A Nonlinear Proportional Integral Disturbance Observer and Motion Control Technique for Permanent Magnet Synchronous Motors

Yong Woo Jeong, IEEE Student Member, and Chung Choo Chung†, IEEE Member

Abstract—In this paper, we present a Nonlinear-Proportional Integrator (N-PI) disturbance observer (DOB) to enhance the motion tracking of the performance of a surface-mounted Permanent Magnet Synchronous Motor (SPMSM) in rapidly speed varying regions. By presenting an N-PI-DOB for load torque estimation with torque modulation technique, we show that the tracking error dynamics of angular position/velocity are coupled with tracking errors of currents loop and estimation errors. After analyzing disturbances of currents tracking error dynamics, we design the N-PI-DOB and Lyapunov-based nonlinear current controller to enhance the motion tracking performances. With these N-PI-DOBs and motion controllers, we analyze the stability of motion tracking error dynamics and estimation error dynamics. We experimentally perform a comparative study with/without the N-PI-DOB to verify the effectiveness of the proposed method in the condition of the unknown load torque and rapidly speed-varying.

Index Terms—Disturbance Observer, Permanent Magnet Synchronous Motor, Lyapunov Method, Velocity Tracking.

I. INTRODUCTION

Precision motion control of a Permanent Magnet Synchronous Motor (PMSM) has drawn attention from the manufacturing industry because of its efficiency and compact structure [1]. To achieve precision position/velocity control performances, the construction of two connected systems, i.e., electro-mechanical system, is essential. One is an outer-loop control system related to the motor shaft dynamics, i.e., mechanical system. The other is an inner-loop control system related to the inverter dynamics, i.e., an electrical system [2]. The outer-loop control system generates desired torque reference once desired motion reference of the motor shaft is given. Then, the inner-loop control system, such as either a field orient control (FOC) or direct torque control (DTC), generates phase voltages to achieve motor torque as the desired torque reference [3]. In the outer/inner-loop control system, a feedback controller along with a feed-forward controller is utilized. However, disturbance terms in outer/inner-loop control systems, caused by un-modeled nonlinear dynamics, hinder PMSMs from having robust position/velocity tracking performances.

For that reason, a linear DOB with a feedback controller for PMSMs is presented in [4], and it ensures the local exponential stability by taking parameter fluctuations and the external load torque as the lumped disturbance. Further, an extended state-based DOB with the sliding mode motion controller has been applied at the servo-tracking system and showed comparative studies with/without the DOB [5]. In addition, a high-order terminal sliding mode-based DOB was implemented in [6] to estimate the mechanical parameters of PMSMs. Later, a super-twisting sliding mode-based disturbance observer for PMSMs was developed in [7], showing its outperformance by presenting the comparative study. In [8], disturbances in inverter dynamics are considered lumped disturbances. Moreover, a generalized proportional integral observer (GPIO) was introduced to estimate the lumped disturbance. Although the outer-loop controllers presented in [4]–[7] estimate and compensate disturbance, only related with the outer-loop system, they do not consider and compensate disturbances, related to inner-loop control system caused by nonlinear properties of inverter dynamics. The disturbances existing in the inner-loop control system can hinder not only performances in steady-state speed operating regions but also in transient speed operating regions.

To compensate for the disturbances of the inner-loop control system, [9] have applied a sliding mode disturbance compensation and predictive current controller. However, the structure of DOB has limits in that it does not contain the integration term of estimation errors which has been reported that enhance the performance of the control/estimation system. Recently, a Lyapunov-based nonlinear inner-loop controller is presented to enhance the motion tracking performance of a surface-mounted Permanent Magnet Synchronous Motor (SPMSM) in [10], and it shows the comparative study of the motion tracking performances with the conventional inner-loop controllers. Although this paper shows that the velocity tracking performance can be enhanced by enhancing the inner-loop control performance, there was no disturbance-observer-based feed-forward compensation technique in both outer/inner-loop control systems.

For that reason, in this paper, we present a nonlinear Proportional-Integrator (N-PI) outer/inner-loop disturbance observer (DOB) and precision motion controller to enhance the speed tracking performances in rapidly speed varying regions for SPMSM. The main contributions of this paper can be
summarized as follows:

1) A new N-PI DOB is proposed to estimate the outer/inner-loop disturbance, i.e., unknown load torque, nonlinearity in the inverter system. Then, a Lyapunov redesign-based nonlinear currents controller is presented.

2) After deriving the tracking error dynamics of SPMSM, we proved that the disturbances in the inner-loop system and current tracking errors converge to the bounded ball. Further, we show that the load torque estimation error and velocity tracking error converge to bounded value.

3) Finally, with the experimental validation results, we show that the proposed N-PI-DOBs in the outer/inner loop system guarantee the uniform motion tracking performances not only in steady-state region but also in rapidly speed-varying regions.

To describe the pre-mentioned contribution of this paper, in section II, we describe the modeling of SPMSM, torque modulation, and N-PI-DOB for load torque estimation. After then, we derive tracking error dynamics of SPMSM and discuss the possible disturbances which hinder convergence of motion tracking errors. In Section III, we present the N-PI-DOB for inner-loop disturbance estimation and stability analysis of tracking error dynamics of SPMSM. After presenting the experimental validation results, a conclusion will follow.

II. SPMSM Modelling and Torque Modulation and N-PI-DOB for Load Torque Estimation

This section describes SPMSM dynamic equation and the outer-loop controller, including torque modulation and nonlinear proportional integral disturbance observer (N-PI-DOB) for load torque estimation. Based on the torque modulation and N-PI-DOB, we derive tracking error dynamics of SPMSM and discuss possible disturbance terms in inner-loop control system which may hinder the regulation of motion tracking errors.

