Resolvent Estimates for Schrödinger Operators with Potentials in Lebesgue Spaces

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Abstract
We prove resolvent estimates in the Euclidean setting for Schrödinger operators with potentials in Lebesgue spaces: \(-\Delta + V\). The \((L^2, L^r)\) estimates were already obtained by Blair-Sogge-Sire, but we extend their result to other \((L^p, L^q)\) estimates using their idea and the result and method of Kwon-Lee on non-uniform resolvent estimates in the Euclidean space.

1 Introduction

Resolvent estimates for Schrödinger operators has always been of keen interest in the study of harmonic analysis and partial differential equations over the past 30 years or so. This interest originated from the consequential work of Kenig, Ruiz and Sogge \[10\] in 1986 that states:

On \(\mathbb{R}^n\) where \(n \geq 3\), if the pair of exponents \(r\) and \(s\) satisfy the conditions

(a) \(\frac{1}{r} - \frac{1}{s} = \frac{2}{n}\),

(b) \(\min\left\{\left|\frac{1}{r} - \frac{1}{2}\right|, \left|\frac{1}{s} - \frac{1}{2}\right|\right\} > \frac{1}{2n}\),

then there exists a constant \(C\), depending only on \(n\), \(r\) and \(s\), such that the following inequality holds:

\[\|u\|_{L^s(\mathbb{R}^n)} \leq C\|(\Delta + z)u\|_{L^r(\mathbb{R}^n)}, \quad u \in H^{2,r}(\mathbb{R}^n), \quad z \in \mathbb{C}.\]

Since then, efforts have branched into several different directions. A most notable one is the corresponding inequality on manifolds, where the Laplacian operator is replaced by the Laplace-Beltrami operator. In this branch we mention the pioneering work of Dos Santos Ferreira-Kenig-Salo \[4\], and also the prominent work of Bourgain-Sogge-Shao-Yao \[3\], of Shao-Yao \[12\] and of Huang-Sogge \[9\]. In recent years, Frank-Schimmer \[6\] proved the endpoint case of Ferreira-Kenig-Salo \[4\]'s result, and their rediscovery of a method in Gutierrez \[8\] has inspired a few later works.

Another important direction is to prove the resolvent inequality for other pairs of exponents. It was shown in \[10\] that the conditions (a) and (b) are necessary to obtain a uniform bound that
estimates of Kwon-Lee \cite{11}. Lebesgue spaces. Before stating our main theorem, we introduce in detail the paper on resolvent exponent pairs are also of use here and there. What is more, we allow the potential to be in other range of exponent pairs \cite{11} and the method in Blair-Sire-Sogge \cite{1} to prove the inequality as in Blair-Sire-Sogge \cite{1} but for exponent pair $p$. In her result, the non-uniform bound is in the form of a power of $z$. Frank \cite{5} proved some crucial $(L^p, L^q)$ estimates (he actually obtained stronger bounds in Schatten spaces). Recently, in a marvelous work, Y. Kwon and S. Lee \cite{11} found the whole region in the coordinate plane for the function $u$ has to be in $V$ is also an analogous result in the Euclidean case, where the only difference we make is that the projection operator for $\lambda$, where $\lambda$ means the geodesic ball of radius $r$ around $x$, and the integration is with respect to the volume element of the manifold. The Kato class $K(\mathbb{R}^n)$ on $\mathbb{R}^n$ is defined similarly.

\footnote{On manifold $M$, let for $r > 0$

$$h(r) = \begin{cases} |\log r| & \text{if } n = 2 \\ r^{4-n} & \text{if } n \geq 3. \end{cases}$$

A function $V(x)$ on $M$ is said to be in the Kato class $K(M)$ if

$$\lim_{r \searrow 0} \sup_x \int_{B_r(x)} h(d_g(x, y)) |V(y)| dy = 0,$$

where $d_g(x, y)$ denotes the geodesic distance between $x$ and $y$, $B_r(x)$ means the geodesic ball of radius $r$ around $x$, and the integration is with respect to the volume element of the manifold. The Kato class $K(\mathbb{R}^n)$ on $\mathbb{R}^n$ is defined similarly.
2 Kwon-Lee’s Work: Sharp Resolvent Estimates Outside of the Boundedness Range

In the coordinate plane, let $I = \{(x, y) \in [0, 1] \times [0, 1], y \leq x\}$ for notational convenience in the future. Define

$$
\mathcal{R}_0 = \begin{cases} 
\{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1, 0 \leq x - y < 1\} & \text{if } n = 2, \\
\{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1, 0 \leq x - y \leq \frac{2}{n}\} \setminus \{(1, \frac{n-2}{n}), (\frac{2}{n}, 0)\} & \text{if } n > 2.
\end{cases}
$$

(2)

This is the region for exponent pairs for which the resolvent operator norm $\|(-\Delta - z)^{-1}\|_{L^p \to L^q}$ is finite for any given $z \in \mathbb{C}\setminus[0, \infty)$.

