General Security Definition and Composability
for Quantum & Classical Protocols

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Abstract. We generalize the universally composable definition of Canetti to the Quantum World. The basic idea is the same as in the classical world. The main contribution is that we unfold the result in a new model which is well adapted to quantum protocols. We also simplify some aspects of the classical case. In particular, the case of protocols with an arbitrary number of layers of sub-protocols is naturally covered in the proposed model.

1 Introduction

In recent times, the analysis of complex cryptographic protocols has been getting more and more attention. A complex cryptographic protocol is a protocol that has a tree of sub-protocols in which any sub-protocol in the tree calls its children. The leafs are the primitives. In all approaches for the analysis of complex cryptographic protocols, one simply obtains security results for the primitives, then use these security results to obtain security results for the parent sub-protocols, and so on until one reaches the root of the tree. In all approaches, this might be done over a long period of time by many researchers that each publishes their respective results. In all approaches, a result about a given sub-protocol can be used in a large class of (complex) protocols that call this sub-protocol. The different approaches only differ in the type of security results that are obtained for these individual sub-protocols.

Essentially, Canetti’s universally composable definition [11] states that we can replace the sub-protocol with an associated ideal protocol together with a simulator, and the environment of the protocol (which includes the parent sub-protocol in the tree) will not notice the difference. The key point that explains the power of this approach is that, whenever we consider a sub-protocol in the tree, the lower level sub-protocols in the tree are already replaced by their associated ideal protocols. So, at every stage of the proof, we only consider sub-protocols that calls ideal protocols, and this makes a big difference when we try to prove the security of a protocol. Note that Canetti’s model only covered a tree of constant dept because of a technical issue that we will explain later.

We unfold this basic principle in a different model that is more adapted to quantum protocols. The idea that the universally composable theorem of Canetti is also valid in the quantum world was previously suggested in [16]. However, no proof of the theorem or even a model to state the theorem in the quantum world was provided. The quantum scenario has lead to many surprises in the past, especially because of the extra difficulty associated with entanglement, and so it was necessary to unfold carefully the universal composability theorem and the associated model in the quantum world. Our work was first reported in [8]. A protocol in our model is a set of circuits executed by distinct participants. A circuit is a set quantum registers and a partially ordered set (POSET) of quantum gates on these registers. As we will see in more details later (see section 3), the POSET can depend on the value of control registers. As in the classical case, the model must also define environments, ideal protocols and simulators.

Just deciding what is a useful security result for a sub-protocol in the tree (really only how to define the result, not how to obtain it) is already a difficult challenge. The principle of simulatability was proposed in the early 80’s to address this challenge [32]. Essentially, this principle says that if the view of the corrupted parties can be efficiently simulated by a machine which interacts with an ideal protocol, then these corrupted parties did not defeat the protocol. So, the idea that the analyzed protocol must be equivalent in someway to an ideal protocol was already there in the simulatability principle. This principle was used in the early 80’s to define and prove the security of multiparty computation protocols [32], but no composability result was proven at the time. The simulatability principle was successfully used again later [21,22,27] and issues such as composition were considered and partial composition theorems were obtained [21,27].
The universally composable definition of Canetti and its associated universal composability theorem only came a few years ago [11]. Independently, people at Zurich IBM obtained composability results in a framework that interestingly can be used with formal methods to automate security proofs [28,29]. Impressively, with independent work and under the inspiration of [11], they recently obtained a universal composability theorem in this framework. At the least at this stage, we use the more informal framework of Canetti. The key element is that these definitions were proven useful (for examples see [12,13,15,29,34,18]).

The notion of security in the quantum world has its own history which we briefly review in Appendix A. Essentially, the notion of security in the quantum world over the last 20 years used an unstructured or unrestricted framework for security proofs. In this unrestricted framework, there is no rule except common sense and mathematics: a security definition can be any property of an analyzed protocol that is potentially useful (when we traverse the tree bottom-up as previously mentioned). The purpose of this work, first reported in [8], is to introduce the more structured framework of universal composability in the quantum world. We will give an example of the difficulties that arise in the unrestricted framework in the next paragraph, but it will never replace the understanding that many researchers gained through the experience of proving or trying to prove the security of complex protocols (for example see [12,13,15,29,34,18].) The point is that the universal composability framework is not just a general proposal for “nice” security definitions. Whether a security definition is nice or not nice is a matter of taste. The universal composability framework is a practical framework that allows security results that seem otherwise difficult to achieve. Moreover, it is less error prone.

Of course, our point is that this general framework is also very much needed in the quantum world, perhaps even more than in the classical word in view of the additional difficulties related to quantum entanglement. As an example, consider the following key degradation problem. Quantum key distribution protocols requires classical authenticated channels. All security proofs for quantum key distribution assume that ideal classical authenticated channels are used. However, in practice, real classical authentication protocols are used, because there is no such a thing as an ideal authenticated classical channel. The difficulty is that classical authentication protocols require a small private key initially and very often this small private key will come from a previous quantum key distribution protocol. Let us assume that sometimes in the past, Alice and Bob met to exchange an ideal small private key and then went into an alternating sequence of authentication and quantum key distribution protocols, each protocol in this sequence calling the previous one. In this case the tree of sub-protocols has only one branch which corresponds to this sequence. The question is whether the key generated by this complex quantum key distribution protocol, this one branch tree, is still secure after \( n \) calls to classical authentication and quantum key distribution. This is just an example. The fact is that quantum key distribution will be used in many other types of application protocols. The universal composability of quantum key distribution is analyzed in [7]. Obviously, a similar question can be raised for other quantum protocols than quantum key distribution.

To summarize, our contribution is to unfold the framework of universal composability in a model that is well adapted to quantum protocols. It is also an interesting alternative model for the universal composability of classical protocols as well.

2 Basic concepts

In this section, we discuss examples of the five main concepts, the concepts of protocol, application, adversary, ideal protocol and simulator, and we give the ideas that are crucial in the composability theorem.

2.1 An example of protocol

We start with an example that is all classical, but it will be enough to explain the basic idea. It is a bit commitment from Alice to Bob that calls an ideal \( (1^2) \)-String-OT from Bob to Alice. In a \( (1^2) \)-String-OT protocol from Bob to Alice, Bob has two strings \( s[1] \) and \( s[2] \) as inputs whereas Alice as a bit \( c \) as input. The objective is that Alice receives the string \( s[c] \), but knows nothing about the other string whereas Bob does not know the bit \( c \), but gets an acknowledgement that Alice has chosen the bit \( c \). An ideal \( (1^2) \)-String-OT protocol from Bob to Alice can be constructed with the help of a trusted party, which we call Charlie. Bob sends the two strings
to Charlie. Charlie receives the bit $c$ from Alice and sends the string $s[c]$ to Alice. Charlie also acknowledges to Bob that Alice has chosen $c$, but he keeps $c$ private. This $(\frac{1}{2})$-String-OT protocol is formally defined in Appendix B.

In the bit commitment protocol, Bob and Alice call the $(\frac{1}{2})$-String-OT protocol. To commit a bit $x$ Alice chooses the string $\hat{s} = s[x]$ and Bob notes the acknowledgement. The acknowledgement (which is output by Bob in the string-OT protocol) is an important part of the commitment phase because the protocol would not be binding otherwise: Alice could wait until much after the end of the commitment phase to choose the bit $x$ in the $(\frac{1}{2})$-String-OT protocol. Later, to open the bit $x$, Alice announces $x$ and the string $\hat{s}$. Bob checks that $\hat{s} = s[x]$ to accept or reject the bit. This bit commitment protocol without the $(\frac{1}{2})$-String-OT protocol is called a module. This module calls the $(\frac{1}{2})$-String-OT protocol, but it does not contain it. This module together with the $(\frac{1}{2})$-String-OT protocol is the (complete) bit commitment protocol. The bit commitment module is formally defined in Appendix B.

Thus far, we used the names Alice, Bob and Charlie to refer to some circuits (also called role-circuits) that are executed inside a protocol. However, a participant executes role-circuits in more than one protocol. For example, Alice executes a circuit in the bit commitment protocol that we call Alice, but different protocols have different circuits with the name Alice. To avoid any confusion, in the future, we will often use the notation $SOT-Alice$, $BC-Alice$, etc. to distinguish these different circuits. This becomes important when we have a complex protocol with many sub-protocols (i.e. many modules), and this is typically the situation where composable security definitions are very useful.

### 2.2 Application protocols

A composable security definition essentially states that the analyzed protocol can be replaced by an ideal protocol and no application will see any significant difference. (We will introduce simulators later.) Let us consider that we ask some individual, a cryptoanalyst, to determine whether or not the proposed bit commitment protocol is as good as an ideal bit commitment. This cryptoanalyst chooses an application and an adversary circuit $A_r$ for every corruptible role-circuit $r$ in the protocol. An adversary circuit $A_r$, also denoted $Adv(r)$, interacts with the application and the protocol, but is only active when the circuit $r$ is corrupted. The circuit $r$ is turned off when it is corrupted. The adversary circuit $A_r$ can act in the name of the corrupted circuit $r$ and it has access to the registers that are normally accessed by $r$. For example, the cryptoanalyst will choose an adversary circuit $Adv(BC-Alice)$ to potentially replace $BC-Alice$ and an adversary circuit $Adv(BC-Bob)$ to potentially replace $BC-Bob$. With the help of a simple mechanism that we will describe later, the application can decide at run time which of these two circuits will be corrupted and replaced, if any.

