Energy spectrum of a 2D Dirac oscillator in the presence of a constant magnetic field

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In this paper we obtain exact solutions of a 2D relativistic Dirac oscillator in the presence of a constant magnetic field. We compute the energy spectrum and discuss its dependence on the spin and magnetic field strength.

I. INTRODUCTION

During the last years the study of electron in two-dimensional systems has become a subject of active research. The rapid advances in microfabrication technology has made possible to confine laterally two dimensional electron systems. These quantum confined few-electron systems are often referred to as artificial atoms where the potential of the nucleus is replaced by an effective potential. A parabolic potential $V = \frac{1}{2} \omega r^2$ is often used as a realistic and at the same time computationally convenient approximation. Despite its simplicity, parabolic quantum dots appear to be a good approximation to complicated quantum dot structures. [1]

The solution of the two-dimensional Schrödinger equation for a single electron in a homogeneous magnetic field $\mathcal{B}$ has been known after the publication of the papers by Fock [2] and Darwin [3] in the 20’s. The inclusion of a parabolic potential to this problem leads to equations that can be solved without significant complications.

Despite the large body of papers discussing the confinement of electrons in low dimensional structures, almost all of them tackle the problem in the framework of the Schrödinger equation. Among the difficulties when we consider relativistic Dirac electrons with parabolic confinement we have that we do not have at hand analytic solutions in terms of special functions and we need to use semi-analytical and numerical methods in order to compute the energy spectrum.

The relativistic extension of a parabolic confining potential can be done with the of the Dirac oscillator [4]. Among the advantages of this approach over the inclusion of Lorentz scalar potential $\frac{1}{2} m \omega^2 r^2$, as confining potential, we have that the first one leads to a non-relativistic quadratic Hamiltonian in both coordinates and momenta and therefore its solution can also be expressed in terms of Laguerre or Hermite polynomials. The stability of the Dirac sea for the Dirac oscillator Hamiltonian [5] also shows that this model does not present difficulties related to the Klein’s Paradox.

In the present article we solve the two-dimensional Dirac oscillator in the presence of a constant magnetic field $\mathcal{B}$ perpendicular to the plane where the electron is confined to move. The article is structured as follows. In Sec. II, we solve the Dirac oscillator in polar coordinates. In Sec. III, we analyze the energy spectrum of the oscillator. In Sec. IV, we discuss our results and the behavior of the energy levels in the non-relativistic limit. Along the paper we adopt the natural units $\hbar = 1$, $c = 1$.

II. SOLUTION OF THE DIRAC OSCILLATOR

The Dirac oscillator proposed by Moshinsky and Szczepaniak [4] includes a type of interaction in the Dirac equation which, besides the momentum, is also linear in the coordinates. The Dirac oscillator reduces, in the non-relativistic limit, to a harmonic oscillator with a very strong spin-orbit coupling term. Namely, the correction to the free Dirac equation

$$i \frac{\partial \Psi_c}{\partial t} = (\beta \gamma \mathbf{p} + \beta m) \Psi_c$$

(1)

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has the form:

\[ p \rightarrow p - \text{i}m\beta r. \]  

(2)

After substituting (2) into (1) we get an Hermitian operator linear in both \( p \) and \( r \). Recently, the Dirac oscillator has been studied in spherical coordinates and its energy spectrum and the corresponding eigenfunctions have been obtained [5]. Since we are interested in studying the Dirac oscillator in a two-dimensional space, a suitable system of coordinates for writing the harmonic interaction are the polar \( \rho \) and \( \vartheta \) coordinates. In polar coordinates \((t, r, \vartheta)\) the metric tensor \( g_{\alpha\beta} \) takes the form

\[ g_{\alpha\beta} = \text{diag}(-1, 1, r^2), \]  

(3)

and the corresponding unitary vectors \( \hat{e}_\rho \) and \( \hat{e}_\vartheta \) are

\[ \hat{e}_\rho = \cos \vartheta \hat{i} + \sin \vartheta \hat{j}, \quad \hat{e}_\vartheta = -\sin \vartheta \hat{i} + \cos \vartheta \hat{j} \]  

(4)

In this case the radial component of the modified linear momentum takes the form: \( p_\rho - \text{i}m\omega \beta \rho \). The vector potential \( \vec{A} \) associated with a constant magnetic field interaction is

\[ \vec{A} = \frac{B\rho}{2} \hat{e}_\vartheta. \]  

(5)

The corresponding \( \vec{B} \) can be written in polar coordinates as follows:

\[ \vec{B} = B\hat{e}_z \]  

(6)

Since we are interested in computing the energy spectrum of a two dimensional confined electron in the presence of a constant magnetic field, in the present article, we analyze the solution of the 2+1 Dirac oscillator in the presence of a constant magnetic field associated with the vector potential (5).

