Quantum construction of a unitary SU(2/1) model of the electro-weak interactions with 2 Higgs doublets.

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Abstract: The interactions and even the number of the Higgs scalar fields are not fixed in the SU(2)U(1) standard model of the electro-weak interactions and the intrinsically chiral nature of the weak interactions is not explained. Embedding SU(2)U(1) into the Lie superalgebra SU(2/1) fills these gaps. The 2 smallest representations of SU(2/1) adequately describe the electron, neutrino, up and down quarks and correlate their chiralities with their U(1) charges, and the Higgs fields have the quantum numbers of the odd generators. But so far, there was an apparent conflict with unitarity, because the quark representation is not Hermitian and the super-Killing metric is not positive definite. We solve this paradox by assuming the existence of 2 complex Higgs doublets minimally coupled to the Fermions via the chiral projections of the odd generators of SU(2/1). We find that Lagrangian induced by the Fermion loops is unitary, thanks to the balance between the leptons and the quarks needed to cancel the triangle anomaly and that the super-Jacobi identity guarantees that the photon remains massless after symmetry breaking. In addition, the Lagrangian has a classical geometric interpretation in terms of the curvature of the corresponding Hermitian algebra. Assuming that the relative strength of the scalar and vector couplings does not depend on the number of families constrains the mass of the Higgs to \( M_{H_1}^2 + M_{H_2}^2 = \frac{32}{9} M_W^2 = 2 \left(107.2 GeV\right)^2 \). Contrary to grand-unified or Wess-Zumino supersymmetric models, the SU(2/1) internal superunification does not predict any unobserved particle besides the 2 Higgs doublets.

Keywords: SU(2/1), superalgebra, chiral Yang Mills, renormalization, Hermitian algebra, super symmetry, Higgs mass, standard model, chirality, neutrino, quark.
1. Background

The Weinberg-Salam $SU(2).U(1)$ model of the electro-magnetic and weak interactions [Wei67, Sa68] is extremely accurate. At high energy, the Fermions are described as massless $SU(2)$ doublets of left spinors: the electron neutrino left doublet $(e_L, \nu_L)$ with $U(1)$ charge (-1), and the up and down left quark doublet $(u_L, d_L)$ with charge $(2/3)$, accompanied by right singlets: the right electron $(e_R)$ with charge (-2) and the right quarks $(u_R, d_R)$ with respective charges $(4/3)$ and (-1/3). The right neutrino is $SU(2)$ and $U(1)$ neutral and drops out from the model.

The corresponding $SU(2)$ and $U(1)$ Yang-Mills gauge fields [YM54] are the $(W^+, W^-)$, the $Z^0$ and the photon. The existence and interactions of the $W^+$, the $W^-$ and the $Z^0$ were beautifully confirmed over the last 30 years. The only component of the model that has not yet been observed is the scalar Higgs field. In the minimal approach, it is predicted to be a scalar $SU(2)$ doublet $\phi$ with $U(1)$ charge (-1). The Higgs potential is assumed to be of the form $V(\phi) = (\phi^2 - v^2)^2$ So, in the vacuum the Higgs field does not vanish. The gauge group breaks down to the electro-magnetic $U(1)$, the maximal subgroup leaving $v$
invariant. The photon and the left neutrino remain massless. All other particles acquire a mass proportional to $v$ (see for example [AL73]).

This construction is superb, but the number of arbitrary parameters is very large. Since the left and right chiral states are independent, we need to choose 10 charges to describe the leptons and the quarks. For example, a massless electrically charged particle, which experimentally does not exist, would be acceptable at the classical level and is only ruled out by the study of the quantum anomalies. Also, since the Higgs field plays such a central role, one would wish to derive it from first principle. The Yang-Mills gauge fields are well understood, but the Higgs potential and even the number of Higgs fields needed in the standard model remain arbitrary.

In 1979, Ne'eman [N79] and Fairlie [F79] have independently proposed to consider the $SU(2/1)$ Lie superalgebra as a basis for describing the weak interactions. Indeed, the fundamental representation, recalled in section 3, exactly corresponds to the lepton triplet if we grade it by chirality. The representation is Hermitian (3.2) and scaling the trace of all the even matrices to a common value seemed to predict a electro-weak angle $\sin^2 \theta_W = 1/4$, a good value at the time, although higher than today’s experimental value. In addition, the odd generators have the quantum numbers of the Higgs fields. The quarks were either left out [F79] or mentioned as a counterargument [N79]. But remarkably, it was found a few weeks later [DJ79, NT80] that $SU(2/1)$ also admits a 4 dimensional representation, recalled in section 4, exactly fitting the quarks, with 2 right singlets and a left doublet with electric charges $2/3$ and $-1/3$.

These results, summarized in a recent Physics Report [NSF05], raised some interest but also some doubts [S92]. A superalgebraic structure seems in direct conflict with unitarity. The supertrace, which is the natural invariant of a superalgebra, yields a negative sign for the propagator of the $U(1)$ vector, the quark representation (4.3) is neither Hermitian nor anti-Hermitian, and even the Hermitian nature of the lepton representation is artefactual, since the odd generators of antilepton representation are anti-Hermitian (3.8). It was therefore often argued that the $SU(2/1)$ superalgebra could not play a role in the quantum field theory. The present paper reverses the situation.

2. Results

By following literally the paradigm of the minimal couplings and turning immediately to the renormalization theory, we predict 2 Higgs doublets $H$ and $K$. This solves the problem of the metric and allows to adjust freely the electro-weak angle. We prove that in the symmetry-breaking vacuum the photon remains massless because of the super-Jacobi identity. For a particular choice of the scalar coupling constants, we find a scalar Ward identity insuring that the 1-loop counterterms do not depend on the number of families and derive the mass relation $M_{H_1}^2 + M_{H_2}^2 = 32/9$ $M_W^2 = 2 (107.2 \text{ GeV})^2$. The whole construction plays on balance between the leptons and the quarks which ensures the cancellation of the triangle anomaly [BIM72]. The theory is however incomplete, because the scalar loops and the gluon loops disturb the superalgebraic structure.
We start in section 5 from the assumption that the adjoint representation of the $SU(2/1)$ superalgebra directly describes the Bosons of the theory, and that the quark and lepton representations, graded by chirality, describe their interactions to the Fermions. The even generators are gauged as usual as Yang-Mills vector fields. But since the odd generators are not Hermitian, we propose (5.3) to separate the left and right interactions and construct 2 sets of scalar fields, $\Phi_L$ and $\Phi_R$ by projecting out the relevant chiral part of the odd matrices, and to define the interactions $\bar{\Psi}_L \Phi_L \Psi_R + \bar{\Psi}_R \Phi_R \Psi_L$. We then construct the rest of the Lagrangian by studying the counterterms induced by the Fermion loops.

As shown in (5.4), the left-trace of the odd matrices induces the propagator of the scalar fields. It is antisymmetric and proportional to the super-Killing metric, as befits a superalgebra. But since the $\Phi_L$ and $\Phi_R$ are distinct, the propagator does not vanish by symmetrization and can be rediagonalized (5.9) in the form of 2 Higgs scalar complex doublets, $H$ and $K$. $H$ is coupled to the negatively charged right particles (electron and down quark) and $K$ to the positively charged right up quark. This re-shuffling depends on the same balance between the leptons and the quarks which cancels the triangle anomaly and the equations are similar. In the later case, one computes the triple vector Fermion loop, which is possibly spoiled by the impossibility to regularize the chiral Fermions in a gauge invariant way, and depends on a supertrace condition (4.5). In the present paper we compute the 2 scalars and 4 scalars Fermion loops, which are a priori spoiled by the non Hermitian nature of the $SU(2/1)$ scalar-Fermion couplings. In all cases, we find that the respective contributions of the leptons and the quarks cancel each other and induce a canonical counterterm.

