High Temperature Limit of the $N = 2$ Matrix Model

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Abstract

The high temperature limit of a system of two D-0 branes is investigated. The partition function can be expressed as a power series in $\beta$ (inverse temperature). The leading term in the high temperature expression of the partition function and effective potential is calculated exactly. Physical quantities like the mean square separation can also be exactly determined in the high temperature limit.

1. INTRODUCTION

The study of string theory at finite temperature has received renewed attention recently [2–17]. In a recent paper one of us [2] attempted to elucidate the nature of the Hagedorn transition [1] using the matrix model and found similarities with the deconfinement transition in gauge theories. This was also investigated in a subsequent paper using the AdS/CFT correspondence [3]. It is clear that recent developments in non-perturbative string theory or

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M-theory [18–22] have some bearing on our understanding of the high temperature behavior of strings. Furthermore, study of high temperature behavior of a system is often a useful probe of the system because many calculations simplify at high temperature. We can thus hope to learn something about M-theory from its high temperature behavior. With this motivation, in this paper, we attempt to study the high temperature behavior of a simple but non-trivial system - the system of two D-0-branes. This is essentially the BFSS matrix model [19] with N=2. This is not big enough to describe M-theory. In particular it would not include for instances processes involving pair-production of D-0 brane - anti-D-0 brane. Nevertheless it is already complicated enough. In particular the nature of the threshold and other bound states that have been studied [23–27] are not fully understood. Furthermore we should keep in mind that while the matrix model reproduces string theory at short distances, the fact that it also does so at long distances seems to be entirely due to the super-symmetric non-renormalization theorems. At finite temperature supersymmetry is broken and perhaps we should not expect this. For all these reasons the study of matrix models at high temperature is worthwhile.

A related model of D-instantons, the IKKT matrix model [20], which is 0+0 dimensional has been solved exactly for N=2 [8]. The D-0-brane action that we are interested in, is a quantum mechanical one (i.e. 0+1 dimensional). However after compactifying the Euclideanised time, if one takes the high temperature limit, it reduces to a 0+0 dimensional model. There is thus a hope of solving this model order by order in $\beta$ but to all orders in $g_s$ using the same techniques as [8]. One can then calculate physical quantities such as the mean square separation of the D-0-branes - a measure of the size of the bound states. This is what is attempted in this paper. We obtain the leading behaviour in $\beta$. We can also estimate, the corrections to the leading result. The noteworthy feature being that each term is exact in its dependence on the string coupling constant.

This paper is organized as follows. In section 2 we set up the problem and in section 3 we describe briefly the solution. The last section contains a summary of the result and some conclusions.
2. THE ACTION

The (0 + 1) dimensional BFSS lagrangian is

\[
L = \frac{1}{2gl_s} \text{tr} \left[ \dot{X}^i \dot{X}_i + 2i \theta \dot{\theta} - \frac{1}{2l_s^4} [X^\mu, X^\nu]^2 - \frac{2}{l_s^2} \dot{\theta} \gamma_\mu [\theta, X^\mu] - \frac{i}{l_s} [X^0, X^i] \dot{X}_i \right]
\] (2.0.1)

where \( i = 1, \ldots, 9; \mu = 0, \ldots, 9 \); \( X^\mu \) and \( \theta \) are \( N \times N \) hermitian matrices. \( X^\mu \) is a 10 dimensional vector and \( \theta \) is a 16 component Majorana-Weyl spinor in 10 dimensional super-Yang Mills theory. For \( N = 2 \), we can re-write \( X^\mu \) and \( \theta \) as

\[
\begin{align*}
X^\mu &= \sum_{a=1}^{3} \frac{1}{2} \sigma^a X^\mu_a, \\
\theta &= \sum_{a=1}^{3} \frac{1}{2} \sigma^a \theta_a
\end{align*}
\] (2.0.2)

where, \( \sigma^a \) are the hermitian Pauli matrices and \( X^\mu_a, \theta_a \) are real fields in terms of which, the Lagrangian (2.0.1) is

\[
L = \frac{1}{4gl_s} \left\{ \sum_{a,b=1}^{3} X^i_a \dot{X}^i_a + 2i \sum_{a=1}^{3} \theta_a \dot{\theta}_a - \frac{2i}{l_s} \sum_{a,b,c=1}^{3} \theta_a \gamma_0 \gamma_\mu \theta_b X^\mu_c e^{abc} \\
+ \frac{e^{abc}}{4l_s^2} X^0_a X^i_b \dot{X}^i_c + \frac{1}{2l_s^4} \sum_{a,b=1}^{3} X^\mu_a X^\nu_b X^\mu_a X^\nu_b - \frac{1}{2l_s^4} \sum_{a,b=1}^{3} X^\mu_a X^\nu_b X^\mu_b X^\nu_a \right\}
\] (2.0.3)

