Composite mapping on hesitant fuzzy soft classes

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Abstract. In this paper, we study composite mapping on hesitant fuzzy soft classes. We also establish some interesting properties of this notion and support them with examples. Also definitions of fuzzy soft image and fuzzy soft inverse image put forward by Kharal and Ahmad [5] have been reintroduced in hesitant fuzzy soft classes.

Keywords: Fuzzy soft sets; Hesitant fuzzy sets; Hesitant fuzzy soft sets; Hesitant fuzzy soft classes; Hesitant fuzzy soft mapping.

AMS subject classification no: 03E72.

1. Introduction

The Hesitant fuzzy set, as one of the extension of Zadeh [20] fuzzy set, allows the membership degree that an element to a set presented by several possible values, and it can express the hesitant information more comprehensively than other extensions of fuzzy set. Torra and Narukawa [15] introduced the concept of hesitant fuzzy set. Xu and Xia [19] defined the concept of hesitant fuzzy element, which can be considered as the basic unit of a hesitant fuzzy set, and is a simple and effective tool used to express the decision makers hesitant preferences in the process of decision making. So many researchers has done lots of research work on aggregation, distance, similarity and correlation measures, clustering analysis, and decision making with hesitant fuzzy information. Babitha and John [2] defined another important soft set Hesitant fuzzy soft sets. They introduced basic operations such as intersection, union, compliment and De Morgan’s law was proved. Wang, et al. [17] applied hesitant fuzzy soft sets in multicriteria group decision making problems.

There are many theories like theory of probability, theory of fuzzy sets, theory of intuitionistic fuzzy sets, theory of rough sets etc. which can be considered as mathematical tools for dealing with uncertain data, obtained in various fields of engineering, physics, computer science, economics, social science, medical science, and of many other diverse fields. But all these theories have their own difficulties. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets, introduced by L.A. Zadeh [20] in 1965. This theory brought a paradigmatic change in mathematics. But there exists difficulty, how to set the membership function in each particular case. The theory of intuitionistic fuzzy sets (see [1]) is a more generalized concept than the theory of fuzzy sets, but this theory has the same difficulties. All the above mentioned theories are successful to some extent in dealing with problems arising due to vagueness present in the real world. But there are also cases where these theories failed to give satisfactory results, possibly due to inadequacy of the

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parameterization tool in them. As a necessary supplement to the existing mathematical tools for handling uncertainty, in 1999, Molodtsov [11] initiated the theory of soft sets as a new mathematical tool to deal with uncertainties while modelling the problems in engineering, physics, computer science, economics, social sciences, and medical sciences. In [12], Molodtsov et al. successfully applied soft sets in directions such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, and theory of measurement. Maji et al. [5] gave the first practical application of soft sets in decision-making problems. Maji et al. [9] defined and studied several basic notions of the soft set theory. Also Çağman et al. [4] studied several basic notions of the soft set theory. In [6], Kharal and Ahmad introduced the notions of a mapping on the classes of soft sets and studied the properties of soft images. In [5], Kharal and Ahmad introduced the notions of a mapping on the classes of fuzzy soft sets. In [13],  V. Torra [14, 15] and Verma and Sharma [16] discussed the relationship between hesitant fuzzy set and showed that the envelope of hesitant fuzzy set is an intuitionistic fuzzy set. A lot of work has been done about hesitant fuzzy sets, however, little has been done about the hesitant fuzzy soft sets. In [8], Kharal and Ahmad introduced the notions of a mapping on hesitant fuzzy soft classes. We also establish some interesting properties of this notion and support them with examples.

2. Preliminary Results

In this section we recall some basic concepts and definitions regarding fuzzy soft sets, fuzzy soft image, fuzzy soft inverse image, hesitant fuzzy set and hesitant fuzzy soft set.

Definition 2.1. [10] Let \( U \) be an initial universe and \( F \) be a set of parameters. Let \( P(U) \) denote the power set of \( U \) and \( A \) be a non-empty subset of \( F \). Then \( F_A \) is called a fuzzy soft set over \( U \) where \( F: A \rightarrow P(U) \) is a mapping from \( A \) into \( P(U) \).

Definition 2.2. [11] \( F_E \) is called a soft set over \( U \) if and only if \( F \) is a mapping of \( E \) into the set of all subsets of the set \( U \).

In other words, the soft set is a parameterized family of subsets of the set \( U \). Every set \( F(\varepsilon), \varepsilon \in E, \) from this family may be considered as the set of \( \varepsilon \)-element of the soft set. Also \( \alpha \) and \( \beta \) be mappings. Then \( f(\varepsilon) \) is a mapping from \( E \) into \( F(\varepsilon) \).

Definition 2.3. [5] Let \( X \) be a universe and \( E \) a set of attributes. Then the collection of all fuzzy soft sets over \( U \) with attributes from \( E \) is called a fuzzy soft class and is denoted by \( (X, E) \).

