Binomial Complexities and Parikh-Collinear Morphisms*

Michel Rigo[0000–0001–7463–8507], Manon Stipulanti[0000–0002–2805–2465], and
Markus A. Whiteland[0000–0002–6006–9902]

Department of Mathematics, University of Liège, Liège, Belgium
{m.rigo,m.stipulanti,mwhiteland}@uliege.be

Abstract. Two words are $k$-binomially equivalent, if each word of length at most $k$ occurs as a subword, or scattered factor, the same number of times in both words. The $k$-binomial complexity of an infinite word maps the natural $n$ to the number of $k$-binomial equivalence classes represented by its factors of length $n$. Inspired by questions raised by Lejeune, we study the relationships between the $k$ and $(k + 1)$-binomial complexities; as well as the link with the usual factor complexity. We show that pure morphic words obtained by iterating a Parikh-collinear morphism, i.e. a morphism mapping all words to words with bounded abelian complexity, have bounded $k$-binomial complexity. In particular, we study the properties of the image of a Sturmian word by an iterate of the Thue–Morse morphism.

Keywords: Factor complexity · Abelian complexity · Binomial complexity · iterates of Thue–Morse morphism.

1 Introduction

When we are interested in the combinatorial structure of an infinite word $x$ over a finite alphabet $A$, it is often useful to study its language $\mathcal{L}(x)$, i.e. the set of its factors, and in particular to look at factors of a given length $n$. We let $\mathcal{L}_n(x)$ denote $\mathcal{L}(x) \cap A^n$. The usual factor complexity function $p_x : \mathbb{N} \rightarrow \mathbb{N}$ counts the number $\# \mathcal{L}_n(x)$ of words of length $n$ occurring in $x$. For instance, ultimately periodic words are characterized by a bounded factor complexity and Sturmian words are exactly those words satisfying $p_x(n) = n + 1$ for all $n$. For a general reference about word combinatorics, see, for instance, [213]. However, to highlight particular combinatorial properties of the infinite word of interest, other complexity measures such as abelian, $k$-abelian, cyclic, privileged, and $k$-binomial complexities have been introduced. See, for instance, [18,11,19]. In most cases, one considers the quotient of the language $\mathcal{L}(x)$ by a convenient equivalence relation $\sim$ and the corresponding complexity function therefore maps

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\( n \in \mathbb{N} \) to \( \#(\mathcal{L}_n(x)/\sim) \). For instance, a binary (non-periodic) word is balanced if and only if its abelian complexity is equal to the constant function 2. This paper focuses on the binomial complexity introduced in [19] and that is also the central theme of Lejeune’s thesis [10].

A parallel can be drawn between the \( k \)-abelian complexity introduced by Karhumäki et al. [9] and the \( k \)-binomial complexity. In both cases, we have a series of refinements of the abelian equivalence already introduced by Erdős [5]. The fundamental difference is the following one. Let \( k \geq 1 \) be an integer. Two finite words \( u,v \) are \( k \)-abelian equivalent if, for each factor \( w \) of length at most \( k \), we count the same number of occurrences of \( w \) in both words \( u \) and \( v \). For \( k \)-binomial equivalence, we count the number of times each word \( w \) of length at most \( k \) occurs in \( u \) and \( v \) as a subword, i.e., scattered factor. Thus, in the first case, we are interested in sequences of \( k \) consecutive letters, whereas in the second case, we look at subsequences of length \( k \) extracted from a given word.

We will thus make the important distinction between a factor of a word and a subword.

1.1 Binomial Coefficients and Complexity Functions

Let us now give precise definitions and notation. For any integer \( k \), we let \( A^k \) (resp., \( A^{\leq k} \); resp., \( A^{< k} \)) denote the set of words of length exactly (resp., at most; resp., less than) \( k \) over \( A \). We let \( A^* \) (resp., \( A^+ \)) denote the set of finite words (resp., non-empty finite words) over \( A \). We let \( \varepsilon \) denote the empty word. The length of the word \( w \) is denoted by \( |w| \) and the number of occurrences of a letter \( a \) in \( w \) is denoted by \( |w|_a \). Writing \( A = \{a_1, \ldots, a_k\} \) and fixing the order \( a_1 < a_2 < \cdots < a_k \) on the letters, the Parikh vector of a word \( w \in A^* \) is defined as the column vector

\[
\Psi(u) = (|w|_{a_1}, |w|_{a_2}, \ldots, |w|_{a_k})^T.
\]

Let \( u, w \in A^* \). The binomial coefficient of \( u \) and \( w \) is the number of times \( w \) occurs as a subsequence of \( u \), i.e., writing \( u = u_1 \cdots u_n \) with \( u_i \in A \),

\[
\binom{u}{w} = \# \{ i_1 < i_2 < \cdots < i_{|w|} : u_{i_1}u_{i_2} \cdots u_{i_{|w|}} = w \}.
\]

By convention, \( \binom{w}{\varepsilon} = 1 \). For more on these binomial coefficients, see, for instance, [13, Chapter 6]. Let \( k \geq 1 \) be an integer. Two words \( u, v \in A^* \) are \( k \)-binomially equivalent, and we write \( u \sim_k v \), if

\[
\binom{u}{x} = \binom{v}{x}, \quad \forall x \in A^{\leq k}.
\]

Salomaa [20] introduces the \( k \)-spectrum of a word \( u \) which is a formal polynomial in non-commutative variables \( \sum_{w \in A^{\leq k}, \text{ length } w = k} \binom{u}{w} w \). Thus two words are \( k \)-binomially equivalent if and only if they have the same \( k \)-spectrum. Observe that the word \( u \) is obtained as a permutation of the letters in \( v \) if and only if \( u \sim_1 v \). In this case, we say that \( u \) and \( v \) are abelian equivalent.
Definition 1. Let $k \geq 1$ be an integer. The $k$-binomial complexity function of an infinite word $x$ is defined as $b^{(k)}_x : \mathbb{N} \rightarrow \mathbb{N}, n \mapsto \#(\mathcal{L}_n(x)/\sim_k)$.

It is immediate from the definition, that for all $k \geq 1$, $u \sim_{k+1} v$ implies $u \sim_k v$. Thus, for all $n$, we have the inequalities (illustrated by Fig. 1)

$$b^{(1)}_x(n) \leq b^{(2)}_x(n) \leq \cdots \leq b^{(k)}_x(n) \leq b^{(k+1)}_x(n) \leq \cdots \leq p_x(n).$$

1.2 Questions Addressed in This Paper

The $k$-binomial complexity function has been studied for particular infinite words: for $k \geq 2$, the $k$-binomial complexity of Sturmian words coincides with their factor complexity [19] and the same property holds true for the Tribonacci word [12]. Recently, the 2-binomial complexity of generalized Thue–Morse words was also computed [14]. The $k$-binomial complexity of the Thue–Morse word $t$ is bounded by a constant (depending on $k$) [11], and more generally bounded $k$-binomial complexity holds for any fixed point of a prolongable Parikh-constant morphism $\phi$ [19], i.e. $\Psi(\phi(a)) = \Psi(\phi(b))$ for all letters $a, b$.

In this work, we generalize the above property of the fixed points of Parikh-constant morphisms to what we call Parikh-collinear morphisms $\phi$: for all letters $a, b$, there is a rational number $r_{a,b}$ such that $\Psi(\phi(a)) = r_{a,b}\Psi(\phi(b))$. Such morphisms were characterized in [4]; see Theorem 16. In Section 3.1, we provide a new characterization of these morphisms in terms of the binomial complexity: they map all words with bounded $k$-binomial complexity to words with bounded $(k+1)$-binomial complexity. Finally, Corollary 18 shows that fixed points of Parikh-collinear morphisms have bounded $k$-binomial complexity. (See Fig. 1 for an illustration.)
For all $j \geq 1$, the exact value of $b_t^{(j)}(n)$ computed in [11] is given by

$$b_t^{(j)}(n) = \begin{cases} p_t(n) & \text{if } n \leq 2^j - 1; \\ 3 \cdot 2^j - 3, & \text{if } n \equiv 0 \pmod{2^j} \text{ and } n \geq 2^j; \\ 3 \cdot 2^j - 4, & \text{otherwise}. \end{cases} \tag{2}$$

We show in Theorem 23 that such a behavior is not specific to $t$, but appears for a large class of words. Let $\varphi$ be the Thue–Morse morphism. For any aperiodic binary word $y$, the word $x = \varphi^k(y)$ is such that $b_t^{(j)}(n) = b_t^{(j)}(n)$ for all $j \leq k$ and $n \geq 2^j$.

In general, not much is known about the general behavior or the properties that can be expected for the $k$-binomial complexity. In particular, computing the $k$-binomial complexity of a particular infinite word remains challenging, see, for instance, Fig. 1 to grasp the difficulty. It would also be desirable to compare in some ways $k$ and $(k+1)$-binomial complexities of a word. For two functions $f, g: \mathbb{N} \to \mathbb{N}$, we write $f \prec g$ when the relation $f(n) < g(n)$ holds for infinitely many $n \in \mathbb{N}$. Our reflexion is driven by the following questions inspired by Lejeune’s questions [10, pp. 115–117] that are natural to consider in view of (1).

**Question A.** Does there exist an infinite word $w$ such that, for all $k \geq 1$, $b_w^{(k)}$ is unbounded and $b_w^{(k)} \prec b_w^{(k+1)}$? If the answer is positive, can we find a (pure) morphic such word $w$?

From (1), notice that $b_w^{(k)}$ is unbounded, for all $k \geq 1$, if and only if the abelian complexity $b_w^{(1)}$ is unbounded. Even though the Thue–Morse word $t$ is such that, for all $k \geq 1$, $b_t^{(k)} \prec b_t^{(k+1)}$, $b_t^{(k)}$ remains bounded [2]. So $t$ is not a satisfying answer to Question A. However, in Section 2 we provide several positive answers to this question.

Section 4 is about binomial properties of iterates of $\varphi$. Going further than (2), we also study the $(k+1)$- and $(k+2)$-binomial complexity of words of the form $x = \varphi^k(y)$ with $y$ aperiodic. In Section 4.1 we prove Theorem 23 mentioned above. In Section 4.2 we characterize $(k+1)$-binomial equivalence in $x$ with Proposition 34. As a consequence of it, we get that $b_x^{(k)} \prec b_x^{(k+1)}$.

We made these considerations because one can wonder if the factor complexity can be achieved (dismissing the trivial cases of periodic words or fixed points of Parikh-constant morphisms).

**Question B.** For each $\ell \geq 1$, does there exist a word $w$ (depending on $\ell$) such that $b_w^{(1)} \prec b_w^{(2)} \prec \cdots \prec b_w^{(\ell)} \prec b_w^{(\ell)} = p_w$? If the answer is positive, is there a (pure) morphic such word $w$?

