) are connected, while \( B_-(\delta), \overline{B}_+(\delta) \) and \( B_+(\delta) \) have two connected components each.
Lemma 4.3. Let \( a \in S_1^{11} \), let \( \lambda_1(\delta, a) \geq \lambda_2(\delta, a) \geq \ldots \geq \lambda_{12}(\delta, a) \) be the eigenvalues of \( D^2 w_3(a) \) and let \( \mu_1(\delta, a) \geq \mu_2(\delta, a) \geq \mu_3(\delta, a) \) be the roots of \( P_{1,\delta}(T, W(a)) \). Then

(i) \( \lambda_1(\delta, a) = \lambda_2(\delta, a) = \mu_1(\delta, a) \), \( \lambda_{12}(\delta, a) = \lambda_{11}(\delta, a) = \mu_3(\delta, a) \);

(ii) \( \lambda_5(\delta, a) = \lambda_6(\delta, a) = \mu_2(\delta, a) \) for \( \Phi(a) \in \bar{B}_+(\delta) \), \( W = W(a) \geq 0 \);

(iii) \( \lambda_7(\delta, a) = \lambda_8(\delta, a) = \mu_2(\delta, a) \) for \( \Phi(a) \in \bar{B}_+(\delta) \), \( W = W(a) \leq 0 \);

(iv) \( \lambda_6(\delta, a) = \lambda_7(\delta, a) = \mu_2(\delta, a) \) for \( \Phi(a) \in \bar{B}_-(\delta) \).

Proof of Lemma 4.3. Since \( \lambda_1(\delta, a) = \lambda_{12}(\delta, a) \), \( \lambda_{12}(\delta, a) = \lambda_1(\delta, a) \), \( \lambda_6(\delta, a) = \lambda_7(\delta, a) \), \( \lambda_7(\delta, a) = \lambda_6(\delta, a) \), \( \lambda_8(\delta, a) = \lambda_5(\delta, a) \), \( \lambda_5(\delta, a) = \lambda_9(\delta, a) \), \( W(a) = -W(a) \), (iii) is implied by (ii) and, moreover one can suppose without loss that \( \Phi(a) = (r_0, m, n) \in \mathbb{R}_+^3 \). Since in the interior of the domain \( B_+(\delta) \cap \mathbb{R}_+^3 \) (resp. \( B_-(\delta) \cap \mathbb{R}_+^3 \)) the function \( R(c, r_0, m, n) \) does not vanish, it is sufficient to verify the ordering of the roots at a single point in \( B_-(\delta) \cap \mathbb{R}_+^3 \) (resp. at a single point in \( B_+(\delta) \cap \mathbb{R}_+^3 \)). We use \( a_- := (\varepsilon, \varepsilon, \varepsilon) \in B_-(\delta) \cap \mathbb{R}_+^3 \) and \( a_+ := (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} - \varepsilon) \in B_+(\delta) \cap \mathbb{R}_+^3 \) for sufficiently small \( \varepsilon > 0 \). Let \( \nu_1(\delta, a) \geq \nu_2(\delta, a) \geq \ldots \geq \nu_6(\delta, a) \) be the roots of \( P_{2,\delta}(T, W(a)) \). Elementary calculations show that for \( a = a_- \) one has \( W = W(a) = \varepsilon^3 \),

\[
\nu_1(\delta, a) = 1 + O(\varepsilon^3), \quad \nu_2(\delta, a) = O(\varepsilon^3), \quad \nu_3(\delta, a) = -1 + O(\varepsilon^3),
\]

while \( P_{2,\delta}(T, W(a)) = F_1(T, \varepsilon) \cdot F_2(T, \varepsilon) \) where

\[
F_1(T, \varepsilon) = T^2 - 1 + 2\varepsilon^2 + O(\varepsilon^3),
\]

\[
F_2(T, \varepsilon) = T^4 + \varepsilon^3(7 + 6\varepsilon - \varepsilon^2)T^3 + (12\varepsilon^2 + 14\varepsilon^3 - 3\varepsilon^6 + 21\varepsilon^6 + 6\varepsilon^7 + 4\varepsilon^8 - 2\varepsilon^2\varepsilon^2 - 1 - 6\varepsilon^4\varepsilon - 4\varepsilon^2\varepsilon - 9\varepsilon^4)T^2 + \varepsilon^3(1 - 10\varepsilon^2 - \varepsilon^3 - 2\varepsilon^4\varepsilon - 18\varepsilon^4\varepsilon + 10\varepsilon^2\varepsilon^2 - 4\varepsilon^2\varepsilon^2 + 6\varepsilon^4\varepsilon^2 + O(\varepsilon^5))T + \varepsilon^3(1 - \varepsilon^2 - \varepsilon^6(\delta + 1)\varepsilon^2 - 4\varepsilon^2 - 1) + O(\varepsilon^{10})
\]

