Quons Restricted to the Antisymmetric Subspace:
Formalism and Applications

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Abstract

In this work we develop a formalism to treat quons restricted to the
antisymmetric part of their many-body space. A model in which a system
of identical quons interact through a pairing force is then solved within this
restriction and the differences between our solution and the usual fermionic
model solution are then presented and discussed in detail. Possible con-
nections to physical systems are also considered.

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1 Introduction

Quons are particles bearing statistics which interpolate between the fermionic
and the bosonic ones, depending on a deformation parameter $q$ which is de-
finite in the interval between $-1$ (fermionic limit) and $+1$ (bosonic limit). It can
be shown [1, 2] that the space spanned by a system of $n$ identical quons can
be separated into subspaces according to their permutational properties, that is,
the symmetric, antisymmetric and mixed symmetry subspaces. The restriction
of the dynamics to the symmetric subspace was recently investigated [2]. Ap-
plications were done for the harmonic oscillator and the rotor hamiltonians and
the results compared to the better known deformed algebra ones. Besides the
intrinsic interest on these applications, the projection of the dynamics onto the
symmetric subspace brings many formal simplifications and has a great advan-
tage as compared with the solution of boson-like systems in the whole quonic
space. This is the case for a system of bosons that are in fact composed by
fundamental fermions and can display deviations from a true bosonic behavior
under certain conditions. A good example are the excitons in the high density
regime [3], whose deviations come from the fermionic character of the underlying
particles, which can influence the dynamics if the system enters in a regime where
the fermion occupation scheme become important. In principle, we may describe
such systems in a natural way within the quon algebra, if we keep the $q$-values
close (but not equal) to $+1$. 
In the present work we concentrate our attention in the other limit of the interval, i.e., the region of $q$-values close to $-1$. In other words we restrict the dynamics to the antisymmetric subspace, useful when the particles that form the system have half-integer spin and so are fermion-like particles. We then keep the system close to the fermionic behavior, while we expect to cope with possible deviations through the deformation parameter. In section 2, we show the fundamental steps of our formalism for the restriction to the antisymmetric space, which follows the same lines as discussed in [2] for the symmetric case.

As an application, we then consider a pairing interaction model [4], in its one and two level versions. This pairing model, albeit a simple one, exhibits many rich features to test our formalism and has already been investigated with the help of quantum and quon algebras [6]. Our main results are then presented in section 3, together with a careful analysis of the energy gap behavior, characteristic of the pairing interaction. Specific physical cases in which the effects studied here are of interest will be the subject of future investigations.

2 Antisymmetric quon states

In this section we build the antisymmetric basis states for a system of $n$ quons and obtain the action of the annihilation operator of a quon on these states. This operation is crucial to obtain matrix elements of many-body operators. As it is well known the quon algebra is defined as

$$a_i a_j^\dagger - qa_j^\dagger a_i = \delta_{ij},$$

while the bare vacuum $|0\rangle$ satisfies the condition $a_i |0\rangle = 0$. An important quantity in this context is the deformed number, defined as

$$[n]_q = \frac{1 - q^n}{1 - q},$$

As far as we restrict our analysis to the antisymmetric subspace we will need the anti-deformed quantum number, hereafter defined as

$$\{n\}_q = [n]_{-q} = \frac{1 - (-q)^n}{1 + q}.$$  

In order to simplify our notation in future expressions we suppress the parameter $q$ from the above notation, considering it implicitly. Another definition to be used ahead is,

$$\{n\}_q! = \{n\}_q \{n-1\}_q ... \{0\}_q!,$$

with $\{0\}_q! = 1$. 