1 We choose $\prec$ because, e.g., for the period-doubling word $pd$ there exist two subsequences such that $b_{pd}^{(2)(n_i)} = p_{pd}(n_i)$ and $b_{pd}^{(2)(n_i)} < p_{pd}(n_i)$ [10, Prop. 4.5.1].
Putting together results from Sections 4 and 5 we fully answer Question B. Theorem 23 and Proposition 34 give
\[ b^{(1)}(x) \prec b^{(2)}(x) \prec \cdots \prec b^{(k-1)}(x) \prec b^{(k)}(x), \]
while assuming that \( y \) above is Sturmian, we show that \( b^{(k+2)}(x) = p_x \). Iterates of \( \varphi \) applied to Sturmian words are studied (among other words) in [7]. Our construction leads to words with bounded abelian complexity. Question B is then strengthened in Section 5 where we ask for words with unbounded abelian complexity. We give a pure morphic answer when \( \ell = 3 \).

1.3 Preliminaries

We collect some useful results on \( k \)-binomial equivalence. First note that \( \sim_k \) is a congruence, i.e. for \( u,v,xy \in A^* \), \( u \sim_k v \) and \( x \sim_k y \) implies \( ux \sim_k vy \).

Using a classical “length-\( n \) sliding window” argument, one has the following.

Lemma 2 (Folklore). For any binary word \( y \) over \( \{0,1\} \), we have
\[ b^{(1)}(y)(n) = 1 + \max_{u,v \in L_n(y)} \left| |u|_1 - |v|_1 \right|. \]

Lemma 3 (Cancellation property). Let \( u,v,w \) be words over \( A \). We have
\[ v \sim_k w \iff uv \sim_k uw \text{ and } v \sim_k w \iff vu \sim_k wu. \]

We will also need the following result characterizing \( k \)-binomial commutation among words of equal length.

Theorem 4 ([21, Thm. 3.5]). Let \( k \geq 2 \) and \( x,y \in A^* \) such that \( |x| = |y| \). Then \( xy \sim_k yx \) if and only if \( x \sim_{k-1} y \).

A proof of the next result can be conveniently found in [11, Lem. 30]. This could also be proved by induction using Theorem 4 with \( x = \varphi^k(0), y = \varphi^k(1) \).

Theorem 5 (Ochsenschläger [15]). Let \( \varphi: 0 \mapsto 01, 1 \mapsto 10 \) be the Thue-Morse morphism. For all \( k \geq 1 \), we have \( \varphi^k(0) \sim_k \varphi^k(1) \) and \( \varphi^k(0) \not\sim_{k+1} \varphi^k(1) \).

The following result from [11, Lem. 31] will be convenient. This can alternatively be proved using Theorem 4 combined with Ochsenschläger’s result.

Lemma 6 (Transfer lemma). Let \( k \geq 1 \). Let \( u,v,v' \) be three non-empty words such that \( |v| = |v'| \). We have \( \varphi^{k-1}(u)\varphi^k(v) \sim_k \varphi^k(v')\varphi^{k-1}(u) \).

It is an exercise to see that, for an arbitrary morphism \( f: A^* \to B^* \), we have, for all \( u \in A^*, e \in B^* \),
\[ \left( f(u) \right)_e = \sum_{a_1,\ldots,a_{\ell} \in A} \left( u \right)_{a_1 \cdots a_{\ell}} \sum_{e = e_1 \cdots e_{\ell}} \prod_{i=1}^{\ell} \left( f(a_i) \right)_{e_i}. \] (3)

We recall the following lemma that appears in [21]: it is a straightforward generalization of an observation in [20]. We give a proof for the sake of completeness.
Lemma 7. Let $C$ be an abelian equivalence class of non-empty words with Parikh vector $(m_a)_{a \in A}$. Then, for any word $u \in A^*$, we have $\sum_{w \in C} (u |_w) = \prod_{a \in A} (|u|_a)$. 

Proof. The sum on the left counts the number of ways one can choose a subword $w$ of $u$ so that $\Psi(w) = (m_a)_{a \in A}$. On the other hand, for a vector $(m_a)_{a \in A}$, any choice of $m_a$ many distinct $a$’s in $u$ for each $a \in A$ gives rise to a subword of $u$ having Parikh vector $(m_a)_{a \in A}$. The number of distinct such choices is the product on the right. \hfill \Box

Theorem 8 ([19, Thm. 7]). For any Sturmian word $s$, we have $b(2) = p_s$. In particular, the theorem implies that for two distinct equal-length factors $u$, $v$ of a Sturmian word, we have either $u \not\sim_1 v$, or $^{(u)}_{(01)} \neq ^{(v)}_{(01)}$.

2 Several Answers to Question A

One can give a rather direct answer to Question A. Indeed, let $c$ be the binary Champernowne word, that is, the concatenation of the binary representations of the non-negative integers: $011011100110110111 \cdots$. Notice that $c$ contains all binary words. For each $k$, there exist two binary words $u, v$ such that $u \sim_k v$ but $u \not\sim_{k+1} v$ (see, for instance, Theorem 5). Therefore, the same properties hold for $ux$ and $vx$, for all $x \in \{0, 1\}^*$, thus $b_c^{(k)} \prec b_c^{(k+1)}$ for all $k$. Clearly $b_c^{(1)}(n) = n + 1$ is unbounded and so is $b_c^{(k)}$ for $k \geq 2$.

Observe that $c$ is not morphic, nor uniformly recurrent. Therefore in the rest of the section we provide more “structured” words answering Question A.

2.1 A Non-Binary Pure Morphic Answer

Let $\varphi: 0 \mapsto 01, 1 \mapsto 10$ be the Thue–Morse morphism over $\{0, 1\}$. Consider the morphism $g: \{a, 0, 1, \alpha\}^* \to \{a, 0, 1, \alpha\}^*$ defined by

$$a \mapsto a0\alpha, \quad 0 \mapsto \varphi(0), \quad 1 \mapsto \varphi(1), \quad \alpha \mapsto \alpha^2.$$ 

We have $g = g^\omega(a) = a \prod_{j=0}^\infty \varphi(0)\alpha^{2^j}$. We show that the word $g$ answers Question A.

Proposition 9. The abelian complexity of $g$ is unbounded and $b_g^{(k)} \prec b_g^{(k+1)}$ for all $k \geq 1$.

Proof. The abelian complexity of $g$ is (at least) linear, since

$$\{|u|_a : u \in \mathcal{L}_n(g)\} = \{0, \ldots, n\}.$$ 

Furthermore, for each $k \in \mathbb{N}$ there exist infinitely many words $u_n, v_n \in \mathcal{L}(g)$ such that $u_n \sim_k v_n$ but $u_n \not\sim_{k+1} v_n$; by Theorem 5 take $u_n = \varphi^k(0)\alpha^n$ and $v_n = \varphi^k(1)\alpha^n$. Consequently $b_g^{(k)} \prec b_g^{(k+1)}$ for all $k \geq 1$. \hfill \Box
2.2 A Binary Morphic Answer

Consider the word $\tau(g)$, where $g$ is the word defined in the previous subsection, and $\tau$ is the coding $a \mapsto \varepsilon$, $0 \mapsto 0$, $1 \mapsto 1$, and $\alpha \mapsto 1$. We have the following:

**Proposition 10.** The abelian complexity of $\tau(g)$ is unbounded and $b^{(k)}_{\tau(g)} < b^{(k+1)}_{\tau(g)}$ for all $k \geq 1$.

**Proof.** The word $\tau(g)$ has unbounded abelian complexity: it contains arbitrarily long words $u$ for which $|u| = \lceil |u|/2 \rceil$ (take factors of the Thue–Morse word for instance). Similarly it contains arbitrarily long powers of 1. Consequently, the word has unbounded abelian complexity (recall Lemma 2).

To show $b^{(k)}_{\tau(g)} < b^{(k+1)}_{\tau(g)}$ for all $k$, we notice that the same arguments as in the case of $g$ can be applied verbatim with $\tau(u_n)$ and $\tau(v_n)$. \qed

2.3 A Binary Uniformly Recurrent Answer

We note that none of the above words are uniformly recurrent (a word $x$ is uniformly recurrent if for each $x \in \mathcal{L}(x)$ there exists $N \in \mathbb{N}$ such that $x$ appears in all factors in $\mathcal{L}_N(x)$). We recall a particular construction from Grillenberger [8] for uniformly recurrent words having arbitrary entropy. The word of interest is constructed as follows. Define $D_0 = \{0, 1\}$. Assuming $D_k$ is constructed, let $u_k$ be the product of words of $D_k$ in lexicographic order, assuming $0 < 1$. Define then $D_{k+1} := u_k D_k^2$. Now the sequence $(u_k)_{k \in \mathbb{N}}$ converges to a uniformly recurrent word $u = 010001011001111\ldots$.

**Lemma 11.** Let $k \geq 1$. If, for some $j \geq 0$, $D_j$ contains two words $u, v$, such that $u \sim_k v$ and $u \not\sim_{k+1} v$, then $D_{j+1}$ contains words $x, y, z$ and $w$ such that

- $x \sim_k y$ but $x \not\sim_{k+1} y$;
- $z \sim_{k+1} w$ but $z \not\sim_{k+1} w$.

**Proof.** By definition, the set $D_{j+1}$ contains the words $x = u_j uu$, $y = u_j vv$, $z = u_j uw$, and $w = u_j vu$.

We first consider the pair $x, y$. Since $\sim_k$ is a congruence, $x \sim_k y$. To see that $x \not\sim_{k+1} y$, assume the contrary so that this equivalence reduces to $uu \sim_{k+1} vv$ by Lemma 3. For any word $e$ of length $k + 1$, we have

$$
\left( \begin{array}{c} uu \\ e \end{array} \right) - \left( \begin{array}{c} vv \\ e \end{array} \right) = 2 \left( \begin{array}{c} u \\ e \end{array} \right) - 2 \left( \begin{array}{c} v \\ e \end{array} \right) + \sum_{e_1, e_2 \in A^+} \left[ \left( \begin{array}{c} u \\ e_1 \end{array} \right) \left( \begin{array}{c} u \\ e_2 \end{array} \right) - \left( \begin{array}{c} v \\ e_1 \end{array} \right) \left( \begin{array}{c} v \\ e_2 \end{array} \right) \right]
$$

$$
= 2 \left( \begin{array}{c} u \\ e \end{array} \right) - 2 \left( \begin{array}{c} v \\ e \end{array} \right)
$$

because $u \sim_k v$. Since $u \not\sim_{k+1} v$, there exists a word $e$ of length $k + 1$ such that $\binom{e}{u} \neq \binom{e}{v}$ which implies $\binom{e}{u} \neq \binom{e}{v}$, a contradiction.

Next we have $uu \sim_{k+1} vu$ by Theorem 4 and thus $z = u_j uv \sim_{k+1} u_j vu = w$ by Lemma 3. Similarly $z \sim_{k+2} w$ would imply $uv \sim_{k+2} vu$ and thus $u \sim_{k+1} v$ by Theorem 4, a contradiction. The claim follows. \qed
Theorem 12. The abelian complexity of \( u \) is unbounded and \( b_u^{(k)} \prec b_u^{(k+1)} \) for all \( k \geq 1 \).

Proof. First we show that \( b_u^{(1)} \) is unbounded. Assume, for some \( j \geq 0 \), that \( D_j \) contains words \( u, v \) with \( |u| - |v| = 2^j \) (this holds for \( j = 0 \)). Then by definition \( D_{j+1} \) contains the words \( x = u_juu \) and \( y = u_jvv \), for which \( |x| - |y| = 2^j \). This observation suffices for the claim by Lemma 2.

