ON ISOTROPIC BERWALD SCALAR CURVATURE

MING LI

ABSTRACT. In this short paper, we establish a closer relation between the Berwald scalar curvature and the S-curvature. In fact, we prove that a Finsler metric has isotropic Berwald scalar curvature if and only if it has weakly isotropic S-curvature. For Finsler metrics of scalar flag curvature and of weakly isotropic S-curvature, they have almost isotropic S-curvature if and only if the flag curvature is weakly isotropic.

INTRODUCTION

Let \((M, F)\) be an \(n\)-dimensional Finsler manifold. The unit sphere bundle (or indicatrix bundle) is defined as \(SM = \{(x, y) \in TM | F(x, y) = 1\}\) with the natural projection \(\pi : SM \rightarrow M\). It is well known that the tangent bundle \(T(SM)\) admits a natural horizontal subbundle \(H(SM)\) and a Sasaki-type metric \(g^{T(SM)}\). The Hilbert form \(\omega^n = F_{gy} dx^i\) defines a contact structure on \(SM\). The Reeb field is just the spray \(G\).

Let \(\tilde{P}\) be the Berwald curvature, which is the \(hv\)-part of the curvature form of the Berwald connection \(\nabla^B\) on \(H(SM)\). Let \(E = \text{tr} \tilde{P}\) be the mean Berwald curvature or the \(E\)-curvature. We introduce the Berwald scalar curvature as \(e = \text{tr} \tilde{P}\) in [9], which is a function on \(SM\) in general cases. If \(e = \pi^*c\) for some \(c \in C^\infty(M)\), then \(F\) is said to have isotropic Berwald scalar curvature. The aim of this paper is to establish certain relations between the Berwald scalar curvature and the S-curvature of a Finsler manifold.

**Theorem 1.** Let \((M, F)\) be an \(n\)-dimensional Finsler manifold. Then \(F\) has isotropic Berwald scalar curvature \(e\) if and only if \(F\) has weakly isotropic \(S\)-curvature, and

\[
S = \frac{1}{n-1} e + \pi^* \xi(G),
\]

where \(\xi \in \Omega^1(M)\). In both cases the mean Berwald curvature \(E\) is isotropic. Moreover, the following equation holds

\[
\text{tr} [\tilde{R}] = \pi^* d\xi + \frac{1}{n-1} de \wedge \omega^n,
\]

where \(\tilde{R}\) is the \(hh\)-part of the Berwald curvature form.

A well known fact in Finsler geometry is that vanishing \(S\)-curvature implies that \(E = 0\). Whether or not the converse is true is a natural question (c.f. [14], p.67). We show that the converse is true under certain conditions.

---

1 Partially supported by NSFC (Grant No. 11871126).
Theorem 2. Let \((M, F)\) be an \(n\)-dimensional Finsler manifold. If the Berwald scalar curvature is constant and the mean stretch curvature \(\Sigma = 0\), then the \(S\)-curvature of \(F\) satisfies (0.1) with \(d\xi = 0\), i.e., \(S\) is almost constant.

The following corollary can be obtained from Theorem 2 immediately.

Corollary 1. Let \(M\) be a smooth manifold satisfies \(H_{1}^{1}(M; R) = 0\). If a Finsler structure \(F\) on \(M\) satisfies \(J = 0\) and \(e = 0\), then \(S = 0\).

For weakly Landsberg manifold, we have the following relations among the Berwald scalar curvature, \(S\)-curvature and the Cartan type 1-form.

Corollary 2. Let \((M, F)\) be an \(n\)-dimensional Finsler manifold with vanishing mean Landsberg curvature \(J = 0\). Then the following statements are equivalent:

1. \(F\) has constant Berwald scalar curvature \(e\);
2. \(F\) has almost constant \(S\)-curvature;
3. The Cartan type 1-form \(d\eta = cd\omega^n\) for certain constant \(c\).

To study the Finsler metrics of scalar flag curvature, i.e., \(K = K(x, y)\), is an important topic in Finsler geometry. Many authors contribute on this subject. One refers [4, 5, 14, 16, 17] and finds the references therein. Applying Theorem 1 to the Finsler metrics of scalar flag curvature and of weakly isotropic \(S\)-curvature, we obtain the following result.

Corollary 3. Let \((M, F)\) be a \(n\)-dimensional Finsler manifold of scalar flag curvature and weakly isotropic \(S\)-curvature. Then \(F\) has almost isotropic \(S\)-curvature if and only if the flag curvature is weakly isotropic and satisfies

\[
(0.3) \quad K = \frac{3}{n^2 - 1}(\pi^*dc)(G) + \pi^*\sigma,
\]

where the \(c\) and \(\sigma\) are function on \(M\), and \(c\) is the Berwald scalar curvature \(e\) up to a constant.

The necessary part of Corollary 3 first appeared in [3] and plays an important role in the study of Finsler geometry.

In this paper we adopt the index range and notations:

\[1 \leq i, j, k, \ldots \leq n, \quad 1 \leq \alpha, \beta, \gamma, \ldots \leq n - 1, \quad 1 \leq A, B, C, \ldots \leq 2n - 1, \]

and \(\bar{\alpha} := n + \alpha, \bar{\beta} := n + \beta, \bar{\gamma} := n + \gamma, \ldots\).

Acknowledgements. The author would like to thanks Professor Huitao Feng for his consistent support and encouragement.

1. Preliminary results

We will introduce the necessary definitions and background of Finsler geometry in this section (c.f. [1] [5] [10] [14] [16] [17] for more details).

