A DETERMINANT FORMULA FOR RELATIVE CONGRUENCE ZETA FUNCTIONS FOR CYCLOMOTIC FUNCTION FIELDS

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Abstract

Rosen gave a determinant formula for relative class numbers for cyclotomic function fields, which may be regarded as an analogue of the classical Maillet determinant. In this paper, we give a determinant formula for relative congruence zeta functions for cyclotomic function fields. Our formula may be regarded as a generalization of the determinant formula for the relative class number.

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1. Introduction

Let $h_p^-$ be the relative class number of the cyclotomic field of $p$th roots of unity. Carlitz and Olson [CO] computed the number $h_p^-$ in terms of a certain classical determinant, known as the Maillet determinant.

In the cyclotomic function field case, several authors gave analogues of the Maillet determinant. Let $k$ be the field of rational functions over the finite field $\mathbb{F}_q$ with $q$ elements. Fix a generator $T$ of $k$, and let $A$ be the polynomial subring $\mathbb{F}_q[T]$ of $k$. Let $m$ be a monic polynomial of $A$, and $\Lambda_m$ be the set of all $m$-torsion points of the Carlitz module. The field $K_m$ obtained by adjoining the points of $\Lambda_m$ to $k$ is called the $m$th cyclotomic function field. For the definition of the Carlitz module and the basic facts about cyclotomic function fields, see Section 2 below. Let $K_m^+$ be the decomposition field of the infinite prime of $k$ in $K_m/k$, which is called the ‘maximal real subfield’ in $K_m$.

Let $h_m^-$ and $h_m^+$ be the orders of the divisor class group of degree zero for $K_m$ and $K_m^+$. Define the relative class number $h_m^-$ of $K_m$ by $h_m^- = h_m / h_m^+$. This work was supported by a Grant-in-Aid for JSPS Fellows (21-1611). © 2010 Australian Mathematical Publishing Association Inc. 1446-7887/2010 $16.00
Rosen [Ro1] gave a determinant formula for $h_P$ in the case of the monic irreducible polynomial $P$, which is regarded as an analogue of the Maillet determinant. Recently, several authors generalized Rosen’s formula and gave class number formulas (see, for instance, [ACJ, BK]).

Let $\zeta(s, K_m)$ be the congruence zeta function for $K_m$. The function $\zeta(s, K_m)$ can be expressed in the form

$$\zeta(s, K_m) = \frac{Z_m(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})},$$

where $Z_m(X)$ is a polynomial with integral coefficients. Then we have the decomposition

$$Z_m(X) = Z_m^+(X)Z_m^-(X),$$

where $Z_m^+(X)$ is the polynomial corresponding to the congruence zeta function $\zeta(s, K_m^+)$ for $K_m^+$. For the polynomial $Z_m^+(X)$, the author gave the determinant formula in the paper [Sh]. We see that

$$Z_m^-(q^{-s}) = \frac{\zeta(s, K_m)}{\zeta(s, K_m^+)};$$

this is called the relative congruence zeta function for $K_m$.

The main result of this paper is a determinant formula for $Z_m^-(X)$. Since $Z_m^-(1) = h_m$, our formula may be regarded as a generalization of the determinant formula for the relative class number.

As an application of our determinant formula, we will give an explicit formula for some coefficients of low-degree terms for $Z_m^-(X)$.

## 2. Basic facts

In this section, we outline several basic facts about cyclotomic function fields and their zeta functions. For the proofs of these facts, see [GR, Ha, Ro2, Wa].

### 2.1. Cyclotomic function fields. Let $K^{ac}$ be the algebraic closure of $k$. For $x \in K^{ac}$ and $m \in A$, we define the action

$$m \cdot x = m(\varphi + \mu)(x),$$

where $\varphi$ and $\mu$ are the $\mathbb{F}_q$-linear maps of $K^{ac}$ defined by

$$\varphi : K^{ac} \rightarrow K^{ac} \quad x \mapsto x^q,$$

$$\mu : K^{ac} \rightarrow K^{ac} \quad x \mapsto T \cdot x.$$

Under the above action, $K^{ac}$ becomes an $A$-module, called the Carlitz module. Let $\Lambda_m$ be the set of all $x$ such that $m \cdot x = 0$; this is a cyclic sub-$A$-module of $K^{ac}$. Fix a generator $\lambda_m$ of $\Lambda_m$. Then we have the following isomorphism of $A$-modules:

