Stimulated by the importance of noncommutative geometry in recent developments in string theory, D-branes and integrable systems; one intends in this work to present a new insight towards adapting the famous idea of Zeeman effect to noncommutativity à la Moyal and develop an analysis leading to connect our results to the Bigatti-Suskind (BS) formulation.
INTRODUCTION

Noncommutative geometry (NCG) or simply noncommutativity of coordinates, is a very old idea \[1\]. The use of NCG in string theory was initiated for the first time by Witten \[2\] and recently, through the original work carried out by Connes, Douglas and Schwartz \[3, 4\], this idea took a new dash in mathematics and physics. Several important contributions on NCG followed these seminal works \[5, 6, 7\], we refer the readers to the textbooks \[8\] for general references on NCG and string theory.

We are focusing to contribute to this very relevant research subject by using the Moyal momentum noncommutativity \[9, 10\] and its various applications in 2\text{d} integrability and conformal symmetry. We suggest then an interpretation of the noncommutative (\(sl_2\) KdV and \(sl_3\) Boussinesq)-Burger’s mappings \[11, 12\] by implementing the famous idea of Zeeman effect.

This interpretation starts to emit signs of rigor and reliability by means of the BS construction \[13\] which we are engaging in our approach. The original idea to incorporate the Zeeman effect in the formalism of Moyal noncommutativity in connection with certain aspects of 2\text{d} integrable models and string-brane theories requires, undoubtedly, a particular interest. More research’s works in this direction will be focused in future occasions.

BASIC DEFINITIONS

Let’s first start by specifying the nature of objects used in this work. The functions often involved in the two dimensional phase-space are arbitrary functions which we generally indicate by \(f(x, p)\) with coordinates \(x\) and \(p\). With respect to this phase space, we have to define the following objects:

1. The constants \(f_0\) defined such that
   \[
   \partial_x f_0 = 0 = \partial_p f_0. 
   \]

2. The functions \(u_i(x, t)\) depending on an infinite set of variables \(t_1 = x, t_2, t_3, ..., \) with
   \[
   \partial_p u_i(x, t) = 0. 
   \]

The index \(i\), stands for the conformal weight of the field \(u_i(x, t)\). These functions can be considered in the complex language framework as being the analytic (conformal) fields of conformal spin \(i = 1, 2, ...\).

3. Other objects that we will use are given by
   \[
   u_i(x, t) \star p^j 
   \]

These are objects of conformal weight \((i + j)\) living on the non-commutative space parametrized by \(\theta\). Throughout this work, we will use the following convention notations: \([u_i] = i, [\theta] = 0, \text{and} [p] = [\partial_x] = -[x] = 1\), where the symbol \([\ ]\) stands for the conformal dimension of the enclosed object.

4. The star product law defining the multiplication of objects in the non-commutative space is shown to satisfy the following expression
   \[
   f(x, p) \star g(x, p) = \sum_{s=0}^{\infty} \sum_{i=0}^{s} \frac{\theta^s}{s!} (-1)^i c_s^i (\partial_x^i \partial_p^{s-i} f)(\partial_x^{s-i} \partial_p^i g) 
   \]
   with \(c_s^i = \frac{s!}{s!(s-i)!}\).

5. The Moyal bracket is defined as
   \[
   \{f(x, p), g(x, p)\}_\theta = \frac{f \star g - g \star f}{2\theta}. 
   \]

6. In order to distinguish the classical objects from the \(\theta\)-deformed ones, we consider the following convention notations:
   a) \(\Sigma_{m}^{(r,s)}\) is the space of momentum Lax differential operators of conformal spin \(m\) and degrees \((r, s)\) with \(r \leq s\). Typical operators of this space are given by
   \[
   \sum_{i=r}^{s} u_{m-i} \star p^i. 
   \]
b) $\Sigma^{(0,0)}_m$: Is the space of coefficient functions $u_m$ of conformal spin $m$; $m \in \mathbb{Z}$, which may depend on the parameter $\theta$. It coincides in the standard limit, $\theta = 0$, with the ring of analytic fields involved in the construction of conformal symmetry and $W$-extensions.

c) $\Sigma^{(k,k)}_m$: Is the space of momentum operators of type

$$u_{m-k} \ast p^k.$$  

7. The $\theta$-Leibnitz rules

$$p^n \ast f(x,p) = \sum_{s=0}^n \theta^s c_n^s f^{(s)}(x,p)p^{n-s},$$  

and

$$p^{-n} \ast f(x,p) = \sum_{s=0}^\infty (-)^s \theta^s c_{n+s-1}^s f^{(s)}(x,p)p^{-n-s},$$  

where $f^{(s)} = \partial^2 f$ is the prime derivative. Few examples are given by

$$
\begin{align*}
1 \ast f(x,p) &= f, \\
p \ast f(x,p) &= fp + \theta f', \\
p^2 \ast f(x,p) &= fp^2 + 2\theta fp + \theta^2 f'', \\
p^3 \ast f(x,p) &= fp^3 + 3\theta fp^2 + 3\theta^2 fp' + \theta^3 f''',
\end{align*}
$$

and

$$
\begin{align*}
p^{-1} \ast f(x,p) &= fp^{-1} - \theta fp^{-2} + \theta^2 fp^{-3} - \theta^3 fp^{-4} + ..., \\
p^{-2} \ast f(x,p) &= fp^{-2} - 2\theta fp^{-3} + 3\theta^2 fp^{-4} - 4\theta^3 fp^{-5} + ..., \\
p^{-3} \ast f(x,p) &= fp^{-3} - 3\theta fp^{-4} + 6\theta^2 fp^{-5} - 10\theta^3 fp^{-6} + ...
\end{align*}
$$