In section 6, we construct the Higgs potential. A priori, since we have 2 Higgs doublets, there are 3 possible $SU(2)U(1)$ invariant quartic terms $V_1 = (H^2)^2 + (K^2)^2$ which controls the mass of the neutral Higgs fields, a kind of vector product squared $V_2(H,K)$ which controls the mass of the charged Higgs fields, and a scalar product squared $V_3 = (H.K)^2$. We show that $V_2$ vanishes if $H$ and $K$ are parallel as a consequence of the super-Jacobi identity, and that $V_3$ is absent, because of the lepton quark compensation. The potential can then be rewritten in terms of a simple quadratic form $\tilde{G}$ (6.2) which plays the role of the Killing metric and contains a free parameter $\alpha$ later associated (10.3) to the electro-weak angle. This construction gives the structure of the scalar potential, but not the strength of the couplings.

In section 7, we show that for a particular value of the scalar Fermion coupling, we have a Ward identity insuring that the Fermion loops do not modify the relative strength of the scalar and vector couplings and that the scalar potential renormalizes like $g^2$. This indicates that the mass of the neutral Higgs fields should satisfy the relation $M_{H_1}^2 + M_{H_2}^2 = 32/9 M_W^2 = (151.6 \ GeV)^2$. In the symmetric situation, we would have $M_{H_1}^0 = M_{H_2}^0 = 107.2 \ GeV$. However, we also observe that other quantum corrections disrupt the universality of the scalar-Fermion coupling, indicating that the theory is still incomplete.

In section 9, we show that the chiral decomposition of the Higgs scalars, in conjunction with the Dirac 1-forms $\Gamma$ of [TM06], naturally constructs a covariant differential corre-
sponding to the SU(2/1) Hermitian algebra first defined by Sternberg and Wolf [SW78]. The chirality operator and the complex Higgs structure conspire so that the corresponding curvature 2-forms is well defined and indeed valued in the adjoint representation of the SU(2)U(1) even Lie algebra.

In section 10, we use the quadratic form (6.2) and the 2 CP conjugated curvatures (9.12-13) to construct the vector Lagrangian and show that the square of the H-algebra curvature tensor reproduces the Higgs quartic potential induced by renormalization. Within this SU(2/1) framework, the electro-weak angle is free.

The theory is not complete and several important problems are presented in the discussion. But even considering these difficulties, the results so far are new and unexpected. Close to 30 years after the initial proposal [N79, F79], this is the first time that SU(2/1) is shown to play a role in the quantized theory, and the first time that the super-Killing metric is used, both in the construction of the propagators (5.9 and 10.1) and of the potential (6.2,6.4), while explicitly respecting unitarity.

3. The SU(2/1) lepton representation

The smallest nontrivial simple Lie superalgebra is SU(2/1), also called A(1/0) in the Cartan-Kac classification. The Lie sub-algebra $A_0$ is SU(2)U(1), and the odd $A_0$ module $A_1$ is a complex doublet with U(1) charge $-1$. The fundamental representation [SNR77] is of superdimension (2/1) and fits the leptons [N79, F79]. In the $(\nu_L, e_L; e_R)$ basis, the 4 even generators read:

$$
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
$$

(3.1)

the 4 odd generators read:

$$
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},
$$

(3.2)

and the chirality operator is

$$
\chi = diag(1, 1, -1).
$$

(3.3)

By inspection, we find that the anticommutator of the odd matrices close on the even matrices, defining the symmetric structure constants $d^{a}_{ij}$ and that the commutators of the odd matrices close on $\chi$ times the even matrices, defining the skew-symmetric structure constants $f^{a}_{ij}$:

$$
\lambda_i \lambda_j + \lambda_j \lambda_i = d^{a}_{ij} \lambda_a,
$$

$$
\lambda_i \lambda_j - \lambda_j \lambda_i = i \chi f^{a}_{ij} \lambda_a,
$$

(3.4)

$$
a = 1, 2, 3, 8; \quad i, j = 4, 5, 6, 7
$$
in these conventions, the $f_{ij}^a$ and $d_{ij}^a$ constants are real. Finally, the vacuum $v$ is chosen along the $\lambda_6$ direction, whose centralizer is the photon:

$$\text{photon} = -(\lambda_6)^2 = \frac{1}{2} (\lambda_3 - \lambda_8) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ \hspace{1cm} (3.5)

There is no loss of generality in this choice of $v$, it just corresponds to the way we named the particles. If we rotate $v$ we just have to rotate the name of the electron and the neutrino and keep the photon in the direction of $v^2$. Notice that, with respect to the $\text{SU}(2/1)$ supermetric:

$$\hat{g}_{MN} = 1/2 \text{Str}(\lambda_M \lambda_N), \quad M, N = 1, 2...8$$

the photon is on the light-cone of the superalgebra.

The even generator of antilepton representation, in the $(e_R; (e_L)_R, (\nu_L)_R)$ basis, are:

$$\lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ \hspace{1cm} (3.7)

the 4 odd generators:

$$\lambda_4 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the chirality operator is

$$\chi = \text{diag}(1, -1, -1).$$ \hspace{1cm} (3.9)

Equations (3.4-5) are again true in this representation. Notice however that the odd matrices (3.8) are anti-Hermitian, this is necessary to maintain (3.5) since the electric charge of the antielectron is positive. Notice also that the charge and parity (CP) are linked. The doublet and antidoublet must be of opposite parity with respect to $\chi$ (3.3) and (3.9) in order to maintain the commutator (3.4b). This switches the sign of the supermetric (3.6), as befits a CP invariant theory.

4. The $\text{SU}(2/1)$ quark representation

In the standard model [Wei67, Sa68, GIM70, BIM72], each family of quarks consists of 4 states, a left doublet and two right singlets, i.e. $(u_R/(u_L, d_L)/d_R)$ for the 'electron family'. Amazingly, the second smallest irreducible representation of the $\text{SU}(2/1)$ superalgebra corresponds to this chiral decomposition, it contains a free parameter $n$ which allows us to fix the charge of the up quark. From the mathematical point of view [SNR77], the existence of this 4 dimensional representation is a simple consequence of the isomorphism
between $SU(2/1)$ and $OSp(2/2)$, which is a generalization of the well known isomorphisms between the first members of the infinite families of simple Lie algebras. But from the physical point of view, it came as a great surprise. In the original papers [F79, N79], the quarks were respectively left out and listed as a counterargument. The incorporation a few weeks later of the quarks [DJ79, NT80] really appeared as a significant confirmation of the model. The 4 even generators read:

$$\lambda_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(4.1)

$$\lambda_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{n} \begin{pmatrix} -n & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & n-1 \end{pmatrix}.$$  

(4.2)

