ANALYTICAL SOLUTION OF THE WEIGHTED FERMAT-TORRICELLI PROBLEM FOR CONVEX QUADRILATERALS IN THE EUCLIDEAN PLANE: THE CASE OF TWO PAIRS OF EQUAL WEIGHTS

ANASTASIOS N. ZACHOS

Abstract. The weighted Fermat-Torricelli problem for four non-collinear points in $\mathbb{R}^2$ states that:

Given four non-collinear points $A_1, A_2, A_3, A_4$ and a positive real number (weight) $B_i$ which correspond to each point $A_i$, for $i = 1, 2, 3, 4$, find a fifth point such that the sum of the weighted distances to these four points is minimized. We present an analytical solution for the weighted Fermat-Torricelli problem for convex quadrilaterals in $\mathbb{R}^2$ for the following two cases:

(a) $B_1 = B_2$ and $B_3 = B_4$, for $B_1 > B_4$ and (b) $B_1 = B_3$ and $B_2 = B_4$.

1. Introduction

The weighted Fermat-Torricelli problem for $n$ non-collinear points in $\mathbb{R}^2$ refers to finding the unique point $A_0 \in \mathbb{R}^2$, minimizing the objective function:

$$f(X) = \sum_{i=1}^{n} B_i \|X - A_i\|,$$

$X \in \mathbb{R}^2$ given four non-collinear points $\{A_1, A_2, A_3, A_4, ..., A_n\}$ with corresponding positive real numbers (weights) $B_1, B_2, B_3, B_4, ..., B_n$ where $\| \cdot \|$ denotes the Euclidean distance.

The existence and uniqueness of the weighted Fermat-Torricelli point and a complete characterization of the solution of the weighted Fermat-Torricelli problem has been given by Y. S Kupitz and H. Martini (see [5], theorem 1.1, reformulation 1.2 page 58, theorem 8.5 page 76, 77). A particular case of this result for four non-collinear points in $\mathbb{R}^2$, is given by the following theorem:

Theorem 1. [2, 5] Let there be given four non-collinear points $\{A_1, A_2, A_3, A_4\}$, $A_1, A_2, A_3, A_4 \in \mathbb{R}^2$ with corresponding positive weights $B_1, B_2, B_3, B_4$.

(a) The weighted Fermat-Torricelli point $A_0$ exists and is unique.

(b) If for each point $A_i \in \{A_1, A_2, A_3, A_4\}$

$$\|\sum_{j=1, i \neq j}^{4} B_j \overline{u}(A_i, A_j)\| > B_i,$$

for $i, j = 1, 2, 3$ holds, then (b1) the weighted Fermat-Torricelli point $A_0$ (weighted floating equilibrium point)
does not belong to \{A_1, A_2, A_3, A_4\} and
(b_2)

\begin{equation}
\sum_{i=1}^{4} B_i \vec{u}(A_0, A_i) = \vec{0},
\end{equation}

where \( \vec{u}(A_k, A_l) \) is the unit vector from \( A_k \) to \( A_l \), for \( k, l \in \{0, 1, 2, 3, 4\} \) (Weighted Floating Case).

(c) If there is a point \( A_i \in \{A_1, A_2, A_3, A_4\} \) satisfying

\begin{equation}
\| \sum_{j=1, i \neq j}^{4} B_j \vec{u}(A_i, A_j) \| \leq B_i,
\end{equation}

then the weighted Fermat-Torricelli point \( A_0 \) (weighted absorbed point) coincides with the point \( A_i \) (Weighted Absorbed Case).

In 1969, E. Cockayne, Z. Melzak proved in [3] by using Galois theory that for a specific set of five non-collinear points the unweighted Fermat-Torricelli point \( A_0 \) cannot be constructed by ruler and compass in a finite number of steps (Euclidean construction).

In 1988, C. Bajaj also proved in [1] by applying Galois theory that for \( n \geq 5 \) the weighted Fermat-Torricelli problem for \( n \) non-collinear points is in general not solvable by radicals over the field of rationals in \( \mathbb{R}^3 \).

