QUANTUM HAMILTONIAN REDUCTION
OF SUPER KAC-MOODY ALGEBRA II

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ABSTRACT

The quantum Hamiltonian reduction on the OSp(1,2) super Kac-Moody algebra is described in the BRST formalism. Using a free field representation of the KM currents, the super Kac-Moody algebra is shown to be reduced to a superconformal one via the Hamiltonian reduction. This reduction is manifestly supersymmetric because of supersymmetric constraints imposed on the algebra.
1. Introduction

One of interesting properties of conformal field theory has been observed by applying the Hamiltonian reduction to a SL(2,R) Kac-Moody algebra [1,2]; an irreducible representation of the conformal algebra is obtained from one of the SL(2,R) KM algebra by imposing a certain constraint. In this constraint system, which can be described in the BRST formalism like gauge field theories, the KMA algebra is not observable and so is hidden behind a physical symmetry of the conformal algebra. Thus one calls the SL(2,R) KMA a hidden symmetry of conformal field theory. This discovered connection between the KM and conformal algebras explains the appearance of the SL(2,R) KM algebra in the effective action of 2-d gravity in the light-cone gauge [3].

Seeking the hidden symmetry of superconformal field theory, the Hamiltonian reduction is applied to an OSp(1,2) KM algebra [4]. The consistency of imposed constraints forces to introduce a Majorana fermion in addition to the KM currents because the constraints satisfy a nontrivial algebra unlike in the SL(2,R) case. One then succeeds in reducing the combined system of the OSp(1,2) KMA and the fermion to a superconformal field theory. Although that system is the smallest one that is reduced to the superconformal field theory, there are the following unsatisfactory points involved; (i) the formulation is not manifestly supersymmetric and so the relation of the extracted superconformal algebra with the original one is obscure. (ii) Since the fermion has been added to the space of the OSp(1,2) KMA, it is questionable to conclude that the hidden symmetry of superconformal field theory is the OSp(1,2) KMA. We would like to consider that the addition of the fermion into the system suggests a larger hidden symmetry than the OSp(1,2) KMA. In the present paper, thus, we give a supersymmetric formulation of the quantum Hamiltonian reduction on the OSp(1,2) super KMA. (See [5] for the classical one.)

Using a free-field realization of the OSp(1,2) KMA and explicitly solving the constraints on the super KM (SKM) currents, we tried in [6] to show that two unconstrained SKM currents are directly reduced into the supercurrent and stress-energy tensor of superconformal field theory. Although this approach is successful in the SL(2,R) case [7], we did not reach any definite results because of the subtlety
of the normal ordering of higher-order terms in the SKM currents when the solved constraints are substituted into them. To avoid this problem, we apply the BRST formalism to our system of the super KMA with constraints on superspace currents.

This paper is organized as follows. Section 2 reviews the main results of super Kac-Moody algebra. In Section 3 we propose constraints on superspace SKM currents, which allow the OSp(1,2) super KMA to be supersymmetrically reduced into a superconformal algebra, introducing a c-number function \( \zeta(z) \). The function \( \zeta(z) \) plays the same important role as the constant 1 in the constraint \( J^{-}(z) - 1 = 0 \) of the SL(2,R) case. After the introduction of pairs of ghosts and antighosts corresponding to the constraints, a BRST charge of our system is constructed. In Section 4 we show that the proposed constraints really reduce a space of the super KMA into a superconformal field theory, using the technique developed for the proof of the no-ghost theorem in string theory. The same constraints as in the OSp(1,2) case are obtained from our constraints on SKM currents by substituting the fermion constraints into them. In Section 5 we exhibit that a twisted superconformal tensor of the constraint system really becomes in the physical subspace one of superconformal field theory. When the function \( \zeta(z) \) depends on \( z \), the BRST invariance requires some operator to be added to the twisted superconformal tensor. We discuss a physical meaning of the added term. Section 8 contains a brief conclusion and some comments.

2. OSp(1,2) Super Kac-Moody Algebra

Let us start with the OSp(1,2) Kac-Moody algebra with central charge \( k \). This KM algebra has three bosonic currents \( J^{(a)}(z) \) for \( a = 0, \pm 1 \), which generate the SL(2,R) KM subalgebra, and two fermionic currents \( J^{(\alpha)}(z) \) for \( \alpha = \pm \frac{1}{2} \). The KM currents \( J^{(A)} \) for \( A = 0, \pm 1, \pm \frac{1}{2} \) satisfy the following operator product expansion (OPE)

\[
J^{(a)}(z)J^{(b)}(w) \sim \frac{\eta^{AB}k}{(z-w)^2} + \frac{f^{AB}_{\quad C}J^{(c)}(w)}{z-w}
\]

(2.1)

where \( \eta^{AB} \) and \( f^{AB}_{\quad C} \) are the metric and structure constant of the algebra.

A KM algebra can be supersymmetrized by introducing fermions in the adjoint representation which become supersymmetric partners of super KM currents [8].
In the present case we then introduce three anticommuting fermions $\psi^{(a)}$ for $a = 0, \pm 1$ and two commuting fermions $\psi^{(\alpha)}$ for $\alpha = \pm \frac{1}{2}$, corresponding to the bosonic and fermionic currents $J^{(a)}$ and $J^{(\alpha)}$ respectively. The superspace currents $J^{(a)}$ are constructed as

$$J^{(a)}(z, \theta) = \psi^{(a)}(z) + \theta \tilde{J}^{(a)}(z) \tag{2.2}$$

and generate the super KM algebra (SKMA) with central charge $\tilde{k}$, satisfying the following OPE

$$J^{(a)}(z_1, \theta_1) J^{(b)}(z_2, \theta_2) \sim \frac{\tilde{k} \eta^{AB}}{z_{12}} + \frac{\theta_{12} f^{AB} c J^{(c)}(z_2, \theta_2)}{z_{12}} \tag{2.3}$$

where $z_{12} \equiv z_1 - z_2 - \theta_1 \theta_2$ and $\theta_{12} \equiv \theta_1 - \theta_2$. From its operator product with the fermion in the OPE (2.3), one finds that the super KM (SKM) current $\tilde{J}^{(a)}$ consists of two independent parts: a current part $J^{(a)}$ with central charge $k$ and a fermion part $J^{(a)}_f$ with central charge $\frac{3}{2}$

$$\tilde{J}^{(a)}(z) = J^{(a)}(z) + J^{(a)}_f(z). \tag{2.4}$$

The fermion current $J^{(a)}_f$ is given by

$$J^{(a)}_f = \frac{1}{k} f^{A}{}_{BC} \psi^{(A)} \psi^{(B)} :. \tag{2.5}$$

Then the total central charge $\tilde{k}$ of the SKMA is equal to a sum of those central charges $k + \frac{3}{2}$.

