Nonlinear resonance in a three-terminal carbon nanotube resonator

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Received 22 December 2006, in final form 30 December 2006
Published 17 April 2007
Online at stacks.iop.org/Nano/18/195203

Abstract
The RF response of a three-terminal carbon nanotube resonator coupled to RF transmission lines is studied by means of perturbation theory and direct numerical integration. We find three distinct oscillatory regimes, including one regime capable of exhibiting very large hysteresis loops in the frequency response. Considering a purely capacitive transduction, we derive a set of algebraic equations which can be used to find the output power ($S_{21}$-parameters) for a device connected to transmission lines with characteristic impedance $Z_0$.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
Nanoelectromechanical systems (NEMS) are systems where mechanical and electronic degrees of freedom are coupled and whose characteristic length scales are measured in nanometres. While metal or silicon are common material choices for microelectromechanical devices (MEMS), carbon nanotubes (CNTs) may become one of the mainstays in future NEMS technology [1] due to their unique combination of electrical and mechanical properties: small mass, extraordinary stiffness, low mechanical dissipation and electrical properties ranging from semiconducting to conducting [2]. Combined together, these properties allow for NEMS devices operating in the high GHz regime. Several CNT-based NEMS devices have already been demonstrated [3–5].

In [6–8] one specific such carbon-nanotube-based system was considered: a three-terminal nanomechanical relay with a layout similar to the one in figure 1. Such devices have since been successfully fabricated [9–11]. In [6, 7, 9–11] the device was considered mainly operating as a switch and transduction was based on a tunnelling current between the tube tip and the drain terminal. In a subsequent publication [12] the high frequency properties were also considered and nonlinear resonant behaviour was demonstrated.

Nonlinear response to AC drive is a characteristic feature of many MEMS devices as well as in NEMS [13] and has recently received much attention [14–19]. It can typically be modelled by using the Duffing equation. Understanding of the parametric dependence on the behaviour of this response is essential for any potential technological application. Of equal importance for technological applications is to understand how NEMS devices act when embedded in an electronic circuit. In this paper we investigate more carefully the RF response of the three-terminal CNT resonator in figure 1 subjected to a harmonic input signal on the gate electrode, going beyond the simple Duffing equation. In contrast to previous publications [12], we base the transduction not on electron tunnelling between the tube tip and the drain but on the displacement current generated by the time-varying tube–drain capacitance when the device is connected to lossless transmission lines.

This paper consists of several sections. In section 2 we present the system and derive a lumped dynamical model. Then, by means of perturbation theory, we derive in section 3 a set of algebraic equations that can be efficiently used to obtain and classify the frequency response of the device along with expressions for the output power ($S_{21}$-parameter). In section 4 we consider the regime of linear response solutions and in section 5 we characterize the nonlinear behaviour. Finally in section 6 we discuss the domain of validity of the perturbative approach by direct comparison with numerical integration and discuss the delivered power arising from a purely capacitive transduction.
using the action gate and drain electrodes. The motion of the carbon nanotube outer diameter $D$ and the strips have heights $H_G$ and $H_D$ respectively.

The electronic degrees of freedom are most conveniently treated within the circuit model in figure 2. For finite source–drain capacitances at time $t$ and $Q_{G,D}$ the associated capacitated charge. Stationarity of the action with respect to $u(x, t)$ then provides the equation of motion for the beam

$$\rho A \ddot{u}(x, t) + E I \dot{u}(x, t) = \frac{1}{2} \sum_{i=G,D} [\Delta V_i(t)]^2 \frac{\delta C_i[u, t]}{\delta u(x)}$$

where we have introduced the shorthand

$$\frac{\delta C_i[u, t]}{\delta u(x)} = \int \frac{\delta C_i[u, t]}{\delta u(x, t')} dt'$$

to emphasize that the functional derivative only affects the spatial dependence of the capacitance at time $t$ and the potential differences $\Delta V_{G,D} = V_T - V_{G,D}$ between the tube and the gate/drain electrodes.

