Simultaneous Hamiltonian Treatment
of Class A Spacetimes
and Reduction of Degrees of Freedom at the Quantum Level

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Dedication
One of us (T.C.) wishes to dedicate this work to the memory of his late father Ioannis Christodoulakis
whose sudden death during the preparation of an early version of this work in 1994 greatly sorrowed him.

We consider the quantization of a general spatially homogeneous space-time belonging to an arbitrary but fixed Class A Bianchi type. Exploiting the information furnished by the quantum version of the momentum constraints, we use as variables the two simplest scalar contractions of $C_\alpha^\beta\gamma$ and $\gamma_{\alpha\beta}$ as well as the determinant of $\gamma_{\alpha\beta}$; We thus arrive at an equation for the wave function in terms of these quantities. This fact enables us to treat in a uniform manner all Class A cosmologies. For these spacetimes the Langrangian used correctly reproduces Einstein’s equations. We also discuss the imposition of simplifying ansätze at the quantum level in a way that respects the scalings of the quadratic Hamiltonian constraint.

1. Introduction

Since the pioneering work of DeWitt \cite{DeWitt1975}, a lot of minisuperspace models have appeared in the literature \cite{DeWitt1975}. In all these models, a particular Bianchi type symmetry group is selected and a, more or less, simple form for the time dependent matrix $\gamma_{\alpha\beta}(t)$ (appearing in the spatial part of the metric) is adopted; it is not thus easy to determine which of the properties of these quantum models can be expected to hold as more and more degrees of freedom (eventually leading to the full theory) are taken into account. Yet, the investigation of precisely such kind of properties is one of the main reasons for considering Quantum Cosmology. In this work we try to avoid this fragmentation by considering the wavefunction to be a function of the minimum possible number of scalar under the action $\text{GL}(3,\mathbb{R})$ combinations of $C_\alpha^\beta\gamma$ and $\gamma_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ to be (in principle) unrestricted. The paper is organised as follows: In section II we set-up the Wheeler-DeWitt equation for a general spatially homogeneous metric. Thus, exploiting the relations between the curvature invariants and the “scalars” constructed out of $C_\alpha^\beta\gamma$ and $\gamma_{\alpha\beta}$ we present an equation for $\Psi$ in terms of the two simplest independent scalar contractions and $\gamma = \text{det}\gamma_{\alpha\beta}$; As an indicative example we solve the resulting equation for the type II case. In section III we describe a reduction scheme for implementing various simplifying ansätze at the quantum level. Employing two simple ansätze we again give, using this reduction scheme, two indicative examples of reduced minisuperspace models. Finally the results obtained are discussed in section IV.

2. Curvature Invariants and The Reduced Wheeler-DeWitt Equation

Our starting point is the action for pure gravity

$$S_{gr} = -\int d^3x dt \sqrt{-g} \frac{1}{16\pi} R$$

(2.1)

A spatially homogeneous space-time is characterised by the line element

$$ds^2 = -N^2(t) dt^2 + g_{ij}(x,t) dx^i dx^j + 2N_i dx^i dt$$

(2.2)
where
\[ g = \gamma_{\alpha\beta}(t)\sigma^\alpha(x)\sigma^\beta(x) \Leftrightarrow g_{ij} = \gamma_{\alpha\beta}(t)\sigma^\alpha_i(x)\sigma^\beta_j(x) \] (2.3)
\[ N_i = N_\alpha(t)\sigma^\alpha_i(x) \] (2.4)

Lower case Latin indices are world (tensor) indices and range from 1…3 while lower case Greek indices number the different one-forms and take values in the same range. The exterior derivative of any basis one-form (being a two-form) is a linear combination of the wedge product of any two of them, i.e.
\[ d\sigma^\alpha = C^\alpha_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \Leftrightarrow \sigma^\alpha_{i,j} - \sigma^\alpha_{j,i} = 2C^\alpha_{\beta\gamma} \sigma^\beta_j \] (2.5)

The coefficients \( C^\alpha_{\beta\gamma} \) are, in general, functions of the point \( x \). When the space is homogeneous and admits a 3-dimensional isometry group, there exist 3 one-forms such that the \( C \)'s become independent of \( x \) and are then called structure constants of the corresponding isometry group. After substitution of metric (2.2) into action (2.1), the passage to the phase space leads to the following Hamiltonian action [4]:
\[ S = \int dt \left[ N_7 \left( \frac{1}{2} G_{\alpha\beta\gamma\delta} \pi^{\alpha\beta} \pi^{\gamma\delta} + \sqrt{|g|} R \right) - 1N^\rho \left( C^\alpha_{\beta\rho} \gamma_{\alpha\mu} \pi^{\beta\mu} + C^\gamma_{\rho\gamma} \gamma_{\mu\nu} \pi^{\mu\nu} \right) \right] \] (2.6)

where
\[ G_{\alpha\beta\gamma\delta} = \gamma^{-1/2} \left( \gamma_{\alpha\gamma} \gamma_{\beta\delta} + \gamma_{\alpha\delta} \gamma_{\beta\gamma} - \gamma_{\alpha\beta} \gamma_{\gamma\delta} \right), \quad \gamma = det [\gamma_{\alpha\beta}] \] (2.7)
with inverse
\[ G^{\alpha\beta\gamma\delta} = \frac{1}{4} \gamma^{1/2} \left( \gamma^{\alpha\gamma} \gamma^{\beta\delta} + \gamma^{\alpha\delta} \gamma^{\beta\gamma} - 2\gamma^{\alpha\beta} \gamma^{\gamma\delta} \right) \] (2.8)
\[ N^2_0 = -N^2 + \gamma^{\alpha\beta} N_\alpha N_\beta, \quad N^\rho = \gamma^{\rho\delta} N_\delta \] (2.9)

and
\[ R = C^\beta_{\alpha\mu} C^\alpha_{\beta\tau} \gamma_{\alpha\beta} \gamma^{\tau\mu} + 2\gamma^{\beta\gamma} C^\alpha_{\beta\delta} C^\delta_{\nu\alpha} + 4\gamma^{\gamma\lambda} C^\mu_{\nu\lambda} C^\beta_{\beta\lambda} \] (2.10)

is the Ricci scalar of the slice \( t=\text{constant} \). Action (2.0) describes a system of ten degrees of freedom (six \( \gamma_{\alpha\beta} \) plus the Lagrange multipliers \( N_0, N_\alpha \)) among which there are constraints. Treating this system according to Dirac’s [4] theory one arrives at the primary constraints \( \{\Pi^0, \Pi^\alpha\} \approx (0, 0, 0, 0) \) which in turn lead to the secondary:
\[ H_0 := \frac{1}{2} G_{\alpha\beta\gamma\delta} \pi^{\alpha\beta} \pi^{\gamma\delta} + \sqrt{|g|} R \approx 0 \quad H_\alpha := C^\mu_{\alpha\mu} \gamma_{\alpha\beta} \gamma^{\beta\mu} + C^\gamma_{\alpha\gamma} \gamma_{\alpha\tau} \pi^{\mu\tau} \approx 0. \] (2.11)

