REALIZING MODULES OVER THE HOMOLOGY OF A DGA

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ABSTRACT. Let $A$ be a DGA over a field and $X$ a module over $H_\ast(A)$. Fix an $A_\infty$-structure on $H_\ast(A)$ making it quasi-isomorphic to $A$. We construct an equivalence of categories between $A_{n+1}$-module structures on $X$ and length $n$ Postnikov systems in the derived category of $A$-modules based on the bar resolution of $X$. This implies that quasi-isomorphism classes of $A_n$-structures on $X$ are in bijective correspondence with weak equivalence classes of rigidifications of the first $n$ terms of the bar resolution of $X$ to a complex of $A$-modules. The above equivalences of categories are compatible for different values of $n$. This implies that two obstruction theories for realizing $X$ as the homology of an $A$-module coincide.

1. Introduction

Let $A$ be a differential graded algebra over a field $k$ and let $R = H_\ast(A)$ be its homology. We say that an $R$-module $X$ is realizable if there exists a differential graded module $M$ over $A$ with $H_\ast(M) \simeq X$. This paper deals with two obstruction theories for answering the question of whether or not a module is realizable.

One obstruction theory is based on the theory of $A_n$-structures. In [St], Stasheff introduced a hierarchy of higher homotopy associativity conditions for multiplications on chain complexes. An $A_2$-structure is just a bilinear multiplication $m_2$, while an $A_3$-structure is an $A_2$-structure together with a homotopy $m_3$ between the two ways of bracketing a 3-fold product. An $A\infty$-structure consists of a sequence of higher associating homotopies $m_n$ satisfying certain conditions (see Section 2 for the definitions and also [Ke] for an excellent introduction to the theory of $A\infty$-algebras and modules).

Kadeishvili proved [Ka] that there is a $A_\infty$-structure on $H_\ast(A)$ making it quasi-isomorphic to $A$ as an $A_\infty$-algebra. Such an equivalence induces an equivalence of derived categories of $A_\infty$-modules and the derived category of (homologically unital) $A_\infty$-modules over $A$ is equivalent to the usual derived category of DG modules over $A$. This implies that a module $X$ is realizable if and only if it admits the structure of an $A_\infty$-module over $H_\ast(A)$. The $H_\ast(A)$-module structure on $X$ makes it an $A_2$-module over the $A_\infty$-algebra $H_\ast(A)$ and so the problem of realizability is naturally broken down into the problem of extending an $A_n$-module structure on $X$ to an $A_{n+1}$-structure for successive $n$.

Given an $A_n$-structure on $X$, the obstruction to extending the underlying $A_{n-1}$-structure to an $A_{n+1}$-structure lies in $\text{Ext}^{n,n-2}(X,X)$ (see Corollary 3.4). The
original motivation for this paper was the observation that this first obstruction, i.e. the obstruction to extending the given $A_2$-structure to an $A_4$-structure, is the primary obstruction to realizability described by other means in a recent paper of Benson, Krause and Schwede [BKS].

In [BKS Appendix A], the authors describe a general obstruction theory for realizability based on the notion of a Postnikov system (see Definition 5.1). This approach has its roots in stable homotopy theory (see [Wh] for example). The basic idea is the following. Since free modules and maps between them are clearly realizable, we can realize a free resolution for $X$ in the derived category of $A$-modules. The problem is then whether this "chain complex up to homotopy" can be rigidified to an actual chain complex of $A$-modules. This is explained in detail in Appendix A (see also [BKS]). The Postnikov system approach can be applied more generally [BKS Appendix A] to the problem of realizing modules over endomorphism rings of compact objects in triangulated categories.

Benson, Krause and Schwede define a canonical Hochschild class $\gamma_A \in HH^{3,1}(H_*(A))$ and show that the primary obstruction to building a Postnikov system for $X$ is the cup product $\text{id}_X \cup \gamma_A \in \text{Ext}^{3,1}(X, X)$. Looking more closely one sees that the cocycles representing $\gamma_A$ are (up to sign) precisely the $A_3$-structures on $H_*(A)$ which extend to an $A_\infty$-structure quasi-isomorphic to $A$. It turns out that the primary obstruction $\text{id}_X \cup \gamma_A$ to realizing $X$ is the obstruction to putting an $A_4$-module structure on $X$ (over any of these quasi-isomorphic $A_\infty$-structures on $H_*(A)$) and so the primary obstructions to realizing a module coincide from the two points of view. A natural question is then whether the two obstruction theories described above coincide in general. We answer this question in the affirmative.

We construct a functor from $A_n$-modules over $H_*(A)$ to filtered differential graded $A$-modules. This filtration gives rise to an $(n-1)$-Postnikov system for $X$ and our main result is that this functor induces an equivalence of categories.

**Theorem 1.1.** Let $A$ be a differential graded algebra over a field. There is an equivalence of categories between minimal $A_n$-modules over $H_*(A)$ and $(n-1)$-Postnikov systems based on the bar resolution$^1$.

This is proved below as Corollary 5.6 and Theorem 5.8. It implies in particular that for a fixed $H_*(A)$-module $X$, the moduli groupoid of $A_n$-structures on $X$ is equivalent to the groupoid of $(n-1)$-Postnikov systems based on the bar resolution for $X$ with isomorphisms which are the identity on the bar resolution (see Corollary 5.10). These equivalences of categories are compatible with the forgetful functors for varying $n$ and hence yield an equivalence of the two obstruction theories (see Theorem 5.5).

It is a folk theorem in homotopy theory that given a chain complex $B_\bullet$ in a homotopy category $\text{Ho}(\mathcal{C})$, $n$-Postnikov systems based on $B_\bullet$ correspond to rigidifications of the first $n$-terms of $B_\bullet$ to a chain complex in $\mathcal{C}$. We explain this in Appendix A. A consequence of this is the following result which is a direct consequence of Corollary 5.6 and Proposition A.6.

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$^1$The class in [BKS] is actually in $HH^{3,-1}(H^*(A))$. This is because of our convention (which follows [We]) that the $k$-th shift $X[k]$ is the $k$-th desuspension of $X$ in the derived category, while in [BKS] it is the $k$-th suspension.

$^2$See Definition 5.1.
Corollary 1.2. Isomorphism classes of $A_n$-module structures on $X$ are in bijective correspondence with weak equivalence classes of rigidifications of the complex formed by the first $n$ terms of the bar resolution for $X$.

It would be interesting to know to what extent the previous results generalize to an abstract stable homotopy theoretic context.

The equivalence of Theorem 1.1 is reminiscent of the equivalence originally established by Stasheff [St] between the existence of an $A_n$-structure (defined as a certain diagram in the homotopy category) and an $A_n$-form on a topological space $X$. It would be interesting to understand the precise relation.

A related problem is to give a similar characterization of $A_n$-algebra structures on a graded algebra $R$. For such an $A_n$-structure, the obstruction to extending the underlying $A_{n-1}$-structure to an $A_{n+1}$-structure is a class in the Hochschild cohomology group $HH^{n+1,n-1}(R)$ (see Proposition 3.7). It would be nice to have a description of these obstructions in terms of rigidification of diagrams.

1.3. Conventions and Notation. $k$ denotes a ground field. We always deal with (not necessarily bounded) homological complexes of $k$-vector spaces (unlike [Ke] and [BKS] who consider cohomological complexes).

We use the sign conventions of [Ke] so that if $f, g$ are maps of graded modules, then

$$(f \otimes g)(x \otimes y) = (-1)^{|f||x|}f(x) \otimes g(y).$$

In particular, this implies the commutation rule

$$(f \otimes g) \circ (h \otimes j) = (-1)^{|g||h|}(f \circ h) \otimes (g \circ j).$$

If $C$ is a graded module we write $C[n]$ for the $n$-fold desuspension of $C$, i.e.

$$C[n]_k = C_{k+n}.$$ 

If $C$ is a chain complex with differential $d$: $C[1] \rightarrow C$, then $C[n]$ is also a chain complex with differential given by $(-1)^n d$.

If $f: C \rightarrow D$ is a map of chain complexes, the standard model for the homotopy fiber of $f$ is the map $F \xrightarrow{\pi} C$ where $F$ is the complex (the desuspension of the mapping cone) defined by

$$F_k = D_{k+1} \oplus C_k$$

with differential given by the matrix

$$
\begin{bmatrix}
-d & f \\
0 & d
\end{bmatrix},
$$

and $\pi$ is the projection onto the second summand.

We will write $\lfloor x \rfloor$ for the greatest integer less than or equal to $x$.

Our differential graded algebras and modules are all unital.

1.4. Organization of the paper. In Section 2 we review the definition of $A_n$-algebra and module structures. In Section 3 we explain the obstruction to extending an $A_n$-module or algebra structure to an $A_{n+1}$-structure. In Section 4 we define the bar construction for an $A_n$-module. In Section 5 we recall the definition of Postnikov system and use the bar construction of the previous section to produce an equivalence of categories between $A_n$-modules over $H_*(A)$ and Postnikov systems based on the bar construction. There is one appendix where we explain the relation between Postnikov systems and rigidifying complexes in the homotopy category.
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2. **$A_n$-structures**

In this section we recall some basic definitions and notation regarding $A_n$-structures [KS, St] and point out some simplifications of the formulas that take place in our setting.

**Definition 2.1.** For $1 \leq n \leq \infty$, an $A_n$-algebra structure on a graded $k$-module $R$ consists of maps

$$m_k : R^k \rightarrow R[k-2] \quad 1 \leq k \leq n$$

satisfying the following relations for $m \leq n$

$$\sum_{r+s+t=m} (-1)^{r+st} m_{r+t+1} \circ (1^r \otimes m_s \otimes 1^t) = 0.$$  

(2.1)

An $A_n$-algebra is said to be minimal if $m_1 = 0$.

