On the spectral radius of block graphs having all their blocks of the same size

Cristian M. Conde∗, Ezequiel Dratman †, and Luciano N. Grippo ‡

1 Instituto de Ciencias, Universidad Nacional de General Sarmiento, Argentina
2 Consejo Nacional de Investigaciones Científicas y Tecnica, Argentina
3 Instituto Argentina de Matemática "Alberto Calderón" - Consejo Nacional de Investigaciones Científicas y Tecnica, Argentina

Abstract

Let $\mathcal{B}(n, q)$ be the class of block graphs on $n$ vertices having all its blocks of size $q+1$ with $q \geq 2$. In this article we prove that the maximum spectral radius $\rho(G)$, among all graphs $G \in \mathcal{B}(n, q)$, is reached at a unique graph. We profit from this fact to present an tight upper bound for $\rho(G)$. We also prove that if $G$ has at most three pairwise adjacent cut vertices then the minimum $\rho(G)$ is attained at a unique graph. Likewise, we present a lower bound for $\rho(G)$ when $G \in \mathcal{B}(n, q)$.

1 Introduction

The problem of finding those graphs that maximize or minimize the spectral radius of a connected graph on $n$ vertices, within a given graph class $\mathcal{H}$, have attracted the attention of many researchers. Usually, this kind of problems are solved by means of graphs transformations preserving the number of vertices, so that the resulting graph also belongs to $\mathcal{H}$, and having a monotone behavior respect to the spectral radius. We refer to the reader to [12] for more details about this and other techniques. In [9], Lovász and Pelikán prove that the unique graph with maximum spectral radius among the trees on $n$ vertices is the star $K_{1,n-1}$ and the unique graph with minimum spectral radius is the path $P_n$. As far as
we know, this article is the first one within this research line. Since adding edges to a graph increases the spectral radius (see Corollary 1), if $H$ contains complete graphs and paths, then $K_n$ maximizes and $P_n$ minimizes $\rho(G)$ among graphs in $H$, meaning that this two graphs have the minimum and maximum spectral radius among graphs on $n$ vertices when $H$ is the class of all connected graphs. Consequently, several authors have considered the problem when $H$ is a graph class not containing either paths or complete graphs and defined by certain restriction of classical graph parameters. Graphs with a given independence number [10, 6], graphs with a given clique number [13] and graphs with given connectivity and edge-connectivity [7]. It is worth mentioning that the foundation stone that gives place to many late articles in connection with this problem is that of Brualdi and Solheid [3].

About our statement in connection with Lovász an Pelikán result

For concepts and definitions used in this section we referred the reader to Section 2.

In this article we consider the class $B(n,q)$ of block graphs on $n$ vertices having all their blocks on $q + 1$ vertices, for every $q \geq 2$. For results related to the adjacency matrix of block graphs we refer to the reader to [2]. Trees are block graphs with all their blocks on two vertices. In connection with the spectral radius on trees it was obtained the following result.

**Theorem 1.1.** [9] If $T$ is a tree on $n$ vertices, then $2 \cos \left( \frac{\pi}{n+1} \right) = \rho(P_n) \leq \rho(T) \leq \rho(K_{1,n-1}) = \sqrt{n-1}$.

In an attempt to generalize Theorem 1.1 we find the unique graph that reaches the maximum spectral radius in $B(n,q)$ and the unique graph that reaches the minimum spectral radius but in $B(n,q)$ in the case in which the graph has at most three pairwise adjacent cut vertices. We also present for the spectral radius a lower bound and a tight upper bound.

**Theorem 1.2.** If $G \in B(n,q)$, then $\rho(G) \leq \rho(S(n,q))$ and $S(n,q)$ is the unique graphs that maximizes the spectral radius. In addition, if $G$ has at most three pairwise adjacent cut vertices then $\rho(P^b_q) \leq \rho(G)$ and $P^b_q$ is the unique graph that minimizes the spectral radius in the class $B(n,q)$, where $b = \frac{n-1}{q}$.

We have strongly evidence obtained by the aid of Sage software that the hypothesis of having at most three pairwise adjacent cut vertices, in connection with the minimum of the spectral radius, can be dropped.

