Limiting absorption principle and radiation condition for repulsive Hamiltonians

Kyohei Itakura

Abstract

For spherically symmetric repulsive Hamiltonians we prove the Besov bound, the radiation condition bounds and the limiting absorption principle. The Sommerfeld uniqueness result also follows as a corollary of these. In particular, the Hamiltonians considered in this paper cover the case of inverted harmonic oscillator. In the proofs of our theorems, we mainly use a commutator argument invented recently by Ito and Skibsted. This argument is simple and elementary, and does not employ energy cut-offs or the microlocal analysis.

1 Introduction

For any fixed $\epsilon \in (0, 2]$ we consider the repulsive Schrödinger operator

$$H = -\frac{1}{2} \Delta - |x|^\epsilon + q; \quad -\Delta = p_j \delta^{jk} p_k, \quad p_j = -i \partial_{x_j},$$

on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$. Here $q$ is a real-valued function that may grow slightly slower than $|x|^\epsilon$, $\delta^{jk}$ is the Kronecker delta, and we use the Einstein summation convention. Throughout the paper we will use this convention. By the Faris-Lavine theorem (see [RS, II]) the operator $H$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, and we denote the self-adjoint extension by the same letter. For the case $\epsilon = 2$ the Hamiltonian $H$ is called the inverted harmonic oscillator.

In this paper we study properties of the resolvent

$$R(z) = (H - z)^{-1}.$$

We prove the Besov boundedness, the radiation condition bounds, the limiting absorption principle and the Sommerfeld uniqueness result. The Besov boundedness yields the absence of singular continuous spectrum of $H$. In this paper the limiting absorption principle is derived from the Besov boundedness and the radiation condition bounds. The Sommerfeld uniqueness result characterizes the limiting resolvents by the Helmholtz equation and the radiation condition. By using the function spaces

*Graduate School of Science, Kobe University, Hyogo, Japan
in (1.3) below, which are somewhat different from the usual one, we can deal with also the case of inverted harmonic oscillator.

To prove the above results we apply a new commutator argument with some weight inside invented recently by IS. A feature of this argument is a choice of the conjugate operator $A$. As with I, we choose $A$ to be a generator of some radial flow, not of dilations or translations.

Spectral theory for the repulsive Hamiltonians was also studied by BCHM. However, to use the Mourre theory they introduced a new conjugate operator by using the pseudo-differential operator. We do not use the Mourre theory or the pseudo-differential operator. Due to this, our argument is simpler than theirs.

1.1 Basic setting

Choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 
1 & \text{for } t \leq 1, \\
0 & \text{for } t \geq 2,
\end{cases}$$

$\chi' \leq 0$, (1.1)

and set $r \in C^\infty(\mathbb{R}^d)$ and the associated differential operator $\nabla^r$ as

$$r(x) = \chi(|x|) + |x| (1 - \chi(|x|)),$$

$$\nabla^r = (\partial_j r) \delta^j k \nabla_k.$$

Moreover we introduce the function $f \in C^\infty(\mathbb{R}^d)$ and the associated differential operator $\nabla^f$ as

$$f(r) = \begin{cases} 
(r^{1-\epsilon/2} - 1)/(1-\epsilon/2) + 1 & \text{for } 0 < \epsilon < 2, \\
\log r + 1 & \text{for } \epsilon = 2,
\end{cases}$$

(1.2)

$$\nabla^f = (\partial_j f) \delta^j k \nabla_k.$$

We note that the function $f$ is continuous with regard to $\epsilon$ and the following properties hold:

$$r \geq 1, \quad f \geq 1, \quad \nabla^f = r^{-\epsilon/2} \nabla^r.$$

In this paper we use the function $f$ frequently. This is closely related to the classical orbit. In particular, it plays an important role for the case $\epsilon = 2$. We are going to see the details of this in Subsection 1.3.

**Condition 1.1.** The perturbation $q$ is a real-valued function. Moreover, there exists a splitting by real-valued functions:

$$q = q_1 + q_2, \quad q_1 \in C^1(\mathbb{R}^d), \quad q_2 \in L^\infty(\mathbb{R}^d),$$

such that for some $\rho, C > 0$ the following bounds hold globally on $\mathbb{R}^d$:

$$|q_1| \leq \begin{cases} 
Cr^\epsilon f^{-\rho} & \text{for } 0 < \epsilon < 2, \\
Cr^2 f^{-1-\rho} & \text{for } \epsilon = 2,
\end{cases}$$

$$\nabla^f q_1 \leq Cf^{-1-\rho}, \quad |q_2| \leq Cf^{-1-\rho}.$$
We introduce the weighted Hilbert space $\mathcal{H}_s$ for $s \in \mathbb{R}$ by

$$\mathcal{H}_s = f^{-s}\mathcal{H}.$$ 

Note that we introduced the space $\mathcal{H}_s$ using the function $f$, not $r$. Here the classical orbit is related, too. We also denote the locally $L^2$-space by

$$\mathcal{H}_{\text{loc}} = L^2_{\text{loc}}(\mathbb{R}^d).$$

We consider $B_R = \{ f < R \}$ and the characteristic functions

$$F_\nu = F(B_{R_{\nu+1}} \setminus B_{R_{\nu}}), \quad R_\nu = 2^\nu, \ \nu \geq 0,$$

where $F(\Omega)$ denotes sharp characteristic function of a subset $\Omega \subseteq \mathbb{R}^d$. Define the spaces $B$, $B^*$ and $B^*_0$ by

$$B = \{ \psi \in \mathcal{H}_{\text{loc}} \mid \| \psi \|_B < \infty \}, \quad \| \psi \|_B = \sum_{\nu \geq 0} R_\nu^{1/2} \| F_\nu \psi \|_\mathcal{H},$$

$$B^* = \{ \psi \in \mathcal{H}_{\text{loc}} \mid \| \psi \|_{B^*} < \infty \}, \quad \| \psi \|_{B^*} = \sup_{\nu \geq 0} R_\nu^{-1/2} \| F_\nu \psi \|_\mathcal{H},$$

$$B^*_0 = \{ \psi \in B^* \mid \lim_{\nu \to \infty} R_\nu^{-1/2} \| F_\nu \psi \|_\mathcal{H} = 0 \},$$

respectively. We note that $B^*_0$ coincides with the closure of $C_0^\infty(\mathbb{R}^d)$ in $B^*$ and for any $s > 1/2$ the following inclusion relations hold:

$$\mathcal{H}_s \subseteq B \subseteq \mathcal{H}_{1/2} \subseteq \mathcal{H} \subseteq \mathcal{H}_{-1/2} \subseteq B^*_0 \subseteq B^* \subseteq \mathcal{H}_{-s}. \quad (1.4)$$

In [I] we define the spaces $B$ and $B^*$ using the function $r$. However, considering the classical orbit it is natural to define the spaces using the function $f$ as above.

We introduce the conjugate operator $A$ as a maximal differential operator

$$A = \text{Re} \ p^f, \quad p^f = -i\nabla f,$$

with domain

$$\mathcal{D}(A) = \{ \psi \in \mathcal{H} \mid A\psi \in \mathcal{H} \}.$$ 

The conjugate operator $A$ is self-adjoint (cf. [I]) and has the following expressions:

$$A = \text{Re} \ p^f = (p^f)^* + \frac{i}{2}(\Delta f) = p^f - \frac{i}{2}(\Delta f). \quad (1.6)$$

By the definition of $r$, there exist $c > 0, r_0 \geq 1$ such that

$$|\nabla r| \geq c,$$

on $\{ x \in \mathbb{R}^d \mid r(x) > r_0 \}$. We set

$$\eta = 1 - \chi(r/r_0), \quad \tilde{\eta} = \eta|\nabla r|^{-2},$$

3
and introduce the tensor $\ell$ as follows.

$$\ell = \delta - \tilde{\eta}(\nabla r) \otimes (\nabla r).$$

For notational simplicity, we set

$$h = r^{-\epsilon/2-1} \left( \delta - (\nabla r) \otimes (\nabla r) + 2Cf^{-1-\rho}\delta \right).$$

Here we choose $C > 0$ large enough so that

$$h \geq r^{-\epsilon} f^{-1} \ell + Cr^{-\epsilon} f^{-2-\rho}\delta \geq 0,$$

as quadratic forms on fibers of the tangent bundle of $\mathbb{R}^d$. For any open subset $I \subseteq \mathbb{R}$ let us denote

$$I_{\pm} = \{ z = \lambda \pm i\Gamma \in \mathbb{C} \mid \lambda \in I, \ \Gamma \in (0,1) \},$$

respectively. We also use the notation $\langle T \rangle_\psi = \langle \psi, T\psi \rangle$.

