Discrete symmetry and quark, lepton and vector gauge boson mass expressions.

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Some interesting mass expressions for all the quarks, leptons and gauge bosons are presented based on simple discrete symmetry. Precise expressions for the nucleon masses are presented as an example of the calculation techniques employed. The structure of the Higgs sector is explored.

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I. INTRODUCTION

In the standard model the masses of the fermions are generated by interaction with the Higgs field:

\[ \mathcal{L}_{\text{mass}} = g \bar{\psi}_L \psi_R \phi \]

where \( g \) is a (dimensionless) coupling constant and \( \phi \) is the scalar Higgs field of dimension \( l^{-1} \). The Dirac mass so defined is in fact specified by the parameter \( g \) and it is fair to say the expression (1) tells us nothing at all about the masses of the fermions; the detail is hidden in the arbitrary parameter \( g \) and the unspecified mass of \( \phi \). One might reasonably ask whether the Lagrangian field theory approach is even in principle capable of explaining the masses of the fermions. Certainly the conventional perturbative approach to (electro-weak) field theory would seem to be inadequate for the purposes of defining particle masses given the requirement for mass renormalisation in such an approach and even non-perturbative approaches to the strong interaction require the insertion ‘by-hand’ of the current quark masses. These failings certainly suggest that an alternative approach to determining the
origin of the spectrum of particle masses might be required; provided always that consistency with the standard model is maintained as the latter is certainly empirically valid over a wide range of experimental parameters.

The data on particle masses is particularly impressive on two counts; firstly that so much precision data is available now and secondly that the standard model fails so miserably to account for it! It is almost accepted wisdom in contemporary high energy physics that, possibly with the exception of neutrino physics, experiment lags behind theory; yet this is not true of the data that exists on particle masses! It therefore behoves us to look closely at the data for clues that might alert us to underlying symmetry not apparent from the standard model.

This paper presents such an approach; it does so by looking at possible discrete symmetries which might underpin the observed spectrum of particle masses and relating these back to the known continuous symmetries of the standard model. The link between the discrete and continuous symmetry in the model presented is provided by local gauge invariance. Particle rest masses then arise as a relic feature of an otherwise unobservable discrete symmetry underpinning quark and lepton structure.

Thus the model presented here is essentially semi-empirical but there is a long history of discovery in physics based on finding patterns in physical data. Some of the most famous would include the discovery of the periodic table of elements and the Balmer series although one could argue that all basic science evolves ultimately from a study of the patterns observed in nature.

In the framework of the symmetry used to derive fermion masses all interactions are treated in an entirely non-perturbative manner. Radiative corrections to particle masses are calculated using a reasonable non-perturbative ansatz. This gives quite precise mass predictions which can be compared with experiment; including the masses of stable hadrons.

The paper is organised as follows;

Sections II-V Introduce the basic geometric ideas employed throughout. These sections are long and quite ‘wordy’ but the concepts are unfamiliar and a little abstract so it was
thought, at the risk of losing the readers attention, probably worthwhile trying to get the ideas across with as much explanation as possible.

Sections VI-X start on some more concrete analysis and puts the ideas presented in the earlier sections on a footing capable of leading to systematic mass calculations concentrating on first generation hadrons. The material is quite different to a standard QCD approach although the reader will note some overall symmetry features in common with QCD. As mentioned, there is reason to believe that a standard QFT approach cannot (even in principle) determine mass ratios from first principles (i.e. without arbitrary parameters).

In section XI we look at the Higgs sector of the theory from the point of view of geometry leading to the masses of second and third generation objects and massive vector gauge bosons. This section is quite heuristic as only vague hints of the origin of the pattern of Higgs couplings can be gleaned from the values of the couplings themselves - which have been determined by comparison with experiment - although the symmetry of the system leads to many constraints on the values the couplings can take; in particular they can only take certain fixed values which then allows comparison with precision measurements. Lastly we look at the issue of neutrino masses and generalisations of postulate 1 to include more complex models of the space-time continuum.

II. BACKGROUND IDEAS

Experiment has unequivocally established the reality of quarks, behaving as point-like objects, as the constituents of hadrons and also shows no indication that leptons have any sub-structure. Quarks and leptons come in three generations or families;

|               | top  | bottom | charm | strange | up   | down |
|---------------|------|--------|-------|---------|------|------|
| quarks;       | ντ   | τ      | νμ    | μ       | ντ   | e    |
| leptons;      |      |        |       |         |      |      |

and the masses generally decrease from left to right; with the exception of the neutrinos which have much smaller, or zero, mass in comparison with their partner charged lepton. The other pattern evident is that, for each family, the quark masses are greater than the
corresponding charged lepton. Thus the bottom quark, for example, has a mass in the vicinity of 5GeV whilst the tau lepton has a mass in the vicinity of 1.7Gev. In the standard model the masses of the neutrinos are zero; these being left-handed particles only and not possessing a Dirac mass. Current data suggests that neutrino masses are non-zero and we will look at this issue towards the end of the paper. In what follows in the interim we will treat the neutrinos as massless.

The standard model theory of interactions is based on the principle of local gauge invariance. In the case of electromagnetism this is simply a U(1) phase. A renormalisable Lagrangian is generally taken to contain at most the field and its first derivative only. The derivative of the spinor field is not invariant under local phase transformations;

$$\partial_\mu (e^{i\theta(x)} \psi) = \partial_\mu \theta(x) e^{i\theta(x)} \psi + e^{i\theta(x)} \partial_\mu \psi \neq e^{i\theta(x)} \partial_\mu \psi \quad (2)$$

but under a covariant derivative containing the electromagnetic field the fermion field transforms as follows;

$$D_\mu (e^{i\theta(x)} \psi) = i\partial_\mu \theta(x) e^{i\theta(x)} \psi + e^{i\theta(x)} \partial_\mu \psi + ieA_\mu e^{i\theta(x)} \psi \quad (3)$$

$$D_\mu \equiv \partial_\mu + ieA_\mu$$

Now, under the (commuting) U(1) symmetry we let the gauge field transform as follows;

$$A_\mu \rightarrow A'_\mu \equiv A_\mu - \frac{1}{e} \partial_\mu \theta(x)$$

Finally, substituting this definition in eq.(3) gives;

$$D_\mu (e^{i\theta(x)} \psi) = e^{i\theta(x)} \partial_\mu \psi + ieA_\mu e^{i\theta(x)} \psi = e^{i\theta(x)} (D \psi) \quad (4)$$

In the case of the weak interactions the local gauge invariance is more complex but the basic idea is the same; the spinor field is represented by a left-handed doublet and right handed singlet. The corresponding phase factor for the doublet is an S.U.(2) symmetry and local gauge invariance occurs under a covariant derivative involving a trio of gauge fields derived from the adjoint representation of S.U.(2); rather than the single field that arises.
from the U(1) phase factor in electro-magnetism. Thus the right handed singlets do not ‘feel’ the force mediated by the weak vector gauge bosons.

Likewise, in the case of quantum chromodynamics, local gauge invariance under S.U.(3) colour leads to a theory of the strong interaction with 8 gauge bosons reflecting the adjoint representation.

There is now ample evidence to assert that the $S.U.(3) \times S.U.(2)_L \times U(1)_Y$ standard model is a (not necessarily the) correct description of fundamental structure yet nowhere do we find, apart from the subjective criterion of mathematical elegance, any reason why nature has chosen local gauge invariance as the cornerstone of its foundation. In the following sections we explore a geometric model capable of accounting for nature’s predilection for local gauge invariance.

Let us begin by considering the structure of the continuum. In physics we routinely assume that space-time is a real continuum. By this we mean that any interval in space (time) may be represented by the real numbers as the ratio of the size of the interval in question with some standard unit interval (such as a metre).

The structure of the continuum is quite interesting mathematically. In terms of set theory Georg Cantor showed that the continuum may be analysed in terms of cardinality, or order, of sets. This idea will be crucial for our study of particle masses. We may analyse the continuum in terms of a hierarchy of cardinality. The first tier is simply that of a finite set; for example the set of two integers $\{0,1\}$ (which is also the basis of a number field of finite cardinality called the Galois field). The next tier is of transfinite (i.e. completed infinite) cardinality and it is the field of rational numbers (which of course has the same cardinality as the counting numbers). This order of cardinality is represented by the symbol $\aleph_0$. Cantor proved that $\aleph_0$ is not the cardinality of the real numbers and is in fact a smaller quantity than $c$ the cardinality of the real continuum; in spite of the fact that $\aleph_0$ represents a (literal) completed infinity.

A subtlety arises with respect to the definition of the third tier of cardinality due to the fact that it is not possible to prove whether there exists an order of cardinality $\aleph_1$ such that
where \( c \) is the cardinality of the real numbers. One is at liberty to assume the continuum hypothesis, which asserts that no such number \( \aleph_1 \) exists, in which case we define the real number continuum in terms of three tiers of cardinality (i.e. finite, transfinite order \( \aleph_0 \) and transfinite order \( c \)). Alternatively one may consider models of space-time based on a continuum built up from more than three orders of cardinality in which case one considers that the number (or numbers) \( \aleph_1 \) exists. We will consider both possibilities in this paper. Initially, however, we will work with the continuum hypothesis as a postulate;

**Postulate 1**: That there exists an isomorphism between the structure of the real number continuum and the structure of physical space-time which may be represented in terms of three tiers of cardinality (the continuum hypothesis).

**Hypothesis 1**: That the fundamental structure of matter and energy in the universe is related to a deconstruction of the the continuum of space-time.

Hypothesis 1 implies that mass/energy and space-time are made out of the same basic ‘stuff’.

### III. AFFINE SET GEOMETRY

We define *affine-set geometry* as one in which distances and angles are not defined but only the shape is manifest. We define an *affine-set* as a collection of points interconnected by lines. A point is always a termination of a line. Multiple lines may terminate at a single point. No points exist on any line except at the terminations of the line. There are thus always two and only two points associated with any single line.

Affine-set geometries can be used to model local gauge invariance but their significance in this paper is much broader.

Consider for a moment the common geometric idea of a two-dimensional sub-space of our three-dimensional real space (time); that is, a two-dimensional plane. Conventionally we would consider that the physical properties of the continuum (of 3-space) carry over to
any two-dimensional sub-space but in fact we don’t really know if this is true (assuming always that the 3-space of space-time is in fact a real continuum!). The reason we don’t know is simply because such a space is not empirically accessible. Let us consider this statement. For a two-dimensional plane to be truely two-dimensional its’ ‘thickness’ must literally be zero! We can only ever assess the properties of any space indirectly; that is, by observing the behaviour of a test particle in that space where the continuum of an object’s motion, momentum etc. implies the existence of the continuum property of the underlying space-time. In the case of a true two-dimensional space (+time) it would require an infinite amount of energy to confine the wave-function of any massive object to the space so such observations can never be made. (For example, to compress length contraction to zero we would need \( v = c \);

\[
l' = l \sqrt{1 - \frac{v^2}{c^2}}
\]

or that the velocity of the massive object reaches the speed of light which is impossible). Thus we do not know whether such a two-dimensional space would have the continuum property of the real numbers or not! we simply cannot confine test particle to the space.

