Singularities of the closed RW metric in Regge Calculus: a generalized evolution of the 600-cell

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Abstract. An evolution scheme is developed, based on Sorkin algorithm, to evolve the most complex regular tridimensional polytope, the 600-cell. The solution of 600-cell, already studied before, is generalized by allowing a larger number of free variables. The singularities of Robertson–Walker (RW) metric are studied and a reason is given why the evolution of the 600-cell stops when its volume is still far from zero. A fit of 600-cell’s evolution with a continuous metric is studied by writing a metric generalizing Friedmann’s and including the 600-cell evolution too. The result is that the 600-cell meets a causality-breaking point of space–time. We also shortly discuss the way matter is introduced in Regge calculus.

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1. Introduction

Regge calculus is 40 years old and has been extensively studied. In its original form it applies only to an empty spacetime, and the problem of how to introduce matter has always been a major one.

A key test of Regge Calculus in presence of matter which has been studied many times in literature is the Friedmann universe of dust. The first papers on the Friedmann universe did not use simplicial complexes. The authors made use of non simplicial blocks for which they had to give some other information besides the lengths of the edges. This information was deduced by symmetry. Collins and Williams (1973) used three tridimensional regular polytopes to describe the spatial sections with topology $S^3$ of the Friedmann universe of dust. These authors assumed $\rho$, the dust density, to be constant between two spatial slices. Brewin (1987) wrote in the discrete formalism of Regge Calculus the expression of the action. His expression is equivalent to the one that would result by assuming a continuous density $\rho \propto 1/R^3$.

Besides the regular polytopes, Brewin also used other non regular tridimensional polytopes. The evolution of all his polytopes, using Regge equations, stopped before the volume of spatial 3-section became zero: a point was reached where the equations had no solution. He saw that this point approaches the limit value (spatial section of null radius) as the polytope approaches the 3-sphere of the continuum solution. To overcome the obstacle, Brewin continued the evolution reverting to differential equations.

The first authors who used a 4-dimensional simplicial complex for the Friedmann universe of dust were Barrett et al (1997). They evolved a spatial section made of a regular 3-polytope having $n$ vertices, and “discretized” matter by placing one mass point on each timelike edge joining a vertex $i$ of the polytope with its evolute $i'$ of the next spatial section. Thus they found the action:

$$S = \frac{1}{8\pi} \sum_b \varepsilon_b A_b - \frac{M}{n} \sum_i l_{ii'},$$

where the first sum is over the bones $b$ of the manifold and $\varepsilon_b$ is the defect of the bone $b$ with area $A_b$. The second sum is done over every vertex $i$ of the first spatial section. For a regular polytope having $n = 120$, a mass $M/120$ is put in each vertex $i$.

The Regge equations are found by extremizing the action (1):

$$\sum_k \varepsilon_{ijk} \frac{\partial A_{ijk}}{\partial l_{ij}} = \frac{\pi}{15} M \delta_{ij},$$

where the sum is done over all vertices $k$ joined to the edge $[ij]$ of length $l_{ij}$. The bone $[ijk]$ has $\varepsilon_{ijk}$ as defect and $A_{ijk}$ as area. They made use of the evolutive scheme found by Sorkin (1975) by assuming that the vertices of the 600-cell could be subdivided in four classes each composed by vertices not joined together.

We have found that the vertices of the 600-cell actually belong to five classes and have investigated if this difference might lead to different results.
All evolutive schemes described above exhibit a common behaviour: before reaching the “true” singularity of Friedmann universe, the evolution stops because Regge equations have no solution. It is not clear if this happens because of the discretization intrinsic in Regge calculus, or for other reasons (see section 5). We decided to investigate this point in greater detail.

