Universality classes of critical points in constrained glasses

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Abstract. We analyze critical points that can be induced in glassy systems by the presence of constraints. These critical points are predicted by the mean field thermodynamic approach and they are precursors of the standard glass transition in the absence of constraints. Through a deep analysis of the soft modes appearing in the replica field theory, we can establish the universality class of these points. In the case of the ‘annealed potential’ of a symmetric coupling between two copies of the system, the critical point is in the Ising universality class. More interesting is the case of the ‘quenched potential’, where a single copy is coupled with an equilibrium reference configuration, or the ‘pinned particle’ case, where a fraction of the particles is frozen in fixed positions. In these cases we find the random field Ising model (RFIM) universality class. The effective random field is a ‘self-generated’ disorder that reflects the random choice of the reference configuration. The RFIM representation of the critical theory predicts non-trivial relations governing the leading singular behavior of relevant correlation functions, that can be tested in numerical simulations.

Keywords: cavity and replica method, disordered systems (theory), spin glasses (theory), structural glasses (theory)
1. Introduction

Recent times have seen a renewed interest in glassy systems in the presence of constraints. Glassy relaxation in liquids is dominated by the presence of metastable states. According to the mean field picture of the glass transition [1, 2], also known as the random first order transition (RFOT) [3], these states have a well defined thermodynamic meaning and can be probed and stabilized by imposing suitable constraints that modify the Hamiltonian. The simplest procedure consists in considering two copies of the system and introducing an attraction between the particles of the first and the second copy [4]. The free energy as a function of the overlap, which is the conjugate parameter to the strength of the attraction, is often referred to as the ‘annealed potential’ function. A second, more refined procedure consists in fixing a reference configuration and biasing the Boltzmann probability of the system in the direction of this configuration [5]. One can consider an external potential that provides an attraction for the particles of the system to the position they take in the reference configuration. In this case, the free energy as a function of the overlap is called the ‘quenched potential’. Finally, the bias towards the reference configuration can be
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imposed by fixing some of the degrees of freedom—in practice the positions of a fraction of the particles—to the values they take in the reference configuration [6]–[10]. This is called the ‘pinned particle’ method.

The three ways of constraining the system have different advantages and reveal different aspects of metastability. In both the annealed and quenched potentials metastability is revealed by the shape of the potential function. It is well known that many aspects of dynamical mode coupling theory-like transitions [11], including dynamical heterogeneities and growth of correlations, can be studied from the quenched potential construction [12]–[16]. Using the overlap as an order parameter, as is done in the annealed and quenched potential cases, allows us to discriminate between the thermodynamic view of the glass transition, where the overlap among configurations is the relevant order parameter, and a purely kinetic one, where the overlap does not allow us to discriminate different metastable states [17, 18].

The pinned particle method, on the other hand, does not directly use the overlap as an order parameter, but is attractive because it can add stability to metastable states in such a way that the equilibrium state of the system is not perturbed.

A characteristic feature of glassiness, as we know from mean field theory, is the fact that in all three cases the imposed constraint induces new phase transitions in the system [19, 20], [9]. The nature of these transitions differs in the different procedures [21]. In the cases of attractive interactions one finds a first order transition line in the plane of temperature and interaction strength [19, 20]. In the case of pinned particles, the constraint induces a line of phase transition in the plane of temperature and fraction of blocked particles. Here the nature of the transitions depends on the detailed procedure of pinning; one can either find a first order transition line, as in the case of the coupled systems, or instead a line of ideal glass transition with Kauzmann entropy crisis that crosses over to a line of second order glass transition [22]. In all cases but this last one, the line of phase transition terminates at a critical point. The existence of these critical points is a crucial prediction of the thermodynamic mean field approach. In contrast, in purely dynamic theories of glassiness [23] and in exactly solvable kinetically constrained models [24] (such as the Fredrickson–Andersen on random graph or similar models) the lines of thermodynamic phase transition and the critical point are not present [25]. Their existence is therefore one of the few discriminating predictions that are different between the two approaches. The thermodynamic scenario has started to receive confirmations in numerical simulations of liquid systems. In [26] evidence was provided for a coupling induced first order transition in the quenched potential setting. In [27] this result was confirmed, for both the annealed and the quenched case, and it was convincingly shown that the line of phase transition terminates at a critical point. Other numerical results in this sense will be presented soon [28]. In this paper we address the problem of the characterization of the universal properties of these critical points. Simple arguments can be put forward to understand these properties. In the annealed potential case, the only source of overlap fluctuations is the thermal noise. The critical point is described by a quartic field theory and is in the Ising universality class. In the quenched and pinned cases, however, a second source of fluctuations can be identified in the choice of the reference configuration [14]. This acts as a random field in the system and the resulting universality class is that of the random field Ising model (RFIM) [29, 30]. In order to turn these qualitative arguments into an accomplished theory, a deep analysis of the soft
modes emerging at the critical point and the properties of the perturbation theory should be performed. Replica field theory (RFT), in terms of which the constrained free energy can in principle be computed, provides the natural formal setting to frame the problem. The three different procedures are found to correspond to different underlying symmetries and/or analytic continuations in the number of replicas that one should consider. The analysis of perturbation theory of replica field theories describing glassy criticality has been initiated in [14], where it was shown how the description of dynamical heterogeneities in the beta regime, close to a mode coupling (MCT) dynamical transition, could be mapped at a spinodal point of an RFIM with cubic interaction. In a subsequent paper [31] the case of a replica symmetric theory where the leading cubic interaction term vanishes was analyzed. This theory describes higher order glass singularities as well as the critical point of the symmetric pinned particle construction where the pinned particles are blocked from a configuration equilibrated at a temperature equal to that at which the free particles evolve. In that case the universality class of the ($\phi^4$) RFIM was found within a perturbative one loop calculation. Here we extend our analysis to the annealed and the quenched potential and asymmetric pinning where the pinned particles are blocked from a temperature smaller than that of the free particles, that we are able to treat at all orders of perturbation theory. In all cases we find that the expectations from the qualitative argument are met. While the annealed case is attractive for the simplicity of the result and the possibility to verify it in numerical simulations, most interesting from the theoretical point of view, for its implication on the nature of fluctuations and heterogeneities in glassy systems, are the quenched potential case [14] and the asymmetric pinning case, where our analysis shows how the Parisi–Sourlas supersymmetry [32] of the RFIM naturally emerges at criticality.

The plan of the paper is the following: in section 2 we briefly review the theory of glassy systems under constraints. In section 3 we discuss the annealed case. Then in section 4 we state the problem of the critical point for the quenched potential and pinned particle case. We analyze the zero modes of the mass matrix in section 5. In section 6 we derive the RFIM action by dimensional analysis. In section 7 we discuss physical correlation functions and their relations. We finally summarize and conclude the paper. An appendix presents some technical details.

