On parametric resonance in the laser action

A. I. Komech

Faculty of Mathematics, University of Vienna
alexander.komech@univie.ac.at

Abstract

We consider the selfconsistent semiclassical Maxwell–Schrödinger system for the solid state laser. The system consists of the Maxwell equations coupled to $N \sim 10^{20}$ Schrödinger equations for active molecules. The system includes time-periodic pumping and a weak dissipation. We introduce the corresponding Poincaré map $\mathcal{P}$ and consider the differential $D\mathcal{P}(Y^0)$ at suitable ground state $Y^0$.

We conjecture that the stable laser action is due to the parametric resonance (PR) which means that the maximal absolute value of the corresponding multipliers is sufficiently large. The multipliers are defined as eigenvalues of $D\mathcal{P}(Y^0)$. The PR makes the stationary state $Y^0$ highly unstable, and we suppose that this instability maintains the coherent laser radiation. We prove that the spectrum $\text{Spec } D\mathcal{P}(Y^0)$ is approximately symmetric with respect to the unit circle $|\mu| = 1$ if the dissipation is sufficiently small.

More detailed results are obtained for the Maxwell–Bloch system which is a finite-dimensional approximation of the Maxwell–Schrödinger system. The approximation consists of one-mode Maxwell field coupled to $N$ two-level molecules. We calculate the corresponding Poincaré map $P$ by successive approximations. The key role in calculation of the multipliers is played by the sum of $N$ positive terms arising in the second-order approximation for the total current. This fact can be interpreted as the synchronization of the main parts of molecular currents in all active molecules, which is provisionally in line with the superradiance and with the role of stimulated emission in the laser action. The calculation of the sum relies on probabilistic arguments which is one of main novelties of our approach. Other main novelties are i) the calculation of the differential $DP$ in the “Hopf representation” which corresponds to the factorization of the dynamics by the action of the symmetry gauge group, ii) the justification of this representation, iii) the block structure of the differential, and iv) the justification of the “rotating wave approximation” by a new estimate for the averaging of slow rotations.

The main peculiarity of the eigenvalue problem is that $DP$ is a matrix of size $\sim N \times N$ which depends on random distribution of active molecules in the resonance cavity. The block structure of $DP$ allows us to reduce the eigenvalue problem to polynomial equation of degree four. This reduction relies on a novel probabilistic arguments.

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The article addresses the problems of laser and maser radiation. In the existing approaches, time-periodic pumping is not included in the dynamical equations, so the corresponding resonance features are absent. This fact was one of our main motivations.

We consider the selfconsistent semiclassical Maxwell–Schrödinger system for the solid state laser. The system consists of the Maxwell equations coupled to $N \sim 10^{20}$ Schrödinger equations for active molecules with one active electron. The system includes time-periodic pumping and a weak dissipation. We introduce the corresponding Poincaré map $\mathcal{P}$ and consider its differential $D\mathcal{P}(Y^0)$ at suitable ground state $Y^0$. Let $\mu$ denote the multipliers which are eigenvalues of the differential $D\mathcal{P}(Y^0)$. We conjecture that the laser action is due to the parametric resonance, when

$$\max_{\mu \in \text{Spec} D\mathcal{P}(Y^0)} |\mu| > 1.$$  \hspace{1cm} (1.1)

For solutions to the linearized dynamical equation at $Y^0$ with the spectrum satisfying (1.1), the components with $|\mu| \leq 1$ are bounded, while the ones with $|\mu| > 1$ grow exponentially for large times. As a result, almost all solutions grow exponentially which is in line with the conventional interpretation of the laser action as the appearance of exponentially growing solutions of the corresponding linearized dynamics, see ([28, (6.5.12] and (6.6.23)). This is why we expect that under the condition (1.1), similar growth takes place in the case of sufficiently small nonlinear coupling (the growth only occurs for limited time because of nonlinear effects and dissipation).

The conjecture is confirmed by the fact that laser action is possible only for the pumping above a threshold that is quite similar to the parametric resonance [2].

**Remark 1.1.** We suppose that the overwhelming predominance of a single mode, corresponding to the maximal value of $|\mu|$, can be responsible for the *coherent monochromatic laser radiation*. 

The main goal of the present article is an analysis of properties of the multipliers and development of methods for their calculation. Our main results are as follows.

I. We develop the Hamiltonian formalism for the ideal case of the Maxwell–Schrödinger system without dissipation. Using this formalism and the Lyapunov–Poincaré theorem [41, (3.28)], we establish the approximate symmetry of the spectrum \( \text{Spec} D\mathcal{P}(\mathbf{Y}) \) with respect to the unit circle \( |\mu| = 1 \) when the dissipation is sufficiently small (Lemma 4.1 and (4.7)).

More detailed results (II and III below) are obtained for the Maxwell–Bloch system which is a finite-dimensional approximation of the Maxwell–Schrödinger system. The approximation consists of one-mode Maxwell field coupled to \( N \) two-level molecules. The system is time-periodic in the case of periodic pumping.

II. We calculate the corresponding Poincaré map \( P \) by successive approximations. The key role in calculation of the multipliers is played by the sum of \( N \) positive terms (16.2) arising in the second-order approximation for the total current. This fact can be interpreted as the synchronization of the main parts of molecular currents in all active molecules, which is provisionally in line with the role of stimulated emission in the laser action.

III. We calculate suitable approximation for the matrix \( DP(\mathbf{Y}) \) corresponding to material parameters of the ruby laser. The main peculiarity of the corresponding eigenvalue problem is that \( DP(\mathbf{Y}) \) is a matrix of size \( \sim N \times N \) which depends on a random distribution of active molecules in the resonance cavity. Using the block structure of this matrix, we reduce the eigenvalue problem to polynomial equation of degree four.

Our main novelties are as follows: i) calculation of the sum (16.6) and the reduction to polynomial equation of degree six relying on Shnirelman’s Ergodic Theorem [39, 40] and probabilistic arguments (20.17)–(20.23), ii) the block structure of the differential (19.8), and iii) the justification of the rotating wave approximation by a new estimate for averaging of slow rotations (Lemma A.1).

**Remark 1.2.** The condition (1.1) alone is not sufficient for the stable laser/maser action: to be sure that the energy of the Maxwell field increases indeed, we check that the projections of the corresponding eigenvectors onto the Maxwell field do not vanish, see (13.8) and Remark 20.3.

Let us comment on previous theories of laser radiation. The today laser/maser theory exists on the following three levels.

*Rate equations.* This approach is based on balance equations for the numbers of emitted and absorbed photons [1, 14, 28, 33, 42].

*Semiclassical Maxwell–Schrödinger theory.* This theory is based on finite-dimensional approximations of the Schrödinger equations for each active molecule coupled to the classical Maxwell equations [1, 3, 14, 28, 33, 36, 42].

*Quantum Electrodynamics.* This approach is based on quantized Maxwell equations in the Dicke model [9, 14, 28, 33, 42] and [15]–[19].

The theory of the rate equation comes back to famous Einstein’s article [11] developing the theory of stimulated and spontaneous emission and absorption.

The semiclassical approach uses method of slowly varying amplitudes [28, Sections 6.2 and 6.3]. The formation of the coherent monochromatic radiation is explained by an instability in the equation for amplitude of the Maxwell mode under various hypothesis on the density of population of active molecules [28, Sections 6.5 and 6.6]. The criterion of the instability is formulated in terms of presence of an unstable (nonnegative) root of the corresponding characteristic equation (28, (6.5.12) and (6.6.23)).

The quantum theory of laser action relies on quantized Maxwell equations in the Dicke–Haken–Lax model [19]. This theory resulted in successful explanation of many laser phenomena: superradiance, photon echoes, self-induced transparency and others [12, 14, 28, 33, 36, 42]. The results by K. Hepp and E. Lieb [19] establish the phase transitions which occur in the thermodynamical limit as \( N \to \infty \) if the coupling constant is sufficiently large.

Note that the time-dependent pumping with a resonant frequency plays the crucial role sustaining the population inversion. At the same time, the existing theories [1, 3, 14, 27, 28, 33, 36, 42] ignore the time-dependence of the pumping, modelling the maintenance of the population inversion by phenomenological constant terms in the corresponding dynamical equations; see [28, (6.4.11), (6.4.12)] for the semiclassical model and the beginning of Section 7.3 in [28] for the quantum model. So, these models do not include an external periodic pumping, and the resonance features are not considered.

In conclusion, the case of quantum Maxwell field will be considered elsewhere.
2 Laser equations

We use the Heaviside–Lorentz units (unrationalized Gaussian units), in which the main physical constants (electron charge and mass, Planck’s constant, and the speed of light in vacuum) read [43, p. 221]

\[ e = -4.8 \times 10^{-10} \text{esu}, \quad m = 9.1 \times 10^{-28} \text{g}, \quad h = 1.055 \times 10^{-27} \text{ergs}, \quad c = 3.0 \times 10^{10} \text{cm/s}. \] (2.1)

**Laser cavity and active molecules.** The laser resonator is a (usually cylindrical) cavity \( V \subset \mathbb{R}^3 \) with metallic walls (ideal conductor) connected with the output waveguide. Active molecules occupy a subregion. In particular, for the typical cylindrical lamp of length 12 cm and diameter 0.6 cm we have

\[ |V_a| = \frac{12 \pi 0.36}{4} \approx 3.4 \text{cm}^3. \] (2.2)

The active region \( V_a \) is filled with a dielectric medium of electrical conductivity \( \sigma \). For example, the ruby laser is filled with corundum of electrical conductivity

\[ \sigma = 10^{-14} \text{Ohm}^{-1} \cdot \text{cm}^{-1} = 10^{-2} \text{s}^{-1}. \] (2.3)

The active region \( V_d \) is filled with active molecules located at the points \( x_n \) with numbers \( n = 1, \ldots, N \sim 10^{20} \). Suppose, for simplicity of notation, that in each active molecule only one electron is involved in this interaction, and it is subject to an effective static molecular potential

\[ \Phi_n(x) = \Phi(R_n(x - x_n)), \quad x \in \mathbb{R}^3, \quad R_n \in SO(3). \] (2.4)

Here \( \Phi \) is the potential of the ion (or nucleus) with the total charge \( |e| > 0 \). Further we consider two different hypothesis on the distribution of the random values \( x_n \) and \( R_n \) with \( n \in \mathbb{N} := (1, \ldots, N) \).

**H1. Polycrystalline medium (or "glass medium" [37]).** The random values \( x_n \in V \) and \( R_n \in SO(3) \) are uniformly distributed and independent, and almost independent from \( x_n', R_n' \) with large \( |x_n - x_n'| \).

**H2. Crystalline medium.** The values \( x_n \in V \) are uniformly distributed, while \( R_n \) do not depend on \( n \in \mathbb{N} \).

**Maxwell–Schrödinger equations.** Each active molecule is described by the corresponding wave function \( \psi_n(x,t) \), and in Heaviside–Lorentz units. Neglecting spin and scalar potential (both can be easily added), the corresponding Maxwell–Schrödinger system reads (see [4, 5, 7, 13, 22, 23, 24, 25, 29, 32, 38]) for the equations without dissipation

\[
\begin{cases}
\frac{\epsilon}{c} \dot{\mathbf{A}}(x,t) = \Delta \mathbf{A}(x,t) - \frac{\sigma}{c} \mathbf{A}(x,t) + \frac{1}{c} \mathbf{P} \mathbf{j}(t), & x \in V \\
i \hbar \psi_n(t) = H_n(t) \psi_n(t), & n = 1, \ldots, N
\end{cases}
\] (2.5)

where \( \sigma > 0 \) is electrical conductivity of the cavity medium. We assume the Coulomb gauge [20]

\[ \text{div} \mathbf{A}(x,t) \equiv 0, \] (2.6)

and we denote by \( \mathbf{P} \) the orthogonal projection onto divergent-free vector fields in the Hilbert space \( L^2(V) \otimes \mathbb{R}^3 \). Further,

\[ H_n(t) := \frac{1}{2m} D^2(t) + e \Phi_n(x). \] (2.7)

Here

\[ D(t) = -i \hbar \nabla - \frac{e}{c} [\mathbf{A}(x,t) + \mathbf{A}_p(x,t)], \] (2.8)

where \( \mathbf{A}_p(x,t) \) is an external pumping potential, also in the Coulomb gauge and time-periodic:

\[ \text{div} \mathbf{A}_p(x,t) \equiv 0; \quad \mathbf{A}_p(x,t + T) = \mathbf{A}_p(x,t), \quad t > 0, \] (2.9)

where \( T > 0 \). The current density is defined by

\[ \mathbf{j}(x,t) = \frac{e}{m} \sum_n \text{Re} \left[ \overline{\psi}_n(x,t) D(t) \psi_n(x,t) \right]. \] (2.10)
This formula neglects overlapping of the supports of the wave functions since the distance between active molecules is sufficiently large: it is of order 10 molecular diameters when the density of active molecules is of order $10^{20}$.

Note that the pumping field is a solution to the Maxwell equations in the cavity $V$ excited by the discharge currents in the pumping lamp.

**Boundary conditions.** We choose the boundary conditions modelling ideally conducting diamagnetic materials (like cooper, silver, gold, etc). More precisely, we assume that

$$\mathbf{n}(x) \times \mathbf{A}(x,t) = 0, \quad \mathbf{n}(x) \cdot \mathbf{rot} \mathbf{A}(x,t) = 0; \quad \psi_n(x,t) = 0, \quad n \in \mathcal{N}; \quad x \in \Gamma,$$

where $\mathbf{n}(x)$ is the outward normal to $\Gamma$ at a point $x \in \Gamma$.

**Remark 2.1.** The boundary conditions (2.11) must to be valid for the total Maxwell field $\mathbf{A}(x,t) + \mathbf{A}_p(x,t)$.

We require the same boundary conditions for $\mathbf{A}(x,t)$ to separate the own field $\mathbf{A}(x,t)$ from the pumping field $\mathbf{A}_p(x,t)$. In this case the pumping field $\mathbf{A}_p(x,t)$ satisfies the same boundary conditions as $\mathbf{A}(x,t)$.

Under the Dirichlet boundary conditions for the wave functions, the total charge of each active molecule is constant, i.e.,

$$\int_V |\psi_n(x,t)|^2 dx = 1, \quad t > 0; \quad n \in \mathcal{N}. \quad (2.12)$$

**Hamiltonian structure.** In the *ideal case*, when $\sigma = 0$, the system (2.5), under the boundary conditions (2.11), is formally Hamiltonian, with the Hamiltonian functional (which is the energy up to a factor)

$$\mathcal{H}(\mathbf{A}, \Pi, \psi, t) = \frac{1}{2} \| \frac{1}{c} \Pi \|^2 + \| \mathbf{rot} \mathbf{A} \|^2 + \sum_n \langle \psi_n, H_n(\mathbf{A}, t) \psi_n \rangle, \quad (2.13)$$

where $\psi := (\psi_1, \ldots, \psi_N)$, $\| \cdot \|$ stands for the norm in the real Hilbert space $L^2(\mathbb{R}^3) \otimes \mathbb{R}^3$, and the brackets $\langle \cdot, \cdot \rangle$ stand for the inner product in $L^2(\mathbb{R}^3) \otimes \mathbb{C}$. The Schrödinger operators are defined by (2.7), (2.8):

$$H_n(\mathbf{A}, t) := \frac{1}{2m} [-i \hbar \nabla - \frac{e}{c}(\mathbf{A}(x) + \mathbf{A}_p(x,t))]^2 + e\Phi_0(x). \quad (2.14)$$

The system (2.5) with $\sigma \geq 0$ can be written in the Hamiltonian form with variational derivatives,

$$\begin{align*}
\frac{1}{c^2} \dot{\mathbf{A}}(t) &= D_\Pi \mathcal{H}(\mathbf{A}(t), \Pi(t), \psi(t), t), \quad \frac{1}{c} \dot{\Pi}(t) = -D_\mathbf{A} \mathcal{H}(\mathbf{A}(t), \Pi(t), \psi(t), t) - \frac{e}{c^2} \dot{\mathbf{A}}(x,t), \\
\dot{\psi}_n(t) &= \frac{i}{\hbar} D_{\psi_n} \mathcal{H}(\mathbf{A}(t), \Pi(t), \psi(t), t) = H_n(\mathbf{A}(t), \psi(t), t) \psi_n(t),
\end{align*} \quad (2.15)$$

In the last equation, $D_{\psi_n}$ denotes the variational derivative with respect to real and imaginary parts of $\psi_n$, and the factor $\frac{1}{c}$ is due to this identification. The factor $i$ in this case is identified with the corresponding skewsymmetric $2 \times 2$ matrix.

