Reflections on tiles (in self-assembly)

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Abstract We define the Reflexive Tile Assembly Model (RTAM), which is obtained from the abstract Tile Assembly Model (aTAM) by allowing tiles to reflect across their horizontal and/or vertical axes. We show that the class of directed temperature-1 RTAM systems is not computationally universal, which is conjectured but unproven for the aTAM, and like the aTAM, the RTAM is computationally universal at temperature 2. We then show that at temperature 1, when starting from a single tile seed, the RTAM is capable of assembling $n^2$ squares for $n$ odd using only $n$ tile types, but incapable of assembling $n^2$ squares for $n$ even. Moreover, we show that $n$ is a lower bound on the number of tile types needed to assemble $n^2$ squares for $n$ odd in the temperature-1 RTAM. The conjectured lower bound for temperature-1 aTAM systems is $2n - 1$. Finally, we give preliminary results toward the classification of which finite connected shapes in $\mathbb{Z}^2$ can be assembled (strictly or weakly) by a singly seeded (i.e. seed of size 1) RTAM system, including a complete classification of which finite connected shapes can be strictly assembled by mismatch-free singly seeded RTAM systems.

Keywords Self-assembly · Tile assembly model · Molecular computing

1 Introduction

Self-assembly is the process by which disorganized components autonomously combine to form organized structures. In DNA-based self-assembly, the combining ability of the components is implemented using complementary strands of DNA as the “glue”. Winfree (1998) introduced a useful mathematical model of self-assembling systems called the abstract Tile Assembly Model (aTAM) where the autonomous components are described as square tiles with specifiable glues on their edges and the attachment of these components occurs spontaneously when glues match. The aTAM provides a convenient way of describing self-assembling systems and their resulting assemblies, and serves as the underpinning of many studies of the properties of self-assembling systems. For surveys of tile-based self-assembly, including several models other than the aTAM, see Patitz (2014) and Doty (2012).

From the broad collection of results in the aTAM, one property of systems that has been shown to yield enormous power is cooperation. The notion of cooperation captures the phenomenon where the attachment of a new tile to a growing assembly requires it to bind to more than one tile (usually 2) already in the assembly. The requirement for cooperation is determined by a system parameter known as the temperature, and when the temperature is equal to 1 (a.k.a. temperature-1 systems), there is no requirement for cooperation. A long-standing conjecture is that temperature-1 aTAM systems are in fact not capable of universal computation or efficient shape building, although it is well-known that temperature $\geq 2$ systems are. However, in actual laboratory
implementations of DNA-based tiles (Rothemund et al. 2004; Barish et al. 2009; Schulman and Winfree 2007; Mao et al. 2000; Winfree et al. 1998), the self-assembly performed by temperature-2 systems does not match the error-free behavior dictated by the aTAM, but instead, a frequent source of errors is the binding of tiles using only a single bond. Thus, temperature-1 behavior erroneously occurs and currently isn’t completely preventable.

Many models of self-assembly can be thought of as extensions of the aTAM (e.g. Fu et al. 2012; Doty et al. 2013; Demaine et al. 2014; Fekete et al. 2015; Schweller and Summers 2011; Cook et al. 2011), and for these models it is common to study the added power that an extra property or constraint gives the extended model. For example, in Cook et al. (2011), Doty et al. (2013), Fekete et al. (2015) and Schweller and Summers (2011), it is shown that at temperature 1, when the aTAM is appropriately extended, the resulting models are computationally universal and capable of efficiently assembling shapes. In this paper, we take the opposite approach and remove a constraint that the aTAM imposes with the goal of modelling physical systems that may be incapable of enforcing these constraints. Tiles in the aTAM are not allowed to flip or rotate prior to attachment to an existing assembly. While this assumption is a realistic one for many implementations of DNA-based tiles (e.g. Winfree 1998), for certain implementations (e.g. Kim et al. 2011), it is unknown whether or not both of the conditions of this assumption can be physically enforced. [See Han et al. (2013), Ke (2012), Andre (2011) and Santini et al. (2013) for more experimentally produced building blocks and systems.] Therefore, we consider a model based on the aTAM where tiles may nondeterministically flip horizontally and/or vertically prior to attachment.

We introduce the Reflexive Tile Assembly Model (RTAM), which can be thought of as the aTAM with the relaxed constraint that tiles in the RTAM are allowed to flip horizontally and/or vertically. Also, unlike most formulations of the aTAM where complementary strands of DNA are represented with the same glue label, the RTAM explicitly uses complementary glues. This importantly prevents copies of tiles of the same type from being able to flip and bind to each other, and is the actual reality with DNA-based tiles. We then show a series of results within the RTAM. First we show that at temperature 1, the class of directed RTAM systems—systems which yield a single pattern up to reflection and ignoring tile orientation—are only capable of assembling patterns that are essentially periodic. Then, following the thesis set forth in Doty et al. (2011), we conclude that the temperature-1 RTAM is not computationally universal. While the inability of temperature-1 aTAM systems to compute is still only conjectured, we are able to conclusively prove it for RTAM systems, specifically by using techniques developed in Doty et al. (2011) to study temperature-1 aTAM systems (Theorem 1). We also show that like the aTAM at temperature 2, the class of directed temperature-2 RTAM systems is computationally universal (Theorem 2).

We then turn our attention to the self-assembly of squares by singly seeded temperature-1 RTAM systems where we show that for even values of n ∈ N it is impossible to self-assemble any n × n square (Theorem 3). This is exceptional due to the fact that it is the first demonstration of a model of tile assembly in which a finite shape is proven to be impossible to self-assemble in a directed system. Typically, any finite shape can be self-assembled by a trivial system in which a unique tile type is created for each point of the shape. However, due to the ability of tiles in the RTAM to flip, it is not possible for the RTAM systems to effectively constrain the reflections of tiles to produce such even squares without the possibility of tiles growing beyond the boundaries of the squares. However, for odd values of n and m any n × m rectangle can be self-assembled using only $\frac{n+m}{2}$ tile types, thus implying that for n odd, an $n \times n$ square can be self-assembled using only n tile types (Theorem 4). In addition, we also show that for n odd, an $n \times n$ square cannot be self-assembled using less than n tile types (thus, n is the upper and lower bound for square assembly) (Theorem 5). This is in contrast to the aTAM at temperature 1, where the conjectured lower bound for assembling an $n \times n$ square is $2n - 1$, and hints that in certain situations the ability of RTAM tiles to attach in flipped orientations can be effectively harnessed to more efficiently build shapes than systems in the aTAM.

Finally, we give preliminary results (Theorems 6–10) toward the classification of the finite connected shapes in $\mathbb{Z}^2$ that can be assembled (strictly or weakly) by a singly seeded RTAM system, including a complete classification of which finite connected shapes can be strictly assembled by a mismatch-free singly seeded temperature-1 RTAM system. We also show that arbitrary shapes with scale factor 2 can be assembled in the singly seeded temperature-1 RTAM. These combined results show that the ability of tiles to bind in flipped orientations is sometimes provably limiting, while at other times can provide advantages, and they provide a solid framework for the study of self-assembling systems composed of molecular building blocks unable to enforce this constraint of the aTAM.

2 Preliminaries

2.1 Definition of the reflexive tile assembly model

The Reflexive Tile Assembly Model (RTAM) is essentially equivalent to the abstract Tile Assembly Model (aTAM) (Winfree 1998; Rothemund and Winfree 2000; Rothemund 2001; Lathrop et al. 2009) but with the modification that tiles are allowed to possibly “flip” across their horizontal
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and/or vertical axes before attaching to an assembly. Also, in some formulations of the aTAM, it is assumed that glues bind to complementary versions of themselves (so that two tiles of the same type which are flipped relative to each other can’t simply bind to each other along the same but reflected side). We now give a formal definition of the RTAM. Our notation is similar (and where appropriate, identical) to that of Lathrop et al. (2009).

We work in the 2-dimensional discrete space \( \mathbb{Z}^2 \). Define the set \( U_2 = \{ (0,1), (1,0), (0,-1), (-1,0) \} \) to be the set of all unit vectors in \( \mathbb{Z}^2 \). We also sometimes refer to these vectors by their cardinal directions N, E, S, W, respectively. All graphs in this paper are undirected. A grid graph is a graph \( G = (V,E) \) in which \( V \subseteq \mathbb{Z}^2 \) and every edge \( \{a,b\} \in E \) has the property that \( a - b \in U_2 \).

### 2.1.1 Tile types

Intuitively, a tile type \( t \) is a unit square that can be translated and flipped across its vertical and/or horizontal axes, but not rotated. This provides each tile type with a pair of North–South (NS) sides and a pair of East–West (EW) sides, such that either side \( s \in NS \) may be facing north while the other is facing south (and vice versa for the EW glues). For ease of discussion, however, we will talk about tile types as being defined in fixed orientations, but then allow them to attach to assemblies in possibly flipped orientations. Therefore, we define each \( t \) as having a well-defined “side \( u \)” for each \( u \in U_2 \). Each side \( u \) of \( t \) has a “glue” with “label” \( \text{label}_l(u) \)—a string over some fixed alphabet—and “strength” \( \text{str}_r(u) \)—a nonnegative integer—specified by its type \( t \). To each glue \( g \) with label \( l \), we associate a “complementary” glue, which we denote by \( g' \).

We denote the label of the complementary glue \( g' \) by \( \bar{l} \). So that tiles cannot simply reflect and bind to copies of themselves, we will define tile binding to only allow a glue \( g \) to bind to its complement \( g' \).

### 2.1.2 Tile reflection

Let \( R = \{ D, V, H, B \} \) be the set of permissible reflections for a tile which is assumed to begin in the default orientation, where \( D \) corresponds to no change from the default, \( V \) a single vertical flip (i.e. a reflection across the x-axis), \( H \) a single horizontal flip (i.e. a reflection across the y-axis), and \( B \) a single horizontal flip and a single vertical flip. (Note that the ordering of flips for \( B \) does not matter as either ordering results in the same orientation, and also that all combinations of possibly many flips across each axis result only in tiles of the orientations provided by \( R \).) See Fig. 1 for an example of each. Finally, we identify an element \((t, r) \) in \( T \times R \), called an oriented tile, as the tile type \( t \in T \) reflected by \( r \in R \). In addition, for \( r \in R \), we let \( r^* \) denote the reflection of \( \mathbb{Z}^2 \) corresponding to \( r \). Formally, for \((x,y) \in \mathbb{Z}^2 \), if \( r = D \), then \( r^*((x,y)) = (x,y) \), if \( r = V \), \( r^*((x,y)) = (y,x) \), if \( r = H \), \( r^*((x,y)) = (-x,-y) \), and if \( r = B \), \( r^*((x,y)) = (-x,y) \).

### 2.1.3 Tile binding

Let \( S : R \times U_2 \to U_2 \) be a function which takes a type of reflection \( r \in R \) and a side \( s \in U_2 \), and which returns the side of a tile in its default orientation which would appear on side \( s \) of the tile when it has been reflected according to \( r \). (E.g. for the tile type shown in Fig. 1, \( S(H,W) = E \), and \( S(H,N) = N \).) It is important to note that a glue does not have any particular orientation along the edge on which it resides, and so remains unchanged throughout reflections.

