Abstract

This paper is the continuation of the research of the author and his colleagues of the canonical decomposition of graphs. The idea of the canonical decomposition is to define the binary operation on the set of graphs and to represent the graph under study as a product of prime elements with respect to this operation. We consider the graph together with the arbitrary partition of its vertex set into $n$ subsets ($n$-partitioned graph). On the set of $n$-partitioned graphs distinguished up to isomorphism we consider the binary algebraic operation $\circ_H$ ($H$-product of graphs), determined by the digraph $H$. It is proved, that every operation $\circ_H$ defines the unique factorization as a product of prime factors. We define $H$-threshold graphs as graphs, which could be represented as the product $\circ_H$ of one-vertex factors, and the threshold-width of the graph $G$ as the minimum size of $H$ such, that $G$ is $H$-threshold. $H$-threshold graphs generalize the classes of threshold graphs and difference graphs and extend their properties. We show, that the threshold-width is defined for all graphs, and give the characterization of graphs with fixed threshold-width. We study in detail the graphs with threshold-widths 1 and 2.

Key words:
Graph decomposition, canonical decomposition, threshold-width, $H$-threshold graph, finite list of forbidden induced subgraphs

1 Introduction

The decomposition methods are widely and fruitfully used in different areas of combinatorics and graph theory. This paper is the continuation of the previous research of the author and his colleagues of the canonical or algebraic
decomposition of graphs. The idea of the canonical decomposition is to define
the binary operation on the set of graphs and to represent the graph under
study as a product of prime elements with respect to this operation.

Before formulating the idea of the canonical decomposition, let us give some
basic definitions. All graphs considered are finite, undirected, without loops
and multiple edges. At the same time further in this paper the loops (but not
multiple arcs) are allowed in digraphs. The vertex and the edge sets of a graph
\( G \) are denoted by \( V(G) \) and \( E(G) \), respectively. The vertex set and the arc
set of a digraph \( H \) are denoted by \( V(H) \) and \( A(H) \). Further, denote by \( G[X] \)
the subgraph induced by the set \( X \subseteq V(G) \). For the convenience of reading
the edges of graphs will be denoted as \( uv \), and the arcs of digraphs - as \((u, v)\).
Write \( u \sim v \) (resp. \( u \not\sim v \)) if \( uv \in E(G) \) (resp. \( uv \not\in E(G) \)).

A graph \( G \) is called \textit{split} [11], if its vertex set could be partitioned into a
clique \( A \) and a independent set \( B \). The graph \( G \) is \textit{bipartite}, if if its vertex set
could be partitioned into two independent sets \( A \) and \( B \). The vertex set of the
complement of bipartite graph could be partitioned into two cliques \( A \) and \( B \).
The partition \((A, B)\) in all those cases is called \textit{a bipartition}.

If \( X, Y \subseteq V(G) \), we will write \( X \sim Y \) \( (X \not\sim Y) \) if for every \( x \in X \) and \( y \in Y \)
\( x \sim y \) \( (x \not\sim y) \). Let \( N_Y(x) = \{ y \in Y : y \sim x \} \).

The first variant of the canonical decomposition was introduced by R. Tyshke-
vich and A. Chernyak [24] (in Russian) and described in detail in [23]. Consider
\textit{triads} (or \textit{split graphs}) \( T = (G, A, B) \) where \( G \) is a split graph and \((A, B)\)
is some fixed partition of the set \( V(G) \) into clique \( A \) and independent set \( B \) \textit{(bipartition)}. The two triads \( T_i = (G_i, A_i, B_i), i = 1, 2 \), are isomorphic, if there
exists an isomorphism \( \beta : V(G_1) \rightarrow V(G_2) \) of the graphs \( G_1 \) and \( G_2 \) preserv-
ing the bipartition \( (\beta(A_1) = A_2, \ \beta(B_1) = B_2) \). Denote the set of all triads
(graphs) up to isomorphism of triads (graphs) by \( Tr(Gr) \).

The triads from \( Tr \) could be considered as left operators acting on the set \( Gr \),
the action of the operators is defined by the formula

\[
(H, A, B) \circ G = G \cup H + \{ ax : a \in A, x \in V(G) \}. \tag{1}
\]

On the set \( Tr \) the action (1) induces the associative binary algebraic operation
\textit{(the multiplication of triads)}:

\[
(G_1, A_1, B_1) \circ (G_2, A_2, B_2) = ((G_1, A_1, B_1) \circ G_2, \ A_1 \cup A_2, \ B_1 \cup B_2). \tag{2}
\]

A triad \( T \) is called \textit{decomposable} if it can be represented as a product of two
triads. The graph is \textit{decomposable}, if it is a product of a triad and a graph.
Every triad $T$ can be represented as a product

$$T = T_1 \circ T_2 \circ \ldots \circ T_k, \ k \geq 1,$$

(3)

of indecomposable triads $T_i$ (the parentheses in (3) could be omitted because the operation $\circ$ is associative). Analogously, every graph $G$ can be represented a product

$$G = T_1 \circ T_2 \circ \ldots \circ T_k \circ G_0, \ k \geq 1,$$

(4)

of indecomposable triads $T_i$ and indecomposable graph $G_0$. The representations (3) and (4) are called the canonical decomposition of the triad and the graphs, respectively.

The most important property of the canonical decomposition is the following unique factorization theorem:

**Theorem 1** [23]

*The canonical decomposition of the graph is determined uniquely, i.e. two graphs $G$ and $H$ with canonical decompositions (4) and $H = S_1 \circ S_2 \circ \ldots \circ S_l \circ H_0$ are isomorphic if and only if

1) $k = l$;
2) $T_i \cong S_i$, $i = 1, \ldots, k$;
3) $G_0 \cong H_0$."

The unique factorization property also holds for triads:

**Theorem 2** *The canonical decomposition of the triad is determined uniquely, i.e. two triads $T$ and $S$ with canonical decompositions (3) and $S = S_1 \circ S_2 \circ \ldots \circ S_l$ are isomorphic if and only if

1) $k = l$;
2) $T_i \cong S_i$, $i = 1, \ldots, k$;"

The unique factorization theorems makes the canonical decomposition a very strong and useful tool to deal with the problems connected with the isomorphism. In particular, using the canonical decomposition the complete structural characterization of unigraphs (graphs defined up to isomorphism by their degree sequences) was obtained by R. Tyshkevich in [23]. The crucial point of the method of R. Tyshkevich was the fact, that the graph $G$ is a unigraph if and only if all graphs in its canonical decomposition are unigraphs, which follows from the unique factorization theorem. So, to describe the structure of unigraphs it is enough to describe all indecomposable split and indecomposable
non-split unigraphs. The description was found in [23] using the properties of the canonical decomposition and its connections with the degree sequences of graphs.

Another applications of the canonical decomposition are the characterizations and/or enumerations of matroidal [25], matrogenic [22], box-threshold [6], domishold [5], pseudo-split graphs [16][19] (these and another examples could be found in monographs [4] and [17]). The very recent studies of the canonical decomposition and its applications were carried out by M. Barrus and D. West [2],[3]. Among their results the very elegant characterization of decomposable graphs from [3] should be especially mentioned: the graph \( G \) is indecomposable if and only if its so-called \( A_4^{-} \) structure is connected. M. Barrus also applied the canonical decomposition to the antimagic labelings of graphs [2].

The success of the canonical decomposition stimulated author and his colleagues to consider the following problem: how to generalize canonical decomposition keeping all its advantages? The most natural way to do it is to consider all triads \( T = (G, A, B) \), where \( G \) is an arbitrary graph and \( (A, B) \) is some arbitrary partition of its vertex set. The multiplication operations remain the same, as in the case of the canonical decomposition. In this case the representations 3 and 4 are called an operator decomposition of triad and graph, respectively (the name came from the observation, that the set of triads acts like the semigroup of operators on the set of graphs). The operator decomposition was firstly considered in [26] (in Russian) and studied in detail in [20].

It appears, that in general the unique factorization theorem does not hold for graphs, but holds for triads (up to permutations of staying together commutative multipliers) [20]. It is still a very powerful property, which was confirmed by the applications of the operator decomposition to the one of the most old and famous open problems in graph theory – the reconstruction conjecture.

