COSET CONSTRUCTION OF MINIMAL MODELS

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Usual coset construction $\text{SU}(2)_k \times \text{SU}(2)_l / \text{SU}(2)_{k+l}$ of Wess-Zumino conformal field theory is presented as a coset construction of minimal models. This new coset construction can be defined rigorously and allows one to calculate easily correlation functions of a number of primary fields.

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1. Introduction

Coset construction\(^1,2\) is a method of obtaining a new two-dimensional conformal field theory from two old ones, G and H. Let \(g\) and \(h\) be chiral algebras of G and H models respectively, \(h\) be a subalgebra of \(g\). States of the coset theory are identified with orbits of the action of \(h\) in representations of \(g\). Conformal blocks\(^3\) of coset construction can be obtained by some factorizing conformal blocks of \(G\) theory\(^4−7\).

One of the most important examples of coset construction is\(^1\)

\[
N_{kl} \sim \frac{SU(2)_k \times SU(2)_l}{SU(2)_{k+l}}, \quad (1.1)
\]

where \(SU(2)_k\) means Wess–Zumino model\(^8,9\) with \(SU(2)\) group on level \(k = 1, 2, 3, \cdots\). Chiral algebra of Wess–Zumino \(SU(2)_k\) model coincides with \(\hat{sl}(2)_k \simeq \hat{su}(2)^C_k\) affine Kac–Moody algebra

\[
[J^\alpha_{k,m} J^\beta_{k,n}] = f^\alpha_\gamma J^\gamma_{k,m+n} + \frac{k}{2} g^{\alpha\beta} \delta_{m+n,0}. \quad (1.2)
\]

Here \(J^\alpha_{k,n}\) are Laurent components of chiral currents, \(J^\alpha_k(z)\),

\[
J^\alpha_{k,m} = \oint \frac{dz}{2\pi i} z^m J^\alpha_k(z), \quad \alpha = +, -, 0, \quad (1.3)
\]

with operator product expansion (OPE)

\[
J^\alpha_k(z') J^\beta_k(z) = \frac{1}{2} k g^{\alpha\beta} \left( \frac{1}{(z'-z)^2} + \frac{f^\alpha_\gamma J^\gamma_k(z)}{z'-z} \right) + O(1); \quad (1.4)
\]

\(f^\alpha_\beta\) are the structure constants and \(g^{\alpha\beta}\) is the basic metric of \(sl(2)\) algebra with nonzero components

\[
f^{++}_0 = 2, \quad f^{0+}_+ = f^{-0}_- = 1, \quad g^{++} = 2, \quad g^{00} = 1,
\]

\[
f^{\beta\alpha}_\gamma = -f^{\alpha\beta}_\gamma, \quad g^{\beta\alpha} = g^{\alpha\beta}. \quad (1.5)
\]

Chiral currents of the denominator of the coset construction (1.1) are identified with diagonal currents of the numerator

\[
J^\alpha_{k+l}(z) = J^\alpha_k(z) + J^\alpha_l(z). \quad (1.6)
\]

The central charge, \(c\), of the Virasoro algebra

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0} \quad (1.7)
\]
of $N_{kl}$ theory is given by

$$c_{kl} = 3 - \frac{6}{k + 2} - \frac{6}{l + 2} + \frac{6}{k + l + 2}. \quad (1.8)$$

Here $L_m$ are Laurent components of the energy-momentum tensor, $T(z)$,

$$L_m = \oint \frac{dz}{2\pi i} z^{m+1} T(z), \quad (1.9)$$

with OPE

$$T(z')T(z) = \frac{\frac{1}{2}c}{(z' - z)^4} + \frac{2T(z)}{(z' - z)^2} + \frac{\partial T(z)}{z' - z} + O(1). \quad (1.10)$$

A special case of the coset construction (1.1) is a unitary minimal conformal model

$$M_k \sim N_{k1} \sim \frac{\text{SU}(2)_k \times \text{SU}(2)_1}{\text{SU}(2)_{k+1}} \quad (1.11)$$

with central charge

$$c_k = 1 - \frac{6}{(k + 2)(k + 3)}. \quad (1.12)$$

These models have been well investigated independently of their coset structure by the use of bosonic representation.\textsuperscript{10-13} If $l = 2$, the coset (1.1) yields $N = 1$ superconformal models.\textsuperscript{14-16,2} For general $l$ the coset construction $N_{kl}$ can be identified with some bosonic models.\textsuperscript{17-21} Chiral algebras of these models have not yet been found. These models will be referred to as minimal-like ones. In principle, they are well defined by their coset structure, but neither coset construction nor bosonization gives an effective tool of calculating correlation functions. Can we use the information about minimal models to calculate correlation functions of $N_{kl}$ minimal-like models? Let us make a naive transformation of the coset construction (1.1) for $l = 2$:\textsuperscript{22}

$$N_{k2} \sim \frac{\text{SU}(2)_k \times \text{SU}(2)_2}{\text{SU}(2)_{k+2}} \times \frac{\text{SU}(2)_1 \times \text{SU}(2)_1 \times \text{SU}(2)_{k+1}}{\text{SU}(2)_1 \times \text{SU}(2)_1 \times \text{SU}(2)_{k+1}} \sim \frac{\text{SU}(2)_k \times \text{SU}(2)_1}{\text{SU}(2)_{k+1}} \times \frac{\text{SU}(2)_{k+1} \times \text{SU}(2)_1}{\text{SU}(2)_{k+2}} \times \frac{\text{SU}(2)_1 \times \text{SU}(2)_1}{\text{SU}(2)_2} \quad (1.13)$$

In general case

$$N_{kl} \sim \frac{M_k M_{k+1} \cdots M_{k+l-1}}{M_1 M_2 \cdots M_{l-1}} \quad (1.14)$$

(the reason of absence of crosses will become clear later). Thus far it has been nothing more than a trick with symbols. There are two natural questions. Can we
assign any exact meaning to the “fraction” (1.14)? If it is possible, does the usual coset construction yield the same result as (1.14)? In what follows we answer in the affirmative to both questions. The main mathematical trick is argued in Sec. 2, and it is applied to the coset construction (1.14) in Sec. 3. Field identification of the coset construction (1.14) and minimal-like models is done in Sec. 4. Sec. 5 answers in the affirmative to the question of using information about minimal models for calculating correlation functions of minimal-like models.

2. Convolution and Factorization of Conformal Blocks

2.1. Pairing Holomorphic and Antiholomorphic Parts

To make what follows more clear and to fix notations we shall recall now some well known facts concerning connection between conformal blocks and correlation functions. Consider conformal block \( F_{\{\lambda\}}(\lambda_5; z) = F^{\lambda_3\lambda_4,\lambda_2\mu_4}_{\lambda_1\mu_3,\lambda_2\mu_3}(\lambda_5; z), \{\lambda\} = (\lambda_1\mu_1, \lambda_2\mu_2, \lambda_3\mu_3, \lambda_4\mu_4) \) (see Fig. 1) of four fields \( \phi_{\lambda_1\mu_1}(0), \phi_{\lambda_2\mu_2}(z), \phi_{\lambda_3\mu_3}(1), \) and \( \phi_{\lambda_4\mu_4}(\infty) \) through states in s-channel from module \( \mathcal{H}_{\lambda_5} \) with highest weight \( \lambda_5 \). Index \( \mu_i \) labels states in module \( \mathcal{H}_{\lambda_i} \); \( \mu_i \) will be usually omitted later. Conformal blocks are solutions of differential equations imposed by null-vectors of the chiral algebra. \(^3,7,23\) Conformal blocks with different \( \lambda_5 \) permitted by fusion rules form a basis of solution space of the differential equations. Correlation function can be found as\(^3,10,11a\)

\[
\langle \phi_{\lambda_1}(0,0)\phi_{\lambda_2}(z,\bar{z})\phi_{\lambda_3}(1,1)\phi_{\lambda_4}(\infty,\infty) \rangle = \sum_{\lambda_5} X^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(\lambda_5) F^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(\lambda_5; z) \overline{F^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(\lambda_5; z)},
\]

(2.1.1)

where \( X^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(\lambda_5) \) are coefficients, and bar means complex conjugation. There are several equivalent approaches to determing \( X \)-coefficients.

1. Duality.\(^3\) We can consider dual (\( u \)-channel) basis in solution space of differential equations, \( \left\{ F_{\{\lambda\}}(\lambda_5; 1-z) = F^{\lambda_1\lambda_4}_{\lambda_3\lambda_2}(\lambda_5; 1-z) \right\}, \{\bar{\lambda}\} = (\lambda_3, \lambda_2, \lambda_1, \lambda_4) \). Solutions of the old basis can be expressed in terms of solutions of the new one (Fig. 2)

\[
F^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(\lambda_5; z) = \sum_{\lambda_6} \alpha^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(\lambda_5, \lambda_6) F^{\lambda_1\lambda_4}_{\lambda_3\lambda_2}(\lambda_6; 1-z).
\]

(2.1.2)

The coefficients \( \alpha^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(\lambda_5, \lambda_6) \) form so called braiding matrix \( \alpha^{\lambda_1\lambda_4}_{\lambda_3\lambda_2} \). Substituting Eq. (2.1.2) into Eq. (2.1.1) and demanding that the correlation function should

\(^a\) We only consider primary fields of spin 0 for simplicity. Furthermore, we consider each module to coincide with the conjugate module and multiplicities to be equal to 1. It is right for minimal, minimal-like and \( SU(2) \) Wess–Zumino models. Generalizations are straightforward.
be expressed in the same form by dual blocks (Fig. 3), we obtain Dotsenko–Fateev equation\textsuperscript{10,11,24}
\[
\sum_{\lambda_5} X_{\lambda_1 \lambda_2}^{\lambda_3 \lambda_4} (\lambda_5) \alpha_{\lambda_1 \lambda_2}^{\lambda_3 \lambda_4} (\lambda_5, \lambda_6) \alpha_{\lambda_1 \lambda_2}^{\lambda_3 \lambda_4} (\lambda_5, \lambda_6') = \delta_{\lambda_6 \lambda_6'} X_{\lambda_3 \lambda_2}^{\lambda_1 \lambda_4} (\lambda_6). \tag{2.1.3}
\]

We shall consider states from modules \(\mathcal{H}_\lambda \otimes \overline{\mathcal{H}_\lambda}\) with the same \(\lambda\) in holomorphic (\(\mathcal{H}\)) and antiholomorphic (\(\overline{\mathcal{H}}\)) parts to be physical. We can say that all intermediate states in \(s\)-channel in Eq. (2.1.1) are physical. Eq. (2.1.3) ensures that only physical states should appear in \(u\)-channel. In other words Eq. (2.1.3) ensures that there is no nonphysical fields in OPE’s of two physical fields.

