GIBBS FIELDS: UNIQUENESS AND DECAY OF CORRELATIONS. REVISITING DOBRUSHIN AND PECHERSKY

DIANA CONACHE, YURI KONDRATIEV, YURI KOZITSKY, AND TANJA PASUREK

Abstract. We give a detailed and refined proof of the Dobrushin-Pechersky uniqueness criterion extended to the case of Gibbs fields on general graphs and single-spin spaces, which in particular need not be locally compact. The exponential decay of correlations under the uniqueness condition has also been established.

1. Introduction

A random field on a countable set $L$ is a collection of random variables — spins, indexed by $\ell \in L$. Each variable is defined on some probability space and takes values in the corresponding single-spin space $\Xi_\ell$. Typically, it is assumed that each $\Xi_\ell$ is a copy of a Polish space $\Xi$. In a ‘canonical version’, the probability space is $(\Xi^L, B(\Xi^L), \mu)$, where $\mu$ is a probability measure on the Borel $\sigma$-field $B(\Xi^L)$. Then also $\mu$ is referred to as random field. A particular case of such a field is the product measure of some single-spin probability measures $\sigma_\ell$. Gibbs random fields with pair interactions are constructed as ‘perturbations’ of the product measure $\otimes_{\ell \in L} \sigma_\ell$ by the ‘densities’

$$\exp \left( \sum_{\ell \neq \ell'} W_{\ell \ell'} (\xi_\ell, \xi_{\ell'}) \right)$$

where $W_{\ell \ell'} : \Xi \times \Xi \to \mathbb{R}$ are measurable functions — interaction potentials, whereas the sum is taken over the set $E \subset L \times L$ such that $W_{\ell \ell'} \neq 0$ for $(\ell, \ell') \in E$. The latter condition defines the underlying graph $G = (L, E)$. For bounded potentials, the perturbed measures usually exist. Moreover, there is only one such measure if the potentials are small enough and the underlying graph is enough ‘regular’. If the potentials are unbounded, both the existence and uniqueness issues turn into serious problems of the theory. Starting from the first results in constructing Gibbs fields with ‘unbounded spins’ [14], attempts to elaborating tools for proving their uniqueness were being undertaken [5, 8, 17]. However, except for the results of [17] obtained for the potentials and single-spin measures of a special type, and also for

This work was financially supported by the DFG through SFB 701: Spektrale Strukturen und Topologische Methoden in der Mathematik and by the European Commission under the project STREVCMS PIRSES-2013-612669. D. Conache also thanks the support of the IRTG (IGK) 1132 “Stochastics and Real World Models”, Universität Bielefeld.
methods applicable to ‘attractive’ potentials, see [11, 18, 22], there is only one work presenting a kind of general approach to this problem. This work is due to R. L. Dobrushin and E. A. Pechersky [8], which was first published in Russian and later on translated to English. In spite of its great importance, the work remains almost unknown (it has been cited only few times) presumably for the following reasons: (i) the English translation in [8] was made with numerous typos and errors of mathematical nature, whereas the Russian version is inaccessible for the most of the readers; (ii) most of the proofs in [8] are very involved and complex, and essential parts of them are only sketched or even missing. In the present publication, we give a refined and complete proof of the Dobrushin-Pechersky result extended in the following directions: (a) we do not employ the compactness arguments crucially used in [8]; (b) we settle (in Proposition 2.3 below) the measurability issues not even discussed in [8]; (c) instead of the cubic lattice \( \mathbb{Z}^d \) we consider general graphs as underlying sets of the Gibbs fields. The refinement consists, among others, in explicitly calculating the threshold value of \( K \) in (2.14) and the constants in the basic estimates in Lemma 3.7. Due to (a), as the single-spin spaces \( \Xi \) one can consider just standard Borel spaces, e.g., infinite dimensional spaces which are not locally compact, see [11, 18]. Due to (c), one can apply the criterion to various models employing graphs as underlying sets. One can also apply the criterion to the equilibrium states of continuum particle systems, see [22], Chapter 4, and Section 2.3 below.

The structure of this paper is as follows. In Section 2, we give necessary preliminaries and formulate the results in Theorems 2.6 and 2.7. Section 3 contains the proof of these theorems based on the estimates obtained in Lemmas 3.7 and 3.8 respectively, as well as on a number of other facts proved therein. In Section 4, we perform detailed constructions yielding the proof of the mentioned lemmas.

2. Setup and the Result

2.1. Notations and preliminaries. The underlying set for the spin configurations of our model is a countable simple connected graph \((L, E)\). For a vertex \( \ell \in L \), by \( \partial \ell \) we denote the neighborhood of \( \ell \), i.e., the set of vertices adjacent to \( \ell \). The vertex degree \( \Delta_\ell \) is then the cardinality of \( \partial \ell \). The only assumption regarding the graph is that

\[
\sup_{\ell \in L} \Delta_\ell =: \Delta < \infty, \tag{2.1}
\]

i.e., the vertex degree is globally bounded. A given \( V \subset L \) is said to be an independent set of vertices if

\[
\forall \ell \in V \quad \partial \ell \cap V = \emptyset. \tag{2.2}
\]
The chromatic number $\chi \in \mathbb{N}$ is the smallest number such that
\[
L = \bigcup_{j=0}^{\chi - 1} V_j, \quad V_j \text{ - independent, } j = 0, \ldots, \chi - 1.
\] (2.3)

Obviously, $\chi \leq \Delta + 1$. However, by Brook’s theorem, see, e.g., [16], for our graph we have that $\chi \leq \Delta$.

For a measurable space $(E, \mathcal{E})$, by $\mathcal{P}(E)$ we denote the set of all probability measures on $\mathcal{E}$. All measurable spaces we deal with in this article are standard Borel spaces. The prototype example is a Polish space endowed with the corresponding Borel probability measures on $C$. All measurable spaces we deal with in this article are standard Borel spaces. The prototype example is a Polish space endowed with the corresponding Borel $\sigma$-field. For $\sigma \in \mathcal{P}(E)$ and a suitable function $f : E \to \mathbb{R}$, we write
\[
\sigma(f) = \int_E f \, d\sigma.
\]

For our model, the single-spin spaces $(\Xi_\ell, \mathcal{B}(\Xi_\ell))$, $\ell \in L$, are copies of a standard Borel space $(\Xi, \mathcal{B}(\Xi))$. Then the configuration space $X = \Xi^L$ equipped with the product $\sigma$-field $\mathcal{B}(X) = \mathcal{B}(\Xi^L)$ is also a standard Borel space. Likewise, for a nonempty $D \subseteq L$, $\Xi^D$ is the product of $\Xi_\ell$, $\ell \in D$. Its elements are denoted by $x_D = (x_\ell)_{\ell \in D}$, whereas the elements of $X$ are written simply as $x = (x_\ell)_{\ell \in L}$. For $y, z \in X$, by $y_D \times z_D$ we denote the configuration $x \in X$ such that $x_D = y_D$ and $x_D^c = z_D^c$, $D^c := L \setminus D$. For $D \subseteq L$, we denote $\mathcal{F}_D = \mathcal{B}(\Xi^D)$ and write $\mathcal{F}_\ell$ if $D = \{\ell\}$.

**Definition 2.1.** Given $\ell \in L$, let $\pi_\ell := \{\pi_\ell^x : x \in X\} \subset \mathcal{P}(\Xi_\ell)$ be such that the map $X \ni x \mapsto \pi_\ell^x(A) \in \mathbb{R}$ is $\mathcal{F}_\ell$-measurable for each $A \in \mathcal{B}(\Xi_\ell)$. A family $\pi = \{\pi_\ell\}_{\ell \in L}$ of the maps of this kind is said to be a one-site specification.

**Definition 2.2.** A given $\mu \in \mathcal{P}(X)$ is said to be consistent with a one-site specification $\pi$ in a given $D \subseteq L$ if $\mu(\cdot | \mathcal{F}_\ell) = \pi_\ell^x$ for $\mu$-almost all $x$ and each $\ell \in D$. By $\mathcal{M}_D(\pi)$ we denote the set of all $\mu \in \mathcal{P}(X)$ consistent with $\pi$ in $D$. We say that $\mu$ is consistent with $\pi$ if it is consistent in $L$, and write just $\mathcal{M}(\pi)$ in this case.

Obviously, $\mu \in \mathcal{M}(\pi)$ if and only if it satisfies the following equation
\[
\mu(A) = \int_X \int_X \Pi_A(x) \pi_\ell^x(dx_\ell) \prod_{\ell \neq \ell} \delta_{y_{\ell'}}(dx_{\ell'}) \mu(dy) \quad (2.4)
\]
\[
= \int_X \left( \int_{\Xi} \Pi_A(\xi \times y_{\ell c}) \pi_\ell^y(d\xi) \right) \mu(dy),
\]
which holds for every $\ell \in L$ and $A \in \mathcal{B}(X)$. Here, for $\eta \in \Xi$, $\delta_\eta \in \mathcal{P}(\Xi)$ is the Dirac measure centered at $\eta$ and $\Pi_A$ stands for the indicator of $A$.

For a standard Borel space $(E, \mathcal{E})$, let $(E^2, \mathcal{E}^2)$ be the product space. For $\sigma, \zeta \in \mathcal{P}(E)$, let $\varrho \in \mathcal{P}(E^2)$ be such that $\varrho(A \times E) = \sigma(A)$ and $\varrho(E \times A) = \zeta(A)$ for all $A \in \mathcal{B}(E)$. Then we say that $\varrho$ is a coupling of $\sigma$ and $\zeta$. By $\mathcal{C}(\sigma, \zeta)$ we denote the set of all such couplings.
For $\xi, \eta \in \Xi$, we set
\[
v(\xi, \eta) = \begin{cases} 
0, & \text{if } \xi = \eta; \\
1, & \text{otherwise}, 
\end{cases}
\]
which is a measurable function on $\Xi^2$ since $\Xi$ is a standard Borel space. Then we equip $\mathcal{P}(\Xi)$ with the total variation distance
\[
d(\sigma, \varsigma) = \sup_{A \in \mathcal{B}(\Xi)} |\sigma(A) - \varsigma(A)|, 
\]
that, by duality, can also be written in the form
\[
d(\sigma, \varsigma) = \inf_{\varrho \in \mathcal{C}(\sigma, \varsigma)} \int_{\Xi^2} v(\xi, \eta) \varrho(d\xi, d\eta).
\]

**Proposition 2.3.** For each $\ell \in \mathcal{L}$ and $(x, y) \in X^2$, there exists $\varrho_{x,y}^\ell \in \mathcal{C}(\pi^x_\ell, \pi^y_\ell)$ such that: (a) for each $B \in \mathcal{B}(\Xi^2_\ell)$, the map $X^2 \ni (x, y) \mapsto \varrho_{x,y}^\ell(B)$ is $\mathcal{F}^2_\ell$-measurable; (b) the following holds
\[
d(\pi^x_\ell, \pi^y_\ell) = \int_{\Xi^2} v(\xi, \eta) \varrho_{x,y}^\ell(d\xi, d\eta). 
\]

**Proof.** Set
\[
(\pi^x_\ell \wedge \pi^y_\ell)(A) = \min\{\pi^x_\ell(A); \pi^y_\ell(A)\}, \quad A \in \mathcal{B}(\Xi). 
\]
In view of the measurability as in Definition 2.1 the map $X^2 \ni (x, y) \mapsto (\pi^x_\ell \wedge \pi^y_\ell)(A)$ is $\mathcal{F}^2_\ell$-measurable since, given $a \in [0, 1]$, we have that
\[
\{(x, y) : a \leq (\pi^x_\ell \wedge \pi^y_\ell)(A)\} = \{x : a \leq \pi^x_\ell(A)\}^2.
\]
Then both maps $(x, y) \mapsto (\pi^x_\ell - \pi^x_\ell \wedge \pi^y_\ell)(A)$ and $(x, y) \mapsto (\pi^y_\ell - \pi^x_\ell \wedge \pi^y_\ell)(A)$ are $\mathcal{F}^2_\ell$-measurable. By (2.5) also $(x, y) \mapsto d(\pi^x_\ell, \pi^y_\ell)$ is measurable in the same sense.

