Cherednik operators and
Ruijsenaars-Schneider model at infinity

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Abstract. Heckman introduced $N$ operators on the space of polynomials in $N$ variables, such that these operators form a covariant set relative to permutations of the operators and variables, and such that Jack symmetric polynomials are eigenfunctions of the power sums of these operators. We introduce the analogues of these $N$ operators for Macdonald symmetric polynomials, by using Cherednik operators. The latter operators pairwise commute, and Macdonald polynomials are eigenfunctions of their power sums. We compute the limits of our operators at $N \to \infty$. These limits yield a Lax operator for Macdonald symmetric functions.

Introduction

The present article is a continuation of our works [7,8]. By using the Lax operator formalism, in [7] we constructed a family of pairwise commuting operators such that the Jack symmetric functions of the infinitely many variables $x_1, x_2, \ldots$ are their eigenfunctions. We expressed these commuting operators in terms of the power sum symmetric functions $x_1^n + x_2^n + \ldots$ where $n = 1, 2, \ldots$.

The Jack symmetric functions can be regarded as limiting cases of Macdonald symmetric functions. The latter depend on the same variables $x_1, x_2, \ldots$ and also on two parameters $q, t$. In [8] we extended the results of [7] to the latter setting. In particular, by again using the Lax operator formalism we constructed a family of pairwise commuting operators such that the Macdonald symmetric functions are their eigenfunctions. In [8] we expressed these commuting operators in terms of the Hall-Littlewood symmetric functions of the variables $x_1, x_2, \ldots$ and of the parameter $t$. These expressions involve only the Hall-Littlewood symmetric functions corresponding to the partitions with one part, see Subsection 1.2 here.

Shortly after [7] was published, A. N. Sergeev and A. P. Veselov communicated to us their remarkable works [13,14] where in particular they found essentially the same commuting operators as we did in [7]. Their approach was different however. They first computed the limits at $N \to \infty$ of the Heckman operators [3] acting on all polynomials in the variables $x_1, \ldots, x_N$. These $N$ operators do not commute in general. But the restrictions of the power sums of these $N$ operators to the space of symmetric polynomials do commute. Moreover, Jack symmetric polynomials are eigenfunctions of these restrictions. Jack symmetric functions are then eigenfunctions of the limits of these restrictions at $N \to \infty$. 

arXiv:1703.02794v1 [nlin.SI] 8 Mar 2017
Here we extend this approach from Jack to Macdonald symmetric functions. It has been discovered by I. V. Cherednik [2] that the Macdonald polynomials in the variables \(x_1, \ldots, x_N\) are eigenfunctions of power sums of some \(N\) commuting operators, acting on all polynomials in these variables. These operators are called the Cherednik operators, see our Subsection 2.2 for their definition. It has been also known [12] that the Cherednik operators have limits at \(N \to \infty\). However, explicit expressions for these limits are unknown. We offer a solution to this open problem by firstly introducing for the Macdonald polynomials the analogues of non-commuting Heckman operators, see our Subsection 2.4. These analogues act on the rational functions of \(x_1, \ldots, x_N\) and are denoted by \(Z_1, \ldots, Z_N\). They are related to the Cherednik operators by Proposition 2.4. The principal property of the operators \(Z_1, \ldots, Z_N\) is stated as Theorem 2.5, see also Proposition 2.2.

An explanation is needed here regarding our scheme of referring to lemmas, propositions, theorems and corollaries. When referring to these, we indicate the subsections where they respectively appear. There is no more than one of each of these in every subsection, so our scheme should cause no confusion. For example, Proposition 2.2 is the only proposition that appears in Subsection 2.2.

In Subsection 2.6 we reformulate Theorem 2.5 by introducing a certain \(N \times N\) matrix \(Z\) with operator entries acting on the rational functions of the variables \(x_1, \ldots, x_N\). It is closely related to the classical Lax matrix of the trigonometric Ruijsenaars-Schneider model [10]. To be precise, let \(\gamma_i\) be the operator defined by (2.1). In the classical limit \(q \to 1\), when the canonical commutation relation \(\gamma_i x_i = q^{-1} x_i \gamma_i\) degenerates to the Poisson bracket \(\{\gamma_i, x_i\} = -\gamma_i x_i\), our \(Z\) degenerates to this Lax matrix up to a change of variables and up to conjugation by a diagonal matrix. In the same classical limit, the Macdonald determinant (2.3) degenerates to the characteristic determinant of the Lax matrix. Thus we have found a way to derive a quantum analogue of this Lax matrix directly from the Cherednik operators.

It is well known how the quantum Hamiltonians [11] of the trigonometric Ruijsenaars-Schneider model are related to the Macdonald operators (2.4), see for instance [5]. But our generating series (2.32) for quantum Hamiltonians differs from the Macdonald determinant and is new. A similar resolvent-type expression was used for the quantum Calogero-Sutherland model in [15]. The eigenstates of the latter model are the Jack symmetric polynomials. The limit at \(N \to \infty\) of the generalisation of that model to particles with spin has been studied in [4] as another extension of our work [7].

Following the approach of [13, 14] in Subsection 3.1 of our article we compute the limits at \(N \to \infty\) of the operators \(Z_1, \ldots, Z_N\). Then we also compute the limit of the restriction of the operator sum (2.20) appearing in Theorem 2.5 to the space of symmetric polynomials in \(x_1, \ldots, x_N\). This limit is a formal power series in another variable \(u\) with operator coefficients acting on the symmetric functions of \(x_1, x_2, \ldots\). After renormalisation and a change of the variable \(u\), this limit becomes the same generating series of the pairwise commuting operators as we constructed in [8]. For details, see Subsections 3.2 and 3.3 here.

In this article we generally keep to the notation of the book [6] for symmetric functions. When using results from [6] we simply indicate their numbers within the book. For example, the statement (6.9) from Chapter I of the book will be referred to as [I.6.9] assuming it is from [6].
1. Symmetric functions

1.1. Standard symmetric functions. Fix a field \( \mathbb{F} \). For any positive integer \( N \geq 1 \) denote by \( A_N \) the \( \mathbb{F} \)-algebra of symmetric polynomials in \( N \) variables \( x_1, \ldots, x_N \). The algebra \( A_N \) is graded by the polynomial degree. The substitution \( x_N = 0 \) defines a homomorphism \( A_N \rightarrow A_{N-1} \) preserving the degree. Here \( A_0 = \mathbb{F} \). The inverse limit of the sequence

\[
A_1 \leftarrow A_2 \leftarrow \ldots
\]

in the category of graded algebras is denoted by \( A \). Note that we get a canonical homomorphism \( A \rightarrow A_N \). The elements of the algebra \( A \) are called symmetric functions. Following [6] we now will introduce some standard bases of \( A \).

Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be any partition of \( 0, 1, 2, \ldots \). The number of non-zero parts is called the length of \( \lambda \) and is denoted by \( \ell(\lambda) \). If \( \ell(\lambda) \leq N \) then the sum of all distinct monomials obtained by permuting the \( N \) variables in \( x_1^{\lambda_1} \cdots x_N^{\lambda_N} \) is denoted by \( m_\lambda(x_1, \ldots, x_N) \). The symmetric polynomials \( m_\lambda(x_1, \ldots, x_N) \) with \( \ell(\lambda) \leq N \) form a basis of the vector space \( A_N \). By definition, for \( \ell(\lambda) \leq N \)

\[
m_\lambda(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_k \leq N} \sum_{\sigma \in \mathfrak{S}_k} c_\lambda^{-1} \ x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(k)}^{\lambda_k}
\]

where we write \( k \) instead of \( \ell(\lambda) \). Here \( \mathfrak{S}_k \) is the symmetric group permuting the numbers \( 1, \ldots, k \) and

\[
c_\lambda = k_1! \ k_2! \cdots
\]

if \( k_1, k_2, \ldots \) are the respective multiplicities of the parts \( 1, 2, \ldots \) of \( \lambda \). Further,

\[
m_\lambda(x_1, \ldots, x_{N-1}, 0) = \begin{cases} m_\lambda(x_1, \ldots, x_{N-1}) & \text{if } \ell(\lambda) < N; \\ 0 & \text{if } \ell(\lambda) = N. \end{cases}
\]

Hence for any fixed partition \( \lambda \) the sequence of polynomials \( m_\lambda(x_1, \ldots, x_N) \) with \( N \geq \ell(\lambda) \) has a limit in \( A \). This limit is called the monomial symmetric function corresponding to \( \lambda \). Simply omitting the variables, we will denote the limit by \( m_\lambda \). With \( \lambda \) ranging over all partitions of \( 0, 1, 2, \ldots \) the symmetric functions \( m_\lambda \) form a basis of the vector space \( A \). Note that if \( \ell(\lambda) = 0 \) then we set \( m_\lambda = 1 \).