If we restrict attention to the Class A subgroup then \( C^\gamma_{\mu\gamma} = 0 \) and constraints (2.11) are first class, satisfying:
\[ \{H_0, H_\alpha\} = 0, \quad \{H_\alpha, H_\beta\} = -\frac{1}{2} C^\gamma_{\alpha\beta} H_\gamma. \] (2.12)

In deriving (2.12) use has been made of the basic P.B.'s \( \{\gamma_{\alpha\beta}, \pi^{\mu\nu}\} = 0 \). For the Class B case further restrictions on the phase- space variables are introduced by the requirement that constraints (2.11) should be preserved in time, indicating that action (2.6) does not correctly reproduces Einstein's Equations [3]. For this reason it is understood that in the rest of the paper we are mostly concerned with Class A spacetimes.

In order to quantize the classical action (2.6) we follow Dirac’s quantization prescription [4]: Firstly we have to realize constraints (2.11) as (possibly Hermitian) operators acting on some Hilbert space spanned by their common null eigenvectors; secondly, we must check consistency, i.e. we need to verify that these operators form a closed quantum algebra, preferably isomorphic to (2.12); thirdly, we have to find the
\(\Psi\)'s by solving the linear quantum constraints as well as the Wheeler-DeWitt equation. Adopting the Schrödinger representation, i.e.:

\[
\hat{\gamma}^{\alpha \beta} = \gamma^{\alpha \beta}, \quad \hat{\pi}^{\alpha \beta} = -i \frac{\partial}{\partial \gamma^{\alpha \beta}}, \quad \left(\hat{N}_0, \hat{N}_\alpha\right) = \left(N_0, N_\alpha\right) \quad \left(\hat{\Pi}^0, \hat{\Pi}^\alpha\right) = \left(-i \frac{\partial}{\partial N_0}, -i \frac{\partial}{\partial N_\alpha}\right),
\]

we readily see that the quantum analogs of the primary constraints simply inform us that \(\Psi\) is not a function of \(N_0, N_\alpha\). In turning the linear constraints (2.11) into operators we adopt the prescription of having all the \(\pi^{\alpha \beta}\)'s to the far right so that the quantum constraints maintain the classical symmetries. Thus we have:

\[
\hat{H}_\alpha \Psi := C^\mu_{\beta \alpha} \gamma_\nu \pi^{\beta \mu} \Psi = 0
\]

In turning the quadratic constraint (2.11) into an operator annihilating the wave function \(\Psi\) we adopt the quantization prescription of Kuchar and Hajicek [5], which amounts in realizing the “kinetic part” of this constraint as the conformal Laplacian based on the “physical” metric:

\[
\Sigma^{A B} = \frac{\partial x^A}{\partial \gamma^{\alpha \beta}} \frac{\partial x^B}{\partial \gamma^{\gamma \delta}} C^{\alpha \beta \gamma \delta}
\]

induced by \(G_{\alpha \beta \gamma \delta}\) and the solutions \(x^A\) to the linear quantum constraints (2.14):

\[
\nabla^\text{Conf}_\Sigma = \nabla_\Sigma + \frac{D - 2}{4(D - 1)} R_\Sigma
\]

\(D\) is the dimension of the reduced minisuperspace spanned by the \(x^A\)’s. This quantization rule is mandatory if one wants to respect the classical covariance of the action (2.6):

\[
\gamma^{\alpha \beta} \rightarrow \tilde{\gamma}^{\alpha \beta}(\gamma^{\mu \nu}), \quad N \rightarrow \tilde{N} = f(\gamma^{\alpha \beta}) N,
\]

or, equivalently, the freedom to multiply constraint \(H_0\) by an arbitrary function. Thus, the Wheeler-DeWitt equation which \(\Psi\) has to satisfy takes the form:

\[
-\frac{1}{2} \left[ \Sigma^{A B} \frac{\partial^2}{\partial x^A \partial x^B} - \Gamma^A_{A B} \Sigma^{K B} \frac{\partial}{\partial x^K} + \frac{D - 2}{4(D - 1)} R_\Sigma + \sqrt{\gamma} R \right] \Psi = 0
\]

where \(\Gamma^A_{A B}, R_\Sigma\) are the Christoffel symbol’s and the Ricci scalar corresponding to \(\Sigma_{M N}\) (inverse to \(\Sigma^{A B}\)). It is tedious but straightforward to verify (using \(\left[\hat{\gamma}^{\alpha \beta}, \hat{\pi}^{\gamma \delta}\right] = -i \delta^{\gamma \delta}_{\alpha \beta}\)) that the quantum constraints satisfy an algebra completely analogous to (2.12):

\[
\{ \hat{H}_0, \hat{H}_\alpha \} = 0, \quad \{ \hat{H}_\alpha, \hat{H}_\beta \} = -i C^{\gamma \delta}_{\alpha \beta} \hat{H}_\gamma
\]

We next come to the measure of the Hilbert space to be constructed by the solutions to (2.17, 2.14): It is desirable to have a measure \(\mu\) under which the state-vector-defining operators (2.17, 2.14) be Hermitian; the most general Hermitian operator conditions corresponding to the linear constraints (2.11) are [6]:

\[
\hat{X}_\alpha \Psi = \frac{1}{2\mu} \left( \mu C^{\beta}_{\alpha \gamma} \gamma_\beta \hat{\pi}^{\gamma \delta} + \hat{\pi}^{\gamma \delta} C^{\beta}_{\alpha \gamma} \gamma_\beta \mu \right) \Psi = 0
\]

If we want \(\hat{X}_\alpha\)'s to be identical to \(\hat{H}_\alpha\)'s in (2.14) we immediately conclude (bearing in mind that \(C^{\gamma \tau}_{\alpha \beta} = 0\)) that \(\mu\) must also satisfy (2.14), i.e.:

\[
C^{\beta}_{\alpha \gamma} \gamma_\beta \hat{\pi}^{\gamma \delta} \mu = 0
\]

