Comments on Equations of Motion for Pure Spinors in Even Dimensions

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Abstract

A Berkovits type action for pure spinors in even dimensions is considered. The equations of motion for pure spinors are investigated by using explicit parameterizations which solve the pure spinor constraints.

For general interactions, the equations of motions are shown to be modified from the naive ones. The extra terms contain a particular projector $\Delta(\lambda, \omega)$.

If the interactions are restricted to the “ghost number” $u(1)$ and the Lorentz $so(p,q)$ current couplings, the action has a large “gauge symmetry”. In this case, in some “gauges”, the extra terms vanish and the equations of motion for the pure spinors retain the naive form even if the pure spinor constraints are taken into account.
1 Introduction

Last year, hidden infinite-dimensional symmetries were found [1] in the classical Green-Schwarz superstring on the $AdS_5 \times S^5$ background [2]. Some properties of the non-local classical conserved charges are investigated in the pp-wave limit [3]. And recently, it is shown that the hidden symmetries are classical super Yangian [4].

An important problem is to check that this classical integrability survives the quantization or not. But due to the $\kappa$-symmetry, it is difficult to quantize the Green-Schwarz action in the curved background with a Ramond-Ramond flux.

One promising approach to quantize superstring action is the Berkovits’ pure spinor formalism [5–8]. The existence of the hidden integrability is shown also for this formalism [9].

In the pure spinor formalism, nilpotency constraints for the BRST-like charges and the holomorphicity constraints for the associated currents play crucial role in restricting the geometry of the target space [10]. The equivalence of these constraints to the on-shell supergravity constraints is analyzed.

Before going to consider the problem of quantum integrability in the particular superstring background, we would like to point out that there is a subtlety in the holomorphicity constraints, which comes from the pure spinor constraints $\lambda^\alpha(\gamma^a)_{\alpha\beta}\lambda^\beta = 0$.

In the superstring action of the pure spinor formalism, the pure spinor $\lambda^\alpha$ and its pair $\omega_\alpha$ appear in the following form (in the conformal gauge):

$$I = -\frac{1}{\pi\alpha'} \int d^2 \xi \left( \omega_\alpha \bar{\partial} \lambda^\alpha + \omega_\alpha X_{\alpha\beta} \lambda^\beta \right). \tag{1.1}$$

Here $X_{\alpha\beta}$ is a field which takes values in the Clifford algebra with the even degree, and is independent of $\lambda^\alpha$ and $\omega_\alpha$.

In derivation of the holomorphicity constraints [10], the naive equations of motion

$$\bar{\partial} \lambda^\alpha + X_{\alpha\beta} \lambda^\beta = 0, \tag{1.2}$$

$$-\partial \omega_\beta + \omega_\alpha X_{\alpha\beta} = 0, \tag{1.3}$$

are used.\footnote{Some related works, such as extension of pure spinor formalism, can be found in [11–19].}

\footnote{Strictly speaking, the equations of motion for pure spinors themselves are not necessary. The equations of motion for the ghost number current and those for the Lorentz currents are sufficient for deriving the holomorphcicity constraints [27].}
But, it is quite common that for a system with constraints, the equations of motion are different from the naive ones. Let us recall the case of the $O(N)$ invariant non-linear sigma model: $L_{O(N)} = (1/2) \partial_\mu \phi^I \partial^\mu \phi^I$. $N$ real scalar fields $\phi^I$ ($I = 1, 2, \ldots, N$) are required to obey the constraint $\phi^I \phi^I = 1$. With help of the Lagrange multiplier, the equations of motion are obtained as $\partial_\mu \partial^\mu \phi^I - \phi^I (\partial_I \partial_\mu \partial^\mu \phi^I) = 0$. The second term is the effect of the constraint. Or, equivalently, we can derive equations of motion by solving the constraint. If we use a solution $\phi^N = \pm \sqrt{1 - \phi^i \phi^i}$ and substituting it into $L_{O(N)}$, the Lagrangian for $N - 1$ real scalars $\phi^i$ ($i = 1, 2, \ldots, N - 1$) is obtained: $L_{O(N)} = (1/2) G_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j$, where $G_{ij}(\phi) = \delta_{ij} + \phi^i \phi^j / (1 - \phi^k \phi^k)$. By using this Lagrangian, we get the equations of motion: $\partial_\mu \partial^\mu \phi^i + \phi^j G_{jk} \partial_\mu \phi^j \partial^\mu \phi^k = 0$ which are equivalent to the manifestly $O(N)$ invariant ones.

Therefore, it seems important to consider the effect of pure spinor constraints to the equations of motion. In order to make the meaning of the effect from the pure spinor constraints clear, we consider a pure spinor action of the type (1.1) in general even dimension ($D = 2N$).

The approach we take in this paper is an analog of the second one for the case of $O(N)$ invariant non-linear sigma model.