The 4 odd generators are not Hermitian, for any value of $n$, they read:

$$\lambda_4 = \frac{1}{\sqrt{2n}} \begin{pmatrix} 0 & 0 & \sqrt{n+1} & 0 \\ 0 & 0 & 0 & \sqrt{n-1} \\ \sqrt{n+1} & 0 & 0 & 0 \\ 0 & \sqrt{n-1} & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \frac{1}{\sqrt{2n}} \begin{pmatrix} 0 & 0 & i\sqrt{n+1} & 0 \\ 0 & 0 & 0 & -i\sqrt{n-1} \\ i\sqrt{n+1} & 0 & 0 & 0 \\ 0 & i\sqrt{n-1} & 0 & 0 \end{pmatrix},$$

(4.3)

$$\lambda_6 = \frac{1}{\sqrt{2n}} \begin{pmatrix} 0 & \sqrt{n+1} & 0 & 0 \\ -\sqrt{n+1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{n-1} & 0 \\ 0 & 0 & 0 & \sqrt{n-1} \end{pmatrix},$$

$$\lambda_7 = \frac{1}{\sqrt{2n}} \begin{pmatrix} 0 & -i\sqrt{n+1} & 0 & 0 \\ -i\sqrt{n+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sqrt{n-1} \\ 0 & 0 & i\sqrt{n-1} & 0 \end{pmatrix}.$$  

Finally the chirality operator is

$$\chi = \text{diag}(-1, 1, 1, -1).$$

(4.4)

We have normalized the quark matrices (4.1-4.4) so that they have exactly the same supercommutators as the lepton matrices (3.1,3.2). As desired, the super-Killing metric (3.6) is also identical in the lepton and the quark representation. Notice however that the commutators of the odd quark matrices do not close on $\chi$ times the even matrices.
Since $n$ is a free parameter, $SU(2/1)$ does not predict the charge of the quarks, or the number of colors. However, the superalgebra does relate the $U(1)$ charges of the singlets and a number of desired relations (postdictions) are automatically satisfied.

First, for $n = -1$ (or $n = 1$), we recover the lepton (or antilepton) representation with only 3 states. In other words, $SU(2/1)$ implies that if the charge of the electron is equal to the charge of the $W$ vector, the $U(1)$ neutral right-neutrino is also neutral with respect to the odd generators.

Then, we should impose the cancellation of the triangle anomalies.

$$d_{abc} = Str_f ( \lambda_a ( \lambda_b \lambda_c + \lambda_c \lambda_b) ) = 0 , \quad a, b, c = 1, 2, 3, 8 , \quad (4.5)$$

If we consider a lepton family, with one lepton of $U(1)$ charge 1, and $n'$ quarks of $U(1)$ charge $-1/n$, the $SU(2)^2U(1)$ anomaly vanishes if and only if we choose $n = n'$. But although there are no other adjustable parameters, $SU(2/1)$ then implies that the $U(1)^3$ anomaly also vanishes

$$Str_f ( \lambda_3^a ) = 2 - 8 + n ((n + 1)^3 - 2 - (n - 1)^3)/n^3 = 0 \quad (4.6)$$

for any value of $n$. The vacuum and the photon direction were already chosen in the lepton representation. In the same conventions (3.5) where the electric charge of the electron is -1, we find:

$$\text{photon} = - (\lambda_6)^2 = \frac{1}{2} (\lambda_3 - \lambda_8) = \frac{1}{2n} \begin{pmatrix} n + 1 & 0 & 0 & 0 \\ 0 & n + 1 & 0 & 0 \\ 0 & 0 & 1 - n & 0 \\ 0 & 0 & 0 & 1 - n \end{pmatrix} . \quad (4.7)$$

Choosing the correct number of colors $n = 3$ implies the correct electric charges for the up quark $2/3$ and the down quark $-1/3$. Finally, the odd sector antisymmetric structure constants and trace metric also vanish:

$$f_{aij} = Tr_f ( \lambda_a ( \lambda_i \lambda_j - \lambda_j \lambda_i) ) = 0 , \quad a = 1, 2, 3, 8 ,$$
$$g_{ij} = \frac{1}{2} Tr_f ( \lambda_i \lambda_j ) = 0 , \quad i, j = 4, 5, 6, 7 . \quad (4.8)$$

We will now study in the next sections some consequences of these identities.

### 5. The scalar propagator

We would like to construct a quantum field theory, based on the $SU(2/1)$ superalgebra, which would extend the Yang-Mills theory [YM54] associated to its maximal $SU(2)U(1)$ Lie sub-algebra and incorporate in some way the odd matrices. In [TM06], we have shown that we can construct an associative covariant exterior differential, mixing left and right chiral Fermions, if and only if the Fermion form a representation of a Lie superalgebra, graded by the chirality. As we have just seen, these conditions are met exactly by the existing fundamental Fermions: the leptons and the quarks. The connection is then defined as the sum of the usual Yang-Mills 1-form $A_\mu \, dx^\mu$ plus the constant Dirac 1-form $\gamma_\mu \, dx^\mu$ multiplied by scalar field valued in the odd sector of the superalgebra.
The natural idea would therefore be to introduce a scalar-field $\Phi$ with a minimal coupling to the Fermions of the form $\Psi^\dagger \Phi^i \lambda^i \Psi$. However, this is not directly possible since the $\lambda_i$ are not Hermitian. To fix this problem, we postulate that the field $\Phi^i$ is composed of 2 parts, each with intrinsically chiral couplings. Considering the chirality operator $\chi$ (3.3,4.4) with eigenvalue $1$ ($-1$) on the left (right) Fermions, we define the left and right projectors $p_L = (1 + \chi)/2$ and $p_R = (1 - \chi)/2$, and define, for any matrix $M$, the chiral traces.

\[ T_{\text{r}}^L(M) = Tr(p_L M) \text{, } T_{\text{r}}^R(M) = Tr(p_R M) \text{, } Str(M) = Tr(\chi M) \text{.} \tag{5.1} \]

We then decompose the odd $\lambda$ matrices as

\[ \lambda_i = \lambda^i_L + \lambda^i_R \]
\[ \lambda^i_L = p_L \lambda^i = \lambda^i p_L \text{, } \lambda^i_R = p_R \lambda^i = \lambda^i p_R \text{,} \tag{5.2} \]

and assume the existence of 2 scalar fields $\Phi^i_L$ which absorbs right Fermions and emits left ones and $\Phi^i_R$ which absorbs left Fermions and emits right ones, according to the coupling

\[ \Psi^\dagger_L \Phi^i_L \lambda^i_L \Psi + \Psi^\dagger_R \Phi^i_R \lambda^i_R \Psi \text{.} \tag{5.3} \]

To find the correct propagator of the $\Phi$ field, we compute the quantum corrections due to the insertion of a Fermion loop between 2 $\Phi$ states.

