Finite generation conjectures for cohomology over finite fields

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Abstract We relate motivic cohomology, and Weil-etale cohomology, both of which are finitely generated, by an intermediate cohomology theory, which we also conjecture to be finitely generated, and examine the relationship of the three theories.

1 Introduction

Bass conjectured that for a regular scheme $X$ of finite type over the integers, the higher algebraic $K$-groups are finitely generated [1]. Via the localization sequence, this is equivalent to the finite generation of $K_i^n(X)$ for all $X$ of finite type over the integers. In view of the spectral sequence from higher Chow groups to $K$-theory, it is a slightly stronger statement to conjecture the finite generation of higher Chow groups $CH_n(X, i)$ for all $X$ of finite type over the integers. If we restrict ourselves to $X$ of finite type over a finite field, then this is equivalent to the finite generation of motivic cohomology $H_i^M(X, \mathbb{Z}(n))$ for smooth $X$ [21]. Under resolution of singularities, it implies finite generation of motivic cohomology for all $X$ of finite type over a finite field, as one sees from the blow-up long exact sequence.

Still for $X$ of finite type over a finite field, Lichtenbaum’s Weil-etale cohomology $H_i^W(X, \mathbb{Z}(n))$ is defined as the cohomology of $R\Gamma_G R\Gamma_{et}(\bar{X}, \mathbb{Z}(n))$, where $G$ is the Weil-group of the finite field. Using ideas of Kahn, we showed in [5] that the finite generation of $H_i^W(X, \mathbb{Z}(n))$ for smooth and proper $X$ is equivalent to the strong form of Tate’s and Beilinson’s conjecture. However, the groups $H_i^W(X, \mathbb{Z}(n))$ are not finitely generated if $X$ is not smooth or not proper, and need to be modified in this case, see [6].

The purpose of this article is to relate the two finite generation conjectures with the help of an intermediate cohomology theory $H_i^F(X, \mathbb{Z}(n))$, which we
call Frobenius cohomology. It is defined as the cohomology of $R\Gamma_G\mathbb{Z}(n)(\bar{X})$, where $\mathbb{Z}(n)$ is Bloch’s cycle complex shifted appropriately. There are natural maps

$$H^i_M(X, \mathbb{Z}(n)) \xrightarrow{\alpha} H^i_F(X, \mathbb{Z}(n)) \xrightarrow{\beta} H^i_W(X, \mathbb{Z}(n)),$$

which can be studied separately. In fact they lie in a diagram

$$\xymatrix{ \cdots \ar[r] & H^i_M(X, \mathbb{Z}(n)) \ar[r]^\alpha \ar[d]^f & H^i_F(X, \mathbb{Z}(n)) \ar[r]^\beta \ar[d]^g & H^{i-1}_K(X, \mathbb{Z}(n)) \ar[r]^\gamma \ar[d]^h & \cdots }$$

with groups $H^i_K(X, \mathbb{Z}(n)) = H^i(Z(n)(\bar{X}), G)$ called Kato cohomology. They are a generalization of the integral version of Kato homology [15] (which is the case $n = \dim X$) defined in [8]. We conjecture all groups in the upper row of (1) to be finitely generated, and believe they form interesting invariants of $X$. For example, $H^{2n}_K(X, \mathbb{Z}(n)) = CH^n(X,G)$, and under Parshin’s conjecture, the groups $H^i_K(X, \mathbb{Z}(n))$ vanish for $i \leq n$ and are torsion for $i \neq 2n$. The first interesting example $H^3_K(X, \mathbb{Z}(2))$ consists of those elements in the cokernel of the integral cycle map $CH^2(X) \to H^{4}_{et}(X, \mathbb{Z}(2))$ which are in the image over the algebraic closure, or in classical language those elements in the unramified cohomology group $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))$ which vanish in $H^3_{nr}(\bar{X}, \mathbb{Q}/\mathbb{Z}(2))$. Pirutka [18] used an idea of Colliot-Thélène to construct an element in $H^3_{nr}(X, \mathbb{Z}(2))$ for a geometrically rational surface (in all but finitely many characteristics), providing a 2-torsion element in $H^3_K(X, \mathbb{Z}(2))$.

Since the above theories exist for any scheme of finite type over a finite field, we define them in this generality and later specialize to smooth projective $X$.

2 Borel Moore homology theory

We consider separated schemes of finite type over a finite field $\mathbb{F}_q$ of characteristic $p$, and $n \geq 0$ (the case $n < 0$ can be reduced to this case using the homotopy formula). Let $\varphi$ be the geometric Frobenius, and $G = \langle \varphi \rangle$ be the Weil group of $\mathbb{F}_q$. Let $Z^c(n)$ be the cycle complex defined by Bloch shifted by $2n$, so that

$$H_i(Z^c(n)(X)) = CH^i(X, i-2n).$$

This agrees with motivic cohomology $H^{2d-i}_M(X, \mathbb{Z}(d-n))$ for smooth $X$ of pure dimension $d$ by [21].

**Definition 2.1** Let $A$ be an abelian group.

a) We define $H^i_M(X, A(n))$ to be $H_i(Z^c(n)(X) \otimes A)$. 
b) We define Frobenius homology $H^F_i(X, A(n))$ to be the homology of the double complex

$$Z^c_i(n)(\tilde{X}) \otimes A \xrightarrow{\varphi^{-1}} Z^c_i(n)(\tilde{X}) \otimes A.$$ 

Here $\varphi$ acts covariantly in $\tilde{X}$ on cycles on $\tilde{X} \times \Delta^i$; the left and right hand complexes sit in homological degrees 1 and 0, respectively.

c) The Kato homology $H^K_i(X, A(n))$ is defined to be the homology of the complex of coinvariants $(Z^c_i(n)(\tilde{X}) \otimes A)_G$.

