Abstract In this paper, we analyse the JLA data on Supernova observations in the context of \( k \)-essence dark energy model with Lagrangian \( L = V F(X) \), with a constant potential \( V \) and the dynamical term \( X = (1/2)\nabla_\mu \phi \nabla^\mu \phi = \dot{\phi}^2/2 \) for a homogeneous scalar field \( \phi(t) \), in a flat FRW spacetime background. Scaling relations are used to extract temporal behaviour of different cosmological quantities and the form of the function \( F(X) \) from the data. We explore how the parameters of the model, viz. value of the constant potential \( V \) and a constant \( C \) appearing in the emergent scaling relation, control the dynamics of the model in the context of JLA data, by setting up and analysing an equivalent dynamical system described by a set of autonomous equations.

1 Introduction

From the observation of type Ia Supernovae (SNe Ia), it was first reported independently in 1998 by Riess et.al. [1] and Perlmutter et.al. [2] that the present universe is undergoing an accelerated expansion and a transition happened from decelerated to this accelerated phase of expansion during late time phase of cosmic evolution of the universe. The luminosity distances and redshifts of SNe Ia are the key observational ingredients in establishing the features of the late-time cosmic evolution. The present-day data of observed SNe Ia events using diverse probes in different supernova surveys include various compilations corresponding to different redshift regions. The small redshift (\( z > 0.1 \)) projects comprise Harvard–Smithsonian Center for Astrophysics survey [3], the Carnegie Supernova Project [4–6] the Lick Observatory Supernova Search [7] and the Nearby Supernova Factory [8]. SDSS-II supernova surveys [9–13] are mainly focused on the redshift region of (0.05 < \( z < 0.4 \)). Programmes like Supernova Legacy Survey [14,15] the ESSENCE project [16], the Pan-STARRS survey [17,18] correspond to the high redshift regime. Around one thousand SNe Ia events have been discovered through all surveys. In the range between \( z \sim 0.01 \) and \( z \sim 0.7 \), luminosity distance has shown a very high statistical precision. ‘Joint Light-curve Analysis (JLA) data’ [18–20] has been newly released, which contains total of 740 SNe Ia events. This entire data sample with the observed values of luminosity distances and redshifts of SNe Ia events has been analysed in the context of various cosmological studies to obtain features of the late-time cosmic acceleration.

Dark energy, a general label for the source of this late-time cosmic acceleration, has been hypothesised as an unclustered form of energy with negative pressure - the negative pressure leading to the cosmic acceleration by counteracting the gravitational collapse. The phenomenological \( \Lambda \)-CDM model [21] of dark energy, though fits well with cosmological data, is plagued with the fine-tuning problem from viewpoint of particle physics. Alternative approaches aiming construction of models of dark energy include the field theoretic models, viz. quintessence and \( k \)-essence models, in which the cosmic acceleration is driven respectively by scalar fields with slowly varying potentials and kinetic energy associated with the scalar field through the energy-momentum tensor of Einstein Field equations. There are also other viable models of dark energy based on modification of geometric part of Einstein’s equation (\( f(R) \) gravity models [22–27]), scalar tensor theories [28–32], brane world models, etc. [33,34]

In this paper, we consider the interesting phenomenological consequences of the \( k \)-essence model extracted from the SNe Ia data and attempted to link the extracted dynamical features of the model with a dynamical system whose evolution is governed by a set of autonomous equations. The approach of using dynamical systems in the study of cosmology has been discussed in detail in [35–44]. We assume an isotropic and homogeneous spacetime geometry of the universe described by a flat Friedmann–Robertson–Walker (FRW) metric involving the time-dependent scale factor \( a(t) \). The content of the universe during its late-time evolution is approximated to be composed of dark matter and dark energy which is consistent with the observations from Planck collaborations that these two components comprise around 96% of the present-day universe [45]. The dark matter and dark energy are modelled as mutually non-interacting ideal perfect fluids characterised by their respective energy densities and pressures symbolised as \( \rho_{\text{dm}}, \rho_{\text{de}} \) and \( \rho_{\text{de}, p_{\text{de}}} \), with dark matter as non-relativistic dust implying \( \rho_{\text{de}} = 0 \). We consider dark energy to be represented...
by a homogeneous scalar field $\phi(t)$ driven by $k$-essence Lagrangian of the form $L = V(\phi)F(X)$, where $X \equiv \frac{1}{2}g_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi = \frac{1}{2}\dot{\phi}^2$ and the potential $V(\phi) = V$ is taken to be a constant. The constancy of the $k$-essence potential ensures existence of a scaling relation $XF_X^2 = Ca^{-\sigma}$ ($F_X \equiv dF/dX$ and $C$ a constant), which connects the scalar field $\phi(t)$ with scale factor $a(t)$. From the model-independent analysis of the JLA SNe Ia data we obtain the temporal behaviour of the FRW scale factor $a(t)$. We use this in the context of our model to obtain the temporal behaviour of the scalar field $\phi(t)$ and consequently extract the $X$-dependence of the dynamical term $F(X)$ in the $k$-essence Lagrangian.

We then show that cosmological evolution in the context of such a $k$-essence model of dark energy (having constant potential) with the form of $F(X)$ and temporal behaviour of relevant cosmological quantities, as extracted from the analysis of JLA data, can be mapped to the evolution of a dynamical system with properly chosen dimensionless variables $x$ and $y$ in terms of relevant cosmological quantities ($\phi(t)$, $a(t)$ and its derivatives) and parameters (value of the constant potential $V$, the constant $C$ in the scaling relation) of the model. We investigate the behaviour of the dynamical system and analyse its features for different chosen values of involved set of parameters. This provides an indirect approach for realising the effect of the numerical values of the constants $V$ and $C$ in the $k$-essence cosmological model of dark energy with constant potential in the context of JLA data.

The paper is organised as follows. In Sect. 2 we discussed the methodology of analysis of JLA data for obtaining temporal behaviour of different relevant cosmological quantities during the late-time phase of cosmic evolution. In Sect. 3 we briefly discussed the $k$-essence model with constant potential and used the scaling relation to establish the connection between the cosmological quantities and the quantities $X$ and $F(X)$ which governs the dynamics of the $k$-essence Lagrangian. We also presented how we used the temporal dependences of the cosmological quantities as extracted from the analysis of JLA data to reconstruct the form of the function $F(X)$. In Sect. 4 we discussed the mapping of dynamical aspects of the $k$-essence model considered along with observational inputs from JLA data to a two-dimensional dynamical system driven by a set of autonomous equations involving the model parameters $V$ and $C$. The study of fixed points of the system, based on linear stability theory, has been presented in this section and the implications of the values of the parameters ($C$, $V$) in the determination of the fixed points have been investigated. We summarise the conclusions of the paper in Sect. 5.

