Transverse Symmetry Transformations and the Quark-Gluon Vertex Function in QCD

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The transverse symmetry transformations associated with the normal symmetry transformations in gauge theories are introduced, which at first are used to reproduce the transverse Ward-Takahashi identities in the Abelian theory QED. Then the transverse symmetry transformations associated with the BRST symmetry and chiral transformations in the non-Abelian theory QCD are used to derive the transverse Slavnov-Taylor identities for the vector and axial-vector quark-gluon vertices, respectively. Based on the set of normal and transverse Slavnov-Taylor identities, an expression of the quark-gluon vertex function is derived, which describes the constraints on the structure of the quark-gluon vertex imposed from the underlying gauge symmetry of QCD alone. Its role in the study of the Dyson-Schwinger equation for the quark propagator in QCD is discussed.

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I. INTRODUCTION

Gauge symmetry imposes powerful constraints on the basic vertex functions of gauge theories, referred to as the Ward-Takahashi(WT) [1] or the Slavnov-Taylor(ST) identities [2]. They play an essential role in demonstrating the renormalizability of gauge theories and are also important in the nonperturbative studies of gauge theories by using the Dyson-Schwinger equations(DSEs) [3]. In these aspects, the knowledge of the structure of the quark-gluon vertex is essential for understanding the dynamics of quark confinement and chiral symmetry breaking and also plays a key role in bridging the color quarks and gluons and their colorless bound states (hadrons) [3, 4, 5, 6].

The quark-gluon vertex $\Gamma_{\mu \nu}^{\text{qg}}(p_1, p_2)$ together with the quark propagator $S_F(p_1)$ and gluon propagator $D_{\mu \nu}(q)$ enter as the vital ingredients in the DSE for the quark propagator. The kernel of the quark DSE is dominated by $D_{\mu \nu}(q)\Gamma_{\mu \nu}^{\text{qg}}(p_1, p_2)$ with $q = p_1 - p_2$. Using the general form of the gluon propagator in covariant gauge, this kernel is written as $D_{\mu \nu}(q)\Gamma_{\mu \nu}^{\text{qg}}(p_1, p_2) = -\frac{4\pi\alpha_s}{g^2} - (g_{\mu \nu} - \frac{\partial_{\mu} \partial_{\nu}}{q^2}) \frac{Z_g}{q^2} \Gamma_{\mu \nu}^{\text{qg}}(p_1, p_2)$ with $Z_g$ being the gluon dressing function and $\xi$ being the covariant gauge parameter, which can be expressed as $D_{\mu \nu}(q)\Gamma_{\mu \nu}^{\text{qg}}(p_1, p_2) = -\frac{4\pi\alpha_s}{g^2} q_{\mu} q_{\nu} \Gamma_{\mu \nu}^{\text{qg}}(p_1, p_2) - \frac{Z_g}{q^2} q_{\mu} q_{\nu} \Gamma_{\mu \nu}^{\text{\xi}}(p_1, p_2)$. This kernel is clearly separated into the contributions from longitudinal and transverse parts of the quark-gluon vertex. Notice that in Landau gauge with $\xi = 0$ the contribution from the longitudinal part of the vertex to this kernel will disappear and the transverse part of the vertex will dominate this kernel and then the DSE for the quark propagator. The normal ST identity for the quark-gluon vertex determines the longitudinal part of the vertex, but leaving the transverse part unconstrained. It therefore appears highly desirable to determine constraints on the transverse part of the quark-gluon vertex from the gauge symmetry–the transverse ST identities. It is known that the ST identity for the quark-gluon vertex can be derived [5] by using the BRST ( Becchi-Rouet-Stora-Tyutin) symmetry [8]. Natural question is then if there exists a kind of symmetry–the transverse ST identities. It is known that the ST identity for the quark-gluon vertex can be derived [9, 10].

In this paper, the transverse symmetry transformations associated with the normal symmetry transformations in gauge theories are proposed, which are defined by the infinitesimal Lorentz transformation for the normal symmetry transformations. At first, we use the transverse symmetry transformations associated with the normal symmetry transformations and chiral transformations in Abelian theory QED in terms of the path-integral approach to reproduce the transverse WT relations that obtained previously based on the canonical field theory approach [9, 10]. Then we show how the transverse symmetry transformations associated with the BRST transformations and chiral transformations in QCD can be used to derive the transverse Slavnov-Taylor identities for the vector and axial-vector quark-gluon vertices, respectively, by the path-integral approach. Based on the set of the normal and transverse ST identities for the vector and the axial-vector quark-gluon vertices, we further derive the quark-gluon vertex function involving both longitudinal and transverse parts of the vertex, which describes the constraints on the quark-gluon vertex structure imposed from the gauge symmetry of QCD alone.

This paper is organized as follows. In Sec.II, the definition of the transverse symmetry transformation associated with the normal symmetry transformation is introduced. According to the definition, in Sec.III we write the detailed
representations for the transverse symmetry transformations associated with the normal and chiral transformations in QED, respectively, which are used to re-derive the transverse WT relations for the vector and axial-vector vertices in QED by the path-integral approach. Then in Sec.IV we give the transverse symmetry transformations associated with the BRST symmetry and the chiral transformations in QCD, and then derive the transverse ST identities for the vector and axial-vector quark-gluon vertices, respectively. By using the set of the normal and transverse ST identities for the vector and axial-vector quark-gluon vertices, in Sec.V we derive the quark-gluon vertex function in QCD. The conclusions and discussions are given in Sec.VI.

II. TRANSVERSE SYMMETRY TRANSFORMATIONS IN GAUGE THEORIES

Let us begin with introducing the definition and representation of the transverse symmetry transformations associated with normal symmetry transformations in gauge theories. Consider an infinitesimal symmetry transformation

$$\phi^\alpha(x) \rightarrow \phi^\alpha(x) + \delta\phi^\alpha(x).$$

(1)

Now we introduce corresponding infinitesimal transverse symmetry transformation

$$\phi^\alpha(x) \rightarrow \phi^\alpha(x) + \delta_T\phi^\alpha(x),$$

(2)

where \(\delta_T\phi^\alpha(x)\) is defined by the infinitesimal Lorentz transformation for the infinitesimal symmetry transformation \(\delta\phi^\alpha(x)\) in (1):

$$\delta_T\phi^\alpha(x) = \delta_{\text{Lorentz}}(\delta\phi^\alpha(x)) = -i/2 \varepsilon^{\mu\nu} S^{(\phi^\alpha)}_{\mu\nu}(\delta\phi^\alpha(x)).$$

(3)

Here \(S^{(\phi^\alpha)}_{\mu\nu}\) denotes the generator of the intrinsic part for the infinitesimal Lorentz transformation for \(\delta\phi^\alpha(x)\), where \(\phi^\alpha\) may be the spinor, vector or scalar field, and \(\delta\phi^\alpha(x)\) may be composed of these fields. For the spinor field,

$$S^{(\text{spinor})}_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu},$$

(4)

where \(\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]\); for the vector field,

$$(S^{(\text{vector})}_{\mu\nu})^\alpha_\beta = i(\delta^\alpha_\mu g^\beta_\nu - \delta^\alpha_\nu g^\beta_\mu);$$

(5)

and for the scalar field,

$$S^{(\text{scalar})}_{\mu\nu} = 0.$$  

(6)

The physical picture of the transformation (2), with the definition (3), is clear: While the transformation (1) defines a symmetry transformation where the change of variable is along the symmetry direction, the transformation (2) transforms the original symmetry direction, by the infinitesimal Lorentz transformation (3), to its transverse direction. This is why we call the transformation (2) with (3) as the transverse symmetry transformation associated with the symmetry transformation (1), explaining why the transverse WT(ST) identity for the vertex, derived in terms of the corresponding transverse symmetry transformation, can constrain the transverse part of the vertex.

