Reply to the comment by Davison and Haynes: Madden–Julian Oscillation like behaviour does persist at small time steps

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Abstract
In their comment, Davison and Haynes remark on an apparent time-step sensitivity in the model presented by Vallis and Penn, such that in their own simulations the excitability is lost when using a smaller time step. However, this reply shows that if the condensational time-scale is suitably small then excitable behaviour can be achieved over a range of time steps, including very small ones, with very similar energy levels and large-scale behaviour at both large and small time steps. In particular, an eastward-propagating disturbance resembling the Madden–Julian Oscillation can still be reproduced, independent of time step. Certainly there are numerical, mathematical, and physical difficulties in understanding a system in which condensation occurs on short time-scales, and this reply discusses further the applicability of the results to the real atmosphere.

KEYWORDS
Madden–Julian Oscillation, moist dynamics, excitable system, tropics

1 SPECIFIC ISSUES

Vallis and Penn (2020) (henceforth VP) presented a model that comprised the conventional shallow-water equations and an equation for the transport of water, along with a source term, representing evaporation from the surface, and a sink term, representing condensation and rain. Variations of these equations have been used previously and they are generically known as the moist shallow-water equations. In VP, the equations were integrated numerically and some interesting behaviour was observed, namely that in certain circumstances the motion becomes self-sustaining, with convection leading to moisture convergence and triggering further convection. When integrated on the beta-plane, the east–west asymmetry leads to an organization of the convection, with moisture convergence and precipitation occurring slightly to the east of the original disturbance, and that in turn leads to eastward propagation of the system. In their comment, Davison and Haynes (2022) (henceforth DH) argue that if a smaller time step than that in VP is used, then the excitability is lost. However, as will be shown below, if the condensation process is such as to remove excess moisture at the fastest possible resolved rate, rather than occurring on a fixed time-scale, then the excitability remains at small time steps, with energy levels and other properties...
(including eastward propagation of a Madden–Julian Oscillation (MJO) like disturbance) very similar to those found at large time steps.

Difficult problems do arise when dealing with condensation, because the process is rapid, with the microphysical reactions occurring on a shorter time-scale than most other processes in the system and occurring on a spatial scale unlikely to be resolved by any reasonable grid. Given this difficulty, a choice was made in VP to represent the system without any additional temporal or spatial filtering. If some form of parameterization is introduced, the results will depend on the way that is done, inevitably introducing some other parameter. In the so-called fast condensation limit, the condensation occurs almost instantaneously, meaning that the humidity is not allowed to exceed its saturation value. In a numerical model, this effectively means that all of the excess moisture above saturation is removed at each time step. More-or-less equivalently, the time-scale of condensation, \( \tau \), should be similar to that of the time step itself rather than being an independent variable—if we reduce the time step, \( \Delta t \), then we should reduce \( \tau \) by the same factor and this was the procedure employed in VP, if not fully explained there. (The merits of such a procedure, which is the analogue of taking the \( \tau \to 0 \) limit, may certainly be debated, although Frierson et al., (2004) show that solutions to the equations do converge in that limit.) Following this procedure (which is different from that employed by DH, who take a fixed \( \tau \) in their tests), we perform here a sequence of experiments varying the time step by successive factors of two from 800 to 100 s, at the same time reducing the condensation time-scale by the same factor and refining the spatial grid (and reducing the subgrid diffusivities), keeping the ratio of the grid size to the time step constant. The domain is, as in VP, a channel on the beta-plane, 10,000 km by 10,000 km in size, periodic in the \( x \)-direction, and the grid size varies across experiments from about 80 to 10 km (see Appendix for more parameter values).

The results are shown in Figure 1. Evidently, the excitability remains with virtually the same energy level at all time steps, even the smallest one (Figure 1a). The large-scale, bulk properties of the flow are also unaltered at high resolution and small time step. The precipitation around the Equator organizes itself into an eastward-propagating disturbance (a form of front, very similar to those shown in figures 4 and 5 of VP), with almost the same propagation speed in all cases. This can be seen in Figure 1b and c, which show the longitudinal

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**FIGURE 1**

(a) The time series of average specific kinetic energy (over the whole domain) in four integrations of the moist shallow-water equations with time steps of 800, 400, 200, and 100 s, with resolutions of approximately 80, 40, 20, and 10 km. (b) and (c) The location of the maximum precipitation at the Equator as a function of time, in simulations with time steps of 100 and 800 s, as labelled. In both cases, after a period of adjustment, an organized disturbance emerges that propagates eastward at a few metres per second. The apparent discontinuities are due to the disturbance leaving the (periodic) domain to the east and entering to the west. [Colour figure can be viewed at wileyonlinelibrary.com]
2 | More General Considerations

The behaviour described in VP is certainly sensitive to a number of parameters, for example, the dissipation and the parameter representing the latent heat of condensation. This, and the structure of the shallow-water model itself, mean that this behaviour may or may not survive as the model is made more complete. However, it is worth emphasizing that arguably the meteorologically most important result of VP, namely the production of a self-sustained eastward-propagating disturbance, results from a fairly general mechanism that does not depend upon the details of the moist shallow-water equations or the condensation scheme, as described in Vallis (2021). The simple idea is that a region of convection close to the Equator will generate a Gill-like pattern in the geopotential, and this is in fact seen in observations (Kiladis et al., 2005). This pattern will converge low-level moisture toward the convection, with moist air coming from the east along the Kelvin lobe and somewhat drier air coming from the west and around the off-equatorial Rossby lobes. The upshot is that the moisture convergence and region of maximum precipitation occur slightly to the east of the initial convection, thus shifting the pattern to the east, and so on, and the whole system consequently moves eastward.