1.2 Results

**Theorem 1.2.** Suppose Condition 1.1 and let $I \subseteq \mathbb{R}$ be any relatively compact open subset. Then there exists $C > 0$ such that for any $\phi = R(z) \psi$ with $z \in I_{\pm}$ and $\psi \in B$

$$\|\phi\|_{B^*} + \|p^f\phi\|_{B^*} + (p_j h^{jk} p_k)_{\phi}^{1/2} + \|r^{-\tau} p_j \delta^{jk} p_k \phi\|_{B^*} \leq C \|\psi\|_B. \quad (1.7)$$

**Corollary 1.3.** Under Condition 1.1 the operator $H$ has no singular continuous spectrum: $\sigma_{sc}(H) = \emptyset$.

To prove Theorem 1.2 we use the absence of $B^0_0$-eigenfunctions for $H$. Since the space $B^0_0$ of this paper is somewhat different from the one in II for $\epsilon = 2$, we state the version of Rellich’s theorem using in this paper in Appendix A.

The absence of eigenvalue for $H$ follows immediately from Theorem A.1. Therefore by combining Corollary 1.3 with it we obtain that the spectrum of $H$ is purely absolutely continuous under Condition 1.1. The limiting absorption principle does not immediately follow from Besov boundedness (1.7). To show it we impose an additional condition and we establish radiation condition bounds.

**Condition 1.4.** In addition to Condition 1.1 there exist $\tau, C > 0$ such that

$$|\nabla f q_1| \leq Cf^{-1-\tau}, \quad |\ell^{*k} r^{-\epsilon/2} \nabla_k q_1| \leq Cf^{-1-\tau}.$$

Now we choose a smooth decreasing function $r_\lambda \geq 1$ of $\lambda \in \mathbb{R}$ such that

$$\lambda - q_1 + r^\epsilon > 1 \quad \text{for } r \geq r_\lambda,$$

and set asymptotic complex phase $a$: For $z = \lambda \pm i\Gamma \in \mathbb{R} \cup \mathbb{R}_+$

$$a = a_z = \eta_\lambda |\nabla r| r^{-\epsilon/2} \sqrt{2(z - q_1 + r^\epsilon)} \pm \frac{\epsilon}{2} |\nabla r|^2 r^{-\epsilon/2-1}, \quad (1.8)$$

4
Corollary 1.7. Suppose Condition 1.4 and let \( I \subset \mathbb{R} \) be any relatively compact open subset. Then for all \( \beta \in [0, \beta_c) \) there exists \( C > 0 \) such that for any \( \phi = R(z)\psi \) with \( \psi \in f^{-\beta}\mathcal{B} \) and \( z \in I_\pm \)

\[
\|f^\beta(A \mp a)\phi\|_{\mathcal{B}^*} + \|p_if^2\beta h^{ij}p_j\phi\|_{\mathcal{B}^*}^{1/2} \leq C\|f^\beta\psi\|_{\mathcal{B}},
\]

(1.9)

respectively.

By Theorem 1.2 and Theorem 1.5 we obtain the limiting absorption principle.

**Corollary 1.6.** Suppose Condition 1.4 and let \( I \subset \mathbb{R} \) be any relatively compact open subset. For any \( s > 1/2 \) and \( \omega \in (0, \min \{(2s - 1)/(2s + 1), \beta_c/(\beta_c + 1)\}) \) there exists \( C > 0 \) such that for any \( z, z' \in I_+ \) or \( z, z' \in I_- \)

\[
\|R(z) - R(z')\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C|z - z'|^\omega,
\]

\[
\|r^{-\epsilon/2}pR(z) - r^{-\epsilon/2}pR(z')\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C|z - z'|^\omega.
\]

(1.10)

In particular, the operators \( R(z) \) and \( r^{-\epsilon/2}pR(z) \) attain uniform limits as \( I_+ \ni z \to \lambda \in I \) in the norm topology of \( \mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s}) \), say denoted by

\[
R(\lambda \pm i0) = \lim_{I_+ \ni z \to \lambda} R(z),
\]

\[
r^{-\epsilon/2}pR(\lambda \pm i0) = \lim_{I_+ \ni z \to \lambda} r^{-\epsilon/2}pR(z),
\]

(1.11)

respectively. These limits \( R(\lambda \pm i0) \) and \( r^{-\epsilon/2}pR(\lambda \pm i0) \) belong to \( \mathcal{B}(\mathcal{B}, \mathcal{B}^*) \).

Combining Theorem 1.5 and Corollary 1.6 we obtain the radiation condition bounds for real spectral parameters.

**Corollary 1.7.** Suppose Condition 1.4 and let \( I \subset \mathbb{R} \) be any relatively compact open subset. Then for all \( \beta \in [0, \beta_c) \) there exists \( C > 0 \) such that for any \( \phi = R(\lambda \pm i0)\psi \) with \( \psi \in f^{-\beta}\mathcal{B} \) and \( z \in I_\pm \)

\[
\|f^\beta(A \pm a)\phi\|_{\mathcal{B}^*} + \|p_if^{2\beta}h^{ij}p_j\phi\|_{\mathcal{B}^*}^{1/2} \leq C\|f^\beta\psi\|_{\mathcal{B}},
\]

(1.12)

respectively.

Finally, we obtain the Sommerfeld uniqueness result.

**Corollary 1.8.** Suppose Condition 1.4 and let \( \lambda \in \mathbb{R}, \phi \in \mathcal{H}_{\text{loc}} \) and \( \psi \in f^{-\beta}\mathcal{B} \) with \( \beta \in [0, \beta_c) \). Then \( \phi = R(\lambda \pm i0)\psi \) holds if and only if both of the following conditions hold:
(i) \((H - \lambda)\phi = \psi\) in the distributional sense.
(ii) \(\phi \in f^\beta B^*\) and \((A \mp a)\phi \in f^{-\beta}B_0^*\).

As is seen in Appendix A, there is no generalized eigenfunction in \(B_0^*\). We constructed a \(B^*\)-eigenfunction in \([1]\) (see also Appendix A). Therefore in the sense that the inclusion relations \((1.4)\) hold, the space \(B^*\) is the minimal space where a generalized eigenfunction exists. Hence Theorem \([1.2]\) asserts the boundedness of \(R(z)\) between natural and optimal spaces. As far as the author knows, there seem to be no literature on the Besov boundedness for repulsive Hamiltonians so far, and our theorem is new. In particular, by setting the spaces \(B\) and \(B^*\) using the function \(f\) of \((1.2)\), even for the case \(\epsilon = 2\) we obtain the results. In fact if we define the spaces \(B\) and \(B^*\) using the function \(r\), the proof of Theorem \([1.5]\) is not completed.

To prove the theorems and the corollaries we apply a new commutator argument with some weight inside from \([IS]\). In \([IS]\), they consider only potentials decaying at infinity. In order to deal with the repulsive potentials that diverge to \(-\infty\) at infinity we need to choose the appropriate conjugate operator \(A\) as \((1.5)\).

The limiting absorption principle for repulsive Hamiltonians was studied also by \([BCHM]\). However, they did not prove the Besov boundedness. Moreover, as for the decay rate of perturbation at infinity, our assumptions are considerably weaker and includes their setting. In this sense, our results are stronger than theirs.

In case \(\epsilon = 0\), there has been an extensive amount of literature on spectral theory (e.g. \([A, FH, FHH20, HH, IS, ISo]\)). As for the case \(\epsilon = 2\), Ishida studied inverse scattering problem in \([Ishi]\) and borderline of the short-range condition in \([Ishi2]\). Moreover Finster and Isidro discussed the \(L^p\)-spectrum in \([FI]\). Skibsted dealt with the Besov bound and the limiting absorption principle for attractive Hamiltonians in \([Sk]\), whereas we considered the case of repulsive Hamiltonians. We also mention recent works related to the repulsive potentials. Josef studied in \([J]\) the properties of spectrum of two-dimensional Pauli operator with repulsive potential. Lakaev studied in \([L1, L2]\) eigenvalue problem for discrete Schrödinger operator with repulsive potential on the two-dimensional lattice \(Z^2\).

In Section 2 we introduce a commutator with weight inside and discuss its properties. In Section 3 we prove Theorem \([1.2]\) by using a commutator estimate and contradiction. In Section 4 by using Theorem \([1.2]\) we prove Theorem \([1.5]\) and Corollaries \([1.6, 1.8]\). In the proofs of the these results commutator estimates play major roles.