A. Electro-Mechanical Dynamics of SPMSM

To begin with, let us define $\theta$, $\omega$, $i_\alpha$, and $i_\beta$ as state variables of SPMSM where $\theta$ is the rotor’s angular position (rad), $\omega$ is the rotor’s angular velocity (rad/s), and $i_\alpha$, $i_\beta$ are currents (A) of $\alpha\beta$ frame. Then, the dynamics of SPMSM can be represented as

$$\dot{\theta} = \omega,$$

$$\dot{\omega} = -\frac{B}{J}\omega + \frac{1}{J}\tau_m - \frac{1}{J}\tau_L,$$

$$\ddot{i}_\alpha = -\frac{R}{L}i_\alpha + \frac{P\Phi}{L}\sin(\theta)\omega + \frac{1}{L}v_\alpha,$$

$$\ddot{i}_\beta = -\frac{R}{L}i_\beta - \frac{P\Phi}{L}\cos(\theta)\omega + \frac{1}{L}v_\beta,$$

where $B$ is the viscous friction coefficient (N-m/s/rad), and $J$ is the inertia of the motor (kg-m²). $v_\alpha$ and $v_\beta$ are voltages (V). $\tau_L$ is the load torque (Nm). $R$ and $L$ are the resistance of the phase winding ($\Omega$) and the inductance of the phase winding (H). $\Phi$ is the rotor’s magnetic flux (Wb). $P$ is the number of pole pairs. For simplicity of notations, let us define $S := \sin(P\theta)$, and $C := \cos(P\theta)$. Given $i_\alpha$ and $i_\beta$, the motor torque of SPMSM, $\tau_m$, can be represented as

$$\tau_m = -\frac{3}{2}P\Phi S i_\alpha + \frac{3}{2}P\Phi C i_\beta. \quad (2)$$

B. Torque Modulation in the presence of Load Torque

In this subsection, we introduce torque modulation in the presence of the unknown load torque. Given desired position/velocity reference $\theta_d$ and $\omega_d$, and measurements $\theta$ and $\omega$, the torque modulation generates desired current references, $i_d^a$ and $i_d^b$. To begin with, let us define the estimation states for the outer-loop control system such as, $\hat{\omega}$ and $\hat{\tau}_L$. $\hat{\omega}$ is a velocity estimation state which is only used for load torque estimation. $\hat{\tau}_L$ is a load torque estimation state. The detail structure of $\hat{\omega}$ and $\hat{\tau}_L$ will be presented in Sec. II-C. Given SPMSM states in (1), the desired references and estimation states, let us define tracking errors and estimation errors such as

$$e_\theta = \theta_d - \theta, \quad e_\omega = \omega_d - \omega, \quad \hat{\omega} = \hat{\omega} - \omega$$

$$e_i^a = i_d^a - i_\alpha, \quad e_i^b = i_d^b - i_\beta, \quad \hat{\tau}_L = \tau_L - \hat{\tau}_L$$

where $e_\theta$ is the rotor position tracking errors and $e_\omega$ is the rotor velocity tracking errors. $e_i^a$ and $e_i^b$ are currents tracking errors, $\hat{\omega}$ is the estimation error of rotor velocity and $\hat{\tau}_L$ is the estimation error of unknown load torque. With these notations, we can design a desired torque reference, $\tau_m^d$, such as

$$\tau_m^d := (J\dot{\omega}^d + B\omega^d + k_\theta e_\theta + k_\omega e_\omega + \hat{\tau}_L)$$

$$= -\frac{3}{2}P\Phi S i_d^a + \frac{3}{2}P\Phi C i_d^b$$

where $k_\theta$ and $k_\omega$ are control gains for the stabilization of the mechanical system. From the characteristics of trigonometric functions, $i_d^a$ and $i_d^b$ are computed as

$$i_d^a := -\frac{2\tau_m^d}{3P\Phi} S, \quad i_d^b := \frac{2\tau_m^d}{3P\Phi} C. \quad (5)$$

C. Nonlinear-PI-DOB for Outer-Loop Control System

In this subsection, we present a Nonlinear PI-based Disturbance Observer (N-PI-DOB) for outer-loop control system to estimate $\tau_L$. Suppose that mechanical dynamics of SPMSM is given by (1), and both $\theta$ and $\omega$ are measurable. Then, with parameters $l_{p,T}$, $l_{i,T}$, and $\omega_{\text{max}} > 0$, the N-PI-DOB for load torque estimation is as follow:

$$\dot{\hat{\omega}} = \frac{B}{J}\omega + \frac{1}{J}\tau_d^a - \frac{1}{J}\tau_L$$

$$\hat{\tau}_L = -m(\hat{\omega}) - l_{i,T}\int \hat{\omega} dt.$$

The nonlinear function $m(\hat{\omega})$ is $m(\hat{\omega}) = l_{p,T} \frac{\hat{\omega}}{(\sqrt{\hat{\omega}})^{\omega_{\text{max}}} + 1}$, where $l_{p,T}$ is the estimation gain related to the peak value of the $m(\hat{\omega})$ as depicted in Fig. 1. The derivative of $\hat{\tau}_L$ with respective to time is represented as follow:

$$\dot{\hat{\tau}}_L = -\frac{d}{dt} \frac{d\hat{\omega}}{d\hat{\tau}_L} \hat{\omega} - l_{i,T}\hat{\omega}.$$
By defining $\partial \mu_r(\dot{\omega}) := \frac{dp_r(\dot{\omega})}{d\omega}$, $k_m := \frac{1}{2} P \Phi$, from (1), (3) and (6), the estimation error dynamics of $\dot{\omega}$ and $\dot{\tau}_L$ becomes

