Abstract

In an attempt to better understand structural benefits and generalization power of deep neural networks, we firstly present a novel graph theoretical formulation of neural network models, including fully connected, residual network (ResNet) and densely connected networks (DenseNet). Secondly, we extend the error analysis of the population risk for two-layer network [10] and ResNet [8] to DenseNet, and show further that for neural networks satisfying certain mild conditions, similar estimates can be obtained. These estimates are a priori in nature since they depend solely on the information prior to the training process, in particular, the bounds for the estimation errors do not suffer from the curse of dimensionality.

Keywords: nonlinear weighted directed acyclic graph, DenseNet, adjacency matrix, A priori estimates

1 Introduction

A central challenge in theoretical machine learning is to figure out the source of the generalization capabilities of deep neural networks. Traditional statistical learning
theory often fails to provide satisfactory explanations [12]. For this reason, there has been a flurry of recent papers endeavor to analyze the generalization error for neural networks [3, 10, 20, 5, 7, 11, 23, 3, 13, 24, 25, 21]. Since the problems that neural networks usually encounter tend to have very high dimensions, one issue of particular interest is the **curse of dimensionality** (CoD) [4]: The computational cost depends exponentially on the dimension. However, in high dimensional settings, deep neural networks have shown great promise in many applications and do not suffer from the CoD. Hence, we expect to obtain some proper error estimates whose error bounds do not deteriorate as the input dimension grows. In other words, an optimal error bound shall scale at a rate independent of the input dimension [9].

Another interesting phenomenon is that neural networks can be substantially deeper, more accurate, and efficient to train if they contain shortcut connections from early layers to later layers. Most of the state-of-the-art neural networks benefit from such bypassing paths [16, 15, 27, 19, 17]. For instance, the identity skip-connection blocks utilized in ResNet [15] serve as the bypassing paths, and the counter-intuitive stochastic depth method introduced in [17] shortens the effective depth in ResNet by randomly dropping layers during training. Theoretically, Hardt and Ma [14] proved that for any residual linear networks with arbitrary depth, they possess no spurious local optima. E et al. [8] showed that optimal rate of the population risk can be guaranteed for ResNet, and for shallow neural networks, similar results still hold. Venturi and Bruna [29] showed that spurious local minima can be avoided with high probability on overparametrized two-layer network models, E et al. [8] also showed that optimal rate of the population risk can be guaranteed for two-layer network. However, no result is available yet for deep networks without shortcut connections. Apart from them, the advantages of using shortcut connections remain to be discovered.

In this paper, we contribute to further understanding of the above two aspects. Firstly, we introduce a new representation for neural networks, namely the nonlinear weighted Directed Acyclic Graph (DAG). The employment of DAG sheds light on the reasons behind success and failure of various network architectures from the perspective of linear algebra. Daniely [7] also used DAGs to characterize the neural network architectures but in a different way from ours. Using our representation, revealing the network structure becomes a linear algebra problem. In particular, some typical feedforward neural networks such as two-layer network, fully connected network, ResNet [8, 15] and DenseNet [16] can be represented by adjacency matrices.

In addition to representing commonly-used networks, we achieve the error bounds for a wide class of neural networks with DenseNet included. For neural networks
satisfying the assumption of shortcut connections (Assumption 2), \textit{a priori} estimates of the population risks can be established. According to [8, 10], most recent attempts [23, 3, 13, 21, 25, 21] on bounding the generalization error of neural networks should be viewed as \textit{a posteriori} estimates, in that the bounds rely on information acquired in the training process. In comparison with the abovementioned \textit{a posteriori} estimates, the \textit{a priori} bound depends solely on properties of the target function, hence it can be served as a more natural reflection on potential performances of different neural networks. The core of our analysis is a specially designed parameter norm termed the \textit{weighted path norm} [8, 23] that proceeds by balancing between the complexity and the approximation. On one hand, the weighted path norm gives control to complexity of the hypothesis space induced by neural networks (Theorem 5). On the other hand, the target functions can be well approximated by neural networks, whose weighted path norm is dominated by the norm of target function, hence bringing about the \textit{a priori} estimates (Theorem 1). To sum up, the hypothesis space determined by the norm is small enough to have low complexity, but also large enough to have low approximation error.

The organization of the paper is listed as follows. In Section 2, we give some preliminary introduction to our problems. In Section 3, we propose our novel representation for feedforward neural networks. In Section 4, we state our main results. In Section 5, we give the full proof of the theorems and apply our estimates directly to DenseNet. Conclusions are drawn in Section 6 and we compare our results with some related works.

2 Preliminaries

Throughout this paper, we use the following notations. We set \( d \) as the input dimension, and \( n \) as the number of input samples. We set \( \Omega = [0, 1]^d \) as the unit hypercube, and we let \( \sigma(\cdot) \) be the Rectified Linear Unit (ReLU) activation, i.e., \( \sigma(x) = \text{ReLU}(x) := \max\{x, 0\} \). Moreover, we use \( \|\cdot\|_1 \) and \( \|\cdot\|_{\infty} \) to denote the \( l_1 \) and \( l_{\infty} \) norms for vectors, and finally we use \( \|\cdot\|_{1,1} \) to denote the entrywise \( L_{1,1} \) norm for matrices. Specifically, for a matrix \( A = [a_{i,j}]_{1 \leq i \leq p, 1 \leq j \leq q} \) of size \( p \times q \), its entrywise \( L_{1,1} \) norm reads

\[
\|A\|_{1,1} := \sum_{i=1}^{p} \sum_{j=1}^{q} |a_{ij}|.
\]
2.1 Feedforward Neural Networks

In this section, we firstly introduce some commonly-used feedforward neural networks. An artificial neural network (NN) is a feedforward neural network when the connections of its nodes (neurons) do not form a cycle. Some of the nodes are activated by a nonlinear function, and this function is termed the activation function. We use $\sigma(\cdot)$ to signify the ReLU activation, i.e., $\sigma(x) = \max\{0, x\}$. Thus $\sigma(\cdot)$ is 1-Lipschitz because $|\sigma(z) - \sigma(z')| \leq C_L |z - z'|$ with $C_L = 1$ for all $z, z' \in \mathbb{R}$. We denote the output function of a neural network as $f(x; \theta)$, where $x$ is a training sample, and $\theta$ is the vector containing all parameters of the function. We list out some typical examples of feedforward neural network and these examples will be studied later from a different viewpoint. Since nodes of a feedforward neural network do not form a cycle, each network can be analogously treated as a weighted DAG \[22\], see the next section for more details.

1. Two-layer Neural Network

$$f_{2\text{Layer}}(x; \theta) = a^\top \sigma(Wx),$$  \hspace{1cm} (1)

where $W \in \mathbb{R}^{m \times d}$, $a \in \mathbb{R}^m$, and $\theta = \text{vec}\{a, W\}$, where “vec” stands for the standard vectorization operation and it will be used hereafter.

2. Fully Connected Deep Network

$$\begin{cases} 
    h[^0] = x, \\
    h[^l] = \sigma(W[^l]h[^{l-1}]), & l = 1, \cdots, L, \\
    f_{\text{FC}}(x; \theta) = u^\top h[^L], 
\end{cases}$$  \hspace{1cm} (2)

where $W[^l] \in \mathbb{R}^{m_l \times m_{l-1}}$, $m_0 = d$, $u \in \mathbb{R}^{m_L}$, and $\theta = \text{vec}\{\{W[^l]\}_l=1^L, u\}$.

3. Residual Network (ResNet) \[8, 15\]

$$\begin{cases} 
    h[^0] = Vx, \\
    g[^l] = \sigma(W[^l]h[^{l-1}]), & l = 1, \cdots, L, \\
    h[^l] = h[^{l-1}] + U[^l]g[^l], & l = 1, \cdots, L, \\
    f_{\text{Res}}(x; \theta) = u^\top h[^L], 
\end{cases}$$  \hspace{1cm} (3)

where $V \in \mathbb{R}^{D \times d}$, $W[^l] \in \mathbb{R}^{m \times D}$, $U[^l] \in \mathbb{R}^{D \times m}$, $u \in \mathbb{R}^D$, $D \geq d + 1$ and $\theta = \text{vec}\{V, \{W[^l], U[^l]\}_l=1^L, u\}$. 
4. Dense Network (DenseNet) [16]

\[
\begin{aligned}
    h^{[0]} &= Vx, \\
    g^{[l]} &= \sigma(W^{[l]}h^{[l-1]}), \quad l = 1, \ldots, L, \\
    h^{[l]} &= \left( h^{[l-1]} U^{[l]} g^{[l]} \right), \quad l = 1, \ldots, L, \\
    f_{\text{Dense}}(x; \theta) &= u^\top h^{[L]},
\end{aligned}
\]

where \( V \in \mathbb{R}^{k_0 \times d} \), \( W^{[l]} \in \mathbb{R}^{lm \times (k_0 + (l-1)k)} \), \( U^{[l]} \in \mathbb{R}^{k \times lm} \), \( u \in \mathbb{R}^{k_0 + Lk} \), \( k_0 \geq d + 1 \), and \( \theta = \text{vec} \{ V, \{ W^{[l]}, U^{[l]} \}_{l=1}^L, u \} \). For each \( l \), \( h^{[l]} \) is the output of layer \( l \), whose dimension is \( k_0 + lk \), where \( k \geq 1 \). Unlike ResNet, DenseNet uses concatenation instead of direct addition after “going through” the skip connection block. In particular, we observe that the dimension of \( h^{[l]} \) grows linearly with respect to the number of layers, and we term \( k \) the growth rate. Usually, a relatively small growth rate (such as ten or twelve) is sufficient to obtain state-of-the-art results on standard datasets, such as CIFAR-10 and ImageNet.