One can construct an N=1 superconformal algebra (SCA) from superspace KM currents. The super stress-energy tensor $T^{SKM}$ is given by

$$T^{SKM}(z, \theta) = \frac{1}{2} G^{SKM}(z) + \theta T^{SKM}(z) \tag{2.6}$$

where the supercurrent and stress-energy tensor are written in the form

$$G^{SKM}(z) = \frac{1}{k} \left( \eta_{AB} J^{(A)} \psi^{(B)} + \frac{1}{3k} f_{ABC} \psi^{(A)} \psi^{(B)} \psi^{(C)} :. \right)$$

$$T^{SKM}(z) = \frac{1}{k} \left( \eta_{AB} : J^{(A)} J^{(B)} : - \eta_{AB} \psi^{(B)} \partial \psi^{(A)} :. \right) \tag{2.7}$$

Since the metric and structure constant of the OSp(1,2) contain the unfamiliar antisymmetric and symmetric parts caused by fermionic elements, we carefully
checked the ordering of operators in the expressions (2.5) and (2.7). The conformal anomaly $c_{SKM}$ of the SCA is easily obtained as

$$c_{SKM} = \frac{1}{2} + \frac{k}{k + \frac{3}{2}}$$

by calculating the OPE between two supercurrents

$$G^{SKM}(z)G^{SKM}(w) \sim \frac{2}{3} c_{SKM} \frac{1}{(z-w)^3} + \frac{2 T^{SKM}(w)}{z-w}$$

With respect to the stress-energy tensor $T^{SKM}$ of Sugawara form, all the superspace currents (2.2) have the same dimensions $\frac{1}{2}$; $\psi^{(A)}$ of dimension $\frac{1}{2}$ and $\hat{J}^{(A)}$ of dimension 1.

The Hilbert space of a SCA is classified into two sectors, the Ramond sector with expansion modes of $G(z)$ integers ($G(z) = \sum G_n z^{-n-\frac{3}{2}}$) and the Neveu-Schwarz sector with expansion modes half-integers ($G(z) = \sum G_n z^{-n-2}$). In the SKMA, along with the integer mode expansion of the currents $\hat{J}^{(A)}$, the fermions $\psi^{(A)}$ are correspondingly expanded as

$$\psi^{(A)}(z) = \sum \psi^{(A)}_{n+\kappa} z^{-n-\kappa-\frac{1}{2}}$$

with $\kappa = 0$ for the Ramond sector and $\kappa = \frac{1}{2}$ for the Neveu-Schwarz sector.

3. Quantum Hamiltonian Reduction on Super Kac-Moody Algebra

In the Hamiltonian reduction of the SL(2,R) and OSp(1,2) KMA’s, the stress-energy tensor of Sugawara form was modified by the addition of a current linear term [2,4]. For the case of the SKMA, an appropriate super stress-energy tensor can be constructed by twisting one of Sugawara form $T^{SKM}(z, \theta)$ with a superspace current $J^{(0)}(z, \theta)$ as

$$T(z, \theta) \equiv T^{SKM} - \partial J^{(0)}.$$

The twisted super stress-energy tensor $T$ satisfies a SCA with conformal anomaly $c = c_{SKM} - 6\hat{k}$. With respect to $T$, the superspace current $J^{(A)}$ is of dimension $\frac{1}{2} + A$; $\psi^{(A)}$ of dimension $\frac{1}{2} + A$ and $\hat{J}^{(A)}$ of dimension $1 + A$. The significant effect of the shift of the conformal dimension will be discussed later.
In order to extract a SCA in a supersymmetric way, we put the following constraints on superspace currents,

\[
\begin{align*}
J^{(-)}(z, \theta) = 0 \\
J^{(-\frac{1}{2})}(z, \theta) - \sqrt{2k} \zeta(z) = 0
\end{align*}
\]  

which are equivalent to

\[
\begin{align*}
\psi^{(-)}(z) = 0 \\
\psi^{(-\frac{1}{2})}(z) - \sqrt{2k} \zeta(z) = 0
\end{align*}
\]  

Here we have introduced the arbitrary c-number function \( \zeta(z) \) of \( z \). The function \( \zeta \) is required by the first constraint in the second column of (3.3) to be of the same properties as the fermion \( \psi^{(-\frac{1}{2})} \), in particular, commuting with \( \psi^{(\pm \frac{1}{2})} \) and anticommuting with \( \psi^{(a)} \) for \( a = 0, \pm \). Those superspace constraints (3.2) are consistent and closed because the superspace current \( J^{(-\frac{1}{2})} \) is of dimension 0 and only their nontrivial operator product is

\[
( J^{(-\frac{1}{2})}(z_1, \theta_1) - \sqrt{2k} \zeta(z) ) ( J^{(-\frac{1}{2})}(z_2, \theta_2) - \sqrt{2k} \zeta(z) ) \sim -2 \theta_{12} J^{(-)}(z_2, \theta_2). \]  

This operator product also indicates that one can not consistently fix the current \( J^{(-)} \) of dimension 0 to be any nonvanishing constant like in the SL(2,R) case.