$$
\dot{\omega} = -\frac{B}{J} \dot{\omega} + \frac{k_m}{J} \left[ S \begin{array}{c} e_{\alpha} \\ e_{\beta} \end{array} \right] - \frac{1}{J} \dot{\tau}_L
$$

$$
\dot{\tau}_L = \left\{ \begin{array}{c} -\partial \mu_r(\dot{\omega}) \frac{B}{J} + l_i, \tau_1 \\ \partial \mu_r(\dot{\omega}) \frac{k_m}{J} \left[ S \begin{array}{c} e_{\alpha} \\ e_{\beta} \end{array} \right] + \dot{\tau}_L \end{array} \right. \quad (7)
$$

For simplicity of notation, let us define the mechanical tracking error states $e_{\alpha\beta}$, currents tracking error states $e_{\alpha\beta}$, and the estimation error states for outer-loop control system $d_\tau$ as

$$
e_\text{m} = \begin{bmatrix} e_\theta & e_\omega \end{bmatrix}^T, e_{\alpha\beta} = \begin{bmatrix} e_\alpha & e_\beta \end{bmatrix}^T, d_\tau = \begin{bmatrix} \hat{\omega} & \hat{\tau}_L \end{bmatrix}^T. \quad (8)
$$

With these notations, (7) can be represented as

$$
d_\tau = A_r(\dot{\omega})d_\tau + G_r(\dot{\omega})e_{\alpha\beta} + B_r \hat{\tau}_L \quad (9)
$$

where $A_r(\dot{\omega}) = A_{r\omega} + \partial \mu_r(\dot{\omega})A_{r1}$, $B_r = \begin{bmatrix} 0 & 0 \end{bmatrix}^T G_r(\dot{\omega}) = \begin{bmatrix} \frac{k_m}{J} \left[ S \begin{array}{c} e_{\alpha} \\ e_{\beta} \end{array} \right] \end{bmatrix}$, $A_{r\omega} = \begin{bmatrix} \frac{B}{J} & -\frac{1}{J} \end{bmatrix}$, and $A_{r1} = \begin{bmatrix} 0 & \frac{B}{J} \end{bmatrix}$. The value of $\hat{\tau}_L$ is related with the variance of injected $\tau_L$. So, in practice, without loss of generality we can assume that there exists $\delta_\tau$ such as

$$
\delta_\tau = \sup_{t \in [0, \infty)} \| \hat{\tau}_L \|. \quad (10)
$$

D. Tracking Error Dynamics of SPMSM with N-PI-DOB and Torque Modulation

To analyze the stability of motion tracking control with the torque modulation and N-PI-DOB and to design the inner-loop controller, in this subsection, we derive the tracking error dynamics of SPMSM. By subtracting (2) and (4), the following relationship can be derived

$$
\tau_m = \tau_d + k_m(S_e\alpha - C e_\beta). \quad (11)
$$

By plugging equations (1), (4) and (11) into time derivative of (3), tracking error dynamics can be derived as

$$
\dot{e}_\theta = e_\omega
$$

$$
\dot{e}_\omega = -\frac{k_d}{J} e_\theta - \frac{B + k_m}{J} e_\omega - \frac{k_m}{J} (S e_\alpha - C e_\beta) + \frac{1}{J} \dot{\tau}_L
$$

$$
\dot{e}_\alpha = \frac{R}{L} e_\alpha + \frac{1}{L} P \Phi S \omega - \frac{1}{L} v_\alpha
$$

$$
\dot{e}_\beta = \frac{R}{L} e_\beta + \frac{1}{L} P \Phi C \omega - \frac{1}{L} v_\beta. \quad (12)
$$

Tracking error dynamics shows that $e_\alpha, e_\beta$ and $\hat{\tau}_L$ are related to the dynamics of $e_\omega$. Therefore, designing $v_\alpha$ and $v_\beta$ to regulate $e_\alpha$ and $e_\beta$ to be zero is essential. Let us define voltage references, $v_\alpha^d$ and $v_\beta^d$, commanded to inverter systems as

$$
v_\alpha^d = R i_\alpha^d - P \Phi S \omega - u_\alpha, \quad v_\beta^d = R i_\beta^d + P \Phi C \omega - u_\beta. \quad (13)
$$

where $u_\alpha$ and $u_\beta$ are feedback control inputs which will be discussed in Sec. III. Due to the inverter nonlinearity, there exists voltage difference between actual phase voltages and desired voltage reference, and it can be represented as

$$
v_\alpha = v_\alpha^d + e_{\alpha}, \quad v_\beta = v_\beta^d + e_{\beta}. \quad (14)
$$

where $e_{\alpha\beta}$ are the voltage differences between $v_\alpha^d$ and $v_\beta^d$, $j \in \{\alpha, \beta\}$. Further, from (12), we see that $v_\alpha^d$ and $v_\beta^d$ are coupled with the dynamics of $e_\alpha$ and $e_\beta$, and they can be disturbances for convergence of $e_\alpha$ and $e_\beta$. By considering these possible disturbance terms for inner-loop control system, let us define inner-loop disturbances as

$$
d_\alpha := -L_\alpha^d + e_{\alpha}, \quad d_\beta := -L_\beta^d + e_{\beta}. \quad (15)
$$

