Singularity analysis in $A_n$ Affine Toda Theories

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ABSTRACT

The leading and the subleading Landau singularities in affine Toda field theories are examined in some detail. Formulae describing the subleading simple pole structure of box diagrams are given explicitly. This leads to a new and nontrivial test of the conjectured exact S-matrices for these theories. We show that to the one-loop level the conjectured S-matrices of the $A_n$ Toda family reproduce the correct singularity structure, leading as well as subleading, of the field theoretical amplitudes. The present test has the merit of being independent of the details of the renormalisations.
I. Introduction

The affine Toda field theories (ATFTs) are a remarkable class of massive two-dimensional models. Classically integrable, these models possess candidate exact S-matrices conjectured to describe the quantum theory\[1, 2, 3, 4, 5\]. Such S-matrices enable one to study the thermodynamic Bethe ansatz\[6\] and other scaling properties of the theory. Although these S-matrices have passed many nontrivial low order checks, a proof has yet to be given that they are indeed the S-matrices of the theory. These models remain an attractive testing ground for understanding some of the rich and diverse phenomena of quantum field theory. This letter will study some of complicated Landau singularity\[7\] structure of ATFT and provide a new and nontrivial test for the putative exact S-matrices.

The coupling constant $\beta$ dependence of the exact ATFT S-matrices is believed to appear through a single universal function $B(\beta)$ \[1, 2, 3\].

\[
B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{1 + \beta^2/4\pi},
\]

and this has been verified to $\beta^4$ by conventional perturbation theory in the absence of the anomalous threshold singularities\[3, 5\]. Recently the $\beta^6$ term was also confirmed in $A_2$ ATFT using a dispersion relation approach\[9\]. This example is particularly simple for the triangle diagrams that appear are nonsingular and one only needs to consider the effect of renormalisation and not the effect of the (here vanishing) Landau singularity\[10\]. In general at this order delicate cancellations appear between the leading Landau singularities of the relevant box and triangle diagrams \[11\] as well as the subleading terms\[12, 13\]. This letter will further test the conjectured S-matrices by examining the role of these subleading terms when they too are singular. We will focus on $A_n$ Toda theories for two reasons. First, the present singularity analysis up to double poles is complete for the $A_n$ series. Second, the
effects of renormalisation can be clearly separated from those of the Landau singularity in these cases. We show how the subleading Landau singularities conspire to give the correct (in fact vanishing) residues of the simple poles at the general double pole positions of the $A_n$ theory’s S-matrices.

The general methods of extracting subleading singularities developed here are universally applicable to any two-dimensional field theories. However, the complete singularity analysis for the other members of ATFTs, $D_n$ and $E_n$ series, is further complicated by the higher order poles and renormalisation effects.

II. Preliminaries

We will now state our conventions and recall the essential points of ATFT needed for our calculation. The bosonic ATFT\[4, 2\] based upon a Lie algebra $g$ has rank $r$ massive scalar fields $\phi^a$ with exponential interactions. The Lagrangian of the theory is\[1\]

$$
\mathcal{L}(\phi) = \frac{1}{2} \partial^\mu \phi^a \partial^\mu \phi^a - \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i c^{\beta a_i^\alpha \cdot \phi^a},
$$

(2)

where $\alpha_i (i = 1, ⋯, r)$ are the simple roots of $g$ (normalised $\alpha_i^2 = 2$) and

$$
\alpha_0 = - \sum_{i=1}^{r} n_i \alpha_i, \quad n_0 = 1.
$$

The integers $n_i$ are the so called Kac labels for the Lie algebra. Here $m$ sets the mass scale and the real number $\beta$ is the dimensionless coupling constant.

