UNIVERSAL REFLECTIVE-HIERARCHICAL STRUCTURE OF QUASIPERIODIC EIGENFUNCTIONS AND SHARP SPECTRAL TRANSITION IN PHASE

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Abstract. We prove sharp spectral transition in the arithmetics of phase between localization and singular continuous spectrum for Diophantine almost Mathieu operators. We also determine exact exponential asymptotics of eigenfunctions and of corresponding transfer matrices throughout the localization region. This uncovers a universal structure in their behavior governed by the exponential phase resonances. The structure features a new type of hierarchy, where self-similarity holds upon alternating reflections.

1. Introduction

Unlike random, one-dimensional quasiperiodic operators feature spectral transitions with changes of parameters. The transitions between absolutely continuous and singular spectrum are governed by vanishing/non-vanishing of the Lyapunov exponent [35]. In the regime of positive Lyapunov exponents (also called supercritical in the analytic case, with the name inspired by the almost Mathieu operator) there are also more delicate transitions: between localization (point spectrum with exponentially decaying eigenfunctions) and singular continuous spectrum. They are governed by the resonances: eigenvalues of box restrictions that are too close to each other in relation to the distance between the boxes, leading to small denominators in various expansions. Localization is said to be a game of resonances, a statement attributed to P. Anderson and Ya. Sinai (e.g. [22]). Indeed, all known proofs of localization, starting with Fröhlich-Spencer’s multi-scale analysis [19] are based, in one way or another, on avoiding resonances and removing resonance-producing parameters, while all known proofs of singular continuous spectrum and even some of the absolutely continuous one [2] are based on showing their abundance.

For quasiperiodic operators, one category of resonances are the ones determined entirely by the frequency. Indeed, for smooth potentials, large coefficients in the continued fraction expansion of the frequency lead to almost repetitions and thus resonances, regardless of the values of other parameters. Such resonances were first understood and exploited to show singular continuous spectrum for Liouville frequencies in [8, 9], based on [23]. The strength of frequency resonances is measured by the arithmetic

\footnote{According to [42], the fact that the Diophantine properties of the frequencies should play a role was first observed in [42].}
parameter

\[ \beta(\alpha) = \limsup_{k \to \infty} -\frac{\ln ||k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|} \]

where \( ||x||_{\mathbb{R}/\mathbb{Z}} = \inf_{\ell \in \mathbb{Z}} |x - \ell| \). Another class of resonances, appearing for all *even* potentials, was discovered in [33], where it was shown for the first time that the arithmetic properties of the phase also play a role and may lead to singular continuous spectrum even for the Diophantine frequencies. Indeed, for even potentials, phases with almost symmetries lead to resonances, regardless of the values of other parameters.\(^2\) The strength of phase resonances is measured by the arithmetic parameter

\[ \delta(\alpha, \theta) = \limsup_{k \to \infty} -\frac{\ln ||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|} \]

In both these cases, the strength of the resonances is in competition with the exponential growth controlled by the Lyapunov exponent. It was conjectured in 1994 [27] that for the almost Mathieu family- the prototypical quasiperiodic operator - the two above types of resonances are the only ones that appear (as is the case in the perturbative regime of [20]), and the competition between the Lyapunov growth and resonance strength resolves, in both cases, in a sharp way. The frequency half of the conjecture was recently solved [7, 32]. In this paper we present the solution of the phase half. Moreover, our proof of the pure point part of the conjecture uncovers a universal structure of the eigenfunctions throughout the entire pure point spectrum regime, which, in presence of exponentially strong resonances, demonstrates a new phenomenon that we call a *reflective hierarchy*, when the eigenfunctions feature self-similarity upon proper reflections. This phenomenon was not even previously described in the (vast) physics literature. This paper is, in some sense, dual to the recent work [32]. While the universal hierarchical structure governed by the frequency resonances discovered in [32] is conjectured to hold, for a.e. phase, throughout the entire class of analytic potentials, the structure discovered here requires evenness of the function defining the potential, and moreover, in general, resonances of other types may also be present. However, we conjecture that for general even analytic potentials for a.e. frequency only finitely many other exponentially strong resonances will appear, thus the structure described in this paper will hold for the corresponding class, with the \( \ln \lambda \) replaced by the Lyapunov exponent \( L(E) \) throughout.

The almost Mathieu operator (AMO) is the (discrete) quasiperiodic Schrödinger operator on \( \ell^2(\mathbb{Z}) \):

\[ (H_{\lambda, \alpha, \theta} u)(n) = u(n + 1) + u(n - 1) + 2\lambda v(\theta + n\alpha)u(n), \quad \text{with} \quad v(\theta) = 2 \cos 2\pi \theta, \]

where \( \lambda \) is the coupling, \( \alpha \) is the frequency, and \( \theta \) is the phase.

It is the central quasiperiodic model due to coming from physics and attracting continued interest there. It first appeared in Peierls [41], and arises as related, in two different ways, to a two-dimensional electron subject to a perpendicular magnetic field. It plays the central role in the Thouless et al theory of the integer quantum Hall

\(^2\)Symmetry based resonances were first observed in [20] for the almost Mathieu operator in the perturbative regime.
effect. For further background, history, and surveys of results see [14, 16, 28, 36, 40] and references therein.

Frequency $\alpha$ is called Diophantine if there exist $\kappa > 0$ and $\tau > 0$ such that for $k \neq 0$,

$$||k\alpha||_{\mathbb{R}/\mathbb{Z}} \geq \frac{\tau}{|k|^\kappa}.$$  

From here on, unless otherwise noted, we will always assume $\alpha$ is Diophantine. When we need to refer to (4) in a non-quantitative way we will sometimes call it the Diophantine condition (DC) on $\alpha$.

Operator $H$ is said to have Anderson localization if it has pure point spectrum with exponentially decaying eigenfunctions.

We have

**Theorem 1.1.**

1. $H_{\lambda,\alpha,\theta}$ has Anderson localization if $|\lambda| > e^{\delta(\alpha,\theta)}$,

2. $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum if $1 < |\lambda| < e^{\delta(\alpha,\theta)}$.

3. $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum if $|\lambda| < 1$.

**Remark**

(1) We will prove part 2 for all irrational $\alpha$, and general Lipschitz $v$ in (3), see Theorem 4.2.

(2) Part 3 is known for all $\alpha, \theta$ and is included here for completeness.

(3) Parts 1 and 2 of Theorem 1.1 verify the phase half of the conjecture in [27], as stated there. The frequency half was recently proved in [7, 32].

Singular continuous spectrum was first established for $1 < |\lambda| < e^{C\delta(\alpha,\theta)}$, for sufficiently small $C$. One can see that even with tight upper semicontinuity bounds the argument of [33] does not work for $c > 1/4$. Here we introduce new ideas to remove the factor of 4 and approach the actual threshold.

Anderson localization for Diophantine $\alpha$ and $\delta(\alpha,\theta) = 0$ was proved in [34]. The argument was theoretically extendable to $|\lambda| > e^{C\delta(\alpha,\theta)}$ for a large $C$ but not beyond. Therefore, the case of $\delta(\alpha,\theta) > 0$ was completely open before. In fact, the localization method of [34] could not deal with exponentially strong resonances. The first way to handle exponentially strong frequency resonances was developed in [6]. It was then pushed to the technical limits in [38] but that method could not approach the threshold. An important technical achievement of [32] was to develop a way to handle frequency resonances that works up to the very transition and leads to sharp bounds. In this paper we develop the first, and at the same time the sharp, way to treat exponential phase resonances.

We borrow two basic technical ingredients from prior work, that we abstract out as Theorems 3.2 and 3.3 which we prove in the appendices. Otherwise, since frequency and phase resonances are fundamentally different in nature (one is based on the repetitions and the other on reflections), the specific techniques and constructions required

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3It is rather straightforward to extend all the results to the case $\beta(\alpha) = 0$, without any changes in formulations. We present the proof under the condition (4) just for a slight simplification of some arguments.
to achieve sharp results both on the point/upper bounds and singular continuos/lower bounds sides are completely different.

Recently, it became possible to prove pure point spectrum in a non-constructive way, avoiding the localization method, using instead reducibility for the dual model [7] (see also [29]) as was first done, in the perturbative regime in [10]. Coupled with recent arguments [4, 5, 26, 46] that allow to conjugate the global transfer-matrix cocycle into the local almost reducibility regime [5] and proceed by almost reducibility, this offers a powerful technique that led to a solution of the measure theoretic version of the frequency part of the conjecture of [27] by Avila-You-Zhou in [7] and a corresponding sharp result for the supercritical regime in the extended Harper’s model [24]. However, we note that proofs by dual reducibility inherently lose the control over phases (thus can only be measure theoretic), and therefore cannot approach the transitions in phase.

Our proof of localization is based on determining the exact asymptotics of the generalized eigenfunctions in the regime $|\lambda| > e^{\delta(\alpha, \theta)}$.

For any $\ell$, let $x_0$ (we can choose any one if $x_0$ is not unique) be such that

$$|\sin \pi(2\theta + x_0 \alpha)| = \min_{|x| \leq 2\ell} |\sin \pi(2\theta + x_\alpha)|.$$

Let $\eta = 0$ if $2\theta + x_0 \alpha \in \mathbb{Z}$, otherwise let $\eta \in (0, \infty)$ be given by the following equation,

$$|\sin \pi(2\theta + x_0 \alpha)| = e^{-\eta|\ell|}.

Define $f : \mathbb{Z} \to \mathbb{R}^+$ as follows.

Case 1: $x_0 \cdot \ell \leq 0$. Set $f(\ell) = e^{-|\ell| \ln |\lambda|}$.

Case 2. $x_0 \cdot \ell > 0$. Set

$$f(\ell) = e^{-(|x_0| + |\ell - x_0|) \ln |\lambda|} e^{\eta|\ell|} + e^{-|\ell| \ln |\lambda|}.$$

We say that $\phi$ is a generalized eigenfunction of $H$ with generalized eigenvalue $E$, if

$$H\phi = E\phi, \quad \text{and} \quad |\phi(k)| \leq \tilde{C}(1 + |k|).$$

For a fixed generalized eigenvalue $E$ and corresponding generalized eigenfunction $\phi$ of $H_{\lambda, \alpha, \theta}$, let

$$U(\ell) = \begin{pmatrix} \phi(\ell) \\ \phi(\ell - 1) \end{pmatrix}.$$ We have

**Theorem 1.2.** Assume $\ln |\lambda| > \delta(\alpha, \theta)$. If $E$ is a generalized eigenvalue and $\phi$ is the corresponding generalized eigenfunction of $H_{\lambda, \alpha, \theta}$, then for any $\varepsilon > 0$, there exists $K$ such that for any $|\ell| \geq K$, $U(\ell)$ satisfies

$$f(\ell) e^{-\varepsilon|\ell|} \leq ||U(\ell)|| \leq f(\ell) e^{\varepsilon|\ell|}.$$ In particular, the eigenfunctions decay at the rate $\ln |\lambda| - \delta(\alpha, \theta)$.

**Remark**

4For the Diophantine case this is the Eliasson’s regime [17]
For $\delta = 0$ we have that for any $\varepsilon > 0$,

$$e^{-\ln|\lambda|+\varepsilon}\ell \leq f(\ell) \leq e^{-\ln|\lambda|}\ell.$$  

This implies that the eigenfunctions decay precisely at the rate of Lyapunov exponent $\ln|\lambda|$.

For $\delta > 0$, by the definition of $\delta$ and $f$, we have for any $\varepsilon > 0$,

$$f(\ell) \leq e^{-\ln(|\lambda|+\delta)}|\ell|.$$  

By the definition of $\delta$ again, there exists a subsequence $\{\ell_i\}$ such that

$$|\sin \pi(2\theta + \ell_i\alpha)| \leq e^{-\delta}\ell_i.$$  

Then

$$f(\ell_i) \geq e^{-\ln|\lambda|+\delta+\varepsilon}|\ell_i|.$$  

This implies the eigenfunctions decay precisely at the rate $\ln|\lambda| - \delta(\alpha, \theta)$.

If $x_0$ is not unique, by the DC on $\alpha$, we must have that $\eta$ is arbitrarily small. Then

$$e^{-\ln|\lambda|+\varepsilon}\ell \leq ||U(\ell)|| \leq e^{-\ln|\lambda|}\ell.$$  

The behavior described in Theorem 1.2 happens around arbitrary point. This, coupled with effective control of parameters at the local maxima, allows to uncover the self-similar nature of the eigenfunctions. Hierarchical behavior of solutions, despite significant numerical studies and even a discovery of Bethe Ansatz solutions \[1, 45\] has remained an important open challenge even at the physics level. In paper \[32\] we obtained universal hierarchical structure of the eigenfunctions for all frequencies $\alpha$ and phases with $\delta(\alpha, \theta) = 0$. In studying the eigenfunctions of $H_{\lambda, \alpha, \theta}$ for $\delta(\alpha, \theta) > 0$ we obtain a different kind of universality throughout the pure point spectrum regime, which features a self-similar hierarchical structure upon proper reflections.

Assume phase $\theta$ satisfies $0 < \delta(\alpha, \theta) < \ln \lambda$. Fix $0 < \varsigma < \delta(\alpha, \theta)$.

Let $k_0$ be a global maximum of eigenfunction $\phi^5$. Let $K_i$ be the positions of exponential resonances of the phase $\theta' = \theta + k_0\alpha$ defined by

$$||2\theta + (2k_0 + K_i)\alpha||_{\mathbb{R}/\mathbb{Z}} \leq e^{-\varsigma|K_i|},$$  

This means that $|v(\theta' + \ell\alpha) - v(\theta' + (K_i - \ell)\alpha)| \leq Ce^{-\varsigma|K_i|}$, uniformly in $\ell$, or, in other words, the potential $v_n = v(\theta + n\alpha)$ is $e^{-\varsigma|K_i|}$-almost symmetric with respect to $(k_0 + K_i)/2$.

Since $\alpha$ is Diophantine, we have

$$|K_i| \geq c e^{|K_{i-1}|},$$  

where $c$ depends on $\varsigma$ and $\alpha$ through the Diophantine constants $\kappa, \tau$. On the other hand, $K_i$ is necessarily an infinite sequence.

\[5\]Can take any one if there are several.
Let \( \phi \) be an eigenfunction, and \( U(k) = \begin{pmatrix} \phi(k) \\ \phi(k - 1) \end{pmatrix} \). We say \( k \) is a local \( K \)-maximum if \( ||U(k)|| \geq ||U(k + s)|| \) for all \( s - k \in [-K, K] \).

We first describe the hierarchical structure of local maxima informally. There exists a constant \( \tilde{K} \) such that there is a local \( cK_j \)-maximum \( b_j \) within distance \( \tilde{K} \) of each resonance \( K_j \). The exponential behavior of the eigenfunction in the local \( cK_j \)-neighborhood of each such local maximum, normalized by the value at the local maximum is given by the reflection of \( f \). Moreover, this describes the entire collection of local maxima of depth 1, that is \( \tilde{K} \) such that \( K \) is a \( cK \)-maximum. Then we have a similar picture in the vicinity of \( b_j \): there are local \( cK_i \)-maxima \( b_{j,i}, i < j \), within distance \( \tilde{K}^2 \) of each \( K_j - K_i \). The exponential (on the \( K_i \) scale) behavior of the eigenfunction in the local \( cK_i \)-neighborhood of each such local maximum, normalized by the value at the local maximum is given by \( \hat{f} \). Then we get the next level maxima \( b_{j,i,s}, s < i \) in the \( \tilde{K}^3 \)-neighborhood of \( K_j - K_i + K_s \) and reflected behavior around each, and so on, with reflections alternating with steps. At the end we obtain a complete hierarchical structure of local maxima that we denote by \( b_{j_0,j_1,...,j_s} \) with each “depth \( s + 1 \)” local maximum \( b_{j_0,j_1,...,j_s} \) being in the corresponding vicinity of the “depth \( s \)” local maximum \( b_{j_0,j_1,...,j_{s-1}} \approx k_0 + \sum_{i=0}^{s-1} (-1)^i K_{j_i} \) and with universal behavior at the corresponding scale around each. The quality of the approximation of the position of the next maximum gets lower with each level of depth, with \( b_{j_0,j_1,...,j_{s-1}} \) determined with \( \tilde{K}^s \) precision, thus it presents an accurate picture as long as \( \tilde{K}_{j_s} \gg \tilde{K}^s \).

