The $\theta^+$ baryon in soliton models: large $N_c$ QCD and the validity of rigid-rotor quantization

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A light collective $\theta^+$ baryon state (with strangeness +1) was predicted via rigid-rotor collective quantization of SU(3) chiral soliton models. This paper explores the validity of this treatment. A number of rather general analyses suggest that predictions of exotic baryon properties based on this approximation do not follow from large $N_c$ QCD. These include an analysis of the baryon’s width, a comparison of the predictions with general large $N_c$ consistency conditions of the Gervais-Sakita-Dashen-Manohar type; an application of the technique to QCD in the limit where the quarks are heavy; a comparison of this method with the vibration approach of Callan and Klebanov; and the $1/N_c$ scaling of the excitation energy. It is suggested that the origin of the problem lies in an implicit assumption in the that the collective motion is orthogonal to vibrational motion. While true for non-exotic motion, the Wess-Zumino term induces mixing at leading order between collective and vibrational motion with exotic quantum numbers. This suggests that successful phenomenological predictions of $\theta^+$ properties based on rigid-rotor quantization were accidental.

I. INTRODUCTION

A narrow baryon resonance with a strangeness of +1 (i.e. containing one excess strange antiquark) has recently been identified by a number of experimental groups. Such a state is exciting: it is unambiguously exotic in the sense that it cannot be a simple three-quark state. These experiments have spurred considerable theoretical activity. Much of this activity has been aimed at understanding the structure of this exotic state. The most common treatment of this problem has been based on variants of the quark model; the new baryon is identified as a pentaquark. Other approaches treat the $\theta^+$ in terms of meson-baryon binding or as a kaon-pion nucleon state or are based on on QCD sum-rules. While these approaches are all interesting, they are also all highly model dependent and it is difficult to assess in an a priori way their validity. A discussion of some of the issues raised by various models may be found in [1].

The analysis based on the SU(3) chiral soliton model treated with rigid-rotor quantization appears different from other treatments of the $\theta^+$ structure in a number of ways: i) The approach was used to predict the state and its properties. In contrast most of the other treatments were subsequent to the experimental discoveries. ii) The predictions of the mass were very accurate; the width was predicted to be narrow, which is consistent with the widths presently observed. iii) The prediction of the mass is totally insensitive to the dynamical details of the model. The analysis holds regardless of the detailed shape of the profile function that emerges from the dynamics.

While the first two points are certainly of interest, the third point is of far greater significance from the point of view of theory. It was noticed quite some time ago by Adkins and Nappi that a number of relationships between observables in chiral soliton models depend on the structure of the models and their symmetries, but do not depend on the dynamical details such as the parameters in the lagrangian. Moreover, these relations were derived in a fully model-independent way directly from large $N_c$ using consistency relations.

In light of this, it is natural to conjecture that the relationships between masses underlying the analysis in refs. which do not depend on the dynamical details of the model are similarly model-independent consequences of large $N_c$ QCD. If this is true, it greatly aids in our understanding of the $\theta^+$: modulo corrections due to $1/N_c$ effects of higher order effects in SU(3), the $\theta^+$ seems essentially understood. However, it remains quite controversial as to whether this is indeed the case.

In a previous paper, it was argued that the prediction of properties of exotic states based on rigid-rotor quantization are not generic predictions of large $N_c$ QCD. Before discussing the content of this paper, a linguistic issue should be addressed to avoid possible confusion. The analysis is done in a $1/N_c$ expansion so one must be able to consider systems with varying $N_c$. Accordingly throughout this paper the phrases “exotic state” or “exotic baryon” refer to states which are exotic for the given value of $N_c$ which is relevant (and not necessarily for $N_c = 3$); i.e., those baryons which cannot be described in a quark model with $N_c$ quarks. Clearly states with positive strangeness are exotic for any $N_c$.

Two basic arguments were presented in ref. which suggest that the rigid-rotor quantization is not valid for exotic states. The first was based on large $N_c$ consistency rules of the sort discussed above. All of the previously known model independent properties of baryons were derived from such relations. However, such relations do not predict the existence of collective exotic states at large $N_c$. The only collective states predicted are precisely those with the same quantum numbers as those which...
emerge in a large $N_c$ quark model with all quarks in the lowest s-wave orbital. The second argument was that the excitation energies of the exotic state of order $N_c^2$ is inconsistent with the type of scale separation needed to justify collective quantization.

Itzhaki, Klebanov, Ouyang and Rastelli (IKOR)\cite{3} reached similar conclusions about the validity of the rigid rotor approximation for exotic states from a rather different starting point: the treatment of SU(3) symmetry breaking. It is clear that for large symmetry breaking fluctuations into the strangeness direction are of a vibrational character and the appropriate formalism to describe these are the Callan-Klebanov “bound state” approach\cite{24}. For the case of non-exotic baryons it can be shown that as the SU(3) symmetry breaking goes to zero, the predicted energies and spatial distributions in the Callan-Klebanov approach match those of the rigid-rotor approach\cite{20}; the two approaches are compatible for non-exotic states in the regime where both should work. In contrast, at small but nonzero SU(3) symmetry breaking the excitation energy for exotic states such as the $\theta^+$ as calculated via the rigid-rotor treatment does not match smoothly onto the Callan-Klebanov value. IKOR took this as evidence that rigid-rotor treatment, while valid for non-exotic states, gives spurious results for exotic states.

Pobylytsa\cite{21} also concluded that rigid-rotor quantization failed for exotic states based on a study of an exactly solvable toy model of the Lipkin-Meshkov-Glick type\cite{22}.

Despite the arguments in refs.\cite{3,19,21}, it is not universally accepted that rigid-rotor collective quantization of chiral soliton models is invalid for exotic baryon properties. Diakonov and Petrov\cite{23} have recently argued that that rigid-rotor collective quantization is accurate for such states up to $1/N_c$ corrections, and have specifically attempted to rebut the argument in ref.\cite{19}. Both a general analysis as to why the authors believe the argument of ref.\cite{19} is incorrect and a toy model to illustrate the point are given. However, as will be discussed below, the general arguments are flawed and the toy model is not analogous to the problem of interest.

It is hoped that the present paper will end help this controversy by giving convincing evidence that the rigid-rotor collective quantization is not valid at large $N_c$. Given this goal, it is important to state precisely what is meant by this claim so as to minimize possible misunderstandings. The claim is formal. If rigid-rotor quantization were valid at large $N_c$, then predictions based on it would become arbitrarily accurate as $N_c \rightarrow \infty$ (i.e., there are no corrections due to other effects which survive at large $N_c$). Arguments are given here that this is false: even at large $N_c$ there are nonzero corrections. This is in contrast to properties of non-exotic baryons for which calculations based on rigid-rotor quantization do become arbitrarily accurate.

It is important to make quite clear the fundamental nature of the difficulty with the rigid-rotor collective quantization. The issue is not related to whether this treatment based on large $N_c$ analysis is a good approximation to the physical world of $N_c = 3$, but to the nature of exotic states in the large $N_c$ world.

This paper presents evidence for the invalidity of the rigid-rotor quantization for exotic baryons. Several general arguments are presented showing that the rigid-rotor quantization leads to predictions of exotic baryon properties which are inconsistent with known large $N_c$ results. These general arguments strongly suggest an inconsistency between various predictions of exotic baryon properties at large $N_c$ as based on rigid-rotor quantization and the known behavior at large $N_c$. If correct, they imply a flaw in the original derivation of the rigid-rotor collective quantization of refs.\cite{25} for these exotic states. Such a flaw must give spurious results for exotic states while being valid for non-exotic states. An analysis of the derivation indicates where such a flaw may lie. The derivation implicitly assumes the orthogonality of collective and vibrational motion. However, the Wess-Zumino term induces a coupling between vibrational modes and the collective motion associated with exotic excitations which spoils this orthogonality.

This paper is organized as follows: In the following section the salient features of the treatment of exotic baryons based on the rigid-rotor collective quantization\cite{8,9,10,11,12} is briefly presented. The next section discusses the significance of these results in light of the recent experimental reports of exotic $s=1$ baryons.

II. RIGID-ROTOR PREDICTIONS FOR EXOTIC BARYONS

There are a number of important assumptions which go into the predictions of exotic baryon states in refs.\cite{8,9,10,11,12}. These include the assumption that low-order perturbation theory in the strange quark mass is justified for real world values, the assumption that $1/N_c$ expansion truncated at low order is justified for these observables for $N_c = 3$ as well as the assumption that rigid-rotor quantization is valid for exotic states. This paper focuses on the issue of the validity of rigid-rotor quantization. It is worth bearing in mind, however, that these other assumptions are not totally innocuous. For example, prior to the experimental observation of the $\theta^+$, Weigel\cite{24} observed that effects which were higher-order in the strange quark mass induced mixing between the vibrational and collective modes which had nontrivial ef-
fects on predictions of the properties of the exotic states. However, the central questions of principle addressed in this paper are seen at leading order in $1/N_c$ and in the exact SU(3) limit and we review the leading order treatment below.

The analysis is based on a standard treatment of SU(3) chiral soliton models developed in the mid-1980s\cite{23}. The starting point in the analysis is a classical static “hedgehog” configuration in an SU(2) subspace (which for convenience one may take to be the u-d subspace). The profile function of this hedgehog is obtained by minimizing the action for a static configuration subject to the constraint that the baryon number (which is taken to be the topological winding number for the chiral field) is unity. The detailed shape of the profile function depends on the model—the types of couplings included in the values of the parameters, and so on. However, the general structure of the theory is completely model independent. As noted above, for the present purpose it is sufficient to consider the exact SU(3) symmetric limit of the theory. In the absence of symmetry breaking effects there are eight flat directions in which one can rotate to the ground state in a mean-field treatment. However, for SU(3) quark-soliton models developed in the mid-1980s\cite{23}, the state $i = 0$ corresponds to the quark ground state; the factor of $N_c$ reflects the fact that there are $N_c$ quarks in the system, each in the ground state in a mean-field treatment. For example, in models with explicit quark degrees of freedom, one has a nonzero quark contribution to the moments of inertia which can be computed via a standard “cranking” procedure borrowed from many-body physics\cite{27}, and introduced into chiral soliton physics in refs. \cite{28,29}. The method was first used for the SU(3) quark-soliton model by McGovern and Birse\cite{30}. With such a treatment the quark contributions to the moments of inertia are given by

$$F_{1}^{\text{quark}} = \frac{N_c}{2} \sum_{i} |\langle i | \lambda^A | 0 \rangle |^2 \quad A = 1, 2, 3$$

$$F_{2}^{\text{quark}} = \frac{N_c}{2} \sum_{i} |\langle i | \lambda^A | 0 \rangle |^2 \quad A = 4, 5, 6, 7$$  \hspace{1cm} (2.4)$$

where $|i\rangle$ are the single-particle quark eigenstates (eigenenergies) for quarks propagating in the static background of the hedgehog fields. The state $i = 0$ corresponds to the quark ground state; the factor of $N_c$ reflects the fact that there are $N_c$ quarks in the system, each in the ground state in a mean-field treatment. However, for the present purpose the key point is that although the quantization requires a treatment somewhat more sophisticated than that used in eq. (2.1), the sole effect of this additional sophistication is a change in the numerical value of the moments of inertia; the structure of the collective Hamiltonian in eq. (2.2) remains valid provided the central assumption that the collective and rotation degrees of freedom decouple is correct.

