WEI-NORMAN AND BEREZIN’S EQUATIONS OF MOTION ON THE SIEGEL-JACOBI DISK

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ABSTRACT. We show that the Wei-Norman method applied to describe the evolution on the Siegel-Jacobi disk $D^J_1 = D_1 \times \mathbb{C}^1$, where $D_1$ denotes the Siegel disk, determined by a hermitian Hamiltonian linear in the generators of the Jacobi group $G^J_1$ and Berezin’s scheme using coherent states give the same equations of quantum and classical motion when are expressed in the coordinates in which the Kähler two-form $\omega_{D^J_1}$ can be written as $\omega_{D^J_1} = \omega_{D_1} + \omega_{C^1}$. The Wei-Norman equations on $D^J_1$ are a particular case of equations of motion on the Siegel-Jacobi ball $D^J_n$ generated by a hermitian Hamiltonian linear in the generators of the Jacobi group $G^J_n$ obtained in Berezin’s approach based on coherent states on $D^J_n$.

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1. Introduction

The Jacobi groups - $G^J_n = H_n \rtimes \text{Sp}(n, \mathbb{R})_C$ - where $H_n$ denotes the $(2n+1)$-dimensional Heisenberg group, are unimodular, nonreductive, algebraic groups of Harish-Chandra type [13]. The Siegel-Jacobi domains are nonsymmetric domains associated to the Jacobi groups by the generalized Harish-Chandra embedding [15, 33, 53, 54, 12]. The Jacobi group is also an important object in physics, where sometimes it is known under other names, as Hagen [29], Schrödinger [43], or Weylsymplectic group [52]. The Jacobi group describes the squeezed states [48, 38, 30] in Quantum Optics [15] concerning the equations of motion on the Siegel-Jacobi domains [13]. The Jacobi groups - $G^J_n$ - have been studied in connection with the group-theoretic approach to coherent states [44] in [6, 14] in the case $n = 1$, while the case $n \in \mathbb{N}$ has been treated in [7, 8, 13]. We have attached to the Jacobi group $G^J_n$ coherent states based on Siegel-Jacobi ball $D^J_n$ [8], which, as set, consists of the points of $\mathbb{C}^n \times D_n$. The non-compact hermitian symmetric space $\text{Sp}(n, \mathbb{R})_C / U(n)$ admits a matrix realization as a bounded homogeneous domain, the Siegel ball $D_n$, $D_n := \{ W \in M(n, \mathbb{C}) : W = W^t, \Im W - \bar{W}W > 0 \}$. We have determined the $G^J_n$-invariant Kähler two-form $\omega_{D^J_n}$ on $D^J_n$ [8], also investigated by Yang [54]. In [7, 8, 13] the $G^J_n$-invariant Kähler two-form $\omega_{D^J_n}(z, W)$, where $z \in \mathbb{C}^n, W \in D_n$, is written compactly as the sum of two terms, one describing the homogeneous Kähler two-form $\omega_{D^J_n}(W)$ on $D^J_n$, the other one is $\text{Tr}(A^t(\Im W - \bar{W}W)^{-1} \wedge A)$, where $A = d z + d \bar{W} \eta$, and $\eta = (\Im W - \bar{W}W)^{-1}(z + \bar{W}z)$. We have denoted by $FC$ the change of variables $FC : \mathbb{C}^n \times D_n \ni (\eta, W) \rightarrow (z, W) \in D^J_n$, $z = \eta - W \bar{\eta}$. We have shown [13] that the $FC$-transform is a Kähler homogeneous diffeomorphism, and, when expressed in the variables $(\eta, W) \in \mathbb{C}^n \times D_n$ - let us call them FC-variables - the Kähler two-form $\omega_{D^J_n}(\eta, W) = \omega_{D^J_n}(W) + \omega_{\mathbb{C}^n}(\eta)$. We have put in [13] this change of variables in connection with the celebrated fundamental conjecture on homogeneous Kähler manifolds of Gindikin and Vinberg [50, 23] on the Siegel-Jacobi ball $D^J_n$, as we did in [14] for the Siegel-Jacobi disk $D^J_1$. Later, we have underlined in [15] that the $FC$-transform has a deep meaning in the context of Perelomov coherent states [44]: it gives the change of variables from the representation of the normalized to the un-normalized coherent state vector, as is recalled in the Appendix in [8].

The equations of motion on the Siegel-Jacobi ball $D^J_n$ determined by a hermitian Hamiltonian linear in the generators of the Jacobi group $G^J_n$ were studied in [13], generalizing the results presented in [6, 14], obtained from the holomorphic differential representation of the Lie algebra $\mathfrak{g}^J_1$ of the Jacobi group. We recall that linear Hamiltonians in generators of the Jacobi group appear in quantum mechanics, as in the case of the quantum oscillator acted on by a variable external force [26, 46, 32] and in the case of quantum dynamics of trapped ions [28, 40].

The aim of this paper is to compare some of the results obtained in our papers [6, 14, 13] concerning the equations of motion on the Siegel-Jacobi $D^J_n$ determined by a hermitian Hamiltonian linear in the generators of the Jacobi group $G^J_n$ with the results of the paper [27], referring to the equations of motion on $D^J_1$ obtained with the Wei-Norman method [51]. In [27] the dynamics on $D^J_1$ determined by a hermitian Hamiltonian linear in the generators of $G^J_1$ is considered in the general framework of Lie systems [33, 56], as was developed further in a geometric approach in [19, 20].
Wei-Norman equations for the Jacobi group $G_1^J$ in real coordinates have been studied in [21, 22].

In order to establish a correspondence between the formulae of [27] and our notation, we give the following dictionary (firstly are introduced the symbols used in [27]): affine symplectic group $G_{AS} = SL(2, \mathbb{R}) \rtimes \mathbb{R}^2 \leftrightarrow$ Jacobi group $G_1^J := H_1 \rtimes SU(1,1)$; extended Poincaré disk $\mathcal{M} = \mathcal{D} \times \mathbb{R}^2 \leftrightarrow$ Siegel-Jacobi disk $\mathcal{D}_1^J = \mathbb{C} \times \mathcal{D}_1$.

Essentially, the Wei-Norman method (see [51] and the Appendix in §7) consists in representing the solution of the equation
\[
\frac{dU(t)}{dt} = A(t)U(t), \quad U(0) = I,
\]
in the form of product of exponentials
\[
(1.1) \quad U(t) = \prod_{i=1}^{n} \exp(\xi_i(t)X_i),
\]
where $A$ and $U$ are linear operators,
\[
(1.2) \quad A(t) = \sum_{i=1}^{n} \epsilon_i(t)X_i,
\]
$\epsilon_i(t)$ are scalar functions of $t$ and $\{X_i\}_{i=1,\ldots,n}$ are the generators of a Lie algebra $\mathfrak{g}$.