**Symmetry group.** For any $\Theta_n \in \mathbb{R}$, the functions $\mathbf{A}(x,t)$, $e^{i\Theta_n} \psi_n(x,t)$ are solutions of the system (2.5) if $\mathbf{A}(x,t)$, $\psi_n(x,t)$ is a solution. In other words, the dynamics (2.5) under the boundary conditions (2.11) commutes with the action of the *symmetry group* $G := [U(1)]^N$ with the action

$$(e^{i\Theta_1}, \ldots, e^{i\Theta_N})(\mathbf{A}, \psi_1, \ldots, \psi_N) = (\mathbf{A}, e^{i\Theta_1} \psi_1, \ldots, e^{i\Theta_N} \psi_N). \quad (2.16)$$

### 3 Parametric resonance

Let us denote $L^2 = L^2(\mathbb{R}^3)$. The system (2.15) can be written as the dynamical equation on the phase space $\mathcal{Y} = [L^2(\mathbb{R}) \otimes \mathbb{R}^3]^2 \oplus [L^2]^2N$:

$$\dot{Y}(t) = \mathcal{F}(Y(t), t), \quad Y(t) := (\mathbf{A}(t), \Pi(t), \psi(t)). \quad (3.1)$$

This system is $T$-periodic by (2.9), i.e.,

$$\mathcal{F}(Y, t + T) \equiv \mathcal{F}(Y, t). \quad (3.2)$$

The corresponding Poincaré map is defined by

$$\mathcal{P} : Y(0) \mapsto Y(T). \quad (3.3)$$
Let \( Y^0(t) \) be a “stationary state” of the system (3.1) with the zero pumping \( A_p(x,t) = 0 \), i.e., a solution of type
\[
Y^0(t) = (A, \Pi, \psi(0)e^{-i\omega t}),
\]
where \( \omega \in \mathbb{R} \). The following definition is relevant for the system (3.1) with sufficiently small pumping. We will denote by \( Y^0 \) every value of the function (3.4).

**Definition 3.1.** The parametric resonance takes place for \( Y^0 \) when the condition (1.1) holds.

### 4 Symmetry of spectrum

In the **ideal case**, when \( \sigma = 0 \), the dissipation is absent, and the system (3.1) is Hamiltonian by (2.15), so it can be written as
\[
\dot{Y}(t) = JD_Y \mathcal{H}(Y(t)), t,
\]
where \( J \) is a skewadjoint operator. We can make \( J^2 = -1 \) choosing suitable units with \( c = 2\hbar = 1 \) (note that the multipliers do not depend on the choice of units). Denote by \( \mathcal{P} : Y(0) \mapsto Y(T) \) the corresponding Poincaré map.

**Lemma 4.1.** (Lyapunov–Poincaré theorem) In the ideal case, the symmetry holds
\[
\text{Spec} D\mathcal{P}(Y^0) = R(\text{Spec} D\mathcal{P}(Y^0)),
\]
where \( R\mu = \mu^{-1} \) is the inversion, which is the reflection in the unit circle \(|\mu| = 1\).

**Proof.** The differential \( D\mathcal{P}(Y^0) \) admits the representation
\[
D\mathcal{P}(Y^0)Y(0) = Y(T),
\]
where \( Y(t) \) is the solution to the linearized equation
\[
\dot{Y}(t) = K(t)Y(t), \quad K(t) = D_Y \mathcal{F}(Y^0(t), t),
\]
and \( Y^0(t) \) is the solution to (4.1) with initial condition \( Y^0(0) = Y^0 \). However, (4.1) implies that
\[
\mathcal{F}(Y, t) = JD_Y \mathcal{H}(Y, t),
\]
and therefore,
\[
K(t) = JD_Y^2 \mathcal{H}(Y^0(t), t) = JA(t),
\]
where \( A(t) \) is the selfadjoint operator. Denote the map \( U(t) : Y(0) \mapsto Y(t), \) so \( D\mathcal{P}(Y^0) = U(T) \) by (4.3). Finally, the Lyapunov–Poincaré arguments [41, (3.28)] imply that \( \text{Spec} U^* = \text{Spec} U^{-1} \), where \( U = U(T) \).

Hence, the symmetry (4.2) is proved.

Recall the arguments [41, (3.28)]. We have \( \dot{U}(t) = JA(t)U(t) \), so differentiating \( W(t) := U^*(t)JU(t) \), we obtain
\[
\dot{W} = U^*JU + U^*JU = U^*A(-J^2)U + U^*J^2AU = 0
\]
since \( J^2 = -1 \) commutes with \( A \). Therefore, \( U^*(t)JU(t) = J \), and hence \( U^*(t) = JU^{-1}(t)J^{-1} \). This similarity implies that the spectra of \( U^*(t) \) and \( U^{-1}(t) \) coincide.

**Remark 4.2.** The symmetry (4.2) can be considered as a nonlinear version of the Lyapunov–Poincaré theorem, which plays the key role in M.G. Krein’s theory of parametric resonance [8, 41]. The ground for this extension is the Hamiltonian structure of the dynamics (4.1).

Thus, in the ideal case, the set of multipliers is symmetric with respect to the unit circle. In general case, when the dissipation is included, the symmetry (4.2) is broken, though is holds approximately for sufficiently small \( \sigma > 0 \):
\[
\text{Spec} D\mathcal{P}(Y^0) \approx R(\text{Spec} D\mathcal{P}(Y^0)).
\]

Accordingly, the parametric resonance (1.1) is very likely in this case. The required smallness defines the pumping/damping threshold.

**Remark 4.3.** Similar symmetry also holds for the case of the Maxwell–Bloch equations (12.9) which makes the parametric resonance (1.1) very plausible.
5 Dipole approximation

The theory admits a significant simplification in the case when the wavelength of the pumping is large with respect to the size of active molecules $D$.

**H3. The dipole radiation condition.** We will assume that

$$\lambda = c T = 2 \pi c / \Omega_p \gg D,$$

where $\Omega_p = 2 \pi / T$ is the frequency of the pumping.

**Example 5.1.** This condition holds for the ruby laser since the wavelength $\lambda \approx 7 \times 10^{-5} \text{cm}$ while the size of the chromium molecule is

$$D \approx 2.5 \text{pm} = 2.5 \times 10^{-10} \text{cm}.$$

Taking into account the condition (5.1), we approximate the system (2.5) by

$$\begin{cases}
\frac{1}{c^2} \tilde{\mathbf{A}}(x, t) = \Delta \mathbf{A}(x, t) - \frac{e}{c} \mathbf{A}(x, t) + \frac{1}{c} \mathbf{P}_n \mathbf{j}(x, t) \\
\hbar \psi_n(x, t) = \tilde{H}_n(t) \psi_n(t), \quad n = 1, \ldots, N
\end{cases}, \quad x \in V. (5.3)$$

Here

$$\tilde{H}_n(t) := \frac{1}{2m} D_n^2(t) + e \Phi_n(x), \quad D_n(t) = -i \hbar \nabla - \frac{e}{c} [\mathbf{A}(x, t) + \mathbf{A}_p(x_n, t)], (5.4)$$

The current (2.10) in the dipole approximation becomes

$$\mathbf{j}(x, t) \approx \frac{e}{m} \text{Re} \sum_n \mathbf{\bar{A}}_n(x, t) \mathbf{A}^*_n(t) \psi_n(x, t). (5.5)$$

The system (5.3) with $\sigma = 0$ is also Hamiltonian with the Hamilton functional

$$\mathcal{H}(\mathbf{A}, \Pi, \psi, t) = \frac{1}{2} \left[ \| \mathbf{\Pi} \|^2 + \| \mathbf{rot} \mathbf{A} \|^2 \right] + \frac{1}{2} \sum_n (\psi_n, \tilde{H}_n(\mathbf{A}, t) \psi_n), (5.6)$$

where

$$\tilde{H}_n(\mathbf{A}, t) := \frac{1}{2m} D_n^2(\mathbf{A}, t) + e \Phi_n(x), \quad D_n(\mathbf{A}, t) = -i \hbar \nabla - \frac{e}{c} [\mathbf{A}(x, t) + \mathbf{A}_p(x_n, t)], (5.7)$$

6 Gauge transform

The dipole approximation (5.3) allows us to apply the gauge transform [36, (5.1.15)]:

$$\psi_n(x, t) = e^{i \mathbf{x} \cdot (x_n t)} \psi_n(x, t), \quad \mathbf{Z}_n(x, t) := \frac{e}{\hbar c} (x - x_n) [\mathbf{A}(x_n, t) + \mathbf{A}_p(x_n, t)]. (6.1)$$

Now the dipole approximation (5.3) becomes

$$\begin{cases}
\frac{1}{c^2} \tilde{\mathbf{A}}(x, t) = \Delta \mathbf{A}(x, t) - \frac{e}{c} \mathbf{A}(x, t) + \frac{1}{c} \mathbf{P}_n \mathbf{j}(x, t) \\
\hbar \psi_n(x, t) = \tilde{H}_n(t) \psi_n(t), \quad n = 1, \ldots, N
\end{cases}, \quad x \in V. (6.2)$$

The Schrödinger operator now reads

$$\tilde{H}_n(t) = -\frac{1}{2m} (-i \hbar \nabla - \frac{e}{c} [\mathbf{A}(x, t) - \mathbf{A}(x_n, t)]) + e \Phi_n(x) + \hbar \mathbf{Z}_n(x, t) \approx -\frac{\hbar^2}{2m} \Delta + e \Phi_n(x) + \hbar \mathbf{Z}_n(x, t), (6.3)$$

where we used the condition (5.1). The current (5.5) now reads as

$$\mathbf{j}(x, t) \approx \frac{e}{m} \text{Re} \sum_n \mathbf{\bar{A}}_n(x, t) (-i \hbar \nabla - \frac{e}{c} [\mathbf{A}(x, t) - \mathbf{A}(x_n, t)]) \psi_n(x, t) \approx \frac{e}{m} \text{Re} \sum_n \mathbf{\bar{A}}_n(x, t) (-i \hbar \nabla) \psi_n(x, t). (6.4)$$
7 Decoupled system and eigenfunction expansions

**Decoupled system.** The system (3.1) is a weak perturbation of the corresponding “decoupled” system obtained by neglecting the interaction terms with the coupling constant $e/c$:

\[
\begin{align*}
\dot{A}(x,t) &= \mathbf{\Pi}(x,t) \\
\frac{1}{c^2}\dot{A}(x,t) &= \Delta A(x,t) - \frac{\sigma}{c^2}\dot{A}(x,t) \\
\dot{\psi}_n(t) &= H_n^0\psi_n(t), \quad n \in \mathcal{N}
\end{align*}
\]

where

\[
H_n^0 := -\frac{\hbar^2}{2m}\Delta + e\Phi_n(x).
\]

**Schrödinger modes.** The decoupled system (7.1) can be solved explicitly using the orthonormal eigenfunctions of the Schrödinger operators $H_n^0$ under the Dirichlet boundary conditions (2.11):

\[
H_n^0\phi_{nl}(x) = \omega_{nl}\phi_{nl}(x), \quad l = 1, 2, \ldots
\]

Hence, the system (3.1) can be analyzed by perturbation methods. The eigenfunctions $\phi_{nl}$ differ by rotations (2.4), while the eigenvalues $\omega_{nl}$ do not depend on $n$ with a great accuracy. The decoupled system (3.1) admits the “stationary” solutions

\[
Z^0(t) \equiv (0, 0, (e^{-i\omega_{nl}t}\phi_{nl}(x) : n \in \mathcal{N})),
\]

where $l(n) = 1, 2, \ldots$.

**Maxwell modes.** The first equation of the decoupled system (7.1) can be solved using the orthonormal eigenmodes

\[
\Delta \mathbf{X}_\nu(x) = -\frac{\Omega^2}{c^2}\mathbf{X}_\nu(x), \quad \text{div} \mathbf{X}_\nu(x) = 0, \quad x \in V
\]

\[
\mathbf{n}(x) \times \mathbf{X}_\nu(x) = 0, \quad (\mathbf{n}(x) \cdot \text{rot} \mathbf{X}_\nu(x) = 0, \quad x \in \partial V)
\]

and $\Omega_\nu > 0$.

**Example 7.1.** The eigenmodes can be calculated explicitly for the rectangular cuboid $V = [0, l_1] \times [0, l_2] \times [0, l_3]$:

\[
\mathbf{X}_k = C \begin{pmatrix}
a^1_k \cos k_1 x_1 \\
a^2_k \sin k_1 x_1 \\
a^3_k \sin k_2 x_2 \\
a^4_k \cos k_2 x_2 \\
a^5_k \sin k_3 x_3 \\
a^6_k \cos k_3 x_3
\end{pmatrix}, \quad C = \sqrt{\frac{8}{l_1l_2l_3}}, \quad k = (k_1, k_2, k_3), \quad k_j = 1, 2, \ldots
\]

where the amplitudes $a_k = (a^1_k, a^2_k, a^3_k)$ are unit vectors, orthogonal to the wave vectors $k$. In particular, consider the cuboid $V$ with dimensions $l_1 = 12 cm, l_2 = 13 cm$. Note that the random points $x_n \in V$ are distributed uniformly, so “the dispersion” of the “random value” $\mathbf{X}(x_n)$ is

\[
D(\mathbf{X}(x_n)) = \frac{1}{|V|} \int_V \mathbf{X}^2(x)dx \approx \frac{1}{|V||V|} = \frac{1}{|V|} \approx 0.02.
\]

Accodingly, the mean value is

\[
|\mathbf{X}(x_n)| \approx 0.14.
\]

**Eigenfunction expansions.** We will expand $\tilde{\psi}_n(\cdot, t)$ in the orthonormal eigenfunctions (7.3):

\[
\tilde{\psi}_n(x, t) = \sum_{l=1}^{\infty} c_{nl}(t) e^{-i\omega_{nl}t} \phi_{nl}(x), \quad \phi_n(t) := \sum_{l=1}^{\infty} c_{nl}(t) \phi_{nl}(x), \quad \omega_{nl} = \omega_{nl} - \omega_l.
\]

Now each Schrödinger equation in (6.2) with $n \in \mathcal{N}$ is equivalent to the system

\[
\dot{c}_n(t) = -i \sum_{l=1}^{\infty} e^{i\omega_{nl}} e^{i\omega_l} (\chi_n(x, t) \phi_{nl}(x), \phi_{nl}(x)) c_{nl}(t), \quad l \geq 1; \quad \omega_{l,n} = \omega_l - \omega_n.
\]

Similarly, the Maxwell field can be expanded in the orthonormal eigenmodes (7.5):

\[
\mathbf{A}(x, t) = \sum_{\nu} a_\nu(t) \mathbf{X}_\nu(x).
\]

Now the Maxwell equation in (6.2) is equivalent to the system

\[
\ddot{a}_\nu(t) + c\sigma \dot{a}_\nu(t) + \Omega_\nu^2 a_\nu(t) = c(i \mathbf{j}(t), \mathbf{X}_\nu), \quad \nu = 1, \ldots
\]
8 Resonance condition

The resonance condition means that

\[ \Delta = \Omega_p = \Omega_{\varphi}, \quad \Delta := \omega_n - \omega_b, \]  

(8.1)

where \( \omega_b < \omega_n \) are some fixed eigenvalues (7.3), \( \Omega_{\varphi} \) is one of the Maxwell eigenfrequencies (7.5), and \( \Omega_p = 2\pi/T \) is the frequency of the pumping (2.9).