This induces a scalar counterterm of the form

\[ \mathcal{L}_s = \eta_{ij} \partial_\mu \Phi^i_L \partial^\mu \Phi^j_R \tag{5.4} \]

where

\[ \eta_{ij} = \frac{1}{n+1} T_{\text{r}}^L (\lambda^i \lambda^j) = \frac{1}{n+1} T_{\text{r}}^R (\lambda^j \lambda^i) \text{.} \tag{5.5} \]

There are two contributions, one coming from the lepton loop and one from the quark loop

\[ (\eta_{ij})_{\text{lepton}} = \frac{1}{n+1} \delta_{ij} + \frac{i}{n+1} \epsilon_{ij} \text{,} \]
\[ (\eta_{ij})_{\text{quark}} = -\frac{1}{n(n+1)} \delta_{ij} + \frac{i}{n+1} \epsilon_{ij} \text{,} \tag{5.6} \]

where $\epsilon_{45} = -\epsilon_{54} = \epsilon_{67} = -\epsilon_{76} = 1$, all other components being zero, which coincides with the odd part of (3.6). Summing over a whole family $f$ with $n$ colored quarks, we find that the Fermion loop is proportional to the canonical supermetric of $SU(2/1)$:

\[ (\eta_{ij})_f = i \epsilon_{ij} = \tilde{g}_{ij} \tag{5.7} \]

For the moment, we have a weird scalar model, with an antisymmetric propagator and non Hermitian couplings. But now a miracle occurs. If we change variables as follows, the Lagrangian becomes canonical. Consider the two $SU(2/1)$ odd valued scalar fields $H$ and $K$:

\[ \Phi^i_L = a_i + i\epsilon_{ij} b_j \text{, } a_i = \frac{1}{2} \left( H_i + i K_i \right) \text{,} \]
\[ \Phi^i_R = a_i - i\epsilon_{ij} b_j \text{, } b_i = \frac{1}{2} \left( H_i - i K_i \right) \text{,} \tag{5.8} \]
where \( a_i \) and \( b_i \) are convenient intermediate variables which are not reused below. Substituting (5.8) in (5.4) we obtain the canonical propagator

\[
\mathcal{L}_s = i \epsilon_{ij} \partial_\mu \Phi_i^L \partial_\mu \Phi_j^R = \frac{1}{2} \delta_{ij} (\partial_\mu H^i \partial_\mu H^j + \partial_\mu K^i \partial_\mu K^j ).
\]

We then define the \( \mu^\pm \) Hermitian matrices:

\[
\lambda_i = (\mu^-_i - i \mu^+_i), \quad \mu^-_i = \frac{i}{2} (\lambda_i + \lambda_i^\dagger),
\]

\[
\lambda^\dagger_i = (\mu^-_i + i \mu^+_i), \quad \mu^+_i = \frac{i}{2} (\lambda_i - \lambda_i^\dagger),
\]

and rewrite the scalar-Fermion interaction in terms of these new variables:

\[
\Psi^\dagger (\Phi_i^L \lambda_i^L + \Phi_i^R \lambda_i^R) \Psi = \Psi^\dagger (H_i^\dagger \mu^-_i + K_i^\dagger \mu^+_i) \Psi .
\]

The matrices \( \mu_i^\pm \) split according to the electric charge and are Hermitian. The field \( H^i \) interacts only with the negatively charged right singlets (electron, down quark) and the field \( K^i \) just with the positively charged right singlets (up quark). In the lepton representation, the \( \mu^-_i \) matrices are equal to the Hermitian matrices given in (3.2)

\[
\mu^-_4 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad \mu^-_5 = \begin{pmatrix}
0 & 0 & -i \\
0 & 0 & 0 \\
i & 0 & 0
\end{pmatrix},
\]

\[
\mu^-_6 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \mu^-_7 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -i \\
i & 0 & 0
\end{pmatrix}.
\]

In the quark representation, the \( \mu^-_i \) are equal to the right and low Hermitian corner of the matrices (4.3)

\[
\mu^-_4 = \sqrt{\frac{n-1}{2n}} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad \mu^-_5 = \sqrt{\frac{n-1}{2n}} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0
\end{pmatrix},
\]

\[
\mu^-_6 = \sqrt{\frac{n-1}{2n}} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \mu^-_7 = \sqrt{\frac{n-1}{2n}} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & i & 0
\end{pmatrix}.
\]

The top left corner of the \( \mu^+_i \) matrices are proportional to the antielectron matrices (3.8)

\[
\mu^+_4 = \sqrt{\frac{n+1}{2n}} \begin{pmatrix}
0 & 0 & i & 0 \\
0 & 0 & 0 & 0 \\
-i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \mu^+_5 = \sqrt{\frac{n+1}{2n}} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\mu^+_6 = \sqrt{\frac{n+1}{2n}} \begin{pmatrix}
0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \mu^+_7 = \sqrt{\frac{n+1}{2n}} \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
If we sum over a complete family, we find that the matrices associated to $H$ and $K$ have the same trace-metric:

$$Tr(\mu^+_i \mu^-_j) = 0, \quad Tr(\mu^+_i \mu^+_j) = Tr(\mu^-_i \mu^-_j) = (n+1) \delta_{ij}, \quad i, j = 4, 5, 6, 7.$$  \hfill (5.15)

This identity, like (4.8), is a consequence of the vanishing of the sum of the charges of all the right singlets (electron + $n$ up quarks + $n$ down quarks) of a family. It shows that one can compute the renormalization of the scalar propagator either using the $\Phi_L$, $\Phi_R$ fields (5.4) or using the $H$, $K$ fields (5.8), and obtain the same result (5.9).

6. The scalar potential

From now on, we restrict our attention to the case $n = 3$ colors. Consider the structure of the scalar potential induced by a Fermion loop. The cubic terms vanish, since each interaction with a scalar changes the chirality of the Fermions. The term of degree 6 or higher converge. So we only have to compute the quartic potential. If we work in terms of the $\Phi$ fields, the counterterm is proportional to

$$\Phi^i_L \Phi^j_R \Phi^k_L \Phi^l_R \quad Tr(L \lambda_i \lambda_j \lambda_k \lambda_l).$$  \hfill (6.1)

Generalizing (5.6), we find that many unfriendly terms, for example $\Phi^4_L \Phi^4_R \Phi^4_L \Phi^5_R$, are eliminated by the balance between the leptons and the quarks. If we introduce the metric:

$$\tilde{G}_{ab} = \begin{pmatrix} 2 \delta_{ab} - \alpha \tilde{g}_{ab} & -\tilde{g}_{ab} \\ -\tilde{g}_{ab} & 2 \delta_{ab} - \alpha \tilde{g}_{ab} \end{pmatrix},$$  \hfill (6.2)

and the products

$$Z^a_L = (f^a_{ij} - i d^a_{ij}) \Phi^i_L \Phi^j_R,$$
$$Z^a_R = (f^a_{ij} + i d^a_{ij}) \Phi^i_L \Phi^j_R,$$  \hfill (6.3)

where the $d^a_{ij}$ and the $f^a_{ij}$ are defined in (3.4), the direct calculation of the Fermion loop induces, up to an infinite multiplicative constant, a scalar potential of the form

$$V = \frac{1}{8} (Z^a_L Z^a_R \tilde{G}_{ab} \left( Z^a_L \right)).$$  \hfill (6.4)

The $\alpha$ parameter is arbitrary in (6.4), because we have the identity

$$Z^a_L \tilde{g}_{ab} Z^b_L = 0.$$  \hfill (6.5)

