The \textit{H}-theorem in \(\kappa\)-statistics: influence on the molecular chaos hypothesis

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We rediscuss recent derivations of kinetic equations based on the Kaniadakis’ entropy concept. Our primary objective here is to derive a kinetical version of the second law of thermodynamics in such a \(\kappa\)-framework. To this end, we assume a slight modification of the molecular chaos hypothesis. For the \(H\)-theorem, it is shown that the collisional equilibrium states (null entropy source term) are described by a \(\kappa\)-power law extension of the exponential distribution and, as should be expected, all these results reduce to the standard one in the limit \(\kappa \to 0\).

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\section{I. INTRODUCTION}

Over the last few years, a great deal of attention has been paid to nonextensive statistic mechanics based on the deviations of Boltzmann-Gibbs-Shannon entropic measure. Basically, in this extended framework, the key point is to substitute the exponential behaviour of the entropy by a power-law one (see, e.g. \cite{1}). Recently, similar motivations also led at least to two new examples, namely, Abe \cite{2,3} and Kaniadakis entropies \cite{4,5}. In this latter works, by using \(\kappa\)-exponential and \(\kappa\)-logarithm functions (see Eqs. (4) and (5)) and the kinetic interaction principle (KIP), Kaniadakis proposed a statistical framework based on \(\kappa\)-entropy \cite{6,7}

\[ S_\kappa(f) = -\int d^3v f[a_\kappa f^\kappa + a_{-\kappa} f^{-\kappa} + b_\kappa] \]  
(1)

where \(a_\kappa\) and \(b_\kappa\) are coefficients so that in limit \(\kappa \to 0\), Eq. (1) reduces to the standard entropy \(S_{\kappa=0}\). Expression (1) is also the most general one that leads to the \(\kappa\)-framework.

Previous works have already discussed some specific choices for the constants \(a_\kappa\) and \(b_\kappa\). For instance, for the pair \([a_\kappa = 1/2\kappa, b_\kappa = 0]\), it is possible to write the Kaniadakis entropy as \cite{3,4}

\[ S_\kappa = -\int d^3v f \ln f = -\langle \ln_k(f) \rangle, \]  
(2)

which has a perfect analogy with the standard formalism. The second and third choices are, respectively, \([a_\kappa = 1/2\kappa(1 + \kappa), b_\kappa = 0]\) \cite{4} and \([a_\kappa = 1/2\kappa(1 + \kappa), b_\kappa = -a_\kappa - a_{-\kappa}]\) \cite{3} while the fourth one is given by \([a_\kappa = Z^\kappa/2\kappa(1 + \kappa), b_\kappa = 0]\) \cite{4}. In particular, in this latter choice, the \(\kappa\)-entropy is given by \cite{4,3}

\[ S_\kappa = -\int d^3v \left( \frac{z^\kappa}{2\kappa(1 + \kappa)} f^{1+\kappa} - \frac{z^{-\kappa}}{2\kappa(1 - \kappa)} f^{1-\kappa} \right). \]  
(3)

The \(\kappa\)-statistic is defined by the \(\kappa\)-deformed functions, given by

\[ \exp_\kappa(f) = \left( \sqrt{1 + \kappa^2 f^2 + \kappa f} \right)^{1/\kappa}, \]  
(4)
\[ \ln_\kappa(f) = \frac{f^\kappa - f^{-\kappa}}{2\kappa}, \]  
(5)
and

\[ \exp_\kappa(\ln_\kappa(f)) = \ln_\kappa(\exp_\kappa(f)) = f. \]  
(6)

As one may check, the above functions reduce to the standard exponential and logarithm when \(\kappa \to 0\). In particular, this \(\kappa\)-framework leads to a class of one parameter deformed structures with interesting mathematical properties \cite{6}. Recently, a connection with the generalized Smoluchowski equation was investigated \cite{8}, and a fundamental test, i.e., the so-called Lesche stability was also checked in the \(\kappa\)-framework \cite{8}. More recently, it was shown that it is possible to obtain a consistent form for the entropy (linked with a two-parameter deformations of logarithm function), which generalizes the Tsallis, Abe and Kaniadakis logarithm behaviours \cite{3}. In the experimental viewpoint, there exist some evidence related with the \(\kappa\)-statistic, namely, cosmic rays flux, rain events in meteorology \cite{6}, quark-gluon plasma \cite{10}, kinetic models describing a gas of interacting atoms and photons \cite{11}, fracture propagation phenomena \cite{12}, and income distribution \cite{13}, as well as construct financial models \cite{14}. In the theoretical front, some studies on the canonical quantization of a classical system has also been investigated \cite{13}.

