Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $\mathcal{O}$ be a complete discrete valuation ring of characteristic 0 with residue field $k$. Suppose $G$ is a finite group and $V$ is a finitely generated $kG$-module. It is a classical question to ask whether $V$ can be lifted to an $\mathcal{O}$-free $\mathcal{O}G$-module. In [23], Green showed that this is always possible when $\operatorname{Ext}^2_{kG}(V, V) = 0$. However, there are many cases when this Ext group is not zero and $V$ can still be lifted over $\mathcal{O}$. This lifting question can be seen as a special case of a more general deformation question which asks over which complete local commutative Noetherian $\mathcal{O}$-algebras with residue field $k$ the $kG$-module $V$ can be lifted. Since $k$ is algebraically closed, one usually takes $\mathcal{O}$ to be the ring $W = W(k)$ of infinite Witt vectors over $k$. It was shown in [6, Prop. 2.1] that if the stable endomorphism ring of $V$ is isomorphic to $k$, then $V$ has a so-called universal deformation ring $R(G, V)$. This ring is universal in the sense that every isomorphism class of lifts of $V$ over a complete local commutative Noetherian ring $R$ with residue field $k$ is associated to a unique morphism $R(G, V) \to R$ (see Section 2).

Suppose that the stable endomorphism ring of $V$ is isomorphic to $k$. In [6] (resp. [11]), the isomorphism types of $R(G, V)$ were determined for all such $V$ belonging to a cyclic block (resp. to a block with Klein four defect groups). In [2, 3, 8], the rings $R(G, V)$ were determined for all such $V$ belonging to various tame blocks with dihedral defect groups. By [22], these blocks include in particular all blocks that are Morita equivalent to principal blocks with dihedral defect groups. For other tame blocks, however, usually much less is known with respect to their representation theory. For this reason, it still remains to systematically study all blocks of infinite tame representation type with respect to universal deformation rings, and this is the goal of the present paper. The key tools used to determine the universal deformation rings in all of the above cases are results from modular and ordinary representation theory due to Brauer, Erdmann [21], Linckelmann [28, 29], Carlson-Thévenaz [16], and others.

The main motivation for studying universal deformation rings for finite groups is that this case helps understand ring theoretic properties of universal deformation rings for profinite groups $\Gamma$. The latter have become an important tool in number theory, in particular if $\Gamma$ is a profinite Galois group (see e.g. [33, 31, 13, 27] and their references). In [18], de Smit and Lenstra showed that if $\Gamma$ is an arbitrary profinite group and $V$ is a finite dimensional vector space over $\mathcal{O}$ with a continuous $\Gamma$-action which has a universal deformation ring $R(\Gamma, V)$, then $R(\Gamma, V)$ is the inverse limit of the universal deformation rings $R(G, V)$ when $G$ ranges over all finite discrete quotients of $\Gamma$ through which the $\Gamma$-action on $V$ factors. Thus to answer questions about the ring structure of $R(\Gamma, V)$, it is natural to first consider the case when $\Gamma = G$ is finite. When determining $R(G, V)$, the main advantage is

\textbf{1. Introduction}

Let $k$ be an algebraically closed field of positive characteristic, and let $W$ be the ring of infinite Witt vectors over $k$. Suppose $G$ is a finite group and $B$ is a block of $kG$ of infinite tame representation type. We find all finitely generated $kG$-modules $V$ that belong to $B$ and whose endomorphism ring is isomorphic to $k$ and determine the universal deformation ring $R(G, V)$ for each of these modules.
that one can make use of powerful techniques that are not available for arbitrary profinite groups 
\( \Gamma \), such as decomposition matrices, Auslander-Reiten theory and the Green correspondence.

Suppose now that \( B \) is a block of \( kG \) of infinite tame representation type. In [21], Erdmann
gave a list of all possible quivers and relations which determine the basic algebra \( \Lambda \) of \( B \) up to
isomorphism. In the case when the defect groups of \( B \) are dihedral, she moreover showed that
\( \Lambda / \text{soc}(\Lambda) \) is a special biserial algebra. This means that in this case one can give a complete list
of isomorphism classes of indecomposable \( \Lambda \)-modules using so-called strings and bands (see [15]). In
particular, this made it possible in [2, 3, 8] to determine all \( B \)-modules whose stable endomorphism
rings are isomorphic to \( k \) when \( B \) has dihedral defect groups. For arbitrary blocks \( B \) of infinite
tame representation type, one usually cannot give such a complete list. However, we will show that
it is still possible to determine all isomorphism classes of \( B \)-modules whose endomorphism rings are
isomorphic to \( k \).

Our main result is as follows; more precise statements can be found in Lemma 5.2 and Theorem 5.3.

**Theorem 1.1.** Suppose \( G \) is a finite group, \( B \) is a block of \( kG \) of infinite tame representation
type, and \( D \) is a defect group of \( B \) of order \( p^n \). Let \( V \) be a \( kG \)-module belonging to \( B \) whose
endomorphism ring is isomorphic to \( k \), and let \( R(G,V) \) be its universal deformation ring. Let
\( d^1(V) = \dim_k \operatorname{Ext}_B^1(V,V) \). Then \( d^1(V) \in \{0, 1, 2\} \).

(i) If \( d^1(V) = 0 \), then either \( R(G,V) \cong W \) or \( R(G,V) \cong k \).

(ii) If \( d^1(V) = 1 \), then either

(a) \( R(G,V) \cong W[[t]]/(t^{p^n} - p\mu t) \) for some non-zero \( \mu \in W \), or

(b) \( R(G,V) \cong W[[t]]/(t^p, pt) \), or

(c) \( n \geq 4 \) and there exists a monic polynomial \( q_n(t) \in W[t] \) of degree \( p^{n-2} - 1 \), which
depends only on \( D \) and which can be given explicitly, such that either

\( R(G,V) \cong W[[t]]/(q_n(t)) \) or \( R(G,V) \cong W[[t]]/(t q_n(t), p q_n(t)) \).

(iii) If \( d^1(V) = 2 \), then \( R(G,V) \cong W[[t_1, t_2]]/(t_1^p - pt_1, t_2^p - pt_2) \).

In all cases, \( R(G,V) \) is isomorphic to a subquotient algebra of the group algebra \( WD \), giving a
positive answer to [5] Question 1.1.

To prove Theorem 1.1, we first determine all \( B \)-modules \( V \) whose endomorphism rings are isomor-
phic to \( k \) by finding the \( \Lambda \)-modules \( M \) that correspond to \( V \) under the Morita equivalence
between \( B \) and its basic algebra \( \Lambda \). The main idea is to use the description of the projective inde-
composable modules to classify certain \( \Lambda \)-modules that have a short radical series. It turns out
that the \( \Lambda \)-modules \( M \) we need to find have at most 4 composition factors, resulting in a finite
list of isomorphism classes of \( B \)-modules \( V \) whose endomorphism rings are isomorphic to \( k \). We
then determine the universal deformation ring \( R(G,V) \) for each of these modules \( V \). Computing
\( d^1(V) \) shows that the case \( d^1(V) = 2 \) only occurs when \( B \) is local, i.e. when there is a unique
isomorphism class of simple \( B \)-modules. This allows us to use nilpotent blocks to prove part (iii)
of Theorem 1.1. For non-local \( B \), \( R(G,V) \) is determined in two steps: Using the basic algebra \( \Lambda \),
we first determine the universal mod \( p \) deformation ring \( R(G,V)/pR(G,V) \). Using decomposition
matrices and generalized decomposition numbers, we then determine the full universal deformation
ring \( R(G,V) \). In particular, we use the results from [5] to prove part (ii)(c) of Theorem 1.1.

