(p, q)-TH RELATIVE ORDER AND (p, q)-TH RELATIVE TYPE 
BASED SOME GROWTH ANALYSIS OF ENTIRE AND 
MEROMORPHIC FUNCTIONS

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ABSTRACT. In this paper we wish to prove some results related to the growth 
rates of entire and meromorphic functions on the basis of relative (p, q) th order 
and relative (p, q) th type of a meromorphic function with respect to an entire 
function for any two positive integers p and q.

1. Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane \( \mathbb{C} \). The maximum 
modulus function \( M_f (r) \) corresponding to \( f \) is defined on \( |z| = r \) as \( M_f (r) = \max_{|z| = r} |f(z)| \).

If an entire function \( f \) is non-constant then \( M_f (r) \) is strictly increasing and continuous 
and its inverse \( M_f^{-1} : ([f(0)], \infty) \to (0, \infty) \) exists and is such that \( \lim_{s \to \infty} M_f^{-1}(s) = \infty \).

When \( f \) is meromorphic, one may introduce another function \( T_f (r) \) known as Nevan-
linna’s characteristic function of \( f \), playing the same role as \( M_f (r) \).

The integrated counting function \( N_f (r, a) \) (\( N_f (r) \)) of \( a \)-points (distinct \( a \)-points) of \( f \) is defined as

\[
N_f (r, a) = \int_0^r \frac{n_f (t, a) - n_f (0, a)}{t} dt + n_f (0, a) \log r
\]

\[
N_f (r) = \int_0^r \frac{\overline{n_f (t, a)} - \overline{n_f (0, a)}}{t} dt + \overline{n_f (0, a)} \log r
\]

where we denote by \( n_f (t, a) (\overline{n_f(t,a)}) \) the number of \( a \)-points (distinct \( a \)-points) of \( f \) in 
\( |z| \leq t \) and an \( \infty \)-point is a pole of \( f \). In many occasions \( N_f (r, \infty) \) and \( \overline{N_f (r, \infty)} \) 
are denoted by \( N_f (r) \) and \( \overline{N_f (r)} \) respectively. The function \( N_f (r, a) \) is called the 
enumerative function. On the other hand, the function \( m_f (r) \equiv m_f (r, \infty) \) known as the

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proximity function is defined as
\[ m_f (r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \]
where \( \log^+ x = \max(\log x, 0) \) for all \( x \geq 0 \)
and an \( \infty \)-point is a pole of \( f \).

Analogously, \( m_{f_a} (r) \equiv m_f (r, a) \) is defined when \( a \) is not an \( \infty \)-point of \( f \).

Thus the Nevanlinna’s characteristic function \( T_f (r) \) corresponding to \( f \) is defined as
\[ T_f (r) = N_f (r) + m_f (r). \]

When \( f \) is entire, \( T_f (r) \) coincides with \( m_f (r) \) as \( N_f (r) = 0 \).

Moreover, if \( f \) is non-constant entire then \( T_f (r) \) is strictly increasing and continuous functions of \( r \). Also its inverse \( T_f^{-1} : (T_f (0), \infty) \to (0, \infty) \) exist and is such that
\[ \lim_{s \to \infty} T_f^{-1} (s) = \infty. \]

Also the ratio \( T_f (r) \) as \( r \to \infty \) is called the growth of \( f \) with respect to \( g \) in terms of the Nevanlinna’s Characteristic functions of the meromorphic functions \( f \) and \( g \).

Now we state the following notation which will be needed in the sequel:
\[ \log^{[k]} x = \log \left( \log^{[k-1]} x \right) \text{ for } k = 1, 2, 3, \ldots \text{ and} \]
\[ \log^{[0]} x = x; \]
and
\[ \exp^{[k]} x = \exp \left( \exp^{[k-1]} x \right) \text{ for } k = 1, 2, 3, \ldots \text{ and} \]
\[ \exp^{[0]} x = x. \]

Taking this into account, Juneja, Kapoor and Bajpai \[\text{[4]}\] defined the \( (p, q) \)-th order and \( (p, q) \)-th lower order of an entire function \( f \) respectively as follows:
\[ \rho_f (p, q) = \limsup_{r \to \infty} \frac{\log^{[p]} M_f (r)}{\log^{[q]} r} \text{ and } \lambda_f (p, q) = \liminf_{r \to \infty} \frac{\log^{[p]} M_f (r)}{\log^{[q]} r}, \]
where \( p, q \) are any two positive integers with \( p \geq q \). When \( f \) is meromorphic one can easily verify that
\[ \rho_f (p, q) = \limsup_{r \to \infty} \frac{\log^{[p-1]} T_f (r)}{\log^{[q]} r} \text{ and } \lambda_f (p, q) = \liminf_{r \to \infty} \frac{\log^{[p-1]} T_f (r)}{\log^{[q]} r}, \]
where \( p, q \) are any two positive integers with \( p \geq q \). If \( p = l \) and \( q = 1 \) then we write \( \rho_f (l, 1) = \rho_f^l \) and \( \lambda_f (l, 1) = \lambda_f^l \) where \( \rho_f^l \) and \( \lambda_f^l \) are respectively known as generalized order and generalized lower order of \( f \). Also for \( p = 2 \) and \( q = 1 \) we respectively denote \( \rho_f (2, 1) \) and \( \lambda_f (2, 1) \) by \( \rho_f \) and \( \lambda_f \), where \( \rho_f \) and \( \lambda_f \) are the classical growth indicator known as order and lower order of \( f \).

