Supplementary Information to:
“Collective phenomena emerging from the interactions between dynamical processes in multiplex networks”

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1 Derivation of the biased random walk stationary distribution

Here we follow Ref. [1] and derive the stationary distribution of random walkers \(y^*\) for the discrete-time version of the motion rule given by Eq. (4) in the main text, namely:

\[
\dot{y}_i = \frac{1}{\tau_y} \left( \sum_{j=1}^{N} \pi_{ij} - \delta_{ij} \right) y_j
\]  

where:

\[
\pi_{ij} = \frac{a_{ji}^{[2]} \chi_i^\alpha}{\sum_l a_{jk}^{[2]} \chi_l^\alpha}
\]  

is the transition probability from node \(j\) to node \(i\). We note first that the steady-state solution of Eq. (1) is given by

\[
y_i = \sum_{j=1}^{N} \pi_{ij} y_j = \sum_{j=1}^{N} \frac{a_{ji}^{[2]} \chi_i^\alpha}{\sum_l a_{jl}^{[2]} \chi_l^\alpha} y_j,
\]
Importantly, this steady-state represents the same steady-state of the discrete-time random walk

$$y_{i}^{t+1} = \sum_{j=1}^{N} \pi_{ij} y_{j}^{t},$$

(4)

which in turn is given by the dominant eigenvector of the transition matrix $\Pi = [\pi_{ij}]$. To find this stationary distribution, we denote $P_{j\rightarrow i}(T)$ as the probability of moving from $j$ to $i$ in $T$ discrete time steps, so that

$$P_{j\rightarrow i}(T) = \sum_{l_{1},l_{2},...l_{t-1}} \pi_{il_{t-1}} \times \pi_{l_{t-1}l_{t-2}} \times \cdots \times \pi_{l_{1}j},$$

(5)

Next, we define the auxiliary parameter $c_{j} = \sum_{i=1}^{N} a_{ji}^{[2]} \chi_{i}^{\alpha}$, and using the symmetry of the network structure (i.e., $a_{ij}^{[2]} = a_{ji}^{[2]}$), it follows that

$$\frac{P_{j\rightarrow i}(T)}{P_{i\rightarrow j}(T)} = \frac{\chi_{i}^{\alpha} c_{i}}{\chi_{j}^{\alpha} c_{j}}.$$  

(6)

Next we look for a stationary state whereby the fraction $y_{i}^{*}$ of a walker occupying node $i$ is equal to $P_{l\rightarrow i}(T)$ for all $l$ and $T > 0$. Substituting the stationary probabilities and rearranging yields:

$$y_{i}^{*} \chi_{j}^{\alpha} c_{j} = y_{j}^{*} \chi_{i}^{\alpha} c_{i}.$$  

(7)

Summing both sides of Eq. (7) over $j$ and rearranging, we then obtain

$$y_{i}^{*} = \frac{\sum_{j=1}^{N} \chi_{j}^{\alpha} c_{j}}{\sum_{j=1}^{N} \chi_{j}^{\alpha} c_{j}} = \frac{\chi_{i}^{\alpha} \sum_{d=1}^{N} a_{d} \chi_{d}^{\alpha}}{\sum_{j=1}^{N} \chi_{j}^{\alpha} \sum_{l=1}^{N} a_{jl} \chi_{l}^{\alpha}}.$$  

(8)

To complete the analysis, we next assume that the biasing values $\chi_{i}$ evolve according to Equation (6) in the main text and relax to a steady state, defined by $\dot{\chi}_{i} = 0 \forall i$. This yields $\chi_{i} = s_{i}^{*}$, where $s_{i}^{*}$ represents the steady-state dynamical strength $s_{i}^{\text{dyn}}$ in the synchronization layer as defined by the synchronization dynamics. (We note that such a steady-state value $s_{i}^{*}$ is valid provided that the network is large enough so that finite-size fluctuations remain sufficiently small). Since the steady-state dynamical strength is defined as $s_{i}^{*} = r_{i}^{*} \cos(\psi_{i}^{*} - x_{i}^{*})$ we have two cases. If the oscillator dynamics are incoherent then $r_{i} \approx 0$. Moreover, any finite system exhibits finite-size fluctuations around the incoherent state, so we can approximate $r_{i} \approx \xi$ for some $\xi \ll 1$, yielding $\chi_{i}^{*} \approx \text{constant}$. On the other hand, if the oscillator dynamics are synchronized then $r_{i}^{*} \approx k_{i}^{[1]} r^{*}$ and
we can approximate \( \chi_i^* \approx rk_i \). Inserting these expressions into Eq. (8) we finally obtain:

