Super Chern-Simons Quantum Mechanics*

Luca Mezincescu

Department of Physics, University of Miami, Coral Gables, FL 33124, USA

ABSTRACT

The Super Chern-Simons mechanics, and quantum mechanics of a particle, on the coset super-manifolds $SU(2|1)/U(2)$ and $SU(2|1)/[U(1) \times U(1)]$, is considered. Within a convenient quantization procedure the well known Chern-Simons mechanics on $SU(2)/U(1)$ is reviewed, and then it is shown how the fuzzy supergeometries arise. A brief discussion of the supersphere is also included.

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1 mezincescu@server.physics.miami.edu
1  Introduction

The actions corresponding to anti-commuting degrees of freedom contain only first order time derivatives of the corresponding fields. This suggests that such actions may be somehow connected with the pull-back of a one-form. But if the action is to be a pull-back of a one form then the base space must be one dimensional, therefore the corresponding action generated by a one form should be that of a particle moving on a super-manifold. Moreover such a one form may be chosen to be the connection form on some super-manifold, associating a certain “dynamics” with the given “affine geometry”. One then deals with the Super Chern-Simons mechanics. Studies of the Chern-Simons mechanics and its world line super-symmetric generalization have been performed before [1, 2]. In what follows some recent work [3, 4] on particle motion on supermanifolds, whose actions are such Super Chern-Simons terms will be described. In fact for simplicity only various cosets of $SU(2|1)$ will be considered, as they illustrate well, the generic case treated, which is the super Kahler manifold $CP^{(n|m)}$.

The bosonic Chern-Simons actions are prototype actions which lead to fuzzy geometry [5, 6, 7, 8, 9, 10], under quantization. Their counterpart the Super Chern-Simons actions will also be shown to lead to the fuzzy supergeometry (see also [11, 12, 13, 14, 15, 16, 17]). The quantization procedure will be first described for the simple case of the Chern-Simons action on the sphere, then the same procedure will be applied to the Super Chern-Simons actions corresponding to the $U(1)$ connection associated with the supermanifolds $SU(2|1)/U(2)$ and $SU(2)/U(1|1)$. In the process the definition of the supersphere will be also touched upon.

2  Chern-Simons Quantum Mechanics on $S^2$

Let $z$ be the complex coordinate on the sphere defined by stereographic projection to the complex plane, and let $z(t)$ be the classical position of a charged particle on the sphere. The Chern-Simons mechanics for this particle is defined by the Lagrangian

$$L = N (\dot{z} A_z + \dot{\bar{z}} A_{\bar{z}}),$$

(1)

where $2N$ is an integer (at the quantum level), and

$$A_z = -i \partial_z \ln K_0, \quad A_{\bar{z}} = i \partial_{\bar{z}} \ln K_0,$$

(2)

are the complex components of the $U(1)$ connection on the sphere, (which can also be viewed as the gauge potential due to a unit charge monopole at the centre of the sphere) with

$$K_0 = (1 + z \bar{z}).$$

(3)

This Lagrangian is invariant (up to a total derivative) under the $SU(2)$ isometry group of the sphere (in fact it is invariant under the infinite dimensional group of
simplectic diffeomorphisms on the sphere, which is why these models are topological), the infinitesimal $SU(2)$ transformations are obtained from the closure of

$$\delta z = \varepsilon + \bar{\varepsilon} z^2, \quad \delta \bar{z} = \bar{\varepsilon} + \varepsilon \bar{z}^2,$$

(4)

for complex parameter $\varepsilon$, under which:

$$\delta_\varepsilon K_0 = (\bar{\varepsilon} z + \varepsilon \bar{z}) K_0,$$

(5)

and (1) transforms by a total derivative. One could pass to the quantum theory by the usual Dirac method prescription, but another method will be used with some advantages that will hopefully become apparent in what follows. With the standard definition of the canonical momenta, the two second class constraints of this model can be expressed\(^1\) as the complex conjugate pair of constraints $\varphi_z \approx 0$ and $\varphi_{z} \approx 0$, where

$$\varphi_z = p_z - N A_z, \quad \varphi_{z} = p_{z} - N A_{z}.$$

(6)