By plugging (13), (14) and (15) into (12), the we can represent SPMSM error dynamics such as

$$
\dot{e}_\theta = e_\omega
$$

$$
\dot{e}_\omega = -\frac{k_d}{J} e_\theta - \frac{B + k_m}{J} e_\omega - \frac{k_m}{J} (S e_\alpha - C e_\beta) + \frac{1}{J} \dot{\tau}_L
$$

$$
\dot{e}_\alpha = -\frac{R}{L} e_\alpha + \frac{1}{L} u_\alpha - \frac{1}{L} d_\alpha
$$

$$
\dot{e}_\beta = -\frac{R}{L} e_\beta + \frac{1}{L} u_\beta - \frac{1}{L} d_\beta. \quad (16)
$$

For simplicity of notations, let us define $u_{\alpha\beta} = \begin{bmatrix} u_\alpha & u_\beta \end{bmatrix}^T$ and $d_{\alpha\beta} = \begin{bmatrix} d_\alpha & d_\beta \end{bmatrix}^T$. With (8), we can represent (16) as

$$
\dot{e}_m = A_m e_m + G_m e_{\alpha\beta} + B_m d_\tau
$$

$$
d_\tau = A_r(\dot{\omega})d_\tau + G_r(\dot{\omega})e_{\alpha\beta} + B_r \hat{\tau}_L \quad (17)
$$

where

$$
A_m = \begin{bmatrix} \frac{k_d}{J} & -1 \end{bmatrix}, \quad G_m = -\frac{k_m}{J} \begin{bmatrix} S & -C \end{bmatrix},
$$

$$
B_m = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{J} \end{bmatrix}, \quad A_{\alpha\beta} = \begin{bmatrix} -\frac{R}{L} & 0 \\ 0 & -\frac{R}{L} \end{bmatrix}, \quad B_{\alpha\beta} = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{L} \end{bmatrix}.
$$

Based on this tracking error dynamics, we are going to regulate $e_{\alpha\beta}$ by designing N-PI-DOB for inner-loop control system and Lyapunov currents controller. Then, we discuss the stability of the motion tracking controller of SPMSM.
III. N-PI-DOB FOR INNER-LOOP CONTROL SYSTEM, NONLINEAR CURRENTS CONTROLLER AND STABILITY ANALYSIS

In this section, we present a nonlinear-PI disturbance observer for inner-loop control system. Then, we briefly introduce a Lyapunov based nonlinear currents controller. Then, we perform the stability analysis of the proposed motion controller of SPMSM.

A. Nonlinear-PI-DOB for Inner-Loop Control System

In this subsection, we present a Nonlinear PI-based Disturbance Observer (N-PI-DOB) for inner-loop control system to estimate \( d_\alpha \) and \( d_\beta \) in (15). Let us define estimation states of \( e_\alpha, e_\beta \) as \( \tilde{e}_j, j \in \{\alpha, \beta\} \) and estimation states of \( d_\alpha, d_\beta \) as \( \tilde{d}_j, j \in \{\alpha, \beta\} \). Furthermore, let us define the estimation error of \( e_j \) and \( d_j \) such as

\[
\hat{e}_j = e_j - \tilde{e}_j, \quad \hat{d}_j = d_j - \tilde{d}_j, \quad j \in \{\alpha, \beta\}
\]

where, \( \hat{e}_j \) is the estimation error of \( e_j \), \( \hat{d}_j \) is the estimation error of inner-loop disturbance \( d_j \). Then, with parameters \( l_{p.c}, l_{i.e} \), and \( e_{\text{max}} > 0 \), we can design the N-PI-DOB for inner-loop control system as

\[
\dot{\hat{e}}_j = -\frac{R}{L} \hat{e}_j + \frac{1}{L} u_j - \frac{1}{L} \hat{d}_j, \quad j \in \{\alpha, \beta\}
\]

By defining \( \partial \mu_\epsilon(\hat{e}_j) := \frac{\partial \mu_\epsilon(\hat{e}_j)}{\partial \hat{e}_j} \), from (16), (19) and (21), the estimation error dynamics of \( \hat{e}_j \) and \( \hat{d}_j \) becomes

\[
\dot{\hat{e}}_j = -\frac{R}{L} \hat{e}_j - \frac{1}{J} \hat{d}_j \quad \dot{\hat{d}}_j = \left\{-\partial \mu_\epsilon(\hat{e}_j) \frac{R}{L} + l_{i.e}\right\} \hat{e}_j - \frac{\partial \mu_\epsilon(\hat{e}_j)}{L} \hat{d}_j + \hat{d}_j.
\]

For simplicity of notation, let us define the estimation error states for inner-loop control system as \( \mathbf{d}_j = [\hat{e}_j \quad \hat{d}_j]^T \), and (22) can be represented as

\[
\dot{\mathbf{d}}_j = A_\epsilon(\hat{e}_j)\mathbf{d}_j + B_\epsilon \mathbf{d}_j
\]

where \( A_\epsilon(\hat{e}_j) = A_{\epsilon 0} + \partial \mu_\epsilon(\hat{e}_j) A_{\epsilon 1} \), \( B_\epsilon = [0 \quad 1]^T \), \( A_{\epsilon 0} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ l_{i.e} & 0 \end{bmatrix} \) and \( A_{\epsilon 1} = \begin{bmatrix} 0 & 0 \\ -\frac{R}{L} & -\frac{1}{L} \end{bmatrix} \). The value of \( \mathbf{d}_j \) is related with the variance of \( \hat{e}_j \) and \( e_{\text{ic}} \). So, in practice, without loss of generality we can assume that there exists \( \delta_j \) such as

\[
\delta_j = \sup_{t \in [0, \infty)} \|\mathbf{d}_j\|.
\]

Theorem 1. (Exponential convergence of \( \mathbf{d}_j \) to bounded ball \( B_\epsilon \) in finite time \( T_f \)): Suppose there exists \( \delta_j \). Further, suppose that there exist \( e_j \) and \( u_j \). Then, the estimation law (19) guarantees that the estimation errors for inner-loop disturbances, \( \mathbf{d}_j \), converge exponentially to the bounded ball \( B_\epsilon \) within finite time \( T_f \).