1.3 Classical orbit

In this subsection we consider the classical orbit on the Hamiltonian

\[ H = \frac{1}{2}p^2 - |x|^{\epsilon}. \]

The Hamilton equation is given by

\[ \dot{x}(t) = p(t), \quad \dot{p}(t) = -\epsilon |x(t)|^{\epsilon-2} x(t). \]
This yields the following equation:

$$\ddot{x}(t) = \epsilon |x(t)|^{-2} x(t). \quad (1.13)$$

As for the case $\epsilon = 2$ we can compute explicitly:

$$x(t) = \frac{1}{2} \left( x(0) + \frac{1}{\sqrt{2}} \dot{x}(0) \right) e^{\sqrt{2}t} + \frac{1}{2} \left( x(0) - \frac{1}{\sqrt{2}} \dot{x}(0) \right) e^{-\sqrt{2}t}.$$  

Thus, in general, $|x(t)|$ grows exponentially as $t \to \infty$. On the other hand for the case $0 < \epsilon < 2$, $|x(t)|$ grows in the order of $t^{1/(1-\epsilon/2)}$ in general. In fact, we set $x(t) = t^{1/(1-\epsilon/2)} y$ for $y \in \mathbb{R}^d$ with $|y| = (2^{-1}(2-\epsilon)^2)^{1/(2-\epsilon)}$, and then the function $x(t)$ satisfies (1.13). By these observations if we define the new position function

$$y(t) = \begin{cases} |x(t)|^{1-\epsilon/2} (x(t)/|x(t)|) & \text{for } 0 < \epsilon < 2, \\ \log |x(t)| (x(t)/|x(t)|) & \text{for } \epsilon = 2, \end{cases}$$

we have $|y(t)| = \mathcal{O}(t)$ as $t \to \infty$ similarly to the case $\epsilon = 0$. Hence it is natural to define the spaces $\mathcal{B}$ and $\mathcal{B}^*$ using the function $f$ rather than $r$.

## 2 Preliminaries

In this section we are going to prepare some lemmas and properties to prove the results that are stated in Section 1. For simplicity, we omit the proofs of these (cf. [Sig]).

**Lemma 2.1.** Let $H^2(\mathbb{R}^d)$ be the Sobolev space of second order, and set

$$H^2_{\text{comp}}(\mathbb{R}^d) = \{ \psi \in H^2(\mathbb{R}^d) | \text{supp } \psi \text{ is compact} \}.$$  

Then the following inclusion relations hold.

$$H^2_{\text{comp}}(\mathbb{R}^d) \subset \mathcal{D}(H) \subset \mathcal{D}(A). \quad (2.1)$$

We consider commutators with a weight $\Theta$ inside:

$$[H, iA]_\Theta := i(H\Theta A - A\Theta H).$$

Let $\Theta = \Theta(f)$ be a non-negative smooth function with bounded derivatives. More explicitly, if we denote its derivatives in $f$ by primes such as $\Theta'$, then

$$\Theta \geq 0, \quad |\Theta^{(k)}| \leq C_k, \quad k = 0, 1, 2, \ldots \quad (2.2)$$

We define the quadratic form $[H, iA]_\Theta$ on $C^\infty_0(\mathbb{R}^d)$, and then extend it to $\mathcal{D}(A)$ according to the following lemma.
Lemma 2.2. Suppose Condition 1.1 and let \( \Theta \) be a non-negative smooth function with bounded derivatives. Then, as quadratic forms on \( C^\infty_0(\mathbb{R}^d) \),

\[
[H, iA]_\Theta = p_j (\nabla^2 f)^{jk} \Theta p_k + (p^f)^* \Theta' p^f + \frac{1}{2} \text{Re} \left( (\Delta f) \Theta p^i \delta^{ij} p_j \right)
- \frac{1}{2} \Re (|\nabla f|^2 \Theta' \nabla f') - \Re \left( |\nabla f|^4 \Theta'' \right)
- \Re (|\nabla f|^2 |\nabla f|^2 \Theta'' - \frac{1}{4} |\nabla f|^4 \Theta'').
\]

In particular noting the formulae below and using the Cauchy-Schwarz inequality, \([H, iA]_\Theta\) restricted to \( C^\infty_0(\mathbb{R}^d) \) extends to a bounded form on \( D(A) \). Here, we regard \( D(A) \) as the Banach space with graph norm.

We have the following formulae (cf. (1.2)): for \( r \geq 2 \)

\[
|\nabla r|^2 = 1, \quad (\nabla^2 f)^{jk} = r^{-\epsilon/2-1} \delta^{jk} - \left( \frac{\epsilon}{2} + 1 \right) r^{-\epsilon/2-1} (\nabla r)^j (\nabla r)^k,
\]

\[
(\nabla r)^j = x^j r^{-1}, \quad \Delta f = (d - \frac{\epsilon}{2} - 1) r^{-\epsilon/2-1}, \quad \Delta r = (d - 1) r^{-1}.
\]

On the other hand, throughout the paper we shall use the notation

\[
\text{Im}(A \Theta H) = \frac{1}{2i} (A \Theta H - H \Theta A)
\]

as a quadratic form defined on \( D(H) \), i.e. for \( \psi \in D(H) \)

\[
\langle \text{Im}(A \Theta H) \rangle_\psi = \frac{1}{2i} \left( \langle A \psi, \Theta H \psi \rangle - \langle H \psi, \Theta A \psi \rangle \right).
\]

Note that by the embedding 2.1 the above quadratic form is well-defined. Obviously the quadratic forms \([H, iA]_\Theta\) and \( 2 \text{Im}(A \Theta H) \) coincide on \( C^\infty_0(\mathbb{R}^d) \), and hence we obtain

\[
[H, iA]_\Theta = 2 \text{Im}(A \Theta H) \quad \text{on } D(H).
\]

Finally using the function \( \chi \) of (1.1) we define \( \chi, \bar{\chi} \) for \( n \geq 0 \) by

\[
\chi_n = \chi(f/R_n), \quad \bar{\chi}_n = 1 - \chi_n.
\]

3 Besov bound

In this section we discuss the locally uniform Besov bound for the resolvent \( R(z) \). Lemma 3.3 and Proposition 3.1 in Subsection 3.1 is a key to prove Theorem 1.2. In Subsection 3.2 we prove Theorem 1.2 by Proposition 3.1 and contradiction.
3.1 Commutator estimate

We introduce the regularized weight

$$\Theta = \Theta^\delta = \int_0^{f/R^\nu} (1 + f/R^\nu)^{-\delta} ds = \left[ 1 - (1 + f/R^\nu)^{-\delta} \right] / \delta; \; \delta > 0, \nu \geq 0, \quad (3.1)$$

and compute derivatives in $f$:

$$\Theta' = (1 + f/R^\nu)^{-1-\delta}/R^\nu, \quad \Theta'' = -(1 + \delta)(1 + f/R^\nu)^{-2-\delta}/R^{2\nu}. \quad (3.2)$$

**Proposition 3.1.** Suppose Condition 1.1, let $I \subseteq \mathbb{R}$ be any relatively compact open subset, and fix any $\delta \in (0, \min\{1, \rho, \epsilon^c\})$ in the definition (3.1) of $\Theta$. Then there exist $C > 0$ and $n \geq 0$ such that for all $\phi = R(z)\psi$ with $z \in I_\pm$ and $\psi \in \mathcal{B}$ and for all $\nu \geq 0$

$$\|\Theta^{1/2}\phi\|^2 + \|\Theta^{1/2}A\phi\|^2 + \langle p_j h^{jk}\Theta p_k \rangle \phi$$

$$\leq C \left( \|\phi\|_{B^*} \|\psi\|_B + \|A\phi\|_B \|\psi\|_B + \|\chi_n \Theta^{1/2}\phi\|^2 \right). \quad (3.3)$$

We first note that $\Theta$ defined by (3.1) has following properties.

**Lemma 3.2.** Suppose Condition 1.1 and fix any $\delta > 0$ in the definition (3.1) of $\Theta$. Then there exist $c, C, C_k > 0$, $k = 2, 3, \ldots$ such that uniformly in $\nu \geq 0$

$$c/R^\nu \leq \Theta \leq \min\{C, f/R^\nu\},$$

$$c(\min\{R^\nu, f\})^\delta f^{-1-\delta} \leq \Theta' \leq f^{-1} \Theta,$$

$$0 \leq (-1)^{k-1} \Theta^{(k)} \leq C_k f^{-k} \Theta.$$

**Proof.** By the definition of $\Theta$ in (3.1) and expressions of derivatives of it as (3.2), the asserted estimates are clearly hold except for the first estimate in the second line. But this estimate follows by using the last estimate of the first line and the following inequality:

$$(\min\{R^\nu, f\})^\delta f^{-1-\delta} (\min\{R^\nu, f\}/R^\nu) \left( (1 + f/R^\nu)^{1+\delta} R^\nu \right) \leq C.$$

The following lemma is a key to prove Theorem 1.2.

