Reduction and a concentration-compactness principle for energy-Casimir functionals

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Abstract
Energy-Casimir functionals are a useful tool for the construction of steady states and the analysis of their nonlinear stability properties for a variety of conservative systems in mathematical physics. Recently, Y. Guo and the author employed them to construct stable steady states for the Vlasov-Poisson system in stellar dynamics, where the energy-Casimir functionals act on number density functions on phase space. In the present paper we construct natural, reduced functionals which act on mass densities on space and study compactness properties and the existence of minimizers in this context. This puts the techniques developed by Y. Guo and the author into a more general framework. We recover the concentration-compactness principle due to P. L. Lions [8] in a more specific setting and connect our stability analysis with the one of G. Wolansky [13].

1 Introduction
The purpose of the present paper is to investigate the compactness properties and existence of minimizers of certain functionals which appear naturally in the stability analysis of various systems in kinetic theory. Given a large ensemble of particles which interact by gravitational attraction we consider energy-Casimir functionals which are defined on the space of phase space
density functions, and certain reduced versions of these which are defined on the space of spatial density functions. This reduction procedure should put the techniques developed in [1, 2, 3, 4, 5, 10, 11] into a more general framework and make them applicable to problems outside kinetic theory. However, to be specific we start by recalling the Vlasov-Poisson system which describes the time evolution of a large ensemble of particles interacting by the gravitational field which they create collectively:

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f - \nabla U \cdot \nabla_v f &= 0, \\
\Delta U &= 4\pi \rho, \quad \lim_{|x| \to \infty} U(t, x) = 0, \\
\rho(t, x) &= \int f(t, x, v) dv.
\end{align*}
\]

Here the dynamic variable is the number density \( f = f(t, x, v) \) of the ensemble in phase space, \( x, v \in \mathbb{R}^3 \) denote position and velocity, \( \rho = \rho(t, x) \) is the spatial mass density induced by \( f \), and \( U = U(t, x) \) is the induced gravitational potential. It is straightforward to check that

\[
\iint Q(f(t, x, v)) dv dx + \frac{1}{2} \iint |v|^2 f(t, x, v) dv dx - \frac{1}{2} \iint \frac{\rho(t, x) \rho(t, y)}{|x - y|} dx dy
\]

is conserved along solutions for any suitable scalar function \( Q \). The first part, which is conserved by itself, is a so-called Casimir functional, the second is the kinetic and the third part the potential energy of the system. When viewed as a functional on phase space densities \( f = f(x, v) \geq 0 \) we denote this functional by \( \mathcal{H}_C \). It is fairly straightforward to see that any minimizer of this functional subject to the constraint

\[
\iint f(x, v) dv dx = M
\]

with prescribed total mass \( M > 0 \) is a steady state of the Vlasov-Poisson system. It is far less obvious that such minimizers exist and that they are nonlinearly stable. When analyzing the minimization problem

\[
\mathcal{H}_C(f_0) = \inf \left\{ \mathcal{H}_C(f) | f \geq 0, \iint f dv dx = M \right\}, \quad (1.1)
\]

one needs to make sure that one can pass to the limit in the (quadratic) potential energy along a minimizing sequence. Obviously, the potential energy
is not a functional of $f$ itself, but of the induced spatial density $\rho$, and the crucial question is how along a minimizing sequence the spatial density can or cannot split into parts or spread uniformly in space. This was analyzed in the context of the Vlasov-Poisson system and with various variations in [1, 2, 3, 4, 5, 10, 11].

In the present paper we want to bring out the basic mechanism more clearly and in a framework not restricted to kinetic theory. To this end we construct in the next section a reduced version $\mathcal{H}_r^C$ of the energy-Casimir functional $\mathcal{H}_C$, which will be defined on spatial densities $\rho$,

$$\mathcal{H}_r^C(\rho) = \int \Phi(\rho(x)) \, dx - \frac{1}{2} \int \int \frac{\rho(x) \rho(y)}{|x-y|} \, dx \, dy$$

with $\Phi$ a function determined by $Q$, which is convex if $Q$ is convex. Then we explore the relation between the variational problems

$$\mathcal{H}_r^C(\rho_0) = \inf \left\{ \mathcal{H}_r^C(\rho) \mid \rho \geq 0, \int \rho \, dx = M \right\}$$

(1.2)

and (1.1), in particular we will show how a minimizer of the reduced problem (1.2) induces a minimizer of (1.1). In the third section we reformulate the techniques developed for (1.1) in the framework of (1.2) and obtain the existence of a minimizer $\rho_0$ under appropriate conditions on $\Phi$. In particular, we prove the essential part of the concentration-compactness principle due to P.-L. Lions [8] by a more direct method based on scaling and splitting. In the last section we discuss the role of symmetries in the problem and point out some applications and extensions of our results. An example of a function $\Phi$ which satisfies all the necessary assumptions is $\Phi(\rho) = \rho^{1+1/n}$ with $0 < n < 3$. In this case the potential $U_0$ induced by a minimizer $\rho_0$ is a solution of the semilinear elliptic problem

$$\Delta U_0 = (E_0 - U_0)_+^n, \quad \lim_{|x| \to \infty} U_0(x) = 0$$

where $(\cdot)_+$ denotes the positive part and $E_0$ is some constant. This equation is sometimes referred to as the Emden-Fowler equation and appears naturally in the study of self-gravitating fluid balls. Throughout this paper we restrict ourselves to the case of space dimension 3; extending these techniques to other space dimensions by adjusting various exponents is easy.
2 Reduction of energy-Casimir functionals

For a measurable function \( f = f(x,v) \) we define
\[
\rho_f(x) := \int f(x,v) \, dv, \quad x \in \mathbb{R}^3,
\]
and
\[
U_f := -\rho_f * \frac{1}{|\cdot|}.
\]
Next we define
\[
E_{\text{kin}}(f) := \frac{1}{2} \int \int |v|^2 f(x,v) \, dv \, dx
\]
\[
E_{\text{pot}}(f) := -\frac{1}{8\pi} \int |\nabla U_f(x)|^2 \, dx = -\frac{1}{2} \int \int \frac{\rho_f(x)\rho_f(y)}{|x-y|} \, dx \, dy,
\]
\(-E_{\text{pot}}\) can equally well be viewed as a functional of \( \rho \) instead of \( f \), and
\[
\mathcal{H}_C(f) := C(f) + E_{\text{kin}}(f) + E_{\text{pot}}(f),
\]
where
\[
C(f) := \int \int Q(f(x,v)) \, dv \, dx
\]
and \( Q \) is a given function satisfying the following

**Assumption on \( Q \):**
\( Q \in C^1([0,\infty[) \) is strictly convex, \( Q(0) = Q'(0) = 0 \), and \( Q(f)/f \to \infty, f \to \infty \).
In particular, this implies that \( Q \geq 0 \) and \( Q':[0,\infty[ \to [0,\infty[ \) is one-to-one and onto.