We now describe the hierarchical structure precisely.

**Theorem 1.3.** Assume sequence \( K_i \) satisfies \((11)\) for some \( \zeta > 0 \). Then there exists \( \tilde{K}(\alpha, \lambda, \theta, \zeta) < \infty \) such that for any \( j_0 > j_1 > \cdots > j_k \geq 0 \) with \( K_{j_k} \geq \tilde{K}^{k+1} \), for each \( 0 \leq s \leq k \) there exists a local \( \frac{2}{\ln \lambda} K_{j_s} \)-maximum \( b_{j_0,j_1,...,j_s} \) such that the following holds:

**I:** \( |b_{j_0,j_1,...,j_s} - k_0 - \sum_{i=0}^{s} (-1)^i K_{j_i}| \leq \tilde{K}^{s+1} \).

**II:** For any \( \varepsilon > 0 \), if \( CK^{k+1} \leq |x - b_{j_0,j_1,...,j_k}| \leq \frac{2}{\ln \lambda} |K_{j_k}| \), where \( C \) is a large constant depending on \( \alpha, \lambda, \theta, \zeta \) and \( \varepsilon \), then for each \( s = 0, 1, ..., k \),

\[
f((-1)^{s+1} x_s) e^{-\varepsilon |x_s|} \leq \frac{||U(x)||}{||U(b_{j_0,j_1,...,j_s})||} \leq f((-1)^{s+1} x_s) e^{\varepsilon |x_s|},
\]

where \( x_s = x - b_{j_0,j_1,...,j_s} \).

**Remark 1.4.** Actually \((12)\) holds for \( x \) with \( CK^{k+1} \leq |x - b_{j_0,j_1,...,j_k}| \leq (\frac{2}{\ln \lambda} - \varepsilon)|K_{j_k}| \) for any \( \varepsilon > 0 \).

Thus the behavior of \( \phi(x) \) is described by the same universal \( f \) in each \( \frac{2}{\ln \lambda} K_{j_k} \) window around the corresponding local maximum \( b_{j_0,j_1,...,j_k} \) after alternating reflections. The positions of the local maxima in the hierarchy are determined up to errors that at all but possibly the last step are superlogarithmically small in \( K_{j_s} \). We call such a structure reflective hierarchy.

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6\( \tilde{K} \) depends on \( \theta \) through \( 2\theta + ka \), see \((9)\).

7Actually, it can be a local \((\frac{2}{\ln \lambda} - \varepsilon)K_{j_s}\)-maximum for any \( \varepsilon > 0 \).
We are not aware of previous results describing the structure of eigenfunctions for Diophantine $\alpha$ (The structure in the regime $\beta > 0$ is described in [32]). Certain results indicating the hierarchical structure in the corresponding semi-classical/perturbative regimes were previously obtained in the works of Sinai, Helffer-Sjostrand, and Buslaev-Fedotov (see [18, 25, 44], and also [48] for another model). We were also informed [21] that for strongly Diophantine $\alpha$ the fact that many eigenfunctions of box restrictions for analytic $v$ in [8] can only “bump up” at resonances, can be obtained from the avalanche principle expansion of the determinants, an important method developed in [22].

reflective self-similarity of an eigenfunction

Fig.1: This depicts reflective self-similarity of an eigenfunction with global maximum at 0. The self-similarity: $I'$ is obtained from $I$ by scaling the $x$-axis proportional to the ratio of the heights of the maxima in $I$ and $I'$. $II'$ is obtained from $II$ by scaling the $x$-axis proportional to the ratio of the heights of the maxima in $II$ and $II'$. The behavior in the regions $I'$, $II'$ mirrors the behavior in regions $I$, $II$ upon reflection and corresponding dilation.
Our final main result is the asymptotics of the transfer matrices. Let $A_0 = I$ and for $k \geq 1$,

$$A_k(\theta) = \prod_{j=k-1}^{0} A(\theta + j\alpha) = A(\theta + (k-1)\alpha)A(\theta + (k-2)\alpha) \cdots A(\theta)$$

and

$$A_{-k}(\theta) = A_k^{-1}(\theta - k\alpha),$$

where $A(\theta) = \begin{pmatrix} E - 2\lambda \cos 2\pi \theta & -1 \\ 1 & 0 \end{pmatrix}$. $A_k$ is called the (k-step) transfer matrix. As is clear from the definition, it also depends on $\theta$ and $E$ but since those parameters will be usually fixed, we omit this from the notation.

We define a new function $g : \mathbb{Z} \to \mathbb{R}^+$ as follows.

Case 1: If $x_0 \cdot \ell \leq 0$ or $|x_0| > |\ell|$, set

$$g(\ell) = e^{|\ell| \ln |\lambda|}.$$

Case 2: If $x_0 \cdot \ell \geq 0$ and $|x_0| \leq |\ell| \leq 2|x_0|$, set

$$g(\ell) = e^{(|\ln \lambda - \eta| |\ell|} + e^{2|x_0 - \ell| \ln |\lambda|}. $$

Case 3: If $x_0 \cdot \ell \geq 0$ and $|\ell| > 2|x_0|$, set

$$g(\ell) = e^{(|\ln \lambda - \eta| |\ell|}. $$

We have

**Theorem 1.5.** Under the conditions of Theorem 1.2, we have

$$g(\ell)e^{-\varepsilon |\ell|} \leq ||A_\ell|| \leq g(\ell)e^{\varepsilon |\ell|}. $$

Let $\psi(\ell)$ denote any solution to $H_{\lambda, \alpha, \theta} \psi = E\psi$ that is linearly independent with $\phi(\ell)$. Let $\tilde{U}(\ell) = \begin{pmatrix} \psi(\ell) \\ \psi(\ell - 1) \end{pmatrix}$. An immediate counterpart of (13) is the following

**Corollary 1.6.** Under the conditions of Theorem 1.2, vectors $\tilde{U}(\ell)$ satisfy

$$g(\ell)e^{-\varepsilon |\ell|} \leq ||\tilde{U}(\ell)|| \leq g(\ell)e^{\varepsilon |\ell|}. $$

Our analysis also gives

**Corollary 1.7.** Under the conditions of Theorem 1.2, we have,

i) $$\limsup_{k \to \infty} \frac{\ln ||A_k||}{k} = \limsup_{k \to \infty} \frac{-\ln ||U(k)||}{k} = \ln |\lambda|,$$

ii) $$\liminf_{k \to \infty} \frac{\ln ||A_k||}{k} = \liminf_{k \to \infty} \frac{-\ln ||U(k)||}{k} = \ln |\lambda| - \delta,$$

iii) outside a sequence of lower density $1/2$,

$$\lim_{k \to \infty} \frac{-\ln ||U(k)||}{|k|} = \ln |\lambda|,$$
iv) outside a sequence of lower density 0,

\[
\lim_{k \to \infty} \frac{\ln ||A_k||}{k} = \ln |\lambda|.
\]

Thus our analysis presents the second, after [32], study of the dynamics of Lyapunov-Perron non-regular points, in a natural setting. It is interesting to remark that (16) also holds throughout the pure point regime of [32]. As in [32], the fact that \( g \) is not always the reciprocal of \( f \) leads to exponential tangencies between contracted and expanded directions with the rate approaching \(-\delta\) along a subsequence. Tangencies are an attribute of nonuniform hyperbolicity and are usually viewed as a difficulty to avoid through e.g. the parameter exclusion (e.g. [11, 13, 47]). Our analysis allows to study them in detail and uncovers the hierarchical structure of exponential tangencies positioned precisely at resonances. This will be explored in the future work. Finally we mention that the methods developed in this paper have made it possible to determine the \textit{exact} exponent of the exponential decay rate in expectation for the two-point function [31], the first result of this kind for any model.

The rest of this paper is organized as follows. We list the definitions and standard preliminaries in Section 2. Section 3 is devoted to the upper bound on the generalized eigenfunction in Theorem 1.2 establishing sharp upper bounds for any eigensolution in the resonant case. This part of the proof has two technical ingredients similar to the arguments used previously to prove localization in presence of exponential frequency resonances. We present a universal version of these statements in Theorem 3.2 (a uniformity statement for any Diophantine \( \alpha \)) and Theorem 3.3 (a resonant block expansion theorem for any one-dimensional operator), proved correspondingly in Appendices A and B. Those statements can be of use for proving localization for other models. The rest of the argument is based on new ideas specific to the phase resonance situation. In Section 4 we prove the sharp transition - Theorem 1.1 and lower bound on the generalized eigenfunctions in Theorem 1.2. The part on the singular continuous spectrum, in particular, requires a new approach to the palindromic argument in order to remove a factor of four inherent in the previous proofs, and the sharp lower bound in the localization regime requires an even more delicate approach. In Section 5 we use the local version of Theorem 1.2 and establish reflective hierarchical structure of \textit{resonances} to prove the reflective hierarchical structure Theorem 1.3. In Section 6 we study the growth of transfer matrices and prove Theorem 1.5 and Corollaries 1.6 and 1.7. Except for the (mostly standard) statements listed in the preliminaries and Lemma A.1 this paper is entirely self-contained.

2. Preliminaries

Without loss of generality, we assume \( \lambda > 1 \) and \( \ell > 0 \). If \( 2\theta \in \alpha\mathbb{Z} + \mathbb{Z} \), then \( \delta(\alpha, \theta) = 0 \), in which case the result follows from [30]. Thus in what follows we always assume \( 2\theta \notin \alpha\mathbb{Z} + \mathbb{Z} \).

For any solution of \( H_{\lambda, \alpha, \theta} \phi = E \phi \), we have for any \( k, m \),

\[
\begin{pmatrix}
\phi(k + m) \\
\phi(k + m - 1)
\end{pmatrix} = A_k(\theta + m\alpha) \begin{pmatrix}
\phi(m) \\
\phi(m - 1)
\end{pmatrix}.
\]
The Lyapunov exponent is given by

\[ L(E) = \lim_{k \to \infty} \frac{1}{k} \int_{\mathbb{R}/\mathbb{Z}} \ln \| A_k(\theta) \| d\theta. \]

The Lyapunov exponent can be computed precisely for \( E \) in the spectrum of \( H_{\lambda,\alpha,\theta} \). We denote the spectrum by \( \Sigma_{\lambda,\alpha} \) (it does not depend on \( \theta \)).

**Lemma 2.1.** \([15]\) For \( E \in \Sigma_{\lambda,\alpha} \) and \( \lambda > 1 \), we have 

\[ L(E) = \ln \lambda. \]

Recall that we always assume \( E \in \Sigma_{\lambda,\alpha} \), so by upper semicontinuity and unique ergodicity, one has

\[ \ln \lambda = \lim_{k \to \infty} \sup_{\theta \in \mathbb{R}/\mathbb{Z}} \frac{1}{k} \ln \| A_k(\theta) \|, \]

that is, the convergence in (19) is uniform with respect to \( \theta \in \mathbb{R} \). Precisely, \( \forall \varepsilon > 0 \),

\[ \| A_k(\theta) \| \leq e^{(\ln \lambda + \varepsilon)k} \text{ for } k \text{ large enough.} \]

We start with the basic setup going back to \([34]\). Let us denote

\[ P_k(\theta) = \det(R_{[0,k-1]}(H_{\lambda,\alpha,\theta} - E)R_{[0,k-1]}). \]

It is easy to check that

\[ A_k(\theta) = \begin{pmatrix} P_k(\theta) & -P_{k-1}(\theta + \alpha) \\ P_{k-1}(\theta) & -P_{k-2}(\theta + \alpha) \end{pmatrix}. \]

By Cramer’s rule for given \( x_1 \) and \( x_2 = x_1 + k - 1 \), with \( y \in I = [x_1, x_2] \subset \mathbb{Z} \), one has

\[ |G_I(x_1, y)| = \left| \frac{P_{x_2-y}(\theta + (y + 1)\alpha)}{P_k(\theta + x_1\alpha)} \right|, \]

\[ |G_I(y, x_2)| = \left| \frac{P_{y-x_1}(\theta + x_1\alpha)}{P_k(\theta + x_1\alpha)} \right|. \]

By (20) and (21), the numerators in (22) and (23) can be bounded uniformly with respect to \( \theta \). Namely, for any \( \varepsilon > 0 \),

\[ |P_k(\theta)| \leq e^{(\ln \lambda + \varepsilon)k} \]

for \( k \) large enough.

**Definition 2.2.** Fix \( \tau > 0 \). A point \( y \in \mathbb{Z} \) will be called \((\tau, k)\) regular if there exists an interval \([x_1, x_2]\) containing \( y \), where \( x_2 = x_1 + k - 1 \), such that

\[ |G_{[x_1,x_2]}(y, x_i)| < e^{-\tau|y-x_i|} \text{ and } |y-x_i| \geq \frac{1}{40}k \text{ for } i = 1, 2. \]

It is easy to check that for any solution of \( H_{\lambda,\alpha,\theta} \phi = E \phi \),

\[ \phi(x) = -G_{[x_1,x_2]}(x_1, x)\phi(x_1 - 1) - G_{[x_1,x_2]}(x, x_2)\phi(x_2 + 1), \]

where \( x \in I = [x_1, x_2] \subset \mathbb{Z} \).
Definition 2.3. We say that the set $\{\theta_1, \ldots, \theta_{k+1}\}$ is $\epsilon$-uniform if

$$\max_{x \in [-1,1]} \left| \max_{i=1,\ldots,k+1} \prod_{j=1, j \neq i}^{k+1} \frac{|x - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \theta_j|} \right| < e^{k\epsilon}. \quad (26)$$

Let $A_{k,r} = \{ \theta \in \mathbb{R} \mid P_k(\cos 2\pi (\theta - \frac{1}{2}(k - 1)\alpha)) \leq e^{(k+1)r} \}$ with $k \in \mathbb{N}$ and $r > 0$. We have the following Lemma.

Lemma 2.4. (Lemma 9.3, [6]) Suppose $\{\theta_1, \ldots, \theta_{k+1}\}$ is $\epsilon_1$-uniform. Then there exists some $\theta_i$ in set $\{\theta_1, \ldots, \theta_{k+1}\}$ such that $\theta_i \notin A_{k,\ln \lambda - \epsilon}$ if $\epsilon > \epsilon_1$ and $k$ is sufficiently large.