The electronic degrees of freedom are most conveniently treated within the circuit model in figure 2. For finite source–drain resistance $R_S$ we have three relevant degrees of freedom $Q_G(t) \equiv Q_G(t) + Q_{G,D}(t)$, $Q_D(t) \equiv Q_D(t) + Q_{G,D}(t)$ and $Q_T(t) \equiv Q_D(t) + Q_{G,D}(t) - Q_G(t)$. With only one incoming signal on the gate, $V_g(t)$, these charges obey the equations of motion

$$\dot{\alpha}_i Q_i = \frac{1}{Z_0} [2V_i(t) - V_G(t) + V_{G,D}^0]$$

$$\dot{\alpha}_i Q_D = \frac{1}{Z_0} [V_D(t) - V_{G,D}^0]$$

$$\dot{\alpha}_i Q_T = \frac{1}{R_S} [V_S - V_T(t)]$$

Here $u(x, t)$ is the instantaneous deviation of the tube towards the drain electrode, at a distance $x$ from the tube support. $E$ is the effective Young modulus [20–22] of the beam and $I = \pi(D^2 - D_0^2)/64$ the moment of inertia. The cross-section area is $A = \pi(D^2 - D_0^2)$ and $\rho$ is the density of the tube. The external forces acting on the tube, i.e. actuation and transduction, arise from capacitive coupling between the tube and the electrodes. These forces depend on the instantaneous charge distribution and geometrical configuration of the tube. This gives rise to an additional part of the action for the tube which can be written

$$S_{el.-mech.} = - \int_0^{\tau_e} dt \sum_{i=G,D} \frac{Q_i(t)^2}{C_i[u(\cdot, \cdot), t]}$$

2. Model

The system is depicted in figure 1. Mounted on a conducting support (source) of height $H$ is a nanotube of length $L$ with outer diameter $D_o$ and an inner diameter $D_i$. Below the tube two conducting strips of heights $H_{G,D}$ and widths $W_{G,D}$ act as gate and drain electrodes. The motion of the carbon nanotube can be described as a simple elastic beam deflecting in only one direction [20] using the action

$$S_{beam} = \int_0^{\tau_e} dt \int_0^L dx \frac{\rho A}{2}[\dot{u}(u(x, t))]^2 - \frac{E I}{2}[\ddot{u}(u(x, t))]^2.$$
where we have allowed for constant DC-bias offsets on each of the electrodes and assumed the gate and drain to be connected to lossless transmission lines with a real impedance $Z_0$. For the tube diameters and signal frequencies we consider, the kinetic inductance of the CNT [23, 24] is insignificant in relation to other impedances. The capacitances give us a linear relationship between voltages $(V_G, V_D, V_T)$ and charges $(Q_G, Q_D, Q_T)$,

$$Q_G = (C_G + C_{GD}) V_G - C_G V_T - C_{GD} V_D$$

(5)

$$Q_D = C_D V_T + C_{GD} V_G - (C_D + C_{GD}) V_D$$

(6)

$$Q_T = (C_T^i + C_S + C_D) V_T - C_S V_S - C_D V_D - C_G V_G$$

(7)

ensuring that we have a closed set of equations for the dynamics consisting of equations (1)-(7).

The full PDE in (1) is not very tractable as it stands. We will assume that only the lowest lying fundamental vibration mode is excited by the incoming signal. In this approximation the deformation of the tube can be written by the displacement $x_T$ of the cantilever tip from the static equilibrium position (for details see appendix)

$$\ddot{x}_T + \Gamma \dot{x}_T + \Omega_0^2 x_T = \frac{1}{M_{\text{eff}}} \sum_{n=0}^l \Delta V_n(t)^2 C_n^i(\xi_T)$$

(8)

where $M_{\text{eff}} = \rho AL/5.684$ and $\Omega_0 = 3\times10^6 L^{-2} \sqrt{ET/\rho A}$. Also, in (8) a factor $\Gamma \dot{x}_T$ has been incorporated in a phenomenological way to account for dissipation [22, 25].