We study the following variational problem: Minimize \( \mathcal{H}_C \) over the set
\[
\mathcal{F}_M := \left\{ f \in L^1_+(\mathbb{R}^6) \mid C(f) + E_{\text{kin}}(f) < \infty, \rho_f \in L^{6/5}(\mathbb{R}^3), \int \int f = M \right\}, \quad (2.1)
\]
where \( M > 0 \) is prescribed and \( L^1_+(\mathbb{R}^6) \) denotes the set of a.e. nonnegative functions in \( L^1(\mathbb{R}^6) \). Note that since \( \rho_f \in L^{6/5}(\mathbb{R}^3) \) the convolution defining \( U_f \) exists in \( L^6(\mathbb{R}^3) \) with \( \nabla U_f \in L^2(\mathbb{R}^3) \) according to the extended Young’s inequality, and the potential energy of \( \rho_f \) is finite.

In order to guarantee the existence of a minimizer we will require additional growth conditions on \( Q \) to be introduced later; at the moment \( E_{\text{pot}}(f) \)
could be minus infinity for \( f \in \mathcal{F}_M \). A typical example of a function for which there exists a minimizer is \( Q(f) = f^{1+1/k} \) with \( 0 < k < 3/2 \).

To obtain a reformulation in terms of spatial densities \( \rho \) which captures the essential properties of this variational problem we proceed as follows. For \( r \geq 0 \) we define

\[
\mathcal{G}_r := \left\{ g \in L^1_+([\mathbb{R}^3]) \mid \int \left( \frac{1}{2} |v|^2 g(v) + Q(g(v)) \right) dv < \infty, \int g(v) dv = r \right\}
\]

(2.2)

and

\[
\Phi(r) := \inf_{g \in \mathcal{G}_r} \int \left( \frac{1}{2} |v|^2 g(v) + Q(g(v)) \right) dv.
\]

(2.3)

In addition to the variational problem of minimizing \( \mathcal{H}_C \) over the set \( \mathcal{F}_M \) we consider the problem of minimizing the functional

\[
\mathcal{H}_C^r(\rho) := \int \Phi(\rho(x)) dx + E_{\text{pot}}(\rho)
\]

over the set

\[
\mathcal{F}_M^r := \left\{ \rho \in L^{6/5} \cap L^1_+([\mathbb{R}^3]) \mid \int \Phi(\rho(x)) dx < \infty, \int \rho(x) dx = M \right\}
\]

(2.5)

The relation between the minimizers of \( \mathcal{H}_C \) and \( \mathcal{H}_C^r \) is the main theme of this section, and a remark on how we passed from \( \mathcal{H}_C \) to \( \mathcal{H}_C^r \) can be found at the end of the section.

**Theorem 1**

(a) For every function \( f \in \mathcal{F}_M \),

\[
\mathcal{H}_C(f) \geq \mathcal{H}_C^r(\rho_f),
\]

and if \( f = f_0 \) is a minimizer of \( \mathcal{H}_C \) over \( \mathcal{F}_M \) then equality holds.

(b) Let \( \rho_0 \in \mathcal{F}_M^r \) be a minimizer of \( \mathcal{H}_C^r \) with induced potential \( U_0 \). Then there exists a Lagrange multiplier \( E_0 \in \mathbb{R} \) such that a. e.,

\[
\rho_0 = \begin{cases} (\Phi')^{-1}(E_0 - U_0) & , U_0 < E_0 \\ 0 & , U_0 \geq E_0. \end{cases}
\]

Denote by

\[
E = E(x,v) := \frac{1}{2} |v|^2 + U_0(x)
\]
the energy of a particle at position $x$ with velocity $v$, and define

$$f_0 := \begin{cases} (Q')^{-1}(E_0 - E), & E < E_0, \\ 0, & E \geq E_0. \end{cases}$$

Then $f_0 \in \mathcal{F}_M$ is a minimizer of $\mathcal{H}_C$.

(c) Now assume that $\mathcal{H}_C^r$ has at least one minimizer in $\mathcal{F}_M^r$. Then the following holds: If $f_0 \in \mathcal{F}_M$ is a minimizer of $\mathcal{H}_C$ then $\rho_0 := \rho_{f_0} \in \mathcal{F}_M^r$ is a minimizer of $\mathcal{H}_C^r$, this map is one-to-one and onto between the sets of minimizers of $\mathcal{H}_C$ in $\mathcal{F}_M$ and of $\mathcal{H}_C^r$ in $\mathcal{F}_M^r$ respectively, and is the inverse of the map $\rho_0 \mapsto f_0$ described in (b).

Remark. This theorem does not exclude the possibility that $\mathcal{H}_C$ has a minimizer but $\mathcal{H}_C^r$ has none. In the next section we show that under appropriate assumptions on $\Phi$ the reduced functional $\mathcal{H}_C^r$ does have a minimizer, and then the theorem guarantees that we recover all minimizers of $\mathcal{H}_C$ in $\mathcal{F}_M$ by “lifting” the ones of $\mathcal{H}_C^r$ as described in (b).

Before we prove this theorem, we investigate the relation between $Q$ and $\Phi$; for a function $h : \mathbb{R} \to ]-\infty, \infty]$ we denote by

$$h^*(\lambda) := \sup_{r \in \mathbb{R}} (\lambda r - h(r))$$

its Legendre transform. Some of the results of the lemma below will be relevant for the next section.

**Lemma 1** Let $Q$ be as specified above, let $\Phi$ be defined by (2.2), (2.3), and extend both functions by $+\infty$ to the interval $]-\infty, 0[$.

(a) For $\lambda \in \mathbb{R}$,

$$\Phi^*(\lambda) = \int Q^* \left( \lambda - \frac{1}{2} |v|^2 \right) dv,$$

and in particular, $Q^* (\lambda) = 0 = \Phi^*(\lambda)$ for $\lambda < 0$.

(b) $\Phi \in C^1([0, \infty[)$ is strictly convex, and $\Phi(0) = \Phi'(0) = 0$.

(c) Let $k > 0$ and $n = k + 3/2$. As in the rest of the paper, constants denoted by $C$ are positive, may depend on $Q$ or $M$, and may change from line to line (or within one line).
If $Q(f) = C f^{1+1/k}$, $f \geq 0$, then $\Phi(\rho) = C \rho^{1+1/n}$, $\rho \geq 0$.

(ii) If $Q(f) \geq C f^{1+1/k}$, $f \geq 0$ large, then $\Phi(\rho) \geq C \rho^{1+1/n}$, $\rho \geq 0$ large.

(iii) If $Q(f) \leq C f^{1+1/k}$, $f \geq 0$ small, then $\Phi(\rho) \leq C \rho^{1+1/n}$, $\rho \geq 0$ small.