In quantizing the collective Hamiltonian in eq. (2.2), a constraint plays an essential role:

$$J_{K}^{z} = -\frac{N_c B}{2\sqrt{3}},$$  \hspace{1cm} (2.5)$$

where $B$ is the baryon number. In the context of Skyrme-type models this quantization condition is deduced from the topology of the Wess-Zumino term \cite{23}. As noted by Witten\cite{31}, this constraint can be understood in analogy to the constraint on the body-fixed angular momentum which arises when quantizing a charged particle in the field of a magnetic monopole.

The masses which emerge from eqs. (2.2) and (2.5) may...
be found easily. Using the fact that

$$\sum_{A=1}^{8} (\hat{J}_A)^2 = \sum_{A=1}^{8} (\hat{J}_A)^2 = C_2$$

(2.6)

where $C_2$ is the quadratic Casimir operator and $\hat{J}_A$ is a generator in the space-fixed frame, one can rewrite the collective Hamiltonian as

$$H_{\text{rot}} = M_0 + \frac{1}{2I_2} \sum_{A=1}^{8} \hat{J}_A^2 + \frac{I_2-I_1}{2I_1I_2} \sum_{A=1}^{3} \hat{J}_A^2 - \frac{1}{2I_2} \hat{J}_2^2.$$  

(2.7)

Equation (2.6) can be used to replace the last term. Moreover, the intrinsic SU(2) subspace satisfies the usual SU(2) soliton rule that $I = J$. Together these relations allow one to express the eigenstates of $H_{\text{rot}}$, i.e., the physical masses:

$$M = M_0 + \frac{C_2}{2I_2} + \frac{(I_2-I_1)J(J+1)}{2I_1I_2} - \frac{N_2^2}{24I_2},$$

with $C_2 = (p^2 + q^2 + pq + 3(p + q))/3$,  

(2.8)

where $C_2$, the quadratic Casimir, is expressed in terms of the traditional labels $p, q$ which denote the SU(3) representation. The quantization condition in eq. (2.5) greatly limits the possible SU(3) representations which can be associated with physical states: those SU(3) representations which do not contain states with hypercharge equal to $N_c/3$ are clearly unphysical: if the hypercharge in a body-fixed frame satisfies eq. (2.5), then that representation will of necessity include a state with that hypercharge. Angular momentum also limits the physically allowed representations. In the body-fixed frame the SU(2) manifold has $I = J$ and $S = 0$, which implies that the number of angular momentum states associated with representation, $(2J+1)$, must equal the number of states in the representation with $S = 0$. This whole procedure is rigid-rotor collective quantization. The moments of inertia are treated as constants independent of the rotational state of the system and in that sense corresponds to a rigid rotor.

There is a practical issue about how one chooses to implement this procedure. One natural approach would be to choose to quantize the theory at large $N_c$ and then to treat systematically all $1/N_c$ corrections. An alternative approach would be to fix $N_c = 3$ at the outset when implementing the quantization condition of eq. (2.5). If the approach is valid and if $N_c = 3$ can be considered large, it ought not make any difference which of these approaches is used. The choice of taking $N_c = 3$ at the outset has been the one typically made. Making this choice, it is straightforward to see that the lowest-lying states are:

$$J = 1/2 \quad (p, q) = (1, 1) \quad \text{(octet)}$$

$$J = 3/2 \quad (p, q) = (3, 0) \quad \text{(decuplet)}$$

$$J = 1/2 \quad (p, q) = (0, 3) \quad \text{(anti-decuplet).}$$

(2.9)

Equation (2.6) can be used to find the mass splitting of the decuplet and the anti-decuplet relative to the octet:

$$M_{10} - M_8 = \frac{3}{2I_1},$$

(2.10)

$$M_{10} - M_8 = \frac{3}{2I_2}.$$  

(2.11)

The prediction of an anti-decuplet representation is at the heart of the issue. The anti-decuplet contains a state with $s = +1$ (which has been identified with the $\theta^+$). Such a state is necessarily exotic, even in the large $N_c$ limit.

In outlining the rigid-rotor quantization procedure, large $N_c$ QCD considerations appeared explicitly only when discussing the quantization constraint of eq. (2.5). In fact, large $N_c$ considerations are at the core of the method and have been used implicitly throughout in two essential ways. In the first place, large $N_c$ is necessary for the justification of the classical static hedgehog configurations in an underlying quantum theory. Standard large $N_c$ scaling rules for couplings ensure that effects of quantum fluctuations around the hedgehogs are suppressed by $1/N_c$. Large $N_c$ also plays a central role in justifying the semi-classical treatment in the rigid rotor collective quantization which requires the decoupling of the collective motion from the vibrational motion around the static of the hedgehog. This is also suppressed by $1/N_c$, at least in certain situations. It should be clear from the previous comment, however, that the validity of the rigid-rotor collective approach depends on restricting its application to those modes which decouple from the vibrational ones. This issue is at the heart of the present paper.

Most treatments of SU(3) solitons identify the octet and decuplet states with the known $N_c = 3$ octets and decuplets. Until fairly recently, the anti-decuplet was often assumed to be a large artifact of large $N_c$ QCD and hence ignored for essentially the same reason that $I=J=5/2$ baryons are ignored in SU(2) soliton models. The central point of ref. is that the anti-decuplet should be taken seriously. The authors of ref. distinguish the situation of anti-decuplet for SU(3) solitons from the $J=I=5/2$ baryons in SU(2) in terms of their widths. The $J=I=5/2$ baryon width is predicted to be so wide with real world parameters that the state can not be observed, while the computed $\theta^+$ width turns out to be quite small.

III. LARGE $N_c$ SCALING OF MASS SPLITTINGS

The analysis presented above is based on making the choice to fix $N_c = 3$ at the outset when implementing the constraint of eq. (2.5). This is not a reasonable way of doing phenomenology but may obscure the large $N_c$ scaling of the system. Consider the scaling of the splitting as given in eq. (2.11). Both $I_1 \sim N_c$ and $I_2 \sim N_c$, eq. (2.11) and this appears to imply that $(M_{10} - M_8) \sim 1/N_c$ and
Thus the exotic states appear to behave similarly with the non-exotic states at large $N_c$. However, this is misleading. It has been known for some time that the exotic states have mass splittings relative to the ground state of order $N_c^0$ and not $N_c^{-1}$ \cite{4,23,34}. To see how this arises we explore the implementation of eqs. (2.5) and (2.8) for $N_c$ arbitrary and large.

In studying large $N_c$ baryons it is useful to restrict attention to the case of odd $N_c$; this ensures that the baryons are fermions. The lowest-lying representation for odd $N_c$, consistent with the quantization condition in eq. (2.5) is $(p, q) = (1, \frac{N_c - 1}{2})$; this representation can easily be shown to have $J = 1/2$ using the method described in sect. IV. The Young tableau for this representation is given in diagram a) of fig. 1. Note that this representation does not correspond to any of the usual representations at $N_c = 3$, and, in particular, is not an octet. However, the states in this representation do include those in the usual octet. Accordingly it is natural to take this representation to be the large $N_c$ generalization of the octet. This representation may be denoted “8”; the quotation marks act as reminders that this is not the octet but its large $N_c$ generalization.

Similarly, the next representation has $(p, q) = (3, \frac{N_c - 2}{2})$ which has $J = 3/2$. The Young tableau for this representation is given in diagram b) of fig. 1. This representation contains all the states in the usual decuplet and can be regarded as the large $N_c$ generalization of the decuplet; accordingly, this representation is denoted by “10”. The mass relation in eq. (2.8) then gives the mass splitting of the “10” from the “8”:

$$M_{10} - M_8 = \frac{3}{2I_1} N_c.$$ (3.1)

The splitting obtained at large $N_c$ is thereby identical to the analogous result for the decuplet-octet splitting in eq. (2.10) which was obtained with the assumption $N_c = 3$. The $N_c$ scaling of this splitting is found to scale as $N_c^{-1}$ (since $I_1$ scales as $N_c$). Thus the representations become degenerate as $N_c$ goes to infinity. This is for a deep reason: all of these non-exotic collective states are part of one contracted $SU(2 N_f)$ representation which emerges at large $N_c$ (as will be discussed in subsection IV.C).

To study of exotic states we need the large $N_c$ generalization of the 10 representation. The key feature of the 10 is that it is the lowest-lying representation that contains a state with strangeness $+1$. Accordingly, its large $N_c$ analog should be the lowest-lying representation that includes an exotic state with strangeness $+1$. This is readily seen to be $(p, q) = (0, \frac{N_c + 3}{2})$ and has $J = 1/2$. This representation is associated with the Young tableau c) in fig. 1 and is denoted as “10”. The excitation energy of this representation is obtained via eq. (2.8):

$$M_{10} - M_8 = \frac{3 + N_c}{4I_2}.$$ (3.2)

By construction, eq. (3.2) agrees with eq. (2.11) when

$$M_{10} - M_8 = \frac{3}{N_c}.$$ (3.3)

In the large $N_c$ limit the “10” does not become degenerate with the “8”. It is easy to see that this behavior is generic for exotic representations. That is, any representation which contains at least one manifestly exotic state will have a splitting from the ground state which is finite as $N_c \to \infty$. It is interesting to note that this behavior is characteristic of typical vibrational excitations\cite{53}.

The study of the excitation energies as calculated via rigid-rotor quantization reveals a fundamental difference between exotic and non-exotic representations and the same fundamental difference is also seen via large $N_c$ consistency rules. The non-exotic representations become degenerate with the ground state at large $N_c$ while the exotic representations do not—they remain split from the ground state at leading order in the $1/N_c$ expansion.