The paper is organized as follows. In Section 2 we review the basic definitions and results
concerning universal deformation rings of modules for finite groups. In Section 3 we let \( B \) be a
block of infinite tame representation type and set up the notation for the remainder of the paper.
We also deal with the case when \( B \) is a local block (see Lemma 5.2). For the remainder of the paper,
we let \( B \) be non-local. In Section 4 we determine all \( B \)-modules \( V \) whose endomorphism rings are
isomorphic to \( k \) (see Proposition 4.1). In Section 5 we then determine the universal deformation
ring \( R(G,V) \) for each such module \( V \) (see Theorem 5.3). This, together with Lemma 5.2 proves
Theorem 1.1.
2. Preliminaries

In this section, we give a brief introduction to deformation rings and deformations. For more background material, we refer the reader to [30] and [18].

Let $k$ be an algebraically closed field of characteristic $p > 0$, and let $W = W(k)$ be the ring of infinite Witt vectors over $k$. Let $\mathcal{C}$ be the category of all complete local commutative Noetherian rings with residue field $k$. Note that all rings in $\mathcal{C}$ have a natural $W$-algebra structure. The morphisms in $\mathcal{C}$ are continuous $W$-algebra homomorphisms which induce the identity map on $k$.

Suppose $G$ is a finite group and $V$ is a finitely generated $kG$-module. A lift of $V$ over an object $R$ in $\mathcal{C}$ is a pair $(M, \phi)$ where $M$ is a finitely generated $RG$-module which is free over $R$, and $\phi : k \otimes_R M \to V$ is an isomorphism of $kG$-modules. Two lifts $(M, \phi)$ and $(M', \phi')$ of $V$ over $R$ are isomorphic if there is an isomorphism $f : M \to M'$ with $\phi = \phi' \circ (k \otimes f)$. The isomorphism class $[M, \phi]$ of a lift $(M, \phi)$ of $V$ over $R$ is called a deformation of $V$ over $R$, and the set of all such deformations is denoted by $\text{Def}_G(V, R)$. The deformation functor

$$\tilde{F}_V : \mathcal{C} \to \text{Sets}$$

is a covariant functor which sends an object $R$ in $\mathcal{C}$ to $\text{Def}_G(V, R)$ and a morphism $\alpha : R \to R'$ in $\mathcal{C}$ to the map $\text{Def}_G(V, R) \to \text{Def}_G(V, R')$ defined by $[M, \phi] \mapsto [R' \otimes_{R, \alpha} M, \phi_\alpha]$, where $\phi_\alpha = \phi$ after identifying $k \otimes_{R'} (R' \otimes_{R, \alpha} M)$ with $k \otimes_R M$.

Suppose there exists an object $R(G, V)$ in $\mathcal{C}$ and a deformation $[U(G, V), \phi_U]$ of $V$ over $R(G, V)$ with the following property: For each ring $R$ in $\mathcal{C}$ and for each lift $(M, \phi)$ of $V$ over $R$ there exists a morphism $\alpha : R(G, V) \to R$ in $\mathcal{C}$ such that $\tilde{F}_V(\alpha)([U(G, V), \phi_U]) = [M, \phi]$, and moreover $\alpha$ is unique if $R$ is the ring of dual numbers $k[[e]]/(e^2)$. Then $R(G, V)$ is called the versal deformation ring of $V$ and $[U(G, V), \phi_U]$ is called the versal deformation of $V$. If the morphism $\alpha$ is unique for all $R$ and all lifts $(M, \phi)$ of $V$ over $R$, then $R(G, V)$ is called the universal deformation ring of $V$ and $[U(G, V), \phi_U]$ is called the universal deformation of $V$. In other words, $R(G, V)$ is universal if and only if $R(G, V)$ represents the functor $\tilde{F}_V$ in the sense that $\tilde{F}_V$ is naturally isomorphic to the Hom functor $\text{Hom}_\mathcal{C}(R(G, V), -)$.

By [30], every finitely generated $kG$-module $V$ has a versal deformation ring $R(G, V)$. By [6] Prop. 2.1, if the stable endomorphism ring $\overline{\text{End}}_{kG}(V)$ is isomorphic to $k$, then $R(G, V)$ is universal.

Note that the above definition of deformations can be weakened as follows. Given a lift $(M, \phi)$ of $V$ over a ring $R$ in $\mathcal{C}$, define the corresponding weak deformation to be the isomorphism class of $M$ as an $RG$-module, without taking into account the specific isomorphism $\phi : k \otimes_R M \to V$. In general, a weak deformation of $V$ over $R$ identifies more lifts than a deformation of $V$ over $R$ that respects the isomorphism $\phi$ of a representative $(M, \phi)$. However, if the stable endomorphism ring $\overline{\text{End}}_{kG}(V)$ is isomorphic to $k$, these two definitions of deformations coincide (see [4] Remark 2.1).

3. Tame blocks

We make the following assumptions for the remainder of the paper:

**Hypothesis 3.1.** Let $k$ be an algebraically closed field of positive characteristic $p$, and let $W = W(k)$ be the ring of infinite Witt vectors over $k$. Suppose $G$ is a finite group, $B$ is a block of $kG$ of infinite tame representation type, and $D$ is a defect group of $B$ of order $p^n$.

It follows from [9] [12] [24] that $p = 2$, $n \geq 2$, and $D$ is dihedral, semidihedral or generalized quaternion. In particular, we have $n \geq 2$ if $D$ is dihedral, $n \geq 3$ if $D$ is generalized quaternion, and $n \geq 4$ if $D$ is semidihedral. By [10] [11] [31], it follows that there are at most three isomorphism classes of simple $B$-modules.

We first consider the case when $B$ in Hypothesis 3.1 is local, i.e. when there is precisely one isomorphism class of simple $B$-modules. We obtain the following result.

**Lemma 3.2.** Assume Hypothesis 3.1 and that $B$ is local. Let $S$ be a simple $kG$-module belonging to $B$. Then $S$ is, up to isomorphism, the only $kG$-module belonging to $B$ whose endomorphism ring
is isomorphic to $k$. We have $\dim_k \Ext^1_G(S, S) = 2$ and $R(G, S) \cong W[\mathbb{Z}/2 \times \mathbb{Z}/2]$. In particular, $R(G, S) \cong W[\{1, t_2\}]/(t_2^2 - p t_1, t_1^2 - p t_2)$ and $R(G, S)$ is isomorphic to a subquotient algebra of $WD$.

**Proof.** Recall that $p = 2$. Since $B$ is local and of infinite tame representation type, it follows that $B$ is nilpotent in the sense of [14] (see e.g. [26 Sect. 2.5]). Let $\hat{B}$ be the block of $WG$ corresponding to $B$. Then $\hat{B}$ is also nilpotent. The main result of [32] implies that $B$ is Morita equivalent to $kD$ and $\hat{B}$ is Morita equivalent to $WD$ (see [32 Sect. 1.4]). Since every non-zero $B$-module has a non-zero socle and a non-zero radical quotient, it is immediate that, up to isomorphism, the only $kG$-module belonging to $B$ whose endomorphism ring is isomorphic to $k$ is $S$. Using the Morita equivalence between $\hat{B}$ and $WD$, it follows for example from [1 Prop. 2.5] that $R(G, S) \cong R(D, k)$ when $k$ denotes the trivial simple $kD$-module (which corresponds to $S$ under the Morita equivalence). By [30 Sect. 1.4], $R(D, k)$ is isomorphic to the group ring over $W$ of the maximal abelian $p$-quotient of $D$. Since $p = 2$ and $D$ is dihedral, semidihedral or generalized quaternion, the maximal abelian $2$-quotient of $D$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, which proves Lemma 3.2.

