Radon-Nikodým property and thick families of geodesics

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Abstract. Banach spaces without the Radon-Nikodým property are characterized as spaces containing bilipschitz images of thick families of geodesics defined as follows. A family $T$ of geodesics joining points $u$ and $v$ in a metric space is called thick if there is $\alpha > 0$ such that for every $g \in T$ and for any finite collection of points $r_1, \ldots, r_n$ in the image of $g$, there is another $uv$-geodesic $\tilde{g} \in T$ satisfying the conditions: $\tilde{g}$ also passes through $r_1, \ldots, r_n$, and, possibly, has some more common points with $g$. On the other hand, there is a finite collection of common points of $g$ and $\tilde{g}$ which contains $r_1, \ldots, r_n$ and is such that the sum of maximal deviations of the geodesics between these common points is at least $\alpha$.

1 Introduction

The Radon-Nikodým property (RNP) is one of the most important isomorphic invariants of Banach spaces. We refer to [BL00, Bou79, Bou83, DU77, Pis11] for systematic presentations of results on the RNP, and [DM13], [DM07] for recent work on the RNP.

In the recent work on metric embeddings a substantial role is played by existence and non-existence of bilipschitz embeddings of metric spaces into Banach spaces with the RNP, see [CK06, CK09, LN06]. At the seminar “Nonlinear geometry of Banach spaces” (Texas A & M University, August 2009) Bill Johnson suggested the problem of metric characterization of reflexivity and the Radon-Nikodým property [Tex09, Problem 1.1]. Some work on this problem was done in [Ost11] and [Ost13+]. The purpose of this paper is to continue this work. More precisely, we are going to characterize the RNP using thick families of geodesics defined in the following way.

Definition 1.1. Let $u$ and $v$ be two elements in a metric space $(M, d_M)$. A $uv$-geodesic is a distance-preserving map $g : [0, d_M(u, v)] \rightarrow M$ such that $g(0) = u$ and $g(d_M(u, v)) = v$ (where $[0, d_M(u, v)]$ is an interval of the real line with the distance inherited from $\mathbb{R}$). A family $T$ of $uv$-geodesics is called thick if there is $\alpha > 0$ such that for every $g \in T$ and for any finite collection of points $r_1, \ldots, r_n$ in the image of $g$, there is another $uv$-geodesic $\tilde{g} \in T$ satisfying the conditions:

- The image of $\tilde{g}$ also contains $r_1, \ldots, r_n$.

Therefore there are $t_1, \ldots, t_n \in [0, d_M(u, v)]$ such that $r_i = g(t_i) = \tilde{g}(t_i)$.
There are two sequences \( \{q_i\}_{i=1}^m \) and \( \{s_i\}_{i=1}^{m+1} \) in \([0, d_M(u, v)]\) which are listed in non-decreasing order and satisfy the conditions:

1. \( \{q_i\}_{i=1}^m \) contains \( \{t_i\}_{i=1}^n \)
2. Points \( s_1, \ldots, s_{m+1} \) satisfy
   \[
   0 \leq s_1 \leq q_1 \leq s_2 \leq q_2 \leq \cdots \leq s_m \leq q_m \leq s_{m+1} \leq d_M(u, v).
   \]
3. \( g(q_i) = \tilde{g}(q_i) \) for all \( i = 1, \ldots, m \), and
   \[
   \sum_{i=1}^{m+1} d_M(g(s_i), \tilde{g}(s_i)) \geq \alpha.
   \]

The main purpose of this paper is to prove the following result.

**Theorem 1.2.** For each non-RNP Banach space \( X \) there exists a metric space \( M_X \) containing a thick family \( T_X \) of geodesics which admits a bilipschitz embedding into \( X \).

This result complements the following two results of [Ost13+]:

**Theorem 1.3** ([Ost13+]). If a Banach space \( X \) admits a bilipschitz embedding of a thick family of geodesics, then \( X \) does not have the RNP.

**Theorem 1.4** ([Ost13+]). For each thick family \( T \) of geodesics there exists a Banach space \( X \) which does not have the RNP and does not admit a bilipschitz embedding of \( T \) into \( X \).

