On the indecomposability in the category of representations of a quiver and in its subcategory of orthoscalar representations

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Representations of quivers of the finite and tame types are classified up to equivalence in the papers [1, 2].

It is naturally to classify representations of quivers in the category of Hilbert spaces up to the unitary equivalence [3, 4]. One can regard the category of such representations as a subcategory in the category of all representations, and at that objects, which are indecomposable in the subcategory, become in general decomposable in the “larger” category. It does not happen with indecomposable locally scalar representations [4] of a quiver.

A proof of this fact and its applications are in present paper.

1. A quiver $Q$ with the set of vertices $Q_v$, $|Q_v| = N$ and the set of arrows $Q_a$ is called separated if $Q_v = \hat{Q} \sqcup \check{Q}$, and for any $\alpha \in Q_a$ its tail $t_\alpha \in \hat{Q}$ and head $h_\alpha \in \check{Q}$. A quiver $Q$ is single if all its arrows are single (i. e. if $\alpha \neq \beta$ then either $t_\alpha \neq t_\beta$ or $h_\alpha \neq h_\beta$). Vertices from $\hat{Q}$ are called even, and from $\check{Q}$ are odd.

Let $m = |\check{Q}|$, $n = |\hat{Q}|$, $\check{Q} = \{i_1, i_2, \ldots, i_m\}$, $\hat{Q} = \{j_1, j_2, \ldots, j_n\}$. A representation $T$ of a quiver $Q$ associates a finite-dimensional linear space $T(i)$ to a vertex $i \in Q_v$, and a linear map $T_{ij}: T(j) \to T(i)$ to an arrow $\alpha: j \to i$, $\alpha \in Q_a$.

Representation $T$ of single separated quiver for fixed bases of spaces $T(i)$, $i \in G_v$ can be associated with a matrix, separated by $m$ horizontal and $n$ vertical bars, i. e. the matrix

$$T = \left[ T_{i_l,j_k} \right]_{l=1}^{m} \quad \text{and} \quad \tilde{T} = \left[ \tilde{T}_{i_l,j_k} \right]_{l=1}^{m}$$

Here we assume that $T_{i_l,j_k} = 0$, if there does not exist any such $\alpha \in Q_a$ that $t_\alpha = j_k$, $h_\alpha = i_l$.

Let $\text{Rep}Q$ be the category of representations of a quiver $Q$, which objects are representations, and morphism of a representation $T$ to a representation $\tilde{T}$ is defined as a family of linear maps $C = \{C_i\}_{i \in Q_v}$ such that for each $\alpha \in Q_a$ with $t_\alpha = j$, $h_\alpha = i$ the diagram

$$\begin{array}{ccc}
T(j) & \xrightarrow{T(\alpha)} & T(i) \\
\downarrow C_j & & \downarrow C_i \\
\tilde{T}(j) & \xrightarrow{\tilde{T}(\alpha)} & \tilde{T}(i)
\end{array}$$

is commutative, i. e. $C_i T_{ij} = \tilde{T}_{ij} C_j$.

Let representations $T$, $\tilde{T}$ of a separate single quiver are defined in the matrix form by the matrices

$$T = \left[ T_{i_l,j_k} \right]_{l=1}^{m} \quad \text{and} \quad \tilde{T} = \left[ \tilde{T}_{i_l,j_k} \right]_{l=1}^{m}$$

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Let $C : T \to \tilde{T}$ be a morphism of representations, $C = \{C_i\}_{i \in Q_v}$. Let us introduce the matrices $A = \text{diag}\{C_{i_1}, C_{i_2}, \ldots, C_{i_m}\}$, $B = \text{diag}\{C_{j_1}, C_{j_2}, \ldots, C_{j_n}\}$. Then commutativity of the diagrams (1) implies
\[ AT = \tilde{T}B \quad (2) \]
We will further say that $C = (A, B)$.

