Classification of integro-differential $C^*$-algebras

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Abstract
The integro-differential algebra $\mathcal{F}_{N,M}$ is the $C^*$-algebra generated by the following operators acting on $L^2([0,1)^N \to \mathbb{C}^M)$: 1) operators of multiplication by bounded matrix-valued functions, 2) finite differential operators, 3) integral operators. We give a complete characterization of $\mathcal{F}_{N,M}$ in terms of its Bratteli diagram. In particular, we show that $\mathcal{F}_{N,M}$ does not depend on $M$ but depends on $N$. At the same time, it is known that differential algebras $\mathcal{H}_{N,M}$, generated by the operators 1) and 2), do not depend on both dimensions $N$ and $M$, they are all $*$-isomorphic to the universal UHF-algebra. We explicitly compute the Glimm-Bratteli symbols (for $\mathcal{H}_{N,M}$ it was already computed earlier)
$$n(\mathcal{F}_{N,M}) = \prod_{n=1}^{\infty} \left( \frac{n}{n-1} \right)^N \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad n(\mathcal{H}_{N,M}) = \prod_{n=1}^{\infty} n,$$
which characterize completely the corresponding AF-algebras.

Keywords: representation of integro-differential operators

1. Introduction
Discrete and continuous analogues of integro-differential algebras are actively used in various applications, for example, in the development of computer algorithms for symbolic and numerical solving of integro-differential equations, see, e.g., [1, 2, 3, 4]. On the other hand, differential algebras are closely related to the rotation $C^*$-algebras well studied in, e.g., [5, 6, 7, 8]. In contrast to the rotation algebras, the integro-differential algebras contain operators of multiplication by discontinuous functions and integral operators. Nevertheless, the integro-differential algebras are AF-algebras and, hence, they admit a classification in terms of, e.g., the Bratteli diagrams.

2. Characterization of AF-algebras. Preliminary results.
Let us recall some facts about Bratteli diagrams. It is well known that any finite-dimensional $C^*$-algebra is $*$-isomorphic the direct sum of simple matrix algebras. Up to the order of terms, this direct sum is determined uniquely. It is convenient to use the following notation for finite-dimensional $C^*$-algebras. Let $p = (p_j)_{j=1}^n \in \mathbb{N}^n$, then
$$\mathcal{M}(p) = \mathbb{C}^{p_1 \times p_1} \oplus \ldots \oplus \mathbb{C}^{p_n \times p_n}. \quad (1)$$
Any $*$-homomorphism from $\mathcal{M}(p)$ to $\mathcal{M}(q)$ with $p \in \mathbb{N}^n$, $q \in \mathbb{N}^m$ is internally (inside each $\mathbb{C}^{q_j \times q_j}$) unitary equivalent to some canonical $*$-homomorphism. Any canonical $*$-homomorphism is completely and uniquely determined by the matrix of multiplicities of partial embeddings $E \in \mathbb{Z}^{m \times n}_+$ (E-matrix) satisfying $E(p_j)_{j=1}^n = (\tilde{q}_j)_{j=1}^m$, where $\tilde{q}_j \leq q_j$. For example, the canonical $*$-homomorphism

$$\varphi : \mathcal{M}(2, 2, 3) \to \mathcal{M}(4, 4), \quad A \oplus B \oplus C \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \oplus (0)$$

has the E-matrix

$$E = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

For simplicity, we can write

$$\mathcal{M}(2, 2, 3) \xrightarrow{E} \mathcal{M}(4, 4).$$

If a canonical $*$-homomorphism is unital then there are no zero rows and columns in E-matrix, and we should replace the above mentioned condition $\tilde{q}_j \leq q_j$ with $\tilde{q}_j = q_j$. For example, the unital embedding

$$\mathcal{M}(2, 2, 3) \xrightarrow{E} \mathcal{M}(4, 5), \quad E = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

has the form

$$A \oplus B \oplus C \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \oplus \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}. $$

The AF-algebra is a separable $C^*$-algebra, any finite subset of which can be approximated by a finite-dimensional $C^*$-sub-algebra. For convenience, we will consider unital AF-algebras only. This is not a restriction because the unitalization of an AF-algebra is obviously an AF-algebra. It is well known, that for any unital AF-algebra $\mathcal{A}$ there is a family of nested finite-dimensional $C^*$-subalgebras $\mathcal{A}_n \subseteq \mathcal{A}$, satisfying

$$\mathbb{C}^{1 \times 1} \cong \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq ... \quad \mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n. \quad (2)$$

Since $\mathcal{A}_n$ are nested finite-dimensional $C^*$-algebras, they are isomorphic to some canonical algebras $\mathcal{M}(p_n)$, where $p_n \in \mathbb{N}^{M_n}$, $M_n \in \mathbb{N}$, and the inclusions (2) can be written as

$$\mathcal{M}(p_0) \xrightarrow{E_0} \mathcal{M}(p_1) \xrightarrow{E_1} \mathcal{M}(p_2) \xrightarrow{E_2} ... \quad (3)$$

where $p_0 = 1$, $M_0 = 1$, and $E_n \in \mathbb{Z}^{M_{n+1} \times M_n}$. Moreover, due to the unital embeddings $\mathcal{M}(p_n) \subseteq \mathcal{M}(p_{n+1})$, all E-matrices have no zero rows and columns and they satisfy $E_n p_n = p_{n+1}$. Because $p_0 = 1$, we obtain

$$p_{n+1} = E_n \ldots E_1 E_0 = \prod_{i=0}^{n} E_n. \quad (4)$$
We will always assume the right-to-left order in the product $\prod$. Using (3)-(4), we conclude that the matrices $\{E_n\}_{n=0}^{+\infty}$ determine completely the structure of the unital AF-algebra $\mathcal{A}$. It is useful to note that the choice of E-matrices is not unique. For example, E-matrices $\{E'_n\}_{n=0}^{+\infty}$, where $E'_n = E_{2n+1}E_{2n}$, determine the same algebra $\mathcal{A}$. This is because the composition of two embeddings has the E-matrix equivalent to the product of E-matrices corresponding to the embeddings. It is possible to describe the class of all E-matrices determining the same unital AF-algebra.