Proof. Let us define the Lyapunov function candidate \( V_j(\mathbf{d}_j) \) as

\[
V_j(\mathbf{d}_j) = \mathbf{d}_j^T P_j \mathbf{d}_j, \quad j \in \{\alpha, \beta\}
\]

where \( P_j = P_j^T > 0 \). The time derivative of \( V_j \) becomes

\[
\dot{V}_j(\mathbf{d}_j) = \mathbf{d}_j^T (A_{\epsilon 0}^T P_j + P_j A_{\epsilon 0}) \mathbf{d}_j + \partial \mu_\epsilon(\hat{e}_j) \mathbf{d}_j^T (A_{\epsilon 1}^T P_j + P_j A_{\epsilon 1}) \mathbf{d}_j + 2\mathbf{d}_j^T P_j B_\epsilon \mathbf{d}_j.
\]

Since we can select \( l_{i.e} > 0 \) satisfying \( \sigma(A_{\epsilon 0}) \subset \mathbb{C}^n_{\omega} \), for any \( Q_{\epsilon 0} = Q_{\epsilon 0}^T > 0 \), there exists \( P_j = P_j^T > 0 \) such that [11]

\[
A_{\epsilon 0}^T P_j + P_j A_{\epsilon 0} = -Q_{\epsilon 0}.
\]
In addition, let us define \( Q_{e1} := A_{e1}^T P_j + P_j A_{e1} \). Here, we use the two norm of vector \( x \) as \( \|x\| \) and the induced matrix norm of matrix \( X \) as \( \|X\| \). Then, we see that (26) becomes
\[
\dot{V}_j(d_j) = -d_j^T Q_{e1} d_j + \partial \mu(\tilde{e}_j) d_j^T Q_{e1} d_j + 2d_j^T P_j B_d \tilde{d}_j.
\]
Let us define the minimum/maximum eigenvalues of matrix \( X \) as \( \lambda_{\min}(X) \), \( \lambda_{\max}(X) \). Since \( Q_{e1} \) can be indefinite and \( \|B_e\| = 1 \), there exists \( \varepsilon > 0 \) such that
\[
V_j(d_j) \leq -\lambda_{\min}(Q_{e1}) \|d_j\|^2 + |\partial \mu(\tilde{e}_j)| \|Q_{e1}\| \|d_j\|^2 + 2\|P_j\| \|d_j\|^2 
= \left\{ \frac{\lambda_{\min}(Q_{e1})}{\lambda_{\max}(P_j)} - \frac{\lambda_{\min}(P_j) |\partial \mu(\tilde{e}_j)|}{\lambda_{\min}(P_j)} \right\} V_j(d_j) + 2\|P_j\| \|d_j\|^2 \varepsilon^{-1} - 2\|P_j\| \|d_j\|^2 \varepsilon^{-1} \varepsilon_j 
\leq -\gamma_\varepsilon(\tilde{e}_j)V_\delta(d_j), 
\]
where,
\[
\gamma_\varepsilon(\tilde{e}_j) = \frac{\lambda_{\min}(Q_{e1})}{\lambda_{\max}(P_j)} - \frac{\lambda_{\min}(P_j) |\partial \mu(\tilde{e}_j)|}{\lambda_{\min}(P_j)} - 2\|P_j\| \|d_j\|^2 \varepsilon^{-1} \varepsilon_j. \tag{28}
\]
Although \( \gamma_\varepsilon(\tilde{e}_j) \) varies due to \( |\partial \mu(\tilde{e}_j)| \), it is straightforward that \( |\partial \mu(\tilde{e}_j)| \) has upper bounded values from (20) and Fig. 1(b) such as
\[
\mu_{\varepsilon} = \frac{\mu_{\varepsilon}}{\mu_{\varepsilon}} = \frac{1}{\mu_{\varepsilon}} \text{ and } (l_p,e) = \frac{l_p,e}{l_p,e},
\]
where \( D_e = \{ \tilde{e}_j : |\tilde{e}_j| \leq \varepsilon_{\max} \} \) and \( \varepsilon_{\max} > 0 \). Therefore, given \( \varepsilon > 0 \), \( Q_{e0} \), \( l_p,e \) and \( l_{e} \), satisfying \( \sigma(Q_{e0}) \subset D_e \), there exists a constant \( \gamma^*_e \) such that \( V_j(d_j) \leq -\gamma^*_e V_j(d_j), \forall \|d_j\| \geq \varepsilon \) as such
\[
\gamma^*_e = \inf_{\tilde{e}_j} \frac{|\partial \mu(\tilde{e}_j)|}{\lambda_{\min}(P_j)} \|d_j\|^2. \tag{29}
\]
We need to show the existence of tuple \( (l_p,e,l_{e}) \) ensuring \( \gamma^*_e > 0 \), and a numerical example will be presented in Sec. IV. Next, we will show that \( d_j \) converges to the bounded ball, \( B_\varepsilon = \{ \|d_j\| < \varepsilon \} \) within finite time \( T_f \). With the positive inﬁmum value \( \gamma^*_e \), using the Gronwall-Bellman Inequality [11], we can get the inequality of \( V_d(T_f) \) as
\[
V_d(T_f) = \|d_j(T_f)\|^2 = \varepsilon^2 \leq V_d(T_0)e^{-\gamma_\varepsilon(T_f-T_0)} \tag{30}
\]
where
\[
T_f \leq T_0 + \frac{\log\varepsilon V_d(T_0) - 2\|d_j\|^2}{\gamma_\varepsilon}
\]
Thus, each \( d_j \) gets into \( B_\varepsilon \) within finite time \( T_f \).