$$A/(m) \rightarrow \Lambda_m \quad a \mod m \mapsto a \cdot \lambda_m,$$

where $(m)$ is the principal ideal $mA$ generated by $m$. Let $(A/(m))^\times$ be the group of units of $A/(m)$, and $\Phi(m)$ be the order of $(A/(m))^\times$. Let $K_m$ be the field obtained by
adjoining all the elements of $\Lambda_m$ to $k$. We call $K_m$ the $m$th cyclotomic function field. The extension $K_m/k$ is abelian, and the following isomorphism is valid:

$$\left(\frac{A}{(m)}\right)^\times \longrightarrow \text{Gal}(K_m/k) \quad a \mod m \mapsto \sigma_a \mod m, \quad (2.1)$$

where $\text{Gal}(K_m/k)$ is the Galois group of $K_m/k$, and $\sigma_a \mod m$ is the isomorphism given by $\sigma_a \mod m(\lambda_m) = a \cdot \lambda_m$. By using isomorphism (2.1), we find that the extension degree of $K_m/k$ is $\Phi(m)$. We see that $\mathbb{F}_q^\times$ is contained in $\left(\frac{A}{(m)}\right)^\times$. Let $K_m^+$ be the subfield of $K_m$ corresponding to $\mathbb{F}_q^\times$. Again from isomorphism (2.1), we find that the extension degree of $K_m^+/k$ is $\Phi(m)/(q-1)$. Let $P_\infty$ be the unique prime of $k$ which corresponds to the valuation $v_\infty$ with $v_\infty(T) < 0$. The prime $P_\infty$ splits completely in $K_m^+/k$, and any prime of $K_m^+$ over $P_\infty$ is totally ramified in $K_m/k$. Hence $K_m^+ = K_m \cap k_\infty$ where $k_\infty$ is the completion of $k$ by $v_\infty$. The field $K_m^+$ is called the maximal real subfield of $K_m$; it is an analogue of the maximal real subfield of a cyclotomic field.

Next, we review some basic facts about Dirichlet characters. For a monic polynomial $m \in A$, let $X_m$ be the group of all primitive Dirichlet characters of $\left(\frac{A}{(m)}\right)^\times$. Let $X_m^+$ be the set of all characters in $X_m$ such that $\chi(a) = 1$ for any $a \in \mathbb{F}_q^\times$. Put

$$\tilde{K} = \bigcup_{\text{monic}} K_m$$

where $m$ runs through all monic polynomials of $A$. Let $\mathbb{D}$ be the group of all primitive Dirichlet characters. By the same argument as in [Wa, Ch. 3], we have a one-to-one correspondence between finite subgroups of $\mathbb{D}$ and finite subextension fields of $\tilde{K}/k$. The following theorem is useful for obtaining information about primes.

**Theorem 2.1** (See [Wa, Theorem 3.7]). Let $X$ be a finite subgroup of $\mathbb{D}$, and $K_X$ the associated field. For an irreducible monic polynomial $P \in A$, put

$$Y = \{\chi \in X \mid \chi(P) \neq 0\}, \quad Z = \{\chi \in X \mid \chi(P) = 1\}.$$

Then the following hold.

- $X/Y$ is isomorphic to the inertia group of $P$ in $K_X/k$.
- $Y/Z$ is isomorphic to the cyclic group of order $f_P$; where $f_P$ is the residue class degree of $P$ in $K_X/k$.
- $X/Z$ is isomorphic to the decomposition group of $P$ in $K_X/k$.

**2.2. The relative congruence zeta function.** Our next task is to investigate congruence zeta functions for cyclotomic function fields. Let $K$ be a geometric extension of $k$ of finite degree. We define the congruence zeta function of $K$ as

$$\zeta(s, K) = \prod_{P \text{ prime}} \left(1 - \frac{1}{N/P^s}\right)^{-1}$$

where $P$ runs through all primes of $K$, and $N/P$ is the number of elements of the residue class field of the prime $P$. We see that $\zeta(s, K)$ converges absolutely when $\text{Re}(s) > 1$. 
**Theorem 2.2 (See [Ro2, Theorem 5.9]).** Let $g_K$ be the genus of $K$ and $h_K$ be the order of the divisor class group of degree zero. Then there is a polynomial $Z_K(X) \in \mathbb{Z}[X]$ of degree $2g_K$ satisfying

$$\zeta(s, K) = \frac{Z_K(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})},$$

(2.2)

and $Z_K(0) = 1$ and $Z_K(1) = h_K$.