Definition 2.4. [5] Let \( (X, E) \) and \( (Y, E') \) be classes of fuzzy soft sets over \( X \) and \( Y \) with attributes from \( E \) and \( E' \) respectively. Let \( p: U \rightarrow V \) and \( q: E \rightarrow E' \) be mappings. Then a mappings \( f = (p, q): (X, E) \rightarrow (Y, E') \) is defined as follows. For a fuzzy soft set \( F_A \) in \( (X, E) \), \( f(F_A) \) is a fuzzy soft set in \( (Y, E') \) obtained as follows: for \( \beta \in q(E) \subset E' \) and \( y \in Y, \)

\[
f(F_A)(\beta)(y) = \begin{cases} \bigcup_{\varepsilon \in p^{-1}(y)} \bigcup_{\alpha \in q^{-1}(\beta) \cap A} F_A(\alpha)(x) & \text{if } p^{-1}(y) \neq \emptyset, q^{-1}(\beta) \cap A \neq \emptyset \\ 0 & \text{otherwise} \end{cases}
\]

\( f(F_A) \) is called a fuzzy soft image of a fuzzy soft set \( F_A \).

Definition 2.5. [5] Let \( p: U \rightarrow V \) and \( q: E \rightarrow E' \) be mappings. Let \( f: (X, E) \rightarrow (Y, E') \) be a mapping and \( G_B \), a fuzzy soft set in \( (Y, E') \), where \( B \in E' \). Then \( f^{-1}(G_B) \), is a fuzzy soft set on \( (X, E) \), defined as follows. For \( \alpha \in q^{-1}(B) \subset E \) and \( x \in U, \)
\[
f^{-1}(G_B)(\alpha)(x) = \begin{cases} 
G_B(q(\alpha))(p(x)), & \text{if } q(\alpha) \in B \\
0, & \text{otherwise}
\end{cases}
\]

\(f^{-1}(G_B)\) is called a fuzzy soft inverse image of \(G_B\).

**Definition 2.6.** [1] Given a fixed set \(X\), then a hesitant fuzzy set (shortly HFS) in \(X\) is in terms of a function that when applied to \(X\) return a subset of \([0,1]\). We express the HFS by a mathematical symbol:

\(F = \{< h, \mu_F(x) > : h \in X \}\), where \(\mu_F(x)\) is a set of some values in \([0,1]\), denoting the possible membership degrees of the element \(h \in X\) to the set \(F\). \(\mu_F(x)\) is called a hesitant fuzzy element (HFE) and \(H\) is the set of all HFEs.

**Definition 2.7.** [1] Given an hesitant fuzzy set \(F\), define below it lower and upper bound as

lower bound \(F^- (x) = \min F(x)\).

upper bound \(F^+ (x) = \max F(x)\).

**Definition 2.8.** [1] Let \(\mu_1, \mu_2 \in H\) and three operations are defined as follows:

1. \(\mu_1^C = \cup_{\gamma_1 \in \mu_1} \{1 - \gamma_1\}\);
2. \(\mu_1 \cup \mu_2 = \cup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} \max\{\gamma_1, \gamma_2\}\);
3. \(\mu_1 \cap \mu_2 = \cap_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} \min\{\gamma_1, \gamma_2\}\).

**Definition 2.9.** [17] Let \(U\) be an initial universe and \(E\) be a set of parameters. Let \(\tilde{F}(U)\) be the set of all hesitant fuzzy subsets of \(U\). Then \(F_E\) is called a hesitant fuzzy soft set (HFSS) over \(U\), where \(\tilde{F} : E \rightarrow \tilde{F}(U)\).

A HFSS is a parameterized family of hesitant fuzzy subsets of \(U\), that is, \(\tilde{F}(U)\). For all \(\epsilon \in E\), \(F(\epsilon)\) is referred to as the set of \(\epsilon\)-approximate elements of the HFSS \(F_E\). It can be written as \(F(\epsilon) = \{< h, \mu_{F(\epsilon)}(x) > : h \in U\}\).

Since HFSE can represent the situation, in which different membership function are considered possible (see [14]), \(\mu_{F(\epsilon)}(x)\) is a set of several possible values, which is the hesitant fuzzy membership degree. In particular, if \(F(\epsilon)\) has only one element, \(F(\epsilon)\) can be called a hesitant fuzzy soft number. For convenience, a hesitant fuzzy soft number (HFSN) is denoted by \(\{< h, \mu_{F(\epsilon)}(x) > \}\).

**Example 2.10.** Suppose \(U = \{a, b\}\) be an initial universe and \(E = \{e_1, e_2, e_3, e_4\}\) be a set of parameters. Let \(A = \{e_1, e_2\}\). Then the hesitant fuzzy soft set \(F_A\) is given as \(F_A = \{F(e_1) = \{< a, \{0.6, 0.8\} >, < b, \{0.8, 0.4, 0.9\} >\}, F(e_2) = \{< a, \{0.9, 0.1, 0.5\} >, < b, \{0.2\} >\}\}\).