**Corollary 1.5** (of Theorems 1.2 and 1.3). A Banach space \( X \) does not have the RNP if and only if it admits a bilipschitz embedding of some thick family of geodesics.

**Remark 1.6.** Theorem 1.4 shows that the metric space \( M_X \) and the family \( T_X \) in Theorem 1.2 should depend on the space \( X \).

We use some standard definitions of the Banach space theory and the theory of metric embeddings, see [Ost13].

## 2 Proof of the main result

**Proof of Theorem 1.2.** We use the characterization of the RNP in terms of bushes (see [BL00] bottom of page 111) or [Bou83, Theorem 2.3.6]).

**Definition 2.1.** Let \( Z \) be a Banach space and let \( \varepsilon > 0 \). A set of vectors \( \{z_{n,j}\}_{n=0}^\infty \) in \( Z \) is called an \( \varepsilon \)-bush if for every \( n \geq 1 \) there is a partition \( \{A_n\}_{k=1}^{m_n-1} \) of \( \{1, \ldots, m_n\} \) such that
   \[
   ||z_{n,j} - z_{n-1,k}|| \geq \varepsilon
   \] (1)
for every $j \in A_k$, and

$$z_{n-1,k} = \sum_{j \in A_k} \lambda_{n,j} z_{n,j}$$

(2)

for some $\lambda_{n,j} \geq 0$, $\sum_{j \in A_k} \lambda_{n,j} = 1$.

The mentioned characterization of the RNP is:

**Theorem 2.2 ([BL00, Bou83]).** A Banach space $Z$ does not have the RNP if and only if it contains a bounded $\varepsilon$-bush for some $\varepsilon > 0$.

In this theorem and below we may and shall assume that $m_0 = 1$.

It is easy to see that the direct sum of two Banach spaces with the RNP has the RNP. Because of this a subspace of codimension 1 in a non-RNP Banach space also does not have the RNP. Let $x^* \in X^*$, $\|x^*\| = 1$ be a functional which attains its norm on $x \in X$, $\|x\| = 1$. By Theorem 2.2 we can find a bounded $\varepsilon$-bush in $\ker x^*$. Shifting this bush by $x$ we get a bush $\{x_{n,j}\}_{n=0, j=1}^{\infty, m_n}$ satisfying the condition $x^*(x_{n,j}) = 1$ for all $n$ and $j$. Consider the closure of the convex hull of the set $B_X \cup \{-\pm x_{n,j}\}_{n=0, j=1}^{\infty, m_n}$, where $B_X$ is the closed unit ball of $X$. It is clear that this set is the unit ball of $X$ in an equivalent norm and that in this new norm

$$\|x_{n,j}\| = 1$$

(3)

Since the property of $X$ which we are going to establish is clearly an isomorphic invariant, it suffices to consider the case where (3) is satisfied.

We are going to use this $\varepsilon$-bush to construct a thick family of geodesics in $X$ joining $0$ and $x_{0,1}$. First we construct a subset of the desired set of geodesics, this subset will be constructed as the set of limits of certain broken lines in $X$ joining $0$ and $x_{0,1}$. The constructed broken lines are also geodesics (but they do not necessarily belong to the family $T_X$).

The mentioned above broken lines will be constructed using representations of the form $x_{0,1} = \sum_{i=1}^{m} z_i$, where $z_i$ are such that $\|x_{0,1}\| = \sum_{i=1}^{m} \|z_i\|$. The broken line represented by such finite sequence $z_1, \ldots, z_m$ is obtained by letting $z_0 = 0$ and joining $\sum_{i=0}^{k} z_i$ with $\sum_{i=0}^{k+1} z_i$ with a line segment for $k = 0, 1, \ldots, m - 1$. Vectors $\sum_{i=0}^{k} z_i$, $k = 0, 1, \ldots, m$ will be called vertices of the broken line.

