Payoff Dynamic Models and Evolutionary Dynamic Models: Feedback and Convergence to Equilibria

Shinkyu Park, Nuno C. Martins, and Jeff S. Shamma

Abstract—We consider that at every instant each member of a population, which we refer to as an agent, selects one strategy out of a finite set. The agents are nondescript, and their strategy choices are described by the so-called population state vector, whose entries are the portions of the population selecting each strategy. Likewise, each entry constituting the so-called payoff vector is the reward attributed to a strategy. We consider that a general finite-dimensional nonlinear dynamical system, denoted as payoff dynamical model (PDM), describes a mechanism that determines the payoff as a causal map of the population state. A bounded-rationality protocol, inspired primarily on evolutionary biology principles, governs how each agent revises its strategy repeatedly based on complete or partial knowledge of the population state and payoff. The population is protocol-homogeneous but is otherwise strategy-heterogeneous considering that the agents are allowed to select distinct strategies concurrently. A stochastic mechanism determines the instants when agents revise their strategies, but we consider that the population is large enough that, with high probability, the population state can be approximated with arbitrary accuracy uniformly over any finite horizon by a so-called (deterministic) mean population state. We propose an approach that takes advantage of passivity principles to obtain sufficient conditions determining, for a given protocol and PDM, when the mean population state is guaranteed to converge to a meaningful set of equilibria, which could be either an appropriately defined extension of Nash’s for the PDM or a perturbed version of it. By generalizing and unifying previous work, our framework also provides a foundation for future work. We specialize our results for well-known protocols and a class of PDM that includes as particular cases the payoff mechanisms proposed in previous work to model learning, inertia, and anticipation.

Index Terms—Population games, game theory, passivity, Nash equilibrium, evolutionary dynamics, Lyapunov stability, optimization

I. INTRODUCTION

We report on a theoretical framework and analytic methods to determine the time-evolution of the strategy choices, out of a finite set with \( n \) elements, by the members of a large population in response to a payoff mechanism. Although we assume that there is a single population, the techniques and methods put forth in this article can be readily adapted to the multi-population case.

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We adopt the evolutionary dynamic paradigm well-documented in [1]-[3] according to which the members of the population, which we call agents, repeatedly revise their strategy choices according to a given mechanism modeled by a so-called revision protocol (protocol for short). The population is protocol-homogeneous because all of its agents follow the same revision mechanism, but is otherwise strategy-heterogeneous considering that the agents are allowed to concurrently select distinct strategies. Although mixed strategies are not allowed as each agent chooses exactly one strategy at every instant, the protocol may be probabilistic when the revision mechanism involves randomization. The identity of each agent is unimportant, and consequently all the information that is relevant, at any given instant, is encapsulated in the so-called population state vector whose entries quantify the portions of the population adopting each strategy. Henceforth, we refer to the set of all possible population states, or equivalently the state space, as the strategy profile set. Hence, every strategy profile is a vector with \( n \) nonnegative entries adding up to the so-called population mass \( n \).

In most related prior work, a so-called population game [4] determines at any given instant the \( n \)-dimensional payoff vector, which quantifies the reward associated with each strategy, as a function of the population state. A population game may represent a pricing scheme that is implemented by a coordinator, or it may model a payoff mechanism resulting from the interactions among the agents and with the environment. Examples include congestion population games for traffic assignment [5], [6] and others, such as with wars of attrition [7], that are obtained by matching of a stage population game. The set of Nash equilibria of a population game can be functionally defined in the usual way, and may be interpreted in the mass-action sense discussed in [8], which was originally suggested in [9].

A. Population games, protocols and stability

Our work focuses on the protocol classes surveyed in [2], [3], which are inspired, to a great extent, on elementary bounded rationality mechanisms of evolutionary biology [10]. As is explained in [2] Section 2.3] and [11], the protocol models how much each agent knows about the payoff vector and the population state, and how it uses the available information to revise its strategy. Consequently, under the widely-used assumption that agents revise their strategies at times determined by a Poisson process, the resulting population state can be viewed as the Markov process associated with the system formed by the feedback interconnection between
the evolutionary dynamics specified by the protocol and the population game. The analysis in [12], which shows that the evolutionarily stable strategies (ESS) characterized in [13] are local attractors for the expectation of the population state, motivated the subsequent early work summarized in [4] exploring a Markovian approach to establish in a probabilistic sense that in the long run the population state remains near, and in some cases converges to, certain Nash equilibria. Furthermore, the concept of stochastically stable set was defined and characterized in [14] to account for the effect of persistent stochastic perturbations and [15] extended these notions to examine equilibrium selection.

B. Mean population state and deterministic payoff

In this article, we consider the case in which the population size is large enough that Khinchin’s law of large numbers can be invoked to conclude that realizations of the population state at each instant will remain close to its expected value with high probability. This justifies our decision to focus on the so-called mean population state, which akin to the approach in [12] represents the expectation of the population state that can be obtained as the deterministic solution of a system of ordinary differential equations.

Likewise, we restrict our analysis to deterministic payoffs that are obtained in terms of the mean population state according to a soon to be described payoff dynamical model (PDM) of which population games are a particular case. As we argue precisely in Section III, the deterministic payoff will approximate the payoff of an associated finite population scenario with increasing fidelity as the size of the population tends to infinity.

1) Nash equilibria of population games and long term behavior of the mean population state: Before we outline our main contributions, we proceed to surveying a few existing concepts and convergence results for the particular case in which a population game governs the deterministic payoff. In this context, the set of ordinary differential equations governing the mean population state, and consequently also the payoff, is commonly referred to as mean dynamic.

As is discussed in [16] for a general class of protocols and population games, and further refined in [17], results in [18] guarantee that the mean population state approximates the population state with arbitrary accuracy uniformly over any given finite time horizon with probability approaching one as the number of agents tends to infinity, which highlights the importance of analyzing the long-term evolution of the mean population state and characterizing the associated globally attracting sets. Notably, the time horizon in the uniform approximation results reported in [16], [17] can be selected large enough to guarantee that with high probability the population state of a sufficiently large population will tend to the smallest globally attractive set. As noted in [4], this conclusion has special relevance when a subset of Nash equilibria is globally attractive. It demonstrates the interesting fact that, even though the agents follow simple bounded rationality protocols that do not require knowledge of the structure of the population game, the population is still capable of self-organizing in a way that the population state tends to a subset of Nash equilibria. This not only reinforces the importance of the mass-action interpretation of Nash equilibria, but it also justifies regarding the globally attractive subsets of Nash equilibria as reliable predictors of the long term behavior of the population.

In many cases of interest it is possible to construct a Lyapunov function [19] establishing that the entire set of Nash equilibria is globally asymptotically stable and hence also attractive. The historical perspective and analysis in [20] popularized and fostered very significant work that uses the Lyapunov approach in conjunction with the positive correlation and Nash stationarity concepts, which are particularly insightful in the context of evolutionary dynamics, to establish global attractivity or, in some cases, global asymptotic stability of the Nash equilibrium set for widely-used protocols and population game classes, such as potential [21], [22] and contractive [1] games [24]. Of key importance in [21], [24] is the fact that, for a given constant payoff, the mean population state governed by a Nash stationary protocol is also constant if and only if it is a best response, which establishes an equivalence between the rest points of the mean dynamic and the Nash equilibria of the population game. The concepts of Nash stationarity and positive correlation are also explained in detail in [2] Chapter 5, where they are viewed as properties of the deterministic mean dynamic associated with each protocol. Instead, here we adopt the convention, which will be useful later on, of viewing Nash stationarity and positive correlation as properties of the protocol.

The Theorems [2] 12.B.3 and [2] 12.B.5, which are derived from work in [23] and [26], offer yet another rationale associating a globally asymptotically stable set of equilibria of the mean dynamic, when one exists, with the long term behavior of the population state for large populations. More specifically, these results ascertain under unrestrictive conditions that the measure, with respect to the stationary distribution of the population state, tends to one within any open set containing a globally asymptotically stable set as the population size tends to infinity. This implies that as the population grows, the stationary distribution of the population state tends to concentrate around the smallest globally asymptotically stable set.

C. Outline of main contributions

From a dynamical systems theory perspective [19], [27], a population game would be qualified as memoryless because it acts as an instantaneous map from the mean population state to the deterministic payoff vector. Consequently, population games cannot capture dynamics in the payoff mechanism such as when there is inertia or anticipation effects in the agents’ perception of the reward for each strategy.

The following are the main contributions of this article:

(i) In Section II-A, we propose a class of payoff dynamical models (PDM) that includes as particular cases the dy-

1This class of population games is also referred to as “stable” in [23]. Following the recent naming convention in [1], we prefer to use the qualifier “contractive” to avoid confusion with other concepts of stability used throughout the article. Similarly, we will use “strictly contractive” to qualify a population game that would be “strictly stable” according to [1].
nically modified payoffs analyzed in [28], of which the so-called smoothed and anticipatory payoff modifications modeling inertia and anticipatory effects, respectively, are examples. According to our formulation, each PDM is associated with a so-called stationary population game that determines the deterministic payoff in the stationary regime characterized by when the mean population state converges as time tends to infinity.

(ii) Given a PDM and a protocol satisfying Nash stationarity, we obtain sufficient conditions similar to (i) for perturbed best response protocols, for which Nash stationary does not hold. In this case, the conditions determine when a perturbed equilibrium set, which can be viewed as an approximation of Nash’s, is either globally asymptotically stable or globally attractive. We also specialize our results for the classes of integrable excess payoff target and impartial pairwise comparison protocols, of which the Brown-von Neumann-Nash and Smith protocols are, respectively, well-known examples.

(iii) In Section VI we obtain sufficient conditions similar to those in (ii) for perturbed best response protocols, for which Nash stationary does not hold. In this case, the conditions determine when a perturbed equilibrium set, which can be viewed as an approximation of Nash’s, is either globally asymptotically stable or globally attractive.

(iv) In Section VII we determine the parameters of a PDM class, whereof the smoothed and anticipatory dynamically modified payoff considered in [28] are particular cases, under which the sufficient conditions outlined (i) and (ii) are satisfied. This includes cases in which the stationary population game is either affine, but unlike what is assumed in [28] may not be strictly contractive, or has a strictly concave potential.

1) Outline of our technical approach: We establish the results in (i) - (iv) by recognizing that in our paradigm the mean population state and the deterministic payoff are governed by the mean closed loop model, which is precisely defined in Section II-C as the feedback interconnection between the PDM and the so-called evolutionary dynamical model (EDM) that captures the strategy revision dynamics as specified by the protocol.

As we discussed in Section I-B1 for the case in which the payoff is determined by a population game, Lyapunov functions [19] are often used to establish convergence towards certain equilibria of the mean dynamic. The typical argument follows techniques derived from the classical principles in [29]. However, these techniques are not immediately applicable to our formulation because the deterministic payoff is no longer a memoryless function of the mean population state, which can no longer be characterized by the mean dynamic. Instead we resort to a well-known compositional approach [30] rooted on passivity principles with which convergence properties for the mean closed loop model can be ascertained by separately establishing certain passivity properties of the EDM and the PDM. More specifically, the search for a Lyapunov function, which was central for establishing convergence to equilibria of the mean dynamic, is superseded in our context by the construction of certain types of storage functions for the EDM and the PDM.

Although the aforementioned passivity-based methodology, including the construction of storage functions for the individual blocks of a feedback loop to ascertain its stability [31], has been widely used [32], we propose our own form of passivity inspired on the approach in [28], which is both equilibrium independent and is compatible with the EDM and PDM classes we are interested in. In Section IV we introduce the passivity concepts used throughout the article. It includes in Section IV-D a comparison with the related concepts of equilibrium independent passivity [30], differential passivity [33], [34] and incremental passivity [35].

Most of the key results presented here date back to [36], and preliminary versions of the results in this article were reported in [37], [38].

D. Structure of the article

Besides the introduction, the article has six sections briefly described as follows:

- Section II starts by describing key preliminary concepts, including the detailed definitions of PDM, EDM and the mean closed loop model. It concludes with an overarching problem formulation that motivates the results derived throughout the paper.
- Section III justifies, via an explicit finite population construction, that the mean population state and deterministic payoff resulting from solutions of the mean closed loop model indeed approximate the population state and the payoff with arbitrary fidelity over any finite time horizon, as the number of agents tends to infinity.
- Section IV introduces the main concepts used throughout the article to characterize relevant passivity properties for any given PDM and EDM. More specifically, we define $\delta$-passivity allowing a possible surplus and $\delta$-antipassivity admitting a possible deficit for an EDM and PDM, respectively. A weak version of $\delta$-antipassivity is also proposed and invoked in Lemma 1 to ascertain sufficient conditions that will be used in Sections V and VI to establish global attractivity results.
- Section V presents a detailed analysis for the case in which the EDM satisfies Nash stationarity, leading to results establishing when the set of Nash equilibria of the stationary game associated with a given PDM is globally attractive or globally asymptotically stable. We also specialize our results for the integrable excess payoff target (EPT) and impartial pairwise comparison (IPC) protocols.
- Section VI establishes for perturbed best response (PBR) protocols results that are analogous to Section V. In addition, it establishes a fundamental trade-off according to which a perturbed equilibria set is globally attractive even when the PDM has a $\delta$-antipassivity deficit provided that it is no larger than the $\delta$-passivity surplus of the EDM. This trade-off is not possible in the context of Section V because, as is shown in [38] Corollary IV.3, any EDM stemming from an EPT or IPC protocol never have a positive $\delta$-passivity surplus.
- Section VII proposes the so-called smoothing-anticipatory PDM class and characterizes its $\delta$-antipassivity properties. It also includes examples
illustrating how the δ-antipassive properties of the aforementioned PDM can be used in conjunction with the results in Sections IV or V to establish global stability of the Nash equilibria of the stationary game or a perturbed version, respectively.

II. PRELIMINARY CONCEPTS AND PROBLEM FORMULATION

Before we specify the class of problems considered throughout this article, we proceed to defining and discussing basic properties of the sets and maps needed for our analysis.

In Section II-A we will propose a class of payoff dynamic models (PDM) comprising a state that evolves in (continuous) nonnegative real-valued time $T = \mathbb{R}_+$. In our single-population framework, the strategy profile set is specified as follows:

$$\mathcal{X} \overset{\text{def}}{=} \left\{ z \in \mathbb{R}_+^n \mid \sum_{j=1}^n z_j = m \right\}$$

where $n$ is the number of strategies and $m$ is the population mass. Every element of $\mathcal{X}$ is interpreted as a n-tuple representing the portions of the population adopting each strategy.

For any vector $r \in \mathbb{R}^n$, we specify the following norm:

$$\|r\| \overset{\text{def}}{=} \max_{1 \leq i \leq n} |r_i|$$

For any trajectory $v : T \to \mathbb{R}^n$, we adopt the following norm:

$$\|v\| \overset{\text{def}}{=} \sup_{t \in T} \|v(t)\|$$

We restrict our analysis to trajectories in the following set:

$$\mathcal{R}^n \overset{\text{def}}{=} \{\text{All differentiable maps from } T \to \mathbb{R}^n\}$$

Definition 1. We reserve $x : T \to \mathcal{X}$ to denote the so-called mean population state trajectory. At a particular time $t$ the mean population state is represented by $x(t)$. The set of admissible mean population state trajectories $\mathcal{X}$ is defined as follows:

$$\mathcal{X} \overset{\text{def}}{=} \left\{ x \in \mathcal{R}^n \mid \|\dot{x}\| < \infty \text{ and } x(\tau) \in \mathcal{X}, \tau \in T \right\}$$

Definition 2. We use $p : T \to \mathbb{R}^n$ to denote the so-called deterministic payoff trajectory. The deterministic payoff at time $t$ is represented by $p(t)$, whose entries ascribe a reward to each strategy. The set of possible deterministic payoff trajectories $\mathcal{P}$ is specified as:

$$\mathcal{P} \overset{\text{def}}{=} \{ p \in \mathcal{R}^n \mid \|p\| < \infty \text{ and } \|\dot{p}\| < \infty \}$$

In Section III, we describe a scenario in which the mean population state and deterministic payoff are high fidelity approximations of the population state and payoff of a finite population, respectively, provided that the number of agents is sufficiently large.

