MINKOWSKI DECOMPOSITION OF ASSOCIAHEDRA AND RELATED COMBINATORICS

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Abstract. Realisations of associahedra with linear non-isomorphic normal fans can be obtained by alteration of the right-hand sides of the facet-defining inequalities from a classical permutahedron. These polytopes can be expressed as Minkowski sums and differences of dilated faces of a standard simplex as described by Ardila, Benedetti & Doker (2010). The coefficients \( y_I \) of such a Minkowski decomposition can be computed by Möbius inversion if tight right-hand sides \( z_I \) are known not just for the facet-defining inequalities of the associahedron but also for all inequalities of the permutahedron that are redundant for the associahedron.

We show for certain families of these associahedra:

- how to compute the tight value \( z_I \) for any inequality that is redundant for an associahedron but facet-defining for the classical permutahedron. More precisely, each value \( z_I \) is described in terms of tight values \( z_J \) of facet-defining inequalities of the corresponding associahedron determined by combinatorial properties of \( I \).

- the computation of the values \( y_I \) of Ardila, Benedetti & Doker can be significantly simplified and depends on at most four values \( z_{a(I)}, z_{b(I)}, z_{c(I)} \) and \( z_{d(I)} \).

- the four indices \( a(I), b(I), c(I) \) and \( d(I) \) are determined by the geometry of the normal fan of the associahedron and are described combinatorially.

- a combinatorial interpretation of the values \( y_I \) using a labeled \( n \)-gon. This result is inspired from similar interpretations for vertex coordinates originally described by J.-L. Loday and well-known interpretations for the \( z_I \)-values of facet-defining inequalities.

1. Introduction

A. Postnikov defined in [18] generalised permutahedra as a subfamily of all convex polytopes that have the following H-description:

\[ P_n(\{z_I\}) := \{ x \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = z[n] \text{ and } \sum_{i \in I} x_i \geq z_I \text{ for } \emptyset \subset I \subset [n] \} \]

where \([n]\) denotes the set \([1, 2, \cdots, n]\). The classical \((n - 1)\)-dimensional permutahedron, as described for example by G. M. Ziegler, [29], corresponds to \( z_I = \frac{|I|(|I|+1)}{2} \) for \( \emptyset \subset I \subseteq [n] \) (we distinguish between \( \subset \) and \( \subseteq \)). Obviously, some of the above inequalities may be redundant for \( P_n(\{z_I\}) \) and, unless the value \( z_I \) is tight, sufficiently small increases and decreases of \( z_I \) for a redundant inequality do not change the combinatorial type of \( P_n(\{z_I\}) \). Although the encoding by all values \( z_I \) is not efficient, Proposition 1.2 below gives a good reason to specify tight values \( z_I \) for all \( I \subseteq [n] \). The subfamily of generalised permutahedra is now characterised by the additional requirement that \( P_n(\{z_I\}) \) is an element of the deformation cone of the classical permutahedron. Equivalently, this means that the normal fan of the generalised permutahedron is a coarsening of the normal fan of the classical permutahedron or that no facet-defining hyperplane of the permutahedron is moved past any vertices, compare A. Postnikov, V. Reiner, and L. Williams, [19]. This fine distinction and additional condition is easily overlooked but essential. For example, Proposition 1.2 does not hold for arbitrary polytopes \( P_n(\{z_I\}) \), we illustrate this by a simple example in Section 5. Fundamental examples are dilations of the standard simplex \( \Delta_n = \text{conv}\{e_1, e_2, \cdots, e_n\} \) where \( e_i \) denotes the \( i \)-th standard basis vector of \( \mathbb{R}^n \).

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For any two polytopes \( P \) and \( Q \), the Minkowski sum \( P + Q \) is defined as \( \{ p + q \mid p \in P, q \in Q \} \). In contrast, we define the Minkowski difference \( P - Q \) of \( P \) and \( Q \) only if there is a polytope \( R \) such that \( P = Q + R \). For more details on Minkowski differences we refer to [23]. We are interested in decompositions of generalised permutahedra into Minkowski sums and differences of dilated faces of the \((n-1)\)-dimensional standard simplex \( \Delta_n \), where the faces \( \Delta_I \) of \( \Delta_n \) are given by \( \text{conv} \{ e_i \mid i \in I \} \) for \( I \subseteq [n] \). If a polytope \( P \) is the Minkowski sum and difference of dilated faces of \( \Delta_n \), we say that \( P \) has a Minkowski decomposition into faces of the standard simplex. The following two results are known key observations.

**Lemma 1.1** ([11] Lemma 2.1). \( P_n(\{z_I\}) + P_n(\{z'_I\}) = P_n(\{z_I + z'_I\}) \).

If we consider the function \( I \mapsto z_I \) that assigns every subset of \([n]\) the corresponding tight value \( z_I \) of \( P_n(\{z_I\}) \), then the Möbius inverse of this function assigns to \( I \) the coefficient \( y_I \) of a Minkowski decomposition of \( P_n(\{z_I\}) \) into faces of the standard simplex:

**Proposition 1.2** ([11] Proposition 2.3). Every generalised permutahedron \( P_n(\{z_I\}) \) can be written uniquely as a Minkowski sum and difference of faces of \( \Delta_n \):

\[
P_n(\{z_I\}) = \sum_{I \subseteq [n]} y_I \Delta_I
\]

where \( y_I = \sum_{J \subseteq I} (-1)^{|I \setminus J|} z_J \) for each \( I \subseteq [n] \).

In particular, we also have \( z_I = \sum_{J \subseteq I} y_I \). A basic example is the classical permutahedron: it is known to be a zonotope and it is the Minkowski sum of the edges and vertices of \( \Delta_n \). The reader is invited to check that the corresponding \( z_I \)-values obtained by this formula yield precisely the right-hand sides mentioned earlier.

We will study Minkowski decompositions of generalised permutahedra that have the same normal fan as \( \mathbb{A}^c_{n-1} \). Two 3-dimensional examples of \( \mathbb{A}^c_3 \) (with distinct normal fans) are shown in Figure 1. We describe their construction in detail in Section 2. The normal fans of these polytopes are determined by a Coxeter element \( c \) of the symmetric group, but we will avoid the explicit use of Coxeter elements and use a partition \( U_c \cup D_c \) of \([n]\) induced by \( c \) instead. The main result is that the relation between \( z_I \) and \( y_I \)-coordinates of Proposition 1.2 simplified significantly: each \( y_I \) can be computed from at most four values \( z_I \) which depend on \( I \) and the normal fan of the polytope (or, equivalently, the Coxeter element \( c \) or the corresponding partition of \([n]\)). Moreover, we give an explicit combinatorial description how to determine these terms \( z_I \). If the we further restrict to the realisations \( \mathbb{A}^c_{n-1} \) as described by C. Hohlweg and C. Lange in [11], we show that the coefficients \( y_I \) can be described as signed product of path-lengths of a labeled polygon.

![Figure 1](image-url)
In other words, we extract combinatorial core data for the Möbius inversion of the function which are tight right-hand sides for certain facet-defining inequalities as specified in the theorem.

We now give examples of Minkowski decompositions of realisations of 2-dimensional associahedra \( As^c_2 \) into faces of the standard simplex is actually a Minkowski sum of some faces of a standard simplex.

We could stop here and be fascinated how the Möbius inversion relates the description by half spaces to the Minkowski decomposition. Nevertheless, for associahedra with the same normal fan as \( As^p_{n-1} \), we go beyond the alternating sum description for \( y_I \) of Ardila, Benedetti & Doker. In Theorem 4.2 we significantly simplify the alternating sum for each \( y_I \). In fact, each \( y_I \) can be expressed as an alternating sum of at most four non-zero values \( z_{a(I)} \), \( z_{b(I)} \), \( z_{c(I)} \) and \( z_{d(I)} \) which are tight right-hand sides for certain facet-defining inequalities as specified in the theorem. In other words, we extract combinatorial core data for the Möbius inversion of the function \( z_I \).
and answer the question which subsets $I$ of $J$ are essential to compute $y_I$ if the associahedron’s normal fan is the normal fan of $A_{c-1}$. These sets $J$ for fixed $I$ depend on the choice $c$ that determines the normal fan. Figure 9 illustrates how Theorem 4.2 can be used to compute the coefficients $y_{I}$ for one of the two examples shown in Figure 1. If the associahedron coincides with some $A_{c-1}$, Theorem 4.3 states a purely combinatorial interpretation of the values $y_{I}$. To illustrate this theorem, we recompute $y_{I}$ for $A_{c-1}$ and $A_{c}$ in Examples 4.6 and 4.7.

The outline of the paper is as follows. Section 2 summarises necessary known facts about $A_{c-1}$ and indicates some occurrences of the realisations considered here in the mathematical literature. In Section 3, we introduce the notion of an up and down interval decomposition for subsets $I \subseteq [n]$. This decomposition depends on the choice of a Coxeter element $c$ (or equivalently on a partition of $[n]$ induced by $c$) and is essential to prove Proposition 3.8. This proposition gives a combinatorial characterisation of all tight values $z_{I}$ for $A_{c-1}$ needed to evaluate $y_{I}$ using Proposition 1.2. The main results, Theorem 4.2 and Theorem 4.3, are then stated in Section 4. The proof of Theorem 4.2 is long and convoluted and deferred to Sections 6 and 7. While Theorem 4.3 is proved under the assumption of Theorem 4.2 in Section 4, to show that Proposition 1.2 and Theorem 4.2 do not hold for polytopes $P_{n}(\{z_{I}\})$ that are not contained in the deformation cone of the classical permutahedron, we briefly study a realisation of a 2-dimensional cyclohedron in Section 5.

About the same time as some of these results were achieved, V. Pilaud and F. Santos showed that the associahedra $A_{c-1}$ are examples of brick polytopes, [16, 17]. One of their results is that any brick polytope can be expressed as a Minkowski sum of other brick polytopes. As a consequence, we have two Minkowski decompositions of $A_{c-1}$ that are extremal in the following sense. The first decomposition of $A_{c-1}$ has a relatively complicated structure with respect to the coefficients $y_{I}$ (possibly negative numbers) but is very simple with respect to the polytopes used (faces of a standard simplex). On the other hand, the second decomposition of $A_{c-1}$ has a simple structure in terms of its coefficients (they are either 0 or 1) but is more complicated with respect to the polytopes used (brick polytopes). At the time of writing, the exact relationship of these two decompositions is not properly understood and remains a joint project of V. Pilaud with the author.

2. ASSOCIAHEDRA AS GENERALISED PERMUTAHEDRA

Associahedra form a family of combinatorially equivalent polytopes and can be realised as generalised permutahedra. Since the combinatorics of a polytope is encoded in its face lattice, we define an associahedron as a polytope with a face lattice that is isomorphic to the lattice of sets of non-crossing proper diagonals of a convex and plane $(n+2)$-gon $Q$ ordered by reversed inclusion. This description immediately tells us that the set of $k$-faces is in bijection to the set of triangulations of $Q$ with $k$ proper diagonals removed. In particular, vertices correspond to triangulations and facets correspond to proper diagonals. Since associahedra turn out to be simple polytopes, a result of R. Blind and P. Mani-Levitska with an elegant proof due to G. Kalai, [3, 13], guarantees that the face lattice is already determined by the 1-skeleton, so it suffices to specify the vertex-edge graph to determine the combinatorics of the face-lattice. This graph is known as the flip graph of triangulations of $Q$. In 2004, J.-L. Loday published a beautiful combinatorial description for the vertex coordinates of associahedra constructed earlier by S. Shnider, S. Sterberg and J. Stasheff, [23, 25]. Loday’s description is in terms of labeled binary trees dual to the triangulations of $Q$. The construction of S. Shnider, S. Sterberg and J. Stasheff as well as Loday’s vertex description was subsequently generalised by C. Hohlweg and C. Lange, [11]. The latter construction explicitly describes realisations $A_{c-1}$ of $(n-1)$-dimensional associahedra and exhibits them as generalised permutahedra. The construction depends on the choice of a Coxeter element $c$ of the symmetric group $\Sigma_{n}$ on $n$ elements.

A proper diagonal is a line segment connecting a pair of vertices of $Q$ whose relative interior is contained in the interior of $Q$. A non-proper diagonal is a diagonal that connects vertices adjacent in $\partial Q$ and a degenerate diagonal is a diagonal where the end-points are equal.
We now outline the construction of $A_{n-1}$ and avoid the explicit use of Coxeter elements. Nevertheless, we use Coxeter elements in our notation to distinguish between different realisations. It is known that the Coxeter elements are in bijection to the certain partitions $D_c\sqcup U_c$ of $[n]$. We will use these partitions to obtain labelings $Q_c$ of $Q$ and refer to $D_c$ as down set and to $U_c$ as up set. The partitions are

$$D_c = \{d_1 = 1 < d_2 < \cdots < d_t = n\} \quad \text{and} \quad U_c = \{u_1 < u_2 < \cdots < u_m\},$$

so $n = \ell + m$, $|D_c| = \ell \geq 2$ and $|U_c| = m$. We now obtain the c-labeling $Q_c$ of $Q$ with label set $[n+1] \cup \{0\}$ as follows. Pick two vertices of $Q$ which are the end-points of a path with $\ell + 2$ vertices on the boundary of $Q$, label the vertices of this path counter-clockwise increasing using the label set $D_c := D_c \cup \{0, n+1\}$ and label the remaining path clockwise increasing using the label set $U_c$. The labeling $Q_c$ has the property that the label set $D_c$ is always on the right-hand side of the diagonal $\{0, n+1\}$ oriented from 0 to $n+1$. To illustrate these c-labelings $Q_c$, observe that there are four distinct partitions $D_c \sqcup U_c$ for $n = 4$ which yield the four labeled hexagons $Q_c$ shown in Figure 4. We derive values $\tilde{z}_I$ for some subsets $I \subseteq [n]$ using oriented proper diagonals of $Q_c$ as follows. Orient each proper diagonal $\delta$ from the smaller to the larger labeled end-point of $\delta$, associate to $\delta$ the set $R_\delta$ that consists of all labels on the strict right-hand side of $\delta$, and replace the elements $0$ and $n+1$ by the smaller respectively larger label of the end-points contained in $U_c$ if possible. For each proper diagonal $\delta$ we have $R_\delta \subseteq [n]$ but obviously not every subset of $[n]$ is of this type if $n > 2$. Now set

$$\tilde{z}_I := \frac{(2|I|+1)}{2} - \infty \quad \text{if } I = R_\delta \text{ for some proper diagonal } \delta,$$

else,

compare Table 1 for the two associahedra $A_5^c$ depicted in Figure 1 that correspond to two different c-labelings of a hexagon. In [11] it is shown that $P_n(\{\tilde{z}_I\})$ is in fact an associahedron of dimension $(n-1)$ realised in $\mathbb{R}^n$ for every choice of $c$. In other words, to obtain these associahedra

$$D_c = \{1, 3, 4\} \quad \text{and} \quad U_c = \{2\},$$

$$D_c = \{1, 4\} \quad \text{and} \quad U_c = \{2, 3\}.$$
from the classical permutahedron, we make all inequalities redundant that do not correspond to a proper diagonal of \(Q_c\). Of course, the right-hand sides \(\tilde{z}_j = -\infty\) are not tight. Proposition 3.8 shows how we can compute the tight values for \(\tilde{z}_j\) using finite values \(\tilde{z}_j\) of facet-defining inequalities only. Throughout this manuscript and for any choice \(c\), the reader may refer to this set of tight values \(\{\tilde{z}_j\}\) to illustrate the results. But we emphasise that this specific choice \(\{\tilde{z}_j\}\) is only assumed for Statements 4.3–4.5. All other results are valid for the larger class of \(z_j\)-coefficients where is polytope \(P_n(\{z_j\})\) is an associahedron with the same normal fan as some \(A_{n-1}^c\). Proposition 3.8 and Theorem 4.2 can be applied to this more general situation to obtain tight values for the redundant values \(z_j\) and to obtain the coefficients \(y_I\) of the Minkowski decomposition into faces of the standard simplex.