3. Localization

Let $\alpha$ be Diophantine and $\delta(\alpha, \theta)$ be given by (2). Suppose $\ln |\lambda| > \delta(\alpha, \theta)$. Recalling that for $E$ a generalized eigenvalue of $H_{\lambda,\alpha,\theta}$ and $\phi$ the corresponding generalized eigenfunction, we denote $U(\ell) = \begin{pmatrix} \phi(\ell) \\ \phi(\ell - 1) \end{pmatrix}$. In this part we will prove the localization part of Theorem 1.1 and the upper bound of Theorem 1.2. That is

Theorem 3.1. For any $\epsilon > 0$, there exists $K$ such that for any $|\ell| \geq K$, $U(\ell)$ satisfies

$$||U(\ell)|| \leq f(\ell)e^{\epsilon|\ell|}. \quad (27)$$

In particular, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization, and the following upper bound holds for the generalized eigenfunction,

$$||U(\ell)|| \leq e^{-(\ln \lambda - \delta - \epsilon)|\ell|}. \quad (28)$$

By Schnol’s Theorem [12] if every generalized eigenfunction of $H$ decays exponentially, then $H$ satisfies Anderson localization. Thus, by Remark 1.2, in order to prove Theorem 3.1, it suffices to prove the first part of Theorem 3.1.

Without loss of generality assume $|\phi(0)|^2 + |\phi(-1)|^2 = 1$. Let $\psi$ be any solution of $H_{\lambda,\alpha,\theta}\psi = E\psi$ linear independent with $\phi$, i.e., $|\psi(0)|^2 + |\psi(-1)|^2 = 1$ and

$$\phi(-1)\psi(0) - \phi(0)\psi(-1) = c,$$

where $c \neq 0$.

Then by the constancy of the Wronskian, one has

$$\phi(y + 1)\psi(y) - \phi(y)\psi(y + 1) = c. \quad (29)$$

We also will denote by $\varphi$ an arbitrary solution, so either $\psi$ or $\phi$, and denote by $U^\varphi(y) = \begin{pmatrix} \varphi(y) \\ \varphi(y - 1) \end{pmatrix}$. Let $U(y) = \begin{pmatrix} \phi(y) \\ \phi(y - 1) \end{pmatrix}$ and $\tilde{U}(y) = \begin{pmatrix} \psi(y) \\ \psi(y - 1) \end{pmatrix}$.

Below $\epsilon > 0$ is always sufficiently small and $\lfloor \frac{p_n}{q_n} \rfloor$ is the continued fraction expansion of $\alpha$.

We will make a repeated use of the following two Theorems that can be useful also in other situations. The first theorem is an arithmetic statement that holds for any Diophantine $\alpha$. 

We have the following Lemma.
Theorem 3.2. (Uniformity Theorem)

Let $I_1, I_2$ be two disjoint intervals in $\mathbb{Z}$ such that $\# I_1 = s_1q_n$ and $\# I_2 = s_2q_n$, where $s_1, s_2 \in \mathbb{Z}^+$. Suppose $|j| \leq Csn$ for any $j \in I_1 \cup I_2$ and $|s| \leq q_n^C$, where $s = s_1 + s_2$. Let $\gamma > 0$ be such that

$$e^{-\gamma sq_n} = \min_{i,j \in I_1 \cup I_2} |\sin \pi(2\theta + (i+j)\alpha)|.$$  

Then for any $\varepsilon > 0$, $\{\theta_j = \theta + j\alpha\}_{j \in I_1 \cup I_2}$ is $\gamma + \varepsilon$ uniform if $n$ is large enough (not depending on $\gamma$).

The second theorem holds for any one-dimensional (not necessarily quasiperiodic or even ergodic) Schrödinger operator. It is the technique to establish exponential decay with respect to the distance to the resonances.

Theorem 3.3. (Block Expansion Theorem)

Fix $\gamma > 0$. Let $r_y^\varphi = \max_{|\sigma| \leq 10\gamma} |\varphi(y + \sigma k)|$. Suppose $y_1, y_2 \in \mathbb{Z}$ are such that $y_2 - y_1 = k$. Suppose there exists some $\tau > 0$ such that for any $y \in [y_1 + \gamma k, y_2 - \gamma k]$, $y$ is $(\tau, k_1)$ regular, for some $\frac{1}{20} k < k_1 \leq \frac{1}{2} \min\{|y - y_1|, |y - y_2|\}$. Then for large enough $k$,

$$r_y^\varphi \leq \max\{r_{y_1}^\varphi \exp\{-\tau(|y - y_1| - 3\gamma k)\}, r_{y_2}^\varphi \exp\{-\tau(|y - y_2| - 3\gamma k)\}\},$$

for all $y \in [y_1 + 10\gamma k, y_2 - 10\gamma k]$.

These two theorems are similar in spirit to the statements in [32] with the ones related to Theorem 3.2 being in turn modifications of the ones appearing in [6, 38, 39]. While these techniques were developed specifically to treat the non-Diophantine case, these ideas turn out to be relevant for the case of phase resonances as well. Theorem 3.3 is essentially the block-expansion technique of multiscale analysis, e.g. [20], coupled with certain extremality arguments, an idea used also in [32]. We expect Theorem 3.2 to be useful for various one-frequency quasiperiodic problems, and Theorem 3.3 for general one-dimensional models. We present the proofs in Appendix A and B respectively.

The following Lemma establishes the non-resonant decay.

Lemma 3.4. Suppose $k_0 \in [-2Ck, 2Ck]$ is such that

$$|\sin \pi(2\theta + \alpha k_0)| = \min_{|x| \leq 2Ck} |\sin \pi(2\theta + \alpha x)|,$$

where $C \geq 1$ is a constant. Let $\gamma, \varepsilon$ be small positive constants. Let $y_1 = 0, y_2 = k_0, y_3 = y'$. Assume $y$ lies in $[y_i, y_j]$ (i.e., $y \in [y_i, y_j]$) with $|y_i - y_j| \geq k$. Suppose $|y_i|, |y_j| \leq Ck$ and $|y - y_i| \geq 10\gamma k$, $|y - y_j| \geq 10\gamma k$. Then for large enough $k$,

$$r_y^\varphi \leq \max\{r_{y_i}^\varphi \exp\{-(\ln \lambda - \varepsilon)(|y - y_i| - 3\gamma k)\}, r_{y_j}^\varphi \exp\{-(\ln \lambda - \varepsilon)(|y - y_j| - 3\gamma k)\}\}.$$  

Proof. By the DC on $\alpha$, there exist $\gamma', \kappa' > 0$ such that for any $x \neq k_0$ and $|x| \leq 2Ck$,

$$|\sin \pi(2\theta + \alpha x)| \geq \frac{\gamma'}{k\kappa'}.$$  

Fix $y'$. For any $p$ satisfying $|p - y'| \geq \gamma k$, $|p| \geq \gamma k$ and $|p - k_0| \geq \gamma k$, let

$$d_p = \frac{1}{10} \min\{|p|, |p - k_0|, |p - y'|\}$$

Fix $y$. For any $p$ satisfying $|p - y| \geq \gamma k$, $|p| \geq \gamma k$ and $|p - k_0| \geq \gamma k$.
Let \( \frac{p_n}{q_n} \) be the continued fraction expansion of \( \alpha \). Let \( n \) be the largest integer such that

\[
2q_n \leq d_p,
\]
and let \( s \) be the largest positive integer such that \( 2sq_n \leq d_p \). Notice that \( 2q_{n+1} > d_p \) and by the Diophantine condition on \( \alpha \), we have \( s \leq q_n^C \).

Case 1: \( 0 \leq k_0 < p \). We construct intervals

\[
I_1 = [-2sq_n, -1], \quad I_2 = [p - 2sq_n, p + 2sq_n - 1].
\]

Case 2: \( 0 \leq p < k_0 \).

If \( p \leq \frac{k_0}{2} \), we construct intervals

\[
I_1 = [-2sq_n, 2sq_n - 1], \quad I_2 = [p - 2sq_n, p - 1].
\]

If \( p > \frac{k_0}{2} \), we construct intervals

\[
I_1 = [-2sq_n, 2sq_n - 1], \quad I_2 = [p, p + 2sq_n - 1].
\]

Case 3: \( p < k_0 \leq 0 \).

We construct intervals

\[
I_1 = [0, 2sq_n - 1], \quad I_2 = [p - 2sq_n, p + 2sq_n - 1].
\]

Case 4: \( k_0 < p < 0 \).

If \( p \leq \frac{k_0}{2} \), we construct intervals

\[
I_1 = [-2sq_n, 2sq_n - 1], \quad I_2 = [p - 2sq_n, p - 1].
\]

If \( p > \frac{k_0}{2} \), we construct intervals

\[
I_1 = [-2sq_n, 2sq_n - 1], \quad I_2 = [p, p + 2sq_n - 1].
\]

Case 5: \( k_0 \leq 0 < p \).

\[
I_1 = [0, 2sq_n - 1], \quad I_2 = [p - 2sq_n, p + 2sq_n - 1].
\]

Case 6: \( p < 0 \leq k_0 \).

\[
I_1 = [-2sq_n, -1], \quad I_2 = [p - 2sq_n, p + 2sq_n - 1].
\]

Using the small divisor condition (33) and the construction of \( I_1, I_2 \), we have in any case

\[
\min_{i,j \in I_1 \cup I_2} |\sin \pi(2\theta + (i + j)\alpha)| \geq \frac{\tau'}{k^{c'}}.
\]

By Theorem 3.2 for any \( \varepsilon > 0 \), we have, in each case \( \{\theta_j = \theta + j\alpha\}_{j \in I_1 \cup I_2} \) is \( \varepsilon \) uniform. Combining with Lemma 2.4 there exists some \( j_0 \) with \( j_0 \in I_1 \cup I_2 \) such that \( \theta_{j_0} \notin A_{6sq_n - 1, \ln \lambda - \varepsilon} \).

We have the following simple Lemma that will be used repeatedly in the rest of the paper.

**Lemma 3.5.** Let \( a_n \to \infty \), \( 0 < t < 1 \). Then for sufficiently large \( n \) and \( |j| < ta_n \) we have \( \theta_j = \theta + j\alpha \in A_{2a_n - 1, \ln \lambda - \varepsilon} \).
Proof: Assume for some $|j| < ta_n$, $\theta_j \notin A_{2a_n - 1, \ln \lambda - \varepsilon}$.
Let $I = [j - a_n + 1, j + a_n - 1] = [x_1, x_2]$. We have $x_1 < 0 < x_2$ and
\begin{equation}
|x_i| > (1 - t)a_n.
\end{equation}
By (22), (23) and (24), one has
\begin{equation}
|G_I(0, x_i)| \leq e^{(\ln \lambda + \varepsilon)(2a_n - 1 - |x_i|) - (2a_n - 1)(\ln \lambda - \varepsilon)}.
\end{equation}
Using (25), we obtain
\begin{equation}
|G_I(0, x_i)| \leq e^{(\ln \lambda + \varepsilon)(2a_n - 1 - |x_i|) - (2a_n - 1)(\ln \lambda - \varepsilon)}.
\end{equation}
Using (25), we obtain
\begin{equation}
|G_I(0, x_i)| \leq e^{(\ln \lambda + \varepsilon)(2a_n - 1 - |x_i|) - (2a_n - 1)(\ln \lambda - \varepsilon)}.
\end{equation}
where $x'_1 = x_1 - 1$ and $x'_2 = x_2 + 1$. Because of (34), (35) implies $|\phi(-1)|, |\phi(0)| \leq e^{-(1-t-\varepsilon)\ln \lambda a_n}$. This contradicts $|\phi(-1)|^2 + |\phi(0)|^2 = 1$. □

Lemma (35) implies that $j_0$ must belong to $I_2$.
Set $I = [j_0 - 3sq_n + 1, j_0 + 3sq_n - 1] = [x_1, x_2]$. By (22), (23) and (24) again, one has
\begin{equation}
|G_I(p, x_i)| \leq e^{(\ln \lambda + \varepsilon)(6sq_n - 1 - |x_i|) - (6sq_n - 1)(\ln \lambda - \varepsilon)} \leq e^{q \ln \lambda} e^{-(p - x_i)\ln \lambda}.
\end{equation}
Notice that $|p - x_1|, |p - x_2| \geq sq_n - 1$. Thus for any $p \in [y_i + \gamma k, y_j - \gamma k]$, $p$ is $(6sq_n - 1, \ln \lambda - \varepsilon)$ regular. Block expansion (Theorem 3.3) now implies the Lemma. □

Remark 3.6. Recall that $U^\varphi(y) = \left( \begin{array}{c} \varphi(y) \\ \varphi(y - 1) \end{array} \right)$. By (17) and (20), we have
\begin{equation}
Ce^{-(\ln \lambda + \varepsilon)k_1 - k_2} ||U^\varphi(k_2)|| \leq ||U^\varphi(k_1)|| \leq Ce^{(\ln \lambda + \varepsilon)k_1 - k_2} ||U^\varphi(k_2)||.
\end{equation}
Thus (32) implies
\begin{equation}
||U^\varphi(y)|| \leq \max\{||U^\varphi(y_i)|| \exp\{-(\ln \lambda - \varepsilon)(|y - y_i| - 14\gamma k)\}, ||U^\varphi(y_j)|| \exp\{-(\ln \lambda - \varepsilon)(|y - y_j| - 14\gamma k)\}\}
\end{equation}

Lemma 3.7. Fix $0 < t < \ln \lambda$. Suppose
\begin{equation}
|\sin \pi(2\theta + \alpha k)| = e^{-t|k|}.
\end{equation}
Then for large $|k|
\begin{equation}
||U^\varphi(k)|| \leq \max\{||U^\varphi(0)||, ||U^\varphi(2k)||\}e^{-(\ln \lambda - t - \varepsilon)|k|}.
\end{equation}
Proof. Without loss of generality assume $k > 0$. By the DC on $\alpha$, we have
\begin{equation}
|\sin \pi(2\theta + \alpha k)| = \min_{|x| \leq 8k} |\sin \pi(2\theta + \alpha x)|.
\end{equation}
Furthermore, there exist $\tau', \kappa' > 0$ such that for any $x \neq k$ and $|x| \leq 8k$,
\begin{equation}
|\sin \pi(2\theta + x\alpha)| \geq \frac{\tau'}{\kappa'}.
\end{equation}
Let $\gamma$ be any small positive constant and define $\tau_y^\varphi = \max_{|\sigma| \leq 10\gamma} |\varphi(y + \sigma k)|$. Let $\frac{q_n}{\theta_n}$ be the continued fraction expansion of $\alpha$. Let $n$ be the largest integer such that
\begin{equation}
\frac{t + Ce}{\ln \lambda - t - C\varepsilon} + 1)q_n \leq \frac{k}{2},
\end{equation}
where $C$ is a large constant depending on $\lambda, t$. Let $s$ be the largest positive integer such that $sq_n \leq \frac{1}{2}k$. Then $s > \frac{t+C\varepsilon}{\ln \lambda - t - C\varepsilon}$. Combining with the fact $(s+1)q_n \geq \frac{1}{2}k$, we obtain

\begin{equation}
2sq_n/k > \frac{t}{\ln \lambda} + C\varepsilon.
\end{equation}

Construct intervals

$I_1 = [-sq_n, sq_n - 1], I_2 = [k - sq_n, k + sq_n - 1]$.

Let $\theta_j = \theta + j\alpha$ for $j \in I_1 \cup I_2$. The set $\{\theta_j\}_{j \in I_1 \cup I_2}$ consists of $4sq_n$ elements.

By Theorem [32] and [33], we have $\{\theta_j\}_{j \in I_1 \cup I_2}$ is $\frac{tk}{4sq_n} + \varepsilon$ uniform. Combining with Lemma [2,4] there exists some $j_0$ with $j_0 \in I_1 \cup I_2$ such that $\theta_{j_0} \notin A_{4sq_n - 1, \ln \lambda - \frac{tk}{4sq_n} - \varepsilon}$.