In this connection we just recall the following definition:
Definition 1. [1] An entire function $f$ is said to have index-pair $(p, q)$, $p \geq q \geq 1$ if $b < \rho_f (p, q) < \infty$ and $\rho_f (p - 1, q - 1)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ if $p > q$. Moreover if $0 < \rho_f (p, q) < \infty$, then
\[
\rho_f (p, q - n) \rightarrow \infty \quad \text{for} \quad n < p, \quad \rho_f (p - q - n) = 0 \quad \text{for} \quad n < q \quad \text{and}
\]
\[
\rho_f (p + n, q + n) = 1 \quad \text{for} \quad n = 1, 2, \ldots.
\]
Similarly for $0 < \lambda_f (p, q) < \infty$, one can easily verify that
\[
\lambda_f (p, q - n) \rightarrow \infty \quad \text{for} \quad n < p, \quad \lambda_f (p - q - n) = 0 \quad \text{for} \quad n < q \quad \text{and}
\]
\[
\lambda_f (p + n, q + n) = 1 \quad \text{for} \quad n = 1, 2, \ldots.
\]
An entire function for which $(p, q)$-th order and $(p, q)$-th lower order are the same is said to be of regular $(p, q)$-growth. Functions which are not of regular $(p, q)$-growth are said to be of irregular $(p, q)$-growth.

Analogously one can easily verify that the Definition 1 of index-pair can also be applicable for a meromorphic function $f$.

In order to compare the growth of entire functions having the same $(p, q)$-th order, Juneja, Kapoor and Bajpai [2] also introduced the concepts of $(p, q)$-type and $(p, q)$-th lower type in the following manner:

Definition 2. [2] The $(p, q)$-th type and the $(p, q)$-th lower type of entire function $f$ having finite positive $(p, q)$-th order $\rho_f (p, q)$ $(b < \rho_f (p, q) < \infty)$ $(p, q)$ are any two positive integers, $b = 1$ if $p = q$ and $b = 0$ for $p > q$ are defined as:
\[
\sigma_f (p, q) = \limsup_{r \to \infty} \frac{\log \left( \left[ p \right] \right)}{\log \left( \left[ q \right] \right) r} \quad \text{and} \quad f_f (p, q) = \liminf_{r \to \infty} \frac{\log \left( \left[ p \right] \right)}{\log \left( \left[ q \right] \right) r} \quad 0 \leq f_f (p, q) \leq \sigma_f (p, q) \leq \infty.
\]

When $f$ is meromorphic one can easily verify that
\[
\sigma_f (p, q) = \limsup_{r \to \infty} \frac{\log \left( \left[ p \right] \right)}{\log \left( \left[ q \right] \right) r} \quad \text{and} \quad f_f (p, q) = \liminf_{r \to \infty} \frac{\log \left( \left[ p \right] \right)}{\log \left( \left[ q \right] \right) r} \quad 0 \leq f_f (p, q) \leq \sigma_f (p, q) \leq \infty.
\]

Likewise, to compare the growth of entire functions having the same $(p, q)$-th lower order, one can also introduced the concepts of $(p, q)$-th weak type in the following manner:

Definition 3. The $(p, q)$ th weak type of entire function $f$ having finite positive $(p, q)$ th tower order $\lambda_f (p, q)$ $(b < \lambda_f (p, q) < \infty)$ is defined as:
\[
\tau_f (p, q) = \liminf_{r \to \infty} \frac{\log \left( \left[ p \right] \right)}{\log \left( \left[ q \right] \right) r} \quad \text{where} \quad p, q \text{ are any two positive integers, } b = 1 \text{ if } p = q \text{ and } b = 0 \text{ for } p > q.
\]
Similarly one may define the growth indicator $\tau_f(p,q)$ of an entire function $f$ in the following way:

$$\tau_f(p,q) = \limsup_{r \to \infty} \frac{\log^{[p-1]} M_f(r)}{(\log^{[q-1]} r)^{\lambda_f(p,q)}}, \quad b < \lambda_f(p,q) < \infty$$

where $p, q$ are any two positive integers, $b = 1$ if $p = q$ and $b = 0$ for $p > q$.

When $f$ is meromorphic one can easily verify that

$$\tau_f(p,q) = \liminf_{r \to \infty} \frac{\log^{[p-2]} M_f(r)}{(\log^{[q-1]} r)^{\lambda_f(p,q)}}, \quad b < \lambda_f(p,q) < \infty,$$

It is obvious that $0 \leq \tau_f(p,q) \leq \tau_f(p,q) < \infty$.

Given a non-constant entire function $f$ defined in the open complex plane $\mathbb{C}$ its maximum modulus function and Nevanlinna’s characteristic function are strictly increasing and continuous. Hence there exists its inverse functions $M_f^{-1}(r): (|f(0)|, \infty) \to (0, \infty)$ with $\lim_{s \to \infty} M_f^{-1}(s) = \infty$ and $T_f^{-1}(r): (|f(0)|, \infty) \to (0, \infty)$ with $\lim_{s \to \infty} T_f^{-1}(s) = \infty$.

In this connection, Bernal [1] introduced the definition of relative order of an entire function $f$ with respect to another entire function $g$, denoted by $\rho_g(f)$ as follows:

$$\rho_g(f) = \inf \{ \mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0 \}$$

$$= \limsup_{r \to \infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

The definition coincides with the classical one [6] if $g(z) = \exp z$. Similarly one can define the relative lower order of an entire function $f$ with respect to another entire function $g$ denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r \to \infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Extending this notion, Ruiz et. al. [5] introduced the definition of $(p,q)$ th relative order of a entire function with respect to an entire function in the light of index pair which is as follows:

**Definition 4.** [5] Let $f$ and $g$ be any two entire functions with index-pairs $(m,q)$ and $(m,p)$ respectively where $p, q, m$ are positive integers such that $m \geq \max(p, q)$. Then the relative $(p,q)$-th order of $f$ with respect to $g$ is defined as

$$\rho_g^{(p,q)}(f) = \limsup_{r \to \infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{(\log^{[q]} r)^{\lambda_f(p,q)}}.$$
Analogously, the relative \((p, q)\)-th lower order of \(f\) with respect to \(g\) is defined by:

\[
\chi_{g}^{(p, q)} (f) = \liminf_{r \to \infty} \frac{\log^{[p]} M_{g}^{-1} M_{f} (r)}{\log^{[q]} r}.
\]

