A RECURSION FORMULA FOR THE IRREDUCIBLE CHARACTERS OF THE SYMMETRIC GROUP

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Abstract. The branching theorem expresses irreducible character values for the symmetric group $S_n$ in terms of those for $S_{n-1}$, but it gives the values only at elements of $S_n$ having a fixed point. We extend the theorem by providing a recursion formula that handles the remaining cases. It expresses these character values in terms of values for $S_{n-1}$ together with values for $S_n$ that are already known in the recursive process. This provides an alternative to the Murnaghan-Nakayama formula.

0. Introduction

Let $n$ be a nonnegative integer and denote by $S_n$ the symmetric group of degree $n$. For a partition $\alpha$ of $n$ (written $\alpha \vdash n$), denote by $\zeta^\alpha$ the corresponding irreducible character of $S_n$, and denote by $\zeta^\alpha_\beta$ the value of $\zeta^\alpha$ at the conjugacy class of $S_n$ corresponding to $\beta \vdash n$.

Assume that $n > 0$. Let $\alpha, \beta \vdash n$ and let $\beta_m$ be the last (nonzero) part of $\beta$. If $\beta_m = 1$, then the conjugacy class of $S_n$ corresponding to $\beta$ contains a permutation that fixes $n$, so the branching theorem [JK81, 2.4.3, p. 59] expresses $\zeta^\alpha_\beta$ in terms of irreducible character values for $S_{n-1}$. The main result of this paper (Theorem 5.1) is the following recursion formula for $\zeta^\alpha_\beta$ in the remaining case $\beta_m \neq 1$:

$$
\zeta^\alpha_\beta = (\beta_m - 1)^{-1} \left[ \sum_{i=1}^{l(\alpha)} (\alpha_i - i)\zeta^{\alpha - \varepsilon_i}_{\beta - \varepsilon_m} - \sum_{j=1}^{m-1} \mu_j \beta_j \zeta^\alpha_{\beta + \varepsilon_j - \varepsilon_m} \right].
$$

Here, $l(\alpha)$ is the length of $\alpha$, $\varepsilon_i$ is the sequence with 1 in the $i$th position and zeros elsewhere, $\mu_j$ is the multiplicity of $\beta_j$ in the partition $\beta - \varepsilon_m$, and $\beta_0 := \infty$.

For each $i$ in the first sum, $\alpha - \varepsilon_i$ is a partition of $n - 1$, so $\zeta^{\alpha - \varepsilon_i}$ is an irreducible character of $S_{n-1}$. For each $j$ in the second sum, $\beta + \varepsilon_j - \varepsilon_m$ is less than $\beta$ with respect to the reverse lexicographical ordering of partitions. It follows that the character values can be found recursively using the formula.

2010 Mathematics Subject Classification. 20C30, 20C15, 20C40.

Key words and phrases. symmetric group, character, recursion.
The character tables for $S_n$, $2 \leq n \leq 10$, generated (with the aid of a computer) using the recursion formula (1), together with the branching theorem, are in agreement with the character tables appearing in [JK81, Appendix I].

The Murnaghan-Nakayama formula [JK81, 2.4.7, p. 60] also expresses the irreducible character values $\zeta^\beta_\alpha$ recursively. It does so by expressing such a value for $S_n$ in terms of character values for $S_{n-r}$, where $r$ is a (nonzero) part of the partition $\beta$. The computation involves the removal of the various “rims” of length $r$ from the Young diagram of the partition $\alpha$. The recursion formula (1) may allow for proofs and computations in situations where the Murnaghan-Nakayama formula cannot be easily applied.

1. General notation and background

Let $n \in \mathbb{N} := \{0, 1, 2, \ldots \}$. Denote by $S_n$ the symmetric group on the set $\mathbb{N} := \{1, 2, \ldots, n\}$ and by $\varepsilon$ the identity element of $S_n$.

For $l \in \mathbb{N}$, put

$$\Gamma_l = \{ \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_l, 0, 0, \ldots) \mid \gamma_i \in \mathbb{Z} \}$$

and $\Gamma_l^+ = \{ \gamma \in \Gamma_l \mid \gamma_i \geq 0 \ \forall i \}$. Further, put $\Gamma = \bigcup \Gamma_l$ and $\Gamma^+ = \bigcup \Gamma_l^+$.

For $\gamma \in \Gamma$, denote by $l(\gamma)$ (length of $\gamma$) the least $l \in \mathbb{N}$ for which $\gamma \in \Gamma_l$. If $\gamma \in \Gamma$ is nonzero, then $l(\gamma)$ is the least $l \in \mathbb{Z}^+$ for which $\gamma_l \neq 0$.

An element $\gamma$ of $\Gamma^+$ is a partition of $n$, written $\gamma \vdash n$, if $|\gamma| := \sum \gamma_i = n$ and $\gamma_i \geq \gamma_{i+1}$ for each $i$.

The cycle structure of a permutation $\sigma \in S_n$ is the partition of $n$ obtained by writing the lengths of the cycles in a disjoint cycle decomposition of $\sigma$ in nonincreasing order (followed by zeros). Two elements of $S_n$ are conjugate if and only if they have the same cycle structure [JK81, p. 9].

