THE GEODESIC MOTION IN
TAUB-NUT SPINNING SPACE

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Abstract

The geodesic motion of pseudo-classical spinning particles in the Euclidean Taub-NUT space is analysed. The generalized Killing equations for spinning space are investigated and the constants of motion are derived in terms of the solutions of these equations. A simple exact solution, corresponding to trajectories lying on a cone, is given.
1 Introduction

Spinning particles can be described by pseudo-classical mechanics models involving anti-commuting c-numbers for the spin-degrees of freedom. The configuration space of spinning particles (spinning space) is an extension of a (pseudo-) Euclidean manifold, parametrized by local co-ordinates \( \{ x^\mu \} \), to a graded manifold parametrized by local co-ordinates \( \{ x^\mu, \psi^\mu \} \), with the first set of variables being Grassmann-even (commuting) and the second set Grassmann-odd (anti-commuting) [1-9].

In spite of the fact that the anti-commuting Grassmann variables do not admit a direct classical interpretation, the Lagrangians of these models turn out to be suitable for the path-integral description of the quantum dynamics. The pseudo-classical equations acquire physical meaning when averaged over inside the functional integrals [1,10]. In the semi-classical regime, neglecting higher-order quantum correlations, it should be allowed to replace some combinations of Grassmann spin-variables by real numbers. Using these ideas the motion of spinning particles in external fields have been studied in Refs. [1,11-13].

On the other hand, generalizations of Riemannian geometry based on anti-commuting variables have been proved to be of mathematical interest. Therefore the study of the motion of the spinning particles in curved space-time is well motivated.

In the present paper we investigate the motion of pseudo-classical spinning point particles in the Euclidean Taub-NUT space. The Kaluza-Klein monopole of Gross and Perry [14] and of Sorkin [15] was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional Kaluza-Klein theory. Remarkably the same object has re-emerged in the study of monopole scattering. In the long distance limit, neglecting radiation, the relative motion of the monopoles is described by the geodesics of this space [16,17]. Slow Bogomolny-Prasad-Sommerfield monopoles move along geodesics in the Euclidean Taub-NUT space. The dynamics of well-separated monopoles is completely soluble, but not trivial [16-22]. The problem of geodesic motion in this metric has therefore its own interest, independently of monopole scattering.

For all of these reasons, the extension of the Euclidean Taub-NUT space with additional fermionic dimensions, parametrized by vectorial Grassmann co-ordinates \( \{ \psi^\mu \} \) follows naturally. At last the spinning Taub-NUT space would be a relevant manifold to investigate the properties of the Killing-Yano tensors [9].

The plan of this paper is as follows. In Sec. 2 we summarize the relevant equations for the motions of spinning points in curved spaces. For a systematic treatment of these subjects, the reader is referred to the original literature, and especially, to Refs. [6-9]. In Sec. 3 we analyze the motion of pseudo-classical spinning particles in the Euclidean Taub-NUT space. We examine the
generalized Killing equations for this spinning space and describe the derivation of constants of motion in terms of the solutions of these equations. The contribution of the spin to the angular momentum is contained in the Killing scalars. In Sect. 4 we solve the equations given in the previous Section for the special case of motion on a cone. This case represents an extension of the scalar particle motions in the usual Taub-NUT space in which the orbits are conic sections [18-21]. An explicit exact solution is given. In spite of its simplicity, this solution is far from trivial. Our comments and concluding remarks are presented in Sec. 5.

2 Motion in spinning space

An action for the geodesics of spinning space is:

$$S = \int_a^b d\tau \left( \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x) \psi^\mu \frac{D\psi^\nu}{D\tau} \right).$$

(1)

Here and in the following the overdot denotes an ordinary proper-time derivative $d/d\tau$, whilst the covariant derivative of $\psi^\mu$ is defined by

$$\frac{D\psi^\mu}{D\tau} = \dot{\psi}^\mu + \dot{x}^\lambda \Gamma_{\lambda\nu}^\mu \psi^\nu.$$

(2)

The trajectories, which make the action stationary under arbitrary variations $\delta x^\mu$ and $\delta \psi^\mu$ vanishing at the end points, are given by:

$$\frac{D^2 x^\mu}{D\tau^2} = \ddot{x}^\mu + \Gamma_{\lambda\nu}^\mu \dot{x}^\lambda \dot{x}^\nu = \frac{1}{2i} \psi^\kappa \psi^\lambda R_{\kappa\lambda\nu}^\mu \dot{x}^\nu$$

