On Goldbach’s Conjecture

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Abstract

It is shown that if every odd integer $n > 5$ is the sum of three primes, then every even integer $n > 2$ is the sum of two primes. A conditional proof of Goldbach’s conjecture, based on Cramér’s conjecture, is presented. Theoretical and experimental results available on Goldbach’s conjecture allow that a less restrictive conjecture than Cramér’s conjecture be used in the conditional proof. A basic result of the Maier’s paper on Cramér’s model is criticized.

1 Introduction

In 1742 Goldbach wrote a letter to Euler conjecturing that every integer greater than 2 is the sum of three prime numbers. Euler replied that this conjecture breaks up into two: every even integer is the sum of two primes; every odd integer is the sum of three primes. The conjecture “every even integer $n > 2$ is the sum of two primes” is now known as Goldbach’s conjecture and the conjecture “every odd integer $n > 5$ is the sum of three primes” is known as “the weaker”, “the odd” or “the ternary” Goldbach’s conjecture.

The ternary Goldbach’s conjecture, abbreviated here as “ternary GC”, is considered the easiest of the two cases. In 1937 Vinogradov proved that the ternary GC is true for sufficiently large odd number. In 1956 Borodzkin showed that odd numbers greater than $3^{315}$ are sufficiently large in Vinogradov’s proof. This bound was reduced to $e^{11563} (\approx 10^{43000})$ by Chen and Wang in 1989 and to $e^{9715} (\approx 10^{7194})$ in 1996.

The Goldbach’s conjecture is known to be true up to $10^{16}$. Deshouillers, te Riele and Saouter have checked it up to $10^{14}$, Richstein up to $4 \times 10^{14}$ and Silva up to $1 \times 10^{16}$.

Deshouillers et al. outlined a proof in which if the Generalized Riemann Hypothesis holds, then the ternary GC is true. As far as we are concerned, there is not an analogous conditional proof for Goldbach’s conjecture.

2 The same truth value

Let us notice that if Goldbach’s conjecture is true then the ternary GC is true. In the case Goldbach’s conjecture is true if the ternary GC is true, the conditional proof of Deshouillers et al. can be used to conditionally prove Goldbach’s conjecture.

Theorem 2.1 If every odd integer $n > 5$ is the sum of three primes, then every even integer $n > 2$ is the sum of two primes.

Proof:

Let us assume that exists an even integer $m$ greater than 2 that can not be expressed as the sum of two primes, that is,

$$\forall p \forall q \ [p + q \neq m] \quad (1)$$

where $p$ and $q$ belong to the set of prime numbers. The formula (1) can be rewritten as

$$\forall p \forall q \ [(p + 1) + q \neq m + 1] \quad (2)$$
The integers \( p \) and \( q \) are not equal to 2 because if one is equal to 2 the other must be equal to 2 and this contradicts the hypothesis made about \( m \). The number \( p + 1 \) is an even integer and it can be expressed as the sum of two odd integers \( j \) and \( k \), that is,

\[
(p + 1) + q = j + k + q \neq m + 1
\]

If the ternary GC is true there are three prime numbers \( a \), \( b \) and \( c \) such that

\[
a + b + c = m + 1
\]

Since that \( m + 1 \) is an odd integer we have the alternatives: (i) \( a \), \( b \) and \( c \) are odd integers or (ii) \( a \) and \( b \) are equal to 2 and \( c \) is an odd integer. Considering the alternative (i) and comparing (3) with (2) and (3) we obtain a contradiction. Considering the alternative (ii) we have

\[
3 + c = m
\]

Comparing (3) with (6) we again obtain a contradiction. The reason of the contradictions is the hypothesis that “there is an even integer \( m \) that can not be expressed as the sum of two primes”. Therefore if every odd integer \( n > 5 \) is the sum of three primes, then every even integer \( n > 2 \) is the sum of two primes.

### 3 A conditional proof of the ternary GC

Let \( A \) be the well-ordered set of odd integers greater than 5. Let us denote by \( B \) the finite well-ordered subset of \( A \) such that (i) each element of \( B \) is a value for which is unknown if the ternary GC is true, (ii) each element of \( A - B \) is a value for which the ternary GC is true and (iii) the set of elements of \( A \) less than \( \alpha \) is not empty.

\[
B = \{\alpha, \alpha + 2, \alpha + 4, \ldots, \beta - 4, \beta - 2, \beta\}
\]

If Goldbach’s conjecture is true for all even integers less than \( \alpha \) and if any element \( n \) of \( B \) satisfies an equation of the form

\[
n = p + r
\]

where \( p \) is a odd prime number and \( r \) belongs to the well-ordered subset

\[
\{4, 6, 8, \ldots, \alpha - 5, \alpha - 3, \alpha - 1\}
\]

of \( A \), we have that the ternary GC is true for all elements of \( B \).

