BOUNDARY REGULARITY ESTIMATES FOR NONLOCAL ELLIPTIC EQUATIONS IN $C^1$ AND $C^{1,\alpha}$ DOMAINS

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Abstract. We establish sharp boundary regularity estimates in $C^1$ and $C^{1,\alpha}$ domains for nonlocal problems of the form $Lu = f$ in $\Omega$, $u = 0$ in $\Omega^c$. Here, $L$ is a nonlocal elliptic operator of order $2s$, with $s \in (0,1)$.

First, in $C^{1,\alpha}$ domains we show that all solutions $u$ are $C^s$ up to the boundary and that $u/d^s \in C^\alpha(\Omega)$, where $d$ is the distance to $\partial\Omega$.

In $C^1$ domains, solutions are in general not comparable to $d^s$, and we prove a boundary Harnack principle in such domains. Namely, we show that if $u_1$ and $u_2$ are positive solutions, then $u_1/u_2$ is bounded and Hölder continuous up to the boundary.

Finally, we establish analogous results for nonlocal equations with bounded measurable coefficients in non-divergence form. All these regularity results will be essential tools in a forthcoming work on free boundary problems for nonlocal elliptic operators [CRS15].

1. Introduction and results

In this paper we study the boundary regularity of solutions to nonlocal elliptic equations in $C^1$ and $C^{1,\alpha}$ domains. The operators we consider are of the form

$$Lu(x) = \int_{\mathbb{R}^n} \left( \frac{u(x+y) + u(x-y)}{2} - u(x) \right) \frac{a(y/|y|)}{|y|^{n+2s}} \, dy,$$

with

$$0 < \lambda \leq a(\theta) \leq \Lambda, \quad \theta \in S^{n-1}. \quad (1.2)$$

When $a \equiv ctt$, then $L$ is a multiple of the fractional Laplacian $-(\Delta)^s$.

We consider solutions $u \in L^\infty(\mathbb{R}^n)$ to

$$\begin{cases}
Lu = f & \text{in } B_1 \cap \Omega \\
u = 0 & \text{in } B_1 \setminus \Omega,
\end{cases} \quad (1.3)$$

with $f \in L^\infty(\Omega \cap B_1)$ and $0 \in \partial\Omega$.

When $L$ is the Laplacian $\Delta$, then the following are well known results:

(i) If $\Omega$ is $C^{1,\alpha}$, then $u \in C^{1,\alpha}(\overline{\Omega} \cap B_{1/2})$.

(ii) If $\Omega$ is $C^1$, then solutions are in general not $C^{0,1}$.

Still, in $C^1$ domains one has the following boundary Harnack principle:

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(iii) If $\Omega$ is $C^1$, and $u_1$ and $u_2$ are positive in $\Omega$, with $f \equiv 0$, then $u_1$ and $u_2$ are
comparable in $\overline{\Omega} \cap B_{1/2}$, and $u_1/u_2 \in C^{0,\gamma}(\overline{\Omega} \cap B_{1/2})$ for some small $\gamma > 0$.
Actually, (iii) holds in general Lipschitz domains (for $\gamma$ small enough), or even in less regular domains; see [Dah77] [BBB91]. Analogous results hold for more general second order operators in non-divergence form $L = \sum_{i,j} a_{ij}(x) \partial_{ij} u$ with bounded measurable coefficients $a_{ij}(x)$ [BB94].

The aim of the present paper is to establish analogous results to (i) and (iii) for nonlocal elliptic operators $L$ of the form (1.1)-(1.2), and also for non-divergence operators with bounded measurable coefficients.

1.1. $C^{1,\alpha}$ domains. When $L = \Delta$ in (1.3) and $\Omega$ is $C^{k,\alpha}$, then solutions $u$ are as regular as the domain $\Omega$ provided that $f$ is regular enough. In particular, if $\Omega$ is

$C^\infty$ and $f \in C^\infty$ then $u \in C^\infty(\Omega)$.

When $L = -(-\Delta)^s$, then the boundary regularity is well understood in $C^{1,1}$ and in $C^{\infty}$ domains. In both cases, the optimal H"older regularity of solutions is

$u \in C^{s}(\overline{\Omega})$, and in general one has $u \notin C^{s+\epsilon}(\overline{\Omega})$ for any $\epsilon > 0$. Still, higher order estimates are given in terms of the regularity of $u/d^s$: if $\Omega$ is $C^\infty$ and $f \in C^\infty$ then

$u/d^s \in C^{\infty}(\overline{\Omega})$; see Grubb [Gru15] [Gru14]. Here, $d(x) = \text{dist}(x, \mathbb{R}^n \setminus \Omega)$.

We prove here a boundary regularity estimate of order $s + \alpha$ in $C^{1,\alpha}$ domains. Namely, we show that if $\Omega$ is $C^{1,\alpha}$ then $u/d^s \in C^{\alpha}(\overline{\Omega})$, as stated below.

We first establish the optimal Hölder regularity up to the boundary, $u \in C^s(\overline{\Omega})$.

**Proposition 1.1.** Let $s \in (0, 1)$, $L$ be any operator of the form (1.1)-(1.2), and $\Omega$ be any bounded $C^{1,\alpha}$ domain. Let $u$ be a solution of (1.3). Then,

$$\|u\|_{C^s(\overline{B_{1/2}})} \leq C \left( \|f\|_{L^\infty(\Omega \cap \partial \Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)} \right).$$

The constant $C$ depends only on $n$, $s$, $\Omega$, and ellipticity constants.

Our second result gives a finer description of solutions in terms of the function $d^s$, as explained above.

**Theorem 1.2.** Let $s \in (0, 1)$ and $\alpha \in (0, s)$. Let $L$ be any operator of the form (1.1)-(1.2), $\Omega$ be any $C^{1,\alpha}$ domain, and $d$ be the distance to $\partial \Omega$. Let $u$ be a solution of (1.3). Then,

$$\|u/d^s\|_{C^{\alpha}(\overline{B_{1/2} \cap \Omega})} \leq C \left( \|f\|_{L^\infty(\Omega \cap \partial \Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)} \right).$$

The constant $C$ depends only on $n$, $s$, $\alpha$, $\Omega$, and ellipticity constants.

The previous estimate in $C^{1,\alpha}$ domains was only known for the half-Laplacian $(-\Delta)^{1/2}$; see De Silva and Savin [DS14]. For more general nonlocal operators, such estimate was only known in $C^{1,1}$ domains [RS14b].

The proofs of Proposition 1.1 and Theorem 1.2 follow the ideas of [RS14b], where the same estimates were established in $C^{1,1}$ domains. One of the main difficulties in the present proofs is the construction of appropriate barriers. Indeed, while any $C^{1,1}$ domain satisfies the interior and exterior ball condition, this is not true anymore.
in $C^{1,\alpha}$ domains, and the construction of barriers is more delicate. We will need a careful computation to show that

$$|L(d^s)| \leq Cd^{\alpha-s} \text{ in } \Omega.$$ 

In fact, since $d^s$ is not regular enough to compute $L$, we need to define a new function $\psi$ which behaves like $d$ but it is $C^2$ inside $\Omega$, and will show that $|L(\psi^s)| \leq Cd^{\alpha-s}$; see Definition 2.1.

Once we have this, and doing some extra computations we will be able to construct sub and supersolutions which are comparable to $d^s$, and thus we will have

$$|u| \leq Cd^s.$$ 

This, combined with interior regularity estimates, will give the $C^s$ estimate of Proposition 1.1.

Then, combining these ingredients with a blow-up and compactness argument in the spirit of [RS14b, RS14], we will find the expansion

$$|u(x) - Q(z)d^s(x)| \leq C|x - z|^{s+\alpha}$$

at any $z \in \partial \Omega$. And this will yield Theorem 1.2.

1.2. $C^1$ domains. In $C^1$ domains, in general one does not expect solutions to be comparable to $d^s$. In that case, we establish the following.

**Theorem 1.3.** Let $s \in (0,1)$ and $\alpha \in (0,s)$. Let $L$ be any operator of the form (1.1)-(1.2), and $\Omega$ be any $C^1$ domain.

Then, there exists $\delta > 0$, depending only on $\alpha$, $n$, $s$, $\Omega$, and ellipticity constants, such that the following statement holds.

Let $u_1$ and $u_2$, be viscosity solutions of (1.3) with right hand sides $f_1$ and $f_2$, respectively. Assume that $\|f_i\|_{L^\infty(B_1\cap \Omega)} \leq C_0$ (with $C_0 \geq \delta$), $\|u_i\|_{L^\infty(\mathbb{R}^n)} \leq C_0$, $f_i \geq -\delta$ in $B_1 \cap \Omega$, and that

$$u_i \geq 0 \text{ in } \mathbb{R}^n, \quad \sup_{B_{1/2}} u_i \geq 1.$$ 

Then,

$$\|u_1/u_2\|_{C^\alpha(\Omega \cap B_{1/2})} \leq CC_0, \quad \alpha \in (0,s),$$

where $C$ depends only on $\alpha$, $n$, $s$, $\Omega$, and ellipticity constants.

We expect the range of exponents $\alpha \in (0,s)$ to be optimal.

In particular, the previous result yields a boundary Harnack principle in $C^1$ domains.

**Corollary 1.4.** Let $s \in (0,1)$, $L$ be any operator of the form (1.1)-(1.2), and $\Omega$ be any $C^1$ domain. Let $u_1$ and $u_2$, be viscosity solutions of

$$\begin{cases}
L u_1 = L u_2 = 0 & \text{in } B_1 \cap \Omega \\
u_1 = u_2 = 0 & \text{in } B_1 \setminus \Omega,
\end{cases}$$

where

$$|u_i| \leq C d^s.$$
Assume that
\[ u_1 \geq 0 \quad \text{and} \quad u_2 \geq 0 \quad \text{in} \quad \mathbb{R}^n, \]
and that \( \sup_{B_{1/2}} u_1 = \sup_{B_{1/2}} u_2 = 1 \). Then,
\[ 0 < C^{-1} \leq \frac{u_1}{u_2} \leq C \quad \text{in} \quad B_{1/2}, \]
where \( C \) depends only on \( n, s, \Omega, \) and ellipticity constants.

Theorems 1.3 and 1.2 will be important tools in a forthcoming work on free boundary problems for nonlocal elliptic operators [CRS15]. Namely, Theorem 1.3 (applied to the derivatives of the solution to the free boundary problem) will yield that \( C^1 \) free boundaries are in fact \( C^{1,\alpha} \), and then thanks to Theorem 1.2 we will get a fine description of solutions in terms of \( d^s \).

1.3. Equations with bounded measurable coefficients. We also obtain estimates for equations with bounded measurable coefficients,
\[
\begin{align*}
M^+ u &\geq -K_0 \quad \text{in} \quad B_1 \cap \Omega \\
M^- u &\leq K_0 \quad \text{in} \quad B_1 \cap \Omega \\
u &= 0 \quad \text{in} \quad B_1 \setminus \Omega.
\end{align*}
\] (1.4)

Here, \( M^+ \) and \( M^- \) are the extremal operators associated to the class \( L_* \), consisting of all operators of the form (1.1)-(1.2), i.e.,
\[
M^+ := M^+_{L_*} u = \sup_{L \in L_*} Lu, \quad M^- := M^-_{L_*} u = \inf_{L \in L_*} Lu.
\]

Notice that the equation (1.4) is an equation with bounded measurable coefficients, and it is the nonlocal analogue of
\[
a_{ij}(x)\partial_{ij} u = f(x), \quad \text{with} \quad \lambda I \leq (a_{ij}(x))_{ij} \leq \Lambda I, \quad |f(x)| \leq K_0.
\]

For nonlocal equations with bounded measurable coefficients in \( C^{1,\alpha} \) domains, we show the following.

Here, and throughout the paper, we denote \( \tilde{\alpha} = \tilde{\alpha}(n, s, \lambda, \Lambda) > 0 \) the exponent in [RS14, Proposition 5.1].

**Theorem 1.5.** Let \( s \in (0, 1) \) and \( \alpha \in (0, \tilde{\alpha}) \). Let \( \Omega \) be any \( C^{1,\alpha} \) domain, and \( d \) be the distance to \( \partial\Omega \). Let \( u \in C(B_1) \) be any viscosity solution of (1.4). Then, we have
\[
\|u/d^s\|_{C^{\alpha}(B_{1/d^s};\mathbb{R}^n)} \leq C \left( K_0 + \|u\|_{L^\infty(\mathbb{R}^n)} \right).
\]

The constant \( C \) depends only on \( n, s, \alpha, \Omega, \) and ellipticity constants.

In \( C^1 \) domains we prove:

**Theorem 1.6.** Let \( s \in (0, 1) \) and \( \alpha \in (0, \tilde{\alpha}) \). Let \( \Omega \) be any \( C^1 \) domain.

Then, there exists \( \delta > 0 \), depending only on \( \alpha, n, s, \Omega, \) and ellipticity constants, such that the following statement holds.
Let $u_1$ and $u_2$, be functions satisfying

$$
\begin{align*}
M^+(au_1 + bu_2) &\geq -\delta(|a| + |b|) & \text{in } B_1 \cap \Omega \\
u_1 = u_2 &= 0 & \text{in } B_1 \setminus \Omega,
\end{align*}
$$

for any $a, b \in \mathbb{R}$. Assume that

$$u_i \geq 0 \quad \text{in } \mathbb{R}^n,$$

$$\|u_i\|_{L^\infty(\mathbb{R}^n)} \leq C_0, \text{ and that } \sup_{B_{1/2}} u_i \geq 1. \text{ Then, we have}$$

$$\|u_1/u_2\|_{C^{\alpha}(\Omega \cap B_{1/2})} \leq C,$$

where $C$ depends only on $\alpha$, $n$, $s$, $\Omega$, and ellipticity constants.

