Statistics of skyrmions in quantum Hall systems

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Abstract

We analyze statistical interactions of skyrmions in the quantum Hall system near a critical filling fraction in the framework of the Ginzburg-Landau model. The phase picked up by the wave-function during an exchange of two skyrmions close to $\nu = 1/(2n+1)$ is $\pi[S + 1/(2n+1)]$, where $S$ is the skyrmion’s spin. In the same setting an exchange of two fully polarized vortices gives rise to the phase $\pi/(2n+1)$. Skyrmions with odd and even numbers of reversed spins have different quantum statistics. Condensation of skyrmions with an even number of reversed spins leads to filling fractions with odd denominators, while condensation of those with an odd number of reversed spins gives rise to filling fractions with even denominators.

Recently skyrmion excitations near the filling factor $\nu = 1$ have drawn remarkable attention [1, 2, 3]. There are two kinds of topological excitations in single-layer quantum Hall systems. When the system is fully polarized, the relevant charged quasiparticles are topological vortices. At $\nu = 1/(2n+1)$ the charge of such a vortex is $e/(2n+1)$ and the spin is $1/2(2n+1)$.

It has been suggested [1] that for a weak Zeeman coupling the lowest energy quasiparticle is a skyrmion or a slowly varying topological texture. Its electric charge is still $e/(2n+1)$ but its total spin $S$ can be substantially larger. Such large spin but moderate charge excitations can explain the observed depolarization as the filling factor slightly deviates from $\nu = 1/(2n+1)$ [4].

It has been proposed to describe skyrmion excitations in terms of an effective ferromagnetic model [1]. The model explains the dependence of the skyrmion size on relative strength of Coulomb and Zeeman interaction. However, it does not predict correctly skyrmion energies. The energy of skyrmion and antiskyrmion seem to be the same. Microscopic calculations [2] show that it is not the case. The effective model has been derived [1, 5] from the Ginzburg-Landau model of the quantum Hall system [6, 7]. In the derivation, among other things, the charge density was assumed to be constant. This leaves polarized vortices outside the reach of the effective model. In terms of the Ginzburg-Landau model the difference between a vortex and a skyrmion is that a vortex is fully polarized while inside the skyrmion core there is some number of electrons with reversed spins. Such a depolarization helps to minimize the total Coulomb energy. It costs some Zeeman energy but it is not an obstacle when the Zeeman coupling is sufficiently weak.

In this paper we consider some topological properties of skyrmions and vortices in the framework of the Ginzburg-Landau model. We show that the Magnus force acting on vortices and skyrmions is the same. We also consider statistical interactions of skyrmions. We will show that during a clockwise exchange of two skyrmions the wave-function picks up the geometrical phase

$$\Gamma = [S + 1/(2n+1)] \int_{t_1}^{t_2} \frac{d}{dt} \text{Arg}[\xi_1(t) - \xi_2(t)] ,$$

where $S$ is the skyrmion’s spin. This formula applies also to vortices when we put $S = S_{\text{vortex}} = 1/(2n+1)$. Thus the Ginzburg-Landau treatment shows that the statistical interaction is much stronger for skyrmions than for vortices. It is proportional to the number of reversed spins.

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1 The Ginzburg-Landau model and topological defects

We wish to investigate the statistical interaction of skyrmions in the quantum Hall effect. We will describe this system by the Zhang-Hansson-Kivelson model (3) generalized by Kane and Lee (6) to describe unpolarized quantum Hall systems.