8. The Ring of analytic functions:

A convenient description consists in using the complex language notation in which we define the two dimensional Euclidean space parametrized by $z = t + ix$ and $\bar{z} = t - ix$. In this notation, the currents of conformal weights $k$ are simply written as $u_k(x,t) \equiv u_k(z)$.

It’s then the time to introduce the space of analytic functions of arbitrary conformal spin. This is the space of completely reducible infinite dimensional $so(2)$ Lorentz representations that can be written as

$$\Sigma^{(0,0)} = \bigoplus_{k \in \mathbb{Z}} \Sigma^{(0,0)}_k,$$

where the $\Sigma^{(0,0)}_k$‘s are one dimensional $so(2)$ spin $k$ irreducible modules. The upper indices $(0,0)$ carried by the space $\Sigma^{(0,0)}_k$ are special values of the general indices $(p,q)$ describing the lowest and highest degrees of Lax operators type $\sum_{n=p}^q u_{m-k} \ast p^l$. The generators belonging to the space $\Sigma^{(0,0)}_k$ are given by the spin $k$ analytic fields. They may be viewed as analytic maps $u_k$ which associate to each point $z$ on the unit circle the fields $u_k(z)$. For $k \geq 2$, these fields can be thought of as the higher spin currents involved in the construction of $w_o$-algebras. The particular example is given by the spin-2 current $u_2(z)$ intimately associated to the Virasoro algebra $T(z)$.

9. The classical limit

Since we are interested in the $\theta$-deformation case, we have to add that the spaces $\Sigma^{(0,0)}_k$ are $\theta$-dependent; the corresponding $w_0$-algebra is shown to exhibit new properties related to the $\theta$-parameter and reduces to the standard $w$-algebra once some special limits are performed. As an example, consider for instance the $w_3$-algebra generalizing the Zamolodchikov algebra. The conserved currents of this extended algebra are shown to take the following form

$$
\begin{align*}
w_2 &= u_2, \\
w_3 &= u_3 - \theta u'_2,
\end{align*}
$$

which coincides with the classical case once the limit $\theta = \frac{1}{2}$ is performed. It is the convenient limit since we must consider in field theory and integrable systems to assure compatibility with the extended conformal symmetry (Zamolodchikov algebra) since the standard limit $\theta = 0$ doesn’t respect this objective.
ZEEMAN EFFECT AND THE BURGER-\( sl_n \) KDV MAPPING

We present in this section an original approach to interpret some results previously established and that concern the mapping between the NC Burger’s integrable system and the NC deformation of the \( sl_n \)-KdV integrable hierarchies in the Moyal momentum space \( \mathbb{1} \mathbb{1} \mathbb{1} \mathbb{1} \mathbb{2} \). This approach consists in adapting, artfully, the famous idea of Zeeman effect to the Moyal algebra of NC Lax operators to provide an alternative issue and interpretation of some properties that are encountered in the study of NC integrable hierarchies.

Let’s underline in this context that the presence of the deformation parameter \( \theta \) in the Moyal algebra is important in the sense that it can leads to identify it with the inverse of the magnetic field \( B \), as it’s well known, such that

\[
\theta \sim B^{-1}
\]  

Before considering such an application, it’s useful to recall some essential basic notions of the Zeeman effect as well as the importance of the magnetic field in this context.

Zeeman Effect: Basic ideas

Definitions:

1. Knowing that an atom can be characterized by a unique set of discrete energy states, when excited, the atom makes transitions between these quantized energy states and emits light. The emitted light is shown to form a discrete spectrum, reflecting the quantized nature of the energy levels. In the presence of a magnetic field, these energy levels can shift, this is the Zeeman effect.

2. Analogously to the Stark effect characterized by the splitting of a spectral line into several components in the presence of an electric field, the Zeeman effect is defined as been the splitting of a spectral line into several components in the presence of a magnetic field.

Origin of the Zeeman Effect:

The origin of the Zeeman effect can be simply presented as follows: Let’s consider an atomic energy state such that an electron orbits around the nucleus of the atom. This electron has a magnetic dipole moment associated with its angular momentum. In the presence of a magnetic field, the electron acquires an additional energy and consequently the original energy level is shifted. The energy shifted may be positive, zero, or even negative, depending on the angle between the electron magnetic dipole moment and the field.