Such a propagation mechanism seems straightforward, but it does rely on a two-way interaction between the convection and the circulation and some kind of instability process to maintain that. The subcritical (i.e., conditional) nature of the convection may help organize this process, for if the Tropics were linearly unstable everywhere then the smallest trigger would lead to convection occurring nearly randomly over the whole domain. However, in a conditionally unstable atmosphere (for example, in the presence of low-level convective inhibition or CIN) the presence of finite perturbations then becomes important in triggering convection. That is, the presence of a lift mechanism (or some other finite perturbation to saturate the fluid), in addition to the temperature profile itself, will play a role in determining where precipitation occurs (Schultz et al., 2000). If convection can provide that perturbation, then convective aggregation may occur while other parts of the domain lie quiescent. The beta-plane provides an additional organizing mechanism, because it will enhance the moisture convergence, drawing in moisture from far afield and localizing the region of instability. Other processes are likely to be important in contributing to the process—wind-enhanced evaporation may aid the moisture convergence and cloud radiative effects may enhance the instability—and the importance of these remains uncertain.

3 | Conclusion

To conclude, the phenomena shown in VP do, in fact, persist at small time steps, and with a fine spatial grid, if the condensation time is refined correspondingly. Energy levels and large-scale behaviour remain almost the same across a sequence of experiments with time steps varying by almost an order of magnitude. Still, recognizing the limitations of the moist shallow-water equations, it may be best to regard the results in VP as a hypothesis, namely that the presence of moist convection can contribute to the process by which convection occurs nearby, both by enhancing moisture convergence and by providing a finite-amplitude perturbation for the additional convection. These processes can lead to an organized convective disturbance and, on the beta-plane, to eastward propagation of that disturbance.

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APPENDIX . EQUATIONS AND PARAMETER VALUES

The moist shallow-water equations are, in fairly standard notation,

\[
\frac{\partial \mathbf{u}}{\partial t} + f \mathbf{k} \times \mathbf{u} = -g \nabla h + \nu_u \nabla^2 \mathbf{u}, \tag{A1}
\]

\[
\frac{\partial h}{\partial t} + H \nabla \cdot \mathbf{u} = -\gamma C - \lambda_r h + \nu_h \nabla^2 h, \tag{A2}
\]

\[
\frac{\partial q}{\partial t} + \nabla \cdot (\mathbf{u} q) = E - C + \nu_q \nabla^2 q, \tag{A3}
\]

where \(E = H(q_0 - q)(q_0 - q)A/\tau_e\) is the evaporation (and \(H\) is the Heaviside function), \(C = H(q - q_e)(q - q_e)/\tau\) is the condensation, and \(q_e = q_0 \exp(-ah/H)\) is the model Clausius–Clapeyron relation. \(A\) is a parameter representing wind effects on evaporation and in the integrations shown as \(A = (|\mathbf{u}| + U)/(u_0)\), where \(U = -1\) and \(u_0 = 1\).

The equations are solved on an Arakawa C-grid in a 10,000 km \(\times\) 10,000 km domain, periodic in the \(x\)-direction and with meridional walls at nominal latitudes of about 45°. The Coriolis parameter is given by \(f = f_0 + \beta y\), where \(f_0 = 0\) and \(\beta = 2 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}\). Resolution varies from 128\(^2\) to 1024\(^2\) grid points, corresponding to a grid-size varying from about 80 to 10 km. At the lowest resolution used here (i.e., 128\(^2\) and \(\Delta t = 800\) s), the values of the diffusive parameters are as follows:

\[
\nu_u = 1 \times 10^4, \quad \nu_h = 1 \times 10^4, \quad \nu_q = 2 \times 10^4
\]

(all with units of \(\text{m}^2 \cdot \text{s}^{-1}\)). \tag{A4}

For each halving of horizontal grid length, the diffusivities are reduced by a factor of four and the time step by a factor of two. Other parameters in the equations are as follows:

\[
H = 30 \text{ m}, \quad g = 10 \text{ m} \cdot \text{s}^{-2}, \tag{A5}
\]

\[
\lambda_r = 1.1 \times 10^{-5} \text{ s}^{-1} \approx \text{1 day}^{-1}, \tag{A6}
\]

\[
\tau_e = 1 \times 10^6 \text{ s} \approx 11 \text{ days}, \tag{A6}
\]

\[
q_0 = 3, \quad q_e = 3, \quad \alpha = 2 \text{ m}^{-1}, \quad \gamma = 5 \text{ m}
\]

(only the product \(\gamma q_0\) is relevant.) \tag{A7}