Line defects in three dimensional mirror symmetry beyond ADE quivers

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ABSTRACT: Understanding the map of line defects in a Quantum Field Theory under a given duality is generically a difficult problem. This paper is the second in a series which aims to address this question in the context of 3d \( \mathcal{N} = 4 \) mirror symmetry. A general prescription for constructing vortex defects and their mirror maps in quiver gauge theories beyond the A-type was presented by the author in an earlier paper [1], where specific examples involving D-type and affine D-type quivers were discussed. In this paper, we apply the aforementioned prescription to construct a family of vortex defects as coupled 3d-1d systems in quiver gauge theories beyond the ADE-type, and study their mirror maps. Specifically, we focus on a class of quiver gauge theories involving unitary gauge nodes with edge multiplicity greater than 1, i.e. two gauge nodes in these theories may be connected by multiple bifundamental hypermultiplets. Quiver gauge theories of this type arise as 3d mirrors of certain Argyres-Douglas theories compactified on a circle. Some of these quiver gauge theories are known to have a pair of 3d mirrors, which are themselves related by an IR duality, discussed recently in [2]. For a concrete example where a pair of 3d mirrors do exist, we study how the vortex defects constructed using our prescription map to Wilson defects in each mirror theory.

KEYWORDS: Field Theories in Lower Dimensions, Supersymmetric Gauge Theory, Supersymmetry and Duality, Wilson, ’t Hooft and Polyakov loops

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1 Introduction and summary of results

1.1 Background and the basic idea of the paper

Line defects constitute a very important class of observables in Quantum Field Theories, and encode a wealth of information about the physics of the system. In particular, they are the charged operators under the one-form symmetries. In supersymmetric theories, the BPS line defects are associated with a rich set of algebraic and geometric structures that have been studied in various space-time dimensions.

A rather ubiquitous feature of Quantum Field Theories is the existence of UV/IR dualities — a generic name for the phenomena where a set of theories with completely
different descriptions at a certain energy scale flow to the same physical theory at some other energy scale. Among other things, these dualities provide an extremely useful tool to probe strongly-coupled physics from a field theory perspective.

Given the importance of these two widely studied subjects in our understanding of Quantum Field Theories, one is naturally led to the question as to how line defects transform across a given duality. It turns out that determining the map of line defects in a Quantum Field Theory under a given duality is generically a difficult problem. For example, the map of a gauge Wilson defect, which is labelled by a representation of the gauge group, to a dual defect may be extremely non-trivial, given the fact that the dual theory in general has a different gauge group. The current work is the second paper (after [1]) in a series which addresses this question in the context of 3d $\mathcal{N} = 4$ mirror symmetry [3] for generic quiver gauge theories. For this specific IR duality, half-BPS Wilson defects are expected to map to half-BPS vortex defects, although writing down the precise map of the operators in a generic dual pair is a challenging task. Even for the case of $A$-type quivers (i.e. linear quivers with unitary gauge groups), for which mirror symmetry has long been understood as $S$-duality in a simple Type IIB setting [4], the problem of mapping half-BPS defects was solved fairly recently [5]. The Type IIB construction of [4] was suitably extended in [5] to incorporate the defects.

In a recent paper [1], the author presented a general procedure for constructing vortex defects in quiver gauge theories beyond the $A$-type and studying their maps under 3d mirror symmetry to Wilson defects. The construction involves a certain generalization of the $S$-operation of Witten [6], which was defined as an $S$-type operation in [7]. As discussed in [1], these $S$-type operations can be used to construct vortex defects and the associated mirror maps in a large class of non-linear quiver gauge theories, starting from the vortex defects in the $A$-type (linear) quivers. The procedure in question is entirely field theoretic, and does not rely on any String Theory realization of the quiver gauge theory. This procedure was used in [1] to study the line defects and the associated mirror maps in quiver gauge theories of the $D$-type and the affine $D$-type.

In the present paper, we initiate the study of half-BPS line defects and their mirror maps in quiver gauge theories beyond the $ADE$ type. For the sake for specificity, we will focus on a class of quiver gauge theories involving unitary gauge nodes, such that a pair of nodes maybe connected by multiple bifundamental hypermultiplets. We will refer to such theories as quivers with edge multiplicity greater than one. Quiver gauge theories of this type are known to arise as mirrors of certain 3d SCFTs [8–12], which are obtained by the circle-compactification of Argyres-Douglas theories [13–15] and flowing to the IR. We construct a certain family of vortex defects in these theories as coupled 3d-1d systems, and study their mirror maps.

It has been shown recently [2] that there is generically a pair of 3d mirrors for a given quiver of the above type (i.e. with edge multiplicity greater than one) associated with an Argyres-Douglas theory.1 One of the mirrors is a unitary quiver gauge theory of non-

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1In other words, the 3d SCFT obtained by the circle reduction of the Argyres-Douglas theory has two different Lagrangian descriptions, in addition to having a Lagrangian 3d mirror.
ADE type with matter in fundamental/bifundamental representation as well as certain hypermultiplets in the determinant/anti-determinant representation of the gauge group — we will refer to this theory as the “unitary mirror” (or, $U$-mirror). The other mirror is a linear chain of unitary and special unitary gauge nodes with fundamental/bifundamental matter [16, 17] — we will refer to this theory as the “unitary-special unitary mirror” (or, $U - SU$-mirror). The two mirrors are related by an IR duality, which was studied in [2]. We will discuss how the vortex defects that we construct in specific examples map to Wilson defects in the $U$-mirror as well as the $U - SU$-mirror.

The paper is organized as follows. In section 2.1, we summarize the quiver notation and briefly review the vortex defects for the $A$-type quivers. We then introduce the 3d-1d coupled quivers that realize vortex defects for gauge nodes connected by edges with higher multiplicity, in section 2.2. In section 3, we construct these vortex defects in an Abelian quiver gauge theory using the $S$-type operations and derive their mirror maps. The Abelian quiver in question is a generalized version of the complete graph quiver associated with the $(A_2, A_{3p - 1})$ theory. Next, we construct vortex defects in a non-Abelian quiver gauge theory which is associated with the $D_9(SU(3))$ Argyres-Douglas theory. This theory admits a $U$-mirror as well as a $U - SU$-mirror. In section 4.1–4.2, we study the mirror maps of the vortex defects in the $U$-mirror, and extend them to the $U - SU$-mirror in section 4.3. A brief summary of the $S$-type operations is given in appendix B, while various details of the Witten index and sphere partition function computations can be found in appendix A, appendix C, and appendix D.

1.2 Summary of results

The main results of the paper are summarized as follows:

**Coupled quivers with higher edge multiplicity.** Consider a generic 3d $\mathcal{N} = 4$ quiver $\mathcal{T}$ with unitary gauge nodes and at least a single edge of multiplicity $p > 1$.

![Diagram](image)

The thick edge connecting the $U(N)$ and $U(M)$ gauge nodes labelled by an integer $p$ denotes $p$ hypermultiplets in the bifundamental representation of $U(N) \times U(M)$. We introduce a class of vortex defects for the $U(N)$ gauge node in $\mathcal{T}$ which are realized by the following pair of coupled 3d-1d quivers $(\mathcal{T}[V_{Q,R,p'}^-])$ and $(\mathcal{T}[V_{Q,R,p'}^+])$, where the 1d quiver is a $(2,2)$ gauged SQM.
The coupled quivers above have some additional ingredients, compared to the more familiar quivers in [5]. Consider the quiver \((\mathcal{T}[V_{Q,R,p'}^{(I)}])\) for example. We show in red the 1d (2,2) multiplets as well as the \(p'\) 3d bifundamental hypermultiplets which couple to the 1d chiral multiplets by a cubic superpotential of the form (2.7). A red, thin directed line denotes a single 1d chiral multiplet, while a red, thick directed line labelled by an integer \(p'\) denotes \(p'\) chiral multiplets. The red node \(\Sigma\) denotes a 1d linear quiver of the form given in figure 5.

The coupled quiver is realized by identifying the \([U(N) \times U(Q) \times U(M) \times U(p')]_{1d}\) flavor symmetry of the SQM to some \([U(N) \times U(Q) \times U(M) \times U(p')]_{3d}\) flavor/gauge symmetry of the 3d theory, via the cubic superpotential. The identification of the \(U(N) \times U(Q) \times U(M)\) factor works in the standard fashion [5], while \(U(p')_{1d}\) is identified with the \(U(p')_{3d} \subset U(p)_{3d}\) flavor symmetry associated with \(p'\) of the 3d bifundamental hypers. Finally, the superscripts \(\mp\) in the quivers \((\mathcal{T}[V_{Q,R,p'}^{(I)}])\) and \((\mathcal{T}[V_{Q,R,p'}^{(II)}])\) imply that the 1d FI parameters should be chosen to be negative definite and positive definite respectively.

For a detailed description of the quiver notation and the associated superpotential, we refer the reader to section 2.1–2.2.

**Map of defects under mirror symmetry.** We construct vortex defects in specific examples of quivers with edges of higher multiplicity. The generalities of the construction, which involves implementing a sequence of the \(S\)-type operations on vortex defects in \(A\)-type quivers, is summarized in appendix B. We determine the map of these vortex defects to Wilson defects under mirror symmetry. For simplifying our presentation, we will restrict ourselves to examples of 3d quivers \(\mathcal{T}\) where \(M = N = 1\) (see the figures above), and the other gauge nodes being generically non-Abelian.

The first example we study is an Abelian quiver pair of the form:

\[
\begin{align*}
1 & \quad \cdots \quad 1 \\
\cdots & \quad \cdots \quad \cdots \\
1 & \quad \cdots \quad 1 \\
\end{align*}
\]

For the special case of \(n = 2p\) and \(l = p\), the quiver \(X'\) reproduces the complete graph quiver with three vertices and edge multiplicity \(p\) — the 3d mirror of the \((A_2, A_{3p-1})\) AD
theory reduced on a circle. The mirror theory $Y'$ was first found in [7], and discussed further in [2].

We find a concrete example of a vortex defect of the form described above and its mirror dual in this case, as shown below.

The dual defect in this case is a Wilson defect of charge $k$ for the U(1) gauge shown in the figure on the right. The details of the computation for the above result and the related notation can be found in section 3.

The second example we study is a non-Abelian quiver pair of the following form (the numbers above the gauge nodes are labels with no physical significance).

The theory $X'$ is a non-ADE-type quiver gauge theory with unitary gauge groups and hypermultiplets in fundamental/bifundamental representation of the gauge group. In particular, the gauge nodes U(1)$_1$ and U(1)$_2$ are connected by an edge of multiplicity 2. The dual theory $Y'$ is also a non-ADE-type quiver gauge theory built out of unitary gauge nodes, and fundamental/bifundamental matter. In addition, $Y'$ has a single hypermultiplet which transforms in the determinant representation of the U(2) gauge group and has charge 1 under one of the adjacent U(1) gauge nodes, as denoted by the blue line in the figure.

The theory $X'$ is the 3d mirror associated with the circle reduction of the AD theory $D_9(SU(3))$. The quiver $Y'$ is the $U$-mirror of $X'$ (as described above), which was first found in [7] and discussed further in [2].

For the dual pair $(X', Y')$, we first construct a vortex defect in $X'$ that does not involve the bifundamental hypers of the edge with multiplicity 2. The coupled quiver for the vortex defect and its mirror dual are given as follows.
The vortex defect in this case is labelled by a representation $R$ of $U(2)$ which is encoded in the 1d quiver $\Sigma$ (see section 2.1 for details). The dual is a Wilson defect for the $U(2)$ gauge node in the theory $Y'$ labelled by the same representation $R$.

Next, we construct a vortex defect in $X'$ that involves the $U(1)_{1} \times U(1)_{2}$ bifundamental hypermultiplets. The defect and its dual are shown below. The dual involves a Wilson defect of charge $k$ and $-k$ for the gauge nodes labelled $U(1)_{1}$ and $U(1)_{3}$ respectively.

The details of the computation and the related notation are given in section 4.1–4.2.

**Hopping duality for the vortex defects.** A vortex defect can be realized by multiple 3d-1d quivers. At the level of supersymmetric observables, like the sphere partition function for example, the matrix models associated to such quivers can be mapped to one another by a simple change of variables. This is known as *hopping duality* [5]. The generic coupled systems $(T[V_{Q,R,g'}^{(I)}])$ and $(T[V_{Q,R,g'}^{(II)}])$, described above, are therefore hopping duals. For the examples of vortex defects discussed above, the associated hopping dualities are listed below.
Map of defects under IR duality. The non-Abelian quiver $X'$, discussed above, has a second Lagrangian mirror, which we refer to as the $U - SU$ mirror $Y''$.