We can renumerate the eigenfunctions \( \varphi_{n,j} \) in such a way that \( b = 1 \) and \( a = 2 \). Then the decoupled system (7.1) admits solutions (7.4) with \( l(n) \equiv 1 \):

\[ Y^0(t) = (0, 0, (e^{-i\omega_1 t}, \varphi_{n,1} : n \in \mathbb{N})), \]

(8.2)

Let us call as “ground states” all values of these solutions: in particular,

\[ Y^0 = (0, 0, (\varphi_{n,1} : n \in \mathbb{N})). \]

(8.3)

The perturbation theory allows us to calculate solutions to the coupled system (2.5) with initial states close to (8.3). We expect that these solutions are rapidly growing due to the resonance condition (8.1) and the parametric resonance (1.1).

9 The harmonic pumping

We consider the Maxwell–Schrödinger system (2.5) with the pumping \( \mathbf{A}_p(x,t) \), which is switched on at \( t = 0 \) and is monochromatic, i.e.,

\[ \mathbf{A}_p(x,t) = \mathbf{a}_p(x) \sin \Omega_p t, \]

(9.1)

where \( \Omega_p \) is the pumping frequency. For example, for the ruby laser the wavelength and the resonant pumping frequency are

\[ \lambda = 694.3nm = 694.3 \times 10^{-9}m \approx 7 \times 10^{-5}cm, \quad \Omega_p = \frac{2\pi c}{\lambda} = \frac{19 \times 10^{10}}{7 \times 10^{-5}} \approx 3 \times 10^{15}s^{-1}. \]

(9.2)

The magnitude of pumping. The pumping field in a neighborhood of any active molecule is approximately a plane wave

\[ \mathbf{A}_p(x,t) \approx \mathbf{a}_p \sin(\Omega_p t - kx), \quad |k| = |\Omega_p|/c, \]

(9.3)

or a mixture of such waves. Let us estimate the pumping field in a typical cylindrical cavity of length 12cm and diameter 0.6cm illuminated by the pumping lamp of the typical power \( W = 1Kwt = 10^{10} \text{ erg/s} \). The side surface \( S \) of the cavity is about \( 12\pi0.6cm^2 \approx 23cm^2 \). The intensity \( I \) of the plane wave (9.3) is \( \frac{ck^2|\mathbf{a}_p|^2}{2} \text{ erg/cm}^2 \text{s}^{-1} \), so

\[ W = IS = \frac{ck^2|\mathbf{a}_p|^2}{2} 12\pi0.6 = 10^{10}. \]

(9.4)

Substituting \( |k| = |\Omega_p|/c \), we find

\[ \frac{\Omega_p^2|\mathbf{a}_p|^2}{2c} 23 = 10^{10}. \]

(9.5)

For example, for the ruby laser with frequency (9.2), we have

\[ |\mathbf{a}_p|^2 \approx \frac{c}{2\Omega_p^2} \times 10^9 = 0.3 \times 10^{-11}. \]

(9.6)

Finally,

\[ |\mathbf{a}_p| \approx 1.7 \times 10^{-6} \text{ esu/cm}. \]

(9.7)

H4. Distribution of pumping. We assume that i) \( \mathbf{a}_p(x_n) \) is the random vector distributed uniformly on the sphere \( |\mathbf{a}_p(x)| = a_p = 1.7 \times 10^{-6} \), and ii) \( \mathbf{a}_p(x_n) \) and \( \mathbf{a}_p(x_{n'}) \) are almost independent at large distances \( |x_n - x_{n'}| \).
10 The Maxwell–Bloch equations

Now we consider the finite-dimensional Faedo–Galerkin approximation of the Maxwell–Schrödinger system (3.1). Namely, we will consider solutions

\[ \mathbf{A}(x,t) = a(t)\mathbf{X}(x), \quad \mathbf{\psi}(x,t) = c_{n,1}(t)\varphi_{n,1}(x) + c_{n,2}(t)\varphi_{n,2}(x) \]

(10.1)

which satisfy the system consisting of two equations (7.10) with \( n = 1 \) and \( n = 2 \) for each \( n \in \mathbb{N} \), and one equation of (7.12) with \( \nu = \nu' \). Below, we will write \( a(t), \Omega(\tau) \) and \( \mathbf{X}(x) \) instead of \( a(\tau), \Omega(\tau) \) and \( \mathbf{X}(\tau) \) respectively. Thus, the approximation of the Maxwell–Schrödinger equations reads as the Maxwell–Bloch equations

\[ \ddot{a}(t) + c\sigma\dot{a}(t) + \Omega^2a(t) = c(j(t), \mathbf{X}), \]

(10.2)

\[ \begin{aligned}
\dot{c}_{n,1}(t) &= -i\sum_{l'=1}^{2} e^{-i\nu_l \nu_{l'}} (\mathcal{K}_{n}(x,t) \varphi_{n,l'}(x), \varphi_{n,1}(x)) c_{n,l'}(t) , \\
\dot{c}_{n,2}(t) &= -i\sum_{l'=1}^{2} e^{-i\nu_l \nu_{l'}} (\mathcal{K}_{n}(x,t) \varphi_{n,l'}(x), \varphi_{n,2}(x)) c_{n,l'}(t)
\end{aligned} \]

(10.3)

\[ n \in \mathbb{N}. \]

11 Molecular currents

By (5.1), the matrix entries in (10.3) read as

\[ b_{n}^{ll'}(t) := \langle \mathcal{K}_{n}(x,t) \varphi_{n,l'}(x), \varphi_{n,l}(x) \rangle \approx \frac{e}{\hbar c} [\hat{\mathbf{A}}(x,t) + \hat{\mathbf{A}}(x,t)] \langle (x-x_{n}) \varphi_{n,l}(x), \varphi_{n,l}(x) \rangle \]

\[ = \frac{e}{\hbar c} [\dot{a}_{1}(t)\mathbf{X}(x_{n}) + \mathbf{a}(x_{n})\Omega_{l} \cos \Omega_{l} t] \mathbf{P}_{n}^{ll'} \]

\[ = \frac{\mathbf{P}_{n}^{ll'}}{\hbar c} [\dot{a}_{1}(t)\mathbf{X}(x_{n}) + \mathbf{a}(x_{n})\Omega_{l} \cos \Omega_{l} t], \quad l, l' = 1, 2, \]

(11.1)

where \( \mathbf{P}_{n}^{ll'} \) are the dipole momenta

\[ \mathbf{P}_{n}^{ll'} := e \langle \varphi_{n,l'}(x), (x-x_{n}) \varphi_{n,l}(x) \rangle. \]

(11.2)

These momenta are real vectors since we have chosen real eigenfunctions \( \varphi_{n,l}(x) \).

Remark 11.1. In the case of the ruby laser, the matrix of type (11.2) corresponds to the dipole momenta of chromium ions \( \text{Cr}^{3+} \). For definiteness of calculations, we accept

\[ |\mathbf{P}| \approx 4 \text{Debye} = 4 \times 10^{-18} \text{esu} \cdot \text{cm}, \]

(11.3)

which is the permanent dipole moment of chromium oxide molecules \( \text{CrO} \); see [10, p. 10]. However, actual magnitude of the matrix elements (11.2) in the case of the ruby laser must be revised.

From now on, we will assume that the potential \( \Phi(x) \) is an even function, i.e.,

\[ \Phi(-x) = \Phi(x), \quad x \in \mathbb{R}^3. \]

(11.4)

In this case, all eigenfunctions \( \varphi_{n,l} \) can be chosen odd or even, and hence,

\[ \mathbf{P}_{n}^{11} = \mathbf{P}_{n}^{22} = 0. \]

(11.5)

Therefore, the system (10.3) reads as

\[ \begin{aligned}
\dot{c}_{n,1}(t) &= -ib_{n}(t)e^{-i\Delta \nu c_{n,2}(t) ,} \\
\dot{c}_{n,2}(t) &= -ib_{n}(t)e^{i\Delta \nu c_{n,1}(t)}
\end{aligned} \]

(11.6)

where we denote

\[ b_{n}(t) := \frac{1}{\hbar c} [\dot{a}_{1}(t)\mathbf{X}_{1}(x_{n}) + \mathbf{a}_{n}(x_{n})\Omega_{l} \cos \Omega_{l} t], \quad \mathbf{P}_{n} := \mathbf{P}_{n}^{12}. \]

(11.7)
The matrix of this system is skew-symmetric, which corresponds to the electron charge conservation in each active molecule i.e.,
\[ |c_{n,1}(t)|^2 + |c_{n,2}(t)|^2 = 1, \quad t > 0, \quad n \in \mathbb{N}. \] (11.8)

We now calculate the current (6.4) corresponding to the wave functions (10.1), and the right hand side of (7.12) with \( X_v = X \): taking into account the dipole approximation (5.1), we obtain that
\[
\langle j(t), X \rangle \approx \frac{e \hbar}{m} \sum_n X(x_n) \text{Im} \left\{ e^{-i\Delta} \int \nabla \varphi_{n,2}(x) dx + e^{i\Delta} \int \nabla \varphi_{n,1}(x) dx \right\}
\]

\[ = 2\Delta \sum_n P_n X(x_n) \text{Im} \{ e^{-i\Delta} \}, \] (11.9)

since
\[
\int \nabla \varphi_{n,2} dx = \frac{\Delta m}{\hbar} \int \nabla \varphi_{n,1} \varphi_{n,2} dx = \frac{\Delta m P_n}{\hbar e} \] (11.10)

by [35, (44.20)].

12 Time scaling

Now the Maxwell–Bloch equations (10.2)–(10.3) become
\[
\dot{a}(t) + \Omega^2 a(t) + c\sigma \dot{a}(t) = j(t) := 2c\Delta \sum_n P_n X(x_n) \text{Im} \left\{ e^{-i\Delta} \right\}, \]
(12.1)

\[
\begin{cases}
\dot{\epsilon}_{n,1}(t) = -ib_n(t)e^{-i\Delta} \epsilon_{n,2}(t) \\ 
\dot{\epsilon}_{n,2}(t) = -ib_n(t)e^{i\Delta} \epsilon_{n,1}(t)
\end{cases}, \quad n \in \mathbb{N}. \] (12.2)

Let us transform the system (12.1) to the natural time scale
\[ \tau := \Omega_p t \approx 3 \times 10^{15} t, \] (12.3)
in which the period of pumping is \( 2\pi \). Below, we denote any function \( f(t) \) in the units of time (12.3) also as \( f(\tau) \). In particular, (11.7) becomes
\[
b_n(\tau) = \frac{\Omega_p}{\hbar c} \partial_\tau a(\tau)P_n X(x_n) + \partial_\tau a_p(x_n) \cos \tau.
\] (12.4)

Now the Maxwell–Bloch equations (12.1), (12.2) read as
\[
\begin{align*}
\partial_\tau^2 c_n(\tau) + \sigma_1 \partial_\tau c_n(\tau) + \frac{\Omega_p^2}{\hbar c} c_n(\tau) = j(\tau),
\end{align*}
\]
\[
\begin{cases}
\partial_\tau c_{n,1}(\tau) = -ib_n(\tau)e^{-i\Delta} c_{n,2}(\tau) \\ 
\partial_\tau c_{n,2}(\tau) = -ib_n(\tau)e^{i\Delta} c_{n,1}(\tau)
\end{cases}, \quad n \in \mathbb{N} \] (12.5)

where
\[
j(\tau) := \frac{2c\Delta}{\Omega_p^2} \sum_n P_n X(x_n) \text{Im} \left\{ e^{-i\Delta} \right\}
\] (12.6)

and
\[
\sigma_1 := \frac{c\sigma}{\Omega_p} \approx 10^{-5} \approx 10^{-7}. \] (12.7)

Let us denote the real constants
\[
\alpha_n = \frac{2c}{\Omega_p} P_n X(x_n), \quad \beta_n = \frac{P_n X(x_n)}{\hbar c}, \quad \gamma_n = \frac{P_n a_p(x_n)}{\hbar c}. \] (12.8)

Then the system (12.5) becomes
\[
\begin{align*}
\dot{a}(\tau) + \sigma_1 \dot{a}(\tau) + a(\tau) = j(\tau),
\end{align*}
\]
\[
\begin{cases}
\dot{c}_{n,1}(\tau) = -i\omega_n(\tau)c_{n,2}(\tau) \\ 
\dot{c}_{n,2}(\tau) = -i\overline{\omega_n(\tau)}c_{n,1}(\tau)
\end{cases}, \quad n \in \mathbb{N}, \] (12.9)
where we used (8.1), \( \dot{a} := \partial_\tau a \), etc., and
\[
j(\tau) = \sum_n \alpha_n \text{Im} \left\{ c_{n,1}(\tau)c_{n,2}(\tau)e^{-i\tau} \right\}, \quad \omega_n(\tau) = |\beta_n a(\tau) + \gamma_n\cos \tau|e^{-i\tau}.
\]
(12.10)

The matrix of system (12.9) is skew-adjoint, and the charge conservation holds: similarly to (11.8),
\[
|c_{n,1}(\tau)|^2 + |c_{n,2}(\tau)|^2 = 1, \quad \tau \in \mathbb{R}.
\]
(12.11)

All coefficients \( \alpha_n, \beta_n \) and \( \gamma_n \) are small. In particular, for the ruby laser with the chromium dipole moment (11.3), and the pumping frequency (9.2), we have
\[
\frac{c|\mathbf{P}_n|}{\Omega_p} \sim 12 \times 10^{10} \frac{10^{-18}}{3 \times 10^{15}} \sim 4 \times 10^{-23}, \quad \frac{|\mathbf{P}_n|}{hc} \sim \frac{4 \times 10^{-18}}{10^{-27}3 \times 10^{10}} \sim 1.3 \times 10^{-1}.
\]
(12.12)

Now (7.8) and (9.7) imply that
\[
|\alpha_n| \sim 10^{-23}, \quad |\beta_n| \sim 0.2 \times 10^{-2}, \quad |\gamma_n| \sim 0.7 \times 10^{-7}.
\]
(12.13)

In all calculations below we will consider these values.

**Remark 12.1.** By our hypotheses H1–H4, the parameters \( \alpha_n, \beta_n, \gamma_n \) and \( \alpha'_n, \beta'_n, \gamma'_n \) are almost independent for sufficiently large distances \( |x'_n - x_n| \).

### 13 Parametric resonance for the Maxwell–Bloch equations

Let us reformulate the concept of parametric resonance (1.1) for the system (12.9). In the case of zero pumping \( a_p = 0 \), this system admits “ground states”, which are stationary solutions corresponding to (8.2):
\[
X^0(t) \equiv X^0 := (a^0, b^0, (\dot{c}^0_n : n \in \mathbb{N}))
\]
(13.1)

with
\[
a^0 = b^0 = 0; \quad \dot{c}^0_n = (e^{i\Theta_n}, 0), \quad n \in \mathbb{N}.
\]
(13.2)

The solutions describe the laser dynamics before the pumping is switched on. Let us write the system (12.9) as
\[
\dot{a}(\tau) = b(\tau), \quad \dot{b}(\tau) + \sigma_1 b(\tau) + a(\tau) = j(\tau), \quad \dot{c}_n(\tau) = -i\Omega_n(\tau)c_n(\tau), \quad n \in \mathbb{N},
\]
(13.3)

where we denote
\[
c_n(\tau) = (c_{n,1}(\tau), c_{n,2}(\tau)), \quad \Omega_n(\tau) = \begin{pmatrix} 0 & \omega_n(\tau) \\ \overline{\omega}_n(\tau) & 0 \end{pmatrix}, \quad \omega_n(\tau) = |\beta_n b(\tau) + \gamma_n\cos \tau|e^{-i\tau}.
\]
(13.4)

The system (13.3) can be written as
\[
\dot{X}(\tau) = F(X(\tau), \tau), \quad X(\tau) := (a(\tau), b(\tau), c(\tau)), \quad c(\tau) := (c_n(\tau) : n \in \mathbb{N}).
\]
(13.5)

This system (13.5) is \( 2\pi \)-periodic, i.e.,
\[
F(X, \tau + 2\pi) \equiv F(X, \tau).
\]
(13.6)

The corresponding Poincaré map is defined by
\[
P : X(0) \to X(2\pi).
\]
(13.7)

The following definition is relevant for the system (13.5) in the case of sufficiently small pumping.