We introduce two superspace ghost systems, \((b(z, \theta), c(z, \theta))\) of dimensions \((-\frac{1}{2}, 0)\) and \((\beta(z, \theta), \gamma(z, \theta))\) of dimensions \((0, \frac{1}{2})\), corresponding to the superspace constraints (3.2), to complete the standard quantization process of constraint system. The superspace ghosts are expanded in \( \theta \) as

\[
\begin{align*}
b(z, \theta) &= \beta^+ + \theta b \\
c(z, \theta) &= c_z + \theta \gamma_z \\
\beta(z, \theta) &= \tilde{b} + \theta \beta_+ \\
\gamma(z, \theta) &= \gamma_+ + \theta \tilde{c}_z
\end{align*}
\]  

The pairs of ghosts, \((b, c_z)\) and \((\tilde{b}, \tilde{c}_z)\) both of dimensions \((0, 1)\), are anticommuting and the pairs, \((\beta^+, \gamma_z^+)\) of \((-\frac{1}{2}, \frac{3}{2})\) and \((\beta_+, \gamma_+)\) of \((\frac{1}{2}, \frac{1}{2})\), are commutting. Their
total super stress-energy tensor with conformal anomaly $c_{\text{ghost}} = 6$ is written in a supersymmetric form

$$T_{\text{ghost}}(z, \theta) = (c D^2 b - \frac{1}{2} Dc \cdot Db + \frac{1}{2} D^2 c \cdot b) + \frac{1}{2}(\gamma D^2 \beta - D\gamma \cdot D\beta)$$

(3.6)

where $D \equiv \frac{\partial}{\partial \theta} + \theta \partial$ is a covariant derivative in the superspace.

Combining the superspace constraints (3.2) with the superspace ghost systems (3.5) and using its closed subalgebra (3.4), a BRST charge of our system is constructed in the standard manner,

$$Q_B = \oint dzd\theta \{ c J^{(-)} + \gamma (J^{(-\frac{1}{2})} - \sqrt{2k} \zeta) - \gamma^2 b \}$$

(3.7)

which is written after the integration of $\theta$ as

$$Q_B = \oint d\{ c_z \tilde{J}^{(-)} + \gamma_+ \tilde{J}^{(-\frac{1}{2})} + \gamma_{z+} \psi^{(-)} + \tilde{c}_z (\psi^{(-\frac{1}{2})} - \sqrt{2k} \zeta) - \gamma^2 b - 2\gamma_+ \tilde{c}_z \beta^+ \}.$$

(3.8)

One can easily verify the nilpotency of the BRST charge $Q_B$. The BRST charge $Q_B$ plays a central role in the BRST formalism, giving a criterion for a physical operator $O_{\text{phys}}$ as $[Q_B, O_{\text{phys}}] = 0$. Since $Q_B$ sums over all generators of gauge symmetry multiplied by the corresponding antighosts, this implies that the physical operator does not contain any gauge freedoms. Furthermore, the physical operator can be written in the form of the summation over a genuine and a BRST trivial parts as

$$O_{\text{phys}} = O_G + [Q_B, *]$$

(3.9)

where $O_G$ can not be expressed in the form of the (anti)commutation with $Q_B$ and * indicates some operator [9]. Also, a physical subspace $H_{\text{phys}}$ is defined as a quotient space

$$H_{\text{phys}} \equiv \{ |v> : Q_B|v> = 0 \} / \{ |*> : <*|*> = 0 \text{ and } Q_B|*> = 0 \}.$$

(3.10)

We will find out the physical symmetry algebra which is extracted from the SKMA through the above Hamiltonian reduction.
Let us first mention the fate of the SKMA. Of the SKM currents, only $J(\alpha)$ commutes with the BRST charge $Q$, but it has no genuine part because $J(\alpha)(z, \theta) = \{Q, b(z, \theta)\}$. Thus the SKMA is not any symmetry algebra in the physical subspace. Since the ghost systems have been introduced into the Hilbert space of the SKMA, a total super stress-energy tensor $T_{\text{total}}$ of the whole system is a sum of ones for the twisted SKMA and ghosts, (3.1) and (3.6),

$$T_{\text{total}}(z, \theta) = (T^{\text{SKM}} - \partial J(\alpha)) + T^{\text{ghost}}$$

with conformal anomaly $c_{\text{total}} = (c_{\text{SKM}} - 6\hat{k}) + c_{\text{ghost}}$. Some algebraic calculation shows that

$$\{Q, T_{\text{total}}(z, \theta)\} = 0.$$  (3.12)

The total super stress-energy tensor $T_{\text{total}}$ also contains a nonvanishing genuine part, as we will show later. Thus the superconformal algebra generated by $T_{\text{total}}(z, \theta)$ is the extracted symmetry of the physical subspace $\mathcal{H}_{\text{phys}}$ from the SKMA.

The currents and fermions $\hat{J}(\alpha)$ and $\psi(\alpha)$ of twisted dimensions $1 + \alpha$ and $\frac{1}{2} + \alpha$ are expanded as

$$\psi(\alpha)(z) = \sum_{n=-\infty}^{\infty} \psi^{(\alpha)}_{n+\kappa} z^{-n-\kappa-\frac{1}{2}-\alpha}, \quad \hat{J}(\alpha)(z) = \sum_{n=-\infty}^{\infty} \hat{J}^{(\alpha)}_{n} z^{-n-1-\alpha}. \quad \text{(3.13)}$$

In the expansion (3.13), the bosonic currents are even under the transformation $z \rightarrow e^{2\pi i} z$ but the fermionic odd. This indicates that the expansion (3.13) breaks the OSp(1,2) (global) symmetry. On the other hand, to keep the symmetry, we may use the other expansion of the currents and fermions, in which the fermionic currents are twisted as $\hat{J}^{(\alpha)}(z) = \sum \hat{J}^{(\alpha)}_{n+\frac{1}{2}} z^{-n-\frac{1}{2}-1-\alpha}$ and the corresponding fermions are expanded as $\psi^{(\alpha)}(z) = \sum \psi^{(\alpha)}_{n+\tilde{\kappa}} z^{-n-\tilde{\kappa}-\frac{1}{2}-\alpha}$ where $\tilde{\kappa} = \frac{1}{2}$ for the Ramond sector and $\tilde{\kappa} = 0$ for the Neveu-Schwarz sector [10]. In the present paper the former expansion (3.13) will be used. The constraint $\psi^{(-\frac{1}{2})}(z) - \sqrt{2\tilde{\kappa}} \zeta = 0$, thus, requires the function $\zeta$ of $z$ to have integer power in $z$ for the Ramond sector and half-integer for the Neveu-Schwarz sector. For simplicity the function $\zeta$ will be chosen to be