Organization of the article

This article is organized as follows. In Section 2 we present some definitions and preliminary results. In Section 3 are presented two graph transformations having
a monotone behavior with respect to the spectral radius. Section 4 is devoted to put all previous results together in order to prove our main result. In Section 5, a tight upper bound and a lower bound for the spectral radius are presented. Finally, Section 6 contains a short summary of our work and two conjectures are posted.

2 Preliminaries

2.1 Definitions

All graphs, mentioned in this article, are finite, have no loops and multiple edges.

Let $G$ be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of $G$, respectively. A graph on one vertex is called trivial graph. Let $v$ be a vertex of $G$, $N_G(v)$ (resp. $N_G[v]$) stands for the neighborhood of $v$ (resp. $N_G(v) \cup \{v\}$), if the context is clear the subscript $G$ is omitted. We use $d_G(v)$ to denote the degree of $v$ in $G$, or $d(v)$ provided the context is clear. By $\overline{G}$ we denote the complement graph of $G$. Given a set $F$ of edges of $G$ (resp. of $G$), we denote by $G - F$ (resp. $G + F$) the graph obtained from $G$ by removing (resp. adding) all the edges in $F$. If $F = \{e\}$, we use $G - e$ (resp. $G + e$) for short.

Let $X \subseteq V(G)$, we use $G[X]$ to denote the graph induced by $X$. By $G - X$ we denote the graph $G[V(G) \setminus X]$. If $X = \{v\}$, we use $G - v$ for short. Let $G$ and $H$ two graphs, we use $G + H$ to denote the disjoint union between $G$ and $H$, and $G^+$ stands for the graph obtained by adding an isolated vertex to $G$. Let $A, B \subseteq V(G)$ we said that $A$ is complete to (resp. anticomplete to) $B$ if every vertex in $A$ is adjacent (resp. nonadjacent) to every vertex of $B$. We denote by $P_n$ and $K_n$ to the path and the complete graph on $n$ vertices.

We denote by $A(G)$ the adjacency matrix of $G$, and $\rho(G)$ stands for the spectral radius of $A(G)$, we refer to $\rho(G)$ as the spectral radius of $G$. If $x$ is the principal eigenvector of $A(G)$, which is indexed by $V(G)$, we use $x_u$ to denote the coordinate of $x$ corresponding to the vertex $u$. We use $P_G(x)$ to denote the characteristic polynomial of $A(G)$; i.e., $P_G(x) = \det(xI_n - A(G))$. It is easy to prove that $P_{Kn}(x) = (x - n + 1)(x + 1)^{n-1}$.

A vertex $v$ of a graph $G$ is a cut vertex if $G - v$ has a number of connected components greater than the number of connected components of $G$. Let $H$ be a graph. A block of $H$, also known as 2-connected component, is a maximal connected subgraph of $H$ having no cut vertex. A block graph is a connected graph whose blocks are complete graphs. We use $\mathcal{B}(n, q)$ to denote the family of block graphs on $n$ vertices whose blocks have $q + 1$ vertices. Notice that if $B \in \mathcal{B}(n, q)$ and $b$ is its number of blocks then $b = \frac{n - 1}{q}$. Let $G$ be a block graph, a leaf block is a block of $G$ such that contains exactly one cut vertex of $G$. We use $S(n, q)$ to denote the block graphs in $\mathcal{B}(n, q)$ having $b$ blocks with only one cut vertex. By $P^n_b$ we denote the block graph in $\mathcal{B}(bq + 1, q)$ with at most two leaf blocks when $n - 1 > q$ and no cut vertices when $n - 1 = q$, called $(q, b)$-path-block.
2.2 Preliminaries results

This subsection is split into two parts. In the first one we present the results needed to deal with the minimum spectral radius in $B(n, q)$, and in the second one we developed the tools used to prove the result in connection with the maximum spectral radius in this class.

Tools for finding the minimum

We will introduce a partial order on the class of graphs. We will use it to deal with the graph transformations used to prove our main result. This technique was pioneered by Lovász and Pelikán [9].

Definition 1. Let $G$ and $H$ be two graphs. We denote by $G \prec H$, if $P_H(x) > P_G(x)$ for all $x \geq \rho(G)$.