If we Euclideanize and compactify time on a circle of circumference \( \beta \) the action becomes

\[
S = i \int_0^\beta L dt
\] (2.0.4)

and the fields satisfy the boundary conditions

\[
X^\mu(0) = X^\mu(\beta), \quad \theta(0) = -\theta(\beta)
\]

Considering these boundary condition, we can expand the fields \( X^\mu, \theta \) in modes as

\[
\begin{align*}
X^\mu_a(t) &= \sum_{n=-\infty}^{\infty} X^\mu_{a,n} e^{\frac{2\pi in}{\beta} t}, \\
\theta_a(t) &= \sum_{r=-\infty}^{\infty} \theta_{a,r} e^{\frac{2\pi i r}{\beta} t}
\end{align*}
\] (2.0.5)

where, \( n \) is an integer and \( r \) is a half-integer.

So, in terms of these modes, the action reduces to
\[ S = \frac{i\beta}{4gt} \left\{ -\sum_{n=-\infty; n\neq 0}^{\infty} \frac{4\pi^2 n^2}{\beta^2} \frac{X_{a,n}X_{a,-n}^i}{4} + \sum_r \frac{4\pi r}{\beta} \theta_{a,r} \theta_{a,-r} \right. \\
- \frac{2i}{l_s^2} \sum_{r,s,l=-\infty; s+r+l=\theta}^{\infty} \theta_{a,r} \gamma_0 \gamma_\mu \theta_{b,s} X_{c,l}^\mu \epsilon^{abc} + \frac{n\pi \epsilon^{abc}}{2l_s^2 \beta} \sum_{n+l+p=0}^{\infty} X_{a,l}^0 X_{b,m}^i X_{c,n}^i \\
+ \frac{1}{2l_s^4} \sum_{n,m,l,p=-\infty; n+m+l+p=0}^{\infty} X_{a,n}^\mu X_{b,m}^\nu X_{c,n}^\mu X_{b,p}^\nu \left\} \right. \\
\] (2.0.6)

\[ n, m, l, p \text{ are integers and } r, s \text{ are half-integers.} \]

Here, \( X_{a,n}^\mu, \beta \) have the dimension of length (L) and \( \theta \) has the dimension \( \sqrt{L} \). We scale \( X_{a,n}^\mu, \beta \) with a factor of \( \frac{1}{l_s} \), and \( \theta \) with a factor \( l_s^{-\frac{1}{2}} \) to make them dimensionless, which is equivalent to replacing all the \( l_s \) in the action by 1.

The first and the second terms in the action give the masses of the modes of the vector and the spinor fields respectively, namely, \( \frac{2\pi n}{\beta} \) and \( \frac{2\pi s}{\beta} \).

The partition function with this action is

\[ Z = \int e^{-iS} \] (2.0.7)

In the next section we will try to calculate this.

### 3. THE PARTITION FUNCTION AND THE EFFECTIVE POTENTIAL.

#### 3.1. Pfaffian

The fermionic terms in the action are of the form \( (\theta_{a,r} \gamma_0 \gamma_\mu \theta_{b,s} X_{c,n}^\mu)_{n+r+s=0} \) and \( \theta_{a,s} \theta_{b,-r} \). In \( \beta \to 0 \) limit, the first term in the action (2.0.6) is dominant and \( (X_{a,n}^\mu)_{n\neq 0} \) is of the order \( \sqrt{\beta} \). So, in large temperature limit the first term contributes to the partition function in the leading order of \( \beta \) only for \( n = 0 \) \( (\theta_{a,r} \gamma_0 \gamma_\mu \theta_{b,-r} X_{c,0}^\mu) \). With these terms the action takes the form

\[ S_f = \frac{i\beta}{4gt} \sum_{r=0}^{\infty} \left\{ \frac{4\pi r}{\beta} \theta_{a,r} \theta_{a,-r} - \frac{4\pi r}{\beta} \theta_{a,-r} \theta_{a,r} - 2i\theta_{a,r} \gamma_0 \gamma_\mu \theta_{b,-r} X_{c,0}^\mu \epsilon^{abc} - 2i\theta_{a,-r} \gamma_0 \gamma_\mu \theta_{b,r} X_{c,0}^\mu \epsilon^{abc} \right\} \]

Now, we will try to find out the pfaffian
\[ Z_f = \int \prod_{a=1}^{3} d^{16} \theta_{a,r}^* d^{16} \theta_{a,-r} e^{-iS_f} \]  