**Definition 2.2.** Let $R$ be an $A_n$-algebra. A right $A_t$-module $X$ over $R$ with $l \leq n$ consists of a graded $k$-module $X$, together with maps

$$m_k^X : X \otimes R^k \rightarrow X[k-2] \quad 1 \leq k \leq l$$

satisfying, for each $m \leq l$,

$$\sum_{r+s+t=m} (-1)^{r+st} m_{r+t+1}^X (1^r \otimes m_s \otimes 1^t) = 0$$

(2.2)

where $m_s$ denotes $m_s^X$ whenever $r = 0$ and $m_s^R$ otherwise.

A left $A_t$-module $X$ over $R$ consists of maps $m_k^X : R^k \otimes X \rightarrow X[k-2]$ satisfying (2.2) with $m_s$ now denoting $m_s^X$ whenever $t = 0$ and $m_s^R$ otherwise.

An $A_t$-module is said to be minimal if $m_1^X = 0$.

Note that $m_1 m_1 = 0$ so that an $A_t$-module is a complex.

**Definition 2.3.** Let $R$ be an $A_n$-algebra and $l \leq n$. A morphism of $A_t$-modules over $R$, $f : X \rightarrow Y$, consists of maps of $k$-modules

$$X \otimes R^k \rightarrow Y[k-1] \quad 1 \leq k \leq l$$

satisfying the following equation in $\text{Hom}(X \otimes R^m, Y[m-2])$ for each $m \leq l$:

$$\sum_{r+s+t=m} (-1)^{r+st} f_{r+t+1} (1^r \otimes m_s \otimes 1^t) = \sum_{i+j=m} (-1)^{(i+1)j} m_{j+1} (f_i \otimes 1^j)$$

(2.3)

where again $m_s$ denotes $m_s^X$ whenever $r = 0$ and $m_s^R$ otherwise.

A morphism is called a quasi-isomorphism if $f_1$ is a quasi-isomorphism.

**Definition 2.4.** A morphism of $A_n$-algebras $f : A \rightarrow B$ consists of maps

$$A^k \rightarrow B[k-1] \quad 1 \leq k \leq n$$

satisfying the following equation in $\text{Hom}(A^m, B[m-2])$ for each $1 \leq m \leq n$:

$$\sum_{r+s+t=m} (-1)^{r+st} f_{r+t+1} (1^r \otimes m_s \otimes 1^t) = \sum (-1)^v m_u (f_{i_1} \otimes \ldots \otimes f_{i_u})$$

(2.4)

where, on the right hand side, the sum is over all decompositions $i_1 + i_2 + \ldots + i_u = m$ and $v = (u-1)(i_1-1) + (u-2)(i_2-1) + \ldots + 2(i_{u-1}-1) + (i_{u-1}-1)$. 

A morphism is called a quasi-isomorphism if \( f_1 \) is a quasi-isomorphism.

The \( A_\infty \)-algebras we will be dealing with arise from the following result.

**Theorem 2.5** (Kadeishvili). Let \( A \) be a differential graded algebra over a field \( k \). There is an \( A_\infty \) structure on \( H_*(A) \) together with a quasi-isomorphism of \( A_\infty \)-algebras

\[
f : H_*(A) \to A.
\]

Moreover \( m_{1}^{H_*(A)} = 0 \) and \( m_{2}^{H_*(A)} \) is the associative multiplication induced by the multiplication on \( A \).

The \( A_\infty \)-structure on \( H_*(A) \) and the quasi-isomorphism \( f \) are constructed inductively. The induction is started by picking a splitting \( f_1 \) for the projection \( Z(A) \to H_*(A) \) and then making use of the following simplification of \( [\mathbf{2.4}] \).

**Lemma 2.6.** Let \( B \) be a minimal \( A_\infty \)-algebra and \( A \) a differential graded algebra. Then an \( A_\infty \)-morphism \( f : B \to A \) consists of maps \( f_n : B \to A[n-1] \) satisfying the following formulas for all \( n \):

\[
m_1 f_n = f_1(m_n) + \sum_{0 < u + t < n - 1} (-1)^{u + t(n-u-t)} f_{u+t+1}(1^\otimes u \otimes m_{n-u-t} \otimes 1^\otimes t) + \\
+ \sum_{i=1}^{n-1} (-1)^{i} m_2(f_i \otimes f_{n-i})
\]

**Remark 2.7.** In the situation of **Theorem 2.5** writing \( d \) for the differential and \( \mu \) for the multiplication on \( A \), the maps

\[
m_n^A = (-1)^n \mu \circ (f_{n-1} \otimes 1) : H_*(A)^{\otimes n-1} \otimes A \to A[n-1] \quad \text{for } n \geq 2
\]

together with \( m_1 = d \) make \( A \) a left \( A_\infty \)-module over \( H_*(A) \).

Given a differential graded algebra \( A \), we fix an \( A_\infty \)-algebra structure on \( H_*(A) \) and a quasi-isomorphism \( f : H_*(A) \to A \) given by **Theorem 2.5** for the rest of the paper.

A graded module \( X \) over the associative graded algebra \( H_*(A) \) has a natural \( A_2 \)-structure with \( m_{1}^{X} = 0 \) and \( m_{2}^{X} \) given by the action of \( H_*(A) \). Moreover, an arbitrary map \( m_{3}^{X} : X \otimes H_*(A)^{\otimes 2} \to X[1] \) gives \( X \) the structure of an \( A_3 \)-module over \( H_*(A) \).

The formulas \( [\mathbf{2.2}] \) defining an \( A_n \)-module simplify for modules over \( H_*(A) \) whose underlying \( A_2 \)-structure arises from the situation described in the previous paragraph. For example, an \( A_2 \)-module structure consists of a map \( m_{2}^{X} : X \otimes H_*(A) \to X \) satisfying no hypothesis and an \( A_3 \)-structure consists of an associative action \( m_{3}^{X} \) together with a map \( m_{3}^{X} : X \otimes H_*(A)^{\otimes 2} \to X[1] \) satisfying no hypothesis. More generally, an \( A_n \)-structure puts no restriction on the map \( m_{n}^{X} \). We state here the simplified formulas here as they will be used repeatedly.

**Lemma 2.8.** Let \( X \) be a graded vector space. An \( A_n \)-module structure on \( X \) over \( H_*(A) \) with \( m_{1}^{X} = 0 \) consists of maps

\[
m_{k}^{X} : X \otimes H_*(A)^{\otimes (k-1)} \to X[k-2] \quad 2 \leq k \leq n
\]

satisfying the following equations for \( 2 \leq k < n \):

\[
\sum_{\substack{2 \leq r+t+1 \leq k \\\ r+s+t = k+1}} \left(-1\right)^{r+s+t+1} m_{r+t+1}^{X} \circ \left(1^{r} \otimes m_{s} \otimes 1^{t}\right) = 0.
\]
3. THE OBSTRUCTION TO EXTENDING AN $A_n$-STRUCTURE TO AN $A_{n+1}$-STRUCTURE

Let $R$ be a minimal $A_\infty$-algebra over the field $k$ (so $m_2$ makes $R$ an associative algebra) and let $X$ be a minimal right $A_n$-module over $R$ (so if $n \geq 3$, $X$ is in particular a module over the associative algebra $R$ in the usual sense). In this section we describe the set of $A_{n+1}$-structures on $X$ extending the given $A_n$-structure. We show that the obstruction to the existence of an $A_{n+1}$-structure extending the underlying $A_{n-1}$-structure is an element in $\text{Ext}^n_{R}(X,X)$.

The exact same computation shows that if $S$ is a graded algebra and one considers $A_n$-algebra structures on $S$ extending the underlying $A_2$-structure then the obstructions to extension are classes in the Hochschild cohomology $HH^{n,n-2}(S)$.

This obstruction theory for minimal $A_n$-algebras is also described in [Le, Appendix B.4]. Lefèvre also discusses obstruction theory for non-minimal $A_n$-algebras and modules in [Le, Appendix B] but this does not seem relevant to the minimal case we consider here.

Recall the bar resolution of a module $M$ over a graded algebra $R$ (see for instance [Wc, 8.6.12]):
\[
\cdots \rightarrow M \otimes_R R \otimes_R R \rightarrow M \otimes_R R \rightarrow M.
\]
This is a free resolution of $M$ as a right $R$-module. If $N$ is another right $R$-module we write
\[
(\text{Bar}^\ast, \ast((M,N), \partial) = \text{Hom}^\ast(M \otimes R^{\otimes (\ast+1)}, N)
\]
for the induced cochain complex of graded $k$-modules. The cohomology of this complex is $\text{Ext}_{R}^{\ast}(M,N)$.

If $R$ is a graded $k$-algebra, $M$ a graded $k$-module, $N$ a right $R$-module, and $f : M \rightarrow N$ a map of graded $k$-modules, we write
\[
f \ast 1 : M \otimes_R R \rightarrow N
\]
for the canonical extension of $f$ to a map of $R$-modules.

Given $R$ and $X$ satisfying our standing assumptions, the map
\[
m_n^X : X \otimes R^{\otimes (n-1)} \rightarrow X[n-2]
\]
yields a map
\[
m_n^X \ast 1 \in \text{Bar}^{n-1,n-2}(X,X).
\]

In this section we will write
\[
\phi_n = -m^X_2(1 \otimes m_n) + \sum_{2 \leq r + s + t \leq n+1} (-1)^{r+s+t}m^X_{r+s+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}).
\]

Note that the condition $\phi_n = 0$ is precisely the $(n+1)$-st condition for $(m^X_2, \ldots, m^X_{n-1}, 0, 0)$ to be an $A_{n+1}$-structure on $X$ (see Definition [2.2]).

