RIGIDITY OF COMPLETE SELF-SHRINKERS WHOSE TANGENT PLANES OMIT A NONEMPTY SET

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Abstract. In this paper we prove rigidity results for the sphere, the plane and the right circular cylinder as the only self-shrinkers satisfying a classic geometric assumption, namely the union of all tangent affine submanifolds of a complete self-shrinker omits a non-empty set of the Euclidean space. This assumption lead us to a new class of submanifolds, different from those with polynomial volume growth or the proper ones. We also prove an analogous result for self-expanders.

1. Introduction

A $n$-dimensional submanifold $X : \Sigma^n \to \mathbb{R}^{n+k}$, $n \geq 2$, $k \geq 1$, is called a self-shrinker if it satisfies

$$\mathbf{H} = -\frac{1}{2} X^\bot,$$

where $\mathbf{H} = \sum_{i=1}^n \alpha(e_i, e_i)$ is the mean curvature vector field of $\Sigma^n$ and $X^\bot$ is the part of $X$ normal to $\Sigma^n$.

Self-shrinkers are self-similar solutions of the mean curvature flow and plays an important role in the study of this flow since they are type I singularities of the flow, see [11]. The simplest examples of self-shrinkers are the round spheres, planes and cylinders. Moreover, there are many results which present these examples as the only self-shrinkers satisfying some geometric restrictions, see [11], [6], [4], [13], and [14]. In common, all these results have the assumption that, when the self-shrinker is not compact, it must have polynomial volume growth or it must be proper. In [9], Cheng and Zhou proved that...
a self-shrinker has polynomial volume growth if and only if it is proper. Recently, jointly with Vieira, see [10], they generalized this result for submanifolds with bounded weighted mean curvature in a wide class of shrinking gradient Ricci solitons, which includes the Gaussian soliton. In particular, Cheng-Vieira-Zhou result gives that, for a surface with bounded $H + \frac{1}{2}X^\perp$, polynomial volume growth is equivalent to the properness of the submanifold (see Theorems 1.3 and 1.4 of [10]).

In this paper, we prove the rigidity of the sphere, the cylinders and the affine subspaces passing through the origin as the only self-shrinkers under another classic geometric assumption we describe below. Here and elsewhere, we identify the tangent spaces $T_p\Sigma^n$ with the affine subspace $X(p) + dX_p(T_p\Sigma^n)$, tangent to $X(\Sigma)$ at $X(p)$.

Let us denote by

$$W = \mathbb{R}^{n+k} \setminus \bigcup_{p \in \Sigma^n} T_p\Sigma^n$$

the set omitted by the union of the affine subspaces tangent to $X(\Sigma^n) \subset \mathbb{R}^{n+k}$. Here, we purpose to classify the self-shrinkers with nonempty $W$. The study of submanifolds of the Euclidean space with non-empty $W$ started with Halpern, see [17], who proved that compact and oriented hypersurfaces of the Euclidean space have nonempty $W$ if and only if it is embedded, diffeomorphic to the sphere and it is the boundary of a star-shaped domain of $\mathbb{R}^{n+1}$. Therefore, since the only self-shrinker with these characteristics are the round spheres of radius $\sqrt{2n}$ (see [19]), the case of compact self-shrinkers of codimension one with nonempty $W$ is completely solved.

In the non-compact case there are many examples of hypersurfaces with nonempty $W$. In fact, cylinders and paraboloids have open and nonempty $W$ and the one sheet hyperboloid has $W = \{0\}$. Surprisingly, in dimension two, if $\Sigma^2$ is a minimal surface of $\mathbb{R}^3$, then Hasanis and Koutrofiotis, see [18], proved that $W \neq \emptyset$ if and only if $\Sigma^n$ is a plane (in fact the result holds for arbitrary codimension, provided $X^\perp/\|X^\perp\|$ is parallel at the normal bundle). For higher dimensions, Alencar and Frensel, see [3], proved that the same result holds in higher dimension hypersurfaces of $\mathbb{R}^{n+1}$ (i.e., $\Sigma^n$ is a hyperplane), assuming in addition that $W$ is open. Other rigidity results involving $W$ and geometric assumptions can be found in [2], [3], [22] and [23].

First, we present two rigidity results for complete $n$-dimensional self-shrinkers in $\mathbb{R}^{n+1}$ with open and nonempty $W$. 

Theorem 1.1. Let $\Sigma^n$ be a complete, $n$-dimensional, self-shrinker of $\mathbb{R}^{n+1}$. If the set $W$ is open and nonempty, and the squared matrix norm $\|A\|^2$ of the second fundamental form $A$ of $\Sigma^n$ satisfies

$$\|A\|^2 \leq \frac{1}{2},$$

then $\Sigma^n = S^p(\sqrt{2p}) \times \mathbb{R}^{n-p}$, $0 \leq p \leq n$.

If $\|A\|^2 \geq 1/2$ and, additionally, we assume the mean curvature $H \geq 0$, we have

Theorem 1.2. Let $\Sigma^n$ be a complete, $n$-dimensional, self-shrinker of $\mathbb{R}^{n+1}$. If the set $W$ is open, $0 \in W$, and the squared matrix norm $\|A\|^2$ of the second fundamental form $A$ of $\Sigma^n$ satisfies

$$\|A\|^2 \geq \frac{1}{2},$$

then $\Sigma^n = S^p(\sqrt{2p}) \times \mathbb{R}^{n-p}$, $1 \leq p \leq n$.