One now quantizes, as if there were no constraints, by setting

$$p_z = \frac{1}{i} \frac{\partial}{\partial z}, \quad p_{\bar{z}} = \frac{1}{i} \frac{\partial}{\partial \bar{z}},$$

(7)

and then imposes the physical state condition

$$\varphi_z \Psi (z, \bar{z})_{\text{phys}} = 0,$$

(8)

which has the solution

$$\Psi (z, \bar{z})_{\text{phys}} = (1 + z \bar{z})^{-N} \Phi (z).$$

(9)

for holomorphic function $\Phi(z)$. The $SU(2)$-invariant inner product of two wave-functions $\Psi$ and $\Omega$ corresponding, respectively, to the holomorphic functions $\Phi(z)$ and $\Upsilon(z)$ is

$$(\Psi, \Omega) = \int \frac{dzd\bar{z}}{(1 + z \bar{z})^2} \bar{\Psi}(\bar{z}, z) \Omega(z, \bar{z}) = \int \frac{dzd\bar{z}}{(1 + z \bar{z})^{2(N+1)}} \bar{\Phi}(\bar{z}) \Upsilon(z).$$

(10)

Normalizability of $\Psi$ and $\Omega$ in this inner product requires $\Phi(z)$ and $\Upsilon(z)$ to be polynomials in $z$ of maximum degree $2N$. Correspondingly the physical Hilbert space is $2N + 1$ dimensional.

For fermionic constraints, this alternative method of dealing with second-class constraints can be traced back to the 1976 papers of Casalbuoni [18] and papers in the early 1980s of Azcárraga et al. [19, 20] and Lusanna [21]. A clear statement of it can be found, again for fermionic constraints, in a 1986 paper of de Azcárraga and Lukierski [22], who called it ‘Gupta-Bleuler’ quantization by analogy with the

\(^1\)One uses the symbol $\approx$ to denote weak equality in the sense of Dirac.
procedure of that name for covariant quantization of electrodynamics. It was also called Gupta-Bleuler quantization in the 1991 book of Balachandran et al., where it is explained for particle mechanics models with bosonic constraints. The justification for this method sketched above arose in independent work on general models with bosonic second-class constraints that can be separated into two sets of real constraints, each in involution. In this context the method has become known as the method of 'gauge-unfixing'.

As the Lagrangian (1) is invariant (up to a total time derivative) under the SU(2) isometry group of the sphere, there should be an action of this group induced on the physical Hilbert space. Allowing for operator ordering ambiguities, the Noether charge operators that generate the infinitesimal transformations (4) are

\[ J_+ = i \left( P_z + z^2 P \bar{z} + i \alpha z \right), \quad J_- = i \left( P_z + z^2 P \bar{z} - i \beta \bar{z} \right) \]

for some constants \( \alpha \) and \( \beta \), leading to the transformation property of \( \Psi \),

\[ \delta \Psi = (\epsilon J_+ + \bar{\epsilon} J_-) \Psi. \]

A necessary condition for these charges to take physical states into physical states is that they commute (weakly) with \( \varphi \), and this fixes \( \beta = N \). The constant \( \alpha \) remains undetermined by this requirement, however when constructing a representation of \( SU(2) \) one must demand that \( N + \alpha \) is an integer and then the representation is \( N + \alpha + 1 \) dimensional. It is therefore possible to choose a natural value \( \alpha = \beta = N' \), with \( 2N' = N + \alpha \), without loss of generality. Then:

\[ J_- = -J_+^\dagger, \]

with respect to the inner product (10). Thus,

\[ J_+ = \left( \partial_z + z^2 \partial_{\bar{z}} \right) - Nz, \quad J_- = \left( \partial_z + z^2 \partial_{\bar{z}} \right) + N\bar{z}. \]

Note that

\[ J_\pm \Psi = (1 + z\bar{z})^{-N} j_\pm \Phi (z), \]

where

\[ j_- = \partial_z, \quad j_+ = z^2 \partial_{\bar{z}} - 2Nz \]

are the charge operators acting on holomorphic functions. These have the commutator

\[ [j_-, j_+] = 2j_3, \quad j_3 = z\partial_z - N, \]

and \( (j_-, j_+, j_3) \) span the Lie algebra of \( SU(2) \). Monomials in \( z \) are eigenfunctions of \( j_3 \), with eigenvalues that range between \( -N \) for constant \( \Phi \) and \( +N \) for \( \Phi \propto z^{2N} \).

