A BSDEs approach to pathwise uniqueness for stochastic evolution equations

Federica Masiero
*Università di Milano Bicocca*

joint works with Davide Addona *Università di Parma*
and with Enrico Priola *Università di Pavia*

June 30\textsuperscript{th} 2022

BSDEs in infinite dimensional control

*Organizer: Giuseppina Guatteri*

The 9th International Colloquium on BSDEs and Mean Field Systems
PLAN

1. Setting of the problem: stochastic wave equation

2. On regularization by noise for non degenerate SPDEs

3. Regularizing properties of the transition semigroup-wave equation
   - The related forward-backward system
   - Well posedness for the stochastic wave equation
   - Regularizing properties of the transition semigroup-damped wave equation
   - The related forward-backward system and the PDE

4. Well posedness for the stochastic damped wave equation

5. A unified BSDE approach for evolution equations
Setting of the problem

**Stochastic wave equation**

\[
\begin{aligned}
\frac{\partial^2}{\partial \tau^2} y (\tau, \xi) &= \frac{\partial^2}{\partial \xi^2} y (\tau, \xi) + b (\tau, \xi, y (\tau, \xi)) + \varepsilon \dot{W} (\tau, \xi), \\
y (\tau, 0) &= y (\tau, 1) = 0, \\
y (0, \xi) &= x_0 (\xi), \\
\frac{\partial y}{\partial \tau} (0, \xi) &= x_1 (\xi), \quad \tau \in (0, T], \quad \xi \in [0, 1],
\end{aligned}
\]

- \(\dot{W} (\tau, \xi)\) space-time white noise, \(f: (e_k)_{k \geq 1}\) o.n. basis in \(L^2([0, 1])\),

\[
W (\tau, \xi) = \sum_{k \geq 1} \beta_k (\tau) e_k (\xi);
\]

- \(b\) bounded measurable, \(\beta\)-Hölder continuous in \(y\), \(\beta \in (2/3, 1)\).
- **Without noise** \((\varepsilon = 0)\) equation (1) not well posed.

\[
b (\xi, y) = 56 \sqrt[4]{\sin \xi |y|^3} \cdot I_{\{|y|<2T^8\}} + |y| \cdot I_{\{|y|<2T^8\}} + 56 \sqrt[4]{8T^{24}} \sin \xi \cdot I_{\{|y|\geq2T^8\}} + 2T^8 I_{\{|y|\geq2T^8\}}.
\]
Stochastic wave equation: Abstract reformulation

\( \Lambda = -\frac{d^2}{dx^2} \) with Dirichlet boundary conditions: in \( U = L^2([0, 1]) \)

\( \mathcal{D}(\Lambda) = H^1_0([0, 1]) \cap H^2([0, 1]), \mathcal{D}(\Lambda^{1/2}) = H^1_0([0, 1]), \mathcal{D}(\Lambda^{-1/2}) = H^{-1}([0, 1]). \)

Set \( X^{0,x}_\tau = (y(\tau), \frac{dy}{d\tau}(\tau)) \), \( y \) solution to (1), \( x = (x_0, x_1) \).

**Wave operator:** \( A = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix} \) in \( H = L^2([0, 1]) \times H^{-1}([0, 1]) \)

\( \Lambda \) positive self-adjoint operator on \( U \leadsto H = U \times \mathcal{D}(\Lambda^{-1/2})' \)

\[
\begin{align*}
dX^{0,x}_\tau &= AX^{0,x}_\tau d\tau + GB(\tau, X^{0,x}_\tau) d\tau + GdW_\tau, \quad \tau \in [0, T], \\
& \quad X^{0,x}_0 = x \in H.
\end{align*}
\]

where

\[
GdW_\tau = \begin{pmatrix} 0 \\ dW_\tau \end{pmatrix}, \quad GB(\tau, X_\tau) = \begin{pmatrix} 0 \\ B(\tau, X_\tau) \end{pmatrix}
\]
Setting of the problem

• $W_A(\tau) := \int_0^\tau e^{(\tau-s)A}GdW_s$ not well defined in $K = \mathcal{D}(\Lambda^{1/2}) \times U$ even if $B = 0$

• $X$ evolves in $H = U \times \mathcal{D}(\Lambda^{-1/2}) = L^2([0, 1]) \times H^{-1}([0, 1])$.

• $B : [0, T] \times H \to U$ Borel, bounded and $\alpha$-Holder continuous, $\alpha \in (2/3, 1)$: $|B(t, x + h) - B(t, x)|_U \leq C|h|_H^\alpha$, $x, h \in H$, $t \in [0, T]$, $\alpha \in (2/3, 1)$.

\begin{align*}
B &\in B_b([0, T]; C^\alpha_b(H, U)), \quad GB \in B_b([0, T]; C^\alpha_b(H, H)).
\end{align*}

• Existence of a weak solution: by the Girsanov Theorem

We prove **pathwise uniqueness** $\leadsto$ **strong existence** by the Yamada-Watanabe principle (see [Ondreját, Dissertationes Math. 2004]).
Overview on regularization by noise

- A.K. Zvonkin: Mat. Sb. (N.S.) (1974) [\(b \in L^\infty(\mathbb{R}): d = 1\)]
- A.J. Veretennikov: Mat. Sb. (N.S.) (1980) [\(b \in L^\infty(\mathbb{R}^d), d \geq 1\)].

