A homogeneous polynomial associated with general hypergraphs and its applications

Yuan Hou\textsuperscript{a,b,}† An Chang\textsuperscript{b}, Lei Zhang\textsuperscript{b}

\textsuperscript{a}Department of Computer Engineering, Fuzhou University Zhicheng College, Fuzhou, Fujian, P.R. China
\textsuperscript{b}Center for Discrete Mathematics and Theoretical Computer Science, Fuzhou University, Fuzhou, Fujian, P.R. China

Abstract: In this paper, we define a homogeneous polynomial for a general hypergraph, and establish a remarkable connection between clique number and the homogeneous polynomial of a general hypergraph. For a general hypergraph, we explore some inequality relations among spectral radius, clique number and the homogeneous polynomial. We also give lower and upper bounds on the spectral radius in terms of the clique number.

AMS: 15A42; 05C50.

Keywords: Spectral radius, general hypergraph, Adjacency tensor, clique number

1 Introduction

A general hypergraph is a pair $H = (V, E)$ consisting of a vertex set $V$ and an edge set $E$, where each edge is a subset of $V$. If $H' = (V', E')$ is a hypergraph such that $V' \subseteq V$ and $E' \subseteq E$, then $H'$ is called a subhypergraph of $H$. For a vertex $i$, we denote the family of edges containing $i$ by $E(i)$. Two vertices $i$ and $j$ are said to be adjacent, denoted by $i \sim j$, if there exists an edge $e$ such that $\{i, j\} \subseteq e$. Otherwise, we call $i$ and $j$ are nonadjacent.

The set $R = \{|e| : e \in E\}$ is called the set of edge types of $H$. We also say that $H$ is an $R$-graph. In particular, if $R$ contains only one positive integer $r$, then an $\{r\}$-graph is just an $r$-uniform hypergraph, which is simply written as $r$-graph. Consequently, a 2-graph is referred to a usual graph. A hypergraph is non-uniform if it has at least two edge types. For a vertex $i$, let $R(i)$ be the multiset of edge types in $E(i)$. For example,
assume $E(1) = \{\{1, 3\}, \{1, 2, 3\}, \{1, 3, 4\}\}$, then $R(1) = \{2, 3, 3\}$. The rank of $H$, denoted by $\text{rank}(H)$, is the maximum cardinality of the edges in the hypergraph. For example, if $R = \{1, 4\}$, we say that $H$ is a $\{1, 4\}$-graph with $\text{rank}(H) = 4$.

For an integer $n$, let $[n]$ denote the set $\{1, 2, \cdots, n\}$. For a set $S$ and integer $i$, let $\binom{S}{i}$ be the family of all $i$-subsets of $S$. An $R$-graph $H$ with vertex set $[n]$ and edge set \[ \bigcup_{i \in R} \binom{[n]}{i} \] is called a complete $R$-graph. A complete $R$-subgraph in $H$ is called a clique of $H$. A clique is said to be maximal if it is not contained in any other clique, while it is called maximum if it has maximum cardinality. The clique number of a hypergraph $H$, denoted by $\omega(H)$, is defined as the number of vertices of a maximum clique. In other words, the clique number of a hypergraph $H$ is the number of vertices of its maximum complete $R$-subgraph in $H$. In particular, when the edge set of $H$ is empty, we delimit $\omega(H) = 1$ for the purpose of complying with mathematical logic.

In 1967, Wilf [23] first used spectral graph theory for computing bounds on the chromatic number of graphs.

**Theorem 1.1 (Wilf Result [23]).** Let $G$ be a 2-graph with chromatic number $\chi(G)$ and spectral radius $\rho(G)$. Then

$$ \chi(G) \leq \rho(G) + 1. $$

By above Wilf’s result, it is immediately to obtain that $\omega(G) \leq \rho(G) + 1$. Later in 1986, Wilf [24] introduced a lower spectral bound on clique number which was inspired by an elegant result due to Motzkin and Straus [11].

In 1965, Motzkin and Straus gave an answer to the following problem proposed in [10].

Given a graph $G = (V, E)$ with vertex set $V = \{1, 2, \cdots, n\}$. Let $S$ be the simplex in $\mathbb{R}^n$ given by $x_i \geq 0, \sum_{i=1}^{n} x_i = 1$. What is max $\sum_{(i,j) \in E} x_i x_j$?

Let $L(G, x) = \sum_{\{i,j\} \in E} x_i x_j$. The Motzkin and Straus Theorem establishes a link between the problem of finding the clique number of a graph $G$ and the problem of optimizing a homogeneous polynomial $L(G, x)$ of $G$ over the simplex $S = \{(x_1, x_2, \cdots, x_n) | \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \cdots, n.\}$.

