Magueijo-Smolin Transformation as a Consequence of a Specific Definition of Mass, Velocity, and the Upper limit on Energy

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Abstract

We consider an alternative approach to double special-relativistic theories. The point of departure is not $\kappa$-deformed algebra (or even group-theoretical considerations) but rather 3 physical postulates defining particle’s velocity, mass, and the upper bound on its energy in terms of the respective classical quantities. For a specific definition of particle’s velocity we obtain Magueijo-Smolin (MS) version of the double special-relativistic theory. It is shown that this version follows from the $\kappa$-Poincare algebra by the appropriate choice of on the shell mass, such that it is always less or equal Planck’s mass. The $\kappa$-deformed Hamiltonian is found which invalidates the recent arguments about unphysical predictions of the MS transformation.

A recent research (e.g. [1], [2], [3], [4], [5], [6], [7]) on the so-called double special relativity not only reexamined its relation to $\kappa$-deformed kinematics, but in one specific example [4] also subjected to criticism physical predictions of one of these theoretical constructs [7].

It should be mentioned that as early as in 1994, J.Lukierski with collaborators [8] demonstrated that there exist an infinite set of transformations reducing the $\kappa$-deformed Casimir in Majid-Ruegg basis [9] (used in all the double-special relativistic theories, e.g. [10]) to the diagonal form. In fact, it is possible to show that any of these transformations correspond to a different choice of what one can consider as a definition of the deformed mass. This makes it difficult, without any additional assumptions, to choose a unique physical theory corresponding to the respective transformation. This difficulty is emphasized [4] by what looks like apparent non-physical predictions of one of these constructions [7].

Here we revisit the latter work [7], more specifically its treatment of the energy-momentum domain, this time departing not from the group-theoretical point of view, but rather from certain physically justified restrictions (postulates)

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imposed on the classically-defined physical quantities, namely energy, mass, and velocity. An analogous approach was used in [3] for a more narrowly defined goal: a study of possible definitions of \( \kappa \)-deformed velocities and their addition laws.

We begin by introducing the postulates defining

i) the velocity of a particle,

ii) its mass to be the same for any scale (from classical to Planck scale) and therefore based on the relations provided by the momentum sector of classical relativity,

iii) the existence of the upper bound (the Planck energy) on the values of both energy and momentum.

We also retain the upper bound (speed of light \( c \)) on a particle velocity.

In what follows we use units where \( c = 1 \), Planck constant \( \hbar = 1 \), and Boltzmann constant \( k = 1 \). We denote the Planck energy (momentum) by \( \kappa \) which in these units is equal to the inverse of the Planck length \( \lambda (\kappa = 1/\lambda) \). The classical relation between energy \( P_0 \) and momentum \( P_i \) in these units has the following dimensionless form:

\[
\Pi_0^2 - |\Pi|^2 = \mu^2 
\]  

where

\[ \mu \equiv m/\kappa, \ \Pi_i = P_i/\kappa, \ \Pi_0 = P_0/\kappa, \ i = 1, 2, 3. \]

and \( m \) is particle’s mass.

Quite analogously we introduce the dimensionless expressions for the physical energy \( p_0 \) and momentum \( p \) (different from the above energy \( \Pi_0 \) and momentum \( \Pi_i \)) applicable in the region of Planck-scale physics

\[ \pi_0 = p_0/\kappa, \ \pi_i = p_i/\kappa, \ i = 1, 2, 3 \]

Following [2] we write the general functional relation between the classical energy-momentum \( \Pi_\alpha, \alpha = 0, 1, 2, 3 \) (not physical anymore in the Planck-scale phenomena) and its Planck-scale counterpart \( \pi_\alpha, \alpha = 0, 1, 2, 3 \):

\[
\pi_0 = f(\Pi_0), \pi_i = g(\Pi_0)\Pi_i
\]

where the functions \( f(\Pi_0) \) and \( g(\Pi_0) \) to be defined.

