Two mechanisms for quantum natured gravitons to entangle masses

Sougato Bose, Anupam Mazumdar, Martine Schut, and Marko Toroš

1Department of Physics and Astronomy, University College London, Gower Street, WC1E 6BT London, UK
2Van Swinderen Institute, University of Groningen, 9747 AG, The Netherlands
3Bernoulli Institute, University of Groningen, 9747 AG, The Netherlands
4School of Physics and Astronomy, University of Glasgow, Glasgow, G12 8QQ, UK

This paper points out the importance of the quantum nature of the gravitational interaction with matter in a linearized theory of quantum-gravity-induced-entanglement of masses (QGEM). We will show how a quantum interaction entangles two test masses placed in harmonic traps and how such a quantum matter-matter interaction emerges from an underlying quantum gravitational field. We will rely upon quantum perturbation theory highlighting the critical assumptions for generating a quantum matter-matter interaction and showing that a classical gravitational field does not render such an entanglement. We will consider two distinct examples; one where the two harmonic oscillators interact via an exchange of a virtual spin-2 and spin-0 components of the graviton, and the other where the harmonic oscillators interact with the gravitons of the gravitational waves with two helicity states. The quantum nature of the gravitons interacting with the harmonic oscillators are responsible for creating an entangled state with the ground and the excited states of harmonic oscillators as the Schmidt basis. We will compute the concurrence as a criterion for the above entanglement and compare the two ways of entangling the two harmonic oscillators.

I. INTRODUCTION

The classical theory of general relativity (GR) is outstanding in matching the observations on large scales, especially from the solar system tests to the observations from the detection of the gravitational waves [1]. Despite these successes, the classical theory fails at very short distances and early times. The classical GR predicts black hole and cosmological singularity where the notion of space-time breaks down [2].

Although it is believed that the quantum theory of gravity will alleviate some of these challenges, however, we still do not know whether gravity is indeed quantum or not. Moreover, there are also many candidates for a quantum theory of gravity [3]. From an effective field theory perspective and at low energies, it is believed that the gravitational interaction is being mediated by a massless spin-2 graviton, which can be canonically quantised [4–7]. Although the perturbative quantum theory of gravity also possesses many challenges, such as the issues of renormalisability at very high energies and the issue of finiteness, at low energies where the day to day experiments are performed, it is still a very good effective field theory description of nature [8].

Given the feeble interaction strength of gravity, it is extremely hard to detect a graviton in a detector by the momentum transfer [9]. Indirect detection of the quantum properties of the graviton remains elusive in the primordial nature of the gravitational waves (GWs) [10, 11]. Astrophysical and cosmological uncertainties shroud any validation of the quantum nature of space-time by modifying the photon dispersion relationship [12]. Moreover, the strict constraint on the graviton mass indirectly arising from the propagation of the GWs detected by the LIGO observatory hints no departure from GR in the infrared [13].

Given all these challenges, it is worth asking how to test the quantum nature of a graviton in a laboratory at low energies. Recently, there has been a proposal to test the quantum nature of gravity by witnessing the spin entanglement between the two quantum superposed test masses, known as quantum gravity induced entanglement of masses (QGEM) [14, 15]. The idea is to create a spatial quantum superposition of two test masses and bring them adjacent to each other in a controlled environment such that their only dominant interaction that remains is the exchange of a massless graviton. It is possible to realise such a daunting experiment but there are many challenges needed to be overcome.

In this paper we will review the conceptual underpinnings of the QGEM mechanism. The entanglement of the two masses emerges from “Local Operation and Quantum Communication (LOQC)” whereas no entanglement would occur by “Local Operations and Classical Communication (LOCC)” [11]. The LOCC principle states that the two quantum states cannot be entangled via a classical channel if they were not entangled to begin with, or entanglement cannot be increased by local operations and classical communication. The classical communication is the critical ingredient which can be put to test when it comes to graviton mediated interaction between the two masses. If the graviton is quantum, it would mediate the gravitational attraction between the two masses and it would also entangle them, hence confirming the QGEM proposal [14, 15].

1The detailed analysis of the demanding nature of the experiment (such as creating Schrödinger cat states with massive test masses along with achieving the required coherence life time required to detect the entanglement) has been discussed already in [14]. A related idea was also proposed in [16]. These initial works [14, 15] garnered extensive interest in the research community [17, 18].
The aim of the current paper is to sharpen the argument of LOCC for the purpose of QGEM, and highlight the role of the quantum nature of the interactions for entangling two quantum systems. We will use basic quantum mechanics and perturbation theory to show how the perturbed wave functions of the matter systems become entangled *solely* by the virtue of the quantum nature of a graviton. We will furthermore highlight the relevant degrees of freedom of the graviton which interacts with the quantised matter, and are responsible for the entanglement.

We will study this problem in the number state basis of two harmonic oscillator states, and we will show that the perturbed state is an entangled state even at the first order in a quantum perturbation theory \[12\]. The quantum interaction between the two matter systems emerges from the change in the graviton vacuum energy due to the presence of the two quantum harmonic oscillators. In the QFT community this is a well known way to understand interactions, albeit often presented in a different formalism than here – they say that the presence of two sources changes the vacuum energy \[40\]. We will show that in a static limit this change in the vacuum energy is the same as that of the Newton’s potential at the lowest order in the Newton’s constant, which appears at the second order in the perturbation theory. Furthermore, the Newtonian potential is the energy shift of the gravitational vacuum. In this case the relevant gravitational degrees of freedom required to be quantized is comprised of both the spin-2 and spin-0 components \[4, 15\].

In particular, if the matter is quantized then the energy shift in the gravitational field becomes an *operator valued* interaction. Since we have the quantum superpositions for the matter systems – then the energy shift in the gravitational field will not be a real number, resulting in the gravitational field itself being a non-classical entity.

The above arguments can be extended to the gravitational waves (GWs). We will show that if the two harmonic oscillators are excited to create the gravitational waves (GWs), then the two harmonic oscillator states are also entangled provided the GWs are quantum, i.e. provided the two *helicity states* of graviton are quantised. This is a new channel for the generation of the entanglement based on on-shell graviton exchange and the quantized quadrupole radiation. We will calculate the *concurrency* \[43\] as a way to measure the entanglement between the two harmonic oscillators and show that the concurrency is always positive for the quantum interaction between the graviton and the matter states\[44\].

