Unified dynamical symmetries in the symplectic extension of the Interacting Vector Boson Model

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Abstract. The algebraic Interacting Vector Boson Model (IVBM) is extended by exploiting three new subgroup chains in the reduction of its highest symplectic dynamical symmetry group Sp(12,R) to the physical angular momentum subgroup SO(3). The corresponding exactly solvable limiting cases are applied to achieve a description of complex nuclear collective spectra of even-even nuclei in the rare earth and actinide regions up to states of very high angular momentum.

First we exploit two reductions in which collective modes can be mixed, and obtain successful descriptions of both positive and negative parity band configurations. The structure of band-head configurations, whose importance is established in the first two limits, is examined in a third reduction, that also provides important links between the subgroups of the other limits.

1. Introduction

With the intensive development of the experimental facilities a lot of new data on the collective spectra of nuclei throughout the nuclear chart is accumulated [1]. It reveals the complicated character of the nuclear motions, yielding different degrees of mixing of the basic rotational and vibrational collective modes. In turn this requires a corresponding degree of complication of the nuclear structure models, but still they should remain analytically solvable and so providing the tool for the interpretation of the observed data.

The algebraic models like IBM [2], based on the notion of the dynamical symmetries have been widely used for the description of low spin states. An other example of such successful applications [3] for the description of the low-lying collective rotational spectra of the even-even medium and heavy mass nuclei, is the number of bosons preserving version of the phenomenological algebraic Interacting Vector Boson Model /IVBM/ [4].

At the same time in the relatively early stages of the development of the algebraic methods in the nuclear structure theory [5, 6, 7] the separation of the collective degrees of freedom from the intrinsic ones, their development and mixing was attempted in the framework of the symplectic geometry of the nuclear many-body problem. The advantages of the application of symplectic structures, are based on their property to change the number of phonons building the collective states and so to provide larger basis spaces to incorporate the more and more complex nuclear spectra.
With the aim to extend the earlier applications of the IVBM [4], and in order to incorporate the new experimental data on the states with higher spins and on the various new excited bands, we explore the symplectic extension of the model, with the $Sp(12, \mathbb{R})$ as a group of dynamical symmetry. The extension is realized and has its physical interpretation over the basis of its maximal compact subgroup $U(6) \subset Sp(12, \mathbb{R})$, which was interpreted as the rotational limit [3] of the number of boson’s preserving model. This naturally leads to the additional description [8] of positive and negative parity states with mixed collective vibrational and rotational modes.

In the larger infinite boson spaces of the dynamical $Sp(12, \mathbb{R})$ there are also new chains of subgroups, starting with noncompact symplectic subalgebras. We investigate the reduction $Sp(2, \mathbb{R}) \otimes SO(6) \subset Sp(12, \mathbb{R})$, which allows the inclusion of a 6-dimensional Davidson potential [9], that is known to allow mixing of the rotational and vibrational modes, so in applications it reproduces very well the transitional behavior in nuclear spectra. The other reduction [10] we consider $Sp(4, \mathbb{R}) \otimes SO(3) \subset Sp(12, \mathbb{R})$ allows the selection of states with fixed angular momentum $L$, given by the $SO(3)$ irreducible representations (irreps). The physics behind this dynamical symmetry outlines the importance of the structure of the band-head configurations in the development of the nuclear spectra.

In this work, the above mentioned dynamical symmetries are unified in a generalized reduction scheme for the symplectic extension of the IVBM, which contains a lot of relations between the subgroups from the different chains. These clarify the physical meaning and motivation of the model and lead to rather successful applications for the description of the vast amount of new experimental data on the nuclear structure, while still retaining the advantages of the use of dynamical symmetries, namely the exact analytic solutions for the energy spectra of nuclei.

2. The symplectic extension of the IVBM dynamical symmetry

The algebraic structure of the IVBM is realized in terms of creation (annihilation) operators $u_m^\dagger(\alpha)(u_m(\alpha))$, in a 3-dimensional oscillator potential $m=0, \pm 1$ of two types of bosons differing by the value of the ”pseudo-spin” projection $\alpha = 1/2(p)$ or $\alpha = -1/2(n)$. The later are related with the cyclic coordinates $x_{\pm 1}(\alpha) = \pm \frac{1}{\sqrt{2}}(x_1(\alpha) \pm ix_2(\alpha)), x_0(\alpha) = x_3(\alpha)$ and their associated momenta $q_m(\alpha) = -\partial/\partial x_m(\alpha)$, in the standard way

$$u_m^\dagger(\alpha) = \frac{1}{\sqrt{2}}(x_m(\alpha) - iq_m(\alpha)), \quad u_m(\alpha) = (u_m^\dagger(\alpha))^\dagger, \quad (1)$$

where $x_i(\alpha) i = 1, 2, 3$ are Cartesian coordinates of a quasi-particle vectors with an additional index - the projection of the ”pseudo-spin” $\alpha = \pm \frac{1}{2}$. The bilinear products of the creation and annihilation operators of the two vector bosons (1) generate the boson representations of the non-compact symplectic group $Sp(12, \mathbb{R})$ [4]:

$$F_M^L(\alpha, \beta) = \sum_{k,m} C^{LM}_{1k1m} u_k^\dagger(\alpha) u_m^\dagger(\beta), \quad (2)$$

$$G_M^L(\alpha, \beta) = \sum_{k,m} C^{LM}_{1k1m} u_k(\alpha) u_m(\beta), \quad (3)$$

$$A_M^L(\alpha, \beta) = \sum_{k,m} C^{LM}_{1k1m} u_k^\dagger(\alpha) u_m(\beta), \quad (4)$$

where $C^{LM}_{1k1m}$, the usual Clebsch-Gordon coefficients for $L = 0, 1, 2$ and $M = -L, -L + 1, ..., L$, define the transformation properties of (2) and (3) under rotations. The commutation relations between the pair creation and annihilation operators (2) and the number preserving operators (3) are calculated in [4].
2.1. Reduction through the compact $U(6)$