In order to refine the above growth scale, now we intend to introduce the definitions of another growth indicators, such as relative \((p, q)\) -th type and relative \((p, q)\) -th lower type of entire function with respect to another entire function in the light of their index-pair which are as follows:

**Definition 5.** Let \(f\) and \(g\) be any two entire functions with index-pairs \((m_1, q)\) and \((m_2, p)\) respectively where \(m_1 = m_2 = m\) and \(p, q, m\) are all positive integers such that \(m \geq \max \{p, q\}\). The relative \((p, q)\) -th type and relative \((p, q)\) -th lower type of entire function \(f\) with respect to the entire function \(g\) having finite positive relative \((p, q)\) th order \(\rho_{g}^{(p, q)} (f) \left(0 < \rho_{g}^{(p, q)} (f) < \infty\right)\) are defined as :

\[
\sigma_{g}^{(p, q)} (f) = \limsup_{r \to \infty} \frac{\log^{[p-1]} M_{g}^{-1} M_{f} (r)}{\left(\log^{[q-1]} r \right)^{\rho_{g}^{(p, q)} (f)}} \quad \text{and} \quad \tau_{g}^{(p, q)} (f) = \liminf_{r \to \infty} \frac{\log^{[p-1]} M_{g}^{-1} M_{f} (r)}{\left(\log^{[q-1]} r \right)^{\rho_{g}^{(p, q)} (f)}}.
\]

Analogously, to determine the relative growth of two entire functions having same non zero finite relative \((p, q)\) -th lower order with respect to another entire function, one can introduced the definition of relative \((p, q)\) -th weak type of an entire function \(f\) with respect to another entire function \(g\) of finite positive relative \((p, q)\) -th lower order \(\lambda_{g}^{(p, q)} (f)\) in the following way:

**Definition 6.** Let \(f\) and \(g\) be any two entire functions having finite positive relative \((p, q)\) th lower order \(\lambda_{g}^{(p, q)} (f) \left(0 < \lambda_{g}^{(p, q)} (f) < \infty\right)\) where \(p\) and \(q\) are any two positive integers. Then the relative \((p, q)\) th weak type of entire function \(f\) with respect to the entire function \(g\) is defined as :

\[
\tau_{g}^{(p, q)} (f) = \liminf_{r \to \infty} \frac{\log^{[p-1]} M_{g}^{-1} M_{f} (r)}{\left(\log^{[q-1]} r \right)^{\lambda_{g}^{(p, q)} (f)}}.
\]

Similarly one can define another growth indicator \(\tau_{g}^{(p, q)} (f)\) in the following way:

\[
\tau_{g}^{(p, q)} (f) = \limsup_{r \to \infty} \frac{\log^{[p-1]} M_{g}^{-1} M_{f} (r)}{\left(\log^{[q-1]} r \right)^{\lambda_{g}^{(p, q)} (f)}}.
\]

In the case of relative order, it therefore seems reasonable to define suitably the relative \((p, q)\) th order of meromorphic functions. Debnath et. al. \[2\] also introduced such definition in the light of index pair in the following manner:

**Definition 7.** \[2\] Let \(f\) be any meromorphic function and \(g\) be any entire function with index-pairs \((m_1, q)\) and \((m_2, p)\) respectively where \(m_1 = m_2 = m\) and \(p, q, m\) are all
positive integers such that $m \geq p$ and $m \geq q$. Then the relative $(p, q)$ th order of $f$ with respect to $g$ is defined as

$$\rho_{g}^{(p,q)}(f) = \limsup_{r \to \infty} \frac{\log^{[p]} T_{g}^{-1}T_{f}(r)}{\log^{[q]} r}.$$  

Similarly, one can define the relative $(p, q)$ th lower order of a meromorphic function $f$ with respect to an entire function $g$ denoted by $\lambda_{g}^{(p,q)}(f)$ where $p$ and $q$ are any two positive integers in the following way:

$$\lambda_{g}^{(p,q)}(f) = \liminf_{r \to \infty} \frac{\log^{[p]} T_{g}^{-1}T_{f}(r)}{\log^{[q]} r}.$$

Now we state the following two definitions relating to the meromorphic function which are similar to Definition [6] and Definition [7] respectively.

**Definition 8.** Let $f$ be a meromorphic function and $g$ be an entire function with index-pairs $(m_{1}, q)$ and $(m_{2}, p)$ respectively where $m_{1} = m_{2} = m$ and $p, q, m$ are all positive integers such that $m \geq \max \{p, q\}$. The relative $(p, q)$ th type and relative $(p, q)$ th lower type of meromorphic function $f$ with respect to the entire function $g$ having finite positive relative $(p, q)$ th order $\rho_{g}^{(p,q)}(f) \left(0 < \rho_{g}^{(p,q)}(f) < \infty\right)$ are defined as:

$$\sigma_{g}^{(p,q)}(f) = \limsup_{r \to \infty} \frac{\log^{[p-1]} T_{g}^{-1}T_{f}(r)}{\log^{[q-1]} r} \rho_{g}^{(p,q)}(f)$$

and

$$\overline{\sigma}_{g}^{(p,q)}(f) = \liminf_{r \to \infty} \frac{\log^{[p-1]} T_{g}^{-1}T_{f}(r)}{\log^{[q-1]} r} \rho_{g}^{(p,q)}(f).$$

**Definition 9.** Let $f$ be a meromorphic function and $g$ be an entire function having finite positive relative $(p, q)$ th lower order $\lambda_{g}^{(p,q)}(f) \left(0 < \lambda_{g}^{(p,q)}(f) < \infty\right)$ where $p$ and $q$ are any two positive integers. Then the relative $(p, q)$ th weak type of meromorphic function $f$ with respect to the entire function $g$ is defined as:

$$\tau_{g}^{(p,q)}(f) = \liminf_{r \to \infty} \frac{\log^{[p-1]} T_{g}^{-1}T_{f}(r)}{\log [q-1]} \lambda_{g}^{(p,q)}(f).$$

