An Algorithmic Study of the Hypergraph Turán Problem

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Abstract

We propose an algorithmic version of the hypergraph Turán problem (AHTP): given a \( t \)-uniform hypergraph \( H = (V,E) \), the goal is to find the smallest collection of \( (t-1) \)-element subsets of \( V \) such that every hyperedge \( e \in E \) contains one of these subsets. In addition to its inherent combinatorial interest—for instance, the \( t = 3 \) case is connected to Tuza’s famous conjecture on covering triangles of a graph with edges—variants of AHTP arise in recently proposed reductions to fundamental Euclidean clustering problems.

AHTP admits a trivial factor \( t \) approximation algorithm as it can be cast as an instance of vertex cover on a structured \( t \)-uniform hypergraph that is a “blown-up” version of \( H \). Our main result is an approximation algorithm with ratio \( t^2 + o(t) \). The algorithm is based on rounding the natural LP relaxation using a careful combination of thresholding and color coding.

We also present results motivated by structural aspects of the blown-up hypergraph. The blown-up is a simple \( t \)-uniform hypergraph with hyperedges intersecting in at most one element. We prove that vertex cover on simple \( t \)-uniform hypergraphs is as hard to approximate as general \( t \)-uniform hypergraphs. The blown-up hypergraph further has many forbidden structures, including a “tent” structure for the case \( t = 3 \). Whether a generalization of Tuza’s conjecture could also hold for tent-free \( 3 \)-uniform hypergraphs was posed in a recent work. We answer this question in the negative by giving a construction based on combinatorial lines that is tent-free, and yet needs to include most of the vertices in a vertex cover.

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1 Introduction

What is the largest number of edges in a $n$ vertex graph without a copy of the $r$-clique $K_r$? This question is answered by Turán’s theorem [Tur41] that among the graphs with no copy of $K_r$, the graph obtained by partitioning $n$ into $(r-1)$ equal or nearly equal parts and adding all edges across the partitions has the largest number of edges. A hypergraph analog of the above question is the following: what is the largest number of edges in a $n$ vertex $r$-uniform hypergraph without a copy of $K_r(3)$, the complete $3$-uniform hypergraph on $4$ vertices? Even though Turán’s theorem is one of the earliest results in extremal combinatorics, the above hypergraph problem is still open.

More generally, given a $r$-uniform hypergraph $F$, finding the largest number of edges in a $r$-uniform hypergraph that doesn’t have $F$ as a subhypergraph is referred to as the Hypergraph Turán problem. We refer the reader to an excellent survey due to Keevash [Kee11] on this topic. In this paper, we restrict ourselves to the case when $F$ is the complete $r$-uniform hypergraph on $(r+1)$ vertices. In this case, the question can be equivalently posed as finding the minimum size of a family $\mathcal{F} \subseteq \binom{[n]}{t}$ of subsets of $[n]$ with cardinality $(t-1)$ such that for every subset $S$ of $[n]$ of size $t$, there exists a $T \in \mathcal{F}$ such that $T$ is a subset of $S$.

The best known upper bound is due to [Sid97]: there exists a family $\mathcal{F} \subseteq \binom{[n]}{\frac{t}{t-1}}$ such that for every subset $S$ of $[n]$ of size $t$, there exists a subset $T \in \mathcal{F}$ such that $T$ is contained in $S$. On the other hand, the lower bound situation is rather dire, with only second-order improvements [CL99, LZ09] over the trivial $\frac{t}{t-1} \binom{n}{t}$ lower bound. Towards this, de Cain [dC94] conjectured that $t \binom{n}{\frac{t}{t-1}} |\mathcal{F}| \to \infty$ for any such $\mathcal{F} \subseteq \binom{[n]}{\frac{t}{t-1}}$. In this paper, we study an algorithmic version of the above problem.

**Problem 1. (Algorithmic Hypergraph Turán Problem(AHTP))** Given a $t$-uniform hypergraph $G = (V = [n], E)$, find the minimum size of a family $\mathcal{F} \subseteq \binom{[n]}{\frac{t}{t-1}}$ of subsets of $V$ of size $(t-1)$ such that for every hyperedge $e \in E$, there exists $T \in \mathcal{F}$ such that $T$ is a subset of $e$.

The problem is a generalization of the minimum vertex cover on graphs, which corresponds to the case when $t = 2$. It seems fundamental and is interesting in its own right from a combinatorial perspective. Our main computational motivation comes from recent work on the hardness of clustering in Euclidean metrics [CCL20]. In this paper, the authors obtained several hardness results for $k$-median and related problems on Euclidean metric spaces under a hardness assumption called the Johnson Coverage Hypothesis. In this problem, we are given a $t$-uniform hypergraph $H = ([n], E)$ and a positive integer $k$. The goal is to pick $k$ subsets $S_1, S_2, \ldots, S_k \in \binom{[n]}{\frac{t}{t-1}}$ so as to maximize the fraction of hyperedges $f \in E$ such that there is an $i \in [k], S_i \subseteq f$. They conjectured that this coverage problem cannot be approximated to a factor better than $1 - \frac{1}{t} + \epsilon$ for every $\epsilon > 0$ for large enough $t$. AHTP is the covering version of the above problem, and we believe understanding the computational complexity of AHTP is a natural and important step towards resolving the Johnson Coverage Hypothesis.

Note that the AHTP gets harder with increasing $t$, and thus the problem is NP-hard for every $t \geq 2$. We can view this problem as a special case of the hypergraph vertex cover on the blown-up hypergraph $H = G^{(t-1)}$ whose vertices are the set of all $(t-1)$ sized subsets that are contained in at least one edge of $G$, and corresponding to every edge $e$ in $G$, all the $(t-1)$-sized subsets of $e$ form an edge in $H$. Note that this blown-up hypergraph is a $t$-uniform hypergraph as well. Thus, there is a trivial factor $t$ approximation algorithm to the AHTP.

The original hypergraph Turán problem corresponds to the case when the input $G$ for the AHTP is the complete $t$-uniform hypergraph on $[n]$. There is a gap of $\Omega(\log t)$ between the current best upper bound and the best lower bound for the hypergraph Turán problem. Thus, a polynomial
time algorithm achieving \(o(\log t)\) approximation factor for AHTP would imply a breakthrough in the hypergraph Turán problem, either by improving the state of the art on the lower bound front or by improving the upper bound of [Sid97].

Our main result is that unlike the general hypergraph vertex cover, we can get an improved approximation algorithm for AHTP.