We now proceed with defining the main dynamic models and concepts needed to complete the description of our framework.

A. Payoff Dynamic Model (PDM)

A population game [2] is specified by a continuous map $\mathcal{F} : \mathcal{X} \to \mathbb{R}^n$. It determines the deterministic payoff trajectory as a memoryless function of the mean population state according to $p(t) = \mathcal{F}(x(t))$, for $t \geq 0$.

In contrast with the conventional formulation, we consider that a payoff dynamic model (PDM), as defined below, governs the deterministic payoff trajectory in terms of the mean population state trajectory.

Definition 3. (PDM) We consider that a payoff dynamic model (PDM) is defined as follows:

$$\begin{align*}
\dot{q}(t) &= \mathcal{G}(q(t), u(t)) \\
p(t) &= \mathcal{H}(q(t), u(t)),
\end{align*}$$

where $\mathcal{G} : \mathbb{R}^n \times \mathcal{X} \to \mathbb{R}^n$ is Lipschitz continuous and $\mathcal{H} : \mathbb{R}^n \times \mathcal{X} \to \mathbb{R}^n$ is continuously differentiable and Lipschitz continuous. The state-space representation (7) specifies a map from the input $u$ and the initial condition $q(0)$ to the state $q$ and output $p$. Henceforth, we identify a PDM by the pair $(\mathcal{G}, \mathcal{H})$ that specifies it.

The PDM is a finite-dimensional dynamical system that, as we will see in Section II-C, will be used in a closed loop configuration to generate the output $p$ in terms of the input $u$, which is set to $x$.

In addition, we require that the PDM satisfies the following assumption for reasons that will become clear in Proposition 1.

Assumption 1. (PDM Boundedness) In this article we consider that every PDM is a bounded map in the following sense:

$$\sup_{u \in \mathcal{X}} \|q\| < \infty, \quad q(0) \in \mathbb{R}^n$$

where $q$ is the solution of (1) for input $u$ and initial condition $q(0)$.

Remark 1. Boundedness of a PDM guarantees that for each initial condition $q(0)$ in $\mathbb{R}^n$ there is a finite upper-bound for $\|q\|$ that holds for all inputs $u$ in $\mathcal{X}$. In addition, because $\mathcal{G}$ is Lipschitz continuous, $\mathcal{H}$ is continuously differentiable, and the elements of $\mathcal{X}$ are bounded, we infer that if the PDM is bounded then $p$, $\dot{q}$, and $\dot{p}$ are also bounded, for each initial condition $q(0)$.

We also assume that every PDM is associated with a so-called stationary population game according to the following assumption.

Assumption 2. (Stationary Population Game of a PDM) In this article, we assume that for every PDM there is a continuous map $\mathcal{F} : \mathcal{X} \to \mathbb{R}^n$ for which the following implication holds for any initial condition $q(0)$ in $\mathbb{R}^n$:

$$\lim_{t \to \infty} \|u(t)\| = 0 \implies \lim_{t \to \infty} \|p(t) - \mathcal{F}(u(t))\| = 0, \quad u \in \mathcal{X}$$

in addition, we require that the following set

$$\{(z, s) \in \mathcal{X} \times \mathbb{R}^n \mid \mathcal{H}(z, s) = \mathcal{F}(z)\}$$

is either a compact subset of or the entire set $\mathcal{X} \times \mathbb{R}^n$. We refer to $\mathcal{F}$ as the stationary population game of the PDM.
Given a PDM, the set of Nash equilibria of its stationary population game is defined as follows:

$$\text{NE}(\mathcal{F}) \equiv \{ z \in \mathbb{X} \mid z^T \mathcal{F}(z) \geq \bar{z}^T \mathcal{F}(z), \bar{z} \in \mathbb{X} \}$$

Although our results are valid for any PDM satisfying the assumptions above, in Sections [V-B1] and [VII] we define and analyze two cases of interest.

B. Evolutionary Dynamic Model (EDM)

Our model for how the state of the population evolves over time is specified as follows.

**Definition 4. (EDM)** The evolutionary dynamic of the population is specified by an evolutionary dynamic model (EDM) of the following form:

$$\dot{z}(t) = \mathcal{V}(z(t), w(t)), \quad x(0) \in \mathbb{X}, \quad t \geq 0, \quad w \in \mathfrak{X} \tag{4}$$

where \( \mathcal{V} : \mathbb{X} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a given Lipschitz continuous map that also satisfies the following constraint:

$$\mathcal{V}(z, r) \in \mathcal{TX}(z), \quad z \in \mathbb{X}, \quad r \in \mathbb{R}^n \tag{5}$$

where \( \mathcal{TX}(z) \) is the cone of viable displacement from \( z \) specified as:

$$\mathcal{TX}(z) \equiv \{ \alpha (\bar{z} - z) \mid \bar{z} \in \mathbb{X} \text{ and } \alpha \geq 0 \}, \quad z \in \mathbb{X}$$

Here, \( z(t) \) is the state, which is also the output of the model, \( x(0) \) is the initial condition, and \( w \) is the input that is typically taken to be a given deterministic payoff trajectory.

Inspired by the notation in [2], we also represent the subspace tangent to \( X \) as follows:

$$\mathcal{TX} \equiv \left\{ \bar{z} \in \mathbb{R}^n \mid \sum_{i=1}^{n} \bar{z}_i = 0 \right\}$$

**Remark 2. (“EDM” versus “mean dynamic”)** We have decided to use “evolutionary dynamic model”, or EDM, to refer to [4] to distinguish it from the concept of “mean dynamic” as defined in [2], which refers to a given map from the set of population games to a certain class of differential equations. The latter is not adequate for our formulation in which we allow for a dynamic population game described by its own PDM.

In this article, we will consider a subclass of EDM that stems from the following class of revision protocols.

**Definition 5. (Protocol)** A revision protocol\(^2\) or protocol for short, is specified by a Lipschitz continuous map \( \mathcal{T} : \mathbb{R}^n \times \mathbb{X} \rightarrow \mathbb{R}^{n \times n} \). It models how members of the population revise their strategies in response to the population state and payoff. In particular, it determines the EDM as follows:

$$\nu_i(z, r) \equiv \sum_{j=1}^{n} z_j \mathcal{T}_{ji}(r, z) - \left( \sum_{j=1}^{n} \mathcal{T}_{ij}(r, z) \right) z_i, \quad \text{where} \quad z \in \mathbb{X}, \quad r \in \mathbb{R}^n, \quad 1 \leq i \leq n \tag{6}$$

\(^2\)Henceforth, in order to streamline our discussion, we will refer to a revision protocol simply as protocol.

In Section [I-C] we model the feedback mechanism that is established when the PDM is actually played by a population characterized by a given protocol and corresponding EDM. We also show in Section [III] how the mean population state approximates the population state of a population with arbitrary accuracy as the number of agents tends to infinity.

Most of our basic notation is summarized in Tables I and II while most acronyms and notation associated with EDM and PDM, some of which will be introduced later on, are summarized in Table III.

**C. Solutions of the Mean Closed Loop Model**

We wish to study the trajectories of the deterministic payoff and mean population state when the PDM is interconnected in feedback with an EDM. That is to say that we consider that the input of the PDM is the mean population state and, in turn, the resulting deterministic payoff is the input of the

\[
\text{PDM: payoff dynamic model.}
\]

\[
\text{q: state trajectory of the PDM.}
\]

\[
\text{q*: map specifying the differential equation for q.}
\]

\[
\text{H: map specifying p based on q and input to the PDM.}
\]

\[
\text{F: stationary population game of a PDM.}
\]

\[
\text{NE(F): set of Nash equilibria of F.}
\]

\[
\text{EDM: evolutionary dynamic model.}
\]

\[
\text{V: map specifying the differential equation modeling the EDM.}
\]

\[
\text{EPT: EDM associated with an excess payoff target protocol.}
\]

\[
\text{IPC: EDM associated with impartial pairwise comparison protocol.}
\]

\[
\text{PBR: EDM associated with perturbed best response protocol.}
\]

\[
\text{Logit: particular case of PBR EDM.}
\]

\[
\text{TABLE III}
\]

**PARTIAL LIST OF MOST ACRONYMS AND NOTATION ASSOCIATED WITH PDM AND EDM.**

| \text{Acronym} | \text{Definition} |
|----------------|------------------|
| \text{PDM}    | payoff dynamic model. |
| \text{q}      | state trajectory of the PDM. |
| \text{q*}     | map specifying the differential equation for q. |
| \text{H}      | map specifying p based on q and input to the PDM. |
| \text{F}      | stationary population game of a PDM. |
| \text{NE(F)}  | set of Nash equilibria of F. |
| \text{EDM}    | evolutionary dynamic model. |
| \text{V}      | map specifying the differential equation modeling the EDM. |
| \text{EPT}    | EDM associated with an excess payoff target protocol. |
| \text{IPC}    | EDM associated with impartial pairwise comparison protocol. |
| \text{PBR}    | EDM associated with perturbed best response protocol. |
| \text{Logit}  | particular case of PBR EDM. |
EDM (see Fig. 1). A state-space closed loop model of this feedback interconnection is constructed as follows.

**Definition 6. (Mean Closed Loop Model)** Given a PDM and an EDM, the associated mean closed loop model is obtained by using $x$ as an input to (7) and $p$ as an input to (4), or equivalently, setting $u = x$ and $w = p$. The following is the state space representation of the mean closed loop model:

$$
\begin{align*}
\dot{q}(t) &= G(q(t), x(t)) \\
\dot{x}(t) &= \mathcal{V}^H(q(t), x(t)), \quad t \geq 0, \; (q(0), x(0)) \in \mathbb{R}^n \times \mathbb{X}, \\
p(t) &= \mathcal{H}(q(t), x(t)) \\
\end{align*}
$$

where $\mathcal{V}^H : \mathbb{R}^n \times \mathbb{X} \rightarrow \mathbb{R}^n$ is defined as:

$$
\mathcal{V}^H(s, z) \overset{\text{def}}{=} \mathcal{V}(z, \mathcal{H}(s, z)), \quad s \in \mathbb{R}^n, z \in \mathbb{X}
$$

Notice that the state of the mean closed loop model is $(q(t), x(t))$. The mean closed loop model has no input, and the state trajectory $(q, x)$ is the solution of the associated initial value problem in response to the preselected initial condition $(q(0), x(0))$.

In Section III, we show that solutions of (7) approximate uniformly, over any finite time horizon, with arbitrary accuracy the population state of a finite population subjected to the same protocol and PDM with probability tending to one as the number of agents tends to infinity.

It is clear from our PDM and EDM definitions that $G$ and $\mathcal{V}^H$ are Lipschitz continuous, which, from Picard-Lindelöf Theorem, guarantees the existence and uniqueness of solutions of the initial value problem in (7). Also the boundedness of the PDM from Assumption 1 and Remark 1 guarantees that $p$ and $x$ remain in $\mathbb{P}$ and $\mathbb{X}$, respectively. We assert these facts in the following proposition, which is an immediate consequence of the Picard-Lindelöf Theorem and Remark 1.

**Proposition 1.** Consider that a mean closed loop model is formed by an EDM and a PDM. For each initial condition $(q(0), x(0))$ in $\mathbb{R}^n \times \mathbb{X}$ there is a unique solution $(q, x)$. The solution is such that $p$ and $x$ are in $\mathbb{P}$ and $\mathbb{X}$, respectively.

Furthermore, under the conditions of Proposition 1, an immediate adaptation of the argument in [2] Theorem 4.A.3 can be used to show that the map $\phi_t : (q(0), x(0)) \mapsto (q(t), x(t))$ is Lipschitz continuous for any $t$ in $\mathbb{T}$.

**D. Background on the Approach of [28]**

As we discussed earlier, standard population games are specified by a continuous map $\mathcal{F} : \mathbb{X} \rightarrow \mathbb{R}^n$ that determines the deterministic payoff at time $t$ according to $p(t) = \mathcal{F}(x(t))$. The map is memoryless because the (deterministic) payoff at time $t$ depends solely on the (mean) population state also at time $t$.

An important class of population games is defined below:

**Definition 7. (Contractive Population Game)** A given population game $\mathcal{F} : \mathbb{X} \rightarrow \mathbb{R}^n$ is qualified as contractive if it satisfies the following condition:

$$(z - \bar{z})^T (\mathcal{F}(z) - \mathcal{F}(\bar{z})) \leq 0, \quad z, \bar{z} \in \mathbb{X}$$

The population game is said to be strictly contractive if the inequality above is strict for $z \neq \bar{z}$.

As shown in [24], when a contractive population game $\mathcal{F}$ is played according to an EDM derived from a target protocol that is integrable, or equivalently has a revision potential, the trajectory of the mean population state is guaranteed to have global convergence properties, which are established using Lyapunov-like functions that are constructed from the revision potentials.

Work in [28] proposed a framework to extend the contractive population game methodology to certain dynamically modified payoffs, which in our formulation could be modeled as PDMs. The underlying idea in [28] is to use techniques related to $\delta$-passivity for establishing sufficient conditions that determine when the time derivative of the state trajectory of (7) is $\mathcal{L}_2$ stable.

While, in [24], contractivity was a requirement for convergence in the case of population games, the approach by [28] for dynamically modified payoffs requires that the PDM satisfies an inequality defining a condition called $\delta$-antipassivity. Interestingly, [28] shows that $\delta$-antipassivity can be rightly viewed as a generalization of contractivity because any contractive population game is also $\delta$-antipassive. Notably, the analysis in [28] proves that when the PDM is $\delta$-antipassive and EDM results from an integrable target protocol then the revision potential can be used to construct a storage function to establish $\mathcal{L}_2$ stability of the solutions of (7). In general, when such a storage function can be constructed, the EDM is shown to be $\delta$-passive, or equivalently, it satisfies an inequality that is the antisymmetric of the one used to define $\delta$-antipassivity. We will revisit these concepts in Section IV.

**E. Problem Statement and Comparison with [28] and [24]**

The analysis in [28] investigates stability of the derivative of the mean population state in the $\mathcal{L}_2$ sense. Remarkably, stability guarantees of the aforementioned sort do not imply convergence of the mean population state towards any specific equilibria as was done, for instance, in [24]. In particular, we
cannot use existing versions of Barbalat’s lemma \(^{[39]}\) because they presume constraints on the second derivative of the mean population state that may not be verified in our framework in which protocols may not be differentiable. In addition, even if some version of Barbalat’s lemma could be constructed to determine when the derivative of the mean population state tends to zero, it would not suffice to ascertain whether Nash equilibria set, or perturbed versions of it, are globally attractive or globally asymptotically stable.

**Problem 1. (Main Problem)** Consider that a PDM is given and that \( \mathcal{F} \) is its stationary population game. For the EDM classes associated with the protocol classes considered in \([24]\), we seek to determine the conditions on the PDM to guarantee that a set of equilibria \( \mathbb{E}(\mathcal{F}) \), which in this article is either \( \mathbb{NE}(\mathcal{F}) \) or a perturbed version of it, is globally attractive or a perturbed version of it, is globally attractive and \( \mathcal{P}(\mathcal{F}) \) is stable. The stability of a set \( \mathcal{E} \) is either stable, stable, or globally attractive or globally asymptotically stable.