First assume $j_0 \in I_2$.

Set $I = [j_0 - 2sq_n + 1, j_0 + 2sq_n - 1] = [x_1, x_2]$. By (22), (23) and (24), one has

\[|G_1(k, x_i)| \leq e^{(\ln \lambda + \varepsilon)(4sq_n - 1 - |k - x_i| - (4sq_n - 1)(\ln \lambda - \frac{tk}{4sq_n} - \varepsilon)}.
\]

Using (25), we obtain

\begin{equation}
|\varphi(k - 1)|, |\varphi(k)| \leq \sum_{i=1,2} e^{(t+\varepsilon)k} |\varphi(x'_i)| e^{-|k - x_i| \ln \lambda},
\end{equation}

where $x'_1 = x_1 - 1$ and $x'_2 = x_2 + 1$.

Fix small $\gamma = \frac{\varepsilon}{2}$, where $C$ is a large constant depending on $\lambda, t$.

If $x'_i \in [-10\gamma k, 10\gamma k]$, $x'_i \in [k - 10\gamma k, k + 10\gamma k]$ or $x'_i \in [2k - 10\gamma k, 2k + 10\gamma k]$, we bound $\varphi(x'_i)$ in (41) by $r_0^\varphi, r_k^\varphi$ or $r_{2k}^\varphi$ respectively. In other cases, we bound $\varphi(x'_i)$ in (41) with (32) using $k_0 = k, y = x'_i$ and $y' = -k, 2k$ or $3k$. Then we have

\[|\varphi(k - 1)|, |\varphi(k)| \leq \max\{r_{k\pm 2k}^\varphi \exp\{-2(\ln \lambda - t - C\gamma - \varepsilon)k\}, r_k^\varphi \exp\{-(\ln \lambda - t - C\gamma - \varepsilon)k\}, r_{2k}^\varphi \exp\{-(\ln \lambda - \varepsilon)2sq_n + (t + C\gamma)k\}\}.
\]

However by (10), we have that

\[|\varphi(k - 1)|, |\varphi(k)| \leq r_k^\varphi \exp\{-(\ln \lambda - \varepsilon)2sq_n + (t + C\gamma)k\}\]

\[\leq e^{-\varepsilon k\tau_k^\varphi}
\]

cannot happen, so we must have

\begin{equation}
|\varphi(k - 1)|, |\varphi(k)| \leq \exp\{-(\ln \lambda - t - C\gamma - \varepsilon)k\} \max\{r_{k\pm 2k}^\varphi, e^{-k\ln \lambda \tau_k^\varphi \pm 2k}\}.
\end{equation}

Notice that by (36), one has

\[r_{k\pm 2k}^\varphi \leq e^{(\ln \lambda + C\gamma)k} r_{k\pm k}^\varphi.
\]

Then (42) becomes

\[||U^\varphi(k)|| \leq \exp\{-(\ln \lambda - t - C\gamma - \varepsilon)k\} \max\{r_0^\varphi, r_{2k}^\varphi\} \max\{r_{k\pm 2k}^\varphi, e^{-k\ln \lambda \tau_k^\varphi \pm 2k}\}.
\]

By (36) again, one has

\[r_{k\pm 2k}^\varphi e^{-(\ln \lambda + \varepsilon)10\gamma k} \leq ||U^\varphi(y)|| \leq r_y^\varphi e^{(\ln \lambda + \varepsilon)10\gamma k}.
\]
Thus
\[ ||U^\varphi(k)|| \leq \max\{||U^\varphi(0)||, ||U^\varphi(2k)||\}e^{-(\ln \lambda - t - C\gamma - \varepsilon)|k|} \]
(43)
\[ \leq \max\{||U^\varphi(0)||, ||U^\varphi(2k)||\}e^{-(\ln \lambda - t - \varepsilon)|k|}. \]
This implies (39). Thus in order to prove the lemma, it suffices to exclude the case \( j_0 \in I_1 \).

Suppose \( j_0 \in I_1 \). Notice that \( I_1 + k = I_2 \) (i.e., \( I_2 \) can be obtained from \( I_1 \) by moving by \( k \) units). Following the proof of (15), we get (move \( -k \) units in (43))
\[ ||U^\varphi(0)|| \leq \max\{||U^\varphi(-k)||, ||U^\varphi(k)||\}e^{-(\ln \lambda - t - \varepsilon)|k|}. \]
This contradicts \( ||U^\phi(0)|| = 1 \).

**Proof of Theorem 3.1** Without loss of generality, assume \( \ell > 0 \).

For any \( \varepsilon > 0 \), let \( \gamma = \frac{\lambda}{C} > 0 \), where \( C \) is a large constant that may depend on \( \lambda \) and \( \delta \). Let \( x'_0 \) (we can choose any one if \( x'_0 \) is not unique) be such that
\[ |\sin \pi(2\theta + x'_0\alpha)| = \min_{|x| \leq 4\ell} |\sin \pi(2\theta + x\alpha)|. \]

Let \( \eta' \in (0, \infty) \) be given by the following equation,
\[ |\sin \pi(2\theta + x'_0\alpha)| = e^{-\eta'|\ell|}. \]
(44)

Case 1: \( |\sin \pi(2\theta + x'_0\alpha)| \neq |\sin \pi(2\theta + x_0\alpha)| \). This implies \( |x'_0| > 2\ell \). In this case for any \( \varepsilon > 0 \), we have \( \eta \leq \varepsilon \) if \( \ell \) is large enough by the Diophantine condition. Let \( y = \ell, C = 2, k = 2\ell, \) and \( y' = 2\ell \) in Lemma 3.4. Then \( k_0 = x'_0 \) and we obtain
\[ |\phi(\ell)|, |\phi(\ell - 1)| \leq e^{-(\ln \lambda - C\gamma)|\ell|}. \]
This implies the right inequality of (7) in this case.

Case 2: \( |\sin \pi(2\theta + x'_0\alpha)| = |\sin \pi(2\theta + x_0\alpha)| \), so \( \eta = \eta' \).

If \( x_0 \leq 0 \), let \( y = \ell, C = 2, k = 2\ell \) and \( y' = 2\ell \) in Lemma 3.4. Then Theorem 3.1 holds by (32).

Now we consider the case \( x_0 > 0 \).

We split the proof into two subcases.

Subcase i: \( \eta \leq \gamma \).

Fix some \( y \in [\gamma\ell, 2\ell - \gamma\ell] \). Let \( n \) be such that \( q_n \leq \frac{1}{20} \min\{y, 2\ell - y\} < q_{n+1} \), and let \( s \) be the largest positive integer such that \( sq_n \leq \frac{1}{20} \min\{y, 2\ell - y\} \). We construct intervals
\[ I_1 = [-2sq_n, 2sq_n - 1], I_2 = [y - 2sq_n, y - 1]. \]
By the definition of \( \eta', \eta \) and construction of \( I_1, I_2 \), we have
\[ \min_{i,j \in I_1 \cup I_2} |\sin \pi(2\theta + (j + i)\alpha)| \geq e^{-\eta|\ell|} = e^{-\eta'}. \]
By Theorem 3.2 we get \( \{\theta_j = \theta + j\alpha\}_{j \in I_1 \cup I_2} \) is \( 2\gamma \) uniform. As in the proof of Lemma 3.4 we obtain there exists some \( j_0 \) with \( j_0 \in I_2 \) such that \( \theta_{j_0} \notin A_{6sq_n - 1, \ln \lambda - 3\gamma} \). Thus \( y \) is \( (\ln \lambda - 3\gamma, 6sq_n) \) regular. By block expansion (Theorem 3.3 with \( y_1 = 0, y_2 = 2\ell, \tau = \ln \lambda - 3\gamma \), we get
\[ |\phi(\ell)|, |\phi(\ell - 1)| \leq e^{-(\ln \lambda - C\gamma)|\ell|}, \]
This implies the right inequality of (7).

Subcase ii: \( \eta \geq \gamma \).

By the definition of \( \delta(\alpha, \theta) \) and the fact that \( \delta(\alpha, \theta) < \ln \lambda \), we must have

\[
\frac{\gamma}{\ln \lambda} \ell \leq |x_0| \leq 2\ell.
\]

Applying Lemma 3.7 with \( k = x_0 \) to the generalized eigenfunction \( \phi(k) \), we have

\[
||U(x_0)|| = ||U\phi(x_0)|| \leq e^{-\ln \lambda e^{\eta \ell}}.
\]

Applying Lemma 3.4 with \( y = \ell, k = 2\ell, C = 2, y' = 2\ell, k_0 = x_0, \varphi = \phi \), considering \( \ell > x_0 \) and \( \ell \leq x_0 \) separately, and using (46), we obtain Theorem 3.1.

Remark 3.8. By (20), we have

\[
||U(\ell)|| \geq ||A\ell||^{-1}||U(0)|| \geq e^{-(\ln \lambda + \varepsilon)(k_i)}.
\]

This already implies the left inequality of (7), except for the Subcase ii.

4. PALINDROMIC ARGUMENTS

4.1. Singular continuous spectrum. We first show that if \( \ln |\lambda| < \delta(\alpha, \theta) \), then \( H_{\lambda, \alpha, \theta} \) has purely singular continuous spectrum, which is the second part of Theorem 4.1. That is

**Theorem 4.1.** Let \( H_{\lambda, \alpha, \theta} \) be an almost Mathieu operator with \( |\lambda| > 1 \). For any irrational number \( \alpha \) and \( \theta \in \mathbb{R} \), define \( \delta(\alpha, \theta) \in [0, \infty] \) by (2). Then \( H_{\lambda, \alpha, \theta} \) has purely singular continuous spectrum if \( \ln |\lambda| < \delta(\alpha, \theta) \).

Actually, we can prove a more general result.

**Theorem 4.2.** Let \( H_{v, \alpha, \theta} \) be a discrete Schrödinger operator,

\[
(H_{v, \alpha, \theta}u)(n) = u(n + 1) + u(n - 1) + v(\theta + n\alpha)u(n),
\]

where \( v : \mathbb{T} \to \mathbb{R} \) is an even Lipchitz continuous function. For any irrational number \( \alpha \) and \( \theta \in \mathbb{R} \), define \( \delta(\alpha, \theta) \in [0, \infty] \) by (2). Then \( H_{v, \alpha, \theta} \) has no eigenvalues in the regime \( \{E \in \mathbb{R} : L(E) < \delta(\alpha, \theta)\} \), where \( L(E) \) is the Lyapunov exponent.

Theorem 4.1 follows directly from Theorem 4.2, Lemma 2.1 and Kotani theory.

By the definition of \( \delta(\alpha, \theta) \), for any \( \varepsilon > 0 \) there exists a sequence \( \{k_i\}_{i=1}^{\infty} \) such that

\[
||2\theta + k_i\alpha||_{\mathbb{R}/\mathbb{Z}} \leq e^{-(\delta - \varepsilon)|k_i|}.
\]

Without loss of generality assume \( k_i > 0 \).

**Proof of Theorem 4.2.**

Suppose not. Let \( u \) be an \( \ell^2(\mathbb{Z}) \) solution, i.e., \( H_{v, \alpha, \theta}u = Eu \), with \( L(E) < \delta(\alpha, \theta) \). Without loss of generality assume

\[
||u||_{\ell^2} = \sum_n |u(n)|^2 = 1.
\]

We let \( u_i(n) = u(k_i - n) \), \( V(n) = v(\theta + n\alpha) \) and \( V_i(n) = v(\theta + (k_i - n)\alpha) \). Then by (47), evenness and Lipchitz continuity of \( v \), one has for all \( n \in \mathbb{Z} \),

\[
|V(n) - V_i(n)| \leq Ce^{-(\delta - \varepsilon)|k_i|}.
\]
We also have

\begin{equation}
(49) \quad u(n + 1) + u(n - 1) + V(n)u(n) = Eu(n)
\end{equation}

and

\begin{equation}
(50) \quad u_i(n + 1) + u_i(n - 1) + V(n)u_i(n) = Eu_i(n).
\end{equation}

Let \( W(n) = W(f, g) = f(n + 1)g(n) - f(n)g(n + 1) \) be the Wronskian, as usual, and let

\[ \Phi(n) = \begin{pmatrix} u(n) \\ u(n - 1) \end{pmatrix}; \Phi_i(n) = \begin{pmatrix} u_i(n) \\ u_i(n - 1) \end{pmatrix}. \]

By a standard calculation using (48), (49) and (50), we have

\[ |W(u, u_i)(n) - W(u, u_i)(n - 1)| \leq |V(n) - V_i(n)||u(n)u_i(n)| \leq Ce^{-(\delta - \varepsilon)|k|}|u(n)u_i(n)|. \]

This implies for any \( m > 0 \) and \( n \),

\[ |W(u, u_i)(n + m) - W(u, u_i)(n - 1)| \leq Ce^{-(\delta - \varepsilon)|k|}\sum_{j=0}^{m-1} |u(n + j)u_i(n + j)| \leq Ce^{-(\delta - \varepsilon)|k|}, \]

(51)

where the second inequality holds by the fact \( ||u||_{\ell^2} = ||u_i||_{\ell^2} = 1 \).

Notice that \( \sum_n |W(u, u_i)(n)| \leq 2 \). Thus for some \( n \), one has

\[ |W(u, u_i)(n)| \leq Ce^{-(\delta - \varepsilon)|k|}. \]

By (51), we must have that

\begin{equation}
(52) \quad |W(u, u_i)(n)| \leq Ce^{-(\delta - \varepsilon)|k|}
\end{equation}

holds for all \( n \).

Now we split \( k_i \) into odd or even to discuss the problem.

