THE PSEUDO-EFFECTIVE CONE OF THE KNOWN IRREDUCIBLE
HOLOMORPHIC SYMPLECTIC MANIFOLDS

FRANCESCO DENISI

ABSTRACT. Thanks to Kovács’ work, it is known that the pseudo-effective cone $\text{Eff}(S)$ of a smooth projective $K3$ surface $S$ of Picard number $\rho(S) \geq 3$ is either circular or equals $\sum_{C \in \text{Eff}(S)} \mathbb{R}^{\geq 0}[C]$. On a higher dimensional (projective) irreducible holomorphic symplectic (IHS) manifold, the structure of the pseudo-effective cone is quite similar to that of a smooth projective surface, due to the existence of the Beauville-Bogomolov-Fujiki form, and the smooth rational curves are naturally replaced by the prime exceptional divisors. In this note we show that, in some sense, Kovács’ result still holds true if $S$ is a smooth projective surface, due to the existence of the Beauville-Bogomolov-Fujiki form, and the OG10 and OG6 deformation classes, discovered by O’Grady (see [OGr99] and [OGr03] respectively). Thanks to the existence of a quadratic form $q_X$ on $H^2(X, \mathbb{C})$, known as the Beauville-Bogomolov-Fujiki (BBF) form, the structure of the pseudo-effective cone of a projective IHS manifold $X$ is similar to that of a smooth complex projective surface. Indeed, for a surface $S$ we have

$$\text{Eff}(S) = \mathcal{C}_S + \sum_{C \in \text{Neg}(S)} \mathbb{R}^{\geq 0}[C]$$

(see [Kol99, Theorem 4.13]), where $\text{Neg}(S)$ is the set of irreducible and reduced curves of negative self-intersection in $S$, and $\mathcal{C}_S$ is the positive cone of $S$, i.e. the connected component of the set $\{\alpha \in N^1(S)_{\mathbb{R}} \mid \alpha^2 > 0\}$ containing the ample cone $\text{Amp}(S)$; while for a projective IHS manifold $X$ we have

$$\text{Eff}(X) = \mathcal{C}_X \cap N^1(X)_{\mathbb{R}} + \sum_{E \in \text{Neg}(X)} \mathbb{R}^{\geq 0}[E],$$

where $\text{Neg}(X)$ is the set of prime exceptional divisors in $X$, i.e. the prime divisors with negative BBF square (see [Den21, Corollary 3.8]), and $\mathcal{C}_X$ is the positive cone of $X$, i.e. the connected component of the set $\{\alpha \in H^{1,1}(X, \mathbb{R}) \mid q_X(\alpha) > 0\}$ containing the Kähler cone $\mathcal{K}_X$. We are mainly interested in the algebraic part of $\mathcal{C}_X$ (i.e. $\mathcal{C}_X \cap N^1(X)_{\mathbb{R}}$). With some abuse of notation, from now on we will denote $\mathcal{C}_X \cap N^1(X)_{\mathbb{R}}$ by $\mathcal{C}_X$. For a projective $K3$ surface (over an arbitrary algebraically closed field of any characteristic), Kovács obtained a refinement for the structure of the pseudo-effective cone. In particular, he proved the following.

**Theorem 1.1** ([Kov13], Theorem 1.1). Let $S$ be a $K3$ surface of Picard number at least 3 over an algebraically closed field of arbitrary characteristic. Then either $\text{Neg}(S) = \emptyset$ and

...
\[ \text{Eff}(S) = C_S \text{ is circular, or } \text{Eff}(S) = \sum_{C \in \text{Neg}(S)} R^{\geq 0}[C], \text{ where } \text{Neg}(S) \text{ is the set of smooth rational curves contained in } S. \]

Note that Kovács first proved the above result over \( \mathbb{C} \) (cf. [Kov93, Theorem 1]). See also [Kov93, Theorem 2] for a complete description of the pseudo-effective cone in this case.

Replacing the smooth rational curves with the prime exceptional divisors, by virtue of all the above, it is natural to ask whether the above result can be generalized to the case of an arbitrary projective IHS manifold, and we could answer affirmatively for all the 4 known deformation classes.

**Theorem 1.2.** Let \( X \) be a projective IHS manifold of Picard number greater or equal than 3, belonging to one of the 4 known deformation classes. Then either \( \text{Neg}(X) = \emptyset \) and \( \text{Eff}(X) = C_X \text{ is circular, or } \text{Eff}(X) = \sum_{E \in \text{Neg}(X)} R^{\geq 0}[E], \text{ where } \text{Neg}(X) \text{ is the set of prime exceptional divisors in } X. \)

The main difficulty in trying to generalize Theorem 1.1 adopting the same strategy of Kovács is that, a priori, the class \( \alpha \) constructed starting from a prime exceptional divisor \( E \), and appearing in equation (6) of Lemma 3.2 is not effective. Indeed, while in the case of \( K3 \) surfaces the effectivity of \( \alpha \) directly follows from the Riemann-Roch Theorem for surfaces, in the higher dimensional case the Riemann-Roch Theorem does not tell us anything about the effectivity of \( \alpha \).

A way to obtain information about the effectivity of \( \alpha \) is to pass through the description of the monodromy group and of the stably exceptional classes, which we do not have for any IHS manifold. Hence we had to restrict to the known cases, where the monodromy and the stably exceptional classes are well known, thanks to recent work (cf. [Mar13], [MO22], [MR20]). In particular, the idea is that a stably exceptional class stays (up to a sign) stably exceptional under the action of the monodromy group, and a stably exceptional class is effective. Hence, if we are able to construct an isometry sending \([E]\) to \( \alpha \), we are done, because \([E]\) is stably exceptional, hence also \( \alpha \) is stably exceptional (up to a sign), and so \( \alpha \) is effective (up to a sign).

The effectivity of \( \alpha \) plays a crucial role in the proof of Theorem 1.2. In particular, if the pseudo-effective cone of an IHS manifold has a circular part \( C \) and the class \( \alpha \) is effective, \( C \) must contain an isotropic integral class, and this is the key point in proving Theorem 1.2.

Theorem 1.2 gives a quite fine description of the pseudo-effective cone of the known IHS manifolds of Picard number at least 3. In the case of Picard number 2 we have the following result of Oguiso.

**Theorem 1.3** ([Ogu14], item (2) of Theorem 1.3). Let \( X \) be a projective IHS manifold of Picard number 2. Then either both the boundary rays of the movable cone \( \text{Mov}(X) \) are rational and \( \text{Bir}(X) \) is a finite group or both the boundary rays of \( \text{Mov}(X) \) are irrational and \( \text{Bir}(X) \) is an infinite group.

From Theorem 1.3 one can easily deduce the following description of the pseudo-effective cone for projective IHS manifolds of Picard number 2.

**Corollary 1.4.** Let \( X \) be a projective IHS manifold with \( \rho(Y) = 2 \). Then either both the boundary rays of \( \text{Eff}(X) \) are rational and \( \text{Bir}(X) \) is a finite group or both the boundary rays of \( \text{Eff}(X) \) are irrational and \( \text{Bir}(X) \) is an infinite group.

We recall that the reflection associated with a prime exceptional divisor \( E \) is the isometry

\[ R_E: H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}), \alpha \mapsto \alpha - \frac{2q_X([E], \alpha)}{q_X([E])}[E]. \]

Actually, a priori, it is not obvious that the reflection \( R_E \) is integral. This has been proven by Markman (cf. [Mar13, Corollary 3.6, item (1)])

**Definition 1.5.** We define \( W_{\text{Exc}} \) as the group generated by the reflections \( R_E \), where \( E \) is any prime exceptional divisor of \( X \).
Putting all together one obtains the following result, which is a higher dimensional analogue of a result due to Pjateckii-Šapiro and Šafarevič, and Sterk (cf. [Huy15, Corollary 4.7]).

**Corollary 1.6.** Let $X$ be a projective IHS manifold belonging to one of the 4 known deformation classes. Consider the following statements.

1. $\text{Eff}(X)$ is rational polyhedral.
2. $\text{Bir}(X)$ is finite.
3. $O(N^1(X))/\text{Exc}$ is finite.
4. $X$ carries finitely many prime exceptional divisors.

Then the statements (1),(2),(3) are equivalent and imply (4). Further, if $X$ is of Picard number at least 3 and carries a prime exceptional divisor, all the above statements are equivalent.

We believe that the experts are aware of the equivalence between (1), (2) and (3) in Corollary 1.6. The result is interesting on its own, and we included it for the sake of completeness.

**Remark 1.7.** Note that, in Corollary 1.6, the implications (1) $\Rightarrow$ (2), (2) $\Leftrightarrow$ (3) and (1) $\Rightarrow$ (4) hold true for any projective IHS manifold (see the proof of the corollary).

**Acknowledgements**

This work will be part of my PhD thesis. First of all, I would like to thank my doctoral advisors: Prof. Gianluca Pacienza and Prof. Giovanni Mongardi. This work benefited a lot from their suggestions and hints and I am very happy I can learn from them every day. I would like to thank Prof. F. Bastianelli, Prof. A. F. Lopez and Prof. F. Viviani, for suggesting me Example 2.12, during the conference "Riposte Armonie: Algebraic Geometry in Cetraro 2021". I would like to thank Prof. K. Yoshioka for sending the letter sent to him by Prof. E. Markman, containing the proof of Proposition 2.7, and Prof. E. Markman for letting me include it in this paper. I would like to thank Prof. A. Höring for pointing out Remark 3.12. To conclude, I would like to thank F. Anella for reading a preliminary version of this paper. His comments helped me to improve the exposition of the paper.

**2. Preliminaries**

In this section we collect the main definitions, tools and results needed in the rest of the paper.

**2.1. Generalities on IHS manifolds.** For a general introduction to IHS manifolds, we refer the reader to [GHJ03]. Let $X$ be an IHS manifold. Thanks to the work [Bea83] of Beauville, there exists a quadratic form on $H^2(X, \mathbb{C})$ generalizing the intersection form on a surface. In particular, choosing the symplectic form $\sigma$ in such a way that $\int_X (\sigma \sigma) = 1$, one can define

$$q_X(\alpha) := \frac{n}{2} \int_X (\sigma \sigma)^{n-1} \alpha^2 + (1 - n) \left( \int_X \sigma \sigma^{n-1} \alpha \right) \cdot \left( \int_X \sigma^{n-1} \sigma^n \alpha \right),$$

for any $\alpha \in H^2(X, \mathbb{C})$. The quadratic form $q_X$ is non-degenerate, and is known as the Beauville-Bogomolov-Fujiki form (BBF form in what follows). Up to a rescaling $q_X$ is integral and primitive on $H^2(X, \mathbb{Z})$. Also, $q_X$ is invariant by deformations. A prime divisor in $X$ will be a reduced and irreducible hypersurface. We say that a prime divisor $E$ of $X$ is exceptional if $q_X(E) < 0$.

