Non-real zeros of linear differential polynomials *

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Abstract

Let \( f \) be a real entire function with finitely many non-real zeros, not of the form \( f = Ph \) with \( P \) a polynomial and \( h \) in the Laguerre-Pólya class. Lower bounds are given for the number of non-real zeros of \( f'' + \omega f \), where \( \omega \) is a positive real constant.

1 Introduction

This paper concerns non-real zeros of linear differential polynomials in real entire functions with real zeros. For each non-negative integer \( p \) the class \( V_{2p} \) [17, 18, 28] consists of all entire functions
\[
f(z) = g(z) \exp(-az^{2p+2}),
\]
where \( a \geq 0 \) is real and \( g \) is a real entire function with real zeros of genus at most \( 2p + 1 \) [14, p.29]. The classes \( U_{2p}, p \geq 0 \), are then given by \( U_0 = V_0 \) and \( U_{2p} = V_{2p} \setminus V_{2p-2} \) for \( p \geq 1 \). Moreover, \( U_{2p}^* \) is the class of entire functions \( f = Ph \), where \( h \in U_{2p} \) and \( P \) is a real polynomial without real zeros [8], so that every real entire function of finite order with finitely many non-real zeros belongs to \( U_{2p}^* \) for some \( p \geq 0 \). It is well known [20] that \( U_0 = LP \), where \( LP \) is the Laguerre-Pólya class of entire functions which are locally uniform limits of real polynomials with real zeros.

The following results established conjectures of Wiman [1, 2] and Pólya [27] respectively. Here all counts of zeros should be understood to be with respect to multiplicity, and the same convention will be maintained throughout the paper unless explicitly stated otherwise.

Theorem 1.1 ([8, 28]) Let \( p \in \mathbb{N} \) and let \( f \in U_{2p}^* \). Then \( f'' \) has at least \( 2p \) non-real zeros.

Theorem 1.2 ([5]) Let \( p \) be a positive integer and let \( f \in U_{2p}^* \). Then the number of non-real zeros of the \( k \)th derivative \( f^{(k)} \) tends to infinity with \( k \).

Theorem 1.3 ([6, 22]) If \( f \) is a real entire function of infinite order then \( f f^{(k)} \) has infinitely many non-real zeros, for every \( k \geq 2 \).

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The present paper addresses the following related problem: if \( f \) is a real entire function with finitely many non-real zeros, must a linear differential polynomial
\[
\Psi = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_0 f
\]
with constant real coefficients \( a_j \) have non-real zeros, and if so how many? This question will be resolved for \( k = 2 \), in which case in view of the standard transformation
\[
f(z) = e^{-a_1 z/2} g(z), \quad \Psi(z) = e^{-a_1 z/2} (g''(z) + (a_0 - a_1^2/4) g),
\]
it may be assumed with no loss of generality that \( a_1 = 0 \).

**Theorem 1.4** Let \( f \) be a real entire function with finitely many non-real zeros, and let \( \omega \) be a positive real number. If \( f \in U^*_{2p} \) for some \( p \in \mathbb{N} \) then \( f'' + \omega f \) has at least \( 2p \) non-real zeros. If \( f \) has infinite order then \( f'' + \omega f \) has infinitely many non-real zeros.

It evidently suffices to prove Theorem 1.4 for \( \omega = 1 \), but the following examples show that the theorem fails for \( \omega < 0 \). If \( f \) is defined by
\[
f'(z) f(z) = a + e^{-2a z}, \quad f''(z) = a^2 + e^{-4a z},
\]
then \( f \) and \( f'' - a^2 f \) have no zeros at all in the plane. For an example of finite order define a zero-free function \( f \in U_2 \) by setting
\[
f'(z) f(z) = -16 z^2 + 8 z + 2, \quad f''(z) - 12 f(z) = 256 z^3 (z - 1),
\]
so that \( f'' - 12 f \) has only real zeros.

The proof of Theorem 1.4 will use machinery developed in [25, 28] for the Wiman conjecture, and refinements from [5, 6, 22], but will depart from the earlier methods in several significant steps. The aim is to construct an auxiliary function having finitely many critical points in \( \mathbb{C} \setminus \mathbb{R} \), and this will be done in Lemma 4.4, but in contrast to [5, 6, 22, 25, 28] the resulting function may have a finite non-real asymptotic value. Moreover for the present problem the normal families arguments used successfully in [6, 22] seem difficult to apply, since the condition
\[
f(z)(f''(z) + f(z)) \neq 0
\]
is not invariant under a change of variables \( w = Rz \). It also seems worth observing that for \( f \) in \( U^*_{2p} \), whereas every derivative of \( f \) has finitely many non-real zeros (see e.g. [8, Corollary 2.12]), this need not be the case for \( f'' + f \), as the simple example \( f(z) = 1 + \sin(z/2) \in U_0 \) shows. For further remarks and contrasts see \S 13.

## 2 Preliminaries

**Definition 2.1** For \( a \in \mathbb{C} \) and \( 0 \leq s < r < R \leq +\infty \) set
\[
D(a, r) = \{ z \in \mathbb{C} : |z - a| < r \}, \quad S(a, r) = \partial D(a, r), \quad A(s, R) = \{ z \in \mathbb{C} : s < |z| < R \}
\]
and
\[
H = \{ z \in \mathbb{C} : \text{Im } z > 0 \}, \quad D^+(0, r) = D(0, r) \cap H, \quad A^+(s, R) = A(s, R) \cap H.
\]
Lemma 2.1 ([32]) Let $u$ be a non-constant continuous subharmonic function in the plane. For $r > 0$ let $\theta^r(r)$ be the angular measure of that subset of $S(0, r)$ on which $u(z) > 0$, except that $\theta^r(r) = \infty$ if $u(z) > 0$ on the whole circle $S(0, r)$. Then, for $r > 0,$

$$B(r, u) = \max\{u(z) : |z| = r\} \leq 3T(2r, u) = \frac{3}{2\pi} \int_0^{2\pi} \max\{u(2re^{it}), 0\} \, dt$$

and, if $r \leq R/4$ and $r$ is sufficiently large,

$$B(r, u) \leq 9\sqrt{2}B(R, u) \exp\left(-\pi \int_{2r}^{R/2} \frac{ds}{s\theta^r(s)}\right).$$

□

Lemma 2.2 Let $A, B, M, r \in (0, \infty)$ with $A < B$, and suppose that $D_1, D_2, \ldots, D_N$ are pairwise disjoint simply connected domains, each lying in $\mathbb{C} \setminus \{0\}$ and satisfying

$$\int_{rA}^{rB} \frac{\pi \, dt}{t\theta_{D_j}(t)} \leq M \log r,$$

where $\theta_{D_j}(t)$ denotes the angular measure of $D_j \cap S(0, t)$. Then $N(B - A) \leq 2M$.

Proof. This is completely standard. The Cauchy-Schwarz inequality gives

$$N^2 = \left(\sum_{j=1}^N 1\right)^2 \leq \left(\sum_{j=1}^N \theta_{D_j}(t)\right) \left(\sum_{j=1}^N \frac{1}{\theta_{D_j}(t)}\right) \leq 2\pi \sum_{j=1}^N \frac{1}{\theta_{D_j}(t)}.$$ 

Hence

$$N^2(B - A) \log r \leq \sum_{j=1}^N \int_{rA}^{rB} \frac{2\pi \, dt}{t\theta_{D_j}(t)} \leq 2NM \log r.$$

□

The proof of Theorem 1.4 requires the characteristic function in a half-plane as developed in [25, 31] (see also [6, 12]). Let $g$ be meromorphic in a domain containing the closed upper half-plane $\overline{H} = \{z \in \mathbb{C} : \text{Im} z \geq 0\}$. For $t \geq 1$ let $n(t, g)$ be the number of poles of $g$ in $\{z \in \mathbb{C} : |z - it/2| \leq t/2, |z| \geq 1\}$, and for $r \geq 1$ set

$$\mathfrak{N}(r, g) = \int_1^r \frac{n(t, g)}{t^2} \, dt, \quad m(r, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\log^+ |g(r \sin \theta)|}{r \sin^2 \theta} \, d\theta. \quad (2.1)$$

The Tsuji characteristic $\mathfrak{T}(r, g)$ is then given by $\mathfrak{T}(r, g) = m(r, g) + \mathfrak{N}(r, g)$.

Lemma 2.3 ([25]) Let $g$ be meromorphic in $\overline{H}$ such that

$$m(r, g) = O(\log r) \quad \text{as} \quad r \to \infty,$$

where $m(r, g)$ is given by (2.1). Then, as $R \to \infty,$

$$\int_R^{\infty} \frac{m_0(r, g)}{r^3} \, dr \leq \int_R^{\infty} \frac{m(r, g)}{r^2} \, dr = O\left(\frac{\log R}{R}\right), \quad m_0(r, g) = \frac{1}{2\pi} \int_0^{\pi} \log^+ |g(re^{i\theta})| \, d\theta.$$
The next lemma involves direct transcendental singularities of the inverse function \([4, 26]\). Let \(a \in \mathbb{C}\) be an asymptotic value of the transcendental meromorphic function \(g\), so that \(g(z) \to a\) as \(z \to \infty\) along a path \(\gamma\) tending to infinity. Then the inverse function \(g^{-1}\) is said to have a transcendental singularity over \(a\). For each \(\varepsilon > 0\) there exists a component \(C = C(a, \varepsilon, g)\) of the set \(\{z \in \mathbb{C} : |g(z) - a| < \varepsilon\}\) with the property that \(C\) contains an unbounded subpath of \(\gamma\). Two asymptotic paths \(\gamma, \gamma'\) on which \(g(z) \to a\) determine distinct singularities if the corresponding components \(C(a, \varepsilon, g), C'(a, \varepsilon, g)\) are distinct for some \(\varepsilon > 0\).

The singularity of \(g^{-1}\) corresponding to \(\gamma\) is called indirect if \(C(a, \varepsilon, g)\), for every \(\varepsilon > 0\), contains infinitely many zeros of \(g - a\) \([4]\), and direct otherwise, in which case \(C(a, \varepsilon, g)\), for all sufficiently small \(\varepsilon > 0\), contains no zeros of \(g - a\). With a slight abuse of notation, such a singularity will be referred to as lying in the upper half-plane \(H\) if \(C(a, \varepsilon, g) \subseteq H\) for sufficiently small positive \(\varepsilon\). Transcendental singularities over \(\infty\) are defined and classified analogously.

**Lemma 2.4** Let \(g\) be a meromorphic function in the plane such that \(\Im(r, g) = O(\log r)\) as \(r \to \infty\). Then there is at most one direct singularity of \(g^{-1}\) lying in \(H\).