**Definition 13.1.** The parametric resonance occurs for the ground state (13.1) when the condition (1.1) holds with \( DP(X^0) \) instead of \( D \mathcal{X}^0 \).

We also need to check the additional condition for all the corresponding eigenvectors \( X := (a, b, (c_n : n \in \mathbb{N})) \):
\[
(a, b) \neq 0,
\]
(13.8)

which guarantees an exponential increment of the Maxwell field. We have verified this condition in the case of ruby laser; see Remark 20.3.
14 Successive approximations

We will construct the Poincaré map (13.7) calculating successive approximations for the equations (13.3) on the interval $[0, 2\pi]$ with initial data

$$a(0), \quad b(0); \quad c_n(0), \quad n \in \mathbb{N},$$

(14.1)

which are close to the set of the ground states (13.2), i.e.,

$$|a(0)| + |b(0)| + \sum_n |c_{n,1}(0) - 1| + |c_{n,2}(0)| \leq \varepsilon.$$  

(14.2)

Here $\varepsilon > 0$ is sufficiently small since we must to study instability of solutions which are close to these ground states. The successive approximations are defined as follows.

**The Schrödinger amplitudes.** We define the approximations $c_n^{(k)}$ with $k = 1, \ldots$ as solutions of equations

$$c_{n}^{(k)}(\tau) = -i\Omega_{n}^{(k-1)}(\tau)c_n^{(k)}(\tau), \quad n \in \mathbb{N},$$

(14.3)

with the initial data (14.1),

$$c_n^{(k)}(0) = c_n(0), \quad n \in \mathbb{N}.$$  

(14.4)

Here we denote

$$\Omega_{n}^{(k-1)}(\tau) = \left( \begin{array}{cc} 0 & \omega_{n}^{(k-1)}(\tau) \\ \overline{\omega_{n}^{(k-1)}(\tau)} & 0 \end{array} \right), \quad \omega_{n}^{(k-1)}(\tau) = [\beta_n b^{(k-1)}(\tau) + \gamma_n \cos \tau]e^{-i\tau}. \quad (14.5)$$

To solve the equations (14.3), we replace them by the corresponding averaged equations

$$c_{n}^{(k)}(\tau) = -i\bar{\Omega}_{n}^{(k-1)}c_n^{(k)}(\tau), \quad n \in \mathbb{N},$$

(14.6)

where

$$\bar{\Omega}_{n}^{(k-1)} := \frac{1}{2\pi} \int_{0}^{2\pi} \Omega_{n}^{(k-1)}(\tau)d\tau.$$  

(14.7)

The matrix functions $\Omega_{n}^{(k-1)}(\tau)$ are sufficiently small by (12.13), so the replacement is justified by Lemma A.1. All the matrices (14.5) and (14.7) are skewsymmetric, and hence,

$$c_n^{(k)}(\tau) = U_n^{(k)}(\tau)c_n(0),$$

(14.8)

where $U_n^{(k)}(\tau)$ are unitary matrices.

**The Maxwell amplitude.** We define the approximations $a^{(k)}(\tau)$ with $k = 1, \ldots$ as solutions of the equations

$$a^{(k)}(\tau) + \sigma_1 a^{(k)}(\tau) + a^{(k)}(\tau) = j^{(k)}(\tau), \quad j^{(k)}(\tau):= \sum_n \alpha_n \text{Im} \left\{ c_{n,1}^{(k)}(\tau)c_{n,2}^{(k)}(\tau)e^{-i\tau} \right\},$$

(14.9)

with the initial data (14.1):

$$a^{(k)}(0) = a(0), \quad \dot{a}^{(k)}(0) = b(0). \quad (14.10)$$

Let us denote $\varkappa := \sigma_1 / 2 \approx 10^{-7}$. The solution of the Maxwell equation (14.9) is given by the convolution

$$a^{(k)}(\tau) = a_0(\tau) + \int_{0}^{\tau} j^{(k)}(\tau')E(\tau - \tau')d\tau', \quad a_0(\tau) := a(0)E(\tau) + [\dot{a}(0) + 2\varkappa a(0)]E(\tau), \quad (14.11)$$

where $E(\tau)$ is the retarded fundamental solution,

$$\tilde{E}(\tau) + 2\varkappa \tilde{E}(\tau) + E(\tau) = \delta(\tau), \quad \tau \in \mathbb{R}; \quad E(0) = 0, \quad \tau < 0.$$  

(14.12)

This fundamental solution is expressed as follows. Denote by $\lambda_{\pm}$ the roots of characteristic equation

$$\lambda^2 + \varkappa \lambda + 1 = 0,$$

(14.13)

which are

$$\lambda_{\pm} = -\varkappa \pm \sqrt{\varkappa^2 - 1} = -\varkappa \pm i + O(\varkappa^2). \quad (14.14)$$

Thus, $\text{Re} \lambda_{\pm} = -\varkappa < 0$, and neglecting terms which are $O(\varkappa^2)$, we obtain that the functions

$$a_1(\tau) = \frac{\lambda_+ e^{\lambda_+ \tau} - \lambda_- e^{\lambda_- \tau}}{\lambda_+ - \lambda_-} = e^{-\varkappa \tau} \cos \tau + O(\varkappa^2), \quad a_2(\tau) = \frac{e^{\lambda_+ \tau} - e^{\lambda_- \tau}}{\lambda_+ - \lambda_-} = e^{-\varkappa \tau} \sin \tau + O(\varkappa^2), \quad (14.15)$$

are solutions of the homogeneous Maxwell equation (14.9) with initial data (1, 0) and (0, 1) respectively. Finally, the fundamental solution is the real-valued function

$$E(\tau) = \theta(\tau)a_2(\tau) = \theta(\tau)e^{-\varkappa \tau} \sin \tau + O(\varkappa^2), \quad \tau \in \mathbb{R}. \quad (14.16)$$
14.1 The first-order approximation

The Schrödinger amplitudes. Formulas (14.5)–(14.7) with \( k = 1 \) give the averaged matrices

\[
\tilde{\Omega}_{n}^{(0)}(\tau) = \begin{pmatrix}
0 & -i\frac{\kappa}{2} \\
-i\frac{\kappa}{2} & 0
\end{pmatrix}.
\]  

(14.17)

Hence, equation (14.3) with \( k = 1 \) reads as the system in (12.9) with \( \tilde{\Omega}_{n}(\tau) \) instead of \( \omega_{n}(\tau) \):

\[
\begin{cases}
\gamma_{n,1}(\tau) &= -i\frac{\gamma_{n}}{2}c_{n,2}(\tau), \\
\gamma_{n,2}(\tau) &= -i\frac{\gamma_{n}}{2}c_{n,1}(\tau),
\end{cases}
\]  

(14.18)

Its solutions are

\[
c_{n,1}(\tau) = d_{n,1}(\tau)\cos\frac{\kappa\tau}{2} + d_{n,2}(\tau)\sin\frac{\kappa\tau}{2}, \quad c_{n,2}(\tau) = -id_{n,1}(\tau)\cos\frac{\kappa\tau}{2} + id_{n,2}(\tau)\cos\frac{\kappa\tau}{2},
\]

(14.19)

where

\[
d_{n,1}(\tau) = c_{n,1}(0), \quad d_{n,2}(\tau) = -ic_{n,2}(0).
\]

(14.20)

Hence, the matrix in the formula (14.8) with \( n = 1 \) reads

\[
U_{n}^{(1)}(\tau) = \begin{pmatrix}
\cos\frac{\kappa\tau}{2} & -i\sin\frac{\kappa\tau}{2} \\
-i\sin\frac{\kappa\tau}{2} & \cos\frac{\kappa\tau}{2}
\end{pmatrix} + O(\gamma_{n}^{2})
\]

(14.21)

by Lemma A.1.

The current. Substituting (14.19) and (14.20) in the second formula of (14.9) with \( k = 1 \), we obtain

\[
j^{(1)}(\tau) = \sum_{n}\alpha_{n}\text{Im}\left\{ c_{n,1}(\tau)c_{n,2}(\tau)e^{-i\tau}\right\}
\]

\[
= \sum_{n}\alpha_{n}\text{Im}\left\{ d_{n,1}(\tau)\cos\frac{\kappa\tau}{2} + d_{n,2}(\tau)\sin\frac{\kappa\tau}{2} \right\} + i\sum_{n}\alpha_{n}\gamma_{n}^{2} + O(10^{-17}), \quad \tau \in [0, 2\pi],
\]

(14.22)

since \( \sum_{n}\alpha_{n}\gamma_{n}^{2} \sim 10^{-17} \) by (12.13).

Notation. For each number \( \varepsilon > 0 \), the equality \( r = O(\varepsilon) \) means that \( |r| \leq 10\varepsilon \).

The Maxwell amplitude. Now (14.11) with \( k = 1 \) becomes

\[
a^{(1)}(\tau) = a_{0}(\tau) + \int_{0}^{\tau} j^{(1)}(\tau')E(\tau - \tau')d\tau'.
\]

(14.23)

Substituting here the current (14.22), we obtain

\[
a^{(1)}(\tau) = a_{0}(\tau) + \sum_{n}\alpha_{n}\left[\text{Im}\left\{ c_{n,1}(0)c_{n,2}(0)i(\tau)\right\} + |c_{n,2}(0)|^{2} - |c_{n,1}(0)|^{2}\right] + O(10^{-17}),
\]

(14.24)

where we denote

\[
I_{1}(\tau) := \int_{0}^{\tau} e^{-i\tau'}E(\tau - \tau')d\tau' = -\frac{i}{2}\sin\tau - \frac{\tau}{2}e^{-i\tau} + \frac{\tau^{2}e^{-i\tau}}{4i} + O_{2}.
\]

\[
I_{2}(\tau) := \int_{0}^{\tau} \frac{\tau}{2}\cos\tau'E(\tau - \tau')d\tau' = \frac{1}{4}\left[\frac{\tau^{2}}{2}\sin\tau + \frac{\tau}{2}\cos\tau - \frac{1}{2}\sin\tau\right] + O_{2},
\]

where \( O_{2} := O(\tau^{2}) = O(10^{-14}) \). The details of calculation can be found in Appendix B.
14.2 The second-order approximation

The Schrödinger amplitudes. Formulas (14.5) with \( k = 2 \) give

\[
\Omega_{n}^{(1)}(\tau) = \begin{pmatrix} 0 & \omega_{n}^{(1)}(\tau) \\ \omega_{n}^{(1)}(\tau) & 0 \end{pmatrix}, \quad \omega_{n}^{(1)}(\tau) = |\beta_{n}|^{2}(\tau) + \gamma_{n} \cos \tau e^{-i\tau}. \tag{14.25}
\]

Let us calculate the averaged matrices \( \tilde{\Omega}_{n}^{(1)} \). First, (14.24) and (14.11) imply that

\[
\mathbf{v} := \frac{1}{2\pi} \int_{0}^{2\pi} \hat{a}^{(1)}(\tau)e^{-i\tau}d\tau = \mathbf{v}_{1} + \mathbf{v}_{2}, \tag{14.26}
\]

where

\[
\mathbf{v}_{1} := \frac{1}{2\pi} \int_{0}^{2\pi} \hat{a}_{0}(\tau)e^{-i\tau}d\tau = \frac{1}{2\pi} \int_{0}^{2\pi} \{ a(0)\dot{E}(\tau) + [\dot{a}(0) + 2\kappa a(0)]E(\tau) \} e^{-i\tau}d\tau,
\]

\[
\mathbf{v}_{2} := \frac{1}{2\pi} \int_{0}^{2\pi} \sum_{n} \alpha_{n} \left[ \Im \left\{ \tau_{n,1}(0)c_{n,2}(0)J_{1}(\tau) e^{-i\tau} \right\} + [\{ c_{n,2}(0) \}^{2} - \{ c_{n,1}(0) \}^{2}] \gamma_{n} J_{2}(\tau)e^{-i\tau} \right]d\tau.
\]

First, let us calculate \( \mathbf{v}_{1} \). Neglecting errors \( O(\kappa^{2}) = O(10^{-14}) \), we obtain:

\[
\int_{0}^{2\pi} \dot{E}(\tau)e^{-i\tau}d\tau \approx i \int_{0}^{2\pi} E(\tau)e^{-i\tau}d\tau \approx i \int_{0}^{2\pi} e^{-i\kappa \tau} \sin \tau e^{-i\tau}d\tau = \frac{1}{2} \int_{0}^{2\pi} e^{-i\kappa \tau}(e^{i\tau} - e^{-i\tau})e^{-i\tau}d\tau
\]

\[
= -\left[ \frac{1}{2\pi} + \frac{1}{2\pi} \right] (e^{-2\pi i} - 1) = -\frac{2\pi + 2\sigma}{\pi} - \kappa \pi + (\pi \sigma)^{2} + O(\kappa^{3})
\]

\[
\approx -[1 + \frac{\kappa}{2\pi}](\pi - \sigma^{2}) \approx -[1 + \frac{\kappa}{2\pi}](\pi + \sigma^{2}) \approx \pi + \kappa \left[ \frac{\pi}{2\pi} - \pi^{2} \right]. \tag{14.27}
\]

Similarly, using (14.12), (14.16), and (14.27), we obtain

\[
\int_{0}^{2\pi} \dot{E}(\tau)e^{-i\tau}d\tau = \int_{0}^{2\pi} [-2\kappa E(\tau) - E(\tau)]e^{-i\tau}d\tau \approx -(i2\pi + 1) \int_{0}^{2\pi} E(\tau)e^{-i\tau}d\tau
\]

\[
\approx i(2\pi + 1)(\pi + \kappa \left[ \frac{\pi}{2\pi} - \pi^{2} \right]) = (-2\pi + i)(\pi + \kappa \left[ \frac{\pi}{2\pi} - \pi^{2} \right])
\]

\[
\approx i\pi + \kappa(-2\pi + \pi^{2} - i\pi^{2}) = i\pi - \kappa\left[ \frac{3\pi}{2\pi} + i\pi^{2} \right]. \tag{14.28}
\]

As a result,

\[
\mathbf{v}_{1} \approx a(0) \left[ \frac{\pi}{2\pi} - \kappa \left( \frac{\pi}{2\pi} + i\pi^{2} \right) \right] + \{ \dot{a}(0) + 2\kappa a(0) \} \left[ \frac{\pi}{2\pi} + \kappa \left( \frac{\pi}{2\pi} - \pi^{2} \right) \right] = \mathbf{v}_{11} + \mathbf{v}_{12}, \tag{14.29}
\]

\[
\mathbf{v}_{11} = \frac{\pi}{2\pi} a(0) + \frac{\pi}{2\pi} (1 - \kappa \pi) \dot{a}(0), \quad \mathbf{v}_{12} = \frac{\pi}{2\pi} (1 - \kappa \pi) a(0) - \frac{\pi}{2\pi} \dot{a}(0).
\]

It remains to note that

\[
\mathbf{v}_{2} := \sum_{n} \alpha_{n} \left[ \Im \left\{ \tau_{n,1}(0)c_{n,2}(0)J_{1} \right\} + [\{ c_{n,2}(0) \}^{2} - \{ c_{n,1}(0) \}^{2}] \gamma_{n} J_{2} \right], \tag{14.30}
\]

where \( J_{1} \) and \( J_{2} \) are the integrals

\[
J_{1} := \frac{1}{2\pi} \int_{0}^{2\pi} J_{1}(\tau)e^{-i\tau}d\tau \approx 0, \quad J_{2} := \frac{1}{2\pi} \int_{0}^{2\pi} J_{2}(\tau)e^{-i\tau}d\tau \approx \frac{\pi^{2}}{12}. \tag{14.31}
\]