If the restriction to large respectively small values of $f$ can be dropped, then the corresponding restriction for $\rho$ can be dropped as well.

**Proof.** By definition,

$$
\Phi^*(\lambda) = \sup_{r \geq 0} \left[ \lambda r - \inf_{g \in \mathcal{G}} \int \left( \frac{1}{2} |v|^2 g(v) + Q(g(v)) \right) dv \right]
$$

$$
= \sup_{r \geq 0} \sup_{g \in \mathcal{G}} \int \left[ \left( \lambda - \frac{1}{2} |v|^2 \right) g(v) - Q(g(v)) \right] dv
$$

$$
= \sup_{g \in L^1_+(\mathbb{R}^3)} \int \left[ \left( \lambda - \frac{1}{2} |v|^2 \right) g(v) - Q(g(v)) \right] dv
$$

$$
= \int \sup_{y \geq 0} \left[ \left( \lambda - \frac{1}{2} |v|^2 \right) y - Q(y) \right] dv = \int Q^* \left( \lambda - \frac{1}{2} |v|^2 \right) dv.
$$

As to the last-but-one equality, observe that both sides are obviously zero for $\lambda \leq 0$. If $\lambda > 0$ then for any $g \in L^1_+(\mathbb{R}^3)$,

$$
\int \left[ \left( \lambda - \frac{1}{2} |v|^2 \right) g(v) - Q(g(v)) \right] dv \leq \int \sup_{y \geq 0} \left[ \left( \lambda - \frac{1}{2} |v|^2 \right) y - Q(y) \right] dv.
$$

If $|v| \geq \sqrt{2\lambda}$ then $\sup_{y \geq 0} \cdots = 0$, and for $|v| < \sqrt{2\lambda}$ the supremum of the term in brackets is attained at $y = y_v := (Q')^{-1} \left( \lambda - \frac{1}{2} |v|^2 \right)$. Thus with

$$
g_0(v) := \begin{cases} y_v, & |v| < \sqrt{2\lambda}, \\ 0, & |v| \geq \sqrt{2\lambda}, \end{cases}
$$

we have

$$
\int \sup_{y \geq 0} \left[ \left( \lambda - \frac{1}{2} |v|^2 \right) y - Q(y) \right] dv = \int \left[ \left( \lambda - \frac{1}{2} |v|^2 \right) g_0(v) - Q(g_0(v)) \right] dv
$$

$$
\leq \sup_{g \in L^1_+(\mathbb{R}^3)} \int \left[ \left( \lambda - \frac{1}{2} |v|^2 \right) g(v) - Q(g(v)) \right] dv,
$$

and part (a) is established.
Since $Q$ is strictly convex and lower semi-continuous as a function on $\mathbb{R}$ with $\lim_{|f| \to \infty} Q(f)/|f| \to \infty$, $Q^* \in C^1(\mathbb{R})$, cf. [1] Prop. 2.4. Obviously, $Q^*(\lambda) = 0$ for $\lambda \leq 0$, in particular, $(Q^*)'(0) = 0$. Also, $(Q^*)'$ is strictly increasing on $[0, \infty]$ since $Q'$ is strictly increasing on $[0, \infty]$ with range $[0, \infty]$. Since for $|\lambda| < \lambda_0$ with $\lambda_0 > 0$ fixed the integral in the formula for $\Phi^*$ extends over a compact set we may differentiate under the integral sign to conclude that $\Phi^* \in C^1(\mathbb{R})$ with derivative strictly increasing on $[0, \infty]$. This in turn implies the assertion of part (b).

Finally, $Q(f) \geq C f^{1+1/k}$, $f \geq 0$ large, implies that $Q(f) \geq C f^{1+1/k} - C'$, $f \geq 0$. Thus

$$Q^*(\lambda) \leq C' + \sup_{f \geq 0} \left( f \lambda - C f^{1+1/k} \right) = C' + \frac{1}{1+k} \left( \frac{k}{CQ(1+k)} \right)^k \lambda^{1+k}, \lambda \geq 0,$$

and

$$\Phi^*(\lambda) \leq C \lambda^{3/2} + C \int_{|v| \leq \sqrt{2\lambda}} \left( \lambda - \frac{1}{2} |v|^2 \right)^{1+k} dv = C \lambda^{3/2} + C \int_0^\lambda E^{1+k} \sqrt{\lambda - E} dE = C' + C \lambda^{k+5/2} = C' + C \lambda^{1+n}, \lambda \geq 0.$$

This in turn yields the assertion on $\Phi$ in (ii). The assertion in (i) is now obvious. As to (iii) note first that for $\lambda \geq 0$ and small the corresponding supremum is attained at small $f$'s, and thus

$$Q^*(\lambda) \leq \sup_{f \geq 0} \left( \lambda f - C f^{1+1/k} \right) = C \lambda^{1+k}.$$

Thus still for $\lambda \geq 0$ small, $\Phi^*(\lambda) \geq C \lambda^{1+n}$, which in turn implies the assertion for $\Phi$. \qed

We now prove the theorem above.

**Proof of Theorem [1].**

**Proof of the inequality in part (a).** For $\rho \in \mathcal{F}_M^r$ define

$$\mathcal{F}_{\rho} := \{ f \in \mathcal{F}_M | \rho_f = \rho \}. \quad (2.6)$$

Clearly, for $\rho = \rho_f$ with $f \in \mathcal{F}_M$,

$$C(f) + E_{\text{kin}}(f) \geq \inf_{\tilde{f} \in \mathcal{F}_\rho} (C(\tilde{f}) + E_{\text{kin}}(\tilde{f}))$$

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\[\geq \inf_{f \in F} \int \left[ \inf_{g \in \mathcal{G}_{\rho(x)}} \int \left( \frac{1}{2} |v|^2 g(v) + Q(g(v)) \right) \, dv \right] \, dx \]
\[= \int \left[ \inf_{g \in \mathcal{G}_{\rho(x)}} \int \left( \frac{1}{2} |v|^2 g(v) + Q(g(v)) \right) \, dv \right] \, dx \]
\[= \int \Phi(\rho(x)) \, dx, \quad (2.7)\]

and the inequality in part (a) is established.

*An intermediate assertion.* We claim that if \( f \in F \) is such that up to sets of measure zero,
\[Q'(f) = E_0 - E > 0, \text{ where } f > 0, \]
\[E_0 - E \leq 0, \text{ where } f = 0. \quad (2.8)\]

with \( E \) defined as in (b) but with \( U_f \) instead of \( U_0 \) and \( E_0 \) a constant then equality holds in part (a).

