A COMPACTNESS RESULT FOR FANO MANIFOLDS AND KÄHLER RICCI FLOWS

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Abstract. We obtain a compactness result for Fano manifolds and Kähler Ricci flows. Comparing to the more general Riemannian versions in Anderson [An] and Hamilton [Ha], in this Fano case, the curvature assumption is much weaker and is preserved by the Kähler Ricci flows. One assumption is the boundedness of the Ricci potential and the other is the smallness of Perelman’s entropy. As one application, we obtain a new local regularity criteria and structure result for Kähler Ricci flows. The proof is based on a Hölder estimate for the gradient of harmonic functions, which may be of independent interest.

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1. Introduction

Compactness theorems have been useful tools in the study of geometric objects such manifolds and their evolutions under certain equations. Well known examples include the Cheeger-Gromov compactness theorem in the case of bounded curvature, diameter and volume lower bound ([Ch], [G]), as well as M. Anderson’s extension to the case of bounded Ricci curvature, diameter and injectivity lower bound ([An]). In both cases, the compactness is in the \(C^{1,\alpha}\) topology for any \(\alpha\) in \((0,1)\). M. Anderson also mentioned that if one replaces the \(L^\infty\) bound of the Ricci curvature by its \(L^p\) bound with \(p > n/2\), then the compactness holds in \(C^\alpha\) topology for some \(\alpha \in (0,1)\). See also [PW].

In this paper we prove a similar compactness result for Fano manifolds. It is shown that in this special case, the assumption on the Ricci curvature can be much weaker and preserved by the normalized Ricci flow. Then we will apply this result to study Kähler Ricci flows.

In order to present the results precisely, We will use the following notations and definitions. We use \(M\) to denote a \(n\) real dimensional compact Fano manifold, i.e. Kähler manifold with positive first Chern class. Denote by \(g\) be a stationary metric and by
The metric at time $t$; $d(x,y,t)$ is the geodesic distance under $g(t)$; $B(x,r,t) = \{y \in M \mid d(x,y,t) < r\}$ is the geodesic ball of radius $r$, under metric $g(t)$, centered at $x$, and $|B(x,r,t)|_{g(t)}$ is the volume of $B(x,r,t)$ under $g(t)$; $dg(t)$ is the volume element. We also reserve $R = R(x,t)$ as the scalar curvature under $g(t)$. When the time variable $t$ is not explicitly used, we may also suppress it in the notations mentioned above. In this paper, when mentioning a single manifold, it is always an $n$ real dimensional compact Fano manifold $M$ which satisfies the following

**Basic assumptions:**

Assumption 1. $L^2$ Sobolev inequality: there is a positive constant $s_0$ such that

$$\left(\int_M v^{2n/(n-2)} dg \right)^{(n-2)/n} \leq s_0 \left( \int_M |\nabla v|^2 dg + \int_M v^2 dg \right)$$

for all $v \in C^\infty(M)$.

Assumption 2. There exist positive constants $\kappa$ and $D$, such that

$$\kappa r^n \leq |B(x,r)| \leq \kappa^{-1} r^n, \quad \forall x \in M, \quad 0 < r < \text{diam}(M) \leq D.$$

Assumption 3. Let $u$ be the Ricci potential i.e. $R_{ij} = -u_{ij} + g_{ij}$. Then $u$ satisfies $\|u\|_{C^2(M)} \leq E$. Moreover the scalar curvature satisfies $\|R\|_\infty \leq E$.

One motivation for the basic assumptions is that they are satisfied, with uniform constant, by metrics along a normalized K"ahler Ricci flow. See Property KRF below. Also the volume lower bound is redundant since the Sobolev inequality implies it.

Parts of our results are also related to Perelman’s $W$ entropy by

**Definition 1.1.** Let $D$ be a domain in $M$. The Perelman $W$ entropy (functional) with parameter $\tau > 0$ is the quantity

$$W(g,f,\tau) = \int_D [\tau (R + |\nabla f|^2) + f - n] e^{-\frac{f}{4\tau}} (4\pi \tau)^{-n/2} dg, \quad f \in C^\infty_c(D).$$

The $\mu$ invariant with parameter $\tau$ is the infimum of the $W$ entropy, given by

$$\mu(g,\tau,D) = \inf \{ W(g,f,\tau) \mid f \in C^\infty_c(D), (4\pi \tau)^{-n/2} \int_D e^{-f} dg = 1 \}.$$  

It is clear that $\mu(g,\tau,D) \geq \mu(g,\tau,M)$. According to Perelman [P], the latter is monotone non-decreasing under Ricci flow if the parameter $\tau$ satisfies $\frac{d\tau}{dt} = -1$. If no confusion arises, we will drop the argument $M$ and simply write $\mu(g,\tau,M)$ as $\mu(g,\tau)$.

The first main results of the paper is the following theorem.

**Theorem 1.2.** (compactness of Fano manifolds) (a). Let $SP = SP(s_0,\kappa,D,E,i_0)$ be the space of compact Fano manifolds $M$ satisfying Assumptions 1, 2, 3 and that $\inf_M i \geq i_0$. Then there exists a positive number $\alpha \in (0,1)$ such that $SP$ is compact in the $C^{1,\alpha}$ topology.

(b). Let $SP = SP(s_0,\kappa,D,E,\eta_0,\tau_0,\rho_0)$ be the space of compact Fano manifolds $(M,g)$ satisfying Assumptions 1, 2, 3 and that $\sup_{\tau \in (0,\tau_0),p \in M} \mu(g,\tau,B(p,\tau_0)) \geq -\eta_0$. Here $\eta_0,\tau_0,\rho_0$ are any given positive numbers. If $\eta_0$ is sufficiently small, then there exists a positive number $\alpha \in (0,1)$ such that $SP$ is compact in the $C^{1,\alpha}$ topology.

Next, we apply the previous theorem to the study the (normalized) K"ahler Ricci flows

$$\partial_t g_{ij} = -R_{ij} + g_{ij} = \partial_i \partial_j u, \quad t > 0,$$
on a compact, Kähler manifold $M$ of complex dimension $m = n/2$, with positive first Chern class. We always assume that the initial metric is in the canonical Kähler class $2\pi c_1(M)$.

Given initial Kähler metric $g_{ij}(0)$, H. D. Cao [Ca] proved that (1.3) has a solution for all time $t$. Recently, many results concerning long time and uniform behavior of (1.3) have appeared. For example, when the curvature operator or the bisectional curvature is nonnegative, it is known that solutions to (1.3) stays smooth when time goes to infinity (see [CCZ], [CT1] and [CT2] for examples). In the general case, Perelman (cf [ST]) proved that the scalar curvature $R$ is uniformly bounded, and the Ricci potential $u(\cdot, t)$ is uniformly bounded in $C^1$ norm, with respect to $g(t)$. When the complex dimension $m = 2$, let $(M, g(t))$ be a solution to (1.3), it is proved in ([CW]) that the flow sequentially converges to an orbifold. When $m = 3$, the authors of [TZz] proved sequential convergence except on a small singular set. For the general case, in the paper [TZq], we proved that a Gromov-Hausdorff limit of the flow is a metric space with volume doubling and $L^2$ Poincaré inequality and $L^1$ isoperimetric inequality. So it is a PI space in the sense of Cheeger. There are many more interesting papers on the convergence issue of Kähler Ricci flow. See for example, [CLW], [MS], [PS1], [PSSW1], [PSSW2], [Sc], [So1], [Sz], [SW], [To], [TZhu1], [TZhu2], [TZZZ], [Zl], [Zz] and [Zhu] and references therein.

The following is a local regularity criteria for Kähler Ricci flows, which can be regarded as a strengthened pseudolocality theorem for Kähler Ricci flow. Note our condition on the Ricci curvature is preserved under the flow. Hence essentially the result does not require any direct assumption on Ricci curvature or curvature tensor except on the initial metric. Some applications will be given in the form of corollaries, including the compactness result.

**Theorem 1.3.** (local regularity criteria for Kähler Ricci flows) Let $(M, g(t))$ be a normalized Kähler Ricci flow on a compact Fano manifold whose initial metric satisfies the basic assumptions. There exists positive numbers $r_0$ and $\eta$ such that the following statement is true.

Suppose either one of the conditions holds for $r \in (0, r_0]$.

1. the geodesic ball $B(x, r, 0)$ is almost Euclidean, i.e. for a sufficiently small positive number $\eta$, any $y \in B(x, r, 0)$ with $B(y, \rho, 0) \subset B(x, r, 0)$, it holds

\[
|B(y, \rho, 0)|_{g(0)} \geq (1 - \eta)\omega_n \rho^n
\]

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$;

2. for a sufficiently small positive number $\eta > 0$ and a given number $\tau_0 > 0$, the infimum of the $W$ entropy on $B(x, r, 0)$ satisfies

\[
\sup_{\tau \in (0, \tau_0]} \mu(g(0), \tau, B(x, r, 0)) \geq -\eta.
\]

3. the injectivity radius of the ball $B(x, r, 0)$ is bounded from below by a number $i_0 > 0$.

Then, exists a number $\epsilon > 0$ such that

\[
|Rm(y, t)| \leq t^{-1} + (\epsilon r)^{-2}
\]

for all $t \in (0, (\epsilon r)^2]$ and $y$ such that $d(x, y, t) \leq \epsilon r$. 

Remark 1.4. (a) The result under condition 1 is an improvement to Theorem 4.2 in [TZ2] where the same conclusion is reached under the extra assumption that $|\nabla^2 u|$ is $L^\infty$ uniformly in time.

(b) The result under condition 2 is similar in spirit to the new $\epsilon$ regularity result for Ricci flow in [HN] where the authors proved boundedness of the curvature tensor in a space-time cube under the condition that a heat kernel weighted entropy is close to zero. Since the heat kernel in that result is coupled to the Ricci flow, that condition is a space-time one. In contrast our condition is applied on the initial value.

(c) The result under condition 2 is also related to the Pseudo-locality Theorem 3.1 in [TW] where a Ricci lower bound is assumed for general normalized Ricci flows.

(d) The result also holds for un-normalized Ricci flows by scaling and choosing $\epsilon$ sufficiently small.

(e) One may prove the result under condition 3 by different methods.

One application of the theorem is the following compactness result for Kähler Ricci flows.

Corollary 1.5. (compactness of Kähler Ricci flows) Let $(M_k, g_k(t), p_k)$ be a sequence of normalized compact Kähler Ricci flows in the time interval $[0, T]$. Then the following conclusion holds.

(a) Suppose the initial metrics $(M_k, g_k(0))$ satisfies the basic assumptions and that the injectivity radius of $(M_k, g_k(0))$ is bounded from below by a positive constant $i_0$. Then, there exists a positive constant $T_1 \leq T$ such that $(M_k, g_k(t), p_k)$, $t \in (0, T_1]$, is compact in $C^\infty_{loc}$ topology.

(b) Suppose the initial metrics $(M_k, g_k(0))$ satisfies the basic assumptions. Let $\{t_j\}$ be a partition of the time interval such that $\sup(t_j - t_{j-1}) \leq T_1$. Suppose the injectivity radii of $(M_k, g_k(t_j))$ are bounded from below by a positive constant $i_0$. Then $(M_k, g_k(t), p_k)$, $t \in (0, T]$, is compact in $C^\infty_{loc}$ topology.

Remark 1.6. Recall that Hamilton's compactness result for the general Ricci flow in forward time requires uniform bound for the curvature tensor through out space time and the positive lower bound of the injectivity radii at initial time. By the classical volume comparison theorem of Gunther, the initial manifolds are $\kappa$ non-collapsed. Now, by Perelman [P], we know that the Ricci flows are also $\kappa$ non-collapsed in the time interval $[0, T]$. Hence the injectivity radius are bounded from below in all space time by the bound on the curvature tensor. Therefore, Hamilton’s compactness theorem for general Ricci flows implicitly requires both curvature bound and injectivity lower bound in all space time.

In contrast, part (a) of the corollary for Kähler Ricci flow only requires lower bound of injectivity radius and the basic assumptions on the initial metric. Part (b) requires lower bound of injectivity radius for some discrete times and the basic assumptions on the initial metric. So, in the Kähler Ricci flow case, we have essentially removed the direct assumption on curvature tensor from compactness result.