**Proof.** Assume that \(g^{-1}\) has at least two direct singularities over \(a_1, a_2\) in \(H\). Here \(a_1, a_2\) need not be distinct but may be assumed finite. Hence for some \(\varepsilon \in (0,1)\) and for \(j = 1, 2\) there exists an unbounded component \(D_j \subseteq H\) of the set \(\{z \in \mathbb{C} : |g(z) - a_j| < \varepsilon\}\), such that \(g(z) \neq a_j\) on \(D_j\). The functions \(u_1, u_2\) defined by

\[
  u_j(z) = \log |\varepsilon/(g(z) - a_j)| \quad (z \in D_j), \quad u_j(z) = 0 \quad (z \notin D_j),
\]

are then non-constant and subharmonic in the plane with disjoint supports \(D_1, D_2 \subseteq H\). Since \(\Im(r, 1/(g - a_j)) = O(\log r)\) as \(r \to \infty\), Lemmas 2.1 and 2.3 lead to, for large positive \(R\),

\[
  \frac{B(R/2, u_j)}{2R^2} \leq \int_R^\infty \frac{B(r/2, u_j)}{r^3} \, dr \leq 3 \int_R^\infty \frac{m_0(r, 1/(g - a_j))}{r^3} \, dr = O\left(\frac{\log R}{R}\right), \tag{2.2}
\]

and hence \(B(R, u_j) = O(R \log R)\) as \(R \to \infty\). But applying Lemma 2.1 again and using the Cauchy-Schwarz inequality as in the proof of Lemma 2.2 as well as the fact that the \(D_j\) are disjoint and lie in \(H\) now yields

\[
  4 \leq \pi \sum_{j=1}^2 \frac{1}{\theta_{D_j}(s)}
\]

and, for \(r_0\) large,

\[
  4 \log R \leq \int_{r_0}^R \sum_{j=1}^2 \frac{\pi ds}{s \theta_{D_j}(s)} + O(1) \leq \sum_{j=1}^2 \log B(2R, u_j) + O(1) \leq (2 + o(1)) \log R
\]

as \(R \to \infty\), which is plainly a contradiction. \(\square\)

The following lemma is the well known Carathéodory inequality \([24, \text{Ch. I.6, Theorem 8’}]\) for analytic self-mappings of the upper half-plane \(H\).
Lemma 2.5 Let \( \psi : H \to H \) be analytic. Then
\[
\frac{|\psi(i)| \sin \theta}{5r} < |\psi(re^{i\theta})| < \frac{5r|\psi(i)|}{\sin \theta} \text{ for } r \geq 1, \theta \in (0, \pi).
\] (2.3)

The proof of Theorem 1.4 will require some elementary inequalities for the hyperbolic metric on the upper half-plane \( H = \{ z = x + iy : x \in \mathbb{R}, y > 0 \} \), on which the hyperbolic density is \( 1/y \). Hence if \( \gamma \) is a curve in \( H \) joining \( i \) to \( z = x + iy \) then the hyperbolic length of \( \gamma \) is
\[
\int_\gamma \frac{1}{\text{Im} \zeta} |d\zeta| \geq \left| \int_1^y \frac{1}{t} \, dt \right| = |\log y| = \left| \log \left( \frac{1}{\text{Im} z} \right) \right|.
\] (2.4)

On the other hand \( i \) may be joined to \( z \) by the line segment \( \gamma_1 \) from \( i \) to \( x + i \) followed by the line segment \( \gamma_2 \) from \( x + i \) to \( z \), which gives the upper bound
\[
[i, z]_H \leq \left( \int_{\gamma_1} + \int_{\gamma_2} \right) \frac{1}{\text{Im} \zeta} |d\zeta| \leq |\text{Re} z| + \left| \log \left( \frac{1}{\text{Im} z} \right) \right|.
\] (2.5)

The imaginary parts of \( T = \tan z \) and \( z \) will now be compared for \( z \in H \). It is clear that \( T = \tan z = M(u) \) for \( z \in H \), where \( M : D(0,1) \to H \) is a Möbius transformation with \( M(0) = i \), and \( u = e^{2iz} \) maps \( H \) into \( D(0,1) \). If \( \text{Im} T \) is small then evidently so is \( y = \text{Im} z \), and
\[
\log \left( \frac{1+|u|}{1-|u|} \right) = [u, T]_{D(0,1)} = [i, T]_H \geq \log \left( \frac{1}{\text{Im} T} \right),
\]
using (2.4). Hence
\[
2y \sim 1 - e^{-2y} = 1 - |u| \leq 2 \text{Im} T, \quad T = \tan z,
\] (2.6)
uniformly in \( x = \text{Re} z \) as \( y = \text{Im} z \) tends to 0.

3 Components and critical points

If an analytic function is a proper mapping between domains each of finite connectivity then the Riemann-Hurwitz formula [29, p.7] links the valency of the mapping with the number of critical points and the connectivities of the domains. To apply this formula requires that the function map boundary to boundary in the sense of [29, p.4].

The function \( f(z) = ze^z \) has a direct transcendental singularity over 0, and a critical point at \(-1\), and the interval \((-\infty, 0]\) lies in a component \( C \) of the set \( \{ z \in \mathbb{C} : |f(z)| < 1 \} \). The function \( f \) is infinite-valent on \( C \), but the number of zeros of \( f \) in \( C \) is equal to the number of critical points of \( f \) in \( C \). The proof of Theorem 1.4 will require a relation between zeros and critical points for components of this type, and this will be obtained by transforming the function in question to one of form \( R(z) \exp(az) \) with \( R \) a rational function and \( a \in \mathbb{C} \).

Lemma 3.1 Let \( b \) be a positive real number and let \( R \) be a rational function such that \( |R(x)| = 1 \) for all \( x \in \mathbb{R} \). Assume that \( f(z) = R(z)e^{ibz} \) is such that \( f \) has no critical values \( w \) with \( |w| = 1 \).
Let \( A \subseteq H \) be an unbounded component of the set \( \{ z \in \mathbb{C} : |f(z)| < 1 \} \), and let \( p \) be the connectivity of \( A \). Let \( m \) be the number of zeros of \( f \) in \( A \) and \( n \) the number of zeros of \( f' \) in \( A \). Then \( m - n = 1 - p \).
Proof. It is evident that such a component $A$ exists, because $|f(iy)| < 1$ for all large positive real $y$ and $|f(x)| = 1$ for $x \in \mathbb{R}$. The set $X = \{z \in \mathbb{C} : |f(z)| = 1\}$ consists of pairwise disjoint Jordan curves, and Jordan arcs tending to infinity in both directions, one of which is the real axis. Since, as $z = re^{i\theta} \to \infty$,

$$\log |f(re^{i\theta})| = -br \sin \theta + O(1), \quad \frac{\partial \log |f(re^{i\theta})|}{\partial \theta} = -\text{Im} \left( \frac{zf'(z)}{f(z)} \right) = -br \cos \theta + o(1),$$

it follows that if $z \in X$ is large then $z \in \mathbb{R}$. Hence the finite boundary $\partial A$ consists of the real axis and $p - 1$ pairwise disjoint Jordan curves $\Gamma_j$ in $H$. Let $\Gamma_j$ be the reflection of $\Gamma_j$ in the real axis. Let $t$ be large and positive and let $\gamma$ be the cycle consisting of the circle $S(0, t)$ described once counter-clockwise and each of the $\Gamma_j$ and $\overline{\Gamma_j}$ described once clockwise. Since $f(z) = 0$ if and only if $f(z) = \infty$, and since the multiplicities coincide, the net change in $\text{arg} f(z)$ as $z$ describes $\gamma$ is 0.

Because $t$ is large it follows that $f'(z) \sim ibf(z)$ on $S(0, t)$ and so the net change in $\text{arg} f'(z)$ as $z$ describes $S(0, t)$ agrees with that of $\text{arg} f(z)$. Moreover, the net change in $\text{arg} f'(z)$ as $z$ describes one of the $\Gamma_j$ or $\overline{\Gamma_j}$ clockwise exceeds that of $\text{arg} f(z)$ by $2\pi$ [30, p.122]. Hence if $N$ is the number of zeros minus the number of poles of $f'$ which lie inside $\gamma$ (i.e. which have winding number 1 relative to $\gamma$), then $N = 2(p - 1)$.

Now the only zeros and poles of $f$ which lie inside $\gamma$ are the the zeros of $f$ in $A$ and their reflections across $\mathbb{R}$, which are poles. Let these zeros of $f$ in $A$ be denoted by $z_j$, with multiplicities $p_j$. Then $z_j$ and $\overline{z}_j$ together contribute $p_j - 1 - (p_j + 1) = -2$ to $N$.

Next consider the critical points of $f$. Since

$$f(\overline{w}) = \frac{1}{f(w)}$$

it follows that $w$ is a multiple point of $f$ if and only if $\overline{w}$ is. Let $w_k$ be the zeros of $f'$ in $A$ which are not zeros of $f$, and denote their multiplicities by $q_k$. Then $w_k$ and $\overline{w}_k$ together contribute $2q_k$ to $N$. Let $r$ be the number of distinct zeros $z_j$ of $f$ in $A$. Then summing over the $z_j$ and $w_k$ gives

$$2(m - n) = 2\left( r - \sum q_k \right) = -N = 2(1 - p).$$

Recall next some standard facts from [26, p.287], albeit in slightly more general form. Let the function $G$ be transcendental and meromorphic in the plane, with no asymptotic values in

$$V_1 = \{ v \in \mathbb{C} : 0 < |v| < 1 \},$$

and assume further that $G'$ has finitely many zeros $z$ with $G(z) \in V_1$. Let $\Gamma$ be a simple piecewise analytic arc, starting at $v_1 \in S(0, 1)$ but otherwise lying in $V_1$, such that all critical values $v \in V_1$ of $G$ lie on $\Gamma$. Choose a branch of the logarithm defined near to $v_1$ and let $\gamma = \log \Gamma$, so that $\gamma$ is a simple piecewise analytic arc and $e^\gamma = \Gamma$. For $k \in \mathbb{Z}$ let $\gamma_k$ be the translation by $k2\pi i$ of $\gamma$; these $\gamma_k$ are then pairwise disjoint. Now let

$$V_0 = V_1 \setminus \Gamma, \quad U_0 = K(0) \setminus \bigcup_{k \in \mathbb{Z}} \gamma_k, \quad \text{where} \quad K(t) = \{ u \in \mathbb{C} : \text{Re} u < t \}. \quad (3.1)$$
Lemma 3.2

Let \( G \) be a component of \( G^{-1}(V_0) \), and choose \( z_0 \in C \) and \( u_0 \in U_0 \) with \( G(z_0) = v_0 = \exp(u_0) \). Let \( g \) be the branch of \( G^{-1} \) mapping \( v_0 \) to \( z_0 \). Then

\[
h(u) = g(e^u) = G^{-1}(e^u)
\]

extends by the monodromy theorem to be analytic on \( U_0 \), with \( h(U_0) \subseteq C \). Indeed if \( z \in C \) then \( z_0 \) may be joined to \( z \) by a path \( \lambda_1 \) in \( C \) and there exists a path \( \lambda_2 \) in \( U_0 \) starting at \( u_0 \) such that \( \exp(\lambda_2) = G(\lambda_1) \subseteq V_0 \). Then \( \lambda_1 = h(\lambda_2) \subseteq h(U_0) \), since \( \lambda_1 \) and \( h(\lambda_2) \) both start at \( z_0 \) and have the same image under \( G \). Hence \( h(U_0) = C \).

Suppose first that \( h \) is univalent on \( U_0 \). Then for \( t < 0 \) with \(|t| \) large the image of the line \( \Re u = t \) under \( h \) is a level curve \( |G(z)| = e^t \) which tends to infinity in both directions. Hence \( h(u) \to \infty \) as \( u \to \infty \) in \( K(t) \), and \( C \) is an unbounded simply connected domain containing a path tending to infinity on which \( G(z) \to 0 \), and will be referred to as type I component of \( G^{-1}(V_0) \). Note also that since \( C \) is simply connected there cannot be a zero of \( G \) on \( \partial C \) and so \( C \) is also a component of the set \( G^{-1}(V_0 \cup \{0\}) \).

If the finite boundary \( \partial C \) of a type I component contains no critical point \( z \) of \( G \) with \( 0 < |G(z)| < 1 \) then \( h \) may be continued analytically along each \( \gamma_k \) to be univalent on \( K(0) \), and \( C \) lies in a component \( B = h(K(0)) \) of \( \{ z \in \mathbb{C} : |G(z)| < 1 \} \) which contains no zeros of \( G \).

Suppose next that \( h \) is not univalent on \( U_0 \). Then there exist distinct \( u_1, u_2 \in U_0 \) with \( h(u_1) = h(u_2) \) and hence \( e^{u_1} = e^{u_2} \). Take the least \( k \in \mathbb{N} \) for which there exist \( u_3, u_4 \in U_0 \) with \( u_3 = u_4 + k\pi i \) and \( h(u_3) = h(u_4) \). Then \( h \) has period \( k\pi i \) by the open mapping theorem and

\[
F(\zeta) = g(\zeta^k) = h(k \log \zeta)
\]

extends to be analytic in \( Z_k = \{ \zeta \in \mathbb{C} : \zeta^k \in V_0 \} \), mapping \( Z_k \) univalently onto \( C \). Moreover, \( z_1 = \lim_{\zeta \to 0} F(\zeta) \) exists, and must be finite, since otherwise every large \( z \in \mathbb{C} \) is \( F(\zeta) \) for some \( \zeta \in Z_k \) and satisfies \( G(z) = \zeta^k \in V_0 \), contradicting the assumption that \( G \) is transcendental. Hence \( z_1 \) is a zero of \( G \) and \( C \cup \{ z_1 \} \) is a component of the set \( G^{-1}(V_0 \cup \{0\}) \), mapped \( k \)-valently onto \( V_0 \cup \{0\} \) by \( G \). This time \( C \) will be called type II. Here if \( \partial C \) contains no zero \( z \) of \( G' \) with \( 0 < |G(z)| < 1 \) then \( F \) may be analytically continued to \( D(0,1) \), with the extended function univalent by the open mapping theorem, and \( C \) lies in a component \( B = F(D(0,1)) \) of \( \{ z \in \mathbb{C} : |G(z)| < 1 \} \) which contains the zero \( z_1 \) and is such that \( G \) is \( k \)-valent on \( B \).

