Meron Pairs and Fermion Zero Modes

James V. Steele and John W. Negele

Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics
Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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Merons, conjectured as a semiclassical mechanism for color confinement in QCD, have been described analytically by either infinite action configurations or an Ansatz with discontinuous action. We construct a smooth, finite action, stationary lattice solution corresponding to a meron pair. We also derive an analytical solution for the zero mode of the meron pair Ansatz, show that it has the qualitative behavior of the exact zero mode of the lattice solution, and propose the use of zero modes to identify meron gauge field configurations in stochastic evaluations of the lattice QCD path integral.

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Chiral symmetry breaking and confinement of color charge, the two essential features of low-energy QCD, cannot be understood perturbatively in the coupling constant. Two complementary nonperturbative approaches are analytical approximation using semiclassical analysis of the QCD path integral (analogous to the WKB approximation in quantum mechanics) and numerical solution using lattice gauge theory. Combining physical understanding from the semiclassical approximation with systematic rigor of the lattice has provided valuable insight into hadron structure and our goal is to use this combined approach to study confinement.

Most semiclassical treatments of QCD have focused on instantons, leading to a qualitative, and in some cases even quantitative, understanding of chiral symmetry breaking and the structure of the QCD vacuum. However, simple configurations of instantons, such as a dilute gas or a random superposition, do not confine color charges. Of alternative semiclassical mechanisms proposed for confinement, including monopoles and center vortices, we focus on merons, which are solutions to the classical Yang-Mills field equations with one-half unit of localized topological charge. They are the (3+1)-dimensional nonabelian extension of the ’t Hooft-Polyakov monopole configuration, which has been shown to lead to confinement in 2+1 dimensions. Ultimately, we seek to understand if merons generate area and perimeter laws characterizing the behavior of confined and screened color sources.

A purely analytic study of merons has proven intractable for several reasons. Unlike instantons, no exact solution exists for more than two merons, gauge fields for isolated merons fall off too slowly (\(A \sim 1/x\)) to superpose them, and the field strength is singular. A patched Ansatz configuration that removes the singularities does not satisfy the classical Yang-Mills equations, preventing even calculation of gaussian fluctuations around a meron pair.

This letter lays the groundwork for using lattice techniques to investigate the presence and role of merons in QCD. We construct a smooth, finite action configuration corresponding to a meron pair, which with its associated quantum fluctuations, should be present in Monte Carlo lattice QCD calculations. Similar work with instantons has shown fermionic zero modes are a powerful tool for identifying topological objects on the lattice. Therefore, we also analytically derive the zero mode of the continuum meron pair Ansatz and compare with exact numerical results. For simplicity, we restrict our discussion to SU(2) color without loss of generality.

Two known solutions to the classical Yang-Mills equations in four Euclidean dimensions are instantons and merons. Both have topological charge, can be interpreted as tunneling solutions, and can be written in the general form (for the covariant derivative \(D_\mu = \partial_\mu - iA_\mu^a\sigma^a/2\) and ’t Hooft symbol \(\eta_{\mu
u}\))

\[
A_\mu^a(x) = -\eta_{\mu
u}\partial_\nu \ln \Pi(x^2),
\]

with \(\Pi(x^2) = 1/(x^2 + \rho^2)\) for an instanton and \(\Pi(x^2) = 1/\sqrt{x^2}\) for a meron.

Conformal symmetry of the classical Yang-Mills action, in particular under inversion \(x_\mu \rightarrow x_\mu/x^2\), shows that in addition to a meron at the origin, there is a second meron at infinity. These two merons can be mapped to arbitrary positions, which we define to be the origin and \(d_\mu\). After a gauge transformation, the gauge field for the two merons takes the simple form

\[
A_\mu^a(x) = \eta_{\mu\nu} \left[ \frac{x_\nu}{x^2} + \frac{(x-d_\nu)}{(x-d)^2} \right].
\]

Similar to instantons, a meron pair can be expressed in singular gauge by performing a large gauge transformation about the mid-point of the pair, resulting in a gauge field that falls off faster at large distances (\(A \sim x^{-3}\)). A careful treatment of the singularities shows that the topological charge density is

\[
Q(x) \equiv \frac{\text{Tr}}{16\pi^2} \left( F_{\mu
u} F^{\mu\nu} \right) = \frac{1}{2} \delta^4(x) + \frac{1}{2} \delta^4(x-d),
\]

yielding total topological charge \(Q = 1\), just like the instanton.
An instanton can therefore be interpreted as consisting of a meron pair. This complements the fact that instantons and antiinstantons have dipole interactions between each other. If there exists a regime in QCD where meron entropy contributes more to the free energy than the logarithmic potential between pairs, like the Kosterlitz–Thouless phase transition, instantons could break apart into meron pairs. The intrinsic size of the instanton caps would then be determined by the instanton scale $\rho$.