Before formulating that results, let us introduce some notions. A pair of graph classes \((P, Q)\) is called closed hereditary, if they are hereditary, \( P \) is closed with respect to the operation of join and \( Q \) is closed with respect to the operation of disjoint union. Graph \( G \) is \((P, Q)\)-split, if there exists a partition \( V(G) = A \cup B \) such, that \( G[A] \in P \) and \( G[B] \in Q \). The set \( M \subseteq V(G) \) is called a homogeneous set, if every vertex \( v \in V(G) \setminus M \) is adjacent either to all vertices of \( M \) or to none of them. Denote the sets of vertices of the first and the second type by \( A(M) \) and \( B(M) \), respectively. The main result of [20] is the following. Suppose that the graph \( G \) have a homogeneous set \( M \) such that for some closed hereditary pair of classes \((P, Q)\) \( G[A] \in P \), \( G[B] \in Q \) and \( G[M] \) is not \((P, Q)\)-split. Then \( G \) is reconstructible. Note, that the property of the closed hereditariness of a pair \((P, Q)\) is not very restrictive (there are many well-known graph classes, which form such a pair), and so
the reconstruction result is rather general. Another applications of the unique
factorization theorem for triads in this area includes proof of the reconstruction
conjecture for \( P_4 \)-disconnected and \( P_4 \)-tidy graphs [18].

The machinery behind the reconstruction results above is based on the unique
factorization theorem for the operator decomposition of triads.

The further development of the theory of decomposition and its applications
requires further generalization. The natural next step is the consideration of
an arbitrary algebraic operation and turning the set of graphs into semigroup
with respect to this operation. In this paper we study such operations.

Consider the graph together with some arbitrary partition of its vertex set
into \( n \) subsets. Let us call this object \( n \)-partitioned graph. The isomorphism
of \( n \)-partitioned graphs is naturally defined as the isomorphism of corresponding
graphs preserving the partitions. On the set of all \( n \)-partitioned graphs
distinguished up to an isomorphism define the binary algebraic operation \( \circ_H \)
\((H\text{-product of graphs})\) determined by the digraph \( H \) with \( V(H) = \{1, \ldots, n\} \).
For the two \( n \)-partitioned graphs \( T = (G, A_1, \ldots, A_n) \) and \( S = (F, B_1, \ldots, B_n) \)
\((V(G) \cap V(F) = \emptyset)\) their product \( S = T \circ_H S \) is the \( n \)-partitioned graph
\((R, A_1 \cup B_1, \ldots, A_n \cup B_n), \) where \( A_i \) and \( B_j \) are completely adjacent in \( F, \) if
\((i, j) \) is an arc of \( H, \) and completely nonadjacent, otherwise. The representation
of the \( n \)-partitioned graph as an \( H \)-product of prime factors is called
its \( H \)-decomposition. Within this approach the operator decomposition is \( H_0 \)-
decomposition, where the digraph \( H_0 \) is shown on the figure 1.

\[
H_0 = \begin{array}{c}
1 \\
2
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\]

Fig. 1. The digraph \( H_0 \)

The algebraic properties of the operation \( \circ_H \) for 2-vertex digraphs \( H \) were
studied before. The fact, that for every \( H \) with \( |H| = 2 \) the operation \( \circ_H \)
defines the unique factorization of 2-partitioned graph up to the permutation
of staying together commutative multipliers, follows from the results of [15].
Independently, the same fact for \( H^* \) with \( A(H^*) = \{(1, 2)\} \) was proved in [21].
Moreover, in [21] the multiplication \( \circ_{H^*} \) of a bipartite graph with the fixed
bipartition and a graph was considered (analogously with the multiplication
of a splitted graph and a graph above), and it was proved, that in this case the
unique factorization property also holds for the decomposition of graphs, with
the exception of the simple and well-described graph family. This unique factorization theorem was used to prove, that for the graphs decomposable with
respect to \( \circ_{H^*} \) the reconstruction conjecture is true. The last result is naturally
related to the old and well-known open problem: to prove the reconstruction conjecture for bipartite graphs.

In fact, this kind of operations was already introduce in the theories of clique-
width [8] and NLC-width [27]. This two notions are similar and in some sense equivalent, so let us quote the definition of NLC-width and the corresponding decomposition. For a given integer $k$ consider the set of all labeled graphs $(G, l)$, where $l$ is a mapping $l : V(G) \rightarrow \{1, ..., k\}$. The class $NLC_k$ is recursively defined as follows [27]:

1) the one-vertex labeled graphs $(K_1, l)$ belongs to $NLC_k$;

2) if $(G_1, l_1), (G_2, l_2) \in NLC_k (V(G_1) \cap V(G_2) = \emptyset)$ and $S$ is some binary relation on the set $\{1, ..., k\}$, then the following labeled graph $(H, p)$ belongs to $NLC_k$:

$$V(H) = V(G_1) \cup V(G_2),$$

$$E(H) = E(G_1) \cup E(G_2) \cup \{uv : (l_1(u), l_2(v)) \in S\}$$

$$p(u) = l_i(u), u \in V(G_i), i = 1, 2.$$  (7)

3) If $(G, l) \in NLC_k$ and $\alpha : \{1, ..., k\} \rightarrow \{1, ..., k\}$ is a function, then $(G, \alpha l) \in NLC_k$ (here $\alpha l$ is the composition of functions).

NLC-width of a graph $G$ is the minimal $k$ such, that $G \in NLC_k$.

Clearly, the operation in 2) is exactly the operation $\circ H$. But it was introduced with completely different purposes, and its algebraic properties in general case have not been studied before. We consider the decomposition idea from the different point of view - as the study of binary algebraic operation. Since we want to obtain the decomposition tool useful for the problems connected with isomorphism (especially for the reconstruction conjecture), the main questions, which we are interested in, is the existence of the unique factorization property.

We also would like to note, that $H$-decomposition is related to another well-known graph-theoretical notion – the idea of $M$-partitions introduced by T. Feder, P. Hell, S. Klein and R. Motwani in [9]. Suppose that $M$ is the $n \times n$ symmetric matrix with the elements from the set $\{0, 1, *\}$. An $M$-partition of the graph $G$ is a partition $V(G) = A_1 \cup ... \cup A_n$ such that each $A_i$ is either a clique (if $M_{i,i} = 1$), or independent set (if $M_{i,i} = 0$), or an arbitrary set (if $M_{i,i} = *$); and $A_i$ and $A_j$ are either completely adjacent (if $M_{i,j} = 1$), or completely nonadjacent (if $M_{i,j} = 0$), or can have arbitrary set of edges between them (if $M_{i,j} = *$). The matrix $M$ could be considered as an adjacency matrix of a trigraph [10], which consists of the set of $n$ vertices $\{v_1, ..., v_n\}$, any two vertices $v_i, v_j$ are connected either by a non-edge (if $M_{i,j} = 0$), or weak edge (if $M_{i,j} = *$), or strong edge (if $M_{i,j} = 1$). In this terms our decomposable graphs are $M$-partitionable graphs, where $M_{i,i} = *$ for all $i$ and the graph formed by strong edges and non-edges of the trigraph defined by $M$ is complete bipartite with the parts of equal size (or, in other terms, our decomposable graphs are the graphs admitting homomorphism to trigraphs with the above-
This paper consists of 3 parts. In the first part we define the $H$-product. We show, that for every digraph $H$ the unique factorization property of $H$-product of $n$-partitioned graphs holds. Namely, for every digraph $H$ every $n$-partitioned graph has the unique $H$-decomposition up to the permutation of staying together commutative factors.

In the second part we define and study $H$-threshold graphs and the corresponding dimension of graphs – the objects based on the binary algebraic operations defined in the first part. The idea came both from the well-known notion of threshold graph [7] and from the theory of clique-width and NLC-width.

Threshold graphs is the important and well-studied graph class with many interesting properties and applications. There is a number of different equivalent definitions of threshold graph. The most important and well-known of them are summarized in the following theorem (those and another characterizations, properties and applications of threshold graphs could be found in the monograph [17])

**Theorem 3 [17]**

The following definitions of the threshold graph $G$ are equivalent:

a) There exist nonnegative weights $(\alpha_v : v \in V(G))$ and a threshold $t$ such, that $U \subseteq V(G)$ is an independent set if and only if $\sum_{v \in U} \alpha_v \leq t$.

b) There exist nonnegative weights $(\beta_v : v \in V(G))$ and a threshold $s$ such, that $uv \in E(G)$ if and only if $\beta_u + \beta_v \geq s$.

c) For every $u, v \in V(G)$ either $N(u) \subseteq N(v) \cup \{v\}$ or $N(v) \subseteq N(u) \cup \{u\}$

d) $G$ is split with a bipartition $(A, B)$, and the sets $\{N_A(b) : b \in B\}$ and $\{N_B(a) : a \in A\}$ are ordered by inclusion.

e) $G$ is $(2K_2, C_4, P_4)$-free.

f) All factors in the canonical decomposition of $G$ are one-vertex.

g) $G$ is split with a bipartition $(A, B)$, and all factors in the canonical decomposition of the triad $(G, A, B)$ are one-vertex

So, according to f) and g) threshold graphs are the graphs with the simplest canonical decompositions. The question is: what are the simplest graphs defined by the $H$-decomposition? Following this idea, we define $H$-threshold graphs as graphs, which could be represented as the product $\circ_H$ of one-vertex
factors. We show, that every graph is $H$-threshold for some digraph $H$. So, it is natural to look for such representation with the digraph $H$, which is as small as possible. We define threshold-width of the graph $G$ as a minimum size of a digraph $H$ such, that $G$ is $H$-threshold.

