On the Funk-Radon-Helgason Inversion Method in Integral Geometry

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Dedicated to Professor Sigurdur Helgason on the occasion of his 85th birthday

ABSTRACT. The paper deals with totally geodesic Radon transforms on constant curvature spaces. We study applicability of the historically the first Funk-Radon-Helgason method of mean value operators to reconstruction of continuous and $L^p$ functions from their Radon transforms. New inversion formulas involving Erdélyi-Kober type fractional integrals are obtained. Particular emphasis is placed on the choice of the differentiation operator in the spirit of the recent Helgason’s formula.

1. Introduction

Inversion of Radon transforms is one of the central topics of integral geometry [GGG, GGV, He]. The method of mean value operators, when the unknown function is reconstructed from its spherical mean, was suggested by Funk [F11, F13] for circular transforms on the 2-sphere. It was adapted by Radon [R] for hyperplane transforms, and extended by Helgason [He] to totally geodesic transforms on arbitrary constant curvature space in any dimension.

In most publications the method of mean value operators is applied to infinitely differentiable rapidly decreasing functions. Below we investigate applicability of this method to arbitrary continuous and $L^p$ functions. As in the original works by Funk and Radon, we invoke Abel type integrals, which are basic objects of Fractional Calculus [Ru96, SKM]. Using tools of this branch of analysis, we show that the Funk-Radon-Helgason method can be successfully applied to

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derive new inversion formulas, which work well when “standard” procedures are inapplicable because of insufficient decay of functions at infinity or lack of smoothness. Importance of fractional integration in integral geometry was pointed out in a series of publications; see, e.g., [Gi, Ru98].

A customary ingredient of the method of mean value operators for the Radon transform \( \varphi = Rf \) over \( k \)-dimensional totally geodesic submanifolds is differentiation of form \( \partial/\partial r^2 \). For example, if \( k \) is even, then

\[
(1.1) \quad f(x) = \pi^{-k/2} \left( - \frac{\partial}{\partial r^2} \right)^{k/2} (R^*_x \varphi)(r) \bigg|_{r=0};
\]

cf. [Ru04b, p. 113] for the \( k \)-plane transform on \( \mathbb{R}^n \) or [He, p. 128] (with minor changes) for the similar transform on the hyperbolic space. Here \( (R^*_x \varphi)(r) \) is a certain mean value of \( \varphi \), which is called the shifted dual Radon transform [Rou] (precise definitions will be given later). A remarkable observation by Helgason [He, p. 116] is that for an even \( k \), to reconstruct \( f \) from the \( k \)-plane transform \( \varphi = Rf \), it suffices to differentiate in \( r \), rather than in \( r^2 \). Helgason’s result reads as follows:

\[
(1.2) \quad f(x) = c_k \left( \frac{\partial}{\partial r} \right)^k (R^*_x \varphi)(r) \bigg|_{r=0}, \quad c_k = \text{const.} \quad k \text{ even.}
\]

In the present article we show that compositions of the Erdélyi-Kober type fractional integrals with power weights yield more inversion formulas with “usual” differentiation \( (\partial/\partial r)^k \). This is done for totally geodesic transforms in all dimensions on arbitrary constant curvature space and under minimal assumptions for \( f \).

Pointwise inversion of Radon-like transforms of nonsmooth functions with minimal assumptions at infinity was studied in [BR04, Ru02a, Ru02b, Ru04a, Ru04b, So, Str], where one can find further references. Methods of these papers mainly deal with different kinds of singular integrals, wavelet transforms, and Riesz potentials. To the best of our knowledge, applicability of the method of mean value operators (which is historically the first) to such “rough” functions was not investigated before. The differentiation issue related to (1.2) is especially appealing. Some Radon inversion formulas with “usual” differentiation, but for smooth rapidly decreasing functions, were obtained in [AR, M]. The method of these papers differs from ours in principle.

Plan of the paper. Section 2 contains necessary preliminaries from Fractional Calculus. Sections 3, 4, and 5 deal with totally geodesic Radon transforms on the Euclidean space \( \mathbb{R}^n \), the \( n \)-dimensional hyperbolic space \( \mathbb{H}^n \), and the unit sphere \( S^n \), respectively. In Section 6 we give detailed proof of the Helgason’s formula (1.2), evaluate the
constant $c_k$ (that was not done in [He]), discuss related results and open problems.

**Notation and conventions.** In the following $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$; $e_1, \ldots, e_n$ are coordinate unit vectors; $o$ is the origin of $\mathbb{R}^n$. We say that $f$ is a locally integrable function on $\mathbb{R}_+ = (0, \infty)$ (resp., on $\mathbb{R}^n \setminus \{o\}$), if it is Lebesgue integrable on any interval $(a, b)$, $0 < a < b < \infty$ (resp., on any shell $0 < a < |x| < b < \infty$). The letter $c$ stands for a constant, which can be different at each occurrence. More notation will be introduced in due course.

2. **Preliminaries from Fractional Calculus**

We recall basic facts about fractional integration and differentiation with the main focus on integral-geometric applications in subsequent sections. More information can be found in [Ru96, SKM].

2.1. **Riemann-Liouville fractional integrals.** For a sufficiently good function $f$ on $\mathbb{R}_+ = (0, \infty)$, we consider two types of the Riemann-Liouville fractional integrals of order $\alpha > 0$:

$$(I^\alpha_+ f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s) \, ds}{(t-s)^{1-\alpha}}, \quad (I^\alpha_- f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{f(s) \, ds}{(s-t)^{1-\alpha}}.$$  

Existence of $I^\alpha_- f$ essentially depends on the behavior of $f$ at infinity.

**Lemma 2.1.** Let $f$ be a locally integrable function on $\mathbb{R}_+$. Then $(I^\alpha_- f)(t)$ is finite for almost all $t > 0$ provided

$$(I^\alpha_- f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{f(s) \, ds}{(s-t)^{1-\alpha}}.$$  

(2.1) $\int_1^\infty |f(s)| \, s^{\alpha-1} \, ds < \infty.$

If $f$ is non-negative and (2.1) fails, then $(I^\alpha_- f)(t) = \infty$ for every $t \geq 0$.

**Proof.** Note that the lower limit of integration in (2.1) can be replaced by any number $a > 0$. To prove the first statement, it suffices to show that

$$(I^\alpha_- f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (I^\alpha_- |f|)(t) \, dt < \infty$$

\footnotetext{1}{In many publications $I^\alpha_- f$ is called the Weyl fractional integral. However, Weyl’s publication is dated by 1917 and focused on periodic functions, while Liouville’s work, dealing with such integrals, was published in 1832; see historical notes in [SKM].}
for any $0 < a < b < \infty$. This can be done by changing the order of integration and using (2.1). To prove the second statement, we assume the contrary, that is, $(I_\alpha^f)(t)$ is finite, but (2.1) fails. Choose any $a > t$ and $N > 0$, and let first $\alpha \leq 1$. Since $(s-t)^{1-\alpha} \leq s^{1-\alpha}$, then

$$\int_t^{a+N} \frac{f(s) \, ds}{(s-t)^{1-\alpha}} > \int_t^{a+N} \frac{f(s) \, ds}{s^{1-\alpha}} > \int_0^{a+N} \frac{f(s) \, ds}{s^{1-\alpha}}$$

$$= \left( \int_1^{a+N} - \int_1^a \right) \frac{f(s) \, ds}{s^{1-\alpha}}.$$

If $N \to \infty$, then, by the assumption, the left-hand side remains bounded whereas the right-hand side tends to infinity. For $\alpha > 1$, we proceed as follows:

$$\int_t^{2a+N} (s-t)^{\alpha-1} f(s) \, ds > \int_2^{2a+N} (s-t)^{\alpha-1} f(s) \, ds$$

$$> 2^{1-\alpha} \int_2^{2a+N} s^{\alpha-1} f(s) \, ds$$

(note that $s-t > s-a > s/2$). The rest of the proof is as above. \(\square\)

Changing the order of integration, we easily get

$$I_\pm^\alpha I_\pm^\beta f = I_\pm^{\alpha+\beta} f, \quad I_\pm^\alpha t^{\alpha-\beta} I^\beta_- f = t^{-\beta} I_\pm^{\alpha+\beta} t^{-\alpha} f,$$

provided that integrals in either side exist in the Lebesgue sense. Here and on, powers of $t$ stand for the corresponding multiplication operators (instead of $t$ there may be another letter). The second equality can be alternatively derived from $I_\pm^\alpha I_\pm^\beta f = I_\pm^{\alpha+\beta} f$ if we replace variables by their reciprocals.

Fractional derivatives $D_\pm^\alpha \varphi$ of order $\alpha > 0$ are defined as left inverses of the corresponding fractional integrals, so that

$$D_\pm^\alpha I_\pm^\alpha f = f.$$

Operators $D_\pm^\alpha$ may have different analytic forms, depending on the class of functions. For example, if $\alpha = m + \alpha_0$, $m = [\alpha]$, $0 \leq \alpha_0 < 1$, then

$$D_\pm^\alpha \varphi = (\pm d/dt)^{m+1} I_\pm^{1-\alpha_0} \varphi.$$

The existence of the fractional derivative and the equality (2.3) must be justified at each occurrence.
2.2. Modified Erdélyi-Kober fractional integrals. There exist many modifications and generalizations of the Riemann-Liouville integrals, for instance,

\[
(I_{\pm,2}^\alpha f)(t) = \frac{2}{\Gamma(\alpha)} \int_0^t f(s) s^{1-\alpha} ds, \quad (I_{\pm,-2}^\alpha f)(t) = \frac{2}{\Gamma(\alpha)} \int_t^\infty f(s) s^{1-\alpha} ds.
\]

We call \(I_{\pm,2}^\alpha f\) the modified Erdélyi-Kober fractional integrals. They differ from the classical Erdélyi-Kober integrals, as in \([SKM, Sn]\), by weight factors. Clearly,

\[
(I_{\pm,2}^\alpha f)(t) = A^{-1} I_{\pm}^\alpha Af, \quad (Af)(t) = f(\sqrt{t}).
\]

The following statement is a consequence of Lemma 2.1:

**Lemma 2.2.** Let \(f\) be a locally integrable function on \(\mathbb{R}_+\). Then \((I_{\pm,2}^\alpha f)(t)\) is finite for almost all \(t > 0\) provided

\[
\int_1^\infty |f(s)| s^{2\alpha-1} ds < \infty.
\]

If \(f\) is non-negative and (2.6) fails, then \((I_{\pm,2}^\alpha f)(t) = \infty\) for every \(t \geq 0\).