Remark 1. In practice, every neuron has a bias and a layer is calculated as \( \sigma(\hat{W}\hat{x} + b) \), where \( \hat{W} \) is the parameter matrix, \( \hat{x} \) is the original data, and \( b \) is the bias vector. We point out that our formulations for Fully Connected Deep Network above can represent the counterpart with biased neurons by considering the extended data \( x = (\hat{x}, 1) \) and parameter matrix

\[
W = \begin{bmatrix}
\hat{W} & b \\
0 & 1
\end{bmatrix}.
\]

2.2 Barron Space, Path Norm, and Rademacher Complexity

Inspired by [8, 10] and the references therein, we study a specific type of target functions. Recall that \( \Omega = [0, 1]^d \) is the unit hypercube, and we consider target functions with domain \( \Omega \).

Definition 1 (Barron function and Barron space). A function \( f : \Omega \to \mathbb{R} \) is called a Barron function if \( f \) admits the following expectation representation:

\[
f(x) = \mathbb{E}_{(a, w) \sim \rho} [a \sigma(w^\top x)], \tag{5}
\]

where \( \rho \) is a probability distribution over \( \mathbb{R}^{d+1} \).

For a Barron function, we define the Barron norm as

\[
\| f \|_B := \inf_{\rho \in \mathcal{P}_f} \mathbb{E}_{(a, w) \sim \rho} [a \| w \|_1], \tag{6}
\]
where $\mathcal{P}_f = \{ \rho \mid f(x) = \mathbb{E}_{(a,w) \sim \rho}[a\sigma(w^\top x)] \}$. 

Equipped with the Barron Norm (6), the Barron space $\mathcal{B}$ is the set of Barron functions with finite Barron norm, i.e.,

$$\mathcal{B} = \{ f : \Omega \to \mathbb{R} \mid \| f \|_B < \infty \}. \quad (7)$$

Normally, a Barron space contains functions with low complexity, such as sufficiently smooth functions in the Sobolev space $H^s(\mathbb{R}^d)$ for $s > \frac{d}{2} + 1$. It also contains non-smooth functions like those represented by two-layer neural networks. Moreover, Barron space is strictly bigger than the Reproducing Kernel Hilbert Space (RKHS) induced by the Neural Tangent Kernel (NTK) [18], and one may refer to [30, 31] for detailed discussions. On the other hand, generalization of neural networks in the NTK regime has been studied in [2, 6].

For a feedforward neural network, we define a parameter-based norm as an analog of the path norm of two-layer neural networks [10], and the $l_1$ path norm of the residual networks [8, 23]. We term it the weighted path norm. In order to describe the norm, we shall introduce firstly the concept of path. To start with, a path is an ordered sequence of scalar operations in the computational process of neural networks that originates from the input and ends at the output, which could also be viewed as a connected chain of edges in the computation graph. For example, a path in two-layer neural networks contains one parameter from the input layer, one nonlinear activation, and one parameter from the output layer connecting this activation to the output. For deep neural networks, a path contains a collection of linear operations (including trainable parameters and fixed parameters) and nonlinear counterparts. Let $\mathcal{P}$ be a path, and we denote the number of linear operations in the path $\mathcal{P}$ by $\text{len}(\mathcal{P})$, and $\text{nl}(\mathcal{P})$ for the number of nonlinearities the path goes through. Finally, let $\{w_i^{\mathcal{P}}\}_{i=1}^{\text{len}(\mathcal{P})}$ be the parameters associated with linear operations throughout the path $\mathcal{P}$, and we define the weighted path norm as follows.

**Definition 2 (Weighted path norm).** Given a network $f(\cdot; \theta)$, we define the weighted path norm of $f$ as

$$\| f \|_\mathcal{P} = \| \theta \|_\mathcal{P} = \sum_\mathcal{P} 3^{\text{nl}(\mathcal{P})} \prod_{l=1}^{\text{len}(\mathcal{P})} \| w_i^{\mathcal{P}} \|. \quad (8)$$

Heuristically speaking, the weighted path norm tends to take large account of the paths that undergoes more nonlinearities, i.e., by assigning bigger weights to paths going through more nonlinearities. The weight characterizes the increased complexity of the hypothesis space induced by nonlinearities. In particular, the weight factor
3\textsuperscript{nl}(P) in \cite{8} was first taken in \cite{8} on the a priori estimate for ResNet, it was taken for the convenience of analysis and may not be optimal. In the latest version of \cite{8}, the base number is reduced to 2. By using the symbol for the adjacency matrix representation proposed in Section 3.1, we come up with a more handy-but-equivalent characterization for the weighted path norm in Proposition 1, a cornerstone upon which some useful estimates are derived.

Finally, to bound the generalization gap, we recall the definition of Rademacher complexity.

**Definition 3 (Rademacher complexity).** Given a family of functions \( H \) and a set of samples \( S = \{z_i\}_{i=1}^n \), the (empirical) Rademacher complexity of \( H \) with respect to \( S \) is defined as

\[
\text{Rad}_S(H) = \frac{1}{n} \mathbb{E}_\tau \left[ \sup_{h \in H} \sum_{i=1}^n \tau_i h(z_i) \right],
\]

where the \( \{\tau_i\}_{i=1}^n \) are i.i.d. random variables with \( \mathbb{P}\{\tau_i = 1\} = \mathbb{P}\{\tau_i = -1\} = \frac{1}{2} \).

# 3 Nonlinear Weighted DAG and Adjacency Matrix Representation

In this section, we systematically present our novel representation for feedforward neural networks. We discuss several properties obtained from the incorporation of this new representation in Section 3.2 and some concrete examples are given out in Section 3.3 using the above-mentioned networks in Section 2.1.

## 3.1 Adjacency Matrix Representation and Symbols for DAG

We start this section by bringing out the definitions of the directed graph and neural network.

**Definition 4 (Directed graph).** A directed graph \( G = (V, E) \) is an ordered pair of sets. Here \( V \) is called the set of nodes (or vertices or neurons), and \( E \subset V \times V \) is called the set of edges (or more precisely, directed edges). For vertices \( v_i, v_j \in V \), if \( (v_j, v_i) \in E \), then the edge is denoted by \( e_{i \leftarrow j} \) and said to be directed from the tail \( v_j \) to the head \( v_i \). A cycle is a finite sequence of nodes \( v_0, v_1, \cdots, v_k \) such that \( v_0 = v_k \) and \( (v_i, v_{i+1}) \) is an edge for all \( i = 0, 1, \cdots, k - 1 \). A directed acyclic graph (DAG) is a directed graph that has no cycles.
Figure 1: a particular nonlinear weighted DAG for ResNet

**Definition 5** (Neural network). A neural network NN consists of an architecture (a directed graph) $G = (V, E)$ with the partition of edges $E := E_{\text{fix}} \sqcup E_{\text{para}} \sqcup E_{\text{non}}$, a collection of fixed weights/parameters (a real-valued function on $E_{\text{fix}}$) $\gamma : E_{\text{fix}} \to \mathbb{R}$, a collection of trainable weights/parameters (a real-valued function on $E_{\text{para}}$) $\theta : E_{\text{para}} \to \mathbb{R}$, and an activation (usually, a nonlinear function) $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$. Here $\sqcup$ is the disjoint union of sets. We write $\text{NN}(G, \theta, \gamma, \sigma)$ to signify this neural network with

\begin{align}
\gamma &:= \text{vec}\{\gamma_{ij} := \gamma(e_{i \leftarrow j}) \mid e_{i \leftarrow j} \in E_{\text{fix}}\}, \\
\theta &:= \text{vec}\{\theta_{ij} := \theta(e_{i \leftarrow j}) \mid e_{i \leftarrow j} \in E_{\text{para}}\}.
\end{align}

We also define $w : E_{\text{fix}} \sqcup E_{\text{para}} \to \mathbb{R}$ by setting $w(e_{i \leftarrow j}) := \gamma(e_{i \leftarrow j})$ on $E_{\text{fix}}$ and $w(e_{i \leftarrow j}) := \theta(e_{i \leftarrow j})$ on $E_{\text{para}}$. We further set $N := |E|$ as the number of edges, and $N_{\text{fix}} := |E_{\text{fix}}|$, $N_{\text{para}} := |E_{\text{para}}|$, and $N_{\text{non}} := |E_{\text{non}}|$. Obviously, $N = N_{\text{fix}} + N_{\text{para}} + N_{\text{non}}$.

We remark that $\gamma$ in (10) is pre-determined and fixed, while $\theta$ in (11) is trainable. For the existence results in main theorems, we refer to the existence of $\theta$ after the network architecture $G = (V, E)$, activation $\sigma$ and fixed weights $\gamma$ are given (See Theorem 1 and Theorem 3).

**Definition 6** (Feedforward neural network). A feedforward neural network is a neural network $\text{NN}(G, \theta, \gamma, \sigma)$ in which the graph $G = (V, E)$ contains no cycles. An input/source neuron (or output/sink neuron) is a vertex in $V$ that is not the head (or tail) of any edge in $E$. The set of input neurons and output neurons are denoted by $V_{\text{in}}$ and $V_{\text{out}}$ respectively. The vertices in $V_{\text{hid}} = V \setminus (V_{\text{in}} \cup V_{\text{out}})$ are called the hidden neurons. The input dimension and output dimension of the network are $d = |V_{\text{in}}|$ and $d' = |V_{\text{out}}|$.

Note that for all four examples given in Section 2.1, $d' = 1$.

