FOURIER TRANSFORM INVERSION: BOUNDED VARIATION, POLYNOMIAL GROWTH, HENSTOCK–STIELTJES INTEGRATION

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Abstract. In this paper we prove pointwise and distributional Fourier transform inversion theorems for functions on the real line that are locally of bounded variation, while in a neighbourhood of infinity are Lebesgue integrable or have polynomial growth. We also allow the Fourier transform to exist in the principal value sense. A function is called regulated if it has a left limit and a right limit at each point. The main inversion theorem is obtained by solving the differential equation \( df(t) - i\omega f(t) = g(t) \) for a regulated function \( f \), where \( \omega \) is a complex number with positive imaginary part. This is done using the Henstock–Stieltjes integral. This is an integral defined with Riemann sums and a gauge. Some variants of the integration by parts formula are also proved for this integral. When the function is of polynomial growth its Fourier transform exists in a distributional sense, although the inversion formula only involves integration of functions and returns pointwise values.

1. Introduction

If \( f: \mathbb{R} \to \mathbb{R} \) then its Fourier transform is \( \hat{f}(s) = \int_{-\infty}^{\infty} e^{-ist} f(t) \, dt \). A sufficient condition for existence of \( \hat{f} \) on \( \mathbb{R} \) is that \( f \in L^1(\mathbb{R}) \); and then \( \hat{f} \) is uniformly continuous on \( \mathbb{R} \). Under the same hypothesis, the Riemann–Lebesgue lemma says \( \hat{f}(s) \) has limit 0 as \( |s| \to \infty \). We also use the fact that if \( f \) is positive and decreases to 0 then the integral \( \int_0^\infty e^{-ist} f(t) \, dt \) exists for \( s \neq 0 \). However, under this condition we need not have \( \hat{f}(s) \to 0 \) as \( s \to \infty \).

The process of recovering \( f \) from \( \hat{f} \) is known as Fourier inversion. A basic theorem is that of Jordan. If \( f \in L^1(\mathbb{R}) \) and is also of bounded variation on the compact interval \([a, b] \) then for each \( x \in (a, b) \)

\[
\frac{f(x-) + f(x+)}{2} = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} e^{ixs} \hat{f}(s) \, ds. \tag{1.1}
\]
Here, the inversion is given by a principal value integral. The usual proofs use the Riemann–Lebesgue lemma and properties of the Dirichlet kernel. See, for example, [3], [4] and [20, §1.9]. These references also supply a suitable background on Fourier transforms.

We prove Fourier inversion theorems under the following assumptions on \( f \).

If the inversion is to hold at \( x \in (a, b) \) then \( f \) is assumed to be of bounded variation on \( [a, b] \). On each compact interval that does not intersect \( (a, b) \), the function \( f \) is locally Lebesgue integrable or has a principal value integral. In a neighbourhood of infinity, \( f \) is Lebesgue integrable, is of bounded variation, or is asymptotic to a polynomial. In the latter two cases, the Fourier transform may only exist as a distribution and yet the inversion only involves integration of functions and yields pointwise values.

There are of course many criteria besides local bounded variation in the literature applied to behaviour of \( f \) at the inversion point. For example, Dini’s test or degrees of differentiability. See [3] and [20]. Koekoek [9] has provided a simple proof of the inversion theorem that applies at points of differentiability or for Lipschitz or Hölder continuous functions.

The outline of the paper is as follows. For a regulated function (one having a left limit and a right limit at each point) the following identity is proved in Lemma 2.2:

\[
\frac{f(x^-) + f(x^+)}{2} = \int_{-\infty}^{\infty} H(x-t)e^{i\omega(x-t)}[df(t) - i\omega f(t) \, dt].
\]

Here \( H \) is the Heaviside step function. The above identity is proved using the Henstock–Stieltjes integral, which is described in Section 6. Some formulas related to integration by parts are proved here. An integral representation of the Heaviside step function is given in Lemma 2.1. Inserting this in (1.2) and changing the orders of integration completes the proof of Jordan’s theorem (Theorem 3.1). Pringsheim’s theorem, for which \( f \) is of bounded variation with limit 0 at \( \pm \infty \) but need not be in \( L^1(\mathbb{R}) \), follows as Corollary 3.6. In Pringsheim’s theorem, when the condition that \( f \) vanishes at infinity is dropped, the Fourier transform may only exist as a distribution (generalised function). However, a pointwise inversion theorem, in which only functions need to be integrated, is given in Theorem 4.1. Similarly in Theorem 4.3 for functions that are asymptotic to a polynomial in a neighbourhood of infinity. Functions for which the Fourier transform exists in a principal value sense are considered in Theorem 5.1.

The Henstock–Stieltjes integral is defined using a gauge that controls function evaluations in Riemann sums. Unlike the Riemann–Stieltjes integral, integration is defined over the entire real line without resorting to improper integrals. Also, the function and integrator can have coincident jump discontinuities. In Example 6.1 it is shown how to compute (1.2) as a Henstock–Stieltjes integral and that the corresponding Riemann–Stieltjes integral can fail to exist, even allowing for improper integrals.

Our results assume the function is of bounded variation in a neighbourhood of the point \( x \) in (1.1). For other Fourier analysis results for functions of bounded variation see [3], [4] and [20, §1.9].
variation, see [10], [11] and [15]. This latter paper uses improper Riemann–Stieltjes integrals.

Some of our results can be proved using summability techniques for distributions developed by R. Estrada and J. Vindas. See [17], Section II.5.2 and references therein.

2. Preliminary results

The two main parts of our proof of the inversion theorem are a contour integral for the Heaviside step function and a representation of a function as a Henstock–Stieltjes integral.

The Heaviside step function is

\[ H(x) = \begin{cases} 
1, & \text{if } x > 0 \\
1/2, & \text{if } x = 0 \\
0, & \text{if } x < 0.
\end{cases} \] (2.1)

And, the signum function is \( \text{sgn}(x) = 2H(x) - 1 \).

Lemma 2.1. Let \( p \) be a real number and let \( \omega \) be a complex number with positive imaginary part. Then

\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ips}}{s - \omega} \, ds = H(p)e^{ip\omega}. \] (2.2)

Carrying out the same calculation with \( \omega = 0 \) shows that \( \int_{0}^{\infty} \frac{\sin(px)}{x} \, dx = (\pi/2)\text{sgn}(p) \).

If \( p \neq 0 \), the integral in (2.2) converges conditionally. If \( p = 0 \) or \( \omega \) is real, the integral exists in the principal value sense. It is sometimes called Perron’s lemma, although it seems to have been first derived by Cauchy (cf. [22, p. 123]). We quote some estimates from the proof in [19] in order to justify some limit operations in the proof of Theorem 3.1.

