Interacting electrons on a half line coupled to impurities

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Abstract

We generalize the bosonization methods for systems in the half line that we discussed elsewhere, to study the effects of interactions on electronic systems coupled to impurities. We introduce a model for a quantum wire coupled with a quantum dot by a purely capacitive interaction. We show that the effect on the quantum wire is a dynamical boundary condition which reflects the charge fluctuations of the quantum dot. Special attention is given to the role of the fermion boundary conditions and their interplay with the electron interactions on the bosonized effective theory.

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I. INTRODUCTION

The Luttinger model describes the most important qualitative features of the low energy phenomena of the one-dimensional electron gases in quantum wires. This physical description is a very powerful tool in situations in which the wire is effectively infinite or cases in which external gates give rise to constrictions which effectively split the wire in two semi-infinite systems. This picture has been exploited with great success by Kane and Fisher [1]. Using a renormalization group analysis, they showed that for repulsive interactions, the scattering potential due to an isolated impurity scales to infinity in the low energy regime and the system is effectively cut into two different semi-infinite systems. In other words, the problem turns out to be equivalent to an open boundary problem. There are other reasons why one-dimensional systems with open boundary conditions have drawn much attention lately. From the theoretical point of view, it is interesting to check that as in higher dimensional problems, the boundary critical exponents are different from the bulk ones. From an experimental point of view, the advances in the experimental techniques have allowed the fabrication of quantum wires in which, due to the small dimensions, boundary effects and the coupling of the system to the external probes can be important in the determination of the physical properties [2,3]. In the last few years, as well as a combination of bosonization and renormalization group techniques [1], the method of conformal field theory applied to the Bethe ansatz soluble models has been used to deal with systems with open boundary conditions [4].

In two previous papers [5,6], we considered a non-interacting system of fermions on a half line interacting with both boundary degrees of freedom and with impurities. The fermions coupled to the boundary degrees of freedom through dynamical boundary conditions, and to the impurities through local forward and backward scattering terms in the hamiltonian. In this paper we generalize the results obtained in [5] to the case of interacting fermions living in the half line coupled to an impurity at the boundary. We consider a Luttinger liquid coupled to a boundary degree of freedom $\theta$ through the boundary condition $\psi_R = -e^{2i\theta} \psi_L$. 
as well as to forward scattering processes. We will show that this model gives a good physical picture of a quantum wire coupled with a quantum dot by a purely capacitive interaction. The fluctuations of the degree of freedom \( \theta \) will turn out to be determined by the fluctuations of the charge on the quantum dot. We will also show that the bosonized currents are modified by the electron-electron interactions as well as by the presence of the boundary degree of freedom, \( \theta \). We calculate the one-particle Green’s function in two cases, when \( \theta \) is a classical degree of freedom and when it is a dynamical variable. We study the values of this Green’s function far away and close to the boundary. As expected, we see that the exponents describing critical phenomena at the surfaces differ from those of the bulk.

This paper is organized as follows. In section II we introduce a physically intuitive model of a quantum wire coupled to a quantum dot and derive the boundary condition. In section III we derive the effective action for the bosonic degree of freedom and use it to calculate the form of bosonized currents in the system with a boundary degree of freedom. In section IV we introduce a model that describes the dynamics of the boundary degrees of freedom. In section V we use these results to calculate the current correlation functions both for a system with classical and quantum mechanical boundary degrees of freedom. In section VI we calculate the one particle Green’s function and discuss its asymptotic limits. Section VII is devoted to the conclusions. In Appendix A we give details of the Green function of fermions with singular gauge fields.

II. BOUNDARY DEGREES OF FREEDOM IN LUTTINGER LIQUIDS

Consider a quantum wire coupled to a quantum dot. For all practical purposes we will regard the dot as a gate with charge \( Q \) confined in a region of linear size \( \ell \). The charge \( Q \) of the dot can change due to fluctuations caused, for example, by a nearby reservoir (such as the tip of a scanning tunneling microscope). Thus, \( Q \) can in principle vary in time and, in the appropriate circumstances, it can also be regarded as a quantum mechanical degree of freedom. In general \( Q \) will fluctuate on time scales that are usually longer than those of
the fermions in the wire. We will assume that the dot is sitting on top of the wire and that it is close enough to be electrically coupled to it but far enough that there is no electron tunneling between dot and wire. Under these circumstances, the electrons on the wire will see the dot as a gate. We will show in this section that the qualitative effects of charge fluctuations on the dot can be represented by a boundary condition for the electrons on the wire. The boundary condition will be parametrized by a phase shift angle $\theta$. We will see that the fluctuations of the charge cause fluctuations of the phase shift $\theta$.

A simple model for electrons moving in a wire interacting with a gate consists of a system of (for the time being free) fermions moving in the potential of the gate. We will take this potential to be a step function of height $V_0$ acting to the left of $x = 0$. The Fermi energy will be assumed to be well below the step, $E_F \leq V_0$. Thus, the two important parameters are the step height measured from the Fermi energy, $\Delta V = V_0 - E_F$, and the Fermi energy $E_F$ itself. Clearly, $\Delta V$ controls the amplitude for backscattering whereas $E_F$ controls the amplitude for forward scattering. An elementary quantum mechanical calculation shows that the reflection (or backscattering) amplitude $R(p_F)$ at the Fermi wavevector $p_F$ is given in terms of the phase shift $\theta$

$$R(p_F) = -e^{-2i\theta(p_F)} \quad (2.1)$$

where

$$\tan(2\theta(p_F)) = \sqrt{\frac{E_F}{\Delta V}} \quad (2.2)$$

Thus, changes in the local chemical potential determine the changes in the phase shift. We will show that this argument can be turned into a boundary condition for the fermions in a Luttinger model.

Let $\Psi(x)$ be the Fermi field for the electrons. The low energy physics of the interacting one-dimensional Fermi system is described by the right and left moving components of $\Psi(x)$ in the standard form

$$\Psi(x) = e^{ip_Fx}R(x) + e^{-ip_Fx}L(x) \quad (2.3)$$
The Fermi energy is $E_F$ and the Fermi velocity is $v_F$. We will set the zero of the energy at the Fermi energy. Let us denote by $A_0(x)$ and $M(x)v_F^2$ the forward scattering and the backscattering amplitudes of the potential generated by the dot seen by the electrons on the wire. In what follows we will assume that $A_0(x)$ varies slowly in the region of the dot and that it vanishes sharply away from it. Similarly, we will assume that the backscattering amplitude $M(x)$ is very large and slowly varying close to the dot and that it also drops to zero rapidly away from it. We will assume that $M(x)$ and $\ell$ are large enough so that the wire has effectively split in two semi-infinite pieces with no direct tunneling between them. Hence we will consider the interaction of just one half of the wire with the dot. However, if the charge $Q$ of the dot is regarded as a quantum mechanical degree of freedom both pieces of the wire will see each other through the fluctuations of $Q$.

In addition to the coupling to a quantum dot, a quantum wire usually has imperfections in shape. It can also be coupled to different sorts of impurities, both static and dynamical, which will generally cause the fermions to scatter. Examples of such impurities are two-level systems. Notice that a weakly coupled quantum dot interacting with the electrons somewhere along the wire can also be regarded as a quantum impurity. We will model the interactions between the electrons and these impurities (quantum mechanical or not) in terms of the forward and backward scattering amplitudes for the electrons. In particular, the forward scattering amplitude terms are

$$H_{\text{forward}} = A_0^{\text{imp}}(x)(R^\dagger(x)R(x) + L^\dagger(x)L(x)) \quad (2.4)$$

while the backward scattering amplitude terms will be given by

$$H_{\text{backward}} = M^{\text{imp}}(x)v_F^2(R^\dagger(x)L(x) + L^\dagger(x)R(x)) \quad (2.5)$$

Impurities of this type will be discussed in the next sections. In this section we will consider only the effects of the edge coupled to the quantum dot.

For a system of non-interacting fermions, the effective Hamiltonian (density) for the low-energy excitations of the fermions can be written in the form (in units of $\hbar = 1$)
\[ \mathcal{H} = -v_F R^\dagger(x)i\partial_x R(x) + v_F L^\dagger(x)i\partial_x L(x) + A_0(x)(R^\dagger(x)R(x) + L^\dagger(x)L(x)) + M(x)v_F^2(R^\dagger(x)L(x) + L^\dagger(x)R(x)) \] (2.6)

Clearly, this is effectively a system of free Dirac fermions in 1 + 1 dimensions. Here \( v_F \) is the Fermi velocity of the electrons and the forward and backward scattering amplitudes \( A_0(x) \) and \( M(x) \) can be regarded as a time component (scalar) of a vector potential and a (varying) mass respectively. For a quantum dot we choose the functions \( A_0^{\text{dot}}(x) \) and \( M^{\text{dot}}(x) \) to have the form

\[ A_0^{\text{dot}}(x) = A_0 \Theta(-x) \] (2.7)
\[ M^{\text{dot}}(x) = M \Theta(-x) \] (2.8)

where \( \Theta(x) \) is the step function. The amplitudes \( A_0 \) and \( M \) are determined by the properties of the dot, such as the charge \( Q \), and are slowly varying functions of the time \( t \). We will take \( M \) to be a constant and \( A_0 \) to decay very slowly, with a characteristic scale \( \ell \), the linear size of the dot.

