General covariant \(xp\) models and the Riemann zeros

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Abstract

We study a general class of models whose classical Hamiltonians are given by

\[ H = U(x)p + V(x)/p, \]

where \(x\) and \(p\) are the position and momentum of a particle moving in one dimension, and \(U\) and \(V\) are positive functions. This class includes the Hamiltonians \(H_I = x(p + 1/p)\) and \(H_{II} = (x + 1/x)(p + 1/p)\), which have been recently discussed in connection with the nontrivial zeros of the Riemann zeta function. We show that all these models are covariant under general coordinate transformations. This remarkable property becomes explicit in the Lagrangian formulation which describes a relativistic particle moving in a \((1+1)\)-dimensional spacetime whose metric is constructed from the functions \(U\) and \(V\). General covariance is maintained by quantization and we find that the spectra are closely related to the geometry of the associated spacetimes. In particular, the Hamiltonian \(H_I\) corresponds to a flat spacetime, whereas its spectrum approaches the Riemann zeros on average. The latter property also holds for the model \(H_{II}\), whose underlying spacetime is asymptotically flat. These results suggest the existence of a Hamiltonian whose underlying spacetime encodes the prime numbers, and whose spectrum provides the Riemann zeros.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In 1999, Berry and Keating conjectured that an appropriate quantization of the classical Hamiltonian \(H = xp\) of a particle moving on the real line could provide the long sought spectral realization of the Riemann zeros [1, 2]. These authors were led to this idea by the similarity between the semiclassical spectrum of a regularized version of the \(xp\) model and the average distribution of the Riemann zeros. The regularization introduces the constraints \(|x| \geq \ell_x\) and \(|p| \geq \ell_p\) in position and momentum such that the product of their minimal values is equal to the Planck constant \((\ell_x\ell_p = 2\pi\hbar)\). This proposal was made in the framework of
quantum chaos and spectral statistics [3–5]. About the same time, Connes proposed another
regularization of $xp$ based on the constraints $|x| \leq A$ and $|p| \leq A$, where $A$ is a common
cutoff [6]. In the limit where $A$ is sent to infinity, one obtains a continuum spectrum where the
Riemann zeros are absorption spectral lines, according to Connes. This interpretation underlies
the adelic approach to the Riemann hypothesis. These results have motivated several works in
the last few years on the $xp$ model, and related quantum mechanical models, for their possible
connection with the Riemann zeros [7–21] (see [22] for a review on physical approaches to
the Riemann hypothesis).

Particularly relevant to this paper are the recent works [18, 19], which propose two
different modifications of the $xp$ Hamiltonian in order to have bounded classical trajectories
and a discrete quantum spectrum. In [18], the classical Hamiltonian is $H_1 = x(p + \ell^2_p/p)$,
which adds to $xp$ a nonstandard term $x\ell^2_p/p$, where $\ell_p$ is a constant. The latter term implements,
in a dynamical way, the constraint $|p| \geq \ell_p$, but one still needs the constraint $x \geq \ell_x$. The
classical Hamiltonian $H_1$ can be quantized in terms of a self-adjoint operator whose spectrum
agrees asymptotically with the first two terms of the Riemann–Mangoldt formula that counts
the number of Riemann zeros [23]. The Hamiltonian $H_1$ breaks the symmetry between $x$ and $p,$
which is an appealing feature of the $xp$ model. This fact led Berry and Keating to propose a new
Hamiltonian $H_2 = (x + \ell^2_x/x)(p + \ell^2_p/p)$, which restores the $x$–$p$ symmetry and implements
dynamically both constraints on $x$ and $p$, as can be seen from the appearance of the constants
$\ell_x, \ell_p$ in it.

The aim of this paper is to generalize the previous models, considering Hamiltonians
of the form $H = U(x)p + V(x)/p$, where $U(x)$ and $V(x)$ are positive functions defined on
intervals of the real line. This class of Hamiltonians have the remarkable property of being
generally covariant, which means that they maintain their form under general coordinate
transformations, i.e. diffeomorphisms $x' = f(x)$. These transformations change the functions
$U$ and $V$, according to prescribed laws, but the physical observables, such as energies, remain
unchanged. General covariance is a signature of gauge symmetry, as it occurs in general
relativity. Indeed, we will show that the present models describe the motion of a relativistic
particle moving in a (1+1)-dimensional spacetime whose metric can be constructed in terms
of the functions $U$ and $V$. The classical trajectories of the Hamiltonian $H = U(x)p + V(x)/p,$
being the geodesics of that metric. Hence these generalized $xp$ models acquire a geometrical
interpretation which gives new insights into their quantum properties, and in particular their
spectrum.

The organization of the paper is as follows. In section 2, we introduce the classical
models and show their general covariance. In section 3, we pass from the Hamiltonian to the
Lagrangian formulation and present a relativistic spacetime interpretation, which is illustrated
with several examples. In section 4, we discuss the classical trajectories in the Hamiltonian
and Lagrangian formulations. In section 5, we analyse the semiclassical spectrum of the
models introduced in section 3. We quantize the models in section 6 and show that general
covariance is maintained. Finally, we present our conclusions. We have included in appendix
A the derivation of the inverse of the semiclassical quantization formula, and in appendix B
the quantization of the Hamiltonian $H = p + \ell^2_p/p$.

2. The classical Hamiltonian

Let us consider a general class of Hamiltonians of the form

$$H = U(x)p + \frac{V(x)}{p}, \quad x \in D,$$

(1)
where \( x \) and \( p \) are the position and momentum of a particle moving in an interval \( D \) of the real line, and \( U(x) \) and \( V(x) \) are positive functions in \( D \). We will be mainly concerned with intervals that are halflines, \( D = (\ell, \infty) \), and eventually with segments i.e. \( D = (\ell_x, \ell_y) \). The two examples discussed in the introduction correspond to \([18, 19]\)

\[
H_{\text{I}} = x \left( p + \frac{\ell^2}{x} \right), \quad D = (\ell_x, \infty) \tag{2}
\]

\[
H_{\text{II}} = \left( x + \frac{\ell^2}{x} \right) \left( p + \frac{\ell^2}{x} \right), \quad D = (0, \infty). \tag{3}
\]

Berry and Keating also studied the model (3) on the whole real line, but we will not consider this case here because the corresponding functions \( U \) and \( V \) are not positive. The positivity conditions on \( U \) and \( V \) are necessary, in order to have bounded classical trajectories, but not sufficient, as shown by the example \( H = p + \ell^2/p \) (see section 3 and appendix B). It is convenient to write \( U \) and \( V \) as

\[
U(x) = u^2(x), \quad V(x) = v^2(x), \tag{4}
\]

where \( u(x) \) and \( v(x) \) will also be positive functions. The Hamiltonians (1) change their sign under the time reversal transformation, i.e.

\[
x \to x, \quad p \to -p \implies H \to -H, \tag{5}
\]

which implies that if \( \{x(t), p(t)\} \) is a classical trajectory with energy \( E \), so is \( \{x(t), -p(t)\} \) with energy \(-E\). Upon quantization, the spectrum will contain time conjugate pairs \( \{E_n, -E_n\} \) for appropriate boundary conditions related to the self-adjoint extensions of (1). The breaking of the time reversal symmetry is suggested by the statistical properties of the Riemann zeros that are described by the Gaussian unitary ensemble distribution (GUE) \([24, 25]\).

The Hamiltonian (1) is covariant under general coordinate transformations of the variable \( x \). Indeed, let us consider the infinitesimal canonical transformation

\[
x' = x + \epsilon(x), \quad p' = (1 - \partial_x \epsilon(x))p, \quad |\epsilon(x)| \ll 1 \tag{6}
\]

that preserves the Poisson bracket

\[
\{x, p\} = 1 \implies \{x', p'\} = 1 + O(\epsilon^2). \tag{7}
\]

Substituting these equations into (1), one obtains

\[
H(x, p) = [U(x') - \epsilon(x')\partial_x U(x') + (\partial_x \epsilon(x'))U(x')]|p' + [V(x') - \epsilon(x')\partial_x V(x') - (\partial_x \epsilon(x'))V(x')]|p' + O(\epsilon^2)
\]

\[
= H(x', p') + O(\epsilon^2), \tag{8}
\]

which has the same form as (1) for redefined functions

\[
U'(x') = U(x') - \epsilon(x')\partial_x U(x') + (\partial_x \epsilon(x'))U(x'),
\]

\[
V'(x') = V(x') - \epsilon(x')\partial_x V(x') - (\partial_x \epsilon(x'))V(x'). \tag{9}
\]

These equations are the infinitesimal version of the transformation laws of one-dimensional tangent and cotangent vectors

\[
U'(x') = \left( \frac{dx}{dx'} \right)^{-1} U(x(x')), \quad V'(x') = \frac{dx}{dx'} V(x(x')). \tag{10}
\]

The momentum \( p \) also transforms as a cotangent vector (i.e. one form). Another way to state these transformation laws is by saying that the products \( U(x)dx^{-1}, V(x)dx, pdx \) and also
$u(x)(dx)^{-1/2}, v(x)(dx)^{1/2}$ are invariant under reparametrizations of $x$. The latter comment is based on an analogy with conformal field theory, where the fields are characterized in a similar fashion [26]. Indeed, a chiral primary field, $\phi_h(z)$, with conformal weight $h$ satisfies that $\phi_h(z)(dz)^h$ is invariant under conformal transformation $z \rightarrow w = w(z)$. Thus, for example, $h = 1$ corresponds to a current (vector), while $h = 1/2$ corresponds to a fermion (spinor), etc. To preserve the positivity of the new functions $U'$ and $V'$, we will restrict ourselves to diffeomorphisms $x' = f(x)$ such that $df(x)/dx > 0$. These diffeomorphisms form the group denoted as $\text{Diff}^+$. The interval $D$ is mapped into the new interval $D' = f(D)$. All the models related by diffeomorphisms are equivalent at the classical level. We will organize them into equivalent classes described by the quotient

$$\mathcal{M}_g = \{U, V, D\}/\text{Diff}^+.$$ 

Each class can be uniquely characterized by a Hamiltonian which has a particularly simple form

$$w(x) = U(x) = V(x) \implies H = w(x) \left( p + \frac{1}{p} \right).$$

(12)

Any other Hamiltonian can be brought into this form by a convenient reparametrization. For example, models (2) and (3) correspond to

$$H_0 \rightarrow w_I(x) = x, \quad D = (h, \infty), \quad h = \ell_x \ell_p$$

(13)

$$H_II \rightarrow w_{II}(x) = x + \frac{\hbar^2}{x}, \quad D = (0, \infty), \quad h = \ell_x \ell_p,$$

(14)

where we made the change of variables $x' = \ell_p x$ in both cases. The function $w(x)$ is unique, up to the shift $x \rightarrow x + \text{cte}$. We will call the canonical form (12) the symmetric gauge. Other gauges are possible, as for example $U(x) = 1$, which will be briefly discussed at the end of the next section. $w(x)$ is a scalar function that can be computed in any coordinate system as

$$w(x) = u(x) v(x).$$

(15)