Let \((M, F)\) be an \(n\)-dimensional Finsler manifold. Let \(SM = \{F(x, y) = 1\}\) be the unit sphere bundle (or indicatrix bundle) with the natural projection \(\pi : SM \to M\).
Let \( \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n-1}\} \) be a local adapted orthonormal frame with respect to the Sasaki-type Riemannian metric
\[
g^{T(SM)} = \sum_i \omega^i \otimes \omega^i + \sum_{\bar{\alpha}} \omega^{\bar{\alpha}} \otimes \omega^{\bar{\alpha}} =: g + \dot{g}
\]
on \( T(SM) \), where \( e_n = G \) is the Reeb vector field. Let \( \theta = \{\omega^1, \ldots, \omega^n, \omega^{n+1}, \ldots, \omega^{2n-1}\} \) be the dual frame, then \( \omega^n = F_y dx^i \) is the Hilbert form. \( \omega^n \) defines a contact structure on \( SM \). Let \( I \) be the trivial bundle generated by \( G \). Let \( D \) be the contact distribution \( \{\omega^n = 0\} \). We define an almost complex structure on \( D \) by
\[
J = -\omega^{\bar{\alpha}} \otimes e_{\bar{\alpha}} + \omega^{\bar{\alpha}} \otimes e_{\bar{\alpha}}.
\]
We will extend \( J \) to be an endomorphism \( T(SM) \) by defining \( J(e_n) = 0 \). According to the definition of a contact metric structure in \( [2] \), one only need to verify that \( (SM, J, G, \omega, g^{T(SM)}) \) gives rise a contact metric structure.

Let \( \mathcal{F} := \mathcal{V}(SM) \) be the integrable distribution given by the tangent spaces of the fibers of \( SM \). Let \( p : T(SM) \to \mathcal{F} \) be the natural projection. Then the tangent bundle \( T(SM) \) admits a splitting
\[
T(SM) = I \oplus J\mathcal{F} \oplus \mathcal{F} =: H(SM) \oplus \mathcal{F}.
\]
The horizontal subbundle \( H(SM) = I \oplus J\mathcal{F} \) is spanned by \( \{e_1, \ldots, e_n\} \) on which the Chern connection is defined. And \( \{e_{n+1}, \ldots, e_{2n-1}\} \) gives a local frame of \( \mathcal{F} \).

Let \( \nabla^{Ch} : \Gamma(H(SM)) \to \Omega^1(SM; H(SM)) \) be the Chern connection, which can be extended to a map
\[
\nabla^{Ch} : \Omega^*(SM; H(SM)) \to \Omega^{*+1}(SM; H(SM)),
\]
where \( \Omega^*(SM; H(SM)) := \Gamma(\Lambda^*(T^*SM) \otimes H(SM)) \) denotes the horizontal valued differential forms on \( SM \). It is well known that the symmetrization of Chern connection \( \tilde{\nabla}^{Ch} \) is the Cartan connection. The difference between \( \tilde{\nabla}^{Ch} \) and \( \nabla^{Ch} \) will be referred as the Cartan endomorphism,
\[
H = \tilde{\nabla}^{Ch} - \nabla^{Ch} \in \Omega^1(SM, \text{End}(H(SM))).
\]
Set \( H = H_{ij} \omega^j \otimes e_i \). By Lemma 3 and Lemma 4 in \( [3] \), \( H_{ij} = H_{ji} = H_{ij} \gamma \omega^\gamma \) has the following form under natural coordinate systems
\[
H_{ij} \gamma = -A_{ijk} u^i_k u^j_\gamma u^k_\gamma,
\]
where \( A_{ijk} = \frac{1}{4} F[F^2]_{y^i y^j y^k} \) and \( u^i_\gamma \) are the transformation matrix from adapted orthonormal frames to natural frames.

Set \( \eta = \text{tr}[H] \in \Omega^1(SM) \). It is referred as the Cartan-type form in \( [5] \). The Cartan-type form has the following local formula
\[
\eta = \sum_{i=1}^n H_{ii} \gamma \omega^\gamma =: H_{\gamma} \omega^\gamma.
\]
Let \( \omega = (\omega^j) \) be the connection matrix of the Chern connection with respect to the local adapted orthonormal frame field, i.e.,
\[
\nabla^{Ch} e_i = \omega^j_i \otimes e_j.
\]
Lemma 1 (\[1, 5, 10, 14\]). The connection matrix $\omega = (\omega^i_j)$ of $\nabla^{\text{Ch}}$ is determined by the following structure equations,

$$
\begin{align*}
  d\vartheta &= -\omega \wedge \vartheta, \\
  \omega + \omega^t &= -2H,
\end{align*}
$$

where $\vartheta = (\omega^1, \ldots, \omega^n)^t$. Furthermore,

$$
\omega^a_n = -\omega^a_n = \omega^a, \quad \text{and} \quad \omega^n_n = 0.
$$

Remark 1. In \[6\], we proved that the Chern connection is just the Bott connection on $H(SM)$ in the theory of foliation (c.f. \[19\]).

Let $R^{\text{Ch}} = (\nabla^{\text{Ch}})^2$ be the curvature of $\nabla^{\text{Ch}}$. Let $\Omega = (\Omega^i_j)$ be the curvature forms of $R^{\text{Ch}}$. From the torsion freeness, the curvature form has no pure vertical differential form

$$
\Omega^i_j := d\omega^i_j - \omega^k_j \wedge \omega^i_k = \frac{1}{2} R^i_{jk} \omega^k \wedge \omega^i + P^i_{jk} \omega^k \wedge \omega^j.
$$

The Landsberg curvature is defined as

$$
L := L^i_{j \gamma} \omega^\bar{\gamma} \otimes \omega^j \otimes e_i = -P^i_{j n} \omega^\bar{\gamma} \otimes \omega^j \otimes e_i,
$$

the mean Landsberg curvature is defined by

$$
J = \text{tr} L = J^i_{\gamma} \omega^\bar{\gamma} = -P^i_{j n} \omega^\bar{\gamma}.
$$

The following formula is well known (c.f. \[1, 5, 10, 14\])

$$
P^i_{j k \gamma} = -H^i_{k \gamma} = -L^i_{j k \gamma}.
$$

If a Finsler manifold satisfies $P = 0$, $L = 0$ or $J = 0$, then it is called a Berwald, Landsberg or weak Landsberg manifold, respectively.