The Lagrangian of the model is

\[ L = i\phi^\dagger (\partial_t + ia_0)\phi - \frac{1}{2m} |(\partial_k + ia_k + eA_k)\phi|^2 + \frac{1}{4\Theta} \varepsilon_{µσ} a_µ \partial_σ a_σ - \frac{1}{2} \lambda [\phi^\dagger \phi(\vec{x}) - \rho_0]^2 + \gamma [\phi^\dagger \phi - \phi^\dagger \phi]\ . \]  

Greek indices denote 0, 1, 2, while Latin indices take values 1, 2. When they are repeated, summation is understood. We use the signature (+, −, −). φ is a two-component complex scalar field, φ = (φ₁, φ₂). a_µ is a statistical gauge field while \( A_k = -\frac{1}{2} B \varepsilon_{kl} x^l \) is a gauge potential of the external uniform magnetic field B directed down the z-axis. \( e\rho_0 \) is the positive background charge density, related to the external magnetic field by \( eB = 2\Theta\rho_0 \). For the boson field to represent fermions the parameter Θ must take one of the values \( \Theta = (2n + 1)\pi \), where n is a nonnegative integer. m is the effective electronic mass and γ is the effective Zeeman coupling. μ is a chemical potential, chosen so that the ground state of the system (4) is, up to a gauge transformation, the solution \( \phi_1 = \sqrt{\rho_0}, \phi_2 = 0 \) with the statistical gauge field “screening” the external magnetic field, \( a_k + eA_k = 0 \). The chemical potential has to be \( \mu = -\gamma \). In this ground state the system is fully polarized. The Lagrangian (2) should be supplemented by the Coulomb interaction term

\[ -\frac{1}{2} \int d^2 x' |\phi^\dagger \phi(\vec{x}) - \rho_0| \frac{e^2/\varepsilon}{|\vec{x} - \vec{x}'|} |\phi^\dagger \phi(\vec{x}') - \rho_0| \ , \]  

where \( \varepsilon \) is a dielectric constant of the host material.

There are two types of relevant topological excitations in the model. One of them is simply a vortex or a fully polarized quasihole. This configuration is given by the Ansatz

\[ \phi_1 = f_v(r)e^{-i\theta} \ , \]
\[ \phi_2 = 0 \ , \]
\[ a_\theta = \frac{eB}{2} r + a_v(r) \ , \]
\[ a_0 = b_v(r) \ . \]  

The modulus \( f_v(r) \) has to interpolate between \( f_v(0) = 0 \) and \( f_v(\infty) = \sqrt{\rho_0} \). The asymptote of \( a_v(r) \) at infinity is \( a_v(r) \approx \frac{1}{r} \).

The vortex is by definition restricted to the lower spin component. This restriction is consistent with field equations of the model (3) for any value of the Zeeman coupling γ. Such a solution indeed exists as is well known from the studies on vortices in the Ginzburg-Landau model of the fully polarized quantum Hall effect (3). The total charge is proportional to the statistical magnetic flux and equals to \( e/(2n + 1) \). The distribution of the electronic density is such that it vanishes in the very center of the vortex and tends to \(-e\rho_0\) outside the vortex core. This is not the most favourable charge distribution from the point of view of both the Coulomb interaction and the quartic self-interaction term. While the total charge is fixed, the Coulomb interaction tends to minimize the integral of the charge density squared. In other words it tends to make the charge distribution as diluted as possible. When we restrict to the spin-down component, the deviation of the charge density from \(-e\rho_0\) must equal to \( e\rho_0 \) at the center of the vortex. This is a topological obstruction. At the same time the total charge is quantized so there is not much freedom left to make the charge distribution more dilute. However, for sufficiently small Zeeman coupling the configuration may find a way to minimize its Coulomb energy by nucleation of the spin-up component. A more general configuration, with more freedom to minimize the Coulomb energy, is

\[ \phi_1 = f_s(r)e^{-i\theta} \ , \]
\[ \phi_2 = g(r) \ , \]
\[ a_\theta = \frac{eB}{2} r + a_s(r) \ , \]
\[ a_0 = b_s(r) \ . \]  