In the following Sec.III and IV, we will use the definition (3) to give the explicit representations for the transverse symmetry transformations in the Abelian gauge theory QED and in the non-Abelian gauge theory QCD, respectively, and then derive the corresponding transverse WT identities in QED and transverse ST identities in QCD.

III. TRANSVERSE WARD-TAKAHASHI RELATIONS IN QED

The Ward-Takahashi(WT) identities are well-known to derive by using the canonical field theory approach or the path-integral approach. The transverse WT identities have been derived already in the canonical field theory approach\[9, 10]. As a check to the transverse symmetry transformation approach, in this section we provide the re-derivations of the transverse WT identities in terms of transverse symmetry transformations by using the path-integral approach. The procedure and results will be helpful also to understand the derivations of transverse ST identities in QCD given in next section.

In the path-integral approach, the origin of the WT(ST) identities lies in the gauge invariance of the generating functional of QED(QCD). If making the infinitesimal gauge transformations of the fields in QED:

$$\psi(x) \rightarrow \psi'(x),$$

where \(\psi(x)\) may be the spinor, vector or scalar field.
The transverse WT identities can be derived by the parallel procedure. To recover the gauge invariant expression, we use the standard procedure by introducing the line integral, i.e., the vertex in coordinate space [1, 11]: the QED lagrangian transforms according to the ST relations in QCD can be derived by the parallel way. But the new operator including the nonlocal current is not gauge invariant. For this purpose, we first write \( \bar{\psi}(x) \gamma^\mu \psi(x) \), which in momentum space gives the familiar expression

\[
q\mu \Gamma^\mu_V(p_1, p_2) = S^{-1}_F(p_1) - S^{-1}_F(p_2),
\]

where \( q = p_1 - p_2 \), \( S_F(p) \) is the full fermion propagator, and \( \Gamma^\mu_V \) is defined by

\[
\int d^4x d^4x' d^4x'' e^{i(p_1 - p_2 - q \cdot x)} \langle 0| T j^\mu(x) \bar{\psi}(x_1) \psi(x_2)| 0 \rangle \delta^4(x - x_2 - x_1)
\]

The transverse WT identities can be derived by the parallel procedure.

**A. Transverse WT identity for the fermion-boson vertex from transverse symmetry transformations**

The infinitesimal transverse symmetry transformations associated with the symmetry transformations given by Eq.(8) can be written by the definition (3):

\[
\delta_T \psi(x) = \frac{1}{4} g \alpha(x) \epsilon^{\mu\nu} \sigma_{\mu\nu} \psi(x), \quad \delta_T \bar{\psi}(x) = \frac{1}{4} g \alpha(x) \epsilon^{\mu\nu} \bar{\psi}(x) \sigma_{\mu\nu},
\]

without the corresponding transformation term for \( A_\mu \). Under such infinitesimal transverse symmetry transformations, the QED lagrangian transforms according to \( L_{QED} \rightarrow L_{QED} + \delta_T L_{QED} \), where

\[
\delta_T L_{QED} = \frac{i}{4} g \alpha(x) \epsilon^{\mu\nu} \bar{\psi}(x) S_{\lambda\mu\nu} (\partial_x^\lambda - \partial_x^\nu) \psi(x) + \frac{1}{2} g^2 \alpha(x) \epsilon^{\mu\nu} \bar{\psi}(x) S_{\lambda\mu\nu} A_\lambda \psi(x) - \frac{1}{2} m g \alpha(x) \epsilon^{\mu\nu} \bar{\psi}(x) \sigma_{\mu\nu} \psi(x) - \frac{1}{4} g \epsilon^{\mu\nu} (j_\mu(x) \partial_\nu \alpha(x) - j_\nu(x) \partial_\mu \alpha(x)).
\]
The form of Eq.(14) insures that each term that can be related to a definite Green’s function. Since the measure of functional integral is invariant under the transformations given by Eq.(12), such transformations then lead to the following identity from Eq.(7) for the functional integral over two fermion fields

$$0 = \int D[\bar{\psi}, \psi] e^{i\int d^4x L_{QED}} \{ i \int d^4x (\delta T L_{QED}) \bar{\psi}(x_1) \bar{\psi}(x_2) + \delta T(\bar{\psi}(x_1) \bar{\psi}(x_2)) \}.$$  \hspace{1cm} (15)

Substituting Eqs.(12) and (14) into Eq.(15) and integrating the term involving $\partial \alpha$ by parts, and then taking the coefficient of $\alpha$ and dividing the generating functional $Z[J = 0]$, we obtain the transverse WT identity for the fermion-boson (vector) vertex in coordinate space:

$$\partial^\mu \langle 0 | T j^\nu(x) \bar{\psi}(x_1) \bar{\psi}(x_2) | 0 \rangle - \partial^\nu \langle 0 | T j^\mu(x) \bar{\psi}(x_1) \bar{\psi}(x_2) | 0 \rangle = i \sigma^{\mu\nu} \langle 0 | T \bar{\psi}(x_1) \bar{\psi}(x_2) | 0 \rangle \delta^4(x_1 - x) + i \langle 0 | T \bar{\psi}(x_1) \bar{\psi}(x_2) | 0 \rangle \sigma^{\mu\nu} \delta^4(x_2 - x) + 2m \langle 0 | T \bar{\psi}(x_1) \sigma^{\mu\nu} \bar{\psi}(x_1) \bar{\psi}(x_2) | 0 \rangle + \lim_{x' \rightarrow x} i(\partial^\mu \bar{\psi}(x') \gamma_5 U_{p'}(x', x) \psi(x_1) \bar{\psi}(x_2) | 0 \rangle.$$  \hspace{1cm} (16)

The result is same as that obtained previously by the canonical field theory approach\cite{9}.

By carefully computing the Fourier transformation of Eq.(16), we obtain the transverse WT identity for the fermion-boson vertex function in momentum space:

$$i q^\mu \Gamma_V(p_1, p_2) - i q^\nu \Gamma_V(p_1, p_2) = S_F^{-1}(p_1) \sigma^{\mu\nu} + \sigma^{\mu\nu} S_F^{-1}(p_2) + 2m \Gamma^{\mu\nu}(p_1, p_2) + (p_{1\lambda} + p_{2\lambda}) \xi^{\lambda\mu\nu\rho} \Gamma_{A\rho}(p_1, p_2) - \int \frac{d^4k}{(2\pi)^4} 2k_\lambda \xi^{\lambda\mu\nu\rho} \Gamma_{A\rho}(p_1, p_2; k),$$  \hspace{1cm} (17)

where the integral term involves $\Gamma_{A\rho}(p_1, p_2; k)$ with the internal momentum $k$ of the gauge boson appearing in the Wilson line. $\Gamma_{A\rho}(p_1, p_2; k)$ is defined by

$$\int \frac{d^4k}{(2\pi)^4} 2k_\lambda \xi^{\lambda\mu\nu\rho} \Gamma_{A\rho}(p_1, p_2; k) \Gamma_{A\rho}(p_1, p_2; k).$$

where $q = (p_1 - k) - (p_2 - k)$. Using this definition, we can write the explicit expression of $\Gamma_{A\rho}(p_1, p_2; k)$ in the perturbation theory order by order. For example, we have at one-loop order

$$\int \frac{d^4k}{(2\pi)^4} 2k_\lambda \xi^{\lambda\mu\nu\rho} \Gamma_{A\rho}(p_1, p_2; k) = g^2 \int \frac{d^4k}{(2\pi)^4} 2k_\lambda \xi^{\lambda\mu\nu\rho} \Gamma_{A\rho}(p_1, p_2; k) \Gamma_{A\rho}(p_1, p_2; k)$$

where $k^\mu = k^\mu$, and $\xi$ is the covariant gauge parameter. The last two terms in the right-hand side of Eq.(19) are the one-loop self-energy contributions accompanying the vertex correction. It hence has been checked that the transverse WT identity for the fermion-boson vertex, Eq.(17), holds by the explicit computations of terms in this transverse WT relation to one-loop order\cite{12} \cite{13} \cite{14} \cite{15} \cite{16} \cite{17}. In Ref.\cite{14} Pennington and Williams also discussed the possibility to construct consistent non-perturbative Feynman rules by using the transverse WT identity (17).