For \( x \in \mathbb{Z}^2 \) and \( u \in U_2 \), two tiles \( t \) and \( t' \) that are placed at the points \( x \) and \( a + u \) and reflected by \( r \in R \) and \( r' \in R \), respectively, bind with strength \( \text{str}_r(S(r,u)) \) if and only if \((\text{label}_l(S(r,u)), \text{str}_r(S(r,u))) = (\text{label}_{\bar{l}}(S(r',-u)), \text{str}_r(S(r',-u)))\). That is, the glues on adjacent edges of two tiles bind if they have complementary labels and the same strength. In the aTAM, glues may bind if they have matching glues (and appropriate strength). Note that in the RTAM, a glue \( g \), say, can only bind to the glue \( g' \) that is the complement of \( g \). This is important for the following reason. Consider the RTAM, but without this condition and let \( T = (T, \sigma, \tau) \) be a system in this hypothetical model. Then, for any tile \( t \in \sigma \) that exposes a strength \( \tau \) glue, a tile with the same type as \( t \) but with a flipped orientation can bind to \( t \). As a result, one can see that systems in such a model can only produce semilinear sets. That is, the proof of Theorem 1 is nearly trivial. As tile binding models interaction of complementary DNA strands, it is reasonable to add the condition that glues in the RTAM only bind to glues with a complementary label (not glues with the same label).

### 2.1.4 Tile assemblies

In the subsequent definitions, given two partial functions \( f, g \), we write \( f(x) = g(x) \) if \( f \) and \( g \) are both defined and equal on \( x \), or if \( f \) and \( g \) are both undefined on \( x \).

![Fig. 1 Left to right (1) Default orientation of an example tile type \( t \), \( 2 \) \( t \) flipped vertically, \( 3 \) \( t \) flipped horizontally, \( 4 \) \( t \) flipped across both axes](image)
Fix a finite set $T$ of tile types. A $T$-assembly, sometimes denoted simply as an assembly when $T$ is clear from the context, is a partial function $\alpha : \mathbb{Z}^2 \to T \times R$ defined on at least one input, with points $x \in \mathbb{Z}^2$ at which $\alpha(x)$ is undefined interpreted to be empty space, so that dom $\alpha$ is the set of points with oriented tiles. Note the divergence from the definition of the aTAM here, where the range of an assembly consists of tile types. Informally, the idea behind defining an assembly to map to oriented tiles rather than just tiles is that an assembly contains all of the information about where a tile is located in an assembly and how the tile is reflected in the assembly. While this leads to a more complicated definition of an assembly, as we will see, this helps keep the definition of an assembly sequence cleaner and more intuitive.

When the context is clear, we refer to oriented tiles simply as tiles. We write $|\alpha|$ to denote $|\text{dom } \alpha|$, and we say $\alpha$ is finite if $|\alpha|$ is finite. For a given location $v \in \mathbb{Z}^2$, we denote the tile in $\alpha$ at location $v$ by $\alpha(v)$ (if no tile exists there, $\alpha(v)$ is undefined).

For assemblies $\alpha$ and $\alpha'$, we say that $\alpha$ is a subassembly of $\alpha'$, and write $\alpha \subseteq \alpha'$, if dom $\beta \subseteq$ dom $\alpha'$ and $\beta(x) = \alpha'(x)$ for all $x \in \text{dom } \beta$. The binding graph of an assembly $\alpha$ is the grid graph whose vertices are the tiles of $\alpha$ and whose weighted edges represent positive strength glue bonds between adjacent sides of tiles in $\alpha$ where the weight of an edge is the strength of the glue it represents. For some $\tau \in \mathbb{N}$, an assembly $\alpha$ is $\tau$-stable if every cut of the binding graph of $\alpha$ has weight at least $\tau$. When $\tau$ is clear from context, we say $\alpha$ is stable.

### 2.1.5 Assembly process

Self-assembly begins with a seed assembly $\sigma$, in which each tile has a specified and fixed orientation, and proceeds asynchronously and nondeterministically, with tiles in any valid reflection in $R$ adsorbing one at a time to the existing assembly in any manner that preserves $\tau$-stability at all times. A tile assembly system (TAS) is an ordered triple $T = (T, \sigma, \tau)$, where $T$ is a finite set of tile types, $\sigma$ is a seed assembly with finite domain in which each tile is given a fixed orientation, and $\tau \in \mathbb{N}$ is the temperature. We write $A[T]$ for the set of all assemblies that can arise (in finitely many steps or in the limit) from $T$. An assembly $\alpha \in A[T]$ is terminal, and we write $\alpha \in A_{\tau}[T]$, if no tile can be $\tau$-stably added to it. It is clear that $A_{\tau}[T] \subseteq A[T]$.

An assembly sequence in a TAS $T$ is a (finite or infinite) sequence $\alpha = (\alpha_0, \alpha_1, \ldots)$ of assemblies in which each $\alpha_{i+1}$ is obtained from $\alpha_i$ by the addition of a single tile. The result $\text{res}(\alpha)$ of such an assembly sequence is its unique limiting assembly. (This is the last assembly in the sequence if the sequence is finite.) The set $A[T]$ is partially ordered by the relation $\alpha \to \alpha'$ defined by $\alpha \to \alpha'$ iff there is an assembly sequence $\alpha = (\alpha_0, \alpha_1, \ldots)$ such that $\alpha_0 = \alpha$ and $\alpha' = \text{res}(\alpha)$. Here we are able to see why an assembly is defined as a map to oriented tiles rather than tiles without orientation. If the definition of an assembly did not capture tile orientation, then describing a valid assembly sequence becomes much more difficult.

For a tile set $T$, we let $p_T : T \times R \to T$ be the projection map onto $T$ (i.e. $p_T((t, r)) = t$). A configuration given by an assembly $\alpha$ is defined to be the map from $\mathbb{Z}^2$ to $T$ given by $p_T \circ \alpha$. This should not be confused with the notion of configuration sometimes defined for the aTAM. We say that $T$ is directed if and only if for all $\alpha, \beta \in A_{\tau}[T]$, $p_T \circ \alpha = p_T \circ \beta$. In other words, all of the assemblies of a directed systems are the same modulo the orientation of individual tiles. That is, an RTAM system is directed if all terminal assemblies give the same configuration. Note that the definition of directed in the RTAM is similar to the definition of directed in the aTAM. In the RTAM, we note that “directed” only concerns tile types at tile locations of terminal assemblies and ignores orientations (in the aTAM, it is assumed that tiles have a single fixed orientation) of individual tiles of these terminal assemblies. The reason for ignoring individual tile orientations the definition of directed for RTAM systems is as follows. Consider any occurrence of a single tile attachment where a tile attaches via interaction of strength $\tau$. Such a tile can attach in two orientation. If the definition of directed for RTAM systems does not ignore tile orientations, then, directed systems would be very limited in terms of their capability for computation (especially at temperature-1 as all binding is $\tau$ strength binding). RTAM systems that produce a single terminal assembly rely heavily on cooperation and must be finite when the seed is a finite assembly. To see this latter fact, notice that to assemble an infinite assembly from a finite seed, there must be tiles that bind via a strength-$\tau$ glue, and such a tile can bind in two orientation; hence, any infinite assembly assembled from a finite seed assembly produces many terminal assemblies (again, when orientation of individual tiles is taken into account). Therefore, one would be at a loss in trying to give an RTAM system that starts from a finite seed and produces a single terminal assembly that is capable of computation. However, it seems reasonable that an RTAM system could be capable

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1 It is interesting to note that there are interesting temperature-2 RTAM systems that assemble a single assembly even when individual tile orientations are taken into account. For example, rectilinear tile assembly systems in the RTAM at temperature-2 can produce a single terminal assembly (even when taking tile orientations into account) when the seed assembly of the system is $L$-shaped. Note that the terminal assembly cannot have width or height exceeding that of the seed. See Kari et al. (2015) for interesting examples of such rectilinear tile assembly systems with $L$-shaped seeds.
of computation by taking a finite seed as input and producing a single configuration that encodes the output of the said computation.

2.2 Configurations and self-assembly of sets in \( \mathbb{Z}^2 \)

Given a reflection \( r \in R \) and a translation vector \( v \in \mathbb{Z}^2 \), let \( F_{r,v} \) be a function which takes as input an assembly \( a \) and returns the assembly equal to \( a \) reflected according to \( r \) and translated by \( v \). More formally, if \( t_{r,v} : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2 \) is the map defined by translation by \( -v \) and \( r^* \) is the reflection of the points in \( \mathbb{Z}^2 \) corresponding to \( r \) (defined above), \( F_{r,v}(a) \) is defined to be the assembly \( a \circ t_{r,v} \circ r^* \).

A set \( X \subseteq \mathbb{Z}^2 \) weakly self-assembles if there exists a TAS \( T = (T, \sigma, \tau) \) and a set \( B \subseteq T \) such that for each \( x \in A_{\mathbb{Z}}[T] \) there exists a reflection \( r \in R \) and a translation \( v \in \mathbb{Z}^2 \) such that \( x_r = F_{r,v}(x) \) and \( x_r^{-1}(B) = X \) holds. Essentially, weak self-assembly can be thought of as the creation (or “painting”) of a pattern of tiles from \( B \) (usually taken to be a unique “color” such as black) on a possibly larger “canvas” of un-colored tiles. A set \( X \) strictly self-assembles if there is a TAS \( T \) such that for each assembly \( a \in A_{\mathbb{Z}}[T] \) there exists a reflection \( r \in R \) and a translation \( v \in \mathbb{Z}^2 \) such that \( x_r = F_{r,v}(x) \) and \( x_r = X \). Essentially, strict self-assembly means that tiles are only placed in positions defined in \( X \). Note that if \( X \) strictly self-assembles, then \( X \) weakly self-assembles. \( X \) in the definition of strict or weak self-assembly is called a shape in \( \mathbb{Z}^2 \). In this paper, we also consider scaled-up versions of shapes. Formally, if \( X \) is a shape and \( m \in \mathbb{N} \), then an \( m \)-scaling of \( X \) is defined as the set \( X^m = \{ (x, y) \in \mathbb{Z}^2 \mid (\lfloor \frac{x}{m} \rfloor, \lfloor \frac{y}{m} \rfloor) \in X \} \). Intuitively, \( X^m \) is the shape obtained by replacing each point in \( X \) with a \( m \times m \) block of points. We refer to the natural number \( m \) as the scale factor.

2.3 Paths in the binding graph and as assemblies

Given an assembly \( a \) and locations \( x \) and \( y \) such that \( x, y \in \text{dom} \ a \), we define a path in \( a \) from \( x \) to \( y \) (or simply a path from \( x \) to \( y \)) as a simple path in the binding graph of \( a \) with the first location being \( x \) and the last \( y \). We refer to such a path as \( \pi_x^y \), and for \( k = |\pi_x^y| \) (i.e. \( k \) is the length of, or number of tiles on, \( \pi_x^y \)) \( 0 \leq i < k \), let \( \pi_x^y(i) \) be the \( i \)-th location of \( \pi_x^y \). Thus, \( \pi_x^y(0) = x \), and \( \pi_x^y(k - 1) = y \). We can thus refer to the \( i \)-th tile on \( \pi_x^y \) and its reflection as \( \pi_x^y(i) \), and as shorthand will often refer to locations and/or tiles along a path. Regardless of the order in which the tiles of \( \pi_x^y \) were placed in \( a \), we define input and output sides for each tile in \( \pi_x^y \) (except for the first and last, respectively) in relation to their position on \( \pi_x^y \). The input side of the \( i \)-th tile of \( \pi_x^y \), \( \pi_x^y(i) \), is that which binds to \( x(\pi_x^y(i + 1)) \). We denote these sides as \( \text{IN}(\pi_x^y(i)) \) and \( \text{OUT}(\pi_x^y(i)) \), respectively. (Thus, \( x(\pi_x^y(0)) \) has no input side, and \( x(\pi_x^y(k - 1)) \) has no output side.) Note that in a temperature-1 system, an assembly \( x' \) whose binding graph is \( \pi_x^y \) would be able to grow solely from \( x(\pi_x^y(0)) \), in the order of \( \pi_x^y \), with each tile having input and output sides as defined for \( \pi_x^y \). Finally, for convenience, we will refer to the subassembly whose binding graph is a path as a path.