**Lemma 3.1.** Let $\partial$ be the coboundary operator in the complex $\text{Bar}^{\ast, \ast}(X,X)$. There is an $A_{n+1}$-structure on $X$ extending a given $A_n$-structure if and only if the following equation holds:
\[
\partial(m^X_n \ast 1) + \phi_n \ast 1 = 0.
\]

Furthermore, when an extension exists, the set of extensions is in 1-1 correspondence with the set of $k$-module maps $m^X_{n+1} : X \otimes R^{\otimes n} \rightarrow X[n-1]$. 
Proof. By Lemma 2.8 we need to check that

\[(3.3) \quad \partial(m_n^X \ast 1) = (-1)^n m_2^X (m_n^X \otimes 1) \ast 1 + \sum_{i=0}^{n-1} (-1)^i m_n^X (1^\otimes i \otimes m_2 \otimes 1^\otimes (n-i-1)) \ast 1.\]

The map,

\[\partial(m_n^X \ast 1) : X \otimes R^{\otimes (n+1)} \to X[n-2]\]

is given by the formula

\[\partial(m_n^X \ast 1)(x, \zeta_1, \ldots, \zeta_{n+1}) = m_n^X (x \zeta_1, \ldots, \zeta_n) \zeta_{n+1} - m_n^X (x, \zeta_1 \zeta_2, \ldots, \zeta_n) \zeta_{n+1} + \ldots
\]
\[+ (-1)^{n-1} m_n^X (x, \zeta_1, \ldots, \zeta_{n-1} \zeta_n) \zeta_{n+1} + (-1)^n m_n^X (x, \zeta_1, \ldots, \zeta_{n-1}) \zeta_n \zeta_{n+1}.\]

On the other hand, applying the right hand side of (3.3) to \((x, \zeta_1, \ldots, \zeta_{n+1})\) we get

\[(-1)^n m_n^X (x, \zeta_1, \ldots, \zeta_{n-1}) \zeta_n \zeta_{n+1} + m_n^X (x \zeta_1, \ldots, \zeta_n) \zeta_{n+1} + \ldots + (-1)^{n-1} m_n^X (x, \zeta_1, \ldots, \zeta_{n-1} \zeta_n) \zeta_{n+1}.\]

\[\square\]

Lemma 3.2. If \(X\) is an \(A_n\)-module over \(R\), \(\phi_n \ast 1 \in \text{Bar}^{n,n-2}(X,X)\) is a cocycle.

Proof. It suffices to check the condition \((\phi_n \ast 1) \circ \partial = 0\) on module generators. Thus we need to check that

\[(3.4) \quad \sum_{i=0}^{n} (-1)^i \phi_n (1^\otimes i \otimes m_2 \otimes 1^\otimes (n-i)) + (-1)^{n+1} \phi_n \ast 1 = 0.\]

In the course of this proof, through (3.10), we omit

\[\sum_{r+s+t=n+1 \atop 2 \leq r+t+1 < n \atop (r,s,t) \neq (0,n,1)}\]

which is understood to precede each sum.

Expanding the left summand in expression (3.4) we obtain

\[(3.5) \quad \sum_{0 \leq i < r} (-1)^{r+s+t+i} m_{r+s+t+1} (1^\otimes i \otimes m_2 \otimes 1^\otimes (r-i-1) \otimes m_s \otimes 1^\otimes t) +\]

\[(3.6) \quad \sum_{r \leq i < r+s} (-1)^{r+s+t+i} m_{r+s+t+1} (1^\otimes r \otimes m_s (1^\otimes i-r \otimes m_2 \otimes 1^\otimes (r+s-i-1)) \otimes 1^\otimes t) +\]

\[(3.7) \quad \sum_{r+s \leq i \leq n} (-1)^{r+s+t+i} m_{r+s+t+1} (1^\otimes r \otimes m_s \otimes 1^\otimes (r+s-i) \otimes m_2 \otimes 1^\otimes (r+s-i-1)).\]

Using the \(A_{s+1}\)-structure on the middle term (3.6) we find it is equal to

\[(3.8) \quad \sum_{j+k+l=s+1 \atop 2 \leq j+l+1 < s} (-1)^{j+l+k+1} m_{j+l+1} (1^\otimes j \otimes m_k \otimes 1^\otimes l) \otimes 1^\otimes t)\]

Separating out the term where \((j,k,l) = (1,s,0)\) and combining it with the term (3.3) we obtain

\[\sum_{0 \leq i \leq r} (-1)^{r+s+t+i} m_{r+s+t+1} (1^\otimes i \otimes m_2 \otimes 1^\otimes (r+t-i))(1^\otimes (r+1) \otimes m_s \otimes 1^\otimes t).\]
Similarly combining the term of (3.8) where \((j, k, l) = (0, s, 1)\) with the term (3.7) we obtain

\[
\sum_{r\leq t\leq r+t} (-1)^{r+s+t+s-1}m_{r+t+1}(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes r+t-i})(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t+1}).
\]

Together the last two expressions yield

\[\sum_{0\leq t\leq r+t} (-1)^{r+s+t+s-1}m_{r+t+1}(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes r+t-i})(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t+1}).\tag{3.9}\]

Using the \((r + t + 2)\)-module structure in (3.9) yields

\[
\sum_{a+b+c=r+t+2 \atop b \geq 2} (-1)^{r+s+t+a+b}m_{a+c+1}(1^{\otimes a} \otimes m_b \otimes 1^{\otimes c})c(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t+1}).\tag{3.10}\]

The terms in (3.10) corresponding to \((a, b, c) = (1, r + t + 1, 0)\) cancel with those terms in (3.8) that remain (i.e. \((j, k, l) \in \{(1, s, 0), (0, s, 1)\}\)) and satisfy \((r, s) = (1, n)\). On the other hand, the terms in (3.10) corresponding to \((a, b, c) = (0, r + t + 1, 1)\) cancel with the term \((-1)^{n+1}\phi_n * 1\) in (3.4).

We are left with showing that the terms

\[
\sum_{r+s+t=n+1 \atop 2<s<n} \sum_{a+b+c=r+t+2 \atop 2\leq b < r+t} (-1)^{r+s+t+a+b}m_{a+c+1}(1^{\otimes a} \otimes m_b \otimes 1^{\otimes c})c(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t+1}). \tag{3.11}
\]

left from (3.10) and

\[
\sum_{r+s+t=n+1 \atop 2<s<n} \sum_{a+b+c=r+t+2 \atop 2\leq k < s} (-1)^{n+1+j+k}m_{r+t+1}(1^{\otimes r} \otimes m_j \otimes 1^{\otimes l} \otimes 1^{\otimes t}). \tag{3.12}
\]

left from (3.8) add up to zero. In (3.11), the terms of the form \(m_{a+c+1}(1^{\otimes a} \otimes m_b \otimes 1^{\otimes r-a-b} \otimes m_s \otimes 1^{\otimes t+1})\) appear twice with opposite signs. Finally, the remaining terms of (3.8), which are of the form \(m_{a+c+1}(1^{\otimes a} \otimes m_b (1^{\otimes r-a-b} \otimes m_s \otimes 1^{\otimes t+1-c}) \otimes 1^{\otimes c})\), cancel with (3.12).

**Remark 3.3.** We have observed that, by definition, \((m_2^X, \ldots, m_{n-1}^X, 0)\) extends to an \(A_{n+1}\)-structure if and only if \(\tilde{\phi}_n = 0\). Writing \(\tilde{\phi}_n = 0\) for the \(A_{n+1}\)-structure equation that \((m_2^X, \ldots, m_{n-1}^X, m_n^X)\) must satisfy in order to extend, it is easy to check that \((\tilde{\phi}_n - \phi_n) * 1\) is a cocycle in the bar complex. Thus, given an \(A_n\)-structure on \(X\), the previous lemma implies that \(\tilde{\phi}_n * 1\) is also a cocycle.

We summarize the previous arguments in the following statement.

**Proposition 3.4.** Let \(X\) be an \(A_n\)-module over \(R\).

(a) The underlying \(A_{n-1}\)-structure on \(X\) can be extended to an \(A_{n+1}\)-structure if and only if the class \([\phi_n * 1] \in \text{Ext}_R^{n-2}(X, X)\) vanishes.

(b) If \([\phi_n * 1] = 0\), the set of \(A_{n+1}\)-structures on \(X\) extending the underlying \(A_{n-1}\)-structure is in bijective correspondence with pairs of \(R\)-module maps

\[
\psi: X \otimes R^{\otimes n} \to X[n-2], \quad \xi: X \otimes R^{\otimes (n+1)} \to X[n-1]
\]

such that

\[
\partial(\psi) = \phi_n * 1
\]
Proof. Statement (a) is the content of Lemmas 3.1 and 3.2. Statement (b) follows immediately from the fact that equation (6.2) is the only equation involving $m^X_3$ among the equations defining an $A_{n+1}$-structure on $X$. \hfill $\square$

Remark 3.5. Let $A$ be a differential graded algebra. In [BKS] the authors consider the problem of deciding whether an $H_*(A)$-module $X$ is the homology of an $A$-module. They define a Hochschild cohomology class $\gamma_A \in HH^{3,1}(H_*(A))$ and show that the first obstruction is

$$1_X \cup \gamma_A \in \text{Ext}^{3,1}_H(A)(X, X)$$

(see [BKS] Corollary 6.3). The choice of a cocycle representing $\gamma_A$ precisely corresponds to the choice of $m^{H_*(A)}_3$ in the inductive proof of Kadeishvili’s theorem (compare Lemma 2.6 with [BKS] Construction 5.1 and Remark 5.8).