The Atiyah–Singer index theorem states that a fermion in the presence of a gauge field with topological charge $Q$, described by the equation

$$
\mathcal{D}\psi = \lambda \psi, \quad \text{with} \quad \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix},
$$

has $n_R$ right-handed and $n_L$ left-handed zero modes (defined by $\lambda = 0$) such that $Q = n_L - n_R$. This can sometimes be strengthened by a vanishing theorem. Applying $\mathcal{D}$ twice to $\psi$ decouples the right- and left-hand components, and focusing on the equation for $\psi_R$, gives

$$
\left(D^2 + \frac{1}{2} \bar{\eta}_{\alpha\beta} \sigma^a F^{\alpha\beta} \right) \psi_R = 0,
$$

where $\sigma^a$ acts on the spin indices of $\psi_R$ and $F^{\mu\nu}$ acts on the color indices. For a self-dual gauge field (like an instanton), the second term in Eq. (3) is zero; and since $D^2$ is a negative definite operator, there are no normalizable right-handed zero modes, implying $Q = n_L$. Although the meron pair is not self-dual, the second term can be shown to be negative definite as well, leading to the same conclusion.

The finite action $Q = 1$ gauge field Eq. (4) has a fermion zero mode given by

$$
\psi = \begin{pmatrix} 0 \\ \phi \end{pmatrix}, \quad \phi^a = \mathcal{N} \Pi^{3/2} \epsilon^a,
$$

with normalization $\mathcal{N}$ and $\epsilon = i \sigma_2$ coupling the spin index $a$ to the color index $a$ (both of which can be either 1 or 2) in a singlet configuration. The gauge field for the meron pair with instanton caps corresponds to

$$
\Pi_i(x) = \begin{cases} 
\frac{2 \xi_i d^2}{x^2 + \xi_i^2 (x - d)^2} & \text{for regions } i = I, III, \\
\frac{d^2}{\sqrt{x^2 (x - d)^2}} & \text{region II,}
\end{cases}
$$

with $\xi_i = r/d$ and $\xi_{III} = R/d$. The normalized solution to the zero mode is then Eq. (5) with $\mathcal{N}^{-1} = 2 \pi d^2 \left[2 - \sqrt{r/R} \right]^{1/2}$. Note that the unregulated meron pair ($r \to 0, R \to \infty$) has a normalizable zero mode itself.

$$
\phi^a_\alpha(x) = \frac{d \bar{\epsilon}^a}{2 \sqrt{2 \pi} (x^2 (x - d)^2)^{3/4}}.
$$
The gauge-invariant zero mode density \( \psi^\dagger \psi(x) \) has a bridge between two merons that falls off like \( x^{-3} \) in contrast to the \( x^{-6} \) fall-off in all other directions. This behavior can be used to identify merons when analyzing their zero modes on the lattice, similar to what was done for instantons in Ref. [13].

As mentioned above, the patching of instanton caps to obtain an explicit analytic solution has unavoidable and unphysical discontinuities in the action density. Therefore, in order to study this solution further, we put the gauge field on a space-time lattice of spacing \( a \) in a box of size \( L_0 \times L_1 \times L_2 \times L_3 \). The gauge-field degrees of freedom are replaced by the usual parallel transport link variable, \( U_\mu(x) = P \exp \left[ -i \int_x^{x+ \vec{e}_\mu} A_\nu(z) dz^\nu \right] \). The exponentiated integral can be performed analytically within the instanton caps, producing arctangents. Outside of the caps, the integral is evaluated numerically by dividing each link into as many sub-links as necessary to reduce the \( O(a^3) \) path-ordering errors below machine precision.