Note next that if \( C^* \) is a component of the set \( G^{-1}(V_0 \cup \{0\}) \) then \( C^* \) contains a component \( C \) of \( G^{-1}(V_0) \) and by the above discussion \( C^* \) is either \( C \) or \( C \cup \{ z_1 \} \) for some \( z_1 \) and in either case is simply connected. Thus the classification into types I and II transfers automatically to \( C^* \).

Now let \( A \) be any component of the set \( \{ z \in \mathbb{C} : |G(z)| < 1 \} \) and let \( C \subseteq A \) be a component of the set \( G^{-1}(V_0) \). If \( \partial C \) contains no zero \( z \) of \( G' \) with \( 0 < |G(z)| < 1 \) then \( C \) is the only component of \( G^{-1}(V_0) \) contained in \( A \), and \( G \) has at most one zero in \( A \), possibly multiple. In the general case, it follows from the fact that \( G' \) has finitely many zeros \( z \) with \( 0 < |G(z)| < 1 \) that \( A \) contains finitely many components \( C \) of \( G^{-1}(V_0) \) and finitely many zeros of \( G \). Moreover if \( A \) does not contain any type I components \( C \) of \( G^{-1}(V_0) \) nor any zeros of \( G' \) then \( A \) contains one simple zero of \( G \) and \( G \) is univalent on \( A \).

**Lemma 3.2** With \( G \) as above and \( V_0 \) defined as in (3.1) let \( A \) be a component of the set \( \{ z \in \mathbb{C} : |G(z)| < 1 \} \) containing precisely one type I component \( C \) of \( G^{-1}(V_0) \). Then the number of zeros of \( G \) in \( A \) is at most the number of zeros of \( G' \) in \( A \).
Proof. Choose \( z_0 \in C \) such that \( t = |G(z_0)| \) is small, and join \( z_0 \) to each zero of \( G \) in \( A \) by a path in \( A \). The union of these finitely many paths forms a compact connected set \( E \subseteq A \) with

\[
\max\{|G(z)| : z \in E\} < 1,
\]

and \( E \) is contained in a component \( \tilde{A} \subseteq A \) of the set \( \{z \in \mathbb{C} : |G(z)| < 1 - \delta\} \) for some small positive \( \delta \). Set \( G = G/(1-\delta) \). Then a set \( \tilde{V}_0 \) may be defined corresponding to \( \tilde{G} \) in the same way as \( V_0 \) was defined for \( G \), and since \( C \) contains a path tending to infinity on which \( G(z) \to 0 \) it is clear that \( \tilde{A} \) contains at least one type I component of \( \tilde{G}^{-1}(\tilde{V}_0) \).

Suppose on the other hand that \( W_1, W_2 \) are distinct type I components of \( \tilde{G}^{-1}(\tilde{V}_0) \) contained in \( \tilde{A} \). Choose \( w_j \in W_j \) with \( G(w_j) \) small and hence \( w_j \) large. Then \( w_1, w_2 \) must both lie in \( C \) and may be joined in \( C \) by a path \( \sigma \) on which \( G(z) \) is small. But then \( \sigma \) lies in a component of \( \tilde{G}^{-1}(\tilde{V}_0) \) and this is a contradiction.

These observations show that in order to prove Lemma 3.2 there is no loss of generality in assuming there exists a small positive \( \eta \) such that \( G \) has no asymptotic values \( v \) with \( 0 < |v| \leq 1 + \eta \), and that \( G' \) has finitely many zeros \( z \) with \( 0 < |G(z)| \leq 1 + \eta \), and none with \( |G(z)| = 1 \), since otherwise \( G \) may be replaced by \( \tilde{G} \).

Choose \( u_0 \in U_0 \) with \( \exp(u_0) = v_0 = G(z_0) \) and define \( h \) as in (3.2) using the branch of \( G^{-1} \) mapping \( v_0 \) to \( z_0 \). Since \( h \) extends to be univalent on \( U_0 \) and \( G' \) has finitely many zeros \( z \) with \( 0 < |G(z)| \leq 1 \), it follows that if \( |k| \) is large then \( h \) may be continued along the arc \( \gamma_k \) and the extended function is still univalent. Indeed, if \( S \) is large enough then \( h \) extends to be analytic and univalent on the set \( U_1 = \{u \in \mathbb{C} : \text{Re } u \leq 0, |u| \geq S\} \).

Since there are no asymptotic values \( v \) of \( G \) with \( 0 < |v| \leq 1 \), all type II components of \( G^{-1}(V_0) \) are bounded, and since there are finitely many of these contained in \( A \), labelled \( D_j \) say, it follows that there exists \( R > 0 \) such that \( E \) and all the \( D_j \) lie in \( D(0,R) \). Moreover \( R \) may be chosen so large that \( |h(u)| < R \) for all \( u \in U_0 \cap D(0,2S) \).

The components of the finite boundary \( \partial A \) are pairwise disjoint level curves \( |G(z)| = 1 \), each either a Jordan curve or a Jordan arc tending to infinity in both directions. If \( \Lambda \) is an unbounded component of \( \partial A \) then each large \( z \in \Lambda \) must belong to \( \partial C \) and so must be \( h(is) \) for some real \( s \) with \( |s| \) large. Hence there is precisely one unbounded component \( \Lambda \) of \( \partial A \). Moreover all but finitely many 1-points of \( G \) in \( \partial A \) lie on \( \Lambda \) and \( \partial A \) has finitely many components.

Let \( \Omega \) be the component of \( \mathbb{C} \setminus \Lambda \) which contains \( A \), and let \( z = p(w) \) map the upper half-plane \( H \) conformally onto \( \Omega \). Then the function \( q \) defined by

\[
q(w) = G(p(w)) \quad (w \in H), \quad \frac{1}{q(w)} = \frac{1}{q(w)}
\]

extends by the reflection principle to a meromorphic function on the plane, which must have the form \( q(w) = R(w)e^{iS(w)} \) with \( R \) a rational function such that \( R(\infty) = 1 \), and \( S \) an entire function which must be real since \( |q(w)| = |R(w)| = 1 \) on \( \mathbb{R} \). Further, the level curves \( |q(w)| = 1 \) have no multiple points, and if \( w \) is large and \( |q(w)| = 1 \) then \( w \) is real, since \( \partial A \) has finitely many components, of which only \( \Lambda \) is unbounded. Since \( A \) contains a path tending to infinity on which \( G(z) \to 0 \) it follows that \( |q(w)| < 1 \) and \( \text{Re } (iS(w)) \leq o(1) \) for all large \( w \in H \). It now follows from the Wiman-Valiron theory [16] that \( S \) is a polynomial, which must be of form \( S(w) = aw + b \) with real constants \( a, b \) and \( a > 0 \). Since \( p^{-1}(A) \subseteq H \) is a component of the set \( \{w \in \mathbb{C} : |q(w)| < 1 \} \), the result now follows from Lemma 3.1. \( \square \)
4 Proof of Theorem 1.4: first steps

Let \( f \) be a real transcendental entire function and assume that \( f \) and \( f'' + f \) have finitely many non-real zeros, and that either \( f \) has infinite order or \( f \in U_{2p}^* \) for some positive integer \( p \).

**Lemma 4.1** Set \( L = f'/f \). Then \( L \) satisfies
\[
\Xi(r, L) = O(\log r) \quad \text{as} \quad r \to \infty.
\]

**Proof.** This uses a modified Tumura-Clunie argument [14, p.69] (see also [19]). Write
\[
M = L' + L^2 + 1 = f'' + f, \quad M' = L'' + 2LL' = \frac{M'}{M}(L' + L^2 + 1).
\]

Then
\[
2PL = Q = \frac{M'}{M}(L' + 1) - L'', \quad \text{where} \quad P = L' - \frac{M'}{2M} L.
\]

But \( L \) has finitely many non-real poles, and \( M \) has finitely many non-real zeros. Since the lemma of the logarithmic derivative holds for the Tsuji characteristic [25, p.332], so does a direct analogue of Clunie’s lemma [14, p.68], which on combination with (4.2) gives
\[
\Xi(r, P) + \Xi(r, M'/M) = \mathcal{G}(r, L),
\]
where \( \mathcal{G}(r, L) \) denotes any quantity which satisfies
\[
\mathcal{G}(r, L) \leq o(\Xi(r, L)) + O(\log r)
\]
as \( r \to \infty \), possibly outside a set of finite measure. Now write
\[
U = L + \frac{M'}{4M}, \quad M = L^2 + \frac{M'}{2M} L + P + 1 = U^2 + R, \quad \Xi(r, R) = \mathcal{G}(r, L),
\]
using (4.2), (4.3) and (4.4). Thus
\[
M' = 2UU' + R', \quad UV = \frac{M'}{M} R - R', \quad \text{where} \quad V = 2U' - \frac{M'}{M} U,
\]
and, using (4.5) and (4.6) and Clunie’s lemma,
\[
\Xi(r, U) = \Xi(r, L) + \mathcal{G}(r, L), \quad \Xi(r, V) + \Xi(r, UV) = \mathcal{G}(r, L).
\]

If \( V \equiv 0 \) then (4.7) gives
\[
\Xi(r, L) \leq \Xi(r, U) + \mathcal{G}(r, L) \leq \Xi(r, UV) + \Xi(r, V) + \mathcal{G}(r, L) = \mathcal{G}(r, L),
\]
which gives (4.1). Assume henceforth that \( V \equiv 0 \). Then there exists a constant \( d \) such that
\[
M = dU^2, \quad (d - 1)U^2 = R,
\]
and it may be assumed that \( d = 1 \), since otherwise (4.5) and (4.7) give (4.1). Thus, by (4.5),
\[
L = W + cM^{1/2}, \quad \text{where} \quad W = -\frac{M'}{4M} \quad \text{and} \quad c^2 = 1.
\]
This gives
\[ L' = W' + \frac{1}{2} c M^{-1/2} M' = W' - 2 W c M^{1/2} \]
and
\[ M = L^2 + L' + 1 = M + 2 W c M^{1/2} + W^2 + L' + 1 = M + W^2 + W' + 1. \]
It follows using (4.8) and (4.9) that
\[ 0 = W^2 + W' + 1, \quad W(z) = -\tan(z + A), \quad U(z) = B \sec^2(z + A), \quad A, B \in \mathbb{C}. \]
But \( e^{iz} \) is bounded in \( H \) and so \( \Im(r, U) = O(1) \), from which (4.1) follows using (4.7) again. \( \square \)

The next step is the Levin-Ostrovskii factorisation [25] of \( L = f'/f \), which will be developed following [28] but using refinements from [5], slightly modified.

**Lemma 4.2** The logarithmic derivative \( L = f'/f \) has a factorisation
\[ L = \frac{f'}{f} = \phi \psi \]  
(4.10)
in which \( \phi \) and \( \psi \) are real meromorphic functions satisfying the following:
(i) either \( \psi \equiv 1 \) or \( \psi(H) \subseteq H; \)
(ii) \( \psi \) has a simple pole at each real zero of \( f \), and no other poles;
(iii) \( \phi \) has finitely many poles, none of them real;
(iv) on each component of \( \mathbb{R} \setminus f^{-1}(\{0\}) \) the number of zeros of \( \phi \) is either infinite or even;
(v) if \( f \in U_{2p} \), then \( \phi \) is a rational function, and if in addition \( f \) has at least one real zero then the degree at infinity of \( \phi \) is even and satisfies
\[ \deg_{\infty}(\phi) = \lim_{z \to \infty} \frac{\log |\phi(z)|}{\log |z|} \geq 2p. \]  
(4.11)

Here a meromorphic function \( g \) on \( \mathbb{C} \) is called real if \( g(\mathbb{R}) \subseteq \mathbb{R} \cup \{\infty\}. \)

**Proof.** Suppose first that \( f \) has no real zeros. Then \( L = f'/f \) has finitely many poles, and if \( f \in U_{2p} \), then \( L \) is a rational function by the lemma of the logarithmic derivative. If the number of real zeros of \( L \) is infinite or even, set \( \psi = 1 \) and \( \phi = L \). On the other hand if \( L \) has an odd number of real zeros \( b \), choose such a zero \( b \) and write \( \psi(z) = z - b \) and \( \phi(z) = L(z)/(z-b) \).