The idea of threshold-width is naturally agreed with the theories of NLC-width and cliquewidth. In particular, the class $\text{lin} - NLC_k[14]$ is defined as the set of graphs which could be constructed by the sequence of the operations 2), 3), where at least one multiplier is one-vertex, and linear NLC-width of the graph $G$ is the minimal $k$ such that $G \in \text{lin} - NLC_k$. In the case of threshold-width, the operation is fixed.

Another important graph class of graphs related to the threshold graphs, is the class of difference graphs [12] (another name is bipartite chain graphs [28]).

**Theorem 4** [12] The following definitions of the difference graph $G$ are equivalent:

a) There exist real weights $(\beta_v : v \in V(G))$ and a threshold $s$ such, that $|\beta_v| \leq t, v \in V(G)$ and $uv \in E(G)$ if and only if $|\beta_u - \beta_v| \geq s$.

b) $G$ is bipartite with a bipartition $(A,B)$, and the sets $\{N_A(b) : b \in B\}$ and $\{N_B(a) : a \in A\}$ are ordered by inclusion.

In [12] authors emphasize, that properties of difference graphs are very similar to properties of threshold graphs. We show, that it is not the coincidence, because difference graphs are $H$-threshold for the particular $H$. So, graphs with fixed threshold-width are direct generalizations of both threshold and difference graphs, and we show, that they extend another properties of those classes. In particular, we show, that graphs with fixed threshold-width are also characterized in terms of vertex partitions into cliques and independent sets and the orderings of vertex neighborhoods, though the characterization become much more complicated. More precisely, we prove, that a graph $G$ has threshold-width at most $k$ if and only if

a) $V(G)$ could be partitioned into $k$ cliques and independent sets $V_1,\ldots,V_k$;

b) for every $i, j = 1,\ldots,k$, $i \neq j$ the sets $\{N_{V_j}(b) : b \in V_i\}$ are ordered by inclusion;

c) those orderings for different $i$ and $j$ are coordinated in the following sense: we can associate with the orderings the graph $R$ and the digraph $F$ such, that $R$ is bipartite and $F$ is acyclic.

In the third part of the paper we consider the graphs with small threshold-width. By the definition the only graphs with threshold dimension 1 are com-
complete and empty graphs. Threshold graphs have threshold-width at most 2, but there are non-threshold graphs with this property. We give the structural characterization and the characterization by the finite list of forbidden induced subgraphs for the class of graphs with threshold-width at most 2.

In particular, we show, that graph $G$ has threshold-width at most 2 if and only if $G$ or $\overline{G}$ is either threshold or difference. It is interesting to compare this characterization with the characterization of the graphs with small linear $NLC$-width from [13]: a graph $G$ has linear $NLC$-width 1 if and only if $G$ is threshold.

2 $H$-product of graphs

Let $H$ be a digraph with the vertex set $V(H) = \{1, ..., n\}$ and the arc set $A(H)$. The $n$-partitioned graph is a $(n + 1)$-tuple $T = (G, A_1, ..., A_n)$, where $G$ is a graph and $(A_1, ..., A_n)$ is a partition of its vertex set into disjoint subsets: $V(G) = A_1 \cup ... \cup A_n$, $A_i \cap A_j = \emptyset$ for all $i \neq j$. Some of sets $A_i$ could be empty. $G$ is called the basic graph of $T$. Denote the set of vertices and the set of edges of $T$ by $V(T)$ and $E(T)$, respectively.

The isomorphism $f$ of $n$-partitioned graphs $T$ and $S = (F, B_1, ..., B_n)$ is an isomorphism of $G$ and $F$ such that $f(A_i) = B_i$ for every $i = 1, ..., n$. Let $\Sigma_n$ be the set of all $n$-partitioned graphs distinguished up to isomorphism.

On the set $\Sigma_n$ consider a binary algebraic operation $\circ_H : \Sigma_n \times \Sigma_n \to \Sigma_n$ ($H$-product of $n$-partitioned graphs) as follows:

\[(G, A_1, ..., A_n) \circ_H (F, B_1, ..., B_n) = (R, A_1 \cup B_1, ..., A_n \cup B_n),\]  

where $V(R) = V(G) \cup V(F)$ (we assume without lost of generality that $V(G) \cap V(F) = \emptyset$), $E(R) = E(G) \cup E(H) \cup \{xy : x \in A_i, y \in B_j, (i, j) \in A(H)\}$.

For the convenience we will further sometimes denote the operation $\circ_H$ simply by $\circ$, if it is clear, what digraph $H$ we mean. The operation, which was introduced and studied in [20], is the particular case of $\circ_H$ for a digraph $H = H_0$ shown in the figure 1.

It is easy to check, that for every digraph $H$ the operation $\circ_H$ is associative. So, the set $\Sigma_n$ with the operation $\circ_H$ is a semigroup.

The digraph $H$ is symmetric, if $(i, j) \in A(H)$ whenever $(j, i) \in A(H)$. It is clear that the operation $\circ_H$ is commutative if and only if $H$ is symmetric.
The $n$-partitioned graph $T \in \Sigma_n$ is called $H$-decomposable, if $T = T_1 \circ_H T_2$, $T_1, T_2 \in \Sigma_n$, and $H$-prime, otherwise. It is clear, that every $n$-partitioned graph $T \in \Sigma_n$ could be represented as a product $T = T_1 \circ_H ... \circ_H T_k$, $k \geq 1$, of $H$-prime factors. Such a representation is called an $H$-decomposition of $T$.

**Theorem 5 (unique factorization theorem for the operation $\circ_H$)** For every $n$-vertex digraph $H$ every $n$-partitioned graph $T \in \Sigma_n$ has the unique $H$-decomposition up to the permutation of staying together commutative factors.

**PROOF.** It is evident, that if two $n$-partitioned graphs have the $H$-decompositions, which differ only by some permutations of staying together commutative multipliers, then they are isomorphic. So let us prove the inverse proposition. It is evident for prime $n$-partitioned graphs. Further apply the induction by the number of vertices.

Let

$$U = T_1 \circ ... \circ T_k, \ W = R_1 \circ ... \circ R_l,$$

(9)

$U \cong W$; $k, l \geq 2$. Let

$$U = (G, X_1, ..., X_n), \ W = (F, Y_1, ..., Y_n).$$

We may assume that $X_i \cup Y_i \neq \emptyset$ for all $i = 1, ..., n$.

Let $f : V(U) \to V(W)$ is the isomorphism of $U$ and $W$. We will use the following notation. For the set $X \subseteq V(U)$ let $f(X) = \{f(x) : x \in X\}$, for the subgraph $G'$ of $G$ let $f(G') = W[f(V(G'))]$ and for the $n$-partitioned graph $T = (G', A_1, ..., A_n)$, where $G'$ is a subgraph of $G$, let $f(T) = (f(G'), f(A_1), ..., f(A_n))$.

Setting $S = T_2 \circ ... \circ T_k = (G'', S_1, ..., S_n), \ Q = R_2 \circ ... \circ R_l = (F'', Q_1, ..., Q_n)$, we have

$$U = T_1 \circ S, \ W = R_1 \circ Q.$$  

(10)

Let $T_1 = (G', A_1, ..., A_n)$, $R_1 = (F', B_1, ..., B_n)$. By the definition of the isomorphism $f(A_i \cup S_i) = B_i \cup Q_i$.

Suppose that there exists $i \in \{1, ..., n\}$ such that $f(A_i) \cap B_i \neq \emptyset, f(A_i) \cap Q_i \neq \emptyset$. Then

$$f(T_1) = T' \circ T'',$$

where

$$T' = (F[f(V(T_1)) \cap V(R_1)], f(A_1) \cap B_1, ..., f(A_n) \cap B_n),$$

$$T'' = (F[f(V(T_1)) \cap V(Q)], f(A_1) \cap Q_1, ..., f(A_n) \cap Q_n).$$
Here \( V(T'), V(T'') \neq \emptyset \) by the assumption. It contradicts the fact that \( T_1 \) is prime.

Analogously, the existence of \( i \in \{1, ..., n\} \) such that \( f^{-1}(B_i) \cap A_i \neq \emptyset \), \( f^{-1}(B_i) \cap S_i \neq \emptyset \) contradicts the fact, that \( R_1 \) is prime.

So, further we can assume that for every \( i = 1, ..., n \) \( f(A_i) \subseteq B_i \) or \( f(A_i) \subseteq Q_i \).