**Definition 2.11.** The union of two hesitant fuzzy soft sets \(F_A\) and \(G_B\) over \((U, E)\), is the hesitant fuzzy soft set \(H_C\), where \(C = A \cup B\) and \(\forall \epsilon \in C\),

\[
\mu_{H(\epsilon)} = \begin{cases}
\mu_{F(\epsilon)}, & \text{if } \epsilon \in A - B; \\
\mu_{G(\epsilon)}, & \text{if } \epsilon \in A - B; \\
\mu_{F(\epsilon)} \cup \mu_{G(\epsilon)}, & \text{if } \epsilon \in A \cap B.
\end{cases}
\]

We write \(F_A \cup G_B = H_C\).

3. **Mapping on hesitant fuzzy soft classes**

**Definition 3.1.** Let \(U\) be a hesitant fuzzy soft universe and \(E\) a set of attributes. Then the collection of all hesitant fuzzy soft sets over \(U\) with attributes from \(E\) is called a hesitant fuzzy soft class and is denoted by \((U, E)\).

**Definition 3.2.** Let \((U, E)\) and \((V, E')\) be classes of hesitant fuzzy soft sets over \(U\) and \(V\) with attributes from \(E\) and \(E'\) respectively. Let \(p : U \rightarrow V\) and \(q : E \rightarrow E'\) be mappings.
Then a hesitant fuzzy soft mappings \( f = (p, q) : (U, E) \rightarrow (V, E') \) is defined as follows:

For a hesitant fuzzy soft set \( F_A \) in \((U, E)\), \( f(F_A) \) is a hesitant fuzzy soft set in \((V, E')\) obtained as follows: for \( \beta \in q(E) \subseteq E' \) and \( y \in V \), \( f(F_A)(\beta)(y) = \bigcup_{\alpha \in q^{-1}(\beta) \cap A, s \in \wp^{-1}(y)(\alpha)} \mu_s \) \( f(F_A) \) is called a hesitant fuzzy soft image of a hesitant fuzzy soft set \( F_A \). Hence \((F_A, f(F_A)) \in f \) where \( F_A \subseteq (U, E) \), \( f(F_A) \subseteq (V, E') \).

**Definition 3.3.** If \( f : (U, E) \rightarrow (V, E') \) be a hesitant fuzzy soft mapping, then hesitant fuzzy soft class \((U, E)\) is called the domain of \( f \) and the hesitant fuzzy soft class \( \{G_B \subseteq (V, E') : G_B = f(H_A), \text{for some } H_A \subseteq (U, E)\} \) is called the range of \( f \). The hesitant fuzzy soft class \((V, E')\) is called co-domain of \( f \).

**Definition 3.4.** Let \( f = (p, q) : (U, E) \rightarrow (V, E') \) be a hesitant fuzzy soft mapping and \( G_B \), a hesitant fuzzy soft set in \((V, E')\), where \( p : U \rightarrow V, q : E \rightarrow E' \) and \( B \subseteq E' \). Then \( f^{-1}(G_B) \) is a hesitant fuzzy soft set in \((U, E)\) defined as follows: for \( (\alpha, s) \in q^{-1}(B) \subseteq E \) and \( x \in U \), \( f^{-1}(G_B)(\alpha)(x) = (q(\alpha)) \mu_{p(x)} \) \( f^{-1}(G_B) \) is called a hesitant fuzzy soft inverse image of \( G_B \).

**Example 3.5.** Let \( U = \{a, b, c\} \) and \( V = \{x, y, z\} \), \( E = \{e_1, e_2, e_3, e_4\} \), \( E' = \{e_1', e_2', e_3'\} \) and \((U, E), (V, E')\) classes of hesitant fuzzy soft sets. Let \( p(a) = y, p(b) = x, p(c) = y \) and \( q(e_1) = e_2', q(e_2) = e_1', q(e_3) = e_2, q(e_4) = e_3' \). Let \( A = \{e_1, e_2, e_4\} \). Let us consider a hesitant fuzzy soft set \( F_A \) in \((U, E)\) as:

\[
F_A = \{e_1 = \{< a, 0.6, 0.8 >, < b, 0.8, 0.4, 0.9 >, < c, 0.3 >\}, e_2 = \{< a, 0.9, 0.1, 0.2 >, < b, 0.5 >, < c, 0.2, 0.4, 0.6 >\}, e_4 = \{< a, 0.3 >, < b, 0.2, 0.6 >, < c, 0.4, 0.8 >\}\}.
\]

Then the hesitant fuzzy soft image of \( F_A \) under \( f = (p, q) : (U, E) \rightarrow (V, E') \) is obtained as:

\[
f(F_A)(e_1')(x) = \bigcup_{\alpha \in q^{-1}(e_1') \cap A, s \in \wp^{-1}(x)(\alpha)} \mu_s
\]