(3.1.1)

for this action

\[ Z_f = \prod_{r=0}^{\infty} \int \prod_{a=1}^{3} d^{16} \theta_{a,r} d^{16} \theta_{a,-r} \exp \left[ -\frac{\beta}{2g} \left\{ \theta_{a,r} \left( \frac{2\pi r}{\beta} \delta^{ab} - i X_{c,0}^\mu e^{abc} \gamma_0 \gamma_\mu \right) \theta_{a,-r} \right. \\
- \theta_{a,-r} \left( \frac{2\pi r}{\beta} \delta^{ab} + i X_{c,0}^\mu e^{abc} \gamma_0 \gamma_\mu \right) \theta_{b,r} \left. \right\} \right] \]  

(3.1.2)

We rotate \( X_{c,0}^\mu \) by a Lorentz transformation so that only \( X_{c,0}^0, X_{c,0}^1 \) and \( X_{c,0}^2 \) are nonzero. We take the representation of the Gamma matrices, in which

\[ \gamma_0 = i \sigma_2 \otimes 1_8, \quad \gamma_1 = \sigma_3 \otimes 1_8, \quad \gamma_2 = -\sigma_1 \otimes 1_8 \]  

(3.1.3)

With this choice of representation we can write

\[ S_f = \frac{i \beta}{g} \sum_{r=\infty}^{\infty} \left\{ \theta_{a,r} \left( \frac{2\pi r}{\beta} \delta^{ab} - i e^{abc} \left( X_{c,0}^0 1_2 + X_{c,0}^1 \sigma_1 + X_{c,0}^2 \sigma_3 \right) \right) \otimes 1_8 \theta_{b,-r} \right\} \]  

(3.1.4)

So, the pfaffian will be

\[ Z_f = \left[ \int \prod_{a,b=1}^{3} d\theta_{a,r}^* d\theta_{b,-r} \exp \left[ \frac{\beta}{g} \left\{ \theta_{a,r} \left( \frac{2\pi r}{\beta} \delta^{ab} - i e^{abc} \left( X_{c,0}^0 1_2 + X_{c,0}^1 \sigma_1 + X_{c,0}^2 \sigma_3 \right) \right) \theta_{b,-r} \right\} \right] \right]^8 \]

where \( \theta_{a,r} \) and \( \theta_{b,-r} \) are the spinors in three dimension, and has two components.

\[ Z_f = \frac{2^{16} \pi^{48}}{g^{34}} \prod_{r>0} \left( 16r^6 + \frac{8r^4 \beta^2}{\pi^2} \left( X_{a,0}^\mu \right)^2 + \frac{4 \pi^4}{\beta^4 r^2} \left( X_{a,0}^\mu \right)^2 - 4 \left( X_{a,0}^0 X_{b,0}^0 X_{a,0}^1 X_{b,0}^1 \right) \right) \]

\[ -4 \left( X_{a,0}^0 X_{b,0}^0 X_{a,0}^2 X_{b,0}^2 \right) + \frac{\beta^6}{\pi^6} \left( \epsilon_{abc} \epsilon^{\mu\nu\gamma} \left( X_{a,0}^\mu X_{b,0}^\nu X_{c,0}^\gamma \right)^2 \right) \]  

(3.1.5)

We can see that the above expression has \( SO(3) \) symmetry in spinor indices, and \( SO(2,1) \) symmetry in the vector indices. The \( 16r^6 \) term gives the free fermionic contribution. Note that it is temperature independent as the Hamiltonian is identically zero for free fermions in \( 0+1 \) dimensions.

### 3.2. Free Bosonic sector

After doing the fermionic integral, the partition function is
\[ Z = \prod_{a=1}^{3} \int d^{10} X_{a,n}^{\mu} \left[ 2^{16/24} \frac{\pi^{24}}{g^{24}} \prod_{r} \left( 4r^{3} + \frac{\beta^{3}}{\pi^{3}} \epsilon_{abc} e^{\mu \nu \gamma} X_{a,n}^{\mu} X_{b,n}^{\nu} X_{c,n}^{\gamma} + \frac{r^{2}}{\pi^{2}} X_{a,n}^{\mu} X_{a,n}^{\mu} \right)^{16} \right] \]

\[
\exp \left[ -\frac{\beta}{4g} \left\{ - \sum_{n=-\infty; n \neq 0}^{\infty} \frac{4\pi^{2}n^{2}}{\beta^{2}} X_{a,n}^{i} X_{a,-n}^{i} - \frac{n\pi \epsilon_{abc}}{2\beta} \sum_{n=l+p=0} X_{a,l}^{0} X_{b,m}^{i} X_{c,n}^{i} \right\} \right]
\]

\[ + \frac{1}{2} \sum_{n,m,l,p=-\infty, n+m+l+p=0} \left( X_{a,n}^{\mu} X_{b,m}^{\nu} X_{a,l}^{\mu} X_{b,p}^{\nu} - \frac{1}{2} \sum_{n,m,l,p=-\infty, n+m+l+p=0} X_{a,n}^{\mu} X_{b,m}^{\nu} X_{a,l}^{\mu} X_{b,p}^{\nu} \right) \right] \] (3.2.1)