The situation with a truely one-dimensional sub-space is even worse for, in addition to requiring an infinite amount of energy to confine the wave-function of a massive object to a space of zero transverse dimension, the line would need to be of infinite length to accommodate it! (It is important here to realise that a quantum object with non-zero rest mass cannot be squeezed into a point of zero dimension!).

These statements, however, are not necessarily true for massless objects such as photons and gravitons (although, once again, if they have non-zero energy they cannot be squeezed into a space of zero dimension; i.e. they cannot be squeezed to a true geometric ‘point’).

One of the central tenets of this work is that any one or two-dimensional sub-space of our physical 3-space is not in fact a real continuum and that the continuum property of space-time only exists at the level of three space dimensions. Put another way, such sub-spaces do not exist! But there is a caveat to this assertion; such sub-spaces may exist...
as the underpinning of the structure of fundamental particles themselves; not so much as a kind of ‘internal’ space as a partial ‘deconstruction’ of continuum space. The role that local gauge invariance will play is to adjust the cardinality of these one and two-dimensional subspaces, that are postulated to underpin the structure of fundamental matter, to that of the continuum. The non-observability of any local phase then becomes equivalent to the unobservable nature of the subspaces ‘building’ the particle structure.

This is the use to which we wish to put affine geometry. Consider the idea of embedding affine-set geometry into a continuum space; that is, we want to see what the geometrical consequences are of trying to wed affine-set geometry and continuous geometry.

For example, consider the following two (affine-set) triangles embedded on a continuum circle;

![Diagram of two triangles](image)

The two triangles represented by these diagrams are, in affine-set geometry, equivalent. This is because the side lengths (the three ‘lines’ of the triangle) and the angles subtended by the lines where they form points are not measurable by definition for an affine-set. They do not represent a physical observable. The transformation which takes one into the other is a local $U(1)$ gauge transformation on the embedding (continuum) circle however as the vertices shift around the margin of the circle by different amounts (by contrast a global transformation would preserve triangle angles and side lengths). Thus an affine-set triangle embedded in a circle defines a $U(1)$ local gauge symmetry in the sense that such a triangle is invariant under both global and local $U(1)$ gauge transformations only if the geometry defining the triangle is an affine-set. In this sense we could in fact use local $U(1)$ gauge symmetry as an alternative definition of affine-set geometry (we don’t have to use a triangle
of course - we could use any kind of affine-set geometry). For example, from the set (infinite but countable) \( \mathbb{N}_0 \) of points on the circle we choose an infinite set of sets of points each with three elements \( \mathbb{N}_3^i \) where the superscript \( i \) indicates the (countable) \( i \)th set of three points (the cardinality of this set is of course also \( \mathbb{N}_0 \)). Now, assume that each \( i \)th set of points is unique; that is

\[
\mathbb{N}_3^i \cap \mathbb{N}_3^j = 0; \quad \forall \ i, j
\]

This defines an infinite, but countable, set of geometric triangles inscribed within the circle. Now the transformation which converts any given triangle into any other is always a local U(1) transformation (global U(1) transformations are a subset of the set of local transformations). For the total geometry to be invariant with respect to such U(1) transformations requires that the geometry be affine-set and can be used as a formal definition.

The union of the sets \( \mathbb{N}_3^i \) defines the set of points in the bounding circle\(^1\). Let \( C_3^i \) be a given affine-set triangle. Define the transformation \( C_3^i \rightarrow C_3^j \) as \( C_3^{ij} \) then;

\[
\sum_{i,j} (C_3^{ij}) = e^{i\theta(x)} \tag{5}
\]

gives us a compact way of expressing the equivalence of a local gauge transformation and an affine-set transformation on the circle.

We could add a requirement of ‘smoothness’ to the affine-set transformations to eliminate singularities produced by vertices ‘crossing-over’ each other in the transformation process (as two vertices pass each other they define a line not a triangle). Does equation (5) remain valid with the elimination of such singularities? Indeed, it would seem that, if \( \theta(x) \) is required to be smoothly differentiable then such transformations must in fact be eliminated and it is a hidden assumption of eq.(4) that the local phase transformation \( e^{i\theta(x)} \), like the global one, is smoothly differentiable. It is not at all clear if local gauge invariance is maintained if the transformation is not a smoothly differentiable manifold!

\(^1\)treating the circle not as a true continuum of points since cardinality \( c \neq \mathbb{N}_0 \)
A way of thinking of this is in terms of an order of the points in the circle. Treating the points in the circle as an ordered set (although infinite) we know that we can still squeeze or stretch any segment of the circle to an arbitrary degree without changing the order of the points and thus preserving the differentiability of the manifold. This follows from the work of Georg Cantor who proved that there is a one-to-one mapping of the points on any two finite length line segments (and indeed onto an infinite length line segment from a finite one!). That a global rotation is a one-to-one order-preserving mapping is obvious. It is not so obvious that this is the case for a local U(1) transformation however since singularities can be introduced by such a transformation as mentioned.

Affine-set geometries, because the ‘length’ of sides of the geometry can only ever be defined as ‘unity’, can be represented by discrete group symmetry. Discrete groups can be ‘spinorial’ or ‘bosonic’ in the sense that they contain (res. do not contain) improper rotations. An example of an improper rotation in two dimensional space would be:

![Diagram](image)

where the vertices ‘c’ and ‘b’ have interchanged as well as shifted around the circle with a local U(1) transformation. In this case it is not clear that the transformation could be represented in the form $e^{i\theta(x)}$ with continuously differentiable $\theta$ as the set of transformations is not one-to-one and onto and not well defined (we would not know, for example, how many points in the circle are crossing over other points during such a transformations or indeed if the continuum remained well defined if a (?completed) infinite number of cross-overs occurred). This is obviously not the case however if the order of the points on the triangle
(a,b,c clockwise or anti-clockwise) is preserved since we may define the transformation in that case as purely a stretching or compression of the continuum which preserves the order of all points in the circle. Thus, if we take only the $C_3$ generator of the triangle as being manifest, and allow it to become a \textit{continuous} rotation, we could use the affine-set triangle as a model of a quantum object of integral spin satisfying a local U(1) symmetry invariance.

\textbf{IV. DECONSTRUCTING SPACE-TIME TO MAKE MATTER}

We will do this geometrically by making geometric models of orders of cardinality which we will then use simultaneously as models of the space-time continuum and as models of particle structure.

We start with an object to represent the Galois Field of finite cardinality. This is the discontinuous set of two digits $\{0,1\}$ which we may represent as a discrete one-dimensional interval. Its’ only definable ‘length’ is 1; in units of itself (i.e.; it is unmeasurable) and the interval possesses an $S_2$ discrete symmetry because the choice of labelling its’ ends with the digits 0 and 1 is an arbitrary choice. Only two geometric points are associated with the interval; those at the terminations of the interval. There are no ‘points’ along the line itself; this is an affine-set of finite cardinality (cardinality 2) and the ‘line’ is absolutely discontinuous. If we embedded such an object in conventional space-time it would have a true ‘gauge’ invariance in the more literal meaning of the word; that is, it would have the property of \textit{scale} invariance. (It is also interesting to note that the $S_2$ generator, expressed as a continuous geometric symmetry, would render the object as spin 2).

To make a geometric model of the next highest order of cardinality, that of $\aleph_0$, we must make some further assumptions. Intuitively we might look at the continuum as rather like a complex structure built up from simpler ‘building blocks’; the simpler building blocks should underpin the more complex structures. We are working under the assumption that these ‘building blocks’ are the tiers of cardinality implicit in the structure of the continuum. Thus we want to use the Galois geometry to ‘generate’ a space with cardinality $\aleph_0$. If we regard
the Galois geometry as a ‘separation’ of two points the obvious geometric option is to iterate this process and separate two Galois intervals to create a two-dimensional space bounded by Galois intervals. Such a process can define (at least) two types of objects; triangles and squares each object bounded by discontinuous Galois intervals. We now assume that the two-dimensional space contained within the bounding intervals has the cardinality $\aleph_0$.

Finally to create a model of the continuum we iterate the process again. This allows us to generate cubes, tetrahedrons etc. The contained space is assumed to be a geometric representation of the real number continuum whilst the boundaries of the geometries remain discontinuous. Of course, no time dimension is defined by this procedure; only a three dimensional continuous but finite space is defined contained ‘within’ the boundary of the defining geometry. Let us call these three-dimensional geometries ‘Cantor’ geometries.

**V. OVERVIEW OF PHENOMENOLOGICAL INTERPRETATION OF THE GEOMETRY LINKED TO LOCAL GAUGE INVARIANCE**

**Hypothesis 2** That the Galois interval is the geometry of the graviton.

**Hypothesis 3** That the Cantor triangle is the geometry of the photon.

**Hypothesis 4** That the Cantor cube is the geometry of a quark.

**Hypothesis 5** That the Cantor tetrahedron is the geometry of a lepton.

**Hypothesis 6** The boundary of an object constitutes its’ associated gauge fields.

**Hypothesis 7** Any geometry defining a real-continuum of space has a mass specified by the symmetry of the object.

**Hypothesis 8** Objects which do not define an intrinsic continuum to be embedded in space-time must elevate their cardinality to that of the continuum.

The are compelling empirical reasons for making these identifications as we shall see but at this stage our chief motivation is to account for nature’s obsession with local gauge invariance.

Hypothesis 7 is the basis of rest mass calculations. We will form matrix representations of
the symmetry groups implied by the affine-set; which in general is a permutation symmetry of a labelling of the points of the set. We will then sum over all matrix elements needed to describe the geometry, and, in the case of the strong interaction, any matrix elements describing the geometry of the interaction, and form ratios of particle masses by taking ratios of sums of matrix elements.

In summary we have assumed that the continuum of space-time has a complex structure in terms of tiers of cardinality and that material objects in the universe are a reflection of that structure. *Time* must be assumed to exist outside the boundaries of the geometries in question and may be added as an additional hypothesis if desired (which is reasonable given that the existence of time is always assumed in any physical theory!). We could alternatively define time in terms of a relationship between different geometries embedded in a larger three-dimensional space. The time dimension is of course assumed to satisfy the continuum criteria. However, the cardinality of four-dimensional *space-time* is not assumed to be different from that of the three-dimensional space as defined above (in section XI we will look at alternative models of the continuum where this latter assumption does not hold).

