ON CENTRALIZERS IN AZUMAYA DOMAINS

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Abstract. We prove a positive characteristic analogue of the classical result that the centralizer of a nonconstant differential operator in one variable is commutative. This leads to a new, short proof of that classical characteristic zero result, by reduction modulo $p$.

1. Introduction

The theory of commuting differential operators in one variable goes back at least a century [4]. Of special interest is the following fundamental result:

Theorem. Let $P = a_n d^n + \cdots + a_1 d + a_0, a_i \in \mathbb{C}[x]$, be a differential operator of positive degree. Then the algebra of differential operators that commute with $P$ is commutative.

The modern version of this result is attributed to Flanders in [1] and has been given at least two rather different proofs [1, 2]. In this note, we are interested in an analogous statement in positive characteristic. Namely for the first Weyl algebra $A_1(k)$ over a field $k$ of positive characteristic. We prove a generalization of the following:

Theorem. Let $k$ be a field of positive characteristic and let $P \in A_1(k)$ be noncentral. Then the centralizer of $P$ is a commutative algebra.

We also prove that the fraction field of the centralizer is isomorphic to the fraction field of $k[P]$, where $Z$ is the center of $A_1(k)$. Note that this is simpler than the analogous assertion for a complex differential operator $Q$, for which the fraction field of the centralizer is in general only a finite extension of the field of rational functions in $Q$, see [1, Corollary 1]. Unfortunately, neither the elementary methods from [1] (because of operators of degree divisible by the characteristic), nor those from [2] (because all operators are algebraic over the center, which is of dimension 2) do adapt to the positive characteristic. Nevertheless, we present here a simple argument based on a dimension count. We note finally that, in the spirit of [6] and [3], this provides a new and very short proof of the classical characteristic zero theorem above, by reduction modulo primes.

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2. Main result

Here is the main result of this note.

Lemma 2.0.1. Let $A$ be a domain that is a finitely generated module of rank $p^2$ over its center $Z$, for $p$ a prime number. Then the centralizer $B_a$ of every noncentral element $a$ is a commutative ring. Moreover, the natural embedding $Z[\alpha] \subseteq B_a$ induces an isomorphism of fields of fractions.
Proof. Let $a \in A$ be an element which does not belong to the center $Z$, and let $B = B_a$ be the centralizer of $a$ in $A$. We let $K$ be the fraction field of $Z$, $L$ be the fraction field of the center $R$ of $B$, and $L'$ be the fraction field of $Z[a]$. Note that $L' \subseteq L$.

Consider $A_K := K \otimes_Z A, B_L := L \otimes_R B$, and $B_{L'} := L' \otimes_Z [a] B$. Since these are all localizations of domains, they are domains. Note that $A_K$ is of finite dimension $p^2$ over its central subfield $K$. Hence it is a division ring, as multiplication by a nonzero element is injective and hence invertible, by the finite dimension. Thus there are natural inclusions of the localizations $B_{L'} \subseteq B_L \subseteq A_K$, and $B_L$ and $B_{L'}$ are also division rings, since they are finite dimensional over their central subfield $K$. Moreover, the dimension $\dim_K(B_L)$ (resp. $\dim_K(B_{L'})$) divides $\dim_K(A_K) = p^2$.

Thus $\dim_K(B_L)$ (resp. $\dim_K(B_{L'})$) is 1, $p$ or $p^2$. But the inclusions $K \subseteq B_L \subseteq A_K$ (resp. $K \subseteq B_{L'} \subseteq A_K$) are proper, since $a$ is a noncentral element. Hence $\dim_K(B_L) = p = \dim_K(B_{L'})$.

Thus $B_{L'} = B_L$.