### Definition 8. (Stability Concepts)
Let an EDM and a PDM be given. The stability of a set \( \mathbb{E}(\mathcal{F}) \), which in our formulation is either \( \mathbb{NE}(\mathcal{F}) \) or a perturbed version of it, is classified as follows:

1. **(Global Attractiveness)** The set \( \mathbb{E}(\mathcal{F}) \) is globally attractive if for every initial condition \( (q(0), x(0)) \) in \( \mathbb{R}^n \times \mathbb{X} \), the solution trajectory \( (q, x) \) of the mean closed loop model \(^{[2]}\) is such that the following limits hold:

\[
\lim_{t \to \infty} \left( \inf_{z \in \mathbb{E}(\mathcal{F})} \| x(t) - z \| \right) = 0 \quad (8a) \\
\lim_{t \to \infty} \| p(t) - \mathbb{F}(x(t)) \| = 0 \quad (8b)
\]

2. **(Lyapunov Stability)** The set \( \mathbb{E}(\mathcal{F}) \) is Lyapunov stable if for every open set \( \mathcal{O} \) in \( \mathbb{R}^n \) that contains \( \mathbb{A} \) defined as:

\[
\mathbb{A} = \{ (q, z) \in \mathbb{R}^n \times \mathbb{X} | z \in \mathbb{E}(\mathcal{F}) \text{ and } \mathcal{H}(q, z) = \mathbb{F}(z) \}
\]

there is another open set \( \mathcal{O}' \) that contains \( \mathbb{A} \) for which the following holds:

\[
(q(0), x(0)) \in \mathcal{O}' \cap (\mathbb{R}^n \times \mathbb{X}) \implies (q(t), x(t)) \in \mathcal{O}, \ t \geq 0 \quad (9)
\]

3. **(Globally Asymptotically Stable)** The set \( \mathbb{E}(\mathcal{F}) \) is globally asymptotically stable if it is globally attractive and Lyapunov stable.

This definition stated above adapts to our framework analogous concepts used in \([24]\).

4. **Our Approach to Solving Problem [\(1\)]** In addition to leveraging the use of and generalizing the passivity techniques proposed in \([28]\) to solve Problem [\(1\)] in Section [\(V\)] we introduce new key concepts that ultimately lead to Lemma [\(1\)]. The latter is a central convergence result that will allow us to propose solutions to Problem [\(1\)] for the two important EDM classes studied in Sections [\(V\)] and [\(VI\)]. It will follow from our analysis in these sections that, in the particular case when the PDM is a memoryless map specifying a population game, our sufficient conditions for global attractiveness and global asymptotic stability are analogous to those in \([24]\).

In Section [\(VII\)] we also extend and adapt to our context an important PDM class proposed in \([28]\). We also provide extensions of \([28\) Theorem 4.6].

### III. Relating The Mean Closed Loop Model With The Population State and Payoff Of Large Populations

The framework in \([16]\) uses classical results \([18]\) to show that, as the size of the population tends to infinity, the solutions of the mean dynamic approximate with arbitrary accuracy, in the sense of \([16\) Theorem 4.1], the realizations of the population state. Furthermore, such an analysis, which can also be found in \([2\) Chapter 10], is further refined in \([17\) Lemma 1].

#### A. Finite population framework

Likewise, in this section, we proceed to outlining the construction of a finite-population framework whose population state and payoff can be approximated with arbitrary accuracy uniformly over any given finite time interval by the solution of the mean closed loop model \(^{[7]}\) with probability approaching one as the population size tends to infinity. Our approach is to modify the framework in \([16\), \([17]\) and \([2\) Chapter 10] to comply with ours in which a PDM governs the deterministic payoff.

Inspired by the construction in \([16]\), we consider that the population state of a single population with \(N\) agents is represented by a right-continuous jump-process \(X^N\) taking values in \(\mathbb{X}^N\) defined as:

\[
\mathbb{X}^N = \frac{m}{N} \mathbb{R}^n \cap \mathbb{X}, \ N \geq 1, \ m > 0
\]

A Poisson process with unit-rate governs the jump-times, which represent the instants at which some agent is allowed to revise its strategy according to a protocol \(\mathcal{T}\). Given a pair \((G, \mathcal{H})\) satisfying the conditions of Definition [\(3\] the payoff vector at time \(\tau\) is represented by \(P^N(\tau)\), and is obtained in terms of \(X^N\) and a pre-determined initial condition \(Q^N(0)\) in \(\mathbb{R}^n\) as the unique solution of:

\[
\dot{Q}^N(t) = G(Q^N(t), X^N(t)) \\
P^N(t) = H(Q^N(t), X^N(t)), \ t \geq 0
\]

In contrast to \([16\), \([17]\), here we assume that \(\mathbb{F}(X^N(\tau))\) is replaced with \(P^N(\tau)\) in the implementation of the protocol that models the strategy revision process. More specifically, \(X^N\) is governed by the following probability transition law:

\[
P\left( X^N_j(\tau) = z_i \bigg| \mathbb{T} \right) = z_i T_{ij}(r, z), \ i, j \in \{1, \ldots, n\}
\]

\[
P\left( X^N(\tau) = z \bigg| \mathbb{T} \right) = 1 - \sum_{i=1}^{n} \sum_{j \neq i} z_i T_{ij}(r, z)
\]

\(^{3}\)See also \([2\) Observation 10.1.2].
where \( \Sigma \) is the event that there are consecutive jumps at times \( \tau^- \) and \( \tau \), and \( (P^N(\tau^-), X^N(\tau^-)) = (r, z) \) holds for pre-specified \( r \) and \( z \) in \( \mathbb{X}^N \) and \( \mathbb{R}^n \), respectively.

Given that, for any consecutive jump times \( \tau^- \) and \( \tau \), \( X^N(t) \) is constant for \( t \) in \([\tau^-, \tau)\), from (10) we can deduce the following update rule:

\[
Q^N(\tau) - Q^N(\tau^-) = \int_{\tau^-}^{\tau} G(Q^N(\gamma), X^N(\tau^-)) \, d\gamma \tag{12}
\]

where \( Q^N : [\tau^-, \tau) \to \mathbb{R}^n \) is the solution of (10) starting with the initial condition \( Q^N(\tau^-) \).

From (11), we also infer that \( X^N(\tau) \) is unchanged if we replace \( P^N \) and \( Q^N \) with the right-continuous jump processes \( \hat{P}^N \) and \( \hat{Q}^N \) that get updated at every jump time \( \tau \) according to

\[
\hat{P}^N(\tau) = P^N(\tau) \quad \text{and} \quad \hat{Q}^N(\tau) = Q^N(\tau).
\]

Consequently, the update rules specified in (11) and (12) imply that the pair \((X^N, \hat{Q}^N)\) is a right-continuous Markov jump process that satisfies the conditions of the framework in [18]. In addition, for every \((z, s)\) in \( \mathbb{X}^N \times \mathbb{R}^n \) and \( t \geq 0 \), the following holds:

\[
\lim_{\delta \to 0} E \left[ \frac{1}{\delta} \left( X^N(t + \delta) - z \right) \right] = \mathcal{G}(s, z) \tag{13a}
\]

\[
\lim_{\delta \to 0} E \left[ \frac{1}{\delta} \left( \hat{Q}^N(t + \delta) - s \right) \right] = \mathcal{G}(s, z) \tag{13b}
\]

B. Approximation in the limit of large populations

Hence, using [18] Theorem 2.11, we conclude from (13) that given any positive \( \bar{t} \) and \( \delta \), and initial conditions \((x(0), q(0))\) in \( \mathbb{X} \times \mathbb{R}^n \), the following holds for every sequence of initial states \( \{(x^N(0), q^N(0))\}_N \) that converges to \((x(0), q(0))\):

\[
\lim_{N \to \infty} \mathbb{P} \left( \sup_{0 \leq t \leq \bar{t}} \left\| (X^N(t), \hat{Q}^N(t)) - (x(t), q(t)) \right\| > \delta \right) = 0 \tag{14}
\]

where \((x(t), q(t))\) is the solution of (7) and we assume that for each \( N \) the process \((X^N, \hat{Q}^N)\) is initialized with \( \hat{Q}^N(0) = q^N(0) \) and \( X^N(0) = x^N(0) \). Since \( \mathcal{H} \) is Lipschitz continuous, from (14), we also conclude that the following holds:

\[
\lim_{N \to \infty} \mathbb{P} \left( \sup_{0 \leq t \leq \bar{t}} \left\| (X^N(t), \hat{P}^N(t)) - (x(t), p(t)) \right\| > \delta \right) = 0 \tag{15}
\]

As we discussed in Section I-B1 for the particular case in which the PDM is a population game (memoryless), Theorems 2.12.B.3 and 2.12.B.5, which are derived from work in [25] and [26], ascertain under unrestricted conditions that as the population grows, the stationary distribution of the population state tends to concentrate around the smallest globally asymptotically stable set. Although it is beyond the scope of this article, we believe that immediate extensions of Theorems 2.12.B.3 and 2.12.B.5 to our context would show that the stationary distribution of \((X^N(t), \hat{Q}^N(t))\), when it exists, will tend to concentrate around a globally asymptotic stable set of the mean closed loop model as \( N \) tends to infinity.

IV. EDM \( \delta \)-passivity, PDM \( \delta \)-antipassivity, and Main Supporting Lemma

We start this section by defining key EDM and PDM properties, which we will use later on to state the conditions under which our convergence results hold. Subsequently, in Section IV-C we state a key supporting lemma that we will use to establish the convergence results presented in Sections V through VI.

A. EDM \( \delta \)-passivity and Informative Storage Functions

Given an EDM with input \( w \) and state \( x \), the following inequality is central for the definition of \( \delta \)-passivity [28]:

\[
S(x(t), w(t)) - S(x(t_0), w(t_0)) \leq \int_{t_0}^{t} \left[ \dot{x}(\tau) \cdot \eta \mathcal{H}(\tau) - \eta \dot{x}(\tau) \right] \, d\tau, \quad t \geq t_0, \quad t, t_0 \in \mathbb{T}, \quad x(t_0) \in \mathbb{X}, \quad w \in \mathbb{P} \tag{16}
\]

where \( \eta \) and \( S : \mathbb{X} \times \mathbb{R}^n \to \mathbb{R}_+ \) are nonnegative real constant and a map, respectively.

Definition 9. (EDM \( \delta \)-passivity)

Given an EDM, we adopt the following \( \delta \)-passivity concepts:

- The EDM is said to be \( \delta \)-passive if there is a continuously differentiable \( S \) for which (16) is satisfied with \( \eta = 0 \).
- If the EDM is \( \delta \)-passive, let \( \eta^* \) be the supremum of all \( \eta \) for which there is a continuously differentiable \( S \) satisfying (16). If \( \eta^* \) is positive then the EDM is qualified as \( \delta \)-passive with surplus \( \eta^* \).

For either case, the map \( S \) is referred to as a \( \delta \)-storage function. We refer to the EDM generally as strictly output \( \delta \)-passive when it is \( \delta \)-passive with some positive \( \eta^* \).

Notice that the larger \( \eta^* \) the more stringent the requirement for strict output \( \delta \)-passivity. When it is positive, we view such \( \eta^* \) as a measure of \( \delta \)-passivity “surplus”.

As is discussed in [28], an EDM is \( \delta \)-passive when the following augmented system with input \( w^\delta \) and output \( x^\delta \) is passive according to its standard definition [32]:

\[
\dot{w}(t) = w^\delta(t), \quad w(0) \in \mathbb{R}^n, \quad w^\delta \in \mathbb{P}^\delta \tag{17a}
\]

\[
\dot{x}(t) = \mathcal{V}(x(t), w(t)), \quad x(0) \in \mathbb{X} \tag{17b}
\]

\[
x^\delta(t) = \mathcal{V}(x(t), w(t)) \tag{17c}
\]

where \( \mathbb{P}^\delta \) \text{ def } = \{w | w \in \mathbb{P}\}. Notably, \((x(t), w(t))\) is the state of the augmented system and \( S \) is a storage function for it.

As we will see later in Section IV-C if a \( \delta \)-storage function is informative, according to the following definition, then we
can use it to establish convergence results for the mean closed loop model.

**Definition 10. (Informative S)** Let $S: X \times \mathbb{R}^n \to \mathbb{R}^+$ be a $\delta$-storage function for a given EDM specified by $\mathcal{V}$. We say that $S$ is informative if it satisfies the following two conditions:

$$
\mathcal{V}(z^*, r^*) = 0 \implies S(z^*, r^*) = 0, \quad \text{(18a)}
$$

$$
\nabla_z^T S(z^*, r^*) \mathcal{V}(z^*, r^*) = 0 \implies \mathcal{V}(z^*, r^*) = 0 \quad \text{(18b)}
$$

for every $z^*$ and $r^*$ in $X$ and $\mathbb{R}^n$, respectively.

The implication in (18a) suggests that, for a constant deterministic payoff, every equilibrium point of the EDM minimizes $S$. In addition, from $\frac{d}{dt} S(x(t), r) = \nabla_x^T S(x(t), r) \mathcal{V}(x(t), r)$, we could conclude from (18b) that, for a constant deterministic payoff $r$, $S(x(t), r)$ is constant only if $x(t)$ remains at an equilibrium point of the EDM.

**B. PDM $\delta$-antipassivity and weak $\delta$-antipassivity**

The following conditions will be used in the definition of $\delta$-antipassivity for a given PDM with input $u$, state $q$, and output $p$:

$$
\mathcal{L}(z, s) = 0 \iff \mathcal{H}(s, z) = \bar{F}(z), \quad z \in X, \ s \in \mathbb{R}^n \quad \text{(19a)}
$$

$$
\mathcal{L}(u(0), q(0)) - \mathcal{L}(u(t), q(t)) \geq \int_0^t [p^T(\tau) \dot{u}(\tau) - \nu u^T(\tau) \dot{u}(\tau)] \, d\tau, \quad t \geq 0, \ q(0) \in \mathbb{R}^n, \ u \in X \quad \text{(19b)}
$$

where $\nu$ is a nonnegative constant, $\bar{F}$ is the stationary population game of the PDM, and $\mathcal{L}: X \times \mathbb{R}^n \to \mathbb{R}^+$ is a map.

**Definition 11. (PDM $\delta$-antipassivity)** Given a PDM, we consider the following cases:

- **The PDM is $\delta$-antipassive if there is a continuously differentiable $\mathcal{L}$ for which (19) is satisfied for $\nu = 0$.**
- **The PDM is $\delta$-antipassive with deficit $\nu^* > 0$ if there is a continuously differentiable $\mathcal{L}$ for which (19) is satisfied for every $\nu > \nu^*$.**

A map $\mathcal{L}$ that satisfies either case is referred to as a $\delta$-antistorage function.

Notice that there is an antisymmetry between (16) and (19b) that is obtained from changing signs of certain terms and swapping the output with the input. This correspondence could be further strengthened by viewing $\mathcal{L}$ as a $\delta$-antistorage function that would be the antisymmetric equivalent of $S$. An analogy similar to (17) is done in [28] to compare $\delta$-antipassivity with standard notions of passivity.

Given a PDM with input $u$, state $q$, and output $p$, the following inequality is central to characterizing a weaker notion of $\delta$-antipassivity:

$$
A(q(0), \|\dot{u}\|) \geq \int_0^t [p^T(\tau) \dot{u}(\tau) - \nu u^T(\tau) \dot{u}(\tau)] \, d\tau, \quad t \geq 0, \ q(0) \in \mathbb{R}^n, \ u \in X \quad \text{(20)}
$$

where $\nu$ and $A: \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$ are a nonnegative real constant and a map, respectively.

**Definition 12. (PDM Weak $\delta$-antipassivity)** Given a PDM, we consider the following cases:

- **The PDM is said to be weak $\delta$-antipassive if there is $A$ for which (20) is satisfied for $\nu = 0$.**
- **The PDM is weak $\delta$-antipassive with deficit $\nu^* > 0$ if there is $A$ for which (20) is satisfied for every $\nu > \nu^*$.**

Unlike $\delta$-antipassivity, which requires the existence of a continuously differentiable $\delta$-antistorage function $\mathcal{L}$, weak $\delta$-antipassivity only requires the existence of a map $A$ satisfying (20). Although well-known results [31], [40], [41] indicate that a so-called “available” storage function can be constructed when (20) is satisfied, there are no guarantees that it will be continuously differentiable or satisfy (19a). The following remark outlines an argument to establish that $\delta$-antipassivity indeed implies weak $\delta$-antipassivity.