We then calculate the action density \( S(x) \) for a meron pair with instanton caps using the improved action of Ref. [14] and the topological charge density \( Q(x) \) using products of clovers. These (open circles) are compared with the patched Ansatz results (solid line) in Fig. 2 for \( r = 9 \) and \( d = 20.8 \) (in units of the lattice spacing) on a \( 32^3 \times 40 \) lattice. We use the Arnoldi method to solve for the zero mode of this gauge configuration and hence the density \( \psi^\dagger \psi(x) \), which is also compared with the analytic result in the same figure, showing excellent agreement.

Two important features of the patched Ansatz also evident with the lattice representation are the discontinuities in the action density at the boundary of the instanton caps and vanishing of the topological charge density outside of the caps as given by Eq. (3). The discontinuity in the action density is unphysical and can be smoothed out by using a relaxation algorithm to iteratively minimize the lattice action [15]. On a sweep through the lattice, referred to as a cooling step, each link is chosen to locally minimize the action density. Since a single instanton is already a minimum of the action, this algorithm would leave the instanton unchanged (for a suitably improved lattice action). The result for the regulated meron pair after ten cooling steps is represented in Fig. 3 by the filled circles, showing the discontinuities in the action density have already been smoothed out. In Fig. 3 we also plot side-by-side the cooled \( Q(x) \) and \( \psi^\dagger \psi(x) \) in the \((z,t)\)-plane for three different meron pair separations. As the separation vanishes, the zero mode goes over into the well-known instanton zero mode result.

Like instanton-antiinstanton pairs, however, a meron pair is not a strict minimum of the action, since it has a weak attractive interaction, Eq. (5), and under repeated relaxation will coalesce to form an instanton. Nevertheless, just as it is important to sum all the quasi-stationary instanton-antiinstanton configurations to obtain essential nonperturbative physics [16], meron pairs may be expected to play an analogous role. A precise framework for including quasi-stationary meron pairs is to express the partition function as an integral over a suitable collective variable \( q \) with a constraint \( Q[A] \) as follows.
The meron pair is then a true minimum of the effective action in Eq. (13), allowing for a semiclassical treatment of $Z_q$. We choose $Q[A]$ to be the quadrupole moment $3z^2 - x^2 - y^2 - t^2$ of the topological charge.

The criterion for a good collective variable $q$ of the system is that the gradient in the direction of $q$ is small compared to the curvature associated with all the quantum fluctuations. In this case, we have an adiabatic limit in which relaxation of the unconstrained meron pair slowly evolves through a sequence of quasi-stationary solutions, each of which is close to a corresponding stationary constrained solution. Detailed comparison of the quasi-stationary and constrained solutions shows this adiabatic limit is well satisfied. Therefore, we will present here the action of a meron pair as it freely coalesces into an instanton. To compare with the patched Ansatz, we need the meron separation $d$ and radii $r$ and $R$ of the lattice configuration. We find these by first measuring the separation of the two maxima in the action density $\Delta \equiv |M_I - M_{III}|$ (using cubic splines) and their values (which are the same in the symmetric case, denoted by $S_{\text{max}} = 48/g^2 w^4$), and then using Eq. (6) to give

$$d^2 = \Delta^2 + 4w^2, \quad \{r, R\} = \frac{2wd}{d^2 + \Delta}.$$  

In Fig. 4, we plot the total action of a meron pair as a function of $d/r$ (solid line), which for the analytic case is given by Eq. (13) with $R = d^2/r$. Also shown are cooling trajectories for four lattice configurations with different initial meron pair separations. Each case starts with a patched Ansatz of a given separation and the lattice action agrees with that of the analytic Ansatz.

The first few cooling steps primarily decrease the action without changing the collective variable (which is now effectively $d/r$), as observed above in Fig. 2. Further cooling gradually decreases the collective variable, tracing out a new logarithmic curve for the total action (dashed line in Fig. 3), which is about 0.25 lower than the analytic case. This curve represents the total action of the smooth adiabatic or constrained lattice meron pair solutions. The essential property of merons that could allow them to dominate the path integral and confine color charge is the logarithmic interaction Eq. (5) which is weak enough to be dominated by the meron entropy. Hence, the physical result from Fig. 3 is the fact that smooth adiabatic or constrained lattice meron pair solutions clearly exhibit this logarithmic behavior.

In summary, we have found stationary meron pair solutions, shown that they have a logarithmic interaction, and shown that they have characteristic fermion zero modes localized about the individual merons. Hence the presence and role of merons in numerical evaluations of the QCD path integral should be investigated, and the study of fermionic zero modes and cooling of gauge configurations provide possible tools to do so.

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