### IV. RIGID-ROTOR QUANTIZATION VERSUS LARGE $N_c$ QCD

This section demonstrates that the predictions of rigid-rotor quantization do not appear to follow from the known behaviors of such states in large $N_c$ QCD. This is seen from a wide variety of perspectives. Five rather general arguments are given all of which imply that the

\begin{figure}[h]
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\caption{Young tableau for arbitrary but large $N_c$: a) the “8” representation with $(p, q) = (1, \frac{N_c - 1}{2})$; b) the “10” representation with $(p, q) = (3, \frac{N_c - 2}{2})$; c) the “10” representation with $(p, q) = (0, \frac{N_c + 3}{2})$. The Young tableau in a) and b) have $N_c$ boxes; the tableau in c) has $N_c + 3$ boxes.}
\end{figure}
results of rigid-rotor quantization do not follow with the known behavior of large $N_c$ QCD.

A. Exotic baryon widths

This subsection focuses on the $N_c$ scaling of the width of the exotic resonance. The essence of this argument is quite simple and requires two things be demonstrated. The first is that rigid-rotor quantization, if valid, must predict widths which vanish at large $N_c$; the second, that the width of exotic states with rigid-rotor quantization is, in fact, of order $N_c^0$ implying that it is invalid. Before demonstrating both of these points, it should be noted that this argument is formal. The issue of relevance is not whether the width of the $\theta^+$ is numerically large or small, but how it scales with $N_c$. The key point is that if the width does not vanish as $N_c \to \infty$, as a matter of principle, the quantization prescription is presumably wrong.

Let us begin with the first point. As noted in the introduction, the question of whether the rigid-rotor quantization is valid at large $N_c$ has a clear formal definition. It is valid, if and only if, the properties computed via rigid-rotor quantization become exact as $N_c \to \infty$. The excitation energy of the exotic state at large in rigid-rotor quantization is given by eq. \((3.2)\); the issue is whether this becomes exact as $N_c \to \infty$. It is not exact if that there is a non-zero width in the large $N_c$ limit. This can be understood in two complementary ways. One way is simply to note that unless the width goes to zero at large $N_c$, the state is not well defined at large $N_c$—it does not correspond to an asymptotic state of the theory; it has no well-defined mass and eq. \((3.2)\) does not give the exact value for the mass at large $N_c$. An complimentary perspective is to assign the width to be an imaginary part of the mass. However, if the width remains finite at large $N_c$, then the mass has a non-vanishing imaginary part at large $N_c$, in which case eq. \((3.2)\) does not give the exact result for the mass. From either perspective, eq. \((3.2)\) is not exact at large $N_c$ unless the width vanishes.

Note, for comparison that for non-exotic collective baryons such as the $\Delta$, the widths do go to zero at large $N_c$ and therefore the prediction of eq. \((3.1)\) can indeed become exact. To see this, one simply notes that the non-exotic collective excitation energies as given by eq. \((3.1)\) scale as $1/N_c$ while meson masses scale as $N_c^0$. Thus, at large $N_c$ non-exotic collective space have no phase-space for decay and, hence, have zero width.

Let us now turn to the case of the exotic collective baryons which is quite different. Recently, Praszalowicz demonstrated that the width of the $\theta^+$ is of order $N_c^0$ [36]. His result is summarized below.

Equation \((3.3)\) implies that the excitation energy of the “$10^+$” representation does not vanish at large $N_c$. Thus, in general phase space it does not inhibit decay. Of course, phase space is not the only reason the width can go to zero. It is also logically possible that the coupling between the initial exotic baryon and the final state of meson plus baryon could vanish at large $N_c$. The issue boils down to whether or not this happens.

As shown by Praszalowicz [36], in the context of rigid-rotor quantization, there are two possible structures for the coupling of collective baryon states to mesons which contribute in leading order to the width:

$$\hat{O}_\kappa = -i \frac{3}{2M_B} \left( G_0 \hat{O}^0_{\kappa j} + G_1 \hat{O}^1_{\kappa j} \right) p_j \quad (4.1)$$

with

$$G_0 \hat{O}^0_{\kappa j} = D^\kappa_{rj} \quad \text{and} \quad G_1 \hat{O}^1_{\kappa j} = d_{jkl} D^8_{nkj} \hat{S}_l \quad (4.2)$$

where $\kappa$ indicates the meson species and $\hat{S}$ is the spin operator. While the values of the coupling constants, $G_0$ and $G_1$, are model dependent, their $N_c$ scaling is known:

$$G_0 \sim N^{-3/2}_c \quad G_1 \sim N^{-1/2}_c. \quad (4.3)$$

Using the collective SU(3) wave functions for the “$8$” and “$10$” representations one finds:

$$|\langle \theta^+, s=\downarrow | \hat{O}_{K3} | N, s=\downarrow \rangle |^2 = \frac{3}{(M_N + M_{\theta^+})^2} \left( \frac{3(N_c + 1)}{(N_c + 3)(N_c + 7)} \right)^2 \left[ G_0 - \frac{N_c + 1}{4G_1} \right]^2 p^2. \quad (4.4)$$

The $G_1$ term enhanced by a factor of $N_c$ (coming from the integral over the collective wave function) compensates for the fact that it is characteristically smaller than $G_0$ by a factor of $N_c$. Using the scaling in eq. \((4.3)\) and the fact the phase space implies $p \sim N_c^0$, one sees the matrix element is of order $N_c^0$. Combining this with the phase space one sees that

$$\Gamma_{\theta^+} \sim N_c^0. \quad (4.5)$$

The scaling in eq. \((4.5)\) completes the demonstration. The width as computed using states from rigid-rotor quantization shows that the rigid rotor mass formula of eq. \((3.2)\) does not correctly give the mass at large $N_c$: rigid-rotor quantization does not appear to be self-
consistent in large $N_c$ QCD.

**B. Time scales for collective and vibrational motion**

The key to the success of collective quantization is a clear separation of time scales between the collective motion and the intrinsic motion. As a general principle, to quantize the theory one must first enumerate the degrees of freedom. The easiest way to do this is via the study of small classical fluctuations around the classical soliton: each classical mode associated with small amplitude fluctuations is a degree of freedom which can be quantized. In studying these small amplitude fluctuations one finds that although the typical modes have a vibrational frequency which scales as $N_c^0$, of necessity there will also be zero frequency modes. Such zero modes arise because symmetries associated with the underlying theory are broken by the soliton configurations.

Once these modes have been identified at the classical level, they can be canonically quantized. As a practical matter, a parametric expansion is used as a systematic way to organize things. It can be shown self-consistently that for properties of low-lying states the neglect of these couplings leads to a $1/N_c$ correction to the soliton mass as compared to a leading order value which goes as $N_c$, and a correction due to the quantized zero modes which goes as $N_c^0$. The quantization of the collective degrees of freedom associated with the zero modes can be handled separately for exactly the same reason—the effects of the coupling of the collective to the vibrational degrees of freedom on the soliton mass is suppressed by $1/N_c$. The collective degrees of freedom are those which at the classical level correspond to zero modes; modes having a finite frequency are quantized as vibrations. As will be discussed in sect. the modes associated with the exotic excitations turn out not to be collective in this sense; they are associated with modes which have finite frequencies at small amplitude and, thus, act as vibrations.

The effect of the quantization of the collective degrees of freedom on the soliton mass is typically of order $1/N_c$. At the classical level the degrees of freedom are flat and have no natural time scale—the classical motion associated with them can occur arbitrarily slowly. The quantization of these collective modes keeps them slow—the typical time scales are of order $N_c$. The characteristic angular speed for non-exotic collective coordinates is $1/N_c$; it goes as the inverse of the moment of inertia since the associated angular momentum is typically of order unity and the motion is of large amplitude (order $N_c^0$ typically circumnavigating the collective space), so the characteristic time, $\tau$, is of order $N_c$. In semi-classical motion one has a general energy-time uncertainty relation:

$$\Delta E \sim 1/\tau,$$

where $\Delta E$ is the typical splitting between levels. This naturally gives energy splittings of order $1/N_c$. It is straightforward to see that this behavior reproduces what is seen in eq. (5.1).

The time scale for the non-exotic collective motion ($N_c$) scales in a qualitatively different way from that of the internal vibrational motion (shown below to scale as $N_c^0$). This difference is critical for the success of the quantization program, particularly for the separate treatment of the collective and vibrational degrees of freedom. Note that there exist vibrational modes with the same quantum numbers as those of the collective degrees of freedom. One generally expects degrees of freedom with identical quantum numbers to couple. However, two degrees of freedom with identical quantum numbers naturally remain weakly coupled if the characteristic times of the two degrees of freedom are widely separated. In this case there is a Born-Oppenheimer type separation of the dynamics of the modes. For the case of collective rotational degrees of freedom this is precisely what happens and the modes are effectively decoupled for low-lying states.

Let us now turn to the behavior of the exotic states. It is apparent from eq. (3.2) that excitations with exotic quantum numbers have excitation energies of order $N_c^0$, implying that even for the lowest-lying exotic state, $\Delta E \sim N_c^0$. This in turn implies that $\tau \sim N_c^0$, where $\tau$ is the characteristic time scale of the motion. This is highly problematic from the viewpoint of collective motion. In the first place, this is manifestly inconsistent with the motion being collective in the sense it corresponds to large amplitude motion. Equation (2.2) implies that that excitation energies of order $N_c^0$ correspond to motion with $J_A^2/I_2 \sim N_c^0$ (with $A=4,5,6,7$). This, along with the fact that $I_c \sim N_c$, further implies $J_A \sim N_c^{1/2}$, and the relevant angular velocities are of order $N_c^{-1/2}$. Therefore the motion is slow—the angular velocity goes to zero as $N_c$ goes to infinity. However, we also know that the characteristic time for this motion is of order $N_c^0$. Clearly this is not possible unless the typical angular displacement associated with this motion is of order $N_c^{-1/2}$. Thus, this motion is confined to a region of angular space which goes to zero at large $N_c$: the motion does not explore the full collective space. In this sense the motion is clearly not collective. This should be contrasted to the case of low-lying non-exotic states which have a characteristic time scale of $N_c^1$, a characteristic angular velocity of $N_c^{-1}$ and, hence, a characteristic angular displacement of $N_c^0$, which is of large amplitude and subsequently collective.

Given the fact that motion is not collective, a collective quantization as in the rigid-rotor approach is presumably invalid. Moreover, it is easy to understand why it can fail. The time scale for even the lowest-lying exotic states, is of order $N_c^0$; this is the as for vibrational motion. In the absence of such a scale separation, there is nothing to prevent the ostensibly collective mode from mixing strongly with vibrational motion with the same quantum numbers and a priori that is exactly what one expects to
happen. When this occurs the rigid-rotor approximation based on the separate dynamical treatment of the two types of motion clearly fails.

\textbf{C. Predictions from large }N_c\text{ consistency rules}

The introduction stressed the remarkable fact that the prediction of the mass in refs. \[8, 9, 10, 11, 12\] was independent of dynamical details of the model. The key model independent relations were that the SU(3) symmetry breaking used for the exotic states was identical to that used for the ground state band. It was largely because of this model independence that the prediction should be taken seriously. Model-independent predictions are special; the very fact that they do not depend on model details suggests that they may well be reflecting the underlying structure of large }N_c\text{ QCD.

