Nilpotent groups and universal coverings of smooth projective varieties

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1 Introduction

Characterizing the universal coverings of smooth projective varieties is an old and hard question. Central to the subject is a conjecture of Shafarevich according to which the universal cover $\tilde{X}$ of a smooth projective variety is holomorphically convex, meaning that for every infinite sequence of points without limit points in $\tilde{X}$ there exists a holomorphic function unbounded on this sequence.

In this paper we try to study the universal covering of a smooth projective variety $X$ whose fundamental group $\pi_1(X)$ admits an infinite image homomorphism

$$\rho : \pi_1(X) \rightarrow L$$

into a complex linear algebraic group $L$. We will say that a nonramified Galois covering $X' \rightarrow X$ corresponds to a representation $\rho : \pi_1(X) \rightarrow L$ if its group of deck transformations is $\text{im}(\rho)$.

Definition 1.1 We call a representation $\rho : \pi_1(X) \rightarrow L$ linear, reductive, solvable or nilpotent if the Zariski closure of its image is a linear, reductive, solvable or nilpotent algebraic subgroup in $L$. We call the corresponding covering linear, reductive, solvable or nilpotent respectively.

The natural homomorphism $\pi_1(X, x) \rightarrow \hat{\pi}_{\text{uni}}(X, x)$ to Malcev’s pro-unipotent completion will be called the Malcev representation and the corresponding covering the Malcev covering.
One may ask not only if the universal covering of \( X \) is holomorphically convex but also if some special intermediate coverings that correspond to representations \( \rho : \pi_1(X) \rightarrow L \) are holomorphically convex.

In case \( X \) is an algebraic surface and \( \rho : \pi_1(X) \rightarrow L \) is a reductive representation this question has been answered in [22]. The author and M. Ramachandran proved there that if \( X' \rightarrow X \) is a Galois covering of a smooth projective surface corresponding to a reductive representation of \( \pi_1(X) \) and such that \( \text{Deck}(X'/X) \) does not have two ends, then \( X' \) is holomorphically convex. The proof is based on two major developments in Kähler geometry that occured in the last decade. The first is a correspondence, established through the work of Hitchin [19], Corlette [10] and Simpson [28], between Higgs bundles, representations of the fundamental group of a smooth projective variety \( \rho : \pi_1(X) \rightarrow G \) (here \( G \) is a linear algebraic group over \( \mathbb{C} \)) and \( \rho \) equivariant harmonic maps from the universal covering of \( X \) to the corresponding symmetric space for \( G \). This correspondence is called now - non-abelian Hodge theory. The second is the theory of harmonic maps to buildings developed by Gromov and Schoen [15]. This theory gives the \( p \)-adic version of the theory of Higgs bundles developed by Corlette, Hitchin and Simpson and can be thought as of a \( p \)-adic non-abelian Hodge theory.

These two ideas are used simultaneously in [22] in order to get more information about \( \pi_1(X) \). The proof in [22] uses also some very powerful ideas of Lasell, Ramachandran [21] and Napier [26], which can be interpreted as a non-abelian strictness property. These ideas provide a bridge and make the Nonabelian Hodge theory suitable for questions related to the Shafarevich conjecture.

In this paper we elaborate further on the idea that the answer to certain uniformization questions depends heavily on the fundamental group of the variety. We study the question if solvable or nilpotent coverings \( X' \rightarrow X \) are holomorphically convex for \( X \) smooth projective variety.

First we prove the following:

**Theorem 1.1** The Malcev covering of any smooth projective \( X \) is holomorphically convex.

As an immediate consequence of this statement we get:

**Theorem 1.2** Let \( X \) be a smooth projective variety with a virtually nilpotent fundamental group. Then the Shafarevich conjecture is true for \( X \).

(Recall that a finitely generated group is nilpotent if its lower central series has finitely many terms. A group is virtually nilpotent if it has a finite index subgroup which is nilpotent.)

The proof of Theorem 1.1 uses the functorial Mixed Hodge Structure (MHS) on \( \pi_1(X) \) combined with some new ideas of János Kollár from [24] and [25]. At the end of section 2 we give a different proof of Theorem 1.2, which combined with the strictness property for the nonabelian Hodge theory seems to be a very promising idea (see [21]). Observe that what helps us prove Theorem 1.1 is the use of all nilpotent representations of \( \pi_1(X) \) at the same time.

We can ask even more basic question than the Shafarevich conjecture:

**Question 1.** Are there any nonconstant holomorphic functions on the universal covering \( X \) of any smooth projective variety?
Clearly it is enough to restrict ourselves to the case when \( \pi_1(X) \) is an infinite group.

To study this question in bigger generality we add some more Hodge theoretic tools - the results of Arapura \([1]\), Beauvile \([3]\), Green, Lazarsfeld \([4]\) and Simpson \([27]\) about characterizing the absolute sets in the moduli space of rank one local systems. We also need the following variant of the result of Arapura and Nori \([2]\) saying that linear solvable Kähler groups are nilpotent.

**Theorem 1.3** Let \( \Gamma \) be a quotient of a Kähler group \( \pi_1(X) \) so that \( \Gamma \) is a \( \mathbb{Q} \)-linear solvable group, then there are two possibilities - either \( \Gamma \) is virtually nilpotent or \( \pi_1(X) \) surjects onto the fundamental group of a curve of genus bigger than zero.

The above theorem gives a way of constructing new examples of non-Kähler groups. In particular any group \( \Gamma \) with infinite \( H^1([\Gamma, \Gamma], \mathbb{Q}) \) possessing a solvable linear quotient defined over \( \mathbb{Q} \) that is not virtually nilpotent cannot be Kähler.

Unfortunately we could not prove a solvable variant of the theorem 1.1. The maximum we were able to say is how much the solvable coverings differ from the nilpotent ones. We show the following.