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Following [2] we build the antisymmetric state for \( n \) quons through an inductive procedure, starting from the antisymmetric state for \( n = 2 \) and \( n = 3 \). In the first case, the normalized antisymmetric state can be written as

\[
|i j; A >_N = \frac{1}{\sqrt{2(1 - q)}}(a_i^\dagger a_j^\dagger - a_j^\dagger a_i^\dagger)|0 >.
\]

Note that the subscript \( N \) means that the state is already normalized and the \( A \) inside the ket means that the state is completely antisymmetric under permutation of any particle label. The action of the annihilation operator on the above state gives

\[
a_i|i j; A >_N = \frac{1}{\sqrt{2(2)}}(a_i a_i^\dagger a_j^\dagger - a_j^\dagger a_j^\dagger)|0 > = \sqrt{\frac{2}{2}}a_j^\dagger|0 >,
\]

where equation (5) and the fact that \( i \neq j \) were used.

Following the same steps, we can show that for \( n = 3 \),

\[
a_i|i j k; A >_N = \frac{1}{\sqrt{3!\{3\}}!}a_i(a_i^\dagger a_j^\dagger a_k^\dagger - a_k^\dagger a_j^\dagger a_i^\dagger + a_i^\dagger a_j^\dagger a_k^\dagger - a_k^\dagger a_i^\dagger a_j^\dagger - a_j^\dagger a_k^\dagger a_i^\dagger - a_i^\dagger a_k^\dagger a_j^\dagger)|0 >
\]

or simply

\[
a_i|i j k; A >_N = \sqrt{\frac{3}{3}}|j k; A >_N.
\]

Continuing for larger numbers of quons, we define

\[
|i_1 i_2 ...i_n; A > = A a_{i_1}^\dagger a_{i_2}^\dagger ...a_{i_n}^\dagger|0 >,
\]

with \( A \) being the usual anti-symmetrizer operator which makes all the permutations of quon operators with the appropriate sign. We then finally find,

\[
a_t|i_1 i_2 i_3 ...i_n; A >_N = \sqrt{\frac{n!}{n}}|i_1 i_2 i_3 ...i_t ...i_{t+1} ...i_n; A >_N.
\]

The above equations have a complete analogy with the ones obtained in [3] for the symmetric quonic state and the general proof [4] follows the same lines presented in the Appendix of that reference. With this result in hand we may now obtain any necessary matrix element in order to get the observables of the theory.
3 Applications to the Pairing Model

Next, we want to apply the above results to solve the pairing model in a completely antisymmetric quonic basis. We analyze the one and two level pairing model. We present below the two level hamiltonian which can easily be reduced to the one level case. The model that we consider consists of two Ω fold degenerate levels (Ω = j + 1/2) whose energy difference is $\epsilon$. All the particles have angular momentum $j$, the lower level has single-particle states labelled $jm_2$ and the upper level has single-particle states labelled $jm_1$. The pairing Hamiltonian then reads:

$$
H = \epsilon (\hat{N}_1 - \hat{N}_2) - \frac{G \Omega}{2} \left[ (\hat{A}_1^\dagger + \hat{A}_2^\dagger)(\hat{A}_1 + \hat{A}_2) + (\hat{A}_1 + \hat{A}_2)(\hat{A}_1^\dagger + \hat{A}_2^\dagger) \right],
$$

(10)

where $N_k$ is half the usual number operator, and measures the number of pairs in each level $\{k\}$. We then have,

$$
\hat{N}_k = \frac{1}{2} \sum m_k a_{m_k}^\dagger a_{m_k},
$$

(11)

and $\hat{A}_k^\dagger$ creates a zero angular momentum pair

$$
\hat{A}_k^\dagger = \frac{1}{2\sqrt{\Omega}} \sum m_k (-1)^{j - m_k} a_{m_k}^\dagger a_{-m_k},
$$

(12)

where $k = 1, 2$.