Of course, just because a particular relation obtained in a chiral soliton model does not depend on the detailed dynamics of the model, need not imply the result is truly model independent results of large }N_c\text{ QCD. There is strong evidence, however, these relations are, in fact, truly model independent. All of these relations seen to date (\textit{with the exception of those involving exotic baryons computed via rigid-rotor quantization}, are known to be correct model-independent predictions of large }N_c\text{ QCD}\[13, 14, 15, 16, 17, 18\]. These include relations of standard static observables (such as magnetic moments or axial couplings) as considered in ref. \[14\], as well as the rather esoteric quantities, such as the non-analytic quark mass dependence of observables near the chiral limit\[35\], or relations between meson-baryon scattering observables in different spin and flavor channels \[38, 40\]. All of these can similarly be derived in a “model-independent” manner in the chiral soliton model and are also derivable by a truly model-independent way.

The basis for demonstrating these model-independent predictions is the use of large }N_c\text{ consistency rules\[15, 16, 17, 18\]: the large }N_c\text{ predictions for various related quantities are not self consistent unless various relations are imposed. For example, according to Witten’s large }N_c\text{ counting rules generically meson-baryon coupling constants scale as }N_c^{1/2}\[33\]. Thus, two insertions of coupling constants will yield a contribution to meson-baryon scattering of order }N_c^1\text{, but unitarity requires the scattering to be of order }N_c^0\text{. This can only be satisfied if the baryons form towers of states which are degenerate at large }N_c\text{ with the values of the coupling constants related to one another by geometrical factors (up to }1/N_c\text{ corrections}\[15, 14, 17, 18\]. These factors turn out to be precisely the ones found in the chiral soliton models. Other quantities can be derived via similar means.

The results of this type of analysis are well known: A contracted SU(2)\[N_f\text{) symmetry emerges in the large }N_c\text{ limit. Baryon states fall into multiplets of this contracted SU(2)\[N_f\text{), and the low-lying states in these multiplets are split from the ground state by energies of order }1/N_c\text{—these excitations with the SU(2)\[N_f\text{) multiplets are collective. In the space of these collective states all operators can ultimately be expressed in terms of generators of the group and from this, relations can be obtained. The key issue here is simply that the multiplet of low-lying baryons has been explicitly constructed—it coincides exactly with the low spin states of a quark model with }N_c\text{ quarks confined to a single s-wave orbital}\[18\]. It is well known that at large }N_c\text{ there are no low-lying collective baryon states (\textit{i.e., states with excitation energies of order }1/N_c\text{) with exotic quantum numbers. This neatly mirrors the analysis of sect. \textit{III} exotic states have excitation energies of order }N_c^0\text{.

This situation is quite problematic, however, if one wishes to assert that relations for the exotic states are truly model independent. The difficulty is simply that the exotic states are not in the same contracted SU(2)\[N_f\text{) multiplet as the ground band baryons. This means that group theory alone cannot relate matrix elements in the ground band to matrix elements involving exotic states: the standard large }N_c\text{ consistency rules do not allow one to relate any property of the exotic states to the ground band states.

At a minimum this implies that the apparently model-independent predictions of exotic state properties seen from rigid-rotor quantization have not been shown to be truly model independent. This is characteristically different from the properties of non-exotic states which are related to one another by large }N_c\text{ consistency rules. Of course, just because the large }N_c\text{ consistency conditions do not give the relations seen for exotic states in rigid rotor computation does not by itself mean that these relations are wrong; it merely means we do not know them to be correct. However, it does mean that the principle reason to take the predictions of the chiral soliton model seriously—the apparently model-independent status—has no known basis in QCD.

\textbf{D. Solitons for large }N_c\text{ in a world with heavy quarks}

The predictions in refs. \[8\] were based on chiral soliton models. However, this is somewhat misleading. The basic arguments of rigid-rotor quantization depends critically on the SU(3) flavor symmetry of the underlying theory and the fact that the classical solution breaks both rotational and flavor symmetries in a correlated way resulting in a hedgehog configuration. However chiral symmetry plays role only through the chiral anomalies encoded in the Wess-Zumino term which leads to the constraint in eq. \[25\]. However, as shown in appendix \[\textit{A}, eq. \[26\] can be derived directly at the quark level with no reference to anomalies. Thus, the logic underlying collective quantization applies to all SU(3) symmetric theories which have hedgehog mean-field solutions. In particular, it applies to models of QCD with three degenerate flavors in the limit where all quark masses are
heavy; \( m_u = m_d = m_d \equiv m_q \) with \( m_q \gg \Lambda \) where \( \Lambda \) is the QCD scale (provided that such models produce a hedgehog at the mean-field level). In this section it is shown that the application of the rigid-rotor quantization in such a regime gives results which are manifestly wrong suggesting that something is wrong with the underlying logic.

In fact, in this regime one need not consider a model of QCD, but rather QCD itself. Recall that in Witten’s original derivation of baryon properties in large \( N_c \) QCD the case of heavy quarks was considered for simplicity (it was later argued that the conclusions are valid for the case of light quarks)\(^{33}\). The derivation is based on the fact that in this limit the quarks are non-relativistic and can be described via the many-body potentials arising from gluon exchange. Witten then demonstrated that the Hartree mean-field approximation becomes valid in the large \( N_c \) limit, that all \( N_c \) quarks are in the same single particle wave function (modulo their color degree of freedom) and that the size of this orbital is independent of \( N_c \). In doing this analysis, the role of flavor and spin degrees of freedom was not highlighted.

Consider the role played by spin and flavor in this system. From the analysis of Dashen, Jenkins and Manohar we know that for states with \( J, I \sim N_c^0 \) at leading order in the \( 1/N_c \) expansion the only interactions which contribute are either spin and isospin independent or both spin dependent and isospin independent\(^{15}\). The spin dependence enters solely through the magnetic coupling of the gluons to the quarks. Note, however, that the underlying magnetic interaction is small for large quark masses since the quark magnetic moment goes as \( 1/m_q \). Thus all spin-flavor dependent interactions are small in the combined large \( N_c \) and heavy quark limits.

The heavy quark limit implies that the spatial shape of the single-particle levels does not depend on spin and flavor. Assuming that rotational symmetry is not broken at this order, the orbitals will be s-wave. However, the Hartree state is highly degenerate at this order: any spin and flavor orientation is equivalent. If one includes the leading \( 1/m_q \) correction, this degeneracy is broken (but at this order the spatial part of the wave functions are unchanged) and one simply chooses the single-particle state to have a spin-flavor orientation which minimizes the spin-flavor interaction at order \( 1/m_q \). The precise form of this spin-flavor orientation depends on the sign of the interaction which either favors a ground of the baryon with maximal spin or minimal spin. If it favors maximal spin, then all quarks will have the same well-defined spin and flavor projections. In contrast, if it favors minimal spin (as is believed to happen in nature, which will be assumed here) the spin-flavor part of the state takes the conventional hedgehog form:

\[
|h\rangle = 2^{-1/2} (| \uparrow \uparrow \rangle - | \downarrow \downarrow \rangle)
\]

(4.7)

where the arrows indicate spin projection and the letters indicate flavor. It is assumed for simplicity that the hedgehog is in the u-d subspace with grand-spin of zero.

It is easy to see that there are low-lying single-particle excited states for quarks propagating in the background of the Hartree potential generated by the hedgehog state. There are six distinct spin-flavor states, one of which, the state, \( | h \rangle \), is, by construction, the lowest-lying state for the Hartree potential. These states differ from the lowest lying state in energy by an amount of order \( \Lambda^2/m_q \), where the \( 1/m_q \) is due to the spin dependence as noted above and the factors of \( \Lambda^2 \) follow from dimensional analysis. Furthermore, all flavor generators (except for \( \lambda_8 \)) when acting on the hedgehog, will produce a superposition of these low-lying excited single-particle states. Thus, if one computes the moments of inertia using eqs.\(^{15}\) one has the following scaling in \( N_c \) and \( m_q \):

\[
I_1^{-1} \sim \frac{N_c \Lambda^2}{m_q}, \quad I_2^{-1} \sim \frac{N_c \Lambda^2}{m_q}.
\]

(4.8)

Equation (4.8) has profound implications for excitation energies assuming that rigid-rotor quantization is legitimate. Consider first the behavior of the non-exotic states, such as those in the “10” representation. Equations \(^{15}\) along with eq. (4.8) imply that scaling of the excitation energy with \( m_q \) and \( N_c \) goes as

\[
M_{10^−} - M_{8^−} \sim \frac{\Lambda^2}{m_q N_c}.
\]

(4.9)

This behavior is precisely what one expects. It is suppressed by \( 1/N_c \) since it involves the excitation of a single quark and it goes as \( 1/m_q \) since it relies on a magnetic gluon-quark coupling.

In a similar way one can compute the scaling of the excitation energy of the exotic states such as the “1[τ]”. Equations \(^{15}\) and eq. (4.8) imply that this excitation scales as

\[
M_{10^−} - M_{8^−} \sim \frac{\Lambda^2}{m_q}.
\]

(4.10)

However, this scaling is quite problematic on physical grounds. To construct an exotic state such as the “10” one must include an extra quark-antiquark pair relative to the “8”. Recall that by assumption the quarks are heavy enough to be non-relativistic; the binding energies of the quarks are much smaller than the mass. The excitation energy of the exotic state must then be \( 2m_q \) plus small corrections; it grows with \( m_q \). This is in contradiction with eq. (4.10) which has the excitation energy decreasing with increasing \( m_q \) assuming that rigid-rotor quantization is valid. The contradiction implies that the assumption is false: rigid-rotor quantization is not valid for the exotic states. As noted above, the logic underlying rigid-rotor quantization does not depend on the quark mass and the failure for the present case indicates that the logic is flawed.

As pointed out recently by Pobylitsa\(^{21}\), the issue is even more stark if presented in the context of an SU(3) symmetric non-relativistic quark model. In this case
model space can be constructed to have only quark—and no anti-quark—degrees of freedom. However, the interactions in such a theory can be chosen to reproduce the same $N_c$ scaling seen in QCD. Again, as $N_c$ becomes large the Hartree approximation becomes increasingly well justified and again the Hartree minimum will be of a hedgehog configuration (provided that the ground state orbitals turn out to be s-waves). The justification for doing rigid-rotor quantization is identical to that in the chiral soliton case. In such a model both $I_1$ and $I_2$ can be computed using the standard expressions in eq. and rigid-rotor quantization can then be implemented. Such a quantization implies the existence of exotic states with an excitation energy of order $N_c$, yet by construction such states are not in the Hilbert space from which the model was constructed. The rigid-rotor quantization is wrong for exotic states in such models. Of course, one might argue that such a model does not represent QCD in a particularly realistic way. However, that should be irrelevant to the central issue of the justification of rigid-rotor quantization: the derivation of this approach has always been based on general considerations and not on the detailed structure of QCD.

