A SUPPORT PROPERTY FOR INFINITE DIMENSIONAL INTERACTING DIFFUSION PROCESSES.

UNE PROPRIÉTÉ DE SUPPORT POUR DES PROCESSUS DE DIFFUSION EN DIMENSION INFINIE AVEC INTERACTION.

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Abstract. The Dirichlet form associated with the intrinsic gradient on Poisson space is known to be quasi-regular on the complete metric space \( \tilde{\Gamma} = \{ \text{\(\mathbb{Z}^+\)}-valued Radon measures on \(\mathbb{R}^d\) \} \). We show that under mild conditions, the set \( \tilde{\Gamma} \setminus \Gamma \) is \( E \)-exceptional, where \( \Gamma \) is the space of locally finite configurations in \( \mathbb{R}^d \), that is, measures \( \gamma \in \tilde{\Gamma} \) satisfying \( \sup_{x \in \mathbb{R}^d} \gamma(\{x\}) \leq 1 \). Thus, the associated diffusion lives on the smaller space \( \Gamma \). This result also holds for Gibbs measures with superstable interactions.

Résumé. Il est connu que la forme de Dirichlet associée au gradient intrinsèque sur l’espace de Poisson est quasi-régulière sur l’espace métrique complet \( \tilde{\Gamma} = \{ \text{mesures de Radon sur } \mathbb{R}^d \text{ à valeurs dans } \mathbb{Z}^+ \} \). Sous des conditions faibles, on montre que l’ensemble \( \tilde{\Gamma} \setminus \Gamma \) est \( E \)-exceptionnel, où \( \Gamma \) désigne l’ensemble de toutes les configurations localement finies dans \( \mathbb{R}^d \), c’est-à-dire, les mesures \( \gamma \in \tilde{\Gamma} \) qui vérifient \( \sup_{x \in \mathbb{R}^d} \gamma(\{x\}) \leq 1 \). La diffusion associée prend donc ses valeurs dans l’espace des phases \( \Gamma \). Ce résultat est valable même si \( \mu \) est une mesure de Gibbs associée à un potentiel superstable.

Version française abrégée

Soit \( \Gamma \) l’ensemble de toutes les configurations localement finies dans \( \mathbb{R}^d \). \( \Gamma \) est vu comme un sous-ensemble de l’espace des mesures de Radon sur \( \mathbb{R}^d \) muni de la topologie vague. Soit \( \sigma(dx) = \rho(x)m(dx) \), \( m \) étant la mesure de Lebesgue sur \( \mathbb{R}^d \) et \( \rho^{1/2} \in H_{loc}^{1,2}(\mathbb{R}^d) \), \( \rho(x) > 0 \) m-presque tout \( x \in \mathbb{R}^d \). Soit \( \pi_\sigma \) la mesure de Poisson d’intensité \( \sigma \) sur \( \Gamma \). On appellera mesure de Poisson mixte toute mesure \( \mu \) de la forme \( \mu := \int_{\mathbb{R}^d} \pi_\sigma \lambda(dz) \). Définissons l’espace des fonctions cylindriques lisses et bornées

\[ \mathcal{FC}_b^\infty := \{ u \mid u(\gamma) = g(\langle f_1, \gamma \rangle, \langle f_2, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle) \} , \]

où \( f_i \in C_0^\infty(\mathbb{R}^d) \), \( g \in C_b^\infty(\mathbb{R}^n) \), et \( \langle f, \gamma \rangle = \sum_{x \in \gamma} f(x) \). D’après Albeverio, Kon- dratiev, et Röckner nous définissons le gradient intrinsèque d’une fonction \( u \in \mathcal{FC}_b^\infty \) par

\[ (\nabla^\Gamma u)(\gamma; x) := \sum_{i=1}^n \frac{\partial g}{\partial x_i}(\langle f_1, \gamma \rangle, \langle f_2, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle) \nabla f_i(x) . \]