Using the transformation laws of $u(x)$ and $v(x)$, one can verify that $w'(x') = w(x)$, as claimed above. To find the coordinate transformation that brings a model into the symmetric gauge, consider the equation

$$\frac{v'(x')}{w'(x')} dx' = \frac{v(x)}{u(x)} dx.$$ 

(16)

In the symmetric gauge $u'(x') = v'(x')$, so integrating (16) yields the mapping $x' = f(x)$, i.e.

$$x' - \ell_x = f(x) = \int_{\ell_x}^{x} dy \frac{v(y)}{u(y)}$$

(17)

which is invertible, $x = f^{-1}(x')$, since $v(x)/u(x) > 0$. The constant $\ell_x$ is left undetermined by this map, so it can be chosen at will. $w'(x')$ is obtained using equation (15) and the inverse of (17) as

$$w'(x') = w(x) = u(x) v(x) = u(f^{-1}(x')) v(f^{-1}(x')).$$

(18)

An application of equations (17) and (18) is to show that apparently different models may turn out to be equivalent, as shown by the following example. Consider the Hamiltonian

$$H_{III} = xp + \ell_x^2 \frac{p}{x} + \ell_p \frac{x}{p}, \quad D = (0, \infty)$$

(19)
which corresponds to \( u(x) = \sqrt{x + \ell_x^2} \), \( v(x) = \sqrt{\ell_x^2} \). Let us next transform this model into the symmetric gauge. Using equation (17), one finds

\[
x' - \ell_x = \int_0^x dy \frac{\ell_p y}{\sqrt{y^2 + \ell_x^2}} \rightarrow x' = \ell_p \sqrt{x^2 + \ell_x^2}, \quad \ell'_x = \ell_x \ell_p, \tag{20}
\]

which plugged into (18) yields

\[
u_H(x') = \ell_p \sqrt{x^2 + \ell_x^2} = x', \tag{21}
\]

so that we recover model I given in (13).

In the definition of the family of Hamiltonians (1), we have imposed the positivity condition on \( U(x) \) and \( V(x) \). Let us suppose for a while that \( V(x) = 0 \). One can see that a reparametrization \( x \rightarrow x' \) can bring \( U(x) \) to \( x' \) and so all the Hamiltonians of the form \( U(x)p \) are equivalent to \( xp \).

### 3. Lagrangian formulation: relativistic spacetime picture

An essential feature of general relativity is that the fundamental equations of the theory take the same form in all coordinate systems. As shown in the previous section, this is also a feature of the models defined by the Hamiltonians (1), with respect to the coordinate \( x \). Henceforth, one may suspect the existence of a general relativistic theory lying behind these models, which would provide them with a spacetime interpretation. In this section, we will show that this is indeed the case via the Lagrangian formulation.

The Lagrangian associated with the Hamiltonian (1) is given by

\[
L = px - H = px - U(x)p - \frac{V(x)}{p}. \tag{22}
\]

In standard classical mechanics, the Lagrangian can be expressed solely in terms of \( x \) and \( \dot{x} = dx/dt \), as \( L = m\dot{x}^2/2 - V(x) \), where \( m \) is the mass of the particle and \( V(x) \) is the potential. To find \( L(x, \dot{x}) \) in our case, we use the Hamilton equation of motion

\[
\dot{x} = \frac{\partial H}{\partial p} = U(x) - \frac{V(x)}{p^2} \tag{23}
\]

to eliminate \( p \) in terms of \( x \) and \( \dot{x} \). This gives two solutions

\[
p = \eta \sqrt{\frac{V(x)}{U(x) - \dot{x}}}, \quad \eta = \pm 1 \tag{24}
\]

that depend on the sign of the momenta, \( \eta = \text{sign } p \), which is a conserved quantity. The positivity of \( U(x) \) and \( V(x) \) imply that the velocity \( \dot{x} \) must never exceed the value of \( U(x) \) for the momentum not to become an imaginary number. Substituting (24) into (22) yields a Lagrangian

\[
L_\eta(x, \dot{x}) = -2\eta \sqrt{V(x)(U(x) - \dot{x})} \tag{25}
\]

for each value of \( \eta \). Note that equation (24) is singular if \( V(x) = 0 \), so that the Lagrangian cannot be expressed in terms of \( x \) and \( \dot{x} \). This is precisely the situation of the usual \( xp \) Hamiltonian, whose Lagrangian, \( L = p(\dot{x} - x) \), has to be considered as a function of the three variables \( x, \dot{x} \) and \( p \). Later on we will give an interpretation of this peculiar fact.

At the classical level, we can restrict the motion of the particle to a definite value of \( \eta \), but not at the quantum level, where both signs would be required. The action \( S \) corresponding to (25) is (we choose \( \eta = 1 \))

\[
S = \int dt L_1 = -2 \int \sqrt{U(x)V(x)(d\dot{x})^2} - V(x) d\dot{x} dx, \tag{26}
\]
and it coincides with the action of a particle moving in (1+1)-dimensional spacetime with the metric \( g_{\mu\nu} \), i.e.

\[
S = -\int d\sigma \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} = -\int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{dx^\tau} \frac{dx^\nu}{dx^\tau}},
\]

(27)

where \( \sigma \) parametrizes the worldline (we have set the mass of the particle to 1). Making the identifications

\[
x^0 = t, \quad x^1 = x,
\]

(28)

the metric tensor becomes

\[
g_{00} = -4U(x) V(x) \equiv -4W(x), \quad g_{01} = g_{10} = 2V(x), \quad g_{11} = 0.
\]

(29)

In our conventions, the square of a line element \( dx^\mu \) will be defined as

\[
(dx)^2 = g_{\mu\nu} dx^\mu dx^\nu,
\]

(30)

so that a time-like distance corresponds to \( (dx)^2 < 0 \), and a space-like distance to \( (dx)^2 > 0 \). Equation (29) imply that, under general transformations of the coordinate \( x \), the function \( W(x) \) is a scalar, while the \( V(x) \) is a cotangent vector, in agreement with the results of section 2. The determinant of metric (29), given by

\[
g \equiv det g_{\mu\nu} = -4V^2(x),
\]

implies that \( g_{\mu\nu} \) is a nondegenerate Minkowski metric since \( V(x) > 0 \).

To gain further insight into the spacetime structure underlying the \( x^\mu p^\nu \) models, we will employ the light-cone formalism, which we now describe. Any two-dimensional metric is conformally equivalent to a flat metric. This means that it can be written as

\[
g \equiv e^{2\chi} \mu_{\nu} \nu dx dx^\nu,
\]

(31)

where \( x^{\pm} \) are the light-cone variables and \( e^\chi \) is the conformal factor. To find the transformation from the variables \( x^0, x^1 \) to the light-cone variables \( x^+, x^- \), we use the transformation law of the metric tensor

\[
g'_{\alpha\beta}(x') = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}(x),
\]

(33)

where the light-cone metric corresponds to

\[
g_{++} = g_{--} = 0, \quad g_{+-} = g_{-+} = \frac{1}{2} e^{2\chi}.
\]

(34)

Equations (29) and (34) allow us to write (33) as

\[
0 = \partial_+ x^0 [W(x^1) \partial_+ x^0 - V(x^1) \partial_+ x^1]
\]

(35)

\[
0 = \partial_- x^0 [W(x^1) \partial_- x^0 - V(x^1) \partial_- x^1]
\]

(36)

\[
e^\chi = -8W(x^1) \partial_+ x^0 \partial_- x^0 + 4V(x^1) [\partial_+ x^0 \partial_- x^1 + \partial_+ x^1 \partial_- x^0].
\]

(37)

Let us suppose, for a while, that \( x^0 \) depends nontrivially on \( x^+ \) and \( x^- \), i.e. \( \partial_\pm x^0 \neq 0 \). Hence, equations (35) and (36) would imply that

\[
\partial_\pm x^0 = \frac{\partial x^1}{U(x^1)} = \partial_\pm \int_{x_1}^{x^1} \frac{dy}{U(y)} \Rightarrow x^0 = \int_{x_1}^{x^1} \frac{dy}{U(y)} + \text{cte},
\]

(38)

so that \( x^0 \) would be a function of \( x^1 \), which is a contradiction since they are independent variables. We will make the choice that \( x^0 \) only depends on \( x^+ \):

\[
x^0 = f(x^+).
\]

(39)
Equation (36) is fulfilled automatically, and equation (35) becomes
\[
\partial_+ x^0 = \frac{\partial_+ x^1}{U(x^1)} = \partial_+ \int_{t_1}^{t_1} \frac{dy}{U(y)} \implies x^0 = \int_{t_1}^{t_1} \frac{dy}{U(y)} - g(x^-),
\]
and so
\[
\int_{t_1}^{t_1} \frac{dy}{U(y)} = f(x^+) + g(x^-),
\]
where \( f(x^+) \) and \( g(x^-) \) are generic functions. \( x^1(x^+,x^-) \) is given, in an implicit way, by equation (41). Finally, equations (37) and (41) provide the conformal factor
\[
e^{2\chi} = 4W(x^1)\partial_+ f(x^+)\partial_- g(x^-).
\]
Equations (39) and (41) give the map from \( x^{0,1} \) to \( x^\pm \). However, the map is not unique due to the freedom in choosing \( f(x^+) \) and \( g(x^-) \). This simply reflects the invariance of the metric (32) under general conformal transformations, \( x^+ \to f(x^+) \) and \( x^- \to g(x^-) \).

In the conformal gauge (34), the tensors and connections simplify considerably. The Christoffel symbols, \( \Gamma_{\mu\nu}^{\rho} \), have only ± nonvanishing components:
\[
\Gamma^{++} = g^{i\pm}\partial_i g_{\pm} = 2\partial_+ \chi, \quad \Gamma^{--} = g^{i\pm}\partial_i g_{\pm} = 2\partial_- \chi,
\]
so that the equations of the geodesics \( x^\pm(s) \) read
\[
d^2 x^\pm/ds^2 + 2 \left( \frac{dx^\pm}{ds} \right)^2 \partial_\pm \chi = 0,
\]
where \( s \) is the proper time, i.e. \( (ds)^2 = -e^{2\chi} \, dx^+ \, dx^- \). The Ricci tensor, \( R_{\mu\nu} \), becomes
\[
R_{++} = R_{--} = -2\partial_+ \partial_- \chi, \quad R_{+} = R_{-} = 0,
\]
and the Ricci scalar, \( R = g^{\mu\nu}R_{\mu\nu} \),
\[
R = -8e^{-2\chi}\partial_+ \partial_- \chi.
\]
It is not difficult to show that
\[
R(x) = -\frac{1}{V(x)} \partial_+ \left[ \frac{\partial_+ V(x)}{V(x)} \right],
\]
In the symmetric gauge (12), one has \( V(x) = w(x) \) and \( W(x) = w^2(x) \), so (47) becomes
\[
R(x) = -\frac{2}{w(x)} \frac{\partial^2 w(x)}{w(x)}.
\]
We shall use this formula to relate different \( xp \) models to the underlying spacetime geometries.