Let $H$ be the category of unitary (finite-dimensional Hilbert) spaces. Denote as $\text{Rep}(Q, H)$ the subcategory in $\text{Rep} Q$ which objects are representations $T$ for which $T(i)$ are unitary spaces ($i \in Q_v$) and morphisms $C : T \to \tilde{T}$ are those of morphisms in $\text{Rep} Q$ for which, except (1), the following diagrams are also commutative
\[ T(j) \xleftarrow{T(\alpha)^*} T(i) \]
\[ \begin{array}{c}
C_j \\
\downarrow \\
\tilde{T}(j)
\end{array} \quad \begin{array}{c}
\downarrow \\
\tilde{T}(i)
\end{array} \]
\[ \tilde{T}(\alpha)^* \tilde{T}(i) \xleftarrow{T(\alpha)^*} \tilde{T}(j) \quad (3) \]
i. e. the following equality holds:
\[ BT^* = \tilde{T}^* A \quad (4) \]
Two representations $T, \tilde{T}$ from $\text{Rep} Q$ are equivalent in $\text{Rep} Q$, if there exists an invertible morphism $C : T \to \tilde{T}$ in $\text{Rep} Q$.

Two representations $T, \tilde{T}$ from $\text{Rep}(Q, H)$ are equivalent $\text{Rep}(Q, H)$, if there exists an invertible morphism $C : T \to \tilde{T}$ in $\text{Rep}(Q, H)$. It can be shown that $T, \tilde{T}$ are equivalent in $\text{Rep}(Q, H)$ if and only if they are unitary equivalent (see, for instance, [3]), i. e. the invertible morphism $C$ can be chosen as consisting of unitary matrices $C_i$.

Denote
\[ \tilde{T}_i = \begin{array}{c}
T_{i, j_1} \\
\vdots \\
T_{i, j_n}
\end{array}, \quad T_i = \begin{array}{c}
T_{1, j} \\
\vdots \\
T_{m, j}
\end{array} \]
\[ \tilde{T}_i : \sum_{k=1}^n \oplus T(j_k) \to T(i), \]
\[ T_i^j : T(j) \to \sum_{l=1}^m \oplus T(i_l). \]

Representation $T$ of a separate single quiver $Q$ from the category $\text{Rep}(Q, H)$ is called orthoscalar (in [4] such representations are called locally scalar) if each $i \in Q_v$ is associated with a real positive number $\chi_i$, and the following conditions hold:
\[ \tilde{T}_i \cdot \tilde{T}_i^* = \chi_i I_i \text{ for } i \in Q, \]
\[ T_j^* \cdot T_j^\dagger = \chi_j I_j \text{ for } j \in Q, \quad (5) \]
here $I_i$ is the matrix of identity operator in $T(i)$.

Define the category $\text{Rep}_{\text{os}}(Q, \mathcal{H})$ as a full subcategory in $\text{Rep}(Q, \mathcal{H})$, which objects are orthoscalar representations of a quiver $Q$.

Let us associate two $N$-dimensional vectors ($N = m + n$) to an orthoscalar representation $T$ of a separate single quiver $Q$: a dimension of representation $T$: $d = \{d(j)\}_{j \in Q_v}$, where $d(j) = \dim T(j)$, and a character of representation $T$: $\chi = \{\chi(j)\}_{j \in Q_v}$, $\chi(j) = \chi_j$ are defined above in (5). It is easy to see that

$$\sum_{i=1}^m d(i)\chi(i) = \sum_{k=1}^n d(j)\chi(j) =$$

(the sum of squares of the lengths of rows in a matrix of a representation $T$ equals to the sum of squares of the lengths of columns).