On the other side, we have developed in [3, 4] an algebraic method to obtain a representation of a Lie algebra $\mathfrak{g}$ of a Lie group $G$ as first order holomorphic differential operator on $M = G/H$ when $M$ is a hermitian symmetric manifold. Later we have applied the method to a larger class of Lie groups, advancing the hypothesis that for the coherent type groups [37, 42], i.e. Lie groups for which the $n$-dimensional homogeneous manifold $M$ admits an holomorphic embedding in a projective Hilbert space $M \hookrightarrow \mathbb{P}(\mathcal{H}^{\infty})$, the generators of the Lie algebra $\mathfrak{g}$ of the Lie group $G$ admit a holomorphic differential representation
\[
(1.3) \quad \mathfrak{g} \ni X \mapsto \mathfrak{k}(z) = P_X(z) + \sum_i Q_X^i(z) \frac{\partial}{\partial z_i},
\]
where $P_X(z)$, and $Q_X^i(z)$ are polynomials defined on $M = G/H$. We have verified [6, 8, 13] this hypothesis in the case of the Jacobi group $G_n^J$, which is a coherent type group [42, 43]. Following a method advanced in [3, 4], which uses Perelomov coherent states [43] and a dequantization method developed by Berezin [16, 17], we have determined the equations of motion on $M = \mathcal{D}_n^J$ when the Hamiltonian $H$ is linear in the generators of the Jacobi group in the case $n = 1$ in [6, 14] and in [13] for $n \in \mathbb{N}$. In general, for groups $G$ for which the representation (1.3) of the Lie algebra $\mathfrak{g}$ of the Lie group $G$ is true, the equations of motion on $M = G/H$ depend on the coefficients $\epsilon_i$ in front of the generators $X_i$ of the group $G$ which appear in $H$ of the form (1.2) and the polynomials $Q^i$ which appear in (1.3), as it is recalled in Proposition [4]. To shorten the expression, we call the equations of motion obtained with this method, Berezin’s equations of motion. We have shown that for a Hamiltonian linear in the generators of $G_n^J$, the motion on $\mathcal{D}_n$ is described by a matrix Riccati equation, while the motion in $z \in \mathbb{C}^n$ is a first order differential equation, with coefficients depending also on $W \in \mathcal{D}_n$. It was proved
in \[14\] for \(G^I_1\) and in \[13\] for \(G^J_n\), \(n \in \mathbb{N}\) that, when the FC-transform is applied, the first order differential equation in the variable \(\eta\) becomes decoupled from the motion on the Siegel ball. These are exactly the equations of motion obtained in \[27\] in the case of the Siegel-Jacobi disk \(D^I_1\) using the Wei-Norman method and we want in the present paper to draw attention to the fact that apparently such different methods lead to the same result.

The paper is laid out as follows. \[2\] recalls the definition of the Jacobi algebra \(g^I_1\) adopted in \[6\]. In \[3\] the unitary operators associated with the Jacobi group \(G^I_1\) are recalled. To a linear operator \(A\) it is associated the operator \(A(\xi) := T^{-1}(\xi)AT(\xi)\) \[9\ \[10\ \[11\], where \(T(\xi) = D(\alpha)S(w)\), \(D(\alpha)\) is the unitary displacement operator associated to the Heisenberg group \(H_1\), \(S(w)\) is the positive discrete series representation associated to the group \(SU(1, 1)\), and \(\xi = (\alpha, w) \in \mathbb{C} \times D_1\). We take the expressions of \(\hat{a}(\alpha, w)\), \(\hat{K}_0(\alpha, w)\) and \(\hat{K}_-(\alpha, w)\) from our papers \[9\ \[10\ \[11\]. Then we apply the Wei-Norman method for the Jacobi group \(G^I_1\) in complex, calculating \(T^{-1}\frac{dT}{dt}\). In \[4\] we determine the equations of motion associated to a Hamiltonian \(H_0\) linear in the generators of the Jacobi group \(G^I_1\). Following \[31\], we introduce the quasienergy operator \(E\) associated to the Hamiltonian \(H_0\). In \[5\] we change the coordinates from complex to real. The main results of our paper are contained in Propositions \[1\] and \[2\]. In brief, the Berezin’s quantum and classical equations of motion on the Siegel-Jacobi disk determined by a hermitian Hamiltonian linear in the generators of Jacobi group \(G^I_1\), expressed in the FC-coordinates, are the same as the equations obtained via the Wei-Norman method. The Wei-Norman equations on \(D^I_1\) are a particular case of Berezin’s equations of motion on the Siegel-Jacobi ball \(D^J_n\) generated by a hermitian Hamiltonian linear in the generators of the Jacobi group \(G^J_n\). In a short remark in \[6\] are compared the phases which appears in the method of Wei-Norman \[27\] and in Berezin’s equations of motion on the Siegel-Jacobi disk \[13\]. For self-containment, in an Appendix in \[7\] we briefly recall the Wei-Norman method. In another Appendix in \[8\] are mentioned the main definitions of coherent states \[14\]. In \[8\ \[1\] it is recalled our method for obtaining Berezin’s equations of motion. The construction of coherent states on the Siegel-Jacobi disk \(D^I_1\) is summarized in \[8\ \[2\] and the equations of motion on \(D^I_1\) obtained in \[14\ \[13\] are reproduced in \[8\ \[3\] in order to make the comparison with the results of \[27\].

**Notation.** In this paper the Hilbert space \(\mathfrak{H}\) is endowed with a scalar product \(\langle \cdot, \cdot \rangle\) antilinear in the first argument, i.e. \(\langle \lambda x, y \rangle = \bar{\lambda} \langle x, y \rangle\), \(x, y \in \mathfrak{H}, \lambda \in \mathbb{C} \setminus 0\). \(\mathbb{R}, \mathbb{C}\) and \(\mathbb{N}\) denotes the field of real, complex numbers, respectively the ring of the integers. We denote the imaginary unit \(\sqrt{-1}\) by \(i\), and the Real and Imaginary part of a complex number by \(\Re\) and respectively \(\Im\), i.e. we have for \(z \in \mathbb{C}\), \(z = \Re z + i \Im z\), and \(\bar{z} = \Re z - i \Im z\), but also we use the notation \(cc(z) := \bar{z}\) for \(z \in \mathbb{C}\) or \(cc(A) = A^\dagger\) for an operator \(A\). We denote by \(M_n(\mathbb{F})\) the set of \(n \times n\) matrices with entries in the field \(\mathbb{F}\). If \(A \in M_n(\mathbb{F})\), then \(A^t (A^\dagger)\) denotes the transpose (respectively, the hermitian conjugate) of \(A\). \(I\) denotes the unit operator, while \(I_n\) denotes the unit matrix of \(M_n(\mathbb{F})\). If \(A \in M_n(\mathbb{F})\), we denote by \(A^s := \frac{1}{2}(A + A^\dagger)\). If \(A\) is a matrix, then \(\text{Tr}(A)\) denotes the trace of the matrix \(A\). We use Einstein convention that repeated indices are implicitly summed. We denote the differential by \(d\). If \(\pi\) is an unitary irreducible representation of a Lie group \(G\) with Lie algebra \(\mathfrak{g}\) on a complex separable Hilbert space \(\mathfrak{H}\), then we denote for the derived representation \(X := d\pi(X), X \in \mathfrak{g}\).
2. The Lie algebra $\mathfrak{g}^J_1$ of the Jacobi group

The Heisenberg group is the group with the 3-dimensional real Lie algebra

$$\mathfrak{h}_1 = \mathfrak{h}_1 \equiv \langle i s I + \alpha a + \bar{\alpha}a \rangle_{s \in \mathbb{R}, \alpha \in \mathbb{C}};$$

where the boson creation (respectively, annihilation) operators $a^\dagger$ ($a$) verify the canonical commutation relation (2.4a).

We consider the Lie algebra of the group $\text{SU}(1, 1)$:

$$\mathfrak{su}(1, 1) = \langle 2i \theta K_0 + y K_+ - \bar{y} K_- \rangle_{\theta \in \mathbb{R}, y \in \mathbb{C}},$$

where the generators $K_0, K_\pm$ verify the standard commutation relations (2.4b).