Idea of the method: ODEs

- A variant of the Zvonkin-Veretennikov approach: the Ito-Tanaka trick for SDEs (cf. Flandoli-Gubinelli-Priola 2010):

\[b: \mathbb{R} \to \mathbb{R}\] be an irregular function (it could be Hölder continuous).

\[
X_t = x + \int_0^t b(X_s)ds + W_t, \quad t \geq 0, \quad x \in \mathbb{R}.
\]

Write

\[
X_t - x - W_t = \int_0^t b(X_s)ds.
\]
Overview on regularization by noise

Let $v$ be a “regular” solution of

$$\lambda v - Lv = b \quad \text{on } \mathbb{R}, \quad \lambda > 0,$$

$L = \frac{1}{2} \frac{d^2}{dx^2} + b(x) \cdot \frac{d}{dx}$ then by Itô’s formula:

$$v(X_t) = v(x) + \int_0^t v'(X_s) dW_s + \int_0^t Lv(X_s) ds$$

and so

$$v(X_t) = v(x) + \int_0^t v'(X_s) dW_s + \int_0^t (\lambda v(X_s) - b(X_s)) ds$$

and

$$X_t + v(X_t) = x + v(x) + W_t + \int_0^t v'(X_s) dW_s + \lambda \int_0^t v(X_s) ds$$

$$\Rightarrow \quad \text{uniqueness thank to the regularity of } v$$
Overview on regularization by noise

**Idea of the method:** non degenerate SPDEs

\[
dX^0,x_\tau = AX^0,x_\tau d\tau + B(\tau, X^0,x_\tau) d\tau + \sqrt{Q} dW_\tau, \quad \tau \in [0, T], \quad X^0,x_0 = x \in \mathcal{H}.
\]

**Assumptions**

- \( A : \mathcal{D}(A) \in \mathcal{H} \to \mathcal{H}, \ Ae_n = -\alpha_n e_n, \ (\alpha_n)_{n \geq 1} \) non-decreasing

- \( B \in C([0, T]; C^\alpha_b(H, H)) \), set \( B_n := \langle B, e_n \rangle \).

- either \( T r(Q) < \infty \) or

\[
\sum_{n=1}^{\infty} \frac{\|B_n\|_\alpha}{\alpha_n} < \infty.
\]

- \( Q_t = \int_0^t e^{sA}Qe^{sA^*} ds \) and \( e^{tA}(H) \in Q_t^{1/2}(H), \ t > 0 \)

We mainly refer to Da Prato-Flandoli, JFA 2010.
Overview on regularization by noise

Idea of the method

• **Kolmogorov PDEs:**

\[
\frac{\partial U_n(t, x)}{\partial t} + \mathcal{L}_t[U_n(t, \cdot)](x) = GB_n(t, x), \quad x \in H, U_n(T, x) = 0, \quad n \geq 1
\]

\[
\mathcal{L}_t[f](x) = \frac{1}{2} Tr GG^* \nabla^2 f(x) + \langle Ax, \nabla f(x) \rangle + <GB(t, x), \nabla U_n(t, x)>.
\]

• **U(t, x) = \sum_{n=1}^{\infty} U_n(t, x)e_n \rightsquigarrow Ito-Tanaka trick: Ito formula to U(t, X_t)**

\[
B(t, X_t) = dU(t, X_t) - \nabla U(t, X_t)\sqrt{Q}dW_t
\]

• **mild form for X_t:**

\[
X_t = e^{tA}(x - U(0, x)) + U(t, X_t) + \int_0^t A e^{(t-s)A} U(s, X_s) ds
\]

\[
+ \int_0^t e^{(t-s)A} \nabla U(s, X_s) dW_s - \int_0^t e^{(t-s)A} \sqrt{Q} dW_s.
\]
Wave transition semigroup

Coming back to our setting...

wave equation
\[ dX_t^x = AX_t^x d\tau + GB(\tau, X_t^x) d\tau + GdW_\tau, \quad \tau \in [t, T], \quad X_t^x = x \in H. \]

Ornstein Uhlenbeck process for the wave equation
\[ d\Xi_0^x = A\Xi_0^x d\tau + GdW_\tau, \quad \tau \in [0, T], \quad \Xi_0^x = x \in H. \]

Wave transition semigroups
\[ P_\tau [\phi] (x) = \mathbb{E} \phi (\Xi^0_\tau, x), \phi \in B_b(H, \mathbb{R}), \quad R_\tau [\Phi] (x) = \mathbb{E} \Phi (\Xi^0_\tau), \Phi \in B_b(H, H) \]

\((R_\tau)_{\tau \geq 0} \): \(H\)-valued transition semigroup

- Regularizing properties: from \(B_b\) functions, to differentiable and \(G\)-differentiable functions.

\[ \nabla^G_a F(x) = \nabla_{Ga} F(x) = \lim_{s \to 0} \frac{F(x + sGa) - F(x)}{s} = \nabla_{Ga} F(x), \quad a \in U, \quad x \in H. \]
Wave transition semigroup

**Regularizing properties**

\[
\begin{aligned}
\dot{w}(t) &= Aw(t) + Gu(t), \\
w(0) &= k \in H,
\end{aligned}
\]

null controllable \(\leftrightarrow\) \(\text{Im} e^{tA} \subset \text{Im} Q_t^{1/2}\)

- \(|Q_t^{-1/2}e^{tA}h|_H \leq \frac{c}{t^{3/2}}|h|_H, \quad h \in H;\)

- \(\begin{aligned}
\dot{w}(t) &= Aw(t) + Gu(t), \\
w(0) &= k \in \text{Im}(G),
\end{aligned}\)

null controllable \(\leftrightarrow\) \(\text{Im} e^{tAG} \subset \text{Im} Q_t^{1/2}\)

- \(|Q_t^{-1/2}e^{tAG}a|_H \leq \frac{c}{t^{3/2}}|\Lambda^{-1/2}a|_U, \quad a \in U;\) (M. AMO 2005, M-Priola JDE 2017)
Damped wave + wave transition semigroup