2. Glassy systems under constraints

In this section we briefly review the use of constraints to unveil glassiness and metastability. Let us consider a system described by the Hamiltonian $H(X)$, where $X$ specifies the configuration of all the particles in the system. We suppose a priori that there is no quenched disorder in $H$, even though this could be included. As we will see, in all cases the computation of the constrained free energy can be tackled through the use of the replica method. The order parameter of the theory is an overlap matrix, and fluctuations will be described by a Landau expansion of the free energy around a saddle point. Specifically, the problems differ in the number of replicas, which is two in the annealed potential problem and $n \rightarrow 1$ in the quenched potential and the pinned particle case, and in the symmetry of the saddle point: $S_{n-1}$ in the quenched potential and asymmetric pinning; $S_n$ in the symmetric pinning.
2.1. Annealed potential construction

The simplest setting consists in considering two copies in the system interacting through an attraction [4, 20]

\[ H_2(X, Y) = H(X) + H(Y) - N\epsilon q(X, Y) \]  

(1)

where for a system with \( N \) particles, the overlap \( q(X, Y) \) between two configurations \( X = \{x_1, \ldots, x_N\} \) and \( Y = \{y_1, \ldots, y_N\} \) can be defined in terms of a short range attractive interaction potential \( w(x) \) as

\[ q(X, Y) = \frac{1}{N} \sum_{i,j} w(x_i - y_j). \]  

(2)

Space dependent overlap fields \( q(x; X, Y) \) can be defined restricting the sum in (2) to the particles in some neighborhood of \( x \).

The free energy of the system \( F(\epsilon, T) \) involves a sum over the configurations of the two copies or replicas of the system. One can see this sum as a particular case of a replicated system where the number of replicas \( n \) here is just equal to two, and, just as in the case of uncoupled systems, the study of liquid phases can be addressed without need for analytic continuations. Conversely, the study of glassy phases requires analytic continuations in the number of replicas, but since, as we will see, the critical point in which we are interested lies in a liquid region we will not need to consider these continuations.

The Legendre transform of \( F(\epsilon, T) \), \( W(q, T) = F(\epsilon, T) + \epsilon q \) is called the annealed potential function. We refer broadly to this procedure of symmetric coupling as annealed potential construction.

In mean field models with a glass transition, e.g. \( p \)-spin or Potts spin glasses [5, 33] or liquids in the HNC approximation [34], the coupling induces temperature dependent phase transitions in the system. A typical phase diagram is presented in figure 1. In the temperature–coupling plane, one finds both a line of ordinary liquid–glass transition where the overlap between the two copies is non-singular [33], and a line of first order phase transition that separates a low overlap or deconfined phase, where the two copies are weakly correlated, from a high overlap or confined phase, where the two replicas stay close to each other. The first order transition line, which departs from the ideal Kauzmann transition temperature \( T_k \) for \( \epsilon = 0 \), terminates at a critical point \( (T_{Cr}, \epsilon_{Cr}) \). Interestingly, this line and the glass transition line meet at a point, and while the deconfined phase is always a liquid, depending on the temperature, the confined phase can be either a liquid or a glass. What is important for us is that a whole part of the line, which includes the critical point, marks the border of a liquid–liquid transition. The critical point lies at a finite distance from the line of glass transition and a description with just two replicas is appropriate.

The inset of figure 1 shows the typical isothermal lines and coexistence curve in the overlap–coupling plane for temperatures close to the critical point. Notice the similarity to the isotherm of the gas–liquid phase transition in the \( V-p \) plane. As in this case, the critical fluctuations can be described by expanding the free energy with respect to the local fluctuation of the order parameter around its (space homogeneous) average value \( q^* \). We can notice the similarity of figure 1(a) to the coexistence diagram recently obtained in numerical simulations of a realistic liquid model by Berthier [27].

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Figure 1. Typical mean field phase diagram in the $T-\epsilon$ plane for the annealed and the quenched constructions. Left: the annealed construction phase diagram. The central (red) line marks the first order phase transition between a confined high overlap phase above and an unconfined phase below. The green line is the line of dynamical (MCT-like) glass transition. The confined phase is a glass to the left of the brown point where the red line and the green line meet and it is a liquid to its right. The blue lines are the spinodal lines of the confined (lower curve) and unconfined (upper curve) phases. In the inset we show the typical isothermal and coexistence lines in the $q-\epsilon$ plane. From top to bottom we have an isotherm in the single phase region $T > T_{C_C}$, the critical isotherm $T = T_{C_C}$ and an isotherm in the two phase region $T < T_{C_C}$; the horizontal line corresponds to the Maxwell construction. The cyan line is the Widom line: the locus of points where the potential has an inflection, $g_3 = 0$ and the susceptibility $\chi_4 = d\langle p \rangle/d\epsilon$ has a maximum. Right: phase diagram of the quenched construction. The transition line and its corresponding spinodal are similar to those of the annealed case; however, here the coupling does not induces new glass transitions. Glassiness appears at the dynamical glass transition temperature $T_d$ of the unconstrained model (vertical green line). Notice the different scales in the two panels. The quenched critical point lies at lower temperature and coupling than the annealed one. We have used here the spherical $p$-spin model [35] for $p = 3$, for which the dynamical (MCT) transition temperature is $T_d = 0.612$ and the Kauzmann transition temperature is $T_k = 0.586$.

2.2. Quenched potential construction

The symmetric coupling between replicas in the annealed procedure introduces strong biases to the equilibrium. In order to faithfully explore the vicinity of typical equilibrium states at a temperature, one considers instead a quenched procedure where one fixes a reference configuration $X_0$ extracted with Boltzmann probability at a temperature $T_{ref}$, and uses the particle positions in the reference configuration to define an external potential in which the particles of the constrained system $X$ evolve [5, 19, 20]. The Hamiltonian of the system for fixed $X_0$ is

$$H'(X) = H(X) - N eq(X, X_0).$$

The fundamental asymmetry between $X_0$, which is just a random equilibrium configuration, and the system $X$, which feels an attraction towards $X_0$, should be noted.
The reference configuration can be considered as a sort of quenched disorder in which the system evolves. The constrained free energy $F_Q(\epsilon, T, T_{ref})$ is defined as

$$F_Q(\epsilon, T, T_{ref}) = -\frac{T}{N} \frac{1}{Z(T_{ref})} \sum_X e^{-\beta_{ref} H(X_0)} \log \left( \sum_X e^{-\beta(H(X) - N\epsilon q(X,X_0))} \right). \quad (4)$$

One usually chooses $T_{ref} = T$, but the case in which the temperature of the system is different from the temperature has also been considered [20, 36] to study the evolution of metastable states with temperature. The quenched average over the distribution of the reference configuration is usually dealt with using the replica method. One needs to replicate the system $X$ a number $n'$ of times and perform a continuation $n' \to 0$ at the end of the computations. Noticing that the reference configuration can be seen as an additional replica, the total number of replicas is $n = 1 + n'$, which should be set to 1. Due to the asymmetry in the interaction between the reference configuration and the system, replica number 0 turns out to be privileged. In fact the effective Hamiltonian reads

$$H_{eff}(X_0, X_1, \ldots, X_{n-1}) = \sum_{a=0}^{n-1} H(X_a) - \epsilon \sum_{a=1}^{n-1} q(X_0, X_a) \quad (5)$$

(notice that the index $a$ runs over different ranges in the two sums). Instead of possessing the familiar symmetry $S_n$ under permutations of all the $n$ replicas, the problem is symmetric only under the permutations $S_{n-1}$ of replicas with index $a > 0$. Analogously to the annealed case, mean field theory predicts the existence of a line of phase transition in the $\epsilon$–$T$ plane that terminates at a critical point [20, 34, 37] and separates a confined phase with high overlap with the reference configuration from a deconfined phase with low overlap.

