Addendum

Addendum to "Subgroup commutativity degrees of finite groups"
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Abstract

We indicate a natural generalization of the concept of subgroup commutativity degree of a finite group and a list of open problems on these new concepts.

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1 Introduction

The starting point for our discussion is given by [4], where the subgroup commutativity degree of a finite group $G$ has been introduced and studied. This new quantity is defined by

$$sd(G) = \frac{1}{|L(G)|^2} \left| \{(H,K) \in L(G)^2 \mid HK = KH\} \right| = \frac{1}{|L(G)|^2} \left| \{(H,K) \in L(G)^2 \mid HK \in L(G)\} \right|$$

and measures the probability that two subgroups of $G$ commute, or equivalently the probability that the product of two subgroups of $G$ be a subgroup in $G$. It was inspired by the well-known commutativity degree $d(G)$ of $G$. Since for $d(G)$ there is a natural generalization, namely the relative commutativity degree
degree of $G$ (see [2]), a similar one can be introduced for $sd(G)$. So, we define the relative subgroup commutativity degree of a subgroup $H$ of $G$

\[
sd(H, G) = \frac{1}{|L(H)||L(G)|} |\{(H_1, G_1) \in L(H) \times L(G) \mid H_1G_1 = G_1H_1\}|,
\]

and, more generally, the relative subgroup commutativity degree of two subgroups $H$ and $K$ of $G$

\[
sd(H, K) = \frac{1}{|L(H)||L(K)|} |\{(H_1, K_1) \in L(H) \times L(K) \mid H_1K_1 = K_1H_1\}|.
\]

It is obvious that $sd(G) = sd(G, G)$, for any finite group $G$, and that the above two notions also have a probabilistic significance. In the following we shall focus on some basic properties of the relative subgroup commutativity degree and on its connections with the classical subgroup commutativity degree.

On the other hand, in the final section of [4] some further research directions and three open problems on subgroup commutativity degrees have been indicated. Since this concept, as well as its above generalizations are very new, we think that a more large list of open problems can be useful.

2 Relative subgroup commutativity degrees of finite groups

Let $G$ be a finite group and $H$ be a subgroup of $G$. Then

\[0 < sd(H, G) \leq 1.\]

Obviously, the equality $sd(H, G) = 1$ holds if and only if all subgroups of $H$ are permutable in $G$, or equivalently if and only if $H$ is modular and subnormal in $G$ (see Theorem 5.1.1 of [3]).

If $H \neq G$, then the set $\{(H_1, G_1) \in L(H) \times L(G) \mid H_1G_1 = G_1H_1\}$ contains the union of the disjoint sets $\{(H_1, G_1) \in L(H)^2 \mid H_1G_1 = G_1H_1\}$ and $\{(H_1, G) \mid H_1 \in L(H)\}$. This shows that $sd(H, G)$ and $sd(H)$ satisfy the inequality

\[sd(H, G) \geq \frac{|L(H)|}{|L(G)|} sd(H) + \frac{1}{|L(G)|}.\]

In the following, for every $H_1 \in L(H)$, we shall denote by $C(H_1)$ the set of subgroups of $G$ which commute with $H_1$ and by $I(H_1)$ the set of subgroups of $G$ strictly containing $H_1$. One obtains

\[sd(H, G) = \frac{1}{|L(H)||L(G)|} \sum_{H_1 \in L(H)} |C(H_1)|.\]
Clearly, $N(G)$ is contained in each set $C(H_1)$, which implies that

$$sd(H, G) \geq \frac{|N(G)|}{|L(G)|}.$$  

Since $L(H_1) \cup I(H_1) \subseteq C(H_1)$, for all $H_1 \in L(H)$, we also infer that

$$sd(H, G) \geq \frac{1}{|L(H)||L(G)|} \left( \sum_{H_1 \in L(H)} |L(H_1)| + \sum_{H_1 \in L(H)} |I(H_1)| \right).$$

Moreover, if $H_1 \in N(G)$, then we find the following inequality between the relative subgroup commutativity degrees of $H$ and of $H/H_1$:

$$sd(H, G) \geq \frac{|L(H/H_1)||L(G/H_1)|}{|L(H)||L(G)|} sd(H/H_1, G/H_1).$$

We remark that the permutability of the subgroups $(H_1, G_1) \in L(H) \times L(G)$ is equivalent to the permutability of the subgroups $(H_1^x, G_1) \in L(H^x) \times L(G)$, for every $x \in G$. This leads to the following proposition.

**Proposition 2.1.** Any two conjugate subgroups of a finite group have the same relative subgroup commutativity degree.

In the following let $(G_i)_{i=1}^k$ be a family of finite groups having coprime orders. Then the subgroup lattice of the direct product $\prod_{i=1}^k G_i$ is decomposable, that is every subgroup $H$ of $\prod_{i=1}^k G_i$ can (uniquely) be written as $H = \prod_{i=1}^k H_i$ with $H_i \leq G_i$, for all $i = 1, \ldots, k$. A result similar with Proposition 2.2 of [4] is now obtained for the relative subgroup commutativity degree.