Case 1. \( k_i \) is even. Let \( m_i = \frac{k_i}{2} \), then

\[ \Phi(m_i) = \begin{pmatrix} u(m_i) \\ u(m_i - 1) \end{pmatrix}; \Phi_i(m_i) = \begin{pmatrix} u(m_i) \\ u(m_i + 1) \end{pmatrix}. \]

Applying (52) with \( n = m_i - 1 \), we have

\[ |u(m_i)||u(m_i + 1) - u(m_i - 1)| \leq Ce^{-(\delta - \varepsilon)|k_i|}. \]

This implies

\begin{equation}
(53) \quad |u(m_i)| \leq Ce^{-\frac{1}{2}(\delta - \varepsilon)|k_i|},
\end{equation}

or

\begin{equation}
(54) \quad |u(m_i + 1) - u(m_i - 1)| \leq Ce^{-\frac{1}{2}(\delta - \varepsilon)|k_i|}.
\end{equation}

If (53) holds, by (49), we also have

\begin{equation}
(55) \quad |u(m_i + 1) + u(m_i - 1)| \leq Ce^{-\frac{1}{2}(\delta - \varepsilon)|k_i|}.
\end{equation}
Putting (53) and (55) together, we get
\[(56) \quad ||\Phi(m_i) + \Phi_i(m_i)|| \leq Ce^{-\frac{1}{2}(\delta-\epsilon)|k_i|}.\]
If (53) holds, we have
\[(57) \quad ||\Phi(m_i) - \Phi_i(m_i)|| \leq Ce^{-\frac{1}{2}(\delta-\epsilon)|k_i|}.\]
Thus in case 1 there exists \(\iota \in \{-1, 1\}\) such that
\[||\Phi(m_i) + \iota\Phi_i(m_i)|| \leq Ce^{-\frac{1}{2}(\delta-\epsilon)|k_i|}.\]
Let \(T_i^1\) and \(T_i^2\) be the transfer matrices with the potentials \(V\) and \(V_i\) respectively, taking \(\Phi(m_i), \Phi_i(m_i)\) to \(\Phi(0), \Phi_i(0)\).
By (20), (45), the usual uniform upper semi-continuity and telescoping, one has
\[||T_i^1||, ||T_i^2|| \leq C e^{(L(E)+\epsilon)m_i}.\]
and
\[||T_i^1 - T_i^2|| \leq C e^{(L(E)-2\delta+\epsilon)m_i}.\]
Then
\[\|\Phi(0) + \iota\Phi_i(0)|| = ||T_i^1\Phi(m_i) + \iota T_i^2\Phi_i(m_i)||
= ||T_i^1\Phi(m_i) + \iota T_i^2\Phi_i(m_i) - \iota T_i^1\Phi_i(m_i) - \iota T_i^2\Phi_i(m_i)||
\leq ||T_i^1||\|\Phi(m_i) + \iota\Phi_i(m_i)|| + ||T_i^1 - T_i^2||\|\Phi_i(m_i)||
\leq e^{-(\delta-L(E)-\epsilon)m_i} + e^{(L(E)-2\delta+\epsilon)m_i}
\leq e^{-(\delta-L(E)-\epsilon)m_i}.\]
This implies \(||\Phi(0)|| - ||\Phi(2m_i + 1)|| \to 0\). This is impossible because \(u \in \ell^2(\mathbb{Z}).\)
Case 2. \(k_i\) is odd. Let \(\bar{m}_i = \frac{k_i - 1}{2}\), then
\[\Phi(\bar{m}_i + 1) = \left(\begin{array}{c} u(\bar{m}_i + 1) \\ u(\bar{m}_i) \end{array}\right); \Phi_i(\bar{m}_i + 1) = \left(\begin{array}{c} u(\bar{m}_i) \\ u(\bar{m}_i + 1) \end{array}\right).\]
Applying (52) with \(n = \bar{m}_i\), we have
\[|u(\bar{m}_i) + u(\bar{m}_i + 1)||u(\bar{m}_i) - u(\bar{m}_i + 1)| \leq Ce^{-(\delta-\epsilon)|k_i|}.\]
This implies
\[|u(\bar{m}_i) + u(\bar{m}_i + 1)| \leq Ce^{-\frac{1}{2}(\delta-\epsilon)|k_i|},\]
or
\[|u(\bar{m}_i + 1) - u(\bar{m}_i)| \leq Ce^{-\frac{1}{2}(\delta-\epsilon)|k_i|}.\]
Thus in case 2, there also exists \(\iota \in \{-1, 1\}\) such that
\[||\Phi(\bar{m}_i + 1) + \iota\Phi_i(\bar{m}_i + 1)|| \leq Ce^{-\frac{1}{2}(\delta-\epsilon)|k_i|}.\]
Thus by the arguments of the case 1, we can also get a contradiction. \(\square\)
4.2. Lower bound on the eigenfunctions. Now we turn to the proof of the left inequality in (7). Our key argument for the lower bound is

Lemma 4.3. Suppose for some \( k > 0 \) and \( 0 < t < \ln \lambda \),

\[
\|2\theta + k\alpha\| = e^{-tk}.
\]

Then for any \( \varepsilon > 0 \), we must have for large \( k \),

\[
\|U(k)\| \geq e^{-(\ln \lambda - t + \varepsilon)k}.
\]

Proof. We let \( \hat{\phi}(n) = \phi(k - n) \), \( V(n) = 2\lambda \cos 2\pi(\theta + n\alpha) \) and \( \hat{V}(n) = 2\lambda \cos 2\pi(\theta + (k - n)\alpha) \). Then by the assumption, one has for all \( n \in \mathbb{Z} \),

\[
|V(n) - \hat{V}(n)| \leq C e^{-tk}.
\]

We also have

\[
\phi(n + 1) + \phi(n - 1) + V(n)u(n) = E\phi(n)
\]

and

\[
\hat{\phi}(n + 1) + \hat{\phi}(n - 1) + \hat{V}(n)\hat{\phi}(n) = E\hat{\phi}(n).
\]

Let

\[
\hat{U}(n) = \left( \begin{array}{c} \hat{\phi}(n) \\ \hat{\phi}(n - 1) \end{array} \right).
\]

Suppose for some small \( \sigma > 0 \),

\[
\|U(k)\| \leq e^{-(\ln \lambda - t + \varepsilon)k}.
\]

By Lemma 3.4 and (36) \( (k_0 = k, y = n, y' = 2n) \), we have for any \( k \leq |n| \leq Ck \),

\[
\|U(n)\| \leq e^{-|n-k|} \ln \lambda e^{\varepsilon |n|} \|U(k)\| + e^{-(\ln \lambda - \varepsilon) |n|} \leq e^{-(\ln \lambda - \varepsilon) |n|} e^{(t - \sigma)k}.
\]

By Lemma 3.4 again, we have for \( |n| \leq k \),

\[
\|U(n)\| \leq \max \{ e^{-|n|} \ln \lambda, e^{-|n-k|} \ln \lambda \|U(k)\| \} e^{\varepsilon k} + e^{-(\ln \lambda - \varepsilon) |n|} \leq e^{-|n|} \ln \lambda e^{\varepsilon k} + e^{-(2k - |n|) \ln \lambda} e^{(t - \sigma + \varepsilon)k}.
\]

This implies for \( |n| \leq Ck \),

\[
|\hat{\phi}(n)||\phi(n)| = |\phi(k - n)||\phi(n)| \leq e^{-(\ln \lambda - t + \sigma - \varepsilon)k} + e^{-(\ln \lambda - \varepsilon)k}.
\]

By a standard calculation using (59), (61) and (62), we have for any \( |n| \leq C|k| \),

\[
|W(\phi, \hat{\phi})(n) - W(\phi, \hat{\phi})(n - 1)| \leq |V(n) - \hat{V}(n)||\phi(n)||\hat{\phi}(n)| + e^{-tk} |\phi(n)||\hat{\phi}(n)| + e^{-(\ln \lambda + \sigma' - \varepsilon)k}.
\]
where $\sigma' = \min\{\sigma, t\}$. This implies for any $0 < m \leq Ck$ and $|n| \leq Ck$,

$$|W(\phi, \hat{\phi})(n + m) - W(\phi, \hat{\phi})(n - 1)| \leq \sum_{j=0}^{m-1} e^{-(\ln \lambda + \sigma' - \varepsilon)k} \leq e^{-(\ln \lambda + \sigma' - \varepsilon)k}. \tag{63}$$

By (28), for some $n_0 = Ck$, we must have

$$|\phi(n_0)|, |\phi(n_0 - 1)| \leq e^{-(\ln \lambda - \delta - \varepsilon)n_0} \leq e^{-(\ln \lambda + \sigma')k}.$$

This implies,

$$|W(\phi, \hat{\phi})(n_0)| \leq e^{-(\ln \lambda + \sigma')k}.$$

Combining with (63), we must have that

$$|W(\phi, \hat{\phi})(n)| \leq e^{-(\ln \lambda + \sigma' - \varepsilon)k} \tag{64}$$

holds for all $|n| \leq Ck$.

Now we split $k$ into odd or even to discuss the problem.

Case 1. $k$ is even. Let $m = \frac{k}{2}$, then

$$U(m) = \begin{pmatrix} \phi(m) \\ \phi(m - 1) \end{pmatrix}; \hat{U}(m) = \begin{pmatrix} \phi(m) \\ \phi(m + 1) \end{pmatrix}.$$

Applying (64) with $n = m - 1$, we have

$$|\phi(m)||\phi(m + 1) - \phi(m - 1)| \leq e^{-(\ln \lambda + \sigma' - \varepsilon)k}.$$

This implies

$$|\phi(m)| \leq e^{-\frac{1}{2}(\ln \lambda + \sigma' - \varepsilon)k}, \tag{65}$$

or

$$|\phi(m + 1) - \phi(m - 1)| \leq e^{-\frac{1}{2}(\ln \lambda + \sigma' - \varepsilon)k}. \tag{66}$$

If (65) holds, by (61), we also have

$$|\phi(m + 1) + \phi(m - 1)| \leq e^{-\frac{1}{2}(\ln \lambda + \sigma' - \varepsilon)k}. \tag{67}$$

Putting (65) and (67) together, we get

$$||U(m) + \hat{U}(m)|| \leq e^{-\frac{1}{2}(\ln \lambda + \sigma' - \varepsilon)k}. \tag{68}$$

If (66) holds, we have

$$||U(m) - \hat{U}(m)|| \leq e^{-\frac{1}{2}(\ln \lambda + \sigma' - \varepsilon)k}. \tag{69}$$

Thus in case 1 there exists $\iota \in \{-1, 1\}$ such that

$$||U(m) + \iota\hat{U}(m)|| \leq e^{-\frac{1}{2}(\ln \lambda + \sigma' - \varepsilon)k}.$$

Let $T$ and $\hat{T}$ be the transfer matrices associated to potentials $V$ and $\hat{V}$, taking $U(m), \hat{U}(m)$ to $U(0), \hat{U}(0)$ correspondingly.

By (20), (60), the usual uniform upper semi-continuity and telescoping, one has

$$||T||, ||\hat{T}|| \leq e^{(\ln \lambda + \varepsilon)m}.$$
and
\[ \|T - \hat{T}\| \leq e^{(\ln \lambda - 2t + \varepsilon)m}. \]
By the right inequality of (7) (\( \ell = m, x_0 = k \)), it is easy to see that
\[ (70) \quad \|\hat{U}(m)\| \leq e^{-(\ln \lambda - \varepsilon)m}. \]
Then, as in (58), we have
\[ \|U(0) + i\hat{U}(0)\| \leq \|T\|\|U(m) + i\hat{U}(m)\| + ||T - \hat{T}||\|\hat{U}(m)\| \]
\[ \leq e^{(\ln \lambda + \varepsilon)m}e^{-\frac{1}{2}(\ln \lambda + \sigma' - \varepsilon)k} + e^{(\ln \lambda - 2t + \varepsilon)m}e^{-m \ln \lambda}. \]
This implies \( ||U(0)|| - ||U(2m + 1)|| \to 0 \). This is impossible because \( \phi \in \ell^2(\mathbb{Z}) \).

Case 2. \( k \) is odd. Let \( \tilde{m} = \frac{k - 1}{2} \), then
\[ U(\tilde{m} + 1) = \begin{pmatrix} \phi(\tilde{m} + 1) \\ \phi(\tilde{m}) \end{pmatrix}; \tilde{U}(\tilde{m} + 1) = \begin{pmatrix} \phi(\tilde{m}) \\ \phi(\tilde{m} + 1) \end{pmatrix}. \]
Combining with (64), we have
\[ |\phi(\tilde{m}) + \phi(\tilde{m} + 1)||\phi(\tilde{m}) - \phi(\tilde{m} + 1)| \leq e^{-(\ln \lambda + \sigma' - \varepsilon)k}. \]
This implies
\[ |\phi(\tilde{m}) + \phi(\tilde{m} + 1)| \leq e^{-\frac{1}{2}(\ln \lambda + \sigma' - \varepsilon)k} \]
or
\[ |\phi(\tilde{m} + 1) - \phi(\tilde{m})| \leq e^{-\frac{1}{2}(\ln \lambda + \sigma' - \varepsilon)k}. \]
Thus in case 2, there also exists \( \iota \in \{-1, 1\} \) such that
\[ ||U(\tilde{m} + 1) + i\hat{U}(\tilde{m} + 1)|| \leq Ce^{-(\ln \lambda + \sigma' - \varepsilon)k}. \]
As before, we also get a contradiction. \( \Box \)

**Proof of the left inequality of (7)**

*Proof.* The left inequality of (7) already follows except for Subcase ii in the proof of Theorem 3.1, by Remark 3.8.

Thus we only need to consider the case when \( \eta \geq \gamma = \frac{\varepsilon}{C} \). Letting \( t = \eta \frac{\ell}{|x_0|} \) and \( k = x_0 \) in Lemma 4.3, we obtain
\[ ||U(x_0)|| \geq e^{-(\ln \lambda + \varepsilon)|x_0|} e^{\eta|\ell|}. \]
Combining with (39), this completes the proof. \( \Box \)

5. **Universal reflective hierarchical structure**

We first present the local version of Theorem 1.2. The definition of \( f(\ell) \) in Theorem 1.2 depends on \( \theta \) and \( \alpha \). Thus sometimes we will write \( f_{\alpha, \theta}(\ell) \) to make clear what \( \theta \) is used.
Theorem 5.1. Fix $\delta$ with $0 < \delta < \ln \lambda$. Suppose $\alpha$ is Diophantine. Let $\varepsilon > 0$ be small enough. Then there exists $L_0 = L_0(\lambda, \alpha, \delta, C)$ such that if for all $k$ with $L_1 \leq |k| \leq C|\ell|$, 
\begin{equation}
||2\theta + 2s_0\alpha + k\alpha||_{\mathbb{R}/\mathbb{Z}} \geq e^{-(\delta + \varepsilon)|k|},
\end{equation}
and the solution of $H\phi = E\phi$ satisfies \textsuperscript{[5]} for all $k$ with $|k - s_0| \leq C|\ell|$ and $||U(s_0)|| = 1$, where $C = C(\alpha, \delta, \lambda)$ is a large constant and $L_0 \leq L_1 \leq \frac{|\ell|}{C}$, then the following statement holds:

Let $x_0$ (we can choose any one if $x_0$ is not unique) be such that 
$$|\sin \pi(2\theta + 2s_0\alpha + x_0\alpha)| = \min_{|x| \leq 2|\ell|} |\sin \pi(2\theta + 2s_0\alpha + x\alpha)|.$$ 
Then if $|x_0| \geq L_1$, we have
\begin{equation}
 f_{\alpha, \theta + s_0\alpha}(\ell)e^{-|\ell|} \leq ||U(\ell)|| \leq f_{\alpha, \theta + s_0\alpha}(\ell)e^{\varepsilon|\ell|}.
\end{equation}
If $|x_0| \leq L_1$, we have
\begin{equation}
e^{-\ln \lambda|\ell|}e^{-\varepsilon|\ell|} \leq ||U(\ell)|| \leq e^{-\ln \lambda|\ell|}e^{\varepsilon|\ell|}.
\end{equation}

Proof. Case 1: $|x_0| \geq L_1$. In sections 3 and 4, we completed the proof of Theorem \textsuperscript{1.2}. It is immediate that if we shift the operator by $s_0$ units and replace the definition of the generalized eigenfunctions $\phi$ with the assumption of \textsuperscript{[6]} only on the scale $C|\ell|$, our arguments will hold for \textsuperscript{[72]} directly. In order to avoid repetition, we omit the proof.

Case 2: $|x_0| \leq L_1$. \textsuperscript{[73]} follows directly from Lemma \textsuperscript{3.4} by shifting the operator $s_0$ units.

Remark 5.2. In order to obtain \textsuperscript{[72]}, we only need condition \textsuperscript{[71]} on scale $\frac{|\ell|}{C} \leq |k| \leq C|\ell|$ and condition \textsuperscript{[6]} on scale $|k| \leq C|\ell|$. Moreover, if we assume for $|k| \leq L_1$, 
\begin{equation}
||2\theta + 2s_0\alpha + k\alpha||_{\mathbb{R}/\mathbb{Z}} \geq e^{-(\delta + \varepsilon)|k|},
\end{equation}
then \textsuperscript{[73]} and \textsuperscript{[72]} imply
\begin{equation}
f_{\alpha, \theta + s_0\alpha}(\ell)e^{-\varepsilon|\ell|} \leq ||U(\ell)|| \leq f_{\alpha, \theta + s_0\alpha}(\ell)e^{\varepsilon|\ell|}.
\end{equation}
in both cases.

