On the Generalization of Stochastic Gradient Descent with Momentum

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Abstract

While momentum-based methods, in conjunction with stochastic gradient descent (SGD), are widely used when training machine learning models, there is little theoretical understanding on the generalization error of such methods. In this work, we first show that there exists a convex loss function for which the stability gap for multiple epochs of SGD with standard heavy-ball momentum (SGDM) becomes unbounded. Then, for smooth Lipschitz loss functions, we analyze a modified momentum-based update rule, i.e., SGD with early momentum (SGDEM), and show that it admits an upper-bound on the generalization error. Thus, our results show that machine learning models can be trained for multiple epochs of SGDEM with a guarantee for generalization. Finally, for the special case of strongly convex loss functions, we find a range of momentum such that multiple epochs of standard SGDM, as a special form of SGDEM, also generalizes. Extending our results on generalization, we also develop an upper-bound on the expected true risk, in terms of the number of training steps, the size of the training set, and the momentum parameter. Our experimental evaluations verify the consistency between the numerical results and our theoretical bounds. SGDEM improves the generalization error of SGDM when training ResNet-18 on ImageNet in practical distributed settings.

1 Introduction

Stochastic gradient descent (SGD) is one of the most popular techniques in training deep neural networks, which involves a huge amount of data (Krizhevsky et al., 2012). This algorithm is scalable, robust, and widely adopted in a broad range of problems. To accelerate the convergence of SGD, a momentum term is often added in the iterative update of the stochastic gradient (Goodfellow et al., 2016). This approach has a long history, with proven benefits in various settings. The heavy-ball momentum method was first introduced by Polyak (1964), where a weighted version of the previous update is added to the current gradient update. Polyak motivated his method by its resemblance to a heavy ball moving in a potential well defined by the objective function. Momentum methods have been used to accelerate the back-propagation algorithm when training neural networks (Rumelhart et al., 1986). Recently, momentum methods are used for training deep neural networks with non-convex loss functions (Sutskever et al., 2013). Intuitively, adding momentum accelerates convergence by circumventing sharp curvatures and long ravines of the sub-level sets of the objective function (Wilson et al., 2018). Ochs et al. (2015) present an illustrative example to show that the momentum can potentially avoid local minima.
In addition to convergence, the generalization of machine learning algorithms is a fundamental problem in learning theory. A classical framework used to study the generalization error in machine learning is PAC learning (Vapnik and Chervonenkis, 1971; Valiant, 1984). However, the associated bounds using this approach can be conservative. Instead, the connection between stability and generalization has been studied in the literature (Bousquet and Elisseeff, 2002; Shalev-Shwartz et al., 2010; Hardt et al., 2016).

According to the definition of Bousquet and Elisseeff (2002), uniform stability requires the algorithm to generate almost the same predictions for all datasets that are different in only one example. Recently, this notion of uniform stability is leveraged to analyze the generalization error of SGD (Hardt et al., 2016). Hardt et al. (2016) have derived the stability bounds for SGD and analyzed its generalization for different loss functions. This is a substantial step forward, since SGD is widely used in many practical systems. However, the algorithms studied in these works do not include momentum.

In this work, we study SGD with momentum (SGDM). Although momentum methods are well known to improve the convergence in SGD, their effect on the generalization error is not well understood. Even though momentum is not studied in (Hardt et al., 2016), it is conjectured therein that momentum might speed up training but adversely impact generalization.

By providing a counter example, we show that the stability gap for multiple epochs of SGDM can become unbounded even for convex loss functions. This motivates us to consider a modified momentum-based update rule, called SGD with early momentum (SGDEM) where a momentum term is added in the earlier training steps. We show that SGDEM is guaranteed to generalize for smooth Lipschitz loss functions and any momentum. To the best of our knowledge, stability and generalization of SGDEM have not been considered in the existing literature. As Figure 1 shows in a practical and distributed setting on ImageNet, while validation loss remains unaffected, the minimum generalization error happens if the momentum is applied for 50 epochs, which indicates that tuning momentum is useful to achieve the best generalization error.

We study the generalization error and true risk of SGDEM. In order to find an upper-bound on the expected generalization error of SGDEM, we use the framework of uniform stability (Bousquet and Elisseeff, 2002; Hardt et al., 2016).

1.1 Main contributions

In Section 3, we show that there exists a convex loss function for which the stability gap for multiple epochs of SGDM becomes unbounded. We introduce SGDEM and show that it is guaranteed to generalize for smooth Lipschitz loss functions. We obtain a bound on the generalization error of SGDEM that decreases inversely with the size of the training set. Our results show that the number of iterations can grow as $n^l$ for a small $l > 1$ where $n$ is the sample size, which explains why complicated models such as deep neural networks

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Validation loss and generalization error of SGDEM when training ResNet-18 (He et al., 2016) on ImageNet (Deng et al., 2009) in a distributed setting with 4 GPUs under tuned learning rate and global minibatch size of 128. For each $t_d$, the momentum is set to $\mu_d = 0.9$ in the first $t_d$ epochs and then zero for the next $90 - t_d$ epochs. SGDM is a special form of SGDEM with $t_d = 90$. The details are provided in Section 5 and Appendix N.}
\end{figure}
can be trained for multiple epochs of SGDEM while their generalization errors are limited. We also analyze the convergence of SGDEM in terms of a bound on the expected norm of the gradient. Our analysis confirms the usefulness of adding momentum in terms of convergence and when considered along with our stability bound, it captures the inherent trade-off between optimization and generalization.

In Section 4, we focus on the special case of strongly convex loss functions. We note that the class of strongly convex loss functions appears in several important machine learning problems, including linear and logistic regression with a weight decay regularization term. In this case, we show that one can obtain a bound on the generalization error of standard SGDM, which suggests that this special form of SGDEM suffices for generalization. Our bound is independent of the number of training iterations and decreases inversely with the size of the training set. Finally, we establish an upper-bound on the expected true risk of SGDM as a function of various problem parameters.

Furthermore, we note that, in practice, training for multiple epochs of SGDEM is typically considered, where each training example is used multiple times. Our generalization bounds for both strongly convex and smooth Lipschitz loss functions tend to zero as the number of samples increases. In addition, our results confirm that using a momentum parameter, \( \mu \approx 1 \), for the entire training improves optimization error. However, it adversely affects the generalization error bounds. Hence, it is crucial to establish an appropriate balance between the optimization error associated with the empirical risk and the generalization error.

Finally, our experimental results show that SGDEM outperforms both vanilla SGD and SGDM in terms of test error on CIFAR10 and generalization error on ImageNet.

1.2 Related work

Studies on the generalization of momentum methods are scarce in the literature. As explained above, momentum is not considered in (Bousquet and Elisseeff, 2002; Hardt et al., 2016). While the generalization error of SGDM is studied in (Ong, 2017) and (Chen et al., 2018), their analysis is limited to the special case of quadratic loss functions. In this work, we show that unlike SGDM, multiple epochs of SGDEM is guaranteed to generalize for smooth Lipschitz loss functions. A similar hybrid method has been shown to generalize better than both vanilla SGD and Adaptive Moment Estimation (Adam) in deep learning practice (Keskar and Socher, 2017). However, it remains unclear why such hybrid method generalizes better. Our work sheds theoretical light on this question.

Convergence of first-order methods with momentum has been studied in (Polyak, 1964; Ochs et al., 2014, 2015; Ghadimi et al., 2015; Lessard et al., 2016; Yang et al., 2016; Wilson et al., 2018; Gadat et al., 2018; Orvieto et al., 2019; Can et al., 2019). Most of these works consider the deterministic setting for gradient update (Polyak, 1964; Ochs et al., 2014, 2015; Ghadimi et al., 2015; Lessard et al., 2016; Wilson et al., 2018). Only a few works have analyzed convergence in the stochastic setting (Yang et al., 2016; Gadat et al., 2018; Orvieto et al., 2019; Can et al., 2019). In (Yang et al., 2016), a unified convergence analysis of SGDM has been studied for both convex and non-convex loss functions with bounded variance. Gadat et al. (2018) have studied the almost sure convergence results of the stochastic heavy-ball method with non-convex coercive loss functions and provided a complexity analysis for the case of quadratic strongly convex. In (Orvieto et al., 2019), differential equation-based analysis is used to study convergence of SGDM. Can et al. (2019) have obtained linear convergence rates for SGDM under a particular momentum for the special case of quadratic loss functions. In this paper, we introduce early momentum for
the class of smooth Lipschitz loss functions, which requires unique convergence analysis as shown in Section 3.

We further note that Lessard et al. (2016) have provided a specific loss function for which the heavy-ball method does not converge. This loss function does not contradict our convergence analysis. The loss function in (Lessard et al., 2016) has been carefully constructed and does not satisfy the assumptions considered in this paper.

In addition to Polyak’s heavy-ball momentum method, Nesterov (1983) has proposed an accelerated gradient descent, which converges as $O(1/k^2)$ in a deterministic and convex setting where $k$ is the number of iterations. Convergence rates are obtained for Nesterov’s accelerated gradient method in various settings (Su et al., 2014; Laborde and Oberman, 2020; Assran and Rabbat, 2020). However, the Netstrov momentum does not seem to improve the rate of convergence for stochastic gradient settings (Goodfellow et al., 2016, Section 8.3.3). Therefore, in this work we focus on the heavy-ball momentum.

2 Problem, assumptions, and definitions

We consider a general supervised learning problem, where $S = \{z_1, \cdots, z_n\}$ denotes the set of samples of size $n$ drawn i.i.d. from some space $Z$ with an unknown distribution $D$.

We assume a learning model described by parameter vector $w$. Let $f(w; z)$ denote the loss of the model described by parameter $w$ on example $z \in Z$.

Our ultimate goal is to minimize the true or population risk given by

$$R(w) \overset{\Delta}{=} \mathbb{E}_{z \sim D}[f(w; z)].$$

(1)

Since the distribution $D$ is unknown, we approximate this objective by the empirical risk during training, i.e., $R_S(w) \overset{\Delta}{=} \frac{1}{n} \sum_{i=1}^{n} f(w; z_i)$. We assume $w = A(S)$ for some potentially randomized algorithm $A$.

2.1 Generalization error and stability

In order to find an upper-bound on the true risk of algorithm $A$, we consider the generalization error, which is the expected difference of empirical and true risk:

$$\epsilon_g \overset{\Delta}{=} \mathbb{E}_{S, A}[R(A(S)) - R_S(A(S))].$$

(2)

In order to find an upper-bound on the generalization error of algorithm $A$, we consider the uniform stability property.

Definition 1. Let $S$ and $S'$ denote two datasets from space $Z^n$ such that $S$ and $S'$ differ in at most one example. Algorithm $A$ is $\epsilon_s$-uniformly stable if for all datasets $S$ and $S'$, we have

$$\sup_{z} \mathbb{E}_{A}[f(A(S); z) - f(A(S'); z)] \leq \epsilon_s.$$

(3)

It is shown that uniform stability implies generalization in expectation:

Theorem 1 (Hardt et al. (2016)). If $A$ is an $\epsilon_s$-uniformly stable algorithm, then the generalization error of $A$ is upper-bounded by $\epsilon_s$.

Our notation and proofs are provided in the appendix.
Theorem 1 suggests that it is enough to control the uniform stability of an algorithm to bound the generalization error.

In our analysis of the stability of SGDM, we will consider the following two properties of the growth of the update rule. Let \( \Omega \) denote the model parameter space. Consider a general update rule \( G \) which maps \( w \in \Omega \) to another point \( G(w) \in \Omega \). Our goal is to track the divergence of two different iterative sequences of update rules with the same starting point.

**Definition 2.** An update rule \( G \) is \( \eta \)-expansive if \( \sup_{v, w \in \Omega} \|G(v) - G(w)\|/\|v - w\| \leq \eta \).

**Definition 3.** An update rule \( G \) is \( \sigma \)-bounded if \( \sup_{w \in \Omega} \|w - G(w)\| \leq \sigma \).

### 2.2 Stochastic gradient descent with momentum

The update rule for SGDM is given by

\[
    w_{t+1} = w_t + \mu(w_t - w_{t-1}) - \alpha \nabla_w f(w_t; z_i)
\]

where \( \alpha > 0 \) is the learning rate, \( \mu > 0 \) is the momentum parameter, \( i_t \) is a randomly selected index, and \( f(w_t; z_i) \) is the loss evaluated on sample \( z_i \). In SGDM, we run the update (4) iteratively for \( T \) steps and let \( w_T \) denote the final output. Note that there are two typical approaches to select \( i_t \). The first is to select \( i_t \in \{1, \cdots, n\} \) uniformly at random at each iteration. The second is to permutate \( \{1, \cdots, n\} \) randomly once and then select the examples repeatedly in a cyclic manner. Our results are valid for both approaches.