Zeeman Effect And The Burger - \( Sl_n \) KdV Mapping

We have shown in previous works \( \mathbb{1} \mathbb{1} \mathbb{2} \) how it’s possible to establish a correspondence between the \( sl_n \)-KdV NC integrable hierarchies and the NC Burger’s system. This issue of mapping exhibits a particular interest because it allows us to install the first steps toward a possible unification of NC integrable models in the Moyal momentum framework.

Presently, we are interested to study another aspect in relation with the Zeeman effect. The crucial point resides in the NC deformation that provides the possibility to join the parameter of non commutativity \( \theta \) with the inverse of the magnetic field \( B \). On the basis of this relation, and also on the use of the NC version of the Miura transformation which rests on the idea of mapping between Burger and \( sl_n \)-KdV systems, we will see explicitly how a strong analogy emerge between the Zeeman effect and what the mapping in question exhibits as consequences.

We guess that the incorporation of the Zeeman effect in this context is not a coincidence, we think that some important physical properties are behind. The starting steps in planting this idea comes from several observations.
FIG. 1: The $sl_2$-KdV-Burger mapping

of the behavior of different expressions and also from the primordial role of the NC Burger’s system and of the $\theta$ parameter whose weight increases proportionally with the order of the $sl_n$-KdV hierarchy.

We are going to illustrate these ideas for two particular examples namely the NC KdV and Boussinesq’s systems.

The $sl_2$ KdV- Burger’s Case

Let us take again the NC KdV-Burger’s mapping discussed before, we have the following equation

$$L_{KdV}(u_2) = p^2 + u_2 = (p + u_1) \star (p - u_1)$$

or equivalently $u_2 = -u_1^2 - 2\theta u_1'$.

Whereas the NC KdV current $u_2$ depends explicitly on the parameter $\theta$, the Burger’s current does not have this property since the associated Lax operator $L_{Burger}(u_1)$ does not admit a Lax operator, whose conformal weight is integer, as a root. This is also due to the fact that the quantities of non integer conformal weights are not authorized in this framework.

Thus from now on, the NC Burger’s current is regarded as being a fundamental current in term of which all the other currents of the $sl_n$-KdV hierarchy are expressed.

Proposition 1:

Given the NC Miura Transformation, binding the NC Burger and KdV systems as follows:

$$L_{KdV}(u_2) = L_{Burg}(u_1) \star L_{Burg}(-u_1),$$

We can represent this mapping graphically as given by fig. 1:

Convention notations:

For this purpose, we adopt the following diagrammatic representation:

1 We represent symbolically the NC KdV Lax operator by a line indexed by the NC KdV current $u_2$. The splitting of $L_{KdV}$, with respect to the Moyal star product, into a pair of NC Burger’s operators is schematized by two parallel lines which leave, starting from a vertex, the initial KdV line. The two parallel lines are considered

---

1 Because of the fact that the NC KdV operator admits the NC Burger Lax operator as a square root, see eq (2)
to be associated to the pair of NC Burger’s operators \((L_{Burg}, L_{Burg})\) who appear on the right hand side of eq.(16).

2. The position of the two emitted lines relative to the Burger’s operators depends on the sign of the spin 1 Burger’s currents. The upper line is associated with \(L_{Burg}(+u_1)\) while the lower one, associated to negative sign of the NC Burger’s current \(-u_1\), is for \(L_{Burg}(-u_1)\).

3. The initial NC KdV line is thus the result of the star product of the parallel Burger’s lines.

4. We then specify two zones; the left one, characterized by the NC KdV initial line where the two Burger’s levels are in coincidence (degenerated twice). The other zone on the right is given by two separated lines describing a broken degeneracy. The passage from the single KdV level to both Burger’s lines, via the Miura transformation or the \(sl_2\) KdV-Burger’s mapping, is identical to a lifting of the degeneracy which means also the passage from a configuration with star product to a configuration without star product:

\[
L_{KdV}(u_2) \leftrightarrow (L_{Burg}(u_1), L_{Burg}(-u_1))
\]

or equivalently

\[
(p + u_1) \star (p - u_1) \leftrightarrow ((p + u_1), (p - u_1))
\]

5. In other words, the transition from a single KdV level degenerated twice to a pair of two Burger’s levels without degeneracy is equivalent to the passage from a phase with a \(\theta \equiv B^{-1}\) predominance to a phase with magnetic \(B \equiv \theta^{-1}\) predominance \(^2\). At this point, we have to underline the striking analogy with the Zeeman effect since it is the presence of a magnetic field which breaks the degeneracy of the initial KdV level.

