Remarks on homotopy equivalence of configuration spaces of a polyhedron

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Abstract
We show that the configuration space $F_n(M)$ of $n$ particles in a compact connected PL manifold $M$ with nonempty boundary $\partial M$ is homotopy equivalent to the configuration space $F_n(\text{Int } M)$ where $\text{Int } M = M \setminus \partial M$. Actually we prove some generalization of this result for polyhedra. Similar results recently have been obtained independently for topological manifolds by Zapata (Collision-free motion planning on manifolds with boundary, 2017. arXiv:1710.00293), using different techniques. We also address the question of whether a compact PL manifold $M$ can be approximated up to homotopy type by discrete configuration spaces defined combinatorially via a simplicial subdivision of $M$.

Keywords Configuration space · PL manifold · Deformation retraction · Equivariant map · Polyhedron · Collar · Discrete configuration space

Mathematics Subject Classification 57Q91

1 Introduction
Let $X$ be a topological space and $X^k$ its $k$-fold Cartesian product, $k \geq 2$. Define the diagonal $D$ of $X^k$ as follows: $D = \{(x_1, \ldots, x_k) \in X^k : x_i = x_j \text{ for some } i \neq j\}$.

For a given topological space $X$, denote by $F_k(X)$ the space $X^k \setminus D$, the configuration space of $k$ particles in $X$ without collisions. The symmetric group $\Sigma_k$ acts freely on $F_k(X)$ by permuting coordinates of $X^k$. The topology of classical configuration spaces $F_k(\mathbb{R}^n)$ was studied by many authors (see, for example, [5,8] for the background). A fundamental work on this topic is the monograph by Fadell and Husseini...
[7], in which the case of sphere \( X = S^m \) is also treated. The homology structure of \( F_k(\mathbb{R}^n) \) was described, for example, in [4]. It is also known that configuration spaces are not homotopy invariant even for closed manifolds (see [11]).

In this paper, we prove that if \((Q, P)\) is a pair of compact polyhedra, the subpolyhedron \( P \) has a collar in \( Q \), and the homotopy equivalence of the space \( Q \setminus P \) and the polyhedron \( Q' \) is given by a deformation retraction of one onto another inside a collar of the subpolyhedron \( P \), then it extends to a retraction of corresponding configuration spaces. It follows that if \( M \) is a compact piecewise linear (PL) manifold and the homotopy equivalence of manifolds \( M \setminus \partial M \) and \( M' \), where \( M' \subset M, M' \cong M \), is given by a deformation retraction of the first one onto the other one inside a collar of the boundary \( \partial M \), then it descends to a deformation of corresponding configuration spaces.

2 Configuration spaces of polyhedra and compact manifolds with boundary

Let \((Q, P)\) be a pair of polyhedra such that \( P \) is a compact subpolyhedron of \( Q \) that has a PL collar in \( Q \). In this section, we compare the configuration space of the polyhedron \( Q \) with the configuration space of the “open” subspace \( Q \setminus P \). In particular, we will show that if \( M \) is a compact PL manifold with nonempty boundary \( \partial M \), then the configuration spaces \( F_k(\text{Int } M) \) and \( F_k(M) \) are \( \Sigma_k \)-equivariantly homotopy equivalent.

Before proving a general result, we first demonstrate how our approach works in the particular case, when \( M \) is a closed unit disk of the Euclidean space. Let \( D^n \) be a closed \( n \)-dimensional disc in \( \mathbb{R}^n \). The proof of the following lemma uses the techniques developed by Crowley and Skopenkov in [6].

**Lemma 2.1** For each positive integer \( k \) the space \( F_k(D^n) \) is a deformation retract of the space \( F_k(\mathbb{R}^n) \). Moreover, there is a \( \Sigma_k \)-equivariant deformation retraction of \( F_k(\mathbb{R}^n) \) onto \( F_k(D^n) \).

**Proof** Let \( S^n \) be the \( n \)-dimensional sphere, \( S^n = \mathbb{R}^n \cup \{ \infty \} \). Decompose \( S^n \) into two half-spheres, \( S_0 \) and \( S_\infty \), where \( S_0 = \{ w \in \mathbb{R}^n : |w| \leq 1 \} \) and \( S_\infty = \text{cl}(S^n \setminus S_0) \). Consider the subspace \( R = S^n \setminus \{0\} \) which is obviously homeomorphic to \( \mathbb{R}^n \). There is a \( \Sigma_k \)-equivariant deformation retraction \( g_t \) of \( F_k(R) \) on \( F_k(S_\infty) \). To show this, consider in \( \mathbb{R}^n \subset S^n \) a closed disc \( D_2 \) of radius 2 centered at 0. The half-sphere \( S_0 \) is identified with a closed unit disc \( D_1 \).

