New self-consistent effective one-body theory for spinless binaries based on the post-Minkowskian approximation

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The effective one-body theories, introduced by Buonanno and Damour, are novel approaches to constructing a gravitational waveform template. By taking a gauge in which $\varphi^0$ and $\varphi^3$ vanish, we find a decoupled equation with separable variables for $\varphi^3$ in the effective metric obtained in the post-Minkowskian approximation. Furthermore, we set up a new self-consistent effective one-body theory for spinless binaries, which can be applicable to any post-Minkowskian orders. This theory not only releases the assumption that $v/c$ should be a small quantity but also resolves the contradiction that the Hamiltonian, radiation-reaction force, and waveform are constructed from different physical models in the effective one-body theory with the post-Minkowskian approximation. Compared with our previous theory [Sci. China-Phys. Mech. Astron. 65, 260411 (2022)], the computational effort for the radiation-reaction force and waveform in this new theory will be tremendously reduced.

Hamilton equations, coalescing compact object binary system, self-consistent effective one-body theory

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1 Introduction

The investigation on gravitational waves (GWs) has attracted much attention since 1918 [1-8], and many direct detections of GWs have been recently announced by LIGO, Virgo, and KAGRA [9-18]. The success of direct detections is based on the development of technology and theoretical research. In theoretical studies, a gravitational waveform template (GWT) plays a central role. The basis of the GWT is to study the late dynamical evolution of a coalescing compact object binary system.

The effective one-body (EOB) theory based on the post-Newtonian (PN) approximation introduced by Buonanno and Damour [19] is a novel approach to studying the late dynamical evolution of a coalescing binary system. The EOB theory can provide an estimate of the gravitational waveform emitted throughout the inspiral, plunge, and coalescence phases for spinless and spin binaries [20-23]. By calibrating the EOB model to numerical relativity simulations, EOB waveforms were improved, which are applied to the GW data analysis [24-41].

After the great success of the EOB theory based on PN approximation, in 2016, Damour [42] developed the EOB theory with post-Minkowskian (PM) approximation, in which
the assumption that $v/c$ should be a small quantity was released. Since then, the correlational research based on the EOB theory with PM approximation has attracted great attention [43–59].

The Hamilton equations [20] of an EOB system based on the PN/PM approximation show that, for a self-consistent effective one-body (SCEOB) theory, all quantities appearing in equations should be constructed from the same physical model. Moreover, as we will show in this paper, the waveform should also be based on the same physical model. That is, the Hamiltonian, radiation-reaction force (RFF), and waveform should be constructed in terms of the same effective spacetime.

To determine the expressions of the RFF and waveform for the “plus” and “cross” modes of GWs, we should first find the decoupled and variable separable equation for $\phi_4^B$ in the effective spacetime. Recently, we [60] have successfully derived the decoupled equations of $\phi_4^B$ for even and odd particles in the Regge-Wheeler gauge [61] by dividing the perturbation part of the metric into odd and even parts. The decoupled equations can be used to study the RFF and waveform in the effective spacetime and to set up an SC EO B theory. The explicit formulas for this model, however, are arduous tasks because we have to simultaneously solve two equations for the odd and even parities. Moreover, the equation for the odd parity is a third-order differential equation.

In this study, we derive another new decoupled and variable separable equation for $\phi_4^B$ in the effective spacetime and obtain the corresponding formal solution. To do so, we first take a gauge in which $\phi_4^B$ and $\phi_5^B$ vanish. This task can be done because, in a linear perturbation theory, $\phi_5^B$ and $\phi_4^B$ are gauge invariant, whereas $\phi_4^B$ and $\phi_5^B$ are not [62]. In this gauge, the decoupled equation for $\phi_4^B$ is obtained. Then, we separate variables for the decoupled equation of $\phi_4^B$ and obtain a formal solution. Based on this solution, we present the formulas for the RFF and waveform. Then, we set up an SC EO B theory for the spinless binaries, which is appropriate for any PM order. The theory not only releases the assumption that $v/c$ should be a small quantity but also resolves the contradiction that the Hamiltonian, RFF, and waveform are constructed from different physical models in the EOB theory with PN approximation. Compared with our previous SC EO B theory [60], the computational effort for the RFF and waveform in the new SC EO B theory was tremendously reduced.