It is known that realisations \(A_{n-1}^{c_1}\) and \(A_{n-1}^{c_2}\) can be linear isometric for certain choices \(c_1\) and \(c_2\) and values \(z_I\). [2]. While the two associahedra depicted in Figure 1 are neither linear isometric nor do they have the same normal fan, we remark that the associahedra \(A_{n}^{c_1}\) and \(A_{n}^{c_2}\) discussed in the previous section are linear isometric and the isometry is a point reflection \(\Phi\) in the hyperplane \(\sum_{i \in [n]} x_i = 6\). Although the \(z_I\) and \(y_I\)-values differ for both realisations, they transform according to this isometry. If we consider a Minkowski decomposition of \(A_{n}^{c_2}\) with respect to the faces of \(\Phi(\Delta_n)\), we obtain precisely the Minkowski coefficients of \(A_{n}^{c_1}\) with respect to the faces of the normal fan of the \(\Delta_n\).

We end this section relating \(A_{n}^{c_1}\) to earlier work. Firstly, we indicate a connection to cambrian fans, generalised associahedra and cluster algebras and secondly to convex rank texts and semi-graphoids in statistics. Thirdly, we mention some earlier appearances of specific instances of \(A_{n}^{c_1}\) up to linear isomorphism.

S. Fomin and A. Zelevinsky introduced generalised associahedra in the context of cluster algebras of finite type, [3], and it is well-known that associahedra are generalised associahedra.
associated to cluster algebras of type A. The construction of \([11]\) was subsequently generalised by C. Hohlweg, C. Lange, and H. Thomas to generalised associahedra, \([12]\), and depends also on a Coxeter element \(c\). The geometry of the normal fans of these realisations are determined by combinatorial properties of \(c\) and the normal fans are \(c\)-Cambrian fans (introduced by N. Reading and D. Speyer in \([20]\)). Reading and Speyer conjectured the existence of a linear isomorphism between \(c\)-Cambrian fans and \(g\)-vector fans associated to cluster algebras of finite type with acyclic initial seed (the notion of a \(g\)-vector fan for cluster algebras was introduced by Fomin and Zelevinsky \([9]\)). In \([21]\), Reading and Speyer describe and relate Cambrian and \(g\)-vector fans in more detail and prove their conjecture up to an assumption of another conjecture of \([9]\). S.-W. Yang and A. Zelevinsky gave an alternative proof of the conjecture of Reading and Speyer in \([28]\).

S. Stella recently recovered the the realizations of generalized associahedra for finite type of \([12]\) and describes the relationship to cluster algebras in detail, \([27]\).

Generalised permutahedra and therefore the associahedra \(A^{c}_{n-1}\) are closely related to the framework of convex rank tests and semigraphoids from statistics as discussed by J. Morton, L. Pachter, A. Shi, B. Sturmfels, and O. Wienand \([15]\). The semigraphoid axiom characterises the collection of edges of a permutahedron that can be contracted simultaneously to obtain a generalised permutahedron. The authors also study submodular rank tests, its subclass of Minkowski sum of simplices tests and graphical rank tests. The latter one relates to graph associahedra of M. Carr and S. Devadoss \([4]\). Among the associahedron studied in this manuscript, Loday’s realisation fits and \(S\). Devadoss \([5]\). Among the associahedra studied in this manuscript, Loday’s realisation fits and \(S\). Stella recently recovered the the realizations of generalized associahedra for finite type of \([12]\) and describes the relationship to cluster algebras in detail, \([27]\).

Some instances of \(A^{c}_{n-1}\) have been studied earlier. For example, the realisations of J.-L. Loday, \([14]\), and of G. Rote, F. Santos, and I. Streinu, \([22]\), related to one-dimensional point configurations, are affine equivalent to \(A^{c}_{n-1}\) if \(U_{c} = \emptyset\) or \(U_{c} = [n] \setminus \{1, n\}\). For \(U_{c} = \emptyset\), the Minkowski decomposition into faces of a standard simplex is described by Postnikov in \([13]\). Moreover, G. Rote, F. Santos, and I. Streinu point out in Section 5.3 that their realisation is not affine equivalent to the realisation of F. Chapoton, S. Fomin, and A. Zelevinsky, \([7]\). It is not difficult to show that the realisation described in \([7]\) is affine equivalent to \(A^{c}_{1}\) if \(U_{c} = \{2\}\) or \(U_{c} = \{3\}\).

3. Tight values for all \(z_{r}^{c}\) for \(A^{c}_{n-1}\)

Since the facet-defining inequalities for \(A^{c}_{n-1}\) correspond to proper diagonals of \(Q_{c}\), we know precisely the irredundant inequalities for the generalised permutahedron \(P_{c}(\{z_{r}^{c}\})\). In this section, we determine tight values \(z_{r}^{c}\) for all \(I \subseteq [n]\) corresponding to redundant inequalities in order to be able to compute the coefficients \(y_{I}\) of the Minkowski decomposition of \(A^{c}_{n-1}\) as described by Proposition \([12]\). The concept of an up and down interval decomposition induced by the partitioning \(D_{U}\cup U_{c}\) (or, equivalently, induced by \(c\)) of a given interval \(I \subseteq [n]\) is a key concept that we introduce first, it allows us to describe any \(I \subseteq [n]\) in terms of unions and intersections of sets \(R_{s}\) for certain proper diagonals determined by this decomposition (or, equivalently, as unions of set differences of certain sets \(R_{s}\) and their complements).

Definition 3.1 (up and down intervals).

Let \(D_{c} = \{d_{1} = 1 < d_{2} < \cdots < d_{n} = n\}\) and \(U_{c} = \{u_{1} < u_{2} < \cdots < u_{m}\}\) be the partition of \([n]\) induced by a Coxeter element \(c\).

(a) A set \(S \subseteq [n]\) is a non-empty interval of \([n]\) if \(S = \{r, r + 1, \ldots, s\}\) for some \(0 < r \leq s < n\).

We write \(S\) as closed interval \([r, s]\) (end-points included) or as open interval \((r, s)\) (end-points excluded). An empty interval is an open interval \((k, k + 1)\) for some \(1 \leq k < n\).

(b) A non-empty open down interval is a set \(S \subseteq D_{c}\) such that \(S = \{d_{r} < d_{r+1} < \cdots < d_{s}\}\) for some \(1 \leq r \leq s < \ell\). We write \(S\) as open down interval \((d_{r+1}, d_{s})D_{c}\) where we allow \(d_{r+1} = 0\) and \(d_{s+1} = n + 1\). For \(1 \leq r \leq \ell - 1\), we also have the empty down interval \((d_{r}, d_{r+1})D_{c}\).

(c) A closed up interval is a non-empty set \(S \subseteq U_{c}\) such that \(S = \{u_{r} < u_{r+1} < \cdots < u_{s}\}\) for some \(1 \leq r \leq s \leq \ell\). We write \([u_{r}, u_{s}]_{U_{c}}\).

We often omit the words open and closed when we consider down and up intervals. There will be no ambiguity, because we are not going to deal with closed down intervals or open up
intervals. Up intervals are always non-empty, while down intervals may be empty. It will be useful to distinguish the empty down intervals \((d_r, d_{r+1})_D_e\) and \((d_s, d_{s+1})_D_e\) if \(r \neq s\) although they are equal as sets.

It might be helpful to read the following definition of the up and down interval decomposition in combination with the following Examples 3.3 and 3.5.

**Definition 3.2** (up and down interval decomposition).

Let \(D_c = \{d_1 = 1 < d_2 < \cdots < d_t = n\}\) and \(U_c = \{u_1 < u_2 < \cdots < u_m\}\) be the partition of \([n]\) induced by a Coxeter element \(c\) and \(I \subset [n]\) be non-empty. The up and down interval decomposition of type \((v, w)\) of \(I\) is a partition of \(I\) into disjoint up and down intervals \(I^U_1, \cdots, I^U_v\) and \(I^D_1, \cdots, I^D_d\) obtained by the following procedure:

1. Suppose there are \(v\) non-empty inclusion maximal down intervals of \(I\) denoted by \(I^D_i = (\bar{a}_i, \bar{b}_i)_{D_e}\), \(1 \leq i \leq (v)\), with \(\bar{b}_i \leq \bar{a}_{i+1}\) for \(1 \leq k < v\). Consider also all empty down intervals \(E^D_i = (d_r, d_{r+1})_{D_e}\) with \(b_r \leq d_r < d_{r+1} \leq \bar{a}_{k+1}\) for \(0 \leq k \leq (v)\) where \(b_0 = 1\) and \(\bar{a}_v = n\). Denote the open intervals \((\bar{a}_i, \bar{b}_i)\) and \((d_r, d_{r+1})\) of \([n]\) by \(\bar{I}_i\) and \(E_i\) respectively.

2. Consider all inclusion maximal up intervals of \(I\) contained in some interval \(\bar{I}_i\) or \(E_i\) obtained in Step 1 and denote these up intervals by

\[
I^U_i = [\alpha_1, \beta_1]_{U_e}, \cdots, I^U_v = [\alpha_w, \beta_w]_{U_e}.
\]

Without loss of generality, we assume \(\alpha_i \leq \beta_i < \alpha_{i+1}\).

3. A down interval \(I^D_i = (a_i, b_i)_{D_e}, 1 \leq i \leq (v)\), is a down interval obtained in Step 1 that is either a non-empty down interval \(I^D_i\) or an empty down interval \(E^D_i\) with the additional property that there is some up interval \(I^U_j\) obtained in Step 2 such that \(I^D_i \subseteq E_j\). Without loss of generality, we assume \(b_i < a_{i+1}\) for \(1 \leq i < v\).

**Example 3.3.**

We describe the up and down interval decomposition for three subsets of \([4]\) which is partitioned into \(D_c = \{1, 3, 4\}\) and \(U_c = \{2\}\) and encourage the reader to sketch the steps.

i) Consider \(J_1 = \{2, 3\}\).

The only non-empty inclusion maximal down interval of \(J_1\) is \(I^D_1 = (1, 4)_{D_e} = \{3\}\); there are no empty down intervals \(E^D_i\) to be considered. As inclusion maximal up intervals of \(J_1\) contained in \(\bar{I}_1 = (1, 4)_{D_e} = \{2, 3\}\), we identify \(I^U_1 = [2, 2]_{U_e} = \{2\}\). It follows that the up and down interval decomposition of \(J_1\) is \(\{1, 4\}_{D_e} \cup [2, 2]_{U_e}\). Its type is \((1, 1)\).

ii) Consider \(J_2 = \{2\}\).

There is no non-empty inclusion maximal down interval of \(J_2\) to be considered, but there is one empty down interval \(I^D_1 = (1, 3)_{D_e}\) such that \(E_1 = (1, 3)_{D_e} = \{2\}\) contains one inclusion maximal up interval \(I^U_1 = [2, 2]_{U_e} = \{2\}\) of \(J_2\). It follows that the up and down interval decomposition of \(J_2\) is \(\{1, 3\}_{D_e} \cup [2, 2]_{U_e}\). Its type is \((1, 1)\).

iii) Consider \(J_3 = \{2, 4\}\).

The only non-empty inclusion maximal down interval of \(J_3\) is \(I^D_1 = (3, 5)_{D_e} = \{4\}\); there is one empty down interval \(I^D_1 = (1, 3)_{D_e}\) such that \(E_1 = (1, 3)_{D_e} = \{2\}\) contains an inclusion maximal up interval of \(J_3\), this is the up interval \(I^U_1 = [2, 2]_{U_e} = \{2\}\). There is no non-empty inclusion maximal up interval contained in \(I^D_1\). It follows that the up and down interval decomposition of \(J_3\) is \(\{1, 3\}_{D_e} \cup [2, 2]_{U_e} \cup \{3, 5\}_{D_e}\). Its type is \((2, 1)\).

**Definition 3.4** (nested up and down interval decomposition, nested components).

Let \(D_c = \{d_1 = 1 < d_2 < \cdots < d_t = n\}\) and \(U_c = \{u_1 < u_2 < \cdots < u_m\}\) be the partition of \([n]\) induced by a Coxeter element \(c\) and \(I \subset [n]\) be non-empty.

(a) The up and down interval decomposition of \(I\) is nested if its type is \((1, w)\).

(b) A nested component of \(I\) is an inclusion-maximal subset \(J\) of \(I\) such that the up and down decomposition of \(J\) is nested.

The definition of a nested up and down interval decomposition can be rephrased as follows: all up intervals are contained in the interval \((a_i, b_i)\) of \([n]\) obtained from the unique (empty or non-empty) down interval \(I^D_1 = (a_1, b_1)_{D_e}\). The following example describes the up and down
interval decompositions of $I = R_\delta$ for all proper diagonals $\delta$ of $Q_c$. The situation is illustrated in Figure 6. As a consequence, we observe that the up and down interval decomposition for $R_\delta$ is always nested if $\delta$ is a proper diagonal.

**Example 3.5.**

Let $D_c = \{d_1 = 1 < d_2 < \cdots < d_\ell = n\}$ and $U_c = \{u_1 < u_2 < \cdots < u_m\}$ be the partition of $[n]$ induced by a Coxeter element $c$. The proper diagonals $\delta = \{a, b\}$, $a < b$, of the $c$-labeled polygon $Q_c$ are in bijection to certain non-empty proper subsets $I \subset [n]$ that have an up and down interval decomposition of type $(1, 0)$, $(1, 1)$, or $(1, 2)$. More precisely, we have

(a) $R_\delta = (a, b)_{D_c}$ if $R_\delta$ has an up and down decomposition of type $(1, 0)$.
(b) $R_\delta = (0, b)_{D_c} \cup \{u_1, a\}_{U_c}$ or $R_\delta = (a, n+1)_{D_c} \cup \{b, u_m\}_{U_c}$ if $R_\delta$ has a decomposition of type $(1, 1)$

\[ \text{compare Figure 6 for an illustration of these two cases.} \]

(c) $R_\delta = [0, n+1]_{D_c} \cup \{u_1, a\}_{U_c} \cup \{b, u_m\} \cup \delta$ if $R_\delta$ has an up and down decomposition of type $(1, 2)$.