The normalization of four-point correlation functions is given by the requirement
\[
\langle \phi_{\lambda_1} (0, 0) \phi_{\lambda_2} (z, \overline{z}) \phi_{\lambda_3} (1, 1) \phi_{\lambda_4} (\infty, \infty) \rangle \approx \langle \phi_{\lambda_1} (0, 0) \phi_{\lambda_2} (z, \overline{z}) \phi_{\lambda_m} (\infty, \infty) \rangle \langle \phi_{\lambda_m} (0, 0) \phi_{\lambda_3} (1, 1) \phi_{\lambda_4} (\infty, \infty) \rangle \quad \text{if} \quad |z| \ll 1,
\tag{2.1.4}
\]
where \(\phi_{\lambda_m}\) corresponds to the permitted intermediate state with the lowest conformal dimension.

If conformal blocks are normalized as
\[
F_{\{\lambda\}} (\lambda_5; z) \approx z^\delta, \quad z \ll 1, \tag{2.1.5}
\]
the \(X\)-coefficients can be decomposed in product\textsuperscript{3,24}
\[
X_{\lambda_1 \lambda_2}^{\lambda_3 \lambda_4} (\lambda_5) = C_{\lambda_1, \lambda_2, \lambda_3} C_{\lambda_3, \lambda_4}, \tag{2.1.6}
\]
where \(C_{\lambda_1, \lambda_2, \lambda_3}\) are structure constants of the operator algebra. Structure constants determine three-point correlation functions
\[
\langle \phi_{\lambda_1} (z_1, \overline{z}_1) \phi_{\lambda_2} (z_2, \overline{z}_2) \phi_{\lambda_3} (z_3, \overline{z}_3) \rangle = C_{\lambda_1, \lambda_2, \lambda_3} (z_1 - z_2) \Delta_3 - \Delta_1 - \Delta_2 (z_1 - z_3) \Delta_2 - \Delta_1 - \Delta_3 (z_2 - z_3) \Delta_1 - \Delta_2 - \Delta_3 \\
\times (\overline{z}_1 - \overline{z}_2) \overline{\Delta}_3 - \overline{\Delta}_1 - \overline{\Delta}_2 (\overline{z}_1 - \overline{z}_3) \overline{\Delta}_2 - \overline{\Delta}_1 - \overline{\Delta}_3 (\overline{z}_2 - \overline{z}_3) \overline{\Delta}_1 - \overline{\Delta}_2 - \overline{\Delta}_3, \tag{2.1.7}
\]
where \(\Delta_i\) and \(\overline{\Delta}_i\) are holomorphic and antiholomorphic conformal dimensions of the state \(|\lambda_\mu_i \overline{\mu}_i\rangle\).

2. Monodromy invariance.\textsuperscript{10,11} A conformal block can be decomposed in powers of \(z\) for \(|z| < 1\)
\[
F_{\{\lambda\}} (\lambda_5; z) = z^{\Delta_5 - \Delta_1 - \Delta_2} \sum_{n=0}^{\infty} \beta_n z^n. \tag{2.1.8}
\]
Therefore
\[
F_{\{\lambda\}} (\lambda_5; e^{2\pi i} z) = e^{2\pi i (\Delta_5 - \Delta_1 - \Delta_2)} F_{\{\lambda\}} (\lambda_5; z). \tag{2.1.9}
\]
Thus the monodromy invariance of the correlation function (2.1.1) under moving $z$ round zero is evident. To investigate monodromy properties under moving $z$ round one we can substitute Eq. (2.1.2) into Eq. (2.1.1). The requirement of monodromy invariance yields again Dotsenko–Fateev equation (2.1.3).

3. Vertex operators, quantum group etc. Instead of conformal blocks vertex operators,

$$(\phi_{\lambda\mu}(z))^\lambda_{\lambda'} : \mathcal{H}_\lambda \longrightarrow \mathcal{H}_{\lambda'},$$

(2.1.10)
can be considered. The primary field, $\phi_\lambda(z, \bar{z})$, is expressed in terms of vertex operators as

$$\phi_\lambda(z, \bar{z}) = \bigoplus_{\lambda' \lambda''} X_\lambda(\lambda'', \lambda') (\phi_\lambda(z))^\lambda_{\lambda'} (\phi_\lambda(z))^\lambda_{\lambda''}. \quad (2.1.11)$$

The direct sum means that each term acts in its own pair of modules. Four-point conformal blocks are given by

$$F_{\{\lambda\}}(\lambda_5; z) = \left\{ (\phi_{\lambda_4}(\infty))^0_{\lambda_4} (\phi_{\lambda_2}(1))^\lambda_{\lambda_5} (\phi_{\lambda_2}(z))^\lambda_{\lambda_2} (\phi_{\lambda_1}(0))^0_{\lambda_1} \right\}. \quad (2.1.12)$$

index 0 means the vacuum module. Transposing vertex operators in a conformal block we change basis in the solution space of the differential equations. Therefore

$$(\phi_{\lambda_3}(z'))^\lambda_{\lambda_5} (\phi_{\lambda_1}(z))^\lambda_{\lambda_2} = \sum_{\lambda_6} R^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(\lambda_5, \lambda_6) (\phi_{\lambda_1}(z))^\lambda_{\lambda_6} (\phi_{\lambda_3}(z'))^\lambda_{\lambda_2}. \quad (2.1.13)$$

It is easy to prove that

$$\alpha^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(\lambda_5, \lambda_6) = R^{\lambda_2\lambda_5}_{\lambda_1\lambda_2}(\lambda_1, \lambda_2) R^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(\lambda_5, \lambda_6) R^{\lambda_3\lambda_6}_{\lambda_2\lambda_0}(\lambda_2, \lambda_3). \quad (2.1.14)$$

Braiding matrix and R-matrix coincide up to a normalization factor. $X$-factors for conformal blocks are given by

$$X^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(\lambda_5) = X_{\lambda_1}(0, \lambda_4) X_{\lambda_3}(\lambda_4, \lambda_5) X_{\lambda_2}(\lambda_5, \lambda_1) X_{\lambda_1}(\lambda_1, 0). \quad (2.1.15)$$

If conformal blocks are normalized as in Eq. (2.1.5), then

$$X_{\lambda_1}(\lambda_3, \lambda_2) = C_{\lambda_1\lambda_2\lambda_3}, \quad X_{\lambda}(\lambda, 0) = X_{\lambda}(0, \lambda) = 1. \quad (2.1.16)$$

It can be proved that R-matrices of SU(2)$_k$ model coincide with those of representations of $U_{q(k)}(sl(2))$ quantum group (more precisely, quantum enveloping algebra) with

$$q(k) = \exp\left(\frac{2\pi i}{k + 2}\right), \quad (2.1.17)$$

and R-matrices of the minimal model $M_k$ coincide with those of representations of $U_{q(k)}(sl(2)) \times U_{g(k+1)}(sl(2))$ quantum group (see also Appendix A). Generators of quantum group act in the spaces spanned by vertex operators.
2.2. Convolution of Conformal Blocks and Vertex Operators

Consider two conformal theories \( \Psi \) and \( \Phi \) with conformal blocks \( \Psi_{\{l,\lambda,m\}}(l_5 \lambda_5; z) \) and \( \Phi_{\{\lambda,\lambda,L,M\}}(\lambda_5 L_5; z) \) respectively. Indices \( m \) and \( M \) label vectors in spaces respectively \( \mathcal{H}_\lambda^\Psi \) and \( \mathcal{H}_\lambda^\Phi \), each module is labelled by two indices. Suppose braiding matrices \( \alpha \) of \( \Psi \) theory and \( \beta \) of \( \Phi \) theory to be decomposed as

\[
\alpha_{\{l,\lambda\}}(l_5 \lambda_5, l_6 \lambda_6) = A_{\{l\}}(l_5, l_6)K_{\{\lambda\}}(\lambda_5, \lambda_6),
\]
\[
\beta_{\{\lambda,L\}}(\lambda_5 L_5, \lambda_6 L_6) = K_{\{\lambda\}}(\lambda_5, \lambda_6)B_{\{L\}}(L_5, L_6).
\]

In the language of quantum groups it means that the quantum group of the theory \( \Psi \) is the direct product \( U_{q_1} \times W_{q_3} \) with deformation parameters \( q_1 \) and \( q_3 \), and the quantum group of the theory \( \Phi \) is \( W_{q_3} \times V_{q_2} \). If we forgot for a while about \( A \) and \( B \) (\( U \) and \( V \)) factors we would see that two holomorphic conformal blocks behave just as holomorphic and antiholomorphic parts of correlation function. More precisely, consider the solution, \( Z_{\{\lambda\}}(\lambda_5) \), of the Dotsenko–Fateev equation for \( K \)-factor

\[
\sum_{\lambda_5} Z_{\lambda_5}^{\lambda_3 \lambda_4}(\lambda_5)K^{\lambda_3 \lambda_4}(\lambda_5; \lambda_6)K^{\lambda_3 \lambda_4}(\lambda_5; \lambda_6') = \delta_{\lambda_5 \lambda_6}Z_{\lambda_5 \lambda_4}(\lambda_6).
\]

We construct new conformal blocks\(^5\text{-}7,21,22\)

\[
F_{\{l,\lambda,L\}}(l_5 L_5; z) = \sum_{\lambda_5} Z_{\{\lambda\}}(\lambda_5)\Psi_{\{l,\lambda,\lambda,M\}}(l_5 \lambda_5; z)\Phi_{\{\lambda,\lambda,L\}}(\lambda_5 L_5; z).
\]

Note that \( \lambda_5 \) is absent in the l.h.s. The braiding matrix, \( \gamma \), of new conformal blocks is given by

\[
\gamma_{\{l,L\}}(l_5 L_5, l_6 L_6) = A_{\{l\}}(l_5, l_6)B_{\{L\}}(L_5, L_6).
\]

Respective quantum group is \( U_{q_1} \times V_{q_2} \).