We will be also using another standard basis of the vector space \( A \). For each \( n = 1, 2, \ldots \) denote \( p_n(x_1, \ldots, x_N) = x_1^n + \cdots + x_N^n \). When the index \( n \) is fixed the sequence of symmetric polynomials \( p_n(x_1, \ldots, x_N) \) with \( N = 1, 2, \ldots \) has a limit in \( A \), called the power sum symmetric function of degree \( n \). We will denote it by \( p_n \). More generally, for any partition \( \lambda \) put

\[
p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}
\]

where \( k = \ell(\lambda) \) as above. The elements \( p_\lambda \) form another basis of \( A \). Equivalently, the elements \( p_1, p_2, \ldots \) are free generators of the commutative algebra \( A \) over \( \mathbb{F} \).

In this article we will be using the natural ordering of partitions. By definition, here \( \lambda \geq \mu \) if \( \lambda \) and \( \mu \) are partitions of the same number and

\[
\lambda_1 \geq \mu_1, \ \lambda_1 + \lambda_2 \geq \mu_1 + \mu_2, \ \ldots .
\]

This is a partial ordering. Note that by [I.6.9] any monomial symmetric function \( m_\mu \) is a linear combination of the symmetric functions \( p_\lambda \) where \( \lambda \geq \mu \).
Define a bilinear form $\langle \ , \ \rangle$ on $\Lambda$ by setting for any two partitions $\lambda$ and $\mu$

$$\langle p_\lambda, p_\mu \rangle = k_\lambda \delta_{\lambda\mu} \quad \text{where } \quad k_\lambda = 1^{k_1} k_1! 2^{k_2} k_2! \ldots$$

in the above notation. This form is obviously symmetric and non-degenerate. We will indicate by the superscript $\perp$ the operator conjugation relative to this form. In particular, by (1.1) for the operator conjugate to the multiplication in $\Lambda$ by $p_n$ with $n \geq 1$ we have

$$p_n^\perp = n \partial / \partial p_n. \quad (1.2)$$

Next put

$$e_n(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \ldots < i_n \leq N} x_{i_1} \ldots x_{i_n}. \quad (1.3)$$

By taking logarithms of the left and right hand side of the above display and then exponentiating,

$$E(v) = \exp \left( - \sum_{n \geq 1} \frac{p_n}{n} (-v)^n \right). \quad (1.4)$$

Also put

$$h_n(x_1, \ldots, x_N) = \sum_{\ell(\lambda) \leq N} m_\lambda(x_1, \ldots, x_N)$$

where the sum is taken over partitions $\lambda$ of $n$. Then the sequence of symmetric polynomials $h_n(x_1, \ldots, x_N)$ with $N = 1, 2, \ldots$ has a limit in $\Lambda$, denoted by $h_n$ and called the \textit{elementary symmetric function} of degree $n$. We will also use a formal power series in another variable $v$,

$$H(v) = 1 + h_1 v + h_2 v^2 + \ldots \quad \text{by \cite{I.2.6} for the series}$$

we have the relation

$$E(-v) H(v) = 1. \quad (1.5)$$

Hence (1.4) implies

$$H(v) = \exp \left( \sum_{n \geq 1} \frac{p_n}{n} v^n \right). \quad (1.6)$$

The elements $h_1, h_2, \ldots$ as well as the elements $e_1, e_2, \ldots$ are free generators of the commutative algebra $\Lambda$ over the field $\mathbb{F}$. We will also use the \textit{vertex operator}

$$H^\perp(v) = 1 + h_1^\perp v + h_2^\perp v^2 + \ldots \quad \text{by \cite{I.2.6}}$$

It follows from (1.2) and (1.7) that for any $n = 1, 2, \ldots$ we have the equality

$$H^\perp(v) p_n = v^n + p_n. \quad (1.8)$$

It also follows from (1.2) and (1.7) that $H^\perp(v) : \Lambda \to \Lambda[v]$ is a homomorphism of $\mathbb{F}$-algebras. See \cite{I.5, Example 29} for both of the last two statements. Hence by applying $H^\perp(v)$ to any symmetric function in the variables $x_1, x_2, \ldots$ we get the same symmetric function but in the variables $v, x_1, x_2, \ldots$. 

\begin{equation}
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\end{equation}
1.2. Hall-Littlewood symmetric functions. Let $F$ be the field $\mathbb{Q}(t)$ with $t$ a formal parameter. The Hall-Littlewood symmetric functions [III.2.11] are labelled by all partitions of $0, 1, 2, \ldots$ and constitute another remarkable basis of the vector space $\Lambda$ over $F$. In the present article we will use only the elements of this basis corresponding to the single part partitions $(1), (2), \ldots$. These elements will be denoted by $Q_1, Q_2, \ldots$ respectively. Their generating series is

$$Q(v) = E(-tv)H(v) = 1 + Q_1v + Q_2v^2 + \ldots.$$ 

By using (1.3) and (1.5) we get the relation

$$Q(v) = \prod_{i \geq 1} \frac{1 - tx_i v}{1 - x_i v}$$

while by using (1.4) and (1.6) we get the relation

$$Q(v) = \exp \left( \sum_{n \geq 1} \frac{1 - t^n}{n} p_n v^n \right).$$

1.3. Macdonald symmetric functions. Now let $F$ be the field $\mathbb{Q}(q, t)$ where $q$ and $t$ are formal parameters. Then define a bilinear form $\langle \cdot, \cdot \rangle_{q,t}$ on $\Lambda$ by setting

$$\langle p_\lambda, p_\mu \rangle_{q,t} = k_\lambda \delta_{\lambda \mu} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}$$

for any partitions $\lambda$ and $\mu$. This form is symmetric and non-degenerate. If $q = t$, it specializes to the form defined by (1.1). We will indicate by the superscript $^*$ the operator conjugation relative to $\langle \cdot, \cdot \rangle_{q,t}$. In particular, by (1.2) and (1.11)

$$p_n^* = \frac{1 - q^n}{1 - t^n} p_n^\perp$$

for any $n \geq 1$. Hence by using (1.10) we get

$$Q^*(v) = 1 + Q_1^* v + Q_2^* v^2 + \ldots = \exp \left( \sum_{n \geq 1} \frac{1 - q^n}{n} p_n^\perp v^n \right).$$

Note that by using (1.6), the latter identity can be rewritten as

$$Q^*(v) = H^\perp(vq)^{-1}H^\perp(v).$$

Similarly to $H^\perp(v)$ the map $Q^*(v) : \Lambda \to \Lambda[v]$ is a homomorphism of $\mathbb{F}$-algebras.

By [VI.4.7] there exists a unique family of elements $P_\lambda \in \Lambda$ such that

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad \text{for} \quad \lambda \neq \mu$$

and such that any $P_\lambda$ equals $m_\lambda$ plus a linear combination of the elements $m_\mu$ with $\mu < \lambda$ in the natural partial ordering. The elements $P_\lambda \in \Lambda$ are called the Macdonald symmetric functions.