Set $D_\ell = \{(\xi, \xi) : \xi \in \Xi_\ell\}$. Since $\Xi_\ell$ is a standard Borel space, the map $\xi \mapsto \psi(\xi) = (\xi, \xi) \in D_\ell$ is measurable. Then, for each $B \in \mathcal{B}(\Xi^2_\ell)$, we have that $\psi^{-1}(B \cap D_\ell) \in \mathcal{B}(\Xi_\ell)$, which allows us to define $\omega_{x,y}^\ell \in \mathcal{P}(\Xi^2_\ell)$ by setting
\[
\omega_{x,y}^\ell(B) = (\pi^x_\ell \wedge \pi^y_\ell)(\psi^{-1}(B \cap D_\ell)). 
\]
The coupling for which (2.6) holds has the form, see [15] Eq. (5.3), page 19,
\[
\varrho_{x,y}^\ell = \omega_{x,y}^\ell + (\pi^x_\ell - \pi^x_\ell \wedge \pi^y_\ell) \otimes (\pi^y_\ell - \pi^x_\ell \wedge \pi^y_\ell)/d(\pi^x_\ell, \pi^y_\ell).
\]
Then the $\mathcal{F}^2_\ell$-measurability of the maps $(x, y) \mapsto \varrho_{x,y}^\ell(A_1 \times A_2)$, $A_1, A_2 \in \mathcal{B}(\Xi_\ell)$, follows by the arguments given above. This yields the proof of claim (a) as $\mathcal{B}(\Xi^2_\ell)$ is a product $\sigma$-field. \(\square\)

Let $\varpi$ be the family of $\varpi_{x,y}^\ell = \{(x, y) \in X^2\}$, $\ell \in \mathcal{L}$, such that each $\varpi_{x,y}^\ell$ is in $\mathcal{P}(\Xi^2_\ell)$ and, for any $B \in \mathcal{B}(\Xi^2_\ell)$, the map $(x, y) \mapsto \varpi_{x,y}^\ell(B)$ is $\mathcal{F}^2_\ell$-measurable. Then $\varpi$ is a one-point specification in the sense of Definition
which determines the set $\mathcal{M}(\pi)$ of $\nu \in \mathcal{P}(X^2)$ consistent with $\pi$. Like in (2.4), $\nu \in \mathcal{M}(\pi)$ if and only if it satisfies

$$
\nu(B) = \int_{X^2} \int_{X^2} \mathbb{I}_B(x,y) \mathbb{A}_x \mu^y (dx\ell, d\bar{x}\ell) \prod_{\ell' \neq \ell} \delta_{y_{\ell'}}(dx_{\ell'}) \delta_{\bar{y}_{\ell'}}(d\bar{x}_{\ell'}) \nu(dy, d\bar{y}),
$$

which holds for all $\ell \in \mathbb{L}$ and $B \in \mathcal{B}(X^2)$.

Proposition 2.4. Suppose that $\mathbb{A}_x \in C(\pi^x_\ell, \pi^x_\ell')$ for all $\ell \in \mathbb{L}$ and $x, x' \in X$. Then each $\nu \in \mathcal{M}(\pi)$ is a coupling of some $\mu_1, \mu_2 \in \mathcal{M}(\pi)$.

Proof. The equality $\mu_1(A) = \nu(A \times X)$, $A \in \mathcal{B}(X)$, determines a probability measure on $X$. Thus, for $A \in \mathcal{B}(X)$, by (2.7) we get

$$
\mu_1(A) = \int_{X^2} \int_{X^2} \mathbb{A}_x \pi^y_\ell (dx\ell, d\bar{x}\ell) \prod_{\ell' \neq \ell} \delta_{y_{\ell'}}(dx_{\ell'}) \delta_{\bar{y}_{\ell'}}(d\bar{x}_{\ell'}) \nu(dy, d\bar{y})
$$

$$
= \int_{X^2} \int_{X} \mathbb{A}_x \pi^y_\ell (dx\ell) \prod_{\ell' \neq \ell} \delta_{y_{\ell'}}(dx_{\ell'}) \nu(dy, d\bar{y})
$$

$$
= \int_{X^2} \int_{X} \mathbb{A}_x \pi^y_\ell (dx\ell) \prod_{\ell' \neq \ell} \delta_{y_{\ell'}}(dx_{\ell'}) \nu(dy, d\bar{y})
$$

$$
= \int_{X^2} \int_{X} \mathbb{A}_x \pi^y_\ell (dx\ell) \prod_{\ell' \neq \ell} \delta_{y_{\ell'}}(dx_{\ell'}) \mu_1(dy).
$$

Therefore, $\mu_1$ solves (2.4) and hence $\mu_1 \in \mathcal{M}(\pi)$. The same is true for the second marginal measure $\mu_2$. \hfill \square

2.2. The results. Our main concern is under which conditions imposed on the family $\pi$ the set $\mathcal{M}(\pi)$ contains one element at most. If each $\pi^x_\ell$ is independent of $x$, the unique element of $\mathcal{M}(\pi)$ is the product measure $\otimes_{\ell \in \mathbb{L}} \pi^x_\ell$, which readily follows from (2.4). Therefore, one may try to relate the uniqueness in question to the weak dependence of $\pi^x_\ell$ on $x$, formulated in terms of the metric defined in (2.6). Thus, let us take $x, y \in X$ such that $x = y$ off some $\ell' \in \partial \ell$, and consider $d(\pi^x_\ell, \pi^y_\ell)$. If this quantity were bounded by a certain $\kappa_{\ell\ell'}$, uniformly in $x$ and $y$, this bound (Dobrushin’s estimator, cf. [4, pp. 20, 21]) could be used to formulate the celebrated Dobrushin uniqueness condition in the form

$$
\sup_{\ell \in \mathbb{L}} \sum_{\ell' \in \partial \ell} \kappa_{\ell\ell'} =: \bar{\kappa} < 1. \tag{2.8}
$$

However, in a number of applications, especially where $\Xi$ is a noncompact topological space, the mentioned boundedness does not hold. The way of treating such cases suggested in [5] may be outlined as follows. Assume that
there exists a matrix \((\kappa_{\ell\ell'})\) with the property as in (2.8) such that, for each \(\ell \in L\), the following holds

\[
d(\pi^x_\ell, \pi^y_\ell) \leq \sum_{\ell' \in \partial \ell} \kappa_{\ell\ell'} v(x_{\ell'}, y_{\ell'}),
\]

for \(x\) and \(y\) belonging to the set

\[
X_\ell(h,K) := \{x \in X : h(x_{\ell'}) \leq K \text{ for all } \ell' \in \partial \ell\}.
\]

Here \(K > 0\) is a parameter and \(h : \Xi \to [0, +\infty)\) is a given measurable function. Clearly, if \(h\) is bounded, then \(X_\ell(h,K) = X\) for big enough \(K\), and hence (2.9) turns into the mentioned Dobrushin condition. Thus, in order to cover the case of interest we have to take \(h\) unbounded and \(\pi^x_\ell\)-integrable, with an appropriate control of the dependence of \(\pi^x_\ell(h)\) on \(x\).

Namely, we shall assume that, for each \(\ell \in L\) and \(x \in X\), the following holds

\[
\pi^x_\ell(h) \leq 1 + \sum_{\ell' \in \partial \ell} c_{\ell\ell'} h(x_{\ell'}),
\]

for some matrix \(c = (c_{\ell\ell'})\), which satisfies

\[
(a) \quad c_{\ell\ell'} \geq 0; \quad (b) \quad \sup_{\ell \in L} \sum_{\ell' \in \partial \ell} c_{\ell\ell'} =: \bar{c} < 1/\Delta^x.
\]

In the original work [8], the first summand on the right-hand side of (2.11) is a constant \(C > 0\), the value of which determines the scale of \(K\), see (2.10).

We thus take it as above for the sake of convenience.

**Definition 2.5.** Let \(h, K, \kappa, \) and \(c\) be as in (2.8) – (2.12). Then by \(\Pi(h,K,\kappa,c)\) we denote the set of one-site specifications \(\pi\) for which both estimates (2.9), (2.10) and (2.11) hold true for each \(\ell \in L\).

Given \(\mu \in \mathcal{M}(\pi)\), the integrability assumed in (2.11) does not yet imply that \(h\) is \(\mu\)-integrable. For \(\pi\) satisfying (2.11), by \(\mathcal{M}(\pi,h)\) we denote the subset of \(\mathcal{M}(\pi)\) consisting of those measures for which the following holds

\[
\mu(h) := \sup_{\ell \in L} \int_X h(x_\ell) \mu(dx) < \infty.
\]

In a similar way, we introduce the set \(\mathcal{M}_D(\pi,h)\) for a given \(D \subset L\), cf. Definition 2.2.

From now on we fix the graph, the function \(h\), and the matrices \(c\) and \(\kappa\). Thereafter, we set

\[
K_* = \max \left\{ \frac{4\Delta^x+1}{\bar{c}(1-\bar{\kappa})}, \frac{2\Delta^x+1}{(1-\bar{\kappa})^2(1-\bar{c}\Delta^x)} \right\}.
\]

**Theorem 2.6.** For each \(K > K_*\) and \(\pi \in \Pi(h,K,\kappa,c)\), the set \(\mathcal{M}(\pi,h)\) contains at most one element.