E. The Callan-Klebanov approach at zero SU(3) breaking

The discussion in this paper has focused on states as computed in the limit of zero SU(3) symmetry breaking. As was argued above, the fundamental issues of principle interest here can be most cleanly addressed if detailed numerical questions, such as whether the SU(3) breaking effects are too large to justify perturbation theory, aside. One powerful way to focus on the problem is to consider how to treat the problem in the presence of SU(3) symmetry breaking and then consider the limit as this breaking goes to zero.

The basic formalism of how to treat chiral solitons in the presence of SU(3) breaking was developed long ago by Callan and Klebanov [20]. The logic underlying this approach is very simple. If $m_s - m_q$ is greater than zero (where $m_q$ is the light quark mass), then at the classical level the minimum energy configuration is a hedgehog in the u-d subspace. A flavor rotation in the 4, 5, 6 or 7 directions will yield a classical configuration with higher energy. Thus these directions are not flat collective ones but are unambiguously of a vibrational character. The formalism is based on the straightforward quantization of these vibrational degrees of freedom along with the collective quantization of the SU(2) degrees of freedom in the presence of the Wess-Zumino term. It is often called the “bound state approach” since states with negative strangeness are viewed as a bound state of an SU(2) skyrmion with an anti-kaon. Here it will be referred to as the Callan-Klebanov approach in order to avoid confusion when discussing states of positive strangeness which are unbound. It should be noted that a principal reason why this approach was introduced was to deal with circumstances where SU(3) symmetry breaking was large enough so that simple perturbative treatments were potentially unreliable. But, as stressed in ref. [2], the formalism should be valid regardless of the size of the symmetry breaking and, in particular, holds as $m_s - m_q \rightarrow 0$.

The key issue here was discussed in the recent paper by IKOR [2]. Consider the Callan-Klebanov formalism as $m_s - m_q \rightarrow 0$. For states with non-exotic quantum numbers it is possible to show analytically that as this limit is approached, the excitation energy of the bound state goes to zero (as it is in rigid-rotor quantization) and the structure of the vibrational mode goes over precisely to the spatial distribution seen in the collective moment of inertia [20]. This supports the view that the Callan-Klebanov formalism applies regardless of the size of SU(3) symmetry breaking and remains valid to the exact SU(3) limit, and that rigid-rotor quantization is valid for non-exotic collective states.

However, the situation is radically different in the case of exotic $s=+1$ excitations. In that case, there is no analytic demonstration that the results of the Callan-Klebanov treatment goes over to those of rigid-rotor quantization as the SU(3) limit is approached. Of course, it is logically possible that the two approaches are, in fact, equivalent in the SU(3) limit but that a mathematical demonstration of this equivalence has not yet been found. However, there is strong numerical evidence in ref. [2] that this is not the case. In particular, for small values of $m_K$ (i.e., for small SU(3) symmetry breaking), if the two approaches were equivalent there ought to be a resonant vibrational mode whose frequency is near to that predicted in rigid-rotor quantization for all Skyrme-type models. In fact, no such mode is seen. Indeed, for the standard Skyrme model, there are no exotic resonances at all. This strongly suggests that as the exact SU(3) limit is approached the Callan-Klebanov approach does not become equivalent to the rigid-rotor quantization as $m_s - m_q \rightarrow 0$.

If one accepts that the standard derivation of the Callan-Klebanov formalism is valid for this problem, then the inequivalence between rigid-rotor quantization and the Callan-Klebanov method for exotic states implies that the rigid-rotor quantization is not valid. There is a mathematical subtlety associated with this. We know the derivation of Callan and Klebanov in ref. [20] is valid when SU(3) breaking is large enough so that motion in the strange direction is of a vibrational nature. As a formal matter, this is guaranteed to happen in the large $N_c$ limit for any finite value of $(m_s - m_q)$. As long as the physical quantities of interest have a uniform limit as $N_c \rightarrow \infty$ and $(m_s - m_q) \rightarrow 0$, then the Callan-Klebanov derivation is automatically valid in the large $N_c$ limit for exact SU(3) symmetry. However, if the two limits do not commute, then taking the SU(3) limit prior to the large $N_c$ limit (as is implicitly done in rigid-rotor quantization) would give results which differ from those when taking the large $N_c$ limit first (as is implicitly done in
the Callan-Klebanov approach). If this is the case, then it remains possible mathematically that both approaches are valid in particular regimes but that their domains of validity do not overlap.

However, on physical grounds this mathematical possibility seems quite unlikely. In the first place, it seems implausible \textit{a priori} that the large \( N_c \) and SU(3) limits commute for non-exotic states and then fail to exotic states. More importantly, there is considerable experience with cases where the large \( N_c \) limit does not commute with some other limit, and in these cases the lack of commutativity can be traced to a clear physical origin which is apparent at the hadronic level. For example, it is well known that the chiral limit and the large \( N_c \) limit do not commute for baryon quantities which diverge in the chiral limit (such as isovector charge or magnetic radii). In these cases the physical origin is easily traced to the role of the \( \Delta \) resonance whose excitation energy is anomalously low parametrically (it scales as \( 1/N_c \)) compared to typical excited baryons and which therefore leads to a class of infrared enhancements in loop graphs. Thus, it is the interplay between the two light scales in the problem—the chiral scale, \( m_\pi \) and the light scale induced at large \( N_c \), \( M_\Delta - M_N \)—which leads to the non-commuting behavior for problems that are infrared singular in the combined limit. There is nothing analogous to this when considering exotic baryons near the SU(3) limit: the SU(3) symmetry breaking scale is the only relevant low scale in the problem since the low-lying decuplet type excitations play no special role. Given these physical arguments it is highly unlikely that the two limits do not commute; as note above, this implies that rigid-rotor quantization is invalid.

Apart from the physical grounds discussed in the previous paragraph there is a clear mathematical way to understand why numerical work of IKOR fails to find a vibrational mode whose properties match those predicted in rigid-rotor quantization for exotic states. This is the mixing between the “collective” mode in rigid-rotor quantization and ordinary vibrational modes which is discussed in detail in sect. \textbf{V}. This mixing occurs at leading order in both the large \( N_c \) expansion and in SU(3) breaking regardless of the ordering of limits, and thus demonstrates mathematically that the physical arguments given above are correct: the two limits do commute, but the rigid-rotor quantization is not valid.

\section{V. RIGID-ROTOR QUANTIZATION REVISITED}

\subsection{A. General considerations}

The previous section provides strong evidence that rigid-rotor quantization is not valid for the description of states with exotic quantum numbers. This can be seen as somewhat paradoxical since the derivation of rigid-rotor quantization \cite{23} closely paralleled the derivation for SU(2) skyrmions, by Adkins, Nappi and Witten (ANW) \cite{24}, which is generally agreed to be correct. How can this method work for SU(2) solitons and for non-exotic states in SU(3) solitons, yet fail for exotic states?

It is important to begin by recalling the fundamental assumption made in the derivation in sect. \textbf{II} namely, that the collective motion and the intrinsic motion are dynamically separate. The issue is whether this is true; as will be seen in this section, it does not appear to be true generally for exotic motion even at large \( N_c \).

The ANW procedure amounts to putting an ansatz for a class of allowed motion into the Lagrangian thereby obtaining a proposed collective Lagrangian. This is done in eq. \text{eq. (2.1)} where the ansatz made is that the motion corresponds to an overall time-dependent rotation of the static soliton. As a general rule, the insertion of an ansatz into a Lagrangian yields a legitimate collective Lagrangian if, and only if, all classical solutions of the collective Lagrangian so obtained are also solutions of the full equations of motion. Only after the collective Lagrangian has been isolated at the classical level can the collective Hamiltonian be found and then quantized.

As alluded to in sec. \textbf{II} the ANW treatment is strictly valid at large \( N_c \) only for models whose Lagrangians have no first derivatives in time (such as the original Skyrme model). The reason for this is that the ansatz in eq. \text{eq. (2.1)} only corresponds to an approximate solution of the full equations of motion in such cases. It has long been known that the method needs to be modified for models where first derivatives in time are present (such as in soliton models with explicit quark degrees of freedom) \cite{25,26}. For the SU(2) model with explicit quarks one can find an appropriate ansatz which corresponds to a solution of the mean-field (classical) equations; the cranking equations provide such an ansatz. The reason this issue becomes central here is the role of the Wess-Zumino term. This term has an explicit time derivative, and \textit{a priori} one ought not to expect the ANW method to work without modification.

First consider models such as the original SU(2) Skyrme model, which only has pion degrees of freedom and has no first derivatives in time. In this case it is easy to find such families of solutions which become exact at large \( N_c \). In particular,

\begin{equation}
U(\vec{r},t) = Ae^{i\vec{x} \cdot \vec{A}/2}U_0(\vec{r})e^{-i\vec{x} \cdot \vec{A}/2}A^\dagger \quad \text{(5.1)}
\end{equation}

is an approximate time-dependent solution of the classical equations of motion provided that \( U_0 \) is a static solution and \( \vec{X} \) is an angular velocity which is small at large \( N_c \) (typically going as \( N_c^{-1} \)). The parameters that specify the motion are the initial angles given in \( A \) and the angular velocities in \( \vec{X} \). This is an allowable approximate time-dependent solution since the effect of the second derivative with respect to time on the field configuration (which is neglected when using a rotating soliton) is of order \( \sim 1/N_c^2 \) down relative to contributions to the static solution. Thus, the neglected shifts in the fields are of...
relative order $N_c^{-2}$. This in turn implies a neglected shift in the angular momentum of order $N_c^{-1}$ (since the angular momentum is intrinsically of order $N_c$); this implies that the neglected shift in the moment of inertia is of order $N_c^0$ and may be neglected at large $N_c$ compared to the leading order contribution of order $N_c^1$.

The ANW ansatz of eq. (2.1) contains all the solutions of the form of eq. (5.1). Moreover, all solutions of the classical equations of motion which emerge from the collective Lagrangian are of this form. Thus, the ANW ansatz gives a legitimate collective Lagrangian on the SU(2) Skyrmion. This Lagrangian can then be quantized.

