Renormalization-scale-invariant continuation of truncated QCD (QED) series – an analysis beyond large-$\beta_0$ approximation

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Abstract

An approximation algorithm is proposed to transform truncated QCD (or QED) series for observables. The approximation is a modification of the Baker–Gammel approximants, and is independent of the renormalization scale ($\mu$) – the coupling parameter $\alpha(\mu)$ in the series and in the resulting approximants can evolve according to the perturbative renormalization group equation (RGE) to any chosen loop order. The proposed algorithm is a natural generalization of the recently proposed method of diagonal Padé approximants, the latter making the result RScl–invariant in large-$\beta_0$ approximation for $\alpha(\mu)$. The algorithm described below can extract large amount of information from a calculated available truncated perturbative series for an observable, by implicitly resumming large classes of diagrams.

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I. INTRODUCTION

Padé approximants (PA’s) can be regarded as an improvement to truncated perturbative series (TPS’s), since they have the same formal accuracy as the latter when expanded in powers of the perturbative coupling parameter. In addition, PA’s act as a kind of analytical continuation to TPS’s, and thus contain explicitly some additional information not explicitly (but implicitly) contained in the TPS’s. PA’s have been shown to have interesting applications in statistical physics and quantum field theory (QFT) [1] and in QCD [2]. In the latter works [2], comparisons of the PA method with other methods which look for optimization of the TPS result via a judicial choice of the renormalization scale (RScl) and scheme (RSch) were performed and showed good numerical agreement. These other methods include the principle of minimal sensitivity [3], the BLM approach [4] and its extensions [5]–[7], and the effective charge approach [8]. A novel RSch–invariant method, based partly on the effective charge approach, has recently been developed [9], and its precise relations with the PA methods, as well as with the method presented here, remain to be investigated. Another new approach [10] reduces the RScl- and RSch-dependence by a method of analytic continuation from the Euclidean to the time-like region of a kinetic variable. In this context, we mention also a review of the role of power expansions in QFT [11].

Recently, Gardi [12] noted that the diagonal Padé approximants (dPA’s) to truncated series for observables are invariant under the change of the renormalization scale (RScl) $\mu$ in the large-$\beta_0$ approximation, i.e., when the QCD coupling parameter $\alpha(\mu)$ is assumed to evolve according to the one-loop renormalization group equation (RGE). His observation was based on the invariance of dPA’s under the homographic transformations of the dPA argument (Ref. [13], Part I): $z \mapsto az/(1+bz)$. Since the (full knowledge of) observables must be RScl–independent, Gardi’s observation strongly suggested that the dPA method correctly sums up certain classes of multi–loop Feynman diagrams. Since the obtained dPA result is invariant in the large–$\beta_0$ approximation, it was conjectured that the summed-up diagrams are one–gluon exchange diagrams where the gluon propagator contains bubble–type of radiative corrections. Later, it was explicitly shown [14] that the latter conjecture in terms of Feynman diagrams was correct. The authors of [14] stressed that an extension of the approximation beyond the large–$\beta_0$ approximation, i.e., beyond the dPA’s, remains a major outstanding question in the strive to extract large amount of possible physical information from a limited number of available (calculated) terms in the formal QCD perturbation series of an observable.

In this paper, we present an algorithm which goes beyond the large–$\beta_0$ approximation. It it based on a modified version of the diagonal Baker–Gammel approximants (dBGA’s) and results in an expression which has the following properties:

1. it reproduces the given truncated series up to any even order;

2. it is invariant under the change of the RScl $\mu$, where the QCD coupling parameter $\alpha(\mu)$ appearing in the expression can be taken to evolve under the RGE to any chosen

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1 For the conventional Baker–Gammel approximants, see for example Part II of Ref. [13]. See also later discussion.
loop order.

The presented algorithm allows one to extract a large amount of physical information – in the sense of the mentioned two points – from a given QCD TPS for any specific QCD observable if the latter is available up to (and including) an even order. However, it should be mentioned that the approximants of the approach of the present paper, being relatively closely related to the usual PA’s, may well not be able to discern in QCD nonperturbative behavior arising from (UV and IR) renormalons – see discussions in Refs. [2], [12], [14] for the case of PA’s. The latter behavior is expected to manifest itself via a factorial growth of coefficients in TPS’s. The presented method may prove useful also in areas of physics other than QCD or QED.

II. GENERAL METHOD

By scaling and raising to a power, it is possible to redefine a generic QCD (or QED) observable \( S \) in such a way that its (formal) perturbative series acquires the following dimensionless form

\[
S \equiv z_1 f^{(1)}(z_1) = z_1 \left[ 1 + r_1^{(1)} z_1 + r_2^{(1)} z_1^2 + \cdots + r_n^{(1)} z_1^n + \cdots \right],
\]

where \( z_1 = \alpha(Q_1^2)/\pi, Q_1 \) is a chosen renormalization scale (RScl), and superscript in coefficients \( r_n^{(1)} \) implies that they also depend on the RScl \( Q_1^2 \): \( r_n^{(1)} \equiv r_n(Q_1^2) \). Since the RScl is chosen arbitrarily, the full result \( S \) is RScl–independent. However, as a rule, only a very limited number of coefficients \( r_j^{(1)} \) are available (calculated), i.e., only a truncated perturbation series

\[
S_n^{(1)} \equiv z_1 f^{(1)}(z_1; n) = z_1 \left[ 1 + r_1^{(1)} z_1 + r_2^{(1)} z_1^2 + \cdots + r_n^{(1)} z_1^n \right]
\]

is available, and it is therefore explicitly RScl–dependent as indicated by the superscripts: \( S_n^{(1)} \equiv S_n(Q_1^2) \). Hence, there appears naturally the question of how to construct, on the basis of (2), an expression which satisfies the two previously mentioned points. We will now present an algorithm for constructing just such approximations to \( S_n^{(1)} \)'s (and therefore to \( S \)), and subsequently show with two theorems and their proofs that the constructed approximations satisfy the two mentioned points.