Now we admit a nonzero spin-up component. We want it to be nonzero in the vortex core, \( g(0) \neq 0 \), and to cost a minimal gradient energy so the phase of the spin-up component has to be constant. For a finite energy configuration \( g(\infty) = 0 \). Excitation of the upper component costs some Zeeman energy so it is not energetically favourable except for small effective Zeeman couplings γ, where the gain in Coulomb energy can prevail the loss in Zeeman energy. Note that with the Ansatz (4) the average spin points up in the middle of the soliton while it points down outside the core. This configuration is a skyrmion but in the language of the untruncated model (3). As compared to a uniform background with charge density \( \rho_0 \), the ↑ component adds a negative charge \( Q_↓ \),

\[ Q_↓ = -2\pi e \int_0^\infty r dr \ g^2(r) \ . \]
while the ↓ component contributes a positive electronic charge deficit
\[ Q_\downarrow = -2\pi e \int_0^{\infty} r dr \left[ f^2(r) - \rho_0 \right]. \] (7)

The total charge is the same as for the fully polarized vortex: the two contributions add up to \( Q_\uparrow + Q_\downarrow = \frac{e}{2n+1} \).

Thus there is an interplay of two factors, namely the Coulomb and the Zeeman energy. For strong Zeeman coupling vortices are the relevant quasiparticles. At weak Zeeman coupling vortices are still solutions of field equations but they have higher energy than skyrmions. In this limit skyrmions become the relevant quasiparticles.

2 Magnus force acting on skyrmions and vortices

We apply the adiabatic approximation to investigate the dynamics of vortices and skyrmions. The only terms in the Lagrangian \( \mathcal{L} \), which can contribute to the terms in the effective mechanical Lagrangian which are linear in velocity, are
\[ L_{eff}^{(1)} = \int d^2x \left[ i\phi^\dagger \partial_t \phi - \frac{1}{4\Theta} \varepsilon^{kl} \partial_k \phi \partial_l a_i \right]. \] (8)

The prescription for the adiabatic approximation is as follows. Take the static vortex or skyrmion solution described by the fields \( \{ \phi(\vec{x} - \vec{\xi}), a_\mu(\vec{x} - \vec{\xi}) \} \) and located at an arbitrary position \( \vec{\xi} \). Promote the parameter \( \vec{\xi} \) to the role of a time-dependent collective coordinate \( \vec{\xi}(t) \). In this way the static fields become time-dependent. The final step is to substitute such a time-dependent field configurations to the Lagrangian and integrate out their spatial dependence. After the spatial integration one should be left with a purely mechanical Lagrangian being a functional of the trajectory \( \vec{\xi}(t) \).

The first term in \( \mathcal{L} \) does not contribute to the effective Lagrangian. The static field is in the Coulomb gauge, \( \partial_\mu a_\mu = 0 \), so that the gauge field can be expressed as \( a_\mu = \varepsilon_{\mu l} \partial_l U \), where \( U \) is an auxiliary potential. In the adiabatic approximation we replace \( U(\vec{x}) \to U[\vec{x} - \vec{\xi}(t)] \). The second term in \( \mathcal{L} \) becomes
\[ \int d^2x \left[ -\frac{1}{4\Theta} \varepsilon^{kl} \varepsilon_{km} \partial_\mu \partial_n U(\varepsilon_{ln} \partial_n U) \right] = \int d^2x \left[ -\frac{1}{4\Theta} \varepsilon_{mn} \partial_\mu \partial_n U \right]. \] (9)

It can be shown, by an integration by parts, that this term does not contribute to the effective Lagrangian. Thus the only contribution to the part of the effective mechanical Lagrangian which is linear in velocity comes from
\[ L_{eff}^{(1)} = \int d^2x \ i\phi^\dagger \partial_t \phi . \] (10)

When we introduce moduli and phases of the scalar fields, \( \phi_A = \sqrt{\rho_A} e^{i\chi_A} \) with \( A = \uparrow, \downarrow \), the effective Lagrangian will split into contributions from spin-up and spin-down components
\[ L_{eff}^{(1)} = -\int d^2x \left[ \rho_\uparrow \partial_t \chi_\uparrow + \rho_\downarrow \partial_t \chi_\downarrow \right] . \] (11)