**Definition 2.1.** Let $E$ be the set of sequences of matrices $\{E_n\}_{n=0}^{+\infty}$, where $E_n \in \mathbb{Z}_+^{M_{n+1} \times M_n}$ have no zero rows and columns, and $M_0 = 1$, $M_n \in \mathbb{N}$ are some positive integer numbers. Let us define the equivalence relation on $E$. Two sequences $\{A_n\}_{n=0}^{+\infty} \sim \{B_n\}_{n=0}^{+\infty}$ are equivalent if there is $\{C_n\}_{n=0}^{+\infty} \in E$ such that

$$
C_0 = \prod_{i=0}^{r_1-1} A_i, \quad C_{2n+1}C_{2n-2} = \prod_{i=m_n-1}^{m_n-1} B_i, \quad C_{2n}C_{2n-1} = \prod_{i=r_n}^{r_n+1-1} A_i, \quad n \geq 1, \quad (5)
$$

where $0 = r_0 < r_1 < r_2 < ...$ and $0 = m_0 < m_1 < m_2 < ...$ are some monotonic sequences of integer numbers. The corresponding set of equivalence classes is denoted by $E := E / \sim$.

It is convenient to denote the equivalence classes as

$$
\{E_n\}_{n=0}^{+\infty} = \prod_{n=0}^{+\infty} E_n
$$

because, see (5),

$$
\prod_{n=0}^{+\infty} A_n = \prod_{n=0}^{+\infty} \prod_{i=r_n}^{r_n+1-1} A_i = \prod_{n=0}^{+\infty} C_n = \prod_{n=0}^{+\infty} \prod_{i=m_n}^{m_n+1-1} B_i = \prod_{n=0}^{+\infty} B_n.
$$

In other words, we can perform the standard manipulations in the product of matrices without leaving the equivalence class. Of course, the manipulations should not go beyond $E$, i.e. all the resulting matrices should have non-negative integer entries and should not have zero rows and zero columns.

Let $\mathcal{A}$ be some unital AF-algebra. Following (2)-(4), there is a Bratteli diagram $\{E_n\}_{n=0}^{+\infty}$ represented $\mathcal{A}$. Let us define the mapping

$$
n : \mathcal{A} \mapsto \prod_{n=0}^{+\infty} E_n. \quad (6)
$$

Because the Bratteli diagram is not unique, the correctness of the mapping $n$ should be checked. It is already done in the main structure theorem for Bratteli diagrams.

**Theorem 2.2.** i) The relation $\sim$ defined in (5) is the equivalence relation. ii) Let $\mathfrak{A}$ be the set of classes of non-isomorphic unital AF-algebras. Then $n : \mathfrak{A} \to E$ is 1-1 mapping.
Note that the inverse mapping \( n^{-1} \) has a more explicit form than \( n \). For example, 
\[
\mathbf{E}_n = n^{-1}(\prod_{n=0}^{\infty} \mathbf{E}_n) \quad \text{is the } C^*\text{-algebra } \mathcal{A} \text{ given by the inductive limit (3)}.
\]

The proof of Theorem 2.2 follows from the similar results formulated for the graphical representations of Bratteli diagrams, see, e.g., [9, 10], and Theorem 3.4.4 in [11]. The equivalence relation \( \sim \) defined in (5) is the analogue of telescopic transformations of Bratteli diagrams. While the Bratteli diagram is not unique, it provides a kind of classification tool. Other types of classification of AF algebras, including the efficient K-theoretic Elliott classification, are discussed in [12–15]. The infinite product \( n(\mathcal{A}) \) representing the Bratteli diagram for AF-algebra \( \mathcal{A} \) can be called as the Glimm-Bratteli symbol. Using supernatural symbols (numbers), when \( E_n \) are natural numbers in (6), J. Glimm provides the classification of uniformly hyper-finite algebras in [16].

Let us consider some examples of AF-algebras.

1. **Compact operators.** Let \( \mathcal{K} \) be the \( C^* \)-algebra of compact operators acting on a separable Hilbert space. Let \( \mathcal{K}_1 = \text{Alg}(\mathcal{K}, 1) \) be its unitalization. It is well known that the Bratteli diagram for \( \mathcal{K}_1 \) is

   ![Diagram](image)

   where the nodes represent simple matrix sub-algebras, and the edges show the multiplicity of embedding: one line means the multiplicity equal to 1. The first node is always \( \mathbb{C}^{1\times1} \). The dimensions of nodes are determined by the dimensions of nodes connected on the left and by the multiplicities of embedding. The corresponding Glimm-Bratteli symbol is

\[
n(\mathcal{K}_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot (1 \ 1).
\]

Combining terms in the infinite product, we can write the another form of the Glimm-Bratteli symbol

\[
n(\mathcal{K}_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \left( \prod_{n=1}^{\infty} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \left( \prod_{n=1}^{\infty} \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

which leads to the labeled Bratteli diagram

![Labeled Diagram](image)

In the labeled Bratteli diagram, the edge numbers are the multiplicities of embedding. The multiplicity 1 is usually omitted.

2. **CAR algebra.** For the CAR algebra \( \mathcal{C} \), which is a UHF-algebra, we have the Glimm-Bratteli symbol \( n(\mathcal{C}) = 2^\infty \). At the same time,

\[
n(\mathcal{C}) = 2^\infty = \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \cdot \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \cdot \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \cdot \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \cdot \ldots
\]

The corresponding Bratteli diagrams are as follows

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3. Direct sum of AF algebras. Above, we already used the notation $\oplus$ for the direct sum of matrix algebras and for their elements. We will use the same symbol in a little bit different context, namely for the direct sum of not necessarily square E-matrices. Suppose that $C = A \oplus B$ is the standard direct sum of two AF-algebras. If $\mathfrak{n}(A) = \prod_{n=0}^{\infty} A_n$ and $\mathfrak{n}(B) = \prod_{n=0}^{\infty} B_n$ then it can be shown that

$$\mathfrak{n}(C) = \left( \prod_{n=0}^{\infty} C_n \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{where } C_n = A_n \oplus B_n = \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix}.$$

