On Quantum Inequalities

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Abstract

Building on the “quantum inequalities” introduced by Ford, I argue that the negative local energies encountered in quantum field theory can only be observed by detectors with positive energies at least as great in magnitude. This means that operationally the total energy density must be non-negative. Like reasoning shows that, in a similar operational sense, the dominant energy condition must hold: any timelike component of the four-momentum density is positive.

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Introduction. A startling prediction of relativistic quantum field theory is that, while the total energy of a system should be positive or zero, the energy density, and hence the energy of a subsystem, can be negative [1]. And indeed this possibility is present generically. Even for a Klein–Gordon field on Minkowski space, for any smooth compactly supported bump function $B$, the expectation values

$$\langle \hat{T}_{00}B d^3x \rangle,$$

for $\hat{T}_{00}$ the renormalized Hamiltonian density, are unbounded below [2]. Thus states with arbitrarily negative energy densities are always available. The set of states with this expectation value equal to $-\infty$ is dense in the Hilbert space.

Although negative total energy densities have never been directly observed, they have received extensive theoretical investigation because they contravene a basic tenet of classical physics. Indeed, if there were no restrictions on the negative energies achievable, there would be gross macroscopic consequences: an ordinary particle could absorb a negative energy and become a tachyon; an isolated patch of negative energy would give rise to a repulsive gravitational field; one could violate the second law of thermodynamics by using negative energies to cool systems without an increase of entropy [3–6]; and the general–relativistic effects might include traversable wormholes, “warp drives” and time machines [7–10]. Too, it is something of a puzzle why such states do not interfere with the dynamics of the quantum fields: why do not perturbations (which are always present) send the field cascading through these negative–energy states, with a corresponding release of positive–energy radiation? It is a matter of common experience that such effects do not occur, or at least not often, and therefore there must be some mechanism restricting the production of negative energy densities, their magnitudes, durations, or interactions with other matter.

Such a mechanism was first proposed by Ford, and has been investigated by him and Roman [3,5,11]. They argue that (for free Bose fields in Minkowski space) the negative energy $-\Delta E$ localizable in a time of order $\Delta t$ should satisfy a quantum inequality

$$\Delta E \Delta t \gtrsim \hbar.$$

These inequalities are powerful; they evidently limit the occurrences of negative energies considerably. However, they do not as they stand seem to be a full explanation. For one thing, the argument for (2) depends on a certain “coherence” assumption, which, as we shall discuss below, is not generally valid.\(^1\) For another,\(^2\) On the other hand, a weaker bound that Ford establishes (our inequality (4), below), which is suggestive of the averaged weak energy condition in general relativity [12], does not depend on this assumption.

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it is not clear that simply restricting the occurrences of large negative energies to short times is enough to rule out unphysical effects. Indeed, explicit analyses of attempts to violate the second law of thermodynamics indicate that while the quantum inequalities play a key role, an equally important one is played by limitations on the measuring devices. (See refs. [3–6]. These also point out that identifying the characteristic time $\Delta t$ which is relevant for a particular physical problem may not be an easy matter.)

In this paper we shall re-examine the derivation of the quantum inequalities, repair the “coherence” assumption, and argue further that a device constructed to measure or capture a local negative energy $-\Delta E$ must itself have energy at least $\Delta E$. We may say briefly that operationally the total energy must be non-negative. We believe that this principle explains why there are no gross consequences of negative energy density states, and underlies the more ad-hoc analyses which have been offered to show that particular attempts to use negative energy densities to violate the second law of thermodynamics fail.

Finally, we shall extend the reasoning to show that operationally the dominant energy condition must hold, that is, the four-momentum density must be future-pointing. This gives a new perspective on questions of how violations of the energy conditions affect the usual general relativistic singularity and positivity-of-energy theorems, as well as Hawking’s process for the evaporation of black holes, but these applications will be discussed separately. Apart from a few comments, the present paper is restricted to special-relativistic quantum field theory.

Our metric signature is $++--$.