In this letter, we aim at rediscussing the \(H\)-theorem in the context of Kaniadakis entropy framework, Eq. (4). However, instead of using the KIP introduced in Ref. \cite{8}, we propose a different route to the \(\kappa\)-statistic which follows similar arguments of Ref. \cite{16}. In reality, the main result is to obtain the equilibrium velocity \(\kappa\)-distribution of a slight modification of the kinetic Boltzmann \(H\)-theorem, where the central idea follows by modifying the molecular chaos hypothesis and generalization of the local entropy formula in accordance with a \(\kappa\)-statistic.
II. THE MOLECULAR CHAOS HYPOTHESIS

As is widely known, the Boltzmann’s kinetic theory (BKT) relies on two statistical ingredients, namely 17, 18:

- The local entropy which is expressed by Boltzmann’s logarithmic measure (Boltzmann’s constant is an unity)

\[ H[f] = -\int f(\mathbf{r}, \mathbf{v}, t) \ln f(\mathbf{r}, \mathbf{v}, t) \, d^3v. \]  

(7)

- The hypothesis of molecular chaos (“Stosszahlansatz”), i.e., the two point correlation function of the colliding particles can be factorized

\[ f(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, t) = f(\mathbf{r}_1, \mathbf{v}_1, t) f(\mathbf{r}_2, \mathbf{v}_2, t). \]  

(8)

There is some controversy associated with the second assumption, Eq. 8. In particular, Burbury 19, was the first to point out that this hypothesis provides the fundamental role within the BKT. Physically, the equation 8 represents that colliding molecules are uncorrelated, i.e., the velocities and positions of pairs of molecules are statistically independents. The irreversibility associated with the Boltzmann’s equation can be traced back to this assumption. Indeed, as the molecules which has been assumed to be uncorrelated before a collision, become correlated after the collision, this clearly represents a time asymmetric hypothesis 20. In particular, this hypothesis may not always be valid for real gases (see 21 22 for details).

In this investigation, we introduce a consistent generalization of this hypothesis, expression 8, within the \( \kappa \)-statistic proposed by Kaniadakis. We remark that equation 8 implies that the logarithm of the joint distribution \( f(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, t) \) describing colliding molecules is given by

\[ \ln f(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, t) = \ln f(\mathbf{r}_1, \mathbf{v}_1, t) + \ln f(\mathbf{r}_2, \mathbf{v}_2, t), \]  

(9)

where each term involve only the marginal distribution associated with one of the colliding molecules. The new hypothesis assumed here is to consider that a power of the joint distribution (instead of the logarithm) be equal to the sum of two terms, each one depending on just one of the colliding molecules. Considering the \( \kappa \)-logarithm function, the condition above can be formulated in a way that extend the standard hypothesis, Eq. 8.

III. H-THEOREM AND \( \kappa \)-STATISTICS

Let us now introduce a spatially homogeneous gas of \( N \) hard-sphere particles of mass \( m \) and diameter \( s \), under the action of an external force \( \mathbf{F} \), and enclosed in a volume \( V \). In BKT the state of a non-relativistic gas is characterized by the one-particle distribution function \( f(\mathbf{r}, \mathbf{v}, t) \), which is defined in such a way that \( f(\mathbf{r}, \mathbf{v}, t) \, d^3\mathbf{v} \) gives at a time \( t \), the number of particles in the volume element \( d^3\mathbf{v} \) around the particle position \( \mathbf{r} \) and velocity \( \mathbf{v} \). Let us consider that distribution function is a solution of the \( \kappa \)-Boltzmann equation, given by

\[ \frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \frac{\partial f}{\partial \mathbf{v}} = C_\kappa(f), \]  

(10)

where \( C_\kappa \) defines the \( \kappa \)-collisional term. In expression 10, the left-hand-side (LHS) is just the total time derivative of the distribution function, thus it is reasonable to consider that the unique possible modification describing \( \kappa \)-statistic must be associated with the collisional term, which is clearly a hypothesis of work. Basically, \( C_\kappa(f) \) may be calculated through the laws of elastic collisions, where the standard assumptions are also valid 17, 18.