The one-particle states of energy \( E \) of this system obey the eigenvalue equation

\[ ER(x) = -iv_F \partial_x R(x) + A_0(x)R(x) + M(x)v_F^2L(x) \]
\[ EL(x) = +iv_F \partial_x L(x) + A_0(x)L(x) + M(x)v_F^2R(x) \] (2.9)

For potentials \( A_0(x) \) and \( M(x) \) of the form of Eq. (2.8), and for \( M \) very large, the solution of Eq. (2.9) is straightforward. For \( x > 0 \), where \( A_0 = M = 0 \), the solutions have eigenvalue \( E = pv_F \) and are just plane waves

\[ \tilde{R}(x) = R_+ e^{ipx} \]
\[ \tilde{L}(x) = L_+ e^{-ipx} \] (2.10)

For \( x < 0 \), the solutions with \( |E| < Mv_F^2 \) decay exponentially fast as \( x \to -\infty \). Thus we write

\[ \tilde{R}(x) = R_- e^{ix} \]
\[ \tilde{L}(x) = L_- e^{-ix} \] (2.11)
The allowed value of $\kappa$ is

$$\kappa = \sqrt{(Mv_F)^2 - \left(\frac{A_0}{v_F} - p\right)^2}$$  \hspace{1cm} (2.12)$$

while the amplitudes $R_-$ and $L_-$ satisfy

$$\frac{R_-}{L_-} = -\frac{Mv_F^2}{A_0 - pv_F - i\sqrt{(Mv_F)^2 - (A_0 - pv_F)^2}}$$  \hspace{1cm} (2.13)$$

Since continuity of the wave function at $x = 0$ demands that $\frac{R_+}{L_+} = \frac{R_-}{L_-}$, we find that the phase shift $\theta(p)$, defined by

$$\frac{R_+}{L_+} = -e^{2i\theta(p)}$$  \hspace{1cm} (2.14)$$

is given by

$$\tan(2\theta(p)) = \text{sgn}(A_0 - pv_F)\sqrt{\left(\frac{Mv_F^2}{A_0 - pv_F}\right)^2 - 1}$$  \hspace{1cm} (2.15)$$

Here we are only interested in the limit of small momentum $|p| \ll \inf(Mv_F, A_0/v_F)$, and potentials $|A_0| < Mv_F^2$. In this regime the phase shift becomes

$$\tan(2\theta(0)) = \text{sgn}(A_0)\sqrt{\left(\frac{Mv_F^2}{A_0}\right)^2 - 1}$$  \hspace{1cm} (2.16)$$

From the point of view of the states to the right of $x = 0$, if we restrict ourselves to states with small momentum, the presence of the dot becomes equivalent to the boundary condition

$$R(0, t) = -e^{2i\theta(0)} L(0, t)$$  \hspace{1cm} (2.17)$$

Thus, all knowledge of $A_0$ and $M$, which are the qualitatively important parameters of the quantum dot as far as the fermions on the wire are concerned, is expressed in the phase shift $\theta(0)$. Clearly, the fermions to the right of $x = 0$ know about the properties of the dot since their wavefunctions leak-out to negative values of $x$ and feel the voltage $A_0$.

Consistent with this conventions, we will choose the domain of the phase shift to be in the interval $-\frac{\pi}{2} < 2\theta \leq \frac{\pi}{2}$. As far as this one-particle analysis goes all the branches of the function arctangent lead to physically equivalent results. However, in the full second
quantized theory, the phase shift will determine a dynamical boundary condition for the fermions. In that situation, moving the boundary angle from one branch to the next is no longer physically innocuous since it changes the total number of fermions (namely, the total charge). This physical picture is clearly not related to the model we are discussing here since in the absence of direct tunneling between the wire and the dot, the total charge of the wire cannot change.

However, this does not mean that there cannot be charge fluctuations near the end of the wire. Quite to the contrary, in physically interesting situations the quantum dot can exhibit charge fluctuations. We can now represent phenomenologically these charge fluctuations $Q(t)$ in the form of a slow time dependence of the voltage $A_0(t) \propto Q(t)$. Hence, by this mechanism, the boundary condition acquires dynamics. We define a boundary degree of freedom $\theta(t)$ as the (adiabatic extension of) phase shift $\theta(0)$. The boundary condition for the fully second quantized right and left moving fields is now

$$R(0,t) = - e^{2i\theta(t)} L(0,t)$$

(2.18)

As the charge $Q(t)$ on the dot changes, so does the potential $A_0(t)$. Consequently, the electrons on the wire are pushed further to the right (left) of $x = 0$ as $A_0(t)$ grows larger (smaller). Hence, a fluctuating boundary condition causes the charge density of the fermions at the edge of the wire to fluctuate. Physically this means that the position of the physical edge of the wire changes with time and it becomes a dynamical variable with quantum fluctuations of its own. Hence, the fluctuations of $\theta$ “push” the Luttinger liquid as a whole and, as a result, they are insensitive to the electron-electron interactions in the liquid. In the next sections we will show that this is precisely what happens. In contrast, the fluctuations of a forward scattering quantum impurity in the bulk (even those close to the edge) are affected by the electron-electron interactions in the liquid.
III. LUTTINGER MODEL WITH A LOCALIZED FORWARD SCATTERING IMPURITY

In this section we give a description of a Luttinger model of interacting electrons on a half-line, coupled to impurities located very close to the edge of the line. We consider two types of impurities: (a) a boundary degree of freedom (of the type discussed in section II), which couples with the fermions through the boundary conditions as in Eq. (2.18), and (b) forward and backward scattering impurities. In this section the impurity will be regarded as a classical degree of freedom. In section IV the impurity will be quantized. The approach that we use here follows closely the methods that we developed in references [5] and [6] for non-interacting electrons.

We begin with the Lagrangian for the massless Luttinger model on the half-line (i.e. 1 + \(\frac{1}{2}\) dimensions). Throughout we will work in two-dimensional euclidean space (imaginary time). This means that we work on a space-time \(\Omega\) which is an infinitely long cylinder, with its axis along the space direction \(x_1\) and periodic (antiperiodic for fermi fields) boundary conditions along the imaginary time direction \(x_2\) of perimeter \(T\), with \(T \to \infty\). The boundary will be denoted by \(\partial \Omega\) and it is the set of points with \(x_1 = 0\).

The imaginary time (euclidean) Lagrangian for the pure system (coupled to sources) is

\[
\mathcal{L} = \bar{\psi} \left( i \partial_\tau \psi - \frac{g^2}{2} \left( \bar{\psi} \gamma_\mu \psi \right)^2 + A_\mu J_\mu \right)
\]

where \(A_\mu\) is a source (a background gauge field) which couples to the fermion current \(J_\mu = \bar{\psi} \gamma_\mu \psi\). We will follow the procedure discussed in section II and impose dynamical boundary conditions for the fermionic field of the form of Eq. (2.18). In Euclidean space the boundary condition becomes (see reference [5])

\[
R(0, x_2) = -e^{2\theta(x_2)} L(0, x_2)
\]

The impurity forward scattering term is

\[
\mathcal{L}_{\text{imp}} = \bar{\psi} \gamma_\mu \psi A^{\text{imp}}_\mu (x)
\]
where $A_\mu^{\text{imp}}$ has the form

$$A_\mu^{\text{imp}}(x) = \delta(x_1) \tilde{A}_\mu(x_2) \quad (3.4)$$

for systems with a (possibly time-dependent) degree of freedom localized at $x_1 = 0$.

Next we proceed to follow the usual path integral bosonization method. First we decouple the four-fermi interaction by means of a vector (Hubbard-Stratonovich) auxiliary field $a_\mu$. In this form, the full euclidean action for the Luttinger model coupled to forward-scattering impurities reads,

$$\mathcal{L} = \bar{\psi} iD \psi + \frac{a_\mu^2}{2g^2} \quad (3.5)$$

where the covariant derivative $D_\mu$ is

$$D_\mu \equiv \partial_\mu - i \left( a_\mu + A_\mu + A_\mu^{\text{imp}}(x) \right) \quad (3.6)$$

which couples the fermions to the auxiliary field $a_\mu$, to the external source $A_\mu$ and to the vector impurity amplitude $A_\mu^{\text{imp}}$. The dynamics of the fermions is described by the functional integral

$$Z = \int D\bar{\psi} D\psi Da_\mu \exp \left( - \int d^2x \mathcal{L}_F \right) \quad (3.7)$$

We are interested in the properties of this system when the fermions are restricted to move on a half-line with the impurity located close to the edge. Since one plausible effect of electron-electron interactions is to induce (or to screen) boundary degrees of freedom, we must define carefully the system near the edge. We write the external sources as a sum of bulk and boundary pieces of the form $A_\mu = B_\mu + \tilde{b}_\mu(x_2)\delta(x_1)$. Clearly, $b_\mu$ is the value of $A_\mu$ at the boundary, and $B_\mu$ is its value in the bulk. We will require $B_\mu$ to vanish at the boundary, i.e. $B_\mu(x_1 = 0) = 0$. For convenience, we define $s_\mu(x) = (\tilde{A}_\mu^{\text{imp}} + \tilde{b}_\mu)(x_2)\delta(x_1) \equiv \tilde{s}_\mu(x_2)\delta(x_1)$. For technical reasons, it will also be convenient to define the impurity to be at finite (but small) distance from the edge and to let $a_\mu$ vanish at the edge $x_1 = 0$ but not at the location of the impurity.
The functional integral now reads

\[ Z[B_\mu, b_\mu] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}a_\mu \exp \left( -S_F(\bar{\psi}, \psi, a_\mu) \right) \]  

(3.8)

where, after shifting \( a_\mu + B_\mu \to a_\mu \), the action \( S_F \) takes the form

\[ S_F(\bar{\psi}, \psi, a_\mu) = \int_\Omega d^2x \bar{\psi}(i/\partial + \hat{\phi} + \hat{\psi})\psi + \int_\Omega d^2x \frac{1}{2g^2}(a_\mu - B_\mu)^2 \]  

(3.9)

The next step in the bosonization of the system is to decouple the fermions from the regular gauge field \( a_\mu \). This is accomplished by means of a combination of (suitably chosen) smooth, single-valued gauge and chiral transformations of the form

\[ \psi(x) = e^{i\eta(x) + \gamma_5 \phi(x)} \chi(x) \]
\[ \bar{\psi}(x) = \bar{\chi}(x) e^{-i\eta(x) + \gamma_5 \phi(x)} \]  

(3.10)

The fermions decouple if the vector potential \( a_\mu(x) \) are smooth and can be written in the form

\[ a_\mu(x) = \partial_\mu \eta(x) - \epsilon_{\mu\nu} \partial_\nu \phi(x) \]  

(3.11)

This condition can be met everywhere except at the boundary where the configurations have a \( \delta \)-function support and become singular and will be treated separately. The boundary condition \( a_\mu(0, x_2) = 0 \) is satisfied if we impose Dirichlet boundary conditions on \( \phi \), i.e., \( \phi(0, x_2) = 0 \), and von Neumann boundary conditions on \( \eta \), i.e., \( \partial_1 \eta(0, x_2) = 0 \).