**Lemma 2.3.** The following formulas hold provided that integrals on the right-hand side exist in the Lebesgue sense:

\[
(I_{\pm,2}^\alpha I_{\pm,2}^\beta f) = I_{\pm,2}^{\alpha+\beta} f,
\]

\[
(I_{-2}^\alpha t^{-2\alpha-2\beta} I_{-2}^\beta f) = t^{-2\beta} I_{-2}^{\alpha+\beta} t^{-2\alpha} f.
\]

\[
(t I_{-2}^\alpha t^{-2\alpha-2} I_{-2}^\alpha f) = 2^{2\alpha} I_{-2}^{2}\alpha f, \quad I_{+2}^{\alpha-\gamma} A I_{+2}^{\alpha} f = 2^{2\alpha} I_{+2}^{2\alpha} t f.
\]

**Proof.** Owing to (2.5), the first two formulas are immediate consequences of (2.2). To prove the first formula in (2.9), we change the order of integration and get

\[
l.h.s. = \frac{4}{\Gamma(\alpha)} \int_1^\infty f(s) I(s, t) ds,
\]

where

\[
I(s, t) = st \int \frac{(s^2 - r^2)^{\alpha-1}(r^2 - t^2)^{\alpha-1}r^{-2\alpha} dr}{(s^2 - t^2)^{\alpha-1}}
\]

(set \(\eta = (r^2 - t^2)/(s^2 - t^2)\))

\[
= \frac{s(s^2 - t^2)^{2\alpha-1}}{2^{2\alpha}} \int_0^1 \eta^{\alpha-1}(1 - \eta)^{\alpha-1} \left(1 - \eta \left(1 - \frac{s^2}{t^2}\right)\right)^{-\alpha-1} d\eta.
\]
The last integral represents the hypergeometric function and can be evaluated using formulas 2.1.3(10) and 2.8.(6) from [E]. This gives

\[ I(s, t) = \frac{2^{2\alpha - 2} \Gamma^2(\alpha)}{\Gamma(2\alpha)} (s - t)^{2\alpha - 1}, \]

and the result follows. The second formula in (2.9) can be obtained from the first one if we replace variables by their reciprocals. □

**Remark 2.4.** Formulas in (2.9), which express compositions of Erdélyi-Kober type integrals through the usual Riemann-Liouville integrals, are not well-known. They occur in the more general context related to Gegenbauer transformations; cf. [D, p. 120, formula (4.19)], [vBE, Theorem 2.2]. These formulas play an important role in our consideration; see Remark 2.6.

Fractional derivatives of the Erdélyi-Kober type can be defined as the left inverses \( D_{\pm}^\alpha = (I_{\pm}^\alpha)^{-1} \). By (2.5),

\begin{equation}
D_{\pm}^\alpha \varphi = A^{-1} D_{\pm}^\alpha A \varphi, \quad (Af)(t) = f(\sqrt{t}),
\end{equation}

where the Riemann-Liouville derivatives \( D_{\pm}^\alpha \) can be chosen in different forms, depending on our needs. For example, if \( \alpha = m + \alpha_0, \quad m = [\alpha], \quad 0 \leq \alpha_0 < 1 \), then, formally, (2.4) yields

\begin{equation}
D_{\pm}^\alpha \varphi = (\pm D)^{m+1} I_{\pm}^{1-\alpha_0} \varphi, \quad D = \frac{1}{2t} \frac{d}{dt}.
\end{equation}

Inversion of \( I_{\pm}^\alpha \) may cause difficulties. Let, for instance, \( \alpha = m + \alpha_0, \quad m = [\alpha], \quad 0 < \alpha_0 < 1 \). Then the standard complementation procedure, as in (2.11), can be inapplicable. Indeed, this formula assumes convergence of the integral \( I_{\pm}^{1-\alpha_0} \varphi = I_{\pm}^{1-\alpha_0} I_{\pm}^{m+\alpha_0} f = I_{\pm}^{m+1} f \)

or, equivalently, \( \int_1^\infty f(t) t^{2m+1} dt < \infty \). The latter is not guaranteed by (2.6), however, this obstacle can be circumvented.

**Theorem 2.5.** Let \( \varphi = I_{\pm}^\alpha f \), where \( f \) is a locally integrable function on \( \mathbb{R}_+ \), satisfying (2.6). Then \( f(t) = (D_{\pm}^\alpha \varphi)(t) \) for almost all \( t \in \mathbb{R}_+ \), where \( D_{\pm}^\alpha \varphi \) has one of the following forms.

(i) If \( \alpha = m \) is an integer, then

\begin{equation}
D_{\pm}^\alpha \varphi = (-D)^m \varphi, \quad D = \frac{1}{2t} \frac{d}{dt}.
\end{equation}

(ii) If \( \alpha = m + \alpha_0, \quad m = [\alpha], \quad 0 \leq \alpha_0 < 1 \), then

\begin{equation}
D_{\pm}^\alpha \varphi = t^{2(1-\alpha+m)} (-D)^{m+1} t^{2\alpha} \psi, \quad \psi = I_{\pm}^{1-\alpha+m} t^{-2m-2} \varphi.
\end{equation}
Alternatively,

\[
D_{\alpha}^{-2} \varphi = 2^{-2\alpha} D_{\alpha}^{-2} t I_{\alpha}^{-2} t^{-2\alpha-1} \varphi,
\]

where \(D_{\alpha}^{-2}\) denotes the Riemann-Liouville derivative of order \(2\alpha\), which can be computed according to (2.4).

(iii) If, moreover, \(\int_1^{\infty} |f(t)| t^{2m+1} dt < \infty\), then

\[
D_{\alpha}^{-2} \varphi = (-D)^m \Gamma^{-\alpha+m} \varphi.
\]

**Proof.** (i) is obvious. To prove (2.13), we swap \(\alpha\) with \(\beta\) in (2.8) to get

\[
I_{\alpha}^{\alpha+\beta} t^{-2\beta} f = t^{2\alpha} I_{\alpha}^{\beta} t^{-2\alpha-2\beta} I_{\alpha}^{-1} f.
\]

The existence of \(I_{\alpha}^{\alpha+\beta} t^{-2\beta} f\) is guaranteed by (2.6). Choosing \(\beta = 1 - \alpha + m\), we obtain (2.13). To prove (2.14), we observe that the existence conditions for the Erdélyi-Kober type integral \(I_{\alpha}^{m} f\) and the Riemann-Liouville integral \(I_{\alpha}^{-m} f\) coincide. Hence, (2.9) yields the result. Formula (2.15) follows from the semigroup property \(I_{\alpha}^{m+1} f = I_{\alpha}^{m} I_{\alpha}^{-1} f\) owing to (i). □

**Remark 2.6.** An advantage of the inversion formula (2.14), which follows from (2.9), in comparison with (2.12), (2.13), and (2.15), is that it employs the derivative \(d/dt\) rather than \(D = (2t)^{-1} d/dt = d/dt^2\).

Similarly, the second formula in (2.9) yields

\[
D_{\alpha}^{\alpha} \varphi = 2^{-2\alpha t^{-1}} D_{\alpha}^{2\alpha} I_{\alpha}^{-2} t^{-2\alpha} \varphi.
\]

In particular, if \(\alpha = k/2\), \(k \in \mathbb{N}\), then

\[
D_{\alpha}^{k/2} \varphi = 2^{-k} \left( \frac{d}{dt} \right)^k I_{\alpha}^{k/2} t^{1-k} \varphi,
\]

\[
D_{\alpha}^{-k/2} \varphi = 2^{-k} \left( -\frac{d}{dt} \right)^k I_{\alpha}^{k/2} t^{-k-1} \varphi.
\]

This observation will be used in inversion formulas for operators of integral geometry in the next sections.

### 3. The k-plane transforms

Let \(\Pi_{n,k}\) be manifold of all non-oriented \(k\)-planes \(\tau\) in \(\mathbb{R}^n\); \(G_{n,k}\) is the Grassmann manifold of \(k\)-dimensional linear subspaces \(\zeta\) of \(\mathbb{R}^n\); \(1 \leq k \leq n - 1\). Each \(k\)-plane \(\tau\) is parameterized by the pair \((\zeta, u)\) where \(\zeta \in G_{n,k}\) and \(u \in \zeta^\perp\) (the orthogonal complement of \(\zeta\) in \(\mathbb{R}^n\)). Clearly, \(\Pi_{n,k}\) is a bundle over \(G_{n,k}\) with an \((n-k)\)-dimensional fiber. The manifold \(\Pi_{n,k}\) is endowed with the product measure \(d\tau = d\zeta du\).
where $d\zeta$ is the $SO(n)$-invariant measure on $G_{n,k}$ of total mass 1, and $du$ denotes the usual volume element on $\zeta^\bot$; cf. [Matt, Chapter 3].

The $k$-plane transform $Rf$ of a function $f$ on $\mathbb{R}^n$ is defined by

$$(Rf)(\tau) \equiv (Rf)(\zeta,u) = \int f(u + v) \, dv$$

provided that this integral is meaningful. According to the general Funk-Radon-Helgason scheme, to reconstruct $f$ from $\varphi = Rf$ we need the following mean value operators

$$(3.1) \quad (\mathcal{M}_xf)(r) = \int_{SO(n)} f(x + r\gamma e_n) \, d\gamma, \quad r > 0,$$

$$(3.2) \quad (R^*_k \varphi)(r) = \int_{SO(n)} \varphi(\gamma\mathbb{R}^k + x + r\gamma e_n) \, d\gamma.$$

Here $\mathcal{M}_xf(r)$ is the usual spherical mean of $f$, $\mathbb{R}^k = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_k$, and $(R^*_k \varphi)(r)$ averages $\varphi$ over all $k$-planes at distance $r$ from $x$.