Assume henceforth that \( f \) has at least one real zero. Then the function \( \psi \) is defined as a product as follows [5, 28]. First, if \( a \) is a real zero of \( f \) but not the greatest real zero of \( f \), then there exists a bounded component \( (a, b) \) of \( \mathbb{R} \setminus f^{-1}(\{0\}) \). Since \( L \) has positive residues at \( a \) and \( b \) the number of zeros of \( L \) in \( (a, b) \) is odd. Choosing such a zero \( c = c_a \in (a, b) \) of \( L \), the factor corresponding to \( a \) is then
\[ p_a(z) = \frac{c - z}{a - z} \quad (\text{if } ac \leq 0), \quad \frac{1 - z/c}{1 - z/a} \quad (\text{if } ac > 0), \]  
(4.12)
and \( \arg p_a(z) \) for \( z \in H \) is the angle between the line segments from \( z \) to \( a \) and \( c \) respectively.

Suppose next that \( a \) is the greatest real zero of \( f \). If the number of zeros \( c \) of \( L \) in \( (a, \infty) \) is finite but odd, choose such a zero \( c \) and form a factor \( p_a(z) \) as in (4.12). On the other hand if \( L \) has an infinite or even number of zeros in \( (a, \infty) \), take the factor \( q_a(z) = 1/(a - z) \), so that
for $z \in H$ the argument $\arg g_a(z)$ is the angle between the line segment from $z$ to $a$ and the horizontal line from $z$ in the direction of $+\infty$.

Finally, if there is a least real zero $a$ of $f$ and the number of zeros $c$ of $L$ in $(-\infty, a)$ is finite but odd, then an extra factor $r_c(z) = z - c$ is included, and $\arg r_c(z)$ for $z \in H$ is the angle between the line segment from $z$ to $c$ and the horizontal line from $z$ in the direction of $-\infty$.

The function $\psi$ is then the product of the terms $p_a(z)$ and (if required) $g_a(z)$ and $r_c(z)$, and satisfies $\arg \psi(z) \in (0, \pi)$ for $z \in H$. Moreover, if there are infinitely many real zeros $a$ of $f$ then $ac_a > 0$ for $|a|$ large and so the product converges by the alternating series test. Furthermore, $\phi$ is defined by (4.10) and it is evident from the construction that (i), (ii), (iii) and (iv) are satisfied.

To establish (v), assume that $f \in U_{2p}^*$ and recall that by assumption $f$ has at least one real zero $a_j$. Then (i), (2.3) and the lemma of the logarithmic derivative give

$$m(r, \phi) \leq m(r, L) + m(r, 1/\psi) = O(\log r),$$

so that $\phi$ is a rational function, using (iii). Since $\phi$ clearly has an even number of non-real zeros and poles, and has an even number of real zeros by (iv), the degree at infinity of $\phi$ must be even. Suppose then that

$$d_0 = \deg_\infty(\phi) \leq 2p - 2, \quad \phi(z) \sim c_0 z^{d_0} \quad \text{as} \quad z \to \infty, \quad c_0 \neq 0. \quad (4.13)$$

Since $\psi(H) \subseteq H$ there exists $c \geq 0$ such that

$$\psi(z) = cz + o(|z|) \quad \text{as} \quad z \to \infty, \quad \pi/4 < \arg z < 3\pi/4, \quad (4.14)$$

using the series representation for $\psi$ [5, 24]. Combining (4.13) and (4.14) gives

$$L(z) = c_0 cz^{d_0+1} + o(|z|^{d_0+1}) \quad \text{and} \quad \log f(z) = \frac{c_0 cz^{d_0+2}}{d_0 + 2} + o(|z|^{d_0+2}) = O(|z|^{2p}) \quad (4.15)$$

as $z \to \infty$ with $\pi/4 < \arg z < 3\pi/4$. Write

$$f = P_0 \Pi \exp(P_1), \quad (4.16)$$

where $P_0$ is a real polynomial with no real zeros, $\Pi$ is the canonical product formed with the real zeros $a_j$ of $f$, and $P_1$ is a real polynomial. If $m_j$ is the multiplicity of the zero of $f$ at $a_j$ and $A_j$ is the residue of $\psi$ there then, again since $\psi(H) \subseteq H [5, 24],

$$0 < m_j = A_j \phi(a_j), \quad A_j < 0, \quad \sum_{a_j \neq 0} \frac{|A_j|}{a_j^2} < \infty. \quad (4.17)$$

Hence it follows from (4.13) and (4.17) that

$$\sum_{a_j \neq 0} \frac{m_j}{a_j^{2p}} = \sum_{a_j \neq 0} \frac{A_j \phi(a_j)}{a_j^{2p}} \leq 2|c_0| \sum_{a_j \neq 0} \frac{|A_j|}{a_j^2} + O(1) < \infty.$$

In particular the product $\Pi$ in (4.16) has genus at most $2p - 1$ and growth at most order $2p$, minimal type. Since $f \in U_{2p}^*$ the polynomial $P_1$ in (4.16) must therefore have degree $d_1 \geq 2p$, and if $d_1 = 2p$ then the coefficient $c_1$ of $z^{d_1}$ in $P_1$ is positive. This gives

$$\log |f(z)| = c_1 \Re (z^{d_1}) + o(|z|^{d_1}) \quad \text{as} \quad z \to \infty, \quad \pi/4 < \arg z < 3\pi/4. \quad (4.18)$$
Comparing (4.15) and (4.18) and recalling that \( d_0 \leq 2p - 2 \) then forces \( d_1 = 2p = d_0 + 2 \) and \( c_0c > 0 \), and since \( c \geq 0 \) both \( c_0 \) and \( c \) must be positive. Thus \( \phi(x) > 0 \) for large positive \( x \), and so \( f \) has a greatest real zero \( a \) by (4.17). But then again by (4.17) the function \( \phi \) satisfies \( \phi(a) < 0 \) and so has an odd number of zeros in \((a, \infty)\), which contradicts (iv).

\[ \square \]

**Lemma 4.3** The functions \( \phi \) and \( f \) satisfy

\[ T(r, \phi) + \log T(r, f) = O(r \log r), \tag{4.19} \]

as \( r \to \infty \), and if \( f \) has infinite order then \( \phi \) is transcendental. Moreover, there exist \( c_1 > 0 \) and a set \( E_0 \subseteq [1, \infty) \) of finite logarithmic measure such that

\[ \left| \frac{f'(z)}{f(z)} \right| \leq \exp(c_1 r \log r) \tag{4.20} \]

for large \(|z| = r \) outside \( E_0 \).

**Proof.** The estimate (4.19) and the fact that \( \phi \) is transcendental if \( f \) has infinite order are proved exactly as in [22, §6 and §7] (see also [6, p.982, pp.989-990]): in particular the bound for \( T(r, \phi) \) follows from (2.3), (4.1), (4.10) and Lemma 2.3. The estimate (4.20) is now an immediate consequence of standard inequalities due to Gundersen [13]. \( \square \)

**Lemma 4.4** Write

\[ L = \frac{f'}{f}, \quad T = \tan z, \quad F = \frac{TL - 1}{L + T}. \tag{4.21} \]

Then for any set \( X \subseteq \mathbb{C} \setminus \mathbb{R} \) the number of zeros of \( F' \) in \( X \) is at most the number of distinct zeros of \( f \) in \( X \) plus the number of zeros of \( f'' + f \) in \( X \), and in particular is finite. Next, let

\[ H = \{ z \in \mathbb{C} : \text{Im} z > 0 \}, \quad W = \{ z \in H : F(z) \in H \}, \quad Y = \{ z \in H : L(z) \in H \}. \tag{4.22} \]

Then \( Y \subseteq W \) and the closure of \( Y \) contains no real zeros of \( f \). Moreover if \( C \) is a component of \( Y \) then either \( \partial C \) contains a non-real zero of \( f \) or \( C \) is unbounded and satisfies

\[ \limsup_{z \to \infty, z \in C} \text{Im} L(z) = +\infty. \tag{4.23} \]

Finally,

\[ L - i \text{ and } F - i \text{ have the same zeros with the same multiplicities.} \tag{4.24} \]

**Proof.** Differentiation of (4.21) gives

\[ F' = \frac{(1 + T^2)(L' + L^2 + 1)}{(L + T)^2} = \frac{(1 + T^2)(f'' + f)}{(L + T)^2 f}, \]

using (4.2). Hence non-real zeros of \( F' \) can only arise from non-real zeros of \( f'' + f \) and non-real zeros of \( f \), each of which is a simple pole of \( L \) and hence of \( L + T \). It is obvious that a non-real zero of \( f'' + f \) which is not a zero of \( f \) is a zero of \( F' \) of at most the same multiplicity. Suppose
Lemma 4.5

Let \( f \) be a non-real zero of \( f' \) and is not a zero of \( f'' + f \) of multiplicity \( n \geq 0 \). Since \((L + T)^2\) has a double pole at \( z \) it follows that \( f' \) cannot have a zero at \( z \) of multiplicity greater than \( n - m + 2 \). If \( m = 1 \) this gives \( n + 1 \), which equals the contribution of \( z \) to the number of distinct non-real zeros of \( f \) plus the number of non-real zeros of \( f'' + f \). If \( m = 2 \) then \( f''(z) \neq 0 \) and \( n = 0 \), while if \( m \geq 3 \) then \( n = m - 2 \), and both these cases give \( n - m + 2 = 0 \).

Next, (4.21) gives
\[
F = \frac{(TL - 1)(\bar{L} + \bar{T})}{|L + T|^2} = \frac{T|L|^2 + L|T|^2 - \bar{L} - \bar{T}}{|L + T|^2},
\]
and since \( z \in Y \) gives
\[
T \in H, \quad -\bar{T} \in H, \quad -\bar{L}(z) \in H,
\]
it follows that \( Y \subseteq W \).

Moreover, poles of \( L \) coincide with zeros of \( f \), and a real pole of \( L \) has positive residue and so is not in the closure of \( Y \) [6, p.987]. If \( C \) is a component of \( Y \) such that \( \partial C \) does not contain a non-real zero of \( f \) it follows that \( C \) is unbounded by the maximum principle, and the function
\[
u_C(z) = \text{Im} L(z) \quad (z \in C), \quad \nu_C(z) = 0 \quad (z \notin C),
\]
is non-constant and subharmonic in the plane, from which (4.23) follows.

It remains to prove (4.24), which will follow from the fact that (4.21) gives
\[
F - i = \frac{(L - i)(T - i)}{L + T}.
\]
If \( L(z) = i \) then \( z \) is non-real and is not a zero of \( L + T \), since \( T \) omits the value \(-i\), and so \( F(z) = i \). Further, the multiplicities coincide since \( T \) omits \( i \). Similarly, if \( F(z) = i \) then \( z \) is non-real and is not a zero of \( T - i \) nor a pole of \( L \), and so \( L(z) = i \).

\[\square\]

**Lemma 4.5** Let \( a \in \mathbb{C} \setminus \mathbb{R} \) and set
\[
s_a = \frac{T(F - a)}{T - F}, \quad T = \tan z.
\]
Then
\[
s_a(z) = (\sin^2 z - a \cos z \sin z)L(z) - \cos z \sin z - a \sin^2 z
\]
and \( s_a \) has finitely many poles in \( H \).

Next, let \( M \) and \( N \) be positive real numbers, and let \( b \in \mathbb{C} \setminus \mathbb{R} \) satisfy \( b \neq a \). If \( z \) is large with
\[
|T - a| > |z|^{-M} \quad \text{and} \quad |F(z) - a| < |z|^{-M-N-1}
\]
then
\[
|s_a(z)| < |z|^{M+1}|F(z) - a| < |z|^{-N} \quad \text{and} \quad |s_b(z)| < |z|^{M+1}.
\]
Finally, for any \( Q > 0 \) there exists \( \eta_0 > 0 \) such that if
\[
|s_a(z)| + |s_b(z)| \leq \eta_0
\]
then
\[
\max\{|\tan z|, |\cot z|\} \geq Q.
\]
Proof. The definition (4.21) of $F$ gives
\[ T - F = \frac{1 + T^2}{L + T}, \quad s_a = \frac{T(L + T)(F - a)}{1 + T^2} = \frac{T(TL - 1 - a(L + T))}{1 + T^2} \] (4.30)
and hence
\[ s_a(z) = \cos z \sin z ((\tan z - a)L(z) - 1 - a \tan z), \]
which is (4.27).