Remark 1.1. Analyzing geometrically, one can see that self-shrinkers of the form $\Sigma^n = \Gamma \times \mathbb{R}^{n-1}$, where $\Gamma$ is a closed Abresch-Langer curve (see [1] and [16]) satisfies $W$ open and $0 \in W$. But all these examples satisfies

$$\min \|A\|^2 < \frac{1}{2} < \max \|A\|^2$$

after normalization.

Remark 1.2. The bound $1/2$ seems natural for self-shrinkers. In [6], Cao and Li proved that the only complete $n$-dimensional self-shrinkers of $\mathbb{R}^{n+k}$ with polynomial volume growth and such that $\|A\|^2 \leq 1/2$ are $S^p(\sqrt{2p}) \times \mathbb{R}^{n-p}$, $0 \leq p \leq n$. Cheng and Peng, see [7], and Rimoldi, see [25], with the aim to remove the hypothesis of polynomial volume growth, proved that the only self-shrinker of $\mathbb{R}^{n+1}$ with $\sup_{\Sigma^n} \|A\|^2 = 1/2$ (but with $\|A\|^2 < 1/2$) is a hyperplane. On the other hand, there are other rigidity results where the bound $\|A\|^2 \geq 1/2$ appears see, for example, [21], [13], [8], and [27]. Again, in common, all the last four references assumes polynomial volume growth or the immersion is proper.

Remark 1.3. Submanifolds with $W \neq \emptyset$ is a class of submanifolds distinct from those with polynomial of volume growth or those which are proper. In fact, cylinders over curves $\Gamma$ in $\mathbb{R}^2$ parametrized by $\Gamma(t) = b(t)(\cos t, \sin t)$, where $b(t) = 1 + e^{-t}$, or the family

$$b(t) = d + \frac{m}{\pi} \left( \frac{\pi}{2} - \arctan(at) \right), \quad d > 0, \quad m > 0, \quad 0 < a \leq 1. \quad (1.1)$$
Some straightforward calculation can prove that $\Sigma^n = \Gamma \times \mathbb{R}^{n-1}$ satisfies $W = D^2(d) \times \mathbb{R}$, and thus, nonempty. Here $D^2 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq d^2\}$ is the closed disk of radius $d$. Moreover, this hypersurface is non proper, since it is asymptotic to the cylinder of radius $d$ (and, thus, with volume growth bigger than polynomial, by the results of Cheng-Vieira-Zhou, see [10]). Since the curvature of $\Gamma$ lies in the interval $((d+m)^{-1}, d^{-1})$, we can chose these hypersurfaces satisfying the assumptions on $\|A\|^2$ of Theorems 1.1 and 1.2.

If $\Sigma^2$ has dimension two, then we can consider arbitrary codimension. We will assume further that $H \neq 0$ and that $H/\|H\|$ is parallel at the normal bundle.

**Theorem 1.3.** Let $\Sigma^2$ be a complete, two-dimensional, self-shrinker of $\mathbb{R}^{2+k}$, $k \geq 2$, with mean curvature vector $H \neq 0$ and such that $H/\|H\|$ is parallel at the normal bundle. If the set $W$ is open and nonempty and the squared matrix norm $\|A\|^2$ of the second fundamental form $A$ of $\Sigma^n$, relative to $H/\|H\|$, satisfies one of the following conditions:

i) $\|A\|^2 \leq 1/2$;

ii) $\|A\|^2 \geq 1/2$ and $0 \in W$;

then $\Sigma^2 = S^p(\sqrt{2p}) \times \mathbb{R}^{2-p}$, $1 \leq p \leq 2$.

**Remark 1.4.** If we assume that $\Sigma^2$ is compact without boundary in Theorem 1.3, then the hypothesis that $W$ is open can be removed. We point out that Smoczyk, see [26], proved that the only compact self-shrinkers, without boundary, of $\mathbb{R}^{n+p}$ with $H \neq 0$ and $H/\|H\|$ parallel in the normal bundle are minimal surfaces of the sphere $S^{n+p-1}(\sqrt{2n})$.

**Remark 1.5.** Drugan and Kleene in [15] proved the existence of infinitely many rotational self-shrinkers of each topological type of $S^n$, $S^{n-1} \times S^1$, $\mathbb{R}^n$ and $S^{n-1} \times \mathbb{R}$. Analyzing geometrically the picture of the profile curves presented there, we can see that the rotational self-shrinkers obtained by the rotation of those profile curves have empty $W$. We also remark that all these examples are not embedded since Kleene and Møller, see [20], proved that the sphere of radius $\sqrt{2n}$, the plane, and the right cylinder of radius $\sqrt{2(n-1)}$ are the only embedded rotational self-shrinkers of their respective topological type.