Two orthoscalar representations $T$ and $\tilde{T}$ are equivalent if there exist such unitary matrices $U = \text{diag}\{U_{i1}, U_{i2}, \ldots, U_{im}\}$ and $V = \text{diag}\{V_{j1}, V_{j2}, \ldots, V_{jn}\}$ that

$$UT = \tilde{T}V, \text{ or } \tilde{T}_{ij} = U_{ii} T_{ij} V_{jk}^*.$$ (6)

Here the matrices $U_{ii}$ have dimension $d_{ii} \times d_{ii}$, and $V_{jk}$ — dimension $d_{jk} \times d_{jk}$.

**Remark 1.** Note that for an arbitrary quiver $Q$ we can construct $\ast$-quiver $\hat{Q}$, adding to each arrow $\alpha_{ij}: j \to i$ an additional arrow $\alpha_{ij}^\ast: i \to j$. At that the category $\text{Rep}_{\text{os}}(Q, \mathcal{H})$ and the category of orthoscalar $\ast$-representations of a $\ast$-quiver $\hat{Q}$ are identified.

**Remark 2.** Let $C = (A, B)$ be a morphism of a representation $T$ to a representation $\tilde{T}$ in the category $\text{Rep} Q$, i. e. the equality (2) holds:

$$AT = \tilde{T}B$$

and $A, B$ are unitary operators, then $C = (A, B)$ is also a morphism of representation $T$ to representation $\tilde{T}$ in the category $\text{Rep}(Q, \mathcal{H})$, i. e. the equality (4) holds:

$$BT^* = \tilde{T}^*A$$ (8)

Indeed, (2) implies $T^*A^* = B^*\tilde{T}^*$ or, considering the unitarity of $A$ and $B$, we have $T^*A^{-1} = B^{-1}\tilde{T}^*$. Therefore $BT^* = \tilde{T}^*A$.

**Remark 3.** Let $C = (A, B)$ be a morphism of a representation $T$ to representation $T$ (endomorphism of representation $T$) in the category $\text{Rep} Q$, i. e.

$$AT = TB$$ (7)

and $A, B$ are self-adjoint operators, then $C = (A, B)$ is also an endomorphism of representation $T$ in the category $\text{Rep}(Q, \mathcal{H})$, i. e.

$$BT^* = T^*A$$ (8)

Indeed, (8) is obtained from (7) by adjunction.
2. Suppose that $T$ is a representation of a quiver $Q$ that is \textit{faithful} if $T(i) \neq 0$ for all $i \in Q_v$ ($d(i) \neq 0$). The \textit{support} of representation $T$ is the set $Q_v^T = \{i \in Q_v | T(i) \neq 0\}$. The character of orthoscalar representation is defined uniquely on the support $Q_v^T$ (and not uniquely out of it). If a representation is faithful then the character of orthoscalar representation is uniquely defined and denoted as $\chi_T$; in general we denote as $\{\chi_T\}$ the set of all characters of representation $T$. Evidently, if $T$ and $\overline{T}$ are equivalent, then $\{\chi_T\} = \{\chi_{\overline{T}}\}$.

Decomposable representations are defined in the natural way (at that, if $T = T_1 \oplus T_2$ in the category $\text{Rep}(Q, \mathcal{H})$, then $T_1(i) \oplus T_2(i)$ means orthogonal sum of unitary spaces), and if $T, T_1, T_2$ are faithful orthoscalar representations, then $\chi_{T_1} = \chi_{T_2} = \chi_T$.

Representation $T$ is called \textit{Schur representation} in the category $\text{Rep} Q$ (resp. $\text{Rep}(Q, \mathcal{H}), \text{Rep}_{os}(Q, \mathcal{H})$), if its ring of endomorphisms is one-dimensional (resp. is isomorphic to $\mathbb{C}$).

Obviously, if $T$ is Schur representation then $T$ is indecomposable (in the respective category).

\textbf{Remark 4.} If $T$ is indecomposable representation in the category $\mathcal{R} = \text{Rep}(Q, \mathcal{H})$ then $T$ is Schur representation.