We then prove the second part of the statement. Observe that \( D_1 \) contains the words \( 0101 \) and \( 0110 \), which are abelian equivalent, but not 2-binomially equivalent (as \( (0101) = 3 \) and \( (0110) = 2 \)). The above lemma then implies that for all \( k \geq 1 \), the set \( D_j \) contains words that are \( k \)-binomially equivalent, but not \( (k+1) \)-binomially equivalent. The claim follows. \( \Box \)

3 An Interlude: Parikh-Collinear Morphisms

Definition 13 (Parikh-collinear morphisms). A morphism \( f : A^* \to B^* \) is said to be Parikh-collinear if, for all letters \( a, b \in A \), there is \( r_{a,b} \in \mathbb{Q} \) such that \( \Psi(f(b)) = r_{a,b}\Psi(f(a)) \).

In this section, we show that, given an infinite fixed point of a prolongable Parikh-morphism, its \( k \)-binomial complexity is bounded for each \( k \).

Remark 14. Given a morphism \( f : A^* \to B^* \), its adjacency matrix \( M_f \) is the matrix of size \( |B| \times |A| \) defined by \( (M_f)_{b,a} = |f(a)|_b \) for all \( a \in A, b \in B \). Observe that \( f \) is a Parikh-collinear morphism if and only if \( M_f \) has rank 1 (unless it is totally erasing). We observe that for any word \( u \in A^* \), we have that \( \Psi(f(u)) = M_f \Psi(u) \).

Example 15. The morphism \( f \) defined by \( 0 \mapsto 000111 \); \( 1 \mapsto 0110 \) is Parikh-collinear since \( \Psi(f(1)) = \frac{3}{2}\Psi(f(0)) \). The first three binomial complexities are graphed in Fig. 1.

Theorem 16 ([4, Thm. 11]). A morphism \( f : A^* \to B^* \) maps all infinite words to words with bounded abelian complexity if and only if it is Parikh-collinear.

We extend the above theorem to the following one, where 0-binomial complexity has to be understood as the “equal length” equivalence relation.

Theorem 17. A morphism \( f : A^* \to B^* \) maps, for all \( k \geq 0 \), all words with bounded \( k \)-binomial complexity to words with bounded \( (k+1) \)-binomial complexity if and only if it is Parikh-collinear.

Before proving this result in Section 3.2 let us mention a straightforward consequence, which generalizes [19] Thm. 13 from Parikh-constant to Parikh-collinear morphisms. For example, the Thue–Morse morphism is Parikh-constant and thus Parikh-collinear but the morphism of Example 15 is Parikh-collinear but not Parikh-constant.
Corollary 18. Let \( z \) be a fixed point of a Parikh-collinear morphism. For any \( k \geq 1 \) there exists a constant \( C_{z,k} \in \mathbb{N} \) such that \( h^{(k)}_z(n) \leq C_{z,k} \) for all \( n \in \mathbb{N} \).

Proof. Let \( f: A^* \to A^* \) be a Parikh-collinear morphism whose fixed point is \( z \). Since \( f(z) = z \), Theorem 16 implies that \( z \) has bounded abelian complexity. For any \( k \geq 1 \), we have that \( z = f(f^{k-1}(z)) \) implying that \( z \) has bounded \( k \)-binomial complexity by induction and the previous theorem. \( \square \)

Remark 19. We cannot relax the assumption on the rank of the adjacency matrix \( M_f \). The morphism \( f: \{0, 1, 2\}^* \to \{0, 1, 2\}^* \) defined by \( 0 \mapsto 0^32^3, 1 \mapsto 0^31^32, 2 \mapsto 2^40^61^3 \) has an adjacency matrix of rank \( 2 \). The fixed point starting with \( 0 \) is aperiodic as \( f^n(0) \) is readily seen to be right special for all \( n \geq 0 \). Yet, its adjacency matrix has eigenvalues \( 0 \) and \( 5 \pm \sqrt{13} \), the latter two of which are strictly greater than \( 1 \). This means that the word has unbounded abelian complexity. Indeed, this follows from a deep result of Adamczewski [1, Thm. 1(ii)] combined with an observation in [17, Lem. 2.2]. Hence the word has unbounded \( b^{(k)} \) for all \( k \geq 1 \).

3.1 A Characterization of Parikh-Collinear Morphisms

To prove Theorem 17, we give further characterizations of Parikh-collinear morphisms. To this end, we require the following lemma where is defined a map \( g_e \) which is constant on any abelian equivalence class. Such a map is natural to consider in view of [3].

Lemma 20. Let \( A, B \) be finite alphabets with \( |A| \geq 2 \). Let \( f: A^* \to B^* \) be a Parikh-collinear morphism. For a word \( e = e_1 \cdots e_n \) of length \( n \) over \( B \), define \( g_e: A^n \to \mathbb{N} \) by

\[
g_e(a_1 \cdots a_n) := \prod_{i=1}^{n} \binom{f(a_i)}{e_i}.
\]

Then, for all words \( w, w' \in A^n \) with \( w \sim_w w' \), we have \( g_e(w) = g_e(w') \).

Proof. Write \( w = a_1 \cdots a_n \) with \( a_i \in A \) for all \( i \in \{1, \ldots, n\} \). For all \( \alpha \in A \) and \( \beta \in B \), define \( I(\alpha, \beta) := \{i \in \{1, \ldots, n\} \mid a_i = \alpha \text{ and } e_i = \beta\} \). We get

\[
g_e(w) = \prod_{\alpha \in A} \prod_{\beta \in B} \binom{f(\alpha)}{\beta}.
\]

The claim is trivial if \( f \) maps all words to \( \varepsilon \), so let \( 0 \in A \) be a letter for which \( |f(0)| \neq 0 \). Since the morphology \( f \) is Parikh-collinear, for all \( \alpha \in A \) and all \( \beta \in B \),
there exists $r_\alpha \in \mathbb{Q}$ such that $(f(\alpha))_\beta = r_\alpha (f(0))_\beta$. We now get

$$g_e(w) = \prod_{\alpha \in A} \prod_{i \in I(\alpha, \beta)} (f(\alpha))_\beta = \prod_{\alpha \in A} \prod_{i \in I(\alpha, \beta)} r_\alpha (f(0))_\beta$$

$$= \left( \prod_{\alpha \in A} \prod_{i \in I(\alpha, \beta)} (f(0))_\beta \right) \left( \prod_{\alpha \in A} \prod_{i \in I(\alpha, \beta)} r_\alpha \right).$$

For any letter $\beta \in B$, the definition of $I(\alpha, \beta)$ gives

$$\prod_{\alpha \in A} \prod_{i \in I(\alpha, \beta)} (f(0))_\beta = (f(0))^{(e\alpha)}.$$

Similarly, for any letter $\alpha \in A$, the definition of $I(\alpha, \beta)$ yields

$$\prod_{i \in I(\alpha, \beta)} r_\alpha = r^{(w\alpha)}_\alpha.$$

Thus

$$g_e(w) = \left( \prod_{\beta \in B} (f(0))^{(e\beta)} \right) \left( \prod_{\alpha \in A} r^{(w\alpha)}_\alpha \right).$$

Observe that the first factor in this product only depends on (the Parikh vector of) $e$ — in particular, not on $w$ — as the morphism $f$ is fixed. Similarly, the second factor in the product depends solely on the Parikh vector of $w$, not on the word $w$ itself. The desired result follows. \hfill \Box

**Proposition 21.** Let $f : A^* \rightarrow B^*$ be a morphism. The following are equivalent.

(i) For all $k \geq 2$ and $u, v \in A^*$, $u \sim_{k-1} v$ implies $f(u) \sim_k f(v)$.

(ii) There exists an integer $k \geq 2$ such that for all $u, v \in A^*$, $u \sim_{k-1} v$ implies $f(u) \sim_k f(v)$.

(iii) For all $u, v \in A^*$, $u \sim_1 v$ implies $f(u) \sim_2 f(v)$.

(iv) $f$ is Parikh-collinear.

**Proof.** Clearly (i) implies (ii). We show that (ii) implies (iii). There is nothing to prove if (ii) holds for $k = 2$, so assume that $k \geq 3$. We show that $f$ also satisfies (ii) with $k - 1$ instead of $k$, and hence, by repeating the argument, $f$ satisfies (ii) with $k = 2$. Assume to the contrary that there exists a pair $u, v$ such that $u \sim_{k-2} v$ but $f(u) \not\sim_{k-1} f(v)$. Since $u$ and $v$ are abelian equivalent ($k - 2 \geq 1$) they have equal length, so by [Theorem 4] we have that $uv \sim_{k-1} vu$. Then, since $f$ has the property for $k$, we have $f(u)f(v) \sim_k f(v)f(u)$. Furthermore, $f(u)$ and $f(v)$ have the same length (due to $u \sim_1 v$). This implies that $f(u) \sim_{k-1} f(v)$ by the converse part of [Theorem 4], contrary to what was assumed.
Assuming (iii), we show that (iv) holds. Let \( x, y \) be distinct letters from \( A \). Since \( xy \sim_1 yx \), we have \( f(xy) \sim_2 f(yx) \) by assumption. In other words, for all \( s, t \in B \) we have, applying (3),

\[
0 = \left( \frac{f(xy)}{st} \right) - \left( \frac{f(yx)}{st} \right) = \sum_{a_1, \ldots, a_\ell \in A} \left( \binom{xy}{a_1 \cdots a_\ell} - \binom{yx}{a_1 \cdots a_\ell} \right) \sum_{i=1}^\ell \prod_{b_i \in B^+} \binom{f(a_i)}{b_i} = \sum_{a_1, a_2 \in A} \left( \binom{xy}{a_1 a_2} - \binom{yx}{a_1 a_2} \right) \left( \binom{f(a_1)}{s} \binom{f(a_2)}{t} \right) = \left( \frac{f(x)}{s} \right) \left( \frac{f(y)}{t} \right) - \left( \frac{f(y)}{s} \right) \left( \frac{f(x)}{t} \right),
\]

where in the third equality we use \( \binom{xy}{a} = \frac{\binom{yx}{a}}{t} \) for all \( a \in A \) (since \( xy \sim_1 yx \)). Summing over \( s \in B \), we get \( |f(x)||f(y)| = |f(y)||f(x)| \) for all \( t \in B \). Now \( x \) and \( y \) were chosen arbitrarily from the alphabet \( A \). If \( |f(x)| = 0 \) for all \( x \in A \), then \( f \) is clearly Parikh-collinear. If there is a letter \( x \) for which \( |f(x)| > 0 \), we may write \( \left( \frac{f(y)}{t} \right)_{t \in B^*} = \left( \frac{f(y)}{f(x)} \right)_{t \in B^*} \left( \frac{f(x)}{t} \right)_{t \in B^*} \) for each \( y \in A \). In other words, \( f \) is Parikh-collinear.