Lemma (first & second order regularization) \( \Phi \in C_b(H, H) \ \forall \ t > 0 \)

\[
\sup_{x \in H} |\nabla_k R_t[\Phi](x)| \leq \frac{c}{t^2} \|\Phi\|_\infty |k|_H; \\
\sup_{x \in H} |\nabla^G_a R_t[\Phi](x)| \leq \frac{c}{t^{\frac{1}{2}}} \|\Phi\|_\infty \Lambda^{-1/2} a|_U, \ \sup_{x \in H} \|\nabla^G R_t[\Phi](x)\|_{L_2(U,H)} \leq \frac{c}{t^{\frac{1}{2}}} \|\Phi\|_\infty. \\
\sup_{x \in H} \|\nabla_k (\nabla^G R_t[\Phi])(x)\|_{L_2(U,H)} \leq \frac{c |k|_H}{t^2} \|\Phi\|_\infty, \\
\lim_{x \to 0} \sup_{|k| = 1} \sup_{y \in H} \|\nabla_k (\nabla^G R_t[\Phi])(x + y) - \nabla_k (\nabla^G R_t[\Phi])(y)\|_{L_2(U,H)} = 0. \\
\]

Lemma (interpolation)

\( \Phi \in C^\alpha_b(H, H), \ k \in H, \ \forall \ t > 0 \)

\[
\sup_{x \in H} |\nabla_k R_t[\Phi](x)|_H \leq \frac{c}{t^\frac{3}{2}(1-\alpha)} \|\Phi\|_\alpha |k|_H; \\
\sup_{x \in H} \|\nabla_k (\nabla^G R_t[\Phi])(x)\|_{L_2(U,H)} \leq \frac{c}{t^{\frac{4-3\alpha}{2}}} \|\Phi\|_\alpha |k|_H. \\
\]
Forward-Backward system (FBSDE)

\[
\begin{align*}
    d\Xi_{t,x}^\tau &= A\Xi_{t,x}^\tau d\tau + GdW^\tau, \quad \tau \in [t, T], \\
    \Xi_{t,x}^t &= x, \\
    -dY_{t,x}^\tau &= -AY_{t,x}^\tau + GB(\tau, \Xi_{t,x}^\tau) d\tau - Z_{t,x}^\tau B(\tau, \Xi_{t,x}^\tau) d\tau - Z_{t,x}^\tau dW^\tau, \quad \tau \in [0, T], \\
    Y_{T,x}^t &= 0,
\end{align*}
\]

A wave operator \( \sim -A \) generator of a semigroup of operators

**FBSDE in mild formulation**

\[
Y_{t,x}^\tau = \int_T^\tau e^{-(s-\tau)A} GB(s, \Xi_{s,x}^t) ds - \int_T^\tau e^{-(s-\tau)A} Z_{s,x}^t B(s, \Xi_{s,x}^t) ds - \int_T^\tau e^{-(s-\tau)A} Z_{s,x}^t dW_s,
\]

**Existence and regularity results** (Hu-Peng SAP 1991, Guatteri JAMSA 2007) \( \exists \) a unique solution \((Y, Z)\) s.t.

\[
\mathbb{E} \sup_{\tau \in [0,T]} |Y_{t,x}^\tau|_H^2 + \mathbb{E} \int_0^T |Z_{\tau}|_{L^2(U,H)}^2 \leq C \sup_{t \in [0,T], x \in H} |B(t, x)|_U.
\]
If moreover $x \mapsto B(\tau, x), \ H \to U$, differentiable, for a.a. $\tau \in [0, T]$, then $x \mapsto (Y^{t,x}, Z^{t,x})$ is also differentiable.

Definition of $vv(t, x) := Y^{t,x}_t \leadsto x \mapsto v(t, x)$ differentiable.

Identification of $Z$ with $\nabla^G v$

- If $x \mapsto B(\tau, x), \ H \to U$, differentiable,
  $$\nabla^G_k v(\tau, \Xi^{t,x}_\tau) = Z^{t,x}_\tau k \text{ in } H \ \mathbb{P} \text{ a.s. for a.a. } \tau \in [t, T]$$
- If $x \mapsto B(\tau, x), \ H \to U$, Hölder continuous: by an approximation procedure (see Peszat-Zabczyk AoP 1995) on $B$

$$B^n(\tau, x) = \int_{\mathbb{R}^n} \rho_n(y - Q_nx) B \left( \tau \sum_{i=1}^n y_ig_i \right) dy,$$

to $B^n$ it is associated $v^n$

$$\nabla^G_k v^n(\tau, \Xi^{t,x}_\tau) = Z^{n,t,x}_\tau k \text{ in } H \ \mathbb{P} \text{ a.s. for a.a. } \tau \in [t, T]$$

let $n \to \infty$.  