In infinite temperature limit \( i.e. \beta \to 0 \), the first term dominates. To see the comparative \( \beta \) dependence of the other terms;

1) we set \((X_{a,n}^{\mu})_{n \neq 0} \to \sqrt{\beta} X_{a,n}^{\mu} \)

2) keep the terms contributing to the leading order of the partition function in \( \beta \to 0 \) limit (this is justified in Appendix A).

3) transform back \( \sqrt{\beta} X_{a,n}^{\mu} \to (X_{a,n}^{\mu})_{n \neq 0} \)

and we can write the partition function up to a numerical factor

\[ Z_{boson} = \frac{1}{g^{24}} \int d^{D} X_{a,n}^{i} d^{D} X_{a,-n}^{i} \exp \left[ \frac{\beta}{4g} \left\{ - \sum_{n=-\infty; n \neq 0}^{\infty} \frac{4\pi^{2}n^{2}}{\beta^{2}} (X_{a,n}^{i} X_{a,-n}^{i}) \right\} \right] \]

\[ \int d^{D} X_{0} \prod_{s} d^{D} \theta_{s} \exp \left[ \frac{\beta}{4g} \left\{ \frac{1}{2} (X_{a,0}^{i} X_{a,0}^{i}) (X_{b,0}^{i} X_{b,0}^{i}) - \frac{1}{2} (X_{a,0}^{i} X_{b,0}^{i}) (X_{b,0}^{i} X_{a,0}^{i}) \right\} \right] \] (3.2.2)

Thus \( Z_{boson} = e^{-iS} = Z_{free} Z_{0} \) where the first part of the partition function \( Z_{free} \) is just the free bosonic particle partition function (per unit volume), which is

\[ Z_{free} = \prod_{n} \prod_{a=1}^{3} \prod_{i=1}^{9} \int d^{D} X_{a,n}^{i} d^{D} X_{a,-n}^{i} \exp \left[ \frac{\beta}{4g} \left\{ - \sum_{n=-\infty; n \neq 0}^{\infty} \frac{4\pi^{2}n^{2}}{\beta^{2}} (X_{a,n}^{i} X_{a,-n}^{i}) \right\} \right] = \prod_{n \neq 0} \left( \frac{\sqrt{g} \pi}{\pi n} \right)^{54} \]

Now, using \( \sum_{n=1}^{\infty} n^{-s} = \zeta(s) \) and \( \frac{d}{ds} \zeta(s) = -\sum_{n=1}^{\infty} n^{-s} \log n \) and \( \zeta(0) = 1, \zeta'(0) = -\frac{1}{2} \log(2\pi) \)

(where \( \zeta(s) \) is the Riemann Zeta function), we get

\[ Z_{free} = \left( \frac{1}{\beta g} \right)^{27} \] (3.2.3)

Note that the free fermionic contribution was discussed in the previous section. \( Z_{0} \) is calculated below.
3.3. Leading Interaction Term.

Now we try to calculate the effect of the interactions. As argued earlier the leading $\beta$ dependence is given by the zero modes, so in the first approximation we drop the terms in the action involving the higher modes. Thus we get,

$$ S_0 = \frac{i\beta}{8g} \left\{ X_{\mu,0} X_{\nu,0} X_{\mu,0} X_{\nu,0} - X_{\mu,0} X_{\nu,0} X_{\mu,0} X_{\nu,0} \right\} $$

$$ = \frac{i\beta}{8g} \left\{ (X_{1,0})^2 (X_{2,0})^2 + (X_{2,0})^2 (X_{3,0})^2 + (X_{3,0})^2 (X_{1,0})^2 \right. $$

$$ \left. - (X_{1,0} X_{2,0})^2 - (X_{2,0} X_{3,0})^2 - (X_{3,0} X_{1,0})^2 \right\} $$

(3.3.1)

We would like to first calculate the leading order contribution to $Z$ that is

$$ Z_0 = \int dX_{\mu,0} e^{-iS_0} $$

(3.3.2)

Now, consider the parametrisation

$$ X_{1,0} = (x_1, \vec{r}_1), \quad X_{2,0} = (x_2, \vec{r}_2), \quad X_{3,0} = (l, 0) $$

(3.3.3)

If we had considered the original action with all the modes the action would not be Lorentz invariant, for example the terms $X_{a,l} X_{b,m} X_{c,n}$ and $X_{a,n} X_{c,-n}$ in the original action are not Lorentz invariant. However, expression (3.3.1) has Lorentz invariance, and we are justified in using this parametrisation in order to evaluate (3.3.2).