The rest of the paper is organized as follows. In sect. 2, we derive a decoupled and variable separable equation for $\phi_4^B$ and determine the corresponding formal solution in the effective spacetime. In sect. 3, we set up the SC EO B theory for spinless binaries based on PM approximation. In sect. 4, the final conclusions are presented.

2 Formal solution for $\phi_4^B$

In an SC EO B theory, $\phi_4^B$ plays a central role in determining the RFF and waveform for the “plus” and “cross” modes of GWs. Therefore, we will first derive the decoupled and variables separable equation of motion and then present the corresponding formal solution for $\phi_4^B$ in the effective spacetime, which is described by [60]

$$
\frac{d^2 x^a}{d\tau^2} = \gamma_{ab} \frac{\Delta q}{r^2} \frac{d^2 x^b}{d\tau^2} - \frac{r^2}{\Delta r} \frac{\partial}{\partial \tau} \left( \frac{r^2}{\Delta r} \frac{\partial x^a}{\partial \tau} \right),
$$

with

$$
\Delta r = r^2 - 2GM_0 r + \sum_{i=2}^{\infty} \frac{a_i (GM_0)^i}{r^{i-2}},
$$

where the definition for all parameters can be found in ref. [60]. In particular, the effective spacetime (1) is type D [63].

2.1 Decoupled equation for $\phi_4^B$

To decouple $\phi_4^B$ in the effective spacetime, we take the null tetrads for the spacetime (1) as:

$$
l^\mu = \left\{ \frac{\tau}{\Delta r}, 1, 0, 0 \right\},
$$

$$
n^\mu = \left\{ \frac{1}{\sqrt{2}} \sqrt{\frac{\Delta r}{r^2}}, 0, 0, 0 \right\},
$$

$$
m^\mu = \frac{1}{\sqrt{2}r} \left\{ 0, 0, 1, -\frac{i}{\sin \theta} \right\}.
$$

Then, we have

$$
\rho = -\frac{1}{r^2}, \quad \mu = -\frac{\Delta r}{2r^2}, \quad \gamma = \frac{\Delta r}{4r^2} + \mu,
$$

$$
\alpha = -\cot \theta, \quad \beta = \cot \theta
$$

$$
\frac{1}{2\sqrt{2}r}, \quad \frac{\gamma - \Delta r}{2\sqrt{2}r},
$$

here and hereafter, ’ represents a derivative with respect to $r$, and all other spin coefficients are equal to zero.

Ref. [63] shows that there are three equations related to $\phi_4$:

$$
\Delta \lambda - \bar{\omega} \psi_4 = -(\mu + \bar{\beta}) \bar{\psi}_4 - (3\gamma - \bar{\omega}) \lambda + (3\alpha + \bar{\beta}) \nu - \psi_4,
$$

$$
\Delta \phi_4 - \phi_4 + \bar{\phi}_4 + \bar{\Delta} \phi_2 - \Delta \phi_2 = 3\lambda \psi_4 - 2\alpha \psi_3 - \rho \psi_4 - 2\psi_{10},
$$

$$
+ 2\lambda \psi_{11} + (2\gamma - \bar{\omega} + \bar{\mu} \phi_{20} - 2\alpha \phi_{21} - \bar{\sigma} \phi_2),
$$

$$
\Delta \phi_4 - \phi_4 + \bar{\phi}_4 + \bar{\Delta} \phi_2 - \Delta \phi_2 = 3\nu \psi_4 - 2\gamma - 2\alpha \nu \psi_3 + 4\beta \phi_4
$$

$$
- 2\psi_{11} + \bar{\nu} \phi_{20} + 2\lambda \phi_{12} + 2\nu (\gamma + \bar{\mu} \phi_{21} - 2\bar{\beta} + \alpha \phi_{22}).
$$

We [60] have shown that, in the effective background spacetime, the gravitational perturbation described by

$$
\gamma_{\mu \nu} = \gamma_{\mu \nu} + \epsilon \eta_{\mu \nu},
$$

(7)
can be achieved by perturbing all null tetrad quantities. Then, from eqs. (4)-(6), we derive the following perturbation equations:

\[
\begin{align*}
\psi_2^B + (\Delta + 3\gamma - \gamma + \mu + \tilde{\mu})\psi_2^B - (\Delta + 3\alpha + \tilde{\alpha})\psi_2^B &= 0, \\
\psi_2^B(3\psi_2 - 2\phi_{11}) - (\Delta + 2\gamma + 4\mu)\psi_2^B &= 0, \\
\psi_2^B(3\psi_2 + 2\phi_{11}) - (\Delta + 2\gamma + 4\mu)\psi_2^B &= 0,
\end{align*}
\]

where all quantities without and with the superscript \(B\) represent the background and perturbation quantities, respectively.

