Integrable cases of gravitating static isothermal fluid spheres

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It is shown that different approaches towards the solution of the Einstein equations for a static spherically symmetric perfect fluid with a $\gamma$-law equation of state lead to an Abel differential equation of the second kind. Its only integrable cases at present are flat spacetime, de Sitter solution and its Buchdahl transform, Einstein static universe and the Klein-Tolman solution.

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I. INTRODUCTION

The Einstein equations for static and spherically symmetric perfect fluids have been investigated by many authors [1], [2], [3]. An abundance of solutions has been found when no equation of state is prescribed, since then the unknown functions are more than the equations. The problem becomes very difficult when the fluid’s pressure $p(r)$ is defined as a known function of the fluid’s density $\rho(r)$, both depending only on the radial coordinate $r$. A realistic equation of state is provided by the Newtonian polytropes $p = \frac{1}{n} \rho^{n+1/k}$ which have been studied for many years [4], [5]. In the limit when $k \to \infty$ a softer isothermal equation of state emerges

$$
\rho = np,
$$

usually called the $\gamma$-law, because it is traditionally written as $p = (\gamma - 1) \rho$. Physically realistic perfect fluid solutions have $1 \leq n \leq \infty$. Important special cases include dust ($n = \infty$), incoherent radiation ($n = 3$) and stiff fluid ($n = 1$) where the speed of sound equals the speed of light.

Even for such a simple linear relation, few analytic solutions have been found. When $n = \infty$ and $\rho \neq 0$ the pressure vanishes, giving the case of dust. In fact, the density also vanishes and what remains is trivial flat spacetime. When $n = -1$ the pressure and the density are constant and the solution is equivalent to a vacuum solution with a cosmological constant found by de Sitter. We show in this paper that the case $n = -5$ is connected to it by a Buchdahl transformation [6], [7], [8] and is also soluble. The case $n = -3$ leads to the Einstein static universe [9]. It is clear that the unphysical cases ($n < 1$) should be studied too since they produce cosmological solutions or are connected to the physical ones by a general transformation, valid also when no equation of state is prescribed.

A simple solution for a general $n$ was hidden among the solutions found by Tolman [10]. In fact, he studied the field equations under simplifying assumptions for the metric components, without imposing an equation of state. The pressure and the density in some of his solutions happen to satisfy Eq. (1) when certain constants are sent to zero or to infinity [11]. The first who systematically investigated relation (1) was O. Klein. He rediscovered the Tolman solution first for $n = 3$ and later for arbitrary $n$ [12], [13]. His approach and results, published in a not-readily available journal, remained unnoticed for a long time. Even in Ref. [14] his second work is mentioned as referring to the polytropic equation of state, which is true only for its beginning. For a third time this Klein-Tolman (KT) solution was found by Misner and Zopolsky [15]. The radial dependence of the density, however, was omitted due to a misprint, causing additional confusion. Nevertheless, this paper became a standard reference towards which further rediscoveries [14], [16] were directed [17]. Klein also found numerically a regular solution, starting in phase space from Minkowski spacetime and spiralling towards the KT solution. His work was based on a certain first order differential equation. This solution was also rediscovered later by numeric studies of two and three-dimensional autonomous dynamical systems [18], [19], [20] and further analysed.

The question whether the regular solution, parameterized by $n$, has an explicit expression has never been answered in a satisfactory way. The persistent closure of analytic methods on the irregular KT solution suggests that there are no more integrable cases except it and the few solutions mentioned above. On the other hand, similar problems have been completely solved explicitly. For example, static dust solutions are not possible. The dust must be non-static or charged. In both cases the general solution has been found [19], [21], [22]. The cylindrically symmetric static case has been solved in simple functions, using two different gauges [23], [24]. The relation between them was clarified in Ref. [24]. The planar case follows easily from the cylindrical one [25] or from the spherical one [17], [26].

The purpose of this paper is to derive the integrable cases in a unified manner and to elucidate the mathematical difficulty of the problem. We show that in its heart stands the Abel equation of the second kind, whose normal form is
\[ \frac{w w_z - w}{w} = f(z). \] (2)

Its integrable cases depend on the shape of \( f(z) \) and are tabulated in Ref. [26]. The functions that emerge are in general transcendental, but simplify for special values of \( n \). The integrable cases that we find are given by \( n = -5, -3, -1, \infty \) and the KT solution with \( n \) outside the interval \((-5.83, -0.17)\).

In the following three sections three different approaches are discussed which invariably lead to Eq. (2) with functions \( f(z) \) possessing the same general structure but with different coefficients. In Sec. II we start from the well-known Tolman-Oppenheimer-Volkoff (TOV) equation [8], [27] written in a general spherical metric. In Sec. III the starting point is a differential equation derived by Klein in curvature coordinates. In Sec. IV we utilize the approach of Haggag and Hajj-Boutros (HH) described in Ref. [14] for a stiff perfect fluid in isotropic coordinates. Sec. V is dedicated to the Buchdahl transformation which supplies the integrable case \( n = -5.83 \).

Finally, Sec. VI contains some discussion and conclusions.