Under this parametrisation, the action (3.3.1) takes the form

$$ S_0 = \frac{i\beta}{8g} \left\{ r_1^2 r_2^2 \sin^2 \alpha + x_1^2 r_2^2 + r_1^2 x_2^2 + r_2^2 l^2 + l^2 r_1^2 - 2x_1 x_2 r_1 r_2 \cos \alpha \right\} $$

(3.3.4)

The partition function is

$$ Z_0 = \int d^{10} X_{1,0} d^{10} X_{2,0} d^{10} X_{3,0} e^{-iS_0} $$

(3.3.5)

At this stage, we can find the temperature dependence of the partition function and the mean square separation of two D-0 branes from simple scaling argument. As we have seen the leading order \textit{i.e.} the zero mode contribution of the partition function comes from the
term $\frac{1}{2g^2}[X^\mu, X^\nu]^2$ in the Lagrangian \((3.3.4)\). And the above parametrisation we have used here is also valid for $SU(N)$ matrix model, only the functional form of $\frac{1}{2g^2}[X^\mu, X^\nu]^2$ in terms of this parametrisation will be different for different $N$, but in each case the function will be homogeneous in $l, r_i, x_i$ where $0 < i < N^2 - 2$ and of order 4. So, in each case we need to scale these variables by $\beta^{-\frac{1}{2}} g^{\frac{3}{2}}$ to scale out the $\beta$ from the exponent. And the temperature dependence of $\langle l^2 \rangle$ will be $\beta^{-\frac{1}{2}} g^{\frac{3}{2}}$ in the leading order. Under the scaling above the measure in $Z_0$ will pick up a $\beta^{-\frac{1}{2}} g^{\frac{3}{2}}$ factor for $SU(2)$, which comes from $(3 \times 10) X^\mu_{a,n}$. In general for $SU(N)$ in $D$ dimension there will be $D(N^2 - 1) X^\mu_{a,n}$ in the measure. So, the partition function $Z_0$ has temperature dependence $\beta^{-\frac{D(N^2-1)}{4}} g^{\frac{D(N^2-1)}{4}}$. And $Z_{\text{free}}$ will be proportional to $(\beta g)^{(D-1)(N^2-1)}$.

Now we evaluate the partition function for this action.

$$Z_0 = \int d\alpha \Omega(l^9) \int dx_1 dx_2 \int dr_1 dr_2 d\Omega_1(8) d\Omega_2(7) \alpha r_1^8 r_2^7 \sin^8 \alpha e^{-\beta S}$$

Using $f d\Omega(1) = 2^{n-1} \pi$, and after integration over $x_1$, $x_2$ and $r_2$ the partition function is

$$Z_0 = \frac{2^{37} g^5 \pi^4 \Gamma(4)}{\beta^5} \int_{-\infty}^{\infty} dl^9 \int_{0}^{\infty} dr_1 r_1^7 \exp \left[ -\frac{\beta}{8g} \left\{ l^2 r_1^2 \right\} \right] \int_{0}^{\pi} d\alpha \sin^6 \alpha \left\{ (r_1^2 + l^2) \sin^2 \alpha + l^2 \cos^2 \alpha \right\}^{-4} \tag{3.3.7}$$

Integrating over $\alpha$ \cite{28} the partition function can be rewritten as,

$$Z_0 = \frac{2^{34} 15 g^5 \pi^5}{\beta^5} \int_{-\infty}^{\infty} dl^9 \int_{0}^{\infty} \frac{dr_1 r_1^7}{(r_1^2 + l^2)^{7/2}} \exp \left[ -\left( \frac{\beta l^2}{8g} \right) r_1^2 \right] \tag{3.3.8}$$

If we scale $r_1$ and $l$ by a factor $\beta^{-\frac{1}{2}} g^{\frac{3}{2}}$, the integral reduces to

$$Z_0 = \frac{2^{34} 15 g^{12} \pi^5}{\beta^{12}} \int_{-\infty}^{\infty} dl^9 \int_{0}^{\infty} \frac{dr_1 r_1^7}{(r_1^2 + l^2)^{7/2}} \exp \left[ -\left( \frac{l^2}{8} \right) r_1^2 \right] \tag{3.3.9}$$