Assume Hypothesis 3.1 and that $B$ is non-local. From Erdmann’s classification of all blocks of tame representation type in [21], it follows that the quiver and relations of the basic algebra of $B$ can be given explicitly and that, up to Morita equivalence, there are 24 families of non-local blocks $B$. We use the description of these families as given in [5 Sect. 4], where Erdmann’s results in [20 21] and [25 Prop. 4.2] were combined with Eisele’s results in [19], giving the list in Figure 1. Note that $D$, or $SD$, or $Q$ in the name indicates that the defect groups of $B$ are dihedral, or semidihedral or generalized quaternion, respectively.

**Figure 1.** The list of basic algebras from [5 Sect. 4] that are Morita equivalent to a non-local block $B$ satisfying Hypothesis 3.1.

- $D(2A), D(2B), D(3A)_1, D(3B)_1, D(3C)$;
- $SD(2A)_1(c), SD(2A)_2(c), SD(2B)_1(c), SD(2B)_2(c), SD(2B)_4(c), SD(3A)_1, SD(3B)_1, SD(3B)_2, SD(3C)_2, SD(3D), SD(3H)_1, SD(3H)_2$;
- $Q(2A)_1(c), Q(2B)_1(c), Q(2B)_2(p, a, c), Q(3A)_2, Q(3B), Q(3C)$.

We will also make use of the decomposition matrix for each non-local block $B$, including the order of the ordinary irreducible characters, as given in [5 Appendix]. Note that $B$ always contains exactly 4 ordinary irreducible characters of height 0 and, unless $D$ is quaternion of order 8, exactly $2^n - 2 - 1$ ordinary irreducible characters of height 1. If $D$ is quaternion of order 8, $B$ contains exactly 3 ordinary irreducible characters of height 1. If $n \geq 4$ then the family of $2^{n-2} - 1$ ordinary irreducible characters of height 1 all define the same Brauer character on restricting to the 2-regular conjugacy classes of $G$. If $D$ is generalized quaternion or semidihedral, there may be additional ordinary irreducible characters of height $n - 2$. In the decomposition matrices in [5 Appendix], the 4 ordinary irreducible characters of height 0 are listed first, followed by the family of $2^{n-2} - 1$ ordinary irreducible characters of height 1, and finally the ordinary irreducible characters of height $n - 2$ if they exist.

For each algebra $\Lambda$ in Figure 1, we use the following notation for certain modules of small length.

**Definition 3.3.** Assume Hypothesis 3.1 and that $B$ is non-local. Let $\Lambda = kQ/I$ be a basic algebra such that $B$ is Morita equivalent to $\Lambda$, where we assume $\Lambda$ is one of the algebras in Figure 1. For each vertex $j$ in $Q$, let $S_j$ denote a simple $\Lambda$-module corresponding to $j$.

(a) Let $v_1, v_2, \ldots, v_r$ be (not necessarily distinct) vertices of $Q$. If there exists, up to isomorphism, a unique uniserial $\Lambda$-module with descending composition factors $S_{v_1}, S_{v_2}, \ldots, S_{v_r}$,
Proposition 4.1. Assume Hypothesis algebra such that $B$ generated equivalence between equal to either $SD(2^A)$ in [5, Sect. 4]. We illustrate the main arguments of the proof by considering the cases when $\Lambda$ is Proposition 4.1 is proved using the description of the basic algebras $\Lambda$ in Figure 1, as provided Proof. Let $E$.

(b) Let $u, v, w$ be (not necessarily distinct) vertices of $Q$. If there exists, up to isomorphism, a unique indecomposable $\Lambda$-module with descending radical factors $S_u, S_v \oplus S_w$, we denote such a $\Lambda$-module by

$$T_{u,v \oplus w} = \begin{pmatrix} S_u \\ S_v \\ S_w \end{pmatrix}$$

If there exists, up to isomorphism, a unique indecomposable $\Lambda$-module with descending radical factors $S_v \oplus S_w, S_u$, we denote such a $\Lambda$-module by

$$T_{v \oplus w,u} = \begin{pmatrix} S_v \\ S_w \\ S_u \end{pmatrix}$$

4. Modules with endomorphism ring $k$

We assume Hypothesis [3,1] and that $B$ is non-local. In this section, we determine all finitely generated $B$-modules whose endomorphism ring is isomorphic to $k$.

**Proposition 4.1.** Assume Hypothesis [3,1] and that $B$ is non-local. Let $\Lambda = kQ/I$ be a basic algebra such that $B$ is Morita equivalent to $\Lambda$, where we assume $\Lambda$ is one of the algebras in Figure 1. Let $\mathcal{E}$ be a complete set of representatives of non-isomorphic $kG$-modules $V$ belonging to $B$ with $\text{End}_{kG}(V) \cong k$. Let $\mathcal{E}_\Lambda$ be a set of $\Lambda$-modules that correspond to the modules in $\mathcal{E}$ under the Morita equivalence between $B$ and $\Lambda$. Using the notation from Definition 8.3, $\mathcal{E}_\Lambda$ is given as follows:

(i) If $Q \in \{2A, 2B\}$ and $\Lambda \notin \{SD(2B)_{4}(c), Q(2B)_{2}(p, a, c)\}$, then

$$\mathcal{E}_\Lambda = \{S_0, S_1, S_{01}, S_{10}, S_{001}, S_{100}\}.$$ If $\Lambda \in \{SD(2B)_{4}(c), Q(2B)_{2}(p, a, c)\}$, then

$$\mathcal{E}_\Lambda = \{S_0, S_1, S_{01}, S_{10}\}.$$ (ii) If $Q \in \{3A, 3B\}$ and $\Lambda \neq SD(3B)_{1}$, then

$$\mathcal{E}_\Lambda = \{S_0, S_1, S_{2}, S_{01}, S_{10}, S_{02}, S_{20}, S_{102}, S_{201}, S_{0201}, S_{1020}, T_{0,1@2}, T_{1@2,0}\}.$$ If $\Lambda = SD(3B)_{1}$, then

$$\mathcal{E}_\Lambda = \{S_0, S_1, S_{2}, S_{01}, S_{10}, S_{02}, S_{20}, S_{102}, S_{201}, S_{0201}, S_{1020}, T_{0,1@2}, T_{1@2,0}\}.$$ (iii) If $Q = 3C$, then

$$\mathcal{E}_\Lambda = \{S_0, S_1, S_{2}, S_{01}, S_{10}, S_{02}, S_{20}, S_{102}, S_{201}, T_{0,1@2}, T_{1@2,0}\}.$$ (iv) If $Q = 3D$, then

$$\mathcal{E}_\Lambda = \{S_0, S_1, S_{2}, S_{01}, S_{10}, S_{02}, S_{20}, S_{102}, S_{201}, S_{0201}, S_{1020}, T_{0,1@2}, T_{1@2,0}\}.$$ (v) If $Q = 3H$, then

$$\mathcal{E}_\Lambda = \{S_0, S_1, S_{2}, S_{01}, S_{10}, S_{02}, S_{20}, S_{102}, S_{201}, S_{0201}, T_{1@2,0}, T_{1@0@2}, T_{0@2,1}, T_{2@0@2}\}.$$ (vi) If $Q = 3K$, then

$$\mathcal{E}_\Lambda = \{S_0, S_1, S_{2}, S_{01}, S_{10}, S_{02}, S_{20}, S_{102}, S_{201}, T_{0,1@2}, T_{1@2,0}, T_{0@2,1}, T_{2,0@1}, T_{0@2,1}\}.$$
(a) Suppose first that \( \Lambda = SD(2A)_1(c) = k[2A]/I_{SD(2A)_1,c} \) for some \( c \in k \), where the quiver \( 2A \) and the ideal \( I_{SD(2A)_1,c} \) are as in Figure 2. Note that \( n \geq 4 \).