13
$$v(t, x) = \int_t^T e^{-(s-t)A} G(s, \Xi_{s,x}^t) \, ds - \int_t^T e^{-(s-t)A} Z_{s,x}^t B(s, \Xi_{s,x}^t) \, ds + \int_t^T e^{-(s-t)A} Z_{s,x}^t \, dW_s$$

$$= \int_t^T e^{-(s-t)A} G(s, \Xi_{s,x}^t) \, ds - \int_t^T e^{-(s-t)A} \nabla^G v(s, \Xi_{s,x}^t) B(s, \Xi_{s,x}^t) \, ds + \int_t^T e^{-(s-t)A} \nabla^G v(s, \Xi_{s,x}^t) \, dW_s,$$

taking expectation

$$v(t, x) = \mathbb{E} \int_t^T e^{-(s-t)A} G(s, \Xi_{s,x}^t) \, ds - \mathbb{E} \int_t^T e^{-(s-t)A} \nabla^G v(s, \Xi_{s,x}^t) B(s, \Xi_{s,x}^t) \, ds$$

$$= \int_t^T R_{s-t} \left[ e^{-(s-t)A} G(s, \cdot) \right] (x) \, ds - \int_t^T R_{s-t} \left[ e^{-(s-t)A} \nabla^G v(s, \cdot) B(s, \cdot) \right] (x) \, ds$$

$$\sup_{t \in [0,T_0], x \in H} \| \nabla v(t, x) \| \leq 1/2; \ \nabla_k (\nabla^G u(t, x)) \in B_b([0,T]; L^2(U, H)), \ k \in H.$$
$$dX_{\tau}^{t,x} = AX_{\tau}^{t,x} \, d\tau + GB(\tau, X_{\tau}^{t,x}) \, d\tau + GdW_\tau, \quad \tau \in [t, T], \quad X_t^{t,x} = x,$$

mild formulation

$$X_{\tau}^{t,x} = e^{(\tau-\cdot)A} x + \int_t^\tau e^{(\tau-s)A} GB(s, X_s) \, ds + \int_t^\tau e^{(\tau-s)A} GdW_s, \quad \tau \in [t, T].$$

Set: \( \tilde{Y}_{\tau}^{t,x} := v(\tau, X_{\tau}^{t,x}), \quad \tilde{Z}_{\tau}^{t,x} := \nabla^G v(\tau, X_{\tau}^{t,x}). \) \( \leadsto \) solution to the BSDE

$$-d\tilde{Y}_{\tau}^{t,x} = -A\tilde{Y}_{\tau}^{t,x} \, d\tau + GB(\tau, X_{\tau}^{t,x}) \, d\tau - \tilde{Z}_{\tau}^{t,x} \, dW_\tau, \quad \tilde{Y}_{T}^{t,x} = 0.$$

mild formulation of the BSDE

$$\tilde{Y}_{\tau}^{t,x} = \int_\tau^T e^{-(s-\tau)A} GB(s, X_s^{t,x}) \, ds - \int_\tau^T e^{-(s-\tau)A} \tilde{Z}_s^{t,x} \, dW_s, \quad \tau \in [0, T].$$
Link between the FBSDEs: \( v(t, x) = Y_{t,x}^t = \tilde{Y}_{t,x}^t \)

**Proposition** (M-Priola, JDE 2017) For \( t = 0 \) the mild of the stochastic wave equation can be rewritten as

\[
X_{\tau}^{0,x} = e^{\tau A} x + e^{\tau A} v(0, x) - v(\tau, X_{\tau}^x) + \int_0^\tau e^{(\tau-s)A} \nabla^G v(s, X_s^x) \, dW_s + \int_0^\tau e^{(\tau-s)A} G dW_s.
\]

The “bad” term \( B \) has been removed

**Proof** For \( \tau \in [0, T] \)

\[
e^{-\tau A} \tilde{Y}_{\tau}^{0,x} = e^{-\tau A} \int_\tau^T e^{-(s-\tau)A} GB(s, X_s^{0,x}) \, ds - e^{-\tau A} \int_\tau^T e^{-(s-\tau)A} \tilde{Z}_s^{0,x} \, dW_s.
\]

\( \tau = 0 : \) \( \tilde{Y}_{0}^{0,x} = v(0, x) = \int_0^T e^{-sA} GB(s, X_s^{0,x}) \, ds - \int_0^T e^{-sA} \tilde{Z}_s^{0,x} \, dW_s. \)

So

\[
\int_0^\tau e^{(\tau-s)A} GB(s, X_s^x) \, ds = e^{\tau A} v(0, x) - v(\tau, X_{\tau}^x) - \int_0^\tau e^{(\tau-s)A} \nabla^G v(s, X_s^x) \, dW_s
\]
Theorem (M-Priola, JDE 2017) For the stochastic wave equation (1) pathwise uniqueness holds. \( \exists c > 0 \) s. t.

\[
\sup_{\tau \in [0,T]} E|X_\tau^{x_1} - X_\tau^{x_2}|_H^2 \leq c|x_1 - x_2|_H^2, \quad x_1, x_2 \in H
\]

Proof \( X^1, X^2 \) starting at \( x_1, x_2 \). By the proposition

\[
X_\tau^1 - X_\tau^2 = e^{\tau A}(x_1 - x_2) + e^{\tau A}[v(0, x_1) - v(0, x_2)]
\]

\[-[v(\tau, X_\tau^1) - v(\tau, X_\tau^2)] + \int_0^\tau e^{(\tau-s)A}[\nabla G v(s, X_s^1) - \nabla G v(s, X_s^2)] \, dW_s.
\]

by regularity properties of \( v \)

\[
|e^{\tau A}(x_1 - x_2)|_H + |e^{\tau A}[v(0, x_1) - v(0, x_2)]|_H + |v(\tau, X_\tau^1) - v(\tau, X_\tau^2)|_H
\]

\[
\leq C|x_1 - x_2|_H + \frac{1}{2}|X_\tau^1 - X_\tau^2|_H
\]
by Ito isometry

$$
\mathbb{E} \left| \int_0^\tau e^{(\tau-s)A} \left[ \nabla^G v(s, X_s^1) - \nabla^G v(s, X_s^2) \right] \, dW_s \right|^2 
\leq \mathbb{E} \int_0^\tau \left\| \nabla^G v(s, X_s^1) - \nabla^G v(s, X_s^2) \right\|_{L_2(U,H)}^2 ds
$$