Under the change of variables of Eq. (3.10) the integration measure of the fermion path integral acquires a jacobian \( J_F \) of the form

\[ J_F = e^{-\frac{1}{2} \int_\Omega (\partial_\nu \phi)^2 + \frac{1}{2} \int_\Omega \epsilon_{\mu\nu} \partial_\nu \phi} \]  

(3.12)

Hence, the fermion part of the path integral becomes

\[ \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left( - \int_\Omega d^2x \bar{\psi}(i\hat{\phi} + \hat{\psi})\psi \right) = J_F \int \mathcal{D}\bar{\chi} \mathcal{D}\chi \exp \left( - \int_\Omega d^2x \bar{\chi}(i\hat{\phi} + \hat{\psi})\chi \right) \]  

(3.13)

From Eq. (3.11), it follows that \( \epsilon_{\mu\nu} \partial_\nu a_\mu = \partial^2 \phi \). Therefore the chiral angle \( \phi \) is given by
\[ \phi(x) = \int_\Omega d^2y \ G_D(x, y) \epsilon_{\mu \nu} \partial_\mu a_\nu(y) \]  

(3.14)

where \( G_D(x, y) \) is the Dirichlet Green’s function in \( \Omega \) which satisfies

\[
\begin{aligned}
\partial^2_x G_D(x, y) &= \delta(x - y) \\
G_D(x_1, x_2; y)|_{x_1=0} &= 0
\end{aligned}
\]  

(3.15)

The solution of Eq. (3.15) is

\[ G_D(x, y) = \frac{1}{4\pi} \ln \frac{(x_1 - y_1)^2 + (x_2 - y_2)^2 + a^2}{(x_1 + y_1)^2 + (x_2 - y_2)^2 + a^2} \]  

(3.16)

where \( a \) is a short distance cut-off.

Using these properties, the jacobian \( J_F \) becomes

\[ J_F = \exp \left( \frac{1}{2\pi} \int_\Omega d^2x d^2y \ a_\mu(x) \ \Gamma_{\mu\nu}(x, y) \ a_\nu(y) \right) \]  

(3.17)

where the kernel \( \Gamma_{\mu\nu}(x, y) \) is

\[ \Gamma_{\mu\nu}(x, y) = (\partial_\alpha^{(x)} \partial_\alpha^{(y)} \delta_{\mu\nu} - \partial_\mu^{(y)} \partial_\nu^{(x)}) G_D(x, y) \]  

(3.18)

After substituting these results back into Eq. (3.8), the partition function takes the form

\[
\begin{align*}
Z[B_\mu, b_\mu] &= \int \mathcal{D}\bar{\chi} \mathcal{D}\chi \exp \left( -\int_\Omega d^2x [\bar{\chi} (i\partial + \not{s}) \chi] \right) \int \mathcal{D}a_\mu \exp \left( -\int_\Omega d^2x \frac{1}{2g^2}(a_\mu - B_\mu)^2 \right) \\
&\quad \times \exp \left( \frac{1}{2\pi} \int_\Omega d^2x d^2y \ a_\mu(x) \Gamma_{\mu\nu}(x, y) a_\nu(y) \right)
\end{align*}
\]  

(3.19)

The transformation Eq. (3.10) does not decouple the singular gauge fields (with support at the boundary) from the fermions. In reference [5] we showed that the result of this path integral is also a fermion determinant but for operators with different boundary conditions (parametrized by a phase shift) which can be computed using Forman’s method [14]. The result is

\[
\begin{align*}
\int \mathcal{D}\bar{\chi} \mathcal{D}\chi \exp \left( -\int_\Omega d^2x [\bar{\chi} (i\partial + \not{s}) \chi] \right) &= \\
&= \text{Det}(i\partial)_{R=-L} \exp \left( \frac{1}{2\pi} \int dx_2 dy_2 (\theta + \bar{s}_2)(y_2)K(x_2, y_2)(\theta + \bar{s}_2)(x_2) \right)
\end{align*}
\]  

(3.20)

The kernel \( K(x_2, y_2) \) is equal to
\[ K(x_2, y_2) = \left( -\frac{1}{\pi} \frac{1}{(x_2 - y_2)^2} + a^2 + \frac{1}{a} \delta(x_2 - y_2) \right). \]  

(3.21)

where \( a \) is the short distance cut-off introduced above. It is easy to show that the exponential in Eq. (3.20) can be written in the standard Caldeira-Leggett form \[16\]

\[
\frac{1}{2\pi} \int dx_2 \int dy_2 (\theta + \bar{s}_2)(y_2) K(x_2, y_2)(\theta + \bar{s}_2)(x_2)
= \frac{1}{4\pi} \int dx_2 \int dy_2 \frac{((\theta + \bar{s}_2)(x_2) - (\theta + \bar{s}_2)(y_2))^2}{(x_2 - y_2)^2}
\]

(3.22)

The integral over the Hubbard-Stratonovich field \( a_\mu \) in Eq. (3.19) can be performed and the result is

\[
\int \mathcal{D}a_\mu \exp \left( \frac{1}{2\pi} \int \Omega \int d^2x \int d^2y a_\mu(x)[\Gamma_{\mu\nu}(x, y) - \frac{\pi}{g^2} \delta_{\mu\nu} \delta^2(x - y)]a_\nu(y) + \frac{1}{\pi} \int \Omega \int d^2x \int d^2y s_\mu(x) \Gamma_{\mu\nu}(x, y)a_\nu(y) + \frac{1}{g^2} \int \Omega \int d^2x a_\mu(x)B_\nu(x) \right)
= \exp \left( \frac{1}{2} \int \Omega \int d^2x \int d^2y B_\mu(x)\left[ \frac{1}{\pi + g^2} \Gamma_{\mu\nu}(x, y) + \frac{1}{g^2} \delta_{\mu\nu} \delta^2(x - y) \right]B_\nu(y) + \int \Omega \int d^2x \int d^2y \frac{1}{\pi + g^2}\left[ s_\mu(x) \Gamma_{\mu\nu}(x, y)B_\nu(y) - \frac{g^2}{2\pi} s_\mu(x) \Gamma_{\mu\nu}(x, y)s_\nu(y) \right] \right)
\]

(3.23)

Here we have used that the inverse of the kernel in the quadratic term in \( a_\mu \) is

\[
\left( \frac{\Gamma_{\mu\nu}(x, y)}{\pi} - \frac{\delta_{\mu\nu} \delta^2(x - y)}{g^2} \right)^{-1} = -g^2 \left( \frac{g^2}{\pi + g^2} \Gamma_{\mu\nu}(x, y) + \delta_{\mu\nu} \delta^2(x - y) \right)
\]

(3.24)

and that \( \int d^2y \Gamma_{\mu\nu}(x, y)\Gamma_{\nu\alpha}(y, z) = -\Gamma_{\mu\alpha}(x, z) \). Substituting the results for the integrals over the fermionic fields and the integral over \( a_\mu \) back into Eq. (3.19), and using that

\[
\int \Omega \int d^2x \int d^2y s_\mu(x) \Gamma_{\mu\nu}(x, y)s_\nu(y) = \int \Omega \int dx_2 \int dy_2 \bar{s}_2(y_2)K(x_2, y_2)\bar{s}_2(x_2)
\]

(3.25)

we obtain the following expression for the partition function

\[ Z[B_\mu, b_\mu] = e^{-S_{\text{ind}}(A_\mu^{\text{imp}}, b_\mu, \theta)} \]

(3.26)

The induced action \( S_{\text{ind}} \) given by

\[
S_{\text{ind}}(A_\mu^{\text{imp}}, b_\mu, \theta) = -\frac{1}{2(g^2 + \pi)} \int \Omega \int d^2x \int d^2y (B_\mu(x) + A_\mu^{\text{imp}} + b_\mu)\Gamma_{\mu\nu}(x, y)(B_\nu + A_\nu^{\text{imp}} + b_\nu)(y) - \frac{1}{2\pi} \int dx_2 \int dy_2 \theta(x_2)K(x_2, y_2)\theta(y_2) - \frac{1}{\pi} \int dx_2 \int dy_2 \theta(x_2)K(x_2, y_2)(A_2^{\text{imp}} + b_2)(y_2)
\]

(3.27)
The induced action $S_{\text{ind}}(\bar{A}^\text{imp}_\mu, \bar{b}_\mu, \theta)$ for the boundary degrees of freedom is explicitly gauge invariant, and it has a smooth dependence in the coupling constant $g$. It also reproduces exactly our results for free fermions of reference [5].

The induced action of Eq. (3.27) has a clear physical meaning. Recall that the degrees of freedom $\theta$ and $\bar{A}_2$ represent, respectively, the dynamical boundary condition and a quantum mechanical forward scattering impurity close to the boundary. In contrast to the case of free fermions [5], once interactions are present, $\theta$ and $\bar{A}_2$ are no longer physically equivalent. In particular, Eq. (3.27) shows that while the interactions modify the amplitude of the terms that contain $\bar{A}_2$ they do not enter in the terms that involve $\theta$. Physically, the renormalization of the amplitude of the terms involving $\bar{A}_2$ is just the screening of the forward scattering impurity by the fluctuations of the fermions. Since the Luttinger model has strictly short range interactions, screening is not complete but, instead, it leads to a finite reduction of the forward scattering amplitude. On the other hand, the boundary degree of freedom $\theta$, which represents the coupling to the quantum dot, is “outside” the system it is not affected by screening effects.

To obtain the bosonization formula for the full partition function, we define a bosonic field $\omega$ which obeys Dirichlet boundary conditions $\omega(0, x_2) = 0$. Following the same steps as in [5] we find that the partition function of the Luttinger model on a half-line is equal to the following bosonic partition function

$$Z[B_\mu, b_\mu] = K \int \mathcal{D}\omega \exp \left( -\frac{1}{2\pi} \int (\partial_\mu \omega)^2 + \frac{i}{\pi} \sqrt{\frac{\pi}{\pi + g^2}} \varepsilon_{\mu\nu} \partial_\nu \omega (B_\mu(x) + \bar{A}^\text{imp}_\mu + \bar{b}_\mu) \right) + \frac{1}{2\pi} \int dx_2 \int dy_2 \theta(x_2) K(x_2, y_2) \theta(y_2)$$

$$+ \frac{1}{\pi} \int dx_2 \int dy_2 \theta(x_2) K(x_2, y_2)(\bar{A}^\text{imp}_2 + \bar{b}_2(y_2)) \right)$$

(3.28)

This is the bosonized form of the theory. It gives an explicit description of the system in terms of the fluctuations of the Dirichlet bos field $\omega$. It also includes the effects of both the boundary and forward scattering degrees of freedom $\theta$ and $\bar{A}_2$.