In terms of vertex operators we construct new vertices \( f_{l,\lambda,L}(z) \) from old ones \( \psi_{l,\lambda}(z) \) and \( \phi_{\lambda,L}(z) \) as

\[
(f_{l,\lambda,L}(z))_{\nu'' \nu'}^{\nu'''' \nu'''} = \bigoplus_{\lambda'' \lambda'} Z_{\lambda''}(\lambda'', \lambda') (\psi_{l,\lambda}(z))_{\nu'' \nu'} (\phi_{\lambda,L}(z))_{\lambda'' \lambda'}^{\nu'''' \nu'''},
\]

where \( Z_{\{\lambda\}}(\lambda_5) = Z_{\lambda_5}(0, \lambda_4)Z_{\lambda_3}(\lambda_4, \lambda_5)Z_{\lambda_2}(\lambda_5, \lambda_1)Z_{\lambda_1}(\lambda_1, 0) \). We shall designate such “convolution” of vertices as

\[
f_{l,\lambda,L}(z) = \psi_{l,\lambda}(z) \phi_{\lambda,L}(z).
\]

It will not cause confusion, because the usual tensor product of vertices will not occur in this paper.
Note that Eq. (2.1.11) can be written in these designations as
\[
\phi_\lambda(z, \bar{z}) = \phi_\lambda(z) \phi_\lambda(z).
\] (2.2.8)

At last, there is a relationship between structure constants
\[
C^F_{\ell_1 \lambda_1 L_1, \ell_2 \lambda_2 L_2, \ell_3 \lambda_3 L_3} = C^\Psi_{\ell_1 \lambda_1 L_1, \ell_2 \lambda_2 L_2, \ell_3 \lambda_3 L_3}
\] (2.2.9)
of theories $F$, $\Psi$ and $\Phi$ respectively.

Consider examples. Quantum groups of models $M_k$ and $SU(2)_{k+1}$ are
$U_q(k) (sl(2)) \times U_{q(k+1)} (sl(2))$ and $U_{q(k+1)} (sl(2))$ respectively. Their convolution
gives vertex operators of a theory with the quantum group $U_q(k) (sl(2))$ or, taking
into account that $U_q(1) (sl(2))$ corresponds to trivial monodromy (or braiding)
properties (Appendix B), with the quantum group $U_q(k) (sl(2)) \times U_q(1) (sl(2))$. But
it is just the quantum group of the numerator of the coset construction (1.11).

The second example is convolution of $M_k$ and $M_{k+1}$ models with quantum
groups $U_q(k) (sl(2)) \times U_{q(k+1)} (sl(2))$ and $U_{q(k+1)} (sl(2)) \times U_{q(k+2)} (sl(2))$ respectively.
This convolution is the most important in this work.

2.3. Factorization

Now we shall consider the reverse to convolution procedure of factorizing conformal
blocks. Suppose that $F$ and $\Phi$ conformal blocks from previous subsection are
known, and (2.2.4) is considered as an equation in unknown $\Psi$. \(^5\)\(^7\) Can it be solved?
How many solutions does it have? It turns out that there is a unique solution if chiral
currents of $\Phi$ theory form a subalgebra of chiral algebra of $F$ theory, and
if Virasoro generators, $L_m^\Phi$, from this subalgebra commute with $L_n^\Psi = L_n^F - L_n^\Phi$,
where $L_n^F$ are Virasoro generators of the theory $F$. \(^7\) This is the coset construction
condition. Now we shall sketch the proof. To avoid too cumbersome designations
we shall omit indices $\lambda_1, \cdots, \lambda_4$ and all $l_i, L_i$ and keep indices $M_i$ labelling basic
vectors in the space $H^\Phi_{\lambda_i L_i}$. Eq. (2.2.4) is written as
\[
F_{\{M\}}(z) = \sum_{\lambda_5} Z(\lambda_5) \Psi(\lambda_5; z) \Phi_{\{M\}}(\lambda_5; z).
\] (2.3.1)

Let us introduce matrix designations $F_i(z) = F_{\{M\}}(z)$, $i = \{M\}$; $\Psi_i(z) = \Psi(\lambda_5; z)$,
$i = \lambda_5$; $\Phi_{ij}(z) = \Phi_{\{M\}}(\lambda_5; z)$, $i = \{M\}$, $j = \lambda_5$; $Z_{ij} = \delta_{\lambda_5, \lambda_5} Z(\lambda_5)$, $i = \lambda_5$, $j = \lambda_5$.
We see that
\[
F = \Phi Z \Psi.
\] (2.3.2)

Now we must restrict region of varying of index $\{M\}$ to a finite number of values
which is equal to the number of values of $\lambda_5$ permitted by fusion rules. The restriction
must allow other rows of $\Phi$ or $F$ to be restored using Ward identities\(^3\)\(^9\) for
chiral currents of $\Phi$ theory. Columns of restricted matrix $\Phi$ are linearly independent
solutions of the differential equations. Hence its determinant is a Wronskian
of linearly independent solutions, and
\[
\det \Phi_{\text{restr}}^{\text{restr}} \neq 0, \text{ if } z \neq 0, 1, \infty.
\] (2.3.3)
By definition $\det Z \neq 0$, and we obtain
\[
\Psi = Z^{-1} \Phi^{-1} F
\]  
for restricted matrices. This proves uniqueness.

Every chiral current of $\Phi$ theory can be presented as differential operator which acts on a conformal block. Let us replace there each derivative by the operator $[L^\Phi_{-1}]$. We obtain the same relationships between rows of both full matrices $F$ and $\Phi$. It assures $\Psi$ from Eq. (2.3.4) to be independent of restriction. It means that Eq. (2.3.1) has a solution.

3. Minimal-Like Models as Coset Constructions of Minimal Models

The minimal model $M_k$ is described by vertex operators
\[
\left( \phi^{(k)}_{(p,q)} (z) \right)_{m,n} : \mathcal{H}_{(p_1,q_1)} \rightarrow \mathcal{H}_{(p_1+p_1-2m,p_1+p_1-2n)},
\]
\[
p = 1, 2, \cdots, k + 1; \quad q = 1, 2, \cdots, k + 2;
\]
\[
m = 0, 1, \cdots, p - 1; \quad n = 0, 1, \cdots, q - 1
\]
with conformal dimensions
\[
\Delta_{(p,q)} = \frac{[(k + 3)p - (k + 2)q]^2 - 1}{4(k + 2)(k + 3)}.
\]

There is an equivalence
\[
\mathcal{H}_{(k+2-p,k+3-q)} \sim \mathcal{H}_{(p,q)},
\]
\[
\left( \phi^{(k)}_{(k+2-p,k+3-q)} (z) \right)_{p_1-1-m,q_1-1-n} \bigg|_{\mathcal{H}_{(p_1,q_1)}} \sim \left( \phi^{(k)}_{(p,q)} (z) \right)_{m,n} \bigg|_{\mathcal{H}_{(p_1,q_1)}}.
\]

It means that any state corresponds to two vectors: one from $\mathcal{H}_{(p,q)}$ and one from $\mathcal{H}_{(k+2-p,k+3-q)}$.

Variables $p$, $q$, $m$, $n$ possess quantum group sense. Namely, “momenta” (components of highest weight) $J_1$, $J_2$ and “projections of momenta” (components of weight) $M_1$, $M_2$ for an irreducible representation of $U_q(k) (sl(2)) \times U_{q(k+1)} (sl(2))$ realized by vertex operators for given $(p,q)$ are given by
\[
J_1 = \frac{1}{2} (p - 1), \quad M_1 = J_1 - m,
\]
\[
J_2 = \frac{1}{2} (q - 1), \quad M_2 = J_2 - m.
\]

The minimal-like model $N_{kl}$ is described by vertices
\[
\left( \phi^{(k,l)}_{pp'} (z) \right)_{mm'n} : \mathcal{H}_{pp',q_1} \rightarrow \mathcal{H}_{p_1+p_1-2m,p_1+p_1-2n},
\]
\[
p = 1, 2, \cdots, k + 1; \quad p' = 1, 2, \cdots, l + 1; \quad q = 1, 2, \cdots, k + l + 1;
\]
\[
p + p' - q - 1 \in 2\mathbb{Z};
\]
\[
m = 0, 1, \cdots, p - 1; \quad m' = 0, 1, \cdots, p' - 1; \quad n = 0, 1, \cdots, q - 1.
\]
There is an equivalence
\[ \mathcal{H}_{k+2-p,l+2-p',k+l+2-q} \sim \mathcal{H}_{pp'q}, \]
\[ \left( \phi_{l+2-p,l+2-p',k+l+2-q}(z) \right)_{p_1-1-m,p_1'q_1-1-m'} \bigg|_{\mathcal{H}_{p_1p_1'q_1}} \]
\[ \sim \left( \phi_{pp'q}(z) \right)_{mm'n} \bigg|_{\mathcal{H}_{p_1p_1'q_1}}. \]

If \( l = 1 \), the minimal-like theory \( N_{kl} \) reduces to \( M_k \):
\[ \left( \phi_{(p,q)}(z) \right)_{m,n} \sim \sum_{m'} \alpha_{m'} \left( \phi_{pp'q}(z) \right)_{mm'n}, \]
\[ p' - 1 = p - q \text{(mod 2)}. \]

Bosonic representations for minimal and minimal-like theories, conformal dimensions of \( \phi_{pp'q}(z) \) and rules for calculating three-point correlation functions are presented in Appendix C.