**Theorem 2.** For every integer \(t \geq 3\), there is a randomized polynomial time algorithm for AHTP that on any given \(t\)-uniform hypergraph \(G\) outputs a vertex cover of \(H = G^{(t-1)}\) whose expected size is at most \(\frac{t}{2} + 2\sqrt{t \ln t}\) times the optimal solution.

We round the standard Linear Programming relaxation to the vertex cover problem on the blown-up hypergraph \(H = G^{(t-1)} = (V(H), E(H))\). First, we use thresholding to ensure that the LP solution does not have any variables that are assigned values greater than \(\frac{2}{t}\). Let \(S\) be the set of vertices of \(H\) that are assigned non-zero LP value. The thresholding procedure ensures that every hyperedge \(e \in E(H)\) intersects with \(S\) in at least \(\frac{2}{t}\) vertices. We can bound the cardinality of \(S\) from above by \(\tau_{OPT}\) using the dual matching LP, where \(\text{OPT}\) is the cost of the optimal LP solution. Thus, our goal is to find an algorithm that outputs a vertex cover with cardinality at most \(\frac{|S|}{2}\). We achieve this by a color-coding technique: we randomly assign a color from \(\{0, 1\}\) to each vertex of \(G\) independently. Once we assign the colors to the vertices of \(G\), we argue that most of the hyperedges of \(G\) satisfy a certain uniformity property in the sense that each color appears at least 0.99 \((\frac{1}{3})\) times in the edge. We then use this uniformity property to find a small vertex cover in \(H\).

As mentioned earlier, AHTP is a special case of vertex cover on \(t\)-uniform hypergraphs. In fact, the blown-up hypergraph \(G^{(t-1)}\) satisfies a stronger property: any two edges intersect in at most one vertex. This is simply because any two distinct \(t\)-sized subsets of \([n]\) intersect in at most one \((t-1)\)-sized subset. A hypergraph in which any two edges intersect in at most one vertex is known as a simple hypergraph.\(^1\) Simple hypergraphs have been well studied in Graph Theory, especially in the context of Erdős-Faber-Lovász conjecture [Erd81, Erd88] and Ryser’s conjecture [FHMW17]. A natural question to ask is whether our algorithm can be extended to obtain better approximation algorithms for vertex cover on simple hypergraphs. We prove that this is not the case, and in fact, vertex cover on simple hypergraphs is as hard as vertex cover on general \(t\)-uniform hypergraphs.

**Theorem 3.** (Vertex cover on simple hypergraphs) For every \(\epsilon > 0\), Unless \(\text{NP} \subseteq \text{BPP}\), no polynomial time algorithm can approximate vertex cover on simple \(t\)-uniform hypergraphs within a factor of \(t - 1 - \epsilon\).

The blown-up hypergraph is also studied in a recent work [AZ20] on a generalization of Tuza’s conjecture. Tuza’s conjecture [Tuz81, Tuz90] states that \(\tau(H) \leq 2\nu(H)\) where \(\tau(H), \nu(H)\) are minimum vertex cover and the maximum matching respectively of the \(3\)-uniform hypergraph \(H\) obtained with edges of \(G\) as the vertices, and triangles of \(G\) as the edges of \(H\). Aharoni and Zerib [AZ20] conjectured that more generally, for any \(t\)-uniform hypergraph \(H\), the minimum vertex cover \(\tau(H')\) of \(H' = H^{(t-1)}\) is at most \(\lceil \frac{t}{2} \rceil\) of the maximum matching \(\nu(H')\). Tuza’s conjecture is a special case of their conjecture when \(t = 3\) and \(H\) has hyperedges corresponding to the triangles in a graph.

Krivelevich [Kri95] proved the fractional version of Tuza’s conjecture that \(\tau(G^{(2)}) \leq 2\tau^*(G^{(2)})\) for any \(3\)-uniform hypergraph \(G\), where \(\tau^*(H)\) denotes the minimum fractional vertex cover of a hypergraph \(H\). Note that \(\nu(H) \leq \tau^*(H) \leq \tau(H)\) for any hypergraph \(H\), and thus the fractional version is a necessary step towards proving the Tuza’s conjecture. As our algorithm is based on

\(^1\)Simple hypergraphs are also referred to as linear hypergraphs.
rounding standard LP relaxation, we obtain an upper bound on the integrality gap of the standard LP relaxation of AHTP. In other words, we prove a generalization of [Kri95] to the setting of Aharoni and Zerbib’s conjecture:

**Corollary 4.** For any \( t \)-uniform hypergraph \( G \),

\[
\tau(H) \leq \left( \frac{t}{2} + 2\sqrt{\ln t} \right) \tau^*(H)
\]

where \( \tau \) and \( \tau^* \) are the minimum vertex cover and minimum fractional vertex cover of the blown-up hypergraph \( H = G^{(t-1)} \) respectively.

As mentioned earlier, a key property of the blown-up hypergraphs \( G^{(t-1)} \) is that they are simple hypergraphs. In addition, they have more structural properties. One such property is the absence of “tent” subhypergraphs. Aharoni and Zerbib [AZ20] studied this property and asked if we can prove a generalization of Tuza’s conjecture to hypergraphs without tents. In this paper, we answer this problem in negative and prove that there are hypergraphs on \( n \) vertices without tents where \( \tau = (1 - o(1))n \).

**Theorem 5.** For every \( \epsilon > 0 \), there exists a \( 3 \)-uniform hypergraph \( H \) without a tent such that 

\[
\tau(H) \geq (3 - \epsilon)\nu(H).
\]

Our counterexample is the hypergraph \( H \) with vertex set \( [3]^n \) for large enough \( n \) and edges being the set of combinatorial lines. By the density Hales Jewett Theorem [FK91,Pol12], there is no large independent set in \( H \), and using the structure of combinatorial lines, we can prove that \( H \) does not have any tent.

**Related work.** \( \frac{t}{2} \)-factor approximation algorithms have been obtained for the vertex cover problem on several other families of \( t \)-uniform hypergraphs: Lovász [Lov75] gave an algorithm to round the natural LP relaxation to get a \( \frac{t}{2} \)-factor approximation algorithm for vertex cover on \( t \)-uniform \( t \)-partite hypergraphs. This algorithm is shown to be optimal under the Unique Games Conjecture by Guruswami, Sachdeva, and Saket [GSS15] (and an almost matching NP-hardness is also shown). Aharoni, Holzman, and Krivelevich [AIK96] generalized the above algorithmic result to other class of hypergraphs which have a partition of vertices obeying certain properties. A factor \( \frac{t}{2} \) approximation algorithm has also been obtained on subdense regular \( t \)-uniform hypergraphs [CKSV12].