With possible application in mind, we single out another consequence of Theorem 1.3 as

Corollary 1.7. ($C^{1,\alpha}$ structure of almost Euclidean region). Let $(M, g(t))$ be a normalized Kähler Ricci flow whose initial metric $g(0)$ satisfies the basic assumptions. There exist positive numbers $\eta \in (0,1)$, $\delta$ and $\tau_0$ such that the following statement holds.
For any $t > 0$, $x \in \mathbf{M}$ and $r \in (0, r_0]$, suppose the ball $B(x, r, g(t))$ is almost Euclidean in volume, i.e. for all $B(y, r, g(t)) \subset B(x, r, g(t))$,

\begin{equation}
|B(y, r, g(t))|_{g(t)} \geq (1 - \eta)\omega_n r^n,
\end{equation}

where $w_n$ is the volume of $n$ dimensional Euclidean unit ball. Then,

\begin{equation}
r_{g(t)}^{\alpha, \theta}(x) \geq \delta r
\end{equation}

Here $r_{g(t)}^{\alpha, \theta}(x)$ is the $C^{1,\alpha}$ harmonic radius at $x$ defined in Definition 4.1 in Section 4. The $C^{1,\alpha}$ norm of the metric is bounded within the radius.

The proof follows from part 2 of Lemma 4.2 which is the core of Theorem 1.3 and the fact that the basic assumptions are preserved under the Kähler Ricci flow. See the Property KRF below.

**Remark 1.8.** We mention that a similar $C^\alpha$ structure result is given by Theorem 2.35 in [YZ], under the assumption that the $L^p$, $p > n/2$ norm of the Ricci curvature is uniformly bounded. That result plays an important role in their proof of convergence result in complex dimension 3.

The proof of Theorem 1.3 is based on the following properties for Kähler Ricci flow on a compact manifold with positive first Chern class.

**Property KRF.** Let $(\mathbf{M}, g(t))$ be a Kähler Ricci flow (1.3) on a compact manifold with positive first Chern class. There exist uniform positive constants $C$ and $\kappa$ depending only on $g(0)$ so that

1. $|R(g(t))| \leq C$,
2. $\text{diam}(\mathbf{M}, g(t)) \leq C$,
3. $\|u(\cdot, t)\|_{C^1} \leq C$.
4. $B(x, r, t)|_{g(t)} \geq \kappa r^n$, for all $t > 0$ and $r \in (0, \text{diam}(\mathbf{M}, g(t)))$.
5. $B(x, r, t)|_{g(t)} \leq \kappa^{-1} r^n$ for all $r > 0$, $t > 0$.
6. There exists a uniform constant $S_2$ so that the following $L^2$ Sobolev inequality holds:

\begin{equation}
\left(\int_{\mathbf{M}} v^{2n/(n-2)} dg(t)\right)^{(n-2)/n} \leq S_2 \left(\int_{\mathbf{M}} |\nabla v|^2 dg(t) + \int_{\mathbf{M}} v^2 dg(t)\right)
\end{equation}

for all $v \in C^\infty(\mathbf{M}, g(t))$.

7. (a). Let $\Gamma$ be the Green’s function on $(\mathbf{M}, g(t))$. Then there exists a uniform constant $C$ such that

\begin{equation}
|\Gamma(x, y)| \leq \frac{C}{d(x, y)^{n-2}}, \quad |\nabla \Gamma(x, y)| \leq \frac{C}{d(x, y)^{n-1}}.
\end{equation}

(b). Let $p = p(x, y, s)$ be the (stationary) heat kernel for $(\mathbf{M}, g(t))$. There exist positive constants $a_1$ and $a_2$, depending only on $g(0)$ such that

\begin{equation}
a_1 \frac{1}{s^{n/2}} e^{-a_2 d(x,y)^2/4s} \leq p(x, y, s) \leq \frac{1}{a_1 s^{n/2}} e^{-d(x,y)^2/(a_2 s)}, \quad s \in (0, 1];
\end{equation}

\begin{equation}
|\nabla p(x, y, s)| \leq \frac{1}{a_1 s^{(n+1)/2}} e^{-d(x,y)^2/(a_2 s)}, \quad s \in (0, 1].
\end{equation}
8. uniform $L^2$ Poincaré inequality: for any $v \in C^\infty(B(x,r))$ where $B(x,r)$ is a proper ball in $(M, g(t))$, there is a uniform constant $C$ such that

\begin{equation}
\int_{B(x,r)} |v - v_B|^2 dg \leq C r^2 \int_{B(x,r)} |\nabla v|^2 dg.
\end{equation}

Here all quantities are with respect to $g = g(t)$ and $v_B$ is the average of $v$ in $B(x,r)$.

Property A 1-4 is due to Perelman (c.f. [ST]); Property 5 can be found in [Z11] and also [CW2]; Property 6 was first proven in [Z07] (see also [Ye], [Z10]). Properties 7 and 8 are in [TZq]. Bounds on the Green’s function and heat kernel are well known if the metrics have uniform $L^\infty$ lower bound for the Ricci curvature ([LY]), a condition that is unavailable here.

The rest of the paper is organized as follows. In Section 2, we will prove some integral bounds for the Hessian of the Ricci potential, which implies that the Ricci curvature is actually small in certain Morrey or Kato type norm. In Section 3, we show that the gradient of the Ricci potential is Hölder continuous within a harmonic coordinate. In Section 4, we will prove a lower bound for the harmonic radius under three separate conditions. One involves the injectivity radius, another involves volume of balls and third one relates with Perelman’s W entropy. The theorems and corollaries will be proven in Section 5. Sometimes we need to switch between real and complex coordinates in computations, which may result in extra harmless constants in the Laplacian, Ricci curvature, etc.

2. SOME INTEGRAL ESTIMATE ON THE HESSIAN OF THE RICCI POTENTIAL

In order to prove the theorems, in this section we state and prove an a priori integral estimate for the Hessian of the Ricci potential, which essentially means that it is a sub-critical quantity comparing with the Laplacian. In fact this simple estimate already shows that in integral sense $|Ric|$ scales like $1/r$ instead of the usual $1/r^2$. Here $r$ is the distance function. It is the reason that one gains one order of regularity in the Theorems.

**Lemma 2.1.** Let $u$ be the Ricci potential. Let $B(x,r)$ be a proper geodesic ball with $r \leq 1$. Then there exists a uniform constant $A_0$ depending only on the parameters in the basic assumptions of the manifold and $\|\nabla u\|_\infty$ and $\|R\|_\infty$ such that

\[ \int_{B(x,r)} |\text{Hess } u|^2 dg \leq A_0 r^{n-2}. \]

**Proof.** Starting from the equation, in real variable form

\begin{equation}
\Delta u = n - R,
\end{equation}

and applying the Bochner’s formula, we know that

\begin{equation}
\Delta |\nabla u|^2 = 2 |\text{Hess } u|^2 + 2 R_{ij} u_j u_i - 2 \nabla R \nabla u.
\end{equation}
Here $\Delta$ is the real Laplacian. Let $\phi$ be a standard Lipschitz cut-off function such that $\phi = 1$ on $B(x, r)$, $\phi = 0$ on $B(x, 2r)^c$ and $|\nabla \phi| \leq C/r$. After integration, we have

$$
2 \int |Hess \ u| \phi^2 \, dg
= \int \Delta |\nabla u|^2 \phi^2 \, dg - 2 \int R_{ij} u_i u_j \phi^2 \, dg + \int 2 R \nabla u \nabla \phi \, dg
= -4 \int \nabla |\nabla u| \cdot \nabla \phi \, dg + 2 \int (\partial_i \partial_j u - g_{ij} u_i u_j) \phi^2 \, dg
- 2 \int R(n - R) \phi^2 \, dg - 4 \int R \nabla u \nabla \phi \, dg.
$$

(2.3)

Note that $|\nabla |\nabla u|| \leq |Hess \ u|$ and, by Perelman (cf [ST]), that $|\nabla u|$ and $|R|$ are bounded. We can apply Cauchy-Schwarz inequality to deduce

$$
\int |Hess \ u| \phi^2 \, dg \leq C \int_{B(x, 2r)} (|\nabla u|^2 + |\nabla u|^4) \, dg + 2 \int_{B(x, 2r)} |R(n - R)| \, dg + \frac{C}{r} \int_{B(x, 2r)} |R| |\nabla u| \, dg \leq Cr^{n-2}.
$$

To get the last inequality, we have used the $\kappa$ non-inflating property. □

The next lemma provides a lower bound on the Green’s function $\Gamma = \Gamma(x, y)$. Since $M$ is compact, we know that $\Gamma$ changes sign. So the lower bound holds only when $x$ and $y$ are close.

**Lemma 2.2.** Let $\Gamma$ be the Green’s function of the Laplace operator on scalar functions. There exist positive numbers $r_0$ and $C$, depending only on the parameters of the basic assumptions on $M$, such that

$$
\Gamma(x, y) \geq \frac{C}{d(x, y)^{n-2}}
$$

provided that $d(x, y) \leq r_0$.

**Proof.**

Since $M$ is a compact manifold, it is well known that

$$
\Gamma(x, y) = \int_0^\infty G(x, t, y) - \frac{1}{|M|} \, dt
$$

(2.6)

where $G$ is the heat kernel on $M$. We mention that the time $t$ here is not the time in the Ricci flow. From Section 2 in [TZq], the following inequalities hold for $G$. There exist positive constants $c_1, \ldots, c_4$ and $\beta$ such that

$$
\left| \int_0^\infty [G(x, t, y) - \frac{1}{|M|}] \, dt \right| \leq c_1 \int_0^\infty e^{-c_2 t} \, dt \leq c_1/c_2;
$$

(2.7)

Also, if $t \in (0, \beta)$, then

$$
G(x, t, y) \geq \frac{c_3}{t^{n/2}} e^{-c_4 d^2(x, y)/t}.
$$

(2.8)
Our next task is to bound each term \( I \)

Using the relation (2.12)

Here and later we have used the notation for the Kato norm of an integrable function \( F \),

Substituting these into (2.6), we find that

Hence, there exists \( r_0 > 0 \) such that

provided that \( d(x, y) \leq r_0 \).

**Lemma 2.3.** Let \( u \) be the Ricci potential. Then there exists a uniform constant \( A_1 \) depending only on the parameters of the basic assumptions of the manifold such that

Here and later we have used the notation for the Kato norm of an integrable function \( F \),

Recall from (2.2) that

Let \( a \) be the average of \( |\nabla u|^2 \) over \( M \). Then, from the definition of Green’s function, we have

Using the relation \( \partial_i \partial_j u = g_{ij} - R_{ij} \) and applying integration by parts, we see that

Our next task is to bound each term \( I_k, k = 1, ..., 7 \).
According to Perelman $|\nabla u|$ and $R$ are bounded. Hence $|I_1| \leq C$ and

\begin{equation}
|I_2| \leq 2 \int |\nabla u| |n - R| |\Gamma(x, y)| dg(y) \leq C \int \frac{1}{d(x, y)^{n-2}}dg(y).
\end{equation}

Here we just used the bound $|\Gamma(x, y)| \leq \frac{C}{d(x, y)^n}$, which was proven in [TZ]. From (2.14) and the volume non-inflating property, it is easy to see that $|I_2| \leq C$. Therefore,

\begin{equation}
|I_1| + |I_2| \leq C.
\end{equation}

Using the bound on $\Gamma(x, y)$ again, we have

\begin{equation}
|I_3| \leq C \int |\nabla u|^2 |Hess u| \frac{1}{d(x, y)^{n-2}}dg(y) \leq C.
\end{equation}

Applying the gradient bound $|\nabla \Gamma(x, y)| \leq \frac{C}{d(x, y)^n}$ in [TZ], we find

\begin{equation}
|I_4| \leq C \int \frac{|\nabla u|^3}{d(x, y)^{n-1}}dg(y) \leq C
\end{equation}

where we have used the volume non-inflating property again.