**Lemma 3.1.** Let $f$ be a locally integrable function on $\mathbb{R}^n \setminus \{0\}$. If $f$ is radial, i.e., $f(x) \equiv f_0(|x|)$, then

$$(3.3) \quad (Rf)(\tau) = \pi^{k/2} (I_{-\frac{k}{2}} f_0)(r), \quad r = \text{dist}(o,\tau).$$

More generally,

$$(3.4) \quad (R^*_k Rf)(r) = \pi^{k/2} (I_{-\frac{k}{2}} \mathcal{M}_xf)(r).$$

These equalities hold provided that expressions on either side are finite when $f$ is replaced by $|f|$.

Formulas (3.3) and (3.4) can be found in [Ru04b, p. 98, 110] and [He, p. 118], where the notation for the Erdélyi-Kober operators is not used. The first formula is a particular case of the second one, corresponding to $x = 0$ and $f$ radial.

**Theorem 3.2.** Let $f$ be a locally integrable function on $\mathbb{R}^n \setminus \{0\}$. If

$$(3.5) \quad \int_{|x|>1} \frac{|f(x)|}{|x|^{n-k}} \, dx < \infty,$$

then $(Rf)(\tau)$ is finite for almost all $\tau \in \Pi_{n,k}$. If $f$ is nonnegative, radial, and (3.5) fails, then $(Rf)(\tau) = \infty$ for every $\tau \in \Pi_{n,k}$.
PROOF. Let $f_0(r) = \int_{SO(n)} f(\gamma e_n) d\gamma$. Owing to (3.5), we have $\int_1^\infty |f_0(r)| r^{k-1} dr < \infty$ and, therefore, by Lemma 2.2, $(I_{-2}^{k/2} f_0)(r)$ is finite for almost all $r > 0$. On the other hand,

$$\int_{SO(n)} (Rf)(\gamma (\mathbb{R}^k + r e_n)) = (R_x^* Rf)(r) \big|_{x=0} = \pi^{k/2} (I_{-2}^{k/2} f_0)(r).$$

Hence, $(Rf)(\gamma (\mathbb{R}^k + r e_n)) < \infty$ for almost all pairs $(\gamma, r) \in SO(n) \times \mathbb{R}_+$. It means that $(Rf)(\tau)$ is finite for almost all $\tau \in \Pi_{n,k}$. If, for nonnegative $f \equiv f_0(|x|)$, (3.5) fails, then $\int_1^\infty f_0(r) r^{k-1} dr \equiv \infty$. Hence, Lemma 2.2 and (3.3) yield $\pi^{k/2} (I_{-2}^{k/2} f_0)(r) = (Rf)(\tau) \equiv \infty$. \hfill $\square$

According to the general Funk-Radon-Helgason formalism, our next aim is to reconstruct $(\mathcal{M}_x f)(r)$ from the equality

$$\pi^{k/2} (I_{-2}^{k/2} \mathcal{M}_x f)(r) = \pi^{-k/2} (R_x^* Rf)(r)$$

(cf. (3.4)), and then pass to the limit as $r \to 0$. We consider two classes of functions $f$ satisfying (3.5). The first one, denoted by $C_\mu(\mathbb{R}^n)$, consists of continuous functions of order $O(|x|^{-\mu})$. If $\mu > k$, then $(Rf)(\tau)$ is finite for every $\tau \in \Pi_{n,k}$. The second class is $L^p(\mathbb{R}^n)$. If $1 \leq p < n/k$, then by Hölder’s inequality, (3.5) is satisfied, and therefore, $(Rf)(\tau)$ is finite for almost all $\tau \in \Pi_{n,k}$.\footnote{Different proofs of this statement can be found in [Ru04b, So, Str].} The conditions $\mu > k$ and $1 \leq p < n/k$ are sharp. It means that there are functions $f_1 \in C_\mu(\mathbb{R}^n)$ and $f_2 \in L^p(\mathbb{R}^n)$ such that $Rf_1$ and $Rf_2$ are identically infinite if $\mu \leq k$ and $p \geq n/k$, respectively. For example, in the second case one can take

$$f_2(x) = \frac{(2 + |x|)^{-n/p}}{\log^{1/p+\delta}(2 + |x|)}, \quad 0 < \delta < 1/p', \quad 1/p + 1/p' = 1.$$ \hfill (3.7)

For this function, the integral on the right-hand side of (3.3) diverges.

**Lemma 3.3.**

(i) If $f \in C_\mu(\mathbb{R}^n)$, $\mu > k$, then for every $x \in \mathbb{R}^n$ and $r > 0$, the spherical mean $(\mathcal{M}_x f)(r)$ can be recovered from $\varphi = Rf$ by the formula

$$\pi^{k/2} (\mathcal{M}_x f)(r) = \pi^{-k/2} (\mathcal{D}_-^{k/2} R_x^* \varphi)(r),$$

where the Erdélyi-Kober differentiation operator $\mathcal{D}_-^{k/2}$ can be computed by (2.12), (2.13), or (2.14). Under the stronger assumption $\mu > 2 + 2[k/2]$ ($> k$), $\mathcal{D}_-^{k/2}$ can also be computed by (2.15).

(ii) If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < n/k$, then (3.8) holds for every $r > 0$ and almost all $x \in \mathbb{R}^n$ with $\mathcal{D}_-^{k/2}$ computed by (2.12), (2.13), or (2.14). If,
moreover, \( p < n/(2 + 2[k/2]) \) \( ( < \frac{n}{k}) \), then \( D^{k/2}_{-2} \) can also be computed by (2.15).

**Proof.** We need to justify (a) the validity of (3.6) for our classes of functions, and (b) applicability of Theorem 2.5.

(a) It suffices to verify convergence of \((I_{-2}^{k/2}M_xf)(r)\) in (3.6) for nonnegative \( f \). If \( f \in C^\mu_\mu(\mathbb{R}^n) \), \( \mu > k \), then

\[
(I_{-2}^{k/2}M_xf)(r) = \frac{2}{\Gamma(k/2) \sigma_{n-1}} \int_0^\infty (t^2 - r^2)^{k/2-1} t \int_{S^{n-1}} f(x - t\theta) d\theta dt
\]

where \( c_x \) is finite. Hence, \((I_{-2}^{k/2}M_xf)(r) < \infty\) for every \( x \) and \( r \). If \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p < \frac{n}{k} \), then

\[
(I_{-2}^{k/2}M_xf)(r) = \frac{2}{\Gamma(k/2) \sigma_{n-1}} \left( \int_{r<|y|<2r} + \int_{|y|>2r} \right) f(x - y)(|y| - r^2)^{k/2-1} \frac{dy}{|y|^{n-2}}.
\]

The first integral is finite for almost all \( x \), because it has a finite \( L^p \)-norm (use Minkowski’s inequality for integrals). The second integral does not exceed

\[
c \int_{|y|>2r} f(x - y) \frac{dy}{|y|^{n-k}}, \quad c = c(t, \alpha).
\]

By Hölder’s inequality, it is bounded for all \( x \) when \( p < \frac{n}{k} \). Thus, \((I_{-2}^{k/2}M_xf)(r) < \infty\) for every \( r > 0 \) and almost all \( x \).

We have proved that (3.6) holds for every \( r > 0 \) and all or almost all \( x \), depending on whether \( f \in C^\mu_\mu \) or \( f \in L^p \).

(b) To obey Theorem 2.5, we have to show that \((M_xf)(t)\) is locally integrable on \( \mathbb{R}_+ \) and

\[
\int_1^\infty |(M_xf)(t)| t^\lambda dt < \infty,
\]

where \( \lambda = k - 1 \) for formulas (2.12), (2.13), (2.14), and \( \lambda = 2[k/2] + 1 \) for (2.15). If \( f \in C^\mu_\mu(\mathbb{R}^n) \), both statements are immediate consequences of (3.9) and the assumptions for \( \mu \). If \( f \in L^p(\mathbb{R}^n) \), \( p \neq 1 \), then, for any
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0 < a < b < ∞ and all x,
\[
\int_a^b |(\mathcal{M}_x f)(t)| \, dt \leq \frac{1}{\sigma_{n-1}} \int_{|y|<b} \frac{|f(x-y)|}{|y|^{n-1}} \, dy
\]
\[
\leq \frac{||f||_p}{\sigma_{n-1}} \left( \int_{|y|>1} \frac{dy}{|y|^{(n-\lambda)p'}} \right)^{1/p'} < \infty.
\]
Similarly,
\[
\int_1^\infty |(\mathcal{M}_x f)(t)| t^\lambda \, dt \leq \frac{||f||_p}{\sigma_{n-1}} \left( \int_{|y|>1} \frac{dy}{|y|^{(n-\lambda)p'}} \right)^{1/p'} < \infty
\]
in view of the assumptions for \( \lambda \) and \( p \). If \( p = 1 \) the changes in this reasoning are obvious.

To complete the proof, we note that Theorem 2.5 gives us \((\mathcal{M}_x f)(r)\) only for almost all \( r \). However, if \( f \in C_\mu(\mathbb{R}^n) \), then \((\mathcal{M}_x f)(r)\) is continuous (in both variables), and if \( f \in L^p(\mathbb{R}^n) \), then \((\mathcal{M}_x f)(r)\) is an \( L^p \)-valued continuous function of \( r \). It follows that the inversion formula (3.8) holds for every \( r > 0 \) for both \( C_\mu \) and \( L^p \) spaces. However, in the first case it is valid for all \( x \in \mathbb{R}^n \), and in the second case for almost all \( x \).