An important characteristic of the states \(\mu \in \mathcal{M}(\pi)\) is the decay of correlations. Fix two distinct vertices \(\ell_1, \ell_2 \in L\) and consider bounded functions
$f, g : X \to \mathbb{R}_+$, such that $f$ is $\mathcal{B}(\Xi_{\ell_1})$-measurable and $g$ is $\mathcal{B}(\Xi_{\ell_2})$-measurable. Set
\[ \text{Cov}_\mu(f; g) = \mu(fg) - \mu(f)\mu(g), \]
and let $\delta$ denote the path distance on the underlying graph.

**Theorem 2.7.** Let $\pi$ and $K$ be as in Theorem 2.6, and $\mathcal{M}(\pi, h)$ be nonempty and hence contain a single state $\mu$. Let also $f$ and $g$ be as just described and $\| \cdot \|_\infty$ denote the sup-norm on $X$. Then there exist positive $C_K$ and $\alpha_K$, dependent on $K$ only, such that
\[ |\text{Cov}_\mu(f; g)| \leq C_K \| f \|_\infty \| g \|_\infty \exp \left( -\alpha_K \delta(\ell_1, \ell_2) \right). \] (2.15)

2.3. **Comments and applications.** Let us make some comments to the above results. For further comments related to the proof of these results see the end of Section 1.

- According to [20, Section 8], the elements of $\mathcal{M}(\pi)$ as in Definition 2.2 are one-site Gibbs states. In [9, Theorem 1.33, page 23] and [20, Section 8], there are given conditions under which the elements of $\mathcal{M}(\pi)$ are 'usual' Gibbs states, e.g., in the sense of [9, Definition 1.23, page 16]. This, in particular, holds if $\pi$ is a subset of the set of all local kernels $\Pi_D$ defined for all finite $D \subset \mathbb{L}$, which determine the states. In this case, Theorem 2.6 yields the existence and uniqueness of the usual states, see [22].
- The condition in (2.13) is usually satisfied for tempered measures, i.e., for those elements of $\mathcal{M}(\pi)$ which are supported on tempered configurations, cf., e.g., [14].
- As mentioned above, we do not require that $h$ be compact in the sense of [8]. This our extension gets important if one deals with single-spin spaces which are not locally compact, e.g., with spaces of Hölder continuous functions as in [1, 11, 18].
- In contrast to [8, Theorem 1], in (2.14) we give an explicit expression for the threshold value $K^*$, which depends only on the parameters of the underlying graph and on the norms $\bar{c}$ and $\bar{\kappa}$.
- The novelty of Theorem 2.7 consists in the following. The decay of correlations under the uniqueness condition was proven only for compact single-spin spaces, see [12], where the classical Dobrushin criterion can be applied. For 'unbounded spins', the corresponding results are usually obtained by cluster expansions, see, e.g., [21], where the correlations are shown to decay due to 'weak enough' interactions' and no information on the number of states is available.
- The parameters $C_K$ and $\alpha_K$ in (2.15) are also given explicitly, see below.

Now we turn to briefly outlining possible applications of Theorems 2.6 and 2.7. A more detailed discussion of this issue can be found in [22], see also the related parts of [18]. Further results in these directions will be published in forthcoming articles.
By means of Theorems 2.6 and 2.7 the uniqueness of equilibrium states and the decay of correlations can be established in the following models:

- Systems of classical $N$-dimensional anharmonic oscillators described by the energy functional
  \[ H(x) = \sum_{\ell \in L} V(\xi_\ell) + \sum_{(\ell, \ell') \in E} W_{\ell\ell'}(\xi_\ell, \xi_{\ell'}), \quad \xi_\ell \in \mathbb{R}^N, \ N \in \mathbb{N} \]

- Systems of quantum $N$-dimensional anharmonic oscillators described by the Hamiltonian
  \[ H = \sum_{\ell \in L} H_\ell + \sum_{(\ell, \ell') \in E} W_{\ell\ell'}(q_\ell, q_{\ell'}), \]
  where $q_\ell = (q^{(1)}_\ell, \ldots, q^{(N)}_\ell)$ is the position operator and $H_\ell$ is the one-particle Hamiltonian defined on the corresponding physical Hilbert space. States of such models are constructed in a path integral approach as probability measures on the products of continuous periodic functions, which are not locally compact, see [1, 11, 18].

- Systems of interacting particles in the continuum (e.g. $\mathbb{R}^d$), including the Lebowitz-Mazel-Presutti model [13], and systems of ‘particles’ lying on the cone of discrete measures introduced in [10]. Note that to continuum systems the original version [8] of the Dobrushin-Pechersky criterion was used in [3, 19].

3. The Proof of Theorems 2.6 and 2.7

3.1. The ingredients of the proof. First we introduce the notion of locality. By writing $D \in \mathcal{L}$ we mean that $D$ is a nonempty finite subset of $\mathcal{L}$. For such $D$, elements of $\mathcal{B}(\Xi^D) \subset \mathcal{B}(X)$ are called local sets. A function $f : X \to \mathbb{R}$ is called local if it is $\mathcal{B}(\Xi^D)$-measurable for some $D \in \mathcal{L}$. Likewise, $B \in \mathcal{B}(X^2)$ is local if $B \in \mathcal{B}((\Xi \times \Xi)^D)$ for such $D$. Locality of functions $f : X^2 \to \mathbb{R}$ is defined in the same way.

Lemma 3.1. Given a one-site specification $\pi$ and $\mu_1, \mu_2 \in \mathcal{M}(\pi)$, suppose there exists $\nu_* \in \mathcal{C}(\mu_1, \mu_2)$ such that
  \[ \int_{X^2} v(x_\ell, y_\ell) \nu_*(dx, dy) = 0, \] (3.1)
  holding for all $\ell \in \mathcal{L}$. Then $\mu_1 = \mu_2$.

Proof. Local sets $A \subset X$ are measure defining, that is, $\mu_1, \mu_2 \in \mathcal{P}(X)$ coincide if they coincide on local sets. For $A \in \mathcal{B}(\Xi^D)$ and the indicator $\mathbb{1}_A$, we have
  \[ \|\mathbb{1}_A(x) - \mathbb{1}_A(y)\| \leq \sum_{\ell \in D} v(x_\ell, y_\ell), \]
  and then
  \[ |\mu_1(A) - \mu_2(A)| \leq \sum_{\ell \in D} \int_{X^2} v(x_\ell, y_\ell) \nu_*(dx, dy) = 0, \]
which yields the proof. □

The proof of Theorem 2.6 will be done by showing that, for each \( \mu_1, \mu_2 \in \mathcal{M}(\pi, h) \), the set \( \mathcal{C}(\mu_1, \mu_2) \) contains a certain \( \nu \) such that (3.1) holds. This coupling \( \nu \) will be obtained by taking the limit in the topology of local setwise convergence, cf. [9], which we introduce as follows.

**Definition 3.2.** A net \( \{\nu_\alpha\}_{\alpha \in I} \subset \mathcal{P}(X^2) \) is said to be convergent to a \( \nu_* \in \mathcal{P}(X^2) \) in the topology of local setwise convergence (\( \mathcal{L} \)-topology, for short), if \( \nu_\alpha(B) \to \nu_*(B) \) for all local \( B \in \mathcal{B}(X^2) \). Or, equivalently, \( \nu_\alpha(f) \to \nu_*(f) \) for all bounded local functions. The same definition applies also to nets \( \{\mu_\alpha\}_{\alpha \in I} \subset \mathcal{P}(X) \).

Note that the \( \mathcal{L} \)-topology is Hausdorff, but not metrizable if \( \Xi \) is not a compact topological space.

**Lemma 3.3.** Given \( \mu_1, \mu_2 \in \mathcal{P}(X) \), let \( \{\nu_\alpha\}_{\alpha \in I} \subset \mathcal{C}(\mu_1, \mu_2) \) be convergent to a certain \( \nu \in \mathcal{P}(X^2) \) in the \( \mathcal{L} \)-topology. Then \( \nu \in \mathcal{C}(\mu_1, \mu_2) \).

The proof of this lemma is rather obvious. The coupling in question \( \nu_* \) will be constructed within a step-by-step procedure based on the mapping

\[
(R_\ell \nu)(f) = \int_{X^2} \left( \int_{\Xi^2} f(\xi \times x(\ell), \eta \times y(\ell)) g^{x,y}_\ell(d\xi, d\eta) \right) \nu(dx, dy), \tag{3.2}
\]

where \( \ell \in \mathcal{L} \), \( g^{x,y}_\ell \) is as in (2.6), and \( f : X^2 \to \mathbb{R} \) is a function such that both \( \nu(f) \) and the integral on the right-hand side of (3.2) exist.

**Lemma 3.4.** For each \( \ell \in \mathcal{L} \), the mapping (3.2) has the following properties: (a) if \( \nu \in \mathcal{C}(\mu_1, \mu_2) \) for some \( \mu_1, \mu_2 \in \mathcal{M}(\pi) \), then also \( R_\ell \nu \in \mathcal{C}(\mu_1, \mu_2) \); (b) if \( f \in \mathcal{F}_\ell(X^2) \)-measurable and \( \nu \)-integrable, then \( (R_\ell \nu)(f) = \nu(f) \).

*Proof.* Claim (a) is true since \( g^{x,y}_\ell \in \mathcal{C}(\pi^x, \pi^y) \) for all \( x, y \in X \). Claim (b) follows by the fact that the considered \( f \) in (3.2) is independent of \( \xi \) and \( \eta \), and that \( g^{x,y}_\ell \) is a probability measure. □

Given \( \ell \in \mathcal{L} \), we set

\[ Y_\ell = \{(x_1^1, x_2^1) \in X^2 : \nu(x_1^2, x_2^2) \leq \sum_{\ell' \in \partial \ell} \nu(x_1^1, x_2^1)\}. \]

**Lemma 3.5.** For each \( \nu \in \mathcal{P}(X^2) \) and \( \ell \in \mathcal{L} \), it follows that \((R_\ell \nu)(Y_\ell) = 1 \).