We showed through this first example that the KdV-Burger’s mapping is accompanied by a rupture of degeneracy that exhibits the initial NC KdV level. This lifting of degeneracy is due to the emergence of the magnetic field \(B\) corresponding to a weakness of the NC deformation parameter \(\theta\) during the transition from the single KdV level to the pair of Burger’s levels. This behavior is identical to the Zeeman effect.

\[\text{The } sl_3 \text{ Boussinesq-Burger’s Case}\]

In a similar way, the \(sl_3\) Boussinesq-Burger’s mapping deals with the following equation

\[
p^3 + u_2 \star p + u_3 = (p + u_1) \star (p + v_1) \star (p - u_1 - v_1)
\]

Proposition 2:

Given the NC Miura Transformation, binding the NC Burger and \(sl_3\) Boussinesq’s system as follows

\[
L_{Bouss}(u_2, u_3) = L_{Burg}(u_1) \star L_{Burg}(v_1) \star L_{Burg}(-u_1 - v_1),
\]

We can represent this mapping graphically as shown by fig.2:

Convention notations:

In the same way as for the NC KdV- Burger’s mapping, we will adopt the following diagrammatic representations for the \(sl_3\) Boussinesq-Burger’s splitting:

---

\(^2\) At the level of the single NC KdV line, where the degeneracy is of order 2, the \(\theta\) parameter acquires a power conversely to the magnetic field \(\theta^{-1}\) which becomes relevant with emission of the pair of NC Burger’s levels
FIG. 2: The $sl_3$-Boussinesq-Burger mapping

1. We represent symbolically the NC $sl_3$ Boussinesq Lax operator by a line indexed by the pair of currents $(u_2, u_3)$. The splitting of $L_{Bouss}$, with respect to the star product, into a triplet ($L_{Burg}(u_1), L_{Burg}(v_1), L_{Burg}(-u_1 - v_1)$) of NC Burger’s operators is schematized by three parallel lines leaving, from a vertex, the initial $sl_3$ Boussinesq line.

2. The position of the three emitted Burger’s lines depends on the sign of spin 1 Burger’s currents. The upper lines are chosen arbitrarily to be associated to $L_{Burg}(+u_1)$ and $L_{Burg}(+v_1)$ while the lower line is being associated to the negative sign of the NC Burger’s current $-u_1 - v_1$ namely to $L_{Burg}(-u_1 - v_1)$.

3. As it’s shown in eq(19), the initial NC $sl_3$ Boussinesq line is the result of the star product of the three parallel Burger’s lines.

4. The initial single line corresponding to the NC $sl_3$ Boussinesq’s Lax operator is characterized by a degeneracy of order 3. This is due to the fact that the three NC Burger’s levels on the right of the vertex fig. 2 coincide at the level of the NC $sl_3$ Boussinesq line. When the degeneracy of order three is broken, the three NC Burger’s levels are emitted. This emission procedure is given by

$$L_{Bouss}(u_2, u_3) \mapsto (L_{Burg}(u_1), L_{Burg}(v_1), L_{Burg}(-u_1 - v_1))$$

(21)

or equivalently

$$(p + u_1) \star (p + v_1) \star (p - u_1 - v_1) \mapsto ((p + u_1), (p + v_1), (p - u_1 - v_1))$$

(22)

5. The contact with Zeeman effect is done as follows:

The transition from the initial single NC $sl_3$ Boussinesq’s level, of degeneracy three, to a triplet of NC Burger’s levels without degeneracy is equivalent to the passage from a phase with a $\theta \equiv B^{-1}$ predominance to a phase with magnetic $B \equiv \theta^{-1}$ predominance. At this point, we have to underline the striking analogy with the Zeeman effect since it is the presence of a magnetic field who breaks the degeneracy of the initial $sl_3$ Boussinesq’s level.

We showed once again through this second example that the $sl_3$ Boussinesq-Burger’s mapping is accompanied by a rupture of degeneracy, of order three, that exhibits the initial NC Boussinesq’s level. This lifting of degeneracy is due to the emergence of the magnetic field $B \sim \theta^{-1}$ accompanied by an annihilation of the NC deformation parameter $\theta$ during the emission. This is a clear manifestation of the Zeeman effect.
Before closing this section, we present the Zeeman splitting for the general case of $sl_n$ KdV-Burger’s mapping as given by fig.3

**THE BS CONSTRUCTION: A REVIEW**

It’s point out that gauge theories on noncommutative spaces [4, 5] are relevant to the quantization of D-branes in background $B_{\mu\nu}$ fields [8]. Such theories are shown to be similar to ordinary gauge theory except that the usual product of fields is replaced by a “star product” that is well defined through the deformation parameter $\theta_{\mu\nu}$ which is an antisymmetric constant tensor. The authors of [8] show on a successful way, by means of simple examples and of a convenient mathematical formulation, that the noncommutativity is once again appearing as a rephrasing of well known physics, namely the physics of the magnetic field.

The principal focus of this section is to recall a part of the BS construction before considering an adaptation of their results to our study. For this propose, we will begin by recalling this construction based first on a simple quantum mechanical system, a fundamental steps, before considering the string theory in the presence of a D$3$-brane and a constant large $B_{\mu\nu}$ field.

**The classical model**

It deals with a system of two unit charges of opposite sign connected by a spring with elastic constant $K$. The system is embedded in a magnetic field $B$ in the regime where the Coulomb and the radiation terms are negligible. The charges are associated to particles of masse $m$ and charge $|q| = e$ and they are localized at the coordinates $\vec{x}_1$.
and \( \vec{x}_2 \) or in component form \( x_1^i \) and \( x_2^i \). The Lagrangian of this system is given by

\[
\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3
\]  

(23)

The first term

\[
\mathcal{L}_1 = \frac{m}{2} \left( (\dot{x}_1)^2 + (\dot{x}_2)^2 \right)
\]  

(24)

is the kinetic energy of the charges.