In the case where the parameter space \( \Omega \) is a compact and convex set, we consider the update rule for projected SGDM:

\[
    w_{t+1} = P(w_t + \mu(w_t - w_{t-1}) - \alpha \nabla_w f(w_t; z_i))
\]

where \( P \) denotes the Euclidean projection onto \( \Omega \).

The key quantity of interest in this paper is the generalization error given by \( \epsilon_g = \mathbb{E}_{S,A}[R(w_T) - R_S(w_T)] = \mathbb{E}_{S,i_0, \cdots, i_{T-1}}[R(w_T) - R_S(w_T)] \) since the randomness in \( A \) arises from the choice of \( i_0, \cdots, i_{T-1} \).

### 2.3 Assumptions on the loss function

In our analysis, we will assume that the loss function satisfies the following properties, which are used also in (Hardt et al., 2016).

**Definition 4.** A function \( f : \Psi \to \mathbb{R} \) is \( L \)-Lipschitz if for all \( u, v \in \Psi \) we have \( |f(u) - f(v)| \leq L \|u - v\| \).

**Definition 5.** A function \( f : \Psi \to \mathbb{R} \) is \( \beta \)-smooth if for all \( u, v \in \Psi \) we have \( \|\nabla f(u) - \nabla f(v)\| \leq \beta \|u - v\| \).

### 3 Smooth Lipschitz loss

We first show that there exists a convex loss function for which the stability gap for multiple epochs of SGDM becomes unbounded. For the case of smooth Lipschitz loss functions, we introduce SGDEM and show that machine learning models can be trained for multiple epochs of SGDEM while their generalization errors are bounded.
In SGDEM, the momentum \( \mu \) is set to some constant \( \mu_d \) in the first \( t_d \) steps and then zero for \( t = t_d + 1, \ldots, T \). Thus, the update rule for SGDEM is given by

\[
\mathbf{w}_{t+1} = \mathbf{w}_t + \mu_d \mathbb{1}\{t \leq t_d\} (\mathbf{w}_t - \mathbf{w}_{t-1}) - \alpha_t \nabla_w f(\mathbf{w}_t; z_i) \tag{6}
\]

where \( \mathbb{1} \) denotes the indicator function, and the projected version can be similarly obtained based on (5).

### 3.1 SGDM is not stable

**Example 1.** Let \( w \in [-1, 1] \) denote a parameter. Consider the one-dimensional and convex loss function \( f(w; z) = Lzw + c_z \) where \( L_z \in \{L, -L\} \) depending on \( z \in \mathcal{Z} \) and \( c_z \geq 0 \) is constant w.r.t \( w \). For \( S \) with \( R_S(w) = 1/n \sum_{i=1}^n (Lw + c_i) = Lw + \sum_{i=1}^n c_i/n \), the optimal parameter minimizing the empirical risk is \( w^*_S = -1 \).

Both SGDM and SGDEM can find the optimal solution of our convex empirical risk minimization problem. We first establish a lower-bound on the stability gap when SGDM is run for multiple epochs, which shows that the gap can be unbounded.

**Theorem 2.** Let \( z \in \mathcal{Z} \). Suppose that the SGDM update (4) is executed for \( T \) steps with momentum \( \mu \) on Example 1. There exist datasets \( S \) and \( S' \) such that \( \mathbb{E}_A[f(A(S); z) - f(A(S'); z)] \) is lower bounded by \( \Omega\left(\frac{T}{n} (1 + k\mu^k )\right) \) with \( k = \lceil \log(T) \rceil \).

Note that the stability lower-bound increases monotonically with \( \mu \). For the same example, we can show that the stability gap for multiple epochs of SGDEM goes to zero.

**Theorem 3.** For Example 1 and datasets described in Theorem 2, the stability gap for SGDEM goes to zero as \( n \to \infty \) with \( T = kn \) for \( k > 1 \) as long as \( \sum_{j=1}^T \alpha_j = o(T) \) and \( t_d \left( \sum_{j=1}^T \alpha_j^2 \right)^{1/3} = o(T^{2/3}) \).

Furthermore, to highlight the importance of early momentum on bounding the stability gap, in Appendix C, we show that the stability gap for multiple epochs of SGDM may become unbounded for any learning rate schedule. This includes \( \alpha_1 = 1 \) and \( \alpha_j = 0 \) for \( j > 1 \), i.e., the gradient term is added only in the first iteration. We also establish an \( \Omega\left(\frac{T}{n}\right) \) lower bound for SGDM on Example 1 even with a time-decaying learning rate, which shows that the learning rate schedule, momentum, and the structure of loss play roles in establishing uniform stability.

**Corollary 1.** For Example 1 with datasets described in Theorem 2 and for time-decaying learning rate, the stability gap for SGDM is lower bounded by \( \Omega\left(\frac{T}{n}\right) \) for any momentum \( \mu > 0 \).

We remark that, since uniform stability is only a sufficient condition for generalization, our result here does not necessarily imply that SGDM does not generalize. However, it does suggest the challenges in terms of the need to find alternative analytical approaches. As shown in the next section, unlike SGDM, SGDEM is stable and thus guaranteed to generalize for smooth Lipschitz loss functions and any momentum.

### 3.2 Generalization analysis of SGDEM

Since generalization is predicated on the convergence of a learning algorithm, we first show that SGDEM is guaranteed to converge to a local minimum for general and possibly non-convex problems. Then, we show that SGDEM is guaranteed to generalize for any \( \mu_d \), when
\( t_d \) is chosen appropriately. Our analysis captures the inherent trade-off between optimization and generalization.

For smooth Lipschitz loss functions, we study the convergence of SGDEM by developing a bound on the expected norm of the gradient.

**Theorem 4.** Suppose that the SGDEM update (6) is executed for \( T \) steps with constant learning rate \( \alpha < 2(1 - \mu_d) \) and momentum \( \mu_d \) in the first \( t_d \) steps. Then, for any \( S \) and \( 0 < t_d \leq T \), we have

\[
\min_{t=0,\ldots,T} \epsilon(t) \leq \frac{W + J_2}{W_1}
\]

where \( \epsilon(t) \) is the suboptimality gap of the non-convex loss function. Theorem 4.

In Appendix G, we study the convergence bound for the non-vanishing term in the convergence bound to become a monotonically decreasing function of \( \mu_d \).

**Remark 1.** In Appendix F, we provide a sufficient condition for the non-vanishing term in the convergence bound to become a monotonically decreasing function of \( \mu_d \).

We now study the upper-bound (7) as a function of \( t_d \) for a given \( \mu_d \). Note that the first term in the upper-bound vanishes as \( T \rightarrow \infty \).

**Remark 2.** We note that optimizing the bounds provided in Theorems 4 and 5 over \( t_d \) will not provide much intuition on the optimal \( t_d \) in terms of training error since we cannot guarantee the actual suboptimality gap (optimization error) of non-convex loss functions. In practice, we need to tune \( t_d \) when training, e.g., neural networks. Our experimental results show that a non-trivial \( t_d \) can be optimal in terms of test error.

We now establish convergence guarantees for SGDEM with time-decaying learning rate.

**Theorem 5.** Suppose that the SGDEM update (6) is executed for \( T \) steps with time-decaying learning rate \( \alpha_t = \frac{\alpha_0}{t+1} \) for \( t = 0, 1, \ldots, T \) with \( \alpha_0 \leq 2(1-c)(1-\mu_d) \) for some \( 0 < c < 1 \) and momentum \( \mu_d > \exp(-1) \) in the first \( t_d \) steps. Then, for any \( S \) and \( 0 < t_d \leq T \), we have

\[
\min_{t=0,\ldots,T} \epsilon(t) \leq \frac{W + \tilde{J}_2}{\tilde{W}_1}
\]

where \( \tilde{J}_2 = \beta \left( \frac{\alpha_0 L}{1-\mu_d} \right)^2 + \frac{\beta}{2} \left( \frac{\alpha_0 L}{1-\mu_d} \right)^2 \left( \frac{\alpha_0 \beta L \mu_d}{1-\mu_d} \right)^2 \), \( \tilde{W}_1 = \frac{\ln(t_d+1)\alpha_0}{1-\mu_d} + \ln \left( \frac{T}{t_d+1} \right) \alpha_0 (1-c)(1-\mu_d) \), \( \tilde{\epsilon}_t = \min \{ \frac{1}{1-\mu_d}, 2 - 1/t, \mu_d^2 (\mu_d^2 + I(t)) \} \), and \( I(t) = \int_1^{\mu_d^2} \frac{2u - 2u^2}{u} \, du \).

We establish convergence guarantees for SGDEM with another time-dependent learning rate.
Theorem 6. Suppose that the SGDEM update (6) is executed for $T$ steps with learning rate $\alpha_t = \alpha_0/t$ and some constant $\mu_d$ in the first $t_d$ steps. Then, for any $1 \leq \bar{t} < t_d \leq T$, SGDEM satisfies $\epsilon_s$-uniform stability with

$$
\epsilon_s \leq \frac{2\alpha_0L^2}{n}T^n h(\mu_d, t_d) + \frac{\bar{t}M}{n} + \frac{2L^2}{\beta(n-1)} \left( \frac{T}{\bar{t}} \right)^u
$$

where $h(\mu_d, t_d) \triangleq \exp \left( 2\mu_d(t_d - \bar{t}) \right) \ln \left( 1 + \frac{1}{2\mu_d} \right) - \frac{1}{2} \ln \left( 1 + \frac{1}{\mu_d} \right)$, $u = (1 - \frac{1}{\bar{t}})\alpha_0\beta$, and $M = \sup_{w, z} f(w; z)$.

Theorem 6 suggests that the stability bound decreases inversely with the size of the training set. It increases as the momentum parameter $\mu_d$ increases.

Remark 3. We can show that our stability bound in Theorem 6 holds for the projected SGDEM since Euclidean projection does not increase the distance between projected points.

Corollary 2. Assume we set $t_d = \bar{t} + K$ where $\bar{t}^* = \left( \frac{2\alpha_0L^2}{M} \right)^{\frac{1}{n-1}} T^\frac{1}{n-1}$ for some constant $K$. Provided that $\alpha_0\beta < 1$, the number of stochastic gradient steps can grow as $n^l$ for a small $l > 1$ while still allowing $\epsilon_s \to 0$ as $n \to \infty$.

To complete our generalization analysis, in the following, we further show that SGDEM updates may not satisfy uniform stability depending on how $t_d$ is set.

Corollary 3. Suppose, in Theorem 6, we set $t_d = \rho T$ and $\bar{t} = \rho T - K$ for some $0 < \rho \leq 1$ and $K < \rho T$. Then SGDEM updates do not satisfy uniform stability for multiple epochs $T = \kappa n$ and the asymptotic upper-bound on the penalty of generalization error is given by $\rho \kappa M$, i.e.,

$$
\lim_{n \to \infty; T = \kappa n} \epsilon_g \leq \rho \kappa M.
$$

Corollary 3 suggests that increasing $t_d$ worsens the generalization penalty when $t_d$ is linear in $T$. Furthermore, increasing $T$ improves the convergence bound. However, the stability upper-bound increases as $T$ increases, which is expected.

4 Strongly convex loss

While we have discussed in the previous section the generalization of SGDEM for smooth Lipschitz loss functions, in this section, we focus on the important class of strongly convex loss functions. We show that it suffices to consider the case $t_d = T$, i.e., where SGDEM becomes SGD, to achieve generalization.

Definition 6. A function $f : \Psi \to \mathbb{R}$ is $\gamma$-strongly convex if for all $\mathbf{u}, \mathbf{v} \in \Psi$ we have $f(\mathbf{u}) \geq f(\mathbf{v}) + \nabla f(\mathbf{v})^T (\mathbf{u} - \mathbf{v}) + \frac{\gamma}{2} \| \mathbf{u} - \mathbf{v} \|^2$.

An example for $\gamma$-strongly convex loss function is Tikhonov regularization, where the empirical risk is given by $R_S(\mathbf{w}) = \sum_{i=1}^n f(\mathbf{w}; z_i) + \frac{\gamma}{2} \| \mathbf{w} \|^2$ with a convex $f(\cdot; z)$ for all $z$. In the following, we assume that $f(\mathbf{w}; z)$ is a $\gamma$-strongly convex function of $\mathbf{w}$ for all $z \in \mathcal{Z}$.

To satisfy the $L$-Lipschitz property of the loss function, we further assume that the parameter space $\Omega$ is a compact and convex set. Since $\Omega$ is compact, the SGDM update has to involve projection.