*Proof.* If \((x^1, x^2)\) is in \( Y_\ell \), then \( \nu(x_1^1, x_2^1) = 1 \) and \( \nu(x_1^2, x_2^2) = 0 \) for all \( \ell' \in \partial \ell \), which follows by the fact that \( \nu \) takes values in \{0, 1\}. This means that \( x_1^1 \neq x_2^1 \) and \( x_1^2 = x_2^2 \) for all \( \ell' \in \partial \ell \). For such \((x^1, x^2)\), the definition of \( \pi \) implies that \( \pi^{x^1} = \pi^{x^2} \), and hence

\[
\int_{\Xi^2} \nu(\xi, \eta) g^{x^1, x^2}_\ell(d\xi, d\eta) = d(\pi^{x^1}, \pi^{x^2}) = 0,
\]

which by (3.2) yields \((R_\ell \nu)(Y_\ell) = 0 \). □
The proof of Theorem 2.6 will be done by showing that, for each \( \mu_1, \mu_2 \in \mathcal{M}(\pi, h) \), there exists \( \nu \in \mathcal{C}(\mu_1, \mu_2) \), for which (3.1) holds. To this end we construct a sequence \( \{\hat{\nu}_n\}_{n \in \mathbb{N}_0} \subset \mathcal{C}(\mu_1, \mu_2) \) such that

\[
\gamma(\hat{\nu}_n) := \sup_{\ell \in \mathbb{L}} \int_{X^2} \nu(x_1^1, x_2^1)\hat{\nu}_n(dx^1, dx^2) \to 0, \quad n \to +\infty. \tag{3.3}
\]

This sequence will be obtained by a procedure based on the mapping (3.2) and the estimates which we derive in the next subsection. The proof of Theorem 2.7 will be obtained as a byproduct.

### 3.2. The main estimates

In the sequel, we use the following functions indexed by \( \ell \in \mathbb{L} \)

\[
I_\ell(x_1^1, x_2^1) = \nu(x_1^1, x_2^1), \quad H_\ell^i(x_1^1, x_2^1) = h(x_i^1), \quad i = 1, 2. \tag{3.4}
\]

By claim (b) of Lemma 3.4, we have that

\[
(R_\ell \nu)(I_\ell^1) = \nu(I_\ell^1), \quad (R_\ell \nu)(I_\ell^1 H_\ell^1) = \nu(I_\ell^1 H_\ell^1) \quad \text{for} \quad \ell \neq \ell_1, \ell \neq \ell_2,
\]

whenever \( H_\ell^1 \) is \( \nu \)-integrable. We recall that \( \hat{\nu}_\ell^{x,y} \) in (3.2) is a coupling of \( \pi_\ell^x \) and \( \pi_\ell^y \), for which (2.9) and (2.11) hold true.

**Lemma 3.6.** Let \( \nu \in \mathcal{P}(X^2) \) be such that the integrals on both sides of (3.2) exist for \( f = H_\ell^i, \ell \in \mathbb{L} \) and \( i = 1, 2 \). Then the following estimates hold

\[
(R_\ell \nu)(I_\ell) \leq \sum_{\ell' \in \partial \ell} \kappa_{\ell \ell'} \nu(I_{\ell'}) + K^{-1} \sum_{i=1,2 \atop \ell_1, \ell_2 \in \partial \ell} \nu(I_{\ell_1} H_{\ell_1}^i), \tag{3.5}
\]

\[
(R_\ell \nu)(I_\ell H_\ell^1) \leq \nu(I_{\ell_1}) + \sum_{\ell_2 \in \partial \ell} c_{\ell \ell_2} \nu(I_{\ell_1} H_{\ell_2}^1), \tag{3.6}
\]

\[
(R_\ell \nu)(I_\ell H_{\ell_1}^i) \leq \sum_{\ell_2 \in \partial \ell} \nu(I_{\ell_2} H_{\ell_1}^i), \quad \ell_1 \neq \ell, \tag{3.7}
\]

\[
(R_\ell \nu)(I_\ell H_{\ell_1}^i) \leq \sum_{\ell_1 \in \partial \ell} \nu(I_{\ell_1}) + \sum_{\ell_1, \ell_2 \in \partial \ell} c_{\ell \ell_2} \nu(I_{\ell_1} H_{\ell_2}^i). \tag{3.8}
\]

**Proof.** The proof of (3.7) readily follows by Lemma 3.5. Let us prove (3.5).

By (2.6) and (3.2), we have

\[
(R_\ell \nu)(I_{\ell}) = \int_{X^2} d(\pi_\ell^{x_1}, \pi_\ell^{x_2}) \nu(dx^1, dx^2)
\]

\[
= \int_{X^2} 1_\ell(x^1) d(\pi_\ell^{x_1}, \pi_\ell^{x_2}) \nu(dx^1, dx^2)
\]

\[
+ \int_{X^2} [1 - 1_\ell(x^1)] d(\pi_\ell^{x_1}, \pi_\ell^{x_2}) \nu(dx^1, dx^2),
\]
where \(1_\ell\) is the indicator of the set defined in (2.10). By (2.9), we have
\[
\int_{X^2} 1_\ell(x^1) 1_\ell(x^2) d(\pi_{x^1_\ell}, \pi_{x^2_\ell}) \nu(dx^1, dx^2) \leq \sum_{\ell' \in \partial \ell} \kappa_{\ell\ell'} \nu(I_{\ell'}) ,
\]
which yields the first term of the right-hand side of (3.5). By (2.10), we have
\[
[1 - 1_\ell(x^1) 1_\ell(x^2)] \leq \sum_{i=1,2} \sum_{\ell_1 \in \partial \ell} [1 - \mathbb{I}_{h \leq K}(x^i_{\ell_1})] ,
\]
where \(\mathbb{I}_{h \leq K}\) is the indicator of \(\{\xi \in \Xi : h(\xi) \leq K\}\). Then the second term of the right-hand side of (3.5) cannot exceed the following
\[
\sum_{i=1,2} \sum_{\ell_1 \in \partial \ell} \int_{X^2} \left( \mathbb{I}_{h \leq K}(x^i_{\ell_1}) \right) d(\pi_{x^1_\ell}, \pi_{x^2_\ell}) \nu(dx^1, dx^2) \leq K^{-1} \sum_{i=1,2} \sum_{\ell_1 \in \partial \ell} \nu(I_{\ell_2} H^i_{\ell_1}) .
\]
The latter line has been obtained by (3.7).

Let us prove now (3.6). By (3.2) and the fact that \(\varrho_{x,y}^\ell \in \mathcal{C}(\pi_{x_\ell}, \pi_{y_\ell})\), we have
\[
(R_\ell \nu)(I_{\ell_1} H^i_{\ell_1}) = \int_{X^2} \left( \int_{\Xi} h(\xi) \pi_{x^i_\ell} (d\xi) \right) \nu(x^1_{\ell_1}, x^2_{\ell_1}) dx^1_\ell, dx^2_\ell \leq \text{RHS}(3.6) ,
\]
where we have used (2.11). To prove (3.8) we employ Lemma 3.5, by which we get
\[
\text{LHS}(3.8) \leq \sum_{\ell_1 \in \partial \ell} (R_\ell \nu)(I_{\ell_1} H^i_{\ell_1}) \leq \text{RHS}(3.8) ,
\]
where the latter estimate follows by (3.6).

From the lemma just proven it follows that along with the parameter \(\gamma(\nu)\) defined in (3.3) one has to control also the following
\[
\lambda(\nu) = \max_{i=1,2} \sup_{\ell, \ell' \in \ell} \nu(I_{\ell_1} H^i_{\ell_1}) ,
\]
where \(\nu \in \mathcal{C}(\mu_1, \mu_2), \mu_1, \mu_2 \in \mathcal{M}_h(\pi)\), and \(\pi \in \Pi(h, K, \kappa, c)\), see Definition 2.5.\]
3.3. The proof of Theorem 2.6. The proof is based on constructing a sequence with the property (3.3). Given \( \mu_1, \mu_2 \in \mathcal{M}(\pi, h) \) with \( \pi \in \Pi(h, K, \kappa, c) \), we take an arbitrary \( \nu_0 \in \mathcal{C}(\mu_1, \mu_2) \) and construct \( \nu \in \mathcal{C}(\mu_1, \mu_2) \) by applying the mapping defined in (3.2) to the initial \( \nu_0 \) with \( \ell \) running over the set \( L \). Each time we use the estimates derived in Lemma 3.6. Then the first two elements of the sequence in question are set \( \hat{\nu}_0 = \nu_0 \) and \( \hat{\nu}_1 = \nu \). Afterwards, we produce \( \hat{\nu}_2 \) from \( \hat{\nu}_1 \), etc.

Recall that the underlying graph is supposed to have the property defined in (2.1) and \( \chi \leq \Delta \) is its chromatic number. Set

\[
A = \frac{2\Delta \chi + 1}{1 - \bar{\kappa}}.
\]

(3.10)

Then, for \( K > K_\ast \), see (2.14), the following holds

\[
K^{-1} < \frac{\bar{c}(1 - \bar{\kappa})}{4\Delta \chi + 1}, \quad AK^{-1} < \bar{c}/2.
\]

(3.11)

Lemma 3.7. For \( K > K_\ast \), take \( \pi \in \Pi(h, K, \kappa, c) \) and \( \mu_1, \mu_2 \in \mathcal{M}(\pi, h) \). Then for each \( \nu_0 \in \mathcal{C}(\mu_1, \mu_2) \) there exists \( \nu \in \mathcal{C}(\mu_1, \mu_2) \) for which the following estimates hold

\[
\gamma(\nu) \leq [\bar{\kappa} + AK^{-1}] \gamma(\nu_0) + 2AK^{-1}\lambda(\nu_0),
\]

(3.12)

\[
\lambda(\nu) \leq \Delta^{-1}\gamma(\nu_0) + \bar{c}\Delta \lambda(\nu_0).
\]

(3.13)

The proof of the lemma will be given in the subsequent parts of the paper.

Proof Theorem 2.6. As already mentioned, we let \( \hat{\nu}_1 \in \mathcal{C}(\mu_1, \mu_2) \) and \( \hat{\nu}_0 \in \mathcal{C}(\mu_1, \mu_2) \) be the measures on the left-hand sides and right-hand sides of (3.12) and (3.13), respectively. Then we apply to \( \hat{\nu}_1 \) the same reconstruction procedure and obtain \( \hat{\nu}_2 \in \mathcal{C}(\mu_1, \mu_2) \), for which both estimates (3.12), (3.13) hold with \( \hat{\nu}_1 \) on the right-hand sides. We repeat this due times and obtain \( \hat{\nu}_n \in \mathcal{C}(\mu_1, \mu_2) \) such that

\[
\begin{pmatrix}
\gamma(\hat{\nu}_n) \\
\lambda(\hat{\nu}_n)
\end{pmatrix} \leq [M(K)]^n \begin{pmatrix}
\gamma(\nu_0) \\
\lambda(\nu_0)
\end{pmatrix},
\]

(3.14)

where \( M(K) \) is the matrix defined by the right-hand sides of (3.12) and (3.13). Its spectral radius is

\[
r_K = \frac{1}{2} \left[ \bar{\kappa} + AK^{-1} + \bar{c}\Delta + \sqrt{(\bar{\kappa} + AK^{-1} - \bar{c}\Delta)^2 + 8\Delta AK^{-1}} \right].
\]

(3.15)

For \( K > K_\ast \), see (2.14), we have \( r_K < 1 \), which by (3.14) yields (3.3) and thereby completes the proof.