By similar calculations we get:

\[
f(F_A)(e_1)' = \{< x, 0.5 >, < y, 0.2, 0.4, 0.9 >, < z, 0.0 >\}
\]

\[
e_2' = \{< x, 0.8, 0.4, 0.9 >, < y, 0.6, 0.8 >, < z, 0.0 >\}
\]

\[
e_4' = \{< x, 0.2, 0.6 >, < y, 0.4, 0.8 >, < z, 0.0 >\}.
\]

Again consider a hesitant fuzzy soft set \( F_B' \) in \((V, E')\) as:

\[
F_B' = \{e_1' = \{< x, 0.2, 0.4 >, < y, 0.3, 0.1, 0.8 >, < z, 0.4 >\}, e_2 = \{< x, 0.5, 0.3, 0.7 >, < y, 0.7 >, < z, 0.7, 0.8, 0.2 >\}, e_4' = \{< x, 0.9 >, < y, 0.6, 0.8 >, < z, 0.8, 0.3 >\}\}.
\]

Therefore,

\[
f^{-1}(F_B')(e_1)(a) = (q(e_1)) \mu_{p(a)} = (e_2') \mu_y = 0.7
\]

\[
f^{-1}(F_B')(e_1)(b) = (q(e_1)) \mu_{p(b)} = (e_2') \mu_y = 0.5, 0.3, 0.7
\]

\[
f^{-1}(F_B')(e_1)(c) = (q(e_1)) \mu_{p(c)} = (e_2') \mu_y = 0.7
\]

By similar calculations, we get

\[
f^{-1}(F_B') = \{e_1 = \{< a, 0.7 >, < b, 0.5, 0.3, 0.7 >, < c, 0.7 >\}\}
\]
Definition 3.6. Let $f = (p, q)$ be a hesitant fuzzy soft mapping of a hesitant fuzzy soft class $(U, E)$ into a hesitant fuzzy soft class $(V, E')$. Then

(i) $f$ is said to be a one-one (or injection) hesitant fuzzy soft mapping if for both $p: U \rightarrow V$ and $q: E \rightarrow E'$ are one-one.

(ii) $f$ is said to be an onto (or surjection) hesitant fuzzy soft mapping if for both $p: U \rightarrow V$ and $q: E \rightarrow E'$ are onto.

If $f$ is both one-one and onto then $f$ is called a hesitant one-one onto (or bijective) correspondence of hesitant fuzzy soft mapping.

Theorem 3.7. Let $f = (p, q): (U, E) \rightarrow (V, E')$ and $g = (r, t): (U, E) \rightarrow (V, E')$ are two hesitant fuzzy soft mappings. Then $f$ and $g$ are equal if and only if $p = r$ and $q = t$.

Proof. Obvious. □

Theorem 3.8. Two hesitant fuzzy soft mappings $f$ and $g$ of a hesitant fuzzy soft class $(U, E)$ into a hesitant fuzzy soft class $(V, E')$ are equal if and only if $f(F_A) = g(F_A)$, for all $F_A \subseteq (U, E)$.

Proof. Let $f = (p, q): (U, E) \rightarrow (V, E')$ and $g = (r, t): (U, E) \rightarrow (V, E')$ are two hesitant fuzzy soft mappings. Since $f$ and $g$ are equal, this implies $p = r$ and $q = t$.

Let $\beta \in g(E) \subseteq E'$ and $y \in V$,

$$f(F_A)(\beta)(y) = \bigcup_{\alpha \in q^{-1}(\beta) \cap A, s \in p^{-1}(y)}(\alpha)\mu_s$$

$$= \bigcup_{\alpha \in t^{-1}(\beta) \cap A, s \in r^{-1}(y)}(\alpha)\mu_s$$

$$= g(F_A)(\beta)(y).$$

Hence $f(F_A) = g(F_A)$.

Conversely, Let $f(F_A) = g(F_A)$, for all $F_A \subseteq (U, E)$.

Let $(R, T) \subseteq f$, where $R \subseteq (U, E)$ and $T \subseteq (V, E')$.

Therefore $T = f(R) = g(R)$, this gives $(R, T) \subseteq g$.

Therefore $f \subseteq g$.

Similarly, it can be proved that $g \subseteq f$. Hence $f = g$. □

Definition 3.9. If $f = (p, q): (U, E) \rightarrow (V, E')$ and $g = (r, t): (V, E') \rightarrow (W, E'')$ are two hesitant fuzzy soft mappings, then their composite $g \circ f$ is a hesitant fuzzy soft mapping of $(U, E)$ into $(W, E'')$ such that for every $F_A \subseteq (U, E)$, $(g \circ f)(F_A) = g(f(F_A))$, we defined as for $\beta \in t(E') \subseteq E''$ and $y \in W$, $g(f(F_A))(\beta)(y) = \bigcup_{\alpha \in t^{-1}(\beta) \cap f(A), s \in r^{-1}(y)}(\alpha)\mu_s$.

