Off-shell $\mathcal{N} = (1, 0), D = 6$ supergravity from superconformal methods

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Abstract

We use the superconformal method to construct the full off-shell action of $\mathcal{N} = (1, 0), D = 6$ supergravity, which has apart from the graviton and the gravitino, a 2-form gauge field, a dilaton and a symplectic Majorana spinor. We give detailed formula for superconformal expressions that can be useful for extensions of the theory to more matter multiplets or gauged supergravity.

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1 Introduction

In [1] Salam and Sezgin argued that $\mathcal{N} = (1,0), D = 6$ Poincaré supergravity coupled to a U(1) vector multiplet spontaneously compactifies on $\mathcal{M}_4 \times S^2$ giving rise to chiral fermions and breaking supersymmetry down to $\mathcal{N} = 1$. In view of extending this model and to obtain a more realistic compactification scheme, general couplings of $\mathcal{N} = (1,0), D = 6$ supergravity to different types of matter multiplets were constructed in [2]. These couplings were constructed by using the method of superconformal tensor calculus. This method, developed in [3, 4, 5, 6, 7, 8], provides a very convenient framework to study general matter couplings to supergravity theories. After writing down the superconformal theory, suitable gauge conditions (breaking superconformal symmetries down to super-Poincaré symmetries) give rise to different formulations of Poincaré supergravity.

The minimal field content of $\mathcal{N} = (1,0), D = 6$ supergravity contains an antisymmetric tensor with a self-dual field strength, often called chiral 2-forms. This leads to difficulties for a Lagrangian formulation, similar to those for type IIB supergravity. We refer to [9] for a discussion on the possibilities for the Lagrangians of theories with chiral 2-forms. The action in [2] is constructed by combining the anti-self dual field strength with a self-dual field strength that is present in a tensor multiplet. This will lead to supergravity with a physical (non-chiral) 2-form, avoiding the chirality problems. We will denote this theory as ‘off-shell $\mathcal{N} = (1,0), D = 6$ supergravity’.

In [2] only the bosonic part of this action was explicitly constructed. In this article we will discuss this method in detail and construct the full (fermionic plus bosonic) action for minimal 6 dimensional supergravity with auxiliary fields using superconformal tensor calculus.

The off-shell Poincaré action has several applications. When obtaining this action by gauge fixing the redundant conformal symmetries of the superconformal action, the SU(2) R-symmetry group is broken down to U(1). A vector multiplet action can then be added to the off-shell Poincaré action, gauging this U(1) R-symmetry. Hence, one obtains an off-shell theory that is dual to the Salam-Sezgin model, which is 6-dimensional gauged Maxwell-Einstein supergravity.

In a next step, curvature squared terms ($R^2$-terms) can be added to this dual Salam-Sezgin model, and the influence of these terms to solutions of the model (like the $\mathcal{M}_4 \times S^2$ solution or brane solutions) can be studied. A convenient trick to construct supersymmetric $R^2$-terms was developed in [10, 11]. This $R^2$-action is off-shell, hence justifying the need for a full off-shell formulation of Poincaré supergravity.

Another application is related to [12]. In this article a three-parameter family of massive $\mathcal{N} = 1$ supergravities in $D = 3$ is obtained from the $S^3$ reduction of an off-shell $D = 6$ Poincaré supergravity that includes an $R^2$-term. Only the bosonic terms of these actions are obtained via compactification. The fermionic part is constructed afterwards via the Noether method. From a full off-shell Poincaré action in six dimensions, also the fermionic terms of these three dimensional theories can be reached directly from compactification.

As said above the aim of this article is to construct an off-shell action for Poincaré supergravity using superconformal methods. The fact that the minimal field content of
\( \mathcal{N} = (1,0), D = 6 \) supergravity contains an anti-self dual field strength is reflected in the superconformal theory by the fact that the Standard Weyl multiplet that is known from \( D = 4 \) \[^{[5]}\), and \( D = 5 \) \[^{[13,14,15]}\) contains an anti-self dual tensor \( T_{abc} \). We denote this here as the Weyl 1 multiplet. Combining this multiplet with a superconformal tensor multiplet, which has a self-dual tensor leads to a superconformal multiplet with a physical (non-chiral) tensor (Weyl 2 multiplet). To obtain meaningful superconformal actions, one needs also a further ‘compensating multiplet’. Two choices have been discussed in \[^{[2]}\): a hypermultiplet and a linear multiplet. In order to get an off-shell Poincaré theory, the linear multiplet is required. The different multiplets and couplings will be reviewed in detail in section 2.

This section also contains different tables in which the reader can follow the set of independent fields in each step of the construction. They contain also the counting of degrees of freedom (dof) without use of field equations (‘off-shell counting’). The final super-Poincaré theory contains 48+48 off-shell dof. After use of the field equations they are reduced to 16+16: graviton, scalar and 2-rank antisymmetric tensor on the bosonic side, and gravitino and a simple spinor on the fermionic side.

In Section 3 the action is constructed explicitly. First the superconformal action is built by coupling a linear multiplet to the Weyl 1 multiplet. After gauge fixing and expressing the action in terms of the Weyl 2 multiplet, the Poincaré action is obtained.

In Section 4 we write down our conclusions. Appendix A sets out our notation and conventions and appendix B summarizes the commutation relations of the \( \mathcal{N} = (1,0), D = 6 \) superconformal algebra. Finally, in Appendix C we discuss the construction of a vector multiplet from the components of a linear multiplet, which is used to construct an action for the linear multiplet in Section 3.

## 2 Outline of the procedure

### 2.1 The \( \mathcal{N} = (1,0), D = 6 \) Weyl multiplet

In this section we will construct the \( \mathcal{N} = (1,0), D = 6 \) Weyl multiplet. We will mainly follow the outline of \[^{[2]}\). We begin our discussion with the superconformal algebra in 6 dimensions: OSp(6,2|1). The bosonic generators of this algebra are the generators of the conformal group (translations \( P_a \), rotations \( M_{ab} \), dilatations \( D \) and special conformal transformations \( K_a \)) plus an SU(2) triplet generator \( U_i^j \). The fermionic generators are the supersymmetries \( Q_i^\alpha \) and the ‘special’ supersymmetries \( S_i^a \). The nonzero commutators between these generators and their properties are given in appendix B.

To each generator \( T_A \) of the superconformal algebra we assign a gauge field \( h_A^\mu \) in the following way:

\[
h_A^\mu T_A = e_{[a}^\mu P_a + \frac{1}{2} \omega_{\mu}^{ab} M_{ab} + b_\mu D + f_{\mu}^a K_a + \bar{\psi}_i^\mu Q_i + \bar{\phi}_i^a S_i + \gamma_{\mu}^{ij} U_i^j, \quad (2.1)
\]

with \( \psi_\mu^i \) and \( \phi_\mu^i \) SU(2) Majorana-Weyl spinors of positive and negative chirality, respectively. Using the structure constants \( f_{AB}^C \) of the superconformal algebra (given in appendix B) and
the basic rules

\[
\delta h^A_{\mu} = \partial_\mu \epsilon^A + \epsilon^C h^B_{\mu} f_{BC}^A, \\
R^A_{\mu
u} = 2\partial_{[\mu} h^A_{\nu]} + h^C_{\mu} h^B_{\nu} f^A_{BC},
\]

one can easily write down the (linear) transformation rules and the curvatures \( R^A_{\mu
u} \) of the superconformal gauge fields given in (2.1). Also, in order to achieve maximal irreducibility of the superconformal gauge field configuration one imposes a maximal set of conventional constraints \([16, 17]\) on these curvatures\(^1\). These constraints determine the (dependent) gauge fields \( \omega^{ab}, \phi^i, \) and \( f^a_\mu \) in function of the (independent) gauge fields \( e^a_\mu, \psi^i_\mu, b_\mu, \nu^{ij}_\mu \).

Counting the number of bosonic and fermionic degrees of freedom of these gauge fields, one finds that they do not match. Hence, the algebra does not close. Additional matter fields \( T^{abc}_\mu \), \( \chi^i \) and \( D \) must be added to the gauge fields in order to obtain a closed multiplet \([5, 6]\). \( T^{abc}_\mu \) is an antisymmetric tensor of negative duality, \( \chi^i \) is an SU(2) Majorana-Weyl spinor of negative chirality and \( D \) is a real scalar.

Starting from the linear transformation rules of the superconformal gauge fields, the curvatures \( R^A_{\mu
u} \) and the matter fields \( T^{abc}_\mu, \chi^i \) and \( D \) we can construct the full nonlinear \( \mathcal{N} = (1, 0), D = 6 \) Weyl multiplet by applying an iterative procedure outlined in \([2]\). The results are\(^2\)

\[
\delta \epsilon^a_\mu = \frac{1}{2} \epsilon^a_\mu \psi^i_\mu - \lambda_D e^a_\mu - \lambda^{ab} e_{\mu b}, \\
\delta \psi^i_\mu = \partial_\mu \psi^i + \frac{1}{2} b_\mu \epsilon^i + \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} \epsilon^i + \nu^{i j}_\mu \epsilon^j + \frac{1}{24} \gamma \cdot T \gamma_{\mu} \epsilon^i + \gamma_{\mu} \eta^i - \frac{1}{2} \lambda_D \psi^i - \lambda^{i} j \psi^j - \frac{1}{4} \lambda^{ab} \gamma_{ab} \psi^j, \\
\delta b_\mu = \partial_\mu \lambda_{D} - \frac{1}{2} \epsilon^i \eta^i - \frac{1}{2} \epsilon^i \psi^i, \\
\delta \nu^{ij}_\mu = \partial_\mu \nu^{ij} + 2 \lambda^{i k}_\mu \nu^j_k + 2 \epsilon^{i} \eta^{j} + 2 \bar{\eta}^{ij} \chi^j + \frac{1}{6} \bar{\epsilon}^{i} \gamma_{\mu} \chi^j, \quad (2.3)
\]

for the independent gauge fields of the multiplet. The transformation rules of the matter fields are

\[
\delta T^{abc}_\mu = 3 \lambda^d_{\mu} T^{abc d}_\mu + \lambda_D \mu T^{abc}_\mu - \frac{1}{32} \bar{\epsilon} \gamma^d \gamma_{abc} \hat{R}_{de}(Q) - \frac{7}{96} \bar{\epsilon} \gamma_{abc} \chi, \\
\delta \chi^i = -\frac{1}{4} \lambda^{ab} \gamma_{ab} \chi^i - \lambda^{i} j \chi^j + \frac{3}{2} \lambda_D \chi^i + \frac{1}{8} \nu^{i} j \gamma_{\mu} \epsilon^j - \frac{3}{2} \gamma \cdot T \gamma_{\epsilon} \chi^j + \frac{1}{4} \nu^{i} j \gamma_{\mu} \epsilon^j + \frac{1}{2} \gamma \cdot T^{-} \eta^j, \\
\delta D = 2 \lambda_D \epsilon^i + \bar{\epsilon} \gamma^i \partial_\mu \chi^i - 2 \bar{\eta} \chi^i, \quad (2.4)
\]

\(^1\)The full expression of the linear transformation rules and curvatures and a more detailed discussion on the conventional constraints is given in \([2]\).