As in the canonical H-theorem, our main goal here is to show that \( C_\kappa(f) \) leads to a nonnegative expression for the time derivative of the \( \kappa \)-entropy, and does not vanish unless the distribution function assumes the equilibrium form for a \( \kappa \)-distribution which has been recently proposed 4. Here, we define

\[ C_\kappa(f) = \frac{s^2}{2} \int |\mathbf{V} \cdot \mathbf{e}| R_\kappa d\omega d^3v_1, \]  

(11)

where \( d^3v_1 \) is an arbitrary volume element in the velocity space, \( \mathbf{V} \) denotes the relative velocity before collision, \( \mathbf{V} = \mathbf{v}_1 - \mathbf{v} \), \( \mathbf{e} \) denotes an arbitrary unit vector, \( d\omega \) is an elementary solid angle such that \( s^2 d\omega \) is the area of the “collision cylinder” (for details see Refs. 17, 18), finally \( R_\kappa(f, f') \) is a difference of two correlation functions (just before and after collision). In this approach, such expression is assumed to satisfy a \( \kappa \)-generalized form of molecular chaos hypothesis, which is given by

\[ R_\kappa = \exp_\kappa (\ln_\kappa z' f' + \ln_\kappa z f), \]  

(12)

where \( \exp_\kappa \) refers to the distribution function after collision, \( z, z' \) are arbitrary constant and \( \exp_\kappa(f), \ln_\kappa(f) \), are defined by Eqs. (2) and (3). Note that \( \kappa \rightarrow 0 \) the above expression reduces to \( R_0 = (z' f')(z f') - (z f)(z' f) \), which is one standard to the molecular chaos hypothesis.

In the present framework, we adopt Kaniadakis formula for local entropy

\[ H_\kappa = -\int d^3v \left( \frac{z^{\kappa}}{2\kappa(1+\kappa)} f^{1+\kappa} - \frac{z^{-\kappa}}{2\kappa(1-\kappa)} f^{1-\kappa} \right). \]  

(13)

Here, in order to obtain the source term, we need the partial time derivative of \( H_\kappa \)

\[ \frac{\partial H_\kappa}{\partial t} = -\int d^3v \ln_\kappa f z \frac{\partial f}{\partial t}, \]  

(14)
and combining with the $\kappa$-Boltzmann equation \(^{10}\), one may rewrite the expression in the form of a balance equation

$$
\frac{\partial H_\kappa}{\partial t} + \nabla \cdot S_\kappa = G_\kappa(\mathbf{r}, t),
$$

(15)

where the $\kappa$-entropy flux vector related to $H_\kappa$ is given by

$$
S_\kappa = -\int d^3\nu \left( \frac{z^\kappa}{2\kappa(1+\kappa)} f^{1+\kappa} - \frac{z^{-\kappa}}{2\kappa(1-\kappa)} f^{1-\kappa} \right),
$$

(16)

and the source term reads

$$
G_\kappa = \frac{-s^2}{2} \int |\mathbf{V} \cdot \mathbf{e}| \ln_\kappa f z \ R_\kappa d\omega d^3v_1 d^3v.
$$

(17)

At this point, it is convenient to rewrite $G_\kappa$ in a more symmetrical form by using some elementary symmetry operations which also take into account the inverse collisions. First we notice that by interchanging $\mathbf{v}$ and $\mathbf{v}_1$ the value of the integral is preserved. This happens because the magnitude of the relative velocity vector and the scattering cross section are invariants \(^{18}\). In addition, the value of $G_\kappa$ is not altered if we integrate with respect to the variables $\nu'$ and $\mathbf{v}_1'$. Actually, although changing the sign of $R_\kappa$ in this step (inverse collision), the quantity $d^3\nu d^3v_1$ is also a collisional invariant \(^{18}\).