**Theorem 1.4** If \( \Gamma \) is a quotient of a Kähler group \( \pi_1(X) \) so that \( \Gamma \) is a complex linear solvable group, then there are two possibilities - either \( \Gamma \) is deformable to a virtually nilpotent representation of \( \pi_1(X) \) or \( \pi_1(X) \) surjects onto the fundamental group of a curve of genus bigger than zero.

This theorem gives a way of constructing new examples of non-Kähler groups. In particular any group \( \Gamma \) with infinite \( H^1([\Gamma, \Gamma], \mathbb{Q}) \) possessing a solvable linear quotient defined over \( \mathbb{Q} \) that is not virtually nilpotent cannot be Kähler. In \([21]\) it is proved that the linear covering are holomorphically convex for \( X \) an algebraic surface. Of course this implies a solvable variant of the theorem 1.1 for algebraic surfaces. The above theorem implies immediately:

**Corollary 1.1** Let \( \rho : \pi_1(X) \rightarrow S(\mathbb{C}) \) be a Zariski dense representation of the fundamental group of a smooth projective variety \( X \) to the complex points of an affine solvable group defined over \( \mathbb{Q} \). Then the image of \( \pi_1(X) \) in the Malcev completion of \( \pi_1(X) \) is infinite.

In particular this implies that the first Betti number of \( X \) is nonzero so the universal covering of \( X \) \( \tilde{X} \) admits nonconstant holomorphic functions. The above corollary can be proved of course in a different way too.

If we restrict ourselves to the case when \( X \) is an algebraic surface we get a stronger statement.

**Theorem 1.5** Let \( X \) be a smooth projective surface with an infinite complex linear representation of its fundamental group. Then there exist non-constant holomorphic functions on \( \tilde{X} \).

In some sense the above theorem says that the universal coverings are different from arbitrary coverings. The well known example of Cousin (see e.g. \([26]\)) gives a \( \mathbb{Z} \)-covering of the two dimensional torus which does not admit holomorphic functions. The Theorem 1.5 raises a natural question:

**Question 2.** Are there examples of infinite \( \pi_1(X) \) without any infinite linear representation?
There are known examples of groups with this properties, e.g. Higman’s four generator group. The question is if they can be fundamental groups. Even more interesting question was asked by J. Kollár and C. Simpson.

**Question 3.** Are there examples of infinite residually finite $\pi_1(X)$ without any infinite linear representation?

As it was pointed out to me by S. Gersten the answer of this question is positive if we are looking for an arbitrary group not for $\pi_1(X)$. There are the groups of Grigorchuk and Gupta-Sidki which are finitely generated infinite torsion groups. These groups are known to be residually finite (see e.g. [4]).

A negative answer to this question could have a great impact on the answer to Shafarevich conjecture for residually finite groups (see [21], [23]). From another side a recent paper of Bogomolov and the author [3] shows that things can get quite exotic even for $\pi_1(X)$. In some sense the examples constructed in [3] indicate that if the answer of **Question 2.** is negative then the statement of Theorem 1.5 could be the best statement in such a generality.

Theorem 1.5 and Corollary 1.1 suggest the following:

**Conjecture 1.1** Let $X$ be a smooth projective variety with an infinite linear representation of its fundamental group. Then there exist non-constant holomorphic functions on $\tilde{X}$.

All of this strongly suggests that Hodge theory has a lot to offer in studying uniformization questions. We stop at the border line, before we introduce the next level of Hodge theoretic considerations, the theory of Nonabelian Mixed Hodge Structures- a theory that is giving us a way of working with all linear representations at the same time to get maximal information about $\pi_1(X)$. The first steps in this theory are done in [29], [30], [31], [32] and [23] and it is far from being sufficiently developed. In any case it has fast consequences even on a very primitive level. Using these very first steps we prove in [21] the Shafarevich conjecture for surfaces with linear fundamental groups. The same method implies that the coverings that correspond to any linear representation are holomorphically convex. The proof uses basically only the mixed Hodge structure on the relative completion of $\pi_1(X)$ with respect to some complex variation of mixed Hodge structures. Our feeling is that this is just the beginning.

**Acknowledgements:** I would like to thank A. Beilinson F. Bogomolov, J. Carlson, K. Corlette, R. Donagi, M. Gromov, S. Gersten, M. Larsen, M. Nori, T. Pantev, C. Simpson, D. Toledo and S. Weinberger for the useful conversations and H. Clemens, P. Deligne, R. Hain, J. Kollár and M. Ramachandran for teaching me all ingredients of the technique used in this paper. Special thanks to Professor J. Kollár for inviting me to visit University of Utah, where most of this work was done.

2 The Malcev covering

In this section we prove Theorem 1.1. and give some applications.

We start with some ideas of János Kollár from [24] and [25].

In [24] Kollár observed that the Shafarevich conjecture is equivalent to:
1) There exists a normal variety $\text{Sh}(X)$ and a proper map with connected fibers $\text{Sh}: X \to \text{Sh}(X)$, which contracts precisely the subvarieties $Z$ in $X$ with the property that $\text{im}[\pi_1(Z') \to \pi_1(X)]$ is finite. Here $Z'$ denotes a desingularization of $Z$.

2) $\text{Sh}(\tilde{X})$ is a Stein space. Here we denote by $\text{Sh}(\tilde{X})$ the Grauert-Remmert reduction of $\text{Sh}(\tilde{X})$. In our notations $\text{Sh}(X) = \text{Sh}(\tilde{X})/\pi_1(X)$. The action of $\pi_1(X)$ may have fixed points on $\text{Sh}(\tilde{X})$ but we can still take a quotient.

One can consider also a relative version of condition 1). Let $H \triangleleft \pi_1(X)$ be a normal subgroup. We will say that a subgroup $R \subset \pi_1(X)$ is almost contained in $H$ if the intersection $R \cap H$ has finite index in $R$ and we will write $R \lhd H$. We have the following condition.

