1-1-2017

Extension properties of asymptotic property C and finite decomposition complexity

Susan Beckhardt
University at Albany, State University of New York, sbeckhardt@albany.edu

The University at Albany community has made this article openly available. Please share how this access benefits you.

Follow this and additional works at: https://scholarsarchive.library.albany.edu/legacy-etd

Part of the Physical Sciences and Mathematics Commons

Recommended Citation
Beckhardt, Susan, "Extension properties of asymptotic property C and finite decomposition complexity" (2017). Legacy Theses & Dissertations (2009 - 2024). 1782. https://scholarsarchive.library.albany.edu/legacy-etd/1782

This Dissertation is brought to you for free and open access by the The Graduate School at Scholars Archive. It has been accepted for inclusion in Legacy Theses & Dissertations (2009 - 2024) by an authorized administrator of Scholars Archive. Please see Terms of Use. For more information, please contact scholarsarchive@albany.edu.
Extension Properties of Asymptotic Property C and Finite Decomposition Complexity

by

Susan Beckhardt

A Dissertation
Submitted to the University at Albany, State University of New York
in Partial Fulfillment of
the Requirements for the Degree of
Doctor of Philosophy

College of Arts & Sciences
Department of Mathematics and Statistics
2017
Abstract

We prove extension theorems for several geometric properties such as asymptotic property C (APC), finite decomposition complexity (FDC), straight finite decomposition complexity (sFDC) which are weakenings of Gromov’s finite asymptotic dimension (FAD).

The context of all theorems is a finitely generated group $G$ with a word metric and a coarse quasi-action on a metric space $X$. We assume that the quasi-stabilizers have a property $P_1$, and $X$ has the same or sometimes a weaker property $P_2$. Then $G$ also has property $P_2$.

We show some sample applications, discuss constraints to further generalizations, and illustrate the flexibility that the weak quasi-action assumption allows.
Acknowledgments

As I complete this chapter of my mathematical career and get ready for the next adventure, I take some time to reflect on and thank the many teachers, mentors, and friends who have helped me find my way.

To my wonderful advisor Dr. Goldfarb and the rest of my committee, Dr. Varisco, Dr. Tchernev, and Dr. Srivastav: You opened my eyes up to a fascinating area of study and mentored me through so many challenges, from researching to giving seminars to the writing of this dissertation. You held me to a high standard and trusted me to achieve it. Without your encouragement, patience, and guidance I would not be where I am today.

To Joan, JoAnna, Rose, and Stacy: Thank you for your tireless efforts to help my fellow graduate students and me navigate through this PhD program. I could always count on you for a friendly word, and to have my back when I made mistakes.

To my undergraduate advisor, Prof. Brenda Johnson: Thank you for giving me my first taste of mathematical research and introducing me to the joys of topology.

To my eighth grade algebra teacher, Miss Frankle: I was convinced that math was boring, but you saw my potential and challenged me to find the fun in math. Thanks to you, I’ve been rising to meet that challenge ever since!

To my friends Elizabeth, Umber, Kim, Alix, Florie, and so many others: Thank you all for keeping me grounded during times of stress, refueling my creativity when I felt stuck, and supporting me with love, laughter, and extra yarn when I needed it most.

To Mom and Dad: From the time you used a pizza box to teach me about logarithms at the dinner table, to our use of carrots to demonstrate a problem of intersecting cylinders, you have inspired me to value learning and take every opportunity to discover something new.

A: My first and best teachers.

Q: Who are Kathy and Dave Beckhardt?
# Contents

1. Introduction ................................................. 1
2. Background .................................................... 3
   2.1 Coarse Geometry ......................................... 3
   2.2 Geometric Group Theory .................................. 5
   2.3 Asymptotic Dimension and Asymptotic Property C ........ 11
   2.4 Finite Decomposition Complexity ....................... 14
   2.5 Property A .............................................. 18
   2.6 Permanence theorems ..................................... 20
3. Extension theorems for FAD, APC, FDC, and sFDC .......... 24
4. Applications and discussion .................................. 28
   4.1 Applications ............................................. 28
   4.2 Fibered properties ....................................... 29

Bibliography .................................................... 32
1 Introduction

A large number of geometric conditions of metric spaces or groups with the word metric have been introduced, beginning with Gromov’s finite asymptotic dimension (FAD) in [15], inspired by the covering dimension in topology. More recently, weaker variants such as asymptotic property C (APC), finite decomposition complexity (FDC), and straight finite decomposition complexity (sFDC) have been introduced; all are coarse invariants which describe the asymptotic or large-scale behavior of a space. All are stronger than Yu’s property A. These invariants have risen to prominence for their applications in the work of many authors on the Novikov and Borel conjectures in manifold topology, L- and K-theory.

A well known example of a group with infinite asymptotic dimension, the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$, has both APC and FDC. Further examples of groups with FDC are established in [17]. For other significant infinite-dimensional groups such as Thompson’s group, the answer is unknown. A natural question arises of how to construct further examples of groups and spaces with these properties.

In Section 2 we present background material on coarse geometry and the various coarse properties we have mentioned above, including a number of variations on FDC, and the relationships between them.

Section 3 is devoted to our main theorem, a generalization of a theorem by Bell and Dranishnikov about groups acting on finite asymptotic dimensional spaces [1, Theorem 2].

We refer the reader to the next section for the precise definition of a coarse quasi-action on a metric space. It is a significant and useful generalization of actions by isometries and quasi-isometries. Quasi-actions by quasi-isometries in particular are the framework for some fundamental questions in geometric group theory; cf. [11, 19, 21].

Main Theorem. Let $G$ be a finitely generated group with a coarse quasi-action on a metric space $X$. If $X$ and all quasi-stabilizers of the action have FAD (respectively, FDC or sFDC) then $G$ has FAD (respectively, FDC or sFDC) with respect to a word metric. If $X$ has APC and the quasi-stabilizers have asymptotic dimension bounded from above by some number $n \geq 0$ then $G$ has APC.
The last statement in the theorem is a new geometric fact even for isometric actions, and it has applications to extensions of groups via standard constructions. We discuss these results in the last section.
2 Background

2.1 Coarse Geometry

Let $X$ and $Y$ be metric spaces with metric functions $d_X$ and $d_Y$. We will assume that the metrics are proper, in the sense that closed bounded subsets of $X$ and $Y$ are compact.

Given a subset $S$ of a metric space $X$ and $r > 0$, we will use the notation $S[r]$ for the $r$-enlargement of $S$, that is the subset $\{ x \in X \mid d(x, s) \leq r \text{ for some } s \in S \}$. In particular, the metric ball centered at $x$ with radius $r$ is denoted by $x[r]$.

If there exists some $r > 0$ such that $S[r] = X$, then we say that $S$ is commensurable with $X$. For example, $\mathbb{Z}$ is a commensurable subset of $\mathbb{R}$ (using $r = \frac{1}{2}$).

**Definition 1.** A function $f : X \to X$ is bounded if there exists some $D > 0$ such that for every $x \in X$, $d(x, f(x)) \leq D$.

**Definition 2.** Let $\lambda > 0$. A function $f : X \to Y$ is said to be $\lambda$-Lipschitz if for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2).$$

A function $f : X \to Y$ is quasi-Lipschitz if there exist $\lambda, C > 0$ such that for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + C.$$

Notice that while Lipschitz maps are uniformly continuous, quasi-Lipschitz maps are in general not continuous. In fact, we can view quasi-Lipschitz maps (as well as coarse maps, defined below) as a kind of large-scale analogue of continuity: where continuous maps respect the small-scale structure of a space (its topology), a quasi-Lipschitz map is a map that respects certain large-scale properties of a metric space.

**Definition 3.** A map $f : X \to Y$ is a quasi-isometry if there exist $\lambda, C > 0$ such that:

1. for all $x_1, x_2 \in X$, $\frac{1}{\lambda} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + C$, and

2. for all $y \in Y$, there exists $x \in X$ such that $d_Y(f(x), y) \leq C$.

Note that the second condition is equivalent to saying that the image of $f$ is commensurate with $Y$. 

3
**Example 4.** An example of a quasi-isometry is the inclusion of a commensurable subspace into a larger metric space, e.g. the inclusion of \( \mathbb{Z} \) into \( \mathbb{R} \). In this case \( \lambda = 1 \) and \( C = \frac{1}{2} \).

**Definition 5.** A map \( f: X \to Y \) between proper metric spaces is called uniformly expansive (or bornologous) if there is a non-decreasing function \( \ell: [0, \infty) \to [0, \infty) \) such that

\[
d_Y(f(x_1), f(x_2)) \leq \ell(d_X(x_1, x_2)) \quad \text{for all } x_1, x_2 \in X.
\]

The same kind of map is called proper if \( f^{-1}(S) \) is a bounded subset of \( X \) for each bounded subset \( S \) of \( Y \). We say \( f \) is a coarse map if it is both uniformly expansive and proper.

A slight strengthening of the second condition results in a coarse embedding:

**Definition 6.** A map \( f: X \to Y \) is effectively proper if there exists a proper, nondecreasing function \( \delta: [0, \infty) \to [0, \infty) \) such that

\[
\delta(d_X(x_1, x_2)) \leq d_Y(f(x_1), f(x_2)) \quad \text{for all } x_1, x_2 \in X.
\]

A coarse embedding is a map that is both uniformly expansive and effectively proper.