(3)

$$\frac{D\psi^\mu}{D\tau} = 0.$$

(4)

The anti-symmetric tensor

$$S^{\mu\nu} = -i \psi^\mu \psi^\nu$$

(5)

can formally be regarded as the spin-polarization tensor of the particle [1-8]. The equations of motion can be expressed in terms of this tensor and in particular eq.(4) asserts that the spin is covariantly constant

$$\frac{DS^{\mu\nu}}{D\tau} = 0.$$

(6)

The concept of Killing vector can be generalized to the case of spinning manifolds. For this purpose it is necessary to consider variations of $x^\mu$ and $\psi^\mu$
which leave the action $S$ invariant modulo boundary terms. Let us assume the following forms of these variations:

$$
\delta x^\mu = R^\mu(x, \dot{x}, \psi) = R^{(1)\mu}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\nu_1} \cdots \dot{x}^{\nu_n} R^{(n+1)\mu}_{\nu_1 \cdots \nu_n}(x, \psi)
$$

$$
\delta \psi^\mu = S^\mu(x, \dot{x}, \psi) = S^{(0)\mu}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\nu_1} \cdots \dot{x}^{\nu_n} S^{(n)\mu}_{\nu_1 \cdots \nu_n}(x, \psi) \tag{7}
$$

and the Lagrangian transforms into a total derivative

$$
\delta S = \int_a^b d\tau \frac{d}{d\tau} \left( \delta x^\mu p_\mu - \frac{i}{2} \delta \psi^\mu g_{\mu\nu} \psi^\nu - J(x, \dot{x}, \psi) \right) \tag{8}
$$

where $p_\mu$ is the canonical momentum conjugate to $x^\mu$

$$
p_\mu = g_{\mu\nu} \dot{x}^\nu + \frac{i}{2} \Gamma_{\mu\nu;\lambda} \psi^\lambda \psi^\nu = \Pi_\mu + \frac{i}{2} \Gamma_{\mu\nu;\lambda} \psi^\lambda \psi^\nu \tag{9}
$$

$\Pi_\mu$ being the covariant momentum.

From Noether’s theorem, if the equations of motion are satisfied, the quantity $J$ is a constant of motion.

If we expand $J$ in a power series in the covariant momentum

$$
J(x, \dot{x}, \psi) = J^{(0)}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \Pi^{\mu_1} \cdots \Pi^{\mu_n} J^{(n)}_{\mu_1 \cdots \mu_n}(x, \psi) \tag{10}
$$

then $J$ is a constant of motion if its components satisfy a generalization of the Killing equation [6, 7] :

$$
J^{(n)}_{(\mu_1 \cdots \mu_n; \nu_{n+1})} + \frac{\partial J^{(n)}_{(\mu_1 \cdots \mu_n)}}{\partial \psi^\sigma} \Gamma^{\nu_{n+1}}_{\mu_1 \cdots \nu_n} \psi^\lambda = \frac{i}{2} \psi^\sigma \psi^\lambda R_{\sigma\lambda\nu(\mu_{n+1})} J^{(n+1)\nu}_{\mu_1 \cdots \mu_n}. \tag{11}
$$

In general the symmetries of a spinning-particle model can be divided into two classes. First, there are conserved quantities which exist in any theory and these are called generic constants of motion. The second kind of conserved quantities, called non-generic, depend on the explicit form of the metric $g_{\mu\nu}(x)$.

In Refs. [6, 7] it was shown that for a spinning particle model defined by the action (1) there are four generic symmetries:

1. Proper-time translations and the corresponding constant of motion is the Hamiltonian

$$
H = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu \tag{12}
$$

2. Supersymmetry generated by the supercharge

$$
Q = \Pi_\mu \psi^\mu \tag{13}
$$
3. Chiral symmetry generated by the chiral charge

\[ \Gamma_\ast = \frac{1}{4!} \sqrt{-g} \varepsilon_{\mu\nu\lambda\sigma} \psi^\mu \psi^\nu \psi^\lambda \psi^\sigma \]  

(14)

4. Dual supersymmetry, generated by the dual supercharge

\[ Q^\ast = \frac{1}{3!} \sqrt{-g} \varepsilon_{\mu\nu\lambda\sigma} \Pi^\mu \psi^\nu \psi^\lambda \psi^\sigma. \]  

(15)

In the next Section we shall apply these results and we shall discuss specific solutions of the generalized Killing equations to the case of a spinning particle moving in a Euclidean Taub-NUT space.