We recall that for \( n = 4 \), Fagnano proved that the solution of the unweighted Fermat-Torricelli problem \( (B_1 = B_2 = B_3 = B_4) \) for convex quadrilaterals in \( \mathbb{R}^2 \) is the intersection point of the two diagonals and it is well known that the solution of the weighted Fermat-Torricelli problem for non-convex quadrilaterals is the vertex of the non-convex angle. Extensions of Fagnano result to some metric spaces are given by Plastria in [6].

In 2012, Roussos studied the unweighted Fermat-Torricelli problem for Euclidean triangles and Uteshev studied the corresponding weighted Fermat-Torricelli problem and succeeded in finding an analytic solution by using some algebraic system of equations (see [4] and [9]).

Thus, we consider the following open problem:

**Problem 1.** Find an analytic solution with respect to the weighted Fermat-Torricelli problem for convex quadrilaterals in \( \mathbb{R}^2 \), such that the corresponding weighted Fermat-Torricelli point is not any of the given points.

In this paper, we present an analytic solution for the weighted Fermat-Torricelli problem for a given tetragon in \( \mathbb{R}^2 \) for \( B_1 > B_4, B_1 = B_2 \) and \( B_3 = B_4 \), by expressing the objective function as a function of the linear segment which connects the intersection point of the two diagonals and the corresponding weighted Fermat-Torricelli point (Section 2, Theorem 2).

By expressing the angles \( \angle A_1 A_0 A_2, \angle A_2 A_0 A_3, \angle A_3 A_0 A_4 \) and \( \angle A_4 A_0 A_1 \) as a function of \( B_1, B_4 \) and \( a \) and taking into account the invariance property of the weighted Fermat-Torricelli point, we obtain an analytic solution for a convex quadrilateral having the same weights with the tetragon (Section 3, Theorem 3).

Finally, we derive that the solution for the weighted Fermat-Torricelli problem for a given convex quadrilateral in \( \mathbb{R}^2 \) for the weighted floating case for \( B_1 = B_3 \).
and $B_2 = B_4$ is the intersection point (Weighted Fermat-Torricelli point) of the two diagonals (Section 4, Theorem 4).

2. The weighted Fermat-Torricelli problem for a tetragon: The case $B_1 = B_2$ and $B_3 = B_4$.

We consider the weighted Fermat-Torricelli problem for a tetragon $A_1A_2A_3A_4$, for $B_1 > B_4$, $B_1 = B_2$ and $B_3 = B_4$.

We denote by $a_{ij}$ the length of the linear segment $A_iA_j$, $O$ the intersection point of $A_1A_3$ and $A_2A_4$, $y$ the length of the linear segment $OA_0$ and $\alpha_{ijk}$ the angle $\angle A_iA_kA_j$ for $i,j,k = 0,1,2,3,4, i \neq j \neq k$ (See fig. 1) and we set $a_{12} = a_{23} = a_{34} = a_{41} = a$.

Problem 2. Given a tetragon $A_1A_2A_3A_4$ and a weight $B_i$ which corresponds to the vertex $A_i$, for $i = 1,2,3,4$, find a fifth point $A_0$ (weighted Fermat-Torricelli point) which minimizes the objective function

$$f = B_1a_{01} + B_2a_{02} + B_3a_{03} + B_4a_{04} \quad (2.1)$$
for \( B_1 > B_4, \ B_1 = B_2 \) and \( B_3 = B_4 \).

**Theorem 2.** The location of the weighted Fermat-Torricelli point of \( A_1A_2A_3A_4 \) for \( B_1 = B_2, \ B_3 = B_4 \) and \( B_1 > B_4 \) is given by:

\[
y = \frac{1}{2} \sqrt{\frac{a^2}{4} + r - \frac{1}{2}} - \frac{a^2}{4} - \frac{t^{1/3}}{24} 2^{2/3} q^{1/3} - \frac{25pq^{1/3}}{32^{2/3} t^{1/3} (B_1^2 - B_4^2)} + \frac{a^2 B_2^2 - a^2 B_4^2}{12 (B_1^2 - B_4^2)} - \frac{-a^3 B_2^2 - a^3 B_4^2}{2 r (B_1^2 - B_4^2)}
\]