**Remark 3. ($\delta$-antipassivity implies weak $\delta$-antipassivity)** Given a PDM, the following holds:

- **If the PDM is $\delta$-antipassive then it is weak $\delta$-antipassive and we can select $A$ as:**

$$
A(q(0), \cdot) = \max_{z \in X} \mathcal{L}(z, q(0)) \quad \text{(21)}
$$

where $\mathcal{L}$ is the $\delta$-antistorage function of the PDM.

- **This choice for $A$ also allows us to conclude that if the PDM is $\delta$-antipassive with deficit $\nu^*$ then it is also weak $\delta$-antipassive with deficit $\nu^*$.**

As we will see later in Sections [V] and [VI] where we study convergence of the mean population state to $\mathcal{NE}(\bar{F})$, as well as perturbed versions of it, $\delta$-antipassivity and weak $\delta$-antipassivity are prerequisites for global asymptotic stability and global attractiveness, respectively. This confirms, as one should expect, that because the $\delta$-antipassivity condition is stricter than its weak version, it leads to stronger stability guarantees.

1) **The Simplest PDM Example:** At this point, we define and characterize $\delta$-antipassivity of the simplest PDM type, which we call memoryless. Notably, as we will state in Proposition 2, a memoryless PDM is $\delta$-antipassive if and only if the stationary population game is contractive in the sense of Definition 7. In Section [VI], we define a significantly more interesting PDM class and investigate the conditions under which it is $\delta$-antipassive.

**Definition 13. (Memoryless PDM)** Let $F: X \to \mathbb{R}^n$ be a continuously differentiable map that defines a population

\[\text{\footnotesize Notice that since } X \text{ is compact, continuous differentiability of } F \text{ implies that it is also Lipschitz continuous.}\]
game. We can then form the simplest memoryless PDM $p(t) = F(u(t))$. It suffices to choose $H(s,z) = F(z)$ and, although it is not necessary, adopt $G(s,z) = F(z) - s$ to conclude that this is a valid PDM according to Definition 2. It also follows immediately from this construct that the stationary population game $F$ is $F$.

Proposition 2. Let $F : \mathbb{X} \rightarrow \mathbb{R}^n$ be a given continuously differentiable map satisfying the following inequality:

$$\bar{\varepsilon}^T D_F(z) \bar{\varepsilon} \leq \nu^* \bar{\varepsilon}^T \bar{\varepsilon}, \quad z \in \mathbb{X}, \bar{\varepsilon} \in \mathbb{R}^n$$

(22)

Here, $\nu^*$ is the least nonnegative real constant for which the inequality holds. The following holds for the memoryless PDM obtained from $F$.

- If $\nu^*$ is zero then the PDM is $\delta$-antipassive.
- If $\nu^*$ is positive then the PDM is $\delta$-antipassive with deficit $\nu^*$.

Proof. Recall that in this case $\bar{F} = F$ and choose $L(z,s) = 0$. Clearly, since $H(s,z) = F(z)$, Assumption 2 and (19a) hold. Also the fact that $\dot{p}^T(t) \dot{u}(t) = \dot{u}^T(t) D_F(u(t)) \dot{u}(t)$ and (22) imply that (19b) holds.

C. Main Supporting Lemma and Outline of Main Convergence Results

We proceed with stating a lemma that ascertains conditions on the EDM and PDM under which key stability properties for the mean closed loop are guaranteed. The lemma will be used as an important building block of the stability results in Sections VII and VIII where we analyze well-known EDM classes.

Lemma 1. (Main Supporting Lemma)

Let a PDM and an EDM be given. We consider the following two cases.

- (Case I) The PDM is weak $\delta$-antipassive ($\nu^* = 0$) and the EDM is $\delta$-passive ($\eta^* = 0$) with respect to an informative $\delta$-storage function $S$.
- (Case II) The PDM is weak $\delta$-antipassive with deficit $\nu^* > 0$ and the EDM is $\delta$-passive with surplus $\eta^* > \nu^*$ with respect to an informative $\delta$-storage function $S$.

If either Case I or Case II is true then the following holds:

$$\lim_{t \to \infty} S(x(t), p(t)) = 0, \quad (x(0), q(0)) \in \mathbb{X} \times \mathbb{R}^n$$

(23)

where the trajectory $(x,p)$ is determined from the unique solution of initial value problem for $\mathcal{W}$.

A proof of Lemma 1 is given in Appendix A. This lemma will enable us to use $S$ to proceed in a manner that is analogous to how Lyapunov functions were used in [24] to establish convergence of the mean population state to meaningful equilibria of contractive population games, for various classes of EDM.

Remark 4. (Trade-off in Case II) Notice that Case II of the lemma allows for a PDM that is weak $\delta$-antipassive with deficit $\nu^* > 0$ at the expense of requiring that the EDM is $\delta$-passive with surplus $\eta^* > \nu^*$. That is to say that a less stringent $\delta$-antipassivity requirement on the PDM can be counterbalanced by an appropriately stricter $\delta$-passivity condition on the EDM.

D. Comparison with related notions of passivity

In dynamical system theory, there are other notions of passivity, namely, incremental passivity [35], differential passivity [33], [34] and equilibrium-independent passivity [30]. For a certain class of dynamical system models, e.g., linear system models, incremental passivity and differential passivity are equivalent to $\delta$-passivity; however, the equivalence would not hold for nonlinear system models such as EDM considered in this paper. On the other hand, as we briefly explain below, the qualification for equilibrium-independent passivity is basically different from that of $\delta$-passivity even for linear system models.

In what follows, we compare these passivity notions with $\delta$-passivity in terms of their roles in establishing the stability of dynamical systems, and discuss the motivation for our selection of $\delta$-passivity. Essentially, incremental passivity and differential passivity are used to analyze the so-called incremental stability defined as the pairwise contraction of the state trajectories of a dynamical system - a concept relevant in synchronization and consensus problems. On the other hand, inextricably related to Lyapunov stability, $\delta$-passivity is used to ascertain the convergence towards certain equilibria, which in our context would typically be Nash or perturbed equilibria of the stationary population game of a PDM. Hence, in comparison with incremental passivity and differential passivity, $\delta$-passivity is more adequate for the stability analysis provided in this article.

Although the $\delta$-passivity concept predates that of equilibrium-independent passivity, the latter could alternatively be applied to an EDM by considering that $w(t)$ and $x(t)$ are the input and output, respectively. Notably, the work in [42] used equilibrium-independent passivity to investigate, for certain higher-order learning rules, the convergence of either the mixed strategies for a finite population, or the mean population state for an infinite population, towards the perturbed equilibrium of a negative-monotone payoff.

According to its definition, equilibrium-independent passivity would hold if for each $w^* \in \mathbb{R}^n$, there would be a continuously differentiable function $S_{w^*} : \mathbb{X} \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ for which the following condition is satisfied:

$$S_{w^*}(x(t), w(t)) - S_{w^*}(x(t_0), w(t_0)) \leq \int_{t_0}^{t} \left(x(\tau) - x^*)^T \left(w(\tau) - w^*\right)\right) d\tau,$$

$$t \geq t_0, \quad t, t_0 \in \mathbb{T}, \quad x(t_0) \in \mathbb{X}, \quad w \in \mathcal{W}$$

(24)

where $x^*$ would be a mean population state satisfying $V(x^*, w^*) = 0$. Furthermore, the standard definition of equilibrium-independent passivity would require that, for every $w^* \in \mathbb{R}^n$, there is a continuously differentiable $S_{w^*}$ satisfying (24), for which $x^*$ is the unique state satisfying $V(x^*, w^*) = 0$. Unfortunately, as can be inferred by analyzing Examples 1 and 2 this uniqueness requirement is not satisfied by important EDM classes, which further justifies our adoption of $\delta$-passivity to develop the approach reported in this article.
V. Nash Stationarity and Convergence to $\text{NE}(\mathcal{F})$

We start by defining a key property called Nash Stationarity, which will allow us to associate equilibria of an EDM with the set of best responses to a deterministic payoff vector.

Definition 14. (Nash Stationarity) A given EDM specified by $V : \mathbb{X} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies Nash stationarity if the following equivalence holds:

$$\forall (z, r) = 0 \iff z \in \arg \max_{z \in \mathbb{X}} z^T r, \quad z \in \mathbb{X}, \ r \in \mathbb{R}^n \ (25)$$

In the framework of [24], Nash stationarity of the mean dynamic is crucial for establishing that the mean population state converges to the set of Nash equilibria of an underlying contractive population game. Not surprisingly, it will also be essential in our analysis, as is evidenced by the following lemma. Fortunately, as we discuss in Remarks 5 and 7, the EDM classes considered throughout this section satisfy Nash stationarity.

Lemma 2. Consider that a mean closed loop model is formed by a given PDM and an EDM that is Nash stationary, $\delta$-passive, and has an informative $\delta$-storage function $S$. If the PDM is weak $\delta$-antipassive then $\text{NE}(\mathcal{F})$ is globally attractive. If the EDM is $\delta$-antipassive then $\text{NE}(\mathcal{F})$ is globally asymptotically stable.

A proof of Lemma 2 is given in Appendix A. Notice that Lemma 2 is restricted to the case in which the EDM is $\delta$-passive and the PDM is either weak $\delta$-antipassive or $\delta$-antipassive. This stands in contrast with Case II of Lemma 1 which allows a PDM to be weak $\delta$-antipassive with positive deficit $\nu^\ast$ at the expense of restricting the EDM to be $\delta$-passive with surplus $\eta^\ast > \nu^\ast$. This level of generality is not viable for the EPT and IPC EDM defined below because, as we show in [38, Corollary IV.3], they are not strictly output $\delta$-passive.

A. Integrable Excess Payoff Target (EPT) EDM

We start by defining Excess Payoff Target (EPT) EDM by specifying the properties that the associated protocol must satisfy.

Definition 15. (Excess Payoff Target (EPT) EDM) A given protocol $T$ yields an EPT EDM if it can be written as:

$$T_j(r, z) = \hat{T}^\text{EPT}(\hat{r}), \quad \hat{r}_i \overset{\text{def}}{=} r_i - \frac{1}{\sum_i r_{ij}} \sum_i r_{ij} z_i, \quad r \in \mathbb{R}^n, \ z \in \mathbb{X} \ (26)$$

where $\hat{T}^\text{EPT} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is a Lipschitz continuous map, $\hat{r}$ is the vector of excess payoff relative to the population average and $\mathbb{R}_+^n = \mathbb{R}^n - \text{int}(\mathbb{R}^n)$ is the set of possible excess payoff vectors. In addition, $\hat{T}^\text{EPT}$ must satisfy the following acuteness condition:

$$\hat{r}^T \hat{T}^\text{EPT}(\hat{r}) > 0, \quad \hat{r} \in \text{int}(\mathbb{R}_+^n) \ (27)$$

When agents follow an EPT protocol they are likely to switch to strategies whose payoff is higher than the average payoff for the population. The higher the excess payoff for a given strategy, relative to the average, the more likely an agent will select it. A comprehensive analysis and motivation for this protocol class is provided in [33].

Remark 5. (EPT EDM is Nash Stationary) In order to simplify the structure of our article, and given that there is no significant disadvantage in doing so, we adopt the convention that every EPT EDM satisfies acuteness [27]. A trivial adaptation to our formulation of [2, Theorem 5.5.2] for excess payoff target dynamic shows that our acuteness assumption guarantees that every EPT EDM is Nash stationary, which is crucial for the results in this section. In comprehensive discussions of the excess payoff target dynamic, such as in [2], acuteness may not be the part of the definition, and is, instead, viewed as a property that identifies a subclass.

Definition 16. (Integrable EPT EDM) A given EPT protocol $\mathcal{T}^\text{EPT} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is integrable if there is a continuously differentiable function $\bar{\mathcal{T}}^\text{EPT} : \mathbb{R} \to \mathbb{R}$ such that the following holds:

$$\mathcal{T}^\text{EPT}(\hat{r}) = \nabla \bar{\mathcal{T}}^\text{EPT}(\hat{r}), \quad \hat{r} \in \mathbb{R}_+^n \ (28)$$

We refer to $\bar{\mathcal{T}}^\text{EPT}$ as the revision potential of $\mathcal{T}^\text{EPT}$.

We can now proceed with establishing that any given EPT EDM with integrable protocol is $\delta$-passive and has an informative $\delta$-storage function. This key step will allow us to use Lemma 2 to ascertain in Theorem 1 that the mean population state of a mean closed loop model converges globally to $\text{NE}(\mathcal{F})$.

Proposition 3. If a given EPT EDM is integrable with revision potential $\bar{\mathcal{T}}^\text{EPT}$ then it is $\delta$-passive and there is a constant $\gamma$ for which $S^\text{EPT}$ given below is an informative $\delta$-storage function:

$$S^\text{EPT}(z, r) = \bar{\mathcal{T}}^\text{EPT}(\hat{r}) - \gamma, \quad z \in \mathbb{X}, \ r \in \mathbb{R}^n \ (29)$$

In addition, the following equivalence holds:

$$S^\text{EPT}(z, r) = 0 \iff z \in \arg \max_{z \in \mathbb{X}} z^T r, \quad z \in \mathbb{X}, \ r \in \mathbb{R}^n \ (30)$$

A proof of Proposition 3 is given in Appendix B. Notice that Proposition 3 extends [28, Theorem 4.4] in two ways.

- Unlike Proposition 3, each $\delta$-storage function in [28, Theorem 4.4] is constructed for a given upper-bound on $\|r\|$, which must be known a priori (see also [28, Eq. (62)]). For each integrable EPT EDM, our construction provides a unique $\delta$-storage function without any such assumptions.

- More importantly, Proposition 3 guarantees that a constant $\gamma$ exists for which (30) holds. This is a key fact in proving Theorem 1 and consequently Corollary 2 at the level of generality we have here.

The following specifies an important subclass of integrable EPT EDM for which a $\delta$-storage function can be readily constructed.

Definition 17. (Separable EPT EDM) A given EPT EDM is separable if its protocol $\mathcal{T}^\text{EPT}$ can be written as:

$$\mathcal{T}^\text{EPT}_j(\hat{r}) = \mathcal{T}^\text{SEPT}_j(\hat{r}), \quad \hat{r} \in \mathbb{R}_+^n, \ j \in \{1, \ldots, n\} \ (31)$$
where $T_j^{\text{SEPT}} : \mathbb{R} \to \mathbb{R}_+$ is a Lipschitz continuous map for each $j$ in $\{1, \ldots, n\}$.

The following corollary follows from Proposition 3 and the fact that a separable EPT protocol is also integrable.

**Corollary 1.** If a given EPT EDM is separable with protocol $T^{\text{SEPT}}$ then it is $\delta$-passive and $S^{\text{SEPT}}$ given below is an informative $\delta$-storage function:

$$S^{\text{SEPT}}(z, r) = \sum_{i=1}^{n} \int_{0}^{\bar{r}_i} T_i^{\text{SEPT}}(r) \, dr, \quad z \in \mathbb{X}, \ r \in \mathbb{R}^n \quad (32)$$

The following is a widely used example of EPT protocol, which was originally introduced in [44] to prove key properties of two-player zero-sum games.