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Donc on a l’opérateur carré du champ
\[ \Gamma(u, v)(\gamma) := \int_{\mathbb{R}^d} \langle (\nabla_\Gamma u)(\gamma; x), (\nabla_\Gamma v)(\gamma; x) \rangle_{\mathbb{R}^d} \gamma(dx). \]

La forme de Dirichlet engendrée par cet opérateur carré du champ est la fermeture \((\mathcal{E}, D(\mathcal{E}))\) de la forme \((\mathcal{E}, \mathcal{F}C^\infty_b)\) définie pour \(u, v \in \mathcal{F}C^\infty_b\) par
\[ \mathcal{E}(u, v) := \int_{\mathbb{R}^d} \Gamma(u, v)(\gamma) \mu(d\gamma). \]

On sait que cette forme est quasi-régulière sur l’espace des phases complet \(\tilde{\Gamma}(\supset \Gamma)\). On va prouver ici que \((\mathcal{E}, D(\mathcal{E}))\) est quasi-régulière sur \(\Gamma\); pour cela il est suffisant de voir que l’ensemble \(\tilde{\Gamma} \setminus \Gamma\) est \(\mathcal{E}\)-exceptionnel. On a le résultat suivant:

**PROPOSITION 1.** Si \(d \geq 2\), \(\rho \in L^2_{\text{loc}}(dx)\), et \(\int_{\mathbb{R}^+} z^2 \lambda(dz) < \infty\), l’ensemble \(\tilde{\Gamma} \setminus \Gamma\) est \(\mathcal{E}\)-exceptionnel.

Ce résultat est valable même si \(\mu\) est une mesure de Gibbs associée à un potentiel superstable au sens de Ruelle, puisque l’existence des points avec masse \(\geq 2\) est un événement local et une telle mesure de Gibbs est localement absolument continue par rapport à une mesure de Poisson.

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Let \(\Gamma\) be the space of locally finite configurations in \(\mathbb{R}^d\),

\[ \Gamma := \{ \gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty \text{ for every compact } K \}. \]

A configuration \(\gamma\) will be identified with the Radon measure \(\sum_{x \in \gamma} \varepsilon_x\). The space \(\Gamma\) will be given the topology of vague convergence of measures, and measures on \(\Gamma\) are defined on the corresponding Borel sets. Define

\[ \mathcal{F}C^\infty_b := \{ u \mid \gamma \mapsto u(\gamma) = g(\langle f_1, \gamma \rangle, \langle f_2, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle) \text{ for some } f_i \in C^\infty_0(\mathbb{R}^d) \text{ and } g \in C^\infty_b(\mathbb{R}^n) \}. \]

(2)

where \(\langle f, \gamma \rangle = \sum_{x \in \gamma} f(x)\).

For \(u \in \mathcal{F}C^\infty_b\), we define the gradient \(\nabla_\Gamma u\) at the point \(\gamma \in \Gamma\) as an element of the “tangent space” \(T_\Gamma(\Gamma) := L^2(\mathbb{R}^d \to \mathbb{R}^d; \gamma)\) by

\[ x \mapsto (\nabla_\Gamma u)(\gamma; x) := \sum_{i=1}^n \partial g \left( \langle f_1, \gamma \rangle, \langle f_2, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle \right) \nabla f_i(x). \]

(3)

Here \(\nabla\) refers to the usual gradient on \(\mathbb{R}^d\). It is not hard to prove that \(\nabla_\Gamma u\) is well-defined, even though the representation of \(u\) as a cylinder function is not unique.

**DEFINITION 1.** For \(u, v \in \mathcal{F}C^\infty_b\) define the square field operator \(\Gamma\) as the real-valued function on \(\Gamma\) given by

\[ \gamma \mapsto \Gamma(u, v)(\gamma) := \langle \nabla_\Gamma u, \nabla_\Gamma v \rangle_{T_\Gamma(\Gamma)} \]

(4)

\[ = \int_{\mathbb{R}^d} \langle (\nabla_\Gamma u)(\gamma; x), (\nabla_\Gamma v)(\gamma; x) \rangle_{\mathbb{R}^d} \gamma(dx). \]
We will often use the abbreviation \( \Gamma(u) := \Gamma(u, u) \).