(2.2)

where

\[
t = 2000 a^6 B_1^6 - 2544 a^6 B_4^4 B_1^2 B_4^2 + 2544 a^6 B_1^2 B_4^4 - 2000 a^6 B_4^6 + 192 \sqrt{3} a^6 B_1^2 B_4^2 (B_1^2 - B_4^2)^2 (125 B_1^4 - 142 B_1^2 B_4^2 + 125 B_4^4),
\]

(2.3)

\[
p = a^4 B_1^4 - 2 a^4 B_4^4 B_1^2 + a^4 B_4^4,
\]

(2.4)

\[
q = B_6^6 - 3 B_1^4 B_4^4 + 3 B_1^2 B_4^2 - B_4^6
\]

(2.5)

and

\[
r = \frac{t^{1/3}}{24} 2^{2/3} q^{1/3} + \frac{25pq^{1/3}}{32^{2/3} t^{1/3} (B_1^2 - B_4^2)} - \frac{a^2 B_2^2 - a^2 B_4^2}{12 (B_1^2 - B_4^2)}
\]

(2.6)

**Proof of Theorem 2:** Taking into account the symmetry of the weights \( B_1 = B_4 \) and \( B_2 = B_3 \) for \( B_1 > B_4 \) and the symmetries of the tetragon the objective function (2.16) of the weighted Fermat-Torricelli problem (Problem 2) could be reduced to an equivalent Problem by placing a wall to the midperpendicular line from \( A_1A_2 \) and \( A_3A_4 \) which states that: Find a point \( A_0 \) which belongs to the midperpendicular of \( A_1A_2 \) and \( A_3A_4 \) and minimizes the objective function

\[
f = \frac{1}{2} B_1 a_{01} + B_4 a_{04}.
\]

(2.7)

We express \( a_{01}, a_{02}, a_{03} \) and \( a_{04} \) as a function of \( y \):

\[
a^2_{01} = \left( \frac{a}{2} \right)^2 + \left( \frac{a}{2} - y \right)^2
\]

(2.8)

\[
a^2_{02} = \left( \frac{a}{2} \right)^2 + \left( \frac{a}{2} - y \right)^2
\]

(2.9)

\[
a^2_{03} = \left( \frac{a}{2} \right)^2 + \left( \frac{a}{2} + y \right)^2
\]

(2.10)

\[
a^2_{04} = \left( \frac{a}{2} \right)^2 + \left( \frac{a}{2} + y \right)^2
\]

(2.11)

By replacing (2.8) and (2.11) in (2.7) we get:
Figure 2. The weighted floating equilibrium point (weighted Fermat-Torricelli point) $A_0$ for a tetragon $B_1 = B_2$ and $B_3 = B_4$ for $B_1 > B_4$

\[ B_1 \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} - y\right)^2} + B_4 \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} + y\right)^2} \rightarrow \min. \quad (2.12) \]

By differentiating (2.12) with respect to $y$, and by squaring both parts of the derived equation, we get:

\[ \frac{B_1^2 \left(\frac{a}{2} - y\right)^2}{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} - y\right)^2} = \frac{B_4^2 \left(\frac{a}{2} + y\right)^2}{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} + y\right)^2} \quad (2.13) \]

or

\[ 8 \left(B_1^2 - B_4^2\right) y^4 + 2a^2 \left(-B_1^2 + B_4^2\right) y^2 - 2a^3 \left(B_1^2 + B_4^2\right) y + a^4 \left(B_1^2 - B_4^2\right) = 0. \quad (2.14) \]

By solving the fourth order equation with respect to $y$, we derive two complex solutions and two real solutions (Ferrari’s solution, see also in [8]) which depend on $B_1, B_4$ and $a$. One of the two real solutions with respect to $y$ is (2.2). From (2.2), we obtain that the weighted Fermat-Torricelli point $A_0$ is located at the interior of $A_1 A_2 A_3 A_4$ (see fig. 2).
The Complementary Fermat-Torricelli problem was stated by Courant and Robbins (see in [4, pp. 358]) for a triangle which is derived by the weighted Fermat-Torricelli problem by placing one negative weight to one of the vertices of the triangle and asks for the complementary weighted Fermat-Torricelli point which minimizes the corresponding objective function.