Similarly,

\begin{equation}
|I_5| + |I_6| + |I_7| \leq C + C \int \frac{|R(n - R)(y)|}{d(x, y)^{n-2}} + C \int \frac{|R \nabla u|}{d(x, y)^{n-1}}dg(y) \leq C.
\end{equation}

Substituting (2.15), (2.16), (2.17) and (2.18) into (2.13), we deduce

\begin{equation}
2 \int \Gamma(x, y)|Hess u|^2 dg(y) \leq C \int |\nabla u|^2 |Hess u| \frac{1}{d(x, y)^{n-2}}dg(y).
\end{equation}

On the other hand, by Lemma 2.2, there are uniform constants $r_0, C > 0$ such that $\Gamma(x, y) \geq C/d(x, y)^{n-2}$ when $d(x, y) \leq r_0$. Hence

\begin{equation}
\int_{d(x, y) \leq r_0} \frac{|Hess u(y)|^2}{d(x, y)^{n-2}} \leq \frac{1}{C} \int_{d(x, y) \leq r_0} |Hess u(y)|^2 \Gamma(x, y) dg(y)
\end{equation}

\[= \frac{1}{C} \int_{M} |Hess u(y)|^2 \Gamma(x, y) dg(y) - \frac{1}{C} \int_{d(x, y) \geq r_0} |Hess u(y)|^2 \Gamma(x, y) dg(y) \]
\[\leq \frac{1}{C} \int_{M} |Hess u(y)|^2 \Gamma(x, y) dg(y) + C \int_{d(x, y) \geq r_0} \frac{1}{d(x, y)^{n-2}} dg(y).\]

Here we just used the upper bound $|\Gamma(x, y)| \leq C/d(x, y)^{n-2}$ in [TZ]. This bound and (2.19) then imply that

\begin{equation}
\int_{d(x, y) \leq r_0} \frac{|Hess u(y)|^2}{d(x, y)^{n-2}} \leq C \int_{d(x, y) \geq r_0} \frac{|Hess u(y)|^2}{d(x, y)^{n-2}} + C \int_{M} |\nabla u|^2 |Hess u(y)| \frac{1}{d(x, y)^{n-2}} dg(y).
\end{equation}
This implies
\[
\int_M \frac{|\text{Hess } u(y)|^2}{d(x,y)^{n-2}} \, dg(y)
\leq (C + 1) \int_{d(x,y) \geq r_0} \frac{|\text{Hess } u(y)|^2}{d(x,y)^{n-2}} + C \int_M \frac{|
abla u|^2 \left| \text{Hess } u(y) \right|}{d(x,y)^{n-2}} \, dg(y)
\]
\[
\leq (C + 1)r_0^{-(n-2)} \int_M |\text{Hess } u(y)|^2 + C \int_M \frac{|
abla u|^2 \left| \text{Hess } u(y) \right|}{d(x,y)^{n-2}} \, dg(y).
\]

In Lemma 2.1 we take \( r = r_0 \). By the uniform diameter bound of \( M \), we know that
\[
\int |\text{Hess } u|^2 \, dg < C,
\]
which together with (2.22), show that
\[
(2.23) \quad \int_M \frac{|\text{Hess } u(y)|^2}{d(x,y)^{n-2}} \leq (C + 1)r_0^{-(n-2)} \int_M |\text{Hess } u(y)|^2 + C \int_M \frac{|
abla u|^2 \left| \text{Hess } u(y) \right|}{d(x,y)^{n-2}} \, dg(y).
\]

After using Cauchy-Schwarz inequality, we deduce
\[
(2.24) \quad \int_M \frac{|\text{Hess } u(y)|^2}{d(x,y)^{n-2}} \leq C + C \int_M \frac{|
abla u|^4}{d(x,y)^{n-2}} \, dg(y) \leq C.
\]
This proves the lemma. \( \square \)

The next lemma is an embedding result which implies that \(|\text{Hess } u|^2\), regarded as a potential function or inhomogeneous term in an equation, is dominated by the Laplacian.

**Lemma 2.4.** Let \( V \) be a smooth function on \( M \), \( p \) be a point on \( M \) and \( r \) be a positive number such that \( r \leq \text{diam}(M)/2 \). Then for any smooth function \( f \) on \( M \), the following embedding result holds
\[
\int_{B(p,r)} |V(x)| f^2(x) \, dg(x)
\leq C \sup_{x \in B(p,2r)} \int_{B(x,2r)} \frac{|V(x)|}{d(x,y)^{n-2}} \, dg(x) \left( \|\nabla f\|_{L^2(B(p,2r))}^2 + r^{-2}\|f\|_{L^2(B(p,2r))}^2 \right).
\]
Here \( C \) is a positive constant depending only on the parameters in the basic assumption for \( M \).

In particular, for the Ricci potential \( u \), it holds
\[
(2.26) \quad \int_{B(p,r)} |\text{Hess } u|^2 f^2(x) \, dg(x) \leq CK\left(|\text{Hess } u|^2\right) \left( \|\nabla f\|_{L^2(B(p,2r))}^2 + r^{-2}\|f\|_{L^2(B(p,2r))}^2 \right).
\]
Here \( K(|\text{Hess } u|^2) \) is defined in (2.10).

**Proof.**

With the gradient bound for the Green’s function and volume lower and upper bound in hands, the lemma and its proof are essentially known in the literature. See [Si] and [CGL] for example. The only difference is that we are dealing with compact manifolds which create one extra term from the Green’s formula. Therefore we will just sketch the proof.
Without loss of generality we assume \( V \geq 0 \). Let \( \phi \) be a Lipschitz cut-off function in \( B(p, 2r) \) such that \( \phi = 1 \) on \( B(p, r) \) and that \( |\nabla \phi| \leq C/r \). Then from the Green’s formula, we have

\[
(2.27) \quad f \phi(x) - \text{ave}(f \phi) = - \int \Gamma(x, y) \Delta(f \phi)(y) dg(y)
\]

where \( \text{ave}(f \phi) \) is the average of \( f \phi \) over \( M \), and \( \Gamma \) is the Green’s function. After integration by parts, this becomes

\[
(2.28) \quad f \phi(x) = \text{ave}(f \phi) + \int \nabla_y \Gamma(x, y) \nabla(f \phi)(y) dg(y).
\]

By the bound for \( \nabla \Gamma \) in [18], this infers

\[
(2.29) \quad |f \phi(x)| \leq |\text{ave}(f \phi)| + \int C|\nabla f \phi(y)| d(x, y)^{n-1}dg(y) + \int C|f \nabla \phi(y)| d(x, y)^{n-1}dg(y).
\]

For any smooth test function \( \eta \) supported in \( B(p, r) \), we have, from the previous inequality,

\[
\int \sqrt{V} f \eta(x) dg(x)
\]

\[
\leq C \int \sqrt{V} \eta(x) \int \frac{|\nabla f \phi(y)|}{d(x, y)^{n-1}} dg(y) dg(x) + C \int \sqrt{V} \eta(x) \int \frac{|f \nabla \phi(y)|}{d(x, y)^{n-1}} dg(y) dg(x)
\]

\[
+ \int \sqrt{V} \eta(x) \text{ave}(f \phi) dg(x)
\]

\[
\equiv I_1 + I_2 + I_3.
\]

Following the argument in [14] and [CGL], one can show that

\[
(2.31) \quad I_1^2 \leq C\|\nabla f\|_{L^2(B(p, 2r))}^2 \|\eta\|_{L^2(B(p, r))}^2 \sup_{z \in B(p, 2r)} \int \frac{V(x)}{d(x, z)^{n-2}} dg(x).
\]

\[
(2.32) \quad I_2^2 \leq \frac{C}{r^2} \|\nabla f\|_{L^2(B(p, 2r))}^2 \|\eta\|_{L^2(B(p, r))}^2 \sup_{z \in B(p, 2r)} \int \frac{V(x)}{d(x, z)^{n-2}} dg(x).
\]

Next

\[
(2.33) \quad |I_3| \leq \int \sqrt{V} \eta(x) \left( \frac{1}{|M|} \right) \int |f \phi(y)| dg(y) dg(x)
\]

\[
\leq \int \sqrt{V} \eta(x) \left( \frac{\text{diam}(M)^{n-1}}{|M|} \right) \int \frac{|f \phi(y)| dg(y)}{d(x, y)^{n-1}} dg(x)
\]

\[
\leq C \int \sqrt{V} \eta(x) \int \frac{|f \phi(y)| dg(y)}{d(x, y)^{n-1}} dg(x),
\]

where we have used Perelman’s bound on the diameter and volume non-collapsing property. Just like the case for \( I_1 \), we can now deduce

\[
(2.34) \quad |I_3|^2 \leq C\|f\|_{L^2(B(p, 2r))}^2 \|\eta\|_{L^2(B(p, r))}^2 \sup_{z \in B(p, 2r)} \int \frac{V(x)}{d(x, z)^{n-2}} dg(x).
\]
Substituting (2.31), (2.32) and (2.33) into (2.30), we find that
\[
\int \sqrt{V} f \eta(x) dg(x) \leq C \left[ \left\| \nabla f \right\|_{L^2(B(p,r))} + \frac{1 + r}{r} \left\| f \right\|_{L^2(B(p,2r))} \right] \left\| \eta \right\|_{L^2(B(p,r))} \left( \sup_{z \in B(p,2r)} \frac{V(x)}{d(x,z)^{n-2}} dg(x) \right)^{1/2}.
\]
Since \( \eta \) is arbitrary, the result follows by applying the Riesz theorem. \( \square \)

3. \( C^{1,\alpha} \) bounds for harmonic functions and Ricci potential in harmonic charts

The goal of this section is to prove \( C^{1,\alpha} \) bounds for a harmonic function \( h \) and Ricci potentials in a harmonic coordinate chart. We will use De Giorgi’s method. In doing so, we first need to tackle two technical issues. One is that the De Giorgi method does not work for systems of equations in general. This can be handled since the system for \( dh \) is a weakly coupled one involving the Ricci curvature and by the integral bound for the Ricci curvature in the previous section. The other issue is that we need a weighted \( L^p \) Poincaré type inequality for some \( p \) strictly less than 2.

**Lemma 3.1.** Given a proper geodesic ball \( B = B(p,r) \) and two functions \( v \in C^\infty(B) \) and \( 0 \leq f \in L^\infty(B) \), there exist a positive number \( p_0 < 2 \) and constant \( C > 0 \) such that
\[
\int_B |v - v_f|^{p_0} dg \leq C r^{p_0} \left[ \sup_{f_{p_0}} \frac{|B|}{|f|_{L^1(B)}} \right] \left[ \int_B |\nabla v|^{p_0} dg \right].
\]
Here \( v_f \) is the average of \( v \) under weight \( f \), i.e. \( v_f = \int_B v f dg / \int_B f dg \).

**Proof.**

Under the basic assumptions, as shown in Section 2 of [TZq], the following unweighted \( L^2 \) Poincaré inequality is true: for all smooth \( v \) on \( B \)
\[
\int_B |v - v_B|^2 dg \leq C r^2 \int_B |\nabla v|^2 dg.
\]
Here \( v_B \) is the average of \( v \) on the ball \( B \). Since the manifold is volume doubling, using the general result of [KZ], one can find a positive number \( p_0 < 2 \) such that the unweighted \( L^{p_0} \) Poincaré inequality holds: for all smooth \( v \) on \( B \)
\[
\int_B |v - v_B|^{p_0} dg \leq C r^{p_0} \int_B |\nabla v|^{p_0} dg.
\]
Here the constant \( C \) may have changed. Let us mention that the self improvement property of the Poincaré inequalities was stated for the so called \( 1 - q \) type, which means the left hand side is in \( L^1 \) norm. However, as pointed out in that paper, this property also holds for the above \( 2 - 2 \) type Poincaré inequality.