It is immediate that if $G \prec H$ then $\rho(H) < \rho(G)$. The spectrum radius is nondecreasing respect to the subgraph partial order.

We repeatedly use the following Lemma to deal with the subgraph partial order previously defined.

Lemma 2.1. If $H$ is a proper subgraph of $G$ then $\rho(H) < \rho(G)$.

The reader is referred to [1] for a proof of the above lemma. In particular, adding edges to a graph increases the spectral radius.

Corollary 1. If $G$ is a graph such that $uv \notin E(G)$, then $\rho(G) < \rho(G + uv)$.

The following technical lemma is a useful tool to develop graph transformations.

Lemma 2.2. [8] If $H$ is a spanning subgraph of the graph $G$ then $P_G(x) \leq P_H(x)$ for all $x \geq \rho(G)$. In addition, if $G$ is connected then $G \prec H$.

Let $G$ and $H$ be two graphs. If $g \in V(G)$ and $h \in V(H)$, the coalescence between $G$ and $H$ at $g$ and $h$, denoted $G \cdot^h_g H$, is the graph obtained from $G$ and $H$, by identifying vertices $g$ and $h$ (see Fig. 1). We use $G \cdot H$ for short. Notice that any block graph can be constructed by recursively using the coalescence operation between a block graph and a complete graph.

In the 70s Schwenk published an article containing useful formulas for the characteristic polynomial of a graph [11]. The part corresponding to minimizing
the spectral radius of the main result of this research is based on the following Schwenk’s formula, linking the characteristic polynomial of two graphs and the coalescence between them.

**Lemma 2.3.** Let $G$ and $H$ be two graphs. If $g \in V(G)$, $h \in V(H)$, and $F = G \cdot H$, then

$$P_F(x) = P_G(x)P_{H-h}(x) + P_{G-g}(x)P_H(x) - xP_{G-g}(x)P_{H-h}(x).$$

More details on Lemmas 2.2 and 2.3 can be found in [5]. The following two technical lemmas will play an important role in order to prove the main result of this article.

**Lemma 2.4.** Let $H$ be a graph, let $v, w \in V(H)$ such that $H - w \prec H - v$ and let $G$ be a connected graph. If $G_1$ and $G_2$ are the graphs obtained by means of the coalescence between $G$ and $H$ at $u \in V(G)$, and $v$ or $w$ respectively, then $G_1 \prec G_2$.

**Proof.** By Lemma 2.3, the characteristic polynomial of $G_1$ and $G_2$ are

$$P_{G_1}(x) = P_{G-u}(x)P_H(x) + (P_G(x) - xP_{G-u}(x))P_{H-v}(x)$$

and

$$P_{G_2}(x) = P_{G-u}(x)P_H(x) + (P_G(x) - xP_{G-u}(x))P_{H-w}(x)$$

respectively, and thus

$$P_{G_2}(x) - P_{G_1}(x) = (P_G(x) - xP_{G-u}(x))(P_{H-w}(x) - P_{H-v}(x)). \tag{1}$$

By Lemmas 2.1 and 2.2 $G \prec (G - u)^+$. Therefore, since $H - w \prec H - v$, by Lemma 2.1 and (1) we have $G_1 \prec G_2$. \qed

**Lemma 2.5.** Let $H_1, H_2$ be two graphs such that either $H_1 = H_2$ or $H_1 \prec H_2$, let $v_i \in V(H_i)$ for each $i = 1, 2$ such that $H_2 - v_2 \prec H_1 - v_1$, and let $G$ be a connected graph. If $G_i$ is the graph obtained by means of the coalescence between $G$ and $H_i$ at $v \in V(G)$ and $v_i$ for each $i = 1, 2$, then $G_1 \prec G_2$.