$(e_k)_k$ basis in $U$

$$
\mathbb{E} \int_0^\tau \left\| \nabla^G v(s, X_s^1) - \nabla^G v(s, X_s^2) \right\|_{L_2(U,H)}^2 ds = \sum_{k \geq 1} \mathbb{E} \int_0^\tau \left| \nabla^G_{e_k} v(s, X_s^1) - \nabla^G_{e_k} v(s, X_s^2) \right|_H^2 ds
\leq \sup_{t,x} \sup_{|k|_H = 1} \left\| \nabla_k \nabla^G v(t, x) \right\|_{L_2(U,H)}^2 \int_0^\tau \mathbb{E} |X_s^1 - X_s^2|_H^2 ds.
$$

Gronwall Lemma: for $x_1 = x_2 \sim$ uniqueness.
For $x_1, x_2 \in H$, $x_1 \neq x_2 \sim$ Lipschitz continuous dependence on the initial data.

By the Yamada-Watanabe Theorem: there exists a strong solution to (1).
Stochastic damped wave equation

\[
\begin{cases}
\frac{\partial^2 y}{\partial t^2}(t) = -\Lambda y(t) - \rho^\alpha \frac{\partial y}{\partial t}(t) + b \left( t, y(t), \frac{\partial y}{\partial t}(t) \right) + \dot{W}(t), & t \in (0, T], \\
y(0) = y_0, \\
\frac{\partial y}{\partial t}(0) = y_1,
\end{cases}
\]

(2)

- \( \rho > 0, \alpha \in [0, 1) \)
- \( b \) bounded measurable, \( \beta \)-Hölder continuous in \( y, \beta \in (\beta_\alpha, 1) \) with

\[
\beta_\alpha = \begin{cases} 
\frac{2}{3}, & \alpha \in \left[0, \frac{3}{4}\right], \\
2 - \frac{1}{\alpha}, & \alpha \in \left[\frac{3}{4}, 1\right), 
\end{cases}
\]

- Without noise (\( \varepsilon = 0 \)) equation (2) not well posed.
Stochastic damped wave equation

Stochastic damped wave equation: Abstract reformulation

\[ \Lambda = -\frac{d^2}{dx^2} \] with Dirichlet boundary conditions: in \( U = L^2([0,1]) \)

\[ D(\Lambda) = H^1_0([0,1]) \cap H^2([0,1]), \quad D(\Lambda^{1/2}) = H^1_0([0,1]), \quad D(\Lambda^{-1/2}) = H^{-1}([0,1]). \]

Set \( X^{0,x}_\tau = (y(\tau), \frac{dy}{d\tau}(\tau)) \), \( y \) solution to (2), \( x = (x_0, x_1) \).

Damped wave operator: \( A_{\alpha,\rho} := \begin{pmatrix} 0 & I \\ -\Lambda & -\rho\Lambda\alpha \end{pmatrix} \), in \( H := L^2([0,1]) \times H^{-1}([0,1]) \)

\[ dX^{0,x}_\tau = A_{\alpha,\rho}X^{0,x}_\tau d\tau + GB(\tau, X^{0,x}_\tau) d\tau + GdW_\tau, \quad \tau \in [0,T], \quad X^{0,x}_0 = x \in H. \]

where

\[ G := \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad GdW_\tau = \begin{pmatrix} 0 \\ dW_\tau \end{pmatrix}, \quad GB(\tau, X_\tau) = \begin{pmatrix} 0 \\ B(\tau, X_\tau) \end{pmatrix} \]
Setting of the problem

\[
\begin{align*}
    dX_t &= \mathcal{A}_{\alpha,\rho}X_t dt + GdW_t, \quad t \in [0, T], \\
    X_0 &= (x_0^1, x_0^2)^T \in H.
\end{align*}
\]

Well posedness in $H$ in mild sense

\[
X_t = e^{t\mathcal{A}_{\alpha,\rho}}x + \int_0^t e^{(t-s)\mathcal{A}_{\alpha,\rho}}GdW_s, \quad t \in [0, T].
\]

\(\leadsto\) study the stochastic convolution

\[
W_{\mathcal{A}_{\alpha,\rho}}(t) := \int_0^t e^{(t-s)\mathcal{A}_{\alpha,\rho}}GdW_s, \quad t \in [0, T],
\]

\(\leadsto\) prove that $\int_0^T \|e^{t\mathcal{A}_{\alpha,\rho}}G\|_{L_2(U,H)}^2 dt < +\infty$: assume $\Lambda^{-1}$ trace class operator $U \rightarrow U$.

Covariance operator $Q_t^{\alpha,\rho} := \int_0^t e^{s\mathcal{A}_{\alpha,\rho}}GG^*e^{s\mathcal{A}_{\alpha,\rho}^*}ds$ trace class operator

spectral decomposition of $\mathcal{A}_{\alpha,\rho}$: Chen-Russel 1981, Triaggiani 1988, Lasiecka-Triggiani 2003
Damped wave transition semigroup

Damped wave equation

\[ dX_t^x = A_{\alpha,\rho}X_t^x \, d\tau + G B(\tau, X_t^x) \, d\tau + G dW_\tau, \quad \tau \in [t, T], \quad X_t^x = x \in H. \]

Ornstein Uhlenbeck process for the damped wave equation with \( \alpha \in \left[0, \frac{3}{4}\right) \)

\[ d \Xi^0 = A_{\alpha,\rho} \Xi^0 \, d\tau + G dW_\tau, \quad \tau \in [0, T], \quad \Xi^0 = x \in H. \]