II. ANALYSIS OF THE TOV EQUATION

The metric of a static spherically symmetric spacetime reads [3], [28]

\[ ds^2 = e^{2\nu} dt^2 - e^{\lambda} dr^2 - R^2 d\Omega^2, \] (3)

where \( d\Omega^2 \) is the metric on the two-sphere and \( \nu, \lambda, R \) depend on \( r \). The Einstein equations are written as

\[ \rho = \frac{1}{R^2} - e^{-\lambda} \left( \frac{2R''}{R} + \frac{R'^2}{R^2} \right) - (e^{-\lambda})' \frac{R'}{R}, \] (4)

\[ p = -\frac{1}{R^2} + e^{-\lambda} \left( \frac{R'}{R} + 2\nu' \right), \] (5)

\[ p = e^{-\lambda} \left[ R'' + \nu'' + \left( \nu' - \frac{\lambda'}{2} \right) \left( \nu' + \frac{R'}{R} \right) \right], \] (6)

where the prime means a derivative with respect to \( r \) and units are used with \( 8\pi G = c = 1 \). The contracted Bianchi identity

\[ p' = - (\rho + p) \nu', \] (7)

follows from Eqs. (4)-(6) and usually replaces Eq. (6). Thus Eqs. (4), (5) and (7) determine \( \nu, \lambda, R, \rho \) and \( p \). One of the metric functions is redundant and can be used to fix the gauge. Different coordinate systems have been introduced in the literature. Curvature coordinates (called also Schwarzschild coordinates) are obtained when \( R = r \). Isotropic coordinates have \( R = re^{\lambda/2} \) while polar gaussian coordinates set \( e^\lambda = 1 \). Other coordinates are known as well. Fixing the gauge and the equation of state equals the number of unknowns and equations. For the time being, we proceed in full generality to derive the TOV equation. Let us define the so-called mass function \( m(r) \):

\[ m(r) = \frac{1}{2} R \left( 1 - e^{-\lambda} R^2 \right). \] (8)

Then Eq. (4) may be written as

\[ \rho = \frac{2m'}{R^3 R'}, \] (9)

which integrates to

\[ m(R) = \frac{1}{2} \int_0^R \rho R^2 dR. \] (10)

Passing from \( r \) to \( R \) dependence, inserting \( e^{-\lambda} \) from Eq. (8) and \( \nu_R \) from Eq. (7) into Eq. (5) results in the general TOV equation.
\[ p_R = -\frac{(\rho + p) (2m + pR^3)}{2R (R - 2m)}. \]  

(11)

In the case of dust \( p = 0 \). Then Eq. (11) yields \( \rho m = 0 \), which combined with Eqs. (9)-(10) gives \( \rho = 0 \). There is no matter and the solution is trivial flat spacetime. In the general case when \( p = p(\rho) \), Eq. (9) shows that Eq. (11) is a differential equation for \( m(R) \). Let us now introduce the variables \( M = m/R, D = \frac{1}{2} pR^2 \) and \( P = \frac{1}{2} pR^2 \). Then Eq. (11) becomes

\[ R D_R = 2D - \frac{(D + P) (M + P)}{(1 - 2M) \rho}. \]  

(12)

Specializing to the \( \gamma \)-law equation of state and introducing \( \tau = \ln R \) we obtain from Eqs. (12) and (9) the autonomous system

\[ (2M - 1) D_\tau = D \left[ (n + 5) M - 2 + \frac{n + 1}{n} D \right], \]  

(13)

\[ D = M_\tau + M. \]  

(14)

The case \( n = 0 \) is excluded because it gives vanishing density. This system was derived in polar gaussian coordinates by Collins [5] and further analysed by him in Refs. [17], [29]. Now, let us insert Eq. (14) into Eq. (13) and define \( x = M - 1/2 \). The result is

\[ x x_{\tau\tau} = \frac{n + 1}{2n} x^2 + \left[ \frac{(n + 1)(n + 2) + 2n}{2n} x + \frac{(n + 1)(n + 2)}{4n} \right] x_\tau + A(x), \]  

(15)

\[ A(x) \equiv \left( x + \frac{1}{2} \right) \left[ \frac{(n + 1)^2 + 4n}{2n} \left( x + \frac{1}{2} \right) - 1 \right]. \]  

(16)

Eq. (8) indicates that \( x < 0 \). The solution of Eq. (15) determines \( M \) and consequently \( m \) as functions of \( R \). Then \( D \) is given by Eq. (14) which determines respectively \( \rho(R) \). The pressure is given by Eq. (1), while \( \nu \) follows from Eq. (7) written as

\[ (\rho + p) \nu_R = -pR. \]  

(17)

Finally, \( e^{-\lambda R^2} \) is found from Eq. (5), written as

\[ e^{-\lambda R^2} = \frac{1 + pR^2}{1 + 2R\nu_R}. \]  

(18)

In order to determine \( \lambda (r) \) one should specify \( R(r) \), the most simple choice being \( R = r \). In polar gaussian coordinates Eq. (18) yields an equation for \( R(r) \) with separated variables. The same is true in isotropic coordinates.

Eq. (15) simplifies enormously when \( x \) is constant, becoming

\[ M \left[ \frac{(n + 1)^2 + 4n}{2n} M - 1 \right] = 0. \]  

(19)

The choice \( M_1 = 0 \) leads to \( m = p = \rho = 0 \), i.e. to flat spacetime. The choice

\[ M_2 = \frac{2n}{(n + 1)^2 + 4n}, \]  

(20)

gives via Eq. (14) \( D = M = \text{const} \) and \( \lambda = \text{const} \), \( \rho = 2D/R^2 \) which is exactly the KT solution. At the centre the density and the pressure have poles and diverge. This solution does not exist when \(-5.83 = -3 - 2\sqrt{2} < n < -3 + 2\sqrt{2} = -0.17 \) because then \( x_2 = M_2 - 1/2 \) is positive or vanishes (when \( n = -1 \)). In the intervals \(-0.17 < n < 0 \) and \( n < -5.83 \) the solution exists but \( \rho \) and \( p \) have different signs.