These theories may be described equally well by listing the masses and multipoint coupling constants. Classically the mass\(^2\) are given by the eigenvalues of the matrix

$$
(M^2)_{ab} = m^2 \sum_{i=0}^{r} n_i \alpha_i^a \alpha_i^b
$$

(3)

\(^1\)For various reasons we restrict to ATFTs based on simply laced algebras.
and the three-point couplings may be obtained from

\[ c_{abc} = m^2 \beta \sum_{i=0}^{r} n_i \alpha_i^a \alpha_i^b \alpha_i^c. \]  

(4)

The four-point and higher point couplings are determined by this information\[10\]. For the \( A_n \) theories the classical masses are

\[ m_a = 2m \sin \frac{a\pi}{h}, \quad a = 1, \ldots, n, \]  

(5)

(where \( h = n + 1 \) is the Coxeter number for \( A_n^{(1)} \)) and the nonzero multipoint couplings are neatly given in terms of these by

\[ C_{a_1 \ldots a_p} = (-1)^{\sum a_k} (-i)^p \left( \frac{\beta^2}{m^2 h} \right)^{\frac{p}{2} - 1} \prod_{k=1}^{p} m_{a_k}, \quad \text{if } \sum_{k=1}^{p} a_k \equiv 0 \pmod{h}. \]  

(6)

The two-particle elastic S-matrices for the \( A_n^{(1)} \) theories are conjectured to be

\[ S_{ab} = \prod_{|a-b|+1 \text{ step2}}^{a+b-1} \{p\}, \]  

(7)

where, using the notation of ref.\[2\],

\[ \{x\} = \frac{(x-1)(x+1)}{(x-1+B)(x+1-B)}, \quad (x) = \frac{\sinh\left(\frac{\theta}{2} + \frac{ix}{2h}\right)}{\sinh\left(\frac{\theta}{2} - \frac{ix}{2h}\right)}, \]  

(8)

and \( B(\beta) \) is given by Eq.(4). These S-matrices have maximally double poles.

Finally we summarise the relevant Feynman rules adopted

(a) For each (three point) vertex : \(-i(2\pi)^2 c_{abc}\),

(b) For each propagator : \( \frac{i}{(2\pi)^2 (p_a^2 - m_a^2)} \),

(c) Loop integration : \( \int d^2 l \).

(9)

In order to obtain the S-matrix element from the Feynman amplitude we need the flux normalisation factor or Jacobian coming from the change of variables from the linear momentum to the rapidity [10]. With \( p_a = m_a (\cosh \theta_a, \sinh \theta_a) \) and \( \theta = \theta_a - \theta_b \) this behaves

\[ \text{For an interesting compact representation of the S-matrices in terms of vertex operators see [15].} \]
near the singular point $\theta = i\theta_0$ as

$$\frac{1}{(2\pi)^24m_am_b\sinh\theta} = \frac{-i}{(2\pi)^24m_am_b\sin\theta_0}[1 + i\cot\theta_0(\theta - i\theta_0) + \cdots].$$

(10)

For future reference we also note the following relationship between the Mandelstam variable $s = (p_a + p_b)^2$ and rapidity $\theta$ near the singularity

$$\frac{1}{s - s_0} = \frac{-i}{2m_am_b\sin\theta_0(\theta - i\theta_0)}[1 + i\cot\theta_0(\theta - i\theta_0) + \cdots].$$

(11)

### III. The Subleading Singularity

A general Feynman amplitude may be expanded about some point $i\theta_0$ in the rapidity as follows

$$\text{Amplitude} = \frac{R_{-p}}{\theta - i\theta_0} + \frac{R_{-p+1}}{(\theta - i\theta_0)^{p-1}} + \cdots + R_0 + R_1(\theta - i\theta_0) + \cdots.$$  

(12)

Here $p$ is the maximal order of the singularity and the tool most often used in the analysis of these singular terms is the so called ‘scaling’ method. We shall now describe how this works for the leading and subleading terms of the uncrossed box diagram (a) and the crossed box diagrams (b) and (c) of figure 1.

If we denote the $i$-th internal propagator momentum as $Q_i$, the momentum integral for the general box diagram is

$$L(s) = \int d^2k \frac{1}{[Q_A^2 - m_A^2][Q_B^2 - m_B^2][Q_C^2 - m_C^2][Q_D^2 - m_D^2]},$$

(13)

$$Q_i = q_i + k = q_i + (s - s_0)l, \quad i = A, B, C, D.$$

When this diagram has a leading Landau singularity (that is, when all the internal propagators become on-shell simultaneously) one finds that in the vicinity of the singular point $s = (p_a + p_b)^2$. 


