Supertranslations: redundancies of horizon data and global symmetries at null infinity

K Sousa\(^1\), G Miláns del Bosch\(^1\) and B Reina\(^2,3\)

\(^1\) Instituto de Física Teórica UAM-CSIC Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain
\(^2\) School of Mathematical Sciences, Dublin City University, Glasnevin, Dublin 9, Ireland
\(^3\) Departamento de Física Teórica e Historia de la Ciencia, University of the Basque Country UPV/EHU, Apartado 644, 48080 Bilbao, Spain

E-mail: kepa.sousa@csic.es, guillermo.milans@csic.es and borja.reina@dcu.ie

Received 15 August 2017, revised 26 October 2017
Accepted for publication 27 October 2017
Published 25 January 2018

Abstract

We characterise the geometrical nature of smooth supertranslations defined on a generic non-expanding horizon (NEH) embedded in vacuum. To this end we consider the constraints imposed by the vacuum Einstein’s equations on the NEH structure, and discuss the transformation properties of their solutions under supertranslations. We present a freely specifiable data set which is both necessary and sufficient to reconstruct the full horizon geometry, and is composed of objects which are invariant under supertranslations. We conclude that smooth supertranslations do not transform the geometry of the NEH and that they should be regarded as pure gauge. Our results apply both to stationary and non-stationary states of a NEH, the latter ones being able to describe radiative processes taking place on the horizon. As a consistency check we repeat the analysis for Bondi–Metzner–Sachs (BMS) supertranslations defined on null infinity, \(\mathcal{I}\). Using the same framework as for the NEH we recover the well-known result that BMS supertranslations act non-trivially on the free data on \(\mathcal{I}\). The full analysis is made in exact, non-linear, general relativity.

Keywords: black holes, horizon, asymptotic symmetries, supertranslations, BMS group, null infinity

1. Introduction

The subject of asymptotic symmetries in gravitational theories has been an active field of research in recent years. One of the main motivations to study the asymptotic structure of the spacetime boundary is the characterisation of candidate theories of quantum gravity. A prominent example is the result in [1, 2], that any consistent theory of quantum gravity on a
spacetime which is asymptotically AdS$_3$ should be a conformal field theory. This result has led to major developments in the understanding of the microscopic origin of black hole entropy, as in the case of the Bañados–Teitelboim–Zanelli (BTZ) black hole [3–5], and for extremal Kerr black holes in four dimensions [6]. The success of this approach has inspired many attempts to extend these results to the case of astrophysical—non-extremal—black holes in asymptotically flat spacetimes (see [7–20] and references therein).

The symmetry group of four-dimensional asymptotically flat spacetimes at null infinity $\mathcal{I}$ is the so called Bondi–Metzner–Sachs (BMS) group [21–23]. This group consists of the semidirect product of the Lorentz group times an infinite dimensional abelian normal subgroup which generalises translations, the so-called supertranslations. Supertranslations act on future (past) null infinity by shifting the advanced (retarded) time independently for each point of the sphere at infinity. These diffeomorphisms are particularly interesting because they act non-trivially on the geometric data at null infinity, which encodes the gravitational degrees of freedom of gravitational radiation. More specifically, the radiative vacua of asymptotically flat spacetimes is infinitely degenerate, and supertranslations act transitively on it, i.e. all radiative vacua are connected to each other by supertranslations. The implications of this symmetry group on the gravitational S-matrix were first studied in the framework of asymptotic quantization [24–26] (see also [27, 28]), and the relation between supertranslations and Weinberg’s soft graviton theorem [29] has been explored in [30–34].

Recently, it has been argued that in black hole spacetimes, if the event horizon is regarded as an inner boundary, it is appropriate to enhance the asymptotic symmetry group (ASG) with those diffeomorphisms which leave invariant the near horizon geometry$^4$ [8, 10, 14, 15, 20, 33–42]. For stationary black holes the corresponding set of diffeomorphisms has been shown to be a reminiscence of the BMS group on $\mathcal{I}$. As in the case of null infinity, the ASG on the horizon is enhanced with respect to the isometry group of the background with the addition of supertranslations, which in this case shift the advance (retarded) time of the future (past) horizon. Following the analogy with the BMS group on null infinity, it has been conjectured that the horizon supertranslations may act non-trivially on the black hole geometry, transforming the black hole to a physically inequivalent one. If this was the case, the ‘supertranslation hair’ could provide some insight on the microscopic degrees of freedom associated to the black hole entropy. Although these ideas are certainly appealing, the physical nature of the asymptotic symmetry group defined on non-extremal horizons is still unclear, and the proposal remains controversial [43–45].

On the one hand, there are still some discrepancies on the structure of the ASG found in different analyses [8, 10, 14, 15, 20, 36, 46]. These differences could be attributed to alternative choices of boundary conditions for the metric tensor near the horizon, but the geometric interpretation of the discrepancies is not well understood. The main difficulty in comparing different analyses is that they are based on a coordinate dependent approach. In practice, the boundary conditions are defined as restrictions on an explicit coordinate expression of the metric tensor in the neighbourhood of the horizon. This complicates the comparison between the various works as, in general, a given set of boundary conditions does not retain the same form in different coordinate systems.

On the other hand, it remains an open question to determine the effect of the horizon ASG on the geometry, and in particular, whether these diffeomorphisms act non-trivially on the physical state of the black hole. This problem was recently addressed in [34], where the authors performed a Hamiltonian analysis to characterise the phase space of a Schwarzschild spacetime. According to this study the phase space of the Schwarzschild spacetime is infinite

$^4$ A more complete list of related works can be found in [34].
dimensional, and supertranslations act non-trivially on it. This conclusion contrasts with the classical result that there is only a three-parameter family of stationary black hole solutions in Einstein–Maxwell theory. Actually, the Hamiltonian analysis of spacetimes containing isolated horizons, such as the Schwarzschild spacetime, had been considered before in [47–50]. In those works it was shown that the corresponding phase space could be infinite dimensional in the presence of radiation, but in the stationary case it was argued that the physical state is completely determined by the standard quantities: ADM mass, angular momentum and electric charge.

In the present paper we will study horizon supertranslations defined on a generic non-expanding horizon (NEH) which is embedded in vacuum [47–51]. A non-expanding horizon is a generalisation of a killing horizon which admits gravitational radiation propagating arbitrarily close to it, and even on the NEH itself (but not crossing it). Our main objective is to provide a coordinate invariant definition of horizon supertranslations, and then to characterise their effect on the NEH geometry. For this purpose we have taken as a guide the geometric method used by Geroch [52] and Ashtekar [24–26, 53, 54] to study the structure and dynamics of null infinity. Following these works, we describe the horizon in terms of an abstract three-dimensional manifold separated from the spacetime, and which is diffeomorphically identified with the horizon. In this framework, the information about the intrinsic and extrinsic geometry of the horizon is encoded in tensor fields living on the abstract manifold. The advantage of this method is that the geometric data of the horizon is isolated from the rest of the spacetime, and moreover, all the gauge redundancies are well characterised.

To clarify the geometrical nature of supertranslations we have studied the set spacetime diffeomorphisms which preserve the horizon as a set of points, and leave invariant the metric tensor on it. Note that these diffeomorphisms, which we call for short hypersurface symmetries, are defined in a coordinate invariant way, i.e. without involving an explicit coordinate expression for the spacetime metric tensor. After checking that horizon supertranslations belong to this class of diffeomorphisms, we have studied the behaviour of the complete horizon geometry (including the extrinsic geometry) under an arbitrary hypersurface symmetry. Our analysis shows that the effect of these diffeomorphisms on the horizon can be identified with a gauge redundancy of the description. In other words, hypersurface symmetries, and in particular supertranslations, leave invariant both the intrinsic and the extrinsic geometry of the horizon up to a gauge redundancy of the description. Note, however, that this result is not sufficient to claim that supertranslations act trivially on the geometry of the horizon. Indeed, supertranslations could be large gauge transformations, i.e. global symmetries, which can change the dynamical state of a system [1, 2, 55]. Thus, we still need to identify the dynamical degrees of freedom of the horizon, or equivalently, a data set which is both necessary and sufficient to reconstruct the full NEH geometry, and then we have to determine how it transforms under supertranslations.

In order to identify the dynamical degrees of freedom of the NEH we follow the geometric method of [56] and [50] (see also [57, 58]), which consists of studying the constraints imposed by the vacuum Einstein’s equations on its geometric data. By solving these constraint equations it is possible to extract a set of freely specifiable quantities which contain all the information necessary to reconstruct the NEH geometry [50]. Thus, the resulting free horizon data set encodes the dynamical degrees of freedom of the horizon, but in general it also involves some gauge redundancies. In the present work we have reconsidered the analysis in [50] for non-expanding horizons, discussing in detail the treatment of the gauge redundancies, and specifically of supertranslations.

The main result of this work is the identification of a free data set which does not involve any unfixed gauge degree of freedom and, in particular, which is composed of objects
which are invariant under supertranslations. An immediate consequence of our result is that supertranslations do not affect the NEH geometry, as it can be encoded entirely in quantities which are invariant under these diffeomorphisms. In particular, the stationary state of the horizon is completely determined by its intrinsic geometry and its angular momentum aspect, and neither of the two transform under horizon supertranslations. The supertranslation invariant data set is also sufficiently general to represent non-stationary states of the NEH, and thus, it can describe radiative processes occurring at the horizon. It is important to stress that, to avoid excluding physically allowed configurations of the horizon, we have not eliminated the freedom to perform supertranslations using gauge fixing conditions. Instead, guided by the treatment of null infinity [53], we have dealt with the redundancy describing the NEH geometry in terms of variables which are invariant under supertranslations. Our conclusions are directly applicable to the non-singular horizon supertranslations discussed in [8, 15, 20, 35, 36, 39, 41, 46]. The supertranslations studied in [34] cannot be described as hypersurface symmetries, and thus we will consider this case in a separate publication [59].

As a consistency check we have repeated the analysis of null infinity as in [53] using the same framework as for non-expanding horizons. In particular, following [53], we have characterised the solutions to the constraint equations of null infinity in terms of variables invariant under BMS supertranslations, and we have reproduced the proof of the degeneracy of the radiative vacuum of asymptotically flat spacetimes. In other words, we find that the solution space of the constraint equations of $\mathcal{Z}$ in the absence of radiation is infinite dimensional. Since the analysis is made in terms of variables which are free of any gauge redundancies, these degenerate vacua must be regarded as physically distinct, and yet they can be shown to be connected to each other by supertranslations. Therefore, we recover the well known result that BMS supertranslations—contrary to the case of horizons—act non-trivially on the free data of null infinity, i.e. they represent a global symmetry of the constraint equations.

This article is organised as follows. In section 2 we review the formalism to describe the geometry of null hypersurfaces, together with the constraint equations that restrict the corresponding geometric data. In section 3 we characterise in detail the gauge redundancies inherent to our description, and we discuss the effect of supertranslations on the horizon geometry. In section 4 we analyse the constraint equations of a non-expanding horizon, and present a free data set to describe its geometry which is composed of quantities invariant under supertranslations. In section 5 we consider the constraint equations for null infinity, and we reproduce the proof of the degeneracy of the radiative vacuum of asymptotically flat spacetimes using our framework. Finally in section 6 we discuss our results.

2. Dynamics of null hypersurfaces

In this section we will review the geometry and dynamics of null hypersurfaces, what will also serve to present the relevant formulae. A more detailed overview of this subject can be found in [58, 60], while the specific framework used here is based on [57].

We begin setting our notation and general conventions. We will work with $(3 + 1)$-dimensional spacetimes $(\mathcal{M}, g)$, described by a manifold $\mathcal{M}$ equipped with a metric tensor $g$ with signature $(-, +, +, +)$. We will denote the spacetime coordinates by $\{x^\mu\}$, with the index running over $\mu = 0, 1, 2, 3$. The Riemann tensor is defined in terms of the Ricci identity as follows

$$\nabla_{\mu} \nabla_{\nu} V^\sigma = \nabla_{\nu} \nabla_{\sigma} V^\mu - \nabla_{\sigma} \nabla_{\mu} V^\nu = R^\sigma_{\rho \mu \nu} V^\rho. \quad (2.1)$$
where $V^\mu$ is an arbitrary vector field\(^5\), the Ricci tensor is given by $R_{\mu\nu} = R'_{\mu\nu\rho\sigma}$, and the scalar curvature by $R = R'_\mu$. The Riemann curvature can be split in its trace part, characterised by the Schouten tensor $S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, and its traceless part, encoded in the Weyl tensor $C_{\mu\nu\rho\sigma}$.

\[ R_{\sigma\rho\mu\nu} = C_{\sigma\rho\mu\nu} + \frac{1}{2}(g_{\sigma[\mu}S_{\rho]\nu] - g_{\rho[\mu}S_{\sigma]\nu]). \tag{2.2} \]

We will use geometrized units $c = G = 1$ so that the Einstein’s equations read

\[ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{2.3} \]

In regions where the spacetime geometry is consistent with the vacuum Einstein’s equations, $R_{\mu\nu} = 0$, the Schouten tensor must be zero and thus the Riemann curvature is completely determined by the Weyl tensor $R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}$.

### 2.1. Geometric data of null hypersurfaces

In this section we will review the geometry and dynamics of a null hypersurface $\mathcal{H}$. Bearing in mind the case of black hole horizons and null infinity, we will assume the hypersurface to have the topology $\mathcal{H} \cong \mathbb{R} \times S^2$. We will describe the hypersurface as the embedding of an abstract three dimensional manifold $\Sigma \cong \mathbb{R} \times S^2$ on the spacetime via the diffeomorphism $\Phi : \Sigma \to \mathcal{M}$, so that $\Phi(\Sigma) = \mathcal{H}$ (see [57]). The manifold $\Sigma$ acts as a diffeomorphic copy of $\mathcal{H}$ detached from the spacetime, and it is introduced for convenience in order to isolate the dynamical degrees of freedom (i.e. the free geometric data) of the hypersurface.

To characterise the intrinsic and extrinsic geometry of the hypersurface it is convenient to introduce a basis of the spacetime tangent space adapted to $\mathcal{H}$. For this purpose let us first define a coordinate system for the abstract manifold $\{\xi^a\}$, with the index running over $a = 1, 2, 3$. The corresponding coordinate basis of the tangent space $T_p\Sigma$ is then given by $\mathcal{B} \equiv \{\xi^a = \partial_{\xi^a}\}$, with $p \in \Sigma$. The elements of the basis $\mathcal{B}$ are identified with a set of linearly independent spacetime vectors tangent to the hypersurface $e_a \equiv\Phi(\xi^a)$, via the pushforward map $\Phi$ associated to $\Phi$. Then, we can form a basis $\mathcal{B} = \{e_a, \ell\}$ of the spacetime tangent space completing the set of vectors $\{e_a\}$ with any vector $\ell$ transverse to $\mathcal{H}$, the so called rigging.

The spacetime metric over the hypersurface can be characterised in terms of a set of tensor fields over $\Sigma$ which, by definition, have the following components on the basis $\mathcal{B}$.

\[ g_{ab} \equiv g(e_a, e_b)\Phi(p), \quad e_a \equiv g(\ell, e_a)\Phi(p), \quad \ell(2) \equiv g(\ell, \ell)|_{\Phi(p)}. \tag{2.4} \]

These fields encode the scalar products of the elements in the spacetime basis $\mathcal{B} = \{e_a, \ell\}$, and in particular $g_{ab}$ represents the induced metric on $\mathcal{H}$. This set of fields is known as the *hypersurface metric data*.

In order to reduce the large degree of gauge freedom in this description, namely the choice of coordinates on $\Sigma$ and the specification of the rigging vector $\ell$, it is useful to introduce some simplifying conventions. The normal one-form $n$ and the normal vector $n$ to the hypersurface are determined by the conditions $n(e_a) = 0$ and $n = g^{-1}(n, \cdot)$, respectively, and thus they are defined up to $n \rightarrow \lambda n$ and $n \rightarrow \lambda n$, where $\lambda$ is a scalar field on $\mathcal{H}$. Since the normal vector to a null hypersurface is null, i.e. $g(n, n) = n(n) = 0$, $n$ is also tangent to $\mathcal{H}$, and we will choose it to be future directed. Therefore we can partially fix the coordinate system on the abstract

\(^5\)We will use the shorthand $W_{[\mu\nu]} = W_{\mu\nu} - W_{\nu\mu}$ and $W_{(\mu\nu)} = W_{\mu\nu} + W_{\nu\mu}$ to denote the symmetrisation and anti-symmetrisation of indices.
manifold $\Sigma$ defining $\xi^1$ so that $e_1 = n$, and parametrising the $S^2$ component of the hypersurface with the coordinates $\xi^M$, where $M = \{2,3\}$. Moreover we will require the rigging vector $\ell$ to be null $g(\ell, \ell) = 0$, and we will fix its normalisation and direction with respect to the vectors $\{e_a\}$ imposing the conditions $n(\ell) = 1$ and $g(\ell, e_M) = 0$ everywhere on $\mathcal{H}$, what can be expressed equivalently as $g(\ell, e_a) = \delta_a^1$. With these choices the explicit coordinate expressions for the hypersurface metric data in the basis $\mathcal{B} = \{e_1, e_M\}$ read

$$\gamma_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & q_{MN} \end{pmatrix}, \quad \ell_a = (1,0,0), \quad \ell^{(2)} = 0. \tag{2.5}$$

Here $q_{MN} \equiv g(e_M, e_N)_{\Phi(\rho)}$ represents the induced metric on the spatial sections $S_{\xi^1} = \Sigma|_{\xi^1}$ of the horizon, which are defined by the level sets of the null parameter $\xi^1$. In the following we will denote the elements of the basis of the spacetime tangent space by $\mathcal{B} = \{n, \ell, e_M\}$. For later convenience, we also write here the following identity satisfied by the elements of $\mathcal{B}$

$$g^{\mu\nu} = \ell^{(\mu} n^{\nu)} + q^{MN} e^\mu_N e^\nu_M. \tag{2.6}$$

The Levi-Civita connection at points of the hypersurface can be characterised specifying its action on the elements of the basis $\mathcal{B}$

$$\nabla_n n = \kappa n, \quad \nabla_M n = \Omega_M n + \Theta^N_M e_N, \quad \nabla_a e_M = \Omega_M n + \Theta^N_M e_N, \quad \nabla_M e_N = -\Theta_{MN} \ell + \Xi_{MN} n + \Gamma^L_{MN} e_L, \quad \nabla_\ell \ell = -\kappa \ell - \Omega^M e_M, \quad \nabla_M \ell = -\Omega_M \ell + \Xi^N_M e_N, \tag{2.7}$$

where the indices $M, N$ are raised and lowered with $q_{MN}$ and its inverse $q^{MN}$, $\nabla_n \equiv n^\mu \nabla_\mu$ and $\nabla_M \equiv e_M^\mu \nabla_\mu$. This is the most general form of the connection coefficients consistent with our conventions (2.5), as it can be easily derived in the framework of [57]. In particular, it can be seen that the integral curves of the normal vector $n$ are null geodesics parallel to the hypersurface, and the inaffinity parameter $\kappa$ is referred as the surface gravity in the case of horizons. The surface gravity, together with the Hajicek one-form $\Omega_M$ and $\Xi_{MN}$ can be conveniently encoded in the tensor field $Y_{ab}$ defined in terms of the basis $\mathcal{B}$ of $\mathcal{T}_p\Sigma$ as follows

$$Y_{ab} \equiv \frac{1}{2} \mathcal{L}_\ell g(e_a, e_b)_{\Phi(\rho)} = \begin{pmatrix} -\kappa & -\Omega_M \\ -\Omega_N & \Xi_{MN} \end{pmatrix}. \tag{2.8}$$

The set of coefficients $\Xi_{MN} = \frac{1}{2} e^\mu_M e^\nu_N \nabla_{(\mu} \ell_{\nu)}$ characterises the components of the Levi-Civita connection associated to directions which are all transverse to the normal vector $n$, and thus we will refer to it as the transverse connection. For later convenience we will also introduce the rotation one-form $\omega_\alpha$ which is defined by

$$\omega_\alpha = -Y_{ab} \hat{n}^b = (\kappa, \Omega_M). \tag{2.9}$$

where $\hat{n}$ is the vector on $T_p\Sigma$ which is identified with the null normal via the embedding $d\Phi(\hat{n}) \equiv n$. The remaining connection coefficients, $\Theta_{MN}$ and $\Gamma^C_{BA}$, are fully determined by the intrinsic geometry via the equations

$$\frac{1}{2} \partial_a q_{MN} = \Theta_{MN}, \quad \Gamma^L_{MN} = \frac{1}{2} q^{LP} (\partial_M q_{NP} + \partial_N q_{MP} - \partial_P q_{MN}), \tag{2.10}$$

where $\partial_a q \equiv \partial_{e_a} q$. Thus, $\Gamma^C_{BA}$ represents the Levi-Civita connection compatible with $q_{MN}$, and the quantity $\Theta_{MN}$ is known as the second fundamental form.
Summarising, the intrinsic and extrinsic geometry of a null hypersurface can be fully encoded in the following set of fields defined on $\Sigma$

\[
\text{Hypersurface data : } \mathcal{D} \equiv (q_{MN}, \kappa, \Omega_M, \Xi_{MN}).
\] (2.11)

Our choice of coordinates for $\Sigma \cong \mathbb{R} \times S^2$, with $\xi^1$ running along the null direction $n$ of the hypersurface, and $\xi^M$ parametrising the sections with constant $\xi^1$, $\mathcal{S}_{\xi^1} \cong S^2$, allows one to picture the data $\kappa, \Omega_M$ and $\Xi_{MN}$ as tensor fields living on a manifold with the topology of a sphere and a Riemannian metric $q_{MN}$. In this picture, the dependence of these fields on the null coordinate $\xi^1$ is interpreted as a “temporal” evolution [61–63]. As we shall see in the next section, the evolution of these fields along the null direction is not completely free, as it is restricted by the geometry of the ambient space. Moreover, as we shall see in sections 3 and 4, this description still involves some residual gauge redundancies, which lead to further constraints on (2.11) after gauge fixing.

In the following we will often identify the abstract manifold $\Sigma$ with the hypersurface $\mathcal{H}$, and we will leave implicit the pull-back $\Phi^*$ operation in the formulae to simplify the notation.

2.2. Constraint equations of null hypersurfaces

The spacetime connection on the hypersurface, given by (2.7), must be consistent with the geometry of the ambient space where $\mathcal{H}$ is embedded. This requirement leads to the hyper-surface constraint equations which relate the connection coefficients in (2.7) with certain projections of the Ricci tensor $R_{\mu\nu}$ at points of the hypersurface $\mathcal{H}$. When expressed in terms of the hypersurface data $\mathcal{D}$ (2.11) these mathematical identities take the form of a set of equations of motion, which we will now review. A formal analysis of these equations can be found in [57, 64], and their application to null hypersurfaces is reviewed in detail in [58]. Since our conventions do not match the ones in these references, for completeness we have included a derivation of these formulae in appendix A. The relevant equations are:

- Raychaudhuri equation:

\[
\partial_n \theta - \theta \kappa + \Theta_{MN} \Theta^{MN} = J_{nn}.
\] (2.12)

- Damour–Navier–Stokes equations:

\[
\partial_n \Omega_M - \partial_M \kappa + \partial_\ell \Omega_{\ell} + D_N \Theta^N_M - D_M \theta = -J_{nM}.
\] (2.13)

- Equation for the transverse connection:

\[
\partial_n \Xi_{MN} = -\frac{1}{2}D_M \Omega_N - \Omega_M \Omega_N - (\kappa + \frac{1}{2} \theta) \Xi_{MN} + \Xi_{P(M} \Theta^{P}_N) - \frac{1}{4} \Theta_{MN} \theta^\ell + \frac{1}{4} \mathcal{R} q_{MN} + \frac{1}{2} J_{MN}.
\] (2.14)

Here $\theta \equiv \Theta^0_M$ is the expansion of the null hypersurface, and $\theta^\ell \equiv \Xi^\ell_M$. The symbols $D_M$ and $\mathcal{R}$ denote the Levi-Civita connection of $q_{MN}$ and the associated Ricci scalar, respectively. The equations also involve the tensor $J_{ab}$ defined on $\Sigma$ in terms of its components in the basis $\mathcal{B} = \{ \hat{e}_a \}$

\[
J_{nn} \equiv -R(n,n)|_{\phi(p)}, \quad J_{nM} \equiv -R(n,e_M)|_{\phi(p)}, \quad J_{MN} \equiv -R(e_M,e_N)|_{\phi(p)}.
\] (2.15)
where $\mathbf{R}(e_a, e_b) = R_{\mu\nu}e^\mu_a e^\nu_b$ represent projections of the spacetime Ricci tensor. The constraint equations (2.12)–(2.14) simplify considerably in the particular case of non-expanding horizons which are embedded in vacuum. If the Ricci tensor $R_{\mu\nu}$ is consistent with Einstein’s field equations (2.3), the quantities (2.15) can be associated to projections of the energy–momentum tensor on the basis $\mathcal{B}$. Therefore they are all vanishing $J_m = J_M = J_{MN} = 0$ in vacuum $T_{\mu\nu} = 0$. By definition, a non-expanding horizon is a null hypersurface which has a vanishing expansion $\theta$ [47, 50] (see also [58]), and then, due to the vacuum Raychaudhuri equation (2.12), we must have

$$\text{Non-expanding horizon : } \theta = 0 \implies \frac{1}{2} \partial_n q_{MN} = \Theta_{MN} = 0.$$  \hspace{1cm} (2.16)

As a consequence, the spatial metric $q_{MN}$ induced on the sections of a non-expanding horizon $\mathcal{S}_\xi$ is independent on the null coordinate $\xi^1$. Moreover, for a NEH the equations (2.13) and (2.14) reduce to

$$\partial_n \Omega_M = \partial_M \kappa,$$  \hspace{1cm} (2.17)

$$\partial_n \Xi_{MN} = -\frac{1}{2} D(M \Omega_N) - \kappa \Xi_{MN} - \Omega_M \Omega_N + \frac{1}{4} q_{MN} R,$$  \hspace{1cm} (2.18)

where we have already imposed the vacuum Einstein’s equations.