The infinite set of broken lines which we construct is labelled by vertices of the infinite binary tree $B$ in which each vertex is represented by a finite (possibly empty) sequence of 0 and 1, two vertices in $B$ are adjacent if the sequence corresponding to one of them is obtained from the sequence corresponding to the other by adding one term on the right. (For example, vertices corresponding to $(1, 1, 1, 0)$ and $(1, 1, 1, 0, 1)$ are adjacent.)

The broken line corresponding to the empty sequence $\emptyset$ is represented by the one-element sequence $x_{0,1}$, so it is just a line segment joining $0$ and $x_{0,1}$. 

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We have 

\[ x_{0,1} = \lambda_{1,1} x_{1,1} + \cdots + \lambda_{1,m_1} x_{1,m_1}, \]

where \( ||x_{1,j} - x_{0,1}|| \geq \varepsilon \) (recall that we assumed \( m_0 = 1 \)). We introduce the vectors

\[ y_{1,j} = \frac{1}{2} (x_{1,j} + x_{0,1}). \]

For these vectors we have

\[ x_{0,1} = \lambda_{1,1} y_{1,1} + \cdots + \lambda_{1,m_1} y_{1,m_1}, \]

\[ ||y_{1,j} - x_{1,j}|| = ||y_{1,j} - x_{0,1}|| \geq \frac{\varepsilon}{2}, \text{ and } ||y_{1,j}|| = 1. \]

As a preliminary step to the construction of the broken lines corresponding to one-element sequences (0) and (1) we form a broken line represented by the points

\[ \lambda_{1,1} y_{1,1}, \ldots, \lambda_{1,m_1} y_{1,m_1}. \] (4)  

We label the broken line represented by (4) by \( \emptyset \).

The broken line corresponding to the one-element sequence (0) is represented by the sequence obtained from (4) if we replace each term \( \lambda_{1,j} y_{1,j} \) by a two-element sequence

\[ \frac{\lambda_{1,j}}{2} x_{0,1}, \frac{\lambda_{1,j}}{2} x_{1,j}. \] (5)  

The broken line corresponding to the one-element sequence (1) is represented by the sequence obtained from (4) if we replace each term \( \lambda_{1,j} y_{1,j} \) by a two-element sequence

\[ \frac{\lambda_{1,j}}{2} x_{1,j}, \frac{\lambda_{1,j}}{2} x_{0,1}. \] (6)  

It is easy to see that if we let \( g \) and \( \tilde{g} \) be the broken lines corresponding to (0) and (1), respectively, parameterized by \([0,1]\) using either the distance to 0 or values of \( x^* \); and pick \( s_1, \ldots, s_{m+1} \) in such a way that they correspond to ends of line segments determined by \( \frac{\lambda_{1,j} \varepsilon}{2} x_{0,1} \) for (0) and \( \frac{\lambda_{1,j} \varepsilon}{2} x_{1,j} \) for (1) (see (5) and (6)), we get

\[ \sum_{i=1}^{m+1} ||g(s_i) - \tilde{g}(s_i)|| \geq \frac{\varepsilon}{2}. \] (7)  

Broken lines corresponding to 2-element sequences are also formed in two steps. To get the broken lines labelled by (0, 0) and (0, 1) we apply the described procedure to the geodesic labelled (0), to get the broken lines labelled by (1, 0) and (1, 1) we apply the described procedure to the geodesic labelled (1).

In the preliminary step we replace each term of the form \( \frac{\lambda_{1,k}}{2} x_{0,1} \) by a multiplied by \( \frac{\lambda_{1,k}}{2} \) sequence (4). We replace a term of the form \( \frac{\lambda_{1,k}}{2} x_{1,k} \) by the multiplied by \( \frac{\lambda_{1,k}}{2} \) sequence

\[ \{ \lambda_{2,j} y_{2,j} \}_{j \in A^2_k}, \] (8)
ordered arbitrarily, where \( y_{2,j} = \frac{x_{1,k} + x_{2,j}}{2} \), \( \lambda_{2,j} \), \( x_{2,j} \), and \( A^2_k \) are as in the definition of the \( \varepsilon \)-bush (observe that (3) implies that \( ||y_{2,j}|| = 1 \)). We label the obtained broken lines by (0) and (1), respectively.