Chandrasekhar pointed out that, for a linear perturbation given by eq. (7), \(\psi_2^B\) and \(\psi_2^B\) are gauge invariant, whereas \(\psi_2^B\) and \(\psi_2^B\) are not [62]. Thus, we can choose a gauge in which \(\psi_2^B\) and \(\psi_2^B\) vanish without affecting \(\psi_2^B\) and \(\psi_2^B\). In this gauge, from eqs. (9) and (10), we get \(\psi_2^B\) and \(\lambda^B\):

\[
\begin{align*}
\psi_2^B &= -(\Delta + 2\gamma + 2\psi_{11})(3\psi_2 - 2\phi_{11}) - (\Delta + 4\alpha)\psi_2^B + (\Delta + 2\gamma + 2\psi_{11})\psi_{22}^B, \\
\lambda^B &= \frac{(\Delta + 2\gamma + 2\psi_{11})(3\psi_2 - 2\phi_{11}) - (\Delta + 2\gamma + 2\psi_{11})\psi_{22}^B}{3\psi_2 - 2\phi_{11}}.
\end{align*}
\]

Substituting eqs. (11) and (12) into eq. (8), we obtain the decoupled equation for \(\psi_2^B\):

\[
(\Delta + 3\gamma - \gamma + \mu + \tilde{\mu} - F_1)(\Delta + 4\alpha - F_2)\psi_2^B = T_4^B,
\]

(13)

with

\[
T_4^B = \left(\Delta + 3\gamma + \mu - \gamma + \mu - F_1\right)\psi_2^B - F_2(3\psi_2 - 2\phi_{11}) - \psi_2(3\psi_2 - 2\phi_{11})
\]

(14)

where the functions \(F_i\) \((i = 1, 2, 3, 4)\) are defined as:

\[
\begin{align*}
F_1 &= -\frac{\Delta}{2r}(3\psi_2 + 2\phi_{11}), \\
F_2 &= \frac{\psi_2 + 2\phi_{11}}{3\psi_2 - 2\phi_{11}}, \\
F_3 &= \frac{1}{\sqrt{2}(3\psi_2 - 2\phi_{11})}\psi_{11}, \\
F_4 &= 3\psi_2 + 2\phi_{11}.
\end{align*}
\]

(15)

To solve eq. (13), we first introduce new derivative operators:

\[
\begin{align*}
\mathcal{D}_n &= \partial_r - \frac{\gamma r^2}{\Delta} + \frac{\psi_2}{\Delta} + \frac{\mu}{\Delta}, \\
\mathcal{D}_n &= \partial_r - \frac{\gamma r^2}{\Delta} + \frac{\psi_2}{\Delta}, \\
\mathcal{D}_n &= \partial_r + \frac{\gamma r^2}{\Delta} + \frac{\psi_2}{\Delta}, \\
\mathcal{D}_n &= \partial_r - \frac{\gamma r^2}{\Delta} + \frac{\psi_2}{\Delta}.
\end{align*}
\]

(16)

Then, the intrinsic derivatives, appearing in eqs. (13) and (14), may be rewritten as:

\[
D = \mathcal{D}_0, \quad \Delta = -\frac{\Delta}{2r^2}, \quad \mathcal{D}_0^+ = \frac{\mathcal{D}_0}{\sqrt{2}}, \\
\delta = \frac{1}{\sqrt{2r}} \mathcal{D}_0^+.
\]

(17)

From eqs. (3) and (17) and by taking \(\psi_2^B = r^{-4}\phi_2^B\), the decoupled GW equation (13) becomes

\[
\begin{align*}
\Delta_r\left[2 \mathcal{D}_0^+(\mathcal{L}_{-1} - \sqrt{2r}F_3)\mathcal{L}_1^+\right] + 2r^2F_4\phi_2^B &= \mathcal{D}_4,
\end{align*}
\]

(18)

where

\[
\mathcal{D}_4 = -2\mathcal{D}_0 T_4^B
\]

(19)

Of note, although we decouple \(\psi_2^B\) by taking the gauge in which \(\psi_2^B\) and \(\psi_2^B\) vanish, eq. (18) with \(a_i = 0\) \((i \geq 2)\) is the same as that of the Schwarzschild spacetime obtained in ref. [64].