Now suppose that $z$ is large and satisfies (4.28). If
\[ |T| > c_a = 2|a| + 1 \] (4.31)
then
\[ |s_a(z)| = \left| \frac{F(z) - a}{1 - F(z)/T} \right| \leq 2|F(z) - a| \quad \text{and} \quad |s_b(z)| = \left| \frac{F(z) - b}{1 - F(z)/T} \right| \leq 2|F(z) - b| \]
and (4.29) is obvious, while if (4.31) fails then writing
\[ |T - F(z)| = |T - a - (F(z) - a)| \geq \frac{1}{2} |z|^{-M} \]
leads to
\[ |s_a(z)| \leq 2c_a|z|^M|F(z) - a| \quad \text{and} \quad |s_b(z)| \leq 2c_a|z|^M|F(z) - b|, \]
which again gives (4.29).

Finally, suppose that there exists a sequence $(z_n)$ such that
\[ |s_a(z_n)| + |s_b(z_n)| \to 0, \quad \max\{|\tan z_n|, |\cot z_n|\} < Q. \]
Then $|F(z_n)| = O(1)$, because otherwise writing
\[ T = -\frac{s_a(1 - T/F)}{1 - a/F} \]
gives a subsequence with $\tan z_n = o(1)$, an immediate contradiction. Hence (4.26) yields
\[ |F(z_n) - a| = O(|s_a(z_n)|) = o(1), \quad |F(z_n) - b| = O(|s_b(z_n)|) = o(1), \]
which is obviously impossible. \qed

5 Direct transcendental singularities

Recall the classification of transcendental singularities summarised prior to Lemma 2.4.

Lemma 5.1 If $F^{-1}$ has a direct transcendental singularity over $a \in \mathbb{C} \setminus \mathbb{R}$ then $a = \pm i$. Moreover the function $L$ has finitely many asymptotic values in $\mathbb{C} \setminus \mathbb{R}$, and $L^{-1}$ cannot have a direct transcendental singularity over $a \in \mathbb{C} \setminus \mathbb{R}$.
Proof. Let \( g \) be \( F \) or \( L \), and assume that \( g^{-1} \) has a direct transcendental singularity over \( a \in \mathbb{C} \setminus \mathbb{R} \), with \( a \neq \pm i \) if \( g = F \). Then there exist a small positive \( \delta_1 \) and a component \( D \) of the set \( \{ z \in \mathbb{C} : |g(z) - a| < \delta_1 \} \) such that \( g(z) \neq a \) on \( D \). Moreover the function

\[
v(z) = \log \frac{\delta_1}{|g(z) - a|} \quad (z \in D), \quad v(z) = 0 \quad (z \in \mathbb{C} \setminus D),
\]

is subharmonic in \( \mathbb{C} \). Since \( g \) is real meromorphic it may be assumed that \( D \subseteq \mathcal{H} \). But \( \mathcal{S}(r, g) = O(\log r) \) as \( r \to \infty \) by (4.1) and (4.21), and so the same argument as in Lemma 2.4 shows that \( B(r, v) = O(r \log r) \) as \( r \to \infty \) (compare (2.2)). In particular \( v \) has order at most 1.

Let \( \delta \) be small and positive and suppose first that \( f \in \mathcal{U}_{2p}^* \). If \( f \) has at least one real zero then (2.3), (4.10), (4.11) and (4.21) show that

\[
L(z) \to \infty \quad \text{and} \quad F(z) \to i \quad \text{as} \quad z \to \infty, \quad \delta < |\arg z| < \pi - \delta. \tag{5.1}
\]

On the other hand if \( f \) has no real zeros then evidently \( L \) is a rational function, with a pole at infinity since \( p \) is positive, and again (5.1) holds. Hence for large \( r \) the angular measure of \( S(0, r) \cap D \) is at most \( 2\delta \), and by a standard application of Lemma 2.1 the order of the subharmonic function \( v \) is at least \( \pi/2\delta \). Since \( \delta \) may be chosen arbitrarily small this is a contradiction.

Suppose next that \( f \) has infinite order. Here a different argument is required since (5.1) is not available, and instead a contradiction will be obtained by showing that \( v \) has lower order greater than 3/2. The function \( \phi \) in (4.10) is transcendental of order at most 1, by Lemma 4.3, and there exists a rational function \( R_1 \) with at most a simple pole at infinity such that

\[
\phi_1(z) = \frac{\phi(z) - R_1(z)}{z^2}
\]

is entire and transcendental of order at most 1. For large \( z \) it follows using (2.3) again that

\[
\text{if} \quad |\phi_1(z)| > 1, \quad \delta < |\arg z| < \pi - \delta \quad \text{then} \quad 1/L(z) = o(1), \quad |g(z) - a| \geq \delta_1. \tag{5.2}
\]

Let \( C \) be a component of the set \( \{ z \in \mathbb{C} : |\phi_1(z)| > 1 \} \) and for \( s > 0 \) let \( \theta_C(s), \theta_D(s) \) denote the angular measure of \( C \cap S(0, s), D \cap S(0, s) \) respectively. Since \( g^{-1} \) also has a direct transcendental singularity over \( \bar{a} \), it follows from (5.2) that, for large \( s \),

\[
\theta_C(s) + 2\theta_D(s) \leq 2\pi + 4\delta. \tag{5.3}
\]

Let \( \theta_C^*(s) = \infty \) if \( S(0, s) \subseteq C \) and \( \theta_C^*(s) = \theta_C(s) \) otherwise. Then

\[
9 \leq \left( \frac{1}{\theta_C^*(s)} + \frac{2}{\theta_D(s)} \right) (2\pi + 4\delta), \tag{5.4}
\]

for large \( s \), using (5.2), (5.3), the Cauchy-Schwarz inequality and the fact that \( \delta \) is small. Integrating (5.4) from \( r_0 \) to \( r \), where \( r_0 \) is large, and using Lemma 2.1 yields, as \( r \to \infty \),

\[
\frac{9 \log r}{2 + 4\delta/\pi} - O(1) \leq \int_{r_0}^{r} \left( \frac{1}{\theta_C^*(s)} + \frac{2}{\theta_D(s)} \right) \frac{\pi ds}{s} \leq (1 + o(1)) \log r + 2 \int_{r_0}^{r} \frac{\pi ds}{s\theta_D(s)}, \tag{5.5}
\]

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since \( \phi_1 \) has order at most 1. Applying Lemma 2.1 to \( v \), and using (5.5) and the fact that \( \delta \) is small by assumption now shows that the lower order of \( v \) is at least
\[
\frac{1}{2} \left( \frac{9}{2 + 4\delta / \pi} - 1 \right) > \frac{3}{2}.
\]

Finally, the assertion that \( L \) cannot have infinitely many asymptotic values \( a \in \mathbb{C} \setminus \mathbb{R} \) is proved by observing that in the contrary case \( L^{-1} \) would have at least two direct transcendental singularities over \( \infty \) lying in \( H \), which by (4.1) contradicts Lemma 2.4.

\[\square\]

6 Indirect transcendental singularities

The following proposition uses again the terminology summarised prior to Lemma 2.4.

**Proposition 6.1** There does not exist \( \alpha \in \mathbb{C} \setminus \mathbb{R} \) such that the inverse function \( F^{-1} \) has an indirect transcendental singularity over \( \alpha \).

**Proof.** To establish this proposition will require the whole of this section and a number of intermediate lemmas. Assume that there exists \( \alpha \in \mathbb{C} \setminus \mathbb{R} \) such that \( F^{-1} \) has an indirect transcendental singularity over \( \alpha \). Since \( F \) is real it may be assumed that the corresponding path and components lie in \( H \). The key idea will be to show that there exist paths \( \Gamma_j \) tending to infinity in \( U_j \) on which \( F(z) \) tends to distinct values \( \beta_j \in H \), and to use the fact that for most large \( z \) on \( \Gamma_j \) it follows from Lemma 4.5 that the function \( s_{\beta_j}(z) \), as defined by (4.26), is small. A contradiction will then arise from an argument of Phragmén-Lindelöf type, using the fact that these functions \( s_{\beta_j}(z) \) have finitely many poles in \( H \). Unfortunately, however, complications arise because in principle the \( \Gamma_j \) may pass close to points where \( \tan z = \beta_j \), and near these points Lemma 4.5 cannot be applied.

**Lemma 6.1** Let \( N \) be a large positive integer. There exist pairwise distinct complex numbers
\[
\beta_j \in \mathbb{C} \setminus \mathbb{R}, \quad \beta_j \neq \pm i, \quad j = 1, 2, \ldots, N,
\]
with \( |\beta_j - \alpha| = \eta_j \) small and positive, and pairwise disjoint simply connected domains \( U_j \subseteq H \) with the following properties:
(i) \( F \) maps \( U_j \) univalently onto the disc \( D(\alpha, \eta_j) \);
(ii) there exists a simple path \( \Gamma_j \) tending to infinity in \( U_j \), mapped by \( F \) onto the half-open line segment \( [\alpha, \beta_j] \), such that \( F(z) \to \beta_j \) as \( z \) tends to infinity on \( \Gamma_j \).

**Proof.** The existence of \( \beta_j, U_j \) and \( \Gamma_j \) with \( 0 \neq |\beta_j - \alpha| \to 0 \) as \( j \to \infty \) follows from Lemma 4.4 and the definition [4] of an indirect singularity (see also [22, Lemma 10.3, p.370]), and in particular \( \beta_j \neq \pm i \) for \( j \) sufficiently large.

**Lemma 6.2** Let \( r \) be large and positive. Then the domains \( U_j \) and paths \( \Gamma_j \) of Lemma 6.1 may be labelled so that
\[
\int_{\theta_1/32}^{\theta_1/16} \pi dt / t \theta_{U_j}(t) > 2048 \pi \log r \tag{6.1}
\]
for \( j = 1, \ldots, 500 \), where \( \theta_{U_j}(t) \) is as defined in Lemma 2.2.
\textbf{Lemma 6.3} The function $F$ satisfies, for $j = 1, \ldots, 500,$
\begin{equation}
|F(z) - \beta_j| \leq dr^{-512} \leq d|z|^{-64} \quad \text{for } z \in \Gamma_j, \ r^{1/8} \leq |z| \leq r^8. \tag{6.2}
\end{equation}

\textit{Proof.} This uses the argument of [22, p.371]. Let $G$ be that branch of the inverse function $F^{-1}$ mapping $D(\alpha, \eta_j)$ onto $U_j.$ For $u \in \Gamma_j$ the distance from $u$ to $\partial U$ is at most $|u|\theta_U(|u|)$ and so Koebe’s theorem implies that
\begin{equation*}
|(v - \beta_j)G'(v)| \leq 4|u|\theta_U(|u|) \quad \text{for } u = G(v), \ v \in [\alpha, \beta_j).
\end{equation*}

Hence, for $z \in \Gamma_j$ with $r^{1/8} \leq |z| \leq r^8$ writing $w = F(z)$ and $u = G(v)$ for $v \in [\alpha, w]$ gives, using (6.1),
\begin{align*}
\log \left| \frac{\beta_j - \alpha}{\beta_j - F(z)} \right| &= \int_{\alpha}^{w} \frac{|dv|}{|\beta_j - v|} = \int_{G(\alpha)}^{z} \frac{|du|}{|(\beta_j - v)G'(v)|} \geq \int_{G(\alpha)}^{z} \frac{|du|}{4|u|\theta_U(|u|)} \\
&\geq \int_{r^{1/4}}^{r^{1/16}} \frac{dt}{4t\theta_U(t)} > 512 \log r \geq 64 \log |z|.
\end{align*}

\textbf{Lemma 6.4} For $1 \leq j \leq 500$ pick an arc $\lambda_j$ of $\Gamma_j$ joining $S(0, r^{1/8})$ to $S(0, r^8)$ and, apart from its endpoints, lying in $A(r^{1/8}, r^8).$ By re-labelling if necessary it may be assumed that these arcs $\lambda_j$ separate the half-annulus $A^+(r^{1/8}, r^8)$ in counter-clockwise order. For $1 \leq j \leq 500$ let $W_j$ be the part of $A^+(r^{1/8}, r^8)$ separating $\lambda_j$ from $\lambda_{j+1}.$ Then there exists $k$ such that
\begin{equation}
\int_{r^{1/4}}^{r^{1/3}} \frac{\pi dt}{t\theta_{W_k}(t)} > 16 \log r \quad \text{and} \quad \int_{r^3}^{r^4} \frac{\pi dt}{t\theta_{W_k}(t)} > 16 \log r. \tag{6.3}
\end{equation}

\textit{Proof.} The existence of $k$ satisfying (6.3) follows since Lemma 2.2 shows that the first inequality of (6.3) fails for at most $2 \cdot 16 \cdot 12 = 384$ of the $W_j$ and the second for at most 32 of them.