**B. Lyapunov-based Nonlinear Currents Control**

In this subsection, we present a nonlinear currents control law combined with N-PI-DOB for inner-loop control system. To make currents tracking errors, \( e_{a,b} \) in (8), stay within the bounded ball, let us deﬁne Lyapunov function candidate for the currents tracking as follow:
\[
V(e_{a,b}) = e_{a,b}^T e_{a,b}. \tag{31}
\]
Then, we can design a nonlinear inputs \( u_{a,b} \) such as
\[
u_{a,b} = \left[ -\eta_1 \frac{\varepsilon_{a,b}}{\varepsilon_{a,b} + \eta_2} + \hat{\theta}_a - \eta_1 \frac{\varepsilon_{\beta}}{\varepsilon_{a,b} + \eta_2} + \hat{\theta}_\beta \right]^T. \tag{32}
\]
where \( \eta_1 \) and \( \eta_2 \) are control gains and positive.

**Theorem 2. (Uniform Convergence of Currents Tracking Errors in Finite Time , \( T_f \)):** Suppose that the tracking error dynamics of SPMSM is given by (17). With \( \eta_1 > 0 \) and \( \eta_2 > 0 \), \( V(e_{a,b}) \), and \( u_{a,b} \), the currents tracking errors, \( e_{a,b} \), converge into bounded ball.

**Proof.** The derivative of Lyapunov function candidate with respect to time, \( t \), becomes
\[
\frac{d}{dt} V(e_{a,b}) = e_{a,b}^T e_{a,b} + e_{a,b}^T \eta_1 e_{a,b} + \eta_1 e_{a,b}^T \eta_1 e_{a,b} 
= -2\eta_1 \frac{\varepsilon_{a,b}}{\varepsilon_{a,b} + \eta_2} + \frac{\varepsilon_{a,b}}{\varepsilon_{a,b} + \eta_2} - 2\eta_1 \frac{\varepsilon_{a,b}}{\varepsilon_{a,b} + \eta_2} \eta_1 e_{a,b} - \frac{\varepsilon_{a,b}}{\varepsilon_{a,b} + \eta_2} \eta_1 e_{a,b} 
\leq \frac{-2\eta_1}{\varepsilon_{a,b} + \eta_2} |e_{a,b}|^2 \leq \frac{2}{\varepsilon_{a,b} + \eta_2} |e_{a,b}|^2. \tag{33}
\]
By plugging (32) into (33), then, we see that (33) becomes
\[
\frac{d}{dt} V(e_{a,b}) \leq -\frac{2\eta_1}{L} V(e_{a,b}) + \frac{2}{\varepsilon_{a,b} + \eta_2} |e_{a,b}|^2 \tag{34}
\]
For Theorem 1, we prove that \( \delta_{a,b} \) becomes convergence to bounded ball within finite time \( T_f \). Therefore, given \( \varepsilon_{\max} > 0 \), there exists
\[
\delta_{a,b} = \sup_{e_{a,b}, \eta_2 < 0} \frac{2}{\varepsilon_{a,b} + \eta_2} |e_{a,b}|^2 \tag{35}
\]
where \( D_e = \{ e_{a,b}, \|e_{a,b}\| \leq \varepsilon_{\max} \} \). Let \( f(x) = \frac{x}{x^2+\eta_2} \). As shown in Fig. 1 of [10], given \( \eta_2 > 0 \) we see that \( \lim_{x \to \infty} f(x) = 1 \). Thus given \( \eta_2 > 0 \) and for \( x \geq \varepsilon_{a,b} \), we see that there exists the infimum of \( f(x) \) for \( x \geq \varepsilon_{a,b} \). With \( \Xi = \{ x \mid f(x) = \frac{\varepsilon_{a,b}}{\varepsilon_{a,b} + \eta_2}, \forall x \} \), let \( \rho(\eta_2) = \inf_{x \in \Xi} \frac{x}{x^2+\eta_2} \). As a result, we have
\[
V(e_{a,b}) \leq -\frac{2\eta_1}{L} \rho(\eta_2) + \delta_{a,b}. \tag{36}
\]
Therefore, it is clear that the control law (32) ensures currents tracking errors converge to bounded ball.

**C. Stability of Outer-Loop Control System**

In this subsection, we analyse stability of the N-PI-DOB for outer-loop control system after the currents tracking errors converged to bounded ball. After that, we will show that the motion tracking system has Input-to-State Stability (ISS) properties [11].