Note that $H^F_i(X, \mathbb{Z}/m(n))$ is isomorphic to $H^{1-i}(\text{Gal}(\mathbb{F}_q), \mathbb{Z}/m(n)(\tilde{X}))$ because with torsion coefficients, Galois cohomology can be calculated by the complex above. The following lemma follows from the definitions.

**Lemma 2.2** The groups $H^F_i(X, A(n)), H^F_i(X, A(n))$ and $H^K_i(X, A(n))$ vanish for $i < 2n$. We have $H^F_{2n}(X, A(n)) \cong CH_n(X) \otimes A$ and

$$H^F_{2n}(X, A(n)) \cong H^F_{2n}(X, A(n)) \cong (CH_n(X) \otimes A)_G.$$ 

For all $i$, there are short exact sequences

$$0 \to H^F_i(\tilde{X}, A(n))_G \to H^F_i(X, A(n)) \to H^F_{i-1}(\tilde{X}, A(n))_G \to 0. \quad (2)$$

**Proposition 2.3** We have an exact sequence

$$\cdots \to H^F_i(X, A(n)) \to H^F_{i+1}(X, A(n)) \to H^K_{i+1}(X, A(n)) \to H^F_{i-1}(X, A(n)) \to \cdots \quad (3)$$

All three theories are covariantly functorial for proper maps, contravariantly functorial for quasi-finite flat maps, and have localization long exact sequences.

**Proof.** This comes from the short exact sequence of double complexes

$$\begin{array}{ccc}
(Z^c_i(n)(\tilde{X}) \otimes A)_G & \longrightarrow & Z^c_i(n)(\tilde{X}) \otimes A \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Z^c_i(n)(\tilde{X}) \otimes A \\
\downarrow & & \downarrow \\
0 & \longrightarrow & (Z^c_i(n)(\tilde{X}) \otimes A)_G
\end{array}$$

and $Z^c_i(n)(X) \otimes A \cong (Z^c_i(n)(\tilde{X}) \otimes A)_G$. Functoriality and the localization property are well-known for cycle complexes, and this carries over to $(Z^c_i(n)(X) \otimes A)_G$ by an easy diagram chase. \hspace{1cm} Q.E.D.

For a finitely generated field $K$ of transcendence degree $d$ over $\mathbb{F}_q$, define $H^F_i(K, A(j))$ to be colim $H^F_{2d+1-i}(U, A(d-j))$, where $U$ runs through smooth schemes with function field $K$. By (2), the cohomology groups of $K$ lie in short exact sequences.
0 \to H_{\mathcal{M}}^{s-t-1}(K \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, A(s-n))_G \to H_{\mathcal{F}}^{s-t}(K, A(s-n)) \to \\
\quad H_{\mathcal{M}}^{s-t}(K \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, A(s-n))^G \to 0, \quad (4)

where $K \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ is a finite product of fields.

**Proposition 2.4** We have functorial spectral sequences

\[
E_{s,t}^1 = \bigoplus_{X(o)} H_{\mathcal{M}}^{s-t}(k(x), A(s-n)) \Rightarrow H^{s+t}_c(X, A(n));
\]

\[
\bar{E}_{s,t}^1 = \bigoplus_{X(o)} H_{\mathcal{F}}^{s-t}(k(x), A(s-n)) \Rightarrow H^{s+t+1}_F(X, A(n)).
\]

**Proof.** This follows with the niveau filtration, together with the fact that

\[H^{s+t}_c(K, A(n)) := \operatorname{colim} H^{s+t}_c(U, A(n)) \cong H_{\mathcal{M}}^{s-t}(K, A(s-n))\]

and $H^{s+t+1}_F(K, A(n)) \cong H_{\mathcal{F}}^{s-t}(K, A(s-n))$ for a field $K$ of transcendence degree $s$ over the base field. \[Q.E.D.\]

### 2.1 Integral coefficients

**Conjecture 2.5** The groups $H_c^i(X, \mathbb{Z}(n))$, $H_F^i(X, \mathbb{Z}(n))$ and $H^K_i(X, \mathbb{Z}(n))$ are finitely generated for any $i, n$ and $X$.

By localization, it suffices to consider the case of smooth and proper $X$ (assuming that every finitely generated field over $\mathbb{F}_q$ has a smooth and proper model). This case will be considered in detail below.

Recall from [1] Parshin’s conjecture $P_n$, stating that $CH_n(X, i)$ is torsion for $i \neq 0$ and $X$ smooth and projective.

**Proposition 2.6** Assume Conjecture $P_n$. Then

\[H_{\mathcal{M}}^{s-t}(k, \mathbb{Z}(s-n)) \cong H_{\mathcal{F}}^{s-t}(k, \mathbb{Z}(s-n))\]

for $t \geq n$. For $t = n - 1$, the left hand side vanishes, and $H_{\mathcal{F}}^{s-n+1}(k, \mathbb{Z}(s-n)) \cong K_{s-n}^M(k \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q)_G$. For $t < n - 1$, both sides vanish.

**Proof.** It suffices to show that the second map and the composition

\[H_{\mathcal{M}}^{s-t}(k, \mathbb{Z}(s-n)) \to H_{\mathcal{F}}^{s-t}(k, \mathbb{Z}(s-n)) \to H_{\mathcal{F}}^{s-t}(k, \mathbb{Z}(s-n))\]

are isomorphism for $t \geq n$. The total composition is an isomorphism for $t \geq n - 1$ by the Beilinson-Lichtenbaum conjectures. For the second map,
we compare to the analog short exact sequence for etale cohomology. By Parshin's conjecture, the groups $H^i_M(k, \mathbb{Z}(s-n))$ are torsion for $i < s-n$, so that Galois cohomology and cohomology of the Weil-group $G$ agree for those groups. The result now follows with the Beilinson-Lichtenbaum conjectures by comparing with the Hochschild-Serre spectral sequence for Galois cohomology (for the finite product of fields $k \otimes \overline{\mathbb{F}}_q$).