When \( x_\tau \neq 0 \) we can decrease the order of Eq. (15) by the standard change of variables \( x_\tau \equiv -y(x) \):

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\[xy_{xx} = \frac{n+1}{2n}y^2 - \left[\frac{(n+1)(n+2)+2n}{2n}x + \frac{(n+1)(n+2)}{4n}\right]y + A(x). \tag{21}\]

Eq. (21) falls in the class

\[\left[g_1(x)y + g_0(x)\right]y_x = f_2(x)y^2 + f_1(x)y + f_0(x). \tag{22}\]

There is a standard procedure for the solution of such equations [24]. It consists of two changes of variables which bring them to the Abel equations of the second kind given by Eq. (2) or by

\[ww_\zeta = g(\zeta)w + 1. \tag{23}\]

This procedure is much easier when \(g_0 = 0\) and \(f_2 = \text{const}\) as in Eq. (21). Namely, we have \(w = yE, f(z) = f_0E/f_1, g(\zeta) = f_1/f_0E\) and

\[E = \exp\left(-\int \frac{f_2}{g_1} dx\right), \tag{24}\]

\[z(x) = \int \frac{f_1}{g_1} E dx, \tag{25}\]

\[\zeta(x) = \int \frac{f_0}{g_1} E^2 dx. \tag{26}\]

Since \(f_1\) is a simpler polynomial than \(f_0\) we shall use Eqs. (2) and (25). The use of Eqs. (23) and (26) is more complicated, but does not bring additional integrable cases.

Applied to Eq. (21) the chosen alternative of the general method yields

\[w = yx^{-\frac{n+1}{2n}}, \tag{27}\]

\[z = -\frac{1}{4n} \int \left[(n^2+5n+2)2x + (n+1)(n+2)\right]x^{-\frac{n+1}{2n}-1} dx. \tag{28}\]

The case \(n = 1\) (stiff fluid) is special because a logarithmic term appears in \(z\)

\[z = -4 \ln|x| + \frac{3}{2x}, \tag{29}\]

\[f(z) = \frac{(2x+1)(4x+1)}{x(8x+3)}. \tag{30}\]

The relation \(z(x)\) is transcendental and throws \(f(z)\) out of the tables with integrable cases present in Ref. [26]. The case \(n = -1\) also leads to a logarithmic term but its coefficient vanishes. This case is integrable. In fact, we may go back directly to Eq. (13) which becomes \(D_x = 2D\) if \(M \neq 1/2\) and yields \(\rho = -p = \text{const}\). This is the well-known de Sitter solution. When \(M = 1/2\) Eq. (13) is satisfied identically and we obtain formally the KT solution (20) with \(n = -1\), but it has \(e^{\lambda R^2} = 0\) which is unacceptable.

In the generic case \(n \neq \pm 1\) and Eq. (28) integrates to

\[z = -\frac{x^{-\frac{n+1}{2n}}}{2(n-1)} \left[(n^2+5n+2)2x - (n-1)(n+2)\right], \tag{31}\]

\[f(z) = \frac{(n-1)(2x+1)\left[(n^2+6n+1)2x + (n+1)^2\right]z}{\left[(n^2+5n+2)2x - (n-1)(n+2)\right]\left[(n^2+5n+2)2x + (n+1)(n+2)\right]} \tag{32}\]

The \(z-x\) connection in Eq. (31) is transcendental, except for special values of \(n\), and Eq. (2) is non-integrable in general. Let us investigate the special cases.
When \( n = -3 \), \( f_1 \) divides \( f_0 \) and \( f (z) = 8z \), \( z = -\frac{1}{3} (2x + 1)x^{-1/3} \). This is an integrable case, corresponding to the Einstein static universe. It is discussed in more details in the next section. There is also the formal \( n = -3 \) case of the KT solution (20). It does not exist since \( x = 1/4 > 0 \).

One may try to simplify Eq. (31) by nullifying the coefficients on the right. The condition \( n^2 + 5n + 2 = 0 \) gives \( n = \frac{1}{2} (-5 \pm \sqrt{17}) \). This is of no good since the radical enters the power of \( x \). The other possibility \( n = -2 \) looks more promising. Then Eq. (31) becomes \( x^{3/4} = -3z^4/4 \) and Eq. (32) reduces to

\[
\frac{f (z)}{z} = \frac{21}{16} z - \frac{9}{16} \left( \frac{4}{3} \right)^{4/3} z^{-1/3} + \frac{3}{64} \left( \frac{4}{3} \right)^{8/3} z^{-5/3}.
\]

This function leads to an integrable equation when the coefficient in front of \( z \) is \(-3/16\) which is not true here.

There are several \( n \) which convert Eq. (31) into an algebraic equation for \( x \) of fourth order or lower. It can be solved explicitly for \( x (z) \) and the answer replaced in Eq. (32). Third and fourth order equations appear when \( n = \pm 1/5, \pm 1/7, -1/2, -3/5 \). The radical structure of \( f (z) \), however, is incompatible with the tables with integrable \( f (z) \). Second order equations appear when \( n = \mp 1/3 \) respectively:

\[
4x = -5 \pm \sqrt{25 + 48z},
\]

\[
6zx = 17 \pm \sqrt{17 + 42z}.
\]

Unfortunately, the only integrable functions with square roots include the term \( \sqrt{z^2 + z_0} \) which is not present in the above relations.