Lemma 3.3 and Theorem 2.5 imply the following theorem, containing the main inversion results for \( Rf \).

**Theorem 3.4.** A function \( f \in C_\mu(\mathbb{R}^n), \mu > k, \) can be recovered from \( \varphi = Rf \) by the formula
\[
f(x) = \lim_{t \to 0} \pi^{k/2}(\mathcal{D}^{k/2}_{-2} R^*_x \varphi)(t),
\]
where the limit is uniform on \( \mathbb{R}^n \) and the Erdélyi-Kober differential operator \( \mathcal{D}^{k/2}_{-2} \) can be computed as follows.

(i) If \( k \) is even, then
\[
\mathcal{D}^{k/2}_{-2} F = (-D)^{k/2} F, \quad D = \frac{1}{2t} \frac{d}{dt}.
\]
(ii) For any \( 1 \leq k \leq n - 1, \)
\[
\mathcal{D}^{k/2}_{-2} F = t^{2-k+2m}(-D)^{m+1} t^k \psi, \quad \psi = R^{1-k/2+m}_{-2} t^{-2m+2} F,
\]
where \( m = \lfloor k/2 \rfloor \). Alternatively,
\[
\mathcal{D}^{k/2}_{-2} F = 2^{-k} \left( -\frac{d}{dt} \right)^k t^{k/2} t^{-k-1} F.
\]
Under the stronger assumption \( \mu > 2 + 2[k/2] (> k) \), \( \mathcal{D}_{-2}^{k/2} \) can also be computed as

\[
\mathcal{D}_{-2}^{k/2} F = (-D)^{m+1} \mathcal{I}_{-2}^{1-a+m} F.
\]

Note that (3.14) employs usual differentiation \( d/dt \) rather than \( D \).

The next theorem contains similar results for \( L^p \)-functions.

**Theorem 3.5.** A function \( f \in L^p(\mathbb{R}^n), 1 \leq p < n/k \), can be recovered from \( \varphi = Rf \) at almost every \( x \in \mathbb{R}^n \) by the formula

\[
f(x) = \lim_{t \to 0} \pi^{-k/2} (\mathcal{D}_{-2}^{k/2} R_x^* \varphi)(t),
\]

where the limit is understood in the \( L^p \)-norm. Here \( \mathcal{D}_{-2}^{k/2} \) is computed as in Theorem 3.4, where (3.15) is applicable under the stronger assumption \( 1 \leq p < n/(2 + 2[k/2]) \).

## 4. The Hyperbolic Space

Let \( E^{n,1}, n \geq 2 \), be the pseudo-Euclidean space of points \( x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \) with the inner product

\[
[x, y] = -x_1y_1 - \ldots - x_ny_n + x_{n+1}y_{n+1}.
\]

We realize the \( n \)-dimensional hyperbolic space \( X \) as the upper sheet of the two-sheeted hyperboloid

\[
\mathbb{H}^n = \{ x \in \mathbb{E}^{n,1} : [x, x] = 1, x_{n+1} > 0 \}.
\]

Let \( \Xi \) be the set of all \( k \)-dimensional totally geodesic submanifolds \( \xi \) of \( X \), \( 1 \leq k \leq n - 1 \). As usual, \( e_1, \ldots, e_{n+1} \) denote the coordinate unit vectors. We set \( \mathbb{R}^{n+1} = \mathbb{R}^{n-k} \oplus \mathbb{R}^{k+1} \), where

\[
\mathbb{R}^{n-k} = \mathbb{R} e_1 \oplus \ldots \oplus \mathbb{R} e_{n-k}, \quad \mathbb{R}^{k+1} = \mathbb{R} e_{n-k+1} \oplus \ldots \oplus \mathbb{R} e_{n+1},
\]

and identify \( \mathbb{R}^{k+1} \) with the pseudo-Euclidean space \( E^{k,1} \).

In the following \( x_0 = (0, \ldots, 0, 1) \) and \( \xi_0 = \mathbb{H}^n \cap \mathbb{E}^{k,1} = \mathbb{H}^k \) denote the origins in \( X \) and \( \Xi \) respectively; \( G = SO_0(n, 1) \) is the identity component of the pseudo-orthogonal group \( O(n, 1) \) preserving the bilinear form (4.1); \( K = SO(n) \) and \( H = SO(n-k) \times SO_0(k, 1) \) are the isotropy subgroups of \( x_0 \) and \( \xi_0 \), so that \( X = G/K, \ \Xi = G/H \). One can write \( f(x) \equiv f(gK), \varphi(\xi) \equiv \varphi(gH), g \in G \). The geodesic distance between points \( x \) and \( y \) in \( X \) is defined by \( d(x, y) = \cosh^{-1}[x, y] \). Each \( x \in X \) can be represented in the hyperbolic polar coordinates as

\[
x = \zeta \sinh \omega + e_{n+1} \cosh \omega,
\]
where $\zeta$ is a point of the unit sphere $S^{n-1}$ in the plane $x_{n+1} = 0$ and $0 \leq \omega < \infty$. In these coordinates the Riemannian measure $dx$ on $X$ has the form $dx = \sinh^{n-1}\omega d\omega d\sigma(\zeta)$, so that

$$\int_X f(x) dx = \int_0^\infty \sinh^{n-1}\omega d\omega \int_{S^{n-1}} f(\zeta \sinh \omega + e_{n+1} \cosh \omega) d\sigma(\zeta).$$

In particular, if $f$ is $K$-invariant (or zonal), that is, $f(x) \equiv f_0(\cosh \omega) = f_0(x_{n+1})$, then

$$\int_X f(x) dx = \sigma_{n-1} \int_1^\infty f_0(s)(s^2 - 1)^{n/2 - 1} ds.$$

The space $L^p(X)$ (with respect to $dx$ above) is defined in a standard way; $C(X)$ is the space of continuous functions on $X$; $C_0(X)$ denotes the space of continuous functions on $X$ vanishing at infinity. We also define

$$C_\mu(X) = \{ f \in C(X) : f(x) = O(x_{n+1}^{-\mu}) \}.$$

Of course, it might be natural to compare functions at infinity with powers of the geodesic distance $d(x_0, x)$ (as in the case of $\mathbb{R}^n$), however, it is technically more convenient to use powers of $x_{n+1} = \cosh d(x_0, x)$.

For $x \in X$ and $\xi \in \Xi$, we denote by $r_x$ and $r_\xi(\in G)$ arbitrary hyperbolic rotations satisfying $r_x x_0 = x$, $r_\xi x_0 = \xi$, and write

$$f_\xi(x) = f(r_\xi x), \quad \varphi_\xi(\xi) = \varphi(r_\xi \xi).$$

The totally geodesic Radon transform $(Rf)(\xi)$ of a sufficiently good function $f$ on $X = \mathbb{H}^n$ is defined by

$$(Rf)(\xi) = \int f(x) d_\xi x \equiv \int_{SO_0(k,1)} f_\xi(\gamma x_0) d\gamma, \quad \xi \in \Xi.$$

The first question is for which functions the integral (4.5) exists.

**Theorem 4.1.**

(i) If $f \in L^p(X)$, $1 \leq p < (n - 1)/(k - 1)$, then $(Rf)(\xi)$ is finite for almost all $\xi \in \Xi$.

(ii) If $f \in C_\mu(X)$, $\mu > k - 1$, then $(Rf)(\xi)$ is finite for all $\xi \in \Xi$.

**Proof.** (i) We make use of the equality

$$\int_\Xi \frac{(Rf)(\xi)}{\cosh^n d(x_0, \xi)} d\xi = \int_X f(x) \frac{dx}{x_{n+1}^{n-k}},$$

$$\int_X f(x) dx = \int_0^\infty \sinh^{n-1}\omega d\omega \int_{S^{n-1}} f(\zeta \sinh \omega + e_{n+1} \cosh \omega) d\sigma(\zeta).$$

The space $L^p(X)$ (with respect to $dx$ above) is defined in a standard way; $C(X)$ is the space of continuous functions on $X$; $C_0(X)$ denotes the space of continuous functions on $X$ vanishing at infinity. We also define

$$C_\mu(X) = \{ f \in C(X) : f(x) = O(x_{n+1}^{-\mu}) \}.$$

Of course, it might be natural to compare functions at infinity with powers of the geodesic distance $d(x_0, x)$ (as in the case of $\mathbb{R}^n$), however, it is technically more convenient to use powers of $x_{n+1} = \cosh d(x_0, x)$.

For $x \in X$ and $\xi \in \Xi$, we denote by $r_x$ and $r_\xi(\in G)$ arbitrary hyperbolic rotations satisfying $r_x x_0 = x$, $r_\xi x_0 = \xi$, and write

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**Proof.** (i) We make use of the equality

$$\int_\Xi \frac{(Rf)(\xi)}{\cosh^n d(x_0, \xi)} d\xi = \int_X f(x) \frac{dx}{x_{n+1}^{n-k}},$$
which holds provided that either of these integrals is finite when $f$ is replaced by $|f|$; see [BR04, p. 48]. By H"older’s inequality the righthand side does not exceed $c \|f\|_p$, where, by (4.4),

$$\varphi' = \int_X \frac{dx}{x^{(n-k)p'}} = \sigma_{n-1} \int_1^{\infty} \frac{(s^2 - 1)^{n/2-1}}{s^{(n-k)p'}} ds.$$

The latter is finite if $1 \leq p < (n-1)/(k-1)$.

