Model independent results for the inflationary and reheating epochs

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We address the problem of determining inflationary characteristics in a model independent way and then study constraints for reheating. We start from a recently proposed equation which allows to accurately calculate the value of the inflaton at horizon crossing $\phi_k$. We then use an equivalent form of this equation to write a formula that relates the tensor-to-scalar index $r$ to the number of e-folds during inflation $N_k$, hence a general bound for $N_k$ follows. In particular, at present $r < 0.063$ implies $N_k < 56.3$. We also give an upper bound to the size of the universe, during the inflationary epoch, that gave rise to the current observable universe. The reheating epoch is discussed and a bound is given for the effective number of relativistic degrees of freedom $g_{re}$ which translates into a bound for the reheating temperature. From here bounds for the number of e-folds during reheating and also during the radiation dominated epoch follow. A criteria to know whether the constraint for the effective number of degrees of freedom exists is given in terms of the ratio $V_e/V_k$ where $V_e$ is the potential at the end of inflation and $V_k$ at the horizon crossing scale $k$. Finally we study two particular models: Starobinsky model, which was studied before and is mostly used here for comparison, and a Mutated Hilltop Inflation (MHI) model. Tables II and III show results for the two specific models of inflation.

I. INTRODUCTION

During the last several years we have seen an extraordinary advance in our knowledge of the universe, its composition, geometry and evolution. The idea of an inflationary universe remains solid some 40 years after its inception [1], [2], (for reviews see e.g., [3], [4], [5]), however the existence of a plethora of models [6] constantly reminds us that our knowledge of that epoch is imprecise, and even more so when we consider the time of reheating after inflation ends, for reviews on reheating see e.g., [7], [8], [9]. Numerous works have been done in our attempt to better understand the reheating era with varying degrees of success [10] - [23]. In this work, we initially address the problem of determining important inflationary characteristics in a model independent way and then study possible constraints for reheating.

The organization of the article is as follows: in Section I we first start from a recently proposed equation [24] which allows us to accurately calculate the value of the inflaton at horizon crossing $\phi_k$. We then use an equivalent form of this equation to write a formula that relates the tensor-to-scalar index $r$ to the number of e-folds during inflation $N_k$, hence a general bound for $N_k$ follows. In particular, at present $r < 0.063$ implies $N_k < 56.3$. We end the section by calculating a bound to the size of the universe, during the inflationary epoch, that gave rise to the current observable universe. In Section II we discuss the reheating epoch and a bound is given for the effective number of relativistic degrees of freedom $g_{re}$ which translates into a bound for the reheating temperature [25] and bounds for the number of e-folds during reheating and also during the radiation dominated epoch. The constraint is given by Eq. (20) as a bound for the ratio $V_e/V_k$ where $V_e$ is the potential at the end of inflation and $V_k$ at the horizon crossing scale $k$. In Section III we study two particular models: Starobinsky model, which was studied before [24] and is used now to compare with a Mutated Hilltop Inflation (MHI) model [33], [34] where several interesting aspects occur. Tables II and III show results for the two specific models of inflation. Finally in Section IV we give our conclusions on the most important points discussed in the article.

II. RESULTS FOR THE INFLATIONARY EPOCH

The equation which determines the inflaton field $\phi$ at horizon crossing is [24]

$$\ln \left( \frac{a_\phi H_k}{k_*} \right) = 2N_k , \quad (1)$$

where $k$ is the horizon scale during inflation $k_*$ is the pivot scale and $a_\phi$ is the corresponding scale factor. The Hubble function at $k$ is given by $H_k = \sqrt{8\pi^2\epsilon_k A_\phi}$ and $N_k \equiv \ln \frac{a_k}{a_\phi}$ is the number of e-folds from $\phi_k$ up to the end of inflation at $\phi_*$. Notice that the Hubble function introduces the scalar power spectrum amplitude given here by $A_\phi$. Eq. (1) is a model independent equation although its solution for $\phi_k$ requires specifying a model of inflation; $H_k$ and $N_k$ are model dependent quantities. Thus, after finding $\phi_k$, we can proceed to determine all inflationary parameters and observables.