The special case of Proposition 3.4 when $n = 3$ says that an $R$-module $X$ has an $A_1$-structure if and only if the map

$$(m^X_2(1 \otimes m_3)) * 1$$

is a coboundary in $\text{Bar}^{3,1}(X, X).$ Thus the obstruction described in [BKS] is exactly the obstruction to the existence of an $A_1$-structure on $X$.

Example 3.6. This example amplifies on the example considered in [BKS] 7.3,7.4,7.6. Let $L = k[z]/z^n$ be the truncated polynomial algebra of height $n$ over a field $k$. Let $A$ be the endomorphism DGA of the complete resolution $\hat{P}$ of the trivial $L$-module $k$. $\hat{P}$ is defined by $\hat{P}_i = L$ for each $i \in Z$ with differentials $d_i: \hat{P}_i \to \hat{P}_{i-1}$ given by the formulas

$$d_i = \begin{cases} \text{multiplication by } -z^{n-1} & \text{if } i \text{ is even}, \\ \text{multiplication by } z & \text{otherwise.} \end{cases}$$

Note that if $k$ has characteristic $p$ and $n$ is a power of $p$, $L$ is isomorphic to the group algebra of the cyclic group $C_n$ and then the homology of $A$ is the Tate cohomology of $C_n$.

The homology algebra of $A$ is [BKS] Theorem 7.3:

$$H_*(A) = \begin{cases} k[x^{\pm1}] & \text{if } n = 2, \\ \Lambda(x) \otimes k[y^{\pm1}] & \text{if } n > 2, \end{cases}$$

where $\Lambda(x)$ denotes the exterior algebra on $x$, and $|x| = -1$ and $|y| = -2$. In the proof the authors define the first two maps $f_1: H_*(A) \to A$ and $f_2: H_*(A) \to A[1]$ in a quasi-isomorphism of $A_\infty$-algebras $f: H_*(A) \to A$ (cf. Theorem 2.6) and use this to find the $A_3$-structure on $H_*(A)$, $m_3: H_*(A)^{\otimes 3} \to H_*(A)[1]$ (in their terminology this is the Hochschild cocycle $m$ representing the canonical class as in the previous remark).

For $n \neq 3$, $m_3$ vanishes, while for $n = 3$, it is given by the formula

$$m_3(a, b, c) = \begin{cases} 0 & \text{if } |a|, |b|, \text{ or } |c| \text{ is even}, \\ y^{i+j+k+1} & \text{if } a = xy^i, b = xy^j, c = xy^k. \end{cases}$$

Proceeding as in the proof of [BKS] Theorem 7.3 we can inductively find formulas for the remaining maps $f_i: H_*(A) \to A[i-1]$. From this we see that, in
general, the $A_\infty$-structure on $H_*(A)$ consists of only $m_2$ and $m_n$ with all other $m_k$'s vanishing and $m_n$ given by the formula
\[
m_n(a_1, \ldots, a_n) = \begin{cases} 0 & \text{if one of the } |a_i|'s \text{ is even}, \\ y^{i_1 + \cdots + i_n + 1} & \text{if } a_i = xy^{i_i}. \end{cases}
\]

In the case when $n = 3$ the authors show in [BKS, Example 7.6] that the realizable $H_*(A)$-modules are precisely the free ones. For $n > 3$, any $H_*(A)$-module $X$ admits a trivial $A_n$-structure with $m_k^X = 0$ for $2 < k \leq n$. The argument of [BKS, Example 7.6] shows more generally that for this trivial $A_n$-structure to extend to an $A_{n+1}$-structure, $X$ must be a free module. However, it is no longer true that only free modules are realizable. In fact, for $n > 3$, all modules are direct summands of realizable modules (by the previous calculation together with the main result of [BKS]) but they are certainly not all direct summands of free modules.

For example, for $n > 3$,
\[
H_*(\text{Hom}(\hat{P}, k[z]/z^2)) = k[y^{\pm 1}] \oplus k[y^{\pm 1}][1]
\]
(with $x$ acting trivially) is obviously realizable.

On the other hand, the $H_*(A)$-module
\[
X = k[y^{\pm 1}] = H_*(A)/xH_*(A)
\]
is not realizable. Indeed, for any choice of $m_k^X : X \otimes H_*(A)^{\otimes(k-1)} \to X[k-2]$ and any $a \in X$ we have
\[
m_k^X(a, x, \ldots, x) = 0
\]
since $X$ is concentrated in even degrees. It follows that $X$ can’t be given an $A_{n+1}$-structure: when we evaluate
\[
\sum_{2 \leq r + r+1 \leq n, r + s + t = n+1} (-1)^{r+s+t} m_r^X (1 \otimes m_s \otimes 1 \otimes t)(a, x, \ldots, x)
\]
all terms except $am_n(x, \ldots, x) = ay$ vanish (either because $x$ acts trivially on $X$, $x^2 = 0$, $m_k^X(a, x, \ldots, x) = 0$, or because $m_k = 0$ for $k < n$).

The algebra $\text{Ext}^{n*}(X, X)$ is a polynomial algebra on $k[y^{\pm 1}]$ on a generator in bidegree $(1, -1)$ ($y$ has degree $(0, -2)$). The obstruction to extending the trivial $A_n$-module structure on $X$ to an $A_{n+1}$-structure must therefore be a generator of $\text{Ext}^{n,n-2}_H(A, X, X)$.

It is somewhat surprising that for the realizable module $Y = k[y^{\pm 1}] \oplus k[y^{\pm 1}][1]$ we can’t choose $m_k^Y$ to vanish for $k < n$. One can check that an $A_\infty$-structure on $Y$ can be defined in the following way. Let $a$ and $b$ be module generators for $Y$ in degrees 0 and $-1$ respectively.

If $n$ is even, set (for $k \geq 3$)
\[
m_k^Y(m, xy^{i_2}, \ldots, xy^{i_k}) = \begin{cases} by^{i_1 + \cdots + i_k} & \text{if } m = ay^{i_1} \text{ and } k = \frac{n}{2} - 1, \\ ay^{i_1 + \cdots + i_k} & \text{if } m = by^{i_1} \text{ and } k = \frac{n}{2} - 1, \\ 0 & \text{otherwise}. \end{cases}
\]

If $n$ is odd, set
\[
m_k^Y(m, xy^{i_2}, \ldots, xy^{i_k}) = \begin{cases} by^{i_1 + \cdots + i_k} & \text{if } m = ay^{i_1}, k \in \left\{ \frac{n+1}{2}, \frac{n+3}{2} \right\}, \text{ and } k \text{ is even,} \\ ay^{i_1 + \cdots + i_k} & \text{if } m = by^{i_1}, k \in \left\{ \frac{n+1}{2}, \frac{n+3}{2} \right\}, \text{ and } k \text{ is odd,} \\ 0 & \text{otherwise}. \end{cases}
\]
The analog for algebras of Proposition 3.4 is the following. Consider the Hochschild complex

\[ (C^n,m(S) = \text{Hom}^{n}_{S \otimes S^{op}}(S^{\otimes (n+2)}, S), \partial_H). \]

An element in \( C^n,m(S) \) is represented by a map of vector spaces \( f : S^{\otimes n} \to S \) of degree \( m \), and, in these terms, the differential is given by the formula

\[ \partial_H(f) = m_2(1 \otimes f) - \sum_{j=0}^{n-1} (-1)^j f \circ (1^{\otimes j} \otimes m_2 \otimes 1^{\otimes n-j-1}) + (-1)^n m_2(f \otimes 1). \]

The analog of (3.2) is that an \( A_n \)-algebra structure on \( S \) can be extended to an \( A_{n+1} \)-algebra structure iff the following equation is satisfied:

\[ \partial_H(m_n) = \sum_{2 < r + t + s + r^{\otimes t} = n + 1} (-1)^{r+st} m_r + t + 1 (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) \]

and the same computations as in the proofs of Lemmas 3.7 and 3.8 give the following analog of Proposition 3.4 which can also be found in [Le, Lemma B.4.1].

**Proposition 3.7.** Let \( S \) be a graded algebra. Given an \( A_n \)-structure on \( S \) extending the given \( A_2 \)-structure, the underlying \( A_{n-1} \)-structure can be extended to an \( A_{n+1} \)-structure iff the Hochschild cocycle

\[ \partial_H(m_n) = \sum_{2 < r + t + s + r^{\otimes t} = n + 1} (-1)^{r+st} m_r + t + 1 (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) \]

represents the trivial class in \( HH^{n+1,n-2}(S) \).

If \( S \) is an \( A_\infty \)-algebra with \( m_1 = m_3 = \ldots = m_{n-1} = 0 \) then \( m_n : S^{\otimes n} \to S \) is a Hochschild cocycle. The primary obstruction to realizing a module \( X \) is then the obstruction to giving \( X \) an \( A_{n+1} \)-structure, namely the class \( 1_X \cup [m_n] \in \text{Ext}_S^{n,n-2}(X,X) \). In fact, any \( S \)-module \( X \) can be given an \( A_{n-1} \)-module structure with \( m_3^X = \ldots = m_{n-1}^X = 0 \) and one can extend this to an \( A_{n+1} \)-module structure on \( X \) if and only if

\[ (m_3^X (1 \otimes m_n)) \ast 1, \]

which is a cocycle representing \( 1_X \cup [m_n] \), is a coboundary in \( \text{Bar}^{n,n-2}(X,X) \). This is exactly the situation for the non-realizable module \( X \) in Example 3.8.