Finally, we have that $\dim_L(B_L)$ (resp. $\dim_{L'}(B_{L'})$) divides $\dim_K(B_L) = p$. Thus $\dim_L(B_L)$ (resp. $\dim_{L'}(B_{L'})$) is either 1 or $p$. But the inclusion $K \subseteq L$ (resp. $K \subseteq L'$) is proper as $a$ is not central, hence $\dim_L(B_L) < \dim_K(B_L)$ (resp. $\dim_{L'}(B_{L'}) < \dim_K(B_{L'})$). We conclude that $\dim_L(B_L) = 1 = \dim_{L'}(B_{L'})$, thus $B \subseteq L$ is commutative and the fraction field of $B$ is $B_L = B_{L'} = L'$.

Remark 2.0.2. If $A$ is as in the lemma and is also an algebra over a field $k$, then $k[a]$ is included in $B_a$. However $B_a$ is not necessarily finite over $k[a]$. For example, in the first Weyl algebra over a field $k$ of positive characteristic $p$ with coordinate $x$, we have $B_x = k[x, (x^n y)]$. This algebra is not a finitely generated module over $k[x]$.

Remark 2.0.3. In case of a domain of higher rank $p^n$ over its center, the centralizer of a noncentral element is not necessarily commutative. For example, the centralizer of $x$ in the second Weyl algebra $A_2(k)$, with coordinates $x$ and $y$, is generated over the positive characteristic $p$ field $k$ by $x, \hat{e}_x, y$ and $\hat{e}_y$, and is thus not commutative. Nevertheless, we believe that the rank of a centralizer over the center is a useful invariant here too, and hope to consider it in a future work.

3. A COROLLARY

The lemma applies in particular to the first Weyl algebra over a field of positive characteristic. This leads to a proof by reduction modulo $p$ of the following classical result [1, 2].

Theorem 3.0.1. Let $k$ be a field of characteristic zero. Then the centralizer of every nonconstant polynomial differential operator in one variable over $k$ is a commutative ring.

Proof. For an arbitrary commutative ring $k'$, we denote by $A(k')$ the ring of polynomial differential operators in one variable over $k'$, i.e. the first Weyl algebra over $k'$. Moreover, for all $a \in A(k')$, we let the total degree $\text{tot}(a)$ of $a$ be the degree of the total symbol of $a$ as a $k'$-polynomial in 2 variables.

Let $a \in A(k)$ be nonconstant and let $P, Q \in A(k)$ be operators commuting with $a$. We want to show that the commutator $C := [P, Q]$ vanishes. We let $S$ be the ring generated by the coefficients of $a$, $P$, and $Q$, lying in $k$. We have $a, P, Q, C \in A(S) \subseteq A(k)$.

Let $n$ be the total degree of $a = \Sigma_{i+j \leq n} a_{i,j} x^i y^j$, and let $u = N \times \Pi_{i+j = n} a_{i,j}$, where $N$ is the factorial of $n$ and the product $\Pi$ is taken over nonzero coefficients only. We note that the ring $S[\frac{1}{u}]$ is Jacobson [5, Cor.10.4.6]. Hence the closed points of $\text{Spec}(S[\frac{1}{u}])$ form a dense subset, and for each closed point $s$, the residue field $k(s)$ is a finite field. Let us denote by $a_s, P_s, Q_s,$ and $C_s$ the images of $a, P, Q,$ and $C$ in $A(k(s))$, respectively. Then by the choice of $u$, for all closed point $s$ of $\text{Spec}(S[\frac{1}{u}])$, the element $a_s$ is of positive total degree which is prime to the characteristic of $k(s)$. Hence $a_s$ is not central, $P_s$ and $Q_s$ commute with $a_s$, and $C_s = [P_s, Q_s]$.
Since the ring \( A(k(s)) \) is a domain of rank \( p_s^2 \) over its center \( \mathbb{Q} \) Thm. 2 and \( \S 4 \), for \( p_s \) the characteristic of \( k(s) \), we can apply the lemma. Hence we conclude that \( C_s = 0 \) for all closed points \( s \) of \( \text{Spec}(S[\frac{1}{p}]) \). This holds generically since the closed points of \( \text{Spec}(S[\frac{1}{p}]) \) are dense. Thus \( C = 0 \). □

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