To find these functions we use the above postulates (i)-(iii). The dimensionless velocity of a particle \( v \leq 1 \) (compatible with its classical definition in terms of the energy-momentum) is defined as follows

\[
v^2 = (\frac{\Pi}{\Pi_0})^2 = (\frac{\pi}{\pi_0})^2; \]

\[
\pi^2 = \sum_{i=1}^{3} \pi_i \pi_i, \ \Pi^2 = \sum_{i=1}^{3} \Pi_i \Pi_i
\]  

(3a)
where \( p^2 = p_i p^i \). Note that in this definition the velocity \( v_i \) looks as the one used in the classical case, except that now this velocity

\[
v^2 \neq (\partial \pi_0 / \partial \pi)^2,
\]

while in the classical case

\[
v^2 = (\Pi / \Pi_0)^2 = (\partial \Pi_0 / \partial \Pi)^2
\]

(3b)

Next we use the second postulate (ii) which defines particle’s mass \( \mu \) to be the same in all the regions (from classical to Planck scale), and independent of the velocity definition (3a)-(3b).

\[
\frac{1}{\mu} = 2 \lim_{\Pi^2 \to 0} \frac{\partial \Pi_0}{\partial \Pi^2} = 2 \lim_{\pi^2 \to 0} \frac{\partial \pi_0}{\partial \pi^2}
\]

(4)

Finally we require (postulate iii) that

\[
|\pi| \leq 1, \quad |\pi_0| \leq 1
\]

(5)

where the equality signs correspond to \( \Pi_0, |\Pi| \to \infty \)

We begin with the velocity definition according to Eq. (3a). Upon substitution of this equation into Eq.(2) we obtain

\[
f(\Pi_0) = g(\Pi_0)\Pi_0
\]

(6)

This means that

\[
\pi_0 = g(\Pi_0)\Pi_0,
\]

\[
d\pi_0 = \Pi_0 \frac{dg}{d\Pi_0} d\Pi_0 + g(\Pi_0) d\Pi_0,
\]

\[
d\pi^2 = \Pi^2 \frac{dg^2}{d\Pi_0} d\Pi_0 + g^2 d\Pi^2
\]

(7)

Inserting Eq.(7) into the definition of mass Eq.(4) we arrive at the following differential equation:

\[
\Pi_0 \frac{dg}{d\Pi_0} + g(\Pi_0) - [g(\Pi_0)]^2 = 0
\]

(8)

Its solution is:

\[
g(\Pi_0) = \frac{1}{1 \mp A\Pi_0}
\]

(9)

where the integration constant \( A \) to be determined on the basis of the above postulates.
As a result, according to Eqs. (2), (6) the energy-momentum $\pi_\alpha (\alpha = 0, 1, 2, 3)$ is:

$$\pi_i = \frac{\Pi_i}{1 \mp A\Pi_0} \quad (10a)$$

$$\pi_0 = \frac{\Pi_0}{1 \mp A\Pi_0} \quad (10b)$$

The value of the integration constant $A$ and the choice of the respective sign in the obtained solution (10a), (10b) are dictated by our postulate (iii), Eq. (5).

To determine both, we notice that since in the classical limit $\pi_\alpha \to \Pi_\alpha$ the positive (negative) values of $\Pi_0$ should correspond to positive (negative) values of $\pi_0$ respectively. This means the following:

$$\pi_0 = \frac{\Pi_0}{1 + A\Pi_0}, \quad \Pi_0 > 0, \pi_0 > 0 \quad (11a)$$

$$\pi_0 = \frac{\Pi_0}{1 - A\Pi_0}, \quad \Pi_0 < 0, \pi_0 < 0 \quad (11b)$$

Taking the limit $\Pi_0 \to +\infty (-\infty)$ of Eq. (11a), (11b) and using our postulate iii) (Eq. 5) we get

$$A = 1 \quad (12)$$

Inserting this value of $A$ into Eqs. (10a), (11a), (11b) we obtain the explicit expressions for $\pi_i$ and $\pi_0$

$$\pi_0 = \frac{\Pi_0}{1 + \Pi_0}, \quad \Pi_0, \pi_0 > 0 \quad (13)$$

$$\pi_0 = \frac{\Pi_0}{1 - \Pi_0}, \quad \Pi_0, \pi_0 < 0 \quad (14)$$

These expressions reproduce the results obtained in [7] with the only difference that here $\pi_0$ is the antisymmetric function of $\Pi_0$ in contradistinction to [7].