Comparison between Eqs. (31)-(32) and Eq. (20) shows how complex must be the numerical regular solution, which starts in \( x, y \) coordinates from flat spacetime and focuses on the KT solution, following a spiral around it. The innocent parameter \( n \), introduced in Eq. (1), proliferates like cancer in the process of solution, ending with the intricate coefficients in \( z (x) \) and \( f (z) \). It even determines the transcendental or algebraic nature of \( z (x) \). In conclusion, the only integrable cases found within this approach are \( n = \infty \) (trivial dust solution), \( n = -1 \) (de Sitter solution), \( n = -3 \) (Einstein static universe) and \( M = \text{const} \) (the KT solution).

Finally, let us discuss for comparison the case of planar symmetry, which is solvable. Going to polar gaussian coordinates, the metric element reads

\[
ds^2 = e^{2\nu}dt^2 - dr^2 - R^2 \left( dx_2^2 + F (x_2)^2 \right),
\]

where \( F (x_2) = \sin x_2 \) for spherical symmetry and \( F (x_2) = 1 \) for planar symmetry [17, 23]. It is possible to generalize the TOV equation to encompass both cases. Instead of Eq. (13) one should write

\[
(2M - K) D_{\tau} = D \left[ -2K + (n + 5) M + \frac{n + 1}{n} D \right],
\]

where \( K = 1 \) or \( 0 \), corresponding to spherical or planar symmetry respectively. In the second case Eq. (37) simplifies

\[
2MM_{\tau \tau} = \frac{n + 1}{n} M_\tau^2 + bMM_\tau + aM^2,
\]

where \( a = n + 1/n + 6 \), \( b = n + 2/n + 5 \). This is the analog of Eq. (15). Proceeding in the same way we obtain again the Abel equation (2) with \( y = -M_\tau \) and

\[
w = yM^{-\frac{n+1}{n}},
\]

\[
z = \frac{bn}{1 - n} M^{-\frac{n-1}{n}},
\]

\[
f (z) = \frac{(n - 1) a}{ab^2}z.
\]

As mentioned before, Eq. (2) with \( f(z) = \alpha z + \beta \), where \( \alpha \) and \( \beta \) are constants, is integrable. The solution, in parametric form, reads
\[ z = C e^T - \frac{\beta}{\alpha}, \quad (42) \]

\[ w = C \sigma e^T, \quad (43) \]

\[ T = - \int \frac{\sigma d\sigma}{\sigma^2 - \sigma - \alpha}, \quad (44) \]

with \( C \) being an arbitrary constant. In Ref. [17] the problem was solved in a different way, by introducing the variable \( \tilde{D} = D/M \). Then Eqs. (14) and (37) are equivalent to

\[ 2\tilde{D}_\tau = \tilde{D} \left( n + 7 + \frac{1 - n}{n} \tilde{D} \right). \quad (45) \]

This is a Bernoulli equation and is easily solved. Further details may be found in Ref. [17] where also a connection with earlier work [30], [31] on the particular cases \( n = 1 \) and \( n = 3 \) is established.

### III. THE APPROACH OF KLEIN

This approach was developed in curvature coordinates where Eqs. (4), (5) and (7) simplify to

\[ p = \frac{n^2}{r} \nu e^{-\lambda} - \frac{1}{r^2} \left( 1 - e^{-\lambda} \right), \quad (46) \]

\[ \rho = \frac{1}{r} \lambda e^{-\lambda} + \frac{1}{r^2} \left( 1 - e^{-\lambda} \right), \quad (47) \]

\[ p' = - (\rho + p) \nu'. \quad (48) \]

Imposing the \( \gamma \)-law, we can integrate Eq. (48):

\[ p = p_0 e^{-(n+1)\nu}. \quad (49) \]

Like before, the case of dust \( n = \infty, p = 0 \) gives a trivial solution with constant \( \nu \) and \( \lambda \). Supposing that \( p \neq 0 \), let us introduce the variables \( s = p_0 r^2, \xi = e^{(n+1)\nu}/s \) and

\[ x = \xi e^{-\lambda} = e^{-\lambda} \frac{e^{-\lambda}}{p r^2}. \quad (50) \]

Obviously \( x \) and \( \xi \) always have the same sign and are positive when the pressure is positive.

We shall derive an equation for \( x \), similar to Eq. (15). In Sec. II, III and IV the main functions will be denoted by \( x \) although they are different, in order to simplify notation and to stress the role of Eq. (22) in the whole problem. Let us multiply Eq. (46) by \( (n + 1)/4 \), Eq. (47) by \(-1/2\) and sum. The result is

\[ 4 (sx)_x - (n + 3) \left( 1 - e^{-\lambda} \right) \xi = 1 - n. \quad (51) \]

Let us introduce \( 2\tau = \ln s \). It differs by a constant from the variable in the previous section, but this is not important since the final equation will be autonomous. Laying temporarily aside the special case \( n = -3 \), we can express \( \xi \) from Eq. (51) as

\[ \xi = \frac{1}{n + 3} \left[ 2x_\tau + (n + 7) x + n - 1 \right]. \quad (52) \]