**B. Transverse WT identity for the axial-vector vertex from the transverse chiral transformations**

The infinitesimal chiral transformations in QED are known as

$$\psi(x) \rightarrow \psi(x) + ig \alpha(x) \gamma^5 \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x) + ig \alpha(x) \bar{\psi}(x) \gamma^5,$$  \hspace{1cm} (20)
where \( \delta^{(5)}(\psi(x) = ig\gamma^5\psi(x) \), \( \delta^{(5)}(\bar{\psi}) = ig\gamma^5\bar{\psi}(x) \), which lead to the axial-vector WT identity (i.e. the WT identity for the axial-vector vertex in coordinate space):

\[
\partial^{\mu}_{\kappa}(0|T_{j^{\mu}_{\kappa}}(x)|\bar{\psi}(x_1)\psi(x_2)|0) = -\delta^\mu_{\kappa}(x_1 - x_2)\gamma^5(0|T\psi(x_1)|\bar{\psi}(x_2)|0) - \delta^\mu_{\kappa}(x - x_2)\langle 0|T\psi(x_1)|\bar{\psi}(x_2)|0\rangle \gamma^5
- \frac{e^2}{16\pi^2} \varepsilon^{\mu\nu\lambda\rho} \langle 0|T\psi(x_1)\bar{\psi}(x_2)F_{\mu\nu}(x)F_{\lambda\rho}(x)|0\rangle
\]

(21)

with \( j^{\mu}_{\kappa}(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) \). The last term in Eq.(21) denotes the contribution of the axial anomaly [18, 19], which arises from the change of functional integration measure under the chiral transformations in the path-integral approach [17].

The transverse axial-vector WT identity in momentum space is given by Fourier transforming Eq.(26) and the infinitesimal transverse chiral-transformations associated with the chiral transformations given by Eq.(20) can be obtained by using the definition (3):

\[
\delta^{(5)}_T(\psi(x) = \frac{i}{4}g\alpha(x)\epsilon^{\mu\nu\lambda\rho}S_{\mu\nu} \gamma^5 \psi(x), \quad \delta^{(5)}_T(\bar{\psi}) = -\frac{i}{4}g\alpha(x)\epsilon^{\mu\nu\lambda\rho}\sigma_{\mu\nu} \gamma^5.
\]

(24)

Under such infinitesimal transverse chiral-transformations, the QED lagrangian transforms according to \( L_{QED} \rightarrow L_{QED} + \delta^{(5)}_T L_{QED} \), where

\[
\delta^{(5)}_T L_{QED} = \frac{i}{4}g\alpha(x) \lim_{x' \rightarrow x} \epsilon^{\mu\nu\lambda\rho}(\partial^\mu_{x'} - \partial^\mu_{x})\bar{\psi}(x')S_{\lambda\nu} \gamma_5 U_{\mu}(x', x)\psi(x)
- \frac{i}{4}g\epsilon^{\mu\nu}(j_{5\nu}(x)\partial^\mu\alpha(x) - j_{5\nu}(x)\partial\nu\alpha(x))
\]

(25)

with \( j_{5\nu}(x) = \bar{\psi}(x)\gamma_\nu\gamma_5\psi(x) \). The calculation shows that the measure of functional integration is invariant under the transformations given by Eq.(24). Hence the transverse chiral transformations lead to a similar identity as given by Eq.(15) where \( \delta_T \) is replaced with \( \delta^{(5)}_T \). Substituting Eqs.(24) and (25) into such identity, we obtain the transverse WT identity for the axial-vector vertex in coordinate space:

\[
\partial^{\mu}_{\kappa}(0|T_{j^{\mu}_{\kappa}}(x)|\bar{\psi}(x_1)\psi(x_2)|0) - \partial^{\mu}_{\kappa}(0|T_{j^{\mu}_{\kappa}}(x)\psi(x_1)|\bar{\psi}(x_2)|0)
= i\sigma^{\mu\nu}(0|T\psi(x_1)|\bar{\psi}(x_2)|0) \delta^\nu(x_1 - x) + i(0|T\psi(x_1)|\bar{\psi}(x_2)|0) \sigma^{\mu\nu} \gamma_5 \delta^\nu(x_1 - x)
+ i \langle d^4x' \delta(x' - x)(\partial^\mu_{x'} - \partial^\mu_{x})\epsilon^{\lambda\nu\mu\rho} \langle 0|T\bar{\psi}(x')\gamma_{\mu}U_{\nu}(x', x)\psi(x_1)|\bar{\psi}(x_2)|0\rangle \rangle,
\]

(26)

which is also same as the result obtained by the canonical field approach [18, 19], which is consistent with the fact that the measure of functional integration is invariant under the transverse chiral transformations.

The transverse axial-vector WT identity in momentum space is given by Fourier transforming Eq.(26) and the result is

\[
ig^{\mu}\Gamma^\nu_A(p_1, p_2) - ig^{\nu}\Gamma^\mu_A(p_1, p_2)
= S_{-1}F(p_1)\sigma^{\mu\nu} \gamma_5 - \sigma^{\mu\nu} \gamma_5 S_{-1}(p_2)
+(p_{1\lambda} + p_{2\lambda})\epsilon^{\lambda\nu\mu\rho}\Gamma_{V\rho}(p_1, p_2) - \frac{d^4k}{(2\pi)^4} 2k_\lambda \epsilon^{\lambda\nu\mu\rho}\Gamma_{V\rho}(p_1, p_2; k),
\]

(27)

where \( \Gamma_{V\rho}(p_1, p_2; k) \) is defined by the Fourier transformation of the last matrix element in Eq.(26), which can be found elsewhere [18, 19].
Now there are normal WT identities, which impose the constraints on longitudinal parts of the vector and axial-vector vertices given by Eqs.(10) and (22), respectively, and transverse WT relations given by Eqs.(17) and (27), which constrain the transverse parts of these vertices. The full constraint relations for the vector and axial-vector vertex functions imposed from the gauge symmetry alone then can be derived in terms of this set of longitudinal and transverse WT relations in QED. It also has been checked that, by the explicit computations, such a full fermion-boson vertex in QED holds to one-loop order.