**Definition 7.** The map \( f \) is a coarse equivalence if there is a coarse map \( g: Y \to X \) such that \( f \circ g \) and \( g \circ f \) are bounded maps. The map \( g \) is said to be a coarse inverse of \( f \).

The notions of quasi-isometry and coarse equivalence are equivalence relations for metric spaces.

**Examples.** Any bounded function \( f: X \to X \) is coarse. In fact, it is a coarse equivalence with \( f \) as its own coarse inverse, using \( \ell(r) = r + 2D \).

The isometric embedding of a metric subspace is a coarse map. An isometry, which is a bijective isometric map, is a coarse equivalence. Quasi-Lipschitz maps are exactly those uniformly expansive maps for which the bounding function \( \ell \) is linear.

Quasi-isometries are coarse equivalences, but not every coarse equivalence is a quasi-isometry. For a counterexample, consider the spaces \( X = \{2^n \mid n = 0, 1, 2, \ldots \} \) and \( Y = \{4^n \mid n = 0, 1, 2, \ldots \} \), considered as metric subspaces of \( \mathbb{R} \), and let \( f: X \to Y \) be the bijection given by \( f(x) = x^2 \).

**Proposition 8.** \( f \) is a coarse equivalence, but not a quasi-isometry.
Proof. To see that $f$ is not quasi-Lipschitz (therefore not a quasi-isometry), suppose there exist $\lambda, C > 0$ such that for all $x_1, x_2 \in X$, we have $|x_1^2 - x_2^2| \leq \lambda |x_1 - x_2| + C$. Letting $x_1 = 1$, the inequality becomes $x_2^2 - 1 \leq \lambda(x_2 - 1) + C$. But it is clear that for any values of $\lambda$ and $C$, we can always find $x_2$ large enough that the inequality is false.

On the other hand, to see that $f$ is a coarse map, let the bounding function $\ell(r) = 3r^2$ and let $x_1, x_2 \in X$ (supposing without loss of generality that $x_1 \leq x_2$. We need to show that

$$x_2^2 - x_1^2 \leq 3(x_2 - x_1)^2.$$ 

If $x_1 = x_2$ this is trivial, since both sides of the inequality equal 0. Otherwise, notice that $2x_1 \leq x_2$, since the elements of $X$ are powers of 2. Therefore

$$x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1) \leq (x_2 - x_1)(x_2 + x_1 + 2x_2 - 4x_1) = (x_2 - x_1)(3x_2 - 3x_1) = 3(x_2 - x_1)^2.$$ 

One easily checks that $f$ is a proper map, and that the coarse inverse of $f$ is $f^{-1}$, also a coarse map with bounding function $\ell(r) = r$; therefore $f$ is a coarse equivalence. \qed

2.2 Geometric Group Theory

For what follows, let $G$ be a finitely generated group equipped with finite generating set $S$. In all cases we will suppose that $S$ is symmetric; that is, for any $s \in S$, $s^{-1} \in S$ as well.

Recall that a left action of $G$ on a set $X$ is an assignment of a bijection $f_g : X \to X$ for each group element $g \in G$ (with $f_g(x)$ denoted $gx$), such that the following are satisfied:

1. $f_{id}(x) = idx = x$ for all $x \in X$, and

2. for all $g, h \in G$, $f_g \circ f_h = f_{gh}$. 

In particular, any group $G$ acts on itself by left multiplication. If $X$ is a metric space and each $f_g : X \to X$ is an isometry, then we say that $G$ acts by isometries on $X$.

The orbit of an element $x \in X$, denoted $Gx$, is the set $\{gx \mid g \in G\}$. The distinct orbits form a partition of $X$ and the set of all orbits $\{Gx \mid x \in X\}$ is the quotient of the action, denoted $X/G$. 

5
The action of $G$ on $X$ is called \textit{transitive} if there is only one orbit; i.e., for all $x_1, x_2 \in X$, there exists $g \in G$ such that $gx_1 = x_2$. If $X$ is a metric space (or more generally a topological space) and $X/G$ is compact, then $G$ is said to act \textit{cocompactly} on $X$.

The group $G$ can be regarded as a metric space by the following construction: For any $g \in G$, the \textit{length} of $g$, $\text{len}(g)$, is the smallest $n$ such that $g = s_1 \ldots s_n$ where $s_1, \ldots, s_n \in S$. Then the word-length metric on $G$ is given by $d_S(g, h) = \text{len}(g^{-1}h)$ for any $g, h \in G$. This metric makes $G$ a proper metric space with a left action by isometries by $G$ via left multiplication. (Notice that although $G$ also has a right action on itself given by right multiplication, the action is not by isometries unless $G$ is abelian.)

Although the word-length metric on $G$ is dependent on the choice of generating set, the metrics induced by two different generating sets are quasi-isometric and therefore coarsely equivalent.

\textbf{Remark 9.} The above construction can be used to induce a metric on any group, even one that is infinitely generated. However, the metric fails to be proper in such cases since there exist bounded subsets which are not finite.

\textbf{Examples.} Consider the infinite cyclic group $\mathbb{Z}$ with generating set $S = \{1, -1\}$. Then the metric induced by $S$ is precisely what we would expect it to be: $d_S(m, n) = |m - n|$ for all $m, n \in \mathbb{Z}$. On the other hand, taking the generating set $S' = \{2, 3, -2, -3\}$, we obtain a different metric:

$$d_{S'}(m, n) = \begin{cases} 2 & \text{if } |m - n| = 1 \\ \lceil |m - n| / 3 \rceil & \text{otherwise.} \end{cases}$$

The word-length metric on $G$ always produces a topologically discrete metric space since distinct elements are separated by integer distances. Frequently it is desirable to have a geodesic metric space to work with instead; thus we define the Cayley graph of a group.

\textbf{Definition 10.} Let $G$ be a group with finite, symmetric generating set $S$. The \textit{Cayley graph of $G$ with respect to $S$}, denoted $\Gamma(G, S)$, is a graph with

1. one vertex $v_g$ for each $g \in G$,
2. an edge between $v_g$ and $v_h$ if and only if $gs = h$ for some $s \in S$. 

6
Notice that condition (2) appears to describe a directed edge from a vertex \( v_g \) to a vertex \( v_h \). However, since the generating set \( S \) is symmetric, whenever \( gs = h \) for some \( s \in S \), we can also say that \( hs^{-1} = g \) where \( s^{-1} \in S \). Thus we need not be concerned with the directions of the edges and we will regard \( \Gamma(G, S) \) as an undirected graph. In the examples below, edges appear as directed edges only for the sake of clarifying the associated generator.

The Cayley graph of a group is given the path metric, with each edge having length 1, making it a path-connected, geodesic metric space. The map from \((G, d_S)\) to \(\Gamma(G, S)\) given by \(g \mapsto v_g\) is a quasi-isometry, with \(\lambda = 1, C = \frac{1}{2}\).

**Example 11.** Let \( G = \mathbb{Z} \) with generating set \( S = \{1, -1\} \). Then the Cayley graph \( \Gamma(\mathbb{Z}, S) \) is isometric to the real line \( \mathbb{R} \) as in Figure 1 below.

\[
\begin{array}{cccccccc}
\cdots & -2 & 1 & -1 & 0 & 1 & 1 & 2 & 1 & 3 & 1 & 4 & 1 & \cdots
\end{array}
\]

Figure 1: \( \Gamma(\mathbb{Z}, \{1, -1\}) \)

**Example 12.** Choosing a different generating set \( S' = \{2, 3, -2, -3\} \) for \( \mathbb{Z} \) results in the Cayley graph in Figure 2 shown below.

\[
\begin{array}{cccccccc}
\cdots & -2 & -1 & 0 & 1 & 2 & 2 & 3 & 2 & 4 & 2 \cdots
\end{array}
\]

Figure 2: \( \Gamma(\mathbb{Z}, \{2, 3, -2, -3\}) \)

Notice that although the graphs in 1 and 2 are clearly non-isomorphic, they both appear to have a similarly “linear” structure if viewed from a great distance. This reflects the fact that the two graphs are quasi-isometric to one another (and to the two metrics on \( \mathbb{Z} \) induced by their respective generating sets).

**Example 13.** Let \( G = \mathbb{Z} \times \mathbb{Z} \), with the two generators \( a = (1, 0) \) and \( b = (0, 1) \) and their inverses. The Cayley graph is a grid as seen in Figure 3, which is quasi-isometric to \( \mathbb{R}^2 \).
Definition 14. A tree is an acyclic graph (either finite or infinite).

Example 15. Let $G = \mathbb{Z} \ast \mathbb{Z}$ be the free group on two generators $a$ and $b$. The Cayley graph of $G$ is the infinite tree shown in Figure 4. In general, the Cayley graph of the free group on $n$ generators is the infinite tree with each vertex having degree $2n$.

The following well-known basic fact due to Shvarts and Milnor is known as “Milnor’s lemma”.

Theorem 16. Suppose $X$ is a path metric space and $G$ is a group acting properly and cocompactly by isometries on $X$. Then $G$ is coarsely equivalent to $X$.

The coarse equivalence is given by the map $g \mapsto gx_0$ for any point $x_0$ of $X$.