**Lemma 3.3.** Suppose Condition 1.1, let $I \subseteq \mathbb{R}$ be any relatively compact open subset, and fix any $\delta \in (0, \min\{1, \rho, \epsilon^c\})$ in the definition (3.1) of $\Theta$. Then there exist $c, C > 0$ and $n \geq 0$ such that uniformly in $z \in I_\pm$ and $\nu \geq 0$, as a quadratic forms on $D(H)$,

$$\text{Im}(A\Theta(H - z))$$

$$\geq c\Theta' + cA\Theta' + c p_j h^{jk}\Theta p_k - C \chi_n^2 \Theta - \text{Re}(\gamma(H - z)), \quad (3.4)$$

where $\gamma = \gamma_{z, \nu}$ is a uniformly bounded complex-valued function: $|\gamma| \leq C$. 

9
Proof. Let $I$ and $\delta$ be as in the assertion. First using Lemmas 2.2, 3.2, 2.3, 2.4, the Cauchy-Schwarz inequality and a general identity holding for any $g \in C^\infty(\mathbb{R}^d)$:

\[ p_i g \delta^{ij} p_j = \text{Re} (gp_i \delta^{ij} p_j) + \frac{1}{2} (\Delta g), \]

we can bound uniformly in $z = \lambda \pm i \Gamma \in I_\pm$ and $\nu \geq 0$

\[
\text{Im}(A\Theta(H - z)) \\
\geq \frac{1}{2} p_j r^{-\epsilon/2-1} \delta^{jk} \Theta p_k - \left( \frac{\epsilon}{4} + \frac{1}{2} \right) (p^r)^{\epsilon r^{-\epsilon/2-1}} \Theta p^r + \frac{1}{2} (p^f)^{\epsilon \Theta' p^f} \\
- \frac{1}{2} \text{Re} \left( |\nabla f|^2 \Theta'(H - z) \right) + \Gamma \text{Re}(A\Theta) + \frac{\epsilon}{2} r^{\epsilon/2-1} \Theta - C_1 Q \\
\geq \frac{1}{2} p_j r^{-\epsilon/2-1} \left( \delta^{jk} - (\nabla r)^j (\nabla r)^k + 2 C f^{-1} r^p \delta^{jk} \right) \Theta p_k + \frac{1}{2} (p^f)^{\epsilon \Theta' p^f} \\
- \text{Re} \left( \left( \frac{1}{2} |\nabla f|^2 \Theta' + \frac{\epsilon}{2} r^{\epsilon/2-1} \Theta \right) (H - z) \right) + \Gamma \Theta^{1/2} A \Theta^{1/2} - C_2 Q \\
\geq \frac{1}{4} p_j h^{jk} \Theta p_k + \frac{1}{2} \Theta' + \frac{1}{4} A \Theta' A + \Gamma \Theta^{1/2} A \Theta^{1/2} \\
- \text{Re} \left( \left( \frac{1}{2} |\nabla f|^2 \Theta' + \frac{\epsilon}{2} r^{\epsilon/2-1} \Theta - \frac{1}{2} r^{-\epsilon} \Theta' \right) (H - z) \right) - C_3 Q,
\]

where

\[ Q = f^{-1-\min\{1, \rho, \epsilon\}} \Theta + p_j r^{-\epsilon} f^{-1-\min\{1, \rho, \epsilon\}} \Theta \delta^{jk} \Theta p_k. \]

To the fourth term of (3.6) we apply the Cauchy-Schwarz inequality, Lemma 3.2 and the general identity holding for any real functions $g, h \in C^1(\mathbb{R}^d)$:

\[ h \text{Re} (gp^2) h = \text{Re} (h^2 gp^2) + \frac{1}{2} (\partial_j h) \delta^{jk} (\partial_k gh). \]

Then it follows that

\[
\pm \Gamma \Theta^{1/2} A \Theta^{1/2} = \pm \Gamma \Theta^{1/2} \left( \text{Re} p^f \right) \Theta^{1/2} \\
\geq -C_4 \Gamma \Theta^{1/2} r^{-\epsilon/2} (H - \lambda) r^{-\epsilon/2} \Theta^{1/2} - C_5 \Gamma \\
= -C_4 \Gamma \text{Re} \left( \Theta^{1/2} r^{-\epsilon/2} (H - z) r^{-\epsilon/2} \Theta^{1/2} \right) \pm C_5 \text{Im}(H - z) \\
\geq -\text{Re} \left( (C_4 \Gamma r^{-\epsilon} \Theta \pm i C_5) (H - z) \right) - C_6 Q.
\]

We substitute the estimate (3.7) into (3.6), and obtain

\[
\text{Im}(A\Theta(H - z)) \\
\geq \frac{1}{4} p_j h^{jk} \Theta p_k + \frac{1}{2} \Theta' + \frac{1}{4} A \Theta' A - C_7 Q \\
- \text{Re} \left( \left( \frac{1}{2} |\nabla f|^2 - r^{-\epsilon} \right) \Theta' + \frac{\epsilon}{2} r^{-\epsilon/2-1} \Theta + C_4 \Gamma r^{-\epsilon} \Theta \pm i C_5 \right) (H - z). \tag{3.8}
\]
Using (3.5) and Lemma 3.1 we can combine and estimate the second and fourth terms of (3.8): For large $n \geq 0$

\[
\frac{1}{2} \Theta' - C_7 Q \geq \frac{1}{4} \Theta' - C_8 X_n^2 \Theta - 2C_7 \text{Re} \left( r^{-\epsilon} f^{-1-\min\{1, \rho, \rho'\}} \Theta(H - z) \right).
\]

(3.9)

Hence by (3.8) and (3.9), if we set

\[
\gamma = \frac{1}{2} \left( |\nabla f|^2 - r^{-\epsilon} \right) \Theta' + \frac{\epsilon}{2} r^{-\epsilon/2 - 1} \Theta + C_4 r^{-\epsilon} \Theta \pm iC_5 - 2C_7 r^{-\epsilon} f^{-1-\min\{1, \rho, \rho'\}} \Theta,
\]

then the assertion follows.

Proof of Proposition 3.1. The assertion follows immediately from Lemma 3.3.

3.2 Besov boundedness

Now we prove Theorem 1.2 by Proposition 3.1 and contradiction.

Proof of Theorem 1.2. Let $I \subset \mathbb{R}$ be any relatively compact open subset. We prove the assertion only for the upper sign.

Step 1. We assume that for $C_1 > 0$ large enough

\[
\| \phi \|_{B^*} \leq C_1 \| \psi \|_B,
\]

(3.10)

then the bound (1.7) holds. In fact, the last term on the left-hand side of (1.7) clearly satisfies the desired estimate by the identity

\[
r^{-\epsilon} p_j \delta^j k p_k \phi = r^{-\epsilon} \psi + r^{-\epsilon} (|x|^\nu - q + z) \phi
\]

and Condition 1.1. Hence it suffice to consider the second and third terms of (1.7).

Fix any $\delta \in (0, \min\{1, \rho, \rho'\})$. Then by Proposition 3.1 and (3.10) there exists $C_2 > 0$ such that for any $\phi = R(z) \psi$ with $z \in I_+$ and $\psi \in B$ uniformly in $\epsilon_1 \in (0, 1)$ and $\nu \geq 0$

\[
\| \mathcal{O}^{1/2} A \phi \|^2 + \| p_j h^j k \Theta p_k \phi \| \leq \epsilon_1 \| A \phi \|^2_{B^*} + \epsilon_1^{-1} C_2 \| \psi \|^2_B.
\]

(3.11)

In the first term on the left-hand side of (3.11) for each $\nu \geq 0$, noting the expression of $\mathcal{O}'$ in (3.2), we restrict the integral region to $B_{R_{\nu+1}} \setminus B_{R_\nu}$. As for the second term on the same side we look at the estimate (3.11) for any fixed $\nu \geq 0$, say $\nu = 0$. Then we have the following inequality.

\[
c_1 \| A \phi \|^2_{B^*} + c_1 \| p_j h^j k \Theta p_k \phi \| \leq 2\epsilon_1 \| A \phi \|^2_{B^*} + 2\epsilon_1^{-1} C_2 \| \psi \|^2_B.
\]

If we let $\epsilon_1 \in (0, c_1/2)$, the rest of (1.7) follows from this estimate and (1.6). Hence (1.7) reduces to (3.10).