To prove this, observe that since \( Q \) is convex, we have for a.e. \( x \in \mathbb{R}^3 \) and every \( g \in \mathcal{G}_{\rho_f(x)} \),
\[\frac{1}{2} |v|^2 g(v) + Q(g(v)) \geq \frac{1}{2} |v|^2 f(x,v) + Q(f(x,v)) \]
\[+ \left( \frac{1}{2} |v|^2 + Q'(f(x,v)) \right) (g(v) - f(x,v)) \text{ a.e.} \]

Now by (2.8),
\[\int \left( \frac{1}{2} |v|^2 + Q'(f) \right) (g - f) \, dv = \int_{\{f>0\}} \ldots + \int_{\{f=0\}} \ldots \]
\[= (E_0 - U_f(x)) \int_{\{f>0\}} (g - f) \, dv + \int_{\{f=0\}} \frac{1}{2} |v|^2 g \, dv \]
\[= -(E_0 - U_f(x)) \int_{\{f=0\}} (g - f) \, dv + \int_{\{f=0\}} \frac{1}{2} |v|^2 g \, dv \]
\[= \int_{\{f=0\}} (E - E_0) g \, dv \geq 0; \]

observe that \( g \geq 0 \) and \( \int g \, dv = \int f \, dv \) so \( \int (g - f) \, dv = 0 \). Thus we see that
\[\Phi(\rho_f(x)) \geq \int \left( \frac{1}{2} |v|^2 f + Q(f) \right) \, dv \]
\[\geq \inf_{g \in \mathcal{G}_{\rho_f(x)}} \int \left( \frac{1}{2} |v|^2 g + Q(g) \right) \, dv = \Phi(\rho_f(x)) \text{ a.e.} \]
and the proof of our intermediate assertion is complete.

Proof of the equality assertion in (a). If \( f_0 \in \mathcal{F}_M \) is a minimizer of \( \mathcal{H}_C \) then the Euler-Lagrange equation of the minimization problem implies that \( (2.8) \) holds for some Lagrange multiplier \( E_0 \); this can be proved as in [5, Thm. 2]. Thus equality holds in (a) by the intermediate assertion, and the proof of part (a) is complete.

Proof of part (b). Let \( \rho_0 \in \mathcal{F}_M^r \) be a minimizer of \( \mathcal{H}_C \). Then the Euler-Lagrange equation yields the relation between \( \rho_0 \) and \( U_0 \). Let \( f_0 \) be defined as in (b). Then up to sets of measure zero,

\[
\int f_0(x,v) \, dv = \int_{|v| \leq \sqrt{2(E_0-U_0(x))}} (Q')^{-1} \left( E_0 - U_0(x) - \frac{1}{2} |v|^2 \right) \, dv
\]

\[
= (\Phi')^{-1}(E_0-U_0(x)) = (\Phi')^{-1}(E_0-U_0(x)) = \rho_0(x)
\]

where \( U_0(x) < E_0 \), and both sides are zero where \( U_0(x) \geq E_0 \). Thus \( \rho_0 = \rho_{f_0} \), in particular, \( f_0 \in \mathcal{F}_M \). By definition, \( f_0 \) satisfies the Euler-Lagrange relation \( (2.8) \) and thus by our intermediate assertion \( \mathcal{H}_C(f_0) = \mathcal{H}_C^r(\rho_0) \). Therefore again by part (a),

\[
\mathcal{H}_C(f) \geq \mathcal{H}_C^r(\rho_f) \geq \mathcal{H}_C^r(\rho_0) = \mathcal{H}_C(f_0), \quad f \in \mathcal{F}_M,
\]

so that \( f_0 \) is a minimizer of \( \mathcal{H}_C \), and the proof of part (b) is complete.

Proof of part (c). Assume that \( \mathcal{H}_C^r \) has a minimizer \( \rho_0 \in \mathcal{F}_M^r \) and define \( f_0 \) as above. Then part (a), the fact that each \( \rho \in \mathcal{F}_M^r \) can be written as \( \rho = \rho_f \) for some \( f \in \mathcal{F}_M \), and our intermediate assertion imply that

\[
\inf_{f \in \mathcal{F}_M} \mathcal{H}_C(f) \geq \inf_{f \in \mathcal{F}_M} \mathcal{H}_C^r(\rho_f) = \inf_{\rho \in \mathcal{F}_M^r} \mathcal{H}_C^r(\rho) = \mathcal{H}_C^r(\rho_0) = \mathcal{H}_C(f_0) \geq \inf_{f \in \mathcal{F}_M} \mathcal{H}_C(f).
\]

Now take any minimizer \( g_0 \in \mathcal{F}_M \) of \( \mathcal{H}_C \). Then by \( (2.8) \) and part (a),

\[
\inf_{\rho \in \mathcal{F}_M^r} \mathcal{H}_C^r(\rho) = \inf_{f \in \mathcal{F}_M} \mathcal{H}_C(f) = \mathcal{H}_C(g_0) = \mathcal{H}_C^r(\rho_{g_0}),
\]

that is, \( \rho_{g_0} \in \mathcal{F}_M^r \) minimizes \( \mathcal{H}_C^r \), and the proof of part (c) is complete. \qed

Remark. If we define an intermediate functional

\[
\mathcal{P}(\rho) := \inf_{f \in \mathcal{F}_C} \int \left( \frac{1}{2} |v|^2 f(x,v) + Q(f(x,v)) \right) \, dv \, dx
\]

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with $F_\rho$ as defined in (2.6) then (2.7) shows that

\[ H_C(f) \geq P(\rho_f) + E_{pot}(\rho_f) \geq \int \Phi(\rho_f(x)) \, dx + E_{pot}(\rho_f) = H^*_C(\rho_f) \]

with equality for minimizers. Note that $P(\rho)$ is obtained by minimizing the positive contribution to $H_C$, which also happens to be the part depending on phase space densities $f$ directly, over all $f$'s which generate a given spatial density $\rho$. Then in a second step one minimizes for each point $x$ over all functions $g = g(v)$ which have as integral the value $\rho(x)$.

These constructions are borrowed from [13] where they appear for the special case $Q(f) = f^{1+1/k}$. In [13] the resulting functional of $\rho$ is investigated under the assumption of spherical symmetry by rewriting it as a functional of $m_\rho(r) := 4\pi \int_0^r s^2 \rho(s) \, ds$ where $r := |x|$. While minimizers of the present variational problems are spherically symmetric a posteriori, the a priori restriction to spherical symmetry implies that any stability result derived from their minimizing property is restricted to spherically symmetric perturbations, which is undesirable. Moreover, in the last section we will comment on some extensions of the present techniques to situations where the minimizers are not spherically symmetric.