3. Perturbation theory

Although direct numerical integration of the lumped model is a straightforward task, it is time consuming due to the multiple timescales involved. It is instead our aim in this paper to derive a set of algebraic equations which can be used to classify or to quickly determine the response of a given geometric or biasing configuration. We do this through a perturbative analysis by means of the averaging method [26]. We will thus assume that we have a single incoming signal on the gate electrode $V_s(t) = V_s \cos(\Omega t)$ and cast the system into a dimensionless form by writing $t = \tau/\Omega_0, x_T = \xi H, C_i = c_i C_0, V_i = v_i V_0, \dot{Q}_i = q_i V_0 C_0, \gamma = \Gamma/\Omega_0$ and $\omega = \Omega/\Omega_0$;

$$\ddot{\xi} + \gamma \dot{\xi} + \xi = \frac{\sigma}{2} [v_0(\xi) - v_T(\xi)]^3 c_G^i(\xi)$$

$$+ \frac{\sigma}{2} [v_0(\tau) - v_T(\tau)]^3 c_D^i(\xi)$$

$$\dot{q}_G(\tau) = e^{-1} [2v_0(\tau) \cos \omega \tau - v_G(\tau) + v_0^G]$$

$$\dot{q}_D(\tau) = e^{-1} [v_0(\tau) - v_0^D]$$

$$\dot{q}_T(\tau) = e^{-1} \frac{Z_0}{R_S} [v_S - v_T(\tau)].$$

Here we have defined $\sigma \equiv V_0^2 C_0/M_{\text{eff}} \Omega_0^2 H^2 = \xi_c/\xi_s$ and $\epsilon \equiv \Omega_0 Z_0 C_0$. For $C_0$ we chose $C_0 \equiv \sqrt{C_{GD}^i C_D^i}$ where $C_{GD}^i$ and $C_D^i$ are the gate–tube and drain–tube capacitances when the tube is undeflected. Typically the capacitances in nanoscale devices lie in the attofarad range, which implies that for a single device the impedance mismatch is large. Hence, $\epsilon \ll 1$, providing a good starting point for perturbation theory. The linear relationships between charges and voltages imply that we can expand in $\epsilon$

$$q_i = \sum_{n=0}^\infty \epsilon^n q_i^{(n)}(\tau), \quad v_i = \sum_{n=0}^\infty \epsilon^n v_i^{(n)}(\tau), \quad i = G, D, T.$$

(9)

To zeroth order in $\epsilon$ we then have

$$v_G^{(0)}(\tau) = 2v_s \cos \omega \tau + v_0^G$$

$$v_D^{(0)}(\tau) = 0, \quad v_T^{(0)}(\tau) = v_S.$$

Inserting these zeroth order solutions into the dynamic equation for the tip motion we find

$$\ddot{\xi} + \gamma \dot{\xi} + \xi = \frac{\sigma}{2} [2v_s \cos \omega \tau - (v_S - v_0^G)]^3 c_G^i(\xi)$$

$$+ \frac{\sigma}{2} [v_0^D - v_S]^3 c_D^i(\xi).$$

We now make the ansatz of an oscillatory solution with slowly changing parameters,

$$\xi = x_0(\tau) + r(\tau) \cos(\omega \tau + \phi),$$

and assume that the quantities $\dot{r}/\omega, \dot{\phi}/\omega \phi \ll 1$ keeping only the lowest order terms

$$\gamma \ddot{x}_0 + x_0 - (2\gamma r \omega + \gamma r \omega) \sin(\omega \tau + \phi) + r(1 - \omega^2 - 2\omega \phi) \cos(\omega \tau + \phi) = \frac{\sigma}{2} K(x_0, r, \phi, \tau).$$