One begins by writing the Dirac equation (1) in a given representation of the gamma matrices. Since we are dealing with two component spinors it is convenient to introduce the following representation in terms of the Pauli matrices

\[ \beta \gamma_1 = \sigma_1, \quad \beta \gamma_2 = \sigma_2, \quad \beta = \sigma_3 \]  

(7)

It is worth mentioning that Dirac matrices expressed in the Cartesian tetrad gauge take the form

\[ \tilde{\gamma}^\rho = \vec{\gamma} \cdot \hat{e}_\rho = \gamma^1 \cos \vartheta + \gamma^2 \sin \vartheta \]  

(8)

\[ \tilde{\gamma}^\vartheta = \vec{\gamma} \cdot \hat{e}_\vartheta = -\gamma^1 \sin \vartheta + \gamma^2 \cos \vartheta \]  

(9)

the similarity transformation \( S(\vartheta) \) which reduces the Dirac matrices \( \tilde{\gamma}^\rho \) and \( \tilde{\gamma}^\vartheta \) to \( \gamma^1 \) and \( \gamma^2 \) is

\[ S(\vartheta) = \cos \frac{\vartheta}{2} - \gamma^1 \gamma^2 \sin \frac{\vartheta}{2} \]  

(10)

with

\[ S^{-1}(\vartheta)\tilde{\gamma}^\rho S(\vartheta) = \gamma^1, \quad S^{-1}(\vartheta)\tilde{\gamma}^\vartheta S(\vartheta) = \gamma^2. \]

Using the matrix representation (5) and the similarity transformation (10), we obtain that the Dirac oscillator in the presence of a constant magnetic field (3) has the form

\[ iE\Psi = H\Psi = \left[ \sigma^1 \partial_\rho + \sigma^2 \left( \frac{ik_\vartheta}{\rho} - m\omega - \frac{eB\rho}{2} \right) + i\sigma^3 m \right] \Psi, \]  

(11)

with,

\[ \Psi = \Psi_0(\rho)e^{i(k_\vartheta \vartheta - Et)}, \]  

(12)

and

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\[ \Psi_0 = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \]  

where the spinor \( \Psi \) is expressed in the (rotating) diagonal gauge. It is related to the Cartesian (fixed) spinor by means of the transformation \( S(\rho, \vartheta) \)

\[ \Psi = \sqrt{\rho} S(\vartheta)^{-1} \Psi_c \]  

with

\[ S(\vartheta) \Psi_d = \Psi_c \]  

The matrix transformation \( S(\vartheta) \) can be written as follows

\[ S(\vartheta) = \exp(-\frac{\vartheta}{2} \gamma^1 \gamma^2) = \exp(-\frac{\vartheta}{2} \sigma_3) \]  

The \( \sqrt{\rho} \) factor has been introduced in (14) in order to eliminate the term \( S(\vartheta) S(\vartheta)^{-1} \rho \). Noticing that \( S(\vartheta) \) satisfies the relation

\[ S(\vartheta + 2\pi) = -S(\vartheta) \]  

we obtain,

\[ \Psi(\vartheta + 2\pi) = -\Psi(\vartheta) \]  

so we have \( k_\vartheta = N + 1/2 \), where \( N \) is an integer number. The rotating Dirac spinor \( \Psi_d \) can be written in terms of \( \Psi_c \) as:

\[ \Psi_d = \begin{pmatrix} e^{i\vartheta/2} \Psi_1c \\ e^{-i\vartheta/2} \Psi_2c \end{pmatrix}, \quad \text{with} \quad \Psi_c = \begin{pmatrix} \Psi_1c \\ \Psi_2c \end{pmatrix}. \]  

We can label the quantum states of the Dirac oscillator in terms of eigenstates of the parity operator. In fact, analogously to the three dimensional Dirac oscillator [5] we have that the parity operator commutes with the Hamiltonian

\[ [P, H] = 0 \]  

where the parity operator acts on the Dirac spinor in the Cartesian gauge as follows:

\[ P \Psi_c(\rho) = \beta \Psi_c(-\rho) = \sigma_3 \Psi(-\rho) \]  