2.1.1. Algebraic structure The set of operators (3) close under commutation the algebra of the maximal compact subgroup of $U(6) \subset Sp(12, R)$ [11]. The linear invariant of $U(6)$ is the number operator:

$$N = \sqrt{3}(A^0(p, p) + A^0(n, n)) = N_+ + N_-, \tag{4}$$

that counts the total number of bosons. Being the first order invariant of $U(6)$, the operator (4) splits the boson representations of $Sp(12, R)$ into a countless number of symmetric unitary irreducible representations /UIR/ of the type $[N, 0, 0, 0, 0, 0] = [N]_0$, where $N = 0, 2, 4,...$ for the even UIR and $N = 1, 3, 5,...$ for the odd ones. The rest of the operators of the physical observables introduced in this limit, which define the algebra of $SU(3)$ are the truncated (“Elliott’s”) [12] quadrupole operator $Q_M = \sqrt{6}\sum_{\alpha, \alpha}A^2_M(\alpha, \alpha)$, $M = 0, \pm 1, \pm 2$ and the angular momentum operator with components $L_M = -\sqrt{2}\sum_{\alpha, \alpha}A^1_M(\alpha, \alpha)$, $M = 0, \pm 1$ that generate its $SO(3)$ subalgebra. These operators are obtained exactly from the quantization of the classical momenta in the coordinate representation and further motivate the use of the vector bosons (1) as building blocks of the model. The scalar operators:

$$A^0(p, n) = \sqrt{\frac{2}{3}}T_+ \quad A^0(n, p) = -\sqrt{\frac{2}{3}}T_- \quad A^0(p, p) = \sqrt{\lambda}N_+ \quad A^0(n, n) = \sqrt{\lambda}N_- \tag{5}$$

are the Weyl generators of the algebra of $U(2)$. We can use the equivalent set of infinitesimal operators containing in addition to the raising $T_+$ and lowering $T_-$ components of the pseudospin (see (5)) the Cartan operators $N$ (4) and

$$T_0 = -\frac{\sqrt{3}}{2}[A^0(p, p) - A^0(n, n)] \tag{6}$$

the third projection of the pseudospin operator $T$. The operators $T_0, T_\pm$ satisfy the commutation relations $[T_0, T_\pm] = \pm T_\pm, \quad [T_+, T_-] = 2T_0, \quad [N, T_{0, \pm}] = 0$ of the pseudospin algebra $su(2)$. These operators play an important role in the consideration of the nuclear system as composed by two interacting subsystems. Obviously (5) commute with the $SU(3)$ generators $Q_M$ and $L_M$, so that the algebras of the two groups are mutually complementary.

As a result of the above considerations, the rotational limit [3] of the number preserving version of the model defined by the chain:

$$Sp(12, R) \supset U(6) \supset SU(3) \otimes U(2) \supset SO(3) \otimes U(1) \tag{7}$$

$$[N] \quad (\lambda, \mu) \quad (N, T) \quad K \quad L \quad T_0 \tag{8}$$

is extended to the $Sp(12, R)$- group of dynamical symmetry. The labels below the subgroups are the quantum numbers (8) corresponding to their irreducible representations. Since the reduction from $U(6)$ to $SO(3)$ is carried out by the direct product of the groups $SU(3)$ and $U(2)$, their quantum numbers are related in the following way: $T = \frac{1}{2}, N = 2\mu + \lambda$. Making use of the latter we can write the basis as

$$| [N]_0; (\lambda, \mu); K, L, M; T_0 \rangle = | (N, T); K, L, M; T_0 \rangle \tag{9}$$

The ground state of the system is the vacuum state $|0\rangle$ with $N = 0, T = 0, K = 0, L = 0, M = 0, T_0 = 0$. The basis states associated with the even irreducible representation of the $Sp(12, R)$ can be constructed by the application on it of powers of raising generators $F_M^0(\alpha, \beta)$ of the same group. Each raising operator will increase the number of bosons $N$ by two. The $Sp(12, R)$
classification scheme for the $SU(3)$ boson representations for even value of the number of bosons $N$ is shown on Table 1. Each row (fixed $N$) of the table corresponds to a given irreducible representation of the $U(6)$. Then the possible values for the pseudospin $T = \frac{N}{2} , \frac{N}{2} − 1, . . . , 0$ are given in the column next to the respective value of $N$. Thus when $N$ and $T$ are fixed, $2T + 1$ equivalent representations of the group $SU(3)$ arise, for which $\lambda$ and $\mu$ are obtained from their relations with $N$ and $T$. Each of them is labelled by the eigenvalues of the operator $T_0 : −T, −T + 1, . . . , T$, defining the columns of Table 1. The same $SU(3)$ representations $(\lambda, \mu)$ arise for the positive and negative eigenvalues of $T_0$.

Finally in order to obtain the values of the angular momenta contained in a given $SU(3)$ representation $(\lambda, \mu)$ we use the standard reduction rules [12] for the chain $SU(3) \supset SO(3)$. The multiplicity index $K$ appearing in this reduction is related to the projection of $L$ in the body fixed frame and is used with the parity ($\pi$) to label the different bands ($K^\pi$) in the energy spectra of the nuclei. We define the parity of the states as $\pi = (−1)^T$, which allows us to describe both positive and negative bands.

The Hamiltonian, corresponding to the considered limit of IVBM, is expressed in terms of the first and second order invariant operators of the different subgroups in the chain (7). As a result of the complementarity of the $SU(3)$ and $SU(2)$ groups, the Casimir operator of $SU(3)$ with eigenvalue $(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu)$, is expressed in terms of the operators $N$ and $T$, so we use the expression:

$$H = aN + bN^2 + \alpha_3T^2 + \beta_3\pi_3 + \alpha_1T_0^2,$$

where $\pi_3$ is the $SO(3)$ second order Casimir operator. $H$ (10) is obviously diagonal in the basis (9) labelled by the quantum numbers of the subgroups of the chain (7). Its eigenvalues are the energies of the basis states of the boson representations of $Sp(12, R)$:

$$E((N, T), L, T_0) = aN + bN^2 + \alpha_3T(T + 1) + \beta_3L(L + 1) + \alpha_1T_0^2$$

The subspaces $[N, 0, 0, 0, 0, 0] = [N]_6$, where $N = 0, 2, 4, . . .$ given in Table 1 are of the finite dimension, which simplifies the problem of diagonalization. Therefore the complete spectrum of the system is calculated trough the diagonalization of the Hamiltonian in the subspaces of all the UIR of $U(6)$, belonging to a given UIR of $Sp(12, R)$, which further clarifies its role of a group of dynamical symmetry.
2.1.2. Description of the ground and octupole bands energies 

The most important application of the above limit of the model is the possibility to describe both even and odd parity bands up to very high angular momentum. In order to do this we first have to identify these experimentally observed bands with the sequences of basis states for the even representation of $Sp(12, R)$ given in Table 1. As we deal with the symplectic extension of boson representations of the number preserving $U(6)$ symmetry we are able to consider all even eigenvalues of the number of vector bosons $N$ with the corresponding set of pseudospins $T$. For the description of the ground band, we choose the $SU(3)$ multiplets $(0, \mu)$ from the different $U(6)$ irreps given by the numbers of bosons, $N = 0, 4, 8, \ldots$ and pseudospin $T = 0$ in the column labelled by $T_0 = 0$ of Table 1.