**Proof.** By applying Lemma 2.3 as in Lemma 2.4 we obtain

$$P_{G_2}(x) - P_{G_1}(x) = (P_G(x) - xP_{G-v}(x))(P_{H_2-v_2}(x) - P_{H_1-v_1}(x)) + (P_{H_2}(x) - P_{H_1}(x))P_{G-v}(x). \tag{2}$$

By Lemmas 2.1 and 2.2 $G \prec (G - v)^+$. Therefore, since either $H_1 = H_2$ or $H_1 \prec H_2$ and $H_2 - v_2 \prec H_1 - v_1$, by (2) and Lemma 2.1 we conclude that $G_2 \prec G_1$. \qed
Tools for finding the maximum

In the following lemma we consider a set of vertices \( u_1, \ldots, u_\ell \) of a graph \( G \), where \( x_i \) stands for \( x_{u_i} \), the corresponding coordinate of \( u_i \) in the principal eigenvector, for every \( 1 \leq i \leq \ell \).

**Proposition 1.** [10, 4] Let \( G \) be a connected graph and let \( u_1, \ldots, u_k, u_{k+1}, \ldots, u_\ell \) vertices of \( G \) such that \( \sum_{i=1}^k x_i \leq \sum_{i=k+1}^{\ell} x_i \), and let \( W \subseteq V(G) \setminus \{u_1, \ldots, u_\ell\} \). If \( \{u_1, \ldots, u_k\} \) is complete to \( W \) and \( \{u_{k+1}, \ldots, u_\ell\} \) is anticomplete to \( W \), then \( \rho(G) < \rho(G - \{wu_i : w \in W \text{ and } 1 \leq i \leq k\} + \{wu_i : w \in W \text{ and } k+1 \leq i \leq \ell\}) \).

Proposition 1 has also shown to be of help to find the unique graph with maximum spectral radius of block graphs with prescribed independence number [4].

3 Graph transformations

To ease the reading of the next proposition we recommend to see Fig. 2.

**Proposition 2.** Let \( G \) be a connected graph, and let \( u \in V(G) \) such that \( G - u \) is connected. Let \( H \) be the graph obtained from \( S(k(q - 1) + 1, k) \) by adding for all \( 1 \leq i \leq k \) one pendant \((q,b_i)\)-path-block (possible empty, i.e., \( b_i = 0 \)) to each leaf block, let \( v \in V(H) \) be the vertex of degree \( k(q - 1) \), and let \( w \in V(H) \) be any vertex in leaf block of \( H \). If \( H_1 \) is the graph obtained by the coalescence between \( G \) and \( H \) at \( u \) and \( v \), \( H_2 \) is the graph obtained by the coalescence between \( G \) and \( H \) at \( u \) and \( w \), then \( H_1 \prec H_2 \).

**Proof.** Observe that \( H - w \) is connected and \( H - v \) is a disconnected spanning subgraph of \( H - w \). Thus, by Lemma 2.2 \( H - w \prec H - v \). Therefore, the result follows from Lemma 2.4.

The following proposition play a central role to prove the main result of this article. We use \( G(r,s) \) to denote the graph obtained by means of the coalescence between \( G \) and a copy of \( K_r \) at \( u \in V(G) \) and any vertex of the complete graph, and between \( G \) and \( K_s \) at \( v \in V(G) \) and any vertex of the complete graph (see the example depicted in Fig. 3).
Figure 3: From left to right $G$, $G(4,1)$ and $G(3,3)$.

**Proposition 3.** Let $G$ be a connected graph and let $u, v \in V(G)$. If $r$ and $s$ are two integers such that $1 \leq r \leq s - 2$, $G-u \prec G-v$ or $G-u = G-v$, then $G(r,s) \prec G(r+1,s-1)$.

**Proof.** By applying Lemma 2.3 to $G(r,s)$ at $v$ we obtain

$$P_{G(r,s)}(x) = (x+1)^{s-2}[(x-s+2)P_{G(r,s)-K_{s-1}}(x) - (s-1)P_{G(r,s)-K_r}(x)].$$