Indeed, the algebra $\mathfrak{A} = \text{End}_{\mathcal{R}} T$ is finite-dimensional $*$-algebra. If $C = (A, B) \in \text{End}_{\mathcal{R}} T$ then $C^* = (A^*, B^*)$. If $C \in \text{Rad} \mathfrak{A}$ then $CC^* = (AA^*, BB^*) \in \text{Rad} \mathfrak{A}$ and $CC^*$ is nilpotent element, so $AA^*$ and $BB^*$ are nilpotent and positive. Therefore $C = (0, 0)$, and the algebra $\mathfrak{A}$ is semisimple. On the other hand, the algebra $\mathfrak{A}$ is local as an algebra of endomorphisms of indecomposable representation. Hence, $\mathfrak{A} \cong \mathbb{C}$.

\textbf{Lemma 1.} Let $Z = \begin{bmatrix} z_{ij} \end{bmatrix}_{i=1}^{m}^{j=1} \in \mathcal{M}_{m,n}$, $W = \begin{bmatrix} w_{ij} \end{bmatrix}_{i=1}^{m}^{j=1} \in \mathcal{M}_{m,n}$ be matrices over the field $\mathbb{C}$, which has equal positive lengths $|z^2|$, $|y^2|$ of corresponding rows and corresponding columns. Let $A = \text{diag}\{a_1, a_2, \ldots, a_m\}$, $B = \text{diag}\{b_1, b_2, \ldots, b_n\}$ be matrices over $\mathbb{R}$, $a_i > 0$, $b_j > 0$ for $i = 1, m$, $j = 1, n$, and let $AZ = WB$. Then $Z = W$.

\textit{Proof.} Denote $K$ — the number of nonzero elements in the matrices $Z$ and $W$ ($AZ = WB$ implies that this number is the same for $Z$ and $W$). Obviously, $K \geq \max(m,n)$, since the matrices $Z, W$ have no zero rows or columns.

We will use induction by the triples of numbers $(m,n,K)$: assume that $(m_1, n_1, K_1) < (m_2, n_2, K_2)$ if $m_1 \leq m_2$, $n_1 \leq n_2$ and at least one inequality is strict, or if $m_1 = m_2, n_1 = n_2$ but $K_1 < K_2$.

a) The base of induction is obtained when $m = 1$, either $n = 1$, or $K = \max(m,n)$.

Let $m = 1$. From $[a_1 z_{i1}, a_1 z_{i2}, \ldots, a_1 z_{in}] = [w_{i1} b_1, w_{i2} b_2, \ldots, w_{in} b_n]$ and $|z_{ij}| = |w_{ij}|$ (the last follows from the equality of lengths of corresponding columns of $Z$ and $W$), where $z_{ij} \neq 0$ and $w_{ij} \neq 0$ for $j = 1, n$, we obtain that $a_1 = b_1 = b_2 = \cdots = b_n$, and then $[z_{i1}, z_{i2}, \ldots, z_{in}] = [w_{i1}, w_{i2}, \ldots, w_{in}]$.

The case $n = 1$ is analogous.

Now let for the sake of definiteness $1 < m \leq n$. Then $K \geq n$.

Let $K = n$. In this case each column of the matrices $Z$ and $W$ contains precisely one nonzero element. Consider corresponding nonzero elements $z_{ij}$ and $w_{ij}$ of the matrices $Z$ and $W$. $AZ = WB$ implies $a_i z_{ij} = w_{ij} b_j$. Since $|z_{ij}| = |w_{ij}|$ then $a_i = b_j$, and then $z_{ij} = w_{ij}$ and, consequently, $Z = W$.

b) Let $K > n \geq m > 1$.  