This paper is organised in the following way. We will first briefly recap the known results, i.e. the two quantum harmonic oscillators (Sec. II), and show how the quantum interaction is responsible for generating the entanglement (Sec. III). We will then quantify the degree of entanglement using concurrence which we will compute using perturbation theory. We will discuss the special case where the interaction potential is generated by the gravitational field in the regime of weak gravity (Sec. IV). In particular, we will first show how the $\hat{T}_{00}$ component of the stress-energy tensor generates entanglement – $|00\rangle$ and $|11\rangle$ are the Schmidt basis of the entangled state, where $|nN\rangle \equiv |n\rangle |N\rangle$, and $|n\rangle$ (|$N\rangle$) denote the number state of the first (second) harmonic oscillator (Sec. IV A). In addition, we find that the $\hat{T}_{ij}$ components of the stress-energy tensor (which give rise to the GWs) generate entanglement – $|00\rangle$ and $|22\rangle$ are the Schmidt basis of the entangled state, in line with the quadrupole nature of the gravitational radiation (Sec. IV B). We will finally conclude with the consequences for the classical/quantum communication (Sec. V).

II. TWO QUANTUM HARMONIC OSCILLATORS

Let us consider two matter systems, denoted by A and B, which are placed in harmonic traps located at $\pm d/2$. We suppose the harmonic oscillators are well-localised such that

\[\hat{x}_A = -\frac{d}{2} + \delta \hat{x}_A, \quad \hat{x}_B = \frac{d}{2} + \delta \hat{x}_B.\]  
(1)

where $\hat{x}_A$, $\hat{x}_B$ are the positions, and $\delta \hat{x}_A$, $\delta \hat{x}_B$ denote small displacements from the equilibrium. The Hamiltonian for the two harmonic oscillators is given by:

\[H_{\text{matter}} = \frac{\hat{p}_A^2}{2m} + \frac{\hat{p}_B^2}{2m} + \frac{m\omega_m^2}{2} \delta \hat{x}_A^2 + \frac{m\omega_m^2}{2} \delta \hat{x}_B^2.\]  
(2)

where $\hat{p}_A$, $\hat{p}_B$ are the conjugate momenta, and $\omega_m$ is the harmonic frequency of the two traps (assumed equal for the two particles for simplicity). We now introduce the adimensional mode operators for the matter by writing

\[\delta \hat{x}_A = \sqrt{\frac{\hbar}{2m\omega_m}} (\hat{a} + \hat{a}^\dagger), \quad \delta \hat{x}_B = \sqrt{\frac{\hbar}{2m\omega_m}} (\hat{b} + \hat{b}^\dagger),\]  
(3)

\[\hat{p}_A = i\sqrt{\frac{\hbar m\omega_m}{2}} (\hat{a}^\dagger - \hat{a}), \quad \hat{p}_B = i\sqrt{\frac{\hbar m\omega_m}{2}} (\hat{b}^\dagger - \hat{b}).\]  
(4)

In this paper we will consider only pure states to highlight the conceptual points, but the analysis could be readily extended to more realistic situations with mixed states to account for the internal/external noise sources and environmental decoherence.

The entanglement features of harmonic oscillators in presence of the interaction are quite well-known in the quantum optics literature, see for example \[43\]. Typically, the quantum nature of the photon plays the role of the quantum interaction. However, our aim here is to concentrate on the quantum nature of the graviton, especially highlighting the graviton’s dynamical degrees of freedom which are responsible for the quantum interaction in enabling the entanglement feature of the quantum harmonic oscillators. These dynamical degrees of freedom of the graviton are very different in nature compared to the photon, and also differ whether the localised matter is static or not.
which satisfy the usual canonical commutation relationships (the only nonzero commutators are given by \([a, a^\dagger] = 1\), and \([b, b^\dagger] = 1\)). Using this notation the Hamiltonian can be written succinctly as:

\[
\hat{H}_{\text{matter}} = \hat{H}_A + \hat{H}_B,
\]

where \(\hat{H}_A = \hbar \omega_m a^\dagger a\) and \(\hat{H}_B = \hbar \omega_m b^\dagger b\). We now want to investigate the steady-state when the system is perturbed by an interaction Hamiltonian \(\hat{H}_{AB}\). In particular, we will show that in general any quantum interaction will entangle the two harmonic oscillators.

### III. QUANTUM INTERACTION INDUCES ENTANGLEMENT

Let us assume that the initial state of the matter-system is given by

\[
|\psi_i\rangle = |0\rangle_A |0\rangle_B,
\]

where \(|0\rangle_A\) (\(|0\rangle_B\)) denote the ground state of the first (second) harmonic oscillator (in the following we will omit the subscripts A, B for the states to ease the notation). Suppose we now introduce an interaction potential \(\lambda \hat{H}_{AB}\) between the two matter systems, where \(\lambda\) is a small bookkeeping parameter. The perturbed state is given by

\[
|\psi_t\rangle \equiv \frac{1}{\sqrt{N}} \sum_{n,N} C_{nN} |n\rangle |N\rangle,
\]

where \(|n\rangle\), \(|N\rangle\) denote the number states, and the overall normalisation is given by \(N = \sum_{n,N} |C_{nN}|^2\). We have that \(C_{00} \equiv 1\) (coefficient of the unperturbed state), while the other coefficients are given by

\[
C_{nN} = \frac{\langle n| \langle N| \hat{H}_{AB} |0\rangle |0\rangle}{2E_0 - E_n - E_N},
\]

where \(E_0\) is the ground-state energy for the harmonic oscillators (equal for the two harmonic oscillators as we have assumed the same trap frequency), and \(E_n, E_N\) denote the energies of the excited states.

Here we note the role of \(\hat{H}_{AB}\) being a quantum operator. If \(H_{AB}\) were classical, it would have an associated c-number (complex number), which would yield \(\langle n| \langle N| \hat{H}_{AB} |0\rangle |0\rangle = 0\), by virtue of the orthogonality of the ground and the excited states (\(|0\rangle |0\rangle\) and \(|n\rangle |N\rangle\)) of the two quantum harmonic oscillators, as \(n, N > 0\) in Eq. (5). By the same argument, interactions acting as operators on only one of the two quantum systems (i.e., without products of operators acting on the two matter systems) cannot entangle the two systems. It is thus instructive to rewrite the state in Eq. (7) in the following way

\[
|\psi_t\rangle \sim (|0\rangle + \sum_{n>0} A_n |n\rangle)(|0\rangle + \sum_{N>0} B_N |N\rangle) + \sum_{n,N>0} (C_{nN} - A_n B_N) |n\rangle |N\rangle,
\]

where \(A_n \equiv C_{n0}\) and \(B_N \equiv C_{0N}\). The first line in Eq. (9) would yield a separable state, while the second line is responsible for entanglement of the two matter systems. We can already see the stark difference between Local Operation Quantum Communication (LOQC) and Local Operation Classical Communication (LOCC). The non-trivial part of a LOQC mechanism is now encoded in the terms of the interaction Hamiltonian \(\hat{H}_{AB}\) producing the second line in Eq. (9). On the other hand, a LOCC mechanism could produce the first line of Eq. (9), but not the second line, as a classical interaction cannot entangle the two the quantum states if they were not entangled to begin with.