Applying again Lemma 2.3 to $P_{G(r,s)-K_{s-1}}(x)$ and $P_{G(r,s)-K_r}(x)$ we obtain

$$P_{G(r,s)}(x) = (x+1)^{s+r-4}[(x-s+2)((x-r+2)P_{G(r,s)}(x) - (r-1)P_{G-u}(x)]$$

$$- (s-1)[(x-r+2)P_{G-v}(x) - (r-1)P_{G-\{u,v\}}(x)]].$$

By symmetry

$$P_{G(r+1,s-1)}(x) = (x+1)^{s+r-4}[(x-s+3)((x-r+1)P_{G(r+1,s-1)}(x) - rP_{G-u}(x)]$$

$$- (s-2)[(x-r+1)P_{G-v}(x) - rP_{G-\{u,v\}}(x)].$$

Hence

$$P_{G(r+1,s-1)}(x) - P_{G(r,s)}(x) = (x+1)^{s+r-4}[(s-r-1)[P_{G(r+1,s-1)}(x)]$$

$$+ P_{G-v}(x) + P_{G-\{u,v\}}(x)] + (x+1)(P_{G-v}(x) - P_{G-u}(x)).$$

Therefore, if $1 \leq r \leq s - 2$, by \[4\] and Lemma 2.1 $G(r,s) \prec G(r+1,s-1)$. \qed

A $(q,b)$-path-blocks in $B(n,q)$ have blocks $B_1, \ldots, B_b$ such that $V(B_i) \cap V(B_{i+1}) = \{v_i\}$ for every $1 \leq i \leq b-1$ and $V(B_i) \cap V(B_j) = \emptyset$ whenever $1 \leq i < j \leq n$ and $|i-j| > 1$. Let $G$ be a graph and let $v, w \in V(G)$ be two adjacent vertices. We use $G[q,k,\ell]$ to denote the graph obtained by adding a pendant $(q,\ell)$-path-block at $v$ and a pendant $(q,k)$-path-block at $w$, where $1 \leq \ell \leq k$, and $G[q,r,0]$ stands for the graph obtained from $G$ by adding just a $(q,r)$-path block at $w$. By “pendant at $v$” we mean identifying a noncut vertex from one of the two leaf blocks with a noncut vertex $v \in V(B_1)$ (see Fig. 4).

**Proposition 4.** Let $G \in B(n,q)$ with at least one cut vertex. If $\ell$ and $k$ are two positive integers such that $1 \leq \ell \leq k$ and $G[q,k,\ell]$ has at most three adjacent cut vertices, then $G[q,k,\ell] \prec G[q,k+1,\ell-1]$.

**Proof.** We use $B_1, \ldots, B_\ell$ and $B'_1, \ldots, B'_k$ to denote the blocks of the $(q,\ell)$-path-block, denoted $P_{\ell}$, and the $(q,k)$-path-block, denoted $P_k$, respectively, $v = v_0$, \[7\]
Analogously, applying Lemma 2.3 to \( G'[q, k, \ell] \) we derive the next identity.

\[
P_{G[q,k,\ell]}(x) = (x+1)^{q-1}((x-q+1)P_{G[q,k,\ell-1]}(x) - qP_{G'[q,k,\ell-1]}(x)). \quad (5)
\]

Analogously

\[
P_{G[q,k,\ell]}(x) = (x+1)^{q-1}((x-q+1)P_{G[q,k-1,\ell]}(x) - qP_{G'[q,k-1,\ell]}(x)). \quad (6)
\]

By combining (5) and (6) we obtain

\[
P_{G[q,k+1,\ell-1]}(x) - P_{G[q,k,\ell]}(x) = q(x+1)^{q-1}(P_{G'[q,k,\ell-1]}(x) - P_{G'[q,k-1,\ell-1]}(x)). \quad (7)
\]

Again, by using properly Lemma 2.3, we derive the next identity

\[
P_{G'[q,k,\ell]}(x) = (x+1)^{2(q-3)}((x-q+1)[(x-q+2)P_{G[q,k-1,\ell-1]}(x)
- (q-1)P_{G'[q,k-1,\ell-1]}(x)] + q((q-1)P_{G'[q,k-1,\ell-1]}(x)
- (x-q+2)P_{G'[q,k-1,\ell-1]}(x)]).
\]

Analogously,

\[
P_{G''[q,k,\ell]}(x) = (x+1)^{2(q-3)}((x-q+1)[(x-q+2)P_{G[q,k-1,\ell-1]}(x)
- (q-1)P_{G''[q,k-1,\ell-1]}(x)] + q((q-1)P_{G''[q,k-1,\ell-1]}(x)
- (x-q+2)P_{G''[q,k-1,\ell-1]}(x))].
\]