IV. TRANSVERSE SLAVNOV-TAYLOR RELATIONS IN QCD

Now we study how the transverse symmetry transformations associated with the BRST symmetry and the chiral transformations enable us to derive the constraint relations for the transverse parts of the vector and axial-vector quark-gluon vertices in QCD. To do these, we begin with the BRST transformations in QCD:

\[ \delta \psi = i g \omega c_\mu a^a \psi, \quad \delta \bar{\psi} = -i g \bar{\psi} t^a \omega c_a, \quad \delta A^a_\mu = \omega D^b_\mu c_b, \quad \delta A^a_0 = -\frac{1}{2} g \omega f^{abc} c_b c_c, \quad \delta c^a = \frac{\omega}{\xi} \partial^\mu A^a_\mu, \]  

(28)

where \( \psi, A^a_\mu \) and \( c^a \) denote the quark, gluon and ghost fields, respectively, \( t^a \) is the generator of \( SU(N_c) \) with \( f^{abc} \) being the corresponding structure constants, \( \omega \) is an infinitesimal Grassmann number, \( \xi \) is the covariant gauge parameter, \( D_\mu = \partial_\mu - i g t_\mu A^a_\mu \) and \( D^{ab}_\mu c_b = \partial_\mu c^a - g f^{abc} A^c_\mu c_b \). The QCD action

\[ \int d^4 x L_{QCD} = \int d^4 x \{ \bar{\psi}(x)(i \gamma^\mu D_\mu - m)\psi(x) - \frac{1}{4} F^{a\mu\nu}(x) F_{a\mu\nu}(x) \}
\]

is invariant under the BRST transformations, which can be used to derive some useful ST identities. The Slavnov-Taylor identity for the quark-gluon vertex in coordinate space reads:

\[ \frac{1}{\xi} \langle 0 | T \bar{\psi}(x_1) \bar{\psi}(x_2) \partial^\mu A_\mu(x) | 0 \rangle = -ig \langle 0 | T \bar{\psi}(x_1) \bar{\psi}(x_2) F^{a\mu}(x_1) \psi(x_2) c^a(x_2) | 0 \rangle + ig \langle 0 | T \bar{\psi}(x_1) \bar{\psi}(x_2) c^b(x_2) c^a(x) | 0 \rangle t^b. \]  

(30)

Defining

\[ \int d^4 x d^4 x_1 d^4 x_2 e^{i(p_1 - p_2 - q - 2 x)} \langle 0 | T \bar{\psi}(x_1) \bar{\psi}(x_2) A^a_\mu(x) | 0 \rangle \]

\[ = (2\pi)^4 \delta^4(p_1 - p_2 - q) i S_F(p_1) i g \bar{\psi} F^{a\mu} i S_F(p_2) i D^{b\mu}_a(q), \]  

(31)

Fourier transforming Eq.(30) and using the ST identity \( q_\mu D_{ab}^{a\mu} = -\xi \delta_{ab} q^\mu / q^2 \), one then obtains the ST identity for the quark-gluon vertex \( V_\mu^{ab} \) in momentum space:

\[ q_\mu V_\mu^{ab}(p_1, p_2, q) = [S_F^{-1}(p_1) (t^a - B^a_2(p_1, p_2)) - (t^a - B^a_2(p_1, p_2))] G(q^2), \]  

(32)

where \( q = p_1 - p_2, G(q^2) \) is the ghost dressing function relating to the ghost propagator by

\[ D_G^{ab}(q) = -\delta^{ab} G(q^2) / q^2, \]  

(33)

and \( B^a_2(p_1, p_2) \) is the 4-point quark-ghost scattering kernel defined by

\[ g t^a \int d^4 x d^4 x_1 d^4 x_2 e^{i(p_1 - p_2 - q - 2 x)} \langle 0 | T \bar{\psi}(x_1) \bar{\psi}(x_2) F^{a\mu}(x_1) \psi(x) | 0 \rangle \]

\[ = (2\pi)^4 \delta^4(p_1 - p_2 - q) g(t^a - B^a_2(p_1, p_2)) i S_F(p_2) i D_G^{b\mu}(q). \]  

(34)

A. Transverse Slavnov-Taylor identity for the quark-gluon vertex

The infinitesimal transverse symmetry transformations of the fields associated with the BRST transformations given by Eq.(28) can be written by using definition (3). The result is

\[ \delta \psi = \frac{1}{2} \delta \bar{\psi} + \frac{1}{2} \omega c^a \bar{\psi} t^a \psi, \quad \delta \bar{\psi} = -\frac{1}{2} \bar{\psi} t^a \omega c_a \psi, \quad \delta \bar{\psi} = \frac{1}{2} \bar{\psi} t^a \omega c_a \psi, \quad \delta A^a_\mu = \omega D^b_\mu c_b, \quad \delta A^a_\mu = -\frac{1}{2} g \omega f^{abc} c_b c_c, \quad \delta c^a = \frac{\omega}{\xi} \partial^\mu A^a_\mu, \]  

(35)
We thus obtain ghost parts:

\[ \langle \bar{\psi} \rangle_{\text{local gauge invariance as pointed out in the QED case given in Sec.III. Note that collecting Eqs. (36)-(39) gives the U symmetry transformations given by Eq.(35), such transform ations then lead to following identity (from the identity \[ d \rangle \]

\[ \int d^4x \{ \partial_\alpha A_\mu^\alpha(x) - g f_{abc} A_\alpha^a D_\mu^b c_b(x) \}, \]

\[ \int d^4x \{ \partial_\alpha A_\mu^\alpha(x) - g f_{abc} A_\alpha^a D_\mu^b c_b(x) \}. \]

Here the Wilson line \( U_\mu(x', x) = P \exp(ig \int_{x'}^{x'} dy \rho^\mu A_\mu^a(y)) \) is introduced in order that the operator recovers the locally gauge invariance as pointed out in the QED case given in Sec.III. Note that collecting Eqs. (36)-(39) gives the expression of \( \int d^4x \delta_T L_{QCD} \) which is purely kinematical. To take into account the dynamics of the system\( \mathbb{Q} \) and also to simplify the expression of \( \int d^4x \delta_T L_{QCD} \), we use the QCD equation of motion for gluon fields

\[ \frac{\partial}{\partial \alpha} F_\mu^a + g f_{abc} F_\mu^b A_\alpha^a + g \bar{\psi} \gamma^\alpha t^a \psi + \frac{1}{\xi} \partial_\mu (\partial_\alpha A_\mu^\alpha) + g f_{abc}(\partial_\mu t^b) c^c = 0. \]

We thus obtain

\[ \int d^4x \delta_T L_{QCD} \]

\[ = \int d^4x \frac{1}{2} \omega^{\mu\nu} \left\{ \frac{i}{4} \lim_{\eta \to 0} (\partial_\eta - \partial_\eta') \bar{\psi}(x') S_{\lambda \mu \nu} \{ U_\mu(x', x), t^a \} \psi(x) \right\} + m \bar{\psi}(x) \sigma_{\mu \nu} t^a \psi(x) \psi(x) - 3 g f_{abc} \bar{\psi}(x) \gamma_\mu t^a \psi(x) A_\mu^b c^a(x), \]

\[ \int d^4x \frac{1}{2} \omega^{\mu\nu} \left\{ \frac{i}{4} \lim_{\eta \to 0} (\partial_\eta - \partial_\eta') \bar{\psi}(x') S_{\lambda \mu \nu} \{ U_\mu(x', x), t^a \} \psi(x) \right\} + m \bar{\psi}(x) \sigma_{\mu \nu} t^a \psi(x) \psi(x) - 3 g f_{abc} \bar{\psi}(x) \gamma_\mu t^a \psi(x) A_\mu^b c^a(x), \]

\[ \int d^4x \delta_T L_{\text{ghost}} = \omega^{\mu\nu} \int d^4x f_{abc} \partial_\mu c^c(x) c^d(x) D_\mu^{ab} c_b(x). \]

Since the generating functional of QCD and the measure of functional integral are both invariant under the transverse symmetry transformations given by Eq.(35), such transformations then lead to following identity (from the identity similar to Eq.(7)) for the functional integral over two fermion and one anti-ghost fields:

\[ 0 = \int \mathcal{D}[\bar{\psi}, \psi, A, c, \bar{c}] e^{i \int d^4x L_{QCD}(x)} \langle \psi(0) \bar{\psi}(0) \rangle \}