Let $\beta \in \Gamma^+$ with $|\beta| = n$. For each $j \in \mathbb{Z}^+$, put $\sigma^\beta_j = (s_{j1}, s_{j2}, \ldots, s_{j\beta_j}) \in S_n$, where $s_{ji} = i + \sum_{k=1}^{j-1} \beta_k$. (If $\beta_j = 0$, then $\sigma^\beta_j = (\varepsilon)$. ) Put $\sigma^\beta = \prod_j \sigma^\beta_j$.

For instance, if $\beta = (4, 2, 2, 1, 0, 0, \ldots)$, then

$$\sigma^\beta = (1, 2, 3, 4)(5, 6)(7, 8)(9) \in S_9.$$

It follows from the preceding paragraph that $\{\sigma^\beta \mid \beta \vdash n\}$ is a complete set of representatives of the conjugacy classes of $S_n$.

Let $\alpha \in \Gamma$ with $|\alpha| = n$. Denote by $T^\alpha$ the set of all sequences $t = (t_1, t_2, \ldots)$ with the $t_i$ pairwise disjoint subsets of $\mathbb{N}$ such that $|t_i| = \alpha_i$ for each $i$ (and note that this final condition implies that $T^\alpha = \emptyset$ if $\alpha \notin \Gamma^+$). An element $t$ of $T^\alpha$ is called an $\alpha$-tabloid; it can be regarded as an $\alpha$-tableau with unordered rows (cf. [JK81, p. 41]).

Assume that $\alpha \in \Gamma^+$. Put $t^\alpha = (\sigma^\alpha_1, \sigma^\alpha_2, \ldots) \in T^\alpha$, where we use the notation $\sigma := \{s_1, s_2, \ldots, s_m\}$ for a cycle $\sigma = (s_1, s_2, \ldots, s_m)$ in $S_n$. For
instance, if \( \alpha = (3, 4, 2, 0, 0, \ldots) \), then
\[
\alpha = (\{1, 2, 3\}, \{4, 5, 6, 7\}, \{8, 9\}, \emptyset, \ldots) \in T^\alpha.
\]
An action of the symmetric group \( S_n \) on the set \( T^\alpha \) is given by \( \sigma t = (\sigma(t_1), \sigma(t_2), \ldots) \) \((\sigma \in S_n, \ t \in T^\alpha)\). The Young subgroup \( S_\alpha \) of \( S_n \) corresponding to \( \alpha \) is given by
\[
S_\alpha = \{ \sigma \in S_n \mid \sigma t_\alpha = t^\alpha \}.
\]

2. Reciprocity for permutation characters

For \( \alpha \in \Gamma \), define \( \xi^\alpha : S_{|\alpha|} \rightarrow \mathbb{Z} \) by
\[
\xi^\alpha = \begin{cases} 
1_{S^\alpha} \uparrow S_{|\alpha|}, & \text{if } \alpha \in \Gamma^+, \\
0, & \text{otherwise}.
\end{cases}
\]
Here \( 1_{S^\alpha} \uparrow S_{|\alpha|} \) denotes the character of \( S_{|\alpha|} \) induced from the trivial character of the Young subgroup \( S_\alpha \) of \( S_{|\alpha|} \); it is a permutation character. (See [Ser77] for general character theory.)

For \( \alpha \in \Gamma \) and \( \beta \in \Gamma^+ \), put
\[
\xi^\alpha_\beta = \begin{cases} 
\xi^\alpha(\sigma_\beta), & \text{if } |\alpha| = |\beta|, \\
0, & \text{otherwise}.
\end{cases}
\]
The goal in this section is to establish Theorem 2.3 below, which is a reciprocal relationship involving these values \( \xi^\alpha_\beta \).

For \( \alpha \in \Gamma \) and \( \sigma \in S_{|\alpha|} \), put
\[
T_\alpha^\beta = \{ t \in T^\alpha \mid \sigma t = t^\beta \}.
\]
The following result was observed in [JK81, p. 41]. We provide a proof for the convenience of the reader.

2.1 Lemma. Let \( \alpha \in \Gamma \) and put \( n = |\alpha| \). For every \( \sigma \in S_n \), we have \( \xi^\alpha(\sigma) = |T_\sigma^\alpha| \).

Proof. Let \( \sigma \in S_n \). If \( \alpha \notin \Gamma^+ \), then \( T^\alpha \) (and hence \( T_\sigma^\alpha \)) is empty and the equality holds. So, without loss of generality, we assume that \( \alpha \in \Gamma^+ \).
The character \( \xi^\alpha \) is afforded by the permutation representation \( \rho \) of \( S_n \) corresponding to the \( S_n \)-set \( C = \{ \tau S_\alpha \mid \tau \in S_n \} \) with action given by left multiplication, so the character value \( \xi^\alpha(\sigma) \), which is the trace of \( \rho(\sigma) \), equals the cardinality of the set \( \{ c \in C \mid \sigma c = c \} \). Now the \( S_n \)-set \( C \) is isomorphic to the \( S_n \)-set \( S_n t^\alpha = T^\alpha \) via \( \tau S_\alpha \mapsto \tau t^\alpha \), so \( \xi^\alpha(\sigma) = |\{ t \in T^\alpha \mid \sigma t = t^\alpha \} | = |T_\sigma^\alpha| \). \( \square \)

Let \( \alpha \in \Gamma^+ \) and \( t \in T^\alpha \). Let \( \sigma \in S_{|\alpha|} \) and write \( \sigma = \prod_{j=1}^m \sigma_j \) with the \( \sigma_j \) disjoint cycles.
implies $\dot{\beta}$.