In cylindrical coordinates, the reflection respecting to the origin \( \rho \rightarrow -\rho \) is obtained via the rotation

\[ \rho \rightarrow \rho \quad \vartheta \rightarrow \vartheta + \pi \]  

Taking into account the relation (14) between \( \Psi \) and \( \Psi_c \) as well as (12) we readily obtain

\[ \Psi_c(\rho) = \begin{pmatrix} e^{i(k_\vartheta - 1/2)\vartheta} \Psi_1(\rho) \\ e^{-i(k_\vartheta + 1/2)\vartheta} \Psi_2(\rho) \end{pmatrix} \]  

and consequently

\[ P \Psi_c(\rho) = (-1)^{k_\vartheta - 1/2} \Psi_c(\rho) \]  

therefore, the parity of the energy eigenfunctions is given by \((-1)^{k_\vartheta - 1/2}\). Reminding that the total angular momentum \( J \) expressed in the Cartesian gauge takes the form

\[ J = -i \left( \frac{\partial}{\partial \vartheta} + \frac{\gamma^1 \gamma^2}{2} \right), \]  

and taking into account the representation (7) for the Dirac matrices we have that
\[ J \Psi_c(r) = (k_\vartheta - \frac{1}{2} \sigma^3) \Psi_c(r) \]  

(26)

then, the eigenvalues of the total angular momentum are

\[ j = k_\vartheta \mp \frac{1}{2} \]  

(27)

Using the representation (6), the spinor equation (11) can be written as system of two first order coupled differential equations,

\[ i(E - m) \Psi_1(\rho) = \left( \frac{d}{d\rho} + \frac{k_\vartheta}{\rho} - \rho \bar{\omega} \right) \Psi_2(\rho), \]  

(28)

\[ i(E + m) \Psi_2(\rho) = \left( \frac{d}{d\rho} - \frac{k_\vartheta}{\rho} + \rho \bar{\omega} \right) \Psi_1(\rho), \]  

(29)

where the frequency \( \bar{\omega} \) can be written in terms of the Larmor frequency as follows:

\[ \bar{\omega} = \omega + s \omega_L = \omega + \frac{eB}{2m} \]  

(30)

substituting (29) into (28) and vice-versa we arrive at

\[ \left[ \frac{d^2}{d\rho^2} - \frac{(k_\vartheta)(k_\vartheta + 1)}{\rho^2} + m\bar{\omega}(2k_\vartheta \pm 1) - m^2 \bar{\omega}^2 \rho^2 + (E^2 - m^2) \right] \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 0 \]  

(31)

It is not difficult to see that the solution of the second order equation (31) for \( \Psi_1 \) can be expressed in terms of associated Laguerre polynomials \( L^\mu_n(x) \) as follows

\[ \Psi_1 = c_1 \exp(-x/2)x^{k_\vartheta/2} L^\mu_n(x) \]  

(32)

where we have made the change of variables

\[ x = m\bar{\omega}\rho^2 \]  

(33)

the parameter \( \mu \) is

\[ \mu = \pm(k_\vartheta - 1/2) \]  

(34)

and the natural number \( n \) satisfies the relation

\[ \frac{E^2 - m^2}{m\bar{\omega}} + (1 \mp 1)(2k_\vartheta - 1) = 4n \]  

(35)

Since the function \( \Psi_1 \) must be regular at the origin, we obtain that the sign of \( \mu \) in (32) is determined by the sign of \( k_\vartheta \). In fact, for \( k_\vartheta > 0 \) we have that \( \Psi_1 \) reads

\[ \Psi_1 = c_1 \exp(-x/2)x^{k_\vartheta/2} L^{k_\vartheta}_{n-1/2}(x) \]  

(36)

substituting (36) into (29) we arrive at

\[ \Psi_2 = 2ic_1 \frac{(m\bar{\omega})^{1/2}}{E + m} \exp(-x/2)x^{(k_\vartheta+1)/2} L^{k_\vartheta+1/2}_{n-1}(x) \]  

(37)

where \( c_1 \) is an arbitrary constant.