Now we give the sharp bounds for the resolvent operator when $(\frac{1}{p}, \frac{1}{q})$ is in the above just-stated region. For $n \geq 2$, given $(\frac{1}{p}, \frac{1}{q})$, define

$$
\gamma(p, q) := \max\{0, 1 - \frac{n+1}{2} - \frac{1}{p} - \frac{1}{q}, \frac{n+1}{2} - \frac{n}{p} - \frac{n-1}{2}\}.
$$

(3)

To express $\gamma(p, q)$ more clearly, with a little calculation, we divide the region $I$ of $\mathbb{R}^2$ into four parts:

$$
U = \left\{(x, y) \in I, x - y \geq \frac{2}{n+1}, x > \frac{n+1}{2n}, y < \frac{n-1}{2n}\right\},
$$

(4)

$$
V = \left\{(x, y) \in I, 0 \leq x - y < \frac{2}{n+1}, \frac{n-1}{n+1}(1-x) \leq y \leq \frac{n+1}{n-1}(1-x)\right\},
$$

(5)

$$
W = \left\{(x, y) \in I, y < \frac{n-1}{n+1}(1-x), x < \frac{n+1}{2n}\right\},
$$

(6)

and the dual $W'$ of $W$. Set $C = (\frac{1}{2}, \frac{1}{2})$, $B = (\frac{n-1}{2n}, \frac{n-1}{2n})$ and $B' = (\frac{n+1}{2n}, \frac{n+1}{2n})$. For a number of points $L_1, L_2, \cdots, L_k$, let $[L_1, L_2, \cdots, L_k]$ denote the convex hull of them. For two points $M$ and $N$, we also use $[M, N]$ to represent the half open line segment connecting $M$ to $N$, $N$ excluded. $(M, N)$ and $(M, N)$ are defined similarly. With these points and notations, we define

$$
\mathcal{R}_1 = U \cap \mathcal{R}_0,
$$

(7)

$$
\mathcal{R}_2 = V \setminus ([B, C] \cup [B', C]),
$$

(8)

$$
\mathcal{R}_3 = W \cap \mathcal{R}_0.
$$

(9)

We can then express $\gamma_{p, q}$ case by case:

$$
\gamma_{p, q} = \begin{cases} 
0 & \text{if } (\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_1, \\
1 - \frac{n+1}{2} - \frac{1}{p} - \frac{1}{q} & \text{if } (\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_2, \\
\frac{n+1}{2} - \frac{n}{p} & \text{if } (\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_3, \\
\frac{n}{q} - \frac{n-1}{2} & \text{if } (\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_3'.
\end{cases}
$$

(10)

Finally, for $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_3'$ and $z \in \mathbb{C}\setminus[0, \infty)$, we set

$$
\kappa_{p, q}(z) = |z|^\frac{p}{q}(\frac{1}{2} - \frac{1}{q})^{-1} + \gamma_{p, q}\mathrm{dist}(z, [0, \infty))^{-\gamma_{p, q}}.
$$

(11)
Kwon and Lee conjectured in [11] that the following resolvent estimates hold whenever \((\frac{1}{p}, \frac{1}{q}) \in R_1 \cup R_2 \cup R_3 \cup R'_3\):

\[
\|u\|_{L^q(\mathbb{R}^n)} \leq C\kappa_{p,q}(z)\|(-\Delta - z)u\|_{L^p(\mathbb{R}^n)}, \quad z \in \mathbb{C}\setminus[0, \infty), \tag{12}
\]

where the constant \(C\) is independent of the complex number \(z\). (They also conjectured that \(12\) does not hold for \((\frac{1}{p}, \frac{1}{q}) \in [B, C) \cup [B', C)\).) As we pointed out above, they proved this conjecture for almost all pairs \((\frac{1}{p}, \frac{1}{q})\) in \(R_1 \cup R_2 \cup R_3 \cup R'_3\), leaving only two small triangles of higher dimensional cases unsolved. More precisely, let \(P = (\frac{1}{p_1}, \frac{1}{q_1})\) where

\[
\frac{1}{p_1} = \begin{cases} 
\frac{3(n-1)}{2(3n+1)} & \text{if } n \text{ is odd} \\
\frac{3n-2}{2(3n+2)} & \text{if } n \text{ is even} 
\end{cases} \tag{13}
\]

and \(Q = (\frac{1}{q_1}, \frac{1}{q_2})\) where

\[
\left(\frac{1}{q_1}, \frac{1}{q_2}\right) = \begin{cases} 
\frac{(n+3)(n-1)}{2(n+4n-1)} & \text{if } n \text{ is odd} \\
\frac{n^2+3n-6}{2(n^2+4n-2)} & \text{if } n \text{ is even} 
\end{cases} \tag{14}
\]

The complicated \((\frac{1}{p_1}, \frac{1}{q_1})\) and \((\frac{1}{q_1}, \frac{1}{q_2})\) arise from application of the oscillatory integral theorem of Guth, Hickman and Iopoulou [7] and that of the multilinear estimates of Tao [13]. Kwon and Lee showed that \(12\) holds for pairs of exponents \((\frac{1}{p}, \frac{1}{q})\) in the region \(R_1 \cup \bar{R}_2 \cup \bar{R}_3 \cup \bar{R}'_3\), where

\[
\bar{R}_2 = R_2 \setminus ([B, Q, C] \cup [P', Q', C]) \cap \{C\},
\]

and

\[
\bar{R}_3 = R_3 \setminus [B, P, Q].
\]

Note that when \(n = 2\), \(\bar{R}_2 = R_2\) and \(\bar{R}_3 = R_3\), so the two dimensional conjecture is no longer a conjecture now. A picture of the regions \(R_1, \bar{R}_2, \bar{R}_3\) and \(\bar{R}'_3\) in the coordinate plane is provided below (Figure 1).