Finally, the Hermiticity of (2.17) fixes the measure to be:

\[
\mu = \sqrt{\text{det} \Sigma_{A B}}
\]

which can be seen to satisfy (2.20).
Let us now start our investigation of the space of state vectors by considering the solutions to (2.14): Except for the trivial Bianchi type I (see end of the section IV) and type II (which is explicitly solved below), in all other Class A Bianchi types conditions (2.14) comprise a set of 3 independent PDE’s in 6 variables. As a result, we expect the general solution to these equations, if it exists, to be a function of 3 combinations $\gamma_{\alpha\beta}$: the existence of the solutions to (2.14) is guaranteed by Frobenius theorem: The quantum algebra (2.13) satisfied by the $\dot{H}_a$’s (seen as vector fields on the configuration space) is essentially the integrability conditions required by the theorem. As one can easily verify, every combination of $C^a_{\beta\gamma}$ and $\gamma_{\alpha\beta}$ that is scalar under the action GL(3,R) (i.e. has all the frame indices contracted), as well as $\gamma$ is a solution to these equations. Thus the general solution to (2.14) is a function of $\gamma$ and any scalar combination of $C^a_{\beta\gamma}$ and $\gamma_{\alpha\beta}$.

As it is well known [8], the number of independent curvature invariants of a three-dimensional space is 3 combinations $\gamma_i$, $\gamma_{ij}$, $\gamma_{ijk}$; as a result, we expect the general solution to these equations, if it exists, to be a function of $\gamma_1$, $\gamma_2$, $\gamma_3$. Thus the general solution to (2.14) is a function of $\gamma$ and any scalar combination of $C^a_{\beta\gamma}$ and $\gamma_{\alpha\beta}$.

For the case of (the spatial part of) metric (2.2) :

$$R_{ij} = \sigma^i_\beta \sigma^j_\gamma (C^a_{\beta\gamma} \gamma^\theta \gamma^\tau \mu) + 2C^a_{\beta\gamma} C^\tau_\mu \gamma^\beta \gamma^\tau \lambda \mu \lambda - 2C^a_{\beta\gamma} C^\tau_\mu \gamma^\delta \gamma^\beta \gamma^\delta \gamma^\mu \lambda - 2C^a_{\beta\gamma} C^\tau_\mu \gamma^\beta \gamma^\delta \lambda \mu \lambda + 2C^a_{\beta\gamma} C^\tau_\mu \gamma^\beta \gamma^\delta \lambda \mu \lambda )$$

(2.23)

see [8]. From this relation it is evident that the curvature invariants will be complicated combinations of terms involving scalar contractions of $C^a_{\beta\gamma}$ and $\gamma_{\alpha\beta}$ (and thus they will also satisfy (2.14)). Searching for the most convenient contractions to take as independent variables we immediately spot the three simplest contractions appearing in expression (2.10) for the Ricci scalar, namely :

$$C^a_{\lambda\mu} C^\alpha_\gamma \gamma^{\beta\lambda} \gamma^{\tau\mu}, \quad \gamma^{\beta\nu} C^a_{{\beta\gamma}} \gamma^{\delta}, \quad \gamma^{\lambda \nu} C^\alpha_{\mu\nu} C^\beta_{\lambda}$$

However, as one can see in the appendix, the third contraction is a multiple of the second i.e. $\gamma^\mu \gamma^\nu C^a_{\mu\nu} C^\beta_{\lambda} = \lambda \gamma^\beta \gamma^\nu C^a_{\gamma} C^\beta_{\alpha}$. The presence of $\gamma$ in the potential term of (2.17), as well as the fact that $\gamma$ is a solution to (2.14), suggests that the three independent variables can be taken to be :

$$x^1 := C^a_{\lambda\mu} C^\alpha_\beta \gamma^{\beta\lambda} \gamma^{\tau\mu}, \quad x^2 := \gamma^{\beta\nu} C^a_{{\beta\gamma}} \gamma^{\delta}, \quad x^3 := \gamma$$

(2.24)

instead of $R_1$, $R_2$, $R_3$ (see appendix for the exact expressions of $R_1$, $R_2$, $R_3$ in terms of $x^1$, $x^2$, $x^3$). Therefore the general solution to (2.14) is an arbitrary function of $x^1$, $x^2$, $x^3$ say $\Psi = \Psi(x^1, x^2, x^3)$. Note that $x^1, x^2, x^3$ are the “physical coordinates” defined by Kuchar and Hajicek [8]. We now turn our attention to (2.17). In trying to calculate $\Sigma^{AB}$ and thus implement the conclusion that $\Psi = \Psi(x^1, x^2, x^3)$ one is coming across terms quartic in $C$’s which are not manifest combinations of sums of products of $x^1$, $x^2$, $x^3$. One such typical term could, for example, be :

$$\frac{\partial^2 \Psi}{\partial x^2 \partial x^1} \frac{\partial x^1}{\partial \gamma^{a\beta}} \frac{\partial x^1}{\partial \gamma^{a\delta}} \gamma^{a\alpha} \gamma^{b\beta} = \frac{\partial^2 \Psi}{\partial x^2 \partial x^1} (2C^\alpha_{\gamma\tau} C^\theta_\tau C^\gamma_\lambda C^\nu_\beta \gamma^{\mu\gamma} \gamma^{\lambda\delta} \gamma^{\beta\lambda} - C^\alpha_{\gamma\tau} C^\theta_\tau C^\gamma_\lambda C^\nu_\beta \gamma^{\mu\gamma} \gamma^{\lambda\nu} \gamma^{\beta\lambda} )$$

In deriving this relation use has been made of the identity :

$$\frac{\partial \gamma^{a\beta}}{\partial \gamma^{a\delta}} = -\frac{1}{2} (\gamma^{a\gamma} \gamma^{b\delta} + \gamma^{a\beta} \gamma^{b\delta})$$

easily obtainable from the basic relations :

$$\frac{\partial \gamma^{a\beta}}{\partial \gamma^{a\delta}} = \delta^{a\beta}_{a\delta}, \quad \gamma^{a\beta} \gamma^{a\delta} = \delta^{a\delta}_{a\delta}.$$

In order to actually show that all terms of the sort are, indeed, algebraic combinations of $x^1$, $x^2$, $x^3$ we have to count the number of independent “scalar” contractions that can be constructed out of $C^a_{\beta\gamma}$ and $\gamma_{\alpha\beta}$. The procedure we adopt is closely similar to that used in [8] to count the number of independent
curvature invariants in d-dimensions. As it is well known, \( C_{\beta\gamma}^{\alpha} \), being the structure constants of a 3-dimensional Lie group, are antisymmetric in their lower indices and satisfy the Jacobi identities:

\[
T_{\alpha\beta\gamma}^{\delta} := C_{\mu\gamma}^{\delta}C_{\alpha\beta}^{\mu} + C_{\mu\alpha}^{\delta}C_{\beta\gamma}^{\mu} + C_{\mu\beta}^{\delta}C_{\gamma\alpha}^{\mu} = 0
\]