This paper is organized as follows. In section 2 we sketch the results of our calculations. In section 3 we embed the closed Robertson–Walker metric into a five-dimensional Lorentzian space. We find it useful in order to understand the problems related to the evolution of our lattice. The nature of the singularities belonging to the closed Robertson–Walker metric is discussed in section 4. In section 5 we propose a deeper reason why our simplicial approximation of the Friedmann universe ceases to exist before the volume of the spatial 3-section vanishes. By fitting the simplicial solution with a continuous metric (which has the same behaviour with respect to the reaching of singularities) we find that the stop condition is caused by a novel singularity of the metric (section 6). Finally, our conclusions are reported in section 7.

2. Results of the evolution

The vertices of the 600-cell belong to five classes each including only vertices not joined together. Using the classification given by Coxeter (1973), one of these five classes is composed by the following vertices:

$$\alpha = \{A_0, A_{10}, A_{20}, A_{30}, A_{40}, A_{50}, B_7, B_{17}, B_{27}, B_{37}, B_{47}, B_{57},$$

$$C_1, C_{11}, C_{21}, C_{31}, C_{41}, C_{51}, D_8, D_{18}, D_{28}, D_{38}, D_{48}, D_{58}\}.$$ 

The other classes $\beta, \gamma, \delta, \varepsilon$ are obtained by adding 2, 4, 6, 8 to the subscripts.

It should be noted that every evolution consists in solving 5 systems of 4 equations in 4 unknowns each, given the lapse and shift freedom. Furthermore the simplicial sandwich does not have the same grade of symmetry as the initial spatial section, where all vertices were equivalent. So the Sorkin evolutive algorithm we used breaks the symmetry among the vertices $\alpha, \beta, \gamma, \delta, \varepsilon$.

We have generalized the evolution of the 600-cell by imposing that the edges of the same type, like $[\alpha \beta]$, were all equal. In this way only four unknowns and four equations remained when each vertex was evolved. By allowing the edges joining vertices belonging to two different classes to be possibly different one from another, we introduce new degrees of freedom in the lattice. We have verified that this enhancement of freedom causes no problem in the evolution.

We performed the evolution of a 600-cell starting from the instant of time symmetry. Our results are in figure 1. The evolution stops at a finite value $R_m$ of the radius $R$ (the radius of the three-sphere equivalent to the 600-cell) which is approximately 1/4 of the initial value ($R_0 = 4.244\ldots$). For $R < R_m$ the Regge equations give no solution. Further numerical details may be found in De Felice and Fabri (2000).
3. The embedding 5-space

We shall find convenient to embed a space–time with closed Robertson–Walker metric into a 5-dimensional Lorentz space. This will be of help for understanding the behaviour of the Regge equations when applied to the evolution of a 600-cell space section.

The metric under consideration can be written in this way:

\[ ds^2 = -dt^2 + R^2 \left[ d\chi^2 + \sin^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right) \right] , \]  

(2)

where \( R \) depends only on the universal time \( t \).

The manifold with metric (2) can be embedded into a 5-dimensional space endowed with the \((1+4)\) metric

\[ d\sigma^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2 . \]  

(3)

In the following we will refer to \( v \) as “outer time.”

The metric (3) can be rewritten using coordinates \( v, R, \chi, \vartheta, \varphi \) by the substitution

\[
\begin{align*}
  w &= R \cos \chi \\
  x &= R \sin \chi \sin \vartheta \cos \varphi \\
  y &= R \sin \chi \sin \vartheta \sin \varphi \\
  z &= R \sin \chi \cos \vartheta .
\end{align*}
\]  

(4)

Then

\[ d\sigma^2 = -dv^2 + dR^2 + R^2 \left[ d\chi^2 + \sin^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right) \right] . \]  

(5)

A 4-dimensional submanifold of (5) can be assigned by imposing that \( R \) is a function only of the outer time \( v \): then we have (\( \dot{R} \equiv dR/dv \))

\[ ds^2 = - \left( 1 - \dot{R}^2 \right) dv^2 + R^2 \left[ d\chi^2 + \sin^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right) \right] . \]  

(6)