We conclude this paper with a non existence result for self-expanders with $W$ open and nonempty. Recall that a $n$-dimensional submanifold $X : \Sigma^n \to \mathbb{R}^{n+k}$ is called a self-expander if it satisfies

$$H = \frac{1}{2} X^\perp.$$
Theorem 1.4. There is no complete, non compact, \(n\)-dimensional self-expanders of \(\mathbb{R}^{n+k}\), \(k \geq 1\), with principal normal vector field \(H/\|H\|\) parallel in the normal bundle, and such that the set \(W\) is open and \(0 \in W\).

Remark 1.6. Clearly, if the codimension \(k = 1\), the hypothesis that \(H/\|H\|\) is parallel in the normal bundle is automatically satisfied and can be omitted in the statement of Theorem 1.4.

Remark 1.7. Analyzing geometrically, one can see that self-expanders of the form \(\Sigma^n = \Gamma \times \mathbb{R}^{n-1}\), where \(\Gamma\) is a self-expanding curve classified by Halldorsson, see [16], satisfies \(H > 0\) and \(W\) is open and nonempty, but \(0 \not\in W\), which implies that the hypothesis of \(0 \in W\) is crucial for the validity of Theorem 1.4.

Remark 1.8. It is well known, see [3], that there is no compact self-expanders in \(\mathbb{R}^{n+k}\). Thus, Theorem 1.4 does not make sense for \(\Sigma^n\) compact.

2. Preliminaries

Let \(i : \Sigma^n \to M^{n+k}\), \(n \geq 2\), \(k \geq 1\), be an isometric immersion, where \(\Sigma^n\) and \(M^{n+k}\) are Riemannian manifolds and the superscripts denote the dimension. Denote by \(\nabla\) and \(\nabla\) be the connections of \(\Sigma^n\) and \(M^{n+k}\), respectively. We assume here that the immersion admits a conformal vector field, i.e., a vector field \(X \in TM\) such that

\[
\nabla_Y X = \varphi Y,
\]

for some smooth function \(\varphi : M^{n+k} \to \mathbb{R}\), called conformal factor of \(X\), and for every \(Y \in T\Sigma^n\). Decompose \(X\) as

\[
X = X^\top + X^\perp,
\]

where \(X^\top \in T\Sigma^n\) and \(X^\perp \in (T\Sigma^n)^\perp\). Here \((T\Sigma^n)^\perp\) is the normal bundle of the immersion such that \(T\Sigma^n \oplus (T\Sigma^n)^\perp = TM^{n+k}\).

If the codimension is one, then we have \(X^\perp = \langle X, N \rangle N\), where \(N\) is the globally defined unitary normal vector field. If the codimension is at least two, suppose further that \(X^\perp \neq 0\). In both cases we can write

\[
X = X^\top + f\eta,
\]
for $f = \langle X, \eta \rangle$, where $\eta = N$ if the codimension is one and $\eta = X^\perp/\|X^\perp\|$ if the codimension is at least two.

The immersion satisfies

$$\nabla_U V = \nabla_U V + \alpha(U, V) \quad \text{and} \quad \nabla_U \eta = -AU + \nabla_U^\perp \eta,$$

where $\langle AU, V \rangle = \langle \alpha(U, V), \eta \rangle$, $\alpha$ is the second fundamental form of the immersion, and $\nabla^\perp$ denotes the normal connection at the normal bundle $(T\Sigma^\perp)^\perp$.

The next proposition contains the basic calculations needed to prove the main theorems of this paper.

**Proposition 2.1.** Let $M^{n+k}$ be a $(n+k)$-dimensional Riemannian manifold which admits a conformal vector field $X$ with conformal factor $\varphi$. Let $\Sigma^n$ be a submanifold of $M^{n+k}$ and $\{\eta, \eta_2, \ldots, \eta_k\}$ be an orthonormal frame of the normal bundle $(T\Sigma^n)^\perp \subset TM^{n+k}$, where, for $k = 1$, $\eta = N$, the globally defined unitary normal vector field, and for $k \geq 2$, we assume that $X^\perp \neq 0$ and take $\eta = X^\perp/\|X^\perp\|$. If $f = \langle X, \eta \rangle$, then

$$\Delta f + \varphi(\text{trace } A) + f \|A\|^2 + \langle X^\perp, \text{grad}(\text{trace } A) \rangle =$$

$$= -\sum_{\beta=2}^{k} s_{1\beta}(A_{\beta}X^\perp) + \sum_{\beta=2}^{k} s_{1\beta}(X^\perp)(\text{trace } A_{\beta}) + \sum_{i=1}^{n} \langle R(e_i, X^\perp)e_i, \eta \rangle.$$

Here, $A$ and $A_{\beta}$ are the shape operators relative to the normals $\eta$ and $\eta_{\beta}$, $\beta \in \{2, \ldots, k\}$, respectively, $\|A\|^2 = \text{trace}(A^2)$ is the matrix norm of $A$, $s_{1\beta}(X) = \langle \nabla^\perp_X \eta, \eta_{\beta} \rangle$, $R$ is the curvature tensor of $M^{n+k}$, and $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal frame of $\Sigma^n$.