2.2 Separation of the variables for \(\psi_2^B\)

For the spinless case, the non-vanishing trace-free Ricci and Weyl tensors in the null tetrads (eq. (3)) are

\[
\psi_{11} = \frac{1}{8r^2}\left[4\Delta r - 4r\Delta' + r^2\Delta'' + 2r^3\right],
\]

(20)

\[
\psi_2 = \frac{1}{12r^2}\left[12\Delta r - 6r\Delta' + r^2\Delta'' - 2r^3\right].
\]

which indicate that, from eq. (15), \(F_3 = 0\) and all other functions \(F_j\) \((j = 1, 2, 4)\) are only \(r\) dependent, so the variables of \(\phi_2^B\) may be separated. To achieve this goal, we expand \(\phi_2^B\) in terms of \(-2Y_m(\theta)\):

\[
\phi_2^B = \sum_{\alpha} \frac{1}{\sqrt{2\ell}} \int d\omega e^{-i\omega t} \varphi_{\alpha}(\theta) Y_{\ell m}(r),
\]

(21)
where we have taken $\sigma^1 = -\omega$, and the functions $-\mathcal{Y}_m(\theta)$ may be normalized as:

$$\int_0^{\pi} -\mathcal{Y}_m^*(\theta) -\mathcal{Y}_m(\theta) \sin \theta d\theta = 1.$$  \hfill (22)

To compare our results with those obtained by Sasaki and Tagoshi [65], we take the normalized (22) instead of the usually normalized $\int -\mathcal{Y}_m^*(\theta) -\mathcal{Y}_m(\theta) d\Omega = 1$, which shows that $-\mathcal{Y}_m^*(\theta) = -\mathcal{Y}_m(\theta)/\sqrt{2\pi}$. Then, we obtain, from eqs. (18) and (19), the separated radial equation:

$$\left[ r^3 F_4 \Delta \frac{d}{dr} \left( \frac{1}{r^3 F_4 \Delta} \frac{d}{dr} \right) + V(r) \right] T_{\text{lmw}}(r) = T_{\text{lmw}}(r), \quad \hfill (23)$$

with

$$V(r) = \frac{r^2 \omega^2 (r^2 \omega + 2i \Delta \omega)}{\Delta} - i r \omega \left( 5 + \frac{2r^2 F_1}{\Delta} \right) + \frac{3(\Delta r + r \Delta \omega)}{r^2}$$

$$- 6rF_1 + 2r^2 F_2 - \lambda F_2, \quad \hfill (24)$$

$$T_{\text{lmw}}(r) = \frac{1}{2\pi} \int_0^\infty dr \int d\Omega \mathcal{Z}_r e^{i\omega t - m \phi} -\mathcal{Y}_m(\theta) \frac{1}{\sqrt{2\pi}}, \quad \hfill (25)$$

where $\lambda$ is the eigenvalue of $-\mathcal{Y}_m(\theta)$, and

$$\mathcal{Z}_r \equiv 4\pi \left\{ r^3 F_2 \mathcal{L}_0 \left[ \mathcal{Z}_0(T_{\text{wm}}) + \frac{\Delta \omega}{2} F_4 \mathcal{D}_0 \left( \frac{r}{F_4} \right) T_{\text{mm}} \right] \right. + \frac{\Delta \omega}{\sqrt{2}} F_2 \mathcal{D}_0 \left[ \frac{r}{\sqrt{2}} \Delta F_4 \left( \mathcal{L}_0 T_{\text{mm}} \right) \right] + \frac{\Delta \omega}{\sqrt{2}} F_2 \left[ \Delta \frac{d}{dr} \mathcal{L}_1 \left( \frac{r^3}{\Delta F_4} T_{\text{mm}} \right) \right]. \quad \hfill (26)$$

In eq. (26), we introduce $T_{\text{nn}}$, $T_{\text{mm}}$ and $T_{\text{mn}}$, which are

$$T_{\text{nn}} = \frac{1}{4\pi} \phi^\theta_{\text{sw}}, \quad T_{\text{mm}} = \frac{1}{4\pi} \phi^\theta_{\text{sw}}, \quad T_{\text{mn}} = \frac{1}{4\pi} \phi^\theta_{\text{sw}}.$$