Let $f: X \dashrightarrow X'$ be a bimeromorphic map, where $X'$ is another IHS manifold. Recall that $f$ restricts to an isomorphism $f: U \rightarrow U'$, where $\text{codim}_X(X \setminus U)$, $\text{codim}_X'(X' \setminus U') \geq 2$ and $X \setminus U$, $X' \setminus U'$ are analytic subsets of $X$ and $X'$ respectively (see for example [Huy99,
Then there exists an isometry $H^2(X, \mathbb{R}) \cong H^2(U, \mathbb{R}) \cong H^2(U', \mathbb{R}) \cong H^2(X', \mathbb{R})$, and the composition is an isometry with respect to $q_X$ and $q_{X'}$ (see for example [OGr97, Proposition I.6.2] for a proof), hence its restriction to $H^{1,1}(X, \mathbb{R})$ induces an isometry $H^{1,1}(X, \mathbb{R}) \cong H^{1,1}(X', \mathbb{R})$, which we will denote by $f_*$ (push-forward). We will denote the inverse of $f_*$ by $f^*$ (pull-back).

2.2. Some lattice theory. A lattice is a couple $(L, b)$, where $L$ is a finitely generated, free abelian group $L$, and $b$ is a non-degenerate, integral valued, symmetric bilinear form $b : L \times L \to \mathbb{Z}$. If no confusion arises, we will simply write $L$. We say that $L$ is even if $b$ takes only even values. The signature $\text{sign}(b)$ of $b$ is the signature of the natural extension $b_{\mathbb{R}}$ of $b$ to $L \otimes_{\mathbb{Z}} \mathbb{R}$.

The divisibility $\text{div}_L(x)$ of an element $x \in L$ is defined as the positive generator of the ideal $b(x, L)$ in $\mathbb{Z}$. This means that $\text{div}_L(x)$ is exactly the least (positive) integer that can be obtained by multiplying $x$ by the elements of $L$. When no confusion arises, we will write $\text{div}(x)$ instead of $\text{div}_L(x)$.

As $b$ is non-degenerate, one has an injective group homomorphism $L \hookrightarrow L' := \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, by sending an element $x$ of $L$ to the element $b(x, -)$ of $L'$. We will denote $b(x, -)$ by $x$, when no confusion arises. Note that we have the following identification

$$L' = \{ x \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid b(x, y) \in \mathbb{Z}, \text{ for any } y \in L \} \subset L \otimes_{\mathbb{Z}} \mathbb{Q}.$$ 

The discriminant group of $L$ is the finite group $A_L := L' / L$. If $A_L$ is trivial we say that $L$ is unimodular. If $f$ is an element of $L'$ we will denote its class in $A_L$ by $[f]$. We say that an element $x \in L$ is primitive if we cannot write $x = ax'$, where $a \neq 1$, $a \in \mathbb{Z}_{>0}$, and $x' \in L$.

An isometry of $L$ is an automorphism of it (as an abelian group) preserving $b$. The group of isometries of $L$ is denoted by $O(L)$. Suppose now that $L$ has signature $\text{sign}(b) = (3, b_2 - 3)$, where $b_2 \in \mathbb{N}$, and $b_2 > 3$. Define the cone

$$C_L := \{ v \in L \otimes_{\mathbb{Z}} \mathbb{R} \mid b(v) > 0 \}.$$ 

In [Mar11, Lemma 4.1] it has been proven that $C_L$ has the homotopy type of $S^2$, hence $H^2(L, \mathbb{Z}) \cong \mathbb{Z}$. Clearly, any isometry in $O(L)$ induces a homeomorphism of $C_L$, which in turn induces an automorphism of $H^2(C_L, \mathbb{Z}) \cong \mathbb{Z}$. This automorphism can act as 1 or $-1$.

We define $O^+(L)$ as the subgroup of the isometries of $L$ acting trivially on $H^2(C_L, \mathbb{Z})$. The group $SO^+(L)$ is the subgroup of the isometries of $L$ of determinant 1, acting trivially on both $A_L$ and $H^2(C_L, \mathbb{Z}) \cong \mathbb{Z}$.

Remark 2.1. Given a primitive element $x \in L$, we can consider the element $x/\text{div}(x)$ of $L'$, which in turn gives the element $[x/\text{div}(x)]$ of $A_L$. Note that $\text{ord}([x/\text{div}(x)]) = \text{div}(x)$, hence $\text{div}(x)$ divides $|A_L|$. Indeed, suppose that the order is $k \leq \text{div}(x)$, then $kt = \text{div}(x)$ for some positive integer $t$. This implies that there exists an element $y \in L$ such that $x = ty$, and the primitivity of $x$ forces $t$ to be 1, and so $k = \text{div}(x)$. Also, note that if $M$ is the maximum among the orders of the elements of $A_L$, then $\text{div}(x) < M$. Indeed, the element $x/\text{div}(x)$ has order $\text{div}(x)$, thus $\text{div}(x) \leq M$.

The following is a very known result of Eichler, which will be very useful in this article.

Lemma 2.2 (Eichler’s criterion). Let $L'$ be an even lattice and $L = U^2 \oplus L'$. Let $v, w \in L$ be two primitive elements such that the following hold:

- $b(v) = b(w)$.
- $[v/\text{div}(v)] = [w/\text{div}(w)]$ in $A_L$.

Then there exists an isometry $\iota \in SO^+(L)$, such that $\iota(v) = w$.

The above version of Eichler’s criterion has been taken from [MR20, Lemma 2.6].
Also whose central fiber is a fixed IHS manifold \( 3.6, \text{item } (1) \) Markman proves that any reflection of the universal family. Also, let \( \text{Exc} \) be a distinguished point of \( X \), and in this case the signature of \( q_X \) is \((1, \rho(X) - 1)\), where \( \rho(X) \) is the Picard number of \( X \). We refer the reader to [GHJ03, Corollary 23.11] for a proof of these facts.

### 2.3. Monodromy operators

**Definition 2.3.**

1. Set \( \pi^{-1}(\gamma(0)) = X \) and \( \pi^{-1}(\gamma(1)) = X' \). The parallel transport operator \( T_{\gamma, \pi}^k \) associated with the path \( \gamma \) and \( \pi \) is the isomorphism \( T_{\gamma, \pi}^k : H^k(X, \mathbb{Z}) \to H^k(X', \mathbb{Z}) \) between the stalks at 0 and 1 of the sheaf \( R^k \pi_* \mathbb{Z} \), induced by the trivialization of \( \gamma^{-1}(R^k \pi_* \mathbb{Z}) \). The isomorphism \( T_{\gamma, \pi}^k \) is well defined on the fixed endpoints homotopy class of \( \gamma \).

2. If \( \gamma \) is a loop, we obtain an automorphism \( T_{\gamma, \pi}^k : H^k(X, \mathbb{Z}) \to H^k(X, \mathbb{Z}) \). In this case \( T_{\gamma, \pi}^k \) is called a monodromy operator.

3. The \( k \)-th group of monodromy operators on \( H^k(X, \mathbb{Z}) \) induced by \( \pi \) is

\[
\text{Mon}^k(X)_\pi := \{ T_{\gamma, \pi}^k \mid \gamma(0) = \gamma(1) \}
\]

With the above definition we can define the monodromy groups of an IHS manifold \( X \).

**Definition 2.4.** The \( k \)-th monodromy group \( \text{Mon}^k(X) \) of an IHS manifold \( X \) is defined as the subgroup of \( \text{Aut}_\mathbb{Z}(\mathbb{H}^k(X, \mathbb{Z})) \) generated by the subgroups of the type \( \text{Mon}^k(X)_\pi \), where \( \pi : \mathcal{X} \to S \) is a smooth and proper family of IHS manifolds over a connected analytic (possibly singular, non reduced, reducible) base \( S \), whose central fiber is a fixed IHS manifold \( X \). Let \( R^k \pi_* \mathbb{Z} \) be the \( k \)-th higher direct image of \( \mathbb{Z} \). The space \( S \) is locally contractible, and the family \( \pi \) is topologically locally trivial (see for example [GHJ03, Theorem 14.5]). The sheaf \( R^k \pi_* \mathbb{Z} \) is the sheafification of the presheaf

\[
U \mapsto H^k(\pi^{-1}(U), \mathbb{Z}), \quad \text{for any open subset } U \subset S,
\]

which is a constant presheaf, by the local contractibility of \( S \) and the local triviality of \( \pi \). Then \( R^k \pi_* \mathbb{Z} \) is a locally constant sheaf, i.e. a local system, for every \( k \in \mathbb{N} \). Now, let \( \gamma : [0, 1] \to S \) be a continuous path. Then \( \gamma^{-1}(R^k \pi_* \mathbb{Z}) \) is a constant sheaf for every \( k \).

We are interested in the group \( \text{Mon}^2(X) \), of which we will need the characterization for some of the known IHS manifolds. Notice that \( \text{Mon}^2(X) \subseteq O^+(\mathbb{H}^2(X, \mathbb{Z})) \). We will mostly use three subgroups of \( \text{Mon}^2(X) \). The first is \( \text{Mon}^2_{\text{Bir}}(X) \), defined as the elements of \( \text{Mon}^2(X) \) induced by the birational self maps of \( X \). Indeed, any birational self map induces a monodromy operator on \( H^2(X, \mathbb{Z}) \), by a result of Huybrechts (cf. [Huy03, Corollary 2.7]). The second is \( \text{Mon}^2_{\text{Hdg}}(X) \), namely the subgroup of the elements of \( \text{Mon}^2(X) \) which are also Hodge isometries. The last one is \( W_{\text{Exc}} \) (see Definition 1.5). Indeed, in [Mar13, Corollary 3.6, item (1)] Markman proves that any reflection \( R_E \) (associated with a prime exceptional divisor \( E \)) is an element of \( \text{Mon}^2_{\text{Hdg}}(X) \).

Let \( \text{Def}(X) \) be the Kuranishi deformation space of any IHS manifold \( X \) and \( \mathcal{X} \to \text{Def}(X) \) the universal family. Also, let \( 0 \) be a distinguished point of \( \text{Def}(X) \) such that \( \mathcal{X}_0 \cong X \). Set \( \Lambda = H^2(X, \mathbb{Z}) \). By the local Torelli Theorem, \( \text{Def}(X) \) embeds holomorphically into the period domain

\[
\Omega_\Lambda := \{ p \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid q_X(p) = 0 \text{ and } q_X(p + \overline{p}) > 0 \}
\]

as an open (analytic) subset, via the local period map

\[
\mathcal{P} : \text{Def}(X) \to \Omega_\Lambda, \ t \mapsto [H^2,0(X_t)].
\]
Now, let $L$ be a holomorphic line bundle on $X$. The Kuranishi deformation space of the couple $(X, L)$ is defined as $\text{Def}(X,L) \cap \mathcal{P}^{-1}(c_1(L)^±)$, where $c_1(L)^±$ is a hyperplane in $\mathbf{P}(\Lambda \otimes \mathbb{Z} \mathbf{C})$. The space $\text{Def}(X,L)$ is the part of $\text{Def}(X)$ where $c_1(L)$ stays algebraic. Up to shrinking $\text{Def}(X)$ around 0, we can assume that $\text{Def}(X)$ and $\text{Def}(X,L)$ are contractible. Let $s$ be the flat section (with respect to the Gauss-Manin connection) of $R^2\pi_*\mathbb{Z}$ through $c_1(L)$ and $s_t \in H^1(X_t, \mathbb{Z})$ its value at $t \in \text{Def}(X,L)$.