The situation is quite different if there are first-order time derivatives. In that case the neglected effect on the fields is proportional to $\lambda \sim N_c^{-1/2}$ yielding a neglected shift in the fields proportional of relative order $N_c^{-1/2}$ which in turn implies that the neglected shift in the angular momentum is of order $N_c^{1/2}$. The neglected shift in the moment of inertia is then of order $N_c$. These effects cannot be neglected since the neglected contribution is of the same order as the contribution which is kept. There is a simple way to incorporate these effects in SU(2) solitons containing quarks. In that case, the key point is that one needs an ansatz for a time-dependent solution which corresponds to the rotating soliton. If such a corresponding solution exists, it is equivalent to a static solution calculated in a rotating frame and one obtains the cranking result of eq. (2.4).

To summarize the general situation, ANW quantization, while agreeing with the generally correct method for the case where it was introduced, does not directly apply to models with first-order time derivatives. In these cases one needs to find families of approximate classical time-dependent solutions which are decoupled from the remaining degrees of freedom.

In the case of exotic motion in SU(3) solitons the Wess-Zumino term plays a dynamical role and a priori there is no reason to believe it should be valid in such a case. In contrast, for non-exotic motion the Wess-Zumino term is inert (as can be seen in eq. (2.4)). In principle, it is sufficient to stop the argument here—rigid-rotor quantization has never been correctly derived for exotic motion. The fact that the derivation was not shown to be correct does not logically mean the result is wrong. However, the fact that rigid-rotor quantization gives inconsistent results as seen in the previous section indicates that the result is, in fact, not correct.

It is useful to understand why the approach fails in a bit more dynamical detail. The key point is that there is no family of classical solutions corresponding to the exotic collective as given by the ansatz.

Before discussing the full problem it is useful to gain some intuition about how things work by considering a couple of “toy” problems.

B. A charged particle in the field of a magnetic monopole

To begin consider the following simple toy model: the non-relativistic motion of a charged particle of mass $m$ and charge $q$ confined on a sphere of radius $R$ with a magnetic monopole of strength $g$ at its center. This problem was introduced by Witten [31] to illustrate the role of the Wess-Zumino term and was used to motivate the treatment of rigid-rotor quantization for SU(3) solitons [23]. The monopole case is essentially similar to the Wess-Zumino case in two fundamental aspects: a) its effect is first order in time derivatives, and b) it is essentially topological in nature imposing topological quantization rules (the Dirac condition for monopoles, the quantization of the Wess-Zumino term for chiral soliton models [32]). To simulate the case of SU(3) solitons at large $N_c$ one has the following scaling rules:

$$q \sim N_c^0 \quad R \sim N_c^0 \quad g \sim N_c^1 \quad m \sim N_c^1; \quad (5.2)$$

the Dirac quantization condition implies that $2gg$ is an integer. This problem is exactly solvable. The key physical point is that in addition to the usual kinetic contribution to the moment of inertia, there is an angular momentum associated with the magnetic field energy which is given by $L_{\text{field}} = qg^*$. Mathematically, the key issue is the unphysical nature of a rotation about the axis linking the charge and the monopole which then imposes the Dirac condition and the restriction $L \geq |gg|$. The wave functions may be expressed in terms of three Euler angles; the independence of the the physical results on the third angle is what gives rise to the Dirac quantization condition.

The spectrum for this problem is

$$E = \frac{L(L+1) - (gg)^2}{2mR^2} \quad \text{with} \quad L \geq |gg|. \quad (5.3)$$

The analogy with the SU(3) soliton is the following. The smallest allowable $L$ corresponds to the non-exotic baryons with the $(2L+1)$ allowable $m$’s corresponding to the different non-exotic baryon states. The various states with $L > gg$ correspond to exotic states. To make this manifest we follow Diakonov and Petrov [23] and express the energy in terms of an “exotonicness”:

$$E = \frac{e^2 + e(2|gg| + 1) + |gg|}{2mR^2} \quad \text{with} \quad e \equiv L - |gg| \quad (5.4)$$

If one focuses on the low-lying exotic states (those states for which $e \sim N_c^0$ which are the states of interest in hadronic physics one finds the excitation energies are given by

$$E_e - E_0 = \frac{e|gg|}{mR^2} + O(N_c^{-1}) \quad (5.5)$$

where the $N_c$ scaling is fixed from eq. (5.4).

Conventional treatments of rigid-rotor quantization model their treatment of the Wess-Zumino term on this
exact treatment of this toy problem \[\text{[23]}\]. The present purpose is different: Here the goal is to understand which collective degrees of freedom can be isolated in a treatment which will ultimately be semi-classical in nature. Of course, this toy problem has no internal degrees of freedom—by construction it is rigid-rotor quantization. However, it is useful to illuminate the underlying physics of this toy problem in a manner which transparently can be used in the case where collective and vibrational modes mix. In particular, it is quite instructive to derive the result in eq. \[\text{[5.6]}\] from a semi-classical treatment which can then be generalized.

We need to first describe the classical motion of the particle. Let us consider the particle at rest at the north pole and we can describe all solutions of the equations of motion relative to this. Because this problem has spherical symmetry, there are “zero modes”. One can rotate the charge from the north pole and leave it in a displaced static position. A static rotation of this type is equivalent in the soliton case to a non-exotic “excitation” (although in this toy model all non-exotic states are degenerate with the ground state so “excitation” is a bit of a misnomer).

Classical behavior associated with the exotic degrees of freedom necessarily involves motion. Were there no velocity dependent forces present due to the monopole, the existence of flat directions would imply the existence of dynamical rotational modes with the system slowly rotating around the entire sphere. However, the magnetic monopole fundamentally alters this. As soon as the particle starts moving, the magnetic force acts to bend the particle into a curved orbit and this curvature can be on a scale much smaller than the radius of our sphere \(R \sim N_c^0\); indeed one can see self-consistently that the characteristic size of such an orbit for states with small “exoticness” (where \(e = L - |qg|\)) will scale as \(N_c^{-1/2}\) and, hence, does not go fully around the sphere. If the orbit is localized in a region of this size then at large \(N_c\) it effectively stays in a region small compared to \(R\), the curvature of the sphere (which is order \(N_c^0\)). In this case the classical motion is effectively that of a charged particle in a magnetic field, \(\vec{B} = qg/R^2\), moving on a plane. Consider an orbit centered around the north pole (one can always rotate your coordinate labels to do this). The position of the particle is then given approximately by

\[
x \approx r \cos(\omega t + \delta) \quad y \approx r \sin(\omega t + \delta) \quad \omega \approx \frac{qg}{mR} \quad \text{(5.6)}
\]

where \(r\) and \(\delta\) are fixed from the initial conditions and the corrections are of order \(1/N_c\).

Now suppose one wishes to quantize this classical circular motion. This is the familiar problem of Landau levels. The energy spectrum of such a system is of precisely the form of a harmonic oscillator \[\text{[11]}\]:

\[
E = (n + 1/2) \frac{|qB|}{m} = (n + 1/2) \frac{|qg|}{mR^2} \quad \text{(5.7)}
\]

Equation \[\text{[5.7]}\] can be derived using the Bohr-Sommerfeld formula. In doing this one finds that the radius of the orbits are quantized to have

\[
r^2 \approx \frac{n + 1/2}{|qB|} \approx \frac{(n + 1/2)R^2}{|qg|} ; \quad \text{(5.8)}
\]

the scaling rules in eq. \[\text{[5.2]}\] then imply that the radius scales as \(r \sim N_c^{-1/2}\). This in turn, justifies treating the problem as motion in the plane. Let us now look at the excitation energies predicted from this semi-classical quantization:

\[
E_n - E_0 = n \frac{|qg|}{mR^2} . \quad \text{(5.9)}
\]

The significant point is that eq. \[\text{[5.9]}\] gives the same excitation spectrum as in eq. \[\text{[5.5]}\] provided one identifies the “exoticness”, \(e\), with the index \(n\).

Of course, this toy model corresponds to rigid-rotor quantization since there is a fixed moment of inertia. It nevertheless teaches us several things of importance which can be generalized to situations where the moment of inertia is dynamical. i) The classical motion associated with the exotic excitation is not a zero mode; no matter how slow the velocity the period of the orbit is of order \(N_c^0\). ii) The classical motion is bounded; for energies corresponding to low exoticness (\(e \sim N_c^0\)) the typical size of the collective orbit is \(N_c^{-1/2}\). iii) In the large \(N_c\) limit the excitation spectrum can be understood by purely semi-classical means. Points i) and ii) indicate that as far as the scales are concerned this problem “looks like” vibrational motion.

One aspect of point i) ought to be stressed. The fact that this dynamical mode is not a zero mode is essential. An apparently odd feature is that the problem has two flat directions and one might naively think there ought to be two zero modes. Where then does the nonzero mode come from given the fact that there are only two degrees of freedom in the problem? Actually, the issue is largely semantic. Consider a typical harmonic vibrational mode in one dimension associated with the equation of motion \(\ddot{x} = -\omega^2 x\). The motion is given by \(x = A \cos(\omega t) + B \sin(\omega t)\) and is parameterized by two numbers which are fit by two initial conditions, \(A = x(0)\) and \(\beta = \dot{x}(0)/\omega_0\).

One typically describes this as one mode of oscillation despite the need for two parameters to specify the motion since one can always rewrite it as \(x = A \cos(\omega t + \delta)\), with \(A = \sqrt{A^2 + B^2}\) and \(\delta = \tan^{-1}(\beta/\alpha)\). Thus, the path followed depends only on one parameter \(A\), and the second parameter merely serves to induce a phase shift in the single mode of motion. One might make the following, somewhat pedantic, description: the motion consists of two distinct modes; the general motion is then the superposition of these two distinct modes. From this perspective one can write the motion as two coupled first-order differential equations for \(x\) and \(\dot{x}\) and define modes as solutions where \(x\) and \(\dot{x}\) each evolve as \(e^{i\omega t}\) with the same omega. By defining modes this way it can be seen that there are two modes in the harmonic problem given
above—one with $\omega = \omega_0$ and the other with $\omega = -\omega_0$—which form a pair. For the problem of a single harmonic oscillator, calling this a pair of modes with equal and opposite frequencies may seem a bit artificial. However, in the toy problem of a particle on a spherical shell moving in the field of a magnetic monopole, this distinction is important. There are two degrees of freedom and thus, in the sense given above, one expects four modes to come in two pairs of equal and opposite $\omega$. This is precisely what happens. One pair of modes corresponds to fully static configurations (one moves the charge in either the $x$ or $y$ direction to a new position and leaves it there at rest) and these correspond to two zero modes which we view as a single pair of modes. The other pair of modes correspond to the charge orbiting either clockwise or counterclockwise corresponding to $\omega = \pm |q|/mR^2$. However, while we still have two pairs of modes the existence of a velocity dependent force means that the pairs do not correspond to a mode associated with initial displacement in a direction paired with an initial velocity in the same direction. This means that although this problem has two zero modes associated with displacements, it does not have two pairs of zero modes but only one.