In section 2, we explain our choice of the pure spinor action which has “gauge symmetries”. The explicit parametrization for $\lambda$ and $\omega$ are introduced. Transformation properties of these parameters for the finite Lorentz transformation $SO(p, q)$ are summarized.

Because it is rather difficult to treat the pure spinor constraints by the Lagrange multiplier method, we use an explicit parametrization of pure spinors which solves the constraints. Using the explicit local coordinates ($\lambda, u_{rs}$) and ($\beta, v^r$) for a pure spinor $\lambda$ and its pair $\omega$, the equations of motion for the pure spinors are derived in the simple “bosonic ghost” gauge.

The case for the action without the gauge symmetry is discussed in section 3. In this case, $\omega$ is also a pure spinor. The equations of motion for pure spinor pairs ($\lambda, \omega$) are derived for the pure “bosonic ghost” parametrization.

Section 4 is devoted to discussions.

In Appendix A, some useful properties of pure spinors in even dimensions are summarized.
2 Pure Spinor Action and Equations of motion

2.1 Pure spinor action with a “gauge symmetry”

In this section, we consider a pure spinor action of the Berkovits type (1.1) for a target space with even dimension \( D = 2N \). Let us consider the following Lagrangian for a (bosonic) pure spinor \( \lambda \) and its (bosonic) pair spinor \( \omega \) (in the conformal gauge):

\[
\mathcal{L} = \langle \omega^T \mathcal{C} (\overline{\partial} + X) | \lambda \rangle. \tag{2.1}
\]

Here \( \langle \omega^T \mathcal{C} | = (|\omega\rangle)^T \mathcal{C} \), \( \mathcal{C} \) is the charge conjugation matrix : \( (\Gamma^a)^T = C \Gamma^a C^{-1} \) \( (a = 1, 2, \ldots, 2N) \) and \( X \) is a field which takes values in the Clifford algebra with even degree.

We choose the chirality of the pure spinor \(|\lambda\rangle\) to be positive. Then, the chirality of the spinor \(|\omega\rangle\) equals to \((-1)^N\).

We consider the pure spinor \( \lambda \) “near” the Fock vacuum \(|+\rangle\) and take the local coordinate \((\gamma, u_{rs})\) as follows (for details, see appendix A):

\[
|\lambda\rangle = \gamma e^U |+\rangle, \quad U = \frac{1}{2} u_{rs} A_r^{(-)} A_s^{(-)}. \tag{2.2}
\]

We choose \(|-\rangle := A_1^{(-)} A_2^{(-)} \cdots A_N^{(-)} |+\rangle\). Let us denote the Hermitian conjugated and transposed states of \(|\pm\rangle\) by \( \langle \pm | = (|\pm\rangle)^T \) and \( \langle \pm, T | = (|\pm\rangle)^T \) respectively. The normalization of the charge conjugation matrix \( \mathcal{C} \) is fixed by the condition: \( \langle -, T | \mathcal{C} = \langle + | \). Then, \( \langle +, T | \mathcal{C} = \varepsilon_N \langle - \). Here \( \varepsilon_N := (-1)^{(1/2)N(N-1)} \).

Note that \( U \) is a linear combination (with coefficients in \( \mathbb{C} \)) of the Lorentz generators in the spinor representation: \((1/2)\Gamma^{ab}\). So, it may be possible to interpret the relation \( e^{-U} \Gamma^a e^U = \Gamma^b \Lambda^a_{b}(u) \) as a “complexified” Lorentz transformation.

We can easily see that

\[
\langle \lambda^T \mathcal{C} | \Gamma^{a_1 a_2 a_n} | \lambda \rangle = \varepsilon_N \gamma^2 \langle - | e^{-U} \Gamma^{a_1 a_2 a_n} e^U | + \rangle = 0, \quad \text{for} \quad 0 \leq n \leq N-1. \tag{2.3}
\]

Similarly,

\[
\langle \lambda^T \mathcal{C} | \Gamma^{a_1 a_2 a_n} \overline{\partial} | \lambda \rangle = 0, \quad \text{for} \quad 0 \leq n \leq N-3. \tag{2.4}
\]

Therefore, the kinetic part \( \mathcal{L}_{\text{kin}} = \langle \omega^T \mathcal{C} | \overline{\partial} | \lambda \rangle \) is invariant under “gauge transformations”:

\[
|\omega\rangle \rightarrow |\omega'\rangle + \xi |\lambda\rangle, \tag{2.5}
\]

where

\[
\xi = \sum_{0 \leq n \leq N-3} \frac{1}{n!} \epsilon_{a_1 a_2 \cdots a_n} \Gamma^{a_1 a_2 \cdots a_n}. \tag{2.6}
\]
The restriction on $N - n$ comes from the chirality condition: $\Gamma |\omega'\rangle = (-1)^N |\omega'\rangle$.

