THE LIMIT OF THE YANG-MILLS-HIGGS FLOW ON HIGGS BUNDLES

JIAYU LI AND XI ZHANG

Abstract. In this paper, we consider the gradient flow of the Yang-Mills-Higgs functional for Higgs pairs on a Hermitian vector bundle \((E, H_0)\) over a compact Kähler manifold \((M, \omega)\). We study the asymptotic behavior of the Yang-Mills-Higgs flow for Higgs pairs at infinity, and show that the limiting Higgs sheaf is isomorphic to the double dual of the graded Higgs sheaves associated to the Harder-Narasimhan-Seshadri filtration of the initial Higgs bundle.

1. Introduction

Let \((E, H_0)\) be a Hermitian vector bundle over a compact Kähler manifold \((M, \omega)\), \(A_{H_0}\) be the space of connections of \(E\) compatible with the metric \(H_0\), and \(A_{1,1}^{H_0}\) be the space of unitary integrable connections of \(E\) (i.e. one whose curvature is of type \((1, 1)\)). Given a unitary integrable connection \(A\) on \((E, H_0)\), then \(D_A^{(0,1)} = \overline{\partial}_A\) defines a holomorphic structure on \(E\), and in fact, \(A\) is the Chern connection on the holomorphic bundle \((E, \overline{\partial}_A)\) with respect to \(H_0\).

A pair \((A, \phi)\) \(\in A_{H_0}^{1,1} \times \Omega^{1,0}(\text{End}(E))\) is called a Higgs pair if the relations \(\overline{\partial}_A \phi = 0\) and \(\phi \wedge \phi = 0\) are satisfied. Let \(B_{(E, H_0)}\) denote the space of all Higgs pairs on Hermitian vector bundle \((E, H_0)\). Given a Higgs pair \((A, \phi)\), then \((E, \overline{\partial}_A, \phi)\) is a Higgs bundle, i.e. \((A, \phi)\) determines a Higgs structure on \(E\). Let’s consider the Yang-Mills-Higgs functional which is defined on \(B_{(E, H_0)}\):

\[
YMH(A, \phi) = \int_M (|F_A + [\phi, \phi^*]|^2 + 2|\partial_A \phi|^2) \, dV_g.
\]

We call \((A, \phi)\) a Yang-Mills Higgs pair if it is the critical points of the above Yang-Mills-Higgs functional. Equivalently, the pair \((A, \phi)\) satisfies the following Yang-Mills-Higgs equations:

\[
\begin{align*}
D_A^{*} F_A + \sqrt{-1} (\partial_A \Lambda_\omega - \overline{\partial}_A \Lambda_\omega) [\phi, \phi^*] &= 0, \\
\sqrt{-1} \Lambda_\omega (F_A + [\phi, \phi^*]), \phi &= 0,
\end{align*}
\]

where the operator \(\Lambda_\omega\) is the contraction with \(\omega\), and \(\phi^*\) denotes the dual of \(\phi\) with respect to the given metric \(H_0\).

1991 Mathematics Subject Classification. 53C07, 58E15.

Key words and phrases. Higgs bundle, Higgs pair, Harder-Narasimhan-Seshadri filtration, Yang-Mills-Higgs flow.

The authors were supported in part by NSF in China, No.11071212, No.11131007, No.10831008 and No. 11071236.
If \((A, \phi)\) satisfies the following Hermitian-Einstein equation

\[
\sqrt{-1} \Lambda_\omega (F_A + [\phi, \phi^*]) = \lambda I d_E,
\]

then it must satisfy the above Euler-Lagrange equation (1.2). By Chern-Weil theory, in fact it is the absolute minima of the above Yang-Mills-Higgs functional.

On Hermitian vector bundle \((E, H_0)\), the Yang-Mills flow, as the gradient flow of the Yang-Mills functional, was first suggested by Atiyah-Bott in [1]. Donaldson [14] proved the global existence of the Yang-Mills flow in a holomorphic bundle, and proved the convergence of the flow at infinity in the case that the initial holomorphic structure is stable, then he used it to establish the correspondence between existence of the Hermitian-Einstein metric and stability of the holomorphic structure over complex algebraic surfaces. This correspondence starts by Narasimhan and Seshadri [33] in the case of compact Riemann surfaces, and is also called Hitchin-Kobayashi correspondence. The general Kähler manifolds case was proved by Uhlenbeck and Yau [39] by using the method of continuity.

Without the stability assumption, Atiyah-Bott [1] point out that there should be a correspondence between the Yang-Mills flow and the Harder-Narasimhan filtration over Riemann surfaces, and this is also conjectured by Bando and Siu [7] for higher dimensional case. This correspondence was proved by Daskalopoulos [11] in the case of Riemann surfaces, by Daskalopoulos and Wentworth [12] in the case of Kähler surfaces. In higher dimensional case, Hong and Tian [20] study the asymptotic behavior of the Yang-Mills flow, they proved that there is a subsequence along the Yang-Mills flow, modulo gauge transformations, which converges smoothly to a limiting Yang-Mills connection away from the bubbling set \(\Sigma_{\text{an}}\) of Hausdorff codimension 4. Recently, Jacob in [23] and Sibley in [35] studied the above correspondence for higher dimension case.

Let’s consider the following gradient flow of the Yang-Mills-Higgs functional of Higgs pairs, which is also called the Yang-Mills-Higgs flow. A regular solution is given by a family of \((A(x, t), \phi(x, t)) \in \mathcal{B}(E, H_0)\) such that

\[
\begin{align*}
\frac{\partial A}{\partial t} &= -D_A^* F_A - \sqrt{-1} (\partial A \Lambda_\omega - \overline{\partial A} \Lambda_\omega) [\phi, \phi^*], \\
\frac{\partial \phi}{\partial t} &= -[\sqrt{-1} \Lambda_\omega (F_A + [\phi, \phi^*]), \phi].
\end{align*}
\]

(1.4)

It is interesting to study the Higgs pairs version of Atiyah-Bott’s (or Bando-Siu’s) conjecture, i.e. there should be a correspondence between the Yang-Mills-Higgs flow for Higgs pairs and the Harder-Narasimhan-Seshadri filtration of Higgs bundle. In Riemann surface case, Wilkin [30] develops the analytic results needed to construct a Morse theory for the Yang-Mills-Higgs functional on the space of all Higgs pairs, and proves the Higgs pairs version of Atiyah-Bott’s conjecture. In [30], the authors study the bubbling phenomena of the Yang-Mills-Higgs flow, on Kähler surface case, and prove that the limit can be extended across the bubbling set \(\Sigma_{\text{an}}\) (a finite collection of points) to a smooth Higgs bundle which isomorphic to the the double dual of the graded object of the Harder-Narasimhan-Seshadri filtration of the initial Higgs structure.

A Higgs bundle \((E, \overline{\partial}_E, \phi)\) is called stable (semi-stable), if for every \(\phi\)-invariant coherent sub-sheaf \(E' \hookrightarrow (E, \overline{\partial}_E)\) of lower rank, it holds:

\[
\mu(E') = \frac{\deg(E')}{\text{rank} E'} < (\leq) \mu(E) = \frac{\deg(E)}{\text{rank} E'},
\]

(1.5)
where \( \mu(E') \) is called the slope of \( E' \). A Hermitian metric \( H \) on \( (E, \nabla_E, \phi) \) is called Hermitian-Einstein if it satisfies the Hermitian-Einstein equation (1.3), where \( A \) is the Chern connection \( A_H \) with respect to the metric \( H \). Higgs bundles first emerged twenty years ago in Hitchin’s [19] reduction of self-dual equation on \( \mathbb{R}^4 \) to Riemann surface. Higgs bundles have a rich structure and play a role in many different areas including gauge theory, Kähler and hyperkähler geometry, group representations and nonabelian Hodge theory. In [36], Simpson generalized it to higher dimensional case and proved that a Higgs bundle admits a Hermitian-Einstein metric iff it is Higgs poly-stable. This is a Higgs bundle version of the Donaldson-Uhlenbeck-Yau theorem. This correspondence has several interesting and important generalizations and extensions where some extra structures are added to the holomorphic bundles, see references: [19], [36], [4], [17], [5], [2], [3], [8], [26], [27], [28], [32], [38].

To a Higgs bundle \( (E, \nabla_E, \phi) \) of rank \( R \), as that for holomorphic bundles, one can associate a filtration by \( \phi \)-invariant holomorphic subsheaves, which is called the Harder-Narasimhan filtration, whose successive quotients are Higgs semi-stable. The topological type of the pieces in the associated graded objects is encoded into an \( R \)-tuple \( \vec{\mu} = (\mu_1, \cdots, \mu_R) \) of rational numbers called the Harder-Narasimhan type (abbr, HN-type) of the Higgs bundle \( (E, \nabla_E, \phi) \). For every semi-stable Higgs sub-sheaf, one can associate a Seshadri filtration, whose successive quotients are Higgs stable. Then, we have a double filtration which is called the Harder-Narasimhan-Seshadri filtration (abbr, HNS-filtration) of the Higgs bundle, and we write \( \text{Gr}^{HNS}(E, \nabla_E, \phi) \) for the associated graded object (i.e. the direct sum of the stable quotients) of the HNS filtration.

Now, we consider the asymptotic behavior of the Yang-Mills-Higgs flow (1.4) of Higgs pairs for higher dimensional case. In [30], we have proved the global existence and uniqueness of the solution for the Yang-Mills-Higgs flow, and we obtained many basic properties of the flow, including the energy inequality, Bochner-type inequality, monotonicities of certain quantities and a small action regularity estimate, these properties also valid in higher dimensional case. Following Hong-Tian’s argument in [20] and using the small action regularity estimates in [30], we conclude that there exists a sequence of the solution which converges, modulo gauge transformations, to a limiting Yang-Mills-Higgs pair in \( C^\infty_{\text{loc}} \) topology outside the bubbling set \( \Sigma_{\text{an}} \subset M \), where \( \Sigma_{\text{an}} \) is a closed set of Hausdorff codimension 4. In this paper, we show that the limiting \( (E_\infty, A_\infty, \phi_\infty) \) can be extended to the whole \( M \) as a reflexive Higgs sheaf, and prove that this extended reflexive Higgs bundle is isomorphic to the the double dual of the graded object of the Harder-Narasimhan-Seshadri filtration of the initial Higgs bundle \( (E, A_0, \phi_0) \), i.e. we prove the Higgs version of Atiyah-Bott’s (or Bando-Siu’s) conjecture for higher dimension case. We obtain the following theorem.

**Theorem 1.1.** Let \( (E, H_0) \) be a Hermitian vector bundle on a compact Kähler manifold \( (M, \omega) \), and \( (A(t), \phi(t)) \) be a global smooth solution of the Yang-Mills-Higgs flow (1.4) with smooth initial Higgs pair \( (A_0, \phi_0) \). Then:

1. There exists a sequence \( \{t_j\} \) such that, as \( t_j \to \infty \), \( (A(t_j), \phi(t_j)) \) converges, modulo gauge transformations, to a Hermitian-Einstein Higgs pair \( (A_\infty, \phi_\infty) \) on Hermitian vector bundle \( (E_\infty, H_\infty) \) in \( C^\infty_{\text{loc}} \) topology outside a closed set \( \Sigma_{\text{en}} \subset M \), where \( \Sigma_{\text{en}} \) is a closed set of Hausdorff codimension at least 4.
(2) The limiting \((E_\infty, \overline{\nabla_{A_\infty}}, \phi_\infty)\) can be extended to the whole \(M\) as a reflexive orthogonal splitting

\[
(E_\infty, H_\infty, A_\infty, \phi_\infty) = \bigoplus_{i=1}^l (E^i_\infty, H^i_\infty, A^i_\infty, \phi^i_\infty),
\]

where \(H^i_\infty\) is an admissible Hermitian-Einstein metrics on the reflexive Higgs sheaf \((E^i_\infty, A^i_\infty, \phi^i_\infty)\).

(3) Moreover, the extended reflexive Higgs sheaf is isomorphic to the double dual of the graded Higgs sheaves associated to the HNS-filtration of the initial Higgs bundle, i.e. we have

\[
(E_\infty, \overline{\nabla_{A_\infty}}, \phi_\infty) \simeq \Gr^{HNS} (E, \overline{\nabla_{A_0}}, \phi_0)^{**}.
\]

We now give an overview of our proof. We ([30]) have shown that there is an uniform \(C^0\) bound on Higgs fields \(\phi(t)\), so the basic idea in [12] for the Yang-Mills flow in Kähler surface case can be used. But there are two points where we need new argument for higher dimensional case. The first one is to prove that the HN type of the limiting Higgs sheaf is in fact equal to the type of the initial Higgs bundle; and second one is to construct a non-zero holomorphic map from any stable quotient Higgs sheaf in HNS filtration of initial Higgs bundle to the limiting Higgs sheaf.

The first one is closely related to the existence of an \(L^p\)-approximate critical Hermitian metric (as defined in [12]). Under the semi-stability assumption of the initial Higgs bundle \((E, A_0, \phi_0)\) in [31], we prove the existence of \(L^\infty\)-approximate metric by the heat flow method. For general case, we will use the cut-off argument by Daskalopoulos and Wentworth in [12]. We use the resolution of singularities theorem of Hironaka [18], and write \(\pi : \tilde{M} \rightarrow M\) as the composition of blow-ups involved in resolution, then the pullback bundle \(\pi^* E\) has a filtration by Higgs subbundles, which is precisely the HNS-filtration of the initial Higgs bundle away from the exceptional divisor \(\tilde{\Sigma}\). The metric \(\pi^* \omega\) is degenerated along the divisor \(\tilde{\Sigma}\), and it can be approximated by a family of Kähler metrics \(\omega_\epsilon\) on \(\tilde{M}\) as that in [7]. Since every pullback quotient bundle is stable with respect to Kähler metrics \(\omega_\epsilon\) for small \(\epsilon\), one can use Simpson’s theorem (in [30]) to take the direct sum of the Hermitian-Einstein metrics on quotient Higgs subbundles in the resolution. If one can get a flat Hermitian metric on neighborhood of singularities, then one may use Daskalopoulos and Wentworth’s cut-off argument (where the singularity set is a collection of finite point), after getting some uniform estimates, and modifying this metric, one can show that its Hermitian-Einstein tensor is close to the HN type in the \(L^p\) norm. By pushing this metric down, one can obtain a smooth \(L^p\)-approximate critical Hermitian metric on the Higgs bundle \((E, A_0, \phi_0)\). However, since the singularity set for the filtration is complex codimension 2 which is not necessary the collection of finite point, in general we can not get a flat Hermitian metric on a neighborhood of singularities. But we should point out that this is not a essential question. Using Sibley’s good observation ([35], or see Lemma 5.8) and by choosing any fixed Hermitian metric on a neighborhood of singularities, we can also obtain an uniform estimate, this is enough to obtain a smooth \(L^p\)-approximate critical Hermitian metric, see proposition 5.11. for details.

For the second one, we use Donaldson’s idea to construct a nonzero holomorphic map to the limiting bundle as the limit of the sequence of gauge transformations.
(by rescaled) defined by the flow. The difficulty is to prove that the limiting map is in fact non-zero, because we have no uniform $L^\infty$ bound on the mean curvature (i.e. $|\sqrt{-1} \Lambda_\omega F_A|$) for subsheaves. If the singularities are finite points, i.e. Kähler surface case, we can follow the argument in [12] by a complex analytic argument to get uniform $C^0$ estimate, and then prove the limiting holomorphic map is non-zero. This argument is not suitable for higher dimensional case, since we do not know whether the complement of the singular set has a strictly pseudo-concave boundary. Using resolution of singularities, we consider the pullback bundle which has a filtration by subbundles. Evolving Hermtian metric by the Donaldson heat flow with respect to Kähler metric $\omega_\epsilon$, by the result in [7], we get uniform $L^\infty$ bound on the mean curvature of $H(t)$ for positive $t$. By uniform local $C^0$ estimate of the evolved Hermitian metrics and using the standard elliptic estimates, we can construct a nonzero holomorphic map which we need (see proposition 4.1. for detail). In [23] and [35], the authors studied the same question for holomorphic vector bundles, they have good observations there. We should point out that our argument is different from the ones they used.

This paper is organized as follows. In Section 2, we recall some basic estimates of the Donaldson’s heat flow and the Yang-Mills-Higgs flow, and prove the first and second part of theorem 1.1, see theorem 2.4 and proposition 2.7. In section 3, we consider the resolution of the HNS filtration of Higgs bundle. In section 4, we construct a non-zero holomorphic map between Higgs sheaves, where proposition 4.1 is the key technical part in the proof of theorem 1.1. In section 5, we use the cut-off argument to obtain $L^p$-approximate critical Hermitian metric, and prove that the Harder-Narasimhan type of the limiting Higgs sheaf is in fact equal to the type of the initial Higgs bundle. In section 6, we complete the proof of theorem 1.1 by inductive argument.

## 2. Analytic preliminaries and basic estimates

Suppose $H(t)$ is a solution of the following Donaldson’s heat flow with initial metric $H_0$, 

\begin{equation}
H^{-1} \partial H = -2(\sqrt{-1} \Lambda_\omega (F_H + [\phi, \phi^*H]) - \lambda d_E).
\end{equation}

Let $h(t) = H_0^{-1}H(t)$, using the identities

\begin{align}
\partial_H - \partial_{H_0} &= h^{-1} \partial_{H_0} h; \\
F_H - F_{H_0} &= \partial_E (h^{-1} \partial_{H_0} h); \\
\phi^{*H} &= h^{-1} \phi^{*H_0} h,
\end{align}

then we can rewrite (2.1) as

\begin{equation}
\frac{\partial h}{\partial t} = -2\sqrt{-1} h \Lambda_\omega (F_{H_0} + \partial_E (h^{-1} \partial_{H_0} h) + [\phi, h^{-1} \phi^{*H_0} h]) + 2\lambda h.
\end{equation}

In [36], Simpson proved the existence of long time solution of the heat flow (2.1), the following lemma is essentially proved by Simpson ([36] Lemma 6.1).

**Lemma 2.1.** Let $H(t)$ be a solution of the heat flow (2.1) with initial metric $K$, then we have:

\begin{equation}
(\partial_t - \Delta)tr(\Lambda_\omega (F_H + [\phi, \phi^*H])) = 0
\end{equation}
and
\begin{equation}
(2.5) \quad \left(\frac{\partial}{\partial t} - \Delta\right)\Lambda_\omega(F_H + [\phi, \phi^{*H}]) = -4|D_\phi''(\Lambda_\omega(F_H + [\phi, \phi^{*H}]))|^2_H,
\end{equation}
where $D_\phi'' = \bar{\partial}E + \phi$.