**Outline.** In Section 2, we introduce some notation and definitions. In Section 3, we describe our algorithm and prove Theorem 2. Then, in Section 4, we consider simple hypergraphs and prove Theorem 3. Finally, in Section 5, we study the structural properties of the blown-up hypergraphs.
2 Preliminaries

Notation. We use \([n]\) to denote the set \(\{1, 2, \ldots, n\}\). We use \(\mathbb{Z}_n\) to denote the set \(\{0, 1, \ldots, n-1\}\). For a set \(S\) and an integer \(1 \leq k \leq |S|\), we use \(\binom{S}{k}\) to denote the family of all the \(k\)-sized subsets of \(S\). A hypergraph \(H' = (V', E')\) is called a subhypergraph of \(H = (V, E)\) if \(V' \subseteq V\) and \(E' \subseteq E'\). For a hypergraph \(H = (V, E)\), we use \(\tau(H), \nu(H)\) to denote the size of the minimum vertex cover and the maximum matching respectively. Similarly, we use \(\tau^*(H)\) to denote the minimum fractional vertex cover of \(H\):

\[
\tau^*(H) = \min \left\{ \sum_{v \in V} x_v : x_v \in \mathbb{R}_{\geq 0} \forall v \in V, \sum_{v \in e} x_v \geq 1 \forall e \in E \right\}
\]

We define the \(k\)-blown up hypergraph formally:

**Definition 6.** For a \(t\)-uniform hypergraph \(G = (V, E)\) and for an integer \(1 \leq k < t\), we define the \(k\)-blown up hypergraph \(H = G^{(k)} = (V', E')\) as follows:

1. The vertex set \(V' \subseteq \binom{V}{k}\) is the set of all \(k\)-sized subsets of \(V\) that are contained in an edge of \(G\):
   \[
   V' = \{ U : U \subseteq V, |U| = k, \exists e \in E : U \subseteq e \}
   \]

2. For every edge \(e \in E\), we include in \(E'\) all the \(k\)-sized subsets of \(e\), so that
   \[
   E' = \left\{ e' : e' = \binom{e}{k}, e \in E \right\}
   \]

We will need the following Chernoff bound:

**Lemma 7.** (Multiplicative Chernoff bound) Suppose \(X_1, X_2, \ldots, X_n\) are independent random variables taking values in \(\{0, 1\}\). Let \(X = X_1 + X_2 + \ldots + X_n\), and let \(\mu = \mathbb{E}[X]\). Then, for any \(0 \leq \delta \leq 1\),

\[
Pr(X \leq (1 - \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}
\]

3 Proof of Theorem 2

In this section, we present our algorithm for the AHTP. Given a \(t\)-uniform hypergraph \(G\) as an input to the AHTP, let \(H = G^{(t-1)}\) be the \((t-1)\)-blown-up hypergraph of \(G\).

3.1 Color-coding based small vertex cover

We first prove a lemma that in any \((t-1)\)-blown-up hypergraph \(H = ([n], E)\), there is a vertex cover of size at most \(O\left(\frac{\log k}{t}\right)n\) using a color-coding argument. This lemma illustrates the color-coding idea well, and is also useful later in the context of structural characterization (Conjecture 21) of the blown-up hypergraphs. This lemma is not used in the main algorithm, and the reader can skip to Section 3.2 for the algorithm.

**Lemma 8.** Suppose \(G = ([n], E(G))\) is a \(t\)-uniform hypergraph and \(H = G^{(t-1)} = (V(H), E(H))\). Then, there exists a randomized polynomial time algorithm that outputs a vertex cover of \(H\) with expected size at most \(|V|\left(\frac{2 \ln t}{t} + O\left(\frac{1}{t}\right)\right)\).
Proof. Our algorithm is based on the color-coding technique used to get upper bounds for the hypergraph Turán problem [KR83, Sid95]. Let \( P = \left\lceil \frac{t-1}{2m} \right\rceil \). Color each vertex of \( G \) with \( c : [n] \rightarrow [P] \) uniformly independently at random. For \( v \in V(H) \) and \( i \in [P] \), let \( C_i(v) \) denote the number of nodes of \( v \) that are colored with \( i \) i.e.

\[
C_i(v) := |\{j \in v : c(j) = i\}|
\]

We define a function \( f : V(H) \rightarrow \mathbb{Z}_P \) as

\[
f(v) = C_1(v) + 2C_2(v) + \ldots + (P - 1)C_{(P-1)}(v) \mod P
\]

For an element \( i \in \mathbb{Z}_P \), let \( f^{-1}(i) \) denote the set \( \{v \in V(H) : f(v) = i\} \). Let \( p \in \mathbb{Z}_P \) be such that \( |f^{-1}(p)| \leq |f^{-1}(i)| \) for all \( i \in \mathbb{Z}_P \). Note that by definition, \( |f^{-1}(p)| \leq \frac{|V|}{P} \). Let \( U \subseteq V(H) \) be defined as follows:

\[
U = \{v : v \in V(H), \exists i \in [P] \text{ such that } C_i(v) = 0\}
\]

We claim that \( S = f^{-1}(p) \cup U \) is a vertex cover of \( H \). Consider an arbitrary edge \( e = \{v_1, v_2, \ldots, v_t\} \in E(H) \). Let the corresponding edge in \( G \) be equal to \( e(G) = \bigcup_{j \in [t]} v_j = (u_1, u_2, \ldots, u_t) \in E(G) \) where \( u_1, u_2, \ldots, u_t \) are elements of \( [n] \). Without loss of generality, let \( v_j = e(G) \setminus \{u_j\} \). For a color \( i \in [P] \), let \( C_i(e) = |\{j \in [t] : c(u_j) = i\}| \). We consider two cases separately:

1. First, if there exists a color \( i \in [P] \) such that \( C_i(e) = 0 \), then for every \( j \in [t] \), \( C_i(v_j) = 0 \), and thus, for every \( j \in [t] \), \( v_j \subseteq U \), and thus, \( e \cap S \neq \phi \).