### 2.3. Particle pinning

Just as in the quenched potential case, in the case of particle pinning one fixes a reference configuration in $X_0$ from the equilibrium distribution at a temperature $T_{ref}$, but then one considers configurations $X$ in which a fraction $\theta$ of the variables are fixed to the values they take in $X_0$ [6]–[10]. Also in this case mean field theory predicts that the reduction of degrees of freedom induces new phase transitions in the system. Interestingly, the nature of the phase transitions depends on the details of the pinning procedure. As discussed in much detail in [22], if $T_{ref} = T/\alpha$ with $\alpha > 1$ ($T_{ref} < T$) one finds a pattern of phase transition similar to that of the annealed and quenched potential: there is a line of confinement first order phase transition in the $\theta$–$T$ plane that terminates at a critical point. Conversely, if $\alpha < 1$ ($T_{ref} > T$) one finds a line of ideal RFOT Kauzmann-like transition of the discontinuous 1RSB kind that crosses over into a line of second order glass transition of the continuous 1RSB kind [9, 22]. The natures of the terminating point of the first order transition in the first case and the RFOT transition in the second case are rather different, as we will discuss in section 3.

Within the replica method this procedure still requires an analytic continuation in the number of replicas $n$, which tends to 1. If the temperature of the reference configuration $T_{ref}$ is different from the temperature of the non-pinned particles, still the reference configuration is singled out and the symmetry is $S_{n-1}$. In the important case $T_{ref} = T$,
however, one can show that the unpinned particles remain at equilibrium [38, 39]. As a consequence, within the replica formalism there is full $S_n$ replica symmetry. The problem of the critical point in $n \to 1$, $S_n$ symmetric replica field theories has been addressed in [31]. In that case, through the analysis of the soft modes of the replica field theory the critical point was shown to belong to the RFIM universality class. In this paper we extend our analysis to the case of $n \to 1$, $S_{n-1}$ symmetric theories.

3. The annealed critical point

As we stated in section 2.3, in the annealed potential case, if we describe liquid phases, the complexity of the replica method is reduced to minimal terms. There are just two replicas and the $n \times n$ overlap order parameter matrix $q_{a,b}$ which appears in the replica method has here a single independent entry $q$ with $a \neq b$. This represents the overlap between the two copies; it is the only order parameter of the problem. The resulting Landau free energy is a functional of a single field $\phi(x) = q(x) - q^*$ representing the fluctuation of the overlap around its average,

$$F[\phi] = \int dx \frac{1}{2} k(\nabla \phi(x))^2 + V(\phi(x))$$

$$V(\phi) = \frac{1}{2} m_0 \phi^2 + g_3 \phi^3 + g_4 \phi^4.$$  

The coefficients $m_0$, $g_3$ and $g_4$ as well as $q^*$ smoothly depend on the control parameters $T$ and $\epsilon$. Away from the critical point, where $g_3 \neq 0$, the quartic term is irrelevant in perturbation theory; however, as in the gas–liquid transition case, at the critical point, both $m_0$ and $g_3$ vanish. We find therefore that the critical point is described by an ordinary scalar field theory with $\phi^4$ interaction and is in the universality class of the ordinary Ising model. This is coherent with the recent analysis of [40, 41].

4. The quenched and pinned critical points

In order to be defined, we consider the context of quenched potential; however, the main ingredients within replica field theory being the number replicas and the relative symmetry, with little modification, that we will specify on the way, one can treat the pinned particle construction with $T_{\text{ref}} < T$. Replicas can be used to average over the choice of the reference configuration, and we would like to describe the class of universality of the critical point within replica field theory. The starting point will be a replica field theory with $n \to 1$ replicas over a space dependent $n \times n$ space dependent matrix $Q_{ab}(x)$ of the kind

$$F[Q_{ab}(x), \epsilon] = F_0[Q_{ab}(x)] - \epsilon \sum_{a=1}^{n-1} \int dx Q_{0a}(x).$$  

The line $T_{\text{ref}} = T$ can be seen as a symmetric line analogous to the Nishimori line familiar in spin glass theory [48]. Mean field theory is based on models with quenched disorder of the family of the spherical $p$-spin model. There the so-called ‘annealed approximation’, where the average partition function rather than the average free energy is evaluated, turn out to be exact and allows us to compute exactly the potential.
The term $F_0[Q_{ab}(x)]$ is symmetric under permutation of all replicas. The last term in the action breaks this symmetry; in fact replica number 0, which corresponds to the reference state, is privileged with respect to the others. The symmetry $S_{n-1}$ under permutations of replicas $1, \ldots, n-1$ remains unbroken. In the pinning particle construction the $\epsilon$-coupling term is absent; however, if the temperature of the reference configuration $T_{\text{ref}}$ is different from the temperature $T$ at which the free particles evolve, the reference configuration is singled out and again $F[Q_{ab}(x)]$ contains terms that break $S_n$ into $S_{n-1}$.

As usual, we will start from a Landau expansion of the free energy close to the critical point, supposing that the non-diagonal elements of the matrix $Q_{ab}(x)$ can be written as

$$Q_{ab}(x) = Q_{ab}^* + \phi_{ab}(x)$$

where $Q_{ab}^*$ is the saddle point value of the matrix order parameter. The diagonal elements, which are related to the structure factor, in general are non-critical and regular across the transition point, and will not be discussed here.

The saddle point matrix $Q_{ab}^*$ is homogeneous in space, and for the critical points we consider here has the replica symmetric form

$$Q_{a,b}^* = q + (\delta_a^0 + \delta_b^0)(p - q).$$

The parameters $p \neq q$ represent respectively the overlap between the system and the reference configuration and the self-overlap of the system with itself. Instead of trying to write the most general $S_{n-1}$-invariant polynomial expansion of the free energy in terms of $\phi_{ab}(x)$, our strategy will consist first of analyzing the properties of an $S_{n-1}$ invariant mass matrix close to criticality and then, after identifying the soft modes, in writing directly the generic field theory describing their interaction disregarding completely the massive modes. As a preliminary, let us study longitudinal fluctuations, i.e. just fluctuations of $p$ and $q$. At the saddle point level, the free energy as a function of $\epsilon$ reads

$$\Gamma[\epsilon] = \frac{\partial}{\partial n} \bigg|_{n=1} F[Q_{ab}^*, \epsilon]$$

$$\Gamma[\epsilon] = N(W[p, q] - \epsilon p) \quad \frac{\partial W}{\partial q} = 0 \quad \frac{\partial W}{\partial p} = \epsilon.$$  

As usual one can interpret the effective potential $V(p, \epsilon) = W[p, q(p)] - \epsilon p$ at the point $q(p)$ defined by $\partial W/\partial q = 0$ as the value of the free energy when the system to reference configuration overlap takes the value $p$. The physical value of $p$ is fixed by the stationary condition $V'[p] = 0$. At a critical point terminating a first order line one should have in addition that the second and the third derivatives of $V$ vanish, $V''[p] = V'''[p] = 0$, conditions that generically fix the values of $T$ and $\epsilon$. Let us remark that close to a generic saddle point value, away from the critical point, the function $W[p, q]$ must admit an expansion of the kind

$$W[p + \delta p, q + \delta q] - W[p, q] = \frac{1}{2} \left[ \tilde{M}_{pp} \delta p^2 + 2 \tilde{M}_{pq} \delta p \delta q + \tilde{M}_{qq} \delta q^2 \right] + \sum_{r=0}^{3} C_r \delta p^r \delta q^{3-r} + O(\delta p^4).$$