We will now prove Theorem \textsuperscript{1.3}

Theorem 5.3. Fix $\varsigma_1 > 0$, $0 < \delta < \ln \lambda$ and $s_0 \in \mathbb{Z}$. Then there exists a constant $L_0 = L_0(\alpha, \lambda, \delta, \varsigma_1)$ such that the following statement holds. Let $L_1 \geq L_0$. Suppose $K$ satisfies $|K| \geq CL_1$ and 
\begin{equation}
||2\theta + 2s_0\alpha + K\alpha||_{\mathbb{R}/\mathbb{Z}} \leq e^{-\varsigma_1|K|},
\end{equation}
and for all $k$ with $L_1 \leq |k| \leq C|K|
\begin{equation}
||2\theta + 2s_0\alpha + k\alpha||_{\mathbb{R}/\mathbb{Z}} \geq e^{-(\delta + \varepsilon)|k|},
\end{equation}
\footnote{We omit the dependence on $\varepsilon$ whenever $\varepsilon$ is (implicitly) present in the statement.}
and $s_0$ is a $CK$-local maximum, where $C = C(\alpha, \lambda, \delta, \varsigma_1)$ is a large constant. Then there exists a $3\varsigma_1 \frac{3}{4\ln \lambda}$-local maximum $b_K$ such that
\begin{equation}
|b_K - K - s_0| \leq 2L_1.
\end{equation}

**Proof.** By shifting the operator, we can assume $s_0 = 0$. Let $\epsilon$ be such that
\[ ||2\theta + 2s_0\alpha + K\alpha||_{\mathbb{R}/\mathbb{Z}} = e^{-\epsilon |K|}. \]
Then $\varsigma_1 \leq \epsilon \leq \delta + \epsilon$.

By Theorem 5.1 with $\ell = x_0 = K$, one has
\begin{equation}
\sup_{||} = \frac{||U(s_0 + K)||}{||U(s_0)||} \leq e^{-\epsilon \ln \lambda - \epsilon |K|}. 
\end{equation}

By Theorem 5.1 again, one has
\begin{equation}
\sup_{||} = \sup_{||} \leq e^{-\epsilon \ln \lambda - \epsilon |K|}. 
\end{equation}

Thus there exists a $3\varsigma_1 \frac{3}{4\ln \lambda}$-local maximum $b_K$ such that
\begin{equation}
|b_K - K| \leq \epsilon |K|. 
\end{equation}

Suppose (78) does not hold. Then there exists $k_0$ with $2L_1 \leq |k_0| \leq \epsilon K$ such that
\begin{equation}
||U(K + k_0)|| = \sup_{||} = \sup_{||}. 
\end{equation}

where $L_1$ is such that (76), (77) hold.

Case 1. $\min_{|k| \leq 2|k_0|} ||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}} \geq e^{-\epsilon |k_0|}$.

Let $\frac{b_n}{q_n}$ be the continued fraction expansion of $\alpha$. For $\gamma > 0$ (we will let $\gamma = \frac{\epsilon}{C}$), let $n$ be the largest integer such that
\[ 2q_n \leq \gamma |k_0|, \]
and let $s$ be the largest positive integer such that $2sq_n \leq \gamma |k_0|$.

Construct intervals $I_1 = [sq_n, sq_n - 1]$ and $I_2 = [K + k_0 - sq_n, K + k_0 + sq_n - 1]$.

**Claim 1:** We have
\begin{equation}
\min_{i, i' \in I_1 \cup I_2} ||2\theta + (i + i')\alpha||_{\mathbb{R}/\mathbb{Z}} \geq e^{-\epsilon |k_0|}
\end{equation}

and for any $i \neq i'$, $i, i' \in I_1 \cup I_2$,
\begin{equation}
||i - i'\alpha||_{\mathbb{R}/\mathbb{Z}} \geq e^{-\epsilon |k_0|}.
\end{equation}

By Theorem 5.2 and the DC condition on $\alpha$, $\{i_0 \mid i_0 \in I_1 \cup I_2\}$ is $\epsilon$-uniform. Combining with Lemma 2.4, there exists some $i_0$ with $i_0 \in I_1 \cup I_2$ such that $\theta_{i_0} \notin A_{4sq_n - 1 \ln \lambda - \epsilon \varsigma}$. By Lemma 3.5, $i_0$ cannot be in $I_1$ so must be in $I_2$. Set $I = [i_0 - 2sq_n + 1, i_0 + 2sq_n - 1] = [x_1, x_2]$. By (22), (23) and (24) again, one has
\[ |G_{I}(K + k_0, x_i)| \leq e^{\ln \lambda + \epsilon)(4sq_n - 1 - |K + k_0 - x_i|)(4sq_n - 1)(\ln \lambda + \epsilon) \leq e \epsilon sq_n e^{-|K + k_0 - x_i| \ln \lambda}. \]

$93/4$ can be replaced with $1 - \epsilon$ for any $\epsilon > 0$.

$10$ $s_0$ is a local maximum so that $\tilde{C}$ in (90) is 1, thus the largeness in Theorem 5.1 does not depend on $\tilde{C}$.
Notice that $|K + k_0 - x_1|, |K + k_0 - x_2| \geq s^2q_n - 1$. By [25] and [82],

$$|\phi(K + k_0)| \leq e^{-(ln \lambda - \varepsilon)sq_n} (|\phi(x_1)| + |\phi(x_0)|) \leq e^{-(ln \lambda - \varepsilon)sq_n} ||U(K + k_0)||.$$

Similarly,

$$|\phi(K + k_0 - 1)| \leq e^{-(ln \lambda - \varepsilon)sq_n} ||U(K + k_0)||.$$

The last two inequalities imply that

$$\frac{\phi(K + k_0 - 1)}{\phi(K + k_0)} \leq e^{-(ln \lambda - \varepsilon)sq_n} ||U(K + k_0)||.$$

Since $2(s + 1)q_n \geq \gamma |k_0|$ and $|k_0| \geq 2L_1$, (85) is impossible.

Case 2. $\min_{|k| \leq 2|k_0|} ||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}} \leq e^{-\varepsilon |k_0|}$ for some $\varepsilon > 0$.

In this case, we construct, as before, intervals $I_1$ around 0 and $I_2$ around $K + k_0$.

Suppose $i \in I_1$. For $i' \in I_2$, we have

$$(86) \quad ||2\theta + (i + i')\alpha||_{\mathbb{R}/\mathbb{Z}} \geq ||(i + i' - K)\alpha||_{\mathbb{R}/\mathbb{Z}} - ||2\theta + K\alpha||_{\mathbb{R}/\mathbb{Z}}$$

and

$$(87) \quad ||(i - i')\alpha||_{\mathbb{R}/\mathbb{Z}} \geq ||2\theta + (i - i' + K)\alpha||_{\mathbb{R}/\mathbb{Z}} - ||2\theta - K\alpha||_{\mathbb{R}/\mathbb{Z}}$$

Suppose $i \in I_2$.

For $i' \in I_1$, we have

$$(88) \quad ||2\theta + (i + i')\alpha||_{\mathbb{R}/\mathbb{Z}} \geq ||(i - K + i')\alpha||_{\mathbb{R}/\mathbb{Z}} - ||2\theta + K\alpha||_{\mathbb{R}/\mathbb{Z}}$$

and

$$(89) \quad ||(i - i')\alpha||_{\mathbb{R}/\mathbb{Z}} \geq ||2\theta - (i - K - i')\alpha||_{\mathbb{R}/\mathbb{Z}} - ||2\theta + K\alpha||_{\mathbb{R}/\mathbb{Z}}$$

For $i' \in I_2$, we have

$$(90) \quad ||2\theta + (i + i')\alpha||_{\mathbb{R}/\mathbb{Z}} \geq ||-2\theta + (i - K + i' - K)\alpha||_{\mathbb{R}/\mathbb{Z}} - ||4\theta + 2K\alpha||_{\mathbb{R}/\mathbb{Z}}$$

and

$$(91) \quad ||(i - i')\alpha||_{\mathbb{R}/\mathbb{Z}} = ||(i - K - (i' - K))\alpha||_{\mathbb{R}/\mathbb{Z}}.$$
we get

\begin{align}
\|U(K + k_0)\| & \leq e^{-(\ln \lambda - \epsilon)|x_1 - K - k_0|}\|U(x_1)\| + e^{-(\ln \lambda - \epsilon)|x_2 - K - k_0|}\|U(x_2)\| \\
(93) & \leq e^{-(\ln \lambda - \epsilon)|k_0|}\|U(x_1)\| + e^{-(\ln \lambda - \epsilon)|k_0|}\|U(x_2)\| \\
(94) & \leq e^{-(\ln \lambda - \epsilon)|k_0|}\|U(K + k_0)\|,
\end{align}

where the third inequality holds because \(K + k_0\) is the local maximum. \((95)\) is also impossible for \(|k_0| \geq 2L_1\).

Case 2.2. \(|k_0 + x_0| \leq \epsilon|k_0|\). In this case, \(|x_0| \geq \frac{1}{2}|k_0| \geq L_1\) so that the condition \((77)\) holds for all \(|k| \geq |x_0|\). By the small divisor conditions \((86)\) to \((92)\) again, and following the proof of \((39)\), we get (using \((82)\))

\[\|U(K + k_0)\| \leq \|U(K + k_0)\| e^{-(\ln \lambda - \epsilon)|k_0|}.\]

This is also impossible.

\[\square\]

**Proof of Claim 1.**

**Proof.** Without loss of generality assume \(i \in I_1\). For \(i' \in I_2\), by the DC condition on \(\alpha\) we have

\[\|2\theta + (i + i')\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \|(i + i' - K)\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta + K\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-\epsilon|k_0|}\]

and

\[\|(i - i')\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \|2\theta + (i - i' + K)\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta - K\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-\epsilon|k_0|}\]

For \(i' \in I_1\), the proof is trivial. \(\square\)

**Proof of Theorem 1.3**

**Proof.** Without loss of generality, assume \(k_0 = 0\). Let \(\bar{K} = L_0(\alpha, \lambda, \delta, \varsigma)\) in Theorem 5.3.

By Theorem 5.3 with \(s_0 = 0\), \(K = K_{j_0}\), \(\varsigma_1 = \varsigma\) and \(L_1 = \bar{K}\), there exists a local \(\frac{3\varsigma}{4\ln \lambda}K_{j_0}\) maximum \(b_{j_0}\) such that \(|b_{j_0} - K_{j_0}| \leq 2\bar{K}\). Let \(b_{j_0} - K_{j_0} = b'_{j_0}\) with \(|b'_{j_0}| \leq 2\bar{K}\).

Shifting the operator \(H_{\lambda, \alpha, \theta}\) by \(b_{j_0}\) units, we get the operator \(H_{\lambda, \alpha, \theta + b_{j_0}\alpha}\). By the conditions of Theorem 1.3 \(\varsigma < \delta + \epsilon < \ln \lambda\), we have

\begin{align}
\|2(\theta + b_{j_0}\alpha) + k\alpha\|_{\mathbb{R}/\mathbb{Z}} & \geq \|2\theta - (2b'_{j_0}\alpha + k\alpha)\|_{\mathbb{R}/\mathbb{Z}} - \|4\theta + 2K_{j_0}\alpha\|_{\mathbb{R}/\mathbb{Z}} \\
& \geq \|2\theta - (2b'_{j_0}\alpha + k\alpha)\|_{\mathbb{R}/\mathbb{Z}} - 2e^{-(\varsigma + \epsilon)|K_{j_0}|} \\
& \geq e^{-(\delta + \epsilon)|k| + 2\bar{K}} \\
& \geq e^{-(\delta + \epsilon)|k|},
\end{align}

\[(96)\]
for all \( \frac{1}{2} \hat{K}^2 \leq |k| \leq \frac{\lambda}{2} \max |K_{j_0}|. \) Similarly, we have
\[
\|2(\theta + b_{j_0}\alpha) + (-K_{j_1} - 2b'_{j_0})\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \|2\theta + K_{j_1}\alpha\|_{\mathbb{R}/\mathbb{Z}} + \|4\theta + 2K_{j_0}\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta + K_{j_1}\alpha\|_{\mathbb{R}/\mathbb{Z}} - 2e^{-(\gamma + \epsilon)|K_{j_0}|} \\
\leq e^{-\frac{\lambda}{2}|K_{j_1} - 2b'_{j_0}|},
\]
(97)

By Theorem 5.3 with \( s_0 = b_{j_0}, \ K = -K_{j_1} - 2b'_{j_0}, \ s_1 = \frac{\lambda}{2}, \) and \( L_1 = \frac{1}{2} \hat{K}^2, \) we get there exists a local \( \frac{K_{j_1}}{10 \ln X} K_{j_1 - 1} \) maximum \( b_{K_{j_0}, K_{j_1}} \) such that \( |b_{j_0, j_1} - b_{j_0} - (-K_{j_1} - 2b'_{j_0})| \leq \hat{K}^2. \) This implies \( b_{j_0, j_1} = K_{j_0} - b'_{j_0} - K_{j_1} + b'_{j_1} \) with \( |b'_{j_1}| \leq \hat{K}^2. \)

Shifting the operator \( H_{\lambda, \alpha, \theta} \) by \( b_{j_0, j_1} \) units, we get the operator \( H_{\lambda, \alpha, \theta + b_{j_0, j_1} \alpha} \). Thus we have
\[
\|2(\theta + b_{j_0, j_1} \alpha) + k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \|2\theta - 2b'_{j_0} \alpha + 2b'_{j_1} \alpha + k\alpha\|_{\mathbb{R}/\mathbb{Z}} - 2\|2\theta + K_{j_0} \alpha\|_{\mathbb{R}/\mathbb{Z}} - 2\|2\theta + K_{j_1} \alpha\|_{\mathbb{R}/\mathbb{Z}} \\
\geq \|2\theta + (-2b'_{j_0} + 2b'_{j_1} + k) \alpha\|_{\mathbb{R}/\mathbb{Z}} - 4e^{-(\gamma + \epsilon)|K_{j_1}|} \\
\geq e^{-(\gamma + \epsilon)(|k| + 2\hat{K} + 2\hat{K}^3)} \\
\geq e^{-(\gamma + \epsilon)|k|} 
\]
(98)

for all \( \frac{1}{2}(\hat{K} + \hat{K}^2) \hat{K} \leq |k| \leq \frac{\lambda}{2} \max |K_{j_1} - 1| \). Similarly, we have
\[
\|2(\theta + b_{j_0, j_1} \alpha) + (K_{j_1} - 2b'_{j_0} - 2b'_{j_1} - 2b_{j_2} \alpha)\|_{\mathbb{R}/\mathbb{Z}} \leq \|2\theta + K_{j_2} \alpha\|_{\mathbb{R}/\mathbb{Z}} + 2e^{-(\gamma + \epsilon)|K_{j_1}|} \\
\leq e^{-\frac{\lambda}{2}|K_{j_2} + 2b'_{j_0} - 2b'_{j_1}|}.
\]
(99)