2. Suppose that for every color \( i \in [P] \), \( C_i(e) > 0 \). We define \( f(e) \in \mathbb{Z}_P \) as

\[
f(e) = C_1(e) + 2C_2(e) + \ldots + (P - 1)C_{(P-1)}(e) \mod P
\]

Note that for every \( j \in [t] \), we have

\[
f(v_j) = f(e) - c(u_j) \mod P
\]

As the cardinality of \( \{c(u_1), c(u_2), \ldots, c(u_t)\} \) is equal to \( P \), the cardinality of \( \{f(v_1), f(v_2), \ldots, f(v_t)\} \) is equal to \( P \) as well. Thus, there exists a \( j \in [t] \) such that \( f(v_j) = p \) which implies that \( v_j \in S \).

Thus, our goal is to upper bound the expected value of \( |S| \). Note that \( P \leq \frac{t-1}{\ln t} \). By taking union bound over all the colors, we get

\[
\mathbb{E}[U] \leq P \left(1 - \frac{1}{P}\right)^{t-1} |V|
\]

\[
\leq \frac{t-1}{\ln t} e^{-2\ln t |V|}
\]

\[
\leq \left(\frac{1}{\ln t}\right) |V| = O \left(\frac{1}{t}\right) |V|
\]

Thus, the expected value of \( S \) is at most \( |f^{-1}(p)| + \mathbb{E}[|U|] \) which is at most \( \left(\frac{2\ln t}{t} + O\left(\frac{1}{t}\right)\right) |V| \).
3.2 LP rounding based algorithm for AHTP

Consider the standard LP relaxation for vertex cover in $H$:

Minimize $\sum_{v \in V(H)} x_v$

such that $\sum_{v \in e} x_v \geq 1 \forall e \in E(H)$

$x_v \geq 0 \forall v \in V(H)$

Let $x$ be an optimal solution to the above Linear Program, and let $\text{OPT} = \sum_{v \in V(H)} x_v$. In general, $\text{OPT}$ could be much smaller than $|V(H)|$, and thus we cannot use Lemma 8 directly. We now describe a randomized algorithm to round the above LP to obtain an integral solution whose expected size is at most $(\frac{t^2}{2} + 2\sqrt{t\ln t}) \cdot \text{OPT}$.

For ease of notation, let $t' = \frac{t}{2} + 2\sqrt{t\ln t}$. Our first step is to round all the variables above a certain threshold to 1 (Algorithm 1). However, we need to do it recursively to ensure that we can bound the optimal value of the remaining instance.

**Algorithm 1** Recursive thresholding for AHTP

1: Let $\gamma = \frac{1}{t'}$.
2: Let $\overline{x}$ be an optimal solution of the LP and let $V' = \{v : \overline{x}_v \geq \gamma\}$.
3: Let $U = V'$.
4: while $V'$ is non-empty do
5: Delete $V'$ from $V(H)$, and delete all the edges $e \in E(H)$ that contain at least one vertex $v \in V'$.
6: Solve the LP with updated $H$. Update $\overline{x}$ to be the new LP solution.
7: Update $V' = \{v \in V(H) : \overline{x}_v \geq \gamma\}$. Update $U \leftarrow U \cup V'$.
8: Output $U$ and the updated $H$.

Let the final updated hypergraph $H$ when Algorithm 1 terminates be denoted by $H'$. Let the optimal cost of the solution $\overline{x}$ for the vertex cover on $H'$ be denoted by $\text{OPT}'$. We prove that the size of the vertex cover output by the algorithm is not too large:

**Lemma 9.** When the above recursive thresholding algorithm (Algorithm 1) terminates, we have $|U| \leq t' \cdot (\text{OPT} - \text{OPT}')$.

**Proof.** We will inductively prove the following: after line 6 in the while loop of the algorithm, $|U| \leq t' \cdot (\text{OPT} - \text{OPT}_{\text{new}})$ where $\text{OPT}_{\text{new}}$ is the cost of the current optimal solution $\overline{x}$. Let $\overline{x}'$ be the optimal solution before deleting $V'$ from $H$. Let $\text{OPT}_{\text{old}}$ be the cost of the solution $\overline{x}'$. By inductive hypothesis, we have $|U| - |V'| \leq t' \cdot (\text{OPT} - \text{OPT}_{\text{old}})$.

We claim that $|V'| \leq t' \cdot (\text{OPT}_{\text{old}} - \text{OPT}_{\text{new}})$. As $\overline{x}$ is an optimal vertex cover of $H$, we have that $\overline{x}$ restricted to $H$ has cost at least $\text{OPT}_{\text{new}}$. This implies that $\sum_{v \in V'} \overline{x}_v \geq \text{OPT}_{\text{old}} - \text{OPT}_{\text{new}}$. As each $\overline{x}_v, v \in V'$ is at least $\frac{1}{t'}$, we obtain the required claim.

We are now ready to state our main algorithm for the AHTP. The input to the algorithm is a $t$-uniform hypergraph $G$, and the output is a vertex cover for the hypergraph $H = G(t-1)$. 
Algorithm 2 Main algorithm

1: Apply Algorithm 1 to obtain $U$ and let $H' = (V(H'), E(H'))$ be the updated $H$. Let $\overline{x}$ be an optimal solution of the vertex cover LP on $H'$ with $\overline{x}_v \leq \gamma$ for all $v \in V(H')$.
2: Let $S \subseteq V(H')$ be defined as $S = \{v : V(H') : \overline{x}_v \geq 0\}$.
3: Let $\delta = \sqrt{\frac{4 \ln t}{t-1}}$.
4: Color the vertices $[n]$ of $G$ using $c : [n] \to \{0, 1\}$ uniformly and independently at random.
5: For a vertex $v \in S$ and a color $i \in \{0, 1\}$, let $C_i(v)$ denote the number of nodes that are colored with the color $i$ i.e.

$$C_i(v) = |\{j \in v : c(j) = i\}|$$

6: Let $S' \subseteq S$ be defined as the set of vertices in $S$ where the discrepancy between two colors is high:

$$S' = \left\{ v \in S : \exists i \in \{0, 1\} : C_i(v) \leq (1 - \delta) \frac{t - 1}{2} \right\}$$

7: We now define a function $f : S \to \{0, 1\}$ as $f(v) = C_1(v) \mod 2$.
8: For $i \in \{0, 1\}$, let $f^{-1}(i)$ denote the set of all the vertices $v \in S$ such that $f(v) = i$.
9: Let $p \in \{0, 1\}$ be such that $|f^{-1}(p)| \leq |f^{-1}(1 - p)|$.
10: Let $T \subseteq S$ be defined as $T = S' \cup f^{-1}(p)$.
11: Output $T \cup U$.