4. Tensor product of AF-algebras. It is useful to note the following property of the tensor product

$$\mathcal{M}(p) \otimes \mathcal{M}(q) = \mathcal{M}(p \otimes q),$$

where

$$p = (p_i)_{i=1}^{n} \in \mathbb{N}^n, \quad q = (q_j)_{j=1}^{m} \in \mathbb{N}^m, \quad p \otimes q = (p_i q_j)_{i,j=1}^{n,m} \in \mathbb{N}^{nm}.$$

Moreover, it is easy to check that if

$$\mathcal{M}(p_1) \xrightarrow{E_1} \mathcal{M}(q_1), \quad \mathcal{M}(p_2) \xrightarrow{E_2} \mathcal{M}(q_2)$$

then

$$\mathcal{M}(p_1 \otimes p_2) \xrightarrow{E_1 \otimes E_2} \mathcal{M}(q_1 \otimes q_2),$$

where the tensor product of matrices is defined in the standard way

$$(A_{i,j}) \otimes (B_{r,s}) = (C_{(i,r),(j,s)}), \quad C_{(i,r),(j,s)} = A_{i,j} B_{r,s}.$$

Hence, the standard tensor product $C = A \otimes B$ of two AF-algebras is AF-algebra which satisfies

$$\mathfrak{n}(C) = \prod_{n=0}^{\infty} A_n \otimes B_n,$$

where $\mathfrak{n}(A) = \prod_{n=0}^{\infty} A_n, \mathfrak{n}(B) = \prod_{n=0}^{\infty} B_n,$ and the tensor product of matrices is then

$$A \otimes B = \begin{pmatrix} b_{11}A & \cdots & b_{1N}A \\ \vdots & \ddots & \vdots \\ b_{M1}A & \cdots & b_{MN}A \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1N} \\ \vdots & \ddots & \vdots \\ b_{M1} & \cdots & b_{MN} \end{pmatrix}.$$

We will also use the following result.

**Theorem 2.3.** Let $\{A_n\}_{n=1}^{\infty}$ be a commutative (multiplicative) semigroup of square matrices with non-negative integer entries and with non-zero determinants. Let $A_0$ be a matrix-column with positive integer entries such that $A_1 A_0$ is defined. Let $\{B_n\}_{n=1}^{\infty} \subset \{A_n\}_{n=1}^{\infty}$ be
a subset consisting of not necessarily different matrices satisfying the condition (\(\sigma\)) for any \(p \in \mathbb{N}\) there are \(r, s \in \mathbb{N}\) such that \(A_p A_r = \prod_{i=1}^s B_i\). Then

\[
n^{-1}((\prod_{n=1}^\infty B_n)A_0) \cong n^{-1}((\prod_{n=1}^\infty A_n)A_0). \tag{7}
\]

Even if (\(\sigma\)) is not fulfilled, LHS in (7) is a sub-algebra of RHS.

**Remark.** The universal UHF-algebra \(\mathcal{U}\) is the AF-algebra generated by the multiplicative semigroup of natural numbers

\[
\mathcal{U} = n^{-1}(\prod_{n=1}^{+\infty} n) = n^{-1}(\prod_{n=1}^{+\infty} p_n^{\infty}) = n^{-1}(\prod_{n=1}^{+\infty} (p_1...p_n)^n) = n^{-1}(\prod_{n=1}^{+\infty} (p_1...p_n)),
\]

where \(p_1 = 2, p_2 = 3, p_3 = 5, \ldots\) are the prime numbers. Any UHF-algebra is a sub-algebra of \(\mathcal{U}\). The CAR-algebra is the UHF-algebra generated by the multiplicative semigroups \(\{2^n : n \in m\mathbb{N}\}\) for any \(m \in \mathbb{N}\).

There is another useful proposition describing non-isomorphic classes of AF-algebras.

**Theorem 2.4.** Let \(N, M \in \mathbb{N}\). Let \(\{A_n\} \subset \mathbb{Z}_+^{N \times N}, \{B_n\} \subset \mathbb{Z}_+^{M \times M}\) be two sequences of matrices having non-zero determinants. Let \(A_0 \in \mathbb{Z}_+^{N \times 1}, B_0 \in \mathbb{Z}_+^{M \times 1}\) be two matrix-columns without zero entries. If \(N \neq M\) then \(n^{-1}(\prod_{n=0}^{\infty} A_n) \not\cong n^{-1}(\prod_{n=0}^{\infty} B_n)\) are non-isomorphic C*-algebras.

Suppose that we have two AF-algebras \(\mathcal{A}_1\) and \(\mathcal{A}_2\) acting on Hilbert spaces \(H_1\) and \(H_2\) respectively. Suppose that \(\mathcal{A}_1 \cong \mathcal{A}_2\) are isomorphic to each other. When can we construct the unitary \(\mathcal{U} : H_1 \to H_2\) inducing the C*-algebra isomorphism, i.e. \(\mathcal{A}_1 \cong \mathcal{U}^{-1}\mathcal{A}_2\mathcal{U}\)?

Let us start with the simple situation \(\mathcal{A}_1 \cong \mathbb{C}^{n \times n}\). Recall that we always consider unital algebras in this paper. Then, there is a decomposition \(H_1 = \bigoplus_{i=1}^N H_{1i}\), where all \(H_{1i}\) have the same dimension. We may think that all \(H_{1i} = H_{11}\) are the same Hilbert space. Any operator \(\mathcal{A}_1 \ni A \simeq (a_{ij})_{i,j=1}^N\) has the form \(\mathcal{A}_1 = (a_{ij}T)_{i,j=1}^N : \bigoplus_{i=1}^N H_{1i} \to \bigoplus_{i=1}^N H_{1i}\), where \(T : H_{11} \to H_{11}\) is the identity operator. We call any decomposition of \(H\) satisfying the explained property as a decomposition associated with the simple algebra \(\mathcal{A}_1\).