It has been mentioned in Sec. 2.2 that there exists a convolution \( M_k M_{k+1} \) of two minimal models. It consists of vertex operators \( \phi_{(p,s)}(z) \phi_{(s,q)}(z) \). We can consider a chain of such convolutions \( M_k M_{k+1} \cdots M_{k+l-1} \) with vertex operators
\[ \phi_{(p,s,q)}(z) = \phi_{(p,s_1)}(z) \phi_{(s_1,s_2)}(z) \cdots \phi_{(s_{l-1},q)}(z), \ s = (s_1, \cdots, s_{l-1}). \]

This is the numerator of the coset construction (1.14). Its quantum group is \( U_{q(k)}(sl(2)) \times U_{q(k+l)}(sl(2)). \) This quantum group differs from those of \( N_{kl} \) by absence of \( U_{q(l)}(sl(2)) \) factor. We must “glue it up” in \( N_{kl} \) by the denominator of the coset construction. Namely, consider vertex operators
\[ \psi_{t;pq}(z) = \phi_{(1)}(z) \phi_{(2)}(z) \cdots \phi_{(l-1)}(z) \phi_{pp'q}(z), \ t = (t_1, \cdots, t_{l-1}, t_l = p'). \]

We want to prove that vertices \( \psi_{t;pq}(z) \) realize some subtheory \( M_1 M_2 \cdots M_{l-1} N_{kl} \) of the theory \( M_k M_{k+1} \cdots M_{k+l-1} \). To do it we shall find the energy-momentum tensors of \( M_k, M_{k+1}, \cdots, M_{k+l-1} \) models among fields of the theory \( M_1 M_2 \cdots M_{l-1} \cdot N_{kl} \). Note that there is a lot of chiral currents in this theory. Particularly, every field
\[ J_t(z) = \psi_{t,11}(z) \]

can be considered as a chiral current. Indeed, \( J_T(z) \) realizes the unit representation of \( U_{q(1)}(sl(2)) \times U_{q(l)}(sl(2)) \) quantum group. In other words, expansion of the operator product \( J_T(z') \psi_{t;pq}(z) \) gives fields with conformal dimensions which differ from conformal dimension of \( \psi_{t;pq}(z) \) by an integer. This can be easily seen using the formulae for conformal dimensions from Appendix C and standard fusion rules.
produced by quantum group $U_q(sl(2))$. Therefore, if $z'$ goes round $z$ and returns to the initial point, a conformal block which contains this product does not change. It means that $J_t(z)$ has no need of pairing with an antiholomorphic field and it can be included into chiral algebra.

We shall use induction. Eq. (1.14) can be rewritten as

$$N_{kl} \sim \frac{N_{k,l-1}M_{k+l-1}}{M_{l-1}}.$$  \hspace{1cm} (3.11)

We shall not prove this equation thoroughly at each step of induction, but we shall construct energy-momentum tensors of $N_{k,l-1}$ and $M_{k+l-1}$ from fields of $N_{kl}$ and $M_{l-1}$ and prove OPE’s of fields necessary for the following step.

Consider the current of type (3.10)

$$J_3^{(k,l)}(z) \sim \phi^{(l-1)}_{(13)}(z)\phi^{(k,l)}_{131}(z) \hspace{1cm} (3.12)$$

with normalization

$$\langle J_3^{(k,l)}(z')J_3^{(k,l)}(z) \rangle = (z'-z)^{-4}. \hspace{1cm} (3.13)$$

It can be proved using formulae from Appendix C that

$$J_3^{(k,l)}(z')J_3^{(k,l)}(z) = \frac{1}{(z'-z)^4} + \frac{2\theta_3(z)}{(z'-z)^2} + \frac{\theta_3(z)}{z'-z} + O(1),$$

$$\theta_3(z) = \frac{l(l+1)}{(l-1)(l+4)}T_{l-1}(z) + \frac{(k+2)(l+4)(k+l+2)}{3kl(k+l+4)}T_{k,l}(z)$$

$$+ \frac{2(2k+l+4)(l-2)}{\sqrt{3kl(l-1)(l+4)(k+l+4)}}J_3^{(k,l)}(z), \hspace{1cm} (3.14)$$

where $T_k(z)$ and $T_{k,l}(z)$ are energy-momentum tensors of $M_k$ and $N_{kl}$ models respectively. Using Eqs. (1.10), (1.12), (1.8) and (3.14) we obtain that fields

$$T_{k+l-1}(z) = \frac{(l+1)(k+l+4)}{(l+4)(k+l+1)}T_{l-1}(z) + \frac{k+2}{l(k+l+1)}T_{k,l}(z)$$

$$+ \frac{1}{k+l+1}\sqrt{\frac{3k(l-1)(k+l+4)}{l(l+4)}}J_3^{(k,l)}(z), \hspace{1cm} (3.15)$$

$$T_{k,l-1}(z) = \frac{3k}{(l+4)(k+l+1)}T_{l-1}(z) + \frac{(l-1)(k+l+2)}{l(k+l+1)}T_{k,l}(z)$$

$$- \frac{1}{k+l+1}\sqrt{\frac{3k(l-1)(k+l+4)}{l(l+4)}}J_3^{(k,l)}(z) \hspace{1cm} (3.15)$$

obey Eq. (1.10) for $M_{k+l-1}$ and $N_{k,l-1}$ models, and besides

$$T_{k+l-1}(z')T_{k,l-1}(z) = O(1). \hspace{1cm} (3.16)$$
We must now prove that the field $\phi_{131}^{(k,l-1)}(z)$ necessary to construct $J_3^{(k,l-1)}$ for the next step of induction can be expressed in terms of $\phi_{131}^{(k,l)}$. It is enough for our purposes to restore coefficients at $\phi_{131}^{(k,l-1)}(z) = 1$ and $\phi_{131}^{(k,l-1)}(z)$ in expansion of $\phi_{131}^{(k,l-1)}(z')\phi_{131}^{(k,l-1)}(z)$ operator product [see (C.26)]. In the normalization of the bosonic representation we reach it by identification

$$\phi_{131}^{(k,l-1)}(z) = \frac{k}{l+4}\phi_{(3,1)}^{(l-1)}(z)\phi_{111}^{(k,l)}(z) + \frac{1}{l+2}\phi_{(3,3)}^{(l-1)}(z)\phi_{131}^{(k,l)}(z).$$  \hspace{1cm} \text{(3.17)}$$

Now we can, in principle, express all energy-momentum tensors of $M_k, M_{k+1}, \ldots, M_{k+l-1}$ in terms of chiral currents of $M_1M_2 \cdots M_{l-1}N_{k,l}$. This completes the proof.

Now we shall see that the energy-momentum tensors of the denominator of the coset construction (1.14) and of $N_{k,l}$ can be expressed in terms of chiral currents of the numerator.\(^b\) Namely, we have

$$T_{l-1}(z) = \frac{(l-1)(k+l+2)}{(k+2)(k+l-1)}T_{k+l-1}(z) + \frac{k+2}{(l+2)(k+l+3)}T_{k,l-1}(z)$$

$$+ \frac{1}{l+2}\sqrt{\frac{3(k(l-1)(k+l+4)}{(k+l-1)(k+l+3)}}t_{k,l-1}(z),$$

$$T_{k,l}(z) = \frac{3k}{(k+2)(k+l-1)}T_{k+l-1}(z) + \frac{(l+1)(k+l+4)}{(l+2)(k+l+3)}T_{k,l-1}(z)$$

$$- \frac{1}{l+2}\sqrt{\frac{3(k(l-1)(k+l+4)}{(k+l-1)(k+l+3)}}t_{k,l-1}(z),$$

where

$$t_{k,l}(z) \sim \phi_{113}^{(k,l)}(z)\phi_{(3,1)}^{(k,l)}(z)$$  \hspace{1cm} \text{(3.19)}$$

is a chiral current, and

$$t_{k,l}(z')t_{k,l}(z) = \frac{1}{(z'-z)^4} + \frac{2\theta(z)}{(z'-z)^2} + \frac{\partial\theta(z)}{z'-z} + O(1),$$

$$\theta(z) = \frac{(k+l+3)(k+l+4)}{(k+l)(k+l+5)}T_{k+l}(z) + \frac{(k+2)(l+2)(k+l)}{3kl(k+l+4)}T_{k,l}(z)$$

$$+ \frac{2(k-l)(k+l+6)}{\sqrt{3kl(k+l)(k+l+4)(k+l+5)}}t_{k,l}(z).$$  \hspace{1cm} \text{(3.20)}$$