3 Euclidean Taub-NUT spinning space

The Kaluza-Klein monopole [14,15] was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional theory, adding the time coordinate in a trivial way. Its line element is expressed as:

\[ ds_5^2 = -dt^2 + ds_4^2 = -dt^2 + V^{-1}(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2] + V(r)[dx^5 + \tilde{A}(\vec{r}) d\vec{r}]^2 \]  

(16)

where \( \vec{r} \) denotes a three-vector \( \vec{r} = (r, \theta, \varphi) \) and the gauge field \( \tilde{A} \) is that of a monopole

\[ A_r = A_\theta = 0, \quad A_\varphi = 4m(1 - \cos \theta) \]

\[ \vec{B} = \text{rot} \vec{A} = \frac{4m \vec{r}}{r^3}. \]  

(17)

The function \( V(r) \) is

\[ V(r) = \left(1 + \frac{4m}{r}\right)^{-1} \]  

(18)

and the so-called NUT singularity is absent if \( x^5 \) is periodic with period \( 16\pi m \) [23].

It is convenient to make the co-ordinate transformation

\[ 4m(\chi + \varphi) = -x^5 \]  

(19)

with \( 0 \leq \chi < 4\pi \), which converts the four-dimensional line element \( ds_4 \) into

\[ ds_4^2 = V^{-1}(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2] + 16m^2 V(r)[d\chi + \cos \theta d\varphi]^2. \]  

(20)
Spaces with a metric of the form given above have an isometry group $SU(2) \times U(1)$. The four Killing vectors are

$$D^{(\alpha)} = R^{(\alpha)\mu} \partial_\mu, \quad \alpha = 1, \ldots, 4$$  \hspace{1cm} (21)

where

$$D^{(1)} = \frac{\partial}{\partial \chi}$$

$$D^{(2)} = - \sin \varphi \frac{\partial}{\partial \theta} - \cos \varphi \cot \theta \frac{\partial}{\partial \varphi} + \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \chi}$$

$$D^{(3)} = \cos \varphi \frac{\partial}{\partial \theta} - \sin \varphi \cot \theta \frac{\partial}{\partial \varphi} + \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \chi}$$

$$D^{(4)} = \frac{\partial}{\partial \varphi}.$$  \hspace{1cm} (22)

$D^{(1)}$ which generates the $U(1)$ of $\chi$ translations, commutes with the other Killing vectors. In turn the remaining three vectors obey an $SU(2)$ algebra with

$$[D^{(2)}, D^{(3)}] = -D^{(4)}, \text{ etc...}$$  \hspace{1cm} (23)

This can be contrasted with the Schwarzschild space-time where the isometry group at spacelike infinity is $SO(3) \times U(1)$. This illustrates the essential topological character of the magnetic monopole mass [24].

In the purely bosonic case these invariances would correspond to conservation of the so called ”relative electric charge” and the angular momentum [18-22] :

$$q = 16m^2 V(r) (\dot{\chi} + \cos \theta \dot{\varphi})$$  \hspace{1cm} (24)

$$\vec{j} = \vec{r} \times \vec{p} + q \frac{\vec{r}}{r}.$$  \hspace{1cm} (25)

The first generalized Killing eq.(11) has the form

$$B^{(\alpha)} + \frac{\partial B^{(\alpha)}}{\partial \psi^\sigma} \Gamma^\sigma_{\mu \lambda} \psi^\lambda = \frac{i}{2} \psi^\rho \psi^\sigma R^\rho_{\sigma \lambda \mu} R^{(\alpha)\lambda}$$  \hspace{1cm} (26)

which shows that with each Killing vector $R^{(\alpha)}$ there is an associated Killing scalar $B^{(\alpha)}$. Therefore if we limit ourselves to variations (7) to terminate after the terms linear in $\dot{x}^\mu$, we obtain the constants of motion

$$J^{(\alpha)} = B^{(\alpha)} + \dot{x}^\mu R^{(\alpha)}$$  \hspace{1cm} (27)

which asserts that the Killing scalars contribute to the ”relative electric charge” and the total angular momentum.