Denote the complex gauge group (unitary gauge group) of Hermitian vector bundle $(E, H_0)$ by $G^C$ (where $G = \{\sigma \in G^C | \sigma^{*H_0} \sigma = Id\}$). $G^C$ acts on the space of Higgs pairs $B_{(E, H_0)}$ as follows: let $\sigma \in G^C$
\begin{equation}
(2.6) \quad \bar{\partial}((A|_0)\sigma) = \sigma \circ \bar{\partial}A \circ \sigma^{-1}, \quad \bar{\partial}(A) = (\sigma^{*H_0})^{-1} \circ \bar{\partial}A \circ \sigma^{*H_0};
\end{equation}
\begin{equation}
(2.7) \quad \sigma(\phi) = \sigma \circ \phi \circ \sigma^{-1}.
\end{equation}

In [30], we have proved the following proposition.

**Proposition 2.2. (Theorem 2.1 in [30])** Given any Higgs pair $(A_0, \phi_0)$, the Yang-Mills-Higgs flow $(1.4)$ has a unique solution $(A(t), \phi(t))$ in the complex gauge orbit of $(A_0, \Phi_0)$. In fact, $(A(t), \phi(t)) = g(t)(A_0, \phi_0)$, where $g(t) \in G^C$ satisfies $g^{*H_0}(t)g(t) = H_0^{-1}H(t)$, and $H(t)$ is the solution of Donaldson’s flow $(2.7)$ on Higgs bundle $(E, \bar{\partial}A_0, \phi_0)$ with initial metric $H_0$.

Furthermore, we have the uniform $C^0$ estimate of $\phi(t)$.

**Lemma 2.3. (Lemma 2.3. in [30])** Let $(A(t), \phi(t))$ be a solution of the heat flow $(1.4)$ with initial Higgs pair $(A_0, \phi_0)$, then we have
\begin{equation}
|\phi(x, t)|^2_{H_0} \leq C,
\end{equation}
where $C$ is a constant depending only on $\phi_0$ and the geometry of $(M, \omega)$.

For simplicity, we set
\begin{equation}
(2.9) \quad \theta(A, \phi) = \frac{1}{2\pi} \Lambda_\omega(F_A + [\phi, \phi^{*H_0}]_0),
\end{equation}
and
\begin{equation}
(2.10) \quad I(t) = \int_M |D_{A(t)}\theta(A(t), \phi(t))|^2 + 2||\theta(A(t), \phi(t))|t^2|\omega^n/\pi|.
\end{equation}

Let $(A(t), \phi(t))$ be the solution of the heat flow $(1.4)$ on Hermitian bundle $(E, H_0)$ with initial Higgs pair $(A_0, \phi_0)$, and $H(t)$ be the solution of the Donaldson’s flow $(2.7)$ on Higgs bundle $(E, \bar{\partial}A_0, \phi_0)$ with initial metric $H_0$. As above, we know that $(A(t), \phi(t)) = g(t)(A_0, \phi_0)$, where $g(t) \in G^C$ and satisfies $g(t)^{*H_0}g(t) = h(t) = H_0^{-1}H(t)$. By $(2.2), (2.6), (2.7)$, it is easy to check that
\begin{equation}
(2.11) \quad F_{A(t)} = g(t) \circ F_H(t) \circ g(t)^{-1},
\end{equation}
\begin{equation}
(2.12) \quad [\phi(t), \phi(t)^{*H_0}] = g(t) \circ [\phi_0, \phi_0^{*H_0}] \circ g(t)^{-1},
\end{equation}
and
\begin{equation}
(2.13) \quad |\theta(A(t), \phi(t))|_{H(t)} = \frac{1}{2\pi} |\Lambda_\omega(F_H(t) + [\phi_0, \phi_0^{*H(t)}])|_{H(t)},
\end{equation}
where $F_{H(t)}$ is the curvature of the Chern connection on $(E, \bar{\partial}A)$ with respect to the metric $H(t)$.

Direct calculation shows that
\begin{equation}
(2.14) \quad (\triangle - \frac{\partial}{\partial t})|\theta(A(t), \phi(t))|^2 \geq 0,
\end{equation}
\begin{equation}
(2.15) \quad \sum_{i=1}^{n} \left|\int_M \omega^n_{H(t)}(\partial_{\phi_i} \theta(A(t), \phi(t))\right|_{H(t)}^2 \geq 0.
\end{equation}
and
\[(2.15) \quad I(t) \to 0, \quad (t \to \infty),\]
the proof can be found in [30] (page 1384-1386). Furthermore, we have monotonicity inequality (Theorem 2.6 in [30]) for the solution \((A(t), \phi(t))\) of (1.4), and then we can get the \(\epsilon\)-regularity theorem (Theorem 3.1 in [30]). Using the \(\epsilon\)-regularity theorem, and the argument of Hong and Tian [20] for the Yang-Mills flow case, we can analyze the limiting behavior of the heat flow (1.4). We have the following theorem.

**Theorem 2.4. (Theorem 3.2. in [30])** Let \((A(t), \phi(t))\) be a smooth solution of the gradient heat flow (1.4) on a Hermitian vector bundle \((E, H_0)\) over Kähler manifold \((M, \omega)\) with smooth initial data. Then there exists a sequence \(\{t_i\}\) such that, as \(t_i \to \infty\), \((A(t_i), \phi(t_i))\) converges, modulo gauge transformations, to a solution \((A_\infty, \phi_\infty)\) of the Yang-Mills-Higgs equation (1.2) on Hermitian vector bundle \((E_\infty, H_\infty)\) in \(C^\infty_{0, \omega}\) topology outside \(\Sigma_{an} \subset M\), where \(\Sigma_{an}\) is a closed set of Hausdorff codimension at least 4, and there exists an isometry between \((E, H_0)\) and \((E_\infty, H_\infty)\) outside \(\Sigma_{an}\).

**Corollary 2.5. (Corollary 3.12. in [30])** Let \((A(t_i), \phi(t_i))\) be a sequence of Higgs pairs along the gradient heat flow (1.4) with Uhlenbeck limit \((A_\infty, \phi_\infty)\), then:
1. \(\theta(A(t_i), \phi(t_i)) \to \theta(A_\infty, \phi_\infty)\) strongly in \(L^p\) for all \(1 \leq p < \infty\), and consequently,
\[\lim_{i \to \infty} \int_M |\theta(A_i, \phi_i)|^2 = \int_M |\theta(A_\infty, \phi_\infty)|^2;\]
2. \(\|\theta(A_\infty, \phi_\infty)\|_{L^\infty} \leq \|\theta(A(t_j), \phi(t_j))\|_{L^\infty} \leq \|\theta(A_{t_0}, \phi_{t_0})\|_{L^\infty}\) for \(0 \leq t_0 \leq t_j\).

**Remark 2.6.** Although the above theorem and corollary in [30] is only stated for Kähler surface, the result can be proved similarly for higher dimensional case.

From the equation (1.2), we know that
\[(2.16) \quad D_{A_\omega} \theta_\infty = 0, \quad [\theta_\infty, \phi_\infty] = 0,\]
where \(\theta_\infty = \Lambda_\omega (F_{A_\infty} + [\phi_\infty, \phi_\infty^\ast])\). Since \(\theta_\infty\) is parallel and \((\sqrt{-1} \theta_\infty)^* = \sqrt{-1} \theta_\infty\), we can decompose \(E_\infty\) according to the eigenvalues of \(\sqrt{-1} \theta_\infty\). We obtain a holomorphic orthogonal decomposition
\[(2.17) \quad E_\infty = \bigoplus_{i=1}^l E^i_\infty,\]
and
\[(2.18) \quad \phi_\infty : E^i_\infty \to E^j_\infty\]
on \(M \setminus \Sigma_{an}\). Let \(H_\infty \) be the restrictions of \(H_\infty\) to \(E^i_\infty\), \(\phi^i_\infty\) be the restriction of \(\phi_\infty\) to \(E^i\), and \(A^i_\infty = A_\infty|_{E^i}\). Then \((A^i_\infty, \phi^i_\infty)\) is a Higgs pair on \((E^i_\infty, H^i_\infty)\) and satisfies
\[(2.19) \quad \sqrt{-1} \Lambda_\omega (F_{A^i_\infty} + [\phi^i_\infty, (\phi^i_\infty)^\ast]) = 2\pi \lambda_i Id_{E^i_\infty}.\]
So \((A^i_\infty, \phi^i_\infty)\) is a Hermitian-Einstein Higgs pair on \((E^i_\infty, H^i_\infty)\), i.e., \((E^i_\infty, H^i_\infty, A^i_\infty, \phi^i_\infty)\) is a Hermitian-Einstein Higgs bundle on \(M \setminus \Sigma_{an}\).

The Yang-Mills-Higgs functional is decreasing along the gradient flow (1.4), and \(\phi(t)\) is uniformly \(C^0\) bounded (by Lemma 2.3.), then we have
\[(2.20) \quad \int_{M \setminus \Sigma_{an}} |F_{A^\ast}|^2_{H^\ast} \frac{\omega^n}{n!} \leq C < \infty.\]
Since the singularity set $\Sigma_{an}$ is of Hausdorff codimension 4, $\phi_\infty$ is holomorphic and $C^0$ bounded, and every metrics $H^\infty_{\phi_\infty}$ (or the connection $A^\infty_{\phi_\infty}$) satisfies the Hermitian-Einstein equation \cite{13}, a similar argument as that in the proof of Theorem 2 in Bando and Siu’s paper \cite{7} can show that, every $(E^\infty, \overline{\nabla}_{E^\infty})$ can be extended to the whole $M$ as a reflexive sheaf (which is also denoted by $(E^\infty, \overline{\nabla}_{E^\infty})$ for simplicity), $\phi^\infty_{\infty}$ and $H^\infty_{\phi_\infty}$ can be smoothly extended over the place where the sheaf $(E^\infty, \overline{\nabla}_{E^\infty})$ is locally free. So we have the following proposition.

**Proposition 2.7.** The limiting $(E^\infty, \overline{\nabla}_{A^\infty_{\phi_\infty}}) \infty_{\phi_\infty}$ can be extended to the whole $M$ as a reflexive Higgs sheaf with a holomorphic orthogonal splitting

\begin{equation}
(E^\infty, H^\infty_{\phi_\infty}, A^\infty_{\phi_\infty}) = \bigoplus_{i=1}^l (E^i_{\phi_\infty}, H^i_{\phi_\infty}, A^i_{\phi_\infty}),
\end{equation}

where $H^i_{\phi_\infty}$ is an admissible Hermitian-Einstein metrics on the reflexive Higgs sheaf $(E^i_{\phi_\infty}, A^i_{\phi_\infty}, \phi^i_{\phi_\infty})$.

Let $S$ be a $\phi$-invariant torsion free sub-sheaf of $(E, \overline{\nabla}_E, \phi)$ with a Hermitian metric $H$. Since we can view $S$ as a holomorphic vector sub-bundle off the singular set $\Sigma$ where $S$ fails to be locally free, away from $\Sigma$ we have a corresponding orthogonal projection $\pi : E \rightarrow E$ with $\pi(E) = S$. Since $S$ is $\phi$-invariant and holomorphic, on almost everywhere of $M$, we have $\pi^2 = \pi = \pi^*H$; $(Id - \pi)[\overline{\nabla}_E] = 0$; and $(Id - \pi)[\phi, \pi] = 0$. Furthermore, it can be shown that $\pi$ extends to an $L^1_2$ section of $EndE$. Conversely such $\pi$ determines a coherent $\phi$-invariant subsheaf.

**Definition 2.8.** An element $\pi$ is called a weakly $\phi$ holomorphic subbundle if $\pi \in L^1_2(End(E))$ and

\begin{align}
(Id - \pi)[\overline{\nabla}_E] = 0; \\
\pi^2 = \pi = \pi^*H; \\
(Id - \pi)[\phi, \pi] = 0
\end{align}

hold almost everywhere.

In \cite{39}, Uhlenbeck and Yau prove that the above $\pi$ determines a coherent subsheaf $S$, the last term in conditions (2.22) implies that $S$ is $\phi$-invariant. So, we have the following proposition.

**Proposition 2.9.** A weakly $\phi$ holomorphic subbundle $\pi$ determines a $\phi$-invariant coherent subsheaf of the Higgs bundle $(E, \overline{\nabla}_E, \phi)$.

3. Resolution of the HNS filtration

Given a Higgs bundle $(E, A, \phi)$ on a Kähler manifold $(M, \omega)$. A Higgs sub-sheaf of $(E, A, \phi)$ is a $\phi$-invariant coherent analytic subsheaf $S \subset (E, A)$. The $\omega$-slope $\mu(S)$ of a torsion-free sheaf $S \rightarrow M$ is defined by:

\begin{equation}
\mu_\omega(S) = \frac{deg_\omega(S)}{rank(S)} = \frac{1}{rank(S)} \int_M C_1(S) \wedge \frac{\omega^{n-1}}{(n-1)!}.
\end{equation}

For any subsheaf $S$, its singular set $\Sigma_S$ is the set of points where $S$ fails to be locally free. If $S$ is a saturated subsheaf then the singular set $\Sigma_S$ is a closed complex analytic subset of $M$ of complex codimension at least 2. A torsion-free Higgs subsheaf $S$ is said $\omega$-stable (resp. $\omega$-semistable) if for all proper $\phi$-invariant saturated subsheaves $F \subset S$, $\mu_\omega(F) < \mu_\omega(S)$ ($\mu_\omega(F) \leq \mu_\omega(S)$).
In the following, we will give a description of the appropriate Higgs bundle versions of the Harder-Narasimhan filtration and the Harder-Narasimhan-Seshadri filtration, the proof is almost the same as the one used in the holomorphic bundles case ([23], v.7.15, 7.17, 7.18 ), the only difference is that we always consider \( \phi \)-invariant subsheaves instead of usually subsheaves. We omit the details here.

**Proposition 3.1.** Let \((E, A, \phi)\) be a Higgs bundle on Kähler manifold \((M, \omega)\). Then there is a filtration of \( E \) by \( \phi \)-invariant coherent sub-sheaves

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_i = E,
\]
called the Harder-Narasimhan filtration of Higgs bundle \((E, A, \phi)\) (abbr, HN-filtration ), such that \( Q_i = E_i/E_{i-1} \) is torsion-free and Higgs semistable. Moreover, \( \mu(Q_i) > \mu(Q_{i+1}) \), and the associated graded object \( Gr^{hn}(E, A, \phi) = \bigoplus_{i=1}^l Q_i \) is uniquely determined by the isomorphism class of \((E, A, \phi)\).

**Proposition 3.2.** Let \((V, \phi)\) be a semistable Higgs sheaf on Kähler manifold \((M, \omega)\), then there is a filtration of \( V \) by \( \phi \)-invariant subsheaf

\[
0 = V_0 \subset V_1 \subset \cdots \subset V_i = V,
\]
called the Seshadri filtration of \((V, \phi)\), such that \( V_i/V_{i-1} \) is torsion-free and Higgs stable. Moreover, \( \mu(V_i/V_{i-1}) = \mu(V) \) for each \( i \), and the associated graded object \( Gr^s(V, \phi) = \bigoplus_{i=1}^l V_i/V_{i-1} \) is uniquely determined by the isomorphism class of \((V, \phi)\).

For Higgs bundle \((E, A, \phi)\), there is a double filtration , called a Harder-Narasimhan-Seshadri filtration of Higgs bundle \((E, A, \phi)\) (abbr, HNS-filtration ).

**Proposition 3.3.** Let \((E, A, \phi)\) be a Higgs bundle on Kähler manifold \((M, \omega)\). Then there is a double filtration \( \{E_{i,j}\} \) with the following properties: if \( \{E_{i}\}_{i=1}^l \) is the HN filtration of \((E, A, \phi)\), then

\[
E_{i-1} = E_{i,0} \subset E_{i,1} \subset \cdots \subset E_{i,l_i} = E_i
\]

and the successive quotient \( Q_{i,j} = E_{i,j}/E_{i,j-1} \) are Higgs stable torsion-free sheaves. Moreover, \( \mu(Q_{i,j}) = \mu(Q_{i,j+1}) \) and \( \mu(Q_{i,j}) > \mu(Q_{i+1,j}) \), the associated graded object:

\[
Gr^{hns}(E, A, \phi) = \bigoplus_{i=1}^l \bigoplus_{j=1}^{l_i} Q_{i,j}
\]
is uniquely determined by the isomorphism class of \((E, A, \phi)\).

The following lemma can be proved by an argument similar the one used in Chapter 5, V.7.11; 7.12 in [25] for the case of holomorphic bundles.

**Lemma 3.4.** (Lemma 6.3. in [30]) Let \((E_1, A_1, \phi_1)\) and \((E_2, A_2, \phi_2)\) be two semistable Higgs sheaves with same rank and degree. If \((E_1, A_1, \phi_1)\) is Higgs stable, and let \( f : E_1 \rightarrow E_2 \) be a sheaf homomorphism satisfying \( f \circ \phi_1 = \phi_2 \circ f \). Then either \( f = 0 \) or \( f \) is injective.

**Definition 3.5.** (Harder-Narasimhan type ) For a Higgs bundle \((E, A, \phi)\) of rank \( R \), construct a nonincreasing \( R \)-tuple of numbers

\[
\bar{\mu}(E, A, \phi) = (\mu_1, \cdots, \mu_R)
\]

from the HN filtration by setting: \( \mu_i = \mu(Q_j) \), for \( rk(E_{j-1}) + 1 \leq i \leq rk(E_j) \). We call \( \bar{\mu}(E, A, \phi) \) the Harder-Narasimhan type of \((E, A, \phi)\).
Remark: For a pair $\tilde{\mu}, \tilde{\lambda}$ of $R$-tuple’s satisfying $\sum_{i=1}^{R} \mu_i = \sum_{i=1}^{R} \lambda_i$, we define:

$\tilde{\mu} \leq \tilde{\lambda} \iff \sum_{i \leq k} \mu_i \leq \sum_{i \leq k} \lambda_i$

for all $k = 1, \cdots, R$.

It will be convenient to denote the $\phi$-invariant subsheaf $E_i$ in the HN-filtration by $F^k_{\hbar}(E, A, \phi)$ or by $F^k_{\hbar, \omega}(E, A, \phi)$ when we wish to emphasize the role of the Kähler structure. Let $\{E_{i,j}\}$ be the HNS-filtration of the Higgs bundle $(E, A, \phi)$, we set

$\Sigma_{\text{alg}} = \cup_{i,j} \text{Sing}(E_{i,j}) \cup \text{Sing}(Q_{i,j}),$

this is a complex analytic subset of complex codimension at least two. We will refer to it as the singular set of the HNS-filtration. Since the HNS-filtration fails to be given by subbundles on the singular set $\Sigma_{\text{alg}}$, it makes difficult to do analysis. We can use the singularities theorem of Hironaka [18] to resolve the singularities $\Sigma_{\text{alg}}$, and obtain a filtration by subbundles. This idea had been used by Bando and Siu [7] to obtain admissible Hermitian-Einstein metric on reflexive stable sheaf. The following proposition has been proved by Sibley in [35].