\(^2\)Noting that the Lorentz gauge condition cannot be consistently imposed as a physical state condition, Gupta and Bleuler suggested that it be separated into its positive and negative frequency parts (of which \( \varphi \) and \( \bar{\varphi} \) are analogs) and that the positive frequency part be imposed as the physical state condition.
The polynomials of maximal degree $2N$ therefore span the $2N+1$ irrep of $SU(2)$, and hence the $2N+1$-dimensional Hilbert space is a carrier space for this irrep. The operators $J_{\pm}$ generate translations on the sphere, and act on the physical Hilbert space, so they are naturally identified with the momentum operators of the quantum theory. But what are the position operators corresponding to the classical position variable $z$ and its complex conjugate $\bar{z}$? On general grounds one may expect some complications for the definition of the position operators due to the fact that the naive definition does not lead to operators with tensorial properties. Also, $\bar{z}\Psi(z)$ is clearly not a physical state (due to the fact that $\bar{z}$ does not commute with $\varphi_z$) and if $\Phi$ is a polynomial of maximal degree then $z\Psi$ is not a physical state either (despite the fact that $z$ commutes with $\varphi_z$). Thus, the ‘naive’ position space operators do not act on the physical Hilbert space. This could have been anticipated from the fact that $z$ and $\bar{z}$ commute whereas physical position operators are non-commutative in first order systems [1]. However, this non-commutativity must disappear in the $N \to \infty$ limit because this limit is in fact the semiclassical limit of this problem, Therefore the position operator $z$ takes the form:

$$z_{\text{op}} = z + O(1/N),$$

where $z$ is the naive position operator. The $O(1/N)$ corrections must be such as to ensure that $z_{\text{op}}$ and $\bar{z}_{\text{op}}$, act on the physical Hilbert space. There is a unique solution to this problem, with a minimal number of derivative operators, and the result is

$$z_{\text{op}} = z - \frac{1}{N} \left( \partial_z + z^2 \partial_z \right), \quad \bar{z}_{\text{op}} = \bar{z} + \frac{1}{N} \left( \partial_{\bar{z}} + \bar{z}^2 \partial_{\bar{z}} \right). \quad (19)$$

Comparison with (14) shows that

$$z_{\text{op}} = -\frac{1}{N} J_+, \quad \bar{z}_{\text{op}} = \frac{1}{N} J_. \quad (20)$$

That is in terms of (16), one just discards the term which does not lead to a normalizable state upon the action of naive $z$. The position operators are proportional to the momentum operators, and thus span the algebra of the $SU(2)$ isometry group of the sphere. The proportionality between the position operators and the corresponding generators of $SU(2)$ is the reason one calls the corresponding manifold “fuzzy”. It is clear that one can exhibit the fact that the position operators correspond to the quantization of certain observables of the classical theory (which in fact corresponds to a reshuffling of classical phase space variables)

$$z_{\text{op}} = z - \frac{i}{N} \left[ p_z + z^2 p_z \right] = \frac{2z}{1 + z \bar{z}} - \frac{i}{N} \left( z^2 \varphi_z + \varphi_z \right) \approx \frac{2z}{1 + z \cdot \bar{z}}. \quad (21)$$

$$\bar{z}_{\text{op}} = \bar{z} + \frac{i}{N} \left[ p_{\bar{z}} + \bar{z}^2 p_{\bar{z}} \right] = \frac{2\bar{z}}{1 + z \bar{z}} + \frac{i}{N} \left( \bar{z}^2 \varphi_{\bar{z}} + \varphi_{\bar{z}} \right) \approx \frac{2\bar{z}}{1 + \bar{z} \cdot z}. \quad (22)$$
The classical part of the operators $z, \bar{z}$ are in fact the conveniently redefined $x, y$ components of the position vector on the sphere as embedded in the three dimensional space. These classical position operators can be shown to obey Poisson brackets which close to the classical $SU(2)$ algebra. By defining the position operators in this model, one is therefore lead to the natural vector coordinate appearing in the problem. Here a limited set set of observables has been considered, it is clear however that once one knows the quantum system, which is a spin $N$ system, one can introduce additional observables corresponding to the number of hermitian operators which can be built on the corresponding space. Even if the Hamiltonian for the system is absent, the requirement of the existence of the scalar product restricts the space of states as well as leads to a correct definition of the observables for this system.