If we work with the $\mu^\pm$ Hermitian matrices (5.12-14), we find, generalizing (5.15), that the symmetrized quartic traces are identical

$$Tr((\mu^+_i \mu^+_j + \mu^+_j \mu^+_i)(\mu^-_i \mu^-_j + \mu^-_j \mu^-_i))$$
$$= Tr((\mu^-_i \mu^-_j + \mu^-_j \mu^-_i)(\mu^+_i \mu^+_j + \mu^+_j \mu^+_i))$$
$$= 32/9 (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$  \hfill (6.6)
Finally the mixed traces satisfy, for all $i, j, k, l$:
\[
\text{Tr}(\mu^+_i \mu^-_j) = \text{Tr}(\mu^+_i \mu^-_j \mu^+_k \mu^-_l) = \text{Tr}(\mu^+_i \mu^-_j \mu^+_k \mu^-_l) = 0, \tag{6.7}
\]
\[
\text{Tr}(\mu^+_i \mu^+_k \mu^-_l) = 0, \quad \text{for any } i, \text{ no sum implied.}
\]

If we now introduce the symmetric product
\[
(X \ast X)^a = d^a_{ij} X^i X^j; \quad X = H, K, \tag{6.8}
\]
the direct calculation of the Fermion loop induces, up to the same infinite multiplicative constant, a scalar potential of the form
\[
V = \frac{1}{16} \left( (H \ast H)^a (K \ast K)^a \right) \left( \tilde{g}_{ab} \right) \left( (H \ast H)^b (K \ast K)^b \right). \tag{6.9}
\]
The $\alpha$ parameter (6.2) is arbitrary in (6.9), because in any Lie superalgebra, we have the identity
\[
\tilde{g}_{ab}(X \ast X)^a (X \ast X)^{b} = 0 \tag{6.10}
\]
Notice that for the same reason, the mixed $H K$ term vanishes if and only if $H$ and $K$ are parallel. This identity insures that the centralizer of the true vacuum is nontrivial, or in other words, that the photon remains massless.

If we change variable again and define
\[
u_1 = (H_4 + iH_5)/\sqrt{2}, \quad \nu_2 = (H_6 + iH_7)/\sqrt{2}, \quad \nu_1 = (K_4 + iK_5)/\sqrt{2}, \quad \nu_2 = (K_6 + iK_7)/\sqrt{2}, \tag{6.11}
\]
and substitute in (5.9) and (6.9), we recover up to a scale factor $\sigma$ the propagator and potential of Fayet [F74, F75]
\[
\mathcal{L}_s + V = \partial_i \varpi \partial_i u + \partial_i \varpi \partial_i v +
+ \sigma((\varpi u)^2 + (\varpi v)^2) + (\varpi \varpi_2 - \varpi \varpi_1)(u_1 v_2 - u_2 v_1). \tag{6.12}
\]
In these variable, it is immediately clear that, assuming a symmetry breaking negative mass term for the scalars, the potential is minimal when $u$ and $v$ are parallel, and therefore the photon remains massless. The scalar potential (6.9) is invariant under the infinitesimal transformation
\[
\Delta H_i = (H^k \delta_{kl} K^l) \tilde{g}_{ij} K^j - (H^k \tilde{g}_{kl} K^l) \delta_{ij} K^j, \quad \Delta K_i = (H^k \delta_{kl} K^l) \tilde{g}_{ij} H^j - (H^k \tilde{g}_{kl} K^l) \delta_{ij} H^j, \tag{6.13}
\]
or, if we use the variables $u$ and $v$:
\[
\Delta u = (u \varpi) v, \quad \Delta v = -(v \varpi) u, \quad \Delta \varpi = (u \varpi) \varpi, \quad \Delta \varpi = -(v \varpi) \varpi, \tag{6.14}
\]
whereas the term $(u v)(\varpi \varpi)$ which could give a mass to the photon and is absent from our Lagrangian, is not invariant under $\Delta$. 


7. Scalar Ward identities

The study of the counterterms induced by the Fermion loop has given in the previous section the structure of the propagator and of the potential, but not their scale. If we write our Lagrangian as

\[ \mathcal{L} = -\frac{1}{4} (F_{\mu\nu}^a)^2 - \frac{1}{2} ((D_\mu H)^2 + (D_\mu K)^2) + \lambda g^2 V + \bar{\psi} \frac{g}{2} (\gamma^\mu A_\mu + \xi \Phi) \psi , \]  

(7.1)

The factor 2 in \( g/2 \) is needed because our \( SU(2) \) matrices (3.1) are normalized as usual in mathematics, not as the standard spin 1/2 isospin matrices of physics. The weak angle and the coefficients \( \lambda \) and \( \xi \) are not yet known. We will now fix them by requiring that they do not depend on the number of lepton families.

The only contribution of the other families to the coupling of the vectors and scalars to the Fermion of the first family is through the renormalization of the wave function of the vector and the scalar. The counterterms are proportional to

\[ Z(A_8^a) = \frac{2}{3} g^2 a^2 \, Tr_f \, \lambda_8^2 , \quad Z(A_8^b) = t g^2 \theta \, \frac{2}{3} a^2 \, Tr_f \, \lambda_8^2 , \quad Z(H^2) = \xi^2 \, \frac{2}{3} a^2 \, Tr_f \, \mu_8^2 . \]  

(7.2)

The factor 2/3 in the vectors comes from the kinematic contribution of the Dirac matrices. We count 2/3, not the usual 4/3 of a massive Fermion, since here the left and right spinors contribute separately to the trace term. Over a full family, the \( SU(2) \) even matrices (3.1,4.1) are normalized to \( Tr_f(\lambda_a \lambda_b) = 8 \delta_{ab} \). The \( U(1) \) matrix is normalized to \( Tr_f(\lambda_8 \lambda_8) = 40/3 \), and the odd matrices to \( Tr_f(\mu_8^+ \mu_8^-) = 4 \, \delta_{ij} \). To integrate these 3 counterterms in the renormalization of the coupling constant \( g \), and insure that \( t g \theta \) and \( \xi \) are not renormalized, we fix

\[ t g^2 \theta = \frac{3}{5} , \quad sin^2 \theta = \frac{3}{8} , \quad \xi = \frac{4}{\sqrt{3}} . \]  

(7.3)

The weak angle coincides with the value derived from the \( SU(5) \) grand unified theory [GG74]. We then look at the renormalization of the scalar potential. The standard Lie algebra Ward identities ensure that the contribution of the Fermions to the self coupling of the vectors, the \( g^2 A^4 \) term, can be reabsorbed in the renormalization of \( g^2 \). A similar condition holds for the scalar potential \( \lambda g^2 H^4 \).

\[ Z(V) = \frac{1}{3} \, \frac{(g \xi)^4}{\lambda g^2} \]  

(7.4)

The contribution of the Fermions is reabsorbed in the renormalization of \( g^2 \) and therefore \( \lambda \) is not renormalized if we choose

\[ Z(V) = Z(H^2) \implies \lambda = \frac{\xi^2}{3} = \frac{4}{9} \]  

(7.5)

However, we note that the Bosonic counterterms, induced by the vector and scalar 1-loops perturb all the scalar couplings, except the very structure of the scalar potential which guarantees the survival of an unbroken \( U(1) \) and the existence of the massless photon. Furthermore the strong interactions gluon vector fields interact with the quarks, but not with the leptons. The photon-quark coupling is protected by a Ward identity and is not
renormalized by the gluon-quark interaction, but the scalar-quark vertex is not protected. Hence, the balance between quarks and leptons, which is central to our analysis is broken by these counterterms. This problem can either be interpreted as a fatal flaw of the $SU(2/1)$ model or as sign that the quantum theory is still incomplete.