To get the sequence representing the broken line labelled by (0, 0) we do the following operation with the preliminary sequence labelled (0).

- Replace each multiple \( \lambda y_{1,j} \) present in the sequence by the two-element sequence
  \[
  \frac{\lambda x_{0,1}}{2}, \frac{\lambda x_{1,j}}{2}.
  \]  
  (9)

- Replace each multiple \( \lambda y_{2,j} \), with \( j \in A^2_k \), present in the sequence by the two-element sequence
  \[
  \frac{\lambda x_{1,k}}{2}, \frac{\lambda x_{2,j}}{2}.
  \]  
  (10)

To get the sequence representing the broken line labelled by (0, 1) we do the same but changing the order of terms in (9) and (10). To get the sequences representing the broken lines labelled by (1, 0) and (1, 1), we apply the same procedure to the broken line labelled (1).

We proceed in an obvious way. Suppose that we have already constructed all broken lines labelled by sequences of length at most \( p \), and all of these broken lines are represented by sequences consisting of multiples of \( x_{n,j} \) with \( n \leq p \). To get broken lines labelled by \( (a_1, \ldots, a_p, 0) \) and \( (a_1, \ldots, a_p, 1) \), we form an intermediate broken line labelled by \( (a_1, \ldots, a_p) \). It is formed as follows: Each of the vectors \( x_{\ell,k} \), where \( \ell \leq p \) is replaced by the sequence

\[
\{\lambda x_{\ell,k} y_{\ell+1,j}\}_{j \in A^{\ell+1}_k},
\]

where \( y_{\ell+1,j} = \frac{1}{\ell}(x_{\ell,k} + x_{\ell+1,j}) \). The sequence is multiplied by the same scalar by which \( x_{\ell,k} \) is multiplied in the sequence labelled by \( (a_1, \ldots, a_p) \).

To form the sequence labelled by \( (a_1, \ldots, a_p, 0) \) we replace each multiple of \( y_{\ell+1,j} \) by the corresponding multiple of the two-element sequence \( \frac{1}{\ell}x_{\ell,k}, \frac{1}{\ell}x_{\ell+1,j} \). To form the sequence labelled by \( (a_1, \ldots, a_p, 1) \) we replace each multiple of \( y_{\ell+1,j} \) by the corresponding multiple of the two-element sequence \( \frac{1}{\ell}x_{\ell+1,j}, \frac{1}{\ell}x_{\ell,k} \).

**Observation 2.3.** A broken line labelled by a sequence \( (a_1, \ldots, a_p, a_{p+1}) \) passes through the vertices of the broken line labelled by \( (a_1, \ldots, a_p) \). There is a new vertex \( u \) of the broken line labelled by \( (a_1, \ldots, a_p, 0) \) between each pair of consecutive vertices of the broken line labelled by \( (a_1, \ldots, a_p) \). There is a new vertex \( v \) of the broken line labelled by \( (a_1, \ldots, a_p, 1) \) between each pair of consecutive vertices of the broken line labelled by \( (a_1, \ldots, a_p) \). The inequality (7) generalizes in the following way. If we add the distances \( ||u - v|| \), where \( u \) and \( v \) are as above, over all pairs of consecutive vertices of the broken labelled by \( (a_1, \ldots, a_p) \), we get at least \( \frac{\varepsilon}{2} \). Furthermore, if we
add the distances $||u - v||$, where $u$ and $v$ are as above, over some set of pairs of consecutive vertices of the broken labelled by $(a_1, \ldots, a_p)$, we get at least $\frac{\epsilon}{2} \cdot (\text{sum of distances between the consecutive pairs in the selection})$.

The thick family $T_X$ of geodesics whose existence is claimed in Theorem 1.2 is constructed in two steps.