As we shall review in section 5, null infinity $\mathcal{I}$ can be described as a non-expanding hypersurface with $\Theta_{MN} = 0$ using Penrose’s conformal framework, and its structure is also constrained by (2.12)–(2.14), which are mathematical identities satisfied by any null hypersurface. However, non-expanding horizons and null infinity have very different dynamical behaviour, and in particular, the constraints (2.17) and (2.18) are not valid for $\mathcal{I}$. One of the reasons is that the geometric data of $\mathcal{I}$ is only defined up to conformal transformations, what requires introducing appropriate equivalence classes of data sets [53]. The other important difference with NEHs is that the Ricci tensor defined on the conformal completion of spacetime does not satisfy the ordinary Einstein’s equations, and thus a specific treatment is required for $\mathcal{I}$.

2.3. Newman–Penrose null tetrad and Weyl scalars

The set of constraint equations (2.12)–(2.14), ensures the consistency of the connection coefficients in (2.7) with the trace part of the ambient-space Riemann tensor, i.e. the Ricci tensor $R_{\mu\nu}$. Therefore, it is possible to obtain further constraints requiring that the extrinsic geometry of $\mathcal{H}$ to be compatible with the traceless part of the curvature, that is, with the Weyl tensor $C_{\mu\nu\rho\sigma}$. The Weyl tensor has 10 independent components which can be collected in the form of five independent complex scalars $\Psi_n$, with $n = 0, \ldots, 4$, the so called Weyl scalars.

In order to define the Weyl scalars, first we have to introduce a Newman–Penrose null tetrad (see e.g. [65, 66]), what can be done in our framework as follows. At any given point $\xi_0^M$ of the spatial sections $\mathcal{S}_\xi$, it is possible to find a set of coordinates $\xi^M$ such that the spatial metric $q_{MN}$ has the simple form $q_{MN}(\xi_0) = \delta_{MN}$. Note that for NEH horizons this choice is independent of $\xi^1$ as $\partial_n q_{MN} = 0$. In this way we ensure that the two basis vectors $e_M|\xi^M$ are orthogonal to each other and have unit norm. Then, we can construct the Newman–Penrose null tetrad $\mathcal{B}_{\xi^0} = \{ n, \ell, m, \bar{m} \}$ at $\{ \xi^1, \xi_0^M \}$ comprised of the normal vector $n$, the rigging $\ell$, and the two complex null vectors $m|\xi^\ell \equiv \frac{1}{\sqrt{2}}(e_2 + ie_3)|\xi^\ell$, $m|\xi^\ell \equiv \frac{1}{\sqrt{2}}(e_2 - ie_3)|\xi^\ell$.  \hspace{1cm} (2.19)
By defining the $B_{NP}$ in this way we avoid introducing the additional gauge freedom which is always associated to the choice of null tetrad. The set of vectors $B_{NP}$ also forms a basis of the spacetime tangent space, and it is composed of null vectors only. Actually, at $\xi^M_0$ the scalar products of its elements read

$$
\begin{align*}
g(n,n) &= 0, & g(n,\ell) &= 1, & g(n,m) &= 0, & g(n,\overline{m}) &= 0, \\
g(\ell,\ell) &= 0, & g(\ell,m) &= 0, & g(\ell,\overline{m}) &= 0, \\
g(m,m) &= 0, & g(m,\overline{m}) &= 1, & g(\overline{m},\overline{m}) &= 0.
\end{align*}
$$

The Weyl scalars are defined in terms of the Newman–Penrose tetrad by

$$
\begin{align*}
\Psi_0 &= C_{\sigma\rho\mu\nu} n^\sigma m^\rho n^\mu m^\nu, & \Psi_1 &= C_{\sigma\rho\mu\nu} m^\sigma \ell^\rho n^\mu n^\nu, \\
\Psi_2 &= C_{\sigma\rho\mu\nu} n^\sigma m^\rho \ell^\mu \overline{m}^\nu, & \Psi_3 &= C_{\sigma\rho\mu\nu} \ell^\sigma \ell^\rho \overline{m}^\mu m^\nu, \\
\Psi_4 &= C_{\sigma\rho\mu\nu} \ell^\sigma \ell^\rho \ell^\mu \ell^\nu.
\end{align*}
$$

The computation of the Weyl scalars is useful to determine the Petrov type of the gravitational field (see [66]), and to characterise its different contributions [67]. In particular, $\Psi_0$ and $\Psi_4$ encode transverse wave components travelling along the directions $-\ell$ and $n$, respectively. The scalars $\Psi_1$ and $\Psi_2$ represent longitudinal wave components propagating respectively parallel to $-\ell$ and $n$, and $\Psi_2$ can be associated with a Coulomb contribution of the gravitational field [67].

The general form of the Weyl scalars in terms of the hypersurface data (2.11) can be found appendix B.2. We will present the relevant formulae when discussing case of non-expanding horizons and null infinity.

### 3. Gauge redundancies and horizon supertranslations

In the present section we will discuss in detail the gauge redundancies in our description in the case of generic non-expanding null hypersurfaces with $\Theta_{MN} = 0$, which are of interest both for the study of black hole horizons and null infinity. Part of this gauge freedom was already used in the last section to set the hypersurface metric data in the form (2.5). Then, in section 3.2 we will begin our analysis identifying the residual gauge redundancies which are left after imposing the conventions (2.5). Since the constraints (2.12)–(2.14) are a direct consequence of the Ricci identity and (2.5), these residual gauge transformations also leave invariant the form of the constraint equations.

In section 3.3, we will turn our attention to horizon supertranslations. We will define them as spacetime diffeomorphisms preserving the metric tensor on a generic NEH. We will characterise how the geometry of the horizon changes under the action of a supertranslation, and show that the transformation of the hypersurface data (2.11) can be identified with a gauge redundancy of the description. In other words, we prove that horizon supertranslations preserve both the intrinsic and extrinsic geometry of the horizon up to a gauge transformation. As a consequence, we find that supertranslations also leave invariant the form of the constraint equations (2.12)–(2.14).

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6 We use the definitions in [58] up to a difference in the conventions: in [58] the second element of the tetrad $B_{NP}$ is future directed, but in our conventions the rigging $\ell$ is past directed.

7 The vector $-\ell$ is future directed since, by convention, $g(n,\ell) = 1$. 

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It is important to stress that, the fact that two data sets can be related to each other by a
gauge transformation is necessary for them to describe the same NEH geometry. However,
the gauge equivalence of two data sets is not sufficient to prove that they correspond to
the same NEH. For this reason the results derived in this section do not yet prove that two
data sets related by a horizon supertranslation represent the same horizon geometry. Although
this might seem unnatural at first, recall that in the case of null infinity there are geometrically
distinct data sets which can be related to each other by a gauge transformation, namely by BMS
supertranslations. In this sense, BMS supertranslations should be regarded as a large gauge
transformation, i.e. a global symmetry of the constraint equations, rather than pure gauge. We
will discuss this point again in section 5.

3.1. Gauge redundancies of hypersurface data

As we described in section 2.1, the intrinsic and extrinsic geometry of a null hypersurface
can be fully encoded in a set of tensor fields defined on the abstract manifold \( \Sigma \), namely
\( \mathcal{D} = (\gamma_{ab}, \ell_a, (\ell^2), Y_{ab}) \) defined in (2.4) and (2.8) [68]. The description of the geometry of a
non-expanding horizon in terms of these quantities has some ‘built in’ gauge redundancies,
that is, different data sets \( \mathcal{D} \) and \( \mathcal{D}' \) might represent equivalent geometries. The following two
types of redundancies represent all the gauge ambiguities in our framework:

3.1.1. Coordinate freedom on the abstract manifold. We recall that the hypersurface data \( \mathcal{D} \) is
given in terms of tensor fields living on the abstract manifold \( \Sigma \), and thus its definition is unaf-
fected by coordinate reparametrisations of \( \Sigma \). Nevertheless, the explicit coordinate expressions
of these tensor fields will have a different form in different coordinate systems. This implies
that the same hypersurface geometry could be encoded in two different data sets \( \mathcal{D} \) and \( \mathcal{D}' \)
which are related to each other through a diffeomorphism of the abstract manifold \( \Sigma \). These
transformations should be regarded as a gauge freedom in the hypersurface data. Under an
arbitrary diffeomorphism on the abstract manifold \( \Sigma \), with the explicit form \( \zeta : \xi^a \to \zeta^i(\xi^a) \)
and \( i = 1, 2, 3 \), the coordinate representation of the data \( \mathcal{D} = (\gamma_{ab}, \ell_a, (\ell^2), Y_{ab}) \) transforms as
follows
\[
\begin{align*}
\gamma'_{ab} &= \zeta^* \gamma_{ab}, \\
\ell'_a &= \zeta^* \ell_a, \\
(\ell^2)' &= \zeta^* (\ell^2), \\
Y'_{ab} &= \zeta^* Y_{ab}.
\end{align*}
\]
(3.1)

Here \( \zeta^* \) is the pull-back map associated to \( \zeta \) which, for example, acts explicitly on the first
fundamental form as \( \zeta^* \gamma_{ab} = \gamma_{ij}|_{\zeta(\xi)} \zeta^i_\xi \zeta^j_\xi \), with \( \zeta^i_\xi \equiv \partial_\xi \zeta^i \).

At this point it is worth to emphasise that the abstract manifold \( \Sigma \) is a redundant object
detached from the ambient spacetime, which can be seen as a bookkeeping device used to
isolate the geometric data of the hypersurface \( \mathcal{H} \subseteq M \). Therefore the diffeomorphisms on \( \Sigma \)
should not be confused with the diffeomorphisms of the ambient spacetime \( M \).

3.1.2. Choice of rigging. The rigging \( \ell \) is an auxiliary vector field over \( \mathcal{H} \) introduced to spec-
ify a direction transversal to the hypersurface. Given a null hypersurface \( \mathcal{H} \) we could construct
the hypersurface data using two different choices of rigging, leading in general to two different
results \( \mathcal{D} \) and \( \mathcal{D}' \) which obviously represent the same geometry. To characterise the effect
of an arbitrary change of rigging, consider two different choices, \( \ell \) and \( \ell' \), related to each other
by \( \ell' = u(\ell + V) \), where \( u \) is a non-vanishing scalar function on the hypersurface \( \mathcal{H} \), and \( V \) is

\[ \text{For a detailed discussion on this type of gauge redundancies in the context of perturbation theory in general rela-
tivity see e.g. [68].} \]
some vector field tangent to it. Then, from the definition of the hypersurface data it follows that the elements characterising the metric tensor transform as [57]
\[
\gamma'_{ab} = \gamma_{ab}, \quad \ell'_{a} = \hat{u}(\ell_{a} + \hat{V}^{b}\gamma_{ab}), \quad \ell^{(2)'} = \hat{u}^{2}(\ell^{(2)} + 2\hat{V}^{a}\ell_{a} + \gamma_{ab}\hat{V}^{a}\hat{V}^{b})
\]  
(3.2)
where \(\hat{u}\) is a function in \(\Sigma\) defined by \(\hat{u} \equiv \Phi^{*}(u)\) and \(\hat{V} \in T_{p}\Sigma\) is defined by the condition \(d\Phi(\hat{V}) = V\). The tensor \(Y_{ab}\) describing the transverse connection coefficients is sent to
\[
Y'_{ab} = \hat{u}Y_{ab} + \frac{1}{2}(\partial_{a}\hat{u}\ell_{b} + \partial_{b}\hat{u}\ell_{a}) + \frac{1}{2}C_{\hat{u}\hat{V}\gamma_{ab}}.
\]  
(3.3)

3.2. Partial gauge fixing and residual gauge redundancies

In section 2.1 we introduced the conventions (2.5) to reduce the elements of the hypersurface data down to \(\mathcal{G} = (q_{MN}, \kappa, \Omega_{M}, Z_{MN})\), fixing some of the redundancies described above. In addition, we can specify a particular form for the metric \(q_{MN}\) to partially fix the coordinate reparametrisations on the spatial sections of the horizon. However, despite all these conventions there is still some residual gauge freedom. Indeed, we are still allowed to perform a diffeomorphism on \(\Sigma\) (3.1) followed by a change of rigging, (3.2) and (3.3), as long as they preserve our choice of gauge (2.5). Thus, the combined transformations must satisfy
\[
\gamma'_{ab} = \zeta^{*}\gamma_{ab} = \gamma_{ab},
\]  
(3.4)
\[
\ell'_{a} = \hat{u}(\zeta^{*}\ell_{a} + \zeta^{*}\gamma_{ab}\hat{V}^{b}) = \delta_{a}^{1},
\]  
(3.5)
\[
\ell^{(2)'} = \hat{u}^{2}(\zeta^{*}\ell^{(2)} + 2\zeta^{*}\ell_{a}\hat{V}^{a} + \zeta^{*}\gamma_{ab}\hat{V}^{a}\hat{V}^{b}) = 0
\]  
(3.6)
where \(\gamma_{mn}, \ell_{a}\) and \(\ell^{(2)}\) are given by (2.5). To determine the form of the allowed diffeomorphisms we begin solving the first equation (3.4), which reads explicitly
\[
\zeta_{b}^{1}\zeta_{a}^{1}|_{\xi(\zeta)} = \gamma_{ab}|_{\xi} \quad \iff \quad \zeta_{b}^{1}q_{b}^{I}|_{\xi(\xi)} = 0, \quad \zeta_{M}^{1}q_{M}^{I}|_{\xi(\xi)} = q_{MN}|_{\xi},
\]  
(3.7)
where \(I, J = 2, 3\). On the one hand, \(q_{MN}\) is non-degenerate, and thus the first equation on the right implies that the components of the diffeomorphism \(\zeta'(\xi)\) are constant along the null direction, \(\zeta_{b}^{1} = 0\). On the other hand, since we are restricting the analysis to non-expanding horizons \(\partial_{n}q_{MN} = 0\), the previous equations are independent of the null coordinate \(\xi^{1}\). Therefore the last equation in (3.7) implies that \(\zeta'(\xi^{M})\) must define an isometry of the metric \(q_{MN}(\xi^{M})\), while the component \(\zeta^{1}(\xi) \equiv \hat{f}(\xi)\) of the diffeomorphism can be any arbitrary function on \(\Sigma\).

Although the diffeomorphism \(\zeta'(\xi)\) leaves \(\gamma_{ab}\) invariant, without a compensating change of rigging, it leads to a non-trivial transformation of the components \(\ell_{a}\)
\[
\zeta^{*}\ell_{a}(\xi) = \zeta_{b}^{1}\ell_{1}(\xi) \quad \Rightarrow \quad \zeta^{*}\ell_{1} = \partial_{a}\hat{f} \equiv \hat{f}_{a}, \quad \zeta^{*}\ell_{M} = \partial_{M}\hat{f} \equiv \hat{f}_{M},
\]  
(3.8)
while \(\zeta^{*}\ell^{(2)} = 0\), and thus \(\ell^{(2)}\) is unchanged. The appropriate rigging transformation which compensates this change can be found solving the remaining equations (3.5) and (3.6). The solution for \(\hat{u}\) and \(\hat{V}^{a}\) reads
\[
\hat{u} = \frac{1}{f^{a}}, \quad \hat{V}^{a} = -\hat{f}^{a}, \quad \hat{V}^{1} = \frac{1}{2f^{a}}\hat{f}^{a}\hat{f}^{M}.
\]  
(3.9)
This result together with the form of the diffeomorphism, \( \zeta'(\xi) = (\hat{f}(\xi), \zeta'(\xi^M)) \), determines completely the residual gauge redundancies of our description. Then, using (3.1) and (3.3) we can find how the form of the tensor \( Y_{ab}(2.8) \) changes under this gauge transformations

\[
Y'_{ab} = \hat{u}(\zeta') Y_{ab} + \frac{1}{2} (\partial_a \hat{u}(\zeta') \ell_b + \partial_b \hat{u}(\zeta') \ell_a) + \frac{1}{2} \mathcal{L}_{\hat{u}} \gamma_{ab}.
\]

Here we have used the fact that the induced metric is invariant under the action of \( \zeta' \) for the last summand (recall (3.4)). Since the freedom to reparametrise the spatial coordinates \( \xi^M \) has no interest for our discussion, we consider for simplicity diffeomorphisms with \( \zeta' = \xi' \), and we find

\[
\begin{align*}
\kappa'(\xi) &= \kappa|_{\zeta(\xi)} \hat{f}_n + \partial_a \log \hat{f}_n, \\
\Omega_{MN}^\prime(\xi) &= \Omega_{MN}|_{\zeta(\xi)} + \kappa|_{\zeta(\xi)} \hat{f}_M + \partial_M \log \hat{f}_N, \\
\Xi_{MN}(\xi) &= \frac{1}{f_n} \left( \Xi_{MN}|_{\zeta(\xi)} - \Omega_{MN}|_{\zeta(\xi)} \hat{f}_N - \kappa|_{\zeta(\xi)} \hat{f}_M \hat{f}_N - D_M \hat{f}_N \right).
\end{align*}
\]

Thus, the previous transformations represent the gauge freedom left in the hypersurface data after setting the conventions (2.5). As these transformations leave invariant our conventions (2.5), the form of the constraint equations (2.12)–(2.14) is also left unchanged\(^9\). This implies that, if a given data set \( \mathcal{D} \) is a solution to the constraint equations, then the transformed data set \( \mathcal{D}' \) obtained via (3.11) will also satisfy the constraints.

Before we close this discussion let us single out the situation when the diffeomorphism is of the form \( \zeta'(\xi) = (\xi' + A(\xi^M), \xi') \). Then the gauge transformations have the simpler form

\[
\begin{align*}
\kappa'(\xi) &= \kappa|_{\zeta(\xi)}, \\
\Omega_{MN}^\prime(\xi) &= \Omega_{MN}|_{\zeta(\xi)} + \kappa|_{\zeta(\xi)} A_M, \\
\Xi_{MN}(\xi) &= \Xi_{MN}|_{\zeta(\xi)} - \Omega_{MN}|_{\zeta(\xi)} A_N - \kappa|_{\zeta(\xi)} A_M A_N - D_M A_N.
\end{align*}
\]

This case is particularly interesting because, as we will see in the next subsection, it is closely related to horizon supertranslations.

Note that a generic change of gauge also induces a transformation on the elements of the basis \( B = \{ n, \ell, e_M \} \), and in turn, of the components of any tensor which is expressed in terms of \( B \). The new basis elements \( n' = e'_i \) and \( e'_M \) at the point \( \xi^a \) can be obtained from their definition after acting with the pushforward \( d\zeta \) on \( e_i \), that is, \( e'_M = d\Phi(d\zeta(e_m)) = d\Phi(\zeta'\ell M e_i) \), and the new rigging \( \ell' \) from (3.9). In the case of transformations of the form (3.12) the basis elements behave as

\[
n'|_{\zeta(\xi)} = n|_{\zeta(\xi)}, \quad e'_M = e_M|_{\zeta(\xi)} + n|_{\zeta(\xi)} A_M, \quad \ell' = \ell|_{\zeta(\xi)} - e_M|_{\zeta(\xi)} A^M - \frac{1}{2} n|_{\zeta(\xi)} A^M A_M.
\]

\[3.3. \text{Horizon supertranslations and hypersurface data}\]

We will now study the effect of horizon supertranslations on the hypersurface data of a generic non-expanding horizon. We will describe supertranslations as active spacetime diffeomorphisms \( F : \mathcal{M} \to \mathcal{M} \) (as opposed to coordinate transformations), which act on the spacetime metric tensor as \( g \to F^* g \). That is, \( F \) induces a deformation of the metric tensor, while the

\(^9\)The vanishing tensor \( J_{ab} \) appearing in the constraint equations (2.12)–(2.14) can so be shown to be left invariant under (3.11) (see [57]).
coordinate charts are left invariant. More specifically, we will characterise horizon supertranslations as diffeomorphisms which leave invariant the horizon as a set of points, and which preserve the full metric tensor on it (see [8, 15, 20, 35–39], [41, 46]). These two conditions can be expressed in a coordinate invariant way as follows

$$F(\mathcal{H}) = \mathcal{H}, \quad \text{and} \quad F^*g(X,Y)_p = g(X,Y)_p.$$  

(3.14)

for all pairs of vectors $X,Y \in T_p\mathcal{M}$ in the spacetime tangent space at points $p \in \mathcal{H}$ on the hypersurface. Actually, we shall see that diffeomorphisms satisfying (3.14) lead to a more general class of transformations than supertranslations, and we will refer to them as hyper-surface symmetries\(^{10}\). We will prove that the action of these diffeomorphisms on the NEH data can be described by the transformations (3.12). That is, hypersurface symmetries, and in particular supertranslations, leave invariant both the intrinsic and the extrinsic geometry of the hypersurface up to a gauge redundancy of the description.

Previous work has used an infinitesimal version of the definition (3.14), which can be recovered expressing $F$ explicitly in (3.14), i.e. in terms of a coordinate system $F : \mathcal{H} \rightarrow \mathcal{M}$ where $\alpha = 0, \ldots, 3$. Then, setting $y^\alpha(x) \approx x^\alpha + \epsilon k^\alpha(x)$ and working at linear order in $\epsilon \ll 1$ we find that the second condition in (3.14) reduces to $L_\ell g_{\mu\nu}|_{\mathcal{H}} = 0$, which is the definition used in [8, 15, 20, 35–39, 41, 46]. The advantage of (3.14) is that it is coordinate independent, what clarifies the geometrical interpretation of these transformations, and that it can be solved easily leading directly to the finite form of the diffeomorphisms.

It is worth mentioning that hypersurface symmetries have also been studied before in the context of horizon shells [69–71], where they represent the ‘soldering’ freedom between two spacetimes across a non-expanding null hypersurface [72, 73].

3.3.1 Spacetime coordinate system. In order to find the set of diffeomorphisms satisfying the conditions (3.14) we will first define a spacetime coordinate system adapted to $\mathcal{H}$ to simplify the derivation. As the hypersurface $\mathcal{H}$ is diffeomorphically identified with the abstract manifold $\Sigma$ via the embedding map $\Phi : \Sigma \rightarrow \mathcal{M}$, we can use the coordinate system $\{\xi^a\}$ on $\Sigma$ to parametrise points over the hypersurface. Then, we can extend these coordinates away from $\mathcal{H}$ introducing a transverse coordinate $r$, which we define by the conditions $\ell = \partial_\ell$ and $r(\mathcal{H}) = 0$, and requiring the coordinates $\{\xi^a\}$ to be constant along the integral lines of $\ell$. Strictly speaking this procedure would require one to define how the rigging is extended off the hypersurface, but the following calculation is independent of this extension, and thus we will leave it unspecified. The complete spacetime coordinate system is given by $x^\mu \equiv \{u = \xi^1, r, x^M = \xi^M\}$, and therefore it follows that the embedding map is simply

$$\Phi : \xi^a \rightarrow x^\mu = \{u = \xi^1, r = 0, x^M = \xi^M\}.$$  

(3.15)

With these choices the corresponding coordinate basis for the spacetime tangent space $\{\partial_u, \partial_r, \partial_M\}$ coincides with the basis $\mathcal{B} = \{e_1, \ell, e_M\}$ defined in section 2.1

$$n = \partial_u, \quad \ell = \partial_r, \quad e_M = \partial_M.$$  

(3.16)

In order to be consistent with our conventions, which require $n(\ell) = 1$, the normal form to the hypersurface must be given by $n = dr$. In addition, from (2.5) it follows that the metric tensor is of the form

\(^{10}\)As a consistency check for this approach we have also rederived the full BMS group at null infinity—including BMS supertranslations—using similar techniques. See appendix C.1.
\[ g_{\mu\nu}(x)|_{r=0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & q_{MN} \end{pmatrix}, \quad \text{with} \quad q_{MN} = q_{MN}(x^\ell), \] (3.17)

at points on the hypersurface \( \mathcal{H} \), which is located at \( r = 0 \). In the following we will use \( \dot{\ldots} \) to write equations which hold on the hypersurface \( \mathcal{H} \), that is, at \( r = 0 \). Note also that the normal vector is given \( n = g^{-1}(n, \cdot) = e_1 \), in consistency with the setting defined in section 2.1. For later reference, note that the first derivatives of the metric have the form

\[
\partial_{\ell} g_{\mu\nu} \equiv 0, \quad \partial_{\ell} g_{\mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_{\ell} q_{MN} \end{pmatrix}, \quad \frac{1}{2} \partial_{\ell} g_{\mu\nu} \equiv \begin{pmatrix} -\kappa & -\Omega_\text{M} \\ \cdot & \cdot \\ -\Omega_N & \Xi_{MN} \end{pmatrix},
\] (3.18)

where the last equality follows directly from the definition of the tensor \( Y_{\alpha\beta} \), and the empty entries are those which cannot be determined from the hypersurface data alone. With this information we are ready to find those diffeomorphisms \( F \) satisfying the conditions (3.14), and to characterise their action on the hypersurface data.