Since the right-hand side of Equation (2.2) is meromorphic on the whole of $\mathbb{C}$, this equation provides the analytic continuation of $\zeta(s, K)$ to the whole of $\mathbb{C}$.

Next, we explain the zeta function of $\mathcal{O}_K$, which is the integral closure of $A$ in the field $K$. We define the zeta function $\zeta(s, \mathcal{O}_K)$ for the ring $\mathcal{O}_K$ by

$$\zeta(s, \mathcal{O}_K) = \prod_{P} \left(1 - \frac{1}{N(P)^s}\right)^{-1},$$

where the product runs over all primes of $\mathcal{O}_K$. Let $X$ be a finite subgroup of $D$, and $K_X$ be the associated field. By the same argument as in the case of number fields (see [Wa]), we have the $L$-function decomposition

$$\zeta(s, \mathcal{O}_K X) = \prod_{\chi \in X} L(s, \chi),$$

where the $L$-function is defined by

$$L(s, \chi) = \prod_{P} \left(1 - \frac{\chi(P)}{N(P)^s}\right)^{-1},$$

where $P$ runs through all monic irreducible polynomials of $A$. Let $f_\infty$ be the residue class degree of $P_\infty$ in $K_X/k$ and $g_\infty$ be the number of primes in $K_X$ over $P_\infty$. Then

$$\zeta(s, K_X) = \zeta(s, \mathcal{O}_K X)(1 - q^{-s f_\infty} - g_\infty).$$

From now on, we will focus on the cyclotomic function field case. For a monic polynomial $m \in A$, let $K_m$ and $K_m^+$ be the $m$th cyclotomic function field and its maximal real subfield. The relative congruence zeta function for $K_m$ is defined by

$$\zeta(-)(s, K_m) = \frac{\zeta(s, K_m)}{\zeta(s, K_m^+)}.$$  

By Theorem 2.2, there are polynomials $Z_m(X)$ and $Z_m^+(X)$ with integral coefficients such that

$$\zeta(s, K_m) = \frac{Z_m(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})},$$

$$\zeta(s, K_m^+) = \frac{Z_m^+(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}. $$

Put

$$Z_m^-(X) = \frac{Z_m(X)}{Z_m^+(X)};$$
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then

$$\zeta(-s, K_m) = Z_m^{(-)}(q^{-s}).$$

Notice that the fields $K_m$ and $K_m^+$ are associated with $X_m^-$ and $X_m^+$ respectively. Since any prime in $K_m^+$ above $P_\infty$ is totally ramified in $K_m/K_m^+$,

$$Z_m^{(-)}(q^{-s}) = \prod_{\chi \in X_m^-} L(s, \chi)$$  \hfill (2.3)

where $X_m^- = X_m - X_m^+$. The $L$-function associated with the nontrivial character can be expressed by a polynomial of $q^{-s}$ with complex coefficients. Hence, we see that $Z_m^{(-)}(X)$ is a polynomial with integral coefficients.

3. The determinant formula for $Z_m^{(-)}(X)$

In the previous section, we defined the relative congruence zeta function $\zeta(-s, K_m)$ for the $m$th cyclotomic function field, and showed that $\zeta(s, K_m)$ is given by a polynomial $Z_m^{(-)}(X)$ with integral coefficients. The goal of this section is to give a determinant formula for $Z_m^{(-)}(X)$. First, we need some notation to construct the determinant formula. Let $m$ be a monic polynomial of degree $d$. For $\alpha \in (A/(m))^\times$, there is a unique element $r_\alpha \in A$ satisfying

$$r_\alpha = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0 \quad \text{where } n = \deg r_\alpha < d,$$

$$r_\alpha \equiv \alpha \mod m,$$

where $\deg f$ denotes the degree of the polynomial $f$. Then we define

$$\Deg(\alpha) = n, \quad L(\alpha) = a_n \in \mathbb{F}_q^\times,$$

and $c^\lambda(\alpha) = \lambda^{-1}(L(\alpha))$, where $\lambda$ is a character of $\mathbb{F}_q^\times$. Put $N_m = \Phi(m)/(q - 1)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{N_m}$ be all of the elements of $(A/(m))^\times$ such that $L(\alpha) = 1$; these form a complete system of representatives for $R_m = (A/(m))^\times/\mathbb{F}_q^\times$. We put

$$c^\lambda_{ij} = c^\lambda(\alpha_i \alpha_j^{-1}) \quad \forall i, j = 1, 2, \ldots, N_m,$$

$$d_{ij} = \Deg(\alpha_i \alpha_j^{-1}) \quad \forall i, j = 1, 2, \ldots, N_m.$$