\textbf{Lemma 6.5} Choose $k$ satisfying (6.3) and for convenience write
\begin{align*}
a = \beta_k, \quad b = \beta_{k+1}, \quad \lambda_a = \lambda_k, \quad \lambda_b = \lambda_{k+1}.
\end{align*}

Denote by $u_\nu$ the solutions in the annulus $A(r^{1/16}, r^{16})$ of the equations
\begin{equation}
tag{6.4}
tanz = a, b.
\end{equation}
Then the discs $D(u_{\nu}, |u_{\nu}|^{-2})$ are pairwise disjoint. Next, set

$$P(z) = s_a(z)s_b(z).$$

(6.5)

Then $P$ has no poles in $A^+(r^{1/16}, r^{16})$. Finally, the functions $s_a, s_b$ and $P$ satisfy the estimates:

(i) $$|s_a(z)| \leq |z|^{-28} \quad \text{and} \quad |P(z)| \leq |z|^{-14} \quad \text{for all } z \in \lambda_a \setminus \bigcup_{\nu} D(u_{\nu}, |u_{\nu}|^{-12});$$

(6.6)

(ii) $$|s_b(z)| \leq |z|^{-28} \quad \text{and} \quad |P(z)| \leq |z|^{-14} \quad \text{for all } z \in \lambda_b \setminus \bigcup_{\nu} D(u_{\nu}, |u_{\nu}|^{-12});$$

(6.7)

(iii) $$|P(z)| \leq \exp(d|u_{\nu}| \log |u_{\nu}|) \quad \text{for all } z \in D(u_{\nu}, |u_{\nu}|^{-12}).$$

(6.8)

**Proof.** The assertion concerning the discs $D(u_{\nu}, |u_{\nu}|^{-2})$ is obvious since $a \neq b$ and $r$ is large, and $P$ has no poles in $A^+(r^{1/16}, r^{16})$ by (4.27). To prove (i) let $z \in \lambda_a \setminus \bigcup_{\nu} D(u_{\nu}, |u_{\nu}|^{-12})$ and observe first that the choice of the $u_{\nu}$ gives $|\tan z - a| \geq |z|^{-13}$. Recalling (6.2) and applying Lemma 4.5 with $M = 13, N = 28$ now leads at once to (6.6), and (6.7) is obtained using the same argument.

Finally, to prove (iii) suppose that $z \in D(u_{\nu}, |u_{\nu}|^{-12})$. Then $d \leq |\text{Im } z| \leq d'$, since $a$ and $b$ are non-real. Hence (2.3) and (4.10) give

$$|\psi(z)| \leq d|z|^2, \quad |L(z)| \leq d|z|^2M(|z|, \phi),$$

and (6.8) follows using (4.19) and (4.27). This proves Lemma 6.5. \[\square\]

**Lemma 6.6** Let $k, a, b, \lambda_a, \lambda_b$ be as in Lemma 6.5. Then there exist $R_0, R_1, R_2$ satisfying

$$R_0, R_1, R_2 \not\in E_0, \quad r^{1/2} \leq R_0 \leq r^2; \quad r^{1/7} \leq R_1 \leq r^{1/6}; \quad r^6 \leq R_2 \leq r^7;$$

(6.9)

where $E_0$ is the exceptional set of Lemma 4.3, and with the additional properties that

$$S(0, R_\mu) \cap D(u_{\nu}, |u_{\nu}|^{-2}) = \emptyset$$

(6.10)

for $\mu = 0, 1, 2$ and each $\nu$, as well as

$$|\tan z| + |\cot z| \leq d \quad \text{for} \quad |z| = R_0.$$

(6.11)

Finally, there exists $w_k \in W_k \cap S(0, R_0)$ with

$$|s_a(w_k)| = |s_b(w_k)|.$$

(6.12)

**Proof.** First, $R_0, R_1$ and $R_2$ exist because $E_0$ has finite logarithmic measure and $r$ is large, and the discs $D(u_{\nu}, |u_{\nu}|^{-2})$ have sum of radii at most $d$. To prove the existence of $w_k$ observe that since $W_k$ separates $\lambda_a$ from $\lambda_b$ there exists an arc $A_k$ of the circle $S(0, R_0)$ which lies in $W_k$ apart from its endpoints $v_a$ and $v_b$, which satisfy $v_a \in \lambda_a$ and $v_b \in \lambda_b$. It then follows using Lemma 4.5, (6.6) and (6.7) that

$$|s_b(v_a)| > |s_a(v_a)|, \quad |s_a(v_b)| > |s_b(v_b)|.$$

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and so a point \( w_k \in A_k \) satisfying (6.12) exists by continuity.

A contradiction will now be obtained using harmonic measure. Let \( k, W_k \) and \( w_k \) be as in Lemmas 6.5 and 6.6, and let \( D \) be the component of the set \( W_k \cap A^+(R_1, R_2) \) which contains \( w_k \). The function

\[
Q(z) = z^{14} P(z)
\]

is analytic on the closure of \( D \), and evidently

\[
|Q(z)| \leq 1 \quad \text{for all } z \in (\lambda_a \cup \lambda_b) \setminus \bigcup \nu D(u_\nu, |u_\nu|^{-12})
\]

by (6.6) and (6.7). Next,

\[
|Q(z)| \leq R_\mu^{14} \exp(dR_\mu \log R_\mu) \leq \exp(dr^8 \log r) \quad \text{for all } z \in S(0, R_\mu), \mu = 1, 2,
\]

by Lemma 4.3, (4.27) and (6.9), while (6.3) and (6.9) give a harmonic measure estimate

\[
\omega(w_k, D, S(0, R_\mu) \cap \partial D) \leq dr^{-16} \quad \text{for } \mu = 1, 2.
\]

It remains to consider the intersection of \( \partial D \) with the discs \( D(u_\nu, |u_\nu|^{-12}) \). First, (6.8) and (6.13) yield

\[
\log |Q(z)| \leq d|u_\nu| \log |u_\nu| \quad \text{for all } z \in D(u_\nu, |u_\nu|^{-12}).
\]

Suppose then that \( D(u_\nu, |u_\nu|^{-12}) \) meets \( \partial D \), at \( y_\nu \) say. Then it follows that

\[
S(y_\nu, t) \setminus D \neq \emptyset \quad \text{for } |u_\nu|^{-11} \leq t \leq |u_\nu|^{-3}.
\]

For if such a circle \( S(y_\nu, t) \) lies in \( D \) then the closed disc \( E^* \) given by \(|z - y_\nu| \leq t|z - y_\nu| \in \partial D \) lies in each of the simply connected domains \( A^+(R_1, R_2) \) and \( W_k \), and so in some component of the intersection; but then \( E^* \subseteq D \), since \( E^* \) meets \( D \) near \( y_\nu \), which contradicts the fact that \( y_\nu \notin D \) and proves (6.18). But

\[
|w_k - y_\nu| \geq |w_k - u_\nu| - |u_\nu - y_\nu| \geq |u_\nu|^{-2} - |u_\nu|^{-12} \geq |u_\nu|^{-3},
\]

since \( w_k \in S(0, R_0) \). It now follows that the change of variables

\[
\zeta = \frac{1}{z - y_\nu}, \quad \zeta_k = \zeta(w_k) = \frac{1}{w_k - y_\nu},
\]

maps \( D \) to a domain \( D^* \) in \( \mathbb{C} \) such that the circle \( S(0, t) \) meets \( \mathbb{C} \setminus D^* \) for \(|u_\nu|^3 \leq t \leq |u_\nu|^{11} \), while

\[
|\zeta_k| \leq |u_\nu|^3 \quad \text{and} \quad |\zeta(z)| \geq |u_\nu|^{11} \quad \text{for } z \in D(u_\nu, |u_\nu|^{-12}).
\]

This now gives, by conformal invariance of harmonic measure,

\[
\omega(w_k, D, D(u_\nu, |u_\nu|^{-12}) \cap \partial D) \leq d \exp \left( -\int|u_\nu|^3 \frac{\pi dt}{(\theta D)t(t)} \right) \leq d \exp \left( -\int|u_\nu|^3 \frac{dt}{2t} \right) \leq d|u_\nu|^{-3}.
\]
Combining (6.14), (6.15), (6.16), (6.17) and (6.19) leads to
\[
\log |Q(w_k)| \leq d \left( r^{8-16} \log r + \sum_{\nu} |u_{\nu}|^{-2} \log |u_{\nu}| \right) \leq d.
\]

Using (6.5), (6.12) and (6.13) it now follows that
\[
s_a(w_k) = o(1), \quad s_b(w_k) = o(1),
\]
which by (6.11) contradicts Lemma 4.5 and proves Proposition 6.1.

7 Zeros of $\phi$

Lemma 7.1 If $f$ has infinite order then the function $\phi$ has infinitely many zeros.

Proof. Assume that $f$ has infinite order but $\phi$ has finitely many zeros. Then it follows from Lemma 4.2(iii) and Lemma 4.3 that there exist a rational function $R_1$ and a non-zero real constant $c_1$ such that
\[
\phi(z) = R_1(z)e^{c_1z}. \tag{7.1}
\]
Hence it follows using (2.3), (4.10) and (4.21) that $F(z) \to i$ as $z \to \infty$ on each of the rays $L_1, L_2$ given by $\arg z = \pi/2 \pm \pi/16$. Thus each of the rays $L_1, L_2$ gives rise to a transcendental singularity of $F^{-1}$ over $i$, which must be direct by Proposition 6.1. Applying (4.1) and (4.21) in combination with Lemma 2.4 then shows that the two rays $L_1, L_2$ must determine the same direct transcendental singularity of $F^{-1}$, and so there exist a small positive constant $\delta$ and a component $C$ of the set $\{ z \in \mathbb{C} : |F(z) - i| < \delta \}$, on which $F(z) \neq i$, such that $z \in C$ for all large $z$ with $\arg z = \pi/2 \pm \pi/16$. It follows from (4.24) that $L(z) \neq i$ on $C$.

Since $i$ is not a limit point of transcendental singularities of $F^{-1}$, by Lemma 2.4 and Proposition 6.1, nor of critical values of $F$, by Lemma 4.4, the singularity over $i$ is logarithmic. Provided $\delta$ is small enough this implies in particular that the boundary of $C$ consists of one simple curve tending to infinity in both directions [26] (see also §3). Hence all large $z$ with $|\arg z - \pi/2| \leq \pi/16$ are in $C$. But it is evident from (2.3), (4.10), (7.1), the Phragmén-Lindelöf principle and the fact that $c_1$ is real that the equation $L(z) = i$ must have infinitely many solutions near the positive imaginary axis, and this is a contradiction. \hfill \Box

8 The behaviour of $L$ near zeros of $\phi$

The next lemma uses the notation of Definition 2.1, the stated convention that all counts of zeros are with regard to multiplicity unless indicated otherwise, and reasoning similar to [6, p.984] and [22, Lemma 14.1].

Lemma 8.1 If $f \in U_2^*$ and $f$ has $2q$ distinct non-real zeros then for sufficiently small positive $\lambda$ there are at least $p + q$ bounded components $K_j \subseteq H$ of the set $L^{-1}(D^+(0, \lambda))$, each mapped univalently onto $D^+(0, \lambda)$ by $L$, and with a zero of $L$ on $\partial K_j$. If $f$ has infinite order and $M \in \mathbb{N}$ then for sufficiently small positive $\lambda$ there exist at least $M$ such components $K_j$. 

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Proof. Suppose first that \( \zeta \in H \) is a zero of \( L \) of multiplicity \( m \). Then since \( L_{\zeta}(z) = L(z)^{1/m} \) is analytic and univalent near \( \zeta \), it follows that provided \( \lambda \) is small enough there exist \( m \) components \( K_{j} \) in the statement of the lemma.

Next, let \( \zeta \) be a real zero of \( L \) of even multiplicity \( m \). Then for \( \lambda \) small enough \( \zeta \) gives rise to \( m/2 \) components \( K_{j} \), and as \( x \) passes through \( \zeta \) from left to right the sign of \( L(x) \) does not change. Now suppose that \( \zeta \) is a real zero of \( L \) of odd multiplicity \( m \) and that \( \lambda \) is small.

If \( L^{(m)}(\zeta) > 0 \) then \( \zeta \) gives rise to \((m + 1)/2\) components \( K_{j} \), and \( L(x) \) has a positive sign change at \( \zeta \), that is, as \( x \) passes through \( \zeta \) from left to right the sign of \( L(x) \) changes from negative to positive. On the other hand if \( L^{(m)}(\zeta) < 0 \) then \( \zeta \) gives rise to \((m - 1)/2\) components \( K_{j} \), and \( L(x) \) has a negative sign change at \( \zeta \).