## 2 Cosmological parameters from JLA data

The recently released ‘Joint Light-curve Analysis’ (JLA) compilation of SNe Ia data [18–20], as discussed in Sect. 1, consists of luminosity distance and redshift measurements of 740 SNe Ia events. This data set involves a compilation of SNe Ia light curves including SNe Ia data from the three-year SDSS survey, first three seasons of the five-year SNLS survey and 14 data points in the very high redshift $0.7 < z < 1.4$ domain from HST [46]. To take care of the different systematic uncertainties involved in the data we analyse the compilation of the data with flux-averaging technique described in [47–49]. The $\chi^2$-function corresponding to JLA data is given by

$$\chi^2 = \sum_{i,j=1}^{740} (\mu_{\text{obs}}^{(i)} - \mu_{\text{th}}^{(i)})(\sigma^{-1})_{ij}(\mu_{\text{obs}}^{(j)} - \mu_{\text{th}}^{(j)}) ,$$

where $\mu_{\text{obs}}^{(i)}$ denotes the theoretical expression for distance modulus in a flat FRW spacetime background at redshift $z_i$ which is related to the corresponding luminosity distance $d_L$ through

$$\mu_{\text{th}}^{(i)} = 5 \log_{10}[d_L(z_{\text{hel}}, z_{\text{CMB}})/\text{Mpc}] + 25$$

where

$$d_L(z_{\text{hel}}, z_{\text{CMB}}) = (1 + z_{\text{hel}})r(z_{\text{CMB}}) \quad \text{with} \quad r(z) = cH_0^{-1} \int_0^z \frac{dz'}{E(z')}$$

$z_{\text{CMB}}$ and $z_{\text{hel}}$ are SNe Ia redshifts in CMB rest frame and in heliocentric frame, respectively, and $H_0$ is the value of Hubble parameter at present epoch. The observed value of distance modulus $\mu_{\text{obs}}^{(i)}$ at redshift $z_i$ is expressed as

$$\mu_{\text{obs}}^{(i)} = m_B^* - M_B + \alpha X_1(z_i) - \beta C(z_i)$$

in terms of the observed peak magnitude $m_B^*$, the time stretching parameter of the light-curve $X_1$ and supernova colour at maximum brightness, $C$. $\alpha$ is the nuisance parameters and $M_B$ is the absolute magnitude kept fixed at $M_B = -19$ for the analysis performed in [48, 50]. The technical details of different terms involved in $\chi^2$ and their handling in the analysis of the data have been comprehensively discussed in [48–51]. $\sigma_{ij}$ is the comoving matrix as given in Eq. (2.16) of [48]. The systematic uncertainties involved in the covariant matrix, instead of dealing individually, may be handled by a flux averaging technique proposed by Wang in [48], which reduces the effect of systematic uncertainties owing to weak lensing of SNe Ia data. By this technique, one recovers unlensed brightness of SNe Ia events at some redshift resulting from averaging of flux of all SNe Ia events corresponding to that redshift [52]. It has also been shown in [53–55] that the bias in distance estimation of SNe Ia events due to systematic effects can be
controlled and reduced by this technique. The flux averaging technique has been comprehensively discussed in [49], and it involves introduction of a redshift cut-off \( z_{\text{cut}} \) to separate out SN samples with \( z < z_{\text{cut}} \) and \( z \geq z_{\text{cut}} \). For samples with \( z < z_{\text{cut}} \), Eq. (1) has been used to compute the \( \chi^2 \) and for samples with redshifts above \( z_{\text{cut}} \), averaging of distance modulus \( \mu \) and the covariant matrix over all the fluxes of SNe Ia samples has been performed, and the resulting average values have been used to compute the \( \chi^2 \).

As mentioned in Sect. 1, the fact that the phenomenological \( \Lambda \)-CDM model is plagued with the fine-tuning problem of particle physics motivates investigation of alternative models of dark energy. A key feature of a certain class of such models, called varying dark energy models, is the time varying equation of state (EOS) \( w = p/\rho \) of dark energy. \( (p \) is the energy density and \( \rho \) the pressure of dark energy.) This time variation is usually expressed in terms of variation of \( w \) with redshift \( z \). The \( z \) dependence of the EOS parameter \( w(z) \), for the varying dark energy models, may be constrained from the observational data. The approach involves consideration of various functional forms of \( w(z; \omega, \omega) \), involving parameters \( \omega \) and \( \omega \) and subsequently realising the observational constraints on \( w(z) \) in terms of constraints in \( \omega \) and \( \omega \) parameter space. In [50], we have presented the results of comprehensive analysis of joint light-curve analysis data to obtain constraints in \( \omega \) and \( \omega \) parameter space for some benchmark models—CPL [56], JBP [57, 58], BA [59, 60] and Logarithmic model [61]—each depicting a characteristic functional form of \( w(z) \). Also in [48], Wang et al. performed a comparative analysis of JLA data and presented the results of analysis of their model and performed a comparative study of results of various models. When we consider the evolution of universe in a FRW spacetime background during its late time phase cosmic evolution which is primarily governed by its dark matter and dark energy contents (which constitutes 96% of the present-day universe contents), the reduced Hubble parameter \( E(z) \) is given by

\[
E(z) = \frac{H(z)}{H_0} = \sqrt{\Omega_{dm}(1+z)^3 + \Omega_{de}X(z)}
\]  

(5)

where \( \Omega_{dm} \) and \( \Omega_{de} \approx 1 - \Omega_{dm} \) are the fractional densities of dark matter and dark energy. The dark energy density function \( X(z) \) is related to dark energy equation of state parameter \( w \) as

\[
X(z) = \exp \left[ 3 \int_0^z \frac{1 + w(z')} {1 + z'} dz' \right]
\]  

(6)

In Table 1 we have presented the best fit values of the parameters \( \omega_a, \omega_b \) obtained from the analysis of JLA data, in [50] for CPL, JBP, BA models and in [48] for Wang model.

The profile of the equation of state parameter \( w(z) \) is translated to the profile of the reduced Hubble constant \( H(z) \) by virtue of Eqs. (5) and (6). For our analysis, in this work, we consider the Wang model and take the \( z \)-dependence of the function \( E(z) = H(z)/H_0 \) obtained in [48] from \( \chi^2 \)-marginalisation with respect to \( M_B \) and the nuisance parameters, taking flux averaged values of distance modulus and the covariant matrix corresponding to a zero redshift cutoff where \( H(z) \) is the Hubble parameter \( da/\alpha \) expressed terms of redshift and \( H_0 \) being its present epoch ( \( z = 0 \) ) value. We consider the 1σ range of the quantity \( E(z) \) at every \( z \), resulting from the above analysis is shown in Fig. 1. We obtain the average of the \( E(z) \) values in this 1σ range for each \( z \), which is depicted by the dashed line in Fig. 1. We consider this central \( E(z) \) vs \( z \) curve as benchmark for extracting temporal behaviour of other relevant cosmological quantities. We have also obtained the 1σ uncertainties of the cosmological quantities corresponding to the 1σ uncertainties of \( E(z) \) as shown in left panel of Fig. 1. In the same plot, we have also depicted the \( E(z) \) vs \( z \) profile for the other models at corresponding best-fit points \( (\omega_a, \omega_b) \) obtained in [50].

In the right panel of Fig. 1, for comparison, we have also shown the variation of the deceleration parameter \( q = -\ddot{a}/\dot{a}^2 \), which is a dimensionless measure of the cosmic acceleration, as a function of a chosen dimensionless time parameter \( \eta \) [chosen to depict temporal behaviour of relevant cosmological quantities, see Eq. (20)], at the best fit values of \( (\omega_a, \omega_b) \) corresponding to different models of variation of \( w(z) \).

