SIMPSON–MOCHIZUKI CORRESPONDENCE FOR $\lambda$-FLAT BUNDLES

ZHI HU AND PENGFEI HUANG

Abstract. The notion of flat $\lambda$-connections as the interpolation of usual flat connections and Higgs fields was suggested by Deligne and further studied by Simpson. Mochizuki established the Kobayashi–Hitchin-type theorem for $\lambda$-flat bundles ($\lambda \neq 0$), which is called the Mochizuki correspondence. In this paper, on the one hand, we generalize Mochizuki’s result to the case when the base being a compact balanced manifold, more precisely, we prove the existence of harmonic metrics on stable $\lambda$-flat bundles ($\lambda \neq 0$). On the other hand, we study two applications of the Simpson–Mochizuki correspondence to moduli spaces. More concretely, we show this correspondence provides a homeomorphism between the moduli space of (semi)stable $\lambda$-flat bundles over a complex projective manifold and the Dolbeault moduli space, and also provides dynamical systems with two parameters on the latter moduli space. We investigate such dynamical systems, in particular, we calculate the first variation, the fixed points and discuss the asymptotic behaviour.

Résumé. La notion de $\lambda$-connexions plates comme interpolation de connexions plates habituelles et champs de Higgs a été suggérée par Deligne et étudiée plus en détail par Simpson. Mochizuki a établi le théorème de type Kobayashi–Hitchin pour les fibrés $\lambda$-plats ($\lambda \neq 0$), qui s’appelle la correspondance de Mochizuki. Dans cet article, d’une part, nous généralisons le résultat de Mochizuki au cas où la variété de base est une variété équilibrée, plus précisément, nous prouvons l’existence de métriques de harmoniques sur les fibrés $\lambda$-plats stables ($\lambda \neq 0$). D’autre part, nous étudions deux applications de la correspondance de Simpson–Mochizuki aux espaces de modules. Plus concretément, nous montrons que cette correspondance fournit un homéomorphisme entre l’espace des modules des fibrés $\lambda$-plats (semi)stables sur une variété projective complexe et l’espace des modules de Dolbeault, et fournit également des systèmes dynamiques avec deux paramètres sur ce dernier espace des modules. Nous étudions de tels systèmes dynamiques, en particulier, nous calculons la première variation, les points fixes et discutons le comportement asymptotique.

Contents

1. Introduction 2
2. Preliminaries 5
  2.1. Flat $\lambda$-Connections, Pluri-harmonic Metrics 5
  2.2. Example 8
3. Simpson–Mochizuki Correspondence 11
  3.1. Categorical Version 11
  3.2. Moduli version 12
4. Mochizuki Correspondence on Balanced Manifolds 15
5. Dynamical Systems on Dolbeault Moduli Spaces 22
  5.1. Construction 22

2010 Mathematics Subject Classification. 14D20, 14J60, 32G13, 53C07.
Key words and phrases. $\lambda$-Flat Bundles, (Pluri-)harmonic Metrics, Simpson–Mochizuki Correspondence, Moduli Spaces, Dynamical System.
1. Introduction

The notion of flat $\lambda$-connections as the interpolation of usual flat connections and Higgs fields was suggested by Deligne [8], illustrated by Simpson in [32] and further studied in [33, 34]. By applying Simpson’s construction for the moduli space of $\Lambda$-modules [30], one can show the existence of the coarse moduli space of rank $r$ semistable $\lambda$-flat bundles with vanishing Chern classes over a complex projective manifold $X$, which is denoted by $\mathcal{M}_{\text{Hod}}(X, r)$. And this construction can be generalized to the case of principal bundles by applying the Tannakian considerations [32].

It is clear that $\mathcal{M}_{\text{Hod}}(X, r)$ has a fibration over $C$, in particular, the fiber over $\lambda = 0$ is the usual Dolbeault moduli space $\mathcal{M}_{\text{Dol}}(X, r)$, and over $\lambda = 1$ it is the usual de Rham moduli space $\mathcal{M}_{\text{dR}}(X, r)$. Deligne’s motivation is to understand Hitchin’s twistor construction for the moduli space of solutions to Hitchin’s self-duality equations which carries a hyperKähler structure [13]. More precisely, according to Deligne’s perspective, Hitchin’s twistor space can be treated as the gluing of the moduli space $\mathcal{M}_{\text{Hod}}(X, r)$ and the complex conjugate moduli space $\mathcal{M}_{\text{Hod}}(\bar{X}, r)$ by the Riemann–Hilbert correspondence. Simpson interpreted the moduli space $\mathcal{M}_{\text{Hod}}(X, r)$ as the Hodge filtration on the non-abelian de Rham cohomology $\mathcal{M}_{\text{dR}}(X, r)$, and showed the Griffiths transversality and the regularity of the Gauss–Manin connection for this filtration [32]. Since then, this notion attracts many researchers’ attention, for example, flat $\lambda$-connections play a role in compactifying the de Rham moduli spaces [32, 18]; the author of [1] used spectral curves to describe $\lambda$-connections that are formal deformations of Higgs bundles; and recently the authors of [19] applied flat $\lambda$-connections to study the Kapustin–Witten equations.

The non-abelian Hodge correspondence provides a homeomorphism $\mathcal{M}_{\text{Dol}}(X, r) \simeq \mathcal{M}_{\text{dR}}(X, r)$, which is a $C^\infty$-isomorphism over the smooth loci [30]. This homeomorphism is achieved by finding the pluri-harmonic metrics, that is, by constructing the category of harmonic bundles in order to connect the Dolbeault side and the de Rham side. When $X$ is a compact Kähler manifold, such metrics exist for semisimple flat bundles due to Donaldson [9] and Corlette [7], and for polystable Higgs bundles with vanishing Chern classes due to Hitchin [12] and Simpson [26]. For $\lambda \neq 0$, Mochizuki introduced the notion of pluri-harmonic metrics for $\lambda$-flat bundles, and also established the Kobayashi–Hitchin-type theorem for this case [23]. We call this remarkable theorem the Mochizuki correspondence. Moreover, together with the Kobayashi–Hitchin correspondence for Higgs bundles, it is unified into the so-called Simpson–Mochizuki correspondence, which indicates the existence of pluri-harmonic metrics on $\lambda$-flat bundles satisfying certain stability conditions for any $\lambda \in \mathbb{C}$. By this correspondence, one can relate the category (moduli stack, moduli space) of polystable $\lambda$-flat bundles and that of polystable Higgs bundles.

**Remark.** When $\lambda \neq 0$, by multiplying with $\lambda^{-1}$ reduces a stable $\lambda$-flat bundle $(E, D^\lambda)$ (see Definition 2.1) of rank $r$ to a usual flat bundle $(E, \lambda^{-1}D^\lambda)$, then by Corlette’s work, we also have a pluri-harmonic metric. However, this metric is very different from the metric given by the Mochizuki correspondence. For example, given a metric on $(E, \lambda^{-1}D^\lambda)$, there is a $\rho$-equivariant map $f : \tilde{X} \to \text{GL}(r, \mathbb{C})/U(r)$, where $\tilde{X}$ is the universal cover of $X$, here $\rho : \pi_1(X) \to \text{GL}(r, \mathbb{C})$ is a simple representation of the fundamental group $\pi_1(X)$ associated to $(E, \lambda^{-1}D^\lambda)$. Then in general, the energy $K_f = \int_X |df|^2 d\nu_X$, where $d\nu_X$ is the volume element of the Kähler metric on $X$, corresponding to Corlette’s metric is smaller than that to Mochizuki’s metric [9, 7, 28]. In our
opinion, the Simpson–Mochizuki correspondence exhibits more natural interpolation between $\lambda = 1$ and $\lambda = 0$, for example, for a given polystable $\lambda_0$-flat bundle, we have a family$^1$ of polystable $\lambda$-flat bundles (varying $\lambda$) such that they correspond to the same Higgs bundle whenever $\lambda_0 \in \mathbb{C}$.

This paper is a study of the Simpson–Mochizuki correspondence. It is organized as follows.

In Section 2, as a preliminary, we collect some basic materials, simple conclusions, and provide an explicit example.

In Section 3, we discuss the Simpson–Mochizuki correspondence at various levels, including the Kobayashi–Hitchin version, categorical version, and moduli version. In particular, following Simpson’s ideas in [31, 34], we show the following theorem.

**Theorem 1** (Corollary 3.8). Let $X$ be a complex projective manifold, and let $M^\lambda_{\text{Hod}}(X, r)$ be the fiber of the fibration $M_{\text{Hod}}(X, r) \to \mathbb{C}$ over $\lambda \in \mathbb{C}$, then the Simpson–Mochizuki correspondence provides a homeomorphism

$$M^\lambda_{\text{Hod}}(X, r) \simeq M_{\text{Dol}}(X, r).$$

In Section 4, we consider the Mochizuki correspondence under a more general framework. Our generalization includes two aspects.

- Firstly, the base manifold $X$ is relaxed to be a compact balanced manifold, that is, the associated fundamental $(1, 1)$-form $\omega$ satisfies the condition $d(\omega^{\dim \mathbb{C} \cdot X^{-1}}) = 0$. Obviously, this condition is weaker than the Kähler condition $d\omega = 0$, but stronger than the Gauduchon condition $\partial \bar{\partial} (\omega^{\dim \mathbb{C} \cdot X^{-1}}) = 0$.

- Secondly, the pluri-harmonicity condition for the Hermitian metrics on $\lambda$-flat bundles is replaced by the harmonicity condition (see Definition 2.3). Obviously, the latter one is weaker in general. Of course, when $X$ is exactly a Kähler manifold and $\lambda \neq 0$, then these two conditions are fully equivalent (see Proposition 2.5).

Via the standard method of continuity, we show the following theorem.

**Theorem 2** (Theorem 4.11). Let $X$ be a compact balanced manifold, and $((E, \bar{\partial}_E), D^\lambda)$ be a stable $\lambda$-flat bundle over $X$ ($\lambda \neq 0$), then there is a unique harmonic metric on $((E, \bar{\partial}_E), D^\lambda)$ up to constant scalars.

**Remark.** It is known that when $\lambda = 0$, the Kobayashi–Hitchin problem (i.e. the existence of harmonic metrics on stable Higgs bundles with vanishing the first Chern class) can be solved for Gauduchon manifolds [20]. However, the condition of the flatness of $\lambda$-connection ($\lambda \neq 0$) is a more rigid constraint than that of Higgs field, so generally one cannot expect the above theorem to be true for Gauduchon manifolds as the case of Higgs bundles.

The last section, i.e. Section 5, is devoted to an application of the Simpson–Mochizuki correspondence to Dolbeault moduli spaces. More concretely, combining the Simpson–Mochizuki correspondence and $\mathbb{C}^*$-actions on Hodge moduli spaces together, we construct dynamical systems with two parameters on Dolbeault moduli spaces. Here a dynamical system means a continuous self-map $\psi(\lambda, t) : M_{\text{Dol}}(X, r) \to M_{\text{Dol}}(X, r)$ with a pair $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ of parameters. We first study the local property of $\psi(\lambda, t)$ by calculating the first variation. Next we consider the fixed points of this map.

For a given Higgs bundle $u := ((E, \bar{\partial}_E), \theta) \in M_{\text{Dol}}(X, r)$, we define the set of stable parameters

$$C_u = \{ (\lambda, t) \in \mathbb{C} \times \mathbb{C}^* : \psi(\lambda, t)(u) = u \},$$

$^1$Such family is called a twistor line or a preferred section under the context of twistor theory [32].
and for a given pair \((\lambda, t) \in \mathbb{C} \times \mathbb{C}^*\) of parameters, we define the set of fixed points
\[
\mathfrak{F}_{\lambda,t} = \{ u \in \mathbb{M}_{\text{Dol}}(X, r) : \psi_{\lambda,t}(u) = u \}.
\]

Our main results on this topic can be summarized as follows:

**Theorem 3** (Theorem 5.8, Corollary 5.9, Theorem 5.11).

1. Let \(X\) be a Riemann surface, and let \(u \in \mathbb{M}_{\text{Dol}}(X, r)\) represents a decoupled Higgs bundle with nontrivial Higgs field, then \(C \times \{ \mu^m : m = 0, \ldots, l - 1 \} \subseteq C_u \subseteq (C \times \{ \mu^m : m = 0, \ldots, l - 1 \}) \cup \{(\lambda, t) \in \mathbb{C}^* \times \mathbb{C}^* : \lambda \in \lambda_{\text{Hod}} \}\), where \(\mu^m = e^{2\pi i \lambda^m}\). In particular, if the Higgs field satisfies \(\psi_{\lambda,t}(u) = u\) at some point \(x \in X\), then \(C_u = C \times \{ 1 \}\).

2. Let \(\mathfrak{F} = \bigcap_{(\lambda, t) \in \mathbb{C}^* \times \mathbb{C}^*} \mathfrak{F}_{\lambda,t}\). Then \(\mathfrak{F}\) consists of the set of complex variations of Hodge structure\(^2\).

Finally, to investigate the limiting behaviour of this dynamical system when the parameters tend to 0, we introduce the following five limits of a Higgs bundle \(((E, \partial_E), \theta) \in \mathbb{M}_{\text{Dol}}(X, r)\) (now \(X\) is a Riemann surface):

1. \(\psi_{(0,0)}((E, \partial_E), \theta) := \lim_{t \to 0} \psi_{(0,t)}((E, \partial_E), \theta),\)
2. \(\psi_{(0,0)}((E, \partial_E), \theta) := \lim_{t \to 0} \psi_{(\lambda,t)}((E, \partial_E), \theta),\)
3. \(\psi_{(0,0)}((E, \partial_E), \theta) := \lim_{\lambda \to 0} \psi_{(\lambda,0)}((E, \partial_E), \theta),\)
4. \(\psi_{(0,0)}((E, \partial_E), \theta) := \lim_{\lambda \to 0} \lim_{t \to 0} \psi_{(\lambda,t)}((E, \partial_E), \theta),\)
5. \(\psi_{(0,0)}((E, \partial_E), \theta) := \lim_{(\lambda, t) \to (0, 0)} \psi_{(\lambda,t)}((E, \partial_E), \theta),\)

where \(\psi_{(\lambda,0)}\) appeared in the third limit is defined by the Simpson filtration that is closely related to the limits of \(\mathbb{C}^*\)-action on \(\mathbb{M}_{\text{Hod}}(X, r)\). For a general Higgs bundle, it’s quite hard to explicitly describe these limits, we do not even know whether they exist. However, if these limits exist, all are the complex variations of Hodge structure. For some special cases, we discuss these limits.

**Theorem 4** (Theorem 5.17). Let \(X\) be a Riemann surface.

1. If \(((E, \partial_E), \theta) \in \mathbb{M}_{\text{Dol}}(X, r)\) is a complex variation of Hodge structure or a decoupled Higgs bundle, then the above limits exist and coincide.
2. Let \(((E, \partial_E), \theta) \in \mathbb{M}_{\text{Dol}}(X, r)\) and assume the maximal destabilizing subbundle of \((E, \partial_E)\) is preserved by \(\theta_{\lambda}^h\) for the pluri-harmonic metric \(h\) on \(((E, \partial_E), \theta)\), then the limit \(\psi_{(0,0)}((E, \partial_E), \theta)\) exists, and it coincides with the limit \(\psi_{(0,0)}((E, \partial_E), \theta)\).
3. Let \(((E, \partial_E), \theta) \in \mathbb{M}_{\text{Dol}}(X, r)\), then the limit \(\lim_{\lambda \to 0} \psi_{(\lambda,0)}((E, \partial_E), \lambda \theta)\) exists, and it coincides with the limit \(\psi_{(0,0)}((E, \partial_E), \theta)\).

**Acknowledgements.** The author P. Huang would like to thank his thesis supervisor Prof. Carlos Simpson for the kind help and useful discussions. Both authors would like to thank Prof. Takuro Mochizuki, Prof. Kang Zuo and Dr. Ya Deng for their useful discussions on various occasions.

\(^2\)In this paper, we agree with the terminology of [6], namely a complex variation of Hodge structure means a (polystable) system of Hodge bundles in the sense of Simpson’s paper [29].
2. Preliminaries

2.1. Flat $\lambda$-Connections, Pluri-harmonic Metrics.

**Definition 2.1 ([32, 23]).** Let $X$ be a complex projective manifold and $E$ be a holomorphic vector bundle over $X$, with the underlying smooth vector bundle denoted by $\mathcal{E}$. Fix $\lambda \in \mathbb{C}$.

1. A **holomorphic $\lambda$-connection** on $E$ is a $\mathbb{C}$-linear map $D^\lambda : E \to E \otimes \Omega^1_X$ that satisfies the following $\lambda$-twisted Leibniz rule:

$$D^\lambda(f s) = f D^\lambda s + \lambda s \otimes df,$$

where $f$ and $s$ are holomorphic sections of $\mathcal{O}_X$ and $E$, respectively. It naturally extends to a map $D^\lambda : E \otimes \Omega^p_X \to E \otimes \Omega^{p+1}_X$ for any integer $p \geq 0$. If $D^\lambda \circ D^\lambda = 0$, we call $D^\lambda$ a (holomorphic) flat $\lambda$-connection and the pair $(E, D^\lambda)$ is called a (holomorphic) $\lambda$-flat bundle.

2. A **$C^\infty$ $\lambda$-connection** on $E$ is a $\mathbb{C}$-linear map $D^\lambda : E \to E \otimes T^*X$ that satisfies the following $\lambda$-twisted Leibniz rule:

$$D^\lambda(f s) = f D^\lambda s + \lambda s \otimes \partial f + s \otimes \bar{\partial}f,$$

where $f$ is a smooth function on $X$ and $s$ is a smooth section of $E$. It naturally extends to a map $D^\lambda : E \otimes \Lambda^r(T^*X) \to E \otimes \Lambda^{r+1}(T^*X)$ for any integer $r \geq 0$. If $D^\lambda \circ D^\lambda = 0$, we call $D^\lambda$ a ($C^\infty$) flat $\lambda$-connection, and the pair $(E, D^\lambda)$ is called a ($C^\infty$) $\lambda$-flat bundle.

**Remark 2.2.** Obviously, when $\lambda = 1$ and 0, then above definition reduces to that of a usual flat connection and Higgs field, respectively. Giving a holomorphic flat $\lambda$-connection $D^\lambda$ on $E$ is equivalent to giving a $C^\infty$ flat $\lambda$-connection $\mathbb{D}^\lambda$ on $E$. For simplicity, we do not distinguish $E$ and $\mathbb{E}$ when there is no ambiguity, and for a $\lambda$-flat bundle, we have various notations such as $(E, D^\lambda), (E, \mathbb{D}^\lambda), ((E, \bar{\partial}E), D^\lambda)$ or $((E, d''_E), d'_E)$, depending on the different contexts. Additionally, above notions can also work for the category of coherent sheaves, i.e. $\lambda$-flat bundles can be generalized to $\lambda$-flat coherent sheaves without any difficulty.

Now we consider the $\lambda$-connections in the $C^\infty$-category for more general base manifold $X$, namely we assume $X$ is a compact balanced manifold. Fixing $\lambda \in \mathbb{C}$, let $(E, D^\lambda)$ be a $\lambda$-flat bundle over $X$, and let $h$ be a Hermitian metric on $E$. We decompose $\mathbb{D}^\lambda$ into its (1,0)-part $d'_E$ and (0,1)-part $d''_E$ that defines a holomorphic structure on $E$. From $h$ and $d'_E$, we have a (0,1)-operator $\delta''_h$ determined by the condition $\lambda \partial h(u, v) = h(d''_E u, v) + h(u, \delta''_h v)$, similarly, $h$ and $d''_E$ provides a (1,0)-operator $\delta'_h$ via the condition $\bar{\partial} h(u, v) = h(d''_E u, v) + h(u, \delta'_h v)$. One easily checks that $\delta'_h(f v) = f \delta'_h v + \lambda v \otimes \partial f$, and $\delta''_h(f v) = f \delta''_h v + \lambda v \otimes \bar{\partial} f$. We introduce the following four operators

$$\partial_h := \frac{1}{1 + |\lambda|^2} \left( \lambda d'_E + \delta''_h \right), \quad \bar{\partial}_h := \frac{1}{1 + |\lambda|^2} \left( d''_E + \lambda \delta''_h \right),$$

$$\theta_h := \frac{1}{1 + |\lambda|^2} \left( d'_E - \lambda \delta'_h \right), \quad \theta^\dagger_h := \frac{1}{1 + |\lambda|^2} \left( \lambda d''_E - \delta''_h \right).$$

They satisfy

$$d'_E = \lambda \partial_h + \theta_h, \quad d''_E = \bar{\partial}_h + \lambda \theta^\dagger_h,$$

$$\delta'_h = \bar{\partial}_h - \lambda \theta_h, \quad \delta''_h = \partial_h - \lambda \theta^\dagger_h.$$  \hspace{1cm} (2.1)

Now $\partial_h$ and $\bar{\partial}_h$ obey the usual Leibniz rule, $\theta_h \in C^\infty(X, \Omega^1_X \otimes \text{End}(E))$ and $\theta^\dagger_h \in C^\infty(X, \Omega^{0,1}_X \otimes \text{End}(E))$. Moreover, it’s easy to check that $D_h := \partial_h + \bar{\partial}_h, d''_E + \delta'_h$ and $\lambda^{-1}d'_E + \lambda^{-1}\delta''_h$ are unitary connections with respect to the metric $h$, and $\lambda^\dagger_h$ is the adjoint of $\theta_h$ in the sense that $h(\theta_h(u), v) = h(u, \theta^\dagger_h(v))$. 

$$h(\theta_h(u), v) = h(u, \theta^\dagger_h(v)).$$
We also introduce the operators $D_{\lambda}^{\star} = \delta_\lambda' - \delta_\lambda''$ and $G(h, D^{\lambda}) = [D^{\lambda}, D_{\lambda}^{\star}]$, the latter one is called the pseudo-curvature.