The Jacobi algebra is defined as the semi-direct sum [6]

$$\mathfrak{g}^J_1 := \mathfrak{h}_1 \rtimes \mathfrak{su}(1, 1),$$

where $\mathfrak{h}_1$ is an ideal in $\mathfrak{g}^J_1$, determined by the commutation relations (2.4c), (2.4d):

$$[a, a^\dagger] = I, \quad [K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0,$$

$$[a, K_] = a^\dagger, \quad [K_-, a^\dagger] = a, \quad [K_+, a^\dagger] = [K_-, a] = 0,$$

$$[K_0, a^\dagger] = \frac{1}{2} a^\dagger, [K_0, a] = -\frac{1}{2} a.$$

In the conventions of [9], see equation (3), we have:

$$a = \frac{1}{2\sqrt{\mu}}(P - iQ); \quad a^\dagger = -\frac{1}{2\sqrt{\mu}}(P + iQ), \quad [P, Q] = 2R.$$

which are different of the conventions used in equations (4.14)-(4.16) in [9]. In the convention of [9], equation (8), $P = \frac{\partial}{\partial x}, Q = 2i\mu x$, corresponding to the derived representation of the Heisenberg group, $R = i\mu I, m \in \mathbb{R}$. The differential realization of (2.3) corresponds to

$$a = \frac{1}{2\sqrt{\mu}} \frac{d}{dx} + \sqrt{\mu} x; \quad a^\dagger = -\frac{1}{2\sqrt{\mu}} \frac{d}{dx} + \sqrt{\mu} x.$$

3. Unitary representations associated to the Jacobi group $G^J_1$

The unitary displacement operator

$$D(\alpha) := \exp(\alpha a^\dagger - \bar{\alpha} a)$$

$$= \exp(-\frac{1}{2}|\alpha|^2) \exp(\alpha a^\dagger) \exp(-\bar{\alpha} a)$$

$$= \exp(\frac{1}{2}|\alpha|^2) \exp(-\bar{\alpha} a) \exp(\alpha a^\dagger)$$

has the composition property

$$D(\alpha_2) D(\alpha_1) = e^{i\theta(\alpha_2, \alpha_1)} D(\alpha_2 + \alpha_1), \quad \theta(\alpha_2, \alpha_1) := \Im(\alpha_2 \bar{\alpha}_1).$$

Note also that

$$D(\alpha)^\dagger = D(\alpha)^{-1} = D(-\alpha).$$
Let us denote by $S$ the unitary irreducible positive discrete series representation $D^k_+$ of the group $\text{SU}(1,1)$ with Casimir operator $C = K_0^2 - K_1^2 - K_2^2 = k(k - 1)$, where $k$ is the Bargmann index for $D^k_+$. We introduce the notation $S(z) = S(w)$, where $w \in \mathbb{C}$, $|w| < 1$ and $z \in \mathbb{C} \setminus 0$, are related by (3.3d). We have the relations:

\begin{align}
(3.3a) & \quad S(z) := \exp(zK_+ - \bar{z}K_-); \\
(3.3b) & \quad S(w) = \exp(wK_+)\exp(\rho K_0)\exp(-\bar{w}K_-) \\
(3.3c) & \quad = \exp(-\bar{w}K_-)\exp(-\rho K_0)\exp(wK_+); \\
(3.3d) & \quad w = \frac{\bar{z}}{|z|}\tanh(|z|), \quad \rho = \ln(1 - w\bar{w}), \quad z \neq 0,
\end{align}

and $w = 0$ for $z = 0$ in (3.3d). Also, it is easy to observe that $S(w)^\dagger = S(w)^{-1} = S(-w)$.

We introduce the unitary operator $T(\xi)$:

\begin{align}
(3.4) & \quad T(\xi) = D(\alpha)S(w), \quad D_1^J \ni \xi = (\alpha, w) \in \mathbb{C} \times D_1.
\end{align}

Following [9, 10, 11], for any linear operator $A$, we define the operator

\begin{align}
(3.5) & \quad \hat{A}(\xi) := T^{-1}(\xi)AT(\xi), \quad D_1^J \ni \xi = (\alpha, w) \in \mathbb{C} \times D_1.
\end{align}

where $T(\xi)$ was defined in (3.4).

In [10, 11] we have proved that:

\begin{align}
(3.6) & \quad \hat{a}(\alpha, w) = r(a + wa^\dagger) + \alpha, \quad r = (1 - w\bar{w})^{-\frac{1}{2}}, \\
(3.7) & \quad \hat{K}_0(\alpha, w) = r^2[\bar{w}K_- + (1 + |w|^2)K_0 + wK_+] + r\Re[\alpha(a^\dagger + w\bar{a})] + \frac{1}{2}|\alpha|^2, \\
(3.8) & \quad \hat{K}_-(\alpha, w) = r^2[K_- + 2wK_0 + w^2K_+] + \alpha r(a + wa^\dagger) + \frac{1}{2}|\alpha|^2.
\end{align}

With formulae (3.1b) and (3.3b), we can express (3.4) as product of exponentials of the generators as in (1.1), where for the Jacobi group $G_1^J$, $n = 6$, and the generators are numbered as

\begin{align}
(3.9) & \quad X_1 = I; \quad X_2 = a^\dagger; \quad X_3 = a; \quad X_4 = K_+; \quad X_5 = K_0; \quad X_6 = K_-,
\end{align}

while the parameters $\xi_i$ in (1.1) are respectively

\begin{align}
(3.10) & \quad \xi_1 = -\frac{1}{2}|z|^2; \quad \xi_2 = z; \quad \xi_3 = -\bar{z}; \quad \xi_4 = w; \quad \xi_5 = \ln(1 - w\bar{w}); \quad \xi_6 = -\bar{w}.
\end{align}

Note that the operator $T(\xi)$, $\xi = (z, w) \in D_1^J$ (3.4), written as the product (1.1) with the generators (3.9) and the parameters (3.10), has the properties expressed in (7.2).

Now we apply (7.8) to the operator $T(\xi)$ (3.4) in the variables $\xi = (z, w) \in D_1^J$ expressed with (3.9), (3.10), taking into account the commutation relations (2.4) of the Lie algebra $g_1^J$.

For

\begin{align}
Y_2 = e^{-\xi_1\text{ad}X_6}e^{-\xi_3\text{ad}X_5}e^{-\xi_2\text{ad}X_4}e^{-\xi_4\text{ad}X_3}X_2,
\end{align}
we get successively:

\[ I_1 = e^{-\xi_3 X_3} X_2 = a^\dagger - \xi_3; \]

\[ I_2 = e^{-\xi_4 X_4} I_1 = I_1; \]

\[ I_3 = e^{-\xi_5 X_5} I_2 = -\xi_3 + e^{\xi_5} a^\dagger; \]

\[ I_4 = e^{-\xi_6 X_6} I_3 = -\xi_3 + e^{\xi_5}(a^\dagger - \xi_6 a). \]

For

\[ Y_3 = e^{-\xi_5 \text{ad} X_6} e^{-\xi_5 \text{ad} X_5} e^{-\xi_5 \text{ad} X_4} X_3, \]

we get successively:

\[ e^{-\xi_5 \text{ad} X_4} X_3 = a + \xi_4 a^\dagger; \]

\[ e^{-\xi_5 \text{ad} X_5} a = e^{\xi_6} a; \]

\[ e^{-\xi_5 \text{ad} X_6} a^\dagger = e^{-\xi_6} a^\dagger. \]

For

\[ Y_4 = e^{-\xi_5 \text{ad} X_6} e^{-\xi_5 \text{ad} X_5} X_4, \]

we get successively:

\[ J_1 = e^{-\xi_5 \text{ad} X_5} X_4 = e^{-\xi_5} K_+; \]

\[ J_2 = e^{-\xi_6 \text{ad} X_6} J_1 = e^{-\xi_6} (K_+ - 2\xi_6 K_0 + \xi_6^2 K_-). \]

We also have the relations:

\[ Y_5 = e^{-\xi_5 \text{ad} X_6} X_5 = K_0 - \xi_6 K_- \]