For example, if $K$ is a finite complex with the fundamental group $G = \pi_1(K)$, the inclusion of any orbit of $G$ in the universal cover of $K$ is a coarse equivalence for any choice of the generating set of $G$. 
Figure 4: $\Gamma(\mathbb{Z} * \mathbb{Z}, \{a, b, a^{-1}, b^{-1}\})$

Since the action of $G$ on itself by left multiplication extends naturally to a proper and cocompact action by isometries on $\Gamma(G)$, we have the following corollary:

**Corollary 17.** A finitely generated group $G$ is coarsely equivalent to its Cayley graph $\Gamma(G)$.

A *coarse quasi-action* is designed to describe situations where elements of a group act on a metric space via a coarse equivalences. We should point out that it is quite a bit weaker than the categorical notion of a group action in the coarse category.

**Definition 18.** A *coarse quasi-action* of a group $G$ on a metric space $X$ is an assignment of a coarse self-equivalence $f_g: X \to X$ for each element $g \in G$ so that the following conditions are satisfied:

1. all $f_g$ are coarse maps with respect to a uniform choice of the function $\ell$,

2. there is a number $A \geq 0$ such that $d(f_{id}, id_X) \leq A$ in the sup norm,
3. there is a number $B \geq 0$ such that $d(f_g \circ f_h, f_{gh}) \leq B$ in the sup norm for all elements $g$ and $h$ in $G$.

As a particular consequence of (2) and (3), all compositions $f_g \circ f_{g^{-1}}$ are $(A + B)$-close to the identity.

This notion is a generalization of the notion of quasi-action central to the fundamental problem of quasi-isometry classification of finitely generated groups; cf. [11, 19, 21]. Quasi-isometries are coarse equivalences with linear control functions. Quasi-actions appear most naturally in a partial converse to Milnor’s Lemma. This is a curious term because it is a “kind of action” by “quasi”-isometries, so both meanings of “quasi” get (inconveniently?) conflated.

To see a simple example of a quasi-action, consider a metric subset $C$ embedded in $X$. Suppose a group $G$ acts by isometries on $C$. If $C$ is commensurable with $X$, we can extend the action of $G$ on $C$ to a quasi-action on $X$ by the following device. If $B \geq 0$ is a commensurability constant, let $\phi : X \to C$ be any function bounded by $B$. Then we define $f_g(x)$ to be the composition $g(\phi(x))$. All resulting maps are coarse equivalences with $\ell(x) = x + 2B$ and the compositions $f_g \circ f_{g^{-1}}$ all $2B$-close to the identity.

This example illustrates a generalization of a well-known construction from geometric group theory: if a metric space is coarsely equivalent to a finitely generated group $G$ with a word metric then the left multiplication action in $G$ can be quisiconjugated to give a coarse quasi-action on $X$.

Another source of quasi-actions of interesting groups on well-understood geometries is a number of recent constructions of actions of groups on quasi-spaces, that is spaces that are quasi-isometric to familiar geometries. For example, Bestvina, Bromberg, and Fujiwara [5] construct actions of the mapping class group and the outer automorphism group of a free group of rank $> 1$ on quasi-trees, which can be translated as quasi-actions on trees.

The general coarse quasi-actions are gaining prominence in applications to algebraic $K$-theory. In that subject, coarse maps are precisely the natural maps between metric spaces that induce maps of bounded $K$-theory spectra $K(X, R)$ built from free $R$-modules parametrized over the metric space $X$. So $K(X, R)$ is a functor on the coarse category of metric spaces and coarse maps. On the other hand, in section 2.2 of [7] an action by (not necessarily bounded) isometries needed to be converted to an action by bounded coarse maps via a change of metric in $X$. For example,
this general construction converts the left multiplication action on any group with a word metric to a bounded coarse quasi-action. The resulting quasi-action is no longer by isometries unless the group is abelian.

2.3 Asymptotic Dimension and Asymptotic Property C

We give a review of some coarse geometric finiteness conditions mostly inspired by the covering dimension in topology.

**Definition 19.** Let $\mathcal{F}$ be a collection of subsets of a metric space $X$. We say that $\mathcal{F}$ is uniformly bounded if there exists $r > 0$ such that for every $F \in \mathcal{F}$, $\text{diam}(F) < r$.

**Definition 20.** Given a number $r > 0$, say that $\mathcal{F}$ is $r$-disjoint if $d(F_1, F_2) > r$ for any $F_1, F_2 \in \mathcal{F}$.

The asymptotic dimension of a metric space $X$ was first defined by Gromov as a large-scale analogue of the covering dimension.

**Definition 21.** Let $X$ be a metric space and $n \in \mathbb{N}$. We say that $X$ has asymptotic dimension at most $n$ (or $\text{asdim}(X) \leq n$) if for any $r > 0$ there exist $n + 1$ uniformly bounded families $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n$ of subsets of $X$ such that

1. each family $\mathcal{F}_i$ is $r$-disjoint,

2. and $\bigcup_{i=0}^{n} \mathcal{F}_i$ covers $X$.

We say that $X$ has finite asymptotic dimension (FAD) if there exists $n$ such that $\text{asdim}(X) \leq n$.

**Example 22.** Any bounded metric space $X$ has asymptotic dimension 0, since we may take $\mathcal{F}_0 = \{X\}$, which is clearly $r$-disjoint for any choice of $r$. The converse is not true; a space with asymptotic dimension 0 need not be bounded. For a counterexample take $X = \{2^n \mid n = 0, 1, 2, \ldots\}$. For $r > 0$, choose $k$ so that $2^k > r$. Let $F = \{2^n \mid 0 \leq n \leq k\}$, and let $\mathcal{F}_0 = \{F, \{2^{k+1}\}, \{2^{k+2}\}, \ldots\}$. Then $\mathcal{F}_0$ is clearly uniformly bounded, $r$-disjoint, and its union covers $X$ as desired.

**Proposition 23.** $\text{asdim}(\mathbb{R}) = 1$. 

11
Proof. To see that asdim(\(\mathbb{R}\) \(\leq\) 1, fix \(r > 0\). We will subdivide \(\mathbb{R}\) into two families of subsets as follows. For each \(i \in \mathbb{Z}\), let \(F_i = [10ri, 10r(i+1)]\). Let \(\mathcal{F}_0 = \{F_i \mid i = 2k\}\) and \(\mathcal{F}_1 = \{F_i \mid i = 2k+1\}\). The conditions are easily checked.

On the other hand, since \(\mathbb{R}\) is connected and unbounded, it is not possible to subdivide \(\mathbb{R}\) into a single family of bounded, disjoint subsets; therefore asdim(\(\mathbb{R}\) > 0).

\[\square\]

**Proposition 24.** asdim(\(\mathbb{R}^2\)) \(\leq\) 3.

Proof. Let \(r > 0\). We can subdivide \(\mathbb{R}^2\) into four families of subsets by subdividing each copy of \(\mathbb{R}\) into two families as in the previous example, then taking products of the individual subsets. More precisely, for each \(i \in \mathbb{Z}\), let \(F_i = [10ri, 10r(i+1)] \subset \mathbb{R}\) as above. Now for each \(a, b \in \{0, 1\}\), let \(\mathcal{F}_{a,b} = \{F_i \times F_j \mid i = 2k + a, j = 2l + b, k, l \in \mathbb{Z}\}\). (In Figure 5a below, the families \(\mathcal{F}_{0,0}, \mathcal{F}_{1,0}, \mathcal{F}_{0,1}, \mathcal{F}_{1,1}\) are colored green, blue, yellow, and red respectively.)

Clearly each family is uniformly bounded by \(10\sqrt{2}r\), each family is \(10r\)-disjoint, and the families together cover \(\mathbb{R}^2\).

\[\square\]

By a more clever choice of subdivisions (Figure 5b), we can show that asdim(\(\mathbb{R}^2\)) \(\leq\) 2. To see that asdim(\(\mathbb{R}^2\)) \(\geq\) 2, consider that if we remove any uniformly bounded, \(r\)-disjoint family of subsets from \(\mathbb{R}^2\), what remains must still be connected; thus it cannot be subdivided into a second bounded, disjoint family of subsets. This shows that asdim(\(\mathbb{R}^2\)) = 2.

The above method can easily be adapted to show that asdim(\(\mathbb{R}^n\)) \(\leq\) \(2^n - 1\), or indeed that if asdim(\(X\)) \(\leq\) \(n\) and asdim(\(Y\)) \(\leq\) \(m\), then asdim(\(X \times Y\)) \(\leq\) \((n + 1)(m + 1) - 1\). However, a tighter bound on asymptotic dimension of products is given by Bell and Dranishnikov in [2]:

**Theorem 25.** If asdim(\(X\)) \(\leq\) \(n\) and asdim(\(Y\)) \(\leq\) \(m\), then asdim(\(X \times Y\)) \(\leq\) \(n + m\).

**Corollary 26.** asdim(\(\mathbb{R}^n\)) = \(n\).

**Example 27.** Every infinite tree has asymptotic dimension 1. As a consequence of the fact that asymptotic dimension is an invariant under quasi-isometries, this shows that every finitely-generated free group has asymptotic dimension 1.

A remarkable theorem of Dranishnikov and Zarichnyi [9] expands on the relationship between trees and asymptotic dimension.
Theorem 28. Any proper metric space with asymptotic dimension \( \leq n \) can be coarsely embedded into a product of \( n + 1 \) trees.

Asymptotic property C was defined by Dranishnikov [8] by analogy to Haver’s topological Property C, and is a weakening of FAD.