8. The mass of the Higgs

Since we have now fixed the scale of the potential we can derive the mass of the physical scalars, the two neutral fields $H^0_{1/2}$ and the charged field $H^\pm$. Suppressing the Fermions, the Lagrangian reads:

$$\mathcal{L} = -\frac{1}{4}(F^\mu_{\nu})^2 - \frac{1}{2}((D_\mu H)^2 + (D_\mu K)^2) + \lambda V$$

(8.1)

where $V$ is given in (6.9). If we develop the potential and show explicitly the occurrences of the component fields $H^4$. We find:

$$\lambda V = \frac{\lambda}{4} \left( H^4_1 + H^4_1 (2 (H^2_5 + H^2_6 + H^2_7) + K^2_6 + K^2_7) + 2 H_4 H_7 (K_4 K_7 + K_5 K_6) \right) + ...$$

(8.2)

Three points should be noticed. The $\lambda$ parameter is scaled as usual to give $\lambda/4 H^4_1$. There are no terms $H^2_4 K^2_4$ or $H^2_4 K^2_5$, finally the coefficients of $H_4$ are invariant under a $\lambda_8$ rotation of the $K$ fields, which rotates $K_4$ into $K_5$ and $K_6$ into $K_7$.

We now add by hand in the Lagrangian a quadratic symmetry breaking 'imaginary mass' term

$$-\frac{\lambda}{2} v^2 \left( \sin^2 \beta H^2 + \cos^2 \beta K^2 \right).$$

(8.3)

We assume that the vacuum expectation value of $H$ is in the direction $H_6$. As discussed in section 3, this choice just fixes the name of the particles. To minimize the mixed $H^2 K^2$ potential, the $K$ vacuum expectation value must be in the plane $(K_6, K_7)$. The vacuum correctly preserves the photon (3.5, 4.7). We then rotate $< K >$ to $< K_6 >$. This 'phase' invariance corresponds to the fact that one of the 4 neutral states in the hyper-plane $(H_6, H_7, K_6, K_7)$, usually called the 'A\(_0\)' pseudo-scalar, is absent from the quartic potential and its mass can be fixed arbitrarily in the symmetry-breaking term. We minimize the potential and find:

$$< H_6 > = v \sin \beta , \quad < K_6 > = v \cos \beta , \quad M_W = v g/2 ,$$

(8.4)

where $\beta$ is defined modulo $\pi/2$. We then expand the potential around the vacuum to second order in $v$. We find 4 massless states which produce the $A_0$ and the longitudinal components of the massive vector fields $Z$ and $W^\pm$

$$H_7 , \quad K_7 , \quad (\sin \beta H_4 + \cos \beta K_4) \pm i(\sin \beta H_5 + \cos \beta K_5) ,$$

(8.5)

two charged physical scalars

$$H^\pm = (\cos \beta H_4 - \sin \beta K_4) \pm i(\cos \beta H_5 - \sin \beta K_5) , \quad M_{H^\pm} = \sqrt{2 \lambda} M_W = 75.8 GeV$$

(8.6)
and two neutral scalars in the \((H_6, K_6)\) plane, usually called \(H_1^0\) and \(H_2^0\), with masses

\[
M_{H_1^0} = 2 \sqrt{2} \lambda \cos \beta M_W = (\sqrt{2} \cos \beta) 107.2 \text{ GeV} ,
M_{H_2^0} = 2 \sqrt{2} \lambda \sin \beta M_W = (\sqrt{2} \sin \beta) 107.2 \text{ GeV} .
\]  

(8.7)

In the symmetric case \(\beta = \pi/4\) the 2 neutral scalar are at 107.2 GeV.

9. The \(SU(2/1)\) hyper-curvature

Since the scalar fields (5.3) play the role of a gauge field associated to the odd generators of the superalgebra, it seems natural to introduce them as part of the covariant differential and curvature 2-form. Such a construction was proposed in [TM06], where we found that the covariant differential is associative if and only if the Fermions sit in a representation of the superalgebra graded by the chirality. As shown in sections 2 and 3, this condition is met by the observed leptons and quarks. But we found in section 3 the additional complication that the quark representation is non Hermitian, so the scalar field is complex, so effectively, the number of odd generators in the quantum field theory is twice the number of odd generators of the \(SU(2/1)\) superalgebra.

We are going to show that the construction of the curvature can be generalized and yields a structure we would like to call a \(H\)-curvature, by reference to Fayet and Iliopoulos [F74, F75], who were the first to consider, in the context of Hyper-Symmetry, a doubling of the Higgs field introduced by Weinberg [Wei67] and to Sternberg and Wolf [SW78] who introduced the Hermitian algebras.

Following [TM06], we define the Dirac 1-forms

\[
\Gamma = dx^\mu \sigma_\mu + dx^\mu \bar{\sigma}_\mu .
\]

(9.1)

where \(\sigma\) maps the right Fermions on the left Fermions and vice versa. We then define the chiral Yang-Mills 1-form \(\tilde{A}\) and covariant differential \(\tilde{D}\)

\[
\tilde{A} = A + \Phi_L + \Phi_R = dx^\mu A^a_\mu \lambda_a + \Gamma \Phi_L^i \lambda_i L + \Gamma \Phi_R^i \lambda_i R , \tilde{D} = d + \tilde{A} .
\]

(9.2)

Notice that \(\tilde{A}\) is an exterior form of homogeneous degree 1, in contradistinction with the Ne‘eman Quillen connection [TMN82, Q85] sometimes discussed in the \(SU(2/1)\) literature ([NS90, NSF05] and references therein). The curvature is defined as the square of the covariant differential

\[
\tilde{F} = \tilde{D} \tilde{D} = d\tilde{A} + \tilde{A} \tilde{A} .
\]

(9.3)

Since the \(dx^\mu\) anticommute, the classical term

\[
A A = dx^\mu dx^\nu A^a_\mu A^b_\nu \lambda_a \lambda_b
\]

(9.4)

is antisymmetric in \(ab\) and is internal provided the even matrices \(\lambda_a\) close by commutation

\[
\lambda_a \lambda_b - \lambda_b \lambda_a = i f_{abc}^c \lambda_c .
\]

(9.5)
Similarly, the scalar vector term in $\Gamma D\Phi$ generate the covariant differential of the scalars provided the even odd sector close by commutation
\[ \lambda_a \lambda_i - \lambda_i \lambda_a = i f^j_{ai} \lambda_j . \] (9.6)

However, the structure of the scalar-scalar term is more involved. Since $\Phi_L$ and $\Phi_R$ are independent, the $\lambda$ matrices are neither symmetrized nor skew symmetrized and we get
\[ \frac{1+\chi}{2} \Phi^i_L \Phi^j_R \lambda_i \lambda_j + \frac{1-\chi}{2} \Phi^i_R \Phi^j_L \lambda_i \lambda_j . \] (9.7)

We now decompose the matrix product into the symmetric and skew symmetric parts
\[ \lambda_i \lambda_j = \frac{1}{2} (\{\lambda_i, \lambda_j\} + [\lambda_i, \lambda_j]) \] (9.8)

and we find that the scalar-scalar term can be expressed as a linear combination of even generators provided the anticommutator of the odd matrices close on the even matrices and the commutators of the odd matrices close on $\chi$ times the even matrices, defining the skew-symmetric structure constants $f^a_{ij}$:
\[ \lambda_i \lambda_j + \lambda_j \lambda_i = d^a_{ij} \lambda_a , \]
\[ \lambda_i \lambda_j - \lambda_j \lambda_i = i \chi f^a_{ij} \lambda_a . \] (9.9)

Therefore the unsymmetrized product of the odd matrices satisfy
\[ \lambda_i \lambda_j = \frac{1}{2} (d^a_{ij} + i \chi f^a_{ij}) \lambda_a . \] (9.10)

By inspection (3.4), all these conditions are satisfied by the lepton representation (3.1-3). In fact, the construction works with the fundamental representation of any $SU(m/n)$ superalgebra, because all the generators coincide with the generators of $SU(m+n)$ except for the supertraceless $U(1)$ but $\chi U(1)$ is traceless and coincides with the $U(1)$ of $SU(m+n)$.