An easy way to understand Eq. (1) is by connecting the epoch where the scale $k$ left the horizon during inflation to the pivot scale $k_*$ where we measure the horizon reentry of precisely the same scale $k$ thus, $k = k_*$. This can be expressed by

$$\ln \left( \frac{a_*}{a_k} \right) = 2N_k , \quad (2)$$
where $2N_k$ is the number of e-folds from the scale $k$ up to the end of inflation plus the number of e-folds from the end of inflation up to the scale $k_*$, which is also equal to $N_k$. Multiplying the term inside the parenthesis above and below by $H_k$ and setting $a_k H_k \equiv k = k_* \equiv a_* H_k$, we get Eq. (1). An alternative but equivalent way of obtaining Eq. (1) is given in [24]. To find the value of $a_*$ we solve the Friedmann equation for $a_*$

$$k_* = H_0 \sqrt{\frac{\Omega_{md,0}}{a_*} + \frac{\Omega_{cd,0}}{a_*^2} + \Omega_{de} a_*^2},$$

(3)

where $k_* = 0.05/Mpc \approx 1.3105 \times 10^{-58}$ (see Table I to find the numerical values of the other parameters used in our calculations).

Note also that Eq. (1) incorporates knowledge from the present universe, in the determination of $a_*$, of the early universe, when considering the scale $k$ during inflation, and also of the CMB epoch by the presence of the scalar power spectrum amplitude $A_s$ through $H_k$.

From Eq. (1) and $H_k = \sqrt{8\pi^2 k_4 A_s}$ we can get an expression for $r \equiv 16k_4$ in terms of the number of e-folds $N_k$

$$r = \frac{2k_4^2}{\pi^2 a_4^2 A_s} e^{4N_k}.$$

(4)

Imposing a bound $b$ to $r$ we get a general bound for $N_k$

$$r < b \Rightarrow N_k < \frac{1}{4} \ln \left( \pi^2 a_4^2 A_s b \right) \approx 57.016 + \frac{1}{4} \ln b,$$

(5)

for the particular value $b = 0.063$ [26, 27] we get the present bound for $N_k$

$$r < 0.063 \Rightarrow N_k < 56.3.$$

(6)

This is a model independent result, it follows from Eq. (1), phenomenological parameters and the bound for $r$ without specifying any model of inflation.

We can also calculate a model independent bound to the size of the patch of the universe from which our present observable universe originates. We adapt Eq. (2) to this situation

$$\ln \left( \frac{a_0}{a_k} \right) = 2N_k,$$

(7)

where $a_0$ is, as usual, the present scale factor $a_0 = 1$, $k$ is the horizon scale during inflation which gave rise to our observable universe such that $k = k_0$ ($k_0 \equiv a_0 H_0$ is the present scale) and $2N_k$ is the number of e-folds from the scale $k$ up to the end of inflation plus the number of e-folds from the end of inflation up to the scale $k_0$ which is also equal to $N_k$. Note that the $k$ in Eq. (7) has not the same value as the $k$ in Eq. (2). From Eq. (1) and from the bound for $N_k$ follows that at the scale $k$

$$k = a_0 e^{-2N_k} > a_0 e^{-132.7} \approx 2.34 \times 10^{-58}. \quad (8)$$

Note that we have added $N_e \equiv \ln \frac{a_0}{a_k} \approx 10.05$ e-folds to the upper bound of 56.3 for $N_k$ because there are 10.05 e-folds from the pivot scale $k_* = 0.05/Mpc$ up to the present scale $k_0$. If the diameter of the observable universe is $8.8 \times 10^{26}m$ then at the scale $k$ the size of the universe from which ours originates was bigger than $2.059 \times 10^{-31}m$. Thus, at the scale $k$ the universe diameter was at least $1.274 \times 10^4$ times bigger than the Planck length. At the end of inflation it had a size of at least $1.35 cm$.