C. Coupled particles in a monopole field

While the problem in the previous section gives some insights into the scales of the problem and the nature of modes and semi-classical quantization, it cannot answer the fundamental question about whether rigid-rotor quantization is generically valid since by construction the problem is a rigid rotor. Accordingly we need a model where the underlying dynamics is not rigid; i.e., a model where the analog of the soliton has some nontrivial internal dynamics. Moreover, because the central concern is mixing between collective and internal modes, it is important that the internal degrees of freedom have excitations which have the same quantum number numbers as the exotic motion. The simplest model of this sort is a generalization of the model studied in the previous section. Consider a non-relativistic theory of two particles confined to the surface of a sphere. For simplicity we will take them to have equal mass, $m$. At the center of the sphere is a magnetic monopole of strength $g$.

The two particles have different charges—which we will take to be $q$ and zero—so they interact differently with the monopole. Finally, the two particles interact with each other via a quadratic potential $V_{\text{int}} = k|\vec{r}_1 - \vec{r}_2|^2/2$, where $k$ acts as a spring constant. To create an analogy with SU(3) solitons at large $N_c$ the following scaling rules must be imposed for the parameters:

$$q \sim N_c^0 \quad R \sim N_c^0 \quad g \sim N_c^1 \quad m \sim N_c^1 \quad k \sim N_c^1.$$ 

(5.10)

The analog of the classical soliton is a static configuration which solves the classical equations of motion. The solution is simple: the particles are on top of each other and we can take their position to be at the north pole. The rigid-rotor quantization of this system is quite trivial. The classical “soliton” is constrained to move coherently with the two particles in their classical ground state (i.e., with the two particles on top of each other). This reduces immediately to the classical motion of a single particle of mass $2m$ and charge $q$ and one can immediately read off the excitation by making the appropriate substitutions in eq. (5.5):

$$E_{\text{rigid}}^c - E_0 = c\omega_r \quad \text{with} \quad \omega_r = \frac{|q|}{2mR^2},$$

(5.11)

where as before $c$ is a non-negative integer.

On the other hand, we can follow the correct full procedure of finding classical time-dependent solutions and then quantize this motion semi-classically. All small amplitude motion can easily be found via a linearization of the equation of motion; this is sufficient provided the quantization keeps the motion within the regime of validity of linear response. As before, this can be checked a posteriori. If one is in the small amplitude regime the problem again reduces to motion on a plane in the presence of a magnetic field. Thus we can parameterize the two degrees of freedom for each particle by its $x$ and $y$ coordinates. Following the discussion in the previous subsection, it is useful to write first-order equations of motion for the particles and their associated velocities:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-\omega_r^2}{2} & 0 & \frac{\omega_r^2}{2} & 0 & 0 & 2\omega_r & 0 & 0 \\ 0 & \frac{-\omega_r^2}{2} & 0 & \frac{\omega_r^2}{2} & 0 & 0 & 2\omega_r & 0 \\ \frac{\omega_r^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-\omega_r^2}{2} & 0 & -\frac{\omega_r^2}{2} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \end{pmatrix}$$

(5.12)

The quantities $\omega_v$ and $\omega_r$ have simple physical interpretations; $\omega_v$ is the vibrational frequency of the two particles

$$\omega_v = \sqrt{\frac{2k}{m}} \quad \text{and} \quad \omega_r = \frac{qg}{2mR^2}. $$


when the monopole field is turned off and \( \omega_r \) is the Landau frequency assuming that rigid-rotor quantization is valid. From the scaling rules in eq. 5.10, one sees that \( \omega_r \sim N_c^0 \) and \( \omega_v \sim N_c^0 \). The normal mode frequencies are \( -i \) times eigenvalues of the matrix on the right-hand side of eq. 5.12.

The matrix can be diagonalized explicitly and it is found that, as expected, the eight modes group into four sets of pairs of modes with equal and opposite frequencies. There is one pair of zero modes which corresponds to static configurations in which both particles are displaced by the same amount. Analytic expressions can be obtained for the frequencies of the three pairs of non-zero modes. However, these expressions are quite cumbersome. Since the modes depend only on \( \omega_v \) and \( \omega_r \), it is useful to express the frequencies as multiples of \( \omega_r \) and to express results as a function of \( \omega_v/\omega_r \). A plot of the three (positive) frequencies for these pairs is given in fig. 2. These modes can be semi-classically quantized and the lowest-lying excitation associated with the motion is \( \omega \) above the ground state. As before, the quantized orbits correspond to velocities of order \( N_c^{-1/2} \) and displacements of order \( N_c^{-1/2} \) which self-consistently justifies the neglect of the curvature at large \( N_c \).

The results in fig. 2 are quite striking. One sees that the rigid-rotor result is reproduced for one of the modes only in the limit \( \omega_v/\omega_r \to \infty \). This can be verified explicitly: expanding the analytic expression for the appropriate eigenvalue which we denote as \( \omega_c \) as a series in \( \omega_v/\omega_r \) yields

\[
\omega_c = \omega_r \left( 1 - \frac{\omega_v}{\omega_r} \right) + \left( \frac{\omega_v}{\omega_r} \right)^2 - 3 \left( \frac{\omega_v}{\omega_r} \right)^4 + \cdots \)
\]

Clearly this series converges onto \( \omega_c \) as \( \frac{\omega_v}{\omega_r} \to \infty \) and deviates from this asymptotic value at finite values of the ratio. This result is very easy to understand physically. The limit \( \frac{\omega_v}{\omega_r} \to \infty \) corresponds to tight binding; the ratio diverges as \( k \), the strength of the inter-particle interaction goes to \( \infty \) with all other parameters held fixed. Of course, in the tight binding limit the system will act like a single coherent particle of mass \( 2m \) and charge \( q \) and one recovers the rigid-rotor result.

Moving away from the tight binding limit the particles no longer move coherently. Suppose one were to provide an initial condition in which both the particles were given an equal kick so that at \( t = 0 \) they both had the same initial velocity and had no initial separation. As they move, they feel different magnetic forces due to the differing charges and hence begin to move apart. At this point the Hook's law potential between the particle adds a new restoring force and the "collective" and vibrational motion now mix and the modes differ from the rigid-rotor modes.

It is reassuring that the analysis of this simple toy model reproduces the general arguments about time scales discussed in subsection IV.B. The rigid-rotor quantization is seen to be justified only when \( \omega_v \gg \omega_r \). This is the situation when there is a scale separation between the collective and vibrational motion and Born-Oppenheimer reasoning applies. However, as just noted, large \( N_c \) QCD does not imply that the dynamics is in this regime.

This model is also useful in clarifying some key issues. It is well known in typical soliton models that zero modes associated with broken symmetries are orthogonal to vibrational degrees of freedom at small amplitude and do...
not mix\textsuperscript{37}. It was precisely this fact that was used to justify rigid-rotor quantization. However, we see that for this toy problem the rigid-rotor quantization fails. The reason for this failure is easy to isolate. Although there are two flat directions, the presence of velocity-dependent forces induced by the monopole implies that there is only one pair of zero modes and not two. This second “would be” zero mode scales with $N_c$, an ordinary vibrational mode, and acts as an ordinary vibrational mode. Nothing prevents it from mixing with other vibrational modes and indeed it does. This can be seen explicitly by looking at one of the eigenvectors of the “would be” zero mode. Again, the expression is long and cumbersome in general, but can be written in a Taylor series in $\frac{1}{N_c}$. To second order, the motion associated with this normal mode is given by

\[
\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \Re \left( A \exp(i\omega_c t) \begin{pmatrix} 1 \\ i \\ 1 \\ 0 \\ i \\ 0 \\ -i \\ -1 \end{pmatrix} + \begin{pmatrix} \omega_r \\ \omega_v \\ \omega_r \\ \omega_v \end{pmatrix} \begin{pmatrix} -1 \\ i \\ 1 \\ -i \end{pmatrix} + \cdots \right) \tag{5.14}
\]

where $\Re$ indicates the real part, $A$ is a complex constant fixed by initial conditions and $\omega_c$ is the normal mode frequency given approximately in eq. (5.13). Note that although the leading term in the expansion has particles one and two moving together in a coherent manner (making a circular orbit), the correction terms do not. Again, we should recall that the large $N_c$ limit does not drive one to the limit where $\omega_r/\omega_v$ goes to zero so that large $N_c$ does not force these correction terms to be small: at leading order in a $1/N_c$ expansion, the collective motion has mixed with vibrational motion.

Returning to the general considerations at the beginning of this section, we see that for models such as this one which contain first order time derivatives, the classical intrinsic vibrational motion can mix with the collective rotational motion. This in turn implies that the collective motion can not be isolated and quantized separately.

D. SU(3) soliton models

The preceding subsection shows explicitly why rigid-rotor quantization fails for this toy model. But we are interested in SU(3) soliton models. The issue is whether the same type of behavior is seen in SU(3) models. The important point is that the behavior seen in the toy model should be generic. Any model with “exotic” motion of order $N_c^0$ and vibrational motion of the same order can be expected to mix in the absence of some symmetry keeping them distinct.

In fact, we know from the analysis of IKOR that this same behavior is seen for SU(3) solitons\textsuperscript{38}. They study the classical motion around the soliton at quadratic order. At large $N_c$ this quadratic order is sufficient since all nonzero kaon modes when quantized have $K/f_\pi \approx N_c^{-1/2}$ corresponding to localized and nearly harmonic motion. While for the non-exotic states at large $N_c$ and exact SU(3) symmetry, they find a rotational zero mode as seen in collective quantization, they also note that there is no exotic mode with a frequency equal to the excitation given in rigid-rotor quantization. However, if the classical motion really had separated into collective and intrinsic motion as assumed in rigid-rotor quantization there would have been a classical mode at the rigid-rotor frequency. Thus the SU(3) solitons do behave in the same way as the toy model.

VI. THE CRITIQUE OF THE TIME SCALE ARGUMENT

As noted in the introduction, the conclusion that the rigid-rotor quantization for exotic states is not justified by large $N_c$ QCD is not universally accepted. Recently Diakonov and Petrov (DP)\textsuperscript{22} criticized the argument given in ref. 10 (and presented here in subsection IV B of this paper)—i.e., the argument based on time scales; using the logic of this critique they concluded that rigid-rotor quantization was valid for “exoticness” of order $N_c^0$ and only breaks down when the “exoticness” is large enough to substantially alter the moment of inertia (which occurs at order $N_c^1$)\textsuperscript{22}. In this section we will discuss this critique and argue that it is erroneous.