\(^2\)The gauge field and parameter of the SU(2) R-symmetry is normalized with a factor \(-2\) difference w.r.t. \([2]\).
where
\[
D_\mu T_{abc}^- = \partial_\mu T_{abc}^- - 3\omega_\mu \gamma^d T_{bc|d}^+ - b_\mu T_{abc}^- + \frac{1}{32} \bar{\psi}_d \gamma^{de} \gamma_{abc} \hat{R}_{de}(Q) + \frac{7}{96} \bar{\psi}_d \gamma_{abc} \chi,
\]
and
\[
D_\mu \chi^i = \left( \partial_\mu - \frac{3}{2} b_\mu + \frac{1}{4} \omega_\mu \bar{\chi} \chi \right) \chi^i + \nu_{\mu} \chi^i - \frac{1}{8} (D_\nu \gamma \cdot T^-) \gamma^\nu \psi_\mu^i + \frac{3}{8} \gamma \cdot \hat{R}^{ij}(\nu) \psi_{\mu j} - \frac{1}{4} D_\nu \psi_\mu^j - \frac{1}{2} \gamma \cdot T^- \phi_\mu^i.
\]

The relevant modified curvatures \( \hat{R}_{\mu\nu}^A \) have the form
\[
\begin{align*}
\hat{R}_{\mu\nu}^a(P) &= 2 \partial_\mu e^a_\nu + 2 b_\mu e^a_\nu + 2 \omega_\mu \gamma^d e^a_{\nu d} - \frac{1}{2} \bar{\psi}_d \gamma^a \psi_\nu, \\
\hat{R}_{\mu\nu}^i(Q) &= 2 D_\mu \psi_\nu^i, \\
\hat{R}_{\mu\nu}^{ab}(M) &= 2 \partial_\mu \omega_\nu^{ab} + 2 \omega_\mu \gamma^d \omega_\nu^{bc} - 8 \nu_{\mu} \omega_\nu^{ab} + \bar{\psi}_\mu \gamma^{ab} \phi_\nu \\
&+ \bar{\psi}_\mu \gamma^{ab} \hat{R}_{\nu}^{bc}(Q) + \frac{1}{2} \bar{\psi}_\mu \gamma_{\nu} \hat{R}^{ab}(Q) - \frac{1}{6} \nu_{\mu} \bar{\psi}_\nu \gamma^{ab} \chi - \frac{1}{6} \bar{\psi}_\mu \gamma_{\nu} \chi T^{-abc}, \\
\hat{R}_{\mu\nu}^{ij}(V) &= 2 \partial_\mu V^i_\nu + 2 V^i_\nu T^{-abc} - 4 \bar{\psi}_\mu \gamma_{\nu} \chi - \frac{1}{3} \bar{\psi}_\mu \gamma_{\nu} \chi T^{-abc} + \frac{3}{26} \bar{\psi}_\mu \gamma_{\nu} \chi T^{-abc}, \\
\end{align*}
\]

where\(^3\)
\[
D_\mu \psi_\nu^i = \left( \partial_\mu + \frac{1}{2} b_\mu + \frac{1}{4} \omega_\mu \gamma_{\nu} \gamma_\mu \right) \psi_\nu^i + \nu_{\mu} \psi_\nu^i - \frac{1}{24} \gamma \cdot T^- \gamma_{\nu} \psi_\mu^i - \gamma_{\nu} \phi_\mu^i \\
\equiv \hat{D}_\mu \psi_\nu^i - \gamma_{\nu} \phi_\mu^i.
\]

Because of the deformation of the transformation rules and curvatures by the matter fields, the conventional constraints mentioned above must also be adapted. The following set of conventional constraints is chosen:
\[
\begin{align*}
\hat{R}_{\mu\nu}^a(P) &= 0, \\
\hat{R}_{\mu\nu}^{ab}(M) e^\nu_b - T_{\mu bc} T^{-abc} + \frac{1}{12} e^a_\mu D &= 0, \\
\gamma^\nu \hat{R}_{\mu\nu}^i(Q) &= -\frac{1}{6} \gamma_\nu \chi^i.
\end{align*}
\]

These constraints determine \( \omega_\mu \gamma^{ab}, \phi_\mu^{ij} \) and \( f_\mu^a \) in function of the independent gauge fields and the matter fields:
\[
\begin{align*}
\omega_\mu^{ab} &= 2 e^\nu_\mu \partial_\nu e_{\nu b} - e^\nu_\mu e_{\nu b} e_\mu \gamma_{\nu \sigma c} + \frac{1}{4} \left( 2 \bar{\psi}_\mu \gamma_\nu \psi^{b} + \bar{\psi}_\mu \gamma_{\nu} \psi^{b} \right), \\
f_\mu^a &= \frac{1}{8} \left( \hat{R}_\mu^a(M) - \frac{1}{10} e^a_\mu \hat{R}(M) \right) - \frac{1}{8} T_{\mu bc} T^{-acd} + \frac{1}{240} e^a_\mu D, \\
\phi_\mu^{ij} &= -\frac{1}{16} \left( \gamma^{ab} \gamma_{\mu} - \frac{3}{2} \gamma^{ab} \phi_\mu^{ij} \right) \hat{R}_{ab}^i(Q) - \frac{1}{60} \gamma_\mu \chi^i, \\
\end{align*}
\]

\(^3\)We will denote the full superconformal covariant derivative with \( D_\mu \) and use \( \hat{D}_\mu \) when we explicitly removed the term proportional to \( \phi_\mu^{ij} \) from the covariant derivative. The hats on the covariant derivatives \( \hat{D}_\mu \) express the fact that they still contain the terms proportional to the matter fields \( D, \chi \) and \( T^- \).
where \( \hat{R}'^a(M) \equiv \hat{R}'^{ab}(M)e^b \) and \( \hat{R}'(M) \equiv \hat{R}'^a(M)e^a \). The notation \( \hat{R}'(M) \) and \( \hat{R}'(Q) \) indicates that we have omitted the \( f^a \) dependent term in \( \hat{R}(M) \) and the \( \phi^i \) dependent term in \( \hat{R}(Q) \) respectively. Hence
\[
\hat{R}'_{\mu i}^\nu(Q) = 2\hat{\mathcal{D}}_{\mu}\psi_i^\nu,
\] (2.9)
and
\[
\hat{R}'^{a i}(Q) = e^\nu \hat{R}'_{\mu \nu}^i(Q).
\] (2.10)

Useful ‘traces’ of the dependent fields are
\[
e^{a\mu}\omega_{\mu a}^b = e^{-1}\partial_{\mu}(e^{ab}e) + \frac{1}{2}\bar{\psi}^a\gamma_a^b + 5b^b,
\]
\[
\gamma^\mu \phi^i_\mu = \frac{1}{5}\hat{\gamma}^{\mu \nu}\hat{\mathcal{D}}_{\mu}\bar{\psi}^i_\nu - \frac{1}{10}\chi^i,
\] (2.11)
\[
8f^a = \frac{2}{5}\left(-R + 5\bar{\psi}^a\phi_a + 2\bar{\psi}^\mu\gamma^\nu\hat{\mathcal{D}}_{\mu}\psi_\nu + \frac{5}{12}\bar{\psi}^b\gamma^b\chi - \frac{1}{2}\bar{\psi}^b\gamma_c^a\psi_c T^{abc}\right) + \frac{1}{5}D.
\]

For the last expression, we expanded the \( \hat{R}'(M) \) and used
\[
R = e^{\nu \mu}e^\mu \left(2\partial_{\mu}\omega_{\mu \nu} + 2\omega_{\mu \nu}^{ac}\omega_{\nu | c}^b\right).
\] (2.12)

From now on we will denote the multiplet described above as the Weyl 1 multiplet. Its independent components are summarized in Table 1. The middle column indicates the number of off-shell dof (subtracting the gauge invariances that are indicated on the right). From the table it is clear that the Weyl 1 multiplet constitutes a 40+40 off-shell multiplet.

**Table 1**

| Field       | \( w \) | Number of dof | Gauge Invariance |
|-------------|---------|---------------|-----------------|
| \( e_\mu^a \) | -1      | 15            | \( P_a, M_{ab} \) |
| \( b_\mu \) | 0       | 0             | \( K_a \)       |
| \( V_\mu^i \) | 0       | 15            | SU(2)           |
| \( T_{abc} \) | 1       | 10            |                 |
| \( D \) | 2       | 1             |                 |
|            | -1      | 1             |                 |
|            | 40      | 40            |                 |
|            | -1/2    | 40            | \( Q^i \)       |
|            | 3/2     | 40            |                 |
|            | -8      | 40            | \( S^i \)       |

**fermionic dof** 40
2.2 The Weyl multiplet with the tensor gauge field

We cannot use the Weyl 1 multiplet, introduced in the previous section, to write down an off-shell action for \( N = (1, 0), \ D = 6 \) supergravity. E.g. using the linear multiplet as compensator, as we will do below, leads to an action that lacks kinetic terms for the matter fields \( D, \chi^i \) and \( T_{-abc} \). Even worse is the fact that the field equation for the scalar \( D \) gives an inconsistency (we will show this explicitly in section 3.2). We can solve both problems by following the procedure outlined in the Introduction: coupling the Weyl 1 multiplet to a tensor multiplet. The tensor multiplet consists of a real scalar \( \sigma \), an SU(2) Majorana spinor \( \psi^i \) of negative chirality and a self-dual antisymmetric tensor field \( F^+_{abc} \). The superconformal algebra only closes on these fields modulo a number of constraints (to be discussed below). One of these constraints can be solved as a Bianchi identity for a new antisymmetric tensor gauge field \( B_{\mu \nu} \) defined in terms of \( T_{-abc} \) and \( F^+_{abc} \). The fields \( \sigma, \psi^i \) and \( B_{\mu \nu} \) will turn out to have proper kinetic terms and consistent field equations, hence solving the problems mentioned above.