Flat spacetimes: linear and constant potentials

In flat spacetimes the scalar curvature vanishes. Equation (48) provides the function \( w(x) \) corresponding to these cases:
\[
R(x) = 0, \quad \forall x \in D \iff w(x) = \alpha x + c, \quad \alpha \geq 0.
\]
The condition \( \alpha \geq 0 \) comes from the positivity of \( w(x) \). If \( \alpha > 0 \), the shift \( x \to x - c/\alpha \) brings \( w(x) \) to the form
\[
w(x) = \alpha x, \quad \alpha > 0, \quad D = \{h, \infty\}.
\]
For \( \alpha = 1 \), this model coincides with (13). The value of \( w_0 = w(h) = \alpha h \) is independent of reparametrizations. If \( \alpha = 0 \), \( w(x) \) is constant and the Hamiltonian is simply
\[
w(x) = c > 0, \quad H = c \left( p + \frac{1}{p} \right), \quad D = \{0, \infty\},
\]
where we have chosen the origin as the boundary of the interval \( D \) (see appendix B).
**Berry–Keating model**

This model was defined in equation (14). The spacetime has a scalar curvature

\[ w(x) = x + \frac{h^2}{x} \implies R(x) = -\frac{4h^2}{x^2(x^2 + h^2)} \]  

(52)

which is always negative, vanishes asymptotically as \( x^{-4} \) and diverges at the origin as \( x^{-2} \).

**Spacetimes with a constant negative curvature**

Equation (48) admits a solution with a constant negative curvature

\[ R(x) = -|R|, \quad \forall x \in D \iff w(x) = w_0 \cosh(x\sqrt{|R|}/2), \quad w_0 > 0, \]  

(53)

where \( w_0 > 0 \) to guarantee the positivity of \( w(x) \). We will take \( D = (0, \infty) \). There is also a solution of (48) with a positive curvature involving the cosine function, but it requires finite \( D \) domains in order to maintain the positivity of \( w(x) \). We will not consider this case below. The interest of solution (53) is that the semiclassical spectrum coincides with that of the harmonic oscillator (see section 5).

**Linear-log model**

An interesting variation of the linear potential (49) is to add a subleading logarithmic term, i.e.

\[ w(x) = \alpha x + \beta \log x \implies R(x) = \frac{2\beta}{x(\alpha x + \beta \log x)} \to \frac{2\beta}{\alpha x^3}, \]  

(54)

where the curvature decays asymptotically as \( x^{-3} \), with a sign determined by that of \( \beta (\alpha > 0) \).

**Power-like models**

These models are defined by

\[ w(x) = Ax^\alpha \quad (A, \alpha > 0) \iff R(x) = -\frac{2\alpha(\alpha - 1)}{x^2}. \]  

(55)

We impose the condition \( \alpha > 0 \) to have a monotonic increasing function \( w(x) \). If \( \alpha \neq 1 \), the curvature vanishes asymptotically as \( x^{-2} \), and its sign is negative for \( \alpha > 1 \) and positive for \( 0 < \alpha < 1 \). In section 5, we will show that the asymptotic behaviour of the curvature is intimately related to the semiclassical spectrum of the model.

### 3.1. Flat spacetimes

Let us study in more detail the model (50). Since the curvature vanishes, there is a choice of \( f(x^+) \) and \( g(x^-) \) for which the conformal factor \( e^x \) is constant, and therefore the geodesics are straight lines; it is given by

\[ f(z) = g(z) = \frac{1}{2\alpha} \log z, \]  

(56)

which plugged into equations (39) and (41) yields \( (h \equiv \ell_x) \)

\[ x^0 = \frac{1}{2\alpha} \log x^+, \quad x^1 = h(x^+x^-)^{1/2}, \]  

(57)

and the conformal factor (recall (42))

\[ e^x = h. \]  

(58)
Figure 1. The region in shadow represents the universe $\mathcal{U}$ in light-cone variables defined in equation (60). Left: the hyperbola (dotted line) corresponds to the worldline of a given position $x^1 = \sqrt{x^+ x^-}$ = cte, and the vertical (dashed) line corresponds to a light ray emanating at the boundary (equations (61)). Right: the worldline of a particle with constants energy $E$, which bounces off at the boundary.

The line element

$$(dx)^2 = h^2 \, dx^+ \, dx^-$$

implies that the geodesics are straight lines in the $x^\pm$ plane. Not the whole $x^\pm$ plane is available for the motion of the particle because it is constrained to the interval $D = (h, \infty)$. In light-cone coordinates, the spacetime domain, $\mathcal{U}$, can be obtained from equation (57)

$$\mathcal{U} = \{ (x^+, x^-) | x^+ \in (0, \infty), x^+ x^- \geq 1 \}. \quad (60)$$

If $x^+$ and $x^-$ denote the vertical and horizontal axes of the plane, then $\mathcal{U}$ is the region in the first quadrant that is above the hyperbola $x^+ x^- = 1$. This hyperbola is the worldline of the point $x^1 = h$. More generally, the worldlines of any point $x^1 \geq h$, are given by the hyperbolas $x^+ x^- = (x^1/h)^2$. $x^-$ is the light-cone time coordinate and it flows upwards. Eliminating $x^+$ in equations (57) one finds

$$x^1 = h e^{\alpha x^-} (x^-)^{1/2}. \quad (61)$$

Hence, the vertical lines, i.e. constant values of $x^-$, coincide with the classical solutions of the Hamiltonian, $\alpha x p$, namely $x \propto e^{\alpha t}$. The line element (59) vanishes along these trajectories, which therefore represent light rays that start at a point on the boundary $x^1 = h$ and escape to infinity as $x^0 \to \infty$ (see figure 1):

$$x^- = \text{cte} \leftrightarrow \text{light ray}. \quad (62)$$

The line element (59) also vanishes along the horizontal lines, $x^+ = \text{cte}$, but they do not correspond to light rays since the time coordinate $x^0$ is frozen. In this theory, the light rays are right movers. The left-moving light rays are absent. This chirality is a reflection of the time reversal symmetry breaking of the Hamiltonian (1).

The causal cone, i.e. $(dx)^2 < 0$, at each point of $\mathcal{U}$, is given by the second and fourth quadrants, which correspond respectively to the future and past events relative to that point. A particle follows straight lines, with a negative slope that start and end of the boundary $x^1 = h$. 

9
To show this fact explicitly, we solve the classical equations of motion of the Hamiltonian with $w(x) = \alpha x$ for positive energy $E$ (see section 4):

$$
\begin{align*}
x^2 &= \frac{E}{\alpha} e^{2a(t-t_0)} - e^{4a(t-t_0)}, \\
p^2 &= \frac{E}{\alpha} e^{-2a(t-t_0)} - 1, \quad E > 0.
\end{align*}
$$

(63)

In light-cone variables (57), this equation becomes a straight line:

$$
q^{-a} x^+ + q^a x^- = \frac{E}{w_0}, \quad E > 0, \quad q = e^{2a h^{1/\alpha}} > 0, \quad w_0 = h \alpha,
$$

(64)

where $q$ parametrizes the slope that depends on the time $t_0$, where $x(t_0) = |p(t_0)|$. This line ends and starts at the points $A$ and $B$ of the boundary $x^+ x^- = 1$, with coordinates

$$
x_{A,B} = \frac{1}{w_0} (E \pm \sqrt{E^2 - 4w_0^2}) = q^a e^{\pm a \epsilon},
$$

(65)

where

$$
\cosh(a \epsilon) = \frac{E}{2w_0} \geq 1.
$$

(66)

The energy $E$ of the classical orbits are bounded by $2w_0$. The entire worldline of a particle with energy $E$ is given by a polygonal line made of linear segments (64) that come from the horizontal axis, $x^0 \rightarrow -\infty$, and approaches the vertical axis, $x^0 \rightarrow +\infty$ (see figure 1). The value of $q$ that parametrizes each segment can be found matching the initial and final positions of consecutive segments, i.e.

$$
x_{A,n-1} = x_{B,n} \rightarrow q_n = e^{2a} q_{n-1}, \quad n = -\infty, \infty
$$

(67)

which means that the particle in the $(n-1)$th segment bounces off at $x^1 = h$ and starts a new orbit corresponding to the $n$th segment. This polygonal worldline represents a periodic motion, since after a shift $x^0 \rightarrow x^0 + T_E$, i.e. $x^+ \rightarrow e^{2aT_E} x^+$, equation (64) remains invariant if $q \rightarrow e^{2T_E} q$ which, according to (67), gives the period $T_E$ as a function of the energy, i.e

$$
T_E = \epsilon = \frac{1}{\alpha} \cosh^{-1} \frac{E}{2w_0}.
$$

(68)

Let us next study the model defined in equation (51), which also has a vanishing curvature. A choice that leads to a constant conformal factor is

$$
f(z) = g(z) = z.
$$

(69)

which using equations (39) and (41) yields

$$
x^0 = x^+, \quad x^1 = c(x^+ + x^-),
$$

(70)

and

$$
e^\epsilon = 2c.
$$

(71)

The constraint $x^1 \geq 0$ provides the domain of spacetime

$$
\mathcal{U} = \{(x^+, x^-) | x^+ + x^- \geq 0 \},
$$

(72)

which is depicted in figure 2, which also shows the light rays and the worldline of the points. The classical equations of motion have the solutions (we choose $E > 0$)

$$
x = c(t-t_0)(1 - e^\epsilon), \quad p = e^\epsilon,
$$

(73)

$$
x = c(t-t_0)(1 - e^{-\epsilon}), \quad p = e^{-\epsilon},
$$

(74)

where $\epsilon$ is defined as

$$
\cosh \epsilon = \frac{E}{2c}, \quad \epsilon \geq 0.
$$

(75)
Figure 2. The region in shadow represents the universe $\mathcal{U}$ in lightcone variables defined in equation (72). Left: the hyperbola (dotted line) corresponds to the worldline of a given position $x^i = c(x^+ + x^-) = \text{cte}$, and the vertical (dashed) line corresponds to a light ray emanating at the boundary $x^+ + x^- = 0$. Right: worldline of a particle with constant energy $E$, described by equations (76) and (77) with $t_0 = 0$.

The solution (73) describes the particle moving towards the origin, i.e. $\dot{x} < 0$, which is reached at the time $t = t_0$. At that moment it bounces off and starts to move to the right, i.e. $\dot{x} > 0$, as described by equation (74). In the lightcone coordinates (70), equations (73) and (74) become

$$e^{-\varepsilon/2}x^+ + e^{\varepsilon/2}x^- = -2t_0 \sinh(\varepsilon/2), \quad (76)$$

$$e^{\varepsilon/2}x^+ + e^{-\varepsilon/2}x^- = -2t_0 \sinh(\varepsilon/2). \quad (77)$$

Figure 2 illustrates the form of these trajectories.