To simplify notation, we extend the definition of $R_\delta$ to the non-proper diagonals $\delta = \{0, u_1\}$ and $\delta = \{u_m, n+1\}$ by defining $R_{\delta} = R_{\{u_m, n+1\}} = [n]$. An example of the diagonals $\delta_{i,j}$ associated to an up and down interval decomposition defined in the next Lemma is discussed and illustrated in Example 3.7 and Figure 7.

**Lemma 3.6.**

Given the partition $[n] = D_c \cup U_c$ induced by a Coxeter element $c$. Let $I$ be a non-empty proper subset of $[n]$ with up and down interval decomposition of type $(v, w)$ and nested components of type $(1, w_1), \cdots , (1, w_v)$. For $1 \leq i \leq v$ and $1 \leq j \leq w_i$, denote by $[\alpha_{i,j}, \beta_{i,j}]_{U_w}$ the inclusion maximal up intervals contained in the down interval $(\alpha_{i,j}, \beta_{i,j})_{D_c}$, where $\beta_{i,j} < \alpha_{i,j+1}$ and $b_1 \leq a_{i+1}$.

If $w_i = 0$ then associate to the nested component $(1, w_1)$ the diagonal $\delta_{i,1} = (\alpha_{i,j} , b_i)$. If $w_i > 0$ then associate to the nested component $(1, w_i)$ the diagonals

\[ \delta_{i,1} = \{a_{i, j}, \alpha_{i,j}\}, \quad \delta_{i,j} = \{\beta_{i,j-1}, \alpha_{i,j}\} \text{ for } 1 < j \leq w_i, \quad \text{and} \quad \delta_{i,w_i+1} = \{\beta_{i,w_i}, b_i\}. \]

Then the diagonals $\delta_{i,j}$ are non-crossing and

\[ I = \bigcup_{i \in [v]} \bigcup_{j \in [w_i]} R_{\delta_{i,j}} \bigcup_{i \in [v]} \left( R_{\delta_{i,w_i+1}} \setminus \bigcup_{j \in [w_i]} R_{\delta_{i,j}} \right). \]

**Proof.** It follows from the definition of nested components that $\delta_{i,j}$ and $\delta_{i',j'}$ are non-crossing if $i \neq i'$. That $\delta_{i,j}$ and $\delta_{i,j'}$ are non-crossing within a nested component is implied by $\beta_{i,j} < \alpha_{i,j+1}$.

**Figure 6.** The four possible situations for a diagonal $\delta = \{a, b\}$ of Example 3.5.
To see the identities on \( I \), we first remark that \( I = \bigcap_{j \in [w_1 + 1]} R_{\delta_{i,j}} \) follows directly from the the up and down interval decomposition of \( I \) and the definition of \( R_{\delta} \) if \( I \) has only one nested component. If \( I \) consists of more than one nested component, we obtain the claim since it holds for each nested component separately. The second identity is a simple reformulation of the first.

This is easily seen in case of just one nested component: instead of intersecting the sets \( R_{\delta} \) for each nested component separately. The second identity is a simple reformulation of the first.

We briefly discuss the diagonals associated to the up and down interval decomposition for the three examples considered in Example 3.7.

These examples are illustrated in Figure 7.

The associated diagonals \( \delta_{i,j} \) follow directly from the inclusion \( \bigcap_{j \in [w_1 + 1]} R_{\delta_{i,j}} \). This yields \( \bigcap_{j \in [w_1 + 1]} R_{\delta_{i,j}} \).

Example 3.7.

We briefly discuss the diagonals associated to the up and down interval decomposition for the three subsets \( J_1 = \{2, 3\} \), \( J_2 = \{2\} \) and \( J_3 = \{2, 4\} \) of \([4]\) partitioned by \( D_c = \{1, 3, 4\} \) and \( U_c = \{2\} \). These examples are illustrated in Figure 7.

i) We computed \((1, 4)_D \cup [2, 2]_{U_c} \) as up and down interval decomposition for \( J_1 \). We therefore have the associated diagonals \( \delta_{1,1} = \{1, 2\} \) and \( \delta_{1,2} = \{2, 4\} \).

ii) We computed \((1, 3)_D \cup [2, 2]_{U_c} \) as up and down interval decomposition for \( J_2 = \{2\} \). The associated diagonals are \( \delta_{1,1} = \{1, 2\} \) and \( \delta_{1,2} = \{2, 3\} \).

iii) We computed \((1, 3)_D \cup [2, 2]_{U_c} \cup (3, 5)_D \) as up and down interval decomposition for \( J_3 = \{2, 4\} \). The associated diagonals are \( \delta_{1,1} = \{1, 2\} \), \( \delta_{1,2} = \{2, 3\} \) and \( \delta_{2,1} = \{3, 5\} \).

The final proposition of this section resolves the quest for tight values \( z^*_I \) of all redundant inequalities of an associahedron that has the normal fan of \( A_{n-1} \). If we denote this associahedron by \( P_n(\{z^*_I\}) \), then the inequalities that correspond to an index set \( I = R_{\delta} \) for some proper diagonal of \( Q_c \) are precisely the facet defining inequalities and all other inequalities are redundant.

Proposition 3.8.

Given the partition \([n] = D_c \cup U_c \) induced by a Coxeter element \( c \). Let \( I \) be a non-empty proper subset of \([n]\) with up and down interval decomposition of type \((v, w)\) and nested components of type \((1, w_1), \ldots , (1, w_v)\). For \( 1 \leq i \leq v \) and \( 1 \leq j \leq w_i \), denote by \([\alpha_{i,j}, \beta_{i,j}]_{U_c} \) the inclusion maximal up intervals contained in the down interval \((a_i, b_i)_{D_c} \) where \( \beta_{i,j} < \alpha_{i,j+1} \) and \( b_i \leq a_{i+1} \). For non-empty \( I \subseteq [n] \) we set

\[
z^*_I := \sum_{i \in [n]} \left( \sum_{j \in [w_i + 1]} z^c_{R_{\delta_{i,j}}} - w_i z^c_{[n]} \right).
\]

Then \( P(\{z^*_I\}) = P(\{z^c_{[n]}\}) \) and all \( z^*_I \) are tight.

\[
\begin{align*}
J_1 &= \{2, 3\} & J_2 &= \{2\} & J_3 &= \{2, 4\}
\end{align*}
\]

Figure 7. The associated diagonals \( \delta_{i,j} \) for the three examples considered in Example 3.7.
Definition 3.9.
Let $I$ be a non-empty proper subset of $[n]$ with up and down interval decomposition of type $(v,w)$. Following notation of Proposition 3.8, we associate diagonals $\delta_{i,j}$ for $1 \leq i \leq v$ and $1 \leq j \leq w$. The subset $D_I$ of proper diagonals of $\{\delta_{i,j} | 1 \leq i \leq v$ and $1 \leq j \leq w\}$ is called set of proper diagonals associated to $I$. Similarly, we say that $\delta \in D_I$ is a proper diagonal associated to $I$.

We make a few remarks. First, if a non-proper diagonal $\delta = \{0, u_1\}$ or $\delta = \{u_m, n + 1\}$ occurs as a diagonal associated to the first or last nested component, the formula for $z_{ij}^\delta$ in Proposition 3.8 can be simplified by cancelation of the corresponding terms $\tilde{z}_{iv}^\delta$. Second, for any proper diagonal $\delta$ of $Q_c$, we obtain $z\tilde{z}_{R_i}^c = \tilde{z}_{R_j}^c$. An finally, we can characterise the face of $P(\{z_{ij}^\delta\})$ that minimises the linear functional $\sum_{i \in I} x_i$ for a given non-empty and proper subset $I \subset [n]$.

Corollary 3.10.
Associate the linear functional $\varphi_I(x) = \sum_{i \in I} x_i$ to a non-empty proper subset $I \subset [n]$ and denote the facet of $P(\{z_{ij}^\delta\})$ that is supported by $\sum_{i \in R_i} x_i = z\tilde{z}_{R_i}^c$ for the proper diagonal $\delta$ by $F_{R_i}$, then the intersection $\cap_{\delta \in D_I} F_{R_i}$ is the minimizing face of $P(\{z_{ij}^\delta\})$ for $\varphi_I$.

4. Main results and examples

Substitution of Proposition 3.8 into Proposition 1.2 provides a way to compute all Minkowski coefficients $y_I$ since all tight values $z_{ij}^\delta$ for $\mathcal{A}_{n-1}^c = P_n(\{z_{ij}^\delta\})$ are known:

\[(1) \quad y_I = \sum_{J \subseteq I} (-1)^{|I \setminus J|} z_{IJ} = \sum_{J \subseteq I} (-1)^{|I \setminus J|} \sum_{i \in [J]} \sum_{j \in [w_i + 1]} (\tilde{z}_{R_{i,j}}^c - w_i \tilde{z}_{[n]}^c).\]

The goal of this section is to provide two simpler formulae for $y_I$. The first one, given in Theorem 4.2 simplifies Formula (1) to at most four non-zero summands for each $I \subseteq [n]$. The second one, stated in Theorem 4.3, is only valid if the right-hand sides of the facet-defining inequalities satisfy $z_{IJ}^\delta = \frac{|I|(|I|+1)}{2}$. The values $y_I$ are then described as a (signed) product of two numbers that measure certain paths of $Q_c$. Theorem 4.3 can be seen as a new aspect to relate combinatorics of the labeled $n$-gon $Q_c$ to a construction of $\mathcal{A}_{n-1}^c$: the coefficients for the Minkowski decomposition into faces of the standard simplex can be obtained from the combinatorics of $Q_c$. Two other relations of the combinatorics of $Q_c$ to the geometry of $\mathcal{A}_{n-1}^c$ were known before. It is possible to extract the coordinates of the vertices, $[13][14]$, but it is also possible to determine the the facet normals and the right-hand sides for their inequalities, $[11]$.

From now on, we use the following notation and make some general assumptions unless explicitly mentioned otherwise. Let $[n] = D_c \cup U_c$ be the partition of $[n]$ induced by some fixed Coxeter element $c$ with $D_c = \{d_1 = 1 < d_2 < \cdots < d_r = n\}$ and $U_c = \{u_1 < \cdots < u_m\}$. A non-empty subset $I \subseteq [n]$ with up and down interval decomposition of type $(v,w)$ has nested components $(1, w_i)$, $1 \leq i \leq v$, such that the inclusion maximal up intervals $[\alpha_{i,j}, \beta_{i,j}]$ of $u_c$ contained in the
down interval \((a_i, b_i)\) satisfy \(\beta_{i,j} < \alpha_{i,j+1}\) and \(b_i \leq a_{i+1}\). For nested \(I\), that is, if \(v = 1\), we simplify notation and drop one subscript: the up and down interval decomposition is \((a, b)_D \cup \bigcup_{j=1}^n [\alpha_j, \beta_j]_{U_j}\), where \(\alpha_j < \beta_j \leq \alpha_{j+1}\) as before. Nevertheless, we do not drop an index for the associated diagonals \(\delta_j\); introduced in Lemma \[5.6\], we continue to denote them by \(\delta_{i,j}\) or \(\delta_{i,j}\) to avoid a conflict with the diagonals \(\delta_1, \delta_2, \delta_3, \delta_4\) defined next. To that respect, we define \(\gamma\) (respectively \(\Gamma\)) to denote the smallest (respectively largest) element of a nested set \(I\) and associate the following four diagonals of the \(c\)-labeled \((n + 2)\)-gon \(Q_c\) to a nested set \(I\):

\[
\delta_1 = \{a, b\}, \quad \delta_2 = \{a, \Gamma\}, \quad \delta_3 = \{\gamma, b\}, \quad \text{and} \quad \delta_4 = \{\gamma, \Gamma\}.
\]

In general, not all diagonals \(\delta_i\) will be proper diagonals, but it will be useful to consider the subset \(\mathcal{D}_I\) of \(\{\delta_1, \delta_2, \delta_3, \delta_4\}\) that consists of proper diagonals only. We emphasize that the diagonals \(\delta_i\) should be distinguished from the diagonals \(\delta_{i,j}\) introduced in Lemma \[5.6\] and the set \(\mathcal{D}_I\) should be distinguished from \(\mathcal{D}_I\).

**Example 4.1.**

We discuss the four diagonals \(\delta_1, \delta_2, \delta_3, \delta_4\) associated to three subsets of \([4]\) which is partitioned into \(D_c = \{1, 3, 4\}\) and \(U_c = \{2\}\). These associated set \(\mathcal{D}_I\) are illustrated in Figure \[8\].

i) The up and down interval decomposition of \(J_1 = \{2, 3\}\) is \((1, 4)_D, \bigcup [2, 2]_{U_c}\). Moreover, we have \(\gamma = 2\) and \(\Gamma = 3\). It follows that

\[
\delta_1 = \{1, 4\}, \quad \delta_2 = \{1, 3\}, \quad \delta_3 = \{2, 4\} \quad \text{and} \quad \delta_4 = \{2, 3\}.
\]

In this situation, all diagonals \(\delta_i\) except diagonal \(\delta_2 = \{1, 3\}\) are proper diagonals. Therefore, \(\mathcal{D}_J = \{\delta_1, \delta_3, \delta_4\}\).

ii) The up and down interval decomposition of \(J_2 = \{2\}\) is \((1, 3)_D, \bigcup [2, 2]_{U_c}\). Moreover, we have \(\gamma = \Gamma = 2\). This implies

\[
\delta_1 = \{1, 3\}, \quad \delta_2 = \{1, 2\}, \quad \delta_3 = \{2, 3\} \quad \text{and} \quad \delta_4 = \{2, 2\}.
\]

In this situation, the diagonals \(\delta_1, \delta_4\) are not proper while the diagonals \(\delta_2, \delta_3\) are proper. Hence, \(\mathcal{D}_J = \{\delta_2, \delta_3\}\).

iii) The set \(J_3 = \{2, 4\}\) is not nested since its up and down interval decomposition is of type \((2, 1)\).

We do not associate diagonals \(\delta_1, \delta_4\) to \(J_3\), the set \(\mathcal{D}_J\) is empty.

We now extend our definition of \(R_d\) and \(z_{R_d}\) to all non-proper and degenerate diagonals \(\delta\). If \(\delta = \{0, n + 1\}\) and \(U_c = \emptyset\) we set \(R_d := [n]\) and \(z_{R_d} = z_{[n]}\). Otherwise, if \(\delta = \{x, y\}\) is not a proper diagonal (different from \(\delta = \{0, n + 1\}\) and \(U_c = \emptyset\)), we set:

\[
R_d := \begin{cases} \emptyset & \text{if } x, y \in D_c \\ [n] & \text{otherwise}, \end{cases} \quad \text{and} \quad z_{R_d} := \begin{cases} 0 & \text{if } R_d = \emptyset \\ z_{[n]} & \text{if } R_d = [n]. \end{cases}
\]

The main result, Theorem \[1.2\] actually combines two statements. Firstly, there is a more efficient way to compute the coefficients of the Minkowski decomposition of an associahedron \(A_{n-1}^c = P \left( \{z_{[n]}^c\} \right) \) compared to the alternating sum proposed by Proposition \[1.2\]. Secondly, the terms \(z_{[n]}^c\) for redundant inequalities that are needed to compute \(y_I\) are combinatorially characterised and

![Figure 8](image-url)
depend on the choice of \( c \) or equivalently on the normal fan of \( \mathcal{A}_{n−1}^c \). Of course, their precise values depend on the values \( z_Γ^c \) of inequalities that are facet-defining.