While for the octupole band the $SU(3)$ multiplets $(2, \mu - 1)$ for $N = 8, 12, \ldots$ and $T = 1$ from the same column $T_0 = 0$ are used. In terms of $(N, T)$ this choice corresponds to $(N = 2\mu, T = 0)$ for the positive ($K^\pi = 0^+)$ and $(N = 2\mu + 2, T = 1)$ for the negative ($K^\pi = 0^-)$ parity band, respectively.

Further we define the energies of each state with given $L$ as yrast energy with respect to $N$ in the two considered bands. Hence their minimum values are obtained at $N = 2L$ for the ground band, and $N = 2L + 2$ for the octupole band, respectively. In the so defined $SU(3)$ representations for each $N$ the maximal values of $L$ appear for the first time (see Table 1).

For the above basis states $T_0 = 0$ and the experimental data on the ground and octupole bands, the last term in the energy formula (11) vanishes. The phenomenological model parameters $a, b, \alpha_3$, and $\beta_3$ are evaluated by a fit to the experimental data [13] on the considered bands of the even-even deformed nuclei belonging to light actinides and rare earth region.

![Figure 1.](image1.png)

**Figure 1.** Comparison of the experimental and theoretical energies for ground and octupole bands of $^{228}$Th. The later are calculated with the parameters $a = 0.0194, b = -0.0010, \alpha_3 = 0, 015224$ and $\beta_3 = 0.0094$.

![Figure 2.](image2.png)

**Figure 2.** Comparison of the theoretical and experimental staggering functions for the ground and octupole bands of $^{226}$Th

The agreement between the theoretical values obtained with only four model parameters and the experimental data for many nuclei considered in [8] is rather good. Applying the yrast conditions relating $N$ and $L$ the energies (11) for the two considered bands can be rewritten as:

$$E(L) = \beta L(L + 1) + (\gamma + \eta)L + \xi.$$  \hspace{1cm} (12)

The new free parameters are $\beta = 4b + \beta_3, \gamma = 2a - 4b, \eta = 8b, \xi = 2a + 4b + 2\alpha_3$. The values of the parameters in (11) determine the behavior of the energies of the two bands and their
position with respect to each other. The shift between the bands depends on the parameter $\xi$. When they are very close they interact through the $L$-dependent interaction with a strength $\eta$.

From (12) we can see that eigenstates of the first positive and negative bands consists of rotational $L(L+1)$ and vibrational $L$ modes. The rotational interaction is with equal strength $\beta$ in both of the bands. In the nuclei, considered in [8], the obtained values of the parameter $\eta$ are always negative, which means that the negative parity band is less vibrational than the positive one.

Odd-even staggering patterns between ground and octupole bands are a typical feature for this type of collectivity. In order to test further our model we applied on the energies the positive one. As fixed in both of the bands. When they are very close they interact through the "stretched" states. Their domination is determined by the important role of the quadrupole-$L$ modes. The rotational interaction is with equal strength $\beta$ in both of the bands. In the nuclei, considered in [8], the obtained values of the parameter $\eta$ are always negative, which means that the negative parity band is less vibrational than the positive one.

Odd-even staggering patterns between ground and octupole bands are a typical feature for this type of collectivity. In order to test further our model we applied on the energies the staggering function defined as [14]:

$$\text{Stg}(L) = 6\Delta E(L) - 4\Delta E(L-1) - 4\Delta E(L+1) + \Delta E(L+2) + \Delta E(L-2),$$

(13)

where $\Delta E(L) = E(L) - E(L-1)$. The comparison of the calculated and experimental energies and staggering patterns are illustrated in Figures 1 and 2. One can see a good agreement with experiment, as well as the reproduction of the “beat” patterns of the staggering behavior. This is due to the interaction term $\eta L$ in (12) between the positive and negative parity bands, which is a result of the introduced notion of yrast energies in the framework of the symplectic extension of the IVBM.

2.1.3. Description of other excited bands In order to demonstrate further applicability of this dynamical symmetry, we use the theory to describe some additional excited positive and negative parity bands. The next most important bands for determining the collective properties of heavy nuclei are the positive parity $\beta$ ($K^* = 0^+\uparrow$) [15] and $\gamma$ ($K^* = 2^+\uparrow$) collective bands and low-lying negative parity bands with $K^* = 1^-,3^-$. These excited bands have to be mapped on the basis states as well. In general the appropriate [7] subset of $SU(3)$ states are the so called "stretched" states. Their domination is determined by the important role of the quadrupole-quadrupole interactions in the collective excitations. Thus, the most important $SU(3)$ states will be those with maximal weight, i.e. those which have maximal eigenvalues of the second order $SU(3)$ Casimir operator. For the considered chain we have two types of stretched states: $(\lambda,\mu) = (\lambda_0,\mu_0 + i)$ or $(\lambda,\mu) = (\lambda_0 + 2i,\mu_0)$, where $\lambda_0$ and $\mu_0$ fix the starting $SU(3)$ build by $N_0 = \lambda_0 + 2\mu_0$ bosons and $i$ is changing.