We will need parts of the Riemannian geometry of the sphere bundle, especially the geometry of the fibres. The following lemma can be found in \[10\].

Lemma 2. Let $\Theta = (\Theta^A_B)$ be the connection form of the Levi-Civita connection $\nabla^{T(SM)}$ of $g^{T(SM)}$ with respect to the adapted frame $\{e_A\}$. Then we have

$$
\Theta^\gamma_\beta = \omega^\gamma_\beta + H_{\beta \gamma \alpha} \omega^\alpha,
$$

where $\omega^\gamma_\beta$ are the connection forms of the Chern connection and $H_{\beta \gamma \alpha} \omega^\alpha$ are the coefficients of the Cartan endomorphism.

As the restriction the Levi-Civita connection $\nabla^{T(SM)}$ on $\mathcal{F}$, $\nabla^{\bar{\gamma}} := p \nabla^{T(SM)} p$ is the Euclidean connection of the bundle $(\mathcal{F}, \bar{g})$. It is clear that along each fiber of $S_x M$, $x \in M$, $\nabla^{\bar{\gamma}}$ is just the Levi-Civita connection of the Riemannian manifold $(S_x M, \bar{g}_x)$. We also define a connection $\nabla^{\text{Ch}}$ on $\mathcal{F}$ by

$$
\nabla^{\text{Ch}} e_\beta = \omega^\alpha_\beta \otimes e_\alpha.
$$
We introduce some symbols to denote different covariant differentials for conveniences. For example, let $T = T^i_j e_i \otimes \omega^j$ be an arbitrary smooth local section of the bundle $H(SM) \otimes H^*(SM)$ over $SM$. Then

$$\nabla^{Ch} T = (dT^i_j + T^k_j \omega^i_k - T^i_k \omega^j_k) \otimes e_i \otimes \omega^j$$

If we expand the coefficients as one forms on $SM$ in terms of the adapted coframe, then we denote

$$\nabla^{Ch} T^i_j := dT^i_j + T^k_j \omega^i_k - T^i_k \omega^j_k =: T^i_j_\alpha \omega^\alpha + T^i_j n \omega^n + T^i_j \gamma \omega^\gamma.$$ 

Therefore we obtain

$$T^i_j_\alpha = i_{e_i} (\nabla^{Ch} T^i_j)_j, \quad T^i_j n = i_{e_i} (\nabla^{Ch} T^i_j)_j, \quad T^i_j \gamma = i_{e_i} (\nabla^{Ch} T^i_j)_j,$$

where $i_v$ is the notation for the interior multiplication on differential forms by any vector $v$.

According to the splitting (1.1) and using the almost complex structure $J$, we obtain a section of $\mathcal{F} \otimes \mathcal{F}^*$ from $T$ as following

$$\bar{T} = T^\alpha_\beta e_\alpha \otimes \omega^\beta.$$ 

By (1.6), the covariant differential of $\bar{T}$ by using $\nabla^{Ch}$ is given by

$$\nabla^{Ch} \bar{T} = (dT^\alpha_\beta + T^\mu_\beta \omega^\alpha_\mu - T^\alpha_\mu \omega^\beta_\mu) \otimes e_\alpha \otimes \omega^\beta$$

Similarly, we denote that

$$\nabla^{Ch} \bar{T}^\alpha_\beta := dT^\alpha_\beta + T^\mu_\beta \omega^\alpha_\mu - T^\alpha_\mu \omega^\beta_\mu = T^\alpha_\beta n \omega^n + T^\alpha_\beta \gamma \omega^\gamma.$$ 

Therefore, by Lemma 1, one has the following relations

$$T^\alpha_\beta n_\alpha = i_{e_\alpha} (dT^\alpha_\beta + T^\mu_\beta \omega^\alpha_\mu - T^\alpha_\mu \omega^\beta_\mu) = i_{e_\alpha} (dT^\alpha_\beta + T^\mu_\beta \omega^\alpha_\mu - T^\alpha_\mu \omega^\beta_\mu) = T^\alpha_\beta n_\alpha$$

(1.7)

$$T^\alpha_\beta \gamma = i_{e_\gamma} (dT^\alpha_\beta + T^\mu_\beta \omega^\alpha_\mu - T^\alpha_\mu \omega^\beta_\mu) = i_{e_\gamma} (dT^\alpha_\beta + T^\mu_\beta \omega^\alpha_\mu - T^\alpha_\mu \omega^\beta_\mu) = i_{e_\gamma} (T^\alpha_\beta n \omega^n + T^\alpha_\beta \gamma \omega^\gamma).$$

It is obvious that if $T$ is a section $J\mathcal{F} \otimes (J\mathcal{F})^*$, then $T^\alpha_\beta \gamma = T^\alpha_\beta \gamma$. These relations will be used in the following study.

The Berwald connection can be represented as $\nabla^B = \nabla^{Ch} + J_\ast L$, where $J_\ast$ is dual of the almost complex structure $J$. Let $\tilde{\omega} = (\tilde{\omega}^i_j)$ denote the Berwald connection form, then

$$\tilde{\omega}^i_j = \omega^i_j - L^i_j_\alpha \omega^\alpha.$$ 

From the torsion freeness the Chern connection and (1.4), the Berwald connection is torsion free,

$$d\omega^i = \omega^j \wedge \tilde{\omega}^i_j.$$ 

Let $\tilde{\Omega}^i_j$ be the curvature forms of the Berwald connection. By the torsion freeness of the Berwald connection, we have

$$\tilde{\Omega}^i_j := d\tilde{\omega}^i_j = \tilde{\omega}^k_j \wedge \tilde{\omega}^i_k =: \frac{1}{2} \tilde{R}^i_j k_l \omega^k \wedge \omega^l + \tilde{P}^i_j k l \omega^k \wedge \omega^l.$$
where $\hat{R}^i_{jk} = -\hat{R}^i_{kj}$. By using Lemma \[1\] the curvature relations between the Chern connection and the Berwald connection are as following. The $hh$-curvatures satisfy

$$\hat{R}^\alpha_{\beta \gamma \mu} = R^\alpha_{\beta \gamma \mu} - (L^\alpha_{\beta \gamma |\mu} - L^\alpha_{\beta \mu |\gamma}) + (L^\alpha_{\beta |\mu \gamma} - L^\alpha_{\beta \mu |\gamma}),$$

$$\hat{R}^\alpha_{\beta \gamma n} = R^\alpha_{\beta \gamma n} + L^\alpha_{\beta \gamma |n}, \quad \hat{R}^\alpha_{n kl} = -\hat{R}^\alpha_{n kl} = R^\alpha_{n kl}.$$ (1.8)

The mean stretch cuvature

$$\Sigma := 2\text{tr}[\hat{R} - R]$$ (1.9)

is introduced and studied in \[11\].