and thus

\[
\nu_1(\delta, a) \geq \nu_1(\delta, a) = 1 - \varepsilon^2 + O(\varepsilon^3) \geq \nu_2(\delta, a) = 1 - \varepsilon^2(2 + 2\varepsilon - \varepsilon^2) + O(\varepsilon^3),
\]

\[
\nu_3(\delta, a) = (1 - \varepsilon^2)^2 + O(\varepsilon^3) \geq \mu_2(\delta, a) \geq \nu_4(\delta, a) = (1 - \varepsilon^2)^3 + O(\varepsilon^3),
\]

\[
\nu_5(\delta, a) = -1 + \varepsilon^2(2 + 2\varepsilon - \varepsilon^2) + O(\varepsilon^3) \geq \nu_6(\delta, a) = -1 + \varepsilon^2 + O(\varepsilon^3) \geq \mu_3(\delta, a)
\]

which proves the claim in this case.
For \( a = a_+ \) one has \( W = W(a) = (1/\sqrt{3} - \varepsilon)/3 \) and similar calculations give

\[
\mu_1(\delta, a) = \frac{2 - \delta}{3\sqrt{3}} + 3^{-1/4}\sqrt{2}\varepsilon + O(\varepsilon), \quad \mu_2(\delta, a) = \frac{2 - \delta}{3\sqrt{3}} - 3^{-1/4}\sqrt{2}\varepsilon + O(\varepsilon),
\]

\[
\mu_3(\delta, a) = \frac{-7 - \delta}{3\sqrt{3}} + (5/3 + \delta)\varepsilon + O(\varepsilon^2),
\]

while \( P_{2,\delta}(T, W(a)) = G_1(T, \varepsilon) \cdot G_2(T, \varepsilon)^2 \cdot G_3(T, \varepsilon)^2 \)

where

\[
G_1(T, \varepsilon) := T^2 + \frac{(\delta + 1)}{3\sqrt{3}}(5 - \delta)(1 - \sqrt{3}\varepsilon)T
\]

\[
+ \frac{(1 - \delta)}{27}(3\delta^2\varepsilon^2 - 2\sqrt{3}\delta^2\varepsilon - 12\delta\varepsilon^2 + \delta^2 + 8\sqrt{3}\delta\varepsilon - 15\varepsilon^2 + 5\delta + 10\sqrt{3}\varepsilon - 14),
\]

\[
G_2(T, \varepsilon) := T + \frac{4 + \delta}{3\sqrt{3}} - \frac{\varepsilon(\delta + 1)}{3},
\]

\[
G_3(T, \varepsilon) := T - \frac{2 - \delta}{3\sqrt{3}} - \frac{\varepsilon(\delta + 1)}{3},
\]

and thus

\[
\mu_1(\delta, a) \geq \nu_1(\delta, a) = \nu_2(\delta, a) = \frac{2 - \delta}{3\sqrt{3}} + O(\varepsilon) \geq \mu_2(\delta, a) \geq
\]

\[
\nu_3(\delta, a) = \frac{(2 - \delta)(1 - \delta)}{3\sqrt{3}} + O(\varepsilon) \geq \nu_4(\delta, a) = \nu_5(\delta, a) = \frac{-4 + \delta}{3\sqrt{3}} + O(\varepsilon),
\]

\[
\nu_6(\delta, a) = \frac{-7 - \delta}{3\sqrt{3}} + \frac{\varepsilon(\delta - 5)(\delta - 9)(\delta + 1)}{3(9 - 2\delta + \delta^2)} + O(\varepsilon^2) \geq \mu_3(\delta, a)
\]

which finishes the proof of the lemma (note that

\[
\nu_6(\delta, a) - \mu_3(\delta, a) = \frac{2\varepsilon(7 + \delta)(1 - \delta)}{3(9 - 2\delta + \delta^2)} + O(\varepsilon^2) \geq 0
\]