(ii) Consider an arbitrary $k$-geodesic $\xi \in \Xi$. Let $x_\xi$ be the point in $\xi$ at the minimum distance from the origin $x_0$ and let $x$ be an arbitrary point in $\xi$. Then the angle $x_0x_\xi x$ is $90^\circ$ and, by the hyperbolic trigonometry, see, e.g., [Rat, p. 102],

$$x_{n+1} = \cosh d(x_0, x) = \cosh d(x_0, x_\xi) \cosh d(x_\xi, x).$$

If $f \in C_\mu(X)$, then, denoting $\lambda_\xi = \cosh d(x_0, x_\xi)$ and using (4.7), we obtain

$$|(Rf)(\xi)| \leq c \int_\xi \frac{d\xi x}{x_{n+1}^\mu} = c \lambda_\xi^{-\mu} \int_\xi \frac{d\xi x}{\cosh^\mu d(x_\xi, x)}.$$

Let $\gamma \in G$ be a hyperbolic rotation that sends $\xi_0$ to $\xi$ and such that $\gamma x_0 = x_\xi$. Changing variable $x = \gamma y$, and using (4.4), we write the last integral in (4.8) as

$$\int_{\xi_0} \gamma_{n+1}^{-\mu} dy = \sigma_{k-1} \int_1^{\infty} (s^2 - 1)^{k/2-1} s^{-\mu} ds.$$

This integral is finite if $\mu > k - 1$. \[\square\]

**Remark 4.2.** The restrictions $\mu > k - 1$ and $1 \leq p < (n-1)/(k-1)$ are sharp; see Example 4.6 below.

Now we introduce the mean value operators, which will be used in the inversion procedure. Let

$$g_\theta = \begin{bmatrix} \cosh \theta & 0 & \sinh \theta \\ 0 & I_{n-1} & 0 \\ \sinh \theta & 0 & \cosh \theta \end{bmatrix},$$

where $I_{n-1}$ is the unit matrix of dimension $n - 1$. Clearly,

$$g_\theta x_0 = g_\theta e_{n+1} = e_1 \sinh \theta + e_{n+1} \cosh \theta.$$
The shifted dual Radon transform of a function $\varphi$ on $\Xi$, which averages $\varphi$ over all $\xi \in \Xi$ at geodesic distance $\theta$ from $x$, is defined by

$$ (R_x^r \varphi)(r) = \int_{K} \varphi_x(\gamma g_\theta^{-1} \xi_0) \, d\gamma, \quad \sinh \theta = r. $$

The case $r = 0$ corresponds to the usual dual Radon transform.

Given $x \in \mathbb{H}^n$ and $s > 1$, let

$$ (M_x f)(s) = \frac{(s^2 - 1)^{(1-n)/2}}{\sigma_{n-1}} \int_{\{y \in \mathbb{H}^n : [x,y] = s\}} f(y) \, d\sigma(y) $$

be the spherical mean of $f$ on $X = \mathbb{H}^n$, where $d\sigma(y)$ stands for the relevant induced Lebesgue measure.

**Lemma 4.3.** [L, pp. 131-133], [BR99, Lemma 2.1].

(i) $\sup_{s > 1} \| (M_x f)(s) \|_p \leq \| f \|_p$, $f \in L^p(X)$, $1 \leq p \leq \infty$.

(ii) If $f \in L^p(X)$, $1 \leq p < \infty$, then $\lim_{s \to 1} \| (M_x f)(s) - f \|_p = 0$. If $f \in C_0(X)$, then $(M_x f)(s) \to f(x)$ as $s \to 1$, uniformly on $X$.

For our purposes it is convenient to set

$$ (\tilde{M}_x f)(t) = (1 + t^2)^{-1/2} (M_x f)(\sqrt{1 + t^2}). $$

Then $\lim_{t \to 0} (\tilde{M}_x f)(x) = f(x)$ as in Lemma 4.3.

**Lemma 4.4.** Let $f$ be a locally integrable function on $\mathbb{H}^n \setminus \{x_0\}$.

If $f$ is zonal, $f(x) \equiv f_0(\cosh \omega) = f_0(x_{n+1})$, and

$$ \tilde{f}_0(t) = (1 + t^2)^{-1/2} f_0(\sqrt{1 + t^2}), $$

then

$$ (Rf)(\xi) = \frac{\sigma_{k-1}}{(1 + r^2)^{(k-1)/2}} \int_0^\infty \tilde{f}_0(t)(t^2 - r^2)^{k/2 - 1} \, t \, dt $$

$$ (4.12) = \frac{\pi^{k/2}}{(1 + r^2)^{(k-1)/2}} (I_{-2} f_0)(r), \quad r = \sinh \theta. $$

More generally,

$$ (R_x^r \varphi)(r) = \frac{\pi^{k/2}}{(1 + r^2)^{(k-1)/2}} (I_{-2} \tilde{M}_x f)(r). $$

(4.13)

These equalities hold provided that expressions on either side are finite when $f$ is replaced by $|f|$.
Proof. It is known that for $\tau = \cosh d(x_0, \xi)$,

$$ (Rf)(\xi) = \frac{\sigma_{k-1}}{\tau^{k-1}} \int_{\tau}^{\infty} f_0(s)(s^2 - \tau^2)^{k/2-1} ds; $$

see [He, p. 121], [BR04, Lemma 3]. Changing variables

$$ r = \sqrt{\tau^2 - 1}, \quad s = \sqrt{1 + t^2}, $$

we obtain (4.12). Similarly,

$$ (R^*_{x}f)(\sqrt{\tau^2 - 1}) = \frac{\sigma_{k-1}}{\tau^{k-1}} \int_{\tau}^{\infty} (M_xf)(s)(s^2 - \tau^2)^{k/2-1} ds; $$

cf. [He, p. 128] and [Ru02a, formula (2.3)]. This gives (4.13). $\square$

The following statement provides additional information about the existence of the Radon transform.

**Theorem 4.5.** Let $f$ be a locally integrable function on $\mathbb{H}^n \setminus \{x_0\}$ satisfying

$$ \int_{x_{n+1}>2} \frac{|f(x)|}{x_{n-k}^{n-1}} \, dx < \infty. $$

Then $(Rf)(\xi)$ is finite for almost all $\xi \in \Xi$. If, moreover, $f$ is non-negative, zonal, and (4.15) fails, then $(Rf)(\xi) = \infty$ for every $\xi \in \Xi$.

**Proof.** Let

$$ f_0(s) = \int_K f(\gamma(e_1 \sqrt{s^2-1} + e_{n+1} s)) \, d\gamma, \quad \tilde{f}_0(t) = (1+t^2)^{-1/2} f_0(\sqrt{1+t^2}). $$

Then (4.15) is equivalent to

$$ I = \int_{a}^{\infty} |\tilde{f}_0(t)| t^{k-1} \, dt < \infty, \quad \forall a > 0. $$

Indeed, setting $s = \sqrt{1+t^2}$, we get
\[ I = \int_{b}^{\infty} |f_0(s)| \left( s^2 - 1 \right)^{k/2 - 1} ds \quad (b = \sqrt{1 + a^2} > 1) \]

\[ \leq c_b \int_{b}^{\infty} \left| f_0(s) \right| \left( s^2 - 1 \right)^{n/2 - 1} \frac{ds}{s^{n-k}} \]

\[ = c_b \int_{\text{cosh}^{-1} b}^{\infty} \frac{\sinh^{n-1} \omega}{\cosh^{n-k} \omega} d\omega \int_{\kappa}^{\infty} |f(\gamma e_1 \sinh \omega + e_{n+1} \cosh \omega)| d\gamma \]

\[ = \frac{c_b}{\sigma_{n-1}} \int_{x_{n+1} > b}^{\infty} \frac{|f(x)|}{x_{n+1}^{n-k}} dx. \]

Calculations in the opposite direction are similar. Thus, by Lemma 2.2, if (4.15) holds, then \((I_{k/2} \check{f}_0)(r)\) is finite for almost all \(r > 0\). On the other hand, for \(r = \sinh \theta\), Lemma 4.4 yields

\[ \int_{K}^{\infty} (Rf)(\gamma g_0^{-1} \xi_0) d\gamma \equiv (R^{x_0} R f)(r) \]

\[ = \frac{\pi^{k/2}}{(1 + r^2)^{(k-1)/2}} (I_{k/2} \bar{M}_{x_0} f)(r) = \frac{\pi^{k/2}}{(1 + r^2)^{(k-1)/2}} (I_{k/2} \check{f}_0)(r). \]

Since \((I_{k/2} \check{f}_0)(r)\) is finite for almost all \(r > 0\), then \((Rf)(\gamma g_0^{-1} \xi_0) < \infty\) for almost all pairs \((\gamma, \theta) \in K \times \mathbb{R}_+\). It means that \((Rf)(\xi)\) is finite for almost all \(\xi \in \Xi\).

If, for \(f \equiv f_0(x_{n+1}) \geq 0\), (4.15) fails, then \(\int_{1}^{\infty} \check{f}_0(t) t^{k-1} dt \equiv \infty\). Hence, by Lemma 2.2, \((I_{k/2} \check{f}_0)(r) \equiv \infty\) and, by (4.12), \((Rf)(\xi) \equiv \infty\). \(\square\)

The following example of application of Theorem 4.5 shows that the restrictions \(1 \leq p < (n - 1)/(k - 1)\) and \(\mu > k - 1\) in Theorem 4.1 are sharp.

**Example 4.6.** Consider the following functions

\[ f_1(x) = \frac{x^{(1-n)/p}}{\log(1 + x_{n+1})}, \quad f_2(x) = \frac{x_{n+1}^{-\mu}}{\log(1 + x_{n+1})}. \]

By (4.4),

\[ \|f_1\|_p^p = \sigma_{n-1} \int_{1}^{\infty} \frac{(s^2 - 1)^{n/2 - 1}}{s^{n-1} \log(1 + s)} ds < \infty. \]
However, if \( p \geq (n - 1)/(k - 1) \), then (4.15) fails because

\[
\int_{x_{n+1}>2} \frac{f_1(x)}{x^{n-k}} \, dx = \sigma_{n-1} \int_2^\infty \frac{(s^2 - 1)^{n/2-1}}{s^{(n-1)/p} \log(1 + s)} \, ds = \infty.
\]

Similarly, \( f_2 \in \mathcal{C}_\mu(\mathbb{H}^n) \), however, if \( \mu \leq k - 1 \), then

\[
\int_{x_{n+1}>2} \frac{f_2(x)}{x^{n-k}} \, dx = \sigma_{n-1} \int_2^\infty \frac{(s^2 - 1)^{n/2-1}}{s^{n-k+\mu} \log(1 + s)} \, ds = \infty.
\]

As in the preceding section, our next aim is to reconstruct \((\tilde{M}_xf)(r)\) from the equality

\[
(I_{k/2} \tilde{M}_xf)(r) = \pi^{-k/2}(1 + r^2)^{(1-k)/2}(R^*_x f)(r)
\]

(cf. (3.6)) and then pass to the limit as \( r \to 0 \).