Here the kernel $K(x_0, r, \phi, \tau)$ is defined as

$$K \equiv (2v_s \cos \omega \tau - (v_S - v_0^G))^3 c_G^i(x_0 + r \cos(\omega \tau + \phi))$$

$$+ (v_0^D - v_S)^3 c_D^i(x_0 + r \cos(\omega \tau + \phi)).$$

Provided the slow variables $r, \phi$ and $x_0$ do not change appreciably during one period we can average over one period to find

$$\gamma \ddot{x}_0 + x_0 = \frac{\sigma \omega}{4\pi} \int_0^{2\pi} d\tau K(x_0, r, \phi, \tau),$$

$$\gamma \ddot{r} + \dot{r} \omega - \frac{1}{2} \gamma r \omega \phi = \frac{\sigma \omega}{4\pi} \int_0^{2\pi} d\tau K \sin(\omega \tau + \phi),$$

$$\frac{1}{2} r(1 - \omega^2) - r \omega \phi = \frac{\sigma \omega}{4\pi} \int_0^{2\pi} d\tau K \cos(\omega \tau + \phi).$$

For generic capacitances we would not expect to be able to do these integrals exactly. However, for the case when $\omega L/2 \ll l$ the derivative of the capacitances can be approximated by an inverse square law according to the local tube deflection, $c_G^i(\xi) = c_G^{i0} \text{sgn}(gD_G^i - \xi)^2$ with

$$a_G^D \equiv (1 - H_G^D/H) u_0(L)/u_0(Z_G^D/L + W_G^D/2L).$$

Introducing

$$a_G^D \equiv \frac{1}{r} \left( a_G^D - x_0 + \sqrt{(a_G^D - x_0)^2 - r^2} \right)$$

3
and $v_{SG} \equiv v_{SD} - v_{D}$, $v_{SD} \equiv v_{S} - v_{D}$ and $\gamma_{SG} \equiv 2v_{S}^2 + v_{SG}^2$, the results after performing the integrals are

$$\gamma \tilde{x}_0 + x_0 = \frac{2\sigma_0^0 G a G a_G^2}{r^2(a_G^2 - 1)} \left[ \tilde{v}_{SG}^2(a_G^2 + 1) - 8v_I v_{SG} a_G \cos \phi \right] + 2v_{S}^2 (3 - a_G^2) \cos 2\phi] + \frac{2\sigma_0^0 G a_G a_G^2}{r^2(a_G^2 - 1)^3} v_{SG}^2$$

(10)

$$2\gamma \omega + \gamma r \omega = 8\sigma_0^0 G a_G a_G^2 \left[ \tilde{v}_{SG} + \left( v_{S} - v_{SG} (a_G^2 + 4 - a_G^2) \cos \phi \right) + v_{S}^2 (a_G + 2a_G^{-1} - a_G^3) \cos 2\phi \right] + 8\sigma_0^0 G a_G a_G^2 \left[ r(a_G^2 - 1)^3 \right]^2 v_{SG}^2.$$

(11)

The output signal is the displacement current in the drain contact and we will express this in terms of the $S$-parameter $|S_{21}|^2$. In general, the $S$-parameter $S_{ij}$ is a complex quantity relating the amplitude and phase of an incoming signal $V_i$ on port $i$ to an outgoing signal on port $j$ through $S_{ij} \equiv V_j / V_i$. In terms of power we have then $|S_{21}|^2 = P_{in}/P_{out}$, relating the total outgoing RF power on the drain to the incoming RF power on the gate. For the given input signal $v_i \cos(\omega t)$ the average power delivered to the device is $P_{out} = v_i^2 / 2Z_0$ and the total power delivered on the output is $P_{out} = \frac{v_i^2}{2Z_0} \int_0^{2\pi} d\tau (v_0 - v_I(\tau))^2$.

From the linear relation between charges and voltages we have $q_0(\tau) = e v_0 \gamma + e G_{DG} v_{DG} - (cD + cDG) v_D$.