Summarizing (3.11) - (3.14), we obtained the following expressions of \( Y_1 - mbY_6 \):

\[ Y_1 = I; \quad Y_2 = -\xi_3 + e^{\xi_5} (a^\dagger - \xi_6 a); \quad Y_3 = (e^{\xi_5} - \xi_6 e^{-\xi_5}) a + \xi_4 e^{\xi_5} a^\dagger; \]

\[ Y_4 = e^{-\xi_5} (K_+ + 2\xi_6 K_0 + \xi_6^2 K_-); \quad Y_5 = K_0 + \xi_4 K_-; \quad Y_6 = K_-. \]

Now we introduce the expressions (3.13) into (3.15) and we get for \( T^{-1}\dot{T} \) in the variables \( (z, w) \in \mathcal{D}_J \) the expression

\[ T^{-1}\dot{T} = i \Im (\dot{z} \bar{z}) + r[(\dot{z} - \dot{z}w)a^\dagger - cc] + [\dot{w}r^2 K_+ - cc] + 2i \Im (\dot{w}\bar{w})r^2 K_0. \]

4. Equations of motion on the Siegel-Jacobi disk \( \mathcal{D}_J \)

The time-dependent Schrödinger equation is expressed as

\[ \mathbf{H}(t)\psi(t) = i\hbar \frac{d\psi(t)}{dt}. \]

As in [27], we consider the following family of unitary operators

\[ U(\xi, \varphi) := \exp(-i \varphi)T(\xi) \]
where $\xi \in \mathcal{D}_1^J$ and $\varphi$ is a real phase. Let $\tau = \frac{t}{\hbar}$. In accord with \[31\], in \[27\] it was introduced the quasienergy operator $E := i \frac{d}{d\tau} - H$. With (3.5), we get:

\[
\langle 4.3 \rangle \quad \hat{E}(\xi, \varphi) = \frac{d}{d\tau} I + iT(\xi)^{-1} \hat{T}(\xi) - \hat{H}(\xi).
\]

In the notation of \[6, 14\], we consider a hermitian Hamiltonian linear in the generators of the Jacobi group $G_J^1$:

\[
\langle 4.4 \rangle \quad H_0 = \epsilon_a a + \bar{\epsilon}_a a^\dagger + \epsilon_0 K_0 + \epsilon_+ K_+ + \epsilon_- K_-, \quad \bar{\epsilon}_+ = \epsilon_-, \quad \epsilon_0 = \bar{\epsilon}_0.
\]

With equations (3.6)-(3.8), we calculate $\hat{H}_0(\xi)$, where $\xi = (\alpha, w) \in \mathbb{C} \times \mathcal{D}_1$:

\[
\langle 4.5 \rangle \quad \hat{H}_0(\alpha, w) = I_0 + \hat{C}_1 a^\dagger + \hat{C}_1 a + \hat{C}_0 K_0 + \hat{C}_+ K_+ + \hat{C}_- K_-,
\]

where the coefficients in (4.5) have the values:

\[
I_0 = \epsilon_a \alpha + \bar{\epsilon}_a \bar{\alpha} + \frac{1}{2}(\epsilon_0|\alpha|^2 + \epsilon_- \alpha^2 + \epsilon_+ \bar{\alpha}^2),
\]

\[
\frac{C_1}{\rho} = \bar{\epsilon}_a + \epsilon_a w + \frac{\epsilon_0}{2}(\alpha + \bar{\alpha} w) + \epsilon_- \alpha w + \epsilon_+ \bar{\alpha},
\]

\[
\frac{C_0}{\rho^2} = \epsilon_0(1 + |w|^2) + 2(\epsilon_- w + \epsilon_+ \bar{w}),
\]

\[
\frac{C_+}{\rho^2} = \epsilon_0 w + \epsilon_- w^2 + \epsilon_+.
\]

5. The Jacobi group $G_J^1$ in real coordinates

In \[27\] it is used as basis of the Lie algebra $g_J^1$ the real basis from \[18, 9\].

We now list the relations between the operators used in the paper \[27\] and the generators (2.4) of the Lie algebra $g_J^1$:

\[
\langle 5.1 \rangle \quad N_1 = a + a^\dagger; \quad N_2 = i(a - a^\dagger),
\]

with the inverse

\[
\langle 5.2 \rangle \quad a = \frac{1}{2}(N_1 - i N_2); \quad a^\dagger = \frac{1}{2}(N_1 + i N_2);
\]

\[
\langle 5.3 \rangle \quad K_1 = \frac{1}{2}(K_+ + K_-); \quad K_2 = \frac{1}{2i}(K_+ - K_-),
\]

and the inverse

\[
\langle 5.4 \rangle \quad K_+ = K_1 + i K_2; \quad K_- = K_1 - i K_2.
\]

In \[27\] it was considered the unitary operator

\[
\langle 5.5 \rangle \quad T(\xi) = D(x, y)S(u, v), \quad \xi = (u, v, x, y) \in \mathfrak{M},
\]

where

\[
\langle 5.6 \rangle \quad D(x, y) = \exp(i y N_1 + i x N_2), \quad S(u, v) = \exp(i k_1 K_1 + i k_2 K_2),
\]

\[
\langle 5.7 \rangle \quad k_1 = \frac{v}{2s} \ln \frac{1 + s}{1 - s}, \quad k_2 = \frac{u}{2s} \ln \frac{1 + s}{1 - s}, \quad s = (u^2 + v^2)^\frac{1}{2}.
\]
The correspondence between the real parameters in the representation \((5.6)\) and the complex parametrization \((3.4)\) is given by the relations
\[
\alpha = x + iy; \quad w = u + iv.
\]
In \([27]\) the Hamiltonian \((4.4)\) was written down as:
\[
H_0 = 2\varepsilon_0 K_0 + 2\varepsilon_1 K_1 + 2\varepsilon_2 K_2 + 2\nu_1 N_1 + 2\nu_2 N_2.
\]
The correspondence of the real and complex coefficients of the Hamiltonians \((5.9)\) and \((4.4)\) is (see also Proposition \(4)\)
\[
\epsilon_a = \nu_1 + i\nu_2; \quad \epsilon_0 = 2\varepsilon_0; \quad \epsilon_+ = \varepsilon_1 - i\varepsilon_2,
\]
\[
\nu_1 = a; \quad \nu_2 = b; \quad \varepsilon_0 = p; \quad \varepsilon_1 = m; \quad \varepsilon_2 = n.
\]
We express \(T^{-1}\hat{T}\) \((4.16)\) given in the complex coordinates \((z, w) \in D_1^2\) in the real coordinates \((x, y; u, w)\), where \(z = x + iy, w = u + iv\), and in the real operators \((5.1), (5.3)\), and we get:
\[
-i T^{-1} \frac{d T}{d \tau} = (xy - yx) I + r[(1 + u)\dot{y} - \dot{x}v] N_1 + r[(1 - u)\dot{x} - \dot{y}v] N_2 + 2r^2[(\dot{u}v - \dot{u}v) K_0 + \dot{v} K_1 + \dot{u} K_2].
\]
Now we express the operator \(\hat{H}_0(\alpha, w)\) \((4.3)\) in the operators \((5.1), (5.3)\) in real coordinates \((x, y; u, v)\), where \(\alpha = x + iy, w = u + iv\), and we get:
\[
\hat{H}_0(x, y; u, v) = D_0 + D_1 N_1 + D_2 N_2 + F_0 K_0 + F_1 K_1 + F_2 K_2.
\]
We find the following values of the coefficients appearing in \((5.13)\):
\[
\begin{align*}
D_0 &= 2(\nu_1 x - \nu_2 y) + \frac{\varepsilon_0}{2}(x^2 + y^2) + \varepsilon_1(x^2 - y^2) - 2\varepsilon_2 xy, \\
D_1 &= \nu_1(1 + u) - \nu_2 v + \frac{\varepsilon_0}{2}[x(1 + u) + yv] + \varepsilon_1[x(1 + u) - yv] - \varepsilon_2[xv + (1 + u)y], \\
D_2 &= \nu_1 v + \nu_2(u - 1) + \frac{\varepsilon_0}{2}[xv + y(1 - u)] + \varepsilon_1[xv + y(u - 1)] + \varepsilon_2[x(u - 1) - yv], \\
F_0 &= \varepsilon_0(1 + u^2 + v^2) + 4(\varepsilon_1 u - \varepsilon_2 v), \\
F_1 &= \varepsilon_0 u + \varepsilon_1(u^2 - v^2 + 1) - 2\varepsilon_2uv, \\
F_2 &= \varepsilon_0 v + 2\varepsilon_1 uv + \varepsilon_2(-1 + u^2 - v^2).
\end{align*}
\]
We introduce \((5.13)\) and \((5.12)\) into \((4.3)\) and we get the expression
\[
\hat{E}(x, y; u, v) = G_0 I + G_1 N_1 + G_2 N_2 + H_0 K_0 + H_1 K_1 + H_2 K_2,
\]
where:

\[ G_0 = \phi + y \dot{x} - xy - 2(\nu_1 x - \nu_2 y) - \varepsilon_0(x^2 + y^2) - \varepsilon_1(x^2 - y^2) + 2\varepsilon_2 xy, \]

\[-\frac{G_1}{r} = (1 + u) \dot{y} - \dot{x} + \nu_1(1 + u) - \nu_2 v + \varepsilon_0[\nu_1(1 + u) + yv], \]

\[ + \varepsilon_1[\nu_1(1 + u) - yv] - \varepsilon_2[\nu_0(1 + u) + xv], \]

\[-\frac{G_2}{r} = -(1 - u) \dot{x} + \dot{y} + \nu_1 v + \nu_2(u - 1) + \varepsilon_0[\nu_0(1 - u) + xv] \]

\[ + \varepsilon_1[\nu_0(u - 1)] + \varepsilon_2[x(u - 1) - yv], \]

(5.16)

\[ -
\frac{H_0}{2r^2} = \dot{v}u - \dot{u}v + \varepsilon_0(1 + u^2 + v^2) + 2(\varepsilon_1 u - \varepsilon_2 v), \]

\[ -
\frac{H_1}{2r^2} = \dot{v} + \varepsilon_0u + \varepsilon_1(u^2 - v^2 + 1) - 2\varepsilon_2 uv, \]

\[ -
\frac{H_2}{2r^2} = -\dot{u} + \varepsilon_0v + 2\varepsilon_1 uv + \varepsilon_2(u^2 - v^2 - 1). \]

Identifying the coefficients of \( N_1, N_2, \) and respectively \( K_1, K_2, \) we get the equations of motion in real coordinates

\[ \dot{x} = -\varepsilon_2 x + (\varepsilon_0 - \varepsilon_1)y - \nu_2; \]

\[ \dot{y} = -2v(\varepsilon_1 u + \varepsilon_0) - \varepsilon_2(1 - u^2 + v^2), \]

\[ \dot{\nu_1} = 2u(\varepsilon_2 v - \varepsilon_0) - \varepsilon_1(1 + u^2 - v^2). \]

Introducing (5.18) into the equation of \( H_0 \) in (5.16), we have

\[ -
\frac{H_0}{2} = \varepsilon_0 + \varepsilon_1 u - \varepsilon_2 v. \]

Introducing (5.17) into expression of \( G_0 \) in (5.16), we have

\[ G_0 = \phi - (\nu_1 x - \nu_2 y). \]

Starting from the complex representation, we have regained the results of Section 3 in [27] and also our results from [13], reproduced in Proposition 4.

**Proposition 1.** If we consider \( \Phi \in \mathfrak{F} \) such that \( K_0 \Phi = k \Phi, \) then

(5.21)

\[ \Psi(\xi, \varphi) = U(\xi, \varphi)\Phi = e^{-i\varphi}T(\xi)\Phi \]

is a solution of the time dependent Schrödinger equation corresponding to the hermitian Hamiltonian (4.1) (or (5.9)) on the Siegel-Jacobi disk \( \mathcal{D}_1 \), where \( x, y \in \mathbb{R} \) verify (5.17), \( (u, v) \in \mathcal{D}_1 \) verify (5.18), while the phase \( \varphi \) in (4.2) verifies

(5.22)

\[ \dot{\varphi} = \nu_1 x - \nu_2 y + 2k(\varepsilon_0 + \varepsilon_1 u - \varepsilon_2 v). \]

The motion (5.18) on the Siegel disc \( \mathcal{D}_1 \) in the complex variable \( w = u + iv \) is described by the Riccati equation

(5.23)

\[ i \dot{w} = \varepsilon_+ + \varepsilon_0 w + \varepsilon_- w^2. \]
The equations of motion (5.17) in the complex variable $\eta = x + iy$ reads

\[(5.24)\]
\[i\dot{\eta} = \bar{\epsilon}_a + \epsilon_+ \eta + \frac{\epsilon_0}{2} \eta,\]

We remark that the Riccati equation (5.23) on $D_1$ obtained with the Wei-Norman method coincides with the Berezin’s equation of motion (4.8b) or (4.10b) in [14], with the difference of notation $\epsilon_+ \leftrightarrow \epsilon_0 = \bar{\epsilon}_+$. The equation (5.24) for $\eta \in \mathbb{C}$ obtained with the Wei-Norman method is the Berezin’s equation of motion (4.10a) in [14], with the correspondence $\epsilon_a \leftrightarrow \bar{\epsilon}_a$, $\epsilon_+ \leftrightarrow \epsilon_0 = \bar{\epsilon}_+$, see Propositions 4 and 6.

We have proved

**Proposition 2.** The quantum and classical Berezin’s equations of motion on the Siegel-Jacobi disk determined by a hermitian Hamiltonian, linear in the generators of Jacobi group $G^J_1$, expressed in the FC-coordinates, are the same as the equations obtained applying the Wei-Norman method.

The equations of motion (5.23) on $D_1$, (5.24) on $\mathbb{C}$ and (5.17) on $\mathbb{R}^2$, determined by the hermitian Hamiltonian (4.4) or (5.9) linear in the generators of the Jacobi group $G^J_1$ obtained with the Wei-Norman method are a particular case of the Berezin’s equations of motion (8.29a) on $D_n$, (8.30) on $\mathbb{C}^n$ and respectively (8.32) on $\mathbb{R}^{2n}$, determined by the hermitian Hamiltonian (8.23) linear in the generators of the Jacobi group $G^J_n$.

### 6. Phases

In the paper [27] it was calculated the phase $\varphi$ which appears in the solution (4.2) for which we have find the equation (5.22). In the Berezin’s approach to the equations of motion, the solution of the time-dependent Schrödinger equation (4.1) differs from the solution parametrized by Perelomov coherent states by a phase $\varphi$ as it is recalled in Proposition 3 proven in [3, 14, 13]. Comparing the Wei-Norman solution of the Schrödinger equation (1.1) for the Hamiltonian (4.4) (or (5.9)) with the solution (8.6) under the form (8.15), we see that

**Remark 1.** The phases $\varphi$ used in the Wei-Norman method associated to the quasi-energy operator (4.3) and the phase $\varphi$ which appears in Berezin’s approach are different:

\[(6.1)\]
\[- \varphi(\xi) = \phi(\xi) - \frac{1}{2} \Im(\omega \bar{\alpha}^2), \quad \xi = (\alpha, w) \in D_1^J.\]

If we introduce the equations of motions on $D_1^J$ into the-\tau derivative of (6.1), we get

\[(6.2)\]
\[\dot{\phi} = \varphi - \nu_1(ux + vy) + \nu_2(uy - vx) + \frac{1}{4}(\epsilon_- z^2 + \epsilon_+ \bar{z}^2),\]

where $z = \alpha - \omega \bar{\alpha}$ and $(\alpha, w)$ are the coordinates (3.1) on $D_1^J$. This is exactly the formula obtained for $\dot{\varphi}$ summing up the explicit expressions of the dynamical and Berry phases (8.22) and (8.21).