**Proposition 3.6. (Proposition 4.3. in [35])** Let $0 = E_0 \subset E_1 \subset \cdots \subset E_{l-1} \subset E_l = E$ be a filtration of a holomorphic vector bundle $E$ on a complex manifold $M$ by saturated subsheaves and let $Q_i = E_i/E_{i-1}$. Then there exists a finite sequence of blowups along complex submanifolds of $M$ whose composition $\pi : M \to M$ enjoys the following properties. There is a filtration

$0 = \tilde{E}_0 \subset \tilde{E}_1 \subset \cdots \subset \tilde{E}_l = \tilde{E}$

by subbundles such that $\tilde{E}_i$ is the saturation of $\pi^* E_i$. If $\tilde{Q}_i = \tilde{E}_i/\tilde{E}_{i-1}$, then we have exact sequences:

$0 \to E_i \to \pi_* \tilde{E}_i \to T_i \to 0$

and

$0 \to Q_i \to \pi_* \tilde{Q}_i \to T'_i \to 0$

where $T_i$ and $T'_i$ are torsion sheaves supported on the singular sets of $E_i$ and $Q_i$ respectively, and furthermore $\pi_* \tilde{E}_i = E_i$ and $Q_i^{**} = (\pi_* \tilde{Q}_i)^{**}$.

Let $\phi$ be a Higgs field on holomorphic bundle $(E, \overline{\partial}_A)$ and $\tilde{\phi} = \pi^* \phi$ be the pulling back Higgs field on $\tilde{E}$. If the filtration $\{E_i\}_{i=1}^{l}$ is by $\phi$-invariant subsheaves, then the pulling back filtration $\{\tilde{E}_i\}_{i=1}^{l}$ in the above proposition is by $\tilde{\phi}$-invariant subbundles. So, we have the following proposition.

**Proposition 3.7.** Let $\{E_{i,j}\}$ be the HNS-filtration of a Higgs bundle $(E, A, \phi)$ on complex manifold $M$ and let $Q_{i,j} = E_{i,j}/E_{i,j-1}$. Then there exists a finite sequence of blowups along complex submanifolds of $M$ whose composition $\pi : M \to M$ enjoys the following properties. There is a filtration $\{\tilde{E}_{i,j}\}$ by $\phi$-subbundles such that $\tilde{E}_{i,j}$ is the saturation of $\pi^* E_{i,j}$, and $\pi_\ast \tilde{E}_{i,j} = E_{i,j}$ and $Q_{i,j}^{**} = (\pi_\ast \tilde{Q}_{i,j})^{**}$, where $\tilde{\phi} = \pi^* \phi$.

The following proposition is well known, the proof can be found in, for example [10].

**Proposition 3.8.** Let $(M, \omega)$ be a compact Kähler manifold and $\pi : \tilde{M} \to M$ be a blow up with non-singular center. Then $\tilde{M}$ is also Kähler, moreover $\tilde{M}$ possesses
a one family of Kähler metrics given by $\omega_\epsilon = \pi^* \omega + \epsilon \eta$ where $\epsilon > 0$ and $\eta$ is Kähler metric on $M$.

Furthermore, Bando and Siu (7) proved the uniform boundedness of heat kernel for $\omega_\epsilon$.

**Proposition 3.9. (Proposition 2 in (7))** Let $(M, \omega)$ be an $n$-dimensional compact Kähler manifold and $\pi: \tilde{M} \rightarrow M$ be a blow up with non-singular center of co-dimensional at least two. Fix a Kähler metric $\eta$ on $M$ and set $\omega_\epsilon = \pi^* \omega + \epsilon \eta$ for $0 < \epsilon \leq 1$. Let $K_\epsilon(t, x, y)$ be the heat kernel with respect to the metric $\omega_\epsilon$, then we have a uniform estimate $0 \leq K_\epsilon(t, \cdot, \cdot) \leq C(1 + t^{-n})$ with a positive constant $C$.

In the following, we consider the $\omega_\epsilon$ slope of an arbitrary coherent subsheaf of a holomorphic vector bundle $\tilde{E}$ on the blow up $\tilde{M}$. One can show that the $\omega_\epsilon$ slope converges to the $\omega$ slope of the push forward sheaf on the base $M$, and the stability will be preserved for small $\epsilon$. These properties should be well known, see Bando and Siu’s paper (7), more details can be found in Siblèsé’s paper (35).

**Proposition 3.10.** Let $\tilde{S}$ be a coherent subsheaf (with torsion free quotient $\tilde{Q}$) of a holomorphic vector bundle $\tilde{E}$ on $\tilde{M}$, and $\pi: \tilde{M} \rightarrow M$ be a blow up with non-singular center of co-dimensional at least two. Then $\mu_{\omega_\epsilon}(\tilde{S}) \rightarrow \mu_{\omega}(\pi_* \tilde{S})$ and $\mu_{\omega_\epsilon}(\tilde{Q}) \rightarrow \mu_{\omega}(\pi_* \tilde{Q})$ as $\epsilon \rightarrow 0$. Furthermore there is a uniform constant $B$ independent of $S$ such that $\mu_{\omega_\epsilon}(\tilde{S}) \leq \mu_{\omega}(\pi_* \tilde{S}) + \epsilon B$ and $\mu_{\omega_\epsilon}(\tilde{Q}) \geq \mu_{\omega}(\pi_* \tilde{Q}) - \epsilon B$.

**Proof.** By the definition, we have

$$\deg(\tilde{S}, \omega_\epsilon) = \int_{\tilde{M}} c_1(\det(\tilde{S})) \wedge \frac{\omega_{\epsilon}^{n-1}}{(n-1)!}$$

$$= \int_{\tilde{M}} c_1(\det(\tilde{S})) \wedge \frac{(\pi^* \omega)^{n-1}}{(n-1)!} + \frac{1}{(n-1)!} \sum_{i=1}^{\dim \tilde{S}} \epsilon^i \int_{\tilde{M}} c_1(\det(\tilde{S})) \wedge \eta^i \wedge (\pi^* \omega)^{n-1-i}.$$  

(3.5)

Since the blow up set is co-dimensional at least two, so

$$\int_{\tilde{M}} c_1(\det(\tilde{S})) \wedge \frac{\omega_{\epsilon}^{n-1}}{(n-1)!}$$

$$= \int_{\tilde{M}} c_1(\pi_* \det(\tilde{S})) \wedge \frac{\omega_{\epsilon}^{n-1}}{(n-1)!}$$

$$= \int_{\tilde{M}} c_1(\det(\pi_* \tilde{S})) \wedge \frac{\omega_{\epsilon}^{n-1}}{(n-1)!}$$

$$= \deg(\pi_* \tilde{S}, \omega),$$

(3.6)

where we used the isomorphism $\det(\pi_* \tilde{S}) = \pi_* \det(\tilde{S})$. Then (3.5) and (3.6) imply $\mu_{\omega_\epsilon}(\tilde{S}) \rightarrow \mu_{\omega}(\pi_* \tilde{S})$ as $\epsilon \rightarrow 0$.

Let $H$ be a Hermitian metric on $\tilde{E}$, we can view $\tilde{S}$ as a holomorphic sub-bundle off the singular set $\Sigma$ which is co-dimensional at least two, away from $\Sigma$ we has a corresponding orthogonal projection $\pi_S: \tilde{E} \rightarrow \tilde{E}$ with $\pi_S(E) = \tilde{S}$. It is well known that $\pi_S$ can be extended to an $L^2$ section of $EndE$. Using the Gauss-Codazzi equation, we have

$$\int_{\tilde{M}} c_1(\det(\tilde{S})) \wedge \eta^i \wedge (\pi^* \omega)^{n-1-i}$$

$$= \frac{1}{2\pi} \int_{\tilde{M}} tr(\pi_S \circ F_{AH} \circ \pi_S + \bar{\partial}_E \pi_S \wedge \partial_{AH} \pi_S) \wedge \eta^i \wedge (\pi^* \omega)^{n-1-i}$$

(3.7)

where $A_H$ is the Chern connection with respect to $H$. Since $\pi^* \omega$ is nonnegative, the second term in right hand side of the above equality is non-positive. Since the first term is uniform bound, we see that $\int_{\tilde{M}} c_1(\det(\tilde{S})) \wedge \eta^i \wedge (\pi^* \omega)^{n-1-i}$ has an uniform upper bound independent of $\tilde{S}$. By (3.5), (3.6) and (3.7), we know there
is a constant $B$ such that $\mu_{\omega_i}(\tilde{S}) \leq \mu_{\omega}(\pi_*\tilde{S}) + \epsilon B$. For the quotient sheaf $Q$, we can consider the dualised sequence $0 \to Q^* \to \tilde{E}^* \to \tilde{S}^* \to 0$, a similar argument as above implies the statements for $Q$ in the proposition.

\[ \square \]

**Remark. 3.11.** If there is a sequence of blow-ups:
\[
\pi_i : \overline{M}_i \to \overline{M}_{i-1}, \quad i = 1, \ldots, r
\]
and $\pi = \pi_r \circ \cdots \circ \pi_1$, where $\overline{M}_0 = M$ and $\overline{M}_r = \tilde{M}$ and every $\pi_i$ is blow up along a smooth complex submanifold of co-dimensional at least two. On each blow-up $\overline{M}_i$, we have a family of Kähler metrics defined iteratively by $\omega_{\epsilon_1, \cdots, \epsilon_{i-1}} = \pi_i^* \omega_{\epsilon_1, \cdots, \epsilon_{i-1}} + \epsilon_i \eta_i$, where $\eta_i$ is a Kähler metric on $\overline{M}_i$ and $\epsilon_i > 0$. For simplicity, in the following, we will denote $\epsilon = (\epsilon_1, \cdots, \epsilon_r)$, $\omega = \omega_{\epsilon_1, \cdots, \epsilon_r}$, and $\|\epsilon\| = \max_i \epsilon_i$. It is easy to see that the above proposition is also valid for such $\pi$.

**Proposition 3.12.** Let $\pi : \tilde{M} \to M$ be a composition of finitely many blowups along complex submanifolds of co-dimensional at least two, $(\tilde{E}, \tilde{\phi})$ be a Higgs bundle over $\tilde{M}$, and $(E, \phi)$ be a Higgs sheaf over $M$ with $\pi_*\tilde{E} = E$, $\phi(\pi_*X) = \pi_*\tilde{\phi}(X)$ for any $X \in \tilde{E}$. If the Higgs sheaf $(E, \phi)$ is $\omega$-stable, then there is a number $\epsilon_0 > 0$, such that the Higgs sheaf $(\tilde{E}, \tilde{\phi})$ is $\omega_i$-stable for all $0 < \epsilon \leq \epsilon_0$.

**Proof.** By the assumption that $E$ is $\omega$-stable, then there is constant $\delta > 0$ such that $\mu_{\omega}(E) - \mu_{\omega}(S) > \delta$ for any proper Higgs subsheaf $S$. By proposition 3.10, for any proper Higgs subsheaf $\tilde{S} \subset \tilde{E}$ we have

\[
\begin{align*}
\mu_{\omega_i}(\tilde{S}) - \mu_{\omega_i}(\tilde{E}) &\leq \mu_{\omega}(\pi_*\tilde{S}) - \mu_{\omega}(E) + 2\|\epsilon\|B \\
&< -\delta + 2\|\epsilon\|B < 0
\end{align*}
\]

for $\|\epsilon\| < \frac{\delta}{2B}$, where we used that $B$ is independent on $\tilde{S}$. This completes the proof.

\[ \square \]

Let $(S, \phi)$ be a Higgs sheaf on $M$, we define $\mu_{\max,\omega}(S)$ to be the maximum $\omega$-slope of $\phi$-invariant subsheaves of $S$, and $\mu_{\min,\omega}(S)$ to be the minimal $\omega$-slope of $\phi$-invariant torsion free quotient sheaves of $S$. It is easy to check that $\mu_{\min,\omega}(S) = -\mu_{\max,\omega}(S^*)$. By proposition 3.10, we have the following corollary.

**Corollary 3.13.** Let $\pi : \tilde{M} \to M$ be a composition of finitely many blowups along complex submanifolds of co-dimensional at least two, $(\tilde{E}, \tilde{\phi})$ be a Higgs bundle over $\tilde{M}$, and $(E, \phi)$ be a Higgs sheaf over $M$ with $\pi_*\tilde{E} = E$, $\phi(\pi_*X) = \pi_*\tilde{\phi}(X)$ for any $X \in \tilde{E}$. There is a constant $B > 0$ such that:

1. $\mu_{\max,\omega}(\tilde{E}) \leq \mu_{\max,\omega}(E) + \|\epsilon\|B$;
2. $\mu_{\min,\omega}(\tilde{E}) \geq \mu_{\min,\omega}(E) - \|\epsilon\|B$.

Let the filtration
\[
0 = \tilde{E}_0 \subset \tilde{E}_1 \subset \cdots \subset \tilde{E}_{l-1} \subset \tilde{E}_l = \tilde{E} = \pi^*E
\]
be the resolution of the HN filtration of the Higgs bundle $(E, \phi)$. By proposition, we have

\[
\mu_{\omega_i}(\tilde{Q}_i) \to \mu_{\omega}(Q_i)
\]
for all \( i \) as \( \epsilon \to 0 \). Using the properties \( \mu_\omega(Q_i) > \mu_\omega(Q_{i+1}) \), \( \mu_{\min, \omega}(E_i) = \mu_\omega(Q_i) \), \( \mu_{\max, \omega}(E/E_i) = \mu_\omega(Q_{i+1}) \), and Corollary 3.13, we have

\[
\mu_{\min, \omega}(\tilde{E}_i) > \mu_{\max, \omega}(\tilde{E}/\tilde{E}_i).
\]

By the above inequality, it is easy to see that the resolution appears within the HN filtration of the Higgs bundle \((\tilde{E}, \tilde{\phi})\) with respect to \( \omega_\epsilon \), and two successive Higgs bundles in the resolution are separated by the HN filtration of the larger Higgs bundle. By an inductive argument, repeatedly using proposition 3.10 one can show the convergence of the HN type, so we have the following proposition.

**Proposition 3.14.** Let \((E, \phi)\) be a Higgs bundle over \( M \), \( \pi : \tilde{M} \to M \) be a sequence of blow-ups resolving the HNS filtration and \((\tilde{E}, \tilde{\phi}) = \pi^*(M, \phi)\) be the pull back Higgs bundle over \( \tilde{M} \). Let \( \bar{\mu} \) denote the HN type of \((\tilde{E}, \tilde{\phi})\) with respect to \( \omega_\epsilon \) and \( \bar{\mu} \) the HN type of \((E, \phi)\) with respect to \( \omega \), then \( \bar{\mu}_\epsilon \to \bar{\mu} \) as \( \epsilon \to 0 \).

4. Existence of non-zero holomorphic map

In order to prove theorem 1.1, we need construct non-zero holomorphic maps from subsheaves in the HNS filtration of the original Higgs bundle to the limiting reflexive sheaf. For bundle case, we can follow Donaldson’s argument in [13] to construct non-zero holomorphic maps. But in general the HNS filtration is given by subsheaves, so Donaldson’s argument can not be applied directly in our case. The following proposition is the key to construct an isomorphism between \((E_\infty, \mathcal{J}_{A_\infty}, \phi_\infty)\) and the double dual of the stable quotients of the HNS filtration \( \Lambda^{HNS}(E, \mathcal{J}_{A_0}, \phi_0) \).

**Proposition 4.1.** Let \((M, \omega)\) be a Kähler manifold, \((E, A_0, \phi_0)\) be a Higgs sheaf on \( M \) with Hermitian metric \( H_0 \), \( S \) be a Higgs sub-sheaf of \((E, A_0, \phi_0)\), and \((A_j, \phi_j) = g_j(A_0, \phi_0)\) be a sequence of Higgs pairs on \( E \), where \( g_j \) is a sequence of complex gauge transformations. Suppose that there exits a sequence of blow-ups: \( \pi_i : \overline{M_i} \to \overline{M}_{i-1}, i = 1, \ldots, r \) (where \( \overline{M}_0 = M \), every \( \pi_i \) is a blow up with non-singular center; denoting \( \pi = \pi_r \circ \cdots \circ \pi_1 \) ); such that \( \pi^* E \) and \( \pi^* S \) are bundles, the pulling back geometric objects \( \pi^* A_0, \phi_0 \), \( \pi^* g_j \) and \( \pi^* H_0 \) can be extended smoothly on the whole \( M_r \). Assume that \((A_j, \phi_j)\) converges to \((A_\infty, \phi_\infty)\) outside a closed subset \( \Sigma_{An} \) of Hausdorff complex codimension 2, and \(|\Lambda_\omega(F_{A_j})|H_0\) is bounded uniformly in \( j \) in \( L^p(\omega_0) \). Let \( i_0 : (S, \partial A_0) \to (E, \partial A_0) \) be the holomorphic inclusion, then there is a subsequence of \( g_j \circ i_0 \), up to rescale, converges to a non-zero holomorphic map \( f_\infty : (S, \mathcal{J}_{A_0}) \to (E_\infty, \mathcal{J}_{A_\infty}) \) in \( C^\infty\) off \( \Sigma \cup \Sigma_{An} \), and \( f_\infty \circ \phi_0 = \phi_\infty \circ f_\infty \), where \( \Sigma \) is the singular set of \( S \) and \( E \).

**Proof.** On each blow-up \( \overline{M}_i \), we have a family of Kähler metrics defined iteratively by \( \omega_{\epsilon_1, \ldots, \epsilon_i} = \pi^*_i \omega_{\epsilon_1, \ldots, \epsilon_{i-1}} + \epsilon_i \eta_i \), where \( \eta_i \) is a Kähler metric on \( \overline{M}_i \). For simplicity, we write \( \omega_\epsilon = \omega_{\epsilon_1, \ldots, \epsilon_i} \), \( \tilde{E} = \pi^* E \). In the following, we denote geometric objects and their pulling back by the same notation, and \( \tilde{H}_0 = \pi^* H_0 \).