**Lemma 4.7.** Let \( X = \mathbb{H}^n \).

(i) If \( f \in \mathcal{C}_\mu X \), \( \mu > k - 1 \), then for every \( x \in X \) and \( r > 0 \), the spherical mean \((\tilde{M}_xf)(r)\) can be recovered from \( \varphi = Rf \) by the formula

\[
(\tilde{M}_xf)(r) = \pi^{-k/2}(D_{k/2}^\mu(1 + r^2)^{(1-k)/2}R^*_x \varphi)(r),
\]

where the Erdély-Kober differentiation operator \( D_{k/2}^\mu \) is computed by (2.12), (2.13), or (2.14). Under the stronger assumption \( \mu > 2[k/2]+1 \), \( D_{k/2}^\mu \) can also be computed by (2.15).

(ii) If \( f \in L^p(X) \), \( 1 \leq p < (n - 1)/(k - 1) \), then (4.18) holds for every \( r > 0 \) and almost all \( x \in X \) with \( D_{k/2}^\mu \) computed by (2.12), (2.13), or (2.14). If, moreover, \( p < (n - 1)/(2[k/2] + 1) \), then \( D_{k/2}^\mu \) can also be computed by (2.15).

**Proof.** As in Lemma 3.3, we first show (a) convergence of \((I_{k/2} \tilde{M}_xf)(r)\) for nonnegative \( f \), and (b) applicability of Theorem 2.5.

(a) We have

\[
I \equiv (I_{k/2} \tilde{M}_xf)(r) \leq c \int_r^\infty (\tilde{M}_xf)(t)(t^2 - r^2)^{k/2-1} \, t \, dt
\]

\[
= c \int_0^\rho (M_x f)(s)(s^2 - \rho^2)^{k/2-1} \, ds, \quad \rho = \sqrt{1 + r^2}.
\]
Setting $s = \cosh \omega$, $\rho = \cosh \theta$, and denoting $f_x(y) = f(\gamma_x y)$, where $\gamma_x \in G$, $\gamma_x x_0 = x$, we continue:

\[
I \leq c \int_\rho^\infty (\cosh^2 \omega - \cosh^2 \theta)^{k/2-1} \sinh^{2-n} \omega \, d\omega \int_{[x,y] = \cosh \omega} f(y) \, d\sigma(y)
\]

\[
= c \int_\theta^\infty (\cosh^2 \omega - \cosh^2 \theta)^{k/2-1} \sinh \omega \, d\omega \\
\times \int_{S^{n-1}} f_x(\zeta \sinh \omega + e_{n+1} \cosh \omega) \, d\sigma(\zeta).
\]

By (4.3) this gives

\[
(4.19) \quad I \leq \int_{y_{n+1} > \rho} \frac{(y_{n+1}^2 - \rho^2)^{k/2-1}}{(y_{n+1}^2 - 1)^{n/2-1}} f_x(y) \, dy.
\]

This integral has a structure of the hyperbolic convolution

\[
(Kf)(x) = \int_X f(y) k([x,y]) \, dy = \int_X f_x(y) k(y_{n+1}) \, dy,
\]

which can be “lifted” to a convolution operator on $G$. By Young’s inequality [HR, Chapter 5, Theorem 20.18],

\[
(4.20) \quad \|Kf\|_q \leq \|f\|_p \|k\|_r,
\]

where $1 \leq p \leq q \leq \infty$, $1 - p^{-1} + q^{-1} = r^{-1}$,

\[
\|k\|_r^r = \left. \int_1^\infty |k(t)|^r \left( t^2 - 1 \right)^{n/2-1} \, dt \right|_{r-1}.
\]

We split the integral in (4.19) in two pieces $I = I_1 + I_2$, corresponding to $\rho < y_{n+1} < 2\rho$ and $y_{n+1} > 2\rho$, and consider the cases $f \in C_\mu(X)$ and $f \in L^p(X)$ separately.

In the first case, when $f(y) \leq c y_{n+1}^{-\mu}$, we have

\[
I_1 \leq c \int_{\rho < y_{n+1} < 2\rho} \frac{(y_{n+1}^2 - \rho^2)^{k/2-1}}{(y_{n+1}^2 - 1)^{n/2-1}} dy \left[ y_{n+1} x_0, e_{n+1} \right]^\mu.
\]
This integral can be estimated using (4.20) with \( r = 1, \ p = q = \infty \), as follows: \( I_1 \leq cA \), where

\[
(4.21) \quad A = \int_{\rho < y_{n+1} < 2\rho} \frac{(y_{n+1}^2 - \rho^2)^{k/2-1}}{(y_{n+1}^2 - 1)^{n/2-1}} \, dy
\]

\[
= c \sigma_{n-1} \int_{\rho}^{2\rho} (s^2 - \rho^2)^{k/2-1} \, ds < \infty \quad \forall \rho \geq 1
\]

(the last equality holds by (4.4)). To estimate \( I_2 \), we apply (4.20) with \( r = p' \) and \( q = \infty \), to get

\[
(4.22) \quad B = \int_X y_{n+1}^{-p} \, dy, \quad C = \int_{y_{n+1} > 2\rho} \frac{(y_{n+1}^2 - \rho^2)^{(k/2-1)p'}}{(y_{n+1}^2 - 1)^{(n/2-1)p'}} \, dy.
\]

The first integral is finite if \( p > (n-1)/\mu \), whereas the second one is finite (for every \( \rho \geq 1 \)) if \( p < (n-1)/(k-1) \). Since both conditions are consistent when \( \mu > k - 1 \), we are done.

Consider the case \( f \in L^p(X) \) and estimate \( I_1 \) by using (4.20) with \( r = 1, \ p = q \). This gives \( ||I_1||_p \leq A||f||_p \), where \( A \) has the form (4.21). For \( I_2 \), we apply (4.20) with \( r = p' \) and \( q = \infty \), so that \( I_2 \leq c C^{1/p'}||f||_p \), where \( C \) is the same as in (4.22). It follows that, for \( 1 \leq p < (n-1)/(k-1) \), \( I \) is finite for almost all \( x \) and all \( \rho \geq 1 \).

(b) To justify applicability of Theorem 2.5, we have to show that \( (\tilde{M}_x f)(t) \) is locally integrable on \( \mathbb{R}_+ \) and

\[
(4.23) \quad \psi(x) \equiv \int_1^{\infty} ||(\tilde{M}_x f)(t)||_t^\lambda \, dt < \infty,
\]

where \( \lambda = k - 1 \) for formulas (2.12), (2.13), (2.14), and \( \lambda = 2[k/2] + 1 \) for (2.15). Proceeding as in Part (a) of the proof of Lemma 4.7, for any \( 0 < a < b < \infty \) we obtain

\[
\int_a^b ||(\tilde{M}_x f)(t)|| \, dt \leq c \int_{a_1 < y_{n+1} < b_1} |f_x(y)| \, dy,
\]

for some \( 1 < a_1 < b_1 < \infty \) depending on \( a \) and \( b \).

If \( f \in C^\mu_p(X) \), \( \mu > 0 \), the last integral is bounded uniformly in \( x \) because \( |f_x(y)| \leq c [\gamma_x e_{n+1}]^{-\mu} \) and \( [\gamma_x e_{n+1}] \geq 1 \). If \( f \in L^p(X) \), then, by (4.20) (with \( r = 1, \ q = p \)), the \( L^p \)-norm of this integral does not exceed \( c ||f||_p \). Hence \( (\tilde{M}_x f)(t) \) is locally integrable on \( \mathbb{R}_+ \) for almost all \( x \).
Similarly we get
\[ \psi(x) \equiv \int_{1}^{\infty} |(M_x f)(t)| t^\lambda \, dt \leq c \int_{y_{n+1} > \sqrt{2}} (y_{n+1}^2 - 1)^{(\lambda+1-n)/2} |f_x(y)| \, dy. \]
If \( f \in L^p(X) \), then, as above, \( \psi(x) \leq c c_{\lambda}^{1/p'} ||f||_p \), where, for \( p > 1 \),
\[ (4.24) \quad c_{\lambda} = \int_{y_{n+1} > \sqrt{2}} (y_{n+1}^2 - 1)^{(\lambda+1-n)p'/2} \, dy = \sigma_{n-1} \int_{\sqrt{2}}^{\infty} (s^2 - 1)^{k/2} \, ds, \]
\[ \delta = (\lambda + 1 - n)p + n - 2. \] The last integral is finite provided that
\[ p < (n - 1)/\lambda = \begin{cases} (n - 1)/(k - 1), & \text{if } \lambda = k - 1, \\ (n - 1)/(2[k/2] + 1), & \text{if } \lambda = 2[k/2] + 1. \end{cases} \]
If \( p = 1 \), then \( \psi(x) \leq c ||f||_1 \) if \( \lambda \leq n - 1 \). In the case \( \lambda = k - 1 \) this inequality does not restrict the range of \( k \). If \( \lambda = 2[k/2] + 1 \), then \( \lambda \leq n - 1 \) is equivalent to \( [k/2] \leq n/2 - 1 \) and the case \( k = n - 1 \) with \( n \) odd must be excluded.