Definition 29 (APC). A metric space \( X \) has asymptotic property C if for every sequence of positive numbers \( 0 < r_1 \leq r_2 \leq \ldots \) there exists a natural number \( n \) and uniformly bounded families \( F_i, 1 \leq i \leq n \), such that

1. each family \( F_i \) is \( r_i \)-disjoint, and

2. the union of all \( n \) families is a covering of \( X \).

Theorem 30. If \( \text{asdim}(X) < \infty \) then \( X \) has asymptotic property C.

Proof. The proof follows easily from the definitions of FAD and APC.

Several interesting examples of groups that have asymptotic property C but infinite asymptotic dimension are given in [23]:

![Asymptotic dimension of \( \mathbb{R}^2 \)](image)
Example 31. Let $G$ be the countably infinite direct sum $\bigoplus_{i=1}^{\infty} \mathbb{Z}$, with metric given by

$$d(g, h) = \sum_{n=1}^{\infty} n|g_n - h_n|$$

for all $g = (g_1, g_2, \ldots), h = (h_1, h_2, \ldots) \in G$. Then $G$ has asymptotic property C.

Example 32. Recall that the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$ is constructed as the semidirect product $(\bigoplus_{-\infty}^{\infty} \mathbb{Z}) \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on $\bigoplus_{-\infty}^{\infty} \mathbb{Z}$ by translation. In other words, elements of $\mathbb{Z} \wr \mathbb{Z}$ are of the form $(\sum_{x \in \mathbb{Z}} n_x x, k)$, where $n_x, k \in \mathbb{Z}$ and $\sum_{x \in \mathbb{Z}} n_x x$ has finitely many nonzero entries. The group operation is given by $(\sum_{x \in \mathbb{Z}} n_x x, k) \cdot (\sum_{x \in \mathbb{Z}} m_x x, l) = (\sum_{x \in \mathbb{Z}} (n_x x + m_x - k), k + l)$. (One may visualize an element of $\mathbb{Z} \wr \mathbb{Z}$ as an infinite tape with integer entries indexed by $\mathbb{Z}$, together with a pointer $k$ which specifies a particular index on the tape. “Adding” two such elements requires a three-step process: shifting the second tape by $k$ units so that the 0 index on the second tape is aligned with the $k^{th}$ index on the first tape, adding the entries on the two tapes coordinate-wise, and finally updating the location of the pointer.)

It is known that $\mathbb{Z} \wr \mathbb{Z}$ is finitely generated with two generators, $(0, 1)$ and $(\delta_0, 0)$, where $\delta_0 = \sum_{x \in \mathbb{Z}} n_x x$ with $n_x = 0$ if $x \neq 0$, $n_0 = 1$. Using the metric induced by this generating set, $\mathbb{Z} \wr \mathbb{Z}$ has asymptotic property C.

It is interesting to note that in both of the preceding examples, the groups are known to have infinite asymptotic dimension because they contain subspaces of asymptotic dimension $n$ for every $n \in \mathbb{N}$.

2.4 Finite Decomposition Complexity

Finite decomposition complexity and several of its variants were first introduced in [17]. These finiteness conditions are defined for families of metric spaces $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$.

Recall that a family of metric spaces is called uniformly bounded if there is a uniform bound on the diameters of the spaces in the family. We denote the collection of all uniformly bounded families by $\mathfrak{B}$. A subfamily $\mathcal{S}$ of a metric family $\mathcal{X}$ is a family consisting of metric subspaces of the spaces within $\mathcal{X}$; that is, for every $S \in \mathcal{S}$, there exists $X \in \mathcal{X}$ such that $S \subset X$.

A map between metric families $\mathcal{F}: \mathcal{X} \to \mathcal{Y}$ is a collection of maps $f: X_f \to Y_f$ such that each $X_f \in \mathcal{X}$, each $Y_f \in \mathcal{Y}$, and every $X \in \mathcal{X}$ is the domain of at least one $f \in \mathcal{F}$.
If \( B \) is a subfamily of \( \mathcal{Y} \), the preimage \( \mathcal{F}^{-1}(B) \) is the set

\[
\{ f^{-1}(B) \mid f \in \mathcal{F}, B \subset X, \text{ where } X \text{ is the domain of } f \}.
\]

**Definition 33.** A map \( \mathcal{F}: \mathcal{X} \to \mathcal{Y} \) between metric families is called *uniformly expansive* if there is a non-decreasing function \( \ell: [0, \infty) \to [0, \infty) \) such that for every \( f: X \to Y \) in \( \mathcal{F} \),

\[
d_Y(f(x_1), f(x_2)) \leq \ell(d_X(x_1, x_2)) \text{ for all } x_1, x_2 \in X.
\]

Observe that for a map of metric families to be uniformly expansive is a stronger condition than simply requiring each \( f \in \mathcal{F} \) (considered as an individual function) to be uniformly expansive. Rather, \( \mathcal{F} \) is uniformly expansive if and only if each \( f \in \mathcal{F} \) is uniformly expansive *with respect to the same bounding function* \( \ell \).

**Definition 34.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two families of metric spaces. Let \( R > 0 \). The family \( \mathcal{X} \) is called *\( R \)-decomposable over \( \mathcal{Y} \)* if for any space \( X \) in \( \mathcal{X} \) there are collections of subsets \( \{ U_{1, \alpha} \}_{\alpha \in A} \), \( \{ U_{2, \beta} \}_{\beta \in B} \) such that

1. \( \{ U_{1, \alpha} \}_{\alpha \in A} \cup \{ U_{2, \beta} \}_{\beta \in B} \) is a cover of \( X \),
2. each \( U_{i, \gamma} \) is a member of the family \( \mathcal{Y} \), and
3. each of the collections \( \{ U_{1, \alpha} \} \) and \( \{ U_{2, \beta} \} \) is \( R \)-disjoint.

The definition of finite decomposition complexity is in terms of a winning strategy for the following game between two players. The families of metric spaces that appear in the decompositions are families of metric subspaces of the elements of \( \mathcal{X} \). In round number 1 the first player selects a number \( R_1 > 0 \); then the second player has to select a family of metric spaces \( \mathcal{Y}_1 \) and an \( R_1 \)-decomposition of \( \mathcal{X} \) over \( \mathcal{Y}_1 \). In each succeeding round number \( i \) the first player selects a number \( R_i > 0 \), the second player has to select a family of metric spaces \( \mathcal{Y}_i \) and an \( R_i \)-decomposition of \( \mathcal{Y}_{i-1} \) over \( \mathcal{Y}_i \). The second player wins the game if for some finite value \( k \) of \( i \) the family \( \mathcal{Y}_k \) is bounded.

**Definition 35 (FDC).** A metric family \( \mathcal{X} \) has *finite decomposition complexity* if the second player possesses a winning strategy in every game played over \( \mathcal{X} \). A single metric space \( X \) has finite decomposition complexity if \( \{ X \} \) does.
An equivalent definition of FDC is also given in [18].

**Definition 36.** Let $\mathcal{C}$ be a collection of metric families. A family $\mathcal{X}$ is *decomposable over* $\mathcal{C}$ if for every $R > 0$ there exists a family $\mathcal{Y}$ in $\mathcal{C}$ such that $\mathcal{X}$ is $R$-decomposable over $\mathcal{Y}$. A collection $\mathcal{C}$ is said to be *stable under decomposition* if any family $\mathcal{X}$ which is decomposable over $\mathcal{C}$ is contained in $\mathcal{C}$. Guentner et. al. [18] show that there exists a smallest collection $\mathcal{D}$ which contains all bounded metric families and which is stable under decomposition; the families in $\mathcal{D}$ are said to have *finite decomposition complexity*.

**Example 37.** Any space $X$ with finite asymptotic dimension has finite decomposition complexity.

The proof of this theorem is obvious in the case that $\text{asdim}(X) \leq 1$, but the proof of the general case (in [18]) is less obvious than it may seem at first glance as it relies on Theorem 28, that any space with FAD can be coarsely embedded in a product of trees.

**Example 38.** ([18]) The countably infinite direct sum $\bigoplus_{i=1}^{\infty} \mathbb{Z}$, with the metric given in Example 31, has FDC.

Although FAD implies both APC and FDC, the relationship between APC and FDC remains unknown. In the interest of reconciling the two properties, the following property weaker than both APC and FDC was defined by Dranishnikov and Zarichnyi in [10].

**Definition 39** *(sFDC).* A metric family $\mathcal{X}$ has *straight finite decomposition complexity* if, for any sequence $R_1 \leq R_2 \leq \ldots$ of positive numbers, there exists a finite sequence of metric families $\mathcal{V}_1$, $\mathcal{V}_2$, $\ldots$, $\mathcal{V}_n$ such that

1. $\mathcal{X}$ is $R_1$-decomposable over $\mathcal{V}_1$,

2. $\mathcal{V}_{i-1}$ is $R_i$-decomposable over $\mathcal{V}_i$ for all $i > 1$, and

3. the family $\mathcal{V}_n$ is bounded.

A single metric space $X$ has straight finite decomposition complexity if $\{X\}$ does.

There are a number of variations on FDC and sFDC in the literature. Guentner et. al. [18] define weak finite decomposition complexity by relaxing the definition of $R$-decomposability to
allow for a decomposition of a space into arbitrarily (but finitely) many \( R \)-disjoint families at each stage instead of only two.