**Example 3.8.** Let \( S = R[\epsilon]/\epsilon^2 \) where \( R \) is a \( k \)-algebra concentrated in degree 0 and \( |\epsilon| = n - 2 \). If \( \{m_n\} \) is an \( A_\infty \)-structure on \( S \) then for degree reasons \( m_i = 0 \) for \( i \neq 2, n \) and \( m_n \) is determined by a \( k \)-linear map \( R^{\otimes n} \to R \) which must be a Hochschild cocycle. One can check that two \( A_\infty \)-structures on \( S \) are quasi-isomorphic iff the corresponding cocycles represent the same cohomology class (cf. [Ke 3.2]).

4. THE BAR CONSTRUCTION

Recall that we have fixed an \( A_\infty \)-structure on \( H_*(A) \) and a quasi-isomorphism \( f : H_*(A) \to A \), which in turn gives \( A \) the structure of an \( A_\infty-H_*(A) \)-module (see Remark 2.7). The goal of this section is to construct a functor, denoted \( B(-,H_*(A),A) \), from \( A_\infty-H_*(A) \)-modules to differential graded \( A \)-modules. The functor can be written as a directed colimit of functors \( B_{n-1}(-,H_*(A),A) \), from \( A_\infty-H_*(A) \)-modules to differential graded \( A \)-modules.
Given a (minimal) $A_n$-module structure on an $H_*(A)$-module $X$

$$m^X_k : X \otimes H_*(A)^{(k-1)} \to X, \quad 2 \leq k \leq n,$$

let $R_k$ denote the free differential graded $A$-module defined by

$$R_k = X \otimes H_*(A)^{\otimes k} \otimes A.$$

For $1 \leq l \leq k + 1$, let

$$M_{k,l} : R_k \to R_{k-l+1}[l-2]$$

be defined as

$$M_{k,l} = \sum_{i=0}^{k+2-l} (-1)^{i(l-1)} \otimes i \otimes m_l \otimes 1 \otimes k-l-i+2$$

where, in the first term of the sum, $m_l$ stands for $m^X_l$ and, in the last term, $m_l$ stands for $m^X_l = (-1)^l f_{l-1} * 1$ if $l > 1$ and for the differential $d$ on $A$ if $l = 1$ (see Remark 2.7). We will sometimes write $D$ for $M_{k,2}$ and $d$ for $M_{k,1}$.

The formulas in the following definition were obtained when attempting to construct a Postnikov system associated to an $A_{n+1}$-module (see Theorem 5.3).

**Definition 4.1.** Given an $A_{n+1}$-module $X$ over $H_*(A)$ (with $1 \leq n \leq \infty$), the bar construction on $X$ is the right $A$-module $B_n(X, H_*(A), A)$ defined by

$$\oplus_{i=0}^{n}(X \otimes H_*(A)^{\otimes i} \otimes A)[-i] = \oplus_{i=0}^{n} R_i[-i].$$

The differential on $B_n(X, H_*(A), A)$ is defined on the summand $R_i$ by the following formula

\[ (4.1) \quad \partial |_{R_i} = \sum_{i+j+k=l+2} (-1)^{k+j+i+1} \left\lfloor \frac{i+1}{2} \right\rfloor \otimes m_j \otimes 1 \otimes k = \sum_{j=1}^{l+1} (-1)^{l+i} \left\lfloor \frac{j-i}{2} \right\rfloor M_{l,j} \]

We use $\lfloor x \rfloor$ to denote the greatest integer less than or equal to $x$. The following easily checked formula will be used constantly in computations.

**Lemma 4.2.** For any integers $i$ and $j$

$$\left\lfloor \frac{i+1}{2} \right\rfloor + \left\lfloor \frac{j}{2} \right\rfloor \equiv \left\lfloor \frac{j-i}{2} \right\rfloor + ij \mod 2.$$

**Lemma 4.3.** The formulas (4.1) give $B_n(X, H_*(A), A)$ the structure of a differential graded $A$-module.

**Proof.** It’s easy to check that the Leibniz rule holds so it is enough to check that (4.1) defines a differential on $B_n(X, H_*(A), A)$. The projection to $R_m$ of $\partial |_{R_l}$ is given by the formula

\[ (4.2) \quad \sum_{j=1}^{l-m+1} (-1)^{l-j+1} \left\lfloor \frac{j-1}{2} \right\rfloor \left\lfloor \frac{l-j-m+1}{2} \right\rfloor M_{l-j+1,l-j-m+2} M_{l,j}. \]

By Lemma 4.2, the sign in the previous expression is equal to

$$(-1)^{(l-m)(j-1)+\left\lfloor \frac{l-m}{2} \right\rfloor}.$$ 

Since $(-1)^{\left\lfloor \frac{l-m}{2} \right\rfloor}$ is independent of $j$, this factor can be eliminated and the equation $\partial^2 = 0$ then follows from the relations that must be satisfied because $H_*(A)$ is an $A_{n+1}$-algebra and $X$ and $A$ are $A_{n+1}$-modules over $H_*(A)$. \qed

We also need to explain the functoriality of the bar construction.
Proposition 4.4. Let \( g : X \to Y \) be a map of \( A_{n+1} \)-modules (with \( 1 \leq n \leq \infty \)). The map

\[
B_n(g) : B_n(X, H_+(A), A) \to B_n(Y, H_+(A), A)
\]
defined by the matrix with entries

\[
(4.3) \quad B_n(g)_{i,j} = (-1)^{\left\lfloor \frac{j-i+1}{2} \right\rfloor} g_{j-i+1} \otimes 1^i
\]

for \( 1 \leq i \leq j \leq n+1 \), or

\[
\begin{bmatrix}
g_1 \otimes 1 & -g_2 \otimes 1 & -g_3 \otimes 1 & g_4 \otimes 1 & \ldots \\
0 & g_1 \otimes 1 & -g_2 \otimes 1 & -g_3 \otimes 1 & \ldots \\
0 & 0 & g_1 \otimes 1 & -g_2 \otimes 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
is a map of differential graded \( A \)-modules.

Proof. This computation is similar to the one above and hence is omitted. \( \square \)

We also write \( B(g) = B_\infty(g) \).

Proposition 4.5. For \( 1 \leq n \leq \infty \), the assignments

\[
X \mapsto B_n(X, H_+(A), A) \quad \left( X \xrightarrow{g} Y \right) \mapsto B_n(g)
\]
define a functor from \( A_{n+1} - H_+(A) \)-modules to differential graded \( A \)-modules.

Proof. Matrix multiplication precisely corresponds to the composition of \( A_{n+1} \)-module maps as defined on [Ke, p. 15]. \( \square \)

Remark 4.6. The quasi-isomorphism of \( A_\infty \)-algebras \( f : H_+(A) \tilde{\to} A \) makes \( A \) and \( A_\infty-H_+(A)-A \)-bimodule. Although the formula for the differential (4.1) is different, it seems likely that \( B_\infty(-, H_+(A), A) \) is equivalent to the functor \(- \otimes_{H_+(A)} A\) considered in [Le, Section 4.1, p.114].

5. \( A_n \)-structures and Postnikov systems

In this section we describe the obstruction theory to realizing a module based on the notion of a Postnikov system [BKS] and show that the bar construction of the previous section gives us a functor from \( A_{n+1} \)-module structures to \( n \)-Postnikov systems. We then show that the obstructions to extending an \( A_{n+1} \)-structure or its associated \( n \)-Postnikov system agree. It follows by induction that any Postnikov system arising from the bar resolution of \( X \) comes from an \( A_{n+1} \)-structure. Finally we prove that this assignment is fully faithful completing the proof of Theorem 1.1.

In this section, we will often use the following simple formula for the maps in the derived category of \( A \)-modules when the source is free: if \( V \) is a \( k \)-module and \( N \) is a differential graded module over \( A \) then

\[
[V \otimes A, N] = \text{Hom}_{H_+(A)}(V \otimes H_+(A), H_+(N)).
\]
Definition 5.1. Let $A$ be a differential graded algebra and $X$ be an $H_*(A)$-module. An $n$-Postnikov system for $X$ is a commutative diagram in the derived category of $A$-modules

$$
\begin{array}{cccccc}
Y_n & Y_{n-1} & Y_{n-2} & \cdots & Y_1 \\
j_n & i_n & j_{n-1} & i_{n-1} & j_{n-2} \\
C_n & C_{n-1} & C_{n-2} & \cdots & C_1 & i_1 & C_0
\end{array}
$$

satisfying

(i) $j_k$ is the homotopy fiber of $i_k$ (i.e. $Y_k \to C_k \to Y_{k-1}$ is part of a triangle),
(ii) $C_k$ is a free $A$-module,
(iii) there is a map $H_*(C_0) \to X$ such that the following is an exact sequence

$$H_*(C_n) \to \cdots \to H_*(C_0) \to X \to 0.$$ 

Maps of $n$-Postnikov systems are maps of diagrams in the derived category which restrict to maps of triangles.

We say that an $n$-Postnikov system is based on the bar resolution if $H_*(C_\ast)$ is isomorphic to the bar resolution for $X$. A map is based on the bar resolution if the maps $H_*(C_k) \to H_*(C'_k)$ are of the form $g \otimes 1 \otimes (k+1)$ with $g : X \to X'$ a map of $H_*(A)$-modules.

Remark 5.2. The previous definition differs from the definition of $n$-Postnikov system in [BKS, Definition A.6] in that the homotopy fiber of $i_n$ is included in the diagram. This distinction is only relevant when considering maps of Postnikov systems.