**Theorem 4.2.**
Let \( I \) be non-empty subset of \([n]\). Then the Minkowski coefficient \( y_I \) of \( \mathcal{A}_{n−1}^c \) is

\[
y_I = \begin{cases} 
(-1)^{|I|−|R_{a_i}|} & \text{if } I \neq \{u_s\} \subseteq U_c \text{ and } v = 1, \\
(n + 1) − K_y K_Γ & \text{if } I = \{u_s\} \subseteq U_c, \\
0 & \text{if } v \geq 2.
\end{cases}
\]

We prove Theorem 4.2 in Section 6. An example illustrating the theorem for the left associahedron \( \mathcal{A}_2^c \) of Figure 4.4 (\( D_c = \{1, 3, 4\} \) and \( U_c = \{2\} \)) is given in Figure 9 where we also explicitly compute the \( y_I \)-values for this realisation with \( z_Γ^c = \frac{|I|(|I| + 1)}{2} \) for the facet-defining inequalities.

For the rest of this section, we specialise to realisations with this specific choice of \( z_Γ^c \)-values. We obtain a nice combinatorial interpretation the coefficients \( y_I \) in Theorem 4.3 and characterise the vanishing \( y_I \)-values in Corollary 4.5.

If \( I \) has a nested up and down interval decomposition, the signed lengths \( K_Γ \) and \( K_y \) of \( I \) are integers defined as follows. \( |K_Γ| \) is the number of edges of the path in \( \partial Q \) connecting \( a \) and \( \Gamma \) that does not use the vertex labeled \( a \) and \( K_y \) is negative if and only if \( \Gamma \in D_c \). Similarly, \( |K_y| \) is the length of path in \( \partial Q \) connecting \( a \) and \( \gamma \) not using label \( b \) and \( K_y \) is negative if and only if \( \gamma \in D_c \). Equivalently, we have that \( K_y \) (respectively \( K_Γ \)) is a positive integer if and only if \( \gamma \in U_c \) (respectively \( \gamma \in U_c \)) and that \( K_y = −1 \) (respectively \( K_Γ = −1 \)) if and only if \( \gamma \in D_c \) (respectively \( \gamma \in D_c \)). We can now express the coefficients \( y_I \) of \( \mathcal{A}_{n−1}^c \) in terms of \( K_y \) and \( K_Γ \).

The following theorem is an easy consequence of Theorem 4.2.

**Theorem 4.3.**
Let \( K_Γ \) and \( K_y \) be the signed lengths of \( I \) as defined above if \( I \subseteq [n] \) has a nested up and down interval decomposition of type \((1, k)\). Then the Minkowski coefficient \( y_I \) of \( \mathcal{A}_{n−1}^c \) is

\[
y_I = \begin{cases} 
(-1)^{|I|−|a_i b|} K_y K_Γ & \text{if } I \neq \{u_s\} \subseteq U_c \text{ and } v = 1, \\
(n + 1) − K_y K_Γ & \text{if } I = \{u_s\} \subseteq U_c, \\
0 & \text{if } v \geq 2.
\end{cases}
\]

**Proof.** By Theorem 4.2, the claim is trivial if \( I \) has up and down interval decomposition of type \( v > 1 \). We therefore assume \( v = 1 \), set \( K := |R_{a_i}| \), and observe \( K_Γ := |R_{a_i}| − |R_{a_i}| \) and \( K_y := |R_{a_i}| − |R_{a_i}| \). Thus

\[
|R_{a_i}| = \begin{cases} 
K + K_y + K_Γ & \text{if } I \neq \{u_s\}, \\
K + K_y + K_Γ − 1 = n & \text{if } I = \{u_s\},
\end{cases}
\]

as well as

\[
\begin{align*}
z^c_{R_{a_i}} &= \frac{K(K + 1)}{2}, \\
z^c_{R_{a_i}} &= \frac{(K + K_Γ)(K + K_Γ + 1)}{2}, \\
z^c_{R_{a_i}} &= \frac{(K + K_y + K_y)(K + K_y + 1)}{2}, \quad \text{and} \\
z^c_{R_{a_i}} &= \frac{(K + K_y)(K + K_y + 1)}{2} − (n + 1) \quad \text{if } I = \{u_s\}.
\end{align*}
\]

A direct computation shows

\[
z^c_{R_{a_i}} − z^c_{R_{a_i}} = \frac{(K + K_Γ)(K + K_Γ + 1)}{2} = K_Γ K_y.
\]

The claim is now an immediate consequence of Theorem 4.2. \( \square \)

**Corollary 4.4.** For \( n \geq 2 \) and any choice \( D_c \sqcup U_c \), we have \( y_{[n]} = (−1)^{|U_c|} \).
Figure 9. Details for the computation of the Minkowski coefficients $y_I$ of $\mathbb{R}^3_4$ in case $D_c = \{1, 3, 4\}$ and $U_c = \{2\}$. The first line states $I$, the second line's first column pictures the non-crossing proper diagonals associated the up and down interval decomposition of $I$, while the second column gives the values for $a$, $b$, $\gamma$, and $\Gamma$ if there is only one nested component. The third line's first column illustrates the proper diagonals of $D_I \subseteq \{\delta_1, \delta_2, \delta_3, \delta_5\}$, the second column specifies their end-points. We give the value for $y_I$, where $y_{\{1,2,3,4\}} = -1$ since $|U_c| = 1$ is omitted.
Proof. The claim follows directly either from Theorem 4.2 or from Theorem 4.3. To obtain the claim from Theorem 4.2 observe that \([n] \setminus R_{s_1} = U_c\) and \(z_{R_{s_1}} - z_{R_{s_1}} - z_{R_{s_1}}^c + z_{R_{s_1}}^c = 1\). To obtain the claim from Theorem 4.3 we remark that \([n] \setminus R_{s_1} = I \setminus (a, b)_D\) and \(K_\gamma = K_\Gamma = -1\) since \(a = 0, b = n + 1, \gamma = 1, \text{ and } \Gamma = n\).

\begin{corollary}
Let \(n \geq 2\) and \(D_c \cup U_c\) be a partition induced by some Coxeter element \(c\). Then \(y_I = 0\) if and only if \(I\) has an up and down decomposition of type \((v_I, w_I)\) with \(v_I > 1\) or \(n = 3\) and \(I = U_c = \{2\}\).
\end{corollary}

\begin{proof}
Since \(K_\gamma\) and \(K_\Gamma\) are non-zero, Theorem 4.3 implies that \(y_I \neq 0\) if \(I \neq \{u_s\} \subseteq U_c\). So we assume that \(I = \{u_s\} \subseteq U_c\). It now suffices to prove that \(y_I = 0\) if and only if \(n = 3\).

If \(n = 2\) then \(I = \{u_s\} \subseteq U_c\) is impossible, so we have \(n \geq 3\). From \(R_{s_2} \cup R_{s_3} = [n]\) and \(R_{s_2} \cap R_{s_3} = \{u_s\}\) we conclude \(K_\gamma + K_\Gamma = n + 1\). On the other hand, Theorem 4.3 implies that \(y_I = 0\) is equivalent to \(K_\Gamma K_\gamma = n + 1\). By substitution we have

\[K_\Gamma^2 = (n + 1)K_\Gamma + (n + 1) = 0\]

and solving for \(K_\Gamma\) gives

\[K_\Gamma,1/2 = -\frac{(n+1)}{2} \pm \frac{\sqrt{(n+1)^2 - (n+1)}}{2}\]

Since \(K_\Gamma\) is a positive integer, we conclude that \(\sqrt{n^2 - 2n - 3}\) is a positive integer. In particular, \(n^2 - 2n - 3 = (n+1)(n-3)\) must be a square. For \(n = 3\), we conclude \(K_\Gamma = 2\), that is \(I = U_c = \{2\}\). For \(n > 3\) we derive the contradiction \((n+1) = r^2(n-3)\) or \((n-3) = r^2(n-1)\) for some positive integer \(r\).

We now illustrate Theorem 4.3 by recomputing the \(y_I\)-values for \(\text{As}_2^{s_1}\) and \(\text{As}_2^{s_2}\) mentioned in the introduction. For \(n = 3\), there are two possible partitions of \(\{1, 2, 3\}\) that correspond to the two Coxeter elements of \(\Sigma_3\): either \(D_{s_1} = \{1, 2, 3\}\) and \(U_{s_1} = \emptyset\) or \(D_{s_2} = \{1, 3\}\) and \(U_{s_2} = \{2\}\).

\begin{example}
Consider \(D_{s_1} = \{1, 2, 3\}\) and \(U_{s_1} = \emptyset\) which yields Loday’s realisation.

(1) We have \(y_I = 1\) for \(I = \{i\}\) and \(1 \leq i \leq 3\).

The up and down interval decomposition of \(\{i\}\) is \((i - 1, i + 1)_D\) and \(\gamma = \Gamma = i\). It follows that \(K_\gamma = K_\Gamma = -1\) and \(I \setminus (a, b)_D = \emptyset\). Thus \(y_I = 1\).

(2) We have \(y_I = 1\) for \(I = \{i, i + 1\}\) and \(1 \leq i \leq 2\).

Then \(I = (i - 1, i + 2)_D\), \(\gamma = \Gamma = i + 1\). It follows that \(K_\gamma = K_\Gamma = -1\) and \(I \setminus (a, b)_D = \emptyset\). Thus \(y_I = 1\).

(3) We have \(y_I = 0\) for \(I = \{1, 3\}\).

Then \(I = (0, 2)_D \cup (2, 4)_D\), so \(I\) is of type \((2, 0)\) and \(y_I = 0\) by Corollary 4.3.

(4) We have \(y_I = 1\) for \(I = \{1, 2, 3\}\).

Then \(I = (0, 4)_D\), \(\gamma = 1\) and \(\Gamma = 3\) implies \(K_\gamma = K_\Gamma = -1\) and \(I \setminus (a, b)_D = \emptyset\). Thus \(y_I = 1\).

Of course, we could also use Corollary 4.5 instead.

Altogether we have \(y_I \in \{0, 1\}\) and \(\text{As}_2^{s_1}\) is a Minkowski sum of faces of the standard simplex:

\[\text{As}_2^{s_1} = 1 \cdot \Delta_{\{1\}} + 1 \cdot \Delta_{\{2\}} + 1 \cdot \Delta_{\{3\}} + 1 \cdot \Delta_{\{1, 2\}} + 0 \cdot \Delta_{\{1, 3\}} + 1 \cdot \Delta_{\{2, 3\}} + 1 \cdot \Delta_{\{1, 2, 3\}},\]

recall Figure 2 for a visualisation of this equation of polytopes.

\begin{example}
Consider \(D_{s_2} = \{1, 3\}\) and \(U_{s_2} = \{2\}\). The associahedron \(\text{As}_2^{s_2}\) is isometric to \(\text{As}_2^{s_1}\), [2], but it is not the Minkowski sum of faces of a standard simplex as we show now.

(1) We have \(y_I = 1\) for \(I = \{\{1\}\}\) and \(I = \{3\}\).

The up and down interval decomposition is \((0, 3)_D\) and \((1, 4)_D\) respectively. Therefore we have \(\gamma = \Gamma = 1\) and \(\gamma = \Gamma = 3\) respectively. It follows \(K_\gamma = K_\Gamma = -1\) and \(I \setminus (a, b)_D = \emptyset\).

(2) We have \(y_I = 0\) for \(I = \{2\}\).

The up and down interval decomposition is \((1, 3)_D \cup [2, 2]_U\), so \(I\) is of type \((1, 1)\). We have \(\gamma = \Gamma = 2\) which implies \(K_\gamma = K_\Gamma = 2\). Since \(n = 3\), we conclude \(y_I = (3 + 1) - 2 \cdot 2 = 0\). Of course, we could have used Corollary 4.5.
(3) We have \( y_I = 2 \) for \( I = \{i, i+1\} \) and \( 1 \leq i \leq 2 \).
Then \( I = (i-1, i+2)_D \), \( \gamma = i \), and \( \Gamma = i+1 \), that is, \( K_\gamma = -1 \), \( K_\Gamma = 2 \). Moreover, \( I \setminus \{a, b\}_D = \{2\} \) if \( I = \{1, 2\} \) and \( K_\gamma = 2 \), \( K_\Gamma = -1 \), and \( I \setminus \{a, b\}_D = \{2\} \) if \( I = \{2, 3\} \).

(4) We have \( y_I = 1 \) for \( I = \{1, 3\} \).
Then \( I = (0, 4)_D \), \( \gamma = 1 \), and \( \Gamma = 3 \). It follows that \( K_\gamma = K_\Gamma = -1 \) and \( I \setminus \{a, b\}_D = \varnothing \).

(5) We have \( y_I = -1 \) for \( I = \{1, 2, 3\} \).
Then \( I = (0, 4)_D \sqcup [2, 2]_U \) with \( \gamma = 1 \) and \( \Gamma = 3 \). It follows that \( K_\gamma = K_\Gamma = -1 \) and \( I \setminus \{a, b\}_D = \{2\} \). Again, we could have used Corollary \[2\] instead.
Thus, we obtain the following Minkowski decomposition into dilated faces of the standard simplex:

\[
\mathbb{A}^2_\gamma = 1 \cdot \Delta_{\{1\}} + 0 \cdot \Delta_{\{2\}} + 1 \cdot \Delta_{\{3\}} + 2 \cdot \Delta_{\{1, 2\}} + 1 \cdot \Delta_{\{1, 3\}} + 2 \cdot \Delta_{\{2, 3\}} + (-1) \cdot \Delta_{\{1, 2, 3\}},
\]

recall that an illustration of this decomposition is given in Figure 3.

5. A Remark on Cyclohedra

We now show that Proposition \[1,2\] does not hold if we consider a polytope obtained by ‘moving some inequalities of the permutahedron past vertices’. The example is a cyclohedron which is an associahedron associated to a Coxeter group of type \( B \). A Minkowski decomposition of ‘generalised permutahedra of type \( B \)’ (similar to Proposition \[1,2\] for generalised permutahedra) is not known.

The canonical embedding of the hyperoctahedral group \( W_n \) in the symmetric group \( S_{2n} \) induces realisations \( \mathcal{C}y^c_n \) of cyclohedra (also known as Bott-Taubes polytopes or type \( B \) generalised associahedra, \[1, 7, 20\]) using realisations \( \mathbb{A}^c_{2n-1} \) for certain symmetric choices \( c \). To obtain realisations of cyclohedra, we follow \[11\] and intersect \( \mathbb{A}^c_{2n-1} \) with ‘type \( B \) hyperplanes’ \( x_i + x_{2n+1-i} = 2n-1 \) for \( 1 \leq i < n \). A 2-dimensional cyclohedron \( \mathcal{C}y^c_2 \) obtained from \( \mathbb{A}^c_1 \) (with up set \( U_c = \{2\} \)) by intersection with \( x_1 + x_4 = 5 \) is shown in Figure \[10\] (the hyperplane \( x_2 + x_3 = 5 \) is implicitly used since \( \mathbb{A}^c_2 \) is contained in \( x_1 + x_2 + x_3 + x_4 = 10 \)).