Inserting the expressions for the connections and the Riemann curvature components corresponding to the Taub-NUT space in eq.(26) after a long and tedious calculation we obtain for the Killing scalars:
\[ B^{(1)} = - \frac{32 m^3 \cos \theta}{(4m + r)^2} S^{\tau \varphi} - \frac{32 m^3}{(4m + r)^2} S^{\tau x} + \frac{8m^2 r \sin \theta}{4m + r} S^{\theta \varphi} \]

\[ B^{(2)} = 2m \sin \varphi S^{\theta \theta} - \frac{16m^2 r + 2mr^2}{(4m + r)^2} \sin \theta \cos \theta \cos \varphi S^{\tau \varphi} \]

\[ - \frac{4m(8m^2 + 8mr + r^2)}{(4m + r)^2} \sin \theta \cos \varphi S^{\tau x} \]

\[ - \left[ \frac{8m^2 r}{4m + r} \sin^2 \theta \cos \varphi + \frac{4mr(2m + r)}{4m + r} \cos^2 \theta \cos \varphi \right] S^{\theta \varphi} \]

\[ - \frac{4mr(2m + r)}{4m + r} \sin \theta \cos \varphi S^{\theta x} + \frac{4mr(2m + r)}{4m + r} \sin \theta \sin \varphi S^{\tau x} \]

\[ B^{(3)} = - 2m \cos \varphi S^{\theta \theta} - \frac{16m^2 r + 2mr^2}{(4m + r)^2} \sin \theta \cos \theta \sin \varphi S^{\tau \varphi} \]

\[ - \frac{4m(8m^2 + 8mr + r^2)}{(4m + r)^2} \sin \theta \sin \varphi S^{\tau x} \]

\[ - \left[ \frac{8m^2 r}{4m + r} \sin^2 \theta \sin \varphi + \frac{4mr(2m + r)}{4m + r} \cos^2 \theta \sin \varphi \right] S^{\theta \varphi} \]

\[ - \frac{4mr(2m + r)}{4m + r} \cos \theta \sin \varphi S^{\theta x} - \frac{4mr(2m + r)}{4m + r} \sin \theta \cos \varphi S^{\tau x} \]

\[ B^{(4)} = \left( 2m + r \right) \sin^2 \theta + \frac{32m^3 \cos^2 \theta}{(4m + r)^2} S^{\tau \varphi} + \frac{32m^3 \cos \theta}{(4m + r)^2} S^{\tau x} \]

\[ + \frac{8mr^2 + r^3}{4m + r} \sin \theta \cos \theta S^{\theta \varphi} - \frac{8m^2 r}{4m + r} \sin \theta S^{\theta x}. \quad (28) \]

Taking into account the contribution of the Killing scalars to eq. (27) one finds for the constants of the motion \( J^{(\alpha)} \):

\[ J^{(1)} = B^{(1)} + q \]

\[ J^{(2)} = B^{(2)} - (4m + r)r \sin \varphi \cdot \dot{\theta} - r(4m + r) \sin \theta \cos \theta \cos \varphi \cdot \dot{\varphi} + q \sin \theta \cos \varphi \]

\[ J^{(3)} = B^{(3)} + (4m + r)r \cos \varphi \cdot \dot{\theta} - r(4m + r) \sin \theta \cos \theta \sin \varphi \cdot \dot{\varphi} + q \sin \theta \sin \varphi \]

\[ J^{(4)} = B^{(4)} + r(4m + r) \sin^2 \theta \cdot \dot{\varphi} + q \cos \theta. \quad (29) \]

We remark that the ”relative electric charge” \( q \) is no longer conserved contrasting with the purely bosonic case. On the other hand the conserved
total angular momentum is the sum of the orbital angular momentum, the Poincaré contribution and the spin angular momentum:

\[ \mathbf{J} = \mathbf{B} + \mathbf{j} \]  

(30)

with \( \mathbf{J} = (\mathcal{J}^{(2)}, \mathcal{J}^{(3)}, \mathcal{J}^{(4)}) \) and \( \mathbf{B} = (B^{(2)}, B^{(3)}, B^{(4)}) \).