We verify the last part of the Remark. If we add the expression of the Berry phases given by (8.21) in which we introduce the equations of motion (8.18) determined by
the Hamiltonian (4.4) and dynamic phase (8.22), we get the value of the phase \( \phi \) in the complex variables \((z, w) \in \mathbb{C} \times \mathcal{D}_1, z = \alpha - \bar{w}\alpha\):

\[
\begin{align*}
-\phi &= k \phi_1 + \phi_0, \\
\dot{\phi}_1 &= \epsilon_0 + \epsilon_- \bar{w} + \epsilon_+ w, \\
\dot{\phi}_0 &= \frac{1}{4}(\epsilon_- \bar{z}^2 + \epsilon_+ z^2) + \frac{1}{2}(\epsilon_a \bar{z} + \bar{\epsilon}_a z).
\end{align*}
\]

When we express \( z \) in the real and imaginary part in the coordinates (5.8) \((x, y, u, v)\), we have for \( z \) appearing in (6.3c) the value

\[
z = (1 - u)x - yv + i[(1 + u)y - vx],
\]

while

\[
\dot{\phi}_1 = 2(\epsilon_0 + \epsilon_1 u + \epsilon_2 v),
\]

i.e. the expression multiplying \( k \) in (5.22), with the correspondence \( \epsilon_+ \leftrightarrow \epsilon_- = \bar{\epsilon}_+ \), \( \epsilon_a \leftrightarrow \bar{\epsilon}_a \).

**In conclusion**, in [13, 14, 15] we have underlined the utility of the FC-coordinates, the geometric significance of the FC-transform in the context of the fundamental conjecture and also its relevance for coherent states on the Siegel-Jacobi ball. In the present paper we have shown that Berezin’s equations of motion on the Siegel-Jacobi disk expressed in the FC-coordinates, generated by a linear Hamiltonian in the generators of the Jacobi group are identical with the equations of motion furnished by the Wei-Norman method. All the calculation in the present paper refers to the motion on the Siegel-Jacobi disk, generated by a linear Hamiltonian in the generators of the Jacobi group \( G_1^J \), but we believe that Proposition 2 is true in more general situations, for some Lie groups which are semidirect product. We have also underlined that the phases which appears in the two methods are different. Remark 1 is also a direct check of the correctness of our calculation in [14] and compatibility with the calculation in [27].

**7. Appendix: The Wei-Norman method**

We use the following convention of notation for noncommuting operators:

\[
\prod_{i=1}^{n} A_i := A_1 \ldots A_n; \quad \prod_{i=n}^{1} A_i := A_n \ldots A_1.
\]

Let \( \xi = (\xi_1, \ldots, \xi_n) \) be some parameters. We consider an unitary operator which can be expressed in the basis \( \{X_i\}_{i=1,\ldots,n} \) of the Lie algebra \( \mathfrak{g} \) as in (1.1). Then we have

\[
U^{-1}(\xi) = U^\dagger(\xi) = U(-\xi) = \prod_{j=n}^{1} \exp(-\xi_j X_j).
\]
Let $X, Y$ be the free generators of a ring $R$. The Baker-Hausdorff formula (see [39] for a proof) reads:

$$
e^X Y e^{-X} = e^{\text{ad} X} Y = \sum_{n=0}^{\infty} \frac{(\text{ad} X)^n}{n!} Y$$

$$= Y + [X, Y] + \frac{1}{2!} [X, [X, Y]] + \cdots + \frac{1}{n!} \left[ X, \left[ X, \cdots, \left[ X, Y \right] \cdots \right] \right] + \cdots \quad (7.3)$$

Now let us consider that the parameters $\xi$ depend on a variable, let call it $t$. For any $t$-dependent operator $A$, we denote $\dot{A} = \frac{dA}{dt}$. Then we can calculate the derivative of (1.1) as

$$(7.4) \quad \dot{U}(\xi) = \sum_{j=1}^{n} \dot{\xi}_j \left( \prod_{k=1}^{j-1} \exp(\xi_k X_k) \right) \left( \prod_{j=r}^{1} \exp(-\xi_j X_j) \right) = \sum_{k=1}^{n} \eta_{ki} X_k, \quad i, r = 1, 2, \ldots, n,$$

where $\eta_{ki} = \eta_{ki}(\xi_1, \ldots, \xi_r)$ is an analytic function of $\xi_1, \ldots, \xi_r$.

Let us consider a linear operator $A(t)$ of the form (1.2), where $a_i(t)$ are scalar functions of $t$. It can be proved [51] that:

**Lemma 1.** If $\{X_1, \ldots, X_n\}$ is a basis of a Lie algebra $\mathfrak{g}$, then

$$(7.5) \quad \left( \prod_{j=1}^{r} \exp(\xi_j X_j) \right) X_i \left( \prod_{j=r}^{1} \exp(-\xi_j X_j) \right) = \sum_{k=1}^{n} \eta_{ki} X_k, \quad i, r = 1, 2, \ldots, n,$$

where $\eta_{ki} = \eta_{ki}(\xi_1, \ldots, \xi_r)$ is an analytic function of $\xi_1, \ldots, \xi_r$.

The Wei-Norman method [51] is expressed in the theorem:

**Theorem 1.** If $A(t)$ is given by (1.2), then there exists a neighborhood of $t = 0$ in which the solution of the equation

$$(7.6) \quad \dot{U}(t) = A(t)U(t), \quad U(0) = I$$

may be expressed in the form (1.1), where the $\xi_i(t)$ are scalar functions of $t$. Moreover, $\xi^t := (\xi_1, \ldots, \xi_n)$ satisfy the first order differential equation

$$(7.7) \quad \eta \dot{\xi} = \epsilon,$$

which depend only on the Lie algebra $\mathfrak{g}$ and the $\epsilon(t)$'s. $\eta = (\eta_{ki})_{k,i=1,\ldots,n}$ is the matrix of coefficients of (1.2), while $\epsilon^t = (\epsilon_1, \ldots, \epsilon_n)$.

In [51] it was proved that the representation (1.1) is global for any solvable Lie algebra $\mathfrak{g}$ and for any $2 \times 2$ system of equations.

In our calculation in (4.3), instead of $\dot{TT}^{-1}$, we need $T^{-1}\dot{T}$. With the convention (7.1) and the Baker-Hausdorff formula (7.3), we obtain for $U$ defined in (1.1):

$$(7.8) \quad U^{-1}(\xi)\dot{U}(\xi) = \sum_{i=1}^{n} \dot{\xi}_i Y_i, \quad Y_i = \left( \prod_{k=n}^{i+1} \exp(-\xi_k \text{ad} X_k) \right) X_i.$$
8. Appendix: Berezin’s equations on motion

We consider the triplet \((G, \pi, \mathfrak{H})\), where \(\pi\) is a continuous, unitary, irreducible representation of the Lie group \(G\) on the separable complex Hilbert space \(\mathfrak{H}\) \[4\].

We introduce the normalized (unnormalized) vectors \(e_x\) (respectively, \(e_z\)) defined on \(G/H\)

\[ e_x = \exp\left(\sum_{\phi \in \Delta^+} x_\phi X_\phi^+ - \bar{x}_\phi X_\phi^\dagger\right) e_0, \quad e_z = \exp\left(\sum_{\phi \in \Delta^+} z_\phi X_\phi^+\right) e_0, \]

where \(e_0\) is the extremal weight vector of the representation \(\pi\), \(\Delta^+\) are the positive roots of the Lie algebra \(\mathfrak{g}\) of \(G\), and \(X_\phi, \phi \in \Delta\), are the generators. \(X_\phi^\dagger (X_\phi^-)\) corresponds to the positive (respectively, negative) generators. See details in \[4, 5\].