A simple diagram chase shows (see [BKS, Lemma A.12]) that an $n$-Postnikov system yields an exact sequence

$$0 \to X[n-1] \to H_*(Y_{n-1}) \to H_*(C_{n-1}) \to \cdots \to H_*(C_0) \to X \to 0.$$ 

It follows from the proof of this result that the following diagram commutes

$$
\begin{array}{c}
0 \\
\downarrow
\end{array}
\begin{array}{c}
X[n-1] \\
\downarrow
\end{array}
\begin{array}{c}
H_*(Y_{n-1}) \\
\downarrow
\end{array}
\begin{array}{c}
H_*(C_0)[n-1] \\
\downarrow
\end{array}
\begin{array}{c}
0
\end{array}
$$

where $j$ denotes the composite

$$H_*(C_0)[n-1] \to H_*(Y_1)[n-2] \to \cdots \to H_*(Y_{n-2})[1] \to H_*(Y_{n-1}).$$ 

Note that when $C_\ast$ is the bar resolution, the nontrivial vertical map in (5.1) is the multiplication map $X \otimes H_*(A) \to X$.

Theorem 5.3. Let $X$ be an $A_{n+1}$-module over $H_*(A)$, $R_k = X \otimes H_*(A)^{\otimes k} \otimes A$ and $Y_k = B_k(X, H_*(A), A)[k]$. Then the following diagram of $A$-modules projects
to an $n$-Postnikov system for $X$:

\[
\begin{array}{cccccc}
Y_n & Y_{n-1} & Y_{n-2} & \cdots & Y_1 \\
\pi_n & i_n & \pi_{n-1} & i_{n-1} & \pi_{n-2} & \pi_1 \\
R_n & R_{n-1} & R_{n-2} & \cdots & R_1 & R_0.
\end{array}
\]

Here $\pi_k$ denotes the projection onto the last summand and

\[
i_k = \begin{bmatrix}
(-1)^{\left\lfloor \frac{k}{2} \right\rfloor} M_{k,k+1} \\
& \vdots \\
(-1)^{\left\lfloor \frac{j-1}{2} \right\rfloor} M_{k,j} \\
& \vdots \\
M_{k,2}
\end{bmatrix}
\]

This assignment is functorial with respect to maps of $A_{n+1}$-modules.

Proof. By definition

\[Y_k = R_0[k] \oplus \cdots \oplus R_{k-1}[1] \oplus R_k\]

as a graded $A$-module, and the $ij$-th entry ($1 \leq i \leq j \leq k + 1$) of the matrix $\partial Y_k$ is

\[(-1)^{k-j+1} \binom{j}{i} M_{j-1,j-i+1}.
\]

Therefore the differential on $Y_k$ satisfies the following inductive formula

\[\partial Y_k = \begin{bmatrix}
-\partial Y_{k-1} & i_k \\
0 & d
\end{bmatrix}.
\]

It follows that $i_k$ is a map of differential graded modules because this condition is precisely the condition that the upper right hand vector in the matrix $\partial Y_k$ vanishes. Clearly $Y_k$ is the homotopy fiber of $i_k: R_k \to Y_{k-1}$. Finally, functoriality follows from Proposition 4.5. □

**Definition 5.4.** The canonical $n$-Postnikov system associated to an $A_{n+1}$-structure on $X$ is the Postnikov system defined in Theorem 5.3.

**Theorem 5.5.** Let $X$ be an $A_{n+1}$-module over $H_*(A)$. There is a bijective correspondence between the sets of

(i) $A_{n+2}$-structures $(m^X_2, \ldots, m^X_{n+1}, \phi)$ on $X$,

(ii) lifts in the homotopy category

\[
\begin{array}{c}
Y_n \\
\pi_n \\
R_{n+1} \xrightarrow{D} R_n.
\end{array}
\]

The assignment sends $(m^X_2, \ldots, m^X_{n+1}, \phi)$ to the homotopy class of the map $i_{n+1}$ defined in Theorem 5.3 from the $A_{n+2}$-structure.

In other words, an $A_{n+1}$-structure on $X$ extends one stage iff its associated canonical $n$-Postnikov system extends one stage and, in that case, the extensions are in bijective correspondence.
Proof. The canonical $n$-Postnikov system associated to the $A_{n+1}$-structure on $X$ extends if and only if the map $i_n D$ is null. Since $R_{n+1}$ is a free $A$-module, this is equivalent to $H_*(i_n D)$ being the zero map. As $X[n-1] \rightarrow H_*(Y_{n+1})$ is an inclusion, this amounts to the vanishing of the map $i_n D$ in the commutative diagram

$$
\begin{align*}
\xymatrix{H_*(D) 
\ar[r]^{H_*(i_n D)} & H_*(R_{n+1}) 
\ar[r]^{H_*(D)} & H_*(R_n) 
\ar[r]^{H_*(i_n D)} & H_*(Y_{n-1}) 
\ar[r]^{H_*(D)} & H_*(R_{n-1}) 
\ar[r]^{H_*(i_n D)} & H_*(R_{n-2}).
}\end{align*}
$$

We will show that

$$(5.2) \quad i_n D = (-1)^{\lceil \frac{n+1}{2} \rceil + n+1} \left( \sum_{2 \leq r+t+1 \leq n+1} (-1)^{r+t} m_r^X (1 \otimes m_{s+t}) \right) * 1$$

Lemma 3.1 then implies that the canonical $(n+1)$-Postnikov system extends if and only if the $A_{n+1}$-structure extends to an $A_{n+2}$-structure.

To prove (5.2), we need to compute $H_*(i_n D)$. We will add a null homotopy map to $i_n D$ in order to perform the computation. For $n \geq 2$, let $H_n : R_{n+1} \rightarrow Y_{n-1}$ be the map defined by the column vector

$$
\begin{bmatrix}
(-1)^{\lceil \frac{n+1}{2} \rceil + n+1} 1 \otimes m_{n+2}^A \\
(-1)^{\lceil \frac{n+2}{2} \rceil} m_{n+1,n+1} \\
\vdots \\
(-1)^{\lceil \frac{n+2}{2} \rceil} m_{n+1,n+3-j} \\
\vdots \\
M_{n+1,3}
\end{bmatrix}
$$

We now compute the effect of the map

$$i_n D + (\partial Y_{n-1} H_n + H_n d)$$

on homology. We will show first that $i_n D + (\partial Y_{n-1} H_n + H_n d)$ factors through $R_0[n-1]$: For $i \geq 2$ the $i$-th component of this map is

$$
(-1)^{\lceil \frac{n+1}{2} \rceil} M_{n,n+2-i} M_{n+1,2} + \sum_{j=1}^{n-i} (-1)^{n-j+[(i-j)/2]+[(n-j)/2]} M_{j-1,j+1-i} M_{n+1,n+3-j} + \cdots + (-1)^{[(n-i)/2]} M_{n+1,n+3-i} M_{n+1,1}
$$

and this simplifies to

$$
(-1)^{n+\lceil \frac{n+1}{2} \rceil} \sum_{j=1}^{n+2} (-1)^{j(n+i)} M_{j-1,j+1-i} M_{n+1,n+3-j}
$$

which up to sign is exactly the sum (4.2) and therefore vanishes (only the $A_{n+1}$-structure is used).

The first component of $i_n D + (\partial Y_{n-1} H_n + H_n d)$ is

$$(5.3) \quad (-1)^{\lceil \frac{n+1}{2} \rceil} \sum_{j=2}^{n+1} (-1)^{j(n+1)} M_{j-1,j} M_{n+1,n+3-j} + d(1 \otimes m_{n+2}^A) + (-1)^{n+1} 1 \otimes (m_{n+2}^A d)$$
Using the $A_\infty$-$H_*(A)$-module structure on $A$, a computation similar to the proof of Lemma 4.3 shows that this formula simplifies to

$$ (5.4) \quad (-1)^{\lfloor \frac{n+1}{2} \rfloor + n + 1} \left( \sum_{2 \leq r + s + t \leq n + 1} (-1)^{r + s + t} m_r^X \right. $$

$$ \left. \otimes m_s^X \otimes 1^{\otimes t} \right) \otimes 1. $$

By the commutativity of diagram (5.1), the map $H_*(R_{n+1}) \to X[1-1]$ is obtained by composing (5.4) with the multiplication map $X \otimes H_*(A) \to X$. This proves (5.2).

It remains to prove the bijection between $A_{n+2}$-structures extending the given $A_{n+1}$-structure and the extensions of the canonical $n$-Postnikov system when an extension exists. In that case, the $A_{n+2}$-structures are arbitrary $k$-module maps

$$ \phi: X \otimes H_*(A)^{\otimes (n+1)} \to X[n]. $$

On the other hand, a homotopy class of maps $j: R_{n+1} \to Y_n$ lifting $D$ is the same as an $H_*(A)$-module map $H_*(R_{n+1}) \to X[n]$. Writing $i_{n+1}(\phi)$ for the lift associated to a $k$-module map $\phi$, the formula for $i_{n+1}(\phi)$ shows that $i_{n+1}(\phi) - i_{n+1}(0)$ factors through $R_0[n]$ and hence (see diagram (5.1)) the $H_*(A)$-module map associated to $i_{n+1}(\phi) - i_{n+1}(0)$ is

$$ \phi \ast 1: H_*(R_{n+1}) \to X[n]. $$

This shows that homotopy classes of lifts of $D$ are in bijective correspondence with $k$-module maps $X \otimes H_*(A)^{\otimes (n+1)} \to X[n]$ and completes the proof. \qed

**Corollary 5.6.** Any $n$-Postnikov system based on the bar resolution for $X$ is isomorphic to the canonical $n$-Postnikov system associated to an $A_{n+1}$-structure on $X$.