For each \( s = 1, \ldots, k \), define a map \( f_s : F_k(R) \to S_\infty \) as follows:

(i) \( f_s(x_1, \ldots, x_k) = x_s \) if no \( x_i, i = 1, \ldots, k \), is contained in \( \text{Int } D_1 \);

(ii) If \( x = (x_1, \ldots, x_k) \in F_k(R) \setminus F_k(S_\infty) \), take any \( j \) such that \( \min_s |x_s| = |x_j| \).

Denote \( |x_j| \) by \( \rho \). We obviously have \( 0 < \rho < 1 \). Put

\[
 f_s(x_1, \ldots, x_k) = \frac{x_s}{|x_s|} \frac{2 - 2\rho + |x_s|}{2 - \rho} \quad \text{if} \quad x_s \in D_2; 
\]

(iii) \( f_s(x_1, \ldots, x_k) = x_s \) if \( x_s \) is not in \( \text{Int } D_2 \).
Each coordinate function $f_s$ is fixed on points $x \in F_k(S_\infty)$ and on the points $x = (x_1, \ldots, x_s, \ldots, x_k) \in F_k(R) \setminus F_k(S_\infty)$ with $|x_s| \geq 2$. For the points $(x_1, \ldots, x_s, \ldots, x_k) \in F_k(S_\infty)$ with $|x_s| \leq 2$, it acts along the rays in $R^n$ originating at 0. In this case, it looks like a monotone PL function $h : [0, 2] \rightarrow [0, 2]$ with $h(0) = 0, h(2) = 2$ and $h'(\rho) = 1$, which is linear on the intervals $[0, \rho]$ and $[\rho, 2]$.

In particular, if $|x_s| = \rho$, where $x_s$ is the $s$-th coordinate of the point $(x_1, \ldots, x_k)$, we have $|f_s(x)| = 1$. Substituting $|x_s| = 2$ in the formula $x_s = \frac{2-2\rho+|x_s|}{2-\rho}$, we get $f_s(x) = x_s$. Moreover if $\rho < |x_s| < 2$, then $1 < |f_s(x_1, \ldots, x_k)| < 2$. It follows that each function $f_s$ is well defined.

It is clear that $f_s$ is continuous at $x \in F_k(S_\infty)$. If $x \in F_k(R) \setminus F_k(S_\infty)$ and $x_s$ is in the exterior of the disc $D_2$ or in $\partial D_2$, the function $f_s$ depends only on $x_s$ and we have $f_s(x) = x_s$. If $x \in F_k(R) \setminus F_k(S_\infty)$ and $x_s$ is in the interior of the disc $D_2$, the value $f_s(x)$ depends continuously on the parameter $\rho$. On the other hand, the function $\rho$ is the minimum of finite number of continuous functions (the norms $|x_s|$). So within a small neighborhood $U(x)$ the parameter $\rho(x)$ also changes very little. It follows that $f_s$ is continuous at the points $x \in F_k(R) \setminus F_k(S_\infty)$ with $|x_s| < 2$. Finally if $x \in F_k(R) \setminus F_k(S_\infty)$ and $|x_s| = 1$, the above remarks and formula (ii) show that $f_s$ is continuous also at the point $x$.

Define a map $f : F_k(R) \rightarrow (S_\infty)^k$ by the formula $f = (f_1, \ldots, f_s, \ldots, f_k)$. It follows that $f_s$ is the $s$-th coordinate function of $f$ and the map $f$ itself is continuous. Actually $f$ maps the points $x = (x_1, \ldots, x_s, \ldots, x_k)$ with distinct coordinates $x_s$ to the points $y = (y_1, \ldots, y_s, \ldots, y_k)$ with different coordinates $y_s$. This is obvious for the points $x \in F_k(S_\infty)$ and for the points $x = (x_1, \ldots, x_s, \ldots, x_k)$ such that all $x_k$ lie on different rays of the space $R^n$. On the other hand, if some coordinates $x_j$ and $x_j$ of $x \in F_k(R) \setminus F_k(S_\infty)$ are on the same ray, then $|x_j| = |x_j|$ and $|f_j(x)| = |f_j(x)|$, according to the monotonic property of each coordinate function $f_s$. Therefore $f$ maps the configuration space $F_k(R)$ onto the configuration space $F_k(S_\infty)$. By the properties of the coordinate functions $f_s$, $f$ retracts the space $F_k(R)$ onto the space $F_k(S_\infty)$. Moreover it is not difficult to see that $f$ is actually a $\Sigma_k$-equivariant retraction of $F_k(R)$ onto $F_k(S_\infty)$.