Suppose that there exist \( i, j \in \{1, ..., n\} \), \( i \neq j \) such that \( f(A_i) \subseteq B_i \) and \( f(A_j) \subseteq Q_j \). Then \( f(T_1) = T'' \circ T''' \), where \( T'', T''' \) are defined as above. Again the contradiction with the indecomposibility of \( T_1 \) is obtained.

So, there are two possibilities:

1) For every \( i = 1, ..., n \) \( f(A_i) \subseteq B_i \). Then the facts proved above imply, that \( f(A_i) = B_i \), \( f(S_i) = Q_i \) for every \( i = 1, ..., n \). Thus \( T_1 \cong R_1, S \cong Q \). After applying induction assumption to the \( S = T_2 \circ ... \circ T_k \) and \( Q = R_2 \circ ... \circ R_l \), we get, that \( k = l \) and under the respective ordering \( R_2 \cong T_2, ..., T_k \cong R_k \).

2) For every \( i = 1, ..., n \) \( f(A_i) \subseteq Q_i \). Then \( B_i \subseteq f(S_i) \).

Let \( f(S_i) \cap Q_i = \emptyset \) for all \( i = 1, ..., n \). Then \( f(S_i) = B_i \), \( f(A_i) = Q_i \) for every \( i = 1, ..., n \). It means, that \( S \cong R_1, T_1 \cong Q \), and thus

\[
U \cong T_1 \circ R_1 \cong W \cong R_1 \circ T_1.
\]

So, the statement of the theorem is true.

Consider the case, when there exist \( i \in \{1, ..., n\} \) such that \( f(S_i) \cap Q_i \neq \emptyset \). Let

\[
Z = (F[f(V(S))] \cap V(Q)), f(S_1) \cap Q_1, ..., f(S_n) \cap Q_n).
\]

By the assumption \( V(Z) \neq \emptyset \). Then \( f(S) = R_1 \circ Z, Q = f(T_1) \circ Z \) and thus

\[
S \cong R_1 \circ Z, Q \cong T_1 \circ Z.
\]

So, \( T_1 \) is the first factor in some \( H \)-decomposition of \( Q \). Applying the induction assumption to \( Q \), we may assume without lost of generality, that \( T_1 = R_2 \) and \( Z = R_3 \circ ... \circ R_l \). So,

\[
T_2 \circ ... \circ T_k \cong S \cong R_1 \circ R_3 \circ ... \circ R_l.
\]

By the induction assumption applied to \( S \), we have \( k = l \) and under the respective ordering \( R_1 \cong T_2, T_3 \cong R_3, ..., T_k \cong R_k \).

To complete the proof, it remains to show, that \( T_1 \) and \( R_1 \) commutate. To do it, it is sufficient to prove, that for every pair \( i, j \in \{1, ..., n\}, i \neq j \), such
that \((i, j) \in A(H)\) and \((j, i) \not\in A(H)\) one of the following four conditions hold: either \(A_i \cup A_j = \emptyset\), or \(A_j \cup B_j = \emptyset\), or \(A_i \cup B_i = \emptyset\), or \(B_i \cup B_j = \emptyset\).

We have \(f(A_i) \sim f(S_j), f(A_j) \not\sim f(S_i)\) (because \(A_i \sim S_j, A_j \not\sim S_i\) and \(f\) is an isomorphism).

But then, since \(f(A_i) \subseteq Q_i, f(A_j) \subseteq Q_j, B_i \subseteq f(S_i), B_j \subseteq f(S_j)\), we have \(f(A_i) \not\sim f(S_j), f(A_j) \sim f(S_i)\).

This two facts imply, that one of the following is true:

1) \(A_i \cup A_j = \emptyset\);
2) \(A_i \cup S_i = \emptyset\), which implies, that \(B_i = \emptyset\);
3) \(A_j \cup S_j = \emptyset\), which implies, that \(B_j = \emptyset\);
4) \(S_i \cup S_j = \emptyset\), which implies, that \(B_i = \emptyset, B_j = \emptyset\).

The theorem is proved.

3 \(H\)-threshold graphs and the threshold-width of graphs

Denote by \(K^k_i\) the \(k\)-partitioned graph \((K_1, \emptyset, \ldots, \emptyset, \{v\}, \emptyset, \ldots, \emptyset)\) (the only nonempty set of the partition is the \(i\)th set).

Let \(H\) be a digraph on \(k\) vertices. Let us call a graph \(G\) \(H\)-threshold graph, if it is basic for the \(n\)-partitioned graph of the form

\[
K^{k}_{i_1} \circ_H \ldots \circ_H K^{k}_{i_n}.
\] (11)

In this case for the simplicity of the notation we will write \(G = K^{k}_{i_1} \circ_H \ldots \circ_H K^{k}_{i_n}\) (though strictly speaking the left part of this equality is the graph and the right part is \(k\)-partitioned graph). The representation of the graph \(G\) in the form (11) is called a threshold representation of \(G\).

To illustrate the notion of \(H\)-threshold graph, we show the threshold representations of graphs \(P_4\) and \(C_4\) for different 2-vertex digraphs \(H\) on the figure 2 (the 2-partitioned factors \(K^2_i\) are represented by ovals).

By Theorem 3 threshold graphs are exactly \(H_0\)-threshold graphs for the digraph \(H_0\) shown in the figure 1.

**Proposition 6** Every graph \(G\) is \(H\)-threshold for some digraph \(H\).
PROOF. Let $V(G) = \{1, ..., n\}$. Define $H$ as follows: $V(H) = \{1, ..., n\}$, $(i,j) \in A(H)$ if and only if $ij \in E(G)$, $i < j$ (i.e. $H$ is obtained from $G$ by assigning the orientation on every edge of $G$). It is easy to see, that $G = K_1^n \circ_H K_2^n \circ_H ... \circ_H K_n^n$.

The digraph $H$ constructed in the proof of Proposition 6 has $|V(G)|$ vertices. But, for example, threshold graphs are $H$-threshold for the digraph $H$ with only 2 vertices. So, it is natural to consider the minimum order of a digraph, for which a graph $G$ is $H$-threshold. Here we introduce the corresponding graph parameter.

The threshold-width of a graph $G$ is the parameter $ThrWidth(G) = \min\{|H| : G \text{ is } H-\text{threshold}\}$. By the Proposition 6 every graph has the threshold-width. It is clear, that for every graph $G$ on $n$ vertices $ThrWidth(G) \leq n$.

**Proposition 7** For every graph $G$ $ThrWidth(G) = ThrWidth(\overline{G})$.

**PROOF.** Suppose, that $G$ is $H$-threshold for a digraph $H$ with the vertex set $V(H) = \{1, ..., k\}$, i.e. $G = K_{i_1}^k \circ_H ... \circ_H K_{i_n}^k$. Let $\{v_j\} = V(K_{i_j}^k)$, $j = 1, ..., n$. Consider the vertices $v_p$ and $v_q$. Suppose, that $p < q$. Then $v_p \sim v_q$ if and only if one of the following conditions hold:

1) $i_p = i_q$ and $(i_p, i_p) \in A(H)$;
2) $i_p \neq i_q$ and $(i_p, i_q) \in A(H)$.

Define $\overline{H}$ be the complement of $H$, i.e. the digraph with the same vertex set and
with the arc set $A(H) = \{(i, j) : (i, j) \not\in A(H)\}$. Then $G = K^{k}_{i_1} \circ \cdots \circ K^{k}_{i_n}$, where $\{v_j\} = V(K^{k}_{i_j})$, $j = 1, \ldots, n$.

Now we are going to give the characterization of graphs with $ThrWidth(G) \leq k$. But firstly we need some auxiliary definitions and lemmas.

For a digraph $H$ and $v \in V(H)$ let $N_{\text{in}}(v) = \{u \in V(H) \setminus \{v\} : (u, v) \in A(H)\}$ and $N_{\text{out}}(v) = \{w \in V(H) \setminus \{v\} : (v, w) \in A(H)\}$ be the in-neighborhood and the out-neighborhood of $v$, respectively.

Let $H$ be a digraph and let $(v_1, \ldots, v_n)$ be the ordering of its vertices. This ordering is called acyclic ordering or topological sort, if all arcs of $H$ have the form $(v_i, v_j)$, where $i < j$. A digraph is acyclic, if it does not contain directed cycles. The following property of acyclic graphs is well-known.

**Proposition 8** [1]

A digraph is acyclic if and only if there exists an acyclic ordering of its vertices.

Let $S$ be the family of sets $S = \{\{X^{i}_{1}, X^{i}_{2}\}, \ldots, \{X^{p}_{1}, X^{p}_{2}\}\}$, where $X^{i}_{j} \subseteq \{1, \ldots, k\} \setminus \{i\}$, $i = 1, \ldots, k$, $j = 1, 2$ (some of sets $X^{i}_{j}$ could be empty). Let us call $S$ a digraphical family, if there exists a digraph $D$ on the vertex set $V(D) = \{1, \ldots, k\}$ such, that $S = (\{N_{\text{in}}(1), N_{\text{out}}(1)\}, \ldots, \{N_{\text{in}}(k), N_{\text{out}}(k)\})$. $D$ is called a realization of $S$.