To complete the proof, we show that (iv) implies (i). So let \( f \) be a Parikh-collinear morphism and \( u \sim_{k-1} v \) with \( k \geq 2 \). We apply (3): for any word \( e \in B^* \), we have

\[
\left( \frac{f(u)}{e} \right) - \left( \frac{f(v)}{e} \right) = \sum_{a_1, \ldots, a_\ell \in A} \left( \binom{u}{a_1 \cdots a_\ell} - \binom{v}{a_1 \cdots a_\ell} \right) \sum_{i=1}^\ell \prod_{c_i \in B^+} \binom{f(a_i)}{c_i}.
\]

Notice that for words \( e \in B^{<k} \), we have \( \binom{u}{a_1 \cdots a_\ell} = \binom{v}{a_1 \cdots a_\ell} \) since \( u \sim_{k-1} v \), which in turn gives \( \binom{f(u)}{e} = \binom{f(v)}{e} \). So to show that \( f(u) \sim_k f(v) \), it suffices to consider words \( e \in B^k \). By assumption, for \( \ell < k \), we again have \( \binom{u}{a_1 \cdots a_\ell} = \binom{v}{a_1 \cdots a_\ell} \). Therefore, we have \( \binom{f(u)}{e} = \binom{f(v)}{e} \) if and only if

\[
\sum_{a_1, \ldots, a_k \in A} \binom{u}{a_1 \cdots a_k} \prod_{i=1}^k \binom{f(a_i)}{e_i} = \sum_{a_1, \ldots, a_k \in A} \binom{v}{a_1 \cdots a_k} \prod_{i=1}^k \binom{f(a_i)}{e_i}.
\]

Observe here that \( \prod_{i=1}^k \binom{f(a_i)}{e_i} = g_e(a_1 \cdots a_k) \) as defined in Lemma 20. Let \( C \) be an abelian equivalence class of a word in \( A^k \). As the Parikh vector is constant on \( C \), let us write \( \Psi(w) = \Psi_C \) for all words \( w \in C \). We now have

\[
\sum_{w \in A^k} \binom{u}{w} g_e(w) = \sum_{C} \sum_{w \in C} \binom{u}{w} g_e(w)
\]
where $\mathcal{C}$ in the outer sum ranges over the abelian equivalence classes of words in $A^k$. By Lemma 20, $g_e(\cdot)$ is constant on $\mathcal{C}$, so write $g_{c,e}(w) = g_{c,e}$ for all words $w \in \mathcal{C}$. Then we obtain
\[
\sum_{u \in A^k} \left( \frac{u}{w} \right) g_e(w) = \sum_{c} g_{c,e} \sum_{w \in \mathcal{C}} \left( \frac{u}{w} \right) = \sum_{c} g_{c,e} \prod_{a \in A} \left( \frac{|u|_a}{m_{c,a}} \right)
\]
by Lemma 7 where $\Psi_{\mathcal{C}} = (m_{c,a})_{a \in A}$. One obtains the same formula by replacing $u$ with $v$, and equality indeed holds in \[4\] as $|u|_a = |v|_a$ for each letter $a \in A$.

This concludes the proof. \qed

### 3.2 Proof of Theorem 17

The next result essentially appears in the proof of [1, Thm. 12]. We give a proof here for the sake of completeness.

**Lemma 22.** Let $x$ be an infinite word over $A$ with bounded abelian complexity.

Let $f : A^* \to B^*$ be a morphism and assume $y = f(x)$ is an infinite word. Then for all $c \in \mathbb{N}$ there exists $D_{x,c} \in \mathbb{N}$ such that $||f(u)| - |f(v)|| \leq c$, for some $u, v \in \mathcal{L}(x)$, then $||u| - |v|| \leq D_{x,c}$.

**Proof.** Assume without loss of generality that $|u| \geq |v|$ and write $u = u'v'$ with $|v'| = |v|$. Let $M_f$ be the adjacency matrix of $f$. If $||f(u)| - |f(v)|| \leq c$, we have by the reverse triangle inequality
\[
c \geq ||f(u')| - |f(v')|| + ||f(v')| - |f(v)|| = |f(u')| - |f(v')| - |f(v') - f(v)| + |f(v)| - |f(v)| = |f(u')| - |M_f(\Psi(v') - \Psi(v))| \]

where $\langle \cdot, \cdot \rangle$ denotes the inner product of vectors, and $\mathbf{1}$ is the all-ones-vector. Recall that $x$ has bounded abelian complexity if and only if it is $C$-balanced for some $C$ [17]. Hence, as $v$ and $v'$ are factors of the same length, $\Psi(v') - \Psi(v)$ attains finitely many distinct integer points (in particular, belonging to $[-C,C]^{|A|}$). So does $M_f(\Psi(v') - \Psi(v))$. We therefore obtain $|f(u')| \leq D$ for some $D \in \mathbb{N}$. We deduce that $u'$ is bounded in length as well: indeed, let $a \in A$ be a letter occurring infinitely often in $x$ and for which $f(a) \neq \varepsilon$ (such a letter exists because $f(x)$ is infinite). Since $x$ is balanced, we deduce that all long enough factors of $x$ contain more than $|u'|$ occurrences of $a$. We let $D_{x,c}$ be this bound on $|u'|$ to conclude the proof. \qed

We are now ready to prove the main result of this section: A morphism $f : A^* \to B^*$ maps, for all $k \geq 0$, all words with bounded $k$-binomial complexity to words with bounded $(k + 1)$-binomial complexity if and only if it is Parikh-collinear.

**Proof (of Theorem 17).**

If $f : A \to B^*$ maps all words with bounded 0-binomial complexity (i.e., all words) to words with bounded 1-binomial complexity, then $f$ is Parikh-collinear by Theorem 16.
Assume thus that \( f \) is Parikh-collinear. Theorem 16 implies that \( f \) maps all words (i.e., all words with bounded 0-binomial complexity) to words with bounded 1-binomial complexity. Let then \( k \geq 1 \) and let \( x \) be a word with bounded \( k \)-binomial complexity. Let \( n \in \mathbb{N} \). Any length-\( n \) factor of \( f(x) \) can be written as \( pf(u)s \), where the word \( u \) is a factor of \( x \), \( p \) is a suffix of \( f(a) \) and \( s \) is a prefix of \( f(b) \) for some letters \( a, b \in A \). Here \( n - 2m < |f(u)| \leq n \), where \( m := \max_{a \in A} |f(a)| \). The \((k + 1)\)-binomial equivalence class of \( pf(u)s \) is completely determined by the words \( p, s \), and the \( k \)-binomial equivalence class of \( f(u) \), which itself is determined by the \( k \)-binomial equivalence class of \( u \) by Proposition 21.

The former two words \( p \) and \( s \) are drawn from a finite set, as their lengths are bounded by the constant \( m \) (depending on \( f \)). The length of \( u \) can be chosen from an interval whose length is uniformly bounded in \( n \). Indeed, assume we have equal length factors \( w = pf(u)s \) and \( w' = pf(v)s' \). As observed above, \( n \geq |f(u)| \) and \( |f(v)| > n - 2m \), so that \( |f(u)| < |f(v)| < 2m \). Applying Lemma 22 (by assumption, \( x \) has bounded \( k \)-binomial complexity and thus, \( x \) has bounded abelian complexity by (1)) there exists a bound \( D \) such that \( |u| - |v| \leq D \) uniformly in \( n \). Since the number of \( k \)-binomial equivalence classes in \( x \) of each length is uniformly bounded by assumption, and the number of admissible lengths for \( u \) above is bounded, we conclude that the number of choices for the \( k \)-binomial equivalence class of \( u \) is bounded. We have shown that the number of \((k + 1)\)-binomial equivalence classes among factors of length \( n \) in \( f(x) \) is determined from a bounded amount of information (not depending on \( n \)), as was to be shown. 

\section{Binomial Properties of the Thue–Morse Morphism}

In this section, we consider binomial complexities of iterates of the Thue-Morse morphism \( \varphi \) on aperiodic binary words. Repeated application of Theorem 17 shows that, for any \( k \geq 1 \) and any binary word \( y \), the \( k \)-binomial complexity function of the word \( \varphi^k(y) \) is bounded. In Section 4.1, we make this result much more precise:

**Theorem 23.** Let \( j, k \) be integers with \( 1 \leq j \leq k \) and let \( y \) be an aperiodic binary word. Let \( x = \varphi^k(y) \). For all \( n \geq 2^j \), we have \( b_x^{(j)}(n) = b_{\varphi^j}(n) \) which is given by (2) and, for \( n < 2^j \), \( b_x^{(j)}(n) = p_x(n) \).

This is a generalization of [11] Thm. 6, which says that, for all \( j \geq 1 \), the \( j \)-binomial complexity of the Thue–Morse word \( t \) is given by (2). It implies that \( b_x^{(1)} \prec b_x^{(2)} \prec \cdots \prec b_x^{(k)} \). The aim of Section 4.2 is to go one step further and get \( b_x^{(k)} \prec b_x^{(k+1)} \). To do so, we characterize \( k \)-binomial and \((k + 1)\)-binomial equivalence among factors of \( x \) (Theorem 29 and Proposition 34).

### 4.1 The First \( k \) Binomial Complexities

Before proving Theorem 23, we require the following general lemma about aperiodic binary words.
Lemma 24. Let $z$ be an aperiodic binary word. Then for all $n \geq 2$ we have $\mathcal{L}_n(z) \cap L \neq \emptyset$ for each $L \in \{0A^*1, 1A^*0, 0A^*0 \cup 1A^*1\}$. Furthermore, for all $n \geq 2$ and $a \in \{0,1\}$, we have
\[(\mathcal{L}_n(z) \cap aA^*a) \cup (\mathcal{L}_{n+1}(z) \cap \pi A^*a) \neq \emptyset.\]

Proof. If $\mathcal{L}_n(z) \cap aA^*a = \emptyset$ for some $n$, then $z$ is ultimately periodic: for all $m \geq 0$, if $z_m = a$, then $z_{m+kn-1} = a$ for all $k \geq 1$. Consequently, for each $0 \leq m \leq n-1$, the word $(z_{m+kn-1})_{k \geq 1}$ is either $0^\omega$ or $0^\ell 1^\omega$ for some $\ell \geq 0$. It follows that $z$ is eventually periodic. Also, since $z$ is aperiodic, there is a right special factor of length $n-1 \geq 1$ of the form $av$ or $\bar{v}w$, in which case $ava \in \mathcal{L}_n(z) \cap aA^*a \neq \emptyset$ (resp., $\bar{v}w \in \mathcal{L}_n(z) \cap \pi A^*a \neq \emptyset$).