By [VI.4.10] the canonical homomorphism $\Lambda \to \Lambda_N$ maps $P_\lambda \mapsto 0$ if $\ell(\lambda) > N$. If $\ell(\lambda) \leq N$ then the image of $P_\lambda \in \Lambda$ under the homomorphism $\Lambda \to \Lambda_N$ is the Macdonald symmetric polynomial usually denoted by $P_\lambda(x_1, \ldots, x_N)$. All these polynomials with $\ell(\lambda) = 0, 1, \ldots, N$ form a basis of the vector space $\Lambda_N$ over $\mathbb{F}$.
2. Cherednik operators

2.1. Macdonald operators. Let $\mathbb{F} = \mathbb{Q}(q, t)$ as in Subsection 1.3. For $i = 1, \ldots, N$ the inverse $q$-shift operator $\gamma_i$ acts on any rational function $f \in \mathbb{F}(x_1, \ldots, x_N)$ by

\[
(\gamma_i f)(x_1, \ldots, x_N) = f(x_1, \ldots, q^{-1}x_i, \ldots, x_N).
\]  

(2.1)

Denote by $\Delta(x_1, \ldots, x_N)$ the Vandermonde polynomial of $N$ variables

\[
\det \left[ x_i^{N-j} \right]_{i,j=1}^N = \prod_{1 \leq i < j \leq N} (x_i - x_j).
\]

Put

\[
D_N(u) = \Delta(x_1, \ldots, x_N)^{-1} \cdot \det \left[ x_i^{N-j} (1 + u t^{j-1} \gamma_i) \right]_{i,j=1}^N
\]

(2.2)

where $u$ is another variable. The last determinant is defined as the alternated sum

\[
\sum_{\sigma \in S_N} (-1)^\sigma \prod_{i=1}^N \left( x_i^{N-\sigma(i)} (1 + u t^{\sigma(i)-1} \gamma_i) \right)
\]

(2.3)

where as usual $(-1)^\sigma$ denotes the sign of permutation $\sigma$. In every product over $i = 1, \ldots, N$ appearing in the alternated sum all the operator factors pairwise commute, hence their ordering does not matter. Note that $D_N(0) = 1$.

By (2.2) the $D_N(u)$ is a polynomial of degree $N$ in the variable $u$ with operator coefficients. It also follows from (2.2) that these coefficients map the space $\Lambda_N$ to itself. By [VI.3.3] for any $k = 1, \ldots, N$ the coefficient of $D_N(u)$ at $u^k$ equals

\[
\sum_{|I|=k} S_I(x_1, \ldots, x_N) \prod_{i \in I} \gamma_i
\]

(2.4)

where the sum is taken over all subsets $I$ of $\{1, \ldots, N\}$ of size $k$, whereas

\[
S_I(x_1, \ldots, x_N) = t^{k(k-1)/2} \prod_{i \in I} \frac{x_i - t x_j}{x_i - x_j}.
\]

Now for every $k = 1, \ldots, N$ consider the restriction of the operator (2.4) to the space $\Lambda_N$. By [VI.4.16] all these restrictions to $\Lambda_N$ pairwise commute. They are called the Macdonald operators. The Macdonald polynomials $P_\lambda(x_1, \ldots, x_N)$ with $\ell(\lambda) \leq N$ make a common eigenbasis of these operators. By [VI.4.15] the eigenvalue of $D_N(u)$ corresponding to any such eigenvector $P_\lambda(x_1, \ldots, x_N)$ is

\[
\prod_{i=1}^N \left( 1 + u q^{-\lambda_i} t^{i-1} \right).
\]

(2.5)

Note that our definition (2.2) of the $D_N(u)$ differs from [VI.3.2] by changing the parameters $q \mapsto q^{-1}$ and $t \mapsto t^{-1}$. However, by [VI.4.14] the Macdonald polynomials $P_\lambda(x_1, \ldots, x_N)$ are invariant under this change of their parameters. After this change, we also replaced the variable $X$ used in the definition [VI.3.2] by $u t^{N-1}$. The reasons for these alterations will be explained in Subsection 3.1.
2.2. Cherednik operators. For \( i, j = 1, \ldots, N \) with \( i \neq j \) introduce the operator acting on the vector space \( \mathbb{F}(x_1, \ldots, x_N) \)

\[
R_{ij} = 1 + \frac{(1 - t) x_j}{x_i - x_j} (1 - \sigma_{ij}) = \frac{x_i - t x_j}{x_i - x_j} + \frac{(t - 1) x_j}{x_i - x_j} \sigma_{ij} \tag{2.6}
\]

where \( \sigma_{ij} \in S_N \) acts by exchanging the variables \( x_i \) and \( x_j \). It is immediately obvious from the definition (2.6) that the operator \( R_{ij} \) maps polynomials in the variables \( x_1, \ldots, x_N \) to polynomials. Further, one can check that

\[
t R_{ij}^{-1} = t + \frac{(t - 1) x_j}{x_i - x_j} (1 - \sigma_{ij}) = \frac{t x_i - x_j}{x_i - x_j} + \frac{(1 - t) x_j}{x_i - x_j} \sigma_{ij}.
\]

The Cherednik operators \( C_1, \ldots, C_N \) acting on \( \mathbb{F}[x_1, \ldots, x_N] \) are then defined by

\[
C_i = t^{i-1} R_{i,i+1} \cdots R_{i,N} \gamma_i R_{1_i}^{-1} \cdots R_{i-1,i}^{-1}. \tag{2.7}
\]

These operators pairwise commute. In general, they do not map the space \( \Lambda_N \) to itself. However, any symmetric polynomial of the operators \( C_1, \ldots, C_N \) with the coefficients from the field \( \mathbb{F} \) does. Moreover by [2, Subsection 1.3.5] we have

**Proposition.** The action of \( D_N(u) \) on \( \Lambda_N \) coincides with that of the product

\[
\prod_{i=1}^{N} (1 + u C_i). \tag{2.8}
\]

In accord with the remark we made at the end of previous subsection, the operator \( C_i \) differs from the operator defined by [2, Equation 1.3.32] by changing the parameters \( q \mapsto q^{-1} \) and \( t \mapsto t^{-1} \). Our normalisation of \( C_i \) is also different.

2.3. Coherence property. We will use the following property of operators (2.7). For \( k = 1, \ldots, N - 1 \) let

\[
C_1^{(k)}, \ldots, C_{N-k}^{(k)} \tag{2.9}
\]

be the Cherednik operators acting on \( \mathbb{F}[x_{k+1}, \ldots, x_N] \) instead of \( \mathbb{F}[x_1, \ldots, x_N] \).

**Lemma.** The action of

\[
\prod_{i=k+1}^{N} (1 + u C_i) \tag{2.10}
\]

on the space \( \Lambda_N \) coincides with the action of

\[
\prod_{i=1}^{N-k} (1 + u t^k C_i^{(k)}). \tag{2.11}
\]
Proof. First let us prove by the downward induction on \( k = N, N - 1, \ldots, 1, 0 \) that the action of (2.10) on the space \( \Lambda_N \) coincides with the action of the product

\[
\prod_{i=k+1}^{N} \left( 1 + u t^{i-1} R_{i,i+1} \ldots R_{iN} \gamma_i \right) \tag{2.12}
\]

where the factors corresponding to the indices \( i = k + 1, \ldots, N \) are arranged from left to right. If \( k = N \) then neither of the products (2.10) and (2.12) has any factors, so the statement to prove is trivial. Now assume that our statement is already proved for some \( k > 0 \). Consider the product obtained from (2.10) by replacing the index \( k \) by \( k - 1 \). By the induction assumption, the action on \( \Lambda_N \) of the so obtained product coincides with that of

\[
(1 + u C_k) \prod_{i=k+1}^{N} \left( 1 + u t^{i-1} R_{i,i+1} \ldots R_{iN} \gamma_i \right) \tag{2.13}
\]

The last \( k - 1 \) factors of the Cherednik operator \( C_k \) appearing in (2.13)

\[
R_{1k}^{-1}, \ldots, R_{k-1,k}^{-1}
\]

commute with \( R_{i,i+1} \) and \( \gamma_i \) for any \( i = k + 1, \ldots, N \). They also act trivially on \( \Lambda_N \). After removing these \( k - 1 \) factors from \( C_k \) in (2.13) we get the product

\[
(1 + u t^{k-1} R_{k,k+1} \ldots R_{kN} \gamma_k) \prod_{i=k+1}^{N} \left( 1 + u t^{i-1} R_{i,i+1} \ldots R_{iN} \gamma_i \right).
\]

Thus we are making the induction step, and our statement is proved for any \( k \).

