Hadamard type fractional time-delay semilinear differential equations: Delayed Mittag-Leffler function approach

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Abstract

We propose a delayed Mittag-Leffler type matrix function with logarithm, which is an extension of the classical Mittag-Leffler type matrix function with logarithm and delayed Mittag-Leffler type matrix function. With the help of the delayed Mittag-Leffler type matrix function with logarithm, we give an explicit form of solutions to nonhomogeneous Hadamard type fractional time-delay linear differential equations. Moreover, we study existence uniqueness and stability in Ulam-Hyers sense of the Hadamard type fractional time-delay nonlinear equations.

1 Introduction

Mathematical descriptions of the models described through differential equations with derivatives of non-integer orders have proved to be a very useful instrument for modeling of various viscoelasticity cases, stability theory, controllability theory, and other related fields. Time-delays are often related with physico-chemical processes, electric networks, hydraulic networks, heredity in population growth, the economy and other related industries. In general, a peculiarity of the adequate mathematical models is that the rate of change of these processes depends on past history. Differential systems describing these models are called time-delay differential equations. The qualitative theory of linear time-delay equations is well investigated. Recently, the time-delay differential equations has been considered in [1]-[6]. In [7]-[10] authors derived the exact expressions of solutions of linear continuous and discrete delay equations by proposing the concepts of delayed matrix functions. On the other hand, stability concepts and relative controllability problems of linear time-delay differential equations were investigated in [11]-[14].

The unification of differential equations with delay and differential equations with fractional derivative is provided by differential equations, including both delay and non-integer derivatives, so called time-delay fractional differential equations. In applications, this unification is useful for creating highly adequate models of some systems with memory. One can notice that works on this field involve Riemann-Liouville and Caputo type fractional derivatives. Besides these derivatives, there is an other fractional derivative, involving the logarithmic function, so called Hadamard fractional derivative. For the literature on the related field of fractional time-delay equations of Caputo type and Riemann–Liouville type, we refer the researcher to [15]-[25].

It is known that ([24], page 235, [25]) a solution of a Hadamard fractional linear system

\[
\begin{align*}
(H D_{1+}^\alpha y) (t) &= \lambda y (t) + f(t), \quad t \in (1, T], h > 0, \\
(H I_{1+}^{1-\alpha} y) (1^+) &= a \in R, \lambda \in R, 0 < \alpha < 1,
\end{align*}
\]

has the form

\[
y(t) = a (\ln t)^{\alpha - 1} E_{\alpha, \alpha} [\lambda (\ln t)^{\alpha}] + \int_1^t \left( \ln \frac{t}{s} \right)^{\alpha - 1} E_{\alpha, \alpha} \left[ \lambda \left( \ln \frac{t}{s} \right)^{\alpha} \right] f(s) \frac{ds}{s}.
\]
However, we find that there exists only one \cite{26} work on the representation of explicit solutions of Hadamard type fractional order delay linear differential equations. In \cite{26} authors studied the Hadamard type fractional linear time-delay system

\begin{equation}
(H D^\alpha_1 y)(t) = A_1 y(t - h), \quad t \in (1, T], h > 0,
\end{equation}

\begin{equation}
y(t) = \varphi(t), \quad 1 \leq t \leq h,
\end{equation}

\begin{equation}
(H I^{1-\alpha}_1 y)(1^+) = a \in \mathbb{R}^n,
\end{equation}

where $A_0$ is constant $n \times n$ square matrix.

Motivated by the above researches, we investigate a new class of Hadamard-type fractional delay differential equations. We extend to consider an explicit representation of solutions of a Hadamard type fractional time-delay differential equation of the following form by introducing a new delayed M-L type function with logarithm.

\begin{equation}
(H D^\alpha_1 y)(t) = A_0 y(t) + A_1 y \left( \frac{t}{h} \right) + f(t), \quad t \in (1, T], h > 0,
\end{equation}

\begin{equation}
y(t) = \varphi(t), \quad \frac{1}{h} < t \leq 1,
\end{equation}

\begin{equation}
(H I^{1-\alpha}_1 y)(1^+) = a \in \mathbb{R}^n,
\end{equation}

where $(H D^\alpha_1 y)(\cdot)$ is the Hadamard derivative of order $\alpha \in (0, 1)$, $A_0, A_0 \in \mathbb{R}^{n \times n}$ denote constant matrices, and $\varphi : [\frac{1}{h}, 1] \to \mathbb{R}^n$ is an arbitrary Hadamard differentiable vector function, $f \in C([1, T], \mathbb{R}^n)$, $T = h^l$ for a fixed natural number $l$.

The second purpose of this paper is to study the existence and stability of solutions for a Hadamard type fractional delay differential equation

\begin{equation}
(H D^\alpha_1 y)(t) = A_0 y(t) + A_1 y \left( \frac{t}{h} \right) + f(t, y(t)), \quad t \in (1, T], h > 0,
\end{equation}

\begin{equation}
y(t) = \varphi(t), \quad \frac{1}{h} < t \leq 1,
\end{equation}

\begin{equation}
(H I^{1-\alpha}_1 y)(1^+) = a \in \mathbb{R}^n,
\end{equation}

At the end of this section, we state the main contribution of the paper as follows:

(i) We propose delayed M-L type functions $Y_{h, \alpha, \beta}^{-A_0, A_1} (t, s)$ with logarithms, by means of the matrix equations \cite{5}. We show that for $A_1 = \Theta$ the function $Y_{h, \alpha, \beta}^{-A_0, A_1} (t, s)$ coincide with M-L type function with two parameters $(\ln t - \ln s)^{\alpha-1} E_{\alpha, \beta} (A_0 (\ln t - \ln s)^{\alpha})$. For $A_0 = \Theta$ delayed M-L type function $Y_{h, \alpha, \beta}^{-A_0, A_1} (t, s)$ coincide with delayed M-L type matrix function with two parameters $E_{\alpha, \beta}^{A_0, A_1} (\ln t - \ln h)$, introduced in \cite{4}.

(ii) We explicitly write the solution of Hadamard type fractional delay linear system \cite{2} via delayed perturbation of M-L type function with logarithm. Using this representation we study existence uniqueness and Ulam-Hyers stability of nonlinear equation \cite{4}.

\section{Preliminaries}

Let $0 < a < b < \infty$ and $C[a, b]$ be the Banach space of all continuous functions $y : [a, b] \to \mathbb{R}^n$ with the norm $\|y\|_C := \max \{\|y(t)\| : t \in [a, b]\}$. For $0 \leq \gamma < 1$, we denote the space $C_{\gamma, \ln} (a, b)$ by the weighted Banach space of the continuous function $y : [a, b] \to \mathbb{R}$, which is given by

$$C_{\gamma, \ln} (a, b) := \left\{ y(t) : \left( \ln \frac{t}{a} \right)^\gamma y(t) \in C[a, b] \right\},$$

endowed with the norm $\|y\|_{\gamma, \ln} := \sup \{ (\ln \frac{t}{a})^{\gamma} \|y(t)\| : t \in (a, b)\}$.

The following definitions and lemmas will be used in this paper.