The \( z \)-dependence of \( E(z) \) as extracted from the observational data can be exploited to find the temporal behaviour of the FRW scale factor \( a(t) \). The numerical method of obtaining this is briefly described below. The scale factor \( a \) which is normalised to \( a = 1 \)
which on integration gives

\[ \frac{1}{a} = 1 + \frac{1}{z} . \]  

Using this we may write

\[ dt = -\frac{dz}{(1+z)H_0 E(z)} , \]

which on integration gives

\[ \frac{t(z)}{t_0} = 1 - \frac{1}{H_0 t_0} \int_z^0 \frac{dz'}{(1+z')E(z')} \]  

where \( t_0 \) denotes the present epoch. Using the \( E(z) \) vs \( z \) profile as depicted in Fig. 1, obtained from the analysis of JLA data by methodology described above, we perform the above integration numerically to obtain \( z \) dependence of \( t(z) \). Equations (7) and (9) together provide the machinery to numerically compute simultaneous values of \( a \) and \( t \) at any given redshift \( z \). This amounts to obtaining values of \( a(t) \) at corresponding \( t \) eliminating \( z \) from Eqs. (7) and (9) leading to extracting temporal behaviour of the scale factor \( a(t) \) from the observational data.

To perform this, we vary \( z \) from zero (present epoch) to \( \sim 1 \) (i.e. within the accessible domain of \( z \) relevant for JLA data set) in small steps (\( \Delta z = 0.01 \)). We numerically evaluate the integral in Eq. (9) and simultaneously compute value \( a(z) \equiv 1/(1+z) \) [Eq. (7)] at each \( z \)-step, to obtain the sets of values \( \{t(z), a(z)\} \) at each step of values of \( z \) within its above-mentioned range. We consider scale factor \( a(t) \) to be normalised to unity at present epoch (\( z = 0 \) or \( t = 1 \)) and found that, the range \( 0 < z < 1 \) corresponds to \( t \)-range: \( 1 > t(z) > 0.44 \). The obtained set of values of \( \{t(z), a(z)\} \) for the entire \( z \)-range thus gives variation of the scale factor with time over the time range \( 0.44 < t < 1 \). The obtained \( t \)-dependence of the scale factor \( a(t) \) corresponding to the best fit of the Wang model is shown in left panel of Fig. 2. Using the obtained temporal profile of the scale factor \( a(t) \), we used numerical differentiation to obtain the time dependences of the time derivatives of the scale factor, viz. \( \dot{a}(t) \) and \( \ddot{a}(t) \) and the obtained temporal profiles are, respectively, shown in middle panel and right panel of Fig. 2. The transition from decelerated to accelerated phase of expansion during the late time cosmic evolution, as probed by the SNe Ia observations, is signified by the appearance of the minima at \( t \sim 0.52 \) in the time profile of \( \dot{a} \) (middle panel) or, equivalently, by the change of sign of \( \ddot{a} \) (transition from \( \ddot{a} < 0 \) to \( \ddot{a} > 0 \) regime) at the same epoch (\( t \sim 0.52 \)) in the time-profile of \( \ddot{a} \) (right panel). The temporal behaviour of scalar field and its time derivatives are instrumental in determining temporal profiles of various other cosmological parameters like equation of state \( (w) \) and energy density \( (\rho_{\text{tot}} + \rho_{\text{de}}) \) of the total dark fluid, the pressure \( (P_{\text{de}}) \) of the dark energy fluid. All these information, together, provides the necessary observational input for exploring and analysing aspects of the \( k \)-essence model of dark energy (with a constant potential) considered in the context of this paper. The corresponding methodology has been comprehensively discussed in Sects. 3 and 4.

We describe below, how we may exploit the observed temporal behaviour of temporal dependence of the scale factor to extract the temporal behaviour of some cosmological parameters like, equation of state parameter \( \omega \) of the total dark fluid (dark matter plus dark energy), its total energy density and pressure. In this context we indulge in a brief recollection of the fundamental equations of cosmology.
Fig. 2 Plot of scale factor $a(t)$ (left panel), $\dot{a}(t)$ (middle panel) and $\ddot{a}(t)$ (right panel) as a function of $t$ corresponding to the best-fit of Wang’s model. The scale factor is normalised to unity at the present epoch ($t = 1$). The epoch of transition from decelerated phase to accelerated phase of cosmic evolution ($t \sim 0.52$) is marked on the $t$–axis in $\dot{a}$ vs. $t$ and $\ddot{a}$ vs. $t$ plots in middle panel and right panel figures, respectively.

The late-time cosmic evolution in a FRW spacetime background with dark matter and dark energy as the primary content of the universe is governed by the Friedmann equations

$$H^2 = \frac{\kappa^2}{3} (\rho_{\text{dm}} + \rho_{\text{de}})$$

(10)

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6} \left[ (\rho_{\text{dm}} + \rho_{\text{de}}) + 3p_{\text{de}} \right]$$

(11)

where $\kappa^2 \equiv 8\pi G$ ($G$ is the Newton’s Gravitational constant), and both dark matter and dark energy are considered as ideal fluids characterised by their respective energy densities and pressure: $(\rho_{\text{dm}}, p_{\text{dm}})$ for dark matter and $(\rho_{\text{de}}, p_{\text{de}})$ for dark energy. Besides, dark matter is considered as non-relativistic dust implying $p_{\text{dm}} = 0$. We have considered a flat spacetime and ignore contributions from radiation and baryonic matter during late time phase of cosmic evolution. Using the above equations, the equation of state $w$ of the total dark fluid (dark matter plus dark energy) can be expressed in terms of the scale factor and its higher time derivatives as

$$w \equiv \frac{p_{\text{de}}}{\rho_{\text{dm}} + p_{\text{de}}} = -\frac{2}{3} \frac{\ddot{a}}{a^2} - \frac{1}{3}$$

(12)

Combining Eqs. (10) and (11) we obtain the continuity equation

$$(\dot{\rho}_{\text{dm}} + \dot{\rho}_{\text{de}}) + 3H(\rho_{\text{dm}} + \rho_{\text{de}} + p_{\text{de}}) = 0$$

(13)

which represents energy conservation in late time universe comprising dark matter and dark energy. When there is no interaction between dark matter and dark energy, energy conservation is separately respected for both fluids and are represented by following equations.

$$\dot{\rho}_{\text{de}} + 3H(\rho_{\text{de}} + p_{\text{de}}) = 0$$

(14)

$$\dot{\rho}_{\text{dm}} + 3H(\rho_{\text{dm}}) = 0$$

(15)

The solution of Eq. (15) is given by

$$\rho_{\text{dm}} = \rho_{\text{dm}}^0 a^{-3}.$$  

(16)