The second term

\[
\mathcal{L}_2 = \frac{B}{2} \epsilon_{ij} \left( \dot{x}_1^i x_1^j - \dot{x}_2^i x_2^j \right)
\]  

(25)

describes the interaction of the charges with the magnetic field \( B \) and the last term

\[
\mathcal{L}_3 = -\frac{K}{2}(x_1 - x_2)^2
\]  

(26)

is known to be the harmonic potential between the charges.

**Particular limit:**

Concerning the kinetic term of \( \mathcal{L} \), it’s assumed [13] that the magnetic field \( B \) is so large that the available energy is insufficient to excite higher Landau levels [14]. So, the principal focus will be on the lowest Landau level.

By virtue of this particular limit, the previous Lagrangian reduces to

\[
\mathcal{L} = \frac{B}{2} \epsilon_{ij} \left( \dot{x}_1^i x_1^j - \dot{x}_2^i x_2^j \right) - \frac{K}{2}(x_1 - x_2)^2
\]  

(27)

From the classical mechanic’s point of view, the canonical momenta are given by

\[
p_1^i = \frac{\partial \mathcal{L}}{\partial \dot{x}_1^i} = B \epsilon_{ij} x_1^j;
\]

\[
p_2^i = -B \epsilon_{ij} x_2^j;
\]  

(28)

and the corresponding Hamiltonian is deduced from the reduced Lagrangian as follows

\[
\mathcal{H} = 2K(\Delta)^2 = 2K \left( \frac{P}{2B} \right)^2 = \frac{K}{2B^2} P^2
\]  

(29)

once the center of mass and relative coordinates \( X, \Delta \) are defined as:

\[
\dot{X} = (\vec{x}_1 + \vec{x}_2)/2
\]

\[
\dot{\Delta} = (\vec{x}_1 - \vec{x}_2)/2
\]  

(30)

One learns from eq(29) that \( \mathcal{H} \) is the Hamiltonian of a nonrelativistic particle with mass

\[
M = \frac{B^2}{K}
\]  

(31)

and center of mass momentum, conjugate to \( X \), given by

\[
P_i = 2B \epsilon_{ij} \Delta^j = \frac{\partial \mathcal{L}}{\partial X^i}
\]  

(32)

with

\[
\mathcal{L} = 2B \epsilon_{ij} \dot{X}^i \Delta^j - 2K(\Delta)^2
\]  

(33)

As it’s claimed by the authors of [13], this Lagrangian exhibits a particular interest since an identical expression can be extracted from string theory’s point of view. The induced message is that a quantum of NCYM can be described
as an unexcited string relating the particles of opposite charges inside a strong magnetic field.

On the other hand, the interesting consequence of the non locality as expressed by the behavior of open string endpoints on D-brane world volume

\[ [x^i, x^j] = i\theta^{ij} \tag{34} \]

is the existence of dipole excitation, whose content is expressed by \( p_i = 2B\epsilon_{ij}\Delta^j \) eq(32). One learns from this relation that the spatial extension \( |\Delta| \) of the system is proportional to its momentum \( |p| \) in the orthogonal direction. This proportionality shows that the size of the system is growing linearly with the momentum which is also equivalent to boost the system along a direction while it will spread in the orthogonal one leading simply to conclude that the particle is not any more point-like.

**String theory’s framework**

The principal task of this section is to show that string theory analysis reproduces the same behavior of the Lagrangian corresponding to the classical model. In fact, let’s consider a bosonic string theory in the presence of a D3-brane oriented along the coordinates \( x^\mu, \mu = 0, 1, 2, 3 \). We have also a background antisymmetric tensor field \( B_{\mu\nu} \) in the 1, 2 direction in light cone frame defined as follows

\[ x^\pm = x^0 \pm x^3 \tag{35} \]

with the usual light cone choice of world sheet time

\[ \tau = x^+ \tag{36} \]

Focusing on the large limit for the magnetic field, namely \( B \longrightarrow \infty \), the string action can be reduced, upon some rescalings and straightforward manipulations, to the following form \[13\]

\[ \mathcal{L} = \left[ -\frac{2\Delta^2}{L} + \dot{y}\epsilon\Delta \right] \tag{37} \]

This Lagrangian shows a striking resemblance with the one of classical mechanics, see eq(33).

**THE BS CONSTRUCTION ADAPTED TO sl_n KDV-BURGER’S MAPPING**

We guess that a strong link should exists between the \( sl_n \) KdV- Burger’s mapping with the interpretation provided through the Zeeman effect, on one hand, and the BS construction from the other hand.

Indeed, the consideration of the authors of \[13\] of the simple classical model dealing with a pair of particles of opposite charges embedded in a strong magnetic field and moving on the infinite noncommutative 2d plane is of great interest because of the remarkable similarity with the behavior of the bosonic string theory in the presence of a D3-brane.