The evolving QCD (or QED) coupling parameter \( \alpha \) satisfies the following RGE:

\[
\frac{dx}{dt} = -\sum_{j=0}^{\infty} \beta_j x^{j+2} \quad \left[ x \equiv \alpha(p^2)/\pi, \ dt \equiv d\ln p^2 \right],
\]

where \( \beta_j \) are constants (RScl-independent), and \( \beta_0 \) and \( \beta_1 \) are even renormalization–scheme–independent (RSch–independent). We can choose to take into account any number of terms on the right of this RGE. Now we define the quantity

\[ \text{However, } S \text{ does depend on a certain energy scale } Q \text{ typical for the process involved. In scattering processes, } Q \text{ is of the order of the center–of–mass energy.} \]
\[ k(z_1, u_1) \equiv \frac{\alpha(p^2)}{\alpha(Q_1^2)} \left[ z_1 = \alpha(Q_1^2)/\pi, \; u_1 = \ln(p^2/Q_1^2) \right]. \tag{4} \]

Its formal Taylor expansion in powers of \( u_1 \equiv \ln(p^2/Q_1^2) \) is

\[ k(z_1, u_1) = 1 + \sum_{j=1}^{\infty} u_1^j k_j(z_1), \quad \text{where:} \quad k_j(z_1) = \frac{1}{j!} \frac{\partial^j}{\partial u^j} k(z_1, u)|_{u=0}. \tag{5} \]

On the basis of RGE (3), we have

\[ k_j(z_1) = (-1)^j \beta_j z_1^j + O(z_1^{j+1}), \quad k_0(z_1) = 1. \tag{6} \]

Now we simply rearrange the formal series (1) for \( S/z_1 \), which is in powers of \( z_1 \equiv \alpha(Q_1^2)/\pi \), into a related series in \( k_j(z_1) \)

\[ S = z_1 \left[ 1 + \sum_{j=1}^{\infty} f_j^{(1)} k_j(z_1) \right]. \tag{7} \]

The superscript in the coefficients \( f_j^{(1)} \) now again means that they are functions of the RScl \( Q_1^2 \): \( f_j^{(1)} \equiv f_j(Q_1^2) \). We now define the corresponding formal series \( F^{(1)} \) in powers of \( -z_1 \)

\[ z_1 F^{(1)}(z_1) \equiv z_1 \left[ 1 + \sum_{j=1}^{\infty} f_j^{(1)} (-z_1)^j \right], \tag{8} \]

and construct for \( z_1 F^{(1)}(z_1) \) the diagonal Padé approximants (dPA’s)

\[ z_1[M - 1/M]_{F^{(1)}}(z_1) = z_1 \left[ 1 + \sum_{m=1}^{M-1} a_m z_1^m \right] \left[ 1 + \sum_{n=1}^{M} b_n z_1^n \right]^{-1}, \tag{9} \]

\[ z_1 F^{(1)}(z_1) = z_1[M - 1/M]_{F^{(1)}}(z_1) + O\left(z_1^{2M+1}\right). \tag{10} \]

We recall that \( f_j^{(1)} \) is a function of only \( r_{1}^{(1)}, \ldots, r_{j}^{(1)} \), due to relations (3). Therefore, also the above dPA, depending only on \( f_i^{(1)} \) \( (i=1, \ldots, 2M-1) \) by (11), depends only on the first \( 2M-1 \) coefficients \( r_i^{(1)} \) \( (i=1, \ldots, 2M-1) \) of the original series (1). At this point of algorithm, we perform decomposition of the dPA (9) into simple fractions

\[ z_1[M - 1/M]_{F^{(1)}}(z_1) = z_1 \sum_{i=1}^{M} \frac{\tilde{\alpha}_i}{1 + \tilde{u}_i z_1}, \tag{11} \]

where \([-1/\tilde{u}_i] \) are the \( M \) zeros of the denominator polynomial of the dPA (9). As long as the latter polynomial doesn’t have multiple zeros, [this decomposition is possible and unique. Although the sum (11) is real for real \( z_1 \), parameters \( \tilde{u}_i \) and \( \tilde{\alpha}_i \) are in general complex

\footnote{We assume that the exceptional situation of multiple zeros of the denominator does not appear. See also later discussion following Eq. (37).}
numbers. They depend on the RScI $Q_i^2$, because they are functions of $r_1^{(1)}, \ldots, r_{2M-1}^{(1)}$ which themselves are functions of $Q_i^2$. Now we come to the central part of the algorithm, by constructing a modified version of the diagonal Baker–Gammel approximants (dBGA’s) to the observable $S = z_1 f^{(1)}$ of Eq. (1)

$$z_1 G^{[M-1/M]}_{j(1)}(z_1) \equiv z_1 \sum_{i=1}^{M} \tilde{a}_i k(z_1, \tilde{u}_i) .$$  \hspace{1cm} (12)

Function $k(z, u)$, which depends on two arguments and is in our case defined via (4) and (3), can be called the kernel function of the above modified dBGA. As mentioned previously, the dBGA (12) is uniquely determined by the first $(2M-1)$ coefficients $r_i^{(1)} (i = 1, \ldots, 2M-1)$ of the original perturbative series (1), i.e., knowledge of the truncated series $S_{2M-1}$ (4) uniquely determines (12). Now we show, by proving two theorems, that these modified dBGA’s have precisely the two wanted properties mentioned earlier on.