By using this statement in the particular case when \( k = 0 \), the action of (2.8) on the space \( \Lambda_N \) coincides with the action of the product

\[
\prod_{i=1}^{N} \left( 1 + u t^{i-1} R_{i,i+1} \ldots R_{iN} \gamma_i \right).
\]

By applying the latter result to the set of operators (2.9) instead of \( C_1, \ldots, C_N \) we obtain that for \( 0 < k < N \) the action of (2.11) on \( \Lambda_N \) coincides with that of

\[
\prod_{i=1}^{N-k} \left( 1 + u t^{i+k-1} R_{i+k,i+k+1} \ldots R_{i+k,N} \gamma_{i+k} \right).
\]

The last displayed product equals (2.12) by renaming \( i+k \) to \( i \). But we had also proved that the action of (2.10) on \( \Lambda_N \) coincides with the action of (2.12). \( \square \)
2.4. Covariant operators. For $i, j = 1, \ldots, N$ with $i \neq j$ denote

$$A_{ij} = \frac{x_i - t x_j}{x_i - x_j} \quad \text{and} \quad B_{ij} = \frac{(t - 1) x_j}{x_i - x_j}$$

(2.14)

so that by (2.6)

$$R_{ij} = A_{ij} + B_{ij} \sigma_{ij}.$$ Define operators $Z_1, \ldots, Z_N$ acting on $F(x_1, \ldots, x_N)$ by setting

$$W_i = \prod_{l \neq i} A_{il} + \sum_{j \neq i} B_{ij} \left( \prod_{l \neq i, j} A_{jl} \right) \sigma_{ij}.$$ (2.15)

In general, these operators do not map polynomials in the variables $x_1, \ldots, x_N$ to polynomials. But by definition, these operators make a covariant set relative to the action of the symmetric group $S_N$ by permutations of the variables:

$$\sigma^{-1} Z_i \sigma = Z_{\sigma(i)} \quad \text{for} \quad \sigma \in S_N.$$ (2.16)

Note that for $N > 1$ the operators $C_1, \ldots, C_N$ on $F[x_1, \ldots, x_N]$ do not enjoy the covariance property. On the other hand, our $Z_1, \ldots, Z_N$ do not commute.

For $k = 1, \ldots, N - 1$ let $A_N^{(k)} \subset F[x_1, \ldots, x_N]$ be the subspace of polynomials symmetric in the variables $x_{k+1}, \ldots, x_N$. Then

$$A_N \subset A_N^{(1)} \subset \ldots \subset A_N^{(N-1)} = F[x_1, \ldots, x_N].$$

Now consider the Cherednik operator $C_1$ acting on $F[x_1, \ldots, x_N]$. Our definition of the operator $Z_1$ originates from the following proposition.

**Proposition.** The actions of the operators $C_1$ and $Z_1$ on $A_N^{(1)}$ coincide.

**Proof.** We will prove that the action of $C_1$ on $A_N^{(k)}$ coincides with the action of

$$R_{12} \ldots R_{1k} \left( \prod_{k < l \leq N} A_{1l} + \sum_{k < j \leq N} B_{1j} \left( \prod_{k < l \leq N} A_{jl} \right) \sigma_{1j} \right) \gamma_1.$$ (2.17)

We will use the downward induction on $k = N - 1, \ldots, 1$. Our proposition will be then obtained when $k = 1$. If $k = N - 1$ then by the definition (2.7) we have

$$C_1 = R_{12} \ldots R_{1N} \gamma_1 = R_{12} \ldots R_{1,N-1} (A_{1N} + B_{1N} \sigma_{1N}) \gamma_1$$

as required. Now assume that our statement is proved for some $k > 1$. Since

$$A_N^{(k-1)} \subset A_N^{(k)}$$ (2.18)

we then know in particular that the action of $C_1$ on the space $A_N^{(k-1)}$ coincides with the action of the product (2.17). The latter product can be rewritten as

$$R_{12} \ldots R_{1,k-1} \times

(A_{1k} + B_{1k} \sigma_{1k}) \left( \prod_{k < l \leq N} A_{1l} + \sum_{k < j \leq N} B_{1j} \left( \prod_{k < l \leq N} A_{jl} \right) \sigma_{1j} \right) \gamma_1.$$
In its turn, the expression in the last displayed line can be rewritten as

\[
\left( \prod_{k \leq l \leq N} A_{1l} + B_{1k} \left( \prod_{k \leq l \leq N} A_{kl} \right) \sigma_{1k} + \sum_{k < j \leq N} \left( A_{1k} B_{1j} + B_{1k} B_{kj} \sigma_{1k} \right) \left( \prod_{k < l \leq N} A_{jl} \right) \sigma_{1j} \right) \gamma_1.
\]

Here none of the indices of the factor \( A_{jl} \) can be equal to 1 or \( k \), because \( j > k \) and \( l > k \). Further, here \( \sigma_{1k} \sigma_{1j} = \sigma_{1j} \sigma_{jk} \) where the factor \( \sigma_{jk} \) commutes with the operator \( \gamma_1 \) on \( \mathbb{F}[x_1, \ldots, x_N] \) and acts trivially on the subspace (2.18). Thus by the identity

\[
A_{1k} B_{1j} + B_{1k} B_{kj} = B_{1j} A_{jk}
\]

the action of the operator \( C_1 \) on \( \Lambda_{N}^{(k-1)} \) coincides with the action of

\[
R_{12} \ldots R_{1,k-1} \times \left( \prod_{k \leq l \leq N} A_{1l} + B_{1k} \left( \prod_{k \leq l \leq N} A_{kl} \right) \sigma_{1k} + \sum_{k < j \leq N} B_{1j} \left( \prod_{k < l \leq N} A_{jl} \right) \sigma_{1j} \right) \gamma_1.
\]

Thus we have made the downward induction step. \( \square \)

We have already noted that for any \( i = 1, \ldots, N \) the Cherednik operator \( C_i \) maps the polynomials in the variables \( x_1, \ldots, x_N \) to polynomials. On the other hand, the operator \( Z_i \) commutes with those permutations of the variables that preserve \( x_i \). By using these two observations when \( i = 1 \), our proposition implies

**Corollary.** Both operators \( C_1 \) and \( Z_1 \) map the space \( \Lambda_{N}^{(1)} \) to itself.

### 2.5. Main identity

Our main result of the current section is the theorem below.

Define operators \( U_1, \ldots, U_N \) acting on \( \mathbb{F}(x_1, \ldots, x_N) \) by setting

\[
U_i = (t - 1) \left( \prod_{l \neq i} A_{il} \right) \gamma_i.
\]  

(2.19)

Similarly to \( Z_1, \ldots, Z_N \) the operators \( U_1, \ldots, U_N \) make a covariant set relative to the action of the group \( S_N \) by permutations of the variables \( x_1, \ldots, x_N \):

\[
\sigma^{-1} U_i \sigma = U_{\sigma(i)} \quad \text{for} \quad \sigma \in S_N.
\]

**Theorem.** The action of the ratio \( D_N(u t)/D_N(u) \) on \( \Lambda_N \) coincides with the action of the sum

\[
1 + u \sum_{i=1}^{N} U_i (1 + u Z_i)^{-1}.
\]  

(2.20)
Proof. We will relate operators on the space $\mathbb{F}(x_1, \ldots, x_N)$ by the symbol $\sim$ if their actions on the subspace $\Lambda_N$ coincide. In Subsection 2.1 we already noted that the coefficients of the polynomial $D_N(u)$ map the space $\Lambda_N$ to itself. Let us multiply by $D_N(u)$ on the right both the ratio and the sum appearing in our theorem, and then subtract $D_N(u)$ from the results. We get to prove the relation

$$D_N(u t) - D_N(u) \sim u \sum_{i=1}^N U_i (1 + u Z_i)^{-1} D_N(u). \quad (2.21)$$