This function describes longitudinal fluctuations which are constant in Space, and the form of $\phi$ is the same as that of the saddle point. By definition, at the critical point, longitudinal fluctuations for which $\delta q = (dq(p)/dp)\delta p = -\tilde{M}_{pq}/\tilde{M}_{qq}\delta p + O(\delta p^2)$ are long ranged, the
quadratic and the cubic forms vanish and the quartic terms become important. Of course, the points where the quadratic form vanishes but the cubic form remains finite are also critical. These correspond rather to spinodal points than to thermodynamic critical points. A well known example is that of dynamical glass transition points that correspond to $S_n$ symmetric cubic theories [14]–[16]. In the present case, as can be seen in figure 1, there are just two spinodal lines, for the confined and the deconfined phases, that converge into the critical point for $(T, \epsilon) \to (T_{Cr}, \epsilon_{Cr})$. Our analysis shows that generically, despite their different natures [21], both these spinodals belong to the $\phi^4$-RFIM universality class.

If one considers longitudinal fluctuations that are not constant in space, an additional ‘kinetic term’ of the kind

$$K[\delta p(x), \delta q(x)] = \int dx \left[ k_p(\nabla \delta p)^2 + k_q(\nabla \delta q)^2 \right]$$

is present in the longitudinal Landau expansion.

Close to the critical point, the mass of the soft mode can be simply related to the coefficient to the quadratic form, to the lowest order,

$$\hat{m}_0 = \frac{\hat{M}_{pp} \hat{M}_{qq} - \hat{M}_{pq}^2}{\hat{M}_{pp} + \hat{M}_{qq}} + O(\hat{m}_0^2).$$

5. The mass matrix and its eigenspaces

In this section we would like to go beyond longitudinal fluctuations, and identify all the zero modes of the problem in order to build up the suitable critical theory that describes their interaction. The physical meaning of the relevant modes will result from their contribution to the various kinds of correlation function that we discuss in section 8.

Let us study the most general mass matrices of small fluctuations, actually a four-index ‘matroid’, $M[a, b; c, d]$ that is symmetric under the operations $(ab) \to (ba), (ab; cd) \to (cd; ab)$, vanishes if $a = b$ or $c = d$ and respects the $S_{n-1}$ replica symmetry. Such a matrix has at most seven distinct elements that can be parametrized in the following way (all indices are assumed to be distinct from one another and from zero in the next formulas):

$$M[0, a; 0, a] = \frac{m_1}{2} + \frac{\mu_2}{2} + \mu_3; \quad M[0, a; 0, b] = \frac{\mu_2}{4} + \mu_3$$

$$M[0, a; a, b] = \frac{\nu_2}{4} + \nu_3; \quad M[0, a; b, c] = \nu_3$$

$$M[a, b; a, b] = \frac{m_1}{2} + \frac{m_2}{2} + m_3; \quad M[a, b; a, c] = \frac{m_2}{4} + m_3$$

$$M[a, b; c, d] = m_3.$$

The parameters $m_1, m_2, m_3, \mu_2, \mu_3, \nu_2$ and $\nu_3$ can be supposed to be distinct. Notice that the usual $S_n$ symmetric matrix is recovered if one poses $\mu_2 = \nu_2 = m_2$ and $\mu_3 = \nu_3 = m_3$.

We now look at the eigenspaces of $M$ proceeding analogously to the classical De Almeida–Thouless analysis of fully $S_n$ invariant matrices [42]. In full generality the eigenspaces of $M$ can be related to the representation of $S_{n-1}$ over symmetric (two index) matrices with vanishing diagonal elements. These representations are well known and consist in replica symmetric matrices, matrices that break the symmetry privileging one replica and matrices that privilege two replicas. In the following we use the terminology
usually employed in spin glass theory, calling these eigenspaces respectively longitudinal, anomalous and replicon.

5.1. The longitudinal space

The simplest eigenvectors are the longitudinal ones that have the same structure of the saddle point $Q_{ab}$, for $a \neq b$:

$$L_{ab} = (u - v)(\delta_{a0} + \delta_{b0}) + v$$

(13)

to which there correspond the two eigenvalues $\lambda_{LO}^\pm$ that are given in the appendix. One of these, that we call $\lambda_{LO}^L$, vanishes at the critical point, while the other remains finite.

Notice that $u$ and $v$ can be identified respectively with the variations $\delta p$ and $\delta q$ of the previous section. Comparing the quadratic form $\langle L| M | L \rangle = \sum_{ab,cd} L_{ab} M[a,b;c,d] L_{cd}$ in the limit $n \to 1$ with that appearing in (10), one can identify $\hat{M}_{pp}, \hat{M}_{qq}$ and $\hat{M}_{pq}$ as

$$\hat{M}_{pp} = 2m_1 + \mu_2, \quad \hat{M}_{qq} = m_2 - m_1, \quad \hat{M}_{pq} = -\nu_2.$$  

(14)

Strictly speaking the longitudinal eigenvalues of the matrix $M$ do not coincide with the eigenvalues of the quadratic form in (10). This due to the fact that if we write $L_{ab} = uw_{ab}^1 + vw_{ab}^2$ the vectors $w^1$ and $w^2$ are not normalized to 1. However, it is easy to see that if one of the eigenvalues of the quadratic form is zero, so is the corresponding eigenvalue of $M$. Simple linear algebra shows that for $n \to 1$ the small longitudinal eigenvalue $\lambda_{LO}$ reads

$$\lambda_{LO}|_{n=1} \equiv m_0 = \frac{\hat{M}_{pp} + \hat{M}_{qq}}{2\hat{M}_{qq} - \hat{M}_{pp}} m_0 + O(m_0^2).$$

(15)

The corresponding eigenvector should be such that for $n \to 1$, $v = (dq(p)/dp)u = -\hat{M}_{pq}/\hat{M}_{qq}u$. For future reference we introduce the notation $\gamma = dq(p)/dp$. In general it can be expected that $\gamma > 0$, implying strong correlations between the fluctuations of $p$ and those of $q$.

5.2. The anomalous space

The second family of eigenvectors is the so-called anomalous ones $a_{ab}^\mu$, where in addition to replica 0 a replica $\mu > 0$ is privileged: there are four distinct elements (all indices are different among themselves and from 0),

$$a_{a0}^\mu = u_0 + u_1; \quad a_{0a}^\mu = u_1$$

$$a_{a0}^\mu = v_0 + v_1; \quad a_{0a}^\mu = v_1.$$  

(16)

If we impose orthogonality between the anomalous and longitudinal spaces we find

$$u_0 = -(n - 1)u_1$$

$$v_0 = -\frac{1}{2}(n - 1)v_1.$$  

(17)

(18)

This fixes two parameters out of four, and also in this case there are two independent eigenvalues. Notice that $u_0$ and $v_0$, that are responsible for the difference between $a_{ab}^\mu$ and $L$, are of order $n - 1$ relative to $u_1$ and $v_1$; this implies, on a very general basis, that the anomalous and longitudinal eigenvalues form degenerate doublets for $n \to 1$. Their
difference, which should be linear in $u_0$ and $v_0$, is of the order $n - 1$. The values of $v_1$ and $u_1$ become degenerate with the values of the parameters $v$ and $u$ in the corresponding longitudinal eigenvectors. This is confirmed by the explicit computation of eigenvalues and eigenvectors as a function of the mass matrix parameters with Mathematica. As we will see this eigenvalue degeneracy is at the origin of typical random field terms in the action.