The following are the line bundles we will mostly be interested in.

**Definition 2.5.** [Stably exceptional line bundles] A line bundle $L$ on an IHS manifold $X$ is said stably exceptional if there exists a closed analytic subset $Z \subset \text{Def}(X,L)$, of positive codimension, such that the linear system $|L_t|$ consists of a prime exceptional divisor for every $t \in \text{Def}(X,L) \setminus Z$.

Prime exceptional divisors are stably exceptional, by [Mar10, Proposition 5.2]. By making use of parallel transport operators, we can give the definition of "stably exceptional classes".

**Definition 2.6.** [Stably exceptional classes] Let $X$ be a projective IHS manifold. A primitive, integral divisor class $\alpha \in N^1(X)$ (the Néron-Severi group of $X$) is said stably exceptional if $q_X(\alpha, A) > 0$, for some ample divisor $A$, and there exist a projective IHS manifold $X'$ and a parallel transport operator $f : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$, such that $kf(\alpha)$ is represented by a prime exceptional divisor on $X'$, for some integer $k$.

Of course the two above definitions are related, and as we work in the Néron-Severi space, we will make use of Definition 2.6.

For example, a line bundle of self-intersection $-2$, intersecting positively any ample line bundle on a (smooth) projective $K3$ surface is stably exceptional, and its class in the Néron-Severi space is stably exceptional. Note that any stably exceptional class (line bundle) is effective, by the semi-continuity Theorem.

Suppose that $X$ is a projective IHS manifold of $\text{Kum}_n$-type, of dimension $2n$. Let $\mathcal{W} \subset O^+(H^2(X, \mathbb{Z}))$ be the subgroup acting on the discriminant group by $±1$. Let $\chi : \mathcal{W} \to \{±1\}$ be the character corresponding to the action of $\mathcal{W}$ on the discriminant group. Let $\det : \mathcal{W} \to \{±1\}$ be the determinant character. Let $\mathcal{N} \subset \mathcal{W}$ be the kernel of the product character $\det \cdot \chi$. The below proposition is due to Markman, and was sent to Yoshioka via a letter. The proof we provide was taken from the letter sent by Markman to Yoshioka, and is due to Markman.

**Proposition 2.7** (E. Markman). Let $X_i$ be irreducible holomorphic symplectic manifolds deformation equivalent to a generalized Kummer variety of dimension $2n$ and $e_i \in H^2(X, \mathbb{Z})$ primitive classes satisfying the two conditions $q_X(e_i) = 2(n+1)$, $(n+1)|\text{div}(e_i)$, $i \in \{1, 2\}$. There exists a parallel-transport operator $g : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ satisfying $g(e_1) = e_2$, if and only if $\text{div}(e_1) = \text{div}(e_2)$, and $rs(e_1) = rs(e_2)$.

**Proof.** The statement is analogous to part 2 of [Mar13, Proposition 1.8] and the proof is essentially the same. There are two differences.

1) The lattice $H^2(X, \mathbb{Z})$ for the $\text{Kum}_n$-type IHS manifolds is different from that of the $K3[n]$-type IHS manifolds. However, both lattices contain a primitive embedding of a unimodular lattice of co-rank 1, and this unimodular lattice contains, as a direct summand, the direct sum of three copies of the hyperbolic plane. This is all that is used in [Mar13], so the proof goes through verbatim to show that the invariants $\text{div}(e)$ and $rs(e)$ determine the $\mathcal{W}$ orbit of $e$.

2) The monodromy group of $X$ coincides with $\mathcal{N}$ (cf. [Mar18, Theorem 1.4] and [Mon16, Theorem 2.3]), and it is an index 2 subgroup of $\mathcal{W}$ in the $\text{Kum}_n$-type case. It remains to show that each $\mathcal{W}$ orbit consists of only one $\mathcal{N}$-orbit. This is established for one $\mathcal{W}$-orbit in [MM11, Lemma 4.3]. The same argument applies for every $\mathcal{W}$-orbit of a primitive class $e$ satisfying the conditions $q_X(e) = 2(n+1)$, $(n+1)|\text{div}(e)$. 

In the proof of the above proposition, note that the monodromy invariant $rs(e)$ of the class $e$, is defined as in the paragraph in [Mar13] preceding [Mar13, Proposition 1.7], except that we replace $n - 1$ by $n + 1$.

2.4. Cones. Recall that the pseudo-effective cone $\text{Eff}(X)$ of $X$ is the closure of the big cone $\text{Big}(X)$ in the Néron-Severi space $N^1(X)^\mathbb{R}$, which in our case can be defined as $N^1(X)^\mathbb{R} := \text{Pic}(X) \otimes \mathbb{Z} \mathbb{R}$, because the numerical and linear equivalence relations coincide on an IHS manifold. For an excellent account on big divisors and cones in the Néron-Severi space on arbitrary complex projective varieties, we refer the reader to [Laz04].

**Definition 2.8.** The birational Kähler cone $\mathcal{B}K_X$ is defined as
\[
\mathcal{B}K_X = \bigcup_f f^*\mathcal{K}_{X'},
\]
where $f$ varies among all the birational maps $f: X \rightarrow X'$, where $X'$ is another IHS manifold (which must be projective).

We now introduce another cone in the Néron-Severi space.

**Definition 2.9.**
- A movable line bundle on $X$ is a line bundle $L$ such that the linear series $[mL]$ has no divisorsal components in its base locus, for $m \gg 1$. A divisor $D$ is movable if $\mathcal{O}_X(D)$ is.
- We define the movable cone $\text{Mov}(X)$ in $N^1(X)^\mathbb{R}$ as the convex cone generated by the classes of movable divisors.

An important relation among the last two cones we have introduced is
\[
\text{Mov}(X) = \mathcal{B}K_X \cap N^1(X)^\mathbb{R}
\]
(cf. [HT09, Theorem 7]), where the closure of the birational Kähler cone is taken in $H^{1,1}(X, \mathbb{R})$.

We now recall a couple of definitions concerned with the geometry of convex cones.

**Definition 2.10.** Let $K \subset \mathbb{R}^k$ be a closed convex cone with non-empty interior.
- We say that $K$ is locally (rational) polyhedral at $v \in \partial K$ if $v$ has an open neighbourhood $U = U(v)$, such that $K \cap U$ is defined in $U$ by a finite number of (rational) linear inequalities.
- Let $U$ be an open subset of $\partial K$. We say that $U$ is a circular part of $K$ if there is no element in $U$ at which $K$ is locally polyhedral.

**Remark 2.11.** Note that in the papers [Kov93], [Kov13] Kovács adopted another definition of "circular part" of a convex cone, in order to prove his Theorem 1.1. In particular, let $K$ be as in Definition 2.10. We say that $K$ is locally finitely generated at $v \in \partial K$ if there exists a finite set of points $S = \{v_1, \ldots, v_k\} \subset K$ and a (not necessarily convex) subcone $C \subset K$ such that $K$ is generated by $S$ and $C$. For Kovács, an open subset $U \subset \partial K$ is a circular part of $K$ if there is no point of $U$ at which $K$ is locally finitely generated. One can show that if $K$ is locally polyhedral at a point, then $K$ is locally finitely generated at that point. But the vice versa does not hold true, in general (see Example 2.12). However, in Theorem 1.2, a circular part of the pseudo-effective cone (if there is any) with respect to Definition 2.10 is circular with respect to Kovács’ definition too, because such a circular part would be contained in $\overline{\mathcal{K}X}$ (see "Proof of Theorem 1.2")

The following is an example of locally finitely generated cone which is not locally polyhedral (at a point).

**Example 2.12.** Let $C$ be the "ice cream" in Figure 2.4 and $C \subset \mathbb{R}^3 \subset \mathbb{R}^4$ an embedding of $C$ in $\mathbb{R}^4$, such that the vertex $V$ of $C$ does not coincide with the origin of $\mathbb{R}^4$. Let $K$ be the convex cone generated by $C$ in $\mathbb{R}^4$ (via the embedding we have chosen). By our choice,
the interior of this cone is not empty. Then $K$ is locally finitely generated at $V$, but not locally polyhedral at $V$. Indeed, if $S$ is the sphere appearing in Figure 2.4, the cone $K$ is generated by $V$ and the subcone $K'$ of $K$ generated by the sphere $S$ embedded in $\mathbb{R}^4$. The non-local polyhedrality of $K$ at $V$ is clear.

We will need the following Lemma, which is the key tool to produce circular parts in $\text{Eff}(X)$, in the proof of Theorem 1.2.

**Lemma 2.13** ([Kov93], Lemma 2.3). Let $Q$ be a smooth compact quadric hypersurface in $\mathbb{R}^k$ and $C \subset \mathbb{R}^k$ a compact convex set. Assume that $Q \not\subset C$, then there exists a non empty open subset $U$ of $Q$ such that $U \subset \partial(\text{Conv}(Q \cup C))$, where $\text{Conv}(Q \cup C)$ is the convex hull of $Q \cup C$.

Let $\mathcal{P}_{\text{eff}}(X)$ be the analytic pseudo-effective cone of an IHS manifold $X$, as defined for example in [Bou04, Section 2.3]. If $X$ is projective

(2) $\mathcal{P}_{\text{eff}}(X) \cap N^1(X)_{\mathbb{R}} = \overline{\text{Eff}(X)}$

(cf. [Dem92, Proposition 1.4]). The following result of Boucksom (which holds true in a more general context) will be useful.

**Theorem 2.14** (Theorem 3.19 of [Bou04]). The boundary of the pseudo-effective cone $\mathcal{P}_{\text{eff}}(X)$ of an IHS manifold $X$ is locally polyhedral away from $\mathcal{B}X$, with extremal rays generated by (the classes of) prime exceptional divisors.

The following conjecture holds true for all the known deformation classes of (projective) IHS manifolds (cf. [MO22], [MR20] for the O'Grady-type case, [Mat17] for the $\text{Kum}_n$-type and $\text{K3}^{[n]}$-type case), but it is not known in general.

**Conjecture 2.15** (RLF). Let $X$ be a projective IHS manifold and $D$ an integral, isotropic (with respect to the BBF form) divisor on $X$, such that $[D] \in \text{Mov}(X)$. Then $\mathcal{O}_X(D)$ induces a rational Lagrangian fibration, i.e. there exists a birational map $f: X \dashrightarrow X'$, where $X'$ is another projective IHS manifold, such that $f_*\mathcal{O}_X(D)$ induces a fibration (i.e. a surjective morphism with connected fibers) $X' \twoheadrightarrow B$ to a projective $\text{dim}(X)/2$-dimensional base $B$.