### 3.3.2. Hypersurface symmetries.

The first condition on (3.14) requires that the diffeomorphism \( F \) maps the hypersurface to itself. Since the null normal \( n \) of a hypersurface is unique up to a scale, \( F \) can only change the normalisation of \( n \), that is \( F^* n = \lambda_1 n \), where \( \lambda_1 \) is some function on \( \mathcal{H} \). In the coordinate system defined above this condition has the explicit form

\[ y^\alpha_\mu n_\alpha(y(x)) \equiv \lambda_1 n_\mu(x) \iff y^\alpha_\mu \equiv \lambda_1 \delta^1_\mu \iff \lambda_1 \equiv y^1_\mu, \] (3.19)

where we are using the shorthand \( y^\alpha_\mu \equiv \partial_{\ell} y^\alpha \). Moreover, since the metric tensor \( g \) should also be preserved by \( F \), the normal vector \( n = g^{-1}(n, \cdot) \) can only change its normalisation under the action of the diffeomorphism \( F \), i.e. \( dF(n) = \lambda_2 n \), where \( \lambda_2 \) is some function over \( \mathcal{H} \). The explicit form of this condition is

\[ \lambda_2 n^\alpha(y(x)) \equiv y^\alpha_\mu n^\mu(x) \iff \lambda_2 \delta^\alpha_\mu \equiv y^\alpha_\mu \iff \lambda_2 \equiv y^0_\mu. \] (3.20)

The following sequence of identities shows that the transformation of the null normal and the normal vector are related by the condition \( \lambda_1 = \lambda_2^{-1} \)

\[
1 \equiv g(\ell, n) \equiv F^* g(\ell, n) \equiv g(dF(\ell), dF(n)) \equiv \lambda_2 g(dF(\ell), n) \equiv \\
\lambda_2 n(dF(\ell)) \equiv \lambda_2 F^* n(\ell) \equiv \lambda_1 \lambda_2 n(\ell) \equiv \lambda_1 \lambda_2, \] (3.21)

where we have used the transformation properties of the metric tensor, the normal form and the normal vector, and the definition of the pullback map (see e.g. [74]). Summarising, the first condition in (3.14) implies that the diffeomorphism should satisfy the following constraints at points on the hypersurface

\[ \partial_{\ell} y^1_\mu \equiv \partial_{\ell} y^1_\mu \equiv 0, \quad \partial_{\ell} y^0_\mu \equiv 0, \quad \text{and} \quad \partial_{\ell} y^0_\mu \equiv 1/\partial_{\ell} y^1_\mu. \] (3.22)

The second condition in (3.14) is satisfied if and only if the diffeomorphism preserves all the scalar products between the elements of the basis \( \mathcal{B} \). From the results obtained above it is straightforward to check that the requirements

\[ F^* g(n, \ell) \equiv g(n, \ell), \quad F^* g(n, n) \equiv g(n, n), \quad \text{and} \quad F^* g(n, e_\text{M}) \equiv g(n, e_\text{M}) \] (3.23)

are satisfied already without imposing further conditions. In order for \( F \) to preserve the remaining scalar products the following equations must hold
\[ F^*g(e_M, e_N) \equiv g(e_M, e_N) : \quad g_{\alpha \beta}^\gamma \gamma_{\alpha}^\gamma \gamma_{\beta}^\gamma \equiv g_{MN} \iff q_{\alpha \beta} Y_{\alpha}^I Y_{\beta}^J \equiv q_{MN}. \]  

(3.24)

\[ F^*g(\ell, e_M) \equiv g(\ell, e_M) : \quad g_{\alpha \beta}^\gamma \gamma_{\alpha}^\gamma \gamma_{\beta}^\gamma \equiv 0 \iff \gamma_{\alpha}^0 \equiv -\frac{1}{f_u} f^M Y_{\alpha}^M. \]  

(3.25)

\[ F^*g(\ell, \ell) \equiv g(\ell, \ell) : \quad g_{\alpha \beta}^\gamma \gamma_{\alpha}^\gamma \gamma_{\beta}^\gamma \equiv 0 \iff \gamma_{\alpha}^0 \equiv -\frac{1}{2f_u} f^M f_{\alpha}^M. \]  

(3.26)

where we have defined \( f(u, x^M) \equiv \gamma^0(x)|_{u=0}, \ Y^I(x^M) \equiv \gamma^I(x)|_{u=0}, \) and \( f_u \equiv \partial_u f. \) The indices \( MN \) are raised and lowered with the metric \( q_{MN} \) and its inverse \( q^{MN}, \) and we also used a prime to denote quantities evaluated at \( y(x), \) e.g. \( g_{\alpha \beta}^\gamma \equiv g_{\alpha \beta}(y(x)). \)

We can conclude that the action of hypersurface symmetries at points of the hypersurface \( \mathcal{H} \) is completely determined by the following functions

\[ \gamma^0(x) \equiv f(u, x^M), \quad \gamma^I(x) \equiv Y^I(x^M), \]  

(3.27)

where \( f(u, x^M) \) has an arbitrary dependence on its variables, while the components \( Y^I(x^M) \) are independent on \( u \) due to (3.22). Moreover, equation (3.24) implies that \( Y^I \) must define an isometry of the \( u \) -- independent metric \( q_{MN}. \) Note that the conditions (3.14) only constrain the form of hypersurface symmetries at points of the hypersurface \( \mathcal{H}, \) and then, their extension away from \( \mathcal{H} \) is arbitrary.

The diffeomorphisms we just found are more general than the supertranslations discussed in [15, 35–39, 41, 46], which correspond to the case when the functions (3.27) are of the form

\[ f(u, x^M) = u + A(x^M), \quad \text{and} \quad Y^I(x^M) = x^I. \]  

(3.28)

Actually, supertranslations can be singled out noting that, in addition to (3.14), they also also preserve the normalisation of the null normal, namely \( F^*n = n, \) which implies \( f_u = 1. \)

### 3.3.3. Effect on the horizon geometry

To conclude this section we will discuss the effect that hypersurface symmetries have on the data \( \mathcal{D} = \{ \gamma_{ab}, \ell^a, \ell^{(2)}, Y_{ab} \}. \) Note that, since the diffeomorphisms satisfying (3.14) preserve the scalar products on \( \mathcal{H}, \) they also leave invariant the metric data of the horizon \( \{ \gamma_{ab}, \ell^a, \ell^{(2)} \}, \) given by (2.4) and (2.5), and in particular its intrinsic geometry. Therefore all that remains to compute is the effect of these diffeomorphisms on the geometric data encoded in the tensor \( Y_{ab}. \) The form of the tensor \( Y_{ab} \) after a horizon supertranslation can be obtained doing the substitution \( g \to F^*g \) in its definition (2.8)

\[ Y'_{ab} = \frac{1}{2} e_a^\mu e_b^\nu \mathcal{L}_\ell (F^*g)_{\mu \nu} = \frac{1}{2} \varepsilon^\mu e_a^\mu e_b^\nu \partial_b \left( g_{\alpha \beta}(y) Y_{\alpha}^\mu Y_{\beta}^\nu \right) \]

\[ = \frac{1}{2} \left( g_{\alpha \beta} \gamma_I \gamma^I \gamma_{\alpha}^\mu \gamma_{\beta}^\nu + g_{\alpha \beta} \gamma^\mu \gamma_{\alpha}^\nu \gamma_{\beta}^\nu + g_{\alpha \beta} \gamma_{\alpha}^\mu \gamma_{\beta}^\nu \right) e_a^\mu e_b^\nu. \]  

(3.29)

with \( e_a = \{ n, e_M \}. \) The relevant derivatives of the metric tensor have been given in (3.18), the first and second derivatives of \( \gamma^a(x) \) can be determined from (3.22) and (3.24)–(3.26), and the basis vector components \( e_a^\mu \) are defined in (3.16). As in the previous subsection we also assume for simplicity that \( \gamma^I(x^M) = x^I. \) After a long but straightforward computation we obtain

\[ Y'_{mn}(\xi) = -\kappa_{\xi}(\xi) \hat{f}_n - \partial_n \log \hat{f}_m, \]

\[ Y'_{mh}(\xi) = -\Omega_{M}\xi(\xi) - \kappa_{\xi}(\xi) \hat{f}_m - \partial_M \log \hat{f}_n, \]

\[ Y'_{MN}(\xi) = \frac{1}{f_u} (\Xi_{MN}|_{\xi(\xi)} - \Omega_{(M}\xi|_{\xi(\xi)} \hat{f}_N - \kappa_{\xi(\xi)} \hat{f}_M \hat{f}_N - D_M \hat{f}_N) \]

(3.30)
where \( \zeta'(\xi) = \tilde{f}(\xi, \xi') \) is a diffeomorphism of the abstract manifold \( \Sigma \) defined as \( \zeta \equiv \Phi^{-1} \circ F \circ \Phi \), in terms of the embedding map \( \Phi \) of the hypersurface (3.15). Note that the diffeomorphism \( \zeta(\xi) \) is well defined, as \( F \) maps the hypersurface \( \mathcal{H} \) onto itself. It is straightforward to identify the behaviour of the tensor \( Y_{ab} \) under a hypersurface symmetry with the effect of a residual gauge transformation (3.11). In particular, the action of supertranslations (3.28) on the horizon data is identical to (3.12). Thus, in the following we will make no distinction between horizon supertranslations and the gauge transformations acting as (3.12).

The same conclusion can be reached analysing the effect of hypersurface symmetries in the tensor \( Y_{ab} \) in terms of the diffeomorphism \( \zeta \), without making use of the explicit spacetime coordinate system constructed above. Using the well-known property \( L_\ell(F^*g) = F^*(L_{dF(\ell)}g) \) (see e.g. [75]) one finds the following equalities

\[
Y' = \frac{1}{2} \Phi^* L_\ell(F^*g) = \frac{1}{2} (F \circ \Phi)^* L_{dF(\ell)}g = \frac{1}{2} (\Phi \circ \zeta)^* L_{dF(\ell)}g = \zeta^* \left( \frac{1}{2} \Phi^* L_{dF(\ell)}g \right).
\]

The quantity in parentheses in the last term can be regarded as the tensor \( Y|_{\zeta(\xi)} \) changed under a rigging transformation (3.3), which is followed by a diffeomorphism in the abstract manifold. These are the two gauge redundancies considered in section 3.2.

The results presented in this section have two main consequences: on the one hand hypersurface symmetries, and in particular supertranslations, leave invariant both the intrinsic and extrinsic geometries of the horizon up to a gauge redundancy. On the other hand, since the residual gauge redundancies (3.11) leave the equations (2.12)–(2.14) unchanged, diffeomorphisms acting as hypersurface symmetries also preserve the form of the constraint equations of null hypersurfaces. That is, supertranslations are a symmetry of the NEH constraint equations.

The next step is to determine the action of these transformations on the dynamical degrees of freedom on the horizon. In other words, we need to find a free data set necessary and sufficient to describe the full horizon geometry, and then we have to characterise the action of supertranslations on such data set. Depending on whether these diffeomorphisms have a non-trivial action on the NEH free data or not, we will identify them as large gauge transformations (i.e. global symmetries of the constraint equations) or pure gauge redundancies of our description. In section 4 we will consider the constraint equations (2.12)–(2.14) for a NEH in order to single out an appropriate free data set, and characterise the geometric nature of horizon supertranslations. For completeness in section 5 we will perform a similar analysis for BMS supertranslations acting on null infinity.

### 4. Evolution of non-expanding horizons

In the present section we will study the solutions to the constraint equations of non-expanding horizons embedded in vacuum, (2.18) and (2.17). Our main objective is to extract a free data set, \( \mathcal{D}_{\text{free}} \), necessary and sufficient to reconstruct the full NEH geometry, and then to determine the behaviour of the free data set under supertranslations. Our starting point is the data set (2.11) presented in section 2.1, which contains sufficient information to characterise completely the NEH geometry. However the data set (2.11) still involves some residual gauge freedom, which we characterised in section 3.2. Moreover, the data elements of (2.11) are not freely specifiable, as they are subject to the constraints (2.18) and (2.17).

Our strategy will be, first, to reduce the residual gauge freedom (3.11) imposing appropriate gauge fixing conditions, so that the only remaining ambiguity is (3.12), which we associated to supertranslations in section 3.3. Then, we will turn to the resolution of the
constraints (2.18) and (2.17), and we will present a free data set, \( D_{\text{free}} \), composed of quantities invariant under supertranslations. Note that, since this free data set involves no unfixed gauge redundancies, all of its elements are necessary to describe the NEH geometry. Finally, we will check explicitly that our free data set encodes all the information about the spacetime curvature which was contained in the original data set (2.11). With this result we conclude that the supertranslation invariant data set \( D_{\text{free}} \) is both necessary and sufficient to reconstruct the NEH geometry, and that supertranslations act trivially on the dynamical variables of the NEH.

In order to simplify the analysis of the gauge redundancies and constraint equations (2.18) let us introduce a more convenient set of variables than (2.11). The hypersurface data element \( \Omega_M \) defines a one-form on the spatial sections of the horizon \( S_{\xi_1} \approx S^2 \), and therefore we can decompose it uniquely as the sum of an exact part \( \Omega_M^e \) and a divergence free part \( \Omega_M^0 \)

\[
\Omega_M = \Omega_M^e + \Omega_M^0, \quad \Omega_M^0 \equiv \partial_M \eta, \quad D^M \Omega_M^0 = 0
\]  

(4.1)

where \( \eta(\xi) \) is a smooth function of the coordinates. The previous equation represents the Hodge decomposition of \( \Omega_M \) on \( S_{\xi_1} \), which determines \( \Omega_M^0 \) uniquely and the potential \( \eta(\xi) \) is defined up to a shift, \( \eta \rightarrow \eta + \eta_0 \), where \( \partial_M \eta_0 = 0 \). Due to the properties of the Hodge decomposition, the exact and divergence free part of each side of the Damour–Navier–Stokes equations (2.17) are separately equal, leading to

\[
\partial_n \Omega_M^0 = 0, \quad \partial_n \partial_M \eta = \partial_M \kappa.
\]  

(4.2)

In particular, this implies that the divergence free part of the Hajicek one form is constant along the null direction. Moreover, the equation on the right can be solved in terms of \( \eta(\xi) \) requiring it to satisfy \( \kappa = \partial_n \eta \), what determines \( \eta \) uniquely up to an additive constant on the horizon. The potential \( \eta \) can also be defined in a more covariant way in terms of a decomposition of the rotation one-form \( \omega_a = (\kappa, \Omega_M) \), which we defined in (2.9). More specifically, if \( \omega_a \) is a solution of (2.17) it can be decomposed as (see [49, 76])

\[
\omega_a = \partial_a \eta + \omega_a^0, \quad \text{where} \quad \omega^0(\eta) = 0 \quad \text{and} \quad D^M \partial_M \eta = D^M \Omega_M.
\]  

(4.3)

Here \( \omega_a^0 \) is uniquely determined to be \( \omega_a^0 = (0, \Omega_M^0) \), and \( \eta(\xi) \) is defined up to a constant shift. As we shall see in section 4.2 the potential \( \eta \) will play an essential role to solve (2.18) in terms of quantities which are invariant under supertranslations. Finally, let us also introduce the following combination\(^{11}\)

\[
\Sigma_{MN}^0 \equiv \frac{1}{2} D(M \Omega_N) + \Omega_M \Omega_N + \kappa \Xi_{MN},
\]  

(4.4)

together with its trace \( \theta^0 \equiv \Sigma_{MN}^0 \) and its traceless part \( \sigma_{MN}^0 \). From the definitions (4.3) and (4.4), it is straightforward to check that the NEH data (2.11) can be equivalently encoded in a new set of quantities \( \mathcal{D} \)

\[
\mathcal{D} = (q_{MN}, \ \kappa, \ \Omega_M, \ \Xi_{MN}) \quad \longrightarrow \quad \mathcal{D} = (q_{MN}, \ \eta, \ \Omega_M^0, \ \Sigma_{MN}^0).
\]  

(4.5)

As we will see below the new data set \( \mathcal{D} \) has particularly simple transformation properties under supertranslations.

\(^{11}\)The quantity \( \Sigma_{MN}^0 \) can be defined covariantly in terms of the rotation one form \( \omega_a \). Using the connection \( \nabla \) defined by equation (17) of [57] we have \( \Sigma_{MN}^0 \equiv \frac{1}{2} \nabla_{(M} \omega_{N)} + \omega_{(M} \Sigma_{N)0} \).
4.1. Reduction of the gauge freedom

We now introduce the relevant gauge fixing conditions so that the residual gauge freedom (3.11) reduces to the transformations (3.12), which we identified with supertranslations. Given an arbitrary non-expanding horizon with surface gravity $\kappa$, it is always possible to choose a gauge where the surface gravity is a constant $\kappa_0$ over the horizon.

**Gauge condition 1**: $\partial_n \kappa_0 = \partial_M \kappa_0 = 0$ and $\kappa_0 > 0$ for all $\xi \in \Sigma$. \hfill (4.6)

This gauge can be achieved making a transformation of the form (3.11) with the function $\hat{f}(\xi)$ satisfying

$$\kappa(\xi) \hat{f}_n + \partial_n \log \hat{f}_n = \kappa_0,$$ \hfill (4.7)

which can always be solved for $\hat{f}(\xi)$. It is straightforward to check that, in this gauge, the equation (2.17) implies that the full Hajicek one-form $\Omega_M$—and not only $\Omega^0_M$—must be constant along the null direction of the horizon, $\partial_n \Omega_M = 0$.

The previous gauge fixing condition still does not reduce the redundancies down to supertranslations. Indeed, after imposing the condition (4.6) on the data, the remaining gauge freedom can be found solving again equation (4.7), but this time setting $\kappa(\xi) = \kappa_0$, which gives

$$\hat{f}(\xi) = \xi^1 + A(\xi^M) + \frac{1}{\kappa_0} \log \left(1 + B(\xi^M) e^{-\kappa_0 \xi^1}\right),$$ \hfill (4.8)

where $B(\xi^M)$ and $A(\xi^M)$ are smooth functions satisfying $\partial_n A = \partial_n B = 0$. At this point we can already identify $A(\xi^M)$ with the freedom to perform a supertranslation (3.12). Therefore it only remains to find a convention to eliminate ambiguity associated to $B(\xi^M)$, which reflects the fact that the gauge condition (4.7) does not determine completely the normalisation of the null normal $n$.

The analysis in the following sections is independent of the actual method to fix the normalisation of $n$, what was discussed for example in [50, 58, 76]. Here we will follow the strategy of [50], which consists in imposing a gauge fixing condition on $\theta^0 = \Xi^M_M$.

Contracting the constraint equation for $\Xi_{MN}$ (2.18) with $q_{MN}$, and using that the surface gravity $\kappa_0$ and $\Omega_M$ are constant along $\xi^1$, we find

$$\partial_n \theta^0 = -\kappa_0 (\theta^0 - \frac{1}{2} R),$$ \hfill (4.9)

where the quantity $^{12} \theta^0$ was defined in (4.4), and $R$ is the scalar curvature associated to $q_{MN}$. Note that in this equation only $\theta^0$ has a non trivial dependence on $\xi^1$, since $\partial_n q_{MN} = 0$ also implies $\partial_n R = 0$, and thus it can be integrated easily

$$\theta^0(\xi) = (\theta_0^1 |_{\xi_0^1} + \frac{1}{2} R) e^{-\kappa_0 (\xi^1 - \xi_0^1)} + \frac{1}{2} R.$$ \hfill (4.10)

As was discussed in [50], for a subclass of non-expanding horizons, known as **generic non-expanding horizons**, it is possible to perform a transformation of the form (4.8) in order to make $\theta^0$ stationary on the horizon.

**Gauge condition 2**: $\partial_n \theta^0 = 0$ for all $\xi \in \Sigma$. \hfill (4.11)

This condition is trivially satisfied by all black holes in the Kerr family, which are stationary, and thus they have generic horizons (see e.g. appendix D in [58]). As we show in appendix

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12 The quantity $^{12} \theta^0$ should not be confused with the trace of the second fundamental form, the expansion $\theta \equiv \Theta^M_M$, which is zero for non-expanding horizons.
B.1. if the hypersurface data of a NEH satisfies the following condition on a spatial section $S_{\xi}$ of $\Sigma$

$$\Omega_{\xi}^{0M} \sigma_{\xi M}^{0} \leq \frac{1}{2} \mathcal{R} \leq \theta^0 \quad \text{for all } \xi \in S_{\xi}, \tag{4.12}$$

the horizon can be shown to be generic. In other words, it is possible to find a gauge transformation of the form (4.8) which allows one to set $\partial_n \theta^0 = 0$ everywhere on $\Sigma$. Moreover, the residual gauge freedom left after imposing this gauge fixing condition is precisely that of supertranslations (3.12). To the best of our knowledge the condition (4.12) has not been presented before in the literature. In the following we will restrict ourselves to horizons where the gauge (4.11) can be attained.

After imposing the gauge fixing conditions (4.6) and (4.11), the only remaining gauge freedom are supertranslations (3.12). We will now characterise the behaviour under supertranslations of the elements in the data set $\mathcal{D}_t$ (4.5). The transformation properties of the spatial metric $q_{MN}$ were discussed in section 3.2, and it was shown to be invariant under (3.12). Since the surface gravity $\kappa_0$ has been set to a constant, the exact and divergence free parts of the Hajicek one-form transform under supertranslations as

$$\Omega_{\xi}^{0M} \sigma_{\xi M}^{0} = \Omega_{\xi}^{0M} \sigma_{\xi M}^{0} |_{\zeta(\xi)} + \kappa_0 A^M(\xi),$$

where $\zeta^a = (\xi^1 + A(\xi^M), \xi^M)$. Actually, due to (4.2) the divergence free part of the Hajicek one-form satisfies $\partial_a \Omega^{0M}_{\xi} = 0$, and thus $\Omega^{0M}_{\xi}$ is completely invariant. In this gauge the functional form of the potential $\eta(\xi)$ reduces to $\eta(\xi) = \kappa_0 \xi^1 + h(\xi^M)$, with $\partial_a h = 0$, and it behaves under supertranslations as $^{13}$

$$\eta(\xi) \rightarrow \eta'(\xi) = \kappa_0 \xi^1 + h(\xi^M) + \kappa_0 A(\xi^M). \tag{4.14}$$

The previous expression can also be written as $\eta'(\xi) = \eta(\zeta(\xi))$, which means that $\eta$ transforms under a supertranslation as a scalar field. Finally, it is easy to check that the object $\Sigma_{MN}^{0}$, defined in (4.4), also transforms as a scalar under (3.12)

$$\Sigma_{MN}^{0} \rightarrow \Sigma_{MN}^{0} |_{\zeta(\xi)}. \tag{4.15}$$

Thus, from (4.13)–(4.15), it follows that all the elements of the NEH data set $\mathcal{D}_t$ (4.5) are either invariant or transform as a scalar field under supertranslations.

4.2. Resolution of the hypersurface constraint equations

In this subsection we will consider the constraint equations of the NEH, and we will present a free data set $\mathcal{D}_t$ composed of quantities which are all invariant under supertranslations.

The data set $\mathcal{D}_t$ (4.5) is subject to the following complete set of constraint equations and gauge fixing conditions

$$\partial_a q_{MN} = 0, \quad \partial_a \Omega_{M}^{0} = 0, \quad \theta^0 = \frac{1}{2} \mathcal{R}, \quad \partial_a \sigma_{MN}^{0} = -\kappa_0 \sigma_{MN}^{0}, \tag{4.16}$$

and the potential has to be of the form $\eta(\xi) = \kappa_0 \xi^1 + h(\xi^M)$, where $\partial_\xi \kappa_0 = \partial_\xi h = 0$, due to (4.6). Let us recapitulate the origin of these equations from left to right: the first one is a consequence of the non-expanding condition and the Raychaudhuri equation, (2.16); the second one follows from the DNS equation (4.2); the third and fourth ones are obtained expressing the equation (2.18) for the transverse connection in terms of $\Sigma_{MN}^{0}$ (4.4), and then decomposing it

$^{13}$ The behaviour of $\eta(\xi)$ under (3.12) should be derived from its definition in (4.3).
in its trace $\theta^0$, and traceless $\sigma^0_{MN}$ parts. To obtain the last two equations we also used the gauge
fixing conditions (4.6) and (4.11).

The equations (4.16) imply that the full NEH geometry can be reconstructed from an initial
data set specified on a spatial section $S_{\xi^1}$ of the horizon

$$(q_{MN}|_{S_{\xi^1}}, \Omega^0_M|_{S_{\xi^1}}, \sigma^0_{MN}|_{S_{\xi^1}}),$$

(4.17)

and providing, in addition, the scalar potential $\eta(\xi) = \kappa_0 \xi^1 + h(\xi^M)$. This initial data set, and
the quantities $\kappa_0$ and $h(\xi^M)$, can be chosen freely on a given spatial slice $S_{\xi^1}$, and then the
geometry over the entire NEH can be obtained solving (4.16).