Let us introduce more general currents

$$t_{k,l}^{(n)} \sim \phi_{113}^{(k,l)}(z)\prod_{i=0}^{n-1}\phi_{(3,3)}^{(k+l-i)}(z)\cdot\phi_{(3,1)}^{(k+l+n)}(z),$$  \hspace{1cm} \text{(3.21)}$$

$$\langle t_{k,l}(z')t_{k,l}(z) \rangle = (z'-z)^{-4},$$

\(^b\) Recall that minimal-like models with $k,l > 1$ are not minimal ones and are not defined uniquely by Virasoro algebra.
such that the proportionality coefficient should be negative in bosonic representation. In the same way as Eq. (3.17) we obtain

\[ t_{k,l}^{(n)}(z) = \sqrt{\frac{k}{l(k + l - 1)}} t_{k+l-n-1}^{(n)}(z) - \sqrt{\frac{(l-1)(k+l)}{l(k + l - 1)}} t_{k,l-1}^{(n+1)}(z), \]  

(3.22)

where

\[ t_{k}^{(n)} = t_{k,1}^{(n)} \sim \phi_{(1,3)}^{(k)}(z) \prod_{i=1}^{n} \phi_{(3,3)}^{(k+i)}(z) \cdot \phi_{(3,1)}^{(k+n+1)}(z). \]  

(3.23)

Applying Eqs. (3.18), (3.22) repeatedly we have

\[ T_{k,l}(z) = \frac{k(k + l + 4)}{l + 2} \left( \sum_{m=0}^{l-1} \frac{3T_{k+m}(z)}{(k+m)(k + m + 5)} \right) \]

\[ - \sum_{m=1}^{l-1} \sqrt{\frac{3}{(k + m + 4)(k + m + 5)}} \sum_{n=0}^{m-1} \frac{(-)^m t_{k+m-n-1}^{(n)}}{\sqrt{(k + m - n)(k + m - n - 1)}}, \]

\[ T_{l}(z) = T_{k,l-1}(z) + T_{k+l-1}(z) - T_{k,l}(z). \]  

(3.24)

4. Field Identification

Now we can identify some fields of minimal-like models with coset fields. Let us try to find fields which are primary with respect to \( T_{k,l}(z), T_{l-1}(z), \ldots, T_{1}(z) \) in the space spanned by fields \( \phi_{(p,s,q)}(z) = \phi_{(p,s_1)}^{(k)}(z) \cdots \phi_{(s_{l-1},q)}^{(k+l-1)}(z). \)

Consider fusion rule

\[ T(z')\phi_{(p,s,q)}(z) \sim \sum_{s' \forall i: s'_i - s_i = 0, \pm 2} [\phi_{(p,s',q)}(z)], \]  

(4.1)

where \( T(z) \) is one of the fields \( T_{k,l}(z), T_{l-1}(z), \ldots, T_{1}(z); \) brackets mean conformal family of the field. The absence of fields with \( s'_i - s_i \neq 0, \pm 2 \) is related with absence of vertices \( \phi_{(p,q)}^{(k+i)}(z) \) with \( p, q \neq 1, 3 \) in currents \( t_{k+j}^{(n)} \). Operators \( L_m, m > 0 \) annihilate \( \phi_{(p,s,q)}(0) \) if

\[ \Delta_{(p,s,q)} \leq \Delta_{(p,s',q)}, s'_i - s_i = 0, \pm 2, \]  

(4.2)

where conformal dimensions \( \Delta_{(p,s,q)} \) in \( M_k M_{k+1} \cdots M_{k+l-1} \) theory are given by \( (s_0 = p, s_l = q) \)

\[ \Delta_{(p,s,q)} = \sum_{i=0}^{l-1} \Delta_{(s_i,s_{i+1})}^{(k+i)} \]

\[ = \frac{1}{4} k + \frac{3}{2} p^2 + \frac{1}{4} k + \frac{1}{2} l + \frac{1}{2} q^2 + \frac{1}{2} \sum_{i=1}^{l-1} s_i^2 - \frac{1}{2} ps_1 - \frac{1}{2} s_{l-1}q - \frac{1}{2} \sum_{i=1}^{l-2} s_is_{i+1}. \]  

(4.3)
Now we shall consider special cases.

1. \( \Delta_{(p,s,q)} \leq \Delta_{(p,s',q)} - 2, \ s'_i - s_i = 0, \pm 2 \). It is easy to prove that

\[
s_i = p + \frac{q-p}{l}, \quad \frac{q-p}{l} \in \mathbb{Z}.
\] (4.4)

The conformal dimension of the respective primary field calculated using Eq. (3.24) is given by

\[
\Delta_{pq} = \frac{[(k + l + 2)p - (k + 2)q]^2 - l^2}{4l(k + 2)(k + l + 2)}.
\] (4.5)

Eq. (C.25) gives the same result:

\[
\Delta_{pq} = \Delta_{(k,l)}^{(p,p')}, \quad p' - 1 = p - q \text{(mod } 2l) = 1 \text{ or } l + 1.
\] (4.6)

Conformal dimension, \( \Delta_{(p,s,q)} \), of the “numerator field” coincides with \( \Delta_{pp'q}^{(k,l)} \), and hence

\[
\Delta_{(t_i,t_{i+1})} = 0, \quad m = 1, 2, \ldots, l - 1.
\] (4.7)

Finally we obtain

\[
\phi_{pp'q}^{(k,l)}(z) = \phi_{(p,s_1)}^{(k)}(z) \phi_{(s_1,s_2)}^{(k+1)}(z) \cdots \phi_{(s_{l-1},q)}^{(k+l-1)}(z),
\] (4.8)

\[
s_i = p + \frac{i}{l}(q-p), \quad q - p \in l\mathbb{Z}, \quad p' - 1 = p - q \text{(mod } 2l).
\]

2. \( \Delta_{(p,s,q)} \leq \Delta_{(p,s',q)} - 1, \ s'_i - s_i = 0, \pm 2 \). In this case one can find representations of fields \( \phi_{pp'q}^{(k,l)}(z) \) with indices

\[
p' = r + 1, \quad r = p - q \text{(mod } 2l)
\] (4.9a)

or

\[
p' = l - r + 1, \quad l + r = p - q \text{(mod } 2l),
\] (4.9b)

and conformal dimensions

\[
\Delta_{pp'q}^{(k,l)} = \frac{[(k + l + 2)p - (k + 2)q]^2 - l^2}{4l(k + 2)(k + l + 2)} + \frac{r(l - r)}{2l(l + 2)}.
\] (4.10)

Two simplest representations for these fields are

\[
\prod_{m=r}^{l-1} \phi_{(r+1,r+1)}^{(m)}(z) \cdot \phi_{pp'q}^{(k,l)}(z) = \phi_{(p,s_1)}^{(k)}(z) \prod_{i=1}^{l-2} \phi_{(s_i,s_{i+1})}^{(k+i)}(z) \cdot \phi_{(s_{l-1},q)}^{(k+l-1)}(z)
\] (4.11a)

with

\[
s_i = \begin{cases} p + \frac{q-p+r}{l} - 1 & \text{if } i \leq r; \\ p + r + \frac{q-p+r}{l} & \text{if } i \geq r, \end{cases}
\] (4.11b)
\[
\prod_{m=l-r}^{l-1} \phi_{(i-r+1,l-r+1)}^{(m)}(z) \cdot \phi_{pp'q}^{(k,l)}(z) = \phi_{(p,s_1)}^{(k)}(z) \prod_{i=1}^{l-2} \phi_{(s_i,s_i+1)}^{(k+i)}(z) \cdot \phi_{(s_{l-1},q)}^{(k+l-1)}(z)
\]

with
\[
s_i = \begin{cases} p + \frac{q-p+r}{l} i & \text{if } i \leq l - r; \\ p + l - r + \left(\frac{q-p+r}{l} - 1\right) i & \text{if } i \geq l - r,
\end{cases}
\]

(Fig. 4b). If \( r = 0 \) or \( r = l \), we return to the case 1.

Consider \( i \)-dependence of \( s_i \) as in Fig. 5. The plot consists of segments with slopes \( \frac{1}{l}(q-p+r) - 1 \) and \( \frac{1}{l}(q-p+r) \). There is a representative for each such plot
\[
\prod_{m=1}^{l-1} \phi_{(m,m)}^{(m)}(z) \cdot \phi_{pp'q}^{(k,l)}(z) = \phi_{(p,s_1)}^{(k)}(z) \prod_{i=1}^{l-2} \phi_{(s_i,s_i+1)}^{(k+i)}(z) \cdot \phi_{(s_{l-1},q)}^{(k+l-1)}(z),
\]
\[
t_m - 1 = s_m - p - \left(\frac{q-p+r}{l} - 1\right) m \quad \text{if } s_m - s_{m-1} = \frac{q-p+r}{l} - 1,
\]
\[
t_m - 1 = p - s_m - \frac{q-p+r}{l} m \quad \text{if } s_m - s_{m-1} = \frac{q-p+r}{l}.
\]

We let \( t_m = 1 \) if \( t_m > m \) according these formulae. It is easy to prove that
\[
t_{l-1} = p' \quad \text{or} \quad t_{l-1} = l + 2 - p',
\]
\[
t_{m-1} = t_m \quad \text{or} \quad t_{m-1} = m + 2 - t_m.
\]

Taking into account the equivalence (3.3) we see that Eq. (4.14) is consistent with Eq. (3.9).

3. \( \Delta_{(p,s,q)} \leq \Delta_{(p,s',q)} \), \( s'_i - s_i = 0, \pm 2 \). This case is very difficult for calculation and has not been investigated thoroughly. We discuss it only for \( l = 2 \).