III. RESULTS FOR THE REHEATING EPOCH

To establish constraints for the reheating epoch we need in particular a formula for the number of e-folds during reheating. The standard way to proceed is to solve the fluid equation with the assumption of a constant equation of state parameter $\omega$ during reheating, this gives the number of e-folds during reheating in terms of the energy densities as follows

$$N_{re} \equiv \ln \frac{a_{re}}{a_0} = \left[3(1 + \omega)\right]^{-1} \ln \left( \frac{\rho_{re}}{\rho_{re}} \right), \quad (9)$$

where $\rho_e$ is the energy density at the end of inflation and $\rho_{re}$ the energy density at the end of reheating

$$\rho_{re} = \frac{\pi^2 g_{re} T_{re}^4}{30}, \quad (10)$$

with $g_{re}$ the number of degrees of freedom of species at the end of reheating. To proceed we assume entropy conservation after reheating, this assumption establish another expression involving $T_{re}$ which can be substituted in Eq. (10) and then in Eq. (9)

$$g_{s, re} T_{re}^3 = \left( \frac{a_0}{a_{eq}} \right) \left( \frac{a_{eq}}{a_{re}} \right) \left( 2T_0^3 + \frac{7}{8} T_{nu,0}^3 \right), \quad (11)$$

where $g_{s, re}$ is the number of degrees of freedom of species after reheating, $T_0 = 2.725K$ and the neutrino temperature is $T_{nu,0} = (4/11)^{1/3}T_0$. The number of e-folds during radiation domination $N_{rd} \equiv \ln \frac{a_{re}}{a_0}$ follows from Eqs. (9) and (11)

$$N_{rd} = \frac{3(1 + \omega)}{4} N_{re} + \frac{1}{4} \ln \left( \frac{30}{g_{re} \pi^2} \right) + \frac{1}{3} \ln \left( \frac{11 g_{re}}{43} \right) + \ln \left( \frac{a_{eq} \rho_{re}^{1/4}}{a_0 T_0} \right). \quad (12)$$
TABLE I. For easy reference this table collects numerical values of parameters used in the paper. Dimensionless quantities have been obtained by working in Planck mass units, where $M_{pl} = 2.4357 \times 10^{18}$ GeV and set $M_{pl} = 1$, the pivot scale $k_\ast \equiv a_\ast H_\ast = 0.05 \frac{1}{\pi} p_c$, used in particular by the Planck collaboration, becomes a dimensionless number given by $k_\ast \approx 1.3105 \times 10^{-58}$. This can be compared with $k_0 \equiv a_0 H_0 \approx 8.7426 \times 10^{-61} h$. To calculate $a_\ast$, we have to specify $h$ for the Hubble parameter $H_0$ at the present time. We take the value given by Planck $h = 0.67$ for definitiveness and check that no important changes occur for $N_\ast \equiv \ln \frac{a_{\ast}}{a_\ast} = h$ in the interval $0.67 < h < 0.73$. The solution of Eq. (3) for $a_\ast$ is $a_\ast = 4.3000 \times 10^{-5}$ from where we get $N_\ast = 10.05$ for the number of e-folds from $a_\ast$ to $a_0$.

| Parameter | usually given as | Dimensionless |
|-----------|------------------|--------------|
| $H_0$    | $100 h_0 \frac{Mpc}{c}$ | $8.7426 \times 10^{-61} h$ |
| $T_0$    | $2.725 K$ | $9.6235 \times 10^{-32}$ |
| $A_s$    | $2.0968 \times 10^{-9}$ | $2.0968 \times 10^{-9}$ |
| $k_\ast$ | $0.05/Mpc$ | $1.3105 \times 10^{-58}$ |
| $a_\ast$ | $-$ | $4.3000 \times 10^{-5}$ |
| $\Omega_{md,0}$ | $0.315$ | $0.315$ |
| $\Omega_{de,0}$ | $7.9 \times 10^{-5}$ | $7.9 \times 10^{-5}$ |
| $\Omega_{de}$ | $0.685$ | $0.685$ |