More particularly the sequence of geometric construction must be viewed as the generation of space-time itself. Thus we do not view the Galois interval geometry as an object embedded in space-time but rather, in combination with the two and three-dimensional geometries, as the generator of space-time itself. The time-propagating Galois interval will ‘sweep-out’ a sheet of two-dimensional space. Curvature of this ‘swept-out’ space would then be sufficient to elevate the cardinality of the generated space to that of the continuum under the continuum hypothesis (viz. postulate 1). However, the one-dimensional Galois interval, which we have identified with the graviton, does not have a continuum structure itself; nor does the ‘swept-out’ sheet of two-dimensional space produced by its’ propagation. Only the final three-dimensional generated space has the continuum property.

These hypotheses contain a lot of interesting implications. Firstly we note that, since all geometries have Galois intervals in their boundaries, all geometries, including the photon and the graviton itself (by hypothesis 6), have gravitons as an associated field and ‘feel’,
and are the source of, gravitation.

An object such as a Cantor tetrahedron must be electro-magnetically charged since it has a photon (gauge field) in its boundary (i.e. cantor triangles in its’ boundary). Let us look at this geometry from the perspective of local gauge invariance. The Galois intervals defining the Cantor tetrahedron are by definition unmeasurable by any observer. The terminations of the intervals can be anywhere in space leading to a field theory rather than a theory of a point particle. However, there is an inbuilt paradox in the theory because the time dimension cannot be defined within the interior of the tetrahedron; which leads to the object behaving like a geometric point in scattering experiments. In other words, the Cantor tetrahedron is a three-dimensional space object embedded in a four-dimensional space-time. No observer probe can penetrate the timeless interior space of the Cantor tetrahedron (all observations occur in time!) which will always appear smaller than the wavelength of any probe.

Of particular interest is the boundary between the timeless three-dimensional space in the interior of the Cantor tetrahedron and the four-dimensional space-time outside the boundary of the tetrahedron. In the case of the Cantor tetrahedron this boundary is composed of Cantor triangles and Galois intervals defining these triangles. These are the postulated geometries of the photon and graviton respectively. Because of the unmeasurable nature of the ‘lengths’ of these intervals their terminations can be anywhere in space and interaction with them will be probabilistic; they will manifest as fields in space-time (by hypothesis respectively the electro-magnetic and gravitational fields of the Cantor tetrahedron). By virtue of the fact that these bounding geometries in turn define the Cantor tetrahedron so too the Cantor tetrahedron will have the schizophrenic identity of both a non-local field and a geometric point under observation. This is a perfect candidate for a quantum field.

One obtains the concept of local gauge invariance under the more demanding hypothesis 8. Hypothesis 8 is a very natural requirement. In essence it demands that only the continuum is an observable. The space contained within the boundary of the Cantor triangle is NOT a continuum by definition (it has cardinality $\aleph_0$ not $c$) and thus is not an observable space. Its lack of observability is satisfied physically if it propagates at the speed of light so that
the contained space of the Cantor triangle cannot be placed in the same inertial frame as any observer. Thus the Cantor triangle will always generate a three-dimensional space with respect to any observer by virtue of its relative motion sufficient to define a continuum space as required for physical existence under hypothesis 8. The astute reader will have realised that this requirement is equivalent to the principle of special relativity if one identifies the Cantor triangle with the photon.

A flat two-dimensional square, the model for an electron neutrino in this schema, has no triangle in it’s boundary and is thus not electro-magnetically charged. (It nevertheless has a piece of phase information in it’s boundary which induces a weak interaction gauge field as we will later see - we will also look at the issue of different generations of quarks and leptons later).

The principle of special relativity emerges quite naturally from this schema viz. hypotheses 3 and 8. If a triangle is a photon then, to exist in space-time, postulate 8 tells us that it must propagate in time to ‘sweep-out’ a three dimensional space to define a real continuum. To be defined geometrically in this schema it must sweep out a bounded space - which means its' velocity must be finite relative to cubes and tetrahedrons but never zero. If one adds relative motion between different cubes and or different tetrahedrons in a larger space-time embedding then it follows that the photon’s velocity relative to any given cube or tetrahedron is always a finite constant. Of course the same must apply to the Galois interval - the model of the graviton in this schema - it must propagate in time and sweep out a volume to exist; but it does not have to propagate at the same velocity as the photon!
It may of course do so but there is no prima facia reason in this geometry why a graviton must propagate at the speed of light.

A cube in this schema satisfies hypothesis 7 and so should represent an object with a non-zero rest mass. By postulate 3 it should not carry an EM charge but this is under the assumption that the six 4-vertex faces of the cube are two dimensional squares rather than three-dimensional tetrahedrons; allowing for the latter includes the possibility that the cube can carry unit or fractional charge in addition to being uncharged. In the scheme that follows all permutations allowed by the symmetry appear to manifest physically.

VI. SPINS AND OTHER THINGS

Hypothesis 9 Discrete symmetry groups specified by any given geometry manifest as an analogous continuous symmetry when the time dimension is added to the space in which the geometry is embedded by virtue of the generators of the discrete group being converted into continuous rotation generators of the corresponding continuous group.

Thus the $S_2$ symmetry of the Galois interval becomes a continuous spin 2 geometry with the addition of a time dimension. (It can only be spin 2 because the symmetry generator of the interval - a reflection - is invariant with a rotation by $\pi$). The triangle becomes a spin 1 object because the discrete symmetry group of a triangle, $C_3$, has a generator axis orthogonal to the plane of the triangle whose sign changes with rotation by $\pi$ and is unity under a rotation by $2\pi$. For the case of the tetrahedron we will take as its' symmetry group, rather than the 24 element group $T_d$, the unique dihedral subgroup of SU(2) which has 24 elements and two generators. In the continuum limit this becomes SU(2). (The generators of the dihedral group are only two in number but of the three generators of SU(2) only two are linearly independent $[\tau_i, \tau_j] = -i \epsilon_{ijk} \tau_k$).

Hypothesis 10

The mass of a three-dimensional object is found by extracting the time dimension from the embedding space and is the cardinality of the corresponding discrete symmetry group.
that remains after the extraction (modulo any radiative corrections and adjustments for the generators).

Thus, in the case of the tetrahedron, extracting the time dimension means removing the dihedral generators and the associated group elements of SU(2) apart from a residual discrete collection of 22 elements which represent the rest mass of the tetrahedron. (Of course removing the time dimension can only define a rest mass and cannot define a dynamical mass because the latter is meaningless in a timeless space! What we obtain is a definition of the mass of an object at rest irrespective of the lifetime of the object; this may or may not be a measurable quantity! Only in the situation where the lifetime of the object is infinite will it be precisely measurable!).

For a Dirac mass term in standard theory we have coupling of the left and right handed components of the fields;

$$L_{\text{mass}} = g \bar{\psi}^L \psi^R \phi$$

and in the limit of zero mass the left and right handed fields decouple;

$$i \gamma^\mu \partial_\mu \psi^R = m \psi^L$$

The geometric meaning of this in the schema being presented here can be understood by considering the photon geometry (hypothesis 3) even though it is not a fermion. The triangular boundary of the geometry is constructed of three gravitons (in the limit of instantaneous time) but the gravitational gauge field ‘smooths’ this out so that the local triangular phase of the geometry is not an observable. The geometry is then best represented by a circular two-dimensional disc with a ‘fuzzy’ margin and the geometry $C_3$ generator as the continuous U(1) generator of the circle. The left and right handed orientations of the generator (with respect to the direction of motion of the object) are distinct and, because of hypothesis 8, cannot be combined to form a three-dimensional volume for the photon at rest with respect to an object such as a tetrahedron or cube. This means that the photon must be massless (the photon may have finite radiative corrections from is gravitational gauge field which induce a non-zero minimum energy for the photon but its’ rest-mass must nevertheless always
be zero). Note that in this pictorial model the non-‘fuzzy’ part of the ‘disc’ is a quantised space which will always appear smaller than any probe (in the case of a photon the only ‘probe’ would be a graviton); it is not a measurable space. The ‘fuzz’ is due the associated gravitational field of the photon and, in principle, extends transversely out to infinity.

Thus the meaning of a term like $g\psi^L\psi^R\phi$ is that the left and right handed components only combine in the context of a three-dimensional space geometry and that a two-dimensional (or one-dimensional) object in this schema must always have zero rest mass. Similarly, any spinor geometry whose contained space defines an intrinsic continuum, such as a cube or a tetrahedron, will have a Dirac mass.

This is, however, not the complete picture since we will have to add radiative corrections to any Dirac mass due to the fact that the boundary of the electron geometry (by hypothesis that of the tetrahedron) contains photons and gravitons - which must propagate by hypothesis 8 to be defined properly and in so doing will generate additional components of particle mass. In QFT these contributions are infinite due to integration over momentum space but in the present geometric picture they must be strictly finite because the analogue of the ‘bare’ mass in the geometric picture is the order of the underlying discrete group (not including generators); which is strictly finite. (The analogue of an infinite ‘bare’ mass would be the order of the corresponding continuous group; but we can eliminate this by throwing away all the elements other than those which represent the real underlying discrete symmetry).

Thus in the case of the three-dimensional tetrahedron in $O(3,1)$ space-time the continuous group is $SU(2)$ of infinite order but the real mass of the Dirac term is represented by the 24 discrete sub-group elements modulo the two generators of this discrete group left as massless intrinsic spin generators.

The reader at this stage may be concerned that no mass units are specified! How can you have a mass of 22? The answer is that all masses are a relative thing. So, for example, if we define the rest mass of the proton as ‘1’ all other masses could be defined in terms of this unit. Ultimately a unit like a Kg can be expressed in terms of the summed mass of a certain number of nucleons and electrons, modulo binding energies, at a certain temperature. This,
in turn, could be expressed solely in terms of proton masses (provided we know the binding energies, \( \frac{M_p}{M_n} \) and \( \frac{M_p}{M_e} \) mass ratios etc). The beauty of defining mass in terms of proton or electron rest mass is that, as far as we know, these are the only two (?absolutely) stable massive colourless fermions. Using the \( \Delta E \Delta t \geq \hbar \) relation means that, at least in principle, we can measure the mass of a stable object to any desired level of accuracy. In the situation at hand what we are really interested in is calculating quantities such as the rest-mass ratio of the proton to the electron which we shall do by performing a similar calculation for the proton that we have performed for the tetrahedron (here a model of the electron - we will deal with muons and taus later).