For each $j > l$ the left is zero, and for $i > l$ (see Section 1), then\footnote{For each $j$, we have $\dot{\beta}_j = 0$.}

So, without loss of generality, we assume that for $\alpha$ and $\beta$.

Proof. Assume that (i) holds. Let $1 \leq j \leq m$. Due to the disjointness of the cycles, we have $\sigma_j t = t$. Let $k \in \tilde{\sigma}_j$. We have $k \in t_i$ for some $i$, and for every integer $l$ we have $\sigma_j^l(k) \in \sigma_j^l(t_i) = t_i$, so $\tilde{\sigma}_j \subseteq t_i$. Hence (ii) holds.

Now assume that (ii) holds. Let $1 \leq j \leq m$. We have $\tilde{\sigma}_j \subseteq t_i$ for some $i$. Then $\sigma_j(t_i) = t_i$ and for $k \neq i$, $\tilde{\sigma}_j \cap t_k = \emptyset$ so $\sigma_j(t_k) = t_k$ as well. Therefore, $\sigma_j t = t$. It follows that (i) holds. \hfill \square

The set $\Gamma$ is a group under componentwise addition. For $i \in \mathbb{Z}^+$, put $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots) \in \Gamma$.

2.2 Lemma. The following are equivalent:

(i) $\sigma t = t$;
(ii) for each $1 \leq j \leq m$, we have $\tilde{\sigma}_j \subseteq t_i$ for some $i$.

Proof. Before fixing elements, we define in general

$$T_{\sigma}^\alpha(i, X) := \{t \in T_{\sigma}^\alpha \mid t_i = X\}$$

for $\alpha \in \Gamma$, $\sigma \in S_{|\alpha|}$, $i \in \mathbb{Z}^+$, and a set $X$.

Let $\alpha \in \Gamma_l$ and $\beta \in \Gamma_m^+$ and put $n = |\alpha|$. If either $\alpha \notin \Gamma^+$ or $|\beta| \neq n - 1$, then $\xi_{\beta}^{\alpha - \varepsilon_i}, \xi_{\beta + \varepsilon_i}^\alpha = 0$ for every $i$ and $j$, and the equality holds. Also, for $i > l(\alpha)$, we have $\alpha - \varepsilon_i \notin \Gamma^+$ so $\xi_{\alpha - \varepsilon_i}^\alpha = 0$ and the corresponding term on the left is zero, and for $j > l(\beta)$, we have $\beta_j = 0$ and the corresponding term on the right is zero. So, without loss of generality, we assume that $\alpha \in \Gamma^+$, $|\beta| = n - 1$, and $l = l(\alpha)$, and $m = m(\beta)$.

For each $j \in \mathbb{Z}^+$, put $\sigma_j = \sigma_j\beta$, and put $\sigma = \sigma_\beta$ (see Section 1). We have $\sigma = \prod_j \sigma_j$, a product of disjoint cycles with each $\sigma_j$ of length $\beta_j$.

Let $X$ be a set and put

$$J(X) = \{ j \in \mathbb{Z}^+ \mid \tilde{\sigma}_j \subseteq X \}. $$

Fix $i \in \mathbb{Z}^+$.

Step 1: If $T_{\sigma}^{\alpha - \varepsilon_i}(i, X)$ is nonempty, then $\alpha_i - 1 = \sum_{j \in J(X)} \beta_j$.

Assume that $T_{\sigma}^{\alpha - \varepsilon_i}(i, X)$ is nonempty. Then there exists $t \in T_{\sigma}^{\alpha - \varepsilon_i}$ such that $\sigma t = t$ and $t_i = X$. We claim that $X = \bigcup_{j \in J(X)} \tilde{\sigma}_j$. Let $k \in X = t_i$. We have $1 \leq k \leq n - 1$, so $k \in \tilde{\sigma}_j \cap t_i$ for some $1 \leq j \leq m$. Then Lemma 2.2 implies $\tilde{\sigma}_j \subseteq t_i = X$, so that $j \in J(X)$. This gives one inclusion of the
that implies (and recalling that $\alpha$ and $\beta$). Assume that $\alpha_i - 1 = |X| = \sum_{j \in J(X)} |\sigma_j| = \sum_{j \in J(X)} \beta_j$

and Step 1 is complete.

Fix $j \in \mathbb{Z}^+$ and denote by $\sigma[j]$ the permutation in $S_n$ obtained from $\sigma$
by appending $n$ to the $j$th cycle $\sigma_j$. More precisely, if we write $\sigma_j = (s_{j_1}, s_{j_2}, \ldots, s_{j_{\beta_j}})$, then $\sigma[j] = \sigma' j \prod_{k \neq j} \sigma_k$, where $\sigma_j' = (s_{j_1}, s_{j_2}, \ldots, s_{j_{\beta_j}}, n)$.