If \( f \in C_\mu(X) \), then
\[ \psi(x) \leq c_1 \int_{y_{n+1} > \sqrt{2}} (y_{n+1}^2 - 1)^{(\lambda+1-n)/2} \frac{dy}{[\gamma_x y, \epsilon_{n+1}]^\mu} \leq c_1 B^{1/p} c_{\lambda}^{1/p'}, \]
where \( B \) is the integral from (4.22), which is finite if \( p > (n - 1)/\mu \), and \( c_{\lambda} \) is known from (4.24). It is finite if \( p < (n - 1)/\lambda \). Thus, we have to choose \( (n - 1)/\mu < p < (n - 1)/\lambda \), which is possible if \( \mu > \lambda \). If \( \lambda = k - 1 \) we arrive at the “standard” assumption \( \mu > k - 1 \). If \( \lambda = 2[k/2] + 1 \), we get the new restriction \( \mu > 2[k/2] + 1 \), as stated in the lemma.

To complete the proof, we note that Theorem 2.5 gives us \((M_x f)(r)\) only for almost all \( r \). However, if \( f \in C_\mu(X) \), then \((\tilde{M}_x f)(r)\) is continuous (in both variables), and if \( f \in L^p(X) \), then \((\tilde{M}_x f)(r)\) is an \( L^p \)-valued continuous function of \( r \). It follows that the inversion formula (4.18) holds for every \( r > 0 \) for both \( C_\mu \) and \( L^p \) spaces. However, in the first case it is valid for all \( x \in X \) and in the second case for almost all \( x \).

Lemma 4.7 and Theorem 2.5 yield the main results for \( X = \mathbb{R}^n \), which mimic those for \( X = \mathbb{H}^n \).

**Theorem 4.8.** A function \( f \in C_\mu(X) \), \( \mu > k - 1 \), can be recovered from \( \varphi = Rf \) by the formula
\[ (4.25) \quad f(x) = \lim_{t \to 0} \pi^{-k/2}(D_{-k/2}^* R_x^* \varphi)(t), \]
in which the limit is uniform on $X$ and the Erdélyi-Kober differential operator $D_{-2}^{k/2}$ can be computed as follows.

(i) If $k$ is even, then

$$D_{-2}^{k/2}F = (-D)^{k/2}F, \quad D = \frac{1}{2t} \frac{d}{dt}. \quad (4.26)$$

(ii) For any $1 \leq k \leq n - 1$,

$$D_{-2}^{k/2}F = t^{2-k+2m}(-D)^{m+1} t^k \psi, \quad \psi = I_{-2}^{1-k/2+m} t^{-2m-2} F, \quad (4.27)$$

where $m = \lfloor k/2 \rfloor$. Alternatively,

$$D_{-2}^{k/2}F = 2^{-k} \left( -\frac{d}{dt} \right)^k t I_{-2}^{k/2} t^{-k-1} F. \quad (4.28)$$

Under the stronger assumption $\mu > 2[k/2] + 1$, $D_{-2}^{k/2}$ can be also computed as

$$D_{-2}^{k/2}F = (-D)^{m+1} I_{-2}^{1-\alpha+m} F. \quad (4.29)$$

For $L^p$-functions we have the following.

**Theorem 4.9.** A function $f \in L^p(X)$, $1 \leq p < (n-1)/(k-1)$, can be recovered from $\varphi = Rf$ at almost every point $x$ by the formula

$$f(x) = \lim_{t \to 0} \pi^{-k/2}(D_{-2}^{k/2} R^*_x \varphi)(t), \quad (4.30)$$

where the limit is understood in the $L^p$-norm. Here $D_{-2}^{k/2}$ is computed as in Theorem 4.8, where formula (4.29) is applicable under the stronger condition $1 \leq p < (n-1)/(2[k/2] + 1)$.

5. The case $X = S^n$

We recall some basic facts [He, Ru02b]. Let $\mathbb{R}^{n+1} = \mathbb{R}^{k+1} \times \mathbb{R}^{n-k},$

$$\mathbb{R}^{k+1} = \Re e_1 \oplus \ldots \oplus \Re e_{k+1}, \quad \mathbb{R}^{n-k} = \Re e_{k+2} \oplus \ldots \oplus \Re e_{n+1};$$

$$\sigma_n = |S^n| = 2\pi^{(n+1)/2}/\Gamma((n+1)/2); \quad \xi_0 = S^k \text{ is the unit sphere in } \mathbb{R}^{k+1}; \quad d(\cdot, \cdot) \text{ denotes the geodesic distance on } S^n; \quad G = SO(n+1); \quad K = SO(n) \text{ and } K' = SO(k+1) \times SO(n-k) \text{ are stabilizers of } e_{n+1} \text{ and } \xi_0 \text{ respectively.} \quad \text{The set } \Xi \text{ of all } k \text{-dimensional totally geodesic submanifolds } \xi \text{ of } X = S^n \text{ can be identified with the Grassmann manifold } G_{n+1,k+1} = G/K' \text{ of all } (k+1) \text{-dimensional linear subspaces of } \mathbb{R}^{n+1}. \quad \text{The } G \text{-invariant probability measure } d\xi \text{ on } \Xi \text{ is defined in a canonical way.
The totally geodesic Radon transform \( \mathcal{R}f(\xi) \) of a sufficiently good function \( f \) on \( S^n \) is defined by

\[
(5.1) \quad \mathcal{R}f(\xi) = \int f(x) \, d_\xi x, \quad \xi \in \Xi,
\]

where \( d_\xi x \) stands for the usual Lebesgue measure on \( \xi \). Clearly, if \( f \) is an odd function, then \( \mathcal{R}f \equiv 0 \). We will be dealing with \( L^p(X) = L^p(S^n) \), the subspace of even functions in \( L^p(S^n) \).

Another important object is the shifted dual Radon transform, which averages a function \( \varphi \) on \( \Xi \) over all \( k \)-geodesics \( \xi \) at a fixed distance \( \theta \) from \( x \in S^n \). To define this operator, we denote by \( r_x \in SO(n+1) \) an arbitrary rotation satisfying \( r_x e_{n+1} = x \) and set \( \varphi_x(\xi) = \varphi(r_x \xi) \). For \( \theta \in [0, \pi/2] \), let \( g_\theta \) be the rotation in the plane \((e_{k+1}, e_{n+1})\) with the matrix

\[
\begin{pmatrix}
\sin \theta & \cos \theta \\
-\cos \theta & \sin \theta
\end{pmatrix}.
\]

For \( r = \cos \theta \), the shifted dual Radon transform of a function \( \varphi \) on \( \Xi \) is defined by

\[
(5.2) \quad (R^*_x \varphi)(r) = \int_{d(x, \xi) = \theta} \varphi(\xi) \, d\mu(\xi) = \int K \varphi_x(\rho g_\theta^{-1} \xi_0) \, d\rho.
\]

The case \( \theta = 0 \) corresponds to the usual dual Radon transform [He]. This definition is slightly different from the similar one for the hyperbolic space, however, it allows us to avoid unnecessary technicalities.

**Lemma 5.1.** [Ru02b, p. 479] For all \( 1 \leq p \leq \infty \),

\[
\| Rf \|_{(p)} \leq \sigma_n^{1/p} \| f \|_p, \quad \| (R^*_x \varphi)(r) \|_p \leq \sigma_n^{1/p} \| \varphi \|_{(p)},
\]

where \( \| \cdot \|_{(p)} \) and \( \| \cdot \|_p \) denote the \( L^p \)-norms on \( \Xi \) and \( X = S^n \), respectively.

We need one more averaging operator:

\[
(5.3) \quad (M_x f)(s) = \frac{(1 - s^2)^{(1-n)/2}}{\sigma_{n-1}} \int_{\{y \in S^n : x \cdot y = s\}} f(y) \, d\sigma(y), \quad s \in (-1, 1).
\]

The integral (5.3) is the mean value of \( f \) on the planar section of \( S^n \) by the hyperplane \( x \cdot y = s \), and \( d\sigma(y) \) stands for the induced Lebesgue measure on this section. It is known that \( \|(M_x f)(s)\|_p \leq \| f \|_p \) and \( \lim_{s \to 1} (M_x f)(s) = f(x) \) in the \( L^p \)-norm for all \( 1 \leq p \leq \infty \).³

³Here and on we identify \( L^\infty(S^n) \) with the space \( C(S^n) \) of continuous functions.
Lemma 5.2. Let \( f \in L^p(S^n), \ 1 \leq p \leq \infty \). Then
\[
(R^*_x Rf)(r) = \frac{2\sigma_{k-1}}{r^{k-1}} \int_0^r (r^2 - s^2)^{k/2-1}(\mathcal{M}_x f)(s) \, ds
\]
\[
= \frac{2\pi^{k/2}}{r^{k-1}} (I^{k/2}_{+2}g_x)(r), \quad g_x(s) = s^{-1}(\mathcal{M}_x f)(s).
\]

This statement can be found in \([\text{He}, \ p. 140]\) and \([\text{Ru02b}, \ p. 485]\). Note that, by Lemma 5.1, \((R^*_x R)f\) represents a bounded operator on \(L^p\) for every \(r \in (0,1)\). Hence, the integral (5.4) is absolutely convergent for every \(r \in (0,1)\) and represents an \(L^p\)-function of \(x\).

Lemma 5.2 implies the following inversion result.

Theorem 5.3. Let \( X = S^n \). A function \( f \in L^p_c(X), \ 1 \leq p < \infty \), can be recovered from \( \varphi = Rf \) by the formula
\[
f(x) = \lim_{s \to 1} \left( \frac{1}{2 s} \frac{\partial}{\partial s} \right)^k \left[ \frac{\pi^{-k/2}}{\Gamma(k/2)} \int_0^s (s^2 - r^2)^{k/2-1} (R^*_x \varphi)(r) r^k \, dr \right].
\]

In particular, for \(k\) even,
\[
f(x) = \lim_{s \to 1} \frac{1}{2\pi^{k/2}} \left( \frac{1}{2 s} \frac{\partial}{\partial s} \right)^{k/2} [s^{k-1}(R^*_x \varphi)(s)].
\]

Alternatively,
\[
f(x) = \lim_{s \to 1} \left( \frac{\partial}{\partial s} \right)^k \left[ \frac{2^{-k} \pi^{-k/2}}{\Gamma(k/2)} \int_0^s (s^2 - r^2)^{k/2-1} (R^*_x \varphi)(r) \, dr \right].
\]

The limit in these formulas is understood in the \(L^p\)-norm. If \( f \in C_c(X) \), it can be interpreted in the sup-norm.