Notice that the approach to particle structure and groups being employed here is fundamentally different to that of the standard model; the SM symmetry involves relations between different types particles by assuming the spectrum of particles is governed by some symmetry group rather than the intrinsic structure of a given particle being governed by a symmetry group; the latter is what we are doing here. Notice also that there is no conflict between the model of a charged lepton (here a tetrahedron) being built up here and the empirical finding that leptons appear to be point particles in deep inelastic scattering experiments. The reason is that time exists only ‘outside’ the boundary of the tetrahedron - there is no time in the ‘interior space’ of the tetrahedron and hence no dynamics can occur in the interior space which is a three-space not a four-dimensional space-time. Observations of the lepton involve interactions with its’ boundary only and the ‘interior space’ has no measurable size. Experiment can only detect the presence of this 3-space in an on/off sense; the lepton is either detected or it is not. Exactly the same applies to the 2-space defined by the photon; the photon can only be probed with regard to its’ boundary gravitational field and the interior space is unmeasurable apart from the sense that it’s existence can be equilibrated with detection of the photon.

By the same token, because the boundaries of the (timeless) contained spaces are themselves gauge quanta, the boundaries of the geometries will represent propagating fields and thus the whole geometry will represent a field quanta rather than a static geometric point
in space-time. The boundaries of geometries such as affine-set tetrahedrons are thus in this sense non-local in this schema and represent the gauge quanta of the object. A spinorial tetrahedron in this model represents an electron. The gauge quanta (in the case of the tetrahedron the triangular faces and individual lines of the boundary of the geometry) eliminate any observable local phase associated with the boundary of the geometry (‘smoothing-out’ the triangles and intervals) by propagating in space time and by sweeping-out an associated 3-space elevate the cardinality of the boundary of the geometry to the continuum. The affine-set tetrahedron may be idealised as a three-dimensional timeless fuzz-ball embedded in a four-dimensional space-time with the surface ‘fuzz’, extending out to infinity, being the propagating gauge quanta represented by the surface of the geometry propagating at the speed of light. The ‘interior’ time-less space is impenetrable and will always appear smaller that the wavelength of any probe which, by the very nature of a dynamical probe, is constrained to lie within the boundaries of the time dimension and cannot enter the 3-space which defines the underlying geometry. Thus the tetrahedron always appears as a point particle in deep inelastic scattering experiments. This is the link to eqn. (4); the non-observability of local phase arises because one and two-dimensional sub-spaces of continuum 3-space do not represent a real continuum and are not accessible to observation. Intrinsic particle spin in the model is then a necessary prerequisite for the existence of a fundamental quantum object in space-time which leads to trouble incorporating a scalar Higgs in the theory (needed as a ‘free’ particle to preserve unitarity for radiative corrections in the SM for example). We will look at this issue again towards the end of the paper but note at this stage that, whilst it appears we cannot express the Higgs as a free-particle in space-time in this model as so far developed, we can give it physical existence in one space; the interior timeless 3-space of a geometry such as an affine cantor tetrahedron. Indeed, we might seek to identify the 3-space (minus the boundary of the tetrahedron) as the Higgs field itself. This is in fact the route we will take. The Higgs field will be ‘buried outside time’.
VII. CUBES, QUARKS AND FRACTIONAL CHARGE

By hypothesis we identify the Cantor cube with a quark. We will be concerned here with derivation of the proton and neutron masses so we will require a representation theory that encompasses three quarks and the associated gluon field. The following components are calculated in sequence;

1; constituent quark mass
2; gluon energy associated with the constituent quarks
3; current quark masses
4; gluon energy associated specifically with the current quarks.

The constituent quark mass represents the energy associated with the momentum of the quarks and in the nucleons we will find that it represents about one-half of the total mass of the nucleon. The gluon energy, 2 above, represents the energy associated with the momentum of the gluons and again will be found to represent about one-half of the total energy of the nucleon. The current quark mass is a different component and represents the intrinsic rest-mass energy of the individual quarks. Component 4 is something of a mystery but appears to be rather like an instantaneous strong-force analogue of the instantaneous Coulomb interaction. It might be viewed as a potential energy of separation of the current quarks at the energy scale of the calculation.

To study the symmetry properties of the Cantor cube geometry we will need a representation theory which is given below. The details are somewhat more involved than in the case of the Cantor tetrahedron but an essential summary of the ideas is as follows. Hypothesis 8 requires the propagation (at light speed) of all boundaries that do not define a rest mass (which means any geometry that is one or two-dimensional) and hypothesis 9 means that the cube boundary will have a complex structure. Hypothesis 8 will mean that a flat square geometry must be massless; when confined to a colour space (defined below) this will manifest as a massless spin-1 object (i.e. as a gluon) and the representation theory specifies that the cubes are fractionally charged but massive. (The fractional charge results from triangulation
of a fractional part of the boundary of the cube; geometrically this means that some of the square surfaces of the cube become a 3-space rather than a 2-space so that the geometry becomes a tetrahedron rather than a square - the charge of the cube is then represented by the fraction of the topology that is triangulated. The group theory is such that opposite square faces of the cube always triangulate in tandem so that the possible fractional charges are $\pm \frac{1}{3}$ and $\pm \frac{2}{3}$ with the sign dependent on the orientation of the $C_3$ generator).

Essentially all phase information with regard to the boundary of any geometry is eliminated by the gauge fields; the position and direction in space of the vertices and edges of the geometry are not an observable. In the case of a cube the overall ‘cube’ phase information is its’colour and the three axes of the colour may be taken as the three (not necessarily orthogonal in real space but orthogonal in affine-set geometry!) lines running through the centre of opposing ‘square’ faces of the cube. (Observation of the colour of a cube would be analogous to removing the intrinsic spin of a quark or freeze-framing the object by removing the time dimension).

We begin by forming a representation for a spinorial version of a tetrahedral geometry which will form the basis for a representation theory of the cubic geometry. The tetrahedron has special discrete symmetry; a symmetry related to the $24=4!$ elements of a permutation of its vertices. For a spinorial version embedded in a 4-dimensional space-time we seek the unique sub-group of SU(2) with 24 elements which is;
where \( a = \frac{1}{2} \) and \( b = \sqrt{\frac{3}{2}} \). The elements of these matrices are built up from the roots of unity; square roots, fourth roots and sixth roots. These elements close to form a group with 24 elements which we shall designate \( T_r \). The generators of the group may be taken as the \( \gamma_4 \) and \( \delta_4 \) matrices. Note that:

\[(\gamma_4 \cdot \delta_4)^2 = (\delta_4 \cdot \gamma_4)^2 = -I_2 \]  

so that, since SU2 is a double covering of SO3 up to a sign, the square of the product of the generators represents a rotation by \( 2\pi \) so that the \( T_r \) group is a kind of discrete spinor
group. It is analogous to the geometric group $Td$ familiar to physical chemists which is the
group of the tetrahedron in three-dimensions with 24 elements and 2 generators. $Td$ is a
point space group in three dimensions which is spinorial but is not isomorphic to $Tr$.

The $Tr$ group will be used as the fundamental form of a discrete representation for spinor
particles in the theory.

The \textit{rest mass} of $Tr$ is defined as $\mathcal{R}.(4! - 2)$ where $(4!-2)$ is the number of non-generator
elements in the group and $\mathcal{R}$ incorporates any radiative corrections due to the elevation
of the cardinality of the boundary of the geometry to the continuum by the gauge fields.
In any continuous field theory analogue the discrete pair of $Tr$ generators must become
continuous rotations and by hypothesis manifest as the intrinsic spin of the geometry; this
is the reason for defining them as massless. The hypothesis of massless group generators is
testable by comparing resulting mass calculations with experiment over a variety of particle
types. The best experimental evidence for the correctness of this assumption comes from the
massive charged leptons which all have identical $Tr$ group elements with identical radiative
corrections in their mass expressions.

\textbf{VIII. EMBEDDING $T_r$ IN SU(3)}

To calculate mass expressions for cube geometries, which by hypothesis we equate with
quarks, requires a matrix representation of the symmetry group of the cube that parallels
that developed for the tetrahedron.

We can mirror the root structure of SU(3) by embedding three copies of SU(2) in SU(3).
We can embed three copies of the $T_r$ group as follows;

\[
r = \begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda_{11} & \lambda_{12} \\
0 & \lambda_{21} & \lambda_{22}
\end{pmatrix}; \quad g = \begin{pmatrix}
\lambda_{11} & 0 & \lambda_{12} \\
0 & 1 & 0 \\
\lambda_{21} & 0 & \lambda_{22}
\end{pmatrix}; \quad b = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & 0 \\
\lambda_{21} & \lambda_{22} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

and we also form the following \textquote{colour-dual} embeddings;
The three ‘colour’ matrices \( r, g \) and \( b \) plus the ‘colour-dual’ basis \( \bar{r}, \bar{g} \) and \( \bar{b} \) together form a discrete version of the group of \( SU(3)_c \). Also notice that each set of colour matrices forms a group with 24 elements. The ‘dual’ or ‘colour-bar’ matrices do not form a group since the product of any two colour-bar matrices is a colour matrix. However, the combination of \( c_i \pm \bar{c}_i \) (for colour index \( i \)) forms a group with 48 elements and two generators located in the colour-bar set of elements. Let us call this group \( T_c \). It is also a ‘discrete spinor’ since the generators are analogous to the \( T_r \) group;

\[
(i\gamma_4.i\delta_4)^2 = -I_2
\]  

Note that the colour-dual is not the same as the Hermitian conjugate matrix.

We could consider \( T_c \) group as a model for a Skyrmion field of charge 4 i.e. one with cubic symmetry. We will see later how the geometric analogy works but to obtain a picture think of the three colours as represented by the three pairs of opposite faces of the cube; one pair of faces representing the \( T_c \) group with the four vertices of each square face representing each tetrahedral sub-group. To represent the total geometry we form a ‘particle vector’ which is composed of three components. First form a matrix representation of \( T_c \) as follows;

\[
R = \begin{pmatrix} r & 0 \\ 0 & \bar{r} \end{pmatrix}, \quad G = \begin{pmatrix} g & 0 \\ 0 & \bar{g} \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix}
\]

and then form the three-component particle vector \( \vec{P} = (R, G, B) \). (Of course the order of the components here is arbitrary).