The Boundary Harnack principle for nonlocal operators has been widely studied, and in some cases it is even known in general open sets; see Bogdan [Bog97], Song-Wu [SW99], Bogdan-Kulczycki-Kwasnicki [BKK08], and Bogdan-Kumagai-Kwasnicki [BKK15]. The main differences between our Theorems 1.3-1.6 and previous known results are the following. On the one hand, our results allow a right hand side on the equation (1.3), and apply also to viscosity solutions of equations with bounded measurable coefficients (1.4). On the other hand, we obtain a higher order estimate, in the sense that for linear equations we prove that $u_1/u_2$ is $C^\alpha$ for all $\alpha \in (0,s)$. Finally, the proof we present here is perturbative, in the sense that we make a blow-up and use that after the rescaling the domain will be a half-space. This allows us to get a higher order estimate for $u_1/u_2$, but requires the domain to be at least $C^1$.

The paper is organized as follows. In Section 2 we construct the barriers in $C^{1,\alpha}$ domains. Then, in Section 3 we prove the regularity of solutions in $C^{1,\alpha}$ domains, that is, Proposition 1.1 and Theorems 1.2 and 1.5. In Section 4 we construct the barriers needed in the analysis on $C^1$ domains. Finally, in Section 5 we prove Theorems 1.3 and 1.6.

2. Barriers: $C^{1,\alpha}$ Domains

Throughout this section, $\Omega$ will be any bounded and $C^{1,\alpha}$ domain, and

$$d(x) = \text{dist}(x, \mathbb{R}^n \setminus \Omega).$$

Since $d$ is only $C^{1,\alpha}$ inside $\Omega$, we need to consider the following “regularized version” of $d$.

Definition 2.1. Given a $C^{1,\alpha}$ domain $\Omega$, we consider a fixed function $\psi$ satisfying

$$C^{-1}d \leq \psi \leq Cd, \quad C \quad \text{and} \quad |D^2\psi| \leq Cd^\alpha - 1,$$

with $C$ depending only on $\Omega$. 

Remark 2.2. Notice that to construct $\psi$ one may take for example the solution to $-\Delta \psi = 1$ in $\Omega$, $\psi = 0$ on $\partial \Omega$, extended by $\psi = 0$ in $\mathbb{R}^n \setminus \Omega$.

Note also that any $C^{1,\alpha}$ domain $\Omega$ can be locally represented as the epigraph of a $C^{1,\alpha}$ function. More precisely, there is a $\rho_0 > 0$ such that for all $z \in \partial \Omega$ the set $\partial \Omega \cap B_{\rho_0}(z)$ is, after a rotation, the graph of a $C^{1,\alpha}$ function. Then, the constant $C$ in (2.1)-(2.2) can be taken depending only on $\rho_0$ and on the $C^{1,\alpha}$ norms of these functions.

We want to show the following.

**Proposition 2.3.** Let $s \in (0,1)$ and $\alpha \in (0, s)$, $L$ be given by (1.1)-(1.2), and $\Omega$ be any $C^{1,\alpha}$ domain. Let $\psi$ be given by Definition 2.1. Then,

$$|L(\psi^s)| \leq Cd^{\alpha-s} \text{ in } \Omega.$$  

The constant $C$ depends only on $s$, $n$, $\Omega$, and ellipticity constants.

For this, we need a couple of technical Lemmas. The first one reads as follows.

**Lemma 2.4.** Let $\Omega$ be any $C^{1,\alpha}$ domain, and $\psi$ be given by Definition 2.1. Then, for each $x_0 \in \Omega$ we have

$$\left|\psi(x_0 + y) - (\psi(x_0) + \nabla \psi(x_0) \cdot y)_+\right| \leq C|y|^{1+\alpha} \text{ for } y \in \mathbb{R}^n.$$  

The constant $C$ depends only on $\Omega$.

**Proof.** Let us consider $\tilde{\psi}$, a $C^{1,\alpha}(\mathbb{R}^n)$ extension of $\psi|\Omega$ satisfying $\tilde{\psi} \leq 0$ in $\mathbb{R}^n \setminus \Omega$. Then, since $\tilde{\psi} \in C^{1,\alpha}(\mathbb{R}^n)$ we clearly have

$$\left|\tilde{\psi}(x) - \psi(x_0) - \nabla \psi(x_0) \cdot (x - x_0)\right| \leq C|x - x_0|^{1+\alpha}$$

in all of $\mathbb{R}^n$. Here we used $\tilde{\psi}(x_0) = \psi(x_0)$ and $\nabla \tilde{\psi}(x_0) = \nabla \psi(x_0)$.

Now, using that $|a_+ - b_+| \leq |a - b|$, combined with $(\tilde{\psi})_+ = \psi$, we find

$$\left|\psi(x) - (\psi(x_0) + \nabla \psi(x_0) \cdot (x - x_0))_+\right| \leq C|x - x_0|^{1+\alpha}$$

for all $x \in \mathbb{R}^n$. Thus, the lemma follows. $\square$

The second one reads as follows.

**Lemma 2.5.** Let $\Omega$ be any $C^{1,\alpha}$ domain, $p \in \Omega$, and $\rho = d(p)/2$. Let $\gamma > -1$ and $\beta \neq \gamma$. Then,

$$\int_{B_1 \setminus B_{\rho}/2} d^n(p + y) \frac{dy}{|y|^{n+\beta}} \leq C(1 + \rho^{1-\beta}).$$

The constant $C$ depends only on $\gamma$, $\beta$, and $\Omega$.

**Proof.** The proof is similar to that of [RV15, Lemma 4.2].

First, we may assume $p = 0$. 
Notice that, since $\Omega$ is $C^{1,\alpha}$, then there is $\kappa_*>0$ such that for any $t \in (0, \kappa_*)$ the level set $\{d = t\}$ is $C^{1,\alpha}$. Since
\[
\int_{(B_1 \setminus B_\rho) \cap \{d \geq \kappa_*\}} d^\gamma(y) \frac{dy}{|y|^{n+\beta}} \leq C, \tag{2.4}
\]
then we just have to bound the same integral in the set $\{d < \kappa_*\}$. Here we used that $B_r \cap \{d \geq \kappa_*\} = \emptyset$ if $r \leq \kappa_* - 2\rho$, which follows from the fact that $d(0) = 2\rho$.

We will use the following estimate for $t \in (0, \kappa_*)$
\[
\mathcal{H}^{n-1}(\{d = t\} \cap (B_{2^{-k+1}} \setminus B_{2^{-k}})) \leq C(2^{-k})^{n-1},
\]
which follows for example from the fact that $\{d = t\}$ is $C^{1,\alpha}$ (see the Appendix in [RV15]). Note also that $\{d = t\} \cap B_r = \emptyset$ if $t > r + 2\rho$.

Let $M \geq 0$ be such that $2^{-M} \leq \rho \leq 2^{-M+1}$. Then, using the coarea formula,
\[
\int_{(B_1 \setminus B_\rho) \cap \{d < \kappa_*\}} d^\gamma(y) \frac{dy}{|y|^{n+\beta}} \leq \sum_{k=0}^{M} \frac{1}{2^{-k(n+\beta)}} \int_{B_{2^{-k+1}} \setminus B_{2^{-k}}} d^\gamma(y) \frac{dy}{|y|^{n+\beta}} \leq \sum_{k=0}^{M} \frac{1}{2^{-k(n+\beta)}} \int_{0}^{C2^{-k}} t^\gamma dt \int_{(B_{2^{-k+1}} \setminus B_{2^{-k}}) \cap \{d = t\}} \mathcal{H}^{n-1}(y) dt
\]
\[
\leq C \sum_{k=0}^{M} \frac{(2^{-k})^\gamma 2^{-k(n-1)}}{2^{-k(n+\beta)}} = C \sum_{k=0}^{M} 2^{k(\beta-\gamma)} = C(1 + |\log \rho|).
\]
Here we used that $\gamma \neq \beta$ — in case $\gamma = \beta$ we would get $C(1 + |\log \rho|)$.

Combining (2.4) and (2.5), the lemma follows. \hfill \Box

We now give the:

**Proof of Proposition 2.3**

Let $x_0 \in \Omega$ and $\rho = d(x)$.

Notice that when $\rho \geq \rho_0 > 0$ then $\psi^s$ is smooth in a neighborhood of $x_0$, and thus $L(\psi^s)(x_0)$ is bounded by a constant depending only on $\rho_0$. Thus, we may assume that $\rho \in (0, \rho_0)$, for some small $\rho_0$ depending only on $\Omega$.

Let us denote
\[
\ell(x) = (\psi(x_0) + \nabla \psi(x_0) \cdot (x - x_0))_+,
\]
which satisfies
\[
L(\ell^s) = 0 \quad \text{in} \quad \{\ell > 0\};
\]
see [RST14, Section 2].

Now, notice that
\[
\psi(x_0) = \ell(x_0) \quad \text{and} \quad \nabla \psi(x_0) = \nabla \ell(x_0).
\]
Moreover, by Lemma 2.7 we have
\[
|\psi(x_0 + y) - \ell(x_0 + y)| \leq C|y|^{1+\alpha},
\]
and using $|a^s - b^s| \leq C|a - b|(a^{s-1} + b^{s-1})$ for $a, b \geq 0$, we find

$$|\psi^s(x_0 + y) - \ell^s(x_0 + y)| \leq C|y|^{1+\alpha} \left( d^{s-1}(x_0 + y) + \ell^{s-1}(x_0 + y) \right).$$

(2.6)

Here, we used that $\psi \leq Cd$.

On the other hand, since $\psi \in C^{1,\alpha}(\overline{\Omega})$ and $\psi \geq cd$ in $\overline{\Omega}$, then it is not difficult to check that

$$\ell > 0 \quad \text{in} \quad B_{\rho/2}(x_0),$$

provided that $\rho_0$ is small (depending only on $\Omega$). Thanks to this, one may estimate

$$\left| D^2(\psi^s - \ell^s) \right| \leq C\rho^{s+\alpha-2} \quad \text{in} \quad B_{\rho/2},$$

and thus

$$\left| \psi^s - \ell^s \right|(x_0 + y) \leq \| D^2(\psi^s - \ell^s) \| \leq C\rho^{s+\alpha-2}|y|^2 \leq C\rho^{s+\alpha-2}|y|^2 \quad \text{(2.7)}$$

for $y \in B_{\rho/2}$.

Therefore, it follows from (2.6) and (2.7) that

$$\left| \psi^s - \ell^s \right|(x_0 + y) \leq \begin{cases} C\rho^{s+\alpha-2}|y|^2 & \text{for} \ y \in B_{\rho/2} \\ C|y|^{1+\alpha} \left( d^{s-1}(x_0 + y) + \ell^{s-1}(x_0 + y) \right) & \text{for} \ y \in B_1 \setminus B_{\rho/2} \\ C|y|^s & \text{for} \ y \in \mathbb{R}^n \setminus B_1. \end{cases}$$

Hence, recalling that $L(\ell^s)(x_0) = 0$, we find

$$|L(\psi^s)(x_0)| = \left| L(\psi^s - \ell^s)(x_0) \right|$$

$$= \int_{\mathbb{R}^n} \left| \psi^s - \ell^s \right|(x_0 + y) \frac{a(y/|y|)}{|y|^{n+2s}} dy$$

$$\leq \int_{B_{\rho/2}} C\rho^{s+\alpha-2}|y|^2 \frac{dy}{|y|^{n+2s}} + \int_{\mathbb{R}^n \setminus B_1} C|y|^{s} \frac{dy}{|y|^{n+2s}} +$$

$$+ \int_{B_1 \setminus B_{\rho/2}} C|y|^{1+\alpha} \left( d^{s-1}(x_0 + y) + \ell^{s-1}(x_0 + y) \right) \frac{dy}{|y|^{n+2s}}$$

$$\leq C(\rho^{\alpha-s} + 1) + C \int_{B_1 \setminus B_{\rho/2}} \left( d^{s-1}(x_0 + y) + \ell^{s-1}(x_0 + y) \right) \frac{dy}{|y|^{n+2s-1-\alpha}}.$$

Thus, using Lemma 2.5 twice, we find

$$|L(\psi^s)(x_0)| \leq C\rho^{\alpha-s},$$

and (2.3) follows. \qed

When $\alpha > s$ the previous proof gives the following result, which states that for any operator $L_{1.1}$- $L_{1.2}$ one has $L(d^s) \in L^\infty(\Omega)$. Here, as in [Gru15, RST14b, RST14], $d$ denotes a fixed function that coincides with $\text{dist}(x, \mathbb{R}^n \setminus \Omega)$ in a neighborhood of $\partial\Omega$, satisfies $d \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, and it is $C^{1,\alpha}$ in $\Omega$. 
Proposition 2.6. Let \( s \in (0,1) \), \( L \) be given by (1.1)-(1.2), and \( \Omega \) be any bounded \( C^{1,\alpha} \) domain, with \( \alpha > s \). Then,
\[
|L(d^s)| \leq C \quad \text{in } \Omega.
\]
The constant \( C \) depends only on \( n, s, \Omega, \) and ellipticity constants.

To our best knowledge, this result was only known in case that \( L \) is the fractional Laplacian and \( \Omega \) is \( C^{1,1} \), or in case that \( a \in C^\infty(S^{n-1}) \) in (1.1) and \( \Omega \) is \( C^\infty \) (in this case \( L(d^s) \) is \( C^\infty \) (\( \Omega \)); see [Gru15]).

Also, recall that for a general stable operator (1.1) (with \( a \in L^1(S^{n-1}) \) and without the assumption (1.2)) the result is false, since we constructed in [RS 14] an operator \( L \) and a \( C^\infty \) domain \( \Omega \) for which \( L(d^s) \not\in \mathcal{L}^\infty(\Omega) \). Hence, the assumption (1.2) is somewhat necessary for Proposition 2.6 to be true.