**Definition 40.** Let \( \mathcal{X}, \mathcal{Y} \) be two families of metric spaces. Let \( k \in \mathbb{N} \) and \( R > 0 \). We say \( \mathcal{X} \) is \((k, R)\)-decomposable over \( \mathcal{Y} \) if for any space \( X \) in \( \mathcal{X} \) there are collections of subsets \( U_1, \ldots, U_k \) such that:

1. \( U_1 \cup \cdots \cup U_k \) is a cover of \( X \),
2. each \( U_{i,\alpha} \in U_i \) is a member of the family \( \mathcal{Y} \), and
3. each of the collections \( U_i \) is \( R \)-disjoint.

**Definition 41.** Let \( \mathcal{C} \) be a collection of metric families. For a fixed \( k \in \mathbb{N} \), we say \( \mathcal{X} \) is \( k \)-decomposable over \( \mathcal{C} \) if for any \( R > 0 \), there exists a family \( \mathcal{Y} \) in \( \mathcal{C} \) such that \( \mathcal{X} \) is \((k, R)\)-decomposable over \( \mathcal{Y} \).

We say that \( \mathcal{X} \) is weakly decomposable over \( \mathcal{C} \) if \( \mathcal{X} \) is \( k \)-decomposable over \( \mathcal{C} \) for some \( k \in \mathbb{N} \).

**Definition 42.** A collection of metric families \( \mathcal{C} \) is said to be stable under \( k \)-fold decomposition if any space that is \( k \)-decomposable over \( \mathcal{C} \) is contained in \( \mathcal{C} \). Let \( \mathcal{D}^k \) denote the smallest collection of metric families which is stable under \( k \)-fold decomposition and contains \( \mathcal{B} \). The elements of \( \mathcal{D}^k \) are said to have \( k \)-fold finite decomposition complexity (\( k \)-FDC).

Similarly, a collection \( \mathcal{C} \) is said to be stable under weak decomposition if any space that is weakly decomposable over \( \mathcal{C} \) is contained in \( \mathcal{C} \). Let \( wD \) denote the smallest collection of metric families which is stable under weak decomposition and contains \( \mathcal{B} \). The elements of \( wD \) are said to have weak finite decomposition complexity (wFDC).

Notice that with \( k = 2 \), we recover the original definition of \( \mathcal{X} \) decomposable over \( \mathcal{C} \). Thus FDC implies both \( k \)-FDC and wFDC.

Also note that a space \( X \) has asdim(\( X \)) \( \leq n \) if and only if \( \{X\} \) is \((n + 1)\)-decomposable over \( \mathcal{B} \), the collection of all bounded metric families. In fact, we can say that a family \( \mathcal{X} \) has finite asymptotic dimension uniformly if there exists some \( k \) such that \( \mathcal{X} \) is \( k \)-decomposable over \( \mathcal{B} \).

**Theorem 43.** [22] Weak finite decomposition complexity implies straight finite decomposition complexity.
Ramras and Ramsey [22] similarly define a weakening of sFDC called weak straight finite decomposition complexity, as follows.

**Definition 44.** A family $\mathcal{X}$ has *weak straight finite decomposition complexity* if there exists a sequence of natural numbers $(k_1, k_2, \ldots)$ such that for any sequence $0 < R_1 < R_2 < \ldots$, we can find $n \in \mathbb{N}$ and families $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_n$ such that $\mathcal{X} = \mathcal{X}_0$ and for each $1 \leq i \leq n$, $\mathcal{X}_{i-1}$ is $(k_i, R_i)$-decomposable over $\mathcal{X}_i$.

Ramras and Ramsey [22] show that weak straight finite decomposition complexity is in fact equivalent to sFDC.

A stronger variant of FDC is regular finite decomposition complexity (rFDC), defined by Kasprowski et. al. in [20].

**Definition 45.** Let $\mathcal{X}$ be a metric family and $\mathcal{C}$ a collection of metric families. We say that $\mathcal{X}$ *regularly decomposes* over $\mathcal{C}$ if there exists a metric family $\mathcal{Y}$ and a map $F: \mathcal{X} \to \mathcal{Y}$ such that

1. $\mathcal{Y}$ has finite asymptotic dimension uniformly,

2. $F$ is uniformly expansive, and

3. For every uniformly bounded subfamily $\mathcal{B}$ of $\mathcal{Y}$, the preimage $F^{-1}(\mathcal{B})$ is contained in $\mathcal{C}$.

**Definition 46** (rFDC). [20] A collection of metric families $\mathcal{C}$ is said to be *stable under regular decomposition* if any space that is regularly decomposable over $\mathcal{C}$ is contained in $\mathcal{C}$. Let $\mathcal{R}$ denote the smallest collection of metric families which is stable under regular decomposition and contains $\mathcal{B}$. The elements of $\mathcal{R}$ are said to have *regular finite decomposition complexity* (rFDC).

**Theorem 47.** [20] A family $\mathcal{X}$ has finite asymptotic dimension uniformly if and only if $\mathcal{X}$ regularly decomposes over $\mathcal{B}$; thus any space that has FAD also has rFDC.

**Theorem 48.** [20] Regular finite decomposition complexity implies finite decomposition complexity.

### 2.5 Property A

Introduced by Yu in [24], property A is weaker than each of the properties introduced in the previous two sections. A theorem of particular interest to geometers is that any discrete space with property A admits a coarse embedding into a Hilbert space.
Definition 49. Let $X$ be a discrete metric space. We say that $X$ has property A if for all $r > 0$, $\epsilon > 0$, there exist finite sets $A_x \subset X \times \mathbb{N}$, one for each $x \in X$, satisfying the following:

1. $(x, 1) \in A_x$ for each $x \in X$,

2. $\frac{\#(A_x \triangle A_y)}{\#(A_x \cap A_y)} < \epsilon$, for all $x, y \in X$ such that $d(x, y) \leq r$, and

3. There exists $R > 0$ such that whenever $(y, m), (z, n) \in A_x$, $d(y, z) \leq R$.

Note that condition (3) may be rephrased as saying that there is a uniform bound $R$ on the diameters of all $A'_x$, where $A'_x = \{ y \mid (y, n) \in A_x \text{ for some } n \in \mathbb{N} \}$.

Intuitively, one may think of property A as the statement that for any choice of $r$ and $\epsilon$, we can assign to each $x \in X$ a finite subset $A'_x$, together with a coloring of the elements of $A'_x$ (the values of $n$; each element may have more than one color). We can make the sets $A'_x$ large enough that if any two $x, y \in X$ are $r$-close together, the symmetric difference of the sets $A_x, A_y$ is sufficiently “small” relative to the size of their intersection, while still ensuring a uniform bound on the diameters.

Example 50. $\mathbb{Z}^2$ has property A.

Proof. Fix $r > 0$, $\epsilon > 0$. It is always possible to choose $k \in \mathbb{Z}$ such that $4rk < \epsilon(k - r)^2$. Now for each $(x, y) \in \mathbb{Z}^2$, let $A'_{(x,y)}$ be the rectangle $\{x, x+1, \ldots, x+k\} \times \{y, y+1, \ldots, y+k\}$, and let $A_{(x,y)} = A'_{(x,y)} \times \{1\}$.

Conditions (1) and (3) of the definition of property A are clearly satisfied. To verify (2), let $(x_0, y_0), (x_1, y_1) \in \mathbb{Z}^2$, with $d((x_0, y_0), (x_1, y_1)) \leq r$. Notice that $|x_1 - x_0| \leq r$ and $|y_1 - y_0| \leq r$. Via Figure 6, one may easily check that

$$\#(A_x \triangle A_y) \leq 2(|x_1 - x_0 | + |y_1 - y_0 |) \leq 4rk$$

and

$$\#(A_x \cap A_y) \geq (k - |x_1 - x_0 |)(k - |y_1 - y_0 |) \geq (k - r)^2.$$ 

Therefore we have the inequality

$$\#(A_x \triangle A_y) \leq 4rk < \epsilon(k - r)^2 \leq \epsilon\#(A_x \cap A_y).$$

Condition (2) follows. $\square$
Remark 51. The preceding proof can easily be adapted to show that $\mathbb{Z}^n$ has property A for any $n \in \mathbb{N}$. Notice that for these groups it is not even necessary to “color” the elements of each $A_x$ by pairing them with natural numbers.

There are very few groups known to fail property A; it is by far the weakest of the properties we have discussed here. Figure 7 shows the relationships between finite asymptotic dimension, asymptotic property C, the variants of finite decomposition complexity, and property A.

2.6 Permanence theorems

We discuss several permanence characteristics of asymptotic dimension, asymptotic property C, finite decomposition complexity, and straight finite decomposition complexity. These are of particular use in the search for additional examples of groups and spaces with these properties. For this section, assume that $X$ and $Y$ are proper metric spaces and $P$ is any of the properties: $\text{asdim} \leq n$, APC, FDC, or sFDC. Proof of Theorems 52, 58, 59, and 62 may be found in [18] in the case of FDC, and in [10] in the case of sFDC.

Theorem 52 (Coarse Invariance). If $f : X \to Y$ is a coarse embedding and $Y$ has property $P$, then $X$ has $P$. 
Corollary 53. If $X$ and $Y$ are coarsely equivalent, then $X$ has $P$ if and only if $Y$ has $P$.

