Linearized stability analysis of Caputo-Katugampola fractional-order nonlinear systems

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Abstract

In this paper, a linearized asymptotic stability result for a Caputo-Katugampola fractional-order systems is described. An application is given to demonstrate the validity of the proposed results.

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1 Introduction

The integer-order calculus is inconvenient for several physical systems whose true dynamics contain fractional (non-integer) derivatives. To accurately model these systems, the fractional-order differential equations are used. For example heat transfer systems [7], financial systems [8] and electromagnetic systems [10], have been modelled using the fractional-order calculus. In the last years, the use of fractional-order equations in the stability theory has distinctly risen [1, 4, 5, 8, 13, 15, 16, 17, 18, 20, 21], and several works have been done in this context, we cite for example two main methods. Firstly, the Lyapunov’s first method: as the method of linearization, the study

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of the linear equation by means of Lyapunov exponents and the theorem of linearized asymptotic stability [5] or [6]. Secondly, Lyapunov’s Second Method: as the method of Lyapunov functions, for instance, see [3] or [9].

In this paper, we propose a Lyapunov’s first method for the zero solution of the following Caputo-Katugampola fractional-order system of order $\alpha \in (0, 1), \rho > 0$:

$$C D_{t_0}^{\alpha, \rho} x(t) = Ax(t) + f(x(t)), \ t \geq t_0,$$

where $A \in \mathbb{R}^{d \times d}$ and $f$ is a locally Lipschitz continuous function satisfying that

$$f(0) = 0,$$

$$\lim_{r \to 0} \ell_f(r) = 0,$$

where

$$\ell_f(r) := \sup_{x, y \in B_{\mathbb{R}^d}(0, r)} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$  

The asymptotic stability of the zero solution of the linear Caputo-Katugampola fractional-order system:

$$C D_{t_0}^{\alpha, \rho} x(t) = Ax(t)$$

is known to be equivalent to the spectrum $\sigma(A)$ satisfies $\sigma(A) \subset \{\lambda \in \mathbb{C} : |\arg(\lambda)| > \frac{\alpha \pi}{2}\}$ see [14]. It is to be demonstrated that if the zero solution of (4) is asymptotically stable then the zero solution of (1) is asymptotically stable which is our main result of theorem 1.

For the proof of such theorem, two main steps are presented. The first step consist on the transformation of the linear part to a matrix which is "very close" to a diagonal matrix. The second step describes the construction of an appropriate Lyapunov-Perron operator whose aim is to present a family of operators with the property that any solution of the nonlinear system can be interpreted as a fixed point. Finally, an application is given to show the validity of the theoretical result. In the special case $\rho = 1$ Cong et al. [5] developed the Lyapunov’s first method for the zero solution of the Caputo fractional-order differential equations of order $\alpha \in (0, 1)$. Our results presented in this paper mainly extend the work of Cong et al. [5].

The rest of the paper is organized as follows. Useful results in relation with the fractional-order calculus and other preliminaries are presented in section 2. Then, the main contributions, dealing with the linearized asymptotic stability for Caputo-Katugampola fractional differential equations, is given in section 3. Finally, an application to Caputo-Katugampola fractional-order Lorenz system is presented in section 4 to show the efficiency of the proposed approach.

## 2 Preliminaries

In this section, let us revisit some basics of the fractional calculus. We adopt the notations of the Caputo-Katugampola fractional integral and derivative from [2] [3] [11] [12].
Definition 1. (Katugampola fractional integral) Given $\alpha > 0$, $\rho > 0$ and an interval $[a, b]$ of $\mathbb{R}$, where $0 < a < b$. The Katugampola fractional integral of a function $x \in L^1([a, b])$ is defined by

$$I_{a+}^{\alpha, \rho} u(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}u(s)}{(t^\rho - s^\rho)^{1-\alpha}} ds,$$

where $\Gamma$ is the Gamma function.