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Since matrices $Z$ and $W$ has equal lengths of corresponding rows and columns (by the assumption), and matrices $AZ$ and $WB$ — because of the equality of matrices, we obtain the following relations:

$$a_i^2 (|z_{i1}|^2 + \cdots + |z_{in}|^2) = |w_{i1}|^2 b_1^2 + \cdots + |w_{in}|^2 b_n^2,$$
$$|z_{i1}|^2 + \cdots + |z_{in}|^2 = |w_{i1}|^2 + \cdots + |w_{in}|^2,$$
$$i = 1, m;$$

$$a_j^2 |z_{1j}|^2 + \cdots + a_m^2 |z_{mj}|^2 = (|w_{1j}|^2 + \cdots + |w_{mj}|^2) b_j^2,$$
$$|z_{1j}|^2 + \cdots + |z_{mj}|^2 = |w_{1j}|^2 + \cdots + |w_{mj}|^2,$$
$$j = 1, n;$$

$$a_i^2 = \sum_{j=1}^{n} |w_{ij}|^2 b_1^2 + \cdots + \sum_{j=1}^{n} |w_{ijn}|^2 b_n^2, \quad i = 1, m;$$

$$b_j^2 = a_2^2 \frac{|z_{1j}|^2}{\sum_{i=1}^{m} |z_{ij}|^2} + \cdots + a_m^2 \frac{|z_{mj}|^2}{\sum_{i=1}^{m} |z_{ij}|^2}, \quad j = 1, n.$$  

In the right parts of equalities in (11)-(12) the $a_i^2, b_j^2$ has coefficients that $\leq 1$.

We will assume (perhaps, after permutations of rows and columns) that $a_1 \leq a_2 \leq \cdots \leq a_m$ and $b_1 \leq b_2 \leq \cdots \leq b_n$.

Let $w_{1r}$ be the first and $w_{1s}$ be the last nonzero element in the first row in the matrix $W$, $1 \leq r \leq s \leq n$. Then

$$a_1^2 = \sum_{k=r}^{s} \frac{|w_{1k}|}{\sum_{j=r}^{s} |w_{1j}|^2} b_k^2 \geq \sum_{k=r}^{s} \frac{|w_{1k}|^2}{\sum_{j=r}^{s} |w_{1j}|^2} = b_r^2.$$

On the other hand,

$$b_1^2 = \sum_{l=1}^{m} \frac{|z_{11}|}{\sum_{i=1}^{m} |z_{i1}|^2} a_i^2 \geq \sum_{l=1}^{m} \frac{|z_{11}|^2}{\sum_{i=1}^{m} |z_{i1}|^2} = a_1^2,$$

i. e. $a_1^2 \geq b_1^2 \geq b_2^2 \geq \cdots \geq a_m^2$. Therefore $a_1^2 = b_1^2$, and then $AZ = WB$ implies $z_{1r} = w_{1r}$.

Substitute elements $w_{1r}$ and $z_{1r}$ in $W$ and $Z$ by zeros, and if after that matrices $W, Z$ contain zero rows or zero columns, eject them. As a result we obtain matrices $\hat{W}, \hat{Z}$ satisfying the conditions of lemma and having either the less number of zero elements with the same dimension ($m \times n$), or matrices with less number of rows or columns (and at the same time with less number of nonzero elements). By the assumption of induction $\hat{W} = \hat{Z}$, and then, of course, $W = Z$.

Using the lemma, we will prove the following theorem:

**Theorem 1.** Let $Q$ be separated single quiver and $T$ be its indecomposable (in the category $\text{Rep}_{oa}(Q, \mathcal{H})$) orthoscalar representation. Then $T$ is indecomposable Schur representation in the category $\text{Rep} Q$. 

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Proof. Let \( C = (A, B) \) be endomorphism of representation \( T \) in the category \( \text{Rep} Q \), i. e. \( AT = TB \). We may consider, that \( A \) and \( B \) are invertible matrices, adding to \( A \) and \( B \), if it is necessary, the appropriate scalar matrices with the same scalar on the diagonal.