7
constant \( \zeta_0 \) and \( \zeta_{\frac{1}{2}} z^{-\frac{1}{2}} \) for the Ramond and Neveu-Schwarz sectors respectively. The ghost systems \((\beta^+, \gamma_{z+})\) and \((\tilde{b}, \tilde{c}_z)\) corresponding to the fermion constraints in (3.3) are expanded as

\[
\begin{align*}
\beta^+(z) &= \sum_{n=-\infty}^{\infty} \beta^+_{n+\kappa} z^{-n-\kappa+\frac{1}{2}} \\
\gamma_{z+}(z) &= \sum_{n=-\infty}^{\infty} \gamma_{z+n+\kappa} z^{-n-\kappa-\frac{3}{2}} \\
\tilde{b}(z) &= \sum_{n=-\infty}^{\infty} \tilde{b}_{n+\kappa} z^{-n-\kappa} \\
\tilde{c}_z(z) &= \sum_{n=-\infty}^{\infty} \tilde{c}_{z+n+\kappa} z^{-n-\kappa-1}.
\end{align*}
\]

The current constraints in (3.3) can be simplified by using the fermion ones as

\[
0 = \tilde{J}^{(-)} = J^{(-)} + \frac{1}{2} \zeta^2 \\
0 = \tilde{J}^{(-\frac{1}{2})} = J^{(-\frac{1}{2})} - \sqrt{2} \tilde{\psi}^{(0)} \zeta
\]

The reduced constraints (3.15) appeared in [4], which showed that a Hilbert space of the OSp(1,2) KMA and a fermion \( \psi \) are reduced to that of SCA by imposing the constraints (3.15). Since \( \psi \) can be identified as \( \psi^{(0)} \zeta \) in the constraints (3.15), the Hilbert space of the KMA and fermion is obviously only a subspace of our Hilbert space of the SKMA.

4. BRST Analysis on Hilbert Space of SKMA and Ghost Systems

The following calculation on the total conformal anomaly \( c_{\text{total}} \) suggests a connection of the physical subspace with a Hilbert space of the super Coulomb gas [11] which gives a free field realization of superconformal models because a conformal anomaly counts dynamical degrees of freedom in the system. The expression of the anomaly \( c_{\text{total}} \) can be rewritten as

\[
c_{\text{total}} = (c_{\text{SKM}} - 6\hat{k}) + c_{\text{ghost}} \\
= \frac{3}{2} \left\{ 1 - 2 \left( \sqrt{2k+3} - \frac{1}{\sqrt{2k+3}} \right)^2 \right\} \quad (4.1)
\]

The super Coulomb gas is a system of a scalar field and a Majorana fermion with background charge \( \alpha_0 \), whose anomaly is \( c_{\text{SCFT}} = \frac{3}{2}(1 - 8\alpha_0^2) \). The expression
(4.1) of $c_{\text{total}}$ is obtained from the anomaly $c_{\text{SCFT}}$ by replacing $\alpha_0$ with $\alpha_0(k) \equiv \frac{1}{2}(\sqrt{2k + 3} - \frac{1}{\sqrt{2k + 3}})$. This implies that the physical subspace is identified as the super Coulomb gas with background charge $\alpha_0(k)$, as we will see in the following sections. When $k + 1 = \frac{1}{q^2}$ or $-\frac{1}{q^4}$ is satisfied, the corresponding physical subspace gives the free field representation of an unitary superconformal minimal model with anomaly $c_{\text{SCFT}}(q) = \frac{3}{2} \left( 1 - \frac{8}{(q+2)(q+4)} \right)$ for a positive integer $q$ [12].

In the SL(2,R) case, its free field representation is a powerful tool to analyze the structure of a Hilbert space of the KMA with a constraint. To realize the OSp(1,2) KM algebra in terms of free fields, one needs a scalar field $\phi$, a pair of bosonic ghosts $(\beta, \gamma)$ of dimensions $(1,0)$, and a pair of fermionic ghosts $(\psi, \psi^+)$ of dimensions $(0,1)$. The OSp(1,2) currents are expressed [4] as

$$
\begin{align*}
J^{(-)}(z) &= \beta \\
J^{(0)}(z) &= \beta \gamma - \frac{i}{2} \alpha_+ \partial \phi + \frac{1}{2} \beta \psi \psi^+ \\
J^{(+)}(z) &= -\beta \gamma^2 + i \alpha_+ \gamma \partial \phi - \gamma \psi \psi^+ + k \partial \gamma - (k + 1) \psi \partial \psi \\
J^{(1/2)}(z) &= \gamma(\psi^+ - \beta \psi) + i \alpha_+ \psi \partial \phi + (2k + 1) \partial \psi
\end{align*}
$$

(4.2)

where $\alpha_+ = \sqrt{2k + 3}$. From now on we use the free field expressions (4.2) as the current part $J^{(a)}$ of the SKM current (2.4). The total super stress-energy tensor $T_{\text{total}}$ then is rewritten in terms of the free fields only. The pairs of the ghosts $(\beta, \gamma)$ and $(\psi, \psi^+)$ have twisted dimensions $(0,1)$ and $(\frac{1}{2}, \frac{1}{2})$ with respect to $T_{\text{total}}$. From the commutation relation of the SKM currents with the fermions, the pair of the fermionic ghosts $(\psi, \psi^+)$ are commuting with $\psi^{(a)}$ for $a = 0, \pm$ but anticommuting with $\psi^{(a)}$ for $a = \pm \frac{1}{2}$ and they are untwisted because the fermionic currents $J^{(a)}$ have integer modes. The Ramond and Neveu-Schwarz sectors are obtained by untwisting and twisting the superpartners $\psi^{(a)}$ of the SKM currents as the first expression in (3.13).

One can readily see that physical fields (anti)commuting with the BRST charge $Q_B$ are a scalar field $\phi(z)$ and a composite fermion $\psi_{\text{phys}}(z) \equiv \frac{i}{\sqrt{2\beta}} (\beta \psi + \psi^+)$ whose operator product is $\psi_{\text{phys}}(z)\psi_{\text{phys}}(w) \sim \frac{1}{z-w} + o(z-w)$. The factor $\frac{i}{\sqrt{2\beta}}$ has been multiplied for the normalization. Since the field $\beta$ itself has a curious property of commuting ghost, one must carefully define the square root function $\sqrt{\beta}$. From the first constraint in (3.15) and the free field expression of $J^{(-)}$ in (4.2), one
can derive a relation \( \beta = -\frac{1}{2} \zeta^2 + \) [a BRST trivial term] in the physical subspace. This relation gives the precise definition of the square root function as

\[
\sqrt{\beta} \equiv \frac{i}{\sqrt{2}} \zeta + \{Q_B, *\}. \quad (4.3)
\]

Thus the function \( \sqrt{\beta} \) is anticommuting with the fermionic ghosts \( (\psi, \psi^+)^{\ast} \) and the physical fermion \( \psi_{\text{phys}} \) with the fermions \( \psi^{(A)} \).