**Proof of Proposition 2.6.** Let \( x_0 \in \Omega \), and \( \rho = d(x) \).

Notice that when \( \rho \geq \rho_0 > 0 \) then \( d^s \) is \( C^{1+s} \) in a neighborhood of \( x_0 \), and thus \( L(d^s)(x_0) \) is bounded by a constant depending only on \( \rho_0 \). Thus, we may assume that \( \rho \in (0,\rho_0) \), for some small \( \rho_0 \) depending only on \( \Omega \).

Let us denote
\[
\ell(x) = (d(x_0) + \nabla d(x_0) \cdot (x - x_0))_+,
\]
which satisfies
\[
L(\ell^s) = 0 \quad \text{in} \quad \{ \ell > 0 \}.
\]
Moreover, as in Proposition 2.3, we have
\[
|d^s(x_0 + y) - \ell^s(x_0 + y)| \leq C|y|^{1+\alpha} (d^{s-1}(x_0 + y) + \ell^{s-1}(x_0 + y)) \tag{2.8}
\]
In particular,
\[
|d^s(x_0 + y) - \ell^s(x_0 + y)| \leq C\rho^{s-1}|y|^{1+\alpha} \quad \text{for } y \in B_{\rho/2}.
\]
Hence, recalling that \( L(\ell^s)(x_0) = 0 \), we find
\[
|L(\psi^s)(x_0)| = |L(\psi^s - \ell^s)(x_0)|
\]
\[
= \int_{\mathbb{R}^n} |\psi^s - \ell^s|(x_0 + y) \frac{a(y/|y|)}{|y|^{n+2s}} \, dy
\]
\[
\leq \int_{B_{\rho/2}} C\rho^{s-1}|y|^{1+\alpha} \frac{dy}{|y|^{n+2s}} + \int_{\mathbb{R}^n \setminus B_1} C|y|^s \frac{dy}{|y|^{n+2s}} +
\]
\[
+ \int_{B_1 \setminus B_{\rho/2}} C|y|^{1+\alpha} (d^{s-1}(x_0 + y) + \ell^{s-1}(x_0 + y)) \frac{dy}{|y|^{n+2s}}
\]
\[
\leq C(1 + \rho^{s-\alpha}).
\]
Here we used Lemma 2.5. Since \( \alpha > s \), the result follows. \( \square \)

We next show the following.
Lemma 2.7. Let \( s \in (0, 1) \), \( L \) be given by (1.1)-(1.2), and \( \Omega \) be any \( C^{1,\alpha} \) domain. Let \( \psi \) be given by Definition 2.1. Then, for any \( \epsilon \in (0, \alpha) \), we have
\[
L(\psi^{s+\epsilon}) \geq cd^{s} - C \quad \text{in } \Omega \cap B_{1/2},
\]
with \( c > 0 \). The constants \( c \) and \( C \) depend only on \( \epsilon, s, n, \Omega \), and ellipticity constants.

Proof. Exactly as in Proposition 2.3, one finds that
\[
|\psi^{s+\epsilon}(x_0 + y) - \ell^{s+\epsilon}(x_0 + y)| \leq C|y|^{1+\alpha} (d^{s+\epsilon-1}(x_0 + y) + \ell^{s+\epsilon-1}(x_0 + y)),
\]
and
\[
|\psi^{s+\epsilon} - \ell^{s+\epsilon}(x_0 + y) \leq C \rho^{s+\epsilon+\alpha-2} |y|^2
\]
for \( y \in B_{\rho/2} \). Therefore, as in Proposition 2.3,
\[
|L(\psi^{s+\epsilon} - \ell^{s+\epsilon})(x_0)| \leq C(1 + \rho^{\alpha+\epsilon-s}).
\]
We now use that, by homogeneity, we have
\[
L(\ell^{s+\epsilon})(x_0) = \kappa \rho^{s-\epsilon},
\]
with \( \kappa > 0 \) (see [RS14]). Thus, combining the previous two inequalities we find
\[
L(\psi^{s+\epsilon})(x_0) \geq \kappa \rho^{s-\epsilon} - C(1 + \rho^{\alpha+\epsilon-s}) \geq \frac{\kappa}{2} \rho^{s-\epsilon} - C,
\]
as desired. \( \square \)

We now construct sub and supersolutions.

Lemma 2.8 (Supersolution). Let \( s \in (0, 1) \), \( L \) be given by (1.1)-(1.2), and \( \Omega \) be any bounded \( C^{1,\alpha} \) domain. Then, there exists \( \rho_0 > 0 \) and a function \( \phi_1 \) satisfying
\[
\begin{align*}
C^{-1}d^s & \leq \phi_1 \leq Cd^s \quad \text{in } \Omega \\
\phi_1 & = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{align*}
\]
The constants \( C \) and \( \rho_0 \) depend only on \( n, s, \Omega \), and ellipticity constants.

Proof. Let \( \psi \) be given by Definition 2.1 and let \( \epsilon = \frac{\alpha}{2} \). Then, by Proposition 2.3 we have
\[
-C_0 d^{\alpha-s} \leq L(\psi^s) \leq C_0 d^{\alpha-s},
\]
and by Lemma 2.7
\[
L(\psi^{s+\epsilon}) \geq c_0d^{s} - C_0.
\]

Next, we consider the function
\[
\phi_1 = \psi^s - c\psi^{s+\epsilon},
\]
with \( c \) small enough. Then, \( \phi_1 \) satisfies
\[
L\phi_1 \leq C_0d^{\alpha-s} + C_0 - cc_1d^{s} \leq -1 \quad \text{in } \Omega \cap \{d \leq \rho_0\},
\]
for some \( \rho_0 > 0 \). Finally, by construction we clearly have
\[
C^{-1}d^s \leq \phi_1 \leq Cd^s \quad \text{in } \Omega.
\]
and thus the Lemma is proved.

Notice that the previous proof gives in fact the following.

**Lemma 2.9.** Let \( s \in (0, 1) \), \( L \) be given by (1.1)-(1.2), and \( \Omega \) be any bounded \( C^{1, \alpha} \) domain. Then, there exist \( \rho_0 > 0 \) and a function \( \phi_1 \) satisfying

\[
\begin{align*}
L\phi_1 &\leq -d^{k-s} \quad \text{in } \Omega \cap \{d \leq \rho_0\} \\
C^{-1}d^s &\leq \phi_1 \leq Cd^s \quad \text{in } \Omega \\
\phi_1 &\equiv 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{align*}
\]

The constants \( C \) and \( \rho_0 \) depend only on \( n, s, \Omega, \) and ellipticity constants.

**Proof.** The proof is the same as Lemma 2.8; see (2.12).

We finally construct a subsolution.

**Lemma 2.10 (Subsolution).** Let \( s \in (0, 1) \), \( L \) be given by (1.1)-(1.2), and \( \Omega \) be any bounded \( C^{1, \alpha} \) domain. Then, for each \( K \subset \subset \Omega \) there exists a function \( \phi_2 \) satisfying

\[
\begin{align*}
L\phi_2 &\geq 1 \quad \text{in } \Omega \setminus K \\
C^{-1}d^s &\leq \phi_2 \leq Cd^s \quad \text{in } \Omega \\
\phi_2 &\equiv 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{align*}
\]

The constants \( c \) and \( C \) depend only on \( n, s, \Omega, K, \) and ellipticity constants.

**Proof.** First, notice that if \( \eta \in C_c^\infty(K) \) then \( L\eta \geq c_1 > 0 \) in \( \Omega \setminus K \). Hence,

\[
\phi_2 = \psi^s + \psi^{s+\epsilon} + C\eta
\]

satisfies

\[
L\phi_2 \geq -Cd^{\alpha-s} + c_0d^{k-s} - C_0 + Cc_1 \geq 1 \quad \text{in } \Omega \setminus K,
\]

provided that \( C \) is chosen large enough.

\[\square\]

### 3. Regularity in \( C^{1, \alpha} \) Domains

The aim of this section is to prove Proposition 1.1 and Theorem 1.2.

#### 3.1. Hölder regularity up to the boundary

We will prove first the following result, which is similar to Proposition 1.1 but allows \( u \) to grow at infinity and \( f \) to be singular near \( \partial \Omega \).

**Proposition 3.1.** Let \( s \in (0, 1) \), \( L \) be any operator of the form (1.1)-(1.2), and \( \Omega \) be any bounded \( C^{1, \alpha} \) domain. Let \( u \) be a solution to (1.3), and assume that

\[
|f| \leq Cd^{k-s} \quad \text{in } \Omega.
\]

Then,

\[
\|u\|_{C^{s}(B_{1/2})} \leq C \left( \|d^{k-s}f\|_{L^\infty(B_1)} + \sup_{R \geq 1} R^{\delta-2s}\|u\|_{L^\infty(B_R)} \right).
\]

The constant \( C \) depends only on \( n, s, \epsilon, \delta, \Omega, \) and ellipticity constants.
Proof. Dividing by a constant, we may assume that
\[
\|d^{8-\varepsilon} f\|_{L^\infty(B_{1}\cap \Omega)} + \sup_{R \geq 1} R^{8-2s} \|u\|_{L^\infty(B_{R})} \leq 1.
\]

Then, the truncated function \( w = u \chi_{B_{1}} \) satisfies
\[
|Lw| \leq Cd^{s-\varepsilon} \quad \text{in} \quad \Omega \cap B_{3/4},
\]
\( w \leq 1 \) in \( B_{1} \), and \( w \equiv 0 \) in \( \mathbb{R}^{n} \setminus B_{1} \).

Let \( \tilde{\Omega} \) be a bounded \( C^{1,\alpha} \) domain satisfying: \( B_{1} \cap \Omega \subset \tilde{\Omega} \); \( B_{1/2} \cap \partial \Omega \subset \partial \tilde{\Omega} \); and \( \text{dist}(x, \partial \tilde{\Omega}) \geq c > 0 \) in \( \Omega \cap (B_{1} \setminus B_{3/4}) \). Let \( \phi_{1} \) be the function given by Lemma 2.8 satisfying
\[
\begin{cases}
L\phi_{1} \leq -d^{s-\varepsilon} & \text{in} \quad \tilde{\Omega} \cap \{d \leq \rho_{0}\} \\
\phi_{1} \leq 0 & \text{in} \quad \mathbb{R}^{n} \setminus \Omega,
\end{cases}
\]
where we denoted \( \tilde{d}(x) = \text{dist}(x, \mathbb{R}^{n} \setminus \tilde{\Omega}) \).

Then, the function \( \varphi = C\phi_{1} \) satisfies
\[
\begin{cases}
L\varphi \leq -Cd^{s-\varepsilon} & \text{in} \quad \Omega \cap B_{1/2} \cap \{d \leq \rho_{0}\} \\
\varphi \leq C\tilde{d}^{s} & \text{in} \quad \Omega \cap B_{1/2} \\
\varphi \geq 1 & \text{in} \quad \Omega \cap (B_{1} \setminus B_{3/4}) \quad \text{and in} \quad \Omega \cap B_{1/2} \cap \{d \geq \rho_{0}\} \\
\varphi \geq 0 & \text{in} \quad \mathbb{R}^{n}.
\end{cases}
\]

In particular, if \( C \) is large enough then we have \( L(\varphi - w) \leq 0 \) in \( \Omega \cap B_{1/2} \cap \{d \leq \rho_{0}\} \), and \( \varphi - w \geq 0 \) in \( \mathbb{R}^{n} \setminus (\Omega \cap B_{1/2} \cap \{d \leq \rho_{0}\}) \).

Therefore, the maximum principle yields \( w \leq \varphi \), and thus \( w \leq Cd^{s} \) in \( B_{1/2} \). Replacing \( w \) by \(-w\), we find
\[
|w| \leq Cd^{s} \quad \text{in} \quad B_{1/2}. \tag{3.1}
\]

Now, it follows from the interior estimates of [RS14b, Theorem 1.1] that
\[
R^{-s}w_{C^{s}(B_{r}(x_{0}))} \leq C\left(R^{2s} \|Lw\|_{L^\infty(B_{2r}(x_{0}))} + \sup_{R \geq 1} R^{8-2s} \|w\|_{L^\infty(B_{R}(x_{0}))}\right)
\]
for any ball \( B_{r}(x_{0}) \subset \Omega \cap B_{1/2} \) with \( 2r = d(x_{0}) \). Now, taking \( \delta = s \) and using (3.1), we find
\[
R^{-s} \|w\|_{L^\infty(B_{R}(x_{0}))} \leq Cr^{s} \quad \text{for all} \quad R \geq 1.
\]
Thus, we have
\[
[w]_{C^{s}(B_{r}(x_{0}))} \leq C
\]
for all balls \( B_{r}(x_{0}) \subset \Omega \cap B_{1/2} \) with \( 2r = d(x_{0}) \). This yields
\[
\|w\|_{C^{s}(B_{1/2})} \leq C.
\]

Indeed, take \( x, y \in B_{1/2} \), let \( r = |x - y| \) and \( \rho = \min\{d(x), d(y)\} \). If \( 2\rho \geq r \), then using \( |u| \leq Cd^{s} \)
\[
|u(x) - u(y)| \leq |u(x)| + |u(y)| \leq Cr^{s} + C(r + \rho)^{s} \leq C\rho^{s}.
\]
If $2\rho < r$ then $B_{2\rho}(x) \subset \Omega$, and hence
\[ |u(x) - u(y)| \leq \rho^s [u]_{C^s(B_{\rho}(x))} \leq C \rho^s. \]
Thus, the proposition is proved. \qed

The proof of Proposition is now immediate.