We must consider [in addition to fields (4.8), (4.11–13)] fields \( \phi_{(p,s_{\pm},q)}(z) \) with
\[
s_{\pm} = \frac{1}{2}(p + q) \pm 1.
\]

We have
\[
\Delta_{(p,s_{\pm},q)} = \Delta_{(p,s_{-},q)},
\]
and we must look for eigenvectors of \( L_{0}^{(k,2)} = \oint_{\frac{dz}{2\pi i}} T_{k,2}(z) \) in the space spanned by \( \phi_{(p,s_{\pm},q)}(0) \) and \( \phi_{(p,s_{-},q)}(0) \). We find
\[
\phi_{(2,1)}^{(1)}(z) \phi_{pp'q}^{(k,2)}(z) = \sqrt{\frac{1}{2} + y} \phi_{(p,s_{+},q)}(z) - \sqrt{\frac{1}{2} - y} \phi_{(p,s_{-},q)}(z),
\]
\[
y = [(k + 4)p - (k + 2)q]^{-1}, p' - 1 = p - q \pmod{4}, \ p - q \in 2\mathbb{Z},
\]
\[\text{(4.17)}\]
\[ \phi_{pp'q}(z) = \sqrt{\frac{1}{2} - y} \phi_{(p,s+,q)}(z) + \sqrt{\frac{1}{2} + y} \phi_{(p,s-,q)}(z), \]
\[ p' - 3 = p - q (\text{mod } 4). \]

The field of \( N_{k2} \) model in the l.h.s. of Eq. (4.17) has been obtained earlier in Eq. (4.8) in another way. The field in the l.h.s. of Eq. (4.18) is quite new and possess conformal dimension
\[ \Delta_{pp'q}^{(k,2)} = \left[ \frac{(k + 4)p - (k + 2)q}{8(k + 2)(k + 4)} \right]^2 - 4 + \frac{1}{2}. \]

This field belongs to Ramond sector of the superconformal model.

We can guess that in the case of general \( l \) we can obtain all fields \( \phi_{pp'q}^{(k,l)}(z) \) with \( 1 \leq p + q - p' \leq 1 + 2k \) in a similar way.

5. Three-Point Correlation Functions

It is easy to express structure constants for fields (4.13) using Eq. (2.2.9). Consider an example. Let \( l = 2 \) and let us try to calculate the constant \( C_{221,223,133}^{(k,2)} \) of the theory \( N_{k2} \). Eq. (4.13) gives four expressions for \( C_{221,223,133}^{(k,2)} \) by structure constants \( C_{(p_1,q_1)(p_2,q_2)(p_3,q_3)}^{(i)} \) of \( M_i \) models

\[
\begin{align*}
C_{(2,1)(2,2)(1,2)}^{(k)} C_{(1,1)(2,3)(2,3)}^{(k+1)} &= \frac{1}{2}, & \text{(5.1a)} \\
C_{(2,1)(2,3)(1,2)}^{(k)} C_{(1,1)(3,3)(2,3)}^{(k+1)} &= 0, & \text{(5.1b)} \\
C_{(2,2)(2,2)(1,2)}^{(k)} C_{(2,1)(2,3)(2,3)}^{(k+1)} &= 0, & \text{(5.1c)} \\
C_{(2,2)(2,3)(1,2)}^{(k)} C_{(2,1)(3,3)(2,3)}^{(k+1)} &= \frac{1}{2}. & \text{(5.1d)}
\end{align*}
\]

We obtain different values for different representatives! But we can see that the combinations (5.1b) and (5.1c) forbidden by fusion rules of \( M_k \) and \( M_{k+1} \) should be rejected. Indeed the numerator of the coset construction (1.14) contains several exemplars of each state of the product \( N_{k1} \cdot N_{kl} \) (denominator). It means that each vertex operator of \( N_{kl} \) is represented by several vertices which map different exemplars. The situation is similar to that in the bosonic representation of minimal models where each state (or field) is represented by two states (or fields) and some three-point correlation functions of bosonic representatives vanish although respective structure constants are nonzero.

Ultimately, we obtain
\[ C_{221,223,133}^{(k,2)} = \frac{1}{2}. \]
In general case we write out

\[ C_{p_1p_2^{\prime}q_1p_2^{\prime}q_2p_3^{\prime}q_3}^{(k,l)} = \prod_{i=0}^{l-1} C_{(s^{(1)}_i,s^{(1)}_{i+1})}^{(k+i)}(t^{(1)}_i,t^{(1)}_{i+1}) \cdot \prod_{i=1}^{l-1} C_{(s^{(2)}_i,s^{(2)}_{i+1})}^{(i)}(t^{(2)}_i,t^{(2)}_{i+1}) \cdot \prod_{i=1}^{l-1} C_{(s^{(3)}_i,s^{(3)}_{i+1})}^{(l-1)}(t^{(3)}_i,t^{(3)}_{i+1}) \]

\[ p^{\prime}_j = p_j - q_j \quad (\text{mod } 2l), \]

where \( s^{(j)}_i \), \( t^{(j)}_i \) are defined by Eqs. (4.11–13), the representation is chosen so that none of structure constants in (5.3) equal to zero. Structure constants of minimal models can be adopted from Ref. 12.

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**Appendix A. Quantum groups of minimal models**

Following Sadov\textsuperscript{28} we shall show here that quantum group of the minimal model \( M_k \) is the usual direct product \( U_q(\mathfrak{sl}(2)) \times U_q(\mathfrak{sl}(2)) \). We start from the quantum enveloping algebra obtained by Gomez and Sierra\textsuperscript{25}:

\[ k_+ k_- = k_- k_+ , \]
\[ k_{\pm} E_{\pm} = q_{\pm}^{-1/2} E_{\pm} k_{\pm} , \quad k_{\pm} E_{\mp} = - E_{\mp} k_{\pm} , \]
\[ k_{\pm} F_{\pm} = q_{\pm}^{1/2} F_{\pm} k_{\pm} , \quad k_{\pm} F_{\mp} = - F_{\mp} k_{\pm} , \]
\[ E_+ E_- = E_- E_+ , \quad F_+ F_- = F_- F_+ , \]
\[ E_{\pm} F_{\pm} - q_{\pm} F_{\pm} E_{\pm} = \frac{1 - k_4^4}{1 - q_{\pm}^4} , \quad E_{\mp} F_{\mp} = F_{\mp} E_{\mp} , \]
\[ q_+ = q(k) , \quad q_- = q(k+1) , \]

with comultiplication

\[ \Delta k_i = k_i \otimes k_i , \quad \Delta E_i = E_i \otimes 1 + k_i^2 \otimes E_i , \quad \Delta F_i = F_i \otimes 1 + k_i^2 \otimes F_i . \]
It seems that this algebra differs from the direct product by signs minus in Eqs. (A.1c,e). Let us introduce more usual generators\(^{29} \ X_{i}^{\pm}, \ H_{i}\) by formulae

\[
\begin{align*}
  k_{\pm} &= q_{\pm}^{-H_{i}/4} e^{i\pi H_{\mp}}, \\
  E_{i} &= q_{i}^{-H_{i}/4} X_{i}^{+}, \\
  F_{i} &= q_{i}^{-H_{i}/4} X_{i}^{-}.
\end{align*}
\]

Substituting (A.3) in (A.1) we obtain immediately

\[
\begin{align*}
  [X_{i}^{\pm}, H_{i}] &= \mp 2X_{i}^{\pm}, \\
  [X_{i}^{+}, X_{i}^{-}] &= \frac{q_{i}^{H_{i}/4} - q_{i}^{-H_{i}/4}}{q_{i}^{1/2} - q_{i}^{-1/2}}, \\
  [X_{+}^{\alpha}, X_{-}^{\beta}] &= [X_{+}^{\alpha}, H_{-}] = [H_{+}, X_{-}^{\alpha}] = 0.
\end{align*}
\]

We see that it is a direct product of two associative algebras. Substituting (A.3) in (A.2) we see

\[
\begin{align*}
  \Delta H_{i} &= H_{i} \otimes 1 + 1 \otimes H_{i}, \\
  \Delta X_{\pm}^{\alpha} &= X_{\pm}^{\alpha} \otimes q_{\pm}^{H_{\mp}/4} + q_{\pm}^{-H_{\mp}/4} e^{i\pi H_{\mp}} \otimes X_{\pm}^{\alpha}.
\end{align*}
\]

This comultiplication differs from usual one in \( U_{q(k)}(sl(2)) \times U_{q(k+1)}(sl(2)) \) by the factor \( \exp(i\pi H_{\mp}) \) in the second term. To get rid of it we recall that there are two copies of algebra in Eq. (A.5): one to the left of sign \( \otimes \) and one to the right of it. In each of them independently we can make any endomorphism of algebra. Namely, let us make a substitution in the “left” algebra

\[
X_{\pm}^{\alpha} \mapsto e^{i\pi H_{\mp}} X_{\pm}^{\alpha},
\]

and, in the result of comultiplication

\[
\Delta X_{\pm}^{\alpha} \mapsto (e^{i\pi H_{\mp}} \otimes 1) \Delta X_{\pm}^{\alpha}.
\]

We obtain the usual comultiplication

\[
\Delta X_{\pm}^{\alpha} = X_{\pm}^{\alpha} \otimes q^{H_{\mp}/4} + q^{H_{\mp}/4} \otimes X_{\pm}^{\alpha}.
\]

In terms of conformal blocks the substitution (A.6-7) is no more than changing signs of some conformal blocks.

**Appendix B. Quantum group** \( U_{q(1)}(sl(2)) \)

In this Appendix we shall show that quantum group \( U_{q}(sl(2)) \) with \( q = \exp(\frac{2\pi i}{3}) \) corresponds to unique intermediate module in any conformal block. In other words,
product of any two irreducible representations of quantum group algebra is irre-
ducible. Indeed, there are two irreducible representations \([j]\) with “spin” \(j = 0, \frac{1}{2}\).