**Example 1. (BNN EDM)** The Brown-von Neumann-Nash (BNN) EDM, as named in [20], is specified by the following separable EPT protocol:

$$T_j^{\text{BNN}}(\hat{r}) \overset{\text{def}}{=} [\hat{r}_j]_+, \ \hat{r} \in \mathbb{R}_n^r \quad (33)$$

The following is the associated $\delta$-storage function, which is informative:

$$S^{\text{BNN}}(z, r) = \frac{1}{2} \sum_{i=1}^{n} [\hat{r}_i]^2, \quad z \in \mathbb{X}, \ r \in \mathbb{R}^n \quad (34)$$

We can now state our main theorem establishing an important convergence theorem for integrable EPT EDM.

**Theorem 1.** Consider a mean closed loop model formed by an integrable EPT EDM and a PDM. If the PDM is weak $\delta$-antipassive then $\mathcal{N}(\mathcal{F})$ is globally attractive. If the PDM is $\delta$-antipassive then $\mathcal{N}(\hat{\mathcal{F}})$ is globally asymptotically stable.

*Proof.* The proof follows immediately from Proposition 3 and Remark 5.

The following corollary is an immediate consequence of Proposition 2 and Theorem 1.

**Corollary 2.** Consider that a memoryless PDM is specified by a given continuously differentiable contractive population game $\mathcal{F}$. Let a mean closed loop model be formed by an integrable EPT EDM and the memoryless PDM. The set $\mathcal{N}^{\text{p}}(\mathcal{F})$ is globally asymptotically stable.

**Remark 6.** (Corollary 3 extends [24 Theorem 5.1]) Notice that, in what regards the trajectory of the mean population state, when the PDM is memoryless the mean closed loop model is equivalent to the formulation in [24 Eq. (E) of Section 4.3]. Consequently, Corollary 3 extends [24 Theorem 5.1] because the latter guarantees global asymptotic stability only when the protocol is separable or $\mathcal{F}$ has a unique Nash equilibrium. Our more general result is possible, in part, because Proposition 3 asserts the important fact that a $\delta$-storage function satisfying [30] exists for any integrable EPT protocol.

B. Impartial Pairwise Comparison (IPC) EDM: Convergence Properties

We now proceed with defining and characterizing global convergence properties for impartial pairwise comparison (IPC) EDM, whose designation was proposed in [11] for a more general context.

**Definition 18.** (Impartial Pairwise Comparison (IPC) EDM) A given protocol $T$ yields an impartial pairwise comparison (IPC) EDM if it can be written as:

$$T_{ij}(r, z) = T_{j}^{\text{IPC}}(r_j - r_i), \quad r \in \mathbb{R}^n, \ z \in \mathbb{X} \quad (35)$$

where $T^{\text{IPC}} : \mathbb{R}^n \to \mathbb{R}_+^n$ is a Lipschitz continuous map, which also satisfies the following sign preservation condition:

$$\begin{cases} T_{j}^{\text{IPC}}(r_j - r_i) > 0, & \text{if } r_j > r_i, \\ T_{j}^{\text{IPC}}(r_j - r_i) = 0, & \text{if } r_j \leq r_i \end{cases}, \quad r \in \mathbb{R}^n \quad (36)$$

According to an IPC protocol, an agent is likely to switch to strategies offering a higher payoff. Typically, the likelihood of switching to a given strategy increases with its payoff.

**Remark 7.** (IPC EDM is Nash Stationary) It follows from a trivial modification of [2 Theorem 5.6.2] that the IPC EDM also satisfies Nash stationarity.

The following proposition shows that the method to construct the Lyapunov function in [24 Theorem 7.1] can be adapted to obtain an informative $\delta$-storage function for an IPC EDM.

**Proposition 4.** Let a Lipschitz continuous map $T^{\text{IPC}} : \mathbb{R}^n \to \mathbb{R}_+^n$ specify the protocol of an IPC EDM. The IPC EDM is $\delta$-passive and the map $S^{\text{IPC}} : \mathbb{X} \times \mathbb{R}^n \to \mathbb{R}_+$ defined below is an informative $\delta$-storage function:

$$S^{\text{IPC}}(z, r) \overset{\text{def}}{=} \sum_{i=1}^{n} \sum_{j=1}^{n} z_i \int_{0}^{r_j - r_i} T_{j}^{\text{IPC}}(\tau) \, d\tau, \quad z \in \mathbb{X}, \ r \in \mathbb{R}^n \quad (37)$$

A simple example is the so-called Smith EDM defined below, which was originally proposed in [45] to investigate a traffic assignment problem.

**Example 2. (Smith EDM)** The Smith EDM is specified by the following Smith IPC protocol:

$$T_j^{\text{Smith}}(r_j - r_i) \overset{\text{def}}{=} [r_j - r_i]_+, \quad r \in \mathbb{R}^n \quad (38)$$

The following is the associated $\delta$-storage function, which is informative:

$$S^{\text{Smith}}(z, r) \overset{\text{def}}{=} \sum_{i=1}^{n} \sum_{j=1}^{n} z_i [r_j - r_i]^2_+ \quad (39)$$

We can finally make use of Lemma 2 to state our main stability theorem for IPC EDM.

**Theorem 2.** Consider a mean closed loop model formed by an IPC EDM and a PDM. If the PDM is weak $\delta$-antipassive.

6See also [2 Examples 5.5.1 and Exercise 5.5.1]. 7See also [2 Example 5.6.1 and Exercise 5.6.1].
then $\text{NE}(\mathcal{F})$ is globally attractive. If the PDM is $\delta$-antipassive then $\text{NE}(\mathcal{F})$ is globally asymptotically stable.

**Proof.** The proof follows immediately from Proposition 2 and Remark 7.

VI. PERTURBED BEST RESPONSE (PBR) EDM: \hspace{0.5cm} CONVERGENCE TO $\text{PE}(\mathcal{F}, Q)$

In this section, we consider a class of protocols according to which the mean population state is steered towards its best response to a perturbed payoff. The perturbation models imperfections in the perception of the payoff by the agents. In a prescriptive scenario or engineering application, the perturbation could account for sensor noise or limitations of the network disseminating payoff and population state information.

**Definition 19. (Payoff Perturbation)** Let $Q : \text{int}(X) \to R$ be a given map. We deem $Q$ an admissible payoff perturbation if it is twice continuously differentiable and satisfies the following conditions:

$$
\begin{align*}
\tilde{z}^T \nabla^2 Q(z) \tilde{z} > 0 & , \quad z \in X, \quad z \in TX - \{0\} \\
\lim_{z_{\min} \to 0} \|\nabla Q(z)\| &= \infty, \quad \text{where } z_{\min} = \min_{1 \leq i \leq n} z_i
\end{align*}
$$

(40) (41)

The subset of $X$ for which $z_{\min}$ is zero is often referred to as boundary of $X$, and is denoted as $\text{bd}(X)$. For every admissible perturbation $Q$, below we also define the associated perturbed maximizer:

$$
M^Q(r) = \arg \max_{z \in \text{int}(X)} (z^T r - Q(z))
$$

(42)

The so-called choice function can be computed as $C^Q(r) = \frac{1}{n} M^Q(r)$ for $r$ in $\mathbb{R}^n$.

Section 6.2 of [2] includes a comprehensive discussion of the properties of $M^Q$. Notably, it explains why, for each $r$ in $\mathbb{R}^n$, $M^Q(r)$ takes a single value in $\text{int}(X)$, in contrast with best response maps that are in general set valued, and it also discusses analogs of most of the notions we will define below.

The seminal article [46] offers a well-documented justification for the model adopted here that provides an important discrete choice theorem relating $Q$ with the distribution of the additive noise that characterizes a probabilistic formulation of $C^Q$.

**Convention:** We should note that in most published work, the domain of the payoff perturbation is a normalized version of $X$ denoted by $\Delta = \{\frac{1}{m} z | z \in X\}$. However, we find that, in our context, stating the definitions and results consistently in terms of $X$ simplifies our notation.

The following is the general form of the PBR EDM, which was originally proposed in an slightly different but equivalent form in [48].

**Definition 20. (Perturbed best response (PBR) EDM)** Consider that an admissible payoff perturbation $Q$ is given. A given protocol $T$ yields a perturbed best response (PBR) EDM associated with $Q$ if it can be written as:

$$
T_{ij}(r, z) = C^Q(r), \quad r \in \mathbb{R}^n, \quad z \in X
$$

(43)

The following expression is determined according to (43):

$$
\tilde{y}^Q(z, r) = M^Q(r) - z, \quad z \in X, \quad r \in \mathbb{R}^n
$$

(44)

A. Perturbed Stationarity for PBR EDM

Unlike the EPT EDM and IPC EDM described in Section V, the PBR EDM does not satisfy Nash stationarity. However, as pointed out in [2, Observation 6.2.7], it does satisfy Perturbed Stationarity as defined below.

**Definition 21. (Perturbed Equilibrium Set and Virtual Payoff)**

Given a PDM with continuously differentiable $\mathcal{F}$ and an admissible payoff perturbation $Q$, the associated perturbed equilibrium is defined as:

$$
\text{PE}(\mathcal{F}, Q) \triangleq \{ z \in X \mid z = M^Q(\tilde{F}(z)) \}
$$

(45)

An immediate adaptation of [2, Theorem 6.2.8] to our context leads to the conclusion that the perturbed equilibrium can also be specified as the Nash equilibrium of the so-called virtual payoff $\tilde{F}^Q : \text{int}(X) \to \mathbb{R}^n$ defined as:

$$
\tilde{F}^Q(z) \triangleq \tilde{F}(z) - \nabla Q(z), \quad z \in \text{int}(X)
$$

(46)

In summary, we can state the following:

$$
\text{NE}(\mathcal{F}, Q) = \text{PE}(\mathcal{F}, Q)
$$

(47)

**Remark 8. (Perturbed Stationarity)** It is an immediate consequence of (44) that the PBR EDM satisfies the following equivalence also referred to as perturbed stationarity:

$$
\tilde{y}^Q(z, r) = 0 \Leftrightarrow z = M^Q(r), \quad z \in X, \quad r \in \mathbb{R}^n
$$

(48)

In addition, it follows from [46, Theorem 3.1] that if $\mathcal{F}$ is continuously differentiable and contractive then $\text{PE}(\mathcal{F}, Q)$ is a singleton.

B. $\delta$-passivity Characterization for PBR EDM

The following proposition establishes $\delta$-passivity properties for a given PBR EDM, which will allow us to use Lemma 1 to assert in Theorem 3 sufficient conditions under which $\text{PE}(\mathcal{F}, Q)$ is globally attractive and globally asymptotically stable.

**Proposition 5.** Consider that an admissible payoff perturbation $Q$ is given, for which we define the following candidate $\delta$-storage function:

$$
S^{PBR} (z, r) \triangleq \max_{z \in \text{int}(X)} \left( \tilde{z}^T r - Q(z) \right) - (z^T r - Q(z)), \quad z \in X, \quad r \in \mathbb{R}^n
$$

(49)

Let $\eta^*$ be the infimum of all nonnegative constants $\eta$ for which the following holds:

$$
\tilde{z}^T \nabla^2 Q(z) \tilde{z} \geq \eta \tilde{z}^T \tilde{z}, \quad z \in \text{int}(X), \quad \tilde{z} \in TX
$$

(50)

One of the two cases holds:

- **(Case I)** If $\eta^* \geq 0$ then the PBR EDM is $\delta$-passive and $S^{PBR}$ is an informative $\delta$-storage function.
• (Case II) If \( \eta^* > 0 \) then the PBR EDM is \( \delta \)-passive with surplus \( \eta^* \) and \( S^{PBR} \) is an informative \( \delta \)-storage function.

A proof of Proposition 3 is given in Appendix B. Notice that \cite{44} Theorem 3.1] uses a Lyapunov function that is analogous to \( S^{PBR} \) to establish convergence results in the framework of contractive population games.

**Example 3. (Logit EDM)** The Logit EDM is specified by the following protocol\( ^{8} \):

\[
C_t^Q(r) = \frac{e^{\eta t - r_i}}{\sum_{j=1}^{n} e^{\eta t - r_j}}, \quad r \in \mathbb{R}^n
\]

where \( \eta \) is a positive constant. The following is the associated \( \delta \)-storage function, which is informative:

\[
S_{\text{Logit}}(z, r) = \max_{\bar{z} \in \text{int}(\mathbb{R}^n)} (\bar{z}^T r - Q(\bar{z})) - (z^T r - Q(z)), \quad z \in \mathbb{R}^n
\]

where the payoff perturbation \( Q \) is given by \( Q(z) = \eta \sum_{i=1}^{n} z_i \ln z_i \).

**C. Global Convergence to Perturbed Equilibria for PBR EDM**

At this point, we have defined all the key concepts and presented the preliminary results required to state our main theorem establishing the conditions under which we can guarantee global convergence to \( \text{PE}(\bar{F}, Q) \).

**Theorem 3.** Consider a mean closed loop model formed by a \( \delta \)-passive PBR EDM characterized by an admissible payoff perturbation \( Q \) and a PDM with a continuously differentiable \( \bar{F} \). One of the following two cases holds:

• (Case I) If the PDM is weak \( \delta \)-antipassive then \( \text{PE}(\bar{F}, Q) \) is globally attractive. If the PDM is \( \delta \)-antipassive then \( \text{PE}(\bar{F}, Q) \) is globally asymptotically stable.

• (Case II) If the PDM is weak \( \delta \)-antipassive with positive deficit \( \nu^* \) and the PBR EDM is \( \delta \)-passive with surplus \( \eta^* > \nu^* \) then \( \text{PE}(\bar{F}, Q) \) is globally attractive. If the PDM is \( \delta \)-antipassive with positive deficit \( \nu^* \) and the PBR EDM is \( \delta \)-passive with surplus \( \eta^* > \nu^* \) then \( \text{PE}(\bar{F}, Q) \) is globally asymptotically stable.

A proof of Theorem 3 is given in Appendix B.

**VII. SMOOTHING-ANTICIPATORY PDM: DEFINITION AND \( \delta \)-ANTIPASSIVITY PROPERTIES**

In this section, we study the class of so-called smoothing-anticipatory PDM, which extends both the anticipatory and smoothing modified payoff dynamics, which were considered in \cite{28, 49–51} to account for learning dynamics.

We start with defining the smoothing-anticipatory PDM class in a way that is consistent with our formulation. Subsequently, in Sections VII-A and VII-B, we establish sufficient conditions under which a smoothing-anticipatory PDM is \( \delta \)-antipassive for the case when the stationary population game is potential.

**Definition 22. (Smoothing-Anticipatory PDM)** Consider that \( \bar{F} : \mathbb{X} \to \mathbb{R}^n \) is a given continuously differentiable map defining a population game. Given a positive constant \( \alpha \) and non-negative parameters \( \mu_0, \mu_1, \mu_2 \) satisfying \( \mu_0 + \mu_1 = 1 \), the associated smoothing-anticipatory PDM is defined as follows:

\[
\dot{q}(t) = \alpha (\bar{F}(u(t)) - q(t)) \quad (53a)
\]

\[
p(t) = \mu_0 \bar{F}(u(t)) + \mu_1 q(t) + \mu_2 \dot{q}(t) \quad (53b)
\]

for \( t \geq 0, q(0) \in \mathbb{R}^n, \) and \( u \in \mathbb{X} \).

In order to show that (53b) complies with (1), it suffices to notice that we can substitute the expression (53a) for \( \dot{q}(t) \) to get the following alternative formula for \( p(t) \):

\[
p(t) = (\mu_0 + \alpha \mu_2) \bar{F}(u(t)) + (\mu_1 - \alpha \mu_2) q(t)
\]

**Remark 9.** The following are parameter choices leading to existing PDM types:

• When \( \mu_0 = 1 \) and \( \mu_1 = \mu_2 = 0 \) the PDM is memoryless.

• When \( \mu_0 = 1, \mu_1 = 0, \) and \( \mu_2 > 0 \) we obtain an anticipatory PDM, as considered in \cite{28, 49, 51}.