The evident necessary condition for the digraphicity of $S$ is $i \in X^{j}_{1} \cup X^{j}_{2}$ whenever $j \in X^{i}_{1} \cup X^{i}_{2}$. Let us call the family $S$ with this property proper.

Suppose that $S$ is the proper family. Define the graph $R(S)$ as follows: $V(R(S)) = S$, $X^{i}_{q} \sim X^{j}_{p}$ if and only if either $i = j$, $q \neq p$ or $i \in X^{j}_{q}$, $j \in X^{i}_{p}$, $i, j = 1, \ldots, k$, $q, p = 1, 2$.

**Lemma 9** The proper family $S$ is digraphical if and only if the graph $R(S)$ is bipartite.

**PROOF.** Suppose that $D$ is a realization of $S$. Let

$$l(X^{i}_{q}) = \begin{cases} 1, & \text{if } X^{i}_{q} = N_{\text{out}}(i) \\ 2, & \text{if } X^{i}_{q} = N_{\text{in}}(i). \end{cases}$$

By the definition $l(X^{i}_{q}) \neq l(X^{j}_{p})$, $i = 1, \ldots, k$. If $j \in X^{i}_{q} = N_{\text{in}}(i)$, then $i \in X^{j}_{p} = N_{\text{out}}(j)$, and so $l(X^{i}_{q}) \neq l(X^{j}_{p})$. This $l$ is a proper 2-coloring of $R(S)$.
Inversely, let \( l \) be a proper 2-coloring of \( R(S) \). Define the digraph \( D \) on the vertex set \( \{1, ..., k\} \) as follows: \((i, j) \in A(G)\) if and only if \( i \in X^r_p, j \in X^q_i, l(X^i_q) = 1, l(X^j_p) = 2.\)

Since \( l \) is a proper 2-coloring, this definition correctly defines a digraph, and for every \( i = 1, ..., k \) if, for example, \( l(X^i_1) = 1, l(X^i_2) = 2 \), then \( X^i_1 = N_{out}(i), X^i_2 = N_{in}(i). \)

**Corollary 10** If \( D_1 \) and \( D_2 \) are two different realizations of \( S \), then \( D_1 \) could be obtained from \( D_2 \) by the reversal of all arcs of some of its connected components.

For a sequence \( \pi = (a_1, ..., a_n) \) denote by \( inv(\pi) \) the sequence \( (a_n, ..., a_1) \).

Let

\[
V(G) = V_1 \cup ... \cup V_k
\]

is a partition of the vertex set of the graph \( G \), where each \( V_i \) is either a clique or an independent set.

We will say, that the partition (12) satisfies the *neighborhoods ordering property*, if for every \( i = 1, ..., k \) there exists a permutation \( \psi(i) = (u^i_1, ..., u^i_r_i) \) of the set \( V_i \) such, that for every \( j \in [k] \setminus \{i\} \) the set \( \{N_{V_j}(u) : u \in V_i\} \) is ordered by inclusion and this ordering either coincides with \( \psi(i) \) or with \( inv(\psi(i)) \). In other words, for every \( j \in [k] \setminus \{i\} \) either

\[
N_{V_j}(u^i_1) \supseteq N_{V_j}(u^i_2) \supseteq ... \supseteq N_{V_j}(u^i_r_i)
\]

or

\[
N_{V_j}(u^i_1) \subseteq N_{V_j}(u^i_2) \subseteq ... \subseteq N_{V_j}(u^i_r_i)
\]

Assume, that the permutations \( \psi(i) \) are fixed. For every \( i \in [k] \) the set \([k] \setminus \{i\} \) is partitioned into two classes. Let us for convenience denote those classes \( Y^i_1 \) (contains \( j \) satisfying (13)) and \( Y^i_2 = ([k] \setminus \{i\}) \setminus Y^i_1 \) (contains \( j \) satisfying (14))

Let

\[
X^i_r = Y^i_r \setminus \{j : V_i \sim V_j \text{ or } V_i \not\sim V_j\}, \ r = 1, 2
\]

and

\[
S = S(V_1, ..., V_k) = (\{X^1_1, X^1_2\}, ..., \{X^k_1, X^k_2\}).
\]

Suppose that \( S \) is a digraphical family (i.e. by the Lemma 9 \( R(S) = R(V_1, ..., V_k) \) is a bipartite graph) and \( D \) is its realization, \( V(D) = [k] \). Assume without lost
of generality, that \( N_{out}(i) = X_i \) (if it is not the case, replace \( \psi(i) \) by \( inv(\psi(i)) \)). Note also, that by the definition for every \( i, j \in [k], i \neq j \) \( D \) contains at most one arc from the set \( \{(i, j), (j, i)\} \).

Using the digraph \( D \), define the digraph \( F = F(V_1, ..., V_k) = F_D(V_1, ..., V_k) \) as follows: \( V(F) = V(G), A(F) = A_1 \cup A_2 \cup A_3^1 \cup ... \cup A_3^k \), where

\[
A_1 = \{(u, v) : u \in V_i, v \in V_j, uv \in E(G), (i, j) \in A(D); i, j \in [k]\};
\]

\[
A_2 = \{(v, u) : u \in V_i, v \in V_j, uv \notin E(G), (i, j) \in A(D); i, j \in [k]\};
\]

\[
A_3^i = \begin{cases} 
\{(u_1^i, u_2^i), ..., (u_{r_i}^i, u_1^i)\}, & \text{if } X_i = N_{out}(i) \text{ in } D, \quad i = 1, ..., k, \\
\{(u_r^i, u_1^i), ..., (u_2^i, u_1^i)\}, & \text{if } X_i = N_{out}(i) \text{ in } D.
\end{cases}
\]

In other words, the digraph \( F \) is constructed in the following way. Firstly consider every pair \( V_i, V_j \) such, that neither \( V_i \sim V_j \) nor \( V_i \not\sim V_j \). Without lost of generality suppose, that \( (i, j) \in A(D). \) Consider the set \( E_{i,j} \) of edges of the complete bipartite graph with the parts \( V_i \) and \( V_j \). If the edge \( uv \in E_{i,j} \) belongs to \( E(G) \), then orientate it in the direction from \( V_i \) to \( V_j \); otherwise orientate it in the direction from \( V_j \) to \( V_i \). Next turn every set \( V_i \) into the oriented path, the order of vertices of this path is defined either by \( \psi(i) \) or by \( inv(\psi(i)) \) (depending on what of the sets \( X_1^i \) or \( X_2^i \) is the out-neighborhood of \( i \) in \( D \)).

Now we are ready to formulate the characterization of graphs with the threshold-width \( ThrWidth(G) \leq k \).

**Theorem 11** Let \( G \) be a graph. \( ThrWidth(G) \leq k \) if and only if there exists a partition \((12)\) such that

1) it satisfies the neighbourhoods ordering property;
2) the family \( S = S(V_1, ..., V_k) \) is digraphical (i.e. the graph \( R(S) = R(V_1, ..., V_k) \) is bipartite);
3) the digraph \( F = F(V_1, ..., V_k) \) is acyclic.

**Proof.** Let us prove sufficiency first. Suppose, that \( D \) is a realization of \( S \), which defines \( F \). Let us expand \( D \) by adding the set of arcs \( \{(i, i) : V_i \text{ is a clique}\} \cup \{(i, j), (j, i) : V_i \sim V_j\} \). Denote the obtained graph by \( H \).

Let \( (v_1, ..., v_n) \) be an acyclic ordering of the digraph \( F \). We will show, that \( G = K_{v_1}^k \circ_H ... \circ_H K_{v_n}^k \), where \( V(K_{v_j}^k) = \{v_j\}, v_j \in V_i \).
Let \( K_{\ell_i}^k \circ_H ... \circ_H K_{\ell_n}^k = Z \). Consider the edge \( ab \in E(G) \). Let us show, that \( ab \in E(Z) \). If \( V_i \) is a clique in \( G \), then \((i, i) \in A(H)\), which implies, that \( V_i \) is a clique in \( Z \). Analogously, if \( V_i \sim_G V_j \), then \((i, j), (j, i) \in A(H)\), and so by the definition of the operation \( \circ_H V_i \sim_Z V_j \).