End of proof of Proposition 4.1. If \( W(a) \) and \( W(b) \) are of the same sign we get the result applying Lemmas 4.1 and 4.3 with \( K := |b|^{-\delta} \); in the exceptional case \( K = 1 \), \( W(a) = W(b) \) the trace of \( H_\delta(a, b, O) \) vanishes and the claim is valid for \( C_\delta = 11 \). In the case \( W(a) \cdot W(b) < 0 \) we can suppose without loss that \( W(a) > 0 \), \( W(b) < 0 \); if \( \Phi(a) \notin B_+ \) or \( \Phi(b) \notin B_+ \) then Lemmas 4.1 and 4.3 work as well. Thus we can suppose \( \Phi(a) \in B_+ \), \( \Phi(b) \in B_+ \); then

\[
\text{Re}fI(\Phi(b)) \in B_+, \quad W(-b) > 0, \quad \lambda_i(-b) = -\lambda_{13-i}(b), \quad \lambda_i(-\bar{b}) = -\lambda_{13-i}(\bar{b})
\]

and

\[
\text{Tr}(H_\delta(a, b, O)) = -(W(a) + KW(-\bar{b}))(\delta + 1)(15 - \delta) < 0
\]
which implies immediately that $11 \geq -\Lambda_1/\Lambda_{12}$.

Moreover,

$$\Lambda_1 \geq \lambda_6(a) - K\lambda_6(\tilde{b}) = \lambda_6(a) + K\lambda_7(-\tilde{b}) = \mu_2(\delta, a) + K\mu_2(\delta, -\tilde{b}) \geq$$

$$(1 - \delta)(W(a) + KW(-\tilde{b})) = \frac{(1 - \delta)\text{Tr}(H_\delta(a, b, O))}{(\delta + 1)(15 - \delta)} > 0$$

and thus

$$-\Lambda_1/\Lambda_{12} \geq \left(11 + \frac{(\delta + 1)(15 - \delta)}{1 - \delta}\right)^{-1} = \frac{1 - \delta}{26 + 3\delta - \delta^2}$$

which finishes the proof of the proposition.

To deduce Corollary 1.1 we need the map

$$H_\delta : B_1^{12} - \{0\} \rightarrow Q, \ a \mapsto D^2w_\delta(a)$$

where $Q = S^2(\mathbb{R}^{12})$ denotes the space of quadratic forms on $\mathbb{R}^{12}$. The following result is sufficient to conclude using Proposition 4.1 and Lemma 2.2 of [NV1]:

**Lemma 4.4.** Let $\delta \in (0, 1)$. Then the image $H_\delta(B_1^{12} - \{0\}) \subset Q$ is diffeomorphic to the product $V_{11, \delta} \times [1, \infty)$ with a smooth 11-dimensional manifold $V_{11, \delta}$.

**Proof.** Since $D^2w_\delta(a) = D^2w_\delta(a/|a|)|a|^{-\delta}$ it is sufficient to show two facts:

(i) $H_\delta|_{S_1^{11}} : S_1^{11} \rightarrow Q$ is a smooth embedding;

(ii) if $D^2w_\delta(a) = D^2w_\delta(b) \cdot k$ with $k > 0$ then $k = 1$.

Lemmas 4.1 and 4.2 imply (ii). To prove (i) we fix $a \neq b \in S_1^{11}$ and consider $d = \frac{a - b}{|a - b|} \in S_1^{11}$. Let then $e, f \in S_1^{11} \cap a^\perp \cap b^\perp$. Since $e, f \perp a, b$ one has

$$w_{\delta, ee}(a) = P_{ee}(a) - (1 + \delta)P(a), \ w_{\delta, ee}(b) = P_{ee}(b) - (1 + \delta)P(b),$$

$$w_{\delta, ff}(a) = P_{ff}(a) - (1 + \delta)P(a), \ w_{\delta, ff}(b) = P_{ff}(b) - (1 + \delta)P(b)$$

and hence

$$(w_{\delta, ee}(a) - w_{\delta, ee}(b)) - (w_{\delta, ff}(a) - w_{\delta, ff}(b)) =$$

$$(P_{ee}(a) - P_{ee}(b)) - (P_{ff}(a) - P_{ff}(b)) = |a - b|(P_{eed} - P_{ffd}) \geq \frac{2}{\sqrt{3}} |a - b|$$

for suitable vectors $e, f$ as in the proof of Proposition 2 in [NV1, Section 4]. It follows that

$$\max\{|w_{\delta, ee}(a) - w_{\delta, ee}(b)|, |w_{\delta, ff}(a) - w_{\delta, ff}(b)|\} \geq |a - b|/\sqrt{3}$$

which finishes the proof.
5 Isaacs equation

We can then prove Theorem 1.2 as a simple consequence of the results of Section 4. Denote by $K_C \subset S^2(\mathbb{R}^2)$ the cone of positive symmetric matrix with the ellipticity constant $C$, i.e., if $A \in K_C$, $A = \{a_{ij}\}$ then

$$C^{-1}|\xi|^2 \leq \sum a_{ij}\xi_i\xi_j \leq C|\xi|^2.$$ 

Recall the following results from [NV3, Section 5]:

