A SHORT PROOF OF THE KAC–WARD FORMULA

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ABSTRACT. We present a short proof of the Kac–Ward formula for the partition function of the Ising model on planar graphs.

Let \( G = (V, E) \) be a finite graph embedded in the complex plane. For a directed edge \( e = (t_e, h_e) \), its reversion is \( -e = (h_e, t_e) \), and its undirected version is \( \vec{e} = \{t_e, h_e\} \in E \). For two directed edges \( e, g \), the turning angle from \( e \) to \( g \) is

\[
\angle(e, g) = \text{Arg} \left( \frac{h_g - t_g}{h_e - t_e} \right) \in (-\pi, \pi]
\]

(see Figure 1). Let \( x = (x_\vec{e})_{\vec{e} \in E} \) be a vector of real edge weights. The transition matrix is a matrix indexed by the directed edges and given by

\[
\Lambda_{e,g} = \begin{cases} x_{\bar{e}}^\angle(e, g) & \text{if } h_e = t_g \text{ and } g \neq \bar{e}; \\ 0 & \text{otherwise}. \end{cases}
\]

An even subgraph is a set \( H \subset E \) such that the degree of each vertex of \((V, H)\) is even. Let

\[
Z = \sum_{H \text{ even}} \prod_{\vec{e} \in H} x_\vec{e}
\]

be the partition function of even subgraphs, where the product over the empty set is taken to be 1. The main result of this note is a short proof of the following theorem.

**Theorem 1** (Kac–Ward formula).

\[
\det(\text{Id} - \Lambda) = Z^2,
\]

where \( \text{Id} \) is the identity matrix.

For edge weights satisfying \( 0 < x_\vec{e} < 1 \), \( Z \) is the partition function of the Ising model \([5]\) defined on \( G \) (or the dual of \( G \), depending on the chosen expansion procedure). We refer the reader to \([7]\) for more details on the connection with the Ising model.

Many papers appeared in the physics and mathematics literature where the Kac–Ward formula is proved or claimed to be proved. The original work of Kac and Ward \([6]\) famously contained an error. Several papers followed where attempts were made to fix it. We mention the contributions of Sherman \([10]\), Burgoyne \([1]\), and Vdovichenko \([11]\), where loop expansions of the determinant were used. However, these papers still left a lot to wish for in terms of mathematical rigour. In the light of the accessible solution of

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the dimer model due to Kasteleyn [8], the combinatorics involved in the Kac–
Ward formula was considered unnecessarily complicated. This was probably
the reason why the first rigorous proof was given only much later by Dolbilin
et al. [3]. In recent years more papers appeared presenting rigorous but still
combinatorially involved proofs [2,4,7]. We refer to [7] for a longer discussion
on the history of this theorem. We also need to mention that there exists
an unpublished (and possibly even shorter than ours) proof due to Chelkak,
Cimasoni and Kassel, who discovered it while investigating the double Ising
model.

In this note a new short proof based on the loop expansion of the deter-
minal is presented. Like all the previous proofs it relies on cancellations
between certain weighted combinatorial objects. In our case these objects
are loops. The cancellations of loop weights fall into two categories: generic
and specific (Lemma 3 and Lemma 4 respectively). The generic cancella-
tions follow from the general theory of loop-erased walks. The specific cancella-
tions are an easy consequence of the unique sign-reversing property of the
weights induced by the transition matrix. The combinatorial mechanism of
the Kac–Ward formula becomes therefore as transparent as the one of the
loop-erased walk.

It is worth mentioning here that the Ising interface on the hexagonal
lattice can be proved to have exactly the distribution of the loop erasure of
the signed measure induced by the transition matrix. However, we will not
go into the details of this observation in this note.