From Eq. (3.28) we can read-off the boson operators which represent the fermion currents. In the bulk, i.e. for $x_1 > 0$, the current operator becomes
\[ J_\mu(x) \equiv \frac{i}{\pi} \sqrt{\frac{\pi}{\pi + g^2 \epsilon_{\mu\nu} \partial_{\nu} \omega}} \] (3.29)

This is the standard bosonization formula for the current. At the boundary we find

\[ J_\mu(0, x_2) \equiv \left. \frac{i}{\pi} \sqrt{\frac{\pi}{\pi + g^2 \epsilon_{\mu\nu} \partial_{\nu} \omega}} \right|_{x_1=0} + \frac{\delta_{\mu, 2}}{\pi} \int dy_2 K(x_2, y_2) \theta(y_2) \] (3.30)

The first term is the bulk contribution of the current at the boundary while the second term is the contribution of the boundary degree of freedom \( \theta \) and it only affects the charge density at the boundary. This is precisely what we anticipated in section II where we showed that the fluctuations in the phase shift \( \theta \) cause the fermions to fluctuate in and out of the wire at the edge. This is the physical meaning of the last term of Eq. (3.30).

The form of the current operator in Eq. (3.30), combined with the Dirichlet boundary condition for the field \( \omega \), ensures that the space component of the fermion current vanishes exactly at the boundary. This identification of the charge density operator at the boundary implies that, at the operator level, \( \theta \) contributes to the boundary charge while the forward scattering impurity \( \bar{A}_2 \) does not. In fact, the boundary charge operator is precisely the same one that was found in reference [5] for the case of free fermions. Nevertheless, the expectation value of this operator does depend on \( \bar{A}_2 \) due to the fluctuations of \( \omega \) and this dependence involves the value of the coupling constant \( g \). We will discuss this issue in section IV, where we use the partition function computed above to find the response of the system to an external electromagnetic field.

**IV. DYNAMICAL BOUNDARY CONDITIONS AND CHARGE FLUCTUATIONS**

So far we have treated the impurities \( \theta \) and \( \bar{A}_2 \) as classical degrees of freedom. We now proceed to quantize the impurity degrees of freedom. In Section II we showed that the fluctuations of the charge \( Q(t) \) of the quantum dot can be modelled by a dynamical boundary condition with phase shift \( \theta(t) \). We now need to define the dynamics of the charge fluctuations in the quantum dot.
Let $|Q\rangle$ be the ground state of the dot with charge $Q$. We will be interested in the effects of the fluctuations of the total charge $Q$. For each value of $Q$, the dot has a Hilbert space of states with the same charge. These states which are closely spaced in energy, only change the charge distribution inside the dot but, at least qualitatively, do not affect strongly the phase shift. In what follows we will only consider the ground states for sectors with different values of $Q$. These states are in general non-degenerate and can be labelled just by the integer $Q$. In the absence of tunneling the energies of the eigenstates $|Q\rangle$ of the dot are $U(Q)$. This energy $U(Q)$ is determined by Coulomb interactions among the electrons in the dot and by the potentials that stabilize the dot. Although in general this spectrum is rather complex, it must exhibit some general features. For $Q$ large enough the energy $U(Q)$ should have a minimum at some value of the charge $Q_0$ determined by the dot gate voltage. In this regime $Q_0$ is large and the spectrum of charges effectively ranges over all the integers.

Tunneling processes between the quantum dot and a reservoir (for instance a scanning tunneling microscope) will cause the charge to fluctuate. Hence, the tunneling term of the Hamiltonian acts as a kinetic energy and its scale is parametrized by $T$, the tunneling amplitude. Since only one particle can tunnel at a time, the tunneling Hamiltonian must involve an operator which maps the state $|Q\rangle$ to the states $|Q \pm 1\rangle$. Thus, we need an operator which translates $Q$ by one unit at a time. Let $\hat{\phi}$ be an operator which satisfies the commutation relation $[\hat{Q}, \hat{\phi}] = i$. Clearly $\hat{\phi}$ is the momentum canonically conjugate to $\hat{Q}$. Since the spectrum of $Q$ are the integers, the canonical momentum $\hat{\phi}$ is an angle. Hence, we can write the dot Hamiltonian in the form

$$H_{\text{dot}} = -T \cos \hat{\phi} + U(\hat{Q})$$  \hspace{1cm} (4.1)$$

For simplicity we will approximate $U(Q) \approx \frac{1}{2C}(Q - Q_0)^2$, where $C$ is the capacitance. We will be interested in situations in which the capacitance is large or, alternatively, the tunneling amplitude $T$ is large. In this regime the discreteness of $Q$ can be ignored and the spectrum of $Q$ becomes the real line. Let us define the rescaled charge operator $\hat{q} = (\hat{Q} - Q_0)/\sqrt{C}$ and the associated canonical momentum $\hat{P} = \sqrt{C}\hat{\phi}$, which satisfy the commutation relation
\[ [\hat{q}, \hat{p}] = i. \] Up to a shift in the energy, the dot Hamiltonian reduces to a simple harmonic oscillator

\[ H_{\text{eff}} = \frac{T}{2C} \hat{\theta}^2 + \frac{\hat{\varphi}^2}{2} \] (4.2)

which describes the small charge fluctuations in the quantum dot.

However, the quantum dot couples to the fermions only through the phase shift. Hence, what we really need is not the Hamiltonian for the charge \( Q \) but instead the Hamiltonian for the phase shift \( \theta \). In Section II we showed that \( Q \) determines \( A_0 \), the voltage felt by the fermions. In turn, the phase shift \( \theta \) also is determined from \( A_0 \) by Eq. (2.15). The actual details of this relation are not very important (in fact, they depend very strongly on the details of the model). What is important is that \( \theta \) is only defined modulo \( \pi \) (see Eq. (2.18)). Thus, \( U(Q) \) becomes a periodic function of \( \theta \). Also, since \( \theta \) is an angle, the momentum conjugated to \( \hat{\theta} \) must be an angular momentum operator \( \hat{L} \) and it should obey the commutation relation \( [\hat{\theta}, \hat{L}] = i. \) The kinetic energy term should now be given in terms of \( \hat{L} \). Hence, the effective Hamiltonian for \( \theta \) must have the form

\[ H_\theta = \frac{\hat{L}^2}{2K} + U(\hat{\theta}) \] (4.3)

where \( K \propto 1/T \). \( U(\hat{\theta}) \) is periodic with period \( \pi \) and it has a minimum at a value \( \theta_0 \) determined by the charge. The actual parameters of this Hamiltonian depend on details of the model. It is important to note that if the capacitance \( C \) is large, the potential \( U(Q) \) will be very shallow. As a result, \( U(\theta) \) not only will be very shallow but also, it will be always different from zero. Hence, capacitance effects will always tend to suppress the fluctuations of the phase shift. In what follows we will consider a model in which \( U(\hat{\theta}) = G \cos(2\theta) \) with a coupling constant \( G \propto 1/C \). This model is explicitly periodic in \( \theta \). In reference [5] we considered a model of non-interacting fermions on a half-line coupled to an impurity with this type of dynamics. The coupling to a the quantum dot described here is an explicit construction of this model.

There are a number of interesting cases in which the dot confinement potential is quite steep. In this situation the level spacing is large and a continuum approximation for \( Q \)
is qualitatively incorrect. In such cases it is better to think of the system as a two level system in which the charge can take just two values, \( Q_\pm \) with tunneling producing transitions between these two states. Hence, there are only two allowed values of the angle as well. In this case the boundary degree of freedom behaves like an Ising degree of freedom. If we denote the two states by \(|\pm\rangle\) the Hamiltonian now takes the simple form

\[
H_\pm = T\sigma_1 + \frac{1}{C}\sigma_3
\]  

The coupling between the fermions and the impurity still takes the same form as before but the phase shift can take just two values. This problem is known \[18\] to be related to the Kondo problem in metals. We will not consider this case in this paper.

In the rest of this paper we will use the approach of reference \[5\] and use path integrals to describe the quantum mechanics of both the fermions and of the quantum dot, the latter being represented by the phase shift \( \theta \) promoted to a full dynamical degree of freedom at the boundary. Using standard methods one derives the path integral for the boundary degree of freedom \( \theta \). The only subtlety here is the fact that the configuration space of \( \theta \) is the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\) and not the unit circle of length \( 2\pi \). In other words, the histories of the boundary degree of freedom do not include configurations with non zero winding number, namely functions \( \theta(x_2) \) which are periodic with period \( \pi \). Such configurations are important to describe tunneling processes between de wire and the dot. In what follows we will always assume that the configurations of \( \theta \) are bounded by \( \pm \pi/2 \).

The action \( S_{\text{imp}}(\theta) \) for the boundary degree of freedom is

\[
S_{\text{imp}}(\theta) = \int_{-\infty}^{\infty} dx_2 \left[ K \left( \frac{d\theta}{dx_2} \right)^2 + U(\theta(x_2)) \right]
\]  

The full partition function which describes the quantum dynamics of both fermions and the boundary degree of freedom is

\[
Z_{\text{full}} = \int D\bar{\psi} D\psi D\theta \exp(-S_F(\bar{\psi}, \psi) - S_{\text{imp}}(\theta))
\]  

where \( S_F(\bar{\psi}, \psi) \) is given in Eq. \[3.9\]. The coupling between the fermions (interacting or
not) and the quantum dot is implemented through the dynamical boundary condition at the edge \( x_1 = 0 \)

\[
R(0, x_0) = -e^{2i\theta(x_0)} L(0, x_0)
\]  

suitably analytically continued to imaginary time as discussed in Section III. The partition function of Eq. (4.6) has the general qualitative features of the systems discussed by Caldeira and Leggett [16] in their work on Macroscopic Quantum Coherence.