In section 4, we work out the mirror maps for the vortex defects in $X'$, mentioned above, to Wilson defects in the $U - SU$ mirror $Y''$. The dual Wilson defect, in each case, is of the following form.
Figure 1. A quiver gauge theory with $G = U(N_1) \times U(N_2) \times SU(N_3) \times SU(N_4)$ and hypermultiplets in bifundamental and fundamental representations, with the conventions listed on the r.h.s.

2 Coupled quivers for half-BPS vortex defects

2.1 Linear quivers with unitary gauge groups

Let us begin by summarizing the 3d $\mathcal{N} = 4$ quiver gauge theory notation that we will be relevant for the rest of the paper, as given in figure 1.

The quivers we consider will also involve hypermultiplets transforming as powers of the determinant representation (or anti-determinant representation) of certain unitary factors in the quiver gauge group, which we will denote by the following notation:

The blue line connecting the gauge nodes $U(N_1)$, $U(N_2)$ and $U(N_\alpha)$ represents a single hypermultiplet transforming in powers of the (anti)determinant representations of the said
unitary groups. The precise powers are given by the charges \( \{Q_i = \pm k_i N_i\} \), where \( k_i \) is a positive integer, such that the hypermultiplet transforms in the \( k_i \)-th power of the determinant or anti-determinant representation of \( U(N_i) \), depending on whether \( Q_i \geq 0 \). Multiple copies of such a hypermultiplet will be represented by a thick blue line with an integer denoting the multiplicity. Similarly, \( P \) copies of a hypermultiplet transforming under the (anti)determinant representation of a single gauge node (and uncharged under the other gauge nodes) by a blue line connected to a blue flavor node.

Next, let us summarize certain basic facts about a vortex defect in the 3d theory. For linear quivers with unitary gauge groups, there exists a Type IIB construction that realizes a large class of vortex defects [5] in terms of coupled 3d-1d system. We will present a brief description of these coupled systems.

Given a 3d linear quiver \( \mathcal{T} \), there exists a pair of 3d-1d systems \( \mathcal{T}[V_{Q,R}^{r,l}] \), given in figure 2, which realize a vortex defect associated with the gauge node \( U(N) \). The 1d theory, which we will denote as \( \Sigma^{Q,R}_{r,l} \), is a (2,2) SQM and its form is explicitly shown in figure 3. The coupled system \( \mathcal{T}[V_{Q,R}^{r,l}] \) is labelled by an integer \( Q \) and a representation \( R \) of \( U(N) \).
perform the integration, and then finally set
respectively.

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the Witten index is evaluated. The Witten indices for the right and the left SQM associated

A prescription for the signs of the FI parameters determines the specific chamber in which

partition function arising from the 3d bulk fields,

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where the . . . in the superpotential denote additional terms which contain higher derivatives
of the complex scalar X [18], and the indices run over \(a = 1, \ldots, n_p, \beta = 1, \ldots, N_L, \alpha =
1, \ldots, N_R\).\(^2\) The cubic superpotential, which preserves the \((2,2)\) supersymmetry of the
system, reduces the flavor symmetry \((U(N_L) \times U(N_R))_{3d} \times (U(N_L) \times U(N_R))_{3d}\) to the
diagonal subgroup \(G_{3d-1d} = (U(N_L) \times U(N_R))_{3d-1d}\).

In the next step, one promotes the background vector multiplet for a subgroup of

\(G_{3d-1d}\) to a dynamical vector multiplet. For example, in constructing the 3d-1d system

\(T[V^r_{Q,R}]\), where \(N_L = N, N_R = M + Q\), one promotes the background vector multiplet
for \(U(N) \times U(M) \subset G_{3d-1d}\) to a dynamical vector multiplet. In contrast, for the system

\(T[V^l_{Q,R}]\), we have \(N_L = P + K - Q, N_R = N\), and we promote \(U(P) \times U(N) \subset G_{3d-1d}\) to
dynamical vector multiplet.

A given vortex defect may be realized by multiple 3d-1d coupled quivers — this is
referred to as the hopping duality [5]. Using the Type IIB description of the vortex defects
in linear quivers, it was demonstrated in [5] that the two coupled quivers \(T[V^r_{Q,R}]\) and
\(T[V^l_{Q,R}]\) in figure 2 describe the same vortex defect. \(T[V^r_{Q,R}]\) and \(T[V^l_{Q,R}]\) are therefore
hopping duals.

The hopping duality can also be inferred by studying the sphere partition functions of
these defects. The expectation values of the vortex defect \(V^l_{Q,R}\), in the theory \(T\) is given as:

\[
\langle V^l_{Q,R} \rangle_T = W_{b,g.}^{l/r}(t) \times \frac{1}{Z(T)(m; t)} \times \lim_{z \to 1} \int [ds] Z^{(T)}_{\text{int}}(s, m, t) \cdot I^{\Sigma^Q_R}_{l,r}(s, m, z | \xi \geq 0),
\]

where \(W_{b,g.}^{l/r}\) are certain background Wilson defects, \(Z^{(T)}_{\text{int}}\) is the integrand of the sphere
partition function arising from the 3d bulk fields, \(I^{\Sigma^Q_R}_{l,r}\) is the Witten index of the coupled
SQM, and the superscripts \(l, r\) denote the choice of the specific 3d-1d system (left or right).\(^3\)
A prescription for the signs of the FI parameters determines the specific chamber in which
the Witten index is evaluated. The Witten indices for the right and the left SQM associated

\(^2\)We have decomposed the 3d \(\mathcal{N} = 4\) bifundamental hypermultiplet into a pair of \(\mathcal{N} = 2\) chiral multiplets
— \(X\) and \(Y\), which transform in the bifundamental representation of \(U(N_L) \times U(N_R)\) and
\(U(N_R) \times U(N_L)\) respectively.

\(^3\)The correct prescription for taking the limit is to first analytically continue \(z \in i\mathbb{R}\) in the integrand,
perform the integration, and then finally set \(z = 1\).
Figure 5. Note that $N_L = N$, $N_R = M + Q$ for the right SQM. The SQM has a global symmetry $U(p') \times U(N_R) \times U(N_L)$.

with the vortex defect in figure 2 are explicitly given as

$$I^{\Sigma, Q, R} = \sum_{w \in R} \mathcal{F}^{r}(s, z) \frac{N}{\prod_{j=1}^{i=1} M} \cosh \pi (s_j^{(N)} - s_i^{(M)}) \prod_{a=1}^{Q} \frac{\cosh \pi (s_j^{(N)} - m_a)}{\cosh \pi (s_j^{(N)} + iw_j z - m_a)},$$

$$I^{\Sigma, Q, R} = \sum_{w \in R} \mathcal{F}^{l}(s, z) \frac{N}{\prod_{j=1}^{i=1} P} \cosh \pi (s_j^{(N)} - s_i^{(P)}) \prod_{b=1}^{K-Q} \frac{\cosh \pi (s_j^{(N)} - m_b)}{\cosh \pi (s_j^{(N)} - iw_j z - m_b)},$$

where $w = (w_1, \ldots, w_N)$ are the weights of the representation $R$ of $U(N)$. The functions $\mathcal{F}^{r,l}(s, z)$ have poles which give zero residues in the limit $z \to 1$, and therefore can be simply replaced by their respective $z \to 1$ limits (which turn out to be overall signs) in the formula for the expectation value. The background Wilson defects for the left and the right SQM are given by

$$W^{T, b.g.}(t) = e^{2\pi |R| t_r}, \quad W^{L, b.g.}(t) = e^{2\pi |R| t_l},$$

where the FI parameter of the $U(N)$ gauge group is given by $\eta^{(N)} = t_l - t_r$. The integer $|R|$ is the number of boxes in the Young diagram associated with the representation $R$.

The hopping duality can then be understood as a simple change of integration variables in the defect partition function. Consider the partition function of the right 3d-1d quiver, given by (2.2)–(2.3), and implement the following transformation:

$$s_j^{(N)} \to s_j^{(N)} - iw_j z, \quad \forall j = 1, \ldots, N,$$

keeping all the other integration variables fixed. Since we are analytically continuing $z \in i\mathbb{R}$, this transformation does not introduce or remove any new poles. One can readily check that the resultant expression gives the partition function of the left 3d-1d quiver in (2.2)–(2.4).

2.2 Quivers with multiple edges

Next, we describe coupled 3d-1d quivers where the 1d theory is coupled to a pair of 3d gauge nodes connected by multiple bifundamental hypermultiplets. Let us denote a generic 3d theory in this class as $\mathcal{T}$, and zoom in on a part of the quiver consisting of two consecutive gauge nodes — $U(N)$ and $U(M)$, connected by $p$ bifundamental hypers. These bifundamentals are associated with a $U(p)$ flavor symmetry (or, $SU(p)$ flavor symmetry, but this distinction will not be important for our presentation).
Figure 6. A generic coupled 3d-1d system that realizes a vortex defect for the gauge node $U(N)$. The $-^*$ superscript implies that the signs of the 1d FI parameters are all negative. The thick red line connecting the $U(N)$ and $U(M)$ gauge nodes represents the bifundamental hypers that enter the cubic superpotential.

Now, consider an SQM of the form in figure 5 where $N_R = N$, $N_L = Q + M$ and a generic $p' < p$, such that the 1d theory has a global symmetry $U(p') \times U(N_R) \times U(N_L)$. One way of obtaining this SQM is to start from an SQM with the same gauge group and bifundamental matter, but with $p'N_R$ anti-fundamental chirals and $N_L$ fundamental hypers. The global symmetry arising from the anti-fundamental chirals has a subgroup $U(N_R)^{p'}$. Identifying these $p' U(N_R)$ factors to a single $U(N_R)$ factor, we obtain the SQM in figure 5. In appendix A, we demonstrate how this operation can be implemented in terms of the Witten index.

The $U(p') \times U(N_R) \times U(N_L)$ global symmetry is then weakly gauged by 3d background vector multiplets, as before. The $U(p')$ factor is identified with a $U(p') \subset U(p)$ subgroup of the 3d global symmetry arising from the $p'$ bifundamental hypers. This is achieved by turning on the cubic superpotential

$$\tilde{W}_0 = q^a_\alpha X^\beta_{i\alpha} \tilde{q}^{a|i}_{\alpha} |_{\text{def}} + \ldots, \quad (2.7)$$

where $i = 1, \ldots, p'$, and the other indices run over the same range as above. In the final step, one promotes the background vector multiplet for a $U(N) \times U(M)$ subgroup of the $U(p') \times U(N_R) \times U(N_L)$ global symmetry to a dynamical vector multiplet. This leads to the 3d-1d coupled quiver of the form given in figure 6.

We propose that the coupled 3d-1d quivers realize a class of vortex defects associated with the $U(N)$ gauge node in the theory $\mathcal{T}$ is given in figure 6. The resultant defects are denoted as $V_{Q,R,p'}^{(i)}$, where $Q$ is an integer and $R$ is a representation of $U(N)$, while the integer $p' < p$ depends on the number of bifundamental hypers involved in the cubic superpotential (2.7). The negative sign in the subscript implies that all the 1d FI parameters should be taken to be negative.
The expectation value of the vortex defect $\langle V_{Q,R,p'} \rangle$ in the theory $\mathcal{T}$ on a round three sphere is given as:

\[
\langle V_{Q,R,p'} \rangle \big|_\mathcal{T} = W_{b.g.}^{-1}(\eta) \times \frac{1}{Z(\mathcal{T})} \times \lim_{z \to 1} \left[ \int [ds] \frac{z^{(T)}(s,m,\eta)}{Z_{\text{int}}^{(T)}(s,m,\eta)} \mathcal{I}^{Q,R,p'}_{\mathcal{T}}(s,m,z|\xi < 0) \right],
\]

(2.8)

where $W_{b.g.}$ is a background Wilson defect, $Z_{\text{int}}^{(T)}$ is the integrand of the sphere partition function arising from the 3d bulk fields, and $\mathcal{I}^{Q,R,p'}_{\mathcal{T}}$ is the Witten index of the coupled SQM. The latter is given as:

\[
\mathcal{I}^{Q,R,p'}_{\mathcal{T}} = \sum_{w \in R} \prod_{j=1}^{N} \prod_{l=1}^{p} \prod_{u=1}^{M} \frac{\cosh(\pi s_j^{(N)} - s_i^{(M)} - \mu_l)}{\cosh(\pi s_j^{(N)} + iw_j z - s_i^{(M)} - \mu_l)} \times \prod_{j=1}^{N} \prod_{a=1}^{Q} \frac{\cosh(\pi s_j^{(N)} - m_a)}{\cosh(\pi s_j^{(N)} + iw_j z - m_a)},
\]

(2.9)

where $\{\mu_l\}_{l=1,\ldots,p}$ denote real masses of the $p$ bifundamental hypermultiplets, $\{m_a\}$ denote the real masses of the $Q$ hypers in the fundamental representation of $U(N)$, and $R$ is a representation of $U(N)$. Plugging in the expression for the Witten index, the expectation value for the vortex defect $\langle V_{Q,R,p'} \rangle$ can be written as:

\[
\langle V_{Q,R,p'} \rangle \big|_\mathcal{T} = \frac{W_{b.g.}^{-1}(\eta)}{Z(\mathcal{T})} \lim_{z \to 1} \sum_{w \in R} \left[ \int [ds] \left[ \prod_{k<k'} \sinh^2 \frac{\pi s_{kk'}^{(P)}}{2} \prod_{j<j'} \sinh^2 \frac{\pi s_{jj'}^{(N)}}{2} \prod_{i<i'} \sinh^2 \frac{\pi s_{ii'}^{(M)}}{2} \right] \prod_{j,k} \cosh \pi(s_j^{(N)} - s_k^{(P)}) \prod_{l=1}^{K-Q} \cosh \pi(s_j^{(N)} - m_l) \right]
\]

\[
\times \frac{1}{\prod_{j,i} \prod_{l=p'+1}^{p} \cosh \pi(s_j^{(N)} - s_i^{(M)} - \mu_l)} \times \frac{1}{\prod_{l=1}^{p'} \cosh \pi(s_j^{(N)} - s_i^{(M)} - \mu_l + iw_j z)} \prod_{a=1}^{Q} \cosh \pi(s_j^{(N)} - m_a + iw_j z),
\]

(2.10)

where $[\ldots]$ denotes the contribution of the rest of the quiver, and won’t be relevant for the discussion here.