**Figure 2.** The quiver and relations for \( \Lambda = SD(2A)_1(c) = k[2A]/I_{SD(2A)_1,c} \).

\[
2A = \begin{array}{ccc}
0 & \beta & 1 \\
\alpha & & \gamma \\
\end{array}
\]

\[
I_{SD(2A)_1,c} = \langle \alpha^2 - c(\gamma \beta \alpha)^{2n-2}, \beta \gamma \beta - \beta \alpha (\gamma \beta \alpha)^{2n-2} - 1, \\
\gamma \beta \gamma - \alpha \gamma (\beta \alpha \gamma)^{2n-2} - 1, \alpha (\gamma \beta \alpha)^{2n-2} \rangle.
\]

Let \( e_0 \) and \( e_1 \) denote the images of the primitive idempotents of \( k[2A] \) corresponding to the vertices 0 and 1, respectively. Let \( S_0 \) and \( S_1 \) denote representatives of the isomorphism classes of simple \( \Lambda \)-modules. The projective indecomposable \( \Lambda \)-modules are pictured in Figure 3, where we use the short-hand 0, 1 to denote \( S_0, S_1 \), respectively.

**Figure 3.** The projective indecomposable modules for \( \Lambda = SD(2A)_1(c) \).

Suppose \( M \) is a non-simple \( \Lambda \)-module such that \( \text{End}_\Lambda(M) \cong k \). Then \( M/\text{rad}(M) \) and \( \text{soc}(M) \) do not have any composition factors in common. We first prove the following auxiliary statement:

\[
(4.1) \quad \text{If } x \in M, \text{ then } (\beta \gamma) x = 0.
\]

Suppose, by contradiction, that there exists \( x \in M \) such that \( (\beta \gamma) x \neq 0 \). Since \( (\beta \gamma) x = (\beta \gamma) e_1 x \), we replace \( x \) by \( e_1 x \) to be able to assume that \( e_1 x = x \). Suppose first \( (\gamma \beta \gamma) x \neq 0 \). Then it follows from the relations in \( \Lambda \) from Figure 2 that \( \left( \alpha \gamma (\beta \alpha \gamma)^{2n-2} - 1 \right) x \) is also not zero in \( M \). This implies that \( \Delta x \), which is a submodule of \( M \), is isomorphic to \( P_1/\text{soc}(P_1) \) or to \( P_1 \). Since \( M \) cannot be isomorphic to \( P_1 \), we obtain that \( \Delta x \cong P_1/\text{soc}(P_1) \). Therefore, using the notation from Definition 3.3, \( S_{10} \) is isomorphic to a submodule of \( M \). On the other hand, \( \Delta x \) is isomorphic to a submodule of \( P_0 \) and, since \( M \) is not isomorphic to \( P_0 \), \( \Delta x \) is isomorphic to a submodule of \( \text{rad}(P_0) \). This implies that \( x \in M - \text{rad}(M) \) and that \( S_{10} \) is also isomorphic to a quotient module of \( M \). But this means that \( M \) has a non-zero endomorphism factoring through \( S_{10} \), contradicting \( \text{End}_\Lambda(M) \cong k \). Therefore, we must have \( (\gamma \beta \gamma) x = 0 \). This implies that \( (\beta \gamma) x \) lies in the socle of \( M \), which means that \( S_1 \) is a direct summand of \( \text{soc}(M) \). In particular, it follows that \( x \in \text{rad}(M) \), since otherwise \( M \) has a non-zero endomorphism factoring through \( S_1 \). Since \( \text{Ext}^1_\Lambda(S_i, S_1) = 0 \) unless \( i = 0 \),
this means there exists $w \in M$ with $e_0 w = w$ such that $\beta w = x$ modulo $\text{rad}^2(\Lambda w)$. Using the relations in $\Lambda$, we see that this implies $(\beta \gamma \beta) w = (\beta \gamma) x \neq 0$. Using again the relations in $\Lambda$, we obtain that $(\beta \alpha (\gamma \beta \alpha)^{2n-2-1}) w$ is also not zero in $M$. Therefore $\Lambda w$, which is a submodule of $M$, surjects onto a quotient module of $P_0$ of the form

$$
\begin{pmatrix}
0 \\
0 \\
1 \\
1 : \cong \text{rad}(P_1) \\
1 \\
0 \\
1
\end{pmatrix}
$$

Since $M$ cannot be isomorphic to $P_1$, it follows that $w \in M - \text{rad}(M)$. In particular, this implies that $S_{01}$ is a quotient module of $M$. Note that $\Lambda w$ is isomorphic to a quotient module of $P_0/\text{soc}(P_0)$. Considering all the possible quotient modules of $P_0/\text{soc}(P_0)$ that surject onto $\text{rad}(P_1)$, we see that $S_{01}$ is isomorphic to a submodule of each of them. But this means that $M$ has a non-zero endomorphism factoring through $S_{01}$, contradicting $\text{End}_\Lambda(M) \cong k$. This completes the proof of $(4.1)$.

Note that $(4.1)$ implies that the uniserial module $S_{101}$ is not isomorphic to either a submodule or a quotient module of $M$.

Since $M/\text{rad}(M)$ and $\text{soc}(M)$ do not have any composition factors in common, there are two cases: either $M/\text{rad}(M) \cong (S_1)^r$ and $\text{soc}(M) \cong (S_0)^s$, or $M/\text{rad}(M) \cong (S_0)^s$ and $\text{soc}(M) \cong (S_1)^r$, for certain $r, s \in \mathbb{Z}^+$.

We consider the case when $M/\text{rad}(M) \cong (S_1)^r$ and $\text{soc}(M) \cong (S_0)^s$, the other case being similar. We claim that, using the notation from Definition 3.3, $M$ is isomorphic either to $S_{10}$ or to $S_{100}$.

To prove this claim, we use that $\text{Ext}_\Lambda^1(S_1, S_j)$ is one-dimensional unless $(i, j) = (1, 1)$, in which case it is zero. This implies that

$$(4.2) \quad M/\text{rad}^2(M) \cong \left( \begin{array}{c}
S_1 \\
S_0
\end{array} \right)^{r_1} \oplus (S_1)^{r_2},$$

$$(4.3) \quad \text{soc}_2(M) \cong \left( \begin{array}{c}
S_1 \\
S_0
\end{array} \right)^{s_1} \oplus \left( \begin{array}{c}
S_0 \\
S_0
\end{array} \right)^{s_2} \oplus \left( \begin{array}{c}
S_1 \\
S_0
\end{array} \right)^{s_3} \oplus (S_0)^{s_4}$$

for certain non-negative $r_1, s_j$, where $r_1$ and at least one of $s_1, s_2, s_3$ must be positive. Considering $(4.2)$ and $(4.3)$, we see that the $k$-dimension of $\text{End}_\Lambda(M)$ is at least 2 unless $M \cong S_{10}$ or $M \cong S_{100}$.