As in the pure spinor action (1.1), we require that the rest part of the Lagrangian also respects this “gauge symmetry”. In other words, we restrict $X$ as follows:

$$X = m + \frac{1}{2} f_{ab} \Gamma^{ab}. \quad (2.7)$$

It means that the pure spinor coupling to external sources is allowed only through “ghost number” $U(1)$ current $j = \langle \omega^T \mathcal{C} | \lambda \rangle$ and $so(p, q)$ current $M^{ab} = \langle \omega^T \mathcal{C} | \Gamma^{ab} | \lambda \rangle$.

Then, due to the gauge symmetry, the “physical” degree of the freedom of $|\omega\rangle$ becomes $(1/2)N(N - 1) + 1$ and is equal to that of $|\lambda\rangle$.

Let us denote the components of $|\omega\rangle$ by

$$|\omega\rangle = |\omega_-\rangle - \frac{1}{2}\omega^s A^{(+)}_r A^{(+)}_s |\omega_-\rangle + \frac{1}{4}\omega^{pqrs} A^{(+)}_p A^{(+)}_q A^{(+)}_r A^{(+)}_s |\omega_-\rangle + \cdots. \quad (2.8)$$

There are various ways to fix the gauge symmetry. One choice is to set $|\omega\rangle$ to be a pure spinor:

$$\left( A^{(-)}_r + A^{(+)}_s w^{sr} \right) |\omega\rangle = 0. \quad (2.9)$$

In the pure spinor gauge, we can take local coordinates $(\rho, w^{rs})$ for $|\omega\rangle$:

$$|\omega\rangle_p = \rho e^W |\omega_-\rangle, \quad W = \frac{1}{2} w^{rs} A^{(+)}_r A^{(+)}_s. \quad (2.10)$$

Other simple gauge is obtained by setting as many components to be zero as possible:

$$|\omega\rangle_s = |\omega_-\rangle - \frac{1}{2}\omega^s A^{(+)}_r A^{(+)}_s |\omega_-\rangle. \quad (2.11)$$

The components in these two gauge are related as follows:

$$\omega_- = \rho, \quad \omega^{rs} = \rho w^{rs}. \quad (2.12)$$

For a fixed $(\gamma, u^{rs}) \quad (2.2)$, there are special ways to parametrize $\omega$ such that the kinetic term $\mathcal{L}_{\text{kin}}$ becomes that for a collection of bosonic “$\beta\gamma$-ghosts”:

$$\mathcal{L}_{\text{kin}} = \beta \overline{\partial} \gamma + \frac{1}{2} v^r s \overline{\partial} u^{rs} = \beta \overline{\partial} \gamma - \frac{1}{2} \text{tr}(v \overline{\partial} u). \quad (2.13)$$

In the superstring theories of the pure spinor formalism, this “bosonic ghost” gauge plays crucial role in quantization.

Within the pure spinor gauge (2.10), the pure “bosonic ghost” parametrization is given by

$$|\omega\rangle_{p, b} = \beta e^{U \gamma} |\omega_-\rangle = \rho e^W |\omega_-\rangle, \quad (2.14)$$
where

\[ \rho = \beta \det^{1/2}(1_N + uy), \]  

\[ Y = \frac{1}{2} y^{rs} A_r^{(+)} A_s^{(+)}, \quad y^{rs} = (\beta \gamma)^{-1} v^{rs}, \quad w = y(1_N + uy)^{-1} = v(\beta \gamma 1_N + uv)^{-1}. \]  

Within the simple gauge (2.11), the simple “bosonic ghost” parametrization is given by

\[ \omega_- = \beta - \frac{1}{2} \gamma^{-1} v^{rs} u_{rs}, \quad \omega^r = \gamma^{-1} v^r. \]  

This corresponds to the following parametrization of \( \omega \):

\[ \text{s.b.} \langle \omega^T \mathcal{C} \rangle = \langle + \rangle (\beta - \gamma^{-1} V)e^{-U}, \quad V := \frac{1}{2} v^{rs} A_r^{(+)} A_s^{(+)}. \]  

### 2.2 Lorentz transformation properties for various parameters

Before deriving equations of motion, we would like to summarize the Lorentz transformation properties of local coordinates.

Under the Lorentz transformation \( \Lambda \in SO(p, q) \), the isotropic complex vectors transform linearly (see appendix): \( n_a^{(l)} = \Lambda_a^b n_b^{(l)} \). Let \( \Lambda_{\mu}^{\nu} := \omega_\mu \Lambda_\mu^{\nu} \omega^{\nu}_\nu \) (no sum), and \( a_{rs} := \Lambda_{2r-1}^{2s-1}, b_{rs} := \Lambda_{2r-1}^{2s}, c_{rs} := \Lambda_{2r}^{2s-1} \) and \( d_{rs} := \Lambda_{2r}^{2s} \). The condition \( \Lambda \in SO(p, q) \) is converted into \( a^T a + c^T c = 1_N, b^T b + d^T d = 1_N, a^T b + c^T d = 0 \).