**Definition 7** (Adjacency matrix representation). Given any feedforward neural network architecture $G = (V, E)$, activation $\sigma$, and fixed weights $\gamma$, we define its adjacency
matrix representation $A(\cdot, \gamma, \sigma) : \theta \mapsto A(\theta, \gamma, \sigma)$ for each $\theta : \mathcal{E}_{\text{para}} \to \mathbb{R}$, the image $A(\theta, \gamma, \sigma)$ is a matrix whose entries $(A(\theta, \gamma, \sigma))_{ij}$ are operators from $\mathbb{R}$ to $\mathbb{R}$. More precisely, given any $\theta : \mathcal{E}_{\text{para}} \to \mathbb{R}$, we define

$$(A(\theta, \gamma, \sigma))_{ij} = \begin{cases} 
\theta_{ij}, & e_{i\leftarrow j} \in \mathcal{E}_{\text{para}}, \\
\gamma_{ij}, & e_{i\leftarrow j} \in \mathcal{E}_{\text{fix}}, \\
\sigma(\cdot), & e_{i\leftarrow j} \in \mathcal{E}_{\text{non}}, \\
0, & \text{otherwise},
\end{cases}$$

(12)

where $\theta_{ij}$ and $\gamma_{ij}$ are considered as linear operators from $\mathbb{R}$ to $\mathbb{R}$, i.e., multiplication, and $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$ refers specifically to the operation of the activation function applied accordingly to its input.

We remark that Daniely [7] also used DAGs to represent architectures of neural networks. However, the way that the DAGs is used in our paper is different from [7] in which the nodes represent nonlinear operations and edges represent linear operations. In our paper, all the operations, linear or nonlinear, are represented by edges. Thus, all nonlinear activations are explicitly represented by entries of the adjacency matrix, which helps us to study various architectures with skip connections.

For simplicity, $A(\theta, \gamma, \sigma)$ is denoted by $A(\theta, \sigma)$ or even $A$ hereafter with no confusion. We claim that $A(\theta, \gamma, \sigma)$ is a nonlinear operator acting on $N$ dimensional vector-valued functions, where $N$ is given in Definition 5 and pre-determined by network architecture. We impose the nodes consisting of components from the training sample $x$ to be source nodes, and the single node of the output function the sink node. Since all connections are from nodes with smaller index to nodes with bigger index, then without loss of generality, $A$ can be written into a strictly lower triangular matrix. We also point out that there is no nonzero entries on the main diagonal of $A$, since we only study feedforward networks without recurrence. More precisely, if we set the value of the source nodes to be $h_1(x), \ldots, h_d(x)$ and the sink node $h_N(x)$, then $A$ is of size $N \times N$, and the output at $h_i(x)$ reads inductively for $i > d$,

$$h_i(x) = \sum_{j : e_{i \leftarrow j} \in \mathcal{E}_{\text{para}}} \theta_{ij} h_j(x) + \sum_{j : e_{i \leftarrow j} \in \mathcal{E}_{\text{fix}}} \gamma_{ij} h_j(x) + \sum_{j : e_{i \leftarrow j} \in \mathcal{E}_{\text{non}}} \sigma(h_j(x)).$$

(13)

Specifically, we define that

**Definition 8** (Feedforward neural network function). Given any feedforward neural network architecture $G = (V, E)$, activation $\sigma$, and fixed weights $\gamma$, then for each $\theta : \mathcal{E}_{\text{para}} \to \mathbb{R}$, we define its (feedforward) neural network function as a mapping $f(\cdot, \theta) : \mathbb{R}^d \to \mathbb{R}^{d'}$, where $d$ is the input dimension of the network, and $d'$ is the output
Figure 1: Symbol of the ResNet in Figure 1

Figure 2: Symbol of the ResNet in Figure 1

dimension of the network, such that for any training sample $x \in \mathbb{R}^d$,

$$f(x, \theta) := h_N(x) = \sum_{i: e_{N-1} \in E_{\text{para}}} \theta_{Nj} h_i(x) + \sum_{i: e_{N-1} \in E_{\text{fix}}} \gamma_{Nj} h_i(x) + \sum_{i: e_{N-1} \in E_{\text{non}}} \sigma(h_i(x)). \quad (14)$$

Next we define the symbol for the adjacency matrix representation.

**Definition 9** (Symbol). Given $A(\theta, \gamma, \sigma)$, we define the symbol $A(\theta, \gamma, \xi)$ as

$$(A(\theta, \gamma, \xi))_{ij} = \begin{cases} 
\theta_{ij}, & e_{i\leftarrow j} \in E_{\text{para}}, \\
\gamma_{ij}, & e_{i\leftarrow j} \in E_{\text{fix}}, \\
\xi, & e_{i\leftarrow j} \in E_{\text{non}}, \\
0, & \text{otherwise},
\end{cases} \quad (15)$$

where $\xi$ refers to the operation of direct multiplication of the numeric $\xi$ to its input.

In short, the symbol $A(\theta, \gamma, \xi)$ is the DAG representation of a linear neural network with weights $\xi$ where the original network had nonlinearities. We remark that the term symbol in this paper is inspired from the definition of the symbol for pseudo-differential operators [28]. But they are quite different because for the latter $\partial_i$ is replaced by $\xi_i$, not a single variable $\xi$ in all dimensions. Similarly, we denote $A(\theta, \gamma, \xi)$ by $A(\theta, \xi)$ hereafter, and symbol for ResNet illustrated in Figure 1 is shown in Figure 2. We observe that since $\xi$ performs exactly like the fixed weights $\gamma$, hence $\xi$ transforms the whole nonlinear connection elements belonging to set $E_{\text{non}}$ into new elements of set $E_{\text{fix}}$. Therefore, the blue dashed line in Figure 1 shall be replaced correspondingly by a solid line in Figure 2.
3.2 Several Properties of the Adjacency Matrix Representation

Given a feedforward neural network, naturally we obtain its adjacency matrix representation $A(\theta, \sigma)$, and for a specific input sample $x \in \mathbb{R}^d$, we define the series of vectors $\{z_s\}_{s=0}^{\infty}$:

$$
z_0 = (x^\top, 0, \ldots, 0)^\top = \left(\begin{array}{c} x \\ 0 \end{array}\right)_{(N-d) \times 1} \in \mathbb{R}^N$$

$$
z_s = z_0 + A(\theta, \sigma)z_{s-1}, \quad s \geq 1. \tag{16}$$

We observe that $A(\theta, \sigma)$ is of size $N \times N$. Moreover, we define two special vectors in $\mathbb{R}^N$

$$
1_{\text{in}} = \left(1,1,\ldots,1,0,0,\ldots,0\right)^\top = \left(\begin{array}{c} 1_{d \times 1} \\ 0_{(N-d) \times 1} \end{array}\right),
$$

$$
1_{\text{out}} = (0,0,\ldots,0,1)^\top = \left(0_{(N-1) \times 1} \right),
$$

and the projection matrix $P_0$ with respect to the input $P_0 = \text{diag}(1_{\text{in}})$, whose size is also $N \times N$. We list out several properties relating to the adjacency matrix representation $A(\theta, \sigma)$ and its symbol $A(\theta, \xi)$.

**Proposition 1.**

1. *(Nilpotent)* For any $A(\theta, \xi)$, there exists a positive integer $s_0$, such that for all $s \geq s_0$,

$$
A^s(\theta, \xi) = 0.
$$

2. *(\(L^\infty\) bound of the finite difference)* For all $s \geq 1$,

$$
z_{s+1} - z_s = A(\theta, \sigma)z_s - A(\theta, \sigma)z_{s-1},
$$

and

$$
\|z_{s+1} - z_s\|_{\infty} \leq \|A^s(\left|\theta\right|,1)\| \|z_1 - z_0\|_{\infty},
$$

where $|z|$ with $z$ being a vector means taking the absolute values of all the entries of the vector.
3. (Limit) There exists a limit for the series of vectors \( \{ z_s \}_{s=1}^{\infty} \), i.e.,

\[
    z_{\infty} = \lim_{s \to \infty} z_s.
\]

4. (Representation for network output) The output function of the neural network \( f(x) \) reads

\[
    f(x) = 1^T_{\text{out}} z_{\infty}.
\]

5. (Fixed point iteration) Define \( \bar{A} = P_0 + A \), then

\[
    z_s = \bar{A} z_{s-1}.
\]

Thus, \( z_{\infty} = \bar{A}^\infty z_0 := \lim_{s \to +\infty} \bar{A}^s z_0 \) exists and is a fixed point for the operator \( \bar{A}(\theta, \sigma) \).

6. (Alternate expression of weighted path norm) Given \( \theta \), its weighed path norm reads

\[
    \| \theta \|_{p} = 1^T_{\text{out}} \sum_{s=0}^{\infty} A^s(|\theta|, 3) 1_{\text{in}} = 1^T_{\text{out}} (I_{N \times N} - A(|\theta|, 3))^{-1} 1_{\text{in}}
\]

\[
    = 1^T_{\text{out}} \bar{A}^\infty (|\theta|, 3) 1_{\text{in}}. \tag{17}
\]

7. (Number of parameters and nonlinear connections) Given symbol \( A(\theta, \gamma, \xi) \), then

\[
    N_{\text{para}} = \# E_{\text{para}} = \| A(1_{\theta}, 0, 0) \|_{1,1},
\]

\[
    N_{\text{fix}} = \# E_{\text{fix}} = \| A(0, 1_{\gamma}, 0) \|_{1,1},
\]

\[
    N_{\text{non}} = \# E_{\text{non}} = \| A(0, 0, 1) \|_{1,1},
\]

where \( 1_{\theta} \) is obtained by replacing all the components of \( \theta \) by 1, and \( 1_{\gamma} \) is attained similarly by replacing all the components of \( \gamma \) by 1.

Remark 2. The proof of Proposition \( \square \) is given in Appendix \( \square \) and the alternate expression of weighted path norm in \( \square \) is useful in the proof of Lemma \( \square \) and Theorem \( \square \).
3.3 Examples

Individually, the adjacency matrix representation for each feedforward neural network mentioned beforehand in Section 2.1 is presented as follows.