We present a bound on the generalization of SGDM for $\gamma$-strongly convex loss.
**Theorem 7.** Suppose that the SGDM update (5) is executed for $T$ steps with constant learning rate $\alpha$ and momentum $\mu$. Provided that $\frac{\alpha^2}{\beta+\gamma} - \frac{1}{2} \leq \mu < \frac{\alpha^2}{3(\beta+\gamma)}$ and $\alpha \leq \frac{2}{\beta+\gamma}$, SGDM satisfies $\epsilon_s$-uniform stability where

$$\epsilon_s \leq \frac{2\alpha L^2(\beta + \gamma)}{n(\alpha\beta\gamma - 3\mu(\beta + \gamma))}. \quad (10)$$

Theorem 7 implies that the stability bound decreases inversely with the size of the training set. It increases as the momentum parameter $\mu$ increases. These properties are also verified in our experimental evaluation.\(^2\)

The theoretically advocated momentum parameters in (Polyak, 1964; Nesterov, 1983) are based on convergence analysis of gradient descent with momentum, and do not account for generalization. Depending on the condition number of the problem, these values may not satisfy the range of momentum in Theorem 7. Note that these values are not necessarily optimal for SGDM, in terms of our objective of true risk. Our goal in Theorem 7 is to find the tightest bound to satisfy uniform stability for SGDM. We suspect that there are lower-bounds for the class of strongly convex loss functions. As future work, we plan to establish such lower-bounds using data-dependent notions of stability by exploiting properties of the underlying data distribution.

**Remark 4.** Compared with the stability bound in (Hardt et al., 2016) for SGD, both bounds are in $O(1/n)$. Our bound in Theorem 7 holds for $\alpha \leq \frac{1}{\beta+\gamma}$, which is slightly less restrictive than the range of learning rate in (Hardt et al., 2016, Theorem 3.9). By substituting $\mu = 0$ in (10), we note that the constant term of our bound, $\frac{\beta+\gamma}{\beta\gamma}$, is slightly larger than that of (Hardt et al., 2016, Theorem 3.9), which is $1/\gamma$. Compared with (Chen et al., 2018), our bound is independent of $T$ and our work analyzes the case of strongly convex loss.

Classical generalization bounds using Rademacher complexity, which measures the rate of uniform convergence, are obtained for linear predictors with various norm constraints (Shalev-Shwartz and Ben-David, 2014). Those classical generalization bounds are typically $O(1/\sqrt{n})$. For linear predictors with a Lipschitz loss and a strongly convex regularizer, by bounding Rademacher complexity, it has been shown that with high probability, the generalization error is bounded by $O(1/\sqrt{n})$ for all parameters in a certain bounded set (Kakade et al., 2008). Our stability bound for strongly convex loss functions is $O(1/n)$ though it is in expectation.

Our stability analysis captures how the learning algorithm explores the hypothesis class, in particular, how the generalization gap depends on the momentum. More broadly, unlike stability, uniform convergence is not necessary for learning (based on the learnability definition in (Shalev-Shwartz et al., 2010)) in the general learning setting (Shalev-Shwartz et al., 2010).

Our ultimate goal is to minimize the true risk (1). In Appendix M, we study how the uniform stability results in an upper-bound on the true risk in the strongly convex case. We first show that stability results similar to Theorem 7 hold even if the average parameter $\hat{w}_T$ is considered as the output of algorithm $A$. We then decompose the expected true risk into a stability error term and an optimization one. We also compare the final results with SGD with no momentum and we show that one can achieve tighter bounds by using SGDM than simple SGD without momentum.

\(^2\)Our purpose in this work is not to show the superiority of SGDM or SGDEM, in terms of the stability bound, over SGD. Given the known advantage of SGDM in terms of speeding up training, our purpose is to further analyze the stability/generalization properties of SGDM and SGDEM.
5 Experimental evaluation

5.1 Non-convex loss

In this section, we validate the insights obtained in our theoretical results using experimental evaluation. Our main goal is to study how adding momentum affects the generalization and convergence of SGD.

We first investigate the performance of SGDEM when applied to both CIFAR10 (Krizhevsky) and notMNIST datasets for non-convex loss functions. We set $T$ to 50000 and 14000 for CIFAR10 and notMNIST experiments, respectively. For each value of $\mu_d$, we add momentum for 0-10 epochs. For each pair of $(\mu_d, t_d)$, we repeat the experiments 10 times with random initializations. SGDM can be viewed as a special form of SGDEM when the momentum is added for the entire training (i.e., $t_d = T$). For 10 epochs and without data augmentation, we train ResNet-20 on CIFAR10 and a feedforward fully connected neural network with 1000 hidden nodes on notMNIST. For the feedforward fully connected neural network, we use ReLU activation functions, a cross-entropy loss function, and a softmax output layer with Xavier initialization to initialize the weights (Glorot and Bengio, 2010). We set the learning rate $\alpha = 0.01$. The minibatch size is set to 10. We use 10 SGDEM realizations to evaluate the average performance. We compare the test performance of SGD without momentum with that of SGDEM under $\mu_d = 0.5$, $\mu_d = 0.9$, and $\mu_d = 0.99$.

Outperforming both SGD and SGDM. We show the test error and test accuracy versus $t_d$ under SGDEM for CIFAR10 dataset in Figure 2 (left and middle). We observe that adding momentum for the entire training (i.e., $t_d = T$ or SGDM) is useful when the momentum parameter is small. For different $\mu_d$ values, we notice there exists an optimal $t_d$ in Figure 2 (left) when test error is minimized. We plot the test error versus $t_d$ for notMNIST dataset in Figure 2 (right). The test accuracy is shown in Appendix N. We observe an overshooting phenomenon for $\mu_d = 0.99$, which is consistent with our convergence analysis in Theorem 4. We observe similar phenomenon when we train a feedforward fully connected neural network with 1000 hidden nodes on MINIST dataset. In terms of test accuracy, we observe that it is not helpful to use a momentum parameter, $\mu_d \approx 1$, for the entire training. In an online framework with high dimensional parameters, early momentum is particularly useful since we can minimize memory utilization as SGDEM does not require $w_{t-1}$ for the entire iterative updates.

Distributed training on ImageNet. Figure 1 shows validation loss and generalization

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Figure 2: Test error (left) and test accuracy (middle) of ResNet-20 on CIFAR10. Test error of a feedforward fully connected neural network for notMNIST dataset (right).
error of SGDEM at epoch 90 when training ResNet-18 on ImageNet in a practical data-parallel setting with 4 GPUs under tuned learning rates. We observe that the minimum generalization error happens if the momentum is applied for 50 epochs. The accuracy results and hardware details are provided in Appendix N. Our accuracy results are on par with existing results (He et al., 2016).

5.2 Strongly convex loss

We now study the performance of SGDEM for a smooth and strongly convex loss function. We train a logistic regression model with the weight decay regularization on notMNIST and MNIST. The setup’s details are provided in Appendix N. We plot the test error and test accuracy versus $t_d$ under SGDEM for notMNIST and MNIST in Appendix N and observe that, unlike the case of non-convex loss functions, it does not hurt to add momentum for the entire training. We then focus on SGDM and compare the training and generalization performance of SGD without momentum with that of SGDM under $\mu = 0.5$ and $\mu = 0.9$, which are common momentum values used in practice (Goodfellow et al., 2016, Section 8.3.2).

**Hurting generalization error and improving training error.** In Figure 3 (left) and (middle), we plot generalization and training error versus $n$ with fixed $T$ and observe that generalization error decreases as $n$ increases for all values of $\mu$, which is suggested by our stability upper-bound in Theorem 7. In addition, for sufficiently large $n$, we observe that the generalization error increases with $\mu$, consistent with Theorem 7. On the other hand, training error increases as $n$ increases with fixed $T$, which is expected. We can observe that adding momentum reduces training error as it improves the convergence rate.

**Negligible improvement of test accuracy.** In Figure 3 (right), we plot test accuracy versus $T$ with fixed $n$ (See Appendix N for the training error, training accuracy, and test error). As the number of epochs increases, we note that the benefit of momentum on the test accuracy becomes negligible. This happens because adding momentum results in a higher generalization error thus penalizing the gain in training error.

\[ \text{Figure 3: Generalization error (left) and training error (middle) of logistic regression (cross entropy loss) for notMNIST dataset with } T = 1000 \text{ iterations. Test accuracy of logistic regression for notMNIST dataset with } n = 500 \text{ (right).} \]
6 Conclusions and future work

We study the generalization error and convergence of SGDEM under mild technical conditions. We show that there exists a convex loss function for which the stability gap for multiple epochs of SGDM becomes unbounded and investigate a modified momentum-based update rule, i.e., SGDEM. We establish a bound on the generalization error of SGDEM for the class of smooth Lipschitz loss functions. Our results confirm that deep neural networks can be trained for multiple epochs of SGDEM while their generalization errors are bounded. We also study the convergence of SGDEM in terms of a bound on the expected norm of the gradient. Then, for the case of strongly convex loss functions, we establish an upper-bound on the generalization error, which decreases with the size of the training set, and increases as the momentum parameter is increased. We establish an upper-bound on the expected difference between the true risk of SGDM and the global minimum of the empirical risk. Finally, we present experimental evaluation and show that the numerical results are consistent with our theoretical bounds and SGDEM is an effective algorithm for non-convex problems.

Beyond uniform stability analysis, which is sufficient for generalization, developing necessary conditions for generalization of various learning algorithms remains an open problem. In particular, “on-average” stability is a more relaxed notion and depends on the data-generating distribution (Shalev-Shwartz et al., 2010). We conjecture that developing a lower-bound using “on-average” stability requires assumptions on the underlying family of distributions, while an upper-bound may be linked to the true risk at initialization. Finally, practical methods for adaptive training, such as Adam, use a variation of the heavy-ball momentum to accelerate (Kingma and Ba, 2017). Adapting our analysis for such extensions is also an interesting area of future work.

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**Notation:** We use $\mathbb{E}[\cdot]$ to denote the expectation and $\|\cdot\|$ to represent the Euclidean norm of a vector. We use lower-case bold font to denote vectors. We use sans-serif font to denote random quantities. Sets and scalars are represented by calligraphic and standard fonts, respectively.
A Proof of Theorem 2

We consider two neighbouring datasets $S$ and $S'$ with

$$R_S(w) = \frac{1}{n} \sum_{i=1}^{n} (Lw + c_i) = Lw + \bar{c}$$

and

$$R_{S'}(w) = -\frac{1}{n}Lw + c_k + \frac{1}{n} \sum_{i=1,i\neq k}^{n} (Lw + c_i) = \frac{n-2}{n}Lw + \frac{c'_k - c_k}{n} + \bar{c}$$

where $\bar{c} = \frac{1}{n} \sum_{i=1}^{n} c_i$. Suppose we select an index $i_t$ uniformly at random from $\{1, \cdots, n\}$. Then we have $\mathbb{E}_i[\nabla f(w; z_i)] = \nabla R_S(w) = L$ and $\mathbb{E}_i[\nabla f(w; z'_i)] = \nabla R_{S'}(w) = (n-2)L/n$, which holds for all $w \in \Omega$. Let $w_T$ and $w'_T$ denote the outputs of SGDM on $S$ and $S'$, respectively. Suppose $w_0 = w_0$. We can follow the steps of SGDM in (4) on $S$ and obtain

$$\mathbb{E}_w[w_T - w'_T] = -(T + (T-1)\mu + (T-2)\mu^2 + \cdots + \mu^{T-1})\alpha L + \mathbb{E}[w_0].$$

Similarly, we have

$$\mathbb{E}_w[w_T - w'_T] = -(T + (T-1)\mu + (T-2)\mu^2 + \cdots + \mu^{T-1})\frac{(n-2)\alpha L}{n} + \mathbb{E}[w_0].$$

Hence, we have

$$\mathbb{E}_w[w_T - w'_T] = \frac{2\alpha L}{n} (T + (T-1)\mu + (T-2)\mu^2 + \cdots + \mu^{T-1}).$$

Let $z \in Z$. Using Jensen’s inequality, we can show that

$$\mathbb{E}_w[f(w_T; z) - f(w'_T; z)] \geq \mathbb{E}_w[f(w_T; z) - f(w'_T; z)]$$

$$= \frac{2\alpha L^2}{n} \sum_{j=0}^{T-1} (T - j)\mu^j.$$ 

Hence, $\mathbb{E}_w[f(w_T; z) - f(w'_T; z)]$ is lower bounded by $\Omega(T/n(1 + k\mu_2))$ where $k = o(T)$.