In the following we will show that the physical subspace \( \mathcal{H}_{\text{phys}} \) is spanned only by excitations of the scalar field and the fermion, \( \phi \) and \( \psi_{\text{phys}} \), a space of which is referred as \( \mathcal{H}(\phi, \psi_{\text{phys}}) \). Obviously, \( Q_B | \Omega > = 0 \) holds for any state \( | \Omega > \) of the space \( \mathcal{H}(\phi, \psi_{\text{phys}}) \). Then it must be proved that any state \( | \Psi > \) annihilated by \( Q_B \) is written in the form

\[
| \Psi > = \mathcal{P} | \Psi > + Q_B | * > 
\]

where the operator \( \mathcal{P} \) is a projection onto the space \( \mathcal{H}(\phi, \psi_{\text{phys}}) \) from the whole Hilbert space. The vacuum \( | 0 > \) is defined by

\[
\varphi_n | 0 > = 0 \quad \overline{\varphi}_m | 0 > = 0 \quad (4.5)
\]

and

\[
\begin{cases}
\chi_n | 0 > = 0 & \chi_{n+\frac{1}{2}} | 0 > = 0 \\
\overline{\chi}_n | 0 > = 0 & \overline{\chi}_{n+\frac{1}{2}} | 0 > = 0
\end{cases}
\]

in the Ramond sector

\[
\begin{cases}
\chi_n | 0 > = 0 & \chi_{n+\frac{1}{2}} | 0 > = 0 \\
\overline{\chi}_n | 0 > = 0 & \overline{\chi}_{n+\frac{1}{2}} | 0 > = 0
\end{cases}
\]

in the Neveu-Schwarz sector

for \( n \geq 0 \) and \( m \geq 1 \) and where

\[
\varphi = (\beta, \psi^+, b, b^+) \quad \overline{\varphi} = (\gamma, \psi, c, \gamma^+) \\
\chi = (\psi^{(0)}, \psi^{(-)}, \psi^{(-\frac{1}{2})}, \gamma^+, \overline{b}) \quad \overline{\chi} = (\psi^{(+)}, \psi^{(\frac{1}{2})}, \gamma^{z+}, \overline{c}).
\]

Due to the presence of the c-number function \( \zeta \) in \( Q_B \), the physical vacuum \( | 0 >_{\text{phys}} \) is

\[
| 0 >_{\text{phys}} = \begin{cases}
e^{-\overline{\lambda}_{\gamma_0}} e^{-\zeta \psi_0^{(\frac{1}{2})}} | 0 > & \text{for the Ramond sector} \\
e^{-\overline{\lambda}_{\gamma_1}} e^{-\zeta \psi_0^{(\frac{1}{2})}} | 0 > & \text{for the Neveu-Schwarz sector}
\end{cases}
\]

The multiplied factors in the physical vacuum of the Neveu-Schwarz sector involve the nonzero modes \( \psi^{(\frac{1}{2})} \) and \( \gamma_{-1} \) which change the energy of the physical vacuum.
The effect of these factors on the super stress-energy tensor will be discussed in the next section. Taking into account the nontrivial physical vacuum, the explicit form of the projection $P$ is

$$P = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{n_1 < \cdots < n_i (m_1, \ldots, m_j)} \frac{1}{j!} (\psi_{\text{phys}} - n_1 - \kappa \cdots (\psi_{\text{phys}} - n_i - \kappa \phi_{-m_1} \cdots \phi_{-m_j}) |0 >_{\text{phys}}$$

and so it is annihilated by $Q_B$.

In the case of the SL(2,\(R\)), one directly use the same argument on the physical subspace as in gauge theories because its BRST charge consists only of bilinear terms in the mode variables [2]. On the other hand, our BRST charge $Q_B$ contains higher-order terms up to biquadratic ones, as well as bilinear terms. We will use the following device, which has been designed in [13] to prove the no-ghost theorem in string theory based on the BRST formalism. Introducing a parameter $t$, a $t$-dependent BRST charge $Q^t_B$ is constructed from $Q_B$ as

$$Q_B(t) = A + t B + t^2 C$$

where the operators $A$, $B$ and $C$ are

$$A \equiv Q_B^{(0)} \equiv \oint dz \left\{ c_z (\beta + \frac{1}{2} \xi^2) + \gamma_+ (-\frac{1}{2} \xi^2 \psi - \psi^+ - \sqrt{\frac{2}{k}} \psi^{(0)} \zeta) + \gamma z \psi (-) + \zeta \right\}$$

$$B \equiv \oint dz \left\{ \left( \frac{1}{k} \right) \left( 2 c_z \psi^{(0)} \beta^+ - \frac{1}{4} c_z (\psi (-) + \sqrt{2k} \xi) \tilde{b} - \gamma_+ \psi^{(0)} \tilde{b} - \gamma_+ (\xi) \beta^+ \right) \right.$$

$$\left. - \gamma_+ \psi \tilde{b} \right\}$$

$$C \equiv \oint dz \sqrt{\frac{2}{k}} \zeta \left( \gamma_+ c_z \beta^+ + \frac{1}{2} \gamma_+ \tilde{b} \right)$$

The operator $A (= Q_B^{t=0})$ has the same bilinear form as the asymptotic one of the BRST charge in the ordinary gauge theories [9], thus being referred as $Q_B^{(0)}$. The $t$-dependent BRST charge $Q^t_B$ is nilpotent for any value of $t$, because the operators
A, B and C fortunately satisfy the following algebraic relations

\[ A^2 = C^2 = 0 \]
\[ \{A, B\} = \{B, C\} = 0 \]  \hspace{1cm} (4.9)
\[ B^2 + \{A, C\} = 0. \]