The integral over $r_1$ can be done to give

$$Z_0 = \frac{2^{34} 15 g^{12} \pi^5}{\beta^{12}} \int_{-\infty}^{\infty} dl^9 \exp \left( l^4/8 \right) \left\{ a^{-1} \Gamma \left( \frac{1}{2}, l^4/8 \right) - 3l^2 a \Gamma \left( \frac{1}{2}, l^4/8 \right) \right\} \tag{3.3.10}$$

where $\Gamma(\alpha, x)$ is the incomplete Gamma function.
In large \( l \) regime using the asymptotic expression for the incomplete Gamma function \[28\] we can write this expression as

\[
Z_0 = \frac{2^{3415} \frac{g^{15}}{\beta^{12}} \pi^5}{\beta^{12}} \int_{-\infty}^{\infty} dl l^8 e^{-\left(\frac{l^4}{8}\right)} \left\{ \Gamma\left(\frac{1}{2} + m\right) - 3 \frac{\Gamma\left(\frac{3}{2} + m\right)}{\Gamma\left(\frac{3}{2}\right)} + 3 \frac{\Gamma\left(\frac{5}{2} + m\right)}{\Gamma\left(\frac{5}{2}\right)} - \frac{\Gamma\left(\frac{7}{2} + m\right)}{\Gamma\left(\frac{7}{2}\right)} \right\} + O\left(\left(|l^4|/8\right)^{-M}\right)
\]

In large \( l \) approximation this boils down to

\[
Z_0 = \frac{2^{3415} \frac{g^{15}}{\beta^{12}} \pi^5}{\beta^{12}} \int_{-\infty}^{\infty} dl l^8 \left(24576l^{-15} - 2752512l^{-19} + ....\right) \tag{3.3.11}
\]

and in the small \( l \) regime the above partition function becomes \[28\]

\[
Z_0 = \frac{2^{3415} \frac{g^{15}}{\beta^{12}} \pi^5}{\beta^{12}} \int_{-\infty}^{\infty} dl l^8 \sqrt{8l} l^{-1} \exp\left(\frac{l^4}{8}\right) \left\{ \sqrt{\pi} \left\{ 1 - 24l^4 + 256l^8 + \frac{5120}{3} l^{12} \right\} \right.
\]

\[
+ \left\{ 12 \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{8l} (2+2n)}{n!} \left( \frac{8}{(4n^2 - 8n - 5)(4n^2 - 8n + 3)} \right) \right\} \right\} \tag{3.3.12}
\]

and \( l \to 0 \) limit gives

\[
Z_0 = \frac{2^{3415} \frac{g^{15}}{\beta^{12}} \pi^5}{\beta^{12}} \int_{-\infty}^{\infty} dl l^8 \left[ \sqrt{8\pi} l^{-1} - \frac{256}{5} l + \left( \sqrt{\frac{\pi}{3}} \frac{256}{3} l^3 \right) l^3 + .... \right] \tag{3.3.13}
\]

The integrand \( l^8 \int_{0}^{\infty} \frac{dr r^7}{(r^2 + l^2)^{1/2}} \exp\left[-\left(\frac{l^2}{8}\right) r^2\right] \) has \( l^{-7} \) dependence for large \( l \) and \( l^7 \) dependence for small \( l \). Hence, it converges for both large \( l \) and small \( l \) and the integral is non-singular and independent of \( \beta \). So, \( Z_0 \) has a temperature dependence of \( T^{12/5} \).

### 3.4. Non-leading Interaction Term.

In the previous section we have seen that the leading order fermionic contribution comes from the free fermionic terms. Here we will try to estimate the \( \beta \) dependence of the non-leading fermionic contributions.

In terms of the parametrisation in eqns. \([3.3.3]\),

\[
Z_f = \frac{2^{16} \pi^{24}}{g^{24}} \prod_r \left(4r^3 + \frac{2\beta^3}{\pi^3} r_1 r_2 l \sin \alpha + \frac{r_1^2 + r_2^2 + (x_1)^2 + (x_2)^2 + l^2}{\pi^3} \right)^{16} \tag{3.4.1}
\]

which, in \( \beta \to 0 \) can be written as
\[ Z_f = g^{-24} \prod_r \left( r^3 O(\beta^0 g^0) + O(\beta^2 g^2) + r O(\beta^2 g^2) + \ldots \ldots \right) \]
\[ = g^{-24} \left( O(1) + O(\beta^2 g^2) + O(\beta^2 g^2) + r O(\beta^2 g^2) + \ldots \ldots \right) \quad (3.4.2) \]

where the first term is the leading order fermionic partition function we have discussed in subsection (3.2).

So we can see that in the partition function at high temperature the contribution of the zero modes (bosonic) is dominant. We have earlier argued that the higher modes of the bosonic fields will also contribute in non-leading terms.