The relations between the $hv$-curvatures of the Berwald connection and the Chern connection are given by

$$\hat{P}^\alpha_{\beta \gamma \mu} = P^\alpha_{\beta \gamma \mu} + L^\alpha_{\beta \gamma |\mu}, \quad \hat{P}^n_{\alpha \gamma \mu} = 2L^\alpha_{\alpha \gamma |\mu},$$

$$\hat{P}^\alpha_{\beta n \mu} = 0, \quad \hat{P}^n_{\beta k \gamma} = 0, \quad \hat{P}^\alpha_{n \gamma n} = 0.$$ (1.10)

The mean Berwald curvature or the $E$-curvature is defined by

$$E := \text{tr}\hat{P} := E^i_{\gamma \mu} \omega^\gamma \wedge \omega^\mu.$$ Under the local adapted frame, the coefficients of the $E$-curvature is represented as

$$E_{\gamma \mu} = \hat{P}^i_{i \gamma \mu} = \hat{P}^\alpha_{\alpha \gamma \mu} = P^\alpha_{\alpha \gamma \mu} + L^\alpha_{\alpha \gamma |\mu} = P^\alpha_{\alpha \gamma \mu} + J^\alpha_{\gamma |\mu} = P^\alpha_{\alpha |\gamma \mu} + J^\alpha_{\gamma ,\mu}.$$ (1.11)

We deliberately omit the factors $\frac{1}{2}$ and $F^{-1}$ in the usual definition of $E$-curvature as a tensor on $TM_0$ in literatures.

2. THE S-CURVATURE AND THE BERWALD SCALAR CURVATURE

In this section we develop the relations between the $S$-curvature and the Berwald scalar curvature $e$. We first state some necessary definitions.

Let $dV_M$ be any volume form of $M$. On a local coordinate chart $(U; x^i)$, $dV_M = \sigma(x) dx^1 \wedge \cdots \wedge dx^n$. The following important function on $SM$ is well defined,

$$\tau = \ln \frac{\sqrt{\det (g_{ij})}}{\sigma(x)}.$$ $\tau$ is called the distortion of $(M, F)$. The derivative of $\tau$ along the vector field $\mathbf{G}$ will be denoted by $S := \mathbf{G}(\tau)$. The function $S := F \cdot S$ defined on $TM_0$ is called the $S$-curvature. In this paper, we will also call $S$ the $S$-curvature for convenient. These two important concepts were first introduced by Zhongmin Shen in \[15\].

Remark 2. In the study \[8\], we show that the centro-affine differential geometric structure of the fibres of $SM$ naturally fit into the overall geometry of the fibre bundle $SM$. In affine geometry, the Tchebychev potential is the logarithm of the Radon-Nikodym derivative of the one induced by the affine metric and the measure induced by the embedding. One easily shows that the restriction of $-\tau$ on each fiber of $SM$ is just the Tchebychev potential of the fiber. Therefore the distortion can be naturally considered as the family version of the Tchebychev potential. One consults \[7, 12, 18\] for more details about the affine differential geometry.
In this paper, we will use the following definitions.

**Definition 1 (§14).** Let $(M, F)$ be an $n$-dimensional Finsler manifold.

(i) If the $S$-curvature satisfies

$$S = \frac{1}{n-1} \pi^* c + \pi^* \xi(G)$$

for some $c \in C^\infty(M)$ and $\xi \in \Omega^1(M)$, then $F$ is said to have weakly isotropic $S$-curvature;

(ii) If $S$ satisfies (2.1) and $d\xi = 0$, then it is called to be almost isotropic;

(iii) If $S$ satisfies (2.1) and $\xi = 0$, then it is called to be isotropic.

Let $d\tilde{V}_M = e^{-f}dV_M$ be any other volume form on $M$, then the related distortion $\tilde{\tau} = \tau + \pi^* f$, where $f \in C^\infty(M)$. Thus the $S$-curvature related to $d\tilde{V}_M$ is given by

$$\tilde{S} = S + \pi^* df(G).$$

Therefore, the $S$-curvatures of a Finsler manifold are determined up to the exact 1-forms on the base manifold $M$. If $H^1_{dR}(M; \mathbb{R}) = 0$, then the conditions almost isotropic $S$-curvature and isotropic $S$-curvature are equivalent.

Let $f \in C^\infty(SM)$ be any smooth function on $SM$, then $df$ admits the following decomposition with respect to (1.1),

$$df = f_\alpha \omega^\alpha + f_{|n}\omega^n + f_\alpha \omega^\alpha,$$

where we denote $f_{|i} := e_i(f)$, and $f_\alpha := e_\alpha(f)$.