**Theorem 1.2 (Motzkin and Straus Theorem [11]).** Let $G$ be a 2-graph with clique number $\omega(G)$, and $x^*$ a maximizer of $L(G, x)$ over $S$. Then

$$ L(G, x^*) = \frac{1}{2} \left(1 - \frac{1}{\omega(G)}\right). $$
Theorem 1.3 (\cite{24}). Let $G$ be a 2-graph with spectral radius $\rho(G)$ and principal eigenvector $x$. Then

$$\omega(G) \geq \frac{M^2}{M^2 - \rho(G)},$$

where $M$ is the sum of the entries of the principal eigenvector $x$.

Naturally we want similar comfort and convenience for spectra of hypergraphs. So it is a natural thought to generalize the Wilf’s results to general hypergraphs by using the tool of the spectral hypergraph theory.

We first define a homogeneous polynomial of degree $m$ for general hypergraphs.

For a general hypergraph $H$ with $\text{rank}(H) = m$, and an edge $e = \{l_1, l_2, \cdots, l_s\}$ with cardinality $s \leq m$, we define

$$x^e_m = \sum x_{i_1} x_{i_2} \cdots x_{i_m},$$

where the sum is over $i_1, i_2, \cdots, i_m$ chosen in all possible ways from $\{l_1, l_2, \cdots, l_s\}$ with at least once for each element of the set.

Definition 1.4. Let $S = \{(x_1, x_2, \cdots, x_n) | \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \cdots, n.\}$ and $H$ be a general hypergraph with $\text{rank}(H) = m$. For a vector $x$ in $S$, define

$$L(H, x) = \sum_{e \in E} \frac{1}{\alpha(s)} x^e_m,$$

(1)

where $s$ is the cardinality of the edge $e$ and $\alpha(s) = \sum_{k_1, k_2, \cdots, k_s \geq 1, k_1 + \cdots + k_s = m} \frac{m!}{k_1! k_2! \cdots k_s!}$.

Furthermore,

$$L(H) = \max\{L(H, x) : x \in S\}.$$ 

A vector $x \in S$ is called an optimal weighting for $H$ if $L(H) = L(H, x)$.

Observe that if $H$ is a 2-graph, by Equation (1), we have $\alpha(2) = 2$. For an arbitrary vector $x$ in $S$,

$$L(H, x) = \sum_{(i,j) \in E} \frac{1}{2}(x_i x_j + x_j x_i) = \sum_{(i,j) \in E} x_i x_j,$$

which is exactly the homogeneous polynomial for 2-graphs in the Motzkin-Straus theorem.

In this paper, we apply the homogeneous polynomial methods to study some relations between the largest $H$-eigenvalues of adjacency tensor and clique numbers of general hypergraphs. This work is motivated by the classic results for graphs \cite{23,24} and some recent results \cite{6,7,9,12,15,20,22,25}. Notice that the Graph-Lagrangian of a non-uniform hypergraph is a nonhomogeneous polynomial, which is different from our definition here. In this paper, we also adopt some definitions and methods of previous studies.

The rest of this paper is organized as follows. In the next section, we present some definitions and properties on eigenvalues of tensors and hypergraphs. Also we give some
useful tools to complete our proof. In Section 3, we attempt to explore the relationships among the homogeneous polynomial, the spectral radius, and the clique number for general hypergraphs. We also bound the spectral radius and clique number for general hypergraphs based on the results obtained.

2 Preliminary

In 2005, Qi [19] and Lim [17] independently introduced the concept of tensor eigenvalues and the spectra of tensors. An $m$th-order $n$-dimensional real tensor $\mathcal{T} = (\mathcal{T}_{i_1 \cdots i_m})$ consists of $n^m$ real entries $\mathcal{T}_{i_1 \cdots i_m}$ for $1 \leq i_1, i_2, \cdots, i_m \leq n$. Obviously, a vector of dimension $n$ is a tensor of order 1 and a matrix is a tensor of order 2. $\mathcal{T}$ is called symmetric if the value of $\mathcal{T}_{i_1 \cdots i_m}$ is invariant under any permutation of its indices $i_1, i_2, \cdots, i_m$. Given a vector $x \in \mathbb{R}^n$, $\mathcal{T} x^m$ is a real number and $\mathcal{T} x^{m-1}$ is an $n$-dimensional vector. $\mathcal{T} x^m$ and the $i$th component of $\mathcal{T} x^{m-1}$ are defined as follows:

$$\mathcal{T} x^m = \sum_{i_1, i_2, \cdots, i_m \in [n]} \mathcal{T}_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

$$(\mathcal{T} x^{m-1})_i = \sum_{i_2, \cdots, i_m \in [n]} \mathcal{T}_{i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$

Let $\mathcal{T}$ be an $m$th-order $n$-dimensional real tensor. For some $\lambda \in \mathbb{C}$, if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying the following eigenequation

$$\mathcal{T} x^{m-1} = \lambda x^{[m-1]}.$$

Then $\lambda$ is an eigenvalue of $\mathcal{T}$ and $x$ is its corresponding eigenvector, where $x^{[m-1]} := (x_1^{m-1}, x_2^{m-1}, \cdots, x_n^{m-1})^T \in \mathbb{C}^n \setminus \{0\}$.