We calculate these integrals in Appendix B. As a result, the averaged matrix reads

\[
\tilde{\Omega}_{n}^{(1)}(\mathbf{v}) = \begin{pmatrix} 0 & \tilde{\omega}_{n}^{(1)}(\mathbf{v}) \\ \tilde{\omega}_{n}^{(1)}(\mathbf{v}) & 0 \end{pmatrix}, \quad \tilde{\omega}_{n}^{(1)}(\mathbf{v}) = \beta_{n} \mathbf{v} + \frac{\gamma_{n}}{2}. \tag{14.32}
\]
Now equation (14.6) with \( k = 2 \) reads as the system
\[
\begin{align*}
\dot{c}_{n,1}^{(2)}(\tau) &= -i\omega_n^{(1)} c_{n,2}^{(2)}(\tau), & n \in \mathbb{N}, \\
\dot{c}_{n,2}^{(2)}(\tau) &= -i\omega_n^{(1)} c_{n,1}^{(2)}(\tau)
\end{align*}
\] (14.33)

Its solution reads similarly to (14.19):
\[
c_{n,1}^{(2)}(\tau) = d_{n,1}^{(2)} \cos | \omega_n^{(1)} | \tau + d_{n,2}^{(2)} \sin | \omega_n^{(1)} | \tau, \quad c_{n,2}^{(2)}(\tau) = -id_{n,1}^{(2)} \frac{\omega_n^{(1)}}{| \omega_n^{(1)} |} \sin | \omega_n^{(1)} | \tau + id_{n,2}^{(2)} \frac{\omega_n^{(1)}}{| \omega_n^{(1)} |} \cos | \omega_n^{(1)} | \tau,
\] (14.34)

where
\[
d_{n,1}^{(2)} = c_{n,1}(0), \quad d_{n,2}^{(2)} = -i \frac{\omega_n^{(1)}}{| \omega_n^{(1)} |} c_{n,2}(0).
\] (14.35)

Thus, (14.34) can be rewritten as
\[
\begin{align*}
c_{n,1}^{(2)}(\tau) &= c_{n,1}(0) \cos | \omega_n^{(1)} | \tau - is_n c_{n,2}(0) \sin | \omega_n^{(1)} | \tau, \\
c_{n,2}^{(2)}(\tau) &= -is_n c_{n,1}(0) \sin | \omega_n^{(1)} | \tau + c_{n,2}(0) \cos | \omega_n^{(1)} | \tau,
\end{align*}
\] (14.36)

where \( s_n = \frac{\omega_n^{(1)}}{| \omega_n^{(1)} |} \). Hence, the matrix in the formula (14.8) with \( k = 2 \) reads
\[
U_n^{(2)}(\tau) = \begin{pmatrix}
\cos | \omega_n^{(1)} | \tau & -is_n \sin | \omega_n^{(1)} | \tau \\
-is_n \sin | \omega_n^{(1)} | \tau & \cos | \omega_n^{(1)} | \tau
\end{pmatrix}.
\] (14.37)

**The current.** Expanding (14.36) into the Taylor series, we obtain
\[
\begin{align*}
c_{n,1}^{(2)}(\tau) &= c_{n,1}(0) - ic_{n,2}(0) \omega_n^{(1)} \tau + O(| \omega_n^{(1)} |^2) \\
c_{n,2}^{(2)}(\tau) &= -ic_{n,1}(0) \omega_n^{(1)} \tau + c_{n,2}(0) + O(| \omega_n^{(1)} |^2)
\end{align*}
\] (14.38)

Here the error \( O(| \omega_n^{(1)} |^2) \sim 10^{-14} \) uniformly in \( \tau \in [0, 2\pi] \). Indeed, \( \omega_n^{(1)} = \beta_n \nu + \frac{\gamma_n}{2} \), where \( \nu \) is expressed by (14.26) and (14.29)–(14.31):
\[
\begin{align*}
v &= v_1 + iv_2 + v_3 \\
v_1 &\approx \frac{b(0)}{2} + \frac{\nu}{4} [a(0) - 2\pi b(0)], \quad v_2 \approx \frac{a(0)}{2} - \frac{\nu}{4} [2\pi a(0) + b(0)] \\
v_3 &\approx J_2 \sum_n \alpha_n \gamma_n |c_{n,2}(0)|^2 - |c_{n,1}(0)|^2
\end{align*}
\] (14.39)

Hence at the points (13.2), we have \( v_1 = v_{12} = 0 \), so
\[
\beta_n \nu = \beta_n v_2 = -\beta_n J_2 \sum_n \alpha_n \gamma_n \sim 10^{-12}
\] (14.40)

by (12.13) and (14.31). Therefore,
\[
\tilde{\omega}_n^{(1)} = \beta_n \nu + \frac{\gamma_n}{2} \approx \frac{\gamma_n}{2} \sim 10^{-7}.
\] (14.41)

Finally, substituting (14.38) into the second formula of (14.9) with \( k = 2 \), we obtain similarly to (14.22),
\[
\begin{align*}
\int^{(2)}(\tau) &= \sum_n \alpha_n \text{Im} \left\{ \tau_n(1)c_n(2)\tau e^{-i\tau} \right\} \\
&= \sum_n \alpha_n \text{Im} \left\{ \tau_n(0)c_n(2)e^{-i\tau} + i\tilde{\omega}_n^{(1)} |c_{n,2}(0)|^2 - |c_{n,1}(0)|^2 \tau e^{-i\tau} \right\} + O(| \omega_n^{(1)} |^2)
\end{align*}
\] (14.42)

**The Maxwell amplitude.** The last formula of (14.11) with \( k = 2 \) gives
\[
\begin{align*}
a^{(2)}(\tau) &= a_0(\tau) + \int^{(2)}(\tau')E(2\pi - \tau')d\tau' \\
b^{(2)}(\tau) &= \tilde{a}_0(\tau) + \int^{(2)}(\tau')E(2\pi - \tau')d\tau'.
\end{align*}
\] (14.43)
15 The Poincaré map

Let us denote
\[
a^0 := a(0), \quad b^0 := \dot{a}(0), \quad c_n^0 := c_n(0); \quad a := a^2(2\pi), \quad b := \dot{a}^2(2\pi), \quad c_n := c_n^2(2\pi).
\] (15.1)

Then (14.43) gives
\[
a = a_0(2\pi) + \int_0^{2\pi} j^2(\tau')E(2\pi - \tau')d\tau', \quad b = \dot{a}_0(2\pi) + \int_0^{2\pi} j^2(\tau')\dot{E}(2\pi - \tau')d\tau'.
\] (15.2)

Let us calculate these expressions. First, (14.11) and (14.12), (14.16) imply that
\[
a_0(2\pi) = a^0 \dot{E}(2\pi), \quad \dot{a}_0(2\pi) = a^0 \dot{E}(2\pi) + [b^0 + \sigma_j a^0] \dot{E}(2\pi)
\approx -a^0 \dot{E}(2\pi) + b^0 \dot{E}(2\pi) \approx b^0 \dot{E}(2\pi) + O_2,
\] (15.3)

where \(O_2 := O(\varepsilon^2) \sim 10^{-14}\). Second, substituting \(\dot{a}_n^{(1)} = \beta_n \nu + \frac{\nu}{2}\) into (14.42), we obtain
\[
j^2(\tau) \approx \sum_n \alpha_n \text{Im} \left\{ \tilde{c}_n^0 \tilde{c}_n^0 e^{-i\tau} \right\}
+ \sum_n \alpha_n \beta_n |c_{n,2}^0|^2 - |c_{n,1}^0|^2 \text{Im} \left\{ \nabla e^{-i\tau} \right\} + \frac{1}{2} \sum_n \alpha_n \gamma_n |c_{n,2}^0|^2 - |c_{n,1}^0|^2 \tau \cos \tau. \] (15.4)

Introducing this current into (15.2), we obtain the formulas for \(a\) and \(b\) expressed via the following integrals which are calculated in Appendix B:
\[
\begin{align*}
A_1 &:= \int_0^{2\pi} e^{-i\tau} E(2\pi - \tau')d\tau', \quad A_2 := \int_0^{2\pi} \tau e^{-i\tau} E(2\pi - \tau')d\tau', \quad A_3 := \int_0^{2\pi} \tau \cos \tau E(2\pi - \tau')d\tau' = \text{Re} A_2, \\
B_1 &:= \int_0^{2\pi} \tau e^{-i\tau} \dot{E}(2\pi - \tau')d\tau', \quad B_2 := \int_0^{2\pi} \tau^2 e^{-i\tau} \dot{E}(2\pi - \tau')d\tau', \quad B_3 := \int_0^{2\pi} \tau^2 \cos \tau \dot{E}(2\pi - \tau')d\tau' = -\text{Im} B_2.
\end{align*}
\] (15.5)

With this notation, (15.2) becomes
\[
\begin{cases}
\quad a &\approx a_0(2\pi) + \sum_n \alpha_n \text{Im} \left\{ \tilde{c}_n^0 \tilde{c}_n^0 A_1 \right\} \\
&+ \sum_n \alpha_n \beta_n [c_{n,2}^0|^2 - |c_{n,1}^0|^2] \text{Re} \left\{ \nabla A_2 \right\} + \frac{1}{2} \sum_n \alpha_n \gamma_n [c_{n,2}^0|^2 - |c_{n,1}^0|^2] A_3. \tag{15.6}
\end{cases}
\]

\[
\begin{cases}
\quad b &\approx \dot{a}_0(2\pi) + \sum_n \alpha_n \text{Im} \left\{ \tilde{c}_n^0 \tilde{c}_n^0 B_1 \right\} \\
&+ \sum_n \alpha_n \beta_n [c_{n,2}^0|^2 - |c_{n,1}^0|^2] \text{Re} \left\{ \nabla B_2 \right\} + \frac{1}{2} \sum_n \alpha_n \gamma_n [c_{n,2}^0|^2 - |c_{n,1}^0|^2] B_3.
\end{cases}
\]

Remark 15.1. i) The Poincaré map \(P\) is given by formulas (15.6) and (14.38).

ii) The second-order approximations (15.6) are obtained solving the equation (14.9) with \(k = 2\), initial data (14.1), and the current (15.4). At the “ground states” (13.2), we have \(|c_{n,1}^0|^2 = 1\) and \(|c_{n,2}^0|^2 = 0\), so the formula (15.4) takes the form
\[
j^2(\tau) = -\sum_n \alpha_n \beta_n \tau \text{Im} \left\{ \nabla e^{-i\tau} \right\} - \frac{1}{2} \sum_n \alpha_n \gamma_n \tau \cos \tau. \] (15.7)

16 Synchronisation of molecular currents

The key peculiarity of the perturbative calculations is the appearance of the summation
\[
S = \sum_n \alpha_n \beta_n \] (16.1)
in the second-order approximation of the total molecular current (15.7). This appearance is due to the Taylor approximation (14.38). Also note that the same summation reappears in (20.15). It is of crucial importance that all terms of the sum \( S \) are nonnegative since
\[
\alpha_n \beta_n = \frac{2c}{\Omega_p} P_n X(x_n) \frac{P_n X(x_n)}{\hbar c} \geq 0, \tag{16.2}
\]
and the sum is a non-negligible positive number, see (16.6) and (16.8) below. This means the synchronisation of the main parts of all molecular currents in the total current (15.4), (15.7) independently of orientations of the polarization \( P_n \) and of the Maxwell mode \( X(x_n) \) at the location of the molecules.

**Remark 16.1.** This synchronisation of molecular currents is provisionally in line with the superradiance and with the role of the stimulated emission in the laser action. Note that the conjecture on appropriate kind of synchronisation in the laser action has been put forward repeatedly, see, e.g., [30].

Note that we must obtain the value of the summation \( S \) for the calculation of the Poincaré map, although the summands are random values. Indeed, in (12.8), all the parameters \( P_n, X(x_n), a_p(x_n) \) with \( n \in N \) can be considered as independent random values, and
\[
\mathbb{E} X(x_n) = \mathbb{E} a_p(x_n) = 0. \tag{16.3}
\]
We will calculate the summation \( S \) in both cases of polycrystalline and monocrystalline medium by probabilistic arguments as follows. We will consider two cases of amorphous and crystalline medium separately.

**I. Amorphous medium.** By our assumption, \( |P_n| = |P| \) does not depend on \( n \), so
\[
(P_n \cdot e_1)^2 + (P_n \cdot e_1)^2 + (P_n \cdot e_1)^2 = |P|^2. \tag{16.4}
\]
Moreover, the dipole momenta \( P_n \) are distributed uniformly over the angles, and hence,
\[
\mathbb{E} (P_n \cdot e_1)^2 = \frac{1}{3} |P|^2. \tag{16.5}
\]
By Remark 12.1, we can apply the law of large numbers for weakly dependent random values [21] (recall that \( N \sim 10^{20} \)), and obtain that \( S \approx N \mathbb{E} \alpha_n \beta_n \). Moreover, \( P_n \) and \( X(x_n) \) are independent. Hence, using (5.1), we obtain that
\[
S \approx N \mathbb{E} \alpha_n \beta_n = N \frac{2}{\Omega_p \hbar} \mathbb{E} (P_n X(x_n))^2 = N \frac{2}{\Omega_p \hbar} \frac{\mathbb{E} |P_n|^2}{3} \mathbb{E} X^2(x_n)
\]
\[
= \frac{2}{\Omega_p \hbar} \frac{|P|^2}{3} \sum_n X^2(x_n) \approx \frac{2|P|^2}{3 \Omega_p \hbar} \int_{V_a} X^2(x) dV = \frac{2|P|^2}{3 \Omega_p \hbar} \frac{|V_a|}{|V|}, \quad d := \frac{N}{|V_a|} \tag{16.6}
\]
by the normalization of \( X \). Here the value \(|V_a|/|V|\) for the integral follows from Shnirelman’s Quantum Ergodic theorem [34, 39, 40]. Note that this value obviously holds for the eigenmodes (7.6) with sufficiently large eigenvalues. For the case of ruby laser (12.13) we obtain \( d \approx 3 \times 10^{19} \) and
\[
S \sim 10^{-5}, \tag{16.7}
\]
which agrees with estimate \( S \approx 10^{20} \times 10^{-25} \). This agreement confirms the obtained formula (16.6). Moreover, \( S \) is a Gaussian random value since \( \alpha_n \beta_n \) are independent and identically distributed.

**II. Crystalline medium.** In this case the vectors \( P_n = P \) do not depend on \( n \). We assume additionally that the mode \( X(x) \) is linearly polarised. Now (16.6) changes to
\[
S \approx \frac{2}{\Omega_p \hbar} |P|^2 \cos^2 \phi \sum_n X^2(x_n) \approx \frac{2|P|^2 \cos^2 \phi}{\Omega_p \hbar} \frac{|V_a|}{|V|} \approx 32 \frac{|V_a|}{|V|} 10^{-5} \cos^2 \phi \sim 10^{-5}, \tag{16.8}
\]
where \( \phi \) is the angle between \( P \) and \( X(x) \).
17 Factordynamics and the Hopf fibration

The system (13.3) is invariant with respect to the symmetry gauge group \( G = [U(1)]^N \) acting on the phase space \( \mathbb{X} = \mathbb{R}^2 \oplus \mathbb{C}^{2N} \) by the formula similar to (2.16):

\[
(e^{i\theta_1}, \ldots, e^{i\theta_N})(a, b, (c_{11}, c_{12}), \ldots, (c_{N1}, c_{N2})) = (a, b, e^{i\theta_1}(c_{11}, c_{12}), \ldots, e^{i\theta_N}(c_{N1}, c_{N2})).
\]

(17.1)

This action commutes with the dynamics (13.3), hence the latter induces the corresponding dynamics on the factorspace \( \mathbb{X}_s = \mathbb{X}/G \). This factorspace is the base of the fibration \( \mathbb{X} \to \mathbb{X}_s \). The action (17.1) does not change the Maxwell component \((a, b)\). Moreover, this action commutes with the Poincaré map \( P \). Hence, \( P \) induces the map \( P_s \) on the factorspace \( \mathbb{X}_s \). Accordingly, the instability and the parametric resonance \((1.1)\) must be checked for the induced factordynamics only. So, we must calculate the differential \( DP_s(X^0) \), where the point \( X^0 \) corresponds to \( X^0 \) in this factorization (see (13.1)).