Let us prove the second part of the statement. Assume for a contradiction that $\mathcal{L}_n(z) \cap 0A^*0 = \emptyset = \mathcal{L}_{n+1}(z) \cap 1A^*1$ for some $n \geq 2$. Consider a factor of the form $z = z_1 \cdots z_{n-1} z_n \cdots z_{2n-1}$ of length $2n$. Since $\mathcal{L}_{n+1}(z) \cap 1A^*1 = \emptyset$, we have $z_n = 0$. Further, since $\mathcal{L}_n(z) \cap 0A^*0 = \emptyset$, we have $z_1 = 1$. Repeating the argument we have $z_{n+i-1} = 0$ and $z_i = 1$ for all $i \geq 1$ which is a contradiction when $i = 1$ and $i = n$. \qed

Definition 25. Let $j \geq 0$. For any factor $u$ of $\varphi^j(y)$ of length at least $2^j - 1$ there exist $a,b \in \{0,1\}$ and $z \in \{0,1\}^*$ with $azb \in \mathcal{L}(y)$ such that $u = p\varphi^j(z)s$ for some proper suffix $p$ of $\varphi^j(a)$ and some proper prefix $s$ of $\varphi^j(b)$. (Note that $z$ could be empty.) The triple $(p,\varphi^j(z),s)$ is called a $\varphi^j$-factorization\footnote{We warn the reader that the term $\varphi$-factorization has a different meaning in \cite{11}. Our $\varphi^j$-factorization corresponds to their “factorization of order $j$.”} of $u$. The word $azb$ (resp., $zb; az; z$) is said to be the corresponding $\varphi^j$-ancestor of $u$ when $p,s$ are non-empty (resp., $p = \varepsilon$ and $s \neq \varepsilon; p \neq \varepsilon$ and $s = \varepsilon; p = s = \varepsilon$).

Since the words $\varphi^j(0)$ and $\varphi^j(1)$ begin with different letters, we notice that if $s \neq \varepsilon$ in a $\varphi^j$-factorization of a word, then the letter $b$ is uniquely determined. Similarly the $j$th images of the letters end with distinct letters, whence the letter $a$ is uniquely determined once $p \neq \varepsilon$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{factorization.png}
\caption{A $\varphi^j$-factorization and its $\varphi^j$-ancestor.}
\end{figure}
Proof (of Theorem 23). Let $j \in \{1, \ldots, k\}$. Notice all factors of length at most $2^j - 1$ of $x = \varphi^k(y)$ occur already in the Thue–Morse word $t$: such factors appear in factors of the form $\varphi^j(ab)$, $ab \in L(y)$. Since $\varphi^j(ab)$ appears in the Thue–Morse word for all $a, b \in \{0, 1\}$, it follows from (2) that all such words are pairwise $j$-binomially non-equivalent. Hence we have shown that $b_x^{(j)}(n) = p_x(n)$ for $n \leq 2^j - 1$.

In the remaining of the proof we let $n \geq 2^j$. We show that $L_n(t)/\sim_j = L_n(x)/\sim_j$ by double inclusion, which suffices for the claim since Theorem 23 holds true for $x = t$.

Let $u \in L(x)$; we show that there exists $v \in L(t)$ such that $u \sim_j v$. To this end, let $z = \varphi^k(y)$ so that $x = \varphi^j(z)$. Let $u$ have $\varphi^j$-factorization $p\varphi^j(u')s$ with $\varphi^j$-ancestor $au'b \in L(z)$. The Thue–Morse word contains a factor $av'b$, where $|v'| = |u'|$ (see, e.g., [11, Prop. 33]). It follows that $t$ contains the factor $v := p\varphi^j(u')s$. Now $u \sim_j v$ because $\varphi^j(u') \sim_j \varphi^j(v')$ by Theorem 5.

Let then $u \in L(t)$ have $\varphi^j$-factorization $p\varphi^j(u')s$ with $\varphi^j$-ancestor $au'b \in L(t)$. As before we show that there exists $v \in L(x)$ such that $u \sim_j v$. By the previous lemma, $z$ contains, at each length, factors from both the languages $0A^*1$ and $1A^*0$. Hence, if $a$ and $b$ above are distinct, we may argue as in the previous paragraph to obtain the desired conclusion. Assume thus that $a = b$. Again the previous lemma says that $z$ contains a factor of length $|u'| + 2$ in the language $1A^*1 \cup 0A^*0$. Assume without loss of generality that it contains a factor from $0A^*0$. Then, if $a = b = 0$, we may again argue as in the previous paragraph. So assume now that $a = b = 1$ and $L_{|u'|+2}z \cap 1A^*1 = \emptyset$. Notice that by the previous lemma, $L_{|u'|+2}z \cap 0A^*0 \neq \emptyset$ and, further, $L_{|u'|+2+\epsilon}z \cap 0A^*0 \neq \emptyset$. To conclude with the proof, we have four cases to consider depending on the length of $p$ and $s$ which can be less or equal, or greater than $2^j-1$.

**Case 1:** Assume that $p$ is a suffix of $\varphi^j-1(0)$ and $s$ is a prefix of $\varphi^j-1(1)$. For all $v'$ such that $|v'| = |u'|-1$, $\varphi^j(u') \sim_j \varphi^j(v'1)$ by Lemma 6. By the Transfer Lemma, $\varphi^j(v'1) \sim_j \varphi^j-1(1) \varphi^j(v') \varphi^j-1(0)$. Consequently

$$u \sim_j p\varphi^j-1(1)\varphi^j(v')\varphi^j-1(0)s = v$$

where $p\varphi^j-1(1)$ is a suffix of $\varphi^j(0)$ and $\varphi^j(v')\varphi^j-1(0)s$ is a prefix of $\varphi^j(0)$. Hence $v$ is a factor of $\varphi^j(0v'0)$. Recall that a factor of the form $0v'0$ appears in $z$ by assumption, and thus $\varphi^j(0v'0)$ appears in $x$. To recap, we have shown a factor $v$ of $x$ $j$-binomially equivalent to $u$.

**Case 2:** Assume that $p = p'\varphi^j-1(0)$ where $p'$ is a suffix of $\varphi^j-1(1)$ and $s$ is a prefix of $\varphi^j-1(1)$. For all $v'$ such that $|v'| = |u'|$, applying Theorem 5 and Lemma 6

$$u \sim_j p'\varphi^j(v')\varphi^j-1(0)s = v.$$ 

Hence $v$ is a factor of $\varphi^j(0v'0)$, and such a factor appears in $z$ by assumption. We conclude as above.

**Case 3:** Assume that $p$ is a suffix of $\varphi^j-1(0)$ and $s = \varphi^j-1(1)s'$ where $s'$ is a prefix of $\varphi^j-1(0)$. For all $v'$ such that $|v'| = |u'|$, applying Theorem 5 and
4.2 The $\phi$-Factorization and Prefix/Suffix Relations

The previous subsection was dealing with the $\phi$-factorization of a word and in particular at the associated prefixes and suffixes.

**Lemma 27** ([11, Def. 43]). For all $j \geq 1$, we have $x \sim_{\phi^j} s =: v$ and the conclusion is the same as in the previous case.

**Case 4:** Assume that $p = p'\phi^j -1(0)$ and $s = \phi^j -1(1)\phi^j (u')s'$ where $p'$ is a suffix of $\phi^j -1(1)$ and $s'$ is a prefix of $\phi^j -1(0)$. For all $v'$ such that $|v'| = |u'| + 1$, applying Theorem 5 and Lemma 6,

$$u \sim_{\phi^j (0w'0)} v$$

Hence $v$ is a factor of $\phi^j (0w'0)$ and the conclusion is similar to Case 1.

**Remark 26.** If $y$ is an aperiodic infinite word, then for all $a, b \in \{0, 1\}$ and $n \geq 2$ we have $L_n(\phi(y)) \cap A^n b \neq \emptyset$. Indeed, for $a \neq b$ the claim follows from Lemma 24.

For $a = b$, we observe the following: for even length factors $n = 2\ell$, $\ell \geq 1$, a factor $\pi ya$ of $y$ of length $\ell - 1$ (which exists by Lemma 24) gives a factor $\pi a\phi(y)a\pi$ in $z$, hence we have the factor $a^2 a$ with $|z| = 2\ell - 2$. For odd length factors $n = 2\ell + 1$, $\ell \geq 1$, we have that a factor of the form $cyc$, $|y| = \ell - 1$, of $y$ (such a factor exists for some $c \in \{0, 1\}$ by Lemma 24) gives $c\phi(y)c\pi$. Consequently $z$ contains a factor in $A^\ell a$ of length $n$.

Applying this observation to $z$ when $j < k$ in the above proof shows that $L_n(z) \cap 1 A^n 1 \neq \emptyset$ for all $n \geq 2$, and thus some of the arguments are unnecessary in the case $j < k$.

### 4.2 The $(k+1)$-Binomial Complexity

The previous subsection was dealing with the $j$-binomial equivalence in $x = \phi^k(y)$, where $y$ is an aperiodic binary word and $j \leq k$. Here, we are concerned with the $(k+1)$-binomial equivalence in such words. To this end, we need to have more control on the $k$-binomial equivalence in $x$. First, we have a closer look at the $\phi^j$-factorizations of a word and in particular at the associated prefixes and suffixes.

**Definition 27 ([11, Def. 43]).** Let $j \geq 1$. As usual, we let $\Pi$ denote the complementation morphism defined by $\Pi = 1 - a$, for $a \in \{0, 1\}$. Let us define the equivalence relation $\equiv_j$ on $A^{<2^j} \times A^{<2^j}$ by $(p_1, s_1) \equiv_j (p_2, s_2)$ whenever there exists $a \in A$ such that one of the following situations occurs:

1. $|p_1| + |s_1| = |p_2| + |s_2|$ and
   a. $(p_1, s_1) = (p_2, s_2)$;
   b. $(p_1, \phi^j -1(a)s_1) = (p_2\phi^j -1(a), s_2)$;
   c. $(p_2, \phi^j -1(a)s_1) = (p_1\phi^j -1(a), s_1)$;
   d. $(p_1, s_1) = (s_2, p_2) = (\phi^j -1(a), \phi^j -1(\bar{a}))$
2. $|p_1| + |s_1| - (|p_2| + |s_2|) = 2^j$ and
The next lemma is essentially [11] Lem. 40 and 41] (except that with an arbitrary word \( y \) instead of the Thue–Morse word \( t \), we cannot use the fact that \( t \) is overlap-free, so factors such as 10101 may appear in \( y \)). To each \( \varphi^j \)-factorization there is a natural corresponding \( \varphi^j \)-factorization, though two \( \varphi^j \)-factorizations may correspond to the same \( \varphi^j \)-factorization. The next lemma says that in such a case the \( \varphi^j \)-factorizations are related.

**Lemma 28.** Let \( j \geq 1 \). Let \( u \) be a factor of \( \varphi^j(y) \) such that \( |u| \geq 2^j - 1 \) with a \( \varphi^j \)-factorization of the form \((p, \varphi^j(z), s)\) and \( z_0 \cdots z_{n+1} \) being the corresponding \( \varphi^j \)-ancestor (where according to Definition 25 \( z_0, z_{n+1} \) or \( z \) could be empty). The factor \( u \) has a unique \( \varphi^j \)-factorization if and only if the word \( z_0 \cdots z_{n+1} \) contains both letters \( 0 \) and \( 1 \). Otherwise stated, the \( \varphi^j \)-factorization is not unique if and only if \( u \) is a factor of \( \varphi^{j-1}(m) \) with \( m \in (01)^* \cup (10)^* \cup 1(01)^* \cup 0(10)^* \). Moreover, when the \( \varphi^j \)-factorization is not unique, i.e. if there is another \( \varphi^j \)-factorization \((p', \varphi^j(z'), s')\), then \((p, s) \equiv_j (p', s')\).