**Definition 2.16.** Let $X$ be a projective IHS manifold. We define the effective movable cone as $\text{Mov}(X)^e := \overline{\text{Mov}(X) \cap \text{Eff}(X)}$ and $\text{Mov}(X)^+$ as the convex hull of $\text{Mov}(X) \cap N^1(X)_Q$ in $N^1(X)_{\mathbb{R}}$.

Although the following lemma may be well-known to experts, we did not find it in the literature. As it is a useful result in this context, we decided to include it in the article, with a proof.
Lemma 2.17. If $X$ is a projective IHS manifold belonging to one of the known deformation classes, we have the equality $\Mov(X)^+ = \Mov(X)^e$.

Proof. We start by observing that the interiors of the two involved cones coincide, hence we only have to check that everything works well at the level of the boundaries. Let $\alpha$ be an integral divisor class belonging to $\partial\Mov(X)^+$. If $q_X(\alpha) > 0$ we are done, because in that case $\alpha$ is a big class. If $q_X(\alpha) = 0$, the RLF conjecture holds true under our assumptions, so the class $\alpha$ is effective, hence the inclusion $\Mov(X)^+ \subseteq \Mov(X)^+$ is verified. Now, let $\alpha$ be an element of $\Mov(X)^e$ lying on its boundary. If $\alpha$ belongs to some wall $[E]$, for some prime exceptional divisor $E$, and $q_X(\alpha) > 0$, we are done, because $\Mov(X)$ is locally rational polyhedral away from $\partial\Mov(X)$ (cf. [Den21, Corollary 4.8]). It remains to be checked the case $0 \neq \alpha \in \partial\Mov(X)$ (and so $q_X(\alpha) = 0$). We can write $\alpha = \sum_{i=1}^k a_i[D_i]$, where the $D_i$s are prime divisors and the $a_i$s are positive integers. Let $W \subset N_1(X)_\mathbb{R}$ be the rational subspace spanned by the $[D_i]’s$. The class $\alpha$ $q_X$-intersects non-negatively every $D_i$, and $q_X(\alpha) = 0$, hence $q_X(\alpha, D_i) = 0$ for any $i$. By linear algebra, as the signature of $q_X$ restricted to $N_1(X)_\mathbb{R}$ is $(1, \rho(X) - 1)$, it follows that a maximal, totally isotropic linear subspace of $N_1(X)_\mathbb{R}$ (with respect to $q_X$) must have dimension 1. Then $W' = \cap_{i=1}^k ([D_i]^{+} \cap W)$ is a rational subspace of dimension 1. Indeed $W'$ contains $\alpha$, and there cannot be other elements linearly independent from $\alpha$, because otherwise we would have a totally isotropic linear subspace of $N_1(X)_\mathbb{R}$ of dimension greater or equal than 2. We conclude that $\alpha$ is a multiple of a rational, movable class, and we are done. $\blacksquare$

Remark 2.18. Note that the above argument implies the following.

- $\Mov(X) = \Mov(X)^+$ for all the known projective IHS manifolds.
- The inclusion $\Mov(X)^+ \supseteq \Mov(X)^e$ holds true for any projective IHS manifold.

The Kawamata-Morrison movable cone conjecture for projective IHS manifolds predicts the existence of a fundamental domain for the action of $\Bir(X)$ on $\Mov(X)^e$, and the below theorem (proved by Markman) is a slightly weaker version.

Theorem 2.19 ([Mar11], Theorem 6.25). Let $X$ be a projective IHS manifold. There exists a rational convex polyhedral cone $\Pi$ in $\Mov(X)^+$, such that $\Pi$ is a fundamental domain for the action of $\Bir(X)$ on $\Mov(X)^+$.

Remark 2.20. Combining Lemma 2.17 and Theorem 2.19 one deduces that $\Pi$ is also a fundamental domain for the action of $\Bir(X)$ on $\Mov(X)^e$.

2.5. Pell’s equations. A Pell equation is any diophantine equation of the form

$$x^2 - ny^2 = 1,$$

where $n$ is a given positive integer. The fundamental solution of the equation (3) is the (positive) solution $(x_1, y_1)$ minimizing $x$. Once the fundamental solution is known, any other positive solution (and hence all the solutions) can be calculated recursively with the formulas

$$x_{k+1} = x_1x_k + ny_1y_k$$
$$y_{k+1} = x_1y_k + y_1x_k,$$

and this is all we need from the theory of diophantine equations.

3. Proof of the main Theorem and some consequences

In this section we prove the main theorem of the article, namely Theorem 1.2. We start by deducing Corollary 1.4, which characterizes the pseudo-effective cone of a projective IHS manifold of Picard number 2, from Oguiso’s theorem [Ogu14, Theorem 1.3, item (2)].
Proof of Corollary 1.4: We first observe that $\overline{\text{Eff}(X)}$ cannot have both one rational extremal ray and one irrational extremal ray. Indeed, if $X$ does not contain any prime exceptional divisor then $\overline{\text{Eff}(X)} = \overline{\text{Mov}(X)}$ (cf. [Den21, Corollary 3.8]) and we are done by Theorem 1.3. As by Theorem 2.14 and equality (2) the classes of the prime exceptional divisors span extremal rays in $\overline{\text{Eff}(X)}$ (see also [Den21, Corollary 3.4] for an algebraic proof of this fact), we only have to check the case "$X$ contains only one exceptional prime divisor". In such case $\overline{\text{Mov}(X)}$ is rational by Theorem 1.3 and [Mar11, Theorem 6.17], and the extremal ray of $\overline{\text{Mov}(X)}$ spanned by an integral class of BBF square 0 is also extremal for $\overline{\text{Eff}(X)}$ and we are done. To conclude, it is sufficient to observe that $\overline{\text{Eff}(X)}$ is rational (resp. irrational) if and only $\overline{\text{Mov}(X)}$ is rational (resp. irrational). Indeed, the perfect pairing induced by the BBF form gives the duality $\overline{\text{Eff}(X)} = \overline{\text{Mov}(X)}^\perp$, and we are done.

Remark 3.1. We observe that under the assumptions of Corollary 1.4, if $X$ satisfies Conjecture 2.15 (e.g. if $X$ belongs to one of the known deformation classes), whenever $\overline{\text{Eff}(X)}$ is rational we have $\overline{\text{Eff}(X)} = \text{Eff}(X)$, i.e. both the extremal rays of the pseudo-effective cone are spanned by the classes of some integral, effective divisors. Also, note that both the cases in Corollary 1.4 do occur (see for example [Rie20, Lemma 3.2] and [Ogu14, Proposition 5.3]).

To prove Theorem 1.2, we first need the following technical lemma, which is an adaptation of [Kov13, Lemma 3.1] to our case.

Lemma 3.2. Let $X$ be a projective IHS manifold, $D$ an integral divisor such that $[D] \in \mathcal{C}_X$ and $E$ an integral divisor such that $0 \neq [E] \in \partial \overline{\text{Eff}(X)}$. Let $t := \text{div}(E)$ be the divisibility of $E$. Consider the 2-plane $\pi := \langle [E], [D] \rangle \subset N^1(X)_{\mathbb{R}}$.

(a) If $q_X(E) = 0$ there exists a pseudo-effective class $\alpha \in \pi \cap \overline{\text{Eff}(X)}$ represented by an integral divisor such that $q_X(\alpha) = 0$ and $\alpha$ and $[E]$ are on opposite sides of $[D]$.

(b) If $q_X(E) = te < 0$ there exists a class $\alpha$, represented by an integral divisor, such that $q_X(\alpha) = 0$ or $q_X(\alpha) = te$ and $\alpha$ and $[E]$ are on opposite sides of $\mathbb{R}^{\geq 0}[D]$.

Furthermore, only a positive multiple of the divisor defining $\alpha$ can be effective.

Proof. Set $d = q_X(D)$, $bt = q_X(E, D)$ and $te = q_X(E)$. Note that $bt > 0$, for example by [MY15, Lemma 3.1].

(a) Let $\alpha = x[D] - y[E]$ be an element of $\pi$, for $x, y$ real numbers. Then in this case $q_X(\alpha) = dx^2 - 2btxy$, and by choosing $x = 2bt$ and $y = d$, we obtain $\alpha = 2bt[D] - d[E]$ satisfying $q_X(\alpha) = 0$. Also, $q_X(\alpha, D) = bt > 0$, hence, by [MY15, Lemma 3.1], $\alpha$ belongs to $\mathcal{C}_X$ and so is pseudo-effective. As $x[D] = \alpha + y[E]$, clearly $\alpha$ and $[E]$ are on opposite sides of $[D]$.

(b) Let $\alpha = tx[D] - y[E]$ be an element of $\pi$, for $x, y$ real numbers. Then

$$q_X(\alpha) = t^2x^2d + y^2et - 2xyt^2b = te \left( \frac{tx^2d}{e} + y^2 - 2xyt^2b - e \right).$$

If we set $x' = y - \frac{txb}{e}$, $y' = -\frac{e}{t}x'$ and $N = t^2b^2 - de$, we can rewrite $q_X(\alpha)$ as

$$q_X(\alpha) = te \left[ \left( y - \frac{txb}{e} \right)^2 + tde \frac{x^2}{e^2} - t^2b^2 \frac{x^2}{e^2} \right] = te[(x')^2 - N(y')^2].$$

If $N$ is a square, then $q_X(\alpha) = te(x' - \sqrt{N}y')(x' + \sqrt{N}y')$. Choosing $x' = N, y' = \sqrt{N}$, i.e. $x = -\sqrt{N}, y = N - \sqrt{N}tb$, we obtain the element $\alpha = -te\sqrt{N}[D] - (N - \sqrt{N}tb)[E]$ satisfying $q_X(\alpha) = 0$. We observe that $N - \sqrt{N}tb > 0$. If $b = 0$ or $b < 0$ this is trivial. If $b > 0$ and we assume $N - \sqrt{N}tb \leq 0$, we would have $N^2 - N(tb)^2 \leq 0$, which would imply $N - (tb)^2 \leq 0$ and this is a contradiction, because we have $N - (tb)^2 = -tde > 0$. Also, we note that the chosen $\alpha$ belongs to $\mathcal{C}_X$. Indeed, it suffices to pick an element
\[ \beta \in \overline{\text{Mov}(X)} \cap \mathcal{C}_X \cap \langle [E] \rangle \] and to observe that \( q_X(\beta, \alpha) = -\sqrt{N_{\text{teq}}([D], \beta)} > 0 \). For instance, let \( D' \) be an integral divisor whose class lies in \( \text{int}(\text{Mov}(X)) \). One can choose \( \beta := [D'] - \frac{q_X(D', E)}{q_X(E)} [E] \). Clearly \( q_X(\beta, E) = 0 \) and
\[
q_X(\beta) = \frac{q_X(D')q_X(E) - q_X(D', E)^2}{q_X(E)} > 0
\]
(for the latter inequality see for example [Den21, Proposition 1.15]). Clearly, also in this case \( \alpha \) and \( [E] \) are on opposite sides of \([D]\).