(4.4) $$\text{soc}_2(M) \cong \left( \begin{array}{c}
S_0 \\
S_0
\end{array} \right)^{s_2} \oplus (S_0)^{s_4}$$

where $s_2 > 0$. Hence we only need to consider the case when $M$ satisfies both $(4.2)$ and $(4.4)$. Since $\text{Ext}_\Lambda^1(S_i, S_j)$ is one-dimensional when $i = 1$ and zero when $i = 0$, it follows that

$$(4.5) \quad \text{soc}_3(M) \cong \left( \begin{array}{c}
S_1 \\
S_0
\end{array} \right)^{s_5} \oplus \left( \begin{array}{c}
S_0 \\
S_0
\end{array} \right)^{s_6} \oplus (S_0)^{s_7}$$
where \( s_5 > 0 \). Since \( \text{Ext}^1(S_i/S_j) \) is one-dimensional for both \( j = 0 \) and \( j = 1 \), it follows that the possible direct summands of \( M/\text{rad}^3(M) \) are isomorphic to:

\[
\begin{array}{c}
S_1 & S_1 & S_1 & S_1 \\
S_0 & S_0 & S_0 & S_0 \\
S_0 & S_0 & S_0 & S_0 \\
S_0 & S_0 & S_0 & S_0
\end{array}
\]

where at least one summand of radical length 3 occurs. Since by (4.1), \( M \) does not surject onto \( S_{101} \), we obtain

\[
M/\text{rad}^3(M) \cong \left( \begin{array}{c} S_1 \\ S_0 \end{array} \right)^{r_3} \oplus \left( \begin{array}{c} S_1 \\ S_0 \end{array} \right)^{r_4} \oplus \left( \begin{array}{c} S_1 \\ S_0 \end{array} \right)^{r_5} \oplus (S_1)^{r_6}
\]

where either \( r_3 > 0 \) or \( r_4 > 0 \). If \( r_3 > 0 \), then either \( M \cong S_{100} \) or the endomorphism ring of \( M \) has \( k \)-dimension at least 2. Hence we only need to consider the case when

\[
(4.6) \quad M/\text{rad}^3(M) \cong \left( \begin{array}{c} S_1 \\ S_0 \end{array} \right)^{r_4} \oplus \left( \begin{array}{c} S_1 \\ S_0 \end{array} \right)^{r_5} \oplus (S_1)^{r_6}
\]

and \( r_4 > 0 \). Using additional \( \text{Ext}^1 \) arguments, we see that then \( S_0 \oplus S_1 \) has to be a direct summand of \( M/\text{rad}^3(M) \). But this implies that there exists an element \( x \in M \) with \( (\beta \gamma) x \neq 0 \), which contradicts (4.1). Summarizing, if \( M/\text{rad}(M) \cong (S_1)^r \) and \( \text{soc}(M) \cong (S_0)^s \), then \( M \) is isomorphic either to \( S_{10} \) or to \( S_{100} \). This completes the case when \( B \) is Morita equivalent to \( \text{SD}(2A_1) \).

(b) Suppose next that \( B \) is Morita equivalent to \( \Lambda = \text{Q}(3\mathcal{B}) = k[3\mathcal{B}] / I_{Q(3\mathcal{B})} \) where the quiver \( 3\mathcal{B} \) and the ideal \( I_{Q(3\mathcal{B})} \) are as in Figure 4. Note that \( n \geq 4 \).

**Figure 4.** The quiver and relations for \( \Lambda = \text{Q}(3\mathcal{B}) = k[3\mathcal{B}] / I_{Q(3\mathcal{B})} \).

\[
3\mathcal{B} = \begin{array}{c}
0 - \gamma & 0 - \delta \\
\gamma - \beta & 0 - \beta
\end{array}
\]

\[
I_{Q(3\mathcal{B})} = \langle \gamma \beta - \alpha^2 \eta^{n-2} - 1, \alpha \gamma - \gamma \eta \delta (\beta \gamma \eta \delta), \beta \alpha - \eta \delta \beta (\gamma \eta \delta \beta), \\
\delta \eta \delta - \delta \beta (\gamma \eta \delta \beta), \eta \delta \gamma - \beta \gamma (\delta \beta \gamma \eta), \beta \alpha^2, \delta \eta \delta \beta \rangle.
\]

Let \( e_0, e_1 \) and \( e_2 \) denote the images of the primitive idempotents of \( k[3\mathcal{B}] \) corresponding to the vertices 0, 1 and 2, respectively. Let \( S_0, S_1 \) and \( S_2 \) denote representatives of the isomorphism classes of simple \( \Lambda \)-modules. The projective indecomposable \( \Lambda \)-modules are pictured in Figure 5 where we use the short-hand 0, 1, 2 to denote \( S_0, S_1, S_2 \), respectively.

Suppose \( M \) is a non-simple \( \Lambda \)-module such that \( \text{End}_\Lambda(M) \cong k \). Then \( M/\text{rad}(M) \) and \( \text{soc}(M) \) do not have any composition factors in common. We first prove the following two auxiliary statements:

\[
(4.7) \quad \text{If } x \in M, \text{ then } (\delta \eta) x = 0.
\]

\[
(4.8) \quad \text{If } y \in M - \text{rad}(M), \text{ then } \alpha y = 0.
\]

The statement (4.7) is proved similarly to the statement (4.1). To prove (4.8), suppose, by contradiction, that there exists \( y \in M - \text{rad}(M) \) such that \( \alpha y \neq 0 \). Since \( \alpha y = \alpha e_1 y \),
we replace \( y \) by \( e_1 y \) to be able to assume that \( e_1 y = y \). This implies in particular that \( S_1 \) is a direct summand of \( M/\text{rad}(M) \). If \( \alpha^2 y \neq 0 \), then there exists an integer \( a \geq 2 \) with \( \alpha^a y \neq 0 \) and \( \alpha^{a+1} y = 0 \). This means that \( \alpha^a y \) lies in the socle of \( M \), implying that \( S_1 \) is a direct summand of \( \text{soc}(M) \), contradicting \( \text{End}_\Lambda(M) \cong k \). Hence \( \alpha^2 y = 0 \). If \( (\beta \alpha) y = 0 \), then \( \alpha y \) lies in the socle of \( M \), again implying that \( S_1 \) is a direct summand of \( \text{soc}(M) \). Therefore, \( \alpha^2 y = 0 \) and \( (\beta \alpha) y \neq 0 \). But then it follows from the relations in \( \Lambda \) from Figure 4 that \( (\eta \delta \beta (\gamma \eta \delta \beta)) y \) is also not zero in \( M \). Since \( \alpha^2 y = 0 \), this implies that \( \Lambda y \), which is a submodule of \( M \), is isomorphic to a quotient module of \( P_1 \) of the form

\[
\begin{array}{ccc}
1 & 1 & 0 \\
2 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}
\]

Therefore, using the notation from Definition 5.3, \( S_{10} \) is isomorphic to a submodule of \( M \). On the other hand, \( \text{Ext}_\Lambda^i(S_1, \Lambda y) = 0 \) unless \( i = 0 \), which implies that \( S_{10} \) is also a quotient module of \( M \). But this means that \( M \) has a non-zero endomorphism factoring through \( S_{10} \), contradicting \( \text{End}_\Lambda(M) \cong k \). This proves (4.8).

Note that (4.8) implies that \( S_{11} \) is not isomorphic to a submodule of \( M \) and (4.7) implies that \( S_{202} \) is not isomorphic to either a submodule or a quotient module of \( M \).

Depending on which of \( S_0, S_1, S_2 \) are direct summands of \( M/\text{rad}(M) \) and \( \text{soc}(M) \), we obtain different possibilities for \( M \). There are altogether twelve different possibilities for \( M/\text{rad}(M) \) and \( \text{soc}(M) \). To illustrate our arguments, we now consider two of these cases:

(4.9) \( M/\text{rad}(M) \cong (S_1)^r \) and \( \text{soc}(M) \cong (S_0)^s \) for certain \( r, s \in \mathbb{Z}^+ \), and

(4.10) \( M/\text{rad}(M) \cong (S_1)^r \oplus (S_2)^s \) and \( \text{soc}(M) \cong (S_0)^t \) for certain \( r, s, t \in \mathbb{Z}^+ \).