**The sl_2 KdV-Burger’s mapping and the BS mechanical model**

We focus in what follows to use and adapt some important results presented in the BS construction that we reviewed previously. Our principal focus is to be able to interpret the Zeeman approach of the \( sl_n \) KdV Burger’s mapping.

For this reason, we consider the 2d phase-space \( \mathcal{F}_{\theta,2} \) of arbitrary functions \( f(x, p) \) and \( g(x, p) \) subject to the following Moyal \( \theta \)-NC rule

\[ f \star g = \sum_{i,s=0}^{sl} \theta^{i_s}(-)^i c_s^i (\partial_x \partial_p^{s-i} f)(\partial_x^{s-i} \partial_p^i g), \tag{38} \]
such that $\{f, g\}_\theta = \frac{1}{4}(x f y - y f x)$ with $x$ and $p$ describing the space and momentum coordinates respectively.

Next, we introduce a new space $C_{B,2}$ generated by the coordinates $x'_1$ and $x'_2$ or in component form $x'_1$ and $x'_2$. These coordinates are assumed to describe a pair of unit charges $(q_1, q_2)$ of opposite sign $(q_2 = -q_1)$ in a magnetic field $B$ in the regime where the Coulomb and the radiation terms are negligible \cite{13}.

We will give a series of propositions and ansatz with a principal aim to install the link in question. The first step is given by the following result:

**Proposition 3: NC $sl_2$-KdV’s Phase ($B \to 0$):**

Let’s consider the phase I dominated by the noncommutative $\theta$ deformation and characterized by the space $\mathcal{F}_{\theta,2}$ corresponding to the NC $sl_2$ KdV hierarchy with the momentum NC Lax operator $p^2 + u_2(x, t)$. From the point of view of Zeeman effect this hierarchy deals with a phase of degenerated single level, this is the NC $sl_2$-KdV Phase.

**Proposition 4: NC $sl_2$-Burger’s Phase ($B \to \infty$):**

In parallel, we consider a phase II with a magnetic field’s $B$ predominance and associated to the BS charge space $C_{B,2}$ that we call, according to Zeeman effect, the $sl_2$-Burger’s phase. This phase is characterized by a splitting of the degenerated NC $sl_2$ KdV single level $p^2 + u_2(x, t) = (p + u_1) \ast (p - u_1)$ to a set of $2 = 1 + 1$ Burger’s levels under the effect of the strong magnetic field $B$.

**Ansatz:**

By virtue of Proposition 4, the two emitted Burger’s levels, belonging to the space $C_{B,2}$ in a strong magnetic field’s regime, are associated to the opposite spin-one currents $u_{1_1} = u_1$ and $u_{1_2} = -u_1$ which are, in turn, in one-to-one correspondence with the opposite BS charges $q_{1_1} = q_1$ and $q_{1_2} = -q_1$ respectively. We have

$$u_{1_1} \equiv u_1 \Leftrightarrow q_{1_1} = q_1$$
$$u_{1_2} \equiv -u_1 \Leftrightarrow q_{1_2} = -q_1$$

This ansatz gives a possibility to look at the non degenerate Burger’s levels, à la Zeeman, and the associated conformal currents of opposite signs as being the two opposite charges suggested by Bigatti and Susskind in their model \cite{13}.

Furthermore, the force of this ansatz is traced to the fact that these two charges as well as the two emitted Burger’s levels exist in a phase $C_{B,2}$ marked by a strong intensity of the magnetic field.

**Ansatz:** The $(\mathcal{F}_{\theta,2})$ - $(C_{B,2})$ duality

The space $\mathcal{F}_{\theta,2}$ associated to the NC $sl_2$ KdV single level in a phase where $\theta$ predominate is dual to the space $C_{B,2}$ of opposite BS charges in the strong magnetic field’s regime. The duality relation can be expressed formally as follows:

$$\mathcal{F}_{\theta,2} \leftrightarrow C_{B,2}$$
$$\theta \leftrightarrow \theta^{-1} \sim B$$

The general $sl_n$ KdV-Burger’s mapping and the BS string-branes models

The previous results can be easily extended to higher order of the NC KdV hierarchy. Before going into the general case, let’s consider the first non trivial extension namely the third order class of the KdV hierarchy, we have

---

3 As it will be clear later, the lower indices in $\mathcal{F}_{\theta,2}$ and $C_{B,2}$ indicate the $\theta$ and $B$ predominance as well as the order of the hierarchy and the number of emitted Burger’s levels respectively. Later on we will consider $C_{B,2}$ as being the BS space of opposite charges.
Proposition 5: NC $sl_3$ Boussinesq’s Phase ($B \to 0$):
Let’s consider a phase I dominated by the noncommutative $\theta$ deformation and characterized by the space $\mathcal{F}_{\theta,3}$ corresponding to the NC $sl_3$ Boussinesq hierarchy with the momentum NC Lax operator $p^3 + u_2 \star p + u_3$. From the point of view of Zeeman effect this hierarchy deals with a phase of degenerated single level, this is the NC $sl_3$-Boussinesq Phase. The degeneracy of the $sl_3$ Boussinesq single level in this phase is of order 3.