Analogously, we obtain that the regular solutions for \( k_\vartheta < 0 \) are,

\[ \Psi_2 = c_2 \exp(-x/2)x^{-k_\vartheta/2} L^{-k_\vartheta-1/2}_{n}(x) \]  

(38)

\[ \Psi_1 = 2ic_2 \frac{(m\bar{\omega})^{1/2}}{E + m} \exp(-x/2)x^{(1-k_\vartheta)/2} L^{1/2-k_\vartheta}_{n}(x) \]  

(39)

where \( c_1 \) is a normalization constant. The expression (35) can be rewritten as follows

\[ E^2 - m^2 = 4 \left[ n - \Theta(-k_\vartheta)(k_\vartheta - 1/2) \right] (m\omega + \frac{eB}{2}) \]  

(40)

where \( \Theta(x) \) is the Heaviside step function.
III. STUDY OF THE ENERGY SPECTRUM

From the relation (40) it is clear that the energy spectrum of the 2+1 Dirac oscillator depends on the value of $k_\vartheta$. Notice that for positive values of $k_\vartheta$ the energy of the system has the form

$$E^2 = m^2 + 4n(m\omega + \frac{eB}{2})$$  \hspace{0.5cm} (41)$$

For $k_\vartheta < 0$ we observe that the states with $(n \pm \frac{l}{2}, k_\vartheta - \frac{1}{2})$, where $l$ is an integer number, have the same energy. In this direction there are some differences with the spherical Dirac oscillator \[5\]. Despite in both cases bound states are obtained, for the 2+1 Dirac oscillator the energy spectrum presents extra degeneracies only for negative values of $k_\vartheta$. In order to get a deeper understanding of the dependence of the energy spectrum on the spin we can take the nonrelativistic limit of the Dirac equation (11). In order to do that, it is advisable to work with Eq. (31). The Galilean limit is obtained by setting $E = m + \varepsilon$, and considering $\varepsilon \ll m$. Taking into account that the first two terms in Eq. (31) are associated with the operator $P^2$, we obtain in the nonrelativistic limit

$$\frac{P^2}{2m} - (\omega + \frac{eB}{2m})(k_\vartheta \pm \frac{1}{2}) + (\omega + \frac{eB}{2m})^2 m^2 \rho^2 = \varepsilon$$  \hspace{0.5cm} (42)$$

notice that Eq. (42) corresponds to the Schrödinger Hamiltonian of a modified harmonic oscillator with an additional spin dependent term given by $-\bar{\omega}(k_\vartheta \pm \frac{1}{2})$. This contribution is proportional to the frequency of the oscillator plus the Larmor frequency.

The energy spectrum of the 2+1 Dirac oscillator can be computed using the standard derivation followed by Moshinsky and Szczepaniak \[4,11\]. In fact, using the Bjorken and Drell \[12\] representation for the Dirac matrices, the Dirac oscillator coupled to an electromagnetic field reads

$$(E - m)\Psi_1 = \sigma \cdot [(\mathbf{p} - e\mathbf{A}) + im\omega \mathbf{r}]\Psi_2$$  \hspace{0.5cm} (43)$$

$$(E + m)\Psi_2 = \sigma \cdot [(\mathbf{p} - e\mathbf{A}) - im\omega \mathbf{r}]\Psi_1$$  \hspace{0.5cm} (44)$$

taking into account that the vector potential for a constant magnetic field $B$ is given by \[3\] and we are working in 2+1 dimensions, we obtain

$$(E^2 - m^2)\Psi = ((p^2 + m^2 \omega^2 \rho^2) - 2m\bar{\omega} - 4m\bar{\omega}L_z S_z)\Psi$$  \hspace{0.5cm} (45)$$

where

$$L_z = (\mathbf{r} \times \mathbf{p})_z \quad S_z = \frac{\sigma_3}{2}$$  \hspace{0.5cm} (46)$$

finally, recalling the energy spectrum of a non relativistic harmonic oscillator we readily obtain that expression (45) takes the form

$$(E^2 - m^2)\Psi = \left(2m\bar{\omega}(2n + (k_\vartheta + \frac{1}{2}) + 1) - 2m\bar{\omega} - 2m\bar{\omega}(k_\vartheta + \frac{1}{2})\right)\Psi$$  \hspace{0.5cm} (47)$$

$$E^2 - m^2 = [4n + 2(-2k_\vartheta + 1)\Theta(-k_\vartheta)] m\bar{\omega}$$  \hspace{0.5cm} (48)$$

result that coincides with expression (40).