To quantify the degree of entanglement we can compute the concurrence

\[
C \equiv \sqrt{2(1 - \text{tr}[\hat{\rho}_A^2])},
\]

where \(\hat{\rho}_A\) can be computed by tracing away the B state

\[
\hat{\rho}_A = \sum_N \langle N|\psi_t\rangle \langle \psi_t| N\rangle.
\]

We recall that the larger the concurrence \(C\) is, the larger is the degree of entanglement \(- C = 0\) corresponds to a separable state, while \(C = \sqrt{2}\) is obtained for a maximally entangled state. Inserting Eq. (7) into Eq. (10) we find

\[
\hat{\rho}_A = \frac{1}{N} \sum_{n,n',N} C_{nN} C_{n'N}^* |n\rangle |n'\rangle.
\]

We then insert Eq. (11) back into Eq. (10) to eventually find:

\[
C \equiv \sqrt{2(1 - \sum_{n,n',N,N'} C_{nN} C_{n'N}^* C_{n'N'} C_{nN'}^* N^2)}.
\]

In the next sections we will consider the entanglement of two harmonic oscillators induced by the quantum nature of gravitons. For this case, the entanglement will be induced by the terms \(C_{11}\) and \(C_{22}\) at the lowest order in the perturbation theory when the potential \(\hat{H}_{AB}\) is generated by the quantized gravitational field in the regime of weak gravity.

---

4The above discussion, of course, relies on initially pure states evolving unitarily under a fixed Hamiltonian so that they remain pure. The general notion of LOCC [11], as used in quantum information is broader, distinguishing entangled states from classically mutually correlated states. The above discussion of Eq. (9) can, of course, be easily generalized to mixed states and probabilistic operations (simply several repeats of our argument for different initial states and different Hamiltonians with their corresponding probabilities).

5A similar discussion was first adopted in momentum space entanglement in perturbative quantum field theory, Ref. [12], where they argued that the entanglement entropy of and mutual information between subsets of field theoretic degrees of freedom at different momentum scales are natural observables in quantum field theory. Here we will compare the degree of entanglement by computing the concurrence, see the discussion below.
IV. QUANTUM GRAVITATIONAL INTERACTION

We consider the setup of two quantum harmonic oscillators (introduced in the previous sections) in the presence of the gravitational field. In particular, we will work in the regime of small perturbations $|h_{\mu\nu}| \ll 1$ about the Minkowski background $\eta_{\mu\nu}$. The metric is given by:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{(where } \mu, \nu = 0, 1, 2, 3 \text{ and we are using (-, +, +, +) signature throughout).}$$

We will promote the fluctuations into quantum operators,

$$\hat{h}_{\mu\nu} = A \int dk \sqrt{\frac{\hbar}{2\omega_k (2\pi)^3}} (\hat{P}_{\mu\nu}(k)e^{-ik\cdot r} + \text{H.c})$$  \hspace{1cm} (14)

where the prefactor is $A = \sqrt{16\pi G/c^2}$, $G$ is the Newton’s constant, and $\hat{P}_{\mu\nu}$ and $\hat{P}_{\mu\nu}^\dagger$ denote the graviton annihilation and creation operator. We will discuss in detail the properties of the graviton and the relevant degrees of freedom below.

Around the Minkowski background, the graviton coupling to the stress-energy tensor $\hat{T}_{\mu\nu}$ is given by the following operator valued interaction term:

$$\hat{H}_{\text{int}} = -\frac{1}{2} \int dr \hat{h}_{\mu\nu}(r)\hat{T}_{\mu\nu}(r),$$  \hspace{1cm} (15)

where $r$ denotes the 3-vector.

We will now consider separately the coupling induced by the components $\hat{T}_{00}$ in the static limit and by $\hat{T}_{ij}$ in the case where the harmonic oscillators interact with the GWs.

A. Entanglement via virtual graviton

Let us consider two particles of mass $m$ (which will form the two oscillating systems). The two particles are generating the following current in the static limit:

$$\hat{T}_{00}(r) = mc^2 (\delta(r - \hat{r}_A) + \delta(r - \hat{r}_B)),$$  \hspace{1cm} (16)

where $\hat{r}_A = (\hat{x}_A, 0, 0)$, $\hat{r}_B = (\hat{x}_B, 0, 0)$ denote the positions of the two matter systems. The Fourier transform of the current is given by

$$\hat{T}_{00}(k) = \frac{mc^2}{(2\pi)^3} (e^{ik\cdot \hat{r}_A} + e^{ik\cdot \hat{r}_B}),$$  \hspace{1cm} (17)

where $k$ denotes 3-momentum.

Following the canonical quantisation of graviton in a weak field regime \[4\], we decompose $h_{\mu\nu} = \gamma_{\mu\nu} - (1/2)\delta_{\mu\nu}\gamma$ around a Minkowski background (where we use the convention $\gamma \equiv \gamma_{\lambda\lambda}$). The two distinct modes, i.e. the spin-2, $\gamma_{\mu\nu}$, and the spin-0, $\gamma$, can be treated as independent variables. They are promoted as self-adjoint operators, and decomposed into:

$$\hat{\gamma}_{\mu\nu} = A \int dk \sqrt{\frac{\hbar}{2\omega_k (2\pi)^3}} (\hat{P}_{\mu\nu}(k)e^{-ik\cdot r} + \text{H.c}),$$  \hspace{1cm} (18)

$$\hat{\gamma} = 2A \int dk \sqrt{\frac{\hbar}{2\omega_k (2\pi)^3}} (\hat{P}^\dagger(k)e^{-ik\cdot r} + \text{H.c}),$$  \hspace{1cm} (19)

where

$$\left[ \hat{P}_{\mu\nu}(k), \hat{P}_{\lambda\rho}^\dagger(k') \right] = [\delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda}]\delta(k - k'),$$  \hspace{1cm} (20)

$$\left[ \hat{P}(k), \hat{P}^\dagger(k') \right] = -\delta(k - k').$$  \hspace{1cm} (21)

The graviton Hamiltonian is now given by \[4\]:

$$\hat{H}_g = \int dk \hbar \omega_k \left( \frac{1}{2} \hat{P}_{\mu\nu}(k)\hat{P}_{\mu\nu}(k) - \hat{P}_0(k)\hat{P}^\dagger_0(k) \right).$$  \hspace{1cm} (22)

We are interested in computing the change in energy $\Delta H_g$ of the graviton vacuum arising from the interaction with the matter. In the static limit (where we neglect the motion of the two harmonic oscillators), the interaction Hamiltonian can be written in a simple form:

$$\hat{H}_{\text{int}} = \frac{1}{2} \int dr \left[ \hat{\gamma}_{00}(r) - (1/2)\hat{\gamma}(r) \right] \hat{T}_{00}(r).$$  \hspace{1cm} (23)