\textbf{Definition 1} Hadamard fractional integral of order $\alpha \in R^+$ of function $y(t)$ is defined by

$$\left( H I^\alpha_1 y \right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left( \ln \frac{t}{s} \right)^{\alpha-1} y(s) \frac{ds}{s}, \quad 0 < a < t \leq b,$$

where $\Gamma$ is the Gamma function.
Definition 2 Hadamard fractional derivative of order $\alpha \in [n-1,n)$, $n \in \mathbb{Z}^+$ of function $y(t)$ is defined by

$$
(HD^\alpha_{a+} y)(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int^t_a \left( \ln \frac{t}{s} \right)^{n-\alpha+1} y(s) \frac{ds}{s}, \quad 0 < a < t \leq b.
$$

Lemma 3 If $a, \gamma, \beta > 0$ then

1. \( \left( HD^\gamma_{a+} \left( \ln \frac{t}{s} \right)^{\beta-1} \right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta + \gamma)} \left( \ln \frac{t}{s} \right)^{\beta+\gamma-1}. \)
2. \( \left( HD^\gamma_{a+} \left( \ln \frac{t}{s} \right)^{\beta} \right)(t) = \left( \Gamma(\beta) \right)^{\beta+\gamma-1}. \)
3. For $0 < \beta < 1$, \( \left( HD^\beta_{a+} \left( \ln \frac{t}{s} \right)^{\beta-1} \right)(t) = 0. \)

Definition 4 M-L type matrix function with two parameters $e_{\alpha,\beta}(A_0; t): \mathbb{R} \to \mathbb{R}^{n \times n}$ is defined by

$$
e_{\alpha,\beta}(A_0; t) := t^{\beta-1}E_{\alpha,\beta}(A_0; t) := t^{\beta-1} \sum_{k=0}^{\infty} \frac{A_0^{k+\alpha}}{\Gamma(k\alpha + \beta)}, \quad \alpha, \beta > 0, t \in \mathbb{R}.
$$

Definition 5 Two parameters delayed M-L type matrix function $E^{\mathcal{A}_01}_{h,\alpha,\beta}(ln t): \mathbb{R}^+ \to \mathbb{R}^{n \times n}$ with logarithm is defined by

$$
E^{\mathcal{A}_1}_{h,\alpha,\beta}(ln t) := \left\{ \begin{array}{ll}
\Theta, & -\infty < t \leq \frac{1}{h}, \\
\frac{1}{h} - t & \frac{1}{h} < t \leq 1, \\
\frac{1}{h} + \frac{1}{h} - t & h^{p-1} < t \leq h^p. 
\end{array} \right.
$$

Our definition of the two parameters delayed M-L type matrix function with logarithm differs substantially from the definition given in [26].

In order to give a definition of delayed M-L type matrix functions with logarithm, we introduce the following matrices $Y_{\alpha,\beta,0}(t,s)$, $Y_{\alpha,\beta,1}(t,s)$, $Y_{\alpha,\beta,k}(t,s)$ of function $E^{\mathcal{A}_01}_{h,\alpha,\beta}(ln t)$:

$$
Y_{\alpha,\beta,0}(t,s) = \left( \ln \frac{t}{s} \right)^{\beta-1} E_{\alpha,\beta}(A_0; \ln \frac{t}{s}), \\
Y_{\alpha,\beta,1}(t,s) = \int_{s h}^{t} e_{\alpha,\alpha}(A_0; \ln \frac{t}{r}) A_1 \frac{1}{\Gamma(\beta)} \left( \ln \frac{r}{s} \right)^{\beta-1} \frac{dr}{r}, \\
Y_{\alpha,\beta,k}(t,s) = \int_{s h^k}^{t} e_{\alpha,\alpha}(A_0; \ln \frac{t}{r}) A_1 Y_{\alpha,\beta,k-1}(\frac{r}{s}, sh^{k-1}) \frac{dr}{r}.
$$

Definition 6 Let $A_0, A_1 \in \mathbb{R}^{n \times n}$ be fixed matrices and $k \in \mathbb{N} \cup \{0\}$. Delayed M-L type function $Y^{\mathcal{A}_0,\mathcal{A}_1}_{h,\alpha,\beta}(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ with logarithm generated by $A_0, A_1$ is defined by

$$
Y^{\mathcal{A}_0,\mathcal{A}_1}_{h,\alpha,\beta}(t,s) := \sum_{j=0}^{\infty} Y_{\alpha,\beta,j}(t,s) H(t - sh^j) = \left\{ \begin{array}{ll}
\Theta, & -\infty < t < s, \\
I, & t = s, \\
Y_{\alpha,\beta,0}(t,s) + Y_{\alpha,\beta,1}(t,s) + \ldots + Y_{\alpha,\beta,k}(t,s), & sh^k < t \leq sh^{k+1},
\end{array} \right.
$$

where $H(t)$ is a Heaviside function: $H(t) = \left\{ \begin{array}{ll}
1, & t > 0, \\
0, & t \leq 0.
\end{array} \right.$

Lemma 7 Let $a, b > -1$. For $sh^k < t \leq sh^{k+1}$, $k \in \mathbb{N} \cup \{0\}$, one has

$$
\int^t_r \left( \ln \frac{t}{s} \right)^a \left( \ln \frac{s}{r} \right)^b \frac{ds}{s} = \left( \ln \frac{t}{r} \right)^{a+b+1} A_1 [a + 1, b + 1],
$$

$$
\frac{1}{\Gamma(1-\alpha)} \int_{sh^k}^t \left( \ln \frac{t}{r} \right)^{-\alpha} Y_{\alpha,\beta,k}(r, sh^k) \frac{dr}{r} = \int_{sh^k}^t E_{\alpha,1}(A_0; \ln \frac{t}{r}) A_1 Y_{\alpha,\beta,k-1}(\frac{r}{s}, sh^{k-1}) \frac{dr}{r}.
$$
Proof. Let $\ln \frac{s}{r} = \tau \ln \frac{t}{r}$. Then $s = r \left( \frac{t}{r} \right)^\tau$, $ds = r \left( \frac{t}{r} \right)^\tau d\tau$. So we have

$$\ln \frac{t}{s} = \ln \left( \frac{t}{r} \right)^{1-\tau} = (1-\tau) \ln \left( \frac{t}{r} \right),$$

and

$$\int_t^1 \left( \ln \frac{t}{r} \right)^a \left( \ln \frac{s}{r} \right)^b ds = \int_0^1 \left( \ln \frac{t}{r} \right)^a (1-\tau)^a \left( \ln \frac{t}{r} \right)^b \tau \ln \frac{t}{r} d\tau = \left( \ln \frac{t}{r} \right)^{a+b+1} B[a+1, b+1].$$

To prove (8), firstly using (7) we calculate it for $k$.

Case 1: $\alpha, \beta, k \neq 0$.

- For any $Y_{\alpha,\beta,k}$, the proof is based on the inequality (7).
- Similarly, for any $k \in \mathbb{N}$, we have

$$Y_{\alpha,\beta,k}(t, sh^k) = \sum_{n=0}^{\infty} \binom{n+k}{k} A_0^n \frac{(\ln t - \ln s h^k)^{\alpha+k \alpha+\beta-1}}{\Gamma(n \alpha + k \alpha + \beta)}.$$ (10)