Definition 2. (Katugampola fractional derivative) Given $0 < \alpha < 1$, $\rho > 0$ and an interval $[a, b]$ of $\mathbb{R}$, where $0 < a < b$. The Katugampola fractional derivative is defined by

$$D_{a+}^{\alpha, \rho} u(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_a^t \frac{s^{\rho-1}u(s)}{(t^\rho - s^\rho)^\alpha} ds.$$

Definition 3. (Caputo-Katugampola fractional derivative) Given $0 < \alpha < 1$, $\rho > 0$ and an interval $[a, b]$ of $\mathbb{R}$, where $0 < a < b$. The Caputo-Katugampola fractional derivative is defined by

$$C D_{a+}^{\alpha, \rho} u(t) = D_{a+}^{\alpha, \rho}[u(t) - u(a)]$$

$$= \frac{\rho^\alpha}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_a^t \frac{s^{\rho-1}[u(s) - u(a)]}{(t^\rho - s^\rho)^\alpha} ds.$$

Lemma 1. Let $h : [t_0, +\infty) \to \mathbb{R}$ be a continuous function. Then, the semigroup property holds

$$I_{a+}^{\alpha, \rho} I_{a+}^{\beta, \rho} u(t) = I_{a+}^{\alpha + \beta, \rho} u(t), \quad 0 < \alpha, 0 < \beta, 0 < \rho.$$

Lemma 2. The Katugampola fractional integral of $u(t) = \left(\frac{t^\rho - a^\rho}{\rho}\right)\beta$ has the result

$$\frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta + \alpha}, \quad -1 < \beta, 0 < \alpha, 0 < \rho.$$

Lemma 3. If $u$ is a constant, then the fractional derivative of $u$ is $C D_{a+}^{\alpha, \rho} u(t) = 0$.

Since $f$ is a locally Lipschitz continuous function, then we have the existence and uniqueness of solutions of initial value problems \([11]\) (see \([11]\)). Let $x(t, x_0)$ denote the solution of \([11]\) on its maximal interval of existence $I = [t_0, t_{\max}(x_0))$ with $t_0 < t_{\max}(x_0) \leq \infty$.

Definition 4. The equilibrium point $x^* = 0$ of \([11]\) is called:

- stable if for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that for every $\|x_0\| \leq \delta$ we have $t_{\max}(x_0) = \infty$ and $\|x(t, x_0)\| \leq \varepsilon$ for all $t \geq t_0$. 

3
asymptotically stable if it is stable and, furthermore, there exists \( c > 0 \) such that for every \( \|x_0\| \leq c \) we have
\[
\lim_{t \to \infty} x(t, x_0) = 0.
\]

**Definition 5.** The Mittag-Leffler function with two parameters is defined as
\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha + \beta)},
\]
where \( \alpha > 0, \beta > 0, z \in \mathbb{C} \).
When \( \beta = 1 \), one has \( E_{\alpha}(z) = E_{\alpha,1}(z) \).

**Proposition 1.** Let \( \lambda \) be an arbitrary complex number with \( \frac{\alpha \pi}{2} < |\arg \lambda| \leq \pi \). Then, the following statements hold:
(i) There exist a positive constant \( M(\alpha, \lambda) \) and a positive number \( t_1 \) such that
\[
|t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)| \leq \frac{M(\alpha, \lambda)}{t^{\alpha+1}} \quad \text{for any } t > t_1.
\]
(ii) There exists a positive constant \( C(\alpha, \lambda) \) such that
\[
\sup_{t \geq 0} \int_0^t |(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda (t-s)^\alpha)| \, ds \leq C(\alpha, \lambda).
\]