\[
y_i^* \propto \begin{cases} k_i^{[2]} k_i^{[1]} & \text{if } r \approx 0 \\ (k_i^{[1]})^\alpha k_i^{[2]} \langle (k_i^{[1]})^\alpha \rangle & \text{if } r \approx 1,
\end{cases}
\]

as in Equation (7) in the main text.

### 2 Derivation of the synchronization self-consistency equation

Here we derive the self consistency condition given in Eq. (8) of the main text, starting from the governing synchronization dynamics in Eq. (3) of the main text and using a heterogeneous mean-field approach. In particular, we will approximate the entries of the adjacency matrix \( A \) using nodal degrees, specifically

\[
da_{ij}^{[1]} \approx \frac{k_i^{[1]} k_j^{[1]}}{N \langle k_i^{[1]} \rangle},
\]

where \( \langle k_i^{[1]} \rangle = N^{-1} \sum_j k_j^{[1]} \) is the mean degree in the synchronization layer. We also introduce a modified order parameter,

\[
\tilde{r} e^{i\tilde{\psi}} = \frac{1}{N \langle k_i^{[1]} \rangle} \sum_{j=1}^{N} k_j^{[1]} e^{ix_j},
\]

which is similar to the typical order parameter, except for a re-weighting of the angles according to degree. We note that, just like \( r \), the modified order parameter \( \tilde{r} \) varies between zero and one, and tends to be a very strong approximation to \( r \). We introduce the modified order parameter because of its utility in the analysis of the synchronization dynamics. Along with the mean-field approximation of \( A \) [i.e., Eq. (10)], Eq. (11) allows us to simplify the governing dynamics in the synchronization layer to

\[
\dot{x}_i = \omega_i + \lambda \tilde{r} \tilde{k}_i^{[1]} \sin(\tilde{\psi} - x_i).
\]

We note that the accuracy of this mean-field approximation improves as the network topology becomes denser, which is why we use a relatively dense network with mean degree \( \langle k \rangle = 10 \) in the main text.

Next, we look for a synchronized state by first noting that for a strongly synchronized population, where each oscillator becomes entrained, the collective frequency \( \Omega \) of the population is given by the mean natural frequency, i.e., \( \Omega = \langle \omega \rangle = N^{-1} \sum_j \omega_j \). Thus, an oscillator that becomes entrained with the rest of the population evolves with a velocity of \( \dot{x}_i = \Omega \). After entering the
appropriate rotating frame $x_i \mapsto x_i + \Omega t$ and inspecting Eq. (12), it is easy to see that an oscillator $i$ becomes entrained, or locked, if and only if it satisfies

$$|\omega_i - \Omega| \leq \lambda \tilde{r} k_i^{[1]},$$  \hspace{1cm} (13)

and otherwise it continually “drifts” past the synchronized population. In particular, a locked oscillator $i$ then comes to equilibrium at

$$x_i = \arcsin \left( \frac{\omega_i - \Omega}{\lambda \tilde{r} k_i^{[1]}} \right).$$ \hspace{1cm} (14)

To classify the degree of synchronization we inspect the modified order parameter $\tilde{r} e^{i\tilde{\psi}}$. First, we note that with a suitable shift of initial conditions we may set the mean angle $\tilde{\psi} = 0$, so expanding the exponential and keeping the real part (since the imaginary part must be zero) yields

$$\tilde{r} = \frac{1}{N \langle k^{[1]} \rangle} \sum_{j=1}^{N} k_j^{[1]} \cos x_j.$$ \hspace{1cm} (15)