Taking a decomposition \(H_2 = \bigoplus_{i=1}^N H_{2i}\) associated with \(\mathcal{A}_2\), we may state: the isomorphism between C*-algebras \(\mathcal{A}_1 \cong \mathcal{A}_2(\cong \mathbb{C}^{n \times n})\) can be induced by a unitary \(\mathcal{U} : H_1 \to H_2\) if and only if \(\dim H_{1i} = \dim H_{2i}\).

Suppose that we have a finite dimensional algebra

\[
\mathcal{M}(p) = \mathbb{C}^{p_1 \times p_1} \oplus \ldots \oplus \mathbb{C}^{p_r \times p_r}, \quad p = (p_i)_{i=1}^N \tag{8}
\]

acting on a separable Hilbert space \(H\). Then there is a decomposition \(H = \bigoplus_{i=1}^N H_i\) such that each direct term in (8) acts on the corresponding \(H_i\). Hence, we can take a decomposition \(H_i = \bigoplus_{j=1}^{p_i} H_{1i}\) associated with the corresponding simple direct term. Let us define \(q = \ldots\)
(q_i)_{i=1}^N$, where $q_i = \dim H_i$. To show the internal structure of the algebra $\mathcal{M}(p)$, we will write $\mathcal{M}(p; q)$. Let $\mathcal{M}(p_1; q_1)$ be some (unital) sub-algebra with the corresponding embedding

$$\mathcal{M}(p_1; q_1) \xrightarrow{E} \mathcal{M}(p; q) \quad (p = E p_1).$$

Then, it is not difficult to check that the dimensions satisfy

$$q^\top E = q_1^\top.$$

Since the dimensions can be infinite, we should specify the rules:

$$a + b = \infty \text{ if } a = \infty \text{ or } b = \infty, \quad \infty \cdot 0 = 0 \cdot \infty = 0.$$

Any (unital) AF-algebra acting on a separable Hilbert space can be represented through its Bratteli diagram

$$\mathcal{M}(1; q_0) \xrightarrow{E_0} \mathcal{M}(p_1; q_1) \xrightarrow{E_1} \mathcal{M}(p_2; q_2) \xrightarrow{E_2} \ldots,$$

where the dimensions satisfy

$$p_n = E_{n-1} \ldots E_0, \quad q_{n-1}^\top = q_n^\top E_{n-1}, \quad n \geq 1. \quad (10)$$

In order to include the dimensions $q_n$, let us extend Definition 2.1:

**Definition 2.5.** Let $\mathfrak{F}$ be the set of sequences of matrices and dimensions $\{E_n, q_n\}_{n=0}^\infty$, where $E_n \in \mathbb{Z}_M^{M_{n+1} \times M_n}$ have no zero rows and columns, and $M_0 = 1$, $M_n \in \mathbb{N}$ are some positive integer numbers. The dimensions $q_n \in (\mathbb{N} \cup \infty)^M$ satisfy

$$q_{n-1}^\top = q_n^\top E_{n-1}, \quad n \geq 1. \quad (11)$$

Let us define the equivalence relation on $\mathfrak{F}$. Two sequences $\{A_n, a_n\}_{n=0}^\infty \sim \{B_n, b_n\}_{n=0}^\infty$ are equivalent if there is $\{C_n, c_n\}_{n=0}^\infty \in \mathfrak{F}$ such that

$$C_0 = \prod_{i=0}^{r_1-1} A_i, \quad C_{2n-1} C_{2n-2} = \prod_{i=m_{n-1}}^{m_n-1} B_i, \quad C_{2n} C_{2n-1} = \prod_{i=r_n}^{r_{n+1}-1} A_i, \quad n \geq 1 \quad (12)$$

and

$$c_0 = a_0 = b_0, \quad c_{2n-1} = a_{r_n}, \quad c_{2n} = b_{m_n}, \quad n \geq 1, \quad (13)$$

where $0 = r_0 < r_1 < r_2 < \ldots$ and $0 = m_0 < m_1 < m_2 < \ldots$ are some monotonic sequences of integer numbers. The corresponding set of equivalence classes is denoted by $\mathfrak{F} : = \mathfrak{F}/\sim$.

Now, we can complement the results of Theorem 2.2.

**Theorem 2.6.** Let $H$ be a separable Hilbert space. We call two $C^*$-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ equivalent iff there is a unitary $U : H \to H$ such that $\mathfrak{A}_1 = U \mathfrak{A}_2 \mathfrak{A}_2 U^{-1}$. Let $\mathfrak{B}$ be the set of corresponding classes of non-equivalent AF-algebras acting on $H$. Then there is $1 - 1$ mapping between $\mathfrak{B}$ and the set of equivalence classes $\mathfrak{F}$ defined in Definition 2.5.

The proof of this Theorem is the same as the proof of previous Theorem 2.2. We only need to note that Hilbert spaces of the same dimension are unitary isomorphic.
3. Main results

Let $N, M \in \mathbb{N}$ be positive integers. Let $L^2_{N,M} = L^2(\mathbb{T}^N \to \mathbb{C}^M)$ be the Hilbert space of periodic vector valued functions defined on the multidimensional torus $\mathbb{T}^N$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0, 1)$. Everywhere in the article, it is assumed the Lebesgue measure in the definition of Hilbert spaces of square-integrable functions. Let $R^\infty_{N,M} = R^\infty(\mathbb{T}^N \to \mathbb{C}^{M \times M})$ be the $C^*$-algebra of matrix-valued regulated functions with rational discontinuities. The regulated functions with possible rational discontinuities are the functions that can be uniformly approximated by the step functions of the form

$$S(x) = \sum_{n=1}^P \chi_{J_n}(x)S_n,$$

where $P \in \mathbb{N}$, $S_n \in \mathbb{C}^{M \times M}$, and $\chi_{J_n}$ is the characteristic function of the parallelepiped $J_n = \prod_{i=1}^N [p_{in}, q_{in})$ with rational end points $p_{in}, q_{in} \in \mathbb{Q}/\mathbb{Z} \subset \mathbb{T}$. In particular, continuous matrix-valued functions belong to $R^\infty_{N,M}$.