If we use the classical expressions for $\Pi_\alpha, \quad \alpha = 0, 1, 2, 3$ (with the same $v_i$ and $\mu$ in all the regions)

$$\Pi_0 = \mu \gamma, \quad \Pi_i = \mu v_i \gamma, \quad \gamma = \frac{1}{\sqrt{1 - v^2}}$$
and Eqs. (10a), (10b) then we readily obtain (restricting our attention to the positive region of $\pi_0$) the respective expressions (cf. [7]) for $\pi_0$:

$$\pi_0 = \frac{\mu \gamma}{1 + \mu \gamma}$$

$$\pi_i = \frac{\mu \gamma_i}{1 + \mu \gamma}$$

(15)

From (15) follows that the rest energy $\pi_0^0$ less than the mass $\mu$:

$$\pi_0^0 = \frac{\mu}{1 + \mu} \leq \mu$$

(16)

Here the equality sign corresponds to the classical region $\mu = m/\kappa << 1$

From expressions (13), (14) we obtain the inversion formulas:

$$\Pi_0 = \frac{\pi_0}{1 - \pi_0} \quad \Pi_0, \pi_0 > 0$$

$$\Pi_0 = \frac{\pi_0}{1 + \pi_0} \quad \Pi_0, \pi_0 < 0$$

(17)

$$\Pi_i = \frac{\pi_i}{1 - \pi_0} \quad \Pi_0, \pi_0 > 0$$

$$\Pi_i = \frac{\pi_i}{1 + \pi_0} \quad \Pi_0, \pi_0 < 0$$

(18)

If we use classical Casimir and expressions (17) and (18) then the respective Casimirs for the energy-momentum in the Planck region are

$$\frac{\pi_0^2}{(1 - \pi_0)^2} - \frac{\pi_0^2}{(1 - \pi_0)^2} = \mu^2$$

(19a)

$$\frac{\pi_0^2}{(1 + \pi_0)^2} - \frac{\pi_0^2}{(1 + \pi_0)^2} = \mu^2$$

(19b)

where (−) corresponds to the positive values of $\pi_0$ and (+) corresponds to negative values of $\pi_0$. Solving Eqs. (19a), (19b) with respect to $\pi_0$ and choosing the correct signs (according to the positive and negative values of $\pi_0$, remembering that both $|\pi|, |\pi_0| \leq 1$) we arrive at the following relation

$$\pi_0 = \pm \sqrt{\pi^2(1 - \mu^2) + \mu^2}$$

(20)

where the upper(lower) sign corresponds to $\pi_0 > 0(\pi_0 < 0)$ respectively. It is seen that the regions of the positive and negative values of $\pi_0 = F(\pi)$ are the same with accuracy to the sign. The graph of $\pi_0 = F(\pi)$ is shown in Fig.1.

Based on the relation between $\pi_0$ and $\Pi_0$ [Eqs. (11a), (11b), (13), (14)] and on the expressions for classical Lorentz boost of $\Pi_0$ in the $z$-direction with a
velocity $V_3$ (in the units of $c = 1$), we can calculate in an elementary fashion the respective boost relations (found by Magueijo and Smolin in [7] with the help of group-theoretical analysis) energy-momentum $\pi_\alpha$ at the Planck scale. We write them in the dimensionless form:

$$\pi'_0 = \frac{\Gamma(\pi_0 - V_3\pi_3)}{1 - \pi_0 + \Gamma(\pi_0 - V_3\pi_3)}$$  \hspace{1cm} (21a)

$$\pi'_i = \frac{\delta_{3i}\Gamma(\pi_i - V_3\pi_0) + \pi_i(1 - \delta_{3i})}{1 - \pi_0 + \Gamma(\pi_0 - V_3\pi_3)}, \ i = 1, 2, 3$$  \hspace{1cm} (21b)

Here $\delta_{3i}$ is the Kroenecker delta-function and $\Gamma = 1/\sqrt{1 - V_3^2}$. Since particle's velocity has been defined as $v_i = \pi_i/\pi_0$, it is not surprising that Eqs.(21a,21b) yield the velocity addition rule, coinciding with the classical relativistic rule:

$$v'_i = \frac{\delta_{3i}(v_i - V_3)}{1 - v_3V_3} + \frac{v_i(1 - \delta_{3i})}{\Gamma(1 - v_3V_3)}$$  \hspace{1cm} (22)

It was shown in [1] that within the context of $\kappa$-Poinciana algebra various possible doubly-special relativity constructions can be viewed as different bases of this algebra. In particular, Magueijo-Smolin basis [7] is one of such bases. In a more general scheme of things, J.Lukierski and collaborators [8] demonstrated that all possible double special relativistic constructions differ by a suitable choice of what one defines as an effective mass.