Let us now take a closer look at the tensor multiplet. The full transformations under \( Q \) and \( S \) are \( \delta \sigma = \bar{\epsilon} \psi, \delta \psi^i = \frac{1}{48} \gamma \cdot F^+ \epsilon^i + \frac{1}{4} \bar{\epsilon} \sigma \epsilon^i - \sigma \eta^i; \delta F^+_{abc} = \frac{1}{2} \bar{\epsilon} \bar{\rho}_{abc} \psi - 3 \eta \gamma_{abc} \psi; \delta D_{\mu} \sigma = (\partial_{\mu} - 2 b_{\mu}) \sigma - \bar{\psi}_{\mu} \psi; \delta D_{\mu} \psi^i = (\partial_{\mu} - \frac{5}{2} b_{\mu} + \frac{1}{4} \omega_{\mu \ab} \gamma_{ab}) \psi^i + \nu_{\mu} \psi^i - \frac{1}{48} \gamma \cdot F^+ \psi^i - \frac{1}{4} \bar{\epsilon} \sigma \psi^i + \sigma \phi^i. \) (2.13)

The algebra only closes on the fields when the following closed set of independent constraints is imposed

\[
\Gamma^i \equiv \bar{\epsilon} \partial \psi^i - \frac{1}{3} \sigma \chi^i - \frac{1}{12} \gamma \cdot T^- \psi^i = 0, \\
C \equiv (D^a D_a - \frac{1}{6} D) \sigma + \frac{1}{3} F^+ \cdot T^- + \frac{7}{6} \bar{\chi} = 0, \\
G_{ab} \equiv \bar{\epsilon} \sigma^a F^+_{abc} - 2 \sigma T^-_{abc} - \bar{\epsilon} \sigma T^-_{abc} = 0. \) (2.14)

The first two constraints can be used to define \( \chi^i \) and \( D \) as a function of fields of the tensor multiplet\(^4\)

\[
D = \frac{15}{4} \sigma^{-1} \left( \partial^a D_a \sigma - 3 b^a D_a \sigma + \omega_{\ab} D_{b} \sigma - \frac{1}{5} \sigma R + \frac{1}{3} F^+ \cdot T^- + \sigma \bar{\psi}^a \phi_a + \frac{2}{5} \sigma \bar{\psi}^a \gamma^\nu D^a [\nu \psi_\mu] - \frac{1}{10} \sigma \bar{\psi}^a \gamma_{\nu} \psi_\mu T^{-abc} - \bar{\psi}^a D_{\mu} \psi + \frac{1}{24} \bar{\psi} \gamma^\nu \cdot T^- \gamma^\mu \psi_\mu + \psi \gamma^\mu \phi_\mu + \frac{7}{6} \bar{\chi} \psi \right), \) (2.15)

\(^4\)Note that we can only solve the first constraint in the domain \( \sigma \neq 0. \)

\(^5\)To obtain this expression we used the expression for \( f^a_{\mu} \) in (2.8).
\[ \chi^i = \frac{15}{4} \sigma^{-1} \hat{D} \psi^i + \frac{3}{8} \gamma^{ab} \hat{R}^{ai}_{ab}(Q) - \frac{5}{16} \sigma^{-1} \gamma \cdot T^- \psi^i, \]  

(2.16)

where

\[ \hat{D}_\mu \psi^i = \left( \partial_\mu - \frac{5}{2} b_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \right) \psi^i + \nabla_\mu \psi^i - \frac{1}{48} \gamma \cdot F^+ \psi^i - \frac{1}{4} \mathcal{D} \sigma \psi^i. \]  

(2.17)

This implies that the \( S \)-gauge field \( \phi^i_\mu \) in this Weyl multiplet gets the value

\[ 16 \phi^i_\mu = \left( -\gamma^{ab} \gamma^\mu + \frac{1}{2} \gamma^{\mu \rho} \hat{R}^i_{ab}(Q) \right) + \gamma_\mu \sigma^{-1} \left( -\hat{D} \psi^i + \frac{1}{12} \gamma \cdot T^- \psi^i \right). \]  

(2.18)

The following consequence is useful for the calculations below:

\[ 16 \gamma^\mu \phi^i_\mu = \gamma^{ab} \hat{R}^i_{ab}(Q) - 6 \sigma^{-1} \hat{D} \psi^i + \frac{1}{2} \sigma^{-1} \gamma \cdot T^- \psi^i. \]  

(2.19)

The third constraint (2.14) can be solved as a Bianchi identity

\[ F^+_{\mu \rho} + 2 \sigma T^-_{\mu \rho} = 3 \partial_{[\mu} B_{\nu \rho]} + 3 \bar{\psi}_{[\mu} \gamma_{\nu \rho]} \psi + \frac{3}{2} \bar{\psi} [\mu \gamma_{\nu \rho]} \sigma \equiv \hat{F}_{\mu \rho}(B). \]  

(2.20)

\( B_{\mu \nu} \) is a newly introduced antisymmetric tensor gauge field, which transforms as

\[ \delta B_{\mu \nu} = -\bar{\epsilon} \gamma_{\mu \nu} \psi - \bar{\epsilon} \gamma_{[\mu \nu]} \sigma + 2 \partial_{[\mu} \Lambda_{\nu]}, \]  

(2.21)

where \( \Lambda_\mu \) denotes the gauge invariance of \( B_{\mu \nu} \). From this we can determine \( F^+_{\mu \rho} \) and \( T^-_{\mu \rho} \) in terms of \( B_{\mu \nu} \):

\[ F^+_{\mu \rho} = \frac{1}{2} \left( \hat{F}_{\mu \rho}(B) + \tilde{F}_{\mu \rho}(B) \right) = \hat{F}^+_{\mu \rho}(B), \]

\[ T^-_{\mu \rho} = \frac{1}{4} \sigma^{-1} \left( \hat{F}_{\mu \rho}(B) - \tilde{F}_{\mu \rho}(B) \right) = \frac{1}{2} \sigma^{-1} \hat{F}^-_{\mu \rho}(B). \]  

(2.22)

Table 2 summarizes the different fields and constraints of the tensor multiplet and their respective dof. In this way, we coupled the Weyl 1 multiplet to the tensor multiplet and we obtained a different Weyl multiplet, which we will call the Weyl 2 multiplet. The latter has the same gauge fields of the superconformal group but a different matter sector. The Weyl 2 multiplet has matter fields \( \sigma, \psi^i \) and \( B_{\mu \nu} \). The construction of the Weyl 2 multiplet is summarized in Table 3. Each column represents the different bosonic (upper part of the table) and fermionic (lower part) fields of the multiplet denoted at the top of the table. The off-shell dof (modulo the gauge transformations denoted in Table 1) are displayed between the brackets. Note that for the tensor multiplet we also included the constraints (2.14) because they restrict the number of dof. From the table it is clear that the Weyl 2 multiplet is also a 40+40 off-shell multiplet. The dependent fields are

\footnote{Note that the \( \mathcal{D} \psi^i \)-term in \( \Gamma^i \) contains a term proportional to \( \phi^i_\mu \). We replaced \( \phi^i_\mu \) by its solution (2.8) to obtain the expression for \( \chi^i \).}

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Table 2
Fields and constraints of the tensor multiplet. The Weyl weight is given and the counting of off-shell and on-shell dof.

| Field/Constraint | $w$ | Off-shell dof | On-shell dof |
|------------------|-----|---------------|--------------|
| $\sigma$         | 2   | 1             | 1            |
| $F_{abc}^+$       | 3   | 10            | 3            |
| $C$              |     | -1            |              |
| $G_{ab}$         |     | -10           |              |
| bosonic dof      |     | 0             | 4            |
| $\psi^i$         | 5/2 | 8             | 4            |
| $\Gamma^i$       |     | -8            |              |
| fermionic dof    |     | 0             | 4            |

- $\omega_{\mu}^{\ ab}$ as given in (2.8);
- $F_{\mu\nu\rho}^+$ and $T_{\mu\nu\rho}^-$ determined by (2.20);
- $\chi^i$ and $\phi^i_{\mu}$, see (2.16) and (2.18) where $F_{\mu\nu\rho}^+$ and $T_{\mu\nu\rho}^-$ are the expressions of the previous item.
- $D$ as given in (2.15)
- $f_{\mu}^{\ a}$, see (2.8), where all the previous expressions are used.

2.3 The linear multiplet

The superconformal tensor calculus procedure prescribes that, in order to obtain an action for supergravity, we need to couple the Weyl 2 multiplet to a compensator multiplet. Fixing the gauges of the redundant symmetries of the superconformal group ($D$, $K_a$ and $S^i$) then amounts to fixing a number of components of this compensator multiplet. The remaining components will then appear as auxiliary fields in the final off-shell formulation.

We will use a linear multiplet as compensator because this is an off-shell multiplet. The linear multiplet consists of a triplet scalar $L^{ij}$, an SU(2) Majorana spinor $\varphi^i$ of negative chirality and a constrained vector $E_a$. The constraint, to be discussed below, can be solved in terms of an antisymmetric tensor gauge field $E_{\mu\nu\rho\sigma}$. The full $Q$ and $S$ transformations

---

Footnote 7: This is only possible if the linear multiplet is inert under gauge transformations. But as we do not assume any internal gauge symmetry this poses no problem.
Table 3

Construction of the Weyl 2 multiplet from the Weyl 1 multiplet and the tensor multiplet, the numbers in parentheses denote the off-shell dof. The Weyl weight of the fields of the last multiplet are given.

| Weyl 1 | Tensor | Weyl 2 | \(w\) |
|--------|--------|--------|------|
| \(e_{\mu}^a (15)\) |        | \(e_{\mu}^a (15)\) | -1   |
| \(b_\mu (0)\)      |        | \(b_\mu (0)\)     | 0    |
| \(\mathcal{V}_{\mu j}^i (15)\) | \(F^+_{abc} (10)\) | \(G_{ab} (10)\) | \(B_{\mu\nu} (10)\) | 0    |
| \(T^-_{abc} (10)\) | \(C (1)\) | \(D (1)\) | \(\sigma (1)\) | \(\sigma (1)\) | 2    |
| dilatations (-1)   |        |        |      | dilatations (-1) |      |
| 40                 | 0      | 40     |      | 40              |      |
| \(\psi_i^i (40)\)  | \(\Gamma^i (8)\) | \(\psi^i (40)\) | -1/2 |
| \(\chi^i (8)\)    | \(\psi_j^j (8)\) | \(\psi^j (8)\) | 5/2  |
| S-susy (-8)        |        | S-susy (-8) |      |
| 40                 | 0      | 40     |      | 40              |      |

are [2]

\[
\begin{align*}
\delta L^{ij} &= \varepsilon^{(i} \varphi^{j)}, \\
\delta \varphi^i &= \frac{1}{2} \mathcal{D} L^{ij} \varepsilon_j - \frac{1}{4} \gamma^a E_a \varepsilon^i - 4 L^{ij} \eta_j, \\
\delta E_a &= \bar{\varepsilon} \gamma_{ab} D^b \varphi + \frac{1}{24} \bar{\varepsilon} \gamma_a \gamma \cdot T^- \varphi - \frac{1}{3} \bar{\varepsilon} \gamma_a \chi^j L_{ij} - 5 \eta \gamma_a \varphi, \\
\mathcal{D}_\mu L^{ij} &= (\partial_\mu - 4 b_\mu) L^{ij} + 2 \mathcal{V}_{\mu (i} L^{j)k} - \bar{\psi}_{\mu} (i, \varphi^j), \\
\mathcal{D}_\mu \varphi^i &= (\partial_\mu - \frac{9}{2} b_\mu + \frac{1}{4} \omega_{\mu ab} \gamma_{ab}) \varphi^i - \mathcal{V}_{\mu}^{ij} \varphi_j - \frac{1}{2} \mathcal{D} L^{ij} \varphi_{ij} + \frac{1}{4} \gamma^a E_a \psi_j^i + 4 L^{ij} \phi_{ij}. \quad (2.23)
\end{align*}
\]