1. There exists a normal variety $\text{Sh}^H(X)$ and a proper map with connected fibers $\text{Sh}^H: X \to \text{Sh}^H(X)$, which contracts exactly the subvarieties $Z$ in $X$ having the property that $\text{im}[\pi_1(Z') \to \pi_1(X)] \lhd H$. Again $Z'$ denotes a desingularization of $Z$. The relative version of 2) is the following:

2. $\text{Sh}^H(\tilde{X})$ is a Stein space. Here we denote by $\text{Sh}^H(\tilde{X})$ the Grauert-Remmert reduction of $\text{Sh}(\tilde{X})$. In our notations $\text{Sh}^H(X) = \text{Sh}^H(\tilde{X})/(\pi_1(X)/H)$.

This was also independently observed by F. Campana in [7].

Our approach is that if there is a natural candidate for $\text{Sh}(X)$ it is enough to check condition 1) only for $Z$ - an algebraic curve. This certainly is the case when $\pi_1(X)$ is a nilpotent group. In the simplest case when $\pi_1(X)$ is virtually abelian one uses for $\text{Sh}(X)$ the Albanese variety $\text{Alb}(X)$.

It is clear (see e.g. [1]) that for a smooth projective variety $X$ with $\pi_1(X)$ an infinite nilpotent group the Albanese map:

$$\text{Alb}: X \to \text{Alb}(X)$$

has nontrivial image. In other words $\dim_{\mathbb{C}}(\text{im}(\text{Alb})) > 0$.

Moreover if we denote by $S$ the Stein factorization of the Albanese map, then this is a natural candidate for $\text{Sh}(X)$ in case $\pi_1(X)$ is a nilpotent group. Observe that the map

$$X \to S$$

contracts all subvarieties $Z$ with the property that $\text{im}[H_1(Z, \mathbb{Q}) \to H_1(X, \mathbb{Q})]$ is trivial.

Now using that $\pi_1(X)$ is a nilpotent group and the theory of Mixed Hodge Structures on its Malcev completion we show that the fact that $\text{im}[H_1(Z, \mathbb{Q}) \to H_1(X, \mathbb{Q})]$ is trivial is equivalent to the fact that $\text{im}[\pi_1(Z) \to \pi_1(X)]$ is finite for $Z$ an algebraic curve. We finish the proof by reducing the argument for $Z$ of arbitrary dimension to $Z$ an algebraic curve.

To prove Theorem 1.1 we need to show again that there is natural candidate for $\text{Sh}^H(X)$, where $H = \ker(\rho: \pi_1(X) \to \hat{\pi}_{uni}(X, x)$ of the Malcev representation. Again this candidate is $S$ the Stein factorization of the Albanese map. At the end of section we give a different proof of Theorem 1.1, which is basically spelling of the proof we have given already in the language of equivariant harmonic maps.
2.1 Mixed Hodge Structure considerations

In this subsection we explain why if $\pi_1(X)$ is a nilpotent group the theory of Mixed Hodge Structures on it implies that $\text{im}[H_1(Z, \mathbb{Q}) \to H_1(X, \mathbb{Q})]$ is trivial is equivalent to the fact that $\text{im}[\pi_1(Z) \to \pi_1(X)]$ is finite for $Z$ an algebraic curve. For some background one can look at [11], [12] or [16].

For the proof of Theorem 1.1 we need to work with $X$ smooth but for completeness in this section we will require only the MHS on $H^1(X)$ is of weights $> 0$.

**Lemma 2.1** If $Z$ is a compact nodal curve and $f: Z \to X$ is a map to a variety such that MHS on $H^1(X)$ is of weights $> 0$ then the map

$$f_*: L(Z, x) \to L(X, f(x))$$

is trivial if and only if the map

$$f^*: H^1(X, \mathbb{Q}) \to H^1(Z, \mathbb{Q})$$

is trivial. Here $L(Z, x)$ and $L(X, f(x))$ are the corresponding Lie algebras of the unipotent completions $\hat{\pi}_{\text{uni}}(Z, x)$ and $\hat{\pi}_{\text{uni}}(X, f(x))$ of the fundamental groups $\pi_1(Z, x)$ and $\pi_1(X, f(x))$ respectively and $x$ is a point in $Z$.

**Proof.** Observe that the map in unipotent completions determines and is determined by a map on the corresponding Lie algebras:

$$L(Z, x) \to L(X, f(x)).$$

First let us consider the case where $H_1(Z)$ is pure of weight $-1$. This is the case when the dual graph of $Z$ is a tree. By a standard strictness argument the weight filtration on $L(Z, x)$ is its lower central series, and the associated graded Lie algebra is generated by $\text{Gr}_{-1}L(Z, x) = H_1(Z, \mathbb{Q})$.

Since

$$L(Z, x) \to L(X, f(x))$$

is a morphism of MHS, it is non-zero if and only if the map

$$\text{Gr} L(Z, x) \to \text{Gr} L(X, f(x))$$

on weight graded quotients is. Since

$$\text{Gr}_{-1} L(X, f(x)) = H_1(X, \mathbb{Q})/W_{-2},$$

and since $H_1(Z, \mathbb{Q}) \to H_1(X, \mathbb{Q})$ is trivial, it follows that $L(Z, x) \to L(X, f(x))$ is trivial.

To prove the general case, we take a partial normalization

$$Z' \to Z$$

with the property that $Z'$ is connected and such that $H^1(Z')$ is a pure MHS of weight 1.

This can be done as follows. Take a maximal tree $T$ in the dual graph of $Z$ and normalize only those double points corresponding to edges not in $T$. Then $H_1(Z)$ is pure MHS of weight -1. The previous argument implies that

$$L(Z', x) \to L(X, f(x))$$
is trivial.