For any character $\lambda$ of $\mathbb{F}_q^\times$, we define the matrix

$$D_m^{(\lambda)}(X) = (c^\lambda_{ij} X^{d_{ij}})_{i,j=1,2,\ldots,N_m}.$$

This matrix plays an essential role in our argument. Note that $d_{ij} > 0$ when $i \neq j$, and $d_{ij} = 0$ and $c^\lambda_{ij} = 1$ when $i = j$. Thus $D_m^{(\lambda)}(0)$ is the unit matrix. We put

$$D_m^{(-)}(X) = \prod_{\lambda \neq 1} \det D_m^{(\lambda)}(X),$$
where the product runs over all nontrivial characters of $\mathbb{F}_q^\times$. To be able to state the main result, we define the polynomial $J_m^{(-)}(X)$ by

$$J_m^{(-)}(X) = \prod_{\chi \in X_m} \prod_{Q | m} (1 - \chi(Q)X^{\deg Q}),$$

where $Q$ is an irreducible monic polynomial dividing $m$. First, we prove the following proposition.

**Proposition 3.1.** With the notation above,

$$J_m^{(-)}(X) = \prod_{Q | m} \frac{(1 - Xf_Q^{\deg Q}Q)g_Q}{(1 - Xf_Q^{\deg Q}g_Q^+)Q},$$

where $f_Q$ and $f_Q^+$ are the residue class degrees of $Q$ in $K_m/k$ and $K_m^+/k$, and $g_Q$ and $g_Q^+$ are the numbers of primes in $K_m$ and $K_m^+$ over $Q$.

**Proof.** Notice that $X_m$ and $X_m^+$ are associated with the $m$th cyclotomic function field $K_m$ and its maximal real subfield $K_m^+$ respectively. Let $Q$ be an irreducible monic polynomial dividing $m$. Put

$$Y_Q = \{\chi \in X_m \mid \chi(Q) \neq 0\} \quad \text{and} \quad Z_Q = \{\chi \in X_m \mid \chi(Q) = 1\}.$$

From Theorem 2.1,

$$\prod_{\chi \in X_m} (1 - \chi(Q)X^{\deg Q}) = \prod_{\chi \in Y_Q} (1 - \chi(Q)X^{\deg Q}) \prod_{\chi \in Y_Q/Z_Q} \prod_{\psi \in Z_Q} (1 - \chi\psi(Q)X^{\deg Q}) = \left(\prod_{\chi \in Y_Q/Z_Q} (1 - \chi(Q)X^{\deg Q})\right)^{g_Q}.$$

Since $Y_Q/Z_Q$ is a cyclic group of order $f_Q$,

$$\prod_{\chi \in Y_Q/Z_Q} (1 - \chi(Q)X^{\deg Q}) = (1 - X^{f_Q \deg Q}).$$

Hence we obtain the formula

$$\prod_{\chi \in X_m} (1 - \chi(Q)X^{\deg Q}) = (1 - X^{f_Q \deg Q})^{g_Q}.$$

By the same argument,

$$\prod_{\chi \in X_m^+} (1 - \chi(Q)X^{\deg Q}) = (1 - X^{f_Q^+ \deg Q})^{g_Q^+}.$$ 

Noting that $X_m^- = X_m - X_m^+$, we can deduce the proposition from the last two equations. \qed
There are several consequences of this proposition. First of all, by Proposition 3.1, we see that $J_m^{(-)}(X)$ is a polynomial with integral coefficients. Second, if $m$ is a power of an irreducible polynomial $P$, the prime $P$ is totally ramified in $K_m/k$ (see [Ro2]). Hence $J_m^{(-)}(X) = 1$ in this case.

The next theorem is the main result of this paper.