As one may check, such considerations imply that the $\kappa$-entropy source term can be written as

$$
G_\kappa(\mathbf{r}, t) = \frac{s^2}{8} \int |\mathbf{V} \cdot \mathbf{e}| (\ln_\kappa z'_1 f'_1 + \ln_\kappa z' f') -
$$

$$- \ln_\kappa z_1 f_1 - \ln_\kappa z f) \exp_\kappa (\ln_\kappa z f + \ln_\kappa z_1 f_1) + \ln_\kappa z f - \ln_\kappa z_1 f_1) -
$$

$$- \exp_\kappa (\ln_\kappa z f + \ln_\kappa z_1 f_1) d\omega d^3v_1 d^3v.
$$

(18)

Note that the integrand in \(^{18}\) is never negative, because the expressions

$$
(\ln_\kappa z' f' + \ln_\kappa z'_1 f'_1 - \ln_\kappa z f - \ln_\kappa z_1 f_1)
$$

(19)

and

$$
\exp_\kappa (\ln_\kappa z' f' + \ln_\kappa z'_1 f'_1) - \exp_\kappa (\ln_\kappa z f + \ln_\kappa z_1 f_1)
$$

(20)

always have the same signs. Therefore, for values of $\kappa$ on the interval $[-1; 1]$, we obtain

$$
\frac{\partial H_\kappa}{\partial t} + \nabla \cdot S_\kappa = G_\kappa \geq 0,
$$

(21)

which is the mathematical expression for the $H_\kappa$-theorem. This inequality states that the $\kappa$-entropy source must be positive or zero, thereby furnishing a kinetic derivation of the second law of thermodynamics in the $\kappa$-statistic.

Now, in order to make the $H$-theorem and the $\kappa$-statistic compatible, we need to recover the related equilibrium distribution previously obtained by an extremization of $\kappa$-entropy and KIP \(^{6}\). The $H_\kappa$-theorem states that $G_\kappa = 0$ is a necessary and sufficient condition for equilibrium. Since the integrand of \(^{18}\) cannot be negative, this occur if and only if

$$
\ln_\kappa z' f' + \ln_\kappa z'_1 f'_1 = \ln_\kappa z f + \ln_\kappa z_1 f_1
$$

(22)

Therefore, the above sum of $\kappa$-logarithms remains constant during a collision, or equivalently, it is a summation invariant. Indeed, the unique quantities satisfying \(^{18}\) are the particle masses, and the expressions for momentum and energy conservation laws, which lead to the following expression

$$
\ln_\kappa z f = a_0 + a_1 \cdot \mathbf{v} + a_2 \mathbf{v}^2,
$$

(23)

where $a_0$ and $a_2$ are constants and $a_1$ is an arbitrary constant vector. By introducing the barycentric velocity, $\mathbf{u}$, we may rewrite \(^{23}\) as

$$
\ln_\kappa z f = \alpha - \gamma (\mathbf{v} - \mathbf{u})^2,
$$

(24)

with a different set of constants. Thus, we obtain a $\kappa$-distribution

$$
f = \frac{1}{2} \exp_\kappa [\alpha - \gamma (\mathbf{v} - \mathbf{u})^2],
$$

(25)

where $\gamma$ and $\mathbf{u}$ may be functions of the temperature. The expression above is the general form of the $\kappa$-Maxwellian distribution function.

IV. FINAL REMARKS

Summing up, we have discussed a $\kappa$-generalization of Boltzmann’s kinetic equation along the lines defined by $\kappa$-statistic. The main results follow from a slight modification of the main statistical hypothesis underlying Boltzmann’s approach:

1. The $\kappa$-statistic has been explicitly introduced through a new functional formula for the local entropy;

2. A nonfactorizable expression for the molecular chaos hypothesis has been adopted.

Both ingredients are shown to be consistent with the standard laws describing the microscopic dynamics, and reduce to the familiar Boltzmann assumptions in the limit ($\kappa = 0$).

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