In order that equations (2) and (6) may coincide it is necessary that

\[ dt^2 = \left( 1 - \dot{R}^2 \right) dv^2 , \]  

(7)

from which it follows that \( \dot{R}^2 < 1 \). We are interested in the case where \( R(v) \) is an even function, so that \( v = 0 \) is an instant of time symmetry. Integrating equation (7), \( t \) can be found as a function of \( v \).
4. Singularities

In order to study the singularities of metric (6) let us compute the quadratic Riemann invariant:

\[ Q = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{12}{R^2 (1 - \dot{R}^2)^2} \left[ \frac{\ddot{R}^2}{(1 - \dot{R}^2)^2} + \frac{1}{R^2} \right]. \]  

(8)

We will assume as a necessary and sufficient condition for a singularity that \( Q \) becomes infinite. A glance to equation (8) shows that the singular points are those values of \( v \) where \( |\dot{R}| = 1 \) or \( R = 0 \) or \( |\ddot{R}| = +\infty \). Let us discuss the possible cases.

1. *The points \( R = 0 \).* These coincide with the infinite contraction of the universe when the volume of the spatial section goes to 0. In these points the density reaches infinite values. By a volume-vanishing (VV) singularity we will mean one of these points.

2. *The points \( \dot{R} = \pm 1 \).* If we are not in the first case we can say that the radius of the space section increases or decreases with speed tending to that of light. If one tries to extend the manifold beyond these points, and if \( |\dot{R}| \) increases further, the metric becomes positive defined. So it can no longer represent a manifold of General Relativity. This is tantamount to saying that the spacelike sections are not causally connected: the points of two different spatial sections can be joined only by spacelike geodesics. These points will be said to be causality-breaking (CB) singularities. Actually, in a rigorous sense a CB singularity is a border of the manifold, and the extension is meaningless.

3. *The points \( \ddot{R} = \pm \infty \).* If we are not in the previous two cases the singular behaviour is shown by the geodesics deviation, which becomes infinite. We will refer to these points as geodesics-deviation (GD) singularities.

It should be noted that in general the GD points do not represent a singularity according to the definition used in the context of Hawking’s theorem, see for example Wald (1984), where the presence of a singularity means that the universe is geodesically incomplete. An example of a GD point which does not stop any geodesic is the point \( v = 0 \) for the universe having \( R(v) = R_0 + A v^{4/3} \).

These singularities can also occur together in the same point: an example is the hypothetical universe whose radius has the form \( R(v) = v - A v^{3/2} \) (the critical point \( v = 0 \) is a VV-CB-GD singularity). Another interesting case of singularity merging is the Friedmann metric discussed in subsection 4.2 below.

4.1. Matter-limited singularities

From Einstein equations for the closed Robertson–Walker metric we have for the diagonal components of \( T_{\alpha\beta} \) in an orthonormal basis:
\[
\begin{align*}
\rho &= \frac{3}{8\pi} \frac{1}{R^2(1 - \dot{R}^2)} \\
p &= -\frac{1}{8\pi} \frac{2R\dot{R} + 1 - \dot{R}^2}{R^2(1 - \dot{R}^2)^2}.
\end{align*}
\] (9)

It is well known that the stress-energy tensor must satisfy
\[
\rho \geq |p|.
\]

In our case
\[
-2 \leq \frac{R\ddot{R}}{1 - \dot{R}^2} \leq 1.
\] (10)

In general it can happen that condition (10) be not satisfied for values of \(v\) for which the metric still has a “geometrical sense.” In fact a CB or GD or CB-GD singularity does not satisfy equation (10). Instead a VV point presents no problem. It should be noted that we cannot become aware of these new “physical” matter-limited (ML) points, wherein \(\rho = |p|\), by looking only at the expression of the metric.