Let \( \sigma \) be a measure on \( \mathbb{R}^d \) that has a density \( \rho \) with respect to Lebesgue measure satisfying \( \rho > 0 \) almost everywhere, and \( \rho^{1/2} \in H_{\text{loc}}^{1,2}(\mathbb{R}^d) \). Here \( H_{\text{loc}}^{1,2}(\mathbb{R}^d) \) denotes the local Sobolev space of order 1 in \( L^2_{\text{loc}}(\mathbb{R}^d; dx) \). The Poisson measure \( \pi_\sigma \) with intensity measure \( \sigma \) is the probability measure on \( \Gamma \) characterized by
\[
\int_\Gamma \exp(\langle f, \gamma \rangle) \pi_\sigma(d\gamma) = \exp \left( \int_{\mathbb{R}^d} \left( e^{f(x)} - 1 \right) \sigma(dx) \right),
\]
for \( f \in C_0(\mathbb{R}^d) \). A mixed Poisson measure is given by
\[
\mu := \int_{\mathbb{R}^+} \pi_{2\sigma} \lambda(dz),
\]
where \( \lambda \) is a probability measure such that \( \int_{\mathbb{R}^+} z \lambda(dz) < \infty \).

**Definition 2.** For \( u, v \in \mathcal{F}C_0^\infty \) we define the pre-Dirichlet form
\[
\mathcal{E}(u, v) := \int_\Gamma \Gamma(u, v)(\gamma) \mu(d\gamma).
\]

Albeverio, Kondratiev, and Röckner proved an integration by parts formula [3, Corollary 4.1 and Remark 4.3] which implies that the form \( (\mathcal{E}, \mathcal{F}C_0^\infty) \) is closable, and that its closure \( (\mathcal{E}, D(\mathcal{E})) \) is a symmetric, local, Dirichlet form. The quasi-regularity of \( (\mathcal{E}, D(\mathcal{E})) \) has been proven for certain cases by Yoshida [9], and in general by Ma and Röckner [4]. However, \( \Gamma \) is not completely metrizable with respect to the vague topology so it is necessary to use the Polish state space
\[
\tilde{\Gamma} := \{ Z_+-\text{valued Radon measures on } \mathbb{R}^d \}.
\]
Since \( \Gamma \subset \tilde{\Gamma} \) and \( \mathcal{B}(\tilde{\Gamma}) \cap \Gamma = \mathcal{B}(\Gamma) \), we can consider \( \mu \) as a measure on \( (\tilde{\Gamma}, \mathcal{B}(\tilde{\Gamma})) \) and correspondingly \( (\mathcal{E}, D(\mathcal{E})) \) as a Dirichlet form on \( L^2(\tilde{\Gamma}; \mu) \). The associated Markov process \( ((X_t)_{t \geq 0}, (P_\gamma)_{\gamma \in \Gamma}) \) has vaguely continuous sample paths since \( (\mathcal{E}, D(\mathcal{E})) \) is a local form (cf. [3, Chapter V, Theorem 1.11]).

We will use the following lemma from Dirichlet form theory.

**Lemma 1.** Let \( u_n \in D(\mathcal{E}) \) be continuous functions, \( \sup_n \mathcal{E}(u_n, u_n) < \infty \), and \( u_n \to u \) pointwise. Then \( u \) is an \( \mathcal{E} \)-quasi-continuous function, in particular, for \( \mu \)-almost every \( \gamma \in \tilde{\Gamma} \),
\[
P_\gamma (t \to u(X_t) \text{ is continuous}) = 1.
\]

For example, if \( 1_N \) is \( \mathcal{E} \)-quasi-continuous and \( \mu(N) = 0 \), then for \( \mu \)-almost every \( \gamma \in \tilde{\Gamma} \),
\[
P_\gamma (X_t \not\in N \text{ for all } 0 < t < \infty) = 1.
\]
Such a set \( N \) is called \( \mathcal{E} \)-exceptional.