Hence

\[
P_{G'[q,k,\ell]}(x) - P_{G''[q,k,\ell]}(x) = (x+1)^{2(q-1)}(P_{G''[q,k-1,\ell-1]}(x)
- P_{G''[q,k-1,\ell-1]}(x)). \quad (8)
\]

By applying (8) repeatedly we obtain for every \( 1 \leq j \leq \ell \)

\[
P_{G'[q,k,\ell]}(x) - P_{G''[q,k,\ell]}(x) = (x+1)^{2j(q-1)}(P_{G''[q,j-1,\ell-j]}(x)
- P_{G''[q,j-1,\ell-j]}(x)). \quad (9)
\]
Replacing in (7) we obtain that for every $0 \leq j \leq \ell - 1$

\[
P_{G[q,k+1,\ell-1]}(x) - P_{G[q,k,\ell]}(x) = q(x + 1)^{(2j+1)(q-1)}(P_{G'[q,k-j,\ell-j-1]}(x) - P_{G'[q,k-\ell+1,j]}(x)).
\]  

(10)

In particular, if $j = \ell - 1$

\[
P_{G[q,k+1,\ell-1]}(x) - P_{G[q,k,\ell]}(x) = q(x + 1)^{(2\ell-1)(q-1)}(P_{(G-v)[q,k-\ell+1,0]}(x) - P_{G'[q,k-\ell+1,0]}(x)).
\]  

(11)

Thus it suffices to prove that $G'[q,t,0] \prec (G-v)[q,t,0]$ for all $t \geq 1$ (see Fig. 5).

First observe that, since $G$ one cut vertex, there exists a graph $H_1 \in \mathcal{B}(n,q)$ such that $G'[q,t,0]$ is the coalescence between $H_1$ and $P_{t+1}^q - v_t$ at a noncut vertex $x \in V(H_1)$ and a noncut vertex $v_0 \in V(B_1)$ of $P_{t+1}^q - v_t$ respectively. Analogously, $(G-v)[q,t,0]$ is the coalescence between $H_1$ and $P_{t+1}^q - v_0$ at a noncut vertex $x \in V(H_1)$ and a noncut vertex $y \in V(B_1) \{v_0\}$ of $P_{t+1}^q - v_0$ respectively. From this observation combined with Lemma 2.5 and Proposition 3 we conclude that $G'[q,t,0] \prec (G-v)[q,t,0]$.

4 Main result

Let $G \in \mathcal{B}(n,q)$ and let $B$ be a block of $G$. We say that $B$ is a special block of type one if $B$ has at least two pendant path-blocks at $v \in V(B)$ (see Fig. 2, the block whose vertex set is $\{v,x,y\}$ is a special block of type one). We say that $B$ is a special of type two if $B$ has a pendant path-block at $v \in V(B)$ and a pendant path-block at $w \in V(B)$ with $v \neq w$ (see Fig. 4, the block whose vertex set is $\{u,v,w\}$ is a special block of type two). The below lemma, whose proof is omitted, will be used to prove our main result.

**Lemma 4.1.** If $G \in \mathcal{B}(n,q)$, then $G$ either is a $(q,b)$-path-block, or has a special block of type one, or has a special block of type two.

We obtain the graph maximizes the spectral radius within $\mathcal{B}(n,q)$ by applying Proposition 1.
Theorem 4.1. If $G \in B(n,q)$, then $\rho(G) \leq \rho(S(n,q))$. Besides, $S(n,q)$ is the unique graph maximizing $\rho(G)$.