**Proof of Proposition 3.1.** The result is a particular case of Proposition 3.1.

3.2. Regularity for $u/d^s$. Let us now prove Theorem 1.2. For this, we first show the following.

**Proposition 3.2.** Let $s \in (0, 1)$ and $\alpha \in (0, s)$. Let $L$ be any operator of the form (1.1)–(1.2), $\Omega$ be any $C^{1,\alpha}$ domain, and $\psi$ be given by Definition 2.1.

Assume that $0 \in \partial \Omega$, and that $\partial \Omega \cap B_1$ can be represented as the graph of a $C^{1,\alpha}$ function with norm less or equal than 1.

Let $u$ be any solution to (1.3), and let
\[ K_0 = \|d^{s-\alpha}f\|_{L^\infty(B_1 \cap \Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}. \]

Then, there exists a constant $Q$ satisfying $|Q| \leq C K_0$ and
\[ |u(x) - Q\psi^s(x)| \leq C K_0 x^{s+\alpha}. \]

The constant $C$ depends only on $n$, $s$, and ellipticity constants.

We will need the following technical lemma.

**Lemma 3.3.** Let $\Omega$, $\psi$, and $u$ be as in Proposition 3.2, and define
\[ \phi_r(x) := Q_*(r)\psi^s(x), \] (3.2)
where
\[ Q_*(r) := \arg\min_{Q \in \mathbb{R}} \int_{B_r} (u - Q\psi^s)^2 \, dx = \frac{\int_{B_r} u\psi^s}{\int_{B_r} \psi^{2s}}. \]

Assume that for all $r \in (0, 1)$ we have
\[ \|u - \phi_r\|_{L^\infty(B_r)} \leq C_0 r^{s+\alpha}. \] (3.3)

Then, there is $Q \in \mathbb{R}$ satisfying $|Q| \leq C(C_0 + \|u\|_{L^\infty(B_1)})$ such that
\[ \|u - Q\psi^s\|_{L^\infty(B_r)} \leq C C_0 r^{s+\alpha}, \]
for some constant $C$ depending only on $s$ and $\alpha$.

**Proof.** The proof is analogue to that of [RS14b, Lemma 5.3].

First, we may assume $C_0 + \|u\|_{L^\infty(B_1)} = 1$. Then, by (3.3), for all $x \in B_r$ we have
\[ |\phi_{2r}(x) - \phi_r(x)| \leq |u(x) - \phi_{2r}(x)| + |u(x) - \phi_r(x)| \leq C r^{s+\alpha}. \]
This, combined with $\sup_{B_r} \psi^s = cr^s$, gives
\[ |Q_*(2r) - Q_*(r)| \leq C r^{\alpha}. \]
Moreover, we have $|Q_*(1)| \leq C$, and thus there exists the limit $Q = \lim_{r \to 0} Q_*(r)$. Furthermore,
\[ |Q - Q_*(r)| \leq \sum_{k \geq 0} |Q_*(2^{-k}r) - Q_*(2^{-k-1}r)| \leq \sum_{k \geq 0} C2^{-m \alpha}r^\alpha \leq Cr^\alpha. \]
In particular, $|Q| \leq C$.

Therefore, we finally find
\[ \|u - Q\psi\|_{L^\infty(B_r)} \leq \|u - Q_*(r)\psi\|_{L^\infty(B_r)} + Cr^s|Q_*(r) - Q| \leq Cr^{s+\alpha}, \]
and the lemma is proved. \[ \square \]

We now give the:

**Proof of Proposition 3.2.** The proof is by contradiction, and uses several ideas from [RS14b, Section 5].

First, dividing by a constant we may assume $K_0 = 1$. Also, after a rotation we may assume that the unit (outward) normal vector to $\partial \Omega$ at 0 is $\nu = -e_n$.

Assume the estimate is not true, i.e., there are sequences $\Omega_k, L_k, f_k, u_k$, for which:

- $\Omega_k$ is a $C^{1,\alpha}$ domain that can be represented as the graph of a $C^{1,\alpha}$ function with norm is less or equal than 1;
- $0 \in \partial \Omega_k$ and the unit normal vector to $\partial \Omega_k$ at 0 is $-e_n$;
- $L_k$ is of the form (1.1)-(1.2);
- $\|d^{s-\alpha}f_k\|_{L^\infty(B_1 \cap \Omega)} + \|u_k\|_{L^\infty(\mathbb{R}^n)} \leq 1$;
- For any constant $Q$, $\sup_{r>0} \sup_{B_r} r^{-s-\alpha}|u_k - Q\psi_k| = \infty$.

Then, by Lemma 3.3 we will have
\[ \sup_{k} \sup_{r>0} \|u_k - \phi_{k,r}\|_{L^\infty(B_r)} = \infty, \]
where
\[ \phi_{k,r}(x) = Q_k(r)\psi_k, \quad Q_k(r) = \frac{\int_{B_r} u_k \psi_k^s}{\int_{B_r} \psi_k^2}. \]

We now define the monotone quantity
\[ \theta(r) := \sup_{k} \sup_{r\geq r'} (r')^{-s-\alpha}\|u_k - \phi_{k,r'}\|_{L^\infty(B_{r'})}, \]
which satisfies $\theta(r) \to \infty$ as $r \to 0$. Hence, there are sequences $r_m \to 0$ and $k_m$, such that
\[ (r_m)^{-s-\alpha}\|u_{k_m} - \phi_{k_m,r_m}\|_{L^\infty(B_{r_m})} \geq \frac{1}{2}\theta(r_m). \quad (3.4) \]

Let us now denote $\phi_m = \phi_{k_m,r_m}$ and define
\[ v_m(x) := \frac{u_{k_m}(r_mx) - \phi_m(r_mx)}{(r_m)^{s+\alpha}\theta(r_m)}. \]
Note that
\[ \int_{B_1} v_m(x) \psi_k^s(r_mx) dx = 0, \quad (3.5) \]
and also
\[ \|v_m\|_{L^\infty(B_1)} \geq \frac{1}{2}, \quad (3.6) \]
which follows from (3.4).

With the same argument as in the proof of Lemma 3.3 one finds
\[ |Q_{k_m}(2r) - Q_{k_m}(r)| \leq Cr^\alpha \theta(r). \]

Then, by summing a geometric series this yields
\[ |Q_{k_m}(rR) - Q_{k_m}(r)| \leq Cr^\alpha \theta(r) R^\alpha \]
for all \( R \geq 1 \) (see [RS14b]).

The previous inequality, combined with
\[ \|u_m - Q_{k_m}(rR)\psi_{k_m}^s\|_{L^\infty(B_{r_mR})} \leq (r_mR)^{s+\alpha} \theta(r_mR) \]
(which follows from the definition of \( \theta \)), gives
\[ \|v_m\|_{L^\infty(B_R)} = \frac{1}{(r_m)^{s+\alpha} \theta(r_m)} \|u_m - Q_{k_m}(r_m)\psi_{k_m}^s\|_{L^\infty(B_{r_mR})} \]
\[ \leq \frac{(r_mR)^{s+\alpha} \theta(r_mR)}{(r_m)^{s+\alpha} \theta(r_m)} + \frac{C(r_mR)^s}{(r_m)^{s+\alpha} \theta(r_m)} |Q_{k_m}(r_mR) - Q_{k_m}(r_m)| \quad (3.7) \]
for all \( R \geq 1 \). Here we used that \( \theta(r_mR) \leq \theta(r_m) \) if \( R \geq 1 \).

Now, the functions \( v_m \) satisfy
\[ L_m v_m(x) = \frac{(r_m)^{2s}}{(r_m)^{s+\alpha} \theta(r_m)} f_{k_m}(r_m x) - \frac{(r_m)^{2s}}{(r_m)^{s+\alpha} \theta(r_m)} (L \psi_{k_m})(r_m x) \]
in \( (r_m^{-1} \Omega_{k_m}) \cap B_{r_m^{-1}} \). Since \( \alpha < s \), and using Proposition 2.3, we find
\[ |L_m v_m| \leq \frac{C}{\theta(r_m)} (r_m)^{s-\alpha} d_{k_m}^{r_m-s}(r_m x) \quad \text{in} \quad (r_m^{-1} \Omega_{k_m}) \cap B_{r_m^{-1}}. \]

Thus, denoting \( \Omega_m = r_m^{-1} \Omega_{k_m} \) and \( d_m(x) = \text{dist}(x, r_m^{-1} \Omega_{k_m}) \), we have
\[ |L_m v_m| \leq \frac{C}{\theta(r_m)} d_m^{r_m-s}(x) \quad \text{in} \quad \Omega_m \cap B_{r_m^{-1}}. \quad (3.8) \]

Notice that the domains \( \Omega_m \) converge locally uniformly to \( \{ x_n > 0 \} \) as \( m \to \infty \).

Next, by Proposition 3.4 we find that for each fixed \( M \geq 1 \)
\[ \|v_m\|_{C^1(B_{M})} \leq C(M) \quad \text{for all} \quad m \quad \text{with} \quad r_m^{-1} > 2M. \]
The constant \( C(M) \) does not depend on \( m \). Hence, by Arzelà-Ascoli theorem, a subsequence of \( v_m \) converges locally uniformly to a function \( v \in C(\mathbb{R}^n) \).

In addition, there is a subsequence of operators \( L_{k_m} \) which converges weakly to some operator \( L \) of the form \( (1.1)-(1.2) \) (see Lemma 3.1 in [RS14b]). Hence, for
any fixed $K \subset \{x_n > 0\}$, thanks to the growth condition (3.7) and since $v_m \to v$ locally uniformly, we can pass to the limit the equation (3.8) to get

$$Lv = 0 \text{ in } K.$$  

Here we used that the domains $\Omega_m$ converge uniformly to $\{x_n > 0\}$, so that for $m$ large enough we will have $K \subset \Omega_m \cap B_{r_m}$. We also used that, in $K$, the right hand side in (3.8) converges uniformly to 0.

Since this can be done for any $K \subset \{x_n > 0\}$, we find

$$Lv = 0 \text{ in } \{x_n > 0\}.$$  

Moreover, we also have $v = 0$ in $\{x_n \leq 0\}$, and $v \in C(\mathbb{R}^n)$.

Thus, by the classification result [RS14b, Theorem 4.1], we find

$$v(x) = \kappa(x_n)^s$$  

for some $\kappa \in \mathbb{R}$.

Now, notice that, up to a subsequence, $r_m^{-1}\psi_{km}(r_m x) \to c_1(x_n)^s$ uniformly, with $c_1 > 0$. This follows from the fact that $\psi_{km}$ are $C^{1,\alpha}(\overline{\Omega_{km}})$ (uniformly in $m$) and that $0 < c_0d_{km} \leq \psi_{km} \leq C_0d_{km}$.

Then, multiplying (3.5) by $(r_m)^{-s}$ and passing to the limit, we find

$$\int_{B_1} v(x)(x_n)^s \, dx = 0.$$  

This means that $\kappa = 0$ in (3.9), and therefore $v \equiv 0$. Finally, passing to the limit in (3.6) we find a contradiction, and thus the proposition is proved. □

We finally give the:

**Proof of Theorem 1.2.** First, dividing by a constant if necessary, we may assume

$$\|f\|_{L^\infty(B_1 \cap \Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)} \leq 1.$$  

Second, by definition of $\psi$ we have $\psi/d \in C^\alpha(\overline{\Omega} \cap B_{1/2})$ and

$$\|\psi^s/d^s\|_{C^\alpha(\overline{\Omega} \cap B_{1/2})} \leq C.$$  

Thus, it suffices to show that

$$\|u/\psi^s\|_{C^\alpha(\overline{\Omega} \cap B_{1/2})} \leq C.$$  

To prove (3.10), let $x_0 \in \Omega \cap B_{1/2}$ and $2r = d(x_0)$. Then, by Proposition 3.2 there is $Q = Q(x_0)$ such that

$$\|u - Q\psi^s\|_{L^\infty(B_r(x_0))} \leq Cr^{s+\alpha}.$$  

Moreover, by rescaling and using interior estimates, we get

$$\|u - Q\psi^s\|_{C^\alpha(B_r(x_0))} \leq Cr^s.$$  

Finally, (3.11)–(3.12) yield (3.10), exactly as in the proof of Theorem 1.2 in [RS14b]. □
Remark 3.4. Notice that, thanks to Proposition 3.2, we have that Theorem 1.2 holds for all right hand sides satisfying $|f(x)| \leq Cd^{\alpha-s}$ in $\Omega$.

3.3. Equations with bounded measurable coefficients. We prove now Theorem 1.5.

We next show:

**Proposition 3.5.** Let $s \in (0, 1)$, and $\Omega$ be any bounded $C^{1,\alpha}$ domain.

Let $u$ be a solution to

$$
\begin{align*}
M^+ u &\geq -K_0 d^{\alpha-s} \quad \text{in } B_1 \cap \Omega \\
M^- u &\leq K_0 d^{\epsilon-s} \quad \text{in } B_1 \cap \Omega \\
 u &= 0 \quad \text{in } B_1 \setminus \Omega.
\end{align*}
$$

(3.13)

Then,

$$
\|u\|_{C^{\alpha}(B_1/2)} \leq C \left( K_0 + \sup_{R \geq 1} R^{\delta-2s} \|u\|_{L^\infty(B_R)} \right).
$$

The constant $C$ depends only on $n$, $s$, $\epsilon$, $\delta$, $\Omega$, and ellipticity constants.

**Proof.** The proof is very similar to that of Proposition 3.3.

We next show:

**Proposition 3.6.** Let $s \in (0, 1)$ and $\alpha \in (0, \bar{\alpha})$. Let $L$ be any operator of the form (1.1)-(1.2), $\Omega$ be any $C^{1,\alpha}$ domain, and $\psi$ be given by Definition 2.1.