Eq. (46) may be written in the following way:

\[ 4 (s\xi)_s - (n + 1) \left( e^\lambda - 1 \right) \xi = (n + 1) e^\lambda. \quad (53) \]
The usage of Eqs. (50) and (52) transforms this relation into an autonomous second-order equation for \( x \). The change of variables \( x = -y(x) \) applies again, leading to
\[
2(n + 3) xy y_x = 2(n + 1) y^2 + 2(n + 11) xy - (n + 1) (3n + 1) y - 4(n + 7) x^2 + (n^3 + 9n^2 + 11n + 11) x + (n + 1)^2 (n - 1). \tag{54}
\]
This is exactly the equation of Klein derived in Refs. \([10], [11]\), where the notation \( n = 2n_{KL} + 1 \) and \( x = (n_{KL} + 1)^2 x_{KL} \) was used. He studied it by series expansion and numerically, and was the first to find its regular solution (see Fig. (1) from Ref. \([10]\)). When \( x(\tau) \) is known Eq. (52) gives \( \xi \), and Eq. (50) determines both \( \lambda \) and \( p \). Then Eq. (49) becomes an expression for \( \nu \) and finally \( \rho \) is given by Eq. (1). Modulo coefficients, Eq. (54) is the same as Eq. (21) and also falls in the class (22). Therefore, the procedure described in Sec. II can be applied to bring it to the form of Eq. (2) with
\[
w = yx^{-\frac{n+1}{n+3}}, \tag{55}
\]
\[
z = \frac{1}{2} [(n + 11) x + 3n + 1] x^{-\frac{n+1}{n+3}}, \tag{56}
\]
\[
f(z) = -\frac{2 \left[ 4x - (n + 1)^2 \right] [(n + 7) x + n - 1] z}{2(n + 11) x - (n + 1)(3n + 1) [(n + 11) x + 3n + 1]}. \tag{57}
\]
There are no logarithmic terms in \( z \) within this approach. As with Eq. (31), \( x(z) \) is transcendental except in special cases.

Eq. (54) shows that the KT solution is given here by
\[
x_0 = \frac{(n + 1)^2}{4}. \tag{58}
\]
The other root \( x = -\frac{n-1}{n+1} \) leads to \( \xi = 0 \) and \( e^{2\nu} = e^\lambda = 0 \), \( \rho = np = \infty \) which is unacceptable. Now, since \( x_0 \geq 0 \) for any \( n \), we must ensure that \( \xi_0 > 0 \) in order to have positive \( e^{-\lambda} \). Eq. (52) gives \( \xi_0 = n^2 + 6n + 1 \) and we obtain the same conditions for the existence of the KT solution as in the previous section.

In the case \( n = -1 \) Eq. (48) yields \( p = p_0 \), \( \rho = -p_0 \), while the sum of Eqs. (46) and (47) provides the relation \( 2\nu = -\lambda \). Eq. (47) is a linear equation for \( e^{-\lambda} \), its solution being
\[
e^{-\lambda} = 1 - \frac{2m_0}{r} + \frac{p_0}{3} r^2, \tag{59}
\]
where \( m_0 \) is a constant of integration, identified as the gravitating mass. This is precisely the Kottler solution \([3]\). It can be used as a regular interior solution when \( m_0 = 0 \). Then it becomes the de Sitter solution. The KT solution for \( n = -1 \) has \( x_0 = 0 \) and does not exist.

When \( n = -3 \) Eq. (51) determines directly \( x \): \( x = 1 + x_1/s \), while Eq. (49) gives \( p = p_0 e^{2\nu} \). Then we have from Eq. (50) \( e^{-2\nu-\lambda} = x_1 + s \). Eq. (46) becomes a linear equation for \( e^{2\nu} \) when these results are taken into account. There are two possibilities. When the constant \( x_1 = 0 \), \( x = 1 \), which is the formal KT solution (58) for \( n = -3 \). In fact, it does not exist. When \( x_1 \neq 0 \) the solution is
\[
x_1 e^{2\nu} = 1 + C_1 \left( 1 + \frac{x_1}{s} \right)^{1/2}, \tag{60}
\]
where \( C_1 \) is another integration constant. A regular solution is obtained when \( x_1 = 1 \) and \( C_1 = 0 \). Then \( \nu = 0 \) and \( e^{\lambda} = \frac{1}{1 + p_0 r^2} \).
\[
e^{\lambda} = \frac{1}{1 + p_0 r^2}. \tag{61}
\]
The last two equations represent the metric of the Einstein static universe. The pressure and the density are constant, \( p = p_0 \), \( \rho = -3p_0 \).

There are two cases when the coefficients of \( z(x) \) are simplified, namely \( n = -11 \) and \( n = -1/3 \). In the first case Eq. (57) reads

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\[-320 f(z) = 75z + 16^{5/3}z^{-3/5} + 22 \times 16^{4/5}z^{1/5}. \quad (62)\]

The function
\[f(z) = c_1 z + c_2 z^{q_1} + c_3 z^{q_2}, \quad (63)\]
is integrable for a set of \((q_1, q_2)\) but \((-3/5, 1/5)\) is not among them. When \(n = -1/3\) we have
\[f(z) = -\frac{15}{64} z + \frac{7}{96} \left(\frac{16}{3}\right)^{4/3} z^{-1/3} - \frac{1}{192} \left(\frac{16}{3}\right)^{8/3} z^{-5/3}. \quad (64)\]
The set \((-1/3, -5/3)\) is integrable, but only when \(c_1 = -3/16\) which is not the case here.

Finally, there are few \(n\) when Eq. (56) is an algebraic equation up to the fourth order. Only the cases \(n = -5, -2, 1\) are candidates for integrability because they lead to quadratic equations. Of these, \(n = -5, -2\) yield radicals resembling those in Eqs. (34) and (35) and should be rejected. More interesting is the case of stiff fluid when
\[x = \frac{1}{72} \left(z^2 - 24 \pm z \sqrt{z^2 - 48}\right), \quad (65)\]
\[f(z) = -\frac{2z \left[(z^2 - 60) x - 4\right]}{9 \left[(z^2 - 24) x - 8\right].} \quad (66)\]
The radicals in \(f(z)\) are of the necessary type, but its structure is too complex to figure in the tables with integrable cases.