Neglecting the effect of drifting oscillators on the order parameter, we sum over the locked population and insert Eq. (14) into Eq. (15) to obtain

$$\tilde{r} = \frac{1}{N \langle k^{[1]} \rangle} \sum_{|\omega_j - \Omega| \leq \lambda \tilde{r} k_j^{[1]} } \left[ 1 - \left( \frac{\omega_j - \Omega}{\lambda \tilde{r} k_j^{[1]} } \right)^2 \right].$$ \hspace{1cm} (16)

Finally, for large networks the sum in Eq. (16) can be converted to an integral using the joint distribution $P(k^{[1]}, \omega)$:

$$\tilde{r} = \frac{1}{\langle k^{[1]} \rangle} \int \int_{|\omega - \Omega| \leq \lambda \tilde{r} k^{[1]}} P(k^{[1]} , \omega) k^{[1]} \sqrt{1 - \left( \frac{\omega - \Omega}{\lambda \tilde{r} k^{[1]} } \right)^2} d\omega dk^{[1]},$$ \hspace{1cm} (17)

Finally, we arrive at Eq. (8) in the main text after approximating $r \approx \tilde{r}$. We note that this approximation holds extremely well for networks with homogeneous degree distributions, but in principle can lose accuracy as the degree distribution becomes more and more heterogeneous, since $\tilde{r}$ represents a convenient re-weighting of the terms that contribute to $r$. We find in practice, however, that the two values are remarkably close due to the fact that, if a system synchronizes strongly as we see in our system, the re-weighting has little effect.
3 Finite relaxation dynamics

Here we present numerical results for the cases of finite relaxation time scales for the random walker, natural frequency, and bias variables given by $\tau_y$, $\tau_\omega$, and $\tau_\chi$ in Eqs. (4), (5), and (6), respectively, in the main text. Recall that the main results presented in the main text considered the limit of quick relaxation, i.e., $\tau_y, \tau_\omega, \tau_\chi \rightarrow 0^+$. Here we demonstrate that the novel phenomenon we observe in the system is not restricted to this particular choice. We begin by considering a few different choices of time scales and focus on the synchronization dynamics. In Figure 1 we plot the resulting synchronization profiles for the degree of synchronization $r$ vs the coupling strength $\lambda$ with a few different values of the bias parameter $\alpha$ for a few different choices of time scales. In panel (a) we choose all $\tau_y, \tau_\omega, \tau_\chi = 10^{-2}$, in panel (b) we choose $\tau_y = 10^{-2}$ and $\tau_\omega, \tau_\chi = 10^{-1}$, and in panel (c) we choose all $\tau_y, \tau_\omega, \tau_\chi = 10^{-1}$. Results use networks of size $N = 1000$ with SF ($\gamma_\text{SF} = 3$) and ER topologies in the synchronization and transport layers, respectively, each with mean degree $\langle k \rangle = 10$. In each case we observe qualitatively similar behavior as those results presented in the main text, i.e., a second order, continuous transition between incoherence and synchronization for smaller $\alpha$ values and the emergence of a first-order explosive transition between incoherence and synchronization for larger $\alpha$ values. While the results remain qualitatively similar, we make note of a quantitative difference: as the relaxation dynamics become slower, i.e., the time scales become larger, a larger value of $\alpha$ is required for explosive synchronization to emerge. We also note that when the time scales become sufficiently large we observe that explosive synchronization does not occur spontaneously.

![Figure 1: Finite Time Scale Dynamics I. Synchronization profiles of the degree of synchronization $r$ vs the coupling strength $\lambda$ with a few different biasing parameters $\alpha$ for finite relaxation time scales: (a) $\tau_y, \tau_\omega, \tau_\chi = 10^{-2}$, (b) $\tau_y = 10^{-2}$ and $\tau_\omega, \tau_\chi = 10^{-1}$, and (c) $\tau_y, \tau_\omega, \tau_\chi = 10^{-1}$.](image)

To accompany these results we explore the $(\alpha, \lambda)$ parameter space more thoroughly, presenting the numerically-obtained stability diagram for the system with finite relaxation time scales. In particular, in Figure 2 we plot the steady-state degree of synchronization $r$ found as a function of both $\alpha$ and $\lambda$. Results are color coded so that $r = 0$ and $r = 1$ are given by blue and yellow,
respectively, and regions where both incoherent and synchronized states are found to be stable are given by the white regions. In panels (a) and (b) we plot the results for time scale choices $\tau_y, \tau_\omega, \tau_\chi = 10^{-2}$ and $\tau_y, \tau_\omega, \tau_\chi = 10^{-1}$, respectively. Results use networks of size $N = 500$ with SF ($\gamma = 3$) and ER topologies in the synchronization and transport layers, respectively, each with mean degree $\langle k \rangle = 10$. Here we can more clearly see the emergence of a bistable regime for a large enough $\alpha$ value. Moreover, we see that a faster relaxation time scale leads to a larger bistable region.