We can finally obtain an expression for the number of e-folds during reheating $N_{re}$ by combining Eqs. (1) and (12), the result is

$$N_{re} = \frac{4}{1-3\omega} \left( N_k - \frac{1}{3} \ln \left[ \frac{11g_{s, re}}{43} \right] - \frac{1}{4} \ln \left[ \frac{30}{\pi^2 g_{re}} \right] - \ln \left[ \frac{a_\ast p_c^{1/4}}{a_0 T_0} \right] \right).$$

(13)

From this equation we can study the dependence of the degrees of freedom $g_{re}$ on $N_{re}$ and $\omega$. For as long as species have the same temperature and $p \approx \frac{1}{3} \rho$ we have that $g_{s, re} \approx g_{re}$. Thus, we set $g_{s, re} = g_{re}$ in Eq. (13) and proceed to solve for $g_{re}$, the result is

$$g_{re} = g_{re} (\phi_k) e^{-3(1-3\omega)N_{re}},$$

(14)

where

$$g_{re} (\phi_k) = \left( \frac{43}{11} \right)^4 \left( \frac{\pi^2}{30} \right)^3 \frac{H_k}{e^{N_k} p_c^{1/4} k_\ast} \left( \frac{a_0 T_0}{k_\ast} \right)^{12}. \quad (15)$$

The quantity $g_{re} (\phi_k)$ essentially depends on $\phi_k$. Numerical studies of the thermalization phase during reheating suggest that $0 \lesssim \omega \lesssim 0.25$ [28], here we extended our discussion up to $\omega \approx 1/3$. To extract a constraint for the reheating temperature we observe that for $\omega \lesssim \frac{1}{3}$ the exponential in Eq. (14) is always less than one. Thus, $g_{re} (\phi_k)$ gives the largest possible value of $g_{re}$

$$g_{re} \lesssim g_{re} (\phi_k).$$

(16)

In certain models $g_{re} (\phi_k)$ and, as a consequence, $g_{re}$ is less than the number of species of the Standard Model of Particles (106.75). If this is the case [25] then a constraint on the reheating temperature $T_{re}$ follows immediately. We have shown before [24] that this is indeed the case for the Starobinsky model and in the next section we show that it is also the case for a mutated hilltop inflation model.

Let us write in more detail the condition under which a constraint is expected. At the end of inflation when $\omega = -1/3$ an expression for $\rho_e$ follows

$$\rho_e = \frac{3}{2} V_e = \frac{9}{2 V_e} H_k^2. \quad (17)$$

Together with Eq. (1), Eq. (15) can then be written in a form containing only known quantities with the exception of the last factor involving the ratio of the potential at the scale $k$ and at the end of inflation

$$g_{re} (\phi_k) = \left( \frac{43}{11} \right)^4 \left( \frac{2\pi^2}{270} \right)^3 \left( \frac{a_0 T_0}{\sqrt{a_\ast k_\ast}} \right)^{12} \left( \frac{V_k}{V_e} \right)^3, \quad (18)$$

taking all the numbers from the Table I we get

$$g_{re} (\phi_k) \approx 1.7981 \left( \frac{V_k}{V_e} \right)^3 < 106.75, \quad (19)$$
where we have written in the r.h.s. of Eq. (19) the number of degrees of freedom of the Standard Model of Particles as an upper limit for $g_{re}(\phi_k)$. When $g_{re}(\phi_k) > 106.75$ no restriction for the reheating temperature coming from the number of degrees of freedom of species arises. In supersymmetric models this number should be twice as big. From the r.h.s. of Eq. (19) and the obvious requirement $V_e < V_k$ follows that

$$0.2563 < \left(\frac{V_e}{V_k}\right) < 1. \quad (20)$$

Whenever $V_e/V_k$ is within these limiting values a constraint on the reheating temperature arises. Notice that all these equations and bounds are model independent although their solutions require specifying a model of inflation.