If we denote the singular configuration of \( Q_i \) by \( q_i^{(0)} \) and the singular configuration of the external particles’ momenta by \( p_a^{(0)} \) and \( p_b^{(0)} \) then \( \epsilon_i \) here is defined to be the product of the two constants \( a_i \) and \( b_i \) such that

\[
q_i^{(0)} = a_i p_a^{(0)} + b_i p_b^{(0)}, \quad q_i = a_i p_a + b_i p_b, \quad \epsilon_i = a_i b_i.
\]  

(15)

The constants \( a_i \) and \( b_i \) can be computed easily from the dual diagram being the ratios of the area of triangles in the dual diagram.

We are interested in the simple pole contribution from Eq.(14). First we note that Eq.(14) is readily evaluated upon the change of variables

\[
u = 2 q_D \cdot l, \quad v = 2 q_A \cdot l.
\]  

(16)

The corresponding Jacobian can be evaluated by expressing \( q_D \) and \( q_A \) in terms of the external momenta using Eq.(15)

\[
J = \frac{\partial (l_0, l_1)}{\partial (u, v)} = \frac{1}{8i \tilde{\Delta} (a_D b_A - b_D a_A)}.
\]  

(17)

Here \( \tilde{\Delta} = \frac{1}{2} m_a m_b \sin \theta \) and the Jacobian for this change of variable is inversely proportional to the area of the triangle spanned by \( q_D \) and \( q_A \). We now observe that there are two sources to the simple pole contribution of Eq.(14). One comes from the next order term in the expansion of \( \tilde{\Delta} \) in the Jacobian \( J \),

\[
\frac{1}{\Delta} = \frac{1}{\tilde{\Delta}} (1 + i \cot \theta_0 (\theta - i \theta_0) + \cdots) \quad \text{with} \quad \Delta = \frac{1}{2} m_a m_b \sin \theta_0
\]  

(18)
and the second arises when we expand the integrand in terms of \((s - s_0)\),
\[
\prod_i \frac{1}{(\epsilon_i + 2q_i \cdot l + (s - s_0)i^2)} = \prod_i \frac{1}{(\epsilon_i + 2q_i \cdot l)} \times \{1 - \sum_j \frac{(s - s_0)l^2}{(\epsilon_j + 2q_j \cdot l)} + \ldots\}.
\] (19)

The first of these contributions from the Jacobian is the easiest to evaluate as we have simply the product of the double pole residue of the appropriate box diagram\[11\] multiplied by \(i \cot \theta_0\). Thus we have corresponding to the diagrams of figure 1
\[
\begin{align*}
(a) & : \left(\frac{\beta}{\sqrt{2h}}\right)^4 \frac{1}{(\theta - i\theta_0)} \left(\frac{2\Delta'_a}{\Delta}\right) i \cot \theta_0 \times S. \\
(b) & : \left(\frac{\beta}{\sqrt{2h}}\right)^4 \frac{1}{(\theta - i\theta_0)} \left(\frac{(\Delta_b - \Delta'_a)}{\Delta}\right) i \cot \theta_0 \times S. \\
(c) & : \left(\frac{\beta}{\sqrt{2h}}\right)^4 \frac{1}{(\theta - i\theta_0)} \left(\frac{(\Delta_a - \Delta_b)}{\Delta}\right) i \cot \theta_0 \times S.
\end{align*}
\]

Here \(S\) is the symmetry factor that takes into account the possible distinct diagrams giving the same amplitude.