**Theorem 3. (Exponential Convergence of inner-loop Disturbance Estimation Error):** Suppose there exists \( \delta_{a} \) in (10). Given states of SPMSM, torque modulation and N-PI-DOB for outer-loop control system (6), the estimation error dynamics of outer-loop disturbances are given by (9). Then, the estimation error of unknown load torque, \( \hat{d}_r \), starts to converge exponentially to bounded ball \( B_\varepsilon \), once \( e_{a,b} \) converges to bounded ball.
Proof. Let us define Lyapunov function candidate $V(d_r)$ as

$$V(d_r) = d_r^T P_r d_r$$

(36)

where $P_r = P_r^T > 0$. The time derivative of $V$ becomes

$$
\dot{V}(d_r) = d_r^T (A_r^T P_r + P_r A_r) d_r
+ \partial \mu(\tilde{\omega}) d_r^T (A_r^T P_r + P_r A_r) d_r
+ 2 \partial \mu(\tilde{\omega}) d_r^T P_r \dot{r}_L.
$$

(37)

Since we can select $l_{i,\tau} > 0$ satisfying $\sigma(A_{\tau}) \subset C^\omega$, for any $Q_{\tau} = Q_{\tau}^T > 0$, there exists $P = P_r^T > 0$ such that [11]

$$A_{\tau}^T P_r + P_r A_{\tau} = -Q_{\tau}.$$  

(38)

In addition, let us define $Q_{r,1} := A_r^T P_r + P_r A_r$. Here, we use the two norm of vector $x$ as $\|x\|$ and the induced matrix norm of matrix $X$ as $\|X\|$. Then, we see that (26) becomes

$$
\dot{V}(d_r) = -d_r^T Q_{r,1} d_r
+ \partial \mu(\tilde{\omega}) d_r^T Q_{r,1} d_r
+ 2 d_r^T P_r \dot{r}_L.
$$

Let us define the minimum/maximum eigenvalues of matrix $X$ as $\lambda_{\min}(X), \lambda_{\max}(X)$. Since $Q_{r,1}$ can be indefinite and $\|B_r\| = 1$, there exists $\varepsilon_r > 0$ such that

$$
\dot{V}(d_r) \leq
- \lambda_{\min}(Q_{r,1}) \|d_r\|^2
+ \lambda_{\max}(Q_{r,1}) \|d_r\|^2
+ 2 \lambda_{\min}(Q_{r,1}) \|d_r\| \|\delta_r\|
= - \lambda_{\min}(Q_{r,1}) \|d_r\|^2
+ \lambda_{\max}(Q_{r,1}) \|d_r\|^2
+ 2 \lambda_{\min}(Q_{r,1}) \|d_r\| \|\delta_r\|
\leq -\gamma_r(\tilde{\omega}) V(d_r),
\|d_r\| \geq \varepsilon_r
$$

(39)

where,

$$
\gamma_r(\tilde{\omega}) = \lambda_{\min}(Q_{r,1}) - \lambda_{\max}(Q_{r,1}) \|d_r\|^2
- 2 \lambda_{\min}(Q_{r,1}) \|d_r\| \|\delta_r\|.
$$

(40)

It is straightforward that $|\partial \mu(\tilde{\omega})|$ has upper bounded values such as

$$
\mu_{\tilde{\omega}}^{\max} = \sup_{\tilde{\omega} \in D_{\tilde{\omega}}} |\partial \mu(\tilde{\omega})| = l_{p,\tau}
$$

where $D_{\tilde{\omega}} = \{ \tilde{\omega} : |\tilde{\omega}| \leq \tilde{\omega}^{\max} \}$ and $\tilde{\omega}^{\max} > 0$. Therefore, given $\varepsilon_r > 0, Q_{\tau_0}, l_{p,\tau}$ and $l_{i,\tau}$ satisfying $\sigma(A_{\tau_0}) \subset C^\omega$, there exists a constant $\gamma_r^\ast$ such that $\dot{V}(d_r) \leq -\gamma_r^\ast V(d_r), \forall \|d_r\| \geq \varepsilon_r$ such as

$$
\gamma_r^\ast = \inf_{\tilde{\omega} \in D_{\tilde{\omega}}} \gamma_r(\tilde{\omega})
$$

We need to show the existence of tuple $(l_{p,\tau}, l_{i,\tau})$ ensuring $\gamma_r^\ast > 0$, and a numerical example will be presented in Sec. IV. Next, we will show that $d_r$ converges to the bounded ball, $B_{\varepsilon_r} = \{d_r : \|d_r\| < \varepsilon_r\}$ within finite time $T_f$. With the positive inmmum value $\gamma_r^\ast$, using the Gronwall-Bellman Inequality [11], we can get the inequality of $V(d_r)$ as

$$
V(T_f) = \|d_r(T_f)\|^2 \leq \varepsilon_r^2 \leq V(t_0) e^{\gamma_r^\ast(T_f - T_0)}
$$

(41)

Thus, each $d_r$ gets into $B_{\varepsilon_r}$ within finite time $T_f$. 

In summary, from Theorem 2, the currents tracking errors converge to bounded ball. Further, from Theorem 3, the load torque estimation error is bounded within $B_{\varepsilon_r}$. Since $\sigma(A_{\tau_0}) \subset C^\omega$, the dynamics of $e_m$ is an exponentially stable system perturbed by $d_r$ and $e_{\alpha\beta}$. Again, from the Input-to-State Stability (ISS) properties [11] of a tracking error

| Symbol | Values | Unit |
|--------|--------|------|
| $J$    | $4.46 \times 10^{-4}$ | (kg · m²) |
| $B$    | $7 \times 10^{-4}$ | (N · m/s/rad) |
| $\Phi$ | $1.58 \times 10^{-2}$ | (N · m · Α) |
| $f_{ctrl}$ | 100 | (us) |
| $f_{PWM}$ | 20 | (kHz) |
| $k_0$ | 0.4 | - |
| $k_\alpha$ | 0.0214 | - |
| $l_{p,\tau}$ | 0.001 | - |
| $l_{i,\tau}$ | 0.0025 | - |
| $l_{\tau}$ | 10 × $10^4$ | - |
| $\omega_\tau$ | $6 \times 10^4$ | - |
| $\eta_1$ | 52.36 | - |
| $\eta_2$ | 15.6 | - |
| $\eta_3$ | 1300 | - |
dynamics (17), it is obvious that the motion tracking error converges into a small bounded ball.