Proof. The proof is based on the inequality (7). For $k = 1$, we have

$$Y_{\alpha,\beta,1}(t, sh) = \int_{sh}^t e_{\alpha,\alpha} \left( A_0; \ln \frac{t}{r} \right) A_1 Y_{\alpha,\beta,0} \left( \frac{r}{h}, s \right) dr = A_1 \int_{sh}^t \sum_{n=0}^{\infty} A_0^n \left( \ln \frac{t}{r} \right)^{\alpha+n+\alpha-1} \frac{1}{\Gamma(\alpha+n+\alpha)} \sum_{k=0}^{\infty} A_0^k \left( \ln \frac{t}{s h} \right)^{\alpha+k-\beta+\alpha-1} \frac{1}{\Gamma(\alpha+k+\beta)} dr$$

$$= A_1 \int_{sh}^t \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_0^n \left( \ln \frac{t}{r} \right)^{\alpha+k+\alpha-1} \frac{1}{\Gamma(\alpha+k+\alpha)} A_0^k \left( \ln \frac{t}{sh} \right)^{\alpha+n-k+\beta-1} \frac{1}{\Gamma(\alpha+n-k+\beta)} dr$$

$$= A_1 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_0^n \frac{1}{\Gamma(\alpha+k+\alpha) \Gamma(\alpha+n-k+\beta)} \int_{sh}^t \left( \ln \frac{t}{r} \right)^{\alpha+n-k+\beta-1} \left( \ln \frac{r}{sh} \right)^{\alpha+n-k+\beta-1} dr$$

$$= A_1 \int_{sh}^t \frac{1}{\Gamma(\alpha+k+\alpha) \Gamma(\alpha+n-k+\beta)} \left( \ln t - \ln s h \right)^{\alpha+n-k+\beta-1} dr$$

$$= A_1 \frac{1}{\Gamma(\alpha+k+\alpha) \Gamma(\alpha+n-k+\beta)} A_0^1 \left[ \alpha k + \alpha (n-k) + \beta \right]$$

$$= A_1 \sum_{n=0}^{\infty} \binom{n+1}{1} \frac{(\ln t - \ln s h)^{\alpha+n+\alpha+\beta-1}}{\Gamma(\alpha+n+\alpha+\beta)}.$$
For $k = 2$, we get

\[
Y_{\alpha,\beta,2}(t, sh^2) = \int_{sh^2}^{t} e_{\alpha,\alpha} \left( A_0; \ln \frac{r}{\alpha} \right) A_1 Y_{\alpha,\beta,1} \left( \frac{r}{n}, sh \right) dr
\]

\[
= A_1 \int_{sh^2}^{t} \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_0^{2} \left( \frac{r}{n} \right) \frac{k^{\alpha+\beta-1}}{\Gamma(k\alpha+\beta)} Y_{\alpha,\beta,1} \left( \frac{r}{n}, sh \right) dr
\]

\[
= A_1 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( n + 1 - k \right) \frac{1}{\Gamma (k\alpha+\beta)} \left( \ln \frac{r}{sh^2} \right)^{n\alpha-k\alpha+\beta-1} \Gamma (n\alpha-k\alpha+\beta+\beta) \int_{sh^2}^{t} \left( \ln \frac{r}{sh^2} \right)^{n\alpha-k\alpha+\beta-1} \Gamma (n\alpha-k\alpha+\beta+\beta) dr
\]

\[
= A_1 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( n + 1 - k \right) \frac{1}{\Gamma (k\alpha+\beta)} \left( \ln \frac{r}{sh^2} \right)^{n\alpha-k\alpha+\beta-1} \Gamma (n\alpha-k\alpha+\beta+\beta) \int_{sh^2}^{t} \left( \ln \frac{r}{sh^2} \right)^{n\alpha-k\alpha+\beta-1} \Gamma (n\alpha-k\alpha+\beta+\beta) dr
\]

\[
= A_1 \sum_{n=0}^{\infty} \left( n + 2 \right) \frac{1}{\Gamma (k\alpha+\beta)} \left( \ln \frac{r}{sh^2} \right)^{n\alpha-k\alpha+\beta-1} \Gamma (n\alpha-k\alpha+\beta+\beta) \int_{sh^2}^{t} \left( \ln \frac{r}{sh^2} \right)^{n\alpha-k\alpha+\beta-1} \Gamma (n\alpha-k\alpha+\beta+\beta) dr
\]

Using the Mathematical Induction in a similar manner we can get (10).

According to Lemma 3 in the case $\mathcal{A}_0 \mathcal{A}_1 = \mathcal{A}_1 \mathcal{A}_0$ delayed M-L type function $Y_{h,\alpha,\beta}^{\mathcal{A}_0,\mathcal{A}_1}(t, s)$ has a simple form:

\[
Y_{h,\alpha,\beta}^{\mathcal{A}_0,\mathcal{A}_1}(t, s) := \begin{cases} \Theta, & -\infty < t < s, \\ I, & t = s, \\ \sum_{i=0}^{\infty} A_0 \left( \ln t - \ln s \right)^{i\alpha+\beta-1} \frac{1}{\Gamma (i\alpha+\beta)} + \sum_{i=0}^{\infty} \left( \frac{i}{1} \right) A_0^{i-1} \mathcal{A}_1 \left( \ln t - \ln s \right)^{i\alpha+\beta-1} \frac{1}{\Gamma (i\alpha+\beta)} \\ + \cdots + \sum_{i=p}^{\infty} \left( \frac{i}{p} \right) A_0^{i-p} A_1 \left( \ln t - \ln s \right)^{i\alpha+\beta-1} \frac{1}{\Gamma (i\alpha+\beta)} \\ \cdots , & sh^p < t \leq sh^{p+1}. \end{cases}
\]

Next lemma shows some special cases of the delayed M-L type function.

**Lemma 9** Let $Y_{h,\alpha,\beta}^{\mathcal{A}_0,\mathcal{A}_1}(t, s)$ be defined by (6). Then the following holds true:

(i) If $\mathcal{A}_0 = \Theta$ then $Y_{h,\alpha,\beta}^{\mathcal{A}_0,\mathcal{A}_1}(t, 1) = E_{h,\alpha,\beta} (\ln \frac{t}{h})$, $h^{-1} < \frac{\alpha}{\beta} \leq h^k$,

(ii) If $\mathcal{A}_1 = \Theta$ then $Y_{h,\alpha,\beta}^{\mathcal{A}_0,\mathcal{A}_1}(t, s) = \left( \ln \frac{t}{s} \right)^{\beta-1} E_{h,\beta} \left( \mathcal{A}_0 \ln \frac{t}{s} \right)$,

(iii) If $\alpha = \beta = 1$ and $\mathcal{A}_0 \mathcal{A}_1 = \mathcal{A}_1 \mathcal{A}_0$ then $Y_{h,1,1}^{\mathcal{A}_0,\mathcal{A}_1}(t, s) = e^{\mathcal{A}_0 \ln (t-s)} e_{h,1} \left( \ln t - \ln h \right)$, $A_{11} = A_1 e^{-\mathcal{A}_0 \ln h}$.

**Proof.** (i) If $\mathcal{A}_0 = \Theta$, then the formula (5).

For $\alpha, \beta, 0 (t, s) = e_{\alpha,\alpha} \left( \Theta, \ln \frac{t}{s} \right) = \int_{s}^{t} \left( \ln t - \ln s \right)^{\beta-1} \frac{1}{\Gamma (\beta)} \frac{dr}{r}$,

\[
Y_{\alpha,\beta,0}(t, s) = \int_{s}^{t} e_{\alpha,\alpha} \left( \Theta, \ln \frac{t}{r} \right) A_1 Y_0 \left( \frac{r}{h}, s \right) dr = \frac{1}{\Gamma (\alpha) \Gamma (\beta)} A_1 \int_{s}^{t} \left( \ln \frac{t}{r} \right)^{\alpha-1} \left( \ln \frac{r}{sh^2} \right)^{\beta-1} \frac{dr}{r}
\]

\[
= \frac{1}{\Gamma (\alpha) \Gamma (\beta)} A_1 \left( \ln \frac{t}{sh^2} \right)^{\alpha+\beta-1} B[\alpha, \beta] = \frac{1}{\Gamma (\alpha) \Gamma (\beta)} A_1 \left( \ln \frac{t}{sh^2} \right)^{\alpha+\beta-1},
\]