From eqs. (26) we can derive an interesting linear combination of \( \mathcal{J}^{(\alpha)} \), \( \alpha = 1, \ldots, 4 \):

\[ J^{(1)} - \frac{\mathbf{J} \cdot \mathbf{r}}{r} = - r \sin^2 \theta \cos \theta S^{\theta \varphi} + 4m \sin^2 \theta S^{\theta \chi} - r^2 \sin \theta \cos^2 \theta S^{\theta \varphi} + 4mr \sin \theta \cos \theta S^{\theta \chi}. \]  

(31)

In the standard Taub-NUT space, eq.(31) reduces to

\[ \frac{\mathbf{j} \cdot \mathbf{r}}{r} = |\mathbf{j}| \cos \theta = q \]  

(32)

which fixes the relative motion to lie on a cone whose vertex is at the origin and whose axis is \( \mathbf{j} \). Eq. (31) express the fact that the total angular momentum in the radial direction receives contributions from the spin angular momentum, the orbital angular momentum being absent in that direction. Moreover the motion is more complicated than in the bosonic case since in general the angle \( \theta \) is not constant in time.

In addition to these constants of motion there are four universal conserved charges described in the previous Section. Using the notation from this Section they are:

1. The energy

\[ E = \frac{1}{2} V^{(-1)}(r) \left[ \dot{r}^2 + \left( \frac{q}{4m} \right)^2 \right] \]

\[ = \frac{1}{2} \frac{4m + r}{r} \dot{r}^2 + \frac{1}{2} (4m + r) r \dot{\theta}^2 + \frac{1}{2} (4m + r) r \sin^2 \theta \dot{\varphi}^2 + 8m^2 \frac{r}{4m + r} (\cos \theta \dot{\varphi} \dot{\chi} + \dot{\chi})^2 \]  

(33)

2. The supercharge

\[ Q = \frac{4m + r}{r} \dot{r} \psi^r + (4m + r) r \dot{\theta} \psi^\theta \]

\[ + \left[ (4m + r) r \sin^2 \theta \dot{\varphi} \dot{\chi} + q \cos \theta \right] \psi^\varphi + q \psi^\chi \]  

(34)

3. The chiral charge

\[ \Gamma_* = 4m(4m + r) r \sin \theta \psi^r \psi^\theta \psi^\varphi \psi^\chi \]  

(35)
4. The dual supercharge

\[ Q^* = 4m(4m + r)r \sin \theta (\dot{r} \psi^r \psi^x - \dot{\theta} \psi^r \psi^\theta + \dot{\phi} \psi^r \psi^\phi \psi^x - \dot{\chi} \psi^r \psi^\theta \psi^\phi). \]  

Finally, having in mind that \( \psi^\mu \) is covariantly constant, the rate of change of spins is:

\[
\begin{align*}
\dot{\psi}^r &= \frac{2m}{r(4m + r)} \dot{r} \psi^r + \frac{r^2 + 2mr}{4m + r} \dot{\theta} \psi^\theta \\
&\quad + \left( \frac{r^2 + 2mr}{4m + r} \sin^2 \theta + \frac{32m^3 r \cos^2 \theta}{(4m + r)^3} \right) \dot{\phi} \psi^\phi \\
&\quad + \frac{32m^3 r}{(4m + r)^3} \left( \dot{\phi} \psi^x + \dot{\chi} \psi^\phi \right) + \frac{32m^3 r}{(4m + r)^3} \dot{\chi} \psi^x,
\end{align*}
\]

\[
\begin{align*}
\dot{\psi}^\theta &= -\frac{r + 2m}{r(4m + r)} (\dot{r} \psi^\theta + \dot{\theta} \psi^r) + \frac{8mr + r^2}{(4m + r)^2} \sin \theta \cos \theta \dot{\phi} \psi^\phi \\
&\quad - \frac{8m^2}{(4m + r)^2} \sin \theta \left( \dot{\phi} \psi^x + \dot{\chi} \psi^\phi \right),
\end{align*}
\]

\[
\begin{align*}
\dot{\psi}^\phi &= -\frac{r + 2m}{r(4m + r)} (\dot{r} \psi^\phi + \dot{\phi} \psi^r) - \frac{8m^2 + 8mr + r^2 \cos \theta}{(4m + r)^2} \frac{\cos \theta}{\sin \theta} \left( \dot{\theta} \psi^\phi + \dot{\phi} \psi^\theta \right) \\
&\quad + \frac{8m^2}{(4m + r)^2} \frac{1}{\sin \theta} \left( \dot{\theta} \psi^x + \dot{\chi} \psi^\theta \right),
\end{align*}
\]