Recalling the perturbation expansion (9) we get to lowest order in $\epsilon$

$$v_D = v_D^0 + \epsilon ((v_S - v_D^0) / 2 \omega v_D cG \sin \omega t) + O(\epsilon^2).$$

Evaluating the integral in the stationary state ($\dot{x}_0 = \dot{\phi} = 0$) one finds

$$|S_{21}|^2 = 4(Z_0 G_{DG}^2)^2 \left[ 4 \left( \frac{v_S - v_D^0}{v_D} \right)^2 \frac{c_G(a_G^2 + 1)}{r^2(a_G^2 - 1)^3} \cos \phi + c_G \right].$$

(13)

A similar result can be derived for the reflection coefficient $|S_{11}|^2$. Note that the prefactor of $\epsilon^2 = (Z_0 G_{DG} \Omega)^2$ indicates that for mismatched systems ($\epsilon \ll 1$) only a very small amount of the incoming power is actually delivered to the device and that most is reflected back ($|S_{11}|^2 \sim 1$).

In a typical set-up we have $a_G \gg a_D$. In this case we have found that the system (10)--(12) can be simplified considerably by omitting all terms related to the parametric driving and double frequency components and it suffices to solve the simplified set of equations

$$x_0 = \frac{\sigma_0^0 G a_G^2}{2a_G} v_{SG} + \frac{2\sigma_0^0 G a_G^2 a_G^2 (a_G^2 + 1)}{r^2(a_G^2 - 1)^3} v_{SG}^2$$

(14)

$$\gamma r \omega = \frac{5\sigma_0^0 G a_G^2 v_{SG} \sin \phi}{(a_G - x_0) + 2\sigma_0^0 G a_G^2 v_{SG} \cos \phi} + \frac{8\sigma_0^0 G a_G^2 (a_G^2 + 1)}{r^2(a_G^2 - 1)^3} v_{SG}^2.$$

(15)

$$\gamma r \omega = \frac{5\sigma_0^0 G a_G^2 v_{SG} \sin \phi}{(a_G - x_0) + 2\sigma_0^0 G a_G^2 v_{SG} \cos \phi} + \frac{8\sigma_0^0 G a_G^2 (a_G^2 + 1)}{r^2(a_G^2 - 1)^3} v_{SG}^2.$$

(16)

Solving this simplified system produces results which agree quantitatively with the full system (10)--(12) for weak driving and qualitatively for all bias ranges.

4. Static solutions and linear response

We consider first the statics and small amplitude vibrations around equilibrium and derive a formula for the output power in this regime. To this end we expand (10)--(12) to first order in $v_i/(v_{DG} - v_S)$ and $r/(a_G - x_0)$, i.e., we assume the oscillation amplitude to be small compared to the maximum amplitude allowed for a given static deflection and obtain

$$\gamma r \omega = \frac{2\sigma_0^0 G a_G^2 v_{SG} \sin \phi}{(a_G - x_0) + 2\sigma_0^0 G a_G^2 v_{SG} \cos \phi}.$$

(17)

Here $x_0$ is the deflection in the absence of drive, i.e. $v_0 = 0$,

$$x_0 = \frac{\sigma_0^0 G a_G^2 v_{SG}^2}{(a_G - x_0)^3}$$

and $\omega_0$ is the renormalized frequency

$$\omega_0 = 1 - \frac{2\sigma_0^0 G a_G^2 v_{SG}^2}{(a_G - x_0)^3}.$$

We note that while the equation for $x_0$ can have two solutions only the solution with the smaller deflection is stable. The solution with large deflection is always unstable and leads to snap-to-contact. If surface forces are taken into account or if the drain electrode is placed outside the reach of the tip bistable operation can be obtained as in [12]. From the above relations we find in the linear response regime a power transmission of

$$|S_{21}|^2 = 4\epsilon^2 \left[ 2G_{DG} \omega^2 (a_G - x_0)^2 + \omega^2 c_{DG} \right]$$

(17)

$$|S_{11}|^2 = 4\epsilon^2 \left[ 2G_{DG} \omega^2 (a_G - x_0)^2 + \omega^2 c_{DG} \right]$$

with $c_{DG} = c_{DG} G_D (a_G - x_0)^3 / (a_G - x_0)^2$.