\[
Y_{\alpha,\beta,1}(t, sh) = \int_{s}^{t} e_{\alpha,\alpha} \left( \Theta, \ln \frac{t}{r} \right) A_1 Y_1 \left( \frac{r}{h}, sh \right) dr = \frac{1}{\Gamma (\alpha) \Gamma (2\alpha+\beta)} A_1 \int_{s}^{t} \left( \ln \frac{t}{r} \right)^{\alpha-1} \left( \ln \frac{r}{sh^2} \right)^{2\alpha+\beta-1} \frac{dr}{r}
\]

\[
= \frac{1}{\Gamma (\alpha) \Gamma (2\alpha+\beta)} A_1^2 \left( \ln \frac{t}{sh^2} \right)^{2\alpha+\beta-1} B[\alpha, \alpha + \beta] = \frac{1}{\Gamma (\alpha) \Gamma (2\alpha+\beta)} A_1^2 \left( \ln \frac{t}{sh^2} \right)^{2\alpha+\beta-1},
\]

\[
Y_{\alpha,\beta,k}(t, sh^k) = A_1^k \left( \ln \frac{t}{sh^k} \right)^{k\alpha+\beta-1} \frac{1}{\Gamma (k\alpha+\beta)}, \quad k \geq 0.
\]
So \( Y_{h,\alpha,\beta}^{A_0,A_1} (t, 1) \) coincides with \( E_{h,\alpha,\beta}^{A_1} (\ln t - \ln h) \):

\[
Y_{h,\alpha,\beta}^{A_0,A_1} (t, 1) = \sum_{i=0}^{k} A_i \left( \ln \frac{t}{h^i} \right)^{i+\beta-1} \frac{1}{\Gamma (i+\beta)} = \frac{t^{\beta-1}}{\Gamma (\beta)} + A_1 (\ln t - \ln h)^{\alpha+\beta-1} + \frac{t}{\Gamma (k\alpha + \beta)} \]

(i) Trivially, from definition of \( Y_{h,\alpha,\beta}^{A_0,A_1} (t, s) \) we have: if \( A_1 = \Theta \), then

\[
Y_{h,\alpha,\beta}^{A_0,A_1} (t, s) = \left( \ln \frac{t}{s} \right)^{\beta-1} E_{\alpha,\beta} \left( A_0 \left( \ln \frac{t}{s} \right) \right).
\]

(ii) By \( \text{(11)} \) for the case \( \alpha = 1 \) and \( A_0 A_1 = A_1 A_0 \), we have

\[
Y_{h,1,1}^{A_0,A_1} (t, s) = \sum_{i=0}^{\infty} A_i (\ln t - \ln s)^i + \sum_{i=0}^{\infty} \binom{i+1}{1} A_i A_1 (\ln t - \ln s)^{i+1} \]

\[
= e^{A_0 (\ln t - \ln s)} + e^{A_0 (\ln t - \ln s)} A_1 (\ln t - \ln s) + \cdots + e^{A_0 (\ln t - \ln s)} A_1^k \frac{1}{k!} (\ln t - \ln s)^k
\]

\[
= e^{A_0 (\ln t - \ln s)} e^{A_1 (\ln t - \ln h)}.
\]

It turns out that \( Y_{h,\alpha,\beta}^{A_0,A_1} (t, s) \) is a delayed perturbation of the Cauchy matrix with logarithm of the homogeneous equation \( \text{(2)} \) with \( f = 0 \).

**Lemma 10** \( Y_{h,\alpha,\beta}^{A_0,A_1} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n} \) is a solution of

\[
HD_{t}^{\alpha} Y_{h,\alpha,\beta}^{A_0,A_1} (t, s) = \left( \ln \frac{t}{s} \right)^{-\alpha-\beta-1} \frac{1}{\Gamma (-\alpha)} + A_0 Y_{h,\alpha,\beta}^{A_0,A_1} (t, s) + A_1 Y_{h,\alpha,\beta}^{A_0,A_1} \left( \frac{t}{h}, s \right). \tag{12}
\]

**Proof.** According to \( \text{(3)} \) we have

\[
(HD_{t}^{\alpha}, Y_{\alpha,\beta,0} (t, s)) (t) = \frac{1}{\Gamma (1-\alpha)} \left( \frac{d}{dt} \right) \int_{s}^{t} \left( \ln \frac{t}{r} \right)^{-\alpha} Y_{\alpha,\beta,0} (r, s) \frac{dr}{r}
\]

\[
= \frac{1}{\Gamma (1-\alpha)} \left( \frac{d}{dt} \right) \int_{s}^{t} \left( \ln \frac{t}{r} \right)^{-\alpha} e_{\alpha,\beta} (r, s) \frac{dr}{r}
\]

\[
= \left( \frac{d}{dt} \right) e_{\alpha,1-\alpha+\beta} \left( A_0; \ln \frac{t}{s} \right)
\]

\[
= \left( \ln \frac{t}{s} \right)^{-\alpha-\beta-1} \frac{1}{\Gamma (-\alpha + \beta)} + A_0 e_{\alpha,\beta} \left( A_0; \ln \frac{t}{s} \right)
\]

\[
= \left( \ln \frac{t}{s} \right)^{-\alpha+\beta-1} \frac{1}{\Gamma (-\alpha + \beta)} + A_0 Y_{\alpha,\beta,0} (t, s). \tag{13}
\]
On the other hand for any \( k \in \mathbb{N} \):

\[
(H D^\alpha_1, Y_{\alpha,\beta,k} (t, sh^k)) (t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{d}{dt} \right)^{\frac{d}{dt}} \left( \ln \frac{t}{r} \right)^{-\alpha} Y_{\alpha,\beta,k} (r, s) \frac{dr}{r} \\
= \left( \frac{d}{dt} \right)^{\frac{d}{dt}} \int_{sh^k}^{t} E_{\alpha,1} \left( A_0; \ln \frac{t}{r} \right) A_1 Y_{\alpha,\beta,k-1} \left( \frac{r}{h}, sh^{k-1} \right) \frac{dr}{r} \\
= \int_{sh^k}^{t} \left( \frac{d}{dt} \right)^{\frac{d}{dt}} E_{\alpha,1} \left( A_0; \ln \frac{t}{r} \right) A_1 Y_{\alpha,\beta,k-1} \left( \frac{r}{h}, sh^{k-1} \right) \frac{dr}{r} \\
+ A_1 Y_{\alpha,\beta,k-1} \left( \frac{t}{h}, sh^{k-1} \right) \\
= A_0 Y_{\alpha,\beta,k} (t, sh^k) + A_1 Y_{\alpha,\beta,k-1} \left( \frac{t}{h}, sh^{k-1} \right). \tag{14}
\]

From (13) and (14) it follows that for \( sh^k < t \leq sh^{k+1} \)

\[
H D^\alpha_1 Y_{h,\alpha,\beta} (t, s) = H D^\alpha_1 Y_{\alpha,\beta,0} (t, s) + H D^\alpha_1 Y_{\alpha,\beta,1} (t, s) + \ldots + H D^\alpha_1 Y_{\alpha,\beta,k} (t, s) = \left( \frac{t}{s} \right)^{-\alpha+\beta-1} \frac{1}{\Gamma(-\alpha+\beta)} + A_0 Y_{\alpha,\beta,0} (t, s) + A_0 Y_{\alpha,\beta,1} (t, s) + A_1 Y_{\alpha,\beta,0} \left( \frac{t}{h}, s \right) \\
+ \ldots + A_0 Y_{\alpha,\beta,k} (t, s) + A_1 Y_{\alpha,\beta,k-1} \left( \frac{t}{h}, sh^{k-1} \right) \\
= \left( \frac{t}{s} \right)^{-\alpha+\beta-1} \frac{1}{\Gamma(-\alpha+\beta)} + A_0 Y_{h,\alpha,\beta} (t, s) + A_1 Y_{h,\alpha,\beta} \left( \frac{t}{h}, s \right). 
\]