Substituting Eqs. (35) and (41) into this identity and then taking the coefficient of \( \frac{1}{2} \omega^{\mu\nu} \) and dividing the generating functional \( Z[J = 0] \), we obtain the transverse Slavnov-Taylor identity for the quark-gluon vertex in coordinate space:

\[ \langle 0 \mid T g j_\mu^a(x') \bar{\psi}(x') D_\mu^{ab} c_b(x) \rangle = \int_0 \left\{ \delta^4(x_1 - x_2) \right\}, \]

\[ = ig \bar{\psi}(x') \gamma^\mu t^a \psi(x') \psi(x) \psi(x) - 3 g f_{abc} \bar{\psi}(x) \gamma_\mu t^a \psi(x) A_\mu^b c^a(x), \]

\[ + \frac{1}{2} \omega^{\mu\nu} \left\{ \frac{i}{4} \lim_{\eta \to 0} (\partial_\eta - \partial_\eta') \bar{\psi}(x') S_{\lambda \mu \nu} \{ U_\mu(x', x), t^a \} \psi(x) \right\} + m \bar{\psi}(x) \sigma_{\mu \nu} t^a \psi(x) \psi(x) - 3 g f_{abc} \bar{\psi}(x) \gamma_\mu t^a \psi(x) A_\mu^b c^a(x), \]

where \( j_\mu^a(x') = \bar{\psi}(x') \gamma_\mu t^a \psi(x) \).

Note that each term in the transverse ST identity (43) contains a disconnected part plus connected terms due to the quark-ghost scattering. To understand how to make Fourier transformation for such kind of term, let us first discuss a more simple case – the another form of the ST identity for the quark-gluon vertex before making Fourier transformation for Eq.(43). Considering the following transformations:
and using the procedure similar to the derivations of the WT identity (9) and the transverse WT identity (16), we can obtain an expression of the ST identity for the quark-gluon vertex in coordinate space:

\[
\begin{align*}
\langle 0 \mid T g j_{\mu}^a(x) \bar{\psi}(x_1) \bar{\psi}(x_2) (D_{ab}^{\mu} c^b(x)) e^c(0) \mid 0 \rangle \\
= g^{\mu} (0 \mid T \bar{\psi}(x_1) \bar{\psi}(x_2) e^a(x) e^c(0) \mid 0 \rangle \delta^a(x - x_1) - g (0 \mid T \psi(x_1) \bar{\psi}(x_2) t^a c^a(x) \bar{c}^c(0) \mid 0 \rangle t^a \delta(x - x_2),
\end{align*}
\]

which should be equivalent to the ST identity (30). Hence one can give the relation between the quark-gluon vertex defined from \( (0 \mid T \psi(x_1) \bar{\psi}(x_2) A_{\mu}^a(x) \mid 0 \rangle \) and that from \( (0 \mid T j_{\mu}^a(x) \bar{\psi}(x_1) \bar{\psi}(x_2) \mid 0 \rangle \) by the following discussion.

Notice that each term in Eq.(45) contains a disconnected part plus terms due to the quark-ghost scattering. For instance, the first term

\[
\begin{align*}
(0 \mid T j_{\mu}^a(x) \bar{\psi}(x_1) \bar{\psi}(x_2) D_{ab}^{\mu} c^b(x) e^c(0) \mid 0 \rangle \\
= (0 \mid T j_{\mu}^a(x) \bar{\psi}(x_1) \bar{\psi}(x_2) \mid 0 \rangle (0 \mid T D_{ab}^{\mu} c^b(x) e^c(0) \mid 0 \rangle + \text{connected}.
\end{align*}
\]

Then the part \( (0 \mid T j_{\mu}^a(x) \bar{\psi}(x_1) \bar{\psi}(x_2) \mid 0 \rangle \) can be decomposed into the three-point proper vertex and the quark propagator as given by Eq.(11) in the Abelian case. The contribution of the part \( (0 \mid T D_{ab}^{\mu} c^b(x) e^c(0) \mid 0 \rangle \) is given by using the identity

\[
\int d^4x e^{-ix \cdot q} \langle 0 \mid T D_{ab}^{\mu} c^b(x) e^c(0) \mid 0 \rangle = \delta^{ac} q_\mu / q^2.
\]

Thus Fourier transforming the first term of Eq.(45) leads to \( iS_F(p_1) \tilde{\Gamma}^{\mu}_V(p_1, p_2) (1 - B_{(D)6}^{(\mu)}) iS_F(p_2) q^\nu / q^2 \), where \( B_{(D)6}^{(\mu)} \) is the relative 6-point(body) quark-ghost scattering kernel from the connected term in Eq.(46) and \( \Gamma^{\mu}_V \) is the vector vertex defined from \( (0 \mid T j_{\mu}^a(x) \bar{\psi}(x_1) \bar{\psi}(x_2) \mid 0 \rangle \) as given in the Abelian case \([9, 10, 13]\). Hence Fourier transforming Eq.(45) obtains following expression of the ST identity for the quark-gluon vertex in momentum space:

\[
q_\mu \tilde{\Gamma}^{\mu}_V(p_1, p_2; q)(1 - B_{(D)6}^{(\mu)}(p_1, p_2)) = [S_F^{-1}(p_1)(t^a - B_0^{(\mu)}(p_1, p_2)) - (t^a - B_0^{(\mu)}(p_1, p_2)) S_F^{-1}(p_2)] G(q^2).
\]

Because the ST identity (48) is equivalent to Eq.(32), we thus obtain the relation

\[
\Gamma^{\mu}_V = \tilde{\Gamma}^{\mu}_V (1 - B_{(D)6}^{(\mu)}).
\]

On the other hand, integrating the term involving \( \partial_{\mu} c^a(x) \) by part and using the QCD equation of motion for quark field: \( (i\gamma_5 D_\mu - m) \psi = 0 \), we have

\[
\begin{align*}
\int d^4x_1 d^4x_2 d^4x e^{i p_1 \cdot x_1 - i p_2 \cdot x_2 - i q \cdot x} \langle 0 \mid T j_{\mu}^a(x) \bar{\psi}(x_1) \bar{\psi}(x_2) D_{ab}^{\mu} c^b(x) e^c(0) \mid 0 \rangle \\
= i q_\mu \int d^4x_1 d^4x_2 d^4x e^{i p_1 \cdot x_1 - i p_2 \cdot x_2 - i q \cdot x} \langle 0 \mid T j_{\mu}^a(x) \bar{\psi}(x_1) \bar{\psi}(x_2) e^a(x) e^c(0) \mid 0 \rangle \\
= (2\pi)^4 \delta^4(p_1 - p_2 - q) i S_F(p_1) q_\mu \tilde{\Gamma}^{\mu}_V(p_1, p_2; 1 - B_0^{(\mu)}) i S_F(p_2) G(q^2) / q^2,
\end{align*}
\]

where \( B_0^{(\mu)} \) is the 6-body quark-gluon scattering kernel from the connected term in \( (0 \mid T j_{\mu}^a(x) \bar{\psi}(x_1) \bar{\psi}(x_2) e^a(x) e^c(0) \mid 0 \rangle \). We thus find

\[
\Gamma^{\mu}_V(p_1, p_2) = \tilde{\Gamma}^{\mu}_V(p_1, p_2; 1 - B_{(D)6}^{(\mu)}(p_1, p_2)) = \tilde{\Gamma}^{\mu}_V(p_1, p_2; 1 - B_0^{(\mu)}(p_1, p_2)) G(q^2),
\]

where \( \Gamma^{\mu}_V \) is defined by Eq.(31), while \( \tilde{\Gamma}^{\mu}_V \) is defined by a similar equation as given by Eq.(11). The axial-vector vertex and the tensor vertex also satisfy similar relations.