Step 2: For each $X \subseteq \mathbf{n}':= \{1, 2, \ldots, n-1\}$, we have

$$|T^\alpha_{\sigma[j]}(i, X \cup \{n\})| = \begin{cases} |T^\alpha_{\sigma[j]}(i, X)|, & \text{if } j \in J(X), \\ 0, & \text{otherwise.} \end{cases}$$

Let $X \subseteq \mathbf{n}'$. If $\alpha - \varepsilon_i \notin \Gamma^+$, then $\alpha_i = 0$, so both $T^\alpha_{\sigma[j]}(i, X \cup \{n\})$ and $T^\alpha_{\sigma[j]}(i, X)$ are empty and the statement holds. So, without loss of generality, we assume that $\alpha - \varepsilon_i \in \Gamma^+$.

Assume that $j \in J(X)$, so that $\sigma_j \subseteq X$. For an $(\alpha - \varepsilon_i)$-tabloid $t$, denote by $t^+$ the $\alpha$-tabloid with $(t^*)_i = t_i \cup \{n\}$ and $(t^*)_k = t_k, k \neq i$. It follows from Lemma 2.2 that $t \mapsto t^+$ defines a bijection $T^\alpha_{\sigma[j]}(i, X) \rightarrow T^\alpha_{\sigma[j]}(i, X \cup \{n\})$ with inverse given by $t \mapsto t^-$ where $(t^-)_i = t_i \setminus \{n\}$ and $(t^-)_k = t_k, k \neq i$. Therefore, the first case follows.

Now assume that $j \notin J(X)$, so that $\sigma_j \not\subseteq X$. We have $s_{jk} \notin X$ for some $1 \leq k \leq \beta_j$. Let $t$ be an $\alpha$-tabloid with $t_i = X \cup \{n\}$. Since $s_{jk} \neq n$, we have $s_{jk} \notin t_i$. Therefore, $n \in \sigma'_j \cap t_i$, but $s_{jk} \in \sigma'_j \setminus t_i$, so Lemma 2.2 implies that $\sigma(j)t \neq t$. We conclude that $T^\alpha_{\sigma[j]}(i, X \cup \{n\})$ is empty and the second case follows. This completes Step 2.

We are now ready to establish the equality in the theorem. Using Lemma 2.1 (and recalling that $\sigma = \sigma_\beta$) we have

$$\sum_{i=1}^l (\alpha_i - 1)\xi^\alpha_{\sigma[j]} = \sum_{i=1}^l (\alpha_i - 1)|T^\alpha_{\sigma[j]}(i, X)|$$

$$= \sum_{i=1}^l \sum_{X \subseteq \mathbf{n}'} (\alpha_i - 1)|T^\alpha_{\sigma[j]}(i, X)|.$$

For each $1 \leq i \leq l$ and $X \subseteq \mathbf{n}'$ we have, using Step 1 and then Step 2,

$$(\alpha_i - 1)|T^\alpha_{\sigma[j]}(i, X)| = \sum_{j \in J(X)} \beta_j|T^\alpha_{\sigma[j]}(i, X)| = \sum_{j=1}^m \beta_j|T^\alpha_{\sigma[j]}(i, X \cup \{n\})|.$$
Therefore,
\[
\sum_{i=1}^{l}(\alpha_i - 1)\xi_\beta^{\alpha - \varepsilon_i} = \sum_{i=1}^{l} \sum_{X \subseteq n'} \sum_{j=1}^{m} \beta_j |T_{\sigma[j]}^\alpha(i, X \cup \{n\})| \\
= \sum_{j=1}^{m} \beta_j \sum_{i=1}^{l} \sum_{X \subseteq n} |T_{\sigma[j]}^\alpha(i, X)| \\
= \sum_{j=1}^{m} \beta_j |T_{\sigma[j]}^\alpha|.
\]

Using Lemma 2.1 and the fact that \(\xi_\alpha\) is a class function we get
\[
|T_{\sigma[j]}^\alpha| = \xi_\alpha^\alpha(\sigma[j]) = \xi_\alpha^{\alpha}(\sigma_{\beta + \varepsilon_j}) = \xi_{\beta + \varepsilon_j}^\alpha
\]
for every \(1 \leq j \leq m\). This completes the proof. \(\square\)

3. An adjoint pair

In this section, we reformulate the reciprocity relationship of the preceding section, expressing it as an adjoint relationship between a pair of \(\mathbb{Z}\)-linear maps.

Denote by \(A\) the free \(\mathbb{Z}\)-module on the set \(\{x^\alpha \mid \alpha \in \Gamma\}\) and denote by \(B\) the free \(\mathbb{Z}\)-module on the set \(\{x^\beta \mid \beta \in \Gamma^+\}\).

Denote by \((\cdot, \cdot) : A \times B \to \mathbb{Z}\) the \(\mathbb{Z}\)-bilinear map uniquely determined by
\[(x^\alpha, x^\beta) = \xi_\beta^\alpha \quad (\alpha \in \Gamma, \beta \in \Gamma^+).\]

Let \(l, m \in \mathbb{N}\). Denote by \(\delta_i^- : A \to A\) and \(\delta_m^+ : B \to B\) the \(\mathbb{Z}\)-linear maps uniquely determined by
\[\delta_i^-(x^\alpha) = \sum_{i=1}^{l}(\alpha_i - 1)x^{\alpha - \varepsilon_i} \quad (\alpha \in \Gamma)\]
and
\[\delta_m^+(x^\beta) = \sum_{j=1}^{m} \beta_j x^{\beta + \varepsilon_j} \quad (\beta \in \Gamma^+),\]
respectively.