**Definition 2.3.** The Hermitian metric $h$ on a $\lambda$-flat bundle $(E, D^{\lambda})$ is called

1. a harmonic metric if $\Lambda_\omega G(h, D^{\lambda}) = 0$, where $\Lambda_\omega$ stands for the contraction by $\omega$,
2. a pluri-harmonic metric if $G(h, D^{\lambda}) = 0$.

**Proposition 2.4 (Kähler Identities of Flat $\lambda$-Connections, [23]).** Let $(X, \omega)$ be a compact Kähler manifold, then we have

$$
(D^{\lambda})_h^{\star} = -\sqrt{-1}[\Lambda_\omega, D_{\lambda}^{\star}],
$$

$$
(D_{\lambda}^{\star})_h^{\star} = \sqrt{-1}[\Lambda_\omega, D^{\lambda}].
$$

The following property says, for a Hermitian metric on a $\lambda$-flat bundle ($\lambda \neq 0$) over a compact Kähler manifold, it is a pluri-harmonic metric if and only if it is a harmonic metric, or if and only if the $(1,1)$-part of its pseudo-curvature vanishes.

**Proposition 2.5.** Let $\lambda \neq 0$, and let $(X, \omega)$ be a compact Kähler manifold, then all the following conditions are equivalent:

1. $G(h, D^{\lambda}) = 0$,
2. $\Lambda_\omega G(h, D^{\lambda}) = 0$,
3. $(\bar{\partial}_h + \theta_h)^2 = 0$,
4. $(\partial_h + \theta_h^1)^2 = 0$,
5. $\bar{\partial}_h \theta_h = 0$ and $\theta_h^2 = 0$,
6. $\partial_h \theta_h^1 = 0$ and $(\theta_h^1)^2 = 0$,
7. $\bar{\partial}_h \theta_h = 0$,
8. $\partial_h \theta_h^1 = 0$,
9. $\Lambda_\omega \bar{\partial}_h \theta_h = 0$,
10. $\Lambda_\omega \partial_h \theta_h^1 = 0$.

**Proof.** We only give the sketch of the proof of (1) $\iff$ (2), namely $h$ is a pluri-harmonic metric if and only if it is a harmonic metric, more details can be found in the second named author’s thesis [17]. The equivalence of (1), (3), (4), (5), (6) has been shown in [23]. And the equivalence of (5), (6), (7), (8) is recently proved by Mochizuki in [25]. By the flatness of $D^{\lambda}$, we have $(D_{\lambda}^{\star})^2 = 0$, which yields the following Bianchi identities

$$
\overline{D}^{\lambda} G(h, D^{\lambda}) = \overline{D}_{\lambda}^{\star} G(h, D^{\lambda}) = 0.
$$

Therefore, it follows from the identity

$$
G(h, D^{\lambda}) = \frac{\lambda}{1 + |\lambda|^2} \overline{D}^{\lambda} (\lambda^{-1} \theta_h - \theta_h^1)
$$

and the assumption $\Lambda_\omega G(h, D^{\lambda}) = 0$ that

$$
\int_X \langle G(h, D^{\lambda}), G(h, D^{\lambda}) \rangle_{h, \omega} \omega^n = \frac{\lambda}{1 + |\lambda|^2} \int_X (\lambda^{-1} \theta_h - \theta_h^1, (\overline{D}^{\lambda})^{\star} G(h, D^{\lambda}))_{h, \omega} \omega^n = 0,
$$

thus $G(h, D^{\lambda}) = 0$. □

---

3Here we add the notation $^{*}$ to indicate the induced operator on $\text{End}(E) \otimes \Omega^\bullet_X$ from the operator on $E \otimes \Omega^\bullet_X$. 

---
Remark 2.6. Very recently, the authors of [5] introduced $n$-dimensional balanced manifolds of Hodge–Riemann type, namely imposing a further condition
\[
\frac{\omega^{n-1}}{(n-1)!} = \omega_0 \wedge \Omega_0
\]
for certain real $(1,1)$-form $\omega_0$ and $(n-2, n-2)$-form $\Omega_0$ satisfying the Hodge–Riemann bilinear relation. For such special balanced manifolds, the above proposition still holds (cf. [25, Proposition 2.15] and [5, Theorem 5.1]).

Proposition 2.7. Let $\lambda \neq 0$, and let $(E, \mathbb{D}^\lambda)$ be a $\lambda$-flat bundle over a Riemann surface $(X, \omega)$ together with a Hermitian metric $h$, then

1. For any local $\mathbb{D}^\lambda$-flat section $s$ of $E$, we have
   \[
   \Delta_\omega |s|^2_h \geq -\frac{2}{1 + |\lambda|^2} |s|^2_h \Lambda_\omega G(h, \mathbb{D}^\lambda)|_h,
   \]
   where $\Delta_\omega$ denotes the usual Laplacian on $(X, \omega)$.
2. For any local nowhere-vanishing $\mathbb{D}^\lambda$-flat section $s$ of $E$, we have
   \[
   \Delta_\omega \log(|s|^2_h) \geq -\frac{2}{1 + |\lambda|^2} |\Delta_\omega G(h, \mathbb{D}^\lambda)|_h.
   \]

Proof. (1) Let $s$ be a local $\mathbb{D}^\lambda$-flat section, namely we have
\[
d''_E s = (\lambda \partial h + \theta h) s = 0,
\]
\[
d''_E s = (\bar{\partial} h + \lambda \theta^\dagger h) s = 0,
\]
then
\[
\bar{\partial} h(s, s) = h(d''_E s, s) + h(s, \delta''_h s) = h(s, \delta'_h s),
\]
\[
\lambda \partial h(s, s) = h(d''_E s, s) + h(s, \delta''_h s) = h(s, \delta''_h s),
\]
which gives rise to
\[
\lambda \bar{\partial} h(s, s) = \lambda \partial h(s, s') = h(d''_E s, \delta''_h s) + h(s, \delta''_h \delta''_h s) = h(s, \delta''_h \delta''_h s).
\]
By means of the following identities
\[
h(s, \lambda \bar{\partial} h \partial h(s)) = -h(s, \lambda \bar{\partial} h \frac{\theta h}{\lambda}(s)) = -h(s, \frac{\lambda}{\lambda} (\bar{\partial} h \partial h)(s)) - h(s, \lambda \theta h \frac{\theta h}{\lambda}(s))
\]
\[
= -\lambda |\theta h|^2(s)_h + \frac{|\lambda|^2}{\lambda(1 + |\lambda|^2)^2} h(s, G(h, \mathbb{D}^\lambda)) s,
\]
\[
h(s, \theta^\dagger_h \partial h(s)) = -h(s, \bar{\theta}^\dagger_h \frac{\theta h}{\lambda}(s)) = -\frac{1}{\lambda} |\theta h|^2(s)_h
\]
\[
h(s, \lambda^2 \bar{\partial} h \theta h(s)) = h(s, \lambda^2 (\bar{\partial} h \theta h)(s)) + h(s, (\lambda)^2 \theta h \lambda \theta^\dagger_h(s))
\]
\[
= |\lambda|^2 |\theta h|^2(s)_h - \frac{|\lambda|^2}{(1 + |\lambda|^2)^2} h(s, G(h, \mathbb{D}^\lambda)) s,
\]
we obtain
\[
h(s, \delta''_h \delta''_h(s)) = h(s, (\lambda \bar{\partial} h - \theta^\dagger_h)(\bar{\partial} h - \lambda \theta h)(s))
\]
\[
= h(s, \lambda \bar{\partial} h \partial h(s)) - h(s, \theta^\dagger_h \partial h(s)) + h(s, \lambda \theta h \theta h(s)) - h(s, \lambda^2 \bar{\partial} h \theta h(s))
\]
\[
= (1 + |\lambda|^2)(-\lambda |\theta h|^2(s)_h + \frac{1}{\lambda} |\theta h|^2(s)_h) + \frac{\lambda}{1 + |\lambda|^2} h(s, G(h, \mathbb{D}^\lambda)) s.
\]
It follows that

$$-\Delta_\omega|s|^2_h = 2\sqrt{-1} \Lambda_\omega \partial \bar{\partial} |s|^2_h$$

$$= 2\sqrt{-1} \Lambda_\omega[-(1 + |\lambda|^2)|\theta_h^1(s)|^2_h + \frac{1 + |\lambda|^2}{|\lambda|^2}|\theta_h(s)|^2_h + \frac{1}{1 + |\lambda|^2}h(s, G(h, D^\lambda)s)]$$

$$\leq -\frac{2}{1 + |\lambda|^2}h(s, \sqrt{-1} \Lambda_\omega G(h, D^\lambda)s)$$

$$\leq \frac{2}{1 + |\lambda|^2}|s||\Lambda_\omega G(h, D^\lambda)s||_h \leq \frac{\lambda}{1 + |\lambda|^2}|s|^2_h|\Lambda_\omega G(h, D^\lambda)|_h,$$

where we apply the Cauchy–Schwarz inequality for the last two inequalities.

(2) We have

$$\lambda \partial \bar{\partial} \log(|s|^2_h) = \frac{\lambda \partial \bar{\partial} |s|^2_h}{|s|^4_h} - \frac{\lambda \partial |s|^2_h \wedge \bar{\partial} |s|^2_h}{|s|^4_h}$$

$$= \frac{h(s, \delta^0 \delta^0_h)}{|s|^4_h} - \frac{h(s, \delta^0 \delta^0_h \wedge h(s, \delta^0_h)}{|s|^4_h},$$

where the first term on the right hand side of the second equality has been calculated, and the second term can be calculated by the identities

$$h(s, \delta^0_h(s)) = h(s, (\partial_h - \lambda \partial) \theta_h(s)) = h(s, (-\lambda^{-1} - \lambda) \theta_h(s)) = \frac{(1 + |\lambda|^2)}{\lambda}h(s, \theta_h(s)),$$

$$h(s, \delta^0_h(s)) = h(s, (\lambda \partial_h - \theta^{-1}_h(s)) = -(1 + |\lambda|^2)h(s, \theta^1_h(s)).$$

Finally, we arrive at

$$-\Delta_\omega \log(|s|^2_h) = 2\sqrt{-1} \Lambda_\omega \partial \bar{\partial} \log(|s|^2_h)$$

$$= 2\sqrt{-1} \Lambda_\omega[-(1 + |\lambda|^2)|\theta_h^1(s)|^2_h + \frac{1 + |\lambda|^2}{|\lambda|^2}|\theta_h(s)|^2_h$$

$$- \frac{(1 + |\lambda|^2)^2}{|\lambda|^2}h(s, \theta_h(s)) \wedge h(s, \theta^1_h(s)) + \frac{2}{1 + |\lambda|^2}h(s, G(h, D^\lambda)s)|}$$

$$\leq \frac{2}{1 + |\lambda|^2}|\Lambda_\omega G(h, D^\lambda)|_h.$$

We complete the proof. \(\square\)

2.2. Example. Let \(E\) be a Hermitian vector bundle over the punctured unit disk \(\triangle^s = \{z : 0 < |z| < 1\}\) of rank 2 with the local unitary frame \(\{v_1, v_2\}\). In [21], the authors introduced the so-called “fiducial solution” of Hitchin’s equations expressed in terms of the frame \(\{v_1, v_2\}\) as follows

$$A = \frac{1}{8} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} dz - \frac{dz}{z} \\ \frac{dz}{z} \end{pmatrix}$$

$$\theta = \begin{pmatrix} 0 \\ \sqrt{|z|} \end{pmatrix} d\lambda,$$

that solves the decoupled Hitchin’s equations

$$F_A = 0, \quad [\theta, \theta^\dagger] = 0, \quad \bar{\partial}_A \theta = 0,$$

where \(F_A\) denotes the curvature of the connection \(A\), and \(\theta\) is the Higgs field. Let \(\mu \in \mathbb{C}^*\) be a constant, then we have a flat \(\lambda\)-connection \(D^\lambda_{\bar{\partial}_A} = d_E' + d_E''\) with

$$d_E' = \lambda \partial A + \theta, \quad d_E'' = \bar{\partial}_A + \mu \theta^\dagger,$$
then a $\mathbb{D}^\lambda\mu$-flat section $s = \begin{pmatrix} f(z, \bar{z}) \\ g(z, \bar{z}) \end{pmatrix}$ should satisfy the following equations

$$
\begin{align*}
\lambda \frac{\partial f}{\partial z} + \frac{\lambda f}{8z} + \sqrt{|z|} g &= 0, \\
\lambda \frac{\partial g}{\partial z} - \frac{\lambda g}{8z} + \frac{z}{\sqrt{|z|}} f &= 0, \\
\frac{\partial f}{\partial \bar{z}} - \frac{1}{8} f + \frac{\mu \bar{z}}{\sqrt{|z|}} g &= 0, \\
\frac{\partial g}{\partial \bar{z}} + \frac{1}{8} g + \mu \sqrt{|z|} f &= 0.
\end{align*}
$$

(2.3)

Let $z \to 0$ in equations (2.3), we have

$$
\begin{align*}
f(z, \bar{z}) &\to z^{-\frac{1}{8}} z^\frac{1}{8}, \\
g(z, \bar{z}) &\to z^\frac{1}{8} z^{-\frac{1}{8}},
\end{align*}
$$

hence we can assume that

$$
\begin{align*}
f(z, \bar{z}) &= z^{-\frac{1}{8}} z^\frac{1}{8} u(z, \bar{z}), \\
g(z, \bar{z}) &= z^\frac{1}{8} z^{-\frac{1}{8}} v(z, \bar{z})
\end{align*}
$$

with $\lim_{z \to 0} u = \lim_{z \to 0} v = 1$. Then $u(z, \bar{z})$ and $v(z, \bar{z})$ should satisfy the following equations

$$
\begin{align*}
\lambda \frac{\partial u}{\partial z} + \sqrt{z} v &= 0, \\
\lambda \frac{\partial v}{\partial z} + \sqrt{z} u &= 0, \\
\frac{\partial u}{\partial \bar{z}} + \mu \sqrt{z} v &= 0, \\
\frac{\partial v}{\partial \bar{z}} + \mu \sqrt{z} u &= 0,
\end{align*}
$$

which imply

$$
\frac{\partial u}{\partial \left( \frac{z^\frac{3}{2}}{\lambda \mu} \right)} = \frac{\partial u}{\partial \left( \frac{\bar{z}^\frac{3}{2}}{\lambda \mu} \right)}, \quad \frac{\partial v}{\partial \left( \frac{z^\frac{3}{2}}{\lambda \mu} \right)} = \frac{\partial v}{\partial \left( \frac{\bar{z}^\frac{3}{2}}{\lambda \mu} \right)}.
$$

Therefore, we can write

$$
\begin{align*}
u(z, \bar{z}) &= U \left( \frac{z^{\frac{3}{4}}}{\lambda \mu} + \bar{z}^{\frac{3}{4}} \right),
\end{align*}
$$

Introducing the new variable $X = \frac{z^{\frac{3}{4}}}{\lambda \mu} + \bar{z}^{\frac{3}{4}}$, we have

$$
\begin{align*}
3 \frac{\partial U}{2 \mu \partial X} + V &= 0, \\
3 \frac{\partial V}{2 \mu \partial X} + U &= 0,
\end{align*}
$$

which can be solved easily

$$
\begin{align*}
U(X) &= C_1 \exp\left( \frac{2\mu}{3} X \right) + C_2 \exp\left( -\frac{2\mu}{3} X \right), \\
V(X) &= -C_1 \exp\left( \frac{2\mu}{3} X \right) + C_2 \exp\left( -\frac{2\mu}{3} X \right),
\end{align*}
$$
where \( C_1 \) and \( C_2 \) are two constants. Consequently, any local \( \mathbb{D}^\lambda_{\mu} \)-flat section \( s \) is the \( \mathbb{C} \)-linear combination of the following two sections

\[
s_1 = \left( z^{-\frac{1}{3}} \bar{z}^\frac{1}{3} \exp\left( \frac{2}{3} \zeta \bar{z}^\frac{2}{3} \right) - z^\frac{1}{3} \bar{z}^{-\frac{1}{3}} \exp\left( -\frac{2}{3} \zeta \bar{z}^\frac{2}{3} \right) \right),
\]

\[
s_2 = \left( z^{-\frac{1}{3}} \bar{z}^\frac{1}{3} \exp\left( -\frac{2}{3} \zeta \bar{z}^\frac{2}{3} \right) - \bar{z}^\frac{1}{3} z^{-\frac{1}{3}} \exp\left( \frac{2}{3} \zeta \bar{z}^\frac{2}{3} \right) \right).
\]

One easily checks that \( \Delta \log|s|^2_{\lambda,\mu} = 0 \).

Now let \( \lambda' = t \lambda \). We want to find the pluri-harmonic metric \( h_t \) for the \( \lambda' \)-flat bundle \( (E, \mathbb{D}^{\lambda'} = td'_{E} + d''_{E}) \) with \( \mu = \lambda \). Denote the matrix form of \( h_t \) in terms of the frame \( \{v_1, v_2\} \) by \( H_t \). We write \( t \cdot d'_{E} = \lambda' \partial_{A} + t \theta, d''_{E} = \bar{\partial}_{A} + \lambda(1 - |t|^2) \theta \bar{1} + \lambda'(t \theta)^1 \), then one can take \( ((E, \partial_{A} + \lambda(1 - |t|^2) \theta \bar{1}), t \theta) \) as the Higgs bundle by requiring

\[
\begin{align*}
\bar{H}_{t}^{-1} \theta 1 \bar{H}_t &= \theta^1, \\
\partial H_t &= (A^{1,0})^T H_t + H_t(A^{0,1} + \lambda(1 - |t|^2) \theta \bar{1}), \\
\bar{\partial} H_t &= (A^{0,1} + \lambda(1 - |t|^2) \theta \bar{1})^T H_t + H_t A^{1,0}.
\end{align*}
\]

Expressing \( H_t \) as

\[
H_t = \begin{pmatrix}
a(z, \bar{z}) & b(z, \bar{z}) \\
b(z, \bar{z}) & c(z, \bar{z})
\end{pmatrix},
\]

then we have

\[
\begin{align*}
a &= c, \\
b \cdot \bar{z}^\frac{1}{2} &= b \cdot \bar{z}^\frac{1}{2}, \\
\frac{\partial a}{\partial z} &= \bar{\lambda}(1 - |t|^2) b \sqrt{|z|}, \\
\frac{\partial a}{\partial \bar{z}} &= \bar{\lambda}(1 - |t|^2) b \frac{\bar{z}}{\sqrt{|z|}}, \\
\frac{\partial b}{\partial z} &= \frac{b}{4 z} + \bar{\lambda}(1 - |t|^2) a \frac{z}{\sqrt{|z|}}, \\
\frac{\partial b}{\partial \bar{z}} &= -\frac{b}{4 z} + \lambda(1 - |t|^2) a \sqrt{|z|}.
\end{align*}
\]

It can be resolved as follows

\[
a(z, \bar{z}) = f(\bar{z}) \exp\left( \frac{2}{3} \bar{\lambda}(1 - |t|^2) z^\frac{2}{3} \right) + g(\bar{z}) \exp\left( -\frac{2}{3} \bar{\lambda}(1 - |t|^2) z^\frac{2}{3} \right),
\]

\[
b(z, \bar{z}) = \frac{z^\frac{1}{2}}{\sqrt{|z|}} \left( f(\bar{z}) \exp\left( \frac{2}{3} \bar{\lambda}(1 - |t|^2) z^\frac{2}{3} \right) - g(\bar{z}) \exp\left( -\frac{2}{3} \bar{\lambda}(1 - |t|^2) z^\frac{2}{3} \right) \right),
\]

where

\[
f(\bar{z}) = C_1 \exp\left( \frac{2}{3} \bar{\lambda}(1 - |t|^2) \bar{z}^\frac{2}{3} \right) + C_2 \exp\left( -\frac{2}{3} \bar{\lambda}(1 - |t|^2) \bar{z}^\frac{2}{3} \right),
\]

\[
g(\bar{z}) = C_2 \exp\left( \frac{2}{3} \bar{\lambda}(1 - |t|^2) \bar{z}^\frac{2}{3} \right) + C_3 \exp\left( -\frac{2}{3} \bar{\lambda}(1 - |t|^2) \bar{z}^\frac{2}{3} \right),
\]

for constants \( C_1, C_2 \) and \( C_3 \).