**Q.E.D.**

**Definition 2.7** We define the Kato complex $KC_n(X)$ of weight $n$ to be the complex

$$
\bigoplus_{X(n)} \mathbb{Z} \leftarrow \bigoplus_{X(n+1)} (k(x) \otimes \overline{\mathbb{F}}_q) \leftarrow \cdots \leftarrow \bigoplus_{X(s)} K^M_{s-n}(k(x) \otimes \overline{\mathbb{F}}_q) G \leftarrow \cdots,
$$

with the summand indexed by $X(s)$ in degree $s-n$. The differentials are induced by boundary maps of higher Chow groups of discrete valuation rings.

**Corollary 2.8** Assuming conjecture $P_n$, we have

$$H^K_i(X, \mathbb{Z}(n)) \cong H^{i-2n}(KC_n(X)).$$

In particular, $H^K_i(X, \mathbb{Z}(n))$ vanishes unless $2n \leq i \leq n + d$.

**Proof.** Compare the spectral sequences of Proposition 2.4 and use Proposition 2.6. Q.E.D.

### 2.2 Finite coefficients

Considering finite coefficients allows us to remove the hypothesis on Parshin’s conjecture and to compare to etale homology. Recall the sequence

$$
\cdots \rightarrow H^i_c(X, \mathbb{Z}/m(n)) \xrightarrow{\alpha} H^F_i(X, \mathbb{Z}/m(n)) \xrightarrow{\beta} H^K_i(X, \mathbb{Z}/m(n)) \rightarrow \cdots (5)
$$

Conjecture 2.5 implies the weaker

**Conjecture 2.9** All terms in the sequence (5) are finite for all $i, n, X$.

Let $f : X \rightarrow \mathbb{F}_q$ be the structure map.

**Definition 2.10** For $p \nmid m$ we define $H^i_{et}(X, \mathbb{Z}/m(n))$ as the (Borel-Moore) etale homology $H^{-i}(X_{et}, Rf^! \mu_{m}^{\otimes -n}) = H^{-i}(\text{Gal}(\mathbb{F}_q), R\Gamma_{et}(X, Rf^! \mu_{m}^{\otimes -n}))$.

Since $\text{Gal}(\mathbb{F}_q)$ has cohomological dimension 1 and $Rf^! \mu_{m}^{\otimes -n}$ is concentrated in negative degrees, we have

**Lemma 2.11** The groups $H^i_{et}(X, \mathbb{Z}/m(n))$ vanish for $i < -1$.
Writing $\mathbb{Z}^c/m(n)$ for $\mathbb{Z}^c(n) \otimes \mathbb{Z}/m$, we showed in [7]:

**Theorem 2.12** We have $Rf^!\mathbb{Z}/m \cong \mathbb{Z}^c/m(0)$ for any $m$. In particular $H^i_{\text{et}}(X, \mathbb{Z}/m(n))$ agrees with the $i$th etale hypercohomology of $\mathbb{Z}^c/m(n)$ if $p \nmid m$ and $n \leq 0$.

If $p \nmid m$ and $k$ contains the $m$-th roots of unity, then cap-product with $\mathbb{Z}/m(1)(k) \cong \mu_m(k)$, defines a map $f^*\mu_m \otimes \mathbb{Z}^c/m(n) \to \mathbb{Z}^c/m(n-1)$ of complexes of sheaves on $X/k$, inducing a map

$$\mathbb{Z}^c/m(n) \to f^*\mu_m^\otimes -n \otimes \mathbb{Z}^c/m(0) \cong f^*\mu_m^\otimes -n \otimes Rf^!\mathbb{Z}/m \cong Rf^!\mu_m^\otimes -n.$$ 

This map is in general not an isomorphism for $n > 0$; for example, the left hand side vanishes for $n > \dim X$, but the right hand side is periodic in $n$.

This construction induces a map $\mathbb{Z}^c/m(n)(\bar{X}) \to R\Gamma_{\text{et}}(\bar{X}, \mathbb{Z}^c/m(n)) \to R\Gamma_{\text{et}}(\bar{X}, Rf^!\mu_m^\otimes -n)$, hence $\gamma : H_{i+1}^F(X, \mathbb{Z}/m(n)) \to H^i_{\text{et}}(X, \mathbb{Z}/m(n))$ by taking the cone of $\varphi - 1$. The shift in degrees stems from homological notation for Galois cohomology for the former, and cohomological notation for the latter.

**Theorem 2.13** Let $n = 0$. Then Conjecture [2] holds for $p \nmid m$, and for any $m$ under resolution of singularities.

**Proof.** If $n = 0$, then $\gamma$ is an isomorphism by [7], hence Frobenius homology is finite. But by Jannsen-Kerz-Saito [16], Kato homology is finite.

If $m = p^r$, then Kato homology is finite under resolution of singularities. To show finiteness of Frobenius homology, one reduces by the usual device to the case that $X$ is smooth and proper of dimension $d$. In this case, $\gamma$ is isomorphic to the finite group $H^{d-i}_{\text{et}}(X, \nu_d^r)$ by Theorem [2.12] and the isomorphism $\mathbb{Z}^c/p^r(0) \cong \nu_d^r[d]$. Q.E.D.