We can label the eigenstates of the parity operator in terms of the total angular momentum $j$. If we define

$$\epsilon = -1 \text{ if parity is } (-1)^j$$
$$\epsilon = 1 \text{ if parity is } (-1)^{j-1}$$  \hspace{0.5cm} (49)$$

in both cases we have $k_\vartheta = j + \frac{\varepsilon}{2}$, therefore we have that the energy spectrum (48) can be written in terms of the parity $\epsilon$ as follows

$$E^2 - m^2 = [4n + 2(2k_\vartheta \epsilon + 1)\Theta(\epsilon k_\vartheta)] m\bar{\omega}$$  \hspace{0.5cm} (50)$$

From (48) we can see that the energy spectrum of the Dirac oscillator in the presence of a constant magnetic field presents a dependence on $B$ that differs from that obtained with the help of the Pauli equation with a harmonic
potential. In this case the spin couples to the magnetic field via the term $-\frac{eB^2}{2m}$, and there is a contribution of $B^2$ on the energy for positive as well as for negative spin orientations.

From (48) it is straightforward to obtain the energy spectrum $S$ states. For $k_\varphi > 0$ we have

$$E^2 - m^2 = 4nm\tilde{\omega} = 4n(m\omega + \frac{eB}{2}) \tag{51}$$

on the other hand, for $k_\varphi < 0$, the energy spectrum satisfies the relation

$$E^2 - m^2 = 4(n + \frac{1}{2} - k_\varphi)(m\omega + \frac{eB}{2}) \tag{52}$$

from (51) and (52) we observe that the energy spectrum of the Dirac oscillator in the presence of a constant $B$ depends on the spin orientation. For $k_\varphi > 0$ the energy of $S$ ground state satisfies $E = m$. Conversely, for $k_\varphi < 0$ the energy $E$ satisfies the relation $E^2 = m^2 + 4(m\omega + \frac{eB}{2})$ and consequently depends on the magnetic field strength $B$.

**IV. DISCUSSION OF THE RESULTS**

The energy spectrum of a non relativistic two-dimensional electron confined by a parabolic potential in the presence of a static magnetic field $B$ is

$$E_{n'} = (2n' + |l| +1)\hbar\Omega - \frac{1}{2}\hbar\omega_\varphi l \tag{53}$$

where $n' = 0, 1, 2...$, and $l = 0, \mp 1, \mp 2,...$. The renormalized oscillator frequency $\Omega$ reads

$$\Omega = \sqrt{\omega^2 + \frac{\omega_c^2}{4}} \tag{54}$$

The expression (53) does not contain the spin field coupling term appearing in the Pauli equation:

$$g^* \mu_B \vec{B} \cdot \vec{S} = \pm \frac{g^* e}{4m} B \tag{55}$$

that, for the value $g^* = 2$, splits the energy levels by a factor $\pm \omega_c$.

In order to better understand the energy spectrum (51), (52), we proceed to analyze the asymptotic limits of the non minimal coupling introduced in Eq.(2).

The weak oscillator limit can be obtained after taking the limit $\omega \to 0$. The energy spectrum (51), (52) reduces to

$$E^2 = m^2 + 2[n - \Theta(-k_\varphi)(k_\varphi - 1/2)]eB \tag{56}$$

which are the relativistic Landau levels obtained when one solves the Dirac equation. The non relativistic limit can also be obtained in a straightforward way from (56) giving as result the expression (53). The energy spectrum in weak magnetic field limit presents a behavior that is not observed in the non relativistic oscillator case. When we eliminate the magnetic field strength $B$ the energy spectrum remains degenerate but depends on the value of $k_\varphi$ therefore, in this limit the Dirac oscillator does not seem to be a suitable approximation to the problem of an electron confined in a quantum dot.

**V. CONCLUDING REMARKS**

The presence of a constant magnetic field $B$ does not break up the symmetry of the vacuum and therefore we do not have mixing of positive and negative energy solutions. The confining Dirac oscillator potential gives as a result a spin orientation dependent energy spectrum for the electron in the presence of a constant magnetic field. There are extra degeneracies of the quantum states with the same energy when $k_\varphi < 0$.The non relativistic limit of the model presented in this paper shows that $B$ modifies the oscillator frequency $\omega$ via the Larmor term $\omega_L = \frac{eB}{2m}$. The nature of the coupling associated with the Dirac oscillator does not permit one to use this model in the study of an electron confined by a quantum dot.
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