Similarly one can define another growth indicator $\overline{\tau}_{g}^{(p,q)}(f)$ in the following way:

$$\overline{\tau}_{g}^{(p,q)}(f) = \limsup_{r \to \infty} \frac{\log^{[p-1]} T_{g}^{-1}T_{f}(r)}{\log [q-1]} \lambda_{g}^{(p,q)}(f).$$

In this paper we wish to prove some results related to the growth rates of entire and meromorphic functions on the basis of relative $(p, q)$ th order and relative $(p, q)$ th type of a meromorphic function with respect to an entire function for any two positive integers $p$ and $q$. We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [3] and [7].
2. Main Results

In this section we present some results which will be needed in the sequel.

**Theorem 1.** Let \( f, g \) be any two meromorphic functions and \( h, k \) be any two entire functions such that \( 0 < \lambda_{h}^{(m,q)}(f) \leq \rho_{h}^{(m,q)}(f) < \infty \) and \( 0 < \lambda_{k}^{(n,q)}(g) \leq \rho_{k}^{(n,q)}(g) < \infty \) where \( m, n \) and \( p \) are any three positive integers. Then

\[
\frac{\lambda_{h}^{(m,q)}(f)}{\rho_{k}^{(n,q)}(g)} \leq \lim_{r \to +\infty} \frac{\log^{[m]} T_{h}^{-1} T_{f}(r)}{\log^{[n]} T_{k}^{-1} T_{g}(r)} \leq \frac{\lambda_{k}^{(n,q)}(g)}{\rho_{h}^{(m,q)}(f)}.
\]

**Proof.** From the definitions of \( \lambda_{h}^{(m,q)}(f) \) and \( \rho_{k}^{(n,q)}(g) \), we get for arbitrary positive \( \varepsilon \) and for all sufficiently large values of \( r \) that

\[
\log^{[m]} T_{h}^{-1} T_{f}(r) \geq \left( \lambda_{h}^{(m,q)}(f) - \varepsilon \right) \log^{[q]} r \quad \text{(2.1)}
\]

and

\[
\log^{[n]} T_{k}^{-1} T_{g}(r) \leq \left( \rho_{k}^{(n,q)}(g) + \varepsilon \right) \log^{[q]} r \quad \text{(2.2)}
\]

Now from (2.1) and (2.2), it follows for all sufficiently large values of \( r \) that

\[
\frac{\log^{[m]} T_{h}^{-1} T_{f}(r)}{\log^{[n]} T_{k}^{-1} T_{g}(r)} \geq \frac{\left( \lambda_{h}^{(m,q)}(f) - \varepsilon \right) \log^{[q]} r}{\left( \rho_{k}^{(n,q)}(g) + \varepsilon \right) \log^{[q]} r}.
\]

As \( \varepsilon (> 0) \) is arbitrary, we obtain that

\[
\lim_{r \to +\infty} \frac{\log^{[m]} T_{h}^{-1} T_{f}(r)}{\log^{[n]} T_{k}^{-1} T_{g}(r)} \geq \frac{\lambda_{h}^{(m,q)}(f)}{\rho_{k}^{(n,q)}(g)} \quad \text{(2.3)}
\]

Again for a sequence of values of \( r \) tending to infinity, we get that

\[
\log^{[m]} T_{h}^{-1} T_{f}(r) \leq \left( \lambda_{h}^{(m,q)}(f) + \varepsilon \right) \log^{[q]} r \quad \text{(2.4)}
\]

and for all sufficiently large values of \( r \),

\[
\log^{[n]} T_{k}^{-1} T_{g}(r) \geq \left( \lambda_{k}^{(n,q)}(g) - \varepsilon \right) \log^{[q]} r \quad \text{(2.5)}
\]

Combining (2.4) and (2.5), we obtain for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log^{[m]} T_{h}^{-1} T_{f}(r)}{\log^{[n]} T_{k}^{-1} T_{g}(r)} \leq \frac{\left( \lambda_{h}^{(m,q)}(f) + \varepsilon \right) \log^{[q]} r}{\left( \lambda_{k}^{(n,q)}(g) - \varepsilon \right) \log^{[q]} r}.
\]

Since \( \varepsilon (> 0) \) is arbitrary, it follows that

\[
\lim_{r \to +\infty} \frac{\log^{[m]} T_{h}^{-1} T_{f}(r)}{\log^{[n]} T_{k}^{-1} T_{g}(r)} \leq \frac{\lambda_{h}^{(m,q)}(f)}{\lambda_{k}^{(n,q)}(g)} \quad \text{(2.6)}
\]
Also for a sequence of values of $r$ tending to infinity, we get that
\[ \log^n T_k^{-1}T_g (r) \leq \left(\lambda^{(n,q)}_k (g) + \varepsilon\right) \log[q] r . \]  
(2.7)

Now from (2.1) and (2.7), we obtain for a sequence of values of $r$ tending to infinity that
\[ \frac{\log^n T_k^{-1}T_f (r)}{\log^n T_k^{-1}T_g (r)} \geq \frac{\left(\lambda_h^{(m,q)} (f) - \varepsilon\right) \log[q] r}{\left(\lambda_k^{(n,q)} (g) + \varepsilon\right) \log[q] r .} \]

As $\varepsilon (> 0)$ is arbitrary, we get from above that
\[ \lim_{r \to +\infty} \frac{\log^n T_k^{-1}T_f (r)}{\log^n T_k^{-1}T_g (r)} \geq \frac{\lambda_h^{(m,q)} (f)}{\lambda_k^{(n,q)} (g)} . \]
(2.8)

Also for all sufficiently large values of $r$,
\[ \log^n T_h^{-1}T_f (r) \leq \left(\rho_h^{(m,q)} (f) + \varepsilon\right) \log[q] r . \]
(2.9)

So it follows from (2.3) and (2.9), for all sufficiently large values of $r$ that
\[ \frac{\log^n T_k^{-1}T_f (r)}{\log^n T_k^{-1}T_g (r)} \leq \frac{\left(\rho_h^{(m,q)} (f) + \varepsilon\right) \log[q] r}{\left(\lambda_k^{(n,q)} (g) - \varepsilon\right) \log[q] r} \].