3.4. The proof of Theorem 2.7. The proof of this theorem is based on the version of the estimates in Lemma 3.7 obtained in a subset \( D \subset L \). For such \( D \), we define

\[
\partial D = \{ \ell' \in D^c : \partial \ell' \cap D \neq \emptyset \},
\]
Let \( \mu \) be measurable for all \( B \in \mathcal{B} \). Next, for \( \ell_1 \) as in (2.15) and \( N = \delta(\ell_1, \ell_2) \), we set

\[
D_0 = \{ \ell_1 \}, \quad D_k = D_{k-1} \cup \partial D_{k-1}, \quad k = 1, \ldots, N - 1.
\]

Let \( \mu^x(\cdot) \) denote the conditional measure \( \mu(\cdot | \mathcal{F}_{D_{N-1}}(x)) \). For brevity, we say that \( \nu^x \in \mathcal{P}(X^2) \) is \( \mathcal{F}_{D_{N-1}} \)-measurable if the maps \( x \mapsto \nu^x(B) \) are \( \mathcal{F}_{D_{N-1}} \)-measurable for all \( B \in \mathcal{B}(X^2) \). Clearly, \( \nu_0^x = \mu^x \otimes \mu \) possesses this property. The version of Lemma 3.7 which we need is the following statement.

**Lemma 3.8.** Let \( \pi, K, \) and \( \mu \) be as in Theorem 2.7 and \( \nu_0^x = \mu^x \otimes \mu \). Then there exist \( \nu_1^x, \ldots, \nu_{N-1}^x \in \mathcal{C}(\mu^x, \mu) \), all \( \mathcal{F}_{D_{N-1}} \)-measurable, such that for the parameters defined in (3.16) the following estimates hold

\[
\begin{pmatrix}
\gamma_{D_{N-1}}(\nu_s^x)
\lambda_{D_{N-1}}(\nu_s^x)
\end{pmatrix} \leq M(K)
\begin{pmatrix}
\gamma_{D_{N-2}}(\nu_s^x)
\lambda_{D_{N-2}}(\nu_s^x)
\end{pmatrix},
\]

for all \( s = 1, \ldots, N - 1 \) and \( \mu \)-almost all \( x \in X \).

**Proof of Theorem 2.7:** Since \( g \) is \( \mathcal{F}_{D_{N-1}} \)-measurable, we have

\[
\int_X f(x)g(x)\mu(dx) = \int_X f(x) \left( \int_X g(y)\mu^x(dy) \right) \mu(dx),
\]

which yields

\[
\text{Cov}_{\mu}(f; g) = \int_X g(x)\Phi(x)\mu(dx), \tag{3.18}
\]

where

\[
\Phi(x) = \int_{X^2} (f(y) - f(z)) \mu^x(dy)\mu(dz). \tag{3.19}
\]

For each \( \nu_s^x, s = 0, \ldots, N - 1 \), as in Lemma 3.8 we then have

\[
\Phi(x) = \int_{X^2} (f(y) - f(z)) \nu_s^x(dy, dz), \tag{3.20}
\]

and hence

\[
|\Phi(x)| \leq 2\|f\|_{\infty}\nu_{N-1}^x(I_{\ell_1}) = 2\|f\|_{\infty}\gamma_{D_0}(\nu_{N-1}^x). \tag{3.21}
\]

Note that the function defined in (3.19), (3.20) is related to the quantity which characterizes mixing in state \( \mu \), cf. [12, Proposition 2.5].

Let \( \nu_s \) and \( \nu_{s-1} \) denote the column vector on the left-hand and right-hand sides of (3.17), respectively. Set

\[
\xi = \frac{\Delta^{x-1}}{r_K - \bar{c}\Delta^x} = \frac{r_K - \bar{k} - AK^{-1}}{2AK^{-1}} > 0,
\]
and let \( T \) be the \( 2 \times 2 \) diagonal matrix with \( T_{11} = \xi \) and \( T_{22} = 1 \). Then the matrix

\[
\tilde{M}(K) := TM(K)T^{-1},
\]

(cf. [2, Corollary 2.9.4, page 102], is positive and such that both its rows sum up to \( r_K \). Set \( \tilde{v}_s = Tv_s \) and let \( \tilde{v}_s^i, i = 1, 2 \), be the entries of \( \tilde{v}_s \). By (3.17) we then get

\[
\|\tilde{v}_s\| := \max\{\tilde{v}_s^1; \tilde{v}_s^2\} \leq \|\tilde{M}(K)\|\|\tilde{v}_{s-1}\| = r_K \max\{\tilde{v}_{s-1}^1; \tilde{v}_{s-1}^2\},
\]

which yields

\[
\gamma_D(\nu_{N-1}^0) \leq r_K^{-N-1} \max\{\gamma_D(N-1)(\nu_0^0); \xi^{-1} \lambda_D(N-1)(\nu_0^0)\}.
\]

Applying this estimate in (3.21) and then in (3.18) we arrive at (2.15) with, cf. (3.15) and (2.13),

\[
\alpha_K = -\log r_K, \quad C_K = 2r_K^{-1} \max\{1; \xi^{-1} \mu(h)\}.
\]

Let us make now further comments on the above results and their proof.

- The mapping in (3.2), which is the main reconstruction tool, see Section 4 below, was first introduced in another seminal paper by R. L. Dobrushin [7]. In a rather general context, it was used in [6]. The main feature of this mapping, which was not pointed out in [8], is the measurability of the coupling \( \tilde{c}_{x,y}^\ell \) in \((x,y) \in X^2\). A similar property of the couplings in Lemma 3.8 was crucial for the proof of Theorem 2.7.
- We avoid using ‘compactness’ of \( h \), and hence the related topological properties of the single-spin space \( \Xi \), by employing the \( L \)-topology, see Definition 3.2.
- In contrast to the estimates obtained in [8, Lemma 5], our estimate in (3.13) is independent of \( K \). The only constant in (3.12) is given explicitly in (3.10). This allowed us to calculate explicitly the spectral radius (3.15), which was then used to obtain the decay parameter \( \alpha_K \), see (3.24).
- The proof of Lemma 3.8 was performed in the spirit of the proof of Proposition 2.5 of [12]. Our \( \Phi(x) \) in (3.19), (3.20) can be used to prove a kind of mixing in state \( \mu \). However, here we cannot estimate this function uniformly in \( x \), and hence employ its measurable estimate (3.21) which is then integrated in (3.18).
- The transformation used in (3.22) allowed us to find explicitly the operator norm of \( M(K) \) equal to its spectral radius \( r_K \). This then was used to find in (3.23) the exact rate of the decay of correlations in \( \mu \).
4. Proof of Lemmas 3.7 and 3.8

For the partition (2.3) of the set of vertices $L$, which has the property (2.2), we set

$$U_j = \bigcup_{i=0}^j V_i, \quad W_j = L \setminus U_j, \quad j = 0, \ldots, \chi - 1. \quad (4.1)$$

The measure $\nu$ in (3.12), (3.13) will be obtained in the course of consecutive reconstructions with $\ell \in V_j$. The first step is

4.1. Reconstruction over $V_0$. Let $\{\ell_1, \ell_2, \ldots, \}$ be any numbering of the elements of $V_0$. Set

$$V_0^{(n)} = \{\ell_1, \ldots, \ell_n\}, \quad \nu_0^{(n)} = R_{\ell_n} R_{\ell_{n-1}} \cdots R_{\ell_1} \nu_0, \quad n \in \mathbb{N}. \quad (4.2)$$

Our first task is to estimate $\nu_0^{(n)}(I_\ell)$. By claim (b) of Lemma 3.4 we have that

$$\nu_0^{(n)}(I_\ell) = \nu_0(I_\ell), \quad \text{for } \ell \notin V_0^{(n)}. \quad (4.3)$$

For $k \leq n$, by (2.2) and claim (b) of Lemma 3.4, and then by (3.5) and (3.6), we have

$$\nu_0^{(n)}(I_{\ell_k}) = \nu_0^{(k)}(I_{\ell_k}) \leq \sum_{\ell \in \partial \ell_k} \kappa_{\ell_k} \nu_0(I_\ell) + \sum_{i=1}^K \sum_{\ell, \ell' \in \partial \ell_k} \nu_0(I_\ell H^i_{\ell'}) \leq \bar{\kappa} \gamma(\nu_0) + 2\Delta^2 K^{-1} \lambda(\nu_0), \quad (4.4)$$

see also (2.8), (3.3), and (3.9).

Next we turn to estimating $\nu_0^{(n)}(I_{\ell_k} H^i_{\ell_m})$. As in (4.3) we have

$$\nu_0^{(n)}(I_{\ell_k} H^i_{\ell_m}) = \nu_0(I_{\ell_k} H^i_{\ell_m}) \quad \text{for } \ell, \ell' \notin V_0^{(n)}. \quad (4.5)$$

For $k < m \leq n$, by claim (b) of Lemma 3.4, and then by (3.6), (3.5), (4.4), and (3.7), we have

$$\nu_0^{(n)}(I_{\ell_k} H^i_{\ell_m}) = \nu_0^{(m)}(I_{\ell_k} H^i_{\ell_m}) \leq \nu_0^{(k)}(I_{\ell_k}) + \sum_{\ell \in \partial \ell_m} c_{\ell_m} \nu_0(I_{\ell_k} H^i_\ell) \leq \bar{\kappa} \gamma(\nu_0) + 2\Delta^2 K^{-1} \lambda(\nu_0) + \sum_{\ell \in \partial \ell_m} c_{\ell_m} \sum_{\ell' \in \partial \ell_k} \nu_0(I_{\ell_k} H^i_{\ell'}) \leq \bar{\kappa} \gamma(\nu_0) + [\Delta \bar{\kappa} + 2\Delta^2 K^{-1}] \lambda(\nu_0). \quad (4.6)$$

For $k \leq n$, by (3.8) we have

$$\nu_0^{(n)}(I_{\ell_k} H^i_{\ell_k}) = \nu_0^{(k)}(I_{\ell_k} H^i_{\ell_k}) \leq \sum_{\ell \in \partial \ell_k} \nu_0(I_{\ell_k}) + \sum_{\ell, \ell' \in \partial \ell_k} c_{\ell_k} \nu_0(I_{\ell_k} H^i_{\ell'}) \leq \Delta \gamma(\nu_0) + \Delta \bar{\kappa} \lambda(\nu_0). \quad (4.7)$$
Next, for $m < k < n$, by (3.7) and (3.6) we have
\[ \nu_0^{(n)}(I_k H_{i_m}^j) = \nu_0^{(k)}(I_k H_{i_m}^j) \leq \sum_{\ell \in \partial \ell_k} \nu_0(I_{\ell}) + \sum_{\ell' \in \partial \ell_m} c_{\ell_m \ell} \nu_0(I_{\ell_m} H_{i_{\ell}}^j) \]
\[ \leq \Delta \gamma(\nu_0) + \Delta \bar{c} \lambda(\nu_0). \]