Proof. Suppose towards a contradiction that $H \in B(n,q)$ maximizes the spectral radius within $B(n,q)$ and it is not $S(n,q)$. Hence $H$ has two leaf blocks $B$ and $B'$ such that $V(B) \cap V(B') = \emptyset$. Assume that $v$ and $v'$ are their corresponding cut vertices. Hence, $x_v \geq x_{v'}$ or $x_v \leq x_{v'}$, say $x_v \geq x_{v'}$. Let $F$ be the set of edges $v'u$ with $u \in V(B') \setminus \{v\}$. By Proposition 1, $\rho(H) < \rho((H - F) \cup \{uv : u \in V(B')\})$, we reach a contradiction that arose from supposing that $\rho(H)$ is the maximum among all possible $\rho(G)$ with $G \in B(n,q)$. Therefore, if $G \in B(n,q) \setminus \{S(n,q)\}$, then $\rho(G) < \rho(S(n,q))$. \[\square\]

Now we are ready to put all pieces together in order to prove the main result of the article.

Proof of Theorem 1.2. The upper bound follows from Theorem 4.1. Assume that $G \in B(n,q)$ and it is not a path-block. By Lemma 4.1, $G$ has either a special block of type one or a special block of type two. Hence, by Propositions 2 and 4, there exists a graph transformation onto $G$, involving the corresponding pendant-path-blocks, such that the resulting graph $G'$ satisfies $\rho(G') < \rho(G)$ and $G'$ has an special block less than $G$. Therefore, continuing with this procedure as long as $G'$ is a path-block, we conclude that $\rho\left(\frac{P^q_{n-1}}{q}\right) < \rho(G)$ for all $G \in B(n,q)$. \[\square\]

5 Bounds for the spectral radius

Theorem 5.1. Let $G \in B(n,q)$ and let $n - 1 > q$. Then $\rho(G) \leq q - 1 + \sqrt{(q-1)^2 + 4(n-1)}$, $\rho(G) \geq q + \frac{\sqrt{q}}{2}$

for every $2 \leq q \leq 4$, and

$\rho(G) \geq q + \frac{4 + (q - 1)\sqrt{2}}{q + 3\sqrt{2}}$

for every $q \geq 5$. Besides, the upper bound holds if and only if $G = S(n,q)$.

Proof. Since $n - 1 > q$, the number of blocks $b$ of $G$ is at least two. By Theorem 1.2, $S(n,q)$ is the graph in $B(n,q)$ having maximum expected spectral radius. Let us call $B_i$ for each $i = 1, \ldots, b$ to the blocks of $S(n,q)$ and $v$ to its only cut vertex. By symmetry we can assume that all coordinates of the principal eigenvector corresponding to those vertices in $B_i \setminus \{v\}$ are equal to $x$ for each $1 \leq i \leq b$ and let $y$ be the coordinate corresponding to $v$. By Perron-Frobenius theorem we
can assume that $x$ and $y$ are positive real numbers. If $\rho$ is the spectral radius of $S(n,q)$, then

\[(q - 1) \ x + y = \rho x \quad \text{and} \quad bq \ x = \rho y.\]

Hence, $\rho^2 - (q - 1)\rho - n + 1 = 0$ and consequently $\rho = \frac{q - 1 + \sqrt{(q - 1)^2 + 4(n - 1)}}{2}$. By Theorem 1.2 the equality holds if and only if $G = S(n, q)$.

Assume now that $q \geq 2$ and $b \geq 3$. By Lemma 2.1 we know that $\rho(P_3^q) \leq \rho(G)$ for every graph $G \in \mathcal{B}(n, q)$. By simple calculation, using Lemma 2.3 we obtain the characteristic polynomial of $P_3^q$

\[P_{P_3^q}(x) = (x + 1)^{3q - 4} \left( (x - q)(x + 2) + 1 \right) \left( (x - q)(x + 2) + 1 \right) - 2q. \tag{12}\]

Since $(x - q)(x + 2) + 1 - 2q = 0$, we have that $\rho(P_3^q)$ is the greatest root of $f_q(x) := (x - q)(x + 2) + 1 - 2q$. Furthermore, since $f_q(x)$ is an increasing function on $(q, +\infty)$ and $f_q(q) < 0$, we have that $\rho(P_3^q)$ is the unique root of $f_q(x)$ on $(q, +\infty)$.