To have a uniform viewpoint on all adversary circuits, we use the rule that every adversary circuit $A_r$ is associated with a role-circuit $r$ in the protocol. For example, in QKD, we define $Eve = Adv(Auxiliary)$ where Auxiliary is an extra circuit in QKD (not owned by Alice or Bob). Note that, in some QKD protocols, this extra circuit could have an active role in the (non corrupted) protocol, for example, to create and share EPR pairs between Alice and Bob. The set of adversary circuits is called the adversary.

The application is a set of role-circuits that provides inputs to the bit commitment protocol and receive outputs from this bit commitment protocol. The application can also communicate with the adversary circuits that are active. The application together with the adversary is called the environment. So, the environment contains many circuits. Some of these circuits belong to the application. Some others of these circuits belong to the adversary. The environment together with the protocol analyzed is called the overall setting. Intuitively, the criteria is that, for every environment, the distribution of a test bit $Z$ is close to an ideal distribution (associated with an ideal protocol). So, the cryptoanalyst must pick special applications that can dynamically corrupt circuits, interact with the associated adversary circuits and, finally, output a test bit $Z$. For example, it can be a coin toss protocol with additional circuits that corrupt role-circuits (i.e., turn them off and activate their associated adversary circuits individually) in the protocol using a simple method described later. The output bit $Z$ of this application can be the outcome of the coin toss.

As we previously mentioned, for a general composable definition, to make sure that no security aspect is ignored, the cryptoanalyst must consider all possible applications in a large class of applications that may call
this bit commitment. However, to be concrete, let us only test how well the bit commitment protocol does when it is called by a coin toss protocol. In this coin-toss protocol, Alice picks a bit $x$ uniformly at random and commits this bit to Bob. Bob waits for the acknowledgement, then picks a bit $y$ uniformly at random and announces this bit to Alice. Alice opens the bit $x$ and outputs the bit $w = x \oplus y$ (the sum modulo 2). Bob outputs the bit $w = x \oplus y$ if Alice opening is accepted and opens $y$ otherwise. The coin-toss module that calls bit commitment is formally defined in Appendix B. This coin-toss protocol does not corrupt any party and does not have any mean to corrupt a party anyway. So, the cryptoanalyst must pick a variation on this coin toss protocol that corrupt a party.

Let $w_A$ and $w_B$ be the respective output of Alice and Bob in the coin toss. With a simple mechanism that we will describe later, a coin toss application can corrupt either Alice or Bob or neither of them. A coin toss application can corrupt Bob and set $Z = w_A$ to test the security against Bob who attempts to find out $x$ before the opening and create a bias on $w_A$. Another coin toss application can corrupt Alice and set $Z = w_B$ to test the security against Alice who attempts to change the value of $x$ to create a bias on $w_B$. Intuitively, the criteria associated with these two applications, in both cases, is that there should be no bias on $Z$. In this way, we have the two fundamental properties of bit commitment: security against Alice and security against Bob.

2.3 Why we want to consider a general class of applications?

The above discussion suggests that an interesting definition for bit commitment can be obtained even if the cryptoanalyst considers only coin-toss based applications. If the coin toss protocol is as good when the analyzed bit commitment is called as when an ideal bit commitment is called, then we must have an interesting bit commitment. Really, we must have some non-trivial form of bit commitment because otherwise the coins toss would be biased. If the bit commitment is not binding, a corrupted Alice can create a bias on Bob’s output. If it is not concealing, a corrupted Bob can create a bias on Alice’s output. So, why do we suggest to consider all applications? We propose to consider all applications because, even if we have a specific application in mind, to prove the security of an application protocol, it is often useful to have stronger results about the protocols that are called by this application protocol. So, the philosophy behind the composability theory is to use strong security definitions that consider all applications so that at each level the job is easier. As we will see, the job will be easier because the security statement is very strong: it says that secure sub-protocols can be replaced by ideal protocols. This philosophy can be useful even if at every level in the tree we have a specific application in mind, and are not interested in all application protocols. In other words, this is not just a generalization because we would like to cover all possible application protocols. Moreover, a proof that consider all applications is not necessarily more complicated because it can be easier to avoid technical details that are associated with a specific application.

A related point is that we must not assume that a few applications are enough to capture all the security issues associated with a given task. For example, it is not because the two coin-toss applications that are discussed above cannot distinguish between the analyzed bit commitment protocol and an ideal bit commitment that the same is true for all applications. The two coin toss applications are not necessarily representative of all possible applications that might call the analyzed bit commitment protocol. Composability is a strong and natural requirement because it says that no application in a large class of considered applications can tell the difference between the analyzed bit commitment protocol and an ideal bit commitment protocol, not only specific applications such as the coin toss applications. Of course, it is useful to consider large but yet restricted class of environments. This implies a restriction on the applications as well. For example, one may consider computationally restricted environments. Also, typically one must consider environments in which the application can only corrupt some fraction of the circuits. It is also useful to consider other types of restrictions on the environments.

2.4 Why the applications output just a single test bit?

Another subtle issue is the fact that our proposed criteria depends only on the probability distribution of a single test bit $Z$. Our result easily generalizes to the case where a test string $\mathbf{Z}$ is used instead of a single bit as long as the measure $d(P_Z, P_Z^{ideal}) \in [0, 1]$ that is used to compare the actual distribution with an ideal
distribution respects the triangle inequality. However, this generalization is problematic because, to be useful, this generalization will have to be applied systematically for all protocols, and it is more convenient to consider a single bit. Fortunately, we see from the coin toss applications that it seems that a single bit approach does already a very good job. Moreover, even if we were to accept the principle that we should compare the distribution of two strings, in the computational scenario, we will naturally be back to the one bit case. Indeed, it is typical in the computational scenario to compare two string distributions \( P_Z \) and \( P_{Z^{ideal}} \) by considering a polynomial time machine that receives one of these two strings as an input and returns a single test bit \( Z \). If for all these polynomial time machines, the distribution of the test bit \( Z \) is the same whether the string \( Z \) comes from the ideal case or the analyzed case, we say that the two cases are computationally indistinguishable. So, if we consider this polynomial machine as a part of an application, this approach brings us back to the one bit case. In a non computational scenario, the situation is similar: the single bit case can be used as a natural way to implement the string case, that is, to compare the distributions of \( P_Z \) and \( P_{Z^{ideal}} \). More precisely, it can be shown that if, for all \( X \in \{0,1\}^m \), we can consider the application that computes the parity bit \( X \otimes Z \) of the random string \( Z \), and require that the probability distribution \( P_{X \otimes Z} \) behaves as in the ideal case. If all the single bit distributions \( P_{X \otimes Z} \) behave as in the ideal case, so does the distribution \( P_Z \). More precisely, it can be shown that if, for all \( X \in \{0,1\}^m \), \(|P_{X \otimes Z}(0) - P_{Z^{ideal}}(0)| \leq \epsilon \), then \( \|P_Z - P_{Z^{ideal}}\|_2 \leq 2\epsilon \), where \( \|\| \) is the \( L_2 \) norm.

2.5 The model: an example

We have seen two coin toss applications based on the coin toss protocol. Here we reconsider these applications to illustrate in more details the essential of our model and the terminology used. The analyzed protocol is again the bit commitment protocol with two circuits \( BC-Alice \) and \( BC-Bob \) for the bit commitment module and three circuits \( SOT-Alice, SOT-Bob \) and \( SOT-Charlie \) for the string-OT protocol. The application uses two circuits \( CT-Alice \) and \( CT-Bob \) for the coin toss protocol. The adversary contains four adversary circuits \( Adv(BC-Alice), Adv(BC-Bob), Adv(SOT-Alice) \) and \( Adv(SOT-Bob) \) that are associated with the corruptible circuits in the protocol. We recall that \( SOT-Charlie \) is not corruptible. The application has access to the test bit \( Z \) and to four corruption registers: \( C(BC-Alice), C(SOT-Alice), C(BC-Bob) \) and \( C(SOT-Bob) \). In our example, only \( BC-Alice \) will be corrupted. Therefore, only the corruption register \( C(BC-Alice) \) and the adversary circuit \( Adv(BC-Alice) \) are needed, but to illustrate the general situation we included all four corruption registers and all four adversary circuits. With the help of the corruption registers, the application decides when to corrupt a circuit in the protocol and which one will be corrupted. We assume that every individual gate in each of these circuits is conditioned by its associated corruption register.

Because it has full control over the corruption registers, the application can turn these corruptible circuits on and off, and replace them by their respective adversary circuit when they are off. In general, the gates of a corruptible circuit \( r \) in the analyzed protocol are conditioned by a corruption register \( C_r \). The gates of the associated adversary circuit \( A_r \) are also conditioned by the corruption register \( C_r \), but the other way around: when the circuit \( r \) is on, \( A_r \) is off and vice-versa.