**Lemma 5.1.** Let $w \in C^\infty(\mathbb{R}^n \setminus 0)$ be a homogeneous order $\alpha$, $1 < \alpha \leq 2$ function. Assume that for any two points $x, y \in \mathbb{R}^n$, $0 < |x|, |y| \leq 1$, there exists a matrix $A \in K_C$ orthogonal to both forms $D^2w(x), D^2w(y)$,

$$Tr(AD^2w(x)) = Tr(AD^2w(y)) = 0.$$ 

Then $w$ is a viscosity solution to an Isaacs equation.

Recall that a symmetric matrix $A$ is called strictly hyperbolic if

$$\frac{1}{M} < -\frac{\lambda_1(A)}{\lambda_n(A)} < M$$

for a positive $M$.

**Lemma 5.2.** Let $F_1, F_2$ be two quadratic forms in $\mathbb{R}^n$ s.t. the form $\alpha F_1 + \beta F_2$ is strictly hyperbolic for any $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{0\}$. Then there exists a positive quadratic form $Q$ orthogonal to both forms $F_1, F_2$,

$$Tr(F_1Q) = Tr(F_2Q) = 0.$$ 

The results of Section 4 imply that the form $\alpha D^2w_{\delta|H}(x) - \beta D^2w_{\delta|H}(y)$ is strictly hyperbolic for positive $\alpha, \beta$; since the function $w_{\delta}$ is odd, it remains true for any $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{0\}$ and thus Lemmas 5.1 and 5.2 imply Theorem 1.4.

6 Eleven Dimensions

For a unit vector $a \in S_1^{10} \subset \mathbb{R}^{11}$ we continue to denote $D^2w_H(a)$ by $H(a)$.

**Lemma 6.1.** Let $a \in S_1^{10}$ and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{11}$ be the eigenvalues of $A = H(a)$. Then

$$\lambda_6 = \frac{2}{\sqrt{3}} \cos \left( \frac{\arccos(3\sqrt{3}P_H(a)) + \pi}{3} \right) - P_H(a).$$
Proof. This follows from Lemma 3.1 and Lemma 2.3.

Let then \( a \neq b \in S_{10}^1 \) and let \( O \in O(11) \) be an orthogonal matrix s.t. \( H(a,b,O) := H(a) - {}^tO \cdot H(b) \cdot O \neq 0 \). Denote \( \Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_{11} \) the eigenvalues of the matrix \( H(a,b,O) \). As in Section 3 above one gets

**Lemma 6.2.** Let \( A := H(a), B := {}^tO \cdot H(b) \cdot O \).

(i) If \( P_H(a) - P_H(b) \geq 0 \) then \( \text{Tr}(B - A) = 14(P_H(a) - P_H(b)) \leq 14\Lambda_1 \);

(ii) If \( P_H(a) - P_H(b) \leq 0 \) then \( \text{Tr}(B - A) = 14(P_H(a) - P_H(b)) \geq 14\Lambda_{11} \),

which implies

**Proposition 6.1.** Let \( a \neq b \in S_{10}^1 \) and let \( O \in O(11) \) be an orthogonal matrix s.t. \( H(a,b,O) := H(a) - {}^tO \cdot H(b) \cdot O \neq 0 \). Denote \( \Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_{11} \) the eigenvalues of the matrix \( H(a,b,O) \). Then

\[
\frac{1}{24} \leq \frac{\Lambda_1}{-\Lambda_{11}} \leq 24.
\]

Proposition 6.1 and Lemma 2.1 give a proof of Theorem 1.3 exactly as Proposition 3.1 implies Theorem 1.1 in the case \( \delta = 0 \).

**Remark 6.1.** The ellipticity constant \( C \) of thus obtained functional \( F \) verifies

\[
C \leq 4 \cdot 24^2 \sqrt{11} < 10^4.
\]

**Remark 6.2.** One can not directly use the approach of Section 4 to the function

\[
\frac{w_H}{|x|^\delta}
\]

for \( \delta > 0 \) since although the corresponding Hessian \( D^2(w/|x|^\delta) \) always has double eigenvalues, they position in the spectrum is not fixed and can vary from (5,6) to (7,8), see Lemma 4.3 above. It means that after the restriction on a hyperplane \( H \) we lose the property necessary to control the ellipticity and thus can not construct a singular solution in 11 dimensions.