Using the following two facts

\[f_q \left(q + \frac{\sqrt{q}}{2}\right) = \frac{q + 4}{8} \sqrt{q} + \frac{q(q - 6)}{4} \leq 0 \quad \text{and} \quad f_q(q + 1) = 4 - q \geq 0,\]

for every $2 \leq q \leq 4$, and

\[f_q(q + 1) = 4 - q \leq 0 \quad \text{and} \quad f_q(q + \sqrt{2}) = 4 + 3\sqrt{2} \geq 0,\]

for every $q \geq 5$, and taking into account that $f_q''(x) > 0$ for every $x \in (q, +\infty)$ we conclude that

\[\rho(G) \geq q + \frac{\sqrt{q}}{2}\]

for every $2 \leq q \leq 4$, and

\[\rho(G) \geq q + \frac{4 + (q - 1)\sqrt{2}}{q + 3\sqrt{2}},\]

for every $q \geq 5$.

\[\square\]

### 6 Discussions and further research

We have found the unique graph maximizing the spectral radius among all graphs in $\mathcal{B}(n, q)$. We have presented three graphs transformations to deal with the
minimum spectral radius of this class of block graphs, namely Propositions 2, 3 and 4, but the last one has very strong hypothesis on the graph \( G \). We do not know if they can be weakened. Nevertheless, we have collected very strong computational evidence that drives us to following conjecture.

Conjecture 1. If \( G \in B(n, q) \setminus \{P^q_b\} \), then \( P^q_b \prec G \).

Consequently, if this statement were true the following weaker conjecture would be also true.

Conjecture 2. If \( G \in B(n, q) \setminus \{P^q_b\} \), then \( \rho(P^q_b) < \rho(G) \).

We believe that in for proving Conjecture 1 knew graph transformations need to be developed.

Another interesting graph class, related to that considered in this paper, to study the problem of finding the maximum and minimum spectral radius is that formed by those block graphs on \( n \) vertices having exactly \( b \) blocks not necessary all of them with the same size.

Acknowledgments

Cristian M. Conde acknowledges partial support from ANPCyT PICT 2017-2522. Ezequiel Dratman and Luciano N. Grippo acknowledge partial support from ANPCyT PICT 2017-1315.

References

[1] R. B. Bapat. *Graphs and matrices*. Universitext. Springer, London; Hindustan Book Agency, New Delhi, second edition, 2014.

[2] R. B. Bapat and S. Roy. On the adjacency matrix of a block graph. *Linear Multilinear Algebra*, 62:406–418, 2014.

[3] R. A. Brualdi and E. S. Solheid. On the spectral radius of complementary acyclic matrices of zeros and ones. *SIAM J Algebra Discrete Method*, 7:265–272, 1986.

[4] C. M. Conde, E. Dratman, and L. N. Grippo. On the spectral radius of block graphs with prescribed independence number \( \alpha \). *Linear Algebra Appl.*, in press, 2020.

[5] D. Cvetković, P. Rowlinson, and S. Simić. *Eigenspaces of graphs*, volume 66 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1997.

[6] X. Du and L. Shi. Graphs with small independence number minimizing the spectral radius. *Discrete Math. Algorithms Appl.*, 5(3):1350017, 8, 2013.
[7] J. Li, W. C. Shiu, W. H. Chan, and A. Chang. On the spectral radius of graphs with connectivity at most \( k \). J. Math. Chem., 46(2):340–346, 2009.

[8] Q. Li and K. Q. Feng. On the largest eigenvalue of a graph (chinese). Acta Math. Appl. Sinica, 2(2):167–175, 1979.

[9] L. Lovázs and J. Pelikán. On the eigenvalues of trees. Period. Math. Hungar., 3:175–182, 1973.

[10] H. Lu and Y. Lin. Maximum spectral radius of graphs with given connectivity, minimum degree and independence number. J. Discrete Algorithms, 31:113–119, 2015.

[11] A. J. Schwenk. Computing the characteristic polynomial of a graph. In Graphs and combinatorics (Proc. Capital Conf., George Washington Univ., Washington, D.C., 1973), pages 153–172. Lecture Notes in Math., Vol. 406, 1974.

[12] D. Stevanović. Spectral radius of graphs. Academic Press, 2014.

[13] D. Stevanović and P. Hansen. The minimum spectral radius of graphs with a given clique number. Electron. J. Linear Algebra, 17:110–117, 2008.