**Proposition 2.2.** Under the above hypotheses, the following equality holds

$$sd(H, \prod_{i=1}^k G_i) = \prod_{i=1}^k sd(H_i, G_i).$$

Obviously, the above formula can successfully be applied in the case of finite nilpotent groups.

**Corollary 2.3.** Let $G$ be a finite nilpotent group and $(G_i)_{i=1}^k$ be the Sylow subgroups of $G$. Then, for every subgroup $H$ of $G$, we have

$$sd(H, G) = \prod_{i=1}^k sd(H_i, G_i),$$

where $H_i$, $i = 1, 2, \ldots, k$, are the Sylow subgroups of $H$. In particular, we infer that the computation of the relative subgroup commutativity degrees of subgroups of finite nilpotent groups is reduced to $p$-groups.
Our next goal is to establish some connections between $sd(G)$ and the relative subgroup commutativity degrees of the maximal subgroups of $G$, say $M_0, M_1, \ldots, M_r$. Let $H \in L(G)$. Then $L(G) = \{G\} \cup \bigcup_{i=0}^r L(M_i)$ and so $C(H) = \{G\} \cup \bigcup_{i=0}^r M_i(H)$, where $M_i(H) = \{K \in L(M_i) \mid HK = KH\}$, for any $i = 0, r$. By applying the well-known Inclusion-Exclusion Principle, it follows that

$$|C(H)| = 1 + \sum_{s=0}^r (-1)^s \sum_{0 \leq i_0 < i_1 < \ldots < i_s \leq r} |\cap_{j=0}^s M_{i_j}(H)|.$$ 

Since

$$sd(G) = \frac{1}{|L(G)|^2} \sum_{H \in L(G)} |C(H)|$$

and

$$\sum_{H \in L(G)} |\cap_{j=0}^s M_{i_j}(H)| = \sum_{H \in L(G)} |\{K \in L(\cap_{j=0}^s M_{i_j}) \mid HK = KH\}| = |L(G)| |L(\cap_{j=0}^s M_{i_j})| \cdot sd(\cap_{j=0}^s M_{i_j}, G),$$

we have proved the following result.

**Theorem 2.4.** Let $G$ be a finite group and $M_0, M_1, \ldots, M_r$ be the maximal subgroups of $G$. Then

$$sd(G) = \frac{1}{|L(G)|} \left(1 + \sum_{s=0}^r (-1)^s \sum_{0 \leq i_0 < i_1 < \ldots < i_s \leq r} |L(\cap_{j=0}^s M_{i_j})| \cdot sd(\cap_{j=0}^s M_{i_j}, G)\right),$$

Clearly, the above equality allows us to compute the subgroup commutativity degree for all finite groups $G$ whose maximal subgroup structure is known. We also remark that certain supplementary assumptions on the maximal subgroups of $G$ can simplify the right side of (1). One of them consists in asking that the relative subgroup commutativity degree of any intersection of at least two (distinct) maximal subgroups of $G$ be equal to 1. In this case $sd(G)$ will depend only on $sd(M_i, G)$, $i = 0, 1, \ldots, r$.

**Corollary 2.5.** Let $G$ be a finite group and $M_0, M_1, \ldots, M_r$ be the maximal subgroups of $G$. If $sd(\cap_{j=0}^s M_{i_j}, G) = 1$, for any $s = 1, r$ and $0 \leq i_0 < i_1 < \ldots < i_s \leq r$, then we have

$$sd(G) = 1 - \frac{1}{|L(G)|} \sum_{i=0}^r |L(M_i)|(1 - sd(M_i, G)),$$

or equivalently

$$sd(G) = 1 - \frac{1}{|L(G)|^2} \sum_{i,j=0}^r |L(M_i)||L(M_j)|(1 - sd(M_i, M_j)).$$
In \[4\], the explicit value \(sd(A_4) = 16/25\) has been directly computed. Since \(A_4\) satisfies the supplementary condition in the hypotheses of Corollary 2.5, this value can be also obtained by using (2) or (3). The same thing cannot be said in the case of \(S_4\), for which we must apply the general formula (1).