This model originally built for fermions is modified when the quon commutation rule is imposed. This can be explicitly seen when we rewrite the hamiltonian (10), using equation (1),

$$
H = \epsilon (\hat{N}_1 - \hat{N}_2) - \frac{G \Omega}{2} \left[ (\hat{A}_1^\dagger + \hat{A}_2^\dagger)(\hat{A}_1 + \hat{A}_2) + (\hat{A}_1 + \hat{A}_2)(\hat{A}_1^\dagger + \hat{A}_2^\dagger) \right] (1 + q^4) + H',
$$

(13)

and where

$$
H' = \frac{G}{8}(q - 1)[4\Omega + 2(q^2 - \Omega)(\hat{N}_1 + \hat{N}_2)].
$$

(14)

Notice that the modifications come from the commutation rules and reflects the different occupation behavior of the quons. The two body pairing interaction is modified by the presence of the factor $(1 + q^4)/2$. In the fermionic case, the $H'$ term is proportional to the difference between the total number of pairs and the total pair degeneracy,

$$
H'_\text{fermions} = -G[\Omega - (\hat{N}_1 + \hat{N}_2)].
$$

(15)

while in the quonic case it displays clearly the modification on the occupation behavior of the quon particles giving a different weight for the number of pairs. When $q = -1$ we obtain the right fermionic limit for the pairing hamiltonian.
3.1 Two level pairing model solution

Before defining the basis states needed to solve the above Hamiltonian, it is worth mentioning that the operator introduced in equation (12) is a tensor operator of order zero once, as shown in [7], quons obey the usual angular momentum coupling rules. So we can talk about a state formed by \( N \) zero angular momentum quon pairs (or \( s \)-quon pairs), which defines the ground state. In other words, a system of \( N \) \( s \)-quon pairs should be of the form 

\[
|N; A > = \frac{\Omega^{N/2}}{N!} A(\hat{A}^\dagger)^N |0 > .
\]  

(16)

For the two level case, except for a normalization factor, our basis states are 

\[
|N_i, N_k; A > \approx A(\hat{A}_i^\dagger)^{N_i}(\hat{A}_k^\dagger)^{N_k} |0 > ,
\]  

(17)

where \( i \) and \( k \) can be either 1 or 2 and \( N = N_i + N_k \). The calculation of the matrix elements can be done using our results shown in section 2, which is a tedious but straightforward procedure. For the number operator we get 

\[
< A; N'_i, N'_j | \hat{N}_i N_j; A > = \frac{2N}{2N} N_i \delta_{N'_i; N_i} \delta_{N'_j; N_j} .
\]  

(18)

When \( q \neq -1 \) this result is different from \( N_i \), the number of pairs in the level \( i \). Thus the usual pair number operator \( \hat{N} \) when applied to the \( N \) quon pair state gives us a number that reflects the different quonic occupation scheme. We are going to interpret this result as an effective number of pairs or the number of quasi-pairs. Applying now the operator \( \hat{A}_i \) onto \( |N_i, N_k; A > \) yields 

\[
\hat{A}_i |N_i, N_k; A > = \sqrt{\frac{2N}{2N(2N - 1)}} \sqrt{\frac{N_i(\Omega - N_i + 1)}{\Omega}} |N_i - 1, N_k; A > .
\]  

(19)

From the above equations, a general expression for the pairing Hamiltonian matrix elements can be written:

\[
< A; N'_i, N'_2 | H | N_1, N_2; A > = \left[ \epsilon(N_1 - N_2) \frac{2N}{2N} - \frac{G}{2} \frac{2N}{2N(2N - 1)} (1 + q^4) \times \right.
\]

\[
(N_1(\Omega - N_1) + N_2(\Omega - N_2) + \Omega) + < H' > \delta_{N_i, N'_i} \delta_{N_2, N'_2} - \left. \frac{G}{2} \frac{2N}{2N(2N - 1)} (1 + q^4) \times \right]
\]

\[
\left[ \sqrt{(N_1 + 1)(\Omega - N_1)(\Omega - N_2 + 1)e_2} \delta_{N_1 + 1, N'_i} \delta_{N_2 - 1, N'_2} \right.
\]

\[
\left. \right].
\]
\[
\sqrt{\lambda_2 + 1}(\Omega - N_2)(\Omega - N_1 + 1)N_1\delta_{N_1 - 1N'_1}\delta_{N_2 + 1N'_2},
\]

(20)

where

\[
\langle H' \rangle = \frac{G}{8}(q - 1)[4\Omega + 2(q - q^2)N].
\]

(21)

Again, in the limit \( q = -1 \) the hamiltonian matrix reduces to the usual expression. Also, in that limit the term \( \langle H' \rangle \) goes to zero when \( N = \Omega \), which is no longer true for other \( q \) values.