Here we show that the Magueijo-Smolin transformation (obtained here in an elementary fashion with the help of simple physical postulates) contains an additional physical constraint on the mass. This transformation can be derived...
pro-forma from $\kappa$-deformed algebra, and the result explicitly shows that in this case the particle mass $\mu \equiv m/\kappa \leq 1$. To demonstrate that we write the relations of the classical basis (denoted here as $\Pi_\alpha$, $\alpha = 0, 1, 2, 3$) by rearranging the formulas given in [8]:

\[
\Pi_0 = A[e^{\pi_0} - \text{cosh}(\pi)] \equiv Ae^{\pi_0}[1 - e^{\pi_0}\text{cosh}(\pi)] \quad (23)
\]
\[
\Pi_i = Ae^{\pi_0}\pi_i \quad (24)
\]

where $A$ is an arbitrary constant to be determined. The respective Casimirs are

\[
\text{cosh}(\pi_0) - \frac{\pi_0^2}{2} = \text{cosh}(\pi) \quad (25)
\]

and

\[
\Pi_0^2 - \Pi_i^2 = A^2\sinh^2(\pi) \quad (26)
\]

We can symmetrize expressions (23), (24) by introducing the following quantities:

\[
\pi_0 = e^{-\pi_0} \frac{\Pi_0}{A} = [1 - e^{-\pi_0}\text{cosh}(\pi)] \quad (27a)
\]
\[
\pi_i = \pi_i \quad (27b)
\]

Combining (27a), (27b) and (23), (24), we obtain:

\[
\Pi_0 = Acosh(\pi)\frac{\pi_0}{1 - \pi_0} \quad (28a)
\]
\[
\Pi_i = Acosh(\pi)\frac{\pi_i}{1 - \pi_0} \quad (28b)
\]

By comparing Eqs.(28a) and (28b) with Eqs. (13) and (14) we immediately see that Magueijo-Smolin basis follows from $\kappa$-deformed algebra if and only if

\[
A = \frac{1}{\text{cosh}(\pi)} \quad (29)
\]

In this case Casimir (26) reads

\[
(\frac{\pi_0}{1 - \pi_0})^2 - (\frac{\pi}{1 - \pi_0})^2 = (\tanh^2(\pi)) \quad (30)
\]

Comparing Eq.(30) with Casimir given by Eq.(19a) [7] we arrive at the conclusion that the mass $\mu$ used in the latter is

\[
\mu = \tanh(\pi) \leq 1, \text{ Q.E.D} \quad (31)
\]
In addition, if we use (16) then Eq.(31) imposes the following condition on the value of the rest energy $\pi_0$:

$$\pi_0 = \frac{1}{2}$$  \hspace{1cm} (32)

At the first glance this condition looks (predicated on the restriction on particle’s mass $\mu \leq 1$) as overly restrictive. On the other hand, considering Magueijo-Smolin transform as a "free-standing" transformations we are not forced to have an upper bound on the rest energy $\pi_0^0$ in the Planck region where $\pi_0^0 = 1/2$ (that is one half of the upper bound on $\pi_0$). Still, there is a strong argument in favor of adopting the upper bound on particle’s mass. If we would like to be consistent then it seems quite reasonable to expect that all the quantities in the region of Planck scales to be bounded from above by Planck energy $\kappa$, or bounded from below by Planck length $\lambda$.

We have derived Magueijo-Smolin transformation without resorting to group-theoretical approach by simply defining particle velocity $v_i$, its mass $\mu$, and the upper bound on the magnitude of momentum-energy $\pi_\alpha$ in terms of the respective classical quantities. In particular, the velocity $v_i$ is defined as $v_i = \pi_i/\pi_0$ (Eq.3a). If we substitute into this definition the relations between $\pi_i, \pi_0$ and the respective quantities $\pi_i, \pi_0$, Eqs. (27a), (27b) (used in the conventional treatment based on $\kappa$-Poincare algebra) we arrive at the value of the velocity $v_i$ which is exactly the right group velocity $V_i^R$ (obeying classical addition law) introduced in [3]:

$$v_i = V_i^R = \frac{e^{\pi_i/\pi_0} - \cosh(\mu)}{e^{\pi_0} - \cosh(\mu)}$$  \hspace{1cm} (33)

Interestingly enough, the particle velocity is identical in 2 different bases: Magueijo-Smolin basis and the basis used in [3].

To complete our elementary treatment of Magueijo-Smolin transform, we address its critique expressed in [4]. It is argued there that a definition of the particle velocity according to Hamilton equations results in a paradoxical situation where 2 particles of different masses moving in an inertial frame with the same velocity will have different velocities when viewed from another inertial frame. The fallacy of this conclusion is due to the fact that the authors of [4] used a non-deformed hamiltonian formalism.