The algebra closes if \(E_a\) satisfies following \(Q\) and \(S\) invariant constraint

\[
\begin{align*}
\mathcal{D}^a E_a - \frac{1}{2} \bar{\varphi} \chi &= 0, \\
\mathcal{D}_\mu E_a &= (\partial_\mu - 5 b_\mu) E_a + \omega_{\mu ab} E^b - \bar{\psi}_{\mu} \gamma_{ab} \mathcal{D}^b \varphi - \frac{1}{24} \bar{\psi}_{\mu} \gamma_a \gamma \cdot T^- \varphi, \\
&+ \frac{1}{3} \bar{\psi}_{\mu} \gamma_a \chi^j L_{ij} + 5 \bar{\phi}_{\mu} \gamma_a \varphi. \quad (2.24)
\end{align*}
\]
The solution for $E_a$ in terms of $E_{\mu\nu\rho\sigma}$ is

$$E_a = \frac{1}{24} e^{-1} e_{\mu a} \varepsilon^{\mu\nu\rho\sigma\lambda\tau} D_\nu E_{\rho\sigma\lambda\tau}, \quad (2.25)$$

where $e$ is the square root of the metric determinant. Note that there is a gauge invariance

$$\delta_L E_{\mu\nu\rho\sigma} = 4 \partial_{[\mu} \Lambda_{\nu\rho\sigma]} . \quad (2.26)$$

We will also be using the dual 2-index field

$$E_{\mu\nu\rho\sigma} = \frac{1}{2} e \varepsilon_{\mu\nu\rho\sigma\lambda\tau} E^{\lambda\tau},$$

$$E^a = e_{\mu}^a D_\nu E^{\mu\nu},$$

$$\delta E^{\mu\nu} = \tilde{e} \gamma^{\mu\nu} \varphi + \tilde{\psi}^i \gamma^{\mu\nu} e^j L_{ij} + \partial_\rho \left( e^{-1} \tilde{\Lambda}^{\mu\nu\rho} \right),$$

$$D_\nu E^{\mu\nu} = \partial_\nu E^{\mu\nu} - \tilde{\psi}_i \gamma^{\mu\nu}\varphi - \frac{1}{2} \tilde{\psi}^i \gamma^{\mu\nu}\tilde{\psi}^j L_{ij} . \quad (2.27)$$

The different components of the linear multiplet are summarized in Table 4. The construction of Poincaré supergravity using the linear multiplet as compensator is summarized in Table 5.

The two columns on the left denote the Weyl 2 multiplet, discussed in the previous section, coupled to the linear multiplet. The first of these two columns displays the fields and their off-shell dof modulo the gauge transformations which are depicted in the second column. The next two columns describe the gauge fixing. The first of these columns contains the gauge choices leading to a fixing of the symmetries denoted in the next column. The final three columns of the table depict the fields of the final off-shell formulation of Poincaré supergravity, first again with their off-shell dof (i.e. modulo the gauge transformations denoted in the middle column) and finally their on-shell dof. From this table it is clear that we end up with a 48+48 off-shell description of Poincaré supergravity, describing 16+16 on-shell dof.

| Field | $w$ | Off-shell dof | On-shell dof |
|------|-----|---------------|--------------|
| $L_{ij}$ | 4 | 3 | 3 |
| $E_{\mu\nu\rho\sigma}$ | 0 | 5 | 1 |
| bosonic dof | 8 | 4 |
| $\varphi^i$ | $9/2$ | 8 | 4 |
| fermionic dof | 8 | 4 |
Table 5

Construction of Poincaré supergravity from the Weyl 2 multiplet coupled to a linear compensator multiplet. The fields and symmetries of this setup are denoted in the first two columns, the third and fourth columns contain the gauge choices and the respective fixed symmetries, the last columns denote the fields and symmetries of Poincaré supergravity, and the number of on-shell dof.

| Weyl 2 × Linear | Gauge fixing | Poincaré |
|------------------|--------------|----------|
| $e^a_\mu$ (15)   | $P_a, M_{ab}$| $e^a_\mu$ (15) |
| $b_\mu$ (0)      | $K_a$        | *         |
| $\mathcal{V}^{ij}_\mu$ (15) | SU(2)       | $\mathcal{V}^{ij}_\mu$ (17) |
| $\sigma$ (1)     | $\Lambda_\mu$| $\sigma$ (1) |
| $B_{\mu\nu}$ (10) | $L^{ij} = \frac{1}{\sqrt{2}} \delta^{ij}$ | $B_{\mu\nu}$ (10) |
| dilatations (−1) | $D, \text{SU}(2)/\text{SO}(2)$ | *         |
| $E_{\mu\nu\rho\sigma}$ (5) | $\tilde{\Lambda}_{\mu\rho}$ | $E_{\mu\nu\rho\sigma}$ (5) |
| 48               | 48           | 16        |
| $\psi^i_\mu$ (40) | $Q^i$        | $\psi^i_\mu$ (40) |
| $\psi^i$ (8)     | $\varphi^i = 0$ | *         |
| S-susy (−8)      | $S^i$        | $\psi^i$ (8) |
| $\varphi^i$ (8)  | *            | *         |
| 48               | 48           | 16        |

3 Explicit construction of the action

In section 3.1 we will construct a superconformal action for the Weyl 1 multiplet coupled to a linear compensator. After applying the gauge fixing procedure in section 3.2 we will show explicitly the problems related to the Weyl 1 multiplet mentioned in section 2.2. We then go to a formulation in terms of the Weyl 2 multiplet by using the relations (2.15), (2.16) and (2.22). This is discussed in section 3.3.

3.1 Construction of the superconformal action

3.1.1 Density formula

We need an expression constructed from the components of the linear multiplet that can be used as a superconformal action. In 2 a density formula is given for the product of a vector multiplet $W_\mu$, $\Omega^i$, $\mathcal{V}^{ij}$ and a linear multiplet $L^{ij}$, $\varphi^i$, $E_a$:

$$e^{-1} \mathcal{L}_{VL} = Y_{ij} L^{ij} + 2\bar{\Omega} \varphi - L^{ij} \bar{\psi}_i \gamma^\mu \Omega_j + \frac{1}{4} F_{\mu\nu}(W) E^{\mu\nu}.$$  (3.1)
The next step is then to construct a vector multiplet from the components of the linear multiplet

\[ \Omega^i = \Omega^i(L^{ij}, \phi^i, E_a), \]

\[ Y^{ij} = Y^{ij}(L^{ij}, \phi^i, E_a), \]

\[ \hat{F}_{\mu\nu} = \hat{F}_{\mu\nu}(L^{ij}, \phi^i, E_a), \]

and to plug in these components into (3.1) to obtain a superconformal action for the linear multiplet. Explicit expressions for (3.2) were constructed in [2] and are summarized in appendix C (equations (C.5)). Note that \( \hat{F}_{\mu\nu} = F_{\mu\nu} + 2 \bar{\psi}_{[\mu} \gamma_{\nu]} \Omega \) is the supercovariant field strength of \( W_{\mu} \). We also define

\[ L = (L_{ij} L^{ij})^{1/2}. \] (3.3)

### 3.1.2 Bosonic part

Using (3.1) and (C.5) and retaining only the bosonic terms we obtain

\[ e^{-1} L_{\text{bos}} = -D_a D^a L + L^{-1} D_a L^{ij} D^a L_{ij} + L^{-3} L_{ij} L_{kl} \left( D^a L^{k(i} D_a L^{j)} - D_a L^{ij} D^a L \right) \]

\[ + \frac{1}{4} L^{-1} E_a E^a - \frac{1}{3} LD - L^{-3} L_{ij} E_a L^{k(i} D_a L^{j)} k \]

\[ + \frac{1}{2} E^{\mu\nu} \left( -2 \partial_\mu L^{ij} L^k j \partial_\nu L_{ik} L^{-3} - 2 \partial_\mu (E_{\nu|i} L^{-1} - 2 \nu_{\nu ij} L^{ij} L^{-1}) \right). \] (3.4)

We also know from equation (2.27) that

\[ E^a = e^a_\mu D^\mu E^{\mu\nu} = e^a_\mu \partial_\nu E^{\mu\nu} + \text{fermionic terms}. \] (3.5)

After partial integration and using the identity (C.2), we can write the bosonic action as

\[ e^{-1} L_{\text{bos}} = -D_a D^a L + \frac{1}{2} L^{-1} D_a L^{ij} D^a L_{ij} \]

\[ - \frac{1}{3} LD - \frac{1}{4} L^{-1} E_a E^a + E^{\mu} \nu_{\mu ij} L^{ij} L^{-1} - \frac{1}{2} E^{\mu\nu} \partial_\mu L^{ij} L^k j \partial_\nu L_{ik} L^{-3}. \] (3.6)

### 3.1.3 Fermionic part

In the next section we will discuss the gauge fixing procedure. To fix the S-gauge we will choose \( \phi = 0 \). Hence, in order to clarify calculations, we will drop all terms directly proportional to \( \phi \) already at this stage of the calculation. To write down the fermionic part of the superconformal Lagrangian we need the fermionic terms of \( F^{\mu\nu}(L) \). We obtain

\[ F_{\mu\nu}(L) = \text{(fermionic terms)} \]

\[ -8 L^{-1} L_{ij} \bar{\psi}_{[\mu} \gamma_{\nu]} \phi^j - \frac{4}{3} L^{-1} L_{ij} \bar{\psi}_{[\mu} \gamma_{\nu]} \chi^j + 2 L^{-1} \bar{\psi}_{[\mu} \gamma_{\nu]} D_\nu \varphi - 2 L^{-1} \bar{\psi}_{[\mu} \gamma_{\nu]} D_\nu \varphi \]

\[ + \frac{1}{2} L^{-1} \bar{\psi}_{[\mu} \gamma^b \psi_{\nu]} E_b - 2 \bar{\psi}_{[\mu} \gamma_{\nu]} \Omega (L^{ij}, \phi^i, E_a), \] (3.7)
where the first term in the third line is obtained by use of the solutions of the conventional constraints \([2.8]\). By use of \([C.5]\) we can write

\[
-2\bar{\psi}_{[\mu} \gamma_{\nu]} \Omega(L^{ij}, \varphi^i, E_a) = -2L^{-1}\bar{\psi}_{[\mu} \gamma_{\nu]} D_{\varphi} + \frac{4}{3} L^{-1}\bar{\psi}_{[\mu} \gamma_{\nu]} L_{ij} \chi^j. \quad (3.8)
\]