As in Proposition [2.4], we obtain

**Proposition 2.14** For $p \nmid m$ we have a spectral sequence

$$\tilde{E}^1_{s,t} = \bigoplus_{X(\sigma)} H^{s-t}_{\text{et}}(k(x), \mu_m^\otimes s-n) \Rightarrow H^{s+t}_{\text{et}}(X, \mathbb{Z}/m(n))$$

The $\tilde{E}^1$-terms lie in short exact sequences

$$0 \to H^{s-t-1}_{\text{et}}(k(x) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mu_m^\otimes s-n)_G \to H^{s-t}_{\text{et}}(k(x), \mu_m^\otimes s-n) \to H^{s-t-1}_{\text{et}}(k(x) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mu_m^\otimes s-n)_G \to 0.$$
Proposition 2.15 For $k$ a field of transcendence degree $s$ over $\mathbb{F}_q$ and $p \nmid m$,

$$H^s_{\text{et}}(k, \mathbb{Z}(s-n)) \cong H^s_F(k, \mathbb{Z}(s-n)) \cong H^s_{\text{et}}(k, \mu_m^{s-n})$$

for $t \geq n$. The left term vanishes for $t < n$, the middle term vanishes for $t < n - 1$ and

$$H^{s-n+1}_F(k, \mathbb{Z}(s-n)) \cong H^{s-n}_M(k \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Z}(s-n)) \cong H^{s-n}_F(k, \mu_m^{s-n})$$

The right term vanishes for $t < -1$.

Proof. It suffices to show that the maps

$$H^s_{\text{et}}(k, \mathbb{Z}(s-n)) \to H^{s-t}_F(k, \mathbb{Z}(s-n)) \to H^{s-t}_{\text{et}}(k, \mathbb{Z}(s-n))$$

are isomorphism for $t \geq n$. The total composition is an isomorphism by the Beilinson-Lichtenbaum conjectures. Comparing the sequences (4) and (7), it follows that the second map is an isomorphism by the Beilinson-Lichtenbaum conjectures for the finite product of fields $k \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$.

Q.E.D.

Definition 2.16 We define $KC_n/m(X)$ to be the complex

$$\bigoplus_{X(n)} \mathbb{Z}/m \leftarrow \bigoplus_{X(n+1)} H^1_{\text{et}}(k(x) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Z}(1)) \leftarrow \cdots$$

$$\leftarrow \bigoplus_{X(1)} H^{s-n}_{\text{et}}(k(x) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Z}(s-n)) \leftarrow \cdots,$$  \hspace{1cm} (8)

with the summand indexed by $X(s)$ in degree $s-n$.

As in the integral case we obtain:

Corollary 2.17 We have

$$H^K_i(X, \mathbb{Z}/m(n)) \cong H_{i-2n}(KC_n/m(X)),$$

and these groups vanish unless $2n \leq i \leq n + d$.

3 Smooth and proper $X$

In this section we assume that $X$ is smooth and proper over $\mathbb{F}_q$, and discuss what other conjectures (like Parshin’s conjecture) imply for the cohomology groups defined above. We use cohomological notation because readers may be more familiar with it. Let $\mathbb{Z}(n)$ be Bloch’s higher cycle complex indexed by codimension, so that $\mathbb{Z}(n)(X)^i = z^n(X, 2n - i)$ and $H^i_{\text{Zar}}(X, \mathbb{Z}(n)) = \cdots$
$\text{CH}_{d-n}(X, 2n - i)$ is motivic cohomology for smooth $X$ of pure dimension $d$.

If $X$ is smooth of pure dimension $d$, we set

\begin{align*}
H^F_{i}(X, A(n)) &= H^{2d+1-i}_F(X, A(d-n)) \\
H^K_{i}(X, A(n)) &= H^{2d-i}_K(X, A(d-n)).
\end{align*}

The following are reformulations of Lemma 2.2:

**Lemma 3.1** The groups $H^F_{i}(X, A(n))$ vanish for $i > 2n + 1$, and in general there are short exact sequences

\[ 0 \to H^{i-1}_M(\bar{X}, A(n))_G \to H^F_{i}(X, A(n)) \to H^i_M(\bar{X}, A(n))_G \to 0. \]

The groups $H^K_{i}(X, A(n))$ vanish for $i > 2n$, and

\[ H^{2n}_K(X, A(n)) \cong H^{2n+1}_F(X, A(n)) \cong (\text{CH}^n(\bar{X}) \otimes A)_G. \]

**Proposition 3.2** We have a commutative diagram with exact rows

\[
\begin{array}{cccccc}
\longrightarrow & H^1_M(X, Z(n)) & \alpha \longrightarrow & H^1_F(X, Z(n)) & \beta \longrightarrow & H^{i-1}_K(X, Z(n)) \\
\downarrow f & & \downarrow g & & \downarrow h & \\
\longrightarrow & H^1_M(X, Z(n)) & \longrightarrow & H^i_W(X, Z(n)) & \longrightarrow & H^{i-1}_M(X, Q(n)) \longrightarrow
\end{array}
\]

**Proof.** The upper row is exact by Proposition 2.3, and the lower row is the exact sequence relating etale cohomology to Weil-etale cohomology [5]. The maps $f$ and $g$ are induced by the change of topology from the Zariski to the etale site.

**Q.E.D.**

**Remark.** The vertical maps are isomorphisms after tensoring with $Q$, because motivic cohomology and etale hypercohomology of the motivic complex agree with rational coefficients.