A similar construction does not yield a cyclohedron if one starts with the other associahedron of Figure \[11\] where \( U_c = \{2, 3\} \). The tight right-hand sides of this realisation of the cyclohedron are obviously the tight right-hand sides of \( \mathbb{A}^c_2 \) except \( z_{\{1, 4\}}^c = z_{\{2, 3\}}^c = 5 \). The inequalities \( x_1 + x_4 \geq 2 \) and \( x_2 + x_3 \geq 2 \) are redundant for \( \mathbb{A}^c_2 \) and altering the level sets for these inequalities from 2 (for \( \mathbb{A}^c_2 \)) to 5 (for \( \mathcal{C}y^c_2 \)) means that we move past the four vertices \( A, B, C, \) and \( D \), so the realisation of the cyclohedron is not in the deformation cone of the classical permutahedron. We now show by example that Proposition \[1,2\] does not hold in this situation. To this respect, we list the function \( z_I \) of tight right hand-sides for all inequalities of the permutahedron (that is, facet-defining or not for the cyclohedron) and its M"{o}bius inverse \( y_I \), both defined on the boolean lattice:

\[
\begin{align*}
&z_{\{1, 2, 3, 4\}} = 10, &y_{\{1, 2, 3, 4\}} = 5
\end{align*}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c}
 & z_{\{1, 2, 3\}} = 6 & z_{\{1, 2, 4\}} = 4 & z_{\{1, 3, 4\}} = 6 & z_{\{2, 3, 4\}} = 6 & z_{\{1, 2\}} = 3 & z_{\{1, 3\}} = 3 & z_{\{1, 4\}} = 5 \\
y_{\{1, 2, 3\}} = -4 & y_{\{1, 2, 4\}} = -3 & y_{\{1, 3, 4\}} = -2 & y_{\{2, 3, 4\}} = -1 & & & \\
\end{array}
\]

In other words, if Proposition \[1,2\] were true for ‘generalised permutahedra not in the deformation cone of the classical permutahedron’, then the following equation of polytopes has to hold:

\[
\mathcal{C}y^c_2 + (\Delta_2 + 4\Delta_{123} + 3\Delta_{124} + 2\Delta_{134} + \Delta_{234})
= \Delta_1 + \Delta_3 + \Delta_4 + 3\Delta_{12} + \Delta_{13} + 3\Delta_{14} + 5\Delta_{23} + \Delta_{34} + 5\Delta_{1234}.
\]
One way to see that this equation does not hold is to compute the number of vertices of the polytope on the left-hand side (27 vertices) and on the right-hand side (20 vertices) using for example polymake, [10].

6. A Proof of Theorem 4.2

This section is devoted to the proof of Theorem 4.2 under the assumption that Lemma 6.3 holds; Lemma 6.3 is proved in Section 7. The strategy to prove Theorem 4.2 is as follows.

First, we prove Proposition 6.2 which weakens Theorem 4.2 in two senses: we restrict to $I \subset [n]$ with a nested decomposition and we restrict to the situation where $D_I = \{\delta_1, \delta_2, \delta_3, \delta_4\}$, that is, where all four diagonals $\delta_i$ are proper. That the statement of Proposition 6.2 is actually the statement of Theorem 4.2 weakened by these additional assumptions follows from Corollary 6.7.

Lemma 6.3 states precisely which subsets of $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ are sets $D_I$ for some $I \subset [n]$ with a nested up and down interval decomposition. Lemma 6.4 then expresses the Minkowski coefficients $y_I$ using these sets $D_I$ if $I \subset [n]$ has a nested up and down interval decomposition and $|D_I| < 4$. Lemma 6.5 and Lemma 6.6 then imply the claim of Theorem 4.2 when $I \subset [n]$ has a nested decomposition and not all $\delta_i$ are proper. Finally, Lemma 6.8 covers the cases $I \subset [n]$ where $I$ does not have a nested decomposition and Lemma 6.9 settles $I = [n]$.

It will be convenient to rewrite Equation (1) that was obtained at the beginning of Section 4 by combination of Proposition 1.2 and Proposition 3.8:

$$y_I = \sum_{J \subseteq I} (-1)^{|I \setminus J|} \sum_{i \in [v_I]} \left( \sum_{j \in [w_i + 1]} \tilde{z}_{R_{I,j}}^c - w_i \tilde{z}_{[n]}^c \right)$$

$$= \sum_{J \subseteq I} (-1)^{|I \setminus J|} \sum_{i \in [v_I]} \left( \tilde{z}_{R_{I,j}}^c \right) + \sum_{j \in [m_I - 1]} \left( \tilde{z}_{R_{I,j}}^c - \tilde{z}_{[n]}^c \right)$$

where $m_I$ is either $w_I$ or $w_I + 1$ in order to simplify the involved sum.

Suppose now that the proper diagonal $\delta$ occurs in the right-hand side of this rewritten formula for $y_I$, that is, $\delta$ is one of the associated diagonals $\delta_{i,j}^c$ for some $J \subseteq I$. We now distinguish whether $\delta$ occurs as a single summand $\tilde{z}_{R_{I,j}}^c$ or as a compound summand $(\tilde{z}_{R_{I,j}}^c - \tilde{z}_{[n]}^c)$.

We now make the following definition.

**Definition 6.1.** Let $I \subset [n]$ be non-empty.

(1) A proper diagonal $\delta$ (associated to $J \subseteq I$) is of type $\tilde{z}_{R_{I}}^c$ (in the expression for $y_I$), if there exist an index $i \in [v_I]$ such that $\delta = \delta_{i,m_I}^c$. 
(2) A proper diagonal $\delta$ (associated to $J \subseteq I$) is of type $(z^c_{R_n} - z^c_{[n]})$ (in the expression for $y_I$), if there exist indices $i \in [v, j]$ and $j \in [m^I - 1]$ such that $\delta = \tilde{\delta}_{ij}$.

A geometric interpretation of these notions is the following. The proper diagonal $\delta$ has up and down decomposition of type $(\tilde{\delta}, \delta)$ of $R$ under which sets $R$ is of type $(\tilde{\delta}, 0)$ if additionally $\delta = \tilde{\delta}$.

Proposition 6.2.

Let $I$ be a non-empty proper subset of $[n]$ with up and down interval decomposition of type $(1, w)$ and $\mathcal{D}_I = \{\delta_1, \delta_2, \delta_3, \delta_4\}$. Then the Minkowski coefficient $y_I$ of an $P((z^c_{\mathcal{I}}))$ with the normal fan of $\mathcal{A}_{n-1}$ is given by

$$y_I = \sum_{\delta \in \mathcal{D}_I} (-1)^{|I \setminus R_\delta|} z^c_{R_\delta}.$$ 

The proof is not difficult but long and convoluted, so we first outline the proof. The goal is to simplify the rewritten Equation 1 for $y_I$ stated above. To that respect, we first study the potential contribution of a proper diagonal $\delta$ that occurs in the sum on the right-hand side. Given such a diagonal $\delta$, we study which sets $S \subseteq I$ satisfy $\delta \in \mathcal{D}_S$ in order to collect all terms that involve $z^c_{R_\delta}$. We will show that the corresponding sum vanishes often. This result is obtained by a case study that depends on the type of the up and down interval decomposition of $R_\delta$. Since the up and down interval decomposition of $R_\delta$ is of type $(1,0)$, $(1,1)$ or $(1,2)$ for any proper diagonal $\delta$, we study these cases in detail. After the necessary information is deduced for every possible diagonal $\delta$, we further simplify the formula for $y_I$ by another case study that distinguishes whether $\gamma = 1$.

Proof. By assumption, the set $I \subseteq [n]$ has an up and down interval decomposition of type $(1, w)$, that is, $I = (a, b)_D \cup \bigcup_{j=1}^{|I|} |(a_j, \beta_j)|_U$. Let $\delta$ be some diagonal $\delta_{ij}$ that occurs on the right-hand side of the equation for $y_I$. In other words, $\delta$ is a proper and non-degenerate diagonal $\delta_{ij}$ associated to the up and down interval decomposition of type $(v^\gamma, w^\gamma)$ for some $J \subseteq I$. By Example 3.5, the up and down interval decomposition of $R_j$ is either of type $(1,0)$, $(1,1)$ or $(1,2)$. A good understanding which sets $S \subseteq I$ (besides $J$) satisfy $\delta \in \mathcal{D}_S$ is essential for the simplification. The complete proof is basically a case study of these three cases.

1. $R_\delta$ has up and down decomposition of type $(1,0)$, see Figure 11.

Then $R_\delta = (\tilde{a}, \tilde{b})_D \subseteq (a, b)_D$ and we may consider $J = R_\delta \subseteq I$ as witness for the occurrence of $\delta$ in the right-hand side of Equation 1. Let $S \subseteq I$ be a set with $\delta \in \mathcal{D}_S$. Then $J = (a, b)_D$ is necessarily a nested component of type $(1,0)$ of $S$ and all other nested components are subsets of $(a, b) \cap I$ and $(\tilde{a}, \tilde{b}) \cap I$. It follows that $S \subseteq I$ satisfies $\delta \in \mathcal{D}_S$ if and only if $R_\delta \subseteq S \subseteq R_\delta \cup ((a, b) \cap I) \cup ((\tilde{a}, \tilde{b}) \cap I)$.

We now collect all terms for $z^c_{R_\delta}$ in the expression for $y_I$. Since $\delta$ is a proper diagonal, we have $z^c_{R_\delta} \neq 0$ and the resulting alternating sum vanishes if and only if there is more than one term of this type, that is, if and only if $((a, \tilde{a}) \cap I) \cup ((\tilde{b}, b) \cap I) \neq \emptyset$. If $((a, \tilde{a}) \cap I) \cup ((\tilde{b}, b) \cap I) = \emptyset$, we obtain $(-1)^{|I \setminus R_\delta|} z^c_{R_\delta}$ as contribution for $y_I$.

For later use in this proof, we note that $((a, \tilde{a}) \cap I) \cup ((\tilde{b}, b) \cap I) = \emptyset$ guarantees $\delta \in \mathcal{D}_I$. The diagonal $\delta_1$ is always of type $(1,0)$ if the up and down decomposition of $R_\delta$ is of type $(1,0)$. Similarly, we have $\delta_2 \in \mathcal{D}_I$ is of type $(1,0)$ if additionally $\Gamma \subseteq \mathcal{D}_I$, $\delta_3 \in \mathcal{D}_I$ is of type $(1,0)$ if additionally $\gamma \subseteq \mathcal{D}_I$, and $\delta_4 \in \mathcal{D}_I$ is of type $(1,0)$ if additionally $\alpha \subseteq \mathcal{D}_I$.

2. $R_\delta$ has up and down decomposition of type $(1,1)$.

In contrast to Case 1, $R_\delta \subseteq I$ is not true in general any more. We distinguish two cases, either $\delta = \{\tilde{a}, \tilde{b}\}$ with $\tilde{a} < \tilde{b} \subseteq U$ and $\tilde{b} \subseteq U$ or $\delta = \{\tilde{a}, \alpha\}$ with $\tilde{a} < \tilde{a} \subseteq \mathcal{D}_I$ and $\tilde{a} \subseteq U$.

a. $\delta = \{\tilde{a}, \tilde{b}\}$, see Figure 12.

Observe first that $R_\delta = (0, \tilde{b})_D \cup [u_1, \tilde{b}]_U$, with $\tilde{a} < \tilde{a} \subseteq b$. Since we assume that $\delta$ appears in the right-hand side of Equation 1, we have $\tilde{b} \in I$ and may consider $J = R_\delta \cap I$.


If $S \subseteq I$ is a subset with $\delta \in \mathcal{D}_S$ then $\delta$ must be the ‘rightmost’ diagonal of one nested component for $S$. This means that the diagonal $\delta$ associated to $S$ is never of type $(\tilde{z}_{R_3}^c - \tilde{z}_{R_3}^\delta)$ in the expression for $y_I$. Similarly to Case 4 we conclude that the terms $\tilde{z}_{R_3}^\delta$ cancel if and only if

$$((a, \tilde{\beta}) \cap I) \cup ((\tilde{b}, b) \cap I) \neq \emptyset \quad \text{or} \quad \tilde{z}_{R_3}^\delta = 0.$$ Again, $\tilde{z}_{R_3}^\delta \neq 0$ since $\delta$ is a proper diagonal and the terms for $\tilde{z}_{R_3}^\delta$ do not cancel if and only if there is only one subset $S \subseteq I$ with $\delta \in \mathcal{D}_S$, that is, if $((a, \tilde{\beta}) \cap I) \cup ((\tilde{b}, b) \cap I) = \emptyset$.

For later use in this proof, we mention the two possible scenarios if $((a, \tilde{\beta}) \cap I) \cup ((\tilde{b}, b) \cap I) = \emptyset$.

Firstly, if $\gamma \in U_c$ and $\Gamma \in D_c$, then $\delta \in \{\delta_3, \delta_4\}$ and the contribution of $\delta_3$ and $\delta_4$ to $y_I$ is

$$(-1)^{|\Gamma \setminus R_3^\delta|} \tilde{z}_{R_3}^{\delta_3} \quad \text{and} \quad (-1)^{|I \setminus R_3^\delta|} \tilde{z}_{R_3}^{\delta_4}.$$ Secondly, if $\gamma, \Gamma \in U_c$, then $\delta = \delta_3$ and the contribution to $y_I$ is $(-1)^{|I \setminus R_3^\delta|} \tilde{z}_{R_3}^{\delta_3}$.

**Figure 11.** Let $I = \{3, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ with $\gamma = 3$ and $\Gamma = 13$. Its up and down decomposition is $I = (2, 14)_{D_c} \cup [3, 3]_{U_c} \cup [6, 11]_{U_c}$.

The diagonal $\delta = \{5, 12\}$ appears in the right hand side for $y_I$ since $(5, 12)_{D_c} \subseteq (2, 14)_{D_c}$ and the up and down interval decomposition of $R_3$ has type $(1, 0)$. Since $(2, 5) \cap I = \{3\}$ and $(12, 14) \cap I = \{13\}$, $\delta$ is associated to $S \in \{\{8, 10\}, \{3, 8, 10\}, \{8, 10, 13\}, \{3, 8, 10, 13\}\}$, the diagonals associated to the up and down interval decompositions of $S$ form a subset of the dashed diagonals. The contribution of $\delta$ to $y_I$ vanishes.

The only diagonals associated to some $J \subseteq I$ with up and down interval decomposition of type $(1, 0)$ and non-vanishing contribution to $y_I$ are diagonals associated to only one subset $J \subseteq I$, i.e. $\delta_1 = \{2, 14\}$ and $\delta_2 = \{2, 13\}$ in this example.