Now we consider the case where $k < n$ and $\ell \notin V_0^{(n)}$. Then by (3.7) we have
\[ \nu_0^{(n)}(I_k H_{i_k}^j) = \nu_0^{(k)}(I_k H_{i_k}^j) \leq \sum_{\ell' \in \partial \ell_k} \nu_0(I_{\ell'} H_{i_{\ell}}^j) \leq \Delta \lambda(\nu_0). \]

For $k \leq n$ and $\ell \notin V_0^{(n)}$, we also have by (3.6) that
\[ \nu_0^{(n)}(I_k H_{i_k}^j) = \nu_0^{(k)}(I_k H_{i_k}^j) \leq \nu_0(I_{\ell}) + \sum_{\ell' \in \partial \ell_k} c_{\ell_k \ell'} \nu_0(I_{\ell'} H_{i_{\ell}}^j) \]
\[ \leq \gamma(\nu_0) + \bar{c} \lambda(\nu_0). \quad (4.5) \]

Now let us consider the sequence $\{\nu_1^{(n)}\}_{n \in \mathbb{N}_0}$ defined in (4.2). By claim (b) of Lemma 3.4 it stabilizes on local sets $B \in \mathcal{B}(X^2)$, and hence is convergent in the $\mathcal{L}$-topology. Let $\nu_1$ be its limit. By Lemma 3.3 we have that $\nu_1 \in \mathcal{C}(\mu_1, \mu_2)$. At the same time, by (4.1), (4.3), and (4.4) it follows that
\[ \nu_1(I_{\ell}) \leq \begin{cases} \tilde{k} \gamma(\nu_0) + 2 \Delta^2 K^{-1} \lambda(\nu_0), & \text{for } \ell \in V_0; \\ \gamma(\nu_0), & \text{for } \ell \in W_0. \end{cases} \quad (4.6) \]

Similarly, by (4.4) - (4.5) we obtain
\[ \nu_1(I_{\ell'} H_{i_{\ell'}}^j) \leq \begin{cases} \Delta \gamma(\nu_0) + [\Delta \bar{c} + 2 \Delta^2 K^{-1}] \lambda(\nu_0), & \ell, \ell' \in V_0; \\ \Delta \lambda(\nu_0), & \ell \in V_0, \ell' \in W_0; \\ \gamma(\nu_0) + \bar{c} \lambda(\nu_0), & \ell \in W_0, \ell' \in V_0; \\ \lambda(\nu_0), & \ell, \ell' \in W_0. \end{cases} \quad (4.7) \]

These estimates complete the reconstruction over $V_0$.

4.2. Reconstruction over $V_j$: Proof of Lemma 3.7 Here we assume that $\nu_j$ satisfies the following estimates, cf. (4.6), where $A$ is as in (3.10):
\[ \nu_j(I_{\ell}) \leq \begin{cases} \tilde{k} + AK^{-1} \gamma(\nu_0) + 2AK^{-1} \lambda(\nu_0), & \text{for } \ell \in U_{j-1}; \\ \gamma(\nu_0), & \text{for } \ell \in W_{j-1}. \end{cases} \quad (4.8) \]
And also, cf. (4.7),

\[
\nu_j(I_\ell H^\ell_\ell) \leq \begin{cases} \\
\Delta^j \gamma(\nu_0) + \bar{c} \Delta^{j+1} \lambda(\nu_0), & \ell, \ell' \in U_{j-1}; \\
\Delta^j \lambda(\nu_0), & \ell \in U_{j-1}, \ell' \in W_{j-1}; \\
j \gamma(\nu_0) + \bar{c} \lambda(\nu_0), & \ell \in W_{j-1}, \ell' \in V_{j-1}; \\
\lambda(\nu_0), & \ell, \ell' \in W_{j-1}.
\end{cases}
\] (4.9)

Since \( W_{\Delta-1} = \emptyset \), see (4.1), for \( j = \Delta - 1 \) we have just the first lines in (4.8) and (4.9), which yields (3.12) and (3.13), respectively, and thus the proof of Lemma 3.7. Note that (4.6) agrees with (4.8) as \( \Delta^2 < A \), see (3.10). Also (4.7) agrees with (4.9), which follows from the fact that

\[ \bar{c} \Delta + 2 \Delta^2 K^{-1} \leq \bar{c} \Delta + AK^{-1} \leq \bar{c} \Delta + \bar{c}/2 < \bar{c} \Delta^2 \leq \bar{c} \Delta^{j+1}, \quad j = 1, \ldots, \chi - 1, \]

see (3.10) and (3.11).

Thus, our aim now is to prove that the estimates as in (4.8) and (4.9) hold also for \( j + 1 \). Note that the last lines in these estimates follow by claim (b) of Lemma 3.3. As above, we enumerate \( V_j = \{ \ell_1, \ell_2, \cdots \} \) and set

\[ \nu_j^{(n)} = R_{\ell_n} R_{\ell_{n-1}} \cdots R_{\ell_1} \nu_j. \]

For \( k \leq n \), by (3.3) we have, cf. (4.4),

\[ \nu_j^{(n)}(I_{\ell_k}) = \nu_j^{(k)}(I_{\ell_k}) \leq \sum_{\ell \in \partial \ell_k \cap U_{j-1}} \kappa_{\ell_k \ell} \nu_j(I_{\ell}) + \sum_{\ell \in \partial \ell_k \cap W_j} \kappa_{\ell_k \ell} \nu_j(I_{\ell}) + \sum_{i=1}^{K^{-1}} \sum_{\ell, \ell' \in \partial \ell_k \cap U_{j-1}} \nu_j(I_{\ell} H^\ell_\ell) 
\]
\[ + \sum_{i=1}^{K^{-1}} \sum_{\ell \in \partial \ell_k \cap W_j} \sum_{\ell' \in \partial \ell_k \cap W_j} \nu_j(I_{\ell} H^\ell_\ell) 
\]
\[ + \sum_{i=1}^{K^{-1}} \sum_{\ell \in \partial \ell_k \cap W_j} \sum_{\ell' \in \partial \ell_k \cap U_{j-1}} \nu_j(I_{\ell} H^\ell_\ell) 
\]
\[ + \sum_{i=1}^{K^{-1}} \sum_{\ell, \ell' \in \partial \ell_k \cap W_j} \nu_j(I_{\ell} H^\ell_\ell). \]

Now we use the assumptions in (4.8) and (4.9) and obtain herefrom

\[ \nu_j^{(n)}(I_{\ell_k}) \leq \left[ \bar{k} + K^{-1} \left( \bar{k} A + 2 \Delta^j \Delta^2 + 2 j \Delta_j \bar{A}_j \right) \right] \gamma(\nu_0) \] (4.10)

\[ + 2 K^{-1} \left[ \bar{k} A + \bar{c} \Delta^{j+1} \Delta^2 + \Delta^j \Delta_j \bar{A}_j \right] \lambda(\nu_0), \]

where

\[ \Delta_j := |\partial \ell_k \cap U_{j-1}|, \quad \bar{A}_j := |\partial \ell_k \cap W_j|. \]
To prove that
\[ \kappa A + 2\Delta^j \Delta_j^2 + 2j \Delta_j \tilde{\Delta}_j \leq A \]
see the first line in (4.8), we use (3.10), take into account that \( \Delta \geq 2 \) (hence, \( j \leq \Delta^j \), \( j = 1, 2, \ldots, \chi - 1 \)) and obtain
\[ 2\Delta^j \Delta_j^2 + 2j \Delta_j \tilde{\Delta}_j \leq 2\Delta^j \Delta_j \left( \Delta_j + \tilde{\Delta}_j (j/\Delta^j) \right) \leq 2\Delta^{j+2} \leq A(1 - \kappa), \]
where we have taken into account that \( j + 2 \leq \chi + 1 \), see (4.10). To prove that the coefficient at \( \lambda(\nu_0) \) in (4.10) agrees with that in (4.8) we use the following estimates
\[
\begin{align*}
\bar{c} & \Delta^{j+1} \Delta_j^2 + \Delta^j \Delta_j \tilde{\Delta}_j + \bar{c} \Delta_j \tilde{\Delta}_j + \Delta_j^2 \\
& = \bar{c} \Delta^{j+1} \Delta_j \left( \Delta_j + \tilde{\Delta}_j \Delta^{-j} \right) + \Delta^j \Delta_j \left( \Delta_j + \tilde{\Delta}_j \Delta^{-(j+1)} \right) \\
& \leq \Delta^2 + \Delta^{j+2} \leq 2\Delta^{j+2} \leq A(1 - \kappa).
\end{align*}
\]
For \( \ell \in U_{j-1} \) by (4.7) and the first and third lines in (4.9) we obtain
\[
\nu_j^{(n)}(I_{\ell} H^i_{\ell'}) = \nu_j^{(k)}(I_{\ell} H^i_{\ell'}) \leq \sum_{\ell \in \partial \ell' \cap U_{j-1}} \nu_j(I_{\ell} H^i_{\ell'}) + \sum_{\ell \in \partial \ell' \cap W_j} \nu_j(I_{\ell} H^i_{\ell'}) \leq \left[ \Delta^j \Delta_j + j \tilde{\Delta}_j \right] \gamma(\nu_0) + \left[ \bar{c} \Delta^{j+1} \Delta_j + \bar{c} \tilde{\Delta}_j \right] \lambda(\nu_0) \leq \Delta^{j+1} \gamma(\nu_0) + \bar{c} \Delta^{j+2} \lambda(\nu_0),
\]
which yields the first line in (4.9) with \( j + 1 \).