WALKS AND LOOPS

A (non-backtracking) walk $\omega$ of length $|\omega| = n$ is a sequence of directed
edges $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ such that $t_{\omega_{i+1}} = h_{\omega_i}$ and $\omega_{i+1} \neq -\omega_i$ for $1 \leq i \leq n - 1$. Let $W_{e,g}$ be the set of all walks starting at $e$ and ending at $g$. Loops
are walks belonging to $W_{e,e}$ for some directed edge $e$, and are denoted by $\ell$.
For $\omega \in W_{e,g}$ and $\omega' \in W_{g,h}$,

$$\omega \oplus \omega' = (\omega_1, \omega_2, \ldots, \omega_{|\omega|}, \omega'_1, \omega'_2, \ldots, \omega'_{|\omega'|}) \in W_{e,h}$$

Figure 1. The turning angle and an even subgraph of the
hexagonal lattice
is the concatenation of \( \omega \) and \( \omega' \), and \( \omega^{-1} = (-\omega_1, -\omega_{n-1}, \ldots, -\omega_1) \in W_{-g,-e} \) is the reversion of \( \omega \). If \( 1 \leq k \leq l \leq |\omega| \), then \( \omega_{k,l} = (\omega_k, \ldots, \omega_l) \) is the segment of \( \omega \) between the indices \( k \) and \( l \).

The transition matrix induces complex valued weights on walks:

\[
\lambda(\omega) = \prod_{i=1}^{\lfloor |\omega|/2 \rfloor} \Lambda_{\omega_i,\omega_{i+1}} = e^{i\alpha(\omega)} \prod_{i=1}^{\lfloor |\omega|/2 \rfloor} x_{\omega_i \omega_{i+1}}, \quad \text{where } \alpha(\omega) = \sum_{i=1}^{\lfloor |\omega|/2 \rfloor} \angle(\omega_i, \omega_{i+1})
\]

is the total turning angle of \( \omega \). These weights posses three important properties. By the very definition,

\[
\lambda(\omega \oplus \omega') = \lambda(\omega)\lambda(\omega') \quad \text{for } \omega \in W_{e,g}, \omega' \in W_{g,h}. \quad \text{(multiplicativity)}
\]

From the fact that \( \text{Arg}(z/w) = \text{Arg}(z) - \text{Arg}(w) \) (mod \( 2\pi \)) for all nonzero complex numbers \( z, w \), it follows that for \( \omega \in W_{e,g} \), \( \alpha(\omega) = \angle(e, g) \) (mod \( 2\pi \)).

Since the walks are non-backtracking it additionally follows that \( \alpha(\omega) = \lambda^{-1}(\omega^{-1}) \). Hence, for any directed edge \( e \),

\[
\lambda(\omega) = -\lambda(\omega^{-1}) \quad \text{for } \omega \in W_{e,-e}, \\
\lambda(\ell) = \lambda(\ell^{-1}) \in \mathbb{R} \quad \text{for } \ell \in W_{e,e}. \quad \text{(sign change)}
\]

The sign change property is the one accounting for the crucial combinatorial cancellations.

Before proving Theorem 1 we need to define a few more notions. A loop \( \ell \) is rooted at \( e \) if \( \ell \in W_{e,e} \). A loop \( \ell \) is simple if \( \ell_{2,|\ell|} \) visits each undirected edge at most once. The multiplicity of \( \ell \), denoted by \( m_\ell \), is the largest number \( m \), such that \( \ell = (\ell')^\circ m \) for some loop \( \ell' \). The signed loop measure is given by

\[
w(\ell) = \frac{\lambda(\ell)}{|\ell|}.
\]

Unrooted loops are equivalence classes of loops under the cyclic shift relation \( \ell \sim \ell_i \oplus \ell_1 \), and are denoted by \( \ell^0 \). With a slight abuse of notation, if \( f \) is any function on rootd loops invariant under cyclic shifts, then \( f(\ell^0) \) is the function evaluated at any representative of \( \ell^0 \).

If \( L \) is a collection of loops, then \( \lambda(L) = \sum_{\ell \in L} \lambda(\ell), \ w(L) = \sum_{\ell \in L} w(\ell). \) Unnecessary brackets will be omitted from this notation, e.g. \( w(\{\ell \text{ visits } e\}) = w(\{\ell \text{ visits } e\}) \). Note that \( L \) will usually be infinite. It will always be assumed that \( \|x\|_\infty = \max_{x \in E} |x_i| \) is sufficiently small to guarantee that all such power series are absolutely summable. This in particular implies that the order in which the sums are taken is irrelevant. Since the walks are non-backtracking, it is actually enough to take \( \|x\|_\infty < 1/(\Delta - 1) \), where \( \Delta \) is the maximal degree of \( G \), but a precise bound will not be important.