**Proof.** If \( |u| \geq 2^j - 1 \), \( u \) contains at least a factor \( \varphi^{j-1}(a) \) and thus at least one \( \varphi^j \)-factorization of the prescribed form exists with \( z = z_1 \cdots z_n \) and \( n \geq 0 \) (\( n = 0 \) if \( z = \varepsilon \)).

We first prove the claim for uniqueness by induction on \( j \). For \( j = 1 \), assume that \( u = z_0 \varphi(z_1) \cdots \varphi(z_0)z_{n+1} \) with \( z_0, z_{n+1} \in \{0, 1, \varepsilon\} \). Suppose, as in the statement, that both letters \( 0 \) and \( 1 \) occur in \( z_0 \cdots z_{n+1} \). Then we have \( z_i z_{i+1} = 01 \) (or similarly 10) for some \( i \). This means that \( u \) contains the factor 11 forcing uniqueness of this kind of a factorization: \( 11 \notin \{\varphi(0), \varphi(1)\} \). Assume that the property holds true up to \( j - 1 \) and prove it for \( j \geq 2 \). Let \( u = p\varphi^j(z_1) \cdots \varphi^j(z_n)s \) be a \( \varphi^j \)-factorization and assume that \( z_i z_{i+1} = 01 \) for some \( i \). To this factorization, we have a corresponding factorization of the form

\[
u = p\varphi^{j-1}(z_1)\varphi^{j-1}(z_2) \cdots \varphi^{j-1}(z_n)\varphi^{j-1}(z_{n+1})s.
\]

Notice that \( p \) is a suffix of \( \varphi^{j-1}(z_0) \) if \( |p| < 2^{j-1} \) and otherwise, \( p = p'\varphi^{j-1}(z_0) \) with \( p' \) a suffix of \( \varphi^{j-1}(z_0) \). Similarly, \( s \) is a prefix of \( \varphi^{j-1}(z_{n+1}) \) if \( |s| < 2^{j-1} \) and otherwise, \( s = \varphi^{j-1}(z_{n+1})s' \) with \( s' \) a prefix of \( \varphi^{j-1}(z_{n+1}) \). Observe that \( z_1 z_{i+1} z_{i+2} = 0110 \). So by the induction hypothesis, the \( \varphi^{j-1} \)-factorization of \( u \) is unique. There are at most two \( \varphi^j \)-factorizations corresponding to a \( \varphi^{j-1} \)-factorization. But since \( \varphi^{j-1}(1)\varphi^{j-1}(1) \notin \{\varphi^j(0), \varphi^j(1)\} \), the claimed uniqueness follows.

We then prove the claim for non-unique factorizations. Assume that \( z_0 = z_1 = \cdots = z_{n+1} = 0 \). Then

\[
u = p\varphi^j(0) \cdots \varphi^j(0)s = p\varphi^{j-1}(0)\varphi^{j-1}(1) \cdots \varphi^{j-1}(0)\varphi^{j-1}(1)s.
\]

If \( |p| \geq 2^{j-1} \), then \( p = p'\varphi^{j-1}(1) \) with \( p' \) a suffix of \( \varphi^{j-1}(0) \) (and thus, a suffix of \( \varphi^j(1) \)), otherwise set \( p = p\varphi^{j-1}(0) \). Similarly, if \( |s| \geq 2^{j-1} \), then \( s = \varphi^{j-1}(0)s' \).
with \( s' \) a prefix of \( \varphi^{j-1}(1) \), otherwise \( s' = \varphi^{j-1}(1)s \). Notice that the corresponding \( \varphi^{j-1} \)-factorization of \( u \) is unique by the previous part. Now \( u \) can also be written as

\[
p' \varphi^{j-1}(1) \varphi^{j-1}(0) \cdots \varphi^{j-1}(1) \varphi^{j-1}(0)s' = p' \varphi^j(1) \cdots \varphi^j(1)s'.
\]

There are no other \( \varphi^j \)-factorizations due to the uniqueness of the \( \varphi^{j-1} \)-factorization of \( u \). To conclude the claim in this case, a straightforward case analysis shows that \( (p, s) \equiv_j (p', s') \):

- If \( |p| \geq 2^{j-1} \) and if \( |s| \geq 2^{j-1} \), then \((p, s') = (p' \varphi^{j-1}(1), \varphi^{j-1}(0)s')\).
- If \( |p| \geq 2^{j-1} \) and if \( |s| < 2^{j-1} \), then \((p, \varphi^{j-1}(1)s) = (p' \varphi^{j-1}(1), s')\).
- If \( |p| < 2^{j-1} \) and if \( |s| \geq 2^{j-1} \), then \((p \varphi^{j-1}(0), s) = (p', \varphi^{j-1}(0)s')\).
- If \( |p| < 2^{j-1} \) and if \( |s| < 2^{j-1} \), then \((p \varphi^{j-1}(0), \varphi^{j-1}(1)s) = (p', s')\).

We have the following theorem, the proof of which is essentially the proof of [11, Thm. 48]. Indeed, the lemmas leading to its proof do not require that the factors \( u \) and \( v \) are from the Thue–Morse word, only that they have \( \varphi^j \)-factorizations. We note that [11, Thm. 48] is stated for \( j \geq 3 \). The case \( j = 1 \) is trivial. The case \( j = 2 \) is obtained by looking closely at the proof of [11, Thm. 34].

**Theorem 29.** Let \( y \) be an aperiodic binary word. Let \( k \geq j \geq 1 \). Let \( u \) and \( v \) be equal-length factors of \( x = \varphi^k(y) \) with \( \varphi^j \)-factorizations \( u = p_1 \varphi^j(z)s_1 \) and \( v = p_2 \varphi^j(z')s_2 \). Then \( u \sim_j v \) if and only if \((p_1, s_1) \equiv_j (p_2, s_2)\).

We then turn to the \((k+1)\)-binomial equivalence in \( x \). We require some lemmas. A straightforward consequence of (3) together with the identities \( \sum_{x \in A^\ell} \binom{u}{x} = \binom{|u|}{\ell} \), \( \ell \geq 1 \), is the following observation.

**Lemma 30.** Let \( \varphi: 0 \mapsto 01, 1 \mapsto 10 \) be the Thue–Morse morphism. Let \( u \in \{0, 1\}^\ast \). Then

\[
\binom{\varphi(u)}{0} = |u|; \quad \binom{\varphi(u)}{01} = |u|_0 + \binom{|u|}{2}; \quad \binom{\varphi(u)}{011} = \binom{u}{01} + \binom{|u|_0}{2} + \binom{|u|}{3}.
\]

**Proof.** For example, \( \binom{\varphi(a)}{011} = \binom{\varphi(a)}{11} \) for both \( a \in \{0, 1\} \). Similarly \( \binom{\varphi(a)}{b} = 1 \) for letters \( a, b \in \{0, 1\} \). Therefore

\[
\binom{\varphi(u)}{011} = \sum_{x_1, x_2 \in A} \binom{u}{x_1x_2} \sum_{x_1 = e_1, x_2 = e_2} \binom{\varphi(x_1)}{e_1} \binom{\varphi(x_2)}{e_2} + \sum_{|x| = 3} \binom{u}{x}
\]

\[
= \binom{u}{00} + \binom{u}{01} + \binom{|u|}{3},
\]

and the claim follows. \( \square \)

**Lemma 31.** Let \( u, v \) be two binary words of equal length. For \( k \geq 1 \), we have

\[
\binom{\varphi^k(u)}{01^k} - \binom{\varphi^k(v)}{01^k} = 2^{(k-1)(k-2)/2(|u|_0 - |v|_0)}.
\]
In particular, $u \not\sim_1 v$ implies $\varphi^k(u) \not\sim_{k+1} \varphi^k(v)$. Moreover, if $u \sim_1 v$, for $k \geq 1$, we have
\[
(\varphi^k(u)_{01^{k+1}} - \varphi^k(v)_{01^{k+1}}) = 2^{(k-1)(k-2)/2}\left(\binom{u}{01} - \binom{v}{01}\right).
\]
In particular, $u \not\sim_2 v$ implies $\varphi^k(u) \not\sim_{k+2} \varphi^k(v)$.

**Proof.** The case $k = 1$ is deduced from Lemma 30. Then assume $k \geq 2$. We encourage the reader to refer to [11] for details that would be too long to reproduce here. From [11] Rem. 23, we have the following expression
\[
(\varphi^k(u)_{01^k} - \varphi^k(v)_{01^k}) = \sum_{x \in f^k(01^k)} m_{f^k(01^k)}(x) \left[\binom{u}{x} - \binom{v}{x}\right],
\]
where the map $f$ is defined to take into account the multiple ways factors $01$ or $10$ may occur in a word: $f(u)$ is a multiset of words of length shorter than $u$; see [11] Def. 15 and 17. We let the coefficient $m_{f^k(01^k)}(x)$ denote the multiplicity of $x$ as an element of the multiset $f^k(01^k)$. It can be shown that the multiset $f^k(01^k)$ only contains the elements $0$ and $1$. Therefore we obtain
\[
(\varphi^k(u)_{01^k} - \varphi^k(v)_{01^k}) = m_{f^k(01^k)}(0) \left(|u|_0 - |v|_0\right) + m_{f^k(01^k)}(1) \left(|u|_1 - |v|_1\right).
\]
To conclude with the proof, we use two facts. The first is that $|u|_1 - |v|_1 = -(|u|_0 - |v|_0)$ since $u, v$ have equal length. The second is that
\[
m_{f^k(01^k)}(0) - m_{f^k(01^k)}(1) = m_{f^{k-1}(01^k)}(01) - m_{f^{k-1}(01^k)}(10) = 2^{(k-1)(k-2)/2},
\]
which follows from [11] Prop. 28. For the second part, the same reasoning may be applied to obtain
\[
(\varphi^k(u)_{01^{k+1}} - \varphi^k(v)_{01^{k+1}}) = \sum_{x \in f^k(01^{k+1})} m_{f^k(01^{k+1})}(x) \left[\binom{u}{x} - \binom{v}{x}\right].
\]
The multiset $f^k(01^{k+1})$ only contains $0, 1, 00, 01, 10, 11$. But since it is assumed that $u \sim_1 v$, the only (potentially) non-zero terms in the sum correspond to $x \in \{01, 10\}$. Then the observation $\binom{u}{01} - \binom{v}{01} = \binom{v}{10} - \binom{u}{10}$ suffices to conclude. \hfill \square

Next we consider the structure of factors of the image of an arbitrary binary word $y$.

**Definition 32.** For $n \geq 1$ we let $S(n) = L_n(y)$. Further, for all $a, b \in \{ \varepsilon, 0, 1 \}$ such that $ab \not= \varepsilon$, we define $S_{a,b}(n) = L_{n+|ab|}(y) \cap aA^*b$. We call these sets factorization classes of order $n$. 