Let \( A = XU, B = VY \) be polar decompositions of the matrices \( A \) and \( B \), where \( U, V \) are unitary, \( X, Y \) are positive nonsingular matrices (here we may assume that \( U, V, X, Y \) has the same block-diagonal structure, as \( A \) and \( B \)). Let besides

\[
X = U_1^* \tilde{X} U_1, \quad Y = V_1 \tilde{Y} V_1^*,
\]

where \( U_1, V_1 \) are unitary matrices, \( \tilde{X}, \tilde{Y} \) are diagonal matrices with positive numbers on the diagonal. Then

\[
U_1^* \tilde{X} U_1 UT = TVV_1 \tilde{Y} V_1^*
\]
or

\[
\tilde{X}(U_1 UTV_1) = (U_1 TVV_1) \tilde{Y}.
\]

Since the lengths of corresponding rows and columns of matrices \( U_1 UTV_1 \) and \( U_1 TVV_1 \) are equal because of orthoscalarity of representation \( T \), we obtain by the lemma\(^1\)

\[
U_1 UTV_1 = U_1 TVV_1,
\]

and then, after reduction,

\[
UT = TV.
\]

Since \( U, V \) are unitary matrices, then by remark\(^2\) \((U, V)\) is an endomorphism of indecomposable in \( \text{Rep}_{os}(Q, \mathcal{H}) \) orthoscalar representation, and then by remark\(^4\) the matrices \( U \) and \( V \) are scalar with the same scalar on the diagonal.

Reduce the equality

\[
XUT = TVY
\]

by the scalar matrices \( U \) and \( V \), obtain

\[
XT = TY,
\]

where matrices \( X \) and \( Y \) are self-adjoint.

Hence by remark\(^3\) \((X, Y)\) is endomorphism of orthoscalar indecomposable representation \( T \) in the category \( \text{Rep}_{os}(Q, \mathcal{H}) \) and (by remark\(^1\)) \( X \) and \( Y \) are scalar with the same scalar on the diagonal. As a result the same will also be products of scalar matrices \( Xu = A \) and \( Vy = B \), and so the representation \( T \) is Schur representation in the category \( \text{Rep} Q \).

Since, obviously, decomposable representation has nonscalar endomorphism, the representation \( T \) is indecomposable in \( \text{Rep} Q \).

Remark 5. The condition that a quiver is single is inessential, the lemma\(^1\) is easy generalized on the case of a quiver with multiple arrows.

Remark 6. If a quiver \( Q \) is not separated, the theorem\(^1\) will not be true. Indeed, let quiver \( Q \) be the loop

\[\bullet \quad \rightarrow \]
Let \( T \) be its indecomposable orthoscalar representation with matrix

\[
T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.
\]

In this case the condition orthoscalarity is \([4]\)

\[
TT^* + T^*T = 3I.
\]

The indecomposability of \( T \) in \( \text{Rep}_{os}(Q, \mathcal{H}) \) is checked as follows: it easy to calculate that \( CT = TC \) and \( CT^* = T^*C \) imply, that \( C \) is a scalar matrix.

Let \( A = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \); it is easy to check that \( AT = TA \), so that \( A \) defines nonscalar invertible endomorphism of representation \( T \) in \( \text{Rep}_Q \) (and so representation \( T \) is decomposable in \( \text{Rep}_Q \)).

3. Set \( \mathcal{P}_n \) be the following \(*\)-algebra:

\[
\mathcal{P}_n = \mathbb{C}(p_1, p_2, \ldots, p_n \mid p_i = p_i^2 = p_i^*),
\]

and \( \mathcal{K}(\mathcal{P}_n) \) be the category, which objects are \(*\)-representations \( \pi \) of the algebra \( \mathcal{P}_n \) in the category of finite-dimensional Hilbert spaces \( \mathcal{H} \) (\( \pi(p_i) = P_i, P_i : H_0 \to H_0, P_i^2 = P_i^* = P_i \)),

and a morphism \( C_0 \) from representation \( \pi \) in space \( H_0 \) to representation \( \tilde{\pi} \) in space \( \tilde{H}_0 \) is defined as a linear map \( C_0 : H_0 \to \tilde{H}_0 \) with the property

\[
C_0 P_i = \tilde{P_i} C_0 P_i, \ i = 1, n.
\]