**Remark 1.** Let \( \lambda \) be an arbitrary complex number with \( \frac{\alpha \pi}{2} < |\arg \lambda| \leq \pi \). Then
\[
\sup_{t \geq a} \int_a^t \left| \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left( \lambda \left( \frac{t^\rho - s^\rho}{\rho} \right)^\alpha \right) \right| \, ds \leq C(\alpha, \lambda).
\]
Indeed, using the change of variable \( u = \frac{s^\rho}{\rho} \), we obtain
\[
\int_a^t \left| \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left( \lambda \left( \frac{t^\rho - s^\rho}{\rho} \right)^\alpha \right) \right| \, ds
= \int_{\frac{s^\rho}{\rho}}^{\frac{t^\rho}{\rho}} \left| \left( \frac{t^\rho - u^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left( \lambda \left( \frac{t^\rho - u^\rho}{\rho} \right)^\alpha \right) \right| \, du.
\]
It follows from Proposition [1] that
\[
\sup_{t \geq a} \int_a^t \left| \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left( \lambda \left( \frac{t^\rho - s^\rho}{\rho} \right)^\alpha \right) \right| \, ds \leq C(\alpha, \lambda).
\]
We consider the following fractional differential equation

\[ C^\alpha D^{\alpha,\rho}_a u(t) = \lambda u(t) + h(t), \quad u(a) = c, \quad 0 < \alpha < 1, \quad 0 < \rho, \]

where \( h \) is a continuous function on \([a, +\infty[\). The equation can be transferred to an equivalent integral one as

\[ u(t) = u(a) + \lambda I_0^{\alpha,\rho}_a u(t) + I_0^{\alpha,\rho}_a h(t), \quad u(a) = c. \]

To obtain an explicit clear solution, we apply the method of successive approximation. Set \( u_0 = u(a) = c \) and

\[ u_{n+1} = u_0 + \lambda I_0^{\alpha,\rho}_a u_n + I_0^{\alpha,\rho}_a h(t), \quad 0 \leq n, \]

It follows that

\[ u_1 = u_0 + \lambda I_0^{\alpha,\rho}_a u_0 + I_0^{\alpha,\rho}_a h(t) \]

\[ = u_0 + \frac{u_0 \lambda}{\Gamma(\alpha + 1)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^\alpha + I_0^{\alpha,\rho}_a h(t) \]

\[ = \sum_{k=0}^{1} \frac{u_0 \lambda^k}{\Gamma(k\alpha + 1)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{k\alpha} + I_0^{\alpha,\rho}_a h(t), \]

\[ u_2 = \sum_{k=0}^{2} \frac{u_0 \lambda^k}{\Gamma(k\alpha + 1)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{k\alpha} + \lambda I_0^{2\alpha,\rho}_a h(t) + I_0^{\alpha,\rho}_a h(t), \]

and

\[ u_n = u_0 \sum_{k=0}^{n} \frac{\lambda^k}{\Gamma(k\alpha + 1)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{k\alpha} + \sum_{k=0}^{n-1} \lambda^k I_0^{(k+1)\rho}_a h(t) \]

\[ = u_0 \sum_{k=0}^{n} \frac{\lambda^k}{\Gamma(k\alpha + 1)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{k\alpha} \]

\[ + \int_{a}^{t} \sum_{k=0}^{n-1} \frac{\lambda^k}{\Gamma(\alpha k + \alpha)} (t^\rho - s^\rho)^{1-\alpha(k+1)} \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha(k+1)}} h(s) \, ds \]

\[ = u_0 \sum_{k=0}^{n} \frac{\lambda^k}{\Gamma(k\alpha + 1)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{k\alpha} \]

\[ + \int_{a}^{t} \sum_{k=0}^{n-1} \frac{\lambda^k}{\Gamma(\alpha k + \alpha)} s^{\rho-1} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha(k+1)-1} h(s) \, ds \]

\[ = u_0 \sum_{k=0}^{n} \frac{\lambda^k}{\Gamma(k\alpha + 1)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{k\alpha} \]

\[ + \int_{a}^{t} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \sum_{k=0}^{n-1} \frac{\lambda^k}{\Gamma(\alpha k + \alpha)} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha k} h(s) \, ds. \]
For $n \to \infty$, we obtain

$$u(t) = u_0 E_\alpha \left( \lambda \left( \frac{t^\rho - a^\rho}{\rho} \right) \right)$$

$$+ \int_a^t \left( \frac{t^\rho - s^\rho}{\rho} \right)^{a-1} E_{\alpha,\alpha} \left( \lambda \left( \frac{t^\rho - s^\rho}{\rho} \right) \right) s^{\rho-1} h(s) \, ds.$$  