Suppose \( g \in L^1(\mathbb{R}) \) and \( \omega \in \mathbb{C} \). If \( f \) is absolutely continuous and is a solution of the differential equation \( f'(t) - i\omega f(t) = g(t) \) then

\[
f(x) = f(0)e^{i\omega x} + \int_{0}^{x} e^{-i\omega t} g(t) \, dt
= f(0)e^{i\omega x} + \int_{0}^{\infty} H(x-t)e^{i\omega(x-t)}[f'(t) - i\omega f(t)] \, dt.
\] (2.3)

Representations of this type were used in [19] to prove a Fourier inversion theorem. See Corollary 3.2 below.

If \( -\infty \leq a < b \leq \infty \) then the function \( f : (a, b) \to \mathbb{R} \) is regulated if for each \( x \in (a, b] \) and each \( y \in [a, b) \) the one-sided limits exist; \( \lim_{t \to x^-} f(t) = f(x-) \in \mathbb{R} \) and \( \lim_{t \to y^+} f(t) = f(y+) \in \mathbb{R} \). If \( f \) is defined on \( (a, b) \) we will assume its domain has been extended to \( [a, b] \) with the above limits. Regulated functions are bounded and have at most a countable number of finite jump discontinuities. They are thus measurable and locally summable. Denote the space of regulated functions by \( R([a, b]) \). For more on regulated functions see [12] or [14].

An analogue of (2.3) for regulated functions is the following lemma.
Lemma 2.2. Let $-\infty < a < x < b < \infty$. (a) Let $\omega$ be a complex number with positive imaginary part. Let $f \in L^1((-\infty, b])$ and regulated on $[a, b]$. Then

$$\frac{f(x-) + f(x+)}{2} = \int_{-\infty}^{\infty} H(x-t)e^{i\omega(x-t)}[df(t) - i\omega f(t)\, dt].$$

(b) Let $\omega$ be a complex number. Let $f$ be regulated on $[a, b]$. Then

$$\frac{f(x-) + f(x+)}{2} = \int_{a}^{b} H(x-t)e^{i\omega(x-t)}[df(t) - i\omega f(t)\, dt] + f(a)e^{i\omega(x-a)}.$$  

The proof uses the Henstock–Stieltjes integral and variants on the integration by parts formula. See section 6 and 7. The lemma is false for Riemann–Stieltjes integrals. See Example 6.1.

For each $N \in \mathbb{N}$ a partition of $[a, b]$ is a collection of points $a = x_0 < x_1 < \ldots < x_N = b$. If $-\infty \leq a < b \leq \infty$ then function $f$ is of bounded variation on $[a, b]$ if the supremum of $\sum_{n=1}^{N}|f(x_n) - f(x_{n-1})|$ is bounded, where the supremum is taken over all partitions of $[a, b]$. In this case the supremum over all partitions is labeled $Vf$. Functions of bounded variation are regulated. For more on functions of bounded variation see [1] or [12].

3. INVERSION THEOREM

We now use formula (2.4) and the integral representation of the Heaviside step function in Lemma 2.1 to prove an inversion formula for functions of bounded variation. Notice that the proof does not use the Riemann–Lebesgue lemma or the Dirichlet kernel.

**Theorem 3.1** (Jordan). Let $f \in BV([-\infty, \infty]) \cap L^1(\mathbb{R})$. Then for each $x \in \mathbb{R}$,

$$\frac{f(x-) + f(x+)}{2} = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} e^{ixs} \hat{f}(s)\, ds.$$  

**Proof.** With Lemma 2.1 and Lemma 2.2 we have

$$\frac{f(x-) + f(x+)}{2} = \int_{-\infty}^{\infty} H(x-t)e^{i\omega(x-t)}[df(t) - i\omega f(t)\, dt]$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-T}^{T} \frac{e^{i(x-t)s}}{s - \omega}\, ds\, dt$$

$$= \frac{1}{2\pi i} \lim_{T \to \infty} \int_{-T}^{T} \int_{-\infty}^{\infty} \frac{e^{ixs}}{s - \omega} [df(t) - i\omega f(t)\, dt].$$

Since $f \in L^1(\mathbb{R})$ and is of bounded variation, dominated convergence and then the Fubini–Tonelli theorem allow us to bring the limit outside the integral and then change the orders of integration. See equation (14) in [19]. When $t \leq x$ this shows $|I_T| \leq 2\pi$. The magnitude of the residue is $e^{-(x-t)\eta} \leq 1$, where $\eta$ is the imaginary part of $\omega$. When $t > x$ we get the same estimate on $I_T$ when the half circle below the real axis is used. Now the residue is zero. Hence, we can bring the $T$ limit outside the integral. And, $\frac{e^{ixs}}{s - \omega} \leq 1/\eta$. We can then change the orders of integration.
Now consider \( \int_{-\infty}^{\infty} e^{-ist} df(t) \). Since \( f \in L^1(\mathbb{R}) \) and is of bounded variation it necessarily has limit 0 at \( \pm \infty \). Dirichlet’s test then shows existence of the integral for each \( s \in \mathbb{R} \). Defining this as a Henstock–Stieltjes integral requires giving the function \( t \mapsto e^{-ist} \) values at \( \pm \infty \). It is immaterial how this is done because every \( \gamma \)-fine tagged partition of \([-\infty, \infty]\) will contain a term \( e^{-is\infty} [f(x) - f(x_{N-1})] \). No matter what value is given to \( e^{-is\infty} \) this term can be made as small as desired by taking \( x_N \) large enough. Similarly at \(-\infty\). This lets us write

\[
\int_{-\infty}^{\infty} e^{-ist} df(t) = \lim_{Y \to \infty} \int_{X}^{Y} e^{-ist} df(t)
\]

\[
= \lim_{Y \to \infty} \left[ e^{-isY} f(Y) \right] - \lim_{X \to -\infty} \left[ e^{-isX} f(X) \right] - \lim_{Y \to \infty} \int_{X}^{Y} f(t) d[e^{-ist}]
\]

\[
= is \int_{-\infty}^{\infty} e^{-ist} f(t) dt,
\]

where the integration by parts formula (6.2) and Proposition 6.3 are applied on each finite interval \([X, Y]\). □

Since the exponential function is continuous the last integral can also be evaluated as the limit of a Riemann–Stieltjes integral over the compact interval \([X, Y]\) and then limits taken.