Step 2. We prove (3.10) by contradiction. Assume the opposite, and let $z_k \in I_+$ and $\psi_k \in B$ be such that

\[
\lim_{k \to \infty} \| \psi_k \|_B = 0, \quad \| \phi_k \|_{B^*} = 1; \quad \phi_k = R(z_k) \psi_k.
\]

(3.12)
Note that then it automatically follows that
\[\|r^{-\epsilon/2}p_0\phi_k\|_{B^*} + \|r^{-\epsilon}p^2\phi_k\|_{B^*} \leq C_3.\] (3.13)
In fact, arguing similarly to Step 1, we can deduce from (3.12) and Proposition 3.1 that
\[\|A\phi_k\|^2_{B^*} + \langle p_j h^k p_k \phi_k \rangle \leq C_4, \quad p^2 \phi_k = \psi_k + (|x|^r - q + z)\phi_k,
\]
and these combined Condition 1.1, (1.6) and (3.12) imply (3.13). Now, choosing a subsequence and retaking \(I \subseteq \mathbb{R}\) slightly larger, we may assume that \(z_k \in I_+\) converges to some \(z \in I \cup I_+\). If the limit \(z\) belongs to \(I_+\), the bounds
\[\|\phi_k\|_{B^*} \leq \|\phi_k\|_H \leq \|R(z_k)\|_{\mathcal{B}(\mathcal{H})}\|\psi_k\|_H \leq \|R(z_k)\|_{\mathcal{B}(\mathcal{H})}\|\psi_k\|_B\]
and (3.12) contradict the norm continuity of \(R(z) \in \mathcal{B}(\mathcal{H})\) in \(z \in I_+\). Hence we have the limit
\[\lim_{k \to \infty} z_k = z = \lambda \in I.\] (3.14)
Let \(s > 1/2\). By choosing a further subsequence we may assume that \(\phi_k\) converges weakly to some \(\phi \in \mathcal{H}_{-s}\). But then \(\phi_k\) actually converges strongly in \(\mathcal{H}_{-s}\). To see this let us fix \(s' \in (1/2, s)\) and \(g \in C_0^\infty(\mathbb{R})\) with \(g = 1\) on a neighborhood of \(I\), and decompose for any \(n \geq 0\)
\[f^{-s}\phi_k = f^{-s} g(H)(\chi_n f^s)(f^{-s}\phi_k) + (f^{-s} g(H)f^s)(\chi_n f^{s-s})(f^{-s}\phi_k) + f^{-s}(1 - g(H))R(z_k)\psi_k.\]
The last term on the right-hand side converges to 0 in \(\mathcal{H}\) due to (3.12), and the second term can be taken arbitrarily small in \(\mathcal{H}\) by choosing \(n \geq 0\) sufficiently large since \(f^{-s} g(H)f^s\) is a bounded operator. By the compactness of \(f^{-s} g(H)\), for fixed \(n \geq 0\) the first term converges strongly in \(\mathcal{H}\). Therefore \(\phi_k\) converges to \(\phi\) in \(\mathcal{H}_{-s}\), i.e.
\[\lim_{k \to \infty} \phi_k = \phi \quad \text{in } \mathcal{H}_{-s}.\] (3.15)
By (3.12), (3.14) and (3.15) it follows that
\[(H - \lambda)\phi = 0 \text{ in the distributional sense.}\] (3.16)
In addition, we can verify \(\phi \in \mathcal{B}^*_0\). In fact, let us apply Proposition 3.1 with \(\delta = 2s - 1 > 0\) to \(\phi_k = R(z_k)\psi_k\), and take the limit \(k \to \infty\) using (3.12), (3.13), (3.15) and Lemma 3.1 We obtain for all \(\nu \geq 0\)
\[\|\Theta^{1/2}\phi\| \leq \|\chi_n \Theta^{1/2}\phi\| \leq C_n R_{\nu}^{-1/2}\|\chi_n f^{1/2}\phi\|.\] (3.17)
Letting \(\nu \to \infty\) in (3.17), we obtain \(\phi \in \mathcal{B}^*_0\), and then we conclude \(\phi = 0\) by (3.16) and Theorem 3.1 But this is a contradiction, because similarly to Step 1 we have
\[1 = \|\phi_k\|^2_{B^*} \leq C_6 \left(\|\psi_k\|_B + \|\chi_n \phi_k\|^2\right),\]
and, as \(k \to \infty\), the right-hand side converges to 0. Hence (3.10) holds.
4 Radiation condition

Our main purpose in this section is to prove the radiation condition bounds for complex spectral parameters. In Subsection 4.1 we state and prove the key lemma to prove Theorem 1.5. In Subsection 4.2 we prove Theorem 1.5. Corollaries 1.6-1.8 are also proved in the same subsection.

Throughout the section we suppose Condition 1.4, and prove the statements only for the upper sign for simplicity.

4.1 Commutator estimate

We introduced the conjugate operator $B$ as a maximal differential operator

$$B = \text{Re} p^r = \frac{1}{2} (p^r + (p^r)^*) ,$$

with domain

$$\mathcal{D}(B) = \{ \psi \in \mathcal{H} \mid B\psi \in \mathcal{H} \} ,$$

and set associated asymptotic complex phase $b$: For $z = \lambda + i\Gamma \in \mathbb{R} \cup \mathbb{R}_+$

$$b = b_z = \eta_\lambda |\nabla r| \sqrt{2(z - q_1 + r^*)} + i\frac{\epsilon}{4} |\nabla r|^2 r^{-1} . \quad (4.1)$$

We note that the operator $B$ is self-adjoint on $\mathcal{D}(B)$ (cf. [IS]).

**Lemma 4.1.** Let $I \subseteq \mathbb{R}$ be any relatively compact open subset. Then there exists $C > 0$ such that uniformly in $z \in I \cup I_+$

$$|a| \leq C, \quad |b| \leq C r^{c/2}, \quad \text{Im} a \geq \frac{\epsilon}{2} |\nabla r|^2 r^{-c/2-1} , \quad \text{Im} b \geq \frac{\epsilon}{4} |\nabla r|^2 r^{-1} ,$$

$$|\ell_j^{\epsilon} \nabla_j a| \leq C r^{-c/2} f^{-1} r^{-c} , \quad |p^r b + b^2 - 2 |\nabla r|^2 (z - q_1 + r^*)| \leq C f^{-1-\min\{\rho,c',\tau\}} .$$

**Proof.** By the definitions of $a$ and $b$ (see (1.8), (1.11)) the first, second, third and fourth estimates clearly hold. The fifth estimate is also clear by Condition 1.4 and the following equation

$$\ell_j^{\epsilon} \nabla_j a = \ell_j^{\epsilon} (\nabla_j \eta_\lambda |\nabla r|) r^{-c/2} \sqrt{2(z - q_1 + r^*)}$$

$$- \frac{\epsilon}{2} (1 - \eta)(\nabla r)^\ast \eta_\lambda |\nabla r| r^{-c/2-1} \sqrt{2(z - q_1 + r^*)}$$

$$+ \frac{1}{2} \ell_j^{\epsilon} \eta_\lambda |\nabla r| r^{-c/2} ( - \langle \nabla_j q_1 \rangle + \epsilon (\nabla r) r^{-c-1} ) \sqrt{2(z - q_1 + r^*)}$$

$$+ i \ell_j^{\epsilon} \frac{\epsilon}{2} (\nabla_j |\nabla r|^2) r^{-c/2-1} - \frac{\epsilon}{2} \left( \frac{\epsilon}{2} + 1 \right) (1 - \eta)(\nabla r)^\ast |\nabla r|^2 r^{-c/2-2} .$$
Since we can write
\[ p^r b + b^2 - 2|\nabla r|^2(z - q_1 + r^\epsilon) \]
\[ = (p^r \eta_\lambda |\nabla r|) \sqrt{2(z - q_1 + r^\epsilon)} + i \frac{\sqrt{2}}{2} \eta_\lambda |\nabla r|(\nabla^r q_1)/\sqrt{2(z - q_1 + r^\epsilon)} \]
\[ + i \frac{\sqrt{2}}{2} \epsilon |\nabla r|^3 r^{-1} \eta \lambda_1 \lambda_2 (z - q_1) \]
\[ - 2(1 - \eta^2_\lambda)|\nabla r|^2(z - q_1 + r^\epsilon) + \frac{c}{4}(\nabla^r |\nabla r|^2)r^{-1} - \left( \frac{c}{4} + \frac{\epsilon^2}{16} \right) |\nabla r|^4 r^{-2}, \]
by Condition [1.4] the last estimate is also holds.