A conclusion of the results we obtained in this subsection is that the scalar curvature by itself does not fully characterize a model and that the boundary of spacetime may play an essential role.

Before we leave this section we make some remarks.

- The reformulation of the Hamiltonians (1) as relativistic models described by the Lagrangians (25) holds strictly speaking at the classical level. At the quantum level, this relation is more involved. Indeed, one should start from the path integral in phase space and perform the integration over the variable $p$. For Hamiltonians of the form $H = p^2/2m + V(x)$, this integral is Gaussian and one gets the familiar Feynman path integral with the Lagrangian $L = mx'^2/2 - V(x)$. However, for the Hamiltonians (1), the integration over $p$ is not Gaussian and in the saddle approximation, one gets extra terms in addition to the Lagrangian (25). In any case, the quantization of these models will be done in section 6 using the Hamiltonian formulation, where this issue does not arise.

- Given a model in the symmetric gauge (12), one can find a new coordinate $x'$ such that $U(x') = 1$ that we will call the $p$-gauge. The transformation $x' = f(x)$ and the new function $V'(x') \equiv V_p(x')$ can be derived from equation (10) and the scalar nature of $W(x)$:

$$x' - \ell' = f(x) = \int_{\ell(x)}^x \frac{dy}{w(y)}, \quad V_p(x') = w^2(f^{-1}(x')). \quad (78)$$
In the $p$-gauge, the Hamiltonian takes the canonical form
\[ H = p + \frac{V_p(x)}{p}, \]  
whose square
\[ H^2 = p^2 + 2V_p^2(x) + \frac{V_p^2(x)}{p^2} \]
coincides with the standard Hamiltonian with a positive potential $2V_p^2(x)$, except for the $V_p^2(x)/p^2$ term. One finds for example that the functions $V_p(x)$ associated with models (13) and (14) are given by
\[ H_{I} \to V_{p,I}(x) = \frac{\hbar^2 e^{2x}}{2}, \quad D = (0, \infty), \]  
\[ H_{II} \to V_{p,II}(x) = \frac{\hbar^2 e^{2x}}{1 - e^{-2x}}, \quad D = (0, \infty), \]
which seem to have some relation with the Liouville model and the Morse potentials studied in [13] (see references therein), whose spectrum is related to the Riemann zeros on average.

4. Classical trajectories and equations of motion

In the symmetric gauge (12), the classical trajectories in phase space are curves with constant energy $E$:
\[ E = w(x) \left( p + \frac{1}{p} \right). \]  
For each position $x$, there are in general two different values of the momentum $p$ that we denote as
\[ p_{\pm}(x, E) = \frac{1}{2w(x)} (E \pm \sqrt{E^2 - 4w^2(x)}) \]
and satisfy the relation
\[ p_{+}(x, E) = \frac{1}{p_{-}(x, E)}. \]  
In figure 3, we plot the classical trajectories corresponding to models (13) and (14). In all these models, the trajectories are clockwise for positive energy and anticlockwise for negative energy. This is a consequence of the time reversal breaking of the Hamiltonian (1).

In the model (13), the particle starts at $x = h$, with a high momentum, say $p_{\text{high}}$. Then, it moves to the right, while its momentum decreases until $|p| = 1$, where $x$ reaches a maximum $x_{\text{M}}(E)$. Afterwards, the particle moves back towards lower values of $x$ and $|p|$. When the particle reaches the boundary of the interval, i.e. $x = h$, with a momenta $|p_{\text{low}}| < 1$, it bounces off acquiring the original momenta $p_{\text{high}} = 1/p_{\text{low}}$ that satisfy equation (85). The latter relation can be written as $\log |p_{\text{high}}| = -\log |p_{\text{low}}|$ which is the analogue of the reflection of a particle in a wall. After the reflection, the particle restarts the motion from its original position and momentum, so that the classical trajectory is closed and periodic [18].

In the model (14), there is no wall for the motion of the particle, which follows a continuous and differentiable orbit around the point $(h, 1)$ in phase space [19]. In both models, for large values of $x$ and $p$, the classical trajectories approach the hyperbola $E = xp$, but they depart from it whenever $x$ or $|p|$ are of the order of the parameters $h$ or 1.
Figure 3. Classical trajectories of the Hamiltonians (2) (left) and (3) (right) in phase space with $E > 0$. We include for comparison the classical trajectory of the $xp$ model, given by the hyperbola $E = xp$ (dashed lines). The parameters of the latter Hamiltonians are chosen as $\ell_x = \ell_p = 1$. The classical trajectories with negative energy can be obtained from the trajectories with positive energy replacing $p \rightarrow -p$.

The Hamilton equations of motion for the general Hamiltonian (1) are given by

$$\dot{x} = w(x) \left( 1 - \frac{1}{p^2} \right), \quad \dot{p} = -w'(x) \left( p + \frac{1}{p} \right),$$

where $\dot{x} = dx/dt$, $w'(x) = dw(x)/dx$. Computing $\ddot{x}$, and expressing $p$ as a function of $\dot{x}$ and $w(x)$, one finds

$$\ddot{x} = w(x)w'(x) \left( -4 + \frac{6\dot{x}}{w(x)} - \frac{\dot{x}^2}{w^2(x)} \right).$$

In section 3, we formulated our model as a general relativistic theory. Equations (44) for the geodesics are therefore expected to follow from equation (87). Let us show this fact explicitly. It is convenient to choose $f(z) = g(z) = z$, so that equations (39), (41) and (42) become

$$t = x^+, \quad \partial_{x^+} x^+ = w(x), \quad e^{2x^+} = 4w^2(x),$$

where $x^0 = t$ and $x^1 = x$. The proper time $s$ and the time $t$ along the trajectory are related by

$$(dx)^2 = -e^{2x^+} dx^+ dx^- = 4w^2(x) \left( (dt)^2 - \frac{dx}{w(x)} \right),$$

so

$$\frac{dx^+}{ds} = \frac{1}{2w(x)} \left( 1 - \frac{\dot{x}}{w(x)} \right)^{-1/2}.$$ 

Taking a derivative with respect to $t$ in this equation yields

$$\frac{d^2x^+}{ds^2} + \left( \frac{dx^+}{ds} \right)^2 = \left( 1 - \frac{\dot{x}}{w(x)} \right)^{-1} \left[ -w'(x)\dot{x} - \frac{w'(x)(\dot{x})^2}{2w^2(x)} + \frac{\ddot{x}}{2w(x)} \right],$$

which together with (87) leads to

$$\frac{d^2x^+}{ds^2} + \left( \frac{dx^+}{ds} \right)^2 2w'(x) = 0.$$
This equation coincides with (44), for the $x^+$ variable, as follows from
\[
\frac{\partial_t X}{w} = \frac{\partial_x X}{w} = w'(x).
\] (93)

The geodesic equation for $x^-$ can be derived in a similar way.

Let us return now to the model defined by the linear potential (50). Here the solution of the equation of motion (86) is given by
\[
x^2 = \frac{E}{\alpha} e^{2\alpha(t-t_0)} - e^{4\alpha(t-t_0)},
p^2 = \frac{E}{\alpha} e^{-2\alpha(t-t_0)} - \frac{1}{2},
\]
where $t_0$ is the instant where $x = p$. At the initial and final times, $t_i, f$, the particle is at $x = h$, which implies that
\[
e^{2\alpha(t_f - t_0)} = \frac{E}{2\alpha} \pm \sqrt{\left(\frac{E}{2\alpha}\right)^2 - h^2},
\] (95)
so that the period of the trajectory coincides with (68):
\[
T_E = t_f - t_i = \frac{1}{\alpha} \cosh^{-1} \left(\frac{E}{2w_0}\right) \rightarrow \frac{1}{\alpha} \log \frac{E}{w_0} (E \gg w_0).\] (96)

The trajectories of the Berry–Keating model with the Hamiltonian $H_{II}$ (recall equation (14)) are given in terms of elliptic functions [19]. In figure 4, we plot $x(t)$ and $p(t)$ for the Hamiltonians $H_I$ and $H_{II}$ with the same energy. The discontinuity of the curve for the $H_I$ model is in contrast with its continuity for the $H_{II}$ model. Despite this fact, both curves have almost the same period. It is remarkable how fast the particle retraces its trajectory near the origin, which is the reason why the periods $T_E$ in the two models converge for large trajectories.

The general expression of the period of a trajectory with energy $E$ can be obtained integrating equations (86):
\[
T_E = \int_{x_m}^{x_M} \frac{dx}{w(x)} \sqrt{E^2 - 4w^2(x)}.
\] (97)
where $x_m$ and $x_M$ are the turning points. The minimal value $x_m$ can actually coincide with the boundary value of the interval $D$, as for the $H_I$ model.

5. Semiclassical analysis

The number of semiclassical states with energy between 0 and $E > 0$, denoted as $n(E)$, is given by the area in phase space swept by the closed trajectory, measured in units of the Planck constant $2\pi \hbar$. In the symmetric gauge, it reads
\[
n(E) = \frac{1}{2\pi \hbar} \int_{0 < H < E} dx \, dp = \frac{1}{2\pi \hbar} \int_{x_m}^{x_M} \frac{dx}{w(x)} \sqrt{E^2 - 4w^2(x)},\] (98)
where $x_{m,M}$ are the turning points of the trajectory (recall (97)). We have not included in (98) the Maslow phase. The counting formula for the negative energy states is also given by (98) with $|E| = 2w(x_{m,M})$. In the next section, we will see that this symmetry is broken in general by the quantum model. In other gauges, the corresponding formula for $n(E)$ is obtained from (98) by the replacements $dx/w(x) \rightarrow dx/U(x)$ and $w^2(x) \rightarrow U(x)V(x)$, which shows that $n(E)$ is invariant under reparametrizations of $x$ (see section 2). The derivative of the area of the trajectory, $2\pi \hbar n(E)$, with respect to $E$, gives the period (97). In the cases where $w(x)$ is an invertible function, and assuming that $x_m = x_0$, one can invert equation (98) in order to find the function $w(x)$, or rather $x(w)$, that produces a given $n(E)$. The formula is given by (see appendix A for the proof)

\[
\frac{x(w) - x_0}{2\hbar w} = \int_{w_0}^{w} dE \frac{d}{dE} \left( \frac{n(2E)}{E} \right) \frac{1}{\sqrt{w^2 - E^2}}.
\]

(99)

Let us apply equations (98) and (99) to the models studied in section 3.