To complete the proof, note that we have an exact sequence
\[ 1 \to N \to \pi_1(Z, x) \to \pi_1(\Gamma, *) \to 1, \]
where \( \Gamma \) denotes the dual graph of \( Z \) and \( N \) is the normal subgroup of \( \pi_1(Z) \) generated by \( \pi_1(Z', x) \).
After passing to unipotent completions, we obtain an exact sequence
\[ 0 \to (L(Z')) \to L(Z, x) \to L(\Gamma, *) \to 0. \]
This is an exact sequence in the category of Malcev Lie algebras with MHS. The ideal \((L(Z'))\) generated by \( L(Z') \) is exactly \( W_{-1}L(Z) \), so the MHS induced on \( L(\Gamma, *) \) is pure of weight 0.

It follows that the homomorphism \( L(Z, x) \to L(\Gamma, x) \) induces a homomorphism \( L(\Gamma, *) \to L(X, f(x)) \).
This is a morphism of MHS of \((0,0)\) type. It is injective if and only if the map \( L(\Gamma, *) \to \text{Gr}_L(\Gamma, x) \to \text{Gr}_L(X, f(x)) \) is also injective. Since \( H_1(X) \) has weights \(<0\) and \( L(\Gamma, *) \) has weight zero, it follows that \( L(\Gamma, *) = \text{Gr}_L(\Gamma, x) \to \text{Gr}_L(X, f(x)) \) is zero. This proves the statement in general. Namely, we have that for any nodal curve (singular, reducible) the map
\[ f_\ast : L(Z, x) \to L(X, f(x)) \]
is trivial if and only if the map
\[ f^\ast : H^1(X, \mathbb{Q}) \to H^1(Z, \mathbb{Q}) \]
is trivial.

\[ \square \]

**Lemma 2.2** Let \( X \) be a smooth projective variety with a nilpotent fundamental group \( \pi_1(X) \). Then for any algebraic curve \( Z \subset X \) the fact \( \text{im}[H^1(Z, \mathbb{Q}) \to H^1(X, \mathbb{Q})] \) is trivial is equivalent to the fact that \( \text{im}[\pi_1(Z) \to \pi_1(X)] \) is finite.

**Proof.** Since we can always find a partial normalization \( \tilde{Z} \to Z \) with \( \tilde{Z} \)-nodal and \( \pi_1(\tilde{Z}, \tilde{x}) \to \pi_1(Z, x) \) surjective it follows from the previous lemma that the map
\[ f_\ast : L(Z, x) \to L(X, f(x)) \]
is the zero map. Furthermore, if \( \pi_1(X) \) is a torsion free nilpotent group then by definition it embeds in \( \pi_{\text{un}}(X, f(x)) \). It is easy to see that torsion elements of a nilpotent group generate a finite group and hence
\[ \pi_1(X, f(x)) \to \tilde{\pi}_{\text{un}}(X, f(x)) \]
is an embedding up to torsion which proves the lemma. \[ \square \]

We have actually proved more:

**Lemma 2.3** Let \( X \) be a smooth projective variety and \( \rho : \pi_1(X) \to L(X, f(x)) \) be the Malcev representation of \( \pi_1(X) \). Then for any algebraic curve \( Z \subset X \) the fact \( \text{im}[H_1(Z, \mathbb{Q}) \to H_1(X, \mathbb{Q})] \) is trivial is equivalent to the fact that \( \text{im}[\pi_1(Z) \to \pi_1(X)/H] \) is finite. Here \( H \) is the kernel of the Malcev representation.
2.2 A reduction to the case of an algebraic curve

In this section we show how to reduce the argument for $Z$ of arbitrary dimension to $Z$ an algebraic curve.

**Lemma 2.4** Let $F$ be a connected subvariety in $X$ then we can find a curve $Z \subset F$ such that $\pi_1(Z)$ surjects on $\pi_1(F)$.

**Proof.** If $F$ is smooth variety the above lemma is just the Lefschetz hyperplane section theorem. Let $F = F_1 + \ldots + F_i$ be singular and with many components of different dimension. Denote by $n$ the normalization $n : F' \rightarrow F$ of $F$. In every component of $F'$ after additional desingularization we can find finitely many points $x_k, y_k$ such that $n(x_k) = n(y_k)$ and $\pi_1(F'/x_k = y_k)$ surjects onto $\pi_1(F)$. The way to do that is to take the Whitney stratification of $F$ and put the points $x_k, y_k$ in every stratum in a way that all loops that come from singularities pass through these points. Now following [K] (ii, 1.1) we take hypersurfaces with big degrees that pass through the points $x_k, y_k$ and intersect every component of $F'$, $F'_i$ in a curve $Z_i$ such that $Z' = \cup Z_i$ and $\pi_1(Z')$ surjects on $\pi_1(F')$. We make $Z = n(Z')$. Observe that $Z$ might be singular and have many components but it will be connected.

Now we are ready to finish the proof of Theorem 1.1. We start with the Stein factorization of the Albanese map for $X$

$$\text{Alb} : X \rightarrow S \rightarrow \text{im}(\text{Alb}) \subset \text{Alb}(X).$$

Denote by $S'$ the fiber product of the universal covering $\widetilde{\text{Alb}}(X)$ of $\text{Alb}(X)$ and $S$ over $\text{Alb}(X)$. By definition the map

$$S' \rightarrow \widetilde{\text{Alb}}(X)$$

is a covering map and since $\widetilde{\text{Alb}}(X)$ is a Stein manifold $S'$ is a Stein manifold as well. It follows from the definition of the Albanese morphism that the fibers of the map

$$\text{Alb} : X \rightarrow S$$

are all subvarieties $F$ in $X$ for which the map $H_1(F, \mathbb{Q}) \rightarrow H_1(X, \mathbb{Q})$ is trivial. We will show that if the fact that $H_1(F, \mathbb{Q}) \rightarrow H_1(X, \mathbb{Q})$ is trivial implies that $\text{im}[\pi_1(F) \rightarrow \pi_1(X)/H]$ is finite.