Assume that $0 \not\in \partial \Omega$, and that $\partial \Omega \cap B_1$ can be represented as the graph of a $C^{1,\alpha}$ function with norm less or equal than 1.

Let $u$ be any solution to (1.4), and let

$$
K_0 = \|f\|_{L^\infty(B_1 \cap \Omega)} + \|u\|_{L^\infty(R^n)}.
$$

Then, there exists a constant $Q$ satisfying $|Q| \leq CK_0$ and

$$
|u(x) - Q\psi^s(x)| \leq CK_0 |x|^s + \alpha.
$$

The constant $C$ depends only on $n$, $s$, and ellipticity constants.

**Proof.** The proof is very similar to that of Proposition 3.2.

We next show:
\( \Omega_k \) is a \( C^{1,\alpha} \) domain that can be represented as the graph of a \( C^{1,\alpha} \) function with norm is less or equal than 1;

- \( 0 \in \partial \Omega_k \) and the unit normal vector to \( \partial \Omega_k \) at 0 is \( -e_n \);
- \( u_k \) satisfies \( (1.4) \) with \( K_0 = 1 \);
- For any constant \( Q \), \( \sup_{r>0} \sup_{B_r} \frac{r^{-s-\alpha}}{u_k - Q}\psi^s_k = \infty \).

Then, by Lemma 3.3 we will have

\[
\sup_{k} \sup_{r>0} \| u_k - \phi_{k,r} \|_{L^\infty(B_r)} = \infty,
\]

where

\[
\phi_{k,r}(x) = Q_k(r)\psi^s_k, \quad Q_k(r) = \frac{\int_{B_r} u_k \psi^s_k}{\int_{B_r} \psi^{2s}_k}.
\]

We now define \( \theta(r) \), \( r_m \to 0 \), and \( v_m \) as in the proof of Proposition 3.2. Then, we have

\[
\int_{B_1} v_m(x) \psi^s_k (r_m x) dx = 0, \quad (3.14)
\]

\[
\| v_m \|_{L^\infty(B_1)} \geq \frac{1}{2}, \quad (3.15)
\]

and

\[
\| v_m \|_{L^\infty(B_R)} \leq CR^{s+\alpha} \quad \text{for all} \quad R \geq 1. \quad (3.16)
\]

Moreover, the functions \( v_m \) satisfy

\[
M^- v_m \leq \frac{(r_m)^{2s}}{(r_m)^{s+\alpha} \theta(r_m)} + \frac{(r_m)^{2s}}{(r_m)^{s+\alpha} \theta(r_m)} (M^+ \psi_{km})(r_m x) \quad \text{in} \quad (r_m^{-1} \Omega_{km}) \cap B_{r_m^{-1}}.
\]

Using Lemma 2.3 and denoting \( \Omega_m = r_m^{-1} \Omega_{km} \) and \( d_m(x) = \text{dist}(x, r_m^{-1} \Omega_{km}) \), we find

\[
M^- v_m \leq \frac{C}{\theta(r_m)} d_m^{a-s}(x) \quad \text{in} \quad \Omega_m \cap B_{r_m^{-1}}. \quad (3.17)
\]

Similarly, we find

\[
M^+ v_m \geq -\frac{C}{\theta(r_m)} d_m^{a-s}(x) \quad \text{in} \quad \Omega_m \cap B_{r_m^{-1}}.
\]

Notice that the domains \( \Omega_m \) converge locally uniformly to \( \{ x_n > 0 \} \) as \( m \to \infty \).

Next, by Proposition 3.5 we find that for each fixed \( M \geq 1 \)

\[
\| v_m \|_{C^a(B_M)} \leq C(M) \quad \text{for all} \quad M \text{ with } r_m^{-1} > 2M.
\]

The constant \( C(M) \) does not depend on \( m \). Hence, by Arzelà-Ascoli theorem, a subsequence of \( v_m \) converges locally uniformly to a function \( v \in C(\mathbb{R}^n) \).

Hence, passing to the limit the equation (3.17) we get

\[
M^- v \leq 0 \leq M^+ v \quad \text{in} \quad \{ x_n > 0 \}.
\]

Moreover, we also have \( v = 0 \) in \( \{ x_n \leq 0 \} \), and \( v \in C(\mathbb{R}^n) \).
Thus, by the classification result [RS14, Proposition 5.1], we find
\[ v(x) = \kappa(x_n)^s \]
for some \( \kappa \in \mathbb{R} \). But passing (3.14) —multiplied by \((r_m)^{-s}\) — to the limit, we find
\[ \int_{B_1} v(x)(x_n)^s dx = 0. \]
This means that \( v \equiv 0 \), a contradiction with (3.15). \( \square \)

Finally, we give the:

**Proof of Theorem 1.5.** The result follows from Proposition 3.6; see the proof of Theorem 1.2. \( \square \)

### 4. Barriers: \( C^1 \) Domains

We construct now sub and supersolutions that will be needed in the proof of Theorem 1.3. Recall that in \( C^1 \) domains one does not expect solutions to be comparable to \( d^s \), and this is why the sub and supersolutions we construct have slightly different behaviors near the boundary. Namely, they will be comparable to \( d^{s+\epsilon} \) and \( d^{s-\epsilon} \), respectively.

**Lemma 4.1.** Let \( s \in (0, 1) \), and \( e \in S^{n-1} \). Define
\[
\Phi_{\text{sub}}(x) := \left( e \cdot x - \eta |x| \left( 1 - \frac{(e \cdot x)^2}{|x|^2} \right) \right)^{s+\epsilon}
\]
and
\[
\Phi_{\text{super}}(x) := \left( e \cdot x + \eta |x| \left( 1 - \frac{(e \cdot x)^2}{|x|^2} \right) \right)^{s-\epsilon}
\]
For every \( \epsilon > 0 \) there is \( \eta > 0 \) such that two functions \( \Phi_{\text{sub}} \) and \( \Phi_{\text{super}} \) satisfy, for all \( L \in \mathcal{L}_s \),
\[
\begin{cases}
L \Phi_{\text{sub}} \geq c \epsilon d^{s-s} > 0 & \text{in } \mathcal{C}_\eta \\
\Phi_{\text{sub}} = 0 & \text{in } \mathbb{R}^n \setminus \mathcal{C}_\eta
\end{cases}
\]
and
\[
\begin{cases}
L \Phi_{\text{super}} \leq -c \epsilon d^{-s-s} < 0 & \text{in } \mathcal{C}_{-\eta} \\
\Phi_{\text{super}} = 0 & \text{in } \mathbb{R}^n \setminus \mathcal{C}_{-\eta}
\end{cases}
\]
where \( \mathcal{C}_{\pm \eta} \) is the cone
\[
\mathcal{C}_{\pm \eta} := \left\{ x \in \mathbb{R}^n : e \cdot \frac{x}{|x|} > \pm \eta \left( 1 - \left( e \cdot \frac{x}{|x|} \right)^2 \right) \right\}.
\]
The constant \( \eta \) depends only on \( \epsilon, s \), and ellipticity constants.
Proof. We prove the statement for $\Phi_{\text{sub}}$. The statement for $\Phi_{\text{super}}$ is proved similarly.

Let us denote $\Phi := \Phi_{\text{sub}}$. By homogeneity it is enough to prove that $L\Phi \geq c_\epsilon > 0$ on points belonging to $e + \partial C_\eta$, since all the positive dilations of this set with respect to the origin cover the interior of $\tilde{C}_\eta$.

Let thus $P \in \partial C_\eta$, that is,

$$e \cdot P - \eta \left( |P| - \frac{(e \cdot P)^2}{|P|} \right) = 0.$$ 

Consider $\Phi_{P,\eta}(x) := \Phi(P + e + x)$

$$= \left( e \cdot (P + e + x) - \eta \left( |P + e + x| - \frac{(e \cdot (P + e + x))^2}{|P + e + x|} \right) \right)^{s+\epsilon} +$$

$$= \left( 1 + e \cdot x - \eta \left( |P + e + x| - |P| - \frac{(e \cdot (P + e + x))^2}{|P + e + x|} + \frac{(e \cdot P)^2}{|P|} \right) \right)^{s+\epsilon} +$$

$$= \left( 1 + e \cdot x - \eta \psi_P(x) \right)^{s+\epsilon}_+, $$

where we define

$$\psi_P(x) := |P + e + x| - |P| - \frac{(e \cdot (P + e + x))^2}{|P + e + x|} + \frac{(e \cdot P)^2}{|P|}. $$

Note that the functions $\psi_P$ satisfy

$$\psi_P(0) = 0,$$

$$|\nabla \psi_P(x)| \leq C \quad \text{in } \mathbb{R}^n \setminus \{-P - e\},$$

and

$$|D^2 \psi_P(x)| \leq C \quad \text{for } x \in B_{1/2},$$

where $C$ does not depend on $P$ (recall that $|e| = 1$).

Then, the family $\Phi_{P,\eta}$ satisfies

$$\Phi_{P,\eta} \to (1 + e \cdot x)^{s+\epsilon}_+ \quad \text{in } C^2(B_{1/2})$$

as $\eta \searrow 0$, uniformly in $P$ and moreover

$$\int_{\mathbb{R}^n} \frac{|\Phi_{P,\eta} - (1 + e \cdot x)^{s+\epsilon}_+|}{1 + |x|^{n+2s}} \, dx \leq \int_{\mathbb{R}^n} \frac{C(C\eta|x|)^{s+\epsilon}_+}{1 + |x|^{n+2s}} \, dx \leq C\eta^{s+\epsilon}.$$ 

Thus,

$$L\Phi_{P,\eta}(0) \to L((1 + e \cdot x)^{s+\epsilon}_+)(0) \geq c(s, \epsilon, \lambda) > 0 \quad \text{as } \eta \searrow 0$$

uniformly in $P$.

In particular one can chose $\eta = \eta(s, \epsilon, \lambda, \Lambda)$ so that $L\Phi_{P,\eta}(0) \geq c_\epsilon > 0$ for all $P \in \partial \tilde{C}_\eta$ and for all $L \in \mathcal{L}_s$, and the lemma is proved. \qed
5. Regularity in $C^1$ domains

We prove here Theorems 1.3 and 1.6.

**Definition 5.1.** Let $r_0 > 0$ and let $\rho : (0, r_0] \to 0$ be a nonincreasing function with $\lim_{t \downarrow 0} \rho(t) = 0$. We say that a domain $\Omega$ is *improving Lipschitz* at $0$ with inwards unit normal vector $e_n = (0, \ldots, 0, 1)$ and modulus $\rho$ if

$$\Omega \cap B_r = \{(x', x_n) : x_n > g(x')\} \cap B_r$$

for $r \in (0, r_0]$,

where $g : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfies

$$\|g\|_{\text{Lip}(B_r)} \leq \rho(r) \quad \text{for } 0 < r \leq r_0.$$

We say that $\Omega$ is *improving Lipschitz* at $x_0 \in \partial \Omega$ with inwards unit normal $e = e_n = (0, \ldots, 0, 1)$ if the normal vector to $\partial \Omega$ at $x_0$ is $e$ and, after a rotation, the domain $\Omega - x_0$ satisfies the previous definition.

We first prove the following $C^\alpha$ estimate up to the boundary.

**Lemma 5.2.** Let $s \in (0, 1)$, and let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain in $B_1$ with Lipschitz constant less than $\ell$. Namely, assume that after a rotation we have

$$\Omega \cap B_1 = \{(x', x_n) : x_n > g(x')\} \cap B_1,$$

with $\|g\|_{\text{Lip}(B_1)} \leq \ell$. Let $u \in C(B_1)$ be a viscosity solution of

$$M^+ u \geq -K_0 d^{-s} \quad \text{and} \quad M^- u \leq K_0 d^{-s} \quad \text{in } \Omega \cap B_1,$$

$$u = 0 \quad \text{in } B_1 \setminus \Omega.$$

Assume that

$$\|u\|_{L^\infty(B_R)} \leq K_0 R^{2s-\epsilon} \quad \text{for all } R \geq 1.$$

Then, if $\ell \leq \ell_0$, where $\ell_0 = \ell_0(n, s, \lambda, \Lambda)$, we have

$$\|u\|_{C^\alpha(B_{1/2})} \leq CK_0.$$

The constants $C$ and $\alpha$ depend only on $n$, $s$, $\epsilon$ and ellipticity constants.

**Proof.** By truncating $u$ in $B_2$ and dividing it by $CK_0$ we may assume that

$$\|u\|_{L^\infty(\mathbb{R}^n)} = 1$$

and that

$$M^+ u \geq -d^{-s} \quad \text{and} \quad M^- u \leq d^{-s} \quad \text{in } \Omega \cap B_1.$$

Now, we divide the proof into two steps.

**Step 1.** We first prove that

$$|u(x)| \leq C|x - x_0|^\alpha \quad \text{in } \Omega \cap B_{3/4},$$

where $x_0 \in \partial \Omega$ is the closest point to $x$ on $\partial \Omega$. We will prove (5.1) by using a supersolution. Indeed, given $\epsilon \in (0, s)$, let $\Phi_{\text{super}}$ and $C_\eta$ be the homogeneous supersolution and the cone from Lemma 4.1, where $e = e_n$. Note that $\Phi_{\text{super}}$ is a positive function satisfying $M^- \Phi_{\text{super}} \leq -cd^{-\epsilon-s} < 0$ outside the convex cone $\mathbb{R}^n \setminus C_\eta$, and it is homogeneous of degree $s - \epsilon$. 
Then, we easily check that the function \( \psi = C\Phi_{\text{super}} - \chi_{B_1(z_0)} \), with \( C \) large and \( |z_0| \geq 2 \) such that \( \Phi_{\text{super}} \geq 1 \) in \( B_1(z_0) \), satisfies \( M^+\psi \leq -d^{1-s} \) in \( B_{1/4} \cap C_\eta \) and \( \psi \geq \frac{1}{4} \) in \( C_\eta \setminus B_{1/4} \). Indeed, we simply use that \( M^-\chi_{B_1(z_0)} \geq c_0 > 0 \) in \( B_{1/4} \). Note that this argument exploits the nonlocal character of the operator and a slightly more complicated one would be needed in order to obtain a result that is stable as \( s \uparrow 1 \).