Suppose first that \( M \) satisfies (4.9). By (4.8), \( M \) does not surject onto \( S_{11} \), which implies that

(4.11) \( M/\text{rad}^2(M) \cong (S_1/S_0)^{r_1} \oplus (S_1)^{r_2} \),

(4.12) \( \text{soc}_2(M) \cong (S_1/S_0)^{s_1} \oplus (S_2/S_0)^{s_2} \oplus (S_1/S_0)^{s_3} \oplus (S_0)^{s_4} \).
for certain non-negative $r_i, s_j$, where $r_1$ and at least one of $s_1, s_2, s_3$ must be positive. This implies that the $k$-dimension of $\text{End}_A(M)$ is at least 2 unless $M \cong S_{10}$ or

$$\text{soc}_2(M) \cong \left( \begin{array}{c} S_2 \\ S_0 \end{array} \right)^{s_2} \oplus (S_0)^{s_4}$$

(4.13)

where $s_2 > 0$. Hence we can concentrate on the case when $M$ satisfies both (4.11) and (4.13). In this case, we have

$$M/\text{rad}^3(M) \cong \left( \begin{array}{c} S_1 \\ S_0 \end{array} \right)^{r_3} \oplus \left( \begin{array}{c} S_1 \\ S_0 \end{array} \right)^{r_4} \oplus (S_1)^{r_5},$$

where $r_3, s_5 > 0$. By (4.13), this then implies that

$$M/\text{rad}^4(M) \cong \left( \begin{array}{c} S_1 \\ S_0 \\ S_2 \\ S_0 \end{array} \right)^{r_6} \oplus \left( \begin{array}{c} S_1 \\ S_0 \\ S_2 \\ S_1 \end{array} \right)^{r_7} \oplus \text{(modules of length \leq 3)},$$

(4.16)

$$\text{soc}_4(M) \cong \left( \begin{array}{c} S_1 \\ S_0 \\ S_2 \\ S_0 \end{array} \right)^{s_8} \oplus \text{(modules of length \leq 3)}$$

(4.17)

where $s_8$ and at least one of $r_6, r_7$ is positive. Using additional Ext$^1$ arguments, we see that $M$ cannot have a quotient module that has radical length 5 and that surjects onto

$$X = \frac{S_0}{S_2} \frac{S_1}{S_0}.$$  

Since the endomorphism ring of $X$ has $k$-dimension 2, this implies that $r_6$ must be positive. Therefore, it follows that $M \cong S_{1020}$, since $M$ always has a non-zero endomorphism factoring through this module. Summarizing, if $M$ satisfies (4.9), then $M$ is isomorphic either to $S_{11}$ or to $S_{1020}$.

Next suppose that $M$ satisfies (4.10). Since $M$ does not surject onto $S_{11}$ by (4.8), we obtain

$$M/\text{rad}^2(M) \cong \left( \begin{array}{c} S_1 \\ S_0 \end{array} \right)^{t_1} \oplus (S_1)^{r_2} \oplus \left( \begin{array}{c} S_2 \\ S_0 \end{array} \right)^{s_1} \oplus (S_2)^{s_2} \oplus \left( \begin{array}{c} S_1 \\ S_0 \end{array} \right)^{t_3} \oplus (S_0)^{t_4},$$

(4.18)

$$\text{soc}_2(M) \cong \left( \begin{array}{c} S_1 \\ S_0 \\ S_0 \end{array} \right)^{t_1} \oplus \left( \begin{array}{c} S_2 \\ S_0 \\ S_0 \end{array} \right)^{t_2} \oplus \left( \begin{array}{c} S_1 \\ S_0 \\ S_0 \end{array} \right)^{t_3} \oplus (S_0)^{t_4}$$

(4.19)

for certain non-negative $r_1, s_1, t_k, u$, where at least one of $r_1, r_2, u$ and at least one of $s_1, s_2, u$ and at least one of $r_1, s_1, u$ and at least one of $t_1, t_2, u$ must be positive. If $t_3$ is positive, then either $M \cong T_1 \oplus S_0$ or the $k$-dimension of $\text{End}_A(M)$ is at least 2. Hence we can concentrate on the case when $t_3 = 0$. In particular, $M$ has radical length at least 3. By (4.8) and (4.7), it follows that

$$M/\text{rad}^3(M) \cong \left( \begin{array}{c} S_1 \\ S_0 \end{array} \right)^{r_3} \oplus \left( \begin{array}{c} S_2 \\ S_0 \end{array} \right)^{s_3} \oplus \text{(modules of radical length \leq 2)}$$

(4.20)

where at least one of $r_3, s_3$ is positive. Therefore, we see that the $k$-dimension of $\text{End}_A(M)$ is at least 2 unless either $s_3 = 0 = t_1 = t_3$ or $r_3 = 0 = t_2 = t_3$. In the first of these two cases
we can argue similarly as in the case when $M$ satisfies (4.9) to see that $M$ has a non-zero endomorphism factoring through $S_{1020}$. In the second case, additional Ext arguments show that $M$ has a non-zero endomorphism factoring through $S_{2010}$. Summarizing, if $M$ satisfies (4.10), then $M \cong T_{1000}$. This concludes the proof of the two cases when $M$ satisfies either (4.9) or (4.10). Hence this completes the case when $B$ is Morita equivalent to $Q(3B)$.

\[ \Box \]

5. Universal deformation rings

We assume Hypothesis 3.1 and that $B$ is non-local. In this section, we determine the universal deformation ring of every $kG$-module $V$ belonging to $B$ whose endomorphism ring is isomorphic to $k$. In particular, this together with Lemma 3.2 proves Theorem 1.1. We use the lists $\mathcal{E}$ and $\mathcal{E}_A$ obtained in Proposition 4.4. Define the following 4 sublists of $\mathcal{E}$:

1. the sublist $\mathcal{E}_1$ of $\mathcal{E}$ consisting of those modules $V$ such that $\operatorname{Ext}^1_{kG}(V,V) \neq 0$ and the 2-modular character of $V$ is equal to the restriction to the 2-regular conjugacy classes of an ordinary irreducible character of $G$ of height 1;
2. the sublist $\mathcal{E}_2$ of $\mathcal{E}$ consisting of those modules $V$ such that $\operatorname{Ext}^1_{kG}(V,V) \neq 0$ and $V$ does not belong to $\mathcal{E}_1$;
3. the sublist $\mathcal{E}_3$ of $\mathcal{E}$ consisting of those modules $V$ such that $\operatorname{Ext}^1_{kG}(V,V) = 0$ and $V$ belongs to a 3-tube of the stable Auslander-Reiten quiver of $B$;
4. the sublist $\mathcal{E}_4$ of $\mathcal{E}$ consisting of those modules $V$ such that $\operatorname{Ext}^1_{kG}(V,V) = 0$ and $V$ does not belong to $\mathcal{E}_3$.

For $i \in \{1,2,3,4\}$, let $\mathcal{E}_{A,i}$ be the set of $A$-modules in $\mathcal{E}_A$ that correspond to the modules in $\mathcal{E}_i$ under the Morita equivalence between $B$ and $A$.

The following lemma describes the modules in each of these sublists.