We need to state the Complementary weighted Fermat-Torricelli problem for a tetragon, in order to explain the second real solution which have been obtained by (2.14) with respect to $y$.

**Problem 3.** Given a tetragon $A_1A_2A_3A_4$ and a weight $B_i$ (a positive or negative real number) which corresponds to the vertex $A_i$, for $i = 1, 2, 3, 4$, find a fifth point $A_0$ (weighted Fermat-Torricelli point) which minimizes the objective function

$$f = B_1a_{01} + B_2a_{02} + B_3a_{03} + B_4a_{04}$$

(2.15)

for $\|B_1\| > \|B_4\|$, $B_1 = B_2$ and $B_3 = B_4$.

**Proposition 1.** The location of the complementary weighted Fermat-Torricelli point $A'_0$ (solution of Problem 3) of $A_1A_2A_3A_4$ for $B_1 = B_2 < 0$, $B_3 = B_4 < 0$ and $\|B_1\| > \|B_4\|$ coincides with the location of the corresponding weighted Fermat-Torricelli point of $A_1A_2A_3A_4$ for $B_1 = B_2 > 0$, $B_3 = B_4 > 0$ and $\|B_1\| > \|B_4\|$.

**Proof of Proposition 1.** By applying theorem 2 for $B_1 = B_2 < 0$, $B_3 = B_4 < 0$ we derive the weighted floating equilibrium condition (see fig. 3):

$$\vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 = \vec{0}$$

(2.16)

or

$$(-\vec{B}_1) + (-\vec{B}_2) + (-\vec{B}_3) + (-\vec{B}_4) = \vec{0}.$$  

(2.17)

From (2.16) and (2.17), we derive that the complementary weighted Fermat-Torricelli point $A'_0$ coincides with the weighted Fermat-Torricelli point $A_0$. The difference between the figures 2 and 3 is that the vectors $\vec{B}_i$ change direction from $A_i$ to $A_0$, for $i = 1, 2, 3, 4$.

□

**Proposition 2.** The location of the complementary weighted Fermat-Torricelli point $A'_0$ (solution of Problem 3) of $A_1A_2A_3A_4$ for $B_1 = B_2 < 0$, $B_3 = B_4 > 0$ or $B_1 = B_2 > 0$, $B_3 = B_4 < 0$ and $\|B_1\| > \|B_4\|$ is given by:

$$y = \frac{\sqrt{d}}{2} + \frac{1}{2}$$

$$-\left(\frac{2}{3}\frac{2^3}{(\sqrt{s}+z)^{1/3}} + 2^{2/3}(\sqrt{s}+z)^{1/3} + 32a \left(2 + a \left(-2 - \frac{3a}{\sqrt{a}}\right)\right)B_1^2 + 32a \left(-2 + 2a - \frac{3a}{\sqrt{a}}\right)B_4^2\right)$$

$$96 (B_1^2 - B_4^2).$$

(2.18)
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Figure 3. The complementary weighted Fermat-Torricelli point $A_0'$ for a tetragon $B_1 = B_2 < 0$ and $B_3 = B_4 < 0$ for $\|B_1\| > \|B_4\|

\[ z = -1024 (-aB_1^2 + a^2B_1^2 + aB_4^2 - a^2B_4^2)^3 + 27648 (B_1^2 - B_4^2) (a^2B_1^2 + a^2B_4^2)^2 + 9216 (B_1^2 - B_4^2) (-aB_1^2 + a^2B_1^2 + aB_4^2 - a^2B_4^2) (2a^3B_1^2 + a^4B_1^2 - 2a^3B_4^2 - a^4B_4^2) \] (2.19)