**Proposition 1.** If \( d \geq 2 \), \( \rho \in L^1_{\text{loc}}(dx) \), and \( \int_{\mathbb{R}^+} z^2 \lambda(dz) < \infty \), then the set \( \tilde{\Gamma} \setminus \Gamma \) is \( \mathcal{E} \)-exceptional.
Proof. It suffices to prove the result locally, that is, to show that for every positive integer \(a\), the function \(u := 1_N\) is quasi-continuous, where

\[
N := \{\gamma : \text{sup}(\gamma(\{x\})) : x \in [-a, a]^d \geq 2\}.
\]

Our analysis begins with a smooth partition of \(\mathbb{R}^d\) into small pieces. Let \(\phi\) be a \(C_0^\infty(\mathbb{R})\) function satisfying \(1_{[0,1]} \leq \phi \leq 1_{[-1/2, 3/2]}\) and \(|\phi'| \leq 3 \times 1_{[-1/2, 3/2]}\), and for any \(n \in \mathbb{N}\) and \(i = (i_1, \ldots, i_d) \in \mathbb{Z}^d\), define a \(C_0^\infty(\mathbb{R}^d)\) function by

\[
\phi_i(x) := \prod_{k=1}^{d} \phi(nx_k - i_k).
\]

We also let \(I_i(x) := \prod_{k=1}^{d} 1_{[-1/2, 3/2]}(nx_k - i_k)\) and note that \(\phi_i \leq I_i\). Taking the \(j\)th partial derivative of \(\phi_i\) gives

\[
\partial_j \phi_i(x) = n\phi'(nx_j - i_j) \prod_{k \neq j} \phi(nx_k - i_k),
\]

and so \((\partial_j \phi_i(x))^2 \leq 9n^2 I_i(x)\). Adding over \(j\) from 1 to \(d\) gives us

\[
|\nabla \phi_i(x)|^2 \leq 9n^2 d I_i(x).
\]

Let \(\psi\) be a smooth function on \(\mathbb{R}\) satisfying \(1_{[2,\infty)} \leq \psi \leq 1_{[1,\infty)}\) and \(|\psi'| \leq 2 \times 1_{(1,\infty)}\). Choosing \(A := \mathbb{Z}^d \cap [-na, na]^d\), define a continuous element of \(D(\mathcal{E})\) by

\[
u_n(\gamma) := \psi \left( \text{sup}_{i \in A} \langle \phi_i, \gamma \rangle \right).
\]

Then \(u_n \rightarrow u\) pointwise as \(n \rightarrow \infty\), so to apply Lemma 1 we only need prove that \(\text{sup}_n \mathcal{E}(u_n, u_n) < \infty\). We begin by bounding \(\Gamma(u_n)\), the square field operator applied to \(u_n\). First note that

\[
\psi \left( \text{sup}_{i \in A} \langle \phi_i, \gamma \rangle \right)^2 \leq 4 \times 1_{\text{sup}_{i \in A} \langle \phi_i, \gamma \rangle > 1} \leq 4 \times 1_{\text{sup}_{i \in A} \langle I_i, \gamma \rangle \geq 2},
\]

where for the final inequality we use the fact that \(\langle I_i, \gamma \rangle\) is an integer. Therefore, first using the inequality \(\Gamma(u \vee v) \leq \Gamma(u) \vee \Gamma(v)\), and then using \((14)\) and \((16)\), we get

\[
\Gamma(u_n)(\gamma) = \left( \psi \left( \text{sup}_{i \in A} \langle \phi_i, \gamma \rangle \right) \right)^2 \Gamma(\text{sup}_{i \in A} \langle \phi_i, \cdot \rangle)(\gamma)
\]