We have shown this in Lemma 2.3 when $F$ is an algebraic curve. Now if $\text{dim}_{\mathbb{C}}(F) > 1$ we apply Lemma 2.4 to find a curve $Z$ in $F$ such that $\pi_1(Z)$ surjects on $\pi_1(F)$. The argument of Lemma 2.1.2 implies that $\pi_1(F)$ goes to a finite group in $\pi_1(X)/H$ since $\pi_1(Z)$ goes to a finite group in $\pi_1(X)/H$. Observe that the curve $Z$ is also contained in the fiber $F$ of the map

$$\text{Alb} : X \rightarrow S.$$ 

Therefore the map $H_1(Z, \mathbb{Q}) \rightarrow H_1(X, \mathbb{Q})$ is also trivial. To finish the proof of Theorem 1.1 we need to observe that $S$ satisfies the conditions for being the Shafarevich variety of $X$, $S = \text{Sh}^H(X)$. Namely

1) There exists a holomorphic map with connected fibers $X \rightarrow S$, which contracts only the subvarieties $Z$ in $X$ with the property that $\text{im}[\pi_1(Z) \rightarrow \pi_1(X)/H]$ is finite.

2) $\text{Sh}^H(X) = S'$ is a Stein space.

To prove Theorem 1.2 we use the same argument as above but $H$ is a finite group. Actually we have shown more:
Corollary 2.1 Let $X$ be a smooth projective variety with a virtually residually nilpotent fundamental group. Then the Shafarevich conjecture is true for $X$.

2.3 Some examples

In this subsections we give some examples and geometric applications of our method. We start with the following result that was also proved by Campana in [8].

Corollary 2.2 Let $X$ be a smooth projective surface and $\Gamma$ is the image of $\pi_1(X)$ in $L(X, f(x))$. Let as before $S$ be the Stein factorization of the map $X \rightarrow \text{im(Alb}(X))$. After taking an etale finite covering $X'' \rightarrow X$ the homomorphism $\pi_1(X'') \rightarrow \Gamma$ factors through the map $\pi_1(S) \rightarrow \Gamma$.

Proof. According to [25] 4.8 after taking some etale finite covering $X'' \rightarrow X$, $\pi_1(X'')$ is the same as the fundamental group of $\pi_1(S)$. This follows from the fact that residually nilpotent groups are residually finite.

Nilpotent Kähler groups were constructed by Sommese and Van de Ven [33], and Campana [8]. The construction goes as follows:

Start with a finite morphism from an abelian variety $A$ to $\mathbb{P}^n$. Now take the preimage $X$ in $A$ of any abelian d-fold in $\mathbb{P}^n$. A double cover of $X$ has as fundamental group a nonsplit central extension of an abelian group by $\mathbb{Z}$.

Let us following [33] give more explicit example. We start with a four dimensional abelian variety $A$ and a finite morphism $f$ to $\mathbb{P}^4$. Take the Mumford-Horrocks abelian surface $Z$ in $\mathbb{P}^4$ and pull it back to $A$. Let us call the new surface $f^{-1}(Z)$. The following exact sequence was established in [33]

$$\pi_2(A) \oplus \pi_2(Z) \rightarrow \pi_2(\mathbb{P}^4) \rightarrow \pi_1(f^{-1}(Z)) \rightarrow \pi_1(A) \oplus \pi_1(Z) \rightarrow 0.$$ 

In our case this sequence reads as:

$$0 \rightarrow \mathbb{Z} \rightarrow \pi_1(f^{-1}(Z)) \rightarrow \mathbb{Z}^{12} \rightarrow 0$$

and shows that $f^{-1}(Z)$ has a two steps nilpotent fundamental group.

Actually we know more. By theorem of Arapura and Nori [4] all Kähler linear solvable groups are virtually nilpotent. So we have the following:

Corollary 2.3 Let $X$ be a smooth projective variety with a linear solvable fundamental group. Then the universal covering $\tilde{X}$ is holomorphically convex.

Now we will use the technique from Lemmas 2.3 and 2.4 to show that the theorem of Arapura and Nori [4] is the marginal statement meaning that there exists a residually solvable linear group which does not embed in its Malcev completion. By residually solvable we mean a group that embeds in its completion with respect to all finitely generated solvable representations.

The following example came out from a discussion with D. Arapura, János Kollár, M. Nori, T. Pantev, M. Ramachandran and D. Toledo.

Consider nontrivial smooth family of smooth abelian varieties of dimension $N$ over curve $C$. Let us denote this family by $X$. The fundamental group of $X$ is given by the following exact sequence

$$0 \rightarrow \mathbb{Z}^{2N} \rightarrow \pi_1(X) \rightarrow \pi_1(C) \rightarrow 0.$$
The group $\pi_1(X)$ is a semidirect product of the groups $\mathbb{Z}^{2N}$ and $\pi_1(C)$. For generic enough family we can make the monodromy action

$$M : \pi_1(C) \longrightarrow Sp(2N, \mathbb{Z})$$

to be irreducible and from here one can get that the image of $\mathbb{Z}^{2N}$ in $H^1(X, \mathbb{Z})$ is trivial.

The group $\pi_1(X)$ is linear. To see that we consider the morphism

$$l : \pi_1(X) \longrightarrow SL(2, \mathbb{C}) \times [GL(V) \ltimes V].$$

Here $V$ is a vector space over $\mathbb{C}$ of dimension $N$ on which $\mathbb{Z}^{2N}$ acts discretely. It is easy to see that $L$ is an injection.