In the notation of Subsection 2.1 the left hand side of the relation (2.21) equals

$$u \sum_{k=1}^N u^{k-1} (t^k - 1) \sum_{|I|=k} S_I(x_1, \ldots, x_N) \prod_{i \in I} \gamma_i.$$

Now consider the summand at the right hand side of (2.21) with the index $i = 1$. By Proposition 2.2 the action of this summand on $\Lambda_N$ coincides with that of $U_1 (1 + u Z_1)^{-1} \prod_{j=1}^N (1 + u C_j) \sim U_1 (1 + u C_j) \sim U_1 \prod_{j=1}^{N-1} (1 + u t C_j^{(1)})$

where we used Proposition 2.4 and then Lemma 2.3 in the particular case $k = 1$. Hence by applying Proposition 2.2 once again, but to the Cherednik operators $C_1^{(1)}, \ldots, C_{N-1}^{(1)}$ instead of $C_1, \ldots, C_N$ we obtain that the summand at the right hand side of the relation (2.21) with the index $i = 1$ acts on $\Lambda_N$ as

$$U_1 (1 + u Z_1)^{-1} \prod_{j=1}^N (1 + u C_j) \sim U_1 (1 + u C_j) \sim U_1 \prod_{j=1}^{N-1} (1 + u t C_j^{(1)}).$$

Here $J$ ranges over all subsets of $\{2, \ldots, N\}$ of size $k - 1$. It follows that the summands at the right hand side of (2.21) with $i = 2, \ldots, N$ act on $\Lambda_N$ as the operators obtained from (2.22) via conjugation by $\sigma_{12}, \ldots, \sigma_{1N}$ respectively.

Thus the right hand side of (2.21) acts on $\Lambda_N$ as the operator sum of the form

$$\sum_{k=1}^N u^{k-1} \sum_{|I|=k} T_I(x_1, \ldots, x_N) \prod_{i \in I} \gamma_i$$

where $I$ ranges over all subsets of $\{1, \ldots, N\}$ of size $k$, and each $T_I(x_1, \ldots, x_N)$ is a certain rational function of the variables $x_1, \ldots, x_N$ over the field $\mathbb{Q}(t)$. To prove the relation (2.21) it now suffices to demonstrate that for each $I$

$$(t^k - 1) S_I(x_1, \ldots, x_N) = T_I(x_1, \ldots, x_N). \quad (2.23)$$
Moreover, because both sides of (2.21) are invariant under conjugation by the elements of $G_N$, it suffices to verify (2.23) only in the case when $I = \{1, \ldots, k\}$. Note that in the latter case the left hand side of (2.23) equals

$$\left( t^k - 1 \right) t^{k(k-1)/2} \prod_{1 \leq j \leq k \atop k < l \leq N} A_{jl}. \quad (2.24)$$

Now consider the right hand side of (2.23) in the case when $I = \{1, \ldots, k\}$. Let us denote it by $T$ for short. The contribution to $T$ from (2.22) corresponds to the set $J = \{2, \ldots, k\}$ and hence equals

$$t^{k-1} (t - 1) \left( \prod_{l \neq 1} A_{1l} \right) t^{(k-1)(k-2)/2} \prod_{2 \leq j \leq k \atop k < l \leq N} A_{jl} =$$

$$= \left( t - 1 \right) \left( \prod_{1 < l \leq k \atop l \neq i} A_{1l} \right) t^{k(k-1)/2} \prod_{1 \leq j \leq k \atop k < l \leq N} A_{jl}. \quad (2.25)$$

If we conjugate (2.22) by any $\sigma_{1i}$ with $i > 1$, the result will make contribution to $B$ only when $i \leq k$, and this contribution will correspond to $J = \{2, \ldots, k\}$. Indeed, then we will need $J$ in (2.22) such that $\sigma_{1i}(\{1\} \sqcup J) = \{1, \ldots, k\}$. Hence for each index $i = 2, \ldots, k$ we get a contribution to $T$

$$\sigma_{1i} \left( t - 1 \right) \left( \prod_{1 < l \leq k \atop l \neq i} A_{1l} \right) t^{k(k-1)/2} \prod_{1 \leq j \leq k \atop k < l \leq N} A_{jl} =$$

$$= \left( t - 1 \right) \left( \prod_{1 < l \leq k \atop l \neq i} A_{1l} \right) t^{k(k-1)/2} \prod_{1 \leq j \leq k \atop k < l \leq N} A_{jl}. \quad (2.26)$$

By dividing (2.24),(2.25),(2.26) by $(t - 1) t^{k(k-1)/2}$ and by cancelling there all the common factors $A_{jl}$ the relation (2.23) now reduces to the identity

$$\frac{t^k - 1}{t - 1} = \sum_{i=1}^{k} \prod_{1 \leq j \leq k \atop j \neq i} \frac{x_i - t x_j}{x_i - x_j}. \quad (2.27)$$

The latter identity is easy to verify, and we omit the details of verification. \hfill \Box

Consider the operator sum over $i = 1, \ldots, N$ appearing in (2.20). Denote by $I_N(u)$ the restriction of this operator sum to the subspace $A_N \subset F(x_1, \ldots, x_N)$. The $I_N(u)$ expands as a formal power series in $u$ with coefficients acting on $A_N$. Our theorem means that the action of the coefficients of the series $1 + u I_N(u)$ on $A_N$ coincides with the action of the respective coefficients of $D_N(u t)/D_N(u)$. The latter ratio should be also expanded as a formal power series in $u$ here.

The coefficients of the series $I_N(u)$ will be called the quantum Hamiltonians corresponding to the basis of Macdonald polynomials in the vector space $A_N$. In the next subsection we give another expression for $I_N(u)$ by using the resolvent of a certain $N \times N$ matrix with operator entries which act on $F(x_1, \ldots, x_N)$.
2.6. Matrix resolvent. Take any $f \in A_N^{(1)}$ and consider the column vector

$$\mathcal{F} = \begin{bmatrix} f \\ \sigma_{12}(f) \\ \vdots \\ \sigma_{1N}(f) \end{bmatrix}$$

Now define a $N \times N$ matrix $Z$ with operator entries acting on the vector space $\mathbb{F}(x_1, \ldots, x_N)$ as follows. The $i,j$-entry $Z_{ij}$ of the matrix $Z$ is defined by setting

$$Z_{ii} = \left( \prod_{l \neq i} A_{il} \right) \gamma_i,$$

$$Z_{ij} = B_{ij} \left( \prod_{l \neq i, j} A_{jl} \right) \gamma_j \quad \text{for} \quad i \neq j.$$  

Then by using the definition (2.15) we have

$$Z_i = W_i \gamma_i = Z_{ii} + \sum_{j \neq i} Z_{ij} \sigma_{ij}$$

where for $j \neq i$ we also use the relation $\sigma_{ij} \gamma_i = \gamma_j \sigma_{ij}$. It follows from (2.29) that

$$\begin{bmatrix} Z_1(f) \\ Z_2 \sigma_{12}(f) \\ \vdots \\ Z_N \sigma_{1N}(f) \end{bmatrix} = Z \mathcal{F}.$$  

Indeed, in its first entry the vector equality (2.30) holds by (2.29) with $i = 1$. If $i \neq 1$ then by using (2.29) we get

$$Z_i \sigma_{1i}(f) = Z_{ii} \sigma_{1i}(f) + Z_{i1} \sigma_{i1} \sigma_{1i}(f) + \sum_{j \neq 1, i} Z_{ij} \sigma_{ij} \sigma_{1i}(f) =$$

$$Z_{ii} \sigma_{1i}(f) + Z_{i1}(f) + \sum_{j \neq 1, i} Z_{ij} \sigma_{1j}(f) = Z_{i1}(f) + \sum_{j \neq 1} Z_{ij} \sigma_{1j}(f)$$

as required. Here for three pairwise distinct indices $1, i, j$ we also use the relations $\sigma_{ij} \sigma_{1i}(f) = \sigma_{1j} \sigma_{ij}(f)$.