An anonymous referee has pointed out that: “Theorem 3.1 is essentially contained in Theorem 15 from [6] when applied with any \( 1 \leq p < \infty \), since under the author hypothesis the Fourier transform under consideration is \( O(1/|s|) \) at infinity. Alternatively, Theorem 3.1 follows at once by combining Theorem 13 in [6] with the distributional version of Littlewood’s Tauberian theorem from Theorem 4.1 in [7].”

We now present three corollaries that follow with different assumptions on \( f \). In Corollary 3.2 we note that if \( f \) is absolutely continuous and \( f' \in L^1(\mathbb{R}) \) then \( f \in BV([-\infty, \infty]) \). Usual methods with the Dirichlet kernel (for example, [20, §1.9]) in Corollary 3.4 show that bounded variation is only a local requirement. Corollary 3.6 is similar to results due to Pringsheim. See [20, p. 15].

**Corollary 3.2.** Let \( f \in L^1(\mathbb{R}) \) such that \( f \) is absolutely continuous and \( f' \in L^1(\mathbb{R}) \). Then for each \( x \in \mathbb{R} \)

\[
f(x) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} e^{ixs} \hat{f}(s) \, ds.
\]

(3.3)

This was also proved in [19], where the assumptions allowed a proof with the Lebesgue rather than Henstock–Stieltjes integral.

**Example 3.3.** Define \( f \) by

\[
f(x) = \begin{cases} 
0, & x \leq 0 \\
1/\log(x), & 0 < x \leq 1/e \\
-1/(ex)^2, & x \geq 1/e.
\end{cases}
\]
Then $f$ is absolutely continuous, the conditions of the corollary hold and we get (3.3) at each point. Note that $f$ is not differentiable at 0 or $1/e$ and that $f$ is not Lipschitz or Hölder continuous at the origin. Hence, Koekoek’s [9] result does not apply at the origin.

**Corollary 3.4.** Let $f \in BV([a,b])$ for some $-\infty < a < b < \infty$ and $f \in L^1((-\infty,a])$ and $f \in L^1([b,\infty))$. Let $x \in (a,b)$. Then

$$\frac{f(x-) + f(x+)}{2} = \frac{1}{2\pi} \lim_{T \to \infty} \int_T^{\infty} e^{ixs} \int_{-T}^{b} e^{-ist} f(t) \, dt \, ds. \quad (3.4)$$

And,

$$\lim_{B \to -\infty} \int_{A}^{B} e^{ixs} \int_{-\infty}^{\infty} e^{-ist} f(t) \chi_{(-\infty,a] \cup [b,\infty)}(t) \, dt \, ds = 0.$$ 

**Proof.** Applying Theorem 3.1 to $f\chi_{(a,b)}$ gives (3.4).

Let $A < 0$ and $B > 0$. By the Fubini–Tonelli theorem

$$\int_{A}^{B} e^{ixs} \int_{b}^{\infty} e^{-ist} f(t) \, dt \, ds = i \int_{b-x}^{\infty} \frac{f(t+x)e^{-iBt}}{t} \, dt - i \int_{b-x}^{\infty} \frac{f(t+x)e^{-iAt}}{t} \, dt.$$ 

Using the Riemann–Lebesgue lemma, this tends to 0 as $A \to -\infty$ and $B \to \infty$. Similarly for integration over $(-\infty,a]$.

**Example 3.5.** Choose $f : [a,b] \to \mathbb{R}$ to be an increasing function whose derivative vanishes almost everywhere. For example, the Cantor–Lebesgue function. Then $f$ is of bounded variation but is not absolutely continuous. The formula in Corollary 3.4 gives $f(x)$ at each $x \in (a,b)$. The results in [9] do not apply.

**Corollary 3.6** (Pringsheim). Let $f \in BV([a,b])$ such that $\lim_{|x| \to \infty} f(x) = 0$. Then $\hat{f}(s)$ exists for $s \neq 0$. And, for each $x \in \mathbb{R}$,

$$\frac{f(x-) + f(x+)}{2} = \frac{1}{2\pi} \lim_{T \to \infty} \int_{S < |x| < T} e^{ixs} \int_{-\infty}^{\infty} e^{-ist} f(t) \, dt \, ds.$$ 

Suppose the function $x \mapsto e^{-iax} f(x)$ is of bounded variation for some $a \in \mathbb{R}$. Then, for each $x \in \mathbb{R}$,

$$\frac{f(x-) + f(x+)}{2} = \frac{1}{2\pi} \lim_{T \to \infty} \int_{S < |x| < T} e^{ixs} \int_{-\infty}^{\infty} e^{-ist} f(t) \, dt \, ds.$$ 

For other proofs, see [18] and [20, §1.10]. See the latter for a reference to Pringsheim’s original paper.

**Proof.** Existence of $\hat{f}$ is by Dirichlet’s test.

Let $-\infty < a < x < b < \infty$. As in Corollary 3.4 we have (3.4).
Integrating by parts and using the Fubini–Tonelli theorem,
\[
\int_{S<|s|<T} e^{ixs} \int_b^\infty e^{-ist} f(t) \, dt \, ds
\]
\[
= \int_{S<|s|<T} \frac{e^{ixs}}{iS} \left\{ e^{-isb} f(b) + \int_b^\infty e^{-ist} \, df(t) \right\} \, ds
\]
\[
= -2f(b) \int_{(b-x)S}^{(b-x)T} \sin(s) \, \frac{ds}{s} - 2 \int_b^\infty \int_{(t-x)S}^{(t-x)T} \sin(s) \, \frac{ds}{s} \, df(t).
\]
The integral \(\int_b^\infty \sin(s) \, ds/s\) is bounded independent of its limits of integration. It is evaluated in Lemma 2.1. Using dominated convergence we can then take limits as \(S \to 0^+\) and \(T \to \infty\) and this gives
\[
-\pi \left\{ f(b) + \int_b^\infty df(t) \right\} = 0.
\]
The integral over \((-\infty, a)\) is handled similarly. \(\square\)

**Example 3.7.** (a) Let \(f(x) = \text{sgn}(x)/\log|x|\) for \(|x| > e\) and \(f(x) = 0\), otherwise. For \(s \neq 0\), integrate by parts:
\[
\frac{i\hat{f}(s)}{2} = \int_e^\infty \sin(st) \, dt = \frac{\cos(es)}{s} - \frac{1}{s} \int_e^\infty \frac{\cos(st)}{t \log^2(t)} \, dt
\]
\[
\sim \frac{\cos(es)}{s} \quad \text{as} \quad s \to \infty.
\]
The last line is by Riemann–Lebesgue lemma. Hence, \(\hat{f} \notin L^1(\mathbb{R})\). The principal value limit \(T \to \infty\) in Corollary 3.6 is required when \(|x| = e\) but for other values of \(x\) the inversion integral exists as a Henstock–Kurzweil integral.