Note that the worst application (against any analyzed protocol) is the trivial application where every role-circuit simply forwards back and fourth every input/output message between the analyzed protocol and an adversary circuit such as Eve that is always active. This means that when we analyze the security of a protocol and want a security statement that is independent of the application protocol, it is enough to only consider this trivial case. However, to be concrete, here we made the choice to consider a less trivial application protocol that includes the coin toss module with \( CT-Alice \) and \( CT-Bob \), and we will be consistent with this choice.

Note that, in the partial ordering of the gates in the circuit \( CT + BC + String-OT \), the output \( w_A = x \oplus y \) (in \( CT \)) does not have to happen after the opening (in \( BC \)). This is because Bob announces \( y \) before the opening and the generation of the random bit \( x \) does not have to occur after the opening. Note that, in a given execution, \( w_A = x \oplus y \) might occur after the opening, but this is irrelevant because only the constraints that are imposed on the adversary by the partial ordering of the gates that is defined by the circuit \( CT + BC + String-OT \) matters. Here is how the overall setting proceeds. The output bit \( w_A = x \oplus y \) is sent to the environment of \( CT \) as in the honest \( CT \) protocol. If \( w_A = 1 \), the application protocol corrupts \( BC-Alice \), that is, it turns off \( BC-Alice \) and turns on \( Adv(BC-Alice) \) by flipping the corruption register \( C(BC-Alice) \). At this point, the adversary circuit
Adv(BC-Alice) has access to all the registers of the corrupted circuit BC-Alice, i.e., \( x \) and \( \hat{s} = s[x] \), and can execute the opening in the name of Alice. This adversary circuit computes \( x' = x \oplus 1 \), picks a uniformly picked random string \( t \in \{0, 1\}^k \) and announces \((x', t)\) to BC-Bob instead of \((x, s[x])\) as it would normally happen if Alice was not corrupted. The application sets \( Z = w_B \). It is not hard to see that \( \Pr(Z = 0) = (1 + 2^{-k})/2 \).

2.6 The composability security definition

In the previous example, the environment has created a bias of \( 2^{-k}/2 \) on \( w_B \) toward 0. This bias on \( w_B \) toward 0 is not in itself very impressive because we had the power to choose the application that we want in this environment. In particular, in the application, we were free to directly set \( w_B = 0 \). However, this bias is interesting because it allows the environment to distinguish between the (analyzed) bit commitment protocol and an ideal bit commitment protocol. We will define in details what is an ideal protocol later. It is essentially a protocol that executes the task perfectly with the help of a trusted party and perfect channels. The goal of the cryptoanalyst (in other words, the goal of an attack) is not to create a bias on \( w_B \) or \( w_A \) or on any other bit, but to find a test bit \( Z \) that detect any difference between the analyzed protocol and the ideal protocol. Even if Alice is corrupted, as long as Bob is not corrupted, there should be no bias at all in the output \( w_B \) of the cointoss if an ideal bit commitment is used in this coin toss application. On the other hand, we just saw that there exists a bias if the analyzed bit commitment is used. Such a distinction between the ideal case and the analyzed case would not show up if we used an application in which Bob sets \( w_B = 0 \) systematically, because \( w_B = 0 \) would systematically hold on both the analyzed and the ideal sides. This example is also interesting because the set of corrupted parties depends on random values in the protocol. In this example, because the protocol is perfectly secure against Bob, the cryptoanalyst could have corrupted Alice at the beginning — no one else is interesting to corrupt. However, it is not hard to conceive an example where there is an advantage to decide later who should be corrupted.

To detect a bias toward 0 on \( w_B \), we saw that this application corrupts Alice (i.e. gives her some strategy to create a bias on \( \hat{x} \) toward \( \hat{x} = y \)). For this application, the criteria that tolerates no bias is \( \Pr(Z = 0) = 1/2 \). (In practice, \( |\Pr(Z = 0) - 1/2| \leq \epsilon \), for some small \( \epsilon > 0 \) would be fine.) However, we recall that we want to consider all possible application protocols. The cryptoanalyst could have designed a biased coin toss application where Alice picks \( x \) with distribution \((p_0, p_1) = (1/4, 3/4)\) and Bob picks \( y \) with distribution \((p'_0, p'_1) = (3/4, 1/4)\). In this case, even though BC is almost perfect, the distribution of \( w_B \) significantly depends on the adversary circuit. Therefore, the criterion on the distribution of \( Z \), which is used to accept or reject the analyzed bit commitment protocol, must depend on the environment.

The above discussion suggests that a general criterion on the probability distribution of the bit \( Z \) is to require that, for every environment, the probability \( \Pr(Z_{\text{analyzed}} = 0) \equiv \Pr(Z = 0) \) in the analyzed case is close to the probability \( \Pr(Z_{\text{ideal}} = 0) \) in the ideal case:

\[
|\Pr(Z_{\text{Analyzed}} = 0) - \Pr(Z_{\text{Ideal}} = 0)| \leq \epsilon
\]

(1)

where \( \epsilon > 0 \) depends on the security parameter. This criterion depends on the environment because \( \Pr(Z_{\text{Ideal}} = 0) \) depends on both the environment and the ideal protocol. However, we admit that, at this point, this criterion is not entirely defined and may seem too strong, too weak or simply obscure because we have not yet really explained how the reference probability \( \Pr(Z_{\text{Ideal}} = 0) \) can be computed, that is, we have not yet really defined \( Z_{\text{Ideal}} \).

An obvious problem in the definition of \( Z_{\text{Ideal}} \) is that the internal communication in the ideal protocol is usually completely different than the internal communication in the analyzed protocol. To be useful, an ideal protocol should be as simple as possible. An ideal protocol can use hidden (private, authenticated and no traffic analysis) channels, and a useful ideal protocol will typically do so, whereas the analyzed protocol might not have access to hidden channels. We cannot ignore the internal communication because the adversary part of the environment can access it. If the adversary waits for some message and this message does not come because it is not generated by the ideal protocol, the adversary will jam. If say \( Z \) is initially set to 0, it will remain set to 0 whereas, on the analyzed side, the environment can be designed to swap \( Z \) to 1 with some probability. In
this way, the analyzed protocol will be rejected because criteria (1) is not respected. With this naïve approach, even good protocols will be rejected.

The solution to this difficulty is to consider a game in which the players are (1) the environment and (2) a simulator that attempts to generate the information that is not generated by the ideal protocol but is normally generated in the analyzed protocol and expected by the environment. The ideal protocol and the simulator, both together, constitute an “extended ideal protocol”. The environment is the same in both sides. The environment tries to distinguish the analyzed protocol from the extended ideal protocol in accordance with the criteria (1). If the environment wins, the protocol is not secure. The simulator tries to compensate for

- the missing communication that would normally be generated by the non-corrupted parties in the analyzed protocol and
- the registers that are normally owned by the corrupted parties in the analyzed protocol

so that the extended ideal protocol and the analyzed protocol cannot be distinguished. If the simulator wins, the protocol is secure. Note that the missing interaction that would normally occur between the environment and non-corrupted parties in the analyzed protocol can be separated in two categories. First, there is the interaction between the environment and the registers that are exchanged between the non-corrupted parties in the protocol. This first type of information depends on the fact that the channels are not private or not authenticated, etc. Second, there is the interaction that occurs because an adversary circuit \( A_r \) acts on behalf of a corrupted circuit \( r \). This second type of interaction can occur even if the channels are perfect, authenticated, etc. The point here is that the simulator should compensate for these two types of interaction.

An interesting question can be raised. Why not allowing the adversary circuit to be different on the ideal side? It turns out that this alternative definition is almost equivalent to the above definition (See Appendix C for details.)

\[ \text{2.7 The main idea behind composability} \]

Thus far, with the help of an example, we discussed a general approach to define the security of a protocol, but we have not explained the concept of composability. This situation is natural because, historically, the notion of simulator and ideal protocol were first used in security definitions independently of the concept of composability. It is much after that the concept of composability was explained. Here, we use our example to illustrate the basic idea behind composability, and its relationship with the proposed definition.