**Step 1.** Consider all infinite sequences consisting of 0 and 1. For each such sequence we consider a branch in the infinite tree $B$ consisting of vertices corresponding to the finite initial segments of the sequence. The geodesic corresponding to the branch is the limit (in the sense described below) of the sequence of broken lines corresponding to vertices of the branch. To define the limit we observe that all of the geodesics of the sequence are distance preserving, and thus 1-Lipschitz, maps and that for any finite initial segment $I$ all further geodesics (that is, corresponding to segments containing $I$ at their beginning) pass through all vertices of the geodesic corresponding to $I$. In addition, the distance between the two consecutive vertices of the geodesic corresponding to a segment $I$ (that is a finite sequences of 0 and 1) containing $t$ terms does not exceed $(\lambda_{\text{max}})^t$ where $\lambda_{\text{max}} = \max \{\lambda_{n,j} : n, j\}$. On the other hand, using Definition 2.1, equality (3), and some easy inequalities we get that $1 - \lambda_{\text{max}} \geq \frac{\epsilon}{2}$ and thus $\lambda_{\text{max}} \leq 1 - \frac{\epsilon}{2}$. We conclude that the sequence of 1-Lipschitz maps which we consider converges on a dense subset of $[0, 1]$. Hence it converges everywhere on $[0, 1]$ to a limit (this is the limit which we meant at the beginning of this paragraph). Let $T_0$ be the family of all such limits corresponding to all of the branches of $B$.

**Step 2.** For each finite collection of vertices of the broken lines corresponding to finite sequences $(a_1, \ldots, a_p)$ consider all geodesics obtained by pasting together pieces corresponding to geodesics of $T_0$ which join the corresponding vertices (in the right order, so that the result of this pasting is again a geodesic joining 0 and $x_{0,1}$). Denote the resulting family of geodesics by $T_X$.

It remains to show that $T_X$ is a thick family of geodesics. Let $g$ be a geodesic in $T_X$ and let $r_1, \ldots, r_n$ be a set of points on it, let $r_i = g(t_i)$. Let $[0, 1] = \bigcup_{d=1}^w [h_{d-1}, h_d]$ be a partition of $[0, 1]$ for which $0 = h_0 < h_1, \ldots < h_w = 1$ and the on each of the intervals $[h_{d-1}, h_d]$ the geodesic $g$ coincides with one of the geodesics in $T_0$.

For each of the intervals $[h_{d-1}, h_d]$ and the corresponding geodesic $\hat{g}$ in $T_0$ we do the following. Let $(b_i)_{i=1}^\infty$ be the sequence of 0 and 1 corresponding to the geodesic $\hat{g}$. We pick sufficiently large initial segment $(b_i)_{i=1}^L$ of this sequence such that

1. $\hat{g}(h_{d-1})$ and $\hat{g}(h_d)$ are among vertices of the broken line $g^L$ corresponding to $(b_i)_{i=1}^L$.

2. All of the $t_i$ which are in $[h_{d-1}, h_d]$ can be covered by subintervals $[\eta_{h-1}, \eta_i]$ of $[h_{d-1}, h_d]$ with total length $\leq \frac{1}{2}|h_d - h_{d-1}|$ and $\hat{g}(\eta_{h-1}), \hat{g}(\eta_i)$ being vertices of the broken line $g^L$. 

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The desired geodesic $\tilde{g}$ will be picked as follows: on all intervals $[\eta_{i-1}, \eta_i]$ it will coincide with $g$. On all of the complementary intervals inside $[h_{d-1}, h_d]$ we let $\tilde{g}$ to coincide with the geodesic corresponding to the branch of $B$ which starts with $(b_l)_{l=1}^L$, but for which the next term (in the infinite sequence) is different from $b_{L+1}$. Together with Observation 2.3 (see the last sentence of it), this implies that that the sum of the corresponding deviations is at least $\frac{\varepsilon}{2} \cdot \frac{h_d-h_{d-1}}{2}$. Doing the same for all intervals $[h_{d-1}, h_d]$, we get that the total deviation is at least $\frac{\varepsilon}{4}$. This completes the proof.

Remark 2.4. Our proof of Theorem 1.2 shows any Banach space without the RNP has an equivalent norm in which it has a thick family of geodesics.

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4 References

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