**Figure 2: Finite Time Scale Dynamics II.** Stability diagrams obtained numerically for the $(\alpha, \lambda)$ parameter space using finite relaxation time scales. The order parameter $r$ is color coded blue and yellow for $r = 0$ and $r = 1$, respectively, with different relaxation time scales: (a) $\tau_y, \tau_\omega, \tau_\chi = 10^{-2}$ and (b) $\tau_y, \tau_\omega, \tau_\chi = 10^{-1}$.

### 4 Effect of Network Topology

Here we present numerical results that explore the effect of using different network topologies than those presented in the main text, where we assumed a SF topology in the synchronization layer with exponent $\gamma = 3$ and an ER topology in the transport layer. In particular, we show that the results we presented in the main text are qualitatively reproduced for different topologies. We begin by varying the heterogeneity in the synchronization layer via the exponent $\gamma$. In Fig. 3 we present the stability diagrams for the synchronization dynamics for (a) a more heterogeneous topology in the synchronization layer obtained by using $\gamma = 2.6$ and (b) a less heterogeneous topology in the synchronization layer obtained by using $\gamma = 3.4$. In both cases, the topology of the transport layer is ER and both layers have mean degree $\langle k \rangle = 10$. As in previous stability diagrams, we plot the steady-state degree of synchronization $r$ found as a function of both $\alpha$ and $\lambda$. Results are color coded so that $r = 0$ and $r = 1$ are given by blue and yellow, respectively, and regions where both incoherent and synchronized states are found to be stable are given by the white regions. We note that in both cases bistability persists, indicating the existence of explosive synchronization for
sufficiently strong bias in the random walker dynamics.

Figure 3: Effect of Network Topology I. Stability diagrams obtained numerically for the \((\alpha, \lambda)\) parameter space using various network topologies. The order parameter \(r\) is color coded blue and yellow for \(r = 0\) and \(r = 1\), respectively, with SF network topology in the synchronization layer with exponents (a) \(\gamma = 2.6\) and (b) \(\gamma = 3.4\), and ER network topology in the transport layer. In both cases the mean degree in each layer is \(\langle k \rangle = 10\).

Moreover, we also explore the possibility where both networks have SF topologies. In Fig. 3 we present the stability diagrams for the case where the synchronization layer has an SF topology with \(\gamma = 3\) while the transport layer has SF topologies with varying degrees of heterogeneity: (a) \(\gamma = 2.6\), (b) \(\gamma = 3\), and (c) \(\gamma = 3.4\). In each case, both layers have mean degree \(\langle k \rangle = 10\). Again, as in previous stability diagrams, we plot the steady-state degree of synchronization \(r\) found as a function of both \(\alpha\) and \(\lambda\). Results are color coded so that \(r = 0\) and \(r = 1\) are given by blue and yellow, respectively, and regions where both incoherent and synchronized states are found to be stable are given by the white regions. We point out that bistability persists in each case, indicating the existence of explosive synchronization for sufficiently strong bias in the random walker dynamics.

References

[1] Gómez-Gardeñes J, Latora V, Entropy rate of diffusion processes on complex networks, Phys. Rev. E 78, 065102 (2008).
Figure 4: **Effect of Network Topology II.** Numerically obtain stability diagram for the \((\alpha, \lambda)\) parameter space using various network topologies. The order parameter \(r\) is color coded blue and yellow for \(r = 0\) and \(r = 1\), respectively, with SF network topology in the synchronization layer with exponent \(\gamma = 3.0\) and SF network topology in the transport layer with exponents (a) \(\gamma = 2.6\), (b) 3, and (c) 3.4. In each case the mean degree in each layer is \(\langle k \rangle = 10\).