**Example 1** (Two-layer network). The adjacency matrix representation \( A \) of the two-layer network \([1]\) reads

\[
\begin{pmatrix}
0_{d \times d} & W & 0_{m \times m} \\
W & \sigma I_{m \times m} & 0_{m \times m} \\
0_{m \times m} & u^T & 0
\end{pmatrix}.
\]

**Example 2** (Fully connected network). For fully connected deep network \([2]\), the adjacency matrix representation \( A \) takes the matrix form

\[
\begin{pmatrix}
0 & B^{[1]} & 0 \\
B^{[1]} & 0 & 0 \\
& & \ddots \\
& & & \ddots \\
& & & & B^{[L]} & 0 \\
& & & & 0 & u^T
\end{pmatrix},
\]

where for each \( l = 1, \cdots, L \), the matrix block \( B^{[l]} \) reads

\[
B^{[l]} = \begin{pmatrix}
W^{[l]} & 0_{m_l \times m_l} \\
& \sigma I_{m_l \times m_l}
\end{pmatrix}.
\]

For Example 3 and Example 4, the matrix representation \( A \) incorporates the form

\[
\begin{pmatrix}
0 & V & 0 \\
V & 0 & 0 \\
0 & B^{[1]} & 0 \\
B^{[1]} & 0 & 0 \\
& & \ddots \\
& & & \ddots \\
& & & & B^{[L]} & 0 \\
& & & & 0 & u^T
\end{pmatrix},
\]

with \( B^{[l]} \) to be specified for each case.
Example 3 (ResNet). For ResNet (3), each $B^{[l]}$ reads

$$B^{[l]} = \begin{pmatrix} W^{[l]} & 0_{m \times m} \\ \sigma I_{m \times m} & 0_{m \times m} \\ I_{D \times D} & U^{[l]} \end{pmatrix}. $$

Example 4 (DenseNet). For DenseNet (4), each $B^{[l]}$ reads

$$B^{[l]} = \begin{pmatrix} W^{[l]} & 0_{lm \times lm} \\ \sigma I_{lm \times lm} & 0_{lm \times lm} \\ \bar{I}_{(k_0+lk) \times (k_0+(l-1)k)} & U^{[l]} \end{pmatrix},$$

where for each $l = 1, \ldots, L$,

$$\bar{I}_{(k_0+lk) \times (k_0+(l-1)k)} = \begin{pmatrix} I_{(k_0+(l-1)k) \times (k_0+(l-1)k)} \\ 0_{k \times (k_0+(l-1)k)} \end{pmatrix}, \quad U^{[l]} = \begin{pmatrix} 0_{(k_0+(l-1)k) \times lm} \\ \bar{U}^{[l]} \end{pmatrix}. $$

4 Main Results

4.1 Setup

The goal of the supervised learning is to find a network function that fits the training samples and also generalizes well on test data. Our problem of interest is to learn a function from a sample dataset of $n$ examples $S := \{(x_i, y_i)\}_{i=1}^n$ drawn i.i.d from an underlying distribution $D$, where for each $i$, $x_i \in \Omega = [0, 1]^d$, and our target function is $f^* : \Omega \to [0, 1]$ with $y_i = f^*(x_i) \in [0, 1]$. Similar to the cases of ResNet [8] and two-layer [10], a truncation operator shall be defined such that for any function $h : \mathbb{R}^d \to \mathbb{R}$, $T_{[0,1]} h(x) = \min \{\max \{h(x), 0\}, 1\}$. With an abuse of notation, we still use $f$ to denote $T_{[0,1]} f$ henceforth. Consider the truncated square loss

$$\ell(x, \theta) = \frac{1}{2} \left| T_{[0,1]} f(x; \theta) - f(x) \right|^2 \quad (18)$$

in the sequel, then the empirical risk is defined as

$$R_S(\theta) = \frac{1}{n} \sum_{i=1}^n \ell(x_i, \theta), \quad (19)$$

and the population risk is defined as

$$R_D(\theta) = \mathbb{E}_{x \sim \mathcal{D}} \ell(x, \theta). \quad (20)$$

The ultimate goal of our paper is to minimize $R_D(\theta)$. 

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4.2 Main Theorems

We consider a feedforward neural network with its adjacency matrix representation satisfying the following assumptions.

**Assumption 1.** Given any feedforward neural network architecture \( G = (V,E) \), activation \( \sigma \), and fixed weights \( \gamma \), we assume that for any edge \( e_{i\rightarrow j} \) with \( j \leq d \), \( e_{i\rightarrow j} \in E_{\text{para}} \). In other words, we assume that there exists no edge \( e_{i\rightarrow j} \) with \( j \leq d \), such that \( e_{i\rightarrow j} \in E_{\text{fix}} \) or \( e_{i\rightarrow j} \in E_{\text{non}} \).

**Assumption 2 (Shortcut connections).** Given any feedforward neural network architecture \( G = (V,E) \), activation \( \sigma \), and fixed weights \( \gamma \), we assume that for any \( \theta : E_{\text{para}} \rightarrow \mathbb{R} \) its adjacency matrix representation takes either the form

\[
A(\theta, \gamma, \sigma) = \begin{pmatrix}
0 & V & 0 \\
V & 0 & \sigma I \\
\sigma I & 0 & u^\top \\
\end{pmatrix},
\]

or the form

\[
A(\theta, \gamma, \sigma) = \begin{pmatrix}
0 & V & 0 & B[1] & 0 & \cdots & \cdots & \cdots & B[L] & 0 \\
V & 0 & B[1] & 0 & \cdots & \cdots & \cdots & B[L] & 0 \\
B[1] & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & B[L] & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
B[L] & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & B[L] & 0 \\
u^\top & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\end{pmatrix},
\]

with \( V \) taking size \( d_0 \times d \), length of vector \( u \) being \( d_L \), and for each matrix block \( B[l] \):

\[
B[l] = \begin{pmatrix}
W[l] & \sigma I_{p_l \times p_l} \\
S[l] & U[l] \\
\end{pmatrix},
\]

where \( B[l] \) has size \( (2p_l + d_l) \times (2p_l + d_{l-1}) \), its components \( \{W[l], \sigma I_{p_l \times p_l}, S[l], U[l]\} \) respectively has size \( p_l \times d_{l-1}, p_l \times p_l, d_l \times d_{l-1}, \) and \( d_l \times p_l \). Moreover, we assume further that \( S[l] \) is a row permutation matrix of \( \begin{pmatrix} I_{d_{l-1} \times d_{l-1}} \end{pmatrix} \), and for all \( l = 1, \ldots, L \), it holds that \( d_l \geq d_{l-1}, \min\{d_0, d_l, p_l\} \geq d + 1 \).
Evidently, any feedforward neural network satisfying Assumption 2 satisfies Assumption 1. With this in mind, we proceed to the statement of our main theorems.

Theorem 1 (Approximation error). Given any feedforward neural network architecture $G = (V, E)$, activation $\sigma$, fixed weights $\gamma$, suppose that Assumption 2 holds, then for any target function $f^* \in \mathcal{B}$, there exists a feedforward neural network function $f(\cdot; \tilde{\theta})$ with $\|\tilde{\theta}\|_P \leq 6\|f^*\|_B$, such that

$$R_D(\tilde{\theta}) := \mathbb{E}_{x \sim D} \frac{1}{2}(f(x; \tilde{\theta}) - f^*(x))^2 \leq \frac{3\|f^*\|^2_B}{2N_{non}}.$$  \hspace{1cm} (23)

Theorem 2 (A posteriori estimate). Given any feedforward neural network architecture $G = (V, E)$, activation $\sigma$, fixed weights $\gamma$, suppose that Assumption 2 holds, then for any $\delta \in (0, 1)$, with probability at least $1 - \delta$ over the choice of the training sample $S$, the following holds

$$|R_D(\theta) - R_S(\theta)| \leq (\|\theta\|_P + 1) \frac{6\sqrt{2\log(2d)}}{\sqrt{n}} + \frac{1}{2\sqrt{2}} + \frac{1}{2} \sqrt{\frac{\log(\frac{2d}{\delta})}{2n}}.$$  \hspace{1cm} (24)

Theorem 3 (A priori estimate). Given any feedforward neural network architecture $G = (V, E)$, activation $\sigma$, fixed weights $\gamma$, suppose that Assumption 2 holds, $f^* \in \mathcal{B}$, $\lambda = \Omega(\sqrt{\log d})$, and that $\theta_{S,\lambda}$ is an optimal solution for the regularized model

$$J_{S,\lambda}(\theta) := R_S(\theta) + \frac{\lambda}{\sqrt{n}}\|\theta\|_P,$$  \hspace{1cm} (25)

then for any $\delta \in (0, 1)$, with probability at least $1 - \delta$ over the choice of the training sample $S$, the population risk satisfies

$$R_D(\theta_{S,\lambda}) := \mathbb{E}_{x \sim D} \frac{1}{2}(f(x; \theta_{S,\lambda}) - f^*(x))^2 \leq \frac{\|f^*\|^2_B}{N_{non}} + \frac{1}{\sqrt{n}} \left(\lambda(f^* + 1) + \sqrt{\log 1/\delta}\right).$$  \hspace{1cm} (26)

Remark 3. As is shown in Section 3.3, some of the examples mentioned in Section 2.1, i.e., Two-layer Network, ResNet and DenseNet satisfy Assumption 2.