When \( t = 1 \), \( Q_{\phi}^1 \) agrees with the original BRST charge \( Q_\phi \). The definition of a \( t \)-dependent physical space \( \mathcal{H}_{\text{phys}}(t) \) is obtained by replacing \( Q_\phi \) in the definition (3.10) with the \( t \)-dependent \( Q_{\phi}^t \). Physical fields (anti)commuting with \( Q_{\phi}^t \) are

\[ \phi^t = \phi \]
\[ \psi_{\text{phys}}^t = i\mathcal{N}(t)\{(t\beta\psi + \psi^+) + (t - 1)\frac{1}{2}\mathcal{L}^2\psi\} \]

with the normalization factor \( \mathcal{N}(t) = \left\{ 2(\beta + (t - 1)\frac{1}{2}\mathcal{L}^2) \right\}^{-\frac{1}{2}} \). A \( t \)-dependent operator \( P^t \) is a projection onto the \( t \)-dependent space \( \mathcal{H}(\psi^t, \psi_{\text{phys}}^t) \), thus being annihilated by \( Q_{\phi}^t \).

The \( t \)-dependent theory as defined above will become the device for our proof of the general form of physical states. It will be proved in the \( t \)-dependent theory that any state \( |\Phi^t\rangle \) annihilated by \( Q_{\phi}^t \) has the form

\[ |\Phi^t\rangle = P^t |\Phi^t\rangle + Q_{\phi}^t |\ast, t\rangle, \]  \hspace{1cm} (4.10)

using mathematical induction with respect to the power of \( t \). In our induction, all \( t \)-dependent states and operators are expanded in \( t \) as \( |\Phi^t\rangle = \sum_{n=0}^{\infty} |\Phi_n^t\rangle t^n \) and \( \mathcal{O}^t = \sum_{n=0}^{\infty} \mathcal{O}_n^t t^n \), assuming their nonnegative expansions. A starting point in the induction is the \( t = 0 \) theory to which the techniques developed for the unitarity proof in gauge theories [9] can be applied. Since the operators \( A, B \) and \( C \) satisfy the same algebra (4.9) as in string theory, the essential part of our proof for (4.10) has been described in [13]. In the following, we will give only a sketch for our proof and recommend to read them for the details.
In the \( t = 0 \) theory, one can easily find out two physical fields and four Kugo-Ojima (KO) quartets

\[
\phi^{t=0} = \phi \\
(\gamma, \beta + \frac{1}{2} \xi^2, c_z, b) \\
(\psi^+, \psi^-, \gamma_z, \beta^+) \\
(\psi^0, \psi^0, \gamma^0, \beta^+) \\
(\psi^0, \psi^0, \gamma^0, \beta^+) \\
(4.11)
\]

where \( \Psi^{(0)}_\pm \equiv \frac{1}{2} \xi^2 \psi + \psi^0 \pm \sqrt{2} \hat{k} \xi \psi^{(0)} \zeta \). Repeating the same argument for those physical fields and KO quartets as in gauge theories [9], one can show that the general solution for \( Q_B^0 | \Phi_0 > = 0 \) is

\[
| \Phi_0 > = \mathcal{P}^{t=0} | \Phi_0 > + Q_B^{(0)} | *, 0 >
\]

where \( \mathcal{P}^{t=0} \) is a projection operator onto the subspace \( \mathcal{H}(\phi^{t=0}, \psi_{phys}^{t=0}) \). The next step is to show by the induction that the \( n \)th term \( | \Phi_n^t > \) in the expansion \( | \Phi^t > = \sum_{n=0}^{\infty} | \Phi_n^t > t^n \) has the form

\[
| \Phi_n^t > = \sum_{m=0}^{n} \mathcal{P}_{n-m}^t | \Phi_m^t > + A| *, n > + B| *, n - 1 > + C| *, n - 2 >
\]

where \( \mathcal{P}^t = \sum_{n=0}^{\infty} \mathcal{P}_n^t t^n \) and \( | *> = \sum_{n=0}^{\infty} | *, n > t^n \), understanding that \( | *, n > = 0 \) for \( n < 0 \). For \( n = 0 \), the expression (4.13) is satisfied by \( \Phi_0^t = \Phi_0 \) because it agrees with (4.12). The condition for physical state, \( Q_B^t | \Phi^t > = 0 \), leads to a relation

\[
A| \Phi_n^t > + B| \Phi_{n-1}^t > + C| \Phi_{n-2}^t > = 0
\]

The statement (4.13) for \( n = N \) can be proved assuming one for \( n \leq N - 1 \) and using (4.14) and the following relation

\[
A \sum_{m=0}^{n} \mathcal{P}_{n-m}^t | \Phi_m^t > + B \sum_{m=0}^{n-1} \mathcal{P}_{n-1-m}^t | \Phi_m^t > + C \sum_{m=0}^{n-2} \mathcal{P}_{n-2-m}^t | \Phi_m^t > = 0
\]

derived from \( Q_B^t \mathcal{P}^t | \Phi^t > = 0 \). Multiplying the expressions (4.13) by \( t^n \) and summing up them over \( n \), we finally arrive at the general form (4.10) of physical state.
in the $t$-dependent theory to be proved. The general form (4.4) of physical state is obtained by setting $t = 1$ in the expression (4.10) and implies that the physical subspace $\mathcal{H}_{\text{phys}}$ is made up only of the scalar field $\phi$ and the fermion $\psi_{\text{phys}}$; $\mathcal{H}_{\text{phys}} = \mathcal{H} (\phi, \psi_{\text{phys}})$.

5. Physical Superconformal Algebra

In the last section we have suggested from the comparison of their anomalies $c_{\text{total}}$ and $c_{\text{SCFT}}$ that the subspace $\mathcal{H}_{\text{phys}}$ is the super Coulomb gas with background charge $\alpha_0 (k)$. To see it, we will show that the total supercurrent $G_{\text{total}}$ becomes one of the super Coulomb gas [11] in the physical space,

$$G_{\text{SCFT}} (z) = i \partial \phi \psi_{\text{phys}} + 2 \alpha_0 (k) \partial \psi_{\text{phys}}, \quad (5.1)$$

The supercurrent $G_{\text{total}}$, hence, plays a role of supersymmetry generators both in the original space of the SKMA and ghost systems and the physical one $\mathcal{H}_{\text{phys}}$. Given the supercurrent, the corresponding stress-energy tensor $T_{\text{SCFT}}$ is easily derived from the operator product of two supercurrents,

$$T_{\text{SCFT}} = - \frac{1}{2} (\partial \phi)^2 + i \alpha_0 (k) \partial^2 \phi - \frac{1}{2} \psi_{\text{phys}} \partial \psi_{\text{phys}}, \quad (5.2)$$

Both of the operators $G_{\text{SCFT}}$ and $T_{\text{SCFT}}$ generate a superconformal algebra in the physical space.