5. Nonlinear response

We now go beyond the linear response regime and consider the full solutions of (10)--(12). For the general case these equations need to be solved numerically. In order to illustrate the typical resonant behaviour we will consider a specific system with a multi-walled carbon nanotube, $D_0 = 30 \text{ nm}$, $D_1 = 20 \text{ nm}$ extending a length $L = 250 \text{ nm}$ out from the support. The electrode dimensions are (see figure 1) $Z_0 = 100 \text{ nm}$, $Z_D = 225 \text{ nm}$, $W_D = W_D = 50 \text{ nm}$, $H = 25 \text{ nm}$ and $H_G = H_D = 10 \text{ nm}$. Finite element modelling of this structure gives us the capacitances for an unbent configuration.
C_G = 5.4 aF, C_D = 4.1 aF, C_GD = 6.1 aF and C_0 = 4 aF. For the mechanical properties of the nanotubes we have assumed an effective Young modulus of $E = 1$ TPa and a quality factor of $Q = 200$ [27–34].

For a qualitative understanding of the response it is useful to look at the non-averaged equations of motion. If we ignore excitations of other than the fundamental frequency we have
d differential equation

$$\ddot{\xi} + \gamma \dot{\xi} + \frac{d}{d\xi} V_{\text{eff}}(\xi) = \sigma v_s v_{SG} c_D(\xi) \cos \omega t$$

with $V_{\text{eff}} = \frac{1}{2} [\xi^2 - \sigma v_{SG}^2 c_D(\xi) - \sigma v_c^2 c_D(\xi)]$. At a source bias $V_S = 5$ V and with $V_D = V_G = 0$ V this potential has the shape as shown in figure 3 and we can clearly distinguish a few different scenarios. For small excitations we expect to obtain the linear response solution stated above. As driving and amplitude increase we reach a point where deviations from the parabolic potential become manifest, leading to a hysteresis downward in the frequency plane. We expect this regime to have a similar frequency response as the Duffing oscillator. This regime is discussed below in section 5.1. As one increases the driving further the oscillation amplitude is expected to increase. As it increases above the point where the curvature of the effective potential changes sign there will be a qualitative change in the response. This change is most markedly seen in the phase response (see figure 5) as a ‘gap’ around $\phi = \pi/2$. We will discuss this regime in section 5.2.

### 5.1. Onset of hysteresis

For sufficiently small vibrations (10)–(12) can be expanded in terms of $x_0$ and $r$ to obtain the frequency response equation

$$(\gamma \omega)^2 r^2 + \left[ (\omega^2 - \omega_0^2) - \beta^2 r^2 \right]^2 = 4 \left( \frac{\sigma v_s^2 v_{SG}^2}{\sigma G} \right)^2 v_s^2 v_{SG}^2$$

(18)

This third order equation is the same as for the Duffing equation and is adequate for determining the onset of the hysteretic behaviour. In the present case it performs less well to determine the frequency response for intermediate oscillation amplitudes in the hysteretic regime, in which case one needs to solve either (10)–(12) or (14)–(16). We illustrate this regime in figure 4, where the amplitude of oscillation has been obtained.
due to the fact that their peak oscillation amplitude exceeds
and average displacement for one such solution
are shown as the full lines in figure 7.

These solutions cannot be described by the frequency response
equation (18) and are large amplitude solutions. The
corresponding amplitudes and average displacements for these
solutions are shown in figure 6. They are mostly unstable
in the phase response in figure 5 as a gap opens up around
φ = π/2. The above only concerned matters related to the averaging
method. Another issue is related to the assumption of large
impedance mismatch (smallness of ϵ). Typical transmission
line impedances are of the order 50 Ω. With capacitances
in the aF range this means that we have ϵ < 10⁻⁴, which makes the approximation excellent. On the other hand this also
means that transmitted power is very low. The obvious way to
remedy this is to connect several devices in parallel, coupled to
common gate and drain electrodes. For an array of N devices
connected this way one finds that Zᵩ should be replaced by an
effective transmission line impedance of Zᵩₑffective = N/Z₀. Shown
in figure 8 is the S₂₁-parameter for such an effective impedance
of 10 kΩ calculated using (13) for biases corresponding to

\( V ÷ = 0.09, \ldots, 0.17 \) V. Most of the solutions have oscillation
amplitudes too large to be stable. Note, however, that perturbation
theory predicts a stable region for the bias \( V ÷ = 0.17 \) V (red curve),
— , stable solutions; - - - - , unstable solutions.