Example 3.10. From example 3 and consider the hesitant fuzzy soft mapping $g = (r, t): (V, E') \rightarrow (W, E'')$, where $W = \{h_1, h_2, h_3\}$, $E'' = \{e''_1, e''_2, e''_3\}$ and $r(x) = h_2, r(y) = h_3, r(z) = h_2$: $t(e'_1) = e'_3, t(e'_2) = e'_4, t(e'_3) = e'_1$.

Therefore

$$g(f(F_A))(e''_1)(h_1) = \bigcup_{\alpha \in t^{-1}(e''_1) \cap f(A), s \in r^{-1}(h_1)}(\alpha)\mu_s$$

$$= \bigcup_{\alpha \in t^{-1}(e''_1) \cap f(A), s \in r^{-1}(h_1)}(\alpha)\mu_s = \{0.0\}$$

$$g(f(F_A))(e''_2)(h_2) = \bigcup_{\alpha \in t^{-1}(e''_2) \cap f(A), s \in r^{-1}(h_2)}(\alpha)\mu_s$$

$$= \bigcup_{\alpha \in t^{-1}(e''_2) \cap f(A), s \in r^{-1}(h_2)}(\alpha)\mu_s = \{0.2, 0.6\}$$

$$g(f(F_A))(e''_3)(h_3) = \bigcup_{\alpha \in t^{-1}(e''_3) \cap f(A), s \in r^{-1}(h_3)}(\alpha)\mu_s$$

$$= \bigcup_{\alpha \in t^{-1}(e''_3) \cap f(A), s \in r^{-1}(h_3)}(\alpha)\mu_s = \{0.4, 0.8\}.$$
By similar calculations we get
\[(g \circ f)(F_A) = g(f(F_A)) = \{e''_3 = \{< h_1, \{0.0\} >, < h_2, \{0.2, 0.6\} >, < h_3, \{0.4, 0.8\} >\}\]
\[e''_3 = \{< h_1, \{0.0\} >, < h_2, \{0.0\} >, < h_3, \{0.0\} >\}\]
\[e''_3 = \{< h_1, \{0.0\} >, < h_2, \{0.5, 0.8, 0.9\} >, < h_3, \{0.6, 0.8, 0.9\} >\}\].

**Theorem 3.11.** Let \(f = (p, q) : (U, E) \rightarrow (V, E')\) and \(g = (r, t) : (V, E') \rightarrow (W, E'')\) are two hesitant fuzzy soft mappings. Then

(i) If \(f\) and \(g\) are one-one then so is \(g \circ f\).

(ii) If \(f\) and \(g\) are onto then so is \(g \circ f\).

(iii) If \(f\) and \(g\) are both bijections then so is \(g \circ f\).

**Proof.** Let us consider the hesitant fuzzy soft mappings \(f = (p, q) : (U, E) \rightarrow (V, E')\) and \(g = (r, t) : (V, E') \rightarrow (W, E'')\). Let \(U = \{a, b, c\}\) and \(V = \{x, y, z\}\), \(W = \{h_1, h_2, h_3\}\), \(E = \{e_1, e_2, e_3\}\), \(E' = \{e'_1, e'_2, e'_3\}\), \(E'' = \{e''_1, e''_2, e''_3\}\) and \((U, E), (V, E'), (W, E'')\) classes of hesitant fuzzy soft sets. Let \(p(a) = z, p(b) = x, p(c) = y\); \(q(e_1) = e'_1, q(e_2) = e'_2, q(e_3) = e'_3\); and \(r(x) = h_2, r(y) = h_3, r(z) = h_1\); \(t(e'_1) = e''_1, t(e'_2) = e''_2, t(e'_3) = e''_3\). Also we consider a hesitant fuzzy soft set \(L_A\) in \((U, E)\) as
\[L_A = \{e_1 = \{< a, \{0.6, 0.8\} >, < b, \{0.8, 0.4, 0.9\} >, < c, \{0.3\} >\}\]
\[e_2 = \{< a, \{0.0\} >, < b, \{0.0\} >, < c, \{0.0\} >\}\]
\[e_3 = \{< a, \{0.9, 0.1, 0.2\} >, < b, \{0.5\} >, < c, \{0.2, 0.4, 0.6\} >\}\].

Then the hesitant fuzzy soft image of \(F_A\) under \(f = (p, q) : (U, E) \rightarrow (V, E')\) is obtained as
\[f(L_A)(e'_1)(x) = \bigcup_{a \in q^{-1}(e'_1) \cap A, s \in p^{-1}(x)} (\alpha)\mu_s\]
\[= \bigcup_{a \in e_1, e_3 \in \{b\}} (\alpha)\mu_s\]
\[f(L_A)(e'_1)(y) = \bigcup_{a \in q^{-1}(e'_1) \cap A, s \in p^{-1}(y)} (\alpha)\mu_s\]
\[= \bigcup_{a \in e_2, e_3} (\alpha)\mu_s\]
\[f(L_A)(e'_1)(z) = \bigcup_{a \in q^{-1}(e'_1) \cap A, s \in p^{-1}(z)} (\alpha)\mu_s\]
\[= \bigcup_{a \in e_3} (\alpha)\mu_s\]
\[= (\alpha)\mu_a = \{0.9, 0.1, 0.2\}\].