In the first case obviously $\lambda_0 = 2T$ is fixed, which fixes the parity of the bands. We require that $\lambda_0 \leq \mu$ and this relates it to the value of $K$, that labels the bands. For the $K = 0$ bands we have $i = 0, 2, 4, \ldots$ and $N$ changing in steps of 4 and for the $K \neq 0$ the values of $i = 0, 1, 2, 3, \ldots$ and $\Delta N = 2$. In this case the yrast condition that relates the number of bosons $N$ with the angular momenta is $N = 2L + 6T - K$. We note that the states of ground and octupole bands which were selected above by means of the algebraic notion of yrast bands are of this type.

For the second type of stretched states $(\lambda_0 + 2i,\mu_0)$, obviously $T = (\lambda_0 + 2i)/2$ is changing and in order to preserve the parity of the bands we need to have $i = 0, 2, 4, \ldots$ and so $\Delta N = 4$. Now we have $\mu_0 \leq \lambda_0$ fixed and related to the value of $K$, that labels the bands. In this case the yrast condition is defined by the relation $N = 2L + 4\mu_0 - K$.

If we consider $T_0$ as fixed in both cases the stretched $SU(3)$ irreps belong to a column in Table 1 and if they change one moves along a diagonal. In the second case $T_0$ plays a more important role because the value of $T$ is changing with the development of the bands and we have more choices in changing respectively the values of $T_0 = \pm T, \pm(T-1), \ldots, 0$ and hence we can make use of the additional parameter $\alpha_1$ in (11) when fitting the values of the energies. For the excited $\beta$-bands and $\gamma$-bands, we use the sequences of stretched $SU(3)$ irreps determined by the "diagonals" $(\lambda_0 + 2i,\mu_0)$ with $\mu_0 = 0$ or 2.

In our phenomenological approach, because of the symplectic extension of the number preserving model [3], we have a great flexibility in identifying bands in any of the ways described.
above. As a result of the $N \leftrightarrow L$ connections we always get a mixing of the rotational and vibrational collective modes, represented by the $L(L+1)$ and $L$ terms, respectively.

We illustrate these advantages for the well deformed nuclei $^{172}$Yb with $(R_{4/2} = 3.3)$ and $^{152}$Sm $(R_{4/2} = 3.025)$, which is an example of a nucleus at the critical point symmetry $X(5)$ [16]. The values of the Hamiltonian (10) parameters $a, b, \alpha_3, \alpha_1$, and $\beta_3$, are obtained in a fitting procedure for all the states of the considered collective bands. The comparison of the theoretical results with the experimental data is presented on Figures 3 and 4 respectively, that prove the good reproduction of the different positions of the collective bands in the framework of the model which can be used further for studying finer effects in the structure of the nuclear spectra.

In summary with the symplectic extension of the rotational $U(6)-$limit of the number preserving version of the IVBM, we extend the applications to the description of nuclei with more complex features.

2.2. Reduction through the non-compact $Sp(2,R)$

2.2.1. Algebraic construction containing the 6–dimensional Davidson potential. The need for a description of nuclei in which rotational-vibrational interactions are taken into account has led to a search for algebraically solvable potentials and a meaningful set of basis states that make the transitional nature of these systems more transparent. An algebraically solvable theory that can describe systems with rotational-vibrational interactions and which has known algebraic solutions when applied to diatomic molecules, is the one containing the Davidson potential [17]. In an algebraic approach for either the nuclear many-body problem or the
Bohr-Mottelson collective model, the addition of the Davidson potential to the Hamiltonian requires the consideration of a dynamical subgroup chain that starts with the direct product 
\[ Sp(2, R) \otimes SO(n) \subset Sp(2n, R), \] with \( n = 3 \) and 5, respectively [18].

Indeed if the local isomorphism of the \( sp(2, R) \approx su(1, 1) \) algebras is taken into account its relation to the spectrum generating algebra of the many body nuclear system with the Davidson interaction becomes explicit. This provides the motivation for considering this reduction in seeking a description of a more complex modes that includes rotational-vibrational interactions. Within the framework of the phenomenological IVBM [4], the more general case of a 6-dimensional Davidson potential naturally appears. The new reduction chain of its dynamical symmetry group [5], [19], [20]:

\[ Sp(12, R) \supset Sp(2, R) \otimes SO(6). \]  
(14)

extends the applicability of the theory to include mixed modes collective interactions.

As can be deduced from the considerations given in [18], this construction obviously survives the addition of Davidson potential. The infinitesimal generators of the \( Sp(2, R) \) algebra:

\[ F = \sum_{k,m,\alpha} C^{00}_{k1m} u^k_\alpha u_m^\alpha(\alpha) = 2S^+, \quad G = \sum_{k,m,\alpha} C^{00}_{k1m} u_k^\alpha u_m(\alpha) = 2S^- \]
and

\[ A = \sum_{k,m,\alpha} C^{00}_{k1m} u_k^\alpha u_m(\alpha) = \frac{1}{\sqrt{3}} N = 2S^0 - 1 \]
are obtained from the \( Sp(12, R) \) generators (2) and (3) by means of contraction with respect to both the spatial \( m = 0, \pm 1 \) and the "pseudospin" \( \alpha = \pm 1/2 \). It is straightforward to show that the operators \( S^7, \tau = 0, \pm \) commute in a standard way for the \( SU(1, 1) \) algebra generators [21] \[ [S^0, S^\pm] = S^\pm, \quad [S^+, S^-] = -2S^0 \], so the \( sp(2, R) \) and the \( su(1, 1) \) algebras are locally isomorphic with a Casimir operator written as \( C_2(SU(1,1)) = S^0(S^0 - 1) - S^+S^- \).

By construction, the generators \( F, G \) and \( A \) are scalars with respect to 6-dimensional rotations and they commute with the components of the 6-dimensional momentum operators [4],

\[ \Lambda^L_M(\alpha, \beta) = A^L_M(\alpha, \beta) - (-1)^{LM} A^L_M(\beta, \alpha), \]  
(15)

which obey the property \( \Lambda^L_M(\alpha, \beta) = (-1)^L \Lambda^L_M(\beta, \alpha) \) and generate the \( SO(6) \supset U(6) \) algebra. In this way, the direct product of the two groups (14) is realized. The second order invariant for the \( SO(6) \) group is \( \Lambda^2 = \sum_{L,\alpha,\beta} (-1)^{LM} \Lambda^L_M(\alpha, \beta)\Lambda^L_{-M}(\beta, \alpha) \), and is related to the second order invariant of the \( Sp(2, R) \), as in the direct product (14) the two groups are complementary [19], which means that the irreps of the \( SO(6) \) determine those of \( Sp(2, R) \approx SU(1, 1) \) and vice versa.