**Proof of Theorem 1**

The first two lemmas only use the fact that \( \lambda \) is a multiplicative weight.

**Lemma 2** (Loop expansion of the determinant). Let \( L \) be the set of all loops. Then,

\[
det(Id - \Lambda) = \exp \left( -w(L) \right).
\]
Proof. Let \( \lambda_i \) be the eigenvalues of \( \Lambda \). We have
\[
\begin{align*}
    w(\mathcal{L}) &= \sum_{n=1}^{\infty} \sum_{|\ell| = n} \lambda(\ell)/n = \sum_{n=1}^{\infty} \text{tr} \Lambda^n/n = \sum_{i=1}^{\infty} \lambda_i^n/n \\
    &= \ln \prod_i (1 - \lambda_i) = \ln \det(\text{Id} - \Lambda).
\end{align*}
\]

The next lemma is a variant of the one used to prove the exponential formula for the law of the loop-erased walk (see e.g. Lemma 9.3.2 in [9]).

**Lemma 3** (Generic cancellations). Let \( \mathcal{L}^1_e \) be the set of loops which visit \( e \) only once and do not visit \( -e \). Then,
\[
\exp \left( -w(\ell \text{ visits } e \text{ and not } -e) \right) = 1 - \lambda(\mathcal{L}^1_e).
\]

In particular, the left-hand side is a polynomial of degree 1 in \( x_e \).

**Proof.** Let \( \mathcal{L}_e \) be the set of loops which visit \( e \) and do not visit \( -e \), and let \( \mathcal{L}_e^\circ \subset \mathcal{L}_e \) be the set of loops rooted at \( e \). Let \( \mathcal{L}_e^\circ \) be the set of unrooted loops which have a representative in \( \mathcal{L}_e^\circ \). Note that \( \mathcal{L}_e \) is the set of all representatives of the unrooted loops from \( \mathcal{L}_e^\circ \).

By \( k_\ell \) we denote the number of visits of \( \ell \) to \( e \). Note that the number of all representatives of \( \ell^\circ \in \mathcal{L}_e^\circ \) is \( |\ell^\circ|/m_\ell \), and the number of its representatives in \( \mathcal{L}_e^\circ \) is \( k_\ell^\circ/m_\ell \). Grouping the loops by their unrooted versions, the negated logarithm of the left-hand side of the desired equality becomes
\[
\sum_{\ell \in \mathcal{L}_e} \frac{\lambda(\ell)}{|\ell|} = \sum_{\ell \in \mathcal{L}_e} \frac{\lambda(\ell)}{m_\ell} \frac{m_\ell}{|\ell|} = \sum_{\ell^\circ \in \mathcal{L}_e^\circ} \frac{\lambda(\ell^\circ)}{m_\ell} \frac{m_\ell}{k_\ell} = \sum_{\ell \in \mathcal{L}_e^\circ} \frac{\lambda(\ell)}{k_\ell}.
\]

Note that each \( \ell \in \mathcal{L}_e^\circ \) has a unique representation \( \ell = \ell_t^1 \oplus \ell_t^2 \oplus \cdots \oplus \ell_t^k \), where \( \ell_t^l \in \mathcal{L}_e^l \). It follows that \( \mathcal{L}_e^\circ \) is a disjoint union of \( (\mathcal{L}_e^l)^\oplus k \) taken over all \( k \). Therefore, by multiplicativity,
\[
\sum_{\ell \in \mathcal{L}_e^\circ} \frac{\lambda(\ell)}{k_\ell} = \sum_{k=1}^{\infty} \sum_{\ell \in (\mathcal{L}_e^l)^\oplus k} \frac{\lambda(\ell)}{k} = \sum_{k=1}^{\infty} \lambda(\mathcal{L}_e^l)^k/k = -\ln(1 - \lambda(\mathcal{L}_e)). \qedhere
\]

The next lemma is the only place where the sign change property is used.