The total dimension of the anomalous space is $2(n - 2)$, as can be realized taking into account the orthogonality with the longitudinal space.

The meaning of the anomalous vectors can be understood within the replica formalism looking at the projection of the fluctuating field:

$$
\langle \phi|a^\mu \rangle = u_1 \left[ -(n - 1)\phi_{0\mu} + \sum_{b=1}^{n-1} \phi_{0b} \right] + v_1 \left[ -(n - 1)\sum_{a=1}^{n-1} \phi_{\mu a} + \sum_{a,b=1}^{n-1} \phi_{ab} \right].
$$

These are replica symmetry breaking fluctuations where $\phi_{0\mu}$ and $\sum_{a=1}^{n-1} \phi_{\mu a}$ differ from their averages over the index $\mu$. Soft modes in these directions have as physical consequence deep relations among different correlation functions that can be defined, as we discuss in section 8.

5.3. The replicon space

The last family of eigenvectors is that of so-called replicons. These can be characterized in two equivalent ways: either as matrices that besides replica number 0 privilege two replicas $\mu, \nu > 0$ and are orthogonal to the longitudinal and anomalous spaces, or, more simply, as matrices $R_{ab}$ such that

$$
R_{a0} = 0; \sum_{b} R_{ab} = 0 \quad \forall a.
$$

The replicon space concerns fluctuations of the $\phi_{ab}$ which are independent from those of $\phi_{0b}$ and induce replica symmetry breaking of the type familiar from spin glass theory [43]. The dimensionality of the replicon sub-space is $(n - 1)(n - 4)/2$. Together with the longitudinal and anomalous spaces it exhausts the $n(n - 1)/2$ dimensional linear space of symmetric matrices null on the diagonal. It is easy to see, using the form of the mass matrix, that there is a single replicon eigenvalue and it is just given by $m_1$. The behavior of the replicon eigenvalue marks the difference between the critical points terminating the first order transition lines of the quenched construction and the pinned particle one for $T_{\text{ref}} < T$ and the critical points marking the passage from an RFOT Kauzmann (or discontinuous 1RSB) transition to a continuous 1RSB glass transition for the pinned particle construction with $T_{\text{ref}} > T$. Generically, in the former case, $m_1$ does not have reasons to vanish at the critical point. For example, it can be checked that $m_1$ indeed remains finite at the critical point of the spherical $p$-spin model in the quenched potential setting. Conversely, in the latter case $m_1$ vanishes at the transition and is zero on the whole second order glass transition line. In this paper we concentrate on the case where $m_1$ remains positive at criticality, and treat therefore the terminating critical points of the first order lines of the quenched construction and the pinned particle problem with $T_{\text{ref}} < T$. The case $T_{\text{ref}} = T$, where full $S_n$ replica symmetry is recovered, marks the boundary between the two behaviors. This case has been analyzed within a one loop approximation in [31]. The analysis performed there shows that
the additional degeneracy of the small eigenvalue does not change the universality class of the problem. In the replica formalism, we think this could be related to the peculiarities of the $n \to 1$ limit, and one can conjecture that the RFIM universality class also holds for $T_{\text{ref}} > T$. Further work will be needed to extend the analysis of [31] to all orders in perturbation theory. In conclusion, generically, at critical points terminating first order lines, the singularities arise from the fact the longitudinal and anomalous fluctuations go soft at the transition, while the replicon ones remain massive.

We notice that an alternative approach to analyze the eigenspaces of $M$ (that leads to same results) consists in treating separately the first line and column of the matrix $\phi_{a,b}$ $a \neq b = 0, \ldots, n - 1$ as an $n - 1$ dimensional vector $z_a = \phi_{0a}$ in replica space with $a = 1, \ldots, n - 1$ and the remaining part of the $\phi_{ab}$ matrix with $a \neq b = 1, \ldots, n - 1$ as a $n - 1 \times n - 1$ matrix. In this way the problem becomes formally closer to that studied in the paper of De Almeida and Thouless [42].

6. A vectorial representation

To build up the relevant critical theory we can disregard massive directions and consider an interacting field theory for fluctuating fields $\phi_{ab}(x)$ which are linear combinations of the critical modes. We therefore concentrate on the zero mode subsector of the longitudinal and anomalous spaces and ignore the sectors corresponding to hard modes. In order to have theory that keeps explicitly the $S_{n-1}$ symmetry, it is convenient to combine longitudinal and anomalous vectors $L$ and $a^\mu$, that we suppose to be defined up to a normalization to be fixed $a \text{ posteriori}$, into vectors $A^\mu$

$$A^\mu = a^\mu + L$$

that form an orthonormal basis $\langle A^\mu | A^\nu \rangle = \sum_{ab} A^\mu_{ab} A^\nu_{ab} = \delta_{\mu\nu}$. Let us state a few properties of these vectors and fix the normalizations of $a^\mu$ and $L$. Notice that, while replica symmetry implies that $\sum_\mu a^\mu$ should be proportional to $L$, the orthogonality condition $\langle a^\mu | L \rangle = 0$ says that $\sum_\mu a^\mu = 0$. Using again the replica symmetry we can write

$$\langle a^\mu | a^\nu \rangle = \delta_{\mu\nu} (\alpha_{11} - \alpha_{12}) + \alpha_{12}.$$  

(22)

Summing over $\mu$ we obtain that $\alpha_{12} = -\alpha_{11}/(n - 2)$. Imposing orthonormality we have

$$\langle A^\mu | A^\nu \rangle = \alpha_{12} + \langle L | L \rangle = 0 \quad \mu \neq \nu$$

$$\langle A^\mu | A^\nu \rangle = \alpha_{11} + \langle L | L \rangle = 1$$

(23)

which implies $\alpha_{11} = 1 - \langle L | L \rangle = (n - 2)\langle L | L \rangle$ and $\langle L | L \rangle = 1/(n - 1)$.

We can now evaluate the matrix element $\langle A^\mu | M | A^\nu \rangle$. If we write the longitudinal eigenvalue as $\lambda_{LO} = m_0 + (n - 1)\eta_{LO}$ and the anomalous one as $\lambda_{AN} = m_0 + (n - 1)\eta_{AN}$ and define $\eta = \eta_{LO} - \eta_{AN}$, we can easily see that

$$\langle A^\mu | M | A^\nu \rangle = m_0 \delta_{\mu\nu} + \eta + O(n - 1).$$

(24)

Let us now expand the critical field on the basis of the $A^\mu$

$$\phi_{ab}(x) = \sum_{\mu=2}^n \psi_\mu(x) A^\mu_{ab}.$$  

(25)

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We can now formulate the critical theory in terms of the single index fields $\psi_\mu$. This theory should of course be invariant under all permutation of the indices $\mu = 1, \ldots, n - 1$. Generically, the action of the theory could be written as a sum of a local term which is a polynomial in the fields, and a kinetic term sensitive to space fluctuations of the $\psi_\mu$. We are led then to the study of the replica symmetric low order local polynomial invariants of the $n - 1$ component vector of the $\psi_\mu$. These can be built up explicitly, starting from the monomials of lower orders.