V. CHARGE FLUCTUATIONS AND BOUNDARY STATES

In this section we will use the model and the methods presented in the past sections to describe the charge fluctuations near the edge of the quantum wire. We will use the standard linear response approach and compute the fermionic current as a response to an external field. We will begin by considering the effects of classical impurities and boundary degrees of freedom. Later in this section we will also discuss the case of quantum impurities.

The expectation values of the fermionic current when an external field \( A_\mu \) acts on the system is defined as

\[
\langle J_\mu(x) \rangle = -\frac{1}{\mathcal{Z}[A]} \frac{\delta \mathcal{Z}[A]}{\delta A_\mu(x)}
\]

Since, as we have shown in section II, we have to split the external field \( A_\mu \) into two fields, \( B_\mu \) and \( \bar{b}_\mu \), we define two currents, one in the bulk

\[
\langle J_\mu(x) \rangle = -\frac{1}{\mathcal{Z}[B_\mu, \bar{b}_\mu]} \frac{\delta \mathcal{Z}[B_\mu, \bar{b}_\mu]}{\delta B_\mu(x)}
\]

for \( x_1 > 0 \) and the other at the boundary

\[
\langle J_\mu(0, x_2) \rangle = -\frac{1}{\mathcal{Z}[B_\mu, b_\mu]} \frac{\delta \mathcal{Z}[B_\mu, b_\mu]}{\delta b_\mu(x_2)}
\]

Using Eq. (3.28) we obtain

\[
\langle J_\mu(x) \rangle = \frac{1}{(\pi + g^2)} \int_{\Omega} d^2 y \Gamma_{\mu\nu}(x, y) (B_\nu + \bar{A}_\nu^{\text{imp}} + \bar{b}_\nu(y))
\]
in the bulk, and

\[ < J_2(0, x_2) > = \frac{1}{(\pi + g^2)} \int d^2 y \Gamma_{2\mu}(x_2, y) B_\nu(y) \]

\[ + \frac{1}{\pi} \int d y_2 K(x_2, y_2) \left[ \theta + \frac{\pi}{(\pi + g^2)} (\bar{A}_{2}^{\text{imp}} + \bar{b}_2) \right] (y_2) \quad (5.5) \]

at the boundary. The spatial component of the current at the boundary vanishes because \( \Gamma_{1\nu}(x, y)|_{x_1=0} = 0 \).

It is interesting to note that in the expressions above \( \theta \) and \( \bar{A}_2^{\text{imp}} \) play different roles. While \( \theta \) is never multiplied by the coupling constant \( g \), the factor in front of \( \bar{A}_2^{\text{imp}} \) is \( \frac{\pi}{(\pi + g^2)} \). That is, \( \theta \) is not affected by the electron-electron interactions inside the wire. We can understand this by recalling that \( \theta \) represents the phase shift due to the quantum dot. Physically, the electrons inside the wire cannot screen the potential of the dot which lays outside the system. In contrast, a local imperfection of the wire, represented by \( \bar{A}_2^{\text{imp}} \), will see the effects of screening and hence its effects will be affected by the electron-electron interactions. Recall that in the calculation of the fermion determinant in reference \[3\] the dynamical degree of freedom \( \theta \) was placed exactly at the origen, while \( \bar{A}_2^{\text{imp}} \) was placed at \( x_1 = \epsilon \), that is “in the bulk”. That means that \( \theta \) is an “external” boundary condition and \( \bar{A}_2^{\text{imp}} \) acts as an “internal” perturbation to the system, affected by screening effects. There are similar issues involving the current and charge distributions as well. In fact, Maslov and Stone \[17\] have reexamined recently the computation of the conductance in a Luttinger liquid. They showed that the renormalization of the conductance \[1\] \( G = e^2/h/(1 + \frac{\epsilon^2}{\pi}) \) (implied by the bulk bosonization equation \[3.29\]) is absent if the conductance is measured from outside the system (i.e. in the external leads).

It is instructive to calculate the induced total charge and to determine the effects of the interactions on this quantity. As it turns out, the induced charge depends on whether the boundary degree of freedom \( \theta \) is quantum mechanical or not.

We first consider the classical case. We set \( B_\mu = \bar{b}_\mu = 0 \), and calculate \( Q_{\text{ind}}(x_2) = \int_0^\infty dx_1 J_2(x_1, x_2) \). Using Eqs. \( (5.4) \) and \( (5.3) \) the induced charge is
\[ Q_{\text{ind}}(x_2) = \frac{1}{\pi} \int_0^a dx_1 \int_{-\infty}^{+\infty} dy_2 K(x_2, y_2) \left( \theta + \frac{\pi}{(\pi + g^2)} \bar{A}^{\text{imp}}_2(y_2) \right) (y_2) \]

\[ + \frac{1}{(\pi + g^2)} \int_0^a dx_1 \int_{-\infty}^{+\infty} dy_2 \Gamma_{2\nu}(x, y) \bar{A}^{\text{imp}}_{2\nu}(y) \quad (5.6) \]

where the spatial integration is over the right half-plane. Recalling that \( \bar{A}^{\text{imp}}_\mu(x) = \delta(x_1) \tilde{A}_\mu(x_2) \), and that \( \Gamma_{21}(x, y)|_{y_1=0} = 0 \) we obtain

\[ Q_{\text{ind}}(x_2) = \frac{a}{\pi} \int_{-\infty}^{+\infty} dy_2 K(x_2, y_2) \theta(y_2) \]

\[ + \frac{1}{(\pi + g^2)} \int_0^a dx_1 \int_{-\infty}^{+\infty} dy_2 K(x_2, y_2) \bar{A}^{\text{imp}}_2(y_2) \]

\[ + \frac{1}{(\pi + g^2)} \int_a^\infty dx_1 \int_{-\infty}^{+\infty} dy_2 \Gamma_{22}(x, y_2) \bar{A}^{\text{imp}}_2(y_2) \quad (5.7) \]

Since the kernel \( K(x_2, y_2) \) can be written as \( K(x_2, y_2) = \lim_{x_1 \to 0} \Gamma_{22}(x, y_2) \), it is easy to see that the second and the third terms in Eq. (5.7) cancel each other, and the induced charge results

\[ Q_{\text{ind}} = \frac{a}{\pi} \int_{-\infty}^{+\infty} dy_2 K(x_2 - y_2) \theta(y_2) \quad (5.8) \]

For a static boundary degree of freedom, \( \theta = \text{constant} \), the last equation gives \( Q_{\text{ind}} = \frac{\theta}{\pi} \) which is the standard formula for the charge fractionalization in the semiclassical theory of solitons \[12\] \[14\]. It is worth to emphasize that \( Q_{\text{ind}} \) is not due to a non-conservation of charge in the wire. Instead, we will view the induced charge as resulting from the fermions being pushed in and out of the wire, back and forth from the state that leaks to the left of \( x_1 = 0 \), by the fluctuations of the quantum dot. The apparently missing (or excess) charge resulting from this mechanism is compensated by fluctuations of opposite sign at infinity (the other end of the wire). In our construction of the quantum dot coupled to the quantum wire of section \[13\] we found that, in the absence of tunneling between the dot and the wire, \( \theta \) was restricted to the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\). Hence, the total induced charge \( Q_{\text{ind}} \) takes values in the range \(-\frac{1}{2} < Q_{\text{ind}} \leq \frac{1}{2} \). The “extra state” implyed by the leaking of the wave functions of the wire into the regions where the electrons are strongly repelled by the dot is mathematically analogous to the left “half” of the midgap state of a soliton \[20\]. Notice
that if \( \theta \) is allowed to wander beyond its natural range shifted by \( N\pi \) (with \( N \) an integer), the induced charge would instead be shifted by \( N \). Such an effect can only happen in the presence of real tunneling between the wire and the dot.

We consider now \( \theta \) as a quantum degree of freedom with the dynamics discussed in the previous section. From the expression for \( S_{\text{imp}}(\theta) \) it is clear that the inertial term acts like a high frequency cutoff. Therefore, at low frequencies the induced term in the action are dominant and we can neglect the kinetic term. We should (and do) keep the potential term. We will only discuss here the semiclassical approximation to this path-integral. In this approximation the integration range can be extended to infinity if the potential is present. In this limit, the potential can be replaced by a linearized approximation of the form \( U(\theta) \approx G\theta^2 \). Notice, however that as \( G \to 0 \) the issue of the integration range becomes important once again.

In this limit, using Eq. (5.27), integrating \( \theta \) we get

\[
\mathcal{Z}[B_\mu, b_\mu] = \mathcal{K} \exp \left( \frac{1}{2(g^2 + \pi)} \int_{\Omega} d^2x \int_{\Omega} d^2y \ B_\mu(x) \Gamma_{\mu\nu}(x, y) B_\nu(y) \right.
\]

\[
+ \frac{1}{(g^2 + \pi)} \int_{\Omega} d^2x \int_{-\infty}^{+\infty} dy_2 \ B_\mu(x) \Gamma_{\mu2}(x, y_2)(A_2^{\text{imp}} + \bar{b}_2)(y_2)
\]

\[
- \frac{g^2}{2\pi(\pi + g^2)} \int_{-\infty}^{+\infty} dx_2 \int_{-\infty}^{+\infty} dy_2 (A_2^{\text{imp}} + \bar{b}_2)(x_2) K(x_2, y_2)(A_2^{\text{imp}} + \bar{b}_2)(y_2)
\]

\[
+ \frac{G}{2} \int_{-\infty}^{+\infty} dx_2 (A_2^{\text{imp}} + \bar{b}_2)^2(x_2)
\]

\[
- \frac{G^2}{2} \int_{-\infty}^{+\infty} dx_2 \int_{-\infty}^{+\infty} dy_2 (A_2^{\text{imp}} + \bar{b}_2)(x_2) K(x_2, y_2)(A_2^{\text{imp}} + \bar{b}_2)(y_2) \right)
\]