It has been demonstrated (e.g.,[11],[12]) that a velocity definition in a non-commutative space is dictated by an appropriate choice of a \textit{deformed} Hamiltonian formalism (see also [3]). To this end let us consider dimensionless relativistic phase space variables $Y_A = (\xi_\alpha, \pi_\alpha), (\xi_\alpha = x_\alpha/\lambda, \alpha = 0, 1, 2, 3)$ normalized by the appropriate Planck scales and whose commutation relations are:
\[ [\pi_i, \xi_j] = \delta_{ij}, \]
\[ [\pi_0, \xi_0] = -(1 - \pi_0), \]
\[ [\pi_0, \xi_i] = 0, \]
\[ [\xi_0, \xi_i] = \xi_i, \]
\[ [\xi_0, \pi_i] = -\pi_i, \]
\[ [\pi_\alpha, \pi_\nu] = 0 \] \hspace{1cm} (34)

The \( \kappa \)-deformed Hamilton equations then yield (cf.[3]):

\[ \frac{d\xi_i}{ds} = -\frac{\partial H^{(\kappa)}}{\partial \pi_i}, \] \hspace{1cm} (35a)
\[ \frac{d\xi_0}{ds} = -\sum_{1}^{3} \pi_i \frac{\partial H^{(\kappa)}}{\partial \pi_i} + (1 - \pi_0) \frac{\partial H^{(\kappa)}}{\partial \pi_0} \] \hspace{1cm} (35b)

Here one particle Hamiltonian \( H^{(\kappa)} \) is taken to be the \( \kappa \)-invariant Casimir, Eq.(19a):

\[ H^{(\kappa)} \equiv \mu^2 = (\frac{\pi_0}{1 - \pi_0})^2 - (\frac{\pi_i}{1 - \pi_0})^2 \]

If we use this expression in the Hamilton’s equations (35a), (35b) we obtain:

\[ \frac{d\xi_i}{ds} = 2 \frac{\pi_i}{(1 - \pi_0)^2}, \quad \frac{d\xi_0}{ds} = 2 \frac{\pi_0}{(1 - \pi_0)^2} \] \hspace{1cm} (36)

From Eqs.(36) immediately follows the expression for the velocity (37a) which was postulated from the very beginning:

\[ \frac{d\xi_i}{d\xi_0} = \frac{\pi_i}{\pi_0} = v_i \] \hspace{1cm} (37)

Another seemingly unphysical prediction(s) of Magueijo-Smolin (MS) basis, as was pointed out by J.Rembielinski and K.Smolinski [4], is connected with an apparent difficulty in formulating statistical mechanics based on MS basis. It is argued that one-particle partition function is divergent when \( \pi_0 \to 1 \). However this conclusion is based on an assumption that the temperature in the Planck region is the same as in the classical region. This is not true, since the existence of the upper limit on the energy immediately implies that there exist a relation between the temperature ( dimensionless) \( \tau \) in the Planck region and its counterpart \( T \) in the classical region analogous to the relations between energies in these two regions,Eq.(11b). As a result, it is not difficult to demonstrate that the partition function (expressed as an integral) does not have singularities.
Additional criticism of MS transformation is connected to the fact that for a large number $N$ of identical particles, each of energy $\pi_0j, j = 1, 2, ..., N$ their total internal energy in the thermodynamic limit ($N \to \infty$) does not depend on temperature. But this represents not a deficiency of the basis, but on the contrary, its advantage. In fact, since the temperature is bounded from above by $\tau_{max} = 1$, in the limit of infinite number of particles, whose total energy tends to the respective upper boundary ($\pi_0 = 1$) the respective temperature must tend to its maximum that is to 1, which explains an apparent absence of the dependence of the internal energy on temperature. In this case the internal energy is simply equal to the temperature, and both are equal to unity (in the chosen units).

In conclusion we would like to say that MS basis following from very simple and consistent physical postulates introduced here represents an attractive model for a description of phenomena which might be associated with Planck scale physics. In fact, the imposition of upper bound on the energy-momentum, and even mass (if we adopt $\kappa$-Poinciana roots of the basis) which are in agreement with a major postulate of Planck scale phenomena, is the feature which is not present in any other bases. Still, there are some problems with this (and to this matter, with any other $\kappa$-deformed) model(s). In particular, the commutation relation $\{\pi_i, \xi_j\} = \delta_{ij}$ is not consistent with the well-known string uncertainty relation.