**Remark 2.8.** This example exhibits the non-uniqueness of pluri-harmonic metrics on \( \lambda \)-flat bundles over a non-complete manifold.
3. Simpson–Mochizuki Correspondence

3.1. Categorical Version.

Definition 3.1 ([23, 25]).

(1) Let $X$ be a complex projective manifold with a fixed ample line bundle $L$. A $\lambda$-flat bundle $(E, D^\lambda)$ over $X$ is called $\mu_L$-stable (resp. $\mu_L$-semistable)$^4$ if for any $\lambda$-flat subbundle $(V, D^\lambda|_V)$ of $0 < \text{rank}(V) < \text{rank}(E)$, we have the following inequality

$$\mu_L(V) < \mu_L(E) \quad (\text{resp.} \quad \mu_L(V) \leq \mu_L(E)), \tag{1}$$

where $\mu_L(\bullet) = \frac{\text{deg}(\bullet)}{\text{rank}(\bullet)}$ denotes the slope of bundle with respect to $L$. It is $\mu_L$-polystable$^5$ if it decomposes as a direct sum of $\mu_L$-stable $\lambda$-flat bundles with the same slope.

(2) Let $X$ be an $n$-dimensional compact Kähler manifold with a Kähler form $\omega$. A $\lambda$-flat bundle $(E, D^\lambda)$ with a Hermitian metric $h$ over $X$ is called analytically stable (resp. analytically semistable) if for any $\lambda$-flat torsion-free coherent subsheaf $(V, D^\lambda|_V)$ of $0 < \text{rank}(V) < \text{rank}(E)$, we have the following inequality

$$\mu_\omega(V) < \mu_\omega(E) \quad (\text{resp.} \quad \mu_\omega(V) \leq \mu_\omega(E)), \tag{2}$$

where $\mu_\omega(\bullet) = \frac{\int_X S(\bullet) \text{Tr}(G(h|_{D^\lambda(\bullet)})\omega^n)}{\text{rank}(\bullet)}$ denotes the slope of sheaf with respect to $\omega$, for $S(\bullet)$ being the singular locus of the sheaf. It is analytically polystable if it decomposes as an orthogonal direct sum of analytically stable $\lambda$-flat bundles with the same slope.

By the wonderful work of Simpson and Mochizuki, we have the following theorem.

Theorem 3.2 (Simpson–Mochizuki Correspondence, Kobayashi–Hitchin version, [26, 23]).

Fix $\lambda \in \mathbb{C}$.

(1) Let $X$ be a complex projective manifold with a fixed ample line bundle $L$. A $\lambda$-flat bundle $(E, D^\lambda)$ over $X$ is $\mu_L$-polystable with vanishing Chern classes if and only if there is a pluri-harmonic metric $h$ on $(E, D^\lambda)$.

(2) Let $(X, \omega)$ be a compact Kähler manifold, $(E, D^\lambda, h_0)$ be an analytically stable $\lambda$-flat bundle. Then there exists a unique Hermitian metric $h$ such that $\det(h) = \det(h_0)$ and the Hermitian–Einstein condition $\Lambda_\omega G(h, D^\lambda)^\perp = 0$ holds, where $G(h, D^\lambda)^\perp$ denotes the trace-free part of $G(h, D^\lambda)$.

(3) (Uniqueness of pluri-harmonic metric) Let $h_i \ (i = 1, 2)$ be the pluri-harmonic metric on the $\lambda$-flat bundle $(E, D^\lambda)$, then

- we have the decomposition of $\lambda$-flat bundles $(E, D^\lambda) = \bigoplus (E_a, D_a^\lambda)$ which is orthogonal with respect to both of $h_i \ (i = 1, 2)$,
- the restrictions $h_{i,a}$ of $h_i$ to $E_a$ satisfy $h_{1,a} = c_a h_{2,a}$ for positive constants $c_a$.

Remark 3.3. This correspondence still holds for the case of stable parabolic logarithmic $\lambda$-flat bundles over a projective variety with a simple normal crossing divisor by imposing a compatibility condition of pluri-harmonic metric with the parabolic structure (for details see [27, 22, 23]).

As a direct application of the above theorem, we have the following correspondence as the interpolation of the usual Corlette–Simpson correspondence [7, 26, 29] and the Riemann–Hilbert correspondence.

$^4$Sometimes we omit the notation $\mu_L$ when there is no ambiguity.

$^5$When $\lambda \neq 0$, a $\lambda$-flat bundle $(E, D^\lambda)$ is $\mu_L$-stable if and only if it is simple, namely it has no non-trivial proper $\lambda$-flat subbundle, and $(E, D^\lambda)$ is $\mu_L$-polystable if and only if it is semisimple, namely it is a direct sum of simple $\lambda$-flat bundles.
Corollary 3.4 (Simpson–Mochizuki Correspondence, Categorical version, [23, Corollary 5.18]). Let $X$ be a complex projective manifold. Then for any $\lambda \in \mathbb{C}$, there is an equivalence between the category of $\mu_L$-polystable $\lambda$-flat bundles with vanishing Chern classes and the category of semisimple representations of the fundamental group $\pi_1(X)$ into $\text{GL}(r, \mathbb{C})$. This equivalence preserves tensor products, direct sums and duals.

Proof. For the case of $\lambda = 0$, we have the usual Simpson correspondence. So we assume $\lambda \neq 0$. Let $(E, D^\lambda)$ be a $\mu_L$-polystable $\lambda$-flat bundle (with trivial characteristic numbers), then there is a pluri-harmonic metric $h$ on $E$. Therefore, we get

$$0 = (D^\lambda)^2 = (\lambda \partial_h + \bar{\partial}_h + \theta_h + \lambda \theta_h^\dagger)^2 = \lambda (R(h) + [\theta_h, \theta_h^\dagger] + \partial_h \theta_h + \bar{\partial}_h \theta_h^\dagger),$$

where $R(h) = (D_h)^2$ is the curvature of the unitary connection $\mathcal{D}_h$, hence by Proposition 2.5

$$R(h) + [\theta_h, \theta_h^\dagger] = \bar{\partial}_h \theta_h = \bar{\partial}_h \theta_h^\dagger = 0,$$

which implies $((E, \bar{\partial}_h), \theta_h, h)$ is a harmonic Higgs bundle associated with a semi-simple representation $\rho : \pi_1(X) \to \text{GL}(r, \mathbb{C})$ by the Hitchin–Simpson correspondence.

Conversely, if we have a semi-simple representation $\rho : \pi_1(X) \to \text{GL}(r, \mathbb{C})$, then we have a Higgs bundle $((E, \bar{\partial}_h), \theta, h)$ with the pluri-harmonic metric $h$, which gives rise to a flat $\lambda$-connection $\mathbb{D}^\lambda = d'_E + d''_E$ with

$$d'_E = \lambda \partial_{E,h} + \theta, \quad d''_E = \bar{\partial}_E + \lambda \theta_h^\dagger,$$

where $\partial_{E,h}$ is a $(1,0)$-type operator such that $\partial_{E,h} + \bar{\partial}_E$ is a unitary connection with respect to $h$, and $\theta_h^\dagger$ is the adjoint of $\theta$ with respect to $h$. Clearly, $h$ is also a pluri-harmonic metric for the $\lambda$-flat bundle $(E, D^\lambda)$, hence it is polystable with trivial characteristic numbers.

Since pluri-harmonic metrics preserve tensor products, direct sums and duals, the equivalence described as above also preserves them. \hfill \Box

Corollary 3.5. If $X$ is a Riemann surface and $(E, D^\lambda)$ is a stable $\lambda$-flat bundle over $X$ of rank $r \geq 2$ and with vanishing the first Chern class, then there is no non-trivial global $D^\lambda$-flat section of $E$.

Proof. When $\lambda = 0$, the claim follows from [4, Theorem 3.1]. Assume $\lambda \neq 0$. Let $h$ be the pluri-harmonic on the stable $\lambda$-flat bundle $(E, D^\lambda)$, and $s$ be the non-trivial global $D^\lambda$-flat section, then the norm $|s|^2_h$ is a sub-harmonic function by Proposition 2.7. Since $X$ is compact, $|s|^2_h$ is a nonzero constant, hence the section $s$ generates a trivial line subbundle of $(E, D^\lambda)$, which contradicts to the stability of $(E, D^\lambda)$. \hfill \Box

3.2. Moduli version. Let $X$ be a complex projective manifold. Fixing $\lambda \in \mathbb{C}$, denote by $M^\lambda_{\text{Hod}}(X, r)$ the moduli stack of rank $r$ $\lambda$-flat bundles with vanishing Chern classes over $X$, and by $M^\lambda_{\text{Hod}}(X, r)$ the coarse moduli space for the semistable stratum of this stack, which is a quasi-projective variety and parameterizes the isomorphism classes of polystable $\lambda$-flat bundles. Let $M^\lambda_{\text{Hod}}(X, r)$ be the smooth locus of $M^\lambda_{\text{Hod}}(X, r)$, which is a Zariski dense open subset and parameterizes the isomorphism classes of stable $\lambda$-flat bundles. In particular, $M^\lambda_{\text{Hod}}(X, r) = M^\lambda_{\text{DR}}(X, r)$ and $M^\lambda_{\text{Hod}}(X, r) = M^\lambda_{\text{Dol}}(X, r)$. Picking a base point $x \in X$, we have the representation space $R^\lambda_{\text{Hod}}(X, x, r)$, which is the fine moduli space of semistable $\lambda$-flat bundles provided with a frame for the fiber over $x$, in particular, $R^\lambda_{\text{Hod}}(X, r) = R^\lambda_{\text{DR}}(X, r)$, and $R^\lambda_{\text{Hod}}(X, r) = R^\lambda_{\text{Dol}}(X, r)$. The group $\text{GL}(r, \mathbb{C})$ acts on $R^\lambda_{\text{Hod}}(X, x, r)$, and $M^\lambda_{\text{Hod}}(X, r) = R^\lambda_{\text{Hod}}(X, x, r) / \text{GL}(r, \mathbb{C})$ as the universal categorical quotient. We also consider the subset $R^\lambda_{\text{Hod}}(X, x, r) \subset R^\lambda_{\text{Hod}}(X, x, r)$ that consists of those points which admits a pluri-harmonic
metric compatible with the frame at \( x \). Such condition fixes the metric uniquely. The group \( U(r) \) acts on \( R^\lambda_{\text{Hod}}(X,x,r) \), and \( M^\lambda_{\text{Hod}}(X,r) = R^\lambda_{\text{Hod}}(X,x,r)/U(r) \) as the topological quotient.

Let \( N^\lambda_{\text{Hod}}(X,r) \) be the Zariski dense open subset of \( M^\lambda_{\text{Hod}}(X,r) \) that parameterizes \( \lambda \)-flat bundles such that the underlying vector bundles are semistable, which is an affine bundle over the coarse moduli space \( \mathbb{B}(X,r) \) of semistable vector bundles of rank \( r \) with vanishing Chern classes over \( X \).

**Proposition 3.6.** Suppose \( r \geq 2 \).

(1) Let \( X \) be a Riemann surface of genus \( g \geq 2 \). One defines \( \wt{M}^\lambda_{\text{Hod}}(X,r) = M^\lambda_{\text{Hod}}(X,r) \setminus M^\lambda_{\text{Hod}}(X,r) \). If both \( M^\lambda_{\text{Hod}}(X,r) \) and \( \wt{M}^\lambda_{\text{Hod}}(X,r) \) are nonempty, then for the codimension of \( M^\lambda_{\text{Hod}}(X,r) \) in \( \wt{M}^\lambda_{\text{Hod}}(X,r) \) we have

\[
\text{codim}_C \wt{M}^\lambda_{\text{Hod}}(X,r) \geq 2.
\]

(2) Let \( X \) be a Riemann surface of genus \( g \geq 3 \). One defines \( \wt{N}^\lambda_{\text{Hod}}(X,r) = M^\lambda_{\text{Hod}}(X,r) \setminus N^\lambda_{\text{Hod}}(X,r) \). Then for the codimension of \( \wt{M}^\lambda_{\text{Hod}}(X,r) \) in \( \wt{M}^\lambda_{\text{Hod}}(X,r) \) we have

\[
\text{codim}_C \wt{M}^\lambda_{\text{Hod}}(X,r) \geq 2.
\]

**Proof.** (1) For any partition \( \vec{\tau} = (r_1, \cdots, r_k) \in \mathbb{Z}^+^k \) with \( \sum_{i=1}^k r_i = r \) and \( 1 < k \leq r \), we introduce a map

\[
\delta_\vec{\tau} : M^\lambda_{\text{Hod}}(X, \vec{\tau}) := M^\lambda_{\text{Hod}}(X,r_1) \times \cdots \times M^\lambda_{\text{Hod}}(X,r_k) \to \mathbb{M}^\lambda_{\text{Hod}}(X,r)
\]

by \( ((E_1, \theta_1), \cdots, (E_k, \theta_k)) \mapsto (\bigoplus_{i=1}^k E_i, \bigoplus_{i=1}^k \theta_k) \). Since \( \delta_\vec{\tau} \) is injective, we have

\[
\dim C \wt{M}^\lambda_{\text{Hod}}(X,r) = \dim C \bigcup_{\{\vec{\tau}\}} \text{Im} (\delta_\vec{\tau}) = \max \{ \dim C M^\lambda_{\text{Hod}}(X, \vec{\tau}) \}.
\]

Hitchin and Simpson calculated the dimension of moduli space \([12, 29, 30]\)

\[
\dim C M^\lambda_{\text{Hod}}(X,r_i) = \dim C M^\lambda_{\text{Dol}}(X,r_i) = 2r_i^2(g-1) + 2,
\]

then one can easily show that

\[
\max \{ \dim C M^\lambda_{\text{Hod}}(X, \vec{\tau}) \} = 2(g-1)((r-1)^2 + 1) + 4,
\]

which means that \( \text{codim}_C \wt{M}^\lambda_{\text{Hod}}(X,r) = 4(g-1)(r-1) - 2 \geq 2 \).

(2) Let \( N^\lambda_{\text{Hod}}(X,r) = N^\lambda_{\text{Hod}}(X,r) \cap M^\lambda_{\text{Hod}}(X,r), \wt{N}^\lambda_{\text{Hod}}(X,r) = N^\lambda_{\text{Hod}}(X,r) \setminus N^\lambda_{\text{Hod}}(X,r), \) and \( \wt{M}^\lambda_{\text{Hod}}(X,r) = M^\lambda_{\text{Hod}}(X,r) \setminus N^\lambda_{\text{Hod}}(X,r) \). The same argument as (1) shows that

\[
\text{codim}_C \wt{N}^\lambda_{\text{Hod}}(X,r) = 2(g-1)(r-1) - 1 \geq 3.
\]

Therefore, it suffices to prove \( \text{codim}_C \wt{M}^\lambda_{\text{Hod}}(X,r) \geq 2 \).

For a filtration \( E = E^0 \supset E^1 \supset \cdots \supset E^{k-1} \supset E^k = 0 \) of subbundles of a given vector bundle \( E \), the pair \( (\vec{\tau}, \vec{d}) \) is called the type of this filtration, where

\[
\vec{\tau} = (\text{rank}(E^{k-1}), \text{rank}(E^{k-2}/E^{k-1}), \cdots, \text{rank}(E/E^0)),
\]

\[
\vec{d} = (\text{deg}(E^{k-1}), \text{deg}(E^{k-2}/E^{k-1}), \cdots, \text{deg}(E/E^1)).
\]

The moduli space \( M^\lambda_{\text{Hod}}(X,r) \) admits a Harder–Narasimhan stratification

\[
M^\lambda_{\text{Hod}}(X,r) = \coprod_{(\vec{\tau}, \vec{d})} H_{(\vec{\tau}, \vec{d})}(X,r),
\]

where the locally closed subset \( H_{(\vec{\tau}, \vec{d})}(X,r) \) of \( M^\lambda_{\text{Hod}}(X,r) \) parameterizes stable \( \lambda \)-flat bundles such that the underlying vector bundles having Harder–Narasimhan type \( (\vec{\tau}, \vec{d}) \). Due to the boundedness of moduli space \([30]\), there are finitely many Harder–Narasimhan types occur in the
disjoint union. The forgetful map $f : H_{(\gamma,d)}(X,r) \to B_{(\gamma,d)}(X,r)$ via $(E, D^\lambda) \mapsto E$ gives rise to a fibration over the space $B_{(\gamma,d)}(X,r)$ of isomorphism classes of vector bundles with Harder–Narasimhan type $(\gamma, d)$ with fibers as Zariski open dense subsets of an affine space of dimension $d_f$. By Riemann–Roch formula, $d_f$ is given by

$$d_f = \dim_{\mathbb{C}} H^1(X, \text{End}(E)) - \dim_{\mathbb{C}} H^0(X, \text{End}(E)) + 1 = r^2(g - 1) + 1.$$ 

Obviously, we have

$$\bar{M}^\lambda_{\text{Hod}}(X, r) = \left\{ \prod_{\gamma \neq (r), d \neq (0)} H_{(\gamma, d)}(X, r) \prod \hat{H}(X, r),$$

where $\hat{H}(X, r)$ is a subset of $H_{(\gamma, d)=(0)}(X, r)$ consisting of $\lambda$-flat bundles such that the underlying vector bundle is semistable but not stable. Then, by a result of [2] which shows that the dimension of $B_{(\gamma,d)}(X,r)$ is at most $r^2(g-1) - (r-1)(g-2)$ if $\gamma \neq (r), d \neq (0)$, we conclude that

$$\text{codim}_{\mathbb{C}} \prod_{\gamma \neq (r), d \neq (0)} H_{(\gamma,d)}(X, r) \geq 1 + (r - 1)(g - 2) \geq 2.$$ 

And by a result of [3] which asserts that the dimension of $\hat{H}(X, r)$ is at most $(2r^2 - r + 1)(g - 1) + 2$, we have

$$\text{codim}_{\mathbb{C}} \hat{H}(X, r) \geq (r - 1)(g - 1) \geq 2.$$ 

From the above two inequalities the final result follows. \hfill \Box

The proof of the following theorem is after Simpson ([31, Lemma 7.13], [34, Lemma 8.1]) essentially.

**Theorem 3.7.** The natural quotient map $q : R^\lambda_{\text{Hod}}(X, x, r) \to \bar{M}^\lambda_{\text{Hod}}(X, r)$ is proper.