So, it remains to consider the case, when \( a \in V_i \), \( b \in V_j \), \( i \neq j \) and neither \( V_i \sim V_j \) nor \( V_i \not\sim V_j \). In this case \( i \) and \( j \) are connected by an arc in \( D \). Let without loss of generality \((i, j) \in A(D)\). So \((a, b) \in A(F)\) by the definition of \( F \). Then in the acyclic ordering \( a \) goes before \( b \), i.e. \( a = V(K_{\ell_i}^k) \), \( b = V(K_{\ell_j}^k) \), \( r < s \). It together with the fact, that \((i, j) \in A(H)\), implies that \( ab \in E(Z) \).

Conversely, let \( ab \in E(Z) \). Let \( a = v_r \), \( b = v_s \), \( r < s \) (i.e. \( a \) precedes \( b \) in the acyclic ordering), \( a \in V_i \), \( b \in V_j \). So we know, that \( V_i \not\sim V_j \) does not hold. By the definition of the operation \( \circ_H (i, j) \in A(H) \). If \( i = j \), then \( V_i \) is a clique, and so \( ab \in V(G) \). So let further \( i \neq j \) and it is not true, that \( V_i \sim V_j \). Then \((i, j) \in A(D)\) and so \( a \) and \( b \) are adjacent in \( F \). Since \( a \) precedes \( b \) in the acyclic ordering, \((a, b) \in A(F)\). So the arc \((a, b)\) is directed from \( V_i \) to \( V_j \), which implies, that \( ab \in E(G) \).

Now we will prove necessity. Assume, that \( G = K_{\ell_i}^k \circ_H ... \circ_H K_{\ell_n}^k \), where \( \{v_j\} = V(K_{\ell_j}^k) \). Then

\[
V(G) = V_1 \cup ... \cup V_k.
\]  

(15)

where \( V_i = \{v : \{v\} = V(K_{\ell_i}^k)\} \), \( i = 1, ..., k \). If \((i, i) \in A(H)\), then \( V_i \) is a clique, otherwise it is an independent set.

Suppose, that \( V_i = \{v_{l_1}, ..., v_{l_i}\} \), \( l_1 < l_2 < ... < l_i \). If \((i, j) \in A(H)\), then \( N_{V_j}(v_{l_1}) \supseteq N_{V_j}(v_{l_2}) \supseteq ... \supseteq N_{V_j}(v_{l_i}) \), otherwise \( N_{V_j}(v_{l_1}) \supseteq ... \supseteq N_{V_j}(v_{l_i}) \). So, the partition (15) satisfies the neighborhoods ordering property.

Let \( D \) be a digraph obtained from \( H \) by deleting loops and arcs of the set \( \{(i, j) : V_i \sim V_j\} \). Then in the digraph \( D \)

\[
N_{\text{out}}(i) = \{j : N_{V_j}(v_{l_1}) \supseteq N_{V_j}(v_{l_2}) \supseteq ... \supseteq N_{V_j}(v_{l_i}) \text{ and neither } V_i \sim V_j \text{ nor } V_i \not\sim V_j\};
\]

\[
N_{\text{in}}(i) = \{j : N_{V_j}(v_{l_1}) \supseteq ... \supseteq N_{V_j}(v_{l_i}) \text{ and neither } V_i \sim V_j \text{ nor } V_i \not\sim V_j\}.
\]

So, \( D \) is a realization of \( S(V_1, ..., V_k) \).
It remains to show, that \((v_1, \ldots, v_n)\) is the acyclic ordering of \(F = F(V_1, \ldots, V_k)\). All arcs with both ends in \(V_l, l = 1, \ldots, k\), have the form \((v_i, v_{i+1})\). So, let us consider \(v_i \in V_l, v_j \in V_s, l \neq s\) such, that \(v_i\) and \(v_j\) are adjacent in \(F\). By the definition of \(F\) neither \(V_l \sim V_s\) nor \(V_i \not\sim V_s\). Then \(l\) and \(s\) are adjacent in \(H\). Let \((l, s) \in A(H)\). If \((v_i, v_j) \in A(F)\), then \(v_i v_j \in E(G)\), which could be only if \(i < j\). If \((v_j, v_i) \in A(F)\), then \(v_i v_j \not\in E(G)\), which could be only if \(j < i\). The theorem is proved.

Remark 12 If the partition \((12)\) is given, it could be tested in a polynomial time, if it satisfies the conditions of the Theorem 11. In case of the positive answer, the proofs of the Lemma 9 and Theorem 11 contain the algorithm for reconstruction of the graph \(H\) such that \(G\) is \(H\)-threshold graph.

The definition of the digraph \(F(V_1, \ldots, V_k)\) depends on the realization \(D\) of the family \(S(V_1, \ldots, V_k)\). But the family \(S(V_1, \ldots, V_k)\) can have different realizations. The next proposition shows, that from the point of view of the Theorem 11 it does not matter, which realization to choose.

Proposition 13 Let \(D_1, D_2\) be two realizations of \(S(V_1, \ldots, V_k)\) for a partition \((12)\). If \(F_{D_1}(V_1, \ldots, V_k)\) is acyclic, then \(F_{D_2}(V_1, \ldots, V_k)\) is also acyclic.

PROOF. Suppose, that \(F_{D_1}(V_1, \ldots, V_k)\) is acyclic. By the corollary from the Lemma 9 \(D_1\) and \(D_2\) have the same sets of connected components. It follows from the definition, that \(\{i_1, \ldots, i_j\}\) is a connected component of \(D_l\) if and only if \(V_{i_1} \cup \ldots \cup V_{i_j}\) is a connected component of \(F_{D_l}(V_1, \ldots, V_k)\), \(l = 1, 2\). So, the definition of \(F\) and the Corollary 10 imply, that \(F_{D_2}(V_1, \ldots, V_k)\) could be obtained from \(F_{D_1}(V_1, \ldots, V_k)\) by the reversal of all arcs of some of its connected components. So, \(F_{D_2}(V_1, \ldots, V_k)\) is acyclic.

4 Graphs with \(ThrWidth(G) \leq 2\)

It is clear, that graphs with \(ThrWidth(G) = 1\) are exactly complete and empty graphs. For every threshold graph \(G\) \(ThrWidth(G) \leq 2\). But the set of graphs with \(ThrWidth(G) \leq 2\) is not reduced to the threshold graphs. For example, on the figure 2 we can see, that \(C_4\) and \(P_4\) have the threshold-width 2.

Proposition 14 \(ThrWidth(G) \leq 2\) if and only if \(G\) or \(\overline{G}\) is either threshold, or difference.

PROOF. By the Theorem 11 the necessity is straightforward, so let us prove the sufficiency. Let us use the Theorem 11. By the definition there exists the
This partition satisfies the neighbourhoods ordering property. It is clear, that the realization of the family $S(V_1, V_2)$ is either empty digraph (if $V_1 \sim V_2$ or $V_1 \not\sim V_2$) or the digraph $D$ with $A(D) = \{(1,2)\}$.

Let us prove that $F = F(V_1, V_2)$ is acyclic. If $V_1 \sim V_2$ or $V_1 \not\sim V_2$, then $F$ is empty. Otherwise let $A(D) = \{(1,2)\}$.

Let $V_1 = \{u_1,\ldots, u_r\}$, $V_2 = \{v_1,\ldots, v_s\}$, where $N_{V_2}(u_1) \supseteq N_{V_2}(u_2) \supseteq \ldots \supseteq N_{V_2}(u_r)$, $N_{V_1}(v_s) \supseteq N_{V_2}(v_{s-1}) \supseteq \ldots \supseteq N_{V_2}(v_1)$. Then all arcs of $F$ with both ends in $V_1$ have the form $(u_i, u_{i+1})$, $i = 1,\ldots, r-1 ((u_i, v_{i+1})$, $i = 1,\ldots, s-1)$. Therefore if there exists a directed cycle in $F$, it should contain arcs $(u_j, v_l)$, $(v_p, u_i)$, $i \leq j$, $l \leq p$ (since $F$ contains no loops we may assume without loss of generality, that $i \neq j$). By the definition of $F$, it means that $u_i v_l \in E(G)$, $u_i v_p \not\in E(G)$. Since $N_{V_2}(u_i) \supseteq N_{V_2}(u_j)$ we have $u_i v_l \in E(G)$. If $l = p$, then we have the contradiction. If $l \neq p$ then, as $N_{V_2}(v_p) \supseteq N_{V_2}(v_l)$, we again have $u_i v_p \in E(G)$. This contradiction finishes the proof.

**Corollary 15** The class of difference graphs coincides with the class of $H'$-threshold graphs, where $V(H') = \{1,2\}$, $A(H') = \{(1,2)\}$.