By the covariance property (2.16) of the operators $Z_1, \ldots, Z_N$ the column vector at the left hand side of (2.30) has a form similar to $\mathcal{F}$. Namely, it can be obtained by replacing the polynomial $f$ in $\mathcal{F}$ by $Z_1(f)$. Here we use Corollary 2.4. By expanding $(1 + u Z_i)^{-1}$ for every $i = 1, \ldots, N$ as a formal power series in $u$ and by repeatedly using the above arguments, we get the equality

$$\begin{bmatrix} (1 + u Z_1)^{-1}(f) \\ (1 + u Z_2)^{-1} \sigma_{12}(f) \\ \vdots \\ (1 + u Z_N)^{-1} \sigma_{1N}(f) \end{bmatrix} = (1 + u Z)^{-1} \mathcal{F}.$$

(2.31)
Now suppose \( f \in \Lambda_N \) so that the polynomial \( f \) is symmetric in all the variables \( x_1, \ldots, x_N \). Then

\[
    f = \sigma_{12}(f) = \ldots = \sigma_{1N}(f)
\]

so that \( F = \mathcal{E} f \) where \( \mathcal{E} \) is the column vector of size \( N \) with every entry being 1.

Let \( U \) be the row vector of size \( N \) where the \( i \)-entry is the \( U_i \) defined by (2.19). By using (2.31) and the definition of the series \( I_N(u) \) as given in Subsection 2.5

\[
    I_N(u) f = U (1 + u Z)^{-1} \mathcal{E} f.
\]

Thus we have proved that the action of \( I_N(u) \) on \( \Lambda_N \) coincides with the action of

\[
    U (1 + u Z)^{-1} \mathcal{E}.
\]

(2.32)

Hence we now obtain the following corollary to Theorem 2.5.

**Corollary.** The action of the ratio \( D_N(u t)/D_N(u) \) on \( \Lambda_N \) coincides with that of

\[
    1 + u U (1 + u Z)^{-1} \mathcal{E}.
\]

3. Inverse limits

3.1. Limits of covariant operators. Let \( \mathbb{F} = \mathbb{Q}(q, t) \) as before. We will find first the inverse limit at \( N \to \infty \) of the restriction of the operator \( Z_1 \) to the subspace

\[
    \Lambda_N^{(1)} \subset \mathbb{F}(x_1, \ldots, x_N).
\]

(3.1)

By Proposition 2.4 the operator \( C_1 \) has the same restriction to \( \Lambda_N^{(1)} \). The limit will be an operator acting on the space \( A[v] \) and denoted simply by \( Z \). To define the limit extend the canonical homomorphism \( \Lambda \to \Lambda_N \) to a homomorphism

\[
    \pi_N : A[v] \to \Lambda_N^{(1)} : v \mapsto x_1.
\]

Here

\[
    \pi_N : p_m \mapsto p_m(x_1, \ldots, x_N) \quad \text{for} \quad n = 1, 2, \ldots.
\]

We will now define an operator \( Z \) on the vector space \( A[v] \) explicitly. Denote by \( \xi \) and \( \eta \) the automorphisms of the \( \mathbb{F} \)-algebra \( A[v] \) which act trivially on the subalgebra \( A \) but map the variable \( v \) to \( q^{-1} v \) and \( t v \) respectively. Thus \( \xi \) is the inverse \( q \)-shift of \( v \) while \( \eta \) is the usual \( t \)-shift. Next equip the vector space \( \mathbb{F}[v] \) with the standard inner product so that \( 1, v, v^2, \ldots \) form an orthonormal basis. Denote by \( v^0 \) the operator on \( \mathbb{F}[v] \) conjugate to multiplication by \( v \). Explicitly,

\[
    v^0 : v^n \mapsto \begin{cases} v^{n-1} & \text{if } n > 0, \\ 0 & \text{if } n = 0. \end{cases}
\]

Extend the operator \( v^0 \) from \( \mathbb{F}[v] \) to \( A[v] \) by \( A \)-linearity. The extension will still be denoted by \( v^0 \). For every \( f \in A \) extend from \( A \) to \( A[v] \) by \( \mathbb{F}[v] \)-linearity the operator of multiplication by \( f \) and its conjugate operator \( f^\perp \). Recall that the superscript \( \perp \) here indicates conjugation relative to the inner product (1.1). The conjugate \( f^* \) relative to the inner product (1.11) extends from \( A \) to \( A[v] \) in the same way as \( f^\perp \) does. Using these conventions, put \( Z = W \gamma \) where

\[
    \gamma = \xi Q^*(v) \quad \text{and} \quad W = \eta Q(v^0).
\]
Theorem. We have a commutative diagram of $\mathbb{F}$-linear mappings

$$
\begin{array}{ccc}
A[v] & \xrightarrow{Z} & A[v] \\
\downarrow{\pi_N} & & \downarrow{\pi_N} \\
A_N^{(1)} & \xrightarrow{Z_1} & A_N^{(1)}
\end{array}
$$

(3.2)

Proof. We will verify commutativity of the two diagrams obtained from (3.2) by replacing $Z, Z_1$ respectively by $\gamma, \gamma_1$ and $W, W_1$. Our theorem will then follow.

Firstly observe that the extended operator $Q^*(v) : A[v] \rightarrow A[v]$ appearing in the definition of $\gamma$ is a homomorphism of $\mathbb{F}$-algebras, and so is $\xi : A[v] \rightarrow A[v]$. Hence it suffices to show that the compositions $\pi_N \gamma$ and $\gamma_1 \pi_N$ coincide on $v$ and on $p_n$ for $n = 1, 2, \ldots$. By applying the compositions to $v$ we get the same result:

$$
v \xrightarrow{\gamma} q^{-1} v \xrightarrow{\pi_N} q^{-1} x_1 \quad \text{and} \quad v \xrightarrow{\pi_N} x_1 \xrightarrow{\gamma_1} q^{-1} x_1.
$$

To check the coincidence on any generator $p_n$ note that by the identity (1.12)

$$
\gamma = \xi H^\perp(v q)^{-1} H^\perp(v) = H^\perp(v)^{-1} \xi H^\perp(v).
$$

Hence due to (1.8) by applying $\pi_N \gamma$ and $\gamma_1 \pi_N$ to $p_n$ we also get the same result:

$$
p_n \xrightarrow{\gamma} q^{-n} v^n - v^n + p_n \xrightarrow{\pi_N} q^{-n} x_1 + x_2^n + \ldots + x_N^n
$$

and

$$
p_n \xrightarrow{\pi_N} x_1^n + \ldots + x_N^n \xrightarrow{\gamma_1} q^{-n} x_1 + x_2^n + \ldots + x_N^n.
$$

Now consider the compositions $\pi_N W$ and $W_1 \pi_N$. By definition, the extended operator $W : A[v] \rightarrow A[v]$ commutes with multiplication by any $f \in A$. But

$$
W_1 = \prod_{1 \leq l \leq N} A_{1l} + \sum_{1 \leq j \leq N} B_{1j} \left( \prod_{1 \leq l \leq N \atop l \neq j} A_{jl} \right) \sigma_{1j}
$$

(3.3)

by (2.15). In particular, the restriction of the operator $W_1$ to the subspace (3.1) commutes with multiplication by $\pi_N(f) \in A_N$. Hence it suffices to show that the compositions $\pi_N W$ and $W_1 \pi_N$ coincide on the elements $1, v, v^2, \ldots \in A[v]$.

Let us use the generating series of these elements

$$
1 + u v + u^2 v^2 + \ldots = \frac{1}{1 - u v}
$$

(3.4)

in the other variable $u$. By applying $\pi_N W$ to the series (3.4) we get

$$
\frac{1}{1 - u v} \xrightarrow{W} \frac{Q(u)}{1 - u t v} \xrightarrow{\pi_N} \frac{1}{1 - u t x_1} \prod_{i=1}^{N} \frac{1 - u t x_i}{1 - u x_i}.
$$

(3.5)
Here we employed the general fact that for any formal power series \( G(u) \) with the coefficients from \( \mathbb{F} \)
\[
G(u^0) \frac{1}{1-u} = \frac{G(u)}{1-u}.
\]
We have also employed the relation (1.9) with the variable \( v \) replaced by \( u \).