### 2.3 Formal solution for $\phi^\theta_{\text{sw}}$ in the effective spacetime

In this subsection, we solve eq. (23) using the Green function method. The homogeneous solutions for eq. (23) can be expressed as:

$$R_{\text{lmw}}^{\text{in}} \rightarrow \begin{cases} \frac{B_{\text{lmw}}^{\text{in}} \Delta^2 e^{-\omega r}}{r^3 F_4} \mathcal{L}_0 \left[ \mathcal{Z}_0(T_{\text{wm}}) + \frac{\Delta \omega}{2} F_4 \mathcal{D}_0 \left( \frac{r}{F_4} \right) T_{\text{mm}} \right], & \text{for } r \rightarrow r_+ \vspace{1em} \\
\frac{r^3 F_2}{r^3 F_4} \mathcal{L}_0 \left[ \frac{r}{\sqrt{2}} \Delta F_4 \left( \mathcal{L}_0 T_{\text{mm}} \right) \right], & \text{for } r \rightarrow +\infty \end{cases} \quad \hfill (27)$$

$$R_{\text{lmw}}^{\text{op}} \rightarrow \begin{cases} C_{\text{lmw}}^{\text{op}} e^{-\omega r} & \text{for } r \rightarrow r_+ \vspace{1em} \\
C_{\text{lmw}}^{\text{op}} e^{-\omega r} & \text{for } r \rightarrow +\infty \end{cases} \quad \hfill (28)$$

where $\omega$ is the standard tortoise coordinate defined by $\frac{d\omega}{dr} = \int \frac{E}{F_4} dr$. Then, the inhomogeneous solution of the radial equation (23), due to the causality property of waves generated by a source, may be constructed as:

$$R_{\text{lmw}} = \frac{1}{2\omega^2} \mathcal{R}_{\text{lmw}} \int_{r_+}^{\infty} \frac{dr}{r^3 F_4} \mathcal{L}_0 \left[ \mathcal{Z}_0(T_{\text{wm}}) + \frac{\Delta \omega}{2} F_4 \mathcal{D}_0 \left( \frac{r}{F_4} \right) T_{\text{mm}} \right]$$

$$+ \frac{2\omega^2}{r^3 F_4} \int_{r_+}^{\infty} \frac{dr}{r^3 F_4} \mathcal{L}_0 \left[ \mathcal{Z}_0(T_{\text{wm}}) + \frac{\Delta \omega}{2} F_4 \mathcal{D}_0 \left( \frac{r}{F_4} \right) T_{\text{mm}} \right]. \quad \hfill (29)$$

Therefore, the solution for the radial equation (23) at the horizon is

$$R_{\text{lmw}}(r \rightarrow r_+ \rightarrow) \rightarrow \frac{B_{\text{lmw}}^{\text{in}} \Delta^2}{2i \omega^2 r^3 F_4} \int_{r_+}^{\infty} \frac{dr}{r^3 F_4} \mathcal{L}_0 \left[ \mathcal{Z}_0(T_{\text{wm}}) + \frac{\Delta \omega}{2} F_4 \mathcal{D}_0 \left( \frac{r}{F_4} \right) T_{\text{mm}} \right]$$

$$\equiv \frac{B_{\text{lmw}}^{\text{in}}}{2i \omega^2 r^3 F_4} \Delta^2 e^{-\omega r}, \quad \hfill (30)$$

whereas the counterpart at the infinity is

$$R_{\text{lmw}}(r \rightarrow +\infty) \rightarrow \frac{C_{\text{lmw}}^{\text{op}}}{2i \omega^2 r^3 F_4} \int_{r_+}^{\infty} \frac{dr}{r^3 F_4} \mathcal{L}_0 \left[ \mathcal{Z}_0(T_{\text{wm}}) + \frac{\Delta \omega}{2} F_4 \mathcal{D}_0 \left( \frac{r}{F_4} \right) T_{\text{mm}} \right]$$

$$\equiv \frac{C_{\text{lmw}}^{\text{op}}}{2i \omega^2 r^3 F_4} \Delta^2 e^{-\omega r}. \quad \hfill (31)$$