Filling in the above equation into equation \((3.7)\) and using \([2.23]\) we find that all fermionic terms of \(F_{\mu\nu}(L)\) drop. Using the density formula \((3.1)\), we obtain the fermionic part of the action which, at this stage of the calculation, looks like (remember that we have put \(\varphi\) to zero)

\[
e^{-1}L_{\text{ferm.}} = -\frac{1}{4} L^{-1} \bar{\psi}^i_{\mu} \gamma^{\mu\nu} \psi^j_L (i,j) E_\mu + \frac{1}{2} L^{-1} \bar{\psi}^k_L \gamma^{\mu\nu} \psi^l_L \mu L_{ij} - \frac{1}{3} L \bar{\psi}_{\mu} \gamma^\mu \chi
\]

\[
- L^{-1} L_{ij} \bar{\psi}_{\mu} \gamma^{\mu\nu} \left(-\frac{1}{2} \bar{\psi} L^{jk} \psi_{\nu k} + \frac{1}{4} \gamma^\rho E_\rho \psi^j_L + 4 L \phi_{\nu k} \right). \quad (3.9)
\]

Note that the two first terms come from the step done in \((3.5)\). In the last line we also used \([2.23]\).

### 3.1.4 Superconformal action

We will now prepare for the gauge fixing procedure by writing out the covariant derivatives and dependent fields. The bosonic superconformal Lagrangian in \((3.6)\) can be rewritten, using \([C.3]\), in a form that is most convenient for the gauge fixing procedure

\[
e^{-1}L_{\text{bos.}} = L^{-3} L_{ij} D_a L^{kl} D^a L_{ij} - L^{-1} D_a \bar{D}_a L_{ij} D^a L_{ij} - L^{-1} L_{ij} D_a D^a L_{ij} + \frac{1}{2} L^{-1} D_a L_{ij} D^a L_{ij}
\]

\[
- \frac{1}{3} L D - \frac{1}{4} L^{-1} E_\alpha E^\alpha + E^\alpha \gamma^L_{ij} L^L - \frac{1}{2} E^{\mu\nu} \partial_{\mu} L_{ij} L^L + \partial_{\nu} L_{ik} L^{-3}. \quad (3.10)
\]

The fermionic terms in these covariant derivatives will be collected in a new fermionic Lagrangian, which contains not only the terms in \((3.9)\), but also terms from the ‘bosonic’ part \((3.10)\) due to terms quadratic in fermions in covariant derivatives or dependent bosonic fields.

We mentioned that the choice for the S-gauge will be \(\varphi = 0\). We dropped already terms proportional to \(\varphi\). Note that these terms should be restored to get a full superconformal action, but that is not the aim of this paper. As the only fermionic terms in \(D_a L_{ij}\) are proportional to \(\varphi\), the first two terms on the RHS of \((3.10)\) will not contribute any new fermionic terms when we write out the covariant derivatives. Instead we take a closer look at the third term on the RHS of \((3.10)\). Note that \(\varphi = 0\) does not imply the vanishing of \(D_\mu \varphi^i\), see \([2.23]\). We obtain

\[
- L^{-1} L_{ij} D_a D^a L_{ij} = -L^{-1} L_{ij} e^{\alpha} \partial_{\alpha} D_a L_{ij} - L^{-1} L_{ij} e^{\alpha} \omega_{\alpha} \mu D_b L_{ij}
\]

\[
-2 L^{-1} L_{ij} V_{a}^{(i} D^a L_{ij} k) + 8 L f_a^a
\]

\[
+ L^{-1} L_{ij} \bar{\psi} \gamma^L_{ij} (\frac{1}{2} \bar{\psi} L^{jk} \psi_{\nu k} + \frac{1}{4} \gamma^\rho E_\rho \psi^j_L + 4 L \phi_{\nu k})
\]

\[
+ \frac{1}{6} L \bar{\psi}_{\alpha} \gamma^L_{\alpha} \chi. \quad (3.11)
\]
The last two lines will thus be added to \( \mathcal{L}_{\text{form}} \). Using the first relation of (2.11), the first two terms in (3.11) combine into \(-L^{-1}L_{ij}e^{-1}\partial_e(eD^\mu L^{ij})\) and there is a term \(-\frac{1}{2}L^{-1}L_{ij}\bar{\psi}^a\gamma_\alpha \psi_b D_b L^{ij}\) that needs to be added to the fermionic Lagrangian. The bosonic Lagrangian thus becomes

\[
e^{-1}\mathcal{L}_{\text{bos}} = -L^{-3}L_{ij}L_{kl}D_aD_bD^aD^bL^{ij} - \frac{1}{2}L^{-1}D_aL_{ij}D^aL^{ij} - L^{-1}L_{ij}e^{-1}\partial_{e}(eD^\mu L^{ij})
\]

\[
-2L^{-1}L_{ij}\gamma^{(i}D_{a}D^{j)k} + 8Lf_a^a
\]

\[
-\frac{1}{3}LD - \frac{1}{4}L^{-1}E_aE^a + E^\mu \gamma_{\mu ij}L^{ij}L^{-1} - \frac{1}{2}E^{\mu \nu} \partial_\mu L^{ij}L_{j}^{k} \partial_{\nu}L_{ik}L^{-3}. \tag{3.12}
\]

Writing out the covariant derivatives \( D^aL_{ij} \), dropping the fermionic terms (because they are proportional to \( \varphi \) and hence vanish by gauge fixing) and terms proportional to \( b_\mu \) (for the same reason) we can write the bosonic Lagrangian as

\[
e^{-1}\mathcal{L}_{\text{bos}} = 8Lf_a^a + \frac{1}{2}L^{-1}\partial_aL^{ij}\partial^aL_{ij} - 2L^{-1}L_{ij}\gamma_{\mu ij}\partial_aL_{kj} + 2L^{-1}\gamma_{(kL)^j}\gamma_{i} L^{ij}L^{-1} - \frac{1}{2}E^{\mu \nu} \partial_\mu L^{ij}L_{j}^{k} \partial_{\nu}L_{ik}L^{-3}. \tag{3.13}
\]

The fermionic action (with the \( \varphi \) put to zero) looks at this point as

\[
e^{-1}\mathcal{L}_{\text{ferm}} = -\frac{1}{4}L^{-1}\bar{\psi}^{i}_{\mu \rho}\gamma_{\mu \nu}L_{ij}E_{\mu} + \frac{1}{2}L^{-1}L_{ij}\bar{\psi}^{i}_{\mu \nu}L_{kj}\gamma_{\mu \nu}L_{ij} - \frac{1}{3}L\bar{\psi}^{i}_{\mu}\gamma_{\mu}L_{ij}E_{\mu} + \frac{1}{4}L^{-1}L_{ij}\bar{\psi}^{i}_{\mu}L_{kj}\gamma_{\mu \nu}L_{ij}L^{-1}
\]

\[-L^{-1}L_{ij}\bar{\psi}^{i}_{\mu}\gamma_{\mu}L_{ij}E_{\mu} + \frac{1}{4}L^{-1}L_{ij}\bar{\psi}^{i}_{\mu}L_{kj}\gamma_{\mu \nu}L_{ij}L^{-1}
\]

\[+\frac{1}{6}L\bar{\psi}^{i}_{\mu}E_{\mu}L^{-1}L_{ij}\bar{\psi}^{i}_{\mu}L_{kj}\gamma_{\mu \nu}L_{ij}L^{-3}L_{kj}. \tag{3.14}
\]

Note that this fermionic Lagrangian differs from the one in (3.9) by the two last terms, and the \( \gamma^{\mu \nu} \) being replaced by \( \gamma_{\mu \nu} \), which originate from the last two lines of (3.11), and from \( e^{\alpha \mu} \omega_{\mu \alpha}^b \) in (2.11).

The bosonic terms in the expression of \( f_a^a \) in (2.11) lead to

\[
e^{-1}\mathcal{L}_{\text{bos}} = \frac{2}{5}LR + \frac{1}{2}L^{-1}\partial_aL^{ij}\partial^aL_{ij} - 2L^{-1}L_{ij}\gamma_{\mu ij}\partial_aL_{kj}
\]

\[+2L^{-1}\gamma_{(kL)^j}L^{ij}L_{kj}L^{-1} - \frac{1}{2} \frac{L^{-1}E_aE^a + E^\mu \gamma_{\mu j}L^{ij}L^{-1}}{2} - \frac{1}{2}E^{\mu \nu} \partial_\mu L^{ij}L_{j}^{k} \partial_{\nu}L_{ik}L^{-3}. \tag{3.15}
\]

The fermionic terms of \( f_a^a \) lead to terms modifying \( \mathcal{L}_{\text{form}} \) to

\[
e^{-1}\mathcal{L}_{\text{ferm}} = \frac{1}{2}L^{-1}L_{ij}L_{kl}\bar{\psi}^{i}_{\mu \rho}L_{kj}\gamma_{\mu \nu}\psi_bL^{ij}L_{kj}L^{-1}
\]

\[+L\bar{\psi}^{i}_{\mu}L_{ij}E_{\mu} + \frac{1}{4}L^{-1}L_{ij}\bar{\psi}^{i}_{\mu}L_{kj}\gamma_{\mu \nu}L_{ij}L^{-1}
\]

\[-L\bar{\psi}^{i}_{\mu}E_{\mu}L^{-1}L_{ij}\bar{\psi}^{i}_{\mu}L_{kj}\gamma_{\mu \nu}L_{ij}L^{-1}
\]

\[+\frac{1}{6}L\bar{\psi}^{i}_{\mu}E_{\mu}L^{-1}L_{ij}\bar{\psi}^{i}_{\mu}L_{kj}\gamma_{\mu \nu}L_{ij}L^{-3}L_{kj}. \tag{3.16}
\]

\[\text{To fix the } K\text{-gauge, the condition } b_\mu = 0 \text{ will be imposed in the next section.}\]
We used here the SU(2)-covariant derivative,

$$D_\mu L^{ij} = \partial_\mu L^{ij} + 2\mathcal{V}_\mu (i_k L^j)^k,$$

(3.17)

where we already put $b_\mu = 0$ in view of the gauge fixing of special conformal transformations that we will adopt soon.