**Lemma 4** (Specific cancellations). For any directed edge \( e \),
\[
w(\ell \text{ visits } e \text{ and } -e) = 0.
\]

**Proof.** Take \( \ell \) which visits both \( e \) and \( -e \), and let \( l \) be the smallest index such that \( l_t = e \) or \( l_t = -e \). Let \( m \) be the largest index such that \( l_{m,t} = -l_t \). Consider the loop \( \ell' = \ell_1 \oplus (\ell_{m,t})^{-1} \oplus \ell_{m,lt} \). From multiplicativity and the sign change property it follows that \( w(\ell') = -w(\ell) \). It is now enough to notice that the map \( \ell \mapsto \ell' \) is an involution of the set of loops which visit both \( e \) and \( -e \). \( \qed \)

A set \( C \subset E \) is called a cycle if each vertex of the unique non-trivial connected component of \( (V,C) \) has exactly 2 neighbors. We first prove the main theorem in the case when \( G \) is trivalent, i.e., all vertices have at most three neighbors. The only property of trivalent graphs used here is that their even subgraphs are collections of disjoint cycles (see Figure 1). The general case is then reduced to the trivalent one by a vertex decoration method.
The trivalent case. Assume \( G \) is trivalent. By Lemma 2, for any directed edge \( e \),

\[
\det (\text{Id} - \Lambda) = C \exp \left( -w\{\ell \text{ visits } e \text{ or } -e\} \right),
\]

where \( C \) does not depend on \( x_\ell \). By Lemma 3 and the loop reversion property, the signed measure of the above set of loops is equal to

\[
w\{\ell \text{ visits } e \text{ and not } -e\} + w\{\ell \text{ visits } -e \text{ and not } e\}
+ w\{\ell \text{ visits } e \text{ and } -e\} = 2w\{\ell \text{ visits } e \text{ and not } -e\}.
\]

Using Lemma 3, we conclude that \( \det (\text{Id} - \Lambda) \) is a square of a polynomial of degree 1 in \( x_\ell \).

It is now enough to prove that the coefficients of the polynomials given by the square roots of the left- and right-hand sides of (1) are equal. Note that in the second line from below the sum is taken first over ordered and then over unordered collections of cycles.

The non-trivalent case. Assume that \( G \) is not trivalent. The idea is to construct a trivalent graph \( G^\dagger \) which has the same partition functions of even subgraphs and loops as \( G \). This is achieved by taking a vertex \( v \in V \) of degree \( k > 3 \), and replacing it with \( k-1 \) new vertices \( v', v^1, \ldots, v^k \), where each \( v^i \) has a weight \( -\frac{\lambda_i(\ell)}{2|\ell|} \) or \( \frac{\lambda_i(\ell)}{2|\ell|} \) depending on whether \( \ell \) is directed towards or away from \( v \) respectively, and the edges \( \{v^i, v^j\} \) are replaced by new edges \( \{v^i, v^j\} \) and new edges \( \{v^i, v^1\} \) are added for \( i = 1, 2, \ldots, k-1 \). The edges \( \{v^1, v^i\} \) inherit the weights from the edges \( \{v, v^i\} \) and all the edges \( \{v^i, v^1\} \) get weight 1 (see Figure 2). If one repeats this procedure for every vertex with degree larger than 3, one obtains a trivalent graph \( G^\dagger \). Note that there is a bijection between the edges of \( G \) and the edges of \( G^\dagger \) which inherited the weights from \( G \).

It is easy to see that there is a weight-preserving bijection between the even subgraphs of \( G \) and \( G^\dagger \). For an even subgraph of \( G \), it is enough to take
the corresponding edges in $G^\dagger$ and connect them in a unique way using the edges with weight 1. Uniqueness is guaranteed by the construction of $G^\dagger$. There is also a weight-preserving bijection between the loops in $G$ and $G^\dagger$. For a loop in $G$, we can construct the corresponding loop in $G^\dagger$ step by step. If the loop makes a step from $(u^i, v)$ to $(v, u^j)$, then the corresponding loop in $G^\dagger$ traverses the unique path starting at $(u^i, v)$, then following the edges of weight 1, and ending at $(v^j, u^j)$. One can check that this map preserves the weight $\lambda$.

It is now enough to use Lemma 2 and Theorem 1 for trivalent graphs.

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