B Proof of Theorem 3

Let $w_T$ and $w'_T$ denote the outputs of SGDEM on $S$ and $S'$, respectively. Following the steps of SGDEM in (6) on $S$, we have

$$\mathbb{E}_w[w_T] = -L(\alpha_1 + \cdots + \alpha_T) - \alpha_1 L(\mu_d + \cdots + \mu_d^{t_d-1}) - \alpha_2 L^2(\mu_d + \cdots + \mu_d^{t_d-2}) - \cdots$$

$$- \alpha_{t_d-1} L\mu_d + \mathbb{E}[w_0].$$

Similarly, we have

$$\mathbb{E}_w[w'_T] = -((n-2)/n)L(\alpha_1 + \cdots + \alpha_T) - ((n-2)/n)\alpha_1 L(\mu_d + \cdots + \mu_d^{t_d-1})$$

$$- ((n-2)/n)\alpha_2 L^2(\mu_d + \cdots + \mu_d^{t_d-2}) - \cdots - ((n-2)/n)\alpha_{t_d-1} L\mu_d + \mathbb{E}[w_0].$$

Let $z \in Z$. For this particular example, we have $\mathbb{E}_w[f(w_T; z) - f(w'_T; z)] = LE_A[w_T - w'_T].$

---

1We assume $w_0, \alpha, \mu$ are set such that the updated parameter remains in the parameter space.
To highlight the importance of early momentum on bounding the stability gap, in this section,
on Example 1 even with a time-decaying learning rate, which shows that the learning rate
dependent term is added only in the first iteration. We also establish a
any learning rate schedule. This includes

\[ \mu > 0 \]  momentum

we show that the stability gap for multiple epochs of SGDM may become unbounded for
schedule, momentum, and the structure of loss play roles in establishing uniform stability.

We note that the set of realizations with

\[ \text{Example 1 with datasets described in Theorem 2 and for any learning rate} \]

\[ \text{This lemma shows } E_A[w_T - w_T'] = E_A[w'_T] - E_A[w_T] \]. Finally, we have

\[ E_A[|f(w_T; z) - f(w'_T; z)|] = \frac{2L^2}{n} \sum_{j=1}^{T} \alpha_j + \frac{2L^2}{n} \sum_{j=1}^{t_d} \alpha_j (\mu_d + \cdots + \mu_d^{t_d-j}) \]

\[ \leq \frac{2L^2}{n} \sum_{j=1}^{T} \alpha_j + \frac{2L^2}{n} \sum_{j=1}^{t_d} \alpha_j (t_d - j) \]

\[ \leq \frac{2L^2}{n} \sum_{j=1}^{T} \alpha_j + \frac{2L^2}{n} \sqrt{\frac{(t_d^2 - 1)(2t_d - 1)}{6} \sum_{j=1}^{t_d} \alpha_j^2} \]

where the last inequality holds due to Cauchy-Schwarz. This completes the proof.

**C Importance of early momentum on bounding the stability gap**

To highlight the importance of early momentum on bounding the stability gap, in this section,
we show that the stability gap for multiple epochs of SGDM may become unbounded for
any learning rate schedule. This includes \( \alpha_1 = 1 \) and \( \alpha_j = 0 \) for \( j > 1 \), i.e., the gradient
term is added only in the first iteration. We also establish a \( \Omega\left(\frac{T}{n}\right) \) lower bound for SGDM
on Example 1 even with a time-decaying learning rate, which shows that the learning rate
schedule, momentum, and the structure of loss play roles in establishing uniform stability.

**Theorem 8.** For Example 1 with datasets described in Theorem 2 and for any learning rate
schedule, there exists a momentum such that the stability gap for SGDM is lower bounded by
\( \Omega\left(\frac{T}{n}\right) \).

In addition, if \( \alpha_j \geq \alpha_{\min} \) for \( j = 1, 2, \cdots, \eta T \) where \( \alpha_{\min} \) and \( \eta < 1 \) are some constants
that do not depend on \( T \), then the stability gap for SGDM is lower bounded by \( \Omega\left(\frac{T}{n}\right) \) for any
momentum \( \mu > 0 \).
Proof. Following the proofs of Theorem 2 for SGDM with \( \alpha_1 > 0 \), we have
\[
\mathbb{E}_A [||f(w_T; z) - f(w'_T; z)||] \geq \frac{2\alpha_1 L^2}{n} \sum_{j=0}^{T-1} \mu^j.
\]
Substituting \( \mu = 1 \), \( \mathbb{E}_A [||f(w_T; z) - f(w'_T; z)||] \) is lower bounded by \( \Omega(\frac{T}{n}) \).

For the second part of the theorem, suppose \( \alpha_j \geq \alpha_{\min} \) for \( j = 1, 2, \ldots, \eta T \). Then we have
\[
\mathbb{E}_A [||f(w_T; z) - f(w'_T; z)||] \geq \frac{2\alpha_{\min} L^2}{n} \sum_{j=0}^{\eta T-1} (T - j) \mu^j
\]
and \( \mathbb{E}_A [||f(w_T; z) - f(w'_T; z)||] \) is lower bounded by \( \Omega(\frac{T}{n}) \).

Corollary 4. For Example 1 with datasets described in Theorem 2 and for time-decaying learning rate, the stability gap for SGDM is lower bounded by \( \Omega(\frac{T}{n}) \) for any momentum \( \mu > 0 \).

Proof. It follows immediately from the proof of Theorem 3.

D Proof of Theorem 4

We analyze the convergence of SGDEM for a smooth Lipschitz loss function with constant learning rate. To facilitate the convergence analysis, we define \( p_t \overset{\Delta}{=} \frac{\mu}{1-\mu t}(w_t - w_{t-1}) \) with \( p_0 = 0 \). Substituting this into the SGDEM update, the parameter recursion is given by
\[
w_{t+1} + p_{t+1} = w_t + p_t - \frac{\alpha}{1-\mu t} \nabla_w f(w_t; z_i).
\]

We also define \( x_t \overset{\Delta}{=} w_t + p_t \). Note that for a \( \beta \)-smooth function \( f \) and for all \( u, v \in \Psi \), we have
\[
f(u) \leq f(v) + \nabla f(v)^T (u - v) + \frac{\beta}{2} ||u - v||^2.
\]

Since the empirical risk \( R_S \) is a \( \beta \)-smooth function, we have
\[
R_S(x_{t+1}) \leq R_S(x_t) + \nabla_w R_S(x_t)^T (x_{t+1} - x_t) + \frac{\beta \alpha^2}{2(1 - \mu t)} ||\nabla_w f(w_t; z_i)||^2
\]
\[
\leq R_S(x_t) + \frac{\beta}{2} \left( \frac{\alpha L}{1 - \mu t} \right)^2 - \alpha \frac{\mu}{1 - \mu t} \nabla_w R_S(x_t)^T \nabla_w f(w_t; z_i)
\]
(13)
where we use the fact that \( ||\nabla_w f(w_t; z_i)|| \leq L \), due to the \( L \)-Lipschitz property.

Upon taking the expectation w.r.t. \( i_t \) in (13) and defining \( r_t \overset{\Delta}{=} R_S(x_{t+1}) - R_S(x_t) \), we have
\[
E_{i_t}[r_t] \leq -\frac{\alpha}{1 - \mu t} \nabla_w R_S(x_t)^T \nabla_w R_S(w_t) + \frac{\beta}{2} \left( \frac{\alpha L}{1 - \mu t} \right)^2
\]
\[
= -\frac{\alpha}{1 - \mu t} (\nabla_w R_S(x_t) - \nabla_w R_S(w_t))^T \nabla_w R_S(w_t) - \frac{\alpha}{1 - \mu t} ||\nabla_w R_S(w_t)||^2 + \frac{\beta}{2} \left( \frac{\alpha L}{1 - \mu t} \right)^2
\]
\[
\leq \frac{1}{2} ||\nabla_w R_S(x_t) - \nabla_w R_S(w_t)||^2 + \frac{\beta}{2} \left( \frac{\alpha L}{1 - \mu t} \right)^2 + \left( \frac{\alpha^2}{2(1 - \mu t)} - \frac{\alpha}{1 - \mu t} \right) ||\nabla_w R_S(w_t)||^2
\]
(14)
where the last inequality is obtained using $2u^Tv \leq \|u\|^2 + \|v\|^2$. In the following, we obtain an upper-bound on $\|\nabla_w R_S(x_t) - \nabla_w R_S(w_t)\|^2$ in (14). We only need to consider the case $t \leq t_d$, i.e., $\mu_t = \mu_d$, since for $t > t_d$ we have $x_t = w_t$.

Since $R_S$ is $\beta$-smooth, we have

$$\|\nabla_w R_S(x_t) - \nabla_w R_S(w_t)\|^2 \leq \beta^2 \|x_t - w_t\|^2. \quad (15)$$

We also note that $\beta^2 \|x_t - w_t\|^2 = \frac{\beta^2 \mu_d^2}{(1 - \mu_d)^2} \|w_t - w_{t-1}\|^2$.

For notational simplicity, we define $q_t \triangleq \frac{1 - \mu_d}{\mu_d} p_t$, with $q_0 = 0$. Rewriting the SGD update rule, the parameter recursion is given by

$$q_{t+1} = \mu_d q_t - \alpha \nabla_w f(w_t; z_t). \quad (16)$$

Unraveling the recursion (16), we have

$$q_t = -\alpha \sum_{k=0}^{t-1} \mu_d^{t-k} \nabla_w f(w_k; z_k)$$

$$= -\alpha \sum_{k=0}^{t-1} \mu_d^k \nabla_w f(w_{t-1-k}; z_{t-1-k}). \quad (17)$$

We define $\Theta_{t-1} \triangleq \sum_{k=0}^{t-1} \mu_d^{2k} = \frac{1 - \mu_d^2}{1 - \mu_d^2}$. Then we can find an upper-bound on $\|q_t\|^2$ as follows:

$$\|q_t\|^2 = \| -\alpha \sum_{k=0}^{t-1} \mu_d^k \nabla_w f(w_{t-1-k}; z_{t-1-k})\|^2$$

$$= \alpha^2 \| \sum_{k=0}^{t-1} \mu_d^k \nabla_w f(w_{t-1-k}; z_{t-1-k})\|^2$$

$$\leq \alpha^2 \sum_{k=0}^{t-1} \mu_d^{2k} \| \nabla_w f(w_{t-1-k}; z_{t-1-k})\|^2$$

$$\leq \alpha^2 \Theta_{t-1} L^2$$

$$\leq \frac{\alpha^2 L^2}{1 - \mu_d^2}. \quad (18)$$

Substituting the inequality (18) into (15), we obtain the following upper-bound on $\|\nabla_w R_S(x_t) - \nabla_w R_S(w_t)\|^2$:

$$\|\nabla_w R_S(x_t) - \nabla_w R_S(w_t)\|^2 \leq \frac{\alpha^2 \beta^2 L^2 \mu_d^2}{(1 - \mu_d)^2(1 - \mu_d^2)}. \quad (19)$$

Substituting (19) into (14) and taking expectation over $i_0, \cdots, i_t$, we have

$$\mathbb{E}_A[r_t] \leq -\left(\frac{\alpha}{1 - \mu_t} - \frac{\alpha^2}{2(1 - \mu_t)^2}\right) \mathbb{E}_A[\|\nabla_w R_S(w_t)\|^2] + \frac{\beta}{2} \left(\frac{\alpha L}{1 - \mu_t}\right)^2 + \frac{1}{2(1 - \mu_t)} \left(\frac{\alpha \beta L \mu_t}{1 - \mu_t}\right)^2. \quad (20)$$
Summing (20) for \( t = 0, \ldots, T \), we have

\[
J_1 \leq E_A[R_S(x_0) - R_S(x_t)] + J_2 \\
\leq E_A[R_S(w_0) - R_S(w^*_S)] + J_2
\tag{21}
\]

where

\[
J_1 = \left( \frac{\alpha}{1 - \mu_d} - \frac{\alpha^2}{2(1 - \mu_d)^2} \right) \sum_{t=0}^{t_d} E_A[\|\nabla_w R_S(w_t)\|^2] + \left( \alpha - \frac{\alpha^2}{2} \right) \sum_{t=t_d+1}^{T} E_A[\|\nabla_w R_S(w_t)\|^2]
\tag{22}
\]

and

\[
J_2 = (t_d + 1) \beta \left( \frac{\alpha L}{1 - \mu_d} \right)^2 + \frac{1}{2(1 - \mu_d^2)} \left( \frac{\alpha \beta L \mu_d}{1 - \mu_d} \right)^2 + (T - t_d) \frac{\beta}{2} \alpha^2 L^2.
\]

Noting \( \alpha \leq 2(1 - c)(1 - \mu_d) \) for some \( 0 < c < 1 \), we obtain the following lower-bound on \( J_1 \) in (22):

\[
J_1 \geq \frac{(t_d + 1)ac}{1 - \mu_d} \min_{t=0, \ldots, t_d} \epsilon(t) + (T - t_d) \alpha(1 - (1 - c)(1 - \mu_d)) \min_{t=t_d+1, \ldots, T} \epsilon(t) \\
\geq \chi_3 \min_{t=0, \ldots, T} \epsilon(t)
\tag{23}
\]

where \( \chi_3 = \frac{(t_d + 1)ac}{1 - \mu_d} + (T - t_d) \alpha(1 - (1 - c)(1 - \mu_d)). \)

Substituting (23) into (21), we obtain (7), which completes the proof.