If the immersion has codimension one (i.e., $k = 1$), then

$$\Delta f + \varphi H + f \|A\|^2 + \langle X^\perp, \text{grad } H \rangle = \sum_{i=1}^{n} \langle R(e_i, X^\perp)e_i, \eta \rangle,$$

where $H = \text{trace } A$ is the mean curvature of $\Sigma^n$. In particular, if the Ricci curvature of $M^{n+1}$ is constant (i.e., $M^{n+1}$ is an Einstein space), then

$$\Delta f + \varphi H + f \|A\|^2 + \langle X^\perp, \text{grad } H \rangle = 0.$$
Proof. Let $U \in T\Sigma^n$. Since, using (2.3),
\[
\varphi U = \nabla_U X = \nabla_U X^\top + (Uf)\eta + f\nabla_U \eta
\]
\[
= \nabla_U X^\top + \alpha(X^\top, U) + (Uf)\eta - fAU + f\nabla^\perp_U \eta,
\]
we have, taking the tangent and the normal parts,
\begin{equation}
\varphi U = \nabla_U X^\top - fAU \tag{2.7}
\end{equation}
and
\begin{equation}
\alpha(X^\top, U) + (Uf)\eta + f\nabla^\perp_U \eta = 0. \tag{2.8}
\end{equation}

From (2.7) we have
\begin{equation}
\nabla_U X^\top = (\varphi I + fA)U, \tag{2.9}
\end{equation}
which implies
\begin{equation}
\text{div} \ X^\top = n\varphi + f(\text{trace} \ A), \tag{2.10}
\end{equation}
where $\text{div} \ X^\top$ is the divergence of $X^\top$ in $\Sigma^n$. From (2.8) we obtain
\begin{equation}
Uf = -\langle \alpha(X^\top, U), \eta \rangle, \tag{2.11}
\end{equation}
since $\langle \nabla^\perp_U \eta, \eta \rangle = 0$. Therefore,
\begin{equation}
\text{grad} \ f = -AX^\top. \tag{2.12}
\end{equation}

Let $\{\eta_1 = \eta, \eta_2, \ldots, \eta_k\}$ be an orthonormal frame of $(T\Sigma^n)^\perp$ and write
\[
\nabla^\perp_U \eta = \sum_{\beta=2}^k s_{1\beta}(X)\eta_\beta, \text{ where } s_{1\beta}(X) = \langle \nabla^\perp_X \eta, \eta_\beta \rangle.
\]
Taking the inner product of (2.8) with $\eta_\beta$, we have
\[
\langle \alpha(X^\top, U), \eta_\beta \rangle + f s_{1\beta}(U) = 0
\]
i.e.,
\begin{equation}
f s_{1\beta}(U) = -\langle A_\beta X^\top, U \rangle, \tag{2.13}
\end{equation}
where $\langle A_\beta U, V \rangle = \langle \alpha(U, V), \eta_\beta \rangle$. 
Let us calculate the Laplacian of $f$. Since, by (2.11), $Uf = -\langle AX^T, U \rangle$, and using (2.9), we obtain

$$U(Uf) = -U \langle AX^T, U \rangle = -U \langle X^T, AU \rangle$$

$$= -\langle \nabla_U X^T, AU \rangle - \langle X^T, \nabla_U (AU) \rangle$$

$$= -\varphi(U, AU) - f \langle AU, AU \rangle - \langle X^T, \nabla_U (AU) \rangle$$

and $$(\nabla_U f) = -\langle AX^T, \nabla_U U \rangle = -\langle X^T, A(\nabla_U U) \rangle.$$ This implies

$$Hess f(U, U) = -\varphi(U, AU) - f \langle AU, AU \rangle - \langle X^T, (\nabla_U A)(U) \rangle,$$

where $(\nabla_U A)(V) = \nabla_U AV - A(\nabla_U V)$. Taking the trace, we have

$$\Delta f = -\varphi(\text{trace } A) - f(\text{trace}(A^2)) - \sum_{i=1}^n \langle X^T(\nabla_{e_i} A)(e_i) \rangle,$$

where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal frame of $T\Sigma^n$. On the other hand, the Codazzi equation

$$\langle \mathcal{R}(U, V)W, \eta \rangle = \langle (\nabla_V A)(U) - (\nabla_U A)(V), W \rangle$$

$$+ \langle \alpha(V, W), \nabla^\perp_U \eta \rangle - \langle \alpha(U, W), \nabla^\perp_V \eta \rangle$$

and

$$\langle \alpha(V, W), \nabla^\perp_U \eta \rangle = \sum_{\beta=2}^k s_{1\beta}(U) \langle \alpha(V, W), \eta_\beta \rangle = \sum_{\beta=2}^k s_{1\beta}(U) \langle A_\beta V, W \rangle$$

give

$$(\nabla_U A)(V) = (\nabla_V A)(U) + \sum_{\beta=2}^k [s_{1\beta}(U)A_\beta V - s_{1\beta}(V)A_\beta U]$$