We are going to use those complex numbers \(z\) so that the above resolvent estimates are uniform, i.e., the bounds do not depend on \(z\). To this end, we define the region

\[
\mathcal{Z}_{p,q} = \{z \in \mathbb{C}\setminus[0, \infty) : \kappa_{p,q} \leq 1\}. \tag{15}
\]

When \(z \in \mathcal{Z}_{p,q}\), it follows easily that

\[
\|u\|_{L^q(\mathbb{R}^n)} \leq C\|(-\Delta - z)u\|_{L^p(\mathbb{R}^n)}, \tag{16}
\]

where the constant \(C\) is independent of the complex number \(z \in \mathcal{Z}_{p,q}\). We also provide a picture of \(\mathcal{Z}_{p,q}\) in the complex plane when \((\frac{1}{p}, \frac{1}{q})\) belongs to different regions in the above classification (Figure 2). In that figure,

\[
\bar{R}_{3, \pm} := \{(x, y) \in \bar{R}_3 : \pm\left(x + y - \frac{n-1}{n}\right) > 0\},
\]

\[
\bar{R}_{3, 0} := \{(x, y) \in \bar{R}_3 : \left(x + y - \frac{n-1}{n}\right) = 0\}.
\]
3 Statement and Proof of the Main Theorem

We are now in a good position to state our main theorem.

**Theorem 1.** Suppose that \((\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_3'\) and \(z \in \mathcal{Z}_{p,q}\), and denote \(\frac{1}{\sigma} = \frac{1}{p} - \frac{1}{q}\). Suppose also that \(V(x) \in (L^\sigma(\mathbb{R}^n) + L^p(\mathbb{R}^n)) \cap \mathcal{K}(\mathbb{R}^n)\). Then there exists a constant \(C\), depending only on \(n, p\) and \(q\), such that the following inequality holds:

\[
\|u\|_{L^n(\mathbb{R}^n)} \leq C(\|(-\Delta + V - z)u\|_{L^p(\mathbb{R}^n)} + |z|^\frac{1}{2}\|u\|_{L^p(\mathbb{R}^n)}) \quad u \in \text{Dom}(-\Delta + V). \tag{17}
\]

**Proof of the Main Theorem**

To prove the main theorem, we basically follow the ideas of Blair-Sire-Sogge [1]. However, their proof uses the expression and estimates of the kernel of the resolvent operator in Kenig-Ruiz-Sogge’s 1986 paper [10], but the proof of Kwon-Lee [11] is a novel one, applying recent techniques such as the oscillatory integral estimates of Guth-Hickman-Iliopoulou [7] and the bilinear estimates of Tao [13]. For this reason, many lines need to be carefully justified to ensure that Blair-Sire-Sogge [1]’s proof combines well with this new method.

We deal with cases where the dimension \(n \geq 4\) first. In this situation, because \(C_0^\infty(\mathbb{R}^n)\) is an operator core for \(-\Delta + V\), it suffices to prove (17) for \(u \in C_0^\infty(\mathbb{R}^n)\).

Let

\[
F_z(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} \frac{\xi}{|\xi|^2 - z} d\xi \tag{18}
\]

denote the kernel for the resolvent operator \((-\Delta - z)^{-1}\). The idea is to break \(\mathbb{R}^n\) into small cubes on each of which the \(L^\sigma\) norm of \(V\) is small enough and then sum them up. To realize this, we introduce
Figure 2: Regions for the complex number $z$ to get uniform estimates in different cases
fulfill the ideas in Blair-Sire-Sogge \[1\], we need estimates of $H$

Let

| Proposition 1. |
\[
\begin{align*}
\text{be careful.}
\end{align*}
\]

Taking adjoints in the above equality, we get

\[
(\Delta - z)H_z(x, y) = \delta_y(x) + [\eta_h(x, y), \Delta]F_z(x, y).
\]

Taking adjoints in the above equality, we get

\[
I = H_z \circ (\Delta - z) + R_z,
\]

where $H_z$ is the operator whose kernel is $H_z(x, y)$ and $R_z$ corresponds to $-\eta_h(x, y, \Delta)F_z(x, y)$. To fulfill the ideas in Blair-Sire-Sogge \[1\], we need estimates of $H_z$ and $R_z$, and this is where we must be careful.

| Proposition 1. |
\[
\begin{align*}
\text{Let \((1, 1, q)\) and \(z \in \mathbb{C}\) be as in our main Theorem} \[1\]. \text{Then we have} \\
\|H_z f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \quad u \in C^\infty_0(\mathbb{R}^n),
\end{align*}
\]

where the constant $C$ depends only on $n, p$ and $q$ but not on $z$ and $\delta$.