The energy momentum tensor for the EOB theory, i.e., a particle with the mass $m_0$ orbits around a massive black hole described by the effective metric, takes the form [66]

$$T^{\mu\nu} = \frac{m_0}{r^2 \sin \theta dr/d\tau} \left[ \delta(r - r(t))\delta(\theta - \theta(t))\delta(\phi - \varphi(t)) \right], \quad \hfill (32)$$

where $\sigma^0 = (t, r(t), \theta(t), \varphi(t))$ is a geodesic trajectory and $\tau = \tau(t)$ is the proper time along the geodesic. By means of eqs. (26), (32), and (25) and performing the integration by parts, we obtain

$$T_{\text{lmw}}(r) \rightarrow \frac{4 m_0}{\sqrt{2\pi}} \int_{r_+}^{\infty} dr \left[ \delta(r - r(t))\delta(\theta - \theta(t)) \right]$$

$$\times \left\{ -\frac{1}{2} \mathcal{Z}_0 \left[ \frac{1}{2} \mathcal{L}_0 \left( -\mathcal{Y}_m \right) \right] C_{\text{nn}} \delta(r - r(t))\delta(\theta - \theta(t)) \right\}$$

$$\left[ \frac{C_{\text{nn}} \delta(r - r(t))\delta(\theta - \theta(t))}{\Delta r} \right] + \frac{1}{2 \sqrt{2}} \left[ F_4 \frac{\partial}{\partial r} \left( \frac{1}{F_4} \right) \right]$$

$$\left[ \frac{C_{\text{nn}} \delta(r - r(t))\delta(\theta - \theta(t))}{\Delta r} \right] + \frac{1}{\Delta r} -\mathcal{Y}_m \frac{\partial}{\partial r} \left( r C_{\text{mm}} \delta(r - r(t))\delta(\theta - \theta(t)) \right) \right\} \right]. \quad \hfill (33)$$

with

$$C_{\text{nn}} = \frac{\Delta}{4r^2} \left( E + \frac{d\omega}{dr} \right)^2, \quad \hfill (34)$$

$$C_{\text{mm}} = \frac{\Delta}{2 \sqrt{2} r^2} \left( E + \frac{d\omega}{dr} \right) \left( \frac{iL}{\sin \theta} - \frac{\omega}{r^2} \frac{d\theta}{dr} \right), \quad \hfill (34)$$

$$C_{\text{mn}} = \frac{\Delta}{2r^2} \left( r^2 \frac{\partial \theta}{\partial r} - \frac{iL}{\sin \theta} \right)^2.$$


Eq. (33) can be rewritten as:

\[
T_{\text{tiso}}(r) = -m_0 \int_{-\infty}^{\infty} \text{d}r \omega e^{i\omega \theta} \Delta_2 \left[ A_0 \delta(r-r(t)) + A_2 \delta(r-r(t)) \right],
\]

with

\[
A_0 = A_{m0} + A_{mm0} + A_{mm1},
\]

\[
A_1 = A_{m1} + A_{mm1},
\]

\[
A_2 = A_{m2},
\]

where

\[
A_{m0} = -\frac{2r^4}{\sqrt{2\pi} \Delta_2^2} \mathcal{C}_m \mathcal{F}_2 \left[ \mathcal{L}_2 \left( -2Y_{\text{t}(\theta=0)} \right) \right],
\]

\[
A_{mm0} = \frac{r^3}{\sqrt{2\pi}} \mathcal{C}_{mm} \left[ (1 + F_2) \frac{r^2 \omega}{\Delta_2} + \frac{F_2 F_4}{r^2} - F_4 \right] \mathcal{L}_2 \left( -2Y_{\text{t}(\theta=0)} \right),
\]

\[
A_{mm1} = \frac{r^3}{\sqrt{2\pi}} \mathcal{C}_{mm} \begin{pmatrix} 1 + F_2 \end{pmatrix} \mathcal{L}_2 \left( -2Y_{\text{t}(\theta=0)} \right),
\]

\[
A_{mm2} = -\frac{r^2}{\sqrt{2\pi}} \mathcal{C}_{mm} \left( -2Y_{\text{t}(\theta=0)} \right).
\]