By the triangle inequality
\[
\int_B |v - v_f|^{p_0} dg \leq 2^{p_0} \left( \int_B |v - v_B|^{p_0} dg + \int_B |v_B - v_f|^{p_0} dg \right).
\]
Observe that the second term inside the parentheses on the right hand side satisfies
\begin{equation}
\int_B |v_B - v_f|^{p_0} dg = |v_B - v_f|^{p_0} |B|
= \left| \int_B v_B fdg - \int_B v fdg \right|^{p_0} |B|
= |B| \left( \int_B fdg \right)^{p_0} \left( \int_B (v - v_B) fdg \right)^{p_0} |B|
\leq |B| \|f\|_{L^1(B)}^{p_0} \|f\|_{L^\infty(B)}^{p_0} \int_B |v - v_B|^{p_0} dg |B \cap \text{supp } f|^{p_0 - 1}.
\end{equation}

Substituting this to the right hand side of (3.4) and using (3.3), we deduce
\begin{equation}
\int_B |v - v_f|^{p_0} dg \leq C \left[ |B| \|B \cap \text{supp } f|^{p_0 - 1} \|f\|_{L^1(B)}^{p_0} \|f\|_{L^\infty(B)}^{p_0} + 1 \right] r^{p_0} \int_B |\nabla v|^{p_0} dg,
\end{equation}
which proves the lemma. □

The next lemma is a rerun of De Giorgi’s method in our situation. We will closely follow the presentation in [Lie] Chapter VI, Section 12. Since there are certain differences from the Euclidean setting, due to the lack of local Poincaré inequality, we will present a detailed proof. First let us define a De Giorgi class, which is not the most general one, but which is sufficient for our setting.

**Definition 3.2.** Let $D$ be a domain in $M$. A function $v \in W^{1,2}(D)$ is said to be in the De Giorgi class in $D$ if the following inequalities hold. For all numbers $\sigma \in (0,1)$, $k$ and $B(p,r) \subset D$,
\begin{equation}
\int_{B(p,\sigma r)} |\nabla (v - k)^+|^2 dg \leq \frac{\beta_1}{(1 - \sigma)^2 r^2} \int_{B(p,r)} |(v - k)^+|^2 dg + \beta_0 |D_k^+|,
\end{equation}
\begin{equation}
\int_{B(p,\sigma r)} |\nabla (v - k)^-|^2 dg \leq \frac{\beta_1}{(1 - \sigma)^2 r^2} \int_{B(p,r)} |(v - k)^-|^2 dg + \beta_0 |D_k^-|.
\end{equation}

Here $\beta_0, \beta_1$ are positive constants, $(v - k)^+ = \max\{v - k, 0\}$, $D_k^+ = \{x \in B(p,r) \mid v - k > 0\}$; $(v - k)^- = \max\{-(v - k), 0\}$, $D_k^- = \{x \in B(p,r) \mid v - k < 0\}$.

**Lemma 3.3.** Suppose a function $v$ is in the De Giorgi class in $B(p,2r) \subset M$. Then there exist a number $\alpha \in (0,1)$ and a positive constant $C$, depending only on $\beta_0, \beta_1$ and the constants in the basic assumptions, such that, for $\rho \in (0,r)$,
\begin{equation}
\text{Osc}_{B(p,\rho)} v \leq C[(\rho/r)^{\alpha} \text{Osc}_{B(p,\rho)} v + \beta_0 \rho].
\end{equation}
Here $\text{Osc}_{D} v$ is the oscillation of the function $v$ in the region $D$.

**Proof.**

The proof is divided into three steps.

**Step 1.** We prove the following claim: Let $v \geq 0$ be in the De Giorgi class in $B(p,2r)$ and let $\tilde{v} = v + \beta_0 r$. Suppose there exists positive numbers $\alpha < 1$ and $K$ such that
\begin{equation}
|x \in B(p,r) \mid \tilde{v} < K| \leq \alpha |B(p,r)|.
\end{equation}

Then, for any small number $\epsilon \in (0,1)$, there exists a number $\delta \in (0,1)$ such that
\begin{equation}
|x \in B(p,r) \mid \tilde{v} < \delta K| \leq \epsilon |B(p,r)|.
\end{equation}
Here δ depending only on b and the constants in the statement of the lemma. The claim says that if \( \bar{v} \) is larger than \( K \) in a portion of the ball, then the set where \( \bar{v} \) is much smaller than \( K \) has a small measure.

To prove the claim, we introduce two functions on \( B(p, r) \).

\[
(3.12) \quad w = w(x) = \begin{cases} 
0, & \text{if } \bar{v}(x) \geq a^i K, \\
2^{-i}K - \bar{v}(x) & \text{if } a^i K \geq \bar{v}(x) > a^{i+1} K, \\
2^{-i}K - a^{i+1} K & \text{if } \bar{v}(x) \leq a^{i+1} K.
\end{cases}
\]

Here \( i \) is a nonnegative integer.

\[
(3.13) \quad f = f(x) = \begin{cases} 
\{x \in B(p, r) | w(x) = 0\}^{-1} & \text{if } w(x) = 0, \ i.e. \ \bar{v} \geq a^i K, \\
0 & \text{if } w(x) \neq 0, \ i.e. \ \bar{v} < a^i K.
\end{cases}
\]

We also denote by \( A_i \) the subset of \( B(p, r) \) on which \( \bar{v} < a^i K \). Thus \( A_i^c \) is the set where \( w \) vanishes, and \( A_i \) is where \( f \) vanishes. Our assumption implies \( |A_i^c| \geq (1 - a)|B(p, r)| \) and also \( w f = 0 \). So the weighted average \( w f = 0 \), \( \|f\|_{L^1(B(p, r))} = 1 \), \( \|f\|_{L^\infty(B(p, r))} = |A_i^c|^{-1} \).

From these we can apply Lemma \[5.1\] to infer

\[
(3.14) \quad \int_{B(p, r)} w^{p_0} dg \leq C_r p_0 \sup_{p_0} \frac{f[B \cap supp f]^{p_0-1}}{\|f\|_{L^1(B)}^{p_0-1}} \int_B |\nabla w|^{p_0} dg
\]

which shows

\[
(3.15) \quad [a^i(1 - a)K]^{p_0} |A_{i+1}| \leq \int_{B(p, r)} w^{p_0} dg \leq C_r p_0 \int_{A_i - A_{i+1}} |\nabla \bar{v}|^{p_0} dg
\]

By Hölder inequality, this implies

\[
(3.16) \quad [a^i(1 - a)K]^{p_0} |A_{i+1}| \leq C_r p_0 \left( \int_{A_i} |\nabla \bar{v}|^2 dg \right)^{p_0/2} |A_i - A_{i+1}|^{(2-p_0)/2}.
\]

From the assumption that \( v \) is in De Giorgi class in \( B(p, 2r) \), we know, for all real numbers \( k \),

\[
(3.17) \quad \int_{B(p, r)} |\nabla (v - k)|^{-2} dg \leq \frac{\beta_1}{r^2} \int_{B(p, 2r)} |(v - k)|^{-2} dg + \beta_0 |D_k^-|.
\]

Here \( D_k^- = \{x \in B(p, 2r) | v - k < 0\} \). Hence

\[
(3.18) \quad \int_{B(p, r)} |\nabla (v - k - \beta_0 r)|^{-2} dg
\]

\[
= \int_{B(p, r)} |\nabla (v - k)|^{-2} dg \leq \frac{C}{r^2} \int_{\{x \in B(p, 2r) | v(x) < k\}} |k - v|^2 + (\beta_0 r)^2 |dg|
\]

\[
\leq \frac{C}{r^2} \int_{\{x \in B(p, 2r) | v(x) < k + \beta_0 r\}} (|k + \beta_0 r - \bar{v}|^2 + \bar{v}^2) dg.
\]
In the last step, we have used the fact that $\tilde{v} = v + \beta_0 r \geq \beta_0 r$. Taking $k$ so that $k + \beta_0 r = a^j K$, we then deduce
\begin{equation}
(3.19) \quad \int_{A_i} |\nabla \tilde{v}|^2 dg \leq C(a^j K)^2 r^{n-2}.
\end{equation}
Here we have used the volume upper bound of geodesic balls. Substituting this to the right hand side of (3.16), we find that
\begin{equation}
(3.20) \quad [a^i (1 - a) K]^{p_0} |A_{i+1}| \leq C(a^j K)^{p_0} r^{p_0/2} |A_i - A_{i+1}|^{(2-p_0)/2}.
\end{equation}
Cancelling the term $(a^j K)^{p_0}$ and writing $\theta_i = r^{-n} |A_i|$, we infer
\begin{equation}
(3.21) \quad \theta_i^{2/(2-p_0)} \leq C (1 - a)^{-2p_0/(2-p_0)} (\theta_i - \theta_{i+1}).
\end{equation}
Notice that $\theta_i \geq \theta_{i+1}$. By adding the above inequality from $i = 0$ to a positive integer $j - 1$, we conclude that
\begin{equation}
(3.22) \quad j \theta_j^{2/(2-p_0)} \leq C (1 - a)^{-2p_0/(2-p_0)} \theta_0 \leq C,
\end{equation}
which means
\begin{equation}
(3.23) \quad |\{x \in B(p, r) \mid \tilde{v}(x) < a^j K\}| \leq C j^{-1} |B(p, r)|.
\end{equation}
The claim is proven by choosing $j$ large enough so that $C j^{-1} \leq \epsilon$ and write $\delta = a^j$.

**Step 2.** We prove the following assertion: there exists a number $\epsilon \in (0, 1)$ such that if,
\begin{equation}
(3.24) \quad |\{x \in B(p, r) \mid \tilde{v}(x) < K\}| \leq \epsilon |B(p, r)|
\end{equation}
for some $K > 0$, then $\tilde{v} \geq K/2 - C_0 r$ on $B(p, r/2)$. Here $C_0$ is a constant depending only on the basic assumptions for $M$.

Following the proof in the Euclidean case verbatim, since $v$ is in the De Giorgi class, we know that the following mean value inequality is true:
\begin{equation}
(3.25) \quad \sup_{B(p, r/2)} [(K - \tilde{v})^+]^2 \leq C \int_{B(p, r/2)} [(K - \tilde{v})^+]^2 dg + C (\beta_0 r)^2.
\end{equation}
The proof just uses the $L^2$ Sobolev inequality and volume non-collapsing property of geodesic balls. We refer the reader to p43 of [3.20] for a detailed proof. Note that the proof there is for the more general parabolic case. The above mean value inequality shows
\begin{equation}
(3.26) \quad \sup_{B(p, r/2)} (K - \tilde{v})^2 \leq C \frac{r^n}{n} |\{x \in B(p, r) \mid \tilde{v}(x) < K\}| K^2 + C (\beta_0 r)^2 \leq C \epsilon K^2 + C (\beta_0 r)^2.
\end{equation}
If $C \epsilon < 1/8$ then for all $x \in B(p, r/2)$, the preceding inequality implies
\begin{equation}
(3.27) \quad \tilde{v}(x) \geq K/2 - C_0 r,
\end{equation}
which proves the assertion.