Flat spacetimes: linear potential

In the case of the linear potential $w(x) = \alpha x$, $D = (h, \infty)$, one finds [18]

\[
n(E) = \frac{E}{2\pi \hbar \alpha} \left( \cosh^{-1} \frac{E}{2w_0} - \sqrt{1 - \left( \frac{2w_0}{E} \right)^2} \right)
\]

(101)

\[
\underset{E \gg 1}{\longrightarrow} \frac{E}{2\pi \hbar \alpha} \left( \log \frac{E}{w_0} - 1 \right) + O(E^{-1}) \quad (E \gg w_0),
\]

where $w_0 = w(h) = \alpha h$. The leading term, $O(E \log E)$, depends only on $\alpha$, and the next to leading term, $O(E)$, depends on $\alpha$ and $w_0$. Let us compare equation (101), with the Riemann–Mangoldt formula up to a height $t$ [23]:

\[
\langle n(t) \rangle \simeq \frac{t}{2\pi} \left( \log \frac{t}{2\pi} - 1 \right) + \frac{7}{8} + O(t^{-1}), \quad t \gg 1.
\]

(102)

The first two leading terms in this formula agree with those of (101) under the identifications

\[
t = \frac{E}{\hbar \alpha}, \quad h = 2\pi \hbar.
\]

(103)

Setting $\alpha$ to 1, we recover the model I. Note that the constant term in (101) vanishes, unlike in Riemann’s formula where it is given by $7/8$.

Linear-log model

One may ask which modification of $w(x)$ would yield a counting function $n(E)$ similar to (101), but containing a constant term, i.e.

\[
n(E) = \frac{E}{2\pi \hbar} \left( \cosh^{-1} \frac{E}{2w_0} - \sqrt{1 - \left( \frac{2w_0}{E} \right)^2} \right) + \mu.
\]

(104)

To find this potential, we use equation (99):

\[
x - x_0 = w - w_0 + \mu \hbar \log \frac{1 - \left( \frac{w_0}{w} \right)^2}{1 + \left( \frac{w_0}{w} \right)^2}
\]

\[
\simeq w - w_0 + 2\mu \hbar \log \frac{w_0}{2w}, \quad w \gg w_0,
\]

(105)
whose inverse is
\[ w(x) \sim x + 2\mu \hbar \log x, \quad x \gg 1. \] (106)
Based on this result we expect that a function \( w(x) \) reproducing the Riemann zeros would contain a \( \log x \) in its asymptotic expansion.

**The Berry–Keating model**

The semiclassical spectrum associated with the function \( w(x) = x + \hbar^2/x \), defined in the halfline \( D = (0, \infty) \), is given by [19]
\[
 n(E) = \frac{E}{2\pi \hbar} \left[ K \left( 1 - \frac{16\hbar^2}{E^2} \right) - E \left( 1 - \frac{16\hbar^2}{E^2} \right) \right],
\]
where \( K \) and \( E \) are the elliptic integrals
\[
 K(k^2) = \int_0^{\pi/2} dx (1 - k^2 \sin^2 x)^{-1/2}, \quad E(k^2) = \int_0^{\pi/2} dx (1 - k^2 \sin^2 x)^{1/2}.
\]
Choosing \( t = E/\hbar \) and \( \hbar = 2\pi \hbar \), as in (103), one gets an agreement with the first two terms of the Riemann–Mangoldt formula (102), in the large \( E \) limit. The constant term is absent in (107) which we believe is due to the absence of the term \( \log x \) in \( w(x) \) for large values of \( x \). All these features are shared with the linear potential analysed previously.

**Spacetime with constant negative curvature**

The semiclassical spectrum associated with the function \( w(x) \) given by (53) and a domain \( D = (0, \infty) \) is
\[
 E_n = \hbar \omega (n + \mu), \quad n(E) = \frac{E}{\hbar \omega} - \mu,
\]
where \( \mu \) and \( \omega \) are related to the parameters \( w_0 \) and the scalar curvature \( R = -|\mathcal{R}| \) as
\[
 R = -\frac{1}{2(\mu \hbar)^2}, \quad \omega = \frac{2w_0}{\mu \hbar}.
\]
It is an interesting fact that the harmonic oscillator spectrum is associated with a spacetime with constant negative curvature, at least semiclassically. This result does not contradict the chaotic spectrum associated with models defined in two-dimensional Euclidean spaces with constant negative curvature, since they have different spatial dimensions, i.e. one versus two [27].

**Power-like models**

To study the semiclassical spectrum of the models defined in (55), we will distinguish the cases \( \alpha < 1 \) and \( \alpha > 1 \), which have very different properties. For functions \( w(x) \) that grow more slowly than \( x \), one obtains
\[
 0 < \alpha < 1 \implies n(E) \propto E^{\frac{\alpha}{2}}, \quad E_n \propto n^\alpha.
\]
(110)
The density of states \( dn/dE \) behaves as \( E^{\frac{\alpha}{2}-1} \) and diverges in the limit \( \alpha \to 0 \), where it becomes a continuum. The model \( \alpha = 0 \) corresponds to a constant \( w(x) \) and it has a continuum spectrum (see appendix B). For potentials that grow faster than \( x \), one finds
\[
 \alpha > 1 \implies n(E) \propto E + O(E^{\frac{1}{2}}), \quad E_n \propto n + O(n^{\frac{1}{2}})
\]
(111)
In models where\( w(x) \) becomes linear for \( x \gg 1 \), the first two leading terms of \( n(E) \) are of order \( E \log E \) and \( E \). The converse is also true, i.e. the latter asymptotic behaviour of \( n(E) \) forces \( w(x) \) to be linear for \( x \gg 1 \). Correspondingly, the scalar curvature \( R(x) \) vanishes faster than \( x^{-2} \).

The appearance of a constant, in the next leading correction to \( n(E) \), requires a logarithmic term in \( w(x) \), in addition to the linear term.

Based on these arguments, one is led to the conjecture that the function \( w(x) \), whose quantum spectrum yields the exact Riemann zeros, is of the form

\[
    w(x) = x + \mu \log x + w_\theta(x),
\]

(112)

where \( w_\theta(x) \) represents the fluctuation part of \( w(x) \). The role of \( w_\theta(x) \) is to provide the fluctuation term \( n_\theta(t) \) of the Riemann–Mangoldt formula for the exact position of the Riemann zeros:

\[
    n_\theta(t) = \langle n(t) \rangle + n_\theta(t), \quad n_\theta(t) = \frac{1}{\pi} \arg \xi \left( \frac{1}{2} + it \right),
\]

where \( \xi(s) \) is the Riemann zeta function. It is not clear at the moment how to construct \( w_\theta(x) \), or even if it exists.

### 6. Quantization

To quantize the classical Hamiltonian (1), we will choose the normal ordering prescription

\[
    \hat{H} = u(x)\hat{p}u(x) + v(x) - \frac{1}{p} v(x), \quad x \in D = (\ell_x, \infty)
\]

(113)

where \( \hat{p} = -i\hbar/dx \) and \( 1/\hat{p} \) is the one-dimensional Green function

\[
    \langle x | \frac{1}{\hat{p}} | y \rangle = -\frac{i}{\hbar} \theta(y - x).
\]

(114)
written in terms of the Heaviside step function. This normal ordering generalizes the one in [18, 19]. One can check that \(1/\hat{p}\) is the inverse of \(\hat{p}\) acting on wavefunctions \(\psi(x)\) that vanish in the limit \(x \to \infty\):

\[
\left(\hat{p} \frac{1}{\hat{p}} \psi\right)(x) = -\frac{\text{d}}{\text{d}x} \int_{\ell_c}^{\infty} \text{d}y \theta(y - x) \psi(y) = \int_{\ell_c}^{\infty} \text{d}y \delta(y - x) \psi(y) = \psi(x),
\]

\[
\left(\frac{1}{\hat{p}} \hat{p} \psi\right)(x) = -\int_{\ell_c}^{\infty} \text{d}y \theta(y - x) \frac{\partial \psi(y)}{\partial y} = -\lim_{y \to \infty} \psi(y) + \psi(x),
\]

(115)

where we have assumed that \(x > \ell_c\). There exist other possible normal orderings defining \(\hat{H}\), as for example \(\frac{1}{2}(u^2(x) \hat{p} + \hat{p} u^2(x) + v^2(x) \hat{p}^{-1} + \hat{p}^{-1} v^2(x))\) (see also [19]). However, the choice (113) yields a Schrödinger equation which, as we will see below, is equivalent to a second order differential equation, supplemented with a nonlocal boundary condition. Our construction parallels in that way the Schrödinger equation arising from the quantum Hamiltonian \(\hat{H} = \hat{p}^2/2m + V(x)\). The action of (113) on a wavefunction \(\psi(x)\) is given by

\[
(\hat{H} \psi)(x) = -i\hbar u(x) \frac{\text{d}}{\text{d}x} \{u(x) \psi(x)\}
\]

\[
-\hbar^{-1} \int_{\ell_c}^{\infty} \text{d}y \psi(x) \theta(y - x) v(y) \psi(y).
\]

(116)

This operator will have a real spectrum if it is self-adjoint, which requires, first of all, that it be symmetric [29]:

\[
\langle \psi_1 | \hat{H} | \psi_2 \rangle = \langle \hat{H} | \psi_1 | \psi_2 \rangle, \quad \forall \psi_1, \psi_2 \in \mathcal{D}(\hat{H}),
\]

(117)

where \(\mathcal{D}(\hat{H})\) is the domain of the operator \(\hat{H}\). To study this condition, let us define the quantity [30]

\[
\langle \psi_1 | \hat{H} | \psi_2 \rangle - \langle \hat{H} | \psi_1 | \psi_2 \rangle = -i\Omega_{12}.
\]

(118)

Using (116), one finds

\[
\Omega_{12} = \hbar \int_{\ell_c}^{\infty} \text{d}x \frac{\text{d}}{\text{d}x} \{u^2(x) \psi_1^*(x) \psi_2(x)\}
\]

\[
+ \hbar^{-1} \int_{\ell_c}^{\infty} \int_{\ell_c}^{\infty} \text{d}x \text{d}y \psi_1^*(x) v(x) \psi_2(y) \theta(x - y) + \psi_2^*(y) \theta(y - x)
\]

\[
= -hu^2(\ell_c) \psi_1^*(\ell_c) \psi_2(\ell_c) + \hbar^{-1} \int_{\ell_c}^{\infty} \int_{\ell_c}^{\infty} \text{d}x \text{d}y \psi_1^*(x) v(x) \psi_2(y) + \psi_2^*(y) \theta(y - x),
\]

(119)

where we have assumed that \(\lim_{x \to \infty} u(x) \psi_{1,2}(x) = 0\). Hence, \(\hat{H}\) is a symmetric operator iff \(\Omega_{12} = 0\) which, in view of equation (119), is guaranteed if \(\psi_{1,2}\) satisfy the equation

\[
e^{i\vartheta} hu(\ell_c) \psi(\ell_c) + \hbar^{-1} \int_{\ell_c}^{\infty} \text{d}x \psi(x) = 0,
\]

(120)

where \(\vartheta\) parameterizes the self-adjoint extensions of \(\hat{H}\). The Schrödinger equation of the Hamiltonian (116) is given by

\[
-h u(x) \frac{\text{d}}{\text{d}x} \{u(x) \psi(x)\} - i\hbar^{-1} \int_{\ell_c}^{\infty} \text{d}y \psi(x) \theta(y - x) v(y) \psi(y) = E \psi(x),
\]

(121)

which we write as

\[
\hbar u(x) \frac{\text{d}}{\text{d}x} \{u(x) \psi(x)\} - \frac{iE}{v(x)} \psi(x) + \hbar^{-1} \int_{\ell_c}^{\infty} \text{d}y \theta(y - x) v(y) \psi(y) = 0.
\]

(122)
Taking a derivative with respect to $x$ and letting $x = \ell$, one gets
\begin{equation}
\frac{d}{dx} \left( \hbar u(x) \frac{d}{dx} [u(x)\psi(x)] - \frac{iE}{v(x)} \psi(x) \right) - \hbar^{-1} v(x)\psi(x) = 0, \tag{123}
\end{equation}

\begin{equation}
\frac{h}{v(x)} \left( \hbar u(x) \frac{d}{dx} [u(x)\psi(x)] - \frac{iE}{v(x)} \psi(x) \right) + \hbar^{-1} \int_{\ell}^{\infty} dy v(y)\psi(y) = 0. \tag{124}
\end{equation}

It would seem that one has to solve simultaneously equations (123) and (124). However, for wavefunctions that decay sufficiently fast at infinity, the latter equation follows from the integration of the former one, dropping a term at infinity. Hence, the Schrödinger equation (121) is equivalent to the second order differential equation (123), which justifies the normal ordering (113).