**Example 2.6.** It is well-known that \(S_4\) possesses eight maximal subgroups: \(M_0 = A_4, M_i \cong S_3\), for \(1 \leq i \leq 4\), and \(M_i \cong D_8\), for \(5 \leq i \leq 7\). By inspecting \(L(S_4)\), we infer that the intersections of any \(s \geq 5\) distinct maximal subgroups is trivial, while the intersections of \(s \leq 4\) distinct maximal subgroups are isomorphic with \(\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2, S_3, D_8\) or \(A_4\). Then (1) leads to \(sd(S_4) = \frac{1}{30} \left(13 - 24sd(\mathbb{Z}_2, S_4) - 8sd(\mathbb{Z}_3, S_4) - 18sd(\mathbb{Z}_2 \times \mathbb{Z}_2, S_4) + 24sd(S_3, S_4) + 30sd(D_8, S_4) + 10sd(A_4, S_4)\right)\). We easily find: \(sd(\mathbb{Z}_2, S_4) = 2/3\), \(sd(\mathbb{Z}_3, S_4) = 7/12\), \(sd(\mathbb{Z}_2 \times \mathbb{Z}_2, S_4) = 44/75\), \(sd(S_3, S_4) = 4/9\), \(sd(D_8, S_4) = 37/75\) and \(sd(A_4, S_4) = 151/300\). Hence \(sd(S_4) = 1841/4500\).

### 3 Open problems

**Problem 3.1.** Let \(G\) be a finite group and \(H \in L(G)\). Which are the connections between \(sd(G)\) and the classical commutativity degree \(d(G)\), respectively between \(sd(H, G)\) and the classical relative commutativity degree \(d(H, G)\)?

**Problem 3.2.** The relative subgroup commutativity degrees can be obviously computed for finite groups whose subgroup structure is precisely determined. An interesting example of such groups is constituted by the finite groups with all Sylow subgroups cyclic, the so-called ZM-groups. Such a group is of type

\[
\text{ZM}(m, n, r) = \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r \rangle,
\]

where the triple \((m, n, r)\) satisfies the conditions \(\gcd(m, n) = \gcd(m, r - 1) = 1\) and \(r^n \equiv 1 \pmod{m}\). The subgroups of \(\text{ZM}(m, n, r)\) have been completely described in \[1\]. Set

\[
L = \left\{(m_1, n_1, s) \in \mathbb{N}^3 \mid m_1 | m, n_1 | n, s < m_1, m_1 | s^{r^{m_1} - 1} - 1\right\}.
\]

Then there is a bijection between \(L\) and \(L(\text{ZM}(m, n, r))\), namely the function that maps a triple \((m_1, n_1, s) \in L\) into the subgroup

\[
H_{(m_1, n_1, s)} = \bigcup_{k=1}^{n/m_1} (b^{n_1}a^s)^k < a^{m_1} > \text{ of } \text{ZM}(m, n, r).
\]

Give an explicit formula for \(sd(H_{(m_1, n_1, s)}, \text{ZM}(m, n, r))\).

**Problem 3.3.** It is clear that \(sd(A_3, S_3) = 1\). We also have seen in Section 2 that \(sd(A_4, S_4) = 151/300\). These lead to the following two natural asks:
compute \( sd(A_n, S_n) \), for an arbitrary \( n \geq 5 \), and the limit \( \lim_{n \to \infty} sd(A_n, S_n) \).

**Problem 3.4.** By using (1), for a finite group \( G \) we are able to calculate \( sd(G) \) whenever the structure of maximal subgroups of \( G \) and their relative subgroup commutativity degrees are known. Is this true for other remarkable systems of subgroups of \( G \) (as the sets of minimal subgroups, cyclic subgroups or proper terms of a composition series, respectively)?

**Problem 3.5.** Given a finite group \( G \), the following function is well-defined

\[
sd_G : L(G) \rightarrow [0, 1], \quad sd_G(H) = sd(H, G), \quad \text{for all } H \in L(G).
\]

By Proposition 2.1, \( sd_G \) is constant on each conjugacy class of subgroups of \( G \). Remark that the converse fails: take the subgroups \( H_1 = \langle y \rangle \) and \( H_2 = \langle xy \rangle \) of \( D_{2n} \); we have \( sd_G(H_1) = sd_G(H_2) = 9/10 \), but \( H_1 \not\sim H_2 \). Study other properties of \( sd_G \) (e.g. injectivity, monotony, ..., and so on), as well as of the restriction of \( sd_G \) to the set of conjugacy classes of subgroups. Describe the finite groups \( G \) for which these functions satisfy certain conditions.

**Problem 3.6.** Another interesting function can be also associated to a finite group \( G \), namely

\[
sd(-, -) : L(G) \times L(G) \rightarrow [0, 1].
\]

Study this function and its restrictions to some remarkable subsets of type \( L \times L \) of \( L(G) \times L(G) \) (e.g. take \( L = C(G) \), the poset of cyclic subgroups of \( G \)). For an arbitrary \( n \geq 2 \), generalize the above function by defining

\[
sd(-, -, ..., -) : L(G)^n \rightarrow [0, 1], \quad sd(H_1, ..., H_n) =
\]

\[\frac{1}{\prod_{i=1}^{n} |L(H_i)|} \left| \{ (K_1, ..., K_n) \in L(G)^n \mid K_1 \cdots K_n = K_{\sigma(1)} \cdots K_{\sigma(n)}, \forall \sigma \in S_n \} \right| .\]

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