The Quantum Inequalities. Ford and Roman have given several derivations of the quantum inequalities, but the elements which are relevant here are common to all. Consider the quantity

$$F = -\inf_{|\Phi\rangle} \int_{-\infty}^{\infty} \hat{T}_{00}(t, 0, 0, 0) b(t) \, dt |\Phi\rangle,$$

(over normalized states $|\Phi\rangle$ in the Fock space of a Klein–Gordon field in Minkowski space), where $b(t) = t_0/\left[\pi(t^2 + t_0^2)\right]$ is a “sampling function” with integral unity and characteristic width $t_0$.\(^2\) It is shown that

$$F \leq k\hbar c (ct_0)^{-4},$$

\(^2\) The sampling function is peaked over an interval of characteristic width $\sim t_0$, but not supported only there. It is not possible to localize to a sharply demarcated time interval, because the quantity $\langle \Phi|\hat{T}_{00}(t, 0, 0, 0)|\Phi\rangle$ is a distribution, and turns out to involve terms like $\delta'(t)$. It is this which is behind difficulties in identifying the correct $\Delta t$ for physical applications.
where it is known that the numerical constant \( k \leq \frac{3}{(32\pi^2)} \). Up to this point the argument is essentially mathematical.

The next step is physical. If a device were to be constructed to measure or trap this negative energy within an interval of length \( t_0 \), then in order to function coherently the linear dimension of the device must be no larger than \( ct_0 \).\(^3\) Thus the magnitude of the negative energy within the device

\[
\Delta E \leq F \cdot (4\pi/3)(ct_0)^3 \leq (4\pi k/3)\hbar t_0^{-1} \leq \hbar/(8\pi t_0).
\]  

(5)

This hypothesis of coherent functioning deserves closer scrutiny.

The trouble here is that although it may be reasonable to think of an experiment as a whole (including preparation at the start and collection of data at the end) as “coherent” on a time scale \( T_0 \), the scales \( t_0 \) of the components of the experiment may be much smaller. For example, suppose we had \( N \) devices obeying (5), so capable of detecting or trapping a negative energy \(-\varepsilon(4\pi k/3)\hbar t_0^{-1}\) in time \( t_0 \); here \( \varepsilon < 1 \) is the efficiency of the device. These devices are arranged in an array in space, and in a common rest–frame. Each carries a clock which has been synchronized with (say) a master clock in the center of the array. At a preset time, each device operates. Then, if the field is in a suitable configuration, the total negative energy absorbed will be \(-N\varepsilon(4\pi k/3)\hbar t_0^{-1}\) and the interval will be \( t_0 \). (Note that there is no requirement that the devices be near one another, so they can be separated far enough apart that locality considerations guarantee that the quantum field can indeed be in a state which will produce such a negative energy. Also note that while it is true that construction of the array of devices requires a different time scale than \( t_0 \), that time scale is larger, namely the time required to synchronize the devices, greater than \( \sim N^{1/3}t_0 \).) By choosing \( N \) large enough, we can arrange for an arbitrarily large negative energy to be trapped within a time \( t_0 \). Thus even if we start from “coherent” devices, we can create others which violate the quantum inequality (5).

We can repair this by taking into account the energy of the measuring device. A device which measures \( \int_{-\infty}^{\infty} \hat{T}_{00}(t,0,0,0)b(t)dt \) must involve some sort of clockwork mechanism and transducer which function to weight the contributions of \( \hat{T}_{ab} \) at different times by \( b(t) \). This clockwork must be able to resolve time increments of order \( t_0 \). (Actually, in order to treat the function \( b(t) \) as free from quantum indeterminacy, the temporal resolution must be finer.) Now let us recall that a clock mechanism which is accurate to a time of order \( \Delta t \) must have mass \( \gtrsim \hbar/(c^2\Delta t) \) and so energy \( E_{\text{mech}} \gtrsim \hbar/\Delta t \) [13]. For any one clock having energy \( E_{\text{mech}} \), then, controlling a measuring device, the inequality (5) applies with \( t_0 \sim \hbar/E_{\text{mech}} \).