Similar to the case of linear quiver, there may be multiple 3d-1d systems that realize the same vortex defect in the theory $\mathcal{T}$. Generically, there is at least one such hopping dual for the 3d-1d system in figure 6. This system can be read off by implementing the change of variables

\[
s_j^{(N)} \to s_j^{(N)} - iw_j z, \quad \forall j = 1, \ldots, N,
\]

(2.11)

in (2.10) keeping all the other integration variables fixed, which gives:

\[
\langle V_{Q,R,p'} \rangle \big|_\mathcal{T} = \frac{W_{b.g.}^{-1}(\eta)}{Z(\mathcal{T})} \lim_{z \to 1} \sum_{w \in R} \left[ \int [ds] \left[ \prod_{k<k'} \sinh^2 \frac{\pi s_{kk'}^{(P)}}{2} \prod_{j<j'} \sinh^2 \frac{\pi s_{jj'}^{(N)}}{2} \prod_{i<i'} \sinh^2 \frac{\pi s_{ii'}^{(M)}}{2} \right] \prod_{j,k} \cosh \pi(s_j^{(N)} - s_k^{(P)}) \prod_{l=1}^{K-Q} \cosh \pi(s_j^{(N)} - m_l) \right]
\]

\[
\times \frac{1}{\prod_{j,i} \prod_{l=p'+1}^{p} \cosh \pi(s_j^{(N)} - s_i^{(M)} - \mu_l)} \times \frac{1}{\prod_{l=1}^{p'} \cosh \pi(s_j^{(N)} - s_i^{(M)} - \mu_l + iw_j z)} \prod_{a=1}^{Q} \cosh \pi(s_j^{(N)} - m_a + iw_j z),
\]

(2.12)
Figure 7. A coupled 3d-1d quiver which is a hopping dual for the quiver in figure 6, i.e. it realizes the same vortex defect. The $+$ subscript implies that the signs of the 1d FI parameters are all positive.

where $W_{p.g.}^+ (\eta)$ is a background Wilson defect. The r.h.s. of the above equation can be readily identified as the partition function of another 3d-1d quiver $\Sigma^{Q,R,p'}_{(II)}$ (normalized by the 3d partition function) shown in figure 7. The $+$ sign in the subscript implies that the Witten index should be computed with all the 1d FI parameters being positive. We will denote the associated vortex defect as $V^{(II)}_{Q,R,p'}$. The expression on the r.h.s. of (2.12) therefore implies:

$$\langle V^{(I)}_{Q,R,p'} \rangle_T = \langle V^{(II)}_{Q,R,p'} \rangle_T,$$

which shows that the coupled quivers $T[\Sigma^{Q,R,p'}_{(I)}]$ and $T[\Sigma^{Q,R,p'}_{(II)}]$ are related by a hopping duality.

In section 3 and section 4, we will explicitly construct 3d-1d quivers of the form figure 6–7 using $S$-type operations, and check that they indeed map to Wilson defects under mirror symmetry.

3 Defects in an Abelian quiver

The first example is an infinite family of Abelian mirrors shown in figure 8, labelled by three positive integers $(n,l,p)$ with the constraints $n > p > 0$, $l \geq 1$, and $p \geq 2$. The theory $X'$ consists of two gauge nodes connected by multiple edges. The dual theory $Y'$ has the shape of a star-shaped quiver consisting of three linear quivers glued at a common U(1) gauge node. The dimensions of the respective Higgs and Coulomb branches, and the associated global symmetries are shown in table 1.

For the choice $n = 2p$ and $l = p$, the quiver $X'$ reduces to the complete graph quiver — the 3d mirror of the $(A_2, A_{3p-1})$ AD theory reduced on a circle. The quiver $Y'$ — the mirror of the 3d mirror — gives another Lagrangian description of the 3d SCFT.

3.1 Duality of the theories without defects

Let us first consider the duality without any half-BPS defects. The starting point for obtaining the mirror pair $(X', Y')$ by $S$-type operations is a linear mirror pair $(X, Y)$, as

---

4In [7], the quiver $Y'$ was written in a slightly different form. One can readily check that the two quivers are related by a simple redefinition of the Abelian vector multiplets.
A three-parameter family of dual quiver gauge theory pairs with $U(1)$ gauge groups labelled by the integers $(n, p, l)$ with $n > p > 0$, $l \geq 1$, and $p \geq 2$.

Table 1. Summary table for the moduli space dimensions and global symmetries for the mirror pair in figure 8.

| Moduli space data | Theory $X'$ | Theory $Y'$ |
|-------------------|-------------|-------------|
| $\dim \mathcal{M}_H$ | $n + l - 2$ | $2$ |
| $\dim \mathcal{M}_C$ | $2$ | $n + l - 2$ |
| $G_H$ | $SU(p) \times SU(n - p) \times SU(l) \times U(1)$ | $U(1) \times U(1)$ |
| $G_C$ | $U(1) \times U(1)$ | $SU(p) \times SU(n - p) \times SU(l) \times U(1)$ |

Figure 9. A pair of mirror dual Abelian linear quivers.

shown in figure 9. Mirror symmetry of $X$ and $Y$ implies that the partition functions are related in the following fashion:

\[
Z^{(X)}(m, t) = C_{XY}(m, t) Z^{(Y)}(t, -m),
\]

\[
C_{XY}(m, t) = e^{2\pi i (m_1 t_1 - m_n t_2)},
\]

where $C_{XY}$ is a contact term.

The quiver $X'$ can be constructed by implementing an elementary Abelian identification-flavoring-gauging operation $O^a_P$ on the linear quiver $X$:

\[
O^a_P(X) = G^a_P \circ F^a_P \circ I^a_P(X),
\]

where $O^a_P$ is shown explicitly in figure 10. The $p$ $U(1)$ nodes being identified are shown in green. The mass parameters associated with the $U(1)^p$ global symmetry being identified are chosen as:

\[
u_j = m_{n-p+j}, \quad j = 1, \ldots, p.
\]
Figure 10. The $S$-type operation $O_P^\alpha$ implemented on $X$. The green nodes are identified, followed by a flavoring operation at the identified node with $l$ fundamental hypermultiplets, and the identified node is then gauged.

Figure 11. Construction of a pair of mirror dual theories, involving non-ADE quivers.

Note that (3.4) corresponds to a specific choice of the permutation matrix $P$. The resultant quiver $X'$ is shown in figure 10 --- it is a $U(1)_1 \times U(1)_2$ gauge theory with $n - p$ hypers in the fundamental of $U(1)_1$, $l$ hypers in the fundamental of $U(1)_2$, and $p$ hypers in the bifundamental of $U(1)_1 \times U(1)_2$. The associated real masses are denoted as $m^{(1)}$, $m^{(2)}$ and $m^{\text{bif}}$ respectively. In addition, we have the parameters $t$, such that the real FI deformations are given as $\eta_1 = t_1 - t_2$, and $\eta_2 = t_2 - t_3$.

The dual theory can be read off from the dual partition function which can be computed using the prescription in appendix B and is summarized in appendix C. The dual operation $\tilde{O}_P$ and the resultant $Y'$ is shown in figure 11. The quiver $Y'$ can be decomposed into two linear subquivers:

- a linear chain (1) consisting of $n - 1$ $U(1)$ gauge nodes,
- a linear chain (2) consisting of $l - 1$ $U(1)$ gauge nodes,

where the $(n - p)$-th gauge node of chain (1) is connected to the first gauge node of chain (2) by a single bifundamental hypermultiplet.
Let us label the FI parameters of the linear chain (1) of \( n-1 \) nodes as \( \eta_k^{(1)} = t_k^{(1)} - t_{k+1}^{(1)} \), with \( k = 1, \ldots, n - 1 \), and the FI parameters of the linear chain (2) of \( l - 1 \) nodes as \( \eta_b^{(2)} = t_b^{(2)} - t_{b+1}^{(2)} \), with \( b = 1, \ldots, (l - 1) \). The fundamental masses associated with chain (1) are denoted as \( m_1^{(1)} \) and \( m_{n-1}^{(1)} \), while the lone fundamental mass associated with chain (2) is denoted as \( m_{l-1}^{(2)} \).

With this parametrization of masses and FI parameters, the partition functions of \( X' \) and \( Y' \) can be shown to be related as:

\[
Z^{(X')}(\{m_1^{(1)}, m_2^{(1)}, m_{\text{bif}}^{(1)}, t\}) = C_{XY'} \cdot Z^{(Y')}(\{m_1^{(1)}, m_{n-1}^{(1)}, m_{l-1}^{(2)}\}, \{-t^{(1)}, -t^{(2)}\}),
\]

\[
\text{(3.5)}
\]

\[
C_{XY'} = e^{-2\pi i t_3 m_2^{(2)}} e^{2\pi i (m_1^{(1)} - m_{\text{bif}}^{(1)} t_2)},
\]

\[
\text{(3.6)}
\]

where on the r.h.s. of (3.5), we have chosen to write the independent mass parameters in terms of the fundamental masses of \( Y' \), setting the bifundamental masses to zero. The mirror map relating the FI parameters of \( Y' \) to the masses of \( X' \) is given as:

\[
\begin{align*}
t^{(1)} &= \{m_1^{(1)}, \ldots, m_{n-p}^{(1)}, m_{\text{bif}}^{(1)} + m_1^{(2)}, \ldots, m_{p}^{\text{bif}} + m_1^{(2)}\}, \\
t^{(2)} &= \{m_1^{(2)}, \ldots, m_l^{(2)}\},
\end{align*}
\]

\[
\text{(3.7a-3.7b)}
\]

\[\text{while the mirror relating the fundamental masses of } Y' \text{ to FI parameters of } X' \text{ is then given as:}
\]

\[
\begin{align*}
m_1^{(1)} &= t_1, \\
m_2^{(1)} &= t_2, \\
m_{l-1}^{(2)} &= t_3.
\end{align*}
\]

\[
\text{(3.8a-3.8c)}
\]

### 3.2 Vortex defects: Hopping duals and mirror maps

In this section, we construct the 3d-1d coupled quivers of the form given in figure 6 that realize half-BPS vortex defects in the quiver \( X' \). We then study the hopping dualities of these coupled systems and their mirror maps. The starting point of the construction is the coupled quiver in the theory \( X \), as shown on the extreme left of figure 12. In the notation of section 2.1, this quiver realizes a vortex defect \( V_{p',k}^{r'} \), where \( p' \) and \( k \) are positive integers. The superscript \( r' \) implies that the single 1d FI parameter associated with the U(1) gauge node should be chosen to be negative.