Since $\varepsilon (> 0)$ is arbitrary, we obtain that
\[ \lim_{r \to +\infty} \frac{\log^n T_k^{-1}T_f (r)}{\log^n T_k^{-1}T_g (r)} \leq \frac{\rho_h^{(m,q)} (f)}{\lambda_k^{(n,q)} (g)} . \]
(2.10)

Thus the theorem follows from (2.3), (2.6), (2.8) and (2.10). \hfill \Box

**Theorem 2.** Let $f$, $g$ be any two meromorphic functions and $h$, $k$ be any two entire functions such that $0 < \rho_h^{(m,q)} (f) < \infty$ and $0 < \rho_k^{(n,q)} (g) < \infty$ where $m,n,p$ are any three positive integers. Then
\[ \lim_{r \to +\infty} \frac{\log^n T_h^{-1}T_f (r)}{\log^n T_k^{-1}T_g (r)} \leq \frac{\rho_h^{(m,q)} (f)}{\rho_k^{(n,q)} (g)} \leq \lim_{r \to +\infty} \frac{\log^n T_k^{-1}T_f (r)}{\log^n T_k^{-1}T_g (r)} . \]

**Proof.** From the definition of $\rho_k^{(n,q)} (g)$, we get for a sequence of values of $r$ tending to infinity that
\[ \log^n T_k^{-1}T_g (r) \geq \left(\rho_k^{(n,q)} (g) - \varepsilon\right) \log[q] r . \]
(2.11)

Now from (2.9) and (2.11), we get for a sequence of values of $r$ tending to infinity that
\[ \frac{\log^n T_k^{-1}T_f (r)}{\log^n T_k^{-1}T_g (r)} \leq \frac{\left(\rho_h^{(m,q)} (f) + \varepsilon\right) \log[q] r}{\left(\rho_k^{(n,q)} (g) - \varepsilon\right) \log[q] r} . \]
As $\varepsilon (>0)$ is arbitrary, we obtain that
\[
\lim_{r \to +\infty} \frac{\log^{[m]} T_h^{-1} T_f (r)}{\log^{[n]} T_k^{-1} T_g (r)} \leq \frac{\rho_h^{(m,q)} (f)}{\rho_k^{(n,q)} (g)}.
\]

Also for a sequence of values of $r$ tending to infinity, we have
\[
\log^{[m]} T_h^{-1} T_f (r) \geq \left( \rho_h^{(m,q)} (f) - \varepsilon \right) \log^{[q]} r.
\]

So combining (2.12) and (2.13), we get for a sequence of values of $r$ tending to infinity that
\[
\log^{[m]} T_h^{-1} T_f (r) \geq \left( \rho_h^{(m,q)} (f) - \varepsilon \right) \log^{[q]} r.
\]

Since $\varepsilon (>0)$ is arbitrary, it follows that
\[
\lim_{r \to +\infty} \frac{\log^{[m]} T_h^{-1} T_f (r)}{\log^{[n]} T_k^{-1} T_g (r)} \geq \frac{\rho_h^{(m,q)} (f)}{\rho_k^{(n,q)} (g)}.
\]

Thus the theorem follows from (2.12) and (2.13). $\square$

The following theorem is a natural consequence of Theorem 1 and Theorem 2.

**Theorem 3.** Let $f$, $g$ be any two meromorphic functions and $h$, $k$ be any two entire functions such that $0 < \lambda_h^{(m,q)} (f) \leq \rho_h^{(m,q)} (f) < \infty$ and $0 < \lambda_k^{(n,q)} (g) \leq \rho_k^{(n,q)} (g) < \infty$ where $m$, $n$ and $p$ are any three positive integers. Then
\[
\lim_{r \to +\infty} \frac{\log^{[m]} T_h^{-1} T_f (r)}{\log^{[n]} T_k^{-1} T_g (r)} \leq \min \left\{ \frac{\lambda_h^{(m,q)} (f)}{\lambda_k^{(n,q)} (g)}, \frac{\rho_h^{(m,q)} (f)}{\rho_k^{(n,q)} (g)} \right\} \leq \max \left\{ \frac{\lambda_h^{(m,q)} (f)}{\lambda_k^{(n,q)} (g)}, \frac{\rho_h^{(m,q)} (f)}{\rho_k^{(n,q)} (g)} \right\} \leq \lim_{r \to +\infty} \frac{\log^{[m]} T_h^{-1} T_f (r)}{\log^{[n]} T_k^{-1} T_g (r)}.
\]

The proof is omitted.