Each map $f_s$ obviously admits an extension to a homotopy $g_t^s$ via the following formula:

$$g_t^s(x_1, \ldots, x_s, \ldots, x_k) = (1-t)x_s + tf_s(x_1, \ldots, x_s, \ldots, x_k), \quad 0 \leq t \leq 1.$$

For each $s, 1 \leq s \leq k$, and each $x \in F_k(S_\infty)$ the homotopy $g_t^s$ keeps the coordinate $x_s$ of $x$ point-wise fixed. Put $g_t = (g_1^t, \ldots, g_k^t)$ for each $t \in [0, 1]$. $g_t$ is obviously a $\Sigma_k$-equivariant deformation retraction of $F_k(R)$ onto $F_k(S_\infty)$. □

Let $Q$ be a polyhedron and $P$ its compact subpolyhedron which has a collar in $Q$. A closed collar of $P$ in $Q$ is represented by the image of PL embedding $h : P \times [0, 2] \rightarrow Q$ where $h(P \times [0, 1])$ is identified with $P$. It is a regular neighborhood of $P$ in $Q$ [10]. Denote by $U$ a small open collar of $P$ in $Q$ which is identified with the image $h(P \times [0, 1])$. Obviously, $Q \setminus U$ is homeomorphic to $Q$. It follows that $F_k(Q \setminus U)$ and $F_k(Q)$ are homeomorphic in a natural way.

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Theorem 2.2 For each \( k \) the space \( F_k(Q \setminus P) \) deformation retracts onto the subspace \( F_k(Q \setminus U) \). Moreover, the configuration space \( F_k(Q) \) is \( \Sigma_k\)-equivariantly homotopy equivalent to the configuration space \( F_k(Q \setminus P) \).

Proof Let \( R_1, \ldots, R_m \) be the connected components of \( P \). Moreover, let \( C_i, i = 1, \ldots, m \), be the closed collar of \( R_1, \ldots, R_m \), respectively, in \( Q \) where each \( C_i \) is identified with \( R_i \times [0, 2] \), \( i = 1, \ldots, m \), via the PL embedding \( h \) and \( R_i \) is identified with \( R_i \times \{0\} \) and \( C_i \cap C_j = \emptyset \) if \( i \neq j \). We also identify \( U = \cup_{i=1}^m (R_i \times [0, 1]) \) with an open collar of \( P \) in \( Q \) as before. Put \( C = \bigcup_{i=1}^m C_i \).

By the above identification, each \( z \in C \) can be uniquely represented as \( z = (x, \tau) \) where \( x \in R_j \) for some \( j \) and \( 0 \leq \tau \leq 2 \). Now we define a deformation retraction of the space \( F_k(Q \setminus P) \) onto the space \( F_k(Q \setminus U) \) as follows.

For each \( \rho \), \( 0 < \rho < 1 \), take a monotone PL function \( h_\rho : [0, 2] \to [0, 2] \) such that \( h_\rho(0) = 0 \), \( h_\rho(2) = 2 \) and \( h_\rho(\rho) = 1 \). For \( \rho \leq \lambda \leq 2 \), the function \( h_\rho \) can be expressed as follows: \( h_\rho(\lambda) = \frac{2-2\rho+\lambda}{2-\rho} \) (see also the proof of Lemma 2.1).

First, for each \( 1 \leq s \leq k \) we shall define a map \( f_s : F_k(Q \setminus P) \to Q \setminus U \). Let \( y = (y_1, \ldots, y_k) \in F_k(Q \setminus P) \). Let \( V = \{y_1, \ldots, y_s\} \) be the set of coordinates of \( y \) which belong to the open collar \( U \) of \( P \). If \( V = \emptyset \), we put \( f_s(y) = y \). Assume that \( V \neq \emptyset \). We have \( y_{is} = (x_{is}, \tau_s) \) for each \( 1 \leq s \leq r \) where \( x_{is} \in P \). Take any \( l \) such that \( \min \{\tau_s\} = \tau_l \) where \( s \) runs from \( 1 \) to \( r \). Denote \( \tau_l \) by \( \rho \). We obviously have \( 0 < \rho < 1 \). Now, the expression for the coordinate function \( f_s \) is the following:

- \( f_s(y_1, \ldots, y_k) = y_s \) if no \( y_i \) belongs to \( U \), \( i = 1, \ldots, k \);
- \( f_s(y_1, \ldots, y_k) = y_s \) if some \( y_i \) belongs to \( U \), \( i = 1, \ldots, k \), but \( y_s \) does not belong to \( C \setminus P \);
- \( f_s(y_1, \ldots, y_k) = (x_s, h_\rho(\tau_s)) \), if \( (y_1, \ldots, y_k) \in F_k(Q \setminus P) \setminus F_k(Q \setminus U) \) and \( y_s \in C \setminus P \), where \( y_s = (x_s, \tau_s), x_s \in P, 0 < \tau_s \leq 2 \).

Note that if the coordinate \( y_s \) of \( y = (y_1, \ldots, y_k) \) belongs to the collar \( C \), i.e., \( y_s = (x_s, \tau_s) \) where \( x_s \in P \) and \( \rho(y) \leq \tau_s \leq 2 \), then the \( s \)-th coordinate of \( f_s(y_1, \ldots, y_k) \) can be represented as \( (x_s, \lambda_s) \) where \( 1 \leq \lambda_s \leq 2 \). Moreover if \( \tau_s = 2 \), then \( \lambda_s = 2 \) and if \( \tau_s = \rho \), then \( \lambda_s = 1 \).

It follows that for each \( s = 1, \ldots, k \) the map \( f_s : F_k(Q \setminus P) \to Q \setminus U \) is well defined in its domain. The continuity of \( f_s \) is performed along the same line as the one of the coordinate functions in the proof of Lemma 2.1. We omit the details.

Therefore the map \( f = (f_1, \ldots, f_s, \ldots, f_k) : F_k(Q \setminus P) \to (Q \setminus U)^k \) is also continuous. Let \( y = (y_1, \ldots, y_k) \) be any point of the configuration space \( F_k(Q \setminus P) \) and let \( f_i \) and \( f_j \) be two coordinate functions of the map \( f \) where \( i \neq j \). If \( y \in F_k(Q \setminus U) \), we have \( f_i(y) = y_i \neq y_j = f_j(y) \). Now assume that some coordinate \( y_s \) of \( y \) is in the set \( U \). If one of the coordinates \( y_i \) and \( y_j \) is outside the collar \( C \), it follows immediately that \( f_i(y) \neq f_j(y) \). Assume that both \( y_i \) and \( y_j \) belong to the collar \( C \). The coordinates \( y_i \) and \( y_j \) have the following presentation: \( y_i = (x_i, \tau_i) \) and \( y_j = (x_j, \tau_j) \) where \( x_i, x_j \in P \) and \( \rho(y) \leq \tau_i, \tau_j \leq 2 \). If \( x_i \neq x_j \) it follows immediately that \( f_i(y) \neq f_j(y) \). On the other hand, if \( x_i = x_j \), then \( \tau_i \neq \tau_j \). By the monotonic property of the function \( h_\rho \), we get \( h_\rho(\tau_i) \neq h_\rho(\tau_j) \) which implies that \( f_i(y) \neq f_j(y) \). It follows that \( f \) maps \( k \)-tuples \( (y_1, \ldots, y_k) \) with distinct coordinates into \( k \)-tuples \( (z_1, \ldots, z_k) \) with distinct coordinates. Therefore \( f \) is actually a map.
from the configuration space $F_k(Q \setminus P)$ onto the configuration $F_k(Q \setminus U)$. Moreover, by the properties of the coordinate functions $f_s$, $f$ is a $\Sigma_k$-equivariant retraction of the space $F_k(Q \setminus P)$ onto the subspace $F_k(Q \setminus U)$.

The map $f$ can be extended to the deformation retraction $g'_t: F_k(Q \setminus P) \to F_k(Q \setminus P)$, $t \in [0, 1]$, with $g'_0 = \text{id}_{F_k(Q \setminus P)}$ and $g'_1 = f$. The deformation retraction $g'_t$ is defined in the same way as the homotopy $g_t$ in the proof of Lemma 2.1. We omit the details. Since the map $f$ is $\Sigma_k$-equivariant, we can arrange that the deformation retraction $g'_t$ of the space $F_k(Q \setminus P)$ onto the space $F_k(Q \setminus U)$ is also $\Sigma_k$-equivariant. This completes the proof of the theorem. \hfill $\square$

Let $M$ be a connected, compact and smooth or PL manifold with the nonempty boundary $\partial M$. Then $\partial M$ is collared in $M$. Moreover we have the following

**Corollary 2.3** For each $k \geq 1$ the configuration space $F_k(M)$ is $\Sigma_k$-equivariantly homotopy equivalent to the configuration space $F_k(\text{Int } M)$.