• The smoothing PDM considered in \cite{28} is obtained when \( \mu_0 = 0, \mu_1 = 1, \) and \( \mu_2 = 0 \).

The following proposition guarantees that any smoothing-anticipatory PDM satisfies Assumptions 1 and 2.

**Proposition 6.** Let \( \bar{F} : \mathbb{X} \to \mathbb{R}^n \) be a given continuously differentiable map defining a population game. Given positive \( \alpha \) and non-negative \( \mu_0, \mu_1, \mu_2 \) satisfying \( \mu_0 + \mu_1 = 1 \), the associated smoothing-anticipatory PDM satisfies Assumption 4 (boundedness). It also satisfies Assumption 2 and the stationary population game is \( \bar{F} = \bar{F} \).

**Proof.** We start by writing the explicit solution of (53) for a given input \( u \) in \( \mathbb{X} \) and \( q(0) \) in \( \mathbb{R}^n \):

\[
q(t) = \alpha \int_0^t e^{-\alpha(t-\tau)} \bar{F}(u(\tau)) \, d\tau + e^{-\alpha t} q(0), \quad t \geq 0 \quad (54)
\]

Since \( \bar{F} \) is continuous, we conclude from (54) that the following inequality holds and the right-hand side is finite:

\[
\|q\| \leq \max_{z \in \mathbb{X}} \|\bar{F}(z)\| + \|q(0)\|, \quad (55)
\]

which implies that the PDM is bounded in the sense of Assumption 1. In order to prove that \( \bar{F} = \bar{F} \), we use a Lyapunov-like argument based on the following function:

\[
L(z, s) = \frac{1}{2 \alpha} (\bar{F}(z) - s)^T (\bar{F}(z) - s), \quad z \in \mathbb{X}, \ s \in \mathbb{R}^n \quad (56)
\]

Now, we can use this to calculate the following derivative:

\[
\frac{d}{dt} L(u(t), q(t)) = -2 \alpha L(u(t), q(t))
\]

\[
+ \frac{1}{\alpha} \left( F(u(t)) - q(t) \right)^T D \bar{F}(u(t)) \dot{u}(t) \quad (57)
\]
Since $F$ is continuously differentiable, $\|u\| \leq m$, and, as we proved above, $q$ is bounded, we conclude that when $u(t)$ tends to zero the second term on the right-hand side above vanishes, which implies that the following limit holds:
\[
\lim_{t \to \infty} \|q(t) - F(u(t))\| = 0 \quad (58)
\]
As a result, from (53a) we infer that $q(t)$ tends to zero and from (53b) we conclude that $\lim_{t \to \infty} \|p(t) - \mu_0 F(u(t)) - \mu_1 q(t)\| = 0$. Finally, using this fact, that $\mu_0 + \mu_1 = 1$, and (58) we infer that $\lim_{t \to \infty} \|p(t) - F(u(t))\| = 0$.

Also, we note that the set $\{(z, s) \in \mathbb{R} \times \mathbb{R}^n : H(z, s) = F(z)\}$ is equivalent to either a compact subset $\{(z, s) \in \mathbb{R} \times \mathbb{R}^n : s = F(z)\}$ if $\mu_1 \neq \alpha \mu_2$, or the entire set $\mathbb{R} \times \mathbb{R}^n$ otherwise. This means that Assumption 2 is satisfied and the stationary population game is well-defined and is given by $F = \emptyset$.

A. Smoothing-Anticipatory PDM when $F$ is Potential Affine

Notice that $\delta$-antipassivity for the classes of anticipatory and smoothing PDM was studied separately in [28] for the case when $F$ is affine, potential, and strictly contractive. Below, in Proposition 7, we determine sufficient conditions for weak $\delta$-antipassivity for smoothing-anticipatory PDM associated with affine potential $F$ that is not required to be strictly contractive.

**Definition 23. (Projection matrix $\Phi$)** Henceforth, we will use a projection matrix $\Phi$ in $\mathbb{R}^{n \times n}$ defined as follows:
\[
\Phi_{ij} = \begin{cases} 
\frac{n-1}{n} & \text{if } i = j \\
\frac{-1}{n} & \text{otherwise}
\end{cases} \quad (59)
\]

**Definition 24.** Given a real symmetric matrix $M$, the largest eigenvalue of $M$ is represented as $\lambda(M)$.

**Proposition 7.** Let $F$ be an affine population game specified as follows:
\[
F(z) = Fz + \bar{r}, \quad z \in \mathbb{R}
\]
where $F \in \mathbb{R}^{n \times n}$ is such that $\Phi F \Phi$ is symmetric⁹ and $\bar{r}$ is a constant vector in $\mathbb{R}^n$. Consider $F$, a given positive $\alpha$, and non-negative $\mu_0, \mu_1$, and $\mu_2$ satisfying $\mu_0 + \mu_1 = 1$ define a smoothing-anticipatory PDM. Let $\lambda^*$ be selected as:
\[
\lambda^* = \bar{\lambda}(\Phi F \Phi)
\]

The PDM satisfies the following:

i) If $\lambda^* = 0$ then the PDM is weak $\delta$-antipassive.

ii) If $\lambda^* > 0$ and $\mu_0 + \alpha \mu_2 \leq 1$ then the PDM is weak $\delta$-antipassive with deficit $\lambda^*$.

iii) If $\lambda^* > 0$ and $\mu_0 + \alpha \mu_2 > 1$ then the PDM is weak $\delta$-antipassive with deficit $(\mu_0 + \alpha \mu_2)\lambda^*$.

A proof of Proposition 7 is given in Appendix C.

**Proposition 8.** If, in addition to the conditions of Proposition 7 a PDM is specified by a symmetric and negative definite $F$ then it is $\delta$-antipassive.

⁹This implies that $F$ is a potential population game. See [3] for more details.

Proof. The proof mirrors that of [28] Theorem 4.5, Theorem 4.7] by showing that $\mathcal{L}$ given below is a $\delta$-antistorage function satisfying (19).
\[
\mathcal{L}(z, s) = -(Fz + \bar{r} - s)^T F^{-1}(Fz + \bar{r} - s), \quad z \in \mathbb{R}, \quad s \in \mathbb{R}^n
\]

B. Smoothing PDM when $F$ is Potential Nonlinear

In the following proposition, we establish $\delta$-antipassivity of smoothing PDM (see Remark 9). We note that unlike the case considered in Proposition 7 we allow an associated population game $F$ to be nonlinear.

**Proposition 9.** Let $F : \mathbb{R} \to \mathbb{R}^n$ admit a strictly concave potential function $f : \mathbb{R}^n \to \mathbb{R}$ for which $\nabla f = F$ holds on $\mathbb{R}$. Suppose that $f$ is twice continuously differentiable and $\text{Im}(F) \subset \text{int}(\mathbb{D}^*)$ holds, where $\text{Im}(F)$ and $\mathbb{D}^*$ are, respectively, defined as
\[
\text{Im}(F) = \{F(z) \mid z \in \mathbb{R}\}
\]
\[
\mathbb{D}^* = \left\{ s \in \mathbb{R}^n \mid \sup_{y \in \mathbb{R}^n} (f(y) - s^T y) < \infty \right\}
\]

Consider that $F$ and a given positive $\alpha$ define a smoothing PDM for which $q(0) \in \text{int}(\mathbb{D}^*)$ holds. The PDM is $\delta$-antipassive and its $\delta$-antistorage function is given by
\[
\mathcal{L}(z, s) = \alpha \left[ \sup_{y \in \mathbb{R}^n} (f(y) - s^T y) - (f(z) - s^T z) \right]
\]

A proof of Proposition 9 is given in Appendix C.

**Remark 10.** In Proposition 9 we assume that the initial condition of the smoothing PDM satisfies $q(0) \in \text{int}(\mathbb{D}^*)$. In fact, based on Lemma 5 it can be verified that given any $q(0) \in \mathbb{R}^n$ and $u \in \mathbb{R}$, $q(t)$ of the PDM converges to a compact convex subset of $\text{int}(\mathbb{D}^*)$ containing $\text{Im}(F)$. This implies that the state trajectory $q$ enters the set $\text{int}(\mathbb{D}^*)$ in a finite time.

C. Examples

We provide examples, with numerical simulations, to demonstrate how our main results can be used to assess convergence to equilibria of the mean closed loop model when the deterministic payoff is governed by a conventional population game (memoryless PDM) or a smoothing-anticipatory PDM.

**Example 4** (Coordination Population Game). Consider the payoff map given below for a coordination population game:
\[
F(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}
\]

A memoryless PDM defined by (63) is $\delta$-antipassive with strictly positive deficit $\nu^* = 1$ but is not $\delta$-antipassive. In what follows, based on Theorem 4 and numerical simulations, we examine stability of equilibrium points of the PDM in a mean closed loop model configuration formed by the PDM together with either the BNN EDM (see Example 7) or Logit EDM (see Example 3).
Fig. 2. Mean population state trajectories induced by the BNN EDM under a memoryless PDM defined by the coordination population game (63).

Fig. 3. Mean population state trajectories induced by the Logit EDM under a memoryless PDM defined by the coordination population game (63).

Note that the payoff map (63) has multiple Nash equilibria including \( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \), and under the perturbation \( Q(z) = 1.1 \sum_{i=1}^{3} z_i \ln z_i \), it has a unique perturbed equilibrium given by \( \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \). It can be inferred from our results in Section V that the BNN EDM is not strictly output \( \delta \)-passive, which yields that the mean closed loop model formed by the PDM and the BNN EDM violates the conditions of Theorem 1. In contrast, the mean closed loop model formed by the PDM and the Logit EDM with \( \eta = 1.1 \) satisfies the conditions of Theorem 2.

Figs. 2 and 3 compare the mean population state trajectories obtained from both configurations. Notice that under the first configuration the set of Nash equilibria is not asymptotically stable since any state trajectory starting from the neighborhood of the equilibrium \( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) does not converge to the equilibrium, unless the trajectory starts from it. On the other hand, under the second configuration the perturbed equilibrium is asymptotically stable. The numerical simulations illustrate that instability of the equilibria of (63) may occur when our sufficient conditions for stability are violated.

Example 5 (Rock-Paper-Scissors Population Game). Consider the payoff map given below for a Rock-Paper-Scissors (RPS) population game:

\[
F(z) = \begin{pmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2 \\
z_3
\end{pmatrix}
\] (64)

where the population game has a unique Nash equilibrium given by \( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \). Recall that unlike the \( \delta \)-antipassivity conditions for memoryless PDM, those for the anticipatory PDM (see Remark 9) require its stationary population game to be symmetric as stated in Proposition 7. In what follows, using numerical simulations, we demonstrate that violating the symmetry requirement may result in instability of the equilibrium of (64).

Consider a mean closed loop model formed by the anticipatory PDM defined by (64) and the BNN EDM, where we select a parameter choice \( \alpha = 1, \mu_2 = 0.1 \) for the PDM. Note that (64) does not satisfy the symmetry requirement and this violates the conditions of Proposition 7. The plots in Fig. 4 illustrate the resulting mean population state trajectories. Notice that the trajectories do not converge to the Nash equilibrium of (64).

Example 6 (Congestion Population Game). Consider the payoff map given below for a congestion population game:

\[
F(z) = \begin{pmatrix}
e^{-z_1} \\
e^{-z_2} \\
e^{-z_3}
\end{pmatrix}
\] (65)

where the concave potential of \( F \) is given by

\[
f(z) = \sum_{i=1}^{3} (1 - e^{-z_i})
\] (66)
The payoff map has a unique Nash equilibrium given by \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\). Different from the previous examples, \((65)\) is a non-linear payoff map.

Consider the smoothing PDM defined by \((65)\) with a parameter choice \(\alpha = 1\). We note that the domain \(D^*\) of the Legendre conjugate of \(f\) is given by \(D^* = \mathbb{R}^3_+\), which ensures that the condition \(\text{Im}(F) \subset \text{int}(D^*)\) stated in Proposition 9 holds. Fig. 5 illustrates the mean population state trajectories obtained from the mean closed loop model of the PDM and the BNN EDM. Notice that the trajectories converge to the Nash equilibrium of \((65)\).

**APPENDIX**

The following three lemmas are key to the proofs of the main results.

**Lemma 3.** Given an EDM \((4)\) specified by \(\mathcal{V}\), consider the following two relations: For every \(z\) and \(r\) in \(\mathbb{R}^n\), respectively,

\[
\nabla_{z} S(z, r) = \mathcal{V}(z, r) \tag{67a}
\]

\[
\nabla_{z}^{T} S(z, r) \mathcal{V}(z, r) \leq -\eta \mathcal{V}^{T}(z, r) \mathcal{V}(z, r) \tag{67b}
\]

where \(\eta\) and \(S : \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R}_{+}\) are nonnegative constants and a map, respectively. The following two statements are true:

1) The EDM is \(\delta\)-passive if and only if there is a continuously differentiable \(S\) satisfying \((67)\) with \(\eta = 0\).

2) For positive \(\eta\), the EDM is qualified as \(\delta\)-passive with surplus \(\eta\) if and only if there is continuously differentiable \(S\) satisfying \((67)\).

**Proof:** As described in Section [V] the EDM (4) can be viewed as a control-affine nonlinear system (17) with the input \(w^0(t)\), state \((x(t), w(t))\), and output \(x^0(t)\). Using the passivity characterization theorem (see, for instance, [41] Theorem 1) for control-affine systems, we can see that there is a continuously differentiable \(S\) satisfying \((67)\) with \(\eta \geq 0\) if and only if \(S\) satisfies the inequality \((19)\) with the same \(\eta\). The statements 1) and 2) immediately follow from this equivalence.

**Lemma 4.** Given a \(\delta\)-passive EDM \((4)\) with its \(\delta\)-storage function \(S\), let \(S = \{ (z, r) \in \mathbb{R}^{n} | S(z, r) = 0 \}\) be the stationary points of the EDM and \(S^{-1}(0) = \{ (z, r) \in \mathbb{R}^{n} | S(z, r) = 0 \}\) be global minima of \(S\). It holds that \(S^{-1}(0) \subseteq S\) and the equality holds if the EDM satisfies Nash stationarity, where we assume that the set \(S^{-1}(0)\) is nonempty.

**Proof:** The first part of the statement directly follows from the condition \((67a)\) and the fact that at a global minimizer \((z^*, r^*)\) of \(S\), it holds that \(\nabla_{z} S(z^*, r^*) = 0\).

Now suppose that the EDM satisfies Nash stationarity. To prove the second statement, it is sufficient to show that at each equilibrium point \((z_0, r_0)\) of \(4\), it holds that \(S(z_0, r_0) = 0\). To this end, let us consider the payoff map given by \(F_{z_0}(z) = - (z - z_0)\) for a fixed \(z_0\) in \(\mathbb{R}^n\). Notice that \(z_0\) is the unique Nash equilibrium of the population game. In what follows, we show that \(S(z_0, r_0) = 0\) holds for any choice of \((z_0, r_0)\) in \(S\).

Let \((z^*, r^*)\) be a global minimizer of \(S\), i.e., \(S(z^*, r^*) = 0\). By the first part of the statement, \((25)\), and \((67a)\), we have that \(V(z^*, \sigma r^*) = 0\) for all \(\sigma\) in \(\mathbb{R}_+\), and hence it holds that

\[
S(z^*, 0) = S(z^*, r^*) - \int_0^1 (r^*)^T V(z^*, \sigma r^*) \, d\sigma = 0
\]

By the continuity of \(S\), for each \(\epsilon > 0\), there exists \(\delta > 0\) for which \(S(z^*, \delta F_{z_0}(z^*)) < \epsilon\) holds.