**Step 3.** We will finish the proof of the lemma.

Given a number $\rho \in (0, 2r]$, we write $M(\rho) = \sup_{x \in B(p, \rho)} v(x)$, $m(\rho) = \inf_{x \in B(p, \rho)} v(x)$ and $J(\rho) = M(\rho) - m(\rho)$ i.e. the oscillation of $v$ in $B(p, \rho)$. Consider the function $h = h(x) = v(x) - m(2r)$. Then $h$ is a nonnegative function in the De Giorgi class in $B(p, 2r)$. Note that
\begin{equation}
(3.28) \quad \{x \in B(p, r) \mid h(x) \leq J(2r)/2\} \cup \{x \in B(p, r) \mid h(x) > J(2r)/2\} = B(p, r).
\end{equation}
Thus we can assume, without loss of generality, that

$$\tag{3.29} \left| \{ x \in B(p, r) | h(x) \leq J(2r)/2 \} \right| \leq \frac{1}{2} |B(p, r)|,$$

since we can consider $J(2r) - h = M(2r) - v$ otherwise. Write $\tilde{h} = h + \beta_0 r$. Then

$$\tag{3.30} \left| \{ x \in B(p, r) | \tilde{h}(x) \leq J(2r)/2 + \beta_0 r \} \right| \leq \frac{1}{2} |B(p, r)|,$$

Let $A > 8\beta_0$ be a large positive number to be specified later. If $J(2r) \leq Ar$, then we stop and rerun the above process on the ball $B(p, r)$ instead. So we assume $J(2r) > Ar$. In this case, (3.30) implies

$$\tag{3.31} \left| \{ x \in B(p, r) | \tilde{h}(x) \leq 3J(2r)/4 \} \right| \leq \frac{1}{2} |B(p, r)|.$$

By Step 1, for any $\epsilon > 0$, there is $\delta > 0$, such that

$$\tag{3.32} \left| \{ x \in B(p, r) | \tilde{h}(x) \leq \delta J(2r)/4 \} \right| \leq \epsilon |B(p, r)|.$$

Choose $\epsilon$ to be the number in the assertion in Step 2 and $K = \delta J(2r)/4$. Then, for $x \in B(p, r/2)$, we have

$$\tag{3.33} \tilde{h}(x) \geq \delta J(2r)/8 - C_0 r \geq \delta J(2r)/4.$$

In the last inequality we used $J(2r) \geq Ar$ and choose $A$ sufficiently large. Therefore

$$\tag{3.34} v(x) \geq \delta J(2r)/4 - \beta_0 r + m(2r) \geq \delta J(2r)/8 + m(2r).$$

i.e.

$$\tag{3.35} m(r/2) \geq \inf_{x \in B(p, r/2)} v(x) \geq \delta J(2r)/8 + m(2r).$$

This implies

$$\tag{3.36} J(r/2) = M(r/2) - m(r/2) \leq M(2r) - m(2r) - \frac{\delta}{8} J(2r) = (1 - \frac{\delta}{8}) J(2r).$$

To summarize, we have proven that either $J(2r) \leq Ar$ or $J(r/2) \leq (1 - \frac{\delta}{8}) J(2r)$. Repeating this on the balls $B(p, r), B(p, r/2), ...$, we have proven the lemma is true. \(\square\)

Since we are concerned with only local properties when applying the above lemma, we will take the radius $r \leq 1$ in the rest of the section.

**Lemma 3.4.** Suppose the ball $B(p, 2r)$ is contained in a harmonic coordinate $\{ x^1, ..., x^n \}$. We assume that the metric satisfies, in the ball, the following $C^1$ bound. a). $e^{-\theta} I \leq (g_{pq}) \leq e^{\theta} I$, $I = (\delta_{pq});$ b). $\sup_{p,q} (r \| g_{pq} \|_{C^1}) \leq e^{\theta}$. Here $\theta > 0$.

Let $h$ be a harmonic function in $B(p, r)$ and $dh = h_i dx^i$. Then exist positive constants $\alpha_0 \in (0, 1)$ and $C$, which depend only on the parameters of the basic assumptions and $\theta$, such that

$$\tag{3.37} \| h_i \|_{C^{\alpha_0}(B(p, r))} \leq C \left( \frac{1}{r^{\alpha_0}} \| \nabla h \|_{L^\infty(B(p, 2r))} + 1 \right).$$

**Proof.**
We work in the real coordinate system. Since $\Delta h = 0$, we know that $dh$ is a harmonic one form, i.e. $dd^*dh = 0$. In the harmonic system, we write $dh = h_idx^i$. Since $x^i$ is a harmonic function, we know, for any constants $k_i$, that

$$\text{(dd}^* + d^*d)((h_i - k_i)dx^i) = 0.$$  

From this, the Weitzenbock formula implies

$$\Delta(h_i - k_i) - R_{ij}(h_j - k_j) = 0$$

where $\Delta$ is the rough Laplacian on 1 forms. Denoting $\eta = \eta_idx^i$ for the one form $(h_i - k_i)dx^i$, we will write down equation (3.39) for $\eta$ in the local system.

Let $\nabla_i$ be the covariant derivative in the $x^i$ direction, and $\partial_i$ be the partial derivative. Then

$$\nabla_i \eta = (\partial_i \eta_k - \eta \Gamma^l _{ik})dx^k.$$  

Here and later in this paragraph, with a slight abuse of notation, we use $k$ as an index rather than the free constant in the De Giorgi argument. But it should be clear from the context what $k$ means. Further more

$$\nabla_j (\nabla_i \eta) = \left[\partial_j (\partial_i \eta_k - \eta \Gamma^l _{ik}) - (\partial_i \eta_m - \eta \Gamma^l _{jm}) \Gamma^m _{jk}\right] dx^k.$$  

Recall that the local formula for the rough Laplacian of $\eta$ is

$$g^{ij} \nabla_j (\nabla_i \eta) + \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij}) \nabla_j \eta.$$  

See Section 10.1 of [Ni] e.g. In harmonic coordinates, it then takes the form

$$\Delta \eta = g^{ij} \nabla_j (\nabla_i \eta) = g^{ij} \left[\partial_j (\partial_i \eta_k - \eta \Gamma^l _{ik}) - (\partial_i \eta_m - \eta \Gamma^l _{jm}) \Gamma^m _{jk}\right] dx^k.$$  

Substituting this identity into (3.39), we find that

$$g^{ij} \partial_i \partial_j \eta_k - g^{ij} \partial_j (\eta \Gamma^l _{ik}) - g^{ij} \partial_i \eta_m \Gamma^m _{jk} + g^{ij} \eta \Gamma^l _{im} \Gamma^m _{jk} = g^{lm} R_{lk} \eta_m.$$  

Taking $\eta_k$ as a scalar function and $\Delta$ be the scalar Laplacian, then we have, in the harmonic coordinates

$$\Delta \eta_k - g^{ij} \partial_j (\eta \Gamma^l _{ik}) - g^{ij} \partial_i \eta_m \Gamma^m _{jk} + g^{ij} \eta \Gamma^l _{im} \Gamma^m _{jk} - g^{lm} R_{lk} \eta_m = 0.$$  

Let $\phi$ be a Lipschitz cut off function supported in $B(p, 2r)$ such that $\|\nabla \phi\|_\infty \leq C/r$. Next we fix the index $i$ and take $k_i$ to be a free constant $k$, and $k_j = 0$ for $j \neq i$. Denote by $(h_i - k)^+$ the positive part of $h_i - k$. Using $(h_i - k)^+ \phi^2$ as a test function in the equation (3.45), after switching the indices suitably and doing integration by parts, we find that

$$\int |\nabla [(h_i - k)^+ \phi]|^2 dg \leq C[\|\nabla \phi\|_\infty^2 + 1] \int_{\text{supp } \phi} [(h_i - k)^+]^2 + C\|\eta\|_{L^\infty} \|\Gamma^m _{jl}\|_{L^\infty} \int |\nabla [(h_i - k)^+ \phi]| dg + C\|\Gamma^m _{jl}\|_{L^\infty} \int (h_i - k)^+ \phi Hess h|\phi dg + \|\eta\|_{L^\infty} C\|\Gamma^m _{jl}\|_{L^\infty}^2 \int [(h_i - k)^+ \phi] dg + C\|\eta\|_{L^\infty} \int |\text{Ric} (h_i - k)^+ \phi^2| dg.$$
Here $\nabla$ stands for the gradient of a scalar function, $||\Gamma^m_{ij}||_{L^\infty}$ is the $L^\infty$ norm of Christoffel symbols. We have also used the fact

\begin{equation}
|g^{ij}\partial_i\eta_m| \leq C|\text{Hess } h| + Ce^\theta|\nabla h|.
\end{equation}

By the assumed $C^1$ bound for the metric, for any $\epsilon > 0$, we deduce

\begin{align}
\int |\nabla[(h_i - k)^+\phi]|^2 dg \\
\leq C (\|\nabla\phi\|_\infty^2 + 1) \int_{\text{supp } \phi} [(h_i - k)^+]^2 + \epsilon \int |\nabla[\text{Hess } h]|^2 [(h_i - k)^+\phi]^2 dg \\
+ Ce^{-1} \int_{D_k^+} \sum j\neq i (h_j - k_j)^2 dg.
\end{align}

Here $D_k^+ = \{x| h_i - k \geq 0\}$. Using the embedding Lemma 2.4, we see that

\begin{equation}
\int |\nabla[(h_i - k)^+\phi]|^2 dg \\
\leq C (\|\nabla\phi\|_\infty^2 + 1) \int_{\text{supp } \phi} [(h_i - k)^+]^2 + CeK (|\nabla[\text{Ric}]|^2 + |\text{Hess } h|^2) \int |\nabla[(h_i - k)^+\phi]|^2 dg \\
+ CeK (|\nabla[\text{Ric}]|^2 + |\text{Hess } h|^2)r^{-2} \int [(h_i - k)^+\phi]^2 dg + Ce^{-1} \|\nabla h\|_\infty^2 |D_k^+|.
\end{equation}

Here $K (|\nabla[\text{Ric}]|^2 + |\text{Hess } h|^2)$ is defined in (2.10), which is a bounded quantity by Lemma 2.3. Although we only proved $K (|\text{Hess } u|^2)$ is bounded, the proof for the boundedness of $K (|\text{Hess } h|^2)$ is the same, and actually simpler. Here $u$ is the Ricci potential.

Choosing $\epsilon = 2CK (|\nabla[\text{Ric}]|^2 + |\text{Hess } h|^2)|^{-1}$, we reach

\begin{equation}
\int |\nabla[(h_i - k)^+\phi]|^2 dg \leq C \left(\|\nabla\phi\|_\infty^2 + \frac{K^2(|\nabla[\text{Ric}]|^2 + |\text{Hess } h|^2)}{r^2} + 1\right) \int_{\text{supp } \phi} [(h_i - k)^+]^2 \\
+ CK^2 (|\nabla[\text{Ric}]|^2 + |\text{Hess } h|^2) \|\nabla h\|_\infty^2 |D_k^+|.
\end{equation}

In the same manner, we deduce, for $(h_i - k)^- = -\min \{0, h_i - k\}$, that

\begin{equation}
\int |\nabla[(h_i - k)^-\phi]|^2 dg \leq C \left(\|\nabla\phi\|_\infty^2 + \frac{K^2(|\nabla[\text{Ric}]|^2 + |\text{Hess } h|^2)}{r^2} + 1\right) \int_{\text{supp } \phi} [(h_i - k)^-]^2 \\
+ CK^2 (|\nabla[\text{Ric}]|^2 + |\text{Hess } h|^2) \|\nabla h\|_\infty^2 |D_k^-|.
\end{equation}

Here $D_k^- = \{x| h_i - k \leq 0\}$. Inequalities (3.50) and (3.51) mean that $h_i$ is in the De Giorgi class. By Lemma 3.3 we know that $h_i$ is $C^{\alpha_0}$ for some $\alpha_0 \in (0, 1)$ and (3.37) holds.

**Lemma 3.5.** Suppose the ball $B(p, 2r)$ is contained in a harmonic coordinate $\{z^1, \ldots, z^m\}$. Assume the same $C^1$ bound for the metric as in the previous lemma. Let $u$ be the Ricci potential and $u_i$ be the component of $du$ with respect to $dz_i$. Then exist positive constants $\alpha_0 \in (0, 1)$ and $C$, which depend only on the parameters of the basic assumptions and the $C^1$ bound for the metric, such that

\begin{equation}
\|u_i\|_{C^{\alpha_0}(B(p, r))} \leq C \left(\frac{1}{r^{\alpha_0}} \|\nabla u\|_{L^\infty(B(p, 2r))} + 1\right).
\end{equation}
Proof.