**Theorem 1 (“Approximation” theorem):** The modified dBGA (12) of order $(2M-1)$ approximates the observable $S \equiv z_1 f^{(1)}(z_1)$ of Eq. (1) up to (and including) $O(z_1^{2M})$:

$$S = z_1 G^{[M-1/M]}_{j(1)}(z_1) + O \left( z_1^{2M+1} \right) .$$  \hspace{1cm} (13)

**Proof:** We use for $k(z_1, \tilde{u}_i)$ in the definition (12) of the modified dBGA the formal Taylor expansion (3) in powers of $\tilde{u}_i$

$$z_1 G^{[M-1/M]}_{j(1)}(z_1) = z_1 \sum_{i=1}^{M} \tilde{a}_i \sum_{m=0}^{\infty} \tilde{u}_i^m k_m(z_1) = z_1 \sum_{m=0}^{\infty} k_m(z_1) \left[ \sum_{i=1}^{M} \tilde{a}_i \tilde{u}_i^m \right] .$$  \hspace{1cm} (14)

Here we implicitly assume convergence of the above series. We do the same kind of formal expansion for the corresponding dPA (3)–(11)

$$z_1 [M - 1/M]_{j(1)}(z_1) = z_1 \sum_{i=1}^{M} \tilde{a}_i \sum_{m=0}^{\infty} \tilde{u}_i^m (-z_1)^m = z_1 \sum_{m=0}^{\infty} (-z_1)^m \left[ \sum_{i=1}^{M} \tilde{a}_i \tilde{u}_i^m \right] .$$  \hspace{1cm} (15)

According to (14), this expression reproduces (“approximates”) the series (8) up to, and including, the term $z_1 f^{(1)}_{2M-1}(-z_1)^{2M-1} \sim z_1^{2M}$. Therefore,

$$\sum_{i=1}^{M} \tilde{a}_i \tilde{u}_i^m = f_m^{(1)} \hspace{1cm} (m = 0, 1, \ldots, 2M - 1) .$$  \hspace{1cm} (16)

Using this in (14) and comparing with the full formal series (7) for the observable $S \equiv z_1 f^{(1)}(z_1)$, we obtain

$$z_1 G^{[M-1/M]}_{j(1)}(z_1) = z_1 \sum_{m=0}^{2M-1} f_m^{(1)} k_m(z_1) + O (z_1 k_{2M}(z_1)) \quad \Rightarrow$$  \hspace{1cm} (17)

$$S - z_1 G^{[M-1/M]}_{j(1)}(z_1) = z_1 \sum_{m=2M}^{\infty} f_m^{(1)} k_m(z_1) = O (z_1 k_{2M}(z_1)) = O \left( z_1^{2M+1} \right) ,$$  \hspace{1cm} (18)

where at the end we used estimates (3) $[k_j(z_1) \sim z_1^j]$. Relation (18) proves the theorem. We stress, however, that the formal series involved in the proof were implicitly assumed to be
convergent. The question of when the proof survives once we abandon the assumption of convergence is left open.

**Theorem 2 (Invariance under argument transformation):** The modified diagonal Baker–Gammel approximant (dBGA) \( z_1 G^{[M-1/M]}_{f_1}(z_1) \), as defined in Eq. (12), for the observable \( S \equiv z_1 f^{(1)}(z_1) \) of Eq. (1), is invariant under the following transformations of the argument:

\[
z_1 \mapsto z_2(z_1; u_21) \quad \text{such that:} \quad z_1 k(z_1, u_1) = z_2 k(z_2, u_1 - u_21),
\]

where \( u_21 \) is any (arbitrary) fixed complex number, \( u_1 \) is arbitrary complex number, \( z_2(z_1; u_21) \) is independent of \( u_1 \), and the following relations are assumed for the \( k \)-function (and its Taylor coefficients) appearing in the dBGA:

\[
k(z, u) \sim z^0 (= 1), \quad k_j(z) \sim z^j \quad (\Rightarrow z_1 \sim z_2).
\]

**Note to Theorem 2:** The seemingly artificial and complicated set of conditions for argument transformation is motivated by the following QCD (or QED) interpretation of the above parameters:

\[
z_1 \equiv \frac{\alpha(Q_1^2)}{\pi}, \quad u_1 = \ln \left( \frac{p^2}{Q_1^2} \right), \quad k(z_1, u_1) = \frac{\alpha(p^2)}{\alpha(Q_1^2)}, \quad u_21 = \ln \left( \frac{Q_2^2}{Q_1^2} \right).
\]

We see then from here and from (1) that conditions (20) for Theorem 2 are fulfilled in the QCD (or QED) case and that the argument transformation in this case means simply the change of the renormalization scale (RScl) from \( Q_1 \) to \( Q_2 \)

\[
z_1 \equiv \frac{\alpha(Q_1^2)}{\pi} \mapsto z_2 \equiv \frac{\alpha(Q_2^2)}{\pi}, \quad (u_1 - u_21) = \ln \left( \frac{p^2}{Q_2^2} \right) \quad (\equiv u_2), \quad k(z_2, u_2) = \frac{\alpha(p^2)}{\alpha(Q_2^2)},
\]

thus implying RScl–invariance of the dBGA (13).

**Proof of Theorem 2:** Let \( z_1 \) and \( z_2 \) be related by transformation (14). We have two different formal series of \( S \) in \( k_j(z_1) \) and \( k_j(z_2) \)

\[
S(= z_1 f^{(1)}(z_1)) = z_1 \left[ 1 + \sum_{j=1}^{\infty} f_j^{(1)} k_j(z_1) \right] = z_2 \left[ 1 + \sum_{j=1}^{\infty} f_j^{(2)} k_j(z_2) \right] \quad (= z_2 f^{(2)}(z_1)).
\]

[In QCD or QED: \( f_j^{(1)} \equiv f_j(Q_1^2) \) and \( f_j^{(2)} \equiv f_j(Q_2^2) \).] The corresponding two formal series in simple powers of \( z_1 \) and \( z_2 \), respectively, are defined in the described algorithm by (8)

\[
z_1 \mathcal{F}^{(1)}(z_1) \equiv z_1 \left[ 1 + \sum_{j=1}^{\infty} f_j^{(1)} (-z_1)^j \right], \quad z_2 \mathcal{F}^{(2)}(z_2) \equiv z_2 \left[ 1 + \sum_{j=1}^{\infty} f_j^{(2)} (-z_2)^j \right].
\]

Following the algorithm, we construct the usual dPA’s for \( z_1 \mathcal{F}^{(1)}(z_1) \) and \( z_2 \mathcal{F}^{(2)}(z_2) \) in the form of decomposition into simple fractions (14)

\[
z_1 [M-1/M] \mathcal{F}^{(1)}(z_1) = z_1 \sum_{i=1}^{M} \frac{\tilde{\alpha}_i}{(1 + \tilde{u}_i z_1)}, \quad z_2 [M-1/M] \mathcal{F}^{(2)}(z_2) = z_2 \sum_{i=1}^{M} \frac{\tilde{\alpha}_i}{(1 + \tilde{u}_i z_2)},
\]