**Figure 12.** Consider $I$ as in Figure 11. For $\delta = \{6, 13\}$, the up and down interval decomposition of $R_3$ is of the required sub-type of $(1, 1)$. $\delta$ is associated to $S \in \{\{3, 5, 6, 8, 10, 12\}, \{5, 6, 8, 10, 12\}, \{3, 6, 8, 10, 12\}, \{6, 8, 10, 12\}\}$ since $(2, 6) \cap I = \{3, 5\}$ and $(13, 14) \cap I = \emptyset$ and some of the diagonals are associated to the interval decomposition of $S$. The contribution of $\delta$ to $y_I$ vanishes. Diagonals of the required sub-type of $(1, 1)$ and non-vanishing contribution to $y_I$ are the diagonals $\delta$ that are associated to precisely one subset $J \subseteq I$, that is, $\delta_3 = \{3, 14\}$ and $\delta_4 = \{3, 13\}$ in this figure.
b. $\delta = \{\bar{a}, \alpha\}$

Observe first that $R_{\delta} = (\bar{a}, n + 1)_{D_c} \cup \{\bar{a}, u_i\}_{U_c}$, with $a \leq \bar{a} < \alpha$. Since we assume that $\delta$ appears in the right-hand side of (1), we have $\bar{a} \in I$ and may consider $J = R_{\delta} \cap I$.

If $S \subseteq I$ with $\delta \in D_S$, then $\delta$ (associated to $S$) can be of type $R_{\delta}^z$ or $(z_{R_{\delta}^z} - z_{[\alpha]}^z)$ in the expression for $y_I$. The diagonal $\delta$ is of type $z_{\delta}^x$ if and only if $R_{\delta} = R_{\delta} \cap I$ and $S = R_{\delta} \cup M$ for some subset $M \subseteq (a, \bar{a}) \cap I$. The diagonal $\delta$ is of type $(z_{R_{\delta}^z} - z_{[\alpha]}^z)$ for all other subsets $S \subseteq I$ with $\delta \in D_S$. In particular, we conclude $R_{\delta} \supset R_{\delta} \cap S$.

We now distinguish two sub-cases: either $\delta$ (associated to $S$) is of type $(z_{R_{\delta}^z} - z_{[\alpha]}^z)$ (in the expression for $y_I$) for all $S \subseteq I$ with $\delta \in D_S$ or there is a $S \subseteq I$ with $\delta \in D_S$ such that $\delta$ (associated to $S$) is of type $z_{\delta}^x$ (in the expression for $y_I$).

i. $\delta$ is of type $(z_{R_{\delta}^z} - z_{[\alpha]}^z)$ for all $S \subseteq I$ with $\delta \in D_S$, see Figure 13.

As mentioned, we have $R_{\delta} \supset R_{\delta} \cap S$ for all sets $S \subseteq I$ with $\delta \in S$. Moreover, these sets are in bijection to the subsets of $((a, \bar{a}) \cap I) \cup ((\bar{a}, b) \cap I)$:

$$S = (R_{\delta} \cap (I \setminus B)) \cup A$$

for $A \subseteq (a, \bar{a}) \cap I$ and $B \subseteq (\bar{a}, b) \cap I$.

If there is more than one set $S \subseteq I$ with $\delta \in D_S$, then collecting all the summands $(z_{R_{\delta}^z} - z_{[\alpha]}^z)$ in the expression for $y_I$ yields a vanishing alternating sum. If there is only one set $S \subseteq I$ with $\delta \in D_S$ as associated diagonal then $((a, \bar{a}) \cap I) \cup ((\bar{a}, b) \cap I) = \emptyset$ and it follows that $\Gamma = \bar{a} \in U_c$ and $\bar{a} \in \{\alpha, \gamma\} \cap D_c$.

For later use in this proof, we note that $\gamma \in D_c$ implies $\delta \in \{\delta_2, \delta_4\}$. The only possible contributions in the expression for $y_I$ are therefore:

$$(-1)^{|I\setminus R_{\delta_2}|}(z_{R_{\delta_2}}^x - z_{[\alpha]}^z) \quad \text{and} \quad (-1)^{|I\setminus R_{\delta_4}|}(z_{R_{\delta_4}}^x - z_{[\alpha]}^z).$$

But since the corresponding subsets $R_{\delta_2} \cap I$ and $R_{\delta_4} \cap I$ differ by $\gamma$, the effective contribution to $y_I$ is:

$$(-1)^{|I\setminus R_{\delta_2}|}z_{R_{\delta_2}}^x + (-1)^{|I\setminus R_{\delta_4}|}z_{R_{\delta_4}}^x.$$

If $\gamma \in U_c$, then $\delta = \delta_2$ and we obtain

$$(-1)^{|I\setminus R_{\delta_2}|}(z_{R_{\delta_2}}^x - z_{[\alpha]}^z)$$

as contribution for $y_I$.

**Figure 13.** Let $I = \{3, 5, 6, 7, 8, 9, 10\}$ with $\gamma = 3$ and $\Gamma = 10$. Its up and down decomposition is $I = (2, 11)_{D_c} \cup [3, 11]_{U_c} \cup [6, 10]_{U_c}$.

For $\delta = \{2, 8\}$, the up and down interval decomposition of $R_{\delta}$ is of the required sub-type of (1, 1). Since $(2, 2) \cap I = \emptyset$ and $(8, 11) \cap I = \{9, 10\}$, $\delta$ is associated to $S \in \{\{5, 7, 8\}, \{5, 7, 8, 9\}, \{5, 7, 8, 10\}, \{5, 7, 8, 9, 10\}\}$. Thus $\delta$ does not contribute to $y_I$.

The only diagonals $\delta$ associated to $J \subseteq I$ of the required sub-type of (1, 1) that contribute to $y_I$ are diagonals associated to precisely one subset $J \subseteq I$.

In this figure, only $\delta_2 = \{2, 10\}$ contributes $-(z_{R_{\delta_2}}^x - z_{[\alpha]}^z)$. 
ii. There is a $S \subseteq I$ with $\delta \in \mathcal{D}_S$ such that $\delta$ is of type $\tilde{z}_R^c$, see Figure 14.

Since $\delta$ must be the ‘rightmost’ diagonal associated to $S$ if $\delta$ (associated to $S$) is of type $\tilde{z}_R^c$ (in the expression for $y_I$), we conclude $R_\delta = R_\delta \cap I$. In particular, we have $\Gamma = n$ and $b = n + 1$ and thus $(\tilde{\alpha}, b) \cap I \neq \emptyset$ and $(\tilde{\alpha}, b) \cap I = (\tilde{\alpha}, b)$. If $(\tilde{\alpha}, \tilde{\alpha}) \cap I \neq \emptyset$, then collecting the terms for $\tilde{z}_R^c$ and $\tilde{z}_n^c$ in the expression for $y_I$ again yields no contribution. We may therefore assume $(\tilde{\alpha}, \tilde{\alpha}) \cap I = \emptyset$, that is $\tilde{\alpha} \in \{a, \gamma\} \cap \mathcal{D}_c$. First suppose that $\gamma \in \mathcal{D}_c$. Then $\delta$ is either $\delta_a = \{a, \tilde{\alpha}\}$ or $\delta = \{\gamma, \tilde{\alpha}\}$. Now $\delta$ is of type $\tilde{z}_R^c$ in the expression of $y_I$ if and only if $\delta$ is associated to $R_{\delta_a} \cup R_{\delta}$. In all other situations, $\delta$ is of type $(\tilde{z}_R^c - \tilde{z}_n^c)$ in the expression of $y_I$ and is associated to a set $R_{\delta_a} \setminus M$ or $R_{\delta} \setminus M$ with non-empty $M \subseteq (\tilde{\alpha}, n + 1)$. Collecting the terms for $\tilde{z}_{R_{\delta_a}}$, $\tilde{z}_{R_{\delta}}$, and $\tilde{z}_n^c$ yields a vanishing contribution as desired (collecting the terms for $\tilde{z}_n^c$ for fixed $\delta$ does not yield a vanishing contribution, but the terms from $\delta_a$ and $\delta_{\gamma}$ cancel). If $\gamma \in \mathcal{U}_c$ then a similar argument gives

$$(−1)^{|I \setminus R_\delta|} \tilde{z}_n^c \quad \text{for } \delta = \{a, \tilde{\alpha}\} \text{ with } \tilde{\alpha} \in \mathcal{U}_c \text{ and } R_\delta = R_\delta \cap I$$

as contribution for $y_I$.

3. $R_\delta$ has up and down decomposition of type $(1, 2)$.

If $R_\delta$ is of type $(1, 2)$ then $\delta = \{\alpha, \beta\}$ with $\alpha, \beta \in \mathcal{U}_c$ and there is $u \in \mathcal{U}_c$ such that $a < \alpha < u < \beta < b$. This in turn gives

$$R_\delta = (0, n + 1)_{\mathcal{D}_c} \cup \{u_1, \alpha\}_{\mathcal{U}_c} \cup \{\beta, u_m\}_{\mathcal{U}_c}$$

as up and down interval decomposition for $R_\delta$. By arguments as before, we conclude that collecting the terms for $\tilde{z}_R^c$ and $\tilde{z}_n^c$ yields a vanishing contribution to $y_I$ if $(\alpha, \alpha) \cap I \neq \emptyset$. We therefore assume that $(\alpha, \alpha) \cap I = \emptyset$ which is equivalent to $\gamma = \alpha \in \mathcal{U}_c$. As a consequence, $\delta$ is an associated diagonal of $S \subseteq I$ if and only if $S = (R_\delta \cap I) \setminus M$ for some $M \subseteq (\beta, b) \cap I$.

We now distinguish two cases: either there is a $S \subseteq I$ with $\delta \in \mathcal{D}_S$ such that $\delta$ (associated to $S$) is of type $\tilde{z}_R^c$ (in the expression for $y_I$) or not.

![Figure 14](image-url) Consider $I = \{3, 5, 6, 7, 8, 9, 10\}$ with $\gamma = 3$ and $\Gamma = 10$. The up and down interval decomposition is $I = \{2, 11\}_{\mathcal{D}_c} \cup \{3, 3\}_{\mathcal{U}_c} \cup \{6, 9\}_{\mathcal{U}_c}$.

For $\delta = \{2, 6\}$, the up and down interval decomposition of $R_\delta$ has the required sub-type of $(1, 1)$. We have $R_\delta = R_\delta \cap I$, so $J = \{5, 6, 7, 8, 9, 10\}$ is the unique $J \subseteq I$ such that $\delta \in \mathcal{D}_J$ is of type $\tilde{z}_R^c$ (in the expression for $y_I$). For all other $S \subseteq I$ with $\delta \in \mathcal{D}_S$, $\delta$ is of type $(\tilde{z}_R^c - \tilde{z}_n^c)$ in the expression of $y_I$. Since $\gamma \in \mathcal{U}_c$, $\delta$ contributes $(-1)^{|I \setminus R_\delta|} \tilde{z}_n^c = -\tilde{z}_n^c$ to $y_I$. We have other diagonals contributing to $y_I$. In this example (and considering only the specific sub-type) these are the proper diagonals $\delta$ with $R_\delta = R_\delta \cap I$ and end-point $a = 2$, that is, $\delta = \{2, 8\}$ and $\delta = \{2, 9\}$, their contribution is $-\tilde{z}_n^c$ and $\tilde{z}_n^c$. 
a. There is no set \( S \subseteq I \) with \( \delta \in D_S \) such that \( \delta \) (associated to \( J \)) is of type \( z_{R_\delta}^* \), see Figure 15.

If \((\beta, b) \cap I \neq \emptyset\) then collecting the terms \( z_{R_\delta}^* \) and \( z_{[n]}^* \) cancel respectively. If \((\beta, b) \cap I = \emptyset\) then we have \( \Gamma = \beta \in U_c \) and \( \delta = \delta_4 \). In this situation, \( \delta \) has a unique contribution to \( y_I \) which equals \( -(1)^{|I\setminus R_\delta|} z_{[n]}^* \).

b. There is a set \( S \subseteq I \) with \( \delta \in D_S \) such that \( \delta \) is of type \( z_{R_\delta}^* \), see Figure 16.

Since \( \delta \) is the ‘rightmost’ diagonal associated to \( S \subseteq I \) and since \((a, \alpha) \cap I = \emptyset\) we conclude that \( b = n + 1 \) and \( \Gamma = n \in D_c \) (recall that we also have \( \alpha = \gamma \in U_c \)). Now observe that the set \( S \subseteq I \) with \( \delta \in D_S \) such that \( \delta \) (associated to \( S \)) is of type \( z_{R_\delta}^* \) (in the expression for \( y_I \)) is unique: it is \( R_\delta \cap I \). In particular, we have \([\beta, n] \cap I = [\beta, n]\). Collecting the terms \( z_{R_\delta}^* \) for all subsets \( S \subseteq I \) with \( \delta \in D_S \) cancel, but collecting the terms \( z_{[n]}^* \) does not vanish: we have a contribution of \( -(1)^{|I\setminus R_\delta|} z_{[n]}^* \) to \( y_I \). We conclude that every diagonal \( \delta = \{\gamma, \beta\} \) with \( \beta \in U_c \), \([\beta, n] \cap I = [\beta, n]\) and \( \{\gamma, \beta\} \neq \{u_r, u_{r+1}\} \) contributes \( -(1)^{|I\setminus R_\delta|} z_{[n]}^* \) to \( y_I \).

**Figure 15.** Consider \( I = \{3, 5, 6, 7, 8, 9, 10, 11\} \) with \( \gamma = 3 \) and \( \Gamma = 11 \). The up and down interval decomposition is \( I = (2, 12)_{D_b} \cup [3, 3]_{U_c} \cup [6, 11]_{U_c} \).

For \( \delta = \{3, 9\} \), the up and down interval decomposition of \( R_\delta \) is of the required sub-type of \((1, 2)\). Since \((\beta, b) \cap I = (9, 12) = [10, 11] \), the diagonal \( \delta \) is not associated to a unique \( J \subseteq I \) and does not contribute to \( y_I \). Only diagonals of the required sub-type of \((1, 2)\) that are associated to a unique \( J \subseteq I \) contribute to \( y_I \). In this example, only \( \delta_4 = \{3, 11\} \) is of this type.

**Figure 16.** Consider \( I = \{3, 5, 6, 7, 8, 9, 10, 11, 12\} \) with \( \gamma = 3 \) and \( \Gamma = 12 \). The up and down interval decomposition is \( I = (2, 13)_{D_b} \cup [3, 3]_{U_c} \cup [6, 11]_{U_c} \).

For \( \delta = \{3, 9\} \), the up and down interval decomposition of \( R_\delta \) is of the required sub-type of \((1, 2)\) and since \((\beta, b) \cap I = (9, 13) \cap I = \{10, 11, 12\} \neq \emptyset\), the diagonal \( \delta \) is associated to eight sets. The contribution of \( \delta \) to \( y_I \) is \( -z_{[n]}^* \).