The proof is completed. \( \blacksquare \)

**Theorem 11** The solution \( y(t) \) of (3) with zero initial condition has a form

\[
y (t) = \int_{1}^{t} Y_{h,\alpha,\alpha} (t, s) f (s) \frac{ds}{s}, \quad t \geq 0.
\]

**Proof.** Assume that any solution of nonhomogeneous system \( y(t) \) has the form

\[
y (t) = \int_{1}^{t} Y_{h,\alpha,\alpha} (t, s) h (s) \frac{ds}{s}, \quad t \geq 0, \tag{15}
\]

where \( h(s), 1 \leq s \leq t \leq T \) is an unknown continuous vector function and \( y(1) = 0 \). Having Hadamard fractional differentiation on both sides of (15), for \( 1 < t \leq h \) we have

\[
(H D^\alpha_1 y) (t) = A_0 y(t) + A_1 \left( \frac{t}{h} \right) + f(t) \\
= A_0 \int_{1}^{t} Y_{h,\alpha,\alpha} (t, s) h (s) \frac{ds}{s} + A_1 \int_{1}^{t/h} Y_{h,\alpha,\alpha} \left( \frac{t}{h}, s \right) h (s) \frac{ds}{s} + f(t) \\
= A_0 \int_{1}^{t} Y_{h,\alpha,\alpha} (t, s) h (s) \frac{ds}{s} + f(t).
\]

On the other hand, according to Lemma 7 we have

\[
(H D^\alpha_1 y) (t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{d}{dt} \right)^{\frac{d}{dt}} \left( \ln \frac{t}{r} \right)^{-\alpha} Y_{h,\alpha,\alpha} (r, s) \frac{dr}{r} \\
= \frac{1}{\Gamma(1-\alpha)} \left( \frac{d}{dt} \right)^{\frac{d}{dt}} \int_{1}^{t} \left( \ln \frac{t}{r} \right)^{-\alpha} Y_{h,\alpha,\alpha} (r, s) h (s) \frac{dr}{r} \\
= c(t) + \frac{1}{\Gamma(1-\alpha)} \int_{1}^{t} \left( \frac{d}{dr} \right)^{\frac{d}{dr}} \left( \ln \frac{t}{r} \right)^{-\alpha} Y_{h,\alpha,\alpha} (r, s) h (s) \frac{dr}{r} \\
= h(t) + A_0 \int_{1}^{t} Y_{h,\alpha,\alpha} (t, s) h (s) \frac{ds}{s}.
\]
Therefore, $h(t) = f(t)$. The proof is completed. ■

**Theorem 12** Let $p = 0, 1, \ldots, l$. A solution $y \in C((p - 1)h, ph], \mathbb{R}^n)$ of (2) with $f = 0$ has a form

$$y(t) = Y_{h,\alpha,a}^{\alpha_0,A_1} \left( t, \frac{1}{h} \right) a + \int_0^t Y_{h,\alpha,a}^{\alpha_0,A_1} \left( t, s \right) \left( \frac{H^\alpha_s}{\alpha} \varphi \right) (s) - A_0 \varphi (s) \frac{ds}{s}.$$  

**Proof.** We are looking for a solution, which depends on an unknown constant $c$, and an Hadamard differentiable vector function $g(t)$, of the form

$$y(t) = Y_{h,\alpha,a}^{\alpha_0,A_1} \left( t, \frac{1}{h} \right) c + \int_0^t Y_{h,\alpha,a}^{\alpha_0,A_1} \left( t, s \right) g(s) \frac{ds}{s} := \varphi(t), \quad \frac{1}{h} < t \leq 1,$$

Moreover, $y(t)$ satisfies initial conditions

$$y(t) = \left( H^\alpha_s \right) \left( \frac{1}{h} \right) a.$$  

We have

$$a = \left( H^\alpha_s \right) \left( \frac{1}{h} \right) = \lim_{t \to h^+} \left( H^\alpha_s \right) (t)$$

$$= \lim_{t \to h^+} \left( \frac{1}{\Gamma (1 - \alpha)} \int_{\frac{1}{h}}^t (\ln t - \ln s)^{-\alpha} Y_{h,\alpha,a}^{\alpha_0,A_1} \left( \frac{1}{h}, s \right) dt \right)$$

$$= \lim_{t \to h^+} \left( \frac{1}{\Gamma (1 - \alpha)} \int_{\frac{1}{h}}^t (\ln t - \ln s)^{-\alpha} e_{\alpha,a} (A_0, \ln s) c dt \right) = c.$$  

Thus $c = a$. Since $\frac{1}{h} < t \leq 1$, we obtain that

$$Y_{h,\alpha,a}^{\alpha_0,A_1} (t, s) = \begin{cases} (\ln \frac{t}{s})^{\alpha - 1} E_{\alpha,a} (A_0 (\ln \frac{t}{s})^\alpha), & \frac{1}{h} \leq s \leq t \leq 1, \\ \Theta, & t < s \leq h. \end{cases}$$

Consequently, on interval $\frac{1}{h} < t \leq 1$, we can easily derive

$$\varphi(t) = Y_{h,\alpha,a}^{\alpha_0,A_1} \left( t, \frac{1}{h} \right) a + \int_{\frac{1}{h}}^t Y_{h,\alpha,a}^{\alpha_0,A_1} (t, s) g(s) \frac{ds}{s}$$

$$= Y_{h,\alpha,a}^{\alpha_0,A_1} \left( t, \frac{1}{h} \right) a + \int_{\frac{1}{h}}^t Y_{h,\alpha,a}^{\alpha_0,A_1} (t, s) g(s) \frac{ds}{s} + \int_{\frac{1}{h}}^t Y_{h,\alpha,a}^{\alpha_0,A_1} (t, s) g(s) \frac{ds}{s}$$

$$= (\ln th)^{-\alpha} E_{\alpha,a} (A_0 (\ln th)^\alpha) a + \int_{\frac{1}{h}}^t \left( \ln \frac{t}{s} \right)^{-\alpha} E_{\alpha,a} \left( A_0 (\ln \frac{t}{s})^\alpha \right) g(s) \frac{ds}{s}.$$  

Having differentiated (16) in Hadamard sense, we obtain

$$H^\alpha_s \varphi (t) = A_0 (\ln th)^{-\alpha} E_{\alpha,a} (A_0 (\ln th)^\alpha) a + A_0 \int_{\frac{1}{h}}^t \left( \ln \frac{t}{s} \right)^{-\alpha} E_{\alpha,a} \left( A_0 (\ln \frac{t}{s})^\alpha \right) g(s) \frac{ds}{s} + g(t)$$

$$= A_0 \varphi(t) + g(t).$$

Therefore, $g(t) = H^\alpha_s \varphi(t) - A_0 \varphi(t)$ and the desired formula holds. ■

Combining Theorems 11 and 12 together we get the following result.