**Conjecture 3.3** The bold face terms are finitely generated for all $i, n$ and $X$.

Finite generation of $H^i_M(X, Z(n))$ is a generalization of Bass’ conjecture, and finite generation of $H^i_W(X, Z(n))$ is equivalent to Tate’s and Beilinson’s conjecture. Lichtenbaum conjectured [17] also finite generation for the etale cohomology groups, except $H^{2n+2}_{et}(X, Z(n)) \cong H^2(G_k, H^{2n}_{et}(\bar{X}, Z(n)))$:

**Conjecture 3.4** The group $H^i_M(X, Z(n))$ is finite for all $i \neq 2n, 2n + 1$, finitely generated for all $i = 2n$, and of cofinite type for $i = 2n + 2$.

**Lemma 3.5** The map $h : H^i_K(X, Z(n)) \to H^i_M(X, Q(n))$ is bijective for $i < n$ and injective for $i = n$. 
Proof. By the Beilinson-Lichtenbaum conjectures, both the map $f$ as well as the map $g$ are isomorphism for $i \leq n + 1$ and injective for $i = n + 2$. Q.E.D.

### 3.1 The case $n = 0, 1, 2, d$

Conjecture 3.3 holds for $n = 0$. In fact we have $H_i^p(X, \mathbb{Z}) \cong H_i^p(X, \mathbb{Z}) \cong \mathbb{Z}$ for $i = 0, 1$, $H_i^0(X, \mathbb{Z}) \cong H_i^0(X, \mathbb{Z}) \cong \mathbb{Z}$, and all other homology groups vanish.

**Proposition 3.6** Let $n = 1$.

a) For $i = 1$, the four left groups in (11) are isomorphism to $O_X(X)^\times$, and for $i = 2$, they are isomorphic to $Pic(X)$.

b) The groups $H_i^K(X, \mathbb{Z}(1))$ vanish for $i \neq 2$, and $H_2^K(X, \mathbb{Z}(1)) \cong NS(\bar{X})_G$.

c) For $i = 3$,

\[
H_3^M(X, \mathbb{Z}(1)) = 0 \\
H_3^{et}(X, \mathbb{Z}(1)) \cong Br(X) \\
H_3^F(X, \mathbb{Z}(1)) \cong NS(\bar{X})_G,
\]

and $H_3^W(X, \mathbb{Z}(1))$ is an extension of $Br(\bar{X})^G$ by $NS(\bar{X})_G$.

d) All groups in (11) are finitely generated, except possibly $H_3^{et}(X, \mathbb{Z}(1)) \cong Br(X)$ and $H_3^{W}(X, \mathbb{Z}(1))$, whose finiteness is equivalent to Tate’s conjecture for divisors.

**Proof.** This follows from $\mathbb{Z}(1) \cong G_m[-1]$, Lemma 3.1 and $Pic^0(\bar{X})_G = 0$. Q.E.D.

Now consider the case $n = 2$. According to Parshin’s conjecture and Lemma 3.5, the groups $H_2^K(X, \mathbb{Z}(2))$ vanish for $i \leq 2$, and $H_2^K(X, \mathbb{Z}(2)) \cong CH^2(\bar{X})_G$. To describe the remaining group, let $\epsilon : \bar{X}_{et} \to X_{zar}$ be the canonical map of sites, and $\tau$ the composition of the boundary map in the lower row of (11) with the composition

\[
H^4_{et}(X, \mathbb{Z}(2)) \to H^0(X, R^4\epsilon_*\mathbb{Z}(2)) \cong H^0(X, R^4\epsilon_*\mathbb{Q}/\mathbb{Z}(2)) = H_{nr}^3(X, \mathbb{Q}/\mathbb{Z}(2)).
\]

The last term is known under the name unramified cohomology.

**Proposition 3.7** The group $\tau_0 H_2^3(X, \mathbb{Z}(2))$ is isomorphic to the cohomology of the sequence

\[
H^2_M(X, \mathbb{Q}(2)) \to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(\bar{X}, \mathbb{Q}/\mathbb{Z}(2))^G.
\]

**Proof.** The first statement follows from a diagram chase in (11), noting that due to the Beilinson-Lichtenbaum conjecture the maps $f$ and $g$ are injective,
coker $f = H^0(X, R^4\epsilon_*Z(2))$ and
\[ \text{coker } g = \text{coker } (H^3_M(X, Z(2))^G \to H^3_{et}(\overline{X}, Z(2))^G) \subseteq H^0(\overline{X}, R^4\epsilon_*Z(2))^G. \]
Q.E.D.

**Corollary 3.8** Under Parshin’s conjecture there is an exact sequence
\[ 0 \to H^3_K(X, Z(2)) \to H^3_{nr}(X, Q/Z(2)) \to H^3_{et}(\overline{X}, Z(2))^G. \]
Thus $\text{tor } H^3_K(X, Z(2))$ consists of those elements in the cokernel of the integral cycle map, whose pull back to the algebraic closure $H^3_{et}(\overline{X}, Z(2))^G$ lie in the image of the cycle map $\text{CH}^n(\overline{X}) \to H^{2n}_{et}(\overline{X}, Z(n))$.

**Proof.** Under Parshin’s conjecture, $H^3_K(X, Z(2))$ is torsion. Furthermore, coker $f$ is the obstruction to the integral Tate conjecture, and coker $g$ is the cokernel of the map $\text{CH}^2(X)^G \to H^2_{et}(\overline{X}, Z(2))^G$. Q.E.D.

**Corollary 3.9** If $X$ is geometrically rational, then there are isomorphisms $H^3_{nr}(X, Q/Z(2)) \cong H^3_K(X, Z(2))$.

**Proof.** For rational $Z$ over a field $k$, we have $H^3_{nr}(Z, Q/Z(2)) = H^3_{et}(k, Q/Z(2))$, and $H^3_M(Z, Q(2)) \cong H^0(Z, H^2(Q(2))) \cong H^3_M(k, Q(2))$ by [3, 2.1.9], and for $k$ the algebraic closure of a finite field these groups vanish. Q.E.D.