Damped wave transition semigroups

\[ P_\tau \left[ \phi \right] \left( x \right) = \mathbb{E} \phi \left( \Xi^0 \right), \quad \phi \in B_b(H, \mathbb{R}), \quad R_\tau \left[ \Phi \right] \left( x \right) = \mathbb{E} \Phi \left( \Xi^0 \right), \quad \Phi \in B_b(H, H) \]

\( (R_\tau)_{\tau \geq 0} \): \( H \)-valued transition semigroup

- **Regularizing properties**: from \( B_b \) functions, to differentiable and \( G \)-differentiable functions.
Regularizing properties

\[
\begin{cases}
\dot{w}(t) = A_{\alpha,\rho}w(t) + Gu(t), \\
w(0) = k \in H,
\end{cases}
\]

null controllable \(\Leftrightarrow\) \(\text{Im} e^{tA_{\alpha,\rho}} \subset \text{Im} Q_t^{1/2}\)

- \(|Q_t^{-1/2}e^{tA_{\alpha,\rho}}h|_H \leq \frac{c}{t^{3/2}}|h|_H, \ h \in H;\)

\[
\begin{cases}
\dot{w}(t) = A_{\alpha,\rho}w(t) + Gu(t), \\
w(0) = k \in \text{Im}(G),
\end{cases}
\]

null controllable \(\Leftrightarrow\) \(\text{Im} e^{tA_{\alpha,\rho}G} \subset \text{Im} Q_t^{1/2}\)

- \(|Q_t^{-1/2}e^{tA_{\alpha,\rho}}Ga|_H \leq \frac{c}{t^{3/2}}|\Lambda^{-1/2}a|_U, \ a \in U.\) (Addona-M-Priola 2021)
\(-A_{\alpha,\rho}\) damped wave operator \(\rightarrow -A_{\alpha,\rho}\) is not the generator of a semigroup of operators

take \(A_{\alpha,\rho}^j\) Yosida approximants of \(A_{\alpha,\rho}\) and then pass to the limit

**Forward-Backward system (FBSDE)**

\[
\begin{cases}
    d\Xi_{t,x,j}^\tau = A_{\alpha,\rho}^j \Xi_{t,x,j}^\tau d\tau + GdW, & \tau \in [t,T], \\
    \Xi_{t,x,j}^t = x, \\
    -dY_{t,x,j}^\tau = -A_{\alpha,\rho}^j Y_{t,x,j}^\tau + GB(\tau,\Xi_{t,x,j}^\tau) d\tau - Z_{t,x,j}^\tau B(\tau,\Xi_{t,x,j}^\tau) d\tau - Z_{t,x,j}^\tau dW, & \tau \in [0,T], \\
    Y_{T}^{t,x} = 0.
\end{cases}
\]

Set \(v^j(t,x) := Y_{t,x,j}^t\).

\[
v^j(t,x) = \mathbb{E} \int_t^T e^{-(s-t)A_{\alpha,\rho}^j} GB(s,\Xi_s^{t,x}) ds - \mathbb{E} \int_t^T e^{-(s-t)A_{\alpha,\rho}^j} \nabla G v(s,\Xi_s^{t,x}) B(s,\Xi_s^{t,x}) ds
\]

\[
= \int_t^T R_{s-t} \left[ e^{-(s-t)A_{\alpha,\rho}^j} GB(s,\cdot) \right] (x) ds - \int_t^T R_{s-t} \left[ e^{-(s-t)A_{\alpha,\rho}^j} \nabla G v(s,\cdot) B(s,\cdot) \right] (x) ds
\]

\(\rightarrow\) no convergence here if we let \(j \rightarrow +\infty\).

regularity of \(v\) from regularizing properties of \(R_t\)
\[ v^j(t,x) = \int_t^T R_{s-t} \left[ e^{-(s-t)A^{j}_{\alpha,\rho}}GB(s,\cdot) \right] (x) \, ds \quad - \quad \int_t^T R_{s-t} \left[ e^{-(s-t)A^{j}_{\alpha,\rho}}\nabla G v(s,\cdot)B(s,\cdot) \right] (x) \, ds \]

Set \( \tilde{v}^j(t,x) := e^{(T-t)A^{j}_{\alpha,\rho}}Y_t^{t,x,j} \). We can prove that \( \tilde{v}^j \) satisfies

\[ \tilde{v}^j(t,x) := \int_t^T R_{s-t} \left[ e^{(T-s)A^{j}_{\alpha,\rho}}G\tilde{C}(s,\cdot) \right] (x) \, ds \quad + \quad \int_t^{T_0} R_{s-t} \left[ \nabla G \tilde{v}^j(s,\cdot)\tilde{C}(s,\cdot) \right] (x) \, ds, \]

Now we can we let \( j \to +\infty: \tilde{v}(t,x) = \lim_{j \to +\infty} \tilde{v}^j(t,x) \)

\[ \tilde{v}(t,x) := \int_t^T R_{s-t} \left[ e^{(T-s)A^{j}_{\alpha,\rho}}G\tilde{C}(s,\cdot) \right] (x) \, ds \quad + \quad \int_t^{T_0} R_{s-t} \left[ \nabla G \tilde{v}^j(s,\cdot)\tilde{C}(s,\cdot) \right] (x) \, ds, \]
Well posedness-damped wave equation

\[ dX^t,x = A_{\alpha,\rho}X^t,x d\tau + GB(\tau, X^t,x) d\tau + GdW_\tau, \quad \tau \in [t, T], \quad X^t,x = x \in H. \]

**mild formulation**

\[ X^t,x = e^{(\tau-t)A_{\alpha,\rho} x} + \int_t^\tau e^{(\tau-s)A_{\alpha,\rho} GB(s, X_s)} ds + \int_t^\tau e^{(\tau-s)A_{\alpha,\rho} GdW_s}, \quad \tau \in [t, T]. \]