Lemma 4.2. Let \( I \subseteq \mathbb{R} \) be any relatively compact open subset. Then there exist a complex-valued function \( q_3 \) and a constant \( C > 0 \) such that uniformly in \( z \in I \cup I^+ \), as a quadratic forms on \( C^0_{\infty}(\mathbb{R}^d) \),
\[ H - z = \frac{1}{2} (B + b) \eta (B - b) + \frac{1}{2} p_j \ell^{jk} p_k + q_3; \quad |q_3| \leq C f^{-1} \min\{\rho, \epsilon, \tau\}. \]

Proof. Using the following expression:
\[ B = p^r - \frac{i}{2} (\Delta r) = (p^r)^* + \frac{i}{2} (\Delta r), \]
we can write
\[ H - z = \frac{1}{2} (B + b) \eta (B - b) + \frac{1}{2} p_j \ell^{jk} p_k + q_3;
\]
\[ - |x|^\epsilon + q_0 + q_1 + q_2 - z, \]
where
\[ q_0 = \frac{1}{4}(\nabla^r \eta)(\Delta r) + \frac{1}{4} \eta (\nabla^r \Delta r) - \frac{1}{8} \eta (\Delta r)^2. \]
Hence the assertion is obtained by setting
\[ q_3 = \frac{1}{2} \eta \left[(p^r b) + b^2 - 2|\nabla r|^2(z - q_1 + r^\epsilon)\right] - (1 - \eta)(z - q_1 + r^\epsilon) \]
\[ - \frac{1}{2} (\nabla^r \eta)b + (r^\epsilon - |x|^\epsilon) + q_0 + q_2 \]
and using Lemma [4.1] \( \square \)

Let us introduce the regularized weight
\[ \Theta = \Theta^\delta_\nu = \int_0^{f/R_{\nu}} (1 + s)^{-1-\delta} ds = \left[ 1 - (1 + f/R_{\nu})^{-\delta} \right]/\delta; \quad \delta > 0, \nu \geq 0, \]
which is the same weight as (3.1) introduced in Section 3. We denote its derivatives in \( f \) by primes such as (3.2) \( \square \).
Lemma 4.3. Let $I \subseteq \mathbb{R}$ be any relatively compact open subset, and fix any $\delta \in (0, \min \{\rho, \epsilon', \tau\}]$ and $\beta \in (0, 1 + \epsilon/2)$. Then there exist $c, C > 0$ such that uniformly in $z \in I \cup I_+ \cup I_-$ and $\nu \geq 0$, as quadratic forms on $D(H)$

$$\text{Im} \left( (A - a)^* \Theta^{2\beta}(H - z) \right) \geq c(A - a)^* \Theta' \Theta^{2\beta-1}(A - a) + cp_j \Theta^{2\beta}h^{j \bar{k}}p_k$$

$$- C \delta^{1 - \min \{2\rho, 2\epsilon', 2\tau\}} + 2 \delta \Theta^{2\beta} - \text{Re}(\gamma \Theta^{2\beta}(H - z)),$$

where $\gamma$ is a certain function satisfying $|\gamma| \leq C \delta^{1 - \min \{2\rho, 2\epsilon', 2\tau\}} + 2 \delta$.

Proof. Let $I, \delta$ and $\beta$ be as in the assertion. To prove the asserted inequality it suffices to compute as a quadratic forms on $C^\infty_0(\mathbb{R}^d)$. By Lemma 3.2, 4.2 and 4.3 and the Cauchy-Schwarz inequality it follows that uniformly in $z \in I \cup I_+ \cup I_-$ and $\nu \geq 0$

$$\text{Im} \left( (A - a)^* \Theta^{2\beta}(H - z) \right)$$

$$= \frac{1}{2} \text{Im} \left( (A - a)^* \Theta^{2\beta}(B + b) \tilde{\eta} \rho^{\epsilon/2}(A - a) \right) + \frac{1}{2} \text{Im} \left( A \Theta^{2\beta}p_j \ell^{j \bar{k}}p_k \right)$$

$$- \frac{1}{2} \text{Im} \left( a^* \Theta^{2\beta}p_j \ell^{j \bar{k}}p_k \right) + \text{Im} \left( (A - a)^* \Theta^{2\beta} q_3 \right)$$

$$= \frac{1}{2} (A - a)^* \beta \Theta' \Theta^{2\beta-1}(A - a) - \frac{1}{4} (A - a)^* \Theta^{2\beta} \left( \nabla^r \tilde{\eta} \right) \rho^{\epsilon/2}(A - a)$$

$$- \frac{\epsilon}{8} (A - a)^* \Theta^{2\beta} \rho^{\epsilon/2-1}(A - a) + \frac{1}{2} (A - a)^* \Theta^{2\beta} (\text{Im} b) \tilde{\eta} \rho^{\epsilon/2}(A - a)$$

$$+ \frac{1}{2} \left[ p_j \Theta^{2\beta} \ell^{j \bar{k}}p_k, iA \right] + \frac{1}{2} \text{Re} \left( A(1 - \eta) \left( \Theta^{2\beta} \right)' p^j \right)$$

$$- \frac{1}{2} \text{Im} \left( a^* \Theta^{2\beta} p_j \ell^{j \bar{k}}p_k \right) + \text{Im} \left( (A - a)^* \Theta^{2\beta} q_3 \right)$$

$$\geq \frac{1}{2} (A - a)^* \left( \beta \Theta' - f^{-1-2\delta} \Theta \right) \Theta^{2\beta-1}(A - a) + \frac{1}{4} \left[ p_j \Theta^{2\beta} \ell^{j \bar{k}}p_k, iA \right]$$

$$- \frac{1}{2} \text{Im} \left( a^* \Theta^{2\beta} p_j \ell^{j \bar{k}}p_k \right) - C_1 Q,$$

where

$$Q = f^{-1-\min \{2\rho, 2\epsilon', 2\tau\}} + 2 \delta \Theta^{2\beta} + p_j r^{-\epsilon} f^{-1-\min \{2\rho, 2\epsilon', 2\tau\}} + 2 \delta \Theta^{2\beta} \delta^{j \bar{k}}p_k.$$ 

Let us further estimate the terms on the right-hand side of (4.2). By Lemma 3.2 the first term of (4.2) can be bounded as

$$\frac{1}{2} (A - a)^* \left( \beta \Theta' - f^{-1-2\delta} \Theta \right) \Theta^{2\beta-1}(A - a)$$

$$\geq c_1 (A - a)^* \beta \Theta' \Theta^{2\beta-1}(A - a) - C_2 Q.$$ 

To estimate the second term of (4.2) we use the following lemma used also in [IS].
Lemma 4.4. Let $g^{ij} = g^{ij} \in C^\infty(\mathbb{R}^d)$ for $i, j = 1, 2, \ldots, d$. Then, as a quadratic form on $C_0^\infty(\mathbb{R}^d)$,

$$[p_i g^{ij} p_j, iA] = p_i \left\{ g^{ij} \left( \nabla^2 f \right)_j^k + \left( \nabla^2 f \right)_j^i g^{kj} - (\nabla f \tilde{g})^{ik} \right\} p_k$$

$$- \text{Im} \left( \tilde{g}^{jk}(\nabla_k \Delta f) p_j \right).$$

We apply Lemma 4.4 with $\tilde{g}$ as follows. If we choose $\epsilon$ small enough, we have the following inequality

$$\frac{1}{4} [p_j \Theta^{2\beta} \ell^k p_k, iA] = \frac{1}{4} p_j \left\{ \Theta^{2\beta} \ell^j \left( \nabla^2 f \right)_j^k + \left( \nabla^2 f \right)_j^i \Theta^{2\beta} \ell^{kj} - (\nabla f \Theta^{2\beta} \ell) p_k \right\} p_k$$

$$- \frac{1}{4} \text{Im} \left( \Theta^{2\beta} \ell^k (\nabla_k \Delta f) p_j \right) \geq \frac{1}{2} p_j \left\{ h^{jk} \Theta - \beta r^{-\epsilon/2} \Theta' \ell^k \right\} \Theta^{2\beta-1} p_k - C_2 Q.$$  \hspace{1cm} (4.4)

As for the third term of (4.2) using Lemma 4.1 and the Cauchy-Schwarz inequality we can estimate as follows.

$$- \text{Im} \left( a^* \Theta^{2\beta} \ell^k p_k, iA \right)$$

$$= - \frac{1}{2} \text{Im} \left( p_j a^* \Theta^{2\beta} \ell^j p_k + i(\nabla_j a) \Theta^{2\beta} \ell^j p_k + i2\beta(1-\eta)a^* \Theta^{2\beta-1} \Theta' p^j \right)$$

$$\geq \left( \frac{\epsilon}{4} - \epsilon_1 \right) p_j r^{-\epsilon/2-1} \Theta^{2\beta} \ell^j p_k - \epsilon_1^{-1} C_3 Q.$$ \hspace{1cm} (4.5)