### E  Convergence guarantees for SGDEM with time-dependent and time-decaying learning rates

We establish convergence guarantees for SGDEM with time-dependent and time-decaying learning rates as follows.

**Theorem 9.** Suppose that the SGDEM update (6) is executed for \( T \) steps with momentum \( \mu_d \) in the first \( t_d \) steps and time-dependent learning rate \( \alpha = \min\{2(1 - c)(1 - \mu_d), \frac{K}{\max\{\sqrt{t_d+1}, \sqrt{T-t_d}\}}\} \) for some \( 0 < c < 1 \) and \( 0 < K \). Then, for any \( S \) and \( 0 < t_d \leq T \), we have

\[
\min_{t=0, \ldots, T} \epsilon(t) \leq \frac{\hat{T}(W + \hat{J}_2)}{\hat{W}_1}
\tag{24}
\]

where

\[
\hat{J}_2 = \frac{\beta}{2} \left( \frac{KL}{1 - \mu_d} \right)^2 + \frac{1}{2(1 - \mu_d^2)} \left( \frac{K \beta L \mu_d}{1 - \mu_d} \right)^2 + \frac{\beta}{2} K^2 L^2,
\]

\[
\hat{W}_1 = \frac{(t_d + 1)c}{1 - \mu_d} + (T - t_d)(1 - (1 - c)(1 - \mu_d)),
\]

\[
\hat{T} = \max\{\frac{1}{2(1 - c)(1 - \mu_d)}, \frac{\max\{\sqrt{t_d+1}, \sqrt{T-t_d}\}}{K}\}.
\]
Theorem 10. Suppose that the SGDEM update (6) is executed for $T$ steps with time-decaying learning rate $\alpha_t = \frac{\alpha_0}{t+1}$ for $t=0,1,\cdots,T$ with $\alpha_0 \leq 2(1-c)(1-\mu_d)$ for some $0 < c < 1$ and momentum $\mu_d > \exp(-1)$ in the first $t_d$ steps. Then, for any $S$ and $0 < t_d \leq T$, we have
\[
\min_{t=0,\ldots,T} \epsilon(t) \leq \frac{W + \hat{J}_2}{\hat{W}_1}
\] (25)
where
\[
\hat{J}_2 = \beta \left( \frac{\alpha_0}{1-\mu_d} \right)^2 + \frac{\beta}{2} \left( \frac{\alpha_0}{1-\mu_d} \right)^2 \frac{1}{t_d+1} + \sum_{t=1}^{t_d} \tau_t \left( \frac{\alpha_0}{1-\mu_d} \right)^2,
\]
\[
\hat{W}_1 = \frac{\ln(t_d+1)\alpha_0}{1-\mu_d} + \ln \left( \frac{T}{t_d+2} \right) \alpha_0(1-c)(1-\mu_d),
\]
\[
\tau_t = \min \left\{ \frac{1}{1-\mu_d^2}, 2-1/t, \mu_d^2((\mu_d^2 - I(t))) \right\}, \quad I(t) = \int_1^{t} \frac{\mu_d^{-2u}}{u^2} \, du.
\]

Proof. Following the proof of Theorem 4, we have
\[
\mathbb{E}_t |r_t| \leq \frac{1}{2} \|\nabla_w R_S(x_t) - \nabla_w R_S(w_t)\|^2 + \frac{\beta}{2} \left( \frac{\alpha_t}{1-\mu_t} \right)^2 + \left( \frac{\alpha_t^2}{2(1-\mu_t)} - \frac{\alpha_t}{1-\mu_t} \right) \|\nabla_w R_S(w_t)\|^2.
\] (26)
In the following, we obtain an upper-bound on $\|\nabla_w R_S(x_t) - \nabla_w R_S(w_t)\|^2$ in (26). We only need to consider the case $t \leq t_d$, i.e., $\mu_t = \mu_d$, since for $t > t_d$ we have $x_t = w_t$.

Since $R_S$ is $\beta$-smooth, we have
\[
\|\nabla_w R_S(x_t) - \nabla_w R_S(w_t)\|^2 \leq \beta^2 \|x_t - w_t\|^2.
\] (27)
We also note that
\[
\beta^2 \|x_t - w_t\|^2 = \frac{\beta^2 \mu_d^2}{(1-\mu_d)^2} \|w_t - w_{t-1}\|^2.
\]

For notational simplicity, we define $q_t \triangleq \frac{1-\mu_d}{\mu_d} p_t$ with $q_0 = 0$. Rewriting the SGDEM update rule, the parameter recursion is given by
\[
q_{t+1} = \mu_d q_t - \alpha_t \nabla_w f(w_t; z_t).
\] (28)
Unraveling the recursion (28), we have
\[
q_t = -\alpha_0 \sum_{k=0}^{t-1} \frac{\mu_d^{t-1-k}}{k+1} \nabla_w f(w_k; z_k).
\] (29)

Lemma 2. Provided that $\mu_d \geq \exp(-1)$, we have $\|q_t\|^2 \leq \tau_t \alpha_0^2 L^2$ for $t \leq t_d$, where
\[
\tau_t = \min \left\{ \frac{1}{1-\mu_d^2}, 2-1/t, \mu_d^2((\mu_d^2 - I(t))) \right\}, \quad I(t) = \int_1^{t} \frac{\mu_d^{-2u}}{u^2} \, du.
\]

20
Proof. Following the proof of Theorem 4, an upper bound on \( \| \mathbf{q}_t \|^2 \) is given by
\[
\| \mathbf{q}_t \|^2 \leq \alpha_d^2 L^2 \tilde{S}
\]
where
\[
\tilde{S} = \sum_{k=0}^{t-1} \frac{\mu_d^{2(t-1-k)}}{(k+1)^2}.
\]

Note that
\[
\tilde{S} \leq \sum_{k=0}^{t-1} \mu_d^{2k} = \frac{1 - \mu_d^{2t}}{1 - \mu_d^2} \tilde{S} \leq \sum_{k=1}^{t} 1/k^2 \leq 1 + \int_1^t 1/u^2 \, du = 2 - 1/t.
\]

Rewriting \( \tilde{S} \) as \( \tilde{S} = \mu_d^{2t} \sum_{k=1}^{t} \mu_d^{-2k} \) and noting \( f(u) = \mu_d^{-2u}/u^2 \) is convex and non-increasing for \( 1 \leq u \leq t \) due to the lower bound \( \mu_d \geq \exp(-1) \). Therefore, we have \( \tilde{S} \leq \mu_d^{2t} (\mu_d^{-2} + I(t)) \). □

Substituting the upper-bound on \( \| \mathbf{q}_t \|^2 \) into (27), we obtain the following upper-bound on \( \| \nabla_w R_S(\mathbf{x}_t) - \nabla_w R_S(\mathbf{w}_t) \|^2 \):
\[
\| \nabla_w R_S(\mathbf{x}_t) - \nabla_w R_S(\mathbf{w}_t) \|^2 \leq \frac{\alpha_d^2 \beta^2 L^2 \mu_d^2}{(1 - \mu_d)^2}.
\]

Substituting (30) into (26) and taking expectation over \( i_0, \cdots, i_t \), we have
\[
\mathbb{E}_A[r_t] \leq -\left( \frac{\alpha_t}{1 - \mu_d} - \frac{\alpha_t^2}{2(1 - \mu_d)^2} \right) \mathbb{E}_A[\| \nabla_w R_S(\mathbf{w}_t) \|^2] + \frac{\beta}{2} \left( \frac{\alpha_t L}{1 - \mu_t} \right)^2 + \bar{c}_t \left( \frac{\alpha_0 \beta L \mu_t}{1 - \mu_t} \right)^2. \tag{31}
\]

Summing (31) for \( t = 0, \cdots, T \), we have
\[
\hat{J}_1 \leq \mathbb{E}_A[R_S(\mathbf{x}_0) - R_S(\mathbf{x}_{t+1})] + \hat{J}_2 \leq \mathbb{E}_A[R_S(\mathbf{w}_0) - R_S(\mathbf{w}_S^*)] + \hat{J}_2 \tag{32}
\]

where
\[
\hat{J}_1 = \sum_{t=0}^{t_d} \left( \frac{\alpha_t}{1 - \mu_d} - \frac{\alpha_t^2}{2(1 - \mu_d)^2} \right) \mathbb{E}_A[\| \nabla_w R_S(\mathbf{w}_t) \|^2] + \sum_{t=t_d+1}^{T} \left( \alpha_t - \frac{\alpha_t^2}{2} \right) \mathbb{E}_A[\| \nabla_w R_S(\mathbf{w}_t) \|^2]. \tag{33}
\]

Noting \( \alpha_0 \leq 2(1-c)(1-\mu_d) \) for some \( 0 < c < 1 \), we obtain the following lower-bound on \( \hat{J}_1 \) in (33):
\[
\hat{J}_1 \geq \sum_{t=0}^{t_d} \frac{\alpha_t c}{1 - \mu_d} \min_{t=0, \cdots, t_d} \epsilon(t) + \sum_{t=t_d+1}^{T} \alpha_t(1 - (1-c)(1-\mu_d)) \min_{t=t_d+1, \cdots, T} \epsilon(t) \geq \tilde{\chi}_3 \min_{t=0, \cdots, T} \epsilon(t) \tag{34}
\]
\[ \hat{\chi}_3 = \frac{\ln(t_d + 1) \alpha_0 c}{1 - \mu_d} + \ln \left( \frac{T}{t_d + 2} \right) \alpha_0 (1 - c) (1 - \mu_d). \] (35)

Finally, we note that
\[ \frac{\beta}{2} \left( \frac{\alpha_0 L}{1 - \mu_d} \right)^2 \sum_{t=0}^{t_d} \frac{1}{(t + 1)^2} \leq \frac{\beta}{2} \left( \frac{\alpha_0 L}{1 - \mu_d} \right)^2 \sum_{t=t_d+1}^{T} \frac{1}{(t + 1)^2} \leq \frac{\beta (\alpha_0 L)^2}{t_d + 1}. \]

F Sufficient condition for the upper-bound (7)

In the following corollary, we provide a simple sufficient condition for the upper-bound (7) to become a monotonically decreasing function of \( t_d \).

**Corollary 5.** Suppose that the SGDEM update (6) is executed for finite \( T \) steps with constant learning rate \( \alpha < 2c(1 - \mu_d) \) with some \( c < \frac{1}{t - \mu_d} \) and momentum \( \mu_d \) in the first \( t_d \) steps. Then the upper-bound (7) is a monotonically decreasing function of \( t_d \) if the following condition is satisfied:

\[ W > \frac{(K_1 - K_2)(K_3 + TK_4)}{K_3 - K_4} - K_1 - TK_2 \] (36)

where \( K_1 = \beta \left( \frac{\alpha_0 L}{1 - \mu_d} \right)^2 + \frac{1}{2\pi^2 (1 - \mu_d)^2} \), \( K_2 = \beta \alpha^2 L^2 \), \( K_3 = \frac{\alpha}{1 - \mu_d} - \frac{\alpha^2}{\pi^2 (1 - \mu_d)^2} \), and \( K_4 = \alpha - \frac{\alpha^2}{2} \).

**Proof.** Note that we can express the upper-bound (7) as
\[ U(t_d) = \frac{W + K_1 + TK_2 + t_d(K_1 - K_2)}{K_3 + TK_4 + t_d(K_3 - K_4)}. \]

The proof follows by taking the first derivative of \( U \) w.r.t. \( t_d \).

Corollary 5 implies that adding momentum for a longer time is particularly useful when our initial parameter is sufficiently far from a local minimum.

G Understanding the role of momentum on convergence

In order to understand how adding momentum affects the convergence, we study the convergence bound for a special form of SGDEM and show the benefit of using momentum.

