Comparative Analysis of Non-linear Periodic Response of Cross-ply and Angle-ply Laminated Composite Plates

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Abstract. Non-linear periodic response periodic response analysis of laminated structures is necessary for their optimal and safe dynamic design. The present work is intended to explore the influence of lamination scheme and fibre-orientation on the nonlinear forced vibration response of laminated plates. The analysis has been carried out using finite element. In present work, a comparative analysis of forced vibration response between cross-ply and angle-ply laminated composite plates have been explored. The effects of fiber orientation on non-linear forced vibration response characteristics have been analyzed. This finite element analysis has been done using shooting method along with continuation scheme in the time domain. The method employed is capable to get the complete stable and unstable branches past bifurcations. The dynamic behavior is explored using the temporal history of the response and phase plane plots. The method presented in the paper is computationally efficient and does not require apriori assumption on the participating modes, unlike harmonic balance and incremental harmonic balance methods.

Keywords: Forced vibration, Shooting Method, Non-linearity, Laminated plates.

1. Introduction

The application of composites in the fabrication of structural components has significantly increased due to their excellent directional properties and low specific weight. Owing to these properties of composite laminates, they are used particularly in aerospace, automobile and biomedical fields. The components have thinner cross section due to reduced weight and fuel efficient requirements and are exposed to dynamic loads in service resulting in vibrations with larger amplitude and in such cases, the linear analysis overpredicts the displacement and incorporation of the geometric non-linearity in the analysis becomes important for the accurate prediction of the response.

If the time integration methods alone are employed for predicting the steady state nonlinear response it is not possible to trace the unstable portions of the nonlinear frequency response and even for stable portions large number of integration cycle is required to arrive at steady state. The other approaches based on frequency domain rely on assuming the number of participating modes, which may lead to deviation from the actual response, if the assumption on the number of modes is incorrect [1]. The linear forced vibration response is estimated by Gudla and Ganguli [2] using Galerkin method. The shooting method [3] is employed by Ribeiro [4] for the forced vibration analysis of isotropic plate. The forced vibration analysis of rectangular angle-ply composite plates was carried out based on FSDT [5] and the repercussions of the support conditions are evaluated. The Reddy’s higher order theory is employed by Amabili [6] to analyse the skewed modes in steady state response of laminated shells. The geometric nonlinearity is included in their analysis. Amabili and Farhadi [7] explored the role of various
displacement field theories on the non-linear vibration response of plates. Their analysis is carried out in frequency domain and it was observed that the nonlinear vibration response obtained using displacement fields of various theories are nearly same. The forced vibration characteristics of curved isotropic beams [8] and of bimodular composite plates/shells [9-10] employing modified shooting method has been evaluated. It has been observed from the literature that the comparative analysis of forced vibration behaviour of cross and angle-ply composite plates is not addressed competently. The method employed in this paper is based on modified shooting method which does not require the conversion of second order governing differential equation into single order differential equation. This result in preserving the banded nature of the matrix involved which results in greater computational efficiency and does not require assumptions on the modal participation. In the present work, a comparative analysis of forced vibration behaviour of plates with different fibre orientation are explored. The nonlinearity has been included in the present work through the strain-displacement relation. The significant difference in the response characteristics have been obtained between cross- and angle-ply plates. This difference in the response is due to lamination scheme induced bending stretching coupling for angle-ply plate. The comparison of the response characteristics due to changes in the lamination scheme is highlighted using temporal variation of the response and phase plane plots.

2. Formulation and Methodology
The coordinate system and the geometrical parameters of composite plate is represented in Figure 1(a) and the fibre orientation is depicted in Figure 1 (b).

![Figure 1](image)

**Figure 1.** (a) Co-ordinates and geometrical parameters of composite plate (b) Fibre Orientation of Kth Lamina

The displacement of any generic point based on FSDT is given by:

\[
u(x, y, z, t) = u_0(x, y, z, t) + z\varphi_x(x, y, t) \\
v(x, y, z, t) = v_0(x, y, z, t) + z\varphi_y(x, y, t) \\
w(x, y, z, t) = w_0(x, y, t)
\]  

(1)

where, \(u_0, v_0, w_0\) are the mid-plane displacements along \(x, y\) and \(z\) direction; and \(\varphi_x, \varphi_y\) are the rotation of the normal about the \(y\), and \(x\)-axis respectively.