3 Odd Coset Quantum Mechanics

In what follows the above described procedure, will be used to treat systems with anticommuting variables. The simplest purely odd supermanifold is $SU(2|1)/U(2)$ ($SU(2|1)$ superalgebra also has an involution). $SU(2|1)$ superalgebra consist of a even part, which is $U(2)$, and the odd sector which is made of two complex spinors under $SU(2)$ conjugate to each other, and of opposite “baryon number” corresponding to the $U(1)$. The superflag manifold $SU(2|1)/U(2)$ has a “complex” structure like the sphere, and it is purely odd. It is easy to show that a simple generalization of the transformation laws (4), is:

$$\delta_e \xi^i = e^i + \bar{\xi} \cdot \xi^i,$$

(1)

where $e^i, (i = 1, 2)$ are the two odd parameters corresponding to the odd transformations and $\xi^i$ are “local” anticommuting coordinates on the superflag. These transformations close to the superalgebra $SU(2|1)$, and one can guess the super Kahler like connection:

$$A = i (d \xi \cdot \partial \bar{\xi} - d \bar{\xi} \cdot \partial \xi) \ln K_1$$

(2)

with:

$$K_1 = 1 + \bar{\xi} \xi$$

(3)

where $K_1$ transforms under (1) as:

$$\delta_e K_1 = (\bar{\xi} \cdot \xi - e \cdot \bar{\xi}) K_1.$$  

(4)

The corresponding Lagrangian is:

$$L = \frac{i}{2} \bar{\xi} \cdot \xi^{-1} \bar{\xi} \cdot \dot{\xi} + c.c.$$  

(5)

and with the definition of the odd canonical conjugate momentum:

$$\pi_i = i \frac{\partial L}{\partial \dot{\xi}_i}, \quad \bar{\pi}^i = i \frac{\partial L}{\partial \dot{\xi}_i}$$  

(6)
leads to the second class constraints:

$$\varphi_i = \pi_i - \frac{\gamma}{2} \left[1 + \bar{\xi} \cdot \xi\right]^{-1} \bar{\xi}_i, \quad \bar{\varphi}^i = \bar{\pi}^i - \frac{\gamma}{2} \left[1 + \bar{\xi} \cdot \xi\right]^{-1} \xi^i$$  \hspace{1cm} (7)

Using the method described in the previous section, one obtains the larger phase space where the odd quantized momenta are given by:

$$\pi_i = \frac{\partial}{\partial \xi_i}, \quad \bar{\pi}^i = \frac{\partial}{\partial \bar{\xi}^i}.$$  \hspace{1cm} (8)

To take the constraints into account it must be required that physical states be annihilated by the operators $\varphi_i$, this is equivalent to the ‘analyticity’ conditions$^3$, or equivalently, with the condition that the corresponding covariant derivative be vanishing:

$$\frac{\partial \Psi}{\partial \xi^i} = \frac{\gamma}{2} \left[1 + \bar{\xi} \cdot \xi\right]^{-1} \bar{\xi}_i \Psi, \quad i = 1, 2,$$  \hspace{1cm} (9)

on wave-functions $\Psi(\{\xi\}, \{\bar{\xi}\})$. These conditions have the solution

$$\Psi = \left[1 + \bar{\xi} \cdot \xi\right]^{-\frac{\gamma}{2}} \Phi$$  \hspace{1cm} (10)

for anti-analytic $\Phi$, which has the expansion

$$\Phi = a + \bar{\xi}_i b^i + \bar{\xi}_1 \bar{\xi}_2 c.$$  \hspace{1cm} (11)