Now define the following ‘current-quark’ operators;

\[
q_r = \begin{pmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad q_g = \begin{pmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad q_b = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]
Lastly we require what is referred to in the text as a ‘script identity’ \( \mathcal{I} \). It carries a colour index:

\[
\begin{align*}
\mathcal{I}_r &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
\mathcal{I}_g &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
\mathcal{I}_b &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\end{align*}
\tag{15}
\]

The purpose of these constructions will become apparent as follows. We form operators from the \( q_i \)’s and \( \mathcal{I} \)’s as follows (the following examples are red format but the other colours follow suit):

\[
\begin{align*}
U_r D_r &= \\
&= \begin{pmatrix} \mathcal{I}_r & 0 \\ 0 & I_3 \end{pmatrix} \begin{pmatrix} q_r^* & 0 \\ 0 & q_r \end{pmatrix} \\
&= \begin{pmatrix} q_g^* & 0 \\ 0 & q_g^* \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix} \\
&= \begin{pmatrix} q_b^* & 0 \\ 0 & q_b^* \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix}
\end{align*}
\tag{16}
\]

where each operator has three components represented above in column format. The effect of the operators on the colour neutral particle vector is as follows;

\[
D_r(U_g\{U_b \begin{pmatrix} R \\ G \\ B \end{pmatrix}\}) = \begin{pmatrix} \bar{R} \\ \bar{G} \\ \bar{B} \end{pmatrix} = \bar{P}
\tag{17}
\]

Contrast this with the effect of the conjugate operators;

\[
D_r^*(U_g^*\{U_b^* \begin{pmatrix} R \\ G \\ B \end{pmatrix}\}) = \begin{pmatrix} \mathcal{I}_r \bar{R} \\ \mathcal{I}_g \bar{G} \\ \mathcal{I}_b \bar{B} \end{pmatrix} = \mathcal{I} \bar{P}
\tag{18}
\]
which is the definition of unit negative electro-magnetic charge in this scheme; the non-script prefix on the altered P is the definition of unit positive electro-magnetic charge.

Two things need mentioning at this stage; 1; the quark operators \( q_i \) and \( \bar{T}_i \) are assumed confined to their respective colour space \( i \); that is, the \( q_r \) operators only operate on the \( R \) matrix etc, and 2; the order of the application of the operators does not change the result; so long as each operator colour is represented only once (that is \( D_g U_r U_b \) gives the same result as \( U_g D_r U_b \) etc).

**IX. GLUE**

The gluon geometry is represented by a four-vertex flat (and hence massless) spin-one geometry related to the boundary of the cube (the sign of the \( C_4 \) generator of the flat square geometry changes sign with a rotation by \( \pi \) so that the continuous analogue must be a spin-one generator). There are six colour states defined directly from the cube geometry; \( R, \bar{R}, B, \bar{B}, G \) and \( \bar{G} \). (The remaining generators of SU(3) are linear combinations of these).

We can form a representation theory for the gluons and we find that their energy equivalence is constrained by the symmetry allowing their contribution to the rest mass of particles to be estimated by summing over the possible matrix representations in the same way as was done for the \( T_r \) and \( T_c \) groups. There will be an implicit assumption in such a summation that \( \alpha_s = 1 \) because we will equate the energy equivalence of one matrix element of the glue with that of the constituent quarks. In fact this choice is the only one possible since they are calculated from the same basis groups and particle vector.

Discretised gluon operators are represented as follows;

\[
\begin{pmatrix}
I_3 & 0 \\
0 & q_b g_s^* \\
\end{pmatrix}
\begin{pmatrix}
g & 0 \\
0 & \bar{g} \\
\end{pmatrix} =
\begin{pmatrix}
b & 0 \\
0 & \bar{b} \\
\end{pmatrix}
\]

(19)

where * represents the complex conjugate. Similar upper product operators are defined for example;
These are then combined to form operators which operate on the particle vector $\vec{P}$ for example;

$$
\left(\begin{array}{c}
\bar{g} \\
0 \\
0
\end{array}\right) g^* \cdot \left(\begin{array}{c}
0 \\
g \\
0
\end{array}\right) = \left(\begin{array}{c}
0 \\
r \\
0
\end{array}\right)
$$

(20)

when applied to the particle vector $\vec{P} = (R, G, B)$ produces $(G, R, B)$ i.e. this is a R-G gluon.

X. RADIATIVE CORRECTIONS

As was suggested earlier in the text *bare* mass is to be interpreted as the transformation of a geometry over a timeless space or, equivalently, as a symmetry which represents such a transformation. Actually, apart from the absence of the time dimension, this is remarkably like the definition of energy associated with the momentum of a moving object for the motion of an object is nothing other than its' translation over a space in a definable time. Mass and translational energy thus have a similar structure! For the $T_r$ group, which represents a discrete spinorial tetrahedron, there are 24 discrete elements or transformations over its contained (timeless!) 3-space. We have made the assumption that the two generators of the geometry generate its intrinsic spin (in the transition to a field theory) and that these generators are as a consequence massless. The Dirac mass of the tetrahedral group would then be given as;
Eqn. (22) is a finite bare mass. It will be modified by the electro-magnetic properties of the boundary of the geometry. In standard theory component (22) must be generated by the Higgs field.

That the radiative correction from electro-magnetism to the mass of an object such as an electron must be finite and of order $\alpha$ was, I believe, pointed out by Dirac and has been emphasised by authors such as Sakurai [2]. The reason is that, for the electro-magnetic contribution $\delta m$ to contribute most of the mass of the electron $m$, the energy of the virtual photons $\Delta$ must exceed the mass of the entire universe!

$$\delta m = \frac{3\alpha m^2}{2\pi} \log \left\{ \frac{\Delta}{m} \right\}$$

If the E.M. contributions are of order $\alpha M_{\text{bare}}$ and finite then $M_{\text{bare}}$ must also be finite!

Of course in the perturbative approach both $M_{\text{bare}}$ and the radiative correction are infinite and the coupling, $\alpha_{em}$ varies with the photon momentum which is integrated over. The infinities are present even if the momentum integrations are regularized with a cut-off.

By contrast in the current non-perturbative approach we have a finite ‘bare’ Dirac mass. Now, because in the geometric picture presented the gauge fields are compressed onto the boundary of the geometry, we expect on basic symmetry grounds that if the Dirac mass can be calculated from the discrete geometry then so should the radiative corrections! (It might be argued that this is a rather loose argument though!) The basic symmetry argument is as follows. There are four triangles on the surface of the geometry and each has a permutation symmetry of $3! = 6$ elements. Four times six equals twenty-four so the surface of the geometry, in terms of photons as triangle affine-sets, has a group order equal to that of the tetrahedron. We assume however that any mass-equivalence is reduced from unity to the coupling strength of the force and that the amount of energy that the surface triangles can express is exactly equal to the bare mass of the tetrahedron because of the equivalence of the order of the two sets (we assume that the bare mass is converted to the photon energy and that the coupling strength $\alpha_{em}$ is then set at this level).
The simplest and most natural possible ansatz is then given by:

\[
mass_{T} = (4! - 2)(1 + \alpha_{(q^2=m^2)})
\]  

(23)

where \(\alpha\) is the fine structure constant defined at the (finite!) bare-mass energy \(m\) of the particle in question. We do not integrate over photon momentum! We do not use a range of values of \(\alpha_{em}\) but only that value set at the energy scale of the underlying (finite) bare mass. The validity of this ansatz depends on experimental testing. We will see shortly that it is accurate to within current experimental limits. Note that this radiative correction is NOT perturbative.

We next generalise this ansatz to the weak interaction to accommodate weak radiative corrections to the lepton mass defined by eqn.(23) under similar assumptions by adding:

\[
mass_{T} = (4! - 2)(1 + \alpha_{(q^2=m^2)} + G_f)
\]  

(24)

where \(G_f\) is a weak interaction correction expressed as a dimensionless number in terms of relative strength in comparison to the E.M. force i.e. \(G_f \approx 10^{-5}\). We will symbolise the radiative correction as;

\[
\mathcal{R} = (1 + \alpha_{em} + G_f)
\]

But what is the value of the strong coupling for the quarks? Comparison with experiment will show that, for the discrete non-perturbative geometric approach, it always takes the value unity (in dimensionless ‘relative strength’ terms). In conventional QCD only the asymptotic ‘free’ region of phase space is accessible to perturbative calculations and the low-energy region presents difficulties (lattice approaches etc). Actually there is no inconsistency in finding that the coupling always takes the value unity for affine-set calculations; if we are not looking at a quark we really don’t know what it is doing! And if we are measuring the rest mass of a proton (ultimately what we are trying to calculate here) then we definitely don’t know what the quarks are up to for such a measurement at rest energy! There are basic symmetry reasons why the coupling should be unity in the affine-set regime (it has
been called the ‘leggo-block’ approach) based on the idea of ‘stacking’ cubes and taking the interaction energy in terms of the energy equivalence of the contacting faces. Because this energy is expressed as unaltered matrix elements the coupling is automatically unity! We will not explore the issue further but assume that, for systems containing quarks only, the identity in the factor $R$ is an expression of $\alpha_{\text{strong}}$.

Because the model of quarks developed in the previous section is based on units of the $T_r$ group we will make identical assumptions about the mass equivalence of respective $T_c$ mass units; that is, we will equate the mass with the sum of matrix elements. We can write each $6 \times 6 T_c$ matrix two ways:

$$
\begin{pmatrix}
C_i & 0 \\
0 & \bar{C}_j
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
\bar{C}_i & 0 \\
0 & C_j
\end{pmatrix}
$$

(25)

These two possible forms of $T_c$ represent opposite parity assignments and the two generators of the $T_c$ group occur only in the barred matrix. As before we assume that these generators, in the continuum limit, generate the continuous intrinsic spin of the geometry and are massless. The two matrices above will thus generate, for a single colour, $24.22$ elements of mass as a result. Now, summing over three colours gives a multiplier of 3 and there are $3!$ possible permutations of colour order in the particle vector. Thus the total number of possible different matrix elements in an ordered particle vector (with E.M. and weak radiative corrections) is:

$$3.3!.24.22(1 + \alpha_{\text{em}} + G_f) = 6.4!.6.(4! - 2)(1 + \alpha_{\text{em}} + G_f)
$$

(26)

Notice that in eq.(26) a gauge field correction identical to that for the $T_r$ group has been added; the radiative corrections are based on the tetrahedral geometry and $T_c$ is just a sum of such tetrahedral geometries each of which has its boundary ‘smoothed’ by the appropriate gauge field with identical corrections to mass as for the $T_r$ group. It is reasonable to take the value $\alpha_{\text{em}}$ in eq.(26) as the low energy value because the current quark masses are small for the first generation (N.B. $T_c$ does not define the current quark masses - it defines constituent
quark energy-momentum). Notice also that $G_f$ has been coupled to both the right and left-hand parts of the particle vector (expression (25) describes opposite parity states) as was the case for the coupling for the $T_r$ group. If one wished one could couple a value $2G_f$ to half the mass, as a means of representing the parity violating nature of the weak interaction, but the result would be the same so we will stick with the current notation.

Notice also that in (26) the R.H.S. has an interpretation in terms of a cubic geometry with six 4-point geometric surfaces. This analogy will be reinforced later. The double counting implicit here (we have the square of the surface order of the cube; the surface order being $6.4!$ - for six four-point subgeometries - or $6.(4!-2)$ depending on whether generators are included) is a reflection of the dual nature of the $T_c$ which is spinorial and results in a double counting of the geometry surface. Mass (26) will represent the constituent mass of a cubic baryon.