\[ w = 64 (-aB_1^2 + a^2B_1^2 + aB_4^2 - a^2B_4^2)^2 + 192 (B_1^2 - B_4^2) (2a^3B_1^2 + a^4B_1^2 - 2a^3B_4^2 - a^4B_4^2), \] (2.20)

\[ s = -4w^3 + (-1024 (-aB_1^2 + a^2B_1^2 + aB_4^2 - a^2B_4^2)^3 + 27648 (B_1^2 - B_4^2) (a^2B_1^2 + a^2B_4^2)^2 + 9216 (B_1^2 - B_4^2) (-aB_1^2 + a^2B_1^2 + aB_4^2 - a^2B_4^2) (2a^3B_1^2 + a^4B_1^2 - 2a^3B_4^2 - a^4B_4^2))^2 \] (2.21)

and

\[ d = \frac{1}{2} (-a + a^2) + \frac{w}{24 2^{2/3} (\sqrt{s} + z)^{1/3} (B_1^2 - B_4^2)^2} + \frac{(\sqrt{s} + z)^{1/3}}{48 21/3 (B_1^2 - B_4^2)} - \frac{-aB_1^2 + a^2B_1^2 + aB_4^2 - a^2B_4^2}{6 (B_1^2 - B_4^2)}. \] (2.22)
Proof of Proposition 2: Taking into account (2.12) for $B_1 = B_2 < 0$, $B_3 = B_4 > 0$ or $B_1 = B_2 > 0$, $B_3 = B_4 < 0$ and $\|B_1\| > \|B_4\|$ and differentiating (2.12) with respect to $y \equiv OA_0'$, and by squaring both parts of the derived equation, we obtain (2.14) which is a fourth order equation with respect to $y$. The second real solution of $y$ gives (2.18). From (2.18) and the vector equilibrium condition $\vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 = \vec{0}$ we obtain that the complementary weighted Fermat-Torricelli point $A_0'$ for $B_1 = B_2 < 0$, $B_3 = B_4 > 0$ coincides with the complementary weighted Fermat-Torricelli point $A_0''$ for $B_1 = B_2 > 0$, $B_3 = B_4 < 0$ (Fig. 4 and 5). Furthermore, the solution (2.18) yields that the complementary $A_0'$ is located outside the tetragon $A_1A_2A_3A_4$ (Fig. 4 and 5).

Example 1. Given a tetragon $A_1A_2A_3A_4$ in $\mathbb{R}^2$, $a = 2$, $B_1 = B_2 = 1.5$, $B_3 = B_4 = 1$ from (2.2) and (2.18) we get $y = 0.36265$ and $y = 1.80699$, respectively, with five digit precision. The weighted Fermat-Torricelli point $A_0$ and the complementary weighted Fermat-Torricelli point $A_0' \equiv A_0$ for $B_1 = B_2 = -1.5$ and $B_3 = B_4 = -1$ corresponds to $y = 0.36265$. The complementary weighted Fermat-Torricelli point $A_0'$ for $B_1 = B_2 = 1.5$ and $B_3 = B_4 = -1$ or $B_1 = B_2 = -1.5$ and $B_3 = B_4 = -1$ lies outside the tetragon $A_1A_2A_3A_4$ and corresponds to $y = 1.80699$.
We denote by $A_{12}$ the intersection point of the midperpendicular of $A_1A_2$ and $A_3A_4$ with $A_1A_2$ and by $A_{14}$ the intersection point of the perpendicular from $A_0$ to the line defined by $A_1A_4$.

We shall calculate the angles $\alpha_{102}$, $\alpha_{203}$, $\alpha_{304}$ and $\alpha_{401}$.