**Proof.** The cases of $\lambda = 0, 1$ have been proved by Simpson ([31, Corollary 7.12, Corollary 7.15]).

For the case of $\lambda \neq 0, 1$, we consider a sequence $\{(E_i, d^\prime_{E_i}), D^\lambda_i, \beta_i)\}$ lying inside the inverse image of a compact subset $\bar{M}_{\text{Hod}}^\lambda(X, r)$, where $\beta_i$ is a frame on $E_i$, and let $h_i$ be the unique pluri-harmonic metric on $((E_i, d^\prime_{E_i}), D^\lambda_i, \beta_i)$. It suffices to show that the characteristic polynomial of the corresponding Higgs fields $\{h_i\}$ are uniformly bounded in $C_0$-norm. By the map $((E, d^\prime), D^\lambda, \beta) \mapsto ((E, d^\prime), \lambda^{-1}D^\lambda, \beta)$ and the Riemann–Hilbert correspondence, $\bar{M}^\lambda_{\text{Hod}}(X, r)$ is complex analytically isomorphic to $\bar{M}_B(X, r)$, the coarse moduli space of representations $\pi_1(X, x) \to \text{GL}(r, \mathbb{C})$. Let $\rho_i$ be the monodromy representation corresponding to $((E_i, d^\prime_{E_i}), D^\lambda_i, \beta_i)$, then $\{\rho_i\}$ lie over a compact subset of $\bar{M}_B(X, r)$, hence the norms $|\rho_i(\gamma)| = \sqrt{\text{Tr}(\rho_i(\gamma)\rho_i^\dagger(\gamma))}$ are uniformly bounded for any generator $\gamma$ of $\pi_1(X, x)$. Denote by $\rho^{(\infty)}(\gamma)$ the limit point of $\{\rho_i(\gamma)\}$. By virtue of Mochizuki correspondence, each $\rho_i$ produces another simple monodromy representation $\bar{\rho}_i$ of $\pi_1(X, x)$ given by the flat bundle $((E, \bar{\rho}_i), \bar{\theta}_h, \theta_h, \bar{\theta}_h, \beta)$, then the norms $|\bar{\rho}_i(\gamma)|$ are also uniformly bounded. Indeed, we consider a family of flat bundles $((E, \bar{\rho}_i, t^{-1}\bar{\theta}_h, \theta_h, \bar{\theta}_h, \beta))$ parameterized by $t \in \mathbb{C}^*$, and the associated family of monodromy representations is denoted by $\rho^{(i)}(\gamma)$. It is clear that the map $t \mapsto |\rho^{(i)}(\gamma)|$ is continuous. We have the bound $|\rho_i(\gamma)| \leq C$. If $|\bar{\rho}_i(\gamma)|$ tends to infinity, then for any constant $C_1 > C$, there is a sequence $\{t_i\}$ which lie in a curve segment joining $\lambda^{-1}$ to $1$ but not passing through $0$ such that $|\rho_i(\gamma)| = C_1$. By [28, Theorem 1], the map $\rho \mapsto |\rho(\gamma)|$ from $\bar{M}_B(X, r)$ to $\mathbb{R}$ is proper, thus we may assume $\{\rho_i\}$ has a limit point $\rho_\infty$, then $|\rho_\infty(\gamma)| = C_1$. We can also assume the sequence $\{t_i\}$ has the limit point $t_\infty$, then $\rho_\infty(\gamma) = \rho^{(\infty)}(\gamma)$ due to the separatedness
of moduli space, whose norm has a bound $C_2$. If one takes $C_1 > C_2$, we will get a contradiction, which lead to the uniform bound of $\{|\tilde{h}(\gamma)|\}$.

Consequently, by [28, Corollary 6], the $L^2$-norms $\{||\theta_{h_i}||_{L^2}\}$ are uniformly bounded. Since the maximum norm of an eigenvalue of a holomorphic matrix is a subharmonic function, the eigenforms of $\theta_{h_i}$ are uniformly bounded in $C^0$. So far, we prove the claim on the characteristic polynomial of Higgs fields $\{\theta_{h_i}\}$. Therefore, [29, Lemma 2.8], or [31, Proposition 7.9] implies that there is a harmonic bundle $(E, \bar{\partial}, \theta, h, \beta)$, a subsequence $\{i'\}$ and $C^\infty$-automorphisms $g_{i'}$ such that $g_{i'}^*(h_{i'}) = h$ and $g_{i'}^*(\bar{\partial}_{h_{i'}}) - \bar{\partial}, g_{i'}^*(\theta_{h_{i'}}) - \theta$ converge to zero strongly in the operator norm for operators from $L^p_1$ to $L^p$ for $p > 1$, and the frames $g_{i'}^*(\beta_{i'})$ converge to $\beta$. Since the $\lambda$-flat bundle can be treated as certain $\Lambda$-module in the sense of Simpson [33], [30, Theorem 5.12] is valid for this case, hence there is a subsequence $\{((E_{i'}, \bar{\partial}_{E_{i'}}, D^\lambda_{i'}, \beta_{i'}))\}$ converge to a point $((E, \bar{\partial} + \lambda \theta_1^h), \lambda \partial_h + \theta, \beta)$ in $R_{\text{Hod}}^\lambda(X, x, r)$. 

Corollary 3.8 (Simpson–Mochizuki Correspondence, Moduli version). The Simpson–Mochizuki correspondence described in Corollary 3.4 provides a homeomorphism of moduli spaces

$$M^\lambda_{\text{Hod}}(X, r) \simeq M_{\text{Dal}}(X, r).$$

Proof. A key step has been completed in the proof of the above theorem, the remaining arguments are totally parallel to [31, Theorem 7.18], so we omit them here. 

4. Mochizuki Correspondence on Balanced Manifolds

In this section, we always assume $X$ is an $n$-dimensional compact balanced manifold, and $(E, \bar{\partial}_E, D^\lambda)$ is a stable $\lambda$-flat bundle of rank $\text{rk}(E) \geq 2$ over $X$ with fixed $\lambda \neq 0$. We will use the standard method of continuity to show the existence of harmonic metric on $(E, \bar{\partial}_E, D^\lambda)$.

Let $h_0$ be a fixed Hermitian metric on $E$, then both $\bar{\partial}_E + \delta_{h_0}'$ and $\lambda^{-1}D^\lambda + \bar{\lambda}^{-1}\delta''_{h_0}$ are $h_0$-unitary connections, whose curvatures are given by, respectively,

$$\begin{align*}
R_1(h_0) &:= (\bar{\partial}_E + \delta_{h_0}')^2 = -\frac{(1 + |\lambda|^2)}{\lambda} \left( \bar{\partial}_{h_0} \theta_{h_0} + \lambda \theta_{h_0} \right), \\
R_2(h_0) &:= (\lambda^{-1}D^\lambda + \bar{\lambda}^{-1}\delta''_{h_0})^2 = -\frac{(1 + |\lambda|^2)}{\lambda^2} \left( \bar{\lambda} \bar{\partial}_{h_0} \theta_{h_0} + \theta_{h_0} \right).
\end{align*}$$

(4.1)

Let $S(E, h_0) \subset C^\infty(\text{End}(E))$ be the set that consists of $h_0$-self-adjoint endomorphisms of $E$ and $S^+(E, h_0)$ be the subset of $S(E, h_0)$ that consists of positive-definite endomorphisms. Write $h = h_0 \cdot s$ for some $s \in S^+(E, h_0)$, then

$$\begin{align*}
\delta_{h}' &:= \delta_{h_0}' + s^{-1}\delta_{h_0}' s, \quad \delta_{h}'' = \delta_{h_0}''' + s^{-1}\delta_{h_0}''' s,
\end{align*}$$

and we have the corresponding curvatures

$$\begin{align*}
R_1(h) &:= R_1(h_0) + \bar{\partial}_E (s^{-1}\delta_{h_0}' s) , \\
R_2(h) &:= R_2(h_0) + |\lambda|^{-2}D^\lambda (s^{-1}\delta_{h_0}''' s).
\end{align*}$$

If $h$ is a harmonic metric on $(E, \bar{\partial}_E, D^\lambda)$, then $s$ must satisfy the following equation

$$K(h_0) + \sqrt{-1}A_\omega \bar{\partial}_E (s^{-1}\delta_{h_0}' s) - \sqrt{-1}A_\omega D^\lambda (s^{-1}\delta_{h_0}''' s) = 0,$$

where $K(h_0) = \sqrt{-1}A_\omega (R_1(h_0) - |\lambda|^2R_2(h_0))$. We consider the following perturbed equation

$$\Gamma_\varepsilon(h_0, s(\varepsilon)) := K(h_0) + \sqrt{-1}A_\omega (\bar{\partial}_E (s^{-1}(\varepsilon)\delta_{h_0}' s(\varepsilon)) - \bar{\partial}_{s^{-1}(\varepsilon)\delta_{h_0}''' s(\varepsilon)} + \varepsilon \log s(\varepsilon) = 0$$

(4.2)
for some real number $\varepsilon$. One defines the set
\[
J(h_0) = \{ \varepsilon \in (0, 1] : \text{there exists } s(\varepsilon) \in S^+(E, h_0) \text{ such that } \Gamma_\varepsilon(h_0, s(\varepsilon)) = 0 \}.
\]
Given a metric $h$ on $E$, let $h_0 = h \cdot e^{K(h)}$ so that $e^{K(h)} \in S^+(E, h_0)$, then $\Gamma_1(h_0, e^{-K(h)}) = 0$. Namely, one can choose a Hermitian metric $h_0$ on $E$ such that $\omega \in J(h_0)$.

**Lemma 4.1.** $J(h_0)$ is a nonempty open subset of $(0, 1]$.

*Proof.* For a rational number $q$, one introduces the adjoint action $\text{Ad}_s^q$ on an operator $O \in C^\infty(\text{End}(E) \otimes \Lambda^\bullet(T^*X))$ as $\text{Ad}_s^q : O \mapsto s^q O s^{-q}$, which defines a new operator $\text{Ad}_s^q(O) = \text{Ad}_s^q \cdot (O \circ \text{Ad}_s^{-q}) : \text{End}(E) \rightarrow \text{End}(E) \otimes \Lambda^\bullet(T^*X)$. Then we introduce the following notations
\[
P_1(q, s) = \text{Ad}_s^q(\tilde{\omega}_h) \circ \text{Ad}_s^{-q}(\delta_{h_0}^\prime), \quad \tilde{P}_1(q, s) = \text{Ad}_s^q(\delta_{h_0}^\prime) \circ \text{Ad}_s^{-q}(\tilde{\omega}_h)
\]
\[
P_2(q, s) = \text{Ad}_s^q(D^\lambda) \circ \text{Ad}_s^{-q}(\delta_{h_0}^\prime), \quad \tilde{P}_2(q, s) = \text{Ad}_s^q(\delta_{h_0}^\prime) \circ \text{Ad}_s^{-q}(D^\lambda).
\]

In particular, we denote $P_1^s = P_1(\frac{1}{2}, s), P_2^s = P_2(\frac{1}{2}, s)$, $\tilde{P}_1^s = \tilde{P}_1(\frac{1}{2}, s), P_2^s = \tilde{P}_2(\frac{1}{2}, s)$. We calculate
\[
\begin{align*}
\frac{d}{dt} \bigg|_{t=0} \tilde{\omega}_h((s + t\eta)^{-1}\delta_{h_0}^\prime(s + t\eta)) &= -\tilde{\omega}_h(s^{-1}\eta s\delta_{h_0}^\prime s) + \tilde{\omega}_h(s^{-1}\delta_{h_0}^\prime \eta) \\
&= \tilde{\omega}_h(s^{-1} \circ \delta_{h_0}^\prime(\eta s^{-1} \circ s) \\
&= \text{Ad}_s^{-\frac{1}{2}}(P_1^s(\eta^2)),
\end{align*}
\]
where $\eta^2 = s^{-\frac{1}{2}}\eta s^{-\frac{1}{2}}$. Similarly, we have
\[
\frac{d}{dt} \bigg|_{t=0} \tilde{\omega}_h((s + t\eta)^{-1}\delta_{h_0}^\prime(s + t\eta)) = \text{Ad}_s^{-\frac{1}{2}}(P_2^s(\eta^2)).
\]

Therefore, the linearization of the equation (4.2) at $(\varepsilon, s(\varepsilon))$ reads
\[
L_{\varepsilon, s}(\eta) := \frac{d}{dt} \bigg|_{t=0} \Gamma_\varepsilon(h_0, s + t\eta) \\
= \sqrt[-1]{\Lambda_\omega(\text{Ad}_s^{-\frac{1}{2}}(P_1^s(\eta^2)) - \text{Ad}_s^{-\frac{1}{2}}(P_2^s(\eta^2))) + \varepsilon s^{-\frac{1}{2}} \eta \\
\tag{4.3}
\]

Since the connections
\[
d_1^s := \text{Ad}_s^\frac{1}{2}(\tilde{\omega}_h) + \text{Ad}_s^{-\frac{1}{2}}(\delta_{h_0}^\prime), \quad d_2^s := \lambda^{-1}\text{Ad}_s^\frac{1}{2}(D^\lambda) + \lambda^{-1}\text{Ad}_s^{-\frac{1}{2}}(\delta_{h_0}^\prime)
\]
are also $h_0$-unitary, we have
\[
\Delta_\tilde{\omega} \eta^2 |_{h_0} = \sqrt[-1]{\Lambda_\omega(\partial_\omega||\eta^2|_{h_0} = -\sqrt[-1]{\Lambda_\omega(\partial_\omega||\eta^2|_{h_0} = h_0(\sqrt[-1]{\Lambda_\omega} P_1^s(\eta^2) + h_0(\eta^2, -\sqrt[-1]{\Lambda_\omega} P_1^s(\eta^2)) - |\tilde{d}_1^s \eta^2|_{h_0, \omega}^2 \\
&\quad = |\lambda|^{-2} h_0(-\sqrt[-1]{\Lambda_\omega} P_2^s(\eta^2, \eta^2) + |\lambda|^{-2} h_0(\eta^2, \sqrt[-1]{\Lambda_\omega} P_2^s(\eta^2) - |\tilde{d}_2^s \eta^2|_{h_0, \omega}^2),
\tag{4.4}
\]
where $\Delta_\tilde{\omega} = \sqrt[-1]{\Lambda_\omega(\partial_\omega|\partial_\omega)}$ is the Laplacian on $C^\infty(X)$ of $\tilde{\omega}$ with respect to $\omega$. Note that $(P_1^s(\eta^2))^\dagger_{h_0} = \tilde{P}_1^s(\eta^2), (P_2^s(\eta^2))^\dagger_{h_0} = \tilde{P}_2^s(\eta^2)$.

If $L_{\varepsilon, s}(\eta) = 0$, then combining (4.3) and (4.4) together leads to
\[
(1 + |\lambda|^2)\Delta_\tilde{\omega} \eta^2 |_{h_0} + |\tilde{d}_1^s \eta^2|_{h_0, \omega}^2 + |\lambda|^2 |\tilde{d}_2^s \eta^2|_{h_0, \omega}^2 = -2|\eta^2|_{h_0}^2 \leq 0.
\]
Hence, the maximum principle implies $|\eta^2|_{h_0} = 0$, i.e. $\eta = 0$, which means the linear second order elliptic differential operator $L_{\varepsilon, s}$ on $S(E, h_0)$ is injective. Moreover, since the index of $L_{\varepsilon, s}$ is zero, it is also surjective. If for some $\varepsilon_0 \in (0, 1]$ there exists $s_{\varepsilon_0}$ such that $\Gamma_{\varepsilon_0}(h_0, s_{\varepsilon_0}) = 0$, then by implicit function theorem on Banach spaces and elliptic regularity, there is a $S^+(E, h_0)$-valued smooth
function \( s(\varepsilon) \) over a small neighborhood \( U \subset (0,1] \) of \( \varepsilon_0 \) with \( s(\varepsilon_0) = s_{\varepsilon_0} \) such that \( \Gamma_\varepsilon(h_0, s(\varepsilon)) = 0 \) holds for any \( \varepsilon \in U \). The lemma follows.

**Lemma 4.2.** Assume for any \( \varepsilon \in (\varepsilon, 1] \) with \( \varepsilon > 0 \) the equation \( \Gamma_\varepsilon(h_0, s(\varepsilon)) = 0 \) admits a solution \( s(\varepsilon) \in S^+(E, h_0) \). Denote \( \chi(\varepsilon) = \frac{ds(\varepsilon)}{d\varepsilon} \) and \( m_c = \max_X |\chi(\varepsilon)|_{h_0} \), then there exists a positive constant \( C(m_c) \) such that

\[
\max_X |\log s(\varepsilon)|_{h_0} \leq C(m_c)
\]

for any \( \varepsilon \in (\varepsilon, 1] \), where the expression \( C(m_c) \) means this constant depends on \( m_c \) (and other fixed data independent of \( \varepsilon \)).

**Proof.** The equation \( \Gamma_\varepsilon(h_0, s(\varepsilon)) = 0 \) is equivalent to \( \hat{\Gamma}_\varepsilon(h_0, s(\varepsilon)) := s(\varepsilon)\Gamma_\varepsilon(h_0, s(\varepsilon)) = 0 \), then we have

\[
0 = \frac{d\hat{\Gamma}_\varepsilon(h_0, s(\varepsilon))}{d\varepsilon} = s(\varepsilon)(\Ad_s^{-\frac{1}{2}}(P_1^\varepsilon((\chi(\varepsilon))\hat{\omega})) - \Ad_s^{-\frac{1}{2}}(P_2^\varepsilon((\chi(\varepsilon))\hat{\omega})) + \log s(\varepsilon) + \varepsilon\Phi(\varepsilon)),
\]

where \( \Phi(\varepsilon) = \frac{d\log s(\varepsilon)}{d\varepsilon} \). It is known that [20]

\[
h_0(\Ad_s^{\frac{1}{2}} \cdot \Phi(\varepsilon), (\chi(\varepsilon))\hat{\omega}) \geq |(\chi(\varepsilon))\hat{\omega}|_{h_0}^2,
\]

combined with (4.4) yields

\[
(1 + |\lambda|^2)\Delta_\varepsilon(\chi(\varepsilon))\hat{\omega}|_{h_0}^2 + |\tilde{d}_1^\varepsilon(\chi(\varepsilon))\hat{\omega}|_{h_0,\omega}^2 + |\Delta_\varepsilon(\chi(\varepsilon))\hat{\omega}|_{h_0,\omega}^2 + 2\varepsilon |(\chi(\varepsilon))\hat{\omega}|_{h_0}^2 \\
\leq -2h_0(\log s(\varepsilon), (\chi(\varepsilon))\hat{\omega}) \leq 2|\log s(\varepsilon)|_{h_0} |(\chi(\varepsilon))\hat{\omega}|_{h_0}^2.
\]

On the other hand, we have

\[
|\tilde{d}_1^\varepsilon(\chi(\varepsilon))\hat{\omega}|_{h_0,\omega}^2 \geq |\Ad_s^{\frac{1}{2}}(\tilde{\Delta}_E(\chi(\varepsilon))\hat{\omega})|_{h_0,\omega}^2 = |\Ad_s^{\frac{1}{2}} \cdot (\tilde{\Delta}_E(\varepsilon))|_{h_0,\omega}^2 \geq e^{-C_0(m_c)} |\tilde{\Delta}_E(\varepsilon)|_{h_0,\omega}^2,
\]

where \( \varepsilon(\varepsilon) = \Ad_s^{\frac{1}{2}} \cdot (\chi(\varepsilon))\hat{\omega} \), and \( C_0(m_c) \) is a constant. Taking the integration over \( X \) yields

\[
||\tilde{d}_1^\varepsilon(\chi(\varepsilon))\hat{\omega}||_{L^2}^2 \geq ||\tilde{\Delta}_E(\varepsilon)||_{L^2}^2 \geq ||\Delta_\varepsilon(\varepsilon)||_{L^2}^2,
\]

where \( || \cdot ||_{L^2} \) denotes the \( L^2 \)-norm with respect to \( h_0, \omega \). Similarly, we have

\[
||\tilde{d}_2^\varepsilon(\chi(\varepsilon))\hat{\omega}||_{L^2}^2 \geq ||\tilde{\Delta}_E(\varepsilon)||_{L^2}^2 \geq e^{-C_0(m_c)} ||\Delta_\varepsilon(\varepsilon)||_{L^2}^2
\]

Consequently,

\[
||\tilde{d}_1^\varepsilon(\chi(\varepsilon))\hat{\omega}||_{L^2}^2 + |\lambda|^2 ||\tilde{d}_2^\varepsilon(\chi(\varepsilon))\hat{\omega}||_{L^2}^2 \geq e^{-C_0(m_c)} ||\tilde{\Delta}_E(\varepsilon)||_{L^2}^2
\]

where \( a_1 \) is the smallest eigenvalue of the Laplacian \( \Delta_{\varepsilon, h_0} = \tilde{\Delta}_{\varepsilon, h_0} \tilde{\Delta}_{\varepsilon} \) on \( C^\infty(\End(E)) \) of \( \tilde{\Delta}_{\varepsilon} \) with respect to \( h_0, \omega \). One claims \( a_1 > 0 \). If \( \omega = 0 \), namely \( \tilde{\Delta}_{\varepsilon, h_0} \tilde{\Delta}_{\varepsilon} = 0 \), which implies \( \varepsilon(\varepsilon) = c(\varepsilon)\Id \) for some constant \( c(\varepsilon) \) since \( (E, \tilde{\Delta}_{\varepsilon}) \) is a stable \( \lambda \)-flat bundle. However, due to (4.1) and \( \omega \) being balanced, we have

\[
\int_X \Tr((\sqrt{-1}\Delta_{\varepsilon, h_0} R_1(\h))\omega^n) = \int_X \Tr((\sqrt{-1}\Delta_{\varepsilon, h_0} R_2(\h))\omega^n) = 0,
\]

which implies

\[
\frac{d}{d\varepsilon} \int_X \Tr(\log s(\varepsilon))\omega^n = \int_X \Tr((\chi(\varepsilon))\hat{\omega})\omega^n = \int_X \Tr(\varepsilon(\varepsilon))\omega^n = 0.
\]

This means that \( a_1 = 0 \) only happens for \( \chi(\varepsilon) = 0 \), then there exists a positive constant \( C_1(m_c) \) such that

\[
||\tilde{d}_1^\varepsilon(\chi(\varepsilon))\hat{\omega}||_{L^2}^2 + |\lambda|^2 ||\tilde{d}_2^\varepsilon(\chi(\varepsilon))\hat{\omega}||_{L^2}^2 \geq e^{-C_1(m_c)} |(\chi(\varepsilon))\hat{\omega}|_{L^2}^2.
\]

(4.6)
Combining (4.5) and (4.6) together immediately gives rise to
\[
\|(\chi(\varepsilon))^{\frac{2}{\varepsilon}}\|_{L^2} \leq C_2(m_\varepsilon)
\] (4.7)
for some positive constant \(C_2(m_\varepsilon)\).