By similar calculations we get
\[f(L_A) = \{< x, \{0.5\} >, < y, \{0.2, 0.4, 0.6\} >, < z, \{0.9, 0.1, 0.2\} >\}\]
\[e'_2 = \{< x, \{0.8, 0.4, 0.9\} >, < y, \{0.3\} >, < z, \{0.6, 0.8\} >\}\]
\[e'_3 = \{< x, \{0.0\} >, < y, \{0.0\} >, < z, \{0.0\} >\}\].

Again,
\[g(f(L_A))(e''_1)(h_1) = \bigcup_{a \in q^{-1}(e''_1) \cap f(A), s \in p^{-1}(h_1)} (\alpha)\mu_s\]
\[= \bigcup_{a \in e_1, e_3 \in \{b\}} (\alpha)\mu_s\]
\[g(f(L_A))(e''_1)(h_2) = \bigcup_{a \in q^{-1}(e''_1) \cap f(A), s \in p^{-1}(h_2)} (\alpha)\mu_s\]
\[= \bigcup_{a \in e_2, e_3} (\alpha)\mu_s\]
\[g(f(L_A))(e''_1)(h_3) = \bigcup_{a \in q^{-1}(e''_1) \cap f(A), s \in p^{-1}(h_3)} (\alpha)\mu_s\]
\[= \bigcup_{a \in e_3} (\alpha)\mu_s\]
\[= (\alpha)\mu_a = \{0.3\}\].

By similar calculations we get
\[(g \circ f)(L_A) = g(f(L_A)) = \{e''_1 = \{< h_1, \{0.6, 0.8\} >, < h_2, \{0.8, 0.4, 0.9\} >, < h_3, \{0.3\} >\}\]
\[e''_1 = \{< h_1, \{0.0\} >, < h_2, \{0.0\} >, < h_3, \{0.0\} >\}\]
\[e''_1 = \{< h_1, \{0.9, 0.1, 0.2\} >, < h_2, \{0.5\} >, < h_3, \{0.2, 0.4, 0.6\} >\}\].

Therefore
(i) From above example we see that, if \(f\) and \(g\) are one-one then so is \(g \circ f\).
Definition 3.13. A hesitant fuzzy soft mapping \( f : (U, E) \rightarrow (V, E') \) is said to be many one hesitant fuzzy soft mapping if two (or more than two) hesitant fuzzy soft sets in \((U, E)\) have the same hesitant fuzzy soft image in \((V, E').\)

Example 3.14. From example 3.3 and consider the hesitant fuzzy soft set \( M_A \subseteq (U, E), \)
\( M_A = \{e_1 = \{a, \{0.6, 0.5\}, < b, \{0.8, 0.4, 0.9\}, < c, \{0.6, 0.8\}\}\} \)
\( e_2 = \{a, \{0.3, 0.2, 0.7\}, < b, \{0.5\}, < c, \{0.4, 0.1, 0.9\}\} \)
\( e_4 = \{a, \{0.4\}, < b, \{0.2, 0.6\}, < c, \{0.3, 0.8\}\} \).

Then the hesitant fuzzy soft image of \( M_A \) under \( f = (p, q) : (U, E) \rightarrow (V, E') \) is obtained as
\[ f(M_A)(e'(x)) = \bigcup_{\alpha \in \{e_2\}, e \in \{b\}} (\alpha) \mu_s \]
\[ = \{(e_2)\}\mu_\alpha = \{0.5\} \]
\[ f(M_A)(e'(y)) = \bigcup_{\alpha \in \{e_2\}, e \in \{a, c\}} (\alpha) \mu_s \]
\[ = \{(e_2)\}\mu_\alpha = \{0.2, 0.4, 0.9\} \]
\[ f(M_A)(e'(z)) = \bigcup_{\alpha \in \{e_2\}, e \in \{a, c\}} (\alpha) \mu_s \]
\[ = \{(e_2)\}\mu_\alpha = \{0.8, 0.4, 0.9\} \].

By similar calculations we get
\[ f(M_A) = \{e'_1 = \{x; \{0.5\}, < y, \{0.2, 0.4, 0.9\}, < z, \{0.0\}\}\} \]
\[ e'_2 = \{x, \{0.8, 0.4, 0.9\}, < y, \{0.6, 0.8\}, < z, \{0.0\}\} \]
\[ e'_3 = \{x, \{0.2, 0.6\}, < y, \{0.4, 0.8\}, < z, \{0.0\}\} \].