If $x$ is a real eigenvector of $\mathcal{T}$, surely the corresponding eigenvalue $\lambda$ is real. In this case, $\lambda$ is called an $H$-eigenvalue and $x$ is called an $H$-eigenvector associated with $\lambda$. Furthermore, if $x$ is nonnegative and real, we say $\lambda$ is an $H^+$-eigenvalue of $\mathcal{T}$. If $x$ is positive and real, $\lambda$ is said to be an $H^{++}$-eigenvalue of $\mathcal{T}$. The maximal absolute value of the eigenvalues of $\mathcal{T}$ is called the spectral radius of $\mathcal{T}$, denoted by $\rho(\mathcal{T})$.

For nonnegative tensors, we have the Perron-Frobenius theorem, established as

**Theorem 2.1** (Perron-Frobenius theorem for nonnegative tensors [5]).

(1) (Yang and Yang 2010). If $\mathcal{T}$ is a nonnegative tensor of order $k$ and dimension $n$, then $\rho(\mathcal{T})$ is an $H^+$-eigenvalue of $\mathcal{T}$.

(2) (Friedland Gaubert and Han 2011). If furthermore $\mathcal{T}$ is weakly irreducible, then $\rho(\mathcal{T})$ is the unique $H^{++}$-eigenvalue of $\mathcal{T}$, with the unique eigenvector $x \in \mathbb{R}^n_{++}$, up to a positive scaling coefficient.

(3) (Chang Pearson and Zhang 2008). If moreover $\mathcal{T}$ is irreducible, then $\rho(\mathcal{T})$ is the
unique $H^\perp$-eigenvalue of $\mathcal{T}$, with the unique eigenvector $x \in \mathbb{R}^n$, up to a positive scaling coefficient.

In 2012, Cooper and Dutle \cite{4} defined the adjacency tensor of an $r$-graph. Later in 2017, Banerjee et al.\cite{3} defined the adjacency tensor for general hypergraphs as the following.

**Definition 2.2.** Let $H = (V, E)$ be a general hypergraph with $\text{rank}(H) = m$. The adjacency tensor $\mathcal{A}$ of $H$ is defined as follows

$$\mathcal{A} = (a_{i_1i_2\cdots i_m}), 1 \leq i_1, i_2, \ldots, i_m \leq n.$$  

For all edges $e = \{l_1, l_2, \ldots, l_s\} \in E$ of cardinality $s \leq m$,

$$a_{i_1i_2\cdots i_m} = \frac{s}{\alpha(s)}, \text{ where } \alpha(s) = \sum_{k_1\ldots k_s \geq 1, k_1 + \cdots + k_s = m-1} \frac{m!}{k_1!k_2!\cdots k_s!},$$

and $i_1, i_2, \ldots, i_m$ are chosen in all possible ways from $\{l_1, l_2, \ldots, l_s\}$ with at least once for each element of the set. The other positions of the tensor are zeros.

Notice that if $H$ is an $r$-graph, it is known from direct calculations that $a_{i_1i_2\cdots i_r} = \frac{1}{(r-1)!}$ for an arbitrary edge $e = \{i_1, i_2, \ldots, i_r\} \in E$, which is exactly the definition of Cooper and Dutle in \cite{4}.

For an edge $e = \{i, i_2, \ldots, i_s\}$ in a general hypergraph with $\text{rank}(H) = m$, we denote an $m$ order $n$ dimensional symmetric tensor $\mathcal{A}(e)$ by

$$(\mathcal{A}(e)x)_i = \frac{s}{\alpha(s)} \sum_{k_1\ldots k_s \geq 1, k_1 + \cdots + k_s = m-1} \frac{(m-1)!}{k_1!k_2!\cdots k_s!} x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_s}^{k_s},$$

which indicates that

$$x^\top \mathcal{A}(e)x = \frac{s}{\alpha(s)} \sum_{k_1\ldots k_s \geq 1, k_1 + \cdots + k_s = m} \frac{m!}{k_1!k_2!\cdots k_s!} x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_s}^{k_s}.$$  

Then the adjacency tensor $\mathcal{A}$ of order $m$ and dimension $n$ uniquely defines a homogeneous polynomial in $n$ variables of degree $m$ by:

$$F_\mathcal{A}(x) = \mathcal{A}x^m = \sum_{e \in E} x^\top \mathcal{A}(e)x.$$  

Based on the definitions in \cite{3}, Kang et al.\cite{20} obtained the Perron-Frobenius theorem for general hypergraphs. For convenience, let $\rho(H)$ denote the spectral radius of the adjacency tensor of a general hypergraph $H$.  