To calculate the corresponding gluon constituent mass we notice that each gluon operator contains the product $C_i C_j^\dagger$ neither of which contains the $T_c$ generator pair. The order of this product is thus $24^2$. For a given three-component particle vector there are three possible couplings of colour exchange; first component and second, second and third and first and third. This triples the gluon order quite independent of the particle vector. For the particle vector there are three cyclic permutations which cannot be interconverted with the exchange of a single gluon resulting in a further tripling of the gluon order. The gluon has two active colour component in its particle vector analogue and there is a second doubling of the order that arises from coupling to the two different formats in eq. (25). Thus the net gluon matrix order is;

$$\text{gluons } 2.2.3.3.24^2 = 6.4!.6.4!$$

which again has ready interpretation in terms of the surface of a cube.

We can now sum the $T_r$ equivalent order for the basic cubic geometry;

$$\text{cubic mass} = \{6.4!.6(4! - 2) + 6.4!.6.4!\}.(1 + \alpha + G_f)$$
Notice that a radiative correction $\mathcal{R}$ containing $\alpha_{em} + G_f$ has been added to the gluon energy in (28)! This seems completely out of place; an explanation is given later in the text.

**XI. CURRENT QUARK MASSES**

We first examine the current quark content of the proton. The current quarks are built up from the quark operators used previously to define the particle types. The current quark masses are the corresponding masses of these operators plus an extra piece that represents either (finite) gluon radiative corrections to the current quark masses or a potential energy of separation of the current quarks - we really are not sure quite which!

The glue and quark masses calculated in the previous section are related to gluon and quark energy-momentum due to the dynamical motion of these objects in the hadron. In this section we calculate the self-mass of individual quarks; this is the *current quark mass*.

We first study the operator component of mass of the current quarks. The glue corrections are less well understood. Our aim here is to extract the $T_r$ equivalent order (i.e. number of $T_r$ equivalent matrix representations) of the quark operators in the previous sections. We can split up the operators into two pieces; which is convenient for calculations. Each current quark operator consists of a vector composed of 3 square $6 \times 6$ matrices. Each $6 \times 6$ block is composed of two $3 \times 3$ blocks on the diagonal. We separate out the upper $3 \times 3$ blocks as one set of matrices and the lower $3 \times 3$ set of blocks as another and represent them as follows (the representation below is for the proton as the reader can confirm by applying the operators to the particle vector);

$$
\begin{align*}
\text{strong component} &= \begin{pmatrix} I & q^* & I \\ q^* & I & I \\ q^* & q^* & q^* \end{pmatrix}, \quad \text{E.M. component} = \begin{pmatrix} I & q^* & I \\ q^* & I & I \\ q^* & q^* & q \end{pmatrix}
\end{align*}
$$

(29)

The rows of these matrices are the three colours (order red, green and blue in the notation used in this paper from top down) and the columns are the type of quark; the left hand and central column of each matrix above is an up-type quark and the right hand column is a
down-type quark. The up-type quarks have their colour defined by the colour of the single identity they carry. Thus for example the first column of these matrices represents a red up quark. The single q or q* in the down defines its colour so that the last column above is a blue down quark.

The strong components couples to the $C$ colour matrix of the particle vector which does not contain any generators. The E.M. component couples to the $\bar{C}$ matrix with two generators. (This coupling is required to properly define the charge characteristic of the particle in the given representation). The order of a $C$ matrix is 24 elements. The order of a $\bar{C}$ matrix is $(4!-2)$ elements. The masses of the current quark operators are defined in terms of the matrices to which they couple. The exception to the rule applies to the identities in the E.M. component which are spinless and of order 4! (i.e. the generators of $\bar{C}$ are massive under the identity components of the current quarks - the I or the $I$ components). The identities in the strong component act like ‘strong charges’ and carry a relative mass sign. For example, in the left hand matrix of (29) the two identities cancel the two script identities leaving only the net 5 q*'s to contribute to the strong component mass.

It is then a simple matter to calculate the basic current quark masses in a proton; these can be read straight off expression (29). There are 5 components in the proton strong-interaction current quark part and each couples to a $C$ matrix of order 4! The 4 identities in the E.M. component have mass 4.4! and, after the cancellation of a q and q* in the E.M. component (these are considered to be opposite charges) we are left with 3.(4!-2) as the order of the 3 q*'s in the E.M. component. There is a doubling of order due to the fact that the particle vector has two parity permutations as indicated earlier in the text ($C, \bar{C}$) and $(\bar{C}, C)$;

Proton current quark $T_r$ equivalent order $= 2\{9.4! + 3.(4! - 2)\}(1 + \alpha_{em} + G_f)$  

(30) where the same kind of radiative gauge field correction as was done for the other components is given. This radiative correction might seem appropriate for the charged components (the q’s) but seems inappropriate for the scalar identities which are related to the strong
interaction not the electro-magnetic interaction. We will apply a \((1 + \alpha_{em})\) correction to current quark mass generated by both the identity (no em charge) parts of (29) and (31) and the \(q\) and \(q^*\) parts just as an em radiative correction was applied to the constituent gluon energy in (28) in relation to (27). Some justification for this is found later in the text but it is quite speculative. The best that can otherwise be said at this stage is that this approach seems to work in all examples so far studied with this technique (which includes a variety of mesons). Ultimately, however, this situation is quite unsatisfactory and an understanding of the radiative corrections to this theory of masses is one of its’ most important outstanding problems.

For the neutron the calculation is analogous. The appropriate matrices which define a colourless electrically-neutral fermion are;

\[
\begin{align*}
\text{strong component} &= \begin{pmatrix}
I & I & I \\
q^* & q^* & I \\
q^* & I & q^*
\end{pmatrix}, \\
\text{E.M. component} &= \begin{pmatrix}
I & I & I \\
q^* & q & I \\
q^* & I & q
\end{pmatrix}
\end{align*}
\]

(31)

where it is implicit that the strong component is coupling the the \(C\) matrices of the particle vector and that the E.M. component is coupling to the \(\bar{C}\) components. From these one can read-off the mass terms applying exactly the same rules as before;

\[
\text{Neutron current quark } T_r \text{ equivalent order} = 2.\{12.4!\}(1 + \alpha_{em} + G_f)
\]

(32)

One can use eq’s (30) and (32) to estimate the up and down quark current masses but the calculation is yet incomplete because there remains yet another component - the \(\omega\) component in the table at the beginning of section XII - whose origin is something of a mystery. This is calculated precisely in the next section. When this is combined with the above results very precise calculations of the up and down current quark masses can be made. It must be born in mind, however, that these masses change somewhat according to context; they are not exactly the same, for example, in a pion as a nucleon. We will not calculate the pion mass in this paper although the calculation follows the same basic principles given here and values within empirical limits are obtained; in fact the whole spectrum of ground state
spin-0 meson masses can be calculated but there is considerable complexity in calculations involving higher-generation quarks which will not be discussed in this paper.

XII. GLUE ENERGY ASSOCIATED WITH CURRENT QUARK MASSES

| quark type | 'strong-charge' | generation multiplier | strong component |
|------------|-----------------|-----------------------|------------------|
| top        | -               | 3                     | 1.\((4\!-\!2)\) |
| bottom     | +               | 3                     | 2.\((4\!-\!2)\) |
| charm      | -               | 2                     | 1.\((4\!-\!2)\) |
| strange    | +               | 2                     | 2.\((4\!-\!2)\) |
| up         | -               | 1                     | 1.\((4\!-\!2)\) |
| down       | +               | 1                     | 2.\((4\!-\!2)\) |

The above table lists the glue corrections for current quark masses for the three generations. (Why they have been called U(1) components will be discussed later). The glue corrections for the up and down quarks in the nucleons can be calculated from the strong components in expressions (29) for the proton and (31) for the neutron by ‘rotating’ the strong component to couple to the generators and setting \(\alpha_s = 1\). (This ‘rotation’ is an induced phase change when they are moved from the left to right hand matrix in (29) and (31)).

A much easier way to consider the current-quark glue corrections is with the leggo approach. Take two Cantor cubes and stack them;
now, there are only 11 4-point sub-geometries in this structure rather than the 12 in two separate cubes so one 4-point geometry has been lost in the stacking process. The energy equivalence missing is equal to the group order minus the generators; here $(4!-2)$ for an object with spin. If we assume that the cubes only stack on the non-triangulated parts of their surface this means that the down-type quark has twice the surface area for stacking compared to the up-type (this is the source of the factor of two difference in the above table). The strong coupling is strongly attractive in the region $\alpha_s = 1$ so energy has to be put into the system to pull them further apart and there must be some potential energy of separation at this energy scale. We will equate this with the ‘missing’ energy in the stacking model. The odd thing we have here, however, is that the up and down quarks have a sign associated with the non-triangulated parts of their topology; something quite unexpected for the strong force! This sign difference on the identities is only found in the ‘strong’ matrix in (29) and (31) but is essential if the algebra is to be consistent. Actually, the stacking energy is always positive but the up and down-types enter the calculation viz. the ‘strong’ matrix identities in (29) and (31) with a different sign. An example will illustrate the ideas involved.
The calculation for the proton is found by summing over all possible pairings of stacked quarks using the above table (gluons again expressed as $T_r$ equivalent groups);

$$\text{proton} = |\text{up} + \text{up}| + 2|\text{up} + \text{down}| = |-2(4!-2)| + 2 + (4!-2) = 4(4!-2)$$

$$\text{neutron} = |\text{down} + \text{down}| + 2|\text{up} + \text{down}| = |+4(4!-2)| + 2(4!-2) = 6(4!-2)$$

where the arbitrary sign choice on the gluon components has been eliminated in the calculation.

Why glue corrections do not originate from the identities in the second block matrix (called the EM component) in expressions (29) for the proton and (31) for the neutron is unclear but comparison with empirical data shows that they do not contribute. It is possible this is because the two block matrix representation of the current quarks is fermionic and the single block rep. is appropriate for the bosonic gluon corrections.

**XIII. NUCLEON MASSES CONTINUED**

It is reasonable to assert that quantum chromodynamics is well established empirically as the correct theory of the strong interactions. The coupling between the gluons and the quarks displays asymptotic freedom at high energy i.e. at high energy the quarks behave like free partons and the coupling is weak. At low energy however the coupling is strong and perturbative calculations are not possible. Of necessity, calculation of the rest mass of any hadron will place the interaction in the non-perturbative strong coupling region.