Note that the supersolution \( \psi \) vanishes in \( B_{1/4} \setminus C_\eta \). Then, if \( \ell_0 \) is small enough, for every point in \( x_0 \in \partial \Omega \cap B_{3/4} \) we will have \( x_0 + (B_{1/4} \setminus C_\eta) \subset B_{1/2} \setminus \Omega \).

Then, using translates of \( \psi \) (and \( -\psi \)) upper (lower) barriers we get
\[
|u(x)| \leq \psi(x_0 + x) \leq C|x - x_0|^{s-\epsilon},
\]
as desired.

**Step 2.** To obtain a \( C^\alpha \) estimate up to the boundary, we use the following interior estimate from \([CS09]\): Let \( r \in (0, 1) \),
\[
M^+u \geq -r^{1-2s} \quad \text{and} \quad M^-u \leq r^{1-2s} \quad \text{in } B_r(x)
\]
and
\[
|u(z)| \leq r^\alpha \left( 1 + \frac{(z - x)^\alpha}{r^\alpha} \right)
\]
in all of \( \mathbb{R}^n \).

Then,
\[
[u]_{C^\alpha(B_{r/2}(x))} \leq C,
\]
with \( C \) and \( \alpha > 0 \) depending only on \( s \), ellipticity constants and dimension.

Combining this estimate with \((5.1)\), it follows that
\[
\|u\|_{C^\alpha(B_{1/2})} \leq C.
\]

Thus, the lemma is proved. \( \square \)

We will also need the following.

**Lemma 5.3.** Let \( s \in (0, 1) \), \( \alpha \in (0, \bar{\alpha}) \), and \( C_0 \geq 1 \). Given \( \epsilon \in (0, \alpha] \), there exist \( \delta > 0 \) depending only on \( \epsilon, n, s \), and ellipticity constants, such that the following statement holds.

Assume that \( \Omega \subset \mathbb{R}^n \) is a Lipschitz domain such that \( \partial \Omega \cap B_{1/\delta} \) is a Lipschitz graph of the form \( x_n = g(x') \), in \( |x'| < 1/\delta \) with
\[
[g]_{\text{Lip}(B_{1/\delta})} \leq \delta,
\]
and \( 0 \in \partial \Omega \).

Let \( \varphi \in C(\mathbb{R}^n) \) be a viscosity solution of
\[
M^+ \varphi \geq -\delta d^{-s} \quad \text{and} \quad M^- \varphi \leq \delta d^{-s} \quad \text{in } \Omega \cap B_{1/\delta},
\]
\[
\varphi = 0 \quad \text{in } B_{1/\delta} \setminus \Omega,
\]
satisfying
\[
\varphi \geq 0 \quad \text{in } B_1.
\]
Assume that $\varphi$ satisfies

$$\sup_{B_1} \varphi = 1 \quad \text{and} \quad \|\varphi\|_{L^\infty(B_{2l})} \leq C_0 (2^l)^{s+\alpha} \quad \text{for all } l \geq 0.$$ 

Then, we have

$$\int_{B_1} \varphi^2 \, dx \geq \frac{1}{2} \int_{B_1} (x_n)^{2s} \, dx \quad (5.2)$$

and

$$\left(\frac{1}{2}\right)^{s+\epsilon} \leq \frac{\sup_{B_{2l+1}} \varphi}{\sup_{B_{2l}} \varphi} \leq \left(\frac{1}{2}\right)^{s-\epsilon} \quad \text{for all } l \leq 0. \quad (5.3)$$

Proof. Step 1. We first prove that, for $\delta$ small enough, we have (5.2) and

$$\left(\frac{1}{2}\right)^{s+\epsilon} \leq \frac{\sup_{B_{2l+1}} \varphi_k}{\sup_{B_{2l}} \varphi_k} \leq \left(\frac{1}{2}\right)^{s-\epsilon} \quad (5.4)$$

In a second step we will iterate (5.4) to show (5.3).

The proof of (5.4) is by compactness. Suppose that there is a sequence $\varphi_k$ of functions satisfying the assumptions with $\delta = \delta_k \downarrow 0$ for which one of the three possibilities

$$\left(\frac{1}{2}\right)^{s+\epsilon} > \frac{\sup_{B_{2l+1}} \varphi_k}{\sup_{B_{2l}} \varphi_k}, \quad (5.5)$$

or

$$\frac{\sup_{B_{2l+1}} \varphi_k}{\sup_{B_{2l}} \varphi_k} > \left(\frac{1}{2}\right)^{s-\epsilon}, \quad (5.6)$$

holds for all $k \geq 1$.

Let us show that a subsequence of $\varphi_k$ converges locally uniformly $\mathbb{R}^n$ to the function $(x_n)^*_+$. Indeed, thanks to Lemma 5.2 and the Arzela-Ascoli theorem a subsequence of $\varphi_k$ converges to a function $\varphi \in C(\mathbb{R}^n)$, which satisfies $M^+ \varphi \geq 0$ and $M^- \varphi \leq 0$ in $\mathbb{R}^n_+$, and $\varphi = 0$ in $\mathbb{R}^n_-$. Here we used that $\delta_k \to 0$. Moreover, by the growth control $\|\varphi\|_{L^\infty(B_R)} \leq CR^{s+\alpha}$ and the classification theorem [RS14, Proposition 5.1], we find $\varphi(x) = K(x_n)^*_+$. But since $\sup_{B_1} \varphi_k = 1$, then $K = 1$.

Therefore, we have proved that a subsequence of $\varphi_k$ converges uniformly in $B_1$ to $(x_n)^*_+$. Passing to the limit (5.5), (5.6) or (5.7), we reach a contradiction.

Step 2. We next show that we can iterate (5.4) to obtain (5.3) by induction. Assume that for some $m \leq 0$ we have

$$\left(\frac{1}{2}\right)^{s+\epsilon} \leq \frac{\sup_{B_{2m+1}} \varphi_k}{\sup_{B_{2m}} \varphi_k} \leq \left(\frac{1}{2}\right)^{s-\epsilon} \quad \text{for } m \leq l \leq 0. \quad (5.8)$$

We then consider the function

$$\bar{\varphi} = \frac{\varphi(2^{-m}x)}{\sup_{B_{2m}} \varphi}.$$
and notice that
\[ 2^{(s+\epsilon)l} \leq \sup_{B_{2^l}} \varphi \leq 2^{(s-\epsilon)l} \quad \text{for } m \leq l \leq 0. \]

Thus,
\[ M^+ \varphi \geq -\frac{\delta 2^{2sm}}{2^{(s+\epsilon)m}} \geq -\delta \quad \text{in } (2^{-m}\Omega) \cap B_{2^{-m}/\delta} \]
and similarly
\[ M^- \varphi \leq \delta \quad \text{in } (2^{-m}\Omega) \cap B_{2^{-m}/\delta}. \]
Clearly
\[ \bar{\varphi} = 0 \quad \text{in } (2^{-m}C\Omega) \cap B_{2^{-m}/\delta} \]
and
\[ \varphi \geq 0 \quad \text{in } B_{2^{-m}} \supset B_1. \]

Since \( \partial \Omega \) is Lipchitz with constant \( \delta \) in \( B_{1/\delta} \) and \( 2^{-m} \geq 1 \) \((m \leq 0)\) we have that the rescaled domain \( (2^{-m}\Omega) \cap B_{2^{-m}/\delta} \) is also Lipchitz with the same constant \( 1/\delta \) in a larger ball.

Finally, using \((5.8)\) again we find
\[ \sup_{B_{2^l}} \bar{\varphi} = \sup_{B_{2^{l+m}}} \varphi \leq 2^{(s+\epsilon)l} \leq 2^{(s+\alpha)l} \quad \text{for } l \geq 0 \text{ with } l + m \leq 0, \]

For \( l + m > 0 \) we have
\[ \sup_{B_{2^l}} \bar{\varphi} = \sup_{B_{2^{l+m}}} \varphi \leq \frac{C_0 2^{(s+\alpha)(l+m)}}{2^{(s+\epsilon)m}} = C_0 2^{(s+\alpha)l} 2^{(s+\epsilon)m} \leq C_0 2^{(s+\alpha)l}. \]

Hence, using Step 1, we obtain
\[ \left( \frac{1}{2} \right)^{s+\epsilon} \leq \sup_{B_{1/2}} \bar{\varphi} \leq \left( \frac{1}{2} \right)^{s-\epsilon}. \]

Thus \((5.8)\) holds for \( l = m - 1 \), and the lemma is proved. \( \square \)

We next prove the following.

**Proposition 5.4.** Let \( s \in (0, 1) \), \( \alpha \in (0, \bar{\alpha}) \), and \( C_0 \geq 1 \).

Let \( \Omega \subset \mathbb{R}^n \) be a domain that is improving Lipschitz at 0 with unit outward normal \( e \in S^{n-1} \) and with modulus of continuity \( \rho \) (see Definition 5.1). Then, there exists \( \delta > 0 \), depending only on \( \alpha, s, C_0 \), ellipticity constants, and dimension such that the following statement holds.

Assume that \( r_0 = 1/\delta \) and \( \rho(1/\delta) < \delta \). Suppose that \( u, \varphi \in C(\mathbb{R}^n) \) are viscosity solutions of
\[ \begin{cases} 
M^+(au + b\varphi) \geq -\delta ||a|| + ||b|| d^{\alpha-s} & \text{in } B_{1/\delta} \cap \Omega \\
u = \varphi = 0 & \text{in } B_{1/\delta} \setminus \Omega,
\end{cases} \quad (5.9) \]
for all \( a, b \in \mathbb{R} \). Moreover, assume that
\[ \|au + b\varphi\|_{L^\infty(\mathbb{R}^n)} \leq C_0 (||a|| + ||b||) R^{s+\alpha} \quad \text{for all } R \geq 1, \quad (5.10) \]
\[ \varphi \geq 0 \text{ in } B_1, \text{ and } \sup_{B_1} \varphi = 1. \]

Then, there is \( K \in \mathbb{R} \) with \( |K| \leq C \) such that
\[ |u(x) - K \varphi(x)| \leq C |x|^{s+\alpha} \text{ in } B_1, \]
where \( C \) depends only on \( \rho, C_0, \alpha, s, \) ellipticity constants, and dimension.

**Proof.** Step 1 (preliminary results). Fix \( \epsilon \in (0, \alpha) \). Using Lemma 5.3, if \( \delta \) is small enough we have
\[ \int_{B_1} \varphi^2 \, dx \geq \frac{1}{2} \int_{B_1} (x \cdot n)^{2s} \, dx \geq c(n, s) > 0 \quad (5.11) \]
and
\[ \left( \frac{1}{2} \right)^{s+\epsilon} \leq \frac{\sup_{B_{2l}} \varphi}{\sup_{B_l} \varphi} \leq \left( \frac{1}{2} \right)^{s-\epsilon} \text{ for all } l \leq 0. \quad (5.12) \]
In particular, since \( \sup_{B_1} \varphi = 1 \) then
\[ \left( \frac{r}{2} \right)^{s+\epsilon} \leq \sup_{B_r} \varphi \leq 2 \left( \frac{r}{2} \right)^{s-\epsilon} \text{ for all } r \in (0, 1). \quad (5.13) \]

Step 2. We prove now, with a blow-up argument, that
\[ \|u(x) - K_r \varphi(x)\|_{L^\infty(B_r)} \leq C r^{s+\alpha} \quad (5.14) \]
for all \( r \in (0, 1] \), where
\[ K_r := \frac{\int_{B_r} u \varphi \, dx}{\int_{B_r} \varphi^2 \, dx}. \quad (5.15) \]

Notice that (5.14) implies the estimate of the proposition with \( K = \lim_{r \searrow 0} K_r \).
Indeed, we have \( |K_1| \leq C \) — which is immediate using (5.10) with \( a = 1 \) and \( b = 0 \) and (5.11) — and
\[
|K_r - K_{r/2}| (r/2)^{s+\epsilon} \leq \|K_r \varphi - K_{r/2} \varphi\|_{L^\infty(B_r)} \leq \|u - K_r \varphi\|_{L^\infty(B_r)} + \|u - K_{r/2} \varphi\|_{L^\infty(B_r)} \leq C r^{s+\alpha}.
\]

Thus,
\[ |K| \leq |K_1| + \sum_{j=0}^\infty |K_{2^{-j}} - K_{2^{-j-1}}| \leq C + C \sum_{j=0}^\infty 2^{-j(\alpha - \epsilon)} \leq C, \]
provided that \( \epsilon \) is taken smaller than \( \alpha \).

Let us prove (5.14) by contradiction. Assume that we have a sequences \( \Omega_j, e_j, u_j, \varphi_j \) satisfying the assumptions of the Proposition, but not (5.14). That is,
\[ \lim_{j \to \infty} \sup_{r > 0} r^{-s-\alpha} \|u_j(x) - K_{r,j} \varphi_j\|_{L^\infty(B_r)} = \infty, \]
were \( K_{r,j} \) is defined as in (5.15) with \( u \) replaced by \( u_j \) and \( \varphi \) replace by \( \varphi_j \).
Define, for $r \in (0, 1]$ the nonincreasing quantity

$$\theta(r) = \sup_{r' \in (r, 1]} (r')^{-s-\alpha}\|u_j(x) - K_{r', j}\varphi_j\|_{L^\infty(B_r)}.$$  

Note that $\theta(r) < \infty$ for $r > 0$ since $\|u_j\|_{L^\infty(\mathbb{R}^n)} \leq 1$ and that $\lim_{r \searrow 0} \theta(r) = \infty$.