**Theorem 3.2.** Let $m \in A$ be a monic polynomial. Then

$$D_m^{(-)}(X) = Z_m^{(-)}(X)J_m^{(-)}(X).$$

**Proof.** For any $\chi \in X_m$, let the monic polynomial $f_\chi$ be the conductor of $\chi$. Define $\tilde{\chi}$ by

$$\tilde{\chi} = \chi \circ \pi \chi$$

where $\pi : (A/(m))^\times \to (A/(f_\chi))^\times$ is the natural homomorphism. Then

$$L(s, \tilde{\chi}) = L(s, \chi) \cdot \prod_{Q|m} (1 - \chi(Q)q^{-s \deg Q}).$$

Fix a nontrivial character $\lambda$ of $\mathbb{F}_q^\times$ and $\psi \in X_m^-(\psi|_{\mathbb{F}_q^\times} = \lambda)$. Then

$$\psi \cdot X_m^+ = \{ \chi \in X_m^- | \chi|_{\mathbb{F}_q^\times} = \lambda \}.$$

For each character $\chi \in X_m^-(\chi|_{\mathbb{F}_q^\times} = \lambda)$, there is a unique character $\phi \in X_m^+$ with $\chi = \psi \cdot \phi$. By the same argument as in [GR, Lemma 3],

$$L(s, \tilde{\chi}) = \sum_{i=1}^{N_m} \tilde{\chi}(\alpha_i)q^{-\deg(\alpha_i)s}$$

$$= \sum_{i=1}^{N_m} \tilde{\phi}(\alpha_i)\tilde{\psi}(\alpha_i)c^\lambda(\alpha_i)q^{-\deg(\alpha_i)s}.$$  

Notice that $\tilde{\psi}(\alpha)c^\lambda(\alpha)$ and $\deg$ are functions over $\mathcal{R}_m$, and $\tilde{\phi}$ runs through all characters of $\mathcal{R}_m$ when $\phi$ runs through all characters of $X_m^+$. By the Frobenius determinant formula (see [Wa, Lemma 5.26]),

$$\prod_{\chi|_{\mathbb{F}_q^\times} = \lambda} L(s, \tilde{\chi}) = \prod_{\phi \in X_m^+} \sum_{i=1}^{N_m} \tilde{\phi}(\alpha_i)\tilde{\psi}(\alpha_i)c^\lambda(\alpha_i)q^{-\deg(\alpha_i)s}$$

$$= \det(\psi(\alpha_i\alpha_j^{-1})c^\lambda ij q^{-s \delta ij})_{i,j = 1,2,\ldots,N_m}$$

$$= \det D_m^{(\lambda)}(q^{-s}).$$

From the decomposition

$$X_m^- = \bigcup_{\lambda \neq 1} \{ \chi \in X_m | \chi|_{\mathbb{F}_q^\times} = \lambda \}.$$
we see that
\[ D_m^{(-)}(q^{-s}) = \left( \prod_{\chi \in \chi_m} L(s, \chi) \right) \times J_m^{(-)}(q^{-s}). \]

By Equation (2.3), we obtain the formula
\[ D_m^{(-)}(q^{-s}) = Z_m^{(-)}(q^{-s})J_m^{(-)}(q^{-s}). \]

Putting \( X = q^{-s} \), we obtain the desired result. \( \Box \)

We offer two remarks about this theorem. First, \( Z_m^{(-)}(X) = 1 \) when \( m \) is a monic polynomial of degree one. In fact, we can calculate that \( D_m^{(-)}(X) = 1 \) in this case. Second, recall that \( J_m^{(-)}(X) = 1 \) when \( m \) is a power of an irreducible polynomial. Hence \( D_m^{(-)}(X) = Z_m^{(-)}(X) \) in this case.

As a special case of our result, we obtain the following determinant formula for relative class numbers.

**Corollary 3.3** (See [ACJ, BK]). Let \( h_m^- \) be the relative class number of \( K_m \). Put \( f_Q^- = f_Q/f_Q^+ \) and \( g_Q^- = g_Q/g_Q^+ \). Then
\[ \prod_{\lambda, \neq 1} \det(c_{ij}^\lambda), \]
where
\[ W_m^- = \begin{cases} \prod_{Q|m} (f_Q^-)^{g_Q^+} & \text{if } g_Q^- = 1 \text{ for every prime } Q \text{ dividing } m, \\ 0 & \text{otherwise.} \end{cases} \]

**Proof.** Putting \( X = 1 \) in Theorem 3.2, we see that
\[ D_m^{(-)}(1) = \prod_{\lambda, \neq 1} \det(c_{ij}^\lambda), \]
and \( J_m^{(-)}(1) = W_m^- \) by Proposition 3.1. Since \( Z_m^{(-)}(1) = h_m^- \), we obtain the desired result. \( \Box \)

If \( m \) is a power of an irreducible polynomial, we see that \( W_m^- = 1 \). Otherwise, each finite prime in \( K_m^+ \) is not ramified in \( K_m/K_m^+ \). Thus we see that \( f_Q^- = q - 1 \) for a prime \( Q \) with \( g_Q^- = 1 \).