In our example, the \( \text{SOT} \) sub-protocol is an ideal functionality with a trusted circuit \( \text{SOT-Charlie} \). A more complex bit commitment protocol would be obtained if we still used the module \( \text{BC} \), but replaced the \( \text{SOT} \) ideal sub-protocol with a real sub-protocol \( \text{RealSOT} \). Now, suppose that we have successfully shown that \( \text{RealSOT} \) securely realizes the ideal protocol \( \text{SOT} \) in accordance with our proposed security definition. Suppose in addition that we have successfully shown that the analyzed bit commitment protocol which call the ideal \( \text{SOT} \) securely realizes an ideal bit commitment protocol. It is much easier to show the security of \( \text{BC} \) when it calls the ideal sub-protocol \( \text{SOT} \) than if it called the real sub-protocol \( \text{RealSOT} \) instead. Great, but how useful are these security results if we want to obtain the security of the real bit commitment protocol that calls the real sub-protocol \( \text{RealSOT} \)? The main difficulty is that the security criteria says that the real sub-protocol \( \text{RealSOT} \) can be replaced by the ideal sub-protocol \( \text{SOT} \), but with an additional simulator. The simulator is a problem because we have proven the security of \( \text{BC} \) which calls the ideal \( \text{SOT} \) without simulator. The main idea behind the composability definition is that, when we proved the security of bit commitment protocol that calls the ideal \( \text{SOT} \), we considered all environments and, therefore, it is possible to consider that the simulator is a part of the environment. We see that it is important for the composability aspect of the security definition that the simulator can be interpreted as a part of the environment. This is the main requirement for composability. It is also important that the simulator together with the ideal protocol can be interpreted as a protocol, called the extended ideal protocol, because the original environment (without the simulator) expect an interaction with a protocol. However, irrespectively of composability, this second requirement is a minimal requirement for any simulator that is used to prove the security of a protocol \( \text{RealSOT} \). It is not specifically related to the composability aspect of the security definition.
3 Model for Universal Composability

This Section describes the formal model used to define the composability of classical and quantum protocols. The motivation for this model was provided in the previous Sections. It should be pointed out that in practice a protocol corresponds to some source and the protocol in execution is a different concept, a bound protocol. The source code does not have fixed participants. They are determined in the binding. For simplicity, we assume that the deployment of the source code into a bound protocol is already executed and we analyse the bound protocol. This is natural in a circuit based model. To simplify the terminology, because we never use the concept of a source code anyway, the bound protocol that is analysed is simply called a protocol.

3.1 Protocols

A quantum gate is a unitary transformation on a set of registers. A classical gate is a permutation on the set of classical values of the registers, a special kind of quantum gate. (The composability theory works fine if a classical gate is any function, not necessarily a permutation, but then we lose the fact that a classical gate is a special kind of quantum gate.) A classically controlled gate is a quantum gate of the form \( \sum_x |x\rangle^X \langle x| \otimes U^Y_x \). The unitary transformation \( U^Y_x \) is the controlled part of the controlled gate, the classical register \( X \) is the control register and the register \( Y \) is the target register. For simplicity, we assume that the control register is always a single classical bit. In other words, whether the controlled part is executed or not is always precomputed into a single classical bit, the control bit.

A classical register can be accessed by classical gates and by quantum gates but, in this latter case, only as a control register or to store the outcome of a quantum measurement (which can be considered as a kind of quantum gate). A quantum measurement always use a fresh classical register initially set to zero to store its outcome: no other gate can access this register before the quantum measurement. Therefore, a classical register can receive the random outcome of a quantum measurement as initial value, but thereafter it can only be modified deterministically through classical gates.

A circuit is a set of gates with a partial ordering that depends on the initial value of its control registers. For every initial value of the control registers, the partial ordering must respect a constraint that we now describe. For every initial value of every control register, we consider that a controlled gate accesses its target registers only if its controlled part is active, that is, distinct from the identity. The constraint is that, for every value of the control registers, every pair of gates that access a same classical or quantum register must be an ordered pair in the partial ordering. To compute the complexity of a circuit, we only consider controlled gates where the controlled part is active. Note that such a complexity is a value that depends on the initial value of the control registers.

A module-role-circuit is a circuit executed by a given participant. A protocol is the union of many module-role-circuits. Each participant executes one or more module-role-circuits in the protocol. Two distinct module-role-circuits can share a common register called a channel register. As we will see, the adversary can also access these channel registers. A communication gate of a module-role-circuit \( r \) is a gate of \( r \) that accesses a channel register of \( r \) and an activation register associated to this gate. A transmission corresponds to a communication gate of a sender module-role-circuit that accesses a channel register and, next, the activation register of a (thus subsequent) communication gate of a recipient module-role-circuit that accesses the same channel register. Only a communication gate of a module-role-circuit \( r \) or a (non-communication) gate of an adversary circuit can access a channel register. To allow synchronisation there are special gates that are not executed until after an associated activation register initially sets to SLEEP is set to WAKE-UP by another gate. This corresponds to the standard approach used in distributed computing in which a procedure sleeps until it is awaken by another procedure at an appropriate time.

In a well-defined protocol, every channel register has a channel-type and a channel-register-ID that includes the recipient-ID, the sender-ID and any other information necessary to uniquely identify the channel registers. The gates of an adversary circuit can access a channel register, but it must be in accordance with its channel-type. Similarly, we assume that the use of communication gates always respect the constraints imposed by the channel-type of the channel-register. For example, if the channel-register is classical and authenticated, the adversary can read it, but cannot change its value. If the channel register is quantum and authenticated,
the adversary cannot access it. In both cases, the associated activation registers are also authenticated, which means that only the legitimate module-role-circuits can set them to WAKE-UP.

Note that, without channel-types, a protocol specifies no constraint on the relative order of gates that belong in distinct module-role-circuits: the sender-ID and the recipient-ID in the channel-register-ID do not have to respectively correspond to the actual sender and the actual recipient. If the channel registers are authenticated, an order is usually implicitly imposed by the protocol with the help of the activation registers. However, no real channel register is perfectly authenticated. In reality, the order is not really fixed: it is only that some test fails if this implicit order is not respected. Therefore, the union of all module-role-circuits in a real protocol is not really a circuit: two communication gates that access a given channel register do not have a fixed order.

A corruptible circuit $r$ is a module-role-circuit that contains a special bit $C_r$ called the corruption register. Every gate in the corrupted circuit $r$ first reads the corruption register $C_r$ and do nothing else if the corruption register is on. Formally, we consider that every gate $G$ of a corruptible circuit $r$ has the form

$$G = |0\rangle^{C_r} |0\rangle \otimes G^{\text{honest}} + |1\rangle^{C_r} |1\rangle \otimes I.$$ 

As discussed in Appendix D, it is not hard to consider different, more nuanced, types of corruption beyond fully corrupted and not corrupted at all.

A valid environment $E$ for a protocol $P$ is a choice of order for every pair of communication gates of $P$ that share a channel register (so that the protocol becomes a circuit in itself) and an additional circuit with its own set of registers $R_E$ that can also access the channel registers and the corruption registers of $P$ and is such that $E + P$ is a circuit (i.e., for every possible value of the control registers, the union of the two partially ordered set of gates is also a partially ordered set of gates). The environment contains a special output register $Z$ and an adversary circuit $A(r)$ for every corruptible circuit $r$ in $P$. The gates of an adversary circuit $A(r)$ are like the gates of the corruptible circuit $r$ except that they are active when the corruption register $C_r$ is on and inactive otherwise. Formally, every gate $G'$ of an adversary circuit $A_r$ has the form

$$G' = |0\rangle^{C_r} |0\rangle \otimes I + |1\rangle^{C_r} |1\rangle \otimes G^{\text{corrupted}}.$$ 

An (active) adversary circuit $A(r)$ can access all registers in the set registers $R_E$ of the environment and all registers of the corrupted circuit $r$. In the application of the constraints of a channel-type, the circuit $A(r)$ can communicate in the name of $r$ as if it was the circuit $r$. Note that an adversary circuit can eavesdrop a channel without using the identity of corruptible circuit $r$. For example, Eve can eavesdrop the communication between Alice and Bob without using the identity of Alice or Bob. Note also that we use the rule that every adversary circuit must be of the form $A(r)$ for some corruptible circuit $r \in P$. For example, $Eve = A(Auxiliary)$ where Auxiliary is an extra circuit in $P = QKD$.

### 3.2 Ideal protocols and simulators

An ideal protocol $I(P)$ for an analyzed protocol $P$ contains one circuit $I(r)$ for every module-role-circuit $r \in P^{I/O} \subseteq P$. Formally, $P^{I/O}$ can be any subset of $P$. However, the ideal protocol will only be realizable if $P^{I/O}$ contains every circuit $r \in P$ that uses an input or an output channel to communicate with the environment. Note that $P^{I/O}$ will often include the Auxiliary in $P$, even if it does not contribute to the input/output of the protocol. We will come back to the special circuit $I(Auxiliary)$ later. The ideal protocol can, and usually do, contain other circuits beside the circuits $I(r)$, $r \in P^{I/O}$. It typically contains one non corruptible circuit, the so called trusted party.

A circuit $I(r) \in I(P)$ with $r \in P^{I/O}$ is corrupted by the corruption register $C_r$ in the same way as was the circuit $r \in P^{I/O}$ and it can use the input or output communication channels that are normally used by the circuit $r$ as if it was the circuit $r$. However, it does not have access to the internal channel of the circuit $r$. It uses its own internal channels.