In order to define the basis of the system with (14) as a dynamical symmetry that allows one to include the 6-dimensional Davidson potential, we consider the reduction of the \( SO(6) \) algebra to the \( SO(3) \) algebra of the angular momentum through the following chain [4], [22]

\[ SO(6) \supset SU(3)_{\bar{\lambda}, \bar{\mu}} \otimes O(2)_{\nu} \supset SO(3)_{L}, \]  
(16)

which could be defined as the \( \gamma \)-unstable limit of the IVBM. The single infinitesimal operator of \( O(2) \) is proportional to the scalar operator \( A^0(\alpha, \beta) \) from the \( SO(6) \) generators (15),

\[ M_{\alpha\beta} = -\sqrt{3} A^0(\alpha, \beta) = -\sqrt{3}[A^0(p, n) - A^0(n, p)], \]  
(17)

and the generators of \( SU(3) \) [4] are

\[ X^2_M = i(A^2_M(p, n) - A^2_M(n, p)), \quad M = 0, \pm 1, \pm 2, \]  
(18)

\[ Y^1_M = A^1_M(p, p) + A^1_M(n, n) = -\frac{1}{\sqrt{2}} L_M, \quad M = 0, \pm 1. \]  
(19)
Note, that in this case the quadrupole moment \( X \) (18) is the proton-neutron quadrupole interaction, which makes the difference with the subgroup \( SU(3) \subset U(6) \) considered in the previous limit. The second-order Casimir invariants of the two groups in the direct product in (16) are \( 2C_2(O_2) = M^2 = \sum_{\alpha,\beta} M_{\alpha\beta} M_{\beta\alpha} \) and \( C_2(SU(3)) = \sum_M (-1)^M (X_M X_M + Y_M Y_M) \). Furthermore, the following relation between the quadratic Casimir operators \( C_2(SU(3)) \), \( M^2 \) of \( O(2) \) and \( \Lambda^2 \) of \( SO(6) \) holds [23]:

\[
\Lambda^2 = 2C_2(SU(3)) - \frac{1}{3}M^2,
\]

which means that the reduction from \( SO(6) \) to the rotational group \( SO(3) \) is carried out through the complementary groups \( O(2) \) and \( SU(3) \) [19].

As the introduced above \( SO(6) \subset U(6) \), the obtained symmetric representations \([N]_6 \) of \( U(6) \) decompose into fully symmetric \((\omega,0,0)_6 = (\omega)_6 \) irreps of \( SO(6) \) according to the rule

\[
[N]_6 = \bigoplus_{\omega=N,N-2,...,0(1)} (\omega,0,0)_6 = \bigoplus_{i=0}^{\frac{N}{2}} (N-2i)_6,
\]

where \( \frac{N}{2} \) if \( N \) is even and \( \frac{N-1}{2} \) if \( N \) is odd. As a consequence, the Elliott’s notation [12] \((\lambda,\mu)\) of \( SU(3) \) are determined by \((\omega)_6 \) of \( SO(6) \) and by the integer label \((\nu)_2 \) of the associated irrep of \( O(2) \) i.e.

\[
(\omega)_6 = \bigoplus_{\nu=\pm\omega,\pm\omega-2,...,0(\pm1)} (\lambda = \frac{\omega+\nu}{2}, \mu = \frac{\omega-\nu}{2}) \otimes (\nu)_2.
\]

Finally, the convenience of this reduction can be further enhanced through the use of the standard rules for the reduction of the \( SU(3) \subset SO(3) \) chain. Hence the basis, labelled by the quantum numbers classified by the group-subgroup chain (16), can be written as \([N\omega; (\lambda,\mu)\nu; K,L) \), where the reduction rules for obtaining specific values for each state are given earlier. By means of these labels, the basis states can be classified in each of the two irreducible even \( H_+ \) with \( N = 0,2,4,... \), and odd \( H_- \) with \( N = 1,3,5,... \) representations of \( Sp(12,R) \). We illustrate this in Table 2 for the even \( H_+ \) irreducible representation, where \( N \)

### Table 2. Classification scheme of the basis states in the even \( H_+ \) space of \( Sp(12,R) \) according to the decompositions given by (14) and (16).

| \( N\omega \) | \( \nu/ \) | 6 | 4 | 2 | 0 | -2 | -4 | -6 |
|---|---|---|---|---|---|---|---|---|
| 0 0 | | (0,0) | | | | | | |
| 2 0 | 2 2 | (2,0) | (1,1) | (0,0) | (0,2) |
| 4 0 | 4 2 | (4,0) | (3,1) | (2,2) | (1,1) | (0,0) | (0,2) |
| 6 0 | 6 4 | (6,0) | (5,1) | (4,2) | (3,3) | (2,4) | (1,5) | (0,6) |
| : | : | : | : | : | : | : | : | : |
with the set of $\omega$ contained in it (21) label the rows and the values of the quantum number $\nu$ label the columns. The $SU(3)$ quantum numbers $(\lambda, \bar{\mu})$ define the cells of the Table 2 as they are obtained with the help of $\omega$ and $\nu$ (22).

At this point some of the similarities between the reductions through the $U(6)$ (7) and $O(6)$ (16) become apparent. The values of the quantum number $\omega$ are clearly related with the values of the pseudospin $T = \frac{\omega}{2}$, so we can introduce a parity operator defined as $(-1)^{\frac{\omega}{2}}$ like in the $U(6)$ limit (7). Respectively the values of $\nu$ have similar relation to the values of $T_0 = \frac{\nu}{2}$. The important difference in this case is that there is no degeneracy in the values of the $SU(3)$ irreps $(\lambda, \bar{\mu})$, that belong to a given row defined by $\omega$, but in the columns (fixed value of $\nu$) the $SU(3)$ irreps repeat each other except the ones corresponding to the maximal value of $\omega = N$ (the first row for each $N$) which is added for each $\nu = \lambda - \bar{\mu}$. Also in this case $\lambda$ and $\bar{\mu}$ are always both even or odd.