Define the map \( \tilde{\eta}_j : (\mathcal{J}_{A_0}, \partial A_j) \to (\tilde{E}, \partial A_j) \) by \( \tilde{\eta}_j = g_j \circ i_0 \). It is easy to check that

\[
\mathcal{J}_{A_0}, A_j \tilde{\eta}_j = 0, \quad \tilde{\eta}_j \circ \phi_0 = \phi_j \circ \tilde{\eta}_j,
\]

i.e. \( \tilde{\eta}_j \) is a \( \phi \)-invariant holomorphic map. For simplicity, we will denote \( \mathcal{J}_{A_0, A_j} \) by \( \mathcal{J}_{0,j} \), and the trace Laplacian operator on the section of \( S^* \otimes E \) with respect to the connection \( A_0 \otimes A_j \) by \( \Delta_{0,j} \).
Let \( H_{j,\epsilon}(t) \) and \( H^S_{j,\epsilon}(t) \) be the solutions of Donaldson’s flow on holomorphic bundles \((\tilde{E}, \tilde{\mathcal{S}}_{\tilde{A}_0})\) and \((\tilde{S}, \mathcal{S}_{A_0})\) with the fixed initial metrics \( \tilde{H}_0 \) and \( H^S_0 \) and with respect to the metric \( \omega_\epsilon \), i.e. it satisfies the following heat equation

\[
(4.2) \quad H^{-1} \partial H = -2\sqrt{-1}\Lambda_{\omega_\epsilon} F_H.
\]

By the heat flow, we have

\[
(4.3) \quad (\Delta_\epsilon - \frac{\partial}{\partial t})|\Lambda_{\omega_\epsilon}(F_{H_{j,\epsilon}(t)})|_{H_{j,\epsilon}(t)} \geq 0,
\]

The maximum principal implies that, for \( t > 0 \),

\[
(4.4) \quad |\Lambda_{\omega_\epsilon}(F_{H_{j,\epsilon}(t)})|_{H_{j,\epsilon}(t)}(x) \leq \int_{M} K_\epsilon(t, x, y)|\Lambda_{\omega_\epsilon}(F_{\tilde{A}_0})|_{\tilde{H}_0} \frac{\omega^n_\epsilon}{n!},
\]

where \( K_\epsilon(t, x, y) \) is the heat kernel of the Laplacian with respect to \( \omega_\epsilon \). On the other hand, we have

\[
(4.5) \quad \Delta_\epsilon|\tilde{\eta}_j|^2_{H^S_{\epsilon}(t), H_{j,\epsilon}(t)} = 2 |\partial H^S_{\epsilon}(t), H_{j,\epsilon}(t) \tilde{\eta}_j|^2 - 2 < -\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H^S_{\epsilon}(t)}, \eta_j > + 2 \sqrt{-1} \Lambda_{\omega_\epsilon} F_{H^S_{\epsilon}(t)}, \eta_j >,
\]

\[
(4.6) \quad \frac{\partial}{\partial t}|\tilde{\eta}_j|^2_{H^S_{\epsilon}(t), H_{j,\epsilon}(t)} = - 2 < -\sqrt{-1}\Lambda_{\omega_\epsilon} F_{H^S_{\epsilon}(t)}, \eta_j > + 2 \sqrt{-1} \Lambda_{\omega_\epsilon} F_{H^S_{\epsilon}(t)}, \eta_j >,
\]

and then

\[
(4.7) \quad (\Delta_\epsilon - \frac{\partial}{\partial t})|\tilde{\eta}_j|^2_{H^S_{\epsilon}(t), H_{j,\epsilon}(t)} \geq 0.
\]

Using the Maximum principle again, we have

\[
(4.8) \quad |\tilde{\eta}_j|^2_{H^S_{\epsilon}(t_0+t), H_{j,\epsilon}(t_0+t)}(x) \leq \int_{M} K_\epsilon(t, x, y)|\tilde{\eta}_j|^2_{H^S_{\epsilon}(t_0), H_{j,\epsilon}(t_0)} \frac{\omega^n_\epsilon}{n!},
\]

for any \( t_0 \geq 0 \) and \( t > 0 \).

By [7] (Lemma 4), for fixed \( \epsilon' = (\epsilon_1, \cdots, \epsilon_{r-1}) \) the heat kernel \( K_\epsilon(t, x, y) \) has a uniform bound for \( 0 < \epsilon_r \leq 1 \). It is easy to see that \( K_\epsilon \) converges to the heat kernel \( K_{\epsilon'} \) on \( M_{r-1} \) outside the exceptional divisor as \( \epsilon_r \to 0 \). Bando and Siu ([7]) had shown that we could choose a subsequence of \( H_{j,\epsilon}(t) \) (and the same for \( H^S_{\epsilon}(t) \)) which converges to a solution of the Donaldson’s heat flow \([12]\) on \( M_{r-1} \) as \( \epsilon_r \) tends to 0. Then we have

\[
(4.9) \quad |\Lambda_{\omega_{\epsilon'}}(F_{H_{j,\epsilon'}(t)})|_{H_{j,\epsilon'}(t)}(x) \leq \int_{M_{r-1}} K_{\epsilon'}(t, x, y)|\Lambda_{\omega_{\epsilon'}}(F_{\tilde{A}_0})|_{\tilde{H}_0} \frac{\omega^n_{\epsilon'}}{n!},
\]

and

\[
(4.10) \quad |\tilde{\eta}_j|^2_{H^S_{\epsilon'}(t_0+t), H_{j,\epsilon'}(t_0+t)}(x) \leq \int_{M_{r-1}} K_{\epsilon'}(t, x, y)|\tilde{\eta}_j|^2_{H^S_{\epsilon'}(t_0), H_{j,\epsilon'}(t_0)} \frac{\omega^n_{\epsilon'}}{n!},
\]

for all \( x \) outside the exceptional set.

Taking the limit \( \epsilon_{r-1} \to 0 \), and repeating the argument, we have a solution of the heat flow \([12]\) \( H_j(t) \) (and \( H^S(t) \)) on \( M \). We also have

\[
(4.11) \quad |\Lambda_{\omega}(F_{H_j(t)})|_{H_j(t)}(x) \leq \int_{M} K(t, x, y)|\Lambda_{\omega}(F_{A_0})|_{H_0} \frac{\omega^n}{n!},
\]

and

\[
(4.12) \quad |\tilde{\eta}_j|^2_{H^S(t_0+t), H_j(t_0+t)}(x) \leq \int_{M} K(t, x, y)|\tilde{\eta}_j|^2_{H^S(t_0), H_j(t_0)} \frac{\omega^n}{n!},
\]
for all \( x \) outside \( \Sigma \), where \( K(t, x, y) \) is the heat kernel of \((M, \omega)\). Using \( K(t, x, y) \leq C_K(1 + \frac{1}{t}) \), and the uniform \( L^1 \) bound in the assumptions, we have a uniform constant \( C_F \) which is independent on \( j \) such that

\[
2(|\Lambda_\omega(F_{H_j(t)})|_{H^1(t)} + |\Lambda_\omega(F_{H^8(t)})|_{H^8(t)}) (x) \leq C_F
\]

for all \( x \in M \setminus \Sigma \) and \( t \geq t_0 > 0 \).

By (4.10) and (4.13), we have

\[
-C_F \leq \frac{\partial}{\partial t} \ln |\tilde{h}_j\rangle |_{H^8(t), H_j(t)} (x) \leq C_F,
\]

for all \( x \in M \setminus \Sigma \) and \( t \geq t_0 > 0 \). Then

\[
e^{-C_F \delta} \leq \frac{|\tilde{h}_j\rangle |_{H^8(t_0), H_j(t_0)}}{|\tilde{h}_j\rangle |_{H^8(t_0), H_j(t_0)}} (x) \leq e^{C_F \delta},
\]

and

\[
\frac{|\tilde{h}_j\rangle |_{H^8(t_0), H_j(t_0)} (x)}{C_F} \leq e^{C_F \delta} |\tilde{h}_j\rangle |_{H^8(t_0), H_j(t_0)} (x)
\]

(4.16)

\[
\leq C_F e^{C_F \delta} (1 + \delta^{-n}) \int_M |\tilde{h}_j\rangle |_{H^8(t_0), H_j(t_0)} \Omega_n
\]

Denote \( h_{j, \epsilon}(t) = \tilde{h}_0^{-1} H_{j, \epsilon}(t) \), it is easy to check that

\[
(\triangle - \partial^2 \ln \omega \rangle) \ln (tr(h_{j, \epsilon}(t)) + tr(h_{j, \epsilon}(t^{-1})) \geq -2 |\Lambda_\omega(F_{A_j})|_{\tilde{H}_0}
\]

From the above inequality, we have

\[
\int_M \ln (tr(h_{j, \epsilon}(t)) + tr(h_{j, \epsilon}(t^{-1})) \Omega_n \leq e^{-2 |\Lambda_\omega(F_{A_j})|_{\tilde{H}_0} \Omega_n}
\]

Recall the result of Bando and Siu in [7], by choosing a subsequence, we know that \( H_{j, \epsilon} \) converges to a solution \( H_j \) of the heat flow \((4.2)\) on \( M \setminus \Sigma \) as \( \epsilon \to 0 \). Then

\[
\int_M \ln (tr(h_j(t)) + tr(h_j(t^{-1})) \Omega_n \leq e^{-2 |\Lambda_\omega(F_{A_j})|_{\tilde{H}_0} \Omega_n}
\]

(4.17)

On the other hand, we have

\[
\Delta \ln (tr(h_j(t)) + tr(h_j(t^{-1})) \geq -2 |\Lambda_\omega(F_{H_j(t)})|_{H_j(t)} - 2 |\Lambda_\omega(F_{A_j})|_{\tilde{H}_0}
\]

(4.18)

on \( M \setminus \Sigma \), for all \( t > 0 \).

For any compact subset \( \Omega \subset M \setminus (\Sigma \cup \Sigma_{A_n}) \), setting \( d(\Omega) = \inf \{ \rho(x, y) | x \in \Omega, y \in \Sigma \cup \Sigma_{A_n} \} \), where \( \rho \) is the distance function on \((M, \omega)\). Let \( B = \bigcup_{y \in \Sigma \cup \Sigma_{A_n}} B_{\frac{1}{2}d(\Omega)} \) and \( \Omega' = M - B \), then we choose the cut-off function \( \varphi \) such that \( \varphi \equiv 1 \) on \( \Omega \), \( \varphi \equiv 0 \) on \( B \), and \( |d\varphi| \leq \frac{4}{d(\Omega)} \). By the assumption, we known that \( A_j \) are locally bounded in \( C^\infty \) outside \( \Sigma_{A_n} \), so we have

\[
|\Lambda_\omega(F_{A_j})|_{\tilde{H}_0} \leq C_c
\]

on \( \Omega' \), where \( C_c \) is a constant independent of \( j \). Using (4.19), (4.20), (4.13), (4.21), the cut-off function \( \varphi \) and the Moser’s iteration, we have

\[
\sup_{x \in \Omega} \ln (tr(h_j(1)) + tr(h_j(1)^{-1}))
\]

\[
\leq C_c \int_M \ln (tr(h_j(1)) + tr(h_j(1)^{-1})) \Omega_n \leq C_c,
\]

(4.22)
where $C_d$, $C_e$ are constants independent of $j$. In a similar way, we have locally $C^0$ bound on metrics $H^S(1)$, i.e. for any compact subset $\Omega$, there exists a constant $C_f$ such that

$$\sup_{x \in \Omega} \ln(\text{tr}((H^S_0)^{-1}(H^S(1))) + \text{tr}((H^S(1))^{-1}H^S_0)) \leq C_f. \tag{4.23}$$

By (4.16) and rescale $\eta_j$, we have a sequence $\phi$-invariant $\overline{\partial}_{0,j}$-holomorphic map $f_j$ such that

$$|f_j|_{H^S(1),0_j(1)}^2 \leq C_a, \quad \int_M |f_j|_{H^S(1),0_j(1)}^2 \frac{\omega^n}{n!} = 1. \tag{4.24}$$

For any compact subset $\Omega \subset M \setminus (\Sigma \cup \Sigma_{An})$, by (4.24) and the above uniform locally $C^0$ bound on $H_j(1)$ and $H^S(1)$ (i.e. (4.22) and (4.23)), we have

$$\sup_{x \in \Omega} |f_j|_{H^S_0,H_0}^2(x) \leq C(\Omega), \tag{4.25}$$

where $C(\Omega)$ is a constant independent of $j$.

Since $f_j$ is $\overline{\partial}_{0,j}$-holomorphic, we have

$$\Delta_{0,j}f_j = \sqrt{-1}\Lambda_\omega(\overline{\partial}_{0,j}\overline{\partial}_{0,j}f_j) = -\sqrt{-1}\Lambda_\omega(\overline{\partial}_{0,j}\overline{\partial}_{0,j}f_j) + \overline{\partial}_{0,j}f_j = -\sqrt{-1}\Lambda_\omega(F_{A_j}f_j - f_j F_{A_0}), \tag{4.26}$$

and

$$\overline{\partial}_{0,\infty}f_j = \overline{\partial}_{A_\infty} \circ f_j - f_j \circ \overline{\partial}_{A_0} = -\beta_{0,j}^{0,1} \circ f_j, \tag{4.27}$$

where $\beta_{0,j} = A_j - A_\infty$.

By the above uniform locally $C^0$ bound of $f_j$ (i.e. (4.25)) and the assumption that $A_j \to A_\infty$ in $C^\infty_{loc}$ topology outside $\Sigma_{An}$, the elliptic theory implies that there exists a subsequence of $f_j$ (for simplicity, also denoted by $f_j$) such that $f_j \to f_\infty$ in $C^\infty_{loc}$ topology outside $\Sigma \cup \Sigma_{An}$, and

$$\overline{\partial}_{A_0,A_\infty}f_\infty = 0, \quad f_\infty \circ \phi_0 = \phi_\infty \circ f_\infty. \tag{4.28}$$

Since $\Sigma \cup \Sigma_{An}$ is Hausdorff codimension at least 2, for any small $\delta > 0$, we can choose a compact subset $\Omega_\delta \subset M \setminus (\Sigma \cup \Sigma_{An})$ such that

$$\int_{M \setminus \Omega_\delta} \frac{\omega^n}{n!} \leq \delta. \tag{4.29}$$

From (4.24) and (4.29), we have

$$\int_{\Omega_\delta} |f_j|_{H^S(1),0_j(1)}^2 \frac{\omega^n}{n!} \geq 1 - \delta C_a. \tag{4.30}$$

Using the above uniform locally $C^0$ bound on $H_j(1)$ and $H^S(1)$ (4.22) and (4.23) again, we have a positive constant $\tilde{C}(\Omega_\delta)$ such that

$$\int_{\Omega_\delta} |f_\infty|_{H^S_0,H_0}^2 \frac{\omega^n}{n!} \geq \tilde{C}(\Omega_\delta)(1 - \delta C_a) > 0, \tag{4.31}$$

for every $j$. Then, we get $\int_{\Omega_\delta} |f_\infty|_{H^S_0,H_0}^2 \frac{\omega^n}{n!} > 0$, and so $f_\infty$ is a non-zero holomorphic map. \qed
5. The HN type of the Uhlenbeck limit

Let \((A_t, \phi_t)\) be a smooth solution of the gradient heat flow (1.4) over Kähler manifold \((M, \omega)\) with initial data \((A_0, \phi_0)\), and let \((A_\infty, \phi_\infty)\) be an Uhlenbeck limit. From theorem 2.4., we know that \((A_\infty, \phi_\infty)\) is a smooth Yang-Mills Higgs pair on Hermitian bundle \((E_\infty, H_\infty)\) over \(M \setminus \Sigma_{an}\), and \(\theta(A_\infty, \phi_\infty)\) is parallel, then the constant eigenvalues vector \(\vec{\lambda}_\infty = (\lambda_1, \ldots, \lambda_R)\) of \(-\sqrt{-1}\theta(A_\infty, \phi_\infty)\) is just the HN type of the extended Uhlenbeck limit Higgs sheaf \((E_\infty, A_\infty, \phi_\infty)\). Let \(\vec{\mu}\) be the HN type of the initial Higgs bundle \((E, A_0, \phi_0)\). In this section, we will prove that 
\[
\vec{\lambda}_\infty = \vec{\mu}.
\]

Let \(u(R)\) denote the Lie algebra of the unitary group \(U(R)\). Fix a real number \(\alpha \geq 1\), for any \(a \in u(R)\), let \(\varphi_\alpha(a) = \sum_{j=1}^{R} |\lambda_j|^\alpha\), where \(\sqrt{-1}\lambda_j\) are eigenvalues of \(a\). It is easy to see that we can find a family \(\varphi_{\alpha, \rho}\) of smooth convex ad-invariant functions \(0 < \rho \leq 1\), such that \(\varphi_{\alpha, \rho} \to \varphi_\alpha\) uniformly on compact subsets of \(u(R)\) as \(\rho \to 0\). Hence, from [1] (Prop.12.16) it follows that \(\varphi_\alpha\) is a convex function on \(u(R)\). For a given real number \(N\), define the Hermitian-Yang-Mills type functionals as follows:

\[
HYM_{\alpha, N}(A, \phi) = \int_M \varphi_\alpha(\theta(A, \phi) - \sqrt{-1}NId_E) \omega^n_n \]

In the following we assume that \(Vol(M, \omega) = 1\), and set \(HYM_{\alpha, N}(\vec{\mu}) = HYM_{\alpha, N}(\vec{\mu} + N) = \varphi_\alpha(\sqrt{-1}((\vec{\mu} + N)))\), where \(\vec{\mu} + N = \text{diag}(\mu_1 + N, \ldots, \mu_R + N)\). We need the following two lemmas, the proofs can be found in [12] (Lemma 2.23 ; Prop.2.24).

**Lemma 5.1.** The functional \(a \mapsto \int_M \varphi_\alpha(a) \text{dvol}\), defines a norm on \(L^\alpha(u(E))\) which is equivalent to the \(L^\alpha\) norm.

**Lemma 5.2.** (1) If \(\vec{\mu} \leq \vec{\lambda}\), then \(\varphi_\alpha(\sqrt{-1}\vec{\mu}) \leq \varphi_\alpha(\sqrt{-1}\vec{\lambda})\) for all \(\alpha \geq 1\).

(2) Assume \(\mu_R \geq 0\) and \(\lambda_R \geq 0\). If \(\varphi_\alpha(\sqrt{-1}\vec{\mu}) = \varphi_\alpha(\sqrt{-1}\vec{\lambda})\) for all \(\alpha\) in some set \(S \subset [1, \infty)\) possessing a limit point, then \(\vec{\mu} = \vec{\lambda}\).