We can now compute the shift to the energy of the graviton vacuum using perturbation theory. The first order term vanishes\[6\] while the second order term in the perturbation theory yields:

$$\Delta H_g = \int dk k E_k \left( \langle 0 | \hat{H}_{\text{int}} | k \rangle \langle k | \hat{H}_{\text{int}} | 0 \rangle \right) / (E_k - E_0),$$  \hspace{1cm} (24)

where $|k\rangle = (\hat{P}_{00}^\dagger(k) - \hat{P}_0^\dagger(k))|0\rangle$ is the one particle state constructed in the unperturbed vacuum, $E_k = E_0 + \hbar \omega_k$ is the energy of the one-particle state, and $E_0$ is the energy of the vacuum state. The mediated graviton is now off-shell by virtue of the integration of all possible momentum $k$ — and hence does not obeys classical equations of motions. Using Eqs. (14, 15, 18, 19) and (23) we readily find\[7]

$$\langle k | \hat{H}_{\text{int}} | 0 \rangle = \frac{A}{2} \int dr e^{-ik\cdot r} \hat{T}_{00}(r).$$  \hspace{1cm} (25)

\[6\]The first order contribution to the energy is given by $\langle 0 | \hat{H}_{\text{int}} | 0 \rangle = 0$, where $|0\rangle$ denotes the unperturbed graviton vacuum. This is due to the fact that $\hat{H}_{\text{int}}$ depends linearly on $\hat{\gamma}_{\mu\nu}$, $\hat{\gamma}$ which are themselves linear combinations of creation and the annihilation operators, $\hat{P}_{\mu\nu}, \hat{P}_{\mu\nu}^\dagger, \hat{P}_0, \hat{P}_0^\dagger$. Hence $0 | \hat{H}_{\text{int}} | 0 \rangle$ depends only linearly on $\hat{P}_{\mu\nu}^\dagger, \hat{P}_{\mu\nu}, \hat{P}_0^\dagger, \hat{P}_0$ and thus vanishes (as $\langle P|0\rangle = 0$ and $0 | \hat{P}_0^\dagger = 0$ and similarly for the other operators). The non-vanishing contribution will come from the second order term in the perturbation theory [13, 18, 47].

\[7\]Inserting the definition of $\hat{H}_{\text{int}}$ from Eq. (22) (and the definitions of $\hat{\gamma}_{\mu\nu}$ and $\hat{\gamma}$ from Eqs. (13) and (19), respectively) we encounter the following expression $\langle k | (\hat{P}_{00}^\dagger(k') - \hat{P}_0^\dagger(k')) | 0 \rangle$. Using
Using the definition of the Fourier transform we then obtain a simple expression

\[ (0|\hat{H}_{\text{int}}|k)\langle k|\hat{H}_{\text{int}}|0\rangle = \frac{\hbar A^2 T_{00}^{\dagger}(k)T_{00}(k)}{8\omega k}. \]  

(26)

From Eq. (24) we then readily find:

\[ \Delta \hat{H}_g = -A^2 \int dk \frac{T_{00}^{\dagger}(k)T_{00}(k)}{8\pi^2 k^2}. \]  

(27)

Performing the momentum integration using spherical coordinates we then find the result

\[ \Delta \hat{H}_g = -\frac{A^2 m^2 c^2}{16\pi |r_A - r_B|}, \]  

(28)

where we have omitted the self-energy terms of individual particles. We finally insert \( A = \sqrt{16\pi G/c^2} \) into Eq. (28) to find Newton’s potential:

\[ \Delta \hat{H}_g = -\frac{Gm^2}{|\hat{x}_A - \hat{x}_B|}. \]  

(29)

\[ |k\rangle = (\hat{P}_{00}^{\dagger}(k) - \hat{P}^{\dagger}(k))|0\rangle \]  

we then find

\[ \langle 0|\hat{P}_{00}(k)\hat{P}_{00}^{\dagger}(k') + \hat{P}(k)\hat{P}^{\dagger}(k')|0\rangle = \delta(k - k'), \]  

while the other terms vanish as the vacuum state satisfies

\[ \hat{P}_{00}(k)|0\rangle = \hat{P}(k)|0\rangle = 0. \]

The two terms on the right-hand side can then be rewritten as \( \langle 0|\hat{P}_{00}(k), \hat{P}_{00}^{\dagger}(k')|0\rangle \) and \( \langle 0|\hat{P}(k), \hat{P}^{\dagger}(k')|0\rangle \), where we have used the definition of the commutator \( [\hat{O}_1, \hat{O}_2] = \hat{O}_1\hat{O}_2 - \hat{O}_2\hat{O}_1 \) (as well as again the definition of the of the vacuum state). Using now the commutation relations defined in Eq. (22) and summing the two terms we then finally obtain

\[ \langle k|(\hat{P}_{00}^{\dagger}(k') - \hat{P}^{\dagger}(k'))|0\rangle = \delta(k - k'). \]

Note that in the final expression the virtual graviton has no role to play. The virtual excitations are not physical states, they are also not on-shell. They do not satisfy the classical equations of motion. Instead, now the interaction Hamiltonian \( \hat{H}_{AB} \) contains only the operators of the two harmonic oscillators \( \delta \hat{x}_A, \delta \hat{x}_B \). Yet it is critical to realise that the product \( \delta \hat{x}_A \delta \hat{x}_B \) would not have arisen if we had assumed a real-valued shift of the energy of the gravitational field. Indeed, a classical gravitational field is unable to produce the operator-valued shift in Eq. (24) (and hence the quantum interaction potential in Eq. (22)). We must thus conclude that gravitationally induced entanglement is indeed a quantum signature of the gravitational field.

We will now use the modes in Eq. (3) to find

\[ \hat{H}_{AB} \equiv \frac{2Gm^2}{d^3} \delta \hat{x}_A \delta \hat{x}_B. \]  

(32)

It is instructive to compare the obtained results for two harmonic oscillators to the results obtained previously for two interferometers. In both cases, the action is proportional to \( S = Er/h \), where the interaction energy of the system is given by \( E \sim \hat{H}_{AB} \) and \( r \) is the coherence time scale. Considering the setup in Eq. (19), and setting \( \Delta \hat{a} \sim \frac{S}{r} \), we then recover the entanglement phase \( \Delta \phi \sim (2Gm^2/h)(\delta \hat{x}/d)^2 r \), where we have assumed \( \delta \hat{x}_A \sim \delta \hat{x}_B \sim \delta \hat{x}_A \) for the localized spatial superpositions of the two test masses.