We denote by \(FC\) the change of variables \(x \to z\) in formula (8.1) such that

\[ e_x = \tilde{e}_z, \quad \tilde{e}_z := \left( e_z, e_z \right)^{-\frac{1}{2}} e_z, \quad z = FC(x). \]

The reason for calling the transform (8.2) a \(FC\)-transform (fundamental conjecture) \[50, 23\] is explained in Proposition 3 in \[15\]. For a concrete example of \(FC\)-transform, see (8.13).

8.1. Berezin’s approach to classical motion and quantum evolution. Let \(M = G/H\) be a homogeneous manifold with a \(G\)-invariant Kähler two-form \(\omega\)

\[ \omega(z) = i \sum_{\alpha \in \Delta^+} g_{\alpha,\beta} dz_\alpha \wedge d\bar{z}_\beta, \quad g_{\alpha,\beta} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \ln \left< e_z, e_z \right>. \]

Passing on from the dynamical system problem in the Hilbert space \(\mathfrak{H}\) to the corresponding one on \(M\) is called sometimes dequantization, and the dynamical system on \(M\) is a classical one \[3, 4\]. Following Berezin \[16, 17\], the motion on the classical phase space can be described by the local equations of motion \(\dot{z}_\alpha = i \{H, z_\alpha\}, \alpha \in \Delta^+, \) where \(H\) is the classical Hamiltonian (the covariant symbol)

\[ H = \left< e_z, e_z \right>^{-1} \left< e_z | H | e_z \right> \]

attached to the quantum Hamiltonian \(H\), and the Poisson bracket is introduced using the matrix \(g^{-1}\).

We consider an algebraic Hamiltonian linear in the generators \(X_\lambda\) of the group of symmetry \(G\)

\[ H = \sum_{\lambda \in \Delta} \epsilon_\lambda X_\lambda. \]

We look for the solution of the Schrödinger equation of motion \[4, 11\] generated by the Hamiltonian (8.5) as

\[ \psi(t) = e^{i\phi \tilde{e}_z}, \]

where \(\tilde{e}_z\) is the normalized Perelomov coherent state vector defined in (8.1), (8.2).

We extract from \[3, 4, 13\] the following Proposition:
Proposition 3. The classical motion and quantum evolution generated by the linear hermitian Hamiltonian (8.5) are described by Berezin’s equations of motion on $M = G/H$

$$i \dot{z}_\alpha = \sum_{\lambda \in \Delta} \epsilon_\lambda Q_{\lambda, \alpha}, \quad \alpha \in \Delta_+,$$

where the differential action corresponding to the operator $X_\lambda$ in (8.5) can be expressed in a local system of coordinates as a holomorphic first order differential operator with polynomial coefficients ($\partial_\beta = \frac{\partial}{\partial z_\beta}$),

$$X_\lambda = P_\lambda + \sum_{\beta \in \Delta_+} Q_{\lambda, \beta} \partial_\beta, \lambda \in \Delta.$$

The phase $\phi$ in (8.6) is given by the sum $\phi = \phi_D + \phi_B$ of the dynamical and Berry phases, where

$$\dot{\phi}_D = -\mathcal{H}(t);$$

$$\dot{\phi}_B = \frac{i}{2} \sum_{\alpha \in \Delta_+} (\dot{z}_\alpha \partial_\alpha - \dot{\bar{z}}_\alpha \partial_{\bar{\alpha}}) \ln <e_z, e_z> .$$

8.2. Coherent states on the Siegel-Jacobi disk $D^J_1$. We impose to the cyclic vector $e_0$ to verify simultaneously the conditions [6]

$$a e_0 = 0, \quad K_- e_0 = 0, \quad K_0 e_0 = ke_0; \quad k > 0, 2k = 2, 3, ...,$$

and we have considered in the last relation in (8.10) the positive discrete series representations $D_k^+$ of SU(1, 1) [2].

Perelomov’s coherent state vectors associated to the group $G_1^J$ with Lie algebra the Jacobi algebra $g_1^J$, based on Siegel-Jacobi disk $D_1^J = H_1/\mathbb{R} \times SU(1, 1)/U(1) = \mathbb{C} \times D_1$, are defined as

$$e_{z,w} := e^{a^\dagger + w K_+} e_0, \quad z, w \in \mathbb{C}, \quad |w| < 1.$$

We introduce also the normalized (squeezed) CS-vector (see also [18])

$$e_\xi := T(\xi)e_0 = D(\alpha)S(w)e_0, \quad \xi = (\alpha, w) \in \mathbb{C} \times D_1.$$

The normalized squeezed state vector and the un-normalized Perelomov’s coherent state vector are related by the relation (see [6])

$$e_{\eta,w} = (1 - w\bar{w})^k \exp(-\frac{\bar{\eta}}{2} z) e_{z,w}, \quad z = \eta - w\bar{\eta}.$$

We recall [6] [13] that

$$<e_{z,w}, e_{z,w}> = (1 - w\bar{w})^{-2k} \exp(\mathcal{F}),$$

$$2\mathcal{F} = \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2w}{1 - w\bar{w}} = 2|\eta|^2 - \bar{w}\eta^2 - w\bar{\eta}^2 .$$

From (8.13) and (8.14), we get for (8.2) on $D^J_1$ the relation

$$e_{\eta,w} = \exp\left[\frac{1}{4}(w\eta^2 - cc)\right] e_{z,w}, \quad z = FC(\eta) = \eta - w\bar{\eta}. $$
The general scheme [3, 4] associates to elements of the Lie algebra $\mathfrak{g}$ first order holomorphic differential operators with polynomial coefficients $X \in \mathfrak{g} \rightarrow X$ as in (8.8).

The calculation in [6], based on (7.3), gives:

**Lemma 2.** The differential action of the generators of the Jacobi algebra (2.3) is given by the formulas:

\begin{align*}
(8.16a) \quad a &= \frac{\partial}{\partial z}; \quad a^\dagger = z + w \frac{\partial}{\partial z}, \quad z, w \in \mathbb{C}, \quad |w| < 1; \\
(8.16b) \quad K_- &= \frac{\partial}{\partial w}; \quad K_0 = k + \frac{1}{2} z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w}; \\
(8.16c) \quad K_+ &= \frac{1}{2} z^2 + 2kw + zw \frac{\partial}{\partial z} + w^2 \frac{\partial}{\partial w}.
\end{align*}

Applying Proposition 3 to the representation given by Lemma 2, we have obtained in [14] Berezin’s equations of motion:

**Proposition 4.** The equations of motion on the Siegel-Jacobi disk $D^1_1$ generated by the linear Hamiltonian (4.4) are:

\begin{align*}
(8.17a) \quad i \dot{z} &= \epsilon_a + \bar{\epsilon}_a w + (\frac{\epsilon_0}{2} + \epsilon_+ w)z, \quad z, w \in \mathbb{C}, \quad |w| < 1, \\
(8.17b) \quad i \dot{w} &= \epsilon_- + \epsilon_0 w + \epsilon_+ w^2.
\end{align*}

For the $\eta$ defined in the FC$^{-1}$ transform (8.13), the system of first order differential equations (8.17) becomes the system of separate equations

\begin{align*}
(8.18a) \quad i \dot{\eta} &= \epsilon_a + \epsilon_- \bar{\eta} + \frac{\epsilon_0}{2} \eta, \quad \eta \in \mathbb{C}, \\
(8.18b) \quad i \dot{w} &= \epsilon_- + \epsilon_0 w + \epsilon_+ w^2, \quad w \in \mathbb{C}, \quad |w| < 1.
\end{align*}