Like in Sec. II, the only integrable cases found are \(n = -3, -1, \infty\) and \(x = x_0\). The unsuccessful candidates for explicit solutions have in general different values of \(n\) in the two approaches. When they coincide, as is the case \(n = 1\), the reasons for rejection are different - a logarithmic term in the TOV-Collins approach and a complicated \(f(z)\) in the Klein approach.

It is interesting to compare the main variable in this section \(x \equiv x_K\) to the variables in the TOV approach, specialized to curvature coordinates. Eq. (8) becomes
\[M = \frac{m}{r} = \frac{1}{2} \left(1 - e^{-\lambda}\right), \quad (67)\]
and transforms Eq. (50) into
\[x_K = \frac{1 - 2M}{2nD}. \quad (68)\]
Thus, not only \(x = M - 1/2\), but also the above combination satisfy Abel equations of the second kind. The function \(x_K\) resembles \(D\), used in the planar case. It is well-known that when \(D\) satisfies the Bernoulli Eq. (45), \(D^{-1} = M/D\) satisfies a linear equation. This fact stresses once more the conclusion that the spherical case is much more complicated than the planar one.

**IV. APPROACH IN ISOTROPIC COORDINATES**

One can pass from arbitrary to isotropic coordinates in Eqs.(4)-(6) by putting \(R = r e^{\lambda/2}\). Let us make also the change \(s = \ln r\). Then
\[pr^2 e^\lambda = -\lambda s - \frac{1}{4} \lambda_s^2 - \lambda_s, \quad (69)\]
\[pr^2 e^\lambda = \frac{1}{4} \lambda_s^2 + \lambda_s \nu_s + \lambda_s + 2 \nu_s, \quad (70)\]
\[pr^2 e^\lambda = \frac{1}{2} \lambda_{ss} + \nu_{ss} + \nu_s^2. \quad (71)\]
Let us introduce next the variable

\[ \frac{1}{t} = \lambda_s + 2, \]  

and impose the \( \gamma \)-law equation of state. An expression for \( \nu_s \) is obtained from Eqs. (69) and (70):

\[ \nu_s = \frac{t_s}{nt} - \frac{n+1}{4n} \left( \frac{1}{t} - 4t \right). \]  

Next, let us combine Eqs. (70) and (71) and replace in them \( \lambda_s \) and \( \nu_s \) from the above equations. A long, but straightforward computation produces an autonomous equation for \( nt t_s \):

\[ 2n t x = -2 (1 - n) t_s^2 + \frac{1}{2} \left( n^2 + 5n + 2 \right) t_s - 2 (n + 1) (n + 2) t^2 t_s - 2 (n + 1)^2 t^4 \]

\[ + \left[ (n + 1)^2 + 2n \right] t^2 - \frac{n+1}{2} - \frac{(n+1)^2}{8}. \]  

A solution of this master equation determines all characteristics of the metric and the fluid.

When \( n = 1 \) Eq. (74) is exactly Eq. (9) from Ref. [14]. In this section we generalize the HH approach to arbitrary \( n \) and bring it to its logical end - the Abel equation (2). We first lower the order of the polynomial in Eq. (74) by setting \( t^2 = x/4 \) and then perform the change of variables

\[ (\sqrt{x})_s = -\frac{1}{2} y. \]  

The condition \( x \geq 0 \) should be maintained throughout the calculations. Eq. (74) acquires its final form

\[ 2n x y x = (n-1) y^2 + \left[ (n+1) (n+2) x - (n^2 + 5n + 2) \right] y \]

\[ + (x - 1) \left[ 4n + (n+1)^2 - (n+1)^2 x \right]. \]  

It falls in the class (22) and resembles Eqs. (21) and (54), but its coefficients are different functions of \( n \). Proceeding like before, we get

\[ w = y x - \frac{n+1}{x}, \]  

\[ (n-1) z = \left[ (n-1) (n+2) x + n^2 + 5n + 2 \right] x^{-\frac{n+3}{2}}, \]  

\[ f(z) = \frac{(n-1) (x - 1) \left[ 4n + (n+1)^2 - (n+1)^2 x \right] z}{\left[ (n+1) (n+2) x - (n^2 + 5n + 2) \right] \left[ (n-1) (n+2) x + n^2 + 5n + 2 \right]}, \]  

in the generic case \( n \neq \pm 1 \).

The stiff fluid case leads to a logarithmic term in \( z (x) \) which makes

\[ f(z) = -\frac{2 (x - 1) (x - 2)}{3x - 4}, \]  

non-integrable.

The case \( n = -1 \) is pseudo-logarithmic since the coefficient in front of \( \ln x \) vanishes. We have \( x = 1/z \) and \( f(z) = 2z - 2 \). This case is integrable and the solution is given by Eqs. (42)-(44) with \( \alpha = -\beta = 2 \).

The case \( n = -3 \) leads to a gross simplification of \( f(z) \) and is also soluble. One obtains \( f(z) = 2z \) i.e. \( \alpha = 2, \beta = 0 \).

The other candidate cases may be investigated in the same manner as in Secs. II and III. The values \( n = \pm 1/3 \) lead to quadratic equations with radicals of the wrong type. The case \( n = -2 \) yields

\[ f(z) = \frac{21}{16} z + \frac{18}{16} \left( \frac{4}{3} \right)^{4/3} z^{-1/3} - \frac{3}{16} \left( \frac{4}{3} \right)^{8/3} z^{-5/3}. \]  

(81)
This equation belongs to the class (63) but again \( c_1 \neq -3/16 \) and integrability is not gained.