By the definition of \( \tilde{v}^j \) and by BSDEs techniques

\[ \int_0^\tau e^{(\tau-s)A_{\alpha,\rho} GB(s, X_s)} ds = e^{\tau A_{\alpha,\rho} Y^0,x,j} + \int_0^\tau e^{(\tau-s)A_{\alpha,\rho} Z^0,x,j} dW_s \]

\[ = \tilde{v}^j(0, x) + \int_0^\tau \nabla G\tilde{v}^j(s, X^x_s) dW_s, \quad \forall \tau \in [0, T]. \]
Proposition (Addona-M-Priola) For $t = 0$ the mild form of the damped stochastic wave equation can be rewritten as

$$X_{\tau}^{0,x} = e^{\tau A_{\alpha,\rho} x} + \int_0^\tau \left( e^{(\tau-s)A_{\alpha,\rho}} - e^{(\tau-s)A_j^{\alpha,\rho}} \right) B(s, X_s^x) ds + \tilde{v}^j(0, x)$$

$$+ \int_0^\tau \nabla^G \tilde{v}^j(s, X_s^x) dW_s + \int_0^\tau e^{(\tau-s)A_{\alpha,\rho}} G dW_s.$$

The “bad” term $B$ appears through

$$\int_0^\tau \left( e^{(\tau-s)A_{\alpha,\rho}} - e^{(\tau-s)A_j^{\alpha,\rho}} \right) B(s, X_s^x) ds := \delta^j(t) \to 0, \text{ as } j \to \infty$$
Theorem (Addona-M-Priola) For the stochastic damped wave equation (2) pathwise uniqueness holds. \( \exists \ c > 0 \) s. t.

\[
\sup_{\tau \in [0,T]} E|X_\tau^{x_1} - X_\tau^{x_2}|_H^2 \leq c|x_1 - x_2|_H^2, \quad x_1, x_2 \in H
\]

Proof \( X^1, X^2 \) starting at \( x_1, x_2 \). By the proposition

\[
X^1_\tau - X^2_\tau = e^{\tau A_{a,p}}(x_1 - x_2) + e^{\tau A_{a,p}}[\tilde{v}^j(0, x_1) - \tilde{v}^j(0, x_2)]
\]

\[
+ \delta_1^j(t) + \delta_2^j(t) + \int_0^\tau e^{(\tau-s)A_{a,p}}[\nabla G\tilde{v}(s, X^1_s) - \nabla G\tilde{v}(s, X^2_s)] \, dW_s.
\]

by regularity properties of \( v \)

\[
|e^{\tau A_{a,p}}(x_1 - x_2)|_H + |e^{\tau A_{a,p}}[\tilde{v}^j(0, x_1) - \tilde{v}^j(0, x_2)]|_H \leq C|x_1 - x_2|_H
\]

\[
E \left| \int_0^\tau e^{(\tau-s)A} [\nabla G\tilde{v}^j(s, X^1_s) - \nabla G\tilde{v}^j(s, X^2_s)] \, dW_s \right|^2 \leq C E|X^1_s - X^2_s|_H^2 ds
\]

Gronwall Lemma: for \( x_1 = x_2 \leadsto \) uniqueness.

For \( x_1 \neq x_2 \leadsto \) Lipschitz continuous dependence on the initial data.

By the Yamada-Watanabe Theorem: there exists a strong solution to (2).
A unified BSDE approach

- **wave equation**: Study a FBSDE, set \( v(t, x) := Y^t,x_t \) that solves
  \[
  v(t, x) = \int_t^T R_{s-t} \left[ e^{-(s-t)A} GB(s, \cdot) \right] (x) \, ds \\
  - \int_t^T R_{s-t} \left[ e^{-(s-t)A} \nabla^G v(s, \cdot) B(s, \cdot) \right] (x) \, ds
  \]

- **damped wave equation**: Study an approximated FBSDE, set \( \tilde{v}^j(t, x) := e^{(T_0-s)A_{\alpha,\rho}} Y_t, x^j \), \( \tilde{v}^j \) satisfies
  \[
  \tilde{v}^j(t, x) = \int_t^{T_0} R_{s-t} \left[ e^{(T_0-s)A_{\alpha,\rho}} G\tilde{C}(s, \cdot) \right] (x) ds + \int_t^{T_0} R_{s-t} \left[ \nabla^G \tilde{v}^j(s, \cdot) \tilde{C}(s, \cdot) \right] (x) ds
  \]

- **evolution equations of parabolic type**: start from the integral PDE above, with different approximations of the operator \( A \)
A unified BSDE approach

\[
\begin{cases}
    dX_{\tau}^{t,x} = AX_{\tau}^{t,x} d\tau + GB(\tau, X_{\tau}^{t,x}) d\tau + GdW_{\tau}, \quad \tau \in [t, T], \quad 0 \leq t \leq \tau, \\
    X_{t}^{t,x} = x \in H,
\end{cases}
\]

\[X_{\tau}^{t,x} \sim \Xi_{t}^{0,x}, \text{ if } B = 0; \quad R_{t}[\Phi](x) := \mathbb{E}[\Phi(\Xi_{t}^{0,x})], \quad t \geq 0, \quad \Phi \in B_{b}(H, H)\]

\[v(t, x) = \int_{t}^{T} R_{s-t} \left[ e^{(T-s)A_{0}}GBs, \cdot \right] (x) ds + \int_{t}^{T} R_{s-t} \left[ \nabla^{G}v(s, \cdot)B(s, \cdot) \right] (x) ds,\]

\[u_{n}^{T}(t, x) = \int_{t}^{T} R_{s-t} \left[ e^{(T-s)A_{n}}GB(s, \cdot) \right] (x) ds + \int_{t}^{T} R_{s-t} \left[ \nabla^{G}u_{n}^{T}(s, \cdot)B(s, \cdot) \right] (x) ds.\]