We now implement the \( S \)-type operation \( O_p \), as defined in (3.3)–(3.4), on the 3d-1d quiver \( (X[V_{p',k}^{r'}]) \), assuming \( p > p' \). The procedure involves identifying \( p \) U(1) flavor nodes, which are shown in green in figure 12, followed by a flavoring and gauging operation of the identified node. Of these \( p \) U(1) flavor nodes, \( p' \) are coupled to the 1d quiver as shown, while the remaining \( p-p' \) nodes are not. The resultant coupled quiver is shown in the extreme right of figure 12. In the notation of section 2.2, this realizes the vortex defect \( V_{0,k,p'}^{(l)-} \). Following the general prescription of (B.7), the defect partition function can be written as:

\[
Z^{(X'[V_{0,k,p'}^{(l)-}])} = W_{bg}(t_2, k) \lim_{\xi \to 1} \int_0^2 ds_k Z^{(X')}_{\text{int}}(s, \{m^{(a)}\}, m_{bif}^{(1)}, t, \eta_\alpha) \cdot I^{(\Sigma_{p',p'})}_{(l)-} (s, m_{bif}^{(1)}, z | \xi < 0),
\]

\[
\text{(3.9)}
\]
where the Witten index is being computed in the negative FI chamber. The 3d matrix model integrand \( Z_{\text{int}}^{(X')} \) and the Witten index \( \mathcal{I}^{(t)-} \) of the SQM are given as:

\[
\begin{align*}
Z_{\text{int}}^{(X')} &= e^{2\pi i(t_1-t_2)s_1} e^{2\pi i(t_2-t_3)s_2} \prod_{i=1}^{n-p} \cosh \pi(s_1-m_i^{(1)}) \prod_{m=1}^{l} \cosh \pi(s_2-m_m^{(2)}) \prod_{j=1}^{p} \cosh \pi(s_1-s_2-m_j^{\text{bif}}), \\
\mathcal{I}^{(t)-} &= \frac{\prod_{j=p-p'\pm 1}^{p} \cosh \pi(s_1-s_2-m_j^{\text{bif}})}{\prod_{j=p-p'+1}^{p} \cosh \pi(s_1-s_2-m_j^{\text{bif}}+ikz)}, \quad W_{bg}(t_2,k) = e^{2\pi t_2 k}. \tag{3.11}
\end{align*}
\]

The defect partition function can then be simplified to the following form:

\[
Z^{(X'|V_{0,k,p'}^{-})} = W_{bg}(t_2,k) \lim_{z \to 1} Z^{(X')}(m^{(1)}, m^{(2)}, \{m_j^{\text{bif}}\}_{j=1}^{p-p'}, \{m_j^{\text{bif}}-ikz\}_{j=p-p'+1}^{p}; t). \tag{3.12}
\]

A hopping dual of this coupled quiver can be read off from the defect partition function after implementing the change of variable:

\[
s_1 \to s_1 - ikz. \tag{3.13}
\]

Following the analysis in section 2.2, the coupled quiver is shown on the right of figure 13, which is manifestly of the type in figure 7. The associated vortex defect is denoted as \( V_{0,k,p'}^{-} \).

Let us now determine the mirror map of this vortex defect. The dual defect partition function can be written down following the general prescription in (B.18)–(B.19). In the present case, we can simply use the relation (3.5) between the partition functions of \( (X', Y') \), and the mirror map of FI parameters/masses in (3.7a)–(3.7b) to write down the dual defect partition function. This leads to the following relation:

\[
Z^{(X'|V_{0,k,p'}^{-})} = C_{X'|Y'} \cdot \int \prod_{k=1}^{n-1} d\sigma_k \prod_{b=1}^{l-1} d\tau_b \cdot e^{2\pi k\sigma_n-p} Z_{\text{int}}^{(Y')}(\sigma, \tau, m', \eta'), \tag{3.14}
\]

\[
= C_{X'|Y'} \cdot Z^{(Y'|W_{k,n-p})}(m', \eta'), \tag{3.15}
\]
where $W_{k,n-p}^{(1)}$ is a Wilson defect of charge $k$ in the $n-p'$-th $U(1)$ gauge node of the linear subquiver (1) of $Y'$, and $C_{X|Y'}$ is the contact term given in (3.6). We therefore have the mirror map:

$$\langle V_{Q,R,p}' \rangle_{X'} = \langle V_{Q,R,p}' \rangle_{Y'} = \langle W_{k,n-p}^{(1)} \rangle_{Y'} , \quad (3.16)$$

and the dual Wilson defect in the theory $Y'$ is shown in figure 14.

The computation in this section can be easily extended to construct the vortex defects $V_{Q,k,p}'$ with $Q < n$, starting from the 3d-1d coupled quiver $X[\Sigma_{p,k}]$ via a similar $S$-type operation.

4 Defects in a non-Abelian quiver

The second example is an infinite family of Abelian mirrors shown in figure 15, labelled by three positive integers $(p_1, p_2, p_3)$ subject to the constraints $p_1 \geq 1$, $p_2 \geq 1$, and $p_3 > 1$. Two of the gauge nodes in the theory $X'$ are connected by $p_1 \geq 1$ bifundamental hypers. The dual theory $Y'$ is a non-ADE-type quiver gauge theory built out of unitary gauge nodes, and fundamental/bifundamental matter, along with a single hypermultiplet which transforms in the determinant representation of $U(2)$ gauge group and has charge 1 under one of the adjacent $U(1)$ gauge nodes, as denoted by the blue line in the figure. The dimensions of the respective Higgs and Coulomb branches, and the associated global symmetries are shown in table 2.
Figure 15. An infinite family of mirror duals labelled by the positive integers $(p_1, p_2, p_3)$ subject to the constraints $p_1 \geq 1$, $p_2 \geq 1$, and $p_3 > 1$. The blue line denotes a hypermultiplet transforming in the determinant representation of $U(2)$ and having charge 1 under $U(1)_1$.

| Moduli space data | Theory $X'$ | Theory $Y'$ |
|-------------------|-------------|-------------|
| $\dim M_H$        | $1 + p_1 + p_2 + p_3$ | 5           |
| $\dim M_C$        | 5           | $1 + p_1 + p_2 + p_3$ |
| $G_H$             | $SU(p_1) \times SU(p_2) \times SU(p_3) \times U(1)^3$ | $SU(3) \times U(1)^2$ |
| $G_C$             | $SU(3) \times U(1)^2$ | $SU(p_1) \times SU(p_2) \times SU(p_3) \times U(1)^3$ |

Table 2. Summary table for the moduli space dimensions and global symmetries for the mirror pair in figure 15.

Figure 16. Dual theories for $p_1 = p_2 = p_3 = 2$.

For keeping the presentation simple, we will consider the case $p_1 = p_2 = p_3 = 2$. The dual pair $(X', Y')$ are mirror Lagrangians associated with the circle reduction of the AD theory $D_9(SU(3))$.

4.1 Duality of the theories without defects

We first consider the duality without any half-BPS defects. The starting point for obtaining the mirror pair $(X', Y')$ by an $S$-type operation is the linear mirror pair $(X, Y)$, as shown
Figure 17. A pair of dual linear quivers with unitary gauge groups.

Figure 18. The construction of the quiver gauge theory $X'$ of figure 15 by a sequence of three elementary Abelian $S$-type operation. The labels $i = 1, 2, 3$ on the flavor nodes correspond to the mass parameters $u_i$. At each step, the flavor node on which the elementary operation acts is shown in green.

in figure 17. Mirror symmetry of $X$ and $Y$ implies that

\[
Z^{(X)}(\mathbf{m}; \mathbf{t}) = C_{XY}(\mathbf{m}, \mathbf{t}) Z^{(Y)}(\mathbf{t}; -\mathbf{m}),
\]

\[
C_{XY}(\mathbf{m}, \mathbf{t}) = e^{2\pi i t_1 (m_1 + m_2)} e^{-2\pi i t_2 (m_3 + m_4)}.
\]  

where $C_{XY}$ is a contact term.

The quiver $X'$ can be obtained from the linear quiver $X$ by implementing an $S$-type operation $O_p$, which includes three elementary Abelian $S$-type operations, i.e.

\[
O_p(X) = O_{p_3} \circ O_{p_2} \circ O_{p_1}(X),
\]  

where $O_{p_i} (i = 1, 2, 3)$ are elementary Abelian $S$-type operations shown in figure 18. The
mass parameters \( \{u_i\} \) associated the \( S \)-type operation \( O_{P_i} \), are given as

\[
u_1 = m_3, \quad u_2 = m_4, \quad u_3 = m_1, \quad v = m_2.
\] (4.4)

Note that \( O_{P_1} \) is a flavoring-gauging operation, \( O_{P_2} \) is an identification-flavoring-gauging operation, while \( O_{P_3} \) is a gauging operation. At each step, the flavor node(s) on which the \( S \)-type operation acts is marked in green. The resultant quiver \( X' \) is shown in figure 18 — it is a \( U(2)_0 \times \prod_{i=1}^3 U(1)_i \) gauge theory (where the subscripts are node labels) with fundamental and bifundamental matter as shown. In particular, the \( U(1)_1 \) and \( U(1)_2 \) nodes are connected by two bifundamental hypers.

The real mass deformations are labelled as follows. Let \( m^{(0)} \), \( m^{(1)} \), and \( m^{(2)} \) be the masses of fundamental hypers associated with the gauge nodes \( U(2)_0 \), \( U(1)_1 \) and \( U(1)_2 \) respectively, and \( m_{\text{bif}} \) be the masses for the \( U(1)_1 \times U(1)_2 \) bifundamental hypers. The masses of other bifundamental hypers can be set to zero by appropriately shifting the integration variables in the matrix model. The FI parameters are labelled as \( \eta_0 = t_1 - t_2 \), and \( \{\eta_i\}_{i=1}^3 \).

The dual theory can be read off from the dual partition function which can be computed using the prescription in appendix B and is summarized in appendix D. The dual operation \( \tilde{O}_p \) and the resultant \( Y' \) is shown in figure 19. The FI parameters are given as \( \{\eta'_j\}_{j=0}^5 \), where the superscript \( j \) coincides with the node label in the quiver \( Y' \). Similarly, the mass deformations for the fundamental hypers are denoted as \( m'^{(0)} \) and \( m'^{(4)} \), while that for the Abelian hypermultiplet is \( m_{\text{Ab}} \).

With this parametrization of FI parameters and masses, the partition functions of \( X' \) and \( Y' \) are related as

\[
Z^{(X')}(m^{(0)}, m^{(1)}, m^{(2)}, m_{\text{bif}}; \{\eta_i\}) = C_{X'Y'} \cdot Z^{(Y')}(m'^{(0)}, m'^{(4)}, m_{\text{Ab}}; -\{\eta'_j\}_{j=0}^5),
\] (4.5)

\[
C_{X'Y'} = e^{2\pi i (2t_1 + \eta_1)m^{(0)}} e^{2\pi i (\eta_2 - t_2)m^{(2)}_1} e^{2\pi i (\eta_3 - t_2)m^{(4)}_2},
\] (4.6)

where \( C_{X'Y'} \) should be identified as the new contact term. The mirror map relating the FI
parameters of the theory \( Y' \) with the masses of the theory \( X' \) is given as follows:

\[
\begin{align*}
\eta_0' &= (m_0^{(0)} - m_1^{(2)}), \\
\eta_1' &= m_1^{bif}, \\
\eta_2' &= (m_1^{bif} - m_2^{bif}), \\
\eta_3' &= (m_2^{bif} - m_1^{(1)} + m_2^{(2)}), \\
\eta_4' &= (m_1^{(1)} - m_2^{(1)}), \\
\eta_5' &= (m_1^{(2)} - m_2^{(2)}).
\end{align*}
\] (4.7a–f)

The mirror map relating the mass parameters of the theory \( Y' \) with the FI parameters of the theory \( X' \) can also be read off from (D.15), but it won’t be relevant for the rest of this paper.

### 4.2 Vortex defects in \( X' \): mirror maps and Hopping dualities

In this section, we construct two distinct types of vortex defects in the quiver gauge theory \( X' \) as coupled 3d-1d quivers, using \( S \)-type operations. In section 4.2.1, we consider a vortex defect associated with the central \( U(2) \) gauge node of \( X' \), while in section 4.2.2, we consider a defect associated with edge of the quiver with multiplicity 2. In each case, we discuss the hopping duals, and obtain the dual Wilson defects in the quiver \( Y' \).

#### 4.2.1 Vortex defect in the central \( U(2) \) node of \( X' \)

The starting point of the construction is the mirror pair of quiver gauge theories with defects \( (X'[V^r_2, R], Y[W_R]) \), as shown on the top line of figure 20, where \( R \) is a representation of \( U(2) \). The superscript \( r \) implies that the 1d FI parameters should be chosen to be negative. We then implement the \( S \)-type operation (4.3)–(4.4) on the coupled quiver \( X[V^r_2, R] \), which leads to the quiver \( X'[V^{(l)-r}_2, R] \).