- The only linear invariant is
  \[ I_1 = \sum_\mu \psi_\mu. \] (26)

- The quadratic invariants are
  \[ I_{2,1} = I_1^2, \quad I_{2,2} \equiv J_2 = \sum_\mu \psi_\mu^2. \]

- The cubic ones are
  \[ I_{3,1} = I_1^3, \quad I_{3,2} = I_1 I_{2,2}, \quad I_{3,3} \equiv J_3 = \sum_\mu \psi_\mu^3. \]

The higher order invariants can be obviously generated in a recursive way. In general, the only invariant of order $k$ which cannot be expressed as a product of lower order ones is $J_k = \sum_\mu \psi_\mu^k$. In addition to purely local invariants, we should consider the lowest order invariant in $\nabla \psi_\mu$, namely, the kinetic term
\[ K[\psi_\mu] = \sum_\mu (\nabla \psi_\mu)^2. \] (27)

7. Dimensional analysis

The quadratic form $\langle \phi | M | \phi \rangle$ in the bases of the $\psi_\mu$ is readily computed:
\[ \langle \phi | M | \phi \rangle = m_0 \sum_\mu \psi_\mu^2 + \eta \left( \sum_\mu \psi_\mu \right)^2 = m_0 I_{2,2} + \eta I_1^2. \] (28)

We can remark at this point that (28) has the typical form that appears in the replica treatment of the RFIM with random field $\delta$-correlated in space. The coefficient $\eta$ that originates here corresponds in that case to the variance of the random field. Its appearance here stems from the degeneracy of the longitudinal and replicon eigenvalues for $n \to 1$. It can be checked in specific problems that while $m_0 \to 0$ the value of $\eta$ remains positive.

It is well known from the theory of the RFIM that the inversion of the form (28) has single pole and double pole propagators. The field $\psi_\mu$ cannot have a well defined scaling dimension. The same conclusion can be reached observing that this is incompatible with the fact that the two terms in equation (28) are of the same order of magnitude for $m_0 \to 0$ and $\eta$ finite.
In order to use fields with well defined scaling dimension we can make a further change of basis, as originally suggested by Cardy \[44\], and write

\[
\psi_\mu = \psi + \delta_\mu \hat{\psi} + \chi_\mu \\
\chi_1 = 0 \\
\sum_\mu \chi_\mu = 0. 
\] (29)

This is a legitimate change of basis, since for all \(x\) it contains \(n-1\) independent parameters.

We notice that in this basis the field \(\phi\) can be written as

\[
\phi = (n-1)\psi L + \hat{\psi} A^1 + \sum_\mu \chi_\mu A^\mu \\
= [(n-1)\psi + \hat{\psi}] L + \hat{\psi} a^1 + \sum_\mu \chi_\mu a^\mu. 
\] (30)

Purely longitudinal fluctuations correspond to \(\hat{\psi} = \chi_\mu = 0\). Notice that, in this case, the invariants \(J_k\) are of order \(n-1\) (in fact \(J_k = (n-1)\psi^k\)), while all composite invariants are of higher order. Since the effective potential is equal to the derivative of the free energy with respect to \(n\) in \(n = 1\), this implies that the only invariants that enter the effective potential are the \(J_k\). We would like to argue that the same invariants are also the ones that govern fluctuations.

In the basis (29) the linear and quadratic invariants read

\[
I_1 = (n-1)\psi + \hat{\psi} \\
J_2 = (n-1)\psi^2 + 2\psi \hat{\psi} + \sum_\mu \chi_\mu^2 + \hat{\psi}^2 
\] (31)

so that the quadratic form is written

\[
\langle \phi | M | \phi \rangle = m_0 \left[ (n-1)\psi^2 + 2\psi \hat{\psi} + \sum_\mu \chi_\mu^2 + \hat{\psi}^2 \right] + \eta((n-1)\psi + \hat{\psi})^2. 
\] (32)

We now proceed with dimensional analysis, which, as is well known in general, is equivalent to the analysis of the leading singularities in perturbation theory.

Imposing that for \(n \to 1\) the terms \(m_0 \psi \hat{\psi}\), \(m_0 \sum_\mu \chi_\mu^2\) and \(\eta \hat{\psi}^2\) share the same superficial scaling dimension, as \(m_0 \to 0\) we find

\[
[\hat{\psi}] = 2 + [\psi] \\
[\chi_\mu] = 1 + [\psi] 
\] (33)

where we have set \([m_0] = 2\). Among the invariants of order \(k\) the ones of lower scaling dimension are these which contain the lower power of \(\hat{\psi}\) and \(\chi_\mu\) for \(n \to 1\). These are
the terms $J_k = \sum_\mu \psi^k_\mu$, which are the only ones that contain $\psi^{k-1}_\mu \hat{\psi}$ and $\psi^{k-2} \sum_\mu \chi^2_\mu$. This is enough to say that the spinodal lines, where the coefficient of $J_3$ is non-null, belong to the universality class of the $\phi^4$-RFIM theory (the spinodal of the RFIM). At the critical point, by definition the coefficient of $J_3$ in the effective action vanishes; however, in general, the coefficients of the other cubic invariants $I_{3,1}$ and $I_{3,2}$ are non-zero. In order to derive the RFIM, we should argue that close to the upper critical dimension of the RFIM, $D_c = 6$, these invariants have superficial scaling dimension higher than that of $J_4$.

The scaling dimension of $J_4$ is $[\hat{\psi} \psi^3]$, those of $I_{3,2}$ and $I_{3,1}$ are respectively $[\hat{\psi}^2 \psi]$ and $[\hat{\psi}^3]$. Since the dimension of $x$ is $-1$, in order to make the action adimensional we need $[\hat{\psi}] = D/2 - 2$. We see that $[J_4] = 2D - 6$ while $[I_{3,2}] = 3/2D - 2$ and $I_{3,1} = 3/2D$. We find therefore that close to the dimension six, if the coefficient of $J_3$ vanishes, the leading singular term becomes $J_4$ and the critical point is in the RFIM class. Denoting by $g$ the coefficient of $J_4$ in the Landau expansion of the free energy, we can write explicitly, close to the critical point and for $n \to 1$,

$$F[\psi] = \int dx \hat{\psi}(x) \left(-k \Delta \psi(x) + m_0 \psi(x) + g \psi^3(x) + \eta \hat{\psi}(x)\right) + \frac{1}{2} \int dx \sum_\mu (k (\nabla \chi_\mu)^2 + [m_0 + 3g \psi(x)^2] \chi^2_\mu). \quad (34)$$

It is well known that, since there are $n - 3 \to -2$ independent parameters $\chi_\mu$, integration over them is equivalent to a fermionic determinant, and (34) is equivalent to the Parisi–Sourlas action of the RFIM. Analogously to the case of the dynamical transition [14], the fluctuations of the potential with respect to the reference configurations can be effectively parametrized by a random field term, with Gaussian statistics, uncorrelated from site to site.