Typically, the ideal protocol $I(P)$ associated with an analyzed protocol $P$ is much simpler than the protocol $P$. It can accomplish the task in a much simpler way because it can use ideal channels and a Non-corruptible party. In particular, the set of internal channels in the ideal protocol is usually much smaller or less used than...
in the analyzed protocol. Therefore, if we keep the same environment and replace \( P \) by its ideal protocol \( I(P) \), after the corruption of a circuit \( r \), the adversary circuit \( A(r) \) in the environment will wait for ever for messages that will never come, and the output bit \( Z \) will never be set as it is set when the environment interacts with \( P \). Therefore, the two cases will easily be distinguished. So, it is not reasonable to require that \( P \) and \( I(P) \) are not distinguishable in such a direct manner. The solution to this dilemma makes use of a simulator \( S(P) \) that contains one simulator circuit \( \text{Sym}(r) \) for every corruptible circuit \( r \in P \) and one adversary circuit \( A^{\text{ideal}}(r) \) for every corruptible circuit \( r \in I(P) \). The simulator can contain other circuits.

Each simulated circuit \( \text{Sym}(r) \) in the simulator, as it is the case with the circuits of a protocol, is associated with a set of registers. This is necessary to determine which registers can be accessed by the environment when a circuit gets corrupted. Nevertheless, to simplify the design of the simulator, we allow any circuit of the simulator to access any register of any other circuit in the simulator. Through the simulation circuits \( \text{Sym}(r) \), the simulator can provide the registers and the communication that is expected by the adversary circuits \( A(r) \) in the environment. Through the adversary circuits \( A^{\text{ideal}}(r) \), the simulator can eavesdrop the internal communication and impersonate the corrupted circuits in the ideal protocol. In this way, the simulator can be seen as an extension of the environment. The purpose of the circuits \( A^{\text{ideal}}(r) \) is to allow the simulator to obtain information that it needs to accomplish the simulation. The overall purpose of the simulator \( S(P) \) is to extend the ideal protocol \( I(P) \) so that \( E + P \) and \( E + I(P) + S(P) \) are indistinguishable. We have the constraint on \( I(P) + S(P) \) that \( E + I(P) + S(P) \) must be a circuit, that is, the union of the partially ordered sets of gates for \( E \), \( I(P) \) and \( S(P) \) must still be a partially ordered set, every two gates that share a register must be an ordered pair, etc. This suggests the following security definition. In this definition, \( n \) is a security parameter used in the protocol, the simulator and the adversary circuits. For example, \( n \) can correspond to the number of photons sent in the protocol. The simulator and the adversary interact with the protocol, so they must also use the security parameter \( n \).

**Definition 1.** A protocol \( P \) \( \epsilon_P \)-securely realizes an associated ideal protocol \( I(P) \) if, for every “valid” environment \( E \) for \( P \), there exists a “valid” simulator \( S(P) \) such that

\[
| \Pr(Z_{E+P} = 0) - \Pr(Z_{E+I(P)+S(P)} = 0) | \leq \epsilon_P(n, E)
\]

for every “valid” environment \( E \) for \( P \) and \( n \). The set \( X \) is called the access rule.

Of course, in many cases such a simulator only exists when we restrict the power of the environment. For example, it is typical to assume that an environment can only corrupt a set of circuits \( X \in \mathcal{X} \subseteq 2^P \) where \( 2^P \) is the set of subsets of \( P \). The set \( \mathcal{X} \) is called the access rule.

There is something special about the above definition. The value \( \epsilon_P \) is a function of the environment \( E \) which makes it impossible to evaluate because the environment is unknown. Fortunately, this dependence on the environment \( E \) is not needed in the case of unconditional security, i.e., in this case we usually achieve \( \epsilon_P(n, E) = \epsilon_P(n) \). In the case of computational security, there is no way to achieve the above definition with an \( \epsilon_P \) that does not depend on the environment, at the least on its size. This is a problem because an important aspect of universal composability is to have a security statement that hold for a large class of environments where the size is unknown. In practice, the only way out of this problem is simply to assume some upper bound on the size of the environment so that we can evaluate \( \epsilon_P = \epsilon_P(n, |E|) \). However, in the security analysis of a complex protocol with many sub-protocols, each with its associated function \( \epsilon \), keeping track of all these functions \( \epsilon \) can be cumbersome. So, for convenience, we simply require that \( \epsilon_P(n, E) \) is a negligible function \( \epsilon_P(n, |E|) \):

**Definition 2.** A function \( \epsilon(n, |E|) \) is negligible (against any polynomial environment \( E \)) if, for every polynomial \( p(n) \), for every \( f_n \leq p(n) \), we have that, for every polynomial \( q(n) \), for \( n \) sufficiently large, \( \epsilon(n, f(n)) \leq 1/q(n) \) or, better, for some \( \alpha > 0 \), \( \epsilon(n, f(n)) \leq 2^{-\alpha n} \) for \( n \) sufficiently large.

The definition in the computational scenario becomes:
**Definition 3.** A protocol $P$ of polynomial size computationally realizes an associated ideal protocol $I(P)$ of polynomial size (but usually of constant size) if there exist a negligible function $\epsilon_P(n, |E|)$ such that, for every valid environment $E$ for $P$, there exists a valid simulator $S(P)$ of polynomial size such that

$$|\Pr(Z_{E+P} = 0) - \Pr(Z_{E+I(P)+S(P)} = 0)| \leq \epsilon_P(n, |E|)$$

(3)

Note that, even though it is not explicitly stated that the environment must be of polynomial size, we need only be concerned with environment of polynomial size because there is no constraint on $\epsilon_P(n, |E|)$ when $|E|$ is not bounded by a polynomial.

Actually, to obtain a universal composability theorem, very natural conditions are added to this definition. So natural that it maybe pointless to mention them. Consider the case where many copies of $P$ are used inside some complex protocol. The polynomial upper bound on the size of the simulator and the function $\epsilon_P(n, |E|)$ should be the same for all copies of $P$. In other words, the polynomial upper bound on the size of the simulator and the function $\epsilon_P(n, |E|)$ should only depend on the protocol definition, not on where the protocol is used inside an application protocol. Moreover, we restrict the universal composability theorem to complex protocols that use a finite number of distinct protocol definitions (but it is OK that there are polynomially many copies of each of them.)

### 3.3 About the design of an ideal protocol

As we explained, the input/output communication channels that are normally available to a circuit $r$ will not be used by the ideal functionality when $r$ is corrupted. However, as we previously explained, it is nevertheless natural to design an ideal protocol that exchanges with the environment (on the analyzed side) or the simulator (on the ideal side) the information that would normally be exchanged through these corrupted input/output channels. This is not entirely obvious because, formally, the simulator (or the environment) can only use adversary circuits $A_{\text{ideal}}(r)$ that act in the name of corrupted circuits $I(r)$ when they interact with the ideal protocol, and thus, because the circuits $I(r)$ are inside the ideal protocol, the simulator only expects internal communication. Here is how it works. A typical approach to construct an ideal protocol is to use a dummy party $I(r)$ in the ideal protocol for each circuit $r$ that participates in the input/output functionality of the analyzed protocol. A dummy party $I(r)$ just plays the role of a channel between $A_{\text{ideal}}(r)$ and a trusted party $\text{Charlie} \in I(P)$ that computes alone the functionality of the ideal protocol. Any input received by a dummy party is sent to the trusted party and this trusted also use these dummy parties to return an output. In this way, when the dummy circuit $I(r)$ is corrupted, the adversary circuit $A_{\text{ideal}}$ can exchange input/output communication with the trusted circuit in the ideal protocol.

Despite the fact that the simulator can only interact with the ideal protocol in the name of a corrupted party, if it help in the design of the simulator, a circuit in the ideal protocol can exchange any information with the simulator (on the ideal side), even if no circuit in the analyzed protocol is corrupted. This exchange can be done through the circuit $\text{Devil} = A_{\text{ideal}}^{\text{ideal}}(\text{Auxiliary})$ in the simulator, where Auxiliary is an extra circuit in the protocol. For example, this extra circuit can be the same Auxiliary that is used in the definition $\text{Eve} = \text{Adv}(\text{Auxiliary})$ in QKD.

Note that $I(\text{Auxiliary})$ in $I(r)$ does not have to be passive. It can communicate with the trusted circuit in $I(r)$, even if the circuit Auxiliary is passive in $r$. We recall that the environment of the ideal protocol on the ideal side is the simulator. Because a more powerful simulator makes a security proof easier, we want an ideal protocol that gives as much power as possible to the simulator but we must respect the fact that the ideal protocol must be useful when used inside an analyzed protocol. The circuit $I(\text{Auxiliary})$ must be defined accordingly. As a part of a security proof, an analyzed protocol can contain ideal sub-protocols. In this case, the environment of the ideal protocol is the environment of the analyzed protocol. So the environment communicates with these ideal sub-protocols when they are on the analyzed side. If this exchange of information with the ideal protocol is fine in that scenario, then it is a valid communication. So, this exchange of information between the trusted circuit and $I(\text{Auxiliary})$ must respect the spirit of the task. For example, an ideal bit commitment should not unveil the bit that is committed before the opening, but it is fine that the trusted circuit in the ideal bit commitment protocol tells the circuit $A_{\text{ideal}}^{\text{ideal}}(\text{Auxiliary})$ that Alice has decided to open the bit, even if no party is corrupted.
4 The universal composability theorem

Any variation on the universal composability theorem is useful when we consider protocols $P$ that have many sub-protocols. A sub-protocol $Q$ of $P$ has the recursive form

$$Q = M(Q) + \sum_{R \in \mathcal{C}(Q)} R$$

where $\mathcal{C}(Q)$ is the set of sub-protocols of $Q$ that are called by its main module $M(Q)$. If $Q$ is a primitive then there are no sub-protocol $R$ in the sum and $Q = M(Q)$. As we explain in Appendix E, we can rewrite a protocol where the set of modules is a directed acyclic graph, not a tree, as a protocol that is a tree. So, we do not lose generality when we assume that a protocol is a tree of modules. Let $I(Q)$ be the ideal protocol associated with $Q$. Let

$$\tilde{Q} = M(Q) + \sum_{R \in \mathcal{C}(Q)} I(R).$$

Here is the composability theorem.