Pirutka [18] constructed an element in $H^3_{nr}(X, Z/2)$ for $X$ a geometrically birational variety of dimension 5, giving a non-trivial 2-torsion element in $H^3_K(X, Z(2))$.

The following result is a consequence of the work of Jannsen, Kerz and Saito, see [8]:

**Theorem 3.10** If $n = \dim X$, then Conjecture 3.3 is equivalent to conjecture $P_0$. In this case, $H^i_K(X, Z(n)) = 0$ for $i \neq 2n$, and $H^{2n}_{et}(X, Z(n)) \cong \mathbb{Z}^\pi_0(X)$.

### 3.2 Assuming Parshin’s conjecture

The cohomological Parshin conjecture $P^n$ of [9] states that $H^i_M(X, Q(n)) = 0$ for smooth and proper $X$ and $i \neq 2n$. This is equivalent to isomorphisms $H^i_{et}(X, Z(n)) \cong H^i_{W}(X, Z(n))$ for $i \leq 2n$, and injectivity in degree $2n + 1$ by [5]. By Lemma 3.5 Parshin’s conjecture implies that the groups $H^i_K(X, Z(n))$ vanish for $i < n$ if $n > 0$, and that they torsion for $i \leq 2n$. Diagram (11) for $i < 2n$ becomes
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\[ \cdots \rightarrow H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) \rightarrow H^i_F(X, \mathbb{Z}(n)) \rightarrow H^{i-1}_{K}(X, \mathbb{Z}(n)) \rightarrow \cdots \]

\[ H^i_{et}(X, \mathbb{Z}(n)) \rightarrow H^i_W(X, \mathbb{Z}(n)) \]

(12)

In degree \(2n+1\) we get

\[ 0 \rightarrow CH^n(X)_{\varphi} \rightarrow H^{2n}_K(X, \mathbb{Z}(n)) \rightarrow 0 \]

\[ \cdots \]

\[ H^{2n+1}_{et}(X, \mathbb{Z}(n)) \rightarrow H^{2n+1}_W(X, \mathbb{Z}(n)) \rightarrow CH^n(X)_{\mathbb{Q}} \rightarrow H^{2n+2}_{et}(X, \mathbb{Z}(n)) \]

The cokernel of the lower right horizontal map is the (conjecturally finite) group \(H^{2n+2}_W(X, \mathbb{Z}(n))\). In degree \(2n\) we have

\[ H^{2n-2}_K(X, \mathbb{Z}(n)) \rightarrow CH^n(X)_{\alpha} \rightarrow H^{2n}_F(X, \mathbb{Z}(n)) \rightarrow H^{2n-1}_K(X, \mathbb{Z}(n)) \rightarrow 0 \]

\[ H^{2n}_{et}(X, \mathbb{Z}(n)) \rightarrow H^{2n}_W(X, \mathbb{Z}(n)) \]

(13)

We get an exact sequence

\[ H^{2n-2}_K(X, \mathbb{Z}(n)) \rightarrow \ker f \rightarrow \ker g \rightarrow H^{2n-1}_K(X, \mathbb{Z}(n)) \rightarrow \coker f \rightarrow \coker g \rightarrow 0. \]

By Corollary 5.4 below, \(f\) has conjecturally the same kernel and cokernel as the cycle map, whereas \(g\) is related to the cycle map over the algebraic closure.

\[ 0 \rightarrow H^{2n-1}_{\mathcal{M}}(\bar{X}, \mathbb{Z}(n))_{G} \rightarrow H^{2n}_F(\bar{X}, \mathbb{Z}(n)) \rightarrow CH^n(X)^G \rightarrow 0 \]

\[ 0 \rightarrow H^{2n-1}_{et}(\bar{X}, \mathbb{Z}(n))_{G} \rightarrow H^{2n}_W(\bar{X}, \mathbb{Z}(n)) \rightarrow H^{2n}_{et}(\bar{X}, \mathbb{Z}(n))^G \rightarrow 0 \]

Thus Kato homology measures the difference of the failure of the integral Tate conjecture over \(\mathbb{F}_q\) and over its algebraic closure.

4 The algebraic closure of a finite field

As we saw, the map \(H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) \rightarrow H^i_K(X, \mathbb{Z}(n))\) is controlled by \(H^i_K(X, \mathbb{Z}(n))\). The maps \(H^i_F(X, \mathbb{Z}(n)) \rightarrow H^i_W(X, \mathbb{Z}(n))\) and \(H^i_F(X, \mathbb{Z}/m(n)) \rightarrow H^i_{et}(X, \mathbb{Z}/m(n))\) arise by taking Galois descent on the maps

\[ H^i_{\mathcal{M}}(\bar{X}, \mathbb{Z}(n)) \rightarrow H^i_{et}(\bar{X}, \mathbb{Z}(n)). \]
It is thus important to get some ideas of the properties of this map. Since it is an isomorphism rationally, we focus on finite coefficients. Assume that $p \not| m$. By the proper base change theorem, $H^i_{\text{et}}(\bar{X}, \mathbb{Z}/m(n))$ is finite, and replacing $\mathbb{Z}/m(n)$ by $\mu_m^{\otimes n}$, we have a localization long exact sequence (the difference is that the latter is non-trivial for negative $n$). The question is if $H^i_{\text{M}}(\bar{X}, \mathbb{Z}/m(n))$ is finite. There are examples of Schoen showing that for certain threefolds over an algebraically closed field of characteristic 0, the group $\text{CH}^2(\bar{X})/l$ is not finite, which implies that $H^4_{\text{M}}(\bar{X}, \mathbb{Z}/l(2))$ cannot be finite. However, we are not aware of any examples in characteristic $p$.