### 3.2 Gauge fixing

#### 3.2.1 Bosonic part

To arrive at the super Poincaré group, the redundant symmetries of the superconformal algebra need to be broken. The special conformal transformations are fixed by the condition $b_\mu = 0$. The dilatation gauge is fixed by $L = 1$. The SU(2) symmetry cannot be completely broken. A gauge choice $L_{ij} = \sqrt{\frac{1}{2}} \delta_{ij}$ still leaves a remaining $U(1)$ symmetry which will be gauged by the auxiliary $\mathcal{V}_{aij} \delta^{ij}$. For the bosonic part of the gauge-fixed action, we use (3.2) to write

$$2\mathcal{V}_a (\xi^k L^j)^k V^a_{ij} L^i_j = \mathcal{V}'_{aij} \mathcal{V}'_{aij}.$$  

(3.18)

Here $\mathcal{V}'_{aij}$ is the traceless part of $\mathcal{V}_{aij}$:

$$\mathcal{V}'_{aij} = \mathcal{V}_{aij} - \frac{1}{2} \delta_{ij} \delta^{kl} \mathcal{V}_{akl}. $$

(3.19)

Applying the gauge fixing in (3.15), we obtain the bosonic part of the gauge fixed action

$$e^{-1}L_{\text{bos}} = \frac{2}{5} R + \mathcal{V}'_{aij} \mathcal{V}'_{aij}$$

$$- \frac{2}{15} D - \frac{1}{4} E_a E^a + \frac{1}{\sqrt{2}} E^\mu \mathcal{V}_{aij} \delta^{ij}. $$

(3.20)

#### 3.2.2 Fermionic part

We still need to fix the S-gauge. As mentioned in the previous section, this can be done by demanding $\varphi = 0$. The fermionic part of the resulting action after gauge fixing is then

$$e^{-1}L_{\text{ferm}} = -\frac{1}{4} \mathcal{V}_\rho^{\ kl} \left( \delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl} + \epsilon_{kj} \epsilon_{li} \right) \tilde{\psi}_i^j \gamma_i^{\mu \rho} \psi_\nu^j$$

$$- \frac{1}{2} \delta_{ij} \tilde{\psi}_i^j \gamma_i^{\mu \rho} \mathcal{V}_\rho^{\ jk} \delta_{kl} \psi_\nu^k - \frac{1}{2} \tilde{\psi}_i^j \gamma_i^{\mu \rho} \mathcal{V}_\rho^{\ k \ i} \psi_\nu^k$$

$$- \tilde{\psi}_i^j \gamma_i^{\mu} \left( 2 \gamma_\nu \phi_\nu + \frac{1}{3} \chi^i \right) - \frac{4}{5} \tilde{\psi}_i^j \gamma_i^{\nu} \hat{D}^{\nu \rho \gamma} \psi_\rho^j + \frac{4}{5} \tilde{\psi}_i^j \gamma_\rho \psi_\rho^j T^{-abc}.$$  

(3.21)

# We will keep the notation $\hat{D}$ for the covariant derivative, but remember that from now on it denotes $\hat{D}|_{\text{gauge fixed}}$. 

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This can still be simplified using $\delta^{ij}\delta^{kl} - \delta^{il}\delta^{jk} = \epsilon^{ik}\epsilon^{jl}$ and the fact that $\delta^{ij}\bar{\psi}_i^\mu \gamma^a \psi^j_\nu = 0$:

$$
e^{-1} \mathcal{L}_{\text{term}} = -\frac{1}{2} V_{\rho ij} \bar{\psi}_i^\mu \gamma^{\mu \rho} \psi^j_\nu - \bar{\psi}_i^\mu \gamma^\mu \left(2 \gamma^\nu \phi_\nu + \frac{1}{3} \chi^i \right) - \frac{4}{5} \bar{\psi}_i^\mu \gamma^\nu \bar{D}_{[\nu} \psi_{\mu]} + \frac{1}{5} \bar{\psi}_b^c \gamma_c \psi_a T^{abc}. \quad (3.22)$$

Next we use (2.11) to write

$$
e^{-1} \mathcal{L}_{\text{term}} = -\frac{1}{2} V_{\rho ij} \bar{\psi}_i^\mu \gamma^{\mu \rho} \psi^j_\nu - 2 \bar{\psi}_i^\mu \gamma^{\mu \rho} \hat{D}_\mu \psi_\nu + \frac{1}{5} \bar{\psi}_b^c \gamma_c \psi_a T^{abc} - \frac{2}{15} \bar{\psi}_i^\mu \gamma^\mu \chi. \quad (3.23)$$

### 3.3 Off-shell action

As can be seen from the Lagrangians (3.20) and (3.23) the matter fields of the Weyl 1 multiplet, $D$, $\chi^i$ and $T_{abc}$, have no kinetic terms. Also, the field equation for $D$ gives an inconsistency. As mentioned in section 2.2 this problem can be solved by using the Weyl 2 multiplet instead. This can be done by plugging in the expressions (2.15), (2.16) and (2.22) into the Lagrangians. We first write out the full expressions for $\chi^i$ and $D$. From (2.15), (2.16), (2.19) and (2.22) we find

$$\chi^i = \frac{15}{4} \sigma^{-1} \gamma^\mu \left( D_\mu \psi_i^\mu - \frac{1}{48} \gamma \cdot \hat{F} \psi_i^\mu - \frac{1}{4} \hat{D} \sigma \psi_i^\mu \right) + \frac{3}{4} \gamma^{\mu \nu} \left( D_\mu \psi_i^\nu - \frac{1}{48} \sigma^{-1} \gamma \cdot \hat{F} \sigma \psi_i^\mu \right) - \frac{5}{32} \sigma^{-2} \gamma \cdot \hat{F} \psi_i, \quad (3.24)$$

and

$$D = \frac{15}{4} \sigma^{-1} \left[ e^{-1} \partial_\mu (e \hat{D}^\mu \sigma) + \frac{1}{2} \bar{\psi}^a \gamma_\mu \psi^b \hat{D}_b \sigma - \frac{1}{5} \sigma R + \frac{1}{12} \sigma^{-1} \hat{F} \cdot \hat{F} \right] + \frac{2}{5} \sigma \bar{\psi}^\mu \gamma^\nu \left( D_{[\nu} \psi_{\mu]} - \frac{1}{48} \sigma^{-1} \gamma \cdot \hat{F} \gamma_{[\sigma} \psi_{\mu]} \right) - \frac{1}{20} \bar{\psi}_b^c \gamma_c \psi_a \hat{F} - \frac{1}{4} \gamma \sigma^{-1} \gamma \cdot \hat{F} \gamma_\mu \psi_\mu \left( D_\mu \psi_i^\mu - \frac{1}{48} \sigma^{-1} \gamma \cdot \hat{F} \gamma_\mu \psi_\mu \right) - \frac{1}{6} \sigma^{-2} \bar{\psi} \gamma \cdot \hat{F} \psi, \quad (3.25)$$

where the $D_\mu \psi_i^\mu$ and $D_\mu \psi^\mu_i$ are Lorentz and SU(2) covariant derivatives, while $\hat{D}_\mu \sigma$ is a supercovariant derivative:

$$D_\mu \psi_i^\mu \equiv \nabla_\mu \psi_i^\mu + \mathcal{V}_\mu^i \psi_j^j = \left( \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \right) \psi_i^\mu + \mathcal{V}_\mu^i \psi_j^j, \quad D_\mu \psi^\mu_i \equiv \nabla_\mu \psi_i^\mu + \mathcal{V}_\mu^i \psi_j^j = \left( \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \right) \psi^\mu_i + \mathcal{V}_\mu^i \psi_j^j, \quad \hat{D}_\mu \sigma = \partial_\mu \sigma - \bar{\psi}_i^\mu \psi. \quad (3.26)$$

\[^{10}\text{The same is true for } E_\alpha \text{ and } V_\mu^i, \text{ but their field equations are consistent.}\]
We also used (A.18) and the techniques of (A.19).

The expressions for $\chi^i$ and $D$ can still be rewritten by using several gamma matrix manipulations:

$$
\chi^i = \frac{15}{4} \sigma^{-1} \gamma_{\mu} D_{\mu} \psi^i + \frac{3}{4} \gamma_{\mu \nu} D_{\mu} \psi^i - \frac{15}{16} \sigma^{-1} \gamma_{\mu} \hat{D} \sigma \psi^i - \frac{3}{32} \sigma^{-1} \gamma^{\mu \rho \sigma \chi} \hat{F}_{\rho \sigma \chi} \psi^i - \frac{5}{32} \sigma^{-2} \gamma \cdot \hat{F} \psi^i
$$

and

$$
D = \frac{15}{4} \sigma^{-1} \left[ e^{-1} \partial_\mu (e \hat{D}^\mu \sigma) + \frac{1}{2} \bar{\psi} \gamma_\mu \psi \hat{D}_\mu \sigma - \frac{1}{5} \sigma R + \frac{1}{12} \sigma^{-1} \hat{F} \cdot \hat{F} \right] \\
+ \frac{2}{7} \sigma \bar{\psi} \gamma^\nu D_{[\nu} \psi_{\mu]} - \frac{1}{40} \bar{\psi} \gamma^{\mu \rho \sigma \chi} \hat{F}_{\rho \sigma \chi} \psi^i - \bar{\psi} \gamma^\mu D_{\mu} \psi + \frac{1}{40} \bar{\psi} \gamma^\mu \gamma \cdot \hat{F} \psi^i \bigg]
$$

$$
+ \bar{\psi} \gamma^\mu D_{\mu} \psi - \frac{1}{6} \sigma^{-2} \bar{\psi} \gamma \cdot \hat{F} \psi - \frac{1}{6} \sigma^{-1} \bar{\psi} \gamma^{\mu \nu \rho} \hat{F}_{\nu \rho \sigma} \psi^i - \frac{1}{6} \sigma^{-1} \bar{\psi} \gamma^{\nu} \hat{F}_{\nu \rho \sigma} \psi^i
$$

$$
+ 4 \sigma^{-1} \bar{\psi} \hat{D} \psi - \sigma^{-1} \bar{\psi} \gamma^\mu \hat{D} \sigma \psi^i \bigg],
$$

where we also used

$$
\gamma_{\rho} \hat{F}^{-\mu \nu \rho} = \frac{1}{2} \gamma_{\rho} \left( \hat{F}_{\mu \nu} + \frac{1}{6} \epsilon_{\mu \nu \rho \sigma \chi} \hat{F}_{\sigma \chi} \right) = \frac{1}{2} (\gamma_{\rho} \hat{F}_{\mu \nu} + \frac{1}{6} \gamma^{\mu \nu \rho \sigma \chi} \gamma_{\chi} \hat{F}_{\sigma \chi}).
$$

This equation can be proven using the duality relation (A.16).