**PROOF.** All $H'$-threshold graphs are difference graphs by the definition of $\circ_{H'}$ and by Theorem 4. Let us show, that all difference graphs are $H'$-threshold. It is sufficient to consider connected difference graph $G$ with the bipartition $(A, B)$ (if $G$ is disconnected, then it is a disjoint union of a connected difference graph $F$ and $r$ isolated vertices. If $T$ is the threshold representation with respect to $\circ_{H'}$ of $F$, then $G = (K^2_2 \circ_{H'} ... \circ_{H'} K^2_2) \circ_{H'} T$ (r multipliers in parentheses)). If $G$ is complete bipartite, then $G = K_{m,n} = (K^2_1 \circ_{H'} ... \circ_{H'} K^2_1) \circ_{H'} (K^2_2 \circ_{H'} ... \circ_{H'} K^2_2)$ ($m$ and $n$ multipliers in each parentheses). So let further $G$ is not complete bipartite, which implies, that $|A|, |B| \geq 2$. Then if $G$ is $H$-threshold, $|H| \leq 2$, then $H$ has no loops. If $A(H) = \emptyset$, then $G = O_n$, and if $A(H) = \{(1,2), (2,1)\}$, then $G = K_{m,n}$. So, $H = H'$.

**Theorem 16** Let $G$ be a graph. $\ThrWidth(G) \leq 2$ if and only if neither $G$ nor $\overline{G}$ contains one of the graphs from the set $L = \{C_5, P_5, House, P_3 \cup P_2, W_4, Bull, X, Y, Z\}$ as an induced subgraph.

**PROOF.** It is straightforward to check, that every graph from the set $L$ do not satisfy the Proposition 14. So we will prove the sufficiency.

Let us prove firstly, that $G$ is either split, or bipartite, or a complement of bipartite. After that we will prove, that for each its part the neighborhoods
Fig. 3. The set $L$

of its vertices in the another part are ordered by inclusion.

Suppose, that neither $G$ nor $\overline{G}$ is bipartite. We will show, that $G$ is split.

Let $A$ be a maximum clique of $G$ and such, that a subgraph induced by the set $B = V(G) \setminus A$ have the smallest possible number of edges. We will prove, that $B$ is an independent set.

Suppose the contrary, i.e. there exist $x, y \in B$ such, that $x \sim y$. Since $A$ is maximum, there exist vertices of $A$, which are not adjacent to $x$ ($y$). If all vertices of $A$, except, possibly, one vertex $u$, adjacent to both $x$ and $y$, then $A \setminus \{u\} \cup \{x, y\}$ is a clique, which contradicts the maximality of $A$. So, there exist $u, v \in A$ such, that $u \not\sim x$, $v \not\sim y$.

It is easy to see, that $|A| \geq 3$. Indeed, if $|A| = 2$, then $G$ is triangle-free. It, together with the fact, that $G$ is $\{C_5, P_3\}$-free, imply that $G$ doesn’t contain odd cycles.

Let $w \in A \setminus \{u, v\}$. Because $\overline{G}$ is not bipartite, there exists $z \in B \setminus \{x, y\}$ such, that $z \not\sim y$ or $z \not\sim x$. We may assume, that $w \not\sim z$, since $A$ is a maximum clique.

Let us call the induced cycle $C = C_4$ bad, if there exists a vertex $a \in V(G) \setminus C$ such, that $|N(a) \cap C| \geq 2$. By the assumption of the theorem $G$ does not contain bad $C_4$’s.

If $u \sim y$ and $v \sim x$, then $G$ contains bad $C_4$. Therefore the following cases are possible: 1) $u \not\sim y$, $v \not\sim x$ and 2) $u \sim y$, $v \not\sim x$. Consider those cases.

1) $u \not\sim y$, $v \not\sim x$.

Let without lost of generality $z \not\sim y$. If $z \sim x$, then without lost of generality $z \sim v$ (since $G[u, v, y, x, z] \neq P_3 \cup P_2$). As $G[y, x, z, v, w] \neq P_3, C_5$, $w \sim x$. But
then \( \{w, v, z, x\} \) form bad \( C_4 \).

So it is proved, that \( z \not\sim x \). Moreover, it is shown, that for every \( t \in B \setminus \{x, y\} \) \( t \sim \{x, y\} \) or \( t \not\sim \{x, y\} \).

Let \( T_1 = \{t \in B \setminus \{x, y\} : t \sim \{x, y\}\} \), \( T_2 = \{t \in B \setminus \{x, y\} : t \not\sim \{x, y\}\} \). We know from the considerations above, that \( T_2 \neq \emptyset \).

Let \( t \in T_2 \). As \( G[u, v, y, x, t] \neq P_3 \cup P_2 \), without lost of generality \( t \sim v \). Then, since \( G[t, v, u, y, x] \neq P_3 \cup P_2 \), \( t \sim u \). So, we have \( T_2 \sim \{u, v\} \).

**Lemma 17** For every \( q \in A \setminus \{u, v\} \) \( q \sim T_2 \) or \( q \sim \{x, y\} \). Moreover, \( T_2 \) is a clique.

**PROOF.** Suppose, that there exists \( t \in T_2 \) such, that \( q \not\sim t \). The statement, that \( q \sim \{x, y\} \) follows from the fact, that \( G[t, v, q, y, x] \neq P_3 \cup P_2, P_3 \). If there exist \( t_1, t_2 \in T_2 \) such, that \( t_1 \not\sim t_2 \), then \( G[t_1, v, t_2, y, x] = P_3 \cup P_2 \).

Let \( Q_1 = \{q \in A \setminus \{u, v\} : q \sim T_2\} \), \( Q_2 = (A \setminus \{u, v\}) \setminus Q_1 \). By the Lemma 17 \( Q_2 \sim \{x, y\} \). Moreover, as \( A \) is maximal clique, \( Q_2 \neq \emptyset \).

**Lemma 18** \( Q_2 \sim T_1 \). Moreover, \( T_1 \) is a clique.

**PROOF.** Suppose, that there exist \( t_1 t_2 \in T_1 \) such, that \( t_1 \not\sim t_2 \). Since \( G[u, v, y, t_1, t_2] \neq P_3 \cup P_2 \), without lost of generality \( t_2 \sim v \). Then either \( t_2 \sim u \) or \( t_1 \sim v \), because \( G[u, v, t_2, y, t_1] \neq P_3, C_5 \). But \( t_1 \not\sim v \), because otherwise \( v, t_2, x, t_1 \) form bad \( C_4 \). So \( t_1 \not\sim v, t_2 \sim u \). Analogously, it is easy to see, that \( t_1 \not\sim \{u\} \).

By the maximality of the clique \( A \), there exists \( q \in A \setminus \{u, v\} \) such, that \( q \not\sim t_2 \). As \( G[q, v, t_2, y, t_1] \neq P_3, C_5, q \sim x \). But then \( G[q, v, t_2, x] \) is a bad \( C_4 \). So it is proved, that \( T_1 \) is a clique.

Let us show now, that \( T_1 \sim Q_2 \). Suppose the contrary, i.e. let there exist \( t \in T_1, q \in Q_2 \) such, that \( t \not\sim q \). By the definition of \( Q_2 \) there exist \( z \in T_2 \) such, that \( q \not\sim z \). Since \( G[z, u, q, x, t] \neq P_3, C_5 \), \( t \sim u \). But then \( G[u, q, x, t] \) is a bad \( C_4 \).

By Lemma 17 and Lemma 18 \( V_1 = Q_2 \cup T_1 \cup \{x, y\} \) and \( V_2 = Q_1 \cup T_2 \cup \{u, v\} \) are cliques, \( V_1 \cup V_2 = V(G) \). The contradiction with the fact, that \( \overline{G} \) is not bipartite, is obtained. So, the case 1) is considered.

2) \( u \sim y, v \not\sim x \).
Lemma 19 For every \( z \in B \setminus \{x, y\} \) \( z \sim \{x, y\} \) or \( z \not\sim \{x, y\} \).

PROOF. Assume, in contrary, that there are exist \( z \in B \setminus \{x, y\} \) such, that the lemma is not satisfied for it.

Let \( z \sim x \), \( z \not\sim y \). Since \( G[v, u, y, x, z] \neq P_5, C_5 \), then \( z \sim u \). Consider \( w \in A \setminus \{u, v\} \). As \( G[y, x, z, v, w] \neq P_3 \cup P_2 \), there are edges between \( \{v, w\} \) and \( \{x, y, z\} \). But it means, that \( G[w, y, x, z] \) is a bad \( C_4 \).

So, \( z \sim y \), \( z \not\sim x \). Suppose, that \( z \sim u \) (because \( G[z, y, u, v] \) is not a bad \( C_4 \)). Therefore by the maximality of \( A \) there exists \( w \in A \) such that \( w \not\sim z \). For this vertex we have \( w \sim y \) (as \( G[x, y, z, v, w] \neq P_5, C_5 \)), and it implies, that \( G[w, v, z, y] \) is a bad \( C_4 \).

So, \( z \not\sim v \). But then \( z \not\sim u \) (otherwise \( G[z, y, u, v, x] = Bull \)). Since \( G[x, y, z, w, v] \neq P_3 \cup P_2 \), there exist some of the edges from the set \( \{wx, wy, wz\} \).