With the help of this propagator, one can find the gravitational potential, i.e. the non-relativistic scattering due to an exchange of an off-shell graviton. The gravitational potential is given by \( \Phi(r) = -\frac{8\pi Gm^2}{(2\pi)^3} \int dk^3 k_1 T_{00}^{\dagger}(k)T_{00}(r) \) for \( k \sim \frac{\Delta \phi}{2} \), which is the same as what we have obtained in Eq. (22). The only difference here is that we have computed the potential by using the full graviton propagator and the scattering amplitude between the two masses via the interaction of a virtual/off-shell graviton. In the text we have computed the change in the graviton vacuum. However, in the non-relativistic limit both the results give rise to the same conclusion.

We thus find that the change in the graviton energy, \( \Delta \hat{H}_g \), due to the interaction between the graviton and the matter is an operator valued function of the two matter systems, i.e.

\[ \Delta \hat{H}_g \equiv f(\hat{x}_A, \hat{x}_B). \]  

(30)

If the two matter systems do not have a sharply defined positions (such as when placed in a spatial superposition or some other non-classical state) then the change in the graviton energy \( \Delta \hat{H}_g \) will not be a real number, as required in a classical theory of gravity, but rather an operator-valued quantity, a bona fide quantum entity.

We now wish to calculate the excited wave function \( |\psi_I\rangle \) of the two harmonic oscillators to establish the link between entanglement and LOQC discussed in Sec. III. We first use Eq. (11) and expand Eq. (29) to find

\[ \Delta \hat{H}_g \approx -\frac{Gm^2}{d} + \frac{Gm^2}{d^2} (\delta \hat{x}_B - \delta \hat{x}_A) - \frac{Gm^2}{d^3} (\delta \hat{x}_B - \delta \hat{x}_A)^2 \]

(31)

The last term gives the lowest-order matter-matter interaction

\[ \hat{H}_{AB} \equiv \frac{2Gm^2}{d^3} \delta \hat{x}_A \delta \hat{x}_B. \]  

(32)

Note that in the final expression the virtual graviton has no role to play. The virtual excitations are not physical states, they are also not on-shell. They do not satisfy the classical equations of motion. Instead, now the interaction Hamiltonian \( \hat{H}_{AB} \) contains only the operators of the two harmonic oscillators \( \delta \hat{x}_A, \delta \hat{x}_B \). Yet it is critical to realise that the product \( \delta \hat{x}_A \delta \hat{x}_B \) would not have arisen if we had assumed a real-valued shift of the energy of the gravitational field. Indeed, a classical gravitational field is unable to produce the operator-valued shift in Eq. (24) (and hence the quantum interaction potential in Eq. (22)). We must thus conclude that gravitationally induced entanglement is indeed a quantum signature of the gravitational field.

With the help of this propagator, one can find the gravitational potential, i.e. the non-relativistic scattering due to an exchange of an off-shell graviton. The gravitational potential is given by \( \Phi(r) = -\frac{8\pi Gm^2}{(2\pi)^3} \int dk^3 k_1 T_{00}^{\dagger}(k)T_{00}(r) \) for \( k \sim \frac{\Delta \phi}{2} \), which is the same as what we have obtained in Eq. (22). The only difference here is that we have computed the potential by using the full graviton propagator and the scattering amplitude between the two masses via the interaction of a virtual/off-shell graviton. In the text we have computed the change in the graviton vacuum. However, in the non-relativistic limit both the results give rise to the same conclusion.
where we have defined the coupling
\[ g = \frac{Gm}{d^3 \omega_m}. \]  

Using $\hat{H}_{AB}$ as the interaction Hamiltonian in Eq. 3 we find that the only non-zero coefficient emerges from the term $\sim \hat{a} \hat{b}^\dagger$ and is given by:
\[ C_{11} = -\frac{g}{2 \omega_m}. \]  

We note that the $a^1 b^1$ term generates the first excited states in the harmonic oscillators (with energy $E_1 = E_0 + \hbar \omega_m$). In addition, we also have the term $C_{00} = 1$ corresponding to the unperturbed state.

The final state in Eq. 17 thus simplifies to (up to first order in the perturbation theory, and by setting $\lambda = 1$):
\[ |\psi_f\rangle \equiv \frac{1}{\sqrt{1 + (g/(2\omega_m))^2}}[|0\rangle|0\rangle - \frac{g}{2 \omega_m} |1\rangle|1\rangle], \]  

which is an entangled state involving the ground and the first excited states of the two harmonic oscillators. The concurrence in Eq. 13 reduces to
\[ C \equiv \sqrt{2(1 - \frac{1}{1 + (g/(2\omega_m))^2})} \approx \sqrt{2 \frac{g}{\omega_m}}, \]  

which is valid when the parameter $g/\omega_m \ll 1$ is small.

Inserting the coupling from Eq. 34 we find the concurrence is given by:
\[ C = \frac{\sqrt{2Gm}}{d^3 \omega_m}. \]  

We thus see that the degree of entanglement grows linearly with the mass of the oscillator and inversely with the distance between the two oscillators (inverse cubic) as well as with the frequency of the harmonic trap (inverse square).

Let us reiterate the key finding. If the underlying gravitational field were classical (specifically, obeying LOCC), then the final state of the matter components, i.e. the two harmonic oscillator states, would have never evolved to the entangled state $|\psi_f\rangle$, but would have rather remained in an unentangled/separable state. Conversely, if the gravitational field is quantized (and hence obeys LOQC) then we have shown that it can give rise to the entangled state $|\psi_f\rangle$.

\[ [\hat{P}_\lambda(k), \hat{P}_\lambda^\dagger(k')] = \delta(k - k'), \]  

and we have defined the Fourier transform of the $ij$ components of the stress-energy tensor
\[ \hat{T}_{ij}(k) = \frac{1}{\sqrt{(2\pi)^3}} \int d\mathbf{r} e^{-i \mathbf{k} \cdot \mathbf{r}} \hat{T}_{ij}(r). \]  

We now consider the two harmonic oscillators generating the currents with velocities $\mathbf{v}_i, \mathbf{v}_j$.
\[ \hat{T}_{ij}(r) \equiv mv_i v_j (\delta(\mathbf{r} - \mathbf{r}_A) + \delta(\mathbf{r} - \mathbf{r}_B)), \]  

where $\mathbf{r}_A = (\dot{x}_A, 0, 0)$, $\mathbf{r}_B = (\dot{x}_B, 0, 0)$ denote the positions of the two matter systems. We are considering only the motion along the $x$-axis where the only nonzero contribution will be from the term $\hat{T}_{11}$, but we will keep for

\[ \sum_\lambda \sum_n c^\dagger_n(n)c_n(n) = P_{ik} P_{\lambda k} + P_{il} P_{\lambda l} - P_{ij} P_{\lambda j}, \]  

The basis tensor satisfy the completeness relation:

\[ \sum_\lambda \sum_n c^\dagger_n(n)c_n(n) = P_{ik} P_{\lambda k} + P_{il} P_{\lambda l} - P_{ij} P_{\lambda j}, \]  

where $P_{ij} \equiv P_{ij}(n) = \delta_{ij} - n_i n_j$. 

\[ \text{The basis tensor satisfy the completeness relation:} \]
now the more general expressions. The Fourier transform of the current is then given by
\[ \hat{T}_{ij}(k) = \frac{1}{m\sqrt{(2\pi)^3}}(\hat{p}_{A_i}\hat{p}_{A_j}e^{ik\cdot r_A} + \hat{p}_{B_i}\hat{p}_{B_j}e^{ik\cdot r_B}), \]  
where we have used \( p_i = mv_i \).