### 3 Concentration-compactness principle and existence of minimizers

In this section we prove a concentration-compactness principle that will yield a solution to the following variational problem: Minimize the functional

\[ H^*_C(\rho) := \int \Phi(\rho(x)) \, dx + E_{pot}(\rho) \]

over the set

\[ F^*_M := \left\{ \rho \in L^1_+ (\mathbb{R}^3) \mid \int \Phi(\rho) < \infty, \int \rho = M \right\} \]

for $M > 0$ given and $\Phi$ satisfying the following

**Assumptions on $\Phi$:** $\Phi \in C^1([0, \infty])$, $\Phi(0) = 0 = \Phi'(0)$, and

- $(\Phi 1)$ $\Phi$ is strictly convex.
- $(\Phi 2)$ $\Phi(\rho) \geq C\rho^{1+1/n}$, $\rho \geq 0$ large, with $0 < n < 3$,
Note that Lemma 1 tells us that the function $\Phi$ which we constructed from a given $Q$ in Section 2 has these properties, provided $Q$ satisfies the growth conditions corresponding to (Φ2) and (Φ3). The aim of this section is to prove the following result:

**Theorem 2** The functional $H^r_C$ is bounded from below on $F^r_M$. Let $(\rho_i) \subset F^r_M$ be a minimizing sequence of $H^r_C$. Then there exists a sequence of shift vectors $(a_i) \subset \mathbb{R}^3$ and a subsequence, again denoted by $(\rho_i)$, such that for any $\epsilon > 0$ there exists $R > 0$ with

$$
\int_{a_i + B_R} \rho_i(x) \, dx \geq M - \epsilon, \ i \in \mathbb{N},
$$

and

$$
T \rho_i := \rho_i(\cdot + a_i) \rightharpoonup \rho_0 \text{ weakly in } L^{1+1/n}(\mathbb{R}^3), \ i \to \infty,
$$

and

$$
\int_{B_R} \rho_0 \geq M - \epsilon.
$$

Finally,

$$
\nabla U_{T \rho_i} \to \nabla U_0 \text{ strongly in } L^2(\mathbb{R}^3), \ i \to \infty,
$$

and $\rho_0 \in F^r_M$ is a minimizer of $H^r_C$.

Here and in the following we denote for $0 < R < S \leq \infty$,

$$
B_R := \{ x \in \mathbb{R}^3 | |x| \leq R \},
\quad
B_{R,S} := \{ x \in \mathbb{R}^3 | R \leq |x| < S \}.
$$

We split our argument into a series of lemmas. The first thing to note is that $H^r_C$ is bounded from below on $F^r_M$:

**Lemma 2** Under the above assumptions on $\Phi$,

$$
H^r_C(\rho) \geq \int \Phi(\rho) \, dx - C - C \left( \int \Phi(\rho) \, dx \right)^{n/3}, \ \rho \in F^r_M,
$$

in particular,

$$
h^r_M := \inf_{F^r_M} H^r_C > -\infty.
$$
Proof. By the extended Young’s inequality, interpolation, and assumption (Φ2),
\[-E_{\text{pot}}(\rho) \leq C\|\rho\|_{6/5}^2 \leq C\|\rho\|_{1}^{(5-n)/3}\|\rho\|_{1+1/n}^{(n+1)/3}\]
\[\leq C + C\left(\int \Phi(\rho) \, dx\right)^{n/3}, \rho \in F_{M}^{r}.\]

Since \(n < 3\), \(\mathcal{H}_{C}^{r}\) is bounded from below on \(F_{M}^{r}\). □

Corollary 1 Any minimizing sequence for \(\mathcal{H}_{C}^{r}\) in \(F_{M}^{r}\) is bounded in \(L^{1+1/n}(\mathbb{R}^3)\) and therefore has a subsequence which converges weakly in \(L^{1+1/n}(\mathbb{R}^3)\).

Proof. By Lemma 2, \(\int \Phi(\rho)\) is bounded along any minimizing sequence. The assertion follows by (Φ2) and the fact that \(\int \rho = M\) for \(\rho \in F_{M}^{r}\). □

Note that the estimates above show that the definition (3.1) coincides with our earlier definition for the set \(F_{M}^{r}\). We also see that the assumption (Φ2) is quite natural. Next we prove a splitting estimate which will show that along a minimizing sequence the mass cannot vanish:

Lemma 3 Let \(\rho \in F_{M}^{r}\). Then
\[\sup_{a \in \mathbb{R}^3} \int_{a+B_R} \rho(x) \, dx \geq \frac{1}{RM} \left( -2E_{\text{pot}}(\rho) - \frac{M^2}{R} - \frac{C\|\rho\|_{1+1/n}}{R^{(5-n)/(n+1)}} \right), \, R > 1.\]

Proof. We split the potential energy as follows:
\[-2E_{\text{pot}}(\rho) = \int_{[x-y] \leq 1/R} \rho(x) \rho(y) \, dx \, dy + \int_{1/R < |x-y| < R} \ldots + \int_{R \geq |x-y|} \ldots =: I_1 + I_2 + I_3.\]

By Hölder’s inequality and Young’s inequality,
\[I_1 \leq \|\rho\|_{1+1/n} \|\rho \ast (1_{B_{|1/R}^1} / |\cdot|)\|_{n+1} \leq \|\rho\|_{1+1/n}^2 \|1_{B_{|1/R}^1} / |\cdot|\|_{(n+1)/2} \leq C\|\rho\|_{1+1/n}^2 R^{-(5-n)/(n+1)};\]

here \(1_S\) denotes the indicator function of the set \(S \subset \mathbb{R}^3\). The estimates for \(I_2\) and \(I_3\) are straightforward:
\[I_2 \leq R \int_{|x-y| \leq R} \rho(x) \rho(y) \, dx \, dy \leq M R \sup_{a \in \mathbb{R}^3} \int_{a+B_R} \rho(x) \, dx,\]
\[I_3 \leq \ldots\]
and 
\[ I_3 \leq R^{-1}M^2. \]

Putting these estimates together yields the assertion. \qed

Note that to obtain this estimate we actually split the Green’s function \(1/|x|\). To exploit this estimate along minimizing sequences we need to know that \(h_M^r < \infty\). It is here that we need the assumption (Φ3):

**Lemma 4**

(a) For every \(M > 0\) we have \(h_M^r < 0\).

(b) For every \(0 < \bar{M} \leq M\) we have \(h_M^r \geq (\bar{M}/M)^{5/3}h_M^r\).