Filling in the expressions for $D$ and $\chi^i$ into (3.20) and (3.23) and distributing the respective terms over the bosonic and fermionic Lagrangians we get

$$
e^{-1} \mathcal{L}_{bos} = \frac{1}{2} R + \mathcal{V}_{a}^{ij} \mathcal{V}^{a}_{ij} - \frac{1}{4} E_{a} E^{a} + \frac{1}{\sqrt{2}} \epsilon^{\mu \nu ij} \mathcal{V}_{\mu \nu ij} \delta \psi^i \bigg) \bigg]
$$

$$
- \frac{1}{2} \sigma^{-2} \partial_\alpha \sigma \partial^\alpha \sigma - \frac{3}{8} \sigma^{-2} \partial_\sigma B_{\nu \rho} \partial^\mu B^{\nu \rho},
$$

where we used the definition of $\hat{F}(B)$ in (2.20). We then perform gamma matrix manipulations and write out the covariant derivatives (3.26). After dropping a total derivative, the fermionic Lagrangian becomes

$$
e^{-1} \mathcal{L}_{ferm} = -\frac{1}{2} \bar{\psi} \gamma^{\mu \nu \rho \sigma} \phi_{\mu \nu} - 2 \sigma^{-2} \bar{\psi} \hat{D} \psi + \sigma^{-2} \bar{\psi} \gamma^\nu \gamma_{\mu} \psi_\nu \partial_\mu \sigma
$$

$$
+ \frac{1}{16} \sigma^{-1} \bar{\psi} \gamma^{\mu \rho \sigma \chi} \psi_\nu \partial_\rho B_{\sigma \chi} - \frac{3}{8} \sigma^{-1} \bar{\psi} \gamma^{\mu \nu} \psi_\rho \partial_\mu B_{\nu \rho}
$$

$$
+ \frac{1}{4} \sigma^{-2} \bar{\psi} \gamma^{\mu \rho \sigma \chi} \psi_\nu \partial_\rho B_{\sigma \chi} - \frac{3}{4} \sigma^{-2} \bar{\psi} \gamma^{\mu \nu} \psi_\rho \partial_\mu B_{\nu \rho}
$$

$$
+ \frac{1}{4} \sigma^{-3} \bar{\psi} \gamma^{\mu \nu} \psi_\rho \partial_\mu B_{\nu \rho} + (4\text{-fermion terms}),
$$

(3.31)
where

\[
\text{(4-fermion terms)} = \frac{1}{32} \tilde{\psi} \gamma^{\mu \rho \sigma \chi} \psi \tilde{\psi} \gamma^\sigma \psi - \frac{3}{32} \tilde{\psi} \gamma_\mu \psi \tilde{\psi} \gamma^\mu \psi + \frac{1}{16} \sigma^{-1} \tilde{\psi} \gamma^{\mu \rho \sigma \chi} \psi \tilde{\psi} \gamma^\sigma \psi - \frac{3}{8} \sigma^{-1} \tilde{\psi} \gamma_\mu \psi \tilde{\psi} \gamma^\mu \psi - \frac{1}{8} \sigma^{-1} \tilde{\psi} \gamma^{\mu \rho \sigma \chi} \psi \tilde{\psi} \gamma^\sigma \psi + \frac{1}{8} \sigma^{-1} \tilde{\psi} \gamma^{\mu \rho \sigma \chi} \psi \tilde{\psi} \gamma^\sigma \psi - \frac{1}{2} \sigma^{-1} \tilde{\psi} \gamma^{\mu \rho \sigma \chi} \psi \tilde{\psi} \gamma^\sigma \psi + \frac{1}{8} \sigma^{-1} \tilde{\psi} \gamma^{\mu \rho \sigma \chi} \psi \tilde{\psi} \gamma^\sigma \psi.
\]

\[ (3.32) \]

4 Conclusions

In this paper we constructed the full (bosonic and fermionic) off-shell action of minimal Poincaré supergravity in six dimensions. We obtained this action by using the methods of superconformal tensor calculus as suggested in [2]. We constructed a superconformal action by coupling a linear compensator multiplet (which is an off-shell multiplet) to the Weyl 1 multiplet in a density formula. By fixing the redundant symmetries (\(D\), \(K\) and \(S\)) we obtained the Poincaré action. However, this action contained no kinetic terms for the matter fields of the Weyl 1 multiplet and even led to an inconsistent field equation for the \(D\) field. This forced us to consider the matter fields of the Weyl 1 multiplet as functions of those of the Weyl 2 multiplet. Expressing the action in terms of fields of the Weyl 2 multiplet led to an action that contains kinetic terms for the matter fields and has consistent field equations.

In this sense the six dimensional case differs from the four and five dimensional one. In four and five dimensions [5, 13, 14, 15] the Weyl 1 multiplet can be used as an independent multiplet to construct a pure Poincaré action, not leading to inconsistencies and having proper kinetic terms for the matter fields.

Another fact worth mentioning is the following. We constructed the superconformal action by using a density formula for the product of a linear and a vector multiplet. This formula could be used to construct an action for the linear compensator provided we were able to define a vector multiplet from the components of the linear multiplet. One way to do this is given in appendix C in which a linear multiplet is coupled to Weyl 1. Factors of \(L\) are used to compensate for the difference in Weyl weights between the linear and the vector multiplet components. After gauge fixing, the Weyl 1 matter fields were solved in terms of Weyl 2 matter fields. Another possibility to construct a vector multiplet from the components of a linear one is to couple directly to the Weyl 2 multiplet and use factors of \(\sigma\) to account for the different Weyl weights. Both possibilities can be generalized by considering.
a combination $L^\alpha \sigma^{2(1-\alpha)}$ as compensator. We expect, of course, this free parameter $\alpha$ to disappear in the gauge fixing procedure, hence leading to the same off-shell Poincaré action for all $\alpha$.

By performing the gauge fixing, the SU(2) R-symmetry of the superconformal group is broken to a U(1). In the final Poincaré action this U(1) is gauged by the auxiliary $V_\mu^{ij} \delta_{ij}$. As mentioned in the introduction, it would be interesting to add a vector multiplet to the action. This physical vector multiplet can then be coupled to the U(1) symmetry by adding a coupling of the form (3.1) [2], thus obtaining the dual off-shell Salam-Sezgin model. Adding $R^2$-terms to the action in a supersymmetric way (as described in [10, 11]), would be interesting to study the influence of these terms on solutions.

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A Notation and conventions

Note that some conventions differ from the ones used in [2]. When this is the case we will explicitly mention it.

We use the \((- + \ldots +\) metric (as opposed to the Pauli metric \((+ + \ldots +)\) in [2]) and the \(8 \times 8\) Dirac matrices \(\gamma_a\) \((a = 0, \ldots, 5)\) are defined by the property
\[
\gamma_a \gamma_b + \gamma_b \gamma_a = 2 \eta_{ab},
\] (A.1)
and are Hermitian. A complete set of \(8 \times 8\) matrices is given by
\[
O_I = \{1, \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}, \gamma^{(5)}, \gamma^{(6)}\},
\] (A.2)
where we have used the following notation
\[
\gamma^{(n)} = \gamma^{a_1 \ldots a_n} = \gamma^{[a_1} \gamma^{a_2} \ldots \gamma^{a_n]} = \frac{1}{n!} \sum_p (-1)^p \gamma^{a_1} \ldots \gamma^{a_n},
\] (A.3)
where \(\sum_p\) means summation over all permutations. The matrix \(\gamma_*\) is defined by
\[
\gamma_* = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5.
\] (A.4)
This definition ensures that it squares to one (note that this differs from [2]). It can be used to define left and right handed spinors:
\[
P_L = \frac{1}{2} (1 + \gamma_*), \quad P_R = \frac{1}{2} (1 - \gamma_*).
\] (A.5)
The spinors are symplectic. We define the supersymmetries as left-handed, i.e.
\[
\epsilon^i = P_L \epsilon^i, \quad \bar{\epsilon}^i = \bar{\epsilon}^i P_R.
\] (A.6)

For the other fermion fields we use
\[
\psi^i_\mu = P_L \psi^i_\mu, \quad \chi^i = P_L \chi^i, \quad \Omega^i = P_L \Omega^i,
\phi^i_\mu = P_R \phi^i_\mu, \quad \psi^i = P_R \psi^i, \quad \varphi^i = P_R \varphi^i.
\] (A.7)

Indices \(i, j\) are raised and lowered with the \(\varepsilon_{ij}\) in \(NW - SE\) direction, i.e.
\[
\lambda_i = \lambda^j \varepsilon_{ji}, \quad \lambda^i = \varepsilon^{ij} \lambda_j.
\] (A.8)
With the raising and lowering conventions of \(SU(2)\) indices as in (A.8), one has for any two objects \(A, B\): \(A^i B_i = -A_i B^i\). When \(SU(2)\) indices are omitted, a \(NW - SE\) contraction is understood, e.g.
\[
\bar{\lambda} \gamma^{(n)} \psi = \bar{\lambda} \gamma^{(n)} \psi_i.
\] (A.9)
Changing the order of spinors in a bilinear leads to the following signs:
\[
\bar{\lambda} \gamma^{(n)} \psi = t_n \bar{\psi} \gamma^{(n)} \lambda, \quad t_n = \begin{cases} +: n = 1, 2, 5, 6 \\ -: n = 0, 3, 4 \end{cases}
\] (A.10)
An additional sign is needed if the SU(2) indices are contracted, e.g. \( \bar{\lambda} \gamma^a \psi = -\bar{\psi} \gamma^a \lambda \) but \( \bar{\lambda}^i \gamma^a \psi^j = \bar{\psi}^j \gamma^a \lambda^i \).