For $N \geq i$, consider the following diagram

$$
\begin{array}{ccc}
H^i_{\text{M}}(\bar{X}, \mathbb{Z}/m(n)) & \longrightarrow & H^i_{\text{et}}(\bar{X}, \mu_m^{\otimes n}) \\
\cup_{S^n-m} & & \\
H^i_{\text{M}}(\bar{X}, \mathbb{Z}/m(N)) & \longrightarrow & H^i_{\text{et}}(\bar{X}, \mu_m^{\otimes N}).
\end{array}
$$

The lower row is an isomorphism by the Beilinson-Lichtenbaum conjecture, and finiteness of $H^i_{\text{M}}(\bar{X}, \mathbb{Z}/m(n))$ is equivalent to finiteness of the kernel of the cup-product with the Bott-element.

If $p = \text{char } k$, then $H^i_{\text{M}}(\bar{X}, \mathbb{Z}/p(n))$ has no localization long exact sequence. The groups $H^2_{\text{et}}(\bar{X}, \mathbb{Z}/p(1))$ and $H^3_{\text{et}}(\bar{X}, \mathbb{Z}/p(1))$ both contain the non-divisible $p$-torsion of the Brauer group $H^3(\bar{X}_\text{et}, \mathbb{Z}(1))$. For a supersingular abelian surface, this group contains the field $\bar{\mathbb{F}}_q$, hence is not finitely generated (however, its Galois invariants and coinvariants are finite). According to Milne, the unipotent part of $H^i(\bar{X}_\text{et}, \mathbb{Z}/p(n))$ and $H^{2d+1-i}(\bar{X}_\text{et}, \mathbb{Z}/p(d-n))$ are in duality. There is a long exact sequence

$$
\cdots \rightarrow H^i_{\text{M}}(\bar{X}, \mathbb{Z}/p(n)) \rightarrow H^i_{\text{et}}(\bar{X}, \mathbb{Z}/p(n)) \rightarrow H^{i-1-n}_{\text{Zar}}(\bar{X}, R^1\epsilon_*\nu^n) \rightarrow \cdots.
$$

By the above example, $H^0_{\text{Zar}}(\bar{X}, R^1\epsilon_*\nu^1)$ and $H^2_{\text{Zar}}(\bar{X}, R^1\epsilon_*\nu^1)$ are infinite, but we don’t know any example where $H^j_{\text{Zar}}(\bar{X}, R^1\epsilon_*\nu^n)$ is infinite for $j \neq n - 1, n$.

Regarding the integral structure of motivic cohomology, it is an interesting question if there is a presentation of the form

$$
H^i_{\text{M}}(\bar{X}, \mathbb{Z}(n)) \cong \mathbb{Z}^r \oplus (\mathbb{Q}/\mathbb{Z})^c \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{c_p} \oplus \text{finite}
$$

with $r = 0$ unless $i = 2n$. In this case, $c$ would be independent of $n$ as soon as $n \geq i$, and it would be interesting to study the variation of $c$ for $n < i$. The analog statement for etale cohomology is wrong due to unipotent groups appearing, see the Brauer groups above.
5 The integral Tate conjecture

In this section, $X$ is smooth and proper over a finite field $\mathbb{F}_q$.

**Proposition 5.1** Fix an integer $n$ and a scheme $X$, and consider the following statements:

a) Lichtenbaum’s conjecture \[3,4\]

b) The groups $H^i_W(X, \mathbb{Z}(n))$ are finitely generated.

c) Parshin’s conjecture.

Then $a) \iff b) \implies c)$.

**Proof.** $a) \implies c)$ This follows because $H^i_{et}(X, \mathbb{Q}(n)) \cong H^i_M(X, \mathbb{Q}(n))$, and the former vanishes for $i \neq 2n$ by hypothesis.

To show the equivalence of $a)$ and $b)$ we can assume Parshin’s conjecture, and consider the exact sequence of \[5\]

$$
\cdots \to H^i_{et}(X, \mathbb{Z}(n)) \to H^i_W(X, \mathbb{Z}(n)) \to H^i_M^{-1}(X, \mathbb{Q}(n)) \to \cdots.
$$

Then $a)$ and $b)$ imply each other for $i \leq 2n$ and $i > 2n + 2$, and we are left with

$$
0 \to H^{2n+1}_{et}(X, \mathbb{Z}(n)) \to H^{2n+1}_W(X, \mathbb{Z}(n)) \to H^{2n}_M(X, \mathbb{Q}(n))
\to H^{2n+2}_{et}(X, \mathbb{Z}(n)) \to H^{2n+2}_W(X, \mathbb{Z}(n)) \to 0.
$$

By the Weil-conjectures and counting coranks one sees that the corank $C$ of $H^{2n+2}_{et}(X, \mathbb{Z}(n)) \cong H^{2n+1}_{et}(X, \mathbb{Q}/\mathbb{Z}(n))$ and of $H^{2n}_M(X, \mathbb{Q}/\mathbb{Z}(n))$ agree.

a) $\implies$ b): Finiteness of $H^{2n+1}_{et}(X, \mathbb{Z}(n))$ implies that $C$ agrees with the dimension of $H^{2n}_{et}(X, \mathbb{Q}(n))$, and since $H^{2n+2}_W(X, \mathbb{Z}(n))$ is torsion, $H^{2n+1}_W(X, \mathbb{Z}(n))/\mathrm{tor}$ must be a lattice of the same rank. Moreover $H^{2n}_M(X, \mathbb{Q}(n))$ surjects onto the divisible part of $H^{2n+2}_{et}(X, \mathbb{Z}(n))$, hence $H^{2n+2}_W(X, \mathbb{Z}(n))$ is finite.

b) $\implies$ a) We see that $H^{2n+1}_{et}(X, \mathbb{Z}(n))$ is finite, because $H^{2n+1}_W(X, \mathbb{Q}(n)) \cong H^{2n}(X, \mathbb{Q}(n)) \cong \mathbb{Z}$. For the same reason, $H^{2n+2}_{et}(X, \mathbb{Z}(n))$ is an extension of the finite group $H^{2n+2}_W(X, \mathbb{Z}(n))$ and a torsion divisible group of finite corank.