**Theorem 4.** Let $f$, $g$ be any two meromorphic functions and $h$, $k$ be any two entire functions such that $0 < \sigma_h^{(m,q)} (f) \leq \sigma_k^{(m,q)} (f) < \infty$, $0 < \sigma_k^{(n,q)} (g) \leq \sigma_k^{(n,q)} (g) < \infty$ and $\rho_h^{(m,q)} (f) = \rho_k^{(n,q)} (g)$ where $m$, $n$ and $p$ are any three positive integers. Then
\[
\frac{\sigma_h^{(m,q)} (f)}{\sigma_k^{(n,q)} (g)} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \frac{\sigma_h^{(m,q)} (f)}{\sigma_k^{(n,q)} (g)} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \frac{\sigma_h^{(m,q)} (f)}{\sigma_k^{(n,q)} (g)}.
\]
Proof. From the definition of $\sigma_h^{(m,q)}(f)$ and $\sigma_k^{(n,q)}(g)$, we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$ that

$$\log^{[m-1]} T_h^{-1} T_f (r) \geq \left( \sigma_h^{(m,q)}(f) - \varepsilon \right) \left[ \log^{[q-1]} r \right] \rho_h^{(m,q)}(f),$$

(2.15)

and

$$\log^{[n-1]} T_k^{-1} T_g (r) \leq \left( \sigma_k^{(n,q)}(g) + \varepsilon \right) \left[ \log^{[q-1]} r \right] \rho_k^{(n,q)}(g).$$

(2.16)

Now from (2.15), (2.16) and the condition $\rho_h^{(m,q)}(f) = \rho_k^{(n,q)}(g)$, it follows that for all sufficiently large values of $r$ that

$$\log^{[m-1]} T_h^{-1} T_f (r) \leq \frac{\log^{[n-1]} T_k^{-1} T_g (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \geq \frac{\sigma_h^{(m,q)}(f)}{\sigma_k^{(n,q)}(g) + \varepsilon}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain from above that

$$\lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \geq \frac{\sigma_h^{(m,q)}(f)}{\sigma_k^{(n,q)}(g)}.$$  

(2.17)

Again for a sequence of values of $r$ tending to infinity, we get that

$$\log^{[m-1]} T_h^{-1} T_f (r) \leq \left( \sigma_h^{(m,q)}(f) + \varepsilon \right) \left[ \log^{[q-1]} r \right] \rho_h^{(m,q)}(f)$$

(2.18)

and for all sufficiently large values of $r$,

$$\log^{[n-1]} T_k^{-1} T_g (r) \geq \left( \sigma_k^{(n,q)}(g) - \varepsilon \right) \left[ \log^{[q-1]} r \right] \rho_k^{(n,q)}(g).$$

(2.19)

Combining (2.18) and (2.19) and the condition $\rho_h^{(m,q)}(f) = \rho_k^{(n,q)}(g)$, we get for a sequence of values of $r$ tending to infinity that

$$\frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \frac{\sigma_h^{(m,q)}(f) + \varepsilon}{\sigma_k^{(n,q)}(g) - \varepsilon}.$$  

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \frac{\sigma_h^{(m,q)}(f)}{\sigma_k^{(n,q)}(g)}.$$  

(2.20)

Also for a sequence of values of $r$ tending to infinity it follows that

$$\log^{[n-1]} T_k^{-1} T_g (r) \leq \left( \sigma_k^{(n,q)}(g) + \varepsilon \right) \left[ \log^{[q-1]} r \right] \rho_k^{(n,q)}(g).$$

(2.21)

Now from (2.15), (2.21) and the condition $\rho_h^{(m,q)}(f) = \rho_k^{(n,q)}(g)$, we obtain for a sequence of values of $r$ tending to infinity that

$$\frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \geq \frac{\sigma_h^{(m,q)}(f) - \varepsilon}{\sigma_k^{(n,q)}(g) + \varepsilon}.$$


As $\varepsilon (> 0)$ is arbitrary, we get from above that
\[
\lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \geq \frac{\sigma_h^{(m,q)} (f)}{\sigma_k^{(n,q)} (g)}.
\] (2.22)

Also for all sufficiently large values of $r$, we get that
\[
\log^{[m-1]} T_h^{-1} T_f (r) \leq \left( \sigma_h^{(m,q)} (f) + \varepsilon \right) \left[ \log^{[q-1]} r \right] \rho_h^{(m,q)} (f).
\] (2.23)

In view of the condition $\rho_h^{(m,q)} (f) = \rho_k^{(n,q)} (g)$, it follows from (2.19) and (2.23) for all sufficiently large values of $r$ that
\[
\frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \frac{\sigma_h^{(m,q)} (f) + \varepsilon}{\sigma_k^{(n,q)} (g) - \varepsilon}.
\]

Since $\varepsilon (> 0)$ is arbitrary, we obtain that
\[
\lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \frac{\sigma_h^{(m,q)} (f)}{\sigma_k^{(n,q)} (g)}.
\] (2.24)

Thus the theorem follows from (2.17), (2.20), (2.22) and (2.24).

**Theorem 5.** Let $f$, $g$ be any two meromorphic functions and $h$, $k$ be any two entire functions such that $0 < \Delta_h^{(m,q)} (f) < \infty$, $0 < \Delta_k^{(n,q)} (g) < \infty$ and $\rho_h^{(m,q)} (f) = \rho_k^{(n,q)} (g)$ where $m$, $n$ and $p$ are any three positive integers. Then
\[
\lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \frac{\Delta_h^{(m,q)} (f)}{\Delta_k^{(n,q)} (g)} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)}.
\]

**Proof.** From the definition of $\Delta_k^{(n,q)} (g)$, we get for a sequence of values of $r$ tending to infinity that
\[
\log^{[n-1]} T_k^{-1} T_g (r) \geq \left( \Delta_k^{(n,q)} (g) - \varepsilon \right) \left[ \log^{[q-1]} r \right] \rho_k^{(n,q)} (g).
\] (2.25)

Now from (2.23), (2.25) and the condition $\rho_h^{(m,q)} (f) = \rho_k^{(n,q)} (g)$, it follows for a sequence of values of $r$ tending to infinity that
\[
\frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \frac{\Delta_h^{(m,q)} (f) + \varepsilon}{\Delta_k^{(n,q)} (g) - \varepsilon}.
\]

As $\varepsilon (> 0)$ is arbitrary, we obtain that
\[
\lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \frac{\Delta_h^{(m,q)} (f)}{\Delta_k^{(n,q)} (g)}.
\] (2.26)