5 Proof of Theorems and Applications

5.1 Approximation Error

First and foremost, in order to prove Theorem 1 we recall the result obtained in [10, Proposition 2.1].
**Theorem 4** (Approximation error for two-layer networks). For any target function $f^* \in \mathcal{B}$, there exists a two-layer network $f_{2\text{Layer}}(\cdot; \theta_{2\text{Layer}})$ of width $m$, such that

$$
E_{x \sim D} \frac{1}{2} (f_{2\text{Layer}}(x; \theta_{2\text{Layer}}) - f^*(x))^2 \leq \frac{3\|f^*\|^2_{\mathcal{B}}}{2m},
$$

with its parameters $\theta_{2\text{Layer}} = \{a_k, w_k\}_{k=1}^m$ satisfying

$$
\sum_{k=1}^m |a_k| \|w_k\|_1 \leq 2\|f^*\|_{\mathcal{B}},
$$

and the output reads $f_{2\text{Layer}}(x; \theta_{2\text{Layer}}) = \sum_{k=1}^m a_k \sigma(w_k^\top x)$.

**Proof of Theorem 4**. Given a feedforward network $f(\cdot; \tilde{\theta})$ with input dimension $d$ and its adjacency matrix representation:

$$
A(\theta, \gamma, \sigma) = \begin{pmatrix}
0 \\
V & 0 \\
\sigma I & 0 \\
u^\top & 0
\end{pmatrix}.
$$

Representation belonging to this case can be easily proved by observing that the total number of nonlinearities for a two-layer network with width $m$ is exactly $m$, then we may apply Theorem 4 directly to obtain the results.

In the case of its adjacency matrix representation being:

$$
A(\theta, \gamma, \sigma) = \begin{pmatrix}
0 \\
V & 0 \\
B[1] & 0 \\
& B[2] & 0 \\
& & \ddots & \ddots \\
& \& \& B[l] & 0 \\
& \& \& & \ddots & \ddots \\
& \& \& \& \& B[L] & 0 \\
\& \& \& \& \& \& \& u^\top & 0
\end{pmatrix},
$$

Without loss of generality, for all $l = 1, 2, \cdots, L$, set $S[l]$ as

$$
S[l] = \begin{pmatrix}
I_{d_{l-1} \times d_{l-1}} \\
0_{(d_l-d_{l-1}) \times d_{l-1}}
\end{pmatrix}.
$$
From Theorem 4, there exists a two-layer network $f_{2\text{Layer}}(\cdot; \theta_{2\text{Layer}})$ of width $m_L$, with its parameters $\theta_{2\text{Layer}} = \{a_k, w_k\}_{k=1}^{m_L}$ satisfying $\sum_{k=1}^{m_L} \|a_k\| \|w_k\|_1 \leq 2\|f^*\|_B$, and
the output reads $f_{2\text{Layer}}(x; \theta_{2\text{Layer}}) = \sum_{k=1}^{m_L} a_k \sigma(w_k^\top x)$, fulfilling that
\[
\mathbb{E}_{x \sim \mathcal{D}} \frac{1}{2} (f_{2\text{Layer}}(x; \theta_{2\text{Layer}}) - f^*(x))^2 \leq \frac{3\|f^*\|_B^2}{2m_L}.
\]
Set $m_l = \sum_{k=1}^{l} p_k$, and $m_0 = 0$ for the purpose of completion, we notice that $N_{\text{non}} = m_L = \sum_{k=1}^{L} p_k$.

Existence of the feedforward network $f(\cdot; \tilde{\theta})$ shall be proved by construction. For each $l = 1, 2, \cdots, L$, we have
\[
V = \begin{pmatrix}
I_{d \times d} \\
0_{(d_0 - d) \times d}
\end{pmatrix}, \quad u = \begin{pmatrix}
0_{(d_L - 1) \times 1} \\
1
\end{pmatrix},
\]
\[
W[l] = \begin{pmatrix}
w_{(m_l-1)+1}^\top & 0_{1 \times (d_l-1-d)} \\
w_{(m_l-1)+2}^\top & 0_{1 \times (d_l-1-d)} \\
\vdots & \vdots \\
w_{(m_l-1)+p_l}^\top & 0_{1 \times (d_l-1-d)}
\end{pmatrix},
\]
\[
U[l] = \begin{pmatrix}
0_{(d_l-1) \times 1} & 0_{(d_l-1) \times 1} & \cdots & 0_{(d_l-1) \times 1} \\
0_{(m_l-1) + 1} & a_{(m_l-2) + 1} & \cdots & a_{(m_l-1)+p_l}
\end{pmatrix}.
\]
One can easily verify that $\tilde{\theta} = \text{vec}\{V, \{W[l], U[l]\}_{l=1}^{L}, u\}$, with
\[
\|\tilde{\theta}\|_P = 1_{\text{out}}^\top \bar{A}^\infty (|\tilde{\theta}|; 3) 1_{\text{in}} = 3 \sum_{j=1}^{m_L} |a_j| \|w_j\|_1 \leq 6\|f^*\|_B.
\]
Moreover, $f(x; \tilde{\theta}) = f_{2\text{Layer}}(x; \theta_{2\text{Layer}}) = \sum_{j=1}^{m_L} a_j \sigma(w_j^\top x)$, thus
\[
\mathbb{E}_{x \sim \mathcal{D}} \frac{1}{2} \left( f(x; \tilde{\theta}) - f^*(x) \right)^2 \leq \frac{3\|f^*\|_B^2}{2m_L},
\]
since the total number of nonlinearities $N_{\text{non}} = \sum_{k=1}^{L} p_k = m_L$, we finish our proof.

\[
5.2 \text{ Rademacher Complexity}
\]

In this part, we endeavor to bound the (empirical) Rademacher complexity of networks with path norm $\|\theta\|_P \leq Q$. Let $\mathcal{H}^N_Q = \{f(\cdot; \theta) : \|\theta\|_P \leq Q\}$ be the set of
feedforward neural networks satisfying Assumption 1 with a total of $N$ nodes, $N > d$. It is evident that for any fixed $N$ and $Q > 0$, $0 \in \mathcal{H}_Q^N$, where 0 refers to the zero function that maps any input to the numeric 0, i.e., for all $x \in \mathbb{R}^d$, $0(x) \equiv 0$.

**Lemma 1.** For any fixed $N$ and $Q > 0$, $\mathcal{H}_Q^N \subseteq \mathcal{H}_Q^\overline{N}$, for all $\overline{N} > N$. Moreover, $\mathcal{H}_Q^N = Q \mathcal{H}_1^N$.

**Proof.** From the positive homogeneity of ReLU, it is obvious that $\mathcal{H}_Q^N = Q \mathcal{H}_1^N$.

We proceed to prove $\mathcal{H}_Q^N \subseteq \mathcal{H}_Q^\overline{N}$. For any fixed $N$ and $Q > 0$, $\mathcal{H}_N^\overline{N} = \mathcal{H}_N^1$.

Let $A$ be its adjacency matrix representation, then $A$ is of size $N \times N$. Let $E$ be a matrix of size $(N - N) \times N$ with its bottom right entry being 1, and other components equal to zero, i.e.,

$$E = \begin{pmatrix} 0_{(N-N-1) \times (N-1)} & 0_{(N-N-1) \times 1} \\ 0_{1 \times (N-1)} & 1 \end{pmatrix},$$

Moreover, we set

$$\tilde{A} = \begin{pmatrix} A & 0_{N \times (\overline{N} - N)} \\ E & 0_{(\overline{N} - N) \times (\overline{N} - N)} \end{pmatrix}.$$  

Then, $\tilde{A}$ is of size $\overline{N} \times \overline{N}$, and it is the adjacency matrix representation of a feedforward neural network $\tilde{f}(\cdot; \overline{\theta})$ with $\overline{N}$ nodes. Thus, for some $\overline{Q} > 0$, $\tilde{f}(\cdot; \overline{\theta}) \in \mathcal{H}_{\overline{Q}}^\overline{N}$.

Next, we need to compute the path norm of $\tilde{f}(\cdot; \overline{\theta})$. Let

$$1_{\text{in}}^N = \begin{pmatrix} 1_d \times 1 \\ 0_{(N-d) \times 1} \end{pmatrix}, 1_{\text{in}}^\overline{N} = \begin{pmatrix} 1_d \times 1 \\ 0_{(\overline{N} - d) \times 1} \end{pmatrix}, 1_{\text{out}}^{N-1} = \begin{pmatrix} 0_{(N-1) \times 1} \\ 1 \end{pmatrix}, 1_{\text{in}}^{\overline{N}-1} = \begin{pmatrix} 0_{(\overline{N} - 1) \times 1} \\ 1 \end{pmatrix},$$

then directly from (17) obtained in Proposition 1 we have

$$\|\overline{\theta}\|_P = \sum_{k=1}^{\infty} (1_{\text{out}}^\overline{N})^T \begin{pmatrix} A^k & 0_{N \times (\overline{N} - N)} \\ EA^{k-1} & 0_{(\overline{N} - N) \times (\overline{N} - N)} \end{pmatrix} 1_{\text{in}}^\overline{N} = (1_{\text{out}}^{\overline{N}-1})^T \sum_{k=1}^{\infty} A^k 1_{\text{in}}^{\overline{N}} = \|\theta\|_P \leq Q.$$

Hence, it holds that $\tilde{f}(\cdot; \overline{\theta}) \in \mathcal{H}_{\overline{Q}}^\overline{N}$.

Finally, since the function outputs satisfy that for all $x \in \mathbb{R}^d$, $\tilde{f}(x; \overline{\theta}) = f(x; \theta)$, we conclude that $\mathcal{H}_Q^N \subseteq \mathcal{H}_Q^\overline{N}$.

We set $\mathcal{H}_N^N = \bigcup_{Q > 0} \mathcal{H}_Q^N$, then the next lemma gives a decomposition for any network in $\mathcal{H}_N^N$. 