IV. THE STAROBINSKY AND MUTATED HILLTOP INFLATION MODELS

The Starobinsky model revisited. – The potential of the Starobinsky model [29–31] is given by [32]:

$$V = V_0 \left( 1 - e^{-\sqrt{2} \phi} \right)^2, \quad (21)$$

with Hubble function

$$H_k = \sqrt{8\pi^2 e_k A_s} = \sqrt{32 A_s \frac{\pi}{3 e\sqrt{2}\phi_k - 1}}, \quad (22)$$

where $e_k$ is the slow-roll parameter $\epsilon \equiv \frac{1}{2} \left( \frac{V'}{V} \right)^2$ at $\phi = \phi_k$. The number of e-folds $N_k$ follows easily

$$N_k = -\int_{\phi_e}^{\phi_k} d\phi = \frac{1}{2} \left( 3e\sqrt{2}\phi_k - \sqrt{5}\phi_k \right) - \frac{1}{2} \left( 3e\sqrt{2}\phi_k - \sqrt{5}\phi_k \right), \quad (23)$$

where $\phi_e$ denotes the end of inflation: $\phi_e = \sqrt{\frac{2}{5}} \ln \left( 1 + \frac{2}{\sqrt{5}} \right)$. We can solve Eq. (1) for $\phi_k$ with the result [24]

$$\phi_k = 5.365, \quad (24)$$

from here all inflationary parameters and observables follow. Notice that in the Starobinsky model there are no further parameters apart from the overall $V_0$ (which is fixed by the scalar amplitude) and we can obtain precise values for all the quantities of interest during inflation. Thus, $N_k = 55.6$ and we can also determine the size of the primordial universe according to Starobinsky model. The scale factor $a_k$ is

$$a_k = a_0 e^{-2N_k} = a_0 e^{-131.3} \approx 9.49 \times 10^{-58}. \quad (25)$$

Following the previous general discussion around Eq. (8) we have now that at the scale $k$ the size of the patch of the universe from which ours originates was $8.35 \times 10^{-31} m$. At the end of inflation it had a diameter size of $2.71 \text{ cm}$.

![Schematic plot of the mutated hilltop potential given by Eq. (26) as a function of $\phi$ for an inflaton field rolling from the right. Characteristics of inflation and reheating for this model are given in Tables II and III.](image)

FIG. 1.

On the other hand, the ratio $\left( \frac{V_e}{V_k} \right) \approx 0.2945$ is well inside the limiting values given by Eq. (20) and thus, a constraint follows for the number of degrees of freedom which translates into a constraint for the temperature at the end of reheating, from Eq. (15) or Eq. (19) it follows that $g_{re} \approx 70.39$.

In models where there is at least one free parameter (different from an overall parameter which is fixed by the scalar amplitude), one can investigate whether there is a range of values of the parameter such that $\phi_k$ gets closer to $\phi_e$ in such a way that the ratio $V_e/V_k$ falls within the limits of Eq. (20). Of course, the value of the parameter should be consistent with the other requirements of inflation, in particular a tensor-to-scalar ratio $r$ and spectral index $n_s$ within the bounds for these observables. We discuss this possibility with a model of mutated hilltop inflation.