**Corollary 13** A solution $y \in C([1,T] \cap (h^{-1}, ph], \mathbb{R}^n)$ of (2) has a form

$$y(t) = Y_{h,\alpha,a}^{\alpha_0,A_1} \left( t, \frac{1}{h} \right) a + \int_1^t Y_{h,\alpha,a}^{\alpha_0,A_1} (t, s) \left[ \frac{H^\alpha_s}{\alpha} \varphi \right] (s) - A_0 \varphi (s) \frac{ds}{s}$$

$$+ \int_1^t Y_{h,\alpha,a}^{\alpha_0,A_1} (t, s) f(s) \frac{ds}{s}.$$
3 Existence Uniqueness and Stability

In this section, we consider the following equivalent integral form of the nonlinear Cauchy problem for fractional time-delay differential equations with Hadamard derivative \( [3] \):

\[
y(t) = Y^{A_0, A_1}_{h,\alpha,\alpha} \left( t, \frac{1}{h} \right) a + \int_{\frac{1}{h}}^{1} Y^{A_0, A_1}_{h,\alpha,\alpha} (t, s) \left[ \left( H D^\alpha_{\frac{s}{h}} f \right) (s) - A_0 \varphi (s) \right] \frac{ds}{s} \\
+ \int_{1}^{t} Y^{A_0, A_1}_{h,\alpha,\alpha} (t, s) f (s, y(s)) \frac{ds}{s}.
\]

(17)

Let us introduce the conditions under which existence and uniqueness of the integral equation \( [17] \) will be investigated.

(A1) \( f : [1, T] \times \mathbb{R} \to \mathbb{R} \) be a function such that \( f (t, y) \in C_{\gamma, \ln} [1, T] \) with \( \gamma < \alpha \) for any \( y \in \mathbb{R}^n \);

(A2) There exists a positive constant \( L_f > 0 \) such that

\[
\| f (t, y_1) - f (t, y_2) \| \leq L_f \| y_1 - y_2 \|,
\]

for each \((t, y_1), (t, y_2) \in [1, T] \times \mathbb{R}^n \).

From (A1) and (A2), it follows that

\[
\| f (t, y) \| \leq L_f \| y \| + L_2 \quad \text{for some } L_2 > 0.
\]

To prove existence uniqueness and stability of \( Y^{A_0, A_1}_{\alpha,\beta} \) we use the following estimation of \( Y^{A_0, A_1}_{\alpha,\beta} (t, s) \).

Lemma 14 We have for \( s h^p < t \leq s h^{p+1}, p = 0, 1, \ldots \)

\[
\left\| Y^{A_0, A_1}_{h,\alpha,\beta} (t, s) \right\| \leq Y^{\| A_0 \|, \| A_1 \| (t, s) \leq Y^{\| A_0 \|, \| A_1 \| (t, 1).}
\]

Proof. Indeed,

\[
\left\| Y^{A_0, A_1}_{h,\alpha,\beta} (t, s) \right\| \leq Y^{\| A_0 \|, \| A_1 \| (t, s) \leq \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{n+k}{k} \right) ^n \frac{\| A_1 \|^k \| A_0 \|^n}{\Gamma (n\alpha + k\alpha + \beta)} \left( \ln t - \ln h^{k+1} \right)^{\alpha+k\alpha+\beta-1} \\
= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{n+k}{k} \right) ^n \frac{\| A_1 \|^k \| A_0 \|^n}{\Gamma (n\alpha + k\alpha + \beta)} \left( \ln t \right)^{\alpha+k\alpha+\beta-1} \\
= Y^{\| A_0 \|, \| A_1 \| (t, 1).
\]

Our first result on existence and uniqueness of \( Y^{A_0, A_1}_{\alpha,\beta} \) is based on the Banach contraction principle.

Theorem 15 Assume that (A1), (A2) hold. If

\[
L_f \Gamma (1 - \gamma) \left( \ln T \right)^\gamma Y^{\| A_0 \|, \| A_1 \| (T, 1) < 1,
\]

then the Cauchy problem \( [3] \) has a unique solution on \([1, T] \).

Proof. We define an operator \( \Theta \) on \( \mathcal{B}_r := \{ y \in C_{\gamma, \ln} [1, T] : \| y \| \gamma \leq r \} \) as follows

\[
(\Theta y) (t) = Y^{A_0, A_1}_{h,\alpha,\alpha} \left( t, \frac{1}{h} \right) a + \int_{\frac{1}{h}}^{1} Y^{A_0, A_1}_{h,\alpha,\alpha} (t, s) \left[ \left( H D^\alpha_{\frac{s}{h}} f \right) (s) - A_0 \varphi (s) \right] \frac{ds}{s} \\
+ \int_{1}^{t} Y^{A_0, A_1}_{h,\alpha,\alpha} (t, s) f (s, y(s)) \frac{ds}{s},
\]

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where \( r \geq \frac{M_2}{1 - M_1} \),

\[
M_2 := (\ln T)^\gamma Y_{\gamma,0,\alpha}^{h,\gamma+1} (T, 1) \|a\| + \Gamma (1 - \gamma) (\ln T)^\gamma Y_{\gamma,0,\alpha}^{h,\gamma+1} (T, 1) \left\| \left( H D^{\alpha}_{\frac{\gamma}{\alpha}} + \varphi \right) \right\|_{\gamma,ln} + L_2 (\ln T)^{\gamma+1} Y_{\gamma,0,\alpha}^{h,\gamma+1} (T, 1),
\]

\[
M_1 := L_f \Gamma (1 - \gamma) (\ln T)^\gamma Y_{\gamma,0,\alpha}^{h,\gamma+1} (T, 1). \]

It is obvious that \( \Theta \) is well defined due to (A1). Therefore, the existence of a solution of the Cauchy problem (3) is equivalent to that of the operator \( \Theta \) has a fixed point on \( \mathcal{B}_r \). We will use the Banach contraction principle to prove that \( \Theta \) has a fixed point. The proof is divided into two steps.

**Step 1.** \( \Theta y \in \mathcal{B}_r \) for any \( y \in \mathcal{B}_r \).

Indeed, for any \( y \in \mathcal{B}_r \), and any \( \delta > 0 \), by (A3)

\[
\| (\ln t)^\gamma (\Theta y) (t) \| \leq (\ln t)^\gamma Y_{\gamma,0,\alpha}^{h,\gamma+1} (t, s) \left\| \left( H D^{\alpha}_{\frac{\gamma}{\alpha}} + \varphi \right) \right\|_{\gamma,ln} \|A_0\| \leq (\ln t)^\gamma \int_1^t Y_{\gamma,0,\alpha}^{h,\gamma+1} (t, s) \left\| \left( H D^{\alpha}_{\frac{\gamma}{\alpha}} + \varphi \right) \right\|_{\gamma,ln} \|f (s, y (s))\| ds.
\]

Similarly,

\[
(\ln t)^\gamma \int_1^t Y_{\gamma,0,\alpha}^{h,\gamma+1} (t, s) \|f (s, y (s))\| ds \leq (\ln t)^\gamma \int_1^t Y_{\gamma,0,\alpha}^{h,\gamma+1} (t, s) (L_1 \|y (s)\| + L_2) ds \leq L_f \Gamma (1 - \gamma) (\ln t)^\gamma Y_{\gamma,0,\alpha}^{h,\gamma+1} (t, 1) (t, \frac{1}{n}) \|y\|_{\gamma,ln} + L_2 (\ln t)^{\gamma+1} Y_{\gamma,0,\alpha}^{h,\gamma+1} (t, 1) (t, \frac{1}{n}).
\]