On the other hand, by applying \( W_1 \pi_N \) to the series (3.4) we get
\[
\frac{1}{1-u} \xrightarrow{\pi_N} \frac{1}{1-ux_1} \xrightarrow{W_1} \frac{1}{1-ux_1} \prod_{1 \leq i \leq N} \frac{x_1-tx_i}{x_1-x_i} + \sum_{1 < j \leq N} \frac{(t-1)x_j}{(1-ux_j)(x_1-x_j)} \prod_{1 \leq i \leq N, i \neq j} \frac{x_j-tx_i}{x_j-x_i}.
\]

Here we also used (2.14) and (3.3). It easy to verify that the results obtained in (3.5) and in the last two displayed lines are the same. Consider them as rational functions of \( u \) and assume that \( x_1, \ldots, x_N \neq 0 \).

Then both rational functions vanish at \( u = \infty \) and have poles only at \( u = x_1^{-1}, \ldots, x_N^{-1} \).

All these poles are simple, and the corresponding residues of the two functions coincide. □

Note that for any index \( i = 2, \ldots, N \) one can also consider the restriction of the operator \( Z_i \) to the subspace of \( \mathbb{F}(x_1, \ldots, x_N) \) consisting of the polynomials in \( x_1, \ldots, x_N \) symmetric in all the variables but \( x_i \).

By the covariance property (2.16) our Corollary 2.4 implies that the operator \( Z_i \) preserves this subspace. We could have defined the extension \( \pi_N \) of the homomorphism \( \Lambda \to \Lambda_N \) from \( \Lambda \) to \( \Lambda[v] \) by mapping the variable \( v \) to \( x_i \) instead of \( x_1 \).

The image of \( \pi_N \) would be then the latter subspace of \( \mathbb{F}(x_1, \ldots, x_N) \).

The inverse limit of the restriction of \( Z_i \) to that subspace would be then the same operator \( Z \) acting on \( \Lambda[v] \). This coincidence follows immediately from the property (2.16).

It is the change of parameters \( q \mapsto q^{-1} \) and \( t \mapsto t^{-1} \) in the original definition [VI.3.2] that allowed us to state the last theorem in terms of the Hall-Littlewood symmetric functions \( Q_1, Q_2, \ldots \).

Otherwise we would have to change \( t \mapsto t^{-1} \) in the definition of the latter symmetric functions.

The change of the variable \( X \) in [VI.3.2] and the corresponding choice of normalization of the operator \( C_i \) as in (2.7) and as in Proposition 2.2 ensure that every \( Z_i \) has a limit at \( N \to \infty \).

### 3.2. Limits of quantum Hamiltonians

In this subsection we will find the inverse limits at \( N \to \infty \) of the quantum Hamiltonians corresponding to the basis of Macdonald polynomials in \( \Lambda_N \).

These quantum Hamiltonians are defined as the operator coefficients of the series \( I_N(u) \) acting on the vector space \( \Lambda_N \), see the end of Subsection 2.5. We will denote by \( I(u) \) the inverse limit of the series \( I_N(u) \).

The coefficients of the series \( I(u) \) will be certain operators \( \Lambda \to \Lambda[w] \) where \( w \) is yet another formal variable.

We will then eliminate the dependence of the coefficients on \( w \) by renormalising the series \( I(u) \). Hence the coefficients of the renormalised series (3.11) will be operators acting on \( \Lambda \).

Consider the sum (2.20) over \( i = 1, \ldots, N \) appearing in (2.20). By (2.19) the action of this sum on the subspace \( \Lambda_N \subset \mathbb{F}(x_1, \ldots, x_N) \) coincides with that of
\[
V_1 \gamma_1 (1 + u Z_1)^{-1}
\]
where we set
\[ V_1 = (t - 1) \sum_{i=1}^{N} \left( \prod_{1 \leq l \leq N, l \neq i} A_{1l} \right) \sigma_{1i}. \]

Here \( \sigma_{11} = 1 \). We will demonstrate that the operator \( V_1 \) maps the subspace (3.1) to \( \Lambda_N \). At the same time we will determine the inverse limit at \( N \to \infty \) of the restriction of the operator \( V_1 \) to the subspace (3.1). The latter limit will be an operator \( \Lambda[v] \to \Lambda[w] \) denoted simply by \( V \). To determine this limit extend the canonical homomorphism \( \Lambda \to \Lambda_N \) to a homomorphism
\[ \tau_N : \Lambda[w] \to \Lambda_N : w \mapsto t^N. \]

Here \[ \tau_N : p_n \mapsto p_n(x_1, \ldots, x_N) \quad \text{for} \quad n = 1, 2, \ldots. \]

This definition of the homomorphism \( \tau_N \) goes back to [9, Section 6]. Now define \( V \) explicitly as the unique \( \Lambda \)-linear operator \( \Lambda[v] \to \Lambda[w] \) such that
\[ V : v^n \mapsto \begin{cases} -Q_n & \text{if} \quad n > 0, \\ w - 1 & \text{if} \quad n = 0. \end{cases} \]

**Proposition.** We have a commutative diagram of \( \mathbb{F} \)-linear mappings
\[
\begin{array}{ccc}
A[v] & \xrightarrow{V} & A[w] \\
\downarrow{\pi_N} & & \downarrow{\tau_N} \\
A^{(1)}_N & \xrightarrow{\pi_1} & A_N
\end{array}
\]

*Proof.* The operator \( V : \Lambda[v] \to \Lambda[w] \) commutes with the multiplication by any \( f \in \Lambda \). In turn, the restriction of the operator \( V_1 \) to the subspace (3.1) commutes with multiplication by \( \pi_N(f) \in \Lambda_N \). So it suffices to show that the compositions \( \tau_N V \) and \( V_1 \pi_N \) coincide on the elements \( 1, v, v^2, \ldots \in \Lambda[v] \). Let us again use the generating series (3.4) of these elements. By applying \( \tau_N V \) to (3.4) we get
\[
\frac{1}{1 - u v} \xrightarrow{V} w - Q(u) \xrightarrow{\tau_N} t^N - \prod_{i=1}^{N} \frac{1 - u t x_i}{1 - u x_i} \quad (3.8)
\]

where we used (1.9). On the other hand, by applying \( V_1 \pi_N \) to (3.4) we get
\[
\frac{1}{1 - u v} \xrightarrow{\pi_N} \frac{1}{1 - u x_1} \xrightarrow{V_1} \sum_{i=1}^{N} \frac{t - 1}{1 - u x_i} \prod_{1 \leq l \leq N, l \neq i} x_i - x_l. \quad (3.9)
\]

The results obtained in (3.8) and (3.9) are equal to each other. Indeed, consider them as rational functions of \( u \) and assume that \( x_1, \ldots, x_N \neq 0 \). Then both rational functions vanish at \( u = \infty \) and have poles at \( u = x_1^{-1}, \ldots, x_N^{-1} \). These poles are simple, and the corresponding residues of two functions coincide. \( \Box \)
By the surjectivity of $\pi_N$ the last proposition implies that the operator $V_1$ maps the subspace (3.1) to $A_N$. Moreover, it implies that the inverse limit at $N \to \infty$ of the restriction of the operator sum (3.6) to the subspace (3.1) equals

$$V \gamma (1 + u Z)^{-1} = \sum_{n=0}^{\infty} (-u)^n V \gamma Z^n.$$

By the definitions of $Z, \gamma$ and $V$ here for every $n \geq 0$ the composition $V \gamma Z^n$ is an operator $A[v] \to A[w]$. The above stated equality of the inverse limit follows from the commutativity of the diagram

$$\begin{array}{cccccc}
A[v] & \xrightarrow{Z^n} & A[v] & \xrightarrow{\gamma} & A[v] & \xrightarrow{V} & A[w] \\
\downarrow{\pi_N} & & \downarrow{\pi_N} & & \downarrow{\pi_N} & & \downarrow{\pi_N} \\
A_N^{(1)} & \xrightarrow{Z_1^n} & A_N^{(1)} & \xrightarrow{\gamma_1} & A_N^{(1)} & \xrightarrow{V_1} & A_N
\end{array}$$

Here we use the commutativity of (3.2),(3.7) and that of the diagram obtained from (3.2) by replacing $Z, Z_1$ respectively by $\gamma, \gamma_1$. The commutativity of the diagram so obtained has been established as a part of our proof of Theorem 3.1.