One can regard the category \( \mathcal{K}(\mathcal{P}_n) \) \([5]\) as the category of collections of \( n \) subspaces of finite-dimensional Hilbert spaces. If \( \pi \) is a representation of the algebra \( \mathcal{P}_n \) in \( H_0 \) (\( \pi(p_i) = P_i \)), then it associates with \( n \) subspaces \( (H_1, H_2, \ldots, H_n) \) of space \( H_0 \); here \( H_i = \text{Im} P_i, i = 1, n \).

A morphism from \( (H_1, H_2, \ldots, H_n) \) to \( (\tilde{H}_1, \tilde{H}_2, \ldots, \tilde{H}_n) \) is defined naturally as linear map \( C : H_0 \to \tilde{H}_0 \), for which \( C(H_i) \subset \tilde{H}_i \) for \( i = 1, n \). The category \( \mathcal{K}(\mathcal{P}_n) \) and the category of collections of \( n \) subspaces in linear spaces are equivalent. Papers \([6–8]\) and others are dedicated to the classification of collections of \( n \) subspaces in linear spaces.

Representation \( \pi \) algebra \( \mathcal{P}_n \) is called orthoscalar, if there exist such positive numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) that

\[
\sum_{i=1}^n \alpha_i \pi(p_i) = I_0
\]

(\( I_0 \) is identity operator in space \( H_0 \)). Denote as \( \mathcal{L}(\mathcal{P}_n) \) the full subcategory of orthoscalar representations in \( \mathcal{K}(\mathcal{P}_n) \). Denote as \( \mathcal{Q}_n \) the quiver

\[
\begin{array}{c}
1 \\
\gamma_1 \\
2 \\
\gamma_2 \\
3 \\
\gamma_3 \\
\vdots \\
\gamma_n \\
n
\end{array}
\]

Let \( \mathcal{K}(\mathcal{Q}_n) \) be the full subcategory in \( \text{Rep}_{\mathcal{Q}_n} \) of those representations \( T \) for which \( T(\gamma_i)^* T(\gamma_i) \) is a scalar operator, different from zero in space \( T(i) \), and \( \text{Rep}_{os}(\mathcal{Q}_n, \mathcal{H}) \) be the full subcategory of orthoscalar representations in the category \( \mathcal{K}(\mathcal{Q}_n) \).
Lemma 2. Categories $\mathcal{K}(Q_n)$ and $\mathcal{K}(P_n)$ are equivalent. Categories $\text{Rep}_{os}(Q_n, \mathcal{H})$ and $\mathcal{L}(P_n)$ are equivalent.

Proof. Construct a functor $F : \mathcal{K}(Q_n) \rightarrow \mathcal{K}(P_n)$. Let $T$ be a representation from $\mathcal{K}(Q_n)$ and $T(\gamma_i)^*T(\gamma_i) = \alpha_i I_i$. Put $\sqrt{\alpha_i} T(\gamma_i) = \Gamma_i$; then $\Gamma_i^*\Gamma_i = I_i$ and $\Gamma_i\Gamma_i^* = P_i$ is orthogonal projection. Let representation $\pi$ of the algebra $P_n$ be defined uniquely by equalities $\pi(p_i) = P_i$. Put $F(T) = \pi$.