4.2. Friedmann metric

Since \(dt = R \, d\eta\), from equation (7) we obtain
\[
v = \int \left[ R^2 - (dR/d\eta)^2 \right]^{1/2} d\eta.
\]

For the Friedmann metric we have:
\[
v = 2R_0 \sin \frac{\eta}{2}
\]
and
\[
R = R_0 \cos^2 \frac{\eta}{2} = R_0 - \frac{v^2}{4R_0}.
\]

Metric (6) takes the following form:
\[
ds^2 = -\frac{R}{R_0} \, dv^2 + R^2 \left[ d\chi^2 + \sin^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right) \right].
\]

Calling \(v_c\) the value of \(v\) where \(\ddot{R} = -1\) and \(v_\Omega\) the positive value where \(R = 0\), for the Friedmann metric these two singularities occur in the same point, i.e. \(v_\Omega = v_c = 2R_0\). So, in the 5-space, in correspondence of the big crunch the speed of contraction equals the speed of light, i.e. the big crunch is a VV-CB point. This behaviour is shared by all matter satisfying \(p = w \, \rho\), where \(w\) is a constant, \(-1/3 < w \leq 1\), since then \(\ddot{R}^2 = 1 - A^2 R^{1+3w}\), \(A\) a constant. Seeing that \(\ddot{R} = -\frac{1}{2} A^2 (1 + 3w) R^{3w}\), in Zel’dovich’s interval, \(0 \leq w \leq 1\), no GD singularity is present besides the other two.
5. The problem of the 600-cell

What is the nature of the critical point for the 600-cell model? Does it reach a singularity, and in this case which of them? Brewin (1987) gives a partial answer to these questions by noting that the following relation occurs:

\[ \frac{\Delta l}{\tau} \approx 1, \]

where \( \tau \) is the interval of universal time between two consecutive spatial sections and \( \Delta l \) is the difference of the lengths of their edges. The collapse becomes so fast that the vertical edges are to become spacelike. Actually in our model the above fraction exceeds 1 and reaches a not easily interpretable value (approximately 2.6). Barrett et al (1997) saw the position of the stop point is quite independent of the timelike interval between two consecutive spatial sections and advanced the idea that the evolution should be continued by making spacelike the edges of the lattice which were timelike before the stop. However they did not pursue the idea. So what is the true meaning of the above condition, and how can we continue the iterations (if possible)?

In order to answer all these questions let us now reconsider equation (7). For each evolutive step we know \( \Delta t (= \tau) \) and \( \Delta R \). Then we can find an approximate value of \( \Delta v \) by calculating \( \Delta v = (\Delta t^2 + \Delta R^2)^{1/2} \) and compute \( R \) as a function of \( v \) (figure 2). We also give \( \Delta R/\Delta v \approx dR/dv \) as a function of \( R \) in figure 3. We can see that this derivative in the last iterations tends, in absolute value, to the unity. In this sense the 600-cell meets a CB singularity before its volume vanishes. So we expect that for the 600-cell \( v_c < v_\Omega \). This leads us to believe that the evolution cannot be continued beyond this point.

We can get a clearer understanding of the situation examining a case easier to be studied in detail. Consider the following partial differential equation:

\[ \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = -t - x t - \frac{1}{2} x^2, \quad \text{with} \quad y(x, 0) = \frac{1}{2}. \]  

(11)

Its solution is

\[ y = \frac{1}{2} - \frac{1}{2} t^2 - \frac{1}{2} t x^2. \]  

(12)

We can see from equation (12) that the relation \( y = 0 \) is satisfied along two curves. There are also two curves (actually two parabolas) where \( \partial y/\partial t = \pm 1 \). These four curves are tangent two by two in two points, \( x = 0 \) and \( t = \pm 1 \), wherein both conditions hold.