\( T_{\alpha\beta\gamma}^{\delta} \) being fully antisymmetric in its lower indices. The symmetric matrix \( \gamma_{\alpha\beta} \) has 6 independent components. The antisymmetry of \( C_{\beta\gamma}^{\alpha} \) leaves it with 9 components restricted by the 3 independent Jacobi Identities. Thus \( C_{\beta\gamma}^{\alpha} \) has 6 independent components. If we are interested in scalar contractions of \( C_{\beta\gamma}^{\alpha} \) and \( \gamma_{\alpha\beta} \) then the 9 components \( \Lambda_{\beta}^{\alpha} \) of an element of \( GL(3,\mathbb{R}) \) can be chosen arbitrarily. Hence, the 6 independent components of \( \gamma_{\alpha\beta} \) and the 6 independent \( C_{\beta\gamma}^{\alpha} \)'s are subject to 9 restrictions when put together to form scalar combinations. The number of independent such scalar combinations is therefore \( 6 + 6 - 9 = 3 \). Note that 3 is the maximum number of independent scalars; it is achieved only in Bianchi types VIII, IX : In these types \( R_{1} \neq R_{2} \neq R_{3} \neq R_{1} \neq 0 \) (this is why sometimes it is said that these types mimic more faithfully the generic three-space); and there is a number density of weight \(-1\), namely \( m = \text{det}[m^\alpha_\beta] \) (see appendix). This \( m \) can be used to promote \( x^3 \) to a true scalar by taking the combination \( \frac{x^3}{3m} \). In all other Bianchi types \( m = 0 \) and, as it can be seen either by direct calculation or from the relations given in the appendix, the number of independent \( R_{1}, R_{2}, R_{3} \) is less than 3: namely it is two for types VI, VII, VII, VIII one for type II and 0 for type I and are sufficient to describe the corresponding three geometries [5]. Now \( x^3 \) cannot be promoted to a true scalar. Nevertheless, since it explicitly appears in the interaction term of (2.17) and because it is a solution to (2.14), we must include it among the independent variables.

Since \( m \) is a number (and thus transparent to the derivatives with respect to \( \gamma_{\alpha\beta} \)) we conclude that (2.24) are valid variables for all Class A types. In any case the above considerations show that the system of (2.17)-(2.14) is self consistent, as one expects from the first class algebra (2.18). The same conclusion follows from the considerations of Kuchar and Hajicek [5] for the particular case of zero cocycle.

Thus we finally get from (2.17) the equation:

\[
(5x^1x^1 + 16(\lambda - 1)x^2x^2 - 8\lambda x^2x^1 + 128W) \frac{\partial^2\Psi}{\partial x^1\partial x^1} + (x^2x^2 - 16W) \frac{\partial^2\Psi}{\partial x^2\partial x^2} + (-3x^3x^3) \frac{\partial^2\Psi}{\partial x^3\partial x^3} + (2x^1x^2 + 16W) \frac{\partial^2\Psi}{\partial x^1\partial x^2} + (2x^1x^3) \frac{\partial^2\Psi}{\partial x^1\partial x^3} + (2x^2x^3) \frac{\partial^2\Psi}{\partial x^2\partial x^3} A^\Lambda \frac{\partial\Psi}{\partial x^\Lambda} - 2x^3(x^1 + 21 + \lambda)x^3) \Psi = 0
\]

(2.25)

where

\[
W := \epsilon m \sqrt{\frac{x^1 - 2x^2}{2x^3}}
\]

(2.26)

see appendix for the definition of \( m, \epsilon \) and \( A^\Lambda = -F^\Lambda_{MN} \Sigma^{MN} \) is the corresponding linear term of the Laplacian. For all Bianchi types except VIII, IX this term is \( A^\Lambda = (5x^1 - 4\lambda x^2, x^1 - 3x^3) \). Equation (2.25) can be compactly written as:

\[
[\Sigma^{AB} \partial_A \partial_B + A^\Lambda \partial_\Lambda - 2x^3(x^1 + 21 + 2\lambda)x^2)] \Psi = 0.
\]

(2.27)

Note that using the conformal covariance of our operator we have multiplied by \((x^3)^{1/2}\) and thus the “physical” metric in (2.27) is:

\[
\Sigma^{AB} = L_{\mu\nu\kappa\lambda} \partial_{x^A} \partial_{x^B} \gamma_{\mu\nu}, \quad \text{where} \quad L_{\mu\nu\kappa\lambda} = \gamma_{\mu\kappa} \gamma_{\nu\lambda} + \gamma_{\mu\lambda} \gamma_{\nu\kappa} - \gamma_{\mu\nu} \gamma_{\kappa}\lambda.
\]

Of course, \( \lambda \) is zero for all Class A cases. We only include it to show that the above reduction from (2.17) to (2.25) can be made for Class B Bianchi types as well; this is also the reason it appears in the examples at the end of the next section. This equation constitutes our main result; it has, over (2.17)-(2.23), the conceptual advantage of enabling us to treat in a uniform manner each and every particular Class A Bianchi type. It also has the technical advantage of a much more simpler interaction term, while the reduced minisuperspace metric \( \Sigma^{AB} \) is no more complicated than \( G_{\alpha\beta\gamma\delta} \) appearing in (2.17)-(2.21) (except of course for the square root term which, however, vanishes for all but VIII, IX Bianchi types). For any
(Class A) Bianchi type $\Sigma^{AB}$ in (2.27) is flat; one can easily see it for the lower Bianchi types (all except the VIII, IX) by transforming to coordinates:

$$
\begin{align*}
  d^1 &:= \sqrt{\frac{x_1^2 - 2x_2^2}{x_2^2}} \\
  d^2 &:= (x_2^2)^{3/8}(x_3^2)^{-1/8} \\
  d^3 &:= x_2^4 x_3^2
\end{align*}
$$

in which equation (2.28) becomes:

$$
\frac{(d^1)^2}{2(\lambda - 2)} \frac{\partial^2 \Psi}{\partial (d^1)^2} + 2d^2 d^3 \frac{\partial^2 \Psi}{\partial (d^2) \partial (d^3)} + d^3 \frac{\partial \Psi}{\partial d^3} - 2d^2 ((d^1)^2 + 4(\lambda + 1)) \Psi = 0
$$

making explicit the flatness of $\Sigma^{AB}$.