\(^3\) Perhaps one should use \( ct_0/2 \) here. This would make the force of our later arguments somewhat stronger.
This suggests that any device controlled by a clock of energy $E_{\text{mech}}$ can detect or trap negative energies $-\Delta E$ with $-\Delta E + E_{\text{mech}} \geq 0$ only. It should be made clear that this argument is not a mathematical proof. For one thing, the quantities $\Delta t$ and $E_{\text{mech}}$ are only defined as orders of magnitude, and it is in this sense that $E_{\text{mech}} \gtrsim \hbar/\Delta t$ is known to hold. For another, the quantum inequality (5) is only been established for one form of sampling function. Nevertheless, the numerical factor $1/8\pi$ in inequality (5) is far enough below unity that it strongly suggests $E_{\text{mech}} \geq \Delta E$. For the remainder of this paper, we shall assume this is the case.

With this assumption, notice that a collection of measuring or trapping devices deployed and set to function simultaneously (or, more generally, at spacelike separations), as in the example above, will also have total energy in excess of the negative energy it can detect or trap.\footnote{The case of timelike separations requires a deeper analysis, taking into account intermediate reductions of the state vector, and will be taken up elsewhere.}

We may summarize our contention by saying that operationally, the energy must be non–negative, that is, the sum of the measured energy and the energy of the measuring device must be non–negative.

**Generality of the Results.** The analysis above is only for a Klein–Gordon field in Minkowski space, but the result is of such a form that one is led to conjecture that it holds generally. We do not expect much difficulty in extending it to higher–spin linear fields. Non–linear quantum field theories in four dimensions are not well enough understood to be able to derive rigorous estimates like Ford’s $F \leq (3/32\pi^2)\hbar c/(c t_0)^4$. However, we offer a few comments about them.

For electroweak theory (or any other weak coupling renormalizable theory), at any one energy scale the theory is a perturbation of free field theory, and our arguments should apply to this effective theory. One might hope to move beyond this by applying renormalization–group techniques to test whether Ford–type inequalities hold over a finite range of energies.

The situation for quantum chromodynamics is to some extent opposite. There the low–energy theory is highly non–linear. On the other hand, quantum chromodynamics is expected to be asymptotically free, so its short–distance behavior is that of a free theory, and one would expect a Ford–type inequality to hold at least in the limit $t_0 \to 0$.

For quantum fields in curved space–time, there is no generally–accepted renormalization prescription for the stress–energy operator. Still, it is possible one could establish a Ford–type inequality in the limit
$t_0 \to 0$, since in this limit the renormalization ambiguities become irrelevant and the asymptotic short-distance behavior is that of free fields in Minkowski space. On the other hand, it is only the leading term in the asymptotics which agrees with the Minkowskian one; the subdominant terms in curved space-times are more divergent than for Minkowski space \cite{2}. So this case requires closer investigation.

The Dominant Energy Condition. The treatment so far concerns the energy of a finite system, as measured by an inertial observer. The result localizes: even if one tries to separate the clockwork used for measuring $\int_{-\infty}^{\infty} \hat{T}_{a0}(t,0,0,0)b(t)dt$ from the world-line $(t,0,0,0)$, one must still transmit timing signals to the vicinity of this world-line, and these signals must resolve times $\gtrsim t_0$, which means the quanta carrying the signals must have energies $\gtrsim \hbar/t_0$. Thus locally the total energy, of the field plus the measuring device, must be non-negative. We may call this the operational weak energy condition: $T_{ab}^{\text{op}} t^a t^b \geq 0$ for all timelike vectors $t^a$.

It is possible to derive a stronger result, the operational dominant energy condition: $T_{ab}^{\text{op}} t^a u^b \geq 0$ for all future-pointing vectors $t^a$ and $u^a$. The changes needed to the treatment above are as follows.