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Lemma 2. Given an input sample \( \mathbf{x} = (x_1, x_2, \cdots, x_d)^T \), for any \( f^N \in \mathcal{H}^N \), it can be decomposed into linear and nonlinear parts:

\[
f^N(\mathbf{x}) = \sum_{1 \leq i \leq d} a_i x_i + \sum_{d+1 \leq i \leq N-1} a_i \sigma(f^i(\mathbf{x})),
\]

where for each index \( i \), \( a_i \) is a scalar, and \( f^i \in \mathcal{H}^i \).

Moreover, we have

\[
\left( \sum_{1 \leq i \leq d} |a_i| + 3 \sum_{d+1 \leq i \leq N-1} |a_i| \|f^i\|_p \right) \leq \|f^N\|_p.
\]

Lemma 2 are essentially proved using mathematical induction. (29) reveals the components of function \( f^N(\mathbf{x}) \), and (30) is essentially proved by repeatedly using triangle inequality.

Proof. We prove (29) and (30) by induction. Firstly, when \( N = d + 1 \), then directly from Assumption 1, we have for any \( f^{d+1} \in \mathcal{H}^{d+1} \), there exists coefficients \( a_i \) with \( 1 \leq i \leq d \), such that

\[
f^{d+1}(\mathbf{x}) = \sum_{1 \leq i \leq d} a_i x_i,
\]

then (29) holds trivially. Moreover,

\[
\|f^{d+1}\|_p = \sum_{1 \leq i \leq d} |a_i|,
\]

then inequality (30) on path norm also holds trivially.

Secondly, we assume that (29) and (30) holds for \( d + 1, d + 2, \cdots, N \), then we proceed to show that they hold true for \( N + 1 \). For any \( f^{N+1} \in \mathcal{H}^{N+1} \), we have

\[
f^{N+1}(\mathbf{x}) = \sum_{1 \leq i \leq d, \ e_{N+1-i} \in E_{\text{para}}} w(e_{N+1-i}) x_i
\]

\[
+ \sum_{d+1 \leq i \leq N, \ e_{N+1-i} \in E_{\text{para}} \cup E_{\text{fix}}} w(e_{N+1-i}) f^i(\mathbf{x}) + \sum_{d+1 \leq i \leq N, \ e_{N+1-i} \in E_{\text{non}}} \sigma(f^i(\mathbf{x})).
\]

From the induction hypothesis, for any index \( i \) with \( d + 1 \leq i \leq N \), then there exist constants \( a_{i,j} \) with \( d + 1 \leq j \leq i - 1 \), such that

\[
f^i(\mathbf{x}) = \sum_{1 \leq j \leq d} a_{i,j} x_j + \sum_{d+1 \leq j \leq i-1} a_{i,j} \sigma(f^j(\mathbf{x})).
\]

By plugging (32) into (31), we obtain that

\[
f^{N+1}(\mathbf{x}) = \sum_{1 \leq i \leq d, \ e_{N+1-i} \in E_{\text{para}}} w(e_{N+1-i}) x_i
\]
\[ 
\frac{w(e_{N+1-i})}{w(e_{N+1-i})} \left( \sum_{1 \leq j \leq d} a_{i,j} x_j + \sum_{d+1 \leq j \leq i-1} a_{i,j} \sigma(f^j(x)) \right) \\
+ \sum_{d+1 \leq i \leq N, e_{N+1-i} \in E_{\text{non}}} \sigma(f^i(x)) = \sum_{1 \leq i \leq d} a_{i} x_i + \sum_{d+1 \leq i \leq N} a_{i} \sigma(f^i(x)), 
\]

where for \(1 \leq i \leq d\),
\[ 
a_i = \begin{cases} 
    w(e_{N+1-i}) + \sum_{d+1 \leq j \leq N, e_{N+1-i} \in E_{\text{para}} E_{\text{fix}}} w(e_{N+1-j}) a_{j,i}, & e_{N+1-i} \in E_{\text{para}}, \\
    \sum_{d+1 \leq j \leq N, e_{N+1-i} \in E_{\text{para}} E_{\text{fix}}} w(e_{N+1-j}) a_{j,i}, & \text{otherwise},
\end{cases} \tag{33} 
\]

and for \(d+1 \leq i \leq N\),
\[ 
a_i = \begin{cases} 
    1 + \sum_{i+1 \leq j \leq N, e_{N+1-i} \in E_{\text{para}} E_{\text{fix}}} w(e_{N+1-j}) a_{j,i}, & e_{N+1-i} \in E_{\text{non}}, \\
    \sum_{i+1 \leq j \leq N, e_{N+1-i} \in E_{\text{para}} E_{\text{fix}}} w(e_{N+1-j}) a_{j,i}, & \text{otherwise},
\end{cases} \tag{34} 
\]

(33) and (34) guarantee existence of the decomposition (29) for the case \(N + 1\).

Finally, for the norm inequality (30), we notice that from (31),
\[ 
\|f^{N+1}\|_P = \sum_{1 \leq i \leq d, e_{N+1-i} \in E_{\text{para}}} |w(e_{N+1-i})| \\
+ \sum_{d+1 \leq i \leq N, e_{N+1-i} \in E_{\text{para}} E_{\text{fix}}} |w(e_{N+1-i})| \|f^i\|_P \\
+ 3 \sum_{d+1 \leq i \leq N, e_{N+1-i} \in E_{\text{non}}} \|f^i\|_P \\
\geq \sum_{1 \leq i \leq d, e_{N+1-i} \in E_{\text{para}}} |w(e_{N+1-i})| \\
+ \sum_{d+1 \leq i \leq N, e_{N+1-i} \in E_{\text{para}} E_{\text{fix}}} |w(e_{N+1-i})| \left( \sum_{1 \leq j \leq d} |a_{i,j}| + 3 \sum_{d+1 \leq j \leq i-1} |a_{i,j}| \|f^j\|_P \right) \\
+ 3 \sum_{d+1 \leq i \leq N, e_{N+1-i} \in E_{\text{non}}} \|f^i\|_P \\
= \sum_{1 \leq i \leq d} b_i + \sum_{d+1 \leq i \leq N} b_i \|f^i\|_P, 
\]

where for \(1 \leq i \leq d\),
\[ 
b_i = \begin{cases} 
    |w(e_{N+1-i})| + \sum_{d+1 \leq j \leq N, e_{N+1-i} \in E_{\text{para}} E_{\text{fix}}} |w(e_{N+1-j})| |a_{j,i}|, & e_{N+1-i} \in E_{\text{para}}, \\
    \sum_{d+1 \leq j \leq N, e_{N+1-j} \in E_{\text{para}} E_{\text{fix}}} |w(e_{N+1-j})| |a_{j,i}|, & \text{otherwise},
\end{cases} \tag{36} 
\]
then inequality $b_i \geq |a_i|$ holds for all $1 \leq i \leq d$. For $d + 1 \leq i \leq N$,

$$b_i = \begin{cases} 
3 + 3 \sum_{1 \leq j \leq N, e_{N+1-i} \in E_{\text{par}\cup E_{\text{fix}}}} |w(e_{N+1-i})||a_{j,i}|, & e_{N+1-i} \in E_{\text{non}}, \\
3 \sum_{1 \leq j \leq N, e_{N+1-i} \in E_{\text{par}\cup E_{\text{fix}}}} |w(e_{N+1-i})||a_{j,i}|, & \text{otherwise},
\end{cases} \quad (37)$$

then inequality $b_i \geq 3|a_i|$ holds for all $d + 1 \leq i \leq N$. Hence last line of (35) becomes

$$\sum_{1 \leq i \leq d} b_i + \sum_{d+1 \leq i \leq N} b_i \|f^i\|_P \geq \sum_{1 \leq i \leq d} |a_i| + 3 \sum_{d+1 \leq i \leq N} |a_i| \|f^i\|_P, \quad (38)$$

thus, we have

$$\sum_{1 \leq i \leq d} |a_i| + 3 \sum_{d+1 \leq i \leq N} |a_i| \|f^i\|_P \leq \|f^{N+1}\|_P, \quad (39)$$

which completes proof of the norm inequality (30) for the case $N + 1$, and we finish our proof.

Next we bound the Rademacher complexity of $H_Q^N$.

**Theorem 5.** Let $\text{Rad}_S(H_Q^N)$ be the empirical Rademacher complexity of $H_Q^N$ with respect to the samples $S = \{x_i\}_{i=1}^n \subseteq \Omega = [0,1]^d$, then for each $N > d$, we have

$$\text{Rad}_S(H_Q^N) \leq 3Q \sqrt{\frac{2 \log(2d)}{n}}. \quad (40)$$

**Proof.** We shall prove (40) by induction. When $N = d + 1$, by Assumption 1, we have for any $f^{d+1} \in H_Q^{d+1}$ and any sample $z = (z_1, z_2, \ldots, z_d)^T$, there exists coefficients $a_i$ with $1 \leq i \leq d$, such that

$$f^{d+1}(z) = \sum_{1 \leq i \leq d} a_i z_i.$$

We observe that $f^{d+1}(z)$ is $f^{d+1}(z) = a^T z$, with $a = (a_1, a_2, \ldots, a_d)^T$, then directly from Lemma 26.11 of [26], the empirical Rademacher complexity of $G_1 = \{g \mid g(z) = b^T z, \|b\|_1 \leq 1\}$ satisfies

$$\text{Rad}_S(G_1) \leq \max_i \|x_i\|_\infty \sqrt{\frac{2 \log(2d)}{n}} \leq \sqrt{\frac{2 \log(2d)}{n}}.$$

Thus, function $\frac{1}{Q} f^{d+1}(z) = \frac{a^T z}{Q} \in G_1$, since $\frac{1}{Q} \|a\|_1 = \frac{\sum_{1 \leq i \leq d} |a_i|}{Q} \leq 1$, hence,

$$\frac{1}{Q} \text{Rad}_S(H_Q^{d+1}) \leq \sqrt{\frac{2 \log(2d)}{n}},$$

and (40) holds for $N = d + 1$. 