The structure (9.9) was introduced by Sternberg and Wolf [SW78], who call it a Hermitian algebra and discussed by Ne’eman and Sternberg in several subsequent papers [NS90, NSF05], but its exact relevance to quantum field theory remained unclear. The unusual presence of the chirality operator $\chi$ on the right hand side of the commutator (9.9) was not explained, because they did not consider the possibility of splitting the odd generators and doubling the number of Higgs fields. But in our modified framework, if we compare (9.7) and (9.10) and use $\chi^2 = 1$, we see that the $\chi$ operator present in (9.9) disappears from the curvature 2-form which is now correctly valued on the even matrices and can be written as
\[ \tilde{F}_L = \tilde{F}^a_L \lambda_a + \Gamma D\Phi^i_L \lambda_i L + \Gamma D\Phi^i_R \lambda_i R , \]
\[ \tilde{F}_L^a = dA^a + \frac{i}{2} f^a_{bc} A^b A^c + i \frac{1}{2} \Gamma \Phi^i_L \Phi^j_R (f^a_{ij} - i d^a_{ij}) . \] (9.11)

The Bianchi identity $\bar{D} \tilde{F} = (\bar{D} \bar{D}) \tilde{D} - \bar{D} (\bar{D} \bar{D}) = 0$ follows from the associativity of the matrix product and implies the Hermitian Jacobi identity of [SW78]. We can also construct a second solution $\tilde{F}_R$ by switching the sign of the $d$ constants changing the chirality:
\[ \tilde{F}_R^a = dA^a + \frac{i}{2} f^a_{bc} A^b A^c + i \frac{1}{2} \Gamma \Phi^i_R \Phi^j_L (f^a_{ij} + i d^a_{ij}) . \] (9.12)

We recover in (9.11-12) the object $Z_{L/R}$ introduced in (6.3)
\[ \tilde{F}_{L/R} = F + i \frac{1}{2} \Gamma \ Z_{L/R} \] (9.13)
10. The electro-weak angle

From the definition of the classical curvature 2-form given in the previous section, we can construct the Lagrangian in the usual way, using the Hodge adjunction $^*$ and the quadratic form $\tilde{G}_{ab}$ (6.2) which reproduces the scalar potential induced by the Fermion loops. The Lagrangian can be written as:

$$\mathcal{L}_F = \frac{1}{8g^2} Tr \left( ^* \hat{F}_L^a \ast \hat{F}_R^a \tilde{G}_{ab} \right)$$

(10.1)

Since the new bilinear scalar term in the H-curvature (9.13) coincides with (6.3), the scalar potential is proportional to (6.4) which can be rewritten as (6.9). It is curious to see that the combined effect of the leptons and quarks in (6.9) is mimicked in the classical construction (10.1) by combined effect of the 2 chiral ways (9.11-12) of extending the superalgebra $SU(2/1)$ curvature of [TM06] to the H-algebra.

The interesting point is that, using (6.2) and (3.6), the effective metric for the Yang-Mill curvature $F^a$ comes out as

$$2g_{ab} - (1 + \alpha) \tilde{g}_{ab} = \text{diag}(1 - \alpha, 1 - \alpha, 1 - \alpha, 3 + \alpha), \quad a, b = 1, 2, 3, 8$$

(10.2)

and we learn the choice of the $\alpha$ parameter in the definition of the $\tilde{G}$ hyper-metric corresponds to the choice of the electro-weak angle in the classical Lagrangian according to:

$$tg^2\theta = (1 - \alpha)/(3 + \alpha) \Leftrightarrow \alpha = 1 - 4 \sin^2\theta.$$  

(10.3)

Substituting (7.3), we find $\alpha = -1/2$.

11. Discussion

The $SU(2/1)$ model appeared in 1979 as very compelling, but so far no setting had been found that could explain how to extend the Yang-Mills quantum field theory from gauging an internal Lie algebra symmetry to the case of a superalgebra. We believe that the present paper represents a step in this direction.

The original observation [N79, F79] is that, provided we grade the Fermions by their chirality, the smallest Lie superalgebra $SU(2/1)$ exactly corresponds to the standard model. The leptons and the quarks fit the 2 smallest irreducible representations (3.1-2) and (4.1-4) of the superalgebra $SU(2/1)$. The $U(1)$ hyper-charge, which is supertraceless, coincides with the original choice of Weinberg [Wei67] who wanted to avoid a current coupled to the lepton number, and explains the non existence of any massless charged particle. The difficulty is to understand the role of the odd generators.

Our new idea is to take the $SU(2/1)$ matrices at face value, despite the fact that they are not Hermitian (4.3), and to construct the whole Lagrangian by studying the counterterms induced by the Fermions. The caveat is to remember the triangle anomaly (4.5). It is avoided if and only if we always consider as our building block a whole family, constituted of one lepton triplet and three quark quadruplets, for example the electron, the left neutrino and 3 colored copies of the up and down quarks, counting independently.
the left and right chiral states. If we associate a single real scalar to each odd generator
the induced scalar propagator vanishes (4.8). But if we split the odd matrices into their
chiral parts (5.2), and consider 2 sets of scalars $\Phi_L$ and $\Phi_R$ (5.3) the leptons and the
quarks conspire to induce a the scalar propagator (5.7) proportional to the odd part of
the skew-symmetric Killing metric of the superalgebra (3.6). It is then possible to change
variables (5.8) from $\Phi_L$, $\Phi_R$ to $H$, $K$ and recover two Higgs scalar doublets, $H$ coupled
to the electron (5.12) and down quark (5.13), and $K$ coupled to the up quark (5.14). If we
compute in the same way the quartic scalar counterterms we recover (6.12) the potential
first considered by Fayet [F74], a building stone of what later became the Minimal Standard
Supersymmetric Model (MSSM). This is rather surprising, because we have not introduced
any Wess-Zumino supersymmetry in our construction, but may be attributed to the near
unicity of the $SU(2)U(1)$ quartic invariants.