The critique has three parts: i) An argument that relevant time scale for exotic excitations is of order $N_c^1$ and not order $N_c^0$; ii) a general treatment of rotational-vibrational mixing in SU(3) solitons in which it is asserted that the mixing is small at large $N_c$; and iii) a toy model to illustrate the argument.
Let us begin with the discussion of point i). DP argue correctly that in the rigid-rotor quantization the characteristic angular velocity is given by

$$v_{\text{char}} \sim \sqrt{\frac{\sum_{l=4}^7 j_A^2}{I_2}} \sim O\left(\sqrt{\frac{e}{N_c}}\right)$$  \hspace{1cm} (6.1)$$

where $e$ is the “exoticness”. This is slow at large $N_c$ when $e \sim N_c^0$. From the behavior in eq. (6.1), it is asserted that the characteristic time scale is of order $N_c^{-1/2}$ [23], in contradiction to the claims in ref. [19] and subsection LVII that the time scale is of order $N_c^0$. This assertion is, on its face, paradoxical. If true, general semi-classical considerations as seen in Bohr’s correspondence principle would imply that the quantal excitation energies associated with point i) of order $1/N_c$ and not $N_c^0$ as, in fact, they are. DP attempt to resolve this paradox by stating that the rotation is not semi-classical because of the quantization condition in eq. (6.2) which arises from the Wess-Zumino term; since the Wess-Zumino term is a full derivative term it should be thrown out classically and is thus not suitable to semi-classical arguments.

However, this resolution does not hold up. As seen explicitly in subsection LVII the problem of “exotic” motion of a charged particle in a monopole field can be quantized explicitly by quantizing the Landau orbits and the semi-classical result agrees with the exact answer at the analogous order of large $N_c$.

Given the problem with the resolution of this paradox by DP, how is one to reconcile eq. (6.1) with a characteristic time of order $N_c^{-1/2}$? The answer is trivial and was discussed in detail in subsection LVII the velocity can be of order $N_c^{-1/2}$ and the time scale $N_c^0$ provided the motion is localized to a region of order $N_c^{1/2}$. This is precisely what happens and this is verified in the model in subsection LVII. In summary, part i) of this critique appears to be without foundation.

Next consider part ii). The analysis here closely parallels the original derivation of rigid-rotor quantization. Let us recapitulate the salient points.

The analysis is based on a soliton which corresponds to the local minimum of an effective action $S_{\text{eff}}[\pi(x)]$ (where $\pi$ represents a dimensionless pion field—namely, the usual pion field divided by $f_\pi$); the action is proportional to $N_c$. The classical configuration $\pi_{\text{class}}(x)$ which minimizes the effective action gives the soliton profile and the moments of inertia $I_{1,2}$ are computed at this minimum. One finds that the classical soliton mass, $M_0$, and the moments of inertia, $I_{1,2}$, are all proportional to $N_c$.

The effective action may be expanded about the classical minimum at second order in the fields; it is given by:

$$E_{\text{eff}}[\pi_{\text{class}} + \delta \pi] = M_0 + \frac{1}{2} \delta \pi W[\pi_{\text{class}}] \delta \pi + \ldots$$  \hspace{1cm} (6.2)$$

where $W$ is an operator for any given external field $\pi_{\text{class}}$. Since the dimensionless fields scale as $N_c^0$, $W$ is of the same order $S_{\text{eff}}$, namely, $N_c^1$. This in turn implies that the harmonic fluctuations scale as $\delta \pi(x) = O(1/\sqrt{N_c})$. The spectrum of $W$ and its eigenmodes both scale as $N_c^0$. Clearly $W$ has zero modes which are related to symmetry breaking in the classical solution. For these models this includes both translations and rotations.

Up to this point the analysis seems reasonable, although one might quibble that the object on the left-hand side of eq. (6.2) should be considered an effective Lagrangian rather than an energy function since no Legendre transformation has yet been made. In any event, the next steps [23] are based on a set of assumptions which are quite problematic: “The quantization of rotations (which are large fluctuations as they occur in flat zero-mode directions) leads to the rotational spectrum discussed in the previous section [that is rigid-rotor quantization]. The vibrational modes are orthogonal to those zero modes.” Given these assumptions, it is shown that vibration-rotational coupling only becomes important when the moment of inertia changes substantially which occurs at $e \sim N_c^1$.

However, the assumption that the exotic motion is associated with zero modes and thus do not mix with vibrational motion due to orthogonality was assumed without proof to be true in analogy with problems without a Wess-Zumino term. This assumption does not appear to be correct. In the first place, the classical motion associated with exotic motion is not associated with a zero mode due to the presence of the Wess-Zumino term. This was shown in subsection LVII where it was seen that classical zero modes are associated with quantum states that have level spacing which go to zero as the large $N_c$ (classical) limit is taken. Exotic states are not in this class. Moreover, in subsection LVII it was shown explicitly that the classical motion associated with exotic states were not zero modes; rather, they were Landau orbits with a nonzero frequency. Secondly, for exotic quantum numbers, the collective motion does mix with the vibration motion at leading order as is shown explicitly in eq. (5.11). Thus, the assumption underlying this general part of the critique appears to be wrong.

Finally, we consider part iii) of the critique. This consists of studies with a toy model with both vibrational and rotational degrees of freedom and with an analog of a Wess-Zumino term. The model considered was a charged particle moving in the field of a magnetic monopole and subject to a spherically symmetric potential with a minimum at $r = R$. In this model, it is shown that rigid-rotor quantization only becomes inaccurate when the moment of inertia is altered substantially—an effect which occurs when the exoticness is of order $N_c$. However, this model is a very poor analog of the problem of interest. Recall that the danger posed to rigid-rotor quantization is the mixing of the collective modes with vibrational modes which carry the same quantum number. Note that in the toy model considered here the exotic motion carries angular momentum (also note that the exotic states correspond to states with different $J$) while the only vibrations correspond to states with different $J$.
allowed in this model are radial vibrations which do not carry angular momentum. In this model there is nothing for the collective modes to couple to at lowest order and rigid-rotor quantization works. However, as soon as the model is rich enough to include vibrational degrees of freedom with the same quantum numbers as the collective motion (as, for example, in the model of subsection VC), the rigid-rotor quantization fails.

To summarize this section, DP make a three part critique at the analysis of time scales given in subsection VC), the rigid-rotor quantization fails.

IV. B and in ref. [19]. However, the first two parts of the critique are based on faulty assumptions, while the third part is based on a model which is not an analog of the relevant problem.

VII. DISCUSSION

This paper presents strong evidence that rigid-rotor quantization is not justified on the basis of large $N_c$ considerations. The important issue is what this tells us about the nature of the $\theta^+$ and other possible exotic states.

One possibility is that the analysis based on the rigid-rotor quantization is, in fact, well justified despite the arguments given in this paper. Note that this paper does not show that rigid-rotor quantization is necessarily invalid but rather that it is not justified due to large $N_c$ QCD. It remains possible that it is justified due to some other reason. (For example, in the toy model in subsection VC rigid-rotor quantization was justified if the parameters of the model had $\omega_v \gg \omega_r$.) This would be most satisfactory in that the very successful prediction of the phenomenology based on rigid-rotor quantization would remain. However, if correct, it raises a very important theoretical question: namely, what justifies the rigid-rotor quantization? A second possibility is that the rigid-rotor quantization is not justified. In this case, the accurate prediction of the mass and the prediction of a narrow width in rigid-rotor quantization would have to simply be dismissed as fortuitous.

In either case, large $N_c$ QCD by itself does not appear to allow one to understand the structure of this state. In this respect, the $\theta^+$ is quite unlike the $\Delta$ and is like the more typical excited baryons. As with such baryons it may be that the best phenomenological treatments may be based on models whose connections to QCD are quite tenuous.

The lack of validity of the rigid-rotor quantization does have one important phenomenological consequence. If the exotic states were of a collective character one could not justify treating meson-baryon scattering in exotic channels via a simple linear response theory in the context of chiral soliton models. After all, one cannot use linear response to describe pion-nucleon scattering at the $\Delta$ resonance in the Skyrme model [29, 30]. However, because these exotic resonances are of a vibrational character, linear response is justified. This is useful in and of itself to describe scattering. Moreover, by imposing the $I = J$ rule on such scattering one can predict that the $\theta^+$ has low-lying partners (at least) at large $N_c$. These partners are related in much the same way as the $\Delta$ is to the nucleons except that both states have widths which are of order $N_c^0$. The quantum numbers of such partner states are enumerated in ref. [12].

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APPENDIX A: QUARK BASED DERIVATION OF THE WESS-ZUMINO CONSTRAINT

In Skyrme-type models eq. (2.5), which constrains the allowable representations follows directly from the Wess-Zumino term (which topology fixes to have a strength which is integer and can be identified identified with $N_c$ [32]). At a more pedestrian level, it can also be easily understood at the quark level for models with explicit quark degrees of freedom. In the body-fixed frame the baryon number of the unrotated hedgehog is associated with the SU(2) sub-manifold. The body-fixed hypercharge is also associated with this sub-manifold. One can relate the body-fixed hypercharge to the body-fixed SU(3) generator as usual so that $Y' = -2J_8/\sqrt{3}$. The baryon number, hypercharge and strangeness are related linearly. The appropriate relation for arbitrary $N_c$ is

$$Y = \frac{N_c B}{3} + S. \tag{A1}$$

Note that eq. (A1) does not coincide with the familiar relation $Y = B + S$ except for $N_c = 3$. For arbitrary $N_c$ eq. (A1) may be obtained from the known hypercharges of up, down and strange quarks:

$$Y_u = 1/3 \quad Y_d = 1/3 \quad Y_s = -2/3. \tag{A2}$$

(These are the standard hypercharge assignments for quarks at $N_c = 3$. It is straightforward to see that these assignments must hold for any $N_c$ provided hypercharge is isosinglet and traceless in SU(3) and has the property that the hypercharge of mesons is equal to the strangeness.) Each quark carries a baryon number of $1/N_c$ while the strangeness is zero for u and d quarks and -1 for s quarks. The combination of hypercharge, strangeness and baryon number assignments can only be satisfied if eq. (A1) holds. Finally, observe that in the body-fixed frame, the SU(2) sub-manifold by construction has zero strangeness; thus, eq. (A1) requires that $Y' = N_c B/3$ and the quantization condition in eq. (A1) immediately follows.
It has been argued, however, that the widths reported in the experiments may be broadened by experimental issues associated with resolution. Comparison with previous data seems to suggest that the actual width may be much narrower. This argument was first presented in hep-ph/0307370. A more detailed analysis based on this work may be found in hep-ph/0308114.

This is a standard topic in soliton theory. A nice treatment can be found in R. Jackiw, Rev. Mod. Phys. 58 (1986).