Consider now a factor $u$ of $\varphi(y)$. We associate with $u$ some factorization classes as follows. Let $a\varphi(u'b)$ be the $\varphi$-factorization of $u$ with $\varphi$-ancestor $au'b \in \mathcal{L}(y)$. If $ab = \varepsilon$, we associate the factorization class $\mathcal{S}(u')$. For $ab \neq \varepsilon$, we have that $u$ is a factor of $\varphi(au'b)$. In this case we associate the factorization class $\mathcal{S}_{\pi,b}(u')$. If $u$ is associated with a factorization class $\mathcal{T}$, we write $u \models \mathcal{T}$, otherwise we write $u \not\models \mathcal{T}$.

Observe that $u \models \mathcal{S}(n)$ implies that $|u| = 2n$. Also, for $ab \neq \varepsilon$, $u \models \mathcal{S}_{\pi,b}(n)$ implies that $|u| = 2n + |ab|$. Notice also that a factor $u$ of $\varphi(y)$ can be associated with several factorization classes: take, e.g., $(10)^{\ell}1 = (01)^{\ell}1$ which is associated with both $\mathcal{S}_{\pi,1}(\ell)$ and $\mathcal{S}_{\pi,0}(\ell)$, or $(01)^{\ell+1} = 0(10)^{\ell}1$ which is associated with both $\mathcal{S}(\ell + 1)$ and $\mathcal{S}_{1,1}(\ell)$.

**Lemma 33.** For two 2-binomially equivalent factors $u, v \in \mathcal{L}(\varphi(y))$, if $u \models \mathcal{T}$ for some factorization class $\mathcal{T}$, then $v \models \mathcal{T}$. Furthermore, a factor $u$ of $y$ is associated with distinct factorization classes if and only if $u \in L = (01)^* \cup (10)^* \cup 1(01)^* \cup 0(10)^*$.

**Proof.** Even-length factors. Let $u \sim_2 v$ with $|u| = 2n$. If $u \models \mathcal{S}_{\pi,a}(n-1) - a \in \{0, 1\}$, then $u$ is of the form $a\varphi(x)a$ with $|x| = n - 1$, whence $|u|_a = n + 1$. Factors $v' \not\models \mathcal{S}_{\pi,a}(n-1)$ of length $2n$ have $|v'|_a \leq n$ by inspection. Hence also $v \models \mathcal{S}_{\pi,a}(n-1)$. The above arguments also show that $u$ is associated with exactly one factorization class. For the latter claim, we note that $u$ has even length and begins and ends with the same letter, so it cannot appear in the language $L$.

Assume then that $u \not\models \mathcal{S}_{\pi,a}(n-1), a \in \{0, 1\}$. Then $v \not\models \mathcal{S}_{\pi,a}(n-1), a \in \{0, 1\}$ by the previous observation. Notice that we may assume $n \geq 2$ as otherwise we have $|u| = 2$ and the claim is trivial (2-binomial equivalence is equality in this case). We compare the values of $\binom{n}{01}$ for $y$ associated with $\mathcal{S}_{1,1}(n-1), \mathcal{S}_{1,0}(n-1)$, and $\mathcal{S}(n)$, respectively.

**Case 1:** $y \models \mathcal{S}_{1,1}(n-1)$. We have $\binom{n}{01} \geq \binom{n}{2} + n$, and equality holds for $y = (01)^n$. Indeed, say $y = \varphi(x)1$ for some $x \in \{0, 1\}^n-1$. Then we have by Lemma 30

$$\binom{\varphi(x)}{01} = \binom{\varphi(x)}{01} + \binom{\varphi(x)0 + \varphi(x)1}{1} = |x|_0 + \binom{|x|}{2} + 2|x| + 1 = |x|_0 + \binom{n}{2} + n,$$

since $|x| = n - 1$. Equality now holds when $|x|_0 = 0$, i.e., $x = 1^n-1$.

**Case 2:** $y \models \mathcal{S}_{1,0}(n-1)$. We have $\binom{n}{01} \leq \binom{n}{2}$, and equality holds when $y = (10)^n$. Indeed, say $y = 1\varphi(x)0$ for some $x \in \{0, 1\}^n-1$. Then

$$\binom{\varphi(x)}{01} = \binom{\varphi(x)}{01} = |x|_0 + \binom{|x|}{2} = |x|_0 + \binom{n}{2} - (n - 1).$$

Since $|x| = n - 1$, we have $\binom{n}{01} \leq \binom{n}{2}$. Equality holds when $x = 0^n-1$.

**Case 3:** $y \models \mathcal{S}(n)$. We have $\binom{n}{2} \leq \binom{y}{01} \leq \binom{n}{2} + n$. The former equality is attained with $y = (10)^n$ and the latter with $y = (01)^n$. Indeed, say $y = \varphi(x')$ for some $x' \in \{0, 1\}^n$. We have $\binom{y}{01} = \binom{2}{2} + |x'|_0$ from Lemma 30. Therefore, $\binom{y}{01} \leq \binom{n}{01} \leq \binom{n}{2} + n$. The former equality is attained with $x' = 1^n$ and the latter with $x' = 0^n$. 


We conclude that \( u \) and \( v \) are associated with a common factorization class. In fact, the latter claim is also implied from the above: a word can be associated with two (and only two) factorization classes if and only if it appears in \( L \). This concludes the proof in the case of even length factors.

**Odd-length factors.** Assume without loss of generality that \( u \models S_{a,c}(n) \) with \( u = a \varphi(u') \) of length \( 2n+1 \). Recalling that \( |\varphi(u')|_0 = |u'| = n \), if \( u \sim v \) with \( u \) and \( v \) associated with distinct factorization classes, then necessarily \( v \in S_{z,a} \), say \( v = \varphi(v')a \). We show that this is impossible, unless \( u = v \in L \).

Indeed, assuming that we have 2-binomial equivalence, we have

\[
\begin{pmatrix}
  a \varphi(u') \\
  01
\end{pmatrix}
= \begin{pmatrix}
  \varphi(u') \\
  01
\end{pmatrix} + \delta_0(a) \begin{pmatrix}
  \varphi(u') \\
  1
\end{pmatrix}
= |u'|_0 + \left( \frac{n}{2} \right) + \delta_0(a)n
\]

(6)

which is equal to

\[
\begin{pmatrix}
  \varphi(v')a \\
  01
\end{pmatrix}
= \begin{pmatrix}
  \varphi(v') \\
  01
\end{pmatrix} + \delta_1(a) \begin{pmatrix}
  \varphi(v') \\
  0
\end{pmatrix}
= |v'|_0 + \left( \frac{n}{2} \right) + \delta_1(a)n
\]

(7)

where \( \delta_a(b) = 1 \) if \( a = b \), otherwise \( \delta_a(b) = 0 \). Rearranging, we get \( |u'|_0 - |v'|_0 = (\delta_1(a) - \delta_0(a))n \in \{\pm n\} \). This implies, without loss of generality, that \( u' = 0^n \), \( v' = 1^n \), and \( a = 1 \). But then \( u = 1(01)^n = (10)^nn = v \in L \), as claimed. \( \square \)

The next result characterizes \((k+1)\)-binomial equivalence in \( x = \varphi^k(y) \) when \( y \) is an arbitrary binary word.

**Proposition 34.** Let \( u \) and \( v \) be factors of length at least \( 2^k - 1 \) of \( x \) with the \( \varphi^k \)-factorizations \( u = p_1 \varphi^k(z)s_1 \) and \( v = p_2 \varphi^k(z')s_2 \). Then \( u \sim_{k+1} v \) and \( u \neq v \) if and only if \( z \sim_1 z' \), \( z' \neq z \), and \( (p_1, s_1) = (p_2, s_2) \).

Notice that the proposition claims that those factors of \( x \) having more than one \( \varphi^k \)-factorization are \((k+1)\)-binomially equivalent only to themselves (in \( L(x) \)).

**Proof.** The “if”-part of the statement follows by a repeated application of **Proposition 21** on the Thue–Morse morphism together with the fact that the morphism is injective.

Let us assume that \( u \sim_{k+1} v \) for some distinct factors. It follows that \( u \sim_k v \), which implies that \( (p_1, s_1) \equiv_k (p_2, s_2) \) by **Theorem 29**. Next we show that \( (p_1, s_1) = (p_2, s_2) \) and \( z \sim_1 z' \). We have the following case distinction from **Definition 27**.

1. \((1)(a)\): We have that \( (p_1, s_1) = (p_2, s_2) \). By deleting the common prefix \( p_1 \) and suffix \( s_1 \), we are left with the equivalent statement \( \varphi^k(z) \sim_{k+1} \varphi^k(z') \). If \( z \varphi_1 z' \), then we have a contradiction with **Lemma 31**. The desired result follows in this case.

1. \((1)(b)\): Suppose that \( (p_1, s_2) = (p_2 \varphi^{k-1}(a), \varphi^{k-1}(a)s_1) \). Deleting the common prefixes \( p_2 \) and suffixes \( s_1 \), we are left with \( \varphi^{k-1}(a \varphi(z)) \sim_{k+1} \varphi^{k-1}(\varphi(z')a) \). Now \( a \varphi(z) \sim_1 \varphi(z')a \), but \( a \varphi(z) \not\sim_2 \varphi(z')a \) by **Lemma 33** (otherwise \( a \varphi(z) = \).
\[ \varphi(z')a \text{ and thus } u = v \text{ contrary to the assumption].} \]

Lemma 31 then implies that \( \varphi^{k-1}(a\varphi(z)) \not\sim \varphi^{k-1}(\varphi(z')a) \), which is a contradiction.

(1)(c): Suppose that \((p_2, \varphi^{k-1}(a)s_2) = (p_1\varphi^{k-1}(a), s_1)\). This is symmetric to the previous case.

(1)(d): Suppose that \((p_1, s_1) = (s_2, p_2) = (\varphi^{k-1}(a), \varphi^{k-1}(\bar{a}))\). We thus have directly \( \varphi^{k-1}(a\varphi(z)\bar{a}) \sim_{k+1} \varphi^{k-1}(p\varphi(z)a) \). The claim follows by an argument similar to that of in Case (1)(b).

(2)(a): Suppose that \((p_1, s_1) = (p_2\varphi^{k-1}(a), \varphi^{k-1}(\bar{a})s_2)\). After removing common prefixes and suffixes, we are left with \( \varphi^{k-1}(a\varphi(z)\bar{a}) \sim_{k+1} \varphi^{k-1}(\varphi(z')) \). We have that \( a\varphi(z)\bar{a} \sim \varphi(z') \), but by Lemma 33 \( a\varphi(z)\bar{a} \not\sim_2 \varphi(z') \) (otherwise \( z = \bar{a}^{\ell+1} \), implying that \( u = v \), a contradiction). This is again a contradiction by Lemma 31.

(2)(b): Suppose that \((p_2, s_2) = (p_1\varphi^{-1}(a), \varphi^{j-1}(\bar{a})s_1)\). This is symmetric to the previous case.