Let \(0 < s < 1/e\). The second mean value theorem gives
\[
\left| \int_{1/s}^\infty \frac{\cos(st)}{t \log^2(t)} \, dt \right| = \frac{1}{\log(s)} \left| \int_1^\xi \frac{\cos(t)}{t} \, dt \right| \leq \frac{C}{\log(s)}
\]
for some \(\xi \geq 1\), where \(C = \sup_{x \geq 1} |\int_1^x \cos(t) \, dt/t|\).

By Taylor’s theorem, there is \(es \leq \eta(s, t) \leq 1\) such that
\[
\int_e^{1/s} \frac{\cos(st)}{t \log^2(t)} \, dt = \int_e^{1/s} \frac{dt}{t \log^2(t)} - s \int_e^{1/s} \frac{\sin(\eta(s, t))}{t \log^2(t)} \, dt
\]
so that
\[
\left| \int_e^{1/s} \frac{\cos(st)}{t \log^2(t)} \, dt - 1 - \frac{1}{\log(s)} \right| \leq s,
\]
the last estimate being by dominated convergence. These results show that \(i\hat{f}(s)/2 \sim -1/(s|\log|s||)\) as \(s \to 0\). Then \(\hat{f}\) is not integrable in any right neighbourhood of the origin but \(\sin(xs)|\hat{f}(s)|\) is bounded as \(s \to 0^+\) so the principal value limit \(S \to 0^+\) in Corollary 3.6 exists.

(b) An example of a function \(f \in BV([-\infty, \infty])\) that has limit 0 at \(\pm \infty\) but is not in \(L^1(\mathbb{R})\) is \(f(x) = \sin(x^{1/2})/x^{2/3}\) for \(x > 1\) and \(f(x) = 0\) for \(x \leq\)
1. The integral of $f$ then converges conditionally. Fourier transforms of such functions have been studied by J.H. Arredondo, F.J. Mendoza and A. Reyes [2]. Asymptotics of the Fourier–Laplace transform of such functions is considered in [5].

4. DISTRIBUTIONAL TRANSFORMS WITH POINTWISE INVERSION

If $f$ is of bounded variation then it has a limit at $\pm \infty$. Subtracting off this limit leaves a function satisfying the conditions of Corollary 3.6. The functions $H(x)$ and $H(-x)$ have distributional Fourier transforms so $\hat{f}$ exists in this sense as well. However, we can obtain a pointwise inversion formula as with (1.1). This is done in the following theorem. Theorem 4.3 considers the case when $f$ is asymptotic to a polynomial.

An anonymous referee has pointed out that: “Theorem 4.1 and Theorem 4.3 also follow from Theorem 13 and Theorem 5 in [6] because trigonometric integrals (or more precisely principal value special evaluations in the sense of [6]) associated to the second sum are easily seen to be Cesàro summable to $H_p(x) + H(-x) p_-(x)$ and moreover if $G$ is a primitive of $e^{ixs}\hat{g}(s)$, then

$$\lim_{s \to 0^+} [G(s) - G(-s)] = 2i \lim_{s \to 0^+} \int_{-\infty}^{\infty} \frac{\sin(st)}{st} df(t-x) = 2i \int_{-\infty}^{\infty} df(t-x) = 0.$$  

Theorem 4.1. Let $f \in BV([-\infty, \infty])$, where we define $f(\infty) = \lim_{x \to \infty} f(x)$ and $f(-\infty) = \lim_{x \to -\infty} f(x)$. Let $g(x) = f(x) - f(\infty)H(x) - f(-\infty)H(-x)$. Then the conditions of Corollary 3.6 apply to $g$. Let $\delta$ be the Dirac distribution. Then

$$\hat{f}(s) = \hat{g}(s) + f(\infty) \left[ \pi \delta(s) + \frac{1}{is} \right] + f(-\infty) \left[ \pi \delta(s) - \frac{1}{is} \right]. \quad (4.1)$$

And, for each $x \in \mathbb{R}$,

$$\frac{f(x-) + f(x+)}{2} = \frac{1}{2\pi} \lim_{T \to \infty} \int_{S < |s| < T} e^{ixs} \hat{g}(s) ds + f(\infty)H(x) + f(-\infty)H(-x). \quad (4.2)$$

The distribution $T = 1/s$ is given as $\langle T, \phi \rangle = \int_0^\infty \frac{\phi(x) - \phi(-x)}{x} dx$ for each test function $\phi$. See [8, p. 25].

Although $\hat{f}$ is in general a distribution, notice that the inversion formula (4.2) only requires integration of functions.

Limits of integrals, such as in (4.2), can be written in principal value notation. See Definition 5.8, page 277, in [17].

Proof. The following distributional Fourier transforms are well-known,

$$\hat{H}(s) = \pi \delta(s) + \frac{1}{is}, \quad \hat{\text{sgn}}(s) = \frac{2}{is}. \quad (4.3)$$

See, for example, [8, p. 172, 174]. These give the formula for $\hat{f}$ in the statement of the theorem.

Notice that $g \in BV([-\infty, \infty])$ and $\lim_{|x| \to \infty} g(x) = 0$. Applying Corollary 3.6 to $g$ and rearranging for $f$ gives the inversion formula (4.2). □
Example 4.2. Let \( f(x) = \arctan(x) \). Then \( g(x) = \arctan(x) - (\pi/2)\text{sgn}(x) \). Now integrate by parts to get

\[
\hat{g}(s) = -2i \int_{0}^{\infty} \left[ \arctan(t) - \frac{\pi}{2} \right] \sin(st) \, dt = \frac{2i}{s} \left( \frac{\pi}{2} - \int_{0}^{\infty} \frac{\cos(st)}{1 + t^2} \, dt \right).
\]

The cosine integral above can be evaluated with a contour integral as in the proof of Lemma 2.1. Its value is \( \pi e^{-|s|}/2 \) and then \( \hat{g}(s) = i\pi (1 - e^{-|s|})/s \). Equation (4.1) then gives \( \hat{f}(s) = -i\pi e^{-|s|}/s \). The inversion formula gives

\[
\frac{1}{2\pi} \lim_{T \to \infty} \int_{S<|s|<T} e^{ixs} \hat{g}(s) \, ds = \int_{0}^{\infty} e^{-s} \sin(xs) \frac{ds}{s} - \int_{0}^{\infty} \frac{\sin(xs)}{s} \, ds = \arctan(x) - \frac{\pi}{2} \text{sgn}(x).
\]

The principal value inversion formula then returns \( \arctan(x) \) at each \( x \in \mathbb{R} \).