(25)
again assuming that the exceptional case of multiple poles doesn’t appear. The modified dBGA’s are then constructed in both cases according to the algorithm, by Eq. (12)

\[ z_1 G_{f(1)}^{[M-1/M]}(z_1) \equiv z_1 \sum_{i=1}^{M} \tilde{\alpha}_i k(z_1, \tilde{u}_i) , \quad z_2 G_{f(2)}^{[M-1/M]}(z_2) \equiv z_2 \sum_{i=1}^{M} \tilde{\alpha}_i k(z_2, \tilde{u}_i) . \]  

(26)

We will now show that \( \tilde{\alpha}_i = \tilde{\alpha}_i \) and \( \tilde{u}_i = \tilde{u}_i - u_{21} \). Define expression

\[ z_2 \tilde{G}_{f(2)}^{[M-1/M]}(z_2) \equiv z_2 \sum_{i=1}^{M} \tilde{\alpha}_i k(z_2, \tilde{u}_i - u_{21}) . \]  

(27)

Transformation \( z_1 \mapsto z_2 \) and definitions (26) and (27) imply

\[ z_2 \tilde{G}_{f(2)}^{[M-1/M]}(z_2) = z_1 G_{f(1)}^{[M-1/M]}(z_1) . \]  

(28)

Therefore, by Theorem 1 [Eq. (13)] we have

\[ S - z_2 \tilde{G}_{f(2)}^{[M-1/M]}(z_2) = S - z_1 G_{f(1)}^{[M-1/M]}(z_1) \sim z_1^{2M+1} \sim z_2^{2M+1} \sim z_2 k_{2M}(z_2) , \]  

(29)

where at the end we used conditions (20). We now make the formal Taylor expansion of (27) in powers of \( (\tilde{u}_i - u_{21}) \)

\[ z_2 \tilde{G}_{f(2)}^{[M-1/M]}(z_2) = z_2 \sum_{i=1}^{M} \tilde{\alpha}_i \sum_{m=0}^{\infty} (\tilde{u}_i - u_{21})^m k_m(z_2) = z_2 \sum_{m=0}^{\infty} k_m(z_2) \left[ \sum_{i=1}^{M} \tilde{\alpha}_i (\tilde{u}_i - u_{21})^m \right] . \]  

(30)

Comparing (29) and (30), and using (23) for the case of argument \( z_2 \), we obtain relations analogous to those for the case of argument \( z_1 \) (14)

\[ \sum_{i=1}^{M} \tilde{\alpha}_i (\tilde{u}_i - u_{21})^m = f_{m}^{(2)} \quad (m = 0, 1, \ldots, 2M - 1) . \]  

(31)

Therefore, the dPA for \( z_2 F^{(2)}(z_2) \) of (24) is

\[ z_2 [M - 1/M] F^{(2)}(z_2) = z_2 \sum_{i=1}^{M} \tilde{\alpha}_i \left[ 1 + (\tilde{u}_i - u_{21}) z_2 \right] = z_2 \sum_{m=0}^{\infty} (-z_2)^m \left[ \sum_{i=1}^{M} \tilde{\alpha}_i (\tilde{u}_i - u_{21})^m \right] , \]  

(32)

since this dPA expression reproduces the power terms of \( z_2 F^{(2)}(z_2) \) up to (and including) \( \sim z_2^{2M} \), as can be seen from explicit expansions (24), (32) and relations (31)

\[ z_2 F^{(2)}(z_2) - z_2 \sum_{i=1}^{M} \frac{\tilde{\alpha}_i}{1 + (\tilde{u}_i - u_{21}) z_2} = \mathcal{O} \left( z_2^{2M+1} \right) . \]  

(33)

Therefore, this shows that (27) is in fact the modified dBGA \( z_2 \bar{G}_{f(2)}^{[M-1/M]}(z_2) \) (23) to the observable \( S \) for the case of argument \( z_2 \), according to the described algorithm of constructing the modified dBGA’s (12)

\[ [z_2 \bar{G}_{f(2)}^{[M-1/M]}(z_2)] = z_2 \sum_{i=1}^{M} \tilde{\alpha}_i k(z_2, \tilde{u}_i - u_{21}) = z_2 G_{f(2)}^{[M-1/M]}(z_2) . \]  

(34)
This, together with equality (28), shows that the modified dBGA of order $(2M - 1)$, as defined by the described algorithm leading to (12), is really invariant under transformations of argument (19). Thus the proof of the theorem is completed.

Again, it should be emphasized that it is left open under which circumstances the proof remains valid once we abandon the assumption of convergence of the series involved.

In the described approach, the special QCD (or QED) case of one–loop evolution of $\alpha(p^2)$ (large-$\beta_0$ approximation) means: $k(z_1, u_1) = 1/(1 + \beta_0 z_1 u_1)$, $k_j(z) = \beta_j^0 (-z)^j$. Therefore, in the one–loop case, expansion (7) for $S/z_1$ and (8) for $F^{(1)}(z_1)$ are identical if $z_1$ in the latter series is replaced by $\beta_0 z_1$ (rescaling). The dBGA (12) is in this case reduced to the usual dPA (11), with $z_1$ in the denominators replaced by $\beta_0 z_1$.