**Theorem 1.** Computational scenario: If, for every sub-protocol $Q$ of a protocol $P$, $\tilde{Q}$ computationally realizes $I(Q)$, then $P$ computationally realizes $I(P)$. Unconditional scenario: If, for every sub-protocol $Q$ of a protocol $P$, $\tilde{Q}$ $\epsilon_\tilde{Q}$-securely realizes $I(Q)$, then $P$ $\epsilon_P$-securely realizes $I(P)$ where $\epsilon_P = \sum_Q \epsilon_\tilde{Q}$. Instead of directly proving the theorem, we will describe a proof technique that should be used if one wants to keep track of the function $\epsilon$ (see previous section). This will lead us to a variation on the universally composability theorem which is more complicated, less practical, but for which the proof is trivial. Then, it will be easy to explain how the proof must be modified to consider the unconditional scenario, and similarly for the computational scenario.

In the proposed proof technique, we assume that, for every sub-protocol $Q$ (in the tree of sub-protocols) of $P$, we have that $\tilde{Q}$ $\epsilon_\tilde{Q}$-securely realizes $I(Q)$ where $\epsilon_\tilde{Q} = \epsilon_Q(n, E(\tilde{Q}))$. The proof technique does not explain how to obtain this hypothesis. It explains how to use this hypothesis to obtain that $P$ $\epsilon_P$-securely realizes $I(P)$ for some function $\epsilon_P = \epsilon_P(n, E(P))$.

As explained before, any approach uses a bottom-up traversal of the tree of sub-protocols of $P$. The natural way to use 2 or 3 is that, at every node $Q$ in the bottom-up traversal, the protocol $\tilde{Q}$ is replaced by $I(\tilde{Q}) + S(\tilde{Q})$. This is because 2 or 3 essentially says that we can do this replacement. The simulator $S(\tilde{Q})$ for a given the environment $E(\tilde{Q})$ is obtained by the hypothesis that $\tilde{Q}$ $\epsilon_\tilde{Q}$-securely realizes $I(Q)$. For the first $Q$ in the bottom-up traversal, we have $E(\tilde{Q}) = E(P)$, the original environment. However, as we do replacements, simulators and ideal protocols are added in the environment $E(\tilde{Q})$ of the visited protocol $\tilde{Q}$. Nevertheless, for $E$, $P$, $n$ and $Q$ fixed, $E(\tilde{Q})$ is a well defined circuit.

Before the first replacement, the entire setting is $E + P$. After the last replacement, the entire setting is $E + I(P) + \sum_Q S(\tilde{Q})$. Using the triangle inequality for each replacement, we obtain that the simulator $S(P) = \sum_Q S(\tilde{Q})$ is the required simulator such that

$$| \Pr(Z_{E+P} = 0) - \Pr(Z_{E+I(P)+S(P)} = 0) | \leq \epsilon(n, E)$$

(4)

where $\epsilon(n, E) = \sum_Q \epsilon_\tilde{Q}(n, E(\tilde{Q}))$. For this last sum to make sense, we need that, for $P$ fixed, $\epsilon_\tilde{Q}(n, E(\tilde{Q}))$ is a function of $n$, $E$ and $Q$. This is the case because $E(\tilde{Q})$ is a function of $n$, $E$ and $P$: it contains ideal protocols and simulators that can be determined given the security definition of the sub-protocols $\tilde{R}$ with $R < Q$ in the traversal order. So, we have proven the following variation on the universal composability theorem.

**Theorem 2.** If, for every sub-protocol $Q$ of a protocol $P$, $\tilde{Q}$ $\epsilon_\tilde{Q}$-securely realizes $I(Q)$, then $P$ $\epsilon_P$-securely realizes $I(P)$ where $\epsilon_P(n, E) = \sum_Q \epsilon_\tilde{Q}(n, E(\tilde{Q}))$ in which, as explained above, $\epsilon_\tilde{Q}(n, E(\tilde{Q}))$ is a function of $n$, $E$ and $Q$. 12
A problem with this variation on the universal composability theorem and in the way to use it as a proof technique is that the description of the function $\epsilon_{Q}(n, E(\bar{Q}))$ requires that one keeps track of all the simulators and ideal protocols in $E(\bar{Q})$. If researchers were to use this theorem as a proof technique, an intermediary security result for a sub-protocol $\bar{Q}$ would have to include a complete description of the simulator $S(\bar{Q})$ used, and the final security result would be expressed in terms of a summation $\sum_{Q} \epsilon_{Q}(n, E(\bar{Q}))$ which cannot be simplified because each $\epsilon_{Q}$ can depend on different aspects of the environment $E$. Fortunately, this issue disappears in the unconditional scenario case because $\epsilon$ is independent of the environment and theorem\ref{def:universal-composability} gives us theorem\ref{thm:universally-composable-security}. In the computational scenario, if we want to know $\epsilon$, which at some point one will want to know, the only way out of this issue is to restrict $\epsilon$ to be a function of the size (or other simple aspects) of $E$ and then make concrete assumptions so that the different $\epsilon_{Q}$ can be computed.

If we just want to know that $\epsilon$ is negligible as in definition\ref{def:universal-composability} the way to take care of this issue in the computational scenario is subtler. We need to show that $\epsilon_{P}(n, E) = \sum_{Q} \epsilon_{Q}(n, E(\bar{Q}))$ is negligible given that each $\epsilon_{Q}(n, E(\bar{Q}))$ is negligible. We must guarantee that $|E(\bar{Q})|$ is of polynomial size because otherwise the fact that each $\epsilon_{Q}(n, E(\bar{Q}))$ is negligible cannot be used. At this point, we simply use the fact that there are only a constant number of distinct protocol definitions $\bar{Q}$ in the protocol $P$, and each has a simulator with a polynomial upper bound on its size that depends only on the definition of $\bar{Q}$. We obtain that there exists a single polynomial $p(n)$ that bounds the size of each of the polynomially many simulators in $E(\bar{Q})$. Similarly, we must have a single lower bound for all the $\alpha$ in the statement that all $\epsilon_{Q}$ are negligible. Again, we use the fact that there are only a constant number of distinct protocol definitions $\bar{Q}$ in the protocol $P$ and so only a constant number of distinct $\epsilon_{Q}$.

We thank Claude Crépeau and Debbie Leung for useful discussions. This work has been supported in part by the National Science Foundation under Grant No. EIA-0086038.

References

1. M. Backes, B. Pfitzmann and M. Waidner. “A General Composition Theorem for Secure Reactive Systems.” In *Proc. First Theory of Cryptography Conference (TCC)*, pp. 336 – 354, 2004.
2. D. Beaver. “Secure multiparty protocols and zero knowledge proof systems tolerating a faulty minority”. In *Journal of Cryptology*, 4(2), pp. 75 – 122, 1991.
3. H. Bechmann-Pasquinucci and N. Gisin. “Incoherent and coherent eavesdropping in the 6-state protocol of quantum cryptography”. *Phys. Rev. A* 59, pp. 4238 – 4248, 1999.
4. C.H. Bennett and G. Brassard, “Quantum cryptography: Public-key distribution and coin tossing”, In *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing*, Bangalore, India, December 1984, pp. 175 – 179.
5. C.H. Bennett, F. Bessette, G. Brassard, L. Salvail and J. Smolin, ”Experimental quantum cryptography.”, *Journal of Cryptology*, vol. 5, no. 1, 1992, pp. 3 – 28. Preliminary version in *Advances in Cryptology - Eurocrypt ’90 Proceedings*, May 1990, Springer - Verlag, pp. 253 – 265.
6. C.H. Bennett, G. Brassard, C. Crépeau and U. Maurer. “Generalized Privacy Amplification”. In *IEEE Transaction on Information Theory*, 1995, Volume 41, Number 6, pp. 1915 — 1923, November 1995
7. M. Ben-Or, M. Horodecki, D. Leung, D. Mayers and J. Oppenheim, “The Universally Composable Security of Quantum Key Distribution”. (In preparation).
8. D. Mayers and M. Ben-Or. “Composing Quantum and Classical protocols”. Presented at Quantum Information Processing 2003 (December 2002). See http://www.msri.org/publications/lm/msri/2002/qip/mayers/1/index.html
9. E. Biham and T. Mor, “Security of quantum cryptography against collective attacks”, *Physical Review Letters* 78, pp. 2256 – 2259 (1997); Also in Los-Alamos Archive: quant-ph/9605007
10. M. Blum, P. Feldman and S. Micali. “Non-interactive zero-knowledge and its applications”, STOC 88.
11. R. Canetti. “Universally composable security: A new paradigm for cryptographic protocols”. In *Electronic Colloquium on Computational Complexity (ECCC)* (016): (2001). Preliminary version in IEEE Symposium on Foundations of Computer Science, pp. 136 – 145, 2001.
12. R. Canetti and M. Fischlin. “Universally composable commitments”. In *Crypto ’01*, pp. 19 – 40, 2001.
13. R. Canetti and H. Krawczyk. “Universally Composable Notions of Key Exchange and Secure Channels”. In *Eurocrypt ’02*, pp. 337 – 351, 2002.
14. R. Canetti, E. Kushilevitz and Y. Lindell. “On the limitations of universally composable two-party computation without set-up assumptions”. In *Eurocrypt ’03*, pp. 68 – 86, 2003.
A History of security definitions for quantum key distribution