Theorem 4.3. Let \( f \in \text{BV}([a, b]) \) for each \(-\infty < a < b < \infty \). Suppose there are polynomials \( p_{\pm}(x) = \sum_{k=0}^{n} a_{\pm k} x^k \) such that \( f(x) \sim p_{\pm}(x) \) as \( x \to \pm \infty \). And, if \( g(x) = f(x) - H(x)p_{+}(x) - H(-x)p_{-}(x) \) then \( g \in \text{BV}(\mathbb{R}) \) with \( \lim_{|x| \to \infty} g(x) = 0 \). Then, for \( s \neq 0 \),

\[
\hat{f}(s) = \hat{g}(s) + \sum_{k=0}^{n} \left[ \pi i^{k} \delta^{(k)}(s) + \frac{k!}{(is)^{k+1}} \right] + \sum_{k=0}^{n} \left[ \pi i^{k} \delta^{(k)}(s) - \frac{k!}{(is)^{k+1}} \right],
\]

And, for each \( x \in \mathbb{R} \),

\[
\frac{f(x-)}{2} + \frac{f(x+)}{2} = \frac{1}{2\pi} \lim_{T \to \infty} \int_{S<|s|<T} e^{ixs} \hat{g}(s) \, ds + H(x)p_{+}(x) + H(-x)p_{-}(x).
\]

(4.4)

Proof. Let \( h_{n}(x) = H(x)x^{n} \). The distributional Fourier transform

\[
\hat{h}_{n}(s) = \pi i^{n} \delta^{(n)}(s) + \frac{n!}{(is)^{n+1}}
\]

is well-known. For example, [8, p. 172]. Let \( q(x) = p_{-}(-x) \). Then \( \hat{f}(s) = \hat{g}(s) + \hat{H}p_{+}(s) + \hat{H}q(-s) \). And,

\[
\hat{H}p_{+}(s) = \sum_{k=0}^{n} a_{+k} \hat{h}_{n}(s) = \sum_{k=0}^{n} a_{+k} \left[ \pi i^{k} \delta^{(k)}(s) + \frac{k!}{(is)^{k+1}} \right].
\]

The transform of \( Hq \) is similar, using the fact that \( \delta^{(k)}(-s) = (-1)^{k} \delta^{(k)}(s) \). The rest of the proof follows as with Theorem 4.1. \( \square \)

Example 4.4. Let \( p \) be a polynomial. Let \( f(x) = p(x) \tanh(x) \) where \( \tanh(x) = \sinh(x)/\cosh(x) \). Define \( g(x) = f(x) - \text{sgn}(x)p(x) \). Then as \( |x| \to \infty \) we have \( g(x) \sim -2 \text{sgn}(x)p(x)e^{-2|x|} \). Hence, \( g \in L^{1}(\mathbb{R}) \). It follows that \( \hat{g}(s) \) is continuous on \( \mathbb{R} \) and has limit 0 as \( |s| \to \infty \). Formula (4.4) gives \( f(x) \) for each \( x \). If \( p(x) = O(x^{2}) \) as \( x \to 0 \) then \( g \) and \( g' \) are absolutely continuous and \( g, g', g'' \in L^{1}(\mathbb{R}) \).
Integration by parts then shows $\hat{g} \in L^1(\mathbb{R})$. Then for each $x \in \mathbb{R}$ the inversion formula reads
\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixs} \hat{g}(s) \, ds + \text{sgn}(x)p(x).
\]

5. Principal value transforms

The inversion formula (3.4) can be combined with conditions on $f$ outside the interval $[a, b]$. If $[b, \infty)$ is written as a finite union of intervals $[b_0, b_1] \cup [b_1, b_2] \cup \ldots \cup [b_{N-1}, b_N]$ (with $b = b_0 < b_1 < \ldots < b_N = \infty$) then to complete an inversion formula such as (1.1) we require
\[
\lim_{T \to \infty} \int_{-T}^{T} e^{ixs} \int_{b_{i-1}}^{b_i} e^{-ist} f(t) \, dt \, ds = 0
\]
for each $1 \leq i \leq N$. Similarly on $(-\infty, a]$. Use of the Dirichlet kernel in the proof of Corollary 3.4 shows $f \in L^1((b_{i-1}, b_i))$ suffices. The following theorem shows that on finite intervals we can replace this Lebesgue integration condition with a principal value condition.

**Theorem 5.1.** Suppose $f$ is odd about $c$ such that $\int_{c}^{c+\delta} |f(t)| |t - c| \, dt$ exists for some $\delta > 0$. Define the Fourier transform of $f$ by the principal value integral $\hat{f}(s) = \int_{c-\delta}^{c+\delta} e^{-ist} f(t) \, dt$. Suppose $x \not\in [c-\delta, c+\delta]$ and define $T_n = (2n+1)\pi/(2|x-c|)$ for $n \in \mathbb{N}$. Then
\[
\lim_{n \to \infty} \int_{S < |s| < \delta} e^{ixs} \hat{f}(s) \, ds = 0. \tag{5.1}
\]

Note that $T_n \to \infty$ only on a special sequence depending on $x$ while $S \to 0^+$ unrestrictedly through real numbers. M. Mikolás [13] has also considered limits through a special sequence. N. Ortner has defined principal value Fourier transforms as distributions and also considers slowly growing distributions [16].

**Proof.** By translation and reflection we see there is no loss of generality in assuming $f$ is odd about 0 and the integral $\int_{0}^{\delta} |f(t)| \, dt$ exists. The principal value Fourier transform is then $\hat{f}(s) = -2i \int_{0}^{\delta} \sin(st) f(t) \, dt$. Suppose $|x| > \delta$. Then use the Fubini–Tonelli theorem and a trigonometric identity to write
\[
\int_{S < |s| < T_n} e^{ixs} \hat{f}(s) \, ds = -2i \left\{ \int_{-T_n}^{S} + \int_{S}^{T_n} \right\} e^{ixs} \int_{0}^{\delta} \sin(st) f(t) \, dt \, ds
\]
\[
= 4 \int_{S}^{T_n} \sin(xs) \int_{0}^{\delta} \sin(st) f(t) \, dt \, ds
\]
\[
= 4 \int_{0}^{\delta} f(t) \int_{S}^{T_n} \sin(xs) \sin(st) \, ds \, dt
\]
\[
= 2 \int_{0}^{\delta} tf(t) \left[ \frac{\phi_{T_n}(t-x) - \phi_{S}(t-x) - \phi_{T_n}(t+x) + \phi_{S}(t+x)}{t} \right] \, dt, \tag{5.2}
\]
where $\phi_{U}(u) = \sin(uU)/u$. 