**Corollary 6.** Suppose we set \( t_d = T \) with constant learning rate \( \alpha < 2(1 - \mu_d) \). Then, for any \( S \), we have

\[ \min_{t=0, \ldots, T} \epsilon(t) \leq \frac{W}{(T + 1) \left( \frac{\alpha}{1 - \mu_d} - \frac{\alpha^2}{2\pi^2 (1 - \mu_d)^2} \right)} + \frac{\beta \alpha^2 L^2 + (\alpha \beta L \mu_d)^2}{2\alpha(1 - \mu_d) - \alpha^2}. \] (37)
Note that the upper-bound (37) is a function of \( \mu_d \). The first term in the upper-bound vanishes as \( T \to \infty \). In the following corollary, we provide a simple sufficient condition for the non-vanishing term in the upper-bound (37) to become a monotonically decreasing function of \( \mu_d \).

**Corollary 7.** Suppose \( \beta \leq 4/3 \) and we set \( t_d = T \) with \( \alpha \leq 2c(1 - \mu_d) \) for some \( 0 < c < 1 \). Then the non-vanishing term in the upper-bound \( (7) \) is a monotonically decreasing function of \( \mu_d \).

**Proof.** Noting \( \alpha \leq 2c(1 - \mu_d) \) for some \( 0 < c < 1 \), we obtain the following lower-bound on \( J_1 \) in (22):

\[
J_1 \geq \chi_4 \min_{t=0,\ldots,T} \epsilon(t) \tag{38}
\]

where

\[
\chi_4 = \frac{(t_d + 1)\alpha(1 - c)}{1 - \mu_d} + (T - t_d)\alpha(1 - c(1 - \mu_d)).
\]

Substituting (38) into (7), the non-vanishing term in the upper-bound becomes a function of \( \mu_d \) through \( (1 - \mu_d)\left(1 + \beta \frac{\mu_d^2}{1 + \mu_d^2}\right) \). We can prove the proposition by taking the first derivative w.r.t. \( \mu_d \) and noting

\[
\frac{2\mu_d + \mu_d^2}{(1 + \mu_d)^2} \leq \frac{3}{4}.
\]

\[\square\]

**H Proof of Theorem 6**

Let \( S \) and \( S' \) be two sets of samples of size \( n \) that differ in at most one example. Let \( \mathbf{w}_T \) and \( \mathbf{w}'_T \) denote the outputs of SGDM on \( S \) and \( S' \), respectively. We consider the updates \( \mathbf{w}_{t+1} = G_t(\mathbf{w}_t) + \mu_t(\mathbf{w}_t - \mathbf{w}_{t-1}) \) and \( \mathbf{w}'_{t+1} = G'_t(\mathbf{w}'_t) + \mu_t(\mathbf{w}'_t - \mathbf{w}'_{t-1}) \) where \( \mu_t \Delta \mu_d \mathbbm{1}\{t \leq t_d\} \) with \( G_t(\mathbf{w}_t) = \mathbf{w}_t - \alpha_t \nabla f(\mathbf{w}_t; \mathbf{z}_n) \) and \( G'_t(\mathbf{w}'_t) = \mathbf{w}'_t - \alpha_t \nabla f(\mathbf{w}'_t; \mathbf{z}'_n) \), respectively, for \( t = 1, \ldots, T \). We denote \( \delta_t \equiv \|\mathbf{w}_t - \mathbf{w}'_t\| \). Suppose \( \mathbf{w}_0 = \mathbf{w}_0' \), i.e., \( \delta_0 = 0 \).

First, as a preliminary step, we observe that the expected loss difference under \( \mathbf{w}_T \) and \( \mathbf{w}'_T \) for every \( \mathbf{z} \in Z \) and every \( t \in \{1, \ldots, T\} \) is bounded by

\[
\mathbb{E}[\|f(\mathbf{w}_T; \mathbf{z}) - f(\mathbf{w}'_T; \mathbf{z})\|] \leq \frac{iM}{n} + L\mathbb{E}[\delta_T|\delta_t = 0]. \tag{39}
\]

This follows from the argument for a similar claim in (Hardt et al., 2016) and applying it to our expression of SGDM parameter update.

Now, let us define \( \Delta_{t,T} = \mathbb{E}[\delta_t|\delta_t = 0] \). Our goal is to find an upper-bound on \( \Delta_{T,T} \) and then minimize it over \( T \).

At step \( t \), with probability \( 1 - 1/n \), the example is the same in both \( S \) and \( S' \). Hence, we have

\[
\delta_{t+1} = \|\mathbf{w}_t - \mathbf{w}'_t \| - \alpha_t \phi_t \]
\[
\leq (1 + \mu_t)\|\mathbf{w}_t - \mathbf{w}'_t \| + \mu_{t-1}\|\mathbf{w}_{t-1} - \mathbf{w}'_{t-1} \| + \alpha_t\|\phi_t \|
\]
\[
\leq (1 + \mu_t + \alpha_t \beta)\delta_t + \mu_{t-1}\delta_{t-1} \tag{40}
\]
where \( \phi_1 = \nabla_w f(w_t; z_i) - \nabla_w f(w'_t; z_i) \). Note that the last inequality in (40) holds due to the \( \beta \)-smooth property. With probability \( 1/n \), the selected example is different in \( S \) and \( S' \).

In this case, we have

\[
\delta_{t+1} = \|(1 + \mu_t)(w_t - w'_t) - \mu_{t-1}(w_{t-1} - w'_{t-1}) - \alpha_t \phi_2 \|
\leq (1 + \mu_t)\delta_t + \mu_{t-1}\delta_{t-1} + \alpha_t \|\nabla_w f(w_t; z_i)\| + \alpha_t \|\nabla_w f(w'_t; z'_i)\|
\leq (1 + \mu_t)\delta_t + \mu_{t-1}\delta_{t-1} + 2\alpha_t L
\]  

(41)

where \( \phi_2 = \nabla_w f(w_t; z_i) - \nabla_w f(w'_t; z'_i) \).

After taking expectation, for every \( t \geq i \), we have

\[
\Delta_{t+1,i} \leq (1 + \mu_t + (1 - 1/n)\alpha_t \beta) \Delta_{t,i} + \mu_{t-1} \Delta_{t-1,i} + 2\alpha_t L/n.
\]

Let us consider the recursion

\[
\tilde{\Delta}_{t+1,i} = (1 + \mu_t + (1 - 1/n)\alpha_t \beta) \tilde{\Delta}_{t,i} + \mu_{t-1} \Delta_{t-1,i} + 2\alpha_t L/n.
\]

Note that we have \( \tilde{\Delta}_{t+1,i} \geq \tilde{\Delta}_{t,i} \). Then, we have the following inequality:

\[
\tilde{\Delta}_{t+1,i} \leq (1 + \mu_t + \mu_{t-1} + (1 - 1/n)\alpha_t \beta) \tilde{\Delta}_{t,i} + \frac{2\alpha_t L}{n}.
\]

Noting that \( \tilde{\Delta}_{t,i} \geq \Delta_{t,i} \) for all \( t \geq i \), we have \( E[\Delta_{T,i}] \leq S_3 + S_4 \) where

\[
S_3 = \sum_{t=i+1}^{T} \prod_{p=i+1}^{t} \left( 1 + \mu_p + \mu_{p-1} + (1 - \frac{1}{n})\alpha_0 \beta \frac{L}{p} \right) \frac{2\alpha_0 L}{nt}
\]

and

\[
S_4 = \sum_{t=i+1}^{T} \prod_{p=i+1}^{t} \left( 1 + \mu_p + \mu_{p-1} + (1 - \frac{1}{n})\alpha_0 \beta \frac{L}{p} \right) \frac{2\alpha_0 L}{nt}.
\]

Substituting \( \mu_p = \mu_{p-1} = \mu_d \) for \( p = 1, \cdots, t_d \), we can find an upper-bound on \( S_3 \) as follows:

\[
S_3 = \sum_{t=i+1}^{t_d} \prod_{p=i+1}^{t} \left( 1 + \mu_p + \mu_{p-1} + (1 - \frac{1}{n})\alpha_0 \beta \frac{L}{p} \right) \frac{2\alpha_0 L}{nt}
\]

\[
\leq \sum_{t=i+1}^{t_d} \prod_{p=i+1}^{t} \exp \left( \mu_p + \mu_{p-1} + (1 - \frac{1}{n})\alpha_0 \beta \frac{L}{p} \right) \frac{2\alpha_0 L}{nt}
\]

\[
\leq \sum_{t=i+1}^{t_d} \exp \left( 2\mu_d(t_d - t) + (1 - \frac{1}{n})\alpha_0 \beta \ln \left( \frac{T}{t} \right) \right) \frac{2\alpha_0 L}{nt}
\]

\[
\leq \frac{2\alpha_0 L}{n} T^{(1 - \frac{1}{n})\alpha_0 \beta} \exp(2\mu_d t_d) \int_{t}^{t_d} h_1(t) t^{-(1 - \frac{1}{n})\alpha_0 \beta} dt
\]

\[
\leq \frac{2\alpha_0 L}{n} T^{(1 - \frac{1}{n})\alpha_0 \beta} \exp(2\mu_d t_d) \int_{t}^{t_d} h_1(t) dt
\]

\[
= \frac{2\alpha_0 L}{n} T^{(1 - \frac{1}{n})\alpha_0 \beta} \exp(2\mu_d t_d) (E_1(2\mu_d t) - E_1(2\mu_d t_d))
\]
where \( h_1(t) \triangleq \frac{\exp(-2\mu dt)}{t} \) and the exponential integral function \( E_1 \) is defined as

\[
E_1(x) \triangleq \int_x^\infty \frac{\exp(-t)}{t} \, dt. \tag{42}
\]

Note that the following inequalities hold for the exponential integral function for \( t > 0 \) (Abramovitz and Stegun, 1972):

\[
\frac{1}{2} \exp(-t) \ln \left( 1 + \frac{2}{t} \right) < E_1(t) < \exp(-t) \ln \left( 1 + \frac{1}{2} \right). \tag{43}
\]

Applying both upper-bound and lower-bound in (43), we have

\[
S_3 \leq \frac{2\alpha_0 L}{n} T^{(1-\frac{1}{n})\alpha_0 \beta} h(\mu_d, t_d). \tag{44}
\]

We can also find an upper-bound on \( S_4 \) as follows:

\[
S_4 = \sum_{t=t_{d}+1}^{T} \prod_{p=t+1}^{T} \left( 1 + \left( 1 - \frac{1}{n} \right) \alpha_0 \beta \right) \frac{2\alpha_0 L}{n t}
\]

\[
\leq \frac{2L}{\beta(n-1)} \left( \frac{T}{\tilde{t}_d} \right)^{(1-\frac{1}{n})\alpha_0 \beta}
\]

\[
\leq \frac{2L}{\beta(n-1)} \left( \frac{T}{\tilde{t}} \right)^{(1-\frac{1}{n})\alpha_0 \beta}. \tag{45}
\]

Replacing \( \Delta T, \tilde{t} \) with its upper-bound in (39), we obtain (9).

### I Proof of Corollary 2

Note that we can minimize the expression \( \frac{t_d M}{n} + \frac{2L^2}{\beta(n-1)} \left( \frac{T}{T} \right)^{u} \) in (9) by optimizing \( t \), where the optimal \( t \) is given by \( \tilde{t}^* \) as defined in the theorem statement. After substituting the optimal \( \tilde{t}^* \) into (9) and setting \( t_d = \tilde{t}^* + K \) for some constant \( K \), we obtain

\[
\epsilon_s \leq \frac{2\alpha_0 L^2}{n} T^u \chi_1 + \frac{1}{\alpha_0 \beta} \left( \frac{2\alpha_0 L^2}{n} + \frac{2L^2}{\beta(n-1)} \right) \left( \frac{T}{T} \right)^{\frac{u}{\alpha_0 \beta}} \tag{46}
\]

where \( \chi_1 = \exp(2\mu_d K) \ln \left( 1 + \frac{1}{2\mu_d \tilde{t}^*} \right) - \frac{1}{2} \ln \left( 1 + \frac{1}{\mu_d (\tilde{t}^* + K)} \right) \).

### J Proof of Corollary 3

Substituting \( t_d = \rho T \) and \( \tilde{t} = \rho T - K \) into (9), we obtain

\[
\epsilon_s \leq \frac{2\alpha_0 L^2}{n} T^u \chi_2 + \frac{(\rho T - K) M}{n} + \frac{2L^2}{\beta(n-1)} \left( \frac{T}{\rho T - K} \right)^{u} \tag{47}
\]

where

\[
\chi_2 = \exp \left( 2\mu_d K \right) \ln \left( 1 + \frac{1}{2\mu_d (\rho T - K)} \right) - \frac{1}{2} \ln \left( 1 + \frac{1}{\mu_d \rho T} \right).
\]

We can derive the asymptotic penalty by substituting \( T = \kappa n \) into the upper-bound (47), letting \( n \to \infty \), and using Theorem 1.
K Proof of Theorem 7

We track the divergence of two different iterative sequences of update rules with the same starting point. We remark that our analysis is more involved than (Hardt et al., 2016) as the presence of momentum term requires a more careful bound on the iterative expressions.