For any smooth convex ad-invariant functions \(\varphi\), we have

\[
(\Delta - \frac{\partial}{\partial t}) \varphi(\theta(A_t, \phi_t) - \sqrt{-1}NId_E) \geq 0,
\]

whose proof can be found in [30] (Section two). Since we can approximate \(\varphi_\alpha\) by smooth convex ad-invariant functions \(\varphi_{\alpha, \rho} \to \varphi_\alpha\). By (5.2), we know that \(t \mapsto HYM_{\alpha, N}(A_t, \phi_t)\) is nonincreasing along the flow. By Corollary 2.5, we can choose a sequence \(t_j \to \infty\), such that

\[
HYM_{\alpha, N}(A_t, \phi_t) \rightarrow HYM_{\alpha, N}(A_\infty, \phi_\infty).
\]

Then we have the following proposition.

**Proposition 5.3.** Let \((A_t, \phi_t)\) be a solution of the gradient flow (1.4) and \((A_\infty, \phi_\infty)\) be a subsequential Uhlenbeck limit of \((A_t, \phi_t)\). Then for any \(\alpha \geq 1\) and any \(N\), \(t \mapsto HYM_{\alpha, N}(A_t, \phi_t)\) is nonincreasing, and \(\lim_{t \to \infty} HYM_{\alpha, N}(A_t, \phi_t) = HYM_{\alpha, N}(A_\infty, \phi_\infty)\).

**Lemma 5.4** Let \((A_j, \phi_j) = g_j(A_0, \phi_0)\) be a sequence of complex gauge equivalent Higgs pairs on a complex vector bundle \(E\) of rank \(R\) with Hermitian metric \(H_0\). Let \(S\) be a coherent \(\phi_0\)-invariant subsheaf of \((E, A_0)\). Suppose that \(\sqrt{-1}\Lambda_\omega(F_{A_j} + \).

\[\pi_j \rightarrow a \text{ in } L^1 \text{ as } j \rightarrow \infty, \text{ where } a \in L^1(\sqrt{-1}u(E)), \text{ and that eigenvalues } \lambda_1 \geq \cdots \geq \lambda_R \text{ of } \frac{1}{2\pi} a \text{ are constant almost everywhere. Then: } \deg(S) \leq \sum_{i \leq \operatorname{rank}(S)} \lambda_i.\]

**Proof.** Since \(\deg(S) \leq \deg(\operatorname{Sat}_E(S))\), we may assume that \(S\) is saturated. Let \(\pi_j\) denote the orthogonal projection onto \(g_j(S)\) with respect to the Hermitian metric \(H_0\). It is well known that \(\pi_j\) are \(L^2\) sections of the smooth endomorphism bundle of \(E\), and satisfy \(\pi_j^2 = \pi_j = \pi_j^\perp, (Id - \pi_j)\overline{\partial}_A \pi_j = 0\) and \((Id - \pi_j)\phi_j \pi_j = 0\) (since \(g_j(S)\) are \(\phi_j\)-invariant). By the usual degree formula (see Lemma 3.2 in [30]), we have

\[
\deg(S) = \frac{1}{2\pi} \int_M (\text{Tr}(-1A_{\omega}(F_{A_j} + [\phi_j, \phi_j^*])\pi_j) - \overline{\partial}_{A_j + \phi_j} \pi_j)^2
\]

\[\leq \frac{1}{2\pi} \int_M (\text{Tr}(-1A_{\omega}(F_{A_j} + [\phi_j, \phi_j^*])\pi_j) - \text{Tr}(a\pi_j + \frac{1}{\sqrt{2\pi}} \int_M (\sqrt{-1}A_{\omega}((F_{A_j} + [\phi_j, \phi_j^*]) - a)\pi_j)).\]

By a result from linear algebra (Lemma 2.20 in [12]), we have \(\frac{1}{2\pi} \text{Tr}(a\pi_j) \leq \sum_{i \leq \operatorname{rank}(S)} \lambda_i\). So, we have \(\deg(S) \leq \sum_{i \leq \operatorname{rank}(S)} \lambda_i + \frac{1}{2\pi} \|\sqrt{-1}A_{\omega}(F_{A_j} + [\phi_j, \phi_j^*]) - a\|_{L^1}.\) Let \(j \rightarrow \infty\), this completes the proof of the lemma.

\[\square\]

Let \(\tilde{\mu}_0 = (\mu_1, \cdots, \mu_R)\) be the HN type of Higgs bundle \((E, A_0, \phi_0)\), by [24] in Lemma 2.1, we have

\[
\sum_{\alpha=1}^R \mu_\alpha = \deg(E, \overline{\partial}_{A_0}) = \deg(E_{\infty}, \overline{\partial}_{A_{\infty}}) = \sum_{\alpha=1}^R \lambda_\alpha.
\]

Let \(\{E_i\}_{i=1}^l\) be the HN filtration of the Higgs bundle \((E, A_0, \phi_0)\). Using Corollary 2.5 and Lemma 5.4, we have:

\[
\sum_{\alpha \leq \operatorname{rank}E_i} \mu_\alpha = \deg(E_i) \leq \sum_{\alpha \leq \operatorname{rank}E_i} \lambda_\alpha
\]

for all \(i\). By Lemma 2.3 in [12], we have the following proposition.

**Proposition 5.5.** Let \((A_i, \phi_i)\) be a sequence of Higgs pairs along the gradient heat flow (1.6) with Uhlenbeck limit \((A_{\infty}, \phi_{\infty})\). Let \(\tilde{\mu}_0 = (\mu_1, \cdots, \mu_R)\) be the HN type of Higgs bundle \((E, A_0, \phi_0)\), and let \(\tilde{\lambda}_0 = (\lambda_1, \cdots, \lambda_R)\) be the type of Higgs bundle \((E_{\infty}, A_{\infty}, \phi_{\infty})\). Then \(\tilde{\mu}_0 \leq \tilde{\lambda}_0\).

Let \(H\) be a smooth Hermitian metric on the holomorphic bundle \(E = (E, \overline{\partial}_E)\), and let \(F = \{F_i\}_{i=1}^l\) be a filtration of \(E\) by saturated subsheaves: \(0 = F_0 \subset F_1 \subset \cdots \subset F_{l-1} \subset F_l = E\). Associated to each \(F_i\) and the metric \(H\) we have the unitary projection \(\pi_i^H\) onto \(F_i\). It is well known that \(\pi_i^H\) are bounded \(L^2\) Hermitian endomorphisms. For convenience, we set \(\pi_0^H = 0\). Given real numbers \(\mu_1, \cdots, \mu_l\) and a filtration \(F\), we define a bounded \(L^2\) Hermitian endomorphism of \(E\) by \(\Psi(F, (\mu_1, \cdots, \mu_l), H) = \sum_{i=1}^l \mu_i (\pi_i^H - \pi_{i-1}^H)\). Given a Hermitian metric on a Higgs bundle \((E, \phi)\), the Harder-Narasimhan projection, \(\Psi^H(E, \phi, H)\) is the bounded \(L^2\) Hermitian endomorphism defined above in the particular case where \(F\) is the HN filtration \(F_i = F_i^{hn}(E)\) and \(\mu_i = \mu(F_i/F_{i-1})\).
\textbf{Definition 5.6.} Fix \( \delta > 0 \) and \( 1 \leq p \leq \infty \). An \( L^p-\delta \)-approximate critical Hermitian metric on a Higgs bundle \((E, \phi)\) is a smooth \( H \) such that

\[
\left\| \frac{\sqrt{-1}}{2\pi} \Lambda_\omega (F_{AH} + [\phi, \phi^H]) - \Psi^{HN}(E, \phi, H) \right\|_{L^p(\omega)} \leq \delta,
\]

where \( A_H \) is the Chern connection determined by \((\overline{\partial}_E, H)\).

\textbf{Proposition 5.7.} Let \((E, \phi)\) be a Higgs bundle on a smooth Kähler manifold \((M, \omega)\), and let \( F = \{F_i\}_{i=1}^{l} \) be a filtration of \( E \) by saturated subsheaves, where every \( F_i \) is \( \phi \) invariant. Let \( \pi : \overline{M} \rightarrow M \) be a blow-up along a smooth complex manifold \( \Sigma \) of complex co-dimension at least 2, \( \overline{E} = \pi^*E \) be the pull-back bundle and \( \overline{\phi} = \pi^*\phi \). Let \( \eta \) be a Kähler metric on \( \overline{M} \), and set a family of Kähler metrics \( \omega_\varepsilon = \pi^*\omega + \varepsilon \eta \). Suppose that the filtration \( \mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{l} = \{\text{Sat}_{\mathcal{F}_i}(\pi^*F_i)\}_{i=1}^{l} \) of \( \overline{E} \) is given by subbundles, and every quotient \( \overline{Q}_i = \mathcal{F}_i/\mathcal{F}_{i-1} \) is \( \overline{\phi} \)-Higgs \( \omega_\varepsilon \)-stable for \( 0 < \varepsilon < \varepsilon^* \). Then for any \( \delta > 0 \) and any \( 0 < \epsilon < \epsilon^* \), there is a smooth Hermitian metric \( \overline{H} \) on \( \overline{E} \) such that

\[
\left\| \frac{\sqrt{-1}}{2\pi} \Lambda_{\omega_\epsilon} (F(\overline{\sigma}_E, \overline{H}) + [\overline{\phi}, \overline{\phi}_H]) - \Psi(\overline{F}, (\mu_{\epsilon,1}, \ldots, \mu_{\epsilon,l}), \overline{H}) \right\|_{L^\infty} \leq \delta,
\]

where \( (\overline{\partial}_E, \overline{H}) \) denotes the Chern connection with respect to holomorphic structure \( \overline{\sigma}_E \) and metric \( \overline{H} \), and \( \mu_{\epsilon,i} \) is the slope of quotient \( \overline{Q}_i \) with respect to the metric \( \omega_\varepsilon \).

\textbf{Proof.} Let \( \overline{\sigma}_i \) be the induced Higgs field on the quotient \( \overline{Q}_i \). Since Higgs bundles \((\overline{Q}_i, \overline{\sigma}_i)\) are \( \omega_\varepsilon \)-stable for all \( 0 < \epsilon < \epsilon^* \), by Simpson’s result (Theorem 1 in \[\text{[36]}\]), we have a Hermitian-Einstein metric \( \overline{H}_{\epsilon,i} \) on Higgs bundle \((\overline{Q}_i, \overline{\sigma}_i)\) respect to \( \omega_\varepsilon \). In particular:

\[
\frac{\sqrt{-1}}{2\pi} \Lambda_{\omega_\epsilon} (F(\overline{\sigma}_E, \overline{H}_{\epsilon,i}) + [\overline{\phi}, \overline{\phi}_H]) - \mu_{\epsilon,i}Id_{\overline{Q}_i} = 0.
\]

We will use Donaldson’s argument in \[\text{[14]}\]. Recall that \( \mathcal{F}_E \) is isomorphic to \( \oplus_i \overline{Q}_i \) in the sense of vector bundle, we take the direct sum \( \overline{H}_\epsilon = \oplus_i \overline{H}_{\epsilon,i} \). By the equivalence of holomorphic structure and integrable unitary connection, we see that it suffices to show that for a fixed Hermitian metric \( \overline{H} \) there is a smooth complex gauge transformation \( \sigma \) preserving the filtration \( \mathcal{F} \) such that

\[
\left\| \frac{\sqrt{-1}}{2\pi} \Lambda_{\omega_\epsilon} (F(\sigma(\overline{\sigma}_E), \overline{H}) + [\sigma(\overline{\phi}), \sigma(\overline{\phi})]) - \Psi(\overline{F}, (\mu_{\epsilon,1}, \ldots, \mu_{\epsilon,l}), \overline{H}) \right\|_{L^\infty} \leq \delta,
\]

We only consider the case \( l = 2 \), the other case \( l > 2 \) can be solved by inductive argument. We can write the holomorphic structure \( \overline{\partial}_E \) and the Higgs field \( \overline{\phi} \) as

\[
\overline{\partial}_E = \begin{pmatrix} \partial_{\overline{Q}_1} & B \\ 0 & \partial_{\overline{Q}_2} \end{pmatrix}, \quad \overline{\phi} = \begin{pmatrix} \phi_1 & \zeta \\ 0 & \phi_2 \end{pmatrix},
\]

where \( B \) is the second fundamental form. Define the complex gauge transformation \( \sigma_t \) to be the following block diagonal matrix

\[
\sigma_t = \begin{pmatrix} tId_{\overline{Q}_1} & 0 \\ 0 & t^{-1}Id_{\overline{Q}_2} \end{pmatrix}.
\]
Then, we have
\[ \sigma_t(\nabla_{\mathcal{F}}) = \left( \begin{array}{cc} \Phi_1 & t^2 \Phi_2 \\ 0 & 0 \end{array} \right), \quad \sigma_t(\Phi) = \left( \begin{array}{c} \Phi_1 \\ 0 \end{array} \right), \]
\[ F(\sigma_t(\nabla_{\mathcal{F}}), \mathcal{F}) = \left( \begin{array}{cc} F(\nabla_{\mathcal{F}}, \mathcal{F}_{\mathcal{H}}) + t^4 B \wedge \mathcal{H}^* & t^2 \partial B^* \\ 0 & F(\nabla_{\mathcal{F}}, \mathcal{F}_{\mathcal{H}}) + t^4 B^* \wedge \mathcal{H} \end{array} \right), \]
and
\[ [\sigma_t(\Phi), \sigma_t(\Phi^*)] = \left( \begin{array}{c} t^4 \Phi_1 + t^2 \Phi_2 \wedge \Phi_1 \\ t^2 (\Phi_1 \wedge \Phi_2 + \Phi_2 \wedge \Phi_1) \end{array} \right). \]

We can also write \( \Psi(\mathcal{F}, (\mu_1, \mu_2), \mathcal{H}) \) as follows
\[ \left( \begin{array}{cc} \mu_1 I_{\mathcal{Q}_1} & 0 \\ 0 & \mu_2 I_{\mathcal{Q}_2} \end{array} \right). \]

Then, it is easy to see that
\[
\left| \frac{\sqrt{t}}{2\pi} \Lambda_{\omega_t}(F(\sigma_t(\nabla_{\mathcal{F}})), \mathcal{F}) + [\sigma_t(\Phi), \sigma_t(\Phi^*)] \right|_{\mathcal{H}} - \Psi(\mathcal{F}, (\mu_1, \mu_2), \mathcal{H}) \right|_{\mathcal{H}} \\
\leq \sum_{i=1}^2 \left| \frac{\sqrt{t}}{2\pi} \Lambda_{\omega_t}(F(\sigma_t(\nabla_{\mathcal{F}})), \mathcal{F}) + [\sigma_t(\Phi), \sigma_t(\Phi^*)] \right|_{\mathcal{H}} \\
+ f(t, B),
\]
where \( f(t, B) \to 0 \) as \( t \to 0 \). Let \( \mathcal{H}_t = t^2 \mathcal{H}_{1,t} \oplus t^{-2} \mathcal{H}_{2,t} \), then \( \sigma_t \circ \sigma_t = \mathcal{H}^{-1} \mathcal{H}_t \).

On the other hand, we have
\[
\left| \frac{\sqrt{t}}{2\pi} \Lambda_{\omega_t}(F(\sigma_t(\nabla_{\mathcal{F}})), \mathcal{F}) + [\sigma_t(\Phi), \sigma_t(\Phi^*)] \right|_{\mathcal{H}} - \Psi(\mathcal{F}, (\mu_1, \mu_2), \mathcal{H}_t) \right|_{\mathcal{H}} \\
= \left| \frac{\sqrt{t}}{2\pi} \Lambda_{\omega_t}(F(\sigma_t(\nabla_{\mathcal{F}})), \mathcal{F}) + [\sigma_t(\Phi), \sigma_t(\Phi^*)] \right|_{\mathcal{H}} \\
- \Psi(\mathcal{F}, (\mu_1, \mu_2), \mathcal{H}_t) \right|_{\mathcal{H}}.
\]
Choosing \( t \) small enough, we obtain a metric \( \mathcal{H} \) which satisfies (5.17).

The following lemma was proved by Sibley in [35] (Lemma 5.3.), we give a proof for reader’s convenience.

**Lemma 5.8.** Let \((M, \omega)\) be a compact Kähler manifold of complex dimension \( n \), and \( \pi : \overline{M} \to M \) be a blow-up along a smooth complex sub-manifold \( \Sigma \) of complex co-dimension \( k \) where \( k \geq 2 \). Let \( \eta \) be a Kähler metric on \( \overline{M} \), and consider the family of Kähler metrics \( \omega_t = \omega + c \eta \). Then for any \( 0 \leq \gamma < \frac{1}{n-k} \), we have \( \frac{\partial}{\partial r} \in L^\gamma(\overline{M}, \eta) \), and the \( L^\gamma(\overline{M}, \eta) \)-norm of \( \frac{\partial}{\partial r} \) is uniformly bounded in \( \epsilon \), i.e. there is a positive constant \( C^* \) such that
\[ \int_{\overline{M}} \left( \frac{\eta^n}{\omega^n} \right)^{\gamma} \eta^n \leq C^* \]
for all \( \epsilon \).

**Proof.** Since \( \omega \) is only degenerated along the exceptional divisor \( \omega^{-1}(\Sigma) \), on the complement of a neighborhood of \( \omega^{-1}(\Sigma) \) there is a constant \( C \) such that \( C^{-1} \eta \leq \omega \leq C \eta \). So, it is suffices to prove the result in a neighborhood of \( \omega^{-1}(\Sigma) \). One can choose a local coordinate chart \( U \) with coordinates \((z_1, \ldots, z_n)\), such that locally \( \Sigma \) is given by the slice \( \{ z_1 = \cdots = z_k = 0 \} \). On the blow-up \( \overline{M} \) we have local coordinate charts \( \overline{U}_i \subset \omega^{-1}(U) \) where \( \overline{U}_i = \{ z \in U | \Sigma | z_i \neq 0 \} \).
Firstly, we have (5.23) and then (5.22). Note that (5.19)

there exists a positive constant

(5.20)

where

\[ C \in \mathbb{R} \]

\[ \Gamma(U, \omega) \]

\[ \text{normal bundle of } \Sigma \text{ and } 1 \]

\[ k \]

\[ \alpha > 0 \]

\[ \gamma > 0 \]

\[ \pi \]

\[ \text{is degenerated} \]

\[ \text{is compact} \]

\[ \text{is a positive constant} \]

\[ \text{depending only on } \epsilon \]

\[ \text{and } \alpha \]

\[ \text{and } \eta \]

\[ \text{and } \pi \omega \]

\[ \text{such that for any } \text{End}(E) \text{-valued} \]

(1, 1) form

\[ F \]

\[ \|A_{\omega_1} F\|_{L^0(\Sigma, \omega_1)} \leq |\epsilon_1 - \epsilon| \|C(U)\|_{L^2(\Sigma)} \|F\|_{L^2(\Sigma)} \]

(5.21) +

\[ C\|A_{\omega_1} F\|_{L^0(\Sigma, \omega_1)} + (\text{Vol}(U, \omega_1))^{\frac{1}{2}} \|F\|_{L^2(\Sigma)} \]

for all \( 0 < \epsilon \leq \epsilon_1 \leq 1 \), where \( C(\alpha, k) \) is a positive constant depending only on \( \alpha \)

and \( k \).