If the gap between each prime less or equal to \( \beta - (\alpha - 1) \) and its consecutive prime is less than or equal to \( \alpha - 4 \), then some \( p \) that satisfies (7) exists for any element \( n \) of \( B \). Let us assume that Cramér’s conjecture \( 3 \) is true, that is,

\[
\max_{p_n \leq k} (p_{n+1} - p_n) \sim \log^2 k
\]

where \( k \) is an integer and \( p_n \) is the \( n \)-th element of the well-ordered set of prime numbers. Substituting \( k \) by \( \beta \) in (9), we have

\[
\max_{p_n \leq \beta} (p_{n+1} - p_n) \sim \log^2 \beta
\]

Let us consider \( \beta \) equal to \( 10^{7194} \), with this value of \( \beta \) we obtain a maximum gap of

\[
\sim \left( \frac{7194}{\log_{10} e} \right)^2 < 274400000
\]
Theorem 3.1 If

\[
\max_{p_n \leq \beta} (p_{n+1} - p_n) < \log^r \beta \tag{12}
\]

where

\[
\log^r \beta = \alpha \tag{13}
\]

\[
\alpha > 10 \times \log^2 \beta \tag{14}
\]

and the ternary GC is true for odd integers less than \(\alpha\) and is also true for odd integers greater than \(\beta\), then the ternary GC is true.

Proof:

Let us consider \(\alpha = 1 \times 10^{16} \) and \(\beta = 10^{7194} \). With these values we have that the right member of (14) is

\[
\approx 2.744 \times 10^9 \tag{15}
\]

and the value of \(r\) is

\[
r = \log(\log \beta) \alpha \tag{16}
\]

\[
r \approx 3.7921 \tag{17}
\]

Comparing (15) with \(\alpha\) we see that the gap between each prime less or equal to \(\beta - (\alpha - 1)\) and its consecutive prime is less than or equal to \(\alpha - 4\). Therefore exists some \(p\) that satisfies (7) for any element \(n\) of \(B\). Considering that Goldbach’s conjecture is true for all even integers less than \(\alpha\), we can conclude that if (12) holds, then the ternary GC is true.

The statement of theorem 3.1 assumes in (14) that Cramér’s conjecture is true in the worst case. With the values used for \(\alpha\) and \(\beta\), we have a proof of the ternary GC if the following conjecture is true for odd integers less than or equal to \(\beta\)

\[
\max_{p_n \leq \beta} (p_{n+1} - p_n) < \log^{3.7921} \beta \tag{18}
\]

with \(\beta = 10^{7194}\).

4 On Cramér’s model

In 1943 Selberg [10] proved, assuming Riemann’s hypothesis, that

\[
\pi(x + \Phi(x)) - \pi(x) \sim \frac{\Phi(x)}{\log x} \quad (x \to \infty) \tag{19}
\]

is true for almost all \(x\) if

\[
\frac{\Phi(x)}{\log^2 x} \to \infty \quad (x \to \infty) \tag{20}
\]

and, in 1985, Maier [11] concluded that Selberger’s result is true with exceptions. To try to guess the number of primes between \(x\) and \(x + y\), Maier first removed those integers that have a small prime factor (following Eratosthenes), and only then did he apply density arguments (following Gauss) [12]. Maier’s result contradicts what one expected from Cramér’s model.

Let us consider two integers \(x\) and \(y\) such that

\[
x > y \tag{21}
\]

and

\[
x + y \gg p \tag{22}
\]
where $p$ is the greatest prime less than or equal to the square root of $x + y$. Let us assume that the chance that a given integer $n$ be a prime is $\frac{1}{\log n}$. Let $q$ be a prime less than or equal to $p$ and let $F(n, q)$ be a function such that

$$\frac{F(n, q)}{\log n}$$

is the chance that a given integer $n$ belonging to $(x, x + y]$ be prime after crossing out of the interval those integers that are divisible by the primes less than or equal to $q$. For $q = p$ we have

$$\frac{F(n, p)}{\log n} = 1$$

or

$$F(n, p) = \log n$$

In accordance with Friedlander and Granville [13] and Maier [11] the probability of a ‘randomly chosen’ integer $n$ belonging to $(x, x + y]$ being prime, given that it has no prime factors $\leq z$ and being $z$ a ‘small’ prime, is

$$\frac{1}{\prod_{s \leq z} \left(1 - \frac{1}{s}\right)} \log n$$

Considering (23) let $z$ be equal to $p$. Since that

$$\frac{1}{\prod_{s \leq p} \left(1 - \frac{1}{s}\right)} \neq \log n$$

we have a contradiction between (24) and (27). The probability (27) is crucial in Maier’s work [11].

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