As we discussed above, all the elements in the data set $D_s$ transform as scalar fields under
supertranslations (3.12), implying that the initial data (4.17) may still involve gauge depend-
ent quantities. Let us examine the transformation properties of the elements in (4.17) under
supertranslations:

- The gauge freedom (3.12) is defined in terms of active diffeomorphisms on $\Sigma$, which
  transform the NEH data but leave the coordinate system unchanged. In consequence, the
  initial slice $S_{\xi^1}$ (defined by $\xi^1 = \xi^1_0$) does not transform under supertranslations.
- The objects $q_{MN}$ and $\Omega^0_M$ do not depend on the null coordinate due to (4.16), and thus
  $q_{MN}|_{S_{\xi^1}}$ and $\Omega^0_M|_{S_{\xi^1}}$ are invariant under supertranslations.
- The potential $\eta(\xi) = \kappa_0 \xi^1 + h(\xi^M)$ has a non trivial dependence on the null coordinate,
  and so does $\sigma^0_{MN}$ unless it is strictly zero (see equation (4.16)). Therefore, in general,
  both the potential $\eta(\xi)$, and the initial value $\sigma^0_{MN}|_{S_{\xi^1}}$ will transform non-trivially under
  supertranslations.

At this point, we could impose appropriate gauge fixing conditions to eliminate the ambi-
guity associated with the transformations$^{14}$ (3.12). However, following the strategy used for
null infinity in [53], we will deal with this redundancy introducing a supertranslation invariant
free data set, and proving that it contains the same information about the spacetime geometry
as the original data (2.11). This is a rigorous way to ensure that we do not exclude physically
allowed configurations of the NEH.

4.2.1. Supertranslation invariant data. In order to define a free data set which is composed of
quantities invariant under supertranslations it is convenient to parametrise the null direction of
the horizon using the scalar potential $\eta(\xi)$. Note that this parametrisation is well defined, since
$\eta(\xi)$ increases monotonically along the null direction everywhere in $\Sigma$, i.e. $\partial_n \eta = \kappa_0 > 0$.

Then, using the fact that both $\sigma^0_{MN}(\xi)$ and $\eta(\xi)$ transform as scalar fields under supertransla-
tions, we can construct a supertranslation invariant variable expressing the evolution of $\sigma^0_{MN}$
along the null direction in terms of $\eta$. For this purpose, let us write the null coordinate in terms
of $\eta$ as $\xi^1 = H(\eta, \xi^M)$,

$$\eta(\xi) = \kappa_0 \xi^1 + h(\xi^M) \implies H(\eta, \xi^M) = \frac{1}{\kappa_0} \left( \eta - h(\xi^M) \right).$$

(4.18)

As the potential $\eta(\xi)$ changes under supertranslations (4.14), the inverse function $H(\eta, \xi^M)$
needs to transform accordingly

$^{14}$This is the approach used in the membrane paradigm for the description of black holes (e.g. see appendix D in
[62]). Other examples of this method are reviewed in [58].
\[
H(\eta, \xi^M) \rightarrow H'(\eta, \xi^M) = \frac{1}{\kappa_0}(\eta - h(\xi^M)) - A(\xi^M).
\]

(4.19)

Thus, we can characterise the evolution of \(\sigma^0_{MN}\) along the null coordinate in terms of the following supertranslation invariant variable

\[
s_{MN}(\eta, \xi^M) \equiv \sigma^0_{MN}[H(\eta, \xi^M), \xi^M].
\]

(4.20)

To prove that this object is invariant under (3.12) we just need to use its definition in combination with the transformation properties of \(\sigma^0_{MN}\) and the function \(H(\eta, \xi^M)\) under (3.12)

\[
s'_{MN}(\eta, \xi^M) = \sigma^0_{MN}[H'(\eta, \xi^M), \xi^M]
= \sigma^0_{MN}[H(\eta, \xi^M) + A(\xi^M), \xi^M] = s_{MN}(\eta, \xi^M).
\]

(4.21)

Therefore, the transformed form of \(s_{MN}(\eta, \xi^M)\) after a supertranslation is the same function of \(\eta\) as \(s_{MN}(\eta, \xi^M)\), which proves that this object is completely invariant under the action of supertranslations.

Following a similar line of argument it can also be shown that \(s_{MN}\) is invariant under gauge transformations (3.11) with \(\xi^M(\xi) = (\lambda\xi^1, \xi^M)\), where \(\lambda\) is a constant over the horizon.

4.2.2. Solution to the constraint equations. The constraint equation for \(s_{MN}(\eta, \xi^M)\) is obtained expressing the last equation in (4.16) in terms of \(\eta\) and \(s_{MN}\). Using that \(\partial_\eta H = 1/\kappa_0\) we find

\[
\partial_\eta s_{MN} = \partial_\eta H \partial_\eta \sigma^0_{MN} |_{H(\eta)} \implies \partial_\eta s_{MN} = -s_{MN},
\]

(4.22)

which has the general solution

\[
s_{MN}(\eta, \xi^M) = s_{MN}|_{\eta_0} e^{-(\eta - \eta_0)},
\]

(4.23)

and \(\eta_0\) is an arbitrary constant which reflects the ambiguity in the definition of \(\eta\). This ambiguity can also be eliminated imposing an additional condition on the data, e.g. the normalisation

\[
\frac{1}{a_H} \int_{S_\eta=0} d\xi^\alpha q s_{MN} s_{MN} = 1,
\]

(4.24)

where the integral is over the spatial section \(S_{\eta=0}\) of the horizon, \(q \equiv \sqrt{\det(q_{MN})}\) and \(a_H\) is the area of \(S_{\eta=0}\).

The result (4.23), together with the equations (4.16), imply that the full NEH geometry can be encoded in the functional form of the potential \(\eta = \kappa_0 \xi^1 + h(\xi^M)\), combined with the following free data set

\[
\text{Free horizon data : } \mathcal{D}_\text{free} \equiv (q_{MN}|_{S_{\eta_0}}, \quad \Omega^0_M|_{S_{\eta_0}}, \quad s_{MN}|_{S_{\eta_0}}).
\]

(4.25)

which is specified on a spatial slice of the horizon \(S_{\eta_0}\) defined by \(\eta = \eta_0\). The full NEH geometry can be recovered from these quantities using that \(q_{MN}\) and \(\Omega^0_M|_{S_{\eta_0}}\) are constant along the null direction of the horizon and (4.23). In particular, \(q_{MN}|_{S_{\eta_0}}\) determines the intrinsic geometry of the NEH, and \(\Omega^0_M|_{S_{\eta_0}}\) can be associated to its angular momentum aspect when \(q_{MQ}\) admits an \(SO(2)\) isometry (see [58]). It is also interesting to note that \(s_{MN}\) (which is symmetric and traceless) has two independent components, matching the number of radiative degrees of freedom of the gravitational field.

Due to (4.16), the first two elements of (4.25), \(q_{MN}|_{S_{\eta_0}} = q_{MN}|_{S_{\eta_0}}^\alpha\) and \(\Omega^0_M|_{S_{\eta_0}} = \Omega^0_M|_{S_{\eta_0}}^\alpha\), coincide with the first two elements in (4.17), which we argued to be invariant under supertranslations.
Moreover, the third element $s_{MN}|_{S_{\xi}}$ is also invariant under (3.12) due to (4.21), implying that none of the elements in (4.25) involve any unfixed gauge freedom. As a consequence, distinct data sets $\mathcal{D}_{\text{free}}$ generate gauge inequivalent NEH structures, and thus, the corresponding spacetime geometries must be different as well. In other words, all the elements in $\mathcal{D}_{\text{free}}$ are necessary to characterise completely the geometry of the NEH. Note, however, that the potential $\eta = \kappa a\xi^1 + h(\xi^M)$ is also part of the NEH initial data set, and it transforms non-trivially under supertranslations (4.14).

### 4.3. Free horizon data and the spacetime curvature

We will now show explicitly that the free data set $\mathcal{D}_{\text{free}}$ (4.25) encodes all the information about the curvature of the ambient spacetime $\mathcal{M}$ contained in the original data set (2.11). In other words, the supertranslation invariant data set (4.25) is both necessary and sufficient to characterise the entire NEH geometry, and thus no knowledge about the functional form of the potential $\eta(\xi)$ is required. In addition, using the Newman–Penrose formalism, we will argue that $\mathcal{D}_{\text{free}}$ is sufficiently general to describe radiative processes taking place at the horizon.

In the case of horizons embedded in vacuum, $T_{\mu\nu} = 0$, the curvature of the ambient spacetime is completely described by the Weyl tensor $R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}$, or equivalently, by the five Weyl scalars $\Psi_n$, with $n = 0, \ldots, 4$. Since the connection coefficients (2.7) only characterise the spacetime connection along the horizon they only constrain four of the Weyl scalars. Indeed, the computation of $\Psi_4$ requires knowledge about the spacetime connection off the hypersurface, which is not available in (2.7). Therefore, all we need to show is that all the information about the spacetime curvature contained in the Weyl scalars $\Psi_n$, with $n = 0, \ldots, 3$ is also encoded in the free data set $\mathcal{D}_{\text{free}}$ (4.25).

The computation of the Weyl scalars can be done as described in section 2.3. First, without loss of generality, we specify an arbitrary point $\xi_0^M$ on the spatial sections $S_{\xi_0}$ of the horizon, choosing the coordinates $\xi^M$ such that $q_{MN}(\xi_0) = \delta_{MN}$. Then, the Weyl scalars can be obtained from (2.21) contracting the Weyl tensor with the elements of the Newman–Penrose tetrad $B_{\nu\rho} = \{n, \ell, m, \overline{m}\}$. The explicit expressions for the Weyl scalars $\Psi_n(\xi^1, \xi^M)$ are functions of the coordinates $\xi^a$, and therefore they are not invariant under diffeomorphisms of the abstract manifold. However, if we impose the gauge fixing conditions (2.5), (4.6) and (4.11) the only remaining gauge transformations are supertranslations, $\xi^1 \to \xi^1 + A(\xi^M)$. Thus, in order to deal with this freedom we introduce the *gauge corrected* Weyl scalars

$$\Psi^c_n(\eta, \xi^M) \equiv \Psi_n(\eta, \xi^M, \xi^M).$$

In the case of non-expanding horizons embedded in vacuum these quantities read

$$\Psi^c_0 = \Psi_1 = 0, \quad \Psi^c_2 = -\frac{1}{4} R + \frac{i}{2} J,$$

with $J \equiv D_\rho \Omega^\rho_{\ 3}$. (4.27)

and

$$\Psi^c_3 = \frac{1}{\kappa_0 \sqrt{2}} \left[ D_s |_{\Omega_2} e^{-(\eta - m)} + D \Psi^c_2 + 3 \tilde{\Omega}^\rho \Psi^c_2 + 3 \tilde{\Omega}^\sigma \Psi^c_2 \right].$$

(4.28)

Here we have used the notation $D \equiv (D_2 - iD_3)$, and $\tilde{\Omega} \equiv (\Omega_2 - i\Omega_3)$. We have also defined the complex field

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15 This choice determines our gauge fixing conventions (2.5) at $\xi^a = \xi^a$. As discussed in section 3 this choice is preserved by the residual gauge redundancies (3.11), including supertranslations.
\[ Ds(\eta, \xi^M) \equiv D^M S_{M2} + \Omega^{0|M} S_{M2} - i(D^M S_{M3} + \Omega^{0|M} S_{M3}). \]  

(4.29)

The details of the computation can be found in appendix B.2. The Weyl scalars \( \Psi_0^c \) and \( \Psi_1^c \) represent gravitational radiative modes which propagate into the horizon, and their vanishing can be seen as a consistency condition for the horizon to be non-expanding. The scalar \( \Psi_2^c \) encodes the coulomb contribution of the gravitational field and, when \( q_{MN} \) admits an axial killing vector field, \( \mathcal{J} \) characterises the angular momentum aspect of the NEH (see [58]). Finally, the Weyl scalars \( \Psi_3^c \) and \( \Psi_4^c \) can be associated to radiative modes propagating along the horizon.

As we explained above, the structure of the NEH does not constrain the value of the fourth Weyl scalar, and thus \textit{a priori} \( \Psi_4^c \) can take any value on \( \mathcal{H} \). Since we also have \( \Psi_0^c = \Psi_1^c = 0 \) and \( \Psi_2^c \neq 0 \), we can conclude that the Weyl tensor on a NEH will be generically of Petrov type II (see [58, 66]). Then, in general, the gravitational field on the NEH will contain a radiative component [51]. In other words, the NEH structure is sufficiently general to allow for the presence of gravitational radiation on the horizon.

It is straightforward to check that \( \Psi_0^c, \Psi_1^c \) and \( \Psi_3^c \) are invariant under supertranslations, and that \( \Psi_2^c \) can be computed from the elements in \( \mathcal{D}_\text{free} \). However, the expression (4.28) for \( \Psi_3^c \) still involves gauge dependent quantities which are not part of the free data set (4.25), namely, the surface gravity \( \kappa_0 \), which depends on the normalisation of the null normal, \( \alpha \), and the exact part of the Hajicek one form \( \Omega_{\text{Haj}} \) which transforms under supertranslations. We will now show that both quantities can be associated to well known gauge redundancies of the Newman–Penrose formalism, i.e. the freedom to perform rotations of the null tetrad \( B_{\text{NP}} \). That is, neither \( \kappa_0 \) or \( \Omega_{\text{Haj}} \) involve any information about the spacetime geometry. More specifically, we will prove that the expression (4.28) for \( \Psi_3^c \) represents the same spacetime geometry as

\[ \Psi_3^c = \frac{1}{\sqrt{2}} \left[ Ds|_{\eta_0} e^{-(\eta - \eta_0)} + \hat{D}\Psi_2^c + 3 \hat{\Omega}^0 \Psi_2^c \right], \]  

(4.30)

which is completely determined by the elements of \( \mathcal{D}_\text{free} \). Both expressions for the third Weyl scalar (4.28) and (4.30) are projections of the \textit{same} Weyl tensor associated to two different null tetrads \( B_{\text{NP}} \).

We will begin considering the surface gravity \( \kappa_0 \). The Newman–Penrose formalism has an inherent gauge freedom associated to the choice of null tetrad \( B_{\text{NP}} \) which, in a general setting, is only required to satisfy the orthogonality and normalisation conditions (2.20). Thus, when there are no further restrictions, it is possible to perform the following redefinition of the null tetrad \( B_{\text{NP}} \) which preserves (2.20)

\[ n' = \lambda n, \quad \ell' = \lambda^{-1} \ell, \quad m' = m, \quad \bar{m}' = \bar{m}, \]  

(4.31)

where \( \lambda \) is real scalar field on \( \Sigma \). From (2.21) it is immediate to check that if we transform the null tetrad as in (4.31)—keeping the Weyl tensor fixed—the gauge corrected Weyl scalars behave as (see section 8 in [66])

\[ \Psi_0^{\prime c} = \Psi_0^c = 0, \quad \Psi_1^{\prime c} = \Psi_1^c = 0, \quad \Psi_2^{\prime c} = \Psi_2^c, \quad \Psi_3^{\prime c} = \lambda^{-1} \Psi_3^c. \]  

(4.32)

That is, \( \Psi_0^c \) and \( \Psi_0^{\prime c} \) represent contractions of the same Weyl tensor with the elements of two different tetrads, \( \{ n, \ell, m, \bar{m} \} \) and \( \{ n', \ell', m', \bar{m}' \} \), respectively, and thus the two sets of Weyl scalars describe the same spacetime geometry. In our setting, the null tetrad is fully determined by the elements in the basis \( \mathcal{B} \) adapted to \( \mathcal{H} \), and thus the rotations (4.31) must always be associated to a gauge transformation (3.11) for consistency with the definition of \( \mathcal{B} \). Actually, it is possible to implement a rotation of the null tetrad of the form (4.31) performing
a transformation (3.11) with \( \zeta^a(\xi) = (\lambda \xi^1, \xi^M) \), where \( \lambda > 0 \) is an arbitrary positive constant. The corresponding change in the hypersurface data can be derived from (3.11), and the definition of \( s_{MN} \) (4.20)

\[
\mathcal{R}' = \mathcal{R}, \quad \kappa'_0 = \lambda \kappa_0, \quad Ds' = Ds, \quad \hat{\Omega}' = \hat{\Omega}. \tag{4.33}
\]

Then, using this data to compute the transformed Weyl scalars (4.27) and (4.28) it is straightforward to check that the behaviour of \( \Psi^c_n \) under these transformations is precisely (4.32), i.e. it is indistinguishable from the effect of a null tetrad rotation. This proves explicitly that gauge transformations with \( \zeta^a(\xi) = (\lambda \xi^1, \xi^M) \) and \( \partial_0 \lambda = 0 \) leave invariant the spacetime geometry, and thus \( \kappa_0 \) can be set to any arbitrary value in (4.28) without changing the geometric information encoded in \( \bar{\psi}_3^c \).

We will now discuss the role of the exact part of the Hajicek one-form \( \Omega^c_M \) in (4.28). Consider the following redefinition of the null tetrad \( \mathcal{B}_{NP} \) which preserves the scalar products (2.20)

\[
n' = n, \quad m' = m + a n, \quad \bar{m}' = \bar{m} + \bar{a} n, \quad \ell' = \ell - \bar{a} m - a \bar{m} - a\bar{a} n, \tag{4.34}
\]

where \( a = a_2(\xi) + ia_3(\xi) \) is a complex valued function on \( \Sigma \). Under this change of null tetrad, and keeping the Weyl tensor fixed, \( \Psi^c_n \) behave as (see [66])

\[
\Psi^c_{0'} = \Psi^c_0 = 0, \quad \Psi^c_{1'} = \Psi^c_1 = 0, \quad \Psi^c_{2'} = \Psi^c_2, \quad \Psi^c_{3'} = \Psi^c_3 + 3\bar{a} \Psi^c_2. \tag{4.35}
\]

In our framework it can be shown, using (2.19) and (3.13), that supertranslations \( \xi^1 \to \xi^1 + A(\xi^M) \) induce a rotation of the null tetrad \( \mathcal{B}_{NP} \) of the form (4.34) with \( a \equiv (A_2 + iA_3)/\sqrt{\lambda} \). Moreover, from (3.12) it follows that supertranslations act on the data appearing in (4.27) and (4.28) as

\[
\mathcal{R}' = \mathcal{R}, \quad \kappa'_0 = \kappa_0, \quad Ds' = Ds, \quad \hat{\Omega}' = \hat{\Omega} + \kappa_0 \sqrt{\lambda} \pi. \tag{4.36}
\]

These transformations lead precisely to the behaviour of the Weyl scalars described by (4.35) when we apply them to (4.27) and (4.28). Therefore, the effect of a supertranslation in the Newman–Penrose formalism is entirely equivalent to a rotation of the null tetrad, which has no effect on the horizon geometry. As a consequence, the quantity \( \hat{\Omega}' \) could be changed to any value in (4.28) without affecting the information about the spacetime curvature carried by \( \Psi^c_3 \), e.g. it could be eliminated from (4.28) choosing \( \pi = -\hat{\Omega}'/(\kappa_0 \sqrt{\lambda}) \) in (4.34). This concludes our proof that the two expressions (4.28) and (4.30) for the third Weyl scalar \( \bar{\psi}_3^c \) can be identified as projections of the same Weyl tensor expressed in terms of two different null tetrads. As a consequence (4.28) and (4.30) represent the same spacetime geometry, which can be entirely encoded in the supertranslation invariant free data set \( \mathcal{D}_{\text{free}} \).

Summarising, we have argued that the horizon free data set (4.25) is both necessary and sufficient to reconstruct all the information about the spacetime geometry determined by the NEH structure:

- the free data set \( \mathcal{D}_{\text{free}} \) (4.25) involves no gauge degrees of freedom,
- all the information about the spacetime curvature which is contained in (2.11) is also enclosed in \( \mathcal{D}_{\text{free}} \), and
- the corresponding data about the curvature tensor can be recovered using the expressions of the Weyl scalars (4.27) and (4.30), and the relation \( R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} \) which holds in vacuum.

Since the all the elements in the data set \( \mathcal{D}_{\text{free}} \) are invariant under supertranslations, this result completes the proof that horizon supertranslations act trivially on the NEH geometry.
5. Radiative vacua of null infinity

For completeness, in this section we review the role of BMS supertranslations at null infinity using Penrose’s conformal framework, and the intrinsic description of \( \mathcal{I} \) developed in [52, 53] (see also [25, 26, 54]). In particular, we will reproduce the well-known result that the radiative vacuum of asymptotically flat spacetimes is degenerate, and we will discuss the connection of this degeneracy with supertranslations. At null infinity the would-be gauge degree of freedom associated to BMS supertranslations is necessary to have a complete characterisation of the dynamics of \( \mathcal{I} \), i.e. it cannot be gauged away. Thus, BMS supertranslations act non-trivially on the geometric data of \( \mathcal{I} \).

The difference between the dynamical behaviour of a NEH and null infinity can be traced back to three main causes. First, the structure of null infinity is only defined up to conformal transformations, what requires that the dynamical degrees of freedom of \( \mathcal{I} \) are encoded in appropriate equivalence classes of data sets. Second, the Ricci tensor of the conformal completion of the physical spacetime does not satisfy the ordinary Einstein’s equations, implying that the constraint equations we used for NEH’s, (2.17) and (2.18), are no longer valid for null infinity. And finally, contrary to the case of horizons, the boundary conditions for the gravitational field at null infinity allow for gravitational radiation propagating in a transverse direction to reach \( \mathcal{I} \).

The geometry of null infinity will be described using the same formalism as in the case of non-expanding horizons, and thus the following analysis shall serve as a non-trivial consistency check of our approach.

5.1. Asymptotically flat spacetimes

Let us begin recalling the definition of asymptotic flatness and null infinity following [53]. A spacetime \((\hat{\mathcal{M}}, \hat{g})\) is said to be asymptotically flat at null infinity if it is possible to find a spacetime \((\mathcal{M}, g)\), together with an embedding \(\Psi : \hat{\mathcal{M}} \to \mathcal{M}\), and a function \(\Omega\) on \(\mathcal{M}\) such that

(i) \(\Psi^* \hat{g}_{ab} = \Omega^2 \hat{g}_{ab}\) on \(\hat{\mathcal{M}}\),
(ii) \(\mathcal{I} \cong \mathbb{S}^2 \times \mathbb{R}\) is the boundary of \(\Psi(\hat{\mathcal{M}})\) on \(\mathcal{M}\), located at \(\Omega = 0\).
(iii) The normal form is given by \(n_a \equiv \nabla_a \Omega \neq 0\) on \(\mathcal{I}\).
(iv) There is a neighbourhood of \(\mathcal{I}\) on \(\mathcal{M}\), such that \(\hat{g}_{ab}\) satisfies the vacuum Einstein equations, i.e. \(\hat{R}_{ab} = 0\).

The spacetime \((\mathcal{M}, g)\) is called the unphysical spacetime, and the hypersurface \(\mathcal{I} \subseteq \mathcal{M}\), which is null as a consequence of (i), (ii) and (iv), is referred as null infinity. Note that, if the pair \((\Omega, g)\) defines an appropriate conformal completion, so does the pair \((\omega \Omega, \omega^2 g)\) for some smooth positive function \(\omega\) on \(\mathcal{M}\). Two asymptotic completions related in this way are regarded as equivalent, and the freedom to perform such conformal transformations should be considered as a gauge redundancy.

5.2. Hypersurface data of null infinity

To describe the geometry of the null hypersurface \(\mathcal{I}\) we can use the formalism introduced in section 2.1. Thus, we introduce an abstract manifold \(\mathcal{I}\), which acts as a diffeomorphic copy of \(\mathcal{I} \subseteq \mathcal{M}\) detached from the unphysical spacetime, and the identification is performed via the embedding \(\Phi : \mathcal{I} \to \mathcal{M}\), such that \(\Phi(\mathcal{I}) = \mathcal{I}\). We will also choose the coordinate system \(\xi^a\)

\[^{16}\text{the factors of } \omega \text{ are chosen so that the physical metric } \Omega^{-2} g \text{ remains the same.}\]
for $\mathcal{I}$ and the rigging $\ell$ following the conventions in section 2.1, so that the hypersurface data can be represented by the set of quantities (2.11). In the case of null infinity it is possible to simplify the hypersurface data taking advantage of the freedom to perform conformal transformations $(\Omega, g) \rightarrow (\omega \Omega, \omega^2 g)$. Actually, under this change the normal form is rescaled as $n \rightarrow \omega n$, what can be used to require that the normal vector $n^\mu$ satisfies (5.1) (see also [77])

$$\nabla_\nu n^\mu = 0 \text{ on } I \implies \kappa = \Omega_M = \Theta_{MN} = 0,$$

(5.1)

where the conditions on the connection coefficients follow from (2.7). In this gauge, the second fundamental form vanishes, and therefore null infinity $\mathcal{I}$ admits a description as a non-expanding null hypersurface with $\partial_n q_{MN} = 0$. In addition, using the same conformal freedom, we can impose that $q_{AB}$ describes a two dimensional metric of constant scalar curvature $\mathcal{R}$ [52]. In this setting the hypersurface data of $\mathcal{I}$ has the form

$$\gamma_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & q_{MN} \end{pmatrix}, \quad \ell^a = (1, 0, 0), \quad \ell^{(2)} = 0, \quad Y_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \Xi_{MN} \end{pmatrix}.$$  

In the following, to simplify the notation, we will make no distinction between null-infinity $I$ and its abstract copy $\mathcal{I}$.