We now give, as an indicative example, the general solution to (2.29) for the Bianchi II geometry. This Bianchi type is of some importance since it can be considered as the highly anisotropic limit of type IX and it has recently been also treated in [10]. The only non-vanishing structure constants are $C_{23}^1 = -C_{22}^1 = 1$ and one can easily verify that the scalar $x^2$ in (2.24) is identically zero. Thus the wave-function satisfying (2.25) depends only on $x^1 := \chi$ and $x^3 := \gamma$ reducing to the equation:

$$
5\chi^2 \frac{\partial^2 \Psi}{\partial \chi^2} - 3\gamma^2 \frac{\partial^2 \Psi}{\partial \gamma^2} + 2\chi \gamma \frac{\partial^2 \Psi}{\partial \chi \partial \gamma} + 5\chi \frac{\partial \Psi}{\partial \chi} - 3\gamma \frac{\partial \Psi}{\partial \gamma} - 2\gamma \chi \Psi = 0.
$$

Effecting the transformation $v = \gamma \chi$, $u = \gamma^3 \chi$ and assuming $\Psi(u, v) = U(u)V(v)$ we get two ODE’s:

$$
\begin{align*}
  16u^2 \frac{d^2 U}{du^2} + 16u \frac{dU}{du} - CU &= 0 \\
  4v^2 \frac{d^2 V}{dv^2} + 4v \frac{dV}{dv} - (C + 2v) V &= 0
\end{align*}
$$

where C is a separation constant.

The first of these is an Euler equation while the second can be easily transformed into a Bessel equation, further reduction to the one and only true degree of freedom for the type II [11], namely $\chi$, requires considerations of the quantum version of some classical integrals of motion [12]. At this point we deem it important to conclude this section underlining the fact that no extra (by hand) assumption has been used in arriving at (2.23); the only “arbitrariness” involved is the factor ordering of the $x^\alpha \chi^\beta$’s to the far right in (2.14). Thus the theory itself naturally leads us to consider as equivalent any two hexads $\gamma_{\alpha \beta}$, $\gamma_{\alpha \beta}$ iff they correspond to the same triplet $(x_1, x_2, x_3)$ for a given Class A Bianchi type under consideration.

This is hardly a surprise since the linear constraints $H_{\alpha}$ are the generators of the inner automorphisms and therefore the two exads corresponding to the same triplet will be inner automorphically related, i.e. $\gamma_{\alpha \beta}^{(2)} = \Lambda_{\alpha}^\mu \Lambda_{\beta}^\nu \gamma_{\mu \nu}^{(1)}$ with $\Lambda \in \text{InAut}[G]$ [12]. The proper investigation of this point and other related issues is too long to be included here; it has been given in [3]. Finally, we wish to remark that the solutions to (2.23) are intended to form the Hilbert space for the general spatially homogeneous Bianchi cosmology for each and every Class A group type separately: no transitions between different Class A Bianchi types can be considered; indeed the structure constants appearing in (2.24) are merely (mostly) discrete parameters and not dynamical variables. If one wishes to consider such transitions one has to go deeper in Quantum Gravity, beyond the Quantum Cosmology approximation.

3. Reducing the Degrees of Freedom at the Quantum Level

Despite the simplification achieved in the previous section, equation (2.23) is still too difficult to solve in its full generality. We would like thus to study simplified models with less (than six) degrees of freedom. The way to do this, up to now, was to insert a particular choice for $\gamma_{\alpha \beta}$ (say diag(a(t), b(t), c(t)) in the classical action (2.6), and quantize the resulting (further) reduced action. We, on the other hand, are already at the quantum level having as our degrees of freedom the combinations of $\gamma_{\alpha \beta}$ and $C_{\beta \gamma}$ appearing in (2.24); it is thus natural to adopt as simplifying Ansatz a restriction on $x^4$ of the form
f(x^A) = 0 \text{ which, given a particular Bianchi type, can be translated into a choice for the form of } \gamma_{\alpha\beta}.

We are thus presented with the task of defining the “reduced equivalent” of equation (2.25). Although there are standard ways to project an equation on a submanifold of a given manifold (3.1), we have to bear in mind that the metric structure on the manifold spanned by x^A (as well as that corresponding to the manifold spanned by the \gamma_{\alpha\beta}'s) is known only up to a rescaling: we are free to multiply (2.25) by an arbitrary function of x^A. Due to this, the “reduction rule” we are going to adopt must respect the conformal transformations of \Sigma^{AB}; in fact the need to respect the scalings of G_{\alpha\beta\gamma} is the main reason behind Kuchar’s recipe for realizing the kinetic part of H_0 as the conformal Laplacian based on \Sigma^{AB}, as seen in (2.21). In seeking to construct this procedure, let for a moment be general and consider an equation of the form

\[ [\Sigma^{AB}\partial_A\partial_B + A^T\partial_T + V(x^A)]\Psi = 0 \]  

on a D-dimensional space M spanned by x^A, A = 1, 2, \ldots, D. We shall restrict our consideration to the case where \Sigma^{AB} is non-degenerate (det \Sigma^{AB} \neq 0) and the submanifold on which we wish to define the reduced equivalent of (3.1) is non-null, i.e. no one of its tangent vectors has zero length with respect to \Sigma^{AB}. We are to interpret \Sigma^{AB} as a contravariant metric on M but, due to the freedom to arbitrary scale (3.1), we are only required to use in our construction the conformal equivalence class a representative of which is \Sigma^{AB}. Suppose now that we are given the D-L restricting conditions f^i(x^A) = 0, i = D, D - 1, \ldots, L + 1 defining the L-dimensional submanifold N on which we wish to restrict (3.1).