Mutated Hilltop Inflation model. – This model is given by the potential [33, 34]

$$V = V_0 \left( 1 - \sech \left( \frac{\phi}{\mu} \right) \right). \quad (26)$$

The number of e-folds $N_k$ can be calculated in closed form with the result

$$N_k = \mu^2 \left( 2 \ln \frac{\cosh \left( \frac{\phi_k}{\mu} \right)}{\cosh \left( \frac{\phi_e}{\mu} \right)} \right) + \cosh \left( \frac{\phi_k}{\mu} \right) - \cosh \left( \frac{\phi_e}{\mu} \right). \quad (27)$$

The field at the end of inflation $\phi_e$ is given by the solution to the condition $\epsilon = 1$. The solution is very involved and is given by
for discussion following Eqs. (19) and (20). The lower bound for corresponds to the bound on the approximate mass of the particle when annihilation begins. The value of the parameter of on this strategy. Once we have the values of There are, of course, some others equally valid variations for two unknowns: △φ for the model g re (△φ) can be easily obtained by solving Eq. (4) although written in terms of r rather than H k. We solve the equation r = 16ε k = b, where b is an upper bound on r (e.g., b = 0.063 at present). With this solution for φ, let us say \( φ_k(μ, b) \)

\[
φ_k = φ_k(μ, b),
\]

we solve the equation \( N_k = N_k(b) \) where \( N_k(b) \) is the value of \( N_k \) in Eq. (1) at the bound (e.g., \( N_k(0.063) = 56.3 \)). In conclusion, we form a system of two equations for two unknowns: \( φ_k \) and \( μ \), consistently with Eq. (1). There are, of course, some others equally valid variations on this strategy. Once we have the values of \( φ_k \) and \( μ \) at the bound we can investigate from there the behaviour of the solution for various values of the parameter \( μ \). What we find in this particular model is that at the bound \( b = 0.063, ρ = 9.8107 \) with \( φ_k = 12.8844 \). Smaller values of \( μ \) go in the right direction: lowering the value of ρ and, in this case, also lowering the spectral index \( n_s \). Due to Eq. (4) \( N_k \) follows the behaviour of \( r \), also diminishing as \( r \) diminish (see Table II).

From the Table II we see in row number four the case \( μ = 1 \) where \( g_{re} \) satisfies the bounds in Eq. (20)

\[
g_{re}(4.791) = 40.10.
\]

Rows numbers three and five correspond to the limiting values 106.75 and 10.75 for the effective number of degrees of freedom. The first limit saturates the degrees of freedom for Standard Model of Particles and the second corresponds to an approximated lower bound for the reheating temperature of 10 MeV coming from nucleosynthesis considerations although a lower \( T_{re} \) for the lower bound has been discussed in the literature [35]. The solution for \( φ_k(μ) \) can be easily obtained by solving Eq. (19) in the two limiting situations

\[
V_e = \alpha (g_{re}(φ_k)) V_e,
\]

where \( \alpha \approx 3.9011 \) when taking \( g_{re}(φ_k) = 106.75 \) and \( \alpha \approx 1.8149 \) when \( g_{re}(φ_k) = 10.75 \). After having determined

| Model    | \( φ_k \)   | \( r \)   | \( n_s \) | \( α \)  | \( N_k \) | \( g_{re}(φ_k) \) | \( T_{re}(GeV) \) |
|----------|--------------|-----------|-----------|---------|----------|------------------|------------------|
| Starobinsky | 5.3653       | 0.00343   | 0.96534   | -6.81 \times 10^{-4} | 55.6 | 70.39         | 0.57           |
| MHI \( μ = 9.8107 \) | 12.8844 | 0.06300   | 0.96599   | -6.16 \times 10^{-4} | 56.3 | 106.75        | -              |
| MHI \( μ = 1.4118 \) | 5.8665 | 0.00420   | 0.96601   | -5.93 \times 10^{-4} | 55.6 | 106.75        | 10^4           |
| MHI \( μ = 1 \) | 4.7910 | 0.00228   | 0.96539   | -6.08 \times 10^{-4} | 55.5 | 40.10         | 0.18           |
| MHI \( μ = 0.5196 \) | 3.1405 | 0.00067   | 0.96444   | -6.35 \times 10^{-4} | 55.2 | 10.75         | 0.01           |