In the complex basis, the Lorentz transformation of the isotropic vectors can be written as

\[ \begin{pmatrix} U' \\ T' \end{pmatrix} = M(\Lambda) \begin{pmatrix} U \\ T \end{pmatrix} = \begin{pmatrix} A(\Lambda) & B(\Lambda) \\ C(\Lambda) & D(\Lambda) \end{pmatrix} \begin{pmatrix} U \\ T \end{pmatrix}, \]  

where

\[ A_{rs} := \frac{1}{2} (a_{rs} + ib_{rs} - ic_{rs} + d_{rs}), \quad B_{rs} := \frac{1}{2} (a_{rs} - ib_{rs} - ic_{rs} - d_{rs}), \]

\[ C_{rs} := \frac{1}{2} (a_{rs} + ib_{rs} + ic_{rs} - d_{rs}), \quad D_{rs} := \frac{1}{2} (a_{rs} - ib_{rs} + ic_{rs} + d_{rs}). \]  

The \( SO(p, q) \) condition for \( \Lambda \) is converted into the condition

\[ M^T(\Lambda) \begin{pmatrix} 0 & 1_N \\ 1_N & 0 \end{pmatrix} M(\Lambda) = \begin{pmatrix} 0 & 1_N \\ 1_N & 0 \end{pmatrix}. \]  

More explicitly, \( A^T C + C^T A = 0, B^T D + D^T B = 0, A^T D + C^T B = 1_N \).
First, let us discuss the transformation properties of the local coordinate \((\gamma, u_{rs})\) for the pure spinor \(\lambda\). A pure spinor transforms as follows

\[
|\lambda'\rangle = S(\Lambda)|\lambda\rangle, \quad S(\Lambda)\Gamma^a n^{(I)}_a S^{-1}(\Lambda) = \Gamma^a n^{(I)}_a, \quad S(\Lambda) = \exp\left(\frac{1}{4}\theta_{ab}\Gamma^{ab}\right). \tag{2.22}
\]

If \(\det(Cu + D) \neq 0\), then we have

\[
(A_r^{(+)} + A_s^{(-)}u_{sr}') |\lambda\rangle = 0, \quad u' = (Au + B)(Cu + D)^{-1}. \tag{2.23}
\]

So, the parameters \(u = UT^{-1}\) transform fractionally under the Lorentz transformation. The Lorentz transformed pure spinor can be written as

\[
|\lambda'\rangle = \gamma' \exp\left[\frac{1}{2}u_{rs}A_r^{(-)}A_s^{(-)}\right]|+\rangle. \tag{2.24}
\]

To summarize, under the Lorentz transformation \(\Lambda\), the parameters \((\gamma, u_{rs})\) transforms as follows

\[
\gamma \rightarrow \gamma' = \gamma F_+(\Lambda, u), \quad u \rightarrow u' = (Au + B)(Cu + D)^{-1}. \tag{2.25}
\]

Here

\[
F_+(\Lambda, u) := \langle + |S(\Lambda)e^U|+\rangle. \tag{2.26}
\]

The forms of \(F_+\) for special cases are given in Appendix B.

Next, let us consider the transformation properties for \(\omega\):

\[
|\omega'\rangle = S(\Lambda)|\omega\rangle. \tag{2.27}
\]

The pure spinor gauge (2.10) is preserved under the Lorentz transformation:

\[
|\omega'\rangle_p = S(\Lambda)|\omega\rangle_p. \tag{2.28}
\]

And parameters \((\rho, w^{rs})\) transform as follows:

\[
\rho \rightarrow \rho' = \rho F_-(\Lambda, w), \quad w \rightarrow w' = (Dw + C)(Bw + A)^{-1}. \tag{2.29}
\]

Here

\[
F_-(\Lambda, w) = \langle - |S(\Lambda)e^W|-\rangle = \langle + |e^{-W}S^{-1}(\Lambda)|+\rangle. \tag{2.30}
\]
The simple gauge (2.11) is not preserved under the Lorentz transformation. So, in this case, we should consider the accompanying “gauge transformation” (2.3):

\[ S(\Lambda)|\omega\rangle_s = |\omega'\rangle_s + |\xi(\Lambda, \omega_s)\rangle. \quad (2.31) \]

Let us examine the case of the “bosonic ghost” gauge more explicitly.

Note that

\[ \partial u' = [(Cu + D)^T]^{-1} \partial u(Cu + D)^{-1}. \quad (2.32) \]

Let us introduce an \( N \times N \) antisymmetric matrix \( G \) by

\[ G_{rs} := \frac{\partial}{\partial u_{rs}} \log F_+ = \frac{\langle ++|S(A)^{-1}A^eU|+\rangle}{\langle ++|S(A)eU|+\rangle}. \quad (2.33) \]

If the conjugate fields transform as follows:

\[ \beta \rightarrow \beta' = \beta F_+^{-1}, \]
\[ v \rightarrow v' = (Cu + D)(v - \beta \gamma G)(Cu + D)^T, \quad (2.34) \]

then the kinetic part of the action is invariant under the Lorentz transformations:

\[ \beta \overline{\partial} \gamma' - \frac{1}{2} \text{tr}(v' \overline{\partial} u') = \beta \overline{\partial} \gamma - \frac{1}{2} \text{tr}(v \overline{\partial} u). \quad (2.35) \]

### 2.3 Equations of motion for pure spinors

In this subsection, we derive the equations of motion for pure spinors. For simplicity, we take the simple “bosonic ghost” gauge (2.2) and (2.17).