The Hamiltonian with the considered dynamical symmetry (25) is expressed in terms of the first and second order Casimir operators of the different subgroups in its corresponding chain (16):

$$H = aN + bN^2 + \alpha_6\lambda^2 + \alpha_2M^2 + \beta_3L^2.$$  \hspace{1cm} (23)

and it is obviously diagonal in the basis $|N\omega; (\lambda, \bar{\mu})\nu; K, L\rangle$. The second order invariant of $SU(3)$ is dropped in (23), because of its linear dependence on the Casimir operators of the $SO(6)$ and $O(2)$ (20). Then the eigenvalues of the Hamiltonian (23) that yield the spectrum of a system interacting with 6-dimensional Davidson potential are:

$$E(N, \omega, \nu, L) = aN + bN^2 + \alpha_6\omega(\omega + 4) + \alpha_2\nu^2 + \beta_3L(L + 1).$$  \hspace{1cm} (24)

This expression is very similar to the one obtained in the $U(6)$ (7) case (11), with a difference coming only from the $SO(6)$ second order invariant $\Lambda^2$, with the parameter $\alpha_6$. Other differences in the applications to real nuclei follow from the possible choices in mapping the experimentally observed collective states to the basis of this dynamical symmetry.

2.2.2. Application to real nuclei

In the applications of this new dynamical symmetry of the IVBM to real nuclear systems that we present here, we exploit again the “algebraic” definition of yrast states as introduced in [8] and in the previous paragraph. Here we use for the states of the ground band, which are the yrast states of the nucleus, a correspondence to the basis states different from the one in [9], where $N = \omega = T\nu = 2L = 0, 4, 8, \ldots$ ($\Delta N = 4$). At $-\nu = 2L$ we obtain the left to right diagonal $(0, 2L)$ of Table 2 with $\bar{\lambda} = 0$ and changing $\bar{\mu} = 2L = 0, 4, 8, \ldots$. At $\nu = 2L$ the respective right to left diagonal of Table 2 is $(2L, 0)$. Here $T = \frac{\omega}{2} = L$ is always even and we consider the states belonging to the $K^\pi = 0^+$ band. Than the ground state band’s (GSB) energies are obtained with the expression $E_g(L) = \alpha L + \beta L(L + 1)$ where $\alpha = 2a - 4b + 4\alpha_6 - 4\alpha_2$ and $\beta = 4b + 4\alpha_6 + 4\alpha_2 + \beta_3$. It is obvious that in the GSB we will have a certain degree of mixing of the vibrational and rotational modes depending on the values of the parameters $\alpha$ and $\beta$. The later depend on all the Hamiltonian (23) parameters. There is no an additive constant to the energies, like in the $U(6)$ case and we can not describe an octupole $(K^\pi = 0^-)$ band here, because $\bar{\lambda}$ and $\bar{\mu}$ are both always even or odd.

Further, the states of the excited bands are mapped to the theoretical ones, generally in the two ways described in the previous paragraph, namely the stretched states $(\lambda, \mu) = (\bar{\lambda}_0 + k, \bar{\mu}_0)$ or $(\lambda_0, \mu_0 + k)$, where $k$ is changing as $k = 1, 2, 3, \ldots$ which in turn can belong to the left ($\nu$—positive) or right ($\nu$—negative) ”diagonal” of Table 2. The situation is similar to the case described for the ground state band, where $\lambda$ is changing and $\mu$ is fixed or vice versa.

The correct placement of the excited bands in the spectrum strongly depends on their bandhead configurations and in particular, on the number of bosons $N_0 = \bar{\lambda}_0 + \mu_0$, from which they are built [10]. So we define the number of bosons for each state in the band with a given $L$ in
the following way $N = N_0 + \nu + 2L$ and the respective $\omega = \nu + 2L$. In this case the shifted values of $N = \omega = N_0 + k + 2L$ remain the same but we have different sequences of $(\lambda, \bar{\mu})$ multiplets, defining the bands along the diagonals with $\pm \nu = \lambda - \bar{\mu} = \pm(2L - k)$.

The variety of possible choices for the correspondence of the excited bands to sequences of states in the symplectic space and the mixing of the rotational and vibrational degrees of freedom like in the $U(6)$-limit allows us to reproduce correctly the behavior of the excited bands with respect to one another, which can change a lot even in neighboring nuclei [25].

The five free parameters of the Hamiltonian (23), are determined by fitting the theoretical predictions for the energies of the ground and few excited bands to the experimental data [1], using a $\chi^2$-procedure. The bands are developed according to the rules described above for their corresponding $K^\pi$ values. We choose for $N_0$ shifting the band-head states, the value that gives the lowest $\chi^2$, after obtaining the parameters mainly from the ground state band. That is why, we usually choose in the applications, nuclei with long spin sequences in their ground bands.

Here in addition to the basic ground, $\beta-$ and $\gamma-$bands, described in [9], we illustrate the possible description of negative parity bands with $K^\pi \neq 0 = 1^-, 2^-, 3^- \ldots$ like in the $U(6)$-reduction chain and use this application to compare the results of both limits. Hence we present here the fit of the Hamiltonian parameters for this limit for the rotational $^{162}$Dy nucleus (Figure 6), and for another $X(5)$ symmetry nucleus $^{154}$Gd [26]. As shown in Figures 5 and 6, the experimental data is reproduced remarkably well. The energies of the excited bands confirm the correct identification of the experimental states with the basis states. This reveals again the
importance of \( N_0 \), which vary quite a lot in the different bands. From the presented results, it could be seen, that both of the above limits of the symplectic extension of the IVBM are equally appropriate for the description of a rather broad range of nuclei, and most importantly nuclei that display different degree of mixing of the rotational and vibrational degrees of freedom.

2.3. The reduction through the noncompact \( Sp(4, R) \)

2.3.1. The algebraic realization

Finally we will introduce the group - theoretical background [10] of the application of the symplectic IVBM, for the description of the energy distributions of collective excited states with fixed angular momenta. This new reduction further elucidates the importance of considering the structure of the band-head’s configurations in respect to the vibrational degrees of freedom.