The quantity $S/z_1 \equiv f^{(1)}(z_1)$ of (1) can lead, at least in some cases, to nonmodified dBGA’s, i.e., those for which the series (8) has the special property of being a Hamburger series (cf. Ref. 13, Part II). This series is the formal Taylor expansion of a Hamburger function

$$F^{(1)}(z_1) = \int_{-\infty}^{\infty} \frac{d\phi(u_1)}{1 + u_1 z_1}, \quad (35)$$

where $\phi(u_1)$ is increasing with increasing $u_1$, and the coefficients $f_j^{(1)}$ in (8) are the finite moments

$$f_j^{(1)} = \int_{-\infty}^{\infty} u_1^j d\phi(u_1) \quad (j = 0, 1, 2, \ldots), \quad (f_0^{(1)} = 1). \quad (36)$$

In this special case, observable $S$ of (1) has the integral representation

$$S = z_1 f^{(1)}(z_1) = \int_{-\infty}^{\infty} z_1 k(z_1, u_1) d\phi(u_1). \quad (37)$$

In such (special) cases, the parameters in decomposition (11) of the dPA’s are real ($\tilde{\alpha}_i$‘s are positive) (14), and the resulting dBGA’s (12) are therefore manifestly real numbers. Decomposition (11) is in this case always possible, i.e., the dPA (9) has no multiple zeros. The case of (37), reinterpreted in terms of QCD parameters (21), means that in the integral for $S$ (over $du_1 \equiv dp^2/p^2$) there is a positive function $\rho(u_1) = \phi'(u_1)$. The latter can be interpreted as momentum probability distribution (6) (see also (3) and (14)) in diagrams with exchange of one effective virtual gluon, where the gluonic propagator contains radiative corrections. In fact, if the (approximate) $S$ contains only contributions which can be reduced to such classes of one–gluon–exchange diagrams, the authors of (14) conjectured that function $\rho(u_1)$ is then positive definite. Their conjecture would therefore suggest the following: in the cases where the truncated series $S_n^{(1)}$ of (3) contains only contributions of diagrams with at most one gluon exchange in each one–particle–irreducible part, the method of the modified dBGA’s presented here would result in a Hamburger series for $F^{(1)}$ of (8) and hence in nonmodified dBGA’s, i.e., with $\tilde{\alpha}_i$ and $\tilde{u}_i$ in (12) being real ($\tilde{\alpha}_i > 0$). Having a truncated

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4 We note that authors of (13) discuss a somewhat more constrained version of nonmodified (d)BGA’s, i.e., the one for which $F^{(1)}(z_1)$ is a Stieltjes series, corresponding to expression (33) with lower integration bound being zero.
series $S_n^{(1)}$ of (2), with $n = 2M - 1$ odd, it is straightforward to check whether such a case sets in (cf. Part I of [13]): determinants of a limited set of matrices $A[0], \ldots, A[(n-1)/2]$ have to be positive, where $A[m]$ is a $(m+1) \times (m+1)$ matrix whose elements are $A_{ij}[m] \equiv f_{i+j-2}^{(1)}$ ($i, j = 1, \ldots, m+1$). For example, for $n = 1$ ($S_1^{(1)}$) this case always sets in, and for $n = 3$ ($S_3^{(1)}$) it sets in when \[ f_2^{(1)} - (f_1^{(1)})^2 > 0. \]

We decided to use for the result (12) the terminology “modified diagonal Baker-Gammel approximant” primarily due to the previously mentioned indirect connection of our result with the nonmodified BGA’s discussed in [13]. The latter reference (part II, Chapter 1.2) briefly discusses BGA’s, for some specific kernels $k(z, u)$, and from the mathematical and numerical point of view. Theorem 1 of the present paper is indirectly related to the Convergence Theorem 1.2.1 of part II of Ref. [13] for BGA’s, while Theorem 2 of the present paper (RScl–invariance) has no analog in [13]. We stress that the presentation of the present paper is self–contained in the sense that the reader is not required to know anything about BGA’s, but should be reasonably familiar with the usual (d)PA’s. In the previous paragraph (which does not represent an essential part of the paper), some familiarity with the Hamburger series was assumed.

What to do when parameters $\tilde{\alpha}_i$ and $\tilde{u}_i$ in the modified dBGA (12) are not simultaneously real? In that case, modified dBGA could in principle be complex. However, Theorems 1 and 2 are valid not just for the entire modified dBGA’s, but also for their real (and imaginary) parts. Since the observable $S$ is real, we then just take the real part of expression (12). In fact, since $z_1$ and $S$ are real, Theorem 1 [Eq. (13)] shows that the imaginary part of the modified dBGA $z_1 G_{f^{(1)}}^{[M-1/M]}(z_1)$ must be $\sim z_1^{2M+1}$ or even less.

### III. EXPLICIT EXAMPLES

It is instructive to obtain explicit formulas for constructing the dBGA’s $z_1 G_{f^{(1)}}^{[M-1/M]}(z_1)$ of (12) once we know the truncated perturbative QCD (or QED) series $S_n^{(1)}$ of Eq. (2) for the practically possible cases of $M=1$ or $M=2$. This will be done below.

The case $M=1$ in QCD (or QED) is the case investigated already by Brodsky, Lepage and Mackenzie (BLM) [4]. Formalism presented here, under the mentioned QCD (or QED) identifications (21) and (3) for the parameters, gives

\[ S_1^{(1)} \equiv z_1 f^{(1)}(z_1; 1) = z_1 \left[ 1 + r_1^{(1)} z_1 \right] \Rightarrow z_1 G_{f^{(1)}}^{[0/1]}(z_1) = z_1 k(z_1, -r_1^{(1)}/\beta_0). \]  

We note that the above approximation, according to notation (21), is in fact the value of the coupling parameter at a scale $Q^2$

\[ z_1 G_{f^{(1)}}^{[0/1]}(z_1) = \alpha(Q^2)/\pi, \quad Q^2 = Q_1^2 \exp(-r_1^{(1)}/\beta_0). \]  

We note that $\alpha(p^2)$ evolves according to RGE (3) where the number of retained terms (loops) can be arbitrarily chosen if known. The result $z_1 G_{f^{(1)}}^{[0/1]}(z_1)$ and the scale $Q^2$ are independent