Suppose, that \( w \sim x \). Then \( w \sim y \) (because otherwise \( G[w, x, y, u] \) is a bad \( C_4 \)). It implies, that \( w \sim z \) (as \( G[w, y, x, z] \neq Bull \)). But then \( G[x, y, z, u, v, w] = Y \).

Thus \( w \not\sim x \). If \( w \sim z \), then \( w \sim y \) (since \( G[w, u, y, z] \) is not bad \( C_4 \)). It implies, that \( G[w, z, y, v, x] = Bull \).

So, \( w \not\sim z \). Then \( w \sim y \) and \( G[w, u, v, y, z, x] = X \).

Let \( B \setminus \{x, y\} = S_1 \cup S_2 \), \( S_1 = \{z \in B : z \sim \{x, y\}\} \), \( S_2 = \{z \in B : z \not\sim \{x, y\}\} \). Since \( \overline{G} \) is not bipartite, \( S_2 \neq \emptyset \).

Lemma 20 For every \( r \in A \setminus \{u, v\} \) \( r \sim \{x, y\} \) or \( r \not\sim S_2 \).

PROOF. Assume, that there exists \( z \in S_2 \) such, that \( r \not\sim z \).

Let \( z \sim v \). As \( G[z, v, r, y, x] \neq P_3 \cup P_2 \), \( r \sim y \) or \( r \sim x \). The situation, when \( r \sim x \) and \( r \not\sim y \), is impossible, because otherwise \( G[r, u, y, x] \) is a bad \( C_4 \). If \( r \sim y \), then \( r \sim x \) (because \( G[x, y, r, v, z] \neq P_5 \)).

It remains to consider the case, when \( z \not\sim v \). Then \( r \sim y \) or \( r \sim x \), since \( G[r, v, y, x, z] \neq \overline{W_4} \). As above, the case, when \( r \sim x \), \( r \not\sim y \), is impossible. So \( r \sim y \). As \( G[v, u, r, y, x, z] \neq Y \), \( z \sim u \) or \( r \sim x \). The situation, when \( z \sim u \), \( r \not\sim x \) contradicts the fact, that \( G[r, u, y, x, r] \neq Bull \). So \( r \sim x \).
Let $A \setminus \{u, v\} = R_1 \cup R_2$, $R_1 = \{r \in A \setminus \{u, v\} : r \sim S_2\}$, $R_2 = (A \setminus \{u, v\}) \setminus R_1$.

By the Lemma 20 $R_2 \sim \{x, y\}$.

**Lemma 21** $S_2 \sim \{u, v\}$. Moreover, $S_2$ is a clique.

**PROOF.** Let us first the first statement of the lemma. Let $z \in S_2$. Assume, that $z \not\sim v$. We will show, that it is impossible.

Suppose, that there exists $r \in A \setminus \{u, v\}$ such that $r \not\sim y$. By the Lemma 20 $r \sim z$. Then $r \sim x$, since $G[z, r, v, y, x] \neq P_3 \cup P_2$. But then $G[r, u, y, x]$ is a bad $C_4$.

So, it is proved that $y \sim A \setminus \{v\}$. Therefore there exists $s \in B \setminus \{x, y, z\}$ such that $s \sim v$ and $s \not\sim y$. Indeed, if, on the contrary, $N_B(v) \subseteq N_B(y)$, then $A' = (A \setminus \{v\}) \cup \{y\}$ is a maximum clique and for the subgraph, induced by the set $B' = V(G) \setminus A'$, we have $|E(G[B'])| < |E(G[B])|$. It contradicts the definition of the clique $A$.

As $G[x, y, u, v, s] \neq P_5, C_5$, $s \sim u$. Moreover, $s \not\sim x$, (because otherwise $G[x, y, u, s]$ is a bad $C_4$) and $s \sim z$ (because otherwise $G[v, s, y, x, z] = \overline{W_4}$). But then $G[z, s, v, y, x] = P_3 \cup P_2$.

So, $z \sim v$. Then $z \sim u$ (see the proof of the Lemma 20).

Now it is easy to see, that $S_2$ is a clique. Indeed, if there exist $s_1, s_2 \in S_2$ such that $s_1 \not\sim s_2$, then $G[s_1, v, s_2, y, x] = P_3 \cup P_2$.

In particular, Lemma 21 and the maximality of $A$ imply, that $R_2 \neq \emptyset$.

**Lemma 22** $R_2 \sim S_1$. Moreover, $S_1$ is a clique.

**PROOF.** Let there exist $r \in R_2$ and $s \in S_1$ such that $r \not\sim s$. By Lemma 20 $r \sim \{x, y\}$. By the definition there exists $z \in S_2$ such, that $z \not\sim r$. Lemma 21 implies, that $z \sim \{u, v\}$. Since $G[z, v, r, x, s] \neq P_5, C_5$, either $z \sim x$ or $s \sim v$. But in the first case $G[z, v, r, x]$ is a bad $C_4$, and in the second case $G[v, r, x, s]$ is a bad $C_4$. So, it is proved, that $R_2 \sim S_1$.

Let us show now, that $S_1$ is a clique. Suppose that there exist $z_1, z_2 \in S_1$ such, that $z_1 \not\sim z_2$. As $G[z_1, x, z_2, u, v] \neq P_3 \cup P_2$, there exists at least one edge between $\{z_1, z_2\}$ and $\{u, v\}$. At the same time, if $z_1 \sim v$ and $z_1 \not\sim u$, then $G[z_1, v, u, y]$ is a bad $C_4$.

So, without lost of generality $z_1 \sim u$. Then $z_2 \not\sim u$ (because otherwise $G[z_1, x, z_2, u]$ is a bad $C_4$). Since $G[v, u, z_1, x, z_2] \neq P_5, C_5, z_1 \sim v$. It implies,
that $z_2 \not\sim v$ (otherwise $G[v, z_1, x, z_2]$ is a bad $C_4$).

The maximality of $A$ implies the existence of $w \in A$ such, that $w \not\sim z_1$, $w \not\sim x$, as $G[w, v, z_1, x]$ is not a bad $C_4$. But then $G[w, v, z_1, x, z_2] = P_5$ or $C_5$.

By Lemma 21 and Lemma 22 $V_1 = R_2 \cup S_1 \cup \{x, y\}$ and $V_2 = R_1 \cup S_2 \cup \{u, v\}$ are cliques, $V_1 \cup V_2 = V(G)$. The contradiction with the fact, that $\overline{G}$ is not bipartite, is obtained. The case 2) is considered.

So, it is proved, that $G$ or $\overline{G}$ is either split or bipartite. Let $(A, B)$ be the bipartition of $G$. Let us show, that the neighborhoods of vertices from $A(B)$ are ordered by inclusion.

Let us suppose the contrary, i.e. there exist $u, v \in A$, $x, y \in B$ such, that $u \sim x$, $v \not\sim x$, $u \sim y$, $v \not\sim y$.

Suppose, that $G$ is bipartite. If $|V(G)| = 4$, then $\overline{G}$ the statement of the theorem obviously holds. Let there exists $z \in B \setminus \{x, y\}$. Since $G[u, v, x, y, z] \neq W_4, P_3 \cup P_2$, $z \sim u, v$. But then $G[u, v, x, y, z] = P_5$. This contradiction proves the theorem for bipartite graphs.

Taking into account Observation 7, it remains to consider the case, when $G$ is split and neither bipartite nor a complement of bipartite.

The following statements hold:

a) $N(x) \cup N(y) = A$ (since $G$ does not contain Bull);

b) for every $z \in B \setminus \{x, y\}$ $|N(z) \cap \{u, v\}| \leq 1$ (by the same reason as in a));

c) $|A| \geq 3$, $|B| \geq 3$ (otherwise either $G$ or $\overline{G}$ is bipartite).

Let $z \in B \setminus \{x, y\}$, $w \in A \setminus \{u, v\}$, $w \sim x$. As $G[u, v, x, y, w, z] \neq Y, \overline{Z}$, at least one of the edges $zu, zv, zw$ belongs to $E(G)$. If there exists exactly one of this edges, then $G[u, v, w, z, y] = Bull$, $G[u, v, w, x, y, z] = X$, $G[u, v, w, z, y] = Bull$, respectively. Therefore, taking into account b), either $zw, zv \in E(G)$, $zu \notin E(G)$ or $zw, zu \in E(G)$, $zv \notin E(G)$.

In the first case $w \sim y$ (since $G[w, v, z, y, x] \neq Bull$), which implies, that $F = G[u, v, x, y, w, z] = \overline{Y}$. In the second case $w \sim y$ (since $F \neq \overline{X}$), which implies, that $F = \overline{Y}$.

The theorem is proved.
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