\(A_{0}\) generator of a \(C_{0}\)-semigroup; \(A_{n}\) generator of a \(C_{0}\)-group of operators

\[
\sup_{t \in [0,T]} \sup_{n \geq 1} \left\| e^{tA_{n}} \right\|_{L(H)} = K_{T} < \infty, \quad \lim_{n \to \infty} e^{tA_{n}}x = e^{tA}x, \quad x \in H, \quad t \geq 0.
\]

\[
\sup_{x \in H} \left\| u_{n}^{T}(0, x + y) - u_{n}^{T}(0, x) \right\|_{H} \leq C_{T} |y|_{H}, \quad y \in H,
\]

\[
\sup_{x \in H} \left\| \nabla^{G}u_{n}^{T}(t, x + y) - \nabla^{G}u_{n}^{T}(t, x) \right\|_{L_{2}(U; H)}^{2} \leq h(T - t) |y|_{H}^{2}, \quad t \in (0, T), \quad y \in H.
\]
A unified BSDE approach

\[
\begin{align*}
\frac{dX^t_\tau}{0,x} &= AX^t_\tau \frac{d\tau}{0,x} + GB(\tau, X^t_\tau) \frac{d\tau}{0,x} + GdW_\tau \\
X^0_0 \frac{d\tau}{0,x} &= x \in H, \quad \tau \in [0, T], \\
X^x_\tau := X^0_\tau & \quad \forall \tau \in [0, T].
\end{align*}
\]

\[
X^x_\tau = e^{\tau A} x + \int_0^\tau \left( e^{(\tau-s)A} - e^{(\tau-s)A_n} \right) GB(s, X^x_s) ds + \int_0^\tau e^{(\tau-s)A_n} GB(s, X^x_s) ds \]

\[
+ \int_0^\tau e^{(\tau-s)A} GdW_s, \quad \forall \tau \in [0, T].
\]

**Proposition** (Addona-M-Priola) For any \( n \in \mathbb{N} \) and any \( \tau \in [0, T] \) we have

\[
X^x_\tau = e^{\tau A} x + \int_0^\tau \left( e^{(\tau-s)A} - e^{(\tau-s)A_n} \right) GB_s, X^x_s) ds \]

\[
+ u^\tau_n(0, x) + \int_0^\tau \nabla^G u^\tau_n(s, X^x_s) dW_s + \int_0^\tau e^{(\tau-s)A} GdW_s, \quad \mathbb{P}-a.s..
\]

**Theorem** \( \exists \ c = c(T) > 0 \) s.t. \( \forall x_1, x_2 \in H \)

\[
\sup_{t \in [0, T]} \mathbb{E}[|X^x_{t_1} - X^x_{t_2}|^2_H] \leq c|x_1 - x_2|^2_H,
\]

\( \leadsto \) uniqueness + Lipschitz continuous dependence
A unified BSDE approach

• Formally $u^T_n$ solves

$$\begin{cases}
\frac{\partial u^T_n(t, x)}{\partial t} + L_t[u^T_n(t, \cdot)](x) = -e^{(T-t)A_n}GB(t, x), & x \in H, \ t \in [0, T], \\
 u^T_n(T, x) = 0, & x \in H,
\end{cases}$$

where $L_t f(x) := \frac{1}{2} \text{Tr}[GG^*\nabla^2 f(x)] + \langle Ax, \nabla f(x) \rangle + \langle GB(t, x), \nabla f(x) \rangle$

$\rightsquigarrow$ we perform regularization by noise for stochastic heat equation in dimension $d = 3$ not reached in Da Pronto-Flandoli JFA 2010 Da Pronto-Flandoli JFA 2010 $-e^{(T-t)A_n}GB$ is replaced by $GB$.

• $v_n := e^{-(T-t)A_n}u^T_n$ formally solves

$$\begin{cases}
\frac{\partial v^T_n(t, x)}{\partial t} + L_t[v^T_n(t, \cdot)](x) = A_n v^T_n(t, x) - GB(t, x), & x \in H, \ t \in [0, T], \\
 v^T_n(T, x) = 0, & x \in H,
\end{cases}$$

similar to the PDE used for the wave equation
Further developments

- stochastic (damped) wave equations with multiplicative noise: gradient estimates to prove regularizing properties of the transition semigroup

- application to regularization by noise

- application of dilations theorems (van Neerven, Veraar) that give a group of operators in a larger space.

- BSDE approach for other problems of regularization by noise
**Short Biblio**

Da Prato, Flandoli, “Pathwise uniqueness for a class of SDEs in Hilbert spaces and applications.” J. Funct. Anal. 259 (2010), no. 1, 243–267.

Da Prato, Flandoli, Priola, Röckner, “Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift.” Ann. Probab. 41 (2013), 3306-3344.

Da Prato, Flandoli, Priola, Röckner, “Strong uniqueness for stochastic evolution equations with unbounded measurable drift term”. J. Theoret. Probab. 28 (2015) 1571-1600

Ondreját, “Uniqueness for stochastic evolution equations in Banach spaces.” Dissertationes Math. (Rozprawy Mat.) 426 (2004)

M., Priola, “Well-posedness of semilinear stochastic wave equations with Hölder continuous coefficients”, J. Differential Equations 263 (2017)

Addona, M., Priola, “A BSDEs approach to pathwise uniqueness for stochastic evolution equations”, preprint arXiv:2110.01994