**Lemma 3.** From the fact $d^2 f = 0$, one has the following identities,

$$f_{|\alpha|\beta} - f_{|\beta|\alpha} + f_{\gamma} R^n_{\alpha \beta} = 0,$$

$$f_{|\alpha|n} - f_{|n|\alpha} + f_{\gamma} R^n_{\alpha n} = 0,$$

$$f_{|\alpha,\beta} - f_{|n}\delta_{\alpha\beta} + f_{\gamma} P_n^\gamma_{\alpha \beta} - f_{|\beta|\alpha} = 0,$$

$$f_{|\alpha} + f_{|n,\alpha} - f_{|\alpha|n} = 0,$$

$$f_{|\alpha,\beta} - f_{|\beta,\alpha} = 0.$$

The following useful equations can be derived from the above identities

$$f_{|n}^{\alpha|\beta} + (f_{|n})_{|\alpha} = f_{|\alpha|n|} + f_{\gamma} R^n_{\alpha n},$$

$$f_{|n}^{\alpha,\beta} + (f_{|n})_{\delta_{\alpha\beta}} = (f_{\gamma} P_n^\gamma_{\alpha \beta} - f_{|\beta|\alpha}) + f_{|\alpha|n|\beta}.$$

**Proof.** By using the properties of the Chern connection and the notations in Section 1, we have

$$d(f_\alpha \omega^\alpha) = (d(f_\alpha) - f_{|\beta|\omega_\alpha^\beta}) \wedge \omega^\alpha - f_{|\beta} \omega^\alpha_n \wedge \omega^n$$

$$= f_{|\alpha|\beta} \omega^\alpha \wedge \omega^\alpha + f_{|\alpha|n} \omega^n \wedge \omega^\alpha + f_{|\alpha,\beta} \omega^\beta \wedge \omega^\alpha + f_{|\alpha} \omega^\alpha \wedge \omega^n$$

$$= \frac{1}{2} (f_{|\alpha|\beta} - f_{|\beta|\alpha}) \omega^\alpha \wedge \omega^\alpha - f_{|\alpha|n} \omega^n \wedge \omega^n - f_{|\alpha,\beta} \omega^\beta \wedge \omega^\alpha - f_{|\alpha} \omega^n \wedge \omega^\alpha,$$

$$d(f_{|n}\omega^n) = f_{|n}\omega^n \wedge \omega^n + f_{|n,\alpha} \omega^\alpha \wedge \omega^n + f_{|n}\delta_{\alpha\beta} \omega^\alpha \wedge \omega^\beta$$

$$= f_{|n}\omega^n \wedge \omega^n + f_{|n,\alpha} \omega^\alpha \wedge \omega^n + f_{|n}\delta_{\alpha\beta} \omega^\alpha \wedge \omega^\beta.$$
and
\[ d(f_\alpha \omega^\alpha) = d(f_\alpha) \wedge \omega^\alpha + f_\alpha (-\Omega^n_\alpha - \omega^n \wedge \omega_\beta) \]
\[ = (d(f_\alpha) - f_\beta \omega_\alpha^\beta) \wedge \omega^\alpha - f_\alpha \Omega^n_\alpha \]
\[ = f_{\alpha | \alpha} \omega^\alpha \wedge \omega^\alpha + f_{\alpha n} \omega^n \wedge \omega^\alpha + f_{\alpha \beta} \omega^\beta \wedge \omega^\alpha \]
(2.13)
\[ - \frac{1}{2} f_\alpha R^n_{\alpha \beta \gamma} \omega^\beta \wedge \omega_\gamma - f_\alpha R^n_{\alpha \beta n} \omega^n \wedge \omega^n - f_\alpha P^n_{\alpha \beta \gamma} \omega^\beta \wedge \omega_\gamma \]
\[ = - \frac{1}{2} f_\alpha R^n_{\alpha \beta \gamma} \omega^\beta \wedge \omega_\gamma - f_\alpha R^n_{\alpha \beta n} \omega^n \wedge \omega^n + (f_\gamma R^n_{\alpha \beta n} - f_\alpha P^n_{\alpha \beta \gamma}) \omega^\beta \wedge \omega_\gamma \]
\[ + f_{\alpha n} \omega^n \wedge \omega^\alpha - \frac{1}{2} (f_{\alpha \beta} - f_{\beta \alpha}) \omega_\alpha \wedge \omega^\beta \]
By taking the sum of (2.11), (2.12) and (2.13), we obtain
(2.14)
\[ 0 = - \frac{1}{2} (f_{\alpha | \beta} - f_{\beta | \alpha}) \omega^\alpha \wedge \omega^\beta - f_{\alpha n} \omega^n \wedge \omega^n - f_{\alpha \beta} \omega^n \wedge \omega^n - f_\alpha \omega^n \wedge \omega^\alpha \]
\[ + f_{\alpha n} \omega^n \wedge \omega^\alpha + f_{\alpha n} \delta_{\alpha \beta} \omega^\alpha \wedge \omega^\beta - f_{\alpha n} \omega^n \wedge \omega^\alpha \]
\[ - \frac{1}{2} f_\alpha R^n_{\alpha \beta \gamma} \omega^\beta \wedge \omega_\gamma - f_\alpha R^n_{\alpha \beta n} \omega^n \wedge \omega^n + (f_\gamma R^n_{\alpha \beta n} - f_\alpha P^n_{\alpha \beta \gamma}) \omega^\beta \wedge \omega_\gamma \]
\[ + f_{\alpha n} \omega^n \wedge \omega^\alpha - \frac{1}{2} (f_{\alpha \beta} - f_{\beta \alpha}) \omega_\alpha \wedge \omega^\beta \]
\[ = - \frac{1}{2} (f_{\alpha | \beta} - f_{\beta | \alpha}) \omega^\alpha \wedge \omega^\beta - (f_{\alpha n} - f_{\alpha | n} + f_\gamma R^n_{\alpha \beta n}) \omega^\alpha \wedge \omega^n \]
\[ - (f_{\alpha \beta} - f_{\beta \alpha}) \omega^n \wedge \omega^\alpha - f_\alpha \omega^n \wedge \omega^\alpha \]
\[ - \frac{1}{2} (f_{\alpha \beta} - f_{\beta \alpha}) \omega^\alpha \wedge \omega^\beta. \]
Therefore, the coefficients in (2.14) with respect to the local adapted bases of $\Omega^2(SM)$ vanish. The identities (2.7)-(2.8) are proved in this way.