By Theorem 5.3 with \( s_0 = b_{j_0, j_1}, \ K = K_{j_2} + 2b'_{j_0} - 2b'_{j_1}, \ s_1 = \frac{\lambda}{2}, \) and \( L_1 = \frac{1}{2}(\hat{K}^2 + \hat{K}^3), \) we get that there exists a local \( \frac{K_{j_1}}{10 \ln X} K_{j_2} \) maximum \( b_{j_0, j_1, j_2} \) such that \( b_{j_0, j_1, j_2} = K_{j_0} + b'_{j_0} - K_{j_1} - b'_{j_1} + K_{j_2} + b'_{j_2} \) with \( |b'_{j_2}| \leq \hat{K}^2 + \hat{K}^3. \)

Define \( a_n = \hat{K}^2(\hat{K} + 1)^{n-2} \) for \( n \geq 2 \) and \( a_1 = \hat{K} \). Then \( a_n = \hat{K} \sum_{i=1}^{n-1} a_i. \) Notice that by (11)
\[
\sum_{i=0}^{s} \|2\theta + K_{j_1} \alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \sum_{i=0}^{s} e^{-(\gamma + \epsilon)|K_{j_1}|} \leq 2e^{-(\gamma + \epsilon)|K_{j_1}|}.
\]
(100)

We will prove that for any \( 1 \leq s \leq k \) there exists a local \( \frac{K_{j_1}}{10 \ln X} K_{j_2} \) maximum \( b_{j_0, j_1, \ldots, j_s} \) such that
\[
b_{j_0, j_1, \ldots, j_s} = \sum_{i=0}^{s} (-1)^i K_{j_1} + (-1)^{s+1} b'_{j_i}
\]
(101)

with \( |b'_{j_i}| \leq a_{i+1} \) by induction in \( s. \)

Assume that (101) holds for \( s \). We will prove that (101) holds for \( s + 1. \)

Shifting the operator \( H_{\lambda, \alpha, \theta} \) by \( b_{j_0, j_1, \ldots, j_s} \) units, we get the operator \( H_{\lambda, \alpha, \theta + b_{j_0, j_1, \ldots, j_s} \alpha} \). Arguing as in (98) we have
\[
\|2(\theta + b_{j_0, j_1, \ldots, j_s} \alpha) + k\alpha\|_{\mathbb{R}/\mathbb{Z}} \\
\geq \|2\theta + (2 \sum_{i=0}^{s} (-1)^{i+1} b'_{j_i}) \alpha + (-1)^{s+1} k\alpha\|_{\mathbb{R}/\mathbb{Z}} - 2 \sum_{i=0}^{s} \|2\theta + K_{j_i} \alpha\|_{\mathbb{R}/\mathbb{Z}}
\]
\[
\geq ||2\theta + (2 \sum_{i=0}^{s} (-1)^{i+1} b_{j_i}')\alpha + (-1)^{s+1} k\alpha||_{\mathbb{R}/\mathbb{Z}} - 2 \sum_{i=0}^{s} e^{-(\delta+\varepsilon)|K_{j_i}|} \\
\geq e^{-(\delta+\varepsilon)(|k|+2\sum_{i=1}^{s+1} a_i)} \\
\geq e^{-(\delta+\varepsilon)|k|}
\]
for all \(\frac{1}{2}a_{s+2} \leq |k| \leq \frac{\varepsilon}{\ln \lambda}|K_{j_s}|\), since \(\sum_{i=1}^{s+1} a_i = \frac{1}{K} a_{s+2}\). Similarly to (97), we have
\[
||2(\theta + b_{j_0,j_1,\ldots,j_s}\alpha) + ((-1)^{s+1} K_{j_{s+1}} + 2 \sum_{i=0}^{s} (-1)^{s+i+1} b_{j_i}')\alpha||_{\mathbb{R}/\mathbb{Z}} \\
\leq ||2\theta + K_{j_{s+1}}\alpha||_{\mathbb{R}/\mathbb{Z}} + 4e^{-(\delta+\varepsilon)|K_{j_1}|} \\
\leq e^{-\frac{\delta}{2}||(-1)^{s+1} K_{j_{s+1}} + 2 \sum_{i=0}^{s} (-1)^{s+i+1} b_{j_i}'||_{\mathbb{R}/\mathbb{Z}}}.
\]
By Theorem 5.3 with \(s_0 = b_{j_0,j_1,\ldots,j_s}, K = (-1)^{s+1} K_{j_{s+1}} + 2 \sum_{i=0}^{s} (-1)^{s+i+1} b_{j_i}'\), \(s_1 = \frac{3}{4}\varepsilon\) and \(L_1 = \frac{1}{2}a_{s+2}\), we get that there exists a local \(\frac{9}{10}K_{j_{s+1}}\) maximum \(b_{j_0,j_1,\ldots,j_{s+1}}\) such that
\[
b_{j_0,j_1,\ldots,j_{s+1}} = b_{j_0,j_1,\ldots,j_s} + (-1)^{s+1} K_{j_{s+1}} + 2 \sum_{i=0}^{s} (-1)^{s+i+1} b_{j_i}' + b_{j_{s+1}}' \\
= \sum_{i=0}^{s-1} (-1)^i K_{j_i} + (-1)^{s-1} b_{j_i}'.
\]
with \(|b_{j_{s+1}}'| \leq a_{s+2}\).
By the fact
\[
|b_{j_0,j_1,\ldots,j_s} - \sum_{i=0}^{s} (-1)^i K_{j_i}| \leq \sum_{i=0}^{s} |b_{j_i}'| \\
\leq \sum_{i=1}^{s+1} a_i \\
\leq (K + 1)^{s+1},
\]
we complete the proof of I of Theorem 1.3.

Now we start to prove II of Theorem 1.3. Fix some \(0 \leq s \leq k\). Let us consider a local \(\frac{9}{10}K_{j_{s}}\) maximum \(b_{j_0,j_1,\ldots,j_s}\) and shift the operator by \(b_{j_0,j_1,\ldots,j_s}\) units. We get the operator \(H_{\lambda,\alpha,\theta+b_{j_0,j_1,\ldots,j_s}\alpha}\). As in (102), we also have
\[
||2(\theta + b_{j_0,j_1,\ldots,j_s}\alpha) + k\alpha||_{\mathbb{R}/\mathbb{Z}} \\
\leq ||2\theta + (2 \sum_{i=0}^{s} (-1)^{i+1} b_{j_i}')\alpha + (-1)^{s+1} k\alpha||_{\mathbb{R}/\mathbb{Z}} + 2 \sum_{i=0}^{s} e^{-(\delta+\varepsilon)|K_{j_i}|} \\
\]
for all \(a_{s+2} \leq |k| \leq \frac{\varepsilon}{\ln \lambda}|K_{j_s}|\).
Actually, the definition of \( f(\ell) \) in Theorems 1.2 and 5.1 depends on \( \theta \) and \( \alpha \). Thus we will use \( f_{\alpha, \theta}(\ell) \) with \( |\ell| \geq C_{s+2} \). Let \( \ell_0 \) be such that
\[
\sin \pi(2\theta + 2b_{j_0, j_1, \ldots, j_s} \alpha + \ell_0 \alpha) = \min_{|x| \leq 2|\ell|} |\sin \pi(2\theta + 2b_{j_0, j_1, \ldots, j_s} \alpha + x\alpha)|.
\]
By (102) and (105), we have for \( |\ell_0| \geq a_{s+2} \),
\[
(106) \quad e^{-|\ell|}f_{\alpha, \theta}((-1)^{s+1} \ell) \leq f_{\alpha, \theta} + b_{j_0, j_1, \ldots, j_s} \alpha(\ell) \leq f_{\alpha, \theta}((-1)^{s+1} \ell)e^{\varepsilon|\ell|}.
\]
If \( |\ell_0| \leq a_{s+2} \), we have
\[
(107) \quad e^{-|\ell|}e^{-\ln \lambda|\ell|} \leq f_{\alpha, \theta}(\ell) \leq e^{-\ln \lambda|\ell|}e^{\varepsilon|\ell|},
\]
since \( |\ell| \geq C_{s+2} \).

Let \( x_s = x - b_{j_0, j_1, \ldots, j_s} \). If \( |x_s| < [Ca_{s+2} \frac{1}{\ln \lambda}|K_{j_s}|] \), II of Theorem 1.3 follows from Theorem 5.1 and (106) and (107).

If \( |x_s| \in \left[ \frac{1}{16 \ln \lambda}|K_{j_s}|, \frac{3}{16 \ln \lambda}|K_{j_s}| \right] \), II of Theorem 1.3 follows from Lemma 3.4 and the fact that \( b_{j_0, j_1, \ldots, j_s} \) is a local maximum. Notice that in this case
\[
e^{-\varepsilon|x_s|}e^{-\ln \lambda|x_s|} \leq f_{\alpha, \theta}((-1)^{s+1} x_s) \leq e^{-\ln \lambda|x_s|}e^{\varepsilon|x_s|}.
\]

\( \square \)

6. ASYMPTOTICS OF THE TRANSFER MATRICES

**Proof of Theorem 1.5**

Proof. Without loss of generality, we consider \( \ell > 0 \). First assume \( x_0 < 0 \) or \( \eta < 1 < \frac{x_0}{8} \). By Theorem 1.2 in those cases, one has
\[
||U(\ell)|| \leq e^{-(\ln \lambda - \varepsilon)\ell}.
\]
By (17), we have
\[
||A_\ell|| \geq ||U(\ell)||^{-1} \geq e^{(\ln \lambda - \varepsilon)\ell}.
\]
Combining with (20), the proof follows.

Now we turn to the proof of the case when \( x_0 > 0 \) and \( \eta > \gamma \). We will assume \( \ell > 0 \) is large enough. By (15), one has \( x_0 > 0 \) is large enough. Thus below we always assume \( x_0 \) is large.

**Theorem 6.1.** Assume \( jx_0 \leq k < (j + 1)x_0 \) with \( k \geq \frac{x_0}{8} \), where \( j = 0, 1 \). Then we have
\[
(108) \quad ||A_k|| \leq \max\{e^{-|k-jx_0|\ln \lambda}||A_{jx_0}||, e^{-|k-(j+1)x_0|\ln \lambda}||A_{(j+1)x_0}||\}e^{k},
\]
\[
(109) \quad ||A_k|| \geq \max\{e^{-|k-jx_0|\ln \lambda}||A_{jx_0}||, e^{-|k-(j+1)x_0|\ln \lambda}||A_{(j+1)x_0}||\}e^{-\varepsilon k}.
\]

**Proof.** Apply (37) with \( k_0 = x_0, y = k, y' = 2x_0 \) and \( \varphi = \psi \). We have for \( jx_0 \leq k < (j + 1)x_0 \) with \( k \geq \frac{x_0}{8} \),
\[
(110) \quad ||\hat{U}(k)|| \leq \max\{e^{-|k-jx_0|\ln \lambda}||\hat{U}(jx_0)||, e^{-|k-(j+1)x_0|\ln \lambda}||\hat{U}((j+1)x_0)||\}e^{\varepsilon k}.
\]

By Last-Simon’s arguments ((8.6) in [32]), one has
\[
(111) \quad ||A_k|| \geq ||A_k \hat{U}(0)|| \geq c||A_k||.
\]
Then (108) holds by (111) and (110).

(109) holds directly by (20).

**Lemma 6.2.** For any $2x_0 \leq k \leq Cx_0$,

$$e^{-\varepsilon x_0}||A_{2x_0}||e^{\ln \lambda |k-2x_0|} \leq ||A_k|| \leq e^{\varepsilon x_0}||A_{2x_0}||e^{\ln \lambda |k-2x_0|}$$

**Proof.** The right inequality holds directly. It suffices to show the left inequality.

By (39) and noting $t \leq \delta + \varepsilon$, we have

$$||\tilde{U}(x_0)|| \leq \max\{e^{-(\ln \lambda - \delta - \varepsilon)x_0}||\tilde{U}(0)||, e^{-(\ln \lambda - \delta - \varepsilon)x_0}||\tilde{U}(2x_0)||\}$$

Clearly, $||\tilde{U}(x_0)|| \leq e^{-(\ln \lambda - \delta - \varepsilon)x_0}||\tilde{U}(0)||$ can not happen. Otherwise, by the fact $||U(x_0)|| \leq e^{-(\ln \lambda - \delta - \varepsilon)x_0}||U(0)||$, we must have

$$|\phi(x_0)\psi(x_0 - 1) - \phi(x_0 - 1)\psi(x_0)| \leq e^{-(\ln \lambda - \delta - \varepsilon)x_0}.$$

This contradicts (29).

Thus we must have

(112) $$||\tilde{U}(x_0)|| \leq e^{-(\ln \lambda - \delta - \varepsilon)x_0}||\tilde{U}(2x_0)||.$$

Lemma holds directly if $k \leq 2x_0 + \frac{\varepsilon}{2}x_0$. If $k - 2x_0 \geq \frac{\varepsilon}{2}x_0$, by (37) again ($k_0 = x_0, y = 2x_0, y' = k, \gamma = \frac{\varepsilon}{2}$), one has

$$||\tilde{U}(2x_0)|| \leq \max\{e^{-(\ln \lambda - \varepsilon)x_0}||\tilde{U}(x_0)||, e^{-(\ln \lambda - \varepsilon)|k-2x_0|}||\tilde{U}(k)||\}.$$

Combining with (112), we must have

$$||\tilde{U}(k)|| \geq e^{(\ln \lambda - \varepsilon)|k-2x_0|}||\tilde{U}(2x_0)||.$$

Combining with (111), we get the left inequality.

**Lemma 6.3.** The following holds

(113) $$e^{(\ln \lambda - \varepsilon)x_0} \leq ||A_{x_0}|| \leq e^{(\ln \lambda + \varepsilon)x_0},$$

(114) $$e^{(\ln \lambda - \varepsilon)2x_0}e^{-\eta' \lambda} \leq ||A_{2x_0}|| \leq e^{(\ln \lambda + \varepsilon)2x_0}e^{-\eta' \lambda}.$$

**Proof.** We first prove (113). The right inequality holds by (20) directly. Thus it suffices to show the left one. By (37), for any $\frac{\varepsilon}{8} \leq k < x_0$, one has

$$||U(k)|| \leq \max\{e^{-k\ln \lambda}, e^{-|k-x_0|\ln \lambda}||U(x_0)||\}e^{\varepsilon k}.$$

Clearly

(115) $$||A_k|| \geq ||U(k)||^{-1},$$

thus by (108), we must have for any $\frac{x_0}{8} \leq k < x_0$,

(116) $$\max\{e^{-k\ln \lambda}, e^{-|k-x_0|\ln \lambda}||A_{x_0}||\}e^{\varepsilon k} \geq \left(\max\{e^{-k\ln \lambda}, e^{-|k-x_0|\ln \lambda}||U(x_0)||\}\right)^{-1}e^{-\varepsilon k}.$$ 

Recall that by (46) and (59),

(117) $$e^{-(\ln \lambda - \eta' + \varepsilon)x_0} \leq ||U(x_0)|| \leq e^{-(\ln \lambda - \eta' - \varepsilon)x_0},$$
where $\eta' = \frac{\ell}{x_0} \eta$. Let

$$k_0 = x_0 - \frac{\eta'}{2 \ln \lambda} x_0.$$ 

One has $k_0 \geq \frac{x_0}{2}$, thus by (117)

$$\max\{e^{-k_0} e^{-|k_0-x_0|} \ln |U(x_0)|\} \leq e^{-(\ln \lambda - \eta') x_0} e^{\varepsilon k_0}.$$ 

Combining with (116), one has

$$\max\{e^{-k_0} e^{-|k_0-x_0|} \ln |A_{x_0}|\} \geq e^{-(\ln \lambda - \eta') x_0} e^{-\varepsilon k_0}.$$ 

This implies

$$|A_{x_0}| \geq e^{(\ln \lambda - \varepsilon) x_0}.$$ 

Now we start to prove (114). By (7) ($\ell = 2 x_0$), one has

$$e^{-(\ln \lambda + \varepsilon) 2 x_0} e^{\eta' x_0} \leq |U(2 x_0)| \leq e^{-(\ln \lambda - \varepsilon) 2 x_0} e^{\eta' x_0},$$ 

Combining with (115), one has

$$|A_{2 x_0}| \geq e^{(\ln \lambda - \varepsilon) 2 x_0} e^{-\eta' x_0}.$$ 

Thus it remains to prove the right inequality of (114). By (8.5) and (8.7) in [37] we have

$$|A_k U(0)|^2 \leq |A_k|^2 m(k)^2 + |A_k|^{-2},$$ 

where

$$m(k) \leq C \sum_{p=k}^{\infty} \frac{1}{|A_p|^2}.$$ 

If $k \geq C x_0$ ($C$ may depend on $\ln \lambda, \delta$), by Theorem 1.2 we have

$$|A_k| \geq |U(k)|^{-1} \geq e^{(\ln \lambda - \delta - \varepsilon) k}$$

and by (20) we have

$$|A_{2 x_0}| \leq e^{(\ln \lambda + \varepsilon) 2 x_0}.$$ 

Combining with (121), we have

$$|A_k| \geq |A_{2 x_0}| e^{\ln \lambda - \varepsilon - k}.$$ 

for $k \geq C x_0$, where $C$ is large enough.