**Proof.** For $n = 1$ the statement is clearly true. The result follows by induction from Theorem 5.5. \qed

**Lemma 5.7.** Let $(g_1, \ldots, g_k)$ be an $A_k$-map between two $A_{k+1}$-modules $X$ and $X'$. Then the square

$$ \begin{array}{c}
R_k \\
\downarrow \\
R_k'
\end{array} \quad \begin{array}{c}
Y_{k-1} \\
\downarrow \\
y_{k-1}
\end{array} $$

$$ \begin{array}{c}
Y'_{k-1} \\
\downarrow \\
B_{k-1}(g)
\end{array} \quad \begin{array}{c}
Y_{k-1}' \\
\downarrow \\
B_{k-1}(g)
\end{array} $$

commutes up to homotopy if and only if $(g_1, \ldots, g_k, 0)$ is an $A_{k+1}$-map.

**Proof.** Because $g_1$ is a map of $H_*(A)$-modules, the square

$$ \begin{array}{c}
R_k \\
\downarrow \\
R_k'
\end{array} \quad \begin{array}{c}
R_{k-1} \\
\downarrow \\
R_{k-1}'
\end{array} $$

commutes strictly and so the difference on homology lies in the kernel of $Y_{k-1}' \to R_{k-1}'$. This kernel is a desuspension of $X$. We want to compute the map

$$ B_{k-1}(g)i_k - i_k'(g_1 \otimes 1^{\otimes (k+1)}): H_*(R_k) \to X'[k - 1] $$
We will add a null-homotopic map so as to make the factorization of this map through $R'_0[k - 1]$ apparent. The homotopy is given by the formula

$$H_k = \begin{bmatrix}
0 \\
(-1)^{\frac{k}{2}} g_k \otimes 1^\otimes 2 \\
\vdots \\
(-1)^{\frac{k}{2} + i} g_{k-i+2} \otimes 1^\otimes i \\
\vdots \\
-g_2 \otimes 1^\otimes k
\end{bmatrix}.$$  

One computes that

$$B_{k-1}(g)j_k - i_k'(g_1 \otimes 1^\otimes (k+1)) + (\partial Y'_{k-1}, H_k + H_k d)$$

has all components zero except the first one because $(g_1, \ldots, g_k)$ is an $A_k$-map. When composed with the multiplication $m^X: R'_0 \to X$, the first component yields the $(k + 1)$-ary map whose vanishing is synonymous with $(g_1, \ldots, g_k, 0)$ being an $A_{k+1}$-map. □

Using the functor $B_n(\cdot, H_*(A), A)$ from the last section, we can now complete the proof of Theorem 1.1.

**Theorem 5.8.** Let $X$ and $X'$ be (minimal) $A_{n+1}$-$H_*(A)$-modules. There is a bijective correspondence between $A_{n+1}$-maps $g: X \to X'$, and maps between the associated canonical $n$-Postnikov systems based on the bar resolution.

**Proof.** Given $g$, the desuspensions of the maps $B_k(g), 1 \leq k \leq n$, described in Proposition 4.4 give the desired map of Postnikov systems. It is easy to check that this assignment is injective (if two $A_n$-maps first differ on $g_k$, the induced maps $Y_{k-1} \to Y'_{k-1}$ will not be homotopic).

The converse is proved by induction. For $n = 1$, a map of Postnikov systems of the sort described above is determined by a map of $H_*(A)$-modules $g_1: X \to X'$ and a map

$$f_1: Y_1 \to Y'_1$$

such that

$$\begin{array}{ccc}
Y_1 & \xrightarrow{f_1} & R_1 \\
\downarrow & & \downarrow g_1 \otimes 1 \\
Y'_1 & \xrightarrow{R'_1} & R'_0
\end{array}$$

is a map of triangles. Thus $f_1$ can be represented by a matrix

$$\begin{bmatrix}
g_1 \otimes 1 & \tilde{g}_2 \\
0 & g_1 \otimes 1 \otimes 1
\end{bmatrix}.$$  

Since $g_1$ is a map of $H_*(A)$-modules, the matrix above with $\tilde{g}_2 = 0$ also defines a map of triangles. The difference between these two matrices factors as

$$Y_1 \to R_1 \to R'_0[1] \to Y'_1$$

There is a unique representative for the homotopy class of the middle map of the form $g_2 \otimes 1$ and therefore $f_1$ has a unique representative of the form

$$\begin{bmatrix}
g_1 \otimes 1 & -g_2 \otimes 1 \\
0 & g_1 \otimes 1 \otimes 1
\end{bmatrix}.$$
The only requirement for \((g_1, g_2)\) to be a map of \(A_2\)-modules is that \(g_1\) commutes with the multiplication. This completes the proof for \(n = 1\).

Suppose given a map of \(k\)-Postnikov systems based on the bar construction. By induction we know that there is a unique map \(g = (g_1, \ldots, g_k)\) of \(A_k\)-modules such that \(Y_j \to Y'_j\) is \(B_j(g)\) for \(j \leq k - 1\).

There is a commutative square

\[
\begin{array}{ccc}
Y_k & \to & R_k \\
\downarrow f_k & & \downarrow g_1 \otimes 1^{(k+1)} \\
Y'_k & \to & R'_k \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \downarrow B_{k-1}(g) \\
Y'_{k-1} & \to & R'_{k-1} \\
\end{array}
\]

By Lemma 5.7, \((g_1, \ldots, g_k, 0)\) is an \(A_{k+1}\)-module map. Let \(d = f_k - B_{k+1}(g_1, \ldots, g_k, 0)\).

This difference factors as

\[
Y_k \to R_k \to Y'_{k-1}[1] \to Y'_k.
\]

The homotopy class of a map from \(R_k\) is determined by its effect on homology. Since \(R_k \to Y'_k\) factors through \(Y'_{k-1}[1]\), it is 0 along \(R'_k\) and hence its image lies in the kernel of the map \(Y'_k \to R'_k\) which is \(X' [k]\).

Therefore it factors through \(R'_k[k]\) up to homotopy and the homotopy class is therefore represented uniquely by a map of the form \((-1)^k g_{k+1} \otimes 1\). We conclude that the homotopy class of \(f_k\) is equal to that of \(B_{k+1}(g_1, \ldots, g_{k+1})\) (note that any choice of \(g_{k+1}\) will give an \(A_{k+1}\)-map). \(\square\)

The previous Theorem show that the functor sending an \(A_{n+1}\)-structure to its canonical \(n\)-Postnikov system is full and faithful. Corollary 5.6 asserts that this functor is essentially surjective hence it is an equivalence of categories. This completes the proof of Theorem 1.1.

Remark 5.9. It follows from Theorem 5.8 that if \(g : X \to X'\) is an \(A_k\)-map such that \(g_1\) is an isomorphism then \(g\) is an isomorphism.

Let \(X\) be an \(H_*(A)\)-module. The moduli groupoid of \(A_{n+1}\)-structures on \(X\) is the groupoid with objects \(A_{n+1}\)-module structures on \(X\) and quasi-isomorphisms \(g\) between them with \(g_1 = \text{id}\). Note that this is equivalent to the groupoid of \(A_{n+1}\)-modules \(X'\) together with an isomorphism of \(H_*(A)\)-modules \(X' \to X\).

Corollary 5.10. The moduli groupoid of \(A_{n+1}\)-structures on \(X\) is equivalent to the groupoid of \(n\)-Postnikov systems for \(X\) based on the bar resolution and isomorphisms which are the identity on the bar resolution.

Appendix A. Relation between realization of Postnikov systems and chain complexes

In this appendix we explain the relation between Postnikov systems and rigidifying complexes in a homotopy category (see [DKS] for the general theory of realizing diagrams). We explain this in the setting of model categories (see [Ho]). The model category \(\mathcal{C}\) which is relevant for this paper is the category of differential graded modules over a DGA \(A\) with the standard projective model structure (see for example [SS]).
Definition A.1. Let \( C \) be a pointed category. A chain complex in \( C \) is a sequence of maps in \( C \)
\[
\cdots \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} C_0
\]
such that \( dd = * \).

Definition A.2. If \( C \) is a pointed model category, a Postnikov system is a commutative diagram in \( \text{Ho}(C) \)
\[
\begin{array}{ccc}
Y_n & \xrightarrow{i_{n-1}} & Y_{n-1} \\
\downarrow{j_n} & \downarrow{j_{n-1}} & \downarrow{j_1} \\
C_n & \xrightarrow{i_{n-1}} & C_{n-1} \\
\cdots & \cdots & \cdots \\
C_1 & \xrightarrow{i_0} & C_0
\end{array}
\]
where for each \( k \), the sequence
\[
Y_k \xrightarrow{j_k} C_k \xrightarrow{i_{k-1}} Y_{k-1}
\]
is a homotopy fiber sequence (we set \( Y_0 = C_0 \)).

An \( m \)-Postnikov system is a diagram as above but with objects only those \( Y_i \) and \( C_i \) where \( i \geq m + 1 \).

Note that in a Postnikov system, \( C_\bullet \) is a chain complex in \( \text{Ho}(C) \).

Definition A.3. Let \( C \) be a model category, \( \pi : C \to \text{Ho}(C) \) be the canonical functor and \( I \) be a small category. A diagram \( F : I \to \text{Ho}(C) \) is realizable if there exists a diagram \( \tilde{F} : I \to C \) together with a natural isomorphism \( \phi : \pi \tilde{F} \to F \). The diagram \( \tilde{F} \) is then called a realization of \( F \).

If \( C \) is pointed we say that a diagram is strictly realizable if \( F(\alpha) = * \) implies that \( \tilde{F}(\alpha) = * \) and \( \tilde{F} \) is then called a strict realization of \( F \).