Then, the equations of motions for local coordinates are easily obtained as

\[ \overline{\partial}_\gamma = -\gamma \langle ++|Xe^U|+\rangle, \]
\[ \overline{\partial}_\beta = \beta \langle ++|Xe^U|+\rangle, \]
\[ \overline{\partial} u_{rs} = u_{rs} \langle ++|Xe^U|+\rangle - \langle ++|A^{(+)\dagger}A^{(+)\dagger}Xe^U|+\rangle, \]
\[ \overline{\partial} v^{rs} = -v^{rs} \langle ++|Xe^U|+\rangle + \left( \beta \gamma - \frac{1}{2} \epsilon^{pq} u_{pq} \right) \langle ++|Xe^U A^{(-\dagger)} A^{(-\dagger)}|+\rangle - \langle ++|V Xe^U A^{(-\dagger)} A^{(-\dagger)}|+\rangle. \quad (2.36) \]

Using these equations, we can show that the equations of motion for pure spinors (in the simple bosonic ghost gauge) are given by

\[ (\overline{\partial} + X)|\lambda\rangle = \Delta(u)X|\lambda\rangle, \quad (2.37) \]
\[
\overline{\partial} \langle \text{s.b.} | \omega^T C \rangle + \text{s.b.} \langle \omega^T C | X = \text{s.b.} \langle \omega^T C | X \Delta(u), \quad (2.38)
\]

where
\[
\Delta(u) = e^U \left( \sum_{k=2}^{[N/2]} \mathcal{P}^{(2k)} \right) e^{-U} = e^U \left( \frac{1}{2} (1 + \Gamma) - \mathcal{P}^{(0)} - \mathcal{P}^{(2)} \right) e^{-U}. \quad (2.39)
\]

Here \( \mathcal{P}^{(n)} \) is the projector into the sector with a \( U(1) \) charge \((N/2) - n\):
\[
\mathcal{P}^{(n)} := \frac{1}{n!} A_{r_1}^{(-)} A_{r_2}^{(-)} \cdots A_{r_n}^{(-)} |+\rangle \langle +| A_{r_1}^{(+)} \cdots A_{r_2}^{(+)} A_{r_1}^{(+)}.
\quad (2.40)
\]

Here the \( U(1) \) charge is an eigenvalue of the operator \( A_{r_1}^{(+)} A_{r_1}^{(-)} - (N/2) \).

To derive the equations of motion for \( \omega \), we have used the relation
\[
\mathcal{P}^{(n)} U = U \mathcal{P}^{(n-2)}.
\quad (2.41)
\]

We can easily see that \( \Delta(u) \) is a projector: \( \Delta^2(u) = \Delta(u) \), and it has the following properties:
\[
\Delta(u) |\lambda\rangle = 0, \quad \Delta(u) \overline{\partial} |\lambda\rangle = 0, \quad \text{s.b.} \langle \omega^T C | \Delta(u) = 0. \quad (2.42)
\]

To derive (2.37) and (2.38), we have used no properties of the interaction term \( X \).

Now, let us use the ansatz of \( X \) (2.7).

For \( k = 2, 3, \ldots, [N/2] \), we can see that
\[
\langle +| A_{r_{2k}}^{(+)} \cdots A_{r_1}^{(+)} |+\rangle = 0, \quad \langle +| A_{r_1}^{(-)} \cdots A_{r_{2k}}^{(-)} |+\rangle = 0, \quad (2.43)
\]
\[
\langle +| A_{r_{2k}}^{(+)} \cdots A_{r_1}^{(+)} e^{-U} \Gamma^{ab} e^U |+\rangle = 0, \quad \langle +| e^{-U} \Gamma^{ab} e^U A_{r_1}^{(-)} \cdots A_{r_{2k}}^{(-)} |+\rangle = 0. \quad (2.44)
\]

Using these relations, for the special types of interactions \( X \) (2.7), we have
\[
\Delta(u) X |\lambda\rangle = 0, \quad \langle \omega^T C | X \Delta(u) = 0. \quad (2.45)
\]

These relations can be understood as follows: in order to have non-zero \( P^{(2k)} e^{-U} X e^U |+\rangle \) or \( \langle +| e^{-U} X e^U P^{(2k)} \), the operator \( e^{-U} X e^U \) should contain components which change the \( U(1) \)-charge by \( 2k \). But the (complex) Lorentz transformed operator \( e^{-U} X e^U \) changes the \( U(1) \)-charge at most 2. Therefore, additional terms vanish for the restricted interaction (2.7).