Following the general prescription of (B.7), the defect partition function in the theory \( X' \) can be written as:

\[
Z^{(X'[V^{(l)-r}_2, R])} = W_{bg}(t, R) \lim_{z \to 1} \int \frac{d^2 s}{2!} \prod_{i=1}^{3} ds_i \ Z^{(X')}_{\text{int}}(s, m_0^{(0)}, m_1^{(1)}, m_2^{(1)}, m_2^{bif}; t, \{ \eta_i \})
\]

\[
\times \mathcal{I}^{X'[V^{(l)-r}_2, R]}(s, z; \xi < 0),
\] (4.8)

where the 3d matrix model model integrand and the Witten index of the SQM are given as

\[
Z^{(X')}_{\text{int}} = e^{2\pi i \sum_0^2 \eta_i s_i} \prod_{a=1}^2 \cosh \pi (s_1 - m_a^{(1)}) \prod_{b=1}^2 \cosh \pi (s_1 - s_2 - m_b^{bif}) \prod_{c=1}^2 \cosh \pi (s_2 - m_c^{(2)})
\]

\[
\times \prod_{j=1}^2 \prod_{i=1}^2 \cosh \pi (s_0_j - s_i) \cosh \pi (s_0_j - m_0^{(0)}),
\] (4.9)

\[
\mathcal{I}^{X'[V^{(l)-r}_2, R]}(s, z; \xi < 0) = \sum_{w \in R} \prod_{i=1}^2 \cosh \pi (s_0_j - s_i) / \cosh \pi (s_0_j + iw_j z - s_i), \quad W_{bg}(t, R) = e^{2\pi t_3 |R|}.
\] (4.10)
The hopping dual of the system can be obtained by the standard transformation $s_{0j} \rightarrow s_{0j} - i w_j z$, and is shown on the right in figure 21. If one had chosen to implement the $S$-type operation on the “left” quiver $X[V_{2,R}']$ (instead of the “right” quiver $X[V_{2,R}]$), one would obtain the quiver $X'[V_{2,R}'] +$ directly. Note that the superscript $+$ indicates that the Witten index should be computed in the positive chamber.

Let us now determine the mirror map of this vortex defects in figure 21. The dual defect partition function can be written down following the general prescription in (B.18)–(B.19):

$$Z^{(X'[V_{2,R}'])} = C_{X'Y'} \cdot \int \frac{d^2 \sigma}{2!} \prod_{k=1}^5 d\sigma_k \ Z^{(Y')}_{\text{int}}(\sigma, m', \eta') \sum_{w \in R} e^{2\pi \sum_j w_j \sigma_0},$$

$$= C_{X'Y'} \cdot Z^{(Y'\{\tilde{W}_R^{(0)}\})}(m', \eta'),$$

(4.11)

where in the second step one can identify the r.h.s. as the partition function of theory $Y'$ with a Wilson defect for the $U(2)$ gauge node in the representation $R$, up to a contact term.
Figure 22. Dual Wilson defect for the vortex defects realized by the 3d-1d quivers in figure 21.

$C_{X'Y'}$ defined in (4.6). Therefore, the final mirror map may be summarized as

$$\langle V_{2,R}^{(l)} \rangle_{X'} = \langle V_{2,R}^{(l)} \rangle_{X'} = \langle \tilde{W}_R^{(0)} \rangle_{Y'}.$$  (4.12)

and the dual Wilson defect in the theory $Y'$ is shown in figure 26.

4.2.2 Vortex defect associated with the edge of multiplicity 2

The starting point of the construction is the mirror pair of quiver gauge theories without
defects $(X,Y)$, as shown in figure 17. We implement an $S$-type operation on the quiver $X$
of the following form

$$O_{P_i}(X) = O_{P_3} \circ O_{P_2} \circ O_{P_1}(X),$$  (4.13)

where the $O_{P_i}$ ($i = 1, 2, 3$) are elementary Abelian $S$-type operations shown in figure 23.

The mass parameters $\{u_i\}$ associated the $S$-type operation $O_{P_i}$, are given as

$$u_1 = m_3, \quad u_2 = m_4, \quad u_3 = m_1, \quad v = m_2.$$  (4.14)

The first operation $O_{P_1}$ is a flavoring-defect-gauging operation, where the defect operation
involves coupling a $(2,2)$ SQM to the 3d theory, as shown in the top right quiver of figure 23.

The FI parameter of the coupled SQM is chosen to be in the negative chamber, so that the
coupled quiver realizes a vortex defect of charge $k$ for the $U(1)_g$ gauge node. The second
operation $O_{P_2}$ is an identification-flavoring-gauging operation and the third operation $O_{P_3}$
is a gauging operation as shown. The resultant vortex defect is denoted as $V_{0,k,2}$, following
the notation of section 2.2.

Following the general prescription of (B.7), the defect partition function can be written as:

$$Z'(X'_{[V_{0,k,2}^-]}) = \lim_{z \rightarrow 1} \int \frac{d^2 s_0}{2!} \prod_{i=1}^{3} ds_i Z_{int}^{(X')} (s_i, ... ) I^{X_{0,k,2}^-}(s, m^{\text{bif}}, z | \xi < 0),$$  (4.15)

where $Z_{int}^{(X')}$ is given in (4.9), and the Witten index $I^{X_{0,k,2}^-}$ is:

$$I^{X_{0,k,2}^-}(s, m^{\text{bif}}, z | \xi < 0) = \frac{\prod_{b=1}^{2} \cosh \pi (s_1 - s_2 - m^{\text{bif}})}{\prod_{b=1}^{2} \cosh \pi (s_1 - s_2 - m^{\text{bif}} + i k z)}.$$  (4.16)
Figure 23. The construction of the quiver gauge theory $X'$ with a vortex defect by a sequence of three elementary Abelian $S$-type operation. The labels $i = 1, 2, 3$ on the flavor nodes correspond to the mass parameters $u_i$. At each step, the flavor node on which the elementary operation acts is shown in blue.

The hopping dual of the system can be obtained by the standard transformation $s_1 \rightarrow s_1 - ikz$, and is shown on the right in figure 24. We denote the associated vortex defect as $V_{0,k,2}$, where the superscript $+$ indicates that the Witten index should be computed in the positive chamber.

The mirror dual of this vortex defect can be read off from the dual defect partition function as follows. Starting from the vortex defect partition function, we have

$$Z^{(X'|V_{0,k,2})} = \lim_{z \to 1} Z^{(X')}(t, \eta; m^{(0)}, m^{(1)}, m^{(2)}; t, \eta)$$

$$= \lim_{z \to 1} C_{X'|Y'}(m^{\text{bif}} - ikz; t, \eta) Z^{(Y')}(t, \eta; m^{(0)}, m^{(1)}, m^{(2)}; t, \eta),$$

where in the second step we have used the relation between the partition functions of $X'$ and $Y'$ due to mirror symmetry. Since $C_{X'|Y'}$ does not depend on the mass parameters $m^{\text{bif}}$ (as can be seen from the explicit expression in (4.6)), the above equation reduces to

$$Z^{(X'|V_{0,k,2})} = C_{X'|Y'} \cdot Z^{(Y')}(t, \eta; m^{(0)}, m^{(1)}, m^{(2)}; t, \eta),$$

where $C_{X'|Y'}$ is a constant depending on $k$.
Using the mirror map (4.7a)–(4.7f) relating masses of $X'$ to FI parameters of $Y'$, we obtain the following relation:

$$Z^{(X'[V(0,k,2)])}_{X'} = C_{X'Y'} \int \frac{d^2 \sigma_0}{2!} \prod_{k=1}^5 d\sigma_k Z_{\text{int}}^{(Y')}(\sigma, \tau, m', -\eta') Z_{\text{Wilson}}(\sigma_1, k) Z_{\text{Wilson}}(\sigma_3, -k),$$

$$= C_{X'Y'} Z^{(Y'[\tilde{W}^{(1)}_k \cdot \tilde{W}^{(2)}_{-k}])}(m', -\eta').$$

(4.19)

where in the second step one can identify the r.h.s. as the partition function of theory $Y'$ with a Wilson defect of charge $k$ for the gauge node $U(1)_1$ and charge $-k$ for the gauge node $U(1)_3$. This is shown in figure 25. One can therefore summarize the mirror map as:

$$\langle V^{0, k, 2}_{(I)_-} X' \rangle = \langle V^{0, k, 2}_{(II)_+} X' \rangle = \langle \tilde{W}^{(1)}_k \cdot \tilde{W}^{(2)}_{-k} \rangle_{Y'}. \quad (4.20)$$

### 4.3 Mapping the defects to the $U - SU$ mirror

In [2], an IR duality was conjectured where the dual theories $(\mathcal{T}, \mathcal{T}^\vee)$ belong to the following classes of quiver gauge theories respectively:

- Theory $\mathcal{T}$: a linear chain of unitary and special unitary gauge groups with fundamentals and bifundamentals.
• Theory $\mathcal{T}'$: a generically non-linear quiver involving only unitary gauge groups, decorated with Abelian matter in addition to fundamental/bifundamental hypers.

In particular, it was shown that an infinite family of such pairs $(\mathcal{T}, \mathcal{T}')$ arise as two different Lagrangians for the 3d SCFT obtained by circle reduction of a $D_p(SU(N))$ Argyres-Douglas theory. It was also shown that the pair $(\mathcal{T}, \mathcal{T}')$ has a mirror dual which is a generically non-ADE quiver gauge theory with unitary gauge nodes and multiple edges. For the case of $p = 9$, $N = 3$, we have the following IR dual pair:

$$\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {2};
\node (b) at (1,0) {SU(2)};
\node (c) at (2,0) {4};
\node (d) at (3,0) {4};
\node (e) at (4,0) {3};
\node (f) at (5,0) {1};
\draw (a) -- (b);
\draw (b) -- (c);
\draw (c) -- (d);
\draw (d) -- (e);
\draw (e) -- (f);
\node at (2.5,0.5) {$\mathcal{T} - Y'$};
\end{tikzpicture}
\end{array}
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {1};
\node (b) at (1,0) {2};
\node (c) at (2,0) {m};
\node (d) at (3,0) {1};
\node (e) at (4,0) {1};
\node (f) at (5,0) {1};
\node (g) at (2,1) {1};
\node (h) at (2,2) {1};
\draw (a) -- (b);
\draw (b) -- (c);
\draw (c) -- (d);
\draw (d) -- (e);
\draw (e) -- (f);
\draw (b) -- (g);
\draw (g) -- (h);
\node at (2.5,0.5) {$(\mathcal{T}' - Y')$};
\end{tikzpicture}
\end{array}
$$

In the above quivers, we have indicated how the FI parameters on the left map to FI parameters on the right. In [2], this result was explicitly obtained by checking the proposed duality at the level of the three sphere partition function.

The quiver of the class $\mathcal{T}'$ in this case can be evidently identified as the quiver $Y''$ in figure 19, which implies that the theory $\mathcal{T}$ (which we will refer to as $Y''$ henceforth) is mirror dual to the theory $X'$. It is therefore natural to ask which defect operators in $Y''$ do the vortex defects considered in section 4.2.1 and section 4.2.2 map to. Computing the partition functions of the theories $\mathcal{T}$ and $Y'$ with Wilson defects, one can readily show that:

$$\langle \tilde{W}_{R}^{''}(0) \rangle_{Y''} = \langle \tilde{W}_{R}^{''}(1) \rangle_{Y''},$$

where $\tilde{W}_{R}^{(0)}$ is a Wilson defect for the $U(2)$ gauge node in the representation $R$ in the theory $Y''$, while $\tilde{W}_{R}^{(1)}$ is a Wilson defect for the leftmost $U(2)$ gauge node in the theory $Y''$ in the same representation. The vortex defect $V_{2,R}^{(I)-}$ studied in section 4.2.1 therefore maps to the following Wilson defect in the theory $Y''$:

$$\langle V_{2,R}^{(I)-} \rangle_{X''} = \langle V_{2,R}^{(I)+} \rangle_{X''} = \langle \tilde{W}_{R}^{(0)} \rangle_{Y''} = \langle \tilde{W}_{R}^{(1)} \rangle_{Y''},$$

as shown in figure 26.

For the vortex defect $V_{0,k,2}^{(I)-}$ studied in section 4.2.2, the computation proceeds in an analogous fashion. Using the map of FI parameters between the theory $\mathcal{T}$ and $Y'(\mathcal{T}')$ as shown above, one can show that at the level of the sphere partition function:

$$\langle \tilde{W}_{k}^{(3)} \cdot \tilde{W}_{-k}^{(4)} \rangle_{Y''} = \langle \tilde{W}_{k}^{(3)} \cdot \tilde{W}_{-k}^{(4)} \rangle_{\mathcal{T}'},$$

where $\tilde{W}_{k}^{(3)} \cdot \tilde{W}_{-k}^{(4)}$ is the Wilson defect in $Y'$ shown in figure 25, while $\tilde{W}_{-k}^{(4)}$ is a Wilson defect in theory $\mathcal{T}$ of charge $k$ for the $U(1)$ gauge node which is fourth from the left (see the figure on the r.h.s. of figure 27). The vortex defect $V_{0,k,2}^{(I)-}$ studied in section 4.2.2 therefore maps to following Wilson defect in the theory $\mathcal{T}$:

$$\langle V_{0,k,2}^{(I)-} \rangle_{X'} = \langle V_{0,k,2}^{(I)+} \rangle_{X'} = \langle \tilde{W}_{k}^{(3)} \cdot \tilde{W}_{-k}^{(4)} \rangle_{\mathcal{T}'},$$

as shown in figure 27.
Figure 26. A vortex defect in a non-ADE quiver and its dual Wilson defect in the unitary-special unitary mirror.