8. Correlation functions

So far we have proceeded to a formal analysis of the replica soft modes; we found that there are different components of the fluctuations that have different scaling dimensions and we derived the RFIM on the basis of a dimensional analysis. We anticipated in the previous sections that the emergence of the RFIM comes from fluctuations with respect to the choice of the reference configuration. To substantiate this statement we study correlation functions in our system, relating them to corresponding functions in the RFIM. We show that functions that are sensitive to thermal fluctuations relate to thermal fluctuations of the RFIM, while functions that are sensitive to the choice of the reference configuration relate to functions sensitive to the choice of the random field in the RFIM.

We define therefore two averages: we denote by angular brackets $\langle \cdot \rangle$ the thermal average that is conditioned by the choice of the reference configuration and can involve several replicas, and by square brackets $[\cdot]$ the average over the choice of the reference configuration. Moreover, inside the averages, index 0 is assigned to the reference configuration, while indices 1, 2, 3, 4 refer to copies with different realizations of the thermal noise but subject to the attraction to the same reference configuration.
We can write in principle seven distinct two-field (or four-body) correlation functions, whose physical meaning is transparent:

\[
\begin{align*}
\varrho_{0101}(x) &= [(\phi_0(x)\phi_{01}(0))]; & \varrho_{0102}(x) &= [(\phi_0(x)\phi_{02}(0))] = [(\phi_0(x)\langle\phi_0(0)\rangle] \\
\varrho_{0112}(x) &= [(\phi_0(x)\phi_{12}(0))]; & \varrho_{0123}(x) &= [(\phi_0(x)\phi_{23}(0))]
\end{align*}
\]

(35)

To the order of the leading singularity, in which we can use the RFIM, however, starting from the fields \(\psi, \dot{\psi}\) and \(\chi_\mu\) we can construct at most four independent correlations, namely \(\langle\dot{\psi}(x)\psi(0)\rangle\), \(\langle\dot{\psi}(x)\dot{\psi}(0)\rangle\), \(\langle\dot{\psi}(x)\dot{\psi}(0)\rangle\), and \(\sum_\mu \langle\chi_\mu(x)\chi_\mu(0)\rangle\), where we have taken into account the condition \(\sum_\mu \chi_\mu = 0\). This number is reduced to two by replica symmetry, which implies that

\[
\begin{align*}
\sum_{a,b}^{1,n-1} \langle\phi_{ab}(x)\phi_{ab}(0)\rangle &= (n - 1)[\varrho_{0101} - \varrho_{0102}] + O((n - 1)^2) \\
\sum_{a,b,c,d}^{1,n-1} \langle\phi_{ab}(x)\phi_{cd}(0)\rangle &= (n - 1)[-2\varrho_{1212} + 8\varrho_{1213} - 6\varrho_{1234}] + O((n - 1)^2).
\end{align*}
\]

(36)

The explicit expression in terms of \(\psi, \dot{\psi}\) and \(\chi_\mu\), that we have computed with our Mathematica script, shows that (36) are of order \(n - 1\) only if the following identities hold:

\[
\begin{align*}
\langle\dot{\psi}(x)\dot{\psi}(0)\rangle &= 0 \\
\sum_\mu \langle\chi_\mu(x)\chi_\mu(0)\rangle &= -\langle\dot{\psi}(x)\dot{\psi}(0) + \psi(0)\dot{\psi}(x)\rangle.
\end{align*}
\]

(37)

These are known identities in the supersymmetric formalism for the RFIM and related problems [45]–[47] if we identify the \(\chi_\mu\) with fermion fields. It follows that the correlations (35) are linear combinations of \(\langle\dot{\psi}(x)\psi(0)\rangle\) and \(\langle\dot{\psi}(x)\dot{\psi}(0)\rangle\), which in the RFIM represent respectively the thermal fluctuations and the sample to sample fluctuation correlation functions. The coefficients of the combination depend on \(\gamma = dq(p)/dp\), which is the only parameter of the system appearing in the eigenvectors.

The correlations have connected and disconnected components with respect to the angular brackets. Correlations which are connected measure thermal fluctuations. Disconnected correlations measure instead fluctuations with respect to changes of the reference configuration. Omitting the position indices, we can write the connected combinations \(\varrho_{0101} - \varrho_{0102}, \varrho_{0112} - \varrho_{0123}, \varrho_{1212} - \varrho_{1213}\) and \(\varrho_{1212} - \varrho_{1234}\), and the disconnected ones \(\varrho_{0102}, \varrho_{0123}, \varrho_{1213}\) and \(\varrho_{1234}\). Exact expressions can be readily obtained with
Mathematica and read
\begin{align*}
g_{0101}(x) - g_{0102}(x) &= \frac{1}{2 - \gamma^2} \langle \psi(x)\psi(0) + \psi(0)\dot{\psi}(x) \rangle \\
g_{0112}(x) - g_{0123}(x) &= \frac{\gamma}{2}(g_{0101}(x) - g_{0102}(x)) \\
g_{1212}(x) - g_{1213}(x) &= \frac{\gamma^2}{4}(g_{0101}(x) - g_{0102}(x)) \\
g_{1212}(x) - g_{1234}(x) &= \frac{\gamma^2}{2}(g_{0101}(x) - g_{0102}(x)).
\end{align*}

The non-connected components are
\begin{align*}
g_{0102}(x) &= \frac{1}{2 - \gamma^2} \langle \psi(x)\psi(0) \rangle - \frac{4 - \gamma^2}{2(2 - \gamma^2)^2} \langle \psi(x)\dot{\psi}(0) + \psi(0)\dot{\psi}(x) \rangle \\
g_{0123}(x) &= \gamma g_{0102}(x) \\
g_{1213}(x) &= \frac{\gamma^2}{2 - \gamma^2} \langle \psi(x)\psi(0) \rangle - \frac{\gamma^2(6 - \gamma^2)}{4(2 - \gamma^2)^2} \langle \psi(x)\dot{\psi}(0) + \psi(0)\dot{\psi}(x) \rangle \\
g_{1234}(x) &= \gamma^2 g_{0102}(x).
\end{align*}

As announced, we find that the connected correlations only contain \( \langle \psi(x)\psi(0) + \psi(0)\dot{\psi}(x) \rangle \), that in the RFIM represents the thermal correlation function for fixed random field, while the disconnected correlations also contain \( \langle \psi(x)\psi(0) \rangle \), which is the correlation sensitive to random field changes in the RFIM.

The relations (38) and (39) between the different overlap correlation functions are an important consequence of our analysis; they are valid at all orders of perturbation theory and can be tested in numerical simulations. The computation of all correlations (35) requires us to simulate a maximum of four independent replicas besides the reference one. The parameter \( \gamma \) can also in principle be measured in simulations by looking at the dependence on \( p \) of the overlap \( q \) between two distinct replicas that have overlap \( p \) with the reference.