Using Theorem 6.37, pp. 146 in [19], there exists a nonsingular matrix $T \in \mathbb{C}^{d \times d}$ transforming $A$ into the Jordan normal form, i.e.,

$$T^{-1}AT = \text{diag} \left( A_1, A_2, ..., A_n \right),$$

where for $i = 1, 2, ..., n$,

$$A_i = \lambda_i id_{d_i \times d_i} + \eta_i N_{d_i \times d_i},$$

where $\eta_i \in \{0, 1\}$, $\lambda_i \in \left\{ \hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_m \right\}$, and the nilpotent matrix $N_{d_i \times d_i}$ is given by

$$N_{d_i \times d_i} := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{d_i \times d_i}.$$

Let $\delta > 0$. Using the transformation $P_i = \text{diag} \left( 1, \delta, ..., \delta^{d_i-1} \right)$, we get

$$P_i^{-1}A_iP_i = \lambda_i id_{d_i \times d_i} + \delta_i N_{d_i \times d_i},$$

$\delta_i \in \{0, \delta\}$. Therefore, by the transformation $y = (TP)^{-1}x$ system (1) becomes

$$C D^{\alpha,\rho}_{t_0^\rho} y(t) = \text{diag} \left( J_1, J_2, ..., J_n \right) y(t) + h(y(t)), \quad (5)$$

where $J_i = \lambda_i id_{d_i \times d_i}$ for $i = 1, 2, ..., n$, and

$$h(y) = \text{diag} \left( \delta_1 N_{d_1 \times d_1}, \delta_2 N_{d_2 \times d_2}, ..., \delta_n N_{d_n \times d_n} \right) y(t) + (TP)^{-1} f(TPy). \quad (6)$$

Remark 2. Note that the map $x \rightarrow \text{diag} \left( \delta_1 N_{d_1 \times d_1}, \delta_2 N_{d_2 \times d_2}, ..., \delta_n N_{d_n \times d_n} \right) x$ is a Lipschitz continuous function with Lipschitz constant $\delta$. Thus, by [2], we have

$$h(0) = 0, \quad \lim_{r \to 0} \ell_h(r) = \left\{ \begin{array}{l} \delta \quad \text{if there exists } \delta_i = \delta, \\ 0 \quad \text{otherwise}. \end{array} \right.$$  

Remark 3. The type of stability of the zero solution of equations (1) and (2) are the same.
We denote by $C_\infty \left([t_0, +\infty), \mathbb{R}^d\right)$ the space of all continuous functions $\xi : [t_0, +\infty) \rightarrow \mathbb{R}^d$ such that

$$\|\xi\|_\infty = \sup_{t \geq t_0} \|\xi(t)\| < \infty.$$ 

For any $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^d = \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_n}$, we define the operator

$$\mathcal{F}_x : C_\infty \left([t_0, +\infty), \mathbb{R}^d\right) \rightarrow C_\infty \left([t_0, +\infty), \mathbb{R}^d\right)$$

as follows:

$$(\mathcal{F}_x \xi)(t) = \left((\mathcal{F}_x \xi)^1(t), (\mathcal{F}_x \xi)^2(t), \ldots, (\mathcal{F}_x \xi)^n(t)\right) \quad \text{for} \quad t \in [t_0, +\infty).$$

Where for $i = 1, 2, \ldots, n$

$$(\mathcal{F}_x \xi)^i(t) = E_\alpha \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\alpha J_i x^i + \int_{t_0}^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha J_i \left(s^\rho - 1\right)^i (\xi(s)) \, ds$$

**Remark 4.** In the next section we will verify that $\mathcal{F}_x$ is well defined.

**Remark 5.** Let $\xi \in C_\infty \left([t_0, +\infty), \mathbb{R}^d\right)$. Then $\xi$ is a solution of $(5)$ with $\xi(t_0) = x$ if and only if it is a fixed point of the operator $\mathcal{F}_x$.