Equations (123) and (124) exhibit the general covariance discussed in section 2. This can be shown using the transformation laws of $u(x)$, $v(x)$ given in section 2, together with that of $\psi(x)$,
\begin{equation}
\psi'(x')(dx')^{1/2} = \psi(x)(dx)^{1/2}, \tag{125}
\end{equation}

which preserves the form of the scalar product of the Hilbert space under diffeomorphism:
\begin{equation}
\int_{\ell}^{\infty} dx' (\psi'(x'))^* \psi'(x') = \int_{\ell}^{\infty} dx \psi^*(x)\psi(x). \tag{126}
\end{equation}

As was done in the classical and semiclassical analysis, it is convenient to work in the symmetric gauge. Let us transform equation (123) into that gauge. First we write it as
\begin{equation}
\frac{h}{v(x)} \left( \hbar u(x) \frac{d}{dx} [u(x)\psi(x)] - \frac{iE}{u(x) v(x)} u(x)\psi(x) \right) - u(x)\psi(x) = 0
\end{equation}

and make the transformation $x \rightarrow x' (17)$ to the symmetric gauge (i.e $dx' = dx v(x)/u(x)$),
\begin{equation}
\frac{h}{v(x')} \left( \hbar \frac{d\phi(x')}{dx} - \frac{iE}{w(x')} \phi(x') \right) - \phi(x') = 0, \tag{127}
\end{equation}

where $w(x')$ is given by equation (18), and the function $\phi$ is defined as
\begin{equation}
\phi(x') = u(x(x'))\psi(x(x')). \tag{128}
\end{equation}

Note that $\phi(x)$ is a scalar function, i.e. $\phi'(x') = \phi(x)$. Now we rename $x'$ as $z$, and write equations (127) and (120), for an eigenfunction $\phi_{E}(z)$, with energy $E$, as
\begin{equation}
\frac{h}{dz} \left( \hbar \frac{d\phi_{E}(z)}{dz} - \frac{iE}{w(z)} \phi_{E}(z) \right) - \phi_{E}(z) = 0, \quad z \geq z_0 \tag{129}
\end{equation}

\begin{equation}
\hbar e^{i\theta} \phi_{E}(z_0) + \hbar^{-1} \int_{z_0}^{\infty} dz \phi_{E}(z) = 0. \tag{130}
\end{equation}

The norm of the wavefunction $\psi(x)$ in this gauge becomes
\begin{equation}
\langle \psi|\psi \rangle = \int_{\ell}^{\infty} dx \psi^*(x)\psi(x) = \int_{z_0}^{\infty} dz \frac{w(z)}{w(z)} \phi^*(z)\phi(z). \tag{131}
\end{equation}

Let us next discuss the behaviour of the eigenfunctions under the time reversal transformation. The classical model has the property that if $\{x(t), p(t)\}$ is a trajectory with energy $E$, then $\{x(t), -p(t)\}$ is a trajectory with energy $-E$. Under time reversal, the quantum Hamiltonian (113) changes sign, which leads us to expect a relation between eigenfunctions with energies $E$ and $-E$. Indeed, taking the complex conjugate of equations (129) and (130), and comparing them with those for an eigenfunction $\phi_{-E}(z)$, one finds
\begin{equation}
\phi_{E}(z) \propto \phi_{-E}(z) \iff e^{i\theta} = e^{-i\theta} \iff \theta = 0 \text{ or } \pi. \tag{132}
\end{equation}
Hence, if \( \vartheta = 0 \) or \( \pi \), the spectrum displays the time reversal symmetry \( E \leftrightarrow -E \) (this fact was already observed in [19]). The difference between these cases resides in the existence of a zero energy state. The solution of equation (129) for \( E = 0 \) is given by
\[
\phi_{E=0}(z) = A e^{-z/h} + B e^{z/h}. \tag{133}
\]
The exponential growing term \( e^{z/h} \) will typically yield unnormalizable wavefunctions, so we only consider the decaying term. In this case, the nonlocal boundary condition (130) becomes
\[
\hbar (e^{i\vartheta} + 1) e^{-h/\hbar} = 0 \implies \vartheta = \pi \tag{134}
\]
so only for this value of \( \vartheta \) is \( \phi_0 = e^{-z/h} \) an eigenfunction of the Hamiltonian with a norm given by
\[
\langle \psi_0 | \psi_0 \rangle = \int_{z_0}^{\infty} \frac{dz}{w(z)} e^{-2z/h} \tag{135}
\]
which will be finite for a large class of models that includes the ones studied in sections 3 and 5. In summary, when \( \vartheta = 0 \), the spectrum of the Hamiltonian (113) contains time-reversed pairs \( \{E_n, -E_n\} \), excluding the zero energy, while if \( \vartheta = \pi \), in addition to the time-reversed pairs, there is a normalizable zero energy state. Depending on the form of \( w(z) \), the spectrum may, or may not, contain a continuum part.

6.1. The model I

In the following subsection, we will apply the previous formalism to the model with a linear potential \( w(x) [18] \):
\[
w(z) = z, \quad D = (z_0, \infty). \tag{136}
\]
The eigenfunctions of the Hamiltonian (113) are the solutions of the differential equation (129), which in this case read (\( h = 1 \))
\[
\phi''(z) - \frac{iE}{z} \phi'(z) + \left( \frac{iE}{z^2} - 1 \right) \phi(z) = 0. \tag{137}
\]
The solutions of this equation are given essentially by the modified Bessel functions, but only the \( K \)-Bessel function gives a normalizable solution
\[
\phi_\nu(z) = A_\nu z^{1-\nu} K_\nu(z), \quad \nu = \frac{1}{2} - \frac{iE}{2}, \tag{138}
\]
where \( A_\nu \) is the normalization constant. Using equation (128), one obtains
\[
\psi_E(z) = A_\nu z^\frac{iE}{2} K_{1-\frac{iE}{2}}(z), \quad z \geq z_0, \tag{139}
\]
whose asymptotic behaviour is
\[
\psi_E(z) \propto \begin{cases} 
 z^{-\frac{1}{2}+iE}, & z_0 < z \ll E/2 \\
 z^{-\frac{1}{2}+\frac{iE}{2}} e^{-z}, & z \gg E/2,
\end{cases} \tag{140}
\]
which shows that in the region \( z_0 < z \ll E/2 \), the function \( \psi_E \) is given approximately by the eigenfunction of the Hamiltonian \( \hat{H} = (x\hat{p} + \hat{p}x)/2 \) [1, 9, 10]. Note that \( x_M = E/2 \) coincides with the maximal elongation of the classical particle. Beyond this value the wavefunction decays exponentially, as corresponds to the particle entering the classical forbidden region \( x > x_M \). The equation for the eigenenergies can be obtained plugging (138) into (130), and using the integral [28]
\[
\int_{z_0}^{\infty} dz z^{1-\nu} K_\nu(z) = z_0^{1-\nu} K_{\nu-1}(z_0), \tag{141}
\]
with the result [18]
\[ e^{i \vartheta} K_v(z_0) + K_{v-1}(z_0) = 0 \rightarrow e^{i \vartheta} K_{\frac{1}{2} + \vartheta}(z_0) + K_{\frac{1}{2} + \vartheta}(z_0) = 0. \] (142)

The asymptotic behaviour of the \( K \)-Bessel function
\[ K_{\frac{1}{2} + \vartheta}(c) \sim \frac{\sqrt{\pi}}{V} e^{-\pi t^2/4} \left( \frac{t}{z_0} \right)^{\vartheta/2}, \quad t \gg 1, \] (143)
yields the asymptotic limit of (142):
\[ \cos \left( \frac{E}{2} \log \left( \frac{E}{z_0 e} \right) - \frac{\vartheta}{2} \right) = 0, \quad E \gg 1, \] (144)
so that the eigenenergies \( E > 0 \) behave as
\[ n(E) = \frac{E}{2\pi} \left( \log \frac{E}{z_0} - 1 \right) - \frac{\vartheta}{2\pi} - \frac{1}{2} \in \mathbb{Z}. \] (145)

The two leading terms in this equation agree with the Riemann formula (102) and the semiclassical result (101), with the identification \( z_0 = 2\pi \), which is the same as in equation (103) (note that \( w_0 = z_0 = h \)). We must set \( \vartheta = 0 \) to guarantee time-reversed eigenenergies in analogy with the symmetry of the Riemann zeros on the real axis. The quantization of the model brings in a constant factor \(-1/2\) in the counting formula (145) so that the factor 7/8 of Riemann’s formula remains unexplained. In this respect, we recall the comment made in the previous section that adding a term \( \log x \) to \( w(x) \) can give rise to this constant term in the counting formula.

7. Conclusions

In this paper, we have studied the properties of a family of one-dimensional classical models, and their quantized version, that are extensions of the well-known \( xp \) model. A fundamental property of these models is that they are covariant under general coordinate transformations, so that they can be organized into equivalent classes that describe the same physics. This fact suggests some sort of universality that could perhaps be explained using renormalization group arguments, as the ones employed in [8], where the operator \( 1/\hat{p} \) was also considered.