Again by (4.5), we have
\[
(1 + |\lambda|^2)\Delta_\delta |(\chi(\varepsilon))^{\frac{2}{\varepsilon}}|_{h_0} \leq 2\log s(\varepsilon)_{h_0} |(\chi(\varepsilon))^{\frac{2}{\varepsilon}}|_{h_0} \leq m_\varepsilon |(\chi(\varepsilon))^{\frac{2}{\varepsilon}}|_{h_0} + m_\varepsilon.
\]
Then one can apply [20, Lemma 3.3.2] and (4.7) to obtain
\[
\max_\chi \log s(\varepsilon)_{h_0} \leq C_3(m_\varepsilon) \max_\chi |(\chi(\varepsilon))^{\frac{2}{\varepsilon}}|_{h_0}
\]
\[
\leq C_4(m_\varepsilon)\left(\|(\chi(\varepsilon))^{\frac{2}{\varepsilon}}\|_{L^2}^2 + m_\varepsilon\right) \leq C_5(m_\varepsilon)
\]
where \(C_3(m_\varepsilon), C_4(m_\varepsilon), C_5(m_\varepsilon)\) are positive constants. \(\square\)

**Lemma 4.3.** With the same setting as in Lemma 4.2 we have

1. \(m_\varepsilon \leq \frac{1}{\varepsilon} \max_\chi |K(h_0)|_{h_0}\)
2. \(m_\varepsilon \leq \frac{C}{1 + |\lambda|^2} \left(\|\log s(\varepsilon)\|_{L^2} + \max_\chi |K(h_0)|_{h_0}\right)^2\), where \(C\) is a positive constant independent of \(\varepsilon\).

**Proof.** From \(\Gamma_\varepsilon(h_0, s(\varepsilon)) = 0\) it follows that
\[
|K(h_0)|_{h_0} |\log s(\varepsilon)|_{h_0} \geq h_0(\sqrt{-1}A_\omega (\tilde{\partial}_E(s^{-1}(\varepsilon)\tilde{\delta}_{h_0} s(\varepsilon)) - D^\lambda(s^{-1}(\varepsilon)\tilde{\delta}_{h_0} s(\varepsilon))), \log s(\varepsilon)) + \varepsilon |\log s(\varepsilon)|_{h_0}^2
\]
By the same approach as in the proof of [20, Lemma 3.3.4], one can show that
\[
h_0(\sqrt{-1}A_\omega (\tilde{\partial}_E(s^{-1}(\varepsilon)\tilde{\delta}_{h_0} s(\varepsilon)), \log s(\varepsilon)) \geq \frac{1}{2} \sqrt{-1}A_\omega \tilde{\partial}_D |\log s(\varepsilon)|_{h_0}^2,
\]
\[-h_0(\sqrt{-1}A_\omega (D^\lambda(s^{-1}(\varepsilon)\tilde{\delta}_{h_0} s(\varepsilon)), \log s(\varepsilon)) \geq \frac{|\lambda|^2}{2} \sqrt{-1}A_\omega \tilde{\partial}_D |\log s(\varepsilon)|_{h_0}^2.
\]
Therefore, we arrive at
\[
\frac{1 + |\lambda|^2}{2} \Delta_\delta |\log s(\varepsilon)|_{h_0}^2 + \varepsilon |\log s(\varepsilon)|_{h_0}^2 \leq |K(h_0)|_{h_0} |\log s(\varepsilon)|_{h_0},
\] (4.8)
which gives the first estimate in the lemma. The above inequality (4.8) also implies
\[
\Delta_\delta |\log s(\varepsilon)|_{h_0}^2 \leq \frac{2}{1 + |\lambda|^2} |K(h_0)|_{h_0} |\log s(\varepsilon)|_{h_0} \leq \frac{1}{1 + |\lambda|^2} \left(\|\log s(\varepsilon)|_{h_0}^2 + \max_\chi |K(h_0)|_{h_0}^2\right).
\]
Again by [20, Lemma 3.3.2], we get the second estimate in the lemma. \(\square\)

**Lemma 4.4.** The setting is the same as in Lemma 4.2. For all \(p > 1\) and \(\varepsilon \in (\varepsilon, 1]\), there exists positive constants \(C, C'\) indepenent of \(\varepsilon\) such that

1. \(\|\chi(\varepsilon)\|_{L^p} \leq C(1 + \|s(\varepsilon)\|_{L^p}),\)
2. \(\|s(\varepsilon)\|_{L^p} \leq C'.\)

**Proof.** (1) We define the Laplacians as follows
\[
\Delta_1^{h_0} := (\tilde{\partial}_E + \delta_{h_0}^{\ast}) \circ (\tilde{\partial}_E + \delta_{h_0}^\prime) \circ (\tilde{\partial}_E + \delta_{h_0}^{\ast}) \circ (\tilde{\partial}_E + \delta_{h_0}^\prime)^{\ast},\]
\[
\Delta_2^{h_0} := (\lambda^{-1}D^\lambda + \tilde{\lambda}^{-1}\delta_{h_0}^{\ast}) \circ (\lambda^{-1}D^\lambda + \tilde{\lambda}^{-1}\delta_{h_0}^\prime) \circ (\lambda^{-1}D^\lambda + \tilde{\lambda}^{-1}\delta_{h_0}^{\ast}) \circ (\lambda^{-1}D^\lambda + \tilde{\lambda}^{-1}\delta_{h_0}^\prime)^{\ast}.
\]
then for any $\Xi \in C^\infty(\text{End}(E))$, we have

$$\overline{\Delta}_1^{h_0} \Xi = 2\sqrt{-1}  \Lambda\omega \overline{\delta}_E^{\prime \prime} \Xi - [\sqrt{-1} \Lambda\omega R_1(h_0), \Xi],$$

$$\overline{\Delta}_2^{h_0} \Xi = -2|\lambda|^{-2} 2\sqrt{-1}  \Lambda\omega D^{\lambda} \overline{\delta}_E^{\prime \prime} \Xi + [\sqrt{-1} \Lambda\omega R_2(h_0), \Xi].$$

The identity $\frac{d}{d\varepsilon} (\varepsilon^{s(\varepsilon)}) = 0$ gives rise to

$$\overline{\Delta}_1^{h_0} \chi(e) + |\lambda|^2 \overline{\Delta}_2^{h_0} \chi(e) = - [K(h_0), \chi(e)]) + 2\sqrt{-1} \Lambda\omega [\chi(e) \circ s^{-1}(e) \circ D^{\lambda} \delta_s(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} s(e)$$

$$- D^{\lambda} \chi(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} s(e) + D^{\lambda} \delta_s(e) \circ s^{-1}(e) \circ \chi(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} s(e)$$

$$- D^{\lambda} \chi(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} \chi(e) - \chi(e) \circ s^{-1}(e) \circ D^{\lambda} \delta_h^{\prime \prime} s(e)$$

$$- \chi(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} \chi(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} s(e) + \delta_h^{\prime \prime} \chi(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} s(e)$$

$$- \delta_h^{\prime \prime} \chi(e) \circ s^{-1}(e) \circ \chi(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} s(e) + \delta_h^{\prime \prime} \chi(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} \chi(e)$$

$$+ \chi(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} \chi(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} s(e)] + \varepsilon \log s(e) + s(e) \Phi(e). \tag{4.9}$$

The $L^p$-norms of all terms on the right hand side of (4.9) can be estimated, for example

$$\| [K(h_0), \chi(e)] \|_{L^p} \leq C_0 \| \chi(e) \|_{L^p} \leq C_7(m_\varepsilon),$$

$$\| \sqrt{-1} \Lambda\omega [\chi(e) \circ s^{-1}(e) \circ D^{\lambda} \delta_s(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} s(e)] \|_{L^p} \leq C_8(m_\varepsilon) \| D^{\lambda} \delta_s(e) \circ \delta_h^{\prime \prime} s(e) \|_{L^p}$$

$$\leq C_8(m_\varepsilon) \| \delta_h^{\prime \prime} s(e) \|_{L^2},$$

$$\| \sqrt{-1} \Lambda\omega [D^{\lambda} \delta_s(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} \chi(e) \circ s^{-1}(e) \circ \delta_h^{\prime \prime} s(e)] \|_{L^p} \leq C_9(m_\varepsilon) \| D^{\lambda} \delta_s(e) \circ \delta_h^{\prime \prime} s(e) \|_{L^p}$$

$$\leq C_9(m_\varepsilon) \| \chi(e) \|_{L^2} \cdot \| s(e) \|_{L^2},$$

$$\| \sqrt{-1} \Lambda\omega [\chi(e) \circ s^{-1}(e) \circ D^{\lambda} \delta_h^{\prime \prime} s(e)] \|_{L^p} \leq C_{10}(m_\varepsilon) \| s(e) \|_{L^p},$$

$$\| s(e) \Phi(e) \|_{L^p} \leq C_{11} (m_\varepsilon) \| \chi(e) \|_{L^p} \leq C_{12}(m_\varepsilon),$$

where we frequently use Lemma 4.2 and Hölder inequality.

Since the operator $\Delta_1^{h_0} + |\lambda|^2 \Delta_2^{h_0} + \text{id} : L^p \rightarrow L^p$ is self-adjoint and has strictly positive spectrum, there exists a positive constant $C_{12}$ such that

$$\| \chi(e) \|_{L^2} \leq C_{12} (\| \chi(e) \|_{L^p} + \| (\Delta_1^{h_0} + |\lambda|^2 \Delta_2^{h_0}) \chi(e) \|_{L^p}).$$

Then due to Lemma 4.3 (1), we get

$$\| \chi(e) \|_{L^2} \leq C_{13} (1 + \| s(e) \|_{L^2}^2) \leq C_{14} (1 + \| s(e) \|_{L^2}^2).$$

(2) By the approach applied to the proof of [20, Proposition 3.3.5 ii)], we deduce the inequality

$$\| s(e) \|_{L^2} \leq e^{C(1-t)} (1 + \| s(1) \|_{L^2})$$

from (1). This immediately implies (2). □

**Lemma 4.5.**

1. $J = (0, 1)$.
2. If there is a positive constant $C$ such that $\| s(e) \|_{L^2} \leq C$ for all $e \in (0, 1)$, then there exists a solution $s(0)$ of the equation $\Gamma_0(h_0, s(0)) = 0$. 

Proof. (1) Thanks to Lemma 4.1, to show (1) it suffices to prove $J$ is a closed subset of $(0,1]$. By Lemma 4.4 (2), $s(\varepsilon)$ is uniformly bounded in $L^2_0(S^+(E,h_0))$ for $\varepsilon \in (\epsilon,1]$, thus there is a sequence $\{\varepsilon_i\} \subset (\epsilon,1]$ such that the sequence $\{s(\varepsilon_i)\} \subset L^2_0(S^+(E,h_0))$ converges weakly to $s(\varepsilon) \in L^2_0(S^+(E,h_0))$. Since the Sobolev embedding $L^2_0 \hookrightarrow L^p_0$ is compact, we may assume $\{s(\varepsilon_i)\} \subset L^2_0(S^+(E,h_0))$ converges strongly to $s(\varepsilon)$ in $L^1_0(S^+(E,h_0))$. Some rather standard arguments (cf. the proof of [20, Proposition 3.3.6]) show that $s(\varepsilon)$ is actually differentiable and satisfies $L_s(h_0,s(\varepsilon)) = 0$.

(2) If $\|s(\varepsilon)\|_{L^2_0} \leq C$ for all $\varepsilon \in (0,1]$, then by Lemma 4.3 (2) and and Lemma 4.4 (2), $\|s(\varepsilon)\|_{L^2_0}$ is uniformly bounded on $(0,1]$, then same argument as in (1) shows (2). □

Lemma 4.6. The setting is the same as in Lemma 4.2. There is a positive constant $C$ independent of $\varepsilon$ such that

$$\max_{\mathcal{X}} |s(\varepsilon)|_{h_0} \leq C \|s(\varepsilon)\|_{L^1_0}.$$ 

Proof. Again by $\Gamma(\varepsilon,h_0,s(\varepsilon)) = 0$ we have

$$0 = h_0(K(h_0),s(\varepsilon)) + |\Lambda_\omega \tilde{\partial} h_0(s^{-1}(\varepsilon)\tilde{\delta}_{h_0} s(\varepsilon)), s(\varepsilon)) + |\Lambda_\omega h_0(s^{-1}(\varepsilon)\tilde{\delta}_{h_0} s(\varepsilon), \tilde{\delta}_{h_0} s(\varepsilon))|
- \sqrt{\Lambda_\omega} \lambda \partial h_0(s^{-1}(\varepsilon)\tilde{\delta}_{h_0} s(\varepsilon), s(\varepsilon)) - \sqrt{\Lambda_\omega} h_0(s^{-1}(\varepsilon)\tilde{\delta}_{h_0} s(\varepsilon), \tilde{\delta}_{h_0} s(\varepsilon)) + \varepsilon h_0(\log s(\varepsilon), s(\varepsilon)).$$

Then by means of the identities

$$\sqrt{\Lambda_\omega} \tilde{\partial} h_0(s^{-1}(\varepsilon)\tilde{\delta}_{h_0} s(\varepsilon), s(\varepsilon)) = \Delta_\partial Trs(\varepsilon),$$

and the inequalities

$$\sqrt{\Lambda_\omega} h_0(s^{-1}(\varepsilon)\tilde{\delta}_{h_0} s(\varepsilon), \tilde{\delta}_{h_0} s(\varepsilon)) \geq |s^{-\frac{1}{2}}(\varepsilon)\tilde{\delta}_{h_0} s(\varepsilon)|_{h_0}^2,$$

we get

$$\Delta_\partial Trs(\varepsilon) \leq -\frac{1}{1 + |\lambda|^2}(\varepsilon h_0(\log s(\varepsilon), s(\varepsilon)) + h_0(K(h_0),s(\varepsilon)))$$

$$\leq \frac{2}{1 + |\lambda|^2} \max_{\mathcal{X}} |K(h_0)|_{h_0} |s(\varepsilon)|_{h_0} \leq C_{15} Trs(\varepsilon)$$

for some positive constant $C_{15}$, where we have applied Lemma 4.3 (1) for the third inequality. Note that $\max_{\mathcal{X}} |s(\varepsilon)|_{h_0} \leq C_{16} \max_{\mathcal{X}} Trs(\varepsilon)$ for some positive constant $C_{16}$, then applying [20, Lemma 3.3.2] once again gives rise to the lemma. □

Assume $s(\varepsilon)$ is a solution to $\Gamma(\varepsilon,h_0,s(\varepsilon)) = 0$ for some $\varepsilon > 0$. For $x \in \mathcal{X}$, let $e(\varepsilon,x)$ be the largest eigenvalue of $\log s(\varepsilon)|_{x}$, then one defines $E(\varepsilon) = \max_{\mathcal{X}} e(\varepsilon,x) = e^{-E(\varepsilon)}$ and $S(\varepsilon) = e(\varepsilon,s(\varepsilon))$.

Lemma 4.7. The setting is the same as in Lemma 4.2. If $\lim_{\varepsilon \to 0} \|s(\varepsilon)\|_{L^2} = \infty$, then there exists a sequence $\{\varepsilon_i \to 0\}$ such that $\rho(\varepsilon_i) \to 0$ and $S(\varepsilon_i)$ converges weakly in $L^1_0$-norm to $S_{\infty} \neq 0$.

Proof. By definition, $S(\varepsilon) \leq Id_{\mathcal{X}}$ and $\max_{\mathcal{X}}(\rho(\varepsilon)|s(\varepsilon)|_{h_0}) \geq 1$, then by Lemma 4.6 we get

$$1 \leq \max_{\mathcal{X}}(\rho(\varepsilon)|s(\varepsilon)|_{h_0}) \leq C_{17}\rho(\varepsilon)|s(\varepsilon)|_{L^1} \leq C_{18}\|S(\varepsilon)\|_{L^2} \leq C_{19}$$

for some positive constants $C_{17}, C_{18}, C_{19}$. Firstly, from $1 \leq C_{18}\|S(\varepsilon)\|_{L^2}$ it follows that if $S(\varepsilon_i)$ converges to $S_{\infty}$ weakly in $L^1_0$-norm then $S_{\infty} \neq 0$. Secondly, by $\|S(\varepsilon)\|_{L^2} \leq C_{19}$ we see that in
order to obtain the $L^2_1$-bound for $S(\varepsilon_i)$ we only need to estimate $\|(\tilde{\partial} + \delta'_{h_0})S(\varepsilon_i)\|_{L^2}$. The same calculations as in the proof of Lemma 4.6 show that

$$\|(\tilde{\partial} + \delta'_{h_0})S(\varepsilon_i)\|_{L^2} \leq 2\|\delta''_{h_0}S(\varepsilon_i)\|_{L^2} \leq 2(\|S^2(\varepsilon_i\delta''_{h_0}S(\varepsilon_i)\|_{L^2} + \|S^{2}(\varepsilon_i\delta''_{h_0}S(\varepsilon_i)\|_{L^2})$$

$$\leq 4\max X|K(h_0)|_{h_0} \int X|S(\varepsilon_i)|_{h_0}\omega^n \leq C_{19}$$

for some positive constant $C_{20}$, i.e. $S(\varepsilon)$ is uniformly bounded in $L^2_1(S^+(E, h_0))$, which implies $S(\varepsilon_i)$ converges weakly in $L^2_1$-norm. □

We have shown that there is a sequence $\{\varepsilon_i \to 0\}$ such that $S(\varepsilon_i)$ converges weakly to a nonzero $L^2_1$-endomorphism $S_\infty$. Similarly, for $0 < \zeta \leq 1$, there is a sequence $\{\varepsilon_i \to 0\}$ such that $S^\zeta(\varepsilon_i)$ converges weakly to a nonzero $L^2_1$-endomorphism $S^\zeta_\infty$, and there is a sequence $\{\xi_i \to 0\}$ such that $S^\xi_\infty \to S^\xi_\infty$ weakly in $L^2_1$. Then one introduces an $L^2_1$-endomorphism $\Theta = \text{Id}_E - S_\infty$.

**Lemma 4.8.** $\Theta$ satisfies the following identities in $L^1$:

1. $\Theta^2 = \Theta = \Theta_1^\perp_{h_0}$,
2. $(\text{Id}_E - \Theta) \circ \tilde{\partial}\Theta = (\text{Id}_E - \Theta) \circ \tilde{D}\lambda\Theta = 0$.

Therefore, $\Theta$ defines a proper $\lambda$-flat coherent subsheaf $F$ of $E$ with $0 < \text{rank}(F) < \text{rank}(E)$.