Let us try to solving equation (11) by using a numerical approximation. If we use Lax method (see Press et al (1992)), taking \( h \) as the spatial \( x \)-step and \( k \) the \( t \)-step (\( h > k \) for the Courant condition), the above differential equation becomes

\[ \frac{y_{m+1,n} - y_{m-1,n}}{2h} + \frac{2y_{m,n+1} - (y_{m+1,n} + y_{m-1,n})}{2k} = -t_n - x_m t_n - \frac{1}{2} x_m^2. \]
The solution can be easily worked out and we can see that the new four curves are not tangent any longer. Furthermore the region where \(|\partial y/\partial t| \leq 1\) is strictly included into the region \(y \geq 0\). So the two points in \(x = 0\) are splitted and for every \(x\) the solution starts and ends in two “CB points” (\(\partial y/\partial t = \pm 1\)). This discrepancy vanishes as \(h\), the \(x\)-step, reaches smaller and smaller values.

The above argument shows that our 600-cell model does not stop because it is going to reach the singularity \(R = 0\) —a situation which, like in all numerical methods, could give rise to problems. Instead our model stops because it fits a universe that reaches another kind of singularity (\(\dot{R} = -1\))

6. Study of a “new” class of solutions of Einstein equations

We have just seen that the behaviour of the evolution for the 600-cell looks somewhat different from the Friedmann solution, in its reaching a CB singularity before the vanishing of \(R\). So it appears expedient to study a generalisation of Friedmann metric, also having that property. For the 600-cell a good empirical fit of \(R(v)\) is given by

\[
R(v) = R_0 - \frac{a^2 v^2}{4R_0}, \quad \text{with} \quad a^2 \approx 1.128,
\]

whereas \(a = 1\) is the Friedmann metric.

It is useful to re-define

\[
v_c \overset{\text{def}}{=} \frac{2R_0}{a^2},
\]

\[
v_\Omega \overset{\text{def}}{=} \frac{2R_0}{a} = a v_c,
\]

so

\[
R = \frac{1}{2v_c} (v_\Omega^2 - v^2)
\]

\[
\dot{R} = -\frac{v}{v_c}
\]

\[
\ddot{R} = -\frac{1}{v_c}.
\]

- For \(0 < a < 1\) we have that \(-v_\Omega < v < v_\Omega\) and the singularity is of VV-type.
- For \(a > 1\) we have that \(-v_c < v < v_c\) and the singularity is of CB-type. This is the case our model fits.
- For \(a = 1\) the solution is the Friedmann metric. So this case is a watershed between two different behaviours of the general metric.

It is not difficult to find the expression of \(t\) as a function of \(v\):

\[
t = \frac{1}{2} v_c \arcsin(v/v_c) + \frac{1}{2} v \left[1 - (v/v_c)^2\right]^{1/2}
\]

and to show that the graph of \(R(t)\) is still a cycloid, like for Friedmann metric, but scaled in \(R\) by a factor \(1/a^2\) and translated upwards by \(\frac{1}{2} v_c (a^2 - 1)\).
6.1. Behaviour of the metric

Condition (10) implies that the following inequality must be satisfied:

\[ v^2 \leq \tilde{v}^2 \overset{\text{def}}{=} v_c^2 - \frac{1}{3} v_c^2 \left| a^2 - 1 \right|. \]

If \( a < 1 \) this condition causes no problem because \( \tilde{v}^2 > v_\Omega^2 \). If \( a > 1 \) the ML point \( \tilde{v} \) is the first one the metric meets in its evolution. In fact

\[ \tilde{v}^2 = v_c^2 - \frac{1}{3} v_c^2 (a^2 - 1) < v_c^2. \]

Note that in order to have \( \tilde{v}^2 \geq 0 \) the value of \( a^2 \) must not exceed 4. So the conditions for the stress-energy tensor reduce the interval of definition of \( a \) to \( 0 < a \leq 2 \). If \( a < 1 \), the metric ends in a VV singularity reached the more softly the smaller the value of \( a \). If \( a > 1 \), the metric reaches a ML point before the CB one. The Friedmann metric \( (a = 1) \) is more “pathological” because its VV singularity is also a CB point, since \( v_\Omega = v_c \).

