Impact of Non-linear Boson Self-interaction on Dark Stars Properties

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Abstract. We study the properties of dark stars using non-linear boson self-interaction models. These stars are compact objects formed from bosonic dark matter. We focus on the properties of dark stars such as the mass and radius relation, tidal deformation, and moment of inertia. In this work, bosonic dark particle mass is set to be 300 MeV and 400 MeV. The bosonic self-interaction potential models are $|\phi|^4$, cosh-Gordon and Liouville which have the same behavior at low densities but relative difference at high densities. Coupling constants on the corresponding self-interaction potentials are determined by the result of numerical simulations CCDM which requires ratio total cross-section to dark particle mass in the range of $0.1 \text{ cm}^2/\text{g} \leq \sigma/M \leq 1 \text{ cm}^2/\text{g}$.

1. Introduction
The structure of the universe consists of dark and ordinary sectors. Composition of the universe is 96% dark sector and 4% ordinary sector. Dark sector consists of 29% cold dark matter and 67% dark energy [1]. For cold dark matter, the $\Lambda$ cold dark matter model is commonly chosen because this model is compatible with the cosmic triangle [2]. However, $\Lambda$ cold dark matter model has several weaknesses. Weaknesses of $\Lambda$ cold dark matter model has been discussed in details in Ref. [3]. Solution for the weaknesses of $\Lambda$ cold dark matter model could be self-interaction cold dark matter [4]. Self-interaction cold dark matter can form compact objects such as condensate dark matter stars [5, 6, 7]. In this work, we study dark stars properties using self-interaction bosonic dark matter within relativistic mean field approximation. Furthermore, we are going to use $|\phi|^4$, cosh-Gordon and Liouville potentials [8], respectively. Bosonic dark particle mass is set to be 300 MeV and 400 MeV [3].

2. Formalism
This section consist of the models and properties of dark stars. The models describe an equation of state of dark stars. Properties of dark stars such as mass and radius relation, tidal deformation, and moment of inertia can be calculated by using equation of state of dark stars.

2.1. The Models
We start from $|\phi|^4$ potential model. First, we write Lagrangian density for $|\phi|^4$ potential model. Lagrangian density for $|\phi|^4$ potential model is

$$\mathcal{L} = \partial_{\mu}\phi^*\partial^{\mu}\phi - M^2 (\phi^* \phi) - \lambda (\phi^* \phi)^2,$$  \hspace{1cm} (1)
which $M$ and $\lambda$ are dark particle mass and dimensionless coupling constant respectively. $\lambda$ can be determined by the result of numerical simulations CCDM [7-8] i.e.,

$$0.1 \frac{cm^2}{g} \leq \frac{\sigma}{M} \leq 1 \frac{cm^2}{g},$$

while the total cross-section for $|\phi|^4$ potential model is

$$\sigma = \frac{\lambda^2}{64\pi M^2}.$$  

Note that because the non-linear contributions of cosh-Gordon and Liouville models are small in low density limit. The total cross section of both models close to that of $|\phi|^4$ model. Therefore, for all models we can use following range of $\lambda$ coupling:

$$\left(\frac{M}{1 MeV}\right)^{\frac{3}{2}} < \frac{10^{-3}}{|\lambda|} < 3 \left(\frac{M}{1 MeV}\right)^{\frac{3}{2}}.$$  

In mean field approximation, for lowest energy $k = 0$ plane wave, we obtain number density of bosons and equation of motion of this model, respectively

$$n = J_0 = 2\omega (\phi^* \phi),$$

and

$$\omega^3 - M^2 \omega - \lambda n (hc)^3 = 0.$$  

From energy-momentum tensor [9]

$$T^{\mu\nu} = \frac{\partial L}{\partial (\partial_\mu \phi^\alpha)} \partial^\nu \phi^\alpha - \eta^{\mu\nu} L,$$

which $\eta^{\mu\nu}$ is Minkowski metric tensor, we obtain energy density as

$$\epsilon = \frac{1}{2} \omega n + \frac{M^2 n}{2} \frac{1}{\omega} + \frac{\lambda n^2 (hc)^3}{4} \frac{1}{\omega^2}.$$  

By using thermodynamics relation [10]

$$P = n^2 \frac{\partial (\frac{\epsilon}{n})}{\partial n},$$

we obtain pressure as the function of energy density of the dark stars. Solving Eq. (6) using numerical methods with $n$ is varied from 0.1 $fm^{-3}$ until 20 $fm^{-3}$, we obtain $\omega$ value. Inserting $n$ and $\omega$ value to energy density and pressure, we obtain energy density and pressure values, respectively. We parametrize the energy density and pressure data by using polynomial function up to tenth-order to obtain the equation of state.