Inserting (19) and (20) into (18) we get

\[
\| (\ln t)^\gamma (\Theta y) (t) \| \leq (\ln t)^\gamma Y_{\gamma,0,\alpha}^{h,\gamma+1} (t, 1) \|a\| + \Gamma (1 - \gamma) (\ln t)^\gamma Y_{\gamma,0,\alpha}^{h,\gamma+1} (t, 1) \|H D^{\alpha}_{\frac{\gamma}{\alpha}} + \varphi - A_0 \varphi\|_{\gamma,ln} + L_2 (\ln t)^{\gamma+1} Y_{\gamma,0,\alpha}^{h,\gamma+1} (t, 1) + L_f \Gamma (1 - \gamma) (\ln t)^\gamma Y_{\gamma,0,\alpha}^{h,\gamma+1} (t, 1) \|y\|_{\gamma,ln} \leq M_2 + M_1 \|y\|_{\gamma,ln} \leq M_2 + M_1 r \leq r.
\]
Step 2. Let \( y, z \in C_{\gamma, \ln} [1, T] \). Then similar to the estimation \(^{(20)}\) we get

\[
\| (\ln t)^\gamma (\Theta y, t) - (\Theta z, t) \| \leq (\ln t)^\gamma \int_1^t Y_{t, h, \alpha, \alpha} \| A_0 \| \| A_1 \| \| (t, s) \| f (s, y (s)) - f (s, z (s)) \| ds \leq L f (\ln t)^\gamma \int_1^t \sum_{k=0}^P \sum_{n=0}^{\infty} \left( \frac{n + k}{k} \right) \| A_1 \|^k \| A_0 \|^n \frac{(\ln t - \ln s)^{\alpha + k + \alpha - 1}}{(\ln s)^{\alpha + k + \alpha}} \| (s, y (s)) - (s, z (s)) \| ds \leq L f (\ln t)^\gamma \int_1^t \sum_{k=0}^P \sum_{n=0}^{\infty} \left( \frac{n + k}{k} \right) \| A_1 \|^k \| A_0 \|^n \frac{(\ln t)^{\alpha + k + \alpha - \gamma}}{(\ln s)^{\alpha + k + \alpha}} \| y - z \|_{t, \gamma, \ln} \leq L f \Gamma (1 - \gamma) (\ln t)^\gamma Y_{t, 1, \alpha, \alpha - \gamma + 1} (t, 1) \| y - z \|_{t, \gamma, \ln}.
\]

which implies that

\[
\| \Theta y - \Theta z \|_{t, \gamma, \ln} \leq L f \Gamma (1 - \gamma) (\ln t)^\gamma Y_{t, 1, \alpha, \alpha - \gamma + 1} (T, 1) \| y - z \|_{t, \gamma, \ln}.
\]

Hence, the operator \( \Theta \) is contraction on \( B_r \) and the proof is competed by using the Banach fixed point theorem. \( \blacksquare \)

Secondly, we discuss the Ulam-Hyers stability for the problems \(^{(3)}\) by means of integral operator given by

\[
y (t) = (\Theta y) (t),
\]

where \( \Theta \) is defined by \(^{(17)}\).

Define the following nonlinear operator \( Q : C_{\gamma, \ln} ([1, T], \mathbb{R}^n) \to C_{\gamma, \ln} ([1, T], \mathbb{R}^n) \):

\[
Q (y) (t) := \left( h D_{\frac{t}{h}}^\alpha + y \right) (t) - A_0 y (t) - A_1 y \left( \frac{t}{h} \right) - f (t, y (t)).
\]

For some \( \varepsilon > 0 \), we look at the following inequality:

\[
\| Q (y) \|_{t, \gamma, \ln} \leq \varepsilon.
\]

\( \Box \)

**Definition 16** We say that the equation \(^{(17)}\) is Ulam-Hyers stable, if there exist \( V > 0 \) such that for every solution \( y^* \in C_{\gamma, \ln} ([\frac{1}{h}, T], \mathbb{R}^n) \) of the inequality \(^{(22)}\), there exists a unique solution \( y \in C_{\gamma, \ln} ([\frac{1}{h}, T], \mathbb{R}^n) \) of problem \(^{(17)}\) with

\[
\| y - y^* \|_{t, \gamma, \ln} \leq V \varepsilon.
\]

\( \Box \)

**Theorem 17** Under the assumptions of Theorem \(^{(13)}\) the problem \(^{(17)}\) is stable in Ulam-Hyers sense.

**Proof.** Let \( y \in C_{\gamma, \ln} ([\frac{1}{h}, T], \mathbb{R}^n) \) be the solution of the problem \(^{(17)}\). Let \( y^* \) be any solution satisfying \(^{(22)}\):

\[
\left( h D_{\frac{t}{h}}^\alpha + y^* \right) (t) = A_0 y^* (t) + A_1 y^* \left( \frac{t}{h} \right) + f (t, y^* (t)) + Q (y^*) (t).
\]

So

\[
y^* (t) = \Theta (y^*) (t) + \int_1^t Y_{t, h, \alpha, \alpha} A_1 (t, s) Q (y^*) (s) ds.
\]

It follows that

\[
(\ln t)^\gamma \| \Theta (y^*) (t) - y^* (t) \| \leq (\ln t)^\gamma \int_1^t \left\| Y_{t, h, \alpha, \alpha} A_1 (t, s) \right\| \| Q (y^*) (s) \| ds \leq (\ln T)^{\gamma + 1} Y_{t, 1, \alpha, \alpha} \| A_0 \| \| A_1 \| (T, 1) \varepsilon.
\]

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Therefore, we deduce by the fixed-point property (21) of the operator $\Theta$, that

$$
(\ln t)^\gamma \|y(t) - y^*(t)\| \leq (\ln t)^\gamma \|\Theta(y)(t) - \Theta(y^*)(t)\| + (\ln t)^\gamma \|\Theta(y^*)(t) - y^*(t)\|
$$

$$
\leq L_f \Gamma(1 - \gamma) (\ln t)^\gamma Y_{1,\alpha,\alpha-\gamma+1} (t, 1) \|y - y^*\|_{\gamma,\ln} + (\ln T)^{\gamma + 1} Y_{1,\alpha,\alpha} \|A_0\| \|A_1\| (T, 1) \varepsilon,
$$

(24)

and

$$
\|y - y^*\|_{\gamma,\ln} \leq \frac{(\ln T)^{\gamma + 1} Y_{1,\alpha,\alpha} \|A_0\| \|A_1\| (T, 1)}{1 - L_f \Gamma(1 - \gamma)(\ln T)^\gamma Y_{1,\alpha,\alpha-\gamma+1} (T, 1)} \varepsilon.
$$

Thus, the problem (3) is Ulam-Hyers stable with.

$$
V = \frac{(\ln T)^{\gamma + 1} Y_{1,\alpha,\alpha} \|A_0\| \|A_1\| (T, 1)}{1 - L_f \Gamma(1 - \gamma)(\ln T)^\gamma Y_{1,\alpha,\alpha-\gamma+1} (T, 1)}.
$$

4 Example

In this section, we give an example to illustrate the obtained theoretical result. Let $\alpha = 0.3, h = 1.2, k = 4$. Consider

$$
(H D_{1.2}^{0.3} y)(t) = A_1 y \left( \frac{t}{1.2} \right), \quad t \in [1, 2.0736],
$$

$$
y(t) = \varphi(t), \quad \frac{1}{1.2} \leq t \leq 1,
$$

$$
(H D_{1.2}^{0.7} y) \left( \frac{1}{1.2} \right) = a \in R^n,
$$

(25)

where

$$
A_1 = \left( \begin{array}{ccc}
2 & 1 \\
3 & 5
\end{array} \right), \quad a = \left( \begin{array}{c}
1 \\
2
\end{array} \right)
$$