We use symbols with index ‘0’ in superscript to denote corresponding present-epoch values of the quantities referred by the symbols. Using the temporal behaviour of the scale factor as extracted from the JLA data, we may exploit Eq. (12) to obtain the time dependence of the equation of state parameter $w$ of total dark fluid over the time domain accessible in SNe Ia corresponding to the JLA data. The time domain as probed in the JLA data can be expressed in terms of a dimensionless time parameter $\eta$ as $-0.7 < \eta < 0$ where $\eta$ is defined as

$$\eta = \ln a$$

(17)

where $\eta = 0$ corresponds to present epoch (as $a$ at present epoch is normalised to unity). Note that, the equation of state $w$ of the total dark fluid is related to scale factor and its time derivatives by Eq. (12). The obtained temporal profile of the scale factor $a(t)$ and its time derivatives $\dot{a}(t)$ and $\ddot{a}(t)$ (presented in Fig. 2) can be used in Eq. (12) to obtain the temporal behaviour of the equation.
of state $w(t)$. Using the temporal profile of $a(t)$, we may also use Eq. (17) to get the relation between $\eta$ and $t$. Thus using the simultaneous values of $w(t)$ and $\eta$ at any $t$, we can compute values of $w$ corresponding to the value of $\eta$. The $w(\eta)$ profile, thus obtained, also expresses the temporal behaviour of the equation of state of the total dark fluid in terms of our chosen time parameter $\eta$. We have chosen a suitable polynomial to express the obtained $\eta$–dependence of $w$ by fitting the coefficients of the polynomial with the obtained $w(\eta)$ profile. The time-dependence of $w(\eta)$ within its 1σ range, extracted from the analysis of the JLA data are shown in left panel of Fig. 3. We find that the temporal behaviour of $w(\eta)$, corresponding to the central best-fit line in the left panel of Fig. 3, may be fitted with a polynomial of order 5, which we express as

$$w(\eta) = -1 + \sum_{i=0}^{4} B_i \eta^i$$  \hspace{1cm} (18)

with values of the coefficients $B_i$ at best-fit is given by

$$B_0 = -0.70, \quad B_1 = -0.61, \quad B_2 = -0.49, \quad B_3 = -2.29, \quad B_4 = -2.81, \quad B_5 = -0.92, \quad \text{and} \quad B_i = 0 \text{ for } i > 5$$  \hspace{1cm} (19)

Using the temporal behaviour of $w(\eta)$ we also obtain the time dependence of the quantities $(\rho_{\text{dm}} + \rho_{\text{de}})$ and $p_{\text{de}}$ over the late time domain $-0.7 < \eta < 0$. In terms of the parameter $\eta$ the continuity Eq. (13) for the total dark fluid takes the form

$$\frac{d}{d\eta} \ln \left( \rho_{\text{dm}} + \rho_{\text{de}} \right) = -3 \left( 1 + w(\eta) \right)$$  \hspace{1cm} (20)

which on integration gives

$$\frac{(\rho_{\text{de}} + \rho_{\text{dm}})_\eta}{(\rho_{\text{de}} + \rho_{\text{dm}})_0} = \exp \left[ -3 \int_0^\eta (1 + w(\eta'))d\eta' \right]$$  \hspace{1cm} (21)

We use the obtained form of $w(\eta)$ as given in Eq. (18) with the best-fit values of coefficients $B_i$’s [Eq. (19)] and perform the integration in the right-hand side of Eq. (21) numerically, to obtain the total energy density as a function of $\eta$. We find that the obtained dependence can be fitted with an order polynomial of order 5 expressed in the form

$$\frac{(\rho_{\text{de}} + \rho_{\text{dm}})_\eta}{(\rho_{\text{de}} + \rho_{\text{dm}})_0} = \sum_{i=0}^{4} C_i \eta^i$$  \hspace{1cm} (22)

with the best-fit values of the coefficients $C_i$’s given by

$$C_0 = 1, \quad C_1 = -0.89, \quad C_2 = 1.28, \quad C_3 = -0.65, \quad C_4 = 1.36, \quad C_5 = -0.97, \quad \text{and} \quad C_i = 0 \text{ for } i > 5$$  \hspace{1cm} (23)

Also from Eq. (12) we can write

$$\frac{(\rho_{\text{de}})_\eta}{(\rho_{\text{de}} + \rho_{\text{dm}})_0} = w(\eta) \cdot \frac{(\rho_{\text{de}} + \rho_{\text{dm}})_\eta}{(\rho_{\text{de}} + \rho_{\text{dm}})_0}$$  \hspace{1cm} (24)

Using Eqs. (18) and (22) we numerically evaluated the right-hand side of the above equation for any $\eta$ and find that, the obtained dependence fits best with a polynomial of order 4 expressed as

$$\frac{(p_{\text{de}})_\eta}{(p_{\text{de}} + p_{\text{dm}})_0} = \sum_{i=0}^{3} D_i \eta^i$$  \hspace{1cm} (25)

with the coefficients $D_i$’s given as

$$D_0 = 0.71, \quad D_1 = 0.002, \quad D_2 = -0.93, \quad D_3 = -2.27, \quad D_4 = -1.48, \quad \text{and} \quad D_i = 0 \text{ for } i > 4$$  \hspace{1cm} (26)

The obtained time dependence of $(\rho_{\text{dm}} + \rho_{\text{de}})$ and $p_{\text{de}}$ thus extracted from the analysis of the JLA data are shown by dashed lines in middle and right panel of Fig. 3, respectively. The corresponding 1σ ranges of the quantities have also been obtained and are shown in Fig. 3 by shaded regions.

### 3 k-essence model with constant potential

We now try to realise the dynamics of dark energy in terms of a homogeneous scalar field $\phi(t)$ whose dynamics is driven by a k-essence Lagrangian of the form $L = V(\phi)F(X)$. The pressure and energy density of dark energy in this model can be expressed as

$$p_{\text{de}} = V F(X)$$  \hspace{1cm} (27)
The existence of a scaling relation implies the presence of relevant scales in the theory. This simple class of models were first explored as a model of inflation in [67]. As elaborated in [65], such class of models may lead to unified dark matter with same equation of state as that of ordinary dark matter plus a cosmological constant with an effective sound speed which is very small. Throughout the work, we have considered $V$ to be constant so that scaling is preserved. The resulting Eq. (30) establishes a connection between the time derivative of the $k-$essence scalar field ($\dot{\phi}$) and the scale factor $a(t)$ whose temporal behaviour for late time cosmic evolution can be extracted from observed SNe Ia data. The scaling relation is instrumental in extracting functional form of $F(X)$ over certain domain of $X$.

Adding Eqs. (27), (28) and then substituting $F_X$ using (30) we obtain

$$X = \frac{a^6 (\rho_{de} + \rho_{de})^2}{4CV^2}.$$  

after some rearrangement and using Eqs. (27)-(31) may be written as

$$\sqrt{X} = \left(\frac{\rho_{de}^0 + \rho_{dm}^0}{2\sqrt{CV}}\right) a^3 \left[ \frac{(\rho_{de} + \rho_{dm})}{(\rho_{de}^0 + \rho_{dm}^0)} + \frac{\rho_{de}}{(\rho_{de}^0 + \rho_{dm}^0)} - \frac{\rho_{dm}}{(\rho_{de}^0 + \rho_{dm}^0)} \right]$$

$$= \left(\frac{1}{\alpha}\right) \left[ a^3 (1 + w) \frac{(\rho_{de} + \rho_{dm})}{(\rho_{de}^0 + \rho_{dm}^0)} - \Omega_{dm}^0 \right] = \left(\frac{1}{\alpha}\right) g_1(\eta)$$

where we denoted $\alpha = 2\sqrt{CV}/(\rho_{de}^0 + \rho_{dm}^0)$, $g_1(\eta) = \left[ a^3 (1 + w) \frac{(\rho_{de} + \rho_{dm})}{(\rho_{de}^0 + \rho_{dm}^0)} - \Omega_{dm}^0 \right]$ and $\Omega_{dm}^0 = \rho_{dm}^0/(\rho_{de}^0 + \rho_{dm}^0)$, the present-day fractional contribution of dark matter density to the total energy density of universe. For numerical evaluation of various cosmological
parameters we used the observed value $\Omega_{\text{dm}}^0 = 0.268$ from Planck observation [45]. Again using Eqs. (30)–(32) in Eq. (27) we can write