TABLE II. Characteristics of the inflationary and reheating epoch are given for the Starobinsky and Mutated Hilltop Inflation (MHI) models. The second column quotes the value of \( φ_k \) for each model, with an specified model dependent parameter in the case of MHI. From this value of \( φ_k \) follows all inflationary quantities. Three values of the parameter \( μ \) are given such that a constraint to the reheating temperature follow. The constraint follows if Eq. (19), equivalently Eq. (20), is satisfied.

| Model    | \( g_{re}(φ_k) \) | \( T_{re} \) | \( N_{re} \) | \( N_{rd} \) |
|----------|--------------------|-------------|-------------|-------------|
| Starobinsky | 70.39             | 10 MeV < \( T_{re} < 500 \) MeV | 40.84 > \( N_{re} > 36.30 \) | 16.69 < \( N_{rd} < 21.23 \) |
| MHI \( μ = 9.8107 \) | \( \geq 106.75 \) | 10 MeV < \( T_{re} \) | 41.57 > \( N_{re} \) | 16.69 < \( N_{rd} \) |
| MHI \( μ = 1.4118 \) | 106.75           | 10 MeV < \( T_{re} < 10^4 \) GeV | 40.89 > \( N_{re} > 28.61 \) | 16.69 < \( N_{rd} < 28.97 \) |
| MHI \( μ = 1 \) | 40.10             | 10 MeV < \( T_{re} < 180 \) MeV | 40.30 > \( N_{re} > 37.41 \) | 17.13 < \( N_{rd} < 20.02 \) |
| MHI \( μ = 0.5196 \) | 10.75             | 10 MeV     | 40.43       | 16.69       |

\[
\text{sech} \left( \frac{1}{2\sqrt{3}} \right) = \frac{1}{\sqrt{e^{1/2}}} \left( -8\rho^6 + 4\rho^4(-6 + 5 \times 2^{1/3} R_1^{1/3}) - 2\rho^2(9 + 2^{1/3}(-15 + 2^{1/3} R_1) R_2^{1/3}) + 4 \times 2^{1/3} \rho R_2^{2/3} + 3 \times 2^{1/3} R_1 R_2^{2/3} + 2 R_1 R_2^{3/2} \right) .
\]

where \( R_1 = 2^{1/6} \mu \sqrt{12 \mu^2 + 66 \mu^2 - 3} \) and \( R_2 = \mu^3(4\mu(9 + \mu^2) + 3\sqrt{6} \rho^4 + 22 \rho^2 - 1) \). We cannot solve in general Eq. (1) for \( φ_k \) and arbitrary \( μ \); thus we resort to the following strategy: Eq. (4) is equivalent to Eq. (1) although written in terms of \( r \) rather than \( H_k \). We solve the equation \( r = 16ε_k = b \), where \( b \) is an upper bound on \( r \) (e.g., \( b = 0.063 \) at present).
\(\phi_k(\mu)\), Eq. (1) gives the solution for \(\mu\). In Table III we summarize our results for this section.

V. CONCLUSIONS

We have studied model independent results for the inflationary and reheating epochs following from the formulas given by Eqs. (1) and (20). We have, in particular, establish an equation (Eq. (3)) for the tensor-to-scalar ratio in terms of the number of e-folds \(N_k\equiv \ln \frac{a_k}{a_0}\) during inflation. From a bound \(b\) for \(r\) follows a general bound for \(N_k\) (Eq. (4)) which at present is \(r < 0.063\) implying \(N_k < 56.3\). These are all model independent results in the sense that no model of inflation is used to obtain them. At the end of Section II we also give a bound to the size of patch of the universe from where our observable universe comes from. Section III discusses our observable universe comes from. Section I discusses the observable universe comes from. Section II discusses the observable universe comes from. Section III discusses the observable universe comes from. Section IV discusses the observable universe comes from. Section V discusses inflationary reheating.

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