Let us introduce the generating operators for integro-differential algebras. These operators are operators of multiplication by a function $M$, finite differential operators $D$, and integral operators $I$, all of them act on $L^2_{N,M}$:

$$M_S u(x) = S(x)u(x)$$
$$D_{i,h} u(x) = h^{-1}(u(x + xe_i) - u(x)),$$
$$I_i u = \int_0^1 u(x)dx_i,$$

where the function $S \in R^\infty_{N,M}$, the index $i \in \mathbb{N}_N = \{1, ..., N\}$, the step of differentiation $h \in \mathbb{Q}$, the standard basis vector $e_i = (\delta_{ij})_{j=1}^N$, and $\delta_{ij}$ is the Kronecker symbol. The $C^*$-algebra of finite-integro-differential operators is generated by all the operators (15)

$$\mathcal{F}_{N,M} = \text{Alg}_B \{M_S, D_{i,h}, I_i : S \in R^\infty_{N,M}, i \in \mathbb{N}_N, h \in \mathbb{Q}\},$$

where $\mathcal{B} \equiv \mathcal{B}_{N,M} = \mathcal{B}(L^2_{N,M})$ is the $C^*$-algebra of all the bounded operators acting on $L^2_{N,M}$. The typical example of an operator $A$ from $\mathcal{F}_{1,1}$ is

$$Au(x) = \sum_{n=1}^P A_n(x)D_{1,p} u(x) + \int_0^1 K(x,y)u(y)dy, \quad u \in L^2_{1,1}, \quad x \in \mathbb{T},$$

where $A_n \in R^\infty_{1,1}$, $K \in R^2_{2,1}$, and $p \in \mathbb{N}$. Let us provide the characterization of $\mathcal{F}_{N,M}$.

Theorem 3.1. The AF-algebra $\mathcal{F}_{N,M}$ has the following Glimm-Bratteli symbol

$$n(\mathcal{F}_{N,M}) = \left( \prod_{n=2}^\infty \begin{pmatrix} n & 0 \\ n - 1 & 1 \end{pmatrix} \otimes N \right) \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes N \right).$$

In particular, $\mathcal{F}_{N,M}$ and $\mathcal{F}_{N_1,M_1}$ are isomorphic if and only if $N = N_1$. 

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Integro-differential algebras with different number of variables are non-isomorphic. This fact distinguishes these algebras from the differential algebras $H_{N,M}$ generated by $M_S$ and $D_{i,h}$. The algebras $H_{N,M}$ are isomorphic to the universal UHF-algebra $U = \bigotimes_{n=1}^{\infty} C^{n \times n}$ independently on the number of variables $N$ and the number of functions $M$, see [17].

**Example.** Let us consider the $C^*$-algebra of two-dimensional integro-differential operators $F_{2,M}$. We have

$$
\prod_{i=1}^{n} \left( \begin{array}{c} i & 0 \\ i-1 & 1 \end{array} \right)^{\otimes 2} \left( \begin{array}{c} 1 \\ 1 \end{array} \right)^{\otimes 2} = \left( n! \right)^{\otimes 2} \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right)
$$

and

$$
\left( \begin{array}{c} n+1 & 0 \\ n & 1 \end{array} \right)^{\otimes 2} = \left( \begin{array}{cccc}
(n+1)^2 & 0 & 0 & 0 \\
n(n+1) & n+1 & 0 & 0 \\
n(n+1) & 0 & n+1 & 0 \\
n^2 & n & n & 1
\end{array} \right).
$$

Hence, the fragment of the Bratteli diagram for $F_{2,M}$ is

Here, the vertices in the row represent direct summands of the finite dimensional subalgebra, the edges represent partial embeddings into the next finite-dimensional subalgebra appearing in the direct limit, and the edge labels are multiplicities of partial embeddings.

**Remark 1.** Let us consider the algebra of one-dimensional scalar integro-differential operators $F_{1,1}$. The E-matrices for $F_{1,1}$ are given by Theorem 3.1

$$
E_0 = \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \quad E_n = \left( \begin{array}{c} n+1 \\ 0 \\ n \\ 1 \end{array} \right).
$$

It is clear that

$$
E_n = \left( \begin{array}{c} n+1 \\ 0 \\ 1 \\ 1 \end{array} \right)^n.
$$

Thus, there are arbitrary large sequences $(1 0 1)^n$, $n \in \mathbb{N}$ in the direct limit for $F_{1,1}$. Remembering that these sequences correspond to the unitized algebra of compact operators $\mathbb{C}1 + K(L^2_{1,1})$, see above and, e.g., Example 3.3.1 in [11], we can expect that $\mathcal{H}(L^2_{1,1}) \subset \mathbb{C}1 + K(L^2_{1,1})$. 
This is true because any compact operator can be uniformly approximated by finite-dimensional operators in some orthonormal basis of $L^2_{1,1}$. Taking Walsh basis $f_n$, $n \in \mathbb{N}$ consisting of step functions, we see that for any $n, m \in \mathbb{N}$ the one-rank operator $C_{n,m}$ given by $u \mapsto f_m \int_0^1 f_n u$, where $u \in L^2_{1,1}$, belongs to $\mathcal{F}_{1,1}$. Hence, any compact operator belongs to $\mathcal{H}_{1,1}$, since it can be uniformly approximated by linear combinations of $C_{n,m}$.

Finally, note that $\begin{pmatrix} n+1 & 0 \\ 0 & 1 \end{pmatrix} = (n+1) \oplus (1)$. E-matrices $(n+1)$, $n \in \mathbb{N}$ correspond to the universal uniformly hyper-finite algebra $\mathcal{U} = \bigotimes_{n=1}^{\infty} C_n^{n \times n}$ which has the supernatural number $n(\mathcal{U}) = \prod_{n=1}^{\infty} n$. Generated by $\mathcal{M}_S$ and $\mathcal{D}_{n,b}$, see (13), $\mathcal{U}$ is a sub-algebra of $\mathcal{F}_{1,1}$. Roughly speaking, $\mathcal{F}_{1,1}$ is a combination of the universal UHF-algebra $\mathcal{U}$ and the algebra of compact operators $\mathcal{K}$.