3 The RTAM is not computationally universal at \( \tau = 1 \)

In this section, we show that directed RTAM systems are not computationally universal by showing that any shape weakly assembled by a directed RTAM system is “simple”. We will first define our notion of simple. Many of the following definitions can also be found in Doty et al. (2011). Weak self-assembly is defined in terms of a subset \( B \) of tile types and a TAS \( T \). For weak self-assembly of a set \( X \subseteq \mathbb{Z}^2 \), any terminal assembly of \( T \) must contain a tile of type in \( B \) at each location in \( X \). Moreover, any tile location in the domain of a terminal assembly of \( T \) that is not in \( X \) does not contain a tile with type in \( B \). \( B \) is often referred to as the set of “black tiles”. We use \( B \) to denote this set of tile types throughout this section.

**Definition 1** A set \( X \subseteq \mathbb{Z}^2 \) is 2-linear if there exist three vectors \( b, x, y \) in \( \mathbb{Z}^2 \) such that \( X = \{ b + n \cdot x + m \cdot y \mid n, m \in \mathbb{N} \} \). Moreover, when \( x \) and \( y \) are linearly independent, we say that the 2-linear set is non-degenerate.

Less formally, a 2-linear set is a set that repeats infinitely along two vectors (linearly independent vectors in the non-degenerate case), starting at some base point \( b \). Note that \( x \) and \( y \) in Definition 1 can be the zero vector. In other words, a singleton set is also considered to be a 2-linear set.

**Definition 2** A set \( X \subseteq \mathbb{Z}^2 \) is semilinear if it is the finite union of 2-linear sets.

Note that Definition 2 is equivalent to the standard definition of semilinear, where a set is said to be semilinear if it is the finite union of linear sets. Now, let \( T = (T, \sigma, 1) \) refer to a directed, temperature-1 RTAM system. We show that any such \( T \) weakly self-assembles a set \( X \subseteq \mathbb{Z}^2 \) that is semilinear.

**Theorem 1** Let \( T = (T, \sigma, 1) \) be a directed RTAM system. If a set \( X \subseteq \mathbb{Z}^2 \) weakly self-assembles in \( T \), then \( X \) is a semilinear set.
First we give a high-level sketch of the proof of Theorem 1. The basic idea of the proof is as follows. For an RTAM system \( T = (T, \sigma, 1) \) we consider all of the paths of \( n \) tiles (for \( n \) to be defined) that can assemble from each exposed glue of \( \sigma \) such that each consecutive tile that binds forming the path attaches via a north or west glue (and we say that such a path “extends to the north–west”). In order to prevent the path that extends to the north–west from being “blocked” (a tile of an assembly prevents another tile from binding at the same location), we show that we can begin growth of such a path from a tile \( t \) placed outside of a box bounding containing every tile of the existing assembly except \( t \).

The set of tile locations for tiles of type in \( B \) belonging to any finite path is trivially a semilinear set since singleton sets are 2-linear. Then, for \( n \) sufficiently large, a path of \( n \) tiles that extends to the north–west must contain two distinct tiles \( t_1 \) and \( t_2 \) of the same tile type in the same orientation. If for every such path, every two distinct tiles \( t_1 \) and \( t_2 \) of the same tile type in the same orientation lie on a horizontal or vertical line, then it can be argued that the terminal configuration of \( T \) must consist of finitely many infinitely long horizontal or vertical paths connected to \( \sigma \), and the set of tile locations with tiles of type in \( B \) belonging to these paths is a semilinear set. On the other hand, if there is a path such that the two distinct tiles \( t_1 \) and \( t_2 \) of the same tile type in the same orientation do not lie on a horizontal or vertical path, then we argue that the set of tile locations with tiles of type in \( B \) belonging to the terminal assembly of \( T \) is a semilinear set as follows. First, we note that for such a path, \( \pi \) say, the tiles between \( t_1 \) and \( t_2 \) can be repeated indefinitely. This is shown in Fig. 2a.

Then we show how to modify \( \pi \) by reflecting tiles to obtain an infinite family of paths. Examples of such modified paths are shown in Fig. 2b–h. Now, if a tile \( t \) belongs to one of these paths and has location \( l \) say, then, since \( T \) is directed the terminal assembly of \( T \) must contain a tile of the same type as \( t \) at each such location \( l \). Note that these paths assemble in independent assemblies and therefore there is no concern of these paths overlapping (sharing tile locations). Finally, we note that the sets of tile locations with tiles of type in \( B \) belonging to all of these paths taken together form a 2-linear set. This is depicted in Fig. 3a. Continuing this line of reasoning, we show that the set of tile locations with tiles of type in \( B \) belonging to the terminal assembly of \( T \) is a semilinear set. A portion of such an assembly is shown in Fig. 3b.

In the proof of Theorem 1, we will be considering many different assemblies and will use the following definition to form a “union” of the configurations given by these assemblies.

**Definition 3** Two assemblies \( \alpha \) and \( \beta \) are type-consistent iff for each \( l \in \text{dom} \alpha \cap \text{dom} \beta \), \( p_T(\alpha(l)) = p_T(\beta(l)) \).

Recall that \( p_T \) is the projection from \( T \times R \) onto \( T \). Notice that if two assemblies \( \alpha \) and \( \beta \) are type-consistent, then we can give a well-defined partial function \( f \) from \( \mathbb{Z}^2 \) to \( T \) by \( x \mapsto f \) iff either \( x \in \text{dom} \alpha \) and \( \alpha(x) = t \) or \( x \in \text{dom} \beta \) and \( \beta(x) = t \), and otherwise \( f \) is undefined. The following definitions will also be useful in the proof of Theorem 1.

Let \( c \in \mathbb{N} \) and \( v \in \mathbb{Z}^2 \). The box of radius \( c \) centered about the point \( v \) is the set of points defined as \( B_c(v) = \{ (x, y) \mid |x| \leq \text{cand}(|y|) \leq c \} \). Finally, we say that a path in the binding graph turns if there are two nodes \( n_0 = (x_0, y_0) \) and \( n_1 = (x_1, y_1) \) in the path such that \( x_0 \neq x_1 \) and \( y_0 \neq y_1 \). In other words, a path turns if its vertices do not lie on a horizontal or vertical line. A path of tiles in an assembly is said to turn if the corresponding path in the binding graph turns.

First, we show a preliminary lemma.

**Lemma 1** Let \( T = (T, \sigma, 1) \) be a directed RTAM system such that any \( \alpha \in \mathcal{A}[\mathcal{T}] \) is infinite. Then there exists a producible assembly \( \beta \in \mathcal{A}[\mathcal{T}] \) such that \( \beta \) contains an infinite subassembly \( \pi \) that is a path consisting of a subpath assembly repeated indefinitely.

**Proof** Without loss of generality, assume that \( \sigma \) contains the point \( 0 \in \mathbb{Z}^2 \) where \( 0 = (0, 0) \). Let \( c \in \mathbb{N} \) be the minimal number such that \( \sigma \subseteq B_c(0) \) and let \( c' \in \mathbb{N} \) be such that \( c' = 4|\mathcal{T}| + c + 1 \). Now let \( \alpha \in \mathcal{A}[\mathcal{T}] \) be a terminal assembly such that \( \alpha \) and \( \sigma \) are type-consistent. Note that a finite assembly trivially weakly assembles a semilinear set. Therefore, we only consider the case where \( B_{c'}(0) \cap \text{dom} \alpha \neq \emptyset \). Let \( x \in \mathbb{Z}^2 \) be a tile location for some tile in \( B_{c'}(0) \cap \text{dom} \alpha \), and let \( \pi^x_\alpha \) be a path in the binding graph of \( \alpha \), \( G_x \), from a node corresponding to a tile in \( \sigma \) to \( x \). (Notice the abuse of notation here. We are using \( \sigma \) in the notation \( \pi^x_\alpha \) where we should be using the location of the tile in \( \alpha \).) Notice that \( \pi^x_\alpha \) must contain at least \( 4|\mathcal{T}| + 1 \) nodes. Now, at temperature 1, \( \sigma \cup \pi^x_\alpha \) is a producible assembly of \( \mathcal{T} \); denote this assembly by \( \beta \). We will modify \( \beta \) by modifying the path \( \pi^x_\alpha \).

Let \( \beta \) be the assembly sequence for \( \beta \), and let \( t \) be the first tile placed outside of \( B_{c'}(0) \) using this assembly sequence. This tile is marked in Fig. 4. We will denote the path from \( t \) to \( x \) by \( \pi^t_x \), and denote the subpath of \( \pi^t_x \) that does not contain the tiles of \( \pi^x_\alpha \) by \( \pi^t' \). (Again we abuse notion here. We are using \( t \) in the notation \( \pi^t_x \) where we should technically use the location of \( t \).) Without loss of generality, suppose that for \( l = (l_x, l_y) \), the location of \( t \), for all \( (x, y) \in B_{c'}(0) \), \( l_y > y \). In other words, suppose that \( t \) lies to the north of \( B_{c'}(0) \). This is depicted in Fig. 4. Now we modify the path \( \pi^t_x \) as follows.
Fig. 2 Each figure labeled (a) through (h) depicts a possible path that can assemble in $T$. The black tiles make up the seed and the white tiles are a path leading to the repeated tile that allows the grey path to repeat.

Fig. 3 a A configuration that can be thought of as the “union” of the type-consistent assemblies depicted in Fig. 2. b A configuration of tiles that must weakly self-assemble in $T$.

Fig. 4 A depiction of tiles for a path that can assemble in $T$. The original path is on the left, and the right is a modification to that path which must also be able to assemble in $T$. The tile at $t$ is the first along the path with location outside of $B_c(0)$, and the tile at $x$ is the first along the path with location outside of $B_c(0)$'. The tiles between copies of the same tile type at $v_0$ and $v_1$ (labeled 1 through 6) can be repeated indefinitely as depicted in Fig. 2a.
Let \( \alpha = (x_0, x_1, \ldots, x_n) \) be the assembly sequence such that \( x_0 = \sigma \) and \( x_i \) is obtained from \( x_{i-1} \) by the attachment of the tile \( \pi^0_{x_i}(i-1) \). We modify \( \alpha \) as follows. Suppose that \( k \) is such that \( x \) is the tile at the location in \( \sigma \) for \( x_{k-1} \). That is, \( k = |\sigma| \), and so \( t = \pi^0_x(k-1) = \pi^0_x(0) \). Starting from \( x_{k-1} \), orient and attach \( t \) to form \( x'_k \) such that a tile with the same type as \( t_{k+1} = \pi^0_x(1) \) can attach to \( x'_k \) at a location that is north or west of \( t \). That is, attach \( t \) in such a way that the output glue is exposed on the north or west edge of \( t \). Note that this is possible since \( t \) is the first tile located outside of \( B_1(0) \) and the location of \( t \) is to the north of \( B_1(0) \). Then, \( x'_{k+1} \) is obtained from \( x'_k \) by attaching a tile with the same type as \( t_{k+1} \) to the north or west of \( t \) in \( x'_k \) such that the output glue of \( t_{k+1} \) is also exposed on the north or west edge. In general, for a step \( j \) such that \( k + 1 < j < n \), \( x'_j \) is obtained from \( x'_{j-1} \) by attaching a tile with the same type as \( \pi^0_x(j-1) \) to the tile \( t' \) at the location in \( \sigma'_{j-1} \) for \( t' \) north or west of \( t \). In other words, we form a path with the same tile types as \( \pi^0_x \), only as tiles attach, flip them so that the next tile to attach does so at a tile location that is north or west of the previously attached tile. Denote the resulting path as \( \alpha' \). Figure 4 (right) shows such a modified path.