**Proof.** By the definition, we know that

\[ A_{\omega_1} F = \frac{\omega_1^n}{\omega_1} (A_{\omega_1} F + n \frac{\omega_1^{n-1} - \omega_1^{n-1}}{\omega_1}) \]

(5.22)

and then

\[ \|A_{\omega_1} F\|_{L^0(\Sigma, \omega_1)} \leq \|\frac{\omega_1^n}{\omega_1} (A_{\omega_1} F)\|_{L^0(\Sigma, \omega_1)} + \|n \frac{\omega_1^{n-1} - \omega_1^{n-1}}{\omega_1}\|_{L^0(\Sigma, \omega_1)} \]

(5.23)

Firstly, we have

\[ \|\frac{\omega_1^n}{\omega_1} (A_{\omega_1} F)\|_{L^0(\Sigma, \omega_1)} = \int_{\Sigma} |A_{\omega_1} F|^0 (\omega_1^n)^{\frac{\alpha - 1}{p} - \frac{\alpha}{q}} \frac{n}{m!} \]

(5.24)

\[ \leq \left( \int_{\Sigma} |A_{\omega_1} F|^{\alpha - 1 - \frac{\alpha}{q}} \right)^{\frac{1}{\alpha - 1 - \frac{\alpha}{q}}} \left( \int_{\Sigma} (\omega_1^n)^{\frac{\alpha - 1}{p} - \frac{\alpha}{q}} \right)^{\frac{1}{\alpha - 1 - \frac{\alpha}{q}}} \]

where \( \alpha \cdot p = \alpha \cdot \frac{n}{m} < \alpha \cdot \frac{1}{p} + \frac{1}{q} = 1 \), we have \( (\alpha - 1)q < -\frac{1}{m} \).

Since \( \Sigma \) is compact, there is a constant \( C_M \) such that \( \pi^\ast \omega \leq C_M \eta \) on \( \Sigma \). Let \( U \) be a neighborhood of the exceptional divisor \( \pi^{-1}(\Sigma) \), since \( \pi^\ast \omega \) is degenerated
only along $\pi^{-1}(\Sigma)$, we can suppose that $\pi^*\omega \geq C_0 \eta$ on $M \setminus U$ for some positive constant $C_0$. Then, we have

$$
\begin{align*}
\int_{M \setminus U} |F^\phi|_{\omega_{\phi}}^2 - \int_{M \setminus U} |F^\phi|_{\omega_{\phi}}^2 & \leq C(n) C_0^{-(n+1)} |\epsilon_1 - \epsilon| |\int_{M \setminus U} |F|_{\omega_{\phi}}^2 |(C_M + \varepsilon)^{n(\alpha - 1)}|\epsilon_1 - \epsilon| \int_{M \setminus U} |F|_{\omega_{\phi}}^2.
\end{align*}
$$

On the other hand, we have

$$
\begin{align*}
\int_{U} |F^\phi|_{\omega_{\phi}}^2 & \leq C(n) \int_{U} |F|_{\omega_{\phi}}^2 |(\alpha - 1)| \omega_{\phi}^{n} |n^{(\alpha - 1)}| \omega_{\phi}^{n}.
\end{align*}
$$

(5.25)

(5.26)

where $q = \frac{1}{2} \left( \frac{2a - 2}{2 - a} + \frac{1}{\epsilon} \right) \cdot \left( \frac{2a - 2}{2 - a} \right)^{-1}$, and note that by the condition on $\alpha$ we have $\frac{2a - 2}{2 - a} < \frac{1}{\epsilon - 1}$.

Using (5.18) in the previous lemma, we see that the result follows from (5.23), (5.24), (5.25) and (5.26).

\[\square\]

**Proposition 5.10.** Let $(E, \phi)$ be a Higgs bundle on a smooth Kähler manifold $(M, \omega)$, and let $F = \{ F_i \}_{i=1}^l$ be a filtration of $E$ by saturated subsheaves, where every $F_i$ is $\phi$ invariant. Let $\pi : M \to M$ be a blow-up along a smooth complex manifold $\Sigma$ of complex co-dimension $k \geq 2$. $\widehat{E} = \pi^*E$ be the pull-back bundle and $\widehat{\phi} = \pi^*\phi$. Let $\eta$ be a Kähler metric on $M$, and set a family of Kähler metrics $\omega_i = \pi^*\omega + \epsilon_i$, and $\overline{E} = \{ \overline{F}_i \}_{i=1}^{l-1} = \{ \text{Sat}_{\pi}(F^* F_i) \}_{i=1}^{l-1}$ is a filtration of $\overline{E}$ (not necessary given by subbundles). Suppose that for any $\delta > 0$ and any $0 < \epsilon < \epsilon^*$, there is a smooth Hermitian metric $\overline{\eta}$ on $\widehat{E}$ such that

$$
(5.27) \left\| \sqrt{\frac{-1}{2\pi}} \Lambda_{\omega_i} (F_{\overline{F}_{\omega_i}}(\overline{\phi}, \phi, \overline{\phi})) - \Psi(\overline{F}, (\mu_1, \cdot, \mu_l), \overline{\eta}) \right\|_{L^2(\mathcal{M}, \omega_i)} \leq \delta.
$$

Then for any $\delta > 0$ and any $1 \leq p < 1 + \frac{1}{2 - a}$ there are $\epsilon_1 > 0$ and a smooth Hermitian metric $\overline{H}_1$ on $\overline{E}$ such that

$$
(5.28) \left\| \sqrt{\frac{-1}{2\pi}} \Lambda_{\omega_i} (F_{\overline{F}_{\omega_i}}(\overline{\phi}, \phi, \overline{\phi})) - \Psi(\overline{F}, (\mu_1, \cdot, \mu_l), \overline{H}_1) \right\|_{L^p(\mathcal{M}, \omega_i)} \leq \delta,
$$

for all $0 < \epsilon \leq \epsilon_1$, where $\mu_i$ is the $\omega$-slope of sheaf $F_i$.

**Proof.** Let $\epsilon_1 \in (0, \epsilon^*)$, by the condition, we can choose a smooth metric $\overline{H}_1$ satisfies (5.27) for $\epsilon_1$ and $\delta$ which will be chosen small enough later. For simplicity,
we denote $\Theta_2 = \frac{\sqrt{-1}}{2\pi} (F_{\overline{\nabla}_1} + [\phi, \phi_{\overline{1}}])$. Then
\begin{align*}
|\nabla^\phi \Lambda_{\omega_i} (F_{\overline{\nabla}_1} + [\phi, \phi_{\overline{1}}]) |_{L^p(\omega_1)} + |\Lambda_{\omega_i} (\Theta_1 - \frac{\omega_1}{n} \Psi(F, (\mu_1, \cdots, \mu_i), \overline{\nabla}_1)) |_{L^p(\omega_1)} \leq & \ |||\Lambda_{\omega_i} (\Theta_1 - \frac{\omega_1}{n} \Psi(F, (\mu_1, \cdots, \mu_i), \overline{\nabla}_1)) ||_{L^p(\omega_1)} \ |\Lambda_{\omega_i} \|_{L^p(M, \omega_1)} \ \ \ (5.29) \\
+ & |\frac{1}{n} \Lambda_{\omega_i} (\Theta_1 - \frac{\omega_1}{n} \Psi(F, (\mu_1, \cdots, \mu_i), \overline{\nabla}_1)) |_{L^p(\omega_1)} \ \ |\Theta_1 |_{L^1(M, \omega_1)} \ |\Lambda_{\omega_i} \|_{L^p(M, \omega_1)} \ \ \ (5.30) \\
+ & |\Psi(F, (\mu_1, \cdots, \mu_i), \overline{\nabla}_1)) |_{L^p(\omega_1)} \ |\Theta_1 |_{L^1(M, \omega_1)} \ |\Lambda_{\omega_i} \|_{L^p(M, \omega_1)} \ \ \ (5.31)
\end{align*}

For simplicity, we set $\Theta_2 = \Theta_1 - \frac{\omega_1}{n} \Psi(F, (\mu_1, \cdots, \mu_i), \overline{\nabla}_1)$. From the equality
\begin{align*}
\int_M |\Theta_1 |_{L^1(M, \omega_1)} = |\Lambda_{\omega_i} \|_{L^p(M, \omega_1)} \leq & \delta \ \ \ (5.32)
\end{align*}

By (5.19) in Lemma 5.8., it is not difficult to see that $\| \Lambda_{\omega_i} \|_{L^p(M, \omega_1)}$ is uniformly bounded. On the other hand, since $\mu_{i, j} \to \mu_i$ as $\epsilon \to 0$, we may choose $\epsilon_1$ small enough so that the second and third terms in (5.29) are both smaller than $\frac{\delta}{2}$, so (5.29) follows.

\textbf{Proposition 5.11.} Let $(E, \phi)$ be a Higgs bundle on a smooth Kähler manifold $(M, \omega)$, and let $F = \{F_i\}_{i=1}^\infty$ be a filtration of $E$ by saturated subsheaves, where every $F_i$ is $\phi$ invariant. Let $\pi : \overline{M} \to M$ be a blow-up along a smooth complex manifold $\Sigma$ of complex co-dimension $k \geq 2$. Let $\overline{E} = \pi^* E$ be the pull-back bundle and $\overline{\phi} = \pi^* \phi$. Let $\eta$ be a Kähler metric on $\overline{M}$, and set a family of Kähler metrics $\omega_k = \pi^* \omega + \epsilon \eta$, and $\overline{F} = \{\overline{F}_i\}_{i=1}^\infty$ is a filtration of $\overline{E}$ (not necessary given by subbundles). Suppose that for any $\delta > 0$ and any $1 \leq p < 1 + \frac{k}{2}$ there is a smooth metric $\overline{\nabla}_1$ on $\overline{E}$ and $\epsilon_1 > 0$ such that (5.28) hold for all $0 < \epsilon \leq \epsilon_1$. Then for any $\delta' > 0$ and any $1 \leq p \leq 1 + \frac{1}{2k-1}$ there is a smooth metric $H$ on $E$ such that
\begin{align*}
\int_\Sigma \| \sqrt{-1} \Lambda_{\omega_i} (F_{\overline{\nabla}_1} + [\phi, \phi_{\overline{1}}]) |_{L^p(M, \omega_1)} \ \ |\Lambda_{\omega_i} \|_{L^p(M, \omega_1)} \leq & \delta', \\
\text{where } \mu_i \text{ is the } \omega \text{-slope of sheaf } F_i.
\end{align*}

\textbf{Proof} We use a cut-off argument to get the smooth metric on bundle $E$. Since $\Sigma$ is a smooth complex submanifold, the open set $\{ (x, \nu) \in N \Sigma \} |\nu | < R \}$ in the normal bundle $N \Sigma$ of $\Sigma$, is diffeomorphic to an open neighborhood $U_R$ of $\Sigma$ for $R$ sufficiently small. For any small $R$, we may choose a smooth cut-off function $\psi_R$ which supported in $U_R$ and identically 1 on $U_{\frac{R}{2}}$, $0 \leq \psi_R \leq 1$, and furthermore $|\partial \psi_R |^2 + |\partial \overline{\psi}_R | \leq CR^{-2}$, where $C$ is a positive constant independent of $R$. Let
$H_D$ be a smooth Hermitian metric on bundle $E$, and $\overline{H}_1$ be the metric on $\overline{E}$ such that (5.28) holds for all $0 < \epsilon \leq \epsilon_1$ where $\delta \leq \frac{\epsilon}{\epsilon_1}$. Note that $E$ is isomorphic to $\overline{E}$ outsiders $\Sigma$, we can define

$$H_R = (1 - \psi_R)\overline{H}_1 + \psi_R H_D$$

on bundle $E$, and $\overline{H}_R = \pi^*H_R$ on bundle $\overline{E}$.

As above, we denote $\Theta(\overline{H}_R) = \frac{\sqrt{-1}}{2\pi}(F_{\overline{\omega}_R}^{\overline{H}_R} + [\overline{\phi}, \overline{\phi}^{\overline{H}_R}])$. We have

$$\int_{\overline{U}_R} |\Lambda_\omega \Theta(\overline{H}_R) - \Psi(\overline{F}, (\mu_1, \cdots, \mu_1), \overline{H}_R)|_p \frac{\omega^n_H}{n!}$$

$$\leq \int_{\pi^{-1}(U_R)} |\Lambda_\omega \Theta(\pi^*H_D) - \Psi(\overline{F}, (\mu_1, \cdots, \mu_1), \pi^*H_D)|_p \frac{\omega^n_H}{n!}$$

(5.35) $$+ \int_{M \setminus \pi^{-1}(U_R)} |\Lambda_\omega \Theta(\overline{H}_R) - \Psi(\overline{F}, (\mu_1, \cdots, \mu_1), \overline{H}_R)|_p \frac{\omega^n_H}{n!}$$

$$+(C(p) \int_{\pi^{-1}(U_R \setminus \overline{U}_R)} |\Lambda_\omega \Theta(\overline{H}_1 - \Theta(\overline{H}_1))|_p \frac{\omega^n_H}{n!}$$

$$+C(p) \int_{\pi^{-1}(U_R \setminus \overline{U}_R)} |\Psi(\overline{F}, (\mu_1, \cdots, \mu_1), \overline{H}_R) - \Lambda_\omega \Theta(\overline{H}_1))|_p \frac{\omega^n_H}{n!}.$$
where \( C_2 \) and \( C_3 \) are constants independent of \( \epsilon \) and \( R \). On the other hand, we have
\[
\int_{\pi^{-1}(U_R \setminus U_D)} |\Lambda_\omega, (\bar{\tau}, \varphi, \bar{\tau}^R_{H}) | \leq C_4 \frac{\eta^n}{\omega^m},
\]
where \( C_4 \) is a constant which may depend on \( \phi, \eta, \bar{\tau}^D \) and \( \bar{\tau}^1 \), but it is independent of \( \epsilon \) and \( R \). Thus
\[
\int_{\pi^{-1}(U_R \setminus U_D)} |\Lambda_\omega, (\Theta(\bar{\tau}^R_{H}) - \Theta(\bar{\tau}^1)) | \leq C_5 R^{-2p} \int_{\pi^{-1}(U_R \setminus U_D)} \frac{\omega^m}{\omega^m} + C_6 \int_{\pi^{-1}(U_R \setminus U_D)} \frac{\eta^n}{\omega^m},
\]
where \( C_4 \) and \( C_5 \) are constants independent of \( \epsilon \) and \( R \). Similarly, we have
\[
\int_{\pi^{-1}(U_R \setminus U_D)} |\Lambda_\omega, (\Theta(\bar{\tau}^1)) | \leq C_7 \int_{\pi^{-1}(U_R \setminus U_D)} \frac{\eta^n}{\omega^m},
\]
where \( C_7 \) is a constant also independent of \( \epsilon \) and \( R \).

By \( (5.38) \), we have \( \int_{\pi^{-1}(\Sigma)} (\frac{n^m}{\omega^m}) \leq C^* \). By the relation \( \pi^* \omega < \omega < 1 \), it is easy to see that
\[
\int_{\pi^{-1}(U_R)} (\frac{n^m}{\omega^m}) \to 0,
\]
\[
\int_{\pi^{-1}(U_R)} (\frac{n^m}{\omega^m}) \to 0,
\]
\[
\int_{\pi^{-1}(U_R)} (\frac{n^m}{\omega^m}) \to 0,
\]
as \( R \to 0 \), uniformly in \( \epsilon \).

By \( (5.39), (5.40), (5.41), (5.42) \), and choosing \( R_0 \) sufficiently small, we have
\[
\int_{\pi^{-1}(U_R)} \frac{\omega^m}{\omega^m} \leq C_5 R^{-2p} \int_{\pi^{-1}(U_R \setminus U_D)} \frac{1}{\omega^m},
\]
for all \( 0 < \epsilon \leq \epsilon_1 \) and \( 0 < R \leq R_0 \). Let \( \epsilon \to 0 \), we have
\[
\int_{\pi^{-1}(U_R \setminus U_D)} \frac{\omega^m}{\omega^m} \leq C_5 R^{-2p} \int_{U_R} \frac{1}{\omega^m},
\]
Since \( \Sigma \) has Hausdorff dimension at most 2n - 2k, it is easy to see that \( Vol(U_R, \omega) \leq CR^{2k} \) for some uniform constant \( C \). By the assumption of \( p \), we know that \( 2k - 2p > 0 \), choosing \( R \) small enough, then \( (5.33) \) follows.

\[\Box\]

**Theorem 5.12.** Let \((E, A_0, \phi_0)\) be a Higgs bundle on a smooth Kähler manifold \((M, \omega)\), and \((A_t, \phi_t)\) be the smooth solution of the Yang-Mills-Higgs flow \((1.4)\) on the Hermitian vector bundle \((E, H_0)\) with initial data \((A_0, \phi_0) \in B(E, H_0)\). Suppose that for any \( \delta' > 0 \) and any \( 1 \leq p < p_0 \) there is a smooth metric \( H \) on \( E \) such that \( (5.53) \) holds, where \( \bar{\mu}_0 \) is the Harder-Narasimhan type of \((E, A_0, \phi_0)\). Let \((A_\infty, \phi_\infty)\) be an Uhlenbeck limit of \((A_t, \phi_t)\), and \((E_\infty, H_\infty)\) be the corresponding Hermitian vector bundle defined away from \( \Sigma_{an} \). Then
\[
HYM_{a,N}(A_\infty, \phi_\infty) = \lim_{t \to \infty} HYM_{a,N}(A_t, \phi_t) = HYM_{a,N}(\bar{\mu}_0)
\]
for all $1 \leq \alpha < p_0$ and all $N \in \mathbb{R}$; the HN type of the Higgs sheaf $(E_\infty, A_\infty, \phi_\infty)$ is the same as $(E, A_0, \phi_0)$.