**Proposition 3.** The angles $\alpha_{102}$, $\alpha_{203}$, $\alpha_{304}$ and $\alpha_{401}$ are given by:

\[
\alpha_{102} = 2 \arccos \frac{\pi - y(B_1, B_4, a)}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} - y\right)^2}} \quad (2.23)
\]

\[
\alpha_{304} = 2 \arccos \left(\frac{B_1}{B_4 \cos \frac{\alpha_{102}}{2}}\right) \quad (2.24)
\]

and

\[
\alpha_{401} = \alpha_{203} = \pi - \frac{\alpha_{102}}{2} - \frac{\alpha_{304}}{2} \quad (2.25)
\]

**Proof of Proposition** 3 From $\triangle A_1A_{12}A_0$ and taking into account (2.2), we get (2.23).

From the right angled triangles $\triangle A_1A_{12}A_0$, $\triangle A_1A_{14}A_0$ and $\triangle A_4A_{14}A_0$, we obtain:
\[ a_{01} = \frac{a}{2 \sin \frac{\alpha_{102}}{2}}, \quad (2.26) \]
\[ a_{04} = \frac{a}{2 \sin \frac{\alpha_{304}}{2}}, \quad (2.27) \]

and

\[ a_{01} \cos \frac{\alpha_{102}}{2} + a_{04} \cos \frac{\alpha_{304}}{2} = a, \quad (2.28) \]

By dividing both members of (2.28) by (2.26) or (2.27), we get:

\[ \cot \frac{\alpha_{102}}{2} = 2 - \cot \frac{\alpha_{304}}{2}. \quad (2.29) \]

From (2.29) the angle \( \alpha_{102} \) is expressed as a function of \( \alpha_{304} \):

\[ \alpha_{102} = \alpha_{102}(\alpha_{304}). \]

By replacing (2.26) and (2.27) in (2.7) we get:

\[ B_1 \sin \alpha_{102}^2 + B_4 \sin \alpha_{304}^2 \to \min. \quad (2.30) \]

By differentiating (2.29) with respect to \( \alpha_{304} \), we derive:

\[ \frac{d\alpha_{102}}{d\alpha_{304}} = -\frac{\sin^2 \alpha_{102}}{\sin^2 \alpha_{304}}. \quad (2.31) \]

By differentiating (2.30) with respect to \( \alpha_{304} \) and replacing in the derived equation (2.31) we obtain (2.24).

From the equality of triangles \( \triangle A_1 A_0 A_4 \) and \( A_2 A_0 A_3 \), we get \( \alpha_{401} = \alpha_{203} \) which yields (2.25).

\[ \square \]

3. The weighted Fermat-Torricelli problem for convex quadrilaterals: The case \( B_1 = B_2 \) and \( B_3 = B_4 \).

We need the following lemma, in order to find the weighted Fermat-Torricelli point for a given convex quadrilateral \( A_1' A_2' A_3' A_4' \) in \( \mathbb{R}^2 \), which has been proved in [10, Proposition 3.1, pp. 414] for convex polygons in \( \mathbb{R}^2 \).

**Lemma 1.** [10 Proposition 3.1, pp. 414] Let \( A_1 A_2 A_3 A_4 \) be a tetragon in \( \mathbb{R}^2 \) and each vertex \( A_i \) has a non-negative weight \( B_i \) for \( i = 1, 2, 3, 4 \). Assume that the floating case of the weighted Fermat-Torricelli point \( A_0 \) is valid:

\[ \| \sum_{j=1, j \neq i}^{4} B_j \bar{u}(A_i, A_j) \| > B_i. \quad (3.1) \]

If \( A_0 \) is connected with every vertex \( A_i \) for \( i = 1, 2, 3, 4 \) and a point \( A'_i \) is selected with corresponding non-negative weight \( B_i \) on the ray that is defined by the line segment \( A_0 A_i \) and the convex quadrilateral \( A'_1 A'_2 A'_3 A'_4 \) is constructed such that:

\[ \| \sum_{j=1, j \neq i}^{4} B_j \bar{u}(A'_i, A'_j) \| > B_i, \quad (3.2) \]

then the weighted Fermat-Torricelli point \( A'_0 \) of \( A'_1 A'_2 A'_3 A'_4 \) is identical with \( A_0 \).
Let $A'_1 A'_2 A'_3 A'_4$ be a convex quadrilateral with corresponding non-negative weights $B_1 = B_2$ at the vertices $A'_1, A'_2$ and $B_3 = B_4$ at the vertices $A'_3, A'_4$.