In particular, note that if $X$ and $Y$ are coarsely equivalent and have finite asymptotic dimension, then asdim($X$) = asdim($Y$).

Corollary 54. For any $n \in \mathbb{N}$, asdim($\mathbb{Z}^n$) = asdim($\mathbb{R}^n$) = $n$.

Since the metrics induced by different finite generating sets of a group $G$ are quasi-isometric (therefore coarsely equivalent), we can treat asymptotic dimension, etc., as properties of $G$ irrespective of the choice of generating set. Furthermore, since a finitely generated group $G$ is quasi-isometric to its Cayley graph $\Gamma(G)$, we have the following corollary:

Corollary 55. If $G$ is a finitely generated group, then $G$ has property $P$ if and only if $\Gamma(G)$ does.

Corollary 56. If $G$ is a finitely generated free group then asdim($G$) = 1.

The inclusion of a subspace is always a coarse embedding; therefore if $X \subset Y$ and $Y$ has property $P$, then $X$ has $P$. As a particular consequence, we have the following corollary:

Corollary 57. If a space $X$ has subsets with asymptotic dimension $\geq n$ for every $n$, then $X$ does not have finite asymptotic dimension.
The following theorems concern proving that a space has one of the given finiteness properties by breaking it down into subspaces with the desired property, then reassembling them in ways that preserve that property.

**Theorem 58** (Finite Union Theorem). Suppose $Z$ is a metric space such that $Z = X \cup Y$, where $X$ and $Y$ have FAD (resp. APC, FDC, sFDC). Then $Z$ has FAD (resp. APC, FDC, sFDC).

**Theorem 59** (Union Theorem). Suppose a metric space $X$ can be written as a union $X = \bigcup_{\alpha \in I} X_{\alpha}$, where

1. the family $\{X_{\alpha}\}_{\alpha \in I}$ has FDC (resp. sFDC), and

2. for any $r > 0$ there is a set $Y_r \subset X$ such that $Y_r$ has FDC (resp. sFDC) and the family $\{X_{\alpha} \setminus Y_r\}_{\alpha \in I}$ is $r$-disjoint.

Then $X$ has FAD (resp. FDC, sFDC).

Notice that it is insufficient to say that each of the sets $X_{\alpha}$ has finite asymptotic dimension (resp. FDC, sFDC); we require the uniformity provided by the statement that the family $\{X_{\alpha}\}_{\alpha \in I}$ has FAD (resp. FDC, sFDC).

We have been unable to find a version of Theorem 59 for APC in the literature; therefore, we offer the following:

**Definition 60.** Let $\mathcal{X} = \{X_{\alpha}\}_{\alpha \in I}$ be a family of metric spaces. We say that $\mathcal{X}$ has asymptotic property $C$ uniformly if for every sequence of positive numbers $0 < r_1 \leq r_2 \leq \ldots$ there exists a natural number $n$ and families $\mathcal{F}_{i,\alpha}$, for each $1 \leq i \leq n$, $\alpha \in I$, such that

1. each family $\mathcal{F}_{i,\alpha}$ is $r_i$-disjoint,

2. for each $\alpha \in I$, $\bigcup_{i=1}^{n} \mathcal{F}_{i,\alpha}$ is a covering of $X$, and

3. there exists $R > 0$ such that $\text{diam}(F) \leq R$ for all $F \in \mathcal{F}_{i,\alpha}$, $1 \leq i \leq n$, $\alpha \in I$.

**Theorem 61.** Suppose $X = \bigcup_{\alpha \in I} X_{\alpha}$, where $\{X_{\alpha}\}_{\alpha \in I}$ has APC uniformly, and for each $r > 0$ there is a set $Y_r \subset X$ such that $Y_r$ has APC and the family $\{X_{\alpha} \setminus Y_r\}_{\alpha \in I}$ is $r$-disjoint. Then $X$ has APC.
Proof. Let \(0 < r_1 \leq r_2 \leq \ldots\). Since \(\{X_\alpha\}_{\alpha \in I}\) has APC uniformly, there exists \(n\) and families \(C_{i,\alpha}\) of subsets of \(X_\alpha, 1 \leq i \leq n, \alpha \in I\), satisfying Definition ??, with \(R > 0\) a uniform bound on the diameters of the subsets. Now let \(r = r_n\) and select \(Y_{r_n} \subset X\) having APC such that \(\{X_\alpha \setminus Y_{r_n}\}_{\alpha \in I}\) is \(r_n\)-disjoint. For each \(1 \leq i \leq n, \alpha \in I\), define

\[F_i = \{F \setminus Y_{r_n} \mid F \in C_{i,\alpha} \text{ for some } \alpha \in I\}.\]

It is easy to see that each \(F_i\) is \(r_i\)-disjoint, since for any \(F_1 \in C_{i,\alpha}, F_2 \in C_{i,\beta}\), we have \(d(F_1, F_2) > r_i\) if \(\alpha = \beta\), and \(d(F_1 \setminus Y_{r_n}, F_2 \setminus Y_{r_n}) > r_n > r_i\) if \(\alpha \neq \beta\). Furthermore, \(R\) is still a uniform bound on the diameters of elements of all \(F_i\), and the families together cover \(X \setminus Y_{r_n}\). Now since \(Y_{r_n}\) has APC, using the sequence \(r_{n+1}, r_{n+2}, \ldots\) we can subdivide \(Y_{r_n}\) into families \(D_j, 1 \leq j \leq m\), such that each \(D_j\) is \(r_{n+j}\)-disjoint, the families \(D_j\) together cover \(Y_{r_n}\), and \(R'\) is a uniform bound on the diameters of the elements of each family. Renumbering, let \(F_{n+j} = D_j\) for each \(1 \leq j \leq m\). Then the families \(F_1, \ldots, F_n, F_{n+1}, \ldots, F_{n+m}\) together cover \(X\), have a uniform bound of \(\max(R, R')\), and each family \(F_k\) is \(r_k\)-disjoint, for all \(1 \leq k \leq n + m\). Thus \(X\) has APC, as desired.

\[\square\]

**Theorem 62 (Fibering Theorem).** Suppose a finitely generated group \(G\) acts by isometries on a space \(X\) with chosen basepoint \(x_0\). Suppose \(X\) has FAD and for every \(r > 0\), the set

\[W_r(x_0) = \{g \in G \mid d(g(x_0), x_0) \leq r\}\]

has \(\text{asdim} \leq n\) (resp. FDC, sFDC). Then \(X\) has FAD (resp. FDC, sFDC).
3 Extension theorems for FAD, APC, FDC, and sFDC

Throughout this section, $G$ is a finitely generated group with a coarse quasi-action on a metric space $X$.

Let $x_0$ be a chosen base point in $X$. An $R$-quasi-stabilizer $W_R(x_0)$ of $x_0$ is the subset of those elements $g$ in $G$ with the property $d(g(x_0), x_0) \leq R$.

We assume that all quasi-stabilizers of $x_0$ have FAD.

**Theorem 63.** If $X$ has FAD then $G$ has FAD.

**Proof.** The proof follows that of Theorem 2 of [1] with modifications to accommodate weaker assumptions on the action. The orbit $Gx_0$ is a subset of $X$ and so has FAD. This allows us to assume without loss of generality that the action on $X$ is transitive.

Let $\lambda = \max \{ d(s(x_0), x_0) \mid s \in S \}$, where $S$ is a finite generating set for $G$. There is a map $\pi: G \to X$ given by $\pi(g) = g(x_0)$. If the action of $G$ on $X$ is by isometries, this map is $\lambda$-Lipschitz.

In our case, $d(\pi(g), \pi(gs)) = d(g(x_0), gs(x_0)) \leq \ell(d(x_0, s(x_0))) \leq \ell(\lambda)$. So $\pi$ is $\ell(\lambda)$-Lipschitz.

Suppose we have $\text{asdim}(X) \leq k$. Given any $r > 0$, there are $\ell(\lambda)r$-disjoint, $T$-bounded families $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k$ which cover $X$. Let us consider an element $g$ and $x = g(x_0)$. Let $F$ be a $T$-bounded subset from one of the covering families $\mathcal{F}_i$ with $x \in F$. We know that $d(g^{-1}(x), x_0) \leq A + B$ (where $A$ and $B$ are the constants from Definition 18), so we get $g^{-1}(F) \subset g^{-1}(x[T]) \subset g^{-1}(x)[T] < x_0[A+B+T]$. Therefore, straight from the definition of $\pi$, $g^{-1}\pi^{-1}(F)$ is contained in $W_{A+2B+\ell(T)}(x_0) = \pi^{-1}(x_0[A+2B+\ell(T)])$. We will denote this particular quasi-stabilizer simply as $W$.

By our assumption, $\text{asdim}(W) \leq n$ for some $n \geq 0$, so there are $n+1$ families $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ which cover $W$, which are $r$-disjoint for the given $r$, and which are uniformly bounded by some $K \geq 0$.