The relations (2.45) lead to the final form of the equations of motion for pure spinors:
\[
(\overline{\partial} + X) |\lambda\rangle = 0, \quad (2.46)
\]
Therefore, the equations of motion in the bosonic ghost gauge retain the forms even if the effect of the pure spinor constraints are taken into account.

3 Pure spinor action without gauge symmetry

In this section, we consider the pure spinor action with general interaction $X$. Without restriction (2.7), the Lagrangian loses the invariance under the gauge transformations (2.5). Since these gauge symmetries are no longer able to use to keep the “simple gauge”, one natural way to restrict the pair field $\omega$ is to require that it is also a pure spinor. Another natural way is to impose no restriction on $\omega$.

We first consider the case with no restriction on $\omega$:

$$\mathcal{L} = \langle \omega^T C (\overline{\partial} + X) \rangle \langle \lambda \rangle. \quad (3.1)$$

The equations of motion for pure spinor $\lambda$ is simply given by

$$\langle \overline{\partial} + X \rangle \langle \lambda \rangle = 0. \quad (3.2)$$

Using the solution of the pure spinor constraints: $\langle \lambda \rangle = \gamma e^U \langle + \rangle$, we can show that the equations of motion for $\omega$ is given by

$$\left(-\overline{\partial} \langle \omega^T C \rangle + \langle \omega^T C \rangle X \right)e^U P^{(0)} e^{-U} = 0, \quad (3.3)$$

$$\left(-\overline{\partial} \langle \omega^T C \rangle + \langle \omega^T C \rangle X \right)e^U P^{(2)} e^{-U} = 0. \quad (3.4)$$

Due to the pure spinor constraints for $\lambda$, particular components of $-\overline{\partial} \langle \omega^T C \rangle + \langle \omega^T C \rangle$ are required to obey the equations of motion.

The rest of degrees of freedom are not dynamical and play the role of the Lagrange multiplier. Since $\Delta(u)\overline{\partial} \langle \lambda \rangle = 0$, the equation $\langle \overline{\partial} + X \rangle \langle \lambda \rangle = 0$ implies the restriction on $X$: $\Delta(u)X \langle \lambda \rangle = 0$. A solution for this restriction is given by (2.7). So, in this case, the action goes back to the one with the gauge symmetry.

Next, let us consider the case that $\omega$ is a pure spinor. It is convenient to use the pure “bosonic ghost” parametrization (2.14). With this parametrization, the Lagrangian density for the pure spinor pair is given by

$$\mathcal{L} = \beta \overline{\partial} \gamma + \frac{1}{2} v^{rs} \overline{\partial} u_{rs} + \beta \gamma \langle + \rangle e^{-Y} e^{-U} X e^U \langle + \rangle. \quad (3.5)$$
The equations of motion for parameters are given by

\[ \partial \gamma = -\gamma \langle | + (1 + Y) e^{-Y} e^{-U X} e^{U} | + \rangle, \]

\[ \partial u_{rs} = \langle | + \chi^{(+)} A_{r}^{(+)} e^{-Y} e^{-U X} e^{U} | + \rangle, \]

\[ \partial \beta = \beta \langle | + (1 + Y) e^{-Y} e^{-U X} e^{U} | + \rangle, \]

\[ \partial y_{rs} = \langle | + e^{-Y} e^{-U X} e^{U} (\chi^{(-)} A_{r}^{(-)} A_{s}^{(-)}) | + \rangle - \langle | + e^{-Y} e^{-U X} e^{U} (\chi^{(-)} A_{r}^{(-)} A_{s}^{(-)}) | + \rangle. \] (3.6)

Using these relations, we can show that the equations of motion for pure spinors are given by

\[ (\partial + X) | \lambda \rangle = \Delta(\lambda, \omega) X | \lambda \rangle, \] (3.7)

\[ -\partial \langle \chi^{T} C | + \langle \chi^{T} C | X = \langle \chi^{T} C | X \Delta(\lambda, \omega), \] (3.8)

where a projector \( \Delta(\lambda, \omega) \) is given by

\[ \Delta(\lambda, \omega) = e^{U} e^{Y} \left( \sum_{k=2}^{[N/2]} \mathcal{P}^{(2k)} \right) e^{-Y} e^{-U}. \] (3.9)

So, due to the pure spinor constraints for \( \lambda \) and for \( \omega \), the equations of motion contain additional terms with the projector \( \Delta(\lambda, \omega) \).

If we restrict the interaction \( X \) to the form (2.7), the operator \( e^{-Y} e^{-U X} e^{U} e^{Y} \) changes the \( U(1) \)-charge at most 2, therefore, the additional terms vanish.

4 Discussion

In this paper, we considered the Berkovits type action for the pure spinors in even dimensions.