Figure 6. Large amplitude solutions in hysteretic regime
\( (V ÷ = 0.09, \ldots, 0.17 \) V). The above only concerned matters related to the averaging
method. Another issue is related to the assumption of large
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in figure 8 is the S₂₁-parameter for such an effective impedance
of 10 kΩ calculated using (13) for biases corresponding to

amplitude vibrations, including the primary hysteretic regime,
we find excellent agreement between numerical simulations
and perturbation theory. Deviations from the predictions of
perturbation theory are only seen for the large amplitude
vibrations in the regime where there is a 'phase gap'. An
example of such a comparison is shown in figure 7, where we
clearly see the good agreement for small amplitude vibrations
and the deviations at larger amplitudes. By looking at the
detailed motion of the tube tip at large amplitudes one can see
that the approximation by a pure harmonic motion is no longer
a good approximation and higher harmonics have to be taken
into account.

Furthermore, perturbation theory predicts the existence of
disconnected manifolds of stable large amplitude orbits
that cannot be reached by sweeping down in frequency. We
have not been able to detect any such orbits in the numerical
simulations. This may be either due to the solutions being unphysical solutions to the perturbation theory equations or
that appropriate initial conditions have not been used in the
numerical simulations.

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of 10 kΩ calculated using (13) for biases corresponding to

5.2. Large driving, large hysteresis

As the driving increases even more the peak amplitude of the oscillator will eventually reach the point where the curvature of \( V ÷ ð \) changes from positive to negative. This is clearly seen
in the phase response in figure 5 as a gap opens up around
φ = π/2. The corresponding large amplitude vibrations in this regime are rendered unstable when the oscillator amplitude
reaches the point where it 'rolls over the hill' \( (x_0 + r > x_m) \) and the system snaps into contact. The oscillation amplitude and average displacement for one such solution \( V ÷ = 0.23 \) V are shown as the full lines in figure 7.

6. Validity of perturbation theory and output power

We have also compared the analytical expressions with direct
numerical integration of the differential equations. For small

Figure 7. Oscillation amplitude \( r \) and average displacement \( x_0 \) in the
'phase gapped' oscillation regime. Solid lines represent the result
from perturbation theory for a bias of \( V ÷ = 0.23 \) V while circles (O)
mark the result of numerical integration of the full system of ODEs.
For small amplitudes the agreement between numerical integration
and perturbation theory is very good, while perturbation starts to fail
for larger amplitude due to the anharmonicity of the tip motion.
the three different oscillation regimes. The arrows denote the associated jumps in the hysteresis curve.

Comparisons with numerical integration show that treating the system to lowest order in \( \epsilon \) as we have done here is a good approximation as long as \( |S_{21}| \ll 1 \). As \( |S_{21}| \sim 1 \) the effect of dissipation in the tube–source contact and power delivered to the gate/drain result in a loaded \( Q \)-factor exceeding the bare mechanical \( Q \)-factor leading to a broadening of the resonances.

7. Conclusions

We have carried out an investigation of the RF response in a three-terminal carbon nanotube resonator structure. By employing perturbation theory we have reduced the problem of determining and classifying the frequency response to that of solving a set of algebraic equations. We have found three distinct oscillatory regimes: linear response, hysteretic and a ‘phase gapped’ regime, which can lead to large hysteresis loops in the frequency response. Comparisons with direct numerical integration have shown that perturbation theory is qualitatively correct, and quantitatively accurate as long as the amplitude of oscillation is not too large. The perturbative treatment also includes terms related to parametric driving. We find no qualitative change in the behaviour in the RF response that arises from incorporating such terms.