Let $t^a$ be a unit future-pointing vector along the observer’s time axis. Let $u^a$ be another unit future-pointing vector. We shall be concerned with a local measurement by the observer of a component $P_a u^a$ of the momentum in some region.

The following quantum inequality is easily provable by the techniques of ref. [11]: Let

$$\Pi_a = \langle \int_{-\infty}^{\infty} \hat{T}_{a0}(t,0,0,0)b(t)dt \rangle ;$$

then $\Pi_a u^a \geq -(3/32\pi^2)t \cdot u \hbar c/(ct_0)^4$ for any future-pointing vector $u^a$, so $P_a u^a \geq -(1/8\pi)t_a u^a \hbar /t_0$. Equivalently,

$$P_a = -(1/8\pi)(\hbar/t_0)t_a + \text{ a future–pointing vector} .$$

Now consider a clock which may be boosted relative to $t^a$. If the clock is required to have resolution $\Delta t$ in the $t^a$–frame, then its resolution in its own frame must be $\Delta t/\gamma$, with $\gamma$ the usual Lorentz factor. Its mass must be

$$m \gtrsim \hbar \gamma/(c^2 \Delta t) .$$

Let the clock’s four–momentum $P_a$ be $(E, p)$ in the $t^a$–frame, so $E = mc^2 \gamma$ and $p = m c \beta \gamma$. Then

$$E^2 - mc^2 E/\gamma - p^2 c^2 = 0$$

6
from which
\[(E - mc^2/(2\gamma))^2 - p^2c^2 = m^2c^4/(4\gamma^2).\] (10)

This means
\[P_{\text{clock}}^a = mc^2/(2\gamma)t_a + \pi_a,\] (11)

where \(mc^2/(2\gamma)\tilde{=}h/(2\Delta t)\) and \(\pi_a\) is timelike future–pointing with \(\pi_a\pi^a = (mc^2/2\gamma)^2\tilde{=}h/(2\Delta t)^2\).

Combining the results of the two previous paragraphs, we see that for any future–pointing vector \(u^a\), the sum of the expectation value \(P_a u^a\) of the \(u^a\)–component of the momentum and the corresponding component of the momentum of the clock which controls the sampling satisfies
\[(P_a + P_{\text{clock}}^a)u^a \geq mc^2/\gamma - (1/8\pi)\tilde{h}/t_0,\] (12)

which we expect to be positive by (8).

A word about the interpretation of this is in order. Here \(P_a\) is the expectation of \(\hat{T}_{ab}t^b\) smeared over a volume in Minkowski space. The components of this smeared operator do not generally commute (one cannot simultaneously measure the components of the four–momentum in a finite box, because of edge effects). Thus it perhaps too strong to say that the four–momentum is operationally future–pointing, since the four–momentum of the field within a finite box cannot, strictly speaking, be measured. What we have shown is that for any future–pointing vector \(u^a\), the operator \(u^aP_{\text{op}}^a\) is non–negative, where \(P_{\text{op}}^a\) is the sum, of the clock’s four–momentum and the four–momentum operator for the field in a box.

Conclusions. The operational dominant energy condition immediately resolves two of the negative–energy pathologies listed in the introduction. It precludes the conversion of ordinary particles to tachyons. And at the level of Newtonian gravity, it forces gravitational fields to be attractive in the sense that \(\nabla \cdot g \leq 0\), for \(g\) the gravitational acceleration field, since a measurement of \(\nabla \cdot g\) is a measurement of the energy density.

The remaining issues require more extensive discussion than can be given here. We can only comment briefly that Grove [6], in his resolution of the second–law problems raised by Ford [3] and Davies [4] in effect establishes a special case of the operational positivity of energy. We hope to discuss this, and the question of why perturbations do not cause quantum systems to decay into states with patches of negative energy together with positive–energy radiation, in a future publication.

Finally, the applications of this to theory of quantum fields in curved space–time, particularly to the question of what the back–reaction of the fields on the space–time geometry is, will be discussed elsewhere [14].
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