It is evident that
\[
[0] \otimes [0] \sim [0], \\
[0] \otimes [\frac{1}{2}] \sim [\frac{1}{2}], \\
[\frac{1}{2}] \otimes [\frac{1}{2}] \sim [0],
\]
which proves the statement. In terms of monodromy it means that all interme-
diate states have the same fractional parts of conformal dimensions and every
monodromy matrix is diagonal.

**Appendix C. Bosonic representation**

In this Appendix we shall sketch some results concerning bosonization of minimal
\(^{10−13}\) and minimal-like\(^{19,20}\) models.

The minimal model \(M_k\) is described by one free bosonic field \(\varphi(z)\) with cor-
relation function
\[
\langle \varphi(z') \varphi(z) \rangle = -\ln(z' - z),
\]
and energy-momentum tensor
\[
T_k(z) = -\frac{1}{2} : (\partial \varphi)^2 : + \frac{i}{\sqrt{2(k + 2)(k + 3)}} \partial^2 \varphi,
\]
which obeys the OPE (1.10) with central charge (1.12). Colons mean normal
ordering, \(\partial \equiv \partial / \partial z\). Vertex operators (3.1) are given by
\[
\left( \phi^{(k)}_{(p,q)}(z) \right)_{m,n} = V_{(p,q)}(z) \prod_{i=1}^{m} \oint_{C_i} du_i I_+ (u_i) \prod_{j=1}^{n} \oint_{S_j} dv_j I_-(v_j), \\
V_{(p,q)}(z) = : \exp \left( -i \frac{(k + 3)(p - 1) - (k + 2)(q - 1)}{\sqrt{2(k + 2)(k + 3)}} \varphi(z) \right) : , \\
I_+(z) = : \exp \left( i \sqrt{\frac{k + 3}{k + 2}} \varphi(z) \right) : = V_{(-1,1)}(z), \\
I_-(z) = : \exp \left( -i \sqrt{\frac{k + 2}{k + 3}} \varphi(z) \right) : = V_{(1,-1)}(z).
\]

Here \(I_+(z)\) and \(I_-(z)\) are screening fields with conformal dimension 1. Integration
contours\(^1_{13}\) are depicted in Fig. 6. Four-point conformal blocks are given by
\[
\Phi_{\{(p,q)\}} ((p_5, q_5); z) \\
= \left( \left( \phi^{(k)}_{(p_1,q_1)}(0) \right)_{0,0} \left( \phi^{(k)}_{(p_2,q_2)}(z) \right)_{m,n} \left( \phi^{(k)}_{(p_3,q_3)}(1) \right)_{m',n'} \left( \phi^{(k)}_{(p_4,q_4)}(\infty) \right)_{0,0} \right), \\
m = q_5(p_1 + p_2 - p_5 - 1), \quad n = \frac{1}{2}(q_1 + q_2 - q_5 - 1), \\
m' = \frac{1}{2}(p_5 + p_3 - p_4 - 1), \quad n' = \frac{1}{2}(q_5 + q_3 - q_4 - 1),
\]
where charge at infinity\textsuperscript{10} $\sqrt{\frac{2}{(k+2)(k+3)}}$ is implicit and the conjugate field is given by

$$
\left(\bar{\phi}^{(k)}_{(p,q)}(z)\right)_{m,n} \sim \left(\phi^{(k)}_{(k+2-p,k+3-q)}(z)\right)_{m,n}.
$$

(C.5)

Contours in Fig. 6 are not very convenient from the technical point of view and we shall use another normalization of conformal blocks\textsuperscript{10,11}

$$
\Phi\{\{p,q\}\}(p_5,q_5;z) = \int_0^z du_1 \cdots \int_0^{u_{m-1}} du_m \int_0^z dv_1 \cdots \int_0^{v_{n-1}} dv_n
\times \int_1^\infty du'_1 \cdots \int_{u'_{m'-1}}^\infty du'_{m'} \int_1^\infty dv'_1 \cdots \int_{v'_{n'-1}}^\infty dv'_{n'}
\times \left(V_{(p_1,q_1)}(0)V_{(p_2,q_2)}(z)V_{(p_3,q_3)}(1)V_{(p_4,q_4)}(\infty)\right)
\times \prod_{i=1}^m I_+(u_i) \prod_{j=1}^n I_-(v_j) \prod_{i=1}^{m'} I_+(u'_i) \prod_{j=1}^{n'} I_-(v'_j).
$$

(C.6)

Coefficients in Eq. (2.1.1) become\textsuperscript{11}

$$
X_{\{\{p,q\}\}}(p_5,q_5) = X_{\{p\}}(p_5;k)\bar{X}_{\{q\}}(q_5;k+1)
$$

$$
X_{\{p\}}(p_5;k) = \prod_{i=1}^m s(ip) \prod_{i=1}^{m'} s(ip) \prod_{i=0}^{m-1} \frac{s(a + i\rho)s(b + i\rho)}{s(a + b + i\rho)}
\prod_{i=0}^{m'-1} \frac{s(c + i\rho)s(a + b + c + (2m + 2n - 2 - i\rho))}{s(a + c + (2m + i\rho))},
$$

(C.7)

$$
s(x) = \sin \pi x,
$$

$$
m = \frac{1}{2}(p_1 + p_2 - p_5 - 1), \quad m' = \frac{1}{2}(p_5 + p_3 - p_4 - 1),
$$

$$
a = -\frac{p_1 - 1}{k + 2}, \quad b = -\frac{p_2 - 1}{k + 2}, \quad c = -\frac{p_3 - 1}{k + 2}, \quad \rho = \frac{1}{k + 2}.
$$

The sign of complex conjugation is used to make clearer the connection with convolution of conformal blocks.

Using (C.6) and (C.7) the following rule for calculating structure constants can be obtained.\textsuperscript{12} Let us calculate the constants

$$
A_{\{p_1,q_1\},\{p_2,q_2\}}^{(k)\{p_3,q_3\}} = \int_0^1 du_1 \cdots \int_0^{u_{m-1}} du_m \int_0^1 dv_1 \cdots \int_0^{v_{n-1}} dv_n
\times \left(V_{(p_1,q_1)}(0)V_{(p_2,q_2)}(1)V_{(p_3,q_3)}(\infty)\prod_{i=1}^m I_+(u_i) \prod_{j=1}^n I_-(v_j)\right).
$$

(C.8)

$$
m = \frac{1}{2}(p_1 + p_2 - p_3 - 1), \quad n = \frac{1}{2}(q_1 + q_2 - q_3 - 1).
$$

20
The constants $A$ take the form\textsuperscript{11}

$$A_{(p_1,q_1)(p_2,q_2)}^{(k)(p_3,q_3)} = \prod_i \Gamma^{d_i}(a_i),$$

where numbers $d_i$ and $a_i$ can be found in Ref. 11. Consider quantities

$$\tilde{A}_{(p_1,q_1)(p_2,q_2)}^{(k)(p_3,q_3)} = \prod_i \left[ \frac{\Gamma(a_i)}{\Gamma(1-a_i)} \right]^{d_i}.$$  

(C.9)

Structure constants are given by\textsuperscript{12}

$$\left(C^{(k)}_{(p_1,q_1)(p_2,q_2)(p_3,q_3)}\right)^2 = \frac{\tilde{A}_{(p_1,q_1)(p_2,q_2)}^{(k)(p_3,q_3)} \tilde{A}_{(p_2,q_2)(p_3,q_3)}^{(k)(p_1,q_1)} \tilde{A}_{(p_3,q_3)(p_2,q_2)}^{(k)(1,1)}}{\tilde{A}_{(p_2,q_2)(p_3,q_3)}^{(k)(1,1)}}.$$  

(C.10)

Now we cite results concerning OPE’s of currents which appear in convolutions of minimal models. Take, for example, the current $\phi_{(1,s_0)}^{(k)}(z)\phi_{(s_0,1)}^{(k+1)}(z)$, and consider the operator product expansion

$$\phi_{(1,s_0)}^{(k)}(z')\phi_{(s_0,1)}^{(k+1)}(z') \cdot \phi_{(p,s)}^{(k)}(z)\phi_{(s,q)}^{(k+1)}(z)$$

$$= \sum_{s'} (z' - z)^{\alpha_{s'}} B_{s_0,s}^{s'}(p,q)\phi_{(p,s')}^{(k)}(z)\phi_{(s',q)}^{(k+1)}(z) + \text{(decendants)},$$

where $\alpha_{s'}$ are real constants. Note that coefficients $B_{s_0,s}^{s'}(p,q)$ are well defined because all exponents $\alpha_{s'}$ differ by integers. Let us introduce coefficients

$$\hat{A}_{s_0,s}^{s'}(p,q) = KK' \prod_i \Gamma^{d_i}(a_i) \prod_i \Gamma^{-d_i'}(1-a'_i),$$

where

$$A_{(p_1,q_1)(p_2,q_2)}^{(k)(p_3,q_3)} = K \prod_i \Gamma^{d_i}(a_i), \quad A_{(p_1,q_1)(p_2,q_2)}^{(k+1)(p_3,q_3)} = K' \prod_i \Gamma^{d_i'}(a'_i),$$

and $a_i$, $a'_i$ are all arguments of gamma-functions, such that

$$(k + 3)a_i \in \mathbb{Z}, \ (k + 3)a'_i \in \mathbb{Z}.$$  

(C.15)

Then

$$B_{s_0,s}^{s'}(p,q) = \frac{\hat{A}_{s_0,s}^{s'}(p,q)\hat{A}_{s_0,s'}^{s}(p,q)}{\hat{A}_{s_0,s_0}^{1}(1,1)}.$$  

(C.16)