## 3 Linearized stability of Caputo-Katugampola fractional-order systems

In this section we study the asymptotic stability of the system $[1]$.

**Proposition 2.** Assume that $\sigma(A) \subset \{\lambda \in \mathbb{C} : |\arg(\lambda)| > \frac{\alpha \pi}{2}\}$. Then, there exists a constant $C(\alpha, \lambda)$ depending on $\alpha$ and $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that for all $x \in \mathbb{R}^d$ and $\xi, \hat{\xi} \in C_\infty \left([t_0, +\infty), \mathbb{R}^d\right)$, it holds

$$\|\mathcal{F}_x \xi - \mathcal{F}_x \hat{\xi}\|_\infty \leq C(\alpha, \lambda) \ell_h \left(\max \left(\|\xi\|_\infty, \|\hat{\xi}\|_\infty\right)\right) \|\xi - \hat{\xi}\|_\infty$$

and

$$\|\mathcal{F}_x \xi\|_\infty \leq \max_{1 \leq i \leq n} \sup_{t \geq t_0} \left|E_\alpha \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\alpha \right| \|x\|$$

$$+ C(\alpha, \lambda) \ell_h (r) \|\xi\|_\infty.$$  

(7)
Proof. For $i = 1, 2, ..., n$, we have
\[
\left\| (F_x \xi)^i(t) - (F_x \hat{\xi})^i(t) \right\| \leq \ell_h \left( \max \left( \|\xi\|_\infty, \|\hat{\xi}\|_\infty \right) \right) \|\xi - \hat{\xi}\|_\infty \times \int_{t_0}^{t} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha - 1} \left| E_{\alpha,\alpha} \left( \lambda_i \left( \frac{t^\rho - s^\rho}{\rho} \right) \right) \right| s^{\rho - 1} ds.
\]
It follows from Remark 1 that
\[
\left\| (F_x \xi)^i - (F_x \hat{\xi})^i \right\|_\infty \leq \ell_h \left( \max \left( \|\xi\|_\infty, \|\hat{\xi}\|_\infty \right) \right) C(\alpha, \lambda_i) \|\xi - \hat{\xi}\|_\infty.
\]
Thus
\[
\| F_x \xi - F_x \hat{\xi} \|_\infty \leq C(\alpha, \lambda) \ell_h \left( \max \left( \|\xi\|_\infty, \|\hat{\xi}\|_\infty \right) \right) \|\xi - \hat{\xi}\|_\infty,
\]
where
\[
C(\alpha, \lambda) = \max \{ C(\alpha, \lambda_1), C(\alpha, \lambda_2), ..., C(\alpha, \lambda_n) \}.
\]
On the other hand, for $\xi = 0$, We have
\[
(F_x 0)^i(t) = E_{\alpha} \left( \left( \frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) J_i x^i.
\]
Then using (7), we get (8).
\[
\square
\]

Remark 6. It follows from Proposition 2 that $F_x$ is well-defined.

Up to now, we have found that the Lyapunov-Perron operator is well-defined and Lipschitz continuous. Note that the Lipschitz constant $C(\alpha, \lambda)$ is independent of the constant $\delta$ which is hidden in the coefficients of system (5). From now on, we choose and fix the constant $\delta$ as follows $\delta := \frac{1}{2C(\alpha, \lambda)}$.