**Local coordinates.** To calculate \( DP_s(X^0) \), we need suitable local coordinates on the factorspace. The charge conservation (12.11) means that

\[
(c_{n,1}(\tau), c_{n,2}(\tau)) \in S^3,
\]

(17.2)

where \( S^3 \) is the unit sphere in \( \mathbb{C}^2 \). The sphere is the Hopf fibration with fibers

\[
F = (c_1 e^{i\Theta}, c_2 e^{i\Theta}), \quad \Theta \in [0, 2\pi]
\]

(17.3)

and the base diffeomorphic to the unit sphere \( S^2 \). Denote by \((c_1, c_2)_s \in S^2\) the Hopf projection of a point \((c_1, c_2) \in S^3\). We choose the following invariant coordinate on the base:

\[
z := \sqrt[4]{c_1 c_2}.
\]

(17.4)

Using (12.11), it is easy to check that

\[
|c_1| + |c_2| = \sqrt{1 + 2|z|}, \quad |c_1| - |c_2| = \pm \sqrt{1 - 2|z|}.
\]

(17.5)

Hence,

\[
2|c_1| = \sqrt{1 + 2|z|} + \sqrt{1 - 2|z|}, \quad 2|c_2| = \sqrt{1 + 2|z|} - \sqrt{1 - 2|z|}
\]

(17.6)

and

\[
4|c_1|^2 = 2 + 2\sqrt{1 - 4|z|^2}, \quad 4|c_2|^2 = 2 - 2\sqrt{1 - 4|z|^2}.
\]

(17.7)

In particular, \(|c_1|^2\) and \(|c_2|^2\) are smooth functions on the disk \( D := |z| < \frac{1}{2} \), and

\[
\nabla_z |c_l|^2 |z = 0 = 0,
\]

(17.8)

where \( \nabla_z \) denotes the gradient with respect to \( \text{Re} z \) and \( \text{Im} z \).

The coordinate \( z \) maps \( S^3 \) onto the closed disk \( D \), and it is a local coordinate on the region of \( S^3 \) outside the set \(|c_1|^2 = |c_2|^2 = 1/2\). However, the inverse map \( D \to S^2 \) is two-valued. Indeed, denote by \((c_1, c_2)_s \in S^2\) the Hopf projection of a point \((c_1, c_2) \in S^3\). One branch of the inverse map is a diffeomorphism of \( D \) onto the neighborhood \( S^2_+ \subset S^2 \) of the point \((1, 0)_s \), and another branch is a diffeomorphism of \( D \) onto the neighborhood \( S^2_- \subset S^2 \) of the point \((0, 1)_s \), where also \( z = 0 \):

\[
\left\{ \begin{array}{ll}
|c_1| > |c_2| & \text{for } (c_1, c_2)_s \in S^2_+ \\
|c_1| < |c_2| & \text{for } (c_1, c_2)_s \in S^2_-
\end{array} \right.
\]

(17.9)

The base \( S^2 \) is the gluing of \( S^2_\pm \) along their boundaries, where \(|z| = \frac{1}{2}\) and \(|c_1|^2 = |c_2|^2 = \frac{1}{2}\).

Thus, the Hopf representation of the dynamics (12.9) uses gauge-invariant variables \( z_n = \sqrt[4]{c_{n,1} c_{n,2}} \). This reduces the number of variables twice that considerably simplifies the calculations. In these new variables, the current (12.10) and the population invertons \( I_n := |c_{n,2}|^2 - |c_{n,1}|^2 \) read as

\[
J(\tau) = \sum_n a_n \text{Im} \{ z_n(\tau) e^{-i\tau} \}, \quad I_n = -\sqrt{1 - 4|z_n|^2}
\]

(17.10)

by (17.8). So, the system (12.9) becomes

\[
\dot{a} = b, \quad \dot{b}(\tau) + \sigma_1 b(\tau) + a(\tau) = J(\tau), \quad \dot{z}_n = -\text{Im} z_n \partial_\tau \sqrt{1 - 4|z_n|^2}.
\]

(17.11)


18 Differential of the Poincaré map on the factorspace

It remains to calculate the differential (18.4). We choose coordinates (17.4) on the bases of the Hopf fibration for each active molecule, and denote

\[ z_n^0 := (z_{n,1}^0, z_{n,2}^0), \quad z_n := z_{n,1}c_{n,2} = c_{n,2}(2\pi)c_{n,2}(2\pi). \]  
(18.1)

Let us write \( z_n = z_{n,1} + i z_{n,2}, z_n^0 = z_{n,1}^0 + i z_{n,2}^0 \), where \( z_{n,1}, z_{n,2}, z_{n,1}^0, z_{n,2}^0 \in \mathbb{R} \), and

\[ z_n = \begin{pmatrix} z_{n,1} \\ z_{n,2} \end{pmatrix}, \quad z_n^0 = \begin{pmatrix} z_{n,1}^0 \\ z_{n,2}^0 \end{pmatrix}, \quad \frac{\partial z_n^0}{\partial z_n} := \begin{pmatrix} \frac{\partial z_{n,1}^0}{\partial z_{n,1}} & \frac{\partial z_{n,1}^0}{\partial z_{n,2}} \\ \frac{\partial z_{n,2}^0}{\partial z_{n,1}} & \frac{\partial z_{n,2}^0}{\partial z_{n,2}} \end{pmatrix}, \quad n \in \mathbb{N}. \]  
(18.2)

Denote

\[ Z = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}, \quad Z^0 = (z_1^0, \ldots, z_N^0). \]  
(18.3)

Then the differential \( DP_e(X_0^e) \) of the Poincaré map \( P_e \) on the factorspace \( \mathbb{X}/G \) is represented by the matrix

\[ DP_e(X_0^e) = \begin{pmatrix} \frac{\partial a}{\partial \phi} & \frac{\partial a}{\partial \phi'} & \frac{\partial a}{\partial \phi''} \\ \frac{\partial b}{\partial \phi} & \frac{\partial b}{\partial \phi'} & \frac{\partial b}{\partial \phi''} \\ \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \phi'} & \frac{\partial z}{\partial \phi''} \end{pmatrix}. \]  
(18.4)

Using (14.38) with \( \tau = 2\pi \), we obtain that

\[ z_n = (z_{n,1}^0 + i z_{n,2}^0 2\pi \omega n) (-ic_{n,1}^0 2\pi \alpha_n^0 + c_{n,2}^0) + O(|\alpha_n^1|^2) \]

\[ = z_n^0 + 2\pi i \omega_n \left[ |c_{n,2}^0|^2 - |c_{n,1}^0|^2 \right] + O(|\alpha_n^1|^2) \]

\[ = z_n^0 + 2\pi i (\beta_n \nabla + \frac{\gamma_n}{2}) \left[ |c_{n,2}^0|^2 - |c_{n,1}^0|^2 \right] + O(|\alpha_n^1|^2). \]  
(18.5)

We must calculate the matrix (18.4) at the ground states (13.2) which correspond to the single point

\[ X_0^e = (0, 0, 0). \]  
(18.6)

The formulas (15.6) for the Poincaré map become

\[ a \approx a_0(2\pi) + \sum_n \alpha_n \text{Im} \left\{ z_n^0 A_1 \right\} \]

\[ + \sum_n \alpha_n \beta_n \left[ |c_{n,2}^0|^2 - |c_{n,1}^0|^2 \right] \text{Re} \left\{ \nabla A_2 \right\} + \frac{1}{2} \sum_n \alpha_n \gamma_n \left[ |c_{n,3}^0|^2 - |c_{n,1}^0|^2 \right] A_3 \] 

\[ b \approx b_0(2\pi) + \sum_n \alpha_n \text{Im} \left\{ z_n^0 B_1 \right\} \]

\[ + \sum_n \alpha_n \beta_n \left[ |c_{n,2}^0|^2 - |c_{n,1}^0|^2 \right] \text{Re} \left\{ \nabla B_2 \right\} + \frac{1}{2} \sum_n \alpha_n \gamma_n \left[ |c_{n,3}^0|^2 - |c_{n,1}^0|^2 \right] B_3 \]  
(18.7)

Here \( a_0(2\pi) \) and \( a_0(2\pi) \) are given by (14.11), so using (14.16) and neglecting errors \( O(x^2) = O(10^{-14}) \), we obtain

\[ \begin{cases} a_0(2\pi) = a^0E(2\pi) + |b^0 + 2\alpha d^0|E(2\pi) \approx a^0(1 - 2\pi x) \\
\phi_0(2\pi) = a^0E(2\pi) + |b^0 + 2\alpha d^0|E(2\pi) \approx -2\pi a^0 + |b^0 + 2\alpha d^0|(1 - 2\pi x) \approx b^0(1 - 2\pi x) \end{cases} \]  
(18.8)

At the point (18.6), we have by (17.8),

\[ \frac{\partial}{\partial x_I^0} \bigg|_{\delta = 0} |c_{n,1}^0|^2 = \frac{\partial}{\partial x_I^0} \bigg|_{\delta = 0} |c_{n,2}^0|^2 = (0, 0). \]  
(18.9)
Differentiating the formulas (14.39) at the point (18.6), we obtain that neglecting errors \(O(\varkappa^2) = O(10^{-14})\), we have
\[
\frac{\partial \mathbf{V}}{\partial d^0} = -\frac{i}{2} + \frac{\varkappa}{4}(1 + 2\pi i), \quad \frac{\partial \mathbf{V}}{\partial b^0} = \frac{1}{2} - \frac{\varkappa}{4}(2\pi - i) = i\frac{\partial \mathbf{V}}{\partial d^0}.
\] (18.10)

Moreover, formulas (14.39) and (18.9) imply that at the point (18.6), we have
\[
\frac{\partial \mathbf{V}}{\partial z_n'} = (0, 0).
\] (18.11)

Now we can calculate the matrix (18.4) at the point (18.6). Differentiating the first formula of (18.7), and taking into account (18.8)–(18.11), we obtain that the first row of (18.4) is
\[
\frac{\partial \alpha_n}{\partial d^0} \approx 1 - 2\pi \varkappa - S\left[\frac{A_{22}}{2} + \frac{\varkappa}{4}(A_{21} - 2\pi A_{22})\right], \quad \frac{\partial \alpha_n}{\partial b^0} \approx -S\left[\frac{A_{21}}{2} - \frac{\varkappa}{4}(2\pi A_{21} + A_{22})\right], \quad \frac{\partial \alpha_n}{\partial z_n'} \approx \alpha_n(A_{12}, A_{11}), \quad (18.12)
\]
where \(S\) is the summation (16.1). Similarly, using (B.7), we obtain the second row
\[
\frac{\partial \beta_n}{\partial d^0} \approx -S\left[\frac{B_{22}}{2} + \frac{\varkappa}{4}(B_{21} - 2\pi B_{22})\right], \quad \frac{\partial \beta_n}{\partial b^0} \approx 1 - 2\pi \varkappa - S\left[\frac{B_{21}}{2} - \frac{\varkappa}{4}(2\pi B_{21} + B_{22})\right], \quad \frac{\partial \beta_n}{\partial z_n'} \approx \alpha_n(B_{12}, B_{11}), \quad (18.13)
\]
where \(B_{11} = \text{Re} B_1, B_{12} = \text{Im} B_1,\) and similarly for \(B_{21}\) and \(B_{22}\). Finally, differentiating (18.5) at the point (18.6), and using (18.9), (18.10), we get
\[
\frac{\partial z_n}{\partial d^0} \approx 2\pi i \beta_n \frac{\partial \mathbf{V}}{\partial d^0} = 2\pi i \beta_n\left[-\frac{i}{2} + \frac{\varkappa}{4}(1 + 2\pi i)\right] = \pi \beta_n[1 - \frac{\varkappa}{2}(2\pi - i)]
\]
\[
\frac{\partial z_n}{\partial b^0} \approx 2\pi i \beta_n \frac{\partial \mathbf{V}}{\partial b^0} = 2\pi i \beta_n\left[\frac{1}{2} - \frac{\varkappa}{4}(2\pi - i)\right] = \pi \beta_n[i - \frac{\varkappa}{2}(2\pi i + 1)]
\] (18.14)

Similarly, using (18.9) and (18.11), we obtain
\[
\frac{\partial z_n}{\partial z_n'} \approx \delta_{nn}(1, i).
\] (18.15)

The expressions (18.14) and (18.15) can be written as vectors and matrices:
\[
\frac{\partial z_n}{\partial d^0} \approx \pi \beta_n \left(1 - \frac{\varkappa}{2}\right), \quad \frac{\partial z_n}{\partial b^0} \approx \pi \beta_n \left(-\frac{\varkappa}{2}\right), \quad \frac{\partial z_n}{\partial z_n'} \approx \delta_{nn}(1, 0, 1).
\] (18.16)

**19 Block-matrix approximation**

All calculations below essentially rely on the constants (12.13) which correspond to the ruby laser. Let us denote the matrix
\[
M = \begin{pmatrix}
\frac{\partial a}{\partial d^0} & \frac{\partial a}{\partial b^0} \\
\frac{\partial b}{\partial d^0} & \frac{\partial b}{\partial b^0}
\end{pmatrix}
\] (19.1)

Recall that for the typical dipole moment (11.3) we have by (16.7), (16.8) and (12.13), that
\[
S = \sum_n \alpha_n \beta_n \approx 10^{-5}, \quad \gamma_n \approx 2 \times 10^{-7}, \quad \varkappa \approx 10^{-7}.
\] (19.2)

Rewrite the formulas (18.12)–(18.13) substituting the expressions (18.10), and neglecting all terms containing \(\varkappa S \approx 10^{-12}\) and \(\varkappa^2 \approx 10^{-14}\):
\[
\frac{\partial a}{\partial d^0} \approx 1 - 2\pi \varkappa + \frac{1}{2}A_{22}S, \quad \frac{\partial a}{\partial b^0} \approx -\frac{1}{2}A_{21}S, \quad (19.3)
\]
\[
\frac{\partial b}{\partial d^0} \approx -\frac{1}{2}B_{22}S, \quad \frac{\partial b}{\partial b^0} \approx 1 - 2\pi \varkappa - \frac{1}{2}B_{21}S. \quad (19.4)
\]
Hence,
\[
M = M(\kappa, S) \approx \begin{pmatrix}
1 - 2\pi\kappa - \frac{1}{2}A_{22}S & -\frac{1}{2}A_{21}S \\
-\frac{1}{2}B_{22}S & 1 - 2\pi\kappa - \frac{1}{2}B_{21}S
\end{pmatrix}.
\]  
(19.5)

Further, denote the matrices
\[
D_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V_n = \alpha_n \begin{pmatrix} A_{12} & A_{11} \\ B_{12} & B_{11} \end{pmatrix}, \quad W_n = \pi\beta_n \begin{pmatrix} 1 - \kappa\pi & -\frac{\kappa}{2} \\ \frac{\kappa}{2} & 1 - \kappa\pi \end{pmatrix}.
\]  
(19.6)

Using (B.7) and (B.4), we obtain that
\[
V_n \approx \pi\alpha_n \begin{pmatrix} 1 - \kappa\pi & \frac{\kappa}{2} \\ \frac{\kappa}{2} & 1 - \kappa\pi \end{pmatrix}.
\]  
(19.7)

Now formulas (18.12), (18.13), (19.5), and (18.16) give the following block-matrix approximation for the differential (18.4):
\[
DP_n(X_n^0) \approx \begin{pmatrix}
M & V_1 & V_2 & V_3 & \ldots & V_N \\
W_1 & D_1 & 0 & 0 & \ldots & 0 \\
W_2 & 0 & D_2 & 0 & \ldots & 0 \\
W_3 & 0 & 0 & D_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
W_N & 0 & 0 & 0 & \ldots & D_N
\end{pmatrix},
\]  
(19.8)

which is not symmetric generally.