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Next, we assume that (40) holds for \(d + 1, d + 2, \cdots, N\), and we consider the case \(N + 1\). By definition of Rademacher complexity,

\[
n\text{Rad}_S(H_Q^{N+1}) = \mathbb{E}_\tau \sup_{f^{N+1} \in H_Q^{N+1}} \sum_{j=1}^n \tau_j f^{N+1}(x_j),
\]

from (29) in Lemma 2, RHS of the equation reads

\[
\mathbb{E}_\tau \sup_{f^{N+1} \in H_Q^{N+1}} \sum_{j=1}^n \tau_j f^{N+1}(x_j)
\leq \mathbb{E}_\tau \sup_{(C1)} \sum_{j=1}^n \tau_j \left( \sum_{1 \leq i \leq d} a_i(x_j)_i + \sum_{d+1 \leq i \leq N} a_i \sigma(f^i(x_j)) \right)
\]

where condition (C1) reads

\[
C1 : \left\{ \{a_i\}_{i=1}^N \left| \left( \sum_{1 \leq i \leq d} |a_i| + 3 \sum_{d+1 \leq i \leq N} |a_i| \|f^i\|_P \right) \leq Q \right. \right\}.
\]

Then, by taking out the supremum and positive homogeneity of ReLU, we have

\[
n\text{Rad}_S(H_Q^{N+1}) \leq \mathbb{E}_\tau \sup_{(C1)} \sum_{j=1}^n \tau_j \left( \sum_{1 \leq i \leq d} a_i(x_j)_i \right) + \mathbb{E}_\tau \sup_{(C1)} \sum_{j=1}^n \tau_j \left( \sum_{d+1 \leq i \leq N} a_i \sigma(f^i(x_j)) \right)
\leq \left( \sup_{(C1)} \sum_{1 \leq i \leq d} |a_i| \right) n\text{Rad}_S(G_1) + \left( \sup_{(C1)} \sum_{d+1 \leq i \leq N} |a_i| \|f^i\|_P \right) \mathbb{E}_\tau \sup_{g^i \in H^i_1} \left| \sum_{j=1}^n \tau_j \sigma(g^i(x_j)) \right|
\]

since zero function 0 is contained in the set \(H_Q^{N+1}\), then for any \(\tau_1, \tau_2, \cdots, \tau_n\), it holds that

\[
\sup_{g^i \in H^i_1} \sum_{j=1}^n \tau_j \sigma(g^i(x_j)) \geq 0,
\]

hence, we have

\[
\sup_{g^i \in H^i_1} \sum_{j=1}^n \tau_j \sigma(g^i(x_j)) + \sup_{g^i \in H^i_1} \sum_{j=1}^n -\tau_j \sigma(g^i(x_j)) \geq \max \left\{ \sup_{g^i \in H^i_1} \sum_{j=1}^n \tau_j \sigma(g^i(x_j)), \sup_{g^i \in H^i_1} \sum_{j=1}^n -\tau_j \sigma(g^i(x_j)) \right\}
\geq \sup_{g^i \in H^i_1} \left| \sum_{j=1}^n \tau_j \sigma(g^i(x_j)) \right|,
\]

which implies that

\[
\mathbb{E}_\tau \sup_{g^i \in H^i_1} \left| \sum_{j=1}^n \tau_j \sigma(g^i(x_j)) \right| \leq 2\mathbb{E}_\tau \sup_{g^i \in H^i_1} \sum_{j=1}^n \tau_j \sigma(g^i(x_j)) := 2n\text{Rad}_S(\sigma \circ H^i_1),
\]

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then directly from Lemma 26.9 of [26], the empirical Rademacher complexity of \( \text{Rad}_S(\sigma \circ H_i^1) \) satisfies
\[
\text{Rad}_S(\sigma \circ H_i^1) \leq \text{Rad}_S(H_i^1).
\]
From our induction hypothesis, we have that
\[
\text{Rad}_S(H_i^1) \leq 3\sqrt{\frac{2\log(2d)}{n}},
\]
thus
\[
\text{Rad}_S(\sigma \circ H_i^1) \leq 3\sqrt{\frac{2\log(2d)}{n}}.
\]
Finally,
\[
n\text{Rad}_S(H_Q^{N+1}) \\
\leq \left( \sup_{(C1)} \sum_{1 \leq i \leq d} |a_i| \right) n\text{Rad}_S(G_1) + \left( \sup_{(C1)} \sum_{d+1 \leq i \leq N} |a_i||f^i||_p \right) \mathbb{E}_z \sup_{g^i \in H_i^1} \left| \sum_{j=1}^n \tau_j \sigma(g^i(x_j)) \right| \\
\leq \left( \sup_{(C1)} \sum_{1 \leq i \leq d} |a_i| \right) \sqrt{2n \log(2d)} + 6 \left( \sup_{(C1)} \sum_{d+1 \leq i \leq N} |a_i||f^i||_p \right) \sqrt{2n \log(2d)} \\
\leq \left( \sup_{(C1)} \sum_{1 \leq i \leq d} |a_i| + 6 \sum_{d+1 \leq i \leq N} |a_i||f^i||_p \right) \sqrt{2n \log(2d)} \\
\leq (Q + 2Q)\sqrt{2n \log(2d)} = 3Q\sqrt{2n \log(2d)},
\]
which completes the proof of (40) for the case \( N + 1 \), thus we finish our proof.

5.3 A Posteriori and A Priori Estimates

We proceed to prove our main theorems, our proofs are extensions of the proofs in [8, 10]. Firstly, we shall introduce the following theorem introduced in [26].

**Theorem 6.** Fix a hypothesis space \( \mathcal{F} \). Assume that for any \( f \in \mathcal{F} \) and \( z, 0 \leq f(z) \leq B \), then for any \( \delta > 0 \), with probability at least \( 1 - \delta \) over the choice of \( S = (z_1, z_2, \cdots, z_n) \), we have for any function \( f(\cdot) \),
\[
\left| \frac{1}{n} \sum_{i=1}^n f(z_i) - \mathbb{E}_z f(z) \right| \leq 2\mathbb{E}_S\text{Rad}_S(\mathcal{F}) + B\sqrt{\frac{\log(2/\delta)}{2n}}. \tag{41}
\]

**Proof of Theorem 6.** Let \( \mathcal{F}_Q := \{\ell(\cdot; \theta) \mid \|\theta\|_p \leq Q\} \). Note that \( \ell(\cdot; \theta) \) is a 1-Lipschitz function and bounded by 0 and \( \frac{1}{2} \), then directly from Lemma 26.9 of [26],
$$\text{Rad}_S(\mathcal{F}_Q) = \text{Rad}_S(\ell \circ \mathcal{H}_Q) \leq \text{Rad}_S(\mathcal{H}_Q) \leq 3Q \sqrt{\frac{2 \log(2d)}{n}},$$

from the above inequality, combined with Theorem 5 and Theorem 6 leads to the following inequalities with probability at least $1 - \delta_Q$

$$\sup_{\|\theta\|_p \leq Q} |R_D(\theta) - R_S(\theta)| \leq 2E_S'\text{Rad}'(\mathcal{F}_Q) + \frac{1}{2} \sqrt{\frac{\log(2/\delta_Q)}{2n}} \leq 6Q \sqrt{\frac{2 \log(2d)}{n}} + \frac{1}{2} \sqrt{\frac{\log(2/\delta_Q)}{2n}}.$$ 

We use this bound with $Q = 1, 2, \ldots$ and $\delta_Q = \frac{6\delta}{\pi^2Q^2}$. Note that $1 - \sum_{Q=1}^{\infty} \delta_Q = 1 - \delta$ and consider the union bound. Then with probability at least $1 - \delta$, the following inequality holds for all $Q > 0$,

$$\sup_{\|\theta\|_p \leq Q} |R_D(\theta) - R_S(\theta)| \leq 6Q \sqrt{\frac{2 \log(2d)}{n}} + \frac{1}{2} \sqrt{\frac{\log(\pi^2Q^2/3\delta)}{2n}}.$$ 

To further use this inequality, we let $Q = \|\theta\|_p + 1$. Therefore, we have

$$|R_D(\theta) - R_S(\theta)| \leq 6(\|\theta\|_p + 1) \sqrt{\frac{2 \log(2d)}{n}} + \frac{1}{2} \sqrt{\frac{\log(\pi^2/3\delta)}{2n}}$$

where in the last inequality we used the fact that $\log(a) \leq a$, for $a \geq 1$, and $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ for positive $a$ and $b$. \hfill \Box

**Proof of Theorem 3.** Firstly, the population risk can be decomposed into

$$R_D(\theta_{S,\lambda}) = R_D(\hat{\theta}) + [R_D(\theta_{S,\lambda}) - J_{S,\lambda}(\theta_{S,\lambda})] + [J_{S,\lambda}(\theta_{S,\lambda}) - J_{S,\lambda}(\hat{\theta})] + [J_{S,\lambda}(\hat{\theta}) - R_D(\hat{\theta})].$$

Set $\lambda = 3\lambda_0 \sqrt{2 \log(2d)}$, where $\lambda_0 \geq 2 + \frac{1}{12\sqrt{\log(2d)}}$, then by Theorem 2, we have with probability at least $1 - \delta/2$,

$$R_D(\theta_{S,\lambda}) - J_{S,\lambda}(\theta_{S,\lambda}) = R_D(\theta_{S,\lambda}) - R_S(\theta_{S,\lambda}) - 3\lambda_0 \sqrt{\frac{2 \log(2d)}{n}} \|\theta_{S,\lambda}\|_p$$

$$\leq (\|\theta_{S,\lambda}\|_p + 1) \frac{3(2 - \lambda_0) \sqrt{2 \log(2d)} + \frac{1}{2\sqrt{2}}}{\sqrt{n}}$$

$$+ 3\lambda_0 \sqrt{\frac{2 \log(2d)}{n}} + \frac{1}{2} \sqrt{\frac{\log(2\pi^2/3\delta)}{2n}} \leq 3\lambda_0 \sqrt{\frac{2 \log(2d)}{n}} + \frac{1}{2} \sqrt{\frac{\log(2\pi^2/3\delta)}{2n}}. \quad (42)$$
where in the last inequality we used the fact that $2 + \frac{1}{12\sqrt{\log(2d)}} = 2 + \frac{1}{2\sqrt{2\cdot3\cdot2\log(2d)}}$.