\[
\frac{F(X)}{2\sqrt{C}} = \left(1 - \frac{w(\rho_{\text{de}} + \rho_{\text{dm}})}{\rho_{\text{de}}^0 + \rho_{\text{dm}}^0}\right) \left(\frac{1}{\alpha}\right) g_2(\eta)
\]

where $g_2(\eta) = \frac{w(\rho_{\text{de}} + \rho_{\text{dm}})}{\rho_{\text{de}}^0 + \rho_{\text{dm}}^0}$. Note that, temporal behaviour of all the quantities involved in the functions $g_1(\eta)$ and $g_2(\eta)$ occurring in Eqs. (32) and (33), has already been obtained from analysis of JLA data for the late time phase of cosmic evolution, as discussed in Sect. 2. From the above two equations we have \( \frac{F(X)}{\sqrt{X}} = \sqrt{2C} \frac{g_2(\eta)}{g_1'(\eta)} \) which implies that chosen value of the constant $C$ sets a constant scaling to time profile of the quantity $\frac{F(X)}{\sqrt{X}}$. The value of $\frac{F(X)}{\sqrt{X}}$ is, however, independent of $\alpha$ (i.e. $V$). However, by eliminating the time parameter $\eta$ from Eqs. (32) and (33), we may get the functional form of $F(X)$ as a function of $X$. This would require knowledge of the inverse function $g_1^{-1}$ of $g_1$, since inverting Eq. (32) we can write, \( \eta = g_1(\alpha \sqrt{X}) \) and putting it in Eq. (33) we have

\[
\frac{F(X)}{\sqrt{2C}} = \left(\frac{1}{\alpha}\right) g_2 \left( g_1^{-1}(\alpha \sqrt{X}) \right).
\]

So, both $C$ and $\alpha$ (i.e. both $C$ and $V$) plays the roll of parameters in the functional form of $F(X)$. Using our knowledge of the numerical values of the functions $g_1(\eta) = \alpha \sqrt{X}$ and $g_2(\eta) = \sqrt{\frac{F(X)}{\sqrt{2C}}} \frac{g_2(\eta)}{g_1'(\eta)}$ at different values of $\eta$ as extracted from analysis of JLA data discussed above, we can obtain the numerical values of $F(X)/2\sqrt{C}$ for different values of $X$ over a certain domain, for specific choices of the values of constant $\alpha$. To depict the dependence of $F(X)$ on $\sqrt{X}$ as extracted from the observational data for constant potential $k$-essence scenario, we choose three benchmark values of the constant $\alpha$ viz. 0.1, 1, 5. For each choice, we have shown the profile of $F(X)/2\sqrt{C}$ obtained from the analysis of JLA data in Fig. 4. The obtained dependence is found to follow the profile of a polynomial of $\sqrt{X}$ of degree 3

\[
\frac{F(X)}{2\sqrt{C}} = A_0 + A_1(\alpha \sqrt{X}) + A_2(\alpha \sqrt{X})^2 + A_3(\alpha \sqrt{X})^3
\]

where the best-fit values of coefficients, determined for the three benchmark cases and are presented in Table 2.

With the form of $F(X)$ in Eq. (35) along with the values of coefficients $A_i$’s as extracted from the analysis of JLA data, we write down the pressure and energy density of dark energy from Eqs. (27) and (28) in terms of $X$ as

\[
\frac{\rho_{\text{de}}}{\rho_{\text{de}}^0 + \rho_{\text{dm}}^0} = \alpha \left( A_0 + A_1(\alpha \sqrt{X})^2 + A_2(\alpha \sqrt{X})^3 + A_3(\alpha \sqrt{X})^4 \right)
\]
In the context of linear stability theory, the nature of the fixed points of the system can be broadly classified into three categories depending on nature of eigenvalues of the Jacobian matrix, $J$. The fixed points of the system are classified as:

1. Stable fixed point: If all the nonzero eigenvalues evaluated at a fixed point have negative real parts, the nearby trajectories of the fixed point are attracted to it and the fixed point is referred as a stable fixed point or attractor.
2. Unstable fixed point: If all the nonzero eigenvalues evaluated at a fixed point have positive real parts, the nearby trajectories of the fixed point have real parts with mutually opposite signs, it is called a saddle point and in such cases, where any (or more) of the eigenvalues have zero real part the linear stability theory fails to provide information about the nature of the fixed points.
3. Neutral fixed point: There are cases, where any (or more) of the eigenvalues of the Jacobian matrix are purely imaginary, then the fixed point is referred as a neutral fixed point or a center.

In the context of linear stability theory, the nature of the fixed points of the system can be broadly classified into three categories depending on nature of eigenvalues of the Jacobian matrix, $J$ evaluated at the fixed points, provided all the eigenvalues have nonzero real parts. The stationary points may be analysed by the method based on linear stability theory. This involves Taylor expansion of $F_i$, where $F_i$ represents the derivative with respect to the time parameter $\eta$. $F_i(x, y)$ $(i = 1, 2)$ only depend on $x$, $y$ without having any explicit time dependence. The fixed points $(x_0, y_0)$ of the system of Eq. (38) correspond to $F_i(x_0, y_0) = 0$ $(i = 1, 2)$. Stability of the stationary points can be determined by the eigenvalues of the Jacobian matrix $J$ whose elements $J_{ij}$ are the first-order derivatives of $F_i$, given by

$$\frac{\rho_{de}}{\rho_{de}^0 + \rho_{dm}^0} = \alpha \left(-A_0(\alpha) + A_2(\alpha)X + 2A_3(\alpha)X^2\right)$$

(37)

4 Aspects of the $k$-essence model in terms of a dynamical system

The $k$-essence model of dark energy with constant potential considered here involves two parameters $C$ and $V$. We explore how these parameters control the cosmological dynamics of the model in the context of JLA data from the perspective of dynamical system analysis. The technique involves a suitable choice of two dimensionless dynamical variables $x(\eta)$ and $y(\eta)$ in terms of relevant cosmological quantities and parameters of the theory and converting equations representing cosmological dynamics into an autonomous system of ordinary differential equations of the form

$$x' = F_1(x, y) \quad \text{and} \quad y' = F_2(x, y)$$

(38)

where $'$ represents the derivative with respect to the time parameter $\eta$. $F_i(x, y)$ $(i = 1, 2)$ only depend on $x$, $y$ without having any explicit time dependence. The fixed points $(x_0, y_0)$ of the system of Eq. (38) correspond to $F_i(x_0, y_0) = 0$ $(i = 1, 2)$. Stability of the stationary points may be analysed by the method based on linear stability theory. This involves Taylor expansion of $F_i(x, y)$ around the fixed point $(x_0, y_0)$ which requires knowledge of the corresponding Jacobian matrix $J$ whose elements $J_{ij}$ are the first-order derivatives of $F_i(x, y)$, given by

$$J_{ij} = \frac{\partial F_i}{\partial x_j} \quad \text{with} \quad x_1 = x, x_2 = y$$

(39)

In the context of linear stability theory, the nature of the fixed points of the system can be broadly classified into three categories depending on nature of eigenvalues of the Jacobian matrix, $J$ evaluated at the fixed points, provided all the eigenvalues have nonzero real parts. The scenario discussed here, $J$ is a $2 \times 2$ matrix with two eigenvalues. If real parts of all the nonzero eigenvalues of $J$ at a fixed point are all negative, such a fixed point attracts all its nearby trajectories in $x-y$ plane and is referred as a stable fixed point or attractor. If all the nonzero eigenvalues evaluated at a fixed point have positive real parts, the nearby trajectories of the fixed point are repelled from it and the fixed point is referred as an unstable fixed point or repeller. If the two eigenvalues of the $J$ at any fixed point have real parts with mutually opposite signs, it is called a saddle point and in $x-y$ plane it attracts some of its nearby trajectories and repels others. In cases, where any (or more) of the eigenvalues have zero real part the linear stability theory fails to explore the nature of the fixed points.