Proof. By (5.4),
\[
(I^{k/2}_{+2}g_x)(r) = \psi_x(r), \quad \psi_x(r) = 2^{-1\pi^{-k/2}r^{-k-1}}(R^*_x \varphi)(r).
\]

Hence, by the semigroup property (2.7), \(I^{k/2}_{+2}g_x = I^{k/2}_{+2} \psi_x\), and therefore, \(g_x = D^k I^{k/2}_{+2} \psi_x\), \(D = (1/2s)(\partial/\partial s)\). This gives the first two formulas. Furthermore, by (2.17),
\[
(M_x f)(s) = 2^{-k-1\pi^{-k/2}} \left( \frac{\partial}{\partial s} \right)^k (I^{k/2}_{+2} R^*_x \varphi)(s),
\]
and the third formula follows. \(\square\)
We observe that the first two formulas in Theorem 5.3 are well known for infinitely differentiable functions; cf. Theorem 1.22 in [He, p. 141]. The third formula, containing usual derivative \( \partial/\partial s^k \), is new.

6. On Helgason’s formula. Open problem

The following interesting result is due to Helgason [He, p. 116].

**Theorem 6.1.** If \( k \) is even, then the \( k \)-plane transform on \( \mathbb{R}^n \) can be inverted by the formula

\[
(6.1) \quad f(x) = c \left[ \partial^k \left( R_x^* R_f \right)(r) \right]_{r=0}, \quad c = \text{const}, \quad \partial_r = \frac{\partial}{\partial r}.
\]

This formula is much simpler than those in Theorem 5.3. The constant \( c \) in (6.1) was explicitly evaluated in [AR], where Theorem 6.1 has been extended to totally geodesic Radon transforms on arbitrary constant curvature space \( X \). To state this result, we introduce the *distance function*

\[
(6.2) \quad \rho(x, \xi) = \begin{cases} 
  d(x, \xi) & \text{if } X = \mathbb{R}^n, \\
  \sinh d(x, \xi) & \text{if } X = \mathbb{H}^n, \\
  \sin d(x, \xi) & \text{if } X = S^n.
\end{cases}
\]

The corresponding shifted dual Radon transform can be defined by

\[
(6.3) \quad (R_x^* \varphi)(r) = \int_{\rho(x, \xi) = r} \varphi(\xi) d\mu(\xi), \quad x \in X, \quad r > 0,
\]

where \( d\mu(\xi) \) is the relevant normalized canonical measure.

**Theorem 6.2.** Let \( \varphi = Rf \). If \( k \) is even, then

\[
(6.4) \quad \partial^k \lambda_X(r)(R_x^* \varphi)(r) \bigg|_{r=0} = c_X f(x),
\]

where

\[
\lambda_X(r) = \begin{cases} 
  1 & \text{if } X = \mathbb{R}^n, \\
  (1 + r^2)^{(k-1)/2} & \text{if } X = \mathbb{H}^n, \\
  (1 - r^2)^{(k-1)/2} & \text{if } X = S^n,
\end{cases}
\]

\[
c_X = \begin{cases} 
  (-1)^{k/2} (k-1)! \sigma_{k-1} & \text{if } X = \mathbb{R}^n, \mathbb{H}^n, \\
  2(-1)^{k/2} (k-1)! \sigma_{k-1} & \text{if } X = S^n.
\end{cases}
\]

---

\(^4\)Coincidence of the constants in both theorems follows by duplication formula for gamma functions.
In both theorems it was assumed that $f$ is infinitely smooth and (for $X = \mathbb{R}^n, \mathbb{H}^n$) rapidly decreasing. This assumption is redundant. See Appendix, where, for $X = \mathbb{R}^n$, it is shown that the result holds under much weaker assumptions.

**Open Problem.** Extend Theorems 6.1 and 6.2 to non-differentiable functions, e.g., $f \in L^p(X)$.

Helgason’s idea to invoke usual differentiation in place of $d/dr^2$ agrees with the 1927 paper by Mader [M] in the sense that her inversion formula for the hyperplane Radon transform also contains usual differentiation. However, the method of [M] is completely different. For the sake of completeness, we present without proof a generalization of Mader’s result, which was obtained in [AR].

For $r > 0$ and $1 \leq k \leq n - 1$, let

\begin{align}
(6.5) & \quad (L^*_x \varphi)(r) = \int_{\Xi} \varphi(\xi) \rho^{k+1-n} \text{sgn}(\rho - r) \, d\xi, \\
(6.6) & \quad (\tilde{L}^*_x \varphi)(r) = \int_{\Xi} \varphi(\xi) \rho^{k+1-n} \log |\rho^2 - r^2| \, d\xi,
\end{align}

where $\rho = \rho(x, \xi)$ is the distance function (6.2).

**Theorem 6.3.** Let $\varphi = Rf$, where $f$ is a $C^\infty$ function, which is rapidly decreasing in the case $X = \mathbb{R}^n, \mathbb{H}^n$.

(i) If $k$ is even, then

\begin{align}
(6.7) & \quad \partial_r^{k+1} (L^*_x \varphi)(r) \bigg|_{r=0} = d_X f(x),
\end{align}

where

\[ d_X = \begin{cases} 
2(-1)^{(k+2)/2} \sigma_{n-k-1} \sigma_{k-1} (k-1)! & \text{if } X = \mathbb{R}^n, \mathbb{H}^n, \\
2 \sigma_{n-k-1} \sigma_k \sigma_{k-1} (k-1)! / \sigma_n & \text{if } X = S^n.
\end{cases} \]

(ii) If $k$ is odd, then

\begin{align}
(6.8) & \quad \partial_r^{k+1} (\tilde{L}^*_x \varphi)(r) \bigg|_{r=0} = \tilde{d}_X f(x),
\end{align}

where

\[ \tilde{d}_X = \begin{cases} 
\pi (-1)^{(k-1)/2} \sigma_{n-k-1} \sigma_{k-1} (k-1)! & \text{if } X = \mathbb{R}^n, \mathbb{H}^n, \\
2\pi (-1)^{(k-1)/2} \sigma_{n-k-1} \sigma_k \sigma_{k-1} (k-1)! / \sigma_n & \text{if } X = S^n.
\end{cases} \]
Let us prove Theorem 6.2 for the case \( X = \mathbb{R}^n \). We provide more details, than in [AR], and make assumptions for \( f \) more precise. For \( \mu > 0 \) and \( k \in \mathbb{N} \), let
\[
C^k_\mu(\mathbb{R}^n) = \{ f \in C^k(\mathbb{R}^n) : |\partial^\alpha f(x) = O(|x|^{-\mu}) \ \forall \ |\alpha| \leq k \}.
\]

**Proposition 7.1.** Suppose that \( 1 \leq k \leq n-1 \), \( \mu > k \). Then a function \( f \in C^k_\mu(\mathbb{R}^n) \) can be reconstructed from \( \varphi = Rf \) by the formula
\[
f(x) = \lim_{r \to 0} c_k (-\partial_r)^k (R_x^r \varphi)(r), \quad c_k = \frac{(-1)^{k/2}}{(k-1)! \sigma_{k-1}},
\]

where the limit is uniform on \( \mathbb{R}^n \).

**Proof.** Fix \( x \) and write (3.4) in the form
\[
(R_x^r \varphi)(r) = \sigma_{k-1} [A(r) + (-1)^{k/2} B(r)],
\]

where
\[
A(r) = \int_0^\infty (\mathcal{M}_x f)(t) (t^2 - r^2)^{k/2-1} t \, dt,
\]
\[
B(r) = \int_0^r (\mathcal{M}_x f)(t) (r^2 - t^2)^{k/2-1} t \, dt.
\]

Since \( A(r) \) is a polynomial of degree \( k-2 \), then \( \partial_r A(r) = 0 \). Regarding \( B(r) \), we write it as \( B_1 + B_2 \), where
\[
B_1 = f(x) \int_0^r (r^2 - t^2)^{k/2-1} t \, dt = \frac{r^k}{k} f(x),
\]
\[
B_2 = \int_0^r (r^2 - t^2)^{k/2-1} [(\mathcal{M}_t f)(x) - f(x)] t \, dt = r^k h(r),
\]
\[
h(r) = \int_0^1 (1 - t^2)^{k/2-1} [(\mathcal{M}_{rt} f)(x) - f(x)] t \, dt.
\]

Clearly, \( \partial_r B_1 = (k-1)! f(x) \). Furthermore,
\[
\lim_{r \to 0} \frac{\partial_r B_2}{r} = \sum_{j=0}^k c_j \lim_{r \to 0} r^j h^{(j)}(r).
\]
The term corresponding to \( j = 0 \) is obviously zero. Other terms are also zero because \( h^{(j)}(r) \) is uniformly bounded. Indeed,

\[
|h^{(j)}(r)| \leq \int_0^1 (1 - t^2)^{k/2-1} t dt \int_{S^{n-1}} |\partial^\alpha f(x + rt\theta)| d\theta
\]

\[
\leq \sum_{|\alpha| = j} \int_0^1 (1 - t^2)^{k/2-1} |\theta^\alpha \partial^\alpha f(x + rt\theta)| t^{j+1} dt
\]

\[
\leq \tilde{c}_j \sup_x \sup_{|\alpha| = j} |\partial^\alpha f(x)|.
\]

Thus, \( \lim_{r \to 0} \partial_k^k B_2(r) = 0 \), and the result follows. \(\square\)

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