We select $B_1$ and $B_4$ which satisfy the inequalities (3.1), (3.2) and $B_1 > B_4$, which correspond to the weighted floating case of the tetragon $A_1 A_2 A_3 A_4$ and $A'_1 A'_2 A'_3 A'_4$. Furthermore, we assume that $A_0$ is located at the interior of $\triangle A'_1 A'_2 A'_3$.

We denote by $a'_{ij}$, the length of the linear segment $A'_i A'_j$, $\alpha'_{ikj}$, the angle $\angle A'_i A'_k A'_j$ for $i, j, k = 0, 1, 2, 3, 4, i \neq j \neq k$ (See fig. 6).

**Theorem 3.** The location of the weighted Fermat-Torricelli point $A_0$ of $A'_1 A'_2 A'_3 A'_4$ for $B_1 = B_2$ and $B_3 = B_4$ under the conditions (3.3) and (3.4), and $B_1 > B_4$, is given by:

$$a'_{02} = a'_{12} \frac{\sin(\alpha'_{123} - \alpha'_{013})}{\sin \alpha_{102}}$$ \hspace{1cm} (3.3)

and

$$\alpha'_{120} = \pi - \alpha_{102} - (\alpha'_{123} - \alpha'_{013})$$ \hspace{1cm} (3.4)
Proof of Theorem 3: From lemma 1 the weighted Fermat Torricelli point $A$ of the diagonals is the intersection point of the diagonals $\Delta A_1A_2A_3$, $\Delta A_3A_4$, $\Delta A_4A_1$, $\Delta A_1A_2$, $\Delta A_2A_3$, $\Delta A_3A_4$, for the weights $B_1 = B_2$ and $B_3 = B_4$, under the conditions (3.1), (3.2), and $B_1 > B_4$.

Thus, we derive that: $\alpha_{102} = \alpha_{012}^\prime$, $\alpha_{203} = \alpha_{023}$, $\alpha_{304} = \alpha_{034}^\prime$ and $\alpha_{401} = \alpha_{041}^\prime$.

By applying the same technique that was used in [10, Solution 2.2, pp. 412-414] we express $\alpha_{012}^\prime$, $\alpha_{023}$, $\alpha_{034}^\prime$ as a function of $\alpha_{01}^\prime$ and $\alpha_{013}^\prime$ taking into account the cosine law to the corresponding triangles $\Delta A_2A_1A_0$, $\Delta A_3A_4A_0$ and $\Delta A_4A_1A_0$. By differentiating the objective function (2.15) with respect to $\alpha_{01}^\prime$ and $\alpha_{013}^\prime$ and applying the sine law in $\Delta A_2A_1A_0$, $\Delta A_3A_1A_0$ and $\Delta A_4A_1A_0$ we derive (3.5) and solving with respect to $\alpha_{013}^\prime$ we derive (3.4). By applying the sine law in $\Delta A_2A_1A_0$, $\Delta A_3A_1A_0$, $\Delta A_4A_1A_0$, we get (3.3).

Finally, $\alpha_{120}^\prime = \pi - \alpha_{123} - \alpha_{130}^\prime$.

\[ \alpha_{102} = \alpha_{012}^\prime, \quad \alpha_{203} = \alpha_{023}, \quad \alpha_{304} = \alpha_{034}^\prime, \quad \alpha_{401} = \alpha_{041}^\prime. \]

\[ (3.5) \]

\section{4. The Weighted Fermat-Torricelli Problem for Convex Quadrilaterals: The Case $B_1 = B_2$ and $B_3 = B_4$.}

Let $A_1^\prime A_2^\prime A_3^\prime A_4^\prime$ be a convex quadrilateral with corresponding non-negative weights $B_1 = B_3$ at the vertices $A_1^\prime$, $A_2^\prime$ and $B_2 = B_4$ at the vertices $A_3^\prime$, $A_4^\prime$.

We select $B_1$ and $B_4$ which satisfy the inequalities (3.1), such that $A_0$ is an interior point of $A_1^\prime A_2^\prime A_3^\prime A_4^\prime$.