5.3. Residual gauge redundancies

The conventions introduced above do not eliminate all the gauge redundancies of our description. Regarding the conformal transformations, the present setting fixes completely the normalisation of the null vector $n$. However, we are still allowed to perform conformal transformations with a conformal factor that satisfies $\omega|_I = 1$ at null infinity, but takes arbitrary values away from it. It is straightforward to check that under this residual conformal transformations the metric data \{\gamma_{ab}, \ell^a, \ell^{(2)}\} remain invariant, but the transverse components of the tensor $Y_{ab}$ transform as

$$\Xi_{MN}' = \Xi_{MN} + \lambda q_{MN},$$

(5.3)

where $\lambda(\xi) \equiv L_\xi \omega|_\mathcal{I}$ can be any smooth function on $\mathcal{I}$.

In addition to these conformal transformations, our description of null infinity also involves redundancies associated to the freedom to perform diffeomorphisms on the abstract manifold, and the choice of rigging. Actually, the analysis of the gauge redundancies of the hypersurface data that we presented in section 3 is also applicable here, since the condition (5.1) implies that null infinity can be described a non-expanding null hypersurface, and the conventions (5.2) are the same we used to study horizons. Taking the results of section 3 into account, and recalling that the normalisation of the null normal $n = \partial_\xi$ is fixed by our choice of conformal gauge, we find that the only residual freedom of this type are BMS supertranslations (3.12)

$$\Xi_{MN}' = \Xi_{MN} - D_M A_N,$$

(5.4)

which corresponds to a diffeomorphism of the abstract manifold $\mathcal{I}$ acting as $\xi^a(\xi) = (\xi^1 + A(\xi^M), \xi^M)$. Note that, since $\kappa = 0$, these transformations leave invariant all other elements of the hypersurface data. We will denote the group of BMS supertranslations by $\mathcal{S}$.

17 Similarly to the case of horizons, BMS supertranslations can be described as hypersurface symmetries of null infinity $I \subseteq M$ (see appendix C.1).
It might seem surprising that we did not encounter the BMS group when discussing
the gauge freedom at null infinity. The reason is that we fixed the scale of the null normal
(by equation (5.1), and choosing $q_{MN}$ to describe a sphere with curvature $R$) before
classifying the residual gauge freedom of our description. Indeed, the BMS group can be
recovered as the set of gauge transformations that leave invariant the conventions (2.5) up to a
canonical transformation $(\Omega, g) \rightarrow (\omega \Omega, \omega^2 g)$, with $\omega \neq 1$ at $\mathcal{I}$. When the
conformal transformations are
gauge fixed so that only those with $\omega|\mathcal{I} = 1$ are allowed, the remaining residual gauge is given
by the transformations (5.3) and (5.4) that we described above. A derivation of the full BMS
group can be found in appendix C.1, where we perform a similar analysis to that of section 3.3
for non-expanding horizons. In the appendix we show that the BMS group can be described as
the set of diffeomorphisms of the unphysical spacetime which leaves invariant the metric
tensor $g_{\mu\nu}$ and the null normal $n$ of $\mathcal{I}$ up to a conformal transformation, $g \rightarrow \omega^2 g$ and $n \rightarrow \omega n$.

5.4. Constraint equations

As in the case of non-expanding horizons, the hypersurface data of null infinity cannot be
specified freely. It must be consistent with the constraint equations (2.12–2.14), which are
mathematical identities satisfied by any null hypersurface. In the previous paragraphs we have
presented most of the elements involved in these equations, and it only remains to compute
the terms (2.15). The crucial difference with our previous discussion of non-expanding hori-
zons is that, although the physical spacetime is vacuum in a neighbourhood of $\mathcal{I}$ (condition
(iv)), the unphysical Ricci tensor $R_{\mu\nu}$ does not vanish. This is just a direct consequence of the
non-trivial transformation properties of the Ricci tensor under the conformal rescaling of the
metric (see [77]). Therefore, (2.17) and (2.18) are not valid for $\mathcal{I}$.

In the following subsection we will characterise the terms (2.15) of the constraint equa-
tions of null infinity, i.e. the unphysical Ricci tensor $R_{\mu\nu}$, and then we will turn to the resolu-
tion of the constraints in section 5.6.

5.5. The Ricci tensor at null infinity

In order to compute the Ricci tensor at points of null infinity it is convenient to note that the
Weyl tensor $C_{\rho\sigma\mu\nu}$ is vanishing at $\mathcal{I}$. This implies that the unphysical Riemann tensor at null
infinity has the general form [52, 53]

$$R_{\sigma\rho\mu\nu} = \frac{1}{2} \left( g_{\sigma[\mu} S_{\nu]\rho] - g_{\rho[\mu} S_{\nu]\sigma} \right),$$

(5.5)

where the symmetric tensor $S_{\mu\nu}$ is the Schouten tensor defined at the beginning of section 2.
The tensor $S_{\mu\nu}$ has a particularly simple form when expressed in the basis $\mathcal{B} = \{n, e_M\}$
due to our gauge fixing conventions (5.1) and (5.2). Indeed, the divergence free condition
(5.1) can be used in combination with the Ricci identity to prove that the four components
$S_{\mu\nu} \equiv n^\mu e^\rho_\nu S_{\rho\mu\nu}$ vanish at null infinity

$$0 = \ell_\sigma n^\mu e^\rho_\nu \nabla_{[\mu} n^\sigma = \ell_\sigma n^\mu e^\rho_\nu R^\sigma_{\rho\mu\nu} n^\rho = \frac{1}{2} S_{AB},$$

$$0 = q^{AB} e_{A[\sigma} n^\mu e^\rho_\nu \nabla_{[\mu} n^\sigma = q^{AB} e_{A[\sigma} n^\mu e^\rho_\nu R^\sigma_{\rho\mu\nu} n^\rho = -S_{mn}.\quad (5.6)$$

Moreover, it is also possible to show that the components of the Schouten tensor satisfy
$S^\mu_\mu = \mathcal{R}$, where $\mathcal{R}$ is the scalar curvature of $q_{MN}$. This expression can be derived comparing
the result of computing $R_{MANB} q^{AB} q^{MN}$ directly from (5.5), with the outcome of the same
computation using the identity (B.33) (see appendix B.2) together with the gauge conditions (5.1). We can simplify $S_{\mu\nu}$ even further making use of the residual conformal transformations with $\omega|_{\mathcal{I}} = 1$, which act on the Schouten tensor as (see [52])

$$
S'_{ab} = S_{ab}, \quad S'_{ta} = S_{ta} - 2\partial_\mu \lambda, \quad S'_{t\ell} = S_{t\ell} - 2\mu + 4\lambda^2, \quad (5.7)
$$

where $\lambda(\xi) = \nabla_\ell \omega|_{\mathcal{I}}$ and $\mu(\xi) = \ell^\nu \nabla_\nu \omega|_{\mathcal{I}}$ are two arbitrary functions on $\mathcal{I}$. Therefore, we can set $S_{t\ell} = 0$ by a suitable choice of the function $\mu$.

Collecting these results, and using the inverse metric (2.6) to compute the trace of the Schouten tensor, it is possible to derive the Ricci tensor of the unphysical spacetime using $R_{\mu\nu} = S_{\mu\nu} + \frac{1}{2}g_{\mu\nu}$. We find the following non-vanishing components

$$
R_{\alpha\ell} = S_{\alpha\ell}, \quad R_{MN} = S_{MN} + \frac{1}{2}(2S_{\alpha\ell} + \mathcal{R})g_{MN}, \quad (5.8)
$$

and $R_{a\mu} = R_{aMN} = R_{\ell\ell} = 0$. This form for the unphysical Ricci tensor at null infinity is universal for any asymptotically flat spacetime. We will now derive the additional conditions satisfied by $R_{\mu\nu}$ in regions of $\mathcal{I}$ where no outgoing radiation is present.

5.5.1 Geometry of the radiative vacuum. In order to find the relevant boundary conditions we need to consider the leading order contribution $K_{\rho\sigma\mu\nu} = \Omega^{-1}C_{\rho\sigma\mu\nu}$ to the Weyl tensor, since the unphysical Weyl tensor $C_{\rho\sigma\mu\nu}$ always vanishes on $\mathcal{I}$ [52].

The condition that there is no outgoing radiation in a region of $\mathcal{I}$ is most easily expressed in terms of the leading order Weyl scalars, which are defined as components of the tensor $K_{\sigma\rho\mu\nu}$ in the basis $B_{MN} = \{\ell, n, m, \overline{m}\}$ (see [65]). Note that we have changed the order of the first two elements of the null tetrad $B_{MN}$, $\ell$ and $n$, with respect to section 4.3, while $m$ and $\overline{m}$ are defined by (2.19). The relevant Weyl scalars are given by

$$
\Psi^0_2 = K^\mu_\nu, \quad \Psi^0_3 = K^\mu_\nu \ell, \quad \Psi^0_4 = K^\mu_\nu m, \quad \Psi^0_5 = K^\mu_\nu \overline{m}, \quad (5.9)
$$

and the conditions for no outgoing radiation at a region of $\mathcal{I}$ read [26]

$$
\text{Radiative vacuum : } \quad \text{Im}\Psi^0_2 = 0, \quad \Psi^0_3 = 0, \quad \text{and } \Psi^0_4 = 0. \quad (5.10)
$$

The implications of these boundary conditions on the form of the Schouten tensor can be derived from the equations

$$
\text{Bianchi Identities : } \quad \nabla_{[\nu}S_{\sigma]\rho] = -K_{\mu\rho\sigma\nu}n^\mu, \quad (5.11)
$$

which are a direct consequence of Bianchi identities of the unphysical spacetime [26, 52].

In order to solve the previous equations and boundary conditions, it is convenient to express them in the basis $B = \{\ell, n, e_M\}$. Taking contractions on both sides of (5.11) with appropriate combinations of the elements in $B$, and using (2.7) in combination with the gauge conditions (5.1) to simplify the result, we find

$$
\text{Im}\Psi^0_2 = 0 \quad \implies \quad D_{[M}S_{N]\ell} = \Xi^p_{[M}S_{N]p}, \quad (5.12)
$$

$$
\Psi^0_3 = 0 \quad \implies \quad \partial_\mu S_{\mu\ell} = 0, \quad \text{and } \quad D_{[M}S_{N]\ell} = 0. \quad (5.13)
$$

$$
\Psi^0_4 = 0 \quad \implies \quad \partial_\nu S_{M\nu} = 0. \quad (5.14)
$$

A detailed derivation can be found in appendix C.2. The last equation (5.14) implies that the components $S_{MN}$ have to be constant along the null direction of $\mathcal{I}$. Moreover, the form of $S_{MN}$
can be found solving the second constraint in (5.13) in combination with $S^M_L = R$, and it has the unique solution
\begin{equation}
S_{MN} = \frac{1}{2} R q_{MN}. \tag{5.15}
\end{equation}

The original proof can be found in [52], but given that the setting therein is slightly different from ours, for completeness we have written a summary of it in appendix C.2. From the previous relation it follows that the rhs of (5.12) must vanish, which together with the first equation in (5.13), also implies that the components $S_{\alpha \ell}$ take the form $S_{\alpha \ell} = \partial_n S_{\epsilon \ell}$ for some function $S_{\epsilon \ell}$ on $\mathcal{I}$. Thus, from equation (5.7) it is straightforward to check that in the radiative vacuum the components $S_{\alpha \ell}$ are pure conformal gauge.

With these results at hand, we can finally obtain the components of the unphysical Ricci tensor on $\mathcal{I}$ in the absence of outgoing radiation
\begin{equation}
R_{\alpha \ell} = \partial_n S_{\epsilon \ell}, \quad R_{MN} = (\partial_n S_{\epsilon \ell} + \bar{R}) q_{MN}, \tag{5.16}
\end{equation}
and $R_{\alpha \alpha} = R_{\alpha M} = R_{\ell \ell} = 0$. This result will allow us to write down the constraint equations at regions of $\mathcal{I}$ where there is no outgoing radiation, and whose solutions represent the radiative vacua of asymptotically flat spacetimes.

### 5.6. Constraint equations for the transverse connection

Before discussing the constraint equations let us comment on the physical degrees of freedom contained in the transverse connection $\Xi_{MN}$. At the beginning of this section we identified the trace of $\Xi_{MN}$ as a pure conformal gauge (see equation (5.3)), and thus, in order to eliminate this redundancy we will proceed as in [53], identifying those connections related by a conformal transformation. In other words, we will introduce the equivalence relation
\begin{equation}
\Xi_{MN} \approx \Xi'_{MN} \iff \Xi'_{MN} - \Xi_{MN} = \lambda q_{MN}, \tag{5.17}
\end{equation}
where $\lambda(\xi)$ is an arbitrary smooth function on $\mathcal{I}$, and we will work with the resulting equivalence classes. This amounts to neglecting the trace part of the transverse connection $\Xi_{MN}$, leaving as the dynamical field its traceless part $\Xi'_{MN} - \frac{1}{2} \Xi_L^L q_{MN}$. Note that this quantity describes precisely two degrees of freedom, which can be identified with the two radiative degrees of freedom of gravitational radiation [26, 53].

#### 5.6.1 General form of the constraint equations.

We begin discussing the constraint equations for a general situation in the presence of radiation. In particular, we will show that the two components in the traceless part of $\Xi_{MN}$ are both necessary and sufficient to describe the radiative degrees of freedom of the gravitational field at null infinity.

In the presence of radiation at null infinity, the terms (2.15) in the constraint equations can be computed from the Ricci tensor given in (5.8)
\begin{equation}
J_{nm} = J_{nM} = 0, \quad J_{MN} = -S_{MN} - S_{\alpha \ell} q_{MN} - \frac{1}{2} R q_{MN}. \tag{5.18}
\end{equation}
This result together with the gauge conditions (5.1) imply that the Raychaudhuri (2.12) and Damour–Navier–Stokes equations (2.13) are trivially satisfied on $\mathcal{I}$. The only non-trivial equations are those for the transverse components of the connection (2.14), which can be expressed as
\[ \partial_n \Xi_{MN} = \frac{1}{2} (S_{MN} + S_{N\ell} q_{MN}) \approx -\frac{1}{2} N_{MN}, \]  
(5.19)

where we have used the equivalence relation (5.17). Here \( N_{MN} \equiv S_{MN} - \frac{1}{2} R q_{MN} \), is the news tensor, which vanishes in the absence of radiation passing through \( \mathcal{I} \), i.e. when equation (5.15) hold. In addition to the previous equation, the connection must satisfy one more constraint coming from the identity (B.33)

\[ D_{[M} \Xi_{N]} = \frac{1}{2} q_{[M} S_{N]} \ell. \]  
(5.20)

Using the Bianchi identities satisfied by the Schouten tensor (5.11) it can be checked that this condition is consistent with the time evolution given by (5.19) (see appendix C.2). In other words, if the previous equation is satisfied at any given value of the null coordinate, then equation (5.19) ensures that it will hold for all values of \( \xi^1 \).

We can now show that the two components in the traceless part of \( \Xi_{MN} \) encode the radiative modes of gravitational radiation at null infinity. Recall that the information about the outgoing radiative modes at \( \mathcal{I} \) is described by the Weyl scalars \( \Im \Psi_2^0, \Psi_3^0 \) and \( \Psi_4^0 \) [26, 53]. As we review in appendix C.2, using the Bianchi identity (5.11), it is possible to express these scalars as

\[ \Im \Psi_2^0 = -\frac{1}{2} (D_{[3} S_{2]} \ell - \Xi_{[3} S_{2]} M), \]  
(5.21)

\[ \Psi_3^0 = \frac{1}{\sqrt{2}} (D_{[3} S_{2]} 3 - i D_{[2} S_{3]} 2), \]  
(5.22)

\[ \Psi_4^0 = \frac{1}{2} (\partial_n S_{22} - \partial_n S_{33}) - i \partial_n S_{23}, \]  
(5.23)

implying that they are completely determined by the components of the Schouten tensor \( S_{M\ell} \) and \( S_{MN} \), and by the transverse connection \( \Xi_{MN} \). Actually, it is easy to prove that only the traceless part of \( \Xi_{MN} \) contributes in the first equation, and that these expressions are invariant under the conformal transformations (5.3) and (5.7). Then, the constraint equations (5.19) and (5.20) can be solved for \( S_{MN} \) and \( S_{M\ell} \) giving

\[ S_{M}^M = -\frac{1}{2} R, \quad S_{MN} \approx -2 \partial_n \Xi_{MN}, \quad S_{N\ell} = q^{PM} D_{[M} \Xi_{N]} P. \]  
(5.24)

In these equations too only the traceless part of \( \Xi_{MN} \) contains relevant information about the geometry at \( \mathcal{I} \), as the contribution of the trace \( \Xi_{MN}^M \) can be identified as pure conformal gauge. Thus, we can conclude that the traceless part of \( \Xi_{MN} \) is both necessary and sufficient to recover completely the information about the radiative modes at \( \mathcal{I} \) (for a more detailed derivation see [53]).

5.6.2. Degeneracy of the radiative vacuum. Finally, we turn to the discussion of the degeneracy of the radiative vacua in asymptotically flat spacetimes. The constraint equations for regions of \( \mathcal{I} \) with no outgoing radiation can obtained from (5.19) and (5.20) together with the boundary conditions (5.16), which imply \( S_{M\ell} = \partial_n S_{\ell}, \) and \( N_{MN} = 0 \). Using the equivalence relation (5.17) they read

\[ \partial_n \Xi_{MN} \approx 0, \quad D_{[M} \Xi_{N]} P \approx 0. \]  
(5.25)
Note that $\Xi_{MN}$ still transforms under supertranslations. In order to characterise the set of radiative vacua avoiding possible gauge artifacts we need to introduce a new gauge invariant dynamical variable. Due to our choice of conformal gauge the Hajicek one-form and the surface gravity are vanishing, and therefore we cannot proceed as in the case of NEHs and construct a supertranslation invariant variable analogous to $s_{MN}(\eta)$ in equation (4.20). Instead, following [53], we choose a reference vacuum $\hat{\Xi}_{MN}$ and then we consider the differences $\Sigma_{MN} = \Xi_{MN} - \hat{\Xi}_{MN}$, between a generic vacuum connection $\Xi_{MN}$ and the fiducial connection $\hat{\Xi}_{MN}$. It is easy to check that $\Sigma_{MN}$ is invariant under supertranslations. Then, given a fixed fiducial connection $\hat{\Xi}_{MN}$, the set of distinct $\Sigma_{MN}$ consistent with the equations (5.25) is isomorphic to the set of radiative vacua. Since both of the connections $\Xi_{MN}$ and $\hat{\Xi}_{MN}$ describe a radiative vacuum, their difference $\Sigma_{MN}$ also satisfies (5.25) due to the linearity of the equations. The general solution to (5.25), and therefore, the set of radiative vacua of null infinity is characterised by the expression

$$\Sigma_{MN} \approx D_{MNf} - \frac{1}{2} \Delta f_{MN},$$

(5.26)

where $f(\xi^M)$ is any smooth function of the coordinates $\xi^M$ (see appendix C.2). It is important to stress that the smoothness $f(\xi^M)$ is essential for the derivation, which uses the fact that the spatial sections of $I$ are compact and simply connected. We will denote the set of vacuum connections by $\hat{\Gamma}$.

The previous expression (5.26) already indicates clearly that the set of vacuum connections is infinitely degenerate. Comparing (5.26) with (5.4) it is straightforward to check that, given a fiducial vacuum $\hat{\Xi}_{MN}$, the most general vacuum connection is given by

$$\Xi_{MN} \approx \hat{\Xi}_{MN} + D_{MNf} - \frac{1}{2} \Delta f_{MN},$$

(5.27)

and therefore, the difference between any two vacuum connections has the form of a supertranslation. In other words, we can construct the full set $\hat{\Gamma}$ acting on $\hat{\Xi}_{MN}$ with all the elements of the group of BMS supertranslations $\hat{\mathcal{S}}$ (5.4). Note that BMS translations, i.e. the four dimensional subgroup $\mathcal{T} \subseteq \mathcal{S}$ of supertranslations satisfying

$$D_{MNf} - \frac{1}{2} \Delta f_{MN} = 0,$$

(5.28)

acts trivially on the connections of null infinity. Thus, the set of radiative vacua is isomorphic to the group of supertranslations modulo BMS translations $\hat{\Gamma} \cong \mathcal{S} / \mathcal{T}$.

It is interesting to see how the presence of a non-vanishing news tensor induces a change of the radiative vacuum. Consider a solution of (5.19) where the news $N_{MN}$ is non-zero in the interval $\xi^1 \in (\xi^1_i, \xi^1_f)$ and vanishes everywhere else. Then, the initial and final states of the connection, $\Xi_{MN}|_{\xi^1_i}$ and $\Xi_{MN}|_{\xi^1_f}$ respectively, represent radiative vacua of $I$, and have to be of the form (5.27). Integrating (5.19) we find that the difference between the final and initial transverse connections $\Xi_{MN}$ is given by the expression

$$\delta\Sigma_{MN} \equiv \Sigma_{MN}|_{\xi^1_f} - \Sigma_{MN}|_{\xi^1_i} = \Xi_{MN}|_{\xi^1_f} - \Xi_{MN}|_{\xi^1_i} \approx -\frac{1}{2} \int_{\xi^1_i}^{\xi^1_f} d\xi^1 N_{MN},$$

(5.29)

which is invariant under supertranslations. For a generic source of radiation the configuration of the news tensor $N_{MN}$ will be such that $\delta\Sigma_{MN} \neq 0$ and therefore, in general, the initial and final transverse connections correspond to distinct radiative vacua. In particular, it is now clear that if we imposed a gauge fixing condition on $\Xi_{MN}$ to eliminate the freedom to perform supertranslations (5.4) we would be restricting the allowed dynamics at null infinity.
From the discussion in the previous paragraphs we can see that, in contrast to the case of horizons, null infinity supertranslations transform the dynamical variables of $I$. On the one hand, supertranslations have been shown to act non-trivially on the traceless part of the transverse connection $\Xi_{MN}$ (see equation (5.4)). On the other hand, the two components in the traceless part of $\Xi_{MN}$ are both necessary and sufficient to describe the two degrees of freedom of gravitational radiation at $I$. Thus, connections related by a supertranslation cannot be identified with each other, as this would require gauging away one further component of the traceless part of $\Xi_{MN}$. As a consequence, BMS supertranslations must be regarded as large gauge transformations, i.e. as global symmetries of the constraint equations of null infinity, which act non-trivially on the geometric data of $I$.

6. Results and discussion

One of the most interesting features about asymptotically flat spacetimes is the infinite dimensional asymptotic symmetry group at null infinity, the BMS group. The BMS symmetries, and in particular null infinity supertranslations, were originally characterised as diffeomorphisms which preserved certain coordinates conventions in a neighbourhood of null infinity [21–23]. Many years later, the study of the geometrical structure of null infinity led to the isolation of the radiative degrees of freedom of the gravitational field [53], and it was understood that BMS supertranslations act non-trivially on the radiative degrees of freedom. Actually, the radiative vacuum of asymptotically flat spacetime was shown to be infinitely degenerate, and that it was possible to transform each of these vacua into any other with a supertranslation.

Recently, it has been argued that the ASG of spacetimes containing a non-extremal black hole should be enhanced with horizon supertranslations. These diffeomorphisms would transform the state of the black hole horizon in an analogous way as BMS supertranslations act on the geometric data of null infinity. According to this proposal, the multiplicity of black hole states generated by horizon supertranslations could provide a partial explanation for the Bekenstein–Hawking entropy formula.

The task of characterising the ASG of the near horizon geometry for non-extremal black holes has been addressed in many works. However, there is no consensus regarding the structure of the ASG, or the physical interpretation of these diffeomorphisms. In the present paper we have presented a detailed characterisation of the geometric properties of supertranslations defined on a generic non-expanding horizon embedded in vacuum. For this purpose we have used a coordinate independent approach analogous to the one used in [52, 53] to study the structure of null infinity in exact, non-linear, general relativity. In this framework, the intrinsic and extrinsic geometry of the horizon are encoded in tensor fields living on an abstract three-dimensional manifold $\Sigma$, which acts as a diffeomorphic copy of the horizon separated from the physical spacetime. In particular, the corresponding set of tensor fields, known as the horizon data set, contains the dynamical degrees of freedom of the horizon, i.e. the freely specifiable and gauge invariant data of the horizon. In order to extract the dynamical degrees of freedom from the data set, and determine their behaviour under supertranslations, we have followed the strategy described below:

1. First, we have characterised in detail all the gauge redundancies in our description of the NEH.

In particular, we have shown that the action of supertranslations on the horizon data is identical to that of a gauge redundancy: they are associated with a reparametrisation of the null direction of the horizon, and a change of the transversal direction used to define the
extrinsic geometry, (i.e. the rigging). Thus, supertranslations leave invariant both the intrinsic and extrinsic geometry of the horizon up to a gauge redundancy of the description.

2. To determine the free data of the horizon we have solved the constraints imposed by the vacuum Einstein’s equations on the NEH geometry.

As a result of this analysis we have identified the set of geometric quantities which can be freely specified on the horizon, and which encode all the information about the spacetime curvature contained on the NEH geometry. This free data set encodes the dynamical degrees of freedom of the horizon, but typically still involves some gauge redundancies.