The energy-shift of the graviton vacuum \(|0\rangle\) is given by the second-order perturbation theory (while the first order perturbation will vanish),
\[ \Delta \hat{H}_g = \sum_{\lambda} \int dk |\hat{H}_{\text{int}}(k, \lambda)|\langle k, \lambda|\hat{H}_{\text{int}}|0\rangle. \]
where \(|k, \lambda\rangle = \hat{P}_k^\dagger(k)|0\rangle\) is the one particle state constructed on the unperturbed vacuum, \( E_0 \) is the energy of the vacuum state, and \( E_k = E_0 + \hbar\omega_k \) is the energy of the one-particle state. We can then readily evaluate
\[ |k, \lambda\rangle\hat{H}_{\text{int}}|0\rangle = -\frac{A}{2}\sqrt{\frac{g}{2\omega_k}}c_{ij}(k)\hat{T}_{ij}(k). \]
From Eq. (47) we then find:
\[ \Delta \hat{H}_g = -A^2\sum_{\lambda} \int dk c_{ij}^\dagger(k)\hat{T}_{ij}(k)c_{ij'}(k)\hat{T}_{ij'}(k)k^2. \]
We can further simplify the expression by recalling the completeness relation. In addition we now use the fact that the two harmonic oscillators are confined along the x-axis, where we set \( \hat{p}_{A_y} = \hat{p}_{A_z} = \hat{p}_{B_y} = \hat{p}_{B_z} = 0 \), and write \( \hat{p}_A = \hat{p}_{Ax}, \hat{p}_B = \hat{p}_{Bx} \), and \( \hat{r}_A = (x_A, 0, 0) \), \( \hat{r}_B = (x_B, 0, 0) \). After some algebra we find that the expression in Eq. (48) simplifies to
\[ \Delta \hat{H}_g = -\frac{G\hat{p}_A^2\hat{p}_B^2}{c^4m^2|x_A - x_B|}. \]
Note that if \( \hat{p}_A = \hat{p}_B = 0 \) vanishes, the change in energy of the graviton states vanishes in this case. There is no emission of the GWs as the current vanishes in this limit, see Eq. (14). However, a quantum system retains its zero point fluctuations and hence we find \( \langle \hat{p}_A^2 \rangle = \langle \hat{p}_B^2 \rangle \sim \frac{\hbar m\omega_m}{2} \) even for ground states of the two harmonic oscillators (using Eq. 4 and the canonical commutation relations).

Let us make a brief comment on Eq. (19) which emerged by assuming quantized GWs – the change of the graviton vacuum,
\[ \Delta \hat{H}_g \equiv f(\hat{p}_A^2, \hat{p}_B^2, \hat{x}_A, \hat{x}_B), \]
is an operator-valued shift depending on the matter operators. On the other hand, if we would have assumed a classical gravitational field we could have only generated a real-valued shift \( \Delta H_g \) in complete analogy to what we have discussed below Eq. (29) for the case of virtual gravitons. Hence, we must conclude that by placing matter in a non-classical state (which does not have sharp values of position/momentum) we can reveal the quantum nature of GWs.

We will be interested in computing the lowest order correction for the final matter state \(|\psi_f\rangle\). From Eq. (19) we can extract the lowest order non-trivial quantum interaction term\(^\text{12}\):
\[ \hat{H}_{AB} \sim -\frac{G\hat{p}_A^2\hat{p}_B^2}{c^4m^2d} + \cdots. \]
Note that at the lowest order the coupling in \( \hat{x}_A, \hat{x}_B \) does not occur, the interaction Hamiltonian is dominated by the momentum operators \( \hat{p}_A, \hat{p}_B \). We now use the modes in Eq. (4) to find
\[ \hat{H}_{AB} \approx -\hbar g(\hat{a}^\dagger - \hat{a})(\hat{b}^\dagger - \hat{b})^2. \]
where the coupling is given by
\[ g = \frac{G\hbar\omega_m^2}{4c^4d} + \cdots. \]
As we will see the only term that is relevant in our case is \( (\hat{a}^\dagger \hat{b}^\dagger)^2 \), which signifies that the final matter state is a linear combination of \(|0\rangle|0\rangle \) and \(|2\rangle|2\rangle \). Hence, at the lowest order the gravitons carry twice the energy of the harmonic oscillators, i.e. \( \omega_k = 2\omega_m \). This matches the result found previously about quadrupole radiation and graviton emission\(^\text{22}\).

In particular, using \( \hat{H}_{AB} \) as the interaction Hamiltonian in Eq. (8) we find that the only non-zero perturbation coefficient emerges from the term \( \sim \hat{a}^\dagger \hat{b}^\dagger \hat{a} \hat{b} \) and is given by:
\[ C_{22} = \frac{g}{2\omega_m}. \]
Here we have used the fact that energy momentum conservation constrains the energy of the GWs emitted from the source
\[ E_2 = E_0 + 2\hbar\omega_m. \]
Note that it is twice the frequency of the harmonic oscillators. In addition, we also have the term \( C_{00} = 1 \), corresponding to the unperturbed state. The final state in Eq. (11) thus simplifies to (setting \( \lambda = 1 \)):
\[ |\psi_f\rangle \equiv \frac{1}{\sqrt{1 + (g/(2\omega_m))^2}}(|0\rangle|0\rangle + \frac{g}{2\omega_m}|2\rangle|2\rangle) \]
which is an entangled state involving the ground and the second excited states of the harmonic oscillators (up to
\( ^\text{12}\)It is again instructive to estimate the entanglement phase. We find \( \Delta \phi \sim G^2\hbar \tau/(c^2m^2\hbar d) \), where \( \tau \) is the coherence time scale. As expected such effects are thus typically suppressed in comparison to the phase accumulated from the exchange of virtual gravitons\(^\text{22}\).
first order in the perturbation theory). The concurrence in Eq. (13) reduces to
\[ C \equiv \sqrt{2(1 - \frac{1 + (g/(2\omega_m))^2}{1 + (g/(2\omega_m))^4})} \approx \sqrt{\frac{g}{\omega_m}}. \] (57)

which is valid when the parameter \( g/\omega_m \ll 1 \). After inserting the coupling from Eq. (53), we find the concurrence to be:
\[ C = \frac{\sqrt{2G\hbar\omega_m}}{4c^3d}. \] (58)

Note that the degree of entanglement grows linearly with frequency of the harmonic oscillators, does not depend on the mass, and scales inversely with the distance between the two oscillators.