The proof is similar to the previous lemma. One difference is that in complex coordinates, the derivative of $u$ is complex valued. In order to apply the De Giorgi method we need to treat the real and imaginary parts separately. Another difference is that an extra term involving the scalar curvature. Let $\{z^1, ..., z^m\}$ be a complex harmonic coordinate. Recall that $\Delta u = m - R$. Write $du = u_i dz^i$ and let $k_i$ be real constants. Then

\begin{equation}
\Delta(u_i - k_i) = \frac{1}{2} R_{i\bar{j}}(u_j - k_j) - R_i, \quad \Delta(\bar{u}_i - k_i) = \frac{1}{2} R_{i\bar{j}}(u_j - k_j) - R_i,
\end{equation}

Adding these two equations together, we see that

\begin{equation}
\Delta(Re u_i - k_i) = \frac{1}{4} R_{i\bar{j}}(u_j - k_j) + \frac{1}{4} R_{i\bar{j}}(u_j - k_j) - Re R_i
\end{equation}

As in the previous lemma, we fix an index $i$ and take $k_i$ to be a real constant $k$, and $k_j = 0$ for all $j \neq i$. Pick a Lipschitz cut off function $\phi$ supported in $B(p, 2r)$ such that $|\nabla \phi| \leq C/r$. Using $(Re u_i - k_i)^+ \phi^2$ as a test function in (3.54), after carrying out the same local calculation as the previous lemma, we deduce

\begin{equation}
\int |\nabla[(Re u_i - k)^+ \phi]|^2 dg \leq C(\|\nabla \phi\|_\infty^2 + 1) \int_{D_k} [(Re u_i - k)^+ \phi]^2 dg + \int |Hess u_i|^2(Re u_i - k)^+ \phi^2 dg + \left| \int Re R_i(Re u_i - k)^+ \phi^2 dg \right|
\end{equation}

Doing integration by parts on the last term and using Cauchy-Schwarz inequality, we arrive at

\begin{equation}
\int |\nabla[(Re u_i - k)^+ \phi]|^2 dg \leq C(\|\nabla \phi\|_\infty^2 + 1) \int_{D_k} [(Re u_i - k)^+ \phi]^2 dg + \int |Hess u_i|^2(Re u_i - k)^+ \phi^2 dg + 2 \int |R| |\nabla[(Re u_i - k)^+ \phi^2]| dg.
\end{equation}

Here $D_k^+ = \{ x \in B(p, 2r) \mid Re u_i(x) - k \geq 0 \}$. By the embedding Lemma 2.4 we can turn the above inequality into

\begin{equation}
\int |\nabla[(Re u_i - k)^+ \phi]|^2 dg \leq C(\|\nabla \phi\|_\infty^2 + \|\nabla \phi\|_\infty^2 \|R\|_\infty^2 + 1) \int_{D_k} [(Re u_i - k)^+ \phi]^2 dg + C \left[ K^2(\|Hess u_i\|^2)(\|\nabla u\|_\infty^2 + 1) + \|R\|_\infty^2 \right] |D_k^+|.
\end{equation}
Likewise, we also have
\[
\int |\nabla [(Re u_i - k)^{-}\phi]|^2 dg
\]
\[
\leq C(\|\nabla \phi\|^2_{\infty} + \|\nabla \phi\|^2_{\infty} R^2 + 1) \int_{D_k} [(Re u_i - k)^{-}\phi]|^2 dg
\]
\[+ C [K^2(\|Hess u_i\|^2)(\|\nabla u_i\|^2_{\infty} + 1) + \|R\|^2_{\infty} |D_k^1|].\]
Therefore, \(Re u_i\) is in the De Giorgi class. Similarly, the same is true for \(Im u_i\). By Lemma 3.3 again, we now know that \(u_i\) is Hölder continuous, proving the lemma. \(\square\)

4. LOWER BOUND OF HARMONIC RADIUS

In the previous section we proved that the gradient of a harmonic function is Hölder continuous within a harmonic chart. In this section we will show that the harmonic chart can not be too small under certain conditions.

**Definition 4.1.** (maximal harmonic radius) Given numbers \(\theta > 0\) and \(\alpha \in (0, 1)\), and a point \(x \in M\) with metric \(g\). We define \(r^{\theta, \alpha}_g(x)\) to be the maximum radius \(l\) such that there exists, in the ball \(B(x, l)\), a \(C^{1,\alpha}\) harmonic coordinate \(X = (x^1, \ldots, x^n) : B(x, l) \to \mathbb{R}^n\), which satisfies the following properties.

\[
(4.1) \quad \begin{align*}
& a). \quad e^{-\theta} I \leq (g_{pq}) \leq e^{\theta} I, \quad I = (\delta_{pq}); \\
& b). \quad \sup_{p, q} l \|\nabla g_{pq}\|_{C^0} \leq e^{\theta} \\
& c). \quad \sup_{p, q} l^{1+\alpha} \|\nabla g_{pq}\|_{C^\alpha} \leq e^{\theta}.
\end{align*}
\]

**Lemma 4.2.** Let \((M, g)\) be a manifold satisfying the basic assumptions, and \(x\) be a point in \(M\). Let \(\theta > 0\) be any fixed number and \(\alpha \in (0, \alpha_0)\) where \(\alpha_0\) is the Hölder parameter in Lemmas 3.4 and 3.5.

Suppose either one of the following conditions holds.
1. the injectivity radius at each point of \(B(x, r)\) is greater than a number \(\iota_0 > 0\).
2. the geodesic ball \(B(x, r)\) is almost Euclidean, i.e. for a sufficiently small positive number \(\eta\), any \(y \in B(x, r)\) and \(0 < \rho < \text{dist}(y, \partial B(x, r))\), it holds
\[
(4.2) \quad |B(y, \rho)| \geq (1 - \eta) \omega_n \rho^n
\]
where \(\omega_n\) is the volume of the unit ball in \(\mathbb{R}^n\).
3. for a sufficiently small positive number \(\eta > 0\) and a given number \(\tau_0 > 0\), the infimum of the W entropy on \(B(x, r)\) satisfies
\[
(4.3) \quad \sup_{\tau \in (0, \tau_0]} \mu(g, \tau, B(x, r)) \geq -\eta.
\]

Then there exists a number \(\delta \in (0, 1)\) such that
\[
(4.4) \quad r^{\theta, \alpha}_g(y) \geq \delta \text{dist}(y, \partial B(x, r))
\]
for all \(y \in B(x, r)\). In particular the harmonic radius at \(x\) satisfies
\[
(4.5) \quad r^{\theta, \alpha}_g(x) \geq \delta r.
\]
Proof. We will use the blow up method to prove the lemma. The strategy follows that in the paper [An] Main Lemma 2.2, where Anderson proved the same conclusion under the assumption that the Ricci curvature is bounded and the injectivity radius lower bound. In the current situation, we do not know whether the Ricci curvature is bounded. To overcome this difficulty, we will use the regularity result proved in the previous lemmas. Another difference occurs since we assume condition 3 instead of the lower bound on the injectivity radius.

We divide the rest of the proof into three cases.

Case 1. Suppose Condition 1 holds.

The proof of this case consists of most of the work for the lemma, on which the proof of the other 2 cases are built upon.

Assuming the conclusion of the lemma is false, then there exists a sequence of pointed manifolds $(M_i, x_i, g_i)$ which satisfies the basic assumptions, but, for some $p_i \in B(x_i, r, g_i) \subset M_i$, we have

\[(4.6) \quad \frac{r_{g_i}^\theta \alpha (p_i)}{\text{dist}(p_i, \partial B(x_i, r, g_i))} = \inf \left\{ \frac{r_{g_i}^\theta \alpha (y)}{\text{dist}(y, \partial B(x_i, r, g_i))} \mid y \in B(x_i, r, g_i) \right\} = \delta_i \to 0.\]

Notice that $p_i$ are the worst points in the sense that for every other point $y \in B(x_i, r, g_i)$, we have

\[(4.7) \quad \frac{r_{g_i}^\theta \alpha (y)}{\text{dist}(y, \partial B(x_i, r, g_i))} \geq \frac{r_{g_i}^\theta \alpha (p_i)}{\text{dist}(p_i, \partial B(x_i, r, g_i))}.\]

For simplicity, we write

\[(4.8) \quad \lambda_i = \text{dist}(p_i, \partial B(x_i, r, g_i)).\]

From the Definition 4.1, the following statement holds: if $\rho > \delta_i \lambda_i$, then there are no harmonic coordinate systems satisfying bounds (4.1) on balls $B(p_i, \rho, g_i)$.

Now, consider the scaled metric

\[(4.9) \quad h_i = (\delta_i \lambda_i)^{-2} g_i.\]

Then by the above discussion, for every point $y \in B(p_i, \delta_i^{-1}, h_i)$, the ball $B(y, 1, h_i)$ is contained in a harmonic coordinate such that on this same ball and under $h_i$, we have

\[(4.10) \quad e^{-\theta} I \leq ((h_i)_{pq}) \leq e^\theta I \quad \text{and} \quad \sup_{p,q} (\| (h_i)_{pq} \|_{C^{1,\alpha}}) \leq e^\theta.\]

However, for any number $\rho > 1$,

\[(4.11) \quad \text{there is no harmonic coordinate system containing } B(p_i, \rho, h_i) \text{ which satisfies (4.10) on } B(p_i, \rho, h_i) \text{ under the metric } h_i.\]

By Lemma 2.1 in [An], there is a subsequence of the triple $\{B(p_i, \delta_i^{-1}, h_i), p_i, h_i\}$, still denoted by the same notation, which converges in $C^{1,\alpha'}_{loc}$ topology to a complete, pointed manifold $(N, z, h)$. Here $\alpha'$ is any positive number strictly less than $\alpha$. Also $h$ is a $C^{1,\alpha}$ metric. Since the original manifold has uniformly $C^1$ Ricci potential $u_i$, it is easy to see that the limit manifold is Ricci flat. The reason is that the components of Ricci curvature $R_{pq}$ does not change after scaling so, under the metric $h_i$, it becomes

\[(4.12) \quad R_{pq} = -\partial_p \partial_q u_i + (\delta_i \lambda_i)^2 (h_i)_{pq}.\]
But
\begin{equation}
\left| \nabla_{h_i} u_i \right| = \delta_i \lambda_i \left| \nabla_{g_i} u_i \right| \to 0
\end{equation}
as \(i \to \infty\). Therefore the limiting manifold is Ricci flat since its Ricci potential is a constant.

Condition 1 then implies that the injectivity radius at \(z\) is infinity under the metric \(h\). The Cheeger-Gromoll splitting theorem tells us that \((N, z, h)\) is in fact the standard Euclidean space.

We claim that, by choosing a subsequence suitably, we can ensure that the above convergence is actually in the original \(C^{1,\alpha}_{loc}\) topology. Taking the claim for granted, then, for the fixed \(\theta\) and any fixed \(\rho > 1\), we know that (4.14) holds in \(B(p_i, \rho, h_i)\) when \(i\) is sufficiently large. This is a contradiction to (4.11), which would prove the lemma.

Since \(\alpha < \alpha_0\), bounded sets in \(C^{1,\alpha}_{loc}\) topology are compact in \(C^{1,\alpha}_{loc}\) topology. Hence the claim holds if we can prove that, for any ball \(B(w, 1, h_i) \subset B(p_i, 2\delta_i^{-1}, h_i)\), the metric \(h_i\) satisfies, for a number \(\alpha_1 \in (\alpha, \alpha_0)\),
\begin{equation}
\sup_{p,q} \left\| (h_i)_{pq} \right\|_{C^{1,\alpha_1}(B(w, 1/2, h_i))} \leq C
\end{equation}
for a uniform constant \(C = C(\alpha_1)\). So, in order to verify the lemma, we just need to prove (4.14).

For the rest of the proof, we will suppress the index \(i\) in the metric \(h_i\). Since \(B(w, 1, h)\) is contained in a harmonic coordinate, say, \(\{x^1, ..., x^n\}\), we know that the components of \(h\) satisfies
\begin{equation}
h^{jk}_{i} \frac{\partial^2 h_{rs}}{\partial x^j \partial x^k} + Q(h_{pq}, \frac{\partial h_{pq}}{\partial x^m}) = (\text{Ric}_h)_{rs},
\end{equation}
where \(Q\) is a quadratic term involving the first covariant derivative of \(h\); \(\text{Ric}_h\) is the Ricci curvature with respect to \(h\). For simplicity we write the above equation as
\begin{equation}
\Delta h_{rs} + Q(h, \nabla h) = R_{rs}.
\end{equation}
Let \(\phi\) be a Lipschitz cut-off function in \(B(w, 1, h)\) such that \(\phi = 1\) in \(B(w, 2/3, h)\) and \(|\nabla \phi| \leq 4\). Let \(\Gamma\) be the Green’s function on the whole manifold \((M, h)\). Using Green’s formula, it is easy to see that
\begin{equation}
\nabla_{x_j}(\phi h_{rs}(x)) = \int \nabla_{x_j} \Gamma(x, y) Q(\nabla h) \phi(y) dg(y) - 2 \int \nabla_{x_j} \Gamma(x, y) \nabla \phi \nabla h_{rs}(y) dg(y)
- \int \nabla_{x_j} \Gamma(x, y) h_{rs} \Delta \phi dg(y) - \int \nabla_{x_j} \Gamma(x, y) R_{rs} \phi(y) dg(y)
\equiv I_1 + I_2 + I_3 + I_4.
\end{equation}
For \(x \in B(y, 1/2)\) we will prove that \(I_1, ..., I_4\) are uniformly bounded in \(C^{\infty}\) norm. Here and later in the proof, when mentioning balls, distance etc we will suppress the reference to the underlying metric \(h\), unless stated otherwise.