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5 See also explicit discussion of $n=1$ ($M=1$) and $n=3$ ($M=2$) cases later on.
of the choice of the RScl $Q_1^2$ (up to the chosen loop-order) according to Theorem 2, while $S_1^{(1)}[\equiv S_1(Q_1^2)]$ and $S_1^{(2)}[\equiv S_1(Q_2^2)]$ in general differ for RScl’s $Q_1^2 \neq Q_2^2$: $S_1^{(2)} - S_1^{(1)} \sim z_1^3$. In fact, it is straightforward to check directly that $z_1 G_{f(1)}^{(0/1)}(z_1) = z_2 G_{f(2)}^{(0/1)}(z_2)$, i.e., that $Q^2$ of (39) is RScl–independent, because $r_1^{(2)} - r_1^{(1)} \equiv r_1(Q_2^2) - r_1(Q_1^2) = \beta_0 \ln(Q_2^2/Q_1^2) \neq 0$ as can be checked by using (3). This result can be formulated also in a more intuitive manner: when choosing the RScl equal to $Q^2$ of (39) [$z_1 \equiv z(Q_1^2) \mapsto z(Q^2)$, $r_1^{(1)} \equiv r_1(Q_1^2) \mapsto r_1(Q^2)$, $S_1^{(1)} \equiv S_1(Q_1^2) \mapsto S_1(Q^2)$], the next-to-leading (NLO) term [$\sim z^2(Q^2)$] in the perturbative expansion of $S$ is zero: $S \equiv z(Q^2)[1+O(z^2(Q^2))]$. This follows from (39) (i.e., implicitly from RScl–invariance Theorem 2 for $M=1$ case) and from Theorem 1 [Eq. (13)], under inclusion of conditions (20) [$z_1 \sim z(Q^2)$].

The case $M=2$ in QCD (or QED) is algebraically more involved, but can be worked out in a straightforward way using the presented formalism. Expansion functions $k_j(z_1)$ (3) are obtained from RGE (3)
\begin{align}
k_1(z_1) &= -\beta_0 z_1 - \beta_1 z_1^2 - \beta_2 z_1^3 - \ldots , \\
k_2(z_1) &= +\beta_0^3 z_1^2 + (5/2)\beta_0 \beta_1 z_1^3 + \ldots , \\
k_3(z_1) &= -\beta_0^3 z_1^3 - \ldots .
\end{align}
Inverting these relations gives
\begin{align}
z_1^3 &= -\frac{1}{\beta_0^3} k_3(z_1) + O(k_4) , \\
z_1^2 &= +\frac{1}{\beta_0^3} k_2(z_1) + \frac{5}{2} \beta_1 k_3(z_1) + O(k_4) , \\
z_1 &= -\frac{1}{\beta_0} k_1(z_1) - \frac{\beta_1}{\beta_0} k_2(z_1) - \frac{5}{2} \beta_2 \beta_0 k_3(z_1) + O(k_4) .
\end{align}
Inserting this into the truncated series $S_3^{(1)}$ of Eq. (4) (presumed available), results in the rearranged truncated series for $S_3^{(1)}$ of the form (7), with coefficients
\begin{align}
f_1^{(1)} &= -\frac{r_1^{(1)}}{\beta_0} , \\
f_2^{(1)} &= -\frac{\beta_1}{\beta_0^3} r_1^{(1)} + \frac{1}{\beta_0^3} r_2^{(1)} , \\
f_3^{(1)} &= \left( -\frac{5}{2} \beta_2 + \frac{\beta_3}{\beta_0} \right) r_1^{(1)} + \frac{5}{2} \beta_1 \beta_0 r_2^{(1)} - \frac{1}{\beta_0^3} r_3^{(1)} .
\end{align}
Having these coefficients, we construct the truncated series for $z_1 F^{(1)}$ (8), and the dPA $z_1[1/2]_{\alpha_1}(z_1)$ (10) in its decomposed form (11). The resulting expressions for parameters $\tilde{u}_i$ and $\tilde{\alpha}_i$ are
\begin{align}
\tilde{u}_{2,1} &= [f_3 - f_1 f_2] \pm \sqrt{\det} \left[ 2(f_2 - f_1^2) \right]^{-1} , \\
\text{where: } \det &= \left[ f_3 + f_1 (2 f_1^2 - 3 f_2) \right]^2 + 4 (f_2 - f_1^2)^3 , \\
\tilde{\alpha}_1 &= (\tilde{u}_2 - f_1)/(\tilde{u}_2 - \tilde{u}_1) , \\
\tilde{\alpha}_2 &= 1 - \tilde{\alpha}_1 .
\end{align}
The plus sign in (46) corresponds to $\tilde{u}_2$. For simplicity of notation, we omitted the superscripts in the coefficients $f_i^{(1)} \equiv f_i(Q_1^2)$. Note that expressions (14)–(15) should be inserted into (10)–(13) in order to obtain these parameters explicitly in terms of the original coefficients $r_i^{(1)} \equiv r_i(Q_1^2)$ ($i = 1,2,3$). Inserting now these parameters into expression (12) ($M=2$) gives the sought for RScl–invariant approximation to $S_3^{(1)}$. Several subcases should be distinguished (for $M=2$):
1. When \((f_2 - f_1^2) > 0\), then: \(\tilde{u}_i, \tilde{\alpha}_i\) are real \((i = 1, 2)\), \(\tilde{u}_1 \neq \tilde{u}_2\) and \(0 < \tilde{\alpha}_i < 1\).

2. When \((f_2 - f_1^2) < 0\) and \(|f_3 + f_1(2f_1^2 - 3f_2)| > 2\sqrt{(f_1^2 - f_2)^3}\), then: \(\tilde{u}_i, \tilde{\alpha}_i\) are real \((i = 1, 2)\) and \(\tilde{u}_1 \neq \tilde{u}_2\).

3. When \((f_2 - f_1^2) < 0\) and \(|f_3 + f_1(2f_1^2 - 3f_2)| < 2\sqrt{(f_1^2 - f_2)^3}\), then: \(\tilde{u}_i\) are complex, \(\tilde{\alpha}_i\) generally complex \((i = 1, 2)\) and \(\tilde{u}_1 \neq \tilde{u}_2\).

4. When \((f_2 - f_1^2) < 0\) and \(|f_3 + f_1(2f_1^2 - 3f_2)| = 2\sqrt{(f_1^2 - f_2)^3}\) [or when \((f_2 - f_1^2) = 0\) and \(f_3 \neq f_1^3\)], then: the system of equations for \(\tilde{u}_i\) and \(\tilde{\alpha}_i\) is not solvable, i.e., form \((\mathbf{1})\) is not valid, the dPA \((\mathbf{3})\) has a multiple (double) pole.