3.3 Analysis of the algorithm and proof of Theorem 2

We will first prove that Algorithm 2 indeed outputs a valid vertex cover of $H$.

Lemma 10. $T \cup U$ is a vertex cover of $H$.

Proof. It suffices to prove that $T$ is a vertex cover of $H'$.

Consider an arbitrary edge $e = (v_1, v_2, \ldots, v_t) \in E(H')$ corresponding to the edge $e(G) = \bigcup_{j \in [t]} v_j = \{u_1, u_2, \ldots, u_t\} \in E(G)$. Since $\overline{x}_v \leq \gamma$ for all $v \in V(H')$, we can deduce that $|e \cap S| \geq \frac{1}{\gamma} - t'$.

Our goal is to show that there exists $j \in [t]$, such that $v_j \in T$. We consider two separate cases:

1. If there is a color $i \in \{0, 1\}$ such that there are at most $(1 - \delta) \frac{t - 1}{2}$ nodes of color $i$ in $e(G)$, then for all $j \in [t], C_i(v_j) \leq (1 - \delta) \frac{t - 1}{2}$. Since $e \cap S$ is non-empty, there exists $j \in [t]$ such that $v_j \in S$. By definition of $S'$, this implies that $v_j \in S'$ as well, and thus $e \cap T \neq \phi$.

2. Suppose that in the coloring $c$, both the colors 0, 1 occur at least $(1 - \delta) \frac{t - 1}{2}$ times in $e$. Let $e' = e \cap S$ and let $k = |e'| \geq t'$. Without loss of generality, let $e' = \{v_1, v_2, \ldots, v_k\}$. For every
Proof. Lemma 11. The cardinality of $S$ is equal to $t^\cdot OPT'$. We have

$$t - k - (1 - \delta)\frac{t - 1}{2} \leq t - t' - (1 - \delta)\frac{t - 1}{2}$$

$$= \frac{t}{2} - 2\sqrt{t\ln t} - \left(1 - \sqrt{\frac{4\ln t}{t - 1}}\right)\frac{t - 1}{2}$$

$$= \frac{1}{2} \left(t - 4\sqrt{t\ln t} - (t - 1) + 2\sqrt{(t - 1)\ln t}\right)$$

$$\leq \frac{1}{2} \left(1 - 2\sqrt{\ln t}\right) < 0$$

Since each color occurs at least $(1 - \delta)\frac{t - 1}{2}$ times in $e(G)$, using the above, we can infer that $|\{c(u_1), c(u_2), \ldots, c(u_k)\}| \geq 2$.

We define the value $f(e)$ in the same fashion as we have defined $f(v)$ for $v \in S$: For $i \in \{0, 1\}$, let $C_i(e)$ denote the number of nodes $j \in [t]$ such that $c(u_j) = i$, and let $f(e) = C_i(e) \mod 2$. Using this definition, we get

$$f(v_j) = f(e) - c(u_j) \mod 2 \forall j \in [k].$$

As $\{c(u_1), c(u_2), \ldots, c(u_k)\} = \{0, 1\}$, we have $\{f(v_1), f(v_2), \ldots, f(v_k)\} = \{0, 1\}$ as well. Thus, there exists $j \in [k]$ such that $f(v_j) = p$, which proves that $v_j \in f^{-1}(p) \subseteq T$. $\Box$

In order to bound the expected size of the output of the algorithm, we need a couple lemmata. First, we prove that the cardinality of $S$ is not too large:

**Lemma 11.** The cardinality of $S$ is at most $t \cdot OPT'$.

**Proof.** Consider the dual of the vertex cover LP:

Maximize $\sum_{e \in E(H')} y(e)$

such that $\sum_{e \ni v} y(e) \leq 1 \forall v \in V(H')$

$y(e) \geq 0 \forall e \in E(H')$

Let $\overline{y}$ be an optimal solution to the above matching LP. By LP-duality, we get $\sum_{e \in E(H')} \overline{y}_e = OPT'$. Recall that for all $v \in S$, $\overline{x}_v \neq 0$. By the complementary slackness conditions, we get that for all $v \in S$, $\sum_{e \ni v} \overline{y}_e = 1$. Summing over all $v \in S$, we obtain

$$|S| = \sum_{v \in S} \sum_{e \ni v} \overline{y}_e \leq t \sum_{e \in E(H')} \overline{y}_e = tOPT'. \Box$$

Note that the expected number of nodes of each color $i \in \{0, 1\}$ in a vertex $v = (u_1, u_2, \ldots, u_{t-1}) \in S$ is equal to $\frac{t}{\sqrt{2}}$. The set $S'$ is the set of vertices of $S$ where there is a color that occurs much fewer than its expected value. We prove that this happens with low probability:

**Lemma 12.** The expected cardinality of $S'$ is at most $\frac{2}{7}|S|$.
Proof. Let $v = (u_1, u_2, \ldots, u_{t-1}) \in S$ be an arbitrary vertex in $S$, where $u_1, u_2, \ldots, u_{t-1}$ are elements of $[n]$. For a color $i \in \{0, 1\}$, let the random variable $X(i)$ denote the number of nodes $j \in [t-1]$ such that $c(u_j) = i$. We can write $X(i) = \sum_{j \in [t-1]} X(i, j)$, where $X(i, j)$ is the indicator random variable of the event that $c(u_j) = i$. We have $\mu = \mathbb{E}[X(i)] = \frac{t-1}{t}$. Using multiplicative Chernoff bound (Lemma 7), we can upper bound the probability that $X(i) \leq (1 - \delta)\frac{t-1}{2}$ by

$$\Pr \left( X(i) \leq (1 - \delta)\frac{t-1}{2} \right) \leq e^{-\frac{\delta^2(t-1)}{4}}$$

By substituting $\delta = \sqrt{\frac{4\ln t}{t-1}}$, we get that the above probability at most $\frac{1}{t}$. By applying union bound over the two colors and adding the expectation over all the vertices in $S$, we obtain the lemma. \(\square\)

Finally, we bound the expected size of the output of the algorithm:

**Lemma 13.** The expected cardinality of $T \cup U$ is at most $\left(\frac{t}{2} + 2\sqrt{t\ln t}\right) \cdot \text{OPT}$.