In this approximation, the kernel \( \tilde{K}(x_2 - y_2) \) is

\[
\tilde{K}(x_2 - y_2) = e^{\pi G \mu} E_1(\pi G \mu) + e^{\pi G^* \mu} E_1(\pi G^* \mu), \text{ and } \mu = a + i(x_2 - y_2)
\]

where \( E_1(x) \) is the exponential integral function. In the bulk, the induced currents now take the form

\[
<J_{\mu}(x)> = \frac{1}{(\pi + g^2)} \int_{\Omega} d^2y \Gamma_{\mu\nu}(x, y)(B_\nu(y) + \bar{A}_\nu^{\text{imp}} + \bar{b}_\nu)(y)
\]

At the boundary we find instead
\[
\langle J_2(x_2) \rangle = \frac{1}{\pi + g^2} \int_\Omega d^2y \Gamma_\nu(x_2, y) B_\nu(y) - \frac{g^2}{\pi(x + g^2)} \int_{-\infty}^{+\infty} dy_2 K(x_2, y_2) (\bar{A}_2^{\text{imp}} + \bar{b}_2)(y_2)
\]
\[
+ G(\bar{A}_2^{\text{imp}} + \bar{b}_2)(x_2) - G^2 \int_{-\infty}^{+\infty} dy_2 \tilde{K}(x_2 - y_2)(\bar{A}_2^{\text{imp}} + \bar{b}_2)(y_2)
\]

(5.12)

To compute the charge induced by the impurity we set \( B_\mu = \bar{b}_2 = 0 \). Therefore

\[
Q_{\text{ind}}(x_2) \equiv \int_0^a dx_1 \langle J_2(x_2) \rangle + \int_{-a}^{a} dx_1 \langle J_2(x_1, x_2) \rangle - \frac{g^2}{\pi(x + g^2)} \int_0^a dx_1 \int_{-\infty}^{+\infty} dy_2 K(x_2, y_2) \bar{A}_2^{\text{imp}}(y_2) + G \int_0^a dx_1 (\bar{A}_2^{\text{imp}} + \bar{b}_2)(x_2)
\]
\[
- G^2 \int_0^a dx_1 \int_{-\infty}^{+\infty} dy_2 \tilde{K}(x_2 - y_2)(\bar{A}_2^{\text{imp}} + \bar{b}_2)(y_2)
\]
\[
+ \frac{1}{(\pi + g^2)} \int_a^{+\infty} dx_1 \int_{-\infty}^{+\infty} dy_2 \Gamma_\nu(x_2, y_2) \bar{A}_2^{\text{imp}}(y_2)
\]

(5.13)

In this expression we see that \( \lim_{x_1 \to 0} \langle J_2(x) \rangle \neq \langle J_\nu(x_2) \rangle \). This discontinuity gives rise to the induced charge. Replacing \( K(x_2, y_2) = \lim_{x_1 \to 0} \Gamma_{22}(x, y_2) \), and integrating over \( x_1 \) we can write

\[
Q_{\text{ind}}(x_2) = \lim_{a \to 0} \left( \frac{1}{\pi^2} \int_{-\infty}^{+\infty} dy_2 \bar{A}_2^{\text{imp}}(y_2) \frac{a}{a^2 + (x_2 - y_2)^2} + aG(\bar{A}_2^{\text{imp}} + \bar{b}_2)(x_2) - aG^2 \int_{-\infty}^{+\infty} dy_2 \tilde{K}(x_2 - y_2)(\bar{A}_2^{\text{imp}} + \bar{b}_2)(y_2) \right)
\]

(5.14)

According to the regularization procedure that has been used throughout this work, \( \lim_{a \to 0} \frac{a}{a^2 + (x_2 - y_2)^2} = \pi \delta(x_2 - y_2) \). Therefore, after taking the limit \( a \to 0 \) the induced charge results

\[
Q_{\text{ind}}(x_2) = \frac{1}{\pi} \bar{A}_2^{\text{imp}}(x_2)
\]

(5.15)

It is clear that, once the boundary degree of freedom \( \theta \) is taken as a dynamical variable, the induced charge is independent of the interaction. In other words, there is no screening of the impurity due to the interactions.

\section{VI. BOSONIZATION OF THE FERMION OPERATORS}

In this section we derive the one-particle Green’s function from which a set of bosonization rules for the fermion operators for an interacting Fermi system with boundaries can be
derived. For an infinite system, these rules are the well known Mandelstam formulas [19].

In reference [5] we derived a set of rules for non-interacting fermions with boundaries. Here we will follow the same methods and extract this information from the one-particle Green’s function, which is defined as

\[ S_{\alpha\beta}(x,y) = \frac{\int D\bar{\psi}D\psi D\alpha \bar{\psi}_\alpha(x)\bar{\psi}_\beta(y) e^{-S_F(\bar{\psi},\psi,a_\mu)}}{\int D\bar{\psi}D\psi D\alpha e^{-S_F(\bar{\psi},\psi,a_\mu)}} \]  \hspace{1cm} (6.1)

with \( \alpha, \beta = 1, 2 \). \( S_F \) is the action defined in Eq. (3.9) with \( A_\mu \) set to zero, i.e., with no external sources. That is, in Eq. (6.1),

\[ S_F(\bar{\psi},\psi,a_\mu) = \int_\Omega d^2x \bar{\psi}(i/\partial + \bar{a} + A^{\text{imp}})\psi + \int_\Omega d^2x \frac{a_\mu^2}{2g^2} \]  \hspace{1cm} (6.2)

We call \( Z_{\alpha\beta}^{\text{num}} \) the numerator of \( S_{\alpha\beta}(x,y) \). That is

\[ Z_{\alpha\beta}^{\text{num}} \equiv \int D\bar{\psi}D\psi D\alpha \bar{\psi}_\alpha(x)\bar{\psi}_\beta(y) e^{-S_F(\bar{\psi},\psi,a_\mu)} \]  \hspace{1cm} (6.3)

In order to compute \( Z_{\alpha\beta}^{\text{num}} \) we decouple the fermion from the bulk H-S field, \( a_\mu \), through the gauge and chiral transformations given by Eq. (3.10). After the transformation, the partition function becomes

\[ Z_{\alpha\beta}^{\text{num}} = K \det(i/\partial + \bar{A}^{\text{imp}}) S_F^{\alpha\beta}(x,y|\theta,\bar{A}^{\text{imp}}) \]

\[ \int D\eta \exp \left( -\frac{1}{2g^2} \int (\partial_\mu \eta)^2 + \eta(x) - \eta(y) \right) \]

\[ \int D\phi \exp \left( -\frac{1}{2} \left( \frac{1}{\pi} + \frac{1}{g^2} \right) \int (\partial_\mu \phi)^2 \right) \]

\[ + \frac{1}{\pi} \int \epsilon_{\nu\mu} \bar{A}^{\text{imp}}_\mu \partial_\nu \phi + \sigma_\alpha \phi(x) + \sigma_\beta \phi(y) \]  \hspace{1cm} (6.4)

where \( \sigma_1 = 1 \) and \( \sigma_2 = -1 \) and the indices \( \alpha, \beta \) are not contracted. The one-particle Green’s function that appears in Eq. (6.4) is, by definition

\[ S_F^{\alpha\beta}(x,y|\theta,\bar{A}^{\text{imp}}) = \frac{\int D\bar{\chi}D\chi \chi_\alpha(x)\bar{\chi}_\beta(y) \exp\{\int \bar{\chi}(i\bar{\partial} + \bar{A}^{\text{imp}})\chi\}}{\int D\bar{\chi}D\chi \exp\{\int \bar{\chi}(i\bar{\partial} + \bar{A}^{\text{imp}})\chi\}} \]  \hspace{1cm} (6.5)

Note that

\[ \det(i\bar{\partial} + A^{\text{imp}}) = \int D\bar{\chi}D\chi \exp\{\int \bar{\chi}(i\bar{\partial} + A^{\text{imp}})\chi\} \]  \hspace{1cm} (6.6)
The integrations over $\eta$ and $\phi$ in Eq. (6.4) are straightforward and give
\[
I_\eta = \exp \left( \frac{g^2}{2} \int_{\Omega} d^2z \int_{\Omega} d^2w J(z) G_N(z, w) J(w) \right) \tag{6.7}
\]
and
\[
I_\phi = \exp \left( -\frac{\pi}{2(\frac{g}{2} + 1)} \int_{\Omega} d^2z \int_{\Omega} d^2w \left[ J_{\alpha\beta}(z) - \frac{1}{\pi} \epsilon_{\mu\nu} \partial_\nu (\bar{A}_{\mu}^{\text{imp}})(z) \right] G_D(z, w) \right. \\
\left. \left[ J_{\alpha\beta}(w) - \frac{1}{\pi} \epsilon_{\mu\nu} \partial_\nu (\bar{A}_{\mu}^{\text{imp}})(w) \right] \right) \tag{6.8}
\]
with $J(z) = \delta(z - x) - \delta(z - y)$ and $J_{\alpha\beta}(z) = \sigma_\alpha \delta(z - x) + \sigma_\beta \delta(z - y)$. The function $G_N(x, y)$ is the Von Newman Green’s function in $\Omega$ which satisfies
\[
\begin{cases}
\partial_x^2 G_N(x, y) = \delta(x - y) \\
\partial_1 G_N(x_1, x_2; y)|_{x_1=0} = 0
\end{cases} \tag{6.9}
\]
The solution of Eq. (6.9) is
\[
G_N(x, y) = \frac{1}{4\pi} \left( \ln \frac{(x_1 - y_1)^2 + (x_2 - y_2)^2 + a^2}{a^2} + \ln \frac{(x_1 + y_1)^2 + (x_2 - y_2)^2 + a^2}{a^2} \right) \tag{6.10}
\]
where $a$ is the short distance cut-off.