In terms of the introduced boson representations of \( Sp(12, R) \), the third chain of subgroups starts with the reduction [5, 19, 10]:

\[
Sp(12, R) \supset Sp(4, R) \otimes SO(3) \tag{25}
\]

The infinitesimal operators of the \( Sp(4, R) \) algebra are the \( L = 0 \) part of the \( Sp(12, R) \) generators (2) and (3): \( F^0(\alpha , \beta) = \sum_{k,m} C_{1k1m}^{00} u_k^\dagger (\alpha) u_m^+(\beta) \), \( G^0(\alpha , \beta) = \sum_{k,m} C_{1k1m}^{00} u_k(\alpha) u_m(\beta) \) and \( A^0(\alpha , \beta) = \sum_{k,m} C_{1k1m}^{00} u_k(\alpha) u_m(\beta) \) for \( \alpha , \beta = \pm 1/2 \). Hence by construction, all these operators are scalars in respect to the 3-dimensional rotations. Obviously they commute with the components of the angular momentum \( L^A_M \), that generate the \( SO(3) \) algebra, i.e. we have a direct product of the two groups (25). As a result the \( Sp(4, R) \) irreps can be labeled by the quantum numbers of their corresponding \( SO(3) \) ones, namely the angular momentum \( L \).

The maximal compact subalgebra \( u(2) \) of \( sp(4, R) \) is generated by the Weyl generators \( A^0(\alpha , \beta) \) of (5), and is the same one that corresponds to the pseudospin algebra of \( U(2) \) (5) in the direct product with the \( SU(3) \) in (7). The operator \( N \) (4) generates \( u(1) \) and plays role of the first-order invariant of \( u(2) \supset su_T(2) \otimes u_Y(1) \). Hence the following correspondence between the chains of subalgebras of \( sp(12, R) \) – through \( u(6) \) and through \( sp(4, R) \), exists [5]:

\[
\begin{align*}
sp(12, R) & \supset sp(4, R) \otimes so(3) \\
u(6) & \supset u(2) \otimes su(3)
\end{align*} \tag{26}
\]

Each of the \( sp(4, R) \) irreps that is contained in the \( sp(12, R) \) boson representations is of infinite dimension and consists of countless number of \( u(2) \) irreps [11]. A basis for the \( sp(4, R) \) representations is generated by a consecutive application of the symmetrically coupled products of the operators \( F^0(\alpha , \beta) \) to the lowest weight state (lws) with angular momentum \( L \) that labels the considered \( Sp(4, R) \) irrep [5],[24]. Each starting \( u(2) \) configuration is characterized by a totally symmetric representation \([L_2] \) formed by \( L = N_{\text{min}} \) vector bosons. The procedure to obtain the rest of the \( su(2) \) irreps \([k] \) that are contained in a given \( L \) irrep of \( sp(4, R) \) is given in details in [10]. It is illustrated for the cases \( L = 0 \) and \( L = 2 \) with the Tables 3 and 4. The columns are defined by the pseudospin quantum number \( T \) and the rows by the eigenvalues of \( N = N_{\text{min}} + n \) for \( n = 0, 2, 4, 6, \ldots \).

By means of the correspondence (26) between the two considered chains of subgroups of \( Sp(12, R) \) and the relations between the \( U(2) \) and \( SU(3) \) quantum numbers we were able also to present the respective \( (\lambda = k, \mu = (N - k)/2) \) irreps in the cells of the Tables 3, 4. For a given value of \( N \) these could be compared to the classification scheme of the \( SU(3) \) irreps contained in the even \( U(6) \) irreps of \( Sp(12, R) \) given in Table 1. Note that the missing \( su(2) \) irreps on
the above tables do not contain in their corresponding \((\lambda, \mu)\) representations, states with the considered value of \(L\) according to the \(SU(3) \supset SO(3)\) reduction rules [3]. In the obtained decomposition of the \(sp(4, R)\) representations \(L\) into \([k]_2\) - \(su(2)\) ones, except for the \(L = 0\) case in Table 3, there is a multiplicity, denoted as \(\rho\times\) in the appearance of some of the irreps, that shows how many times the respective irrep \([k]_2\) appears for the specified value of \(N\). This multiplicity is exactly equal to the multiplicity of the appearance of the considered value of \(L\) in the reduction of the corresponding \(su(3)\) irrep \((\lambda, \mu)\) to the \(so(3)\) values of \(L\) [5](see \(N = 6\) and \(N = 8\) rows of Table 4).

\[\text{Table 3. } L = 0\]

\[
\begin{array}{cccccc}
T & 4 & T & 3 & T & 2 \\
--- & | & | & | & | & |
0 & 2(0, 0) & N & 0 & 2 & 2(0, 0) \\
| & 2 & | & 2 & 0 \\
| & | & | & 0 & 2(0, 2) \\
| & | & | & 2 & 2(0, 2) \\
| & | & | & 0 & 2(0, 4) \\
| & | & | & 0 & 2(0, 4) \\
| & | & | & 8 & 2(8, 0) \\
| & | & | & 4 & 2(4, 2) \\
| & | & | & 0 & 2(0, 4) \\

\end{array}
\]

\[\text{Table 4. } L = 2\]

\[
\begin{array}{cccccc}
T & 4 & T & 3 & T & 2 \\
--- & | & | & | & | & |
2 & 2(2, 0) & N & 2 & 2 & 2(2, 0) \\
| & | & | & | & 2 & 2(2, 1) \\
| & | & | & | & 0 & 2(0, 2) \\
| & | & | & | & 2 & 2(2, 2) \\
| & | & | & | & 0 & 2(0, 4) \\
| & | & | & | & 2 & 2(2, 3) \\
| & | & | & | & 0 & 2(0, 4) \\
| & | & | & | & 8 & 2(8, 0) \\
| & | & | & | 6 & 2(6, 1) \\
| & | & | & | 2 & 4 & 2(4, 2) \\
| & | & | & | 2 & 2 & 2(2, 3) \\
| & | & | & | 0 & 2(0, 4) \\

\end{array}
\]