Proposition A.4. Let \( C \) be a pointed model category. Let \( C_\bullet \) be a chain complex in \( \text{Ho}(C) \). Then the following are equivalent:

(i) \( C_\bullet \) is strictly realizable,

(ii) \( C_\bullet \) extends to a Postnikov system,

Proof. (i) \( \Rightarrow \) (ii): Replacing \( C_\bullet \) if necessary by an isomorphic complex we may assume that
\[
\cdots \to C_n \xrightarrow{d_{n-1}} C_{n-1} \to \cdots
\]
is a chain complex in \( C \) projecting to \( C_\bullet \).

Replacing the map \( C_1 \xrightarrow{d_0} C_0 \) by a fibration we obtain a diagram
\[
\begin{array}{ccc}
C_3 & \xrightarrow{d_2} & C_2 \\
\downarrow{i_1} & \downarrow{i_0} & \downarrow{\sim} \\
Y_1 & \xrightarrow{Y_1} & C_1' \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\ast & \ast & \ast & \ast
\end{array}
\]
where \( Y_1 \) is the homotopy fiber of \( C_1' \to C_0 \). Since \( d_0 i_1 = * \), there is a canonical factorization \( C_2 \xrightarrow{i_1} Y_1 \). Furthermore, the composite
\[
C_3 \xrightarrow{d_2} C_2 \xrightarrow{i_1} Y_1
\]
is the zero map since its composite with the map \( Y_1 \to C_1' \) is zero by construction of \( i_1 \).
We may apply the same procedure to the sequence of maps

\[ \cdots \rightarrow C_3 \xrightarrow{d_2} C_2 \xrightarrow{i_1} Y_1 \]

and continuing inductively we obtain a Postnikov system which we denote by \( P(C_\bullet) \).

This construction is clearly functorial so we have defined a functor

\[(A.1) \quad P : \mathcal{C} \rightarrow \mathcal{PS} \]

from the category of chain complexes in \( \mathcal{C} \) to the category of Posnikov systems in \( \text{Ho}(\mathcal{C}) \) which sends weak equivalences to isomorphisms.

(ii) ⇒ (i): Let

\[
\begin{array}{cccc}
Y_n & Y_{n-1} & Y_1 \\
\downarrow j_n & \downarrow j_{n-1} & \downarrow j_1 \\
C_n & C_{n-1} & C_1 & C_0
\end{array}
\]

be a Postnikov system in \( \text{Ho}(\mathcal{C}) \).

We will write \( f \) for an arbitrary representative of the map \( f \in \text{Ho}(\mathcal{C}) \) and \( [\psi] \) for the homotopy class of \( \psi \in \mathcal{C} \).

First note we can assume that all the objects \( Y_k \) and \( C_k \) are fibrant and cofibrant.

We will construct a chain complex \( \tilde{C}_\bullet \) in \( \mathcal{C} \) lifting \( C_\bullet \) inductively.

Let \( \tilde{C}_0 = C_0 \). Let

\[
\begin{array}{ccc}
C_1 & \xrightarrow{\phi_1} & \tilde{C}_1 \\
\downarrow d_0 & \nearrow d_0 \\
C_0 & & \\
\end{array}
\]

be a factorization of \( \tilde{d}_0 \) into a trivial cofibration followed by a fibration and

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{j_1} & \tilde{C}_1 \\
\downarrow \psi_1 & & \downarrow [\phi_1] \\
C_1 & \xrightarrow{[j_1]} & \tilde{C}_1
\end{array}
\]

be the inclusion of the fiber of \( \tilde{d}_0 \). Since \( Y_1 \) is the homotopy fiber of \( d_0 \), there is an isomorphism \( \psi_1 : Y_1 \rightarrow \tilde{Y}_1 \) such that

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{\psi_1} & \tilde{Y}_1 \\
\downarrow j_1 & & \downarrow [j_1] \\
C_1 & \xrightarrow{[\phi_1]} & \tilde{C}_1
\end{array}
\]

commutes.

Now factor \( \psi_1 \tilde{j}_1 : C_2 \rightarrow \tilde{Y}_1 \) (which exists because \( C_2 \) is cofibrant and \( \tilde{Y}_1 \) is fibrant) as a trivial cofibration \( \phi_2 \) followed by a fibration \( i_1 \). We get a commutative diagram

\[
\begin{array}{ccc}
C_2 & \xrightarrow{\phi_2} & \tilde{C}_2 \\
\downarrow i_1 & \nearrow i_1 \\
Y_1 & \xrightarrow{\psi_1} & \tilde{Y}_1
\end{array}
\]

Let \( \tilde{d}_1 = \tilde{j}_1 \tilde{i}_1 \). Since \( \tilde{Y}_1 \) is the fiber of \( \tilde{d}_0 \) it follows that the composite \( \tilde{d}_0 \tilde{d}_1 \) is the zero map.
Let \( \tilde{j}_2 : \tilde{Y}_2 \to C_2 \) denote the inclusion of the fiber of \( \tilde{i}_1 \). Since \( Y_2 \to C_2 \to Y_1 \) is a fiber sequence, there is an isomorphism \( \psi_2 : Y_2 \to \tilde{Y}_2 \) in \( \text{Ho}(\mathcal{C}) \) such that
\[
\begin{array}{ccc}
Y_2 & \xrightarrow{\psi_2} & \tilde{Y}_2 \\
\downarrow{j_2} & & \downarrow{[j_2]} \\
C_2 & \xrightarrow{[\phi_2]} & \tilde{C}_2
\end{array}
\]
commutes and we can proceed inductively to obtain a realization \( \tilde{C}_* \) of \( C_* \). □

**Remark A.5.** The statements in Proposition [A.4] are equivalent to the vanishing of the Toda bracket \( \langle d_0, \ldots, d_n \rangle \) for all \( n \geq 2 \). The Toda bracket can be defined in several different ways. We use the following definition: \( \langle d_0, \ldots, d_n \rangle \) is a subset (possibly empty) of \( \text{Ho}(\mathcal{C})(C_{n+1}, \Omega^{n-1} C_0) \) consisting of all possible lifts \( \phi \) in diagrams of the form (A.2), for all choices of \( n \)-Postnikov systems extending \( C_n \to \cdots \to C_0 \).

\[(A.2) \]
\[
\begin{array}{ccc}
\Omega^{n-1} C_0 & \xrightarrow{d_0} & C_n \\
\uparrow & & \downarrow{d_n} \\
\Omega^{n-2} C_0 & \xrightarrow{d_{n-1}} & C_{n-1} \\
\uparrow & & \downarrow{d_{n-2}} \\
\cdots & & \cdots \\
\Omega Y^{n-2} & \xrightarrow{Y_{n-1}} & Y_n \\
\uparrow & & \downarrow{Y_{n-1}} \\
C_{n+1} & \xrightarrow{d_n} & C_n \\
\phi & & \\
\uparrow{i_{n-1}} & & \downarrow{j_{n-1}} \\
Y_{n-1} & \xrightarrow{Y_{n-1}} & Y_1 \\
\uparrow & & \downarrow{d_0} \\
C_0 & \xrightarrow{d_0} & C_0
\end{array}
\]

\( \Omega^j \) denotes the \( j \)-th iterate of the loop functor and the vertical maps belong to the homotopy fiber sequences which end in \( Y_1 \to C_k \to Y_{k-1} \) (see [Ho] Chapter 6).

We say that a Toda bracket vanishes if it contains the zero map. It is clear that the \( n \)-Postnikov system in (A.2) extends one stage if and only if \( \phi \) can be chosen to be zero. Thus, an \( n \)-Postnikov system encodes the vanishing of the Toda bracket of the maps in the underlying chain complex.

The higher order cohomology operations in [Ma, 16.3] are defined as Toda brackets with the above definition. The definition of Toda bracket in [Wh IV.1] is very similar. Whitehead works in a stable setting where cofiber and fiber sequences are equivalent. To define the Toda bracket of \( < d_0, \ldots, d_n > \) he considers all possible
where the sequences $X_i \to C_i \to X_{i-1}$ are cofiber sequences and defines the Toda bracket to be the set of all possible extensions of $d_0 i_1$ along

$$X_1 \to \Sigma X_2 \to \cdots \to \Sigma^{n-1} C_n$$

where $\Sigma$ denotes the suspension functor.

It is possible to check that our definition and Whitehead’s agree by exhibiting both sets as certain choices of $(n-1)$-spheres $\partial \Delta^n \subset \text{Hom}(C_n, C_0)$ in the homotopy function complex from $C_n$ to $C_0$. For more on this perspective, see [BM, Examples 3.10, 3.20].

**Proposition A.6.** Let $\mathcal{C}$ be a pointed model category, $\mathcal{CC}$ be the category of length $n$ chain complexes in $\mathcal{C}$ (with $n \leq \infty$) and $\mathcal{PS}$ be the category of $n$-Postnikov systems in $\text{Ho}(\mathcal{C})$. Then the functor

$$P: \mathcal{CC} \to \mathcal{PS}$$

(see (A.1)) induces a bijection from weak equivalence classes in $\mathcal{CC}$ to isomorphism classes of objects in $\mathcal{PS}$.

**Proof.** A chain complex realizing a Postnikov system will, by definition, realize any equivalent Postnikov system so we have already proved in Proposition A.4 that the functor $P$ is essentially surjective.

On the other hand using the homotopy lifting property for fibrations, the construction of the chain complex from the Postnikov system in Proposition A.4 will also yield lifts of isomorphisms between Postnikov systems to weak equivalences between chain complexes in $\mathcal{C}$ (because the $\tilde{C}_i$ are fibrant and cofibrant, the $Y_i$ are the actual fibers of maps and the maps $\tilde{C}_{i+1} \to Y_i$ are fibrations). \hfill \Box

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3It is easy to check using the limited naturality of triangles that in the definition of Toda bracket in [Wh] we may assume that either the map $i_0$ or $j_n$ are the identity and we are taking $j_n$ to be the identity.
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