A new nonlinear generalization of the Dirac equation

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Abstract

We postulate a new nonlinear generalization of the Dirac equation for an electron. Basic properties of the new equation are considered.

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Dirac equation for an electron. Let $\mathbb{R}^{1,3}$ be the Minkowski space with cartesian coordinates $x^\mu$, $\mu = 0, 1, 2, 3$, with partial derivatives $\partial_\mu = \partial/\partial x^\mu$, and with a metric tensor given by the diagonal matrix $\eta = \text{diag}(1, -1, -1, -1)$. Consider the Dirac equation for an electron

$$i\gamma^\mu(\partial_\mu \psi - ia_\mu \psi) - m\psi = 0,$$

(1)

where $\psi = \psi(x)$ is a Dirac spinor, $a_\mu$ are components of a covector potential.

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of electromagnetic field, and

\[
\gamma^0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}, \\
\gamma^2 = \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
-i & 0 & 0 & 0
\end{pmatrix}, \quad \gamma^3 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix},
\]

are \(\gamma\)-matrices in the Dirac representation. We have

\[
\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu, \quad \mu \neq \nu, \quad (\gamma^0)^2 = 1, \quad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1,
\]

where 1 is the four dimensional identity matrix. Note that \(i\gamma^\mu \in \text{su}(2, 2)\) (see [2]).

Let us recall basic properties of the Dirac equation.

1. The electric charge conservation law:

\[
\partial_\mu j^\mu = 0, \quad \text{where} \quad j^\mu = \bar{\psi} \gamma^\mu \psi = \psi^\dagger \gamma^0 \gamma^\mu \psi,
\]

where \(\dagger\) is the operation of Hermitian conjugation.

2. Lorentz invariance of the Dirac equation:

\[
x^\mu \rightarrow \hat{x}^\mu = P_\nu^\mu x^\nu, \quad \psi \rightarrow \hat{\psi} = S \psi,
\]

where \(P = \|P_\psi\| \in O(1, 3), \ S \in \text{Pin}(1, 3)\) and \(S^{-1} \gamma^\mu S = p^{\nu}\gamma^\nu\).

3. Gauge invariance w.r.t. \(U(1)\) gauge Lie group:

\[
\psi \rightarrow \hat{\psi} = e^{i\lambda} \psi, \quad a_\mu \rightarrow \hat{a}_\mu = a_\mu + \partial_\mu \lambda, \quad \lambda : \mathbb{R}^{1,3} \rightarrow \mathbb{R}.
\]

4. Decomposition of the Klein-Gordon-Fock operator:

\[
(i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu + m) = -(\partial_\mu \partial^\mu + m^2).
\]
Generalized Dirac equation with nonlinearity. Let us take \( I = \gamma^0\gamma^1\gamma^2\gamma^3 \).
We see that \( I^2 = -1 \), \( I^\dagger = -I \). Denote a subalgebra of matrix algebra
\[ N = \{ \alpha 1 + \beta I \in \text{Mat}(4, \C) : \alpha, \beta \in \R \} \simeq \C. \]

Let \( f = f(z) \) be a continuous function \( f : \C \rightarrow \C \) and let \( F = F(Z) \) be
the corresponding function \( F : N \rightarrow N \) such that \( F(Z)|_{1\rightarrow 1, i\rightarrow i} = f(z) \).

Let us postulate the following equation, which depend on the function \( F : N \rightarrow N \):
\[
i\gamma^\mu (\partial_\mu \psi - ia_\mu \psi) - F(Z)\psi = 0, \tag{3}\]
where
\[
Z = (\bar{\psi}\psi)1 - (\bar{\psi}I\psi)I
\]
The first term in the equation (3) is equal to the first term in the Dirac
equation for an electron (1). So, we say that the equation (3) is a generalized
Dirac equation (with a nonlinearity).

Consider basic properties of the generalized Dirac equation (3).

1. The electric charge conservation law:
\[
\partial_\mu j^\mu = 0, \quad \text{where} \quad j^\mu = \bar{\psi}\gamma^\mu \psi = \psi^\dagger \gamma^0\gamma^\mu \psi.
\]

2. Lorentz invariance of the Dirac equation:
\[
x^\mu \rightarrow \dot{x}^\mu = p^\mu x', \quad \psi \rightarrow \dot{\psi} = S\psi,
\]
where \( P = \|p^\mu\| \in \text{SO}_+(1, 3) \), \( S \in \text{Spin}_+(1, 3) \) and \( S^{-1}\gamma^\mu S = p^\mu \gamma^\nu \).

3. Gauge invariance w.r.t. U(1) gauge Lie group:
\[
\psi \rightarrow \hat{\psi} = e^{i\lambda} \psi, \quad a_\mu \rightarrow \hat{a}_\mu = a_\mu + \partial_\mu \lambda, \quad \lambda : \R^{1,3} \rightarrow \R.
\]

4. Decomposition of the second order operator:
\[
(i\gamma^\mu \xi_\mu - F(Z))(i\gamma^\nu \xi_\nu + \overline{F(Z)}) = -(\xi_\mu \xi^\mu + |F(Z)|^2). \tag{4}\]
If \( F(Z) = \sigma 1 + \rho I \), where \( \sigma, \rho \) is functions \( \R^{1,3} \rightarrow \R \), then \( \overline{F(Z)} = \sigma 1 - \rho I \), \( |F(Z)|^2 = \sigma^2 + \rho^2 \), and \( \xi^\mu \) are commutative symbols.
We see two differences in the basic properties of equations (3) and (1).

- The equation (1) is invariant under Lorentz transformations of coordinates from the Lie group $O(1, 3)$, but the equation (3) is invariant under Lorentz transformations of coordinates from the proper orthochronous Lorentz group $SO_+(1, 3)$.

- The decomposition (2) is, generally speaking, different to the decomposition (1).

Consider special cases of the generalized Dirac equation.

If we take $F(Z) \equiv m1$ in (3), then we get the Dirac equation (1). That means the equation (3) is, in fact, a generalization of the equation (1).

Let us remind that the Dirac equation (1) can be derived from the Lagrangian

$$L = \bar{\psi} i\gamma^\mu (\partial_\mu \psi - ia_\mu \psi) - m (\bar{\psi} \psi).$$

If we take $F(Z) \equiv Z = (\bar{\psi} \psi)1 - (\bar{\psi} I \psi)I$, then we get the equation

$$i\gamma^\mu (\partial_\mu \psi - ia_\mu \psi) - ((\bar{\psi} \psi)1 - (\bar{\psi} I \psi)I)\psi = 0,$$

which can be derived from the Lagrangian

$$L = \bar{\psi} i\gamma^\mu (\partial_\mu \psi - ia_\mu \psi) - \frac{1}{2} (\bar{\psi} \psi)^2 + \frac{1}{2} (\bar{\psi} I \psi)^2.$$

Heisenberg’s nonlinear field equation. One can see similarity between the generalized Dirac equation and the Heisenberg nonlinear field equation (1)

$$i\gamma^\mu (\partial_\mu \psi - ia_\mu \psi) - (\bar{\psi} \gamma_\mu \psi)\gamma^\mu \psi - (\bar{\psi} I \gamma^\mu I \psi)\gamma^\mu I \psi = 0. \quad (5)$$

Heisenberg in (1) had made an attempt to create a unified field theory on the basis of his equation (5). So, it will be interesting to develop such a theory on the basis of new generalized Dirac equation (3).

References

[1] Heisenberg W., Introduction to the unified field theory of elementary particles, Intersience publishers, 1966.

[2] Marchuk N., Field theory equations, Amazon, 2012.