In the case where \( f \) has infinite order, it is now clear that the conclusion of the lemma holds if \( L \) has infinitely many non-real or multiple zeros, so assume that all but finitely many zeros of \( L \) are real and simple. Since \( \phi \) has infinitely many zeros by Lemma 7.1, there are two alternatives. The first is that there exists an unbounded open interval \( I \) of \( \mathbb{R} \) containing no poles of \( L \) but infinitely many zeros of \( \phi \) and so of \( L \), in which case \( I \) evidently contains infinitely many zeros \( \zeta \) of \( L \) with \( L'(\zeta) > 0 \), and the conclusion of the lemma follows. The second alternative is that there exist infinitely many bounded open intervals \( I = (a, b) \) lying between adjacent zeros \( a, b \) of \( f \) and containing at least one zero of \( \phi \), in which case \( \psi \) has a zero in \((a, b)\) by construction, or by the fact that \( \psi \) has negative residues, and so \( L \) has at least two zeros \( \zeta \in (a, b) \), at least one of them having \( L'(\zeta) \geq 0 \), so that again the conclusion of the lemma follows.

Suppose now that \( f \in U_{2p}^{*} \). Then \( \phi \) is a rational function. Let \( I \) be a component of \( \mathbb{R} \setminus f^{-1}(\{0\}) \) containing \( \mu_{1} > 0 \) zeros of \( \phi \) and \( m_{1} \) zeros of \( L \). Then \( m_{1} \geq \mu_{1} \) and \( \mu_{1} \) is even by Lemma 4.2(iv). Hence, by the above analysis, if \( \lambda \) is sufficiently small, the interval \( I \) gives rise to \( n_{I} = \frac{m_{1} + s_{I}}{2} \geq \frac{\mu_{1} + s_{I}}{2} \) (8.1) components \( K_{j} \), where \( s_{I} \) is the number of positive sign changes minus the number of negative sign changes undergone by \( L(x) \) on \( I \). Since \( s_{I} \geq -1 \) and \( \mu_{1} \) is even, (8.1) yields \( n_{I} \geq \mu_{1}/2 \).

Let \( 2r \) be the number of non-real zeros of \( \phi \), these coinciding with zeros of \( L \), and let \( 2\nu \) be the number of real zeros of \( \phi \). Summing over all the intervals \( I \) it follows that for small enough \( \lambda \) there are at least \( \nu + r \) components \( K_{j} \) as in the statement of the lemma. But each non-real zero of \( f \) is a simple pole of \( \phi \), and the argument principle gives

\[
2\nu + 2r = 2q + \deg_{\infty}(\phi).
\]

Thus the conclusion of the lemma follows at once from (4.11), except in the case where \( f \) has no real zeros. In this last case, however, \( L \) is a rational function and \( f \) satisfies

\[
f = P_{0} \exp(P_{1})
\]

where \( P_{0} \) is a real polynomial with no real zeros, and \( P_{1} \) is a real polynomial of degree \( d_{1} \geq 2p \). If \( d_{1} \geq 2p + 1 \) then \( \deg_{\infty}(\phi) \geq 2p - 1 \) by Lemma 4.2(i) and (2.3), and again the result follows from (8.2). Suppose finally that \( d_{1} = 2p \). Then the leading coefficient \( c_{1} \) of \( P_{1} \) is positive and \( L(z) \sim 2pc_{1}z^{2p-1} \) as \( z \to \infty \). Here there is one component \( I = \mathbb{R} \), and \( s_{I} = 1 \), and if \( \lambda \) is small then (8.1) and the argument principle applied to \( L \) give at least \( p^{*} \) components \( K_{j} \), where

\[
p^{*} \geq n_{I} + r = \frac{m_{1} + 1}{2} + r = \frac{m_{1} + 2r + 1}{2} = \frac{2p - 1 + 2q + 1}{2} = p + q.
\]

\( \square \)
9 Components of $W$

This section will discuss components of the set $W$ defined in (4.22). Recall from Lemma 4.4 that $F$ has finitely many non-real critical points, and that by Lemma 5.1 and Proposition 6.1 the only possible asymptotic value $w \in H$ of $F$ is $i$.

**Lemma 9.1** Choose a simple polygonal path $\Lambda$ in $(H \cup \{0\}) \setminus \{i\}$, such that $\Lambda$ contains all critical values of $F$ in $H \setminus \{i\}$, and let $H^* = H \setminus \Lambda$. Then all components $A^*$ of the set $W^* = \{z \in H : F(z) \in H^*\}$ are simply connected. Moreover each such component $A^*$ belongs to one of two types:

(a) type I, for which $A^*$ contains no $i$-points of $F$, but a path tending to infinity on which $F(z) \to i$, and $A^*$ is mapped onto $H^* \setminus \{i\}$ by $F$;

(b) type II, for which $A^*$ contains one $i$-point of $F$, of multiplicity $m$, and is mapped $m : 1$ onto $H^*$ by $F$.

There is at most one type I component $A^*$ of $W^*$, and the following properties hold:

(i) each component $A$ of $W$ contains finitely many components $A^*$ of $W^*$ and so finitely many $i$-points of $F$;

(ii) if a component $A$ of $W$ does not contain a type I component $A^*$ of $W^*$ and does not contain any critical points of $F$ then $A$ is mapped conformally onto $H$ by $F$.

**Proof.** Applying the standard transformation

$$u = G(w) = \frac{F(w) - i}{F(w) + i} \quad (9.1)$$

and recalling that $F(\mathbb{R}) \subseteq \mathbb{R} \cup \{\infty\}$ shows that every component of $W$ is a component of the set $\{w \in \mathbb{C} : |G(w)| < 1\}$. Hence the fact that the components $A^*$ are simply connected, their classification as types I or II, and properties (i) and (ii) all follow from the discussion in §3. Since every type I component of $W^*$ gives rise to a direct singularity of $F^{-1}$ lying in $H$, it follows from Lemma 2.4, (4.1) and (4.21) that there is at most one type I component. \hfill \Box

**Lemma 9.2** Let $A$ be a component of $W$ containing a type I component of $W^*$. Then the number of $i$-points of $F$ in $A$ is at most the number of zeros of $F'$ in $A$.

**Proof.** This follows from Lemmas 3.2 and 9.1. \hfill \Box

10 Components of $Y$

Recall from Lemma 8.1 that there exist a small positive $\lambda$ and $M$ components $K_j \subseteq H$ of the set $L^{-1}(D^+(0,\lambda))$, each mapped univalently onto $D^+(0,\lambda)$ by $L$. Here $M = p + q$ if $f \in U_{2q}^*$, where $2q$ is the number of distinct non-real zeros of $f$, and $M$ may be chosen arbitrarily large if $f$ has infinite order.

Each such $K_j$ lies in a component $C_j$ of the set $Y$ defined in (4.22), which in turn lies in a component $A_j$ of $W$, by Lemma 4.4. Here the $C_j$ corresponding to different $K_j$ need not be distinct, and this is also the case for the $A_j$ corresponding to different $C_j$. 

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Lemma 10.1 For each $C_\nu$, the number of $K_j$ contained in $C_\nu$ is at most the number of $i$-points of $F$ in $C_\nu$, and this number is finite.

Proof. Recall first that the number of $i$-points of $L$ in $C_\nu$ equals the number of $i$-points of $F$ in $C_\nu$, by (4.24), and since $C_{\nu}$ lies in some $A_\eta$, this number is finite, by Lemma 9.1. Choose a circular arc $\gamma$ joining 0 to $i$ in the closure of $H$ and passing through no singular values of $L^{-1}$ apart possibly from 0 and $i$ themselves. This is possible by Lemma 5.1. For each $K_j \subseteq C_\nu$ choose $z_j \in K_j$ with $L(z_j) \in \gamma$. Then the inverse function $L^{-1}$ may be continued along the half-open subarc of $\gamma$ joining $L(z_j)$ to $i$, by the choice of $\gamma$, and the image $\gamma_j(w)$ of this continuation starts at $z_j$ and lies in $C_\nu$. If $\gamma_j(w)$ tends to infinity as $w \to i$ this gives a path tending to infinity in $C_\nu$ on which $L(z)$ tends to $i$. But an indirect singularity of $L^{-1}$ over $i$ is excluded since there are finitely many $i$-points of $L$ in $C_\nu$, while a direct singularity is ruled out by Lemma 5.1.

Hence $\gamma_j(w)$ cannot tend to infinity, so that $\gamma_j(w)$ has a finite limit point $z_j^*$ as $w$ tends to $i$ along $\gamma$. Thus $\gamma_j(w)$ tends to $z_j^*$, and $z_j^*$ must be an $i$-point of $L$ in $C_\nu$. Moreover the number of such $\gamma_j$ tending to an $i$-point of $L$ in $C_\nu$ is at most the multiplicity of that $i$-point, which is the same for $L$ as for $F$, by (4.24). This proves the lemma. \hfill \Box

Now choose $\theta' \in (\pi/4,3\pi/4)$ such that the ray $\gamma'$ given by $z = se^{i\theta'}$, $0 < s < \infty$, contains no singular values of $L^{-1}$, again using Lemma 5.1. For each $K_j$ choose $z_j' \in K_j$ with $L(z_j') \in \gamma'$, and continue $L^{-1}$ along $\gamma'$ in the direction of $\infty$. Let $\Gamma_j$ be the image of this continuation starting at $z_j'$. Then $\Gamma_j$ is a path in $C_j$ on which $L(z) \to \infty$, and $\Gamma_j$ tends either to infinity or to a pole of $L$, which must be a zero of $f$ in $H$, by Lemma 4.4. A component $A_\nu$ of $W$ will be called type $(\alpha)$ if there exists $K_j \subseteq C_j \subseteq A_\nu$ such that $\Gamma_j$ tends to infinity, and type $(\beta)$ otherwise.

Lemma 10.2 Let $A_\nu$ be type $(\beta)$. Then the number of $K_j$ contained in $A_\nu$ is at most the number of distinct non-real zeros of $f$ in $A_\nu$.

Proof. For each $K_j$ contained in $A_\nu$ the path $\Gamma_j$ must tend to a zero $v_j$ of $f$ in $H$, and since these are simple poles of $L$ the $v_j$ for different $K_j$ must be distinct. Moreover, (4.21) gives $F(v_j) = \tan v_j \in H$ and so $v_j \in A_\nu$.

\hfill \Box

11 Completion of the proof when $f$ has finite order

Lemma 11.1 Assume that $f \in U_{2p}^*$ and let $A_\nu$ be a type $(\alpha)$ component of $W$. Then the number of $K_j$ contained in $A_\nu$ is at most the number of distinct non-real zeros of $f$ in $A_\nu$ plus the number of zeros of $f'' + f$ in $A_\nu$.

Proof. By Lemmas 4.4 and 10.1 it suffices to show that the number of $i$-points of $F$ in $A_\nu$ is at most the number of zeros of $F'' + f$ in $A_\nu$. This follows in turn from Lemma 9.2 provided that it can be shown that $A_\nu$ contains a type I component of the set $W^*$ defined in Lemma 9.1.

Let the type II components of $W^*$ which are contained in $A_\nu$ be $A_1^*, \ldots, A_N^*$, and let $\eta$ be small and positive. Then since each $A_j^*$ is mapped $m_j : 1$ onto $H^*$ by $F$, for some integer $m_j$, each set

$$B_j^* = \{z \in A_j^* : |F(z) - i| < \eta\}$$

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is bounded. It suffices therefore to show that there exist points \( z \in A_\nu \) with \( |z| \) arbitrarily large and \( |F(z) - i| < \eta \), since these points \( z \) must then lie in a type \( I \) component of \( W^* \).