**Remark 19.1.** Since \(\alpha_n \approx 10^{-23}\), the matrix is very close to its triangle approximation
\[
DP_n(X_n^0) \approx \begin{pmatrix}
M & 0 & 0 & 0 & \ldots & 0 \\
W_1 & D_1 & 0 & 0 & \ldots & 0 \\
W_2 & 0 & D_2 & 0 & \ldots & 0 \\
W_3 & 0 & 0 & D_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
W_N & 0 & 0 & 0 & \ldots & D_N
\end{pmatrix}.
\]  
(19.9)

The corresponding eigenvalues coincide with the ones of the matrices \(M\) and \(D_n\). In particular, for the eigenvalues of \(M\) we have \(|\mu - 1| \approx 10^{-5}\) by (19.5) and (16.7).

### 20 Reduction to polynomial equation

The spectral problem for the multipliers reads
\[
DP_n(X_n^0)\mathbf{V} = \mu\mathbf{V}, \quad \mathbf{V} = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_N \end{pmatrix},
\]  
(20.10)

where all \(v_n \in \mathbb{R}^2\) and \(\mathbf{V} \neq 0\). We will reduce the spectral problem to a polynomial equation. Namely, substituting here the approximation (19.8), we obtain the equivalent system
\[
(M - \mu)v_0 + \sum_{n=1}^{N} V_nv_n = 0, \quad W_nv_0 + (D_n - \mu)v_n = 0, \quad n \in \mathcal{N}.
\]  
(20.11)

Since we seek for eigenvalues with \(|\mu| > 1\), we can write
\[
v_n = (\mu - D_n)^{-1}W_nv_0 \approx \pi\beta_n \begin{pmatrix}
\frac{1 - \kappa\pi}{\mu - 1} & -\frac{\kappa/2}{\mu - 1} \\
\frac{\kappa/2}{\mu - 1 + 2\pi^2r^2} & \frac{1 - \kappa\pi}{\mu - 1 + 2\pi^2r^2}
\end{pmatrix}v_0,
\]  
(20.12)
Thus, the characteristic equation for the matrix \( DP(V) \) is equivalent to

\[
\det \left( \frac{\pi^2}{\mu - 1} S - \frac{\alpha_n \beta_n}{\mu - 1} \sum_{i} \left( 1 - \frac{\pi \beta_n}{2} \right) \right) = 0.
\]

Let us consider the multipliers for which \( 2\pi^2 \frac{2\beta_n}{\mu - 1} \ll 1 \). The existence of such multipliers is suggested by Remark 19.1. For such multipliers, the last two sums in (20.13) can be calculated as follows. First,

\[
\sum_{i} \frac{\alpha_n \beta_n}{\mu - 1 + 2\pi^2 \gamma_n} = \frac{1}{\mu - 1} \sum_{i} \frac{\alpha_n \beta_n}{1 + 2\pi^2 \frac{\pi}{\mu - 1} \gamma_n} \approx \frac{1}{\mu - 1} \sum_{i} \alpha_n \beta_n \left( 1 - 2\pi^2 \gamma_n \frac{\pi}{\mu - 1} \right)
\]

\[
= \frac{S}{\mu - 1} - \frac{2\pi^2}{\mu - 1} \frac{\sum_{i} \alpha_n \beta_n \gamma_n^2}{(\mu - 1)^2} - \frac{2\pi^2}{\mu - 1} \frac{2}{\Omega_p h^2 \pi} \sum_{i} (P_n \cdot X(x_n))^2 (P_n \cdot a_p(x_n))^2.
\]

Second, by Remark 12.1, we can apply the law of large numbers for weakly dependent random values [21], and obtain that \( S \approx \frac{1}{N} \sum \alpha_n \beta_n \).

\[
\sum_{i} (P_n \cdot X(x_n))^2 (P_n \cdot a_p(x_n))^2 = N \Sigma, \quad \Sigma := \mathbb{E} (P_n \cdot X(x_n))^2 (P_n \cdot a_p(x_n))^2.
\]

The last expectation can be calculated using the fact that random values \((P^1, P^2, P^3) = P_n, (X^1, X^2, X^3) = X(x_n)\) and \((a^1, a^2, a^3) = a_p(x_n)\) are independent, so

\[
\Sigma = \mathbb{E} \sum_{i,j=1}^{3} (P^i X^j)^2 (P^i X^j)^2 = \mathbb{E} \sum_{i,j=1}^{3} P^i P^i' P^j P^j' X^i X^i' a^i a^j = \sum_{i,j=1}^{3} \mathbb{E} [P^i P^i' P^j P^j'] \mathbb{E} [X^i X^j] \mathbb{E} [a^i a^j].
\]

By our assumption \( H_4 \), the random vector \( a_p(x_n) \) is distributed uniformly on the sphere of a radius \( a_p \), so

\[
\mathbb{E} [a^i a^j] = \frac{1}{2} \delta_{ij} a_p^2.
\]

Further, let us conjecture that the distribution of the random vector \( X(x_n) \) is close to the one of the eigenmodes (7.6) of rectangular cuboid \( V = [0, l_1] \times [0, l_2] \times [0, l_3] \). In this case, similarly to (16.6),

\[
\mathbb{E} [X^i X^j] = \delta_{ij} \mathbb{E} [X^i]^2 = \frac{\delta_{ij}}{N} \sum_{n=1}^{N} |X^i(x_n)|^2 \approx \frac{\delta_{ij}}{|V_a|} \int_{V_a} |X^i|^2 dx = \frac{\delta_{ij}}{|V|} |a^i|^2,
\]

where \( a^i = a^i_n \) are the amplitudes from (7.6). Substituting (20.18) and (20.19) into (20.17), we obtain that

\[
\Sigma = \frac{1}{9|V|} |a_p|^2 \sum_{i,j=1}^{3} \mathbb{E} [|P^i|^2 |P^j|^2] |a^i|^2
\]

**Remark 20.1.** According to Berry’s conjecture, in the case of ergodic geodesic flow, the distribution of values of Laplacian eigenfunctions with sufficiently large eigenvalues can be Gaussian [6, 26], thus different from the “sinusoidal distribution” of eigenmodes (7.6).
Let us consider the following two cases separately.

I. Polycrystalline medium. Let us calculate \( E[|P|^2|P|^2] \). By our assumption H1, the random vector \((P^1, P^2, P^3)\) is distributed uniformly on the sphere of radius \(|P|\). Hence, integrating in spherical coordinates, we obtain

\[
E[|P|^2|P|^2] = \frac{1}{15}|P|^4, \quad E(P^1)^4 = \frac{1}{5}|P|^4.
\] (20.21)

Hence,

\[
\Sigma = \frac{1}{9|V|}|a_p|^2 \sum_{i=1}^{3} \left( \frac{1}{5} + \frac{2}{15} \right) |a_i|^2 = \frac{1}{27|V|}|a_p|^2 |P|^4
\] (20.22)

since \( a = (a^1, a^2, a^3) \) is the unit vector.

II. Crystalline medium. In this case the vectors \( P_n = P \) do not depend on \( n \). Thus, (20.20) becomes

\[
\Sigma = \frac{1}{9|V|}|a_p|^2 \sum_{i=1}^{3} |P|^2 |P|^2 |a_i|^2 = \frac{a_p^2 |P|^2}{9|V|} \sum_{i=1}^{3} |P|^2 |a_i|^2.
\] (20.23)

In both cases, (20.14) for \( \mu \neq 1 \) is equivalent to the algebraic equation

\[
p(\mu) = 0, \quad p(\mu) = \text{det}((\mu - 1)^2 \mathcal{A}(\mu)),
\] (20.24)

where \( p(\mu) \) is a polynomial of degree six.

Remark 20.2. i) The summation \( S \) reappears in the calculations (20.13) and (20.15).

ii) The degree six of the polynomial can be reduced to three using the approximate symmetry of type (4.7), see Remark 4.3.

Remark 20.3. For every nonzero eigenvector (20.10), its component \( v_0 \), representing the Maxwell field, does not vanish. Hence, if (1.1) holds, then the magnitude of the Maxwell field provisionally increases exponentially for large (but bounded) times.

Remark 20.4. We expect that

i) all the multipliers \( |\mu| \) are simple with probability one;

ii) the multipliers form clusters around the roots of the polynomial \( p(\mu) \);

iii) the condition (1.1) holds for the roots of the polynomial \( p(\mu) \) when

\[
a_p \geq d(\sigma_1),
\] (20.25)

where \( d(\sigma_1) \) is the corresponding threshold, which is an increasing function of \( \sigma_1 \). Physically, the condition (20.25) means that the amplitude of the pumping exceeds the damping losses.

### A On averaging of slow rotations

Here we justify the approximation of the Schrödinger equations from (13.3) by their averaged version (14.6) (“rotating wave approximation” \cite{8, 11, 12, 22, 25, 26, 30, 33, 36, 44}). Let us show that this approximation is very accurate for small matrices \( \Omega_n(\tau) \). Namely, consider the system

\[
\dot{c}(\tau) = -i\Omega(\tau)c(\tau), \quad \tau \in [0, T],
\] (A.1)

where \( \Omega(\tau) \) is a bounded measurable \( 2 \times 2 \) complex matrix-function,

\[
\varepsilon := \sup_{\tau \in [0, T]} |\Omega(\tau)| < \infty.
\] (A.2)

Denote the averaged matrix

\[
\tilde{\Omega} := \frac{1}{T} \int_0^T \Omega(\tau)d\tau
\] (A.3)

and consider the corresponding averaged system

\[
\dot{\tilde{c}}(\tau) = -i\tilde{\Omega}\tilde{c}(\tau), \quad \tau \in [0, T].
\] (A.4)
Lemma A.1. For solutions of (A.1) and (A.4) with identical initial data \( \hat{c}(0) = c(0) = c_0 \), their final values \( c(T) \) and \( \hat{c}(T) \) are very close for small \( \varepsilon \):

\[
|c(T) - \hat{c}(T)| = O(\varepsilon^2), \quad \varepsilon \to 0. \tag{A.5}
\]

Proof. Rewrite (A.1) as

\[
\hat{c}_\varepsilon(\tau) = -i\varepsilon\Omega_1(\tau)c_\varepsilon(\tau), \quad \tau \in [0, T]; \quad c_\varepsilon(0) = c_0, \tag{A.6}
\]

where \( \Omega_1(\tau) := \frac{1}{\varepsilon}\Omega(\tau) \). The corresponding averaged system (A.4) now reads

\[
\hat{\hat{c}}_\varepsilon(\tau) = -i\varepsilon\tilde{\Omega}_1(\tau)\hat{c}_\varepsilon(\tau), \quad \tau \in [0, T]; \quad \hat{c}_\varepsilon(0) = c_0, \tag{A.7}
\]

where \( \tilde{\Omega}_1(\tau) := \frac{1}{T} \int_0^T \Omega_1(\tau) d\tau \). It suffices to prove that

\[
|c_\varepsilon(T) - \hat{c}_\varepsilon(T)| = O(\varepsilon^2), \quad \varepsilon \to 0. \tag{A.8}
\]

The Taylor formula gives

\[
c_\varepsilon(\tau) = c_0 + \varepsilon[\partial_\tau c_\varepsilon(\tau)]|_{\tau=0} + O(\varepsilon^2), \quad \dot{c}_\varepsilon(\tau) = c_0 + \varepsilon[\partial_\tau \dot{c}_\varepsilon(\tau)]|_{\tau=0} + O(\varepsilon^2). \tag{A.9}
\]

Therefore, (A.8) will follow from the identity

\[
[\partial_\tau c_\varepsilon(T)]|_{\tau=0} = [\partial_\tau \hat{c}_\varepsilon(T)]|_{\tau=0}. \tag{A.10}
\]

Indeed, \( \hat{c}_\varepsilon(T) = e^{-i\varepsilon\tilde{\Omega}_1T}c_0 \), so the right-hand side of (A.10) is equal to

\[
[\partial_\tau \hat{c}_\varepsilon(T)]|_{\tau=0} = -i\tilde{\Omega}_1Tc_0 = -i\int_0^T \Omega_1(\tau)c_0 d\tau. \tag{A.11}
\]

It remains to calculate the left-hand side. Denote

\[
b_\varepsilon(\tau) := \partial_\tau c_\varepsilon(\tau). \tag{A.12}
\]

Differentiating equation (A.6) in \( \varepsilon \), we obtain

\[
\dot{b}_\varepsilon(\tau) = -i\Omega_1(\tau)c_\varepsilon(\tau) - i\varepsilon\Omega_1(\tau)b_\varepsilon(\tau), \quad \tau \in [0, T]; \quad b_\varepsilon(0) = 0. \tag{A.13}
\]

In particular, for \( \varepsilon = 0 \) we have

\[
b_0(\tau) = -i\Omega_1(\tau)c_0, \quad \tau \in [0, T]; \quad b_0(0) = 0. \tag{A.14}
\]

Hence,

\[
b_0(T) = -i\int_0^T \Omega_1(\tau)c_0 d\tau. \tag{A.15}
\]

Now (A.10) is proved. \( \square \)

Remark A.2. Equation (A.6) can be rewritten for \( \hat{c}_\varepsilon(s) := c_\varepsilon\left(\frac{s}{\varepsilon}\right) \) as

\[
\partial_s \hat{c}_\varepsilon(s) = -i\Omega_1\left(\frac{s}{\varepsilon}\right)\hat{c}_\varepsilon(s), \quad s \in [0, \varepsilon T], \tag{A.16}
\]

where \( s = \varepsilon \tau \) is the ‘slow time’. Thus, Lemma A.1 is a specific version of averaging principle.
B Some integrals