Lemma 4. Take $r > 0$ such that
\[
q := C(\alpha, \lambda) \ell_h(r) < 1,
\]
and set
\[
r^* = \frac{r (1 - q)}{\max_{1 \leq i \leq n} \sup_{t \geq t_0} \left| E_{\alpha} \left( \lambda_i \left( \frac{t^\rho - t_0^\rho}{\rho} \right) \right) \right|}.
\]
Let $B_{C_{\infty}}(0, r) := \{ \xi \in C_{\infty}([-t_0, +\infty), \mathbb{R}^d) : \|\xi\|_\infty \leq r \}$. Then, for any $x \in B_{\mathbb{R}^d}(0, r^*)$ we have $F_x(B_{C_{\infty}}(0, r)) \subset B_{C_{\infty}}(0, r)$ and
\[
\| F_x \xi - F_x \hat{\xi} \|_\infty \leq q \|\xi - \hat{\xi}\|_\infty \quad \text{for all} \quad \xi, \hat{\xi} \in B_{C_{\infty}}(0, r).
\]
Proof. Let \( x \in \mathbb{R}^d \) with \( \| x \| \leq r^* \). Let \( \xi \in B_{C_{\infty}}(0, r) \). It follows from (8)
\[
\| F_x \|_{\infty} \leq \max_{1 \leq i \leq n} \sup_{t \geq t_0} \left| E_\alpha \left( \lambda_i \left( \frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \right| \| x \| \\
+ C (\alpha, \lambda) \ell_h (r) \| \xi \|_{\infty} \\
\leq (1 - q) r + qr = r,
\]
then \( F_x (B_{C_{\infty}}(0, r)) \subset B_{C_{\infty}}(0, r) \).

Then for any \( x \in B_{\mathbb{R}^d}(0, r^*) \) and \( \xi, \hat{\xi} \in B_{C_{\infty}}(0, r) \), it follows from (7) and (9) that
\[
\left\| F_x \xi - F_x \hat{\xi} \right\|_{\infty} \leq C (\alpha, \lambda) \ell_h (r) \left\| \xi - \hat{\xi} \right\|_{\infty} \\
\leq q \left\| \xi - \hat{\xi} \right\|_{\infty}.
\]
The proof is completed. \( \square \)

Consider system (1):
\[
CD_{t_0}^{\alpha, \rho} x (t) = Ax(t) + f (x(t)), \ t \geq t_0.
\]

We now state the main result of this paper.

**Theorem 1.** Assume that \( \sigma (A) \subset \{ \lambda \in \mathbb{C} : \left| \arg (\lambda) \right| > \frac{\alpha \pi}{2} \} \) and the nonlinear term \( f \) is a locally Lipschitz continuous function satisfying (2). Then, the zero solution of (1) is asymptotically stable.

**Proof.** In reason of Remark 3 it is sufficient to prove the asymptotic stability for the zero solution of system (5). To do that, let \( r^* \) be defined as in (10). Let \( x \in B_{\mathbb{R}^d}(0, r^*) \). Using the Banach fixed point theorem and Lemma 4, there exists a unique fixed point \( \xi \in B_{C_{\infty}}(0, r) \) of \( F_x \). This point is also a solution of (5) with the initial condition \( \xi (t_0) = x \). The zero solution 0 is stable, since the initial value problem for Equation (5) has unique solution. To finish the proof of the theorem, we have to demonstrate that the zero solution 0 is attractive. Let \( x = (x^1, x^2, ..., x^n) \in B_{\mathbb{R}^d}(0, r^*) \) and \( \xi (t) = (\xi^1 (t), \xi^2 (t), ..., \xi^n (t)) \) the solution of (5) which satisfies \( \xi (t_0) = x \).

From Lemma 4 we see that \( \| \xi \|_{\infty} \leq r \). Let \( b := \lim_{t \to \infty} \sup \| \xi (t) \| \) thus \( b \in [0, r] \). Let \( \epsilon > 0 \). Then, there exists \( T (\epsilon) > t_0 \) such that
\[
\| \xi (t) \| < b + \epsilon \quad \text{for all } t \geq T (\epsilon).
\]
Using Proposition \((i)\), we get
\[
\limsup_{t \to \infty} \left\| \int_{t_0}^{T(t)} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha - 1} E_{\alpha, \alpha} \left( \lambda_i \left( \frac{t^\rho - s^\rho}{\rho} \right)^\alpha \right) s^{\rho - 1} h^i(\xi(s)) \, ds \right\|
\]
\[
\leq \max_{t \in [t_0, T(t)]} \left\| h^i(\xi(t)) \right\| \limsup_{t \to \infty} \int_{t_0}^{T(t)} \frac{M(\alpha, \lambda_i)}{(t^\rho - s^\rho)^{\alpha + 1}} s^{\rho - 1} \, ds
\]
\[
\leq \max_{t \in [t_0, T(t)]} \left\| h^i(\xi(t)) \right\| \limsup_{t \to \infty} \int_{t_0}^{T(t)} \frac{\rho^{\alpha + 1} M(\alpha, \lambda_i)}{(t^\rho - u)^{\alpha + 1}} \, du = 0.
\]