Proof. Note that by definition, $|f^{-1}(p)| \leq \frac{|S|}{2}$. We bound the expected size of the output of the algorithm $T \cup U$ as

$$\mathbb{E}[|T \cup U|] \leq \mathbb{E}[|T|] + \mathbb{E}[|U|] \leq \mathbb{E}[|S'|] + \frac{1}{2}|S| + \mathbb{E}[|U|]$$

$$\leq \left(\frac{1}{2} + \frac{2}{t}\right)|S| + \mathbb{E}[|U|] \quad \text{(Using Lemma 12)}$$

$$\leq \left(\frac{t}{2} + 2\right)\text{OPT} + \mathbb{E}[|U|] \quad \text{(Using Lemma 11)}$$

$$\leq \left(\frac{t}{2} + 2\sqrt{t\ln t}\right)\text{OPT} \quad \text{(Using Lemma 9)}. \quad \square$$

Lemma 10 and Lemma 13 together give a proof of Theorem 2.

4 Vertex Cover in Simple Hypergraphs

In this section, we prove Theorem 3. Our hardness result is obtained using a reduction from the general problem of vertex cover on $t$-uniform hypergraphs. In particular, we use the following result:

**Theorem 14.** ([DGKR05]) For every constant $\epsilon > 0$ and $t \geq 3$, the following holds: Given a $t$-uniform hypergraph $G = (V, E)$, it is $\text{NP}$-hard to distinguish between the following cases:

1. Completeness: $G$ has a vertex cover of measure $\frac{1+\epsilon}{t}$.

2. Soundness: Any subset of $V$ of measure $\epsilon$ contains an edge from $E$.

We give a randomized reduction from Theorem 14 to Theorem 3. In particular, we instantiate Theorem 14 with $\epsilon$ replaced by $\epsilon' = \frac{\epsilon}{4}$, and let the resulting hypergraph be denoted by $G$. Now, given this $t$-uniform hypergraph $G = (V, E)$, we output a $t$-uniform hypergraph $H = (V', E')$ as follows: Let $n = |V|, m = |E|$. We have integer parameters $B, P$ depending on $\epsilon, t, n, m$ to be set later. The vertex set of $H$ is $V' = V \times [B]$—we have a cloud of $B$ vertices $v^1, v^2, \ldots, v^B$ in $V'$ corresponding to every vertex $v \in V$. For every edge $e = (v_1, v_2, \ldots, v_l) \in E$, we pick $P$ edges $e^1, e^2, \ldots, e^p$ with
\[ e^i = ((v_1)_i, (v_2)_i, \ldots, (v_t)_i) \text{ and add them to } E', \text{ where for each } j \in [t] \text{ and } i \in [P], (v_j)_i \text{ is chosen uniformly and independently at random from } (v_j)^1, (v_j)^2, \ldots, (v_j)^B. \] Thus, so far, we have added \( mP \) edges to \( E' \).

We first upper bound the expected value of the number of pairs of edges in \( E' \) that intersect in more than one vertex. Order the edges in \( E' \) as \( e_1, e_2, \ldots, e_{mP} \). Let \( X \) denote the random variable that counts the number of pairs of edges in \( E' \) that intersect in more than one vertex. For every pair of indices \( i, j \in [mP] \), let the random variable \( X_{ij} \) be the indicator variable of the event that the edges \( e_i \) and \( e_j \) of \( E' \) intersect in greater than one vertex. Note that the edges in \( E \) corresponding to \( e_i \) and \( e_j \) have at most \( t \) vertices in common. Thus, the probability that \( e_i \) and \( e_j \) intersect in at least two vertices is upper bounded by \( \left( \frac{t}{2} \right) \frac{1}{B^2} \). Summing over all the pairs \( i, j \), we get

\[
E[X] \leq \left( \frac{mP}{2} \right) \left( \frac{t}{2} \right) \frac{1}{B^2} \leq \frac{m^2 t^2 P^2}{B^2}.
\]

By Markov’s inequality, with probability at least \( \frac{9}{10} \), \( X \) is at most \( \frac{10m^2 t^2 P^2}{B^2} \).

We consider all the pairs of edges that intersect in more than one vertex in \( E' \), and arbitrarily delete one of those edges. Let the resulting set of edges be denoted by \( E'' \). The final hypergraph resulting in this reduction is \( H = (V', E'') \). Note that \( H \) is indeed a simple hypergraph. We will prove the following:

1. (Completeness) If \( G \) has a vertex cover of measure \( \mu \), then there is a vertex cover of measure \( \mu \) in \( H \).

2. (Soundness) If every subset of \( V \) of measure \( \epsilon' \) contains an edge from \( E \), then with probability at least \( \frac{4}{5} \), every subset of \( V' \) of measure \( \epsilon \) contains an edge from \( E'' \).

**Completeness.** If \( G \) has a vertex cover of size \( \mu n \), then picking all the vertices in \( V' \) in the cloud corresponding to these vertices ensures that \( H \) has a vertex cover of size \( \mu n B \). Thus, in the completeness case, there is a vertex cover of measure \( \mu \) in \( H \).

**Soundness.** Suppose that every \( \epsilon' \) measure subset of \( V \) contains an edge from \( E \). Our goal is to show that with probability at least \( \frac{4}{5} \), every \( \epsilon \) measure subset of \( V' \) contains an edge from \( E'' \). We first prove the following lemma:

**Lemma 15.** With probability at least \( \frac{9}{10} \) over the choice of \( E' \), the following holds: For every edge \( e = (v_1, v_2, \ldots, v_t) \in E \), and every subset \( S \subseteq V' \) such that for each \( i \in [t] \), \( S \) contains at least \( \frac{t}{4} B \) vertices from \( \{v_1^1, v_1^2, \ldots, v_i^B\} \), there exists an edge \( e' \in E' \) all of whose vertices are in \( S \).

**Proof.** The probability that there exists an edge \( e = (v_1, v_2, \ldots, v_t) \) and a subset \( S \) which contains at least \( \frac{t}{4} B \) vertices from each cloud and does not contain any edge from \( E' \) is at most

\[
m^2 t^B \left( 1 - \left( \frac{e}{4} \right)^t \right)^P \leq m^2 t^B - \log \epsilon' \frac{P}{\epsilon} \leq \frac{1}{10}
\]

when \( P = maB \) where \( a := a(t, \epsilon) = \frac{4^{\log_4 2}}{e^t} \).