By the bounds (4.2), (4.3), (4.4) and (4.5) we obtain

$$\text{Im} \left( (A - a)^* \Theta^{2\beta}(H - z) \right) \geq c_1 (A - a)^* \Theta' \Theta^{2\beta-1} (A - a) - \epsilon_1^{-1} C_3 Q$$

$$+ \frac{1}{2} p_j \left\{ h^{jk} \Theta + \left( \frac{\epsilon}{2} - 2\epsilon_1 \right) r^{-\epsilon/2-1} \Theta' \ell^j \Theta - \beta r^{-\epsilon/2} \Theta' \ell^j \Theta \right\} \Theta^{2\beta-1} p_k. \hspace{1cm} (4.6)$$

If we choose $\epsilon_1 > 0$ small enough, we have the following inequality

$$\frac{1}{2} p_j \left\{ h^{jk} \Theta + \left( \frac{\epsilon}{2} - 2\epsilon_1 \right) r^{-\epsilon/2-1} \Theta' \ell^j \Theta - \beta r^{-\epsilon/2} \Theta' \ell^j \Theta \right\} \Theta^{2\beta-1} p_k \geq c_2 p_j \Theta^{2\beta} h^{jk} p_k. \hspace{1cm} (4.7)$$

Finally we can bound $-Q$ as

$$-Q \geq -C_5 f^{-1-\min\{2\rho, 2\epsilon', 2\tau\} + 2\delta} \Theta^{2\beta}$$

$$- 2 \text{Re} \left( r^{-\epsilon} f^{-1-\min\{2\rho, 2\epsilon', 2\tau\} + 2\delta} \Theta^{2\beta}(H - z) \right). \hspace{1cm} (4.8)$$

By (4.6), (4.7) and (4.8), if we set

$$\gamma = 2\epsilon_1^{-1} C_4 r^{-\epsilon} f^{-1-\min\{2\rho, 2\epsilon', 2\tau\} + 2\delta},$$

then the assertion follows. \hfill \square
4.2 Applications

Proof of Theorem 4.20. Let $I \subset \mathbb{R}$ be any relative compact open subset. For $\beta = 0$ the assertion is obvious by Theorem 1.2 and hence we may let $\beta \in (0, \beta_c)$. We take any $\delta \in (0, \min\{\rho, \epsilon', \tau\} - \beta)$. By Lemma 4.3, the Cauchy-Schwarz inequality and the Theorem 1.2 there exists $C_1 > 0$ such that for any state $\phi = R(z)\psi$ with $\psi \in r^{-\beta}\mathcal{B}$ and $z \in I_+$

$$
\|\Theta^{1/2}\Theta^{3-1/2}(A-a)\phi\|^2 + \langle p_j \Theta^{2\beta} h^k p_k \rangle _\phi \\
\leq C_1 \left[ \| \Theta^\beta (A-a)\phi\|_{\mathcal{B}^*} \| \Theta^\beta \psi\|_{\mathcal{B}} + \| f^{-1/2-\min\{\rho, \epsilon', \tau\}} + \delta \Theta^\beta \phi\|^2 \\
+ \| f^{1/2-\min\{\rho, \epsilon', \tau\}} + \delta \Theta^\beta \psi\|^2 \right]
$$

(4.9)

Here we note that $f^\beta (A-a)\phi \in \mathcal{B}^*$ for each $z \in I_+$ and hence the quantity on the right-hand side of (4.9) is finite. In fact, this can be verified by commuting $R(z)$ and powers of $f$ sufficiently many times and using the fact that $\psi \in f^{-\beta}\mathcal{B}$. Then by (4.9) it follows

$$
R_\nu^{2\beta} \| \Theta^{1/2}\Theta^{3-1/2}(A-a)\phi\|^2 + R_\nu^{2\beta} \langle p_j \Theta^{2\beta} h^k p_k \rangle _\phi \\
\leq C_2 \left[ \| f^\beta (A-a)\phi\|_{\mathcal{B}^*} \| f^\beta \psi\|_{\mathcal{B}} + \| f^\beta \psi\|^2_{\mathcal{B}} \right].
$$

(4.10)

In the first term on the right-hand side of (4.10) we take the supremum in $\nu \geq 0$ noting (3.2), and then obtain

$$
c_1 \| f^\beta (A-a)\phi\|^2_{\mathcal{B}^*} \leq C_2 \left[ \| f^\beta (A-a)\phi\|_{\mathcal{B}^*} \| f^\beta \psi\|_{\mathcal{B}} + \| f^\beta \psi\|^2_{\mathcal{B}} \right],
$$

which implies

$$
\| f^\beta (A-a)\phi\|^2_{\mathcal{B}^*} \leq C_3 \| f^\beta \psi\|^2_{\mathcal{B}}.
$$

(4.11)

As for the second term on the right-hand side of (4.10) we use (4.11), the concavity of $\Theta$ and Lebesgue’s monotone convergence theorem, and then obtain by letting $\nu \to \infty$

$$
\langle p_j f^{2\beta} h^k p_k \rangle _\phi \leq C_4 \| f^\beta \psi\|^2_{\mathcal{B}}.
$$

Hence we are done.

Proof of Corollary 4.21. Let $s > 1/2$ be as in the assertion. Throughout the proof let us fix any $\beta \in (0, \min\{\beta_c, s-1/2\})$ and $s' \in (s-\beta, s)$. We decompose for $m \geq 0$ and $z, z' \in I_+$

$$
R(z) - R(z') = \chi_m R(z) \chi_m - \chi_m R(z') \chi_m \\
+ (R(z) - \chi_m R(z) \chi_m) - (R(z') - \chi_m R(z') \chi_m).
$$

(4.12)
By Theorem 1.2 we can estimate the third term of (4.12) uniformly in \( n > m \) as

\[
\| R(z) - \chi_m R(z) \chi_m \|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \\
\leq \| f^{-s} \chi_m R(z) \chi_m f^{-s} \|_{\mathcal{B}(\mathcal{H})} + \| f^{-s} \chi_m R(z) \chi_m f^{-s} \|_{\mathcal{B}(\mathcal{H})} \\
+ \| f^{-s} \chi_m R(z) \chi_m f^{-s} \|_{\mathcal{B}(\mathcal{H})} \\
\leq C_1 R_m^{s}.
\]

Similarly, we obtain

\[
\| R(z') - \chi_m R(z') \chi_m \|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C_2 R_m^{s}.
\]

As for the first and second terms on the right-hand side of (4.12), using the equation (4.12), (4.13), (4.14) and (4.16), we obtain uniformly in \( n > m \)

\[
(4.17). \text{The Hölder continuity (1.10) for } \mathcal{H}_s, \mathcal{H}_{-s} \text{ follows immediately from (1.10). By Theorems 1.2 and 1.5, we have uniformly in } n > m \geq 0 \text{ and } z, z' \in I_+ \]

\[
\| \chi_m R(z) \chi_m - \chi_m R(z') \chi_m \|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C_3 R_m |z - z'| + C_4 R_m^{s}. \tag{4.16}
\]

By (4.12), (4.13), (4.14) and (4.16), we obtain uniformly in \( n > m \geq 0 \) and \( z, z' \in I_+ \)

\[
\| R(z) - R(z') \|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C_5 R_m^{s} + C_3 R_m |z - z'|.
\]

Now we choose \( n = m + 1 \) and \( R_m \leq |z - z'|^{1/(s' - 1)} \leq R_n \), and then obtain uniformly in \( z, z' \in I_+ \)

\[
\| R(z) - R(z') \|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C_6 |z - z'|^{\omega}
\]

with \( \omega = (s - s')/(s - s' + 1) \). The Hölder continuity (1.10) for \( R(z) \) follows from (4.17). The Hölder continuity (1.10) for \( r^{-s'/2} p R(z) \) follows by using (3.5).