$$+ \sum_{i=1}^n \langle \mathcal{R}(U, V)\eta, e_k \rangle e_k.$$ 

Since $A$ is symmetric, $\nabla_U A$ is symmetric also, and moreover

$$\text{trace}(\nabla_U A) = U(\text{trace } A).$$
These equations give
\[ \sum_{i=1}^{n} \langle X^\top, (\nabla_{e_i} A)(e_i) \rangle = \sum_{i=1}^{n} \langle (\nabla_{e_i} A)(X^\top), e_i \rangle = \sum_{i=1}^{n} \langle (\nabla_{X^\top} A)(e_i), e_i \rangle \\
+ \sum_{\beta=2}^{k} \sum_{i=1}^{n} [s_{1\beta}(e_i) \langle A_\beta X^\top, e_i \rangle - s_{1\beta}(X^\top) \langle A_\beta e_i, e_i \rangle] \\
+ \sum_{i=1}^{n} \langle \bar{R}(e_i, X^\top) \eta, e_i \rangle \\
= \text{trace}(\nabla_{X^\top} A) + \sum_{\beta=2}^{k} \sum_{i=1}^{n} s_{1\beta}(e_i) \langle A_\beta X^\top, e_i \rangle \\
- \sum_{\beta=2}^{k} s_{1\beta}(X^\top) \langle \text{trace} A_\beta \rangle + \sum_{i=1}^{n} \langle \bar{R}(e_i, X^\top) \eta, e_i \rangle \\
= \langle X^\top, \text{grad} \text{trace}(A) \rangle + \sum_{\beta=2}^{k} s_{1\beta}(A_\beta X^\top) \\
- \sum_{\beta=2}^{k} s_{1\beta}(X^\top) \langle \text{trace} A_\beta \rangle + \sum_{i=1}^{n} \langle \bar{R}(e_i, X^\top) \eta, e_i \rangle, \\
\] which implies
\[ \Delta f = -\varphi(\text{trace} A) - f \| A \|^2 - \langle X^\top, \text{grad} \text{trace}(A) \rangle \]
(2.16)
\[ - \sum_{\beta=2}^{k} s_{1\beta}(A_\beta X^\top) + \sum_{\beta=2}^{k} s_{1\beta}(X^\top) \langle \text{trace} A_\beta \rangle + \sum_{i=1}^{n} \langle \bar{R}(e_i, X^\top) \eta, e_i \rangle, \]
where \( \| A \|^2 = \text{trace}(A^2) \) is the matrix norm of \( A \).

In the next consequence of Proposition 2.1, let us assume that there exists \( \varepsilon \in \mathbb{R} \) such that, restricted to \( \Sigma \),
(2.17)
\[ \mathbf{H} = \varepsilon \mathbf{X}^\perp, \]
where \( \mathbf{H} = \sum_{i=1}^{n} \alpha(e_i, e_i) \) is the mean curvature vector field of \( \Sigma \) in \( M^{n+k} \). If \( M^{n+k} = \mathbb{R}^{n+k} \), then \( \Sigma \) is a mean curvature flow soliton, which is called a self-shrinker, if \( \varepsilon < 0 \), and a self-expander, if \( \varepsilon > 0 \). Here, we will adopt one of the canonical normalizations, considering \( \varepsilon = -\frac{1}{2} \) for self-shrinkers and \( \varepsilon = \frac{1}{2} \) for self-expanders.

If \( \Sigma \) is submanifold of \( M^{n+k} \) satisfying (2.17), then
(2.18)
\[ \text{trace} A = \varepsilon f \] and \( \text{trace} A_\beta = 0. \]
Let us define the elliptic operator \( L \) by
\[
L f = \Delta f + \varepsilon \langle X, \text{grad} \ f \rangle.
\]

The next result is a direct consequence of Proposition 2.1, and gives us the main equations to prove our results.

**Corollary 2.1.** Let \( M^{n+k} \) be a \((n+k)\)-dimensional Riemannian manifold which admits a conformal vector field \( X \) with conformal factor \( \varphi \). Let \( \Sigma^n \) be a submanifold of \( M^{n+k} \) such that the mean curvature vector \( H \) of \( \Sigma^n \) satisfies \( H = \varepsilon X \perp \) for some \( \varepsilon \in \mathbb{R} \), and \( \{\eta, \eta_2, \ldots, \eta_k\} \) be an orthonormal frame of the normal bundle \( (T\Sigma^n)\perp \subset TM^{n+k} \), where, for \( k = 1 \), \( \eta = N \), the globally defined unitary normal vector field, and for \( k \geq 2 \), we assume that \( X \perp \neq 0 \) and take \( \eta = X \perp /\|X \perp\| \).

If \( f = \langle X, \eta \rangle \), then
\[
L f + (\|A\|^2 + \varepsilon \varphi) f = -\sum_{\beta=2}^{k} s_{1\beta}(A_{\beta}X^\top \) \sum_{i=1}^{n} (\mathcal{R}(e_i, X^\top e_i, \eta).
\]

Here, \( A \) and \( A_{\beta} \) are the shape operators relative to the normals \( \eta \) and \( \eta_{\beta}, \beta \in \{2, \ldots, k\} \), respectively, \( s_{1\beta}(U) = \langle \nabla^\perp \eta, \eta_{\beta} \rangle \), \( \mathcal{R} \) is the curvature tensor of \( M^{n+k} \), and \( \{e_1, \ldots, e_n\} \) is an orthonormal frame of \( T\Sigma^n \). Moreover, if \( f \neq 0 \), then
\[
L f + (\|A\|^2 + \varepsilon \varphi) f = \frac{1}{f} \sum_{\beta=2}^{k} \|A_{\beta}X^\top\|^2 + \sum_{i=1}^{n} (\mathcal{R}(e_i, X^\top e_i, \eta).
\]