In figure 1 we have plotted the ground state energy obtained from the diagonalization of the pairing hamiltonian as a function of the particle angular momentum \( j \), for some selected values of \( q \). We have chosen \( \epsilon = 1 \) and \( G = 0.3 \) for the calculations and considered the case \( N = \Omega \), which means that in the absence of the interaction the ground state is formed by the lower level completely full. One can immediately see that the system becomes less bound when the deformation is turned on, which is true for any value of the interaction strength \( G \), as we have checked. To better understand the effect of the deformation on the various terms in the hamiltonian, we have plotted in figure 2 the ground state energy when we deform only the kinetic term, the kinetic plus the interaction term and finally disregarding the term \( \langle H' \rangle \) in the hamiltonian matrix. We clearly see that the main responsible for the deformation effects is the interaction term.

A quantity of crucial importance in the pairing interaction theory is the so-called gap energy between the ground state and the two quasiparticle excitation state [8]. In the one-level pairing model, which is mathematically simpler than the two-level version, one can already understand the implications of using quons instead of fermions in the system under consideration. For simplicity, in what follows, we investigate the effects of the quon algebra on the gap energy in the one-level pairing model.

### 3.2 One level pairing interaction and the energy gap

The restriction of our previous results for \( N \) pairs in a single \( j \)-shell can be done taking \( \epsilon = 0 \) and disregarding all the terms with the subscript 2 in the hamiltonian. The ground state energy in the one level pairing model reads:

\[
E_0 = \langle N; A|H|N; A \rangle = -\frac{G}{2} \frac{(2N)(2N - 1)}{2N(2N - 1)} N(\Omega - N + 1)(1 + q^4)
\]

\[
+ \frac{G}{8}(q - 1)[4\Omega + 2(q - q^2)\frac{(2N)}{2N}N].
\]

(22)
Before proceeding with the calculations, it is worthwhile to note the effect of the factor
\[ F(N) = \frac{\{2N\} \{2N - 1\}}{2N(2N - 1)}, \]
where \(\{2N\} \{2N - 1\}\) gives the effective number of interactions among pairs and \(2N(2N - 1)\) gives the number of interactions among pairs. The factor \(F(N)\) measures the deviations due to the fractional quon occupation scheme. This quantity is always smaller than one and it depends on \(q\) and \(N\), being equal to one when \(q = -1\). The presence of this term tends to decrease the pairing interaction.