5
Theorem 2.3 (Perron-Frobenius theorem for general hypergraphs [26]).

1. Let $H$ be a general hypergraph, then $\rho(H)$ is an $H^+$-eigenvalue of $H$.
2. If $H$ is connected, then $\rho(H)$ is the unique $H^{++}$-eigenvalue of $H$, with the unique eigenvector $x \in \mathbb{R}^n_{++}$, up to a positive scaling coefficient.

By Theorem 2.3, if $H$ is connected, there is a unique unit positive eigenvector $x$ corresponding to $\rho(H)$. Let $\|x\|_m = (\sum_{i=1}^n x_i^m)^{\frac{1}{m}}$. The positive eigenvector $x$ with $\|x\|_m = 1$ corresponding to $\rho(H)$ is called the principal eigenvector of $H$. Assume $x$ is the principal eigenvector of an $R$-graph $H$, by the theory of optimization, we have

$$\rho(H) = F_A(x) = A x^m = \sum_{e \in E} x^T A(e) x.$$  \hfill (2)

We give some auxiliary lemmas which will be used in the sequel.

Lemma 2.4 (Maclaurin’s inequality [8]). Let $x_1, x_2, \cdots, x_n$ be positive real numbers. For any $k \in [n]$, define $S_k$ as follows:

$$S_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \left( \begin{array}{c} n \\ k \end{array} \right).$$

Then $S_1 \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \cdots \geq \sqrt[n]{S_n}$ with equality if and only if all the $x_i$ are equal.

For any $k \in [n]$, by Maclaurin’s inequality, we get $S_1 \geq \sqrt[k]{S_k}$, that is,

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \leq \left( \begin{array}{c} n \\ k \end{array} \right) \left( \frac{x_1 + x_2 + \cdots + x_n}{n} \right)^k.$$

The following lemma is the generalization of the Cauchy-Schwarz inequality for more than two vectors.

Lemma 2.5 ([8]). Let $x_1 = (x_1^{(1)}), x_2 = (x_1^{(2)}), \cdots, x_k = (x_1^{(k)})$ be nonnegative vectors of dimension $n$. Then

$$\sum_{i=1}^n \prod_{j=1}^k x_i^{(j)} \leq \|x_1\|_k \|x_2\|_k \cdots \|x_k\|_k.$$  \hfill (3)

Equality holds if and only if all vectors are collinear to one of them.

3 Main results

In this section, we discuss the relations between the clique number and the spectral radius of an $R$-graph. A tight lower bound of the spectral radius of an $R$-graph is presented. Further, we determine the upper bound of the spectral radius based on the clique
number for an \( \{m, m-1\} \)-graph, which derived from a Motzkin-Straus type result due to \( L(H) \) for \( \{m, m-1\} \)-graphs. In the following discussions, without loss of generality, suppose \( m \geq 3 \).

**Theorem 3.1.** Let \( H \) be an \( R \)-graph with clique number \( \omega \). Then

\[
\rho(H) \geq \sum_{s \in R} \left( \frac{\omega - 1}{s - 1} \right)
\]

with equality if and only if \( H \) is a complete \( R \)-graph.

**Proof.** Assume \( H_0 \) be the maximum complete \( R \)-subgraph of \( H \) on \( \omega \) vertices. Suppose \( \text{rank}(H) = m \). Let \( x \) be a nonnegative vector of dimension \( n \) with entries

\[
x_i = \begin{cases} \frac{1}{\sqrt{\omega}} & \text{if } i \in H_0, \\ 0 & \text{otherwise.} \end{cases}
\]

It is easy to verify that \( \|x\|_m^m = 1 \).