**Proof.** For \(\rho \in \mathcal{F}_M^r\) and \(a,b > 0\) we define \(\bar{\rho}(x) := a\rho(bx)\). Then
\[
    \int \bar{\rho} dx = ab^{-3} \int \rho dx,
    
    E_{\text{pot}}(\bar{\rho}) = a^2 b^{-5} E_{\text{pot}}(\rho),
    
    \int \Phi(\bar{\rho}) = b^{-3} \int \Phi(a\rho) dx.
\]

To prove part (a) we fix a bounded and compactly supported function \(\rho \in \mathcal{F}_M^r\) and choose \(a = b^3\) so that \(\bar{\rho} \in \mathcal{F}_M^r\) as well. By (Φ3) and since \(3/n' > 1\),
\[
    \mathcal{H}_C^r(\bar{\rho}) = b^{-3} \int \Phi(b^3 \rho) dx + bE_{\text{pot}}(\rho) \leq C b^{3/n'} + bE_{\text{pot}}(\rho) < 0, \quad b \to 0,
\]

and part (a) is established. As to part (b), we take \(a = 1\) and \(b = (M/\bar{M})^{1/3} \geq 1\). For \(\rho \in \mathcal{F}_M^r\) and \(\bar{\rho} \in \mathcal{F}_{\bar{M}}^r\) rescaled with these parameters we find that
\[
    \mathcal{H}_C^r(\bar{\rho}) = b^{-3} \int \Phi(\rho) dx + b^{-5} E_{\text{pot}}(\rho)
    
    \geq b^{-5} \left( \int \Phi(\rho) dx + E_{\text{pot}}(\rho) \right) = \left( \frac{\bar{M}}{M} \right)^{5/3} \mathcal{H}_C^r(\rho).
\]

Since for the present choice of \(a\) and \(b\) the map \(\rho \mapsto \bar{\rho}\) is one-to-one and onto between \(\mathcal{F}_M^r\) and \(\mathcal{F}_{\bar{M}}^r\) this estimate proves part (b). \qed

**Corollary 2** Let \((\rho_i) \subset \mathcal{F}_M^r\) be a minimizing sequence of \(\mathcal{H}_C^r\). Then there exist \(\delta_0 > 0\), \(R_0 > 0\), \(i_0 \in \mathbb{N}\), and a sequence of shift vectors \((a_i) \subset \mathbb{R}^3\) such that
\[
    \int_{a_i + BR} \rho_i(x) dx \geq \delta_0, \quad i \geq i_0, \quad R \geq R_0.
\]
Proof. By Corollary 1, \( \| \rho_i \|_{1+1/n} \) is bounded. By Lemma 4 (a) we have

\[ E_{pot}(\rho_i) \leq \mathcal{H}^r_c(\rho_i) \leq \frac{1}{2} h^* M < 0, \ i \geq i_0, \]

for a suitable \( i_0 \in \mathbb{N} \). Thus by Lemma 3 there exist \( \delta_0 > 0, \ R_0 > 0 \), and a sequence of shift vectors \( (a_i) \subset \mathbb{R}^3 \) as required. \( \square \)

Finally, we will also need to exploit the well known compactness properties of the solution operator of the Poisson equation:

Lemma 5 Let \( (\rho_i) \subset L^{1+1/n}(\mathbb{R}^3) \) be bounded and

\[ \rho_i \rightharpoonup \rho_0 \text{ weakly in } L^{1+1/n}(\mathbb{R}^3). \]

(a) For any \( R > 0 \),

\[ \nabla U_{1_B R \rho_i} \to \nabla U_{1_B R \rho_0} \text{ strongly in } L^2(\mathbb{R}^3). \]

(b) If in addition \( (\rho_i) \) is bounded in \( L^1(\mathbb{R}^3) \), \( \rho_0 \in L^1(\mathbb{R}^3) \), and for any \( \epsilon > 0 \) there exists \( R > 0 \) and \( i_0 \in \mathbb{N} \) such that

\[ \int_{|x| \geq R} |\rho_i(x)| \, dx < \epsilon, \ i \geq i_0 \]

then

\[ \nabla U_{\rho_i} \to \nabla U_{\rho_0} \text{ strongly in } L^2(\mathbb{R}^3). \]

Proof. As to part (a), take any \( R' > R \). Since \( 1+1/n > 4/3 > 6/5 \), the mapping

\[ L^{1+1/n}(\mathbb{R}^3) \ni \rho \mapsto 1_{B_{R'}} \nabla U_{1_{B_{R'}} \rho} \in L^2(B_{R'}) \]

is compact. Thus the asserted strong convergence holds on \( B_{R'} \). On the other hand,

\[ \int_{|x| \geq R'} |\nabla U_{1_{B_{R'} \rho_i}}|^2 \, dx \leq \frac{C}{R' - R} \| 1_{B_{R} \rho_i} \|_1^2 \leq \frac{C}{R' - R}, \ i \in \mathbb{N} \cup \{0\}, \]

which is arbitrarily small for \( R' \) large. As to part (b), we have for any \( R > 0 \),

\[ \| \nabla U_{\rho_i} - \nabla U_{\rho_0} \|_2 \leq \| \nabla U_{1_{B_{R} \rho_i}} - \nabla U_{1_{B_{R} \rho_0}} \|_2 + \| \nabla U_{1_{B_{R', \infty} \rho_i}} - \nabla U_{1_{B_{R', \infty} \rho_0}} \|_2. \]
Using the extended Young’s inequality, interpolation, and the boundedness of the sequence in $L^{1+1/n}(\mathbb{R}^3)$ we find that
\[
\|\nabla U_{1_{B_{R,\infty}}\rho_i} - \nabla U_{1_{B_{R,\infty}}\rho_0}\|_2 \leq C \left( \|\nabla U_{1_{B_{R,\infty}}\rho_i}\|_{6/5} + \|\nabla U_{1_{B_{R,\infty}}\rho_0}\|_{6/5} \right) \\
\leq C \left( \|\nabla U_{1_{B_{R,\infty}}\rho_i}\|_{(5-n)/6} + \|\nabla U_{1_{B_{R,\infty}}\rho_0}\|_{(5-n)/6} \right).
\]
Given $\epsilon > 0$ we now choose $R > 0$ and $i_0 \in \mathbb{N}$ such that this is less than $\epsilon > 0$ for $i \geq i_0$, and recalling (a) completes the proof. \hfill \Box

We are now ready to prove the main result of this section:

**Proof of Theorem 2.**

We split $\rho \in F_{2}$ into three different parts:
\[
\rho = 1_{B_{R_1}} \rho + 1_{B_{R_1,R_2}} \rho + 1_{B_{R_2,\infty}} \rho =: \rho_1 + \rho_2 + \rho_3;
\]
the parameters $R_1 < R_2$ of the split are yet to be determined. With
\[
I_{lm} := \int \int \frac{\rho_l(x) \rho_m(y)}{|x-y|}, \quad l,m = 1,2,3;
\]
we have
\[
\mathcal{H}_C^r(\rho) = \mathcal{H}_C^r(\rho_1) + \mathcal{H}_C^r(\rho_2) + \mathcal{H}_C^r(\rho_3) - I_{12} - I_{13} - I_{23}.
\]
If we choose $R_2 > 2R_1$ then
\[
I_{13} \leq \frac{C}{R_2}.
\]
Next, we use the Cauchy-Schwarz inequality, the extended Young’s inequality, and interpolation to get
\[
I_{12} + I_{23} = \frac{1}{4\pi} \left| \int \nabla (U_1 + U_3) \cdot \nabla U_2 dx \right| \leq C \|\rho_1 + \rho_3\|_{6/5} \|\nabla U_2\|_2 \\
\leq C \|\rho\|_{(n+1)/6} \|\nabla U_2\|_2.
\]
Using the estimates above and Lemma [b] (b) we find with $M_l = \int \rho_l$, $l = 1,2,3,$
\[
h_M^r - \mathcal{H}_C^r(\rho) \leq \left( 1 - \left( \frac{M_1}{M} \right)^{5/3} - \left( \frac{M_2}{M} \right)^{5/3} - \left( \frac{M_3}{M} \right)^{5/3} \right) h_M^r \\
+ C \left( R_2^{-1} + \|\rho\|_{(n+1)/6} \|\nabla U_2\|_2 \right) \\
\leq \frac{C}{M^2} (M_1 M_2 + M_1 M_3 + M_2 M_3) h_M^r \\
+ C \left( R_2^{-1} + \|\rho\|_{(n+1)/6} \|\nabla U_2\|_2 \right) \\
\leq C h_M^r M_1 M_3 + C \left( R_2^{-1} + \|\rho\|_{(n+1)/6} \|\nabla U_2\|_2 \right); \quad (3.2)
\]

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observe that by Lemma 4 (a) \( h_M^r < 0 \) and that constants denoted by \( C \) are positive and depend on \( M \) and \( \Phi \), but not on \( R_1 \) or \( R_2 \). We want to use (1.2) to show that up to a subsequence and a shift \( M_3 \) becomes small along any minimizing sequence for \( i \) large provided the splitting parameters are suitably chosen.

The sequence \( T \rho_i := \rho_i(\cdot + a_i), \ i \in \mathbb{N} \), is minimizing and bounded in \( L^{1+1/n}(\mathbb{R}^3) \) so there exists a subsequence, denoted by \( (T \rho_i) \) again, such that \( T \rho_i \rightharpoonup \rho_0 \) weakly in \( L^{1+1/n}(\mathbb{R}^3) \), cf. Corollary [1]. Now choose \( R_0 < R_1 \) so that by Corollary [2], \( M_{i,1} \geq \delta_0 \) for \( i \) large. By (3.2),

\[
-C h_M^r \delta_0 M_{i,3} \leq \frac{C}{R_2} + C \| \nabla U_{0,2} \|_2 + C \| \nabla U_{i,2} - \nabla U_{0,2} \|_2 + \mathcal{H}_C^r(T \rho_i) - h_M^r
\]

(3.3)

where \( U_{i,l} \) is the potential induced by \( \rho_{i,l} \) which in turn has mass \( M_{i,l}, \ i \in \mathbb{N} \cup \{0\} \), and the index \( l = 1, 2, 3 \) refers to the splitting. Given any \( \epsilon > 0 \) we increase \( R_1 > R_0 \) such that the second term on the right hand side of (3.3) is small, say less than \( \epsilon/4 \). Next choose \( R_2 > 2R_1 \) such that the first term is small. Now that \( R_1 \) and \( R_2 \) are fixed, the third term in (3.3) converges to zero by Lemma [3] (a). Since \( (T \rho_i) \) is minimizing the remainder in (3.3) follows suit. Therefore, for \( i \) sufficiently large,

\[
\int_{a_i + B_{R_2}} T \rho_i = M - M_{i,3} \geq M - (-C h_M^r \delta_0)^{-1} \epsilon.
\]

(3.4)

Clearly, \( \rho_0 \geq 0 \) a.e.. By weak convergence we have that for any \( \epsilon > 0 \) there exists \( R > 0 \) such that

\[
M \geq \int_{B_R} \rho_0 \, dx \geq M - \epsilon
\]

which in particular implies that \( \rho_0 \in L^1(\mathbb{R}^3) \) with \( \int \rho_0 \, dx = M \). The functional \( \rho \mapsto \int \Phi(\rho) \, dx \) is convex, so by Mazur’s Lemma and Fatou’s Lemma

\[
\int \Phi(\rho_0) \, dx \leq \limsup_{i \to \infty} \int \Phi(T \rho_i) \, dx.
\]

The strong convergence of the gravitational fields now follows by Lemma 3 (b), and in particular,

\[
\mathcal{H}_C^r(\rho_0) \leq \limsup_{i \to \infty} \mathcal{H}_C^r(\rho_i) = h_M^r
\]

so that \( \rho_0 \) is a minimizer of \( \mathcal{H}_C^r \). \( \square \)
4 Applications, symmetries, extensions

Although the main purpose of the present paper is to get a more general understanding of the techniques developed in [1, 2, 3, 4, 5, 10, 11] we want to at least indicate some possible applications of these techniques. First we should mention that [5] differs from the other papers in so far as the Casimir functional is used as part of the constraint under which the total energy is minimized. This made it possible to relax the growth conditions on $Q -- 0 < k \leq 7/2$ is covered in [5], but since in the reduction process we turn $C(f) + E_{\text{kin}}(f)$ into a new functional of $\rho$, [5] seems to be outside the present framework.

We start with the observation, already noted in Theorem 1, that if $\rho_0 \in F_M$ is a minimizer of $H_C$ with induced potential $U_0$ then

$$\rho_0 = (\Phi')^{-1}(E_0 - U_0) := \begin{cases} (\Phi')^{-1}(E_0 - U_0), & U_0 < E_0 \\ 0, & U_0 \geq E_0, \end{cases}$$

and thus

$$\triangle U_0 = 4\pi (\Phi')^{-1}(E_0 - U_0)$$

(4.1)
on $\mathbb{R}^3$. The corresponding minimizer of $H_C$,

$$f_0 = \begin{cases} (Q')^{-1}(E_0 - E), & E < E_0 \\ 0, & E \geq E_0, \end{cases}$$

is a steady state of the Vlasov-Poisson system, since $E = E(x,v) = \frac{1}{2}|v|^2 + U_0(x)$ is a conserved quantity for the characteristics of the Vlasov equation with potential $U_0$ induced by $\rho_0 = \rho_{f_0}$. Steady states obtained in this manner have finite mass $M$, which is a necessary property for physically relevant steady states. We remark that the ansatz $f_0 = \phi(E_0 - E)$ reduces the stationary Vlasov-Poisson system to the semilinear Poisson equation

$$\triangle U_0 = 4\pi \int \phi \left( E_0 - \frac{1}{2}|v|^2 - U_0 \right) dv$$

which is exactly (4.1), provided $Q$ can be chosen such that $(Q')^{-1} = \phi$ on $\mathbb{R}_+$ and $\phi = 0$ on $\mathbb{R}_-$. As far as the existence of steady states is concerned our “reduced” approach allows us to cover $f_0 = (E_0 - E)^k_+$ with $-1 < k < 3/2$ which leads to (4.1) with right hand side $C(E_0 - U_0)_n^k$ with $n = k + 3/2$ in
the permissible range $|0, 3|$; note that the lower bound $k > -1$ is necessary to make the $v$-integral above converge. With the direct approach working with $\mathcal{H}_C$ we were restricted to $0 < k < 3/2$.