In the previous sections a discrete model of QCD was developed employing discrete rather than continuous colour and calculations summing the matrix order of the discrete components of the model were made. The analogy with the tetrahedral group is now to be extended to the definition of mass. From a continuous group (SU(2) in the case of the tetrahedron) a discrete sub-group of elements is extracted which represents a finite bare mass term where the actual finite group involved is dictated by the symmetry of the geometry.
For affine-set geometries there is always a permutation symmetry of the points of the set. To calculate radiative corrections to masses, as in the case of the calculation of the rest mass of the electron (eq.23), we do not sum over boson momenta in the case of cubic quarks. Instead we invoke the same ansatz of equilibrating boson energy to the Dirac mass of the associated fermion multiplied by the coupling constant at that energy scale. In the case at hand one would expect that the scale appropriate is $q^2 \approx m_q^2$ where $m_q$ is the current quark mass but this may not be quite correct because the current quark masses contain both $q/q^*$ charged pieces and $I/I$ electrically neutral pieces and the correct energy scale to set $\alpha_{em}$ may be more like half the current quark mass. In any case it makes little difference since $\alpha_{em} q^2=m_q^2$ is little different from $\alpha_{em} q^2=0$ for the up and down quarks.

The value of the strong coupling is specified by setting the gluon energy in matrix units equal to constituent quark matrix units; this sets the strong coupling at 1 unambiguously for the non-perturbative calculation.

We are now in a position to calculate our nucleon masses. Basically we have four types of components. The first two components are the constituent quark mass and the constituent gluon mass equivalence and these are almost exactly the same (expressions 26 and 27). These terms, in the transition to a field theory, represent the energy associated with the motion of the current quarks and the momentum of the gluons. They dominate the mass expression and indicate that, to a good approximation, about 1/2 the nucleon momentum is carried by the gluons. It is possible to produce an exact calculation. The constituent quark energy is:

$$6.(4!).6!(4! - 2).R$$

and the constituent gluon energy is

$$6.(4!).6!(4!).R$$

where $R = (\alpha_s + \alpha_{em} + G_f)$ and $\alpha_s = 1$ precisely. (A variable $\alpha_s$ coupling constant and asymptotic freedom will only appear in the transition to a field theory with all its attendant complications).
To these components we must add the current quark masses which are in two pieces; the
electro-magnetically charged current quark masses calculated from the tables and the glue
corrections to the current quark masses calculated in the previous section. For the proton
we calculated the current quark mass as;

\[ 2\{9.4! + 3.(4! - 2)\}.R + 4.(4! - 2) \]

and for the neutron we calculated the total current quark mass as;

\[ 2.12.4!.R + 6.(4! - 2) \]

where in each case the glue corrections to the current quarks are the ones without
electro-magnetic or weak radiative corrections. The derived current quark masses are
up\(\approx\)4.6120MeV and down\(\approx\)5.9053MeV. One then obtains the following expressions for the
nucleon masses directly;

\[
\text{Proton Mass} = R.(8! - 12) + 4.(4! - 2) \quad \text{Neutron Mass} = R.8! + 6.(4! - 2) \quad (33)
\]

Notice that the cubic geometry is implicit in the resulting expressions even though each
quark is represented by an individual cubic geometry; they sum to provide a single effective
colourless cube as is represented by the appearance of the 8! which is the order of the \(S_8\)
permutation symmetry group. (The permutation symmetry is characteristic of an affine-set
geometry; in this case of 8 points).

This seems to be the reason why the nucleons are so special; they have the geometry
of quarks but they are not coloured. In this they appear to be unique and it is possible
this exceptional expression of symmetry is the reason why protons and neutrons are so
fundamental to physical structure.

The 12 massless generators in the proton mass expression are the 6 pairs of generators
associated with the surface of the cube (they can be seen explicitly in the current quark
masses expressions; they are associated with the three \(q^*\)’s in the E.M. matrix (right matrix)
of expression (29) - one \(q\) cancels one of the \(q^*\)’s and parity doubling doubles the number of
$q^*$'s to 6 - each of which is associated with two massless generators of an $S_4$ symmetry; the $q$'s in the right-hand matrix in (29) of course have no generators) and explicitly appear in the charged object as would be expected. The neutral cube, although its structure contains analogous generators, does not manifest these generators explicitly (the $q$'s cancel the $q^*$'s in the current quark EM matrix (31)). The mass expressions can be converted into MeV by using the electron mass=$\mathcal{R}(4! - 2)$. (This of course does not mean that the electron has $\alpha_s$ in its gauge field correction; the ‘1’ in this case is the tetrahedral unit of mass which is universal).

For simplicity we set $\alpha_{em; q^2=0}$ in the $\mathcal{R}$ for both nucleons and electron. Setting $G_f \approx 1.4 \times 10^{-5}$ (a value extracted from study of the lepton masses - see later in the text) gives $\mathcal{R} \approx 1.0073115$ and we obtain;

$$ \frac{M_p}{M_e} = 1836.1528(1836.1526675) \quad (39) $$

and

$$ \frac{M_n}{M_e} = 1838.6837(1838.6836550) \quad (40) $$

Both values are a little high, but given the uncertainties in the couplings $\alpha_{em}$ and $G_f$, the results are very good.

**XIV. Why does the gluon momentum related to the constituent quarks have a $\alpha_{EM} + G_F$ radiative correction?**

Good question! As might be expected the gluon correction to the current quark mass only has a multiplier ‘1’ which can be interpreted as $\alpha_s$, but the ‘constituent’ glue energy, (28) has an electro-weak radiative correction! This appears quite out of place; only an $\alpha_s = 1$ should appear with the glue momentum - but if we omit the electro-weak radiative correction the result is well outside the experimental value. The calculation results at the end of the last section clearly suggest that there should be an electro-weak radiative correction to the constituent glue energy, but why?
If we are to have a theory free of any arbitrary parameters then the value of $\alpha_{em}$ must be based on the geometry also. This is dangerous territory (almost a taboo topic in physics) but it must unfortunately be attacked.

There is only one (Dirac) massive geometric object - the tetrahedron - in two (fermionic) incarnations; as lepton and quark (the cube geometry ultimately reduces to tetrahedral units). The fine structure constant appears to represents the ratio of the number of effective tetrahedral groups generated by the $T_c$ (cubic) generators compared with the single tetrahedral geometry generated by the $T_r$ generators (noting that in a unit charge cubic proton there is an amalgam of six effective pairs of $T_c$ tetrahedral generators); the lepton generators are about 137 times more potent at generating electro-magnetic charge than the quark generators. The explicit E.M. component associated with the 6 pairs of $T_c$ generators on the surface of a unit charged cube such as a proton is $6.(4!-2) +$ radiative corrections.

If we strip this off the proton and then remove the constituent gluon energy with only the $\alpha_s = 1$ coupling to the gluon energy - i.e. what we physically expect - (omitting the weak interaction in both cases), divide by six to get the relative charge-generating power of one of the six pairs of $T_c$ generators and then divide out the tetrahedral units we find:

$$\frac{1}{6} \cdot \frac{1}{4!} [(1 + \alpha_{em})\{8! - 12 - 6.(4! - 2)\} - \{6.4!.6.4!\}] = \alpha_{em}^{-1}$$

which gives a value $\alpha^{-1} \approx 137.03596$ which would be an appropriate value for $\alpha_{em}$ at the energy scale involved i.e. not at $q^2 = 0$ but at the electron rest mass ($T_r$) scale of $\approx 0.5 MeV$. Equation (34) tells us that, as expected, the gluons don’t in fact have electro-magnetic radiative corrections; the fact that the gluons carry no electro-magnetic charge is absorbed into the value of the coupling constant in the nucleon mass expressions (33).

Let us try and interpret this expression. The proton as a cubic geometry has the same geometry as a quark; stripped of its current quark interactions and gluon momentum it is a large collection of $S_4$ units with 4! matrix elements each. The surface charge $6.(4!-2)(1+\alpha_{em})$ may be considered an electro-magnetic interaction energy between the charged current quarks. Eq.(34) tells us that, when this latter energy, and the gluon energy, is
eliminated, one pair of $T_e$ generators generates exactly 137.0359... units of 4! matrix elements while the lepton pair $T_r$ generators only generate one $S_4$ group of matrix elements (we are not concerned with mass here but rather with matrix elements; that at least is the justification for using 4! instead of $(4!-2)$ in the divisor of eqn.(34) - mass and charge of course are not the same thing!). So, in this sense, the lepton $T_r$ generators are 137.0359... times more powerful than the quark generators and creating electro-magnetic charge. Geometry generators generate intrinsic spin and so generate the local (gauge) phase on the geometry which generates the gauge fields which generate space-time.....

Thus maybe $\alpha_{em}$ represents a compensating mechanism which allows quarks and leptons to occupy the same space-time with the same electro-magnetic gauge field coupling to both; they have the same electro-magnetic vacuum. All quite speculative to be sure but there must be a reason for everything in nature yes? At least this speculation can give some justification for the radiative correction found in eqn.(28).

Once again the fact that the transition to a formal field theory results in a variable $\alpha_{em}$ with energy scale does not invalidate this concept because the masses also will change with energy in a field theory. What we can assert is that the running coupling $\alpha_{em}; (q^2)$ will evolve with energy in such a way the the quarks and leptons continue to ‘see’ the same space-time as the energy scale evolves. A changing $\alpha_{em}$ indicates that the quarks and lepton masses will not evolve in exactly the same way with increasing energy.

Notice that there is an implicit assumption in eqn.(34) it assumes that the electron and proton have exactly the same absolute value of charge; i.e. charge quantisation.
XV. MASS FORMULAE WITH SOME GENERAL COMMENTS

| Mass Type     | Mass Formula             |
|---------------|--------------------------|
| proton mass   | $R \cdot (8! - 12) + 4 \omega$ |
| neutron mass  | $R \cdot (8!) + 6 \omega$ |
| electron mass | $R \cdot (4! - 2)$        |
| muon mass     | $R \cdot (4! - 2) + 6.6! + 2.5!$ |
| tau mass      | $R \cdot (4! - 2) + 8! + 7.7! + 2.6!$ |

The value $R$ here is considered to be a non-perturbative radiative correction which is assumed to have the following simple form $R = 1 + \alpha + G_f$ where $\alpha$ is the fine structure constant and $G_f$ is the weak coupling constant converted to a dimensionless constant by use of a selected energy scale (we will take a low energy scale whereby it is assumed that, in comparison to the electro-magnetic coupling constant, $G_f$ has a value $\approx 10^{-5}$). The extra bits added on to the muon and tau are Higgs components and we will examine these more closely later.