### 3 Discrete configuration spaces of complexes

Let $K$ be a finite simplicial complex. Denote by $|K|$ the underlying topological space of $K$ which is a polyhedron. For each $k \leq n$ the subcomplex $D_n(K)$ of the cell complex $K^n$ is defined in the following way: $D_n(K) = \bigcup \sigma_1 \times \cdots \times \sigma_n$ where the sum is over all $n$ pairwise disjoint closed cells in $K$ (see [2,3]). The subcomplex $D_n(K)$ is called the discrete configuration space of the complex $K$ with the parameter $n$. This is the largest cell complex that is contained in the product $K^n$ minus its diagonal $\{(x_1, \ldots, x_n) \in |K|^n : x_i = x_j$ for some $i \neq j\}$. The symmetric group $\Sigma_n$ acts naturally on $D_n(K)$ by permuting the cells in the product. The polyhedron $|D_n(K)|$ has natural $\Sigma_n$-equivariant embedding in the configuration space $F_n(|K|)$ for each $n \geq 2$.

A graph $G$ can be considered as a 1-complex. Abrams [1] proved that for each graph $G$ there is a subdivision $G'$ of $G$ such that the discrete configuration space $D_n(G')$ is homotopy equivalent to the usual configuration space $F_n(G)$, $n \geq 2$.

The problem of a cell approximation of the space $F_n(X)$ where $X$ is a polyhedron of dimension $\geq 2$ was considered and studied in [3]. For $n = 2$, Hu [9] showed that the configuration spaces $D_2(K)$ and $F_2(K)$ are homotopy equivalent. Moreover he showed that for any finite simplicial complex $K$ there is a $\Sigma_2$-equivariant deformation retraction of $F_2(K)$ onto $|D_2(K)|$. In general, the problem can be formulated as follows

**Problem** Let $X$ be a compact connected PL manifold of dimension $k \geq 2$ and let $n > 2$. Show that there is a subdivision $K$ of $X$ such that the manifold $F_n(X)$ admits a $\Sigma_n$-equivariant deformation retraction onto the polyhedron $|D_n(K)|$ or give a counterexample.

To the best of our knowledge, for PL manifolds of dimension $k \geq 2$, the question of cell approximation of configuration spaces remains open.

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References

1. Abrams, A.D.: Configuration Spaces of Braid Groups of Graphs. Ph.D. thesis, University of California, Berkeley (2000)
2. Abrams, A., Gay, D., Hower, V.: Discretized configurations and partial partitions. Proc. Amer. Math. Soc. 141(3), 1093–1104 (2013)
3. An, B.H., Drummond-Cole, G.C., Knudsen, B.: Subdivisional spaces and graph braid groups (2017). arXiv:1708.02351v1
4. Bödigheimer, C.-F., Cohen, F., Taylor, L.: On the homology of configuration spaces. Topology 28(1), 111–123 (1989)
5. Cohen, F.R.: Introduction to configuration spaces and their applications. In: Berrick, A.J., et al. (eds.) Braids. Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, vol. 19, pp. 183–261. World Scientific, Hackensack (2009)
6. Crowley, D., Skopenkov, A.: Embeddings of non-simply-connected 4-manifolds in 7-space III. Piecewise-linear classification (preprint)
7. Fadell, E.R., Husseini, S.Y.: Geometry and Topology of Configuration Spaces. Springer Monographs in Mathematics. Springer, Berlin (2001)
8. Fadell, E., Neuwirth, L.: Configuration spaces. Math. Scand. 10, 111–118 (1962)
9. Hu, S.: Isotopy invariants of topological spaces. Proc. Roy. Soc. London Ser. A 255, 331–366 (1960)
10. Hudson, J.F.P.: Piecewise Linear Topology. University of Chicago Lecture Notes prepared with the assistance of J. L. Shaneson and J. Lees. W.A. Benjamin, New York (1969)
11. Longoni, R., Salvatore, P.: Configuration spaces are not homotopy invariant. Topology 44(2), 375–380 (2005)
12. Zapata, C.A.I.: Collision-free motion planning on manifolds with boundary (2017). arXiv:1710.00293

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