To keep the notation uncluttered, we first consider SGDM without projection and defer the discussion of projection to the end of this proof. Let \( S = \{z_1, \cdots, z_n\} \) and \( S' = \{z'_1, \cdots, z'_n\} \) be two samples of size \( n \) that differ in at most one example. Let \( \mathbf{w}_T \) and \( \mathbf{w}'_T \) denote the outputs of SGDM on \( S \) and \( S' \), respectively. We consider the updates \( \mathbf{w}_{t+1} = G_t(\mathbf{w}_t) + \mu(\mathbf{w}_t - \mathbf{w}_{t-1}) \) and \( \mathbf{w}'_{t+1} = G'_t(\mathbf{w}') + \mu(\mathbf{w}'_t - \mathbf{w}'_{t-1}) \) with \( G_t(\mathbf{w}_t) = \mathbf{w}_t - \alpha \nabla w f(\mathbf{w}_t; z_i) \) and \( G'_t(\mathbf{w}') = \mathbf{w}'_t - \alpha \nabla w f(\mathbf{w}_t; z'_i) \), respectively, for \( t = 1, \cdots, T \). We denote \( \delta_t \triangleq \| \mathbf{w}_t - \mathbf{w}'_t \| \). Suppose \( \mathbf{w}_0 = \mathbf{w}'_0, \) i.e., \( \delta_0 = 0 \).

We first establish an upper-bound on \( \mathbb{E}_A[\delta_T] \). At step \( t \), with probability \( 1 - 1/n \), the example is the same in both \( S \) and \( S' \), i.e., \( z_i = z'_i \), which implies \( G_t = G'_t \). Then \( G_t \) becomes \( (1 - \frac{\alpha \beta \gamma}{\beta + \gamma}) \)-expansive for \( \alpha \leq \frac{2}{\beta + \gamma} \) (see, e.g., (Hardt et al., 2016, Appendix A)). Hence, we have

\[
\delta_{t+1} = \| \mu(\mathbf{w}_t - \mathbf{w}'_t) - \mu(\mathbf{w}_{t-1} - \mathbf{w}'_{t-1}) + G_t(\mathbf{w}_t) - G'_t(\mathbf{w}') \|
\leq \mu \| \mathbf{w}_t - \mathbf{w}'_t \| + \mu \| \mathbf{w}_{t-1} - \mathbf{w}'_{t-1} \| + \| G_t(\mathbf{w}_t) - G'_t(\mathbf{w}') \|
\leq \delta_t + \mu \delta_{t-1}
\]

where \( \vartheta = 1 + \mu - \frac{\alpha \beta \gamma}{\beta + \gamma} \). With probability \( 1/n \), the selected example is different in \( S \) and \( S' \). In this case, we have

\[
\delta_{t+1} = \| \mu(\mathbf{w}_t - \mathbf{w}'_t) - \mu(\mathbf{w}_{t-1} - \mathbf{w}'_{t-1}) + G_t(\mathbf{w}_t) - G'_t(\mathbf{w}') \|
\leq \mu \| \mathbf{w}_t - \mathbf{w}'_t \| + \mu \| \mathbf{w}_{t-1} - \mathbf{w}'_{t-1} \| + \varphi_3
\leq \delta_t + \mu \delta_{t-1} + \| G_t(\mathbf{w}_t) - G'_t(\mathbf{w}') \|
\leq \delta_t + \mu \delta_{t-1} + \| \mathbf{w}'_t - G'_t(\mathbf{w}') \| + \| \mathbf{w}'_t - G'_t(\mathbf{w}') \|
\leq \delta_t + \mu \delta_{t-1} + 2\alpha L
\]

where \( \varphi_3 = \| G_t(\mathbf{w}_t) + G_t(\mathbf{w}'_t) - G_t(\mathbf{w}_t) - G'_t(\mathbf{w}') \| \). The last inequality in (49) holds due to the \( L \)-Lipschitz property. Combining (48) and (49), we have

\[
\mathbb{E}_A[\delta_{t+1}] \leq (1 - 1/n)(\vartheta \mathbb{E}_A[\delta_t] + \mu \mathbb{E}_A[\delta_{t-1}]) + 1/n(\vartheta \mathbb{E}_A[\delta_t] + \mu \mathbb{E}_A[\delta_{t-1}] + 2\alpha L)
\]

\[
= \vartheta \mathbb{E}_A[\delta_t] + \mu \mathbb{E}_A[\delta_{t-1}] + \frac{2\alpha L}{n}.
\]

Let us consider the recursion

\[
\mathbb{E}_A[\delta_{t+1}] = \vartheta \mathbb{E}_A[\delta_t] + \mu \mathbb{E}_A[\delta_{t-1}] + \frac{2\alpha L}{n}
\]

with \( \bar{\delta}_0 = \delta_0 = 0 \). Upon inspecting (51) it is clear that

\[
\mathbb{E}_A[\delta_t] \geq \vartheta \mathbb{E}_A[\delta_{t-1}], \quad \forall t \geq 1,
\]

as we simply drop the remainder of positive terms. Substituting (52) into (51), we have

\[
\mathbb{E}_A[\delta_{t+1}] \leq \left( 1 + \mu + \frac{\mu}{\vartheta} - \frac{\alpha \beta \gamma}{\beta + \gamma} \right) \mathbb{E}_A[\delta_t] + \frac{2\alpha L}{n}
\]

\[
\leq (\vartheta + 2\mu) \mathbb{E}_A[\delta_t] + \frac{2\alpha L}{n}
\]

(53)
where the second inequality holds due to \( \mu \geq \frac{\alpha \beta \gamma}{\beta + \gamma} - \frac{1}{2} \).

Noting that \( \mathbb{E}_A[\delta_t] \geq \mathbb{E}_A[\delta_t] \) for all \( t \) including \( T \), we have

\[
\mathbb{E}_A[\delta_T] \leq \frac{2\alpha L}{n} \sum_{t=1}^{T} (\beta + 2\mu) t \leq \frac{2\alpha L(\beta + \gamma)}{n(\alpha \beta \gamma - 3(\beta + \gamma))}
\]

where the second expression holds since \( 0 \leq \mu < \frac{\alpha \beta \gamma}{3(\beta + \gamma)} \).

Applying the \( L \)-Lipschitz property on \( f(\cdot; z) \), it follows

\[
\mathbb{E}_A[\|f(w_T; z) - f(w_T'; z)\|] \leq L \mathbb{E}_A[\delta_T] \\
\leq \frac{2\alpha L^2(\beta + \gamma)}{n(\alpha \beta \gamma - 3(\beta + \gamma))}.
\]

Since this bound holds for all \( S, S' \), and \( z \), we obtain an upper-bound on the uniform stability and the proof is complete.

Our stability bound in above holds for the projected SGDM update (5) because Euclidean projection onto a convex set does not increase the distance between projected points (Rockafellar, 1976). In particular, note that inequalities (48) and (49) still hold under the projected version of SGDM.

**L Convergence bound for strongly convex loss**

In this section, we develop an upper-bound on the optimization error, which is defined as

\[
\epsilon_{\text{opt}} \triangleq \mathbb{E}_{S,A}[R_S(\hat{w}_T) - R_S(w_S^*)]
\]

where \( \hat{w}_T \) denotes the average of \( T \) steps of the algorithm, i.e., \( \hat{w}_T = \frac{1}{T+1} \sum_{t=0}^{T} w_t, R_S(w) = \frac{1}{n} \sum_{i=1}^{n} f(w; z_i) \), and \( w_S^* = \arg \min_w R_S(w) \).

The optimization error quantifies the gap between the empirical risk of SGDM and the optimal empirical risk.

**Theorem 11.** Suppose that the projected SGDM update (5) is executed for \( T \) steps with constant learning rate \( \alpha \) and momentum \( \mu \). Then we have\(^5\)

\[
\epsilon_{\text{opt}} \leq \frac{\mu W_0}{(1 - \mu)T} + \frac{(1 - \mu)W_1}{2\alpha T} - \frac{\gamma W_2}{2} - \frac{\mu \gamma W_3}{2(1 - \mu)} + \frac{\alpha L^2}{2(1 - \mu)}
\]

where \( W_0 = \mathbb{E}_{S,A}[R_S(w_0) - R_S(w_T)], W_1 = \mathbb{E}_{S,A}[\|w_0 - w_S^*\|^2], W_2 = \mathbb{E}_{S,A}[\|\hat{w}_T - w_S^*\|^2], \) and \( W_3 = \frac{1}{T+1} \sum_{t=0}^{T} \mathbb{E}_{S,A}[\|w_t - w_{t-1}\|^2] \).

**Proof.** Again, we first consider SGDM without projection and discuss the extension to projection at the end of this proof. To facilitate the convergence analysis, we define:

\[
p_t \triangleq \frac{\mu}{1 - \mu} (w_t - w_{t-1})
\]

\(^5\)Linear convergence results for SGD can be obtained under a stringent condition (Needell et al., 2014). Such a condition requires that the loss function is simultaneously minimized on each training example, and it does not apply to our setting. Different from (Yang et al., 2016; Ghadimi et al., 2015), we analyze the convergence of projected SGDM for a smooth and strongly convex loss function with constant learning rate.
with $p_0 = 0$. Substituting into the SGDM update, we have

$$\|s_{t+1} - \tilde{w}\|_2^2 = \|s_t - w\|_2^2 + \left(\frac{\alpha}{1 - \mu}\right)^2 \|\nabla_w f(w_t; z_{it})\|_2^2 - \frac{2\alpha}{1 - \mu} (s_t - w)^T \nabla_w f(w_t; z_{it})$$  \hspace{1cm} (58)

where $s_t = w_t + p_t$. Substituting $s_t$, taking the expectation w.r.t. $i_t$, using the $L$-Lipschitz assumption, noting $R_S$ is a $\gamma$-strongly convex function, summing for $t = 0, \cdots, T$, and rearranging terms, we have

$$\varsigma_T \leq \frac{2\alpha\mu}{(1 - \mu)^2} \mathbb{E}_s[R_S(w_0) - R_S(w_T)] + \mathbb{E}[\|w_0 - w\|_2^2] - \frac{\alpha\gamma}{1 - \mu} \sum_{t=0}^{T} \mathbb{E}_A[\|w_t - w\|_2^2]$$

$$+ \frac{2\alpha^2 L^2(T + 1)}{(1 - \mu)^2} - \frac{\alpha\mu\gamma}{(1 - \mu)^2} \sum_{t=0}^{T} \mathbb{E}_A[\|w_t - w_{t-1}\|_2^2]$$  \hspace{1cm} (59)

where $\varsigma_T = \frac{2\alpha}{1 - \mu} \sum_{t=0}^{T} \mathbb{E}_A[R_S(w_t) - R_S(w)]$. Since $\|\cdot\|_2^2$ is a convex function, we have $\|\tilde{w}_T - w\|_2^2 \leq \frac{1}{T+1} \sum_{t=0}^{T} \|w_t - w\|_2^2$ for all $w_T$ and $w$. Furthermore, due to the convexity of $R_S$, we have

$$R_S(\tilde{w}_T) - R_S(w) \leq \frac{1}{T+1} \sum_{t=0}^{T} (R_S(w_t) - R_S(w)).$$  \hspace{1cm} (60)

Taking expectation over $S$, applying the above inequalities, and substituting $w = w_S^*$, we obtain (56).

Our convergence bound in (56) can be extended to projected SGDM (5). Let use denote

$$y_{t+1} \overset{\Delta}{=} w_t + \mu (w_t - w_{t-1}) - \alpha \nabla_w f(w_t; z_{it}).$$

Then, for any feasible $w \in \Omega$, (58) holds for $y_{t+1}$, i.e.,

$$\|\tilde{y}_{t+1} - w\|_2^2 = \|s_t - w\|_2^2 + \left(\frac{\alpha}{1 - \mu}\right)^2 \|\nabla_w f(w_t; z_{it})\|_2^2$$

$$- \frac{2\alpha}{1 - \mu} (s_t - w)^T \nabla_w f(w_t; z_{it})$$  \hspace{1cm} (61)

where $\tilde{y}_t = y_t + \frac{\mu}{1 - \mu} (y_t - w_{t-1})$.