Consider the (D-L) one forms \Phi^i := \frac{\partial f^i(x^A)}{\partial x^A}dx^A spanning the subspace orthogonal (with respect to \Sigma^{AB}) to the cotangent space of N. The hypothesis that N is non-null means that \Sigma^{AB}\Phi^i\Phi^j \neq 0 for all i = D, D - 1, \ldots, L (no summation in i). Let us suppose that f^i are such that the vector fields \xi_i := \Sigma^{AB}\Phi^i\partial_{\gamma^A} are surface forming (i.e. \{\xi_i, \xi_j\} = C_{ij}^k\xi_k where \{\ldots\} stands for the Lie bracket). This implies that the system of first order partial differential equations \xi_i^\gamma\partial_\gamma \Omega(x^A) = 0 admits L functionally independent solutions q^\alpha(x^\Gamma), \alpha = 1, 2, \ldots, L, that is to say the general solution to this system is of the form: \Omega(x^\Gamma) = \Omega(q^\alpha). Furthermore, no one of the f^i's can be functionally dependent on the q^\alpha's since if f^i = f^i(q^\alpha) for some i, then \Sigma^{AB}\Phi^i\Phi_\Delta = 0 contradicting the hypothesis that N is non-null. Therefore the D functions x^\Gamma = (q^\alpha, f^i) constitute a regular coordinate system for M, i.e. the Jacobian matrix \left[\frac{\partial x^\Gamma}{\partial x^\Delta}\right] is non-singular. Thus, via the inverse mapping theorem, we can always (in principle) express x^\Gamma as functions of x^T.

We are now ready to define the reduction scheme: We supplement (3.1) with the requirement that \Phi does not change when we move along the integral curves of each \xi_i; thus we define as reduction of (3.1) the system of partial differential equations:

\[ L_{\xi_i}\Psi = 0, \quad i = D, D - 1, \ldots, L + 1 \]

\[ [\Sigma^{AB}\partial_A\partial_B + A^T\partial_T + V(x^A)]\Psi = 0. \]

Let us see why this system defines an equation on N. As we explained above (3.2) has a general solution \Psi = \Psi(q^\alpha). Inserting this into (3.3) we get:

\[ \Sigma^{AB}\frac{\partial q^\alpha}{\partial x^A}\frac{\partial q^\beta}{\partial x^B} \frac{\partial^2\Psi}{\partial q^\alpha\partial q^\beta} + \left(\Sigma^{AB} A^T\frac{\partial q^\gamma}{\partial x^\Gamma} + A^T \frac{\partial q^\gamma}{\partial x^\Gamma}\right) \frac{\partial\Psi}{\partial q^\gamma} + V\Psi = 0 \]

where the coefficients are, in principle, functions of all x^\Gamma. Using the fact that x^\Gamma = x^\Gamma(q^\alpha, f^i) and restricting on N (i.e. setting f^i = 0), we finally get an equation on N:

\[ [\sigma^\alpha\beta(q)\partial_\alpha\partial_\beta + \alpha^\gamma(q)\partial_\gamma + V(q)] \Psi(q) = 0 \]

In the rest of the section we give examples of the above reduction scheme exhibiting the solutions for a large class of five dimensional minisuperspace models (except Bianchi types VIII, IX). Starting from (2.25) let us reduce the number of independent \gamma_{\alpha\beta} to 5 by imposing the restriction (simplifying Ansatz) d^4 = const. (\Rightarrow x^4 = \mu x^2).

Thus the defining equation for N is:

\[ f(d^1, d^2, d^3) := d^1 - c = 0 \]
Hence the vector $\xi$ is $\xi = ((d^1)^2 - 2(\lambda - 2)\frac{\partial}{\partial d^1})$, implying that (3.2) is satisfied by $\Psi = \Psi(d^2, d^3)$. Inserting this $\Psi$ into (2.28-2.29) and restricting to $\mathcal{N}$ we arrive at the equivalent of (3.5):

$$d^2 \frac{\partial^2 \Psi(d^2, d^3)}{\partial d^2 \partial d^3} - C \Psi(d^2, d^3) = 0 \quad (3.7)$$

with $C := c^2 + 4(\lambda + 1)$. Effecting the transformation:

$$d^2 = e^{s^2}, \quad d^3 = s^3$$

we arrive at a Klein-Gordon equation (in lightcone coordinates):

$$\frac{\partial^2 \Psi}{\partial s^2 \partial s^3} = C \Psi \quad (3.8)$$

with well-known space of solutions. A particular solution can be reached by separation of variables in (3.4) and reads:

$$\Psi(d^2, d^3) = (d^2)^{l'} \exp \left[ C d^3 / l' \right] \quad (3.9)$$

where $l'$ is another separation constant.

Another very interesting simplifying Ansatz is:

$$d^3 = c' (\text{const.}) \Leftrightarrow x^2 x^3 = \text{const.}.$$  

Similarly to the previous case we arrive from (2.29) at the Wheeler-DeWitt equation for $\Psi$ which has the form:

$$\left[ (d^1)^2 - 2(\lambda - 2) \right] \frac{\partial^2 \Psi}{\partial (d^1)^2} + d^1 \frac{\partial \Psi}{\partial d^1} - 2d^3 ((d^1)^2 + 4(\lambda + 1)) \Psi = 0 \quad (3.10)$$

This equation is of the Matthieu Modified type and is equivalent to equation (39) of [14] (under the identification $d^1 = \sqrt{w - 2}$). There, of course, the reduction has been achieved by means of the quantum version of a classical integral of motion and not as a simplifying Ansatz.

4. Discussion

In this work we have investigated the quantization of the general spatially homogeneous geometry for each Class A Bianchi type. Counting the number of independent scalar contractions of $\gamma_{\alpha\beta}$ and $C_{\beta\gamma}$ we were led to identify the minimum number of variables (i.e. degrees of freedom) which $\Psi$ can be assumed to depend upon: $x^1, x^2, x^3$ defined in (2.24). The very fact that this is possible establishes, in our opinion, the conceptual (as well as formal) connection of Quantum Cosmology with full Quantum Gravity: Indeed, adopting the long standing belief that the wave-functional of Quantum Gravity must be a functional of the geometry, let us restrict ourselves within the class of smooth functionals [15], consider some general solution to the full Wheeler-DeWitt equation of the form $\Omega = \Omega(I)$ with $I = \int d^3 x \sqrt{g} F \left( R_1, R_2, R_3 \right)$. If we now insert in $\Omega$ the particular metric (2.2-2.5) we see, because of (2.23) and the relations for $R_1, R_2, R_3$ given in the appendix, that indeed $\Omega$ becomes a function of $x^1, x^2, x^3$ (forgetting, of course, the (possibly infinite) constant $c = \int d^3 x \sigma(x)$ with $\sigma = \det(\sigma^a_i)$ which multiplies the $\sqrt{x^3}$).