In this space an action of $SU(2|1)$ should be introduced. The quantum version of the classical Noether generators corresponding to the transformation (1) is:

$$\hat{S}_i = \frac{\partial}{\partial \xi^i} + \frac{\alpha}{2} \bar{\xi}_i - \bar{\xi}_i \left(\bar{\xi} \cdot \frac{\partial}{\partial \xi}\right),$$

$$\hat{\bar{S}}^i = \frac{\partial}{\partial \bar{\xi}^i} + \frac{\beta}{2} \xi^i + \xi^i \left(\xi \cdot \frac{\partial}{\partial \bar{\xi}}\right).$$  \hspace{1cm} (12)

The coefficients $\alpha$ and $\beta$ are undetermined at this stage due to the quantum ordering ambiguities. The quantum operators $\hat{S}_i$ and $\hat{\bar{S}}^i$ must be however weakly commuting with the constraints (9), and this fixes $\beta = \gamma$ in (12), in fact, one can also choose, with one notable exception, $\alpha = \beta$. The action of these operators on the subspace of anti-analytic superfields can be easily deduced from:

$$\delta \epsilon \Psi \equiv - \left(\epsilon \cdot \hat{\bar{S}} + \bar{\epsilon} \cdot \hat{S}\right) \Psi = \left[1 + \bar{\xi} \cdot \xi\right]^{-\frac{\gamma}{2}} \delta \epsilon \Phi.$$  \hspace{1cm} (13)

where $\delta \epsilon \Phi$ is given by:

$$\delta \epsilon \Phi = - \left[\gamma (\epsilon \cdot \bar{\xi}) + \bar{\epsilon} \cdot \frac{\partial}{\partial \bar{\xi}} - (\epsilon \cdot \bar{\xi}) \bar{\xi} \cdot \frac{\partial}{\partial \bar{\xi}}\right] \Phi$$  \hspace{1cm} (14)

$^3$This is analogous to the chirality condition on 4D chiral superfields, which arises in a similar way from analytic quantization of the 4D superparticle [26, 27].
For component fields in the expansion (11) this transformation implies:

\[
\begin{align*}
\delta_i a &= -\bar{\epsilon}_i b^i, \\
\delta_i b^i &= \gamma \epsilon^i a + \bar{\epsilon}_j \epsilon^{ij} c, \\
\delta_i c &= (\gamma - 1) \epsilon^i \epsilon^{ij} b^j,
\end{align*}
\]

where \(\epsilon^{12} = -\epsilon_{12} = 1\), is the corresponding totally antisymmetric symbol. One can see that for the special values of \(\gamma = 0, 1\) the corresponding transformations are reducible (but not totally reducible), while for the generic values of \(\gamma\) they are irreducible. In fact the case \(\gamma = 0\), corresponds to the zero action, however it can be obtained from a non zero action using the arbitrariness in the definition of the supercharges mentioned before. To obtain the space of states however an invariant inner product must be introduced. This can be accomplished by introducing the \(SU(2|1)\) invariant measure:

\[
\int d\mu = \int d\mu_0 \left[ 1 + \bar{\xi} \cdot \xi \right],
\]

where

\[
\int d\mu_0 = \prod_i \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \bar{\xi}_i}.
\]

Then the following bilinear form is \(SU(2|1)\) invariant

\[
||\Psi(\gamma)||^2 = \int d\mu \left[ 1 + \bar{\xi} \cdot \xi \right]^{-\gamma} |\Phi(\gamma)|^2.
\]

Within this “norm” one can see that for generic \(\gamma\)’s the space of states will consist of four dimensional irreducible multiplet of \(SU(2|1)\), corresponding to \(q = \frac{1}{2}\), [29]. For the special values \(\gamma = 0, 1\) the space of states consists of a degenerate representation of \(SU(2|1)\) [29] and respectively a singlet. In this case therefore the wave vector contains zero norm states. \(\gamma\) is the \(U(1)\) “baryon” charge in the \(SU(2|1)\) algebra, it is not quantized like it usually happens for the WZ-like terms, however simplifications associated with integer values nevertheless appear for certain integer values of it.