Note that since \( \pi^0_x \) contains at least \( 4|\sigma| + 1 \) tiles, so does \( \alpha' \). By the pigeonhole principle, there must be two nodes \( v_0 \) and \( v_1 \) contained in \( \alpha' \) with locations in \( B_1(0) \) that correspond to the same tile type in the same orientation. These are depicted as dark grey tiles in Fig. 4. Let \( \pi^0_{v_0} \) denote the path in \( G_v \) that is the subpath of \( \pi^0_x \) from \( v_0 \) to \( v_1 \). Because each tile of \( \alpha' \) attaches to the north or west of a previous tile in the path, \( \alpha' \) can be modified to extend indefinitely to the north–west since the subpath \( \pi^0_{v_0} \) can be repeated an arbitrary number of times to obtain \( \pi \) as given in the lemma statement. See Fig. 2a for an example of repeating \( \pi^0_{v_0} \).

The next lemma categorizes sets that can be weakly self-assembled in the RTAM at temperature 1.

**Lemma 2** Let \( T = (T, \alpha, 1) \) be a directed RTAM system. If a set \( X \subseteq \mathbb{Z}^2 \) weakly self-assembles in \( T \), then \( X \) is a finite union of a box of constant (which depends only on \( |\alpha| \) and \( |\sigma| \)) radius containing the seed and at most 4 (possibly empty) 2-linear sets. Moreover, each of these 4 2-linear sets \( S_i \) for \( 1 \leq i \leq 4 \) is described as one of the following.

1. \( S_i \) is empty.
2. \( S_i \) is the finite union of \( d \in \mathbb{N} \) horizontal or vertical paths of tiles where \( d \) is bounded by \( |\sigma| \).
3. \( S_i \) is a non-degenerate 2-linear set and the number of such sets in this case is bounded by \( |\alpha| \).

**Proof** Let \( \pi^0_{v_0} \) be the repeatable subpath given by Lemma 1. Now, either \( \pi^0_{v_0} \) turns or it does not. If it does not turn, then \( \pi^0_{v_0} \) can be modified so that the tile types of the subpath \( \pi^0_{v_0} \) are repeated indefinitely starting from \( v_0 \) and this modified path lies on a vertical (or horizontal) line. If each such path \( \pi^0_{v_0} \) from the seed \( \sigma \) to some point \( x \) is such that for each pair of distinct nodes of \( \pi^0_{v_0} \) with corresponding tiles of the same type and orientation, the path segment between these nodes does not turn, then every path containing a node in \( B_1(0) \) can be modified to assemble an arbitrarily long path that lies on a vertical or horizontal line. Then, it can easily be seen that the set of tile locations with tiles of type in \( B \) belonging to \( \alpha \) is a semilinear set. This is due to the fact that in the limit, \( \alpha \) must consist of \( \sigma \) and finitely many infinitely long paths connected to \( \sigma \). The set of tile locations with tiles of type in \( B \) belonging to each of the infinitely long paths is a 2-linear set, and there can only be finitely many paths that lie on vertical or horizontal lines that can assemble outside of \( B(0) \). On the other hand, if there is a \( \pi^0_{v_0} \) such that the subpath \( \pi^0_{v_0} \) turns, then we can use this turn and the non-orientable nature of tiles in the RTAM to show that the set of tile locations with tiles of type in \( B \) belonging to \( \alpha \) is a semilinear set.

Let \( \beta' \) denote \( \sigma \cup \alpha' \). Notice that \( \beta' \in \mathcal{A}(T) \). We now use \( \pi^0_{v_0} \) to form an infinite set of producible assemblies in \( T \) by repeating the tiles of \( \pi^0_{v_0} \) and changing tile orientations by reflecting. Without loss of generality, we assume that output glues of the tiles at locations \( v_0 \) and \( v_1 \) are on the north edges of the tiles at those locations. (An analogous argument holds when the output glues are on the west edges.) Figure 2 depicts this. Figure 2b through h shows seven of the various paths that can assemble in \( T \) from the path depicted in Fig. 2a. We can construct these paths as follows. Since tiles of \( \pi^0_{v_0} \) always attach to the north or west, tile types of \( \pi^0_{v_0} \) can be repeated (in the order that they appear in the path \( \pi^0_{v_0} \) and with the same orientations) indefinitely in the north–west direction to yield a producible assembly. This gives (a) in Fig. 2. Now, as tile types of \( \pi^0_{v_0} \) are repeated, at any arbitrary number of repetitions, we can reflect the tile types of \( \pi^0_{v_0} \) about a vertical axis so that the reflection of the path \( \pi^0_{v_0} \) assembles instead of \( \pi^0_{v_0} \). Let \( \tilde{\pi}^0_{v_0} \) denote the reflection of \( \pi^0_{v_0} \) about a vertical axis. Then, tiles of \( \tilde{\pi}^0_{v_0} \) can be repeated an indefinite number of times to give (b) through (d) in Fig. 2. Note that each path of tiles is producible in \( T \) (though perhaps not in the same assembly). Similarly, we can produce paths (e) through (h) in \( T \) by first forming a path where each successive tile attaches to the north–east of the previous tile. This gives an infinite set of paths such that each path belongs to a producible assembly of \( T \). Denote this set by \( S \). Note that \( S \) is an infinite subset of \( \mathcal{A}(T) \).

Notice that the assemblies in \( S \) must be pairwise type-consistent since \( T \) is directed. Consider the map \( f : \mathbb{Z}^2 \rightarrow \)
\( T \) defined by \( x \mapsto a \) iff \( \exists y \in S \) such that \( p_T(y(x)) = a \). It should be noted that the fact that assemblies in \( S \) are pairwise type-consistent implies that \( \Gamma \) is well-defined. It should also be noted that \( \Gamma \) is equal to the map \( p_T \circ \gamma \mid_{\cup_{c \in S} \text{dom } \gamma} \). Notice that it may be the case that there does not exist an assembly \( /C_138 \) such that \( C = p_T \circ /C_14 \). Since orientations of tiles of the same type at a particular location may differ. Figure 3a depicts a portion of this configuration. The points in the domain of \( C \) consist of the union of a finite number of points and the set of tile locations for tiles of type in \( B \) belonging to the domain of \( \Gamma \) is a 2-linear set. This can be seen by considering the vectors defined by \( u = v_1 - v_0 = (a, b) \) and \( v = (-a, b) \).

Using a similar technique that is used to form the assemblies of \( S \), the path \( /C_0 \) can also repeated (with appropriate reflections) to form type-consistent assemblies containing paths whose domains collectively form a set of points \( W \) such that the set of tile locations for tiles of type in \( B \) belonging to \( W \) is the union of four 2-linear sets. Figure 5 shows six examples of such paths. Because the assemblies whose domains make up \( W \) are type-consistent, we can once again give a well-defined partial map \( p_T \circ \gamma \mid_{\cup_{c \in S} \text{dom } \gamma} \) from \( \mathbb{Z}^2 \) to \( T \). Figure 3b depicts a portion of a configuration given by the map \( p_T \circ /C_14 \) from \( \mathbb{Z}^2 \) to \( T \). Note that any point of \( \mathbb{Z}^2 \) is either in \( W \) or is in some finite region bounded by points of \( W \).

It now suffices to show that the set of all points in \( \text{dom } \gamma \) (not just the points in \( W \)) that contain tiles of type in \( B \) is a semilinear set. First, note that except for possibly a finite number of points, any point in \( \text{dom } \gamma \) is contained in some “parallelogram” region \( R \) bounded by points of \( W \), where these bounding points are contained in a 2-linear set with vectors \( b, u, \) and \( v \) as in Definition 1. Note that for each path whose domain makes up \( W \) (see Fig. 2 for examples of these paths), we can orient the tiles of the path so that the exposed glues of each tile of each path are always oriented the same way. This ensures that if some tile can be placed in the region \( R \), then the same tile can be placed in the region \( R_{n,m} = \{ r + n \cdot u + m \cdot v \mid r \in R \} \) for each \( n, m \in \mathbb{N} \). See Fig. 6 for more details. Therefore, the set of all points in \( \text{dom } \gamma \) containing tiles of type in \( B \) is a semilinear set.

To finish the proof, note that the number of paths \( \pi \) as described here that may assemble from a single glue exposed by \( \sigma \) is bounded by \( l/\ell \). Therefore, there is a constant \( \kappa \) that depends only on \( |\sigma| \) and \( |\ell| \) such that for the terminal assembly \( \alpha \) of \( T \), the set of tile locations for tiles of type in \( B \) belonging to \( \gamma \mid_{\beta_1(0)} \) (the portion of \( \gamma \) outside of a box of radius \( \kappa \)) is the union of 4 2-linear sets. We may take \( \kappa \) to be \( c' \).

**Proof (of Theorem 1)** Theorem 1 is directly concluded from Lemma 2 since finite sets and the set of tile locations for tiles of type in \( B \) belonging to finite union of horizontal or vertical paths of tiles are each semilinear sets.

### 3.1 Universal computation at temperature 1

Semilinear sets are definable by Presburger arithmetic and are known to be decidable (Ginsburg and Spanier 1966). Therefore, the sets which weakly self-assemble in the
temperature-1 RTAM are “simple”. Now we give a corollary that provides alternative evidence to the conclusion that the directed temperature-1 RTAM is not computationally universal. For a tileset $T$, fix a finite configuration $C_T$ over $T$, and let $C_T^v$ be the translation of $C_T$ by the vector $v$. In other words, $C_T$ is a partial function from $\mathbb{Z}^2$ to $T$, and $C_T^v(x) = C_T(x - v)$ for $x \in \{z \in \mathbb{Z}^2 \mid v \in \text{dom}(C_T)\}$. Then, let $A(C_T)$ be the set of all seed assemblies $\sigma$ such that the RTAM system $(T, \sigma, 1)$ is directed and the terminal assembly $a_\sigma$ of the RTAM system $(T, \sigma, 1)$ contains some translation of the configuration $C_T$. More formally, $A(C_T)$ is the set of all seed assemblies $\sigma$ such that the RTAM system $(T, \sigma, 1)$ is directed and the terminal assembly $a_\sigma$ of RTAM system $(T, \sigma, 1)$ has the property that there exists a vector $v$ in $\mathbb{Z}^2$ such that $p_T \circ a_\sigma(x) = C_T^v(x)$ for all $x \in \text{dom}(C_T^v)$.

Intuitively, we are considering the set $C_T$ (and its translations) for the following reason. Let $H = \{(i,x) \mid \text{programhaltswhenrunoninput}\}$ be the halting set and let $M$ be a Turing machine that outputs a 1 iff $(i,x) \in H$. The typical way of expressing the computation of $M$ in tile assembly is as follows. For a fixed tileset $T$, a seed assembly $\sigma$ encodes an “input” to the computation, while the “output” of 1 by $M$ corresponds to some configuration being contained in the terminal assembly $a_\sigma$ of $(T, \sigma, 1)$. For this sense of computation, the following corollary says that the set of seed assemblies that “output” a 1 is a recursive set (not just a recursively enumerable set). This would contradict the fact that the halting set is not recursive.

**Corollary 1** For any tileset $T$ in the RTAM and configuration $C_T$ over $T$, $A(C_T)$ is a recursive set.

**Proof (sketch)** In each of the cases for the sets $S_i$ in Corollary 2, to determine if some $C_T^v$ is contained in one of the $S_i$ subassemblies one need only check whether or not a finite portion of $S_i$ contains a translation of $C_T$, which can be done in a finite number of steps depending only on $|\sigma|$ and $|T|$.