**Proof.** (1) is obvious. For (2), it suffices to show

$$|\Theta \circ \delta_{h_0}^\perp(\text{Id}_E - \Theta)|_{h_0} = |\Theta \circ \delta_{h_0}^\perp(\text{Id}_E - \Theta)|_{h_0} = 0.$$

Indeed, by the same arguments as in the proof of [20, Proposition 3.4.6 iii]), we have

$$\|\Theta \circ \delta_{h_0}^\perp(\text{Id}_E - \Theta)||_{L^2} + \|\Theta \circ \tilde{\delta}_{h_0}^\perp(\text{Id}_E - \Theta)||_{L^2}$$

$$= \lim_{\delta'_{k,j} \to 0} \lim_{\delta'_{k,j} \to 0} \lim_{\delta'_{k,j} \to 0} \left(\|\Theta \circ \delta_{h_0}^\perp(\text{Id}_E - \Theta)||_{L^2} + \|\Theta \circ \tilde{\delta}_{h_0}^\perp(\text{Id}_E - \Theta)||_{L^2}\right)$$

$$\leq \lim_{\delta'_{k,j} \to 0} \lim_{\delta'_{k,j} \to 0} \lim_{\delta'_{k,j} \to 0} \left(\|\Theta \circ \delta_{h_0}^\perp(\text{Id}_E - \Theta)||_{L^2} + \|\Theta \circ \tilde{\delta}_{h_0}^\perp(\text{Id}_E - \Theta)||_{L^2}\right)$$

for some positive constant $C_{21}$, which leads to (2). The existence of $\lambda$-flat coherent sheaf $F$ defined via $\Theta$ is just an application of a classical result due to Uhlenbeck and Yau [35]. The nonvanishing of $S_\infty$ guarantees $\text{rank}(F) < \text{rank}(E)$, and the identity $\int_X \log(\det s(\varepsilon))\omega^n = 0$ implies $0 < \text{rank}(F)$ (cf. the proof of [15, Proposition 3.13 (3)]). □

**Lemma 4.9.** $F$ is defined as in Lemma 4.8, then $\text{deg}(F) \geq 0$.

**Proof.** There is an analytic subset $S \subset X$ with $\text{codim}_X S \geq 2$ such that $F|_{X \setminus S}$ is a holomorphic subbundle of $E|_{X \setminus S}$ and $\Theta|_{X \setminus S}$ is $C^\infty$ [35]. Since

$$\int_{X \setminus S} \text{Tr}(\sqrt{-1}\Lambda\omega R_1(h_0|_F))\omega^n = \frac{1}{1 + |\lambda|^2} \int_{X \setminus S} \text{Tr}(\sqrt{-1}\Lambda\omega G(h_0|_F, D_\lambda|_F))\omega^n$$

$$= \frac{1}{1 + |\lambda|^2} \int_{X \setminus S} (\text{Tr}(K(h_0) \circ \Theta) - |\delta_{h_0}^\perp|_{h_0}^2 - |\delta_{h_0}^\perp|_{h_0}^2)\omega^n,$$
we only need to show
\[ \int_{X \backslash S} \text{Tr}(K(h_0) \circ \Theta) \omega^n \geq \int_{X \backslash S} |\delta_{h_0}^\prime \Theta|_{h_0}^2 + |\delta_{h_0}'' \Theta|_{h_0}^2 \omega^n. \] (4.10)

By the identities \( \int_X \text{Tr}(K(h_0)) \omega^n = 0 \) and \( \Theta = \lim_{1 \leq \zeta_i \to 0} \lim_{\varepsilon_i \to 0} (\text{Id}_E - S_{h_0}^{\zeta_i}(\sigma_i)) \) (strongly in \( L^2 \)), we have
\[ \int_X \text{Tr}(K(h_0) \circ \Theta) \omega^n = - \lim_{1 \leq \zeta_i \to 0} \lim_{\varepsilon_i \to 0} \int_X \text{Tr}(K(h_0) \circ S_{h_0}^{\zeta_i}(\varepsilon_i)) \omega^n. \]

A calculation of reuse shows that
\[ \int_X \text{Tr}(K(h_0) \circ S_{h_0}^{\zeta_i}(\varepsilon_i)) \omega^n \]
\[ = - \int_X \sqrt{-1} \Lambda h_0(S^{-1}(\varepsilon_i) \delta_{h_0}^\prime S(\varepsilon_i), \delta_{h_0}^\prime S^{\zeta_i}(\varepsilon_i)) - h_0(S^{-1}(\varepsilon_i) \delta_{h_0}'' S(\varepsilon_i), \delta_{h_0}'' S^{\zeta_i}(\varepsilon_i))) \omega^n \]
\[ \leq - ||S^{-\frac{1}{2}}(\varepsilon_i) \delta_{h_0}^\prime S^{\zeta_i}(\varepsilon_i)||_{L^2} - ||S^{-\frac{1}{2}}(\varepsilon_i) \delta_{h_0}'' S^{\zeta_i}(\varepsilon_i)||_{L^2} \]
\[ \leq - ||\delta_{h_0}''(\text{Id}_E - S^{\zeta_i}(\varepsilon_i))||_{L^2} - ||\delta_{h_0}''(\text{Id}_E - S^{\zeta_i}(\varepsilon_i))||_{L^2}, \]
where we have also used the assumption that \( X \) being a balanced manifold and the inequality \( \int_X \text{Tr}(\log S(\varepsilon_i) \circ S^{\zeta_i}(\varepsilon_i)) \omega^n \geq 0 \). This immediately gives rise to the desired inequality (4.10). \( \square \)

**Lemma 4.10.** If both \( h_1, h_2 \) are harmonic metrics on \( ((E, \bar{\partial}_E), D^\lambda) \), then there is a positive constant \( C \) such that \( h_2 = C h_1 \).

**Proof.** Write \( h_2 = h_1 s \) with \( s \in S^+(E, h_1) \), then from the proof of Lemma 4.6 we see that
\[ (1 + |\lambda|^2) \Delta \delta \text{Tr}s + |s^{-\frac{1}{2}}(\varepsilon) \delta_{h_1}^\prime s|_{h_1}^2 + |s^{-\frac{1}{2}}(\varepsilon) \delta_{h_1}'' s|_{h_1}^2 \leq 0, \]
which implies \( \bar{\partial}_E s = D^\lambda s = 0 \). Since \( ((E, \bar{\partial}_E), D^\lambda) \) is a stable \( \lambda \)-flat bundle, this makes \( s = C \cdot \text{Id}_E \) for some positive constant \( C \). \( \square \)

In conclusion, we achieve the following theorem:

**Theorem 4.11.** Let \( X \) be a balanced manifold, and \( ((E, \bar{\partial}_E), D^\lambda) \) be a stable \( \lambda \)-flat bundle over \( X \) \( (\lambda \neq 0) \), then there is a unique harmonic metric on \( ((E, \bar{\partial}_E), D^\lambda) \) up to constant scalars.

5. **Dynamical Systems on Dolbeault Moduli Spaces**

5.1. **Construction.** In this section, \( X \) is assumed to be a complex projective manifold. For any \( t \in \mathbb{C}^* \), the \( \mathbb{C}^* \)-action on \( M_{\text{Dal}}(X, r) \) is given by:
\[ t : M_{\text{Dal}}(X, r) \longrightarrow M_{\text{Dal}}(X, r) \]
\[ ((E, \bar{\partial}_E), \theta) \longmapsto ((E, \bar{\partial}_E), t \theta), \]
which plays a crucial role in studying the moduli space. Due to the Simpson–Mochizuki correspondence, we can construct a new action on \( M_{\text{Dal}}(X, r) \) as follows. Fixing some \( (\lambda, t) \in \mathbb{C} \times \mathbb{C}^* \), for any stable Higgs bundle \( ((E, \bar{\partial}_E), \theta) \in M_{\text{Dal}}(X, r) \) with a pluri-harmonic metric \( h \), we have a stable \( \lambda \)-flat bundle \( ((E, d''_E = \bar{\partial}_E + \lambda \theta_h^1), D^\lambda = \lambda \delta_{E,h} + \theta) \in M_{\text{Hod}}(X, r) \), and a stable \( \lambda' \)-flat bundle \( ((E, d''_E), D^\lambda = t \lambda \delta_{E,h} + t \theta) \in M_{\text{Hod}}(X, r) \) for \( \lambda' = t \lambda \), the latter one admits a pluri-harmonic metric \( h_t \), which gives rise to a stable Higgs bundle \( ((E, \bar{\partial}_E, h_t), \theta_h) \in M_{\text{Dal}}(X, r) \) by the Simpson–Mochizuki correspondence. We conclude the above process in the following:

\[
\begin{align*}
M_{\text{Dal}}(X, r) & \longrightarrow M_{\text{Hod}}(X, r) \\
((E, \bar{\partial}_E), \theta) & \longmapsto ((E, \bar{\partial}_E + \lambda \theta_h^1), \lambda \delta_{E,h} + \theta) \longmapsto ((E, \bar{\partial}_E + \lambda \theta_h^1, t \lambda \delta_{E,h} + t \theta)) \longmapsto ((E, \bar{\partial}_E, h_t), \theta_h).
\end{align*}
\]
As a summary, the Simpson–Mochizuki correspondence provides a dynamical system (i.e. a smooth self-map) with two-parameters on the Dolbeault moduli space $M_{\text{Dol}}(X, r)$

$$\psi_{(\lambda, t)} : M_{\text{Dol}}(X, r) \longrightarrow M_{\text{Dol}}(X, r)$$

$$((E, \bar{\partial}_E), \theta) \mapsto ((E, \bar{\partial}_{E,h_t}), \theta_{h_t}),$$

and we also call it the $(\lambda, t)$-action.

**Remark 5.1.**

1. Clearly, $\psi_{(\lambda, t)}$ can also be defined on $\mathbb{M}_{\text{Dol}}(X, r)$ as a continuous self-map.
2. The similar construction proceeding from $\mathbb{M}_{\text{dR}}(X, r)$ provides a dynamical system on $\mathbb{M}_{\text{dR}}(X, r)$.

The following several facts are very obvious.

**Proposition 5.2.**

1. $\psi_{(0, t)}$ is the usual $\mathbb{C}^*$-action by $t$ on $M_{\text{Dol}}(X, r)$, and $\psi_{(\lambda, 1)}$ is the identity morphism,
2. $\psi_{(M_1, t_2)} \circ \psi_{(\lambda, t_1)} = \psi_{(\lambda, t_1 t_2)}$,
3. A stable vector bundle (with zero Higgs field) with vanishing Chern classes is a fixed point of $\psi_{(\lambda, t)}$ for any $\lambda \in \mathbb{C}, t \in \mathbb{C}^*$.

**5.2. The First Variation.** Let $u = ((E, \bar{\partial}_E), \theta) \in M_{\text{Dol}}(X, r)$ with the pluri-harmonic metric $h$, the tangent space of $M_{\text{Dol}}(X, r)$ at $u$ is given by the hypercohomology $H^1(\text{End}(E), \bar{\partial}_E)$ of Higgs complex [29]

$$\text{End}(E) \xrightarrow{\bar{\partial}_E} \text{End}(E) \otimes \Omega^1_X \xrightarrow{\bar{\partial}_E} \cdots.$$ 

By Kähler identities, there is an isomorphism

$$H^1(\text{End}(E), \bar{\partial}_E) \simeq H^1(E, \theta) := \{ (\alpha, \beta) \in \Omega^{0,1}_X(\text{End}(E)) \oplus \Omega^{1,0}_X(\text{End}(E)) \ |

(\bar{\partial}_E h^1 + \bar{\theta}_h^1)(\alpha + \beta) = (\bar{\partial}_E + \bar{\theta})(\alpha + \beta) = 0 \}.$$ 

**Definition 5.3 ([14]).** The pair $(\alpha, \beta) \in H^1(E, \theta)$ is called the infinitesimal deformation of the Higgs bundle $((E, \bar{\partial}_E), \theta)$, in particular, if $\bar{\partial}_E h^1 \alpha = \bar{\partial}_E \beta = 0$, $(\alpha, \beta)$ is called the holomorphic infinitesimal deformation.

Now we assume $X$ is a Riemann surface. Consider the family $u(s) := ((E, \bar{\partial}_{E_s}), \theta_s)$ lying in $M_{\text{Dol}}(X, r)$ with parameter $s$ such that $u(0) = u$ and $\frac{du}{ds}|_{s=0} = (\alpha, \beta) \in H^1(E, \theta)$. The pluri-harmonic metric on the Higgs bundle $((E, \bar{\partial}_{E_s}), \theta_s)$ is denoted by $h(s)$ with $h(0) = h$, and fixing $\lambda, t$, the pluri-harmonic metric on the $\lambda$-flat bundle $((E, d''_{E_s} = \bar{\partial}_{E_s} + \lambda(\theta_s)) \dagger, d''_{E_s} = t(\lambda \bar{\partial}_{E_s, h(s)} + \theta_s))$ is denoted by $h_t(s)$ with $h_t(0) = h_t$, which yields the operators $\delta''_{E_s} := \delta''_{E, h_t(s)}$ and $\delta''_{E_s} := \delta''_{E, h_t(s)}$.

There is an integral curve $\gamma$ in $M_{\text{Dol}}(X, r)$ passing through the point $u$ with tangent vector $(\alpha, \beta)$, the $(\lambda, t)$-action maps this curve to another curve $\gamma'$, we can study its local property at the point $\psi_{(\lambda, t)}(u)$ by calculating the variations of $\psi_{(\lambda, t)}$.

**Proposition 5.4.** Assume the original point $u$ and the parameters $\lambda, t$ are chosen to satisfy $h_t = h$, and assume $\frac{du}{ds}|_{s=0} = (\alpha, \beta)$ is a holomorphic infinitesimal deformation, then

$$\left. \frac{d \psi_{(\lambda, t)} u(s)}{ds} \right|_{s=0} = \left( \alpha + \frac{\lambda(1 - |t|^2)}{1 + |t\lambda|^2} \beta h \right) = \left( \frac{1}{1 + |t\lambda|^2} \right) \frac{d}{ds} \left( \psi_{(\lambda, t)}(u(s)) \right).$$
Proof. We write \( h_t(s) = h_t H_t(s) \), and \( d^\prime_E = d^\prime_{E_0}, d^\mu_E = d^\mu_{E_0}, \delta^\prime_E = \delta^\prime_{E_0}, \delta^\mu_E = \delta^\mu_{E_0} \), then choosing a local \( h_t \)-unitary frame \( \{ e_i \} \) of \( E \), we have

\[
\chi \partial h_t(s)(e_i, e_j) = h_t(d^\prime_E H_t(s) e_i, e_j) + h_t(d^\mu_E H_t(s) e_i, \delta^\prime_E e_j)
\]

which implies that

\[
\partial h_t(s)(e_i, e_j) = h_t(H_t(s) e_i, e_j) + h_t(H_t(s) e_i, \delta^\prime_E e_j)
\]

Due to \([\partial_t) E, \) bundle \( (\partial_t) E \), then by \((5.1)\) we arrive at

\[
\begin{align*}
\delta^\prime_E \left( d^\prime_E \frac{dH_t(s)}{ds} \bigg|_{s=0} \right) & = h_t \left( d^\mu_E \frac{dH_t(s)}{ds} \bigg|_{s=0} \right) - h_t \left( d^\mu_E \frac{d(d^\mu_E)}{ds} \bigg|_{s=0} \right), \\
\delta^\mu_E \left( d\frac{dH_t(s)}{ds} \bigg|_{s=0} \right) & = h_t \left( d\frac{d^\prime_E}{ds} \bigg|_{s=0} \right) - h_t \left( d\frac{d^\mu_E}{ds} \bigg|_{s=0} \right)
\end{align*}
\]

On the other hand, from the pluri-harmonicity of \( h_t(s) \), namely

\[
[d^\mu_E + \lambda^\prime \delta^\mu_E, d^\prime_E - \lambda^\prime \delta^\prime_E] = -\lambda^\prime [d^\mu_E, \delta^\prime_E] + \lambda^\prime [d^\prime_E, \delta^\mu_E] = 0,
\]

it follows that

\[
\begin{align*}
\left( \tilde{\delta}^\prime_E \frac{dH_t(s)}{ds} \bigg|_{s=0} \right) - \tilde{\delta}^\mu_E \frac{dH_t(s)}{ds} & = h_t \left( d\frac{d^\mu_E}{ds} \bigg|_{s=0} \right) + \left( d\frac{d^\prime_E}{ds} \bigg|_{s=0} \right), \\
\left( \tilde{\delta}^\mu_E \frac{dH_t(s)}{ds} \bigg|_{s=0} \right) - \tilde{\delta}^\prime_E \frac{dH_t(s)}{ds} & = h_t \left( d\frac{d^\mu_E}{ds} \bigg|_{s=0} \right) + \left( d\frac{d^\prime_E}{ds} \bigg|_{s=0} \right)
\end{align*}
\]

Due to [6, Proposition 3.2], we have

\[
\left( \frac{d(d^\mu_E)}{ds} \bigg|_{s=0} \right) = \alpha + \lambda \beta \left( h_t \right), \quad \left( \frac{d(d^\prime_E)}{ds} \bigg|_{s=0} \right) = -\lambda^\prime \alpha \left( h_t \right) + \beta.
\]

The condition \( h_t = h \) leads to

\[
\delta^\prime_E = \delta_E - \lambda \theta, \quad \delta^\mu_E = \lambda^\prime \theta - \beta h_t,
\]

then by \((5.1)\), since \((\alpha, \beta)\) is an infinitesimal holomorphic deformation, we arrive at

\[
\begin{align*}
\frac{d\frac{dH_t(s)}{ds}}{ds} \bigg|_{s=0} & = 0,
\end{align*}
\]

for which applying the Kähler identities in Proposition 2.4 implies \( \frac{d\frac{dH_t(s)}{ds}}{ds} \bigg|_{s=0} = 0 \). But \( \lambda \)-flat bundle \( (E, D^\lambda E = d^\prime_E + d^\mu_E) \) is simple, \( \frac{dH_t(s)}{ds} \bigg|_{s=0} \) has to be constant. Therefore, from the calculation of

\[
\frac{d}{ds} \bigg|_{s=0} \left( \frac{1}{1 + |\lambda|^2} (d^\mu_E + \lambda \delta^\mu_E, h_t(s)), \frac{1}{1 + |\lambda|^2} (d^\prime_E - \lambda \delta^\prime_E, h_t(s)) \right),
\]

the desired result immediately follows.
5.3. Fixed Points. In this subsection, we study the fixed points of the dynamical system \( \psi_{(\lambda, t)} \). We first introduce the following definitions.

**Definition 5.5.**

1. For a given Higgs bundle \( u \in M_{\text{Dol}}(X, r) \), one defines the set of stable parameters as
   \[
   C_u = \{(\lambda, t) \in \mathbb{C} \times \mathbb{C}^* : \psi_{(\lambda, t)}(u) = u\}.
   \]
2. For a given pair \((\lambda, t) \in \mathbb{C} \times \mathbb{C}^*\), one defines the set of fixed points as
   \[
   \mathfrak{F}_{(\lambda, t)} = \{u \in M_{\text{Dol}}(X, r) : \psi_{(\lambda, t)}(u) = u\}.
   \]

**Definition 5.6 ([21, 24]).** A Higgs bundle \((E, \tilde{\partial}_E, \theta)\) over \( X \) is called a decoupled Higgs bundle if there is a Hermitian metric \( h \) on \( E \) satisfying \( R(h) = (\partial_{E,h} + \tilde{\partial}_E)^2 = 0 \) and \([\theta, \theta_h^t] = 0\), and in this case, such metric is called a decoupling metric.

**Proposition 5.7.** Let \( X \) be a Riemann surface of genus \( g \geq 2 \), and let \( M_{\text{de}}(X, r) \) be the subset of \( M_{\text{Dol}}(X, r) \) consisting of stable decoupled Higgs bundles. Then \( M_{\text{de}}(X, r) \) is a connected real analytic subvariety of \( M_{\text{Dol}}(X, r) \) with dimension \( 3r^2(g-1) + rg + 3 \).

**Proof.** It is known that the Hitchin moduli space \( M_{\text{Hir}}(X, r) \) defined as the space of irreducible Hitchin pairs (solutions to Hitchin’s self-duality equations with a given Hermitian metric on the complex vector bundle) modulo unitary gauge transformations is diffeomorphic to \( M_{\text{Dol}}(X, r) \). This means \( M_{\text{de}}(X, r) \) can be defined as a subset of \( M_{\text{Hir}}(X, r) \) consisting of irreducible decoupled Hitchin pairs. The forgetful map \((E, \theta) \mapsto E\) provides a fibration \( M_{\text{de}} \to B(X, r) \), where \( B(X, r) \) is the moduli space of rank \( r \) stable bundles with vanishing the first Chern class over \( X \). One locally writes \( \theta = \Theta dz \) for an \( r \times r \) complex matrix \( \Theta \), then the condition \([\Theta, \Theta^t] = 0\) implies that \( \Theta \) is a normal matrix, hence the real dimension of the fibers is given by \( r(r+1)g-(r^2-1) = r^2(g-1)+rg+1 \). It follows from the invariance of \( \mathbb{C}^* \)-action on \( M_{\text{de}}(X, r) \) and the connectedness of \( B(X, r) \) that \( M_{\text{de}}(X, r) \) is connected. \( \square \)

**Theorem 5.8.** Let \( X \) be a Riemann surface and let \( u \in M_{\text{Dol}}(X, r) \) represents a decoupled Higgs bundle of rank \( r \) with nontrivial Higgs field, then \( \mathbb{C} \times \{\mu^m, m = 0, \ldots, l-1\} \subseteq C_u \subseteq (\mathbb{C} \times \{\mu^m, m = 0, \ldots, l-1\}) \cup \{(\lambda, t) \in \mathbb{C}^* \times \mathbb{C}^* : |t| \lambda^2 = 1, |t| \neq 1, t = |t| \mu^k, k = 1, \ldots, l-1\}, \) where \( \mu = e^{\frac{2\pi i}{l}}, \mu^* = e^{\frac{2\pi i}{l'}} \) for some fixed positive integers \( 1 \leq l \leq r, 2 \leq l' \leq r \).