\textbf{Proof.} We are going to show that $H_z$ bears the same estimates as the resolvent operator $F_z$:

\[
\|H_z f\|_{L^p(\mathbb{R}^n)} \leq C|z|^{\frac{1}{2}(\frac{1}{p} - \frac{1}{q}) - 1 + \gamma_{p,q}} \text{dist}(z, [0, \infty))^{-\gamma_{p,q}}\|f\|_{L^p(\mathbb{R}^n)}.
\]

Since we assume $z$ to be in the region

\[
Z_{p,q} = \{ z \in \mathbb{C}\setminus[0, \infty) : \kappa_{p,q} \leq 1 \},
\]

the desired conclusion follows. By dilation, we may assume that $z \in S^2\setminus\{1\}$:

\[
\begin{align*}
\|\mathcal{F}^{-1}\left\{ \frac{1}{|\xi|^2 - z}\right\} \hat{f}(\xi) \|_{L^p(\mathbb{R}^n)} & \leq C\|f\|_{L^p(\mathbb{R}^n)} \\
\|\mathcal{F}^{-1}\left\{ \frac{1}{|\xi|^2 - \xi} \hat{f}(\xi) \right\} \|_{L^p(\mathbb{R}^n)} & \leq C|z|^{\frac{1}{2}(\frac{1}{p} - \frac{1}{q}) - 1}\|f\|_{L^p(\mathbb{R}^n)}.
\end{align*}
\]

Denote $\frac{1}{|\xi|^2 - z} = m(\xi)$ for convenience. We decompose $m(\xi)$ into a part near the unit sphere (the one that makes the operator singular) and a part away from it. To do this, fix a small number $\epsilon > 0$ and choose a $C^\infty_0(\mathbb{R}^n)$ function $\rho(\xi)$ that equals 1 when $1 - \epsilon \leq |\xi| \leq 1 + \epsilon$ and equals 0 when $|\xi| \leq 1 - 2\epsilon$ or $|\xi| \geq 1 + 2\epsilon$. Let

\[
\begin{align*}
m_1(\xi) &= (1 - \rho(\xi))1_{B_1(O)} m(\xi), \\
m_2(\xi) &= (1 - \rho(\xi))1_{B_1(O)^c} m(\xi), \quad \text{and} \\
m_3(\xi) &= \rho(\xi)m(\xi),
\end{align*}
\]

where $B_1(O)$ stands for the unit ball. $H_z$ is broken up into three parts accordingly.
In the first place, since \( m_1(\xi) \) is smooth and compactly supported, it follows that the multiplier operator defined by \( m_1(\xi) \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) for any \( 1 \leq p \leq q \leq \infty \). Multiplying the kernel of this operator by the bounded function \( \eta_2(x, y) \) does not have any essential effect on the boundedness of this operator, so we see that this part of \( H_z \) is bounded from \( L^p \) to \( L^q \), a better result than needed.

Second, notice that \( |\xi|^2 - z | \geq 2 \epsilon + c^2 \) when \( |\xi| \geq 1 + \epsilon \), we have
\[
|\hat{c}_m^2 m_2(\xi)| \leq C_\alpha |\xi|^{-|\alpha| - 2}.
\]

Then we just do a standard work. Pick a Littlewood-Paley bump function \( \beta(t) \) on \( \mathbb{R} \) supported in \( \{ t \in \mathbb{R} : \frac{1}{2} \leq |t| \leq 1 \} \) such that \( \sum_{j=-\infty}^{\infty} \beta(2^{-j}t) = 1 \) for all \( t \neq 0 \). Let \( \beta_0(t) = \sum_{j=-\infty}^{-1} \beta(2^{-j}t) \), so \( \beta_0(t) \) is supported in \( |t| \leq 1 \). Therefore, \( \sum_{j} \beta(2^{-j}|\xi|) = 1 \) on the support of \( m_2(\xi) \). By an easy scaling argument together with the estimate (24), we deduce
\[
\| \mathcal{F}^{-1} \{ \beta(2^{-j}|\xi|) m_2(\xi) \} \|_{L^r(\mathbb{R}^n)} \leq C 2^{(n-\frac{\bar{\alpha}}{2} - 2)j},
\]
where the constant \( C \) is independent of \( j \). From this, we know that the multiplier operator defined by \( m_2(\xi) \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) whenever \( (\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_0 \). Indeed, when \( \frac{1}{p} - \frac{1}{q} < \frac{\alpha}{n} \), we apply Young’s inequality and sum over \( j \). When \( \frac{1}{p} - \frac{1}{q} = \frac{2}{n} \), summing over \( j \) does not work, but we can resort to Bourgain’s interpolation [2] to get restricted weak type estimates at these \( (\frac{1}{p}, \frac{1}{q}) \) first, and then apply real interpolation to get strong inequalities. See the appendix for Bourgain’s interpolation method. Once again, multiplying the kernel of the operator corresponding to \( m_2(\xi) \) by \( \eta_0(x, y) \) does not affect the boundedness of this operator, hence we see that this part of \( H_z \) is also bounded from \( L^p \) to \( L^q \).