1 Addendum

After this paper has been written, paper [13] appeared, where (among some other topics) the addition law for energy-momentum was modified as compared to the one used in [14]. This was done to comply with the physical requirement that a set of particles with even sub-Planckian energies can have an energy much exceeding the Planck energy. The modification was achieved by simply replacing the Planck energy $\kappa(= 1/\lambda$ in our units) for a system of $N$ particles with $N\kappa$.

Here we demonstrate that the modified addition law follows from our scheme, by simply adjusting one of our postulates (postulate iii, [3]). The modification is as follows: we require that for a set of $N$ particles the upper bound on both energy and momentum (normalized by Planck energy $\kappa$) is to be not 1, but $N$. This is equivalent to a postulate that the energy composition law for particles (each having Planck energy) is a simple addition. In turn, postulate (iii) for one particle follows from that as a particular case of $N = 1$.

Thus, if we use thus modified postulate in Eqs. (11a, 11b), we obtain the following value of constant $A$ which now depends on the number of particles $N$: $A(N) = \frac{1}{N}, \; N = 1, 2, 3, ...$

As a result, the energy $\pi_0^{(N)}$ and momentum $\pi_i^{(N)}$ of a set of $N$ particles follow
from Eqs. (11a,11b) (we restrict our attention to the positive values of $\pi_0$):

$$\pi_0^{(N)} = \frac{\Pi_0^{(N)}}{1 + \Pi_0^{(N)}/N}$$  \hfill (38)

$$\pi_i^{(N)} = \frac{\Pi_i^{(N)}}{1 + \Pi_0^{(N)}/N}$$  \hfill (39)

The inverse expressions are

$$\Pi_0^{(N)} = \frac{\pi_0^{(N)}}{1 - \pi_0^{(N)}/N}$$  \hfill (40)

$$\Pi_i^{(N)} = \frac{\pi_i^{(N)}}{1 - \pi_0^{(N)}/N}$$  \hfill (41)

$\Pi_0^{(N)}$ and $\Pi_i^{(N)}$ represent the conventional sums of the respective individual quasi-energies $\Pi_{0k}$ (momenta $\Pi_{ik}$) (as in special relativity):

$$\Pi_0^{(N)} = \sum_k \Pi_{0k}, \quad \Pi_i^{(N)} = \sum_k \Pi_{ik}$$  \hfill (42)

Inserting (42) into (41), (40), we obtain the composition law for energies $\pi_{0k}$ and momenta $\pi_{ik}$ which we write as follows:

$$\pi_0^{N} = N \sum_{k=1}^{N} \frac{\pi_{0k} \prod_{j \neq k} (1 - \pi_{0j})}{\prod_{j \neq k} (1 - \pi_{0j})(1 - \delta_{jk})}$$  \hfill (43)

$$\pi_i^{N} = N \sum_{k=1}^{N} \frac{\pi_{ik} \prod_{j \neq k} (1 - \pi_{0j})}{\prod_{j \neq k} (1 - \pi_{0j})(1 - \delta_{jk})}$$  \hfill (44)

The respective Casimir is found from Eqs. (40) and (41) if we take the mass $\mu$ to be defined the same way in all the regions, from classical to Planck’s:

$$(\Pi_0^{(N)})^2 - (\Pi_1^{(N)})^2 = \left(\frac{\pi_0^{(N)}}{1 - \pi_0^{(N)}/N}\right)^2 - \left(\frac{\pi_i^{(N)}}{1 - \pi_0^{(N)/N}}\right)^2 = (\mu^{(N)})^2$$  \hfill (45)

The composition laws Eqs. (43), (44) are reduced to the conventional addition laws not only if all the individual energies $\pi_{0k}$ (momenta $\pi_{ik}$) are the same $\text{[13]}$, but also if at least one of the values $\pi_{0k} = 1$ ($\pi_{ik} = 1$). These laws take especially simple form for two single particles:

$$\pi_1 \oplus \pi_2 = 2 \pi_1 + \pi_2 - 2 \pi_1 \pi_2$$ \hfill (46)

$$\pi_{k1} \oplus \pi_{k2} = 2 \pi_{k1}(1 - \pi_{02}) + \pi_{k2}(1 - \pi_{01})$$ \hfill (47)
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