For every $m \in \mathbb{N}$, by definition of $\theta$ there exist $r'_m \geq 1/m$, $j_m$, $\Omega_m = \Omega_{j_m}$, and $e_m = e_{j_m}$ such that

$$(r'_m)^{-s-\alpha}\|u_{j_m}(x) - K_{r'_m, j_m}\varphi_{j_m}\|_{L^\infty(B_{r'_m})} \geq \frac{1}{2}\theta(1/m) \geq \frac{1}{2}\theta(r'_m).$$

Note that $r'_m \to 0$. Taking a subsequence we may assume that $e_m \to e \in S^{n-1}$. Denote

$$u_m = u_{j_m}, \quad K_m = K_{r'_m, j_m} \quad \text{and} \quad \varphi_m = \varphi_{j_m}.$$  

We now consider the blow-up sequence

$$v_m(x) = \frac{u_m(r'_m x) - K_m \varphi_m(r'_m x)}{(r'_m)^{s+\alpha}\theta(r'_m)}.$$  

By definition of $\theta$ and $r'_m$ we will have

$$\|v_m\|_{L^\infty(B_1)} \geq \frac{1}{2}. \quad (5.16)$$

In addition, by definition of $K_m = K_{r'_m, j_m}$ we have

$$\int_{B_1} v_m(x) \varphi_m(r'_m x) \, dx = 0 \quad (5.17)$$

for all $m \geq 1$.

Let us prove that

$$\|v_m\|_{L^\infty(B_R)} \leq CR^{s+\alpha} \quad \text{for all } R \geq 1. \quad (5.18)$$

Indeed, first, by definition of $\theta(2r)$ and $\theta(r)$,

$$\frac{\|K_{2r, j}\varphi_j - K_{r, j}\varphi_j\|_{L^\infty(B_r)}}{r^{s+\alpha}\theta(r)} \leq \frac{2^{s+\alpha}\theta(2r)}{\theta(r)} \frac{\|u_j - K_{r, j}\varphi_j\|_{L^\infty(B_{2r})}}{(2r)^{s+\alpha}\theta(2r)} + \frac{\|u_j - K_{r/2, j}\varphi_j\|_{L^\infty(B_{2r})}}{r^{s+\alpha}\theta(r)} \leq 2^{s+\alpha} + 1 \leq 5.$$  

On the one hand, using Step 1 we have

$$\frac{|K_{2r, j} - K_{r, j}|(r/2)^{s+\epsilon}}{r^{s+\alpha}\theta(r)} \leq \frac{|K_{2r, j} - K_{r, j}|\varphi_j|_{L^\infty(B_r)}}{r^{s+\alpha}\theta(r)} = \frac{|K_{2r, j}\varphi_j - K_{r, j}\varphi_j|_{L^\infty(B_r)}}{r^{s+\alpha}\theta(r)} \leq 5,$$

and therefore

$$|K_{2r, j} - K_{r, j}| \leq 10 r^{\alpha-\epsilon}\theta(r), \quad (5.19)$$

which we will use later on in this proof.
On the other hand, by (5.12) in Step 1 we have, whenever $0 < 2^l r \leq 2^N r \leq 1$,
\[
\| \varphi_j \|_{L^\infty(B_{2^N r})} \leq 2^{(s+\epsilon)(N-l)} \| \varphi_j \|_{L^\infty(B_{2^l r})}
\]
and therefore
\[
\| K_{2^{l+1}r,j} \varphi_j - K_{2^l r,j} \varphi_j \|_{L^\infty(B_{2^N r})} \leq 2^{(s+\epsilon)(N-l)} \| \varphi_j \|_{L^\infty(B_{2^l r})}.
\]

Thus,
\[
\| K_{2^N r,j} \varphi_j - K_{r,j} \varphi_j \|_{L^\infty(B_{2^N r})} \leq 2^{(s+\epsilon)N} \sum_{l=0}^{N-1} 2^{l(\alpha-\epsilon)} \leq C 2^{(s+\epsilon)N},
\]
where we have used that $\epsilon \in (0, \alpha)$.

Form the previous equation we deduce
\[
\| K_{R r,j} \varphi_j - K_{r,j} \varphi_j \|_{L^\infty(B_{R r})} \leq C R^{s+\alpha}
\]
whenever $0 < r \leq R r \leq 1$.

Hence,
\[
\| u_m \|_{L^\infty(B_R)} = \frac{1}{\theta(r_m')} \| u_m - K_{r,m} \varphi_j \|_{L^\infty(B_{R r_m'})} \leq 2^{s+\alpha} \| u_m - K_{r_m,j} \varphi_j \|_{L^\infty(B_{R r_m'})} + \| K_{R r_m,j} \varphi_j - K_{r_m,j} \varphi_j \|_{L^\infty(B_{R r_m'})} \leq R^{s+\alpha} + C R^{s+\alpha},
\]
whenever $R r_m' \leq 1$.

When $R r_m' \geq 1$ we simply use the assumption (5.10), namely,
\[
\| a u_m + b \varphi_j \|_{L^\infty(\mathbb{R}^n)} \leq C_0 (|a| + |b|) R^{s+\alpha} \quad \text{for all } R \geq 1,
\]
twice, with \(a = 1, b = -K_{1,j_m}\) and with \(a = 0, b = 1\) to estimate
\[
\|v_m\|_{L^\infty(B_R)} = \frac{1}{\theta(r_m')(r_m')^{s+\alpha}} \|u_m - K_m \varphi_m\|_{L^\infty(B_{Rr_m})}
\leq \frac{R^{s+\alpha} \|u_{j_m} - K_{1,j_m} \varphi_{j_m}\|_{L^\infty(B_{Rr_{j_m}})}}{\theta(r_m')(r_m')^{s+\alpha}} + \frac{\|K_{1,j_m} \varphi_{j_m} - K_{r_m,j_m} \varphi_{j_m}\|_{L^\infty(B_{r_{j_m}})}}{(r_m')^{s+\alpha} \theta(r_m')}
\leq C_0(1 + |K_{1,j_m}|)R^{s+\alpha} + \frac{\|K_{1,j_m} \varphi_{j_m} - K_{r_m,j_m} \varphi_{j_m}\|_{L^\infty(B_1)}}{(r_m')^{s+\alpha} \theta(r_m')}
\leq CR^{s+\alpha} + C(r_m')^{-s-\alpha}(Rr_m')^{s+\alpha} \leq CR^{s+\alpha},
\]
where we have used \(|K_{1,j_m}| \leq C\) (that we will prove in detail in Step 3).

**Step 3.** We prove that a subsequence of \(v_m\) converges locally uniformly to a entire solution \(v_\infty\) of the problem
\[
\begin{cases}
M^+ v_\infty \geq 0 \geq M^- v_\infty & \text{in } \{e \cdot x > 0\} \\
v_\infty = 0 & \text{in } \{e \cdot x < 0\}.
\end{cases}
\tag{5.20}
\]

By assumption, the function \(w = au_m + b \varphi_m\) satisfies
\[
\begin{cases}
M^+ (au_m + b \varphi_m) \geq -\delta(|a| + |b|) d^{a-s} & \text{in } B_1 \cap \Omega_m \\
u_m = \varphi_m = 0 & \text{in } B_1 \setminus \Omega_m,
\end{cases}
\tag{5.21}
\]
for all \(a, b \in \mathbb{R}.

Now, using (5.19) we obtain
\[
\frac{|K_{1,j} - K_{2,-N,j}|}{\theta(2^{-N})} \leq \sum_{l=0}^{N-1} \frac{|K_{2,-N+l+1,j} - K_{2,-N+l,j}|}{\theta(2^{-N})}
= \sum_{l=0}^{N-1} 10 \frac{\theta(2^{-N+l})}{\theta(2^{-N})} 2^{(-N+l)(\alpha-\epsilon)}
\leq 10 \sum_{l=0}^{N-1} 2^{(-N+l)(\alpha-\epsilon)} \leq C,
\]
since \(\alpha - \epsilon > 0\).

Next, using (5.11)—that holds with \(\varphi\) replaced by \(\varphi_{j_m}\)—the definition \(K_{r,j}\), and that \(\|\varphi_j\|_{L^\infty(B_1)} = 1\) while \(\|u_j\|_{L^\infty(B_1)} \leq C_0\), we obtain
\[
|K_{1,j}| = \frac{\int_{B_1} u_j \varphi_j \, dx}{\int_{B_1} \varphi_j^2 \, dx} \leq C.
\tag{5.22}
\]
Thus
\[ \frac{|K_{2^{-N},j}|}{\theta(2^{-N})} \leq \frac{|K_{1,j}|}{\theta(2^{-N})} + \frac{|K_{1,j} - K_{2^{-N},j}|}{\theta(2^{-N})} \leq C \]

Using this control for \( K_{r,j} \) and setting in (5.21)
\[ a = \frac{1}{\theta(r'_m)} \quad \text{and} \quad b = -K_{r'_m,j_m} \]
we obtain
\[ M^+ v_m = \frac{(r'_m)^{2s}}{(r'_m)^{s+\alpha}\theta(r'_m)} M^+ \left( \frac{1}{\theta(r'_m)} u_m - \frac{K_{r'_m,j_m}}{\theta(r'_m)} \varphi_m \right) (r'_m \cdot) \]
\[ \geq -C\delta \frac{d^{s-\alpha}}{\theta(r'_m)} \quad \text{in} \quad B_{(r'_m)^{-1}} \cap (r'_m)^{-1} \Omega_m, \]
where \( d_m(x) = \text{dist}(x, r_m^{-1}\Omega_{k_m}) \). Similarly, changing sign in the previous choices of \( a \) and \( b \) we obtain
\[ -M^- (v_m) = M^+ (-v_m) \geq -C\delta \frac{d^{s-\alpha}}{\theta(r'_m)} \quad \text{in} \quad B_{(v'_m)^{-1}} \cap (r'_m)^{-1} \Omega_m \]
As complement datum we clearly have
\[ v_m = 0 \quad \text{in} \quad B_{(r'_m)^{-1}} \setminus (r'_m)^{-1} \Omega_m. \]

Then, by Lemma 5.2 we have
\[ \|v_m\|_{C^\gamma(B_R)} \leq C(R) \quad \text{for all} \quad m \text{large enough}. \]
The constant \( C(R) \) depends on \( R \), but not on \( m \).

Then, by Arzelà-Ascoli and the stability lemma in [CST11b] Lemma 4.3 we obtain that
\[ v_m \rightarrow v_\infty \in C(\mathbb{R}^n), \]
locally uniformly, where \( v_\infty \) satisfies the growth control
\[ \|v_\infty\|_{L^\infty(B_R)} \leq CR^{s+\alpha} \quad \text{for all} \quad R \geq 1 \]
and solves (5.20) in the viscosity sense. Thus, by the Liouville-type result [RS14, Proposition 5.1], we find \( v_\infty(x) = K(x \cdot e)^{s}_+ \) for some \( K \in \mathbb{R} \).

**Step 4.** We prove that as subsequence of \( \tilde{\varphi}_m \), where
\[ \tilde{\varphi}_m(x) = \frac{\varphi_m(r'_mx)}{\sup_{B_{r'_m}} \varphi_m}, \]
converges locally uniformly to \( (x \cdot e)^s_+ \).

This is similar to Step 3 and we only need to use the estimates in Step 1, and the growth control (5.11), to obtain a uniform control of the type
\[ \|\tilde{\varphi}_m\|_{L^\infty(B_R)} \leq C_0 R^{s+\alpha} \quad \text{for all} \quad R \geq 1. \]
Using the estimates in Step 1 we easily show that
\[
\frac{(r'_m)^{2s}}{\sup_{B'_m} \varphi_m} \downarrow 0.
\]
Thus, we use (5.21) with \(a = 0\) and \(b = (\sup_{B'_m} \varphi_m)^{-1}\) to prove that \(\tilde{\varphi}_m\) converges locally uniformly to a solution \(\tilde{\varphi}_\infty\) of
\[
\begin{cases}
M^+ \tilde{\varphi}_\infty \geq 0 \geq M^- \tilde{\varphi}_\infty & \text{in } \{e \cdot x > 0\} \\
\tilde{\varphi}_\infty = 0 & \text{in } \{e \cdot x < 0\},
\end{cases}
\]
Then, using the Liouville-type result \([RS14, \text{Proposition 5.1}]\) and since
\[
\|\tilde{\varphi}_\infty\|_{L^\infty(B_1)} = \lim_{m \to \infty} \|\tilde{\varphi}_m\|_{L^\infty(B_1)} = \lim_{m \to \infty} 1 = 1
\]
we get
\[
\tilde{\varphi}_\infty \equiv (x \cdot e)_+^s.
\]
Hence, \(\tilde{\varphi}_m(x) \to (x \cdot e)_+^s\) locally uniformly in \(\mathbb{R}^n\).

**Step 5.** We have \(v_m \to K(x \cdot e)_+^s\) and \(\tilde{\varphi}_m \to (x \cdot e)_+^s\) locally uniformly. Now, by (5.17),
\[
\int_{B_1} v_m(x) \tilde{\varphi}_m(x) \, dx = 0.
\]
Thus, passing this equation to the limits,
\[
\int_{B_1} v_\infty(x)(x \cdot e)_+^s \, dx = 0.
\]
This implies \(K = 0\) and \(v_\infty \equiv 0\).