3. The previous two analyses can be combined to characterise the gauge redundancies on the free data set. This allows one to extract a set of quantities which are both necessary and sufficient to reconstruct the full NEH geometry.

This procedure has led us to find a free data set which contains no unfixed gauge degrees of freedom, and in particular, which only involves objects invariant under supertranslations. More specifically, this free data set is composed of quantities defined on a particular spatial section \( S_{\eta_0} \) of the horizon

\[
\text{Free horizon data : } D_\text{free} \equiv (q_{MN}|_{\eta_0}, \Omega_{M}^0|_{\eta_0}, s_{MN}|_{\eta_0}).
\]

where \( q_{MN} \) represents the induced metric on the spatial sections of the horizon, and \( \Omega_{M}^0 \) determines its angular momentum aspect when \( q_{MN} \) has an \( SO(2) \) isometry. In those situations when there is gravitational radiation propagating along the horizon the symmetric traceless tensor \( s_{MN} \) can be associated to radiative degrees of freedom of the gravitational field. Since the elements of the free data set \( D_\text{free} \) are all invariant under supertranslations, we conclude that supertranslations act trivially on the NEH geometry, i.e. they must be regarded as pure gauge. In particular, the stationary state of the NEH, which corresponds to the case \( s_{MN}|_{\eta_0} = 0 \), is uniquely determined by \( q_{MN} \) and \( \Omega_{M}^0 \), and it does not transform under supertranslations.

A fundamental step to obtain the supertranslation-invariant data set \( D_\text{free} \) is the choice of an appropriate parametrisation for the null direction of the horizon. Rather than using an arbitrary coordinate, the null direction is parametrised by the value of a potential \( \eta \), which is defined in a coordinate invariant way (see equation (4.3)). The horizon can be foliated by the level sets of the potential \( \eta \), the spatial sections \( S_{\eta_0} \), and the evolution of the geometry along the null direction can be expressed in terms of the dependence on \( \eta \) of the horizon data. In particular, \( q_{MN} \) and \( \Omega_{M}^0 \) are both constant along the null direction of the horizon, while \( s_{MN} \) behaves as

\[
s_{MN} = s_{MN}|_{\eta_0} e^{-(\eta-\eta_0)},
\]

which shows that any deviation away from the stationary state, i.e. \( s_{MN} = 0 \), relaxes exponentially fast to it. The use of the potential to express the evolution of the horizon data avoids the ambiguity associated to supertranslations, which are related to coordinate reparametrisations of the null direction.

It is important to remark that the present work is restricted to the case the non-expanding horizons embedded in vacuum, and thus we have not considered processes involving matter or radiation falling across the horizon. To check if our results can be extended to more general situations we have considered the possibility of ‘implanting’ supertranslation hair on an event horizon with a non-spherical shock-wave of null matter or radiation, as proposed in [34]. We have found that, consistently with the conclusions of this paper, the shock-wave cannot excite the degree of freedom associated to supertranslations. In other words, the supertranslation degree of freedom cannot encode any ‘memory’ about the energy–momentum tensor of
the shock-wave, which is in harmony with our identification of supertranslations as a gauge redundancy. The corresponding analysis will be presented in a companion paper [59].

Acknowledgments

We would like to thank J Barbón, P Benincasa, J J Blanco-Pillado, G Dvali, A Garcia-Parrado, C Gómez, A Helou, M Mars, J Martín, M Panchenko, A Strominger and R Vera for useful discussions. We are grateful to J M M Senovilla for comments on the draft and for discussions. This work is supported by the Basque Government grants IT-956-16, POS-2016-1-0075 and IT-979-16, the Spanish Government Grant FIS2014-57956-P, the project FPA2015-65480-P, by the Spanish Research Agency (Agencia Estatal de Investigación) through the grants IFT Centro de Excelencia Severo Ochoa SEV-2012-0249 and SEV-2016-0597, and by the ERC Advanced Grant 339169 Selfcompletion.

Appendix A. Constraint equations for null hypersurfaces

In this appendix we will present a derivation of the constraint equations for null hypersurfaces (2.12)–(2.14).

A.1. Raychaudhuri equation

We begin discussing the constraint equation (2.12) for the expansion \( \theta \) of the hypersurface. From the definition of the second fundamental form

\[
\Theta_{MN} = \nabla_{M} e_{N}^{\mu} - \nabla_{N} e_{M}^{\mu},
\]

and using Leibniz rule to expand the derivative we find

\[
\partial_{\nu} \Theta_{MN} = e_{N}^{\mu} \nabla_{M} e_{\nu}^{\mu} + e_{M}^{\mu} \nabla_{N} e_{\nu}^{\mu} + e_{M}^{\mu} e_{N}^{\nu} \nabla_{\mu} n_{\nu},
\]

(A.1)

where \( \nabla_{\nu} = n^{\mu} \nabla_{\mu} \). Substituting the connection coefficients (2.7) we have

\[
\partial_{\nu} \Theta_{MN} = \Theta_{M}^{\mu} \Theta_{N}^{\nu} + e_{M}^{\mu} e_{N}^{\nu} \nabla_{\mu} n_{\nu},
\]

(A.2)

where the second equality follows from using the Ricci identity. Using again the Leibniz rule and (2.7) the last term can be written as

\[
e_{M}^{\mu} e_{N}^{\nu} \nabla_{\mu} n_{\nu} = \partial_{\mu} (e_{N}^{\mu} \nabla_{\nu} n_{\nu}) - \nabla_{M} e_{N}^{\mu} \nabla_{\nu} n_{\nu} - e_{M}^{\mu} \nabla_{N} n_{\nu} \nabla_{\mu} n_{\nu} = \kappa \Theta_{MN} - \Theta_{NP} \theta_{NP},
\]

(A.3)

The contribution from the Riemann tensor can be expressed in terms of the Weyl and Ricci tensors

\[
R_{nMnN} = C_{nMnN} + \frac{1}{2} S_{nMqN} = C_{nMnN} + \frac{1}{2} R_{nMqN},
\]

(A.4)

where we are using the shorthand \( R_{nMnN} = R_{\mu\nu\rho\sigma} n^{\mu} e_{M}^{\rho} n^{\sigma} \), and similar expressions to denote the contraction of spacetime tensors with the elements of the basis \( B = \{ n, \ell, e_{M} \} \). Using the last equation we obtain

\[
\partial_{\nu} \Theta_{MN} = \kappa \Theta_{MN} + \Theta_{M}^{\rho} \Theta_{NP} - C_{nMnN} - \frac{1}{2} R_{nMqN},
\]

(A.5)
which is known as the *tidal force equation* (see [58]). The equation for the expansion $\theta = \Theta^M_M$ can be calculated from the expression

$$\partial_\nu \theta = \partial_\nu (q^{MN} \Theta_{MN}) = \partial_\nu q^{MN} \Theta_{MN} + q^{MN} \partial_\nu \Theta_{MN}. \quad (A.6)$$

Note also that $\partial_\nu q^{MN} = -2\Omega^{MN}$, what follows from differentiating $q_{MN}q^{NP} = \delta^P_M$ with respect to $\xi^1$ and using the relation $\Theta_{MN} = \frac{1}{2} \partial_\nu q_{MN}$. Collecting all these results we arrive to the final expression for the Raychaudhuri equation

$$\partial_\nu \theta - \kappa \theta + \Theta^{MN} \Theta_{MN} = -R_{nn}. \quad (A.7)$$

### A.2. Damour–Navier–Stokes equations

In order to derive the Damour–Navier–Stokes equation (2.13) we will compute the component $R_{nA}$ of the Ricci tensor in terms of the hypersurface data. Using the expression (2.6) for the inverse metric we find

$$R_{nA} = R_{\mu\nu AB}^{\mu\nu} = R_{\text{ent}} + q^{MN} R_{MnNA}. \quad (A.8)$$

The two terms can be rewritten using the Ricci identity

$$R_{\text{ent}} = \ell^\sigma n^\mu e^\nu_M \nabla_\nu \nabla_\mu n_\sigma - \ell^\sigma n^\mu e^\nu_A \nabla_\nu \nabla_\mu n_\sigma, \quad (A.9)$$

$$R_{MnNA} = e^\nu_M e^\mu_N e^\nu_A \nabla_\nu \nabla_\mu n_\sigma - e^\nu_M e^\mu_N e^\nu_A \nabla_\nu \nabla_\mu n_\sigma. \quad (A.10)$$

Each of these four terms can be expressed in terms of the connection coefficients using the definitions (2.7) and the Leibniz rule for the covariant derivative. For example, noting that $\Omega^A_M = \ell^\sigma n^\mu e^\nu_M \nabla_\nu \nabla_\mu n_\sigma$, we have

$$\ell^\sigma n^\mu e^\nu_M \nabla_\nu \nabla_\mu n_\sigma = \partial_\nu \Omega^A_M - \nabla_\nu \kappa^A_M - \nabla_\nu \nabla_\mu n_\sigma \Omega^A_M - \kappa \Omega^A_M + \kappa \Omega^A_M + \Omega^{\nu N} \Theta_{MN}. \quad (A.11)$$

Also since $\kappa = \ell^\sigma n^\nu \nabla_\mu n_\sigma$, we have

$$\ell^\sigma n^\mu e^\nu_M \nabla_\nu \nabla_\mu n_\sigma = \partial_\mu \kappa - \nabla_\mu \nabla_\nu n_\sigma \kappa^A_M - \nabla_\mu \nabla_\nu \kappa^A_M = \partial_\mu \kappa - \nabla_\mu \kappa + \kappa \Omega^A_M. \quad (A.12)$$

Collecting terms we find

$$R_{\text{ent}} = R_{\text{ent}} = \partial_\nu \Omega^A_M - \partial_\mu \kappa + \Theta^A_M \Omega^A_N. \quad (A.13)$$

Similarly it can be shown that

$$R_{MnNA} = D\left[ \Theta^A_M + \Theta^A_M \Omega^N_A \right]. \quad (A.14)$$

Then we have

$$R_{nA} = \partial_\nu \Omega^A_M - \partial_\mu \kappa + D\Theta^A_N - D \theta + \Theta^A_M. \quad (A.15)$$

which can be identified with the Damour–Navier–Stokes equations (2.13).

### A.3. Equation for the transverse connection

We now describe the derivation of the constraint equation (2.14) for the transverse connection $\Xi_{\nu M}$. From the definition of the transverse connection $\Xi_{\nu M} = \frac{1}{2} \ell^\mu e^\nu_M \nabla_\mu \ell_\nu$, and using Leibniz rule to expand the derivative we find
From the Ricci identity we have
\begin{equation}
\nabla e^\mu_\nu = \nabla_\mu e^\nu_\nu + e^\rho_\nu \nabla_\mu e^\rho_\nu + e^\mu_\nu \nabla_\mu e^\rho_\nu.
\end{equation}

Substituting the expressions for the connection coefficients (2.7) we arrive to
\begin{align}
\partial_\nu \Xi_{MN} &= -2\Omega M\Omega_N + \Theta^\rho_{(M}\Xi_{N)\rho} + \frac{1}{2} e^\mu_\rho e^\rho_\sigma \nabla_\mu \nabla_\sigma \ell_{\nu},
\end{align}

where we have also used the Ricci identity to derive the second equality. Using the Leibniz rule for the connection and (2.7) we can rewrite the last term as
\begin{align}
e^\rho_\sigma \nabla_\mu \nabla_\sigma \ell_{\nu} &= \nabla_\mu (\ell\nabla_\sigma \ell_{\nu}) - \nabla_\nu \ell \nabla_\sigma \ell_{\nu} - \nabla_\nu \ell \nabla_\mu \nabla_\sigma \ell_{\nu}
\end{align}

\begin{align}
&= -D_\mu (\ell\Omega_N) - 2\kappa \Xi_{MN} + 2\Omega M\Omega_N - \Theta^\rho_{(M}\Xi_{N)\rho}.
\end{align}

where $D_M$ is the Levi-Civita connection of the spatial metric $q_{MN}$. Thus, substituting the previous expression in it we have
\begin{align}
\partial_\nu \Xi_{MN} &= -\frac{1}{2}D_\mu (\ell\Omega_N) - \Omega M\Omega_N - \kappa \Xi_{MN} + \frac{1}{2} \Theta^\rho_{(M}\Xi_{N)\rho} + \frac{1}{2} \Theta^\rho_{(M}\Xi_{N)\rho}.
\end{align}

Using the symmetries of the Riemann tensor, and the form (2.6) for the inverse metric, we find that
\begin{align}
\frac{1}{2} (R_{N\rho\mu N} + R_{M\rho\mu N}) &= -\frac{1}{2} R_{MN} + \frac{1}{2} R_{MANB} q^{AB},
\end{align}

where $R_{AB} = g^{\mu\nu} R_{\mu\nu AB}$. We will now rewrite $R_{MANB}$ in terms of the connection coefficients. From the Ricci identity we have
\begin{align}
R_{MANB} &= \epsilon_{M|\sigma} \nabla_\mu e^\sigma_\nu e^\rho_\beta = \partial_{[N}(\epsilon_{M|\sigma} \nabla_\beta e^\rho_\nu) - \nabla_{[N}e_{M|\sigma} \nabla_\beta e^\rho_\nu - \nabla_{[N}e^\rho_\mu \nabla_\beta e^\rho_\nu e^{\mu}_A.
\end{align}

Substituting the expressions for the connection coefficients (2.7) we obtain
\begin{align}
R_{MANB} &= \partial_{[N}(\Gamma^L_{A|B} q_{ML}) - \Theta_{M[N|A]B} - \Xi_{M[N|A]B} - \Theta_{M[N\rho|A]B} - \Gamma^L_{M[N|A]B} q_{PL},
\end{align}

which after contracting with $q^{AB}$ can be written as
\begin{align}
q^{AB} R_{MANB} &= R_{MN} + \Xi_{P(M\rho} \Theta^P_{\rho N)} - \Theta_{M|N\rho} - \Theta_{M|N} \Theta^\rho.
\end{align}

Here $R_{MN}$ is the Ricci tensor associated to the spatial metric $q_{MN}$. This result can be used together with (A.19) to express (A.17) as follows
\begin{align}
\partial_\nu \Xi_{MN} &= -\frac{1}{2}D_\mu (\ell\Omega_N) - \Omega M\Omega_N - \kappa \Theta_{MN} + \Theta^\rho_{(M}\Xi_{N)\rho}
\end{align}

\begin{align}
&= -\frac{1}{2} \Theta_{MN} \Theta^\rho + \frac{1}{2} R_{MN} - \frac{1}{2} R_{MN},
\end{align}

which leads to the constraint equation for the transverse connection (2.14) after using the identity $\mathcal{R}_{MN} = \frac{1}{2} R_{MN}$. 

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Appendix B. Calculations for non-expanding horizons

B.1. Fixing the normalisation of the null normal

In this appendix we will show that when the NEH data (4.5) satisfy one of the following conditions on a spatial slice $\Sigma_{\xi}$:

\begin{align}
(i) \quad \Omega^M\Omega^0_M & \leq \frac{1}{2} R \leq \theta^0,
(ii) \quad \Omega^M\Omega^0_M & \leq \theta^0 \leq \frac{1}{2} R,
\end{align}

(B.1)

it is possible to find a gauge transformation (3.11) that sets $\partial_n\theta^0 = 0$ on the horizon $\Sigma$. Moreover, the gauge freedom that remains after imposing this condition is precisely that of supertranslations.

B.1.1. Conditions to set $\partial_n\theta^0 = 0$.

Among the residual gauge freedom (3.11), the transformations $\zeta^a(\xi) = (\hat{f}(\xi), \xi^M)$ that keep $\kappa$ constant (i.e. gauge condition 1 (4.6)) are given by

\begin{align}
\hat{f}(\xi) & = \xi^1 + A(\xi^M) + \frac{1}{\kappa_0} \log \left(1 + B(\xi^M)e^{-\kappa_0\xi^1}\right), \\
\hat{f}_n & = \frac{1}{1 + Be^{-\kappa_0\xi^1}}, \quad \hat{f}_M = \frac{B_M e^{-\kappa_0\xi^1}}{\kappa_0(1 + Be^{-\kappa_0\xi^1})}, \quad \hat{f}_{nM} = -\frac{B_M e^{-\kappa_0\xi^1}}{(1 + Be^{-\kappa_0\xi^1})^2}.
\end{align}

Under these transformations the data changes as follows

\begin{align}
\kappa'(\xi) & = \kappa|_{\zeta(\xi)}, \\
\Omega'_M(\xi) & = \Omega_M|_{\zeta(\xi)} + \kappa_0 A_M, \\
\Xi'_{MN}(\xi) & = (1 + Be^{-\kappa_0\xi^1})(\Xi_{MN}|_{\zeta(\xi)} - \Omega_M|_{\zeta(\xi)} A_N - \kappa_0 A_M A_N - D_M A_N) \\
& \quad - \frac{e^{-\kappa_0\xi^1}}{\kappa_0} (\Omega_M|_{\zeta(\xi)} B_N + \kappa_0 A_M B_N + D_M B_N).
\end{align}

(B.5)

The transformation of the quantity $\Sigma^0_{MN}$, which is invariant under (3.12), can be found by plugging the previous equations into its definition (4.4), giving

\begin{align}
\Sigma^0'_{MN} = \Sigma^0_{MN} + Be^{-\kappa_0\xi^1} \left(\Sigma^0_{MN} - \frac{1}{2} D_M(\Omega'_N - \Omega'_M\Omega'_N) - \Omega'_M\Omega'_N\right) \\
& \quad - e^{-\kappa_0\xi^1} \left(\Omega'_M B_N + D_M B_N\right).
\end{align}

(B.6)

Taking the trace we find

\begin{align}
\theta' = \theta^0 + Be^{-\kappa_0\xi^1} \left(\theta^0 - \Omega^M\Omega^0_M - D^M\Omega^0_M\right) - e^{-\kappa_0\xi^1} \left(2\Omega^M B_M + D^M B_M\right).
\end{align}

(B.7)

This quantity evolves according to (4.9), and thus, if at any given time (e.g. $\xi^1 = 0$) we can set $\theta' = \frac{1}{2} R$, then $\partial_n\theta' = 0$ for all $\xi^1$, where $n'$ is the vector resulting from the transformation of $n$ under the map $\zeta$. This requires to solve
\[ D^M B_M + 2 \Omega^M B_M + (D^M \Omega^M_M + \Omega^M_M - \theta^0) B - (\theta^0 - \frac{1}{2} \mathcal{R}) = 0. \] (B.8)

To simplify this expression we decompose the Hajicek one-form \( \Omega^M \) in its exact \( \Omega^M_e = \partial^M M^\eta \) and divergence free parts \( \Omega^M_0 = \epsilon^M_N \partial_N g^N \) as in (4.1). Here \( \eta(\xi) \) and \( g(\xi) \) are two smooth functions on \( \Sigma \) satisfying \( \partial_n \eta = \partial_n g = 0 \), and \( \epsilon_{MN} \) is the volume one form associated to \( q_{MN} \). The transformed Hajicek one-form \( \Omega^M_\prime \) is then determined by the functions \( \eta^\prime = \eta + A \), and \( g^\prime = g \). With the change of variables \( B = e^{-\eta^\prime} \tilde{B} \) we find

\[ B_M = \tilde{B}_M e^{-\eta^\prime} - \eta^\prime_M \tilde{B} e^{-\eta^\prime}, \]
\[ D^M B_M = D^M \tilde{B}_M e^{-\eta^\prime} - 2 \eta^\prime_M \tilde{B} e^{-\eta^\prime} + \eta^M_M \eta^\prime M \tilde{B} e^{-\eta^\prime} - D^M \eta^\prime M \tilde{B} e^{-\eta^\prime}, \]

which allows us to rewrite (B.8) as

\[ D^M \tilde{B}_M + 2 \epsilon^M_N \tilde{B} M^\eta N + \tilde{B}(g^M M - \theta^0) - (\theta^0 - \frac{1}{2} \mathcal{R}) e^{-\eta} = 0. \] (B.9)

In this form we can easily identify two solutions to this equation

\[ \tilde{B} = -1 \quad A = \log \left( \frac{\theta^0 - g^M M^\eta}{\theta^0 - \frac{1}{2} \mathcal{R}} \right) - \eta, \] (B.10)

and

\[ \tilde{B} = 1 \quad A = \log \left( \frac{g^M M^\eta - \theta^0}{\theta^0 - \frac{1}{2} \mathcal{R}} \right) - \eta, \] (B.11)

provided the argument of the logarithms in these equations are non-negative, and that (B.2) is well defined at \( \xi_1 = 0 \). We find the following possibilities

\[ \tilde{B} = -1 : \quad g^M M^\eta \geq \frac{1}{2} \mathcal{R} \geq \theta^0, \quad \text{or} \quad g^M M^\eta \leq \frac{1}{2} \mathcal{R} \leq \theta^0, \] (B.12)

\[ \tilde{B} = 1 : \quad g^M M^\eta \leq \theta^0 \leq \frac{1}{2} \mathcal{R}, \quad \text{or} \quad g^M M^\eta \geq \theta^0 \geq \frac{1}{2} \mathcal{R}. \] (B.13)

Thus, each of these four sets of conditions (to be met at the spatial slice \( S_{\xi^1 = 0} \)) is sufficient to ensure that the gauge \( \partial_\nu \theta^0 = 0 \) exists.

**B.1.2. Uniqueness of the gauge.** Now we will show that this gauge is unique up to super-translations provided \( g^M M^\eta \leq \frac{1}{2} \mathcal{R} \) on \( \Sigma \). For this, suppose that we are already in this gauge. We would like to know what the possible transformations which maintain this gauge are. They would have to solve (B.9) with \( \theta^0 = \frac{1}{2} \mathcal{R} \),

\[ D^M \tilde{B}_M + 2 \epsilon^M_N \tilde{B} M^\eta N + (g^M M^\eta - \frac{1}{2} \mathcal{R}) \tilde{B} = 0. \] (B.14)

We can multiply by \( \tilde{B} \) and integrate over the sphere. After integrating the first two terms by parts and dropping the boundary terms (the spatial sections of \( S_{\xi^1} \cong S^2 \) are compact and simply connected) we get

\[ \int_{S^2} d^2 \xi (\tilde{B}^M \tilde{B}_M + \frac{1}{2} \mathcal{R} - g^M M^\eta) \tilde{B}^2 = 0. \] (B.15)
This implies that, if
\[ \frac{1}{2} \mathcal{R} \geq g^M g_M \]  
(B.16)
is satisfied the integral is the sum of two positive contributions, so we must have \( B = \tilde{B} = 0 \).
But \( A \) is unconstrained, so the condition \( \partial_0 \theta^0 = 0 \) fixes the gauge up to transformations for which \( f(\xi) = \xi^1 + A(\xi^M) \), i.e. supertranslations. Then we arrive to the following conditions which guarantee that the gauge (3.11) can be reduced down to supertranslations
\[ \tilde{B} = -1 : \quad g^M g_M \leq \frac{1}{2} \mathcal{R} \leq \theta^0, \]  
(B.17)
\[ \tilde{B} = 1 : \quad g^M g_M \leq \theta^0 \leq \frac{1}{2} \mathcal{R}. \]  
(B.18)
Noting that \( g^M g_M = \Omega^0 M \Omega_0 M \) we arrive to (B.1).

A priori it might seem that the first situation, for which \( B < 0 \), is ill-behaved because the domain of \( \zeta(\xi) \) covers only the range
\[ \xi^1 \in \left[ -\frac{1}{\kappa_0} \log(-1/B), \infty \right), \]  
(B.19)
but this is not the case. Actually, the consistency condition that we should impose on \( \zeta(\xi) \) is that given a tensor field defined on \( \Sigma \), e.g. \( \gamma_{ab} \), the coordinate representation of the transformed tensor \( \zeta^* \gamma(\xi) \) contains the same information as the original one \( \gamma(\xi) \). Thus, we must require that the image of \( \zeta \) is the full abstract manifold \( \Sigma \). This condition ensures that scanning over the domain of \( \zeta^* \gamma(\xi) \) we will access the full domain where \( \gamma(\xi) \) is defined. It is straightforward to see that the image of \( \zeta \) for the two cases in (B.1) covers the following ranges of the null coordinate
\[ (i) \quad B < 0 : \quad \hat{f}(\xi) \in (-\infty, \infty), \]  
(B.20)
\[ (ii) \quad B > 0 : \quad \hat{f}(\xi) \in (A + \frac{1}{\kappa_0} \log B, \infty). \]  
(B.21)
Then, only diffeomorphisms satisfying the condition (i) are well-behaved in the sense explained above. This is the condition we presented in the main text (4.12).

**B.1.3. Supertranslation independent condition.** We will now prove that the condition (i) in (B.1) is preserved by supertranslations, even before setting the gauge \( \partial_0 \theta^0 = 0 \). To see this, first note that the condition (B.16) is preserved by supertranslations since both \( \mathcal{R} \) and \( g_M \) transform as scalar fields under supertranslations, but neither of the two depend on the null coordinate, so they are actually invariant. Finally, \( \theta^0 \) also transforms as a scalar under supertranslations \( \theta^0(\xi) \rightarrow \theta^0(\xi) = \theta^0(\zeta(\xi)) \), as it can be checked setting \( B = 0 \) in (B.7). However, due to the form of (4.9), the rhs cannot change sign, implying that if for some \( \xi^1 \) the inequality \( \theta^0 \geq \frac{1}{2} \mathcal{R} \) holds, then it will hold for all \( \xi^1 \), including the supertranslated one \( \hat{f}(\xi) = \xi^1 + A(\xi^M) \). Therefore, it is impossible to cross the bound \( \theta^0 \geq \frac{1}{2} \mathcal{R} \) with a supertranslation.