4 Universal computation at $\tau = 2$

In this section we prove that universal computation is possible in the RTAM at $\tau = 2$.

**Theorem 2** The RTAM is computationally universal at $\tau = 2$. Moreover, the class of directed RTAM systems is computationally universal at $\tau = 2$.

**Proof (sketch)** First we note that given a Turing machine $M$, we use Lemma 7 of Cook et al. (2011) to obtain a tile set which simulates $M$ using a zig-zag system in the aTAM. In fact, as noted in Schweller and Summers (2011), we can find a singly seeded compact zig-zag system $T = (T, \sigma, 2)$ with $A_{\sigma}[\tau] = \{\alpha\}$ which simulates $M$. Then the proof of Theorem 2 relies on showing that any compact zig-zag system in the aTAM at temperature 2 can be converted into a directed RTAM system $S$ that is “almost” compact zig-zag. The RTAM system that we construct differs from a compact zig-zag system in that for each row which grows to a length one greater than that of the preceding row, a strength-2 glue is exposed that allows one tile to bind below that row (which isn’t allowed in actual compact zig-zag systems). This results in the possibility of a single “misplaced” tile in each such row, but nevertheless this is sufficient to simulate a Turing machine (and remain directed). In addition, notice that the orientation of the first tile which binds to the seed tile is such that a strength-2 glue is exposed on either the west or the east. Since the two assemblies obtained from the different binding orientations of this tile are the same up to reflection, this does not affect the assembly which $S$ produces. First, we give an example of how to simulate a zig-zag system.

Figure 7 shows how we convert a tile set used in an aTAM $\tau = 2$ system which simulates a binary counter (Fig. 7a) into a tile set used in an RTAM $\tau = 2$ system.

Essentially, a compact zig-zag system is one in which assembly proceeds along a single possible assembly sequence which grows one row completely from left-to-right, then immediately above that grows the next complete row right-to-left, and so on. Also, each row may grow to a length of one tile longer than the row below it.
which simulates a binary counter (Fig. 7b). Figure 7c shows a portion of the assembly of the RTAM system which simulates the counter.

For the remainder of the proof, we say that two tiles \( t \) and \( t' \) are of the same form provided that they have glues of the same strengths on the same sides. Given the singly seeded compact zig-zag system \( T = (T, \sigma, 2) \) which simulates \( M \), w.l.o.g. we assume that \( T \) grows from south to north. Let \( S = (S, \sigma', 2) \) be an RTAM system defined as follows.

First, for clarity, we rewrite the glues contained on tiles of \( T \) in terms of their complementary glues. We use the convention that if a glue \( g \) appears on the north or east side of a tile, then we simply write \( g \). If a glue \( g \) appears on the south or west of a tile, then we rewrite that glue as \( g' \). Next, we set \( \sigma' \) to be the same as the tile which composes \( \sigma \). It follows from the constructive proof of Lemma 7 in Cook et al. (2011), that for any \( t \in T \) such that \( t \) has a strength 2 glue on its south, it must be of the same form as tile \( A \) shown in Fig. 8a up to reflection (except for the special case tile which attaches directly to the seed and is discussed later). For each \( t \in T \) with a strength 2 glue on the south, we create a tile \( t' \in S \) which consists of the same south glue as the one on \( t \), but has a copy of the east/west glue of \( t \) such that \( t' \) has identical east and west glues. The form of \( t' \) is that of tile \( A' \) shown in Fig. 8b. Henceforth, we refer to tiles of the same form (up to reflection across the vertical axis) as the tile labeled \( B \) in Fig. 8a as corner tiles. For each corner tile \( t \in T \), we form a tile \( t' \in S \) which consists of the same east and west glues as \( t \), but has a copy of the north/south glue of \( t \) such that \( t' \) has identical south and north glues. Thus, in our RTAM tile set \( S \), corner tiles in \( T \) take on the form of the tile labeled \( B' \) in Fig. 8b. Notice that tiles which require cooperation to bind are necessarily oriented upon binding. Consequently, for all tiles \( t \in T \) which bind cooperatively (i.e. those not of the form \( A \) or \( B \) shown in Fig. 8a), we add tile \( t \) to our RTAM tile set \( S \).

Now, we show that the northernmost row of tiles in the assembly obtained from \( S \) contains the final configuration of the tape of \( M \) and its final state just as it is in \( T \). Because tiles which bind cooperatively in \( S \) always orient themselves in the same manner in which they appear in \( T \), we only examine the cases where \( \tau \) strength glues are used in binding. Up to reflection, all of the cases of tiles which bind with strength 2 are of the same form as tiles \( A' = C' \) in Fig. 8b. Tiles that are of the same form as the tile labeled \( A' \) in Fig. 8b do not pose a problem since the extra glue it has relative to its counterpart in \( t \) is strength 1 and exposed on the east/west edges of the assembly in such a way that no other tiles can bind to expose a glue with which it can cooperate.

Next, we examine the binding of a tile of the same form as \( B' \) in Fig. 8b. Upon binding, a tile of the same form as \( B' \) will allow for the binding of a tile which its counterpart in \( T \) does not. That is, it will allow for the binding of a tile to its south. But, the constructive proof of Lemma 7 in Cook et al. (2011) is such that only tiles of the same form as \( A' \) in Fig. 8b can flip and bind to the south of the \( B' \) tile. Now, observe that due to the zig-zag nature of growth in \( S \), the
tile located diagonally to the south-east of the misplaced $A'$
tile is of the same form as the tiles $B'$ or $D'$ in Fig. 8b.
Notice that this means the misplaced $A'$ tile cannot
cooperatively place a tile with any other glue, and thus
the placement of “misplaced” tiles here must halt.

It follows from the compact zig-zag Turing machine
simulation construction used in Schweller and Summers
(2011), that tiles of the form $C'$ appear only once in $x$.
More specifically, tiles of the same form as $C'$ always
attach to the seed tile. Thus, in our case it does not matter
which orientation $C'$ has upon binding since the two
assemblies obtained from allowing $C'$ to bind in its two
orientations are the same up to reflection.

Consequently, the “misplaced” tiles in the assembly
produced by $S$, that is the tiles that differ from the tiles in
the assembly produced by $T$ in location or label, are tiles
that bind to the south of corner tiles. Since this does not
interfere with the simulation of $M$, $S$ is computationally
universal. To prove the second statement of the theorem,
notice that the constructed RTAM system is directed. □

5 Self-assembly of shapes in the RTAM at $\tau = 1$

In this section, we discuss the self-assembly of shapes in the
RTAM, especially the commonly used benchmark of
squares. At temperature 2, using a zig-zag binary counter
similar to that used in Sect. 4, $n \times n$ squares can be built
using the optimal $\log n/\log \log n$ tile types following the
construction of Rothemund and Winfree (2000) with only
trivial modifications. Similarly, many shapes which can be
weakly self-assembled in the temperature-2 aTAM can be
built in the temperature-2 RTAM. On the other hand, shapes
with single-tile-wide branches which are not symmetric are
impossible to strictly self-assemble in the RTAM.

At temperature-1, however, the differences between the
powers of the aTAM and RTAM appear to increase. Here
we will demonstrate that squares whose sides are of even
length cannot weakly (or therefore strictly) self-assemble
in the RTAM at $\tau = 1$, although any square can strictly
self-assemble in the $\tau = 1$ aTAM. We then prove a tight
bound of $n$ tile types required to self-assemble an $n \times n$
square for odd $n$ in the RTAM at $\tau = 1$. (Which is, inter-
estingly, better than the conjectured lower bound of $2n - 1$
for the $\tau = 1$ aTAM.)

5.1 For even $n$, no $n \times n$ square self-assembles
in the $\tau = 1$ RTAM

Theorem 3 For all $n \in \mathbb{Z}^+$ where $n$ is even, there exists
no RTAM system $T = (T, \sigma, 1)$ where $|\sigma| = 1$ and $T$
weakly (or strictly) self-assembles an $n \times n$ square.
overall binding graph of \( \alpha \), \( \pi^x_a \) must somehow also be connected to the tile at \( \mathbf{d} \). Let \( \pi^x_a \) be a path in the binding graph which contains both \( \mathbf{d} \) and some tile in \( \pi^x_a \). Note that this may just be the tile at \( \mathbf{d} \) since \( \pi^x_a \) may contain that location. We now define path \( \pi' \) as the longest of the following two paths formed by possibly combining subpaths of \( \pi^x_a \) and \( \pi^x_d \): (1) a path which connects \( \mathbf{d} \) and \( \mathbf{c} \) and contains a tile in \( B \), or (2) a path which connects \( \mathbf{a} \) and \( \mathbf{d} \) and contains a tile in quadrant \( B \). For ease of discussion, we will talk about \( \pi^x_a \) as being a directed path from \( \mathbf{a} \) to \( \mathbf{c} \), while noting that we don’t know the actual ordering of its growth. Such a \( \pi' \) must exist because the point at which \( \pi^x_d \) intersects \( \pi^x_a \) must be either (1) before it has entered \( B \), which yields case 1 for \( \pi' \), (2) after it has left \( B \), which yields case 2 for \( \pi' \), or (3) within \( B \) in which case either holds. Note that \( \pi^x_a \) could perhaps enter and leave \( B \) multiple times, which would only strengthen our argument, but for simplicity we’ll simply make use of the first time it enters quadrant \( B \) and the last time it leaves quadrant \( B \) for the above argument.

Without loss of generality, we will assume case 1 for path \( \pi' \), and note that we now have found the following path in the binding graph of \( \alpha \). Path \( \pi' \) includes the tile at location \( \mathbf{d} \), contains at least one tile in quadrant \( B \), and also includes the tile at location \( \mathbf{c} \). (See Fig. 10a for an example.) For a square of dimension \( n \), it must be the case that \( \pi' \) contains a minimum of \( n \) pairs of tiles bound to each other on their east/west edges.

Now we check to see if the seed \( \sigma \) is contained within \( \pi' \). If not, we define \( \pi_{\alpha} \) as the minimum path in \( \alpha \)’s binding graph which contains \( \sigma \) and some tile in \( \pi' \), else \( \pi_{\alpha} \) is simply \( \sigma \). Now we define a new valid assembly sequence, \( \alpha' \) for \( T \) as follows.

Let \( \mathbf{x} \) denote the location of the intersection of \( \pi^x_{\mathbf{a}} \) and \( \pi' \), and \( \pi^x_{\mathbf{c}} \) be the portion \( \pi' \) from \( \mathbf{x} \) to \( \mathbf{c} \) and let \( \pi^x_{\mathbf{d}} \) be the portion of \( \pi' \) from \( \mathbf{x} \) to \( \mathbf{d} \). Without loss of generality, we will assume that the tile on \( \pi^x_{\mathbf{a}} \) which binds to \( \sigma \) attaches to the north of \( \sigma \). (If this is not the case, then all following directions can simply be rotated by the same angle as the difference of the direction of the first binding from north.)

Starting from \( \sigma \), \( \alpha' \) attaches one tile at a time from \( \pi^x_{\mathbf{a}} \) so that output glues are exposed on the north or east edge of each tile. In other words, as tiles attach, flip them so that the next tile to attach does so at a tile location that is north or east of the previously attached tile. (Figure 11 shows how this must be possible for each tile.) This causes all inputs along \( \pi^x_{\mathbf{a}} \) to be on the south or west sides of tiles, and outputs on the north and east. Furthermore, place the tile from location \( \mathbf{x} \) so that the exposed glue used for the binding of \( \pi^x_{\mathbf{c}} \) is on its the north or west, and the exposed glue used for the binding of \( \pi^x_{\mathbf{d}} \) is on its south or east. Figure 12 shows how this can be done. (Note that depending on where and from which direction \( \pi^x_{\mathbf{a}} \) intersects \( \pi' \), it could be the case that the output sides which bind to \( \pi^x_{\mathbf{c}} \) and \( \pi^x_{\mathbf{d}} \) are instead south or east, and north or west, respectively. However, again this doesn’t change the argument as the following directions can be rotated appropriately, so we assume the output for \( \pi^x_{\mathbf{c}} \) is on the north or west for ease of discussion.)