7 **Singular solutions with cusp**

Let \( P \) be a linear elliptic operator of the form

\[
P = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j},
\]

defined in a half-ball \( B_+ = \{ x \in B \subset \mathbb{R}^n, x_1 > 0 \} \), \( a_{ij} \in L_\infty(B_+) \) and satisfying the inequalities

\[
C^{-1}||\xi||^2 \leq \sum a_{ij} \xi_i \xi_j \leq C||\xi||^2, \forall \xi \in \mathbb{R}^n.
\]
Let $z \in C^2(B_+)$ and $Pz = 0$ in $B_+$, $z = 0$ on $L$, where $L = \{x \in B, x_1 = 0\}$. Assume that $z < 1$ in $B_+$. Then it is well known, [GT], that

$$|\nabla z(0)| \leq K,$$

where constant $K$ depends on the ellipticity constant $C$.

**Lemma 7.1.** The following inequality holds with positive constants $K, \epsilon$ depending on the ellipticity constant $C$:

$$|z - dz(0)| \leq K|x|^{1+\epsilon},$$

where $dz$ is the differential of the function $z$.

The lemma follows directly from P. Bauman’s boundary Harnack inequality, [B].

**Proof of Theorem 1.4.** We may assume w.r.g. that $F(0) = 0$, otherwise instead of the function $u$ we consider the function $u + c|x|^2$ with a suitable constant $c$.

Set

$$v(r) = \sup_{|x|=r} u(x),$$

$$u_i = u(x_1, ..., -x_i, ..., x_n),$$

$$z_i = u - u_i.$$

Since $u$ is a solution of a Hessian equation the functions $u_i$ are solutions of the same equation as well. Hence functions $z_i$ given as the difference of two solutions of the fully nonlinear elliptic equation are solutions to a linear elliptic equation $Pz_i = 0$ in $B$. Define a linear function $l$ as

$$l = \frac{1}{2} \sum dz_i(0).$$

Set

$$u_0 = u - l.$$

Let $|y| = |y'| = r < 1$. Choose in $\mathbf{R}^n$ an orthonormal coordinate system $y_1, ..., y_n$, such that $y_1 = (y - y')/|y - y'|$. Set

$$u'(y_1, ..., y_n) = u_0(-y_1, ..., y_n),$$

$$v = u - u'.$$

Since $F(u') = 0$ we get $Pv = 0$ in $B$. Moreover

$$\nabla v(0) = 0.$$
Hence by Lemma 7.1,
\[ v(x) = o(|x|^{1+\epsilon}). \]
Therefore
\[ u_0(y) - u_0(y') = o(|y|^{1+\epsilon}). \]
Set
\[ h(r) = \inf_{|x|=r} u_0(x), \]
\[ h_0(r) = \sup_{|x|=r} u_0(x). \]
Then
\[ (7.1) h(|x|) - h_0(|x|) = o(|x|^{1+\epsilon}). \]
Since \( F(0) = 0 \), we may assume without loss that \( u(0) = 0, h'_0(1) > 0 \). Then by the maximum principle \( h_0'(r) \) is a monotone function of \( r \). If \( h(r) = o(|x|^{1+\epsilon/2}) \) we may set \( h \equiv 0 \) and the theorem is proved. Assume that \( h(r) > \epsilon |x|^{1+\epsilon/2} \). Then from (6.1) it follows that \( |h(r)| \) is a positive function for sufficiently small \( r \).

By a direct computation
\[ \lambda(D^2 h(|x|)) = (h'', h'/|x|, ..., h'/|x|). \]
Hence \( h \) has no local minimums and since \( h > 0 \) we get \( h' > 0, h'' < 0 \) for sufficiently small \( r \). Therefore \( h \) is a monotone, concave function for small \( r \).

For any \( 0 < r < 1 \) there exists a point \( x_0, |x_0| = r \) such that \( u_0(x_0) = h(r) \) and since \( h - u_0 \leq 0 \) the quadratic part of the function \( u_0 - h \) is non-negatively defined. Hence from the uniform ellipticity condition for \( F \) we get the inequality
\[ -|x|h''/h' > \delta, \]
on an interval \( (0, a) \) for some \( a > 0 \), where \( \delta \) depends on the ellipticity constant. From the last inequality it follows that
\[ h(r) > r^{1-\delta} \]
on \((0, a)\). Since we can redefine \( h \) on \((a, 1)\) as a monotone, concave function, the theorem is proved.
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