The strain field incorporating nonlinearity in the in–plane membrane strain terms based on von Karman’s assumption, is given by:

\[
\{\varepsilon\} = \begin{bmatrix} \varepsilon^L_L \\ 0 \end{bmatrix} + \begin{bmatrix} Z\varepsilon^E_b \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon^{NL}_P \\ 0 \end{bmatrix}
\]

(2)
where \( \varepsilon_P^L, \varepsilon_P^{NL} \) denotes the linear and nonlinear membrane strain terms, \( \varepsilon_b \) denotes bending strain, \( \varepsilon_s \) represents the shear strain and are expressed as:

\[
\varepsilon_P^L = \begin{cases} 
    \frac{u_{0,x}}{u_{0,y}} \\
    \frac{v_{0,y}}{u_{0,y} + v_{0,x}} 
\end{cases} \\
\varepsilon_b = \begin{cases} 
    \frac{\varphi_{x,x}}{\varphi_{x,y}} \\
    \frac{\varphi_{y,y} + \varphi_{y,x}}{\varphi_{x,y}} 
\end{cases} \\
\varepsilon_s = \begin{cases} 
    \varphi_{x} + w_{0,x} \\
    \varphi_{y} + w_{0,y} 
\end{cases} \\
\varepsilon_P^{NL} = \begin{cases} 
    \frac{1}{2} \left( \frac{w_{0,x}}{w_{0,x} + w_{0,y}} \right)^2 \\
    \frac{1}{2} \left( \frac{w_{0,y}}{w_{0,x} + w_{0,y}} \right)^2 
\end{cases}
\]

(3)

If “\( \rho_k \)” is the mass density of the material of the \( k\)th layer, the total kinetic energy \( T(\delta) \) of the plate is expressed as:

\[
T(\delta) = \frac{1}{2} \int f_k \int_{b_{h_k}} \rho_k \{ \dot{u}_k \ \dot{\psi}_k \ \dot{w}_k \} \{ \ddot{u}_k \ \ddot{\psi}_k \ \ddot{w}_k \}^T \, dz \, dx \, dy
\]

(4)

\[
[U(\delta)] = \frac{1}{2} \int f_k \int_{b_{h_k}} \{ \sigma \}^T \{ \varepsilon \} \, dz \, dx \, dy - \int_A F w_0 \, dA
\]

The total potential energy \( U(\delta) \) is expressed by Equation (5). This consists of the contributions of the strain energy and the potential of the load.

The strain energy is split into linear and non-linear strain energies [11] and the total potential energy is reframed as:

\[
[U(\delta)] = \{ \delta \}^T \left[ \left( \frac{1}{2} \right) K + \left( \frac{1}{6} \right) K_1(\delta) + \left( \frac{1}{12} \right) K_2(\delta) \right] \{ \delta \} - \{ \delta \}^T \{ F \}
\]

(6)

The stiffness matrices \([K], [K_1]\) and \([K_2]\) represent, linear, nonlinear (linearly dependent) and non-linear (quadratically dependent) stiffness matrix, respectively.

The standard finite element procedure is adopted to get the element level governing equations. The element level governing equations are transformed into global equations. The kinetic energy for the element can be obtained as:

\[
[T(\delta)] = \frac{1}{2} \{ \delta \}^T [M] \{ \delta \}
\]

(7)

The generalized governing equation after including the Rayleigh proportional damping is obtained using Hamilton’s principal and is given by:

\[
[M] \{ \ddot{\delta} \} + [C] \{ \dot{\delta} \} + \left[ K + \left( \frac{1}{2} \right) K_1(\delta) + \left( \frac{1}{3} \right) K_2(\delta) \right] \{ \delta \} = \{ F \}
\]

(8)

The damping is modelled using Rayleigh damping model, and is given by:

\[
[C] = \ddot{\alpha} [M] + \beta [K]
\]

(9)

where \( \beta = \frac{\xi}{2\omega_n} \); \( \ddot{\alpha} = 2\xi \omega_n \)

\( \xi \) is damping factor.

For the solution of the governing equation of motion (Equation (8)), a two stage procedure is adopted. In the first stage starting with forcing frequency far away from fundamental frequency the governing equations are solved using modified shooting method [9-10] until the bifurcation point or up to the forcing frequency where it gives the converged steady state. In the second stage when the shooting
method does not give the converged solution, the frequency response is obtained using arc length control employing continuation schemes as can be seen from the work of the author [9-10].