Put \(A_l = \langle x^\alpha \mid \alpha \in \Gamma_l \rangle \leq A\) and \(B_m = \langle x^\beta \mid \beta \in \Gamma_m^+ \rangle \leq B\).

3.1 Theorem. For every \(a \in A_l\) and \(b \in B_m\), we have
\[(\delta_i^-(a), b) = (a, \delta_m^+(b)).\]
Proof. Let $a \in A_l$ and $b \in B_m$. We assume, without loss of generality (due to linearity), that $a = x^\alpha$ and $b = x^\beta$ with $\alpha \in \Gamma_l$ and $\beta \in \Gamma^+_m$. Using Theorem 2.3, we have

$$\langle \delta_l^{-}(a), b \rangle = \left( \sum_{i=1}^{l} (\alpha_i - 1)x^{\alpha - \varepsilon_i}, x^\beta \right) = \left( \sum_{i=1}^{l} (\alpha_i - 1)\xi_{\beta}^{\alpha - \varepsilon_i} \right)$$

$$= \sum_{j=1}^{m} \beta_j \xi_{\beta}^{\alpha} = (x^\alpha, \sum_{j=1}^{m} \beta_j x^\varepsilon_j)$$

$$= (a, \delta_m^+(b)).$$

\[\square\]

4. Reciprocity for irreducible characters

The goal of this section is Theorem 4.5, which provides a reciprocal relationship for the irreducible character values for $S_n$ ($n \in \mathbb{N}$) analogous to the relationship given in Theorem 2.3.

Let $n \in \mathbb{N}$. For $\alpha \vdash n$, denote by $\zeta^\alpha$ the irreducible character of $S_n$ corresponding to $\alpha$ [JK81, 2.2.5, p. 39], and for $\beta \in \Gamma^+$ with $|\beta| = n$, put $\zeta_{\beta}^{\alpha} = \zeta^\alpha(\sigma_\beta)$. The matrix $[\zeta_{\beta}^{\alpha}]_{\alpha,\beta}$ with $\alpha, \beta \vdash n$ (relative to a choice of ordering of the partitions) is the character table of $S_n$.

We write sgn for the sign character of $S_n$, so $\text{sgn}(\sigma)$ is 1 or $-1$ according as the permutation $\sigma$ is even or odd.

For $l \in \mathbb{N}$, put $1_l = (1, 2, 3, \ldots, l, 0, 0, \ldots) \in \Gamma$.

For $\alpha \in \Gamma$, we have $\alpha \in \Gamma_l$ for some $l \in \mathbb{N}$; put

$$\chi^\alpha = \sum_{\sigma \in S_l} \text{sgn}(\sigma)\xi^{\alpha+\sigma-1_l}.$$

(As observed in [JK81, p. 47], this definition is independent of the choice of $l$.) In the sum, the permutation $\sigma \in S_l$ is regarded as the element $(\sigma(1), \sigma(2), \ldots, \sigma(l), 0, 0, \ldots)$ of the group $\Gamma$.

4.1 Theorem [JK81, 2.3.15, p. 52]. For every $\alpha \vdash n$, we have $\zeta^\alpha = \chi^\alpha$. \[\square\]

For $l \in \mathbb{N}$, denote by $D_l : A \to A$ the $\mathbb{Z}$-linear map uniquely determined by

$$D_l(x^\alpha) = \sum_{\sigma \in S_l} \text{sgn}(\sigma)x^{\alpha+\sigma-1_l} \quad (\alpha \in \Gamma),$$

where $1_l = (1, 1, \ldots, l, 0, 0, \ldots) \in \Gamma$.

For $\alpha \in \Gamma$ and $\beta \in \Gamma^+$ with $|\alpha| = |\beta|$, put $\zeta^\alpha_{\beta} = \chi^\alpha(\sigma_\beta)$.
4.2 Lemma. Let $l \in \mathbb{N}$, $\alpha \in \Gamma_l$, and $\beta \in \Gamma^+$, with $|\alpha| = |\beta|$. We have

$$\chi^\alpha_\beta = (D_l(x^{\alpha - \text{id}_l} + 1), x_\beta).$$

Proof. Using the definitions, we have

$$\chi^\alpha_\beta = \sum_{\sigma \in S_l} \text{sgn}(\sigma) x^{\alpha + \sigma - \text{id}_l} = \sum_{\sigma \in S_l} \text{sgn}(\sigma)(x^{\alpha + \sigma - \text{id}_l}, x_\beta)$$

$$= (\sum_{\sigma \in S_l} \text{sgn}(\sigma)x^{\alpha + \sigma - \text{id}_l}, x_\beta) = (D_l(x^{\alpha - \text{id}_l} + 1), x_\beta).$$

\[\square\]

4.3 Lemma. For each $l \in \mathbb{N}$, we have $D_l\delta^+_l = \delta^-_l D_l$.

Proof. Let $l \in \mathbb{N}$ and $\alpha \in \Gamma$. On the one hand,

$$D_l\delta^+_l(x^{\alpha}) = D_l\left(\sum_{i=1}^{l}(\alpha_i - 1)x^{\alpha - \varepsilon_i}\right)$$

$$= \sum_{i=1}^{l}(\alpha_i - 1) \sum_{\sigma \in S_l} \text{sgn}(\sigma)x^{\alpha - \varepsilon_i + \sigma - 1_l}.$$