The contribution coming from expanding the integrand requires a little more work. The integrals are evaluated by closing contours in the lower \((u, v)\) half-planes where, under the condition \(\Delta'_a \geq \Delta_b\) (which avoids the appearance of extra poles), there are poles at \(u = -\epsilon_D - i\epsilon\) and \(v = -\epsilon_A - i\epsilon\). Using the results of \[11\] such as \(\Delta \equiv \Delta_a + \Delta'_a \equiv \Delta_b + \Delta'_b\) and the expressions for \(q_i\) in terms of the triangles and on-shell momenta together with the relevant vertex \(\delta\) and flux \(\pi\) factors we are rewarded with the following nice formulas for the simple pole residues of the box diagrams
\[
\begin{align*}
(a) & : \left(\frac{\beta}{\sqrt{2h}}\right)^4 \frac{i\Delta'_a}{2\Delta^2 \Delta_a \Delta_b} [(p_b \cdot p_b)\Delta_a (\Delta_b - \Delta'_b) + (p_a \cdot p_b) (2\Delta_b \Delta'_b)] \times S. \tag{21} \\
(b) & : \left(\frac{\beta}{\sqrt{2h}}\right)^4 \frac{i\Delta'_a}{2\Delta^2 \Delta_a \Delta_b} [(p_a \cdot p_a)\Delta^2_b \Delta_a + (p_b \cdot p_b)\Delta_a^2 \Delta'_a + (p_a \cdot p_b)\Delta_a \Delta_b (\Delta'_a + \Delta'_b)] \times S. \\
(c) & : \left(\frac{\beta}{\sqrt{2h}}\right)^4 \frac{i\Delta'_a}{2\Delta^2 \Delta_a \Delta_b} [-(p_b \cdot p_b)\Delta^2_b - (p_a \cdot p_b) \Delta_a^2 \Delta'_a + (p_a \cdot p_b) \Delta_a \Delta'_b (\Delta'_a + \Delta'_b)] \times S.
\end{align*}
\]
Again \(S\) is a symmetry factor. In obtaining these formulae we remark that some simplifications may arise. For example, in some cases \(q_i\) and \(q_j\) happen to be the same, so reducing the
the number of integrations to be done. This is the case for the crossed box diagrams (b) and (c) in figure 1, where \( q_A = q_C \) or \( q_B = q_D \), respectively. Finally it is worth observing that the simple pole residue of a crossed box diagram may not vanish even though the corresponding double pole residue vanishes.

IV. Simple pole residue at the double pole position

It is known\(^\text{[11]}\) that the double pole residue of the conjectured S-matrices (7) is in agreement with perturbation theory assuming \( B(\beta) = \frac{\beta^2}{2\pi} + O(\beta^4) \). Using the results of the previous section we will now compare the nonleading simple poles. The following argument shows that, at least to order \( \beta^4 \), the residue of a simple pole of the exact S-matrix at the general double pole position vanishes and so we are left with showing the various contributions from perturbation theory sum to zero.

The conjectured S-matrices for the ATFT are built in terms of the building blocks \( \{x\} \) (8). For \( \theta \) away from a pole and \( B \ll 1 \) these blocks have an expansion \( \{x\} \sim 1 - \frac{\pi B}{2\hbar} \sum_{\alpha,\beta \in \{0,1\}}(-1)^{\alpha+\beta} \cot\left( \frac{-i\theta}{2} + [(-1)^\alpha x + (-1)^\beta] \frac{\pi}{2\hbar} \right) \) while near the pole \( \theta_0 = \frac{\pi}{\hbar}[x \pm 1] \) of \( \{x\} \) we have \( \{x\} \sim [1 \pm \frac{i\pi B}{\hbar}(\theta - i\theta_0)](1 - \frac{\pi B}{2\hbar} \sum_{\alpha,\beta \in \{0,1\}}(-1)^{\alpha+\beta} \cot\left( \frac{-i\theta}{2} + [(-1)^\alpha x + (-1)^\beta] \frac{\pi}{2\hbar} \right) \), where the prime means the singular term of the sum is not to be included. Putting this together means that near a double pole \( i\theta_0 \) the S-matrices (7) have an expansion

\[
S_{ab} = \left( \frac{\pi B}{\hbar} \right)^2 \frac{1}{(\theta - i\theta_0)^2} \times \{c_0 + c_1(\theta - i\theta_0)B + \ldots\}, \quad c_0 = 1 + O(B). \tag{22}
\]

Here \( c_1 \) is a sum of cotangents whose precise form is not important for the present argument. All we must observe is that the residue of the simple pole is proportional to \( B(\beta)^3 \). Using again no more than the tree level result \( B(\beta) = \frac{\beta^2}{2\pi} + O(\beta^4) \) this means that this simple pole vanishes to order \( \beta^4 \).
It now remains to be shown that the order $\beta^4$ contribution to the simple pole residue of the S-matrix vanishes in ordinary perturbation theory. We prove this using the Landau singularity analysis of the subleading singularities already presented, together with the leading singularity analysis of the box and triangle diagrams. There are four kinds of contributions.