Note that the LHS of (61) can be written as

$$\|\tilde{y}_{t+1} - w\|_2^2 = \frac{1}{(1 - \mu)^2} \|y_{t+1} - (\mu w_t + (1 - \mu)w)\|_2^2.$$  \hspace{1cm} (62)

We note that $\tilde{w}_t \overset{\Delta}{=} \mu w_t + (1 - \mu)w \in \Omega$ for any $w \in \Omega$ and $w_t \in \Omega$ since $\Omega$ is convex.

Now in projected SGDM, we have

$$\|w_{t+1} - \tilde{w}_t\|_2^2 = \|P(y_{t+1}) - \tilde{w}_t\|_2^2$$

$$\leq \|y_{t+1} - \tilde{w}_t\|_2^2$$

since projection a point onto $\Omega$ moves it closer to any point in $\Omega$. This shows inequality (59) holds, and the convergence results do not change. \hfill \Box
Theorem 11 bounds the optimization error, i.e., the expected difference between the empirical risk achieved by SGDM and the global minimum. Upon setting $\mu = 0$ and $\gamma = 0$ in (56), we can recover the classical bound on optimization error for SGD (Nemirovski and Yudin., 1983), (Hardt et al., 2016, Theorem 5.2). The first two terms in (56) vanish as $T$ increases. The terms with negative sign improve the convergence due to the strongly convexity. The last term depends on the learning rate, $\alpha$, the momentum parameter $\mu$, and the Lipschitz constant $L$. This term can be reduced by selecting $\alpha$ sufficiently small.

M Upper-bound on true risk

Our ultimate goal is to minimize the true risk (1). We now study how the uniform stability results in an upper-bound on the true risk in the strongly convex case. We also compare the final results with SGD with no momentum and we show that one can achieve tighter bounds by using SGDM than simple SGD without momentum.

The expected true risk estimate under parameter $\hat{w}_T$ can be decomposed into a stability error term and an optimization one. In Appendix L, we present an upper-bound on the empirical risk achieved by SGDM and the global minimum. Upon setting $\mu = 0$ and $\gamma = 0$, we consider the average parameter $\hat{w}_T$ instead of $w_T$. In other words, the same upper-bound holds even if $\hat{w}_T$ is considered as the output of algorithm A.

Lemma 3. Suppose that the SGDM update is executed for $T$ steps with learning rate $\alpha$ and momentum $\mu$. Provided that $\frac{3\alpha^2}{\beta+\gamma} - \frac{1}{2} \leq \mu < \frac{\alpha\beta\gamma}{n(\beta+\gamma)}$ and $\alpha \leq \frac{2}{\beta+\gamma}$, then the average of the first $T$ steps of SGDM satisfies $\epsilon_s$-uniform stability with (10).

Proof. Let us define $\hat{w}_t = \frac{1}{T} \sum_{k=1}^{t} w_k$ and $\hat{\delta}_k \triangleq \|\hat{w}_t - \hat{w}_t\|$ where $\hat{w}_t$ is obtained as specified in the proof of Theorem 7. Following the proof of Theorem 7, we have

$$E[\hat{\delta}_{k+1}] \leq \left(1 + 3\mu - \frac{\alpha\beta\gamma}{\beta+\gamma}\right)E[\hat{\delta}_k] + \frac{2\alpha L}{n}$$

(63)

for $k = 0, \cdots, T$. Defining $\hat{\delta}_t \triangleq \sum_{k=1}^{t} \hat{\delta}_k$, we have $\hat{\delta}_T \leq \hat{\delta}_t$ by the triangle inequality. Summing (63) for $k = 0, \cdots, T$ and dividing by $T$, we have $E[\hat{\delta}_T] \leq E[\hat{\delta}_T] \leq \frac{2\alpha L(\beta+\gamma)}{n(\alpha\beta\gamma - 3\mu(\beta+\gamma))}.

Applying the $L$-Lipschitz property on $f(\cdot, z)$, we have

$$E[|f(\hat{w}_T; z) - f(\hat{w}_T; z)|] \leq \frac{2\alpha L^2(\beta+\gamma)}{n(\alpha\beta\gamma - 3\mu(\beta+\gamma))},$$

(64)

which holds for all $S, S'$, and $z$.

Adding the stability error following Lemma 3, we have

$$E_S[A[R(\hat{w}_T)] \leq E_S[A[R_S(\hat{w}_T)] + \epsilon_s \leq E_S[A[R_S(w_S)]] + \epsilon_{opt} + \epsilon_s$$

(65)

where $\epsilon_{opt} \triangleq E_S[A[R_S(w_T)] - R_S(w_S)].$

Note that there is a tradeoff between the optimization error and stability one. We can balance these errors to achieve reasonable expected true risk.
**Theorem 12.** Suppose that the SGDM update (4) is executed for \( T \) steps with constant learning rate \( \alpha \) and momentum \( \mu \), satisfying the conditions in Theorem 7. Then, for sufficiently small \( \mu \) and setting \( \alpha = \frac{1 - \mu}{L} \sqrt{\frac{W_1}{T}} \), we have:

\[
E_S, A[R(\hat{w}_T)] \leq E_S, A[R_S(w^*_S)] + \frac{\mu W_0}{(1 - \mu)T} + L \sqrt{\frac{W_1}{T}} - \frac{\mu_2}{2(1 - \mu)} + \frac{\mu_3}{2(1 - \mu)} - \frac{\alpha L^2}{2(1 - \mu)} + \frac{2L^2(\beta + \gamma)}{n\beta\gamma C}.
\]

(66)

where \( C \triangleq 1 - \frac{3\mu L(\beta + \gamma)^2}{(1 - \mu)^2 \gamma \sqrt{W_1}} \) and \( W_0, \cdots, W_3 \) are defined in Appendix L.

**Proof.** By our convergence analysis in Theorem 11, we have

\[
\epsilon_{\text{opt}} \leq \frac{\mu W_0}{(1 - \mu)T} + \frac{(1 - \mu)W_1}{2\alpha T} - \frac{\gamma W_2}{2} \quad - \frac{\mu \gamma W_3}{2(1 - \mu)} + \frac{\alpha L^2}{2(1 - \mu)}.
\]

By our stability analysis in Lemma 3, we have

\[
\epsilon_s \leq \frac{2\alpha L^2(\beta + \gamma)}{n(\alpha \beta \gamma - 3\mu(\beta + \gamma))}.
\]

Adding the upper-bounds of \( \epsilon_{\text{opt}} \) and \( \epsilon_s \) above, we have

\[
E_S, A[R(\hat{w}_T)] \leq E_S, A[R_S(w^*_S)] + \frac{\mu W_0}{(1 - \mu)T} + \frac{(1 - \mu)W_1}{2\alpha T} - \frac{\gamma W_2}{2} \quad - \frac{\mu \gamma W_3}{2(1 - \mu)} + \frac{\alpha L^2}{2(1 - \mu)} + \frac{2\alpha L^2(\beta + \gamma)}{n(\alpha \beta \gamma - 3\mu(\beta + \gamma))}.
\]

(67)

Choosing sufficiently small \( \mu \) such that \( \alpha \beta \gamma \gg \mu(\beta + \gamma) \), we have \( C \approx 1 \). Then we can optimize the upper-bound in (67) by setting

\[
\alpha = \frac{1 - \mu}{L} \sqrt{\frac{W_1}{T}}.
\]

(68)

Substituting (68) into (67), we obtain the upper-bound in Theorem 12. \( \Box \)

**Theorem 12** provides a bound on the expected true risk of SGDM in terms of the global minimum of the empirical risk. The bound in (66) is obtained by combining Theorem 7 and Theorem 11 and minimizing the expression over \( \alpha \). The choice of \( \alpha \) simplifies considerably when \( \mu \) is sufficiently small, as stated in Theorem 12. Note that the first two terms in (66) vanish as \( T \) increases. The last term in (66) vanishes as the number of samples \( n \) increases.

**N Additional experiments**

In Figure 4, we plot the test accuracy versus \( t_d \) of SGDEM and SGDM (which is a special case of SGDEM with \( t_d = T \)) for the notMNIST dataset for different \( \mu_d \) values. We observe dramatic decrease in the test accuracy for \( \mu_d = 0.99 \), which is consistent with our convergence analysis in Theorem 4.
We now study the performance of SGDEM for a smooth and strongly convex loss function. We train a logistic regression model with the weight decay regularization using SGDEM for binary classification on the two-class notMNIST and MNIST datasets that contain the images from letter classes “C” and “J”, and digit classes “2” and “9”, respectively. We set the learning rate $\alpha = 0.01$. The weight decay coefficient and the minibatch size are set to 0.001 and 10, respectively. We use 100 SGDEM realizations to evaluate the average performance.

We plot the test error and test accuracy versus $t_d$ under SGDEM for the notMNIST dataset in Figures 5 and 6, respectively. We show the same performance measures for the MNIST dataset in Figures 7 and 8 respectively. We observe that, unlike the case of non-convex loss functions, it does not hurt to add momentum for the entire training. In the following, we focus on SGDM with the classical momentum update rule for a smooth and strongly convex loss function for the notMNIST dataset.

In the following, we focus on SGDM with the classical momentum update rule for a smooth and strongly convex loss function on notMNIST.

We compare the training and generalization performance of SGD without momentum with that of SGDM under $\mu = 0.5$ and $\mu = 0.9$, which are common momentum values used in practice (Goodfellow et al., 2016, Section 8.3.2).

We show in Figure 9 the generalization error (w.r.t. cross entropy) versus the number of training samples, $n$, under SGDM with fixed $T = 1000$ iterations for $\mu = 0, 0.5, 0.9$. In Figure 10, we plot the training accuracy as a function of the number of training samples for the same dataset. First, we observe that the generalization error decreases as $n$ increases for all values of $\mu$, which is also suggested by our stability upper-bound in Theorem 7. In addition, for sufficiently large $n$, we observe that the generalization error increases with $\mu$, consistent with Theorem 7. The training accuracy also improves by adding momentum as illustrated in Figure 10.

In order to study the optimization error of SGDM, we show in Figures 11 and 12, the training error and test error, respectively, versus the number of epochs, under SGDM trained with $n = 500$ samples. We plot the classification accuracy for training dataset in Figure 13. We observe that the training error decreases as the number of epochs increases for all values of
\( \mu \), which is consistent with the convergence analysis in Theorem 11. Furthermore, as expected, we see that adding momentum improves the training error and accuracy. However, as the number of epochs increases, we note that the benefit of momentum on the test error becomes negligible. This happens because adding momentum also results in a higher generalization error thus offsetting the gain in training error.

**Details of ImageNet experiments.** We train ResNet-18 on ImageNet in a distributed setting with 4 GPUs under tuned learning rate. The global minibatch size and weight decay are set to 128 and \( 5 \times 10^{-5} \), respectively. For each \( t_d \), the momentum is set to \( \mu_d = 0.9 \) in the first \( t_d \) epochs and then zero for the next \( 90 - t_d \) epochs. In Figure 14, we plot validation accuracy and generalization gap of SGDEM at epoch 90. Similar to the loss results, we observe that the minimum generalization error happens if the momentum is applied for 50 epochs. We use a cluster with 4 NVIDIA 2080 Ti GPUs with the following CPU details: Intel(R) Xeon(R) CPU E5-2650 v4 @ 2.20GHz; 48 cores; GPU2GPU bandwidth: unidirectional 10GB/s and bidirectional 15GB/s.
Figure 6: Test accuracy of logistic regression for notMNIST dataset.

Figure 7: Test error of logistic regression for MNIST dataset.
**Figure 8:** Test accuracy of logistic regression for MNIST dataset.

**Figure 9:** Generalization error (cross entropy) of logistic regression for notMNIST dataset with $T = 1000$ iterations.
Figure 10: Training accuracy of logistic regression for notMNIST dataset with $T = 1000$ iterations.

Figure 11: Training error (cross entropy) of logistic regression for notMNIST dataset with $n = 500$. 

\[ \text{train error} = 0.9 \quad \mu = 0.9 \]
\[ \mu = 0.5 \]
\[ \mu = 0 \]
Figure 12: Test error (cross entropy) of logistic regression for notMNIST dataset with \( n = 500 \).

Figure 13: Training accuracy of logistic regression for notMNIST dataset with \( n = 500 \).
Figure 14: Validation accuracy and generalization gap of SGDEM when training ResNet-18 on ImageNet in a distributed setting with 4 GPUs under tuned learning rate and global minibatch size of 128. For each $t_d$, the momentum is set to $\mu_d = 0.9$ in the first $t_d$ epochs and then zero for the next $90 - t_d$ epochs. SGDM is a special form of SGDEM with $t_d = 90$. 