With the change of function $w = XY^{-1}$, the Riccati equation (8.17b) became the linear Hamiltonian system

\begin{equation}
(8.19) \quad \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = h_c \begin{pmatrix} X \\ Y \end{pmatrix}, \quad h_c = i \begin{pmatrix} -\frac{\epsilon_0}{2} & -\epsilon_- \\ \epsilon_+ & \frac{\epsilon_0}{2} \end{pmatrix} \in \mathfrak{sp}(1, \mathbb{R})_{\mathbb{C}}.
\end{equation}

If in (8.18a) we make the change of variables $\eta = \xi - i \zeta$, then we get the system of linear differential equations in real ($\epsilon_a = b + i a$)

\begin{equation}
(8.20) \quad \dot{Z} = h_r Z + F, \quad Z = \begin{pmatrix} \zeta \\ \eta \end{pmatrix}, \quad F = \begin{pmatrix} a \\ b \end{pmatrix}, \quad h_r = \begin{pmatrix} n & m-p \\ m+p & -n \end{pmatrix} \in \mathfrak{sp}(1, \mathbb{R}).
\end{equation}

We also reproduce the results concerning the phases obtained in [14]:

**Proposition 5.** The Berry phase on the Siegel-Jacobi disk $D^1_1$ expressed in the variables $(\eta, w)$, $z = \eta - w\bar{\eta}$, reads

\begin{equation}
(8.21) \quad \frac{2}{i} d\phi_B = (\frac{2kw}{1-w\bar{w}} - \frac{\eta^2}{2})dw + (\bar{\eta} + \bar{w}\eta)d\eta - \text{cc}.
\end{equation}
The energy function (8.4) attached to the Hamiltonian (1.4) in the coherent state representation (8.11) can be written as $H = \mathcal{H}_\eta + \mathcal{H}_w$, where

\begin{align}
\mathcal{H}_\eta &= \epsilon_a \bar{\eta} + \epsilon_a \eta + \frac{1}{2} (\epsilon_+ \eta^2 + \epsilon_- \bar{\eta}^2 + \epsilon_0 \eta \bar{\eta}), \\
\mathcal{H}_w &= k \epsilon_0 + \frac{2k}{1 - \omega \bar{\omega}} (\epsilon_+ w + \epsilon_- \bar{w} + \epsilon_0 w \bar{w}).
\end{align}

8.3. Equations of motion on the Siegel-Jacobi ball $D^J_n$. Following [13], we consider a Hamiltonian linear in the generators of the group $G^J_n$.

\begin{equation}
H = \epsilon_i a_i + \epsilon^\dagger_i a_i^\dagger + \epsilon_{ij}^0 K_{ij}^0 + \epsilon_{ij}^\pm K_{ij}^\pm + \epsilon_{ij}^\pm K_{ij}^\pm.
\end{equation}

The hermiticity condition imposes to the matrices of coefficients $\epsilon_{0,\pm} = (\epsilon_{0,\pm})_{i,j = 1,\ldots,n}$ the restrictions

\begin{equation}
\epsilon_0^\dagger = \epsilon_0; \quad \epsilon_- = \epsilon_-^\dagger; \quad \epsilon_+ = \epsilon_+^\dagger; \quad \epsilon_-^\dagger = \epsilon_+.
\end{equation}

It is useful to introduce the matrices $m, n, p, q \in M(n, \mathbb{R})$ such that

\begin{equation}
\epsilon_- = m + i n, \quad \epsilon_0^\dagger/2 = p + i q; \quad p^t = p; \quad m^t = m; \quad n^t = n; \quad q^t = -q.
\end{equation}

We consider a matrix Riccati equation (8.26a) on the manifold $M$ and a linear differential equation (8.26b) in $z \in \mathbb{C}^n$.

\begin{align}
\dot{W} &= A W + W D + B + W C W, \quad A, B, C, D \in M(n, \mathbb{C}); \\
\dot{z} &= M + N z; \quad M = E + W F; \quad N = A + W C, \quad E, F \in C^n.
\end{align}

Firstly, we recall how to solve the matrix Riccati equation (8.26a) by linearization. If we proceed to the homogenous coordinates $W = X Y^{-1}, X, Y \in M(n, \mathbb{C})$, a linear system of ordinary differential equations is attached to the matrix Riccati equation (8.26a) (cf. [34], see also [4])

\begin{equation}
\begin{pmatrix}
\dot{X} \\
\dot{Y}
\end{pmatrix} = h \begin{pmatrix}
X \\
Y
\end{pmatrix}, \quad h = \begin{pmatrix}
A & B \\
-C & -D
\end{pmatrix}.
\end{equation}

Every solution of (8.27) is a solution of (8.26a), whenever $\det(Y) \neq 0$.

**Proposition 6.** The classical motion and quantum evolution generated by the linear hermitian Hamiltonian (8.23), (8.24) are described by first order differential equations:

a) On $D^J_n$, $(z, W) \in \mathbb{C}^n \times D_n$ verifies (8.26), with coefficients

\begin{align}
A_e &= -\frac{i}{2} \epsilon_0^t, \quad B_e = -i \epsilon_-, \quad C_e = -i \epsilon_+; \quad D_e = A_e^t; \\
E_e &= -i \epsilon, \quad F_e = -i \bar{\epsilon}.
\end{align}

b) Explicitly, the differential equations for $(W, z) \in D^J_n$ are

\begin{align}
i \dot{W} &= \epsilon_- + (W \epsilon_0)^t + W \epsilon_+ W, \quad W \in D_n, \\
i \dot{z} &= \epsilon + W \bar{\tau} + \frac{i}{2} \epsilon_0 z + W \epsilon_+ z, \quad z \in \mathbb{C}^n,
\end{align}
c) Under the FC transform, $z = \eta - W\bar{\eta}$, the differential equations in the variables $\eta \in \mathbb{C}^n$, $W \in \mathcal{D}_n$ become independent: $W$ verifies (8.26a) with coefficients (8.28a) and $\eta$ verifies

\begin{equation}
(8.30) \quad i\dot{\eta} = \epsilon + \epsilon_-\bar{\eta} + \frac{1}{2}\epsilon_0\eta, \quad \eta \in \mathbb{C}^n.
\end{equation}

d) The linear system of differential equations (8.27) attached to the matrix Riccati equation (8.29a) is

\begin{equation}
(8.31) \quad \left( \begin{array}{c} \dot{X} \\ \dot{Y} \end{array} \right) = h_c \left( \begin{array}{c} X \\ Y \end{array} \right), \quad h_c = \left( \begin{array}{cc} -i(\epsilon_0)^t & -i\epsilon_- \\ i\epsilon_+ & i\frac{\epsilon_0}{2} \end{array} \right) \in \mathfrak{sp}(n, \mathbb{R})_\mathbb{C}, \quad W = X/Y \in \mathcal{D}_n.
\end{equation}

e) In (8.30) we introduce $\eta = \xi - i\zeta$, $\xi, \zeta \in \mathbb{R}^n$ and we put $\epsilon = b + ia$, where $a, b \in \mathbb{R}^n$. The first order complex differential equation equation (8.30) is equivalent with a system of first order real differential equations with real coefficients, which we write as

\begin{equation}
(8.32) \quad \dot{Z} = h_r Z + F, \quad Z = \left( \begin{array}{c} \xi \\ \zeta \end{array} \right), \quad F = \left( \begin{array}{c} a \\ b \end{array} \right), \quad h_r = \left( \begin{array}{cc} n + q & m - p \\ m + p & -n + q \end{array} \right) \in \mathfrak{sp}(n, \mathbb{R}).
\end{equation}

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