2.1 Integrals (14.25)

First, (14.16) implies that

\[ I_1(\tau) := \int_0^\tau e^{-i\tau' e^{-\kappa(\tau'-\tau)}} \sin(\tau - \tau') d\tau' = \int_0^\tau e^{-i\tau'} [e^{(-\kappa+\i)(\tau-\tau')} - e^{(-\kappa-\i)(\tau-\tau')} ] d\tau' \]

\[ = \frac{1}{2i} e^{(-\kappa+\i)\tau} \int_0^\tau e^{(\kappa-2\i)\tau'} d\tau' - \frac{1}{2i} e^{(-\kappa-\i)\tau} \int_0^\tau e^{\kappa \tau'} d\tau' \]

\[ = \frac{1}{2i} e^{(-\kappa+\i)\tau} e^{(\kappa-2\i)\tau} - \frac{1}{2i} e^{(-\kappa-\i)\tau} e^{\kappa \tau} - 1 \]

\[ = \frac{e^{-i\tau} - e^{(-\kappa+\i)\tau}}{2i\kappa + 4} - \frac{e^{-i\tau} - e^{(-\kappa-\i)\tau}}{2i\kappa} \]

\[ \approx -\frac{i}{2} \sin \tau - \frac{\tau}{2} e^{-i\tau} + \frac{\tau^2}{4} \left[ -\sin \tau + \tau e^{i\tau} + \frac{\tau^2 e^{-i\tau}}{4} \right], \quad \text{(B.1)} \]

where we have neglected errors \( O(\kappa^2) = O(10^{-14}) \). Similarly,

\[ I_2(\tau) := \frac{1}{2} \int_0^\tau \tau' \cos \tau' e^{-\kappa(\tau'-\tau)} \sin(\tau - \tau') d\tau' = \frac{1}{4} \int_0^\tau \tau' e^{-\kappa(\tau'-\tau)} \left[ \sin \tau + \sin(\tau - 2\tau') \right] d\tau' \]

\[ = \frac{1}{4} e^{-\kappa\tau} \left[ \sin \tau \int_0^\tau \tau' e^{\kappa \tau'} d\tau' + \text{Im} \int_0^\tau \tau' e^{\kappa \tau'} e^{i(\tau-2\tau')} d\tau' \right] \]

\[ = \frac{1}{4} e^{-\kappa\tau} \left[ \sin \tau \left( \frac{e^{\kappa \tau}}{\kappa} - \frac{e^{\kappa \tau'} - 1}{\kappa^2} \right) + \text{Im} \left( e^{i\tau} \tau e^{(-\kappa+\i)\tau} - e^{i\tau} \int_0^\tau \frac{e^{(-\kappa+\i)\tau'}}{\kappa - 2\i} d\tau' \right) \right] \]

\[ = \frac{1}{4} e^{-\kappa\tau} \left[ \sin \tau \left( \frac{\tau e^{\kappa \tau} - e^{\kappa \tau}}{\kappa^2} + \frac{e^{\kappa \tau} - e^{\kappa \tau}}{\kappa^2} \right) + \text{Im} \left( \frac{e^{i(\kappa+1)\tau} (\kappa + 2\i)}{(\kappa + 2\i)^2} - \frac{e^{i(\kappa-1)\tau} (\kappa + 2\i)}{(\kappa^2 + 4\i)^2} \right) \right] \]

\[ \approx \frac{1}{4} e^{-\kappa\tau} \left[ \sin \tau \left( \frac{\tau^2}{2} + \frac{\kappa \tau^2}{2} \right) + \tau e^{i\kappa \tau} \text{Im} (e^{-i\tau} (\kappa + 2\i)) - \frac{1}{16} \text{Im} \left( (e^{i(\kappa-1)\tau} - e^{i\tau}) (\kappa + 2\i)^2 \right) \right] \]

\[ \approx \frac{1}{4} e^{-\kappa\tau} \left[ \sin \tau \left( \frac{\tau^2}{2} + \frac{\kappa \tau^2}{2} \right) + \tau e^{i\kappa \tau} \frac{2 \cos \tau - \kappa \sin \tau}{4} \right. \]

\[ - \frac{1}{16} \left( (e^{i\kappa \tau} \cos \tau - \cos \tau) 4\kappa - 4(-e^{i\kappa \tau} \sin \tau - \sin \tau) \right) \]

\[ \approx \frac{1}{4} e^{-\kappa\tau} \left[ \sin \tau \left( \frac{\tau^2}{2} + \frac{\kappa \tau^2}{2} \right) + \tau e^{i\kappa \tau} 2 \cos \tau - \kappa \sin \tau \right. \]

\[ - \frac{1}{4} (e^{i\kappa \tau} + 1) \sin \tau \]

\[ \approx \frac{1}{4} \left[ \sin \tau \left( \frac{\tau^2}{2} + \frac{\kappa \tau^2}{4} \right) + \frac{\tau}{4} (e^{i\kappa \tau} 2 \cos \tau - \kappa \sin \tau) - \frac{1}{4} (e^{i\kappa \tau} + 1) \sin \tau \right] \]

\[ \approx \frac{1}{4} \left[ \sin \tau \left( \frac{\tau^2}{2} + \frac{\kappa \tau^2}{4} \right) + \frac{\tau}{4} (2 \cos \tau - \kappa \sin \tau) - \frac{1}{4} (1 + e^{-i\kappa \tau}) \sin \tau \right] \]

\[ \approx \frac{1}{4} \left[ \sin \tau \left( \frac{\tau^2}{2} + \frac{\kappa \tau^2}{4} \right) + \frac{\tau}{4} (2 \cos \tau - \kappa \sin \tau) - \frac{1}{4} (2 - \kappa \tau) \sin \tau \right] \]

\[ = \frac{1}{8} \left[ \tau^2 \sin \tau + \tau \cos \tau - \sin \tau \right]. \quad \text{(B.2)} \]
2.2 Integrals (14.31)

Calculate integrals

\[
\begin{align*}
\int_0^{2\pi} e^{it} d\tau &= -\pi i \
\int_0^{2\pi} \tau e^{it} d\tau &= -4\pi^2 i + 4\pi, \
\int_0^{2\pi} \tau^2 e^{it} d\tau &= -2\pi^2 i + \pi, \
\int_0^{2\pi} \tau^2 \sin \tau d\tau &= -4\pi^2 i.
\end{align*}
\]

Then, neglecting errors \(O(\tau^2) = O(10^{-14})\), we can rewrite (14.31) as

\[
\begin{align*}
J_1 &:= \frac{1}{2\pi} \int_0^{2\pi} I_1(\tau)e^{-i\tau} d\tau \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left[ -\frac{1}{2} \cos \tau - \frac{\tau e^{-i\tau}}{2i} + \frac{\tau^2}{4} \right] e^{-i\tau} d\tau \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left[ -\frac{1}{4} + \frac{\tau e^{-2i\tau}}{2} + \frac{2\pi i - 4(2\pi^2 i + \pi)}{8\pi} \right] d\tau = 0, \\
J_2 &:= \frac{1}{2\pi} \int_0^{2\pi} I_2(\tau)e^{-i\tau} d\tau \\
&= \frac{1}{16\pi} \int_0^{2\pi} \left[ \tau \sin \tau + \tau^2 \cos \tau - \sin \tau \right] e^{-i\tau} d\tau \\
&= \frac{1}{32\pi} \int_0^{2\pi} \left[ -i(1 - e^{-2i\tau}) + \tau^2(1 + e^{-2i\tau}) \right] d\tau \\
&= \frac{\pi^2}{12}.
\end{align*}
\]

(B.3)

2.3 Integrals (15.5)

Neglecting errors \(O(\tau^2) = O(10^{-14})\), we obtain

\[
\begin{align*}
A_1 &\approx -\int_0^{2\pi} e^{-i\tau} e^{-\tau(2\pi - \tau)} d\tau' = -\frac{e^{-2\pi \tau}}{2i} \int_0^{2\pi} e^{-i\tau'} e^{\tau' - e^{-i\tau'}} d\tau' \\
&= -\frac{e^{-2\pi \tau}}{2i} \int_0^{2\pi} e^{\tau'} - e^{(\tau - 2\pi)i} d\tau' = -\frac{e^{-2\pi \tau}}{2i} \left[ \frac{e^{2\pi \tau} - 1}{2\pi - 2i} \right] \\
&\approx -\frac{e^{-2\pi \tau}}{2i} \left[ \frac{2\pi \tau + \frac{1}{2}(2\pi \tau)^2}{2\pi - 2i} \right] \approx -\frac{e^{-2\pi \tau}}{2i} \left[ 2\pi + 2\pi^2 - 2\pi i \right] \\
&\approx (1 - 2\pi \tau) \left[ \pi i + \pi^2 i + \frac{\pi^3}{2} \right] \approx \pi i + \pi^2 i + \frac{\pi^3}{2} - \pi^2 2\pi i \\
&= \pi i + \pi \left[ -\pi^2 i + \frac{\pi}{2} \right] = A_{11} + iA_{12}, \quad A_{11} = \frac{\pi}{2}, \quad A_{12} = \pi - \pi^2 i.
\end{align*}
\]

(B.4)
Similarly,

\[ A_2 \approx -\frac{e^{-\pi \sigma_1}}{2i} \int_{0}^{2\pi} \tau'[e^{i\tau'} - e^{i(-2\pi i)\tau'}]d\tau' \]

\[ = -\frac{e^{-\pi \sigma_1}}{2i} \left[ \frac{e^{i\tau} - e^{i\tau}}{\pi^2} - \frac{e^{i\tau} - 1}{\pi^2} \right] \int_{0}^{2\pi} \tau \left( e^{i\tau} - e^{i(-2\pi i)\tau} \right) d\tau' \]

\[ \approx -\frac{e^{-\pi \sigma_1}}{2i} \left[ \frac{\tau \pi - 1\pi} {\pi^2} - \frac{\tau(-2\pi i) - 1\pi} {\pi^2} \right] \int_{0}^{2\pi} \tau \left( e^{i\tau} - e^{i(-2\pi i)\tau} \right) d\tau' \]

\[ = -\frac{e^{-\pi \sigma_1}}{2i} \left[ \frac{(\pi \sigma_1 - 1)\pi + 1}{\pi^2} - \frac{(-2\pi i + 1)\pi}{\pi^2} \right] \]

\[ \approx -\frac{e^{-\pi \sigma_1}}{2i} \left[ \frac{(\pi \sigma_1 - 1)(\pi + 1\pi + 1/2)} {\pi^2} - \frac{(-2\pi i + 1)\pi}{\pi^2} \right] \]

\[ \approx -\frac{e^{-\pi \sigma_1}}{2i} \left[ \frac{2\pi^2 - \pi \sigma_1 + 2/3}{\pi^3} - \frac{(-2\pi i - 2\pi^2)\pi}{\pi^2} \right] \]

\[ = -\frac{e^{-\pi \sigma_1}}{2i} \left[ \frac{\pi^2 + \pi}{\pi^2} - \sigma_1 \left[ \frac{2}{3} \pi^3 - \frac{\pi}{4} + \frac{\pi^2}{2} \right] \right] \]

\[ = \pi^2 + \frac{\pi}{2} - \sigma_1 \left[ \frac{2}{3} \pi^3 - \frac{\pi}{4} + \frac{\pi^2}{2} \right] = A_21 + iA_{22}, \quad A_{21} = \frac{\pi}{2}, \quad A_{22} = \pi^2 - \sigma_1 \left[ \frac{2}{3} \pi^3 + \frac{\pi}{4} \right]. \] (B.5)

Hence,

\[ A_3 = \text{Re} A_2 \approx \pi^2 - \sigma_1 \left[ \frac{2}{3} \pi^3 + \frac{\pi}{4} \right]. \] (B.6)

Finally,

\[
B_1 = -i \int_{0}^{2\pi} e^{-i\tau'} E(2\pi - \tau') d\tau' = -iA_1 \\
B_2 = \int_{0}^{2\pi} [e^{-i\tau'} - i\tau' e^{-i\tau'}] E(2\pi - \tau') d\tau' = A_1 - iA_2 \] \quad (B.7)

\[
B_3 = \text{Re} B_2 = A_{11} + A_{22}
\]
[7] G. M. Coclite, V. Georgiev, Solitary waves for Maxwell–Schrödinger equations, Electronic Journal of Differential Equations 94 (2004), 1–31. arXiv:math/0303142 [math.AP]

[8] Ju.L. Daleckii, M.G. Krein, Stability of Solutions of Differential Equations in Banach Space, Amer Mathematical Society, Providence, Rhode Island, 1974. ISBN-10. 0821832387.

[9] R.H. Dicke, Coherence in Spontaneous Radiation Processes, Phys. Rev. 93 (1954), no. 1, 99–110.

[10] J. Demaison, W. Hütter, Dipole Moments and Related Constants of Diamagnetic Molecules, Springer, Berlin, 1992.

[11] A. Einstein, Quantentheorie der Strahlung (On the quantum theory of radiation) Phys. Z. 18 (1917), 121–128.

[12] B. Garraway, The Dicke model in quantum optics: Dicke model revisited, Phil. Trans. R. Soc. A 369 (2011), 1137–1155.

[13] Y. Guo, K. Nakamitsu, W. Strauss, Global finite-energy solutions of the Maxwell–Schrödinger system, Comm. Math. Phys. 170 (1995), no. 1, 181–196.

[14] H. Haken, Laser Theory, Springer, Berlin, 1984.

[15] K. Hepp, E.H. Lieb, On the superradiant phase transition for molecules in a quantised radiation field: The Dicke Maser model, Ann. Physics 76 (1973), 360–404.

[16] K. Hepp, E.H. Lieb, Phase transitions in reservoir-driven open systems with applications to lasers and superconductors, Helv. Phys. Acta 46 (1973), 573–603.

[17] K. Hepp, E.H. Lieb, Equilibrium statistical mechanics of matter interacting with the quantised radiation field, Phys. Rev. A (3) 8 (1973), 2517–2525.

[18] K. Hepp, E.H. Lieb, The laser: a reversible quantum dynamical system with irreversible classical macroscopic motion. Dynamical systems, theory and applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974), pp. 178–207 in: Lecture Notes in Phys., Vol. 38, Springer, Berlin, 1975.

[19] K. Hepp, E.H. Lieb, Laser models, pp 145-234 in: E. H. Lieb, Condensed Matter Physics and Exactly Soluble Models, Springer, Berlin, 2004.

[20] R. D. Jackson, Classical Electrodynamics, Wiley, New York, 1999.

[21] I. A. Ibragimov, Ju. V. Linnik, Independent and Stationary Sequences of Random Variables, Wolters-Noordhoff, Groningen, 1971.

[22] A. Komech, Quantum Mechanics: Genesis and Achievements, Springer, Dordrecht, 2013.

[23] A.I. Komech, Lectures on Quantum Mechanics and Attractors, World Scientific, Singapore, 2022.

[24] A.I. Komech, On quantum jumps and attractors of the Maxwell–Schrödinger equations, Annales mathematiques du Québec 46 (2022), 139–159. arXiv 1907.04297.

[25] A. Komech, E. Kopylova, Attractors of Hamiltonian Nonlinear Partial Differential Equations, Cambridge University Press, Cambridge, 2021.

[26] T. Letendre, H. Ueberschär, Random moments for the new eigenfunctions of point scatterers on rectangular flat tori, Annales Henri Poincaré, In press, ff10.1007/s00023-021-01032-5 ffhal02308180v2

[27] P. Mandel, Theoretical Problems in Cavity Nonlinear Optics, Cambridge University Press, Cambridge, 1997.

[28] H. Nussenzveig, Introduction to Quantum Optics, Gordon and Breach, London, 1973.

[29] K. Petersen, J. P. Solovej, Existence of travelling wave solutions to the Maxwell–Pauli and Maxwell–Schrödinger systems, Archive: arXiv:1402.3936.
[30] A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization. A Universal Concept in Nonlinear Sciences, Cambridge University Press, Cambridge, 2001.

[31] I.I. Rabi, Space quantization in a gyrating magnetic field, Phys. Rev. 51 (1937), 652–654.

[32] J.J. Sakurai, Advanced Quantum Mechanics, Pearson Education, Incorporated, 2006.

[33] M. Sargent III, M.O. Scully, W.E. Lamb Jr, Laser Physics, Addison Wesley, Reading, 1978.

[34] P. Sarnak, Recent progress on the quantum unique ergodicity conjecture, Bull. AMS 48, no. 2, 211–228.

[35] L.I. Schiff, Quantum Mechanics, McGraw-Hill, New York, 1955.

[36] M. O. Scully, M. S. Zubairy, Quantum Optics, Cambridge University Press, Cambridge, 1997.

[37] W.T. Silfvast, Laser Fundamentals, Cambridge University Press, Cambridge, 2012.

[38] A. Shimomura, Modified wave operators for Maxwell–Schrödinger equations in three space dimensions, Ann. Henri Poincaré 4 (2003), 661–683.

[39] A.I. Shnirelman, Ergodic properties of eigenfunctions, Usp. Mat. Nauk 29 (1974), no. 6, 181–182.

[40] A.I. Shnirelman, On the asymptotic properties of eigenfunctions in the region of chaotic motion, addendum to V.F.Lazutkin, KAM theory and semiclassical approximations to eigenfunctions, Springer 1993.

[41] V.A. Yakubovich, V.M. Starzhinskii, Linear Differential Equations with Periodic Coefficients, Wiley & Sons, New York, 1975.

[42] O. Svelto, Principles of Lasers, Springer, New York, 2010.

[43] G.K. Woodgate, Elementary Atomic Structure, Clarendon Press, Oxford, 2002.