Given input \(w(t) = \delta F_{z_0}(x(t))\), let \(x\) be the mean population state trajectory derived by the EDM. Since the EDM is \(\delta\)-passive, according to \((67)\), the following relation holds for every positive constant \(\delta\):

\[
\frac{d}{dt} S(x(t), \delta F_{z_0}(x(t))) \\
\leq \delta V^{T}(x(t), \delta F_{z_0}(x(t)))DF_{z_0}(x(t))V(x(t), \delta F_{z_0}(x(t))) \\
= -\delta \| V(x(t), \delta F_{z_0}(x(t))) \|^2
\]

(68)

Suppose that the mean population state \(x(t)\) satisfies the initial condition \(x(0) = z^*\). By an application of LaSalle’s theorem \([19]\) and by \((25)\), we can verify that \((x(t), \delta F_{z_0}(x(t)))\) converges to \((z_0, 0)\) as \(t \rightarrow \infty\). In addition, due to \((68)\), we have that

\[
S(z_0, 0) \leq S(z^*, \delta F_{z_0}(z^*)) < \epsilon
\]

Since this holds for every \(\epsilon > 0\), we conclude that \(S(z_0, 0) = 0\). By the fact that \(V(z_0, \sigma r_0) = 0\) for all \(\sigma\) in \(\mathbb{R}_+\) if \((z_0, r_0)\) belongs to \(S\), we can see that the following equality holds for every \(r_0\) for which \((z_0, r_0)\) belongs to \(S\):

\[
S(z_0, r_0) = S(z_0, 0) + \int_0^1 r_0^2 V(z_0, \sigma r_0) \, d\sigma = 0
\]

(69)

Since we made an arbitrary choice of \(z_0\) from \(\mathbb{X}\) in constructing the payoff map \(F_{z_0}\), we conclude that \((69)\) holds for every \((z_0, r_0)\) in \(S\). This proves the lemma.

**Lemma 5.** Consider a differential equation given by

\[
\dot{q}(t) = \alpha (v(t) - q(t))
\]

(70)

where \(\alpha\) is a positive constant and \(v\) is a continuous function that takes a value in a closed convex subset \(F\) of \(\mathbb{R}^n\). The set \(F\) is positively invariant and it holds that \(\lim_{t \rightarrow \infty} (\inf_{s \in F} \|q(t) - s\|) = 0\) for any \(q(0) \in \mathbb{R}^n\).

**Proof:** We first proceed with the case where \(q(0)\) is contained in \(F\) and show that \(F\) is a positively invariant set of \((70)\). By contradiction, suppose that there is time indices \(t_0, t_1\) for which \(q(t_0) \in F\) and \(q(t) \notin F\) for all \(t \in (t_0, t_1]\).

Let us define a piecewise constant function by

\[
v_K(t) \equiv v \left( t_0 + k - \frac{1}{K} (t_1 - t_0) \right)
\]

if \(t \in \left( t_0 + k - \frac{1}{K} (t_1 - t_0), t_0 + \frac{k}{K} (t_1 - t_0) \right)\) for each \(k\) in \(\{1, \ldots, K\}\). Using the function \(v_K(t)\), we define the following:

\[
q_K(t_1) = e^{-\alpha (t_1 - t_0)} b_0 \tag{71}
\]

\[
+ \alpha \sum_{k=1}^{K} \int_{t_0 + k - \frac{1}{K} (t_1 - t_0)}^{t_0 + \frac{k}{K} (t_1 - t_0)} e^{-\alpha (t_1 - \tau)} \, d\tau \, b_k
\]
where \( b_0 = q(t_0) \in \mathbb{F} \) and \( b_k = v \left( t_0 + \frac{k-1}{K} (t_1 - t_0) \right) \in \mathbb{F} \) for \( k \in \{1, \ldots, K\} \). Note that since \( q_K(t_1) \) is a convex combination of \( \{b_k\}_{k=0}^K \), it holds that \( q_K(t_1) \in \mathbb{F} \). Using the fact that \( \lim_{K \to \infty} \| q_K(t_1) - q(t_1) \| = 0 \), we have that \( q(t_1) \in \mathbb{F} \), which contradicts the hypothesis that \( q(t) \notin \mathbb{F} \) for all \( t \in (t_0, t_1) \).

Now consider that \( q(0) \) is not necessarily contained in \( \mathbb{F} \). By explicitly writing a solution to (70), we can derive the following expression:

\[
q(t) = e^{-\alpha t} q(0) + \alpha \int_0^t e^{-\alpha (t - \tau)} v(\tau) \, d\tau = \bar{q}(t) + e^{-\alpha t} (q(0) - \bar{q}(0))
\]

(72)

where \( \bar{q}(t) = e^{-\alpha t} \bar{q}(0) + \alpha \int_0^t e^{-\alpha (t - \tau)} v(\tau) \, d\tau \) with \( \bar{q}(0) \in \mathbb{F} \).

Since the second term in (72) vanishes as \( t \to \infty \), based on the positive invariance of \( \mathbb{F} \), we conclude that

\[
\lim_{t \to \infty} \left( \inf_{s \in \mathbb{F}} \| q(t) - s \| \right) \leq \lim_{t \to \infty} \| q(t) - \bar{q}(t) \| = 0
\]

(73)

\[\]

A. Proofs of Lemmas 1 and 2

a) Proof of Lemma 1: Recall that under Case I, since the \( \delta \)-storage function \( \mathcal{S} \) is informative, we have that

\[
\nabla^T \mathcal{S} (z, r) \nabla (z, r) = 0 \implies \mathcal{S} (z, r) = 0
\]

(74)

Also, using Lemma 5, we note that under Case II the following relation is true:

\[
\nabla^T \mathcal{S} (z, r) \nabla (z, r) + \nu^* \nabla^T (z, r) \nabla (z, r) \leq 0
\]

(75)

Since \( \mathcal{S} \) is informative, we have that

\[
\nabla^T \mathcal{S} (z, r) \nabla (z, r) + \nu^* \nabla^T (z, r) \nabla (z, r) = 0
\]

\[\implies \mathcal{S} (z, r) = 0\]

(76)

Hence, under either Case I (\( \nu^* = 0 \)) or Case II (\( \nu^* > 0 \)), according to (74) and (76), we can see that

\[
\nabla^T \mathcal{S} (z, r) \nabla (z, r) + \nu^* \nabla^T (z, r) \nabla (z, r) = 0
\]

\[\implies \mathcal{S} (z, r) = 0\]

(77)

In what follows, using (77), we prove the statement of the lemma. We proceed with defining an open set given by \( \mathcal{O}_\epsilon \) \( \{t > 0 \mid \mathcal{S}(x(t), p(t)) > \frac{\epsilon}{2} \} \) for a given state trajectory \( (x, p) \) and any constant \( \epsilon > 0 \). According to (77) and Remark 3, there exists \( \delta_1 > 0 \) for which the following holds for all \( t \) in \( \mathcal{O}_\epsilon \):

\[
\nabla^T \mathcal{S}(x(t), p(t)) \nabla(x(t), p(t)) + \nu^* \nabla^T (x(t), p(t)) \nabla(x(t), p(t)) \leq -\delta_1
\]

(78)

Note that using (20), we can derive the following relations:

\[
\mathcal{S}(x(t), p(t)) - \mathcal{S}(x(0), p(0)) - \mathcal{A}(q(0), \| \dot{x} \|)
\]

\[
\leq \int_0^t \left[ \frac{d}{d\tau} \mathcal{S}(x(\tau), p(\tau))
\right.
\]

\[
- \dot{p}(\tau) \dot{x}(\tau) + \nu^* \dot{x}(\tau) \dot{x}(\tau) \right] d\tau
\]

\[
= \int_0^t \left[ \nabla^T \mathcal{S}(x(\tau), p(\tau)) \nabla(x(\tau), p(\tau))
\right.
\]

\[
+ \nu^* \nabla^T (x(\tau), p(\tau)) \nabla(x(\tau), p(\tau)) \left. \right] d\tau
\]

(79)

where we use the fact that \( \nabla_x \mathcal{S}(z, r) = \mathcal{V}(z, r) \) (see Lemma 3). Since \( \mathcal{S} \) is a non-negative function, we can infer that (79) is lower-bounded by \( -\mathcal{S}(x(0), p(0)) - \mathcal{A}(q(0), \| \dot{x} \|) \) for \( t \geq 0 \). In conjunction with (78), this yields that

\[
- \mathcal{S}(x(0), p(0)) - \mathcal{A}(q(0), \| \dot{x} \|)
\]

\[
\leq \int_0^\infty \left[ \nabla^T \mathcal{S}(x(\tau), p(\tau)) \nabla(x(\tau), p(\tau))
\right.
\]

\[
+ \nu^* \nabla^T (x(\tau), p(\tau)) \nabla(x(\tau), p(\tau)) \left. \right] d\tau
\]

\[
\leq \int_{\mathcal{O}_\epsilon} \left[ \nabla^T \mathcal{S}(x(\tau), p(\tau)) \nabla(x(\tau), p(\tau))
\right.
\]

\[
+ \nu^* \nabla^T (x(\tau), p(\tau)) \nabla(x(\tau), p(\tau)) \left. \right] d\tau
\]

\[
\leq -\delta_1 \cdot \mathcal{L}(\mathcal{O}_\epsilon)
\]

(80)

where \( \mathcal{L}(\mathcal{O}_\epsilon) \) is the Lebesgue measure of \( \mathcal{O}_\epsilon \). Hence, we have that \( \mathcal{L}(\mathcal{O}_\epsilon) \leq \frac{\epsilon}{\delta_1} - (\mathcal{S}(x(0), p(0)) + \mathcal{A}(q(0), \| \dot{x} \|)) \). Note that since \( x \in \mathcal{X} \), \( \mathcal{L}(\mathcal{O}_\epsilon) \) is bounded.

We can represent the open set \( \mathcal{O}_\epsilon \) as a union of disjoint open intervals \( \{I_i\}_{i=1}^\infty \), i.e., \( \mathcal{O}_\epsilon = \bigcup_{i=1}^\infty I_i \). Notice that by our construction of \( \mathcal{O}_\epsilon \), by letting \( I_i = (a_i, b_i) \), we have that \( \mathcal{S}(x(a_i), p(a_i)) \leq \frac{\epsilon}{2} \) and \( \mathcal{S}(x(t), p(t)) > \frac{\epsilon}{2} \) for all \( t \) in \( I_i \). Since \( \mathcal{O}_\epsilon \) has finite Lebesgue measure, it holds that \( \lim_{\epsilon \to 0} \mathcal{L}(I_i) = 0 \).

In what follows, we show that for each \( \epsilon > 0 \), there exists \( T_\epsilon > 0 \) for which \( \mathcal{S}(x(t), p(t)) < \epsilon \) holds for all \( t \geq T_\epsilon \), and we conclude that \( \lim_{\epsilon \to 0} \mathcal{S}(x(t), p(t)) = 0 \). By contradiction, suppose that there exist a constant \( \epsilon > 0 \) and an infinite subsequence \( \{I_j\}_{j \in J} \) of \( \{I_i\}_{i=1}^\infty \) for which the following holds: For every \( j \in J \),

\[
\max_{t \in \mathcal{C}(\bar{I})} \mathcal{S}(x(t), p(t)) \geq \epsilon
\]

where \( \mathcal{C}(\bar{I}_j) \) is the closure of \( \bar{I}_j \). Let \( \bar{I}_j \) be for which the following holds:

\[
\mathcal{S}(x(\bar{I}_j), p(\bar{I}_j)) = \max_{t \in \mathcal{C}(\bar{I}_j)} \mathcal{S}(x(t), p(t))
\]

By letting \( I_j = (a_j, b_j) \), we can derive the following:

\[
\mathcal{S}(x(\bar{I}_j), p(\bar{I}_j)) - \mathcal{S}(x(a_j), p(a_j))
\]

\[
= \int_{a_j}^{b_j} \frac{d}{d\tau} \mathcal{S}(x(\tau), p(\tau)) \, d\tau
\]

\[
= \int_{a_j}^{b_j} \mathcal{S}(x(\tau), p(\tau)) \, d\tau
\]

\[
\leq \delta_2 \mathcal{L}(\bar{I}_j)
\]

(81)
The inequality (i) can be derived using the facts that \(\nabla_T S(x(t), p(t))V(x(t), p(t)) \leq 0\) and \(\nabla_p S(x(t), p(t)) = V(x(t), p(t))\) (see Lemma 3). To see that (ii) holds, recall that \(p\) and \(\hat{p}\) are both bounded (see Remark 4), and hence by the Lipschitz continuity of \(V\) (see Definition 4), there is \(\delta_2 > 0\) for which \(p^T(\tau)V(x(\tau), p(\tau)) \leq \delta_2\) holds for \(\tau \geq 0\). This immediately yields (ii).

Since \(S(x(a_j), p(a_j)) \leq \frac{\epsilon}{2}\) for every \(j \in J\) and \(\lim_{t \to \infty} S(t_j, p(t_j)) = 0\), from (81), we can see that \(S(x(t_j), p(t_j)) < \epsilon\) for sufficiently large \(j\) in \(J\). This contradicts the hypothesis that \(S(x(t_j), p(t_j)) \geq \epsilon\) holds for all \(j \in J\). Hence, we can infer that, for each \(\epsilon > 0\), there exists \(T_\epsilon > 0\) for which \(S(x(t), p(t)) < \epsilon\), \(\forall t \geq T_\epsilon\) from which we conclude that \(\lim_{t \to \infty} S(x(t), p(t)) = 0\).

**Proof of Lemma 2** Since the EDM is Nash stationary and has an informative \(\delta\)-storage function \(\mathcal{S}\), according to Lemma 4, the following relations hold:

\[
\mathcal{S}(z, r) = 0 \iff V(z, r) = 0 \iff z \in \arg\max_{z \in R} z^T r
\]

Using Lemma 1 we have that

\[
\lim_{t \to \infty} \mathcal{S}(x(t), p(t)) = 0
\]

According to Remark 1 without loss of generality, we may assume that there is a positive constant \(\rho\) for which the deterministic payoff \(p(t)\) satisfies \(\|p(t)\| \leq \rho\), \(t \geq 0\), and we define the set of stationary points of the EDM as \(\mathbb{S} = \{z, r) \in \mathbb{R}^n \times \mathbb{R}^n | z \in \arg\max_{z \in \mathbb{R}^n} z^T r\text{ and } \|r\| \leq 2\rho\}\). Note that \(\mathbb{S}\) is a closed set, and hence it is compact.