For any antisymmetric tensor \( A^{ij} \) one can write \( A^{ij} = \frac{1}{2} \varepsilon^{ij} A^k_k \). Under charge conjugation (which is equal to complex conjugation on scalar quantities)

\[
\lambda_i = (\lambda^i)^C, \quad \lambda^i = -(\lambda_i)^C. \tag{A.11}
\]

The totally antisymmetric rank 6 tensor is denoted by \( \varepsilon_{abcdef} \) or \( \varepsilon^{abcdef} \) with

\[
\varepsilon_{012345} = 1, \quad \varepsilon^{012345} = -1. \tag{A.12}
\]

It satisfies

\[
\varepsilon_{a_1a_2...a_nb_1...b_6-n} \varepsilon^{a_1a_2...a_n c_1...c_6-n} = -(6 - n)! n! \delta_{[c_1...c_6-n]}^{b_1...b_6-n}. \tag{A.13}
\]

Throughout this paper the dual \( \tilde{T}_{abc} \) of a tensor \( T_{abc} \) is defined by

\[
\tilde{T}_{abc} = -\frac{1}{6} \varepsilon_{abcdef} T^{def}, \tag{A.14}
\]

which means that \( \tilde{\tilde{T}} = T \). In [2] there is an \( i \) factor in (A.14), which is related to the use of the Pauli metric. Positive and negative dual parts are defined by

\[
T_{abc}^\pm = \frac{1}{2} (T_{abc} \pm \tilde{T}_{abc}). \tag{A.15}
\]

Following duality relation will also prove to be useful

\[
\gamma_{a_1a_2...a_r} \gamma_* = -\frac{1}{(6 - r)!} \varepsilon^{a_1a_2...a_r b_1b_2...b_D-r} \gamma_{b_1b_2...b_D-r}. \tag{A.16}
\]

It implies that

\[
\gamma_{abc} \gamma_* = -\gamma^{abc}, \quad \gamma_{abc} P_L = \gamma_{abc}. \tag{A.17}
\]

Finally we remark that in \( D = 6 \) the product of two tensors with opposite duality is non-zero but the product of two tensors of the same duality vanishes:

\[
T_{abc}^+ T^{abc} = \frac{1}{2} T_{abc} T^{abc}, \quad T_{abc}^+ T^{abc} = T_{abc}^+ T^{abc} = 0. \tag{A.18}
\]

This can be combined with (A.17) to write e.g.

\[
\gamma \cdot F \psi^i = \gamma \cdot F P_L \psi^i = \gamma^+ \cdot F \psi^i = \gamma^+ \cdot F^+ \psi^i = \gamma \cdot F^+ \psi^i, \quad \\
\gamma \cdot F \psi^\mu = \gamma \cdot F P_R \psi^\mu = \gamma^- \cdot F \psi^\mu = \gamma^- \cdot F^- \psi^\mu = \gamma \cdot F^- \psi^\mu. \tag{A.19}
\]
B  The superconformal algebra in six dimensions

The superalgebra that we gauge is OSp(8|2). The conformal algebra SO(6, 2) = SO*(8) is

\[ [M_{\mu\nu}, M_{\rho\sigma}] = 4\eta_{[\mu\rho} M_{\sigma\nu]} - \eta_{\mu\nu} M_{\sigma\rho} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma}, \]
\[ [P_{\mu}, M_{\nu\rho}] = 2\eta_{\mu[\nu} P_{\rho]} , \quad [K_\mu, M_{\nu\rho}] = 2\eta_{\mu[\nu} K_{\rho]} , \]
\[ [P_\mu, K_\nu] = 2(\eta_{\mu\nu} D + M_{\mu\nu}) , \]
\[ [D, P_\mu] = P_\mu , \quad [D, K_\mu] = -K_\mu . \]  (B.1)

The SU(2) algebra, can be written as

\[ [U_{ij}, U_k^\ell] = \delta_i^\ell U_k^j - \delta_j^\ell U_k^i . \]  (B.2)

The fermionic generators are symplectic Majorana-Weyl spinors, with the convention

\[ Q^i = P_R Q^i = -\gamma_s Q^i , \quad S^i = P_L S^i = \gamma_s S^i . \]  (B.3)

The commutators between bosonic and fermionic generators are

\[ [M_{ab}, Q^i_\alpha] = -\frac{1}{2}(\gamma_{ab} Q^i)_\alpha , \quad [M_{ab}, S^i_\alpha] = -\frac{1}{2}(\gamma_{ab} S^i)_\alpha , \]
\[ [D, Q^i_\alpha] = \frac{1}{2} Q^i_\alpha , \quad [D, S^i_\alpha] = -\frac{1}{2} S^i_\alpha , \]
\[ [U_{ij}, Q^k_\alpha] = \delta_i^k Q^j_\alpha - \frac{1}{2} \delta_j^k Q^i_\alpha , \quad [U_{ij}, S^k_\alpha] = \delta_i^k S^j_\alpha - \frac{1}{2} \delta_j^k S^i_\alpha , \]
\[ [K_a, Q^k_\alpha] = -\delta_a^k Q^i_\alpha + \frac{1}{2} \delta_i^k Q^i_\alpha , \quad [U_{ij}, S^k_\alpha] = \delta_i^k S^j_\alpha - \frac{1}{2} \delta_j^k S^i_\alpha , \]
\[ [P_a, S^i_\alpha] = - (\gamma_a S^i)_\alpha . \]  (B.4)

The anticommutation relations between the fermionic generators are (with the convention that \( Q_\alpha \) are the components of the spinors \( Q \), and \( Q^a \) those of \( \bar{Q} = Q^TC \)):

\[ \{ Q_{\alpha i}, Q^j_{\beta} \} = -\frac{1}{2} \delta_i^j (\gamma_{\alpha})_\beta P_a , \quad \{ S_{\alpha i}, S^j_{\beta} \} = -\frac{1}{2} \delta_i^j (\gamma_{\alpha})_\beta K_a , \]
\[ \{ Q_{\alpha i}, S^j_{\beta} \} = \frac{1}{2} \delta_i^j \delta_{\alpha}^\beta D + \frac{1}{2} \delta_i^j (\gamma_{ab})_{\alpha}^\beta M_{ab} + 4 \delta_{\alpha}^\beta U_{ij} . \]  (B.5)

For readability of the formulas, we omitted in the right-hand side \( P_L \) or \( P_R \) projection matrices, which follow from the chirality properties of the generators in the left-hand side.

C  Construction of a vector multiplet from the components of a linear multiplet

The linear multiplet was introduced in section 2.3 with the fields \( L^{ij}, \varphi^i \) and \( E_a \). We define

\[ L = (L_{ij}L^{ij})^{1/2} , \]  (C.1)

which implies

\[ L_{ij}L^{jk} = \frac{1}{2} \delta^k_i L^2 , \quad L^i_j L^{jk} = \frac{1}{2} \varepsilon^{ijk} L^2 . \]  (C.2)
The following formula for the second derivative of $L$ (with any type of covariant derivative $\mathcal{D}$) is also useful:

$$
\mathcal{D}_a \mathcal{D}^a L = L^{-1} \mathcal{L}^{ij} \mathcal{D}_a \mathcal{D}^a L_{ij} + L^{-1} \mathcal{D}_a L^{ij} \mathcal{D}^a L_{ij} - L^{-3} \mathcal{L}^{ij} \mathcal{D}_a L_{ij} L^{kl} \mathcal{D}^a L_{kl}. \tag{C.3}
$$

The $\mathcal{N} = (1, 0)$, $D = 6$ abelian vector multiplet consists of a real vector field $W_\mu$, an SU(2) Majorana spinor $\Omega^i$ of positive chirality and a triplet of auxiliary scalar fields $Y^{ij} = (Y^{ij})^*$. The full nonlinear $Q$- and $S$-transformation rules are given by:

$$
\begin{align*}
\delta W_\mu &= -\bar{\epsilon} \gamma_\mu \Omega, \\
\delta \Omega^i &= \frac{1}{8} \gamma \cdot \hat{F}(W) \epsilon^i - \frac{1}{2} Y^{ij} \epsilon_j, \\
\delta Y^{ij} &= -\bar{\epsilon} (i \mathcal{D} \Omega^j) + 2 \bar{\epsilon} (i \Omega^j), \\
\hat{F}_{\mu\nu}(W) &= \mathcal{F}_{\mu\nu}(W) + 2 \bar{\psi}_i [\gamma_\mu \gamma_\nu] \Omega, \\
\mathcal{D}_\mu \Omega^i &= \partial_\mu \Omega^i - \frac{3}{2} b_\mu \Omega^i + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \Omega^i - \frac{1}{2} V^{ij} \Omega^j \\
&\quad - \frac{1}{8} \gamma \cdot \hat{F}(W) \psi^i + \frac{1}{2} Y^{ij} \bar{\psi}_{ij}. \tag{C.4}
\end{align*}
$$

In [2] one constructs a vector multiplet as multiplet of field equations for the fields of the linear multiplet. We summarize the results:

$$
\begin{align*}
\Omega^i &= L^{-1} \mathcal{D} \varphi^i - L^{-3} (\mathcal{D} L^{ij}) L_{jk} \varphi^k + \frac{1}{2} L^{-3} \gamma^a E_a L^{ij} \varphi_j + \frac{2}{3} L^{-1} L^{ij} \chi_j \\
&\quad + \frac{1}{12} L^{-1} \gamma \cdot T \varphi^i + \frac{1}{2} L^{-5} L^{ij} \gamma^a \varphi_j L^{kl} \varphi_k \gamma^a \varphi_l, \\
Y^{ij} &= -L^{-3} \mathcal{D}_a \mathcal{D}^a L^{ij} + L^{-3} \mathcal{D}_a \mathcal{D}^a L^{k(i} \mathcal{D}_a L^{j)} + \frac{1}{4} L^{-3} E_a E^a L^{ij} \\
&\quad - L^{-3} E^a L^{k(i} \mathcal{D}_a L^{j)} - \frac{1}{3} L^{-1} L^{ij} D + \frac{1}{6} L^{-1} \chi^{(ij)} \\
&\quad - \frac{4}{3} L^{-3} L^{k(i} L^{j)} \varphi_k \varphi_l + \frac{1}{4} L^{-3} L^{ij} \varphi^k \mathcal{D} \varphi_k + 2 L^{-3} L^{k(i} \varphi_k \mathcal{D} \varphi^{j)} \\
&\quad - L^{-3} \mathcal{D}_a L^{k(i} \varphi^a \varphi_k - 3 L^{-5} L^{pq} L^{k(i} \mathcal{D}_a L^{j)} \varphi_p \gamma^a \varphi_q \\
&\quad - \frac{1}{12} L^{-3} L^{ij} \varphi^k \mathcal{D} \varphi_k + \frac{1}{4} L^{-3} \mathcal{D} \varphi^{(i} \chi^a \gamma^a \varphi_{ij)} \\
&\quad + \frac{3}{2} L^{-5} L^{k(i} \varphi_k \gamma^a \varphi_l E_a - \frac{1}{2} L^{-5} \varphi^{(i} \gamma^a \varphi_{ij)} L^{kj} \varphi_k \gamma^a \varphi_l \\
&\quad + \frac{5}{4} L^{-7} L^{ij} L^{kl} \varphi_k \gamma_a \varphi_l L^{mn} \varphi_m \gamma^a \varphi_n, \\
\hat{F}_{ab}(W) &= -L^{-1} L^{ij} \mathcal{R}_{a(i} (W) - 2 \mathcal{D}_a (L^{-1} E_b) - 2 L^{-3} L^{k} \mathcal{D}_a L^{k} \mathcal{D}_b L^{ij} \\
&\quad + L^{-1} \mathcal{R}_{ab}^k(Q) \varphi_k - 2 \mathcal{D}_a (L^{-3} L^{ij} \varphi_{i} \varphi_j). \tag{C.5}
\end{align*}
$$
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