Let us apply the procedure to a single vortex or skyrmion. The static configuration is described by \( \vec{\xi}(t) \). In the vortex case we can put formally \( g(r) = 0 \). Let us take the trajectory \( \vec{\xi}(t) \) which passes through the origin at \( t = 0 \). For a very small \( \vec{\xi} \) the fields in \( \mathcal{L} \) can be expanded as
\[
\begin{align*}
\rho_A(\vec{x} - \vec{\xi}) &= \rho_A(r) - (\vec{\xi}^1 \cos \theta + \vec{\xi}^2 \sin \theta) \frac{d\rho_A}{dr}(r) + O(\vec{\xi}^2), \\
\partial_t \chi_A(\vec{x} - \vec{\xi}) &= -n_A(-\vec{\xi}^1 \sin \theta + \vec{\xi}^2 \cos \theta) + O(\vec{\xi}^2). \end{align*}
\] (12)

With this expansion the effective Lagrangian becomes
\[ L_{eff}^{(1)} = -\pi \varepsilon_{kl} \varepsilon^{\kappa l} \sum_A n_A [\rho_A(\infty) - \rho_A(0)] + O(\vec{\xi}^2). \] (13)

The system with just one quasiparticle is translationally invariant. The term proportional to \( \varepsilon_{kl} \xi^k \xi^l \) is the most general term linear in velocity which is, up to a total time derivative, translationally invariant. This proves that the term \( O(\vec{\xi}^2) \) in Eq. (13) vanishes identically.

There are two contributions to the effective Lagrangian. The contribution from the spin-up component vanishes because \( \rho_\uparrow(\infty) = n_1 \rho_\uparrow(0) = 0 \). Thus the only contribution comes from the lower component \( (\rho_\downarrow(0) = 0, \rho_\downarrow(\infty) = \rho_0) \) and amounts to
\[ L_{eff}^{(1)} = \pi \rho_0 \varepsilon_{kl} \varepsilon^{\kappa l} \frac{eB}{2(2n+1)} \xi^k \xi^l. \] (14)
The Magnus force is exactly the same for skyrmions and for vortices. It equals the Lorenz force acting on a particle with the charge \( \frac{e}{2m^*} \). As discussed in [3] this is not a mere coincidence. A quasiparticle is not an independent object but a defect composed out of electrons. As the location of defect moves it act with a Magnus force on the surrounding electrons. They in turn respond with a current which interacts through the Lorentz force with the external magnetic field. Thus the two forces combine to be just one.

If the inertial mass were zero then the Magnus force would prevent a quasiparticle from moving with respect to the condensate. Note that in our derivation we have been slowly varying the location of a quasiparticle without moving the condensate so that the \( \xi \)'s in Eq. (4) are quasiparticle coordinates with respect to the condensate’s frame. The Lorentz force acts on the condensate as a whole. Namely, if the fields \( \{ \phi(t, \vec{x}), a_\mu(t, \vec{x}) \} \) are solutions of field equations of the model (3), then there are also the following boosted solutions

\[
\hat{\phi}(t, \vec{x}) = \phi(t, \vec{x} - \vec{R}(t))e^{\chi_B} , \\
\hat{a}_0(t, \vec{x}) = a_0[t, \vec{x} - \vec{R}(t)] - \vec{R}_k(t)a_k[t, \vec{x} - \vec{R}(t)] , \\
\hat{a}_k(t, \vec{x}) = a_k[t, \vec{x} - \vec{R}(t)] , \\
\chi_B = m\vec{R}_k \cdot k - \frac{1}{2}m[\vec{R}_k \cdot R^k]t - e \int_{t_0}^{t_1} d\tau \vec{R}_k(\tau)A_k[\vec{x} - \vec{R}(\tau)] , \\
\vec{R}(t_0) = 0 , \\
\vec{R}_k = -\frac{eB}{m} \varepsilon_{kl} \vec{R}^l , \\
A_k(\vec{x}) = -\frac{B}{2} \varepsilon_{kl} x^l .
\] (15)

Any solution can be boosted to move along an electronic cyclotron orbit. Even the uniform condensate feels the Lorentz force in spite of the fact that the external magnetic field seems to be screened by the statistical gauge field. If there is a quasiparticle in the original solution then after the boost it follows the cyclotron orbit but it does not move with respect to the condensate. The Magnus force remains zero.