By (82), (83), and Remark 1 it holds that

\[
\lim_{t \to \infty} \left( \inf_{(z,r) \in \mathbb{S}} \left[ \|x(t) - z\| + \|p(t) - r\| \right] \right) = 0
\]

\[
\lim_{t \to \infty} \|\dot{x}(t)\| = 0
\]

and in conjunction with Assumption 2, we have that

\[
\lim_{t \to \infty} \|p(t) - \mathcal{F}(x(t))\| = 0
\]

Also note that in conjunction with (84), (85), using the fact that

\[
(z, \mathcal{F}(z)) \in \mathbb{S} \iff z \in \mathcal{N}(\mathcal{F})
\]

we can derive the following inequality:

\[
\lim_{t \to \infty} \left( \inf_{(z,r) \in \mathbb{S}} \left[ \|x(t) - z\| + \|\dot{x}(t) - r\| \right] \right) \leq \lim_{t \to \infty} \left( \inf_{(z,r) \in \mathbb{S}} \left[ \|x(t) - z\| + \|p(t) - r\| \right] \right) + \lim_{t \to \infty} \|p(t) - \mathcal{F}(x(t))\| = 0
\]

To show global attractiveness of \(\mathcal{N}(\mathcal{F})\), in what follows, we prove that 3 yields

\[
\lim_{t \to \infty} \left( \inf_{(z,\mathcal{F}(z)) \in \mathbb{S}} \left[ \|x(t) - z\| + \|\mathcal{F}(x(t)) - \mathcal{F}(z)\| \right] \right) = 0
\]

and we conclude that

\[
\lim_{t \to \infty} \inf_{z \in \mathcal{N}(\mathcal{F})} \|x(t) - z\| = \lim_{t \to \infty} \inf_{z \in \mathcal{N}(\mathcal{F})} \|x(t) - z\| \leq \lim_{t \to \infty} \left( \inf_{(z,\mathcal{F}(z)) \in \mathbb{S}} \left[ \|x(t) - z\| + \|\mathcal{F}(x(t)) - \mathcal{F}(z)\| \right] \right) = 0
\]

By contradiction, suppose that there is a sequence of increasing time indices \(\{t_n\}_{n=1}^\infty\) for which the sequence \(\{x(t_n)\}_{n=1}^\infty\) converges and it holds that

\[
\lim_{n \to \infty} \left( \inf_{(z,r) \in \mathbb{S}} \left[ \|x(t_n) - z\| + \|\mathcal{F}(x(t_n)) - \mathcal{F}(z)\| \right] \right) = 0
\]

but

\[
\lim_{n \to \infty} \left( \inf_{(z,r) \in \mathbb{S}} \left[ \|x(t_n) - z\| + \|\mathcal{F}(x(t_n)) - \mathcal{F}(z)\| \right] \right) > 0
\]

Since \(\mathbb{S}\) is compact, there is a converging sequence \(\{(z_n, r_n)\}_{n=1}^\infty\) for which its limit point \((z^*, r^*)\) is contained in \(\mathbb{S}\) and it holds that

\[
\|x^* - z^*\| + \|\mathcal{F}(x^*) - r^*\| = 0
\]

where \(x^*\) is the limit of \(\{x(t_n)\}_{n=1}^\infty\). Hence, we have that \((x^*, \mathcal{F}(x^*)) \in \mathbb{S}\).

On the other hand, by a similar argument, from (92), we can show that \(x^*\) is not a Nash equilibrium. This is a contradiction and proves that \(\mathcal{N}(\mathcal{F})\) is globally attractive.

In what follows, we prove Lyapunov stability of \(\mathcal{N}(\mathcal{F})\). Recall that if the PDM is \(\delta\)-anti-passive, there is a continuously differentiable map \(\mathcal{L} : \mathbb{X} \times \mathbb{R}^n \to \mathbb{R}_+\) for which the following two relations are true:

\[
\mathcal{L}(z, s) = 0 \iff \mathcal{H}(s, z) = \mathcal{F}(z)
\]

\[
\frac{d}{dt} \mathcal{L}(x(t), q(t)) \leq -\hat{p}^T(t) \dot{x}(t)
\]

In conjunction with (87) and Lemma 4, using (94) and (95), we have that

\[
\mathcal{S}(z, \mathcal{H}(s, z)) + \mathcal{L}(z, s) = 0
\]

\[
\iff \mathcal{H}(s, z) = \mathcal{F}(z)\text{ and } z \in \mathcal{N}(\mathcal{F})
\]

Let \(\mathbb{A}\) be a subset of \(\mathbb{X} \times \mathbb{R}^n\) defined by

\[
\mathbb{A} = \{(z, s) \in \mathbb{X} \times \mathbb{R}^n | \mathcal{S}(z, \mathcal{H}(s, z)) + \mathcal{L}(z, s) = 0\}
\]

Note that according to Assumption 2 the set \(\mathbb{A}\) is a compact subset of \(\mathbb{X} \times \mathbb{R}^n\), where the set \(\mathbb{F}\) is given by

\[
\mathbb{F} = \{z \in \mathbb{X} | \mathcal{S}(z, \mathcal{F}(z)) = 0\}
\]

Let \(\mathbb{O}\) be a given open set containing \(\mathbb{A}\). For the former case, without loss of generality, suppose that \(\mathbb{O}\) is bounded; otherwise, we can select a bounded open set \(\mathbb{V}\) containing \(\mathbb{A}\) and proceed with the intersection \(\mathbb{O} \cap \mathbb{V}\). Define a constant \(\alpha\) as \(\alpha = \min_{(z,s) \in \partial(\mathbb{O} \cap \mathbb{V})} \{\mathcal{S}(z, \mathcal{H}(s, z)) + \mathcal{L}(z, s)\}\). With a constant \(\beta\) satisfying \(\alpha > \beta > 0\),
we consider an open set defined by $\Omega_\beta = \{(z, s) \in \Omega \cap (X \times \mathbb{R}^n) | S(z, H(s, z)) + L(z, s) < \beta\}$. Note that since $\frac{d}{dt} [S(x(t), H(q(t), x(t)))] + L(x(t), q(t))] \leq 0$, any trajectory starting from $\Omega_\beta$ should remain in $\Omega$; otherwise, there should exist a time index $t_1$ for which $S(x(t_1), H(q(t_1), x(t_1))) + L(x(t_1), q(t_1)) \geq 0$, which is a contradiction. This completes the proof of lemma.

For the later case, the set $\Omega$ can be written as $\Omega = \bigcup_{z \in \mathbb{R}^n} F \cap \{S(z, \bar{F}(1)) < \beta\}$. We conclude that $S$ with Remark 5 and Lemma 4, we show that $\bar{S}$ is non-negative.

Then by defining $\bar{S}$, which is described as follows:

$$\bar{S} = \min_{z \in \mathbb{R}^n} S(z, \bar{F}(z))$$

where $\bar{S}$ is a contradiction. This completes the proof of lemma.

B. Proofs of Propositions 3 and 5 and Theorem 3

a) Proof of Proposition 3

We first note that the acuteness condition (27) implies the so-called Positive Correlation [43] which is described as follows:

$$V(z, r) \neq 0 \implies r^T V(z, r) > 0$$

where $V$ is a map defined by the protocol $T^{EPT}$ as in Definition 5.

Let $T^{EPT}$ be a continuously differentiable function satisfying (28). It can be verified that the gradients of $T^{EPT}$ with respect to $r$ and $z$, respectively, satisfy

$$\nabla_r T^{EPT}(\hat{r}) = V(z, r)$$

$$\nabla_r T^{EPT}(\hat{r}) V(z, r) = - (r^T V(z, r)) \sum_{i=1}^n T_i^{EPT}(\hat{r})$$

Let us select a candidate $\delta$-storage function as $S^{EPT}(z, r) = T^{EPT}(\hat{r}) - \gamma$ for some constant $\gamma$. Due to (99a), the function $S^{EPT}$ satisfies (67a). In contradiction, the fact that $T^{EPT}(\hat{r}) = 0$ implies $V(z, r) = 0$, due to (97a) and (97b), we can see that (67b) holds with $\eta = 0$ and the equality in (67b) holds only if $V(z, r) = 0$.

Suppose that $T^{EPT}$ also satisfies the following inequality for every $\hat{r}$ in $\mathbb{R}^n$:

$$T^{EPT}(\hat{r}) \geq T^{EPT}(0)$$

We first claim that (100) holds for all $(z, r)$ in the set $S^{EPT}$ of stationary points of the EPT EDM. By (99a), for fixed $z$ in $X$, the following equality holds for all $r$ in $\mathbb{R}^n$:

$$T^{EPT}(\hat{r}) - T^{EPT}(0) = \int_0^1 r^T V(z, \sigma r) d\sigma$$

where $\hat{r}_1 = r_1 - \frac{1}{m} \sum_{i=1}^m r_i z_i$. Since the EPT EDM is Nash stationary (see Remark 5), $(z, r) \in S^{EPT}$ implies $(z, \sigma r) \in S^{EPT}$ for all $\sigma$ in $[0, 1)$, and by (101), for each $(z, r)$ in $S^{EPT}$, we have that

$$T^{EPT}(\hat{r}) = T^{EPT}(0)$$

Since (102) holds for every $(z, r)$ in $S^{EPT}$, this proves the claim.

To see that (100) extends to the entire domain $X \times \mathbb{R}^n$, by contradiction, let us assume that there is $(z', r') \notin S^{EPT}$ for which $S^{EPT}(z', r') + \gamma = T^{EPT}(r') < T^{EPT}(0)$ holds, where $\gamma = \gamma(z, r)$. Define $S^{EPT}(z, r)$ as in (98) and (99b), the value of $S^{EPT}(z, r)$ is strictly decreasing unless $V(z, r), V(t) = 0$. By the hypothesis that $S^{EPT}(z', r') + \gamma < T^{EPT}(0)$ and by (102), for every $(z, r)$ in $S^{EPT}$, it holds that $S^{EPT}(z', r') < S^{EPT}(z, r)$ and the state $(x(t), p(t))$ never converges to $S^{EPT}$. On the other hand, by LaSalle’s Theorem [19], since $p(t)$ is constant and the mean population state $x(t)$ is contained in a compact set, $x(t)$ converges to an invariant subset of $\{z \in X | V(z, r) = 0\}$. By (98) and (99b), the invariant subset is contained in $S^{EPT}$. This contradicts the previous argument that the state $(x(t), p(t))$ does not converge to $S^{EPT}$. This proves that $T^{EPT}(\hat{r}) \geq T^{EPT}(0)$ holds for all $(z, r)$ in $X \times \mathbb{R}^n$.

b) Proof of Proposition 5

The analysis used in [47, Theorem 2.1] suggests that the following hold: For all $r$ in $\mathbb{R}^n$, $z$ in $X$, and $\hat{z}$ in $T_X$,

$$\hat{z}^T \nabla_r \max_{\hat{z} \in \text{int}(X)} (r^T \hat{z} - Q(\hat{z})) = \hat{z}^T M^Q(r)$$

(103a)

$$\hat{z}^T \nabla_z Q(z) = \hat{z}^T r$$

if and only if $z = M^Q(r)$

(103b)

Using (103), we can see that

$$\nabla_r S^{PBR}(z, r) = M^Q(r) - z = \hat{V}^Q(z, r)$$

(104)

and

$$\nabla_z S^{PBR}(z, r) \hat{V}^Q(z, r) = - (r - \nabla Q(z))^T \hat{V}^Q(z, r)$$

$$= - (\nabla Q(w) - \nabla Q(z))^T (w - z)$$

(105)

where $w = M^Q(r)$. By the fact that $Q$ is strictly convex, it holds that $\nabla_z S^{PBR}(z, r) \hat{V}^Q(z, r) \leq 0$ where the equality holds only if $\hat{V}^Q(z, r) = 0$, which is equivalent to $S^{PBR}(z, r) = 0$. By Lemma 3 and Remark 8, the PBR EDM is $\delta$-passive and $S^{PBR}$ is an associated informative $\delta$-storage function. This proves Case 1.

Furthermore, if the perturbation $Q$ satisfies $\hat{z}^T \nabla^2 Q(z) \hat{z} \geq \eta \hat{z}^T \hat{z}$ for all $z \in X, \hat{z} \in T_X$, i.e., $Q$ is strongly convex, then it holds that $(\nabla Q(w) - \nabla Q(z))^T (w - z) \geq \eta \hat{z}^T \hat{z}$ for all $z \in X, \hat{z} \in T_X$.
Based on the analysis used in [47, Theorem 2.1], if \( \eta^* \) is time-invariant and characterized by the transfer function matrix \( \frac{\alpha_1 + \mu_2 s}{\alpha_1 + \mu_2 s + \mu_3} \), where \( s \) is the complex Laplace variable and \( \mu \) is defined as \( \mu_0 + \alpha \mu_2 \). At this point we can use Parseval’s Theorem, which guarantees that (110) holds if the following inequality is satisfied:

\[
\begin{align*}
H \left( \frac{\alpha_1 + \mu j\omega}{\alpha_1 + j\omega} \right) F^T z & \leq 2\lambda^* F z, \\
\omega & \in \mathbb{R}, \ z \in \mathcal{T}\mathcal{C} 
\end{align*}
\]

where \( H \) represents the Hermitian operator (transpose conjugate) and \( \mathcal{T}\mathcal{C} \) is defined as \( \{ z \in \mathbb{C} | \sum_{i=1}^{n} z_i = 0 \} \). Since \( (\alpha_1 + j\omega) \) is nonzero for all real \( \omega \), we can re-state the condition as follows:

\[
\begin{align*}
\int_0^t \dot{u}^T(\tau) \dot{p}(\tau) d\tau & \leq \int_0^t \dot{u}^T(\tau) \dot{u}(\tau) d\tau, \quad t \geq 0, \quad u \in \mathcal{X} 
\end{align*}
\]

By proceeding in a way that is analogous to our proof of i) and ii), we can show that (115) is equivalent to:

\[
\begin{align*}
F z & \leq \mu_0 \alpha_2 + \mu_2 z^T z, \quad \omega \in \mathbb{R}, \ z \in \mathcal{T}\mathcal{X} 
\end{align*}
\]

The proof of iii) is concluded once we realize that if \( \lambda^* > 0 \) and \( \mu > 1 \) then the inequality above holds.

b) Proof of Proposition 9

Define \( f^* \) as the Legendre conjugate of \( f \) given by

\[
\begin{align*}
f^*(s) & \overset{\text{def}}{=} \sup_{y \in \mathbb{R}^n} (f(y) - s^T y)
\end{align*}
\]

Note that \( \mathbb{D}^* \) is the domain of \( f^* \) and its interior \( \text{int}(\mathbb{D}^*) \) is an open convex set.
We first show that, for any \( u \in X \), the set \( \text{int}(\mathbb{D}^+) \) is positively invariant, i.e., \( q(t) \in \text{int}(\mathbb{D}^+), \forall t \geq 0 \). Let \( \mathcal{F} \) be a closed convex subset of \( \text{int}(\mathbb{D}^+) \) containing \( \text{Im}(\mathcal{F}) \) and \( q(0) \).

Based on Lemma 5 we can infer that \( q(t) \in \mathbb{F} \subset \text{int}(\mathbb{D}^+) \) for all \( t \geq 0 \). This proves the positive invariance of \( \text{int}(\mathbb{D}^+) \).

Note that it follows from continuous differentiability of \( f \) and [52] Theorem 26.5] that \( \mathcal{L} \) is also continuously differentiable. Hence, to show \( \delta \)-antipassivity of the PDM, according to Definition 11, it suffices to show that

\[
\mathcal{L}(z, s) = 0 \iff \mathcal{F}(z) = s
\]

(117)

holds for all \( z \) and \( s \) in \( \mathbb{X} \) and \( \mathbb{R}^n \), respectively, and

\[
\frac{d}{dt} \mathcal{L}(u, q) \leq -p^T \dot{u}
\]

(118)

holds for all \( q(0) \) in \( \text{int}(\mathbb{D}^+) \) and \( u \) in \( \mathbb{X} \).

To see that (117) is valid, note that the function \( \mathcal{L} \) is defined over \( \mathbb{R}^n \times \mathbb{R}^n \) and \( \mathcal{L}(z, s) = 0 \) implies \( \nabla_u \mathcal{L}(z, s) = 0 \) which, according to (62), ensures that \( \mathcal{F}(z) = s \). Conversely, if \( \mathcal{F}(z) = s \) then, by strict concavity of \( f \), it holds that \( \sup_{u \in \mathbb{R}^n} f(y) - s^T y = f(z) - s^T z \), which ensures that \( \mathcal{L}(z, s) = 0 \).

To show (118), we compute the time-derivative of (62) as follows:

\[
\frac{d}{dt} \mathcal{L}(u, q) = \nabla_u \mathcal{L}(u, q) \dot{u} + \nabla_q \mathcal{L}(u, q) \dot{q}
\]

\[
= -p^T \dot{u} + \alpha^2 (\nabla f^*(q) + u)^T (\mathcal{F}(u) - q)
\]

\[
\leq -p^T \dot{u} + \alpha^2 (u - v)^T (\nabla f(u) - \nabla f(v))
\]

(119)

where to derive (i) and (ii), we use strict concavity of \( f \), the positive invariance of \( \text{int}(\mathbb{D}^+) \), and [52] Theorem 26.5] to show that \( q = \nabla f(v) \) whenever \( \nabla f^*(q) = -v \).
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