Therefore \( f(M_A) = f(F_A) \). Hence \( f \) is many one hesitant fuzzy soft mapping.

Definition 3.15. Let \( i = (p, q) : (U, E) \rightarrow (U, E) \) be a hesitant fuzzy soft mapping, where \( p : U \rightarrow U \) and \( q : E \rightarrow E \). Then \( f \) is said to be a hesitant fuzzy soft identity mapping if both \( p \) and \( q \) are identity mappings.

Remark 3.16. \( i = (p, q) : (U, E) \rightarrow (U, E) \) be a hesitant fuzzy soft identity mapping, then \( i(F_A) = F_A \), where \( F_A \subseteq (U, E) \).

Theorem 3.17. Let \( f = (p, q) : (U, E) \rightarrow (V, E') \) be a hesitant fuzzy soft mapping and let \( i = (i, m) : (U, E) \rightarrow (U, E) \) and \( j = (g, h) : (V, E') \rightarrow (V, E') \) are hesitant fuzzy soft identity mappings then \( f \circ i = f \) and \( j \circ f = f \).

Proof. Consider the following example. We consider \( F_A \) from example 3.3 and consider the hesitant fuzzy soft mappings \( i = (p, q) : (U, E) \rightarrow (U, E) \), where \( p : U \rightarrow U \) and \( q : E \rightarrow E \), such that \( p(a) = a, p(b) = b, p(c) = c ; q(e_1) = e_1, q(e_2) = e_2, q(e_3) = e_3, q(e_4) = e_4 \). Therefore,
By similar calculations we get
\[ i(F_A)(e_1)(a) = \bigcup_{\alpha \in \{e_1, e_2, e_3\}} \{\alpha\} \mu_s \]
\[ = \bigcup_{\alpha \in \{e_1, e_2, e_3\}} \{\alpha\} \mu_s \]
\[ = (e_1)\mu_a = \{0.6, 0.8\} \]
\[ i(F_A)(e_1)(b) = \bigcup_{\alpha \in \{e_1, e_2, e_3\}} \{\alpha\} \mu_s \]
\[ = \bigcup_{\alpha \in \{e_1, e_2, e_3\}} \{\alpha\} \mu_s \]
\[ = (e_1)\mu_b = \{0.8, 0.4, 0.9\} \]
\[ i(F_A)(e_1)(c) = \bigcup_{\alpha \in \{e_1, e_2, e_3\}} \{\alpha\} \mu_s \]
\[ = \bigcup_{\alpha \in \{e_1, e_2, e_3\}} \{\alpha\} \mu_s \]
\[ = (e_1)\mu_c = \{0.3\} \]

By similar calculations we get
\[ i(F_A) = \{e_1 = \{< a, \{0.6, 0.8\} >, < b, \{0.8, 0.4, 0.9\} >, < c, \{0.3\} >\} \]
\[ e_2 = \{< a, \{0.9, 0.1, 0.2\} >, < b, \{0.5\} >, < c, \{0.2, 0.4, 0.6\} >\} \]
\[ e_3 = \{< a, \{0.0\} >, < b, \{0.0\} >, < c, \{0.0\} >\} \]
\[ e_4 = \{< a, \{0.3\} >, < b, \{0.2, 0.6\} >, < c, \{0.4, 0.8\} >\} \]

Hence \( i(F_A) = F_A \Rightarrow f(i(F_A)) = f(F_A) \Rightarrow f \circ i)(F_A) = f(F_A) \Rightarrow f \circ i = f \).
Similarly we get \( f(F_A) \in \mathcal{V}(V, E') \) and \( j(f(F_A)) = f(F_A) \Rightarrow (j \circ f)(F_A) = f(F_A) \).
Hence \( j \circ f = f \). \( \square \)

**Definition 3.18.** A one-one onto hesitant fuzzy soft mapping \( f = (p, q) : (U, E) \rightarrow (V, E') \) is called hesitant fuzzy soft invertable mapping. Its hesitant fuzzy soft inverse mapping is denoted by \( f^{-1} = (p^{-1}, q^{-1}) : (V, E') \rightarrow (U, E) \).

**Remark 3.19.** In a hesitent fuzzy soft invertable mapping \( f : (U, E) \rightarrow (V, E') \), for \( F_A \in \mathcal{V}(U, E), G_B \in \mathcal{V}(V, E') \), \( f^{-1}(G_B) = F_A \), whenever \( f(F_A) = G_B \).