General covariance manifests itself most clearly in the Lagrangian formulation of a relativistic particle moving in a (1+1)-dimensional spacetime. The geometrical properties of this spacetime turn out to be related to the spectral properties of the associated quantum model in a deep way. To wit, when the curvature of spacetime vanishes fast enough at infinity, one obtains a spectrum that coincides with the Riemann zeros on average. It is tempting to think that certain fluctuations of asymptotically flat metrics may cause fluctuations in the spectrum so as to accurately reproduce the Riemann zeros. It is expected that these fluctuations are determined by the prime numbers, but the precise manner in which this might occur is unclear. This general class of models can arise as effective descriptions of the dynamics of an electron moving in the plane under the action of a perpendicular uniform magnetic field and an electrostatic potential of the form \( V(x, y) = U(x)y + V(x)/y \), where \( x \) and \( y \) are the two-dimensional coordinates [15]. Further investigation will be needed to clarify these issues.

We hope the results presented here shed new light on the spectral interpretation of the Riemann zeros, stimulating research into such an interdisciplinary topic.

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Appendix A. Semiclassical Abel-like inversion formula

The aim of this appendix is to derive a formula that, under certain conditions, permits us to construct the potential \( w(x) \) that reproduces a given semiclassical counting formula \( n(E) \). For Hamiltonians of the form \( H = p^2/2m + V(x) \), the answer can be obtained using the Abel inversion method \([29]\). We will also use Abel’s method for Hamiltonians of the form \( H = w(x)(p + 1/p) \). First of all, we will assume that \( w(x) \) is a monotonic increasing function in the domain \( D = (x_0, \infty) \), so that it is invertible, \( x = x(w) \). In this case, the semiclassical formula (98) becomes

\[
n(E) = \frac{1}{2\pi} \int_{x_0}^{x_M} \frac{dx}{w(x)} \sqrt{E^2 - 4w^2(x)}, \quad E = 2w(x_M),
\]

which we write as

\[
\frac{2\pi \hbar n(E)}{E} = \int_{x_0}^{x_M} \frac{dx}{\sqrt{w^2(x) - \left(\frac{2}{E}\right)^2}}.
\]

Making the change of variables (with \( E > 0 \))

\[
r(x) = \frac{1}{w(x)}, \quad s = \frac{2}{E}, \quad r_0 = \frac{1}{w_0} \geq s, \quad w_0 = w(x_0),
\]

and using the inverse \( x(w) \), we transform (A.2) into

\[
f(s) = \pi \hbar s n(s) = \int_{x_0}^{x_M} dx \sqrt{r^2(x) - s^2} = \int_{r_0}^{r_M} dr \frac{dr}{\sqrt{r^2 - s^2}}.
\]

Differentiating with respect to \( s \) yields

\[
\frac{df(s)}{ds} = \int_{r_0}^{r_M} \frac{dr}{\sqrt{r^2 - s^2}} \frac{ds}{dr} = \int_{r_0}^{r_M} \frac{dr}{\sqrt{r^2 - s^2}}.
\]

Next, we use the chain of identities

\[
\int_{y}^{y_0} \frac{ds}{\sqrt{s^2 - y^2}} \frac{df(s)}{ds} = \int_{y}^{y_0} \frac{ds}{dr} \int_{r}^{r_0} \frac{dr}{\sqrt{r^2 - s^2}} = \int_{y}^{y_0} \frac{dr}{dr} \int_{r}^{r_0} \frac{dr}{\sqrt{(y^2 - y^2)(r^2 - s^2)}} = \frac{\pi}{2} \int_{y}^{y_0} \frac{dr}{dr} = \frac{\pi}{2} (x_0 - x(y))
\]

that can be obtained from the Frobenius theorem and the integral

\[
\int_{a}^{b} \frac{dx}{\sqrt{(x^2 - a^2)(b^2 - x^2)}} = \frac{\pi}{2}, \quad 0 < a < b.
\]

Replacing \( f(s) = \pi \hbar s n(s) \) into (A.6), and undoing the change of variables (A.3) yields finally

\[
\frac{1}{2\hbar w} \int_{w_0}^{w} dE \frac{dE}{E} \left( 2E \right) \frac{1}{\sqrt{w^2 - E^2}}.
\]

It is not guaranteed \textit{a priori} that \( x(w) \) is an invertible function. This fact may restrict the functions \( n(E) \) that can be obtained as semiclassical spectrum. Another issue concerning (A.8) is the following. Adding to \( n(E) \) a term linear in \( E \) produces the same function \( w(x) \), as can be easily seen from (A.8). We lost track of this linear term when taking the derivative.
with respect to $s$ in equation (A.4). To recover this term, one has to plug the function $w(x)$, obtained from (A.8), back into (A.1), and read the linear term in $E$.

It is worth comparing these semiclassical formulae with those associated with the standard Hamiltonian $H = p^2 + V(x)$. The analogue of equation (A.1), for an even potential $V(x)$, is given by

$$n(E) = \frac{2}{\pi \hbar} \int_{s_m}^{s} \text{d}x \sqrt{E - V(x)}, \quad |E| = V(s_m),$$

(A.9)

where we have ignored the Maslov phase, and the analogue of (A.8) is given by

$$x(V) = \hbar \int_{V_0}^{V} \frac{\text{d}n(E)}{\text{d}E} \frac{1}{\sqrt{V - E}}.$$  

(A.10)

The latter equation was used by Wu and Sprung to obtain a potential $V(x)$ whose semiclassical spectrum coincides in average with the Riemann zeros ([32]; for a review see [22]):

$$x(V) = \frac{\sqrt{V}}{\pi} \log \left( \frac{2V}{\pi \epsilon^2} \right) \Rightarrow V(x) \propto \left( \frac{x}{\log x} \right)^2, \quad x, V \gg 1.$$  

(A.11)

This result was used as a seed for a numerical reconstruction of a potential whose spectrum matches a large number of Riemann zeros lying at the bottom part of the critical line. Quite interestingly, that potential has a fractal structure whose dimension is nearly the value 1.5. A problem with this approach is that the Hamiltonian is time reversal invariant, a fact which does not agree with the distribution of the Riemann zeros which follow the GUE statistic which is characteristic of time reversal breaking random Hamiltonians (see [22] for a discussion on this issue).

Independently of the previous works, Mussardo employed (A.10) to find a potential whose spectrum behaves, on average, like the prime numbers [33]. The prime number theorem (PNT) states that the number of primes up to $x$ behaves asymptotically as $\pi(x) \sim x / \log x$, so that their density decreases as $\text{d}\pi(x)/\text{d}x \sim 1/\log x$, as conjectured long ago by Gauss and Legendre [23]. The PNT implies that the $n$th prime number $p_n$ grows roughly as $p_n \sim n \log n$. Mussardo noticed that this growth allows one to find a quantum mechanical model whose energies are the prime numbers. Choosing the leading term in the expansion of the Riemann formula for $\pi(x)$,

$$\pi(x) \sim \text{Li}(x) = \int_2^x \frac{\text{d}y}{\log y},$$

(A.12)

he obtained

$$x(V) \sim \frac{\sqrt{V}}{\log V} \Rightarrow V(x) \sim (x \log x)^2, \quad (x, V \gg 1).$$  

(A.13)

This result was also used as a seed for finding a potential whose spectrum matches precisely the lowest prime numbers [22]. As in the case of the Riemann zeros, the prime potential has a fractal structure with dimension near 2. The difference in fractal dimensions, 1.5 versus 2, is consistent with the fact that the Riemann zeros are less random than the primes numbers. The former ones satisfy the GUE statistics and the latter follow an almost Poissonian statistics. The proximity of the prime number/Riemann zeros potentials to the harmonic oscillator potential is rather remarkable. In table A1, we summarize these semiclassical results together with other well-known cases.

Appendix B. Quantization of $H = \hat{p} + \ell^2/\hat{\ell}$

In standard textbooks of quantum mechanics, it is taught that the momentum operator $\hat{p} = -i\hbar d/dx$ is self-adjoint acting in the Hilbert space of square integrable functions, $L^2(D)$,
in two cases: (i) \( D = \mathbb{R} \) is the real line, and (ii) \( D = (a, b) \) is a finite interval of the real line [29, 31]. In case (i), the operator \( \hat{p} \) is essentially self-adjoint, and in case (ii) \( \hat{p} \) admits infinitely many self-adjoint extensions characterized by the boundary condition \( \psi (b) = e^{i\vartheta} \psi (a) \), where \( \vartheta \in [0, 2\pi) \). However, \( \hat{p} \) is not self-adjoint when \( D \) is the halfline, \( D = (0, \infty) \), and therefore its spectrum is not real. A solution of this problem is suggested by the model we have discussed in this paper. Indeed, let us choose the simplest non-vanishing potential \( w(x) \), namely a constant defined on the halfline \( D = (0, \infty) \). This model is equivalent to (51), via a scale transformation which gives the relation \( c = \ell_p \). We will show below that \( \hat{H} \) is self-adjoint and that its spectrum is a continuum and eventually a bound state. This model illustrates in a simple example the more complicated models considered in the main body of the paper. In spite of its simplicity, this model shares several features with the so-called Kondo model in condensed matter physics, which suggests that it may have other applications.

To quantize (B.1), we follow the steps of section 6. The Schrödinger equation is given by

\[
-i\hbar \frac{d\psi(x)}{dx} - \frac{\ell_p^2}{\hbar^2} \int_0^\infty dy \vartheta (y - x) \psi(y) = E \psi(x). \tag{B.2}
\]

Taking one derivative with respect to \( x \) gives

\[
-\hbar^2 \frac{d^2\psi(x)}{dx^2} + iE \hbar \frac{d\psi(x)}{dx} + \ell_p^2 \psi(x) = 0, \tag{B.3}
\]

whose general solution is

\[
\psi_E(x) = A(E) e^{ik_+(E)x} + B(E) e^{ik_-(E)x}, \tag{B.4}
\]

where

\[
k_\pm(E) = \frac{1}{2\hbar} \left( E \pm \text{sign}(E) \sqrt{E^2 - 4\ell_p^2} \right). \tag{B.5}
\]

We assume in (B.5) that \( E \) is real, and for other values we replace \( \text{sign}(E) \) by \( \pm E/|E| \). Using (B.5), one can compute the von Neumann defect indices \( n_\pm \), which give the number of linearly independent solutions of the equation

\[
n_\pm = \dim \{ \psi_\pm | \hat{H} \psi_\pm = \pm i\hbar \psi_\pm, \quad \text{Im} z > 0 \}. \tag{B.6}
\]

Choosing \( z = 2\ell_p c (c > 0) \), we get two normalizable solutions of (B.5), namely

\[
\psi_+(x) = A \exp \left[ -\frac{\ell_p x}{\hbar} (c + \sqrt{c^2 + 1}) \right] \implies n_+ = 1, \tag{B.7}
\]

\[
\psi_-(x) = A \exp \left[ -\frac{\ell_p x}{\hbar} (-c + \sqrt{c^2 + 1}) \right] \implies n_- = 1;
\]

in two cases: (i) \( D = \mathbb{R} \) is the real line, and (ii) \( D = (a, b) \) is a finite interval of the real line [29, 31]. In case (i), the operator \( \hat{p} \) is essentially self-adjoint, and in case (ii) \( \hat{p} \) admits infinitely many self-adjoint extensions characterized by the boundary condition \( \psi (b) = e^{i\vartheta} \psi (a) \), where \( \vartheta \in [0, 2\pi) \). However, \( \hat{p} \) is not self-adjoint when \( D \) is the halfline, \( D = (0, \infty) \), and therefore its spectrum is not real. A solution of this problem is suggested by the model we have discussed in this paper. Indeed, let us choose the simplest non-vanishing potential \( w(x) \), namely a constant defined on the halfline \( D = (0, \infty) \). This model is equivalent to (51), via a scale transformation which gives the relation \( c = \ell_p \). We will show below that \( \hat{H} \) is self-adjoint and that its spectrum is a continuum and eventually a bound state. This model illustrates in a simple example the more complicated models considered in the main body of the paper. In spite of its simplicity, this model shares several features with the so-called Kondo model in condensed matter physics, which suggests that it may have other applications.