Now we assemble the path \( \pi^x_{\mathbf{c}} \) as follows. Starting from the tile of location \( \mathbf{x} \), attach one tile at a time so that the output glues are exposed on the north or west edge of each tile. This results in a version of \( \pi^x_{\mathbf{c}} \) which grows strictly up and to the left. Such growth is possible because of the set of reflections possible and the fact that no tile previously placed at \( \pi^x_{\mathbf{a}} \) or \( \pi^x_{\mathbf{d}} \) can block the placement. The fact that blocking can’t occur is due to the fact that the stretched version of \( \pi^x_{\mathbf{c}} \) grows strictly up and/or to the left so all previous tiles placed for \( \pi^x_{\mathbf{c}} \) cannot block, and after the first tile placed after the final tile of \( \pi^x_{\mathbf{c}} \), the newly forming path is either completely above \( \pi^x_{\mathbf{d}} \) or to the left of the topmost tile of that path. Next we assemble the path \( \pi^x_{\mathbf{d}} \) in an analogous manner, but down and to the right. Starting from the tile of location \( \mathbf{x} \), attach one tile at a time so that the output glues are exposed on the south or east edge of each tile. For the same but reflected reasons as for the growth of the modified version of \( \pi^x_{\mathbf{c}} \), \( \pi^x_{\mathbf{d}} \) can be grown in this way. See Fig. 10b for an example of these “stretched” paths.

The tile types which are at the ends of \( \pi^x_{\mathbf{c}} \) and \( \pi^x_{\mathbf{d}} \) were at locations \( \mathbf{c} \) and \( \mathbf{d} \) in \( \alpha \), so they must be of types which are in \( B \subseteq T \) (i.e. “black” tile types). However, since there are \( \geq n \) pairs of tiles connected via east/west glues in \( \pi' \) and it is now maximally stretched, it must be \( \geq n + 1 \) tiles wide (i.e. from leftmost to rightmost tiles). Thus, this is a producible assembly which has “black” tiles at a distance...

![Fig. 10](a) Example paths in an 8 \times 8 square (i.e. of even dimension). The path from corners \( \mathbf{a} \) and \( \mathbf{c} \) is composed of tiles of different colors, from corner \( \mathbf{d} \) to a point on that path is dark grey, from the seed to that intersection is black, and the path to be stretched out is outlined in black. (b) Example paths \( \pi^x_{\mathbf{a}}, \pi^x_{\mathbf{d}} \), and \( \pi^x_{\mathbf{d}} \) stretched out when assembled by \( \alpha' \).
for all \( n \) necessary to self-assemble a square of odd dimension. We now prove tight bounds on the number of tiles necessary to self-assemble a square of odd dimension.

**Theorem 4** For all \( n \in \mathbb{Z}^+ \) where \( n \) is odd, an \( n \times n \) square strictly self-assembles in an RTAM system \( T = (T, \sigma, 1) \) where \( |T| = n \) and \( |\sigma| = 1 \).

**Proof** To prove Theorem 4, we provide the following construction which demonstrates how to build an RTAM TAS which strictly self-assembles an \( n \times n \) square for any odd \( n \). The basic idea is that we place the seed tile in the middle of the square, and from both its top and bottom it grows copies of a reflected column of tiles which grow to the top and bottom of the square, respectively. This requires one tile type for the seed and \( (n - 1)/2 \) tile types for the column (which grows in both directions). Each of the seed and the tiles of the vertical column have the same glue on both east and west sides. From each of those a row of \( (n - 1)/2 \) unique tile types grows to the left or right boundary of the square. This construction is robust to any reflection of the tiles and strictly self-assembles an \( n \times n \) square using exactly \( 1 + 2(n - 1)/2 = n \) tile types.

In Fig. 13b we show the templates used to generate the necessary tile types to self-assemble an \( n \times n \) square for odd \( n \), and in Fig. 13a we show an example of the terminal assembly for \( n = 3 \). For all odd \( n \), the seed is exactly as in Fig. 13b. Then for each \( m \) where \( 0 < m < (n - 1)/2 \), we create a unique tile type from the tile type template labeled “Vx”. We do this by replacing each “x” (i.e. those of the tile label and the south glue) with the value “m”, and the “y” of the north glue with the value “\( m + 1 \)”. (Note the “prime” markings of some glues which denote they are complementary to the “unprimed” versions, and for all tile types created from these templates we keep those markings unchanged, which is important in restricting the potential interactions.) In this way, we create the tile types for the horizontal rows which grow outward from the central column, for each \( m \) where \( 0 < m < (n - 1)/2 \) we create a unique tile type from the tile type template labeled “Hx”. We do this by replacing each “x” (i.e. those of the tile label and the west glue) with the value “m”, and the “y” of the east glue with the value “\( m + 1 \)”. In this way we create the tile types for the horizontal row between the central column and left or right most locations. To make the top/bottom tile type, we start with the “Vz” template and replace the “z” of the label and south glue with the value \( (n - 1)/2 \). To make the tile types for the horizontal rows which grow outward from the central column, for each \( m \) where \( 0 < m < (n - 1)/2 \) we create a unique tile type from the tile type template labeled “Hx”. We do this by replacing each “x” (i.e. those of the tile label and the west glue) with the value “m”, and the “y” of the east glue with the value “\( m + 1 \)”. In this way we create the tile types for the horizontal row between the central column and left or right most locations. To make the left/right tile type, we start with the “Hz” template and replace the “z” of the label and west glue with the value \( (n - 1)/2 \). This completes the creation of exactly \( n \) tile types: 1 for the seed, \( 2((n - 1)/2 - 1) \) for the interiors of the column and rows, and 1 each for the top/bottom tile type and left/right tile type. The RTAM TAS \( T \) consisting of this tile set, a seed consisting of a copy of the seed tile in any valid orientation at the origin, and \( \tau = 1 \) strictly self-assembles an \( n \times n \) square and is directed. This is because the only tiles which can attach to the north and south of the seed are properly reflected versions of the “V1” tile type, and then to the north/south of those two tiles properly reflected copies of “V2”, etc. until copies of the
\(V(n - 1/2)\) tile type cap off the north and south growth of the column. Regardless of the reflections of the \(Vx\) and \(Vz\) tiles across the vertical axis, the only subassemblies that can grow to the left and right are the arms consisting of the \(Hx\) and \(Hz\) tile types. All producible terminal assemblies of \(T\) are equivalent up to translation and reflection, and in the shape of an \(n \times n\) square.

The proof of Theorem 4 gives a scheme for obtaining the tileset for any given odd \(n\) that exploits the fact that for \(n\) odd, an \(n \times n\) square in \(\mathbb{Z}^2\) is symmetric across a row and column. While Theorem 4 pertains to squares, a simple modification of the proof shows the following corollary.

**Corollary 2** For all \(n, m \in \mathbb{Z}^+\) where \(n\) is odd, an \(n \times m\) rectangle strictly self-assembles in an RTAM system \(T = (T, \sigma, 1)\) where \(|T| = \frac{n + m}{2}\) and \(|\sigma| = 1\).

We also prove that the upper bound of Theorem 4 is tight, i.e., an \(n \times n\) square, where \(n\) is odd, cannot be self-assembled using less than \(n\) tile types.

**Theorem 5** For all \(n \in \mathbb{Z}^+\) where \(n\) is odd, there exists no RTAM system \(T = (T, \sigma, 1)\) where \(|T| < n\) and \(|\sigma| = 1\) such that \(T\) weakly (or strictly) self-assembles an \(n \times n\) square.

**Proof** We prove Theorem 5 by contradiction. Therefore, assume that for some \(n \in \mathbb{Z}^+\) such that \((n \mod 2) = 1\), there exists an RTAM system \(T = (T, \sigma, 1)\) such that \(|\sigma| = 1\) and \(|T| < n\), and \(T\) weakly self-assembles the \(n \times n\) square \(S\). It must therefore be the case that for some subset of tile types \(B \subseteq T\), for all \(x \in A[B]\) there exist \(r \in R\) and \(v \in \mathbb{Z}^2\) such that for \(x = F_r(x)\), \(x^{-1}(B) = S\).

We choose an arbitrary \(x \in A[B]\) and, using the notation of Fig. 9, note that in the binding graph of \(x\), there is a path \(\pi_a^x\) which connects the tiles in locations \(a\) and \(c\) (i.e., opposite corners of the square). Furthermore, the tile types of the tiles at locations \(a\) and \(c\) must be in \(B \subseteq T\), and tiles of types in \(B\) can be no further apart from each other (in the plane) than a Manhattan distance of \(2n - 1\). Since the seed \(\sigma\) may or may not be on \(\pi_a^x\), define the location \(x\) to be either (1) the location of \(\sigma\) if it is on \(\pi_a^x\), or (2) the location of the intersection of \(\pi_a^x\) and the shortest path in the binding graph which contains \(\sigma\) and some tile in \(\pi_a^x\).

Note that such a location \(x\) must exist since \(x\) must be connected. Define the path \(\pi_a^x\) as the path from \(\sigma\) to \(x\) and note that if \(\sigma\) is on \(\pi_a^x\) then \(\pi_a^x\) consists of a single tile. Also, we can now define \(\pi_a^x\) and \(\pi_c^x\) as the paths from \(x\) to \(a\) and \(c\), respectively. (See Fig. 14 for a possible example of such paths in \(x\).)

We will now define a valid assembly sequence \(\gamma\) in \(T\) which builds an assembly \(\gamma\) which consists of modified versions of the paths \(\pi_a^x\) and \(\pi_c^x\), and which places tiles of types in \(B \subseteq T\) (i.e., “black” tiles) at points further apart than \(2n - 1\), proving that \(T\) does not weakly self-assemble \(S\). Let \(k = |\pi_a^x|\) and let \(\gamma\) start from \(\sigma\) and first build a modified version of \(\pi_a^x\) in the following way. If \(k = 1\), stop. Else, without loss of generality assume that \(\sigma(\pi_a^x(1))\) attaches to the north of \(\sigma\) in \(x\). (If it binds on the east, south, or west, for the following argument used to construct \(\gamma\), rotate all directions used clockwise by 90, 180, or 270 degrees respectively.) For each \(i\) where \(0 < i < k\), place a tile of the same type as \(\sigma(\pi_a^x(i))\) but reflected so that its input side is either south or west, and its output side is either north or east. Note that for any pair of input and output sides, it is possible to reflect a tile to meet this criteria (see Fig. 11). Place the tile of that type and orientation so that it binds with the \((i - 1)\)th tile of the newly created path. Note that this also must be possible because, following from \(\sigma\), every tile is oriented so that the side used as its output side in \(\pi_a^x\) is north or east and has a
At this point, \( \mathcal{A}' \) grows assembly \( \mathcal{A}' \) which is a path like that in Fig. 15a that starts from the seed, grows a stretched version of \( \pi_0^a \), then a stretched version of \( \pi_0^c \) (as connected stretched versions of \( \pi_1^a \) and \( \pi_1^c \)) from the end of that path. Since in \( \mathcal{A} \), \( \pi_0^c \) connects the bottom-left corner of the square \( S \) with the top-right corner, it must be the case that \( |\pi_0^c| \geq (2n-1) \). Since \( |T| \leq (n-1) \), and \( 2(n-1) = 2n - 2 \), by the pigeonhole principle we know that along \( \pi_0^c \) (and therefore also its stretched version), it must be the case that at least 3 tiles of some same type appear.