The natural extension of $\mathcal{F}_{1,1}$ (or $\mathcal{F}_{1,M}$) is the AF-algebra $\mathcal{F}_1$ generated by the following commutative semigroup

$$n(\mathcal{F}_1) = \prod_{n=1}^{\infty} \prod_{m=1}^{n} \begin{pmatrix} n & 0 \\ n-m & m \end{pmatrix}.$$

This is the maximal commutative semigroup of $2 \times 2$-matrices from $\mathcal{E}$ having the eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Perhaps, it would be interesting to see the "physical meaning" of extended integro-differential operators from $\mathcal{F}_1$.

**Remark 2.** Note that, while $\mathcal{F}_{N,M} \cong \mathcal{F}_{N,M'}$, there is no unitary $U$ such that $\mathcal{F}_{N,M} = U \mathcal{F}_{N,M} U^{-1}$ for $M \neq M'$. This is because the dimension-vectors, see Definition 2.3, contain one entry $M$ for $\mathcal{F}_{N,M}$ and one entry $M'$ for $\mathcal{F}_{N,M'}$. Other entries of dimension vectors are equal to $\infty$. The finite entries correspond to $M$-dimensional subspace of $L^2_{N,M}$ and $M'$-dimensional subspace of $L^2_{N,M'}$ generated by constant vector-functions.

4. Proof of the main results

**Proof of Theorem 2.3** The conditions of Definition 2.1 will be checked. We set $C_0 = A_0$, $C_1 = B_1$, and $r_1 = 1$, $m_1 = 2$ correspondingly. Next, $B_1 = A_{n_1}$ for some $n_1 \geq 1$. We take $C_2 = \prod_{i=0}^{n_1-1} A_i$, or $C_2 = A_2$ if $n_1 = 1$. In the first case we set $r_2 = n_1 + 1$, in the second case we set $r_2 = 3$.

Anyway, $C_2 \in \{A_n\}_{n=1}^{\infty}$, since this is the semigroup. Hence, for some $C_3 \in \{A_n\}_{n=1}^{\infty}$, we have $(B_1 C_2 )C_3 = \prod_{i=1}^{m_1-1} B_i$ by the condition (3). Thus $C_2 C_3 = \prod_{i=m_1-1}^{r_2-1} B_i$ because $B_1$ is invertible and all the matrices are commute.

By induction, suppose that for some $n > 1$ we already found $1 = r_1 < \ldots < r_n$, and $2 = m_1 < \ldots < m_n$, and $C_i \in \{A_n\}_{n=1}^{\infty}$ satisfying

$$C_{2j-1} C_{2j-2} = \prod_{i=m_{j-1}}^{r_{j-1}-1} B_i, \quad C_{2j-2} C_{2j-3} = \prod_{i=r_{j-1}}^{r_j-1} A_i, \quad 2 \leq j \leq n.$$

Let $\mu(A)$ be the maximal element of the matrix $A$. It is true

$$\mu(A_n A_m) \geq \max(\mu(A_n), \mu(A_m)),$$
since $A_n, A_m$ are matrices with non-negative integer entries, without zero rows and columns. There are two possibilities: (a) $\lim_{p \to \infty} \mu(C_{2n-1}^p) = \infty$, and (b) $\mu(C_{2n-1}^p)$ are bounded. In the case (a), for some sufficiently large $p > 1$ we have $C_{2n-1}^p = A_r$, where $r > r_n$. We set $r_{n+1} = r + 1$, $C_{2n} = C_{2n-1}^{r_{n+1}-1} \prod_{i=r_n}^{r_{n+1}} A_i$. Hence, we obtain

$$C_{2n}C_{2n-1} = \prod_{i=r_n}^{r_{n+1}-1} A_i.$$  \hfill (20)

Note that $C_{2n} \in \{A_n\}^{\infty}_{n=1}$, since this is the semigroup. Another possibility: (b) $\mu(C_{2n-1}^p)$ are uniformly bounded for all $p$. Then $C_{2n-1}^p = C_{2n-1}^s$ for some $s > r_n$ because $\{C_{2n-1}^p\}$ is a sequence of matrices with bounded non-negative integer entries. The existence of inverse matrix $C_{2n-1}^{-1}$ leads to $C_{2n-1}^{p-s} = I$ is the identity matrix. We set $r_{n+1} = r_n + 1$, $C_{2n} = C_{2n-1}^{-1} A_{r_n}$. These values also satisfy (20). Note that $E$-matrices satisfying the condition (b) correspond to a permutation of elements in the Bratteli diagrams.

Again, there are two possibilities: (a) $\lim_{p \to \infty} \mu(C_{2n-1}^p) = \infty$, and (b) $\mu(C_{2n-1}^p)$ are bounded. Consider the first case (a), the second (b) can be treated as above. There is $p \geq 1$ such that

$$\mu(C_{2n}^p) > \mu(\prod_{i=1}^{m_n} B_i).$$  \hfill (21)

Hence, by the condition (σ), taking $A_p = (\prod_{i=1}^{m_n-1} B_i)C_{2n}^p$ (recall that the set $\{A_n\}^{\infty}_{n=1}$ is a semigroup) we have

$$((\prod_{i=1}^{m_n-1} B_i)C_{2n}^p)A_r = \prod_{i=1}^{m_{n+1}-1} B_i$$  \hfill (22)

for some $m_{n+1} > m_n$ because of (21) and (19). We set $C_{2n+1} = C_{2n}^{p-1} A_r$. Using (22), we deduce

$$C_{2n+1}C_{2n} = \prod_{i=m_n}^{m_{n+1}-1} B_i.$$  \hfill (23)

Thus, by induction we prove that $\prod_{n=0}^{\infty} A_n$ and $(\prod_{n=1}^{\infty} B_n)A_0$ are equivalent, see Definition 2.1. By Theorem 2.2, they represent the same algebra.