We have
\[
\frac{\phi_{T_n}(t-x) - \phi_{T_n}(t+x)}{t} = \frac{2[t \sin(xT_n) \cos(tT_n) - x \cos(xT_n) \sin(tT_n)]}{(x^2 - t^2)t}.
\]

Note that \(\sin(tT_n)/t \sim T_n\) as \(t \to 0\). Since this is not bounded as a function of \(T_n\), we cannot, in general, bring the limit \(T_n \to \infty\) inside the integral if \(T_n\) is a real variable. But, with the choice \(T_n = (2n+1)\pi/(2|x|)\) we have
\[
\frac{\phi_{T_n}(t-x) - \phi_{T_n}(t+x)}{t} = \frac{2\sin(xT_n) \cos(tT_n)}{x^2 - t^2} = \frac{2\sgn(x)(-1)^n}{x^2 - t^2} \cos \left( \frac{(2n+1)\pi t}{2x} \right).
\]

Then, by the Riemann–Lebesgue lemma,
\[
\lim_{n \to \infty} \int_0^\delta t f(t) \left[ \frac{\phi_{T_n}(t-x) - \phi_{T_n}(t+x)}{t} \right] dt = 2\sgn(x) \lim_{n \to \infty} (-1)^n \int_0^\delta \frac{t f(t)}{x^2 - t^2} \cos \left( \frac{(2n+1)\pi t}{2x} \right) dt = 0.
\]

Similarly,
\[
\left| \frac{-\phi_S(t-x) + \phi_S(t+x)}{t} \right| = \left| \frac{2[x \sin(tS) \cos(xS) - t \cos(tS) \sin(xS)]}{(x^2 - t^2)t} \right| \leq 2(|x|S + 1) \frac{\delta}{x^2 - \delta^2}.
\]

By dominated convergence,
\[
\lim_{s \to 0^+} \int_0^\delta t f(t) \left[ \frac{-\phi_S(t-x) + \phi_S(t+x)}{t} \right] dt = 0
\]
since we can take the limit \(S \to 0^+\) inside the integral and \(\phi_S(t \pm x) \to 0\) as \(S \to 0\).

\[\square\]

**Example 5.2.** Let \(f(x) = \sgn(x)|x|^{-\alpha}\) for \(0 < \alpha < 2\). Then the principal value Fourier transform is \(\hat{f}(s) = -2i \sgn(s)|s|^{\alpha-1} \int_0^\infty \sin(t) t^{-\alpha} dt\). Note that \(\hat{f}\) is Lebesgue integrable at the origin but integration in a neighbourhood of infinity requires the inversion be a principal value integral. If \(x \neq 0\) we can combine Corollary 3.6 and Theorem 5.1 to get the pointwise inversion
\[
f(x) = \frac{1}{2\pi} \lim_{n \to \infty} \int_{-T_n}^{T_n} e^{ixs} \hat{f}(s) ds,
\]
where \(T_n = (2n+1)\pi/(2|x|)\).

Take \(0 < \epsilon < |x|\). Write \(f = f_1 + f_2\) where \(f_1 = f\chi_{(-\epsilon, \epsilon)}\). Then
\[
\hat{f}_1(s) = \frac{2}{i} \int_0^\epsilon \frac{\sin(st)}{t^{\alpha}} dt = \frac{2}{i} \sgn(s)|s|^{\alpha-1} \int_0^\epsilon \frac{\sin(t)}{t^{\alpha}} dt \sim \frac{2 \epsilon^{2-\alpha}}{i 2^{\alpha}} \quad \text{as } s \to 0
\]
\[
\hat{f}_1(s) \sim \frac{2}{i} \sgn(s)|s|^{\alpha-1} \int_0^\epsilon \frac{\sin(t)}{t^{\alpha}} dt \quad \text{as } |s| \to \infty.
\]
Hence, \( \hat{f}_1 \) is continuous at the origin but the principal value inversion of Theorem 5.1 is required since \( \hat{f}_1 \) is not integrable in a neighbourhood of infinity. Integration by parts and the Riemann–Lebesgue lemma show

\[
\lim_{n \to \infty} \int_{-T_n}^{T_n} e^{ixs} \hat{f}_1(s) \, ds = 0.
\]

And,

\[
\hat{f}_2(s) = \frac{2}{i} \int_{\epsilon}^{\infty} \frac{\sin(st)}{t^\alpha} \, dt = \frac{2}{i} \text{sgn}(s)|s|^{\alpha-1} \int_{\epsilon|s|}^{\infty} \frac{\sin(t)}{t^\alpha} \, dt
\]

\[
\sim \frac{2}{i} \text{sgn}(s)|s|^{\alpha-1} \int_{0}^{\infty} \frac{\sin(t)}{t^\alpha} \, dt \quad \text{as } s \to 0
\]

\[
\hat{f}_2(s) \sim \frac{2 \cos(es)}{i \epsilon^\alpha s} \quad \text{as } |s| \to \infty.
\]

In this case, \( \hat{f}_2 \) is Lebesgue integrable at the origin but the inversion integral will exist conditionally. Corollary 3.6 gives

\[
f_2(x) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} e^{ixs} \hat{f}_2(s) \, ds.
\]

Let \( g(x) = \text{sgn}(x) \sin(ax)|x|^{-\alpha} \) for \( a \in \mathbb{R} \). Then \( \hat{g}(s) = -i(\hat{f}(s-a) - \hat{f}(s+a))/2 \). Let \( h(x) = \text{sgn}(x) \cos(ax)|x|^{-\alpha} \). Then \( \hat{h}(s) = (\hat{f}(s-a) + \hat{f}(s+a))/2 \). There are similar inversion formulas.