Lemma 5.2. Assume Hypothesis 3.1 and that $B$ is non-local. Let $\Lambda = kQ/I$ be a basic algebra such that $B$ is Morita equivalent to $\Lambda$, where we assume $\Lambda$ is one of the algebras in Figure 1. Let $\mathcal{E}$ and $\mathcal{E}_{A,1}, \mathcal{E}_{A,2}, \mathcal{E}_{A,3}, \mathcal{E}_{A,4}$ be as in Definition 5.1.

(i) If $Q = 2A$, then $\mathcal{E}_{A,1} = \{S_{001}, S_{100}\}$ and $\mathcal{E}_{A,2} = \{S_0, S_{01}, S_{10}\}$. If $\Lambda \in \{D(2A), SD(2A)_{1/2}(c)\}$ then $\mathcal{E}_{A,3} = \{S_1\}$, and if $\Lambda \in \{SD(2A)_{1/2}(c), Q(2A)(c)\}$ then $\mathcal{E}_{A,3} = \emptyset$.

If $Q = 2B$ and $A \not\in \{SD(2B)_{1/2}(c), Q(2B)_{1/2}(p,a,c)\}$, then $\mathcal{E}_{A,1} = \{S_1\}$ and $\mathcal{E}_{A,2} = \{S_0, S_{01}, S_{10}\}$. If $\Lambda \in \{D(2B), SD(2B)_{1/2}(c)\}$ then $\mathcal{E}_{A,3} = \{S_{001}, S_{100}\}$, and if $\Lambda \in \{SD(2B)_{1/2}(c), Q(2B)_{1/2}(c)\}$ then $\mathcal{E}_{A,3} = \emptyset$. If $\Lambda \in \{SD(2B)_{1/2}(c), Q(2B)_{1/2}(p,a,c)\}$, then $\mathcal{E}_{A,1} = \{S_{001}, S_{101}, S_{10}\}$, $\mathcal{E}_{A,2} = \{S_0, S_1\}$ and $\mathcal{E}_{A,3} = \emptyset$.

(ii) If $Q = 3A$, then $\mathcal{E}_{A,1} = \emptyset = \mathcal{E}_{A,2}$ in the cases when $n = 2$ or when $n = 3$ and $D$ is quaternion, and $\mathcal{E}_{A,1} = \{S_{0102}, S_{0110}, S_{1001}, S_{1020}\}$ and $\mathcal{E}_{A,2} = \emptyset$ in all other cases. If $\Lambda = D(3A)_1$, then $\mathcal{E}_{A,3} = \{S_1, S_2, S_{0102}, S_{0110}, S_{1001}, S_{1020}\}$ in the case when $n = 2$, and $\mathcal{E}_{A,3} = \{S_1, S_2\}$ in the case when $n \geq 3$. If $\Lambda = D(3A)_1$, then $\mathcal{E}_{A,3} = \{S_1\}$, and if $\Lambda = Q(3A)_2$ then $\mathcal{E}_{A,3} = \emptyset$.

If $Q = 3B$, then $\mathcal{E}_{A,1} = \{S_1\}$. If $Q = 3B$ and $A \neq SD(3B)_{1/2}$, then $\mathcal{E}_{A,2} = \emptyset$. If $\Lambda = D(3B)_1$ then $\mathcal{E}_{A,3} = \{S_2, S_{0102}, S_{0110}, S_{1001}, S_{1020}\}$, and if $\Lambda = SD(3B)_{1/2}$ then $\mathcal{E}_{A,3} = \{S_{0102}, S_{1020}\}$.
and if $\Lambda = Q(3E)$ then $\mathcal{E}_{\Lambda,3} = \emptyset$. If $\Lambda = SD(3E)_2$, then $\mathcal{E}_{\Lambda,2} = \{S_{0102}, S_{2010}\}$ and $\mathcal{E}_{\Lambda,3} = \{S_2\}$.

(iii) If $Q = 3C$, then $\mathcal{E}_{\Lambda,3} = \emptyset$. If $\Lambda = SD(3C)_{2,1}$, then $\mathcal{E}_{\Lambda,1} = \{S_0\}$ and $\mathcal{E}_{\Lambda,2} = \{T_{0,1\oplus 2}, T_{1\oplus 2,0}\}$. If $\Lambda = SD(3C)_{2,2}$, then $\mathcal{E}_{\Lambda,1} = \{S_{0102}, S_{2011}, T_{0,1\oplus 2}, T_{1\oplus 2,0}\}$ and $\mathcal{E}_{\Lambda,2} = \{S_0\}$.

(iv) If $Q = 3D$, then $\mathcal{E}_{\Lambda,1} = \{S_1\}, \mathcal{E}_{\Lambda,2} = \{S_2\}$ and $\mathcal{E}_{\Lambda,3} = \{S_{0102}, S_{2010}\}$.

(v) If $Q = 3H$, then $\mathcal{E}_{\Lambda,3} = \{S_{20}\}$. If $\Lambda = SD(3H)_1$, then $\mathcal{E}_{\Lambda,1} = \{S_{12}, S_{21}\}$ and $\mathcal{E}_{\Lambda,2} = \{S_0\}$.

(vi) If $Q = 3K$, then $\mathcal{E}_{\Lambda,1} = \emptyset = \mathcal{E}_{\Lambda,2}$ in the cases when $n = 2$ or when $n = 3$ and $D$ is quaternion, and $\mathcal{E}_{\Lambda,1} = \{S_{12}, S_{21}\}$ and $\mathcal{E}_{\Lambda,2} = \emptyset$ in all other cases. If $\Lambda = D(3K)$, then $\mathcal{E}_{\Lambda,3} = \{S_{01}, S_{10}, S_{12}, S_{21}, S_{02}, S_{20}\}$ in the case when $n = 2$, and $\mathcal{E}_{\Lambda,3} = \{S_{01}, S_{10}, S_{02}, S_{20}\}$ in the case when $n \geq 3$. If $\Lambda = Q(3K)$ then $\mathcal{E}_{\Lambda,3} = \emptyset$.

In all cases, $\mathcal{E}_{\Lambda,4} = \mathcal{E}_\Lambda - (\mathcal{E}_{\Lambda,1} \cup \mathcal{E}_{\Lambda,2} \cup \mathcal{E}_{\Lambda,3})$. Moreover, $\dim_k \text{Ext}_k^1(M,M) \in \{0,1\}$ for all $M \in \mathcal{E}_\Lambda$.

Proof. Lemma 5.2 is proved using the description of each basic algebra $\Lambda$ in Figure 1 as provided in [5] Sect. 4. Using this description, we can readily determine the $k$-dimension of $\text{Ext}_k^1(M,M)$ for all modules $M \in \mathcal{E}_\Lambda$. In particular, we see that $\dim_k \text{Ext}_k^1(M,M) \in \{0,1\}$ for all such $M$. The modules in $\mathcal{E}_{\Lambda,1}$ have already been determined in [5] Lem. 6.1. Note that the cases when $n = 2$, respectively $n = 3$ and $D$ is quaternion, play a special role, since in these cases $\mathcal{E}_{\Lambda,1} = \emptyset$. The modules $M \in \mathcal{E}_\Lambda - \mathcal{E}_{\Lambda,1}$ with $\text{Ext}_k^1(M,M) \neq 0$ then provide $\mathcal{E}_{\Lambda,2}$. If $\Lambda$ is of quaternion type, the stable Auslander-Reiten quiver $\Gamma_s(\Lambda)$ of $\Lambda$ does not contain any 3-tubes, which implies that $\mathcal{E}_{\Lambda,3} = \emptyset$. If $\Lambda$ is of dihedral type, then $\Gamma_s(\Lambda)$ always contains at least one 3-tube and the modules in $\mathcal{E}_{\Lambda,3}$ have been determined, for example, in [3] Sect. 4 and [8] Sect. 5. If $\Lambda$ is of semidihedral type, we consider the $\Omega^2$ orbit of the $\Lambda$-modules $M \in \mathcal{E}_\Lambda$ with $\text{Ext}_k^1(M,M) = 0$ to determine $\mathcal{E}_{\Lambda,3}$. It is obvious that $\mathcal{E}_{\Lambda,4} = \mathcal{E}_\Lambda - (\mathcal{E}_{\Lambda,1} \cup \mathcal{E}_{\Lambda,2} \cup \mathcal{E}_{\Lambda,3})$, which completes the proof of Lemma 5.2. \end{proof}

Using the sublists of $\mathcal{E}$ from Definition 5.1 we can now determine the universal deformation ring for every module $V$ in $\mathcal{E}$. For the modules $V \in \mathcal{E}_1$, the universal deformation ring depends on whether or not $V$ corresponds to a 3-tube, as defined in [5] Def. 6.3. These $V$ were explicitly determined in [5] Lem. 6.4.