Figure 27. A vortex defect in a non-ADE quiver and its dual Wilson defect in the unitary-special unitary mirror.

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A Witten index of SQMs

A detailed treatment of Witten indices for (2,2) SQMs and 1d dualities can be found in [19]. The computation of Witten indices, relevant for vortex defects in linear quivers, can be found in appendix B of [5]. In this appendix, we briefly review a few points relevant to our discussion in section 2.1 and section 2.2.

Consider a (2,2) SQM of the following form:

The gauge node $U(n_P)$ has $N_L$ fundamental and $N_R$ anti-fundamental chiral multiplets. The integers $\{n_1, n_2, \ldots, n_P\}$ encode a representation $R$ of $U(N_L)$ and a representation $R'$.
of U(N_R). For simplicity, consider the case of P = 1, with n_1 = k. In this case, the weights of the representation R = S_k are in one-to-one correspondence to the N_L-partitions of the integer k, where S_k is the k-th symmetric representation of U(N_L). Similarly, the weights of the representation R' = S'_k are in one-to-one correspondence to the N_R-partitions of the integer k, where S'_k is the k-th symmetric representation of U(N_R).

For a generic P, the representations have the form:

\[ R = \otimes_{i=1}^{P} S_{k_i}, \quad R' = \otimes_{i=1}^{P} S'_{k_i}, \quad (A.1) \]

where the weights of S_{k_i} and S'_{k_i} are in one-to-one correspondence to the N_L-partitions and the N_R-partitions respectively of the integer k_i = n_i - n_{i-1}. Antisymmetric representations can be included in the discussion by appropriately tweaking the SQM [5].

In the chamber where all FI parameters are negative, the JK-residue formula for the Witten index of the SQM [19] gets non-zero residues from poles associated with the fundamental chiral multiplets only, and leads to the following expression:

\[ \mathcal{I}^\Sigma(\bm{\zeta} < 0) = \sum_{w \in R[U(N_L)]} \mathcal{F}(s^{(N_L)}, z) \prod_{j=1}^{N_L} \prod_{i=1}^{N_R} \frac{\cosh \pi(s_j^{(N_L)} - s_i^{(N_R)})}{\cosh \pi(s_j^{(N_L)} + i w_j z - s_i^{(N_R)})}. \quad (A.2) \]

In the chamber where all FI parameters are positive, the index is given as:

\[ \mathcal{I}^\Sigma(\bm{\zeta} > 0) = \sum_{w' \in R'[U(N_R)]} \tilde{\mathcal{F}}(s^{(N_R)}, z) \prod_{j=1}^{N_R} \prod_{i=1}^{N_L} \frac{\cosh \pi(s_j^{(N_R)} - s_i^{(N_L)})}{\cosh \pi(s_j^{(N_R)} + i w'_i z - s_i^{(N_L)})}. \quad (A.3) \]

Note that the two expressions are very different for a generic SQM, and therefore the choice of the FI chamber is an important piece of data in specifying the vortex defect. For a vortex defect associated with a gauge group U(N), one may pick either N_L = N or N_R = N and then gauge the corresponding flavor with dynamical 3d vector multiplets. The first choice will require that one takes \( \bm{\zeta} < 0 \) for the coupled SQM, while the second choice will require that \( \bm{\zeta} > 0 \). For a linear quiver, these choices correspond to the right and the left coupled quivers respectively.

Now, let us write down the Witten index of the SQM in figure 6 in the negative chamber, which allows us to compute the partition function of the 3d-1d quiver. We start from the SQM above, with N_L \rightarrow N, and N_R = p'M + Q. We then identify the flavor chemical potentials (or masses) associated to the anti-fundamental chiral multiplets as follows:

\[ s_{i+nM}^{(N_R)} = s_i^{(M)} + \mu^{n+1}, \quad n = 0, 1, \ldots, p' - 1, \quad i = 1, \ldots, M, \quad (A.4) \]

\[ s_{a+p'M}^{(N_R)} = m_a, \quad a = 1, \ldots, Q. \quad (A.5) \]

The above identification leads to the SQM \( \Sigma_{Q,R,p'}^{(l)} \) of the following form:
and its Witten index in the negative chamber can be written as:

\[
\mathcal{I}^{X,Q,n,s'}_{\gamma}\left(\mathbb{R}[U(N)]\right) = \sum_{w \in \mathbb{R}[U(N)]} \prod_{j=1}^{N} \prod_{i=1}^{N'} \prod_{M} \cosh \pi s_{j}^{(N)} - s_{i}^{(M)} - \mu_{1} \cosh \pi s_{j}^{(N)} + iw_{j}z - s_{i}^{(M)} - \mu_{1} \cosh \pi s_{j}^{(N)} + iw_{j}z - m_{a}) \]

\[\times \prod_{j=1}^{N} \prod_{a=1}^{Q} \cosh \pi s_{j}^{(N)} - m_{a}) \cosh \pi s_{j}^{(N)} + iw_{j}z - m_{a})]. \tag{A.6}\]

### B Construction of 3d-1d coupled quivers from S-type operations

In this section, we briefly review the construction of coupled 3d-1d systems realizing vortex defects using S-type operations. We refer the reader to section 3 of the paper [1] for a more detailed discussion.

#### B.1 S-type operations on a quiver with defects

Consider a class of 3d quiver gauge theories for which the Higgs branch global symmetry has a subgroup \(G^{\text{sub}}_{\text{global}} = \prod_{\gamma} U(M_{\gamma}) \subset G_{H}\). In addition, we will demand that these are good theories in the Gaiotto-Witten sense [20]. We will refer to them as class \(U\). Let \(X[\mathbf{A}, D]\) be a generic theory in this class decorated by a line defect \(D\), where \(\mathbf{A}\) collectively denotes the background vector multiplets. Given such a quiver \(X[\mathbf{A}, D]\), one can define a set of four basic quiver operations, as shown in figure 28:

1. **Gauging** \(G_{\mathcal{P}}^{\alpha}\)
2. **Flavoring** \(F_{\mathcal{P}}^{\alpha}\)
3. **Identification** \(I_{\mathcal{P}}^{\alpha}\)
4. **Defect** \(D_{\mathcal{P}}^{\alpha}\)

The superscript \(\alpha\) specifies the flavor node at which the S-type operation is being implemented. Each operation involves two steps. First, one splits the flavor node \(U(M_{\alpha})\) into two, corresponding to \(U(r_{\alpha}) \times U(M_{\alpha} - r_{\alpha})\) flavor nodes. The \(U(1)^{M_{\alpha}}\) masses are related to the \(U(1)^{r_{\alpha}} \times U(1)^{M_{\alpha} - r_{\alpha}}\) masses by the following map:

\[
\overrightarrow{m}_{\alpha,i} = P_{\alpha,i} \overrightarrow{v}_{i} + P_{\alpha,r_{\alpha}+j} \overrightarrow{v}_{j}, \quad (i_{\alpha} = 1, \ldots, M_{\alpha}, \quad i = 1, \ldots, r_{\alpha}, \quad j = 1, \ldots, M_{\alpha} - r_{\alpha}), \tag{B.1}\]

where \(P\) is a permutation matrix of order \(M_{\alpha}\). The theory deformed by the \(U(r_{\alpha}) \times U(M_{\alpha} - r_{\alpha})\) mass parameters is denoted as \((X[\mathbf{A}, D], P)\). In the second step, one gauges, adds flavor to, or identifies the \(U(r_{\alpha})\) flavor node with other flavor nodes. The operation \(D_{\mathcal{P}}^{\alpha}\) involves turning on a vortex or a Wilson-type defect for the flavor node \(U(r_{\alpha})\) in the theory \((X[\mathbf{A}, D], P)\).

**Definition.** An **elementary S-type operation** \(O_{\mathcal{P}}^{\alpha}\) on \(X\) at a flavor node \(\alpha\), is defined as any possible combination of the identification \((I_{\mathcal{P}}^{\alpha})\), the flavoring \((F_{\mathcal{P}}^{\alpha})\) and the defect \((D_{\mathcal{P}}^{\alpha})\) operations followed by a single gauging operation \(G_{\mathcal{P}}^{\alpha}\), i.e.

\[
O_{\mathcal{P}}^{\alpha}(X) := (G_{\mathcal{P}}^{\alpha}) \circ (D_{\mathcal{P}}^{\alpha})^{n_{3}} \circ (F_{\mathcal{P}}^{\alpha})^{n_{2}} \circ (I_{\mathcal{P}}^{\alpha})^{n_{1}}(X), \quad (n_{i} = 0, 1, \forall i). \tag{B.2}\]
Figure 28. The gauging, flavoring and identification quiver operations on a generic quiver $X$ of class $\mathcal{U}$. The identification operation may involve two or more $U(r_\alpha)$ flavor nodes. The operation $D_\alpha^P$ involves turning on a vortex or a Wilson-type defect for the flavor node $U(r_\alpha)$ in the theory $(X, P)$.

The operation $\mathcal{O}_\alpha^P$ generically maps a quiver gauge theory with defect $X[\hat{A}, D]$ to a new quiver gauge theory with defect $X'[\hat{B}, D']$:

$$\mathcal{O}_\alpha^P : X[\hat{A}, D] \mapsto X'[\hat{B}, D'], \quad (B.3)$$

where the final defect $D'$ is built out of the original defect $D$ and the defect operation $D_\alpha^P$.

One can classify the elementary $S$-type operations into four distinct types, depending on the constituent quiver operations:

- Gauging.
- Flavoring-gauging.
- Identification-gauging.
- Identification-flavoring-gauging.
Each one can be further combined with a defect operation. An elementary Abelian $S$-type operation is one which involves gauging a $U(1)$ global symmetry. Finally, any number of elementary $S$-type operations can be combined to form a generic $S$-type operation.

**B.2 Constructing vortex defects and the dual defects**

The $S$-type operation $O^\alpha_B$ can be realized in terms of supersymmetric observables. In this work, we will focus on the three sphere partition function. We will explicitly discuss the case of a vortex defect which can be realized as a coupled 3d-1d quiver, noting that Wilson defects can be treated in an analogous fashion. The partition function for the coupled 3d-1d quiver associated with $X[\hat{A}, D]$ is given as

$$Z(X[\hat{A}, D], \{P_\beta\})(\{u^\beta\}, \ldots; \eta, z) = \lim_{z \to 1} \int [dz] Z_{\text{int}}^{(X[\hat{A}, D], \{P_\beta\})}(s, \{u^\beta\}, \ldots; \eta, z)$$

(B.4)

$$= \lim_{z \to 1} Z_{\text{int}}^{(X[\hat{A}, D], \{P_\beta\})}(\{u^\beta\}, \ldots; \eta|z),$$

(B.5)

where the function $Z_{\text{int}}^{(X[\hat{A}, D], \{P_\beta\})}$ has the schematic form:

$$Z_{\text{int}}^{(X[\hat{A}, D], \{P_\beta\})} = W_{b.g.}(\eta) \cdot Z_{\text{int}}^{(X[\hat{A}, \{P_\beta\}])}(s, \{u^\beta\}, \ldots; \eta) \cdot \mathcal{I}_{\Sigma^\prime}(s, \{u^\beta\}, \ldots; z|\xi).$$

(B.6)

The $S$-type operation $O^\alpha_B$ can then be implemented in terms of the sphere partition function as follows:

$$Z_{O^\alpha_B}^{(X[\hat{A}, D])}(m^\alpha, \ldots; \eta, \eta_\alpha) = Z_{O^\alpha_B}^{(X[\hat{A}, D])}(m^\alpha, \ldots; \eta, \eta_\alpha)$$