In conclusion, we have shown that if a given data set satisfies the condition (i) in (B.1), then there is always a gauge transformation of the form (B.2) which allows us to set \( \partial_0 \theta^0 = 0 \) everywhere on the horizon. Furthermore, the gauge freedom that remains once we have done so is that of supertranslations.
B.2. Weyl scalars

In the present section we will compute the Weyl scalars $\Psi_n$, with $n = \{0,1,2,3\}$, at a generic point $\xi^a = \xi^a_0$ of a non-expanding horizon which is embedded in vacuum, i.e. $T_{\mu\nu} = 0$. As explained in the main text, the scalar $\Psi_1$ involves information about the spacetime connection off the hypersurface, and thus it cannot be computed from the connection coefficients (2.7).

To be more precise, we will compute the pullback of $\Psi_n$ to the abstract manifold $\Sigma$, but we will keep the pullback operation implicit in order to ease the notation. We will use the setting described in section 2.3: we choose a coordinate system for the abstract manifold such that $q_{\mu\nu}(\xi_0) = \delta_{\mu\nu}$, and we will define the Weyl scalars in terms of the Newman–Penrose tetrad $\mathcal{B}_{NP} = \{n, \ell, m, \overline{m}\}$, where $m$ and $\overline{m}$ are defined by (2.19). The Weyl scalars are given by the expressions

\[
\begin{align*}
\Psi_0 &= C_{\mu\nu\rho\sigma} n^\mu n^\nu n^\rho n^\sigma = \frac{1}{2} (C_{\alpha2n2} - C_{\alpha3n3}) + i C_{n2n3}, \\
\Psi_1 &= C_{\mu\nu\rho\sigma} n^\mu n^\nu \ell^\rho \ell^\sigma = \frac{1}{\sqrt{2}} (C_{n2\ell n} + i C_{n3\ell n}), \\
\Psi_2 &= C_{\mu\nu\rho\sigma} n^\mu n^\nu m^\rho m^\sigma = \frac{1}{2} (C_{\ell2n2} + C_{\ell3n3}) + i \left(\frac{1}{2} (C_{\ell2n2} - C_{\ell3n2})\right), \\
\Psi_3 &= C_{\mu\nu\rho\sigma} n^\mu n^\nu m^\rho n^\sigma = \frac{1}{\sqrt{2}} (C_{\ell2\ell n} - i C_{\ell3\ell n}).
\end{align*}
\]

(B.22)

First we will express the scalars $\Psi_n$ in terms of the connection coefficients, and then we will compute the gauge corrected Weyl scalars, (4.27) and (4.28), which we introduced in section 4.3.

**Lemma B.1.** Let $\mathcal{B} = \{q_{\mu\nu}, n, \Omega_M, \Xi_{MN}\}$ be the hypersurface data of a non-expanding horizon $\mathcal{H}$ embedded in the vacuum, and let $\Psi_n$, $n = 0, 1, 2, 3$, be the Weyl scalars defined with respect to the Newman–Penrose tetrad $\mathcal{B}_{NP} = \{n, \ell, m, \overline{m}\}$ on $\mathcal{H}$. Then, the following equations hold

\[
\begin{align*}
\Psi_0 &= \Psi_1 = 0, \quad \Psi_2 = -\frac{i}{4} \mathcal{R} + \frac{i}{2} \mathcal{J}, \\
\Psi_3 &= \frac{1}{\sqrt{2}} (D_{(M} \Xi_{2)N} + \Omega_{(M} \Xi_{2)N}) q_{MN}.
\end{align*}
\]

(B.23)

and

\[
\begin{align*}
\text{Re}\Psi_3 &= \frac{1}{\sqrt{2}} (D_{(M} \Xi_{2)N} + \Omega_{(M} \Xi_{2)N}) q_{MN}, \\
\text{Im}\Psi_3 &= -\frac{1}{\sqrt{2}} (D_{(M} \Xi_{2)N} + \Omega_{(M} \Xi_{2)N}) q_{MN},
\end{align*}
\]

(B.24)

where $D_{(M} \Omega_{N)} = \epsilon_{MN} \mathcal{J}$. $\mathcal{R}$ is the Ricci scalar of $q_{MN}$ and $\epsilon_{MN}$ the volume form.

**Proof.** Due to the Einstein’s equations, the Ricci tensor vanishes in vacuum $R_{\mu\nu} = 0$, and the Weyl tensor is equal to the Riemann curvature tensor $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}$. Taking this into account we can compute $\Psi_0$ and $\Psi_1$ using the identities (A.5) and (A.13)

\[
\begin{align*}
\text{Re}(\Psi_0) &= \frac{1}{2} \left( -\partial_n (\Theta_{22} - \Theta_{33}) + \kappa (\Theta_{22} - \Theta_{33}) + (\Theta_{22}^2 - \Theta_{33}^2) \right), \\
\text{Im}(\Psi_0) &= -\partial_n \Theta_{22} + \kappa \Theta_{22} + \Theta_{22}^2 \Theta_{33},
\end{align*}
\]

(B.25)

and

\[
\begin{align*}
\text{Re}(\Psi_1) &= \frac{1}{\sqrt{2}} \left( -\partial_n (\Theta_{22} + \Theta_{33}) + \kappa (\Theta_{22} + \Theta_{33}) - (\Theta_{22}^2 + \Theta_{33}^2) \right), \\
\text{Im}(\Psi_1) &= -\partial_n \Theta_{22} - \kappa \Theta_{22} - \Theta_{22}^2 \Theta_{33},
\end{align*}
\]

(B.26)
\[
\text{Re}(\Psi_1) = \frac{1}{\sqrt{2}} (\partial_n \Omega_2 - \partial_2 \kappa + \Theta^2 N \Omega_N), \quad (B.27)
\]
\[
\text{Im}(\Psi_1) = \frac{1}{\sqrt{2}} (\partial_n \Omega_3 - \partial_3 \kappa + \Theta^3 N \Omega_N). \quad (B.28)
\]

Note that, for non-expanding horizons embedded in vacuum all these quantities vanish, \(\Psi_0 = \Psi_1 = 0\), since the second fundamental form is zero \(\Theta_{MN} = 0\) (see section 3), and as a consequence of the Damour–Navier–Stokes equation (2.13). This proves the left equation in (B.23). It is worth mentioning that this result could also have been obtained using the Goldberg–Sachs theorem (see [66]), and noting the existence of a geodesic and shear free null vector, namely the null normal \(n\).

To compute \(\Psi_2\) we can use the following two identities
\[
R_{[\alpha MN]}q^{\alpha M} = (-\partial_\alpha \Xi_{MN} - D_\alpha \Omega_N - \Omega_\alpha \Omega_M - \kappa \Xi_{MN} + \Theta_\alpha^\mu \Xi_{\mu NP})q^{\alpha M},
\]
\[
R_{[\alpha MN]} = D_{[\alpha \Omega_M]} - \Xi_{[\alpha \Theta_{M]L}}. \quad (B.29)
\]
The first one follows from (A.17), and the second one from the Ricci identity \(R_{\ell 23} = \ell_\mu \rho^\mu e_j^\ell \nabla_{[\mu} \nabla_{\nu]} e^\nu_j\). From them we obtain
\[
\text{Re}(\Psi_2) = \frac{1}{2} (-\partial_\ell \theta^\ell - D_\ell \Omega^M - \kappa \theta^\ell - \Omega^A \Omega_\ell + \Theta^{AB} \Xi_{AB}), \quad \text{Im}(\Psi_2) = \frac{1}{2} D_{[\ell 2} \Omega_{3]}, \quad (B.30)
\]
Here we have also used the fact that the real part of \(\Psi_2\) can also be written as \(\text{Re}\Psi_2 = \frac{1}{2} q^{AB} C_{\ell AB}\).

The expressions (4.27) can be recovered when we impose the constraint equations of a non-expanding horizon embedded in vacuum. Setting \(\Theta_{MN} = 0\), and from the constraint equation for the trace of \(\Xi_{MN}\), (2.14) we arrive to
\[
\text{Re}(\Psi_2) = -\frac{1}{4} R, \quad \text{Im}(\Psi_2) = \frac{1}{2} D_{[\ell 2} \Omega_{3]}, \quad (B.31)
\]
At the point where we are evaluating the expressions, \(\xi_0^a\), the spatial metric has the canonical form \(q_{MN} = \delta_{MN}\), and therefore the volume form reduces to the Levi-Civita symbol, which satisfies \(\epsilon_{23} = 1\). Thus \(D_{[\ell 2} \Omega_{3]} \equiv \epsilon_{23} J = J\), which proves (B.23).

To compute the last Weyl scalar \(\Psi_3\) it is convenient to use the equation \(R_{\alpha \ell AE} = R_{\alpha MN} q^{MN}\) which holds in vacuum. It follows from
\[
0 = R_{\alpha A} = R_{\alpha [MN} g^{\mu \nu} = R_{\alpha MN} q^{MN} + R_{\alpha [MN} + R_{\ell \alpha AE} = R_{\alpha MN} q^{MN} - R_{\ell \alpha AE} \quad (B.32)
\]
with \(M \neq A\). Here we used the form for the inverse metric (2.6), and the symmetries of the Riemann tensor, which imply \(R_{\alpha \ell \alpha A} = 0\). Then, the contractions of the Riemann curvature of the form \(R_{\alpha MN}\) can be calculated from the relation
\[
R_{\alpha MN} = D_{[K \Xi_{A|M}} + \Omega_{[K \Xi_{A|M}}, \quad (B.33)
\]
which is a direct consequence of the Ricci identity \(R_{\alpha MN} = L_\sigma e_\sigma^\alpha e_\mu^N \nabla_{[\mu} \nabla_{\nu]} e^\nu_\sigma\), and the definitions of the connection coefficients (2.7). Recalling that the Riemann and Weyl tensors are equal in vacuum we have that \(C_{\alpha \ell AE} = R_{\alpha MN} q^{MN}\), we can obtain (B.24) using (B.33) and the definitions (B.22).
Recall, that the gauge corrected Weyl scalars are given by
\[ \Psi'_\alpha(\eta, \xi^M) \equiv \Psi'_\alpha(H(\eta, \xi^M), \xi), \]
where \( H(\eta, \xi^M) \) is defined in (4.18). Therefore, since the Weyl scalars \( \Psi_0, \Psi_1 \) and \( \Psi_2 \) do not depend on the null coordinate \( \xi^1 \), their gauge corrected expressions are identical to those in (B.23), which proves (4.27).

It only remains to derive the expression (4.28) for the gauge corrected Weyl scalar \( \Psi'_3(\eta, \xi^M) \).

**Proposition B.1.** Let \( \mathcal{G} = \{g_{MN}, \kappa, \Omega_M, \Xi_{MN} \} \) represent the hypersurface data of a generic non-expanding horizon embedded in vacuum, and let \( \Psi'_3(\eta, \xi^M) \) be the gauge corrected Weyl scalar defined in (4.26). Then, in the gauge defined by (4.6) and (4.11), \( \Psi'_3(\eta, \xi^M) \) is given by (4.28).

**Proof.** We will first write \( \Psi_3 \) in terms of the object \( \Sigma^0_{MN} \) defined (4.4). Substituting the definition (4.4) into (B.33), after a straightforward calculation we find
\[
\kappa C_{\alpha\beta} = (D[N\Sigma^0_{A\beta} + \Omega[N\Sigma^0_{A\beta}])q_{NM} = D^M \sigma^0_{AM} + \Omega^M \sigma^0_{AM} - \frac{1}{4} \partial_A \mathcal{R} - \frac{1}{4} \Omega_A \mathcal{R},
\]
(B.34)
where \( \mathcal{R} \) and \( \epsilon_{MN} \) are, respectively, the curvature scalar and volume form of \( q_{MN} \), and \( \mathcal{J} = D[\Omega_M] \). In order to simplify this expression we can use the assumption that the horizon is generic, and that the gauge redundancies (3.11) have been partially fixed by the conventions (4.6) and (4.11). Then, the trace of \( \Sigma^0_{MN} \) satisfies \( \theta^0 = \frac{1}{2} \mathcal{R} \), and thus the first term in the previous equation takes the form
\[
(D[N\Sigma^0_{A\beta} + \Omega[N\Sigma^0_{A\beta}])q_{NM} = D^M \sigma^0_{AM} + \Omega^M \sigma^0_{AM} - \frac{1}{4} \partial_A \mathcal{R} - \frac{1}{4} \Omega_A \mathcal{R},
\]
(B.35)
where \( \sigma^0_{MN} \) is the traceless part of \( \Sigma^0_{MN} \). This leads to
\[
\kappa C_{\alpha\beta} = D^M \sigma^0_{AM} + \Omega^M \sigma^0_{AM} - \frac{1}{4} \partial_A \mathcal{R} + \frac{1}{2} \epsilon_{AM} D^M \mathcal{J} + \frac{3}{2} \epsilon_{AC} \Omega^C \mathcal{J} - \frac{3}{4} \Omega_A \mathcal{R},
\]
(B.36)
Then from the definition of \( \Psi_3 \) we find
\[
\text{Re} \Psi_3 = \frac{1}{\kappa \sqrt{2}} (D^M \sigma^0_{2M} + \Omega^M \sigma^0_{2M} - \frac{1}{4} \partial_2 \mathcal{R} + \frac{1}{2} D_3 \mathcal{J} + \frac{3}{2} \Omega_3 \mathcal{J} - \frac{3}{4} \Omega_2 \mathcal{R})
\]
\[
\text{Im} \Psi_3 = \frac{1}{\kappa \sqrt{2}} (-D^M \sigma^0_{3M} - \Omega^M \sigma^0_{3M} + \frac{1}{4} \partial_3 \mathcal{R} + \frac{1}{2} D_2 \mathcal{J} + \frac{3}{2} \Omega_2 \mathcal{J} + \frac{3}{4} \Omega_3 \mathcal{R}),
\]
(B.37)
or equivalently
\[
\Psi_3 = \frac{1}{\kappa \sqrt{2}} (D\sigma^0 + D\psi_2 + 3 \hat{\Omega} \psi_2),
\]
(B.38)
where we used the shorthands \( D \equiv D_2 - i D_3 \) and \( \hat{\Omega} \equiv \Omega_2 - i \Omega_3 \), and we defined the complex scalar
\[
D\sigma^0 \equiv D^M \sigma^0_{2M} + \Omega^M \sigma^0_{2M} - i (D^M \sigma^0_{3M} + \Omega^M \sigma^0_{3M}).
\]
(B.39)
Then, the gauge corrected Weyl scalar is given by
\[
\Psi^c_3(\eta, \xi^M) = \frac{1}{\kappa \sqrt{2}} (D\sigma^0 + \dot{D}\Psi_2 + 3 \hat{\Omega} \dot{\Psi_2})|_{\xi^I = H(\eta, \xi^M)}
\]
\[
= \frac{1}{\kappa \sqrt{2}} D\sigma^0|_{\xi^I = H(\eta, \xi^M)} + \frac{1}{\kappa \sqrt{2}} (D\Psi_2^c + 3 \dot{\Omega} \dot{\Psi_2}),
\]
where the second equality follows from the fact that neither \(\Psi_2, \Omega_M\), or \(q_{MN}\) depend on \(\xi^I\), and thus their functional form is unchanged after evaluating them in \(\xi^I = H(\eta, \xi^M)\). Therefore we just have to compute the first term on the right in the last equation, which reads
\[
D\sigma^0|_{\xi^I = H(\eta, \xi^M)} = D^M \sigma^0_M|_{\xi^I = H(\eta, \xi^M)} + \Omega^M \sigma^2_M
\]
\[= i(D^M \sigma^0_M|_{\xi^I = H(\eta, \xi^M)} + \Omega^M \sigma^3_M).
\]
In the previous expression we already made the substitution \(\sigma_{MN}^0|_{\xi^I = H(\eta, \xi^M)} = \sigma_{MN}\), using the definition of the supertranslation invariant variable \(\sigma_{MN}\), i.e. (4.20). From the definition (4.20) it is essential to check that the following relation holds
\[
D^M \sigma^2_M = (D^M \sigma^0_M + \partial_\eta \sigma^0_M \partial^M H)|_{\xi^I = H(\eta, \xi^M)}
\]
\[= \left( \begin{array}{c} D^M \sigma^0_M + \tau_{2M}^0 \Omega^1_M \right|_{\xi^I = H(\eta, \xi^M)}
\]
\[= D^M \sigma^0_M|_{\xi^I = H(\eta, \xi^M)} + \tau_{2M}^0 \Omega^1_M + \Omega^M \sigma^3_M + \Omega^2_M \Omega^M \]  \hspace{1cm} (B.42)

where, for the second equality, we have used the equations (4.16) and (4.18), and that the exact part of the Hajicek one form is given by \(\partial_\eta \eta = \Omega_0^M\). The last result allows us to express \(\Psi^c_3\) in terms of the gauge invariant variable \(\sigma_{MN}\) as follows
\[
\Psi^c_3(\eta, \xi^M) = \frac{1}{\kappa \sqrt{2}} (D\sigma + \dot{D}\Psi_2 + 3 \hat{\Omega} \dot{\Psi_2}),
\]
where \(D\sigma\) is now defined as
\[
D\sigma(\eta, \xi^M) \equiv D^M \sigma_{2M} + \Omega^{0|M}_M \sigma_{2M} - i(D^M \sigma_{3M} + \Omega^{0|M}_M \sigma_{3M}).
\]
It can be seen that the exact part of the Hajicek one form \(\Omega_0^M\) has been cancelled out, and thus \(D\sigma\) only involves the divergence free part \(\Omega_M^i\). Substituting the solution to the constraint equations (4.23) in (B.43) we arrive to our final result, which is given by (4.28).

Appendix C. Calculations for null infinity

C.1. Derivation of the BMS group

In this appendix we present a derivation of the BMS group of transformations at null infinity. We show that it can be described as the set of diffeomorphisms of the unphysical spacetime which preserve null infinity as a set of points, and that leave invariant both the metric tensor and the null normal on \(I^\perp\) up to a conformal transformation. The analysis is done in a similar fashion as in section 3.3, where we studied the set of diffeomorphisms preserving the metric tensor at a non-expanding horizon, i.e. the hypersurface symmetries, which include horizon supertranslations. Therefore, the present computation also serves as a check for our approach in section 3.3 to characterise horizon supertranslations.

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We will now characterise the set of diffeomorphisms of the unphysical spacetime \( F : \mathcal{M} \to \mathcal{M} \) which preserve the structure of null infinity implied by the definition of asymptotically flat spacetimes given in section 5, i.e. conditions (i), (ii) and (iii). More specifically, any diffeomorphism \( F \) in this set should satisfy the following conditions:

(i) Leave invariant the scalar products at \( \mathcal{I} \) up to a conformal transformation. That is, denoting \( g'_{\mu\nu} \equiv (F^* g)_{\mu\nu} \), and \( \Omega' \equiv F^* \Omega \) we should have

\[
g'_{\mu\nu} \doteq \omega^2 g_{\mu\nu}, \quad \text{and} \quad \Omega' \doteq \omega \Omega,
\]

so that \( \Omega'' - \Omega'' \doteq g'_{\mu\nu} \doteq g_{\mu\nu} \), where \( \doteq \) denotes equality on \( \mathcal{I} \).

(ii) Map null infinity to itself, \( F(\mathcal{I}) = \mathcal{I} \), or equivalently \( F^* \Omega \doteq \Omega \doteq 0 \).

(iii) Preserve the definition of the null normal, \( n \equiv d\Omega \).

The properties of this set of diffeomorphisms are more easily studied using a coordinate system of the unphysical spacetime adapted to \( \mathcal{I} \). We proceed as in section 3.3 for the case of non-expanding horizons. Since null infinity \( I \) is diffeomorphically identified with the abstract manifold \( \mathcal{I} \) via the embedding map \( \Phi \), we can use the coordinates on the later one, \( \xi^a \), to parametrise the hypersurface \( I \). The coordinate system on \( I \) is then extended off the hypersurface introducing a transverse coordinate \( r \), which is defined in terms of the rigging as \( \ell = \partial_r \), with \( r(\mathcal{I}) = 0 \), and then keeping the coordinates \( \xi^a \) constant along the integral curves of \( \ell \). Thus, the coordinate system for the unphysical spacetime reads \( x^\mu = \{ \xi^1, r, \xi^M \} \), so that the embedding map takes the simple form

\[
\Phi : \xi^a \longrightarrow x^\mu = \{ u = \xi^1, r = 0, x^M = \xi^M \}.
\]

The elements of coordinate basis \( \mathcal{B} = \{ n, \ell, e_M \} \) have the following explicit form

\[
n = \partial_0, \quad \ell = \partial_t, \quad e_M = \partial_M.
\]

Then the null normal has the coordinate form \( n \equiv d\Omega \), what follows from our conventions in section 2.1, \( n(\ell) = 1 \), and the properties of the null normal \( n(e_M) = n(n) = 0 \). Moreover, on this coordinate system the metric tensor has the following form at \( \mathcal{I} \)

\[
g_{\mu\nu}|_\mathcal{I} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & q_{MN}
\end{pmatrix}.
\]

Let us turn to the characterisation of the properties of the diffeomorphisms \( F \) satisfying the conditions (i), (ii) and (iii) above. From the condition (iii) it is immediate to find the required behaviour of the null normal under the pull back \( F^* \). Indeed, since \( \Omega \doteq 0 \), we have

\[
d\Omega' \doteq (\omega d\Omega + \Omega d\omega) \doteq \omega d\Omega \quad \implies \quad n' \doteq \omega n
\]

where \( n' \equiv F^* n \). From this we can also find the transformation of the normal vector \( n = g^{-1}(n, \cdot) \) under the pushforward of \( F \). On the one hand, from the definition of \( n \) we have that for any \( k \in T_p \mathcal{I} \)

\[
F^* g(n, k) \doteq \omega^2 g(n, k) \doteq \omega^2 n(k).
\]

On the other hand, since \( F \) maps \( \mathcal{I} \) to itself, it follows that \( n \) can change at most by a rescaling \( dF(n) = \alpha n \). Then

\[
F^* g(n, k) \doteq g(dF(n), dF(k)) \doteq \alpha g(n, dF(k)) \\
\doteq \alpha n(dF(k)) \doteq \alpha F^* n(k) \doteq \alpha \omega n(k).
\]
Comparing the two previous expressions we find $\alpha = \omega$. Let $y^\alpha = y^\alpha(x)$ be the explicit form for the diffeomorphism $F$, in a given set of coordinates, then from the condition on the pullback of $n$ (C.6) we find the following constraints on the mapping $F$

$$F^* n_\mu \doteq y^\alpha_\mu n_\alpha \implies y^1_\mu \doteq \omega \delta^1_\mu,$$  

where we use the short hands $y^\alpha_\mu = \partial_\mu y^\alpha$. From the condition on the pushforward of $n$ (C.7) we find

$$\operatorname{dF}(n)^\alpha \doteq y^\alpha_\mu n^\mu \implies y^0_\alpha \doteq \omega \delta^0_\alpha.$$  

(C.8)

From the condition on the pushforward of $n$ (C.7) we find

$$dF(n)^\alpha \doteq y^\alpha_\mu n^\mu \implies y^0_\alpha \doteq \omega \delta^0_\alpha.$$  

(C.9)

Collecting both results we have

$$y^0_0 \doteq y^1_1 \doteq \omega, \quad y^0_0 \doteq y^0_1 \doteq 0.$$  

(C.10)

The mapping $F$ satisfies conditions (C.1), if and only if the scalar products on the basis $B$ are mapped as follows

$$(F^* g)(n, n) \doteq 0, \quad (F^* g)(e_M, e_N) \doteq \omega^2 q_{MN}$$  

(C.11)

$$(F^* g)(n, e_M) \doteq 0, \quad (F^* g)(e_M, \ell) \doteq 0$$  

(C.12)

$$(F^* g)(n, \ell) \doteq \omega^2, \quad (F^* g)(\ell, \ell) \doteq 0.$$  

(C.13)

In components they read

$$g_{\alpha\beta} y^\alpha_1 y^\beta_1 \doteq g_{11} \omega^2 \doteq 0$$  

(C.14)

$$g_{\alpha\beta} y^\alpha_M y^\beta_N \doteq q_{MN} y^I_M y^I_N \doteq \omega^2 q_{MN}$$  

(C.15)

$$g_{\alpha\beta} y^\alpha_0 y^\beta_M \doteq \omega y^1_M \doteq 0$$  

(C.16)

The second equation of the first line implies that, on $I$, $y^I(x^M) \equiv y^I(x^M)|_{r=0}$ define a conformal symmetry of the metric $q_{MN}$ with conformal factor $\omega$.

$$q_{IJ} Y^I M Y^J N = \omega^2 q_{MN},$$  

(C.17)

Note that, since $y^I$ are constant along the null coordinate $u$, the conformal factor must satisfy $\mathcal{L}_{n_\omega} \doteq 0$. If we restrict ourselves to globally well-defined transformations, that is, to one-to-one mappings of the spatial sections of $I$ on to themselves, then the functions $Y^I$ generate a group isomorphic to the homogeneous orthochronous Lorentz group (see [23]).