For an arbitrary edge \( e \) in \( H_0 \), suppose \( |e| = s \), we have

\[
x^\top A(e)x = \frac{s}{\alpha(s)} \sum_{\substack{k_1, \ldots, k_s \geq 1, \\ k_1 + \cdots + k_s = m}} \frac{m!}{k_1!k_2! \cdots k_s!} x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s}
\]

\[
= \frac{1}{\omega} \frac{s}{\omega \alpha(s)} \sum_{\substack{k_1, \ldots, k_s \geq 1, \\ k_1 + \cdots + k_s = m}} \frac{m!}{k_1!k_2! \cdots k_s!}
\]

\[
= \frac{s}{\omega}.
\]

After using the Equation (2), we have

\[
\rho(H) \geq F_A(x) = \sum_{e \in E(H_0)} x^\top A(e)x = \sum_{s \in R} \sum_{|e| = s} \frac{s}{\omega} = \sum_{s \in R} \frac{s}{\omega} \left( \frac{\omega}{s} \right) = \sum_{s \in R} \left( \frac{\omega - 1}{s - 1} \right).
\]

If \( \rho(H) = \sum_{s \in R} \left( \frac{\omega - 1}{s - 1} \right) \), that is to say the unit vector \( x \) is a maximizer of \( F_A(x) \) over \( \mathbb{R}_+^n \). Then by Theorem 2.3, \( x \) must be a positive vector. Thus all vertices in \( H \) belong to \( H_0 \), which indicates that \( \omega = n \), i.e., \( H \) is a complete \( R \)-graph. On the other hand, if \( H \) is a complete \( R \)-graph, then \( \omega = n \) and \( x \) is the principal eigenvector. By Equation (2), we have \( \rho(H) = \sum_{s \in R} \left( \frac{\omega - 1}{s - 1} \right) \). Therefore the theorem follows. \( \square \)

Observe that if \( H \) is an \( m \)-graph, Theorem 3.1 indicates that \( \rho(H) \geq \left( \frac{\omega - 1}{m - 1} \right) \) as proved by Yi and Chang in [25].
Spectral methods for 2-graphs reside on a solid ground, with traditions settled both in tools and problems, such as number of edges, independence number. Theorem 3.1 is also a useful tool for general $R$-graphs. Now, we give the lower bound of clique number for $R$-graphs with $R = \{m, m-1\}$.

**Theorem 3.2.** Let $H$ be an $\{m, m-1\}$-graph with clique number $\omega$. Then

$$\omega \leq m - 2 + [(m-1)!\rho(H)]^{\frac{1}{m-1}}.$$  

**Proof.** Since

$$\binom{\omega}{m-1} = \frac{\omega(\omega-1)\cdots(\omega-m+2)}{(m-1)!} \geq \frac{(\omega-m+2)^{m-1}}{(m-1)!}$$

By Theorem 3.1, if $R = \{m, m-1\}$, we have

$$\rho(H) \geq \binom{\omega-1}{m-1} + \binom{\omega-1}{m-2} = \binom{\omega}{m-1} \geq \frac{(\omega-m+2)^{m-1}}{(m-1)!}.$$  

Thus it is easy to verify that $\omega \leq m - 2 + [(m-1)!\rho(H)]^{\frac{1}{m-1}}$. This completes the proof. \qed

**Theorem 3.3.** Let $H$ be an $\{m, m-1\}$-graph with clique number $\omega$. If either $H$ is a complete $\{m, m-1\}$-graph or there exists two nonadjacent vertices $i$ and $j$ such that $R(i) = R(j)$, then

$$L(H) = (\frac{1}{\omega})^m \binom{\omega+1}{m}.$$  

(5)

**Proof.** Assume $H_0$ be the maximum complete $\{m, m-1\}$-subgraph of $H$ on $\omega$ vertices. Let $x$ be a nonnegative vector of dimension $n$ with entries

$$x_i = \begin{cases} \frac{1}{\omega} & \text{if } i \in H_0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $\|x\|_1 = \sum_{i=1}^n x_i = 1$, i.e., $x \in S$.

After using the Definition 1.4 we have

$$L(H) \geq L(H, x) = \sum_{e \in E(H_0)} \frac{1}{\alpha(s)} x_m^e.$$
To prove the opposite inequality, we proceed by induction on \( n \). For \( n \leq m - 2 \), we have \( \omega = 1 \) and \( L(H) = 0 \). Without loss of generality, suppose \( n \) is sufficiently large. Assume the theorem true for \( \{m, m-1\} \)-graphs with fewer than \( n \) vertices. Suppose \( x \) is an optimal weighting for \( H \), where \( x_1 \geq x_2 \geq \cdots \geq x_k \geq x_{k+1} = x_{k+2} = \cdots = x_n = 0 \).

If \( k < n \), there exists one of the entries \( x_i = 0 \). Let \( H' \) be obtained from \( H \) by deleting the corresponding vertex \( i \) and the edges containing \( i \). Since the theorem holds for \( H' \), we have

\[
L(H) = L(H') = (\frac{1}{\omega'})^m \left( \frac{\omega' + 1}{m} \right),
\]

where \( \omega' \) is the clique number of \( H' \). It is clearly that \( \omega' \leq \omega \).