Finally, we deal with \( m_3(\xi) \), the singular part. Fix a small number \( \theta_0 > 0 \). If \( z \) is in the set \( \{ z \in e^{i\theta} : \theta \in [\theta_0, 2\pi - \theta_0] \} \), then we easily have \( |\hat{c}_m^2 m_3(\xi)| \leq C_{\alpha, \epsilon, \theta_0} \), and by the same reasoning as for \( m_1(\xi) \), we settle \( m_3(\xi) \). If \( z \) is close to 1, note that taking Fourier transform, the effect of \( \eta_0(x, y) \) on \( m_3(\xi) \) amounts to convolving \( m_3(\xi) \) with a function whose \( L^1 \) norm is finite (of the form \( \delta^a g(\delta x) \)). To explain our proof, we outline the method of Kwon-Lee [11].

Write \( \xi = (\xi', \xi_n) \) where \( \xi' \in \mathbb{R}^{n-1}, \xi_n \in \mathbb{R} \). After decomposing by a partition of unity, discarding the easy smooth part, rotating (so that we may assume that the multiplier is supported near \( (0, \cdots, 0, -1) \)), and making the change of variable \( \xi_n \to \xi_n - 1 \) (now the multiplier is supported near the origin), we are able to express the essential part of \( m_3(\xi) \) as
\[
m(\xi', \xi_n) = \frac{1}{|\xi|} \phi \left( \frac{r(\xi', \xi_n)(\xi_n - \psi(\xi'))}{|\xi|} \right) \chi_0(\xi', \xi_n),
\]
where
\[
\phi(t) = \frac{1}{2t \pm i}, \quad \psi(\xi') = 1 - \sqrt{1 - |\xi'|^2}, \quad r(\xi', \xi_n) = \frac{1}{2}(\xi_n + \psi(\xi') - 2),
\]
and \( \chi_0 \) is a smooth function supported in a small neighborhood of the origin. By a further affine transformation, we can make \( \psi \in \Ell(N, \epsilon) \) and \( r \in \Mul(N, b) \). Here \( \Ell(N, \epsilon) \) and \( \Mul(N, b) \) are
two function classes defined for a large number $N > 0$, a small number $e > 0$ and some number $b > 0$ (to focus on our proof, we omit their definitions and the affine transformation which can be found in 2.2, 2.3 and Remark 7 in [11]).

Now we break $m(\xi', \xi_n)$ into the following sum using the $\beta$ and $\beta_0$ as in the treatment of $m_2(\xi)$ above:

$$
m(\xi', \xi_n) = \frac{1}{|\epsilon|} \phi \left( \frac{r(\xi', \xi_n)(\xi_n - \psi(\xi'))}{|\epsilon|} \right) \beta_0 \left( \frac{r(\xi', \xi_n)(\xi_n - \psi(\xi'))}{|\epsilon|} \right) \chi_0(\xi', \xi_n) + \frac{1}{|\epsilon|} \sum_{j=1}^{\log^\frac{1}{2}|\epsilon|} \phi \left( \frac{r(\xi', \xi_n)(\xi_n - \psi(\xi'))}{|\epsilon|} \right) \beta \left( \frac{r(\xi', \xi_n)(\xi_n - \psi(\xi'))}{2^{j-1}|\epsilon|} \right) \chi_0(\xi', \xi_n). 
$$

For notational convenience, we denote for $\lambda \geq 1$,

$$
m_0(\xi', \xi_n) = \phi \left( \frac{r(\xi', \xi_n)(\xi_n - \psi(\xi'))}{|\epsilon|} \right) \beta_0 \left( \frac{r(\xi', \xi_n)(\xi_n - \psi(\xi'))}{|\epsilon|} \right) \chi_0(\xi', \xi_n),
\quad m_\lambda(\xi', \xi_n) = \phi \left( \frac{r(\xi', \xi_n)(\xi_n - \psi(\xi'))}{|\epsilon|} \right) \beta \left( \frac{r(\xi', \xi_n)(\xi_n - \psi(\xi'))}{\lambda|\epsilon|} \right) \chi_0(\xi', \xi_n).
$$

Then the question transforms to analyzing the operators $m_0(D)$ and $m_\lambda(D)$, where in accordance with convention, $D = \frac{1}{\epsilon}(\partial_{\xi_1}, \ldots, \partial_{\xi_n})$. The key estimates related to these two operators are:

$$
(i) \|m_0(D)f\|_{L^p(\mathbb{R}^n)} \leq C_\epsilon^{\frac{1-n}{2} + \frac{p}{2}} \|f\|_{L^p(\mathbb{R}^n)},
\quad \|m_\lambda(D)f\|_{L^p(\mathbb{R}^n)} \leq C\lambda^{-1} \epsilon^{\frac{1-n}{2} + \frac{p}{2}} \|f\|_{L^p(\mathbb{R}^n)},
$$

for $p, q$ satisfying $\frac{1}{q} = \frac{n}{p} - \frac{1}{n+1}(1 - \frac{1}{p})$ and $q_2 < q \leq \frac{2(n+1)}{n-1}$. (This is Proposition 2.4 in [11]. Recall that $q_2$ is defined by (14) in Section 2.) The constant $C$ in these inequalities is independent of $\epsilon$, $\lambda$, $\psi \in \text{Ell}(N,e)$, $r \in \text{Mul}(N,b)$ and $f$.

$$(ii) \|m_0(D)f\|_{L^p(\mathbb{R}^n)} \leq C_\epsilon^{\frac{1-n}{2} + \frac{p}{2}} \|f\|_{L^p(\mathbb{R}^n)},
\quad \|m_\lambda(D)f\|_{L^p(\mathbb{R}^n)} \leq C\lambda^{1-\epsilon^{n-1}} \epsilon^{\frac{1-n}{2} + \frac{p}{2}} \|f\|_{L^p(\mathbb{R}^n)},
$$

for $p_1 < p < \infty$. (This is Proposition 2.5 in [11]. Recall that $p_1$ is defined by (13) in Section 2.) Again, the constant $C$ here is independent of $\epsilon$, $\lambda$, $\psi \in \text{Ell}(N,e)$, $r \in \text{Mul}(N,b)$ and $f$.