Using these relations and by the similar procedure for deriving the ST identity (32) from Eq.(30) and for deriving the ST identity (48) from Eq.(45), we can perform the Fourier transformation for the identity (43), which leads to the transverse Slavnov-Taylor identity for the quark-gluon vertex in momentum space:

\[
\begin{align*}
i q_\mu \Gamma_V^{\mu}(p_1, p_2) - i q_\nu \Gamma_V^{\nu}(p_1, p_2) \\
= [S_F^{-1}(p_1) \sigma^{\mu\nu}(t^a - B_0^{(\mu)}(p_1, p_2)) + (t^a - B_0^{(\mu)}(p_1, p_2)) \sigma^{\mu\nu} S_F^{-1}(p_2)] G(q^2)
\end{align*}
\]

\[
+ 2 \mu \gamma^{\mu\nu}(p_1, p_2) + (p_1 + p_2) e c^{\mu\nu}(p_1, p_2) \gamma_5 \partial_{\lambda} \partial_{\lambda} G(q^2),
\]

\[
\Gamma_V^{\mu}(p_1, p_2) = \tilde{\Gamma}_V^{\mu}(p_1, p_2; 1 - B_0^{(\mu)}(p_1, p_2)) G(q^2),
\]

(59)
Here $\Gamma^a_{T\mu} = \hat{\Gamma}^a_{T\mu}(1 - B^\mu_{(5)})G(q^2)$, $\Gamma^a_{Ap} = \hat{\Gamma}^a_{Ap}(1 - B^\mu_{(5)})G(q^2)$ and $\Gamma^a_{Ap}(p_1, p_2; k) = \hat{\Gamma}^a_{Ap}(p_1, p_2; k)(1 - B^\mu_{(5)}(p_1, p_2; k))G(q^2)$, where $B^\mu_{(5)}$ and $B^\rho_{(5)}$ are the 6-body quark-ghost scattering kernels from the relative connected terms, $\Gamma^a_{T\mu}$ and $\Gamma^a_{Ap}$ are respectively the tensor and axial-vector vertices defined as that in the Abelian case \[9, 10\]. The four-point-like non-local axial-vector vertex $\Gamma^a_{Ap}(p_1, p_2; k)$ is defined by the Fourier transformation of the last matrix element in Eq.(43):

$$
\int d^4x d^4x' d^4x'' d^4x''' e^{i(p_1 \cdot x - p_2 \cdot x' + (p_2 \cdot k) \cdot x - (p_1 \cdot k) \cdot x')} \langle 0 \vert T \bar{\psi}(x') \gamma_\rho \gamma_5 \psi(0) \rangle = (2\pi)^4 \delta^4(p_1 - p_2 - q)iS_F(p_1)\Gamma^a_{Ap}(p_1, p_2; k)iS_F(p_2).
$$

(53)

**B. Transverse axial-vector Slavnov-Taylor identity**

The transverse ST identity (52) shows that the transverse part of the quark-gluon vertex is related to the tensor and axial-vector vertices. In the case of $m = 0$, the contribution of $\Gamma^a_{T\mu}$ disappears. Therefore, in this case to constrain completely the quark-gluon vertex, the constraint relation for the transverse part of $\Gamma^a_{Ap}$ (the longitudinal part of $\Gamma^a_{Ap}$ does not contribute to Eq.(52) due to the factor $\epsilon^{a\mu\nu}$) is required to build as well. This can be performed by using the transverse symmetry transformations associated with the chiral transformations of the fields in QCD.

The infinitesimal chiral transformations in QCD can be written as

$$
\delta(\psi) = ig\omega A^\mu a^\mu \gamma_5 \psi, \quad \delta(\bar{\psi}) = ig\bar{\psi} \gamma_5 a^\mu \omega c_a, \quad \delta(a^\mu) = 0, \quad \delta(\bar{c}) = \delta(\bar{c}^a) = 0.
$$

(54)

By using definition (3), we obtain the infinitesimal transverse chiral transformations associated with Eq.(54):

$$
\delta_T(\psi) = \frac{1}{4} ge^{\mu\nu} \omega A^\mu a^\nu \gamma_5 \psi, \quad \delta_T(\bar{\psi}) = -\frac{1}{4} ge^{\mu\nu} \bar{\psi} \gamma_5 \gamma_5 \omega c_a, \quad \delta_T(a^\mu) = \delta_T(\bar{c}) = \delta_T(\bar{c}^a) = 0.
$$

(55)

Such transverse chiral-transformations lead to $\int d^4x L_{QCD} \longrightarrow \int d^4x L_{QCD} + \int d^4x \delta_T(\bar{\psi}^a \psi^a)$, where

$$
\int d^4x \delta_T(\bar{\psi}^a \psi^a) = \int d^4x \frac{1}{4} ge^{\mu\nu} \omega A^\mu a^\nu \frac{i}{2} \lim_{x' \rightarrow x} \left( \delta_T(\bar{\psi}(x')) - \delta_T(\bar{\psi}(x)) \right) S^{a\nu}(U_P(x', x), t^a) \gamma_5 \psi(x)c_a(x)
$$

$$
+ j_{5\mu}^{a}(x) D_{\nu}^{ab} c_b(x) - j_{5\mu}^{a}(x) D_{\nu}^{ab} c_b(x)
$$

(56)

with $j_{5\mu}^{a} = \bar{\psi}(x) \gamma_5 \gamma_5 a^\mu \psi(x)$.

Note that there is no transverse axial-anomaly of QCD, since the axial-anomaly of QCD should be described by the Abelian result, supplemented by an appropriate group theory factor, and there is no transverse axial-anomaly in the Abelian case \[13, 14\]. Correspondingly, the measure of functional integration is invariant under such transverse chiral transformations, which can be checked by the explicit calculation. As a result, the gauge invariance of the generating functional of QCD together with the invariance of the measure of functional integration in such transverse chiral-transformations lead to following identity

$$
0 = \int D[\bar{\psi}, \psi, A, c, \bar{c}] \delta T(\bar{\psi}^a \psi^a) \int d^4x L_{QCD} \left\{ i \int d^4x (\delta_T(\bar{\psi}^a \psi^a)) \bar{\psi}(x_1) \psi(x_2) c^a(0) + \delta_T(\bar{\psi}(x_1) \bar{\psi}(x_2)c^a(0)) \right\}.
$$

(57)

Substituting Eqs.(55) and (56) into this identity and then taking the coefficient of $\frac{1}{4} ge^{\mu\nu}$ and dividing the generating functional $Z[J = 0]$, we obtain the transverse ST identity for the axial-vector quark-gluon vertex in coordinate space:

$$
\frac{1}{4} \lim_{x' \rightarrow x} \left( \delta_T(\bar{\psi}(x')) - \delta_T(\bar{\psi}(x)) \right) \bar{\psi}(x') \gamma_5 \psi(0) \psi_t + 2 \lim_{x' \rightarrow x} \epsilon^{a\mu\nu} \frac{i}{2} \epsilon^{b\rho\sigma} \left( \bar{\psi}(x') \gamma_5 \gamma_5 \gamma_5 \gamma_5 \psi(0) \right)
$$

$$
\int d^4x \delta_T(\bar{\psi}^a \psi^a) \int d^4x L_{QCD} \left\{ i \int d^4x (\delta_T(\bar{\psi}^a \psi^a) \bar{\psi}(x_1) \psi(x_2) c^a(0) + \delta_T(\bar{\psi}(x_1) \bar{\psi}(x_2)c^a(0)) \right\}.
$$

(58)