**4. Some coefficients of the low degree terms of \( D_m^{(-)}(X) \)**

In this section, we will calculate the coefficients of \( D_m^{(-)}(X) \) of degrees one and two, by using the derivative of the determinant. Let \( m \in A \) be a monic polynomial. Noting that \( D_m^{(-)}(0) = 1 \), we see that \( D_m^{(-)}(X) \) may be written in the form
\[ D_m^{(-)}(X) = 1 + a_1 X + a_2 X^2 + \cdots, \]
where each \( a_i \) is an integer \((i = 1, 2, \ldots)\).
**Proposition 4.1.** Let \( m \in A \) be a monic polynomial of degree \( d \), where \( d > 1 \). Then

\[
\begin{align*}
    a_1 &= 0, \\
    a_2 &= 0 \quad \text{if } \deg m > 2, \\
    a_2 &= \frac{N_m}{2} \{ (q - 1)(1 - C_m) + N_m - 1 \} \quad \text{if } \deg m = 2,
\end{align*}
\]

where

\[
C_m = \#\{ i = 1, 2, \ldots, N_m \mid L(\alpha_i^{-1}) = 1 \}.
\]

Here \( \#A \) is the number of elements of a set \( A \).

By Proposition 3.1, we can find \( J_{m}^{(-)}(X) \). Hence we can also calculate the coefficients of the low-degree terms of \( Z_{m}^{(-)}(X) \). As a preliminary to Proposition 4.1, we first state the next lemma, which can be proved by simple calculations.

**Lemma 4.2.** Let \( F(X) = (f_{ij}(X))_{i,j} \) be a matrix with coefficients in the ring of functions of one variable. If \( F(X) \) is twice differentiable and invertible when \( X = X_0 \), then

\[
\begin{align*}
    \frac{d \det F(X)}{dX} \bigg|_{X=X_0} &= \det F(X_0) \cdot \mathrm{Tr} \left( F(X_0)^{-1} \frac{dF}{dX}(X_0) \right), \\
    \frac{d^2 \det F(X)}{dX^2} \bigg|_{X=X_0} &= \det F(X_0) \cdot \left\{ \mathrm{Tr} \left( F(X_0)^{-1} \frac{d^2F}{dX^2}(X_0) \right) \\
    &- \mathrm{Tr} \left( F(X_0)^{-1} \frac{dF}{dX}(X_0) F(X_0)^{-1} \frac{dF}{dX}(X_0) \right) \\
    &+ \mathrm{Tr} \left( F(X_0)^{-1} \frac{dF}{dX}(X_0) \right)^2 \right\},
\end{align*}
\]

where \( \mathrm{Tr}(A) \) denotes the trace of the matrix \( A \).

We now prove Proposition 4.1.

**Proof.** Let \( \lambda \) be a nontrivial character of \( \mathbb{F}_q^\times \), and write

\[
\det D_m^{(\lambda)}(X) = 1 + a_1^{\lambda} X + a_2^{\lambda} X^2 + \cdots.
\]

Note that \( D_m^{(\lambda)}(0) \) is the unit matrix, and

\[
\frac{d D_m^{(\lambda)}}{dX}(0) = (l_{ij})_{i,j=1,2,\ldots,N_m},
\]

where

\[
l_{ij} = \begin{cases} 
    0 & \text{if } d_{ij} = 0 \text{ or } d_{ij} > 1, \\
    c_{ij}^{\lambda} & \text{if } d_{ij} = 1.
\end{cases}
\]
By Lemma 4.2, $a_1^λ = 0$ and

$$a_2^λ = \frac{1}{2} \operatorname{Tr} \left( \left( \frac{dD_m^{(λ)}}{dX}(0) \right)^2 \right).$$

Thus we have shown assertion (4.4).