Since the fermionic fields $\bar{\chi}$ and $\chi$ satisfy $\bar{\chi}_R(x_2) = -e^{2\theta(x_2)}\chi_L(x_2)$, the $\text{Det}(i\partial + \bar{A}^{\text{imp}})$ turns out to be (see [3])
\[
\text{Det}(i\partial + \bar{A}^{\text{imp}}) = \exp \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz_2 \int_{-\infty}^{+\infty} dw_2 (\theta + \bar{A}_{\text{imp}}^2)(z_2) K(z_2, w_2)(\theta + \bar{A}_{\text{imp}}^2)(w_2) \right) \tag{6.11}
\]
In Appendix [A] we calculate the one particle Green’s function $S_F(x, y|\theta, \bar{A}^{\text{imp}})$. There we find that it has the form
\[
\begin{align*}
S_F^{\alpha\beta}(x, y|\theta, \bar{A}_\mu^{\text{imp}}) &= \exp \left( \frac{1}{\pi} \int_{-\infty}^{+\infty} dz_2 \bar{A}_{\text{imp}}^2(z_2) \frac{\sigma_\alpha x_1 + i(x_2 - z_2)}{y_1^2 + (x_2 - z_2)^2} \right) \\
S_F^{0,\alpha\beta}(x, y|\theta) &= \exp \left( \frac{1}{\pi} \int_{-\infty}^{+\infty} dz_2 \bar{A}_{\text{imp}}^2(z_2) \frac{\sigma_\beta y_1 - i(y_2 - z_2)}{y_1^2 + (y_2 - z_2)^2} \right) \tag{6.12}
\end{align*}
\]
where $S_F^{0,\alpha\beta}(x, y|\theta)$ is the one particle Green’s function for free fermions in the half line with $R(x_2) = -e^{2\theta(x_2)}L(x_2)$ which was calculated in [3].

In order to simplify the notation, we define $R_{\alpha\beta}(w_2)$ and $R(w_2)$ as
\[ R_{\alpha\beta}(w_2) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dz \frac{z_1}{z_1^2 + (w_2 - z_2)^2} J_{\alpha\beta}(z) \]  
(6.13)

and

\[ R(x_2) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dz \frac{z_1}{z_1^2 + (z_2 - x_2)^2} J(z) \]  
(6.14)

Using these results, the one particle Green’s function of Eq. (6.1) can be shown to have the form

\[ S^{\alpha\beta}(x, y) = S_F^{0,\alpha\beta}(x, y|\theta) \exp \left( \int_{z_2} \left[ R_{\alpha\beta}(z_2) \frac{\pi}{(\pi + g^2)} \tilde{A}^{\text{imp}}_2(z_2) + iR(z_2)\tilde{A}^{\text{imp}}_2(z_2) \right] \right) \]

\[ \exp \left( \int_{z,w} \left[ \frac{g^2}{2} J(z)G_N(z, w)J(w) - \frac{\pi}{2(\pi + 1)} J_{\alpha\beta}(z)G_D(z, w)J_{\alpha\beta}(w) \right] \right) \]  
(6.15)

We can rewrite this one particle Green’s function in a more familiar way as

\[ S^{\alpha\beta}(x, y) = 1 \frac{1}{2\pi} \left( \frac{4x_1^2 + a^2}{a^2} \right)^{\frac{\Delta_1}{2}} \left( \frac{4y_1^2 + a^2}{a^2} \right)^{\frac{\Delta_1}{2}} \left( \frac{H^{11}}{w|\Delta_2|w|\Delta_1|} - \frac{H^{21}}{z|\Delta_2|z|\Delta_1|} \right) \left( \frac{H^{12}}{w|\Delta_2|w|\Delta_1|} - \frac{H^{22}}{z|\Delta_2|z|\Delta_1|} \right) \]  
(6.16)

where \( \Delta_1 = \frac{q^2(2\pi + g^2)}{2\pi(\pi + g^2)} \), \( \Delta_2 = \frac{q^4}{2\pi(\pi + g^2)} \), \( z = (x_1 - y_1) + i(x_2 - y_2) \), \( w = (x_1 + y_1) + i(x_2 - y_2) \)

and \( H^{\alpha\beta}(x, y) \) is

\[ H^{\alpha\beta}(x, y) = \exp \left( \frac{i}{\pi} \int_{-\infty}^{+\infty} dz_2 \left( \theta + \tilde{A}^{\text{imp}}_2(z_2) \right) \left[ \frac{x_2 - z_2}{x_1^2 + (x_2 - z_2)^2} - \frac{y_2 - z_2}{y_1^2 + (y_2 - z_2)^2} \right] \right) \]

\[ + \frac{1}{\pi} \int_{-\infty}^{+\infty} dz_2 \left( \frac{\pi}{\pi + g^2} \tilde{A}^{\text{imp}}_2(z_2) \right) \left[ \sigma^\alpha_{\beta} \frac{x_1}{x_1^2 + (x_2 - z_2)^2} + \sigma^\beta_{\alpha} \frac{y_1}{y_1^2 + (y_2 - z_2)^2} \right] \]  
(6.17)

These results coincide with the one obtained by Fabrizio et. al. \[10\] (in the limit of \( L \to \infty \) and for spinless fermions in that reference) if we set \( \theta = \tilde{A}^{\text{imp}}_\mu = 0 \).

It is interesting to analyze the behavior of the one particle Green’s function away and close to the boundary. The condition on \( z \) and \( w \) to be far from the boundary is given by

\[ |x_1 - y_1| \ll \min(x_1, y_1) \text{ and } (x_2 - y_2) \ll \min(x_1, y_1) \].

This last condition comes from the fact that if it does not hold, the excitations will have time to reach the boundary and “return” with information about it to the points where the Green’s function is being calculated. In
this limit the diagonal terms of Eq (6.16) vanish because $\Delta_1 - (\Delta_2 + 1) < 0$ for any value of $g^2$. Then the one particle Green’s function becomes

$$S^{\alpha \beta}(x, y) = \frac{1}{2\pi} \begin{pmatrix} 0 & \frac{H^{12}}{z|z|^{\Delta_2}} \\ -\frac{H^{21}}{\bar{z}|\bar{z}|^{\Delta_2}} & 0 \end{pmatrix}$$

which is the one particle Green’s function for the whole line if we take $\theta = \bar{A}_2 = 0$ in $H^{12}$ and $H^{21}$. Note also that the diagonal terms vanish as $x_1 \sim \pi g^2$. Therefore they go to zero slower than in the free case. Physically this means that the presence of the boundary can be felt further away when the electrons interact among themselves. That is, the electron interaction “propagate” the boundary into the bulk. The condition for $z$ and $w$ to be close to the boundary is that $x_1, y_1 \ll \min x_2, y_2$. In this limit the components of $S^{\alpha \beta}(x, y)$ become proportional to $|x_2 - y_2|^{-(\frac{g^2}{\pi} + 1)}$. Thus we see here the effect of the interactions is to change the exponent of the power law at the boundary relative to the free electron gas. In particular, it goes to zero faster than in the free case. We remark that the asymptotic behaviors for the one particle Green’s function, far away and close to the boundary is in total agreement with results from Conformal Field Theory [11].

Up to this point we have considered the boundary condition $\theta$ as a classical degree of freedom. To see the effect in the one particle Green’s function of a quantum degree of freedom at the boundary we have to integrate over $\theta$ in eq. (6.16). We follow the same steps described in reference [3] to obtain

$$S^{\alpha \beta}(x, y) = \frac{1}{2\pi} \begin{pmatrix} 0 & \frac{H^{12}}{z|z|^{\Delta_2}} \\ -\frac{H^{21}}{\bar{z}|\bar{z}|^{\Delta_2}} & 0 \end{pmatrix}$$

where

$$\bar{H}^{\alpha \beta}(x, y) = -\frac{g^2}{\pi(\pi + g^2)} \int_{-\infty}^{+\infty} dz_2 \left[ \sigma_\alpha \frac{x_1}{x_1^2 + (x_2 - z_2)^2} + \sigma_\beta \frac{y_1}{y_1^2 + (y_2 - z_2)^2} \right] \bar{A}_2^{\text{imp}}(z_2)$$

At this point, we make the analytic continuation $\bar{A}_2^{\text{imp}} \to i\bar{A}_0^{\text{imp}}$ which set us back in Minkowski space (real time). We can see then that $\bar{H}^{\alpha \beta}(x, y)$ becomes a pure phase. Moreover, for a static impurity , that is when $\bar{A}_0^{\text{imp}}$ is a constant, $\bar{H}^{\alpha \beta}(x, y) \equiv 1$ for $\alpha \neq \beta$. In
other word, the phase shift for $\langle \psi^\dagger_R \psi_R \rangle$ and $\langle \psi^\dagger_L \psi_L \rangle$ vanishes for static impurities. This result can be understood if we think in terms of the analogous result in the whole line. It can be seen that for the whole line and for a forward scattering static impurity $A^\text{imp}_0$

$$\langle \psi^\dagger_R(x)\psi_R(y) \rangle = e^{\frac{-A^\text{imp}_0}{1+\frac{g^2}{\pi^2}}} \langle \psi^\dagger_R(x)\psi_R(y) \rangle |_{\text{Latt}}, \quad x > 0, \quad y < 0 \quad (6.21)$$

where $\langle \psi^\dagger_R(x)\psi_R(y) \rangle |_{\text{Latt}}$ is the one particle Green’s function for the Luttinger liquid in absence of the impurity. If both, $x_1$ and $y_1$ are either to the left or to the right of the impurity, the phase factor vanishes. In the case we are dealing with, $x_1$ and $y_1$ are always to the right of the impurity. Hence, in the absence of boundary backscattering terms, the the only effect of the interactions is a partial screening of the impurity represented by a factor of $\frac{g^2}{\pi+g^2}$ in the phase shift.

VII. CONCLUSIONS

In this paper we studied the Luttinger model in the half line coupled to a quantum impurity at the origin. We generalized the bosonization methods discussed in reference [5] to study the effects of interactions on fermi systems coupled to impurities. We gave special attention to the role of the fermion boundary conditions on the bosonized effective theory. We showed that this model gives a good physical picture of a quantum wire coupled with a quantum dot by a purely capacitive interaction. The fluctuations of the degree of freedom $\theta$ at the boundary turn out to be determined by the fluctuations of the charge on the quantum dot. The screening effect at the boundary has been described. We calculated the induced charge at the boundary in two cases, when the boundary condition is a classical degree of freedom and when it is a quantum one. In the first case, in accordance with semiclassical theories, we obtained charge fractionalization. In the quantum case, the induced charge depends on the impurity as expected. We also showed that the bosonized currents are modified by the electron-electron interactions as well as by the presence of the boundary degree of freedom, $\theta$. Finally we discussed the properties of the one-particle green function.
(relevant for tunneling problems) far away and close to the boundary.