For \( \omega \geq 1 \),

\[
(\frac{1}{\omega})^m \left( \frac{\omega + 1}{m} \right) = \frac{(1 + \frac{1}{\omega})(1 - \frac{1}{\omega})(1 - \frac{2}{\omega}) \cdots (1 - \frac{m-2}{\omega})}{m!} = \frac{(1 - \frac{1}{\omega})(1 - \frac{2}{\omega}) \cdots (1 - \frac{m-2}{\omega})}{m!},
\]

which is an monotonically increasing function of \( \omega \). Thus

\[
L(H) = L(H') \leq (\frac{1}{\omega})^m \left( \frac{\omega + 1}{m} \right).
\]

If \( k = n \), that is \( x_1 \geq x_2 \geq \cdots \geq x_n > 0 \). To proceed our proof, we consider the following two cases.

**Case 1.** There exists two nonadjacent vertices \( i \) and \( j \) such that \( R(i) = R(j) \). Notice that \( R(i) \) and \( R(j) \) are multisets. For a vector \( x \) in \( R_+^m \), we write

\[
L^i(H, x) = \sum_{e \in E(i)} \frac{1}{\alpha(s)} x^e_m;
\]
and

\[ L^i(H, x) = \sum_{e \in E(i)} \frac{1}{\alpha(s)} x^e_m. \]

Define two vectors \( y = (y_k) \) and \( z = (z_k) \) in \( R^m_+ \) as follows. Let \( y_k = x_k \) for \( k \neq i, j \), \( y_i = x_i + x_j \) and \( y_j = 0 \). Clearly, \( y \in S \). Further, \( z_i = x_j \) and \( z_k = x_k \) for \( k \neq i \).

Without loss of generality, assume \( L^i(H, z) \geq L^i(H, x) \). By definition \( l_4 \) and \( R(i) = R(j) \), we have

\[
L(H, y) - L(H, x) = L^i(H, y) - L^i(H, x) - L^j(H, x) = \sum_{e \in E(i)} \frac{1}{\alpha(s)} y^e_m - \sum_{e \in E(i)} \frac{1}{\alpha(s)} x^e_m - \sum_{e \in E(j)} \frac{1}{\alpha(s)} x^e_m.
\]

For an arbitrary positive integer \( k \), it is clearly that \( (x_i + x_j)^k > x_i^k + x_j^k \). Consequently, \( L^i(H, y) > L^i(H, x) + L^i(H, z) \). Thus

\[
L(H, y) - L(H, x) = L^i(H, y) - L^i(H, x) - L^j(H, x) > L^i(H, z) - L^j(H, x) \geq 0,
\]

So that the maximum is attained for the subgraph \( H' \) obtained from \( H \) by deleting the vertex \( j \) and the corresponding edges containing \( j \). Similarly, the theorem is again true by the induction hypothesis.

**Case 2.** If \( H \) is a complete \( \{m, m - 1\} \)-graph, it is obvious that \( \omega = n \) and the maximizer of \( L(H, x) \) over \( S \) must be a positive vector. Otherwise, we can also use the induction hypothesis. Then we have

\[
L(H) = \max_{x \in S} L(H, x) = \max_{x \in S} \sum_{e \in E} \frac{1}{\alpha(s)} x^e_m = \max_{x \in S} \sum_{s \in \{m-1, m\}} \frac{1}{\alpha(s)} \sum_{e \in E, |e| = s} x^e_m
\]

\[
\leq \max_{x \in S} \frac{1}{\alpha(m)} \sum_{e \in E, |e| = m} x^e_m + \max_{x \in S} \frac{1}{\alpha(m-1)} \sum_{e \in E, |e| = m-1} x^e_m.
\]

It is easy to verify that \( \alpha(m) = m! \) and

\[
\sum_{e \in E, |e| = m} x^e_m = \sum_{\{i_1, i_2, \ldots, i_m\} \in E, 1 \leq i_1 < i_2 < \ldots < i_m \leq n} m! x_{i_1} x_{i_2} \cdots x_{i_m},
\]

where the sum is over all edges with cardinality \( m \).

By Maclaurin’s inequality, we have

\[
\frac{1}{\alpha(m)} \sum_{e \in E, |e| = m} x^e_m = \sum_{\{i_1, i_2, \ldots, i_m\} \in E, 1 \leq i_1 < i_2 < \cdots < i_m \leq n} x_{i_1} x_{i_2} \cdots x_{i_m}
\]
The equality holds if and only if \( x = x = \cdots = x_n = \frac{1}{n} \).