(B.7)

$$= \lim_{z \to 1} \int [d \alpha] \cdot Z_{O^\alpha_B}(X)(u^\alpha, \{u^\beta\}_{\beta \neq \alpha}, \eta_\alpha, m^\alpha, z|\Sigma^\prime) \cdot Z^{(X[\hat{A}, D], \{P_\beta\})}(\{u^\beta\}, \ldots; \eta|z),$$

where $\eta_\alpha$ is an FI parameter associated with gauging, and $m^\alpha$ are hypermultiplet masses associated with flavoring and/or identification operations. The operator $Z_{O^\alpha_B}(X)$ can be constituted from the partition function contributions of gauging ($G^\alpha_B$), flavoring ($F^\alpha_B$), identification ($I^\alpha_B$), and defect ($D^\alpha_B$) operations as follows:

$$Z_{O^\alpha_B}(X)(u^\alpha, \{u^\beta\}_{\beta \neq \alpha}, m^\alpha, z|\Sigma^\prime) = Z_{G^\alpha_B}(X) \cdot (Z_{F^\alpha_B}(X))^{n_2} \cdot (Z_{I^\alpha_B}(X))^{n_1},$$

(B.8)

where $n_1, n_2 = 0, 1$, and the constituent operators are given as

$$Z_{G^\alpha_B}(X)(u^\alpha, m^\alpha, \eta_\alpha, z) = W_{b.g.}(\eta_\alpha, \eta) \cdot \mathcal{I}_{\Sigma^\prime}(u^\alpha, m^\alpha, z|\xi),$$

(B.9)

$$Z_{F^\alpha_B}(X)(u^\alpha, m^\alpha) = Z_{\text{vector}}^{(1-\text{loop})}(u^\alpha, m^\alpha) = e^{2\pi \eta_\alpha} \sum_{i<j} u_i^\alpha u_j^\alpha \prod_{i<j} \sinh^2 \pi (u_i^\alpha - u_j^\alpha),$$

(B.10)

$$Z_{I^\alpha_B}(X)(u^\alpha, \{u^\beta\}, \mu) = \int \prod_{j=1}^p du_j^\gamma \int \prod_{j=1}^p \delta(\mu^\gamma) \left(u^\alpha - u_j^\gamma + \mu^\gamma\right).$$

(B.12)

In (B.9), $\Sigma^\prime$ denotes the SQM being coupled to the 3d theory by the defect operation $D^\alpha_B$, with $\mathcal{I}_{\Sigma^\prime}$ being the corresponding Witten index. The parameters $m^\alpha_F$ in (B.11) denote the
masses for the hypers added in the flavoring operation. In (B.12), the identified nodes (we assume that there are \( p \) of them) are labelled as \( \{\gamma_j\}_{j=1,...,p} \), while \( \{\mu^{\gamma_j}\}_{j=1,...,p} \) denote the masses introduced by the identification operation.

The dual of the 3d-1d coupled system \( X'|[\tilde{B},D'] \) is then given as follows. Let the theory dual to \( X[A,D] \) be denoted by \( Y[A,D'^V] \), which is the quiver gauge theory \( Y \) decorated by a Wilson line defect \( D'^V \). The dual Wilson defect \( D'^V \) is of the generic form:

\[
D'^V = \sum_{k} c_k \tilde{W}^{\text{flavor}}_{\tilde{R}_k} \cdot \tilde{W}^\sim_{\tilde{R}_k(\tilde{G})},
\]

where \( \tilde{W}^\sim_{\tilde{R}_k(\tilde{G})} \) is a Wilson defect in a representation \( \tilde{R}_k \) of the gauge group \( \tilde{G} \) of the theory \( Y \), and \( \tilde{W}^{\text{flavor}}_\kappa \) is a background Wilson defect associated with the hypermultiplets.

Given an operation \( \mathcal{O}_\beta \) on \( X[A,D] \), one can define a dual operation on \( Y[A,D'^V] \):

\[
\tilde{\mathcal{O}}_\beta : Y[A,D'^V] \mapsto Y'[\tilde{B},D'^{V'}],
\]

such that the theories \( (X'[\tilde{B},D'],Y'[\tilde{B},D'^{V'}]) \) are IR dual. We can write down the partition function of \( Y'[\tilde{B},D'^{V'}] \) in terms of the theory \( Y[A,D'^V] \). Mirror symmetry implies the following identity involving the defect partition functions:

\[
Z^{X[A,D],(P_\beta)}(\{u^\beta,\ldots;\eta\}) = C_{XY}(\{u^\beta,\ldots;\eta\}) \cdot Z^{Y[A,D'^V],(P_\beta)}(m^Y(\eta);\eta^Y(\{u^\beta,\ldots\}),\ldots),
\]

where \( D \) is a vortex defect and \( D'^V \) is a Wilson defect. However, as discussed in [7], there exists a more refined \( z \)-dependent identity involving the two partition functions i.e.

\[
Z^{X[A,D],(P_\beta)}(\{u^\beta,\ldots;\eta\}) = C_{XY} \cdot \prod_{\gamma'} \int [d\sigma^\gamma] Z^{Y[A,D'^V],(P_\beta)}(\{\sigma^\gamma\},m^Y(\eta),\eta^Y(\{u^\beta,\ldots\}),\ldots),
\]

where the function \( Z^{X[A,D],(P_\beta)}(\ldots) \) is defined in (B.5) in terms of the vortex defect partition function. The function \( Z^{Y[A,D'^V],(P_\beta)}(\ldots) \) has the property that it reduces to the Wilson defect partition function in the limit \( z \to 1 \), i.e.

\[
\lim_{z \to 1} Z^{Y[A,D'^V],(P_\beta)}(m^Y(\eta),\eta^Y(\{u^\beta,\ldots\}),\ldots) = Z^{Y[A,D'^V],(P_\beta)}(m^Y(\eta);\eta^Y(\{u^\beta,\ldots\})).
\]

Using the identity (B.16), the dual defect partition function is given by the following formula (up to certain contact terms):

\[
Z^{\tilde{\mathcal{O}}_\beta(\gamma)}(A,D'^V) = \sum_{k} c_k \lim_{z \to 1} \prod_{\gamma'} \int [d\sigma^\gamma] Z^{\tilde{\mathcal{O}}_\beta(\gamma)}(\{\sigma^\gamma\},m^{\tilde{\mathcal{O}}_\beta(\gamma),\eta,\eta',z} \cdot C_{XY}(\{u^\beta = 0\},\ldots)
\times Z^{\tilde{\mathcal{W}}^{\text{flavor}}_{\tilde{R}_k}}(m^Y(\eta);\eta^Y(\{u^\beta = 0\},\ldots),\ldots),
\]

where the function \( Z^{\tilde{\mathcal{W}}^{\text{flavor}}_{\tilde{R}_k}}(\{P_\beta\}) \) and the function \( Z^{\tilde{\mathcal{W}}^{\text{flavor}}_{\tilde{R}_k}}(\{P_\beta\}) \) can be read off from the second line of (B.16). The functions \( Z^{\tilde{\mathcal{W}}^{\text{flavor}}_{\tilde{R}_k}}(\{P_\beta\}) \) are given by a formal Fourier transformation...
of the operator \( \mathcal{Z}_{\mathcal{O}_p^b(X)} \) defined in (B.8):

\[
Z_{\mathcal{O}_p^b(Y)}^c = \int [du^\alpha] Z_{\mathcal{O}_p^b(X)}(u^\alpha, \{u^\beta\}_{\beta \neq \alpha}, \eta_{\alpha}, m^{\mathcal{O}_p^b}, z|\Sigma') \cdot e^{2\pi i \sum_{i,\beta} \left[ g_{i}^j(\{\sigma\}', \mathcal{P}_\beta) + \sum_{i} b_{i\beta}^j \eta_{i} \right] u_{i}^\beta}.
\]

The functions \( g_{i}^j(\{\sigma\}', \mathcal{P}_\beta) \) and the parameters \( \{b_{i\beta}^j\} \) that enter on the r.h.s. of (B.19) are defined as follows:

\[
C_{X,Y}(\{u^\beta\}, \ldots, \eta) = e^{2\pi i \sum_{i,\beta} b_{i\beta}^j u_{i}^\beta \eta_{i}} \cdot C_{X,Y}(\{u^\beta = 0\}, \ldots, \eta), \quad (B.20)
\]

\[
Z_{\text{int}}^{(X,Y)(\{P_\beta\})} = e^{2\pi i \sum_{i,\beta} \left[ g_{i}^j(\{\sigma\}', \mathcal{P}_\beta) \right] u_{i}^\beta \cdot Z_{\text{int}}^{(X,Y)(\{P_\beta\})}}(\{\sigma\}', \mathcal{P}_\beta, \eta) \cdot \eta^{(\{u^\beta = 0\}, \ldots)}.
\]  

To summarize: the expression of the dual partition function in (B.18)–(B.19) will serve as the working definition for the dual S-type operation acting on the quiver gauge theory, decorated by a Wilson defect \( D' \). If the dual theory \( Y' \) is Lagrangian, the r.h.s. of (B.18) can be rewritten in the standard form that makes the gauge group and matter content of the theory manifest. The Wilson defect can then be read off from the defect partition function.

C \quad S-type operation: partition function analysis for the Abelian quiver

The partition functions of the theories \( (X,Y) \) are given as

\[
Z^{(X)}(m, t) = \int ds \frac{e^{2\pi i (t_1 - t_2)s}}{\prod_{i=1}^{n} \cosh \pi (s - m_i)} = \int ds Z_{\text{int}}^{(X)}(s, m, t), \quad (C.1)
\]

\[
Z^{(Y)}(t, m) = \int \prod_{k=1}^{n-1} d\sigma_k \frac{\prod_{k=1}^{n-1} e^{2\pi i (m_k - m_{k+1}) \sigma_k}}{\prod_{k=1}^{n-1} \cosh \pi (\sigma_k - \sigma_{k+1}) \cosh \pi (\sigma_{n-1} - t_2)}
\]

\[
=: \int \prod_{k=1}^{n-1} d\sigma_k Z_{\text{int}}^{(Y)}(\{\sigma_k\}, m, t), \quad (C.3)
\]

where \( Z_{\text{int}}^{(X)} \) and \( Z_{\text{int}}^{(Y)} \) are the respective matrix model integrands for \( X \) and \( Y \), and the parameters \( m, t \) are unconstrained. Mirror symmetry of \( X \) and \( Y \) implies that

\[
Z^{(X)}(m, t) = C_{X,Y}(m, t) Z^{(Y)}(t, -m), \quad (C.4)
\]

\[
C_{X,Y}(m, t) = e^{2\pi i (m_1 t_1 - m_n t_2)}, \quad (C.5)
\]

where \( C_{X,Y} \) is a contact term. The partition function of the theory \( X' = \mathcal{O}_p^b(X) \), where the operation \( \mathcal{O}_p^b \) is specified by (3.3)–(3.4), is given as

\[
Z_{\mathcal{O}_p^b(X)} = \int du^\alpha Z_{\mathcal{O}_p^b(X)}(u^\alpha, \eta_{\alpha}, m^f, \mu) \cdot Z^{(X,p)}(u, \ldots; t). \quad (C.6)
\]

The operator \( Z_{\mathcal{O}_p^b(X)} \), constructed from the general prescription of section B.1, is given as:

\[
Z_{\mathcal{O}_p^b(X)} = Z_{\text{F1}}(u^\alpha, \eta_{\alpha}) Z_{\text{hyper}}^{\text{1-loop}}(u^\alpha, m^f_{\mathcal{P}}) \int \prod_{j=1}^{p} du_j \prod_{j=1}^{p} \delta u^\alpha_j = \prod_{a=1}^{c} \cosh \pi (u^\alpha - m_a) \int \prod_{j=1}^{p} du_j \prod_{j=1}^{p} \delta(u^\alpha_j - u^\alpha_j + \mu_j). \quad (C.7)
\]
Therefore, we have,

\[
Z^{(X')} = \int ds \, du^\alpha \frac{e^{2\pi i n_\alpha \eta^\alpha} e^{2\pi i (t_1 - t_2) s}}{\prod_{i=1}^{n-p} \cosh \pi (s - m^{(1)}_i) \prod_{\alpha=1}^p \cosh \pi (u^\alpha - m^{(2)}_\alpha) \prod_{j=1}^p \cosh \pi (s - u^\alpha - m^{\text{bif}}_j)},
\]

where the fundamental masses \(m^{(1)}\), \(m^{(2)}\) and the bifundamental masses \(m^{\text{bif}}\) are defined as:

\[
\begin{align*}
    m^{(1)}_i &= m_i, & i &= 1, \ldots, n - p, \\
    m^{(2)}_a &= m^f_a, & a &= 1, \ldots, l, \\
    m^{\text{bif}}_j &= \mu_j, & j &= 1, \ldots, p.
\end{align*}
\]