\[
\begin{align*}
\dot{\psi}^x &= \frac{\cos \theta}{(4m + r)^2} (\dot{r} \psi^x + \dot{\phi} \psi^r) - \frac{2m}{r(4m + r)} (\dot{r} \psi^x + \dot{\chi} \psi^r) \\
&\quad + \left( \frac{8m^2 + 8mr + r^2 \cos \theta}{(4m + r)^2} \frac{1}{\sin \theta} + \frac{1}{2} \sin \theta \right) \left( \dot{\theta} \psi^\phi + \dot{\phi} \psi^\theta \right) \\
&\quad - \frac{8m^2}{(4m + r)^2} \frac{\cos \theta}{\sin \theta} \left( \dot{\theta} \psi^x + \dot{\chi} \psi^\theta \right).
\end{align*}
\]

As a rule these complicated equations could be integrated to obtain the full solution of the equations of motion for the usual co-ordinates \( \{x^\mu\} \) and Grassmann co-ordinates \( \{\psi^\mu\} \).

For example, it is possible to use an iterative procedure starting with the motion of a spinless point particle in the Taub-NUT space. Moreover in this case there is a conserved vector analogous to the Runge-Lenz vector of the Coulomb problem whose existence is rather surprising in view of the complexity of the equations of motion. This conserved vector is [18]:

\[
\vec{K} = \vec{p} \times \vec{j} + \left( \frac{q^2}{4m} - 4mE \right) \frac{\vec{r}}{r}. \]

8
The trajectories lie hence simultaneously on the cone $\vec{j} \cdot \vec{r}/r = q$ and also in the plane perpendicular to

$$\vec{n} = qK + \left(4mE - \frac{q^2}{4m}\right)\vec{j}, \quad (39)$$

They are thus conic sections.

Starting with a solution of this type, from eqs.(37) it is possible to find the Grassmann variables $\{\psi^\mu\}$ in a first approximation. The next step is to use eq.(3) to get corrections to $\{x^\mu\}$ due the spin variables and so on. The essential point is that the iterative procedure brings to an end after a finite number of steps taking into account that the co-ordinate $\{\psi^\mu\}$ are anti-commuting variables.

Unfortunately, the equations of motion are quite intricate and the general solution is by no means illuminating. Instead of the general solution, in the next Section we shall discuss a special solution which is very simple and far from trivial.

4 Special solution

In this Section we solve the equations given above limiting ourselves to the motion on a cone. This characteristic of the motion of the scalar particles in Taub-NUT spaces can be found for spinning particles only in special cases.

For this purpose let us choose the $z$ axis along $\vec{J}$ so that the motion of the of the particle may be conveniently described in terms of polar co-ordinates

$$\vec{r} = r\vec{e}(\theta, \varphi) \quad (40)$$

with

$$\vec{e} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta). \quad (41)$$

Using the iterative procedure described above, we shall start with the motion of a scalar particle in Taub-NUT space taking $\dot{\theta} = 0$. Eqs.(37) can be written in terms of the anti-symmetric tensor $S^{\mu\nu}$ as follows:

$$\dot{S}^{r\theta} = -\frac{\dot{r}}{4m + r}S^{r\theta} + \frac{8m + r}{2(4m + r)^3}q \sin \theta S^{r\varphi} - \frac{8m^2}{r(4m + r)^3}q \sin \theta \cos \theta S^{r\chi}$$

$$- \frac{r \sin^2 \theta + 2m}{(4m + r)^2} q \cos \theta S^{\theta\varphi} - \frac{2mq}{(4m + r)^2} S^{\theta\chi}$$

$$\dot{S}^{r\varphi} = -\frac{\dot{r}}{2r(4m + r) \sin \theta}S^{r\theta} - \frac{\dot{r}}{4m + r} S^{r\varphi} - \frac{2mq}{(4m + r)^2} S^{r\chi}$$