For each $F \in \mathcal{F}_i$, choose an element $g_F \in \pi^{-1}(F)$. Left multiplication by any element in $G$ is an isometry, so the formula $g_F(g_F^{-1}\pi^{-1}(F) \cap A_j)$ gives families $\mathcal{A}_{F,j}$ which cover $\pi^{-1}(F)$, which are $r$-disjoint, and which are $K$-bounded. Now it is clear that the covering of $G$ by the $r$-disjoint $K$-bounded families

$$W_{i,j} = \{ g_F (g_F^{-1}\pi^{-1}(F) \cap A) \mid F \in \mathcal{F}_i, A \in \mathcal{A}_j \}$$
shows that $\text{asdim}(G) \leq (k + 1)(n + 1) - 1$. \hfill \square

**Remark 64.** The assumptions do not explicitly require that there is a uniform bound on the asymptotic dimension of each quasi-stabilizer. Instead, a geometric property of the action pointed out in the middle of the proof allows to isometrically embed all pullbacks $\pi^{-1}(F)$ of subsets $F$ with diameter bounded by $T$ in the $(A + B + \ell(T))$-quasi-stabilizer $W$ of $x_0$, which is treated as a common “chopping block”. This says in particular that all subsets $\pi^{-1}(F)$ of $G$ have asymptotic dimension bounded from above by the same number $n$.

**Remark 65.** We want to insert a technical comment here related to the last remark. Unlike the situation with the action of $G$ by isometries, it cannot be assumed that for every subset $F$ with diameter bounded by $T$ there is a number $K$ so that $F \subset g(x_0[K])$ for some group element $g$. We were only able to guarantee that $g^{-1}(F) \subset x_0[K]$ for $K = A + B + \ell(T)$.

**Theorem 66.** If for all quasi-stabilizers the asymptotic dimension is uniformly bounded by $n \geq 0$, and $X$ has APC, then $G$ has APC.

**Proof.** Some features of the proof of Theorem 63 should be borrowed without change. So we have the $\ell(\lambda)$-Lipschitz projection $\pi: G \to X$. This time the orbit $Gx_0$ inherits APC, so we can assume $\pi$ is onto.

Let $0 < r_0 < r_1 < r_2 < \ldots$ be a sequence of real numbers. This allows to generate a new sequence $0 < \ell(\lambda)r_{n+1} < \ell(\lambda)r_{2(n+1)} < \ell(\lambda)r_{3(n+1)} < \ldots$ Since $X$ has APC, we can choose finitely many uniformly $T$-bounded families $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_m$ which cover $X$ and where each $\mathcal{F}_i$ is $\ell(\lambda)r_{(i+1)(n+1)}$-disjoint for $0 \leq i \leq m$. Now the pulled-back families $\pi^{-1}(\mathcal{F}_i) = \{\pi^{-1}(F) \mid F \in \mathcal{F}_i\}$ cover $G$, and each family $\pi^{-1}(\mathcal{F}_i)$ is $r_{(i+1)(n+1)}$-disjoint, though in general their elements are not bounded as subsets of $G$.

Given a subset $F \in \mathcal{F}_i$ and an element $g_F$ such that $g_F(x_0) \in F$, we have seen that $g_F^{-1}(F) \subset g_F^{-1}(x)[\ell(T)] \subset x_0[A + B + \ell(T)]$. Therefore, $g_F^{-1}\pi^{-1}(F)$ is contained in the quasi-stabilizer $W = W_{A+2B+\ell(T)}(x_0)$. By the special assumption, $\text{asdim}(W) \leq n$, so there are $n + 1$ families $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ which cover $W$, which are $r_{(n+1)(n+1)}$-disjoint and which are uniformly bounded.
We cover $G$ by $(m+1)(n+1) - 1$ families of subsets $W_k$ as follows: for every $0 \leq i \leq m$, $0 \leq j \leq n$, let
\[ W_{i(n+1)+j} = \{g_F (g_F^{-1} \pi^{-1}(F) \cap A) \mid F \in F_i, A \in A_j\}. \]

Notice that

1. For each $k$, $0 \leq k \leq (m+1)(n+1) - 1$, there is exactly one pair $(i,j)$ such that $k = i(n+1)+j$.

2. $\bigcup_k W_k$ is a cover of $G$ because $\bigcup_i \pi^{-1}(F_i)$ is a cover of $G$.

3. Each $W_{i(n+1)+j}$ is uniformly bounded because its elements are subsets of isometric translations of elements of $A_j$, which are uniformly bounded.

4. Each $W_{i(n+1)+j}$ is $r_{i(n+1)+j}$-disjoint by the following argument. Let
\[ g_F (g_F^{-1} \pi^{-1}(F) \cap A) \neq g_{F'} (g_{F'}^{-1} \pi^{-1}(F') \cap A) \in W_{i(n+1)+j}. \]

If $F \neq F'$, then $d_G(\pi^{-1}(F), \pi^{-1}(F')) \geq r_{(i+1)(n+1)} \geq r_{i(n+1)+j}$ since the family $\pi^{-1}(F_i)$ is $r_{(i+1)(n+1)}$-disjoint. Otherwise, if $F = F'$ but $A \neq A'$, then $d_G(g_FA, g_FA') \geq r_{(m+1)(n+1)} \geq r_{i(n+1)+j}$ because the family $A_j$ is $r_{(m+1)(n+1)}$-disjoint.

Therefore $G$ has APC, as desired.

It is remarkable that the extension theorems for FDC and sFDC have fewer geometric demands and are easier than the case of APC. In the case of the action by isometries, a proof for FDC was given by Guentner, Tessera, and Yu [18], a proof for sFDC was given by Bell and Moran [3].

**Theorem 67.** If $X$ has FDC (respectively, sFDC) then $G$ has FDC (respectively, sFDC).

**Proof.** We start with FDC. Since $X$ has FDC, then there is always a winning strategy that allows the second player to win the game on $X$ with a bounded family $Y_k$ in a certain number of steps $k$.

Recall that $\pi: G \to X$ is $\ell(\lambda)$-Lipschitz. Every time player one calls out a number $R_i$ for $1 \leq i \leq k$, player two computes $R_i' = \max\{1, \ell(\lambda)\} R_i$ and uses this number as data for a winning strategy over $X$. At every step, player two returns the following family as the response to $R_i \geq 0$ called by player one:
\[ W_i = \{\pi^{-1}(F) \mid F \in Y_i\}, \quad 1 \leq i \leq k. \]
We have seen that this should give an $R_i$-decomposition of $W_{i-1}$ over $W_i$.

If the members of $Y_k$ are bounded by $T$, the same argument as before shows that for any subset $F \in Y_k$ and an element $g_F$ such that $g_F(x_0) \in F$, we have $g_F^{-1}(F) \subset x_0[A+B+\ell(T)]$. So $g_F^{-1} \pi^{-1}(F)$ is contained in the quasi-stabilizer $W = W_{A+2B+\ell(T)}(x_0)$, and the restriction of $g_F^{-1}$ to $\pi^{-1}(F)$ is an isometry.

We assume the first player goes on producing numbers $R_{k+1}$, $R_{k+2}$, etc. as part of the game. Since $W$ has FDC, there is always a winning strategy that can be played entirely (in second player’s mind) over $W$ starting with a family $A_{k+1}$ so that $\{W\}$ is $R_{k+1}$-decomposable over $A_{k+1}$ and ending with a bounded family $A_{k+n}$ for some $n$. From these auxiliary constructions the second player can produce responses, at every step, to first player’s calls as follows:

$$W_i = \{g_F \left(g_F^{-1} \pi^{-1}(F) \cap A\right) \mid F \in F_k, A \in A_i\}, \quad i > k.$$

The elements $g_F$ act by isometries on $G$, so this gives an $R_i$-decomposition of $W_{i-1}$ over $W_i$ for all $i > k$, and if $A_{k+n}$ is bounded by $U$ then $W_{k+n}$ is bounded by $U$.

The case of sFDC is entirely similar, with the sequence of numbers $R_i$ being given by player one in advance.
4 Applications and discussion

4.1 Applications

The following simple corollary to Theorem 66 illustrates applications to finitely generated groups that are readily available and require only isometric actions.

**Corollary 68.** Let $G$ be a free product of finitely generated groups $A * B$. We assume that all three groups are given word metric with respect to finite generating sets. If $A$ has FAD and $B$ has APC then $G$ has APC.

**Proof.** Every element of the group $G$ can be written as a product of words in $A$ and $B$. We define the action of such words on $B$ as either trivial for words from $A$ or by left multiplication for words from $B$. The quasi-stabilizers in $G$ are commensurable with $A$ (which is the stabilizer of each point). If $\text{asdim}(A) \leq n$ then all quasi-stabilizers, which are coarsely equivalent to $A$, have $\text{asdim}(W_R(x_0)) \leq n$, so the Theorem applies in this situation.

More sophisticated statements can be proven using the strategy in [1, 18].

**Corollary 69.** If $A$ has FAD and $B$ has APC then their extensions and free products with amalgamation have APC.

In view of [1, 18], the details are routine.

There is a recent effort in extending geometric properties of metric spaces to general coarse properties of coarse structures, motivated by applications in controlled $K$-theory of rings and $C^*$-algebras. This was done for asymptotic dimension by Grave [14] and for APC and FDC by Bell, Moran, and Nagórko [4]. The notion of coarse actions is precisely what is needed to formulate group actions on coarse structures, and we expect generalizations of our results to be true when restated for the coarse properties.