For a long time, researchers only considered attacks that interact with a single photon or few photons at a time, the individual attacks (for examples see [4,5,26,20,3,23]). A special interesting case are the collective attacks where the interaction is with a single photon at a time with a different probe for each photon, but there is a final measurement on all the probes together at the end [9]. The security notion for quantum key distribution was simply that, for all (individual or collective) attacks that have an error rate below some finite threshold, Eve’s information must be below some associated finite threshold. A private key could then be extracted with standard privacy amplification techniques [6]. These privacy amplification techniques being taken for granted, researchers considered the disturbance/information trade-off as the basic ingredient in the security analysis of quantum key distribution protocols [19].

This works fine for individual attacks. For non individual attacks, the formal notion of error rate or disturbance is tricky. For example, with probability 1/100 Eve could intercept all photons and otherwise do nothing. The error rate after this attack is 1/100, but yet no privacy amplification techniques can successfully extract a private key. Because of this difficulty and the possible entanglement between photons created by a non individual attack, the security of quantum key distribution against all attacks was difficult to prove.
The first security result against all attacks was obtained in [33] for the quantum oblivious transfer protocol of [17]. Later, this result was adapted in [21] for the quantum key distribution protocol of [4]. Even at the time, without any reference to quantum error correction, the basic result was that, if the number of phase flips (i.e., bit flips in the complementary basis) in the raw key is small, then privacy amplification works. A similar property is needed in the proofs that are based on Quantum Error Correction (for example, see [31]). The main point is that we could use the number of errors in the raw key, including phase flip errors, as a measure of Eve’s information before privacy amplification. A small number of phase flip errors was called the small weight property [33], and later the small sphere property [21].

Note that it is not enough that the expected phase flip error rate is small, below some fixed small threshold. Privacy amplification only works when the actual number of phase flip errors, not just its expected value, is below some small threshold. For example, in the above simple attack, the expected phase flip error is 1/100, but privacy amplification does not work irrespectively of whether or not 1/100 is below any threshold. The solution is to define Eve’s success as the event where the number of bit flips on tested photons is small and the number of phase flips on the other photons is large. This leads to a new notion of security: it is not possible for Eve to both pass the test and have information at the same time. So, the criteria is of the form \( Pr(\text{Pass} \land \text{Info} > \mu) \leq \epsilon \) where \( \text{Info} \) is some variable that determines how much information Eve has.

Equivalently, the above criteria can be written \( Pr(\text{Pass}) \times Pr(\text{Info} > \mu|\text{Pass}) \leq \epsilon, \) for some \( \epsilon, \mu > 0. \) Note that we can set \( \text{Info} = 0 \) when the test fails because no key is generated. So, in a way, the key point is that we must consider separately the case where the test passes and the case where the test fails, and average over the two cases. Note that the case where Eve fails the test corresponds to a key of length zero. This suggests a generalization of the above notion: the key point is to consider each possible length of the key separately, and average over all possible length. This is in accord with the old principle that in key distribution it is understood that we must also have the correctness condition: \( Pr(\text{Pass} \neq \text{Pass}) \leq \beta. \) In this case, conditioning over \( M \) makes no difference because \( \sum_m Pr(M = m)Pr(K_A \neq K_B|M = m) = Pr(K_A \neq K_B). \)

B The formal protocols

In this Section, we formally describe the modules used in this document.

(Ideal) SOT

1. SOT-Bob: For \( i = 0, 1 \) do \{receive input \( s[i] \in \{0, 1\}^k \) from App; send \( s[i] \) to SOT-Charlie; \};
2. SOT-Charlie: For \( i = 0, 1 \) do \{ receive \( s[i] \); \};

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3. SOT-Alice: receive input $c \in \{0,1\}$ from App; send $c$ to SOT-Charlie;
4. SOT-Charlie: receive $c$; send OK = “ok” to SOT-Bob;
5. SOT-Bob: receive OK; send output OK to App;
6. SOT-Charlie: send $\hat{s}:=s[c]$ to SOT-Alice;
7. SOT-Alice: receive $\hat{s}$; send output $\hat{s}$ to App;

In the ideal \((\frac{1}{2})\)-String-OT module, the channels are all hidden. The bit commitment (BC) protocol calls the \((\frac{1}{2})\)-String-OT protocol. Here is the BC module.

**BC on top of SOT**

**Commit Phase**

1. BC-Bob: For $i = 0,1$ do \{ picks $s[i] \in_R \{0,1\}^k$; send input $s[i]$ to SOT-Bob; \}
2. BC-Alice: receive input $x \in \{0,1\}$; send input $x$ to SOT-Alice
   receive output $\hat{s}$ from SOT-Alice;
3. BC-Bob: receive output OK from SOT-Bob; send output OK to App;
   Opening
4. BC-Alice: receive input $FOO \in \{0,1\}$ from App; send $(x, \hat{s})$ to BC-Bob;
5. BC-Bob: receive $(x, \hat{s})$ from BC-Alice; If $(s[x] = \hat{s})$ do \{ $\hat{x} := x$ \} else \{ $\hat{x} := \bot$ \}; send output $\hat{x}$ to App;

In the BC module, all channels are hidden except the unique send/receive channel which is used by BC-Alice to send the opening information $(x, \hat{s})$ to BC-Bob. This is a public but authenticated channel. Here is the CT protocol.

**CT on top of BC**

1. CT-Alice: picks $x \in_R \{0,1\}$; send input $x$ to BC-Alice;
2. CT-Bob: receive output OK from BC-Bob; picks $y \in_R \{0,1\}$; send $y$ to CT-Alice;
3. CT-Alice: receive $y$ from CT-Bob; send output $x \oplus y$ to App; send input “open” to BC-Bob;
4. CT-Bob: receive output $\hat{x}$ from BC-Bob; send output $\hat{x} \oplus y$ to App;

In the CT module, all channels are hidden except the two public but authenticated send/receive channels which are used by BC-Bob and BC-Alice to send $x_B$ and $x_A$, respectively.

C  **Equivalence with an alternative definition**

It may seem that only the application should be the same on both sides, not the adversary. After all, we saw that the only purpose of the ideal side is to determine the probability $\Pr(Z_{Ideal} = 0)$ that is acceptable in [1]. It should not matter that the adversary is different on the ideal side. Consider the alternative definition where, on the ideal side of the game, $\Pr(Z_{Ideal} = 0)$ can be computed with a different adversary than on the analyzed side. This alternative definition reads as follows. The protocol is secure if, for all applications $App$ and adversary $Adv$ for $P$, there exists an ideal adversary $Adv'$ such that $Z_{Analysed}$ and $Z_{Ideal}$ cannot be distinguished, where $Analysed = App + Adv + P$ and $Ideal = App + Adv' + I(P)$. In the original security definition (without the extra existential quantifier on the adversary), we had $Ideal = App + Adv + S(P) + I(P)$. In the alternative definition, the simulator is part of the ideal adversary; two consecutive existential quantifiers can be replaced by a single one. This alternative definition is fine because, if the ideal protocol $I(P)$ is really ideal, no matter what is the ideal adversary $Adv'$ on the ideal side, the probability $\Pr(Z_{Ideal} = 0)$ should still be a good reference value. However, in this document, we consider that the adversary is the same on both sides. Certainly, the fact
that our security definition avoids this extra existential quantifier can only make it stronger (more restrictive on the protocols) because it simply means that we have less freedom for the acceptable value of $\Pr(Z_{\text{Ideal}} = 0)$ in \( I \), that is, less freedom to accept the protocol. In the following, we essentially prove that the fact that our security definition avoids an extra quantifier on the adversary on the ideal side does not make it strictly stronger, and so these two definitions are essentially equivalent.

We say “essentially” because there is a subtle issue. The proof that the alternative definition is equivalent only works if, in this alternative definition, “all environments” means “all environments in which the application $\text{App}$ is allowed to interact with registers that are exchanged between non corrupted circuits in the protocol in accordance with their security type”. In the original definition (without the extra existential quantifier), these two classes of environments have the same power, but we do not know this as a fact in the alternative definition with the existential quantifier on the adversary on the ideal side. Therefore, it seems that the alternative definition (with the existential quantifier on the adversary) might actually be weaker (less restrictive on the protocols) if we do not allow the application to interact with the registers that are exchanged in non secure channels between the non-corrupted parties in the protocol. Because, by definition, the application does not include the adversary circuits, this is a reasonable constraint on the application (even if the channels are public!)