The group $\pi_1(X)$ is also virtually residually solvable. This can be seen as follows:

Choose a prime number $p$. Since the group $\pi_1(X)$ is linear, namely it embeds in $GL(T)$ for some vector space $T$ we can embed it up to a finite index in a series of finite solvable groups $GL(T_pq)$ for $q = 1, 2, \ldots$.

From another point $\pi_1(X)$ does not embed in its Malcev completion up to a finite index. Assume that $\pi_1(X)$ does embed in its Malcev completion. Then lemma 2.3 and lemma 2.4 imply that if the image of $\mathbb{Z}^{2N}$ in $H^1(X, \mathbb{Z})$ is trivial then the image of $\mathbb{Z}^{2N}$ in the Malcev completion of $\pi_1(X)$ is trivial which is not the case in our situation. Therefore our technique does not answer the question if the universal coverings of $X$ or of generic hyperplane sections of it is holomorphically convex. Of course this is true and can be seen as follows:

**Proposition 2.1** The universal covering of any smooth family of Abelian varieties or of any generic hyperplane section of them is holomorphically convex.

**Proof.** It follows from [24], Theorem 6.3 that every smooth family of abelian varieties over a curve has a linear fundamental group since according to 6.3 [24] after a finite etale covering it is birational to a family of a smooth abelian varieties. But the universal covering of a family of smooth abelian varieties or a generic hyperplane section of it is holomorphically convex since it is a Stein space since. It embeds in $(SIEG \times C^N)$, where $SIEG$ is the Siegel upperhalf plane.

It also follows from [21] where more powerful technique, the theory of Nonabelian Mixed Hodge Structures, is used.

We formulate:

**Corollary 2.4** Let $X$ be a smooth projective variety with an infinite virtually nilpotent fundamental group and such that $\text{rank Pic}(X) = 1$ (or better $\text{rank NS}(X) = 1$). Then for every subvariety $Z$ in $X$ we have that $\text{im}[\pi_1(Z) \longrightarrow \pi_1(X)]$ is infinite.

The proof is an easy consequence of [25] (Chapter 1).

We demonstrate a quick application of the above corollary. Denote by $A$ a four dimensional abelian variety. Let us say that $X$ is a hypersurface in it with an isolated singular point $s$ of the following type $xy = zt$. Denote by $X'$ the blow of $X$ in $s$. We glue in $X \mathbb{P}^1 \times \mathbb{P}^1$ instead of $s$. It is easy to see that $X'$ is smooth and that $\mathbb{P}^1$ can be blown down. The new space $X''$, obtained after blowing down one of the above $\mathbb{P}^1$, is smooth, $\text{rank } NS(X'') = 1$ and $\pi_1(X) = \mathbb{Z}^4$. Therefore the Shafarevich conjecture should be true for $X''$. But as we can see $X''$ contains $\mathbb{P}^1$. This contradicts
the above corollary and we conclude that $X''$ is not projective. Of course all this can be seen in many different ways. This is another spelling of the fact that $\mathbb{P}^1$, that remains in $X''$, should be homologically nontrivial if $X''$ is projective.

We give now an idea of an alternative proof of Theorem 1.1 which came from conversations with M. Ramachandran. It is based on the use of $\pi_1(X)$ equivariant harmonic maps to the universal coverings to Higher Albanese varieties defined in [18]. Combined with the strictness property for the nonabelian Hodge theory this seems to be a very promising idea (see [21]).

Denote by $G_s$ the complex simply connected group $\pi_1(X)/\Gamma^{s+1}$, where $\Gamma^i$ are the groups from the lower central series for $\pi_1(X)$ and $\Gamma^s$ is the smallest nontrivial one. The corresponding Lie algebra $g_s$ has MHS. Denote by $F^0G_s$ the closed subgroup in $G_s$ group that corresponds to $F^0g_s$. Since the group $\pi_1(X)/\Gamma^{s+1}$ is unipotent then as it is easy to see we have a free action of the corresponding to $G_s$ lattice $G_s(\mathbb{Z})$ on $G_s/F^0G_s$.

Therefore in the same way as in [22] we obtain a $\pi_1(X)$ equivariant proper horizontal holomorphic map (see [18])

$$\tilde{X} \to G_s/F^0G_s.$$

According to [17] $G_s/F^0G_s$ is biholomorphic to $\mathbb{C}^N$. Therefore $\tilde{X}$ is holomorphically convex.

**Remark 2.1** The above argument is weaker then the argument we have used in the first proof. It cannot be generalized to the case of residually nilpotent groups since in this case $G_s/F^0G_s$ will not be a manifold.

### 3 Solvable coverings

We would like to obtain the analog of Theorem 1.1 for solvable coverings. The analog of Theorem 1.2 for solvable groups - Corollary 2.3 was proved in the previous section as a consequence of the result of Arapura and Nori. We cannot prove solvable analog of theorem 1.1. The maximum we can do is to realize how close the solvable representations come to nilpotent ones. To be able to do so we need to generalize slightly the result of Arapura and Nori.

First we prove Theorem 1.3.

**Proof.** (The idea of the proof was suggested to me by T. Pantev.) Denote by $\Gamma$ the image of the solvable representation $\rho : \pi_1(X) \to L$. We need to show that either $\Gamma$ is virtually nilpotent or there exists a holomorphic map with connected fibers $f : X \to C$ to a smooth curve $C$ of genus $\geq 1$.