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\[
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where

\[
k_\pm(E) = \frac{1}{2\hbar} \left( E \pm \text{sign}(E) \sqrt{E^2 - 4\ell_p^2} \right). \tag{B.5}
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\[
n_\pm = \dim \{ \psi_\pm | \hat{H} \psi_\pm = \pm i\hbar \psi_\pm, \quad \text{Im} z > 0 \}. \tag{B.6}
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Choosing \( z = 2\ell_p c (c > 0) \), we get two normalizable solutions of (B.5), namely

\[
\psi_+(x) = A \exp \left[ -\frac{\ell_p x}{\hbar} (c + \sqrt{c^2 + 1}) \right] \implies n_+ = 1, \tag{B.7}
\]

\[
\psi_-(x) = A \exp \left[ -\frac{\ell_p x}{\hbar} (-c + \sqrt{c^2 + 1}) \right] \implies n_- = 1;
\]
hence, \( n_+ = n_- = 1 \), which implies, by the von Neumann theorem, that the operator \( \hat{H} \) is self-adjoint, with infinitely many extensions parametrized by the group \( U(1) \) [29, 30]. The spectrum of \( \hat{H} \) is given by two intervals whose boundaries are \( \pm 2\ell_p \) and \( \pm \infty \) and a bound state with eigenvalue \( E_0 \):

\[
\text{spec } \hat{H} = \mathcal{C} \cup \{E_0\} = (-\infty, -2\ell_p) \cup (2\ell_p, \infty) \cup \{E_0\}. \tag{B.8}
\]

We denote by \( \mathcal{C} \) the continuum part of the spectrum. It is convenient to parametrize the two branches of the continuum as follows:

\[
E = 2\ell_p \eta \cosh u, \quad \eta = \text{sign}(E) = \pm 1, \quad u > 0, \tag{B.9}
\]

in which case the momenta (B.5) become

\[
k_+(E) = \frac{\eta \ell_p}{\hbar} e^u, \quad k_-(E) = \frac{\eta \ell_p}{\hbar} e^{-u}, \tag{B.10}
\]

and the wavefunction (B.4)

\[
\psi_E(x) = A(E) e^{i\ell_p x e^u/\hbar} + B(E) e^{i\ell_p x e^{-u}/\hbar}. \tag{B.11}
\]

The discrete eigenvalue of \( \hat{H} \) appears for \( |E_0| < 2\ell_p \) with a normalizable eigenfunction corresponding to the momenta \( k_+ \), i.e.

\[
|E_0| < 2\ell_p \implies \psi_{E_0}(x) = Ce^{-k_0 x}, \quad k_0 = -ik_+ = \frac{1}{2\hbar} \left(-iE_0 + \sqrt{4\ell_p^2 - E_0^2} \right). \tag{B.12}
\]

To find the value \( E_0 \), we impose the nonlocal boundary condition (120), which guarantees that \( \hat{H} \) is a Hermitian operator

\[
- e^{i\vartheta} \psi_{E_0}(0) + \frac{\ell_p}{\hbar} \int_0^\infty dx \psi_{E_0}(x) = 0, \tag{B.13}
\]

where \( \vartheta \) is the parameter that characterizes the self-adjoint extension of \( \hat{H} \). In equation (120), we made the shift \( \vartheta \rightarrow \vartheta + \pi \), so that the first term in (B.13) changed its sign. Plugging (B.12) into (B.13), one finds

\[
k_0 = \frac{\ell_p}{\hbar} e^{-i\vartheta}, \tag{B.14}
\]

which gives

\[
E_0 = 2\ell_p \sin \vartheta, \quad \vartheta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{B.15}
\]

The restriction on \( \vartheta \) comes from the relation \( \cos \vartheta \propto \text{Re} k_0 > 0 \). If \( \pi/2 < \vartheta \leq \pi \), equation (B.13) is not satisfied and therefore there is no bound state. We will restrict below to the case \( |\vartheta| < \pi/2 \). If \( \vartheta = 0 \), one gets \( E_0 = 0 \), so that the spectrum is time reversal symmetric. In the limits \( \vartheta \rightarrow \pm \pi/2 \), one has \( E_0 = \pm 2\ell_p \), and \( k_0 = \mp i\ell_p/\hbar \), so that the eigenfunction (B.12), becomes a plane wave. The constant \( C \) in equation (B.12) is fixed by the normalization of the wavefunction

\[
\int_0^\infty dx |\psi_{E_0}(x)|^2 = 1 \implies C = \sqrt{\frac{2\ell_p \cos \vartheta}{\hbar}}. \tag{B.16}
\]

The measure of the size of the bound state is given by the average of \( x \):

\[
\langle x \rangle = \int_0^\infty dx x |\psi_{E_0}(x)|^2 = \frac{\hbar}{2\ell_p \cos \vartheta}, \tag{B.17}
\]

and diverges in the limit \( \vartheta \rightarrow \pm \pi/2 \). The operator \( \hat{H} \) is self-adjoint, and then the spectral theorem implies that its eigenfunctions \( \psi_E \) form an orthonormal basis, namely

\[
\langle \psi_{E_0} | \psi_{E_0} \rangle = \int_0^\infty dx \psi_{E_0}(x) \psi_{E_0}(x) = 1. \tag{B.18}
\]
\[
\langle \psi_E | \psi_E \rangle = \int_0^\infty dx \, \psi^*_E(x) \psi_E(x) = 0, \quad E \in \mathcal{C}, \quad (B.19)
\]

\[
\langle \psi_E | \psi_{E'} \rangle = \int_0^\infty dx \, \psi^*_E(x) \psi_{E'}(x) = \delta(E - E'), \quad E, E' \in \mathcal{C}. \quad (B.20)
\]

(B.18) coincides with (B.16). Equation (B.19) gives the relation between the coefficients \(A(E)\) and \(B(E)\) of the wavefunction (B.11):

\[
\frac{A(E)}{B(E)} = \frac{k_+^* - ik_+(E)}{k_-^* - ik_-(E)} = -\frac{e^{i\theta} - i\eta e^{i\phi}}{e^{i\theta} - \eta e^{i\phi}}, \quad (B.21)
\]

where we have used equations (B.10) and (B.14). Condition (B.20), together with (B.21), fix the form of these coefficients. Using (B.11), one obtains

\[
\langle \psi_E | \psi_{E'} \rangle = A_E^* A_{E'} \left[ \pi \delta(k_+ - k_+) + iP \frac{1}{k_+ - k_-} \right] + B_E^* B_{E'} \left[ \pi \delta(k_- - k_-) + iP \frac{1}{k_- - k_+} \right] + A_E^* B_{E'} \left[ \pi \delta(k_+ - k_-) + iP \frac{1}{k_+ - k_-} \right] + B_E^* A_{E'} \left[ \pi \delta(k_- - k_+) + iP \frac{1}{k_- - k_+} \right],
\]

where \(k_\pm = k_\pm(E), k_\pm' = k_\pm(E')\) and \(P_\pm\) denotes the principal part of \(\frac{1}{\pm}\). To derive this equation, we have used the improper integral

\[
\int_0^\infty dx \, e^{ikx} = \pi \delta(k) + iP \frac{1}{k}, \quad (B.22)
\]

which is the integral version of the distribution identity

\[
\frac{1}{k + i0} = -i\pi \delta(k) + P \frac{1}{k}. \quad (B.23)
\]

Equation (B.20) is satisfied provided

\[
A_E^* A_{E'} \delta(k_+ - k_+) + B_E^* B_{E'} \delta(k_- - k_-) + A_E^* B_{E'} \delta(k_+ - k_-) + B_E^* A_{E'} \delta(k_- - k_+) = \frac{1}{\pi} \delta(E - E'), \quad (B.24)
\]

and

\[
(A_E^*, B_E^*) \, M \, \left( \begin{array}{c} A_E \\ B_E \end{array} \right) = 0, \quad (B.25)
\]

where \(M\) is the matrix

\[
M = \frac{1}{k_+ - k_-} \left( \begin{array}{ccc} 1 & 1 & 1 \\ k_+ - k_- & 1 & 1 \\ k_+ - k_- & 1 & 1 \end{array} \right) = \frac{\hbar}{\ell_p} \left( \begin{array}{ccc} 1 & 1 & 1 \\ \eta e^{i\phi} - \eta' e^{i\theta} & 1 & 1 \\ \eta e^{-i\phi} - \eta' e^{-i\theta} & 1 & 1 \end{array} \right). \quad (B.26)
\]

Using

\[
\delta(k_\pm - k_\pm') = \sqrt{\frac{E^2 - 4\ell_p^2}{|k_\pm|}} \delta(E - E'), \quad \delta(k_\pm - k_\pm') = 0, \quad (B.27)
\]

equation (B.24) becomes

\[
|A_E|^2 \frac{\sqrt{E^2 - 4\ell_p^2}}{|k_+|} + |B_E|^2 \frac{\sqrt{E^2 - 4\ell_p^2}}{|k_-|} = \frac{1}{\pi}, \quad (B.28)
\]
and similarly
\[ |A_E|^2 e^{-u} + |B_E|^2 e^{u} = \frac{1}{2\pi \hbar}. \]  
(B.29)

Finally, equation (B.21) implies that
\[ A(E) = \frac{e^{i\vartheta} - i\eta e^{u}}{\sqrt{8\pi \hbar (\cosh u - \eta \sin \vartheta)}}. \]  
(B.30)

These expressions satisfy equation (B.25), which finally proves that the wavefunctions (B.11), with coefficients given by (B.30), together with the normalizable state (B.12) for a nonorthonormal basis. The operator \(\hat{H}\) has the physical meaning of momentum rather than energy. Its spectrum (B.8) almost coincides with that of the momentum operator \(\hat{p}\) defined on the entire line, except in an interval around the origin \((-2\ell_p, 2\ell_p)\), which is replaced by a bound state localized at the edge of the system. This result is reminiscent of the Kondo model where a bound state is formed between an impurity localized at the origin and the conduction electrons [34]. However, in our model there is no spin and it does not describe a many-body system, so this analogy is for the time being formal.

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