If $2 x_0 \leq k \leq C x_0$, by Lemma 6.2 we have

$$|A_k| \geq |A_{2 x_0}| e^{(\ln \lambda - \varepsilon) k} e^{-\varepsilon x_0}.$$ 

Thus by (120), (122) and (123), we have

$$m(2 x_0) \leq |A_{2 x_0}|^{-2} e^{\varepsilon x_0}.$$ 

Let $k = 2 x_0$ in (119). Then

$$|U(2 x_0)| \leq e^{\varepsilon x_0}.$$
Thus by (118), we obtain
\[
||A_{2x_0}|| \leq e^{(2\ln \lambda - \eta' - \varepsilon)x_0}.
\]
□

Theorem 1.5 for the remaining case ($\eta \geq \gamma = \frac{\varepsilon}{2}$ and $x_0 > 0$) now follows directly from Theorem 6.1 and Lemma 6.2, 6.3.

**Proof of Corollary 1.6**

Proof. The Corollary follows from Theorem 1.5 and (111).
□

**Proof of Corollary 1.7**

Proof. i) and ii) of Corollary 1.7 follow from Theorem 1.5 and Corollary 1.6 directly.

Fix some small $\varepsilon_1, \varepsilon_2 > 0$. By the definition of $\delta$, there exists a sequence $n_j$ (assume $n_j > 0$ for simplicity) such that
\[
e^{-\delta + \varepsilon_1)n_j} \leq ||2\theta + n_j\alpha||_{\mathbb{R}/\mathbb{Z}} \leq e^{-\frac{\delta}{2}n_j}.
\]
By the Diophantine condition on $\alpha$, we have
\[
n_{j+1} \geq e^{\frac{n_j}{\tau_1'}}.
\]

We prove (15) first. By Theorem 1.2, one has for any $|k| \in [\varepsilon_2n_{j+1}, \frac{n_{j+1}}{2}]$,
\[
||U(k)|| \leq e^{-(\ln \lambda - \varepsilon_1)|k|}.
\]
This implies (15) by the arbitrariness of $\varepsilon_1, \varepsilon_2$.

Now we turn to the proof of (16). By Theorem 1.5, one has for any $|k| \in [\varepsilon_2n_{j+1}, n_{j+1}]$,
\[
||A_k|| \geq e^{(\ln \lambda - \varepsilon_1)|k|}.
\]
This implies (16). □

**Appendix A. Uniformity**

The following lemma is critical when we prove Theorem 3.2.

**Lemma A.1.** (Lemma 9.7, [4]) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $x \in \mathbb{R}$ and $0 \leq k_0 \leq q_n - 1$ be such that $|\sin \pi(x + k_0\alpha)| = \inf_{0 \leq k \leq q_n-1} |\sin \pi(x + k\alpha)|$, then for some absolute constant $C > 0$,
\[
(126) \quad -C \ln q_n \leq \sum_{k=0, k \neq k_0}^{q_n-1} \ln |\sin \pi(x + k\alpha)| + (q_n - 1) \ln 2 \leq C \ln q_n.
\]

**Proof of Theorem 3.2**

Proof. Let $i_0, j_0 \in I_1 \cup I_2$ be such that $|\sin \pi(2\theta + (i_0 + j_0)\alpha)| = \min_{i,j \in I_1 \cup I_2} |\sin \pi(2\theta + (i + j)\alpha)|$. By the Diophantine condition on $\alpha$, there exist $\tau', \kappa' > 0$ such that for any $i + j \neq i_0 + j_0$ and $i, j \in I_1 \cup I_2$,
\[
(127) \quad |\sin \pi(2\theta + (i + j)\alpha)| \geq \frac{\tau'}{(sq_n)^{\kappa'}}.
\]
Also for all \( i, j \in I_1 \cup I_2, i \neq j \), we have
\[
(128) \quad |\sin \pi(j - i)\alpha| \geq \frac{\tau'}{(sq_n)^2}.
\]

In (126), let \( x = \cos 2\pi a, k = sq_n - 1 \) and take the logarithm. Then
\[
\ln \prod_{j \in I_1 \cup I_2, j \neq i} \frac{|\cos 2\pi a - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \theta_j|} = \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j| - \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi \theta_i - \cos 2\pi \theta_j|.
\]

First, we estimate \( \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j| \). Obviously,
\[
\sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j|
= \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\sin \pi(a + \theta_j)| + \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\sin \pi(a - \theta_j)| + (sq_n - 1) \ln 2
= \Sigma_+ + \Sigma_- + (sq_n - 1) \ln 2.
\]

Both \( \Sigma_+ \) and \( \Sigma_- \) consist of \( s \) terms of the form of (126), plus \( s \) terms of the form
\[
\ln \min_{j=0,1,\ldots,q_n} |\sin \pi(x + j\alpha)|,
\]
minus \( \ln |\sin \pi(a + \theta_i)| \). Thus, using (126) \( s \) times for \( \Sigma_+ \) and \( \Sigma_- \) respectively, one has
\[
(129) \quad \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j| \leq -sq_n \ln 2 + C s \ln q_n.
\]

If \( a = \theta_i \), we obtain
\[
\sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi \theta_i - \cos 2\pi \theta_j|
= \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\sin \pi(\theta_i + \theta_j)| + \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\sin \pi(\theta_i - \theta_j)| + (sq_n - 1) \ln 2
= \Sigma_+ + \Sigma_- + (sq_n - 1) \ln 2,
\]
where
\[
\Sigma_+ = \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\sin \pi(2\theta + (i + j)\alpha)|,
\]
and
\[
\Sigma_- = \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\sin \pi(i - j)\alpha|.
\]

We will estimate \( \Sigma_+ \). Set \( J_1 = [1, s_1] \) and \( J_2 = [s_1 + 1, s] \), which are two adjacent disjoint intervals of length \( s_1, s_2 \) respectively. Then \( I_1 \cup I_2 \) can be represented as a disjoint union of segments \( B_j, j \in J_1 \cup J_2 \), each of length \( q_n \). Applying (126) to each \( B_j \), we obtain
\[
(131) \quad \Sigma_+ \geq -sq_n \ln 2 + \sum_{j \in J_1 \cup J_2} \ln |\sin \pi \theta_j| - C s \ln q_n - \ln |\sin 2\pi(\theta + i\alpha)|,
\]
where
\begin{equation}
|\sin \pi \hat{\theta}_j| = \min_{\ell \in B_j} |\sin \pi (2\theta + (\ell + i)\alpha)|.
\end{equation}
By (30) and (127), we have
\begin{equation}
\sum_{j \in J_1 \cup J_2} \ln |\sin \pi \hat{\theta}_j| \geq -\gamma sq_n - Cs \ln sq_n.
\end{equation}
Putting (133) in (131), we get
\begin{equation}
\Sigma_+ \geq -sq_n \ln 2 - \gamma sq_n - Cs \ln sq_n.
\end{equation}
Similarly, replacing (30), (127) with (128), and arguing as in the proof of (134), we obtain,
\begin{equation}
\Sigma_- > -sq_n \ln 2 - Cs \ln sq_n.
\end{equation}
From (130), (134) and (135), one has
\begin{equation}
\sum_{j \in I_1 \cup I_2, j \neq i} |\cos 2\pi \theta_i - \cos 2\pi \theta_j| \geq -sq_n \ln 2 - \gamma sq_n - Cs \ln sq_n.
\end{equation}
By (129) and (136), we have
\begin{equation}
\max_{i \in I_1 \cup I_2} \prod_{j \in I_1 \cup I_2, j \neq i} \frac{|x - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \theta_j|} < e^{sq_n(\gamma + C \ln sq_n)}.
\end{equation}
By the assumption \( s \leq q_n^C \), we get for any \( \varepsilon > 0 \) and large \( n \),
\begin{equation}
\max_{i \in I_1 \cup I_2} \prod_{j \in I_1 \cup I_2, j \neq i} \frac{|x - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \theta_j|} < e^{sq_n(\gamma + \varepsilon)}.
\end{equation}
This completes the proof. \( \square \)

**Appendix B. Block Expansion Theorem**

**Proof of Theorem 3.3**

*Proof.* For any \( \hat{y} \in [y_1 + \gamma k, y_2 - \gamma k] \), by the assumption we have there exists an interval \( I(\hat{y}) = [x_1, x_2] \subset [y_1, y_2] \) such that \( \hat{y} \in I(\hat{y}) \) with \( \frac{\gamma}{20} k \leq |I(\hat{y})| \leq \frac{\gamma}{2} \text{dist} (y, \{y_1, y_2\}) \), and
\begin{equation}
\text{dist}(\hat{y}, \partial I(\hat{y})) \geq \frac{1}{40} |I(\hat{y})| \geq \frac{\gamma}{800} k
\end{equation}
and
\begin{equation}
|G_{I(\hat{y})}(\hat{y}, x_i)| \leq e^{-\tau |\hat{y} - x_i|}, \ i = 1, 2,
\end{equation}
where \( \{x_1, x_2\} = \partial I(\hat{y}) \) is the boundary of the interval \( I(\hat{y}) \). For \( z \in \partial I(\hat{y}) \), let \( z' \) be the neighbor of \( z \) (i.e., \( |z - z'| = 1 \)) not belonging to \( I(\hat{y}) \).
If \( x_2 + 1 \leq y_2 - \gamma k \) or \( x_1 - 1 \geq y_1 + \gamma k \), we can expand \( \varphi(x_2 + 1) \) or \( \varphi(x_1 - 1) \) using (25). We can continue this process until we arrive to \( z \) such that \( z + 1 > y_2 - \gamma k \) or \( z - 1 < y_1 + \gamma k \), or the numbers of iterations reach \( \lfloor \frac{1600}{\gamma} \rfloor \). Then, by (25)

\[
\varphi(y) = \sum_{s: z_{i+1} \in B(z') I(z')} G_I(y)(k, z_1) G_I(z') (z_1', z_2) \cdots G_I(z') (z_s', z_{s+1}) \varphi(z'_{s+1}),
\]

(139)

where in each term of the summation one has \( y_1 + \gamma k + 1 \leq z_i \leq y_2 - \gamma k - 1, i = 1, \ldots, s \), and either \( z_{s+1} \notin [y_1 + \gamma k + 1, y_2 + \gamma k - 1] \), \( s + 1 < \lfloor \frac{1600}{\gamma} \rfloor \); or \( s + 1 = \lfloor \frac{1600}{\gamma} \rfloor \). We should mention that \( z_{s+1} \in [y_1, y_2] \).

If \( z_{s+1} \in [y_1, y_1 + \gamma k] \), \( s + 1 < \lfloor \frac{1600}{\gamma} \rfloor \), this implies

\[
|\varphi(z'_{s+1})| \leq r_{y_1}^\varphi.
\]

By (135), we have for such terms

\[
|G_I(y)(k, z_1) G_I(z') (z_1', z_2) \cdots G_I(z') (z_s', z_{s+1}) \varphi(z'_{s+1})|
\]

\[
\leq r_{y_1}^\varphi e^{-\tau(y-z_1) + \sum_{i=1}^{s} |z_i' - z_{i+1}|}
\]

\[
\leq r_{y_1}^\varphi e^{-\tau(y-z_{s+1}) - (s+1)}
\]

\[
\leq r_{y_1}^\varphi e^{-\tau(y - y_1 - \gamma k - \frac{1600}{\gamma})}.
\]

(140)

If \( z_{s+1} \in [y_2 - \gamma k, y_2] \), \( s + 1 < \lfloor \frac{1600}{\gamma} \rfloor \), by the same arguments, we have

\[
|G_I(y)(k, z_1) G_I(z') (z_1', z_2) \cdots G_I(z') (z_s', z_{s+1}) \varphi(z'_{s+1})| \leq r_{y_2}^\varphi e^{-\tau(y-y_2) - \gamma k - \frac{1600}{\gamma}}.
\]

(141)

If \( s + 1 = \lfloor \frac{1600}{\gamma} \rfloor \), using (137) and (138), we obtain

\[
|G_I(y)(k, z_1) G_I(z') (z_1', z_2) \cdots G_I(z') (z_s', z_{s+1}) \varphi(z'_{s+1})| \leq e^{-\frac{\gamma}{\gamma} \frac{1600}{\gamma} k \frac{1600}{\gamma} |\varphi(z'_{s+1})|}.
\]

(142)

Notice that the total number of terms in (139) is at most \( 2^{1600 \gamma} \) and \( |y - y_1|, |y - y_2| \geq 10 \gamma k \). By (140), (141) and (142), we have

\[
|\varphi(y)| \leq \max \{ r_{y_1}^\varphi e^{-\tau(y-y_1 - 3 \gamma k)}, r_{y_2}^\varphi e^{-\tau(y-y_2 - 3 \gamma k)} \}, \max_{p \in [y_1, y_2]} \{ e^{\tau k} |\varphi(p)| \}.
\]

(143)

Now we will show that for any \( p \in [y_1, y_2] \), one has \( |\varphi(p)| \leq \max \{ r_{y_1}^\varphi, r_{y_2}^\varphi \} \). Then (143) implies Theorem 3.3. Otherwise, by the definition of \( r_{y_1}^\varphi \) and \( r_{y_2}^\varphi \), if \( |\varphi(p')| \) is the largest one of \( |\varphi(z)|, z \in [y_1 + 10 \gamma k + 1, y_2 - 10 \gamma k - 1] \), then \( |\varphi(p')| > \max \{ r_{y_1}^\varphi, r_{y_2}^\varphi \} \).

Applying (143) to \( \varphi(p') \) and noticing that \( |p' - y_1|, |p' - y_2| \geq 10 \gamma k \), we get

\[
|\varphi(p')| \leq \max \{ e^{-\tau \gamma k} r_{y_1}^\varphi, e^{-\tau \gamma k} r_{y_2}^\varphi, e^{-\tau k} |\varphi(p')| \}.
\]

This is impossible because \( |\varphi(p')| > \max \{ r_{y_1}^\varphi, r_{y_2}^\varphi \} \).
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