We showed that for general interactions \( X \), both of \( \lambda \) and \( \omega \) are pure spinors and the equations of motions for pure spinors are modified from the naive ones. The extra terms contain a particular projector \( \Delta(\lambda, \omega) \).

If the types of interactions are restricted to the “ghost number” \( u(1) \) and the Lorentz \( so(p, q) \) current couplings, the action has a large “gauge symmetry”. In this case, at least in the pure and simple “bosonic ghost” gauges, the extra terms vanish and the equations of motion for the pure spinors retain the naive form even if the pure spinor constraints are taken into account.
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A Review of Pure Spinor

Let us consider the Clifford algebra \( C(p, q) \) associated to \( V = \mathbb{R}^{p,q} \). The generators of \( C(p, q) \) are \( 2^m \times 2^m \) Gamma matrices

\[
\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}, \quad \eta_{ab} = \text{diag}(+, +, \cdots, +, -, -, \cdots, -), \quad a, b = 1, 2, \ldots, D. \quad (A.1)
\]

Here \( D = p + q \) and \( m = [D/2] \). Further, we require that \((\Gamma^a)\dagger = \Gamma_a\).

A spinor \( |\lambda\rangle \in \mathbb{C}^{2^m} \) is said to be pure \([20, 21]\) if we can take \( m \) linear independent complex vectors \( n^{(I)}(I = 1, 2, \ldots, m) \) in \( V^\mathbb{C} = V \otimes \mathbb{C} = \mathbb{C}^{D} \), the complexification of the vector space \( V \), such that

\[
\Gamma^a n^{(I)}_a |\lambda\rangle = 0, \quad I = 1, 2, \ldots, m. \quad (A.2)
\]

In order to hold \((A.2)\), the complex vectors should satisfy the isotropic (or null) conditions:

\[
n^{(I)}_a n^{(J)}_a = 0, \quad I, J = 1, 2, \ldots, m. \quad (A.3)
\]

So, the subspace \( W = \text{span}\{n^{(1)}, \ldots, n^{(m)}\} \subset \mathbb{C}^{D} \) is a maximal isotropic subspace. Note that the isotropic conditions can be rewritten as follows:

\[
\{\Gamma^a n^{(I)}_a, \Gamma^b n^{(J)}_b\} = 0. \quad (A.4)
\]

In even dimension \((D = 2N)\), the complexification \( V^\mathbb{C} \) decomposes into two isotropic spaces with maximal dimension \( : V^\mathbb{C} = W \oplus \overline{W} \). This fact allows one to solve the pure spinor conditions in terms of \( N \) pairs of fermionic creation and annihilation operators \([22]\):

\[
\{A_r^{(\epsilon_1)}, A_s^{(\epsilon_2)}\} = \delta_{\epsilon_1+\epsilon_2,0} \delta_{rs}, \quad \epsilon_1, \epsilon_2 = \pm 1, \quad (A.5)
\]

where

\[
A_r^{(\pm)} := \frac{1}{2}(\hat{\Gamma}^{2r-1} \pm i\hat{\Gamma}^{2r}), \quad (A_r^{(\pm)})\dagger = A_r^{(\mp)}, \quad r = 1, 2, \ldots, N. \quad (A.6)
\]

Here \( \Gamma^a = \omega_a \hat{\Gamma}^a \) with \((\omega_a)^2 = \eta_{aa}\) (no sum). (So, \((\hat{\Gamma}^a)\dagger = \hat{\Gamma}^a\) and \(\{\hat{\Gamma}^a, \hat{\Gamma}^b\} = 2\delta^{ab}\).
The pure spinor conditions (A.2) can be rewritten as follows:

\[ \Gamma^a \eta^{(l)}_a |\lambda\rangle = (A_1^{(+)})^I T^I + A_1^{(-)} U^I |\lambda\rangle = 0, \]  

(A.7)

where \( T^I = \tilde{n}_{2r-1}^{(l)} + i \tilde{n}_{2r}^{(l)} \) and \( U^I = \tilde{n}_{2r-1}^{(l)} - i \tilde{n}_{2r}^{(l)} \) for \( r = 1, 2, \ldots, N \) and \( I = 1, 2, \ldots, N \). Here \( \tilde{n}^{(l)}_a := \omega_a \eta^{(l)}_a \) (no sum).

The maximal isotropic condition (A.4) becomes

\[ T^I U^J + U^I T^J = (T^T U + U^T T)^IJ = 0. \]  

(A.8)

If \( \det(T^I) \neq 0 \), then the pure spinor conditions can be rewritten as

\[ (A_1^{(+)}) + A_1^{(-)} u_{sr}) |\lambda\rangle = 0, \]  

(A.9)

where \( u_{sr} := (UT^{-1})_{sr} = U^I (T^{-1})_{Ir} \). From (A.8), \( u \) is antisymmetric: \( u_{sr} = -u_{rs} \).