In particular, if \( M^{n+k} \) has constant sectional curvature and \( \nabla^\perp \eta = 0 \), or the immersion has codimension one and \( M^{n+1} \) is Einstein, then
\[
L f + (\|A\|^2 + \varepsilon \varphi) f = 0.
\]

**Proof.** By using (2.18) and (2.19) in (2.4), p. 6, we obtain (2.20). Equation (2.21) comes from replacing (2.13) in the first term of the right hand side of (2.20). To prove (2.22), notice that, if \( M^{n+k} \) has constant sectional curvature \( \kappa_0 \), then
\[
\langle \mathcal{R}(e_i, X^\top e_i, \eta) \rangle = \kappa_0 (\langle e_i, e_i \rangle \langle X^\top, \eta \rangle - \langle X^\top, e_i \rangle \langle e_i, \eta \rangle) = 0,
\]
since \( \langle X^\top, \eta \rangle = 0 = \langle e_i, e_i \rangle \). Moreover, if \( \nabla^\perp \eta = 0 \), then \( s_{1\beta} \equiv 0 \) for every \( \beta \in \{2, \ldots, k\} \), i.e., \( A_{\beta}X^\top = 0 \). On the other hand, if Ric_M = \( \lambda \langle \cdot, \cdot \rangle \), \( \lambda \in \mathbb{R} \), then
\[
\sum_{i=1}^{n} \langle \mathcal{R}(e_i, X^\top e_i, \eta) \rangle = \text{Ric}_M(X^\top, \eta) - \langle \mathcal{R}(\eta, X^\top)\eta, \eta \rangle
\]
\[
= \text{Ric}_M(X^\top, \eta) = \lambda \langle X^\top, \eta \rangle = 0.
\]
In order to prove our results, we will also need the classical Hopf maximum principle for elliptic operators:

**Lemma 2.1** (Hopf’s maximum principle, see [24]). Let

\[ Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u \]

be a strictly elliptic differential operator defined in a open set \( \Omega \subset \mathbb{R}^n \).

(i) If \( c = 0 \), \( Lu \geq 0 \) (resp. \( Lu \leq 0 \)) and there exists \( \max_{\Omega} u \) (resp. \( \min_{\Omega} u \)), then \( u \) is constant.

(ii) If \( c \leq 0 \), \( Lu \geq 0 \) (resp. \( Lu \leq 0 \)) and there exists \( \max_{\Omega} u \geq 0 \) (resp. \( \min_{\Omega} u \leq 0 \)), then \( u \) is constant.

(iii) Independently of the signal of \( c \), if \( Lu \geq 0 \) (resp. \( Lu \leq 0 \)) and \( \max_{\Omega} u = 0 \) (resp. \( \min_{\Omega} u = 0 \)), then \( u \) is constant.

### 3. Proof of the main theorems

Now we are ready to proof our main theorems.

**Proof of Theorem 1.1**. In \( \mathbb{R}^{n+1} \), the position vector is a conformal vector field with conformal factor \( \varphi = 1 \). Since \( \Sigma^n \) is a self-shrinker, we have

\[ (3.1) \quad H = -\frac{1}{2} \langle X, N \rangle = -\frac{1}{2} f, \]

where \( N \) is a unitary normal vector field. Since the codimension is one and \( \Sigma^n \) is a self-shrinker, using Equation (2.22) of Proposition 2.1 for \( \varepsilon = -1/2 \), we obtain

\[ (3.2) \quad \mathcal{L}f + \left( \|A\|^2 - \frac{1}{2} \right) f = 0. \]

By using Newton’s inequality

\[ \frac{(\text{trace } A)^2}{n} \leq \|A\|^2, \]

Equation (3.1), and the hypothesis \( \|A\|^2 \leq 1/2 \), we have

\[ (3.3) \quad f^2 = 4H^2 = 4(\text{trace } A)^2 \leq 4n\|A\|^2 \leq 2n. \]

This implies

\[ -\sqrt{2n} \leq f \leq \sqrt{2n}. \]
and, thus, there exist $m = \inf_{\Sigma^n} f$ and $d = \sup_{\Sigma^n} f$. If $m \leq 0$, then
\[
\mathcal{L}(f - m) + \left(\|A\|^2 - \frac{1}{2}\right)(f - m) = -\left(\|A\|^2 - \frac{1}{2}\right)m \leq 0.
\]

If $m = \min_{\Sigma^n} f$, i.e., if $f$ reaches a minimum, then by the Hopf maximum principle (Lemma 2.1 item (ii)), applied to $f - m$, we can conclude that $f$ is constant. On the other hand, if $m > 0$, then $d = \sup_{\Sigma^n} f > 0$. Thus if $f$ reaches (positive) a maximum, i.e., $d = \max_{\Sigma^n} f$ then, applying the Hopf maximum principle (Lemma 2.1 item (ii)), to equation (3.2), we conclude that $f$ is constant.

On the other hand, Dajczer and Tojeiro, see [12], Theorem 1, p.296, proved that the only hypersurfaces of $\mathbb{R}^{n+1}$ with constant support function $f$ are the cylinders, spheres and hyperplanes. The conclusion that $\Sigma^n = S^p(\sqrt{2p}) \times \mathbb{R}^{n-p}, 0 \leq p \leq n$, comes from Equation (3.1).