**Q.E.D.**

Let us relate the cycle map to the change of topology map for motivic cohomology. Using the functorial identification $\mathbb{Z}/m(n) \cong \mathbb{Z}/m$ of \[11\] and $\mathbb{Z}/p^r(n) \cong \nu^r[-n]$ of \[10\], the $l$-adic cycle map can be factored as follows:

$$
c : H^i_M(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \xrightarrow{\nu} H^i_{et}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \xrightarrow{\nu} H^i_{et}(X, \mathbb{Z}(n))^\wedge l \xrightarrow{\nu} H^i_{et}(X, \mathbb{Z}(n)).
$$

Here we write $A^\wedge l$ for the $l$-adic completion, and $H^i_{et}(X, \mathbb{Z}_l(n))$ for the $l$-adic cohomology $\lim H^i_{et}(X, \mathbb{Z}/l^r(n))$ to distinguish it from the hypercohomology of $\mathbb{Z}(n) \otimes \mathbb{Z}_l$. 


Lemma 5.2 The completion map $v$ is surjective.

Proof. The $\mathbb{Z}_l$-module $H^i_{et}(X, \mathbb{Z}(n))^\wedge \subseteq H^i_{et}(X, \hat{\mathbb{Z}}_l(n))$ is finitely generated and the cokernel of $v$ is $l$-divisible by [12, (4.2)], hence must be trivial. $Q.E.D.$

Proposition 5.3 We have an exact sequence

$$0 \to \ker u \to \ker c \to \text{div} H^i_{et}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \to \coker u \to \coker c \to T_l H^{i+1}_{et}(X, \mathbb{Z}(n)) \to 0.$$  

The groups $\ker u$ and $\coker u$ are torsion.

Proof. This is the kernel-cokernel sequence of the composition $c = (wv) \circ u$. The kernel of $wv$ is the kernel of $v$, which is the group of divisible elements of $H^i_{et}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l$. The cokernel of $wv$ is the Tate module $T_l H^{i+1}_{et}(X, \mathbb{Z}(n))$. $Q.E.D.$

Corollary 5.4 Under Conjecture [3.4], the kernel and cokernel of $u$ are equal to the kernel and cokernel of the cycle map, respectively.

Proof. If $i \neq 2n$, then all groups in the factorization of $c$ are expected to be finite, and $v$ and $w$ are isomorphisms. The cycle map $CH^n(X) \otimes \mathbb{Z}_l \to H^i_{et}(X, \hat{\mathbb{Z}}(n))$ is rationally surjective if and only if $T_l H^{i+1}_{et}(X, \mathbb{Z}(n))$ vanishes, hence Tate’s conjecture is equivalent to $\text{Div} H^{i+1}_{et}(X, \mathbb{Z}(n))$ torsion free. It is rationally injective (Beilinson’s conjecture) if and only if $\text{div} H^{i+1}_{et}(X, \mathbb{Z}(n))$ is torsion. $Q.E.D.$

5.1 The algebraic closure of a finite field

Tate’s original conjecture involved cycles over the algebraic closure and taking invariants under the Galois group. Let $X_r = X \times_{\mathbb{F}_q} \mathbb{F}_{q^r}$, $G_r = Gal(\mathbb{F}_q/\mathbb{F}_{q^r})$ and consider the colimit over the composition
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\[ H^*_M(X_r, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \xrightarrow{\gamma} H^*_M(X_r, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \xrightarrow{v} H^*_\text{et}(X_r, \mathbb{Z}(n))^\wedge \xrightarrow{w} H^*_\text{et}(X_r, \hat{\mathbb{Z}}(n)) \rightarrow H^*_\text{et}(\bar{X}, \hat{\mathbb{Z}}(n))^G_r. \] (14)

It is clear that the first two groups commute with the colimit in \( r \), so if we write \( \gamma_*M = \text{colim} M^G_r \) for the largest continuous submodule of a Galois-module, then the colimit takes the form

\[ \bar{c} : H^i_M(\bar{X}, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \xrightarrow{\bar{u}} H^i_M(\bar{X}, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \xrightarrow{\bar{v}} \text{colim} H^i_\text{et}(X_r, \mathbb{Z}(n))^\wedge \xrightarrow{\bar{w}} \text{colim} H^i_\text{et}(X_r, \hat{\mathbb{Z}}(n)) \rightarrow \gamma_*H^i_\text{et}(\bar{X}, \hat{\mathbb{Z}}(n)). \] (15)

**Corollary 5.5** Under Conjecture \( \Delta \), \( \text{coker} \bar{c} = \text{coker} \bar{u} \).

Note that surjectivity of \( v \) implies that \( \text{colim} H^i_\text{et}(X_r, \mathbb{Z}(n))^\wedge \) surjects onto \( H^i_\text{et}(\bar{X}, \mathbb{Z}(n))^\wedge \), but the example of \( \text{Br}(X) = H^2_\text{et}(X, \mathbb{Z}(1)) \) shows that this surjection is not an isomorphism in general.

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