Similarly, \( \alpha(m - 1) = \sum_{k_1 \cdots k_{m-1} \geq 1, k_1 + \cdots + k_{m-1} = m} \frac{m!}{k_1!k_2! \cdots k_{m-1}!} x_1^{k_1} x_2^{k_2} \cdots x_{m-1}^{k_{m-1}} \)

\[
\sum_{e \in E} x_e = \sum_{|e| = m-1} \sum_{k_1 \cdots k_{m-1} \geq 1, k_1 + \cdots + k_{m-1} = m} \frac{m!}{k_1!k_2! \cdots k_{m-1}!} x_1^{k_1} x_2^{k_2} \cdots x_{m-1}^{k_{m-1}} \]

By Maclaurin’s inequality, we have

\[
\frac{1}{\alpha(m - 1)} \sum_{|e| = m-1} x_e = \frac{1}{m - 1} \left[ x_1^2 \sum_{1 \leq i_2 < \cdots < i_{m-1} \leq n} x_{i_2} x_{i_3} \cdots x_{i_{m-1}} + \cdots + x_n^2 \sum_{1 \leq i_2 < \cdots < i_{m-1} \leq n} x_{i_2} \cdots x_{i_{m-1}} \right] \]

\[
\leq \frac{1}{m - 1} \left( \frac{n - 1}{m - 2} \right) \left[ x_1^2 \left( \frac{n - 1}{n - 1} \right)^{m-2} + \cdots + x_n^2 \left( \frac{n - 1}{n - 1} \right)^{m-2} \right].
\]

The equality holds if and only if \( x_1 = x_2 = \cdots = x_n = \frac{1}{n} \).

Hence, the maximum is attained on \( x_1 = x_2 = \cdots = x_n = \frac{1}{n} \). After setting \( x_1 = x_2 = \cdots = x_n = \frac{1}{n} \), we get

\[
\max_{x \in S} \frac{1}{\alpha(m - 1)} \sum_{|e| = m-1} x_e = \left( \frac{1}{n} \right)^{m-1} \frac{1}{m - 1} \left( \frac{n - 1}{m - 2} \right) \]

\[
= \left( \frac{1}{n} \right)^{m-1} n \left( \frac{n - 1}{m - 2} \right)
\]

11
\[
\binom{\frac{1}{n}}{m} \binom{n}{m-1} = (\frac{1}{n})^m \binom{n}{m-1}
\]

Combining the above inequalities, if \( H \) is a complete \( \{m, m-1\} \)-graph, then \( \omega = n \) and

\[
L(H) \leq (\frac{1}{n})^m \binom{n}{m} + (\frac{1}{n})^m \binom{n-1}{m-1} = (\frac{1}{n})^m \binom{n+1}{m}.
\]

The equality holds if and only if \( x_1 = x_2 = \cdots = x_n = \frac{1}{n} \).

This completes the proof. \( \square \)

**Theorem 3.4.** Let \( H \) be an \( \{m, m-1\} \)-graph with clique number \( \omega \). If either \( H \) is a complete \( \{m, m-1\} \)-graph or there exists two nonadjacent vertices \( i \) and \( j \) such that \( R(i) = R(j) \), then

\[
\rho(H) \leq m(\frac{U}{\omega})^m \binom{\omega + 1}{m}
\]

where \( U \) is the sum of the entries of the principal eigenvector.

**Proof.** Let \( x \) be the principal eigenvector of \( H \). By Equation (2), we have

\[
\rho(H) = \sum_{e \in E} x^\top A(e)x = \sum_{s \in \{m, m-1\}} \frac{s}{\alpha(s)} \sum_{e \in E, \ |e| = s} x_e m.
\]

Set \( y = \frac{x}{U} \), where \( U \) is the sum of the entries of \( x \). It is clearly that \( y \in S \). Apply the Theorem 3.3, then

\[
\frac{\rho(H)}{U^m} = \sum_{s \in \{m, m-1\}} \frac{s}{\alpha(s)} \sum_{e \in E, \ |e| = s} y_e^m m \leq m(\frac{1}{\omega})^m \binom{\omega + 1}{m}
\]

Thus

\[
\rho(H) \leq m(\frac{U}{\omega})^m \binom{\omega + 1}{m}
\]

Therefore the theorem follows. \( \square \)

And further applying Lemma 2.5 it is easy to check the following result.