First we introduce some notations. For a finitely generated group $\Gamma$ denote by $\Sigma(\Gamma)$ the set of all special characters of $\Gamma$. That is

$$\Sigma(\Gamma) := \{ \alpha : \Gamma \to \mathbb{C}^\times \mid H^1(\Gamma, \mathbb{C}_\alpha) \neq 0 \},$$

where $\mathbb{C}_\alpha$ is the one dimensional $\Gamma$-module associated to $\alpha$. Now we recall the following:

**Proposition 3.1 (Arapura-Nori [2])** Let $\Gamma$ be a finitely generated $\mathbb{Q}$-linear solvable group. Then the following are equivalent

1. $\Gamma$ is virtually nilpotent.
2. $\Sigma(\Gamma)$ consists of finitely many torsion characters.
Due to this proposition it is enough to show that either $\Sigma(\Gamma)$ consists of finitely many torsion characters or $X$ has a non-trivial map to a curve of genus bigger than zero. Now, since $\pi_1(X)$ surjects on $\Gamma$ it follows that $\Sigma(\Gamma) \subset \Sigma(\pi_1(X))$ and hence it suffices to show that either $\Sigma(\pi_1(X))$ consists of finitely many torsion characters or $X$ has an irrational pencil.

For a smooth projective variety $X$ denote by $M(X)$ the moduli space of homomorphisms from $\pi_1(X)$ to $\mathbb{C}^\times$. The locus of special characters is a jump locus in $M(X)$ and hence it is a subscheme in a natural way. It turns out that $\Sigma(\pi_1(X))$ is actually a smooth subvariety having very special geometric properties which we are going to exploit. Since the subvariety $\Sigma(\pi_1(X)) \subset M(X)$ is completely canonical one expects it to have an intrinsic description. One way to construct natural subvarieties in $M(X)$ is via pullbacks. Namely, given any surjective morphism $\varphi : X \to Y$ we can pullback the moduli space of characters of $\pi_1(Y)$ to get a subvariety $\varphi^*M(Y) \subset M(X)$. According to \[27\], Lemma 2.1 and Theorem 6.1 every connected component $\Sigma$ of the subvariety $\Sigma(\pi_1(X)) \subset M(X)$ is of this kind. More specifically for every such $\Sigma$ there exists a torsion character $\sigma \in \Sigma$ and a connected abelian subvariety $P \subset \text{Alb}(X)$ so that $\Sigma$ is the translation of $\varphi^*\text{Alb}(X)/P \subset M(X)$ by $\sigma$. Here $\varphi : X \to \text{Alb}(X) \to \text{Alb}(X)/P$ is the composition of the Albanese map and the natural quotient morphism. In particular, $\Sigma(\pi_1(X))$ has a positive dimensional component if and only if its intersection with the set of all unitary characters has a positive dimensional component. Now the Hodge decomposition of the cohomology of a unitary local system implies that unless $\Sigma(\pi_1(X))$ consists of finitely many torsion characters the subvariety of all special line bundles in $\text{Pic}^\sigma(X)$ has a positive dimensional component. Indeed, for a unitary character $\alpha$ denote by $L_\alpha$ the corresponding rank one local system and by $L_\alpha = L_\alpha \otimes \mathcal{O}_X$ the corresponding holomorphic line bundle. Now by the Hodge theorem

$$h^1(\pi_1(X), \mathbb{C}_\alpha) = h^1(X, L_\alpha) = h^1(X, L_\alpha) + h^0(X, \Omega^1_X \otimes L_\alpha) = 2h^1(X, L_\alpha),$$

i.e. $\alpha$ is a special character iff the line bundle $L_\alpha$ is special.

Furthermore a theorem of Beauville (\[3\], Proposition 1) asserts that the subvariety of $\text{Pic}^0(X)$ consisting of special line bundles is a union of a finite set and the subvarieties of the form $f^*\text{Pic}^0(B)$ where $f : X \to B$ is a morphism with connected fibers to a curve $B$ of genus $\geq 1$. Thus $X$ possesses irrational pencils which finishes the proof of Theorem 1.3.

The above theorem can be seen as the solvable analog of the theorem of Simpson’s that $\text{SL}(n, \mathbb{Z})$ is not a Kähler group, $n > 2$. This theorem gives a way of constructing new examples of non-Kähler groups. In particular any group $\Gamma$ with infinite $H^1(\Gamma, \mathbb{Q})$ possessing a solvable linear quotient defined over $\mathbb{Q}$ that is not virtually nilpotent cannot be Kähler.

Now we prove Theorem 1.4.

**Proof.** We would like to use theorem 3.1. Therefore we need an infinite solvable representation defined over $\mathbb{Q}$. First we show

**Lemma 3.1** Let $S$ be a connected affine solvable group defined over $\mathbb{Q}$. If $\rho : \pi_1(X) \to S(\mathbb{C})$ is a Zariski dense representation into the group of complex points of $S$, then $\rho$ can be deformed to a representation $\nu : \pi_1(X) \to S(\overline{\mathbb{Q}})$ having an infinite image.

**Proof.** Denote by $\Lambda$ the image of $\rho : \pi_1(X) \to S$. Let $\phi : \Lambda \to S(\mathbb{C})$ be a homomorphism from $\Lambda$ to the group of complex points of $S$ with an infinite image. We want to find a subgroup $\Lambda'$ of $\Lambda$ and
of finite index and a homomorphism \( \phi' : \Lambda' \to S(\mathbb{Q}) \) with infinite image, such that \( \phi' \) is arbitrarily close to \( \phi \).

Let \( S_0 := S \), and consider the (upper) derived series \( S_{i+1} := [S_i, S_i] \) for \( S \). We choose the maximal \( i \) such that \( \Lambda \cap S_i(\mathbb{C}) \) is of finite index of \( \Lambda \). We replace \( G \) by \( \Lambda' = \Lambda \cap S_i(\mathbb{C}) \). The image of \( \Lambda' \) in \( S_i(\mathbb{C})/S_{i+1}(\mathbb{C}) \) is infinite. The group \( A = S_i/S_{i+1} \) is either a torus or a vector space group. Let \( X \) be the affine variety of homomorphisms \( \Lambda' \to A \), \( Y \) the affine variety \( \text{Hom}(\Lambda', S_i) \), \( X' \) the image of \( Y \) in \( X \). Thus \( X' \) is the affine subvariety of \( X \) consisting of homomorphisms which factor through \( S_{i+1} \). We want to find points on \( X'(\mathbb{Q}) \) arbitrarily close in \( X'(\mathbb{C}) \) to the point defined by \( \phi \).