### 3.5. Mean-square Separation of the D-0 branes.

As we are working in Euclidean metric and since for zero mode calculation we have Lorentz symmetry, we can identify \( l \) as one of the spatial components and hence as the separation between two D-0 branes.

Now, we try to see the temperature dependence of the expectation value of \( l^{2n} \)

\[ \langle l^{2n} \rangle = \frac{\int e^{-iS} l^{2n}}{Z} \quad (3.5.1) \]

i.e.

\[ \langle l^{2n} \rangle = (\beta^{-\frac{1}{8}} g^\frac{1}{2})^{2n} \frac{\int_{-\infty}^{\infty} dl l^{2n} \int_{0}^{\infty} \frac{dr_1 r_1^7}{(r_1^2 + l^2)^7/2} \exp \left[ -\left( \frac{l^2}{8} \right) r_1^2 \right]}{\int_{-\infty}^{\infty} dl l^{2n} \int_{0}^{\infty} \frac{dr_1 r_1^7}{(r_1^2 + l^2)^7/2} \exp \left[ -\left( \frac{l^2}{8} \right) r_1^2 \right]} \quad (3.5.2) \]

As \( l \) is the separation of two D-0 branes, we get the mean square separation of two D-0 branes from this, by putting \( n = 1 \) and doing the integral numerically, and restoring \( l_s \), we get

\[ \left\langle \left( \frac{l}{l_s} \right)^2 \right\rangle = 6.385 \left( \frac{\beta}{g l_s} \right)^{-\frac{1}{2}} \quad (3.5.3) \]

If we assume high temperature expression has a finite radius of convergence, we can conclude that the mean square separation is finite for finite temperature. This implies that there is a confining potential that binds the D-0 branes. As argued earlier the scaling argument that gives the \( \beta \) and \( g \) dependence in (3.5.3) is valid for all \( N \). So we can conclude that \( \langle l^2 \rangle \approx \sqrt{\frac{\beta}{g}} \) for all \( N \).
3.6. Effective Potential.

For high temperature we have evaluated the partition function both for large and small $l$ (eqn. 3.3.11, 3.3.13). Up to leading order the effective potential between two D-0 branes is proportional to $-\log l$ and $\log l$ for small and large $l$. We can see that the potential increases at both $l$ ends, though we can not clearly see the nature of the potential in the intermediate region but we can conclude that the potential is a confining potential and binds the D-0 branes.

4. CONCLUSION

In this paper we have attempted to solve the $N = 2$ matrix model in the high temperature limit. The leading nontrivial term of the partition function has been calculated exactly (eqn. 3.3.10). The non-leading terms can also be systematically calculated although we haven’t attempted to work them out in this paper. From a scaling argument we have also determined the $\beta$ and $g$ dependence of the leading term for any $N$. This complements the work of [6], where the one loop partition function was calculated with the entire $\beta$ dependence. We find that $\langle l^2 \rangle \propto \sqrt{g} \beta$ (eqn. 3.5.3) (true for any $N$), the finiteness of which shows that there must be a potential between D-0 branes that binds them. In [6] also a logarithmic and attractive potential were found. The present calculation being exact in $g_s$ is valid at all distances. Thus unlike in [4,6], the (finite temperature) logarithmic potential found here is attractive at long distances and repulsive at short distances so it has a minimum at non-zero separation. Similar issue for low temperature has been discussed in [27]. In [2] it was found that at high temperatures, the configuration with all the D-0-branes clustered at the origin \textit{i.e.} with the zero separation, had lower free energy than the one where they were spread out. However, that was a large $N$ calculation and also restricted to one loop. It is therefore possible that more exact calculation will resolve this issue.

As mentioned in the introduction, describing completely the dynamics of two D-0 branes
in M-theory would require the infinite $N$ model. Whether some high temperature expansion of that model within the $\frac{1}{N}$ approximation scheme can be attempted is an open question.

APPENDIX A: COMPARITIVE $\beta$ DEPENDENCE OF THE TERMS IN ACTION

We are interested in investigating the comparitive $\beta$ dependence of the terms in the action in eqn. (2.0.6). For convenience, in this part we will suppress the isospin and vector indices. The action takes the form

$$S = \frac{1}{g} \left\{ - \sum_{n=\infty}^{\infty} \frac{a_n}{\beta^2} X_n X_{-n} + \sum_{l+m+n=0} \frac{f_n}{\beta} X_l X_m X_n - c \sum_{n,m,l,p=\infty}^{\infty} X_n X_m X_l X_p \right\} \quad (A.1)$$

where $a_n, b_n, c, d$ and $f_n$ are constants and are given by $a_n = \pi^2 n^2, c = \frac{1}{8},$ and $f_n = \frac{n\pi}{4}$