Some interesting results follow from the analysis of the quantum stability of the sym-
metry breaking pattern of the theory. In the Fayet potential, the crucial element is the
form of the mixed term which insures that, in the vacuum, the 2 fields $H$ and $K$ are par-
allel, and therefore leave an unbroken $U(1)$ symmetry, gauged by a massless photon. In
our model, this condition is a consequence of the super-Jacobi identity (6.10) and is stable
against the Fermionic 1-loop quantum corrections, since this is the way we derived the
scalar potential. Furthermore, if fix (7.3) the relative strength of the 2 vectors coupling to
tg$^2\theta = 5/3$ (as in $SU(5)$ grand unified theory [GG74]), the strength of the scalar to $2/\sqrt{3}$,
and the quartic self coupling to $\lambda = 4/9$, these coefficients are stable against the quantum
correction induced by the Fermion loops for any number of families, thanks to the Ward
identities (7.5). These identities are typical of a Yang-Mills minimal coupling and states
that scalar Fermion coupling and the scalar quartic self coupling are indeed proportional
to $g$ and $g^2$ where $g$ is the Yang-Mills $SU(2)$ coupling constant. This fixes (8.7) the mass
of the neutral Higgs at $M_{H^0}^2 + M_{H^0}^2 = 2 \left(4/3 \ M_W^2\right)^2 = 2 \left(107.2 \ GeV\right)^2$. The mass of
the charged Higgs corresponds to the mixed $H K$ term in the potential and gives (8.6):
$M_{H^\pm} = 2\sqrt{2}/3 \ M_W = 75.8 \ GeV$, a value a few GeV too low relative to the minimum
given by the particle data group [Yao06]. Finally the mass of the ’$A_0$’ pseudo-scalar is not
constrained.

On the other hand, we clearly have a difficulty with the mass of the quarks. If we
assume that the 2 Higgs have the same mass and that the $SU(2/1)$ minimal coupling
(5.3) directly gives the mass of the heaviest quark, we predict $M_{top} = M_{H^0} = 4/3 \ M_W =
107.2 \ GeV$ which falls way under the current experimental value $M_{top} = 174.3 \pm 1.8 \ GeV,$
deduced from direct observations [Yao06]. But a worse problem is that the $charm$ and
$up$ quarks, in the two other families should have the same mass, when actually they are
very light (1.25 GeV and 2 MeV). This problem is possibly solved by the $SU(2/1)$ inde-
composable representations which have been shown by Haussling and Scheck to correctly
reproduce the Masakawa quark mixing phenomenology [HS94]. Assuming that the masses
get redistributed, we are not so far from the relation $M_{top}^2 + M_{charm}^2 + M_{up}^2 = 3 \ M_H^2$. This
question needs further studies.

A classical geometric interpretation of the theory is also possible. When expressed in
terms of the $\Phi_L$ and $\Phi_R$ fields, we found that the potential involves both the symmetric $\delta_{ij}$ structure constants of the superalgebra and the skew symmetric $f_{ij}^k$ structure constants of $SU(3)$, times the chirality operator (3.4). This gives the idea of revisiting the Hermitian algebras introduced by Sternberg and Wolf [SW78]. Our new result is to show that this structure exactly fits the chiral decomposition (5.3) of the $\Phi$ fields, allowing us to construct a classical curvature 2-forms (9.12), where the unexpected apparition of the chirality operator $\chi$ in the definition of the Hermitian Lie bracket (9.9) exactly compensates the signs implied when adapting our chiral connection 1-form $\tilde{A}$ [TM06] to the doubling of the $\Phi$ fields. There is also a probable relation with non commutative differential geometry, as discussed in [TM06].

We are now in a position to give a classical geometrical construction of the vector/scalar part of our Lagrangian as the square of the curvature of the $SU(2/1)$ Hermitian algebra. In the earlier work of Ne’eman and others, the central problem was the choice of the quadratic form needed to compute the square of the curvature. The natural choice was the supertrace metric (3.6), but this yields a negative propagator for the $U(1)$ field and breaks unitarity. The phenomenological choice seemed to be the trace of the lepton representation (3.1) which yields $\sin^2\theta = .25$. In 1979, this value was acceptable, but is now too high. Also the trace metric is not a good invariant for a superalgebra, and this choice was often criticized. Our new result is to deduce the relevant quadratic form from the renormalization theory. We found the very elegant hyper metric $\tilde{G}$ (6.2) which is a combination of the trace and the supertrace metric and, thanks to the super-Jacobi identity, contains a free parameter $\alpha$. The 2 rows and columns of $\tilde{G}$ are used to combine the 2 $CP$ conjugated versions of the H-curvature (10.1), and this combination restores unitarity, because in the combined Lagrangian we can switch between the native $SU(2/1)$ geometric but non-Hermitian $\Phi_L$ and $\Phi_R$ (6.4) variables and the good Hermitian quantum field theory scalars $H$ and $K$ (6.9). When we inject this metric in our Lagrangian (10.1), we find that the freedom in the choice of $\alpha$ reflects a freedom in the choice of the electro-weak angle $\alpha = 1 - 4 \sin^2\theta$ (10.3) and that the simplest choice $\alpha = 0$ corresponds to the trace-metric choice, $\alpha = -1/2$ corresponds to $\sin^2\theta = 3/8$.

Let us now recall the distinction between this model and the minimal supersymmetric standard model (MSSM). The geometrical space behind Yang-Mills theory is the principal fiber bundle. The base is space-time, the fiber is the gauge group. In both cases, a superalgebra is introduced but not in the same way. In the MSSM, the Poincare superalgebra is introduced in the base space, acts on Bose/Fermi supermultiplets and naturally leads to supergravity. In our approach, the superalgebra is introduced in the fiber, by embedding the Glashow-Weinberg-Salam $SU(2)U(1)$ Lie algebra in $SU(2/1)$, it acts of left/right Fermionic chiral supermultiplets and groups the Higgs scalars with the Yang-Mills vectors. This construction is not incompatible with supergravity at much higher energy or with superstrings but does not lead to these theories. In both models, (5.11) and [F74], we are led to introduce 2 sets of scalar doublets, $H$ coupled to the electron and down-quark right singlets, and $K$ to the up-quark singlet. The quartic scalar potential has the same structure (6.9-12) and implies the survival of an unbroken $U(1)$ and a massless photon. So the scalar spectrum is similar, 3 states become the longitudinal components of the $W$ and $Z$ vector
fields, 5 survive. In both models the scale of the potential is fixed and the values are in
the same range. But of course the main difference is that the MSSM predicts for every
known particle the existence of supersymmetric partner, like the winos, gluinos, squarks
and sleptons, whereas SU(2/1) fits exactly the known particles but does not predict any
new ones, except the Higgs fields. The existence or non existence of these particles below
1 TeV will allow us to choose between the 2 kinds of supersymmetry.

The SU(2/1) theory is testable, but incomplete in several directions. At the experi-
mental level, the model predicts 2 complex doublets of Higgs fields, rather than 1 as in the
original Weinberg paper, and constrains their masses within the range of the new CERN
collider which opens this year. At the phenomenological level, the mixing of the quark
families is not understood in details, but may be linked to the indecomposable representa-
tions of SU(2/1). At the theoretical level, the balance between the quarks and leptons is
broken by the quantum counterterms induced by the gluons, and all the scalar couplings,
except those ensuring the existence of a massless photon, are perturbed by the Boson loops.
We believe that this problem could indicate the incompleteness of the current quantization
procedure and will discuss it elsewhere.

But even if these difficulties cannot be solved immediately, the 1-loop stability of the
structure of the scalar potential, ensuring a massless photon, and the resulting prediction
of the number and masses of the Higgs fields seem worth reporting as a first step towards
the construction of SU(2/1) Quantum Astheno Dynamics.

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