$$\dot{S}^{r\chi} = \frac{q}{2r(4m + r) \sin \theta \cos \theta} S^{r\theta} + \frac{\dot{r} \cos \theta}{4m + r} S^{r\varphi} + \frac{(r \sin^2 \theta + 2m) \dot{q}}{(4m + r)^2 \cos \theta} S^{r\chi}$$
\[ \dot{S}^{\theta \varphi} = \frac{(2m + r) q}{r^2(4m + r)^2 \cos \theta} S^{r \theta} - \frac{2m + r}{r(4m + r)} \dot{r} S^{\theta \varphi} + \frac{8m^2 \sin \theta q}{r(4m + r)^3 \cos \theta} S^{\phi \chi} \]
\[ \dot{S}^{\theta \chi} = \frac{q}{8m(4m + r)^2} S^{r \theta} + \frac{\dot{r}}{4m + r} \cos \theta S^{\theta \varphi} - \frac{\dot{r}}{r} S^{\theta \chi} + \frac{8m + r}{2(4m + r)^3} q \sin \theta S^{\phi \chi} \]
\[ \dot{S}^{\phi \chi} = \frac{q}{8m(4m + r)^2} S^{r \varphi} - \left( \frac{2m + r}{r^2(4m + r)^2 \cos \theta} \right) S^{r \chi} - \frac{q}{2r(4m + r) \sin \theta \cos \theta} S^{\theta \varphi} - \frac{q}{2r(4m + r) \sin \theta} S^{\theta \chi} - \frac{\dot{r}}{r} S^{\phi \chi}. \quad (42) \]

Since we are looking for solutions with \( \dot{\theta} = 0 \) we have
\[ \frac{d}{dt} \left[ J^{(1)} - \frac{\vec{J} \cdot \vec{r}}{r} \right] = 0 \quad (43) \]

and from eqs. (31) and (42) we conclude that
\[ S^{r \theta} + \frac{2r(2m + r)}{4m + r} \sin \theta S^{\phi \chi} = 0. \quad (44) \]

This relation implies that the special solutions investigated in this Section are situated in the sector with
\[ \Gamma^* = 0. \quad (45) \]

Expressing \( S^{r \theta} \) through \( S^{\phi \chi} \) we can form the following combinations which are equivalent with eqs. (42):
\[ \frac{d}{dt} [(4m + r)S^{r \varphi}] = \frac{r}{4m + r} q S^{\phi \chi} \]
\[ \frac{d}{dt} [\cos \theta S^{r \varphi} + S^{r \chi}] = 0 \]
\[ \frac{d}{dt} [r(4m + r)S^{\theta \varphi}] = -2 \frac{\sin \theta}{\cos \theta} \frac{r}{4m + r} q S^{\phi \chi} \]
\[ \frac{d}{dt} [r \cos \theta S^{\theta \varphi} + r S^{\theta \chi}] = -\frac{\sin \theta}{4m} \frac{r}{4m + r} q S^{\phi \chi}. \quad (46) \]

Thus the equations of motion for \( S^{\mu \nu} \) are written in a more tractable form and the solution follows without difficulties [25]. However the general solution is still quite involved and in what follows we prefer to present explicitly a particular solution. For this purpose we shall satisfy eq.(44) in a trivial way:
\[ S^{r \theta} = S^{\phi \chi} = 0. \quad (47) \]
In spite of this drastic simplification, eqs. (46) have a nontrivial solution:

\[
\begin{align*}
S^{r\varphi} &= \frac{C^{r\varphi}}{4m + r}, \\
S^{r\chi} &= C^{r\chi} - \cos \theta \frac{C^{r\varphi}}{4m + r}, \\
S^{\theta\varphi} &= \frac{C^{\theta\varphi}}{r(4m + r)}, \\
S^{\theta\chi} &= \frac{C^{\theta\chi}}{r} - \cos \theta \frac{C^{\theta\varphi}}{r(4m + r)}
\end{align*}
\] (48)

where \( C^{\mu \nu} \) are Grassmann constants. These constants are not all independent, having two relations between them

\[
C^{r\chi} = \sin \theta \frac{C^{\theta\varphi}}{4m}
\] (49)

\[
\sin \theta \cos \theta C^{r\varphi} - 4m \cos \theta C^{\theta\chi} - 2 \sin^2 \theta C^{\theta\varphi} = 0.
\] (50)

In the case of this particular solution

\[
J^{(1)} = q
\] (51)

and therefore the "relative electric charge" is conserved as in the case of the scalar particle in the usual Taub-NUT space. However the total angular momentum is modified by the spin contribution

\[
J^{(1)} - \frac{\vec{J} \cdot \vec{r}}{r} = J^{(1)} - J \cos \theta = - \sin \theta C^{\theta\varphi}.
\] (52)

Here \( J \) is the magnitude of the total angular momentum and eq. (52) fixes the angle \( \theta \) in terms of the constants \( q, J \) and \( C^{\theta\varphi} \).