**Corollary 3.5.** Let \( H \) be an \( \{m, m-1\} \)-graph with clique number \( \omega \). If either \( H \) is a complete \( \{m, m-1\} \)-graph or there exists two nonadjacent vertices \( i \) and \( j \) such that
\[ R(i) = R(j), \text{ then} \]
\[ \rho(H) \leq \frac{mn^{m-1}}{\omega^m} \left( \frac{\omega + 1}{m} \right) \]

**Proof.** Let \( x \) be the principal eigenvector of \( H \) and \( U \) is the sum of the entries of \( x \). By Lemma 2.5 we have
\[
U = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} 1 \cdot 1 \cdots x_i \leq \left( \sum_{i=1}^{n} 1 \right)^{\frac{m-1}{m}} \left( \sum_{i=1}^{n} x_i^m \right)^{\frac{1}{m}} = n^{\frac{m-1}{m}}.
\]

Obviously, \( U^m \leq n^{m-1} \). Combining with Theorem 3.3 the result follows.

\[ \Box \]

**References**

[1] C. Berge, Hypergraph: Combinatorics of Finite Sets, third edition, North-Holland, Amsterdam, 1973.

[2] A. Bretto, Hypergraph Theory: An Introduction, Springer, 2013.

[3] A. Banerjee, A. Char, B. Mondal, Spectral of general hypergraphs, Linear Algebra Appl. 518 (2017) 14-30.

[4] J. Cooper, A. Dutle, Spectra of uniform hypergraphs, Linear Algebra Appl. 436 (2012) 3268-3292.

[5] Y. Fan, Y. Tan, X. Peng, A. Liu, Maximizing spectral radii of uniform hypergraphs with few edges, Discussiones Math. Graph Theory 36(2016) 845-856.

[6] P. Frankl, Z. Füredi, Extremal problems and the Lagrange function of hypergraphs, Bulletin Institute Math. Academia Sinica 16(1988) 305-313.

[7] R. Gu, X. Li, Y. Peng, Y. Shi, Some Motzkin-Straus type results for non-uniform hypergraphs, J. Comb. Optim. 31(2016) 223-238.

[8] G. Hardy, J. Littlewood, G. Pólya, Inequalities, 2nd edition, Cambridge University Press, 1988.

[9] D. Hefetz, P. Keevash, A hypergraph Turán theorem via lagrangians of intersecting families, J. Combin. Theory Ser. A 120 (2013) 2020-2038.

[10] J. MacDonald Jr., Problem E1643, Amer. Math. Monthly 70 (1963) 1099.

[11] T. Motzkin, E. Straus, Maxima for graphs and a new proof of a theorem of Turán, Canad. J. Math. 17(1965) 533-540.
[12] D. Mubayi, A hypergraph extension of Turans theorem, J. Combin. Theory Ser. B 96 (2006) 122-134.

[13] V. Nikiforov, Analytic methods for uniform hypergraphs, Linear Algebra Appl. 457 (2014) 455-535.

[14] B. Papendieck, P. Recht, On maximal entries in the principal eigenvector of graphs, Linear Algebra Appl. 310 (2000) 129-138.

[15] Y. Peng, H. Peng, Q. Tang, C. Zhao, An extension of Motzkin-Straus Thorem to non-uniform hypergraphs and its applications, Discrete Appl. Math. 200 (2016) 170-175.

[16] L. Qi, Symmetric nonnegative tensors and copositive tensors, Linear Algebra Appl. 439(2013) 228-238.

[17] L. Lim, Singular values and eigenvalues of tensors: a variational approach, in: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP 05) 1(2005) 129-132.

[18] L. Liu, L. Kang, X. Yuan, On the principal eigenvectors of uniform hypergraphs, Linear Algebra Appl. 511(2016) 430-446.

[19] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symb. Comput. 40(2005) 1302-1324.

[20] S. Rota Bulò, M. Pelillo, A generalization of the Motzkin-Straus theorem to hypergraphs, Optim. Lett. 3 (2009) 187-295.

[21] S. Rota Bulò, M. Pelillo, New bounds on the clique number of graphs based on spectral hypergraph theory, Learning and Intelligent Optim. 5851(2009) 45-58.

[22] J. Talbot, Lagrangians of hypergraphs, Combin. Prob. Comput. 11 (2002) 199-216.

[23] H. Wilf., The eigenvalues of a graph and its chromatic number, J. London Math. Soc. 42(1967) 330-332.

[24] H. Wilf., Spectral bounds for the clique and independence numbers of graphs, J. Comb. Theory Series B 40(1986) 113-117.

[25] G. Yi, A. Chang, The spectral bounds for the clique numbers of r-uniform hypergraphs, Manuscript, Fuzhou University, 2013.

[26] W. Zhang, L. Liu, L. Kang, Y. Bai, Some properties of the Spectral radius for general hypergraphs, Linear Algebra Appl. 513(2017) 103-119.