Using the similar procedure for obtaining Eq.(52) from Eq.(43), Fourier transforming Eq.(58) leads to the transverse axial-vector Slavnov-Taylor identity in momentum space:

$$
\begin{align*}
& i q^\mu \Gamma^a_{T\mu}(p_1, p_2) - i q^\mu \Gamma^a_{Ap}(p_1, p_2) \\
& = \left[ S^{-1}_F(p_1) \sigma^{\mu\nu} \gamma_5 \left( t^a - B^a_4(p_1, p_2) \right) - (t^a - B^a_4(p_1, p_2)) \sigma^{\mu\nu} \gamma_5 S^{-1}_F(p_2) \right] G(q^2) \\
& = \left[ \left( p_1 + p_2 \right) \epsilon^{\lambda\mu\nu\rho} \Gamma^a_{Ap}(p_1, p_2) \right] - \int \frac{d^4k}{(2\pi)^4} 2k^\rho \epsilon^{\mu\nu\rho\sigma} \Gamma^a_{T\mu}(p_1, p_2) k^\sigma.
\end{align*}
$$

(59)
where the notations are same as that given in last subsection, and \( \Gamma_{V\rho}^{a}(p_1, p_2; k) \) is defined by the Fourier transformation of the last matrix element in Eq.(58), which is given from Eq.(53) by replacing \( \gamma_p \gamma_5 \) and \( \Gamma_{A\rho}^{a} \) with \( \gamma^p \) and \( \Gamma_{V\rho}^{a} \), respectively.

The transverse Slavnov-Taylor identities (52) and (59) are the primary results derived in terms of the transverse symmetry transformations associated with the BRST symmetry and the chiral transformations in QCD. They together with the ST identity (32) form a complete set of Slavnov-Taylor relations for the quark-gluon vertex in the case of massless fermion.

**V. THE QUARK-GLUON VERTEX FUNCTION IN QCD**

Now let us derive the the quark-gluon vertex function \( \Gamma_{V\mu}^{a}(p_1, p_2) \) by consistently solving this set of ST relations for the vector and the axial-vector quark-gluon vertex functions in the case of massless fermion. To do this, multiplying both sides of Eqs.(52) and (59) by \( iq_{\nu} \), and then moving the terms proportional to \( q_{\nu}\Gamma_{V\nu}^{a} \) and \( q_{\nu}\Gamma_{A\nu}^{a} \) into the right-hand side of the equations, we thus have

\[
q^2 \Gamma_{V\mu}^{a}(p_1, p_2) = q^\mu [q_{\nu}\Gamma_{V\nu}^{a}(p_1, p_2)] + iq_{\nu}[S_{F}^{-1}(p_1)\sigma^{\mu\nu}(t^a - B^a_4) + (t^a - B^a_4)\sigma^{\mu\nu}S^{-1}_F(p_2)]G(q^2) \\
+ i(p_{1\lambda} + p_{2\lambda})q_{\nu}\epsilon^{\lambda\mu\nu\rho}\Gamma_{A\rho}^{a}(p_1, p_2) - iq_{\nu}C_{V\mu\nu}^{a},
\]

(60)

\[
q^2 \Gamma_{A\mu}^{a}(p_1, p_2) = q^\mu [q_{\nu}\Gamma_{A\nu}^{a}(p_1, p_2)] + iq_{\nu}[S_{F}^{-1}(p_1)\sigma^{\mu\nu}\gamma_5(t^a - B^a_4) - (t^a - B^a_4)\sigma^{\mu\nu}\gamma_5 S^{-1}_F(p_2)]G(q^2) \\
+ i(p_{1\lambda} + p_{2\lambda})q_{\nu}\epsilon^{\lambda\mu\nu\rho}\Gamma_{V\rho}^{a}(p_1, p_2) - iq_{\nu}C_{A\mu\nu}^{a},
\]

(61)

where

\[
C_{V\mu\nu}^{a} = \int \frac{d^4k}{(2\pi)^4} 2k_{\lambda}\epsilon^{\lambda\mu\nu\rho}\Gamma_{A\rho}^{a}(p_1, p_2; k),
\]

(62)

\[
C_{A\mu\nu}^{a} = \int \frac{d^4k}{(2\pi)^4} 2k_{\lambda}\epsilon^{\lambda\mu\nu\rho}\Gamma_{V\rho}^{a}(p_1, p_2; k),
\]

(63)

which are non-local vertex terms. Substituting Eq.(61) into Eq.(60) and using the ST identity (32) and following identities

\[
q_{\nu}q_{\alpha}(p_{1\lambda} + p_{2\lambda})\epsilon^{\lambda\mu\rho\sigma}\gamma_5 \\
= i[q_{\nu}(p_1 + p_2)\sigma^{\mu\nu} - q^2(p_1 + p_2)\sigma^{\mu\nu} - q^a q_{\nu}(p_{1\lambda} + p_{2\lambda})\sigma^{\lambda\nu}]
\]

(64)

and

\[
q_{\nu}q_{\alpha}(p_{1\lambda} + p_{2\lambda})(p_{1\beta} + p_{2\beta})\epsilon^{\lambda\mu\rho\sigma}\alpha\beta\delta\Gamma_{V\delta}^{a} \\
= [q^2(p_1 + p_2)^2 - ((p_1 + p_2) \cdot q)^2]\Gamma_{V\delta}^{a} + [(p_1 + p_2) \cdot qq^\mu - q^2(p_1^\mu + p_2^\mu)(p_{1\nu} + p_{2\nu})\Gamma_{V\delta}^{a} \\
+ [(p_1 + p_2) \cdot q(p_1^\mu + p_2^\mu) - (p_1 + p_2)^2q^\mu]q_{\nu}\Gamma_{V\delta}^{a},
\]

(65)

after self-consistent iterating, we finally obtain the quark-gluon vertex function \((m = 0 \text{ case})\) of involving both the longitudinal part of the vertex, \( \Gamma_{V(L)}^{a} \), and the transverse part of the vertex, \( \Gamma_{V(T)}^{a} \):

\[
\Gamma_{V}^{a}(p_1, p_2) = \Gamma_{V(L)}^{a}(p_1, p_2) + \Gamma_{V(T)}^{a}(p_1, p_2),
\]

(66)

\[
\Gamma_{V(L)}^{a}(p_1, p_2) = q^{-2}q_{\nu}[q_{\rho}\Gamma_{V\rho}^{a}(p_1, p_2)] \\
= q^a[S_{F}^{-1}(p_1)(t^a - B^a_4(p_1, p_2)) - (t^a - B^a_4(p_1, p_2))S^{-1}_F(p_2)]G(q^2)/q^2,
\]

(67)
\[
\Gamma^{\alpha \mu}_{V(p_1, p_2)} = q^{-2} i q_\nu [i q^\nu \Gamma^{\alpha \mu}_{V}(p_1, p_2) - i q^\nu \Gamma^{\alpha \mu}_{V}(p_1, p_2)]
\]

where \(C^{\alpha \mu}_{A(p_1, p_2)} = C^{\alpha \mu}_{A(p_1, p_2)} G^{-1}(q^2) = \int \frac{d^4k}{(2\pi)^4} 2k_\nu \epsilon^{\lambda \mu \rho \sigma} T^{\alpha}_{A(p_1, p_2)(1 - B_6^{(p_5)}(p_1, p_2; k))} \) and \(C^{\alpha \beta \gamma}_{V(p_1, p_2)} = C^{\alpha \beta \gamma}_{V(p_1, p_2)} G^{-1}(q^2) = \int \frac{d^4k}{(2\pi)^4} 2k_\nu \epsilon^{\lambda \beta \gamma \rho} T^{\alpha}_{V(p_1, p_2)(1 - B_6^{(p_5)}(p_1, p_2; k))} \). Eqs.(66)-(68) describe the constraints on the structure of the quark-gluon vertex function \((m = 0)\) imposed from the gauge symmetry alone of QCD, showing how the quark-gluon vertex function relates to the quark propagator, the ghost dressing function, the quark-ghost scattering kernels and four-point-like non-local vertex terms. The axial-vector quark-gluon vertex function and the quark-gluon vertex function with fermion mass \((m \neq 0)\) can be similarly derived.