Again for a sequence of values of $r$ tending to infinity that
\[
\log^{[m-1]} T_h^{-1} T_f (r) \geq \left( \Delta_h^{(m,q)} (f) - \varepsilon \right) \left[ \log^{[q-1]} r \right] \rho_h^{(m,q)} (f).
\] (2.27)
So combining (2.16) and (2.27) and in view of the condition \( \rho_h^{[m]}(f) = \rho_k^{(n,q)}(g) \), we get for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log^{[m-1]} T_{h}^{-1} T_f(r)}{\log^{[n-1]} T_{k}^{-1} T_g(r)} \geq \frac{\Delta^{(m,q)}_h(f) - \varepsilon}{\Delta^{(n,q)}_k(g) + \varepsilon}.
\]

Since \( \varepsilon (> 0) \) is arbitrary, it follows that

\[
\lim_{r \to +\infty} \frac{\log^{[m-1]} T_{h}^{-1} T_f(r)}{\log^{[n-1]} T_{k}^{-1} T_g(r)} \geq \frac{\Delta^{(m,q)}_h(f)}{\Delta^{(n,q)}_k(g)}.
\]

Thus the theorem follows from (2.26) and (2.28). \( \square \)

The following theorem is a natural consequence of Theorem 4 and Theorem 5.

**Theorem 6.** Let \( f, g \) be any two meromorphic functions and \( h, k \) be any two entire functions such that \( 0 < \tau_h^{(m,q)}(f) \leq \sigma_h^{(m,q)}(f) < \infty \), \( 0 < \sigma_k^{(n,q)}(g) \leq \sigma_k^{(n,q)}(g) < \infty \) and \( \rho_h^{(m,q)}(f) = \rho_k^{(n,q)}(g) \) where \( m, n \) and \( p \) are any three positive integers. Then

\[
\lim_{r \to +\infty} \frac{\log^{[m-1]} T_{h}^{-1} T_f(r)}{\log^{[n-1]} T_{k}^{-1} T_g(r)} \leq \min \left\{ \frac{\tau_h^{(m,q)}(f)}{\sigma_h^{(n,q)}(g)}, \frac{\sigma_h^{(m,q)}(f)}{\sigma_k^{(n,q)}(g)} \right\} \leq \max \left\{ \frac{\sigma_h^{(m,q)}(f)}{\tau_h^{(n,q)}(g)}, \frac{\sigma_h^{(m,q)}(f)}{\sigma_k^{(n,q)}(g)} \right\} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_{h}^{-1} T_f(r)}{\log^{[n-1]} T_{k}^{-1} T_g(r)}.
\]

Now in the line of Theorem 4, Theorem 5 and Theorem 6 respectively, one can easily prove the following six theorems using the notion of \((p,q)\)-th relative weak type and therefore their proofs are omitted.

**Theorem 7.** Let \( f, g \) be any two meromorphic functions and \( h, k \) be any two entire functions such that \( 0 < \tau_h^{(m,q)}(f) \leq \tau_h^{(m,q)}(f) < \infty \), \( 0 < \tau_k^{(n,q)}(g) \leq \tau_k^{(n,q)}(g) < \infty \) and \( \lambda_h^{(m,q)}(f) = \lambda_k^{(n,q)}(g) \) where \( m, n \) and \( p \) are any three positive integers. Then

\[
\frac{\tau_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_{h}^{-1} T_f(r)}{\log^{[n-1]} T_{k}^{-1} T_g(r)} \leq \frac{\tau_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)}.
\]

**Theorem 8.** Let \( f, g \) be any two meromorphic functions and \( h, k \) be any two entire functions such that \( 0 < \tau_h^{(m,q)}(f) < \infty \), \( 0 < \tau_k^{(n,q)}(g) < \infty \) and \( \lambda_h^{(m,q)}(f) = \lambda_k^{(n,q)}(g) \) where \( m, n \) and \( p \) are any three positive integers. Then

\[
\lim_{r \to +\infty} \frac{\log^{[m-1]} T_{h}^{-1} T_f(r)}{\log^{[n-1]} T_{k}^{-1} T_g(r)} \leq \frac{\tau_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_{h}^{-1} T_f(r)}{\log^{[n-1]} T_{k}^{-1} T_g(r)}.
\]
Theorem 9. Let \( f, g \) be any two meromorphic functions and \( h, k \) be any two entire functions such that \( 0 < \tau_h^{(m,q)}(f) \leq \tau_k^{(m,q)}(f) < \infty \), \( 0 < \tau_k^{(n,q)}(g) \leq \tau_k^{(n,q)}(g) < \infty \) and \( \lambda_h^{(m,q)}(f) = \lambda_k^{(n,q)}(g) \) where \( m, n \) and \( p \) are any three positive integers. Then

\[
\lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \min \left\{ \frac{\tau_h^{(m,q)}(f)}{\tau_k^{(m,q)}(f)}, \frac{\tau_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)} \right\} \leq \max \left\{ \frac{\tau_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)}, \frac{\tau_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)} \right\} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)}.
\]

We may now state the following theorems without their proofs based on \((p,q)\)-th relative type and \((p,q)\)-th relative weak type:

Theorem 10. Let \( f, g \) be any two meromorphic functions and \( h, k \) be any two entire functions such that \( 0 < \sigma_h^{(m,q)}(f) \leq \sigma_h^{(m,q)}(f) < \infty \), \( 0 < \tau_k^{(n,q)}(g) \leq \tau_k^{(n,q)}(g) < \infty \) and \( \rho_h^{(m,q)}(f) = \lambda_k^{(n,q)}(g) \) where \( m, n \) and \( p \) are any three positive integers. Then

\[
\frac{\sigma_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \frac{\sigma_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)}.
\]