But then passing to the limit (5.16) we get
\[
\|v_\infty\|_{L^\infty(B_1)} \geq \frac{1}{2},
\]
a contradiction. \(\square\)

We next prove Theorems 1.3 and 1.6.

**Proof of Theorem 1.6.** **Step 1.** We first show, by a barrier argument, that for any given \(\epsilon > 0\) we have
\[
cd^{s+\epsilon} \leq u_i \leq Cd^{s-\epsilon} \quad \text{in } B_{1/2},
\]
where \(d = \text{dist}(\cdot, B_1 \setminus \Omega)\), and \(c > 0\) is a constant depending only on \(\Omega, n, s\), ellipticity constants.

First, notice that by assumption we have \(M^- u_i = -M^+(-u_i) \leq \delta\) and \(M^+ u_i \geq -\delta\) in \(B_1 \cap \Omega\). Therefore, since \(\sup_{B_{1/2}} u_i \geq 1\), for any small \(\rho > 0\) by the interior Harnack inequality we find
\[
\inf_{B_{3/4} \cap \{d \geq \rho\}} u_i \geq C^{-1} - C\delta \geq c > 0,
\]
provided that \(\delta\) is small enough (depending on \(\rho\)).
Now, let \( x_0 \in B_{1/2} \cap \partial \Omega \), and let \( e \in S^{n-1} \) be the normal vector to \( \partial \Omega \) at \( x_0 \). By the previous inequality,
\[
\inf_{B_{\rho}(x_0 + 2\rho e)} u_i \geq c.
\]
Since \( \Omega \) is \( C^1 \), then for any \( \eta > 0 \) there is \( \rho > 0 \) for which
\[
(x_0 + \mathcal{C}_\eta) \cap B_{4\rho} \subset \Omega,
\]
where \( \mathcal{C}_\eta \) is the cone in Lemma 4.1.

Therefore, using the function \( \Phi_{\text{sub}} \) given by Lemma 4.1, we may build the subsolution
\[
\psi = \Phi_{\text{sub}} \chi_{B_{4\rho}(x_0)} + C_1 \chi_{B_{\rho/2}(x_0 + 2\rho e)}.
\]
Indeed, if \( C_1 \) is large enough then \( \psi \) satisfies
\[
M^- \psi \geq 1 \quad \text{in} \quad (x_0 + \mathcal{C}_\eta) \cap (B_{3\rho}(x_0) \setminus B_{\rho}(x_0 + 2\rho e))
\]
and \( \psi \equiv 0 \) outside \( x_0 + \mathcal{C}_\eta \).

Hence, we may use \( c_2 \psi \) as a barrier, with \( c_2 \) small enough so that \( u_i \geq c_2 \psi \) in \( B_{\rho}(x_0 + 2\rho e) \). Then, by the comparison principle we find
\[
u_i \geq c_2 \psi,
\]
and in particular
\[
u_i(x_0 + te) \geq c_3 t^{s+\epsilon}
\]
for \( t \in (0, \rho) \). Since this can be done for all \( x_0 \in B_{1/2} \cap \partial \Omega \), we find
\[
u_i \geq c d^{s+\epsilon} \quad \text{in} \quad B_{1/2}.
\]
(5.23)

Similarly, using the supersolution \( \Phi_{\text{sup}} \) from Lemma 4.1, we find
\[
u_i \leq C d^{s-\epsilon} \quad \text{in} \quad B_{1/2},
\]
(5.24)
for \( i = 1, 2 \).

Step 2. Let us prove now that
\[
u_1 \leq C \nu_2 \quad \text{in} \quad B_{1/2}.
\]
(5.25)

To prove (5.25), we rescale the functions \( u_1 \) and \( u_2 \) and use Proposition 5.4. Let \( x_0 \in B_{1/2} \cap \partial \Omega \), and let
\[
\theta(r) = \sup_{r' > r} \frac{\|u_1\|_{L^\infty(B_{r'}(x_0))} + \|u_2\|_{L^\infty(B_{r'}(x_0))}}{(r')^{s+\epsilon}}.
\]
Notice that \( \theta(r) \) is monotone nonincreasing and that \( \theta(r) \to \infty \) by (5.23). Let \( r_k \to 0 \) be such that
\[
\|u_1\|_{L^\infty(B_{r_k}(x_0))} + \|u_2\|_{L^\infty(B_{r_k}(x_0))} \geq \frac{1}{2} (r_k)^{s+\epsilon} \theta(r_k),
\]
with \( c_0 > 0 \), and define
\[
v_k(x) = \frac{u_1(x_0 + r_k x)}{(r_k)^{s+\epsilon} \theta(r_k)}, \quad w_k(x) = \frac{u_2(x_0 + r_k x)}{(r_k)^{s+\epsilon} \theta(r_k)}.
\]
Note that
\[ \|v_k\|_{L^\infty(B_1)} + \|w_k\|_{L^\infty(B_1)} \geq \frac{1}{2}. \]
Moreover,
\[ \|v_k\|_{L^\infty(B_R)} = \frac{\|u_1\|_{L^\infty(B_R)}}{(r_k)^{s+\epsilon}\theta(r_k)} \leq \frac{\theta(r_kR)(r_kR)^{s+\epsilon}}{(r_k)^{s+\epsilon}\theta(r_k)} \leq R^{s+\epsilon}, \]
for all \( R \geq 1 \), and analogously
\[ \|w_k\|_{L^\infty(B_R)} \leq R^{s+\epsilon} \]
for all \( R \geq 1 \).

Now, the functions \( v_k, w_k \) satisfy the equation
\[ M^+(av_k + bw_k)(x) = \frac{(r_k)^{2s}}{(r_k)^{s+\epsilon}\theta(r_k)} M^+(av_1 + bu_2)(x_0 + r_kx) \geq -C_0(r_k)^{s-\epsilon}\delta(\|a\| + |b|) \]
in \( \Omega_k \cap B_{r_k^{-1}} \), where \( \Omega_k = r_k^{-1}(\Omega - x_0) \).

Taking \( k \) large enough, we will have that \( \Omega_k \) satisfies the hypotheses of Proposition 5.4 in \( B_{1/\delta} \), and
\[ M^+(av_k + bw_k) \geq -\delta(\|a\| + |b|) \quad \text{in} \quad \Omega_k \cap B_{1/\delta}. \]
Moreover, since \( \sup_{B_1} v_k + \sup_{B_1} w_k \geq 1/2 \), then either \( \sup_{B_1} v_k \geq 1/4 \) or \( \sup_{B_1} w_k \geq 1/4 \). Therefore, by Proposition 5.4 we find that either
\[ |v_k(x) - K_1 w_k(x)| \leq C|x|^{s+\alpha} \]
or
\[ |w_k(x) - K_2 v_k(x)| \leq C|x|^{s+\alpha} \]
for some \( |K| \leq C \). This yields that either
\[ |u_1(x) - K_1 u_2(x)| \leq C|x - x_0|^{s+\alpha} \quad (5.26) \]
or
\[ |u_2(x) - K_2 u_1(x)| \leq C|x - x_0|^{s+\alpha}, \quad (5.27) \]
with a bigger constant \( C \).

Now, we may choose \( \epsilon > 0 \) so that \( \epsilon < \alpha/2 \), and then (5.27) combined with (5.23)-(5.24) gives \( K_2 \geq c > 0 \), which in turn implies (5.26) for \( K_1 = K_2^{-1}, |K_1| \leq C \). Thus, in any case (5.26) is proved.

In particular, for all \( x_0 \in B_{1/2} \cap \partial \Omega \) and all \( x \in B_{1/2} \cap \Omega \) we have
\[ u_1(x)/u_2(x) \leq K_1 + \frac{|u_1(x)|}{|u_2(x)|} - K_1 \leq K_1 + C|x - x_0|^{s+\alpha}/u_2(x). \]
Choosing \( x_0 \) such that \( |x - x_0| \leq Cd(x) \) and using (5.24), we deduce
\[ u_1(x)/u_2(x) \leq K_1 + Cd^{s+\alpha}/d^{s-\epsilon} \leq C, \]
and thus (5.26) is proved.
Step 3. We finally show that $u_1/u_2 \in C^\alpha(\Omega \cap B_{1/2})$ for all $\alpha \in (0, \bar{\alpha})$. Since this last step is somewhat similar to the proof of Theorem 1.2 in [RST14b], we will omit some details.

We use that, for all $\alpha \in (0, \bar{\alpha})$ and all $x \in B_{1/2} \cap \Omega$, we have

$$\left| \frac{u_1(x)}{u_2(x)} - K(x_0) \right| \leq C|x - x_0|^{\alpha - \epsilon}, \quad (5.28)$$

where $x_0 \in B_{1/2} \cap \partial \Omega$ is now the closest point to $x$ on $B_{1/2} \cap \partial \Omega$. This follows from (5.26), as shown in Step 2.

We also need interior estimates for $u_1/u_2$. Indeed, for any ball $B_{2r}(x) \subset \Omega \cap B_{1/2}$, with $2r = d(x)$, there is a constant $K$ such that $\|u_1 - Ku_2\|_{L^\infty(B_r(x))} \leq Cr^{s+\alpha}$. Thus, by interior estimates we find that $[u_1 - Ku_2]_{C^{\alpha - \epsilon}(B_r(x))} \leq Cr^{s+\epsilon}$. This, combined with (5.23)-(5.24) yields

$$[u_1/u_2]_{C^{\alpha - \epsilon}(B_r(x))} \leq C. \quad (5.29)$$

Let now $x, y \in B_{1/2} \cap \Omega$, and let us show that

$$\left| \frac{u_1(x)}{u_2(x)} - \frac{u_1(y)}{u_2(y)} \right| \leq C|x - y|^{\alpha - \epsilon}. \quad (5.30)$$

If $y \in B_r(x)$, $2r = d(x)$, or if $x \in B_r(y)$, $2r = d(y)$, then this follows from (5.29). Otherwise, we have $|x - y| \geq \frac{1}{2} \max\{d(x), d(y)\}$, and then (5.30) follows from (5.28).

In any case, (5.30) is proved, and therefore we have

$$\|u_1/u_2\|_{C^{\alpha - \epsilon}(\Omega \cap B_{1/2})} \leq C.$$

Since this can be done for any $\alpha \in (0, \bar{\alpha})$ and any $\epsilon > 0$, the result follows. \hfill \Box

Proof of Theorem 1.3. The proof is the same as Theorem 1.6, replacing the Liouville-type result [RST14, Proposition 5.1] by [RST14b, Theorem 4.1], and replacing $\bar{\alpha}$ by $s$. \hfill \Box

Remark 5.5. Notice that in Proposition 5.4 we only require the right hand side of the equation to be bounded by $d^{\alpha - s}$. Thanks to this, Theorem 1.3 holds as well for

$$- \delta d^{\alpha - s} \leq f_i(x) \leq C_0 d^{\alpha - s}, \quad \alpha \in (0, s). \quad (5.31)$$

In that case, we get

$$\|u_1/u_2\|_{C^\alpha(\Omega \cap B_{1/2})} \leq CC_0,$$

with the exponent $\alpha$ in (5.31).

Proof of Corollary 1.4. The result follows from Theorem 1.3. \hfill \Box
References

[BBB91] R. Bañuelos, R. Bass, K. Burdzy, Hölder domains and the boundary Harnack principle, Duke Math. J. 64 (1991) 195–200.

[BB94] R. Bass, K. Burdzy, The boundary Harnack principle for non-divergence form elliptic operators, J. Lond. Math. Soc. 50 (1994), 157-169.

[Bog97] K. Bogdan, The boundary Harnack principle for the fractional Laplacian, Studia Math., 123 (1997), 43-80.

[BKK08] K. Bogdan, T. Kulczycki, and M. Kwasnicki, Estimates and structure of α-harmonic functions, Probab. Theory Related Fields 140 (2008), 345-381.

[BKK15] K. Bogdan, T. Kumagai, M. Kwasnicki, Boundary Harnack inequality for Markov processes with jumps, Trans. Amer. Math. Soc. 367 (2015), 477-517.

[CRS15] L. Caffarelli, X. Ros-Oton, J. Serra, Obstacle problems for integro-differential operators: regularity of solutions and free boundaries, preprint arXiv (Jan. 2016).

[CS09] L. Caffarelli, L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math. 62 (2009), 597-638.

[CS11b] L. Caffarelli, L. Silvestre, Regularity results for nonlocal equations by approximation, Arch. Rat. Mech. Anal. 200 (2011), 59-88.

[Dah77] B. Dahlberg, Estimates of harmonic measure, Arch. Rat. Mech. Anal. 65 (1977) 275–288.

[DS14] D. De Silva, O. Savin, Boundary Harnack estimates in slit domains and applications to thin free boundary problems, Rev. Mat. Iberoam., to appear.

[Gru15] G. Grubb, Fractional Laplacians on domains, a development of Hörmander’s theory of µ-transmission pseudodifferential operators, Adv. Math. 268 (2015), 478-528.

[Gru14] G. Grubb, Local and nonlocal boundary conditions for µ-transmission and fractional elliptic pseudodifferential operators, Anal. PDE 7 (2014), 1649-1682.

[RS14] X. Ros-Oton, J. Serra, Boundary regularity for fully nonlinear integro-differential equations, Duke Math. J., to appear.

[RS14b] X. Ros-Oton, J. Serra, Regularity theory for general stable operators, J. Differential Equations, to appear.

[RV15] X. Ros-Oton, E. Valdinoci, The Dirichlet problem for nonlocal operators with singular kernels: convex and non-convex domains, Adv. Math. 288 (2016), 732-790.

[SW99] R. Song, J.-M. Wu, Boundary Harnack principle for symmetric stable processes, J. Funct. Anal. 168 (1999), 403-427.

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