The analytic properties of the fermions $\psi^{(A)}$ determine ones of the function $\zeta (z)$ through the constraint $\psi^{(-\frac{1}{2})} = \sqrt{2k} \zeta = 0$ so that our simple choices of $\zeta (z)$ are the constant $\zeta_0$ for the Ramond sector and $\zeta \sqrt{z^{-\frac{1}{2}}}$ for the Neveu-Schwarz sector. These analytic properties are carried into the fermion $\psi_{\text{phys}}$ in the physical space by the factor $\sqrt{\beta}$ because of its definition (4.3) in terms of $\zeta (z)$. The Ramond and Neveu-Schwarz sectors in the physical space, thus, correspond to ones of the SKMA.
The operator product of the supercurrent $G^{\text{total}}$ and the scalar field $\phi$ gives its superpartner, which would be expected to be the fermion $\psi_{\text{phys}},$

$$G^{\text{total}}(z) \phi(w) \sim \frac{1}{z-w} i \sqrt{\frac{2}{k}} \left( \psi(-)^{\gamma} + \frac{1}{2} \psi(-)^{\frac{1}{2}} - \psi^{(0)} \right)$$

Since it anticommutes with $Q_B,$ the operator in the right-hand side of the above operator product can be written in the general form (3.9) of physical operator. Its genuine part is really a $\phi$'s superpartner in the space $\mathcal{H}_{\text{phys}}$ and it must be made up only of $\phi$ and $\psi_{\text{phys}}$ because of $\mathcal{H}_{\text{phys}} = \mathcal{H}(\phi, \psi_{\text{phys}})$ as we have shown in the last section. The physical operator can be rewritten in succession into a series of BRST equivalent ones, using the BRST charge $Q_B,$ to reach its desired form as described above; for example of such a process,

$$\psi(-)^{\gamma} + \frac{1}{2} \psi(-)^{\frac{1}{2}} - \psi^{(0)} = \hat{k} \zeta \psi - \psi^{(0)} - \frac{1}{2} \gamma_+ \tilde{b} - c_z \beta^+ + \left[ Q_B, \beta^+ \gamma + \tilde{b} \beta \right].$$

As a result of repeating this type of calculation, the following identity is obtained,

$$\sqrt{\frac{2}{k}} (\hat{k} \zeta \psi - \psi^{(0)} - \frac{1}{2} \gamma_+ \tilde{b} - c_z \beta^+) = \frac{i}{\sqrt{2\beta}} (\beta \psi + \psi^+) + \left[ Q_B, ** \right] \quad (5.3)$$

where the definition (4.3) of $\sqrt{\beta}$ has been used and ** denotes some complicated operator. Using those identities, the above operator product is successfully rewritten as

$$G^{\text{total}}(z) \phi(w) \sim \frac{1}{z-w} \left\{ i \psi_{\text{phys}}(w) + \left[ Q_B, ** \right] \right\} + \left[ Q_B, \cdots \right] \quad (5.4)$$

Using the operator product (5.4) and the superconformal algebra of $G^{\text{total}}$ and $T^{\text{total}},$ one can show that $G^{\text{total}}$ also transforms $\psi_{\text{phys}}$ into $\phi,$

$$G^{\text{total}}(z) \psi_{\text{phys}}(w) \sim \frac{\phi}{(z-w)^2} + \frac{\partial \phi}{(z-w)} + \left[ Q_B, \cdots \right] \quad (5.5)$$

with use of the fact that $G^{\text{total}}$ anticommutes with $Q_B.$ Thus, the supercurrent $G^{\text{total}}$ is a supersymmetry generator in the space $\mathcal{H}(\phi, \psi_{\text{phys}}).$
Let us show that the genuine part of the supercurrent $G^{\text{total}}$ is really one of the super Coulomb gas (5.1). The similar but much longer calculation gives the following expression of the supercurrent

$$G^{\text{total}} = (i\partial\phi + 2\alpha_0(k)\partial) \sqrt{\frac{2}{k}} (\hat{k} \zeta \psi - \psi^{(0)} - \frac{1}{2} \gamma b - c z^+ - 2k \alpha_0(k) \partial \zeta \psi) + [Q_B, \cdots].$$

For the Ramond sector with the function $\zeta$ constant, substituting the identity (5.3) into the above expression, the supercurrent has the following form to be expected,

$$G^{\text{total}} = G^{\text{SCFT}} + [Q_B, \ast] \quad (5.6)$$

On the other hand, for the Neveu-Schwarz (and Ramond with $\zeta$ dependent of $z$) sector(s), an additional term $\partial \zeta \psi$, which does not anticommute with $Q_B$, is involved in the expression

$$G^{\text{total}} = G^{\text{SCFT}} + [Q_B, \ast] - 2\sqrt{2k} \alpha_0(k) \partial \zeta \psi.$$

Defining a new supercurrent $\tilde{G}^{\text{total}}$ as

$$\tilde{G}^{\text{total}} \equiv G^{\text{total}} + 2\sqrt{2k} \alpha_0(k) \partial \zeta \psi, \quad (5.7)$$

$\tilde{G}^{\text{total}}$ anticommutes with $Q_B$ and has the same form as (5.6). The corresponding stress-energy tensor $\tilde{T}^{\text{total}}$ is

$$\tilde{T}^{\text{total}} \equiv T^{\text{total}} + \partial \zeta (2\gamma \psi^{(-)} \psi - 2\psi^{(0)} \psi - \psi^{(-\frac{1}{2})} \gamma - \psi^{(\frac{1}{2})}). \quad (5.8)$$