For \( \ell' \in W_j \) and \( k \leq n \), by (3.7) and the second and fourth lines in (4.9) it follows that
\[
\nu_j^{(n)}(I_{\ell} H^i_{\ell'}) = \nu_j^{(k)}(I_{\ell} H^i_{\ell'}) \leq \sum_{\ell \in \partial \ell' \cap U_{j-1}} \nu_j(I_{\ell} H^i_{\ell'}) + \sum_{\ell \in \partial \ell' \cap W_j} \nu_j(I_{\ell} H^i_{\ell'}) \leq \left( \Delta^j \Delta_j + \tilde{\Delta}_j \right) \lambda(\nu_0) \leq \Delta^{j+1} \lambda(\nu_0),
\]
which agrees with the second line in (4.9).
For $\ell \in U_{j-1}$ and $k \leq n$, by (3.6) and the first and second lines in (4.9) we get
\[
\nu_j^{(n)}(I_{\ell}H_{\ell_k}^i) = \nu_j^{(k)}(I_{\ell}H_{\ell_k}^i) \leq \nu_j(I_{\ell}) + \sum_{\ell' \in \partial \ell \cap U_{j-1}} c_{\ell_k, e} \nu_j(I_{\ell}H_{\ell'}^i) + \sum_{\ell' \in \partial \ell \cap W_j} c_{\ell_k, e} \nu_j(I_{\ell}H_{\ell'}^i) 
\]
\[+ \sum_{\ell' \in \partial \ell \cap U_{j-1}} c_{\ell_k, e} \nu_j(I_{\ell}H_{\ell'}^i) \leq [\bar{k} + AK^{-1}] \gamma(\nu_0) + 2AK^{-1} \lambda(\nu_0) + [\Delta^j \gamma(\nu_0) + \bar{c} \Delta^j \lambda(\nu_0)] \sum_{\ell' \in \partial \ell \cap U_{j-1}} c_{\ell_k, e} + \Delta^j(2 + \bar{c}) \sum_{\ell' \in \partial \ell \cap W_j} c_{\ell_k, e}.
\]

In order for this to agree with the first line in (4.9), it is enough that the following holds
\[\bar{k} + AK^{-1} + \Delta^j \sum_{\ell' \in \partial \ell \cap U_{j-1}} c_{\ell_k, e} \leq \Delta^j + 1, \tag{4.11}\]
\[2AK^{-1} + \bar{c} \Delta^j + 1 \sum_{\ell' \in \partial \ell \cap U_{j-1}} c_{\ell_k, e} + \Delta^j \sum_{\ell' \in \partial \ell \cap W_j} c_{\ell_k, e} \leq \bar{c} \Delta^j + 2.
\]

Recall that we assume $\Delta \geq 2$. By (3.11) and (2.12) we get that the left-hand side of the first line in (4.11) does not exceed
\[\bar{k} + \bar{c}/2 + \Delta^{-1} < 2 < \Delta^j + 1, \quad \text{for } j = 1, \ldots, \chi - 1.
\]
Likewise, the left-hand side of the second line in (4.11) does not exceed
\[\bar{c} + \bar{c} \Delta^j \leq \bar{c}(2 + \Delta^j) < \bar{c} \Delta^j + 2 \quad \text{for } j = 1, \ldots, \chi - 1.
\]

For $\ell \in W_j$ and $k \leq n$, by (3.6) and the third and fourth lines in (4.9) we get
\[
\nu_j^{(n)}(I_{\ell}H_{\ell_k}^i) = \nu_j^{(k)}(I_{\ell}H_{\ell_k}^i) \leq \nu_j(I_{\ell}) \leq \gamma(\nu_0) + j \gamma(\nu_0) + \bar{c} \lambda(\nu_0) \sum_{\ell' \in \partial \ell \cap U_{j-1}} c_{\ell_k, e} + \lambda(\nu_0) \sum_{\ell' \in \partial \ell \cap W_j} c_{\ell_k, e} \leq (1 + j \bar{c}) \gamma(\nu_0) + \left(\bar{c} \sum_{\ell' \in \partial \ell \cap U_{j-1}} c_{\ell_k, e} + \sum_{\ell' \in \partial \ell \cap W_j} c_{\ell_k, e}\right) \lambda(\nu_0),
\]
which clearly agrees with the third line in (4.9).
Now we consider the cases where both $\ell, \ell'$ lie in $V_j$. For $k < m \leq n$, by first (3.6) and (3.7), and then by (3.5), we have

\[
\nu_j^n(I_{\ell_k}H_{\ell_m}^i) = \nu_j^m(I_{\ell_k}H_{\ell_m}^i) \leq \nu_j^k(I_{\ell_k}) + \sum_{\ell' \in \partial \ell_m} c_{\ell_m \ell'} \nu_j^k(I_{\ell_k}H_{\ell'_m}^i) \\
\leq \sum_{\ell \in \partial \ell_k} \kappa_{\ell_k \ell} \nu_j(I_{\ell}) + K^{-1} \sum_{s=1}^n \sum_{\ell, \ell' \in \partial \ell_k} \nu_j(I_{\ell}H_{\ell'}^i) \\
+ \sum_{\ell' \in \partial \ell_m} c_{\ell_m \ell'} \nu_j(I_{\ell}H_{\ell'_m}^i).
\]

(4.13)

The next step is to split the sums in (4.13) as it has been done in, e.g., (4.12), and then use (4.8) and (4.9). By doing so we get

\[
\nu_j^n(I_{\ell_k}H_{\ell_m}^i) \leq [(\bar{\kappa} + AK^{-1})\gamma(\nu_0) + 2AK^{-1}\lambda(\nu_0)] \sum_{\ell \in \partial \ell_k \cap U_{j-1}} \kappa_{\ell_k \ell} \\
+ \gamma(\nu_0) \sum_{\ell \in \partial \ell_k \cap W_j} \kappa_{\ell_k \ell} + 2K^{-1} \Delta_{j} \left[ \Delta^j \gamma(\nu_0) + \bar{c} \Delta^{j+1} \lambda(\nu_0) \right] \\
+ 2K^{-1} \Delta j \left[ \Delta^j \lambda(\nu_0) + j \gamma(\nu_0) + \bar{c} \lambda(\nu_0) \right] + 2K^{-1} \Delta^2_{j} \lambda(\nu_0) \\
+ \Delta_{j} \left[ \Delta^j \gamma(\nu_0) + \bar{c} \Delta^{j+1} \lambda(\nu_0) \right] \sum_{\ell' \in \partial \ell_m \cap U_{j-1}} c_{\ell_m \ell'} \\
+ \Delta^j \Delta_j \lambda(\nu_0) \sum_{\ell' \in \partial \ell_m \cap W_j} c_{\ell_m \ell'} + \Delta_j \left[ j \gamma(\nu_0) + \bar{c} \lambda(\nu_0) \right] \sum_{\ell' \in \partial \ell_m \cap U_{j-1}} c_{\ell_m \ell'} \\
+ \Delta_j \lambda(\nu_0) \sum_{\ell' \in \partial \ell_m \cap W_j} c_{\ell_m \ell'}.
\]
In order for this to agree with the first line in (4.9), it is enough that the following two estimates hold:

\[
\begin{align*}
(\bar{\kappa} + AK^{-1}) & \sum_{\ell \in \partial \ell \cap U_{j-1}} \kappa_{\ell, \ell} + \sum_{\ell \in \partial \ell \cap W_j} \kappa_{\ell, \ell} + 2K^{-1} \Delta_j^2 \Delta_j^i + 2K^{-1} \Delta_j \Delta_j^i + 2K^{-1} \Delta_j \Delta_j^i + 2K^{-1} \Delta_j \Delta_j^i & \leq \Delta_j^{i+1} \\
2AK^{-1} & \sum_{\ell \in \partial \ell \cap U_{j-1}} \kappa_{\ell, \ell} + 2K^{-1} \Delta_j^2 \Delta_j^i + 2K^{-1} \Delta_j \Delta_j^i + 2K^{-1} \Delta_j \Delta_j^i + 2K^{-1} \Delta_j \Delta_j^i & \leq \Delta_j^{i+1} \\
+2K^{-1} & \sum_{\ell' \in \partial \ell' \cap W_j} c_{\ell, \ell'} + \Delta_j \sum_{\ell' \in \partial \ell' \cap U_{j-1}} c_{\ell, \ell'} + \Delta_j \sum_{\ell' \in \partial \ell' \cap W_j} c_{\ell, \ell'} + \Delta_j \sum_{\ell' \in \partial \ell' \cap U_{j-1}} c_{\ell, \ell'} + \Delta_j \sum_{\ell' \in \partial \ell' \cap W_j} c_{\ell, \ell'} & \leq \Delta_j^{i+2}.
\end{align*}
\]

Taking into account that \( \bar{\kappa} < 1 \) and (3.11) one can show that the left-hand side of (4.14) does not exceed:

\[
1 + \bar{c}/2 + 2K^{-1} \Delta_j \Delta_j^i \left( \Delta_j + \bar{\Delta}_j(j/\Delta_j^i) \right) + \bar{c} \Delta_j^{i+1}
\leq 1 + \bar{c}/2 + \bar{c}/2 + \bar{c} \Delta_j^{i+1} < 2 + \frac{1}{\Delta^x} < \Delta_j^{i+1}.
\]

To prove (4.15) we use (3.11), (2.12), the inequality \( \Delta_j \bar{\Delta}_j \leq \Delta_j^2/4 \), and perform the following calculations:

\[
\begin{align*}
\text{LHS}(4.15) & \leq 2AK^{-1} \bar{\kappa} + \frac{1}{2} K^{-1} \Delta_j^{i+2} + 2K^{-1} \left( \Delta_j^2 + \bar{c} \Delta_j \bar{\Delta}_j + \bar{\Delta}_j^2 \right) + \Delta_j \sum_{\ell' \in \partial \ell' \cap W_j} c_{\ell, \ell'} + \Delta_j \sum_{\ell' \in \partial \ell' \cap U_{j-1}} c_{\ell, \ell'} \\
& + \bar{\Delta}_j \sum_{\ell' \in \partial \ell' \cap U_{j-1}} c_{\ell, \ell'} + \bar{\Delta}_j \sum_{\ell' \in \partial \ell' \cap W_j} c_{\ell, \ell'}
\end{align*}
\]

which holds even for \( j = 1, \chi = 2, \) and \( \Delta = 2 \).
Next, for $k \leq n$, by (3.8) we have

$$
\nu_j^{(n)}(I_k^j H_k^j) = \nu_j^{(k)}(I_k^j H_k^j) \leq \sum_{\ell, \ell' \in \partial \ell_k} \nu_j(I_{\ell}) + \sum_{\ell, \ell' \in \partial \ell_k} c_{\ell_k \ell'} \nu_j(I_{\ell} H_{\ell'})
$$

(4.16)

As above, we split the sums in (4.16) and then use (4.8) and (4.9), and obtain

$$
\nu_j^{(n)}(I_k^j H_k^j) \leq \Delta_j \left[ \bar{\kappa} + AK^{-1} \right] \gamma(\nu_0) + \Delta_j 2AK^{-1} \lambda(\nu_0) + \bar{\Delta}_j \gamma(\nu_0) + \Delta_j \left[ \Delta^j \gamma(\nu_0) + \bar{c} \Delta^{j+1} \lambda(\nu_0) \right] \sum_{\ell' \in \partial \ell_k \cap U_{j-1}} c_{\ell_k \ell'}
$$

$$
+ \Delta_j \Delta^j \lambda(\nu_0) \sum_{\ell' \in \partial \ell_k \cap W_j} c_{\ell_k \ell'} + \bar{\Delta}_j \left( j \gamma(\nu_0) + c \lambda(\nu_0) \right) \sum_{\ell' \in \partial \ell_k \cap W_{j-1}} c_{\ell_k \ell'}
$$

$$
+ \bar{\Delta}_j \lambda(\nu_0) \sum_{\ell' \in \partial \ell_k \cap W_j} c_{\ell_k \ell'}.
$$