Finally, let us discuss the KT solution in the HH approach. When \( x = \text{const} \) Eq. (76) becomes purely algebraic and has two roots: \( x_1 = 1 \) and

\[
x_2 = 1 + \frac{4n}{(n+1)^2}. \tag{82}
\]

When \( n = -1 \), \( x_2 \) does not exist. In fact, the requirement \( x_2 > 0 \) shows that \( n \) must satisfy the conditions derived in Sec. II. The first root leads to \( t_1 = \pm 1/2 \). In the first subcase Eqs. (72) and (73) give flat spacetime. In the second subcase the following line element is obtained

\[
ds^2 = dt^2 - r^{-4} \left( dr^2 + r^2 d\Omega^2 \right).
\tag{83}
\]

The transformation \( \tilde{r} = 1/r \) converts this element into the usual element for flat spacetime.

The second root \( x_2 \) represents the KT solution in isotropic coordinates, namely \( e^\lambda = r^{\alpha_1} \), \( e^{2\nu} = r^{\alpha_2} \) where

\[
\alpha_1 = \pm \frac{2(n+1)}{\sqrt{4n+(n+1)^2}} - 2, \tag{84}
\]

\[
\alpha_2 = \pm \frac{4}{\sqrt{4n+(n+1)^2}}. \tag{85}
\]

When \( n = 1 \), \( \alpha_1 = \pm \sqrt{2} - 2 \) and \( \alpha_2 = \pm \sqrt{2} \). These values were found in Ref. [14], where the KT solution was discovered for a fifth time.

The integrable cases found in the HH approach coincide with those found in the Klein or the TOV-Collins approaches. In order to make connection with the last one, we must pass in Sec. II to isotropic coordinates. We have

\[
R' = e^{\lambda/2} \left( 1 + \frac{\lambda_s}{2} \right), \tag{86}
\]

\[
2x = 2M - 1 = -R'^2 e^{-\lambda} = -\frac{1}{4t^2}. \tag{87}
\]

We cannot obtain Eq. (74) from Eq. (15) by replacing there just \( x \) from Eq. (87) because \( s = \ln r \), while \( \tau = \ln R = s + \lambda/2 \). The necessary additional relations are

\[
x_\tau = \frac{t_s}{2t^2}, \tag{88}
\]

\[
x_{\tau\tau} = \frac{t_{ss}}{t} - \frac{2t^2}{t^2}. \tag{89}
\]

Therefore, the HH approach combines the TOV equation in isotropic coordinates together with the linear equation (18) for \( \lambda \).

V. BUCHDAHL TRANSFORMATION AND THE CASE \( N = -5 \)

This transformation was found by Buchdahl [6] and rediscovered by Glass and Goldman [7]. Its general formulation refers to static perfect fluid solutions, not necessarily satisfying an equation of state. In the case of spherical symmetry it states that if

\[
ds^2 = e^{2\nu} dt^2 - e^\lambda \left( dr^2 + r^2 d\Omega^2 \right), \tag{90}
\]

is the metric of a perfect fluid solution with pressure \( p \) and density \( \rho \) then there is a reciprocal solution which has \( \nu_b = -\nu, \lambda_b = 4\nu + \lambda \) and
\[ p_b = e^{-4\nu} p, \quad (91) \]

\[ \rho_b = -e^{-4\nu} (\rho + 6p). \quad (92) \]

The transformation is simplest in isotropic coordinates and may be applied to the results obtained in the previous section. When \( p \) and \( \rho \) satisfy the \( \gamma \)-law, the same is true for \( p_b \) and \( \rho_b \) but with a different parameter:

\[ \rho_b = -(n + 6) p_b. \quad (93) \]

Thus, the transformation of the parameter is \( n \to -(n + 6) \). It is clear that when the starting solution is physically realistic (\( n \geq 1 \)), the transformed one is unphysical. \(- (n + 6) \leq -7 \). However, when the starting solution is unphysical and \( n \leq -7 \), then the reciprocal one is physical. When \( n \) is in the interval \(-7 < n < 1 \) both solutions are unphysical.

The existence of this transformation teaches that unphysical solutions should not be neglected a priori and the whole spectrum of \( n \) must be investigated.

Let us apply the transformation to the integrable cases found in Sec. IV. The KT solution is self-reciprocal, i.e. transforms into itself. The reason is that \( 4n + (n + 1)^2 \) is invariant under the transformation. Taking \( \alpha_1^- \) and \( \alpha_2^- \) as the powers of \( r \) in the starting solution, it is easy to show that \( \alpha_1^+ = \alpha_2^- \) and \( \alpha_1^- = \alpha_1^- + 2\alpha_2^- \) when the plus solution is Buchdahl transformed. Solutions with \( n \) outside the interval \(-5.83, -0.17 \) transform between themselves.

The case \( n = -3 \) is also self-reciprocal, as was noticed already by Buchdahl. This explains why \( \nu = 0 \) when the element of the Einstein static universe is written in isotropic coordinates \([32]\). This is necessary to ensure the equality between the starting and the transformed solution.