Thus, it follows from the fact that \(\xi^i(t) = (F_{\alpha} \xi)^i(t)\) and \(\lim_{t \to \infty} E_{\alpha} \left( \lambda_i \left( \frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) = 0\) that
\[
\limsup_{t \to \infty} \left\| \xi^i(t) \right\| = \limsup_{t \to \infty} \left\| \int_{t_0}^{T(t)} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha - 1} E_{\alpha, \alpha} \left( \left( \frac{t^\rho - s^\rho}{\rho} \right)^\alpha \right) s^{\rho - 1} h^i(\xi(s)) \, ds \right\|
\]
\[
\leq \ell_h(r) C(\alpha, \lambda_i) (b + \epsilon).
\]

Therefore,
\[
b \leq \max \left\{ \limsup_{t \to \infty} \left\| \xi^1(t) \right\|, \limsup_{t \to \infty} \left\| \xi^2(t) \right\|, \ldots, \limsup_{t \to \infty} \left\| \xi^n(t) \right\| \right\}
\]
\[
\leq \ell_h(r) C(\alpha, \lambda) (b + \epsilon).
\]

We tend \(\epsilon\) to zero we find,
\[
b \leq \ell_h(r) C(\alpha, \lambda) b.
\]

From the assumption \(\ell_h(r) C(\alpha, \lambda) < 1\), we obtain that \(b = 0\). This ends the proof.

\[\square\]

4 Application

To illustrate the theoretical result and to show its effectiveness, the Caputo-Katugampola fractional-order Lorenz system is given as an example. Such system can be written as:
\[
\begin{aligned}
C D_{t_0}^{\alpha, \rho} x_1(t) &= a(x_1(t) - x_2(t)) \\
C D_{t_0}^{\alpha, \rho} x_2(t) &= b x_1(t) - x_1(t) x_3(t) - c x_2(t) \\
C D_{t_0}^{\alpha, \rho} x_3(t) &= x_1(t) x_2(t) - d x_3(t)
\end{aligned}
\]  

(11)
where $a, b, c$ and $d$ are four parameters and $\alpha, \rho$ are the fractional-orders. It can be rewritten as 

$$\begin{align*}
\mathcal{C}D_{t_0}^{\alpha, \rho}x(t) &= Ax(t) + g(x(t)),
\end{align*}$$

where $x = (x_1, x_2, x_3)^T$, $A = \begin{pmatrix} a -a & 0 \\ b -c & 0 \\ 0 & 0 & -d \end{pmatrix}$ and $g(x) = \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}$.

Let consider $a = -8, b = 26, c = -7, d = 3, \alpha = 0.9$, and $\rho = 1.2$. We design the linear state feedback controller as $u = BKx$ and select:

$$B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & -50 & 0 \end{pmatrix}.$$  

The eigenvalue of the matrix $A + BK$ are $\lambda_1 = -2.8229, \lambda_2 = -48.1771$ and $\lambda_3 = -3$ which make $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \text{ } i \in \{1, 2, 3\}$. Thus, according to Theorem 1 the zero solution of the closed-loop system

$$\begin{align*}
\mathcal{C}D_{t_0}^{\alpha, \rho}x(t) &= Ax(t) + g(x(t)) + Bu(t),
\end{align*}$$

is asymptotically stable.

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