If \( N \) is not a square, the Pell equation
\[ x^2 - Ny^2 = 1 \]
has infinitely many integer solutions (see [IR90], Pell’s equation 17.5.2) and we may choose a solution \((x', y')\) of positive integers. Arguing as above we obtain a class
\[
\alpha = -\text{tey}([D] - (x' - tby')[E])
\]
satisfying \( q_X(\alpha) = t\alpha \). Also in this case we have \( x' - tby' > 0 \). Indeed, if \( b = 0 \) or \( b < 0 \) this is trivial. If \( b > 0 \) and we assume \( x' - tby' \leq 0 \), we would have \( x'^2 - (tb)^2y'^2 = 1 - t\text{dey}'^2 \leq 0 \), which is clearly a contradiction. Also, as above, \( \alpha \) and \([E]\) are on opposite sides of \([D]\). Now, suppose that a multiple of the divisor defining \( \alpha \) (namely \( D' := -\text{tey'}D - (x' - tby')E \)) is effective. By contradiction, without loss of generality, we can assume that \( -D' \) is effective. Then \( -D' - \text{tey}'D = (x' - tby')E \) is big and this is clearly a contradiction, because the latter belongs to the boundary of Big\((X)\). It follows that only a positive multiple of \( D' \) can be effective, and in such case the class \( \alpha \) is effective.

\[\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Picturing Lemma 3.2}
\end{figure}\]

**Lemma 3.3.** Let \( X \) be a projective IHS manifold belonging to one of the known deformation classes. With the notation of Lemma 3.2, suppose that the classes \([E]\) and \([D]\) are primitive in \( H^2(X, \mathbb{Z}) \). Also, suppose that a (positive) multiple of \( E \) is prime exceptional. Then, up to choosing a suitable solution for the Pell equation (5), the class \( \alpha \) constructed in item (b) of Lemma 3.2 is primitive and effective.

**Proof.** We adopt the notation of Lemma 3.2. It is natural to split the proof by distinguishing the deformation type.

- Suppose that \( X \) is of OG10-type. By [MO22, Proposition 3.1] we can only have \( q_X(E) = -2 \) and \( \text{div}(E) = 1 \) or \( q_X(E) = -6 \) and \( \text{div}(E) = 3 \). In the first case the class \( \alpha \) we get is of BBF square \(-2\) and this implies that \( \alpha \) is primitive. Furthermore, its divisibility must be one, because by Remark 2.1 \( \text{div}(\alpha) \) divides \( |A_X| \), and the discriminant group of \( X \) is \( A_X \cong \mathbb{Z}/3\mathbb{Z} \) (the reader is referred to [Rap06] for the computation of the discriminant group of the 4 known deformation classes). It follows by [MO22, Proposition 5.4] and item (b) of Lemma 3.2 that the class \( \alpha \) is stably exceptional, hence effective. In the second case we have \( q_X(\alpha) = -6 \), and this again implies the primitivity of \( \alpha \). Also,
\( \alpha = 6 \gamma'(D) - (x' - 3by')|E| \), and for any \( \gamma \in H^2(X, \mathbb{Z}) \) we have \( q_X(\alpha, \gamma) \equiv 0 \mod 3 \). But \( |A_X| = 3 \), hence \( \text{div}(\alpha) = 3 \). Using again [MO22, Proposition 5.4] and item (b) of Lemma 3.2, we conclude that in this case the class \( \alpha \) is effective.

- Suppose that \( X \) is of OG6-type. By [MR20, Lemma 6.4] it can only be \( q_X(E) = -4 \) and \( \text{div}(E) = 2 \) or \( q_X(E) = -2 \) and \( \text{div}(E) = 2 \). In both cases the class \( \alpha \) must be primitive. Furthermore, in the first case we have \( \alpha = 4 \gamma'(D) - (x' - 2by')|E| \), so that \( q_X(\alpha, \gamma) \equiv 0 \mod 2 \), where \( \gamma \) is any element of \( H^2(X, \mathbb{Z}) \). It follows that \( \text{div}(\alpha) \geq 2 \), and as \( A_X \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), by Remark 2.1, we conclude that \( \text{div}(\alpha) = 2 \). The same argument proves that also in the second case the divisibility of \( \alpha \) is 2. By [MR20, Proposition 5.4] and item (b) of Lemma 3.2 we conclude that the class \( \alpha \) is effective. Notice that until now we have not needed the primitivity of \([D]\).

- Suppose that \( X \) is of \( K3^{[n]} \)-type. By [Mar13, Theorem 1.12] we can have \( \text{div}(E) = 2(n-1) \) or \( \text{div}(E) = n-1 \), and in both cases the BBF square of \( E \) is \(-2(n-1)\).

First, suppose \( \text{div}(E) = 2(n-1) \). We observe that \( \text{div}(\alpha) = 2(n-1) \), because \( \text{div}(\alpha) \geq 2(n-1) \), and \( q_X(\alpha) = 2(n-1) \). In this case we have

\[
\alpha = 2(n-1)\gamma'(D) - [x' - 2(n-1)by']|E|.
\]

Now, we would like to find a solution \((x', y')\) of the equation (5) satisfying \( x' - 2(n-1)by' \equiv 1 \mod 2(n-1) \). We observe that \( x' - 2(n-1)by' \equiv x' \mod 2(n-1) \), for any solution \((x', y')\), hence it suffices to find a solution such that \( x' \equiv 1 \mod 2(n-1) \). Let \((x_1, y_1)\) be the fundamental solution of the equation (5). The "second" solution of the equation is \( x_2 = x_1^2 + Ny_1^2 \), \( y_2 = 2x_1y_1 \), where in this case \( N = 4(n-1)^2b^2 - 2(n-1)d \).

Clearly \( x_2^2 \equiv 1 \mod 2(n-1) \), hence \( x_2 \equiv 1 \mod 2(n-1) \), and \((x_2, y_2)\) is a solution we were looking for. We claim, with respect to this choice of the solution of the Pell equation, the class \( \alpha \) is primitive and \( \text{div}(\alpha) = 2(n-1) \). Indeed, \([D]\) and \([E]\) are linearly independent in \( H^2(X, \mathbb{R}) \). We observe that \( \text{g.c.d.}(2(n-1), x_2 - 2(n-1)by_2) = 1 \), because \( x_2 - 2(n-1)by_2 \equiv 1 \mod 2(n-1) \), and \( \text{g.c.d.}(x_2, y_2) = 1 \), as they satisfy \( x_2^2 - Ny_2^2 = 1 \).

Furthermore

\[
2(n-1)b \cdot y_2 + [x_2 - 2(n-1)by_2] = x_2,
\]

which implies that \( \text{g.c.d.}(y_2, x_2 - 2(n-1)by_2) \) divides \( x_2 \). Then \( \text{g.c.d.}(y_2, x_2 - 2(n-1)by_2) = 1 \), because \( \text{g.c.d.}(x_2, y_2) = 1 \), and this implies the primitivity of \( \alpha \).

We now prove the effectivity of \( \alpha \). Recall that \( q_X(\alpha) = te((x')^2 - N(y')^2) \) (see the equality (4)). To do so, first observe that \( \left[ \frac{\alpha}{2(n-1)} \right] = \left[ -\frac{E}{2(n-1)} \right] \) in \( A_X \cong \mathbb{Z}/2(n-1)\mathbb{Z} \), hence, using Lemma 2.2 and changing the sign of \([-E]\) with the reflection \( R_E \in \text{Mon}^2(X) \), we obtain an isometry \( \iota \in O^+(H^2(X, \mathbb{Z})) \), sending \([E]\) to \( \alpha \), acting as \(-1\) on \( A_X \). Indeed, \( R_E \) acts as \(-1\) on \( A_X \), and the isometry given by Lemma 2.2 acts trivially on \( A_X \). By [Mar10, Lemma 4.2], the isometry \( \iota \) belongs to \( \text{Mon}^2(X) \), and by [Mar13, Proposition 9.16, item (3)], the monodromy orbit of \( \alpha \) is determined by \( \text{div}(\alpha) \) and \( rs(\alpha) \), hence \( rs(\alpha) = rs([E]) \) (see [Mar13] for the definition of the latter invariant). Hence \( \alpha \) is an effective class, by item (2) of Lemma 3.2 and Markman’s characterization of stably exceptional classes on \( K3^{[n]} \)-type IHS manifolds (cf. [Mar11, Theorem 9.17]).

Now, suppose that \( \text{div}(E) = n - 1 \). We would like to find a solution \((x', y')\) of the equation (5) satisfying \( x' - (n-1)by' \equiv 1 \mod 2(n-1) \). Also in this case the "second" solution \((x_2, y_2)\) of the equation (5) yields the conclusion. Indeed \( N = (n-1)^2b^2 - 2(n-1)d \) in this case, and we have

\[
x_2 = x_1^2 + Ny_1^2 \equiv 1 + 2Ny_1^2 \equiv 1 \mod 2(n-1).
\]

But \( y_2 \) is even, hence \( x_2 - (n-1)by_2 \equiv 1 \mod 2(n-1) \). Now, we observe that, with respect of the solution \((x_2, y_2)\) of (5), we have \( \text{div}(\alpha) = n - 1 \). Indeed, \( \text{div}(\alpha) \geq n - 1 \), and we can have either \( \text{div}(\alpha) = n - 1 \), or \( \text{div}(\alpha) = 2(n-1) \), because \( q_X(\alpha) = 2(n-1) \). If \( N \) is even, we have that \( x' \) is odd and \( y' \) is even, for any solution \((x', y')\). If \( N \) is odd, then we can have two possibilities: \( x_1 \) is odd and \( y_1 \) is even or \( x_1 \) is even and \( y_1 \) is odd. In any case we will have that \( x_2 \) is odd and \( y_2 \) is even. By definition of divisibility, there exists an element
\(\gamma \in H^2(X, \mathbb{Z})\) such that \(q_X(E, \gamma) = n - 1\), thus \(q_X(\alpha, \gamma) = -x_2(n - 1) \mod 2(n - 1)\). As \(x_2\) is odd, \(q_X(\alpha, \gamma)\) cannot be divided by \(2(n - 1)\), hence \(\text{div}(\alpha) = n - 1\). Arguing as in the case of divisibility \(2(n - 1)\), we see that \(\alpha\) is primitive. To show that \(\alpha\) is effective in this case, just note that \(\alpha/[\alpha/(n - 1)] = [-E/(n - 1)]\) in \(X\), and, arguing as in the case of divisibility \(2(n - 1)\), we obtain an isometry \(\iota \in O^+(H^2(X, \mathbb{Z}))\), sending \([E]\) to \(\alpha\), and acting as \(-1\) on \(X\). This isometry is a monodromy operator, by [Mar10, Lemma 4.2]. By [Mar13, Proposition 9.16, item (3)], the monodromy orbit of \(\alpha\) is determined by \(\text{div}(\alpha)\) and \(rs(\alpha)\), hence \(rs(\alpha) = rs([E])\). Again, by item (b) of Lemma 3.2 and [Mar13, Theorem 1.12], we conclude that the class \(\alpha\) is effective.