Proposition 6: NC $sl_3$-Burger’s Phase ($B \to \infty$):
In parallel, we consider a phase II with a magnetic field’s $B$ predominance and associated to the “charge” space $C_{B,3}$. This phase is characterized by a splitting of the degenerated NC $sl_3$ KdV single level $p^3 + u_2 \star p + u_3 = (p + u_1) \star (p + v_1) \star (p - u_1 - v_1)$ to a set of $3 = 2 + 1$ Burger’s levels under the effect of the strong magnetic field $B$.

Ansatz:
By virtue of Proposition 6, the three emitted Burger’s levels, belonging to the space $C_{B,3}$ in a strong magnetic field’s regime, are associated to the following set of spin one currents:

$$
\begin{align*}
  u_{1_1} &= u_1 \\
  u_{1_2} &= v_1 \\
  u_{1_3} &= -u_1 - v_1
\end{align*}
$$

(41)

In analogy with previous analysis, these three Burger’s currents $u_{1_i}, i = 1,2,3$ are assumed to be associated to a set of three charges extending the BS opposite charges $(q_1,-q_1)$ in the following way:

$$
\begin{align*}
  q_1 &\equiv u_{1_1} \\
  q_2 &\equiv u_{1_2} \\
  q_3 &\equiv -u_{1_1} - v_1
\end{align*}
$$

(42)

Note that the extended BS charges $q_1, q_2$ are positives while $q_3$ is negative$^4$.

Ansatz: The $(\mathcal{F}_{\theta,3})$ - $(C_{B,3})$ duality

The space $\mathcal{F}_{\theta,3}$ associated to the NC $sl_3$ Boussinesq single level in a phase where $\theta$ predominate is dual to the space $C_{B,3}$ of extended BS charges $q_i, i = 1,2,3$ in the strong magnetic field’s regime. We have: $\mathcal{F}_{\theta,3} \leftrightarrow C_{B,3}$

Proposition 7: The $sl_n$ General case
- The $n$ emitted Burger’s levels, belonging to the space $C_{B,n}$ in a strong magnetic field’s regime, are associated to the following set of spin one currents: $u_{1_1}, u_{1_2}, ..., u_{1_{n-1}}, u_{1_n} \equiv -u_{1_1} - u_{1_2} - ... - u_{1_{n-1}}$.
- The $n = (n - 1) + 1$ Burgers’s currents $u_{1_i}, i = 1,2, ..., n$ are assumed to be associated to a set of $n$ charges extending the BS opposite charges $(q_1,-q_1)$ in the following way:

$$
\begin{align*}
  q_1 &\equiv u_{1_1} \\
  q_2 &\equiv u_{1_2} \\
  ... &\equiv ... \\
  q_{n-1} &\equiv u_{1_{n-1}} \\
  q_n &\equiv -\sum_{i=1}^{n-1} u_{1_i}
\end{align*}
$$

(43)

$^4$ One may think of the positions of these three charges as forming a triangle.
Note that the extended BS charges $q_1, \ldots, q_{n-1}$ belonging to the space $\mathcal{C}_{B,n}$ are positives while $q_n$ is negative.  

- One have the following $(\theta) - (\theta^{-1} \sim B)$ duality: $\mathcal{F}_{\theta,n} \leftrightarrow \mathcal{C}_{B,n}$.

Recapitulating diagram:

$$
\begin{align*}
\Sigma^{(0,2)}/\Sigma^{(1,1)} & \quad \overset{\text{mapping}}{\leftrightarrow} \quad (\Sigma^{(0,1)}_1, \Sigma^{(0,1)}_1) \\
\mathcal{F}_{\theta,2} & \quad \overset{\text{duality}}{\leftrightarrow} \quad \mathcal{C}_{B,2}
\end{align*}
$$

(44)

We have introduced previously some consistent conventional notations in the Moyal momentum algebra’s context to describe the NC $\mathfrak{sl}_2$ KdV Lax operator $L_2 = p^2 + u_2 \equiv (p + u_1) * (p - u_1)$ through the following coset space $\Sigma^{(0,2)}/\Sigma^{(1,1)}$. We proposed a mapping between the $\mathfrak{sl}_2$ KdV system and the NC Burger’s Lax operators.

From the Zeeman effect’s point of view, this mapping is similar to a duality between the spaces $\mathcal{F}_{\theta,2}$ corresponding to the single degenerated KdV level belonging to $\Sigma^{(0,2)}/\Sigma^{(1,1)}$ and $\mathcal{C}_{B,2}$ describing the pair of emitted Burger’s levels. We then conclude that the space $\mathcal{C}_{B,2}$ of Burger’s levels is the same as the space of BS opposite charges (BS dipole).