If $\mu(\prod_{i=1}^{p} B_i)$ are bounded for all $p$ then there is $A_r$ and $1 \leq m_1 < m_2 < \ldots$ such that $\prod_{i=1}^{m_n} B_i = A_r$ for all $n$. Thus, $n^{-1}(\prod_{n=1}^{\infty} B_n)A_0) \cong \mathcal{M}(A_r)A_0$ is a sub-algebra of $\mathcal{M}(\prod_{n=0}^{\infty} A_n)$, which, in turn, is a sub-algebra of $n^{-1}(\prod_{n=0}^{\infty} A_n)$. Now, suppose that $\mu(n^{-1}(\prod_{i=1}^{p} B_i)) \to \infty$. Then we can take $1 = m_1 < m_2 < \ldots$ such that

$$\prod_{i=m_n}^{m_{n+1}-1} B_i = A_{r_n} \quad n \geq 1,$$

where $0 = r_0 < r_1 < r_2 < \ldots$. Denote

$$D_n = \prod_{i=r_{n+1}+1}^{r_n} A_i, \quad E_n = \prod_{j=1}^{n} \prod_{i=r_{j-1}+1}^{r_j-1} A_i,$$
where $E_n = I$ is the identity matrix if $r_n - 2 < r_{n-1}$. Then the following infinite commutative diagrams

\begin{align*}
\mathcal{M}(A_0) &\xrightarrow{D_1} \mathcal{M}(D_1A_0) \xrightarrow{D_2} \mathcal{M}(D_2D_1A_0) & \xrightarrow{\ldots} & n^{-1}(\prod_{n=0}^{\infty} A_n) \\
I &\xrightarrow{1} E_i &\xrightarrow{E_2} \mathcal{M}(A_0) \xrightarrow{A_{r_1}} \mathcal{M}(A_{r_1}A_0) &\xrightarrow{A_{r_2}} \mathcal{M}(A_{r_2}A_{r_1}A_0) & \xrightarrow{\ldots} & n^{-1}(\prod_{n=1}^{\infty} B_nA_0)
\end{align*}

show that $n^{-1}(\prod_{n=1}^{\infty} B_nA_0)$ is the sub-algebra of $n^{-1}(\prod_{n=0}^{\infty} A_n)$. □

**Proof of Theorem 2.4.** Suppose that $N > M$. If $n^{-1}(\prod_{n=0}^{\infty} A_n) \cong n^{-1}(\prod_{n=0}^{\infty} B_n)$ then there is the sequence of matrices $\{C\}_{n=0}^{\infty}$ satisfying (5), namely

$$C_{2n-2}C_{2n-3} = \prod_{i=r_{n-1}}^{r_n-1} A_i, \quad C_{2n-1}C_{2n-2} = \prod_{i=m_{n-1}}^{m_n-1} B_i, \quad C_{2n}C_{2n-1} = \prod_{i=r_n}^{r_{n+1}-1} A_i$$

for some $n > 2$. This yields to

$$\prod_{i=r_n}^{r_{n+1}-1} A_i = C_{2n}(\prod_{i=m_{n-1}}^{m_n-1} B_i)C_{2n-3}. $$

The matrix in LHS has the full rank $N$, while the matrix in RHS has a rank not more than $M$. This is the contradiction. □

**Proof of Theorem 3.1.** Let us start from the 1D case $N = M = 1$. For $h \in \mathbb{Q}$, define the shift operator $S_h = 1 - hD_{1,h}$. Define also the operators of multiplication by the characteristic functions of intervals

$$M_{j,p} \equiv M_{x_{j,p}}, \quad I_j^p = \left[ j, j + \frac{1}{p} \right], \quad j \in \mathbb{Z}_p = \{0, \ldots, p-1\}, \quad p \in \mathbb{N}. \quad (24)$$

The operators satisfy some elementary properties

$$M_{i,p}M_{j,p} = \delta_{ij}M_{i,p}, \quad S_h^*M_{i,p} = M_{i+j,p}S_h^*, \quad S_hS_t = S_{h+t}, \quad M_{i,p}^* = M_{i,p}, \quad S_h^* = S_{-h}, \quad S_hI_1 = I_1S_h = I_1, \quad I_1M_{i,p}I_1 = p^{-1}, \quad (25)$$

where $i, j \in \mathbb{Z}_p$, $h, t \in \mathbb{Q}$, and $p \in \mathbb{N}$. For $i, j \in \mathbb{Z}_p$, define the basis operators

$$B_{i,j}^p = pM_{i,p}I_1M_{j,p}, \quad A_{i,j}^p = M_{i,p}S_{i-j,p} - B_{i,j}^p. \quad (26)$$

Using (25), we can directly check the properties

$$B_{i,j}^pB_{i,m}^p = \delta_{jn}B_{i,m}^p, \quad (B_{i,j}^p)^* = B_{j,i}^p, \quad A_{i,j}^pA_{i,m}^p = \delta_{jn}A_{i,m}^p, \quad (A_{i,j}^p)^* = A_{j,i}, \quad A_{i,j}B_{m,n} = 0. \quad (27)$$

Identities (27) means that

$$\mathcal{H} \equiv \text{Alg}\{A_{i,j}^p, B_{i,j}^p : i, j \in \mathbb{Z}_p\} \cong \mathcal{M}(p) \oplus \mathcal{M}(p) = \mathcal{M}(p, p) \quad (28)$$
with the ∗-isomorphism defined by

\[ A_{ij}^p \mapsto (\delta_{in}\delta_{jm})_{n,m=0}^{p-1} \oplus 0_p, \quad B_{ij}^p \mapsto 0_p \oplus (\delta_{in}\delta_{jm})_{n,m=0}^{p-1}. \]  

(29)

where 0_p is the zero element in 𝕀_p. Let \( q \in \mathbb{N} \) be some positive integer. Using (26) and the identity

\[ M_{i,p} = (i+1)^q - 1 \sum_{n=iq}^{iq} M_{n,pq}, \]

we obtain

\[ A_{ij}^p = (i+1)^q - 1 \sum_{n=iq}^{iq} M_{n,pq} S_{i,p+n}^{pq} - B_{ij}^p = \]

\[ B_{ij}^p = \frac{1}{q} \sum_{n=1q}^{(i+1)q-1} \sum_{m=jq}^{(j+1)q-1} B_{nm}^{pq}, \]

\[ \mathcal{M}_{i,p} = \sum_{n=1q}^{(i+1)q-1} \mathcal{M}_{n,pq}, \]