The next potential model is cosh-Gordon. Interaction Lagrangian density for this model is defined as

$$L_{int} = -\alpha M^2 \left[ \cosh \left( \beta \sqrt{\phi^* \phi} \right) - 1 \right],$$

which $\alpha$ and $\beta$ are constants, respectively. Expanding Eq. (10) into series, and retain only up to second order, we also obtain similar interaction Lagrangian density as

$$L_{int} = -\alpha M^2 \left[ \frac{\beta^2 (\phi^* \phi)}{2} + \frac{\beta^4 (\phi^* \phi)^2}{24} + \ldots \right].$$
By comparing Eq. (11) and Eq. (1), we can obtain following relation

\[ \alpha \beta^2 = 2, \]  

(12)

\[ \beta = \frac{2\sqrt{3}\lambda}{M}, \]  

(13)

which \( \lambda \) can obtain from Eq. (4). In mean-field approximation, for lowest energy \( k = 0 \) plane wave, we obtain equation of motion of cosh-Gordon model as

\[ \omega^2 - \frac{\alpha M^2 \beta}{2} \left\{ \frac{2\omega}{n(h\epsilon)^3} \sinh \left( \beta \sqrt{\frac{n(h\epsilon)^3}{2\omega}} \right) \right\} = 0. \]  

(14)

From energy-momentum tensor, we obtain energy density

\[ \epsilon = \frac{\omega n}{2} + \frac{\alpha M^2}{(h\epsilon)^3} \left[ \cosh \left( \beta \sqrt{\frac{(h\epsilon)^3 n}{2\omega}} \right) - 1 \right]. \]  

(15)

From Eq. (9), we obtain pressure of dark stars. Solving Eq. (14) using numerical methods with \( n \) is varied from 0.1 \( fm^{-3} \) until 20 \( fm^{-3} \), we also obtain \( \omega \) value. Inserting \( n \) and \( \omega \) value to energy density and pressure, we obtain energy density and pressure value respectively. From energy density and pressure value we will obtain equation of state in a expansion form similar to that of \( |\phi|^4 \) model.

The last potential is Liouville model. Interaction Lagrangian density for this model is

\[ \mathcal{L}_{int} = -\alpha M^2 \left[ e^{\beta^2 (\phi^* \phi)} - 1 \right], \]  

(16)

which \( \alpha \) and \( \beta \) are constants, respectively. Expanding Eq. (16) into series until second order, we obtain interaction Lagrangian density as

\[ \mathcal{L}_{int} = -\alpha M^2 \left[ \beta^2 (\phi^* \phi) + \frac{\beta^4 (\phi^* \phi)^2}{2} + \ldots \right]. \]  

(17)

By comparing Eq. (17) and Eq. (1), we will obtain following relation

\[ \alpha \beta^2 = 1, \]  

(18)

\[ \beta = \frac{\sqrt{2}\lambda}{M}, \]  

(19)

which \( \lambda \) can also obtain from Eq. (4). In mean-field approximation, for lowest energy \( k = 0 \) plane wave, we obtain equation of motion of Liouville model as

\[ \omega^2 - \alpha M^2 \beta^2 e^{\frac{n^2 (h\epsilon)^3}{2\omega}} = 0. \]  

(20)

From energy-momentum tensor, we obtain energy density

\[ \epsilon = \frac{\omega n}{2} + \frac{\alpha M^2}{(h\epsilon)^3} \left[ e^{\frac{n^2 (h\epsilon)^3}{2\omega}} - 1 \right]. \]  

(21)

From Eq. (9), we also obtain pressure dark stars. Solving Eq. (20) using numerical methods with \( n \) is varied from 0.1 \( fm^{-3} \) until 20 \( fm^{-3} \), we obtain \( \omega \) value. Inserting \( n \) and \( \omega \) value to energy density and pressure, we obtain energy density and pressure value, respectively. From energy density and pressure value we will also obtain equation of state in a expansion form similar to that of \( |\phi|^4 \) model.
2.2. Properties of Dark Stars

Mass and radius relation of dark stars can be described by Tolman Oppenheimer and Tolkov (TOV) equation \[11, 12\]

\[
\frac{dm}{dr} = 4\pi r^2 \epsilon(r),
\]

\[
\frac{dP}{dr} = \frac{G\epsilon(r) m(r)}{r^2} \left[1 + \frac{P(r)}{\epsilon(r)} \right] \left[1 + \frac{4\pi r^3 P(r)}{m(r)} \right] \left[1 - \frac{2Gm(r)}{r} \right]^{-1}.
\]

In binary system, the moment quadrupole of dark star has relation linear with external potential field \[13\]

\[
Q_{ij} = -\gamma \epsilon_{ij},
\]

which \(\gamma\) is tidal deformability parameter. \(\gamma\) can be found from the equation below

\[
\gamma = \frac{2}{3} k_2 R^5,
\]

which \(k_2\) and \(R\) are electric tidal Love numbers and radius of dark stars, respectively. The dimensionless tidal deformability parameter can be expressed by equation below

\[
\Gamma = \frac{2k_2}{3C^5},
\]

which \(C\) is compactness \((C = \frac{Gm}{M})\). Before we find \(k_2\), we solving first order differential equation \[13, 14\]

\[
ry'' + y'^2 + Fy + r^2Q = 0,
\]