If $\deg m > 2$, there is no pair $(i, j)$ such that $d_{ij} = 1$ and $d_{ji} = 1$. Thus $a_2^λ = 0$ in the case where $\deg m > 2$. Since $a_2 = \sum_{λ≠1} a_2^λ$, we obtain assertion (4.5).

Next we consider the case where $\deg m = 2$. In this case, $l_{ij} = \begin{cases} 0 & \text{if } i = j, \\ c_{ij}^λ & \text{if } i ≠ j. \end{cases}$

Thus

$$\sum_{λ≠1} a_2^λ = \sum_{λ≠1} \left( \frac{N_m}{2} - \frac{1}{2} \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} λ^{-1}(L(α_iα_j^{-1})L(α_jα_i^{-1})) \right) = \frac{N_m(q - 2)}{2} - \frac{1}{2} \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} e_{ij},$$

where

$$e_{ij} = \begin{cases} q - 2 & \text{if } L(α_iα_j^{-1})L(α_jα_i^{-1}) = 1, \\ -1 & \text{otherwise}. \end{cases}$$

For any $i, j ∈ \{1, 2, \ldots, N_m\}$, there exist $γ_{ij} ∈ \mathbb{F}_q^×$ and $β_{ij} ∈ (A/(m))^×$ such that $L(β_{ij}) = 1$ and $α_iα_j^{-1} = γ_{ij}β_{ij}$. Then

$$L(α_iα_j^{-1})L(α_jα_i^{-1}) = L(β_{ij}^{-1}).$$

By noting that

$$\{β_{ij} \mid j = 1, 2, \ldots, N_m\} = \{α_j \mid j = 1, 2, \ldots, N_m\},$$

we see that

$$\sum_{j=1}^{N_m} e_{ij} = (q - 1)C_m - N_m.$$

Thus we have completed the proof of Proposition 4.1.

5. Examples

We consider the case where $m = T^2 + aT + b ∈ A$. If $α = T - c$ satisfies $L(α^{-1}) = 1$, then $c$ is a root of the equation $T^2 + aT + b + 1$. Thus $C_m ≤ 3$.
EXAMPLE 5.1. When $q = 3$ and $m = T^2 + 1$, we see that the extension degree of $K_m/k$ is 8 and $N_m = 4$. Since the polynomial $m$ is irreducible, $D_m(X) = Z_m(X)$.

Put
\[
\alpha_1 = 1, \quad \alpha_2 = T, \quad \alpha_3 = T + 1, \quad \alpha_4 = T + 2.
\]

Then
\[
Z_m(X) = D_m(X)
\]
\[
= \begin{vmatrix}
1 & -X & X & X \\
X & 1 & -X & X \\
X & -X & 1 & -X \\
X & X & X & 1
\end{vmatrix}
= 1 - 2X^2 + 9X^4.
\]

The relative class number $h_m$ of $K_m$ is $Z_m(1) = 8$.

EXAMPLE 5.2. When $q = 3$ and $m = T^3 + T^2$, we see that the extension degree of $K_m/k$ is 12 and $N_m = 6$. Put
\[
\alpha_1 = 1, \quad \alpha_2 = T^2 + 2T + 2, \quad \alpha_3 = T^2 + T + 1,
\]
\[
\alpha_4 = T + 2, \quad \alpha_5 = T^2 + 1, \quad \alpha_6 = T^2 + T + 2.
\]

Then
\[
D_m(X) = \begin{vmatrix}
1 & X & -X^2 & X^2 & X^2 & -X^2 \\
X^2 & 1 & -X^2 & -X^2 & -X^2 & -X \\
X & X^2 & 1 & X & -X^2 & X^2 \\
X^2 & X^2 & X & 1 & X^2 & X^2 \\
X^2 & -X^2 & -X^2 & 1 & X^2 & X \\
X^2 & -X^2 & -X^2 & X^2 & X & 1
\end{vmatrix}
= 1 - 6X^3 - 3X^4 - 6X^5 + 23X^6 + 30X^7 + 6X^8 - 18X^9 - 27X^{10}
\]

and
\[
J_m(X) = 1 + X - X^3 - X^4.
\]

Thus
\[
Z_m(X) = \frac{D_m(X)}{J_m(X)}
= 1 - X + X^2 - 6X^3 + 3X^4 - 9X^5 + 27X^6.
\]

The relative class number $h_m$ of $K_m$ is $Z_m(1) = 16$.

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