The concurrence involving the harmonic oscillators thus behaves very differently for the two cases we have discussed above (virtual excitations of the graviton and GWs). The concurrence in the case of the virtual exchange of a graviton dominates over the interaction of the GWs with the harmonic oscillators, provided
\[ \omega_m^2d^2 < \frac{4mc^4}{\hbar}. \] (59)

If the natural frequency of the oscillators is assumed to be roughly equal to their masses, i.e. \( \omega_m \sim m \), we get \( \omega_m d < 2 \) in natural units where \( c = \hbar = 1 \). The entanglement from the GWs could thus in principle dominate over the off-shell exchange of the graviton provided \( d \gg 1/\omega_m \). Although, at this point it is not clear how to devise an experiment to probe the quantum nature of GWs using the discussed mechanism, it would be nonetheless interesting to explore whether quantized GWs leave some other indelible effect – we leave this for future research.

Let us conclude by highlighting the link between LOCC/LOQC and GWs. If the GWs were treated classically, then the final state of the two harmonic oscillator states would have never evolved to an entangled state like Eq. (50). Indeed, a classical field is unable to give the operator-valued shift of the vacuum energy in Eq. (50) which led to the quantum coupling in Eq. (52) (i.e., a cross-product of matter operators). Hence we again conclude that gravitationally-induced entanglement is a signature of the quantum nature of the gravitational field. For a more detailed discussion about LOCC/LOQC and entanglement see Sec. IV.A where a completely analogous argument applies to the exchange of virtual gravitons.

V. DISCUSSION

We have selected two specific examples to reinforce the importance of the quantum interaction in the QGEM protocol. In the first example, we considered two quantum harmonic oscillators separated by a distance \( d \) interacting via the off-shell/virtual exchange of the graviton comprising of the spin-2 and spin-0 components in a static limit. We have shown that the quantum nature of the graviton (for both spin-2 and spin-0, \( \hat{J}_{\mu\nu} \equiv \hat{\gamma}_{\mu\nu} - (1/2)\hat{\gamma} \)) is essential to create an entangled state with the ground and excited states of the harmonic oscillators forming the Schmidt basis. In the second example, we have shown that harmonic oscillators also emit/absorb GWs. The quantum nature of GWs is crucial to entangle the ground state and the second excited state of the harmonic oscillators. Here the two helicity states (+, x) of the GWs are required to be quantized.

We have obtained all the results relying only on elementary perturbation theory: the wave function was evaluated up to the first order, and the correction to the graviton vacuum was computed up to the second order (to obtain the non-vanishing contribution to the vacuum energy). Both the wave function calculations and the correction to the energy of the vacuum suggest that the quantum interaction between the graviton and the matter is crucial to obtain entanglement, reinforcing that the LOCC can not yield or lead to the increment in the entanglement. For the two cases of interest (GWs and off-shell exchange of the graviton) the calculation followed the same steps. We computed the entanglement concurrence and showed that the concurrence is always positive for a quantum gravitational field (indicating entanglement), but would remain zero for a classical gravitational field (no entanglement).

So far we have kept our investigation limited to the local quantum interaction between matter and the gravitational field – our \( \hat{H}_{\text{int}} \) was strictly local. It would be interesting to study what would happen if the locality in the gravitational interaction is abandoned \([14, 22, 51]\). Giving up local gravitational interaction will help us to further investigate the entanglement in theories beyond GR, and in quantum theories of gravity where non-local interactions enter in various manifestations, see \([3, 56–58]\).

In summary, our results corroborate the importance of the QGEM experiment, which relies on the fact that the two quantum superposed masses kept at a distance can entangle via the quantum nature of the graviton. This would be crucial in unveiling the quantum properties of a spin-2 graviton which is hitherto a hypothetical particle responsible for the fluctuations in the space-time in the context of a perturbative quantum theory of gravity.

ACKNOWLEDGMENTS

MT and SB would like to acknowledge EPSRC grant No.EP/N031105/1, SB the EPSRC grant EP/S000267/1, and MT funding by the Leverhulme Trust (RPG-2020-197). MS is supported by the Fundamentals of the Universe research program within the University of Groningen. AM’s research is funded by the Netherlands Organisation for Science and Research (NWO) grant number 680-91-119.
[1] C. M. Will, “The Confrontation between General Relativity and Experiment,” Living Rev. Rel. 17 (2014), 4 doi:10.12942/lrr-2014-4 [arXiv:1403.7377 [gr-qc]].

[2] S. W. Hawking and G. F. R. Ellis, “The Large Scale Structure of Space-Time,” doi:10.1017/CBO9780511524646

[3] C. Kiefer, Quantum Gravity, pp. 709?722, 2014.

[4] S. N. Gupta, Quantization of Einstein’s Gravitational Field: Linear Approximation Proc. Phys. Soc. A 65 161, 1952.

[5] S. N. Gupta, Gravitation and electromagnetism, Physical Review, vol. 96, no. 6, p. 1683, 1954.

[6] R. P. Feynman, Quantum theory of gravitation, Acta Phys. Polon., vol. 24, pp. 697?722, 1963.

[7] B. S. DeWitt, Quantum Theory of Gravity. 1. The Canonical Theory, Phys. Rev., vol. 160, pp. 1113 1148, 1967. B. S. DeWitt, Quantum Theory of Gravity. 2. The Manifestly Covariant Theory, Phys. Rev., vol. 162, pp. 1195?1230, 1967.

[8] J. F. Donoghue, “General relativity as an effective field theory: The leading quantum corrections,” Phys. Rev. D 50 (1994), 3874-3888 doi:10.1103/PhysRevD.50.3874 [arXiv:gr-qc/9405057 [gr-qc]].

[9] F. Dyson, Is a graviton detectable, in XVIIth International Congress on Mathematical Physics, pp. 670 682, World Scientific, 2014.

[10] A. Ashoorioon, P. S. Bhupal Dev, and A. Mazumdar, Implications of purely classical gravity for inflationary tensor modes, Mod. Phys. Lett. A, vol. 29, no. 30, p. 1450163, 2014.

[11] J. Martin and V. Vennin, Obstructions to Bell CMB Experiments, Phys. Rev. D, vol. 96, no. 6, p. 063501, 2017.