Theorem 11. Let \( f, g \) be any two meromorphic functions and \( h, k \) be any two entire functions such that \( 0 < \sigma_h^{(m,q)}(f) \leq \sigma_h^{(m,q)}(f) < \infty \), \( 0 < \tau_k^{(n,q)}(g) \leq \tau_k^{(n,q)}(g) < \infty \) and \( \rho_h^{(m,q)}(f) = \lambda_k^{(n,q)}(g) \) where \( m, n \) and \( p \) are any three positive integers. Then

\[
\lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \frac{\sigma_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)}.
\]

Theorem 12. Let \( f, g \) be any two meromorphic functions and \( h, k \) be any two entire functions such that \( 0 < \sigma_h^{(m,q)}(f) \leq \sigma_h^{(m,q)}(f) < \infty \), \( 0 < \tau_k^{(n,q)}(g) \leq \tau_k^{(n,q)}(g) < \infty \) and \( \rho_h^{(m,q)}(f) = \lambda_k^{(n,q)}(g) \) where \( m, n \) and \( p \) are any three positive integers. Then

\[
\lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)} \leq \min \left\{ \frac{\sigma_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)}, \frac{\sigma_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)} \right\} \leq \max \left\{ \frac{\sigma_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)}, \frac{\sigma_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)} \right\} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_h^{-1} T_f (r)}{\log^{[n-1]} T_k^{-1} T_g (r)}.
\]
\[ \lambda_{h}^{(m,q)}(f) = \rho_{k}^{(n,q)}(g) \text{ where } m, n \text{ and } p \text{ are any three positive integers. Then} \]
\[ \frac{\tau_{h}^{(m,q)}(f)}{\sigma_{k}^{(n,q)}(g)} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_{h}^{1} T_{f}(r)}{\log^{[n-1]} T_{k}^{1} T_{g}(r)} \leq \frac{\tau_{h}^{(m,q)}(f)}{\sigma_{k}^{(n,q)}(g)} \]
\[ \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_{h}^{1} T_{f}(r)}{\log^{[n-1]} T_{k}^{1} T_{g}(r)} \leq \frac{\tau_{h}^{(m,q)}(f)}{\sigma_{k}^{(n,q)}(g)} . \]

**Theorem 14.** Let \( f, g \) be any two meromorphic functions and \( h, k \) be any two entire functions such that \( 0 < \tau_{h}^{(m,q)}(f) < \infty, 0 < \sigma_{k}^{(n,q)}(g) < \infty \) and \( \lambda_{h}^{(m,q)}(f) = \rho_{k}^{(n,q)}(g) \) where \( m, n \) and \( p \) are any three positive integers. Then
\[ \lim_{r \to +\infty} \frac{\log^{[m-1]} T_{h}^{1} T_{f}(r)}{\log^{[n-1]} T_{k}^{1} T_{g}(r)} \leq \frac{\tau_{h}^{(m,q)}(f)}{\sigma_{k}^{(n,q)}(g)} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_{h}^{1} T_{f}(r)}{\log^{[n-1]} T_{k}^{1} T_{g}(r)} . \]

**Theorem 15.** Let \( f, g \) be any two meromorphic functions and \( h, k \) be any two entire functions such that \( 0 < \tau_{h}^{(m,q)}(f) \leq \tau_{h}^{(m,q)}(f) < \infty, 0 < \sigma_{k}^{(n,q)}(g) \leq \sigma_{k}^{(n,q)}(g) < \infty \) and \( \lambda_{h}^{(m,q)}(f) = \rho_{k}^{(n,q)}(g) \) where \( m, n \) and \( p \) are any three positive integers. Then
\[ \lim_{r \to +\infty} \frac{\log^{[m-1]} T_{h}^{1} T_{f}(r)}{\log^{[n-1]} T_{k}^{1} T_{g}(r)} \leq \min \left\{ \frac{\tau_{h}^{(m,q)}(f)}{\sigma_{k}^{(n,q)}(g)}, \frac{\tau_{h}^{(m,q)}(f)}{\sigma_{k}^{(n,q)}(g)} \right\} \]
\[ \leq \max \left\{ \frac{\tau_{h}^{(m,q)}(f)}{\sigma_{k}^{(n,q)}(g)}, \frac{\tau_{h}^{(m,q)}(f)}{\sigma_{k}^{(n,q)}(g)} \right\} \leq \lim_{r \to +\infty} \frac{\log^{[m-1]} T_{h}^{1} T_{f}(r)}{\log^{[n-1]} T_{k}^{1} T_{g}(r)} . \]

**Remark 1.** The same results of above theorems in terms of Maximum modulus of entire functions can also be deduced if we consider \( f \) and \( g \) be any two entire functions.

**References**

[1] L. Bernal : Orden relative de crecimiento de funciones enteras, Collect. Math., Vol. 39 (1988), pp. 209-229.
[2] L. Debnath, S. K. Datta, T. Biswas and A. Kar: Growth of meromorphic functions depending on \((p,q)\)-th relative order, Facta Universitatis series: Mathematics and Informatics, Vol. 31, No 3 (2016), pp. 691-705.
[3] W.K. Hayman : Meromorphic Functions, The Clarendon Press, Oxford (1964).
[4] O. P. Juneja, G. P. Kapoor and S. K. Bajpai : On the \((p,q)\)-order and lower \((p,q)\)-order of an entire function, J. Reine Angew. Math., Vol. 282 (1976), pp. 53-67.
[5] L. M. S. Ruiz, S. K. Datta, T. Biswas and G. K. Mondal: On the \((p,q)\)-th relative order oriented growth properties of entire functions, Abstract and Applied Analysis, Vol.2014, Article ID 826137, 8 pages, [http://dx.doi.org/10.1155/2014/826137](http://dx.doi.org/10.1155/2014/826137).
[6] E.C. Titchmarsh : The theory of functions, 2nd ed. Oxford University Press, Oxford, 1968.
[7] G. Valiron : Lectures on the general theory of integral functions, Chelsea Publishing Company, 1949.

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