Using the above lemma, we can conclude that with probability at least \( \frac{4}{5} \), \( X \leq \frac{10m^2 t^2 P^2}{B^2} = 10m^4 t^2 a^2 \) and for every edge \( e \in E \) and every subset \( S \subseteq V' \) such that for each \( i \in [t] \), \( S \) contains at
least $\frac{\epsilon}{4} B$ vertices from $\{v_1^1, v_2^1, \ldots, v_i^B\}$, there exists an edge $e' \in E'$ all of whose vertices are in $S$. We claim that this implies that with probability at least $\frac{1}{4}$, every $\epsilon$ measure subset of $V'$ contains an edge of $E''$. Consider an arbitrary subset $U \subseteq V'$ such that $|U| \geq \epsilon n B$. We choose $B$ large enough such that $t(10m^4t^2a^2) \leq \frac{\epsilon}{4} n B$. Thus, the set of the vertices $W$ in the edges deleted from $E'$ to obtain $E''$ has cardinality at most $\frac{\epsilon}{4} n B$.

Let $U' = U \setminus W$. Note that all the edges in $U'$ that are in $E'$ are present in $E''$ as well. As $U'$ has a measure of at least $\frac{\epsilon}{4}$ in $V'$, for at least $\frac{\epsilon}{4} n$ vertices $v$ in $V$, $U'$ should contain at least $\frac{\epsilon}{4}$ fraction of the vertices in the cloud $\{v_1^1, v_2^1, \ldots, v_i^B\}$. Since otherwise, the cardinality of $U'$ is at most $(n - \frac{\epsilon n}{4}) \cdot \frac{a B}{4} + \frac{\epsilon a B}{4} \cdot B < \frac{\epsilon n B}{2}$, a contradiction. By Lemma 15, we can deduce that there exists an edge $e \in E'$ all of whose vertices are in $U'$, which implies that the edge $e$ is in $E''$ as well. This proves that in the soundness case, with probability at least $\frac{1}{5}$, there exists an edge in every $\epsilon$ measure subset of $V'$.

This completes the proof of Theorem 3. Under the Unique Games Conjecture [Kho02], the hardness of vertex cover in $t$-uniform hypergraphs can be improved to $t - \epsilon$. We remark that we can get the same hardness for simple hypergraphs by our reduction.

5 Regarding structural characterization of the blown-up hypergraphs

Since AHTP is the problem of vertex cover on $H = G^{(t-1)}$ for a given $t$-uniform hypergraph $G$, an interesting question is to characterize for which $t$-uniform hypergraph $H$, there exists another $t$-uniform hypergraph $G$ such that $H = G^{(t-1)}$. As mentioned in the previous section, one necessary condition is that the hypergraph $H$ should be simple. However, this is not sufficient—there are simple $t$-uniform hypergraphs that cannot be written as $G^{(t-1)}$. One possible avenue towards resolving this is the following question: Are there a finite set of hypergraphs $\mathcal{F}$ such that every hypergraph without any subhypergraph from $\mathcal{F}$ can be represented as a $G^{(t-1)}$ for some hypergraph $G$?

Aharoni and Zerbib [AZ20] study this question for $t = 3$ in the context of Tuza’s conjecture. They propose a general form of Tuza’s conjecture that $\tau(G^{(2)}) \leq 2 \nu(G^{(2)})$ for all 3-uniform hypergraphs $G$. They suggested that pinning down the exact structural property of the blown-up hypergraphs $H = G^{(2)}$ that leads to $\tau(H) \leq 2 \nu(H)$ could be a way to resolve the conjecture. A candidate characterization they had is the absence of a “tent” which we define below.

Definition 16. A $t$-tent (Figure 1) is a set of four $t$-uniform edges $e_1, e_2, e_3, e_4$ such that

1. $\cap_{i=1}^3 e_i \neq \emptyset$.
2. $|e_4 \cap e_i| = 1$ for all $i \in [3]$.
3. $e_4 \cap e_i \neq e_4 \cap e_j$ for all $i \neq j \in [3]$.

In [AZ20], the authors ask the following problem:

Problem 17. Is it true that for every 3-uniform hypergraph $H$ without a 3-tent, $\tau(H) \leq 2 \nu(H)$?

We answer this question in the negative. Our counterexample is a hypergraph with vertex set $[3]^n$ for large enough $n$ and the edge set is the set of all combinatorial lines that we formally define below:

Definition 18. (Combinatorial lines in $[3]^n$) A set of three distinct vectors $u = (u_1, u_2, \ldots, u_n), v = (v_1, v_2, \ldots, v_n), w = (w_1, w_2, \ldots, w_n) \in [3]^n$ forms a combinatorial line if there exists a subset $S \subseteq [n]$ such that
1. For all $i \in [n] \setminus S$, $u_i = v_i = w_i$.

2. There exist three distinct integers $u', v', w' \in [3]$ such that for all $i \in S$, $u_i = u', v_i = v', w_i = w'$.

We will use the following seminal result about combinatorial lines:

**Theorem 19.** (Density Hales Jewett Theorem [FK91], [Pol12] ) For every positive integer $k$ and every real number $\delta > 0$ there exists a positive integer $\text{DHJ}(k, \delta)$ such that if $n \geq \text{DHJ}(k, \delta)$ and $A$ is any subset of $[k]^n$ of density at least $\delta$, then $A$ contains a combinatorial line.

We now prove Theorem 5.

**Theorem 5.** For every $\epsilon > 0$, there exists a 3-uniform hypergraph $H$ without a 3-tent such that $\tau(H) > (3 - \epsilon)\nu(H)$.

**Proof.** The hypergraph that we use $H = (V, E)$ has $V = [3]^n$ for $n$ large enough to be set later, and the edges are all the combinatorial lines in $[3]^n$. First, we claim that the above defined hypergraph does not have a 3-tent. Suppose for contradiction that there are edges $e_1, e_2, e_3, e_4$ satisfying the properties of Definition 16. Let $u = (u_1, u_2, \ldots, u_n) \in e_4 \cap e_1, v = (v_1, v_2, \ldots, v_n) \in e_4 \cap e_2 = (w_1, w_2, \ldots, w_n) \in e_4 \cap e_3$. Note that $e_4 = \{u, v, w\}$. Thus, there exists a subset $S \subseteq [n]$ such that for all $i \in [n] \setminus S$, $u_i = v_i = w_i$. Without loss of generality, we can also assume that for all $i \in S$, $u_i = 1, v_i = 2, w_i = 3$.