The solution of (25) can be represented by $Y_{1,\alpha,\alpha} A_1 (t, \frac{1}{1.2}) = E_{1,\alpha,\alpha}^{A_1} (\ln t)$

$$
y(t) = E_{1,\alpha,\alpha}^{A_1} (\ln t) a + \int_{\frac{1}{1.2}}^{t} E_{1,\alpha,\alpha}^{A_1} \left( \ln \frac{s}{1.2} \right) \left( H D_{1.2}^{0.7} \phi \right)(s) \frac{ds}{s},
$$

where

$$
E_{h,\alpha,\alpha}^{A_1} (\ln t) = \left\{ \begin{array}{ll}
0, & -\infty < x \leq \frac{1}{1.2}, \\
n \frac{(\ln t + \ln 1.2)^{-0.7}}{\Gamma(0.3)}, & \frac{1}{1.2} < x \leq 1, \\
n \frac{(\ln t + \ln 1.2)^{-0.7}}{\Gamma(0.3)} + A_1 \frac{(\ln t)^{-0.4}}{\Gamma(0.6)}, & 1 < x \leq 1.2, \\
n \frac{(\ln t + \ln 1.2)^{-0.7}}{\Gamma(0.3)} + A_1 \frac{(\ln t)^{-0.4}}{\Gamma(0.6)} + A_2 \frac{(\ln t - \ln 1.2)^{-0.1}}{\Gamma(0.9)}, & 1.2 < x \leq (1.2)^2, \\
n \frac{(\ln t + \ln 1.2)^{-0.7}}{\Gamma(0.3)} + A_1 \frac{(\ln t)^{-0.4}}{\Gamma(0.6)} + A_2 \frac{(\ln t - \ln 1.2)^{-0.1}}{\Gamma(0.9)} + A_3 \frac{(\ln t - 2\ln 1.2)^{0.2}}{\Gamma(1.2)}, & (1.2)^2 < x \leq (1.2)^3, \\
n \frac{(\ln t + \ln 1.2)^{-0.7}}{\Gamma(0.3)} + A_1 \frac{(\ln t)^{-0.4}}{\Gamma(0.6)} + A_2 \frac{(\ln t - \ln 1.2)^{-0.1}}{\Gamma(0.9)} + A_3 ^2 \frac{(\ln t - 2\ln 1.2)^{0.2}}{\Gamma(1.2)} + A_4 \frac{(\ln t - 3\ln 1.2)^{0.5}}{\Gamma(1.5)}, & (1.2)^3 < x \leq (1.2)^4.
\end{array} \right.
$$

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References

[1] Diblík J, Fečkan M, Pospíšil M. Representation of a solution of the Cauchy problem for an oscillating system with two delays and permutable matrices. Ukrainian Math J. 2015;65:58–69.

[2] Diblík J, Khusainov DYa, Baštinec J, Sirenko AS. Exponential stability of linear discrete systems with constant coefficients and single delay. Appl Math Lett. 2016;51:68–73.

[3] Khusainov DYa, Shuklin GV. Linear autonomous time-delay system with permutation matrices solving. Stud Univ Žilina. 2003;17:101–108.

[4] Medveď M, Pospíšil M. Representation of solutions of systems linear differential equations with multiple delays and linear parts given by nonpermutable matrices, Journal of Mathematical Sciences. 2018;228.

[5] Pospíšil M. Representation and stability of solutions of systems of functional differential equations with multiple delays. Electronic Journal of Qualitative Theory of Differential Equations. 2012:54.

[6] Diblík, J., Fečkan, M., Pospíšil, M.: Representation of a solution of the Cauchy problem for an oscillating system with two delays and permutable matrices. Ukr. Math. J. 65, 58–69 (2013)

[7] Diblík, J., Khusainov, D.Y.: Representation of solutions of discrete delayed system $x(k + 1) = Ax(k) + Bx(k - m) + f(k)$ with commutative matrices. J. Math. Anal. Appl. 318, 63–76 (2006)

[8] Diblík, J., Khusainov, D.Y.: Representation of solutions of linear discrete systems with constant coefficients and pure delay. Adv. Differ. Equ. 2006, Article ID 80825 (2006)

[9] Pospíšil M. Representation of solutions of delayed difference equations with linear parts given by pairwise permutable matrices via $Z$-transform. Appl Math Comput. 2017;294:180–194.

[10] Mahmudov, N.I.: Representation of solutions of discrete linear delay systems with non permutable matrices. Appl. Math. Lett. 85, 8–14 (2018)

[11] Khusainov DYa, Shuklin GV. Relative controllability in systems with pure delay, Int J Appl Math. 2005;2:210–221.

[12] Medveď M, Pospíšil M, Škripková L. Stability and the nonexistence of blowing-up solutions of nonlinear delay systems with linear parts defined by permutable matrices. Nonlinear Anal. 2011;74:3903–3911.

[13] Medveď M, Pospíšil M. Sufficient conditions for the asymptotic stability of nonlinear multidelay differential equations with linear parts defined by pairwise permutable matrices, Nonlinear Anal. 2012;75:3348–3363.

[14] Medved’, M., Pospíšil, M.: Sufficient conditions for the asymptotic stability of nonlinear multidelay differential equations with linear parts defined by pairwise permutable matrices. Nonlinear Anal. 75, 3348–3363 (2012)

[15] Li M, Wang JR. Finite time stability of fractional delay differential equations. Appl Math Lett. 2017;64:170–176.

[16] Li M, Wang JR. Exploring delayed M-L type matrix functions to study finite time stability of fractional delay differential equations. Appl Math Comput. 2018;324:254–265.

[17] Li M, Debouche A, Wang JPR. Relative controllability in fractional differential equations with pure delay, Mathematical Methods in the Applied Sciences. 2017;1–9, DOI: 10.1002/mma.4651

[18] Liang C, Wang JR, O’Regan D. Representation of solution of a fractional linear system with pure delay. Appl Math Lett. 2018;77:72–78.
[19] Luo Z, Wang JR. Finite time stability analysis of systems based on delayed exponential matrix. *J Appl Math Comput.* 2017;55:335-351.

[20] Mahmudov, N.I.: Delayed perturbation of M-L functions and their applications to fractional linear delay differential equations. Math. Methods Appl. Sci., 1–9 (2018). https://doi.org/10.1002/mma.5446

[21] Mahmudov, N.I.: A novel fractional delayed matrix cosine and sine. Appl. Math. Lett. 92, 41–48 (2019)

[22] Klimek, M.: Sequential fractional differential equations with Hadamard derivative. Commun. Nonlinear Sci. Numer. Simul. 16, 4689–4697 (2011)

[23] Ma, Q., Wang, R., Wang, J., Ma, Y.: Qualitative analysis for solutions of a certain more generalized two-dimensional fractional differential system with Hadamard derivative. Appl. Math. Comput. 257, 436–445 (2014)

[24] A.A.Kilbas,H.M.Srivastava,J.J.Trujillo,Theory and Applications of Fractional Differential Equations, in:North-Holland Mathematics Studies, vol.204, Elsevier,Amsterdam,2006.

[25] Li M, Wang JR, Analysis of nonlinear Hadamard fractional differential equations via properties of Mittag–Leffler functions

[26] Yang P., Wang JR and Zhou Y., Representation of solution for a linear fractional delay differential equation of Hadamard type, Advances in Difference Equations (2019) 2019:300