On the other hand,

$$\delta^-_l D_l(x^{\alpha}) = \delta^-_l \left(\sum_{\sigma \in S_l} \text{sgn}(\sigma)x^{\alpha + \sigma - 1_l}\right)$$

$$= \sum_{\sigma \in S_l} \text{sgn}(\sigma) \sum_{i=1}^{l}(\alpha_i + \sigma(i) - 2)x^{\alpha + \sigma - 1_l - \varepsilon_i}.$$

Therefore,

$$(\delta^-_l D_l - D_l\delta^+_l)(x^{\alpha}) = \sum_{\sigma \in S_l} \text{sgn}(\sigma) \sum_{i=1}^{l}(\sigma(i) - 1)x^{\alpha + \sigma - 1_l - \varepsilon_i}$$

$$= \sum_{\sigma \in S_l} \text{sgn}(\sigma) \sum_{i=1}^{l}(i - 1)x^{\alpha + \sigma - 1_l - \varepsilon_{\sigma(i)} - 1}$$

$$= \sum_{i=1}^{l}(i - 1) \sum_{\sigma \in S_l} \text{sgn}(\sigma)x^{\alpha + \sigma - 1_l - \varepsilon_{\sigma(i)}}.$$

Let $1 < i \leq l$. Put $\tau = (i - 1, i) \in S_l$, and for $\sigma \in S_l$, put

$$\gamma^\sigma = \alpha + \sigma - 1_l - \varepsilon_{\sigma(i)}.$$
Let $\sigma \in S_l$. For $1 \leq k \leq l$, we have

$$
\gamma_k^\sigma = \begin{cases} 
\alpha_k + i - 2, & \text{if } \sigma(k) \in \{i - 1, i\}, \\
\alpha_k + \sigma(k) - 1, & \text{otherwise}
\end{cases}
= \gamma_k^\tau^\sigma,
$$

and for $k > l$, we have $\gamma_k^\sigma = \alpha_k = \gamma_k^\tau^\sigma$. Therefore, $\gamma^\sigma = \gamma^\tau^\sigma$, giving

$$
\sum_{\sigma \in S_l} \text{sgn}(\sigma) x^{\alpha + \sigma - 1 - \varepsilon_{\sigma^{-1}(i)}} = \sum_{\sigma \in S_l} (x^{\gamma^\sigma} - x^{\gamma^\tau^\sigma}) = 0.
$$

We conclude that $D_l \delta_l^- = \delta_l^- D_l$.

4.4 Lemma [JK81, 2.3.9, p. 48]. For every $\gamma \in \Gamma$ and $i \in \mathbb{Z}^+$, we have $\chi^\gamma = -\chi^\eta$, where

$$
\eta = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1} - 1, \gamma_i + 1, \gamma_{i+2}, \ldots).
$$

The formula in the following theorem is analogous to that in Theorem 2.3, but we point out (lest it be supposed a misprint) that the factor here is $\alpha_i - i$ instead of $\alpha_i - 1$ as earlier.

4.5 Theorem. Let $n \in \mathbb{Z}^+$. For each $\alpha \vdash n$ and each $\beta \in \Gamma^+$ with $|\beta| = n - 1$, we have

$$
\sum_{\alpha_i > \alpha_{i+1}} (\alpha_i - i)\zeta_{\beta}^{\alpha - \varepsilon_i} = \sum_{j=1}^{l(\beta)} \beta_j \zeta_{\beta + \varepsilon_j}^{\alpha}.
$$

Proof. Let $\alpha \vdash n$ and let $\beta \in \Gamma^+$ with $|\beta| = n - 1$.

Fix $1 \leq i \leq l(\alpha)$. We first observe that if $\alpha_i > \alpha_{i+1}$, then $\alpha - \varepsilon_i \vdash (n - 1)$, so $\zeta_{\beta}^{\alpha - \varepsilon_i}$ is defined and equals $\chi_{\beta}^{\alpha - \varepsilon_i}$ by Theorem 4.1. Now assume that $\alpha_i \geq \alpha_{i+1}$. Then $\alpha_i = \alpha_{i+1}$, implying $\gamma_i = \gamma_{i+1} - 1$, where $\gamma = \alpha - \varepsilon_i$. Then, in the notation of Lemma 4.4, we have $\eta = \gamma$. Therefore, by that lemma, $\chi^\gamma = -\chi^\eta$, implying $\chi^{\alpha - \varepsilon_i} = \chi^\gamma = 0$.