5. When \((f_2 - f_1^2) = 0\) and \(f_3 = f_1^3\), then: \(\tilde{u}_i, \tilde{\alpha}_i\) are real \((i = 1, 2)\) and \(\tilde{u}_1 = \tilde{u}_2 = f_1\).

We should note that in QCD, presently available results of perturbative calculations include for various observables \(S\) the coefficients \(r_1^{(1)}\) and \(r_2^{(1)}\) of \((\mathbf{1})\), but not yet \(r_3^{(1)}\). Therefore, at this stage, the algorithm described here still cannot be applied for \(M = 2\) in the case of QCD observables \((r_{3,2M-1}^{(1)} = r_3^{(1)}\) are not available yet\). This contrasts with QED where perturbative coefficients \(r_3^{(1)}\) have been obtained for several QED observables.

One may raise the question of how to construct, e.g. in QCD, an explicit approximate expression for the function \(k(z_1, u_1)\) appearing in the modified dBGAs \((\mathbf{12})\), once we go beyond the large–\(\beta_0\) approximation. In the case of QCD (or QED), \(k(z_1, u_1)\) is defined in \((\mathbf{21})\). For example, the QCD coupling parameter \(z_1 \equiv \alpha_s(Q_1^2)/\pi\) at two–loop level and written as an expansion in inverse powers of \(\ln Q_1^2\), is given in \((\mathbf{15})\)

\[
z_1 \equiv \frac{\alpha_s(Q_1^2)}{\pi} = \frac{1}{\beta_0 \ln(Q_1^2/\Lambda^2)} \left[ 1 - \frac{\beta_1 \ln \left( \frac{Q_1^2}{\Lambda^2} \right)}{\beta_0^2 \ln \left( \frac{Q_1^2}{\Lambda^2} \right)} \right], \tag{49}
\]

where the neglected terms are of order \(\beta_0^2 \ln^2[\ln(Q_1^2/\Lambda^2)]/\ln^3(Q_1^2/\Lambda^2)\), and \(\Lambda\) is the QCD scale which depends (like \(\beta_i\)’s) on the number of effective quark flavors \(n_f\), e.g., \(\Lambda(n_f = 5) \approx 200\) MeV, \(\Lambda(n_f = 4) \approx 280\) MeV. It should be stressed that \(\Lambda\) is renormalization–scheme–dependent (RSch–dependent), hence change of RSch is equivalent to change of RScl when effects of \(\beta_i\)’s \((j \geq 2)\) are neglected. In order to find \(k(z_1, u_1)\) [note: \(u_1 = \ln(p^2/Q_1^2)\)], we can first write \((\mathbf{9})\) for the scale \(p^2\)

\[
z_1 k(z_1, u_1) \equiv \frac{\alpha_s(p^2)}{\pi} = \frac{1}{\beta_0 [\ln(Q_1^2/\Lambda^2) + u_1]} \left[ 1 - \frac{\beta_1 \ln \left( \frac{Q_1^2}{\Lambda^2} \right) + u_1}{\beta_0^2 \ln \left( \frac{Q_1^2}{\Lambda^2} \right) + u_1} \right]. \tag{50}
\]

\(^6\) It can be checked that also in this case the modified dBGAs \((\mathbf{12})\) is real, because \((\mathbf{16})–(\mathbf{18})\) imply: \(\tilde{u}_2 = (\tilde{u}_1)^*\) and \(\tilde{\alpha}_2 = (\tilde{\alpha}_1)^*\).

\(^7\) Note that convention \((\mathbf{3})\) for QCD \(\beta_i\) coefficients here is different from that in Particle Data Book \((\mathbf{15})\): \(\beta_0^{PDB} = 4\beta_0 = 11–2n_f/3; \beta_1^{PDB} = 8\beta_1 = 51–19n_f/3; \beta_2^{PDB} = 128\beta_2 = 2857–5033n_f/9+325n_f^2/27\). Here, \(n_f\) is number of effective quark flavors with mass less than the considered scale \(Q_1\). The form for \(\beta_2\) is in \(\overline{\text{MS}}\) scheme; \(\beta_0\) and \(\beta_1\) are RSch–dependent.
To obtain an explicit form for \( k(z_1, u_1) \), we express \( \ln(Q_1^2/\Lambda^2) \) in (54) by \( z_1 \) via (59). Inverting (59) can be done numerically, for example by iteration

\[
\left[ \ln \frac{Q_1^2}{\Lambda^2} \right]^{(n+1)} = \frac{1}{\beta_0 z_1} \left( 1 - \frac{\beta_1}{\beta_0^2} \ln \left[ \ln \frac{Q_1^2}{\Lambda^2} \right]^{(n)} \right), \quad \left[ \ln \frac{Q_1^2}{\Lambda^2} \right]^{(0)} = \frac{1}{\beta_0 z_1}.
\]  

(51)

If we choose to stop already after the first iteration step, we get

\[
\left[ \ln \frac{Q_1^2}{\Lambda^2} \right]^{(1)} = \frac{1}{\beta_0 z_1} + \frac{\beta_1}{\beta_0^2} \ln(\beta_0 z_1),
\]

(52)

\[
k(z_1, u_1) \equiv \frac{\alpha_s(p^2)}{\alpha_s(Q_1^2)} = \frac{1}{[1 + \beta_0 z_1 u_1 + (\beta_1/\beta_0) z_1 \ln(\beta_0 z_1)]} \left[ 1 + \left( \frac{\beta_1}{\beta_0} z_1 \ln(\beta_0 z_1) - \ln(1 + \beta_0 z_1 u_1 + (\beta_1/\beta_0) z_1 \ln(\beta_0 z_1)) \right) \right].
\]

(53)

This would then be a \( k(z_1, u_1) \) function beyond large-\( \beta_0 \) approximation that could be used in the RScl–invariant expressions for modified dBGA’s (12), with parameters \( \tilde{u}_i \) and \( \tilde{\alpha}_i \) in the case of \( M = 2 \) determined from the original truncated series \( S_3^{(1)} \) by Eqs. (14)–(18). In the latter equations, \( \beta_2 \) would have to be set equal to zero, because (13)–(53) represent solutions of RGE (3) with only \( \beta_0 \) and \( \beta_1 \) retained. The RScl–invariance of the obtained modified dBGA in \( M = 2 \) case would still not be absolutely precise because (59) and (53) represent only an approximate solution to truncated \( \beta_2 = \beta_3 = \ldots = 0 \) RGE (3). To make RScl–invariance precise, we would have to integrate the latter truncated RGE numerically from \( t \equiv \ln(p^2/Q_1^2) = 0 \) to \( t = \tilde{u}_i \) \( (i = 1, 2) \) to obtain \( k(z_1, \tilde{u}_i) \), provided we have at RScl \( Q_1^2 \) a reasonably reliable value of \( \alpha_s(Q_1^2) \equiv \pi z_1 \).