The relations (38) and (39) express the fact that to the leading order the overlap of the system with the reference configuration and the self-overlaps are strongly correlated and the fluctuations verify \( \phi_{12}(x) = \delta q_{12}(x) \sim \gamma \phi_{01}(x) = \delta q_{01}(x) \). The second of (38) for example can be derived observing that both the LHS and the RHS combinations can be expressed as derivatives of local overlap averages with respect to space dependent coupling \( \epsilon(x) \),
\begin{align*}
g_{0101}(x) - g_{0102}(x) &= \frac{1}{T} \frac{\delta[\langle q_{01}(x) \rangle]}{\delta\epsilon(0)}; \\
g_{0112}(x) - g_{0123}(x) &= \frac{1}{2T} \frac{\delta[\langle q_{12}(x) \rangle]}{\delta\epsilon(0)}.
\end{align*}

and using the chain rule for the derivative
\begin{equation}
\frac{\delta[\langle q_{12}(x) \rangle]}{\delta\epsilon(0)} = \frac{dq(p)}{dp} \frac{\delta[\langle q_{01}(x) \rangle]}{\delta\epsilon(0)}.
\end{equation}

It can be noted however that when integrated over space the second of (38) holds exactly, even beyond the level of leading singularity described by the RFIM, and can be used to measure \( \gamma \).

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Finally, we remark that, as is well known in the theory of the RFIM, to the one loop order of Gaussian fluctuations, the non-connected components are more singular than the connected ones. In momentum space, connected correlations behave as \((k^2 + m_0)^{-1}\), while disconnected ones behave as \((k^2 + m_0)^{-2}\).

9. Summary and conclusions

In this paper we have analyzed the universality class of critical points terminating first order transition lines in glassy systems in the presence of constraints. After the analysis of the case of a symmetric coupling between two replicas, for which the Ising universality class is found, we have considered the case in which a coupling with a quenched reference configuration is present. We extend in this way the analysis of the equal temperature pinned particle critical point presented in [31]. This includes the quenched potential construction and the pinned particle construction for \(T_{\text{ref}} < T\). A full analysis of the soft modes within replica field theory at all orders of perturbation theory leads to the universality class of the random field Ising model. The effective random field appearing in the final description parametrizes the randomness in the reference configuration. We have analyzed the various four-body correlation functions that appear in the theory and found that there are only two independent combinations that become dominant close to the critical point. The existence of the critical point, its universality class and the relation between correlation functions constitute important predictions of the thermodynamic theory of glasses based on mean field theory, and now, on its loop expansion. We hope that in the near future they can be tested in numerical simulations of realistic glass forming liquid models. The critical point and line of continuous glass transition present in the particle pinning problem for \(T_{\text{ref}} > T\) are excluded by the present analysis. In that case the replicon modes are critical and their interaction with the longitudinal and anomalous modes that we have seen to give rise to the RFIM should be included. This is a fascinating research project that we leave for the future.

After completion of our work we came to know that G Biroli, C Cammarota, G Tarjus and M Tarzia have also considered the problem of quenched critical points with a similar approach (arXiv:1309.3194).

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Appendix

The algebra of multiplication of the four replica index mass matrix \(M[a, b; c, d]\) by a two replica index vectors \(v_{ab}\), which is necessary to perform the explicit computation of the eigenvalues, can be implemented in Mathematica. In fact, given the structure of \(M\) and its eigenvalues, one just needs to be able to perform sums over replica indices of constants.
and delta functions. A detailed script performing this task was published in [31]. The one we use here is an adaptation of that one. The eigenvalues of the mass matrix can be then explicitly found. At the end, this amounts to solving second order algebraic equations. We present the result of the computation to the order \( (n - 1) \). The longitudinal eigenvalues read

\[
\lambda_{LO}^\pm = \frac{1}{4} \left( 4m_1 - 2m_2 + \mu_2 \pm \sqrt{(\mu_2 + 2m_2)^2 - 8\nu_2^2} \right) \\
+ \frac{1}{4}(n - 1) \left( 2m_2 - 2m_3 + \mu_2 + 4\mu_3 \right) \\
\pm \frac{4\nu_2 (\nu_2 - 4\nu_3) - (\mu_2 + 2m_2) (-\mu_2 - 4\mu_3 + 2m_2 - 2m_3)}{\sqrt{(\mu_2 + 2m_2)^2 - 8\nu_2^2}}.
\]

(A.1)

The zero mode corresponds to the positive determination of the square root \( \lambda_{LO} = \lambda_{LO}^+ \). It can be argued be the condition that in the \( S_n \) symmetric limit \( \mu_2 = \nu_2 = m_2 \) and \( \mu_3 = \nu_3 = m_3 \) the leading order \( m_0 \) become degenerate with the replicon eigenvalue \( \lambda_{RE} = m_1 \). In this way we find

\[
m_0 = \frac{1}{4} \left( 4m_1 - 2m_2 + \mu_2 + \sqrt{(\mu_2 + 2m_2)^2 - 8\nu_2^2} \right) \tag{A.2}
\]

\[
\eta_{LO} = \frac{1}{4} \left( 2m_2 - 2m_3 + \mu_2 + 4\mu_3 \right) \\
+ \frac{4\nu_2 (\nu_2 - 4\nu_3) - (\mu_2 + 2m_2) (-\mu_2 - 4\mu_3 + 2m_2 - 2m_3)}{\sqrt{(\mu_2 + 2m_2)^2 - 8\nu_2^2}}.
\]

(A.3)

We notice that for \( n \to 1 \) the longitudinal eigenvalue becomes degenerate with the replicon if \( \nu_2 = \pm \sqrt{m_2 \mu_2} \). This condition is met at the critical point of the junction of the discontinuous and continuous glass transitions of the pinned particle problem for \( T_{\text{ref}} > T \), and actually on the whole line of continuous glass transition.

Analogously, one can compute the anomalous eigenvalues that turn out to be

\[
\lambda_{AN}^\pm = \frac{1}{4} \left( 4m_1 - 2m_2 + \mu_2 \pm \sqrt{(\mu_2 + 2m_2)^2 - 8\nu_2^2} \right) \\
+ \frac{(n - 1)\nu_2^2}{\sqrt{(\mu_2 + 2m_2)^2 - 8\nu_2^2}} \left( \mu_2 \pm \sqrt{(\mu_2 + 2m_2)^2 - 8\nu_2^2} + 2m_3 \right) - 4\nu_2^2 \right) \\
\pm \frac{4\nu_2 (\nu_2 - 4\nu_3) - (\mu_2 + 2m_2) (-\mu_2 - 4\mu_3 + 2m_2 - 2m_3)}{\sqrt{(\mu_2 + 2m_2)^2 - 8\nu_2^2}}.
\]

(A.4)

with \( \lambda_{AN} = \lambda_{AN}^+ \) and

\[
\eta_{AN} = \frac{\nu_2^2 (\mu_2 + \sqrt{(\mu_2 + 2m_2)^2 - 8\nu_2^2} + 2m_3) - 4\nu_2^2)}{\sqrt{(\mu_2 + 2m_2)^2 - 8\nu_2^2}}.
\]

(A.5)

The expression of \( \eta = \eta_{LO} - \eta_{AN} \) is not particularly illuminating. The consistency of the approach requires that it should remain positive at the critical point.

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