(4.17)

In order for this to agree with the first line in (4.9), it is sufficient that the following two inequalities hold

$$
\Delta_j \left[ \bar{\kappa} + AK^{-1} \right] + \bar{\Delta}_j + \left( \Delta^j \Delta_j + j \bar{\Delta}^j \right) \sum_{\ell' \in \partial \ell_k \cap U_{j-1}} c_{\ell_k \ell'} \leq \Delta^{j+1},
$$

(4.18)

$$
2AK^{-1} \Delta_j + \bar{c} \Delta^{j+1} \Delta_j \sum_{\ell' \in \partial \ell_k \cap U_{j-1}} c_{\ell_k \ell'} + \Delta^j \Delta_j \sum_{\ell' \in \partial \ell_k \cap W_j} c_{\ell_k \ell'}
$$

$$
+ \bar{c} \bar{\Delta}_j \sum_{\ell' \in \partial \ell_k \cap U_{j-1}} c_{\ell_k \ell'} + \bar{\Delta}_j \sum_{\ell' \in \partial \ell_k \cap W_j} c_{\ell_k \ell'} \leq \bar{c} \Delta^{j+2}.
$$

(4.19)

By means of (3.11) we get

$$
\text{LHS}(4.18) \leq \Delta + \Delta AK^{-1} + \bar{c} \Delta^{j+1} < \Delta + \frac{1}{2\Delta^{k-1}} + 1 < \Delta^{j+1}.
$$

Similarly,

$$
\text{LHS}(4.19) \leq \bar{c} \Delta_j + \Delta_j \sum_{\ell' \in \partial \ell_k \cap U_{j-1}} c_{\ell_k \ell'} + \Delta^j \Delta_j \sum_{\ell' \in \partial \ell_k \cap W_j} c_{\ell_k \ell'}
$$

$$
+ \bar{c} \bar{\Delta}_j \sum_{\ell' \in \partial \ell_k \cap U_{j-1}} c_{\ell_k \ell'} + \bar{\Delta}_j \sum_{\ell' \in \partial \ell_k \cap W_j} c_{\ell_k \ell'}
$$

$$
\leq \bar{c} \Delta + \bar{c} \Delta^j \Delta_j + \bar{c} \bar{\Delta}_j < \bar{c} \Delta + \bar{c} \Delta^{j+1} \leq \bar{c} \Delta^{j+2}.
$$
Now we consider the case where \( m < k \leq n \). By (3.7), and then by (3.6), we have
\[
\nu_j^{(n)}(I_{\ell_k} H_{\ell_m}^k) = \nu_j^{(k)}(I_{\ell_k} H_{\ell_m}^k) \leq \sum_{\ell \in \partial \ell_k} \nu_j^{(m)}(I_{\ell} H_{\ell_m}^\ell) \leq \sum_{\ell \in \partial \ell_k} \nu_j(I_{\ell}) + \sum_{\ell \in \partial \ell_m} \sum_{\ell' \in \partial \ell_m} c_{\ell_m} \nu_j(I_{\ell} H_{\ell'}^\ell).
\]
Again we split the sums in (4.20) and then use (4.8) and (4.9), and obtain that
\[
\nu_j^{(n)}(I_{\ell_k} H_{\ell_m}^k) \leq \text{RHS}(4.17).
\]
Thus, we have that (4.9) with \( j + 1 \) holds also in this case. The proof is complete.

4.3. The proof of Lemma 3.8. Assume that we have given \( \nu_{s-1}^x \in C(\mu^x, \mu) \) with the properties in question. Then we split \( D_{N-s-1} \) into independent subsets by taking intersections with the sets \( V_j \), as in (2.3). Let \( \ell^1, \ldots, \ell^m \) be a numbering of \( D_{N-s-1} \cap V_0 \). Set
\[
\nu_0^x = \nu_{s-1}^x \quad \text{and} \quad \nu_k^x = R_{\ell_k} \nu_{k-1}^x, \quad k = 1, \ldots, m,
\]
where \( R_{\ell} \) is defined in (3.2). Thus, \( \nu_m^x \) is \( F_{D_{N-s-1}} \)-measurable, and \( \nu_m^x(I_{\ell}) \) and \( \nu_m^x(I_{\ell} H_{\ell'}^\ell), \ell, \ell' \in D_{N-s-1} \), satisfy the inequalities in (4.6) and (4.7), respectively, in which the right-hand sides contain \( \gamma_{D_{N-s-1}}(\nu_{s-1}^x) \) and \( \lambda_{D_{N-s-1}}(\nu_{s-1}^x) \). Then we perform the reconstruction over the remaining independent subsets of \( D_{N-s-1} \) and obtain an element of \( C(\mu^x, \mu) \), which we denote by \( \nu_s^x \). Its \( F_{D_{N-s-1}} \)-measurability is then guaranteed by construction, and the parameters \( \gamma_{D_{N-s-1}}(\nu_s^x) \) and \( \lambda_{D_{N-s-1}}(\nu_s^x) \) satisfy the first-line inequalities in (4.8) and (4.9), respectively, and hence (3.17) with \( \gamma_{D_{N-s}}(\nu_{s-1}^x) \) and \( \lambda_{D_{N-s}}(\nu_{s-1}^x) \) on the right-hand side. The \( F_{D_{N-s}} \)-measurability of \( \nu_0^x \) is straightforward.

REFERENCES

[1] S. Albeverio, Yu. Kondratiev, Yu. Kozitsky, and M. Röckner, *The statistical mechanics of quantum lattice systems. A path integral approach*. EMS Tracts in Mathematics, 8. European Mathematical Society (EMS), Zürich, 2009.

[2] G. R. Belitskii and Yu. I. Lyubich, *Matrix norms and their applications*. Translated from the Russian by A. Iacob. Operator Theory: Advances and Applications, 36. Birkhäuser Verlag, Basel, 1988.

[3] V. Belitsky and E. A. Pechersky, Uniqueness of Gibbs state for non-ideal gas in \( \mathbb{R}^d \): the case of multibody interaction, *J. Stat. Phys.* 106, 931–955 (2002).

[4] S. A. Bethuelsen, Uniqueness of Gibbs measures, Master Thesis, Utrecht, 2012.

[5] M. Cassandro, E. Olivieri, A. Pellegrinotti, and E. Presutti, Existence and uniqueness of DLR measures for unbounded spin systems, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 41, 313–334 (1977/78).

[6] T. de la Rue, R. Fernández, and A. D. Sokal, How to clean a dirty floor: probabilistic potential theory and the Dobrushin uniqueness theorem, *Markov Process. Related Fields* 14, 1–78 (2008).
[7] R. L. Dobrushin, Description of a random field by means of conditional probabilities and conditions for its regularity. (Russian) Teor. Verojatnost. i Primenen 13, 201–229 (1968).

[8] R. L. Dobrushin and E. A. Pechersky, A criterion of the uniqueness of Gibbsian fields in the noncompact case. Probability theory and mathematical statistics (Tbilisi, 1982), 97–110, Lecture Notes in Math., 1021, Springer, Berlin, 1983.

[9] H.-O. Georgii, Gibbs measures and phase transitions. Studies in Mathematics, 9, Walter de Gruyter, Berlin New York, 1988.

[10] D. Hagedorn, Y. Kondratiev, T. Pasurek, and M. Röckner, Gibbs states over the cone of discrete measures, J. Func. Anal. 264, 2550–2583 (2013).

[11] Y. Kozitsky and T. Pasurek, Euclidean Gibbs measures of interacting quantum anharmonic oscillators, J. Stat. Phys. 127, 985–1047 (2007).

[12] H. Künsch, Decay of correlations under Dobrushin’s uniqueness condition and its applications, Comm. Math. Phys. 84, 2017–222 (1982).

[13] J. L. Lebowitz, A. Mazel, and E. Presutti, Liquid-vapor phase transitions for systems with finite-range interactions, J. Stat. Phys. 94, 955–1025 (1999).

[14] J. L. Lebowitz and E. Presutti, Statistical mechanics of systems of unbounded spins, Comm. Math. Phys. 50, 195–218 (1976).

[15] T. Lindvall, Lectures on the coupling method, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1992.

[16] L. Lovász, Three short proofs in graph theory, J. Combinatorial Theory Ser. B 19, 269–271 (1975).

[17] V. A. Malyshev and I. V. Nickolaev, Uniqueness of Gibbs fields via cluster expansions, J. Stat. Phys. 53, 375–379 (1984).

[18] T. Pasurek, Theory of Gibbs measures with unbounded spins: probabilistic and analytic aspects. Habilitation Thesis, Universität Bielefeld, Bielefeld, 2008; available as SFB 701 Preprint 08101, 2008 at https://www.math.uni-bielefeld.de/sfb701/preprints/view/292.

[19] E. Pechersky and Yu. Zhukov, Uniqueness of Gibbs state for nonideal gas in $\mathbb{R}^d$: the case of pair potentials, J. Stat. Phys. 97, 145–172 (1999).

[20] Ch. Preston, Specifications and their Gibbs states, Universität Bielefeld, 2005, available at http://www.mathematik.uni-bielefeld.de/~preston/rest/gibbs/files/specifications.pdf

[21] A. Procacci and B. Scoppola, On decay of correlations for unbounded spin systems with arbitrary boundary conditions, J. Stat. Phys. 105, 453–482 (2001).

[22] D. Putan, Uniqueness of equilibrium states of some models of interacting particle systems. PhD Thesis, Universität Bielefeld, Bielefeld, 2014; available at http://pub.uni-bielefeld.de/luur/download?func=downloadFile&recordOId=2691509&fileOId=2691511

Fakultät für Mathematik, Universität Bielefeld, Bielefeld D-33615, Germany
E-mail address: dputan@math.uni-bielefeld.de

Fakultät für Mathematik, Universität Bielefeld, Bielefeld D-33615, Germany
E-mail address: kondrat@math.uni-bielefeld.de

Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, 20-031 Lublin, Poland
E-mail address: jkozi@hektor.umcs.lublin.pl

Fakultät für Mathematik, Universität Bielefeld, Bielefeld D-33615, Germany
E-mail address: pasurek@math.uni-bielefeld.de