Thus we need to prove only that $f$ reaches a minimum. The proof that $f$ reaches a maximum is identical.

Since $W \neq \emptyset$, there exists $p_0 \in W$, i.e., $p_0 \not\in \bigcup_{p \in \Sigma^n} T_p \Sigma^n$, which implies that $p - p_0 \not\in T_p \Sigma^n$ for every $p \in \Sigma^n$. Let $\{p_k\}$ be a sequence of points in $\Sigma^n$ such that $f(p_k) \rightarrow m$ when $k \rightarrow \infty$. For each $p_k$ consider $q_k$ the projection of $p_0$ over $T_{p_k} \Sigma^n$ (see Figure 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{projection.png}
\caption{Projection of $p_0$ over $T_{p_k} \Sigma^n$}
\end{figure}

Since
\[
\text{dist}(q_k, p_0) = \|q_k - p_0\| = \|\text{proj}_{N(p_k)}(p_k - p_0)\| = |\langle p_k - p_0, N(p_k) \rangle| = |f(p_k) - \langle p_0, N(p_k) \rangle| \leq |f(p_k)| + |p_0| \leq \sqrt{2n} + |p_0|
\]
where \( \text{proj}_u v \) denotes the projection of the vector \( v \) over the vector \( u \), we have that \( \{q_k\} \) is a bounded sequence in \( \bigcup_{p \in \Sigma^n} T_p \Sigma^n = \mathbb{R}^{n+1} - W \). Moreover, since \( W \) is open, we have that \( \mathbb{R}^{n+1} - W \) is closed. Thus, passing to a subsequence if necessary, we can deduce that \( q_k \) converges to a point \( q_1 \in \mathbb{R}^{n+1} - W \). Let \( p_1 \in \Sigma^n \) such that \( q_1 \in T_{p_1} \Sigma^n \). This implies

\[
f(p_1) = \lim_{k \to \infty} f(p_k) = m,
\]
i.e., \( m \) is a minimum for \( f \).

**Proof of Theorem 1.2.** Since the codimension is one and \( \Sigma^n \) is a self-shrinker, using Equation (2.22) of Proposition 2.1 for \( \varepsilon = -1/2 \), we obtain

\[
\mathcal{L}f = \left( \frac{1}{2} - \|A\|^2 \right) f.
\]

Since \( 0 \in W \), we have that \( 0 \notin \bigcup_{p \in \Sigma^n} T_p \Sigma^n \) which implies that \( p \notin T_p \Sigma^n \) (seen as a vector centered at the origin) for every \( p \in \Sigma^n \). This implies that \( f(p) = \langle p, N \rangle \neq 0 \) for every \( p \in \Sigma^n \), where \( N \) denotes the unit normal vector field of \( \Sigma^n \) in \( \mathbb{R}^{n+1} \). We assume, without loss of generality, that \( f < 0 \) everywhere in \( \Sigma^n \). Thus, if \( \|A\|^2 \geq 1/2 \), then \( \mathcal{L}f \geq 0 \). Since \( f < 0 \), there exists \( d = \sup_{\Sigma^n} f \). Thus, if \( f \) reaches a maximum, i.e., \( d = \max_{\Sigma^n} f \), then by using the Hopf maximum principle (Lemma 2.1 item (i)), we conclude that \( f \) is constant. Therefore, we need to prove only that \( f \) reaches a maximum.

Let \( \{p_k\} \) be a sequence of points in \( \Sigma^n \) such that \( f(p_k) \to d \) when \( k \to \infty \). For each \( p_k \) consider \( q_k \) the projection of \( p_0 \) over \( T_{p_k} \Sigma^n \). Since

\[
dist(q_k, 0) = |q_k| = |\text{proj}_{N(p_k)}(p_k)| = |\langle p_k, N(p_k) \rangle| = -f(p_k)
\]

and \( f(p_k) \) is a bounded sequence (since it converges), we have that \( \{q_k\} \) is a bounded sequence in \( \bigcup_{p \in \Sigma^n} T_p \Sigma^n = \mathbb{R}^{n+1} - W \). Moreover, since \( W \) is open, we have that \( \mathbb{R}^{n+1} - W \) is closed. Thus, passing to a subsequence if necessary, we can deduce that \( q_k \) converges to a point \( q_1 \in \mathbb{R}^{n+1} - W \). Let \( p_1 \in \Sigma^n \) such that \( q_1 \in T_{p_1} \Sigma^n \). This implies

\[
f(p_1) = \lim_{k \to \infty} f(p_k) = d,
\]
i.e., \( d \) is a maximum for \( f \). Thus, \( f = d \) is constant, which implies

\[
\left( \frac{1}{2} - \|A\|^2 \right) d = 0.
\]
If \( d = 0 \), then \( H = \frac{1}{2}(X, N) = 0 \), which gives that \( \Sigma^n \) is a hyperplane, but it contradicts the assumption \( \|A\|^2 \geq 1/2 \). Thus, \( d < 0 \), \( \|A\|^2 = 1/2 \), and \( \Sigma^n = \mathbb{S}^p(\sqrt{2p}) \times \mathbb{R}^{n-p}, 1 \leq p \leq n \), as in the proof of in Theorem 1.1.