The existence of the limits of (4.11) follows immediately from (1.10). By Theorems 1.2 and 1.5 we have \( R(\lambda \pm i 0) \) and \( r^{-s'/2} p R(\lambda \pm i 0) \) actually map into \( \mathcal{B}_s \), and moreover they extend continuously to maps \( \mathcal{B} \to \mathcal{B}_s \) by a density argument. Hence we are done. \( \square \)
Proof of Corollary 1.7. Note the elementary property
\[ \|\psi\|_{B^*} = \sup_{n \geq 0} \|\chi_n \psi\|_{B^*}; \quad \psi \in B^*. \]

Let \( \beta \in [0, \beta_c) \) be as in the assertion. By Theorem 1.5 there exists \( C > 0 \) such that for any \( \Gamma > 0 \) and \( n > 0 \)
\[ \|\chi_n f^\beta (A - a) R(\lambda + i\Gamma) \psi\|_{B^*} \leq C \|f^\beta \psi\|_B, \quad \psi \in C^\infty_0(\mathbb{R}). \]

By taking the limit \( \Gamma \to 0 \) and using Corollary 1.6 and a density argument, we obtain
\[ \|\chi_n f^\beta (A - a) R(\lambda + i0) \psi\|_{B^*} \leq C \|f^\beta \psi\|_B, \quad \psi \in f^{-\beta} B. \]

Finally, by the Lebesgue’s monotone convergence theorem we obtain
\[ \|f^\beta (A - a) R(\lambda + i0) \psi\|_{B^*} \leq C \|f^\beta \psi\|_B, \quad \psi \in f^{-\beta} B. \]

Similarly, we obtain
\[ \langle p_j f^{2\beta} k^j p_k \rangle^{1/2}_{R(\lambda + i0) \psi} \leq C \|f^\beta \psi\|_B. \]

Hence we are done. \( \square \)

Proof of Corollary 1.8. Let \( \lambda \in \mathbb{R} \), \( \phi \in \mathcal{H}_{\text{loc}} \) and \( \psi \in f^{-\beta} B \) with \( \beta \in [0, \beta_c) \). We first assume \( \phi = R(\lambda + i0) \psi \). Then (i) and (ii) of the corollary hold by Corollaries 1.6 and 1.7. Conversely, assume (i) and (ii) of the corollary, and let
\[ \phi' = \phi - R(\lambda + i0) \psi. \]

Then by Corollaries 1.6 and 1.7 it follows that \( \phi' \) satisfies (i) and (ii) of the corollary. In addition, we can verify \( \phi' \in B^*_0 \) by the virial-type argument. In fact noting the identity
\[ 2 \text{Im} \left( \chi_\nu (H - \lambda) \right) = (\text{Re} a) \chi'_\nu + \text{Re} (\chi'_\nu (A - a)), \]

cf. (1.6) and (4.15), we conclude that
\[ 0 \leq \langle (\text{Re} a) \chi'_\nu \rangle_{\phi'} = \text{Re} \langle \chi'_\nu (A - a) \rangle_{\phi'}. \quad (4.18) \]

Taking the limit \( \nu \to \infty \) and using \( \phi' \in f^{\beta} B^* \) and \( (A - a) \phi' \in f^{-\beta} B^*_0 \) in (4.18), we obtain \( \phi' \in B^*_0 \). By Theorem A.1 it follows that \( \phi' = 0 \). Hence we have \( \phi = R(\lambda + i0) \psi \). \( \square \)
A Rellich’s theorem

In the proofs of Theorem 1.2 and Corollary 1.8 the absence of $B_0^*$-eigenfunctions for $H$ plays a major role. This result was studied in [I]. However, the space $B_0^*$ employed in [I] is somewhat different from the one introduced in this paper. Due to this, we actually need a slightly relaxed version of Rellich’s theorem.

For comparison we set

$$B_r^* = \{ \psi \in H_{\text{loc}} | \| \psi \|_r < \infty \}, \quad \| \psi \|_r = \sup_{\nu \geq 0} R_{\nu}^{\epsilon/4-1/2} \| F(\nu \leq r \leq \nu + 1) \psi \|_H,$$

$$B_{r,0}^* = \{ \psi \in B_r^* | \lim_{\nu \to \infty} R_{\nu}^{\epsilon/4-1/2} \| F(\nu \leq r \leq \nu + 1) \psi \|_H = 0 \},$$

$$B_f^* = \{ \psi \in H_{\text{loc}} | \| \psi \|_f < \infty \}, \quad \| \psi \|_f = \sup_{\nu \geq 0} R_{\nu}^{-1/2} \| F(\nu \leq f \leq \nu + 1) \psi \|_H,$$

$$B_{f,0}^* = \{ \psi \in B_f^* | \lim_{\nu \to \infty} R_{\nu}^{-1/2} \| F(\nu \leq f \leq \nu + 1) \psi \|_H = 0 \}.$$

As we can see with ease, for $0 < \epsilon < 2$ the spaces $B_r^*$ and $B_f^*$ are the same, and for $\epsilon = 2$ the following inclusion relations hold:

$$B_r^* \subseteq B_f^*, \quad B_{r,0}^* \subseteq B_{f,0}^*.$$  \hspace{1cm} \text{(A.1)}

In [I] we constructed a $B_r^*$-eigenfunction. Hence by (A.1) a $B_f^*$-eigenfunction certainly exists. Although the absence of $B_{r,0}^*$-eigenfunctions was proved in [I], we use in this paper the absence of $B_{f,0}^*$-eigenfunctions as follows.

**Theorem A.1.** Suppose Condition 1.1 and let $\lambda \in \mathbb{R}$. If a function $\phi \in B_{f,0}^*$ satisfies that

$$(H - \lambda)\phi = 0,$$

in the distributional sense, then $\phi = 0$ in $\mathbb{R}^d$.

We note that for $\epsilon = 2$ we impose weaker assumption than [I], and by the second inclusion relation of (A.1) Theorem A.1 is stronger than [I, Theorem 1.2]. As with [I] we can prove Theorem A.1 using a commutator argument with some weight inside.

In the proof we need modification of weight function $\Theta$. In fact, we need to replace the cut-off function $\chi_{m,n}(r)$ as $\chi_{m,n}(f)$. Since the other points of the proof are almost the same, we omit the details.

**References**

[A] Agmon, S.: Lower bounds for solutions of Schrödinger equations. J. Analyse. Math. 23 (1970), 1-25.

[BCHM] Bony, J. F., Carles, R., Häfner, D., Michel, L.: Scattering theory for the Schrödinger equation with repulsive potential. J. Math. Pures Appl. 84 (2005) 509-579.
[FI] Finster, F., Isidro, J. M.: $L^p$-spectrum of the Schrödinger operator with inverted harmonic oscillator potential. J. Math. Phys. 58 (2017), no. 9, 092104, 9 pp.

[FH] Froese, R., Herbst, I.: Exponential bounds and absence of positive eigenvalues for $N$-body Schrödinger operators. Comm. Math. Phys. 87 (1982/83), no. 3, 429-447.

[FHH2O] Froese, R., Herbst, I., Hoffmann-Ostenhof, M., Hoffman-Ostenhof, T.: On the absence of positive eigenvalues for one-body Schrödinger operators. J. Analyse Math. 41 (1982), 272-284.

[Hö] Hörmander, L.: The Analysis of Linear Partial Differential Operators, vol. II, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1985.

[I] Itakura, K.: Rellich’s theorem for spherically symmetric repulsive Hamiltonians. preprint, 2017.

[IJ] Inoescu, A. D., Jerison, D.: On the absence of positive eigenvalues of Schrödinger operators with rough potentials. Geom. Funct. Anal. 13 (2003), no. 5, 1029-1081.

[IS] Ito, K., Skibsted, E.: Stationary scattering theory on manifolds, I. Preprint, 2016.

[Ishi] Ishida, A.: On inverse scattering problem for the Schrödinger equation with repulsive potentials. J. Math. Phys. 55 (2014), no. 8, 082101, 12 pp.

[Ishi2] Ishida, A.: The borderline of the short-range condition for the repulsive Hamiltonian. J. Math. Anal. Appl. 438 (2016), no. 1, 267-273.

[Iso] Isozaki, H.: A uniqueness theorem for the $N$-body Schrödinger equation and its applications. Spectral and scattering theory (Sanda, 1992), 63-84, Lecture Notes in Pure and Appl. Math., 161, Dekker, New York, 1994.

[J] Josef, M.: On the essential spectrum of two-dimensional Pauli operators with repulsive potentials. Ann. Henri Poincaré 17 (2016), no. 3, 733-755.

[L1] Lakaev, Sh. S.: Asymptotics of the eigenvalue of the discrete Schrödinger operator on a two-dimensional lattice with repulsive potential. Uzbek. Mat. Zh. 2015, no. 3, 54-64.

[L2] Lakaev, Sh. S.: Bound states of a two-particle Hamiltonian with a repulsive potential. Uzbek. Mat Zh. 2015, no. 2, 52-62.

[RS] Reed, M., Simon, B.: Methods of modern mathematical physics II and IV, New York: Academic Press 1975 and 1978.
[Ski] Skibsted, E.: Sommerfeld radiation condition at threshold. Comm. Partial Differential Equations. 38 (2013), 1601-1625.

[Sig] Sigal, I. M.: Stark Effect in Multielectron Systems: Non-Existence of Bound States. Commun. Math. Phys. 122 (1989), 1-22.