\[
\leq \left( \psi \left( \text{sup}_{i \in A} \langle \phi_i, \gamma \rangle \right) \right)^2 \text{sup}_{i \in A} \Gamma(\langle \phi_i, \cdot \rangle)(\gamma)
\]

\[
= \left( \psi \left( \text{sup}_{i \in A} \langle \phi_i, \gamma \rangle \right) \right)^2 \text{sup}_{i \in A} \int |\nabla \phi_i(x)|^2 \gamma(dx)
\]

\[
\leq 4 \times 1_{\text{sup}_{i \in A} \langle I_i, \gamma \rangle \geq 2} 9 n^2 d \text{sup}_{i \in A} \langle I_i, \gamma \rangle
\]

\[
\leq 36n^2 d \sum_{i \in A} 1_{\langle I_i, \gamma \rangle \geq 2} \langle I_i, \gamma \rangle.
\]
From equation (6) we have
\[ \int_{\langle I_i, \gamma \rangle \geq 2} \langle I_i, \gamma \rangle \mu(d\gamma) = \int_{\mathbb{R}^+} z \langle I_i, \sigma \rangle \left( 1 - e^{-z \langle I_i, \sigma \rangle} \right) \lambda(dz) \]
\[ \leq \langle I_i, \sigma \rangle^2 \int_{\mathbb{R}^+} z^2 \lambda(dz), \]
and combined with (17) this gives
\[ \mathcal{E}(u_n, u_n) \leq cn^2 \sum_{i \in A} \langle I_i, \sigma \rangle^2. \]

Although the supports of the indicator functions $I_i$ are not disjoint, each point belongs to at most $2^d$ of the sets $\{I_i = 1\}$ for $i \in A$. Therefore the Cauchy-Schwarz inequality gives us
\[ \sum_{i \in A} \langle I_i, \sigma \rangle^2 = \sum_{i \in A} \left( \int I_i(x) \rho(x) \, dx \right)^2 \]
\[ \leq \sum_{i \in A} \left( \int I_i(x)^2 \rho(x) \, dx \right) \left( \int I_i(x) \, dx \right) \]
\[ \leq 2^d \int_{[-(a+1),a+1]^d} \rho(x)^2 \, dx (2/n)^d, \]
and combining this with (21) we find that
\[ \mathcal{E}(u_n, u_n) \leq cn^{2-d}. \]
Since $d \geq 2$ we see that $\sup_n \mathcal{E}(u_n, u_n) < \infty$.

**COROLLARY 1.** The conclusion of Proposition 1 holds if $\mu$ is replaced by any Gibbs measure $\nu$, in the sense of Ruelle [8], with a pair potential $\Phi$ that is superstable, lower regular, and where $\int |\exp(-\Phi(x)) - 1| \, dx < \infty$.

**Proof.** A Gibbs measure $\nu$ as in the statement admits a system of density distributions that satisfy $\sigma^\Delta(x_1, \ldots, x_n) \leq \xi^n$ for some constant $\xi$ [8, Theorem 5.5].

Any measure $\nu$ on $\Gamma$ admitting such density distributions is locally absolutely continuous with respect to the Poisson measure $\pi_{m^\nu}$. The bounds obtained in the proof of Proposition 1 are valid for this Poisson measure ($\lambda = \varepsilon \xi$, $\sigma = m = \text{Lebegue measure}$) and carry over trivially to the measure $\nu$.

**Remarks.**
1. More detailed analysis shows that the condition $\int z^2 \lambda(dz) < \infty$ in Proposition 1 can be dropped if the measure $\sigma$ satisfies the growth condition $\sigma(S_r) \leq a \exp(br)$ for some $a, b > 0$, where $S_r$ is the sphere of radius $r$ centered at the origin.
2. We point out that the form $(\mathcal{E}, \mathcal{D})$ in Osada [7] extends our $(\mathcal{E}, D(\mathcal{E}))$, but it is not known whether the forms coincide. A form with a larger domain has more exceptional sets, so proofs of exceptionality are easier in Osada’s setting, and do not imply exceptionality in our setting.
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