Inserting eq. (35) into eq. (31), we obtain

\[
\mathcal{Z}_{\text{tiso}} = -\frac{m_0}{\omega \theta} \int_{-\infty}^{\infty} \text{d}r \omega e^{i\omega \theta} \Delta_2 \left[ A_0 \frac{R^{\text{t}(\text{iso})}}{r^3 F_4} - A_1 \left( \frac{R^{\text{t}(\text{iso})}}{r^3 F_4} \right) \right] + A_2 \left( \frac{R^{\text{t}(\text{iso})}}{r^3 F_4} \right),
\]

Because we focus on circular orbits, \( r(t) \) in eq. (38) is not related to the time, so we can take \( r(t) = r_0 \). On the geodesic trajectory, we also have \( \theta(t) = \theta_0 \) and \( \phi(t) = \Omega t \), where \( \Omega \) is the angular velocity. Then, carrying out the integration for eq. (38), we find

\[
\mathcal{Z}_{\text{tiso}} = \frac{\tau \Delta_0}{\omega \theta \mathcal{B}_{\text{tiso}}^{\text{t}(\psi)}} \left[ A_0 \frac{R^{\text{t}(\text{iso})}}{r^3 F_4} - A_1 \left( \frac{R^{\text{t}(\text{iso})}}{r^3 F_4} \right) \right]_{\theta=\theta_0} \cdot \delta(\omega - \omega_0),
\]

where \( \omega_0 = m \Omega \). Then, noting the definition \( \psi_4^B = \psi_4^B / r^4 \) and using eqs. (21), (31) and (39), we find that the formal solution for \( \psi_4^B \) is given by

\[
\psi_4^B = \frac{1}{r} \sum_{n=1}^{\infty} \frac{\tau \Delta_0}{\omega_n B_{\text{tiso}}^{\text{t}(\psi)}} \left[ A_0 \frac{R^{\text{t}(\text{iso})}}{r^3 F_4} - A_1 \left( \frac{R^{\text{t}(\text{iso})}}{r^3 F_4} \right) \right]_{\theta=\theta_0}.
\]

3 Framework of the self-consistent EOB theory for spinless binaries

The basis of GWT is the late dynamical evolution of a coalescing compact object binary system. The dynamical evolution of the system is described by [20, 60]

\[
\frac{dr}{dt} - \frac{\partial H}{\partial P_r} = 0, \quad \frac{d\psi}{dt} - \frac{\partial H}{\partial P_{\psi}} = 0,
\]

\[
\frac{dP_r}{dt} + \frac{\partial H}{\partial \psi_{\psi}} = 0, \quad \frac{dP_{\psi}}{dt} = \mathcal{F}_{\psi}.
\]

Eq. (41) shows that, for an SCEOBI theory, \( H, \mathcal{F}_r, \) and \( \mathcal{F}_{\psi} \) and the waveform should be based on the same physical model.

With the effective line element (1), energy map, and solution for \( \psi_4^B \) (40) at hand, we now set up an SCEOBI theory for a spinless coalescing compact object binary system. That is, we present the expressions of the Hamiltonian, RRF, and waveform for the system in the following.

First, using Buonanno and Damour’s proposals [19, 20] and the energy map eq. (7) in ref. [60], we know that the Hamiltonian in eq. (41) is still described by eq. (10) in ref. [60], which shows that the Hamiltonian is related to the effective spacetime described by eq. (1).

Second, the general relation between the flux of the GW energy emitted to infinity and the RRF is described by \( \frac{dE}{dt} = R \mathcal{F}_R + \psi \mathcal{F}_{\psi} \). Buonanno and Damour [20] showed that, for quasi-circular orbits, \( \mathcal{F}_R \) turns into zero, so we have

\[
\mathcal{F}_{\psi} = \frac{1}{N} \frac{dE}{dt}. \tag{42}
\]

Furthermore, the flux can be shown by [67, 68]

\[
\frac{dE}{dt} = \frac{1}{16\pi G} \int (k_x^2 + k_y^2) r^2 d\Omega. \tag{43}
\]

Noting \( \psi_4^B = \psi_4^B / r^4 \) at infinity and using eq. (40), we can get the gravitational waveform \( h_+ - i h_\times \) at infinity. By substituting \( h_+ - i h_\times \) into eq. (43), we can obtain the expression for the flux of the GW energy emitted to infinity. Then, using eq. (42), we find that the RRF appearing in eq. (41) is given by

\[
\mathcal{F}_{\psi} = \frac{1}{N} \sum_{\ell=2}^{\infty} \sum_{m=1}^{\ell} \frac{2\pi m^2}{B_{\text{tiso}}^{\text{t}(\psi)}} \left[ A_0 \frac{R^{\text{t}(\text{iso})}}{r^3 F_4} - A_2 \left( \frac{R^{\text{t}(\text{iso})}}{r^3 F_4} \right) \right]_{\theta=\theta_0}, \tag{44}
\]
which indicates that the RRF is constructed in terms of the effective spacetime.