2.3.2. Energy distribution of states with fixed \(L\) As established above, because of the correspondence (26) and the relation between the \(SU(3)\) and \(SU(2)\) second order Casimir operators [8], the Hamiltonian (10) and the bases in this case are equivalent to the ones in the \(U(6)\) limit of the model (7). As a result the eigenvalues of the Hamiltonian (10) for the \(N\) case, sequences of \(\lambda\) and \(\mu\) terms in each nuclei. Further, the values of \(N_L\) corresponding to the experimentally observed \(E_L\) and the values of the parameters in (11) are evaluated in a multi-step \(\chi\)-square fitting procedure making use only of the three sets of states with \(J_L^z = 0, 1, 2\). We choose nuclei
with enough of these states to have good statistics in the fit. Most of these states are band heads (all the $0^+$, some of the $2^+$ and $4^+$) of collective bands and as a result the situation of the whole band depends on them. The set of $N_L$ with minimal value of $\chi^2$ determines the distribution (the parameters of the Hamiltonian) of the $L_i$ states energies with respect to the number of bosons $N_L$ that build them. For the set of $0^+$ states ($L = 0$) we chose both $T = T_0 = 0$, so the parameters $a$ and $b$ in (11) are evaluated and fixed. Further, for the $2^+$ we use even $T > 0$, $T_0 = 0$ and so we get the $\alpha_3$ parameter and finally for the $4^+$ we determine $T > 0$ and also a possible value of $T_0 > 0$ in order to obtain $\alpha_1$. The first two parameters $a$ and $b$ ($b < 0$) of the Hamiltonian actually determine the form of the parabolas, and they are the same for each set of states with fixed $L$. The rest of them with the values of the quantum numbers $T$, $T_0$ and $L$, only shift the curves in respect to each other. The predicting power of the model in this limit is related to the possibility to describe any other set of states, once all the parameters of the Hamiltonian are evaluated from the distributions of $0^+, 2^+$ and $4^+$ states, without involving additional parameters.

The examples chosen for the present application are the nuclei $^{146}$Sm (Figure 7), which has a typical vibrational spectra and the other one, $^{162}$Dy, that has typical rotational character (see Figure 8).

For the nucleus with vibrational spectra $^{146}$Sm we apply the procedure described above with values of $T$ that differ quite significantly ($\Delta T = 4$) for the sets with $L = 0, 2, 4$, which allows us to place them on the left hand side of the symmetric parabolas. As a result (see Figure 7) the

**Figure 7.** Comparison of the theoretical and experimental energy distributions of states with fixed $L$ in the spectra of $^{146}$Sm. The later are calculated with the parameters $a = 0.03243$, $b = -0.00006$, $\alpha_3 = -0.00161$, $\alpha_1 = -0.00553$ and $\beta_3 = 0.04134$.

**Figure 8.** Comparison of the experimental and theoretical energies for the listed excited bands of $^{162}$Dy. The later are calculated with the parameters $a = 0.02376$, $b = -0.00005$, $\alpha_3 = 0.03543$, $\alpha_1 = -0.02110$ and $\beta_3 = 0.01288$. 
values of $N_{L_i}$ increase with increasing energy of these states.

The low-lying states of the ground band, $L = 0^+_1, 2^+_1, 4^+_1$, of the typical rotational nucleus \[ ^{162}\text{Dy} \] (Figure 8) require small changes in the number of quanta $N_{L_i}$ that build the corresponding initial states in each sequence. Making use of the latter and the symmetric feature of the second order curves the states with a given $L$ in the rotational spectra are placed on the right-hand-side of the theoretical curves. On a parabola, specified for a fixed $L$, the number of bosons that build the states will decrease with increasing energies.

With the procedure employed, the ordering of the states into different bands can be recognized. In order to avoid nearly degeneracies of the energies in respect to $N_{L_i}$, we use the symmetric feature of the second order curves and place some of the states on the right side of the parabolas.

Hence, if the number of quanta that is required to build a collective state is taken as a measure of collectivity, the states from a rotational spectra are much more collective than vibrational ones, which is the expected result.

3. Generalized reduction scheme for the IVBM

The use of symplectic geometry in the investigation of the nuclear collective motion, relates the later to its microscopic structure [27]. A further elaboration in the problem can be achieved, if we consider the nuclear many body system as consisting of two interacting proton and neutron subsystems. This motivation is behind the algebraic construction of the phenomenological IVBM [4], where $Sp(12, R)$ – the group of linear canonical transformation in a 12-dimensional phase space [19] appears as the group of dynamical symmetry of the model. The three considered above dynamical symmetries can be unified in the reduction scheme (27).

\[
\begin{align*}
U(6) & \supset U(2) \otimes SU(3) \\
N & \cap T \cap (\lambda, \mu) \\
Sp(12, R) & \supset Sp(4, R) \otimes SO(3) \\
\cup K & \cup SO(6) \cap \cup Sp(2, R) \\
SO(6) & \otimes (SU(1, 1)) \\
\cup \omega & \cup \omega \\
SU(3) & \otimes O(2) \\
(\lambda, \pi) & \nu
\end{align*}
\] (27)

In mathematical terms the established relations are based [11] on the appearance of the physically important $U(2)$ [8] group of the pseudospin as the maximal compact subgroup of $Sp(4, R)$ [10], as well as its noncompact counterpart $SU(1, 1)$ [9]. By means of this vertical structure the dynamical symmetries describing the ground and excited bands are connected with the dynamical symmetry describing the sets of states with fixed angular momentum, most of which are band-head configurations. A very important model characteristic leading to the correct description of the experimental energies is the interaction between the proton and neutron subsystems, yielded in the symplectic extension of the model, which still retains the exact analytic solutions in each of the considered applications.

This generalized reduction scheme relates the presented applications of the three dynamical symmetries in their physical interpretations and the conclusions following from each one of them. It was established that the two reduction schemes, that describe the developments of collective bands in various types of nuclear spectra, the one trough $U(6)$ and the one trough $SO(6)$, yield very similar applications for the description of the ground bands and the excited positive and negative parity bands. A common feature of these applications is the possibility to mix with varying strength the two main collective modes - vibrational and rotational, which results in
the accurate description even of nuclei at the critical points of phase/shape transitions in the framework of these exactly solvable cases. The success of this approach is due not only to the easy evaluation of the relatively small number of model parameters by means of fitting to the experiment, moreover only of the first lowest bands like the ground band and the immediately following one or two excited bands. The important predicting power of the model is related to the symplectic extension, which allows only by correctly finding the number of bosons that build the band-head configurations of the other observed excited bands, to evaluate the energies of all states that belong to them.

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