The above key estimates (27), (28) are proved using the recent oscillatory integral theorem of Guth-Hickman-Iliopoulou [7] and the bilinear estimate of Tao [13]. The rest of Kwon-Lee’s proof is then mere interpolation and summation in $j$.

Let

$$
A_0(\xi) = \phi \left( \frac{\tilde{r}(\xi', \xi_n)\xi_n}{|\epsilon|} \right) \beta_0 \left( \frac{\tilde{r}(\xi', \xi_n)\xi_n}{|\epsilon|} \right) \chi(\xi'),
\quad A_\lambda(\xi) = \phi \left( \frac{\tilde{r}(\xi', \xi_n)\xi_n}{|\epsilon|} \right) \beta \left( \frac{\tilde{r}(\xi', \xi_n)\xi_n}{\lambda|\epsilon|} \right) \chi(\xi'),
$$

where $\tilde{r}(\xi', \xi_n) = r(\xi', \xi_n + \psi(\xi'))$. The essential ingredients that the proofs of the two key estimates (27), (28) rely on are the following observations (Lemma 2.6 in [11]):
(i) For every $n-1$ dimensional multi-index $\alpha = (\alpha_1, \cdots, \alpha_{n-1})$ with $|\alpha| \leq N$, we have
\[
|\partial_\xi^\alpha A_0(\xi)| \leq C_\alpha, \quad |\partial_\xi^\alpha A_\lambda(\xi)| \leq C_\alpha \lambda^{-1} \tag{29}
\]

(ii) For every $n-1$ dimensional multi-index $\alpha$, $n$ dimensional multi-index $\zeta = (\zeta', \zeta_n)$ with $|\alpha| + |\zeta| \leq N$, and every natural number $l$, we have
\[
|\partial_\xi^\alpha \partial_{\xi_n}^l((\xi_n)^l \partial_\xi^\alpha A_0(\xi))| \leq C_{\alpha,\zeta}(\epsilon)^{-\zeta_n+1}, \\
|\partial_\xi^\alpha \partial_{\xi_n}^l((\xi_n)^l \partial_\xi^\alpha A_\lambda(\xi))| \leq C_{\alpha,\zeta} \lambda^{-1}(\epsilon \lambda)^{-\zeta_n+1} \tag{30}
\]

In these observations, the constants $C_{\alpha}$, $C_{\alpha,\zeta}$ are independent of $\epsilon$, $\lambda$, $\psi \in \mathbf{Ell}(N, e)$ and $r \in \mathbf{Mul}(N, b)$.

Notice the right sides of the above two estimates (29), (30) do not depend on $\xi$, so convolving with a function with finite $L^1$ norm does not affect these size estimates. And obviously, the affine transformations at the beginning has no impact on these estimates—we simply make the same affine transformations to the convolution integral. Therefore, (29), (30) still hold when the kernel of the operator $m_3(D)$ is multiplied by $\eta_3(x, y)$. It then follows that the key estimates (27) and (28) are also valid for our operator. By Kwon-Lee’s proof, we come to the conclusion that we have the same estimates for this part of $H_z$ as those for the multiplier operator corresponding to $m_3(\xi)$. The proposition is proved thereby.

**Proposition 2.** $(\frac{1}{p}, \frac{1}{q})$ and $z \in \mathbb{C}$ are still as in our main Theorem 7. For $R_z$ we have
\[
\|R_z f\|_{L^q(\mathbb{R}^n)} \leq C_{\delta} |z|^\frac{1}{2} \|f\|_{L^p(\mathbb{R}^n)}, \quad u \in C_0^\infty(\mathbb{R}^n). \tag{31}
\]

The constant $C_{\delta}$ depends on $\delta$ but not on $z$.

**Proof.**
\[
R_z(x, y) = -[\eta_3(x, y), \Delta]F_z(x, y) = (\Delta \eta_3(x, y))F_z(x, y) + 2\nabla \eta_3(x, y) \cdot \nabla F_z(x, y). \tag{32}
\]

Here, all derivatives are with respect to $x$. $(\Delta \eta_3(x, y))F_z(x, y)$ can be handled in exactly the same way as in Proposition 7, so we concentrate on $\nabla \eta_3(x, y) \cdot \nabla F_z(x, y)$.