Finally, by comparing $\psi^B_4$ with the waveform described by [69]

$$ h_+ - i h_\times = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} h_{lm}^m \frac{Y^m_l(\theta, \varphi)}{\sqrt{2\pi}},$$

we find

$$ h_{lm}^m = \frac{1}{r} \omega_{lm}^{(0)} \omega_{lm}^{(0)} \left[ A_0 \left( \frac{R_{lm}^0}{r F_4} \right) + A_1 \left( \frac{R_{lm}^1}{r F_4} \right) \right]^{\prime} \sin(r \gamma) \left( r'/r \right).$$

(46)

Clearly, the calculation for the waveform $h_{lm}^m$ is also based on the effective spacetime.

To improve the waveform, Damour and Nagar [32] proposed that, based on the waveforms obtained using the PN perturbation, multipolar waveforms should be built as:

$$ h_{lm}^m = h_{lm}^{(N_{\text{EOB})}} S_{\text{eff}} T_{lm} \epsilon^{(m)} f_{\text{lm}},$$

where $h_{lm}^{(N_{\text{EOB})}}$ is the leading Newtonian order term, $S_{\text{eff}}$ is the relativistic conserved energy or angular momentum of the effective moving source, $T_{lm}$ assumes an infinite number of leading logarithms entering the tail effect, $\epsilon^{(m)}$ is an additional phase correction, and $f_{\text{lm}}$ is the remaining (essentially nonlinear) PN effects. Furthermore, to reproduce effects in the numerical simulations that go beyond the quasi-circular motion assumption, Pan et al. [34] introduced a non-quasicircular (NQC) effect in $h_{lm}^m$.

We believe that the waveform eq. (46) can be improved using Damour-Nagar-Pan’s proposals [32-35].

4 Conclusions and discussion

The basis of GWT is the late dynamical evolution of a coalescing compact object binary system. To study the dynamical evolution, Buonanno et al. [19] presented, based on the PN/PM approximation, a novel approach to map the two-body problem onto an EOB problem. Based on the Hamilton equations describing the dynamical evolution, an SCCOB theory, the Hamiltonian $H$, RRF $F_q$, and waveform should be constructed from the same physical model.

To build such an SCCOB theory, the key step is to find the decoupled equation with separable variables for $\psi^B_4$ in the effective spacetime background. Chandrasekhar [62] pointed out that, for a linear gravitational perturbation described by eq. (7), $\psi^0_4$ and $\psi^B_4$ are gauge invariant, whereas $\psi^B_4$ and $\psi^0_4$ are not. Therefore, we can take a gauge in which $\psi^0_4$ and $\psi^B_4$ vanish without affecting $\psi^0_4$ and $\psi^B_4$. In this gauge, we obtained the decoupled equation for $\psi^B_4$ in the effective spacetime. Then, we separated variables for the decoupled equation and obtained a formal solution of $\psi^B_4$. Based on the solution, we presented the formulas for the RRF and waveform.

With the Hamiltonian eq. (10) in ref. [60], RRF (eq. (44)) and waveform (eq. (46)), which are based on the same effective spacetime, at hand, we set up an SCCOB theory for spinless binaries, which can be applicable to any PM orders. The theory not only releases the assumption that $v/c$ should be a small quantity but also resolves the contradiction that the Hamiltonian, RRF, and waveform are constructed from different physical models in the EOB theory based on PN approximation. Compared with our previous SCCOB theory [60], the computational effort for the RRF and waveform in the new SCCOB theory will be tremendously reduced.

Generally, the usefulness of an EOB model critically depends on the accuracy of the waveform template. Next, we will construct the GWT based on the SCCOB theory for the spinless binaries and compare it with that of the numerical relativity. We expect that the advantages of the SCCOB theory can improve the accuracy of the GWT. Furthermore, we will extend the study to spin binaries.

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