Taking covariant derivative of (2.7) along $e_n$ and substituting in (2.5) yields (2.9). Taking covariant derivative of (2.7) along $e_\alpha$ and substituting in (2.6) yields (2.10).

Applying Lemma 3 to the distortion $\tau$ yields the following lemma.

Lemma 4. The differential of $\tau$ is given by
(2.15)
\[ d\tau = \tau_i \omega^i + \eta = \tau_\alpha \omega^\alpha + S \omega^n + \eta. \]
The following identities are valid
(2.16) $\tau_{\alpha | \beta} - \tau_{\beta | \alpha} + R^n_{i \alpha \beta} = 0$,
(2.17) $S_\alpha - \tau_{\alpha | n} - R^n_{i \alpha n} = 0$,
(2.18) $\tau_{\gamma | \mu} - S \delta_{\gamma \mu} + P^n_{i \gamma \mu} = 0$,
(2.19) $S_\mu + \tau_{\mu} + P^n_{i \mu} = 0$. 

□
Lemma 5. For any
\begin{equation}
S|_{\alpha} - S.n|_{\alpha} = J_{\alpha}|_{n} + R_{i\alpha n},
\end{equation}
and
\begin{equation}
S_{\gamma\mu} + S\delta_{\gamma\mu} = E_{\gamma\mu}.
\end{equation}

Proof. These identities \((2.16)-(2.21)\) are direct consequences of Lemma \(3\) by using \((2.15), (1.11)\) and some well known Bianchi identities or the following fact
\begin{equation}
d\eta = d(H_{\gamma}\omega^{\gamma}) = -\frac{1}{2} R_{i\ k\ l\ \omega}^{\gamma} \omega^{k} \wedge \omega^{l} - P_{i\ k\ l\ \omega}^{\gamma} \omega^{k} \wedge \omega^{l}.
\end{equation}

Moreover, one has
\begin{equation}
\pi(n,\alpha) = 1 = \pi(n,\alpha).
\end{equation}

Thus the facts \((2.27)-(2.29)\) yield
\begin{equation}
\pi^\ast \xi = \xi_{i} u_{i}^{\alpha} \omega^{\alpha} + \frac{\xi_{i} y_{i}}{F} \omega^{n} =: \xi_{\alpha} \omega^{\alpha} + \xi_{n} \omega^{n}.
\end{equation}

The following lemma gives a representation of \(\pi^\ast d\xi\) on \(SM\).

Lemma 5. For any \(\xi \in \Omega^{1}(M)\), one can express \(\pi^\ast d\xi\) in the following form
\begin{equation}
d\pi^\ast \xi = -\frac{1}{2} (\xi_{i|\beta} - \xi_{\beta|\alpha}) \omega^{\alpha} \wedge \omega^{\beta} - (\xi_{i|n} - \xi_{n|\alpha}) \omega^{\alpha} \wedge \omega^{n}.
\end{equation}

Moreover, one has
\begin{equation}
\xi_{\alpha\beta} - \xi_{n \delta_{\alpha\beta}} = 0.
\end{equation}

Proof. For any sections \(\xi_{i} \alpha \omega^{\alpha} \in \Gamma(J\mathcal{F}^*)\) and \(\xi_{i} n \omega^{n} \in \Gamma(T^*\mathfrak{N})\), one has
\begin{equation}
d(\xi_{i} \alpha \omega^{\alpha}) = (d(\xi_{i}) \alpha - \xi_{i \beta} \omega^{\alpha} \wedge \omega^{\beta} - (\xi_{i} | n) \omega^{\alpha} \wedge \omega^{n}.
\end{equation}

\begin{equation}
\xi_{\alpha\beta} - \xi_{n \delta_{\alpha\beta}} = 0.
\end{equation}

When \(\xi \in \Omega^{1}(M)\), the derivative of \(\xi_{n}\) along the direction \(e_{\alpha} = -u_{i}^{\alpha} F_{\partial y^{i}}\) is given by
\begin{equation}
\xi_{n,\alpha} = e_{\alpha}(\xi_{n}) = -u_{i}^{\alpha} F_{\partial y^{i}}(\frac{\xi_{i} y^{i}}{F}) = -u_{i}^{\alpha} (\delta_{j} - \frac{F_{\partial y^{j}} y^{j}}{F}) \xi_{i} = -\xi_{i} u_{i}^{\alpha} = -\xi_{\alpha}.
\end{equation}

Thus the facts \((2.27)-(2.29)\) yield
\begin{equation}
d\pi^\ast \xi = -\frac{1}{2} (\xi_{i|\beta} - \xi_{\beta|\alpha}) \omega^{\alpha} \wedge \omega^{\beta} - (\xi_{i|n} - \xi_{n|\alpha}) \omega^{\alpha} \wedge \omega^{n} - (\xi_{\alpha\beta} - \xi_{n \delta_{\alpha\beta}}) \omega^{\alpha} \wedge \omega^{\beta}.
\end{equation}
Since \( d\pi^*\xi = \pi^*d\xi \) does not involve vertical differential forms, we obtain the desired formulae from (2.30). \( \square \)

Now we present a proof of the theorems and corollaries.

**Proof of Theorem 1** The necessary part of the theorem has been proved in [9]. The key ingredient is that along each fibre of \( SM \), the elliptic equation

\[
\Delta_g f + \dot{g}(\eta, d^{SM/M}f) + (n - 1)f = 0,
\]

has only linear solutions (c.f. [13, 12]). By using this fact and (2.21), in [9], we prove that if \( e \) is constant along each fiber of \( SM \), then \( S \) is weakly isotropic and

\[
(2.31) \quad S = \frac{1}{n - 1}e + \xi_i \frac{y^i}{F},
\]

where \( \xi = \xi_i(x)dx^i \in \Omega^1(M) \).