With 3 copies of the same tile type appearing along \( \pi_0^c \), it must be the case that a path can be formed which connects two of the copies in such a way that it enters each of the two tiles along different sides of that tile type when it's in its default configuration. (See Fig. 15b for an example where a path can be formed between the top copy and the bottom copy, with the top being entered via the \( a \) side and the bottom via the \( b \) side, while the path between the top and middle copies enters each from the \( a \) side, and the path between the middle and bottom copies enters each from the \( b \) side.) Let \( f \), \( g \), and \( h \) be the locations of the repeated tile type in \( \mathcal{A}' \), and let \( f \) and \( g \) be the locations which can be connected by a path entering the tiles at those locations from different sides, and let \( f \) be the furthest of \( f \) and \( g \) from the final tile of \( \pi_0^c \) (or the southernmost if they tie). Without loss of generality, assume that \( f \) is south of \( g \) (otherwise, rotate the following directions).

We now modify \( \mathcal{A}' \) so that it builds the stretched copies of \( \pi_0^a \), \( \pi_0^c \), and \( \pi_0^e \) until it is about to place the tile before the one at \( f \). At this point, it rotates the tile for that position so that the side which serves as input for the tile at \( g \) is facing either south or east, and places that tile. Then, since the tile at \( g \) was above and/or to the left of that at \( f \), the side it used as output is now oriented so that it can be used as output from the copy at \( g \). We add to \( \mathcal{A}' \) the tile placements which now make a copy of the path \( \pi_0^f \), again orienting the final tile to allow yet another copy. We make \( \mathcal{A}' \) add \( 2n \) copies of
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symmetric with respect to a horizontal line placed makes the endpoints strictly further from each other. not weakly (or therefore strictly) self-assemble neither does \( Z \) assemble in the RTAM. Then we prove 3 theorems about showing that sufficiently symmetric shapes weakly self-assemble in the RTAM. 5.3 Assembling finite general shapes in the RTAM In this section we first give a corollary of Theorem 4 showing that sufficiently symmetric shapes weakly self-assemble in the RTAM. Then we prove 3 theorems about assembling finite shapes in \( \mathbb{Z}^2 \) in the RTAM. These theorems show that the assembly of finite shapes in singly seeded RTAM systems is quite a bit different than the assembly of finite shapes in the \( aTAM \) by singly seeded systems.

Given a shape \( S \) in \( \mathbb{Z}^2 \), let \( \chi_S \) denote the characteristic function of the set \( S \). That is, \( \chi_S(x, y) = 1 \) if \( x \in S \) and \( \chi_S(x, y) = 0 \) otherwise. Then, we say that a shape \( S \) is odd-symmetric with respect to a horizontal line \( y = l \) (respectively, vertical line \( x = l \)) for \( l \in \mathbb{Z} \) iff for all \( (a, b) \in \mathbb{Z}^2 \), \( \chi_S(a, l - b) = \chi_S(a, l + b) \) (respectively, \( \chi_S(l - a, b) = \chi_S(l + a, b) \)). If there exists a line such that a shape, \( S \), is odd-symmetric with respect to this line, we say that the shape is odd-symmetric. Given a shape \( S \), we call the smallest rectangle of points in \( \mathbb{Z}^2 \) containing \( S \) the bounding-box for \( S \).

Let \( R \) denote the bounding box of an odd-symmetric shape \( S \), and let \( n, m \in \mathbb{N} \) be the dimensions of \( R \). A simple modification to the proof of Theorem 4 where a tile is labeled with a label \( B \) if and only if it corresponds to points in \( S \), shows that odd-symmetric shapes can weakly assemble in the RTAM.

Corollary 3 Given a shape \( S \) in \( \mathbb{Z}^2 \), if \( S \) is odd-symmetric, then there exists an RTAS \( T = (T, \sigma, \tau) \) such that \( |\sigma| = 1 \), \( \tau \geq 1 \), and \( T \) weakly self-assembles \( S \).

Additionally, if one is willing to build \( 2 \) mirrored copies of the shape in each assembly, then any finite shape can be weakly self-assembled in the RTAM at \( \tau = 1 \), along with its mirrored copy (at a cost of tile complexity approximately equal to the number of points in the shape) by simply building a central column (or row) from which identical copies of hardcoded rows (or columns) grow, so that each side grows a reflected copy of the shape in hardcoded slices.

We say that a TAS (in either the \( aTAM \) or the RTAM) \( T \) is called mismatch-free if for every producible assembly \( x \in \mathcal{A}[T] \) with two neighboring tiles with abutting edges \( e_1 \) and \( e_2 \), either \( e_1 \) and \( e_2 \) do not have glues or \( e_1 \) and \( e_2 \) have glues with matching labels and strengths. Then, for singly seeded RTAM systems, any finite connected shape can be strictly assembled by a mismatch-free system. However, Theorems 6, 7, and 8 show that assembling shapes in the RTAM is more complex.

Theorem 6 There exists a finite connected shape \( S \) in \( \mathbb{Z}^2 \) that weakly self-assembles in a singly seeded RTAM system such that there exists no singly seeded RTAM system that strictly self-assembles \( S \).

Proof Consider the shape \( S \) depicted in Fig. 16a. By Corollary 3, there is a singly seeded RTAM system that weakly self-assembles \( S \). Therefore, to complete the proof, we show that this shape cannot be strictly self-assembled by a singly seeded system in the RTAM by contradiction. Suppose that \( T = (T, \sigma, \tau) \) is such a system, and let \( x \) in \( \mathcal{A}_\tau[T] \) be a terminal assembly such that \( \text{dom}(x) = S \). Without loss of generality, assume that the seed for \( T \) is at a location shown in dark grey in Fig. 16a. Then, by applying a single tile reflection, we can modify the assembly sequence of \( x \) to obtain a producible assembly \( \beta \) depicted in Fig. 16b, contradicting the fact that \( S \) strictly self-assembles in \( T \).

Theorem 7 There exists a finite shape \( S_2 \) in \( \mathbb{Z}^2 \) that can be strictly self-assembled by a singly seeded RTAM system, but is such that every such system which strictly self-assembles it is not directed.

Consider the shape \( S_2 \) depicted in Fig. 16c. First, to see that \( S_2 \) can be strictly self-assembled in the RTAM by a singly seeded system, consider the tile set shown in Fig. 17a. Then, to complete the proof, we show by contradiction that \( S_2 \) cannot be strictly self-assembled by a directed singly seeded system in the RTAM. Suppose that \( T = (T, \sigma, \tau) \) is a directed singly seeded system that strictly assemblies \( S_2 \), and let \( x \) in \( \mathcal{A}_\tau[T] \) be a terminal assembly such that \( \text{dom}(x) = S_2 \). Without loss of generality, assume that the location of the seed in \( x \) is at a location shown in dark grey in Fig. 16c. Let \( t_i \) denote the tiles of \( x \) located at positions \( L_i \). Note that the south glue of either \( t_1 \) or \( t_5 \) is bound to \( x \) with
strength \( \tau \). Without loss of generality, assume that the south glue of \( t_1 \) is bound to \( x \) with strength \( \tau \). If \( t_1 \) is bound to \( t_2 \) with strength \( \tau \), \( t_2 \) is bound to \( t_3 \) with strength \( \tau \), and \( t_3 \) is bound to \( t_4 \) with strength \( \tau \), then by modifying the assembly sequence for \( x \) and applying the appropriate reflections, we can can give an assembly that is producible in \( T \), and is a total of 11 tiles wide. An example of such an assembly is shown in Fig. 16d. This contradicts the assumption that \( T \) strictly self-assembles \( S \). On the other hand, if the strength of the bond between \( t_i \) and \( t_{i+1} \) is less than \( \tau \) for \( 1 \leq i \leq 4 \), then the south glues of \( t_1 \) and \( t_5 \) must each be bound with strength \( \tau \). Therefore, the assembly \( x' \) depicted as gray tiles in Fig. 17b must be producible in \( T \).

In the following argument, without loss of generality, we assume that for every glue, \( g \), on a tile in \( T \), there is a matching glue \( g' \) on some (possibly the same) tile in \( T \). Then, since two tiles must be able to attach so that they occupy the two tile locations to the west of \( t_1 \), after applying a reflection of \( t_1 \), these same tiles, call them \( a \) and \( b \), must be able to attach with strength \( \tau \) to the east to \( t_1 \) with the eastmost tile, \( b \), located at \( L_3 \). Similarly, (prior to \( a \) and \( b \) binding) three tiles, call them \( a', b', \) and \( c' \), must be able to attach with strength \( \tau \) to the west of \( t_5 \), with \( a' \) located at \( L_4 \), \( b' \) located at \( L_3 \), and \( c' \) located at \( L_2 \). Notice that \( b' \) must have two strength-\( \tau \) glues. Hence, nondeterministically, either \( b \) or \( b' \) will be at location \( L_3 \). Finally, note that \( b \) and \( b' \) cannot have the same tile type. This follows by contradiction; suppose that \( b \) and \( b' \) are of the same type. Then, after binding, \( b \) must expose a strength-\( \tau \) glue. However, when the westermmost tile that attaches to the west of \( t_1 \) is \( b \), this strength-\( \tau \) glue would allow for a tile to bind, giving a producible assembly with a domain that is not contained in \( S \), regardless of reflection and translation. This contradicts the assumption that \( T \) strictly self-assembles \( S \).

Finally, since in any terminal assembly of \( T \), either \( b \) or \( b' \) will be at location \( L_3 \), and \( b \) and \( b' \) do not have the same tile type, we conclude that \( T \) is not directed.

**Theorem 8** There exists a finite shape \( S \) in \( \mathbb{Z}^2 \) such that every singly seeded RTAM system that strictly self-assembles \( S \) is not mismatch-free.

**Proof** Let \( S \) be the shape shown in Fig. 16c. Then, the proof follows from the proof of Theorem 7, noting that if \( b \) is placed at \( L_3 \), then the west edge of \( a' \) and the east edge of \( b \) must have different glues. Similarly, if \( b' \) is placed at \( L_3 \), then the west edge of \( a \) and the west edge of \( b' \) must have different glues. Therefore, any singly seeded RTAM system that strictly self-assembles \( S \) is not mismatch-free.

### 5.4 Mismatch-free assembly of finite general shapes in the RTAM

Given a shape \( S \), i.e. a finite connected subset of \( \mathbb{Z}^2 \), we say that a graph of \( S \) is a graph \( G_S = (V, E) \) with a vertex at the center of each point in \( S \) and an edge between every pair of...
vertices at adjacent points of $S$. A tree of $S$, $T_S$, is a graph of $S$ which is a tree. (See Fig. 18 for examples of $S$, $G_S$, and $T_S$.) Given a graph $G = (V, E)$, we say that an axis of $G$ is a horizontal or vertical line of vertices such that there is an edge between each pair of adjacent points on that line. Notice that two distinct axes can be collinear. Given an axis $a$, an axial branch of $T_S$ is a branch of $T_S$ which contains exactly one vertex $v$ on $a$ and all vertices and edges of $T_S$ which are connected to a vertex that does not lie on $a$ and is adjacent to $v$. We say that the branch begins from $v$. Intuitively, an axial branch is a connected component extending from an axis. (See the wide grey highlighted portion of Fig. 18c for an example axial branch off of the central vertical axis shown in black.)