In this example, the four diagonals \( \delta' \in \{\{3, 6\}, \{3, 7\}, \{3, 9\}, \{3, 11\}\} \) are of the required sub-type of \((1, 2)\) each contributes \( -(1)^{|I\setminus R_{\delta'}|} z_{[n]}^* \) to \( y_I \).
After this analysis for the possible contributions to $y_I$ induced by proper diagonals, we now prove

$$y_I = \sum_{\delta \in \mathcal{D}_I} (-1)^{|I \setminus R_\delta|} \bar{z}_{R_\delta}$$

where we assume that $I$ is a non-empty proper subset of $[n]$ with a nested up and down decomposition and $|\mathcal{D}_I| = 4$. We distinguish the following four cases:

1. $\gamma, \Gamma \in D_c$

   Then $\delta_1, \delta_2, \delta_3, \delta_4$ contribute $(-1)^{|I \setminus R_\delta|} \bar{z}_{R_\delta}$ to $y_I$ according to Case 1 and no other diagonal contributes according to the previous analysis. The claim follows immediately.

2. $\gamma \in D_c$ and $\Gamma \in U_c$

   Then $\delta_1$ and $\delta_3$ do contribute $(-1)^{|I \setminus R_\delta|} \bar{z}_{R_\delta}$ to $y_I$ according to Case 1, while $\delta_2$ and $\delta_4$ contribute $(-1)^{|I \setminus R_{\delta_2}} \bar{z}_{R_{\delta_2}}$ to $y_I$ according to Case 2(b). No other diagonal contributes to $y_I$. The claim follows immediately.

3. $\gamma, \Gamma \in U_c$

   The only diagonals with a contribution to $y_I$ are $\delta_1$ (by Case 1), $\delta_2$ (by Case 2(b)i), $\delta_3$ (by Case 2a) and $\delta_4$ (by Case 2a). Taking their contribution into account, we obtain

   $$y_I = (-1)^{|I \setminus R_\delta_1|} \bar{z}_{R_\delta_1} + (-1)^{|I \setminus R_\delta_3|} \bar{z}_{R_\delta_3} + (-1)^{|I \setminus R_\delta_4|} \bar{z}_{R_\delta_4}.$$ 

   The claim follows since $I \setminus R_{\delta_2}$ and $I \setminus R_{\delta_4}$ differ by $\gamma$.

4. $\gamma \in U_c$ and $\Gamma \in D_c$

   We distinguish the two sub-cases $\Gamma \neq n$ and $\Gamma = n$.

   (a) $\Gamma \neq n$ implies that there is no $u \in U_c$ such that $[u, n] = [u, n] \cap I$.

   In this situation, $\delta_1$ and $\delta_3$ contribute $(-1)^{|I \setminus R_\delta_1|} \bar{z}_{R_\delta_1}$ to $y_I$ according to Case 1 and $\delta_2$ and $\delta_4$ contribute $(-1)^{|I \setminus R_\delta_4|} \bar{z}_{R_\delta_4}$ according to Case 2a. No other diagonal contributes, so the claim follows immediately.

   (b) $\Gamma = n$

   If there is no $u \in U_c$ such that $[u, n] = [u, n] \cap I$ then $\delta_1$ and $\delta_2$ contribute according to Case 1 and $\delta_3$ and $\delta_4$ contribute according to Case 2a. No other diagonal contributes, so the claim follows immediately.

   If there exists $u \in U_c$ such that $[u, n] = [u, n] \cap I$ then denote by $u_{\text{min}}$ the smallest element of $U_c$ such that $[u_{\text{min}}, n] = [u_{\text{min}}, n] \cap I$. Now diagonals $\delta_1$ and $\delta_2$ contribute to $y_I$ according to Case 1 and diagonals $\delta_3, \delta_4$ according to Case 2a. But in this situation, according to Cases 2(b)ii and 3a, we also have contributions of diagonals $\{u, u\}$ and $\{\gamma, u\}$ for $u \in [u_{\text{min}}, u_{\text{max}}]$. This yields

   $$\sum_{\delta \in \mathcal{D}_I} (-1)^{|I \setminus R_{\delta}} \bar{z}_{R_{\delta}} + \sum_{\delta = \{u, \alpha\} \text{ with } \alpha \in [u_{\text{min}}, u_{\text{max}}]} (-1)^{|I \setminus R_{\delta}} \bar{z}_{\delta} + \sum_{\delta = \{\gamma, \alpha\} \text{ with } \alpha \in [u_{\text{min}}, u_{\text{max}}]} (-1)^{|I \setminus R_{\delta}} \bar{z}_{\delta}.$$ 

   But the second and third sum cancel, so we end up with the claim.

In fact, the methods used in the proof of Proposition 5.2 suffice to prove the degenerate cases $\mathcal{D}_I \neq \{\delta_1, \delta_2, \delta_3, \delta_4\}$ as well. But before we try to analyse these cases, we remark that some subsets of $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ never form a set $\mathcal{D}_I$ associated to some $I \subseteq [n]$ and Coxeter element $c$.

**Lemma 6.3.**

Let $n \geq 3$ and $I \subset [n]$ be non-empty with up and down interval decomposition of type $(1, w)$. Then

(a) There is no partition $[n] = D_c \cup U_c$ induced by a Coxeter element $c$ and no non-empty $I \subset [n]$ such that $\mathcal{D}_I$ is one of the following sets:

   $\emptyset, \{\delta_2\}, \{\delta_3\}, \{\delta_4\}, \{\delta_1, \delta_2\}, \{\delta_1, \delta_3\}, \{\delta_2, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}$ or $\{\delta_3, \delta_4\}$.

(b) There is a partition $[n] = D_c \cup U_c$ induced by a Coxeter element $c$ and a non-empty $I \subset [n]$ such that $\mathcal{D}_I$ is one of the following sets:

   $\{\delta_1\}, \{\delta_1, \delta_4\}, \{\delta_2, \delta_3\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_2, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}$ or $\{\delta_1, \delta_2, \delta_3, \delta_4\}$. 

\[\square\]
Lemma 6.4.
Let $n \geq 3$ and $I \subseteq [n]$ be non-empty with up and down interval decomposition of type $(1, w)$ and $|\mathcal{P}_I| < 4$. Then

(a) Suppose that $I$ satisfies one of the following conditions

(i) $\mathcal{P}_I = \{\delta_1\}$ (Lemma 7.2),

(ii) $\mathcal{P}_I = \{\delta_1, \delta_2\}$, that is, Cases (a) and (b) of Lemma 7.3,

(iii) $\mathcal{P}_I = \{\delta_1, \delta_2, \delta_3\}$, (a, b)D = $\{\Gamma\}$, and $\gamma \in \mathbb{N}_c$, that is, Cases (b) and (c) of Lemma 7.4,

(iv) $\mathcal{P}_I = \{\delta_1, \delta_2, \delta_3\}$ and (a, b)D = $\{\gamma\}$, that is, Case (a) of Lemma 7.5,

(v) $\mathcal{P}_I = \{\delta_2, \delta_3, \delta_4\}$ and (a, b)D = $\emptyset$, that is, Lemma 7.7.

Then the Minkowski coefficient $y_I$ of $\mathcal{A}_{n-1}$ is

$$y_I = \sum_{\delta \in \mathcal{P}_I} (-1)^{|R|}z_{R\delta}.$$  

(b) Suppose that $I$ satisfies one of the following conditions

(i) $\mathcal{P}_I = \{\delta_1, \delta_4\}$, that is, Case (a) and (b) of Lemma 7.3,

(ii) $\mathcal{P}_I = \{\delta_2, \delta_4\}$, that is, Case (a) and (b) of Lemma 7.3,

(iii) $\mathcal{P}_I = \{\delta_1, \delta_2, \delta_3\}$ and $\bigcup_{i=1}^k \alpha_i, \beta_i \in \{\Gamma\}$, that is, Case (a) of Lemma 7.5,

(iv) $\mathcal{P}_I = \{\delta_1, \delta_2, \delta_3\}$ and $\bigcup_{i=1}^k \alpha_i, \beta_i = \{\gamma\}$, that is, Case (a) of Lemma 7.5,

(v) $\mathcal{P}_I = \{\delta_2, \delta_3, \delta_4\}$ and $\bigcup_{i=1}^k \alpha_i, \beta_i \in \{\Gamma, \gamma\}$, that is, Case (a) of Lemma 7.6.

Then the Minkowski coefficient $y_I$ of $\mathcal{A}_{n-1}$ is

$$y_I = (-1)^{|\{\gamma, \Gamma\}|}z_{[n]} + \sum_{\delta \in \mathcal{P}_I} (-1)^{|R|}z_{R\delta}.$$  

Proof. The proof of the claim is a study of the 14 mentioned cases that characterise the non-empty proper subsets $I \subseteq [n]$ with $\mathcal{P}_I \neq \{\delta_1, \delta_2, \delta_3, \delta_4\}$. These 14 cases are described in detail in Section 7. The proofs are along the lines of the proof of Proposition 6.2. \hfill \Box

Lemma 6.5.
For $n \geq 3$, let $I$ be non-empty proper subset of $[n]$ with up and down interval decomposition of type $(1, w)$ and $|\mathcal{P}_I| < 4$.

(a) In all cases of Part (a) of Lemma 6.4 we have $R_\delta = \emptyset$ if $\delta \in \{\delta_1, \delta_2, \delta_3, \delta_4\} \setminus \mathcal{P}_I$. Thus

$$y_I = \sum_{i=1}^4 (-1)^{|R_\delta_i|}z_{R_\delta_i}.$$  

(b) In all cases of Part (b) of Lemma 6.4 there is precisely one $\delta \in \{\delta_1, \delta_2, \delta_3, \delta_4\} \setminus \mathcal{P}_I$ with $R_\delta = [n]$:

(i) $R_\delta = [n]$ in Case (a) of Lemma 7.3 and Case (a) of Lemma 7.6 and we have

$$y_I = (-1)^{|R_\delta|}z_{R_\delta} + \sum_{\delta \in \mathcal{P}_I} (-1)^{|\delta|}z_{\delta} = \sum_{i=1}^4 (-1)^{|R_\delta_i|}z_{R_\delta_i}.$$  

(ii) $R_\delta = [n]$ in Case (a) of Lemma 7.3 and Case (a) of Lemma 7.5 and we have

$$y_I = (-1)^{|R_\delta|}z_{R_\delta} + \sum_{\delta \in \mathcal{P}_I} (-1)^{|\delta|}z_{\delta} = \sum_{i=1}^4 (-1)^{|R_\delta_i|}z_{R_\delta_i}.$$  

(iii) $R_\delta = [n]$ in both cases of Lemma 7.3 and in Case (a) of Lemma 7.4 and we have

$$y_I = (-1)^{|R_\delta|+|\{\gamma, \Gamma\}|}z_{R_\delta} + \sum_{\delta \in \mathcal{P}_I} (-1)^{|\delta|}z_{\delta}.$$  

Moreover, we have $\gamma \neq \Gamma$ except Case (a) of Lemma 7.3 when $\gamma = \Gamma \in \mathbb{N}_c$.

Proof. The first case is trivial, since we only add vanishing terms to $\sum_{\delta \in \mathcal{P}_I} (-1)^{|\delta|}z_{\delta}$. The second case is a bit more involved. First observe that $\gamma \neq \Gamma$ except for Case (a) of Lemma 7.3 when $\gamma = \Gamma \in \mathbb{N}_c$. Now, using the description given in Section 7, it is straightforward.
Lemma 6.6.
Let $I$ be non-empty subset of $[n]$ with up and down interval decomposition of type $(1, w)$. Then
\[(−1)^{|I\setminus R_1|} = (−1)^{|I\setminus R_2|} = (−1)^{|I\setminus R_3|}\]
for the first subcase, $(−1)^{|γ, Γ|} = (−1)^{|I\setminus R_4|+1}$ for the second subcase and $(−1)^{|γ, Γ|} = (−1)^{|I\setminus R_4|+|γ, Γ|}$ for the third subcase.

Proof. The claim for $δ_2$ follows from
\[R_{δ_2} ∩ I = \begin{cases} (R_{δ_2} ∩ I) ∪ \{Γ\}, & Γ ∈ U_c, \\ (R_{δ_2} ∩ I) \setminus \{Γ\} & Γ ∈ D_c. \end{cases}\]
The case for $δ_3$ is similar. For $δ_4$ we have to consider
\[R_{δ_4} ∩ I = \begin{cases} (R_{δ_4} ∩ I) ∪ \{Γ, γ\}, & γ, Γ ∈ U_c, \\ ((R_{δ_4} ∩ I) ∪ \{γ\}) \setminus \{Γ\} & γ ∈ U_c, Γ ∈ D_c, \\ ((R_{δ_4} ∩ I) ∪ \{γ\}) \setminus \{Γ\} & Γ ∈ U_c, γ ∈ D_c, \\ (R_{δ_4} ∩ I) \setminus \{Γ, γ\} & γ, Γ ∈ D_c. \end{cases}\]

We combine Proposition 6.2, Lemma 6.5, and Lemma 6.6 to obtain Theorem 4.2 if $I ⊆ [n]$ has an up and down interval decomposition of type $(1, w)$:

Corollary 6.7.
Let $I$ be non-empty proper subset of $[n]$ with up and down interval decomposition of type $(1, w)$ and $D_l ⊆ \{δ_1, δ_2, δ_3, δ_4\}$. Then
\[y_I = (−1)^{|I\setminus R_1|} \left( z_{R_{δ_1}} - z_{R_{δ_2}} - z_{R_{δ_3}} + z_{R_{δ_4}} \right). \]

The techniques to prove Proposition 6.2 also enable us to compute the Minkowski coefficient $y_I$ of $A_{n−1}^c$ if the up and down interval decomposition of $I$ is of type $(v, w)$, $v > 1$, and $I \neq [n]$.

Lemma 6.8.
Let $I$ be a non-empty proper subset of $[n]$ with up and down interval decomposition of type $(v, w)$ with $v > 1$. Then $y_I = 0$ for the Minkowski coefficient of $A_{n−1}^c$.

Proof. For every proper diagonal $δ = \{d_1, d_2\}$ with $d_1 < d_2$ that appears in the expression for $y_I$, there is a nested component $N = (a_i, b_i) \cup \bigcup_{j=1}^{n} [α_{i,j}, β_{i,j}]$ of $I$ such that $a_i ≤ d_1 < d_2 ≤ b_i$. Now $δ$ appears in the expression for $y_I$ for every set $S$ where $R_δ ∩ N ⊆ S ⊆ I$. Since $v > 1$, the diagonal $δ$ never contributes to $y_I$.

We now analyse the remaining case $I = [n]$ and consider $(0, n + 1)_D \cup [u, v, w]_U$ as up and down interval decomposition of $I$.