In the special case when \(\gamma = 1\) it is possible to introduce an alternative invariant norm, if one forces the singlet to vanish by imposing the covariant condition:

\[
c = 0 \iff \frac{\partial^2 \Phi(1)}{\partial \xi_i \partial \bar{\xi}_k} = 0.
\]

The alternative norm is

\[
||\bar{\Phi}(1)||^2 = -\int d\mu_0 \ln(1 + \bar{\xi} \cdot \xi) \bar{\Phi}(1) \bar{\Phi}(1)
= |a|^2 + \bar{b} b^i
\]

and it projects again a degenerate representation of \(SU(2|1)\).
A discussion, similar to that in the previous section, of the position operators
can be performed with the analogous result that the naive odd operators must be
redefined and the newly defined operators are just the odd generators of the $SU(2|1)$,
while the classical odd coordinates are the “projective” coordinates:

$$W^i = \gamma (1 + \xi \cdot \bar{\xi})^{-1} \xi^i. \quad (22)$$

### 4 Supersphere

In what follows Chern-Simons mechanics with both even and odd coordinates will
be considered. One therefore chooses other cosets of $SU(2|1)$. Any coset of $SU(2|1)$
whose body is $S^2$ can be viewed as a graded generalization of the sphere. By considering
an appropriate parametrization of $SU(2|1)/[U(1) \times U(1)]$ it will be shown that
one is lead to a minimal extension of the sphere by spinor coordinates. Indeed it can
be shown [30] that the following transformations:

$$\begin{align*}
\delta z &= \varepsilon + \bar{\varepsilon} z^2 - (\bar{\varepsilon}_2 + z\bar{\varepsilon}_1) (\xi^1 - z\xi^2) \\
\delta \xi^1 &= \varepsilon \xi^2 + \epsilon^1 + (\bar{\varepsilon} \cdot \xi) \xi^1 \\
\delta \xi^2 &= -\bar{\varepsilon} \xi^1 + \epsilon^2 + (\bar{\varepsilon} \cdot \xi) \xi^2
\end{align*} \quad (1)$$

close to the algebra of $SU(2|1)$, these transformations correspond to the isometries
of the $SU(2|1)/[U(1) \times U(1)]$ supermanifold which is a graded generalization of $S^2$.

Then, one can introduce an action of $SU(2|1)$ on a subsupermanifold of the above
supermanifold. Consider the local coordinates $z$ and $\xi = \xi^1 - z\xi^2$, they transform
among themselves under the above transformations:

$$\begin{align*}
\delta z &= \varepsilon + \bar{\varepsilon} z^2 - (\bar{\varepsilon}_2 + z\bar{\varepsilon}_1) \xi \\
\delta \xi &= (\bar{\varepsilon}z) \xi + \epsilon^1 - \epsilon^2 z
\end{align*} \quad (2)$$

What happens is that $z$ and $\xi$ transform linearly, among themselves, under the $\epsilon^2$
transformation which therefore passes into the stability group. The stability group
consists now of the corresponding odd generators, and of the two $U(1)$’s in the
$SU(2|1)$. Therefore it is the $U(1|1)$. The two $U(1)$’s can be redefined, so that one of
them commutes with the odd generators in the stability group, while the other $U(1)$
appears in the anticommutator of the corresponding odd generators. One therefore
has only one abelian connection which transforms by a total derivative under the
motions on the supermanifold, which are those corresponding to $\varepsilon$ and $\epsilon^1$. This supermanifold is an extension of the sphere which maintains its complex structure and it can be easily shown that it is a homogenous symmetric space. It has been called
[4] the supersphere $^4$. It will be shown that the ‘Hilbert’ space of a particle on a

$^4$The term ‘supersphere’ has been used previously for the coset superspace $OSp(1|2)/U(1)$
[31, 32, 33, 34, 35]. Although this superspace is often stated to have real dimension $(2|2)$, its
‘reality’ is defined with respect to a ‘pseudoconjugation’; see e.g. [36] for details. With respect
to standard complex conjugation, it actually has real dimension $(2|4)$ (that is the same dimension
as the $SU(2|1)/U(1) \times U(1)$ example which however is not a symmetric space) since spinors of
$USp(2) \cong SU(2)$ span a vector space of dimension 4 over the reals.
supersphere at fuzziness level $2N$ is a degenerate irrep of $SU(2|1)$, $q = N$ [29] that decomposes with respect to $SU(2)$ into a supermultiplet of $SU(2)$ spins $(N - \frac{1}{2}, N)$.