[12] A. Addazi, et.al, et al. “Quantum gravity phenomenology at the dawn of the multi-messenger era – A review,” arXiv:2111.05659 [hep-ph].

[13] B. P. Abbott et al. [LIGO Scientific and Virgo], “Tests of general relativity with GW150914,” Phys. Rev. Lett. 116 (2016) no.22, 221101 [erratum: Phys. Rev. Lett. 121 (2018) no.12, 129902] doi:10.1103/PhysRevLett.116.221101 [arXiv:1602.03841 [gr-qc]].

[14] S. Bose, A. Mazumdar, G. W. Morley, H. Ulbricht, M. Töröš, A. Mazumdar, and A. Paiger, Large splitting massive schrödinger kittens, arXiv preprint arXiv:2106.10118 [quant-ph].

[15] R. J. Marshman, A. Mazumdar, R. Fölan, and S. Bose, Large splitting massive schrödinger kittens, arXiv preprint arXiv:2105.01094 [2021].

[16] T. W. van de Kamp, R. J. Marshman, S. Bose and A. Mazumdar, “Quantum Gravity Witness via Entanglement of Masses: Casimir Screening,” Phys. Rev. A 102 (2020) no.6, 062807 doi:10.1103/PhysRevA.102.062807 [arXiv:2006.00831 [quant-ph]].

[17] H. C. Nguyen and F. Bernards, Entanglement dynamics of two mesoscopic objects with gravitational interaction, arXiv preprint arXiv:1906.11184 [2019].

[18] J. S. Pedernales, G. W. Morley, and M. B. Plenio, Motional dynamical decopling for matter-wave interferometry, arXiv preprint arXiv:1906.00835 [2019].

[19] R. J. Marshman, A. Mazumdar, R. Fölman, and S. Bose, Large splitting massive schrödinger kittens, arXiv preprint arXiv:2105.01094 [2021].
[34] H. Miao, D. Martynov, H. Yang, and A. Datta, Quantum correlations of light mediated by gravity, Physical Review A, vol. 101, no. 6, p. 063804, 2020.
[35] M. Schut, J. Tilly, R. J. Marshman, S. Bose and A. Mazumdar, “Improving resilience of the Quantum Gravity Induced Entanglement of Masses (QGEM) to decoherence using 3 superpositions,” [arXiv:2101.14695 [quant-ph]].
[36] A. Datta and H. Miao, Signatures of the quantum nature of gravity in the differential motion of two masses, arXiv preprint arXiv:2104.04414 (2021).
[37] T. Krisnanda, G. Y. Tham, M. Paternostro, and T. Paterek, Observable quantum entanglement due to gravity, npj Quantum Information, vol. 6, no. 1, pp. 176, 2020.
[38] T. Weiss, M. Roda-Llordes, E. Torrentegui, M. Aspelmeyer, O. Romero-Isart Phys. Rev. Lett. 127, 023601 (2021).
[39] S. Rijavec, M. Carlesso, A. Bassi, V. Vedral, and C. Marletto, Decoherence effects in non-classicality tests of gravity, New J. Phys., vol. 23, no. 4, p. 173002, 2020.
[40] B. Yi, U. Sinha, D. Home, A. Mazumdar, and S. Bose, Massive spatial qubits for testing macroscopic non-classicality and casimir induced entanglement, [arXiv:2106.11906v2 [quant-ph]].
[41] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Mixed-state entanglement and quantum error correction, Physical Review A, vol. 54, no. 5, p. 3824, 1996.
[42] V. Balasubramanian, M. B. McDermott and M. Van Raamsdonk, “Momentum-space entanglement and renormalization in quantum field theory,” Phys. Rev. D 86 (2012), 045014 doi:10.1103/PhysRevD.86.045014 [arXiv:1108.3568 [hep-th]].
[43] S. Hill and W. K. Wootters, “Entanglement of a pair of quantum bits,” Phys. Rev. Lett. 78 (1997), 5022-5025 doi:10.1103/PhysRevLett.78.5022 [arXiv:quant-ph/9703041 [quant-ph]].
[44] P. Rungta, V. Bužek, C.M. Caves, M. Hillery, and G. J. Milburn, “Universal state inversion and concurrence in arbitrary dimensions,” Phys. Rev. A 64, 042315 (2001).
[45] W. P. Bowen, G. J. Milburn, Quantum Optomechanics, CRC Press, 2016
[46] A. Zee, Quantum field theory in a nutshell. Vol. 7. Princeton university press, 2010.
[47] D.J. Griffiths, Introduction to quantum mechanics. Pearson International Edition (Pearson Prentice Hall, Upper Saddle River, 2005).
[48] C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, Photons and Atoms: Introduction to Quantum Electrodynamics, 1997.
[49] M. D. Scadron, Advanced Quantum Theory and Its Applications Through Feynman Diagrams, 1991.
[50] D. Kafri, J. M. Taylor and G. J. Milburn, New J. Phys. 16 (2014), 065020 doi:10.1088/1367-2630/16/6/065020 [arXiv:1401.0946 [quant-ph]].
[51] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, 1972.
[52] E. T. Tomboulis, Superrenormalizable gauge and gravitational theories, [arXiv:hep-th/9702146 [hep-th]].
[53] T. Biswas, A. Mazumdar and W. Siegel, “Bouncing universes in string-inspired gravity,” JCAP 03 (2006), 009 doi:10.1088/1475-7516/2006/03/009 [arXiv:hep-th/0508194 [hep-th]].
[54] T. Biswas, E. Gerwick, T. Koivisto and A. Mazumdar, “Towards singularity and ghost free theories of gravity,” Phys. Rev. Lett. 108 (2012), 031101 doi:10.1103/PhysRevLett.108.031101 [arXiv:1110.5240 [gr-qc]].
[55] L. Modesto, “Super-renormalizable Quantum Gravity,” Phys. Rev. D 86 (2012), 044005 doi:10.1103/PhysRevD.86.044005 [arXiv:1107.2403 [hep-th]].
[56] C. de Lacroix, H. Erbin, S. P. Kashyap, A. Sen and M. Verma, “Closed Superstring Field Theory and its Applications,” Int. J. Mod. Phys. A 32 (2017) no.28n29, 1730021 doi:10.1142/S0217751X17300216 [arXiv:1703.06410 [hep-th]].
[57] R. Loll, “Quantum Gravity from Causal Dynamical Triangulations: A Review,” Class. Quant. Grav. 37 (2020) no.1, 013002 doi:10.1088/1361-6382/ab57c7 [arXiv:1905.08669 [hep-th]].
[58] R. D. Sorkin, “Does locality fail at intermediate length-scales,” [arXiv:gr-qc/0703099] [gr-qc].