Denote by $\delta$ the embedding of $A$ to $A[v]$ as the subspace of degree zero in $v$. Then we have a commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\delta} & A[v] \\
\downarrow{\pi_N} & & \downarrow{\pi_N} \\
A_N & \xrightarrow{} & A_N^{(1)}
\end{array}$$

where the left vertical arrow is the canonical projection. The bottom horizontal arrow is the natural embedding. It follows that the inverse limit of $I_N(u)$ equals

$$I(u) = V \gamma (1 + u Z)^{-1} \delta. \quad (3.10)$$

By the above definition, every coefficient in the formal power series expansion of $I(u)$ in $u$ is a certain operator $A \to A[w]$. Now consider the series

$$(1 + u) (1 + u w)^{-1} (1 + u I(u)) \quad (3.11)$$

where the summand 1 in front of $u I(u)$ stands for the embedding of $A$ to $A[w]$ as the subspace of degree zero in $w$. This should cause no confusion. In the next subsection we will show that the series (3.11) does not depend on $w$. Hence the coefficients of this series will be operators mapping the vector space $A$ to itself.

Note that by the definition of the homomorphism $\tau_N$ and by the above given arguments, the series (3.11) is equal to the inverse limit at $N \to \infty$ of

$$(1 + u) (1 + u t^N)^{-1} (1 + u I_N(u)). \quad (3.12)$$
But by using the multiplicative formula (2.5), the eigenvalue of $D_N(ut)/D_N(u)$ on the trivial Macdonald polynomial $1 \in A_N$ corresponding to $\lambda = (0, 0, \ldots)$ is

$$(1 + u)^{-1}(1 + ut^N).$$

Hence our Theorem 2.5 implies that the eigenvalue of (3.12) on $1 \in A_N$ equals 1. By taking the limit at $N \to \infty$ the eigenvalue of (3.11) on the trivial Macdonald symmetric function $1 \in A$ also equals 1. This explains the definition of (3.11).

3.3. Truncated space. Here we will use the vector space decomposition

$$A[v] = A \oplus vA[v].$$  \hfill (3.13)

The second direct summand in (3.13) will be called the truncated space. Relative to this decomposition the operator $\gamma$ on $A[v]$ is represented by the $2 \times 2$ matrix with operator entries

$$\begin{bmatrix} 1 & 0 \\ \beta & \alpha \end{bmatrix},$$

where $\beta$ denotes the composition of the restriction of $\gamma$ to the first summand in (3.13) with the projection to the second summand. The map $\gamma$ preserves the second summand, and $\alpha$ denotes the restriction of $\gamma$ to it. Similarly, the operator $W$ on $A[v]$ is represented by the $2 \times 2$ matrix with operator entries

$$\begin{bmatrix} 1 & Y \\ 0 & X \end{bmatrix},$$

where $X$ and $Y$ respectively denote the compositions of the restriction of $W$ to the second summand in (3.13) with the projections to the first and to the second summands. Note that the operator $Z = W\gamma$ is then represented by the product

$$\begin{bmatrix} 1 & Y \\ 0 & X \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} 1 + Y\beta & Y\alpha \\ X\beta & X\alpha \end{bmatrix}. $$  \hfill (3.14)

By definition the operator $V : A[v] \to A[w]$ acts on the first direct summand in (3.13) as multiplication by $w - 1$. The restriction of $V$ to the second direct summand does not depend on $w$. Thus it maps $vA[v]$ to $A$. Moreover, by the definitions of $V$ and $W$ this restriction coincides with the operator $-Y$. Hence relative to (3.13) the operator $V$ is represented by the row with operator entries

$$[w - 1, -Y].$$

Finally denote $L = \alpha X$. This is an operator on the truncated space $vA[v]$. We shall call it the Lax operator for the Macdonald symmetric functions in $A$. This terminology is justified by the following theorem.

**Theorem.** The series (3.11) is equal to $(1 + u)(1 + u + u J(u))^{-1}$ where

$$J(u) = Y (1 + u L)^{-1} \beta.$$

In particular, the series (3.11) does not depend on the variable $w$. 
Proof. Relative to (3.13) the operator $\delta : A \rightarrow A[v]$ is represented by the column with two operator entries

$$
\begin{bmatrix}
1 \\
0
\end{bmatrix}.
$$

Therefore the operator product $(1 + u Z)^{-1} \delta$ appearing in the definition (3.10) of the series $I(u)$ is represented by the first column of the $2 \times 2$ matrix inverse to

$$
\begin{bmatrix}
1 + u + u Y \beta & u Y \alpha \\
u X \beta & 1 + u X \alpha
\end{bmatrix}.
$$

Here we employ the matrix representation (3.14) of the operator $Z$. To find that column we will use a well known formula for the inverse of a $2 \times 2$ block matrix, see [1, Lemma 3.2]. The block matrix is assumed to be invertible too. The first entry of the first column that we find in this way is

$$
(1 + u + u Y \beta - u Y \alpha (1 + u X \alpha)^{-1} u X \beta)^{-1} =
$$

$$
(1 + u + u Y \beta - u^2 Y \alpha X (1 + u \alpha X)^{-1} \beta)^{-1} =
$$

$$
(1 + u + u Y (1 - u L (1 + u L)^{-1}) \beta)^{-1} =
$$

$$
(1 + u + u Y (1 + u L)^{-1} \beta)^{-1} =
$$

$$
(1 + u + u J(u))^{-1}.
$$

(3.15)

The second entry of the first column of the inverse matrix that we find is then

$$
-(1 + u X \alpha)^{-1} u X \beta (1 + u + u J(u))^{-1} =
$$

$$
-u X (1 + u L)^{-1} \beta (1 + u + u J(u))^{-1}.
$$

(3.16)

The product $V \gamma$ in (3.10) is represented by the row with operator entries

$$
[w - 1 , -Y] \begin{bmatrix}
1 \\
\beta \alpha
\end{bmatrix} = [w - 1 - Y \beta , -Y \alpha].
$$

The series $I(u)$ is equal to the product of this row by the column representing $(1 + u Z)^{-1} \delta$. That column has the entries (3.15) and (3.16). Hence $I(u)$ equals

$$
(w - 1 - Y \beta)(1 + u + u J(u))^{-1} + Y \alpha u X (1 + u L)^{-1} \beta (1 + u + u J(u))^{-1} =
$$

$$
(w - 1 - Y (1 - u L (1 + u L)^{-1}) \beta)(1 + u + u J(u))^{-1} =
$$

$$
(w - 1 - Y (1 + u L)^{-1} \beta)(1 + u + u J(u))^{-1} =
$$

$$
(w - 1 - J(u)) (1 + u + u J(u))^{-1}.
$$

Our theorem immediately follows from the last displayed expression for $I(u)$. □

Note that by replacing in the series $- u J(u)$ the variable $u$ by $-u^{-1}$ we get the same generating series for the limits of the quantum Hamiltonians at $N \rightarrow \infty$ as was denoted in [8, Section 2] by $I(u)$. But in the present article the notation $I(u)$ was introduced in (3.10) and has a meaning different from that in [8].
Acknowledgements

We are grateful to I. V. Cherednik for illuminating conversations. The first named author was supported by the EPSRC grant EP/N023919, and by the programme Geometry and Representation Theory at the Erwin Schrödinger Institute. The second named author was supported by Leverhulme Senior Research Fellowship.

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