By Lemma 3.4, we know that for \(z_1, z_2 \in B(w, 1/2)\) and \(y \in B(w, 1)\), the following inequalities hold: if \(d(z_1, y) \leq d(z_2, y)\) then
\begin{equation}
|\nabla_{x_j} \Gamma(z_1, y) - \nabla_{x_j} \Gamma(z_2, y)| \leq C \left[ d(z_1, z_2) / d(z_1, y) \right]^{\alpha_0} d(z_1, y)^{-(n-1)};
\end{equation}
if \(d(z_1, y) \leq d(z_2, y)\), then

\[
|\nabla_x \Gamma(z_1, y) - \nabla_x \Gamma(z_2, y)| \leq C[d(z_1, z_2)/d(z_2, y)]^{\alpha_0} d(z_2, y)^{(n-1)},
\]

(4.19)

Here is the proof. Assume \(d(z_1, y) \leq d(z_2, y)\). If also \(d(z_1, y) \geq 2d(z_1, z_2)\), we just apply Lemma 3.4 on the ball \(B(z_1, 2d(z_1, y)/3)\), which says

\[
|\nabla_x \Gamma(z_1, y) - \nabla_x \Gamma(z_2, y)| \leq C[d(z_1, z_2)/d(z_1, y)]^{\alpha_0} \sup_{z \in B(z_1, 2d(z_1, y)/3)} |\nabla_z \Gamma(z, y)| + C d(z_1, z_2)^{\alpha_0}.
\]

(4.20)

This implies (4.18) by the gradient bound on the Green’s function. On the other hand, if \(d(z_1, y) \leq 2d(z_1, z_2)\), then by this gradient bound again, we have

\[
|\nabla_x \Gamma(z_1, y)| \leq C d(z_1, y)^{-(n-1)} \leq C[d(z_1, z_2)/d(z_1, y)]^{\alpha_0} d(z_1, y)^{-(n-1)},
\]

(4.21)

and, since \(d(z_1, y) \leq d(z_2, y)\) by assumption,

\[
|\nabla_x \Gamma(z_2, y)| \leq C d(z_2, y)^{-(n-1)} \leq C d(z_1, z_2)^{-(n-1)} \leq C[d(z_1, z_2)/d(z_1, y)]^{\alpha_0} d(z_1, y)^{-(n-1)}.
\]

These two inequalities also imply (4.18). Similarly we can prove (4.19). We remark that the controlling parameters for the Ricci curvature etc do not become worse when the metric is magnified. Recall that the metric \(h\), which is \(h_i\) after restoring the index, is an magnification of the original metric. Therefore we have uniform constants through out.

In the above we have also used assumption (4.10) (a) which allows us to bound, in the same time, the Hölder norm of the gradient of \(\Gamma\) and the components of \(d\Gamma\) in the coordinate.

We observe, due to (4.10), that \(Q(h, \nabla h)\) is a bounded function on \(B(m, 1)\). Hence, from (4.18) and (4.19), we see that

\[
\|I_1\|_{C^0(B(m, 1/2))} \leq C \sup_{x \in B(m, 1/2)} \int_{B(m, 1)} d(z, y)^{-n+1-\alpha_0} dg(y) \leq C,
\]

(4.23)

where we have used the volume non-inflating property.

Next comes \(I_2\). Since \(|\nabla \phi| \leq 4\) and \(|\nabla h_{rs}|\) is bounded by (4.10) again, we know, from (4.18) and (4.19) that

\[
\|I_2\|_{C^0(B(m, 1/2))} \leq C \sup_{x \in B(m, 1/2)} \int_{B(m, 1)} d(z, y)^{-n+1-\alpha_0} dg(y) \leq C.
\]

(4.24)

Now we work on \(I_3\) which should be written as

\[
I_3 = \int h_{kl}(y) \nabla_x \nabla_{y_k} \Gamma(x, y) \nabla_{y_l} \phi(y) h_{rs}(y) dg(y)
\]

(4.25)

\[+ \int \nabla_x \Gamma(x, y) h_{kl}(y) \nabla_{y_k} \phi(y) \nabla_{y_l} h_{rs}(y) dg(y) \equiv I_{31} + I_{32}.
\]

For fixed \(y\), as a the scalar function of \(x\), we know that

\[
\nabla_{y_k} \Gamma(x, y) \equiv h(\nabla \Gamma(x, y), \frac{\partial}{\partial y_k})
\]

(4.26)

is a harmonic function of \(x \neq y\). This can be seen by picking a curve in \(c = c(s)\) on the manifold, whose tangent at \(s = 0\) is \(\frac{\partial}{\partial y_k}\) and differentiating with respect to \(s\) on the identity

\[
\Delta_x \Gamma(x, c(s)) = 0.
\]

(4.27)
For $y \in B(m, 2/3)^c$ and $z_1, z_2 \in B(m, 1/2)$, we can apply Lemma 3.4 on $\nabla y_k \Gamma(x, y)$ to deduce
\[
|\nabla x_j \nabla y_k \Gamma(z_1, y) - \nabla x_j \nabla y_k \Gamma(z_2, y)|
\leq C \max \{[d(z_1, z_2)/d(z_1, y)]^\alpha d(z_1, y)^{-n}, [d(z_1, z_2)/d(z_2, y)]^\alpha d(z_2, y)^{-n}\}
\leq C d(z_1, z_2)^{-\alpha_0}.
\]
(4.28)

Here we also used the bound
\[
|\nabla x_j \nabla y_k \Gamma(x, y)| \leq \frac{C}{d(x, y)^n}
\]
(4.29)

which can be proven by using the gradient bound for the Green’s function twice on suitably chosen balls. We mention that $\nabla x_j \nabla y_k \Gamma(x, y)$ means the component in the $x_j$ direction of the gradient of the scalar function $\nabla y_k \Gamma(x, y)$.

Thus $\|I_{31}\|_{C^{\alpha_0}(B(m, 1/2))} \leq C$. Similar to the case of $I_2$, we also have $\|I_{32}\|_{C^{\alpha_0}(B(m, 1/2))} \leq C$. Therefore
\[
\|I_3\|_{C^{\alpha_0}(B(m, 1/2))} \leq C.
\]
(4.30)

Finally we come to $I_4$. Recall that $R_{rs} = -\partial_r \partial_s u + h_{rs}$ where $r$ and $s$ implicitly mean unbarred and barred index. Thus we can write, after integration by parts, (4.31)
\[
I_4 = -\int \nabla x_j \nabla r \Gamma(x, y)(\partial_s u(y) - \partial_s u(q)) \phi(y) dg(y)
= -\int \nabla x_j \Gamma(x, y)(\partial_s u(y) - \partial_s u(q)) \nabla r \phi(y) dg(y)
= -\int \nabla x_j \Gamma(x, y)(\partial_s u(y) - \partial_s u(q)) \phi \partial_r (\ln \sqrt{\det g}) dg(y)
= I_{41} + I_{42} + I_{43} + I_{44},
\]

where $q$ is a point to be chosen later. Also the scalar function $\nabla r \Gamma$ is defined in (4.26) and $\nabla r \phi$ is similarly defined.

Recall that in the ball $B(m, 1)$, $|\nabla u|$ is uniformly bounded, and by (4.10),
\[
e^{-\theta} I \leq (h_{pq}) \leq e^\theta I \quad \text{and} \quad \sup_{p, q}(\|h_{pq}\|_{C^{1, \alpha}}) \leq e^\theta.
\]
(4.32)

We can now follow the case for $I_2$ to show that
\[
\|I_{42}\|_{C^{\alpha_0}(B(m, 1/2))} + \|I_{43}\|_{C^{\alpha_0}(B(m, 1/2))} + \|I_{44}\|_{C^{\alpha_0}(B(m, 1/2))} \leq C.
\]
(4.33)

So we are left with treating $I_{41}$.

Pick $z_1, z_2 \in B(m, 1/2)$ which is divided into two regions
\[
B(m, 1/2) = D_1 \cup D_2 \equiv \{y \mid d(z_1, y) \leq d(z_2, y)\} \cup \{y \mid d(z_2, y) \leq d(z_1, y)\}.
\]
(4.34)

For $y \in D_1$, from (4.28), we have the following bounds. If $d(z_1, z_2) \geq d(z_1, y)$, then
\[
|\nabla x_j \nabla r \Gamma(z_1, y) - \nabla x_j \nabla r \Gamma(z_2, y)| \leq C d(z_1, y)^{-\alpha_0} d(z_1, y)^{-n};
\]
(4.35)

If $d(z_1, z_2) \geq d(z_1, y)$, then
\[
|\nabla x_j \nabla r \Gamma(z_1, y) - \nabla x_j \nabla r \Gamma(z_2, y)| \leq C d(z_1, y)^{-n}.
\]
(4.36)

Similarly
Therefore, for any positive number $\alpha_1 < \alpha_0$, if $d(z_1, z_2) \leq d(z_1, y)$, then
\begin{equation}
|\nabla_{x_j} \nabla_r \Gamma(z_1, y) - \nabla_{x_j} \nabla_r \Gamma(z_2, y)| \leq C[d(z_1, z_2)/d(z_1, y)]^{\alpha_1} d(z_1, y)^{-n};
\end{equation}
If $d(z_1, z_2) \geq d(z_1, y)$, then
\begin{equation}
|\nabla_{x_j} \nabla_r \Gamma(z_1, y) - \nabla_{x_j} \nabla_r \Gamma(z_2, y)| \leq C \frac{d(z_1, z_2)^{\alpha_1}}{d(z_1, y)^{n+\alpha_1}}.
\end{equation}

Similar, for $y \in D_1$, inequalities (4.35) and (4.37) still hold after switching $z_1$ with $z_2$.

Take $q = z_1$ in (4.31). By Lemma 3.5 and (4.32), we also have
\begin{equation}
|\partial_s u(y) - \partial_s u(z_1)| \leq C d^{\alpha_0}(z_1, y).
\end{equation}
From (4.35) and (4.37), we find that
\begin{equation}
|I_{41}(z_1) - I_{41}(z_2)| d(z_1, z_2)^{-\alpha_1}
\end{equation}
\begin{equation}
\leq C \int_{D_1} \frac{1}{d(z_1, y)^{n-\alpha_0+\alpha_1}} dg(y) + C \int_{D_2} \frac{1}{d(z_2, y)^{n-\alpha_0+\alpha_1}} dg(y) \leq C.
\end{equation}
From this and (4.38), we deduce that $\|I_4\|_{C^{\alpha_1}(B(m,1/2))} \leq C$. Hence we have proven that (4.14) is true. This proves the lemma in Case 1.

**Case 2.** Suppose Condition 2 holds.

The proof in this case differs with that of Case 1 only in the paragraph below (4.11).