The proof proceeds in three steps. First, we show that, in the original definition (without the extra existential quantifier), a protocol is accepted against all environments in which the application cannot directly interact with channel registers if and only if it is accepted against all environments in which the application can directly interact with channel registers. In other words, in the original definition, allowing the application to directly interact with channels registers does not make the environment more powerful. For the two other steps, when we say “all environments” we mean “all environments in which the application is allowed to directly interact with channel registers”. In the second step, we show that with the original definition, if a protocol $P$ is accepted against a sub-class of all environments $E'$ that only use a dummy adversary (defined later), then it is accepted against all environment $E$. Third, we show that every protocol $P$ that is accepted by the alternative definition (with the extra existential quantifier) is also accepted by the original definition against the environments with the dummy adversary. There is a subtle point here. This shows the equivalence of the two definitions, but only if in the alternative definition (with the extra existential quantifier) “all environments” means “all environments in which the application is allowed to directly interact with channel registers”.

**First step.** The environment can simulate an interaction between the application and channel registers because the application can send a probe register to the adversary which can use it as an ancilla to interact with a channel register and then return the probe register back to the application. So, allowing a direct interaction between the application and channel registers does not make the environment more powerful.

**Second step.** The strategy for the second step is that, for every environment $E$ for $P$, we construct an environment $E'$ for $P$ with a dummy adversary, obtains the simulator $S$ against $E'$ and show that it works also against $E$. A dummy adversary is an adversary that only contains communication gates that are executed between the application and itself or between itself and the protocol. With these gates, a dummy adversary does nothing except forwarding back and fourth the communication between the application and the protocol. We construct the dummy adversary $\text{Dummy}$ in $E'$ in the following way. Let $P$ be the protocol and $\text{Adv}$ the adversary in $E$. We recall that the protocol $P$ communicates with the adversary $\text{Adv}$. We denote the communication gates in $\text{Adv}$ between $\text{Adv}$ and $P$, the $\text{Adv-P}$ communication gates. An $\text{Adv-P}$ communication gate is a gate of the adversary that access a channel-register that is received by a module-role-circuit of $P$. Let $\text{Dummy}$ be the circuit that contains two $\text{Dummy-P}$ communication gates for each $\text{Adv-P}$ communication gate: one for reception and the other one for forwarding. Let $S\text{Adv}$ be the same as the adversary $\text{Adv}$ except that each $\text{Adv-S}$ communication gate is replaced by a corresponding an $\text{App-Dummy}$ communication gate. So, $S\text{Adv} + \text{Dummy}$ is essentially the same as $\text{Adv}$, except that every $\text{Adv-S}$ communication gate is replaced by three communication gates, one in $S\text{Adv}$ and two in $\text{Dummy}$. The totality of the application in $E'$ is $\text{App}' = \text{App} + S\text{Adv}$. The circuit $S\text{Adv}$ can be interpreted as a simulation of $\text{Adv}$ inside the application. We have constructed $E' = \text{App}' + \text{Dummy}$.

It is useful to picture the situation in terms of a graph where the nodes are the gates in $E + P$ with $E = \text{App} + \text{Adv}$ and the arrows are the ordered pair of gates in the associated POSET. In this picture, we
obtain $E' + P$ where $E' = App' + Dummy$ in the following way. Every arrow between Adv and $P$ is now an
arrow between Dummy and $P$. This is because Dummy is the new adversary that replaces Adv. Moreover, for
each such arrow, we replace the arrow between Adv and $P$ with an arrow between Adv and Dummy, reinterpret
Adv as $SAadv$, and add another arrow inside Dummy from the reception to the forwarding gates. We cannot
create a cycle in this way. So, any valid environment $E$ correspond to a valid environment $E'$.

To end this second step, we must construct a simulator $S$ against $E$. By hypothesis, there exists a simulator
$S$ against $E'$. This simulator that is valid with $E'$ is also valid with $E$ because, replacing any two consecutive
arrows with an intermediary step in the dummy adversary by the original single arrow cannot create a cycle.
Second, with this same simulator, the computation is the same with $E$ or $E'$. So, if $S$ succeeds against $E'$, it
also succeeds against $E$. This concludes the second step.

*Third step.* Consider a protocol $P$ that is accepted by the alternative definition (with the extra quantifier).
We must show that this protocol is also accepted in the original definition against all environments $E'$. So,
for a given $E' = App' + Dummy$, we must find $S$ such that $Z_{E' + P} \approx Z_{E' + I(P) + S}$. By hypothesis, because $E'$ is a valid environment against $P$, the protocol $P$ is accepted against $E'$ in the alternative definition with the extra quantifier. So, we have that there exist an adversary $Adv'$ and a simulator $S'$ so that $Z_{Adv' + Dummy + P} \approx Z_{Adv' + Dummy + S' + I(P)}$. We define $S$ so that $Adv' + S'$ and $App' + Dummy + S$ are equivalent. Essentially, $S$ is identical to $Adv' + S'$, but it communicates with the dummy adversary Dummy instead of directly with $App'$. We have that $Z_{E' + P} \approx Z_{App' + Dummy + S' + I(P)} = Z_{E' + S + I(P)}$. This concludes the proof.

If we go back to our coin toss environment example, this means that we can consider that the adversary on
the real side which creates the small bias $2^{-k}$ is in fact a simulated adversary which communicates with the
real protocol through a dummy adversary, but the simulator on the ideal side is like an adversary that freely
interacts with the ideal protocol and try to play the role of the dummy adversary, and tries to compensate for
differences between the ideal and the real protocols – a difficult job because the dummy adversary is the worst
adversary.

**D** More about the adversary status

Usually, the computational basis states of a corruption register $C(U)$ are the states $|not\ corruptions\rangle$ and
$|fully\ corrupted\rangle$. However, we will see that we can have different levels of corruption such as honest but
curious. The initial state of every register $C(X)$ is $|not\ corruptions\rangle$. The gates of every circuit $U \in \mathcal{P}$ are
always conditioned by the register $C(U)$. If the register $C(U)$ is fully corrupted when $U$ is activated and
ready to execute a gate $G_i$, the gate $G_i$ is ignored. The next time that it will be activated, the circuit $U$ will be
ready to execute the next gate $G_{i+1}$ (which will also be ignored if $C(U)$ remains corrupted). The environment
has access to the registers of a fully corrupted circuit and can communicate in its name. A circuit is honest but
curious if it is not turned off but the environment can read its classical registers. A circuit is fair but curious,
if it can be replaced by an equivalent circuit (no difference can be seen by the environment unless it
reads its classical registers) and the environment can read its classical registers. Also, a circuit might not be
forced to give its registers when corrupted. A circuit that is fully corrupted except that its register are kept
private is corrupted without memory. In the following we will restrict ourselves to non corrupted and fully
corrupted circuits.

**E** The directed acyclic graph case

If the graph of sub-protocols is a rooted directed acyclic graph instead of a tree, an ideal protocol $I(Q)$ for a
sub-protocol $Q$ in the graph can be in the intersection of two protocols $\tilde{R}$ and $\tilde{S}$ with $R < S$ in the traversal
order. After we replace $\tilde{R}$, which includes $I(Q)$, we have the problem that $I(Q)$ is not available anymore for $\tilde{S}$.

Fortunately, this situation will typically occur with a sub-protocol $Q$ that is designed so that it still provide
its functionality $I(Q)$ even after it is called by $R$. We must use this fact to reorganize the protocol so that it
becomes a tree of sub-protocols instead of a directed acyclic graph. For every sub-protocol $Q$ that is called by
many sub-protocols $R_1 < \ldots < R_{n_Q}$, $n_Q > 1$, starting with those with the longest path from the we do root,
we do the following. Without loss of generality, we assume that the nodes \( R_i \) are ordered by the length of the longest path from the root to \( Q \) in which they belong. For \( i = 1, \ldots, n_Q \), we redefine the ideal protocol of \( R_i \) as 
\[
I'(R_i) = I(Q) + \sum_{j=1}^{i} I(R_j),
\]
we let \( R_{i+1} \) call \( R_i \) instead of \( Q \) and every protocol \( T \) that calls \( R_i \) must now call \( R_m \) which is the latest \( R_j \), with \( j \geq i \), that occurs before \( T \) in the traversal order.

The hypothesis of the universal composability for the new tree will have to be proven. This may seem a difficult task because of the apparent complexity of the new ideal protocols \( I'(R_i) \). However, this complexity is only superficial. The protocol \( R_{i+1} \) only uses \( I(Q) \) inside \( I'(R_i) \) and any other protocol that called \( R_i \) but now calls \( R_m \), only uses \( I(R_i) \) inside \( I'(R_m) \). This suggests that whenever we have a directed acyclic graph of sub-protocols, there are good chances that we can reorganize it as a tree and still be able to use the universal composability theorem.