The dual theory can be worked out from the general construction in (B.18)–(B.19), with the function \(Z_{\mathcal{O}_p}(Y)\) being given as:

\[
Z_{\mathcal{O}_p}(Y) = \int du^\alpha Z_{\mathcal{O}_p}(X) \cdot \prod_{j=1}^p e^{2\pi i g'((\sigma_k), \mathcal{P}) + \sum_i b^j_1 t_1} u_j, \\
\prod_{j=1}^p e^{2\pi i g'((\sigma_k), \mathcal{P}) u_j} = \left( \prod_{j=1}^{p-1} e^{2\pi i u_j (\sigma_{n-p+j-1} - \sigma_{n-p+j})} \right) e^{2\pi i u_p \sigma_{n-1}}, \\
\prod_{j=1}^p e^{2\pi i u_j \sum_i b^j_1 t_1} = e^{-2\pi i u_p t_2}.
\]

The dual partition function, therefore, can be written as:

\[
Z_{\mathcal{O}_p}(m'; \eta') = C_{X'Y'} \cdot \int d\sigma_k \prod_{b=1}^{l-1} d\tau_b Z^{(\text{int})}_{1-\text{loop}}(\tau_1, \sigma_{n-p}, 0) Z^{(\text{int})}_{(\tau_{n-1})}(\{\tau_b\}, -m^f, \eta_\alpha - t_2) \\
\times Z^{(Y')}(\{\sigma_k\}, t, \{-m_1, \ldots, -m_{n-p}, -\mu_1, \ldots, -\mu_p\}),
\]

where the mass parameters \(\mu_j = \mu_j + m^f_1\) for \(j = 1, \ldots, p\), and the functions \(C_{X'Y'}\) and \(Z^{(\text{int})}_{(\tau_{n-1})}\) are given as:

\[
C_{X'Y'}(m^f, \mu, t) = e^{2\pi i m^f_1 (\eta_\alpha - t_2)} e^{2\pi i (m_1 t_1 - \mu_2)},
\]

\[
Z^{(\text{int})}_{(\tau_{n-1})}(\{\tau_k\}, m^f, \eta_\alpha - t_2) = \frac{\prod_{b=1}^{l-1} e^{2\pi i \tau_b (m^f_b - m^f_{b+1})}}{\prod_{b=1}^{l-1} \cosh \pi (\tau_b - \tau_{b+1}) \cosh \pi (\tau_{n-1} - \eta_\alpha + t_2)}.
\]

From the r.h.s. of (16), one can read off the gauge group and the matter content of the dual theory, and the dual partition function can then be written as:

\[
Z_{\mathcal{O}_p}(m'; \eta') = C_{X'Y'}(m^f, \mu, t) \cdot Z^{(Y')}(m'_t, \eta'_m; m^f, m^f, \mu),
\]

where \(Y'\) is the quiver gauge theory shown in figure 11, and \(C_{X'Y'}\) is a contact term associated with the new duality. The mirror map can also be read off from the r.h.s. of (16), and can be summarized as follows. Note that the quiver \(Y'\) can be decomposed.
into two linear quivers — chain (1) of $n - 1$ gauge nodes and chain (2) of $l - 1$ gauge nodes, where the $(n - p)$-th gauge node of chain (1) is connected to the first gauge node of chain (2) by a single bifundamental hypermultiplet. Let us label the FI parameters of the linear chain (1) of $n - 1$ nodes as $\eta_{k}^{(1)} = t^{(1)}_k - t^{(1)}_{k+1}$, with $k = 1, \ldots, n - 1$. Also, let us label the FI parameters of the linear chain (2) of $l - 1$ nodes as $\eta_{b}^{(2)} = t^{(2)}_b - t^{(2)}_{b+1}$, with $b = 1, \ldots, (l - 1)$. The mirror maps associated with the parameters $t^{(1)}$ and $t^{(2)}$ are then given as

$$
t^{(1)} = \{m_1, \ldots, m_{n-1}, m_1^{\text{bif}}, \ldots, m_p^{\text{bif}}\} =: \{m_1^{(1)}, m_1^{(2)}\}, \quad (C.20a)
$$

$$
t^{(2)} = \{m_p, \ldots, m_l\} =: \{m_2^{(2)}\}. \quad (C.20b)
$$

Similarly, the hypermultiplet masses of the theory $Y'$ are given in terms of the FI parameters of $X'$. In writing (C.16), we chose to set the masses of the bifundamental hypers to zero, while the masses of the fundamental hypers are non-zero. The mirror map relating the fundamental masses of $Y'$ to FI parameters of $X'$ is then given as:

$$
m_1^{(1)} = t_1, \quad (C.21a)
m_1^{(2)} = t_2, \quad (C.21b)
m_{l-1}^{(2)} = -\eta_a + t_2 = t_3, \quad (C.21c)
$$

where $m_l^{(i)}$ denotes the mass of the fundamental hyper associated with the $l$-th gauge node in the $i$-th linear chain, and $\eta_a = t_2 - t_3$. With this parametrization of masses and FI parameters, the partition functions of $X'$ and $Y'$ are related as

$$
Z^{(X')}\left(\{m_1^{(1)}, m_2^{(2)}, m_1^{\text{bif}}\}, \eta_a, t\right) = C_{X'Y'} \cdot Z^{(Y')}\left(\{m_1^{(1)}, m_1^{(2)}\}, \{-t^{(1)}, -t^{(2)}\}\right), \quad (C.22)
$$

$$
C_{X'Y'} = e^{2\pi i (\eta_a - t_3)} m_1^{(2)} e^{2\pi i (m_1^{(1)} - m_p^{\text{bif}})} t_2. \quad (C.23)
$$

D  $S$-type operation: partition function analysis for the non-Abelian quiver

The partition functions of the theories $(X,Y)$ are given as

$$
Z^{(X)}(m,t) = \int \frac{d^2s}{2!} \frac{e^{2\pi i \text{Tr}s(t_1-t_2)} \sinh^2 \pi (s_1 - s_2)}{\prod_{j=1}^2 \prod_{i=1}^4 \cosh \pi (s_j - m_i)} =: \int \frac{d^2s}{2!} Z^{(X)}(\sigma, m, t), \quad (D.1)
$$

$$
Z^{(Y)}(t;m) = \int d\sigma^1 \left[ d\sigma^2 \right] d\sigma^3 \times e^{2\pi i \sigma_1(m_1-m_2)} e^{2\pi i \sigma_2(m_2-m_3)} e^{2\pi i \sigma_3(m_3-m_4)} \sinh^2 \pi (\sigma_1^2 - \sigma_2^2) \prod_{j=1}^2 \cosh \pi (\sigma_j - \sigma_j^2) \prod_{k=1}^4 \cosh \pi (\sigma_k - t_a) \cosh \pi (\sigma_3 - \sigma_1^2) \quad (D.2)
$$

$$
=: \int d\sigma^1 \left[ d\sigma^2 \right] d\sigma^3 Z^{(Y)}(\{\sigma^2\}, t, m), \quad (D.3)
$$

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where \( Z^{(X)}_{\text{int}} \) and \( Z^{(Y)}_{\text{int}} \) are the respective matrix model integrands for \( X \) and \( Y \), and the parameters \( \mathbf{m}, \mathbf{t} \) are unconstrained. Mirror symmetry of \( X \) and \( Y \) implies that
\[
Z^{(X)}(\mathbf{m}; \mathbf{t}) = C_{XY}(\mathbf{m}, \mathbf{t}) Z^{(Y)}(\mathbf{t}; -\mathbf{m}),
\]
(\text{D.4})
where \( C_{XY} \) is a contact term.

The partition function of the theory \( X' = \mathcal{O}_P(X) \), where the operation \( \mathcal{O}_P \) is specified in (4.3)–(4.4), is then given as
\[
Z^{\mathcal{O}_P(X)} = \int \prod_{i=1}^{3} du_i \mathcal{Z}_{\mathcal{O}_P(X)}(u, \eta, \mathbf{m}', \mathbf{\mu}) \cdot Z^{(X)}(u, v; t).
\]
(\text{D.6})

The operator \( \mathcal{Z}_{\mathcal{O}_P(X)} \), constructed from the general prescription of section B.1, has the form:
\[
\mathcal{Z}_{\mathcal{O}_P(X)} = e^{2\pi i u_3} \left( \prod_{b=1}^{2} \frac{d m'_b}{\Pi_{a=1}^{2} \cosh \pi (u_1 - m'_a)} \right) \cdot \frac{e^{2\pi i u_1}}{\Pi_{a=1}^{4} \cosh \pi (u_1 - m'_a)}.
\]
(\text{D.7})
The partition function of the theory \( X' \) can be then written as
\[
Z^{X'} = \int \prod_{i=1}^{3} du_i \frac{d^2 s}{2!} \cdot e^{2\pi i \sum_{i} \eta_i u_i} \cdot e^{2\pi i \sum_{j=1}^{3} \eta_j u_j} \cdot \frac{e^{2\pi i \text{Tr} (t_1 - t_2)} \sinh^2 \pi (s_1 - s_2)}{\Pi_{j=1}^{2} \cosh \pi (s_j - u_j)} \cdot \cosh \pi (s_j - m^{(0)}),
\]
(\text{D.8})
where the fundamental masses \( m^{(0)}, m^{(1)}, m^{(2)} \) and the bifundamental masses \( m^{\text{bif}} \) are given in terms of mass parameters introduced above in the following fashion:
\[
m^{(0)} = v, \quad (\text{D.9})
m^{(1)}_a = m'_a + 2, \quad a = 1, 2, \quad (\text{D.10})
m^{(2)}_c = m'_c, \quad c = 1, 2, \quad (\text{D.11})
m^{\text{bif}}_b = \mu_b, \quad b = 1, 2. \quad (\text{D.12})
\]
The dual theory can be read off from the general construction in (B.18)–(B.19). The function \( \mathcal{Z}_{\mathcal{O}_P(Y)} \) is given as
\[
\mathcal{Z}_{\mathcal{O}_P(Y)} = \int \prod_{i=1}^{3} du_i \mathcal{Z}_{\mathcal{O}_P(X)} \cdot \prod_{j=1}^{3} e^{2\pi i (g^j((\sigma^k), \mathcal{P}) + b^j t_1) u_j},
\]
(\text{D.13})
\[
\prod_{j=1}^{3} e^{2\pi i (g^j((\sigma^k), \mathcal{P}) + b^j t_1) u_j} = e^{2\pi i u_1 (-\sigma + \text{Tr} \sigma^2 - t_2)} e^{2\pi i u_2 (\sigma^3 - t_2)} e^{2\pi i u_3 (-\sigma^1 + t_1)}.
\]
(\text{D.14})
The dual partition function can then be computed in the standard fashion. After some change of variables, the dual partition function can be written as

\[
Z_{\bar{\mathcal{Y}^{(Y)}}}(m';\eta') = C_{X^{(Y')}} \cdot \int \frac{d^2 \sigma}{2 \pi i} \prod_{k=1}^{5} d\tau_k \sinh^2 \pi (\sigma_1 - \sigma_2) \prod_{k=1}^{5} \cosh \pi (\sigma_i - \tau_1) \cosh \pi (\sigma_i - \tau_1) \cosh \pi (\tau_1 - \tau_2) \cosh \pi (\tau_2 - \tau_3) \cosh \pi (\tau_3 - \tau_4) \cosh \pi (\tau_4 - \tau_5) \cosh \pi (\tau_5 - \tau_1) \cosh \pi (\tau_1 - \tau_2) \cosh \pi (\tau_2 - \tau_3) \cosh \pi (\tau_3 - \tau_4) \cosh \pi (\tau_4 - \tau_5) \cosh \pi (\tau_5 - \tau_1)
\]

The r.h.s. of (D.15) can then be identified as the partition function of the theory \( Y' \) in figure 19, where \( \sigma \) label the matrix integral variables for the U(2) gauge node, while \( \{ \tau_i \}_{i=1,...,5} \) label that for the U(1) gauge nodes (in the order labelled in the figure). Therefore, we have

\[
Z_{\bar{\mathcal{Y}^{(Y)}}}(m';\eta') =: C_{X^{(Y')}} \cdot Z^{(Y')}(m';-\eta'),
\]

where \( C_{X^{(Y')}} \) should be identified as the new contact term. The mirror map relating the FI parameters of the theory \( Y' \) with the masses of the theory \( X' \) is given as follows:

\[
\begin{align*}
\eta_0 &= (m^{(0)} - m^{(2)}), \\
\eta_1 &= m^{1}_{1}, \\
\eta_2 &= m^{1}_{2}, \\
\eta_3 &= m^{1}_{1} + m^{1}_{2}. \\
\eta_4 &= m^{1}_{1} - m^{1}_{2}, \\
\eta_5 &= m^{1}_{2} - m^{1}_{1}.
\end{align*}
\]

The mirror map relating the mass parameters of the theory \( Y' \) with the FI parameters of the theory \( X' \) can also be read off from (D.15).

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