Let \( C \) be a component of \( Y \) with \( C \subseteq A_\nu \) such that \( C \) contains a curve \( \Gamma_j \) as defined in §10 which tends to infinity. Such a component \( C \) exists since \( A_\nu \) is type (H). Choose \( R^*, S^* \in (0, \infty) \) such that all non-real zeros of \( f \) lie in \( D(0, R^*) \) and \( |L(z)| \leq S^* \) on \( S(0, R^*) \). Then \( C \) contains an unbounded component \( C^* \) of the set \( \{ z \in \mathbb{C} : \Im L(z) > 2S^* \} \) with no poles of \( L \) in its closure, using Lemma 4.4. The function \( v_C \) defined in analogy with (4.25) by

\[
v_C(z) = \Im L(z) \quad (z \in C^*), \quad v_C(z) = 2S^* \quad (z \notin C^*),
\]

is non-constant and subharmonic in the plane, and of lower order at least 1 since \( v_C = 2S^* \) on \( \mathbb{C} \setminus H \). On the other hand \( v_C \) has finite order since \( L = f'/f \) and \( f \) has finite order. Hence combining Lemma 2.1 with a result of Hayman [15] shows that there exist positive constants \( d_1, d_2, d_3 \) and arbitrarily large positive \( r \) such that

\[
B(r, v_C) \leq 3T(2r, v_C) \leq d_1 T(r, v_C), \quad v_C(z) > d_2 T(r, v_C) > r^{1-o(1)}
\]
on a subset of \( S(0, r) \) of angular measure at least \( d_3 \). Therefore choosing such points \( z \) with \( d_3/4 \leq |z| \leq \pi - d_3/4 \) gives \( F(z) \sim i \) by (4.21), as required. \( \Box \)

This completes the proof of Theorem 1.4 when \( f \in U_{2p}^* \), since Lemma 8.1 gives \( p + q \) components \( K_j \), but by Lemmas 10.2 and 11.1 the number of \( K_j \) does not exceed the number \( q \) of distinct zeros of \( f \) in \( H \) plus the number of zeros of \( f'' + f \) in \( H \).

12 Completion of the proof when \( f \) has infinite order

Assume now that \( f \) has infinite order. Here the method of Lemma 11.1 is not available, and a different approach is required, based on the notation and results of §10.

**Lemma 12.1** Let \( N \) be a positive integer. Then there exist at least \( N \) distinct components \( A \) of \( W \) with the following properties:

(i) \( A \) is mapped conformally onto \( H \) by \( F \), and \( A \) contains a component \( C = C(A) \) of \( Y \) with no poles of \( L \) on \( \partial C \);

(ii) \( C = C(A) \) contains a path \( \gamma_C \) tending to infinity such that

\[
\Im L(z) \geq |z|^{1/4} \quad \text{as} \; z \to \infty \text{ on } \gamma_C \quad (12.1)
\]

and

\[
\Im z \to 0 \quad \text{as} \; z \to \infty \text{ on } \gamma_C. \quad (12.2)
\]

**Proof.** As in §10 there exist \( M \) distinct components \( K_j \), each contained in a component \( C_j \) of \( Y \) which in turn lies in a component \( A_j \) of \( W \). By Lemmas 9.1 and 10.1 the number of \( K_j \) contained in a given component \( A \) of \( W \) is at most the number of \( i \)-points of \( F \) in \( A \), and this number is finite, while all but finitely many components \( A \) of \( W \) are conformally equivalent to \( H \) under \( F \). Since \( M \) may be chosen arbitrarily large and \( f \) has finitely many non-real zeros assertion (i) follows using Lemma 4.4.
To prove assertion (ii) let $A$ and $C = C(A)$ be as in (i) and observe that the function $u_C$ of (4.25) is non-constant and subharmonic in the plane, but vanishes on $\mathbb{R}$. Thus the existence of a path $\gamma_C$ satisfying (12.1) follows from a result of Barth, Brannan and Hayman [3].

It remains to show that $\gamma_C$ also satisfies (12.2). To prove this assume that the sequence $(w_\nu) \subset \gamma_C$ tends to infinity with $\text{Im} w_\nu \geq \varepsilon > 0$. Take open discs $D_n$ of radius $\varepsilon$ about the poles $\zeta_n = (n + 1/2)\pi$ of $\tan z$. Then $w_\nu \in \gamma_C \setminus \bigcup D_n$ and

$$\tan w_\nu = O(1), \quad F(w_\nu) = \tan w_\nu + o(1) = O(1), \quad (12.3)$$

using (4.30) and (12.1). But $\gamma_C$ tends to infinity in $C \subset A$ and $F$ is univalent on $A$, and so by passing to a subsequence it may be assumed in view of (12.3) that $F(w_\nu) \to x \in \mathbb{R}$. Hence $\tan w_\nu \to x$, so that $\text{Im} w_\nu \to 0$ using (2.6). This contradiction proves (12.2). \hfill \Box

Choose distinct components $A_1, \ldots, A_5$ as in Lemma 12.1 and set $C_j = C(A_j)$, and take a large positive $R$ such that the circle $S(0, R)$ meets $\gamma_C$, for $j = 1, \ldots, 5$. For each such $j$, choose a subpath $\lambda_{C_j}$ of $\gamma_{C_j}$ lying in $|z| \geq R$ and joining $S(0, R)$ to infinity. It may be assumed, after re-labelling if necessary, that in $A^+(R, \infty)$ the path $\lambda_{C_2}$ separates $\lambda_{C_1}$ from $\mathbb{R}$ and $\lambda_{C_3}$ separates $\lambda_{C_2}$ from $\mathbb{R}$ and, in view of (12.2) and Lemma 12.1(i), that $\text{Im} z \to 0$ as $z \to \infty$ in $A_2 \cup A_3$. Denote positive constants by $c$, not necessarily the same at each occurrence.

**Lemma 12.2** The function $F(z)$ satisfies the hyperbolic distance estimate

$$\frac{|i, F(z)|_H}{|z|} \to \infty \quad \text{as } z \to \infty \text{ in } A_2. \quad (12.4)$$

**Proof.** It follows from the fact that $\text{Im} z \to 0$ as $z \to \infty$ in $A_2$ that for large $w$ in $A_2$ the largest disc of centre $w$ which lies in $A_2$ has radius $\tau(w) = o(1)$. Choose $z^*$ in $A_2$ with $F(z^*) = i$ and let $z \in A_2$ be large. Since the function $u = G(w)$ in (9.1) maps $A_2$ conformally onto $\Delta = D(0, 1)$, with inverse function $w = h_1(u)$, the hyperbolic distance $[0, G(z)]_\Delta$ is the infimum over all curves $\Gamma$ in $A_2$ joining $z^*$ to $z$ of

$$\int_{G(\Gamma)} \frac{2}{1 - |u|^2} |du| = \int_{\Gamma} \frac{2|dw|}{|h_1(u)(1 - |u|^2)|} \geq c \int_{\Gamma} \frac{|dw|}{\tau(w)} \geq \frac{|z|}{o(1)},$$

using Koebe’s theorem, and (12.4) follows since $[i, F(z)]_H = [0, G(z)]_\Delta$. \hfill \Box

**Lemma 12.3** The component $C_2$ and its associated path $\lambda_{C_2}$ satisfy

$$\text{Im} z \leq |z|^{-1/16} \quad \text{as } z \to \infty \text{ on } \lambda_{C_2}. \quad (12.5)$$

**Proof.** Suppose that $z \in \lambda_{C_2}$ is large but $\text{Im} z > |z|^{-1/16}$. Then (4.30) and (12.1) give

$$|\tan z| \leq c|z|^{1/16}, \quad |F(z) - \tan z| \leq c|z|^{-1/8}, \quad |F(z)| \leq c|z|^{1/16}. \quad (12.6)$$

Combining the last of these estimates with (2.5) and (12.4) forces

$$\log \left( \frac{1}{\text{Im} F(z)} \right) \geq \frac{|z|}{o(1)}, \quad \text{Im} F(z) \leq \exp(-|z|).$$

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Applying the second estimate of (12.6) again gives
\[ \text{Im} \tan(z) \leq \text{Im} F(z) + c|z|^{-1/8} \leq \exp(-|z|) + c|z|^{-1/8} \leq c|z|^{-1/8}, \]
and using (2.6) this implies that \( \text{Im} z \leq c|z|^{-1/8} \), contrary to assumption. This contradiction proves the lemma. \( \square \)

Now let \( u = u_{C_3} \) be the subharmonic function defined by (4.25) and for large \( t > 0 \) let \( \theta(t) \) be the angular measure of \( S(0, t) \cap C_3 \). Then Lemma 12.3 and the choice of \( C_3 \) give \( \theta(t) \leq ct^{-17/16} \) for large \( t \), and
\[ B(2r, u) \geq c \exp \left( \pi \int_c^r \frac{dt}{t \theta(t)} \right) \geq \exp \left( c t^{17/16} \right) \]
for large \( r \). By (4.21) and (4.25) this contradicts Lemma 4.3, and the proof of Theorem 1.4 is complete.

13 Concluding remarks

This section will outline some analogies and contrasts between the proofs of Theorems 1.1 and 1.3 (for \( k = 2 \)) on the one hand, and of Theorem 1.4 on the other. In the proof of Theorem 1.4 the function \( F \) defined in (4.21) plays a role comparable to that of the Newton function \( z - f(z)/f'(z) \) in [6, 8, 28]. The connection between these apparently unrelated auxiliary functions may be seen as follows. If \( f \) is a real entire function such that \( f \) and \( f'' + f \) have only real zeros then following Frank's method [9, 10, 11] write
\[ f_1(z) = \cos z, \quad f_2(z) = \sin z, \quad W(f_1, f_2) = 1, \quad W(f_1, f_2, f) = f'' + f := \frac{f}{g^2}, \]
which gives, using standard properties of Wronskians [21, p.10],
\[ \frac{1}{fg} = W(f_1/f, f_2/f, 1) = W((f_1/f)', (f_2/f)'), \]
and
\[ W(w_1, w_2) = 1, \quad \text{where} \quad w_j = (fg)(f_j/f)' = f_jg - Lf_jg, \quad L = f'/f. \]
It then follows that \( w_1 \) and \( w_2 \) are analytic in \( H \) and that the quotient \( w_2/w_1 \) has no critical points in \( H \). But
\[ \frac{w_2}{w_1} = \frac{f_2/Lf_2}{f_1/Lf_1} = \frac{f_1 - Lf_2}{-f_2 - Lf_1} = \frac{TL - 1}{L + T} = F, \]
where \( T = \tan z \) and \( F \) is as in (4.21). When \( f \) and \( f'' \) have only real zeros the same calculation with \( f_1(z) = 1, f_2(z) = z \) leads to \( w_2(z)/w_1(z) = z - f(z)/f'(z) \), which is the Newton function.

The method in §11 for the case of finite order is closely related to the proof of Theorem 1.1 [8, 28], which as presented for \( f \in U_{2p}^* \) in [8] is lengthy. It seems appropriate therefore to summarise the main steps for Theorem 1.1 in the context of the present method and to highlight the contrasts with Theorem 1.4. Indeed, suppose that \( f \in U_{2p}^* \) with \( 2q \) distinct non-real zeros, and define \( \phi \) and \( \psi \) as in Lemma 4.2. Then \( \phi \) is a rational function [8], and so \( f' \) has finitely many non-real zeros, as has \( f'' \) by repetition of the same argument. Let \( L = f'/f \) as before
and let $F$ be the Newton function of $f$. Then $F$ has finitely many multiple points in $\mathbb{C} \setminus \mathbb{R}$ and finitely many non-real critical values, but in contrast to the situation of Theorem 1.4 the function $F$ has no asymptotic values in $\mathbb{C} \setminus \mathbb{R}$ (see e.g. [23, Lemma 4]). A standard argument [6, p.987] then shows that $F$ is finite-valent on each component $A$ of $W = \{ z \in H : F(z) \in H \}$. As in $\S$ 10 there are at least $p + q$ components $K_j$ defined as in Lemma 8.1. Each $K_j$ satisfies $K_j \subseteq C_j \subseteq A_j$ for components $C_j$ of $Y = \{ z \in H : L(z) \in H \}$ and $A_j$ of $W$, not necessarily distinct. Moreover each $K_j$ gives rise to a path $\Gamma_j \subseteq C_j$ as in $\S$ 10 on which $L(z) \to \infty$, and $\Gamma_j$ tends either to infinity or to a non-real zero of $f$. Components $A_\nu$ of $W$ may then be classified as type $(\alpha)$ or $(\beta)$ as in $\S$ 10, and Lemma 10.2 applies to the type $(\beta)$ components. Moreover if $\Gamma_j \to \infty$ then $F(z) \to \infty$ on $\Gamma_j$ and so, since each $K_j$ has a zero of $L$ and so a pole of $F$ on its boundary, the valency of $F$ on a type $(\alpha)$ component $A$ of $W$ exceeds the number $\mu_A$ of $K_j$ contained in $A$ by at least 1, so that by the Riemann-Hurwitz formula the number of critical points of $F$ in $A$ is at least $\mu_A$. Since these critical points are either non-real zeros of $f$ or of $f''$, it follows that there are at least $p$ zeros of $f''$ in $H$.

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