Next, by Theorem 2 again, we have with probability at least $1 - \delta/2$,

$$J_{S,\lambda}(\tilde{\theta}) - R_D(\tilde{\theta}) = R_S(\tilde{\theta}) - R_D(\tilde{\theta}) + 3\lambda_0\sqrt{\frac{2\log(2d)}{n}}\|\tilde{\theta}\|_p$$

$$\leq (\|\tilde{\theta}\|_p + 1)\frac{3(2 + \lambda_0)\sqrt{2\log(2d)} + \frac{1}{\sqrt{2}\sqrt{n}}}{\sqrt{n}} - 3\lambda_0\sqrt{\frac{2\log(2d)}{n}} + \frac{1}{2}\sqrt{\frac{\log(2\pi^2/3\delta)}{2n}}$$

Finally, we observe that $J_{S,\lambda}(\theta_{S,\lambda}) - J_{S,\lambda}(\tilde{\theta}) \leq 0$ by optimality of $\theta_{S,\lambda}$, then combined with the fact $\|\tilde{\theta}\|_p \leq 6\|f^*\|_B$ from Theorem 1, our proof is finished by collecting altogether (42), (43) and $R_D(\tilde{\theta}) \leq \frac{3\|f^*\|_B^2}{2N_{\text{non}}}$ from Theorem 1, i.e.,

$$R_D(\theta_{S,\lambda}) = R_D(\tilde{\theta}) + [R_D(\theta_{S,\lambda}) - J_{S,\lambda}(\theta_{S,\lambda})] + [J_{S,\lambda}(\theta_{S,\lambda}) - J_{S,\lambda}(\tilde{\theta})] + [J_{S,\lambda}(\tilde{\theta}) - R_D(\tilde{\theta})]$$

$$\leq \frac{3\|f^*\|_B^2}{2N_{\text{non}}} + 3\lambda_0\sqrt{\frac{2\log(2d)}{n}} + \frac{1}{\sqrt{n}}\sqrt{\frac{\log(2\pi^2/3\delta)}{2n}}$$

$$\leq \frac{3\|f^*\|_B^2}{2N_{\text{non}}} + (6\|f^*\|_B + 1)\frac{3(2 + \lambda_0)\sqrt{2\log(2d)} + \frac{1}{\sqrt{2}\sqrt{n}}}{\sqrt{n}} - 3\lambda_0\sqrt{\frac{2\log(2d)}{n}} + \frac{1}{2}\sqrt{\frac{\log(2\pi^2/3\delta)}{2n}}$$

$$\lesssim \frac{\|f^*\|_B^2}{N_{\text{non}}} + \frac{1}{\sqrt{n}}\left(\lambda(\|f^*\|_B + 1) + \sqrt{\log 1/\delta}\right),$$

which finishes our proof.

\section*{5.4 Applications to DenseNet}

We directly apply our results, especially Theorem 3 to obtain the a priori estimate for DenseNet.

\textbf{Corollary 1} (A priori estimate for DenseNet). Suppose $f^* \in B$, $\lambda = \Omega(\sqrt{\log d})$, and assume that $\theta_{S,\lambda}$ is an optimal solution for the regularized model (25), i.e., $\theta_{S,\lambda} \in \arg\min_{\theta} J_{S,\lambda}(\theta)$, then for any $\delta \in (0, 1)$, with probability at least $1 - \delta$ over the choice of the training sample $S$, we have for any DenseNet $f(\cdot; \theta)$, its population risk satisfies

$$R_D(\theta_{S,\lambda}) := \mathbb{E}_{x \sim \mathcal{D}} \frac{1}{2} (f(x; \theta_{S,\lambda}) - f^*(x))^2$$

$$\lesssim \frac{\|f^*\|_B^2}{L^2m} + \frac{1}{\sqrt{n}}\left(\lambda(\|f^*\|_B + 1) + \sqrt{\log 1/\delta}\right).$$

(44)
Proof. In the case of DenseNet, as is shown already in Example 4, the matrix representation $A$ incorporates the form

$$
\begin{pmatrix}
0 & 0 & \ldots & 0 \\
V & B^{[1]} & \ldots & B^{[L]} \\
& 0 & \ldots & 0 \\
& & \ldots & 0 \\
& & & u^\top \\
\end{pmatrix},
$$

with $V$ taking size $d_0 \times d$, $d_0 = k_0$, length of vector $u$ being $d_L$, $d_L = k_0 + Lk$, and for each matrix block $B^{[l]}$, $l = 1, \ldots, L$:

$$
B^{[l]} = \begin{pmatrix}
W^{[l]} & \sigma I_{p_l \times p_l} \\
\bar{I}_{(k_0 + lk) \times (k_0 + (l-1)k)} & \bar{U}^{[l]}
\end{pmatrix},
$$

where $B^{[l]}$ has size $(2p_l + d_l) \times (2p_l + d_{l-1})$, $p_l = lm$, $d_l = k_0 + lk$, and the size of its components $\{W^{[l]}, \sigma I_{p_l \times p_l}, \bar{I}_{(k_0 + lk) \times (k_0 + (l-1)k)}, \bar{U}^{[l]}\}$ respectively reads $p_l \times d_{l-1}$, $p_l \times p_l$, $d_l \times d_{l-1}$, and $d_l \times p_l$. Specifically, $\bar{I}_{(k_0 + lk) \times (k_0 + (l-1)k)} = \begin{pmatrix} I_{d_{l-1} \times d_{l-1}} \\ 0_{(d_l - d_{l-1}) \times d_{l-1}} \end{pmatrix}$. Therefore, DenseNet satisfies Assumption 2. Moreover, we have

$$
N_{non} = \sum_{l=1}^{L} p_l = \sum_{l=1}^{L} lm = \frac{L(L + 1)}{2} m,
$$

apply Theorem 3 directly, (26) reads exactly as (44).

6 Conclusion

Our main contribution is the introduction of a novel representation of feedforward neural networks, namely the nonlinear weighted DAG. This representation provides further understanding of efficiency of shortcut connections utilized by various networks. We also show in detail how typical examples of feedforward neural networks can be represented using this formulation.

Moreover, we derive a priori estimates in avoidance of the CoD for neural networks satisfying the assumption of shortcut connections (Assumption 2). Our estimates
serve as an extension for the results in [8, 10], and key to our analysis is the employment of weighted path norm. As demonstrated in [8, 10], the weight path norm is capable of bounding the approximation and estimation errors simultaneously for ResNet and two-layer network, and we show that it is also the case for DenseNet.

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A Properties of the Representation and Symbol

*Proof of Proposition 1* 1. As a lower triangular matrix, $A(\theta, \xi)$ is obviously nilpotent.

2. By definition, we have

$$z_{s+1} - z_s = A(\theta, \sigma)z_s - A(\theta, \sigma)z_{s-1}.$$ 

Since $\sigma(\cdot)$ is 1-Lipschitz, then

$$|(Az_s - Az_{s-1})_i| = \left|\sum_{j=1}^{N} (A)_{ij}(z_s)_j - (A)_{ij}(z_{s-1})_j\right| \leq \sum_{j=1}^{N} (A(|\theta|, 1))_{ij}|(z_s - z_{s-1})_j|.$$ 

Thus for all $i = 1, \cdots, N$, $(|z_{s+1} - z_s|)_i \leq (A(|\theta|, 1)|z_s - z_{s-1}|)_i$. Therefore, inductively we have

$$\|z_{s+1} - z_s\|_\infty \leq \|A(|\theta|, 1)|z_s - z_{s-1}|\|_\infty \leq \cdots \leq \|A^s(|\theta|, 1)|z_1 - z_0|\|_\infty.$$ 

3. Since $A(|\theta|, 1)$ is nilpotent, for sufficiently large $s$, we have

$$\|z_{s+1} - z_s\|_\infty \leq \|0_{N \times N}|z_1 - z_0|\|_\infty = 0.$$ 

Hence $\{z_s\}_{s \geq 0}$ is a Cauchy sequence, and its limit exists.

4. This is straightforward by definition of the sequence $\{z_s\}_{s \geq 0}$ as well as the existence of the limit $z_\infty$.

5. Since $z_0 = P_0z_s$ for any $s$, we have $z_s = P_0z_{s-1} + Az_{s-1} = \bar{A}z_{s-1}$. Hence, it holds naturally that $z_\infty = \bar{A}^\infty z_0$. 

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6. A simple calculation is sufficient to show that \( \| \theta \|_P = 1^T_{\text{out}} \sum_{s=0}^\infty A^s(|\theta|, 3)1_{\text{in}} \), and by taking advantage of the property of nilpotency, we have that

\[
\sum_{s=0}^\infty A^s(|\theta|, 3) = (I_{N \times N} - A(|\theta|, 3))^{-1},
\]

then \( \| \theta \|_P = 1^T_{\text{out}}(I_{N \times N} - A(|\theta|, 3))^{-1}1_{\text{in}} \). Moreover, by definition of \( P_0 \), we observe that \( \| \theta \|_P = 1^T_{\text{out}}A^\infty(|\theta|, 3)1_{\text{in}} \).

7. These are straightforward by definition of the entries in the adjacency matrix representation.

\[ \square \]