**Example 3.20.** From proof of the Theorem 3.11. We consider \( f(L_A) = G_B \). Therefore,
\[ f^{-1}(G_B)(e_1)(a) = (q(e_1))\mu_{p(a)} = (e_2')\mu_{s} = \{0.6, 0.8\} \]
\[ f^{-1}(G_B)(e_1)(b) = (q(e_1))\mu_{p(b)} = (e_2')\mu_{s} = \{0.8, 0.4, 0.9\} \]
\[ f^{-1}(G_B)(e_1)(c) = (q(e_1))\mu_{p(c)} = (e_2')\mu_{s} = \{0.3\} \]

By similar calculations we get
\[ f^{-1}(G_B) = \{e_1 = \{< a, \{0.6, 0.8\} >, < b, \{0.8, 0.4, 0.9\} >, < c, \{0.3\} >\} \]
\[ e_2 = \{< a, \{0.0\} >, < b, \{0.0\} >, < c, \{0.0\} >\} \]
\[ e_3 = \{< a, \{0.9, 0.1, 0.2\} >, < b, \{0.5\} >, < c, \{0.2, 0.4, 0.6\} >\} \]

Hence \( f^{-1}(G_B) = L_A \).

**Theorem 3.21.** Let \( f : (U, E) \rightarrow (V, E') \) be a hesitant fuzzy soft invertable mapping. Therefore its hesitant fuzzy soft inverse mapping is unique.

**Proof.** Let \( f^{-1} \) and \( g^{-1} \) are two hesitant fuzzy soft inverse mappings of \( f \). Therefore,
\[ f^{-1}(G_B) = F_A \], whenever \( f(F_A) = G_B \), \( F_A \in \mathcal{V}(U, E), G_B \in \mathcal{V}(V, E') \), and
\[ g^{-1}(G_B) = H_A \], whenever \( g(H_A) = G_B \). Hence \( f^{-1}(G_B) = g^{-1}(G_B) \) i.e \( f^{-1} = g^{-1} \). \( \square \)

**Theorem 3.22.** Let \( f : (U, E) \rightarrow (V, E') \) and \( g : (V, E') \rightarrow (W, E''') \) are two one-one onto hesitant fuzzy soft mappings. If \( f^{-1} : (V, E') \rightarrow (U, E) \) and \( g^{-1} : (W, E''') \rightarrow (V, E') \) are hesitant fuzzy soft inverse mappings of \( f \) and \( g \), respectively, then the inverse of the mapping \( g \circ f : (U, E) \rightarrow (W, E''') \) is the hesitant fuzzy soft mapping \( f^{-1} \circ g^{-1} : (W, E''') \rightarrow (U, E) \).

**Proof.** Obvious. \( \square \)

**Theorem 3.23.** A hesitant fuzzy soft mapping \( f : (U, E) \rightarrow (V, E') \) is invertable if and only if there exists a hesitant fuzzy soft inverse mapping \( f^{-1} : (V, E') \rightarrow (U, E) \) such that \( f^{-1} \circ f = i_{(U, E)} \) and \( f \circ f^{-1} = i_{(V, E')}, \) where \( i_{(U, E)} \) and \( i_{(V, E')} \) is hesitant fuzzy soft identity mapping on \((U, E)\) and \((V, E')\), respectively.
Proof. Let \( f : (U, E) \to (V, E') \) be a hesitant fuzzy soft invertible mapping. Therefore by definition we have 
\[ f^{-1}(G_B) = F_A, \text{ whenever } f(F_A) = G_B, F_A \in (U, E), G_B \in (V, E'). \]
Since \((f^{-1} \circ f)(F_A) = f^{-1}(f(F_A)) = f^{-1}(G_B) = F_A. \) Therefore \( f^{-1} \circ f = i_{(U, E)}. \)
Similarly, we prove that \( f \circ f^{-1} = i_{(V, E')}. \)

\[ \square \]

Theorem 3.24. If \( f : (U, E) \to (V, E') \) and \( g : (V, E') \to (W, E'') \) are two one-one onto hesitant fuzzy soft mapping then \((g \circ f)^{-1} = f^{-1} \circ g^{-1}. \)

Proof. Since \( f \) and \( g \) are one-one onto hesitant fuzzy soft mapping, then there exists \( f^{-1} : (V, E') \to (U, E) \) and \( g^{-1} : (W, E'') \to (V, E') \) such that 
\[ f^{-1}(G_B) = F_A, \text{ whenever } f(F_A) = G_B, F_A \in (U, E), G_B \in (V, E'), \]
and 
\[ g^{-1}(H_C) = G_B, \text{ whenever } g(G_B) = H_C, H_C \in (W, E''), G_B \in (V, E'). \]
Therefore, \((g \circ f)(F_A) = g(f(F_A)) = g(G_B) = H_C. \)

As \( g \circ f \) is one-one onto, therefore \((g \circ f)^{-1} \) exists such that 
\[ (g \circ f)(F_A) = H_C \implies (g \circ f)^{-1}(H_C) = F_A. \]
Also \((f^{-1} \circ g^{-1})(H_C) = f^{-1}(g^{-1}(H_C)) = f^{-1}(G_B) = F_A. \)
Hence \((g \circ f)^{-1}(H_C) = (f^{-1} \circ g^{-1})(H_C) \implies (g \circ f)^{-1} = f^{-1} \circ g^{-1}. \)

\[ \square \]

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