For this metric it is also possible in principle to write down an equation of state of matter

\[ \varrho = \frac{3}{2} \left[ p^2 + \frac{4}{\sqrt{2\pi}} v_c \left( \frac{p}{a^2 - 1} \right)^{3/2} \right]^{1/2} - \frac{3}{2} p, \]

but of course we do not attach any physical meaning to this result.

7. Conclusions

We have seen that the evolution of the 600-cell does not describe the Friedmann universe well. Instead we can think of it as the evolution of a different type of matter. But where does such a difference come from? In our opinion, one should consider the following three points as playing an important role in answering this question.

First, the 600-cell is a coarse approximation to a 3-sphere. For every approximating method, Regge Calculus included, the smaller the step interval the better the fit. When this interval reduces, the two solutions (numerical and analytical) tend to coincide. In Regge Calculus, in order to get a better approximation, one should increase the number of tetrahedra of each space section, i.e. the 600-cell should be substituted by another non-regular polytope nearer to a 3-sphere (Brewin (1987)).

The second aspect is deeper since it is closely related to the grounds of the Regge Calculus itself. We have seen that Regge equations give rise to a different evolution of the universe, ending in a CB point. This happens because they are only an approximation of Einstein equations.

Let us discuss this point by analogy with a related topic. In solving differential equations it is usual that a numerical method works better than another. For example let us take the following equation

\[ \frac{dx}{dt} = f(x, t) \]
and try to find a solution. If we used Euler’s method we should write
\[ \frac{x_{k+1} - x_k}{h} \approx \frac{dx}{dt}(t_k) + \frac{1}{2} h \frac{d^2 x}{dt^2}(t_k) + \ldots = \frac{dx}{dt}(t_k) + O(h), \]
whereas, using the central-differences method, we would obtain
\[ \frac{x_{k+1} - x_{k-1}}{2h} \approx \frac{dx}{dt}(t_k) + \frac{1}{6} h^2 \frac{d^3 x}{dt^3}(t_k) + \ldots = \frac{dx}{dt}(t_k) + O(h^2). \]
So, even if both methods lead to the correct solution when \( h \) approaches to 0, the central-differences method is generally more accurate and converges more quickly than Euler’s. According to the approximating method we choose, we have a smaller or greater truncation error. Furthermore, given that an approximate value for \( \frac{dx}{dt} \) implies the presence of other additional derivative terms of order greater than one, solving a differential equation through a numerical method is more like to solve another differential equation, which, as we have seen in section 5, of course has different solutions which may behave qualitatively different from the one we are looking for: for instance, the approximate solution may exhibit singularities not belonging to the correct solution.

The third point is related to the way matter is taken into account. We followed Barrett et al. (1997) by putting each mass point along the vertical edges, i.e. the “rest” geodesics that pass through the vertices of the simplicial complex. This appears to be the only easy way to introduce matter in a simplicial complex; so doing, however, a further discretization is introduced, not logically related to the original Regge idea. Alternatively, the mass points could be placed along other “rest” geodesics; then the proper lengths would be a function not only of \( \tau \) but also of other edges belonging to the region between two spatial sections. In this case we should obtain a novel expression for the action and of course new Regge equations which might behave differently. Furthermore, we could put more than one single mass point in each tetrahedron, or even think of tetrahedra uniformly filled with dust. It is not clear how this change could reflect on the behaviour of the model.

The undesired halting of the iteration scheme appears to be a property of all approaches attempted thus far to the Friedmann universe of dust, irrespective of their different treatment of matter. It is then reasonable to ascribe such behaviour either to the spatial discretization or to the Regge equations themselves, as explained before. Our opinion — admittedly not entirely justifiable — inclines towards the latter.

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Figure 1. Evolution of the radius $R$ as a function of the universal time $t$ through Regge Calculus and General Relativity
Figure 2. The behaviour of $R$ as a function of $v$. 
Figure 3. The behaviour of $|\Delta R/\Delta v| \approx |\dot{R}|$ as a function of $v$ for the 600-cell