To map the dynamics of $k$-essence model to a system of autonomous equations we define the two dimensionless dynamical variables $x$ and $y$ as

$$x = \dot{\phi} \quad \text{and} \quad y = \frac{\kappa \sqrt{V}}{\sqrt{3}H}$$

(40)

Using Eq. (29) (with $V_\phi = 0$ for constant potential $V$), Eq. (35) and the fact that for a homogeneous field $\phi(t)$, $X = (1/2)\dot{\phi}^2$, the temporal evolution of the quantity $x = \dot{\phi}$ with respect to the time parameter $\eta = \ln(\alpha)$ may be written as

$$x' = -3 \left(\frac{A_1(\alpha) + \sqrt{2}A_2(\alpha)X + \frac{3}{2}A_3(\alpha)X^2}{\sqrt{2}A_2(\alpha) + 3A_3(\alpha)}\right)$$

(41)

Derivative of $y$ with respect to $\eta$ gives $y' = -y\dot{H}/H^2$. Using Eqs. (10), (11) and (36) we may write this time evolution of $y$ as

$$y' = \frac{3}{2} y \left[1 + \frac{y^2\sqrt{C}}{\sqrt{2}} \left(2\sqrt{2}A_0(\alpha) + 2A_1(\alpha)X + \sqrt{2}A_2(\alpha)X^2 + A_3(\alpha)X^3\right)\right]$$

(42)

As discussed in Sect. 2, the analysis of data from SNe IA observations is instrumental in capturing the observable features of late time cosmic evolution of the universe in the parameters $A_i(\alpha)$’s arising in the context of the $k$-essence model considered here. Observed features of temporal behaviour of quantities like $\rho_{de}$, $P_{de}$, $F(X)$, etc., relevant in this context, have been expressed through their
based on this, therefore, contain observational inputs as extracted from the JLA data.

The values of $A_i$ occurring in the scaling relation [Eq. (30)] and the dynamical system are coupled through the parameter $C$. This approach of dynamical system analysis in the context of late time cosmic evolution with dark energy determines the set of autonomous Eqs. (41) and (42) describing the dynamical system.

For illustration, the fixed points and nature of their stability for some benchmark choices of values of the parameter set $(C, \alpha)$ are shown in the $x - y$ plane for $C = 0.1$ (left panel), $C = 1$ (middle panel), $C = 10$ (right panel). Note that, though the $x'$-equation (41) does not contain $y$ explicitly, the two autonomous Eqs. (41) and (42) representing the dynamical system are coupled through the parameter $\alpha$. The value of the parameter $\alpha$ enters in Eqs. (41) and (42) through the coefficients $A_i$'s. Also, the $y'$-equation (42) contains the parameter $C$, apart from $\alpha$. The parameter $\alpha \equiv 2\sqrt{E V / (\rho_{de}^0 + \rho^0_{dm})}$, in turn, is determined by the choice of the value of the constant $C$ occurring in the scaling relation [Eq. (30)] and also on the constant value of the potential $V$ in the k-essence model considered here. So choice of the parameters $(C, V)$ or equivalently $(C, \alpha)$ determines the set of autonomous Eqs. (41) and (42) describing the dynamical system.

The dynamical system governed by set of autonomous Eqs. (41) and (42) represents the motion of a hypothetical particle moving in two dimensions described in terms of Cartesian coordinates $x(\eta)$ and $y(\eta)$. The dynamics of cosmic evolution gets captured into the aspects of motion of the hypothetical particle through defining expressions of $x$ and $y$ in terms of cosmological variables and parameters [Eq. (40)]. This approach of dynamical system analysis in the context of late time cosmic evolution with dark energy dynamics represented by a k-essence scalar field driven by a constant potential offers a way to explore how the features of the late time cosmic evolution depends on the numerical values of the constants $C$ and $V$, within the context of JLA observational data.

For illustration, the fixed points and nature of their stability for some benchmark choices of values of the parameter set $(C, \alpha)$ are presented in Table 3. Our analysis shows that the number of fixed points of the system and their respective nature are insensitive to the values of $C$ and $\alpha$. However, the coordinates of fixed points change with the change of values of these parameters. For different

### Table 3 Fixed points and their stability for the dynamical system represented by set of Eqs. (41) and (42) corresponding to different benchmark choices of $(C, \alpha)$

| $C$ | $\alpha = 0.1$ | $\alpha = 1.0$ | $\alpha = 5.0$ |
|-----|----------------|----------------|----------------|
|     | Fixed points   | Stability      | Fixed points   | Stability      | Fixed points   | Stability      |
| 0.1 | $(-0.11, -0.45)$ | Stable         | $(-0.16, -1.42)$ | Stable         | $(-0.01, -3.22)$ | Stable         |
|     | $(-0.11, 0)$    | Saddle         | $(-0.16, 0)$    | Saddle         | $(-0.01, 0)$    | Saddle         |
|     | $(-0.11, 0.45)$ | Stable         | $(-0.16, 1.42)$ | Stable         | $(-0.01, 3.22)$ | Stable         |
|     | $(1.18, -0.48)$ | Stable         | $(0.47, -1.53)$ | Stable         | $(0.17, -3.4)$  | Stable         |
|     | $(1.18, 0)$     | Saddle         | $(0.47, 0)$     | Saddle         | $(0.17, 0)$     | Saddle         |
|     | $(1.18, 0.48)$  | Stable         | $(0.47, 1.53)$  | Stable         | $(0.17, 3.4)$   | Stable         |
| 1   | $(-0.11, -0.25)$| Stable         | $(-0.16, -0.9)$ | Stable         | $(-0.01, -1.81)$| Stable         |
|     | $(-0.11, 0)$    | Saddle         | $(-0.16, 0)$    | Saddle         | $(-0.01, 0)$    | Saddle         |
|     | $(-0.11, 0.25)$ | Stable         | $(-0.16, 0.9)$  | Stable         | $(-0.01, 1.81)$ | Stable         |
|     | $(1.18, -0.27)$ | Stable         | $(0.47, -0.86)$ | Stable         | $(0.17, -1.91)$ | Stable         |
|     | $(1.18, 0)$     | Saddle         | $(0.47, 0)$     | Saddle         | $(0.17, 0)$     | Stable         |
|     | $(1.18, 0.27)$  | Stable         | $(0.47, 0.86)$  | Stable         | $(0.17, 1.91)$  | Stable         |
| 10  | $(-0.11, -0.14)$| Stable         | $(-0.16, -0.45)$| Stable         | $(-0.01, -1.01)$| Stable         |
|     | $(-0.11, 0)$    | Saddle         | $(-0.16, 0)$    | Saddle         | $(-0.01, 0)$    | Saddle         |
|     | $(-0.11, 0.14)$ | Stable         | $(-0.16, 0.45)$ | Stable         | $(-0.01, 1.01)$ | Stable         |
|     | $(1.18, -0.15)$ | Stable         | $(0.47, -0.48)$ | Stable         | $(0.17, -1.07)$ | Stable         |
|     | $(1.18, 0)$     | Saddle         | $(0.47, 0)$     | Saddle         | $(0.17, 0)$     | Stable         |
|     | $(1.18, 0.15)$  | Stable         | $(0.47, 0.48)$  | Stable         | $(0.17, 1.07)$  | Stable         |