Acknowledgments

This project has been supported by Nokia Research Center (AI), Swedish Foundation for Strategic Research (SSF) (JMK) and the EU through the Nano-RF project FP6-2005-028158 (JMK). We are grateful for stimulating discussions with Eleanor Campbell, Anders Eriksson, Sang-Wook Lee and Jukka Wallinheiro.

Appendix. Derivation of (8)

Here we outline the derivation of the lumped model for tube vibrations used in the paper. We begin by writing the solution to the PDE in the form

\[
    u(x, t) = \Delta_{f_0}(x) + \sum_{n=0}^{\infty} \gamma_n(t)u_n(x).
\]

Here \( u_n(x) \) are the eigenmodes with frequency \( \Omega_n \) satisfying the homogeneous equation \((-\Omega_n^2 + (E1/\rho A)\delta_2^2)u_n(x) = 0\) while the offset \( \Delta_{f_0}(x) \) satisfies \((E1/\rho A)\delta_2^2\Delta_{f_0}(x) = f_0(x)\) for some function \( f_0(x) \) to be determined (typically this is the non-vanishing part of the time averaged force). Inserting the solution into the PDE for \( u(x, t) \) and projecting onto a normal mode \( u_n(x) \) we find

\[
    \ddot{\gamma}_n(t) + \omega_n^2\gamma_n(t) = -L^{-3}\int_0^L dx \ u_n^\dagger(x)f_0(x)
    + \frac{1}{2\rho AL^3}\sum_{i \in i,D} [\Delta V_i(t)]^2 \int_0^L dx \ u_i^\dagger(x) \frac{\delta C_i[\Delta(x) + \gamma_0(T)u_0(x)]}{\delta u(x)}.
\]

We use here the normalization convention \( \int_0^L dx \ u_n^\dagger(x)u_n(x) = L^3\delta_{nm} \). We will typically work with a system where the driving only excites resonances of the fundamental mode \( u_0(x) \). So we can quite safely make the approximation \( u(x, t) \approx \Delta(x) + \gamma_0(T)u_0(x) \). The differential equation then reads

\[
    \ddot{\gamma}_0(t) + \omega_0^2\gamma_0(t) = -L^{-3}\int_0^L dx \ u_0^\dagger(x)f_0(x)
    + \frac{\rho u_0(L)}{2\rho AL^3}\sum_{i \in i,D} [\Delta V_i(t)]^2 \int_0^L dx \ u_i^\dagger(x) [\Delta(x) + \gamma_0(T)u_0(x) + \epsilon(x)] 
    \times \frac{\delta C_i[\Delta(x) + \gamma_0(T)u_0(x) + \epsilon(x)]}{\delta u(x)}.
\]

Here, the static bending has been expressed in terms of the deviation from the shape of the deformation in the lowest mode. \( \Delta(x) = u_{0}(x)\Delta(L)/u_{0}(L) + \epsilon(x) \). Assuming that the statically deformed shape closely resembles the resonance shape, i.e. we set \( \epsilon(x) = 0 \), we arrive at

\[
    \ddot{\gamma}_0(t) + \omega_0^2\gamma_0(t) = \frac{\rho u_0(L)}{2\rho AL^3}\sum_{i \in i,D} [\Delta V_i(t)]^2 
    \times \int_0^L dx \ u_i^\dagger(x) \frac{\delta C_i[\Delta(x) + \gamma_0(T)u_0(x)]}{\delta u(x)}.
\]

Finally, using that \( u_0^\dagger(x) = u^\dagger_0(x) \) one finds

\[
    \ddot{\gamma}_0(t) + \omega_0^2\gamma_0(t) = \frac{1}{2M_{eff}}\sum_{i \in i,D} [\Delta V_i(t)]^2 C_i[\gamma_0(T)]. \tag{A.1}
\]

where \( M_{eff} = ML^2/u_0(L)^2 \approx M/5.684 \) and \( \Omega_0 = 3.516L^{-2}/ET/\rho A \).
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