Now, when we expand the sum over $n, m, l$ and $p$ in last two terms, we will get terms with all of these indices being 0, with two of the indices being 0 and with one of them being 0, so the action can be written in the form (taking one of each type, as the terms of the same type have same $\beta$ dependence).

$$S = -\frac{1}{\beta g} \sum_{n=\infty}^{\infty} a_n X_n^2 - \frac{c\beta}{g} X_0^4 - \frac{c\beta}{g} \sum_{n\neq 0} X_0^2 X_n^2 + \frac{1}{g} \sum_{n\neq 0} f_n X_0 X_{-n} X_n$$

$$- c\beta \sum_{m,l,p\neq 0, m+l+p=0} X_0 X_l X_l X_p - \frac{c\beta}{g} \sum_{n,m,l,p\neq 0, n+m+l+p=0} X_n X_m X_l X_p$$

Now, we set

$$(X_n)_{n\neq 0} = \sqrt{g\beta} X'_n$$

Under this transformation, the action reduces to

$$S = \left\{ -\frac{c\beta}{g} X_0^4 \right\} + \left\{ -a_n \sum_{n\neq 0} X_n' X_n' - c\beta^2 \sum_{n\neq 0} X_n'^2 X_0^2 + \sum_{n\neq 0} f_n X_0 X_{-n} X_n$$

$$- c\beta \sum_{n+m+l=0} X'_n X'_m X'_l X_0 - c\beta^3 \sum_{n+m+l+p=0} X'_n X'_m X'_l X'_p \right\} \quad (A.2)$$

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Again, we can neglect $\beta^2 X'_n X'_m X'_l X_0$ and $\beta^3 X'_n X'_m X'_p X'_r$ terms also, as they have higher order $\beta$ dependence ($\frac{5}{2}$ and 3 respectively). To justify this step, note that since the potential is always greater than zero, it is equivalent to an integral of the form

$$Z = \int dx \exp \left( -ax^2 + \frac{\beta^2 d}{\sqrt{g}} x - \beta^2 cx^3 - cg\beta^3 x^4 \right) = \int dx e^{-ax^2} \exp \left( \frac{\beta^2 d}{\sqrt{g}} x - \beta^2 cx^3 - cg\beta^3 x^4 \right)$$

Being a convergent integral, as $\beta \to 0$, we can use Taylor expansion and write as

$$Z = \int e^{-ax^2} \left( 1 - \frac{\beta^2 d}{\sqrt{g}} x + \beta^2 cx^3 - \frac{\beta^2 d}{4\sqrt{g}} x + \ldots \right) dx \quad (A.3)$$

Neglecting the higher order $\beta$ terms, we can write the partition function for action (A.2),

$$Z = \int dX_0 \exp \left\{ -\frac{c\beta}{g} X_0^4 \right\} \prod_{n \neq 0} \int dX_n \exp \left\{ (-a_n + \beta f_n X_0 - c\beta^2 X_0^2) \frac{1}{g\beta} X_n^2 \right\}$$

In order to justify dropping the $X_0^2 X_n^2$ and $X_0 X_n^2$ terms, we can proceed as follows. After doing the $X_n$ integral the partition function reduces to

$$Z = \sqrt{\pi g \beta} \prod_{n \neq 0} \int dX_0 \exp \left\{ -\frac{c\beta}{g} X_0^4 \right\} \frac{1}{(a_n - f_n \beta X_0 + c\beta^2 X_0^2)^{\frac{1}{2}}} \quad (A.4)$$

As here $4(a_n - 1)c - f_n^2$ is positive, $(a_n - 1) - f_n X_0 + c X_0^2$ is also positive, i.e. $[a_n - f_n X_0 + c X_0^2] > 1$. So

$$\exp \left\{ -\frac{c\beta}{g} X_0^4 \right\} \frac{1}{(a_n - f_n \beta X_0 + c\beta^2 X_0^2)^{\frac{1}{2}}} < \exp \left\{ -\frac{c\beta}{g} X_0^4 \right\} \quad (A.5)$$

and the integral is finite. So in $\beta \to 0$ limit the leading order term can be given by $\int dX_0 a_n^{-\frac{1}{2}} \exp \left\{ -\frac{c\beta}{g} X_0^4 \right\}$ and the corrections will vanish at the positive power of $\beta$. So, in $\beta \to 0$ limit we can ignore those, which is equivalent to ignoring the $f_n \beta X_0 X_n X_m$ and $c\beta^2 X_0^2 X_n^2$ terms in action so it can be written as

$$S = \frac{a}{\beta} \sum_{n=-\infty; n \neq 0}^{\infty} X_n^2 - c\beta X_0^4 \quad (A.6)$$

which is also eqn.(3.3.1).

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