The interesting case is the de Sitter solution \( (n = -1) \) which transforms into an explicit solution with \( n = -5 \). In isotropic coordinates the de Sitter solution reads \([32]\).

\[ e^{2\nu} = \left( \frac{1 + cr^2}{1 - cr^2} \right)^2, \quad (94) \]

\[ e^\lambda = (1 - cr^2)^{-2}, \quad (95) \]

and \( p = 12/c, \rho = -12/c \) where \( c \) is some constant. The transformed solution has

\[ e^{2\nu_b} = \left( \frac{1 - cr^2}{1 + cr^2} \right)^2, \quad (96) \]

\[ e^\lambda_b = \left( \frac{1 + cr^2}{1 - cr^2} \right)^4, \quad \frac{12}{c}, \quad (97) \]

\[ p_b = \left( \frac{1 - cr^2}{1 + cr^2} \right)^4 \frac{12}{c}, \quad (98) \]

and, of course, \( \rho_b = -5p_b \). Eq. (72) supplies the corresponding \( t \):

\[ t = \frac{1 - cr^2}{2(1 + cr^2)}, \quad (99) \]

\[ t_b = \frac{1 - c^2r^4}{2(1 + 10cr^2 + c^2r^4)}. \quad (100) \]

From Eq. (75) we get \( x = 4t^2 \) and \( y = -4rt \). When \( t_b \) is plugged in these relations we obtain the parametric solution \( x(r), y(r) \) of Eq. (76) for \( n = -5 \):

\[ xyy_x = \frac{3}{5} y^2 - \frac{6}{5}xy + \frac{1}{5} y + \frac{2}{5} (4x^2 - 3x - 1). \quad (101) \]

Eqs. (78) and (79) yield in this case
\[-3z = (9x + 1)x^{-3/5}, \quad (102)\]

\[f(z) = \frac{6(x-1)(4x+1)}{(6x-1)(9x+1)}z. \quad (103)\]

Eq. (102) is a fifth order equation for \(x\) and seems to be transcendental. Obviously \(f(z)\) is not among the tabulated integrable functions. Eq. (101) does not coincide also with any of the equations in Sec.1.3.4. from Ref. [23] which belong to the class (22). If we use Eq. (23) instead, \(g(\zeta) = 1/f(z)\) and

\[3\zeta x^{6/5} = 6x^2 + 18x + 1. \quad (104)\]

This equation is even more complicated than Eq. (102) and \(g(\zeta)\) is not among the few integrable cases listed in Sec.1.3.2 from the same handbook. Unless some mistake has been made, Eq. (101) is integrable, but is not covered by Ref. [29]. This situation is not unique. Recently, non-static charged perfect fluid distributions were discovered which too are missing in the handbooks with solutions of differential equations [33].

**VI. DISCUSSION AND CONCLUSIONS**

In this paper we have discussed explicit solutions of the Einstein equations for a static spherically symmetric perfect fluid with the \(\gamma\)-law equation of state. Three approaches may be found in the literature which, at first sight, have nothing in common. The approach of Collins transforms the TOV equation in polar gaussian coordinates into a two-dimensional autonomous system of differential equations. He studies it numerically [6], [11], [29]. This is a representative example of other similar dynamical systems [6]. In his approach Klein finds a second-order autonomous master equation, whose solution determines all characteristics of the metric and the fluid. He lowers its order and studies it by series expansion and numerically. Klein also discovers a simple exact but singular solution. This work is done in curvature coordinates [10], [11]. Haggag and Hajj-Boutros perform a similar study but in isotropic coordinates and only for a stiff fluid [14]. They also obtain an autonomous second-order equation and transform it into another second-order equation (see Eqs. (9) and (14) in their paper). Then they look for polynomial solutions and find either flat spacetime or the KT solution.

We have generalized these three approaches and pursued them further, till their logical end - an equation, whose integrable cases are tabulated in the handbooks, like Ref. [26]. Surprisingly, we invariably reach the Abel equation of the second kind (2). It behaves like a 'strange attractor' and underlines the common features in the different approaches. The question of integrability is answered by the form of \(f(z)\), the known integrable cases being tabulated. For this purpose we have generalized to an arbitrary coordinate system the dynamical system of Collins and to arbitrary \(n\) the HH approach. The functions \(z(x)\) and \(f(x)\) have the same general structure in all approaches but with different coefficients. The relations between the master variables \(x\), \(x_K\) and \(x_{HH}\) have been elucidated, the TOV-Collins approach serving as a basis.

The integrable cases found are one and the same and include \(n = \infty\) (trivial dust solution), \(n = -1\) (de Sitter solution), \(n = -3\) (Einstein static universe) and \(x = \text{const}\), \(n < -5.83\) or \(n > -0.17\) (Klein-Tolman solution). They appear either as exceptional cases, when equations simpler than Eq. (2) are to be solved, or as Eq. (2) with \(f(z) = \alpha z + \beta\), the simplest integrable case. Some candidate \(n\) lead to Eq. (63), but at least one of the constants there has the wrong value.

It has been shown that Eq. (2) stands in the centre of the problem, independent from the approach or the coordinate system and the integrable cases may be derived in a unified manner. The problem is a very strange mixture of simple integrable cases and extremely difficult non-integrable ones. Perhaps this gives some explanation why the de Sitter and the Einstein solutions were found in 1917, two years after the appearance of general relativity, and why the next years have brought only a five-fold discovery of the KT solution.

An important point is that the problem has been pushed to the mathematical realm of Abel differential equations and further progress depends on developments in this field. Using the Buchdahl transformation we have shown that the case \(n = -5\) is integrable and have given its metric and fluid characteristics. However, its master equation in isotropic coordinates (101) does not seem to be present in the handbooks. This leaves this paper with an open end and one may hope that other integrable cases will be found in the future.

Finally, it is interesting to note that all discrete integrable cases \(n = -1, -3, -5\) are regular and fall in the interval where the singular KT solution does not exist. One is tempted to speculate that in this interval there is a regular one-parameter solution, encompassing the discrete cases.
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