The equations (A.9) are easily solved as

\[ |\lambda\rangle = \gamma \exp \left[ \frac{1}{2} u_{rs} A_r^{(-)} A_s^{(-)} \right] |+\rangle = \gamma \prod_{1 \leq r \leq s \leq N} \prod_{1 \leq r \leq s \leq N} \left( 1 + u_{rs} A_r^{(-)} A_s^{(-)} \right) |+\rangle, \]  

(A.10)

where \( |+\rangle \) is the Fock vacuum: \( A_r^{(+)} |+\rangle = 0 \) \( (r = 1, 2, \ldots, N) \).

In general, for \( \epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_N) \), \( \epsilon_i = \pm 1 \), we can rewrite the elements \( \Gamma^a \eta^{(l)}_a \) as follows:

\[ \Gamma^a \eta^{(l)}_a = A_1^{(+)})^I T^I + A_1^{(-)} U^I = A_1^{(\epsilon_i)} T^{(\epsilon_i)} I + A_1^{(-\epsilon_i)} U_{(\epsilon_i)}^I, \]  

(A.11)

where

\[ \left( \begin{array}{c} U^{(\epsilon)} \\ T^{(\epsilon)} \end{array} \right) = \left( \begin{array}{cc} 1_N - I^{(\epsilon)} & I^{(\epsilon)} \\ I^{(\epsilon)} & 1_N - I^{(\epsilon)} \end{array} \right) \left( \begin{array}{c} U \\ T \end{array} \right). \]  

(A.12)

Here

\[ I^{(\epsilon)} = \frac{1}{2} \operatorname{diag}(1 - \epsilon_1, 1 - \epsilon_2, \ldots, 1 - \epsilon_N), \]  

(A.13)

\( (I^{(\epsilon)})_{rs} = \delta_{\epsilon_r, -1} \delta_{rs} \).

If \( \det(T^{(\epsilon)}) \neq 0 \), the pure spinor conditions can be written as

\[ [A_1^{(\epsilon_i)} + A_s^{(-\epsilon_i)} u_{sr}^{(\epsilon_i)}] |\lambda\rangle = 0, \quad u^{(\epsilon)} = U^{(\epsilon)} (T^{(\epsilon)})^{-1}. \]  

(A.14)

Therefore, the pure spinors for \( \det(T^{(\epsilon)}) \neq 0 \) can be expressed as

\[ |\lambda\rangle = \gamma^{(\epsilon)} \exp \left[ \frac{1}{2} u_{rs}^{(\epsilon)} A_r^{(-\epsilon_i)} A_s^{(-\epsilon_i)} \right] |\epsilon\rangle, \]  

(A.15)

where

\[ |\epsilon\rangle := \left( A_1^{(\epsilon_1)}(1/2)(1-\epsilon_1) \right) \left( A_2^{(\epsilon_2)}(1/2)(1-\epsilon_2) \right) \cdots \left( A_N^{(\epsilon_N)}(1/2)(1-\epsilon_N) \right) |+\rangle. \]  

(A.16)
The space of pure spinors are covered by $2^N$ open sets and the local coordinates $(\gamma^\epsilon, u_{rs}^\epsilon)$ parametrize pure spinors “near” $|\epsilon\rangle$. Obvious examples of pure spinors which cannot be expressed in the form (A.10) are $|\epsilon\rangle$ for $\epsilon \neq (+1, +1, \ldots, +1)$. So, we need to consider these $2^N$ local coordinate systems in order to express all pure spinors.

Note that the pure spinor has definite chirality:

$$\Gamma |\lambda\rangle = \left(\prod_{r=1}^{N} \epsilon_r\right) |\lambda\rangle. \quad (A.17)$$

Here the chirality matrix is chosen such that $\Gamma |+\rangle = |+\rangle$. So, there are $2^{N-1}$ choices of $\epsilon = (\epsilon_1, \ldots, \epsilon_N)$ for each chirality.

For other properties of pure spinors, see, for example, [23–26].

## B Explicit form of $F_+(\Lambda, u)$ for special case

For a Lorentz transformation in the $ab$-plane:

$$S(\Lambda^{(ab)}) = \exp \left(\frac{\theta}{2} \hat{\Gamma}^{ab}\right), \quad (B.1)$$

the function $F_+(\Lambda, u)$ (2.26) is given by

$$F_+(\Lambda^{(2r-1,2r)}, u) = e^{i(\theta/2)},$$

$$F_+(\Lambda^{(2r-1,2s-1)}, u) = \cos(\theta/2) - u_{rs} \sin(\theta/2), \quad r \neq s,$$

$$F_+(\Lambda^{(2r,2s-1)}, u) = \cos(\theta/2) + i u_{rs} \sin(\theta/2), \quad r \neq s,$$

$$F_+(\Lambda^{(2r,2s)}, u) = \cos(\theta/2) + u_{rs} \sin(\theta/2), \quad r \neq s. \quad (B.2)$$

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