VI. CONCLUSIONS AND DISCUSSIONS

This paper introduces the transverse symmetry transformations associated with the normal symmetry transformations in gauge theories, which enables us to build the transverse constraints on the vertex functions in gauge theories. This has been tested at first by using such approach to reproduce the transverse WT relations that obtained already in canonical field theory approach in QED. Then by using the transverse symmetry transformations associated with the BRST transformations and chiral transformations in QCD and in terms of the path-integral approach, we have derived the transverse ST identities for the vector and axial-vector quark-gluon vertices, respectively. Based on the set of normal and transverse ST identities, we have further obtained an expression of the quark-gluon vertex function.

It is important to emphasize that, while the BRST symmetry leads to the ST identity which constrains the longitudinal part of the quark-gluon vertex from the gauge symmetry, the transverse symmetry transformations associated with the BRST and the chiral transformations lead respectively to the transverse ST identities for the vector and the axial-vector quark-gluon vertices, which have the potential to constrain the transverse part of the quark-gluon vertex from the gauge symmetry, and hence the expression of the quark-gluon vertex function given by Eqs.(66)-(68) describes the constraints on the quark-gluon vertex structure imposed from the gauge symmetry alone of QCD theory in the massless case. Hence, such a quark-gluon vertex function should be satisfied both perturbatively and non-perturbatively and then has the potential to unravel the non-Abelian structure of the quark-gluon vertex. In these aspects, some comments are given as follows.

At first, it can be checked that the transverse ST identities (52) and (59) and then the quark-gluon vertex function given by Eqs.(66)-(68) should hold in perturbation theory by performing the corresponding one-loop calculations as done in the Abelian theory QED case\([13, 14, 15, 20]\). As shown in the Abelian QED case, the non-local vertex terms in the transverse WT identities are essential to insure that the fermion-boson vertex derived based on the set of normal and transverse WT identities holds to one-loop order, and are responsible for multiplicative renormalizability in perturbation theory. The situation should be similar for non-Abelian QCD case. Besides, the quark-ghost scattering kernels are responsible for the one-loop non-Abelian vertex diagram as shown by the one-loop calculations of 4-point quark-ghost scattering kernel \(B_4^{(p_5)}\). These tedious one-loop calculations for these transverse ST identities and the quark-gluon vertex given by Eqs.(66)-(68) will remain to be performed in the further work.

Second, if the 4-point and 6-point quark-ghost scattering kernels are neglected, the quark-gluon vertex function will reduce to the Abelian-type vertex function\([20]\) multiplying the ghost dressing function. Lattice QCD calculations\([6]\) for the quark-gluon vertex in Landau gauge at two specific kinematic limits (‘asymmetric’ and ‘symmetric’) have found substantial deviations from the Abelian form – which cannot be described by a universal function multiplying the Abelian form with the longitudinal vertex of the Ball-Chiu construction\([22]\) and the transverse vertex of the Curtis-Pennington construction\([23]\) as given in \(24, 27\). Furthermore, recent lattice data lead to an essentially constant ghost dressing function in the infrared limit\([24, 27]\). These lattice results, together with the one-loop calculations, show that the transverse vertex is universal in direction and the transverse symmetry transformations are essential for the description of the quark-ghost scattering kernel.
characterizing the non-Abelian property of the quark-gluon vertex. How to extract the non-trivial information encoded in these quark-ghost scattering kernels is required to be studied further.

Third, the longitudinal and transverse parts of the quark-gluon vertex given respectively by Eqs. (67) and (68) contain several kinematic singularities at \( q^2 = 0 \), which arise from following reasons: (i) The vertex has been separated into longitudinal and transverse parts by such a way: \( \Gamma^{\mu\nu}_{p_1, p_2} = q^{-2}q^\mu[q, \Gamma^{\mu\nu}_{p_1, p_2}] + q^{-2}iq^\nu[q, \Gamma^{\mu\nu}_{p_1, p_2}] = \Gamma^{\mu\nu}_{p_1, p_2} + iq_2 \Gamma^{\nu\mu}_{p_1, p_2} \). In the case that the quark-gluon vertex is used into the quark propagator DSE, this type of factor \( q^{-2} \) might be attributed to the gluon propagator as shown in the introduction by the expression of the kernel of quark DSE. (ii) The itinerant procedure performed by substituting the expression of the axial-vector vertex, Eq. (61), into the expression of the vector vertex, Eq. (60), leads to the appearance of the factor \((p_1 + p_2) \cdot qq^{-2}\) in the transverse part of the vertex. The same situation also appears in the Abelian QED case where the fermion-boson vertex has been expressed in terms of the normal and transverse WT identities for the vector and axial-vector vertices\[20\]. As shown by the explicit calculations, such a fermion-boson vertex to one-loop order leads to the same result as one given in QED perturbation theory\[13, 14\], which does not exhibit particle-like singularity at \( q^2 = 0 \). This result implies that such a fermion-boson vertex should not exhibit the particle-like singularity at \( q^2 = 0 \)[14], and hence in the practical application these kinematic singularities contained in such a fermion-boson vertex should be cancelled by a proper procedure like the Ball-Chiu construction\[22\] as discussed by Pennington and Williams for the Abelian QED case\[14\]. The discussion for non-Abelian case should be analogous.

Finally, let us mention the possible application of the transverse ST identities (52) and (59) and the quark-gluon vertex function given by Eqs. (66)-(68) in the study of the Dyson-Schwinger equation (DSE) for the quark propagator. As shown already in the introduction, the kernel \( D_{\mu\nu}(q)\Gamma^{\mu\nu}_{p_1, p_2} \) of the quark DSE can be naturally separated into the contributions from transverse and longitudinal parts of the quark-gluon vertex, which provides important information: Generally, the transverse and longitudinal parts of the quark-gluon vertex both are essential for the quark DSE; however, in the Landau gauge QCD with \( \xi = 0 \) the contribution from longitudinal part of the vertex to the quark DSE will disappear and then the transverse part of the vertex will dominate the quark DSE. As a consequence, the transverse ST identities, which impose the constraints on the transverse part of the quark-gluon vertex from the gauge symmetry, play the crucial role in the study of the DSE for the quark propagator in Landau gauge QCD. Present work provides the transverse ST identities (52) and (59) for the vector and axial-vector quark-gluon vertices and the transverse part of the quark-gluon vertex function (68) derived from the symmetry relations of QCD. They together with the ST identity for the quark-gluon vertex, Eq. (32), provide the bases of an Ansatz for constructing the quark-gluon vertex being free of kinematic singularity, like that the Ward-Takahashi identity is a base of Ansatz for the Ball-Chiu construction\[22\] of the Abelian vertex in QED, for the practical application in the study of the DSE for the quark propagator. This interesting subject is beyond the scope of the present work and calls for the further study.

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[25] Note that the B-C type’s longitudinal vertex\[22\] has been constructed in a way free of kinematic singularities by satisfying the Ward and WT identities, and the C-P type’s transverse vertex\[23\] has been constructed, by requiring to satisfy multiplicative renormalizability, through a nonperturbative extension for the asymptotic limit contribution of transverse part of one-loop fermion-boson vertex at large fermion momenta ($p_1^2 \gg p_2^2$).
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