- Suppose now that \(X\) is of \(\text{Kum}_n\)-type. By [Yos16, Proposition 5.4] we can have \(\text{div}(E) = 2(n + 1)\) or \(\text{div}(E) = n + 1\), and in both cases the BBF square of \(E\) is \(2(n + 1)\). Recall that \(A_X \cong \mathbb{Z}/2(n + 1)\mathbb{Z}\) in this case. The proof goes exactly as in the \(K^3[n]\)-type case. Also in this case the "second" solution \((x_2, y_2)\) of the equation \((5)\) satisfies the congruence \(x_2 - \text{div}(E)by_2 \equiv 1 \mod 2(n + 1)\), hence \(\alpha = 2(n + 1)[D] - [x_2 - \text{div}(E)by_2][E]\) and \(E\) are such that \(\left[\frac{\alpha}{\text{div}(E)}\right] = \left[-\frac{E}{\text{div}(E)}\right]\) in \(X\). Arguing as in the \(K^3[n]\)-type case, we conclude that \(\text{div}(\alpha) = \text{div}(E)\) (if \(\text{div}(E) = 2(n + 1)\)), this is true for any solution of \((5)\), if \(\text{div}(E) = n + 1\), this is true if we choose the solution \((x_2, y_2)\) of \((5)\), and \(\alpha\) is primitive (with respect to the solution \((x_2, y_2)\)) . Again, arguing exactly as in the \(K^3[n]\)-type case, we obtain an isometry \(\iota \in O^+(H^2(X, \mathbb{Z}))\) of determinant \(-1\), acting as \(-1\) on \(A_X\), and sending \([E]\) to \(\alpha\). By Markman’s and Mongardi’s characterization of \(\text{Mon}^3(X)\) (cf. [Mar18, Theorem 1.4] and [Mon16, Theorem 2.3]), we conclude that the isometry \(\iota\) lies in \(\text{Mon}^3(X)\). By Proposition 2.7, the monodromy orbit of \(\alpha\) is determined by \(\text{div}(\alpha)\) and \(rs(\alpha)\), hence \(rs(\alpha) = rs([E])\). Then the class \(\alpha\) is effective, by item (b) of Lemma 3.2 and Yoshioka’s characterization of the stably exceptional classes on \(\text{Kum}_n\)-type IHS manifolds (cf. [Yos16, Proposition 5.4]).

We are now ready to prove Theorem 1.2. The strategy of the proof is mainly based on the one adopted by Kovács in the proof of Theorem 1.1 and on the slightly modified argument provided by Huybrechts in [Huy15, Chapter 8, Section 3]. The other fundamental tools are: Lemma 3.3, the characterization of the stably exceptional classes and of the monodromy groups.

**Proof of Theorem 1.2:** Suppose that \(\text{Neg}(X) = \emptyset\). In this case \(\text{Eff}(X) \neq \emptyset\), hence \(\text{Eff}(X) = \overline{\text{Eff}}(X)\), hence \(\overline{\text{Eff}}(X)\) is circular.

Suppose that \(\text{Neg}(X) \neq \emptyset\). We first prove that \(\text{Eff}(X)\) does not contain any circulat part. Let \(E\) be a prime exceptional divisor and \(D\) any integral divisor lying in \(\mathcal{C}_X\). Without loss of generality we can assume that the classes \([E]\) and \([D]\) are primitive in \(H^2(X, \mathbb{Z})\). Set \(q_X(E) = \epsilon E\), where \(\text{div}(E) = \epsilon \). Assume by contradiction that \(\text{Eff}(X)\) has a circular part \(\mathcal{C}\). By Theorem 2.14 and equality \((2)\), \(\mathcal{C}\) must be contained in \(\partial \text{Mov}(X) \cap \partial \text{Eff}(X)\). Also, \(\text{Mov}(X)\) is locally polyhedral away from the boundary of \(\text{Big}(X)\) (cf. [Den21, Corollary 4.8]), so that we can assume \(\mathcal{C} = \mathbb{R}^{>0} \mathcal{C} \subset \partial \overline{\mathcal{C}}_X \cap \partial \text{Eff}(X)\). Then there exists a neighbourhood \(U\) of \(\mathcal{C}\) in \(\text{Eff}(X)\) such that \(q_X(\beta) \geq 0\) for any \(\beta \in U\). By Lemma 3.2 and Lemma 3.3, for any primitive integral divisor \(D\) whose class lies in \(\mathcal{C}_X\), on the opposite side of \(D\) (with respect to \(E\)) there exists a class \(\alpha\), represented by an integral divisor, being either pseudo-effective and of BBF square 0 or effective and of BBF square \(\epsilon E\). As \([D]\) (which we always assume being primitive) approaches \(\mathcal{C}\), we can only have \(q_X(\alpha) = 0\), with \(\alpha\) pseudo-effective. This means that \(\mathcal{C}\) contains an integral class \(\alpha\) of BBF square 0. Now, consider the class \(\alpha_\epsilon = (1 - \epsilon)\alpha + \epsilon [E]\), where \(\epsilon > 0\) is rational. Clearly, for \(\epsilon\) small enough \(\alpha_\epsilon\) is effective. Also we note that \(q_X(\alpha, E) > 0\), as otherwise we would have \(q_X(\alpha_\epsilon) = -2\epsilon^2 q_X(\alpha, E) + 2\epsilon q_X(\alpha, E) - 2\epsilon^2 < 0\), which would contradict \(\alpha \in \partial \text{Eff}(X) \cap \partial \overline{\mathcal{C}}_X\). As by assumption \(\rho(X) \geq 3\), we can find an integral class \(\alpha'\) of negative BBF square, lying
in \((\mathbb{R} \beta \oplus \mathbb{R}[E])^+\). We now define

\[\alpha_k := -2k^2 q_X(\alpha') (q_X(\alpha, E))^3 \alpha - 2k (q_X(\alpha, E))^2 \alpha' + [E].\]

An easy computation shows that \(q_X(\alpha_k) = q_X(E) = \ell e\) and that for \(k \gg 0\) \(q_X(\alpha_k, [A]) > 0\), where \(A\) is any ample divisor. We want to prove that for \(k\) large enough the classes \(\alpha_k\) are effective. For this purpose we need to distinguish the different deformation types. Notice that the classes \(\alpha_k\) are primitive independently of the deformation type we have chosen. Indeed, \([E], \alpha\) and \(\alpha'\) are linearly independent in \(H^2(X, \mathbb{R})\), and the class \([E]\) is primitive, whence any \(\alpha_k\) is primitive.

- Suppose that \(X\) is of OG10-type. As remarked in the proof of Lemma 3.3, by [MO22, Proposition 6.4], we have \(q_X(E) = -2\) or \(q_X(E) = -6\) and \(\text{div}(E) = 3\). In the first case \(q_X(\alpha_k) = -2\). By [MO22, Proposition 5.4], \(\alpha_k\) is stably exceptional, hence effective, and we are done. In the second case we have

\[\alpha_k = -54k^2 q_X(\alpha') \alpha - 18kg^2 \alpha' + [E],\]

where \(q_X(\alpha, E) = 3g\), hence \(\text{div}(\alpha_k) = 3\). If \(k\) is large enough, by [MO22, Proposition 5.4], \(\alpha_k\) is a stably exceptional class. Hence, for \(k\) large enough, the classes \(\{\alpha_k\}_k\) are effective.

- Suppose that \(X\) is of OG6-type. As said in Lemma 3.3 we have \(q_X(E) = -2\) and \(\text{div}(E) = -2\) or \(q_X(E) = -4\) and \(\text{div}(E) = 2\). Arguing as in Lemma 3.3 we conclude that in both cases the classes \(\alpha_k\) have divisibility 2 and using [MR20, Proposition 5.4] we conclude that \(\alpha_k\) is stably exceptional, hence effective for \(k\) large enough.

- Suppose that \(X\) is of \(K3^{[\delta]}\)-type. We can have \(\text{div}(E) = -2(n-1)\) or \(\text{div}(E) = n-1\), and in any case \(q_X(E) = -2(n-1)\), which implies \(q_X(\alpha_k) = 2(n-1)\). As the coefficient of \([E]\) in \(\alpha\) is 1, we conclude that \(\text{div}(\alpha) = \text{div}(E)\). Further, arguing as in Lemma 3.3, we see that \([\alpha_k/\text{div}(\alpha_k)] = [E/\text{div}(E)]\) in \(A_X\). Hence we obtain an isometry \(\iota \in \widehat{SO}^+(H^2(X, \mathbb{Z}))\) (so that \(\iota\) belongs to \(\text{Mon}^2(X)\) sending \([E]\) to \(\alpha_k\). By [Mar13, Proposition 9.16, item (3)], the monodromy orbit of \(\alpha_k\) is determined by \(\text{div}(\alpha_k)\) and \(rs(\alpha_k)\), hence \(rs(\alpha_k) = rs([E])\). For \(k\) large enough we have \(q_X(\alpha_k, [A]) > 0\), by [Mar11, Theorem 9.17] we conclude that the \(\alpha_k\) are effective for \(k\) large enough.

- Suppose that \(X\) is of \(\text{Kum}_{\delta}\)-type. We can have \(\text{div}(E) = -2(n+1)\) or \(\text{div}(E) = n+1\), and in any case \(q_X(E) = -2(n+1)\) (cf. [Yos16, Proposition 5.4]), which implies \(q_X(\alpha_k) = 2(n+1)\) (recall that the BBF square of \(\alpha_k\) equals that of \(E\)). As for the \(K3^{[\delta]}\)-type case, the classes \(\alpha_k\) have divisibility \(\text{div}(\alpha_k) = \text{div}(E)\). Again, \([\alpha_k/\text{div}(\alpha_k)] = [E/\text{div}(E)]\) in \(A_X\), hence, by Lemma 2.2, we get an isometry \(\iota \in \widehat{SO}^+(H^2(X, \mathbb{Z}))\) sending \([E]\) to \(\alpha_k\). The isometry \(\iota\) has determinant 1 and acts trivially on \(A_X\), hence, by Markman’s and Mongardi’s characterization of \(\text{Mon}^2(X)\), it lies in \(\text{Mon}^2(X)\). By Proposition 2.7, the monodromy orbit of \(\alpha_k\) is determined by \(\text{div}(\alpha_k)\), \(rs(\alpha_k)\), hence \(rs(\alpha_k) = rs([E])\). We conclude that \(\alpha_k\) is effective for \(k\) large enough, by [Yos16, Proposition 5.4].