Lemma 6.9.
For any partition $D_c \cup U_c = [n]$ induced by some Coxeter element $c$, the Minkowski coefficient $y_{[n]}$ satisfies
\[y_{[n]} = (−1)^{|n\setminus R_1|} \left( z_{R_{δ_1}} - z_{R_{δ_2}} - z_{R_{δ_3}} + z_{R_{δ_4}} \right). \]

Proof. For $I = [n]$, we have $a = 0$, $γ = 1$, $Γ = n$, and $b = n + 1$. We associate to the up and down interval decomposition of $[n]$ precisely one diagonal that is not proper and rewrite the formula for $y_{[n]}$ as
\[y_{[n]} = z_{[n]} + \sum_{J \subseteq [n]} (−1)^{|J\setminus I|} z_J. \]

We are now interested in the contribution of proper diagonals that are associated to $J ⊆ [n]$ and distinguish four cases. To find the contributions in each case, we proceed along the lines of the proof of Proposition 6.2.
(1) $U_c \neq \varnothing$ and $D_c \neq \{1, n\}$.
Then $\mathcal{D}_{[n]} = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ and each diagonal of $\mathcal{D}_{[n]}$ contributes to $y_{[n]}$ as well as all proper diagonals $\{0, u\}$ and $\{1, u\}$ with $u \in U_c$ since $a = 0$ and $\gamma = 1$. Hence we have
\[
\sum_{\delta \in \mathcal{D}_{[n]}} (-1)^{|[n]\setminus R_{\delta}|} z^c_{R_{\delta}} + \sum_{\delta = (0, a) \in \alpha \in [u_2, u_3, u_4, u_c]} (-1)^{|[n]\setminus R_{\delta}|} z^c_{R_{\delta}} + \sum_{\delta = (1, a) \in \alpha \in [u_1, u_2, u_3, u_c]} (-1)^{|[n]\setminus R_{\delta}|} z^c_{R_{\delta}}
\]
for $\sum_{J \subseteq [n]} (-1)^{|J|} z_J$. Since $\{0, u_1\}$ is not a proper diagonal, the second and third sum do not cancel and the term $(-1)^{|[n]\setminus R_{(1, u_1)}|} z^c_{[n]}$ remains. Now, $|[n]\setminus R_{(1, u_1)}| = 1$ and
\[
\sum_{\delta \in \mathcal{D}_{[n]}} (-1)^{|[n]\setminus R_{\delta}|} z^c_{R_{\delta}} = (-1)^{|[n]\setminus R_{(1, u_1)}|} \left(z^c_{R_{d_1}} - z^c_{R_{d_2}} + z^c_{R_{d_3}} + z^c_{R_{d_4}}\right)
\]
implies the claim.

(2) $U_c = \varnothing$ and $D_c \neq \{1, n\}$.
Then $\mathcal{D}_{[n]} = \{\bar{\delta}_2, \bar{\delta}_3, \bar{\delta}_4\}$ and we have
\[
\sum_{J \subseteq [n]} (-1)^{|J|} z_J = \sum_{\delta \in \mathcal{D}_{[n]}} (-1)^{|[n]\setminus R_{\delta}|} z^c_{R_{\delta}}.
\]
The claim follows now from $R_{\bar{\delta}_1} = [n]$ and Lemma 6.6.

(3) $U_c \neq \varnothing$ and $D_c = \{1, n\}$.
We have $\mathcal{D}_{[n]} = \{\delta_1, \bar{\delta}_2, \bar{\delta}_3\}$. Now each diagonal of $\mathcal{D}_{[n]}$ and all proper diagonals $\{0, u\}$ and $\{1, u\}$ with $u \in U_c$ contribute to $y_{[n]}$ since $a = 0$ and $\gamma = 1$. Similar to the first case, a term $(-1)^{|[n]\setminus R_{(1, u_1)}|} z^c_{[n]}$ is not canceled and we obtain
\[
y_{[n]} = (-1)^{|[n]\setminus R_{\delta_1}|} \left(z^c_{R_{d_1}} - z^c_{R_{d_2}} + z^c_{R_{d_3}} + z^c_{R_{d_4}}\right).
\]
Since $z_{R_{d_4}} = z_{c} = 0$, the claim follows.

(4) $U_c = \varnothing$ and $D_c = \{1, n\}$.
We have $\mathcal{D}_{[n]} = \{\bar{\delta}_2, \bar{\delta}_3\}$, $R_{\delta_1} = [n]$ and $R_{\delta_4} = \varnothing$. Hence
\[
y_{[n]} = z_{[n]} + \sum_{\delta \in \mathcal{D}_{[n]}} (-1)^{|[n]\setminus R_{\delta}|} z^c_{R_{\delta}} = (-1)^{|[n]\setminus R_{\delta}|} \left(z^c_{R_{d_2}} - z^c_{R_{d_3}} + z^c_{R_{d_4}}\right).
\]

7. Characterisation of $\mathcal{D}_I \neq \{\delta_1, \delta_2, \delta_3, \delta_4\}$ for $I \subseteq [n]$

As stated in Lemma 6.3, not all 15 proper subsets of $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ appear as set of proper diagonals $\mathcal{D}_I$ for $I \subseteq [n]$ with up and down decomposition of type $(1, w)$ and some Coxeter element c. Certain subsets are never obtained this way, a complete list is given in Lemma 6.6. The proof that they do not appear is not difficult, for example, we can show that if $\mathcal{D}_I$ contains certain diagonal(s) then $\mathcal{D}_I$ is forced to contain certain others. In this section we discuss the second statement of Lemma 6.3 in detail and study the sets $\mathcal{D}_I$ with $|\mathcal{D}_I| < 4$ that do appear. The seven proper subset of $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ that are possible are characterised in Lemma 7.1.

Lemma 7.1. If $\mathcal{D}_I = \{\delta_1\}$, then $I = \{d_r\}$ with $1 \leq r \leq \ell$.

Proof. $\delta_1 \in \mathcal{D}_I$ implies $(a, b)_{D_{\delta_1}} \neq \varnothing$ and $\bar{\delta}_2, \bar{\delta}_3, \bar{\delta}_4 \notin \mathcal{D}_I$ imply $\gamma = \Gamma \in D_c$. 

Lemma 7.2 (compare Figure 17).
If $\mathcal{D}_I = \{\delta_1, \delta_4\}$, then either
(1) $I = \{d_1, u_1\}$ and $u_1 < d_2$, or
(2) $I = \{u_m, d_\ell\}$ and $d_{\ell-1} < u_m$. 

Proof.
Proof. δ₂, δ₃ ̸∈ ℳ implies that {a, Γ} and {γ, b} are (non-degenerate) edges of Q. In particular, neither γ, Γ ∈ D_c nor γ, Γ ∈ U_c is possible.

Firstly, suppose γ ∈ D_c and Γ ∈ U_c. Then δ₂ ̸∈ ℳ implies a = 0, Γ = u₁, and γ = d₁ = 1. Now δ₃ ̸∈ ℳ yields b = d₂ and Γ = u₁ requires u₁ < d₂ and we have shown [b].

Secondly, suppose γ ∈ U_c and Γ ∈ D_c. Then δ₃ ̸∈ ℳ implies b = n + 1, γ = uₘ, and Γ = dₜ = n. Now δ₂ ̸∈ ℳ yields a = dₜ₋₁ and γ = uₘ requires dₜ₋₁ < uₘ. This gives [b]. □

Lemma 7.3 (compare Figure 18).
If δ₁ ∈ ℳ then either
(a) I = {uₜ} with 1 ≤ s ≤ m, or
(b) I = {uₜ, uₜ₋₁} with 1 ≤ s < m.

Proof. From δ₁ ̸∈ ℳ, we obtain (a, b)D_c = Ø, thus a < γ ≤ Γ < b and γ, Γ ∈ U_c. Now δ₁ ̸∈ ℳ implies that {γ, Γ} is either degenerate or an edge of Q. This proves the claim. □

Lemma 7.4 (Compare Figure 19).
If δ₂, δ₃ ̸∈ ℳ, then either
(a) I = {dₜ₋₁, dₜ₋₁} ∪ M with 1 ≤ r < ℓ and M ⊆ [dₜ₋₁, dₜ₋₁] ∩ U_c or
(b) I = M ∪ {uₜ, uₜ₋₁} with 1 ≤ s < m and M = [uₜ, uₜ₋₁] ∩ D_c ≠ Ø.

Figure 17. Schematic illustrations the two cases of ℳ = {δ₁, δ₄} (Lemma 7.2).

Figure 18. Schematic illustrations the two cases of ℳ = {δ₂, δ₃} (Lemma 7.3).

Figure 19. Schematic illustrations the two cases of ℳ = {δ₁, δ₂, δ₃} (Lemma 7.4).
Proof. \( \delta_1 \in \mathcal{D}_I \) implies \( (a, b)_{\mathcal{D}_c} \not= \emptyset \), while \( \delta_4 \not\in \mathcal{D}_I \) implies that \( \{\gamma, \Gamma\} \) is either an edge of \( Q \) or \( \gamma = \Gamma \). Suppose first \( \gamma = \Gamma \). Then \( \gamma = \Gamma \in \mathcal{D}_c \) implies the contradiction \( \mathcal{D}_I = \{\delta_1\} \), while \( \gamma = \Gamma \in \mathcal{U}_c \) implies \( (a, b)_{\mathcal{D}_c} = \emptyset \), contradicting \( \delta_1 \not\in \mathcal{D}_I \). We therefore assume \( \gamma \not= \Gamma \) and only have to distinguish the cases \( \gamma, \Gamma \in \mathcal{D}_c \) and \( \gamma, \Gamma \in \mathcal{U}_c \), the other cases \( \gamma \in \mathcal{D}_c, \Gamma \in \mathcal{U}_c \) and \( \gamma \in \mathcal{U}_c \) and \( \Gamma \in \mathcal{D}_c \) are not possible since \( \delta_4 \not\in \mathcal{D}_I \).

Firstly, suppose \( \gamma, \Gamma \in \mathcal{D}_c \). Then \( \gamma = d_r \) and \( \Gamma = d_{r+1} \) for some \( 1 \leq r \leq \ell - 1 \), since \( \delta_4 \not\in \mathcal{D}_I \). But this implies \( I = \{d_r, d_{r+1}\} \cup ([d_r, d_{r+1}] \cap \mathcal{U}_c) \), which is the claim of (a). Secondly, suppose \( \gamma, \Gamma \in \mathcal{U}_c \). Then \( \gamma = u_s \) and \( \Gamma = u_{s+1} \) for some \( 1 \leq s \leq m - 1 \). But this implies \([u_s, u_{s+1}] \cap \mathcal{D}_c = (d_q, d_r)_{\mathcal{D}_c} \not= \emptyset \) and \( I = ([u_s, u_{s+1}] \cap \mathcal{D}_c) \cup [u_s, u_{s+1}]_{\mathcal{U}_c} \), which proves (b). \( \square \)

Lemma 7.5 is symmetric to Lemma 7.6, their proofs are along the same lines.

**Lemma 7.5** (Compare Figure 20).

If \( \mathcal{D}_I = \{\delta_1, \delta_2, \delta_3\} \), then either

(a) \( I = \{d_{r+1}, \ldots, d_{\ell}\} \cup \{u_m\} \) with \( d_r < u_m < d_{r+1} < d_{\ell} \)

(b) \( I = \{d_{s}\} \cup M \) with \( 1 < s < \ell \) and \( \emptyset \not= M \subseteq [d_r, d_{r+1}] \cap \mathcal{U}_c \)

(c) \( I = \{d_{1}\} \cup M \) with \( M \subseteq \{d_{1}, d_{2}\} \cap \mathcal{U}_c \) and \( M \not= \emptyset \)

Proof. Since \( \delta_1 \not\in \mathcal{D}_I \), we have \( (a, b)_{\mathcal{D}_c} \not= \emptyset \), that is, \( a, b \) are not consecutive numbers in \( \mathcal{D}_c \). From \( \delta_3 \not\in \mathcal{D}_I \), we deduce that \( \{\gamma, b\} \) is an edge of \( Q \) and \( \gamma, \Gamma \in \mathcal{U}_c \) is therefore impossible unless \( \gamma = \Gamma \).

Moreover, \( \delta_3 \not\in \mathcal{D}_I \) implies that \( \gamma = \Gamma \) is impossible. We now have two cases to distinguish.

Firstly, suppose \( \gamma = u_m \) and \( b = n + 1 \). Then \( \Gamma = d_r = n \) and \( \delta_2 \not\in \mathcal{D}_I \) implies \( (a, \Gamma)_{\mathcal{D}_c} \not= \emptyset \). Together with \( a = \max \{d \in \mathcal{D}_c \mid d < u_m\} \), we have \( a = d_r \) for some \( 1 \leq r \leq \ell - 2 \) with \( u_m < d_{r+1} \), and \( I = (d_r, n+1)_{\mathcal{D}_c} \cup [u_m, u_m]_{\mathcal{U}_c} \). This shows (a).

Secondly, suppose \( \gamma = d_r \) and \( b = d_{r+1} \) for some \( 1 \leq r \leq \ell - 1 \), and \( \Gamma \in (\gamma, b) \cap \mathcal{U}_c \). If \( \gamma = 1 \) then \( \delta_2 \not\in \mathcal{D}_I \) implies \( \Gamma \not= u_1 \), so we distinguish the cases \( \gamma = 1 \) and \( \gamma \not= 1 \). Suppose first that \( \gamma = d_r \) with \( r > 1 \). If \( [d_r, d_{r+1}] \cap \mathcal{U}_c \not= \emptyset \) then we immediately have the claim for every non-empty \( M \subseteq [d_r, d_{r+1}] \cap \mathcal{U}_c \). If \( [d_r, d_{r+1}] \cap \mathcal{U}_c = \emptyset \) then \( \gamma = \Gamma \in \mathcal{D}_c \), which is impossible. Thus we have shown (b). Suppose now that \( \gamma = d_1 = 1 \). Then \( a = 0, b = d_2 \), and \( \delta_2 \not\in \mathcal{D}_I \) implies \( \Gamma \in \mathcal{U}_c \setminus \{u_1\} \). This proves (c). \( \square \)

**Figure 20.** Schematic illustrations the three cases of \( \mathcal{D}_I = \{\delta_1, \delta_2, \delta_4\} \) (Lemma 7.5).
Lemma 7.6 (Compare Figure 21).
If $\mathcal{I} = \{\delta_1, \delta_3, \delta_4\}$, then either
(a) $I = \{d_1, \ldots, d_{r-1}\} \cup \{u_1\}$ with $d_1 < d_{r-1} < u_1 < d_r$, or
(b) $I = \{d_r\} \cup M$ with $1 < r < \ell$, and $M \subseteq [d_1, d_r] \cap U_c$ and $M \not= \emptyset$.

Lemma 7.7. If $\mathcal{I} = \{\delta_2, \delta_3, \delta_4\}$, then $I = \{u_s, \ldots, u_t\}$ with $s + 1 < t$ and $(u_s, u_t) \cap D_c = \emptyset$.

Proof. From $\delta_1 \not\in \mathcal{I}$, we obtain $(a, b)_{D_c} = \emptyset$, in particular, $a = d_r$ and $b = d_{r+1}$ for some $1 \leq r \leq \ell - 1$. Thus $\gamma, \Gamma \in U_c$ and because of $\delta_4 \in \mathcal{I}$ we have $\gamma = u_s$ and $\Gamma = u_t$ for some $1 \leq s < s + 1 < t \leq u_m$. But then $I = M$ for some $M \subseteq [u_s, u_t] \cup u_m$ with $u_s, u_t \in M$. \hfill \Box

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