6. The Henstock–Stieltjes Integral

This is an integral that properly extends the Riemann–Stieltjes integral. It is described in detail on the real line in [12] and on compact intervals in [14] and [21]. The name Henstock is sometimes replaced with Kurzweil, Denjoy or Perron. Let \( -\infty \leq a < b \leq \infty \). A tagged partition is a finite collection \( P = \{(x_{n-1}, x_n), z_n\}_{n=1}^{N} \) with tag \( z_n \in [x_{n-1}, x_n] \) and \( a = x_0 < x_1 < \ldots < x_N = b \), for some \( N \in \mathbb{N} \). A gauge is a function \( \gamma \) from \( [a, b] \) to the open intervals in \( [a, b] \) where we consider intervals such as \( [a, x), (x, y) \) and \( (x, b] \) to be open for \( x, y \in (a, b) \). (For \( a = -\infty \) and \( b = \infty \) this provides a two-point compactification of the real line.) Then \( P \) is said to be \( \gamma \)-fine if for each \( 1 \leq n \leq N \) we have \([x_{n-1}, x_n] \subset \gamma(z_n)\). Let \( f, g : [a, b] \to \mathbb{R} \). Then \( f \) is integrable with respect to \( g \) with integral \( \int_{a}^{b} f \, dg \in \mathbb{R} \) if for each \( \epsilon > 0 \) there is a gauge \( \gamma \) such that if the tagged partition \( P \) is \( \gamma \)-fine then

\[
\left| \sum_{n=1}^{N} f(z_n)(g(x_n) - g(x_{n-1})) - \int_{a}^{b} f \, dg \right| < \epsilon. \tag{6.1}
\]

For every \( \epsilon > 0 \) it is always possible to choose a gauge \( \gamma \) so that each \( \gamma \)-fine tagged partition must have \( a \) and \( b \) as tags (of exactly one interval each). We will always assume such a gauge has been chosen.

The Henstock–Stieltjes integral differs from the Riemann–Stieltjes integral in that it can integrate over unbounded intervals and the tags, \( z_n \), are chosen...
more carefully. This allows the functions $f$ and $g$ to have a common point of discontinuity, in which case the Riemann–Stieltjes integral would not exist.

**Example 6.1.** We verify (1.2) for Henstock–Stieltjes integrals and show the integral can fail to exist as a Riemann–Stieltjes integral, even allowing for improper integrals.

Define
\[
f(x) = \begin{cases} 
0, & x < 0 \\
\alpha, & x = 0 \\
1, & x > 0,
\end{cases}
\]
for some $\alpha \in \mathbb{R}$. Take $x = 0$ in (1.2). The left side of this identity is then $(f(0^+) + f(0^-))/2 = 1/2$.

Note that $H(t) = 0$ for $t > 0$, $df = 0$ except in a neighbourhood of the origin and $f = 0$ on $(-\infty, 0)$.

From (1.2),
\[
-i\omega \int_{-\infty}^{\infty} H(x-t)e^{i\omega(x-t)}f(t)\,dt \text{ reduces to } -i\omega \int_{-\infty}^{0} e^{-i\omega t}f(t)\,dt = 0,
\]
the integral existing in the Lebesgue sense.

The only tag that can contribute to the Riemann sum in (6.1) is $z_n = 0$. Let $\epsilon > 0$. Let $\delta > 0$. Define a gauge by $\gamma(0) = (-\delta, \delta)$. For each $x \neq 0$ let $\gamma(x)$ be an open interval that contains $x$ but does not contain the origin. Define $\gamma(-\infty) = [-\infty, -M)$ and $\gamma(\infty) = (M, \infty]$ for some $M > 0$. Combining tagged intervals with coincident endpoints if necessary, every $\gamma$-fine tagged partition of $[-\infty, \infty]$ must then have exactly one tagged interval with $0$ as a tag. This is the only tag that can contribute to the Riemann sum representing $\int_{-\infty}^{\infty} H(x-t)e^{i\omega(x-t)}\,df(t)$. Write this tagged interval as $[\alpha, \beta, 0]$ where $-\delta < \alpha < 0 < \beta < \delta$. The Riemann sum in (6.1) then reduces to $H(0)[f(\beta) - f(\alpha)] = 1/2$. This verifies (1.2) for Henstock–Stieltjes integrals.

Now consider $\int_{-\infty}^{\infty} H(-t)e^{-i\omega t}\,df(t)$ as a Riemann–Stieltjes integral. As noted above, the integrand or integrator vanish except in a neighbourhood of the origin. We can then just integrate over $[-1, 1]$. With Riemann–Stieltjes integrals, the length of intervals allowed in a Riemann sum are controlled by a parameter. Suppose $\delta > 0$ and all intervals must have length less than $\delta$. In a partition of $[-1, 1]$ we are free to choose an interval $[\alpha, \beta]$ where $-1 < \alpha < 0 < \beta < 1$ and $\beta - \alpha < \delta$. We are free to choose any tag $z \in [\alpha, \beta]$. As above, this is the only tag that contributes non-trivially to a Riemann sum. The Riemann sum then reduces to $H(z)e^{-i\omega z}[f(\beta) - f(\alpha)] = H(z)e^{-i\omega z}/2$. If $z = \alpha$ this gives 0. If $z = \beta$ this gives $e^{-i\omega \beta}/2$. Hence, the Riemann–Stieltjes integral does not exist.

The improper Riemann–Stieltjes integral also does not exist. If $0 < \epsilon < 1$ then \( \int_{-\infty}^{1} H(-t)e^{-i\omega t}\,df(t) \) and \( \int_{-\epsilon}^{\infty} H(-t)e^{-i\omega t}\,df(t) \) both vanish since $f$ is constant on the intervals of integration. Hence, any improper integral computed by taking the limit $\epsilon \to 0^+$ will yield 0, not the required value of 1/2.

If one of $f$ and $g$ is regulated and the other is of bounded variation then the integrals $\int_{a}^{b} f\,dg$ and $\int_{a}^{b} g\,df$ exist and they are related by an integration by parts formula. It allows the functions to have discontinuities at the same point. The following formula is proved in [12, p. 199]. Let $-\infty < a < b \leq \infty$. Then
\[
\int_a^b f\, dg = f(b)g(b) - f(a)g(a) - \int_a^b g\, df + \sum [f(c_n) - f(c_n-)][g(c_n) - g(c_n-)] - \sum [f(c_n) - f(c_n+)][g(c_n) - g(c_n+)].
\]

(6.2)

The sums contain all points \(c_n\) at which \(f\) and \(g\) are either discontinuous from the left or discontinuous from the right.

We now prove two related results.

**Proposition 6.2.** Let \(-\infty \leq a < b \leq \infty\). Let \(A, B, C : [a, b] \to \mathbb{R}\) such that two functions are in \(BV([a, b])\) and the other is in \(R([a, b])\). If \(B\) and \(C\) have no common point of discontinuity then

\[
\int_a^b A\, d[BC] = \int_a^b AB\, dC + \int_a^b AC\, dB.
\]

**Proof.** Since \(BV([a, b])\) and \(R([a, b])\) are both closed under pointwise products, the integrals in the proposition exist.