3 Statistical interactions of quasiparticles

We have shown that the Magnus force is the same for both skyrmions and vortices. Now we proceed to their mutual statistical interactions. We will show that the statistical interaction of skyrmions depends on the total number of inversed spins they carry.

Once again we will apply the formula (11) but this time to the system of two distant antiskyrmions. The total phase of the lower component reads

\[
\chi_{\downarrow}(\vec{x}) = -\text{Arg}(\vec{x} - \vec{\xi}_1) - \text{Arg}(\vec{x} - \vec{\xi}_2)
\] (16)

while the upper component phase is constant. Thus, once again, only the \( \downarrow \) component contributes to the effective Lagrangian. The formula (11) simplifies to

\[
L_{\text{eff}}^{(1)} = - \int d^2x \left[ \delta \rho_{\downarrow} \partial_t \chi_{\downarrow} \right] ,
\] (17)

where \( \delta \rho_{\downarrow} = \rho_{\downarrow} - \rho_0 \) is a deviation of the down component’s density from the uniform background. When the distance between the skyrmions is large as compared to their widths the deviation can be approximated by the sum

\[
\delta \rho_{\downarrow}(t, \vec{x}) \approx \delta \rho_{\downarrow}[\vec{x} - \vec{\xi}_1(t)] + \delta \rho_{\downarrow}[\vec{x} - \vec{\xi}_2(t)]
\] (18)

of two nonoverlapping rotationally-symmetric deviations. When the formulas (17) and (15) are substituted to (17), there appear four terms to be integrated out. Two of them are "self-interaction" terms which lead to Magnus forces as discussed in the previous section but in addition there are two mutual-interaction terms

\[
L_{\text{eff}}^{(1)} = \int d^2x \left\{ \delta \rho_{\downarrow}[\vec{x} - \vec{\xi}_1(t)] \partial_t \text{Arg}[\vec{x} - \vec{\xi}_2(t)] + \delta \rho_{\downarrow}[\vec{x} - \vec{\xi}_2(t)] \partial_t \text{Arg}[\vec{x} - \vec{\xi}_1(t)] \right\} .
\] (19)

For very distant skyrmions the density deviations can be approximated by \( \delta \rho_{\downarrow}[\vec{x} - \vec{\xi}_1(t)] \approx -\frac{Q_\downarrow}{e} \delta^{(2)}[\vec{x} - \vec{\xi}_1(t)] \), where \( Q_\downarrow \) is the charge deficit of the spin-down component, compare with (8). In this approximation the integral (14) becomes

\[
L_{\text{eff}}^{(1)} = -\frac{Q_\downarrow}{e} \int d^2x \left\{ \delta^{(2)}[\vec{x} - \vec{\xi}_1(t)] \partial_t \text{Arg}[\vec{x} - \vec{\xi}_2(t)] + \delta^{(2)}[\vec{x} - \vec{\xi}_2(t)] \partial_t \text{Arg}[\vec{x} - \vec{\xi}_1(t)] \right\} = \frac{Q_\downarrow}{e} \frac{d}{dt} \text{Arg}[\xi_1(t) - \xi_2(t)] ,
\] (20)
which is just the statistical interaction we were looking for. The prefactor \( Q_\downarrow / e \) is the total number of electrons missing in the lower component as compared to the uniform ground state. As the number of electrons in the upper component is \( -Q_\uparrow / e \), the total spin of the skyrmion as compared to the uniform background is \( S = \frac{1}{2}(Q_\uparrow - Q_\downarrow) / e \). \( Q_\uparrow = -Q_\downarrow - \frac{2n\pi}{1} \), so that the statistical interaction becomes

\[
\begin{align*}
I^{(1)}_{eff} &= -[S + \frac{1}{2(2n+1)} \frac{d}{dt} \text{Arg}[\xi_1(t) - \xi_2(t)]] ,
\end{align*}
\]

where \( S \) is the total spin of the skyrmion with respect to the uniform polarized background. In the case of the vortex \( S = S_{vortex} = 1/2(2n+1) \).