**Proof of Theorem 1.3.** Since \( \nabla \perp \eta = 0 \), where \( \eta = H/\|H\| \), then \( s_{1, \beta} \equiv 0 \), which implies that \( A_{\beta}X^\top = 0 \) for every \( \beta = 2, \ldots, k \). Since \( \text{trace } A_{\beta} = 0 \) and the dimension is two, we have \( A_{\beta} = 0 \).

On the other hand, in \( \mathbb{R}^{2+k} \) the position vector is a conformal vector with conformal factor \( \varphi = 1 \). In this case, since the codimension is \( k \geq 2 \) and \( \Sigma^2 \) is a self-shrinker, we have

\[
 f = \langle X, \eta \rangle = \|X^\perp\| = 2\|H\| > 0.
\]

Thus, by the Proposition 2.1, we have, for \( c = -1/2 \),

\[
(3.4) \quad \mathcal{L} f = \left( \frac{1}{2} - \|A\|^2 \right) f.
\]

i) If \( \|A\|^2 \leq 1/2 \), then \( \mathcal{L} f \geq 0 \). Since \( f^2 \leq 4 \) (see estimate (3.3) in the proof of Theorem 1.1), there exists \( d = \sup_{\Sigma^2} f \). Thus, if \( f \) reaches a maximum, i.e., \( d = \max_{\Sigma^2} f \), then by using the Hopf maximum principle (Lemma 2.1 item (i)), we conclude that \( f \) is constant.

ii) If \( \|A\|^2 \geq 1/2 \) then \( \mathcal{L} f \leq 0 \). Since \( f > 0 \), there exists \( m = \inf_{\Sigma^2} f \). Thus, if \( f \) reaches a minimum, i.e., \( m = \min_{\Sigma^2} f \), then by using the Hopf maximum principle, (Lemma 2.1 item (i)), we conclude that \( f \) is constant.

The proof that \( f \) reaches a maximum or a minimum is identical to that presented in the proof of Theorem 1.1 and Theorem 1.2.

Thus, in both cases, \( f \) is constant, which implies that \( \|A\| = 0 \) and \( \Sigma \) is plane passing through the origin, or \( \|A\|^2 = 1/2 \). Since the second fundamental \( \alpha \) satisfies

\[
\mathcal{L} \|\alpha\|^2 = 2\|
abla \alpha \|^2 + \|\alpha\|^2 - 2 \sum_{\beta \neq \delta} \|[A_{\beta}, A_{\delta}]\|^2
- 2 \sum_{\beta, \delta} \left( \sum_{i, j=1}^2 \langle \alpha(e_i, e_j), \eta_\beta \rangle \langle \alpha(e_i, e_j), \eta_\delta \rangle \right)^2
\]
\[
\begin{align*}
&= 2\|\nabla \alpha\|^2 + \|\alpha\|^2 - 2\sum_{\beta \neq \delta} \|A_\beta \circ A_\delta - A_\delta \circ A_\beta\|^2 \\
&\quad - 2\sum_{\beta, \delta} \left( \sum_{i,j=1}^2 \langle A_\beta(e_i), e_j \rangle \langle A_\delta(e_i), e_j \rangle \right)^2 \\
&= 2\|\nabla \alpha\|^2 + \|\alpha\|^2 - 2\sum_{\beta \neq \delta} \|A_\beta \circ A_\delta - A_\delta \circ A_\beta\|^2 \\
&\quad - 2\sum_{\beta, \delta} (\text{trace}(A_\beta \circ A_\delta))^2
\end{align*}
\]

(see [13], p.5069, Eq. (2.5)) and \(A_\beta = 0, \beta = 2, \ldots, k\), we have \(\|\alpha\| = \|A\|\) and

\[
\mathcal{L}\|A\|^2 = 2\|\nabla A\|^2 + \|A\|^2 - 2\|A\|^4.
\]

Thus \(\|A\|^2 = 1/2\) implies that \(\|\nabla A\|^2 = 0\). Therefore \(\Sigma^2\) is isoparametric and thus \(\Sigma^2 = S^1(\sqrt{2}) \times \mathbb{R}\) or \(\Sigma^2 = S^2(2)\).

\[\square\]

**Proof of Theorem 1.4.** If \(k \geq 2\), \(\Sigma^n\) is a self-expander such that \(f = 2\|H\| > 0\), and \(H/\|H\|\) is parallel, then, by Proposition 2.1,

\[
\mathcal{L}f + \left( \|A\|^2 + \frac{1}{2} \right) f = 0.
\]

If \(k = 1\) and \(0 \in W\), then \(f \neq 0\) as in the proof of Theorem 1.2 and we can assume \(f > 0\). Thus \(\mathcal{L}f \leq 0\). Since \(f\) is bounded below, there exists \(m = \inf_{\Sigma^n} f\). Since \(W\) is open and \(0 \in W\), reasoning as in the proof of Theorem 1.2, we can prove that \(m\) is actually a minimum. Therefore, by the Hopf maximum principle, we can see that \(f\) is constant, which implies

\[
\left( \|A\|^2 + \frac{1}{2} \right) f = 0,
\]

but it is impossible, since \(f > 0\).

\[\square\]

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