Given \(\epsilon > 0\) there is a gauge \(\gamma\) and a \(\gamma\)-fine tagged partition of \([a, b]\), defined by \(\{[x_{n-1}, x_n], z_n\}_{n=1}^N\), such that

\[
\sum_{n=1}^N A(z_n)[B(x_n)C(x_n) - B(x_{n-1})C(x_{n-1})] - \int_a^b A\, d[BC] < \epsilon
\]

\[
\sum_{n=1}^N A(z_n)B(z_n)[C(x_n) - C(x_{n-1})] - \int_a^b AB\, dC < \epsilon
\]

\[
\sum_{n=1}^N A(z_n)C(z_n)[B(x_n) - B(x_{n-1})] - \int_a^b AC\, dB < \epsilon
\]

\[
\sum_{n=1}^N B(z_n)[C(x_n) - C(x_{n-1})] - \int_a^b B\, dC < \epsilon
\]

\[
\sum_{n=1}^N C(z_n)[B(x_n) - B(x_{n-1})] - \int_a^b C\, dB < \epsilon
\]

\[
\sum_{n=1}^N [B(x_n)C(x_n) - B(x_{n-1})C(x_{n-1})] - \int_a^b d[BC] < \epsilon.
\]

By Henstock’s lemma [12, p. 186],

\[
\sum_{n=1}^N B(z_n)[C(x_n) - C(x_{n-1})] - \int_{x_{n-1}}^{x_n} B\, dC < 2\epsilon
\]

\[
\sum_{n=1}^N C(z_n)[B(x_n) - B(x_{n-1})] - \int_{x_{n-1}}^{x_n} C\, dB < 2\epsilon.
\]
Let

\[ D_n = B(x_n)C(x_n) - B(x_{n-1})C(x_{n-1}) - B(z_n)[C(x_n) - C(x_{n-1})] \]
\[ - C(z_n)[B(x_n) - B(x_{n-1})]. \]

It then suffices to show \( \sum_{n=1}^{N} |A(z_n)D_n| \) is less than a multiple of \( \epsilon \). Notice that since \( B \) and \( C \) have no common points of discontinuity, the integration by parts formula is

\[
\int_{x_{n-1}}^{x_n} d[BC] = B(x_n)C(x_n) - B(x_{n-1})C(x_{n-1}) = \int_{x_{n-1}}^{x_n} B \, dC + \int_{x_{n-1}}^{x_n} C \, dB.
\]

The above inequalities then give

\[
\sum_{n=1}^{N} |D_n| = \sum_{n=1}^{N} \left| D_n - \int_{x_{n-1}}^{x_n} d[BC] + \int_{x_{n-1}}^{x_n} B \, dC + \int_{x_{n-1}}^{x_n} C \, dB \right| < 4\epsilon.
\]

Then \( \sum_{n=1}^{N} |A(z_n)D_n| < 4\epsilon\|A\|_{\infty}. \)

If \( B \) and \( C \) have coincident discontinuities at \( c_n \) then sum terms as in the integration by parts formula need to be added: \( \sum A(c_n)[B(c_n) - B(c_{n-1})][C(c_n) - C(c_{n-1})] \) and \( \sum A(c_n)[B(c_n) - B(c_{n+1})][C(c_n) - C(c_{n+1})] \).

**Proposition 6.3.** Let \( A \in \mathcal{R}([a, b]) \) and \( B \) be absolutely continuous such that \( B' \in L^1([a, b]) \). Then \( \int_{a}^{b} A \, dB = \int_{a}^{b} A(t)B'(t) \, dt \).

**Proof.** If \([a, b]\) is a finite interval then \( B \in \mathcal{BV}([a, b]) \). If \([a, b]\) is not finite we can write

\[
\sum_{n=1}^{N} |B(x_n) - B(x_{n-1})| = \sum_{n=1}^{N} \left| \int_{x_{n-1}}^{x_n} B'(t) \, dt \right| \leq \sum_{n=1}^{N} \int_{x_{n-1}}^{x_n} |B'(t)| \, dt = \|B'\|_1.
\]

Hence, \( B \in \mathcal{BV}([a, b]) \). The rest of the proof is similar to that for Proposition 6.2, starting with a gauge that simultaneously makes Riemann sums within \( \epsilon \) of the integrals \( \int_{a}^{b} A \, dB, \int_{a}^{b} A(t)B'(t) \, dt \) and \( \int_{a}^{b} B'(t) \, dt \).

Notice that in the case of unbounded intervals absolute continuity of \( B \) does not imply \( B' \in L^1([a, b]) \). For example, \( B(x) = x \).
Proof of Lemma 2.2

Proof. (a) Using the cutoff property of the Heaviside step function, the integration parts formula (6.2) and the fact that \( f \) is bounded gives

\[
\int_{-\infty}^{\infty} H(x-t)e^{-i\omega t}[df(t) - i\omega f(t) dt] = \int_{-\infty}^{b} H(x-t)e^{-i\omega t}[df(t) - i\omega f(t) dt]
\]

\[
= H(x-b)e^{-i\omega b}f(b) - \lim_{t \to -\infty} [H(x-t)e^{-i\omega t}f(t)] - \int_{-\infty}^{b} f(t) d[H(x-t)e^{-i\omega t}]
\]

\[
+ e^{-i\omega x} \left\{ [H(0) - H(0+)] [f(x) - f(x-)] - [H(0) - H(0-)][f(x) - f(x+)] \right\}
\]

\[
- i\omega \int_{-\infty}^{b} H(x-t)e^{-i\omega t}f(t) dt
\]

\[
= - \int_{-\infty}^{b} f(t)e^{-i\omega t} dH(x-t) - e^{-i\omega x} \left\{ \left[ \frac{f(x) - f(x-)}{2} \right] + \left[ \frac{f(x) - f(x+)}{2} \right] \right\}
\]

\[
e^{-i\omega x} \left[ \frac{f(x-) + f(x+)}{2} \right].
\]

Since the function \( t \mapsto e^{-i\omega t} \) is absolutely continuous on \((-\infty, b]\) and its derivative is in \(L^1((-\infty, b])\) we can use Propositions 6.2 and 6.3 in the last lines above. A gauge can be chosen so that each \( \gamma \)-fine tagged partition must have \( x \) as a tag. This shows \( \int_{-\infty}^{b} f(t)e^{-i\omega t} dH(x-t) = f(x)e^{-i\omega x} [H(0-) - H(0+)]. \)

Part (b) is similar. \( \square \)

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