**Proof.** Case I: \( \lambda \neq 0, |t| = 1 \).

Let \((E, D^\lambda, h)\) be a stable \( \lambda \)-flat bundle with the pluri-harmonic metric \( h \). The operators \( \delta'_h, \delta''_h, \bar{\delta}_h, \tilde{\bar{\delta}}_h, \delta^t_h, \delta^t_h \) can be defined via \((D^\lambda, h_1)\) and \((D^\lambda', h_1)\), respectively, in order to distinguish them, we add the subscripts \( \lambda, \lambda' \) for them. Then by definition, we have

\[
\delta'_{h, \lambda'} = \delta'_{h, \lambda}, \quad \delta''_{h, \lambda'} = t \delta''_{h, \lambda},
\]

hence

\[
\bar{\delta}_{h, \lambda'} = \frac{1}{1 + |t\lambda|^2} \left(d'_E + |t|^2 \lambda \delta''_{h, \lambda}\right), \quad \delta_{h, \lambda'} = \frac{1}{1 + |t\lambda|^2} \left(|t|^2 \lambda d'_E + \delta'_{h, \lambda}\right),
\]

\[
\theta^t_{h, \lambda'} = \frac{t}{1 + |t\lambda|^2} \left(\lambda d''_E - \delta''_{h, \lambda}\right), \quad \theta_{h, \lambda'} = \frac{t}{1 + |t\lambda|^2} \left(d'_E - \lambda \delta'_{h, \lambda}\right).
\]  \( (5.2) \)

When \( |t| = 1 \), we arrive at

\[
\bar{\delta}_{h, \lambda'} = \bar{\delta}_{h, \lambda}, \quad \delta_{h, \lambda'} = \delta_{h, \lambda},
\]

\[
\theta^t_{h, \lambda'} = t \theta_{h, \lambda}, \quad \theta_{h, \lambda'} = t \theta_{h, \lambda}.
\]
It follows from $\bar{\partial}_{h_t}^2 = \bar{\partial}_{h_t} \theta_{h_t} = \theta_{h_t} \wedge \theta_{h_t} = 0$ that $\bar{\partial}_{h_t}^2 = \bar{\partial}_{h_t} \theta_{h_t} = \theta_{h_t} \wedge \theta_{h_t} = 0$, namely, $h_t$ is also a pluri-harmonic metric on $(E, \mathbb{D}^\lambda)$. Then by the uniqueness of pluri-harmonic metric, we have $h_t = c \cdot h$ for some constant $c$ when $|t| = 1$.

Consequently, the dynamical system $\psi_{(\lambda, t)}$ sends a polystable Higgs bundle $((E, \bar{\partial}_E), \theta)$ to another one $((E, \bar{\partial}_E), t\theta)$, namely, $\psi_{(\lambda, t)}$ is just the usual $S^1$-action by $t$ on $\text{M}_{\text{Dol}}(X, r)$.

Now let $((E, \bar{\partial}_E), \theta)$ be a decoupled Higgs bundle with the Higgs field $\theta$ nonzero. If it is a fixed point of $\psi_{(\lambda, t)}$ for $|t| = 1$, then there is a $C^\infty$-automorphism $g \in \text{Aut}(E)$ such that

$$g \bar{\partial}_E g^{-1} = 0, \quad g\theta g^{-1} = t\theta. \quad (5.3)$$

Since $(E, \bar{\partial}_E)$ is already a polystable bundle, thus $((E, \bar{\partial}_E), \theta)$ is just the usual $S^1$-action by $t$ on $\text{M}_{\text{Dol}}(X, r)$.

We find that $h_t$ is of the following form

$$g = \begin{pmatrix} a_1 \text{Id}_{E_1} & \cdots & a_N \text{Id}_{E_N} \end{pmatrix}$$

for nonzero constants $a_1, \ldots, a_N$. If there exists $i$ such that $\text{pr}_{E_i} \circ \theta|_{E_i} : E_i \to E_i \otimes K_X$ is nonzero, where $\text{pr}_{E_i}$ denotes the projection onto $E_i \otimes K_X$ of $E \otimes K_X$, then the second equation admits a solution for $g$ if and only if $|t| = 1$. If each $\text{pr}_{E_i} \circ \theta|_{E_i}$ vanishes, since $\theta$ is nonzero and $|\theta|_h^2 = 0$, there exist $i_1 \neq i_2 \neq \cdots \neq i_l$ for $1 \leq i_1, \ldots, i_l \leq N$ such that $\text{pr}_{E_{i_{\mu+1}}} \circ \theta|_{E_{i_{\mu}}} : E_{i_{\mu}} \to E_{i_{\mu+1}} \otimes K_X$ for $1 \leq \mu \leq l - 1$ and $\text{pr}_{E_{i_1}} \circ \theta|_{E_{i_1}} : E_{i_1} \to E_{i_1} \otimes K_X$ are all nonzero. Therefore by the equation (5.4) we have

$$a_i = ta_{i-1}, a_2 = ta_1, \ldots, a_{i-1} = ta_i, a_i = ta_i, \quad (5.5)$$

thus $t$ has to be $l$-roots of units. Moreover all components $a_1, \ldots, a_N$ are solved by a series of equations as the form of (5.5).

Case II: $\lambda = 0, t \in \mathbb{C}^*$.

In this case, $(\lambda, t)$-action is just the scalar multiplication on Higgs field by $t$. The same conclusions as above follows.

Case III: $\lambda \neq 0, |t| \neq 1$.

Let $((E, \bar{\partial}_E), \theta)$ be a decoupled Higgs bundle with the Higgs field $\theta$ nonzero and the decoupling metric $h$. One writes

$$td''_E = t\lambda(\bar{\partial}_{E,h} + \frac{t-a}{t\lambda} \theta) + a\theta,$$

$$d''_E = \bar{\partial}_E + \lambda(1 - t\bar{a})\theta_h^1 + t\lambda\bar{a}\theta_h^1,$$

for some $a \in \mathbb{C}$, then $((E, \bar{\partial}_E + \lambda(1 - t\bar{a})\theta_h^1), a\theta)$ is a Higgs bundle. Note that $(\partial_{E,h} - \bar{\lambda}(1 - \bar{a}\theta)) + (\bar{\partial}_E + \lambda(1 - t\bar{a})\theta_h^1)$ is a unitary connection with respect to $h$. If one takes

$$a = \frac{t \frac{1 + |\lambda|^2}{1 + |\lambda|^2},}$$

we find that $h$ is the pluri-harmonic metric both for the $\lambda'$-flat bundle $(E, \mathbb{D}^{\lambda'} = td'_E + d''_E)$ and the Higgs bundle $((E, \bar{\partial}_E + \lambda(1 - t\bar{a})\theta_h^1), a\theta)$. Therefore, by the uniqueness of pluri-harmonic metric we get

$$\psi_{(\lambda, t)}((E, \bar{\partial}_E), \theta) = ((E, \bar{\partial}_E + \lambda(1 - t\bar{a})\theta_h^1), a\theta).$$
Assume there is a $C^\infty$-automorphism $g \in \text{Aut}(E)$ such that
\[ g \bar{\partial}_E g^{-1} - \lambda (1 - t a) \bar{\partial}_h = 0, \quad (5.6) \]
\[ g \bar{\partial}_E g^{-1} - a \theta = 0. \quad (5.7) \]
Since $[\theta, \theta^\perp] = 0$, over a small neighborhood of some point $x \in X$ with $\theta|_x \neq 0$, there is an orthogonal decomposition of $(E, h)$ into Hermitian line bundles as $(E, h) = \bigoplus_{i=1}^r (L_i, h_i)$ such that the Higgs field $\theta$ has the decomposition $\theta = \bigoplus_{i=1}^r \varphi_i \cdot \text{Id}_{L_i}$ with one-forms $\varphi_i$ [24]. From the equation (5.7) it follows that $|a| = 1$, namely $|t| |\lambda|^2 = 1$ $(|\lambda|^2 \neq 1)$, hence $a = \frac{\lambda}{|t|}$. We consider $n$-times iteration of the $(\lambda, t)$-action on $((E, \bar{\partial}_E), \theta)$. The direct calculation shows
\[ \psi^n_{(\lambda, t)}((E, \bar{\partial}_E), \theta) = \begin{cases} ((E, \bar{\partial}_E + n(\lambda - \frac{1}{\lambda}) \bar{\partial}_h), \theta), & t > 0, t \neq 1; \\ ((E, \bar{\partial}_E + (\lambda - \frac{1}{\lambda}) \frac{1-(\frac{1}{|t|})^{n} \theta^\perp}{1-\theta^\perp} \bar{\partial}_h), (\frac{1}{|t|})^n \theta), & \text{other cases}. \end{cases} \]
By assumption, the limit $\lim_{n \to \infty} \psi^n_{(\lambda, t)}((E, \bar{\partial}_E), \theta)$ lies in the isomorphism class of $((E, \bar{\partial}_E), \theta)$, hence $t$ cannot be a positive real number. For the other cases, writing $t = |t| e^{i\alpha}$, we must have $e^{i\alpha n t'} = 1$ for any positive integer $n$, where $2 \leq t' \leq r$ is a fixed positive integer, therefore, $\alpha = \frac{2k\pi}{t'}$ for some $k \in \{1, \ldots, t' - 1\}$.

Combining the three cases together, we complete the proof the theorem. \[ \square \]

**Corollary 5.9.**

1. If $\text{Tr}(\theta)$ is nonzero at some point $x \in X$, then $C_x = \mathbb{C} \times \{1\}$.
2. Fixing $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ with $|t| |\lambda|^2 \neq 1$ and $t \neq 1$, let $\mathcal{H}_{\text{de}}^{\lambda, t} = \mathcal{H}_{\text{de}}^{\lambda, t} \cap M_{\text{de}}(X, r)$, then $B(X, r)$ as a subvariety of $M_{\text{Doal}}(X, r)$ is a connected component of $\mathcal{H}_{\text{de}}^{\lambda, t}$.

**Proof.** (1) is obvious. To show (2), we consider a sequence $\{(E_n, \theta_n)\}$ of stable decoupled Higgs bundles lying in $\mathcal{H}_{\text{de}}^{\lambda, t} \setminus B(X, r)$ parameterized by positive integers $n \in [N, \infty)$ for a large $N$ such that $\lim_{n \to \infty} (E_n, \theta_n) = (E_\infty, 0) \in B(X, r)$. For each $(E_n, \theta_n)$, there is a $C^\infty$-automorphism $g_n \in \text{Aut}(E)$ satisfying the equations (5.6) and (5.7). Since $[\theta_n, (\theta_n)^\perp] = 0$ for the decoupling metric $h_n$, by equation (5.7) $(a \neq 1)$, for any $n \in [N, \infty)$ the automorphism $g_n$ locally has a matrix form as $\begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix}$, where all diagonal elements of the nonzero matrix $B_n$ are zero. On the other hand, from equation (5.6) it follows that $g_\infty = \lim_{n \to \infty} g_n$ is exactly $c \cdot \text{Id}_{E_\infty}$ for some constant $c \in \mathbb{C}^*$, which is a contradiction. Therefore, the desired sequence does not exist, the conclusion follows. \[ \square \]

**Remark 5.10.** Studying $M_{\text{de}}(X, r)$ and $\mathcal{H}_{\text{de}}^{\lambda, t}$ is an interesting and hard problem. For example, what are the smooth (or singular) points of $M_{\text{de}}(X, r)$, and does there exist any other connected components of $\mathcal{H}_{\text{de}}^{\lambda, t}$ except $B(X, r)$?

**Theorem 5.11.** Let $\mathcal{H} = \bigcap_{(\lambda, t) \in \mathbb{C}^* \times \mathbb{C}^*} \mathcal{H}_{\text{de}}^{\lambda, t}$. Then $\mathcal{H}$ consists of the set of complex variations of Hodge structure.

**Proof.** Let $\overline{\mathcal{H}} = \bigcap_{(\lambda, t) \in \mathbb{C}^* \times \mathbb{C}^*} \mathcal{H}_{\text{de}}^{\lambda, t}$. We first show that $\overline{\mathcal{H}}$ consists of the complex variations of Hodge structure. Consider a complex variation of Hodge structure $u = ((E, \bar{\partial}_E), \theta)$ as
\[ \bar{\partial}_E = \begin{pmatrix} \bar{\partial}_{E_1} & \cdots & \bar{\partial}_{E_k} \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 & \cdots & 0 \\ \theta_1 & \cdots & \theta_{k-1} \\ 0 \end{pmatrix}. \]
we only need to show $C_u = \mathbb{C} \times C^\ast$. By the pluri-harmonic metric $h$ on $((E, \bar{\partial}_E), \theta)$ which makes the splitting $E = \bigoplus_{i=1}^k E_i$ being orthogonal, assuming $\lambda \neq 0$, it produces two flat bundles $((E, \bar{\partial}'_E), \nabla')$ and $((E, \bar{\partial}''_E), \nabla'')$ given by

$$
\nabla' = \begin{pmatrix}
    \partial_{E_1,h} & \partial_{E_2,h} & \cdots & \partial_{E_k,h} \\
    \lambda^{-1}\theta_1 & \cdots & \lambda^{-1}\theta_{k-1} & \partial_{E_k,h}
\end{pmatrix}, \quad \bar{\partial}'_E = \begin{pmatrix}
    \partial_{E_1} & \lambda(\theta_1)_h \\
    \cdots & \cdots & \cdots & \cdots \\
    \lambda^{-1}\theta_{k-1} & \cdots & \partial_{E_k,h} \\
    \lambda(\theta_{k-1})_h & \cdots & \cdots & \cdots
\end{pmatrix},
$$

$$
\nabla'' = \begin{pmatrix}
    \partial_{E_1,h} & \partial_{E_2,h} & \cdots & \partial_{E_k,h} \\
    (t\lambda)^{-1}\theta_1 & \cdots & (t\lambda)^{-1}\theta_{k-1} & \partial_{E_k,h}
\end{pmatrix}, \quad \bar{\partial}''_E = \begin{pmatrix}
    \partial_{E_1} & t\lambda(\theta_1)_h \\
    \cdots & \cdots & \cdots & \cdots \\
    (t\lambda)^{-1}\theta_{k-1} & \cdots & \partial_{E_k,h} \\
    t\lambda(\theta_{k-1})_h & \cdots & \cdots & \cdots
\end{pmatrix}.
$$

If these two flat bundles are equivalent to each other, then there is a $C^\infty$-automorphism $g \in \text{Aut}(E)$ such that

$$
g \begin{pmatrix}
    \bar{\partial}_{E_1,h} \\
    \bar{\partial}_{E_2,h} \\
    \cdots \\
    \bar{\partial}_{E_k,h}
\end{pmatrix} = \begin{pmatrix}
    \tilde{\partial}_{E_1} \\
    \tilde{\partial}_{E_2} \\
    \cdots \\
    \tilde{\partial}_{E_k}
\end{pmatrix},
$$

$$
g^{-1} + \lambda^{-1}g \begin{pmatrix}
    0 & 0 & \cdots & 0 \\
    \theta_1 & \theta_{k-1} & \cdots & 0 \\
    \cdots & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & 0
\end{pmatrix} g^{-1} = (t\lambda)^{-1} \begin{pmatrix}
    0 & 0 & \cdots & 0 \\
    \theta_1 & \theta_{k-1} & \cdots & 0 \\
    \cdots & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & 0
\end{pmatrix}.
$$

Obviously, the above equations have a solution

$$
g = \begin{pmatrix}
    \text{Id}_{E_1} & t^{-1}\text{Id}_{E_2} & \cdots & t^{-k+1}\text{Id}_{E_k}
\end{pmatrix}.
$$

It immediately follows that $\psi_{(\lambda,t)}((E, \bar{\partial}_E), \theta) = ((E, \bar{\partial}_E), \theta)$ for any $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^\ast$.

Next we show that $\mathfrak{H}\mathfrak{F} = \mathfrak{F}\mathfrak{H}$. Assume $((E, \bar{\partial}_E), \theta)$ lies in $\mathfrak{F}\mathfrak{H}$, then we have

$$
\lim_{t \to 0, \lambda \to 0} \psi_{(\lambda,t)}((E, \bar{\partial}_E), \theta) = \lim_{t \to 0}((E, \bar{\partial}_E), \theta) = ((E, \bar{\partial}_E), \theta).
$$

On the other hand, let $h$ be a pluri-harmonic metric on $((E, \bar{\partial}_E), \theta)$ and $h_t$ be the pluri-harmonic metric on $\psi_{\lambda,t}((E, \bar{\partial}_E), \theta)$. Writing $h_t = h \cdot s$ with $s = e^\chi$ for $\chi \in \text{End}(E)$, the direct calculation shows that the image of $(\lambda, t)$-action on $((E, \bar{\partial}_E), \theta)$ is given by $\psi_{(\lambda,t)}((E, \bar{\partial}_E), \theta) = ((E, \bar{\partial}_E), \theta')$, where

$$
\bar{\partial}_E(\lambda, t) = \bar{\partial}_E + \frac{\lambda(1-|t|^2)}{1 + |\lambda|^2}\theta_k^h + \frac{\lambda|t|^2}{1 + |\lambda|^2}s^{-1}(\lambda\bar{\partial}_E - \theta_k^h)s,
$$

$$
\theta'(\lambda, t) = \frac{t(1 + |\lambda|^2)}{1 + |\lambda|^2}\theta - \frac{\lambda t}{1 + |\lambda|^2}s^{-1}(\bar{\partial}_E,h - \lambda\bar{\theta})s.
$$

The condition $\bar{\partial}_E \theta' = 0$ gives rise to a equation satisfied by $s$. Then we immediately find that

$$
\lim_{t \to 0} \psi_{(\lambda,t)}((E, \bar{\partial}_E), \theta) = \lim_{t \to 0} \psi_{(0,t)}((E, \bar{\partial}_E), \theta) = \lim_{t \to 0}((E, \bar{\partial}_E), t\theta).
$$
Comparing (5.8) with (5.9) implies \(((E, \bar{\partial}_E), \theta)\) is a complex variation of Hodge structure. 

5.4. Asymptotic Behaviour. In this subsection, \(X\) is always assumed to be a Riemann surface. We first recall Simpson’s beautiful work on the limits of \(C^*\)-action on the Hodge moduli space \(M_{\text{Hod}}(X, r)\) (for more details, see [34, 17, 16]).

**Definition 5.12** ([34]). Let \(E\) be a holomorphic vector bundle over a Riemann surface \(X\) with a holomorphic flat connection \(\nabla : E \to E \otimes_{\mathcal{O}_X} K_X\), where \(K_X\) denotes the canonical line bundle over \(X\). A decreasing filtration \(\{F^*_i\}\) of \(E\) by strict subbundles

\[
E = F^0 \supset F^1 \supset \cdots \supset F^k = 0
\]

is called a Simpson filtration if it satisfies the following two conditions:

- Griffiths transversality: \(\nabla : F^p \to F^{p-1} \otimes_{\mathcal{O}_X} \Omega^1_X\),
- graded-semistability: the associated graded Higgs bundle \((\text{Gr} F(E), \text{Gr} F(\nabla))\), where \(\text{Gr}_F(E) = \bigoplus_p F^p\) with \(F^p = F^p / F^{p-1}\) and \(\text{Gr}_F(\nabla) = \bigoplus_p \theta^p\) with \(\theta^p : E^p \to E^{p-1} \otimes_{\mathcal{O}_X} K_X\) induced from \(\nabla\), is a semistable Higgs bundle.

**Theorem 5.13** ([34, Theorem 2.5, Lemma 4.1, Corallary 4.2, Proposition 4.3]). Let \((E, \nabla)\) be a flat bundle over a Riemann surface \(X\).