The action of the diffeomorphism on the null coordinate at $I$ is determined by the function $f(u, x^M) \equiv y^0(u, x^M)|_{r=0}$, which is constrained by the first equality in (C.10), namely $y^0_0 \doteq \omega$. Thus, the function $f(u, x^M)$ has the general form

$$f(u, x^M) = \omega(x^M) (u + A(x^M),$$  

(C.18)

where $A(x^M)$ can be any smooth function of the spatial coordinates $x^M$. The remaining non-trivial conditions can be solved in terms of $f$ and $Y^I$ to give

$$y^I_1 \doteq - \frac{1}{\omega} f_M q^{MN} Y^I_N, \quad y^0_1 \doteq - \frac{1}{2\omega} f_M q^{MN}.$$

(C.19)
The set of transformations determined by the functions \( f, Y \) given by (C.17) and (C.18) define the BMS group (see e.g. chapter 1 in [65]). Null infinity supertranslations can be identified as those transformations of the BMS group with \( \omega = 1 \), and \( Y_1(x) = x^i \), that is \( f(u, x^M) = x^0 + A(x^M) \) (C.20).

The infinitesimal version of the defining conditions of BMS transformations can be recovered setting \( y_\alpha(x) \approx x_\alpha + \epsilon k_\alpha(x) \), in (C.1) and (C.5), where the vector field \( k_\alpha \) is the corresponding generator and \( \epsilon \ll 1 \) is a small real parameter. We obtain
\[
L_k g_{\mu\nu} = 2\lambda g_{\mu\nu}, \quad L_k n = -\lambda n,
\]
where \( \omega \approx 1 + \lambda \), and \( L_\mu \lambda = 0 \). This is precisely the definition used to characterised the BMS group in the works by Geroch and Ashtekar [26, 52] (see also [54]). For the second equality we have used that the definition of the Lie derivative of a vector field involves the pushforward of the inverse mapping \( F^{-1} \), this explains the minus sign on the second expression.

From the previous equations it is also straightforward to find the action of a supertranslation on the tensor \( Y_{ab} \) at null infinity. Indeed, if we adopt the gauge conventions in section 5, null infinity can be described as a non-expanding null hypersurface. Moreover, since supertranslations preserve exactly the metric tensor and the null normal at null infinity they can be identified with a hypersurface symmetry of \( \mathcal{I} \). Therefore, we can use the results in section 3.3 to find the transformation properties of the tensor \( Y_{ab} \) under null infinity supertranslations, which hold for arbitrary hypersurface symmetries of a generic non-expanding null hypersurface. Setting \( \kappa = \Omega = 0 \) and \( \hat{f}_1 = 1 \) on equation (3.30) we obtain
\[
Y'_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \Xi_{MN} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & D_M A_N \end{pmatrix},
\]
which is the relation we have used in the main text (5.4).

C.2. Boundary conditions for no-outgoing radiation

In this appendix we will derive the equations (5.12)–(5.14), that is, the constraints satisfied by the Schouten tensor \( S_{\mu\nu} \) of the unphysical spacetime in the absence of outgoing gravitational radiation at null infinity. Our starting point are the boundary conditions (5.10) expressed in terms of the leading order Weyl scalars \( \Psi_0^a \), and the Bianchi identities (5.11), relating the leading order unphysical Weyl tensor \( K_{\mu\nu\rho\sigma} \) with \( S_{\mu\nu} \).

We prepare our set up as described in section 2.3. Given a point \( \xi^a_0 \) at null infinity, the Weyl scalars \( \Psi_0^a \) can be expressed in terms of the Newman–Penrose null basis \( \{\ell, n, m, \overline{m}\} \), where \( \ell \) is the rigging vector, \( n \) the null normal vector to \( \mathcal{I} \), and the complex null vectors \( m \) and \( \overline{m} \) are defined as in (2.19). In addition we choose the coordinates on \( \mathcal{I} \) such that, at the point \( \xi^a = \xi^a_0 \), the spatial metric has the canonical form \( q_{MN} = \delta_{MN} \).

C.2.1. Condition on the second Weyl scalar. We begin deriving the constraint on \( S_{\mu\nu} \) which follows from imposing \( \text{Im} \Psi_2^0 = 0 \) at null infinity. The second Weyl scalar \( \Psi_2^0 \) has the form
\[
\Psi_2^0 = K_{\ell m \overline{m}}, \quad \text{Im} \Psi_2^0 = \frac{1}{2} (K_{\ell m \overline{m}} - K_{\ell \overline{m} m}).
\]
Here we will use the notation \( K_{\ell m \overline{m}} = K_{\mu\rho\sigma}^{ij} \ell^i \ell^j m^\rho \overline{m}^\sigma \), and similar expressions for contractions of a tensor with the elements of a basis. The last term can be rewritten using the symmetries of the Weyl tensor, and the first (algebraic) Bianchi identity
\[ \text{Im} \Psi_0^0 = \frac{1}{2} \left( K_{\alpha \beta 
oleft \alpha} - K_{\alpha \beta \beta} \right) = \frac{1}{2} \left( K_{\alpha \beta 3} + K_{\alpha 3 \beta} \right) = \frac{1}{2} K_{\alpha \beta 3} = \frac{1}{2} K_{32\beta}. \]  

(C.24)

Using the Bianchi identity (5.11), together with the definitions for the connection coefficients (2.7), and the gauge conventions (5.1) we find

\[ \text{Im} \Psi_0^0 = \frac{1}{2} K_{32\beta} = -\frac{1}{2} \nabla_{[\mu} S_{\nu]}e^\mu e_2^\nu e_2^\nu = -\frac{1}{2} \left( D_{[3} S_{2]\beta} - \Xi_{[3} S_{2]\beta} \right), \]  

(C.25)

where \( S_{MN} = S_{\mu\nu} e^\mu e_\nu \) and \( S_{MN} = S_{\mu\nu} e^\mu e_\nu \). It is easy to check that only the traceless part of \( \Xi_{MN} \) contributes in the previous equation. The condition \( \text{Im} \Psi_0^0 = 0 \) implies that the previous expression should vanish, what can be written more covariantly as

\[ D_{[M} S_{N]} \ell = \Xi_{[M} S_{N]} \ell. \]  

(C.26)

C.2.2. Condition on the third Weyl scalar. The vanishing of the third Weyl scalar on \( \mathcal{I} \), \( \Psi_3^0 = 0 \), leads to two constraints on the Schouten tensor. First let us write third Weyl scalar as

\[ \Psi_3^0 = K_{\alpha \beta \gamma \delta} = K_{\alpha \beta \gamma \delta} = \frac{1}{\sqrt{2}} (K_{\alpha \beta \gamma \delta} - i K_{\alpha \beta \gamma \delta}). \]  

(C.27)

Using the Bianchi identity (5.11), together with (2.7) and (5.1), we can rewrite this expression as

\[ K_{\alpha \beta \gamma \delta} = -n^\mu e_\mu^\ell \nabla_{[\mu} S_{\nu]e^\ell e_\nu e_\nu = - (\partial_\mu S_{M\ell} - \partial_M S_{\ell \mu}), \]  

(C.28)

and therefore

\[ \Psi_3^0 = -\frac{1}{\sqrt{2}} (\partial_\mu S_{M\ell} - i \partial_\mu S_{M\ell}). \]  

(C.29)

In the absence of outgoing radiation on \( \mathcal{I} \) the previous expression vanishes, or equivalently

\[ \partial_\mu S_{M\ell} = 0. \]  

(C.30)

The second constraint for the Schouten tensor can be found writing the third Weyl scalar as follows

\[ \Psi_3^0 = \frac{1}{\sqrt{2}} (K_{\alpha \beta \gamma \delta} - i K_{\alpha \beta \gamma \delta}) = -\frac{1}{\sqrt{2}} (K_{\alpha \beta \gamma \delta} - i K_{\alpha \beta \gamma \delta}) = \frac{1}{\sqrt{2}} (D_{[3} S_{2]\beta} - i D_{[3} S_{2]\beta}). \]  

(C.31)

The second equality is a consequence of the Weyl tensor being traceless, \( g^{\mu \nu} K_{\mu \nu \alpha \beta} = 0 \). In particular, using equation (2.6) for the inverse metric, with \( \delta_{MN} = \delta_{MN} \), and the symmetries of the Weyl tensor we find

\[ g^{\mu \nu} K_{\mu \nu \alpha \beta} = K_{\alpha \beta \alpha \beta} + K_{\alpha \beta \alpha \beta} + K_{\alpha \beta \alpha \beta} = K_{\alpha \beta \alpha \beta} + K_{\alpha \beta \alpha \beta} = 0. \]  

(C.32)

The last equality in (C.31) is obtained after using the Bianchi identity (5.11), together with (2.7) and (5.1). Thus, the vanishing of the third Weyl scalar also implies the tensor equation

\[ D_{[M} S_{N]} \ell = 0. \]  

(C.33)

C.2.3. Condition on the fourth Weyl scalar. The last constraint on \( S_{\mu \nu} \) is obtained from the vanishing of \( \Psi_4^0 \), which reads

\[ \Psi_4^0 = K_{\alpha \beta \gamma \delta} = \frac{1}{2} (K_{\alpha \beta \gamma \delta} - K_{\alpha \beta \gamma \delta}) = i K_{\alpha \beta \gamma \delta}. \]  

(C.34)
The Bianchi identities (5.11), (2.7) and (5.1) imply $K_{MnN} = \partial_n S_{MN}$, and thus

$$\Psi_4^0 = \frac{1}{2} \left( \partial_n S_{22} - \partial_n S_{33} \right) - i \partial_n S_{23} = 0. \tag{C.35}$$

In addition, the Schouten tensor satisfies $S^M_M = R$, which together with $q_{MN} = \delta_{MN}$, implies $\partial_n R = \partial_n S_{22} + \partial_n S_{33} = 0$. Therefore, we can summarise the constraints which follow from $\Psi_4^0 = 0$ in the tensor equation

$$\partial_n S_{MN} = 0. \tag{C.36}$$

This completes our proof of equations (5.12)–(5.14).

### C.3. Constraint equations at null infinity and radiative vacua

In this appendix we prove various results needed in section 5 to derive the constraint equations of null infinity, and to find their solutions in the absence of outgoing radiation.

#### C.3.1. Consistency check for the constraint equations

In this section we check that the dependence of $\Xi_{MN}$ on the null coordinate $\xi^1$ implied by the constraint equations (5.19) is consistent with the identity (5.20).

**Proposition C.1.** Let $\Xi_{MN}$ be a solution to the constraint equations of null infinity (5.19). Then, if the relation (5.20) is satisfied at a particular value of the null coordinate $\xi^1_0$, then it will be satisfied for all values of $\xi^1$.

**Proof.** The constraint equation for $\Xi_{MN}$ at null infinity can be obtained from (2.14) substituting the gauge fixing conditions (5.1), and the form of the source term (5.18). The result is

$$\partial_n \Xi_{MN} = -\frac{1}{2} (S_{MN} + S_{M}^{\ell} q_{MN}). \tag{C.37}$$

Note that this equation reduces to (5.19) when we express it in terms of the equivalence relation (5.17). The relation (5.20) is satisfied at a given value of the null coordinate $\xi^1_0$; they will be satisfied for all values of $\xi^1$ provided the following expression vanishes on $I$

$$\partial_n \left( D_{[M} \Xi_{N]}^\ell - \frac{1}{2} q_{[M} M_{N]} q_{\ell} N \right) = D_{[M} \partial_n \Xi_{N]}^\ell - \frac{1}{2} q_{[M} M_{N]} q_{\ell} N \partial_n \Xi_{[N]}^\ell, \tag{C.38}$$

where the equality is obtained using that the metric $q_{MN}$ and its Levi-Civita connection are independent of $\xi^1$. Thus, we need to prove that the previous expression is zero. Substituting the constraint equation (C.31) we find

$$D_{[M} \partial_n \Xi_{N]}^\ell - \frac{1}{2} q_{[M} M_{N]} q_{\ell} N = -\frac{1}{2} D_{[M} S_{N]}^\ell + \frac{1}{2} \partial_{[M} S_{N]}^\ell q_{\ell} N - \frac{1}{2} q_{[M} M_{N]} q_{\ell} N \partial_n \Xi_{[N]}^\ell. \tag{C.39}$$

As the indices $M, N, P = \{1, 2\}$ and $N \neq M$, then $P$ must be either equal to $M$ or $N$. Without loss of generality we choose $P = M$. Moreover, in the following we will also assume that we have chosen the coordinates so that $q_{MN} = \delta_{MN}$ locally. We obtain

$$-\frac{1}{2} (D_{[M} S_{N]}^\ell + \partial_{[M} S_{N]}^\ell) = \frac{1}{2} (K_{MNM} + K_{\ell MN}). \tag{C.40}$$
where we have also used the Bianchi identity (5.11) (contracted with elements of the basis \(B = \{\eta, \ell, eM\}\)) in the second equality. Using the symmetries of the Weyl tensor, and that \(q_{MN} = \delta_{MN}\), we find
\[
\frac{1}{2} (K_{MN} + K_{NM}) = \frac{1}{2} (q^{A\ell} K_{AB} + K_{nN} n^{e}),
\]
(C.41)
where \(A, B\) run over \(\{1, 2\}\). The previous expression can be written in the form
\[
\frac{1}{2} (q^{AB} K_{AB} + K_{nN} n^{e}) = \frac{1}{2} g^{\mu\nu} K_{\mu} n^{\nu},
\]
(C.42)
where we used the formula (2.6) for the inverse metric. Summarising, we have found the relation
\[
\partial_{n} (D_{P} \Xi_{M} - \frac{1}{2} q_{MP} S_{N}^{[\ell]} = \frac{1}{2} g^{\mu\nu} K_{\mu} n^{\nu},
\]
(C.43)
which can be easily checked to be always zero, since the contraction of any two indices of the Weyl tensor is always zero.

C.3.2. Schouten tensor at null infinity with no outgoing radiation. We will begin proving the relation (5.15) which is satisfied by the Schouten tensor in the absence of outgoing radiation through \(I\). According to our discussion above, the boundary conditions (5.10) for no outgoing radiation imply the constraints (C.33), (C.36) for \(S_{\mu\nu}\). Moreover, as we proved in section 5, in the divergence free conformal gauge (5.1) the Schouten tensor also satisfies \(S_{M}^{M} = R\).

**Proposition C.2.** The tensor \(S_{MN} = \frac{1}{2} R q_{MN}\) is the unique solution of the constraints (C.33), (C.36) whose trace is given by \(S_{M}^{M} = R\).

**Proof.** The tensor \(S_{MN}\) can be decomposed in its trace and traceless parts as follows
\[
S_{MN} = \sigma_{MN} + \frac{1}{2} R q_{MN},
\]
(C.44)
where \(\sigma_{M}^{M} = 0\). Then \(S_{MN}\) is a solution to the constraints (C.33), (C.36) if and only if the traceless part \(\sigma_{MN}\) satisfies
\[
\partial_{n} \sigma_{MN} = 0, \quad D_{M} \sigma_{N}^{[\ell]} = 0, \quad \sigma_{MN} q^{MN} = 0.
\]
(C.45)
To complete our proof we just need to use a result by Geroch [52], who showed that the unique solution to these equations is \(\sigma_{MN} = 0\).

Since our setting is slightly different to that of [52], we will reproduce here the proof of the last statement:

**Lemma C.1.** The only solution to the system of equations (C.45) is the trivial one, that is \(\sigma_{MN} = 0\).

**Proof.** Let \(\sigma_{MN}\) be a tensor satisfying (C.45) and \(k^{M}\) a Killing vector\(^{18}\) of \(q_{MN}\). For convenience we work in a coordinate system such that \(q_{MN} = \delta_{MN}\) locally. We will begin proving that

\(^{18}\)Recall that, in our conformal gauge, \(q_{MN}\) represents the geometry of a two dimensional sphere with constant curvature.
the spatial tensor \( \lambda_{MN} \equiv D_{[M}(\sigma_{N]k^P) = 0 \) is vanishing. Due to the antisymmetry of \( \lambda_{MN} \) we already have \( \lambda_{11} = \lambda_{22} = 0 \) and \( \lambda_{12} = -\lambda_{21} \), and thus, it only remains to show that \( \lambda_{12} = 0 \). Using (C.45) the it an be checked that the components of \( \lambda_{MN} \) satisfy

\[
\lambda_{MN} = k^P D_{[M}[\sigma_{N]}k^P + \sigma_{[NP}D_{M]}k^P = \sigma_{[NP}D_{M]}k^P,
\]

and therefore we also have

\[
\lambda_{12} = \sigma_1^1 D_2 k_1 + \sigma_2^1 D_1 k_2 - \sigma_2^2 D_1 k_1 - \sigma_1^2 D_2 k_2 = \sigma_1^1 D_2 k_1 - \sigma_2^2 D_2 k_1 = (\sigma_1^1 + \sigma_2^1)D_2 k_1 = 0.
\]

Here we have used \( \sigma^M_M = 0 \), and also the following properties of the killing vector \( k^M \)

\[
D_{(M}k_{N)} = 0 \quad \implies \quad D_1 k_1 = D_2 k_2 = 0, \quad D_1 k_2 = -D_2 k_1,
\]

which are a direct consequence of the killing equation. Therefore we have that \( D_{[M}(\sigma_{N]}k^P) = 0 \) and, since the spatial sections of \( \mathcal{I} \) are topologically equivalent to \( \mathbb{S}^2 \), this implies that \( \sigma_{NP}k^P = \partial_N \alpha \) for some smooth function \( \alpha = \alpha(\xi^M) \). Actually, it is straightforward to check that \( \alpha \) must be harmonic

\[
\Delta \alpha = D^M(\sigma_{MP}k^P) = k^P q^{MN} D_N \sigma_{MP} + \sigma_{MP} D^N k^P = 0.
\]

The last term vanishes because \( D_M k_N \) is antisymmetric and \( \sigma_{MN} \) symmetric, and the first one can be shown to be zero using the second of the equations (C.45) together with \( \sigma_{MN} = \sigma_{NM} \) and \( \sigma^M_M = 0 \)

\[
k^P q^{MN} D_N \sigma_{MP} = -k^P D_P \sigma^M_M = 0.
\]

The only harmonic functions on the spatial sections of \( \mathcal{I} \) (which are by assumption compact and simply connected) are constants, and thus we have \( \sigma_{MN}k^M = \partial_M \alpha = 0 \). Finally, since the killing vectors \( k^M \) of the sphere span all of the tangent space, we can conclude that \( \sigma_{MN} = 0 \). 

\[\square\]

C.3.3. Constraint equations in the absence of outgoing radiation at null infinity. We will now prove that, in the absence of outgoing radiation thought \( \mathcal{I} \), the general solution to the constraint equations (2.12), (2.13) and (2.14) at null infinity is given by (5.26).

As discussed in section 5, in the conformal gauge (5.1) the only non-trivial constraint equation is the one of the transverse connection \( \Xi_{MN} \) (2.14), together with (5.20). The final form of these constraint equations can be found substituting in them the expression for the gauge conditions (5.1), the form of the source term (5.18), and using the fact that in the absence of radiation the Schouten tensor satisfies (5.20) and \( S_{\ell \ell} = \partial_\ell S_\ell \) (see section 5). The result is

\[
\partial_\ell \Xi_{MN} = -\frac{1}{4} (\mathcal{R} + 2 \partial_\ell S_\ell) q_{MN}, \quad\quad D_{[M}[\Xi_{N]P] = \frac{1}{2} q_{P[M} \partial_{N]} S_\ell.
\]

In order to eliminate the redundancy associated with supertranslations we specify a fiducial vacuum connection \( \Xi^0_{MN} \), and then characterise a generic vacuum connection \( \Xi_{MN} \) by the difference \( \Sigma_{MN} \equiv \Xi_{MN} - \Xi^0_{MN} \). It is straightforward to check that this quantity is invariant under supertranslations (5.4). Thus, given a fixed fiducial vacuum \( \Xi^0_{MN} \), we can characterise the full set of radiative vacua at null infinity finding the most general form of \( \Sigma_{MN} \) which is consistent with the constraint equations (C.51).
**Theorem C.1.** Let $\Sigma_{MN} \equiv \Xi_{MN} - \Xi_{MN}^0$ be the difference between two transverse connections, $\Xi_{MN}$ and $\Xi_{MN}^0$, which solve the constraint equations (C.51) in the absence of outgoing radiation through $I$. Then $\Sigma_{MN}$ has the general form

$$\Sigma_{MN} = D_M f_N + \frac{1}{2} R q_{MN} - \frac{1}{2} (S_\ell - S_0^\ell) q_{MN},$$

where $f(\xi)$ is smooth function on $I$ satisfying $\partial_n f = 0$, and the potentials $S_\ell$ and $S_0^{\ell}$ are defined by $S_{\ell\ell} = \partial_\ell S_\ell$ and $S_0^{\ell\ell} = \partial_{0\ell} S_0^\ell$, in terms of the Schouten tensor associated to the connections $\Xi_{MN}$ and $\Xi_{MN}^0$, respectively.

**Proof.** Due to the linearity of the equations (C.51), $\Sigma_{MN}$ should satisfy

$$\partial_n \Sigma_{MN} = -\frac{1}{2} \partial_n (S_\ell - S_0^\ell) q_{MN}, \quad D_M \Sigma_{N|P} = -\frac{1}{2} D_M q_{N|P} (S_\ell - S_0^\ell).$$

With the change of variables $\hat{\Sigma}_{MN} \equiv \Sigma_{MN} + \frac{1}{2} (S_\ell - S_0^\ell) q_{MN}$, the previous equations take the simpler form

$$\partial_n \hat{\Sigma}_{MN} = 0, \quad D_M \hat{\Sigma}_{N|P} = 0.$$  

Contracting the second equation with $q^{MP}$ we also find

$$q^{MP} D_M \hat{\Sigma}_{N|P} = 0 \implies D^n \hat{\Sigma}_{MN} = D_M \hat{\Sigma}_N^N.$$  

In order to solve (C.54) and (C.55) consider the following decomposition of $\hat{\Sigma}_{MN}$ [78, 79]

$$\hat{\Sigma}_{MN} = D_{MN} \chi + D_M A_N + W_{MN} + t q_{MN},$$

where $\chi$ and $t$ are two scalar fields on $I$ satisfying $\partial_n \chi = \partial_n t = 0$, and $t = \hat{\Sigma}_M^M$. The vector $A_M$ and the tensor $W_{MN}$ are both independent of the null coordinate $\partial_n A_M = \partial_n W_{MN} = 0$, and moreover $W_{MN}$ is also transverse and traceless

$$D^N W_{MN} = 0, \quad W^M_M = 0.$$  

The operator $D_{MN} \chi$ is defined by

$$D_{MN} \chi \equiv D_M D_N \chi - (\Delta + \frac{1}{2} R) \chi,$$

and has the property that $D_{MN} \chi$ is transverse for all scalar fields $\chi$. As was proven in [78], in the previous decomposition the scalar $\chi$ can only be determined up to $\chi \to \chi + \lambda$ where $\lambda$ is a solution to $(\Delta + \frac{1}{2} R) \lambda = 0$, and the vector $A_M$ is fixed up to $A_M \to A_M + k_M$, where $k_M$ is a killing vector of $q_{MN}$. The tensor $W_{MN}$ and $t$ are both uniquely determined by $\hat{\Sigma}_{MN}$. Inserting this decomposition (C.56) in the equation (C.55) we find

$$D^n D_M A_N = \partial_M t,$$

which can be solved by $A_M = \partial_M \phi$, where $\phi$ is a scalar satisfying $\Delta \phi + \frac{1}{2} R \phi = \frac{1}{2} t$. The condition $\hat{\Sigma}_M^M = t$ leads to the equation
\[ q^{MN} D_{MN} \chi - 2D^M A_M = 0 \quad \Rightarrow \quad \Delta \chi + \mathcal{R} \chi = 2\Delta \phi. \]  
(C.60)

The decomposition (C.56) is more conveniently expressed in terms of the combination \( f \equiv \chi + 2\phi \), which satisfies \( \Delta f + \mathcal{R} f = 2t \). We obtain the expression

\[ \hat{\Sigma}_{MN} = D_M D_N f + \frac{1}{2} \mathcal{R} f q_{MN} + W_{MN}. \]  
(C.61)

Substituting this result into (C.54) we find the following constraint for \( W_{MN} \)

\[ D_M W_{NP} = 0. \]  
(C.62)

Thus we can see the lemma C.1 applies to \( W_{MN} \), since this tensor is constant along the null direction and traceless, and therefore we can conclude that \( W_{MN} = 0 \). Collecting these results, the final form of the solution to the equations (C.51) is

\[ \Sigma_{MN} = D_M f_N + \frac{1}{2} \mathcal{R} f q_{MN} - \frac{1}{2} (S_{\ell} - S_{\ell}^0) q_{MN}, \]  
(C.63)

which is the expression we were looking for. Note that the freedom to shift \( A_M \) by a killing vector leaves this expression invariant, while the ambiguity to shift \( \chi \to \chi + \lambda \) amounts to shifting \( f \to f + \lambda \).

It is trivial to check that (C.63) reduces to the form of the solution we presented in the main text, equation (5.26), when we express it in terms of the equivalence relation (5.17).

**ORCID iDs**

K Sousa \( \text{https://orcid.org/0000-0001-7755-339X} \)

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