Fig. 5 Fixed points and orbits in the $x - y$ plane for $\alpha = 0.1$ with $C = 0.1$ (left panel), $C = 1$ (middle panel), $C = 10$ (right panel)
choices of values of \((C, \alpha)\) the system is always found to have six different fixed points out of which four are stable fixed points and two are saddle points. The two saddle fixed points (two eigenvalues of the Jacobian having mutually opposite signs) are always found to lie on the \(x\)-axis (\(y=0\)): one on the positive side and the other on the negative side. Both the saddle points approach more closer to the origin with increasing values of \(\alpha\) but do not move with change of \(C\) for a fixed \(\alpha\). This can be seen from Table 3 as well as from Figs. 5, 6 and 7. Each of four stable fixed points lies on each quadrant of the \(x-y\) plane. The stable points in the 1st and 4th quadrant are reflections of each other with respect to \(x\)-axis (same \(y\) values with opposite signs). Same is true for the stable points in 2nd and 3rd quadrants. Each of the stable points move toward the \(x\)-axis with increasing values of \(\alpha\) and move towards the \(x\)-axis with increasing values of \(\alpha\) or a given \(C\).

Analytically we see from Eq. (42) that \(y' = 0\) corresponds to either of the two equations:

\[
y_0 = 0 \tag{43}
\]

\[
y_0^2 \left( 2\sqrt{2}A_0(\alpha) + 2A_1(\alpha)x_0 + \sqrt{2}A_2(\alpha)x_0^2 + A_3(\alpha)x_0^3 \right) = -\frac{\sqrt{2}}{\sqrt{C}} \tag{44}
\]

The saddle fixed points always correspond to the \(y_0 = 0\) solution [Eq. (43)] and lie on the \(x\)-axis. The stable fixed points always correspond to the solution [Eq. (44)] which has a \(y_0 \rightarrow -y_0\) symmetry justifying occurrence of each pair of fixed points as reflections of each other about the \(x\)-axis. Again from Eq. (41), \(x' = 0\) corresponds to the quadratic equation

\[
A_1(\alpha) + \sqrt{2}A_2(\alpha)x_0 + \frac{3}{2}A_3(\alpha)x_0^3 = 0 \tag{45}
\]

whose two solutions are

\[
x_0 = \frac{-\sqrt{2}A_2(\alpha) \pm \sqrt{2A_2(\alpha)^2 - 6A_1(\alpha)A_3(\alpha)}}{3A_3(\alpha)} \tag{46}
\]

For each of two solutions for \(x_0\), the solutions of Eq. (44) result in four stable fixed points of the system in \(x-y\) plane. The positions of the stable points in the \(x-y\) plane are controlled by the value of \(C\) appearing in Eq. (44) and the value of \(\alpha\) entering both the solutions in Eqs. (44) and (46) through the quantities \(A_i(\alpha)\)’s. We also observe that the constant \(C\) appears in the \(y'\) equation (42) as \(\sqrt{C}\). So for any given choice of \(C > 0\), the negative value of its square root has to be considered separately for a complete analysis. For different above-mentioned choices of values of \(C\) and \(\alpha\), consideration of the negative square root \((\sqrt{C} < 0)\) in Eq. (42), gives only two real fixed points, both of which are found to be saddle points corresponding to \(y_0 = 0\) fixed point solution.
Also note that, using Eqs. (12), (27), (35), (40) and \( X = \dot{\phi}^2/2 \), the equation of state of the total dark fluid may be expressed as

\[
\omega = \sqrt{C} y^2 \left[ 2A_0(\alpha) + \sqrt{2} A_1(\alpha) x + A_2(\alpha) x^2 + \frac{1}{\sqrt{2}} A_3(\alpha) x^3 \right]
\]  

(47)

Therefore, for all the saddle fixed points \( (y_0 = 0) \) we have \( w = 0 \) which does not correspond to the acceleration of the present universe as revealed from Supernova Ia observations. For all the other obtained real fixed points, we find, \( w = -1 \) implying all the stable fixed points or attractors correspond to an accelerating universe.

Note that using Eq. (31) we can express the scale factor \( a \) in terms of the kinetic term \( X = \dot{\phi}^2/2 \) of the \( k \)-essence model as

\[
a = \left[ \frac{4C V^2 X}{(\rho_{de} + \rho_{de})^2} \right]^{1/6}
\]

(48)

Using Eqs. (36) and (37) in Eq. (48) and putting \( \alpha = 2\sqrt{C} V/(\rho_{de}^0 + \rho_{dm}^0) \) we get,

\[
a = \left[ \frac{\sqrt{X}}{A_1(\alpha) \sqrt{X} + 2A_2(\alpha) X + 3A_3(\alpha) X^{3/2}} \right]^{1/3}
\]

(49)

Since \( x = \dot{\phi} = \sqrt{2X} \), the above equation establishes connection between the dynamical variable \( x \) and scale factor \( a \) for any chosen values of \( \alpha \) (i.e. the constants \( C \) and \( V \)). The other dynamical variable, \( y \), as defined by Eq. (40), is related to Hubble Parameter \( H \equiv \dot{a}/a = (d/dt) \ln a \) as \( y \sim H^{-1} \) for given values of the constant \( k \)-essence potential \( V \). The temporal behaviour of quantities like \( a, \dot{a}, H \) during the late time cosmic evolution has been extracted from the SNe Ia observations. The dynamical variables \( x \) and \( y \) defined by Eq. (40) capture this observed cosmological dynamics, in the context when the dark energy is realised by a scalar field \( \phi \) dynamically driven by a \( k \)-essence Lagrangian with constant potential \( V \) leading to the scaling relation \( X F_X^2 = C a^{-6} \) involving a constant \( C \).

Also note from Eq. (32) that, in the expression for \( \sqrt{X} \), \( (1/\alpha) \) appears as a constant multiplicative factor with the temporal part known from SNe Ia observation. Since, \( x = \dot{\phi} = \sqrt{2X} \), the constant \( \alpha \) may be absorbed in the definition of the field \( \phi \) (and hence in \( x \)) by the rescaling \( \phi \rightarrow \phi/\alpha \). However, as can be seen from Eq. (35), the effect of the constants \( C \) and \( V \) explicitly shows up in the functional form of \( F(X) \) extracted from the observational data by virtue of the scaling relation. This feature is reflected in the difference in the phase orbit plots (presented in Figs. 5, 6 and 7) of the representative autonomous system for different benchmark choices of values of \( (C, \alpha) \) or \( (C, V) \).