**Proof** Firstly, since the norm $\left( \int_M \varphi(\phi) d\text{vol} \right)^{\frac{1}{\alpha}}$ is equivalent to the $L^\alpha$ norm on $u(E)$, we have

$$
\left| (HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0)) - (HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0)) \right|^\frac{1}{\alpha} \\
\leq \left( \int_M |(\varphi(\sqrt{\Theta((\overline{\partial}_{A_0}, H), \phi_0) + NId_E)})^{\frac{1}{\alpha}} - (\varphi(\overline{\partial}_{A_0}, H), \phi_0))^{\frac{1}{\alpha}} | d\text{vol} \right) \\
\leq \left( \int_M \varphi(\sqrt{\Theta((\overline{\partial}_{A_0}, H), \phi_0) - NId_E)})^{\frac{1}{\alpha}} - (\varphi(\overline{\partial}_{A_0}, H), \phi_0))^{\frac{1}{\alpha}} \right) \\
\leq \left( \int_M \varphi(\sqrt{\Theta((\overline{\partial}_{A_0}, H), \phi_0) - NId_E)})^{\frac{1}{\alpha}} - (\varphi(\overline{\partial}_{A_0}, H), \phi_0))^{\frac{1}{\alpha}} \right) \\
\leq C(\alpha)\|\Theta((\overline{\partial}_{A_0}, H), \phi_0) - (\varphi(\overline{\partial}_{A_0}, H), \phi_0))\|_{L^\infty(M, \omega)}).
$$

By the above inequality and the condition (5.33), we see for any $\delta > 0$ and any $1 \leq \alpha < p_0$ there is $H$ such that

$$
(5.46) \quad HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0) \leq HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0) + \delta.
$$

For fixed $\alpha$, $1 \leq \alpha \leq \alpha_0$, and fixed $N$, since degree of line bundles are discrete, we can define $\delta_0 > 0$ such that

$$
(5.47) \quad 2\delta_0 + HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0) \leq HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0) + \delta,
$$

where $\bar{\mu}$ runs over all possible HN types of Higgs sheaves on $M$ with the rank of $E$.

Let $H$ be a Hermitian metric on the complex bundle $E$, and $(A_t^H, \phi_t^H)$ be the solution to the Yang-Mills-Higgs flow $(1.4)$ on the Hermitian vector bundle $(E, H)$ with initial pair $(A_0^H, \phi_0) \in B(E, H)$ where $A_0^H = (\overline{\partial}_{A_0}, H)$. Let $(A^H_{\infty}, \phi_{\infty})$ be an Uhlenbeck limit along the flow $(1.4)$.

Assume that the $H$ satisfies:

$$
(5.48) \quad HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0)) \leq HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0)) + \delta_0.
$$

By Prop. 5.3 and Prop. 5.5, we obtain:

$$
HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0)) \leq HYM_{\alpha,N}(A_t^H, \phi_\infty) \leq HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0)) + \delta_0.
$$

Hence, we must have $HYM_{\alpha,N}(A_t^H, \phi_\infty) = HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0))$. This shows that the result holds if the metric $H_0$ satisfies (5.48).

We are going to prove that for any metric $H$, for any fixed $\delta$ there is $T \geq 0$ such that:

$$
(5.49) \quad HYM_{\alpha,N}(A_t^H, \phi_t^H) < HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0)) + \delta,
$$

for all $t \geq T$. Without loss of generality, we can assume $0 < \delta \leq \frac{\delta_0}{2}$.

Let us denote by $\mathcal{H}_d$ the set of smooth Hermitian metrics on $E$ with the property that the above inequality (5.49) holds for some $T$. From (5.40) and the discussion above, we know $\mathcal{H}_d$ is nonempty. In [30] (Proposition 2.1'), we have proved that the continuous dependence of the Donaldson’s flow (2.1) on initial conditions. Following the argument in [12] (Lemma 4.3), see also Theorem 5.13 in [30], we can show that $\mathcal{H}_d$ is close and also open. The proof is exactly the same as that in [30] (Theorem 5.13), we omit it. Since the space of smooth metrics is connected, we conclude that every metric is in $\mathcal{H}_d$. Then, we have $\lim_{t \to \infty} HYM_{\alpha,N}(A_t^H, \phi_t^H) = HYM_{\alpha,N}(\overline{\partial}_{A_0}, H), \phi_0))$ for any metric $H$.

Let $\tilde{\lambda}_\infty$ be the HN type of $(E_\infty, A_\infty, \phi_\infty)$, by (5.45) and Proposition 5.5, we have $\varphi(\tilde{\lambda}_\infty + N) = \varphi(\lambda_\infty + N)$ for all $1 \leq \alpha < p_0$ and all $N$. We may choose
Let the following filtration of saturated sheaves
\[(5.50) \quad 0 = E_0 \subset E_1 \subset \cdots \subset E_i = (E, \overline{\partial} A_\alpha)\]
be a HN filtration of the Higgs bundle \((E, \overline{\partial} A_\alpha, \phi_0)\). The action of \(g_j\) produces a sequence of HN filtration
\[(5.51) \quad 0 = E_0^{(j)} \subset E_1^{(j)} \subset \cdots \subset E_i^{(j)} = (E, \overline{\partial} A_j) ,\]
where \(E_\alpha^{(j)} = g_j(E_\alpha), \alpha = 1, \ldots, l\). Let \(\pi_\alpha^{(j)}\) be the orthogonal projection onto \(E_\alpha^{(j)}\), then we have \(\pi_\alpha^{(j)} \in L_2^1\) and satisfies the conditions in (2.22). Using the above Theorem 5.12, by the same argument in [12] (Proposition 4.5), we have the following lemma.

**Lemma 5.13.** Let \((E, A_\alpha, \phi_0)\) be a Higgs bundle on a smooth Kähler manifold \((M, \omega)\), and satisfy the same assumptions as that in Theorem 5.12.

1. Let \(\{\pi_\alpha^{(j)}\}\) be the HN filtration of the reflexive Higgs sheaf \((E_\alpha, A_\alpha, \phi_\alpha)\), then there is a subsequence of HN filtration \(\{\pi_\alpha^{(j)}\}\) converges to a filtration \(\{\pi_\alpha^{(\infty)}\}\) strongly in \(L^p \cap L^1_{1,loc}\) off \(\Sigma_{an}\).

2. Assume the Higgs bundle \((E, A_0, \phi_0)\) is semi-stable and \(\{E_\alpha\}\) is the Seshadri filtration of \((E, A_0, \phi_0)\), then, after passing to a subsequence, \(\{\pi_\alpha^{(j)}\}\) converges to a filtration \(\{\pi_\alpha^{(\infty)}\}\) strongly in \(L^p \cap L^1_{1,loc}\) off \(\Sigma_{an}\), the rank and degree of \(\pi_\alpha^{(\infty)}\) is equal to the rank and degree of \(\pi_\alpha^{(j)}\) for all \(\alpha\) and \(j\).

**Proof** By the formula (5.4) in Lemma 5.4, we have
\[(5.52) \quad \deg(E_\alpha^{(j)}) + \frac{1}{2\pi} \int_M |\overline{\partial} A_{\alpha}, \pi_\alpha^{(j)}|^2 H_0 + ||\phi_j, \pi_\alpha^{(j)}||^2 \omega^\alpha \leq \sum_{i \leq \text{rank} E_\alpha} \lambda_i + ||\theta(A_j, \phi_j) - \theta(A_\alpha, \phi_\alpha)||_{L^1}.\]
By theorem 5.12, we have \(\tilde{\mu}_0 = \tilde{\lambda}_\infty\). Since \(E_\alpha^{(j)}\) is a Higgs sheaf of the Higgs bundle \((E_\alpha, \overline{\partial} A_\alpha, \phi_\alpha)\) and \(\mu(E_\alpha^{(j)}) = \mu(E_\alpha) = \mu(E)\), we have
\[(5.53) \quad \int_M |\overline{\partial} A_{\alpha}, \pi_\alpha^{(j)}|^2 H_0 + ||\phi_j, \pi_\alpha^{(j)}||^2 \omega^\alpha \leq ||\theta(A_j, \phi_j) - \theta(A_\alpha, \phi_\alpha)||_{L^1},\]
as \(j \to \infty\), where we have the property that \(\theta(A_j, \phi_j) \to \theta(A_\alpha, \phi_\alpha)\) strongly in \(L^p\) for all \(p\). After perhaps passing to a subsequence, we have \(\pi_\alpha^{(j)} \to \tilde{\pi}_\alpha^{(\infty)}\) weakly in \(L^1_2\), for some \(L^1_2\) projection \(\tilde{\pi}_\alpha^{(\infty)}\). Since \(\pi_\alpha^{(j)}\) is uniformly bounded, we see \(\overline{\partial} A_{\alpha}, \tilde{\pi}_\alpha^{(\infty)}\) strongly in \(L^p\) for all \(p\). Recall that \(\theta(A_j, \phi_j) = (A_\alpha, \phi_\alpha)\) in \(C_\infty^\infty\) topology on \(M \setminus \Sigma_{an}\) and write
\[(5.54) \quad \overline{\partial} A_{\alpha}, \pi_\alpha^{(j)} = \overline{\partial} A_{\alpha}, \pi_\alpha^{(j)} + (A_{0,1} - A_{0,1}^0) \circ \pi_\alpha^{(j)} - \pi_\alpha^{(j)} \circ (A_{0,1} - A_{0,1}^0),\]
then as in the proof of Lemma 4.5 in [12], we conclude from (5.53) that \(\overline{\partial} A_{\alpha}, \tilde{\pi}_\alpha^{(\infty)} = 0\), and \(\pi_\alpha^{(j)} \to \tilde{\pi}_\alpha^{(\infty)}\) strongly in \(L^p \cap L^1_{1,loc}\) off \(\Sigma_{an}\). On the other hand, it is easy to check that \([\phi_\alpha, \tilde{\pi}_\alpha^{(\infty)}] = 0\) and \((\tilde{\pi}_\alpha^{(\infty)})^2 = \tilde{\pi}_\alpha^{(\infty)} = (\tilde{\pi}_\alpha^{(\infty)})^*\). By Proposition 2.9, we know that
\( \hat{\pi}_\alpha^\infty \) determines a \( \phi_\infty \)-invariant coherent subsheaf \( \hat{E}_\alpha^\infty \) of \( (E_\infty, \Theta_{A_\infty}) \). Furthermore, it is clear \( \text{rank}(\hat{E}_\alpha^\infty) = \text{rank}(E_\alpha) = \text{rank}(\pi^\infty_\alpha) \). Using (5.53), we have

\[
\deg(\hat{E}_\alpha^\infty) = \int_M \text{tr}(\sqrt{-1} \theta(A_\infty, \phi_\infty) \hat{\pi}_\alpha^\infty) \frac{\omega^n}{n!} = \lim_{j \to \infty} \int_M \text{tr}(\sqrt{-1} \theta(A_j, \phi_j) \hat{\pi}_\alpha^\infty) \frac{\omega^n}{n!} = \deg(E^{(j)}_\alpha) + \lim_{j \to \infty} \int_M |\bar{\theta}_{A_j, \pi^{(j)}_\alpha}|^2 \omega_0 + ||\phi_j, \pi^{(j)}_\alpha||^2 \frac{\omega^n}{n!} = \deg(E_\alpha) = \deg(\pi^\infty_\alpha).
\]

(5.55)

So, the rank and degree of \( \hat{\pi}_\alpha^\infty \) is equal to the rank and degree of \( \pi^\infty_\alpha \) for all \( \alpha \). By the uniqueness of the maximal destabilizing subsheaf \( \pi^\infty_1 \) in the HN filtration of Higgs sheaf \( (E_\infty, \Theta_{A_\infty}, \phi_\infty) \), then we have \( \hat{\pi}_1^\infty = \pi_1^\infty \). Proceed by induction, it is easy to conclude that \( \hat{\pi}_\alpha^\infty = \pi_\alpha^\infty \) for all \( \alpha \). This completes the proof of part (1) of the lemma.

For part (2), notice that the argument given above applies to Seshadri filtration as well, where because of the lack of uniqueness of Seshadri filtration we may conclude only that the ranks and degrees of limiting filtration are same with that of the original filtration.

\[ \square \]

**Proposition 5.14.** Let \( (E, A_0, \phi_0) \) be a Higgs bundle on a smooth Kähler manifold \( (M, \omega) \), and satisfy the same assumptions as that in Theorem 5.12. Then given \( \delta > 0 \) and \( 1 \leq p < \infty \), \( (E, A_0, \phi_0) \) has a \( L^p \) \( \delta \)-approximate Hermitian structure.

**Proof.** Let \( (A_t, \phi_t) \) be the solution of the Yang-Mills-Higgs flow (1.4) with initial data \( (A_0, \phi_0) \in B(E, H_0) \), and \( H_t \) be the solution of the heat flow (2.1) on Higgs bundle \( (E, A_0, \phi_0) \) with initial data \( H_0 \). Let \( (A_\infty, \phi_\infty) \) be the Uhlenbeck limiting of some sequence \( (A_t, \phi_t) \). Applying the previous lemma, we have

\[ \Psi^H_\omega ((A_t, \phi_t), H_0) \to \Psi^H_\omega ((A_\infty, \phi_\infty), H_\infty) \text{ strongly in } L^p \text{ for all } 1 \leq p < \infty. \]

By corollary 2.5, we have

\[
\begin{align*}
&\left\| \frac{\sqrt{-1}}{2\pi} A_0 (F_{A_{H(t_j)}} + [\phi_0, \phi_0^H H(t_j)]) - \Psi^H_\omega ((E, A_0, \phi_0), H_0) \right\|_{L^p(\omega)}
= &\left\| [\theta(A_t, \phi_t) - \Psi^H_\omega ((A_t, \phi_t), H_0)] \right\|_{L^p(\omega)}
= &\left\| [\theta(A_t, \phi_t) - \theta(A_\infty, \phi_\infty) \right\|_{L^p(\omega)}
+ &\left\| \Psi^H_\omega ((A_\infty, \phi_\infty), H_\infty) - \Psi^H_\omega ((A_t, \phi_t), H_0) \right\|_{L^p(\omega)} \to 0.
\end{align*}
\]

(5.56)

**Theorem 5.15.** Let \( (E, A_0, \phi_0) \) be a Higgs bundle on a smooth Kähler manifold \( (M, \omega) \). Then given \( \delta > 0 \) and \( 1 \leq p < \infty \), \( (E, A_0, \phi_0) \) has a \( L^p \) approximate Hermitian structure.

**Proof.** By Proposition 3.7, we can resolve the singularity set \( \Sigma_{al} \) by blowing up finitely many times, i.e. we have a sequence of blow-ups:

\[
\pi_i : \overline{M}_i \to \overline{M}_{i-1}, \quad i = 1, \ldots, r
\]

where \( \overline{M}_0 = M \), such that every \( \pi_i \) is blow up along a smooth complex submanifold, every \( E_i = \pi^*(E_{i-1}) \) is bundle, and the pull back filtration \( (\pi_r \circ \cdots \circ \pi_1)^*(F_{E_0}) \) of \( E_0 \) is given by \( \phi_i \)-invariant sub-bundles, where \( \phi_i = (\pi_i \circ \cdots \circ \pi_1)^*(\phi_0) \). On each blow-up \( \overline{M}_i \), we have a family of Kähler metrics defined iteratively by \( \omega_{\epsilon_1, \ldots, \epsilon_r} = \pi^*_{\epsilon_1} \omega_{\epsilon_2, \ldots, \epsilon_r} + \epsilon_i \eta_i \), where \( \eta_i \) is a Kähler metric on \( \overline{M}_i \).
By proposition 5.14., for any fixed small $\epsilon_1, \ldots, \epsilon_{r-1}$, and any $\delta'$, we have a metric $H$ on bundle $E_{r-1}$ such that

$$\|\Theta_{\omega_{\epsilon_1, \ldots, \epsilon_{r-1}}}((\overline{\partial}_{E_{r-1}}, H), \Omega_{\epsilon_{r-1}}) - \Psi_{\omega_{\epsilon_1, \ldots, \epsilon_{r-1}}}((\overline{\partial}_{E_{r-1}}, \Omega_{\epsilon_{r-1}}), H)\|_{L^2(\omega_{\epsilon_1, \ldots, \epsilon_{r-1}})} \leq \delta'.$$

By induction, we can assume that, for any fixed small $\epsilon_1$ and any $\delta'$, we have a metric $H$ on bundle $E_1$ such that

$$\|\Theta_{\omega_1}((\overline{\partial}_{E_1}, H), \Omega_1) - \Psi_{\omega_1}((\overline{\partial}_{E_1}, \Omega_1), H)\|_{L^2(\omega_1)} \leq \delta'.$$

Since $\pi_1 : M_1 \to M$ is the blow-up along a smooth complex submanifold, by Proposition 5.10, then for any $\delta > 0$ and any $1 \leq p < 1 + \frac{1}{2k-1}$ there are $\epsilon_1 > 0$ and a smooth Hermitian metric $\overline{\Theta}_1$ on $E_1$ such that

$$\|\Theta_{\omega_1}((\overline{\partial}_{E_1}, H), \Omega_1) - \Psi_{\omega_1}((\overline{\partial}_{E_1}, \Omega_1), H)\|_{L^p(\omega_1)} \leq \delta'.$$

for all $0 < \epsilon \leq \epsilon_1$. By Proposition 5.11, Theorem 5.12 and Proposition 5.14, we see that for any given $\delta$ and $1 \leq p < \infty$, $(E, A_0, \phi_0)$ has a $L^p$ $\delta$-approximate Hermitian structure.

\[\square\]

Then repeating the argument in Theorem 5.12, we have:

**Theorem 5.16.** Let $(A_t, \phi_t)$ be a smooth solution of the gradient flow $[L_A]$ on the Hermitian vector bundle $(E, H_0)$ with initial condition $(A_0, \phi_0) \in B_{(E, H_0)}$, and $(A_\infty, \phi_\infty)$ be a Uhlenbeck limit. Let $E_\infty$ denote the vector bundle obtained from $(A_\infty, \phi_\infty)$ as that in Proposition 2.7. Then the Harder-Narasimhan type of the extended reflexive Higgs sheaf $(E_\infty, A_\infty, \phi_\infty)$ is same as that of the original Higgs bundle $(E_0, A_0, \phi_0)$.