Notice that Theorem 23 and Proposition 34 have the following corollary:

**Corollary 35.** Let \( x = \varphi^k(y) \), where \( y \) is an arbitrary aperiodic binary word. We have

\[ b^{(1)}_x \prec b^{(2)}_x \prec \ldots \prec b^{(k)}_x \prec b^{(k+1)}_x. \]

**Proof.** Recall that \( y \) contains arbitrarily long factors of the form \( \bar{a}^n a \), \( a \in \{0,1\} \). Therefore \( x \) contains the \( k \)-binomially equivalent (by Lemma 6) factors \( \varphi^{k-1}(a)\varphi^k(z) \) and \( \varphi^k(z)\varphi^{k-1}(a) \). However, by Proposition 34 these factors are either not \((k+1)\)-binomially equivalent, or \( \varphi^{k-1}(a)\varphi^k(z) = \varphi^k(z)\varphi^{k-1}(a) \). The latter happens when \( \varphi^k(z) = \varphi^{k-1}(a)^\ell \) for some \( \ell \geq 0 \), and thus only when \( \ell = 0 \) and \( z = \bar{a} \). (Indeed, it is not hard to prove that if \( w \) is primitive so is \( \varphi(w) \).)

This observation suffices for showing \( b^{(k)}_x \prec b^{(k+1)}_x \). The rest of the claim follows by Theorem 23.

5 Answer to Question B and Beyond

The word \( 0^\omega \) gives \( b^{(1)}_x = \rho \). The Fibonacci word \( f = 0100101001010010100\cdots \), the fixed point of the morphism \( 0 \mapsto 01, 1 \mapsto 0 \), is a pure morphic word such that \( 2 = b^{(1)}_f \prec b^{(2)}_f = \rho f \) by Theorem 8.

**Remark 36.** We notice that \( b^{(1)}_x = \rho x \) cannot be attained for an aperiodic word \( x \) (indeed, there must exist a factor \( ava \), with \( a \in A \) and \( v \) containing a letter different to \( a \), whence \( av \sim \bar{a}v \) with \( av \not\sim va \)). In fact, the only ultimately periodic words over an \( m \)-letter alphabet \( \{a_1, \ldots, a_m\} \) for which the equality holds are of the form \( a_1^{n_1} a_2^{n_2} \cdots a_m^{n_m} \), \( n_i \in \mathbb{N} \) (up to permutation of the letters).

To answer Question B for larger values of \( k \), we take images of a Sturmian word \( s \) by a power of \( \varphi \) and we prove the following result.


Theorem 37. Let \( \varphi \) be the Thue–Morse morphism. Let \( s \) be a Sturmian word. For each \( k \geq 0 \), the word \( s^k := \varphi^k(s) \) has
\[
\mathbf{b}_s^{(1)} \prec \mathbf{b}_s^{(2)} \prec \cdots \prec \mathbf{b}_s^{(k+1)} \prec \mathbf{b}_s^{(k+2)} = p_s.
\]
In particular, putting the Fibonacci word for \( s \) gives a morphic positive answer to Question B.

Proof. Observe that \( s^k \) has bounded \((k+1)\)-binomial complexity as a straightforward application of Theorem 17 (because \( s \) has bounded abelian complexity), and thus \( \mathbf{b}_s^{(k+1)} \prec p_s \). By Corollary 35, we need only to show that \( \mathbf{b}_s^{(k+2)} = p_s \).

Let \( u \) and \( v \) be distinct factors of \( s^k \). Assume they are \((k+2)\)-binomially equivalent. By Proposition 34, we have that \( u = \varphi^k(z)s, v = \varphi^k(z')s \) with \( z \sim_1 z' \). If \( z \neq z' \), then \( z \sim_2 z' \) by Theorem 8. But then Lemma 31 implies that \( \varphi^k(z) \not\sim_{k+2} \varphi^k(z') \), contradicting the assumption. Hence we deduce that \( z = z' \), but then \( u = v \) contrary to the assumption. \( \square \)

Remark 38. In the above proof, since \( s \) is Sturmian, Theorem 8 says distinct factors are not \( 2 \)-binomially equivalent. This means that Theorem 37 applies to and only to aperiodic words \( s \) such that \( \mathbf{b}_s^{(2)} = p_s \). The "only if"-part of the statement follows by a repeated application of Proposition 21 on the Thue–Morse morphism together with the fact that the morphism is injective.

We answered Question B by providing a word with bounded abelian complexity. We can therefore strengthen the question with the following extra requirement.

**Question C.** For each \( \ell \geq 1 \), does there exist a word \( w \) (depending on \( \ell \)) such that \( \mathbf{b}_w^{(1)} \) is unbounded and
\[
\mathbf{b}_w^{(1)} \prec \mathbf{b}_w^{(2)} \prec \cdots \prec \mathbf{b}_w^{(\ell - 1)} \prec \mathbf{b}_w^{(\ell)} = p_w.
\]
If the answer is positive, can we find a (pure) morphic such word \( w \)?

The following word answers the question for \( \ell = 3 \) in the positive.

**Theorem 39.** The word \( h = 0112122122122212222 \cdots \) fixed point of the morphism \( 0 \mapsto 01, 1 \mapsto 12, \) and \( 2 \mapsto 2 \) is such that its abelian complexity \( \mathbf{b}_h^{(1)} \) is unbounded and \( \mathbf{b}_h^{(1)} \prec \mathbf{b}_h^{(2)} \prec \mathbf{b}_h^{(3)} = p_h \).

We obtain the previous theorem by combining the following two results.

**Proposition 40.** The abelian complexity \( \mathbf{b}_h^{(1)} \) of \( h \) is unbounded and \( \mathbf{b}_h^{(1)}(n) < \mathbf{b}_h^{(2)}(n) < p_h(n) \) for all \( n \geq 6 \).

Proof. We claim that \( \mathbf{b}_h^{(1)} \) is of the order \( \Theta(\sqrt{n}) \). Clearly it suffices to show the claim for the word \( h' = 0^{-1}h \), as removing the first zero always removes exactly one abelian equivalence class: the only one that contains a zero. The resulting
word \( h' \) is effectively a binary word; it is evident that the maximal number of 1’s in a word of length \( n \) is attained by the prefix of \( h' \). This value equals the maximal \( m \) for which \( \sum_{i=1}^{m} i = \binom{m+1}{2} \leq n \). Clearly \( m = \Theta(\sqrt{n}) \). By Lemma 2 we conclude that the abelian complexity of \( h \) is \( \Theta(\sqrt{n}) \).

Since the abelian complexity of \( h \) if unbounded, so is its 2-binomial complexity. However, the 2-binomial complexity does not equal the factor complexity at lengths \( n \geq 6 \): \( h \) contains both the factors \( 12^n - 2^1 \) and \( 212^n - 4^1 \) which are readily seen to be 2-binomially equivalent. (One may also invoke a result from [6] for binary alphabets.)

Finally observe that the abelian complexity does not coincide with the 2-binomial complexity either: the factors \( 2^x 12^y \) with \( x + y = n - 1 \) are abelian equivalent but not 2-binomially equivalent. This ends the proof. \( \square \)

**Proposition 41.** We have \( b^{(3)}_h = p_h \).

**Proof.** We may again discard the first 0 of \( h \), as the prefix is the only factor containing a zero. Assume to the contrary that there exist 3-binomially equivalent distinct factors \( u_1 \) and \( u_2 \) in \( h' = 0^{-1}h \). The two factors must contain the same number of 1’s, and hence at least one under the assumption that they are distinct. If the factors are of the form \( u_i = 2^{x_i}12^{y_i} \) with \( x_1 \neq x_2 \), then the factors are not even 2-binomially equivalent. So the words contain at least two 1’s. By the structure of \( h \), we may write \( u_i = 2^{x_i}12^{a_i+1} \cdots 12^{a_i+t} \) for some \( t \geq 0, a_i \in \mathbb{N}, x_i < a_i \) and \( y_i \leq a_i + t + 1 \) for all \( i \in \{1, 2\} \). If \( a_1 = a_2 \), then \( x_1 \neq x_2 \), and we again deduce that the factors are not even 2-binomially equivalent. So we must have \( a_1 < a_2 \) without loss of generality. We show that in this case the factors are not 3-binomially equivalent. Indeed, consider the coefficient \( \binom{v_i}{121} \). For \( i = 1, 2 \), we clearly have

\[
\binom{u_i}{121} = \binom{v_i}{121},
\]

where \( v_i = 12^{a_i+1} \cdots 12^{a_i+t+1} \) is obtained from \( u_i \) by deleting a prefix and a suffix. But, since \( a_1 < a_2 \), notice now that \( v_1 \) is a proper subword of \( v_2 \), meaning that each occurrence of 121 in \( v_1 \) has a corresponding occurrence in \( v_2 \). Clearly \( v_2 \) will have more occurrences of 121. This combined with (8) gives the claim. \( \square \)

### 6 Concluding Remarks

A complete answer to Question C is far from obvious; especially if one wishes to obtain a pure morphic word. Conversely, for a non-periodic morphic word \( w \) which is not the fixed point of a Parikh-collinear morphism, one can wonder about the existence of a minimal value \( m \) for which the binomial and factor complexities would coincide. Does there exists \( m \in \mathbb{N} \) such that \( b^{(m)}_w = p_w \)?

Even with an apparently simple situation, it is far from obvious. As stated in the introduction, computing the \( k \)-binomial complexity of a particular infinite word remains challenging. We can prove that the period doubling word
**Binomial Complexities and Parikh-Collinear Morphisms**

\[ pd = 01000101010001 \cdots \], fixed point of \( \sigma : 0 \mapsto 01, 1 \mapsto 00 \), has the following properties [10]. Its abelian complexity \( b^{(1)}_{pd} \) is unbounded. For the 2-binomial complexity, we can show that \( b^{(2)}_{pd}(2^n) = p_{pd}(2^n) \) for all \( n \), but \( b^{(2)}_{pd}(n) < p_{pd}(n) \) for all \( n \neq 2^m \). Otherwise stated, \( b^{(1)}_{pd} \prec b^{(2)}_{pd} \prec p_{pd} \). Computer experiments suggest that \( b^{(3)}_{pd} \prec b^{(4)}_{pd} = p_{pd} \).

**Proposition 42.** Let \( w \) be the fixed point of an injective morphism \( f \) such that \( M_f \) is invertible. If there exist two distinct factors \( u \) and \( v \) of the same length such that \( u \sim_k v \), then \( b^{(k)}_w \prec p_w \).

**Proof.** One can define an extended Parikh vector \( \Psi_k(u) \) of size \(|A| + |A|^2 + \cdots + |A|^k\) encoding the binomial coefficients for all subwords of length at most \( k \). As in [12, Lemma 9], an extended adjacency matrix \( M'_f \) can be defined accordingly and it satisfies \( M'_f \Psi_k(u) = \Psi_k(f(u)) \). It can be shown that this matrix is block-triangular and the square blocks on the main diagonal are the Kronecker products of \( i \) copies of \( M_f : M_f, M_f \otimes M_f, \ldots, M_f \otimes \cdots \otimes M_f, \) for \( i = 1, \ldots, k \). Since \( M_f \) is invertible, \( M'_f \) is also invertible (its determinant is a power of \( \det(M_f) \)). Using this fact, observe that \( u \sim_k v \) if and only if \( f(u) \sim_k f(v) \). So we have found infinitely many pairwise distinct factors \( f^i(u) \) and \( f^j(u) \) of the same length that are \( k \)-binomially equivalent. \[ \square \]

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