Note that

$$d'_i = d_i, \ a_i + a'_i \in \mathbb{Z}.$$  

(C.17)
Generalizations to more complicated convolutions are evident. We write out clearly a three-point correlation function necessary for results of Sec. 3

\[ \langle \phi(p, s, q)(0, 0) t_{k+m}^{(n)}(1) \phi(p, s, q)(\infty, \infty) \rangle \]

\[ = \frac{(-1)^{n+1}}{2 \sqrt{3(k + m)(k + m + 1)(k + m + n + 5)(k + m + n + 6)}} \]

\[ \times \prod_{i=m+1}^{m+n+1} \frac{\Gamma(s_i - s_{i+1} + 1 + \frac{s_i+1}{k+i+2}) \Gamma(s_{i-1} - s_i + \frac{s_i-1}{k+i+2})}{\Gamma(s_{i-1} - s_i + 1 + \frac{s_i+1}{k+i+2}) \Gamma(s_i - s_{i+1} + \frac{s_i-1}{k+i+2})} \]

(B.18)

Bosonic representation of the minimal-like theory \( N_{kl} \) is given by three free bosons \( \chi(z), \rho(z) \) and \( \varphi(z) \) with nonzero correlation functions

\[ \langle \chi(z') \chi(z) \rangle = \langle \rho(z') \rho(z) \rangle = \langle \varphi(z') \varphi(z) \rangle = -\ln(z' - z), \quad (C.19) \]

and energy-momentum tensor

\[ T_{k,l}(z) = -\frac{1}{2} : (\partial \chi)^2 : -\frac{i}{2} \partial^2 \chi - \frac{1}{2} : (\partial \rho)^2 : + \frac{1}{2} \sqrt{\frac{l}{l+2}} \partial^2 \rho \]

\[ -\frac{1}{2} : (\partial \varphi)^2 : + i \sqrt{\frac{l}{2(k+2)(k+l+2)}} \partial^2 \varphi, \quad (C.20) \]

which obey the OPE (1.10) with central charge (1.8). Screening fields are given by

\[ I_1(z) = \oint_{C_z} \frac{du}{2\pi i} : e^{ix} : (u) : \exp \left[ -2i \chi + \sqrt{\frac{l+2}{l}} \rho + i \sqrt{\frac{2k+l+2}{l(k+2)}} \varphi \right] : (z), \]

\[ I'_1(z) = \oint_{C_z} \frac{du}{2\pi i} : e^{ix} : (u) : \exp \left[ -2i \chi + \sqrt{\frac{l}{l+2}} \rho \right] : (z), \]

\[ I_2(z) = \oint_{C_z} \frac{du}{2\pi i} : \exp \left[ i(l+1) \chi - \sqrt{l(l+2)} \rho \right] : (u) \]

\[ \times : \exp \left[ -il \chi + (l-1) \sqrt{\frac{l+2}{l}} \rho - i \sqrt{\frac{2k+l+2}{l(k+l+2)}} \varphi \right] : (z), \quad (C.21) \]

contour \( C_z \) is a small circle with center \( z \). There are four fields with conformal
dimension 0

\[ Z(z) = \oint_{C_z} \frac{du}{2\pi i} : e^{i\chi} : (u) \exp \left[ -i(l+1)\chi + \sqrt{\frac{l(l+2)}{l}} \rho \right] : (z), \]

\[ \tilde{Z}(z) = \oint_{C_z} \frac{du}{2\pi i} : \exp \left[ i(l+1)\chi - \sqrt{\frac{l(l+2)}{l}} \rho \right] : (u) : e^{-i\chi} : (z), \]

\[ Z'(z) = \oint_{C_z} \frac{du}{2\pi i} : e^{i\chi} : (u) \exp \left[ -i(l+1)\chi + (k+2)\sqrt{\frac{l+2}{l}} \rho \right. \]

\[ + i\sqrt{\frac{2(k+2)(k+l+2)}{l}} \varphi \] : (z),

\[ \tilde{Z}'(z) = \oint_{C_z} \frac{du}{2\pi i} : \exp \left[ i(l+1)\chi - (k+2)\sqrt{\frac{l+2}{l}} \rho \right. \]

\[ - i\sqrt{\frac{2(k+2)(k+l+2)}{l}} \varphi \] : (u) : e^{-i\chi} : (z). \quad (C.22)

For any field \( \phi(z) \) we shall write

\[ W\phi(z) = \oint_{C_z} \frac{du}{2\pi i} \frac{Z(u)}{u} \phi(z), \quad (C.23) \]

and similar for \( \tilde{Z}, Z', \tilde{Z}' \).

Vertex operators are given by\(^{20}\)

\[ \left( \phi_{pp'}^{(k,l)}(z) \right)_{mm'} = V_{pp'}(z) \prod_{i=1}^{m} \oint_{C_z} du_i I_1(u_i) \prod_{i=1}^{m'} \oint_{C'_z} du'_i I'_1(u'_i) \prod_{i=1}^{n} \oint_{S_i} dv_i I_2(v_i); \]

\[ V_{pp'}(z) = W_N \psi_{pp'}^{N,0}(z) \text{ or } \tilde{W}^{-N} \psi_{pp'}^{N,0}(z), \]

if \( -p' + 1 \leq q - p - 2lN \leq -p' + 1 + 2l, \) \( -p + 1 \leq q - p' \leq -p + 1 + 2k; \)

\[ V_{pp'}(z) = W_{N'} \psi_{pp'}^{0,N'}(z) \text{ or } \tilde{W}_{N'}^{-N'} \psi_{pp'}^{0,N'}(z), \]

if \( -p + 1 \leq q - p' - 2kN' \leq -p + 1 + 2k, \) \( -p' + 1 \leq q - p \leq -p' + 1 + 2l; \)

\[ \psi_{pp'}^{N,N'}(z) = \psi_{p-2(k+2)N,p'-2(l+2)N' - q - 2(k+l+2)(N+N')}(z), \]

\[ \psi_{pp'}^{N',N'}(z) = f_{p+p'-q-1}(z) : \exp \left[ \frac{(l + 2)(q - p) - l(p' - 1)}{2\sqrt{l(l+2)}} \right] \rho \]

\[ - i\frac{(k + l + 2)(p - 1) - (k + 2)(q - 1)}{2l(k+2)(k+l+2)} \varphi \] : (z),

\[ f_{2n}(z) = \begin{cases} 
: e^{i\chi} : (z) & \text{if } n \geq 0, \\
\oint_{C_z} \frac{du}{2\pi i} : e^{i\chi} : (u) : e^{i(n-1)\chi} : (z) & \text{if } n \leq 0. 
\end{cases} \quad (C.24) \]
Conformal dimensions are given by
\[
\Delta_{pp'q} = \frac{[(k + l + 2)p - (k + 2)q]^2 - l^2}{4l(k + 2)(k + l + 2)} + \frac{p'^2 - 1}{4(l + 2)} - \frac{t^2}{4l} + \frac{1}{2}(t - p' + 1)\theta(t - p' + 1),
\]
\[q - p - t \in 2l\mathbb{Z}, \quad -p' + 1 \leq t \leq -p' + 1 + 2l, \quad -p + 1 \leq q - p' \leq -p + 1 + 2k.\]

If \(-p + 1 > q - p'\) or \(q - p' > -p + 1 + 2k\), we can use the same formula with substitution \(k \leftrightarrow l, \ p \leftrightarrow p'\).

Note that these vertices do not exhaust all fields primary with respect to Virasoro algebra. Nevertheless, every field of the theory is contained in the Fock space of one of these bosonic vertices.

At last we write out some bosonic three-point correlation functions which are used together with (C.16)-like equations in the proof of Eqs. (3.14), (3.17), (3.20) and (3.22):
\[
\langle \phi_{131}(0)\phi_{131}(1)\tilde{\phi}_{131}(\infty) \int_0^1 du I_1'(u) \rangle = -2k + 4 \frac{\Gamma\left(\frac{2}{l+2}\right)\Gamma\left(-\frac{2}{l+2}\right)\Gamma^2\left(-\frac{1}{l+2}\right)}{\Gamma\left(\frac{1}{l+2}\right)\Gamma\left(-\frac{3}{l+2}\right)},
\]
\[
\langle \phi_{131}(0)\phi_{131}(1)\tilde{\phi}_{111}(\infty) \int_0^1 du_1 \int_0^{u_1} du_2 I_1'(u_1)I_1'(u_2) \rangle = -2k + l \frac{\Gamma\left(\frac{2}{k+l+2}\right)\Gamma\left(-\frac{3}{k+l+2}\right)\Gamma^2\left(-\frac{1}{k+l+2}\right)}{\Gamma\left(-\frac{1}{k+l+2}\right)\Gamma\left(-\frac{2}{k+l+2}\right)\Gamma^2\left(-\frac{1}{k+l+2}\right)},
\]
\[
\langle \phi_{113}(0)\phi_{113}(1)\tilde{\phi}_{113}(\infty) \int_0^1 du I_2(u) \rangle = -2k - l \frac{\Gamma^2\left(-\frac{2}{l+2}\right)}{\Gamma\left(\frac{3}{l+2}\right)\Gamma\left(\frac{4}{l+2}\right)},
\]
\[
\langle \phi_{113}(0)\phi_{113}(1)\tilde{\phi}_{111}(\infty) \int_0^1 du_1 \int_0^{u_1} du_2 I_2(u_1)I_2(u_2) \rangle = -2k + l \frac{\Gamma\left(-\frac{2}{k+l+2}\right)\Gamma\left(\frac{1}{k+l+2}\right)\Gamma^2\left(\frac{2}{k+l+2}\right)}{\Gamma\left(-\frac{1}{k+l+2}\right)\Gamma\left(-\frac{3}{k+l+2}\right)\Gamma^2\left(\frac{1}{k+l+2}\right)}.
\]

(C.26)
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