Taking Fourier transform, the operator $\frac{\partial}{\partial z} F_z(x, y)$ amounts to multiplying on the Fourier transform side by $\frac{\xi}{|\xi|^2 - z}$. By dilation again, we may assume that $z \in S^2 \setminus \{1\}$:
\[
\left\| \mathcal{F}^{-1} \left\{ \frac{\xi}{|\xi|^2 - z} \cdot \hat{f}(\xi) \right\} \right\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \tag{31}
\]

Dilation again, we may assume that $z \in S^2 \setminus \{1\}$:
\[
\left\| \mathcal{F}^{-1} \left\{ \frac{\xi}{|\xi|^2 - z} \cdot \hat{f}(\xi) \right\} \right\|_{L^q(\mathbb{R}^n)} \leq C |z|^{\frac{1}{2} \frac{1}{p} - \frac{1}{2} - \frac{1}{2}} \|f\|_{L^p(\mathbb{R}^n)}. \tag{31}
\]

Note that this time, the power on $|z|$ is $\frac{1}{2}$ more than that in $\kappa_{p,q}$ because of the additional $\xi_i$. With this reduction, we can do as in Proposition 7 making the following modification as we proceed.

When dealing with $m_2(\xi)$ (the estimate becomes $|\partial_\xi^\alpha m_2(\xi)| \leq C_{\alpha} |\xi|^{-|\alpha|-1}$), note that the kernel of the corresponding operator is to be multiplied by a function supported in $\{(x, y) : \frac{\delta}{2} \leq |x-y| \leq \delta\}$
Since raising both sides to the $q$-th power, we have

\[ C \text{ is the constant in (36).} \]

Recalling that $C$ is a constant depending only on $n$.

We have by (21),

\[ u = H_z((-\Delta + V - z)u) + R_z u - H_z(V u). \]  

(34)

Recall that $H_z(x, y)$ is supported in the set \{ $(x, y) : |x - y| \leq \delta$ \} and that $R_z(x, y)$ in the set \{ $(x, y) : \frac{\delta}{2} \leq |x - y| \leq \delta$ \}. Taking $L^q$ norm on both sides of (34), where $q$ is as in the main theorem and applying Propositions 1 and 2, we have for $p, q$ and $z$ as before,

\[ \|u\|_{L^q(Q_j)} \leq C\|(-\Delta + V - z)u\|_{L^p(Q_j^*)} + C\delta|z|^\frac{1}{2}\|u\|_{L^p(Q_j^*)} + C\|Vu\|_{L^p(Q_j^*)}. \]

(35)

Raising both sides to the $q$-th power, we have

\[ \|u\|_{L^q(Q_j)}^q \leq C\|(-\Delta + V - z)u\|_{L^p(Q_j^*)}^q + C\delta|z|^\frac{1}{2}\|u\|_{L^p(Q_j^*)}^q + C\|Vu\|_{L^p(Q_j^*)}^q. \]

(36)

Since $V \in (L^\sigma(\mathbb{R}^n) + L^\infty(\mathbb{R}^n))$, we can choose a $\delta$ so small that $C_nC\|V\|_{L^\sigma(Q_j^*)}^q < \frac{1}{4}$ for any $j$. Here $C$ is the constant in (36). Recalling that $\frac{1}{p} = \sigma + \frac{1}{q}$, we have by Holder’s inequality

\[ \|Vu\|_{L^p(Q_j^*)} \leq \|V\|_{L^\sigma(Q_j^*)}\|u\|_{L^q(Q_j^*)}. \]

(37)

Finally we sum up the balls $Q_j$, and use inequalities (36), (37) and our choice of $\delta$ to get
estimates on $\mathbb{R}^n$:
\[
\|u\|_{L^q(\mathbb{R}^n)}^q \leq \sum_j \|u\|_{L^q(Q_j)}^q
\]
\[
\leq \sum_j \{C\|(-\Delta + V - z)u\|_{L^p(Q_j^q)}^q + C_\delta(|z|^{1/2}\|u\|_{L^p(Q_j^q)})^q + C\|V\|_{L^q(Q_j^q)}\|u\|_{L^q(B_j^q)} \}
\]
\[
\leq \sum_j \{C\|(-\Delta + V - z)u\|_{L^p(\mathbb{R}^n)}^q + C_\delta(|z|^{1/2}\|u\|_{L^p(\mathbb{R}^n)})^q
\]
\[
+ C\|V\|_{L^q(\mathbb{R}^n)}\|u\|_{L^q(\mathbb{R}^n)} \}
\]
(38)

Note that we can sum up the $Q_j^q$ on the right in the above process just as we do on the left because $q > p$.

Moving the $\frac{1}{2}\|u\|_{L^q(\mathbb{R}^n)}$ to the right of (38) and taking $\frac{1}{q}$-th power on both sides gives the desired result.