Also the equations for \( \varphi \) and \( \chi \) are modified:

\[
\begin{align*}
\dot{\varphi} &= \frac{q}{(4m + r) r \cos \theta} - \frac{1}{(4m + r)^2} C^{r\varphi} + \frac{\sin \theta}{(4m + r)^2} C^{\theta\varphi}, \\
\dot{\chi} &= \frac{8m + r}{16m^2(4m + r)} q + \frac{\cos \theta}{(4m + r)^2} C^{r\varphi} - \frac{\sin \theta}{(4m + r)^2} C^{\theta\varphi}.
\end{align*}
\] (53)

Finally, \( \dot{r} \) can be derived from the energy, eq. (33).

Since for this particular solution the motion of the spinning particle is restricted to a cone, it is natural to ask for a conserved vector similar to the Runge-Lenz vector (38). Unfortunately, for the spinning particles this form turns out to be inadequate.
Motivated by the studies of Peres [26] and Holas and March [27], we shall construct a vector $\vec{K}$ which is constant in time, as an appropriate generalization of the Runge-Lenz vector (38). Using the parametrization (40), (41) the velocity vector can be expressed as

$$\dot{\vec{r}} = \dot{r}\vec{e} + r\dot{\varphi}\vec{e}'$$

where

$$\vec{e}' = \frac{d\vec{e}}{d\varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0).$$

The generalized Runge-Lenz vector can be written in a local rotating basis $(\vec{e}(\varphi), \vec{e}', \vec{j})$ as

$$\vec{K} = X_1 \left( \vec{e} - \cos\theta\vec{j}_j \right) + X_2 \vec{e}'.$$

$\vec{K}$ will remain constant in the laboratory frame if it will rotate in the opposite direction with respect to its local basis

$$X_1 = X_0 \cos(\varphi - \varphi_0)$$
$$X_2 = -X_0 \sin(\varphi - \varphi_0)$$

where $X_0, \varphi_0$ are constants which can be chosen as one wishes.

However it is necessary to note that the property of being constant in time is not sufficient for $\vec{K}$ to be an integral of motion. It must be a one-valued function of the state of the particle [22, 27]. Therefore, even restricting to the solution described in this Section, only in some particular cases the orbits are closed paths and the vector $\vec{K}$ (56) may serve as an integral of motion.

## 5 Concluding remarks

In the last time the pseudo-classical limit of the Dirac theory of a spin-1/2 particle in curved space-time is described by the supersymmetric extension of the ordinary relativistic point particle. The spinning space represents the extension of the ordinary space-time with anti-symmetric Grassmann variables to describe the spin degrees of freedom.

Our main concern has been the motion of pseudo-classical spinning particles in Euclidean Taub-NUT space. This space has been considered extensively in the literature in connection with the study of monopole scattering and Kaluza-Klein monopole. The geodesic motion in the ordinary Euclidean Taub-NUT space is integrable and has a remarkably close analogy with motion under a Coulomb force. The existence of a conserved vector analogous to the Runge-Lenz vector of the Coulomb problem is rather surprising in view of the velocity-dependent forces.
In our analysis of the spinning Taub-NUT space we have restricted to the contribution of the spin contained in the Killing scalars $B^\alpha(x, \psi)$ defined by eq.(26). In spite of the complexity of the equations, we are able to present a special solution which is very simple, but not at all trivial. Other particular solutions will be presented elsewhere [25].

Extensions of these results to the Killing-Yano tensors [9,28] are possible and necessary. In general it is desirable to have a deeper understanding of the role of the Runge-Lenz vector (38) for the motion of spinning particles. The existence of this vector can be related to a Killing-Yano 2form [19]. On the other hand some properties of the classical Coulomb (Kepler) motion may be rediscovered in a metric admitting a Killing-Yano tensor of rank four [28].

These generalizations are under investigations.

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