\[
F = \frac{r - 4\pi r^3 [\epsilon - p]}{r - 2m},
\]

\[
Q = \frac{4\pi r \left(5\epsilon + 9p + \frac{r + p}{m} - \frac{l(l+1)}{4\pi r^4} \right)}{r - 2m} + \frac{m + 4\pi r^3 p}{r^2 \left(1 - \frac{2m}{r} \right)} \left(\epsilon - p\right),
\]

with initial boundary condition \(y(0) = 2\). Next, for \(l = 2\) we can calculate \(k_2\) \[13, 15\]

\[
l_2 = \frac{8}{5} \left(1 - 2C^2\right) C^5 \left[2C \left(y_2 - 1\right) - y_2 + 2\right] \left[2C \left(4y_2 + 1\right) C^4 + 6y_2 - 4\right] C^4 + 26 - 22y_2 \right) C^2 + 3 \left(5y_2 - 8\right) C - 3y_2 + 6 \right) - 3 \left(1 - 2C\right)^2 \left(2C \left(y_2 - 1\right) - y_2 + 2\right) \log \left(\frac{1}{1 - 2C}\right)]^{-1}.
\]

To calculate moment of inertia, we start with the line element of slowly rotating compact objects \[16\]

\[
ds^2 = -e^{2\omega} dt^2 + e^{2\lambda} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) - 2\omega(r) r^2 \sin^2 \theta dtd\phi.
\]

which \(\omega(r)\) are frame-dragging effect due to rotation. From the definition of \(\varpi(r) \equiv \Omega - \omega(r)\), we solving second order ordinary differential equation below

\[
\frac{1}{r^4} \frac{d}{dr} \left[r^4 e^{-\nu} \left(1 - \frac{2Gm(r)}{r} \right) \frac{2}{\varpi} \right] + \frac{4}{r^4} \left[\frac{d}{dr} e^{-\nu} \left(1 - \frac{2Gm(r)}{r} \right) \frac{2}{\varpi}\right] \varpi = 0.
\]

After solving equation above, we can obtain the moment of inertia of dark stars from equation below

\[
\frac{d\varpi}{dr} = \frac{6GI\Omega}{R^4}.
\]
Figure 1. (Top) Difference of potentials, red line is $|\phi|^4$ potential model, blue line is Liouville potential model and green line is cosh-Gordon model. (Bottom) Equation of state with dark particle mass $M = 300$ MeV (left) and $M = 400$ MeV (right)

Figure 2. Mass and radius relation dark stars with dark particle mass $M = 300$ MeV (left) and $M = 400$ MeV (right)

3. Result and Discussion
It can be seen from each panel in Fig 1 that at low densities, the three models have same potential values. However, at high densities, all models yield significantly different potential
values. Therefore, the differences in dark stars properties come from the stiffness difference in equation of state at high densities. It can be seen obviously from each panel in Fig 2 that at low densities i.e., very low mass and radius region, mass of dark stars with cosh-Gordon and Liouville potential models are close to those of $|\phi|^4$ potential. However, for large mass and radius region, both models show a different behavior than those of with $|\phi|^4$ potential i.e., increasing radius by softening equation of state. It can be seen also all models predict more less the same maximum mass of dark stars. It can be seen also that the maximum mass of dark stars depends significantly on dark particle mass. It can be seen obviously from each panel in Fig 3 the dependency of tidal deformation of the dark stars to the compactness of the corresponding stars. It can seen

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Tidal deformation dark stars with dark particle mass $M = 300$ MeV (left) and $M = 400$ MeV (right)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Moment of inertia dark stars with dark particle mass $M = 300$ MeV (left) and $M = 400$ MeV (right)}
\end{figure}
also that the stiffness difference of the corresponding stars appears more significantly at relative large compactness values. It can be seen also that the tidal deformation parameter of dark stars does not depend on dark particle mass. Fig. 4 show the moment of inertia predicted by all potentials. at low densities i.e., very low mass and radius region, the moment of inertia of dark stars predicted by cosh-Gordon and Liouville potential models are close to those of $|\phi|^4$ potential. However, significant differences in moment of inertia appear at high densities region which are represented by large mass region. It can be seen also that the moment of inertia of darks stars does not depend on dark particle mass.

4. Conclusion

We study the properties of dark stars using non-linear boson self-interaction models using $|\phi|^4$, cosh-Gordon and Liouville bosonic self-interaction potential models. Bosonic dark particle mass is set to be 300 MeV and 400 MeV. The corresponding potentials have the same stiffness behavior at low densities but relative difference at high densities. Coupling constants on the corresponding self-interaction potentials are determined by the result of numerical simulations CCDM which requires ratio total cross-section to dark particle mass in the range of $0.1 \text{cm}^2/\text{g} \leq \sigma/\mu \leq 1 \text{cm}^2/\text{g}$. We have found that the stiffness difference at high densities are manifested in radii and moment of inertia of large masses dark star, tidal deformation at large masses. While the effect of mass of dark particle appears in maximum mass of dark stars.

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