IV. EXPERIMENTAL RESULTS

This section describes experimental results for validating the proposed outer/inner-loop N-PI-DOB and Lyapunov-based nonlinear currents controller for precision motion control of SPMSM as illustrated in Fig. 2. Figure 3. shows the motor and generator set (MG Set). Control logics for the outer/inner-loop system were implemented in MicroAutobox embedded computer (dSPACE). Furthermore, the 3-phase motor driver system was implemented by RapidPros [1]. Each RapidPro unit includes three half-bridge power stage modules, and the two switches of each half-bridge were driven by complementary signals with some dead time to avoid feedthrough fault. The sample rate of control logics, $f_{\text{ctrl}}$, was 10kHz. The switching frequency of pulse width modulation, $f_{\text{pwm}}$, was 20 kHz. The parameters of SPMSM and control/estimation gains are listed in Table. I. Given PMSM motor parameters, and estimation gains and $\delta_\tau = \delta_j = 1$, $\varepsilon = \varepsilon_\tau = 0.1$, $Q_{\tau0} = Q_{\varepsilon0} = 1000 \times l_2 \times 2$, we see that $\gamma_\tau^* = 0.03 > 0$ and $\gamma_\varepsilon^* = 50.4 > 0$. As shown in Fig. 3, one SPMSM (APM-SB03ADK-9, LS Mecapion & Co) was located between an encoder (2500 pulses per revolution) and a coupler. Then, the torque meter is connected between couplers and another PMSM for generating load torque locate in series. Motor speed was obtained by time stamped method using dSPACE AC Motor Control Solutions. Figure 4 shows the velocity reference and the measured load torque. The desired velocity reference was changed from 500 rpm to 1000 rpm around 13.5 sec. To validate the robustness for the outer/inner-loop N-PI-DOB performances, as illustrated in Fig. 4(b), we intentionally injected the load torque.

Figure 5 shows the estimation results of inner-loop disturbances ($\hat{d}_\alpha, \hat{d}_\beta$) around 10.9 sec and 16.2 sec, respectively. Figure 6 shows the currents tracking performances ($i_\alpha, i_\beta$). Notice that there is no way to measure the exact values of $d_\alpha, d_\beta$. However, we can confirm its performances indirectly by observing current tracking performances as illustrated in Fig. 6. In Fig. 6, the red/blue lines indicate the graph of $\alpha/\beta$-phase signals. We present experimental results in two cases: Case 1) 500 rpm speed region with the load torque was reset from -0.3 to 0 Nm as shown in Fig. 5(a), Case 2) 1000 rpm speed region with the load torque was applied from 0 to 0.3 Nm as shown in Fig. 5(b). Here, we observed that each of
estimated inner-loop disturbances has 33.33 Hz and 66.66 Hz frequency components, identically same as with the electric angular speed. As we expect that disturbances come from the parameter uncertainties and the phase delay of the low-pass filter, the amplitude of disturbances changed as the amplitude of \( i_{dL}, i_{qL}, i_{\alpha}, i_{\beta} \) varied.

Figure 7 shows the comparative studies for analyzing tracking/estimation performances (\( e_{\alpha/\beta}, d_{r}, e_{m} \)) between the methods in [10] and N-PI-DOB with Lyapunov-based nonlinear currents controller. Figure 7(a) shows the currents tracking performances. We expect that the inner-loop N-PI-DOB and Lyapunov-based nonlinear currents controller enhance the currents tracking performances. And we observed that the amplitude of \( \|e_{\alpha/\beta}\|_{2} \) of the proposed method uniformly remains within the bounded ball although the velocity reference changed and load torque was injected. However, the other previous method and without N-PI-DOB for inner-loop control system do not show the uniform currents tracking performances in same conditions. Figure 7(b) shows a comparative study of load torque estimation performance, \( \hat{\tau}_{L} \). Figure 7(c) shows the velocity tracking performances. We expected that the inner-loop disturbance, which has fast dynamics, will degrade the tracking performances in the transient region. And the proposed nonlinear outer/inner-loop N-PI-DOB enhance the tracking performances in the transient region. We observed that the tracking performance of outer/inner-loop N-PI-DOB and outer-loop N-PI-DOB are similar in steady-state region. However, they are different in a transient region such as acceleration duration, 13.5~13.6 sec. We confirmed that the proposed method enhances the motion tracking performances in transient region with the inner-loop disturbance estimation/compensation logics. The experimental results validate the effectiveness of the proposed nonlinear PMSM controller.

V. CONCLUSION

This paper presented the precision motion tracking control with a new nonlinear proportional-integral disturbance observer (N-PI-DOB) for a surface-mounted permanent magnet synchronous motor (SPMSM). Firstly, we introduced the torque modulation technique and presented the N-PI-DOB for load torque estimation to control the desired motion of SPMSM. Then, we showed that the motion tracking error dynamics of the SPMSM can be represented in the form of a 3-cascade system, including a mechanical motion system, load torque estimation system, and currents tracking system. With this 3-cascade representation, we analyzed that the currents tracking errors disturb the convergence of the motion tracking errors. Then, we presented a new nonlinear disturbance observer of inverter dynamics and show the global exponential stability. We performed the experimental comparative studies of SPMSM controller and show that velocity tracking performances in rapidly speed varying region have been improved with the proposed outer/inner-loop N-PI-DOB and motion tracking controller.

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