To obtain the gap energy, as mentioned in last subsection, we need to define the rank-\(J\) tensor in terms of the quon operators:
\[
\hat{B}^\dagger_{JM} = \frac{1}{\sqrt{2}} \sum_{m_1m_2} (jm_1jm_2|JM)a^\dagger_{m_1}a^\dagger_{m_2}.
\]
Equation (23)
This operator creates a pair with angular momentum \(J\) and can be used to define a one broken pair excited state.
\[
|N - 1, 1; A >_N = \mathcal{N}_J \mathcal{A} (\hat{A}^\dagger)^N \hat{B}^\dagger_{JM} |0 >, \tag{24}
\]
\(\mathcal{N}_J\) being a normalization factor. The energy gap is then given by
\[
\Delta = E_2 - E_0 = < N - 1, 1; A | H | N - 1, 1; A > - < N; A | H | N; A > =
\]
\[
\frac{G\Omega}{2} \frac{(1 + q^4)}{2} \frac{\{2N\} \{2N - 1\}}{2N(2N - 1)}, \tag{25}
\]
which can be compared with the usual value [8]. We can see that the energy for breaking a pair is now proportional to the factor \((1 + q^4)/2\) which comes from the symmetrization of the two-body interaction term of the hamiltonian with the quonic commutation rules, and it is also proportional to \(F(N)\). As mentioned before this fraction is always smaller than one and clearly stress the fact that the energy necessary to break a pair of quons in a fully antisymmetric state depends on the total number of particles in the system. In other words, the occupation scheme of the quons introduces a medium dependence on the gap. In order to better understand the effect of the quon occupation scheme it is convenient to redefine the \(q\) parameter as a function of \(\Omega\), the degeneracy of the level, where \(q \sim -1\). We take
\[
|q| = \frac{x}{\Omega}
\]
Equation (26)
so that eq. (3) can be rewritten as
\[
\{n\} = \frac{1 - x^{n/\Omega}}{1 - x^{1/\Omega}}, \tag{27}
\]
with $0 \leq x \leq 1$. Using this definition we have plotted the value of $\Delta/G\Omega$ as a function of the deformation parameter $x$ in figure 3 for a fixed value of $\Omega = 40$ and different number of particles. Reminding that $\Delta/G\Omega = 1$ for the fermionic case ($x = 1$) the departure from the horizontal line exhibits the effect of the quonic deformation. When the number of pairs approaches the degeneracy $\Omega$, the rôle played by the deformation is more evident because the correlations due to the occupation scheme becomes more important. If we choose another value for $\Omega$ we have qualitatively the same trend shown in figure 3.

Finally, we may now establish a link between our results for the energy gap behavior and the results for the ground state energy in the two level model: once the deformation conspires against the formation of Cooper pairs, the binding energy of the system tends to become smaller than in the usual fermionic case.

4 Conclusions

We have extended some of the results obtained in [2] to the permutational antisymmetric part of the whole quonic space, which allows us to restrict the dynamics to a system of fermion-like identical particles, in the same way we have restricted it to boson-like particles using only the symmetric subspace. A general expression for the annihilation of a particle from a $N$ quon antisymmetric state was found, so that any observable can be determined within the corresponding subspace. An application of our results was made to study the behavior of a system of quons interacting through a pairing force. The main conclusion is that the Cooper pair formation, which is characteristic of that type of interaction, can be largely weakened for small deviations from a fermionic system. This behavior can be attributed to the fact that the deformation introduces an extra dependence in the two-body force which becomes more and more important with the increase of the total number of particles in the system and tends to effectively decrease the interaction. This seems to be a property obeyed by interacting quons in general and not only due to the pairing interaction, but this is an investigation that remains to be confirmed.

A possible interesting problem is the consideration of a system formed by a gas of particles with half-integer spin that are composed by many fermions and that interact through the pairing force. In analogy to what was discussed in [3] for composed bosons, the commutation relations obeyed by creation and annihilation operators that define those particles are not the same as the ones obeyed by a fermion (or a boson) and the departure from the fermionic behavior can be described by the quon algebra. This could be, for instance, the case in the recent experiments with "fermion" traps [4] in a high density regime. In that case the fermions are in fact complex atoms and possible pairing type interactions between them [5] are supposed to be responsible for some of measurable behaviors of the gas. These systems can be, in principle, modelled by extensions of the present
results.

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Figure 1: Ground state energy for $G = 0.3$ and $\epsilon = 1$. The curves drawn from bottom to top were obtained respectively with $q = -1$, $q = -0.99$, $q = -0.98$, $q = -0.97$, $q = -0.96$ and $q = -0.95$. The energy is given in arbitrary units.

Figure 2: Ground state energy for $G = 0.3$ and $\epsilon = 1$. The curves drawn from bottom to top were obtained respectively with $q = -1$, $q = -0.97$ deforming only the kinetic term, $q = -0.97$ deforming all terms of the Hamiltonian and $q = -0.97$ with $< H' > = 0$. The energy is given in arbitrary units.
Figure 3: Energy gap in units of $G\Omega$ as a function of the number of pairs $N$ with $\Omega = 40$. The curves drawn from bottom to top are for $x = 0.96, 0.97, 0.98, 0.99$ and 1.0 respectively.