Dividing both the members of (7) by \(2k^2\) we see that the sequence of rays \(\{\mathbb{R}^{>0} \alpha_k\}_k\) converges to \(\mathbb{R}^{>0} \alpha\) and this contradicts the circularity of \(\text{Eff}(X)\) at \(\gamma\) (in particular the fact that locally around \(\gamma\) in \(\text{Eff}(X)\) the BBF form is nonnegative).

Now, using that \(\text{Eff}(X)\) does not contain any circular part, we show the equality \(\text{Eff}(X) = \sum_{E \in \text{Neg}(X)} \mathbb{R}^{>0}[E]\). To do so, as \(\text{Eff}(X)\) does not contain lines (see for example [Den21, Lemma 3.2]), by Minkowski’s Theorem on closed convex cones not containing lines, it suffices to show that \(\sum_{E \in \text{Neg}(X)} \mathbb{R}^{>0}[E]\) contains all the extremal rays of \(\text{Eff}(X)\).

Suppose by contradiction that there exists an extremal ray \(R\) of \(\text{Eff}(X)\) not belonging to \(\sum_{E \in \text{Neg}(X)} \mathbb{R}^{>0}[E]\). Pick a non-zero element element \(v \in R\) (which must therefore satisfy \(q_X(v) = 0\)) and consider the hyperplane

\[H := \{x \in \mathbb{N}^1(X) | q_X(x, A) = q_X(v, A)\}.
\]

Clearly \(Q := H \cap \partial \mathbb{R}X\) is a smooth compact quadric hypersurface in \(H\). Indeed \(q_X(v, A) > 0\) (see for example [Den21, Lemma 3.2]), hence \(H\) does not intersect \(-\partial \mathbb{R}X\), so that
$H \cap \partial \overline{\mathcal{E}}_X$ is non-degenerate. Moreover, up to a rescaling, any element $y \in \partial \overline{\mathcal{E}}_X$ is such that $q_X(y, A) = q_X(v, A)$, hence $Q$ must be compact. Let $N$ be the closure of the convex hull in $H$ of all the elements $y \in H$ belonging to an extremal ray of the type $\mathbf{R}^{\geq 0}[E]$, where $E$ is a prime exceptional divisor of $X$. Clearly $N$ is a compact convex subset of $H$. As $v \in Q \setminus N$, by Lemma 2.13, there exists an open subset $U$ of $Q$ contained in $\partial(\text{Conv}(Q \cup N))$, where $\text{Conv}(Q \cup N)$ is the convex hull of $Q \cup N$. Then $\mathbf{R}^{\geq 0}U$ is a circular part of $\mathcal{E}(X)$, and this contradicts item (1) of the theorem.

**Corollary 3.4.** Let $X$ be a projective IHS manifold of Picard number at least 3, carrying a prime exceptional divisor, and belonging to one of the 4 known deformation classes. Then $\text{card}(\text{Neg}(X)) \geq \rho(X)$ and $\mathcal{E}(X)$ is rational polyhedral if and only if $\text{card}(\text{Neg}(X))$ is finite.

**Proof.** By Theorem 1.2 we have, $\mathcal{E}(X) = \sum_{E \in \text{Neg}(X)} \mathbf{R}^{\geq 0}[E]$, and the latter equality is satisfied only if $\# \text{Neg}(X) \geq \rho(X)$, because the interior of $\sum_{E \in \text{Neg}(X)} \mathbf{R}^{\geq 0}[E]$ (which is non-empty) equals the interior of $\sum_{E \in \text{Neg}(X)} \mathbf{R}^{\geq 0}[E]$. The proof of the second part of the statement is clear.

**Example 3.5.** Let $S$ be a projective $K3$ surface of Picard number 2, carrying one (and only one) smooth rational curve. Then, by Corollary 3.4, $S[n]$ carries at least a prime exceptional divisor coming neither from $S$, nor from the desingularization of $S^{(n)}$, for any $n$.

**Corollary 3.6.** Let $X$ be a projective IHS manifold of Picard number greater than 3, belonging to one of the 4 known deformation classes, and carrying a prime exceptional divisor. Then the nef cone $\text{Nef}(X)$ has not any circular part.

**Proof.** By Theorem 1.2, $\text{Mov}(X)$ has not any circular part, and $\text{Nef}(X)$ is rational polyhedral away from the boundary of $\mathcal{E}(X)$, thanks to a result of Kawamata (cf. [Kaw88, Theorem 5.7]), hence also $\text{Nef}(X)$ does not have any circular part.

In the proof of Corollary 1.6 we will need a theorem by Burnside and a result of Oguiso.

**Theorem 3.7 ([Bur05], Main Theorem).** Let $G$ be a subgroup of $\text{GL}(n, \mathbf{C})$. Assume that there exists a positive integer $d$ such that any element of $G$ has order at most $d$. Then $G$ is a finite group.

**Lemma 3.8 ([Ogu14], Corollary 2.6).** Let $Y$ be a smooth projective variety with trivial canonical bundle, such that $h^1(Y, \mathcal{O}_Y) = 0$. Moreover, let $r : \text{Bir}(Y) \to \text{GL}(N^1(Y)_{\mathbf{R}})$ be the natural representation of $\text{Bir}(Y)$. Then the following hold true.

1. $[\text{Bir}(Y) : \text{Aut}(Y)] = [r(\text{Bir}(Y)) : r(\text{Aut}(Y))]$.
2. If $G$ is a subgroup of $\text{Bir}(Y)$, $G$ is finite if and only if there is a positive integer $d$ such that every element of $r(G)$ as order at most $d$.

**Proof of Corollary 1.6:** We start by proving $(1) \iff (2)$. If $\rho(X) = 2$ we are done, by Corollary 1.4. If $\rho(X) \geq 3$, consider the natural representation $f : \text{Bir}(X) \to \text{GL}(N^1(X)_{\mathbf{R}})$.

We first prove $(1) \to (2)$. As $\mathcal{E}(X)$ is rational polyhedral, $X$ carries a finite number $N$ of prime exceptional divisors, by Corollary 3.4. Without loss of generality we can assume that all the extremal rays of $\mathcal{E}(X)$ are spanned by prime exceptional divisors. Indeed, here the assumption that $X$ belongs to one of the known deformation types can be dropped. Also, we do not need Theorem 1.2, hence, a priori, there could be some extremal rays generated by primitive, integral, isotropic classes. If $s$ is a birational self map of $X$ and $f(s)$ is the automorphism induced on $N^1(X)_{\mathbf{R}}$, we observe that $f(s)$ has order at most $N!$. Indeed, $f(s)$ acts on $\text{Neg}(X)$, hence an automorphism of $N^1(X)_{\mathbf{R}}$ is uniquely determined by $f(s)(\text{Neg}(X))$, because in $\text{Neg}(X)$ we can find a basis for the Néron-Severi space, since
\[ \text{Eff}(X) = \sum_{E \in \text{Neg}(X)} R^0[E] \] by assumption. Now we use Theorem 3.7 and Lemma 3.8 to conclude that Bir\((X)\) is finite.

Now we prove \((2) \Rightarrow (1)\). As Bir\((X)\) is finite, \(\overline{\text{Mov}(X)}^*\) is rational polyhedral, by Theorem 2.19 and Corollary 2.20. Also, the equality \(\overline{\text{Mov}(X)}^* = \text{Mov}(X) = \overline{\text{Mov}(X)}\) holds in this case. Hence, \(\overline{\text{Mov}(X)}^* = \text{Eff}(X) = \text{Eff}(X)\) is rational polyhedral, and we are done. We now prove \((2) \Leftrightarrow (3)\). Let \(O^+(N^1(X))\) be the group of isometries of \(N^1(X)\) preserving the positive cone \(\mathcal{E}_X\). Clearly \(O^+(N^1(X))\) has index 2 in \(O(N^1(X))\). Moreover, by [Mar11, Lemma 6.23], the image of \(\text{Mon}_{\text{Hdg}}^2(X)\) via the natural map \(\rho : \text{Mon}_{\text{Hdg}}^2(X) \to O^+(N^1(X))\) has finite index in \(O^+(N^1(X))\) and is isomorphic to a semi-direct product of \(W_{\text{Exc}}\) and \(\text{Mon}_{\text{Bir}}^2(X)/\ker(\rho)\). Now, the kernel of \(\rho\) is finite by [Ogu14, Proposition 2.4], and looking at the chain of inclusions

\[ W_{\text{Exc}} \subset \text{Im}(\rho) \subset O^+(N^1(X)) \subset O(N^1(X)), \]

the equivalence between \((2)\) and \((3)\) is easily checked.

The implication \((3) \Rightarrow (4)\) is obtained by \((3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (4)\). Suppose now that \(X\) carries a prime exceptional divisor and has Picard number at least 3.

The implication \((4) \Rightarrow (1)\) is nothing but the "only if" part of Corollary 3.4, thus all the statements are equivalent in this case, because we also have \((4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3)\).

\[ \square \]

**Remark 3.9.** Note that, given a projective IHS manifold \(X\) belonging to one of the known deformation classes, replacing Bir\((X)\) with Aut\((X)\) and the Kawamata-Morrison movable cone conjecture with the classical Kawamata-Morrison cone conjecture, stated for \(\text{Nef}^*(X) = \text{Nef}(X) \cap \text{Eff}(X)\) (see [MY15, Theorem 1.3] and [AV17, Section 5.2]), a similar argument to that given in the proof of \((1) \Leftrightarrow (2)\) in Corollary 1.6 can be used to prove that \(\text{Nef}(X)\) is rational polyhedral if and only if Aut\((X)\) is finite.

We recall the following definition.

**Definition 3.10.** Let \(X\) be a normal and \(\mathbb{Q}\)-factorial projective variety. We say that \(X\) is a Mori dream space (MDS) if the following properties hold:

1. Pic\((X)\) is finitely generated (equivalently, \(h^1(\mathcal{O}_X) = 0\));
2. \(\text{Nef}(X)\) is generated by the classes of finitely many semiample divisors;
3. there is a finite collection of small \(\mathbb{Q}\)-factorial modifications \(s_i : X \dashrightarrow X_i\), such that every \(X_i\) satisfies item (2) and

\[ \text{Mov}(X) = \bigcup_i s_i^*(\text{Nef}(X_i)). \]

The corollary below explains when a projective IHS manifold belonging to one of the 4 known deformation classes is a MDS.

**Corollary 3.11.** Let \(X\) be a projective IHS manifold, belonging to one of the 4 known deformation classes. Then the following conditions are equivalent:

1. \(X\) is a MDS,
2. Bir\((X)\) is finite,
3. \(0 < \rho(X) < 3\) and Eff\((X)\) is rational, or \(\rho(X) \geq 3\) and Neg\((X)\) is non-empty and finite.