4 Dual formulation of the model

To get further insight into statistical interactions of vortices we will perform Hubbard-Stratanovich transformation on the model (2). The phase gradients squared in the Lagrangian (2) can be rewritten as

\[
\rho_A^2 \exp i \int d^2x \left[-\frac{\rho_A}{2m} \partial_k \chi_A \partial_k \chi_A \right] = \int [DI_A] \exp i \int d^2x \left[ \frac{I_A^A I_A^A}{2m \rho_A} - \frac{I_A^A \partial_k \chi_A}{m} \right] ,
\]

where the auxiliary fields \( I_A^A \) have been introduced. In the next step the phases can be split into two parts \( \chi_A = \chi_A^0 + \eta_A \). \( \chi^0_A \)'s are multivalued phases due to vortices/skyrmions while \( \eta_A \)'s are singlevalued components. At the present stage the Lagrangian is linearized in phase gradients. Functional integration over \( \eta_A \)'s leads to the conservation laws

\[
\dot{\rho}_A + \partial_k \left[ \frac{I_A^A + \rho_A(a_A^0 + eA_k)}{m} \right] = 0 .
\]

These conservation laws will be identically satisfied when we introduce a pair of dual gauge fields

\[
\{ \rho_A, \frac{1}{m} [I_A^A + \rho_A(a_A^0 + eA_k)] \} = \varepsilon^{\mu\nu\sigma} \partial_\mu B^A_\sigma \Rightarrow \frac{1}{2} \varepsilon^{\mu\nu\sigma} H^A_{\mu\sigma} .
\]

This definition together with the definition of the topological currents, \( 2\pi K_A^\mu = \varepsilon^{\mu\nu\sigma} \partial_\nu \partial_\sigma \chi_A \), leads, after some rearrangement and integration by parts, to the following dual Lagrangian

\[
L_D = L_B + \frac{1}{4\Theta} \varepsilon^{\mu\nu\sigma} \partial_\mu \partial_\nu a_\sigma - \sum_A \left[ \varepsilon^{\mu\nu\sigma}(\rho_A + eA_\mu) \partial_\nu B^A_\sigma + 2\pi B^A_\mu K_A^\mu \right] .
\]

\( L_B \) is the Lagrangian of the dual gauge field

\[
L_B = \sum_A \left\{ \frac{mH^A_{\mu} H^A_{\mu} + \partial_\mu H^A_{\mu} \partial_k H^A_{\mu}}{2H^A_{\mu}} - \frac{\partial_k H^A_{\mu} \partial_k H^A_{\mu}}{8mH^A_{\mu}} \right\} - \frac{\lambda}{2} \left[ \sum_A H^A_{\mu} \right] - \rho_0^2 - 2\gamma H^A_{\mu} .
\]

Let us concentrate now on the minimal coupling between the topical current and the dual gauge field. The topological current due to the antivortex/antiskyrmion moving along the trajectory \( \xi(t) \) is

\[
K_A^0 = -\delta_{\downarrow A} \delta^2[\vec{x} - \vec{\xi}(t)] , \quad K_A^k = -\delta_{\downarrow A} \delta^2[\vec{x} - \vec{\xi}(t)] .
\]

Only the topological current of the \( \downarrow \) component is nonzero. It is like a current of a negatively charged point particle. What is the dual gauge potential which couples to the topological current? We know that the relation between the dual magnetic field and the original density is \( \rho_A = -H^A_{\mu} \). Far from any topological defects the dual magnetic field is just \( H^A_{\mu} = -\delta_{\downarrow A} \rho_0 \) and is directed down the \( z \)-axis. A single vortex/skyrmion moves in a uniform dual magnetic field. This is the origin of the Magnus force. Thus Magnus force is in fact a dual Lorentz force. Topological defects distort the uniform background. Their contribution to the dual gauge potential is

\[
B^k(\vec{x}) = \frac{Q_\downarrow}{2\pi e} \sum_v \varepsilon_{kl} \frac{x^l - \xi^l_v}{|\vec{x} - \vec{\xi}_v|^2} ,
\]