One proceeds in a way similar to the previous section to find the connection forms:

$$A = -i \left( dZ^M \partial_M - d\bar{Z}_M \bar{\partial}^M \right) \ln K_2,$$

with $(Z^M = z, \xi)$ and $(\bar{Z}_M = \bar{z}, \bar{\xi})$, and

$$K_2 = 1 + \bar{z}z + \xi\bar{\xi},$$

where $K_2$ transforms under (2) as:

$$\delta \epsilon K_2 = (\bar{\epsilon}z + \epsilon \bar{z} + \epsilon^1 \bar{\xi} - \bar{\epsilon}^1 \xi) K_2,$$

The Lagrangian is then:

$$L = -iN \left( 1 + Z \cdot \bar{Z} \right)^{-1} \dot{Z} \cdot \bar{Z} + c.c.$$ (6)

and leads to the physical wave function:

$$\Psi = \left[ 1 + Z \cdot \bar{Z} \right]^{-N} \Phi(Z)$$ (7)

for holomorphic superfield $\Phi$, with the corresponding “norm”

$$||\Psi||^2 = \int d\mu_0 \left[ 1 + Z \cdot \bar{Z} \right]^{-(2N+1)} |\Phi|^2$$ (8)

where (allowing for an arbitrary normalization factor $N$)

$$\int d\mu_0 = N \int \int dzd\bar{z} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \bar{\xi}}.$$ (9)

Therefore normalizability of $\Psi$ requires $\Phi$ to be a polynomial in $Z$ of maximum degree $2N$, that is in this case as expected the WZ term is quantized, and $\Phi$ has the expansion:

$$\Phi = \phi_0(z) + \xi \phi_1(z)$$ (10)

where $\phi_0$ and $\phi_1$ are two holomorphic functions of opposite Grassmann parity. After performing the Berezin integrals over $\xi$ and $\bar{\xi}$, one finds that

$$||\Psi||^2 \propto \left[ \int_{S^2} \frac{dzd\bar{z}}{(1 + \bar{z}z)^{2N+1}} |\phi_1|^2 + (1 + 2N) \int_{S^2} \frac{dzd\bar{z}}{(1 + \bar{z}z)^{2N+2}} |\phi_0|^2 \right].$$ (11)

Normalizability of the second integral implies that $\phi_0(z)$ is a polynomial of maximum degree $2N$ and hence that its coefficients transform as spin $N$ under $SU(2)$, while normalizability of the first integral implies that $\phi_1(z)$ is a polynomial in $z$ of maximum degree $(2N - 1)$, and hence that its coefficients transform as spin $N - \frac{1}{2}$ under $SU(2)$.
If the spin-statistics connection is to be respected then \( \Phi \) should be chosen to have Grassmann parity \((-1)^{2N}\). Then the ‘Hilbert’ space that is a supermultiplet with spins \((N - \frac{1}{2}, N)\) carrying a \(2N \oplus (2N + 1)\) representation of \(SU(2)\); this is the decomposition into \(SU(2)\) irreps of the ‘degenerate’ irrep of \(SU(2|1)\) of total dimension \(4N + 1\) \((q = N\) in the notation of \([29]\)). As in the preceding sections the corresponding position operators become proportional with the corresponding charges of \(SU(2|1)\).

In conclusion a very simple setting for the study of Super Chern-Simons quantum mechanics has been outlined. It produces the starting point for the study of fuzzy super-geometries, by generating the corresponding “fuzzy” commutation relations among the coordinates, within a convenient quantization procedure. This approach is a useful companion to the methods of geometric quantization on compact simplectic manifolds. It illustrates within an extended phase space associated with Chern-Simons action, the quantization, of the Poisson brackets associated with these manifolds. The existence of the Chern-Simons action, implies that of a non-degenerate two form, which in turn guarantees the existence of the Poisson bracket mentioned before and vice versa, in order to construct a Poisson bracket one needs a non degenerate closed two form (see e.g. \([37]\)), whose corresponding potential one form generates the Chern-Simons action.

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