(a) Feynman diagrams involving singular box diagram.

(b) Feynman diagrams involving singular triangle diagrams.

(c) Contributions from the expansion of Eq.(10).

(d) Contributions from the expansion of Eq.(11).

We classify the contributing singular box diagrams in figure 1 with participating particles as follows ($k^*$ denotes the antiparticle of $k$).

\[
\begin{array}{cccc}
\text{A} & \text{B} & \text{C} & \text{D} \\
(i) & a - k & a + b - k & b - k & k^* \\
(ii) & a - k & a + b - k & a - k & k^* \\
(iii) & a - k & k^* & b - k & k^* \\
\end{array}
\]

Adding the three contributions from the three box diagrams in Eq.(21) and taking into account the next order term in the Jacobian given by Eq.(20), we get

\[
i(\frac{\beta}{\sqrt{2h}})^4 \frac{1}{(\theta - i\theta_0)} \cot\theta_0 + i(\frac{\beta}{\sqrt{2h}})^4 \frac{1}{(\theta - i\theta_0)} \cot\theta_0 = i(\frac{\beta}{\sqrt{2h}})^4 \frac{1}{(\theta - i\theta_0)} \cot\theta_0 \times 2.
\]  

The possible singular triangle diagrams are enumerated in figure 2. Using the formula for the leading singularity of the triangle diagrams given in [11] together with the relevant vertex factors [1] we easily obtain as the residues of the simple pole

\[
T_1 + T_2 = i(\frac{\beta}{\sqrt{2h}})^4 \frac{1}{\sin\theta_0} \frac{-s_0}{m_am_b},
\]

\[
T_3 + T_4 = i(\frac{\beta}{\sqrt{2h}})^4 \frac{1}{\sin\theta_0} \frac{t_0}{m_am_b}.
\]

9
Invoking the relation between the Mandelstam variables

$$s_0 - t_0 = 4m_am_b\cos\theta_0, \quad (27)$$

we get the sum of the simple pole residues from the singular triangle diagrams

$$T1 + T2 + T3 + T4 = -i\left(\frac{\beta}{\sqrt{2h}}\right)^4 \cot\theta_0 \times 4. \quad (28)$$

Finally the contributions from (c) and (d) are identical, each being given by

$$i\left(\frac{\beta}{\sqrt{2h}}\right)^4 \cot\theta_0 \times 1. \quad (29)$$

Adding the four contributions in Eq.(24,28,29) gives the desired vanishing of the simple pole residue at the general double pole positions,

$$i\left(\frac{\beta}{\sqrt{2h}}\right)^4 \cot\theta_0 \times (2 - 4 + 1 + 1) = 0. \quad (30)$$

V. Conclusions and Discussions

We have in the ATFT a quantum field theory both rich in structure and yet offering the tantalising possibility that it may indeed be solved. The existence of quantum higher spin currents\cite{18} leads to the two-particle factorisation of the theories S-matrices. Together with unitarity, analyticity and crossing, the bootstrap equations of Zamolodchikov have been applied to these theories to produce S-matrices. This bootstrap is still mysterious: we do not know, for example, whether this is a genuine assumption or may be proven within the axioms of field theory. Nonetheless, the bootstrap appears to encode nonperturbative information and the resulting S-matrices may be checked within the context of standard perturbation theory. Thus far they have passed every test applied. This paper has provided a new and nontrivial test. These theories have a complicated Landau singularity structure.
By deriving general expressions for the subleading singularities of the box diagrams we have been able to compare simple pole residues (ordinarily masked behind double poles) from both field theory and the putative S-matrices. For the $A_n$ theories dealt with here for simplicity we find complete agreement. The techniques developed are applicable to the wider class of bosonic and supersymmetric ATFT. The outstanding question remains how the bootstrap encodes the intricate cancellations and structure of the renormalised field theory.

**Acknowledgements**

This research is supported in part by KOSEF and Basic Science Research Institute, Kyung Hee University.
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Figure and Table Captions

Figure 1. Singular box diagrams and their dual diagrams.

Figure 2. Diagrams involving singular triangle subdiagrams.
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Figure 2. Diagrams involving singular triangle subdiagrams.