In fact, numerical integration of the RGE (3) would always work and give us such \( k(z_1, \tilde{u}_i) \)’s in (12) that RScl–invariance would be precise. In this RGE, we should not neglect at least those \( \beta_j \) coefficients which appear as nonzero numbers in the expressions for the coefficients \( f^{(1)}_j \ (j = 1, \ldots, 2M - 1) \) of the rearranged series (4), e.g. for \( M = 2 \) case we should not neglect \( \beta_1 \) and \( \beta_2 \) [cf. Eqs. (14)–(15)]. On the other hand, we can take into account in the RGE (3) more than these minimally required coefficients \( \beta_j \), thus making the result (12) RScl–invariant at such an improved level. We note, however, that only a limited number of perturbative coefficients \( \beta_j \) (namely: \( \beta_0, \ldots, \beta_3 \)) are known in QCD (cf. (14), in \( \overline{\text{MS}} \) scheme) and QED (cf. (17), in \( \overline{\text{MS}}, \text{MOM} \) and in on-shell schemes). Therefore, the RGE (3) will always be truncated at some level.

An alternative way to obtain \( k(z_1, \tilde{u}_i) \), for known \( z_1 \) and \( \tilde{u}_i \), would be to solve (a truncated) RGE (3) by integrating it analytically from \( t = t_1 = \ln(Q_1^2/Q_1^2) = 0 \) to \( t = \tilde{u}_i \), beforehand Taylor–expanding the integrands on the right around the point \( t = 0 \) and repeatedly using the same RGE (3) in evaluating Taylor coefficients. If we ignore threshold effects for simplicity, this procedure leads to the following expression:

\[
k(z_1, \tilde{u}_i) = 1 + z_1(-\beta_0 \tilde{u}_i) + z_1^2(\beta_0^2 \tilde{u}_i^2 - \beta_1 \tilde{u}_i) + z_1^3(-\beta_0^3 \tilde{u}_i^3 + \frac{5}{2} \beta_0 \beta_1 \tilde{u}_i^2 - \beta_2 \tilde{u}_i) + \ldots.
\]

(54)
If we take into account only a limited number of $\beta_i$ coefficients, then (54) may represent a convergent, or a well behaved asymptotic, series for small enough parameters $\tilde{u}_i$. We conjecture that the latter (generally complex) parameters have reasonably small absolute values if our choice of $Q_1^2$ is judicious, i.e., if $Q_1^2$ is close to a typical scale of the process considered.

IV. CONCLUSIONS, OPEN QUESTIONS

We presented a method of constructing approximants to available truncated perturbative series (TPS’s) of QED or QCD observables. The approximants are modified diagonal Baker–Gammel approximants (dBGA’s), with the kernel being a ratio of the gauge coupling parameters at different scales. We showed that these approximants have two favorable properties: they reproduce the TPS to the available order when expanded in powers of the gauge coupling parameter, and they are invariant under the change of the renormalization scale (RScl). The gauge coupling parameter can be taken to evolve with the scale at any chosen loop-order.

There are several questions that the presented work raises. For example, it remains unclear whether the dPA’s (11) could have multiple poles in some exceptional physical cases, and if so, what would have to be changed in the presented formalism in such a case. Furthermore, the formalism offers RScl–invariant approximants to truncated perturbation series $S_n^{(1)}$ of (11) only for odd $n = 2M - 1$. What would be the best approximant in cases of even $n = 2M$ (especially in case $n = 2$)? This question remains open even in the large-$\beta_0$ approximation when BGA’s reduce to the usual PA’s. The question of when the series of modified dBGA’s $z_1 G_{f(1)}^{[M-1/M]}(z_1) (M = 1, 2, \ldots)$, defined by (12), is convergent or asymptotic also remains open, especially since our proofs of Theorems 1 and 2 implicitly assumed convergence of series involved in the proofs. The latter question appears also when dealing with the usual dPA’s – however, in such a case a lot of progress in answering this question has already been achieved – from formal mathematical [13] and from physical/empirical point of view [2]. We also note that the dBGA result (12) becomes at the level $M \geq 2$ in principle RSch–dependent, since parameters in the dBGA involve also $\beta_2$ (and possibly higher $\beta_j$’s) coefficients [cf. (44)–(48)] which are explicitly scheme–dependent. Therefore, it would be instructive to test numerically RSch–dependence of dBGA (12) for $M \geq 2$ cases. We expect that this RSch–dependence (– dependence on $\beta_2$) would be very weak, at least for such choices of RScl $Q_1^2$ for which $z_1 \equiv \alpha_s(Q_1^2)/\pi$ is not large ($z_1 \approx 0.09$), since work of Ref. [2] (third entry – PRD) indicates that this is the case for some PA’s when observable $S$ is the effective charge in the Bjorken sum rule. In any case, one of the next natural steps would be to study efficiency of the presented formalism in cases of actual available truncated perturbative series for QED observables, to compare it numerically with other methods, and to investigate which classes of Feynman diagrams the presented method actually sums up.

8 The presented algorithm probably cannot be trivially extended to the cases of $n = 2M$, because it relies heavily on the decomposition (11) which is valid only for diagonal PA’s.
Furthermore, the method, with some modifications, may prove useful also in areas of physics others than high energy physics.

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Abbreviations used frequently in the article:
(d)BGA – (diagonal) Baker–Gammel approximant; (d)PA – (diagonal) Padé approximant; RSch – renormalization scheme; RScl – renormalization scale; TPS – truncated perturbation series.
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