Let $C = \{C_i\}_{i=0}^{n} : T \rightarrow \widetilde{T}$ be a morphism in the category $\mathcal{K}(Q_n)$:

$$C_0T(\gamma_i) = \widetilde{T}(\gamma_i)C_i, \quad i = \overline{0,n}. \tag{14}$$

Then

$$\sqrt{\alpha_i}C_0\Gamma_i = \sqrt{\alpha_i}\tilde{\Gamma}_iC_i, \tag{15}$$

and since $\tilde{\Gamma}_i\tilde{\Gamma}_i = \tilde{I}_i$ then

$$C_i = \sqrt{\frac{\alpha_i}{\tilde{\alpha}_i}} \tilde{\Gamma}_i^*C_0\Gamma_i. \tag{15}$$

After substitution of $C_i$ in (14) we obtain

$$C_0\Gamma_i = \tilde{\Gamma}_i\tilde{\Gamma}_i^*C_0\Gamma_i,$$

and, after multiplying of equality on the right by $\Gamma_i^*$,

$$C_0P_i = \tilde{P}_iC_0P_i,$$

i. e. $C_0 : \pi \rightarrow \tilde{\pi}$ is a morphism in the category $\mathcal{K}(P_n)$. Put $F(C) = C_0$, then $F$ is a functor from $\mathcal{K}(Q_n)$ to $\mathcal{K}(P_n)$.

Let $C_0 : \pi \rightarrow \tilde{\pi}$ be a morphism in $\mathcal{K}(P_n)$ where $C_0$ is a linear map from $H_0$ to $\tilde{H}_0$. Construct maps $C_i : H_i \rightarrow \tilde{H}_i$ where $H_i = \text{Im} P_i$, $\tilde{H}_i = \text{Im} \tilde{P}_i$ by the formulas (15). If $\Gamma_i, \tilde{\Gamma}_i$ are natural embeddings of $H_i$, resp. $\tilde{H}_i$ in space $H_0$, resp. $\tilde{H}_0$ ($P_i = \Gamma_i\Gamma_i^*$, $\tilde{P}_i = \tilde{\Gamma}_i\tilde{\Gamma}_i^*$), then

$$\tilde{T}(\gamma_i)C_i = \sqrt{\alpha_i}\tilde{\Gamma}_iC_i = \sqrt{\frac{\alpha_i}{\tilde{\alpha}_i}} \tilde{\Gamma}_i^*C_0\Gamma_i = \sqrt{\frac{\alpha_i}{\tilde{\alpha}_i}} \tilde{P}_iC_0P_i\Gamma_i = \sqrt{\alpha_i}C_0P_i\Gamma_i = \sqrt{\alpha_i}C_0\Gamma_i = C_0T(\gamma_i),$$

i. e. $C = \{C_i\}_{i=0}^{n}$ is such morphism in $\mathcal{K}(Q_n)$ that $F(C) = C_0$. Since, using formulas (15) $C_i$ are defined uniquely by $C_0$, $F$ is a complete and univalent functor. Each representation in $\mathcal{K}(P_n)$ is obtained from a certain representation in $\mathcal{K}(Q_n)$, therefore functor $F$ is an equivalence of categories.

Restriction of $F$ on $\text{Rep}_{os}(Q_n, \mathcal{H})$ gives equivalence of $\text{Rep}_{os}(Q_n, \mathcal{H})$ and $\mathcal{L}(P_n)$.

The following theorem is a corollary from lemma 2 and theorem 1.

Theorem 2. If $(H_1, H_2, \ldots, H_n)$ is a collection $n$ subspaces of finite-dimensional Hilbert space, and corresponding collection of $n$ orthogonal projections $(P_1, P_2, \ldots, P_n)$ define an indecomposable representation in the category of orthoscalar representations $\mathcal{L}(P_n)$ ($\sum_{i=1}^{n} \alpha_i P_i = I_0$ for a certain collection of numbers $\alpha_i > 0$, $i = \overline{1,n}$), then this collection of orthogonal projections is a Schur object in the category $\mathcal{K}(P_n)$, and a corresponding collection of subspaces is an indecomposable Schur object in the category of collections of $n$ subspaces of finite-dimensional Hilbert space.
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