where \( v \) runs over topological defects. With this form of the gauge potential we are able to work out the mutual statistical interaction

\[
-2\pi \int d^2x \ K_A^\mu B^A_\mu = \frac{Q_\downarrow}{e} \sum_{v \neq w} \int d^2x \ \varepsilon_{kl} \frac{x^l - \xi^l_v}{|\vec{x} - \vec{\xi}_v|^2} \delta_{k}^{w} \delta(2)(\vec{x} - \vec{\xi}_w) = -\frac{Q_\downarrow}{e} \sum_{v \neq w} \frac{d}{dt} \text{Arg}[\xi^v_v - \xi^w_w] ,
\]

which is the same as Eq. (21).
Even if the inertial mass of quasiparticles is zero there still remains some possibility of motion provided that there are long range potential interactions between them. For nonzero Zeeman coupling the quasiparticles are exponentially localized field configurations. There is thus no long-range potential interaction through the matter fields. However there is still the long range Coulomb interaction which gives rise to the following effective Lagrangian for diluted quasiholes

\[ L_{\text{eff}}^{(1)} = \pi \rho_0 \sum_v \varepsilon_{k\ell} \xi_k^v \xi_{\ell}^v - \frac{e^2}{\varepsilon(2n+1)^2} \sum_{v<w} \frac{1}{|\xi_v^v - \xi_w^w|} - \sum_{v<w} \left[ S + \frac{1}{2(2n+1)} \right] \frac{d}{dt} \text{Arg} \left( \xi_v^v - \xi_w^w \right) \]  

(30)

The last term is a total time derivative and can be neglected in classical considerations. According to the classical part of this Lagrangian the particles move along the trajectories of constant total potential energy. For example a pair of quasiholes performs a strictly circular clockwise motion.

5 Remarks

We have derived statistical interaction of skyrmions in the framework of the time-dependent Ginzburg-Landau model. These interactions can be much stronger than interactions of fully polarized vortices. This conclusion could not be obtained in the effective Heisenberg model. This example shows that the Heisenberg model is not relevant not only to description of quasiparticles’ energies but it also fails to give a correct answer to more basic topological questions.

We did not make any quantitative predictions as to what is the skyrmion’s spin \( S \) which determines the strength of the statistical interaction. Such calculations are possible in the Ginzburg-Landau model but, there exist already microscopic Hartree-Fock techniques to establish the value of \( S \), see [2].

From our calculations it turns out that the geometrical phase picked up by electronic wave-function during a clockwise exchange of two skyrmions is

\[ \Gamma = \pi [S + \frac{1}{2(2n+1)}] \equiv \pi [N_\uparrow + \frac{1}{(2n+1)}] , \]  

(31)

where \( N_\uparrow \) is the number of electrons with reversed spins trapped inside the skyrmion core. This phase determines the quantum statistics of skyrmions. There are two types of anyons. For \( N_\uparrow \) even the phase is \( \pi \) up to an even multiplicity of \( \pi \). This is the same kind of statistics as for fully polarized Laughlin quasiparticles (or quasiholes). Condensation of such skyrmions leads to hierarchy states with odd denominator filling fractions. Condensation of such skyrmions with \( N_\uparrow \) odd the exchange phase \( \pi \) is \( \pi \frac{2n+2}{2n+1} \) up to irrelevant even multiplicities of \( \pi \). This type of anyons differs from the Laughlin quasiparticles by an addition of one flux quantum. Their condensation gives rise to even-denominator filling fractions. To summarize the condensation of skyrmions of the primary state with \( \nu = \frac{1}{2n+1} \) gives rise to the following filling fractions labelled by \( p \)

\[ \nu = \frac{1}{(2n+1) - \frac{\alpha}{p}} , \quad p = 1, 2, 3, \ldots , \]  

(32)

where \( \alpha = +1 \) or \( -1 \) dependent on whether skyrmions or antiskyrmions condense. Note that in the above formula \( p \) is an arbitrary integer so that the filling fractions can have both even and odd denominators.

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