1. There exist Simpson filtrations \(\{F^*_i\}\) on \((E, \nabla)\).
2. Let \(\{F^*_i\}, \{F^*_j\}\) be two Simpson filtrations on \((E, \nabla)\), then the associated graded Higgs bundles \((\text{Gr}_{F_1}(E), \text{Gr}_{F_1}(\nabla))\) and \((\text{Gr}_{F_2}(E), \text{Gr}_{F_2}(\nabla))\) are \(S\)-equivalent.
3. \((\text{Gr}_F(E), \text{Gr}_F(\nabla))\) is a stable Higgs bundle iff the Simpson filtration is unique.
4. \(\lim_{t \to 0} (E, t \cdot \nabla) = (\text{Gr}_F(E), \text{Gr}_F(\nabla))\).

Now we apply Simpson’s theorem to study the asymptotic behaviour of the dynamical system \(\psi_{(\lambda, t)}\). We first introduce the following notations.

**Definition 5.14.** Given a Higgs bundle \(((E, \bar{\partial}_E), \theta) \in M_{\text{Hod}}(X, r)\), we define the following five limits:

1. \(\psi_{(0,0)}((E, \bar{\partial}_E), \theta) := \lim_{t \to 0} \psi_{(0,t)}((E, \bar{\partial}_E), \theta)\),
2. \(\psi_{(0,0)}((E, \bar{\partial}_E), \theta) := \lim_{t \to 0} \psi_{(\lambda,t)}((E, \bar{\partial}_E), \theta)\),
3. \(\psi_{(0,0)}((E, \bar{\partial}_E), \theta) := \lim_{\lambda \to 0} \psi_{(\lambda,0)}((E, \bar{\partial}_E), \theta)\),
4. \(\psi_{(0,0)}((E, \bar{\partial}_E), \theta) := \lim_{\lambda \to 0} \psi_{(\lambda,t)}((E, \bar{\partial}_E), \theta)\),
5. \(\psi_{(0,0)}((E, \bar{\partial}_E), \theta) := \lim_{(\lambda,t) \to (0,0)} \psi_{(\lambda,t)}((E, \bar{\partial}_E), \theta)\),

where \(\psi_{(\lambda,0)}\) is defined by By Simpson’s theorem, namely

\[
\psi_{(\lambda,0)}((E, \bar{\partial}_E), \theta) = \lim_{t \to 0} ((E, \bar{\partial}_E + \lambda \theta \partial_E + t \lambda^{-1} \theta) = (\text{Gr}_{F_{\lambda}}(E, \lambda), \text{Gr}_{F_{\lambda}}(\nabla, \lambda)),
\]

with \(h\) being a pluri-harmonic metric on \(((E, \bar{\partial}_E), \theta), (E, \bar{\partial}_E + \lambda \theta \partial_E, t \lambda^{-1} \theta)\), and \(\{F^*_i\}\) standing for a Simpson filtration on \((E, \bar{\partial}_E, \nabla, \lambda)\).

**Remark 5.15.** The first two limits have been used in the proof of Theorem 5.11, and we have showed that

\[
\psi_{(0,0)}((E, \bar{\partial}_E), \theta) = \psi_{(0,0)}((E, \bar{\partial}_E), \theta) = \lim_{t \to 0} ((E, \bar{\partial}_E), t \theta).
\]

In general, it is not clear whether the last three limits exist, secondly, we also do not know whether these limits coincide if they all exist.
Proposition 5.16. If for a given Higgs bundle \(((E, \bar{\partial}_E), \theta) \in M_{\text{Dol}}(X, r)\), the limit \(\psi^{(0,0)}((E, \bar{\partial}_E), \theta)\) (or \(\psi_{(0,0)}((E, \bar{\partial}_E), \theta)\)) exists, then it must be a complex variation of Hodge structure.

Proof. Let \(\psi^{(0,0)}((E, \bar{\partial}_E), \theta) = ((E, \bar{\partial}_E'), \theta')\), then we calculate
\[
\lim_{t \to 0} \lim_{\lambda \to 0} \lim_{t_i \to 0} \psi_{(\lambda, t)}((E, \bar{\partial}_E'), \theta') = \lim_{t \to 0} \lim_{\lambda \to 0} \lim_{t_i \to 0} \psi_{(\lambda, t)}((E, \bar{\partial}_E), \theta) = \lim_{\lambda \to 0} \lim_{t \to 0} \psi_{(\lambda, t)}((E, \bar{\partial}_E), \theta) = \lim_{t \to 0} ((E, \bar{\partial}_E'), \theta'),
\]
on the other hand, we have
\[
\lim_{t \to 0} \lim_{\lambda \to 0} \lim_{t_i \to 0} \psi_{(\lambda, t)}((E, \bar{\partial}_E'), \theta') = \lim_{t \to 0} \psi_{(0, t)}((E, \bar{\partial}_E'), \theta') = \lim_{t \to 0} ((E, \bar{\partial}_E'), \theta').
\]
Comparing these two results, we find that \(((E, \bar{\partial}_E'), \theta')\) has to be a complex variation of Hodge structure. \(\square\)

Theorem 5.17. Let \(X\) be a Riemann surface.

1. If \(((E, \bar{\partial}_E), \theta) \in M_{\text{Dol}}(X, r)\) is a complex variation of Hodge structure or a decoupled Higgs bundle, then the above limits exist and coincide.

2. Let \(((E, \bar{\partial}_E), \theta) \in M_{\text{Dol}}(X, 2)\) and assume the maximal destabilizing subbundle of \((E, \bar{\partial}_E)\) is preserved by \(\theta^1\) for the pluri-harmonic metric \(h\) on \(((E, \bar{\partial}_E), \theta)\), then the limit \(\psi^{(0,0)}((E, \bar{\partial}_E), \theta)\) exists, and it coincides with the limit \(\psi^{(0,0)}((E, \bar{\partial}_E), \theta)\).

3. Let \(((E, \bar{\partial}_E), \theta) \in M_{\text{Dol}}(X, r)\), then the limit \(\lim_{\lambda \to 0} \psi_{(\lambda, 0)}((E, \bar{\partial}_E), \lambda \theta)\) exists, and it coincides with the limit \(\psi^{(0,0)}((E, \bar{\partial}_E), \theta)\).

Proof. (1) i) Let \(((E, \bar{\partial}_E), \theta) \in M_{\text{Dol}}(X, r)\) be a complex variation of Hodge structure. Since it is a fixed point of \((\lambda, t)\)-action for any \((\lambda, t) \in \mathbb{C} \times \mathbb{C}^*\) by Theorem 5.11, we have
\[
\psi^{(0,0)}((E, \bar{\partial}_E), \theta) = \psi^{(0,0)}((E, \bar{\partial}_E), \theta) = \psi^{(0,0)}((E, \bar{\partial}_E), \theta) = \psi^{(0,0)}((E, \bar{\partial}_E), \theta) = ((E, \bar{\partial}_E), \theta).
\]
Hence we only need to show \(\psi^{(0,0)}((E, \bar{\partial}_E), \theta) = ((E, \bar{\partial}_E), \theta)\). For \(\lambda \neq 0\), we write \((E, \bar{\partial}_E) = \bigoplus_{i=1}^k (E_i, \bar{\partial}_E_i), \theta = \bigoplus_{i=1}^{k-1} \theta_i\) for \(\theta_i : E_i \to E_{i+1} \otimes K_X\), then by virtue of the pluri-harmonic metric \(h\) on \(((E, \bar{\partial}_E), \theta)\), we have a holomorphic flat connection
\[
\nabla = \begin{pmatrix}
\bar{\partial}_{E_1, h} & \lambda^{-1} \theta_1 & \ldots & \ldots & \lambda^{-1} \theta_{k-1} & \bar{\partial}_{E_k, h} \\
\lambda^{-1} \bar{\partial}_1 & \bar{\partial}_{E_2, h} & \ldots & \ldots & \bar{\partial}_{E_{k-1}, h} & \lambda^{-1} \theta_{k-1} \bar{\partial}_{E_k, h}
\end{pmatrix}
\]
with respect to the holomorphic structure
\[
\bar{\partial}_E = \begin{pmatrix}
\bar{\partial}_{E_1} & \lambda(\theta_1)^\dagger_h & \ldots & \ldots & \lambda(\theta_{k-1})^\dagger_h
\end{pmatrix}.
\]
There is a Simpson filtration \(\{F^p\}\) on \(((E, \bar{\partial}_E), \nabla)\) given by \(F^p = \bigoplus_{i=1}^{k-p} E_i\) since one easily checks that
\[
\nabla F^p \subset F^{p-1} \otimes K_X, \quad \bar{\partial}_E F^p = 0.
\]
It follows that \( \psi_{\lambda,0}(E, \partial E, \theta) = ((E, \partial E, \lambda^{-1} \theta)) \) from Simpson’s theorem. Therefore, \( \psi_{(0,0)}((E, \partial E), \theta) = \lim_{\lambda \to 0} ((E, \partial E, \lambda^{-1} \theta)) \).

ii) Let \((E, \partial E, \theta) \in \mathbb{M}_{Dol}(X, r)\) be a decoupled Higgs bundle with decoupling metric \( h \). We can assume \( \theta \) is nonzero. We have seen that \( \psi_{(0,0)}((E, \partial E), \theta) = (E, \partial E) \), meanwhile we can also calculate the limits

\[
\psi_{(0,0)}(E, \partial E, \theta) = \lim_{\lambda \to 0} (E, \partial E + \lambda \theta^\dagger) = (E, \partial E),
\]

and

\[
\lim_{\lambda, t \to 0} \left( \frac{\lambda(1 - |t|^2)}{1 + |t\lambda|^2} \theta^\dagger \right) = \lim_{\lambda \to 0} \left( \frac{\lambda(1 - |t|^2)}{1 + |t\lambda|^2} \theta^\dagger \right) = \lim_{\lambda \to 0} (E, \partial E + \lambda \theta^\dagger) \]

\[
\lim_{t \to 0} \lim_{\lambda \to 0} \left( \frac{\lambda(1 - |t|^2)}{1 + |t\lambda|^2} \theta^\dagger \right) = \lim_{t \to 0} \lim_{\lambda \to 0} \left( \frac{\lambda(1 - |t|^2)}{1 + |t\lambda|^2} \theta^\dagger \right) = \lim_{t \to 0} ((E, \partial E), t\theta) = (E, \partial E).
\]

Consequently, \( \psi_{(0,0)}((E, \partial E), \theta) = \psi_{(0,0)}((E, \partial E), \theta) = \psi_{(0,0)}((E, \partial E), \theta) = (E, \partial E) \).

(2) Consider a family of flat bundles \((E, \partial E + \lambda \theta^\dagger), \partial E, \lambda^{-1} \theta)\). It is divided into two cases.

i) Assume \((E, \partial E + \lambda \theta^\dagger)\) are non-semistable over some small deleted neighborhood \( U \) of \( \lambda = 0 \). Let \( L \) be the maximal destabilizing subbundle of \((E, \partial E)\), and \( L^\perp \) be the orthogonal complement of \( L \) in \( E \) with respect to the pluri-harmonic metric \( h \), namely there are \( C^\infty \)-decompositions \( E \cong L \oplus L^\perp \cong L \oplus E/L \). With respect to the above decomposition, we write

\[
\partial E = \left( \begin{array}{ccc} \theta_1 & 0 \\ 0 & \theta_2 \end{array} \right), \quad \theta = \left( \begin{array}{cccc} \theta_1 & 0 \\ \beta & \theta_2 \end{array} \right),
\]

where \( \beta \) must be non-zero and satisfies \( \partial_2 \beta = 0 \). By assumption \( L \) is preserved by \( \theta^\dagger \). Since the Simpson filtration exactly coincides with the Harder–Narasimhan filtration for the case of rank \( r = 2 \), we get

\[
\psi_{(0,0)}((E, \partial E), \theta) = \lim_{\lambda \to 0} \left( \frac{\alpha + \lambda \theta_1}{\partial_1 + \lambda \theta_1}, \frac{0}{\partial_2 + \lambda \theta_2} \right) = \left( \begin{array}{cccc} \alpha + \lambda^{-1} \beta \end{array} \right).
\]

Choosing a \( C^\infty \)-automorphism \( g = \left( \begin{array}{ccc} 1 & 0 \\ 0 & \lambda \end{array} \right) \in \text{Aut}(E) \), from the identities

\[
g \circ \left( \begin{array}{ccc} \partial_1 + \lambda \theta_1 \\ 0 \\ \partial_2 + \lambda \theta_2 \end{array} \right) \circ g^{-1} = \left( \begin{array}{ccc} \partial_1 + \lambda \theta_1 \\ 0 \\ \partial_2 + \lambda \theta_2 \end{array} \right),
\]

it follows that

\[
\psi_{(0,0)}((E, \partial E), \theta) = \lim_{t \to 0} ((E, \partial E), t\theta),
\]

thus \( \psi_{(0,0)}((E, \partial E), \theta) = \psi_{(0,0)}((E, \partial E), \theta) \).

ii) Assume \((E, \partial E + \lambda \theta^\dagger)\) are semistable over some small deleted neighborhood \( U \) of \( \lambda = 0 \). Then \((E, \partial E)\) is also a semistable bundle. Otherwise, by our assumption, the maximal destabilizing
subbundle \( L \) of \( (E, \bar{\partial}_E) \) is also that of \( (E, \bar{\partial}_E + \lambda \theta^\lambda_h) \), which contradicts the semistability of \( (E, \bar{\partial}_E + \lambda \theta^\lambda_h) \). Therefore, we have
\[
\psi_{(0,0)}((E, \bar{\partial}_E), \theta) = \lim_{\lambda \to 0} (E, \bar{\partial}_E + \lambda \theta^\lambda_h) = (E, \bar{\partial}_E) = \psi_{(0,0)}((E, \bar{\partial}_E), \theta).
\]

(3) It follows from the calculation of so-called conformal limit in [11, 10, 6]. Indeed, the limit
\[
\lim_{c \to 0} ((E, \bar{\partial}_E + |c|^2 \theta^c_{h_c}), \partial_{h_c} + \theta) = ((E, \bar{\partial}_E), \nabla')
\]
exists as a flat bundle, where \( h_c \) is a pluriharmonic metric on the Higgs bundle \( ((E, \bar{\partial}_E), c\theta) \), and it satisfies
\[
\lim_{t \to 0} ((E, \bar{\partial}_E), t\nabla') = \lim_{c \to 0} ((E, \bar{\partial}_E), c\theta).
\]

\[\square\]

References

[1] D. Arinkin, On \( \lambda \)-connections on a curve where \( \lambda \) is a formal parameter, Math. Res. Lett. 12 (2002) 551–565.
[2] U. Bhosle, Picard group of the moduli spaces of vector bundles, Math. Ann. 314 (1999) 245-263.
[3] I. Biswas, V. Muñoz, Torelli theorem for moduli space of \( SL(r, \mathbb{C}) \)-connections on a compact Riemann surface, Commun. Contemp. Math. 11 (2009) 1-26.
[4] S. Cardona, On vanishing theorems for Higgs bundles, Diff. Geom. Appl. 35 (2014) 95-102.
[5] X. Chen, R. Wentworth, The nonabelian Hodge correspondence for balanced hermitian metrics of Hodge–Riemann type, arXiv:2106.09133.
[6] B. Collier, R. Wentworth, Conformal limits and the Bialynicki-Birula stratification of the space of \( \lambda \)-connections, Adv. Math. 350 (2019) 1193-1225.
[7] K. Corlette, Flat \( G \)-bundles with canonical metrics, Jour. Diff. Geom. 28 (1988) 361-382.
[8] P. Deligne, Various letters to C. Simpson.
[9] S.K. Donaldson, Twisted harmonic maps and the self-duality equations, Proc. London Math. Soc. 55 (1987) 127–131.
[10] O. Dumitrescu, L. Fredrickson, G. Kydonakis, R. Mazzeo, M. Mulase, A. Neitzke, From the Hitchin section to opers through nonabelian Hodge, Jour. Diff. Geom. 117 (2021) 223-253.
[11] D. Gaioitio, Opers and TBA, arXiv:1403.6137.
[12] N.J. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. 1 (1987) 59-126.
[13] N.J. Hitchin, A. Karlhede, U. Lindström, M. Roček, Hyper-Kähler metrics and supersymmetry, Comm. Math. Phys. 108 (1987) 535-589.
[14] Z. Hu, P. Huang, Degenerate, strong and stable Yang–Mills–Higgs pairs, Jour. Geom. Phys. 120 (2017) 73-88.
[15] Z. Hu, P. Huang, The Hitchin–Kobayashi correspondence for quiver bundles over generalized Kähler manifolds, Jour. Geom. Anal. 30 (2020), 3641-3671.
[16] Z. Hu, P. Huang, Simpson filtration and oper stratum conjecture, manus. math. 167 (2022) 653-673.
[17] P. Huang, Théorie de Hodge non-Abélienne et des spécialisations, Ph.D. Thesis, Université Côte d’Azur, 2020.

HAL: https://tel.archives-ouvertes.fr/tel-03134917.
[18] M. Inaba, K. Iwashiki, M.-H. Saito, Moduli of stable parabolic connections, Riemann–Hilbert correspondence and geometry of Painlevé equation of type VI, Part I, Publ. Res. Inst. Math. Sci. 42 (2006) 987-1089.
[19] C.-C. Liu, S. Rayan, Y. Tanaka, The Kapustin-Witten equations and nonabelian Hodge theory, Eur. J. Math. to appear, arXiv:2012.06175. doi: 10.1007/s40879-022-00538-4.
[20] M. Lübke, A. Teleman, The Kobayashi–Hitchin correspondence, World Scientific, Singapore, 1995.
[21] R. Mazzeo, J. Swoboda, H. Weiss, F. Witt, Ends of the moduli space of Higgs bundles, Duke Math. J. 165 (2016) 2227-2271.
[22] T. Mochizuki, Kobayashi–Hitchin correspondence for tame harmonic bundles and an application, Astérisque No. 309 (2006) viii+117.
[23] T. Mochizuki, Kobayashi–Hitchin correspondence for tame harmonic bundles, II, Geom. Topol. 13 (2009) 359-455.
[24] T. Mochizuki, Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces, J. Topol. 9 (2016) 1021-1073.
[25] T. Mochizuki, Good wild harmonic bundles and good filtered Higgs bundles, SIGMA Symmetry Integrability Geom. Methods Appl. 17 (2021), 068, 66 pages.
[26] C.T. Simpson, Constructing of variations of Hodge structure using Yang–Mills theory and applications to uniformization, J. Amer. Math. Soc. 1 (1988) 867-918.
[27] C.T. Simpson, Harmonic bundles on noncompact curves, J. Amer. Math. Soc. 3 (1990) 713-770.
[28] C.T. Simpson, A lower bound for the size of monodromy of systems of ordinary differential equations, in: Algebraic geometry and analytic geometry, ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991, pp. 198-230.
[29] C.T. Simpson, Higgs bundles and local systems, Inst. Hautes Études Sci. Publ. Math. 75 (1992) 5-95.
[30] C.T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety I, Inst. Hautes Études Sci. Publ. Math. 79 (1994) 47-129.
[31] C.T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety II, Inst. Hautes Études Sci. Publ. Math. 80 (1994) 5-79.
[32] C.T. Simpson, The Hodge filtration on nonabelian cohomology, in: Algebraic geometry-Santa Cruz 1995, in: Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 217–281.
[33] C.T. Simpson, A weight two phenomenon for the moduli of rank one local systems on open varieties, in: From Hodge theory to integrability and TQFT tt*-geometry, in: Proc. Sympos. Pure Math., vol. 78, Amer. Math. Soc., Providence, RI, 2008, pp. 175-214.
[34] C.T. Simpson, Iterated destabilizing modifications for vector bundles with connection, in: Vector bundles and complex geometry, in: Contemp. Math., vol. 522, Amer. Math. Soc., Providence, RI, 2010, pp. 183–206.
[35] K. Uhlenbeck, S.-T. Yau, On the existence of Hermitian–Yang–Mills connections in stable vector bundles, Comm. Pure Appl. Math. 39 (1986) 257-293.

School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, P.R. China
Department of Mathematics, Mainz University, 55128 Mainz, Germany

Email address: halfask@mail.ustc.edu.cn; huz@uni-mainz.de

School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, P.R. China
Laboratoire J.A. Dieudonné, Université Côte d’Azur, CNRS, 06108 Nice, France
Mathematisches Institut, Ruprecht-Karls Universität Heidelberg, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany

Email address: pfhwangmath@gmail.com