Our results in the case of open boundary conditions, \( i.e. \), for \( \theta = 0 \), coincide with the ones obtained in the literature in that particular case (see for instance [3]). We stress here that with our approach it is possible to deal with a more general case, since we can compute the correlation functions in the case of arbitrary boundary conditions, \( i.e. \), arbitrary \( \theta \). In principle, by changing the dynamics assigned to the boundary degree of freedom one should be able to study different experimental realizations, like for instance a Luttinger liquid coupled to a superconductor [21]. Moreover, this approach can be used to study other situations like infinite systems coupled to impurities by using its representations as a semi-infinite system coupled to an impurity at the origin with properly defined boundary conditions [22].

**VIII. ACKNOWLEDGEMENTS**

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APPENDIX A: GREEN’S FUNCTIONS FOR DIRAC FERMIONS COUPLED TO SINGULAR GAUGE FIELDS

The aim of this section is to compute of the Green’s function

\[
S_F(x, y|\theta, \bar{A}^{\text{imp}}) = \frac{\int \mathcal{D}\bar{\chi} \mathcal{D}\chi \chi_\alpha(x)\bar{\chi}_\beta(y) \ e^{\int \bar{\chi}(i\partial_x + \bar{A}^{\text{imp}})\chi}}{\int \mathcal{D}\bar{\chi} \mathcal{D}\chi e^{\int \bar{\chi}(i\partial_x + \bar{A}^{\text{imp}})\chi}} \tag{A1}
\]

By definition, this Green’s function satisfies the equation

\[
(i\partial_x + \bar{A}^{\text{imp}})S_F(x, y|\theta, \bar{A}^{\text{imp}}) = \delta(x - y) \tag{A2}
\]

What requires a little more analysis is to determine the boundary condition that this Green’s function satisfies at \(x_1 = 0\). At the beginning, the functional space was the one whose functions at the boundary satisfy \(R(x_2) = -e^{2\theta(x_2)}L(x_2)\). But as we showed in reference \[5\], in order to compute the determinant \(\mathcal{D}et(i\partial_x + \bar{A}^{\text{imp}})\) in this space, it is equivalent to work in the functional space whose functions satisfy, at the boundary, \(R(x_2) = -e^{2(\theta + \bar{A}^{\text{imp}})(x_2)}L(x_2)\) and compute on such space, the determinant \(\mathcal{D}et(i\partial_x)\). Since \(\mathcal{D}et(i\partial_x + \bar{A}^{\text{imp}})\) is the denominator of Eq. (A1), and since, obviously, the whole left hand side of Eq. (A1) has to be computed consistently over a single functional space, the resulting Green’s function is the Green’s function over such functional space. Hence, the Green’s function in Eq. (A1) is the one over such functional space. Therefore, the Green’s function in Eq. (A1) is the one over the functional space whose functions satisfy, at the boundary,\(R(x_2) = -e^{2(\theta + \bar{A}^{\text{imp}})(x_2)}L(x_2)\).

Another way of understanding this point is recalling that, in a second quantification language, the denominator of Eq. (A1) being a normalization for the Green’s function, defines the vacuum states. That is the creation and anihilation operators which expand the physical phase space. Hence if we choose such space as the one whose operators satisfy \(R(x_2) = -e^{2(\theta + \bar{A}^{\text{imp}})(x_2)}L(x_2)\), the boundary conditions for the Green’s functions we compute have the same boundary conditions.

Therefore, the aim now is to compute

\[
\begin{cases}
(i\partial_x + \bar{A}^{\text{imp}})S_F(x, y|\theta, \bar{A}^{\text{imp}}) = \delta(x - y) \\
S_{F1}^{11}(0, x_2; y|\theta, \bar{A}^{\text{imp}}) = -e^{2(\theta + \bar{A}^{\text{imp}})(x_2)}S_{F}^{21}(0, x_2; y|\theta, \bar{A}^{\text{imp}})
\end{cases} \tag{A3}
\]
It is easy to see that if we choose $U$ and $V$ such that $A^{\text{imp}}_{\mu}(x_2)\delta(x_1) = \partial_{\mu}V - \epsilon_{\mu\nu}\partial_{\nu}U$ then

\[
S_F(x, y|\theta, \bar{A}^{\text{imp}}) = e^{iV(x)+\gamma_5U(x)}S^0_F(x, y|\theta, \bar{A}^{\text{imp}}, U)e^{-iV(x)+\gamma_5U(x)} 
\]  

where $S^0_F(x, y|\theta, \bar{A}^{\text{imp}}, U)$ satisfies

\[
\begin{aligned}
&\left(\mp\frac{\partial}{\mp}\right)S^0_F(x, y|\theta, \bar{A}^{\text{imp}}, U) = \delta(x - y) \\
&S^0_{F,11}(0, x_2; y|\theta, \bar{A}^{\text{imp}}, U) = -e^{2[(\theta + A^{\text{imp}}_{\alpha\beta}(x_2) - U(0, x_2)]}S^0_{F,21}(0, x_2; y|\theta, \bar{A}^{\text{imp}}, U) 
\end{aligned} 
\]  

As has been shown in [5],

\[
S^0_{F,\alpha\beta}(x, y|\theta, \bar{A}^{\text{imp}}, U) = \exp\left(\frac{1}{\pi} \int dz_2 (\bar{A}^{\text{imp}}_{\alpha\beta} + \theta - U)(z_2)\frac{\sigma_\alpha x_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2}\right) \exp\left(\frac{1}{\pi} \int dz_2 (\bar{A}^{\text{imp}}_{\beta\alpha} + \theta - U)(z_2)\frac{\sigma_\beta y_1 + i(y_2 - z_2)}{y_1^2 + (y_2 - z_2)^2}\right) 
\]  

with $S^0_F(x, y)$ the Green’s function

\[
S^0_F(x, y) = \frac{1}{2\pi} \left( \frac{1}{(x_1 - y_1) + i(x_2 - y_2)} \right) \left( \frac{1}{(x_1 - y_1) - i(x_2 - y_2)} \right) 
\]  

We have found that $U$ and $V$ above are

\[
U(x) = \frac{-1}{\pi} \int dz_2 \bar{A}^{\text{imp}}_1(z_2)\frac{x_1}{x_1^2 + (x_2 - z_2)^2} + \frac{1}{\pi} \int dz_2 \bar{A}^{\text{imp}}_2(z_2)\frac{x_1}{x_1^2 + (x_2 - z_2)^2} 
\]  

and

\[
V(x) = \frac{1}{\pi} \int dz_2 \bar{A}^{\text{imp}}_1(z_2)\frac{x_1}{x_1^2 + (x_2 - z_2)^2} + \frac{1}{\pi} \int dz_2 \bar{A}^{\text{imp}}_2(z_2)\frac{x_1}{x_1^2 + (x_2 - z_2)^2} 
\]  

Using the results above we obtain

\[
\begin{aligned}
S^0_{F,\alpha\beta}(x, y|\theta, \bar{A}^{\text{imp}}) &= \exp\left(\frac{1}{\pi} \int dz_2 (2\bar{A}^{\text{imp}}_\alpha + \theta - U)(z_2)\frac{\sigma_\alpha x_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2}\right) \\
&\quad \exp\left(\frac{1}{\pi} \int dz_2 (\bar{A}^{\text{imp}}_\alpha(z_2)\frac{ix_1 - \sigma_\alpha(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2})S^0_{F,\alpha\beta}(x, y) \\
&\quad \exp\left(\frac{1}{\pi} \int dz_2 (2\bar{A}^{\text{imp}}_\alpha + \theta - U)(z_2)\frac{\sigma_\beta y_1 - i(y_2 - z_2)}{y_1^2 + (y_2 - z_2)^2}\right) \\
&\quad \exp\left(\frac{1}{\pi} \int dz_2 (\bar{A}^{\text{imp}}_\alpha(z_2)\frac{-iy_1 - \sigma_\beta(y_2 - z_2)}{y_1^2 + (y_2 - z_2)^2}\right) 
\end{aligned} 
\]  

It can be shown that
\[
\frac{1}{\pi} \int d\bar{z}_2 U(z_2) \frac{\sigma_\alpha x_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2} = \frac{1}{\pi} \int d\bar{z}_2 \bar{A}_2^{\text{imp}}(z_2) \frac{\sigma_\alpha x_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2} \\
+ \frac{1}{\pi} \int d\bar{z}_2 \bar{A}_1^{\text{imp}}(z_2) \frac{i x_1 - \sigma_\alpha (x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2}
\]

(A11)

Hence the Green’s function \( S_F(x, y|\theta, \bar{A}^{\text{imp}}, \bar{a}) \) becomes

\[
S^{\alpha\beta}_F(x, y|\theta, \bar{A}^{\text{imp}}) = \exp \left( \frac{1}{\pi} \int d\bar{z}_2 (\bar{A}_2^{\text{imp}} + \theta)(z_2) \frac{\sigma_\alpha x_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2} \right) S_F^{0,\alpha\beta}(x, y) \\
\exp \left( \frac{1}{\pi} \int d\bar{z}_2 (\bar{A}_2^{\text{imp}} + \theta)(z_2) \frac{\sigma_\beta y_1 - i(y_2 - z_2)}{y_1^2 + (y_2 - z_2)^2} \right)
\]

(A12)

Note that this expression can be factorized as

\[
S^{\alpha\beta}_F(x, y|\theta, \bar{A}^{\text{imp}}) = \exp \left( \frac{1}{\pi} \int d\bar{z}_2 \bar{A}_2^{\text{imp}}(z_2) \frac{\sigma_\alpha x_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2} \right) S_F^{0,\alpha\beta}(x, y|\theta) \\
\exp \left( \frac{1}{\pi} \int d\bar{z}_2 \bar{A}_2^{\text{imp}}(z_2) \frac{\sigma_\beta y_1 - i(y_2 - z_2)}{y_1^2 + (y_2 - z_2)^2} \right)
\]

(A13)

where \( S_F^0(x, y|\theta) \) satisfies Eq. (A3) with \( \theta \) in the boundary condition.
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