**Theorem 5.3.** Assume Hypothesis 5.1 and that $B$ is non-local. Let $\mathcal{E}$ and $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4$ be as in Definition 5.1.

(a) Suppose $V \in \mathcal{E}_1$, and let $q_n(t) \in W[[t]]$ be the monic polynomial of degree $2^{n-2} - 1$ from [5] Def. 5.3. If $V$ corresponds to a 3-tube, as defined in [5] Def. 6.3, then $R(G,V) \cong W[[t]]/(t q_n(t), 2 q_n(t))$. Otherwise $R(G,V) \cong W[[t]]/(q_n(t))$.

(b) Suppose $V \in \mathcal{E}_2$. If $Q \in \{2A, 2B\}$ then $R(G,V) \cong W[[t]]/(t^2 - 2 t \mu t)$ for some non-zero $\mu \in W$. Otherwise, $R(G,V) \cong W[[t]]/(t^2, 2 t)$.

(c) If $V \in \mathcal{E}_3$ then $R(G,V) \cong k$.

(d) If $V \in \mathcal{E}_4$ then $R(G,V) \cong W$.

In all cases, the ring $R(G,V)$ is isomorphic to a subquotient ring of $WD$.

Proof. Recall that $p = 2$.

Part (a) of Theorem 5.3 follows from [5] Thm. 6.6. Part (c) follows in the case when $D$ is dihedral from [3] Sect. 5.2 and [8] Prop. 6.3, and in the case when $D$ is semidihedral by using similar arguments as in the proof of [5] Prop. 6.5.

To prove parts (b) and (d), let $\Lambda = kQ/I$ be a basic algebra such that $B$ is Morita equivalent to $\Lambda$, where we assume $\Lambda$ is one of the algebras in Figure 1.

To prove part (b), suppose that $V \in \mathcal{E}_2$. Since $\text{Ext}_k^1_{kG}(V,V) \cong k$, $R(G,V)$ is isomorphic to a quotient algebra of $W[[t]]$. Note that each $V$ has either a simple radical quotient or a simple socle. Considering the submodules and quotient modules of the projective indecomposable $B$-modules, we see that there is a unique $B$-module $U$, up to isomorphism, such that we have a short exact sequence

$$0 \to V \xrightarrow{\iota} U \xrightarrow{\pi} V \to 0.$$
Therefore, \( \mathcal{U} \) defines a lift of \( V \) over \( k[t]/(t^2) \) where we let \( t \) act as the composition \( \iota \circ \pi \). Moreover, we see that \( \iota (V) \) is the unique submodule of \( \mathcal{U} \) that is isomorphic to \( V \), and \( \pi \) induces a \( kG \)-module isomorphism \( \varphi : \mathcal{U}/\iota (V) \to V \). Since \( \text{Ext}_{kG}^1(\mathcal{U}, V) = 0 \) and since the kernel of every surjective \( kG \)-module homomorphism \( \mathcal{U} \to V = \text{equal to } \iota (V) \), we can argue as in the proof of \([4]\) Lemma 2.5] to show that \( R(G, V)/2R(G, V) \) is isomorphic to \( k[t]/(t^2) \) and that the universal mod 2 deformation of \( V \) is given by the isomorphism class of \( \mathcal{U} \). Using the decomposition matrices provided in \([5]\) Appendix] together with \([17]\) Prop. (23.7)], we see that \( V \) always has at least one lift over \( W \). Therefore, it follows by \([7]\) Lem. 2.1 that \( R(G, V) \cong W[[t]]/(t(t - 2 \mu), a^{2m}t) \) for certain \( \mu \in W, a \in \{0, 1\} \) and \( m \in \mathbb{Z}^+ \) depending on \( V \).

In the case when \( Q \in \{2A, 2B\} \), the decomposition matrix of \( B \) together with \([17]\) Prop. (23.7)] show that \( V \) has 2 non-isomorphic lifts over \( W \), which implies that \( \mu \neq 0 \) and \( a = 0 \). In other words, \( R(G, V) \cong W[[t]]/(t^2 - 2 \mu t) \) for some non-zero \( \mu \in W \).

On the other hand, if \( Q \notin \{2A, 2B\} \) then the defect groups of \( B \) must be semidihedral. Moreover, \( \mathcal{U} \) lies at the end of a 3-tube and the stable endomorphism ring of \( \mathcal{U} \) is isomorphic to \( k \). If \( a = 0 \) then \( R(G, V) \cong W[[t]]/(t(t - 2 \mu)) \) is free over \( W \). If \( a = 1 \) then \( R(G, V)/2mR(G, V) \cong (W/2mW)[[t]]/(t(t - 2 \mu)) \) is free over \( W/2mW \). Therefore it follows that if \( a = 0 \) (resp. \( a = 1 \)), then there is a lift of \( \mathcal{U} \), when regarded as a \( kG \)-module, over \( W \) (resp. \( W/2mW \)). However, arguing similarly as in the proof of \([3]\) Prop. 6.5], we see that \( R(G, \mathcal{U}) \cong k \), which means we must have \( a = 1 \) and \( m = 1 \). This proves part (b) of Theorem 5.3.

To prove part (d), suppose that \( V \in \mathcal{E}_4 \). Since \( \text{Ext}_{kG}^1(V, V) = 0 \), \( R(G, V) \) is isomorphic to a quotient algebra of \( W \). Since \( V \) is of length at most 4 and has either a simple radical quotient or a simple socle, we can use the decomposition matrix of \( B \) provided in \([5]\) Appendix] together with \([17]\) Prop. (23.7)] to see that \( V \) has a lift over \( W \). This implies \( R(G, V) \cong W \).

The last statement of the theorem is obvious for parts (c) and (d). For part (a), this follows from \([3]\) Lem. 5.5]. For part (b), this follows since \( W[[t]]/(t^2 - 2 \mu t) \) is isomorphic to a subalgebra (resp. quotient algebra) of \( W[\mathbb{Z}/2] \cong W[[t]]/(t^2 - 2t) \).

This completes the proof of Theorem 5.3. \( \square \)

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F.B.: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, IA 52242-1419, U.S.A.
E-mail address: frauke-bleher@uiowa.edu

G.L.: DEPARTMENT OF MATHEMATICS, CSU SAN BERNARDINO, CA 92407-2397, U.S.A.
E-mail address: gllosent@csusb.edu

J.S.: DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, DICKINSON COLLEGE, CARLISLE, PA 17013, U.S.A.
E-mail address: schaej@dickinson.edu