Put $l = l(\alpha)$ and $m = l(\beta)$, and denote by LHS the left-hand side of the equation in the statement. Using the preceding paragraph for the first equality,
and then Lemma 4.2 we have
\[ \text{LHS} = \sum_{i=1}^{l} (\alpha_i - i) \chi_{\beta}^{\alpha_i - i} = \sum_{i=1}^{l} (\alpha_i - i) (D_l(x^{\alpha_i - i, -i} + 1), x_{\beta}) \]
\[ = (D_l(\sum_{i=1}^{l} (\alpha_i - i)x^{\alpha_i - i, -i} + 1), x_{\beta}) = (D_l(\delta_l^{-1}(x^{\alpha_i - i, -i} + 1)), x_{\beta}) \]
\[ = (\delta_l^{-1}(D_l(x^{\alpha_i - i, -i} + 1)), x_{\beta}), \]
where the last equality uses Lemma 4.3. Next, the argument of \( \delta_l^{-1} \) is seen to be in \( A_l \), and \( x_{\beta} \in B_m \), so Theorem 3.1 applies and we get
\[ \text{LHS} = (D_l(x^{\alpha_i - i, -i} + 1), \delta_l^{-1}(x_{\beta})) = (D_l(x^{\alpha_i - i, -i} + 1), \sum_{j=1}^{m} \beta_j(x_{\beta} + \epsilon_j)) \]
\[ = \sum_{j=1}^{m} \beta_j(D_l(x^{\alpha_i - i, -i} + 1), x_{\beta} + \epsilon_j) = \sum_{j=1}^{m} \beta_j \chi_{\beta}^{\alpha} + \epsilon_j \]
\[ = \sum_{j=1}^{m} \beta_j \zeta_{\beta}^{\alpha} + \epsilon_j, \]
again using Lemma 4.2 and then Theorem 4.1.

\[ \Box \]

5. Recursion Formula

The theorem below provides a method for recursively finding the irreducible characters values for \( S_n \) \( (n \in \mathbb{Z}^+) \). Each value \( \zeta_{\beta}^{\alpha} \) is expressed in terms of values \( \zeta_{\delta}^{\gamma} \) with \( \gamma \vdash n - 1 \) or with \( \gamma = \alpha \) and \( \delta < \beta \), where \( < \) is the reverse lexicographical order on the set of partitions of \( n \) (i.e., \( \delta < \beta \) if for some \( k \in \mathbb{Z}^+ \) we have \( \delta_j = \beta_j \) for \( j < k \) and \( \delta_k > \beta_k \)).

Therefore, if the character table for \( S_{n-1} \) is known, then the character table for \( S_n \) can be determined by taking the irreducible characters in turn (in any order) and working through the conjugacy classes of \( S_n \) ordered using the reverse lexicographic ordering of the associated partitions.

The first case in the theorem (\( \beta_m = 1 \)) is the case where the permutation at which the character is being evaluated has a fixed point, so the character value is given by the branching theorem [JK81, 2.4.3, p. 59]. We have included the formula in this case for the sake of completeness.

The statement of the second case (\( \beta_m \neq 1 \)) requires additional notation: For \( \gamma \in \Gamma \), put \( \mu(\gamma) = (\mu_1, \mu_2, \ldots, \mu_m) \), where \( m = l(\gamma) \) and
\[ \mu_j = |\{1 \leq k \leq m \mid \gamma_k = \gamma_j\}| \quad (1 \leq j \leq m). \]

The sole irreducible character value \( \zeta_{0}^{0} = 1 \) for \( S_0 \) begins the recursion.
5.1 Theorem. Let \( n \in \mathbb{Z}^+ \), let \( \alpha, \beta \vdash n \), and put \( m = l(\beta) \).

(i) If \( \beta_m = 1 \), then

\[
\zeta^\alpha_\beta = \sum_{i=1}^{l(\alpha)} \zeta_{\beta - \varepsilon_m}^{\alpha - \varepsilon_i} \quad \text{(the branching theorem)}.
\]

(ii) If \( \beta_m \neq 1 \), then

\[
\zeta^\alpha_\beta = (\beta_m - 1)^{-1} \left[ \sum_{i=1}^{l(\alpha)} (\alpha_i - i) \zeta_{\beta - \varepsilon_m}^{\alpha - \varepsilon_i} - \sum_{j=1}^{m-1} \mu_j \beta_j \zeta_{\beta + \varepsilon_{j-1} - \varepsilon_m}^\alpha \right],
\]

where \( \mu = \mu(\beta - \varepsilon_m) \) and \( \beta_0 := \infty \).

Proof. (i) See [JK81, pp. 58–59].

(ii) Assume that \( \beta_m \neq 1 \). By Theorem 4.5, we have

\[
\sum_{\alpha_i > \alpha_{i+1}}^{l(\alpha)} (\alpha_i - i) \zeta_{\beta - \varepsilon_m}^{\alpha - \varepsilon_i} = \sum_{j=1}^{m-1} \beta_j \zeta_{\beta - \varepsilon_m}^{\alpha - \varepsilon_j} + (\beta_m - 1) \zeta^\alpha_\beta. \tag{5.1.1}
\]

If \( \beta_k = \beta_j \) for some \( 1 \leq k, j \leq m - 1 \), then the permutations \( \sigma_{\beta + \varepsilon_{k-1} - \varepsilon_m} \) and \( \sigma_{\beta + \varepsilon_{j-1} - \varepsilon_m} \) are conjugate, implying \( \zeta_{\beta + \varepsilon_{k-1} - \varepsilon_m}^\alpha = \zeta_{\beta + \varepsilon_{j-1} - \varepsilon_m}^\alpha \). Therefore,

\[
\sum_{j=1}^{m-1} \beta_j \zeta_{\beta - \varepsilon_m + \varepsilon_j}^\alpha = \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} \beta_k \zeta_{\beta + \varepsilon_{j-1} - \varepsilon_m}^\alpha = \sum_{j=1}^{m-1} \sum_{\beta_j < \beta_{j-1}} |M_j| \beta_j \zeta_{\beta + \varepsilon_{j-1} - \varepsilon_m}^\alpha, \tag{5.1.2}
\]

where \( M_j = \{1 \leq k \leq m - 1 \mid \beta_k = \beta_j \} \).