An important new feature of (2.27) is also the fact that it offers the possibility to study the quantum mechanics of spatially homogeneous geometries in a uniform manner for all (Class A) Bianchi types. Thus, the properties common to all these models can be more easily revealed. Our inability to solve (2.25) in its full generality leads us to consider simplified models, but again the advantage over the usual procedure is obvious: In our case the Ansatz is made on the $x^A$'s enabling us to treat large subgroups of simplified models simultaneously. The reduction procedure we adopt has the unique property of being covariant under rescaling of $\Sigma^{AB}$. By this we mean the following: let us imagine that somebody, using a particular $\Sigma^{AB}$ and following our reduction scheme, arrives at a reduced equation, say (3.5). Suppose
now that someone else multiplies \( (3.1) \) with a function \( G(x) \). If he wants to reduce his scaled equation, he has to find the \( q \)'s solving the system corresponding to the conditions \((3.2)\) with \( \Sigma^{AB} \) replaced by \( G(x)\Sigma^{AB} \); it is therefore obvious that he will get the same set of \( q \)'s as the previous person. This fact, together with the form of \((3.4)\), leads to the conclusion that the reduced equation on \( \mathcal{N} \) (for the second person) will be just \((3.3)\) multiplied by \( g(q) = G(x) \bigg|_{\mathcal{N}} \). This establishes the conformal covariance of our reduction scheme. The form of \((3.4) \) makes also explicit the covariance of our scheme under diffeomorphisms of \( \mathcal{M} \) and \( \mathcal{N} \).

We would also like to briefly mention two important issues not touched upon in this work:

The first issue concerns the normalisability of the solutions to \((2.17-2.21)\) and \((2.25)\); as it is well known, \( G^{\alpha\beta\gamma\delta} \) has signature \((-+,+,+,+,+>) \) which implies that the solutions to \((2.17-2.21)\) are not expected to be normalisable with respect to all \( \gamma_{\alpha\beta} \)'s. \( \Sigma^{AB} \) in \((2.25)\) can also be seen to be of hyperbolic nature, implying that the same problem will occur. This non-normalisability is intrinsically connected with the long debated problem of time in Quantum Gravity and Quantum Cosmology. In connection to this problem, our reduction scheme (when it refers to one restricting condition) could be taken as a basis for a satisfactory definition of probability on any space-like hypersurface of \( \Sigma^{AB} \); but clearly much work is needed before something conclusive can be said. Another possibility of improving normalizability is the inclusion of matter fields, mostly half integer spin fields \([16]\). Recent results concerning supersymmetric FRW \([18]\) and/or Bianchi models \([14]\) show substantial improvement in the normalizability of the wavefunctions.

The second issue is related to the question of whether action \((2.6)\) gives equations of motion equivalent to Einstein’s Field equations for the metric \((2.2)\); For unrestricted \( \gamma_{\alpha\beta} \) the answer is that the two sets of equations are equivalent only for Class A spacetimes. The examples cited at the end of section III do contain Class A spacetimes and might therefore correspond to valid classical actions. To see whether they actually do, one would have to test them case by case for the different Bianchi types and the restrictions adopted. We have not done this since it lied beyond the scope of our work.

Finally, we would like to remark that, for the case of Bianchi type I, \( x^1 = x^2 = 0 \) and equation \((2.25)\) reduces to an Euler Equation for the single variable \( x^3 = \gamma \). This indicates that the configuration space for this type is effectively one-dimensional, a number that points to the one-parameter Kasner family of classical solutions. The explanation for this reduction of the seemingly 12-dimensional phase-space \((6 \gamma_{\alpha\beta} \text{ plus } 6 \pi^{\alpha\beta})\) involves considering the action of the 9-parameter automorphism group (GL(3,R)) and is properly given by A. Ashtekar and J. Samuel (see in \([22]\)). The fact that no \( H_{\alpha} \)'s exist (due to the vanishing of the structure constants) is adequately compensated by the existence of many classical integrals of motion; their quantum counterparts can be used to reduce the degrees of freedom to \( \gamma \equiv \text{det}[\gamma_{\alpha\beta}] \) \([20]\).

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5. Appendix : Calculating the Invariants

We give an outline of how the relations between higher order (in $C_{\beta\gamma}^\alpha$) scalars and the basic variables $(x^1, x^2, x^3)$ arises.

Since $C_{\beta\gamma}^\alpha$ is antisymmetric in its lower indices, it can be expressed (in three dimensions) in terms of the totally antisymmetric symbol $\epsilon_{\alpha\beta\gamma}$, a symmetric matrix $m^{\alpha\beta}$ and a vector $a_\gamma$:

$$C_{\beta\gamma}^\alpha = \epsilon_{\beta\gamma\delta} m^{\delta\alpha} + \delta_\gamma^\alpha a_\beta - \delta_\beta^\alpha a_\gamma \Leftrightarrow m^{\alpha\beta} = \frac{1}{2} C_{\gamma\delta}^{(\alpha \beta)}$$

where the parenthesis means symmetrization:

$$a_\gamma = \frac{1}{2} C_{\gamma\alpha}^{\alpha}$$

and

$$\epsilon^{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}.$$  

Using this representation of $C_{\beta\gamma}^\alpha$ (or directly through an algebraic computing facility) one can see that:

$$\gamma_{\nu\lambda} C_{\mu\nu}^{\beta} C_{\beta\lambda}^\gamma = \lambda_{\gamma\lambda}^\nu C_{\mu\nu}^{\alpha} C_{\alpha\mu\nu}^{\delta \epsilon}$$

with the number $\lambda$ given in the following table:

| Bianchi Type | II | III | IV | V | VI | VII | VIII | IX |
|--------------|----|-----|----|---|----|-----|------|----|
| $\lambda$    | any| 1   | 2  | 2 | (h + 1)$^2$ | $h^2$ | $h^2 - 2$ | 0  |

Under the (possibly time dependent) changes:

$$\sigma^\alpha \rightarrow \tilde{\sigma}^\alpha = \Lambda^{\alpha}_{\beta} \sigma^\beta$$

where $\epsilon_{\alpha\beta\gamma}$ is a tensor density of weight 1 and therefore $m^{\alpha\beta}$ has weight -1 (so that $C_{\beta\gamma}^\alpha$ be a tensor under GL(3,R)). This means that $m = \det[m^{\alpha\beta}]$ is a scalar density of weight -1 as well (m is non zero only for Bianchi types VIII, IX). Since $\gamma_{\alpha\beta}$ is a second rank tensor $\gamma := \det[\gamma_{\alpha\beta}]$ is a scalar density of weight -2.