Now we turn to the proof of the two and three dimensional cases, when $C_0^\infty(\mathbb{R}^n)$ is not an operator core for $-\Delta + V$. We get around this difficulty by using test functions and duality. Here instead of using identity (21), we use identity (20):
\[
I = (-\Delta - z) \circ H_z + R_z.
\]  
(39)

To prove (17), it suffices to prove
\[
\int u\psi dx \leq C\|(-\Delta + V - z)u\|_{L^p(\mathbb{R}^n)} + |z|^{1/2}\|u\|_{L^p(\mathbb{R}^n)}) + \frac{1}{2}\|u\|_{L^q(\mathbb{R}^n)},
\]  
(40)

for any $u \in \text{Dom}(-\Delta + V)$ and $\psi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\|\psi\|_{L^{q'}(\mathbb{R}^n)} = 1$ where $q'$ stands for the conjugate exponent of $q$. Applying identity (39), we have
\[
\int u\psi dx \leq |\int u((-\Delta - z) \circ H_z)\psi dx| + |\int u(R_z\psi) dx|
\]
\[
\leq |\int ((-\Delta - z)u)(H_z\psi) dx| + |\int u(R_z\psi) dx|
\]
\[
\leq |\int ((-\Delta + V - z)u)(H_z\psi) dx| + |\int u(R_z\psi) dx| + |\int uV(H_z\psi) dx|.
\]  
(41)

By duality, $H_z$ is bounded from $L^q(\mathbb{R}^n)$ to $L^{q'}(\mathbb{R}^n)$, hence from Holder’s inequality we get
\[
|\int ((-\Delta + V - z)u)(H_z\psi) dx| \leq \|(-\Delta + V - z)u\|_{L^p(\mathbb{R}^n)}\|H_z\psi\|_{L^{q'}(\mathbb{R}^n)}
\]
\[
\leq \|(-\Delta + V - z)u\|_{L^p(\mathbb{R}^n)}\|\psi\|_{L^{q'}(\mathbb{R}^n)} + C\|(-\Delta + V - z)u\|_{L^p(\mathbb{R}^n)}.
\]  
(42)
For the same reason, since \( \|R_z\|_{L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)} = O(|z|^{\frac{1}{2}}) \),
\[
|\int u(R_z\psi)dx| \leq \|u\|_{L^p(\mathbb{R}^n)}\|R_z\psi\|_{L^q(\mathbb{R}^n)} \leq C|z|^{\frac{1}{2}}\|u\|_{L^p(\mathbb{R}^n)} \|\psi\|_{L^q(\mathbb{R}^n)} \leq C|z|^{\frac{1}{2}}\|u\|_{L^p(\mathbb{R}^n)}. \tag{43}
\]

It remains to tackle \( |\int uV(H_z\psi)dx| \). By Sobolev embedding theorem, \( L^\infty(\mathbb{R}^n) \subset \text{Dom}(\Delta + V) \), hence by the definition of the Kato class and the estimates of \( H_z(x,y) \) (see for detail),
\[
H_z(Vu) = \int H_z(x,y)V(y)u(y)dy, \quad u \in \text{Dom}(-\Delta + V)
\]
defines an absolutely convergent integral and is bounded in \( x \). It then follows easily that \( \int uV(H_z\psi)dx \) is absolutely convergent as well. Therefore, we may use Fubini’s theorem to get
\[
|\int uV(H_z\psi)dx| = |\int H_z(Vu)\psi dx| \leq \|H_z(Vu)\|_{L^q(\mathbb{R}^n)}\|\psi\|_{L^\infty(\mathbb{R}^n)} = \|H_z(Vu)\|_{L^q(\mathbb{R}^n)}. \tag{44}
\]

The question now becomes bounding \( \|H_z(Vu)\|_{L^q(\mathbb{R}^n)} \) by \( \frac{1}{2}\|u\|_{L^p(\mathbb{R}^n)} \). This is the same work as the one done in the higher dimensional cases: choose a lattice \( Q_j \) of cubes of side length \( \delta \), then if \( \delta \) is small enough, we have the following string of inequalities which finishes the our proof:
\[
\|H_z(Vu)\|_{L^q(\mathbb{R}^n)}^q \leq \sum_j \|H_z(Vu)\|_{L^q(Q_j)}^q \leq C \sum_j \|Vu\|_{L^p(Q_j)}^q \\
\leq C \sum_j \|V\|_{L^q(Q_j)}^q \|u\|_{L^p(Q_j)}^q \leq 2^{-q} \|u\|_{L^p(\mathbb{R}^n)}^q. \tag{45}
\]

**Appendix**

We state Bourgain’s interpolation method.

**Lemma 1.** Suppose that an operator \( T \) between function spaces is the sum of the operators \( T_j \):
\[
T = \sum_{j=1}^{\infty} T_j.
\]
If for \( 1 \leq p_1, p_2, q_1, q_2 \leq \infty \), there exist \( \beta_1, \beta_2 > 0 \) and \( M_1, M_2 > 0 \) such that each \( T_j \) satisfies
\[
\|T_j\|_{L^{p_1} \to L^{q_1}} \leq M_1 2^{-j\beta_1},
\]
and
\[
\|T_j\|_{L^{p_2} \to L^{q_2}} \leq M_2 2^{j\beta_2},
\]
then we have restricted weak type estimate for the operator \( T \) between two intermediate spaces:
\[
\|Tf\|_{L^{p,q}} \leq C(\beta_1, \beta_2)M_1^{1-\theta}M_2^{\theta}\|f\|_{L^{p,1}},
\]
where
\[
\theta = \frac{\beta_1}{\beta_1 + \beta_2}, \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.
\]
If the power of 2 in the bounds changes linearly as we go from \((\frac{1}{p_1}, \frac{1}{q_1})\) to \((\frac{1}{p_2}, \frac{1}{q_2})\), then the \((\frac{1}{p}, \frac{1}{q})\) in the conclusion is exactly the point at which the power becomes 0.
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