Assuming the conclusion of the lemma is false, then there exists a sequence of pointed manifolds $(M_i, x_i, g_i)$ which satisfies the basic assumptions, but, for some $p_i \in B(x_i, r; g_i) \subset M_i$, we have
\begin{equation}
\frac{r^{\theta, \alpha}_{g_i}(p_i)}{\text{dist}(p_i, \partial B(x_i, r; g_i))} = \inf \left\{ \frac{r^{\theta, \alpha}_{g_i}(y)}{\text{dist}(y, \partial B(x_i, r; g_i))} \mid y \in B(x_i, r; g_i) \right\}
\end{equation}
\begin{equation}
= \delta_i \rightarrow 0.
\end{equation}
Again, write $\lambda_i = \text{dist}(p_i, \partial B(x_i, r; g_i))$ and consider the scaled metric $h_i = (\delta_i \lambda_i)^{-2} g_i$. Then, for every point $y \in B(p_i, \delta_i^{-1}, h_i)$, the ball $B(y, 1, h_i)$ is contained in a harmonic coordinate such that on this same ball and under $h_i$, we have
\begin{equation}
e^{-\theta} I \leq (h_i)_{pq} \leq e^{\theta} I \quad \text{and} \quad \sup_{p,q} \| (h_i)_{pq} \|_{C^{1,\alpha}} \leq e^\theta.
\end{equation}
However, for any number $\rho > 1$,
\begin{equation}
\text{there is no harmonic coordinate system containing } B(p_i, \rho, h_i)
\end{equation}
which satisfies (4.42) on $B(p_i, \rho, h_i)$ under the metric $h_i$.

By Lemma 2.1 in [An], there is a subsequence of the triple $\{B(p_i, 2\delta_i^{-1}, h_i), p_i, h_i\}$, still denoted by the same notation, which converges in $C^{1,\alpha}$ topology to a complete, pointed manifold $(N, z, h)$. Here $\alpha'$ is any positive number strictly less than $\alpha$. Also $h$ is a $C^{1,\alpha}$ metric. Since the original manifold has bounded Ricci potential, we have shown that the limit manifold is Ricci flat.

In the previous case, we use Condition 1 i.e. the lower bound of the injectivity radius and Cheeger-Gromoll splitting theorem to show that the limit manifold $(N, z, h)$ is Euclidean. Now, Condition 2 implies that the ratio between the volume of the balls in the limit manifold and the volume the Euclidean balls with the same radius is close to 1. Therefore
\( (N, z, h) \) must be the Euclidean space by [An]. Afterwards, going exactly as Case 1, we can finish Case 2.

**Case 3.** Suppose Condition 3 holds.

If the conclusion of the lemma is false, then again there exists a sequence of pointed manifolds \((M_i, x_i, g_i)\) which satisfies the basic assumptions, such that
\[
(4.44) \quad \sup_{0 < r \leq \tau_0} \mu(g_i, r, B(x_i, r, g_i)) = \eta_i \to 0.
\]

But, for some \( p_i \in B(x_i, r, g_i) \subset M_i \), we have
\[
(4.45) \quad \frac{r_{g_i}^{\theta, \alpha}(p_i)}{\text{dist}(p_i, \partial B(x_i, r, g_i))} = \inf \{ \frac{r_{g_i}^{\theta, \alpha}(y)}{\text{dist}(y, \partial B(x_i, r, g_i))} | y \in B(x_i, r, g_i) \} = \delta_i \to 0.
\]

Consider the scaled metric \( h_i = (\delta_i \lambda_i)^{-2} g_i \) where \( \lambda_i = \text{dist}(p_i, \partial B(x_i, r, g_i)) \) again. Then, for every point \( y \in B(p_i, \delta_i^{-1}, h_i) \), the ball \( B(y, 1, h_i) \) is contained in a harmonic coordinate such that on this same ball and under \( h_i \), we have
\[
(4.46) \quad e^{-\theta} I \leq ((h_i)_{pq}) \leq e^{\theta} I \quad \text{and} \quad \sup_{p, q} (\|(h_i)_{pq}\|_{C^{1, \alpha}}) \leq e^{\theta}.
\]

However, for any number \( \rho > 1 \),
\[
(4.47) \quad \text{there is no harmonic coordinate system containing } B(p_i, \rho, h_i)
\]
which satisfies \( (4.46) \) on \( B(p_i, \rho, h_i) \) under the metric \( h_i \).

By Lemma 2.1 in [An], there is a subsequence of the triple \( \{B(p_i, \delta_i^{-1}, h_i), p_i, h_i\} \), still denoted by the same notation, which converges in \( C^{1, \alpha'}_{\text{loc}} \) topology to a complete, pointed manifold \((N, z, h)\). Here \( \alpha' \) is any positive number strictly less than \( \alpha \). Also \( h \) is a \( C^{1, \alpha} \) metric. Since the original manifold has bounded Ricci potential, it is easy to see that the limit manifold is Ricci flat. It is well known that the following scaling property holds
\[
(4.48) \quad \mu(h_i, \frac{1}{2}, B(p_i, \delta_i^{-1}, h_i)) = \mu((\delta_i \lambda_i)^2 h_i, \frac{1}{2} (\delta_i \lambda_i)^2, B(p_i, \lambda_i, g_i))
\]
\[
= \mu(g_i, \frac{1}{2} (\delta_i \lambda_i)^2, B(p_i, \lambda_i, g_i)) \geq \mu(g_i, \frac{1}{2} (\delta_i \lambda_i)^2, B(x_i, r, g_i)).
\]

The last inequality holds since \( B(p_i, \lambda_i, g_i) \subset B(x_i, r, g_i) \) by the definition \( \lambda_i = \text{dist}(p_i, \partial B(x_i, r, g_i)) \).

Since \( \tau_0 \) is a fixed number, if \( i \) is sufficiently large, then \( \frac{1}{2} (\delta_i \lambda_i)^2 \leq \tau_0 \). By \( (4.44) \) we know that
\[
(4.49) \quad \mu(h_i, \frac{1}{2}, B(p_i, \delta_i^{-1}, h_i)) \geq -\eta_i
\]

Since \( \delta_i^{-1} \to \infty \) and \( \eta_i \to 0 \) when \( i \to \infty \), this implies
\[
(4.50) \quad \mu(h, \frac{1}{2}, N) \geq 0.
\]

According to Theorem 16.35 in the book [C++], we know that \((N, z, h)\) is in fact the standard Euclidean space. We should mention that this result was attributed to the paper [BCL], Corollary 1.6, where it was stated slightly differently.

Now we can finish the proof of the lemma just like Case 1. \( \square \)
5. PROOF OF THEOREMS

Proof of Theorem 1.2

With Lemma 4.2 in hand, we know that each manifold in the class SP has a harmonic coordinate atlas with uniform lower bound for the radii and uniform $C^{1,\alpha}$ bound for the metric. Hence Theorem 1.2 follows immediately from the general result Lemma 2.1 in [An]. □

Next we give

Proof of Theorem 1.3.

Under either one of the conditions, by Lemma 4.2, for fixed small $\theta > 0$ and each point $p \in M$, there is a fixed number $a > 0$ such that the ball $B(p, ar, 0)$ is contained in a harmonic coordinates. Moreover, the metric $g_{pq}$ satisfies (5.7). This shows, there exists a smaller positive number $b$ such that the isoperimetric constant of the ball $B(p, br, 0)$ is so close to the Euclidean one that Perelman’s pseudolocality theorem [P] Section 10 can be applied. The theorem follows. □

We finish by pointing out a possible smooth convergence result when the initial metric has entropy close to 0.

Observation. Let $(M, g(t))$, $\partial_t g_{ij} = -R_{ij} + g_{ij}$, be a Kähler Ricci flow on a $n$ real dimensional compact, Kähler manifold with positive first Chern class. Suppose also that the initial metric is so scaled that the unnormalized flow blows up exactly at time $\frac{1}{2}$. There exists a positive number $\eta_0$ with the following property. If $\mu(g(0), \frac{1}{2}, M) \geq -\eta_0$, then every sequence $\{(M, g(t_k))\}, t_k \to \infty$, sub-converges in $C^\infty$ topology to a gradient Kähler Ricci soliton. Moreover the curvature tensor is uniformly bounded for all time.

One may wonder for what initial metrics the unnormalized flow blows up exactly at time $\frac{1}{2}$? By Cao’s result [Ca] that the normalized flow exists for all time, after a simple change of time variable, we know that this is always true if $g(0)$ has canonical Kähler class, i.e. $2\pi c_1(M)$, as its Kähler class.

At this moment, we are not sure if there exist manifolds satisfying the conditions of the observation since we do not know the size of $\eta_0$. Since the argument reveals a connection between the entropies at initial time and large time, we present it for the interested reader.

Let $\tilde{g}$ and $\tilde{t}$ denote the metric and time of the corresponding un-normalized Ricci flow, which is made to blow up when $\tilde{t} = 1/2$. Then we have the relation

\begin{align}
(5.1) \quad t &= -\ln(1 - 2\tilde{t}), \quad g(t) = \frac{1}{1 - 2\tilde{t}} \tilde{g}(\tilde{t}).
\end{align}

From [P], $\forall \epsilon > 0$,

(5.2) $\mu(\tilde{g}(\tilde{t}), \epsilon) \geq \mu(g(0), \tilde{t} + \epsilon),$

and therefore

\begin{align}
(5.3) \quad \mu\left(\frac{1}{1 - 2\tilde{t}} \tilde{g}(\tilde{t}), \frac{1}{1 - 2\tilde{t}} \epsilon\right) \geq \mu(g(0), \tilde{t} + \epsilon)
\end{align}

which can be converted to

\begin{align}
(5.4) \quad \mu(g(t), e^t \epsilon) \geq \mu(g(0), \frac{1}{2} - \frac{1}{2} e^{-t} + \epsilon).
\end{align}

Writing $\tau = e^t \epsilon$, then this inequality becomes

\begin{align}
(5.5) \quad \mu(g(t), \tau) \geq \mu(g(0), \frac{1}{2} - \frac{1}{2} e^{-t} + e^{-t} \tau).
\end{align}
Since $(M, g(0))$ is a fixed manifold, given any fixed $\tau_0$, if $\tau \in (0, \tau_0]$, we can then deduce

$$\mu(g(t), \tau) \geq \mu(g(0), \frac{1}{2}) + o(1), \quad t \to \infty.$$  

(5.6)

By our assumption that $\mu(g(0), \frac{1}{2}) \geq -\eta$, with $\eta$ small, we know that the family $\{M, g(t)\}$, $t$ large, satisfies the conditions of Theorem 1.2 (b). This proves sub-convergence in $C^{1,\alpha}$ topology.

Now we prove the convergence in $C^\infty$ topology. Pick a sequence of times $t_k$ which go to 0 when $k \to \infty$. Let $\delta > 0$ be a small number to be chosen later. Then a subsequence of $\{(M, g(t_k - \delta))\}$, still identified by the same notation, converges in $C^{1,\alpha}$ topology. By Lemma 4.2, for fixed $\theta > 0$ and each point $p \in M$, there is a fixed number $r_0$ such that the ball $B(p, r_0, t_k - \delta)$ is contained in a harmonic coordinates. Moreover, the metric $g_{pq}$ in each of the ball satisfies,

$$\text{a). } e^{-\theta} I \leq (g_{pq}) \leq e^\theta I; \quad \text{b). } \sup_{p,q} (\|g_{pq}\|_{C^1} r_0) \leq e^\theta r_0; \quad \text{c). } \sup_{p,q} (\|g_{pq}\|_{C^{1,\alpha} r_0^{1+\alpha}}) \leq e^\theta.$$  

(5.7)

This shows, by choosing $\theta$ small, there exists a small positive number $\epsilon$ such that the isoperimetric constant of the ball $B(p, \epsilon r_0, g(t_k - \delta))$ is so close to the Euclidean one that Perelman’s pseudolocality theorem $[P]$ can be applied. This implies, if one chooses $\delta = c_0(\epsilon r_0)^2$ with $c_0 > 0$ sufficiently small, then the curvature tensor satisfies

$$|Rm(p, t)| \leq C(\epsilon r_0)^{-2}, \quad t \in [t_k - \delta, t_k + \delta].$$  

(5.8)

Since $p$ is arbitrary, we find that $|Rm(\cdot, t)|$, $t \in [t_k - \delta, t_k + \delta]$, is uniformly bounded. Hence the convergence is actually in $C^\infty$ topology. Using the result in $[Se]$, we know that the limit manifold is a gradient Kähler Ricci soliton. See also Lemma 3.5 in $[TZhu2]$ for a detailed proof.

Finally, since $\{t_k\}$ is an arbitrary time sequence, from (5.8), we see that the curvature tensor $Rm$ is uniformly bounded for all time. This confirms the observation.

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