The calculation of the various scalars given below proceeds as follows:

One can use a matrix $\Lambda_1 \in$ GL(3,R) to transform $\gamma_{\alpha\beta}$ to the identity matrix $I_3$. Under such a transformation $m^{\alpha\beta}$ transforms to an arbitrary symmetric matrix $\tilde{m}^{\alpha\beta}$. Subsequently one can use an orthogonal matrix $\Lambda_2$ to diagonalise $\tilde{m}^{\alpha\beta}$ (keeping $\gamma_{\alpha\beta}$ the identity matrix). The product $\Lambda = \Lambda_1\Lambda_2$ transforms $\gamma_{\alpha\beta}$ to $I_3$ and puts $m^{\alpha\beta}$ to diagonal form.

In this way we have the structure constants in terms of the three eigenvalues of $m^{\alpha\beta}$ say $a, b, c$ and the components $d, e, f$ of the vector $a_\gamma$. The Jacobi identities imply that $a_\gamma$ is a null eigenvector of $m^{\alpha\beta}$ and thus $ad = be = cf = 0$.

In this representation one can (through an algebraic computing facility) see that:

$$x^1 = 4 (d^2 + e^2 + f^2) + 2 (a^2 + b^2 + c^2)$$

$$x^2 = 2 (d^2 + e^2 + f^2 - ab - ac - bc)$$

and for example:

$$w_1 := C_{\xi\epsilon}^{\mu} C_{\lambda\delta}^{\tau} C_{\nu\rho}^{\alpha} C_{\kappa\lambda\gamma\lambda}^{\mu \nu} = \left[ 2 (d^2 + e^2 + f^2 - ab - ac - bc) \right]^2 - 8abc(a + b + c).$$

The first term in $w_1$ is $(x^2)^2$. The second term has to be a scalar. Therefore although at first sight it looks like $-8m m^{\alpha\beta} \gamma_{\alpha\beta}$ it is actually the scalar $\frac{8m}{x^3} m^{\alpha\beta} \gamma_{\alpha\beta}$. Thus

$$w_1 := C_{\xi\epsilon}^{\mu} C_{\lambda\delta}^{\tau} C_{\nu\rho}^{\alpha} C_{\kappa\lambda\gamma\lambda}^{\mu \nu} = \left( x^2 \right)^2 - \frac{8m}{x^3} m^{\alpha\beta} \gamma_{\alpha\beta}$$
Working similarly, the following scalars (appearing when one tries to calculate $\Sigma^{AB}$ and thus reach to (2.25)) can be calculated.

\[ w_2 := C_{\xi\lambda} C^{\mu\nu} C_{\mu\nu} \gamma_{\lambda\gamma} = \lambda (x^2)^2 \]

\[ w_3 := C_{\xi\lambda} C_{\mu\nu} C_{\sigma\rho} \gamma_{\lambda\mu\nu\rho} = \frac{\lambda x^1 x^2}{2} \]

\[ w_4 := C_{\xi\lambda} C_{\mu\nu} C_{\mu\nu} \gamma_{\lambda\mu\nu\rho} = -\frac{36 m \alpha^\beta \gamma_{\alpha\beta}}{x^3} \]

\[ w_5 := C_{\xi\lambda} C^{\mu\nu} C_{\mu\nu} \gamma_{\lambda\rho \gamma} = \frac{x^1 x^2}{2} + \frac{2m}{x^3} m^{\alpha\beta} \gamma_{\alpha\beta} \]

\[ w_6 := C_{\xi\lambda} C^{\mu\nu} C_{\mu\nu} \gamma_{\lambda\rho \gamma} = \frac{(x^2)^2}{2} - \frac{(\lambda - 1)^2 (x^2)^2}{2} + \frac{4m}{x^3} m^{\alpha\beta} \gamma_{\alpha\beta} \]

\[ w_7 := C_{\xi\lambda} C^{\mu\nu} C_{\mu\nu} \gamma_{\lambda\rho \gamma} = (x^1)^2 - 2(\lambda - 1)^2 (x^2)^2 + \frac{16m}{x^3} m^{\alpha\beta} \gamma_{\alpha\beta} \]

\[ w_8 := C_{\xi\lambda} C^{\mu\nu} C_{\mu\nu} \gamma_{\lambda\rho \gamma} = (1 - \lambda^2)(x^2)^2 + \lambda x^1 x^2 + \frac{8m}{x^3} m^{\alpha\beta} \gamma_{\alpha\beta} \]

\[ w_9 := C_{\xi\lambda} C^{\mu\nu} C_{\mu\nu} \gamma_{\lambda\rho \gamma} = \frac{x^1 x^2}{2} + \frac{2m}{x^3} m^{\alpha\beta} \gamma_{\alpha\beta} \]

Using (2.23) we find the curvature invariants to be:

\[ R_1 = x^1 + 2(1 + 2\lambda)x^2 \]

\[ R_2 = 3(x^1)^2 + (4\lambda - 1)(x^2)^2 + 4x^1 x^2 + \frac{64m}{x^3} m^{\alpha\beta} \gamma_{\alpha\beta} \]

\[ R_3 = 3(x^1)^3 + 8(-4x^1 + 12x^2 - 6\lambda + 1)(x^2)^3 + 6(1 + 2\lambda)(x^1)^2 x^2 + 12(x^2)^2 x^1 - \frac{192m}{x^3} \left( -\frac{m^{\alpha\beta} \gamma_{\alpha\beta} m^\gamma m^\delta}{x^3} + \frac{2m^{\alpha\beta} \gamma_{\alpha\beta} m^\gamma}{x^3} + 2m \right) \]

It now remains to express the terms involving the contractions of $m^{\alpha\beta}$ and $\gamma_{\alpha\beta}$ as functions of $x^1, x^2, x^3$.

It is straightforward to see that:

\[ \left( \frac{m^{\alpha\beta} \gamma_{\alpha\beta}}{\sqrt{x^3}} \right)^2 = (a + b + c)^2 = \frac{x^1 - 2x^2}{2} \Leftrightarrow \frac{m^{\alpha\beta} \gamma_{\alpha\beta}}{\sqrt{x^3}} = \epsilon \sqrt{\frac{x^1 - 2x^2}{2}}, \quad \epsilon = \text{sign}(m^{\alpha\beta} \gamma_{\alpha\beta}) \]

Analogously one finds that:

\[ \frac{m^\gamma m^\delta}{x^4} = \frac{x^1 - \lambda x^2}{2} \quad \text{and} \quad \frac{m^{\alpha\beta} m_{\gamma\delta} m^\gamma}{x^3 \sqrt{x^3}} = \epsilon \sqrt{\frac{x^1 - 2x^2}{2} \left( \frac{2x^1 + (2 - 3\lambda)x^2}{4} \right) + \frac{3m}{\sqrt{x^3}}} \]

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