\textbf{Theorem 4.} The location of the weighted Fermat-Torricelli point $A_0$ of $A_1^\prime A_2^\prime A_3^\prime A_4^\prime$ for $B_1 = B_3$ and $B_2 = B_4$ under the conditions (3.1), (3.2), is the intersection point of the diagonals $A_1^\prime A_3^\prime$ and $A_2^\prime A_4^\prime$.

Proof of Theorem 4: From the weighted floating equilibrium condition (1.2) of theorem 1 we get:

\[ \bar{B}_1 + \bar{B}_2 = -(\bar{B}_3 + \bar{B}_4) \]  (4.1)

and

\[ \bar{B}_1 + \bar{B}_4 = -(\bar{B}_2 + \bar{B}_3) \]  (4.2)

Taking the inner product of the first part of (4.1) with $\bar{B}_1 + \bar{B}_2$ and the second part of (4.1) with $-(\bar{B}_3 + \bar{B}_4)$, we derive that:

\[ \alpha_{102} = \alpha_{304}. \]

Similarly, taking the inner product of the first part of (4.2) with $\bar{B}_1 + \bar{B}_4$ and the second part of (4.2) with $-(\bar{B}_2 + \bar{B}_3)$, we derive that:

\[ \alpha_{104} = \alpha_{203}. \]
Proposition 4. The location of the complementary weighted Fermat-Torricelli point \( A_0 \) of \( A'_1A'_2A'_3A'_4 \) for \( B_1 = B_3 < 0 \) and \( B_2 = B_4 < 0 \) under the conditions (3.1), (3.2) is the intersection point of the diagonals \( A'_1A'_3 \) and \( A'_2A'_4 \).

Proof of Proposition 4. Taking into account (2.15) for \( B_1 = B_3 < 0 \), \( B_2 = B_4 < 0 \) we derive the same vector equilibrium condition \( \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 = 0 \). Therefore, we obtain that the complementary weighted Fermat-Torricelli point \( A'_0 \) for \( B_1 = B_3 < 0 \), \( B_2 = B_4 < 0 \) coincides with the weighted Fermat-Torricelli point \( A_0 \) of \( A'_1A'_2A'_3A'_4 \) for \( B_1 = B_3 > 0 \), \( B_2 = B_4 > 0 \).

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References

[1] C. Bajaj, The algebraic degree of geometric optimization problems. Discrete Comput. Geom. 3 (1988), no. 2, 177-191.
[2] V. Boltyanski, H. Martini, V. Soltan, Geometric Methods and Optimization Problems, Kluwer, Dordrecht-Boston-London, 1999.
[3] E. Cockayne, Z. Melzak, Euclidean constructibility in graph-minimization problems Math. Mag. 42 (1969) 206–208.
[4] R. Courant and H. Robbins, What is Mathematics? Oxford University Press, New York, 1951.
[5] Y.S. Kupitz and H. Martini, Geometric aspects of the generalized Fermat-Torricelli problem, Bolyai Society Mathematical Studies.6, (1997), 55-127.
[6] F. Plastria, Four-point Fermat location problems revisited. New proofs and extensions of old results. IMA J. Manag. Math. 17 (2006), no. 4, 387–396.
[7] I. Roussos, On the Steiner minimizing point and the corresponding algebraic system. College Math. J. 43 (2012), no. 4, 305–308.
[8] S. L. Shmakov, A universal method of solving quartic equations, International Journal of Pure and Applied Mathematics, 71, no. 2 (2011), 251-259.
[9] A. Uteshev, Analytical solution for the generalized Fermat-Torricelli problem, Amer. Math. Monthly. 2014. 121, no. 4 (2014) 318-331.
[10] A.N. Zachos and G. Zouzoulas, An evolutionary structure of convex quadrilaterals, J. Convex Anal., 15, no. 2 (2008) 411–426.

University of Patras, Department of Mathematics, GR-26500 Rion, Greece
E-mail address: azachos@gmail.com