The main feature of steady states obtained as minimizers in this manner is that their nonlinear stability. Since this is the main point in the investigations cited above, we do not go into this here. Instead, we briefly look at the role of symmetries in our problem. First we note that for any $\rho_0 \in \mathcal{F}^r_M$ its spherically symmetric decreasing rearrangement, denoted by $\rho_0^*$, also lies in $\mathcal{F}^r_M$ and satisfies

$$\int \Phi(\rho_0) = \int \Phi(\rho_0^*), \quad E_{\text{pot}}(\rho_0) \geq E_{\text{pot}}(\rho_0^*)$$

with equality if and only if $\rho_0 = \rho_0^*(\cdot - x^*)$ for some $x^* \in \mathbb{R}^3$, cf. [7, Thms. 3.7, 3.9]. In particular, any minimizer of $\mathcal{H}_C^r$ must be spherically symmetric with respect to some point in $\mathbb{R}^3$. If we are only interested in solving (4.1) or the stationary Vlasov-Poisson system we therefore loose nothing if we restrict ourselves to the set of spherically symmetric functions in $\mathcal{F}^r_M$. The crucial part of the concentration-compactness argument simplifies considerably under this restriction:

**Lemma 6** Define

$$R_0 = -\frac{3M^2}{5h_M^r} > 0.$$ 

Let $\rho \in \mathcal{F}^r_M$ be spherically symmetric, $R > 0$, and

$$m := \int_{\{|x| \geq R\}} \rho.$$

Then the following estimate holds:

$$\mathcal{H}_C^r(\rho) \geq h_M^r + \left[ \frac{1}{R_0} - \frac{1}{R} \right] (M - m) m.$$ 

If $R > R_0$ then for any spherically symmetric minimizing sequence $(\rho_i) \subset \mathcal{F}^r_M$ of $\mathcal{H}_C^r$,

$$\lim_{i \to \infty} \int_{|x| \geq R} \rho_i = 0.$$
Proof. Clearly,
\[ \mathcal{H}^r_C(\rho) = \mathcal{H}^r_C(\rho_1) + \mathcal{H}^r_C(\rho_2) - \int \frac{\rho_1(x) \rho_2(y)}{|x-y|} dx dy, \]
where \( \rho_1 = 1_{B_R \rho}, \rho_2 = \rho - \rho_1 \). Due to spherical symmetry,
\[
\int \frac{\rho_1(x) \rho_2(y)}{|x-y|} dx dy = \frac{1}{4\pi} \int \nabla U_{\rho_1} \cdot \nabla U_{\rho_2} dx
= \int_0^\infty \frac{4\pi}{r^2} \int_0^r \rho_1(s)s^2 ds \frac{4\pi}{r^2} \int_0^r \rho_2(s)s^2 ds dr
= \int_0^\infty \cdots dr \leq \frac{(M-m)m}{R}.
\]
Thus by Lemma 4,
\[
\mathcal{H}^r_C(\rho) \geq h^r_M + \frac{(M-m)m}{R} - \frac{1}{R_0} \left[ \left( \frac{M-m}{M} \right)^{5/3} + \left( \frac{M-m}{M} \right)^{5/3} \right] h^r_M - \frac{(M-m)m}{R}
\geq \left[ 1 - \frac{5}{3} \frac{M-m}{M} \right] h^r_M - \frac{(M-m)m}{R},
\]
which is the first assertion of the lemma; note that the scaling transformations in the proof of Lemma 4 preserve spherical symmetry. Now take \( R > R_0 \) and assume that the second assertion were false so that up to a subsequence,
\[
\lim_{i \to \infty} \int_{|x| \geq R} \rho_i = m > 0.
\]
Choose \( R_i > R \) such that
\[
m_i := \int_{|x| \geq R_i} \rho_i = \frac{1}{2} \int_{|x| \geq R} \rho_i.
\]
By the already established splitting estimate,
\[
\mathcal{H}^r_C(\rho_i) \geq h^r_M + \frac{1}{R_0} - \frac{1}{R_i} \left( M - m_i \right) m_i \geq h^r_M + \left[ \frac{1}{R_0} - \frac{1}{R} \right] (M-m_i)m_i.
\]
and with \( i \to \infty \),

\[
h_M^r \geq h_M^r + \left[ \frac{1}{R_0} - \frac{1}{R} \right] (M - m/2)m/2 > h_M^r,
\]
a contradiction. \( \Box \)

The lemma above now replaces Lemma 3, Corollary 2, and the proof of (3.4) which relied on the fairly lengthy argument via (3.2) and (3.3). In addition, we get a somewhat sharper result on the minimizer:

\[
\text{supp } \rho_0 \subset B_{R_0}.
\]

That spherical symmetry helps with compactness issues was already noted in [12]. The a-priori restriction to the spherically symmetric case is undesirable in view of resulting stability assertions: These would then be restricted to spherically symmetric perturbations. Moreover, the symmetry simplification cannot be used if one does not a priori know that the minimizers will be spherically symmetric. One example for this situation is the construction of steady states with axial symmetry, say with respect to the \( x_3 \)-axis, by making \( Q \) in addition depend explicitly on \( x_1 v_2 - x_2 v_1 \), the angular momentum with respect to the axis of symmetry. Exactly the same reduction procedure as before now gives a function \( \Phi \) that depends in addition on \( r = \sqrt{x_1^2 + x_2^2} \), and minimizers will not be spherically symmetric. An investigation of axially symmetric steady states and their stability will be the content of [6]. Another situation where the minimizers will in general not be spherically symmetric arises if one includes in the Vlasov-Poisson system an exterior gravitational field, say \( U_e = U_{\rho_e} \) with some fixed \( \rho_e \in L^1_+ \cap L^{1+1/n}(\mathbb{R}^3) \). It is quite easy to check that all the analysis carried out in this paper extends to this case; only the potential energy needs to be modified accordingly:

\[
E_{\text{pot}}(\rho) = -\frac{1}{2} \int \int \frac{\rho(x) \rho(y)}{|x-y|} \, dx \, dy - \int \int \frac{\rho(x) \rho_e(y)}{|x-y|} \, dx \, dy
\]

\[
= \frac{1}{2} \int \rho(x)U_{\rho}(x) \, dx + \int \rho(x)U_{\rho_e}(x) \, dx,
\]

and if \( \rho_e \) is not spherically symmetric then neither are possible minimizers. If one wishes to study the minimization of \( H_C^r \) with \( \Phi(\rho) \) generalized to \( \Phi(x,\rho) \) the crucial step in the analysis which restricts the possible dependence on \( x \) is the scaling in Lemma 4.
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