If the original point is defined over \( \overline{\mathbb{Q}} \), we are done. If not, the original point cannot be isolated, since \( X' \) is defined over \( \overline{\mathbb{Q}} \). Thus we have arbitrarily close points defined over \( \overline{\mathbb{Q}} \). To find a \( \overline{\mathbb{Q}} \) point with an infinite image we consider two cases:

1) \( A \) is a vector space group. Let \( g \) be an element of \( \Lambda' \) such that \( \phi(g) \) maps to a non-trivial element of \( A \). Then for every nearby representation \( \phi' \) the element \( \phi'(g) \) is non-trivial in \( A \), therefore it is of infinite order in \( A \).

2) \( A \) is a torus \( T \). We fix an element \( g \) in \( \Lambda' \) such that \( \phi(g) \) maps to a point of infinite order on \( T \). Let \( Z \subset T \) denote the image of \( X' \) in \( T \) under the map which takes each homomorphism \( \Lambda' \to T \) to the image of \( g \) in \( T \). By definition \( X' \to Z \) is a surjective map and it is defined over \( \overline{\mathbb{Q}} \), so every \( \overline{\mathbb{Q}} \)-point of \( Z \) comes from a \( \overline{\mathbb{Q}} \)-point of \( X' \) and therefore from a \( \overline{\mathbb{Q}} \)-point of \( Y \), i.e. an actual \( \overline{\mathbb{Q}} \)-homomorphism from \( \Lambda' \) to \( S_i \). So it is enough to find a \( \overline{\mathbb{Q}} \)-point of \( Z \) which is close to the image of the original homomorphism \( \phi \) but which is also of infinite order.

We prove the following:

**Claim 3.1** Let \( T \) be a torus, \( Z \) a \( \overline{\mathbb{Q}} \)-affine subvariety of \( T \), \( p \) a point in \( Z(\mathbb{Q}) \) of infinite order. Then \( p \) is in the closure of the subset of \( Z(\overline{\mathbb{Q}}) \) consisting of points of infinite order.

**Proof.** If \( p \) is in \( Z(\overline{\mathbb{Q}}) \), we are done. If not, there exists a character \( \chi : Z \to GL(1, \mathbb{C}) \) such that \( \chi(p) \) is not in \( \overline{\mathbb{Q}} \). As \( Z \) and \( \chi \) are defined over \( \overline{\mathbb{Q}} \), and \( GL(1, \mathbb{C}) \) is 1-dimensional, it follows that \( \chi(Z) \) is an open subset of \( GL(1, \mathbb{C}) \). In particular, every neighborhood of \( p \) in \( Z \) maps to a neighborhood of \( \chi(p) \) containing non-torsion elements. If \( q \in Z(\overline{\mathbb{Q}}) \) maps to a non-torsion element, then of course \( q \) is a point of infinite order in \( T(\mathbb{C}) \). This finishes the proof of the claim and the lemma.

To get an infinite solvable representation defined over \( \mathbb{Q} \) consider the affine solvable group \( \tilde{S} \) obtained from \( S \) by restriction of scalars, i.e. \( \tilde{S} := \text{res}_{\mathbb{Q}/\mathbb{Q}} S \). The representation \( \nu \) induces a representation \( \tilde{\nu} : \pi_1(X) \to \tilde{S}(\mathbb{Q}) \) which has an image isomorphic to the image of \( \nu \).

Now to prove Corollary 1.1 we have to consider the following two alternatives

1) The group \( \tilde{\nu}(\pi_1(X)) \) is virtually nilpotent so it has a subgroup of finite index which is nilpotent.

2) There exists an holomorphic map with connected fibers \( f : X \to C \) to a curve of genus one or higher. But then we know that \( \pi_1(C) \) embeds in its Malcev completion.

In both cases there exists a finite étale cover of \( X \) which has a non-trivial Albanese variety, which is what we need.

Theorem 1.5 follows easily from Theorem 1.1 and the result from [22].

**Proof.** Let us start with a complex linear representation \( \text{im}[\pi_1(X) \to L] \). Then we have the following three possibilities.
(a) The image of $\pi_1(X)$ in $L/R^uL$ does not have zero or two ends. Then we can apply [22] to conclude that $\pi_1(X)$ has a non-constant holomorphic function.

(b) The image $\text{im}[\pi_1(X) \to L/R^uL]$ has two ends. Then by the theorem of Hopf and Freudenthal it follows that $\text{im}[\pi_1(X) \to L/R^uL]$ has a subgroup of finite index that is isomorphic to $\mathbb{Z}$. Therefore the abelianization of $\pi_1(X)$ is not finite. This implies that the Malcev representation is not trivial and we apply Theorem 1.1 to finish the proof.

(c) The image of $\pi_1(X)$ in $L/R^uL$ has zero ends. Then $L/R^uL$ is a finite group. So the Malcev representation is not trivial and we are taken applying Theorem 1.1.

Theorem 1.5 follows from [21] as well. To be able to attack conjecture 1.1 we should be able to analyze the real issue, the semisimple representations. Some initial steps in this direction are done in [21].

What should we do if the answer of Question 3 is positive? We hope using [23] to be able to handle the case when the image of $\pi_1(X)$ in its proalgebraic completion is infinite.

What should we do if the answer of Question 2 is positive and the $\pi_1(X)$ in question has finite image in its proalgebraic completion. At the moment this case seems to be out of reach.

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