A tree $T_S$ is symmetric across an axis $a$ if, for every vertex $v$ contained on $a$, the branches of $a$ which begin from $v$ are symmetric across $a$. A tree $T_S$ is off-by-one symmetric across an axis $a$ if, for every vertex $v$ except for at most 1, the branches of $a$ which begin from $v$ are symmetric across $a$. See Fig. 18c for an example of such a tree, with the axis $a$ shown in black.

**Definition 4** A tree $T$ is $\epsilon$-symmetric iff for some axis $a$ of $T$, $T$ is off-by-one symmetric across $a$.

**Definition 5** Given a shape $S$ with graph $G_S$, we say that $S$ is $\epsilon$-symmetric if and only if there exists a spanning tree, $T_S$, of $G_S$ such that $T_S$ is $\epsilon$-symmetric.

For an example of an $\epsilon$-symmetric shape $S$, see Fig. 18a. The tree $T_S$ is off-by-one symmetric across the vertical black axis, the branches off of that axis are symmetric across the horizontal grey axes, and the branches off of those axes are symmetric across the vertical grey axes. The following theorem gives a complete classification of finite connected shapes which can be assembled by temperature-1 singly seeded mismatch-free RTAM systems.

**Theorem 9** Let $S \subseteq \mathbb{Z}^2$ be a finite connected shape. There exists a mismatch-free RTAM system $T = (T, \sigma, 1)$ with $|\sigma| = 1$ that strictly assembles $S$ if and only if $S$ is $\epsilon$-symmetric.

**Proof** First, we show that if $S$ is $\epsilon$-symmetric, then there is a singly seeded mismatch-free RTAM system that strictly assembles $S$. Let $T_S$ be a tree for $S$ which satisfies Definition 5. We use $T_S$ to define a tile set $T$ and give a seed tile $\sigma$ such that the system $T = (T, \sigma, 1)$ strictly self-assembles $S$, and then we argue that for any terminal assembly $x$ in $T$, the domain of $x$ is equal to $S$ (up to translation and reflection). $T$ and $\sigma$ are obtained as follows. Notice that we can give a temperature-1 aTAM system $T' = (T', \sigma', 1)$ that assembles the shape $S$ with the properties (1) $T'$ is a singly seeded directed system, (2) for $x' \in A[\alpha][T']$, the binding graph of $x'$ is exactly the tree $T_S$, and (3) the seed tile $\sigma'$ is the single tile located at some point $v$, and for each point $p$ of $S$, there is a unique tile $t'$ in $T'$ such that $x(p) = t'$. More intuitively, when the seed tile of $T'$ is placed at $v$, then assembly proceeds by the binding of unique tiles equipped with glues specific to each point of $S$.

Now, we define the tiles of $T$ to be copies of the tiles in $T'$ and define $\sigma'$ to be $\sigma$. Notice that in the RTAM, if we assume that no tiles are reflected before binding as assembly proceeds, then the terminal assembly, $x$, that results has a domain equal to $S$. Furthermore, since the binding graph of $x$ is $T_S$, a reflected tile, $t$, of $x$ lies on some axis, $a_t$ say, and corresponds to some vertex $v_t$ of that axis. Then, since $T_S$ is $\epsilon$-symmetric and appropriately reflected tiles of $T'$ can still bind to $t$ even when $t$ is reflected prior to binding, the domain of a terminal assembly of $T$ will either (1) be equal to $S$ (This is the case that the axial branches beginning from $v_t$ are equivalent up to reflection about the axis $a_t$. See Fig. 19b.), or (2) be equal to a reflection of $S$ about the line corresponding to the axis $a_0$. (This is the special case that the axial branches beginning from $v_t$ are not equivalent after a reflection about $a_0$. See Fig. 19c.) In either case, it follows that $S$ strictly assembles in $T$.

Now we show that if there is a singly seeded mismatch-free RTAM system that strictly assembles $S$, then $S$ is $\epsilon$-symmetric. Let $T = (T, \sigma, 1)$ be a singly seeded mismatch-free RTAM system that strictly self-assembles $S$, and let $x \in A[\alpha][T]$ be some terminal assembly of $T$. 

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**Fig. 18** a Example $\epsilon$-symmetric shape $S$, b Graph of $S$, $G_S$, c Tree of $S$, $T_S$
First, we note that the binding graph of \( z \) must be a tree, which we denote by \( T_z \). This follows from the fact that if the binding graph contains a cycle, then there are infinitely many producible assemblies in \( T \). See Fig. 20b for an example. Moreover, \( T_z \) must be \( \epsilon \)-symmetric. This can be shown by contradiction as follows.

Suppose that \( T_z \) is not \( \epsilon \)-symmetric. Then, there exists an axis \( a \) of \( T_z \) such that \( T_z \) is not off-by-one symmetric across \( a \). In other words, there are at least two vertices, which we denote by \( v_1 \) and \( v_2 \), of \( T_z \) which lie on \( a \) such that the branches of \( a \) which begin from \( v_1 \) are not symmetric across \( a \) and the branches of \( a \) which begin from \( v_2 \) are not symmetric across \( a \). Now, we can modify the assembly sequence of \( z \) by reflecting the tile, \( t \), located at \( v_1 \) prior to binding, allowing reflected tiles to assemble reflections of the subassemblies which are bound to \( t \), and keeping the original assembly sequence otherwise. Now, as tiles of the reflected subassemblies originating from \( t \) bind, either a mismatch occurs or the reflected subassemblies completely assemble. If a mismatch occurs, we arrive at a contradiction since \( T \) is mismatch-free. If the reflected subassemblies completely assemble, then we note that no translation and reflection of \( \text{dom}(\beta) \) is equal to \( S \). (This is because we essentially chose a sequence where the two asymmetric branches across \( a \) adopted reflections across \( a \) such that the sides of each which would be on the same side of \( a \) are now on different sides of \( a \).) This contradicts the assumption that \( T \) strictly assembles \( S \). Therefore, \( T_z \) is \( \epsilon \)-symmetric. Hence \( S \) is \( \epsilon \)-symmetric. \( \square \)

While Theorem 9 shows exactly which shapes can be assembled without cooperation or mismatches by singly seeded RTAM systems, the following theorem shows that with cooperation, RTAM systems can assemble arbitrary scale factor 2 shapes.

**Theorem 10** Let \( S \subseteq \mathbb{Z}^2 \) be a finite connected shape, and \( S^2 \) be \( S \) at scale factor 2. There exists a mismatch-free RTAM system \( T = (T, \sigma, 2) \) with \(|\sigma| = 1\) that strictly self-assembles \( S^2 \).

**Proof** Let \( S \) be a finite connected shape in \( \mathbb{Z}^2 \) and \( S^2 \) be \( S \) at scale factor 2. Note that we can give a temperature-1 aTAM system \( T' = (T', \sigma', 1) \) that assembles the shape \( S \) with the properties (1) \( T' \) is a singly seeded directed system, (2) for \( x' \in \mathcal{A}[T]' \), the binding graph of \( x' \) is a tree \( T_{x'} \), (3) the location of the seed tile \( \sigma' \) corresponds to a leaf of \( T_{x'} \), and (4) for each point \( p \) of \( S \), there is a unique tile \( t' \) in \( T' \) such that \( x(p) = t' \). For an example of such an assembly \( x' \) see Fig. 22a.

We now give an RTAM system \( T \) based on \( T' \) such that a terminal assembly \( x \) of \( T \) can be obtained from \( x' \) by replacing single tiles of \( x' \) by \( 2 \times 2 \) blocks of tiles, and thus assembles \( S^2 \). aTAM tiles and corresponding RTAM tiles that give rise to this block replacement scheme are
described in Fig. 21. As $2 \times 2$ blocks assemble, cooperation is used to ensure that the orientations of the tiles of a $2 \times 2$ block are fixed relative to the orientation of the seed. Figure 22b depicts a terminal assembly of the RTAM system $T$ based on the aTAM system $T^0$ whose terminal assembly is shown in Fig. 22a.

### 6 Conclusion

We show in Sect. 3 that the temperature-2 RTAM is computationally universal by showing that any Turing machine can be simulated by a directed temperature-2 RTAM system. On the other hand, we show that directed temperature-1 RTAM systems can only weakly self-assemble semilinear sets (a result that has long been conjectured but has been much more difficult to prove in the aTAM). In Sect. 5.4, we also show that for singly-seeded temperature-1 RTAM systems, $N \times N$ squares can be strictly self-assembled by $N$ tile types when $N$ is odd, but this is not possible when $N$ is even. For even $N$, it is interesting to consider the minimal number of tiles needed in a seed for a system to strictly self-assemble an $N \times N$ square. As one is permitted to fix tile orientation of the seed assembly, it is a simple exercise to give a system with a seed assembly of size $N$ from which an $N \times N$ square can strictly self-assemble. Moreover, one can achieve this using a tile set for this system that contains $2N$ tiles. This tile set and seed size have not been shown to be minimal.

Section 5.4 gives a series of preliminary results toward the classification of which finite connected shapes in $Z^2$ can be strictly self-assembled by a singly-seeded RTAM system. In particular, Theorem 9 classifies exactly which finite connected shapes can be strictly self-assembled by mismatch-free systems. It would interesting to give a classification of finite connected shapes that can be strictly self-assembled by singly-seeded temperature-1 RTAM systems when mismatches are permitted. Also, there are some finite connected shapes that cannot be strictly self-assembled by singly-seeded mismatch-free RTAM systems (see Theorem 8), so one could also attempt to classify finite

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**Fig. 21**

(a) An example seed tile of the aTAM system $T'$. Tiles that represent the seed tile of (a) used to define the RTAM system $T$. The seed of $T$ consists of only the tiles labeled $S$. c An example tile of the aTAM system $T'$. The seed of $T$ consists of only the tile labeled $S$. d Tiles that represent the tile of (e) used to define the RTAM system $T$. Notice that tiles bind in the order $A$, $B$, $C$, and $D$. The tile shown in (e) has “output” glues $g_1$, $g_2$, and $g_3$ exposed. In the case where one or more of these glues is not exposed by a tile, we obtain tiles analogous to those in (d) that represent this single tile as follows. The glues $g_1$, $g_2$, and $g_3$ of the tiles of (d) are exposed if and only if the respective glues of the tile in (e) are exposed. For example, if the tile in (e) exposes a $g_1$ glue, then so will the tile $D$ of (d). Otherwise, $D$ will not expose a $g_1$ glue. Note that the cooperative binding of $B$ fixes the orientation of $B$. Similarly, the cooperative binding of $D$ fixes the orientation of the tile $D$.

**Fig. 22**

(a) An example terminal assembly giving a shape assembled by an aTAM system $T'$, b an example terminal assembly giving a shape assembled by an RTAM system $T$ based on the aTAM system $T'$ whose terminal assembly is shown in (a).
connected shapes that can only be strictly self-assembled by singly-seeded temperature-1 RTAM systems such that every terminal assembly of this system contains a mismatch. Theorem 10 shows that for any finite connected shape, the shape at scale factor 2 can be strictly self-assembled by a singly-seeded temperature-2 RTAM system. For some finite connected shapes (such as the shape in Fig. 22a), this scale factor is likely required. It would be interesting to study which finite connected shapes can be strictly self-assembled by a singly-seeded temperature-2 RTAM system at scale factor 1. Finally, in the proof of Theorem 10, we show how a singly-seeded temperature-2 RTAM system can strictly self-assemble any finite connected shape \( S \) at scale factor 2 by taking an \( aTAM \) system that strictly self-assembles \( S \) at scale factor 1 and then performing a \( 2 \times 2 \) block replacement of the tiles in the \( aTAM \) system by tiles that ultimately define a tile set for an RTAM system. Extending this idea, one could ask if any temperature-1 (or 2) \( aTAM \) system can be simulated by a temperature-2 RTAM system.

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