(30)

Identity (30) shows how \( \mathcal{H}_p \) is embedded into \( \mathcal{H}_{pq} \). Namely, the corresponding ∗-embedding is defined by

\[ A \oplus B \mapsto (A \otimes I_q) \oplus (A \otimes 1_q - A \otimes (q^{-1}1_q) + B \otimes q^{-1}1_q), \]

(31)

where \( I_q \) is the identity matrix in \( \mathcal{M}_q \) and \( 1_q = (1) \in \mathcal{M}_q \) is the matrix, which has all entries equal to 1. The matrix \( q^{-1}1_q \) is the rank-one matrix and, hence, it is unitarily equivalent to the matrix with one non-zero entry

\[ q^{-1}1_q \simeq \begin{pmatrix} 1 & 0 & \ldots \\ 0 & 0 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}. \]

(32)

Using (32) we conclude that ∗-embedding (31) between \( \mathcal{H}_p \cong \mathcal{M}(p,p) \) and \( \mathcal{H}_{pq} \cong \mathcal{M}(pq,pq) \) has E-matrix

\[ E = \begin{pmatrix} q & 0 \\ q-1 & 1 \end{pmatrix}. \]

(33)

The integral operator \( I_1 \) belongs to all \( \mathcal{H}_p \), since

\[ I_1 = \sum_{i \in \mathbb{Z}_p} \sum_{j \in \mathbb{Z}_p} B_{ij}^p \in \mathcal{H}_p. \]

(34)

By definition, any \( S \in R_{1,1}^\infty \) can be uniformly approximated by step functions with rational discontinuities. Thus, the operator of multiplication by the function \( M_S \) can be uniformly approximated by linear combinations of \( M_{i,p} \), \( i \in \mathbb{Z}_p \). On the other hand, using (26), we have

\[ M_{i,p} = A_{i,i}^p + B_{i,i}^p \in \mathcal{H}_p \]

(35)
Hence, for any \( S \in R_{1,1}^{\infty} \), the operator \( \mathcal{M}_S \) can be uniformly approximated by the elements from \( \mathcal{H}_p \) with arbitrary precision when \( p \to \infty \). The identity operator 1 belongs to all \( \mathcal{H}_p \), since

\[
1 = \sum_{j=0}^{p-1} \mathcal{M}_{j,p}.
\]

The shift operators \( \mathcal{S}_h \) (with \( h = q/p \in \mathbb{Q} \)) belongs to \( \mathcal{H}_p \), since

\[
\mathcal{S}_p = \sum_{i \in \mathbb{Z}_p} (A_{i,i-1}^p + B_{i,i-1}^p) \in \mathcal{H}_p, \quad \mathcal{S}_p = \mathcal{S}_p \in \mathcal{H}_p
\]

by (20), (25), and (36). Hence, the finite differentials belong also to \( \mathcal{H}_p \):

\[
\mathcal{D}_{1,h} = h^{-1}(1 - \mathcal{S}_h) \in \mathcal{H}_p.
\]

Using (31), (38), and the mentioned above fact about the approximation of \( \mathcal{M}_S \) (for any \( S \in R_{1,1}^{\infty} \)) by the elements from \( \mathcal{H}_p \), we conclude that \( \mathcal{F}_{1,1} \) is the inductive limit of \( \mathcal{H}_p \) for \( p \to \infty \). In particular, taking \( p_n = n! \) and using (33) for \( \mathcal{H}_{p_n} \subset \mathcal{H}_{p_{n+1}} \) we obtain (17) for \( N = M = 1 \).

The algebra \( \mathcal{F}_{1,M} = \mathbb{M}(M) \otimes \mathcal{F}_{1,1} \) has the same Glimm-Bratteli symbol as \( \mathcal{F}_{1,1} \), since

\[
n(\mathcal{F}_{1,M}) = n(\mathbb{M}(M) \otimes \mathcal{F}_{1,1}) = \left( \prod_{n=2}^{\infty} \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)^M = \left( \prod_{n=2}^{\infty} \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} M \\ M-1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = n(\mathcal{F}_{1,1}), \tag{39}
\]

Let us discuss why the first identity in the last string of (39) is true. The matrices (33) form a commutative (multiplicative) semigroup. Then, the infinite product with one duplicated term in LHS of (39) obviously satisfies the condition (σ) from Theorem 2.3. Hence, \( \mathcal{F}_{1,M} \) and \( \mathcal{F}_{1,1} \) are isomorphic. There is also a more intuitive similarity with supernatural numbers

\[
\left( \prod_{n=2}^{\infty} \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} M \\ M-1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \left( \prod_{n=2}^{\infty} \begin{pmatrix} p_n \\ p_n - 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right),
\]

where \( p_1 = 2, p_2 = 3, p_3 = 5, \ldots \) are prime numbers and \( M = \prod_{j=1}^{K} p_{n_j} \) is the prime factorization of \( M \). This similarity with supernatural numbers is possible because all the matrices are commute.

Consider the case \( N > 1 \). Using the fact that \( L^2_{N,M} = \bigoplus_{j=1}^{M} (L^2_{1,1})^{\otimes N} \) we deduce that

\[
\mathcal{F}_{N,M} = \mathbb{M}(M) \otimes \mathcal{F}_{1,1}^{\otimes N}.
\]

This means that

\[
n(\mathbb{M}(M) \otimes \mathcal{F}_{1,1}^{\otimes N}) = n(\mathcal{F}_{1,M} \otimes \mathcal{F}_{1,1}^{\otimes N-1}) = n(\mathcal{F}_{1,1}^{\otimes N}) = \left( \prod_{n=2}^{\infty} \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix} \right)^{\otimes N} \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)^{\otimes N}
\]

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which proves (17). Thus, the $C^*$-algebras $\mathcal{F}_{N_1, M_1}$ and $\mathcal{F}_{N_2, M_2}$ are isomorphic if and only if $N_1 = N_2$ by Theorem 2.4.

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