3. Results and Discussion

The effect of fibre orientation on the frequency response of cross and angle-ply laminated plates undergoing nonlinear forced vibration is investigated. The plate is under the action of external harmonic force \( F = F_0 \cos \omega t \) and is assumed to be uniformly distributed over the span of the plate. The nonlinear frequency response curves for both cross and angle-ply plates have been obtained.

The material selected for the analysis has the following properties:

\[ Y_1/Y_2 = 25, \quad Y_2 = Y_3 = 1 \text{ GPa}, \]

\[ G_{12}/Y_2 = G_{13}/Y_2 = 0.5, \quad G_{23}/Y_2 = 0.2, \quad \nu_{12} = \nu_{23} = \nu_{13} = 0.25, \quad \rho = 1000 \text{ kg/m}^3. \]

The boundary condition considered in the analysis is all the edges of the laminate plates to fixed (CCCC).

The analysis has been carried out using eight-noded shear flexible serendipity element with five nodal degrees of freedom. The mesh convergence has been done and a mesh size 100 elements has been chosen.

![Figure 2](image1.png)

**Figure 2.** Periodic forced vibration response of cross-ply and angle-ply laminated composite plate

(All edges clamped, \( l/b = 2, \quad b/h = 100, \quad b = 0.5 \text{ m} \)).

![Figure 3](image2.png)

**Figure 3.** Response time history and phase plane plots at the peak response frequency in the response curves of Figure 2.
The comparison between the nonlinear frequency response of symmetrical and anti-symmetrical composite cross- and angle-ply laminates are shown in Fig.2. The solution procedure and the methodology adopted in the present work is computationally efficient and is able to predict the entire response curve of the periodic response. In addition, the bifurcation points have been identified as turning points for the cases considered from the analysis of monodromy matrix. The effect of the fiber orientation on the forced vibration characteristics has been presented in Figure 2. It can be seen that the forcing frequency corresponding to peak amplitude is greater for cross-ply plates compared to angle-ply plates, further the response amplitude is greater for cross-ply plates. This reduction in amplitude for angle-ply plates is due to increase in non-linear restoring forces. The greater hardening non-linearity for cross-ply plates compared to angle-ply plate is due to their different bending-extension coupling and because of coupling between extension and shear for angle-ply plates. The comparison of steady state frequency response of symmetric and anti-symmetric laminated plates reveal greater hardening non-linearity for symmetrical plates compared to anti-symmetric plates. The peak non-dimensional amplitude \((w/h)\) at force amplitude \(F_0 = 100 \text{ Pa}\), for symmetric cross-ply and angle-ply being 1.1879 & 1.0457 respectively and the corresponding forcing frequency ratio \((\omega_0/\omega)\) being 1.557 & 1.160. The peak non-dimensional amplitude \((w/h)\) at force amplitude \(F_0 = 100 \text{ Pa}\), for anti-symmetric cross-ply and angle-ply being 1.0783 & 1.0752 respectively and the corresponding forcing frequency ratio \((\omega_0/\omega)\) being 1.257 & 1.171. The eigen values of the monodromy matrix shows turning point bifurcation for the cases considered.

The time history of the steady-state response (Figure. 3) corresponding to the peak displacement in the forced vibration response shows nearly equal half cycle time for both cross- and angle-ply plates as expected depicting equal time in tension and compression during a cycle. The phase plane plots corresponding to forcing frequency at peak amplitude are plotted and the greater asymmetry for the angle-ply plate reveal greater higher harmonic participation for angle-ply plate compared to cross-ply plate. It is also observed in the frequency response curve (Fig. 2) that there are secondary peaks in the frequency ratio range greater than 1.6. The presence of secondary peak is due to interaction between the fundamental and higher modes.

4. Conclusions
The comparison of the nonlinear vibration response of cross and angle-ply laminated plates undergoing forced vibration is explored using modified shooting method in time domain. The frequency response curves, time history of the steady state response and the phase plane plots are obtained. Unlike methods based on frequency domain the participation of the harmonics is not assumed. Based on the detailed study it can be concluded that:
- The hardening nonlinearity and the peak amplitude is greater for cross-ply plates compared to angle-ply plate. This is due to the difference in their bending-extension coupling and presence of coupling between extension and shear for angle-ply plates.
- The hardening non-linearity is greater for symmetric plates compared to anti-symmetric plates.
- The secondary peaks in the nonlinear response curves are due to interaction between first and higher modes.
- The steady state time history shows nearly equal half cycle time for both cross- and angle-ply plates.

5. References

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