This analyse can be more generally extended to the order $n$ as follows

$$
\begin{align*}
\Sigma^{(0,n)}/\Sigma^{(n-1,n-1)} & \quad \overset{\text{mapping}}{\leftrightarrow} \quad (\Sigma^{(0,1)}_1, \ldots, \Sigma^{(0,1)}_1) \\
\mathcal{F}_{\theta,n} & \quad \overset{\text{duality}}{\leftrightarrow} \quad \mathcal{C}_{B,n}
\end{align*}
$$

(45)

CONCLUDING REMARKS:

This work aims principally to present a new aspect of noncommutative integrable models. We show that the spectrum defined by the Lax pair with spectral parameter for the non commutative deformation of certain $\mathfrak{sl}_n$ KdV integrable hierarchies, namely the KdV and Burger’s systems, exhibits a degeneracy splitting reminiscent of the Zeeman effect.

The originality of this result and its natural aspect come from the crucial use of the physical property $\theta \sim B^{-1}$ giving the known analogy between the non commutative parameter $\theta$ and the magnetic field $B$.

The Zeeman effect in the present context seems to be a natural incorporation for the following reasons:

- The $\theta$-Miura transformation is very significant since its equivalent to a splitting of every $\mathfrak{sl}_n$ KdV hierarchy into $n$ different Burger’s hierarchies.

The associated equation is given by: $L_{(KdV)}(u_2) = p^2 + u_2 = (p + u_1) * (p - u_1)$.

- The KdV hierarchy described by the NC Lax operator $L_{(KdV)}(u_2)$ and the current $u_2$ of conformal spin 2 such that $u_2 = -u_2^2 - 2\theta u_2$ shows an explicit dependence of the KdV current $u_2 = u_2(\theta)$ in terms of the non commutative $\theta$ parameter. Unlike the KdV current, the spin 1 Burger’s current $u_1$ does not have this property since the associated NC Lax operator $L_{(Burger)}(u_1)$ does not admit a Lax operator, whose conformal weight is integer, as a root. This is also due to the fact that the quantities of non integer conformal weights are not authorized in this framework.

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5 By virtue of the Bigatti-Susskind construction, one may think of the positions of these $n$ extended charges as forming an hyperplan, and since the opposite charges of BS are shown to be stretched by a spring (≡ open string), the charges of $\mathcal{C}_{B,3}$ in the triangle are assumed to be stretched by a membrane or a 2-string while the $n$ charges of $\mathcal{C}_{B,n}$ are stretched by a $(n-1)$-brane. Note also the important fact that all these systems of $k$ charges belonging to $\mathcal{C}_{B,k}$, for arbitrary $k = 1, 2, \ldots, n$ are neutral systems.
The NC Burger’s current \( u_1 \) is regarded as being a fundamental current in term of which all the other currents of the \( sl(n) - KdV \) hierarchy are expressed.

The Miura transformation, applied to \( sl(n) \) noncommutative hierarchies, share with the famous Zeeman effect the property of the splitting giving rise in turn to a couple of phase of opposite magnetic field’s dominance.

In fact, from the Zeeman effect point of view, the NC \( sl(n) \) KdV hierarchy is associated to a phase \( \theta \sim B^{-1} \) where noncommutativity dominate, while the Miura transformation leads to a splitting of the Burger’s levels gives rise to a new phase \( B \sim \theta^{-1} \) where the magnetic field dominate.

To make concrete our idea, we have proceed by presenting a graphical scenario similar to the Zeeman effect representation. The essential of our representation for the general case, given in Fig. 3, can be summarized as follows:

1. The original NC \( sl(n) \) Lax operator \( L_{sl(n)-KdV} \) is represented by an horizontal line indexed by a multiplet of currents \( (u_2, u_3, \ldots, u_n) \). These currents indicate that the degree of degeneracy inside this level is of order \( n \). Note that the degree of degeneracy is synonym of the highest degree in the non commutativity whose parameter is \( \theta \).

2. The splitting of \( L_{sl(n)-KdV} \), with respect to the Miura transformation, into a multiplet \((LB_{Burg_1}, LB_{Burg_2}, \ldots, LB_{Burg_n})\) of NC Burger’s operators is schematized by \( n \) parallel lines leaving, from a vertex, the initial \( sl(n) \) KdV line.

3. The position of the \( n \) emitted Burger’s lines depends on the sign of spin 1 Burger’s currents. Uppers lines are for example chosen to be associated to \( LB_{Burg}(\alpha u_1), \alpha > 0 \) while lower lines are associated to the negative sign of the NC Burger’s currents, ie \( LB_{Burg}(\alpha u_1), \alpha < 0 \)

4. When the degeneracy of order \( n \) is broken, the \( n \) NC Burger’s levels are emitted. This emission procedure is given by

\[
LB_{Bouss}(u_2, \ldots, u_n) \leftrightarrow (LB_{Burg}(u_1), \ldots, LB_{Burg}(u_n)) \quad (46)
\]

or equivalently

\[
(p + u_1) \ast \ast \ast (p + u_n) \leftrightarrow ((p + u_1), \ldots, (p + u_n)) \quad (47)
\]

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