Now we prove the sufficient part. Assume that the \( S \)-curvature is weakly isotropic and satisfies (2.1) for some \( c \in C^\infty(M) \) and \( \xi \in \Omega^1(M) \). The condition (2.1) and the representation (2.24) of \( \xi \) shows that

\[
(2.32) \quad \xi_n = S - \frac{1}{n - 1}\pi^*c.
\]

By taking derivative of (2.32) along the fibres, the formula (2.29) implies that

\[
(2.33) \quad \xi_\alpha = -S_{,\alpha}.
\]

Substituting (2.33) and (2.32) in the formula (2.26) yields

\[
(2.34) \quad S_{,\alpha,\beta} + (S - \frac{1}{n - 1}\pi^*c)\delta_{\alpha\beta} = 0.
\]

By (2.21) and (2.34), one obtains that the \( E \)-curvature is isotropic, i.e.,

\[
(2.35) \quad E_{\alpha\beta} = \frac{1}{n - 1}\pi^*c_\delta_{\alpha\beta}.
\]

The equation (2.35) obviously implies the Berwald scalar curvature

\[
(2.36) \quad e = \pi^*c.
\]

Now applying (2.33), (2.32) and (2.36) to the equation (2.23) yields

\[
(2.37) \quad \pi^*d\xi = \frac{1}{2}(S_{,\alpha|\beta} - S_{,\beta|\alpha})\omega^\alpha \wedge \omega^\beta + (S_{,\alpha|n} + (S - \frac{1}{n - 1}e)|_n)\omega^\alpha \wedge \omega^n.
\]

Applying (2.16), (2.17), (2.19) and (2.20) to (2.37), we obtain

\[
(2.38) \quad \pi^*d\xi = -\frac{1}{2}(\tau_{|\alpha|\beta} - \tau_{|\beta|\alpha})\omega^\alpha \wedge \omega^\beta + \frac{1}{2}(J_{|\alpha|\beta} - J_{|\beta|\alpha})\omega^\alpha \wedge \omega^\beta
\]

\[
+ (S_{|\alpha} - \tau_{|\alpha|n} + J_{|\alpha|n})\omega^\alpha \wedge \omega^n - \frac{1}{n - 1}de \wedge \omega^n
\]

\[
+ \frac{1}{2}R_i^i\omega^\alpha \wedge \omega^\beta + \frac{1}{2}(J_{|\alpha|\beta} - J_{|\beta|\alpha})\omega^\alpha \wedge \omega^\beta
\]

\[
+ (R_i^i + J_{|\alpha|n})\omega^\alpha \wedge \omega^n - \frac{1}{n - 1}de \wedge \omega^n.
\]
The relations (1.8) between the $hh$-curvature forms of the Berwald connection and the Chern connection and (2.38) yields
\[
\pi^*d\xi = \text{tr}[\tilde{R}] - \frac{1}{n-1}de \wedge \omega^n = \text{tr}[R] + \frac{1}{2}\Sigma - \frac{1}{n-1}de \wedge \omega^n,
\]
where $\Sigma$ is the mean stretch curvature (1.9).

\textbf{The proof of Theorem 2.} Assume that the Finsler metric $F$ satisfies $de = 0$ and $\overline{\Sigma} = 0$. From (2.39) we obtain
\[
\pi^*d\xi = \text{tr}[R] = \frac{1}{2}H_\gamma R_{\gamma}^a \omega^a \wedge \omega^b .
\]
Since that $\eta = d^{SM/M} \tau$ and the fibres of $SM$ are compact, $\eta$ has zero points on each fibre of $SM$. However $\pi^*d\xi$ is constant along each fibre. We conclude that the both sides of the equation (2.40) vanish. Therefore $d\xi = 0$ as $\pi$ is surjective. And $\text{tr}[R] = 0$. \hfill \square

\textbf{The proof of Corollary 2.} We are going to prove the chain of statements: (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

(1) $\implies$ (2): By the assumption $de = 0$ and $J = 0$, a similar argument as the proof of Theorem 2 shows that the $S$-curvature is almost constant
\[
S = \frac{1}{n-1}e + \pi^*\xi(G),
\]
with $d\xi = 0$.

(2) $\implies$ (3): Assume that $S$ is given by (2.41) with $de = d\xi = 0$. The hypothesis $J = 0$ and (2.39) yields $\text{tr}[R] = 0$. By Theorem 1 the hypothesis $J = 0$ and (2.22), we obtain $d\eta = -\text{tr}[P] = -E = -ed\omega^n$.

(3) $\implies$ (1): By (2.22), the proof is clear. \hfill \square

\textbf{The proof of Corollary 3.} Assume that $F$ is of scalar curvature, the following formula (c.f. [10]) is well known
\[
R_{i}^{i}_{an} + J_{a|n} = -\frac{n+1}{3}K_{,\alpha}.
\]
Assume that $F$ has almost isotropic $S$-curvature. By (2.42) and (2.38), one obtains
\[
0 = \frac{n+1}{3}K_{,\alpha}\omega^\alpha \wedge \omega^n + \frac{1}{n-1}de \wedge \omega^n.
\]
Since $e$ is a function on $M$, (2.33) and (2.7) yields
\[
0 = (\frac{n+1}{3}K - \frac{1}{n-1}e_{[\alpha]} \omega^\alpha \wedge \omega^n).
\]
Therefore $\frac{n+1}{3}K - \frac{1}{n-1}de(G) = \pi^*\sigma$, for some $\sigma \in C^\infty(M)$.