Fix \( 1 \leq j \leq m-1 \) with \( \beta_j < \beta_{j-1} \). For \( 1 \leq k \leq m-1 \), we have \( \beta_k = (\beta - \varepsilon_m)_k \) and \( \beta_j = (\beta - \varepsilon_m)_j \), while

\[
(\beta - \varepsilon_m)_m = \beta_m - 1 < \beta_j = (\beta - \varepsilon_m)_j.
\]

Therefore,

\[
M_j = \{1 \leq k \leq m \mid (\beta - \varepsilon_m)_k = (\beta - \varepsilon_m)_j \},
\]

which gives \( |M_j| = \mu_j \).

We now get the formula in the statement by substituting this into Equation (5.1.2), substituting that result into Equation (5.1.1), and finally solving for \( \zeta^\alpha_\beta \). \( \square \)
5.2 Remark. We can sacrifice readability a bit for the sake of compactness and express both cases in the theorem using a single formula by using the Kronecker delta: Putting $\kappa = 1 - \delta_{\beta_m 1}$, we have

$$
\zeta^\alpha_\beta = (\beta_m - \kappa)^{-1} \left[ \sum_{i=1}^{l(\alpha)} (\alpha_i - i)^\kappa \zeta^{\alpha - \varepsilon_i}_{\beta - \varepsilon_m} - \kappa \sum_{j=1}^{m-1} \mu_j \beta_j \zeta^{\alpha}_{\beta + \varepsilon_j - \varepsilon_m} \right].
$$

As an application we give a proof of the well-known formula for the value of an irreducible character of $S_n$ at an $n$-cycle. In the proof, we use the term “hook,” which refers to a partition of $n$ of the form

$$(n - r, 1^r) := (n - r, 1, \ldots, 1, 0, 0, \ldots)$$

with $r$ ones.

5.3 Corollary [JK81, 2.3.17, p. 54]. Let $n \in \mathbb{Z}^+$, let $\alpha \vdash n$, and put $\beta = (n, 0, 0, \ldots)$. We have

$$
\zeta^\alpha_\beta = \begin{cases} 
(-1)^r, & \text{if } \alpha = (n - r, 1^r), \text{ some } 0 \leq r < n, \\
0, & \text{otherwise.}
\end{cases}
$$

Proof. We proceed by induction on $n$. If $l(\alpha) = 1$, then $\alpha = (n)$ and $\zeta^\alpha_\beta$ is the trivial character, so the claim holds with $r = 0$. Now assume that $l(\alpha) > 1$. In particular, $n > 1$.

We have $m = l(\beta) = 1$ and $\beta_m = n \neq 1$, so part (ii) of Theorem 5.1 applies. In that formula the sum over $j$ is empty, while, for each index $i$ in the first sum, the induction hypothesis applies, since $\alpha - \varepsilon_i \vdash (n - 1)$ and $\beta - \varepsilon_m = (n - 1, 0, 0, \ldots)$, giving $\zeta^{\alpha - \varepsilon_i}_{\beta - \varepsilon_m} = 0$ if $\alpha - \varepsilon_i$ is not a hook.

Assume that $\alpha$ is not a hook and assume that $\alpha - \varepsilon_i$ is a hook for some $1 \leq i \leq l(\alpha)$ with $\alpha_i > \alpha_{i+1}$. Then $i = 2$ and $\alpha_2 = 2$, so that $\alpha_i - i = 0$. Therefore, the theorem gives $\zeta^\alpha_\beta = 0$, which establishes the second case.

Now assume that $\alpha = (n - r, 1^r)$ for some $0 \leq r < n$. Since $l(\alpha) > 1$, we have $r > 0$. If $r = n - 1$, then $\alpha = (1^n)$ and $\zeta^\alpha_\beta$ is the alternating character, so the claim holds. Now assume that $r < n - 1$, so that $n - r > 1$. Then the formula in the theorem has two terms, corresponding to $i = 1$ and $i = r + 1$, respectively, and we get

$$
\zeta^\alpha_\beta = (\beta_1 - 1)^{-1} \left[ (\alpha_1 - 1)\zeta^{\alpha - \varepsilon_1}_{\beta - \varepsilon_1} + (\alpha_{r+1} - (r + 1))\zeta^{\alpha - \varepsilon_{r+1}}_{\beta - \varepsilon_1} \right]
$$

$$
= (n - 1)^{-1} \left[ (n - r - 1)(-1)^r + (1 - (r + 1))(-1)^{(r-1)} \right]
$$

$$
= (-1)^r,
$$
again using the induction hypothesis. This establishes the first case and completes the proof. □

We chose to streamline this proof by using that the cases $\alpha = (n)$ and $\alpha = (1^n)$ correspond to the trivial character and the alternating character, respectively, but this was not necessary since Theorem 5.1 handles these cases as well.

References

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