Let $x = (x_1, x_2, \ldots, x_n) \in e_1 \cap e_2 \cap e_3$. Note that $\{x, u\} \subsetneq e_1, \{x, v\} \subsetneq e_2, \{x, w\} \subsetneq e_3$. Consider an arbitrary element $p \in S$, and without loss of generality, let $x_p = 1$. Thus, we have that $x_p = 1, v_p = 2$ and both $x, v$ share the combinatorial line $e_2$. This implies that there exist a subset $S_2 \subseteq [n]$ such that for all $i \in [n] \setminus S_2$, $x_i = v_i$ and for all $i \in S_2, x_i = 1, v_i = 2$. Similarly, there exists a subset $S_3 \subseteq [n]$ such that for all $i \in [n] \setminus S_3, x_i = w_i$ and for all $i \in S_3, x_i = 1, w_i = 3$.

Note that $S_2 \subseteq S$. Suppose for contradiction that there exists $j \in S_2 \setminus S$. Then, we have $v_j = 2, x_j = 1$. However, since $v_i = w_i$ for all $i \in [n] \setminus S$, we get that $w_j = 2$, and thus, $j \notin S_3$, which implies that $x_j = w_j = 2$, a contradiction. Thus, $S_2 \subseteq S$, and similarly $S_3 \subseteq S$. We can also observe that $S_2 \neq S$ since in that case, $x = u$ which cannot happen since $|e_4 \cap e_2| = 1$. By the same argument on $e_3$, we can deduce that $S_3 \neq S$. As $S_2$ is a strict subset of $S$, there exists $j \in S \setminus S_2$. As $v_i = x_i$ for all $i \in [n] \setminus S_2$, $x_j = v_j = 2$. As $j \in S$, we have $w_j = 3$. However, as $w_j \neq x_j$, this implies that $j \in S_3$, which then implies that $x_j = 1$, a contradiction.

Now, we will prove that for large enough $n$, $\tau(H) > (3 - \epsilon)\nu(H)$. Let $N = 3^n$. Since the cardinality of $V$ is equal to $N$, we have $\nu(H) \leq \frac{N}{3}$. We apply Theorem 19 with $k = 3, \delta = \frac{\epsilon}{3}$, and set $n \geq \text{DHJ}(k, \delta)$. Thus, we can infer that in any subset $T \subseteq V$ of size $\frac{\epsilon}{3}N$, there exists an edge of $H$ fully contained in $T$. Thus, we get that $\tau(H) > (1 - \frac{\epsilon}{3})N$, which gives $\tau(H) > (3 - \epsilon)\nu(H)$.  

One might wonder if the above combinatorial lines based construction can be used as a counterexample to the generalized Tuza’s conjecture of Aharoni and Zerib [AZ20] that $\tau(H) \leq 2\nu(H)$ for all 3-uniform hypergraphs $H$ such that $H = G^{(2)}$ for some $G$. However, the blown-up hypergraphs have stronger structural properties. For example, the below “(2,3)-grid” subhypergraph (Figure 2) is absent in blown-up hypergraphs but is abundant in combinatorial lines.

This raises the question of whether there are finite substructures the exclusion of which fully characterizes the blown-up hypergraphs. Aharoni and Zerib [AZ20] informally ask this question, and we make it formal in the below conjecture:
**Conjecture 20.** (Subhypergraph characterization of the blown-up hypergraphs) For every $t \geq 3$, there is a finite family of $t$-uniform hypergraphs $F_t$ such that: A $t$-uniform hypergraph $H$ can be written as $G^{(t-1)}$ for another $t$-uniform hypergraph $G$ if and only if $H$ does not have any subhypergraph from $F_t$.

Similar to Problem 17, Conjecture 20 also implies (using Lemma 8) the following weaker conjecture on the presence of large independent sets in hypergraphs without certain substructures:

**Conjecture 21.** For every $t \geq 3$, there is a finite family of $t$-uniform hypergraphs $F_t$, and a constant $c_t < 1$ such that:

1. For every hypergraph $H$ such that $H = G^{(t-1)}$, $H$ does not contain any subhypergraph from $F_t$.

2. For every hypergraph $H = ([n], E)$ without any subhypergraph from $F_t$, there is a vertex cover of $H$ with cardinality $c_t n$.

We believe that either a positive or negative resolution to Conjecture 20 and Conjecture 21 would improve our understanding of the blown-up hypergraphs and help in making progress on Tuza’s conjecture and AHTP.

### 6 Conclusion

In this paper, we introduced an algorithmic version of the hypergraph Turán problem (AHTP) for $t$-uniform hypergraphs and gave a factor $\frac{t}{2} + o(t)$ algorithm for it. Our work also raises several natural directions to further explore:

1. Finding better algorithms and improved hardness results for AHTP is still wide open. Especially on the hardness front, the best hardness result (for any $t$) is the factor 2 hardness of the vertex cover problem on graphs, under the Unique Games Conjecture. Towards this, an interesting open problem is to obtain $\omega(1)$ lowerbound on the integrality gap of the standard LP relaxation for AHTP.

2. $(t,k)$-version of AHTP: Throughout the paper, we have studied the problem of vertex cover on $H = G^{(t-1)}$ for a given $t$-uniform hypergraph $G$. An interesting generalization is the problem of vertex cover on $H = G^{(k)}$ for a $t$-uniform hypergraph $G$, for arbitrary $1 \leq k < t$. The case of $k = 1$ is the standard vertex cover on $t$-uniform hypergraphs, and $k = t - 1$ is the AHTP. Of special interest is the case when $k = 2$: finding an $o(t^2)$ factor approximation algorithm or showing NP-hardness of finding one is an interesting open problem.
3. The dual problem to AHTP is the maximum matching problem on $t$-blown-up hypergraphs. For the general case of the maximum matching problem on $t$-uniform hypergraphs, also known as $t$-set packing, Cygan [Cyg13] gave a local search algorithm that achieves an approximation factor of $\frac{1}{3} + \epsilon$ for any $\epsilon > 0$. Can we get better algorithms for the maximum matching problem on $H = G^{(t-1)}$? On the hardness front, by a simple reduction from the independent set problem on graphs with maximum degree $t$ [AKS11, Cha16], it follows that the maximum matching problem on $t$-uniform simple hypergraphs is NP-hard to approximate better than $\Omega\left(\frac{t}{\log^2 t}\right)$.

As mentioned earlier, the hardness of the coverage version of AHTP, Johnson Coverage Hypothesis (JCH) has various implications for fundamental clustering problems in Euclidean metrics. The coverage version of the set cover problem, known as Maximum coverage has $1 - \frac{1}{e} + \epsilon$ hardness which can be proved by a simple reduction from the $\ln n$ hardness of set cover. Whether such progress towards JCH can be made assuming some form of hardness of AHTP is an interesting open problem.

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