Finally we describe the significance of the four stable points occurring for a given choice of the values of \( (C, \alpha) \), in the context of dark energy scenario represented by a scalar field \( \phi \) with a constant potential \( V \) in its Lagrangian. As mentioned earlier the value of equation of state of the total dark fluid \( w \) is -1 at all the stable points implying approach of the phase orbits towards all the stable attractors correspond to acceleration of expanding universe. The four stable points lie in the 4 different quadrants in the \( x-y \) plane. For both the points in 1st and 4th quadrants we have \( x = \dot{\phi} > 0 \). This implies that the orbits approaching towards these two stable points correspond to the scenario where the value of the real scalar \( k \)-essence field increases monotonically with time as the current universe is accelerating. On the other hand, for both the points in 2nd and 3rd quadrants we have \( x = \dot{\phi} < 0 \), implying realisation of approaching of orbits towards these two stable points by a scenario where the value of the \( k \)-essence field decreases monotonically with time. So both scenarios of the homogeneous scalar field \( \phi(t) \) increasing with time and decreasing with time can be accommodated in JLA data. However, each of the two stable points for each of above-discussed scenarios corresponds to \( y > 0 \) (for the point in upper half plane) and \( y < 0 \) (for the lower half plane and also, as discussed earlier, these points are symmetrically positioned about the \( x-y \) axis. The reason for these, as seen from the definition of the dynamical variable \( y = \kappa \sqrt{\sqrt{V}/\sqrt{3} H} \), is that for a given choice of \( V \), both the positive square root \( (\sqrt{V}) \) and the negative square root \( (-\sqrt{V}) \) affects the description of the representive autonomous system through the dynamical variable \( y \). But the dark energy model and its cosmological consequences remain insensitive to the sign of \( \sqrt{V} \), as only \( V \) (and not \( \sqrt{V} \)) appears in the phenomenological form of \( F(X) \) extracted from observation which enters in the equation of motion of the \( k \)-essence field in Eq. (29). The occurrence of the two stable points for a given \( x = \phi \), one in upper half plane (with \( \sqrt{V} > 0 \) and the other in the lower half plane (with \( \sqrt{V} > 0 \)) is thus only an artefact arising from the choice of the specific autonomous system, for the purpose of realising the interplay of the constants \( C \) and \( V \) in the context of JLA data and the dark energy model considered here.

5 Conclusion

In this paper, we considered a homogeneous \( k \)-essence scalar field \( \phi \) with a non-canonical Lagrangian of the form \( L = V F(X), X = (1/2)\dot{\phi}^2 \) with constant potential \( V \) governing the dynamics of the dark energy of the universe during its late time phase of evolution. The constancy of the potential ensures existence of a scaling relation \( X F_X^2 = Ca^{-6} \) (\( C = \text{constant} \)) which connects the dynamical term \( F(X) \) of the Lagrangian with the scale factor \( a(t) \) of the expanding universe, with a flat FRW spacetime metric filled with perfect fluids. Using the Supernova Ia observations (JLA data), which is instrumental in probing features of late time cosmic evolution, we
obtain the temporal behaviour of the cosmological quantities like the scale factor $a(t)$, the equation of state and energy density of the dark fluid content (dark matter dust and dark energy) of the universe. Using these results obtained from the model independent analysis of the JLA data, we obtain the time dependence of the $k$-essence scalar field and reconstructed the $X$-dependence of the dynamical term $F(X)$ in the Lagrangian. In our analysis, we ignore the contribution from radiation and baryonic matter to the energy density of the universe during its late time phase of evolution based on the observations of present-day relative densities of different components from Planck collaboration. The values of the constant potential $V$ and the constant $C$ appearing in the scaling relation are parameters of this model.

To investigate the interplay of the values of the parameters $C$ and $V$ in controlling the cosmological dynamics in the context of the constant potential $k$--essence model and the JLA data, we use the mapping of the model to dynamical system represented by a set of autonomous equations involving two dimensionless variables $x$ and $y$ suitably defined in terms of cosmological quantities and relevant parameters of the model. We investigate impact of the parameter values ($C$, $V$) on the analysis of this dynamical system. For any specific choice of values of parameter set ($C$, $V$), the system contains the observational inputs from JLA data. For convenience of calculations involved, we choose an equivalent parameter set ($C$, $\alpha$) where $\alpha \equiv 2\sqrt{C}/(\rho_{0}^{dm} + \rho_{0}^{de})$, to present the results of our analysis. We used linear stability theory to analyse dynamical features of the system and for any ($C$, $\alpha$) we obtain six real fixed points of the system two of which are saddle points and the other four being stable fixed points. We find that the two saddle fixed points correspond to a fixed point solution $y = 0$ and lie on the $x$--axis. At the saddle points, the equation of state of the total dark fluid $w = -1$ and they do not correspond to the accelerating universe. The four stable fixed points correspond to accelerating universe with $w = -1$ and they may be grouped into two pairs, where points of each pair have $y$--values of same magnitude with opposite signs as evident from the fixed point solution given in Eq. (44) which has a $y \rightarrow -y$ symmetry. We investigated how locations of all these fixed points in $x$ -- $y$ plane change with change in the values of parameters: $C$ and $\alpha$. In the context of $k$--essence cosmological model of dark energy and the observational JLA data, this provides an indirect approach towards realising the dependence of dynamical aspects of the model on the values of the constant potential $V$ and the constant $C$ in the scaling relation.

Note that, the approach is taken here in considering the SNe Ia data and associated errors are simplistic and restricted to a particular class of modes which does not allow uncertainties at smaller redshifts as can be seen from Fig. 1. We avoided a rigorous and comprehensive analysis of the SNe Ia data and took this simplistic approach obtaining a gross profile of some cosmological parameters from observation, as this provides the optimal, necessary observational inputs required to emphasise what we explored in this work.

Our primary focus, in this paper, is to investigate the role of values of two constants viz. $C$ and $V$ in controlling dynamical features of late time dark energy dominated era cosmic evolution, where the constant $C$ appears in the scaling relation arising in the context of $k$--essence model of dark energy driven by a Lagrangian with a constant potential $V$. In this paper we wanted to show the results of such an investigation by mapping dynamical aspects of the $k$--essence model onto a two-dimensional dynamical system driven by a set of autonomous equations involving the model parameters $V$ and $C$. The variation of corresponding resulting stable points and the phase orbits with values of $V$ and $C$ has been shown—which reflects the sensitivity of the dynamical features of the $k$--essence model on $V$ and $C$. While taking the different benchmark values of $V$ and $C$ to depict such variations, we ensured that their chosen values correspond to late time profile of relevant cosmological parameters which are more or less compatible with their extracted profile from JLA data. In this paper, we wanted to show the results of such an investigation by mapping the dynamical aspects of the $k$--essence model onto a two-dimensional dynamical system driven by a set of autonomous equations involving the model parameters $V$ and $C$. The variation of corresponding resulting stable points and the phase orbits with values of $V$ and $C$ has been shown—which reflects the sensitivity of the dynamical features of the $k$--essence model on $V$ and $C$. While taking the different benchmark values of $V$ and $C$ to depict such variations, we ensured that their chosen values correspond to late time profile of relevant cosmological parameters which are compatible with their profile extracted from JLA data. The role of observational input is only up to this extent, in the context of the present work which emphasises on the study of the sensitivity of the dynamical features of the $k$--essence model on above mentioned parameters $V$ and $C$. So we only took with the best-fit values of the different coefficients occurring in different parametrisations used in the paper without considering propagation of uncertainties to the coefficients in equations and evaluating corresponding covariance matrix.

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