**Proof.** (1) \(\Leftrightarrow\) (2). Suppose that \(X\) is a MDS. Then Bir\((X)\) is finite, by definition of MDS. Suppose now that Bir\((X)\) is finite. By [AV17, Theorem 5.3] the BBF square of the integral, primitive, and extremal classes of the Mori cones of the birational models of \(X\) is bounded, hence by [MY15, Corollary 1.5] the number of birational models of \(X\) is finite (up to isomorphism). This implies that \((3)\) in Definition 3.10 is satisfied. The base point free Theorem, the fact that any known IHS manifold satisfies Conjecture 2.15, and Remark 3.9 imply that also (2) in Definition 3.10 is satisfied. Thus \(X\) is a MDS.

(2) \(\Leftrightarrow\) (3). Follows directly from Corollary 1.6. \[ \square \]
The following remark has been pointed out by A. Höring.

**Remark 3.12.** Let $X$ be a projective IHS manifold belonging to one of the 4 known deformation classes, carrying a prime exceptional divisor $E$, with $\rho(X) \geq 3$. Let $X'$ be a birational model of $X$, from which we can contract the strict transform $E'$ of $E$ to a singular symplectic variety $Y'$, via $c : X' \to Y'$, where the exceptional locus of $c$ is $\text{Exc}(c) = E'$. Then $Y'$ carries a uniruled divisor. Indeed, the contractibility of $E$ from a birational model of $X$ follows from [Dru11, Proposition 1.4] (see also [BBP13, Proof of Theorem A]). By Corollary 3.4, $X$ carries a prime exceptional divisor $E_1 \not= E$. Then $c_*(E'_1)$ is a uniruled divisor on $Y'$, where $E'_1$ is the strict transform of $E_1$ via $X' \rightsquigarrow X'$. Indeed, $E'_1$ is a prime exceptional divisor, hence it is uniruled, by [Bou04, Proposition 4.7]. Thus also $c_*(E'_1)$ is uniruled.

We conclude this article with the following conjecture, which looks reasonable in light of our results.

**Conjecture 3.13.** Let $X$ be a projective IHS manifold of Picard number greater than 3. Then, either $\text{Neg}(X) = \emptyset$ and $\text{Eff}(X)$ is circular, or $\text{Neg}(X) \not= \emptyset$ and $\text{Eff}(X) = \sum_{E \in \text{Neg}(X)} R_{\geq 0}[E]$.

**References**

[AV17] Ekaterina Amerik and Misha Verbitsky. “Morrison-Kawamata cone conjecture for hyperkähler manifolds”. In: Ann. Sci. Éc. Norm. Super. 50.4 (2017), pp. 973–993 (cit. on p. 16).

[BBP13] Sébastien Boucksom, Amaël Broustet, and Gianluca Pacienza. “Uniruledness of stable base loci of adjoint linear systems via Mori theory”. In: Math. Z. 275 (2013), pp. 499–507 (cit. on p. 17).

[Bea83] Arnaud Beauville. “Variétés Kähleriennes dont la première classe de Chern est nulle”. In: J. Differential Geom. 18.4 (1983), pp. 755–782 (cit. on pp. 1, 3).

[Bou04] Sébastien Boucksom. “Divisorial Zariski decompositions on compact complex manifolds”. In: Ann. Sci. Éc. Norm. Super. 4th ser. 37.1 (2004), pp. 45–76 (cit. on pp. 8, 17).

[Bur05] William Burnside. “On criteria for the finiteness of the order of a group of linear substitutions”. In: Proc. Lond. Math. Soc. (3) s2-3 (1 1905), pp. 435–440 (cit. on p. 15).

[Dem92] J. P. Demailly. “Regularization of closed positive currents and Intersection theory”. In: J. Algebraic Geom. 1 (1992), pp. 361–409 (cit. on p. 8).

[Den21] Francesco Denisi. “Boucksom-Zariski and Weyl chambers on irreducible holomorphic symplectic manifolds”. 2021. URL: https://arxiv.org/abs/2106.03678 (cit. on pp. 1, 9, 10, 11, 13, 14).

[Dru11] Stéphane Druel. “Quelques remarques sur la decomposition de Zariski divisoriale sur les varietes dont la premiere classe de Chern est nulle”. In: Math. Z. 267.1-2 (2011), pp. 413–423 (cit. on p. 17).

[GHJ03] Mark Gross, Daniel Huybrechts, and Dominic Joyce. Calabi-Yau Manifolds and Related Geometries. Lectures at a Summer School in Nordfjordeid, Norway, June 2001. Springer-Verlag, 2003 (cit. on pp. 3, 5).

[HT09] Brendan Hassett and Yuri Tschinkel. “Moving and ample cones of holomorphic symplectic fourfolds”. In: Geom. Funct. Anal. 19 (2009), pp. 1065–1080 (cit. on p. 7).

[Huy03] Daniel Huybrechts. “The Kähler cone of a compact hyperkähler manifold”. In: Math. Ann. 326.3 (2003), pp. 499–513 (cit. on p. 5).

[Huy15] Daniel Huybrechts. Lectures on K3 surfaces. Available online, here: https://www.math.uni-bonn.de/people/huybrech/K3Global.pdf. Cambridge University Press, 2015 (cit. on pp. 3, 7, 13).
[Huy99] Daniel Huybrechts. “Compact Hyperkähler Manifolds: Basic Results”. In: *Invent. Math.* 135 (1999), pp. 63–113 (cit. on p. 3).

[IR90] K. Ireland and M. Rosen. *A classical introduction to modern number theory*. Springer-Verlag, 1990 (cit. on p. 11).

[Kaw88] Yujiro Kawamata. “Crepant Blowing-Up of 3-Dimensional Canonical Singularities and Its Application to Degenerations of Surfaces”. In: *Ann. of Math.* 2nd ser. 127.1 (1988), pp. 93–163 (cit. on p. 15).

[Kol99] János Kollár. *Rational curves on algebraic varieties*. Springer-Verlag Berlin Heidelberg, 1999 (cit. on p. 1).

[Kov13] Sandor Kovács. *The cone of curves of a K3 surface revisited*. 2013. URL: https://arxiv.org/pdf/1309.5562v1.pdf (cit. on pp. 1, 7, 10).

[Kov93] Sandor Kovács. “The cone of curves of a K3 surface”. In: *Math. Ann.* 300.4 (1993), pp. 681–691 (cit. on pp. 2, 7, 8).

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry I. Classical Setting: Line Bundles and Linear Series*. Springer Nature, 2004 (cit. on p. 7).

[Mar10] Eyal Markman. “Integral constraints on the monodromy group of the hyperkähler resolution of a symmetric product of a K3 surface”. In: *Internat. J. of Math.* 20 (2010), pp. 169–223 (cit. on pp. 6, 12, 13).

[Mar11] Eyal Markman. “A survey of Torelli and monodromy results for holomorphic-symplectic varieties, Proc. of the conference ‘Complex and Differential Geometry’”. In: *Springer Proc. Math.* 8 (2011), pp. 189–220 (cit. on pp. 4, 9, 10, 12, 14, 16).

[Mar13] Eyal Markman. “Prime exceptional divisors on holomorphic symplectic varieties and monodromy reflections”. In: *Springer Proc. Math.* 53.2 (2013), pp. 345–403 (cit. on pp. 2, 5, 6, 7, 12, 13, 14).

[Mar18] Eyal Markman. *The monodromy of generalized Kummer varieties and algebraic cycles on their intermediate Jacobians*. To appear in J. Eur. Math. Soc. (JEMS) 2018. URL: https://arxiv.org/abs/1805.11574 (cit. on pp. 6, 13).

[Mat17] Daisuke Matsushita. “On isotropic divisors on irreducible symplectic manifolds”. In: *Higher Dimensional Algebraic Geometry: In honour of Professor Yujiro Kawamata’s sixtieth birthday* 74 (2017), pp. 291–312 (cit. on p. 8).

[MM11] Eyal Markman and Sukhendu Mehrotra. “Hilbert schemes of K3 surfaces are dense in moduli”. In: *Math. Nachr.* 290.5-6 (Dec. 2011), pp. 876–884 (cit. on p. 6).

[MO22] Giovanni Mongardi and Claudio Onorati. “Birational geometry of irreducible holomorphic symplectic tenfolds of O’Grady type”. In: *Math. Z.* 300 (2022), pp. 3497–3526 (cit. on pp. 2, 8, 11, 12, 14).

[Mon16] Giovanni Mongardi. “On the monodromy of irreducible symplectic manifolds”. In: *Algebr. Geom.* 3 (3 2016), pp. 385–391 (cit. on pp. 6, 13).

[MR20] Giovanni Mongardi and Antonio Rapagnetta. “Monodromy and birational geometry of O’Grady’s sixfolds”. In: *J. Math. Pures Appl.* (9) 146 (2020), pp. 31–68 (cit. on pp. 2, 4, 8, 12, 14).

[MY15] Eyal Markman and Kôta Yoshioka. “A Proof of the Kawamata–Morrison Cone Conjecture for Holomorphic Symplectic Varieties of $K3^{[n]}$ or Generalized Kummer Deformation Type”. In: *Int. Math. Res. Not. IMRN* 2015 (24 2015), pp. 13563–13574 (cit. on pp. 10, 16).

[OGr03] K. G. O’Grady. “A new six dimensional irreducible symplectic variety”. In: *J. Algebraic Geom.* 12 (2003), pp. 435–505 (cit. on p. 1).

[OGr97] K. G. O’Grady. “The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface”. In: *J. Algebraic Geom.* 6 (1997), pp. 599–644 (cit. on p. 4).

[OGr99] K. G. O’Grady. “Desingularized moduli spaces of sheaves on a K3”. In: *J. Reine Angew. Math.* 512 (1999), pp. 49–117 (cit. on p. 1).

[Ogu14] Keiji Oguiso. “Automorphism groups of Calabi-Yau manifolds of Picard number 2”. In: *J. Algebraic Geom.* 23 (2014), pp. 775–795 (cit. on pp. 2, 9, 10, 15, 16).
[Rap06] Antonio Rapagnetta. “On the Beauville form of the known irreducible symplectic varieties”. In: Math. Ann. 340 (2006), pp. 77–95 (cit. on p. 11).

[Rie20] Ulrike Riess. “On the non-divisorial base locus of big and nef line bundles of $K3^{[2]}$-type varieties”. In: Proc. Roy. Soc. Edinburgh Sect. A 151.1 (2020), pp. 52–78 (cit. on p. 10).

[Yos16] Kōta Yoshioka. “Bridgeland’s Stability and the Positive Cone of the Moduli Spaces of Stable Objects on an Abelian Surface”. In: Adv. Stud. Pure Math. 69 (2016), pp. 473–537 (cit. on pp. 13, 14).

Institut Elie Cartan de Lorraine, F-54506 Vandœuvre-lès-Nancy Cedex, France
Email address: francesco.denisi@univ-lorraine.fr