Conversely, $F$ has weakly isotropic flag curvature if and only if
\[
-\frac{n+1}{3}K_{,\alpha} = -\frac{1}{n-1}c_{[\alpha]} = \frac{1}{n-1}c_{[\alpha]},
\]
where \( c \in C^\infty(M) \). Assume that \( F \) has weakly isotropic \( S \)-curvature, (2.38) and (2.45) implies that

\[
\pi^*d\xi = \frac{1}{2}(R_{i\alpha\beta} + (J_{\alpha|\beta} - J_{\beta|\alpha}))\omega^\alpha \wedge \omega^\beta + \frac{1}{n-1}d(c - e) \wedge \omega^n
\]

(2.46)

Taking derivative of (2.46), one obtains

\[
0 = \frac{1}{2}(d\tilde{R}_{i\alpha\beta} \wedge \omega^\alpha \wedge \omega^\beta + \tilde{R}_{i\alpha\beta}d\omega^\alpha \wedge \omega^\beta - \tilde{R}_{i\alpha\beta}\omega^\alpha \wedge d\omega^\beta)
+ \frac{1}{n-1}d(c - e) \wedge \sum_{\alpha} (\omega^\alpha \wedge \omega^\alpha)
\]

(2.47)

\[
= \frac{1}{2}(d\tilde{R}_{j\alpha\beta\gamma\omega^\beta} + \tilde{R}_{j\alpha\beta\gamma\omega^\beta}) \wedge \omega^\alpha \wedge \omega^\beta - 2\tilde{R}_{i\alpha\beta}\omega^n \wedge \omega^\alpha \wedge \omega^\beta
+ \frac{1}{n-1}[(c - e)|\beta\omega^\beta + (c - e)|\beta\omega^n] \wedge \sum_{\alpha} (\omega^\alpha \wedge \omega^\alpha)
\]

therefore

\[
2\tilde{R}_{i\alpha\beta}\omega^n \wedge \omega^\alpha \wedge \omega^\beta = \frac{1}{n-1}(c - e)|\beta\omega^n \wedge \sum_{\alpha} (\omega^\alpha \wedge \omega^\alpha).
\]

Since \( \tilde{R}_{i\alpha\beta} = -\tilde{R}_{i\beta\alpha} \), both sides of (2.47) vanish. By this fact, (2.46) is simplified as

(2.48)

\[
\pi^*d\xi = -\frac{1}{n-1}d(c - e) \wedge \omega^n.
\]

Hence

(2.49)

\[
0 = d(c - e) \wedge (\omega^1 \wedge \omega^{n+1} + \cdots + \omega^{n-1} \wedge \omega^{2n-1}),
\]

which is equivalent to the following system

\[
0 = d(c - e) \wedge \omega^1 \wedge \omega^{n+1},
\]

\[
\cdots
d(\omega^{n-1} \wedge \omega^{2n-1}).
\]

Therefore \( d(c - e) = 0 \) and thus \( d\xi = 0 \).
REFERENCES

[1] David Bao, Shiing-Shen Chern and Zhongmin Shen, *An Introduction to Riemann-Finsler Geometry*. Graduate Texts in Mathematics, Vol. 200, Springer-Verlag, New York, Inc., 2000.

[2] David E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*. (2nd Edition). Birkhäuser, New York, 2010.

[3] Xinyue Chen(g), Xiaohuan Mo and Zhongmin Shen, *On the flag curvature of Finsler metrics of scalar curvature*. J. of London Math. Soc., Vol. 68, No. 2, 2003: 762-780.

[4] Xinyue Cheng and Zhongmin Shen, *Finsler Geometry-an approach via Randers spaces*. Science Press Beijing and Springer-Verlag Berlin Heidelberg, 2012.

[5] Shiing-Shen Chern and Zhongmin Shen, *Riemann-Finsler Geometry*. Nankai Tracts in Mathematics, Vol. 6, World Scientific, 2005.

[6] Huitao Feng and Ming Li, *Adiabatic limit and connections in Finsler geometry*. Communications in Analysis and Geometry, Vol. 21, No. 3, 2013: 607-624.

[7] Anmin Li, Udo Simon and Guosong Zhao, *Global Affine Differential Geometry of Hypersurfaces*. W. de Gruyter, Berlin-New York, 1993.

[8] Ming Li, *Equivalence theorems of Minkowski spaces and applications in Finsler geometry*(in Chinese) Acta Math. Sinica (Chin. Ser.) Vol. 62, No.2, 2019: 177-190. (see arXiv:1504.04475v2 for the English version.)

[9] Ming Li and Lihong Zhang, *Properties of Berwald scalar curvature*. Front. Math. China, 15(6), 2020: 1143-1153.

[10] Xiaohuan Mo, *An Introduction to Finsler Geometry*. Peking University series in Math., Vol. 1, World Scientific Publishing Co. Pte. Ltd., 2006.

[11] Behzad Najafi and Akbar Tayebi, *Weakly stretch Finsler metrics*. Publ. Math. Debrecen, Vol. 91, 2017: 1-12.

[12] Katsumi Nomizu and Takeshi Sasaki, *Affine Differential Geometry*. Cambridge University Press, 1994.

[13] Rolf Schneider, *Zur affinen Differentialgeometrie im Großen*. I. Math. Zeitschr., Vol. 101, 1967: 375-406.

[14] Yibing Shen and Zhongmin Shen, *Introduction to Modern Finsler Geometry*. World Scientific, Singapore, 2016.

[15] Zhongmin Shen, *Volume comparison and its applications in Riemann-Finsler geometry*. Adv. in Math., Vol. 128, 1997: 306-328.

[16] Zhongmin Shen, *Differential Geometry of Spray and Finsler Spaces*. Kluwer Acad. Publ., 2001.

[17] Zhongmin Shen, *Lectures on Finsler Geometry*. World Scientific, 2001.

[18] Udo Simon, Angela Schwenk-Schellschmidt and Helmut Viesel, *Introduction to the Affine Differential Geometry of Hypersurfaces*. Lecture notes, Science University Tokyo, 1991.

[19] Weiping Zhang, *Lectures on Chern-Weil Theory and Witten Deformations*. Nankai Tracts in Mathematics, Vol. 4, World Scientific Publishing Co. Pte. Ltd., 2001.

MING LI: MATHEMATICAL SCIENCE RESEARCH CENTER, CHONGQING UNIVERSITY OF TECHNOLOGY, CHONGQING 400054, P. R. CHINA

Email address: mingli@cqut.edu.cn