6. **Proof of Theorem 1.1.**

Let $\{E_{\alpha, \beta}\}$ be the HNS-filtration of the Higgs bundle $(E, \overline{\partial}_{A_0}, \phi_0)$, the associated graded object $G_{\text{HNS}}(E, A_0, \phi_0) = \oplus_{\alpha=1}^r \oplus_{\beta=1}^r Q_{\alpha, \beta}$ be uniquely determined by the isomorphism class of $(A_0, \phi_0)$, where $Q_{\alpha, \beta} = E_{\alpha, \beta}/E_{\alpha, \beta-1}$. We refer to $\Sigma_{\alpha}$ as the singular set of the double filtration $\{E_{\alpha, \beta}\}$, it is a complex analytic subset of $M$ of complex codimensional at least 2. We will prove the result inductively on the length of the HNS filtration. The inductive hypotheses on a sheaf $Q$ are following:

**Inductive hypotheses:** There is a sequence of Higgs structures $(A_j^Q, \phi_j^Q)$ on $Q$ such that:

1. $(A_j^Q, \phi_j^Q) \to (A_\infty^Q, \phi_\infty^Q)$ in $C^0_{\text{loc}}$ off $\Sigma_{\text{al}} \cup \Sigma_{\text{an}}$;
2. $(A_j^Q, \phi_j^Q) = g_j(A_0^Q, \phi_0^Q)$ for some $g_j \in C^\infty(Q)$;
3. $(Q, \overline{\partial}_{A_j^Q}, \phi_j^Q)$ and $(Q, \overline{\partial}_{A_\infty^Q}, \phi_\infty^Q)$ extended to $M$ as reflexive Higgs sheaves with the same HN type;
4. $\|\phi_j^Q\|_{C^0}$ and $\|\sqrt{-1}A_{\omega}(F_{A_j^Q})\|_{L^1(\omega)}$ is uniformly bounded in $j$.

Let $S = E_{1,1}$ be the first stable Higgs sub-sheaf corresponding to the HNS-filtration of the Higgs bundle $(E, \overline{\partial}_{A_0}, \phi_0)$, $\pi : \tilde{M} \to M$ be the resolution of singularities $\Sigma_{\text{al}}$ then the filtration of $\tilde{E} = \pi^*E$ is given by subbundles $\{E_{\alpha, \beta}\}$, isomorphic to $\{E_{\alpha, \beta}\}$ off the exception divisor $\tilde{\Sigma} = \pi^{-1}(\Sigma_{\text{al}})$. Setting $(\tilde{A}_j, \tilde{\phi}_j) =$
and considering the induced connections on subsheaf \( \pi \) on \( M \) to a nonzero smooth \( \tilde{\omega} \in \Omega \) is the extended reflexive Higgs sheaf). By Theorem 2.4, we know that \( \pi \) converges to \( (\tilde{\omega}, \phi_0) \) in \( C^\infty_{\text{loc}} \) topology outside \( \pi^{-1}(\Sigma_{al} \cup \Sigma_{an}) \). By Corollary 2.5 and the uniform \( C^0 \) bound on \( \phi(t) \) (Lemma 2.3), we have 
\[
|\sqrt{-1}\Lambda_{\omega_0}(F_{\tilde{A}_j})|_{L^\infty}, \text{ specially } ||\sqrt{-1}\Lambda_{\omega_0}(F_{\tilde{A}_j})||_{L^1, j} \text{ is uniformly bounded in } j, \text{ where } \omega_0 = \pi^* \omega.
\]

Using Proposition 4.1, we have a subsequence of \( \tilde{g}_j \circ i_0 \), up to rescale, converges to a nonzero smooth \( \phi \)-invariant holomorphic map \( f_\infty : S \to E_\infty \) off \( \pi^{-1}(\Sigma_{al} \cup \Sigma_{an}) \). Since \( \tilde{S} \) is isomorphic to \( S \) off the exception divisor, then we obtain a subsequence of \( f_j = g_j \circ i_0 \) up to rescale, which converges to a nonzero smooth \( \phi \)-invariant holomorphic map \( f_\infty : S \to (E_\infty, \tilde{\partial}_{A_\infty}) \) in \( C^\infty_{\text{loc}} \) on \( M \setminus \Sigma_{an} \cup \Sigma_{al} \), where \( i_0 : S \to (E, \tilde{\partial}_{A_0}) \) is the holomorphic inclusion. By Hartog’s theorem, \( f_\infty \) extends to a Higgs sheaf homomorphism \( f_\infty : (S, \phi_0) \to (E_\infty, \tilde{\partial}_{A_\infty}, \phi_\infty) \) on \( M \) (where \( (E_\infty, \tilde{\partial}_{A_\infty}, \phi_\infty) \) is the extended reflexive Higgs sheaf).

As above, \( \pi_1^{(j)} \) denotes the projection to \( g_1(S) \). Since \( \pi_1^{(j)} \circ f_j = f_j \), we see that in the limit \( \pi_1^{\infty} \circ f_\infty = f_\infty \). By Lemma 5.13, we know that \( \pi_1^{\infty} \) determines a Higgs subsheaf \( E_1^{\infty} \) of \( (E_\infty, \tilde{\partial}_{A_\infty}, \phi_\infty) \), with \( \text{rank}(E_1^{\infty}) = \text{rank}(S) \) and \( \mu(E_1^{\infty}) = \mu(S) \). Since \( (E_\infty, \tilde{\partial}_{A_\infty}, \phi_\infty) \) and \( (E_0, \tilde{\partial}_{A_0}, \phi_0) \) have the same HN type, thus we have the Higgs subsheaf \( (E_1^{\infty}, \phi_\infty) \) is semistable and

\[
(6.1) \quad f_\infty : S \to E_{1,1}^{\infty}.
\]

Recall that \( S = E_{1,1} \) is Higgs stable. By Lemma 3.4., we see that the non-zero holomorphic map \( f_\infty \) must be injective, then

\[
(6.2) \quad S \simeq E_{1,1}^{\infty} = f_\infty(S)
\]
on \( M \setminus (\Sigma_{al} \cup \Sigma_{an}) \). It is easy to see that \( E_{1,1}^{\infty} \) is a stable Higgs subsheaf of \( (E_\infty, \tilde{\partial}_{A_\infty}, \phi_\infty) \).

Let \( \{e_\alpha\} \) be a local frame of \( S \), and \( H_{j,\alpha,\beta} = \langle f_j(e_\alpha), f_j(e_\beta) \rangle > H_0 \). We can write the orthogonal projection \( \pi_1^{(j)} \) as

\[
(6.3) \quad \pi_1^{(j)}(X) = \langle X, f_j(e_\beta) \rangle > H_{j,\alpha,\beta} f_j(e_\alpha)
\]
for any \( X \in E \), where \( H_{j,\alpha,\beta} \) is the inverse of the matrix \( (H_{j,\alpha,\beta}) \). Because \( f_j \to f_\infty \) in \( C^\infty(\Omega) \), and \( f_\infty \) is injective, then we can prove that \( \pi_1^{(j)} \to \pi_1^{\infty} \) in \( C^\infty_{\text{loc}} \) off \( \Sigma_{an} \cup \Sigma_{al} \).

Let \( Q = E/S \), then we have \( \text{Gr}_{HNS}(E, \tilde{\partial}_{A_0}, \phi_0) = S \oplus \text{Gr}_{HNS}(Q, \tilde{\partial}_{A_0^Q}, \phi_0^Q) \). Write the orthogonal holomorphic decomposition \( (E_\infty, \tilde{\partial}_{A_\infty}, \phi_\infty) = E_1^{\infty} \oplus Q_\infty \), where \( Q_\infty = (E_1^{\infty})^\perp \) because \( H_{1,1} \) is admissible Hermitian-Einstein metric. Using Lemma 5.12 in [11], we can choose a sequence of unitary gauge transformation \( u_j \) such that \( \pi_1^{(j)} = u_j \pi_1^{\infty} u_j^{-1} \) where \( \pi_j(E) = \pi_1^{\infty}(E) = E_1^{\infty} \) and \( u_j \to \text{Id}_E \) in \( C^\infty(\text{loc}) \) on \( M \setminus (\Sigma_{al} \cup \Sigma_{an}) \). It is easy to check that \( u_j(Q_\infty) = (\pi_1^{(j)}(E))^\perp \). Noting the bundle isomorphisms \( p^* : Q \to S^\perp \) and the unitary gauge transformation \( u_0 : Q_\infty \to S^\perp \), and considering the induced connections on \( Q \), we have

\[
(6.4) \quad D_{A_j^0} = u_0 \circ u_j^{-1} \circ \pi_j^\perp \circ D_{A_j} \circ \pi_j \circ u_j \circ u_0^{-1},
\]

\[
(6.5) \quad \phi_j^Q = u_0 \circ u_j^{-1} \circ \pi_j^\perp \circ \phi_j \circ \pi_j \circ u_j \circ u_0^{-1} \in \Omega^{1,0}(\text{End}(Q)),
\]

\[
(6.6) \quad h_j = u_0 \circ u_j^{-1} \circ \pi_j^\perp \circ g_j \in G^C(Q).
\]
Then, we have
\begin{align}
\overline{\mathcal{A}}_j^Q &= u_0 \circ u_j^{-1} \circ \pi_j^1 \circ \overline{\mathcal{A}}_j \circ \pi_j^0 \circ u_j \circ u_0^{-1} \\
&= h_j \circ \overline{\mathcal{A}}_0 \circ h_j^{-1},
\end{align}
(6.7)
where we have used (6.10)
\begin{align}
\partial_A^Q = (h_j^*)^{-1} \circ \partial_A^Q \circ h_j^*.
\end{align}
(6.8)
\begin{align}
\phi_j^Q &= u_0 \circ u_j^{-1} \circ \pi_j^1 \circ g_j \circ \phi_0 \circ g_j^{-1} \circ \pi_j^1 \circ u_j \circ u_0^{-1} \\
&= h_j \circ \phi_0 \circ h_j^{-1},
\end{align}
(6.9)
and
\begin{align}
\overline{\mathcal{A}}_j^Q \phi_j^Q = \pi_0^1 \circ (\overline{\mathcal{A}}_0 \circ \phi_0 + \overline{\mathcal{A}}_0) \pi_0^1 = 0,
\end{align}
(6.10)
where we have used $h_j^{-1} = \pi_0^1 \circ q_j^{-1} \circ u_j \circ u_0^{-1}$. On the other hand, by the definition, it is easy to check that $u_0^* (A_j^Q, \phi_j^Q) \to (A_j^Q, \phi_j^Q)$ in $C_{loc}^\infty$. Now we check the third statement in the inductive hypotheses. Let’s consider the Gauss-Codazzi equation on $(\pi_1^j (E))^\perp = Q_j$.
(6.11)
\begin{align}
F_{A_Q} = (\pi_1^j)^\perp \circ F_{A_j} + \partial_A \pi_1^j \wedge \overline{\mathcal{A}}_j \pi_1^j,
\end{align}
where $D_{A_Q} = (\pi_1^j)^\perp \circ D_{A_j}$. Setting the Higgs field $\phi_Q = (\pi_1^j)^\perp \circ \phi_j$, by (5.53) and (5.56), we have
\begin{align}
&\int_M |\sqrt{-1} \Lambda_\omega (F_{A_Q}^Q + [\phi_Q^Q, (\phi_Q^Q)^*]) - \Psi^h ((A_j^Q, \phi_j^Q), H_0) ||^2 d\omega \\
= &\int_M [\pi_1^j]^\perp \{ |\sqrt{-1} \Lambda_\omega (F_{A_j} + [\phi_j, (\phi_j)^*]) - \Psi^h ((A_j, \phi_j), H_0) ||^2 \\
&+ |\phi_j||[(\phi_j, \pi_1^j)]||^2 \}
\end{align}
(6.12)
\begin{align}
&\to 0.
\end{align}
Since $C^0$ norm of $\phi_j$ is uniformly bounded, then $\|\phi_Q^Q\|_{C^0}$ and $\|\sqrt{-1} \Lambda_\omega (F_{A_Q}^Q)\|_{L^1(\omega)}$ is uniformly bounded in $j$. So, $(Q, A_j^Q, \phi_j^Q)$ satisfy the inductive hypotheses. Since we can resolve the singularity set $\Sigma_{al}$ by blowing up finitely many times with non-singular center, and the pulling back of the HNS filtration is given by sub-bundles. The sheaf $Q$ and every geometric objects which we considered are induced by the HNS filtration, so their pulling back are all smooth. Using Proposition 4.1 again, by induction we have
\begin{align}
E_\infty \simeq Gr_HNS (E, \overline{\mathcal{A}}_0, \phi_0) = \oplus_{i=1}^l \oplus_{j=1}^r Q_{i,j}
\end{align}
(6.13)
on $M \setminus (\Sigma_{al} \cup \Sigma_{an})$. By Proposition 2.7, we know that $(E_\infty, \overline{\mathcal{A}}_\infty, \phi_\infty)$ can be extended to the whole $M$ as a reflexive Higgs sheaf. By the uniqueness of reflexive extension in [37], we know that there exists a sheaf isomorphism
\begin{align}
f : (E_\infty, \overline{\mathcal{A}}_\infty, \phi_\infty) \to Gr_HNS (E, \overline{\mathcal{A}}_0, \phi_0)^{**}
\end{align}
(6.14)
on $M$. This completes the proof of Theorem 1.1.

\[\square\]

References

[1] M. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Phil. Trans. Roy. Soc. London A 308 (1982), 524-615.
[2] L. Alvarez-Consul and O. Garcia-Prada, Dimensional reduction, $SL(2, C)$-equivariant bundles and stable holomorphic chains, Int. J. Math., 2 (2001), 159-201.
[3] O. Biquard, On parabolic bundles over a complex surface, J. London. Math. Soc., 53 (1996), no.2, 302-316.
[4] S. B. Bradlow, Vortices in holomorphic line bundles over closed Kähler manifolds, Comm. Math. Phys. 135 (1990), 1-17.
[5] S. B. Bradlow and O. Garcia-Prada, Stable triples, equivariant bundles and dimensional reduction, Math. Ann. 304 (1996), 225-252.
[6] U. Bruzzo and B. G. Otero, Metrics on semistable and numerically effective Higgs bundles, J. reine ang. Math., 612 (2007), 59-79.
[7] S. Bando and Y. T. Siu, Stable sheaves and Einstein-Hermitian metrics, in Geometry and Analysis on Complex Manifolds, World Scientific, 1994, 39-50.
[8] P. D. Bartolomeis and G. Tian, Stability of complex vector bundles, J. Differential Geometry, 43 (1996), 232-275.
[9] N. M. Buchdahl, Hermitian-Einstein connections and stable vector bundles over compact complex surfaces, Math. Ann. 280 (1988), pp. 625-648.
[10] S. A. H. Cardona, Approximate Hermitian-Yang-Mills structures and semistability for Higgs bundles I: Generalities and the one-dimensional case., arXiv: 1108.2614v3.
[11] G. Daskalopoulos, The topology of the space of stable bundles on a Riemann surface, J. Reine Angew. Math. 575 (2004), 69-99.
[12] G. Daskalopoulos and R. Wentworth, Convergence properties of the Yang-Mills flow on Kähler surfaces, Math. Ann. 360 (2015), 59-126.
[13] S. K. Donaldson, A new proof of a theorem of Narasimhan and Seshadri, J. Differential Geom., 18 (1983), 279-315.
[14] S. K. Donaldson, Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, Proc. London Math. Soc. 50 (1985), 1-26.
[15] S. K. Donaldson, Infinite determinates, stable bundles and curvature, Duke J. Math., 54 (1987) 231-247.
[16] P. Griffiths and J. Harris, Principles of algebraic geometry, Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.
[17] O. Garcia-Prada, Dimensional reduction of stable bundles, vortices and stable pairs, Int. J. Math. 5 (1994), 1-52.
[18] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math. 2 (1964), 109-203.
[19] N. J. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. 55 (1982), 59-126.
[20] M. C. Hong and G. Tian, Asymptotical behaviour of the Yang-Mills flow and singular Yang Mills connections, Math. Ann. 330 (2004), no. 3, 441-472.
[21] A. Jacob, Existence of approximate Hermitian-Einstein structures on semistable bundles, arXiv:1012.1888v2.
[22] A. Jacob, The limit of the Yang-Mills flow on semi-stable bundles, arXiv:1104.4767
[23] A. Jacob, The Yang-Mills flow and the Atiyah-Bott formula on compact Kähler manifolds, arXiv:1109.1550
[24] S. Kobayashi, Curvature and stability of vector bundles, Proc. Japan Acad. Ser. A Math. Sci. 58 (1982) 158-162.
[25] S.Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, 15. Kano Memorial Lectures, 5. Princeton University Press, Princeton, NJ (1987).

[26] J.Y.Li, *Hermitian-Einstein metrics and Chern number inequalities on parabolic stable bundles over Kähler manifolds*, Comm. Anal. Geom. 8(2000), no. 3, 445–475.

[27] J.Y.Li and M.S.Narasimhan, *Hermitian-Einstein metrics on parabolic stable bundles*, Acta Math. Sin. (Engl. Ser.) 15 (1999), no. 1, 93–114.

[28] J.Li and S.T.Yau, *Hermitian-Yang-Mills connection on non-Kähler manifolds*, Mathematical aspects of string theory (San Diego, Calif., 1986), 560–573, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987.

[29] M.Lübke, *Stability of Einstein-Hermitian vector bundles*, Manuscripta Math., 42(1983), 245-257.

[30] J.Y.Li and X.Zhang, *The gradient flow of Higgs pairs*, J. Eur. Math. Soc., 13(2011), 1373-1422.

[31] J.Y.Li and X.Zhang, *Existence of approximate Hermitian-Einstein structures on semi-stable Higgs bundles*, arXiv:1206.6676.

[32] I. Mundet i Riera, *A Hitchin-Kobayashi correspondence for Kähler fibrations*, J.reine angew. Math. 528(2000), 41-80.

[33] M.S.Narasimhan and C.S.Seshadri, *Stable and unitary vector bundles on compact Riemann surfaces*, Ann. of Math., 82 (1965) 540-567.

[34] B.Shiffman, *On the removal of singularities of analytic sets*, Michigan Math.J. 15(1968), 111-120.

[35] B.Sibley, *Asymptotics of the Yang-Mills flow for holomorphic vector bundles over Kähler manifolds: the canonical structure of the limit*, arXiv:1206.5401v1.

[36] C.T.Simpson, *Constructing variations of Hodge structures using Yang-Mills connections and applications to uniformization*, J.Amer.Math.Soc., 1, (1988)867-918.

[37] Y.T.Siu, *A Hartogs type extension theorem for coherent analytic sheaves*, Ann. of Math. (2) 93(1971), no.1, 166-188.

[38] T.Mochizuki, *Kobayashi-Hitchin correspondence for tame harmonic bundles and an application*, Astérisque 309 (2006), ISBN: 978-2-85629-226-6, +117pp.

[39] K.K.Uhlenbeck and S.T.Yau, *On existence of Hermitian-Yang-Mills connection in stable vector bundles*, Comm.Pure Appl.Math., 39S(1986), 257-293.

[40] G.Wilkin, *Morse theory for the space of Higgs bundles*, Comm. Anal. Geom., 16(2)(2008), 283-332.

**School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026,** and **AMSS, CAS, Beijing, 100080, P.R. China,**

*E-mail address: jiayuli@ustc.edu.cn*

**School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026,** P.R. China,

*E-mail address: mathzx@ustc.edu.cn*