These BRST-invariant supercurrent and stress-energy tensor $\tilde{G}^{\text{total}}$ and $\tilde{T}^{\text{total}}$ are ones for the Neveu-Schwarz (and Ramond with $\zeta$ dependent of $z$) sector(s). For our simple choice $\zeta(z) = \zeta \frac{z^+}{2}$, although $L_0^{\text{total}} | 0 >_{\text{phys}} = (\frac{1}{2} \zeta^{(\frac{1}{2})} \psi^{(-\frac{1}{2})} - \hat{k} \zeta^{(2)} \gamma - 0 >_{\text{phys}}$, the energy of the physical vacuum with respect to $\tilde{T}^{\text{total}}$ is zero due to the added term of $\partial \zeta$ in (5.8); $\tilde{L}_0^{\text{total}} | 0 >_{\text{phys}} = 0$. Note that in the SL(2,R) case one must add an $a$-dependent term to the stress-energy tensor to get a BRST-invariant one when a constraint $J^{(-)}(z) = a(z) = 0$ with a $z$-dependent $a(z)$ is imposed.
6. Conclusion and Discussions

Taking the super Kac-Moody algebra of OSp(1,2), we have given the constraints on superspace currents, which allow us to formulate the Hamiltonian reduction in the supersymmetric way, to reduce the SKMA into the superconformal algebra. Constructing the BRST charge from those constraints in the standard way, the familiar BRST formalism has been applied to the above Hamiltonian reduction. With help of the free-field realization of the OSp(1,2) KMA, we have analyzed the structure of the physical subspace and the relation of superconformal algebras both in the SKMA and the physical subspace to show that the subspace is really a space of the super Coulomb gas. This result suggests that the hidden symmetry of superconformal field theory is the OSp(1,2) super Kac-Moody algebra, because the whole space of the SKMA is reduced into one of SCFT through the Hamiltonian reduction.

Finally we will give a few comments. In the present paper we have focused on the relation between two spaces of free fields realizing SCFT and the OSp(1,2) SKMA. Although each of the free field realizations is highly reducible, One can extract an irreducible representation of the algebra from it by using the screening charge as pointed out by Felder [14]. Two screening operators $J_{\text{OSP}}^{\pm}$ of the OSp(1,2) free-field realization are expressed in the superspace

\[
J_{\text{OSP}}^{-}(z, \theta) = \zeta^{-1} \left\{ \frac{1}{\sqrt{k}} \left( \psi(-) \psi - \frac{1}{2\sqrt{2}} \psi(-)^{1/2} \right) + \theta \left( \psi^+ + \beta \psi \right) \right\} e^{i\alpha \phi}
\]

\[
J_{\text{OSP}}^{+}(z, \theta) = \zeta^{-1} \left\{ \frac{1}{\sqrt{k}} \beta^{-k-3} \left( (k+2)\psi(-)^{1/2} + (k+1)\psi(-)^{1/2} \beta + \frac{1}{2\sqrt{2}} \psi(-)^{1/2} \beta \right) + \theta \beta^{-k-2} \left( \beta \psi + \psi^+ \right) \right\} e^{i\alpha \phi}
\]

(6.1)

On the other hand, ones of the super Coulomb gas are [11]

\[
J_{\text{SCFT}}^{\pm}(z, \theta) = (1 + \theta \psi_{\text{phys}}) e^{i\alpha \phi}
\]

(6.2)

These two sets of screening operators are BRST-equivalent to each other,

\[
J_{\text{OSP}}^{\pm}(z, \theta) = J_{\text{SCFT}}^{\pm}(z, \theta) + [Q_B, *].
\]

(6.3)

The presence of the factor $\zeta^{-1}$ in the expression (6.1) has allowed the screening operators $J_{\text{OSP}}^{\pm}$ to satisfy the BRST-equivalent relation (6.3). Us-
ing the BRST-equivalent relation (6.3), one can show in the same way as in [2] that any irreducible representation of the SKMA is reduced into the corresponding one of SCFT via the Hamiltonian reduction.

The present construction can be extended to the case of OSp($N,2$) ($N \geq 2$) SKMA. In $N = 2$ case the SKMA generates not only a $N=1$ superconformal algebra but also a $N=2$ one [15]. Thus one can expect that the OSp(2,2) SKMA is reduced into the $N=2$ SCA in a manifestly $N=2$ supersymmetric way.
REFERENCES

1. A.A. Belavin, in: Quantum String Theory, Springer Proc. in Physics Vol 31, eds. N. Kawamoto and T. Kugo (Springer-Verlag, Berlin, Heidelberg, 1988) p132.

2. M.B. Bershadsky and H. Ooguri, Commun. Math. Phys. 26 (1989) 49.

3. A.M. Polyakov, Mod. Phys. Lett. A2 (1987) 893.

4. M.B. Bershadsky and H. Ooguri, Phys. Lett. 229B (1989) 374.

5. K. Kimura, preprint RIMS-811 (1991).

6. T. Kuramoto, preprint QMW/PH/90/20 ; in: Proc. of KEK workshop ”Superstrings and Conformal Field Theory ” eds. M. Kobayashi and M. Kato (KEK, July 1991) p18.

7. A. Gerasimov, A. Marshakov and A. Morozov, Phys. Lett. 236B (1990) 269.

8. V.G. Kac and T. Todorov, Commun. Math. Phys. 102 (1989) 337.

9. T. Kugo and I. Ojima, Suppl. Prog. Theor. Phys. 66 (1979) 1.

10. T. Kuramoto, Nucl. Phys. B346 (1990) 527.

11. M.A. Bershadsky, V.G. Knizhnik and M.G. Teitelman, Phys. Lett. 151B (1985) 31;
    G. Mussardo, G. Sotkov and M. Stanishov, Phys. Lett. 195B (1987) 397.

12. H. Eichenherr, Phys. Lett. B151 (1985) 26;
    D. Friedan, Z. Qui and S. Shenker, Phys. Lett. B151 (1985) 37.

13. M. Kato and K. Ogawa, Nucl. Phys. B212 (1983) 443;
    M. Ito, T. Morozumi, S. Nojiri and S. Uehara, Prog. Theor. Phys. 75 (1986) 934;
    K. Itoh, Nucl. Phys. B342 (1990) 449.

14. G. Felder, Nucl. Phys. B317 (1989) 215.

15. Y. Kazama and S. Suzuki, Nucl. Phys. B321 (1989) 232;
    C.M. Hull and B. Spence, Phys. Lett. B241 (1990) 357.