**Remark 70.** We could not relax the assumption in Theorem 66. One can see that the proof relies on the bound for the asymptotic dimension of the quasi-stabilizers known a priori. This is likely an indication that the more general statement which assumes only that the quasi-stabilizers have FAD is not true. For much the same reasons, we suspect that the assumptions that $X$ has FAD and quasi-stabilizers have APC do not in general imply that $G$ has APC.
4.2 Fibered properties

There exist special positive results of the general type we just dismissed in Remark 70. Guentner [16] points out in 7.2.7 that weak fibered-type conditions for uniformly expansive maps and weak assumptions on the fiber such as simply FAD are sometimes sufficient for extension results. In his example, the base space $X$ is a simplicial tree, and the action is cofinite.

We would like to offer a different perspective on the extension problem. We will define new \textit{fibred} properties in terms of extensions that are weaker than the absolute analogues but are likely as useful for some purposes. The first observation is that there is a natural generalization of coarse quasi-actions.

**Definition 71.** A (nonuniform) coarse action of a group $G$ on a metric space $X$ is an assignment of a coarse self-equivalence $f_g: X \to X$ to each $g \in G$ so that the following conditions are satisfied:

1. there is a number $A \geq 0$ such that $d(f_{id}, id_X) \leq A$ in the sup norm,

2. for each pair of elements $g$ and $h$ in $G$, there is a number $B_{g,h} \geq 0$ such that $d(f_g \circ f_h, f_{gh}) \leq B_{g,h}$ in the sup norm.

Just as before, all compositions $f_g \circ f_{g^{-1}}$ are $(A + B_{g,g^{-1}})$-close to the identity.

The reason this is a natural definition is that a coarse action induces a naive $G$-equivariant structure on the bounded $K$-theory spectrum $K(X,R)$. Each coarse self-equivalence induces a self-equivalence of $K(X,R)$, and the maps close to identity induce genuine identities on $K(X,R)$. So a coarse action on $X$ induces a genuine $G$-action on $K(X,R)$.

On the other hand, we can now make the following definition.

**Definition 72.** A finitely generated group $G$ has \textit{fibred property} $P_1 \setminus P_2$ if it has a coarse action on a proper metric space $X$ with property $P_1$ so that for all $x \in X$ and all $R \geq 0$ the subsets of $G$ of the form $W_R(x,x_0) = \{g \in G \mid d(x,gx_0) \leq R\}$ have property $P_2$.

We believe that fibered FAD\setminus FAD will be as easy to use in inductive proofs of the integral Novikov Conjecture and the Borel Isomorphism Conjecture in $K$- and $L$-theory as the FAD property itself, cf. [6, 12]. So we ask a question of great interest to us.
Question 73. How large is the class of groups with fibred property FAD\FAD? It includes groups with FAD. Does it also include $\text{Out}(F_n)$ with $n > 0$? Does it include groups with proper isometric actions on a CAT(0) space containing at least one rank-1 element? Such groups are known to act on quasi-trees.

Even when the properties $P_1$ and $P_2$ are both FAD it is unlikely that $G$ has FAD unless one is in the situation of Theorem 63. In the proof of that theorem it was essential that elements of $G$ acted by self-equivalences with the same characteristic function $\ell$. In the nonuniform case, there is no guarantee that every pull-back $\pi^{-1}(x[R]) = W_R(x, x_0)$ has an isometric embedding in $W_K(x_0)$ for some specific number $K$. To illustrate this point, we show an example of a well-known infinite dimensional group which has fibred property FAD\FAD.

Example 74. The restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$ has FAD\FAD.

We will think of the base group as the additive group of the group ring $\mathbb{Z}[[\mathbb{Z}]]$, so the elements of $\mathbb{Z} \wr \mathbb{Z}$ can be written as pairs $(\sum_{x \in \mathbb{Z}} n_x x, k)$. The second factor acts on $\mathbb{Z}$ by translation $k \cdot x = x + k$, so the semidirect product operation is given by $(\sum n_x x, k) \cdot (\sum m_x x, l) = (\sum (n_x + m_{x-k}) x, k + l)$. It is known that $\mathbb{Z} \wr \mathbb{Z}$ is generated by two elements. We define a nonuniform coarse action by $\mathbb{Z} \wr \mathbb{Z}$ on its cobase $\mathbb{Z}$ by the formula $(\sum n_x x, k) \cdot t = t + k + n_i/|n_i| \sum_{x \neq -t, t} |n_x|$, where $i$ stands for the smallest index such that $n_i \neq 0$. One easily checks that $\operatorname{asdim} W_R(t, 0) = Z^{2R+2}$, so there is no uniform bound on the dimension of the quasi-stabilizers. Clearly, $A = 0$. Using the notation $\| (\sum n_x x, k) \| = |k| + \sum |n_x|$, we can choose $B_{g, h} = \| g \| + \| h \|$. Moreover, while the action by each element $(\sum n_g g, k)$ is eventually the translation by $k$ and so is a coarse equivalence, the function $\ell$ depends linearly on the sum $\sum |n_x|$, so this is not a uniform coarse quasi-action.

The notion of fibred property $P_1 \setminus P_2$ does not need to be restricted to groups with geometric actions. There is the following geometric analogue of Definition 72 in terms of uniformly expansive maps as in [18] and [3].

Definition 75. A proper metric space $Y$ has fibred property $P_1 \setminus P_2$ if there is a uniformly expansive map $\pi: Y \to X$ where the metric space $X$ has property $P_1$ and, for all $x \in X$ and all $R \geq 0$, the pull-backs $\pi^{-1}(x[R])$ have property $P_2$. 

30
To point out that this is a useful generalization even when one is interested in geometry of groups, we should recall that the inductive proofs of Novikov and Borel conjectures using controlled algebra are based on the (nonequivariant) Bounded Borel Conjecture which is stated for general metric spaces.

Now the geometric condition can be naturally iterated. For example, there is the evident property
\[ P^n = ((P\setminus P)\setminus P) \ldots \setminus P. \]

**Question 76.** How large is the class of spaces with property FAD^n?
Bibliography

[1] G. Bell and A. Dranishnikov, *On asymptotic dimension of groups*, Alg. Geom. Topol. 1 (2001), 57–71.

[2] G. Bell and A. Dranishnikov, *A Hurewicz-type theorem for asymptotic dimension and applications to geometric group theory*, Trans. Amer. Math. Soc. 358 (2006) no. 11, 4749–4764.

[3] G. Bell and D. Moran, *On constructions preserving the asymptotic topology of metric spaces*, NCJM 1 (2015), 46–57.

[4] G. Bell, D. Moran, and A. Nagórko, *Coarse property C and decomposition complexity*, Topol. Appl. (2016)

[5] M. Bestvina, K. Bromberg, and K. Fujiwara, *Constructing group actions on quasi-trees and applications to mapping class groups*, Publ. Math. I.H.É.S. 122 (2015), 1–64.

[6] G. Carlsson and B. Goldfarb, *The integral K-theoretic Novikov conjecture for groups with finite asymptotic dimension*, Invent. Math. 157 (2004), 405–418.

[7] ———, *Algebraic K-theory of geometric groups*, preprint, 2013, see v.3, 2015; available as arXiv:1305.3349

[8] A. Dranishnikov, *Asymptotic topology*, Russian Math. Surveys 55 (2000), 1085–1129.

[9] A. Dranishnikov and M. Zarichnyi, *Universal spaces for asymptotic dimension*, Topology Appl. 140 (2004), no. 2-3, 203-225.

[10] ———, *Asymptotic dimension, decomposition complexity, and Haver’s property C*, Topology Appl. 169 (2014), 99–107.
[11] C. Drutu, Quasi-isometry rigidity of groups. in Géométries à courbure négative ou nulle, groupes discrets et rigidités, in Sémin. Congr., 18, Soc. Math. France, Paris, 2009, 321–371.

[12] B. Goldfarb, Weak coherence and the K-theory of groups with finite decomposition complexity, to appear in Int Math Res Notices IMRN; available as arXiv:1307.5345

[13] B. Grave, A finitely presented group with infinite asymptotic dimension, preprint, 2003. available from http://www.uni-math.gwdg.de/bgr/

[14] _____, Asymptotic dimension of coarse spaces, NYJM 12 (2006), 249–256.

[15] M. Gromov, Asymptotic invariants of infinite groups, in Geometric group theory, Vol.2, Cambridge U. Press (1993).

[16] E. Guentner, Permanence in coarse geometry, in Recent Progress in General Topology III, Springer (2014), 507–533.

[17] E. Guentner, R. Tessera, and G. Yu, A notion of geometric complexity and its applications to topological rigidity, Invent. Math. 189 (2012), 315–357.

[18] _____, Discrete groups with finite decomposition complexity, Groups Geom. Dyn. 7 (2013), 377–402.

[19] M. Kapovich, Lectures on quasi-isometric rigidity, in Geometric Group Theory, volume 21 of Publications of IAS/Park City Summer Institute, Amer. Math. Soc., Providence, RI, 2014, 127–172.

[20] D. Kasprowski, A. Nicas, and D. Rosenthal, Regular Finite Decomposition Complexity, preprint, 2016; available as arXiv:1608.04516.

[21] L. Mosher, M. Sageev, and K. Whyte, Quasi-actions on trees I. Bounded valence, Annals Math. 158 (2003), 115–164.

[22] D.A. Ramras and B.W. Ramsey, Extending properties to relatively hyperbolic groups, preprint, 2014; available as arXiv:1410.0060.
[23] Y. Wu and J. Zhu, *Asymptotic property C of wreath products*, preprint, 2016; available as arXiv:1607.07599.

[24] G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. **139** (2000), no. 1, 201-240.