Of course $\alpha$ and $G_f$ are scale dependent parameters whose value varies with energy scale but we are not here necessarily implying radiative corrections found by summing over a range of boson energies. Rather it is being assumed that, instead of a perturbation expansion, a finite radiative correction can be applied to a finite ‘bare’ mass; in the case of electromagnetism of order $\alpha$ at $q^2 \approx m_e^2$ (although one must always keep an open mind as it is possible the perturbation expansion has a finite sum for some at present unknown reason; that the electro-magnetic radiative correction to the electron mass, for example, is of order $\alpha$ has been suggested before [2].) The term $\omega$ has the value:

$$\omega = (4! - 2)$$

There is one free parameter in the mass expressions; the energy scale used to render the weak coupling to a dimensionless number but the weak vector gauge boson masses can be calculated. Notice that $R$ is probably the simplest possible conceivable non-perturbative radiative correction one could propose.
The best fit is given by $G_f = 1.4101 \times 10^{-5}$ using $\alpha^{-1} = 137.035989$. This then yields
the following masses (current empirical values in parentheses);

\[ M_\mu/M_e = 206.7682621(206.7682657(63)) \]
\[ M_\tau/M_e = 3477.4006381(3477.60(57)). \]

Both ratios are within one $\sigma$ of the empirical data and are predicted to higher accuracy
than current data and can thus be tested. Since the predicted masses are all within one $\sigma$
of the empirical value the $\chi^2$ value is less than one indicating a very good fit with the data.

Can we discern any patterns in the other parts of the data formulas? The obvious thing
is the presence of factorial numbers is all the expressions. This suggests a discrete symmetry
under permutation. Evidently the charged leptons are built around a symmetry that involves
a permutation of four objects (not necessarily particles!), the nucleons a permutation of eight
objects and the pions seven with a specifically identifiable six-object subgroup (the pions
have not been included here).

Suppose the basic lepton unit mass, $R(4! - 2)$, is related to tetrahedral symmetry; the
permutation group of the vertices of a tetrahedron is isomorphic to the discrete group $T_d$
of the three-dimensional tetrahedron. This group has two generators and 4! elements. An
alternative more realistic choice may be $T_r$ the unique 24 element sub group of SU(2) which
also has two generators; the -2 can then be considered to correspond to the generators of
the group (the masslessness of which we associated with intrinsic spin). This sub-group is
of the non-abelian type. The tetrahedral structure may represent a type of 3-dimensional
brane or orbifold. The idea of such a symmetry underlying physical structure is of some
current interest in field theory and of course has long been a feature of physical chemistry
(tetrahedral symmetry is a feature of some crystal structures for example). Firstly it has
been pointed out that certain soliton models of the Skyrme type (albeit for baryon structure)
have symmetry patterns of a tetrahedral, cubic and icosahedral nature [1]. Secondly there
has been some discussion of tetrahedral symmetry in relation to orbifold gauge field theory
(for example [2], [3], [4], [5] and [6]; for a discussion of permutation symmetry in relation to
orbifolds see [7] and [8]). Of interest for example might be the association of tetrahedral
symmetry with a discrete subgroup of SU(3) which encompasses gauge fields appropriate for the standard model \(\hat{E}_6 \approx T\) and \(\hat{E}_7 \approx O\) where O is the octahedral (cubic) discrete symmetry. We could then try to associate the 8! elements in the nucleons with a cubic type of symmetry. This interpretation works rather neatly for the neutron because the extra \(\omega\) term in the mass expression for the neutron - the 6(4!-2) - could be interpreted as related to the six square faces of a cube each of which could be allocated a tetrahedral-equivalent symmetry of total 4! elements with two generators (each square face of course has four vertices). The appearance of electro-magnetic radiative corrections for the neutron might also be taken to imply charged sub-structure.

Of course what is being studied in the above references such as \[5\], \[6\] and \[7\] is, chiefly, the issue of symmetry (and in particular supersymmetry) groups connecting different types of particles not masses of individual particles. However, there may be a connection. We might, for example, view the discrete symmetry as a kind of symmetry of the unobservable bare particle; a particle devoid of its investing gauge fields. Of course the ‘bare’ particle structure in field theory is not only unobservable it is mathematically inaccessible being plagued by infinities (infinite mass, infinite charge etc.) but this might be an artifact of our ignorance of some underlying discrete structure. The surface geometries of such a discrete structure may form the gauge fields as a worldvolume brane type structure rendering the underlying symmetry unobservable. For example, if the isolated tetrahedron is a ‘bare’ lepton then the triangular surface geometries may be considered to be photons and the individual interval ‘edges’ of the geometry to be gravitons. (A quick review of the geometry of these objects will reveal that the triangle symmetry is appropriate for a discrete spin-1 object and the single interval or ‘edge’ of the geometry appropriate for a spin-2 object). Because of the non-abelian nature of the double-cover tetrahedral group its overall discrete ‘SU(2)-phase’ might then be ‘gauged-away’ by the weak vector gauge bosons so that all phase information in relation to the discrete geometry - the phase of each individual edge, the U(1) phase of each investing triangle and the global SU(2) phase of the total geometry, becomes experimentally unobservable when the gauge fields propagate and sweep-out world-volumes.
that render the underlying discrete symmetry unobservable. Such a scenario intrinsically demands *local* gauge invariance because of the discrete nature of the geometry (perhaps this is why nature chooses local gauge invariance?). A flat ‘square’ version of the tetrahedron would be necessary to account for uncharged neutrinos of course; the absence of any triangular sub-geometry would then correspond to the absence of electric charge - it would of course retain an overall SU(2) weak ‘local’ phase and its gravitational investing gauge field associated with the individual investing intervals of the geometry. There are problems with this interpretation however, because a flat two-dimensional geometry is nominally spin 1 and massless. We will return to this issue later.

The only residue of discreteness would then be the rest mass of the object. The symmetries between particle types may then arise because of this shared underlying discrete structure; we can see from the mass formulas for the hadrons that the tetrahedral symmetry appears to be embedded in the cubic symmetry for example. There is in fact quark-lepton unification here.

**XVI. VECTOR GAUGE BOSON MASSES AND ELECTRO-WEAK MIXING**

Now, $T_r$ (or the 3-geometry analogue $T_d$), is spinorial in the double covering sense so that this group would be appropriate for a spinor particle but not for a boson. The analogous ‘bosonic’ group for a tetrahedron is $T$ which has 12 elements and two generators. Suppose we were to choose this kind of geometry for our vector gauge bosons, since it matches the geometry of the leptons in this model apparently and is triangulated; a requirement for charged vector gauge bosons at least. How can we create a factorial expression out of the $T$ elements; assuming that this is the only way mass is expressible for an affine-set geometry? The easiest and simplest way is to take the non-generator elements of $T$ and assemble them into a permutation group viz;

$$(T - 2)! = 10!$$

and we can then arrange, viz some appropriate choice of decomposition, a model for the
vector boson masses. First, note that any permutation number has a unique decomposition into a sum of permutations;

\[ n! = (n - 1)(n - 1)! + (n - 2)(n - 2)! + \ldots + 1! + 0! \]

so if we take the 10! elements and strip-off a tetrahedral equivalent unit 4! = 3.3! + 2.2! + 1! + 0! to couple to the underlying four-point geometry (and define it physically - a reasonable pre-requisite) we end up with a unique and inevitable decomposition into six components if we are working with affine-set geometries;

\[ 10! = 9.9! + 8.8! + 7.7! + 6.6! + 5.5! + 4.4! + (4!) \]

and we throw away the last 4! Each ‘10!’ should correspond to the mass of one vector gauge boson. At this stage comparison with the empirical data shows that maximal mixing of the charged components on the 8! (cubic) and 4! (tetrahedral) geometry terms occurs. The charge appears only in the \( W^\pm \) so the mass shifts only to the \( Z^0 \) in maximal mixing;

| Boson | 9.9! | 8.8! | 7.7! | 6.6! | 5.5! | 4.4! |
|-------|------|------|------|------|------|------|
| \( W^+ \) | 1 | 1/2 | 1 | 1 | 1 | 1/2 |
| \( W^- \) | 1 | 1/2 | 1 | 1 | 1 | 1/2 |
| \( Z^0 \) | 1 | 2 | 1 | 1 | 1 | 2 |

which sums to 3.10! as required.

In these mass expressions what has happened is that each \( W^\pm \) has ‘donated’ half its' charged \( S_8 \) component of mass to the \( Z^0 \); a 4.8!(1 + \( \alpha_{q^2=M_W^2} \)) from its mass to the \( Z^0 \) so that the \( Z^0 \) has a term (8.8! + (1 + \( \alpha_{q^2=M_W^2} \))8.8!) instead of just 8.8! as its second component of mass; and similarly for the \( S_4 \) terms. In this process we expect each ‘donated’ piece to keeps its' radiative correction. On the basis of the analysis of \( \alpha_{em} \) in section Taking \( \alpha = 128^{-1} \) ‘on-shell’ at \( q^2 = m_W^2 \) instead of \( q^2 \approx 0 \) we obtain;

\[ M_0^2 = 91.1729(91.188(0.007)) GeV \quad \text{and} \quad M_W = 80.5794(80.49(0.14)) GeV \]
which gives, using $\cos \theta = \frac{M_W}{M_Z}$, $\sin^2 \theta = 0.2188$. This is the value with radiative corrections to the masses included. If one omits these one easily obtains $\sin^2 \theta = 0.2299$. It is interesting to note that the weak mixing is related to shifting of mass components from both the charged bosons to the neutral vector gauge boson. For example, if this mixing did not occur the weak mixing angle would be very small and due only to radiative corrections to the W mass. Notice that charge neutrality of the $Z^0$ is maintained in spite of the radiative corrections to its mass since it obtains two equal contributions from opposite charge consignments.

What is most interesting is that 1. the electro-weak mixing is taking place at the $S_8$ symmetry level; exactly the level we used previously to extract a value for the fine structure constant (the proton mass is an $S_8$ symmetry) and this is the reason for adding the $S_4$ mixing even though on the basis of the empirical data this is not so strongly implied, and 2. the mixing is maximal! There is absolutely no arbitrariness at all in the value of the weak mixing angle if one can account for the maximal mixing (which unfortunately we cannot).

**XVII. THE HIGGS FIELD**

Let us focus first on the masses of the charged leptons. Notice that the component with radiative corrections is identical in each of the three species of lepton. This suggests that the component carrying the electro-magnetic charge is identical in each case and this is consistent with charge quantisation. The formulas for the leptons imply that there is a common component in the leptons which is responsible for generating the electro-magnetic charge (and weak interaction properties) of the individual leptons as well as the intrinsic spin (it is the only piece with generators set explicitly massless). Let us then attribute the remaining mass components in the muon and tau to the Higgs scalar field. This is a spinless field and the lack of a radiative correction for these mass components (at least to the level of the current data) implies that, if these components are the result of the Higgs mechanism alone, that the Higgs particle is not electro-magnetically charged because of the absence of any associated radiative correction for the electro-magnetic field. We might also postulate
that the -2 appearing with the charged component of lepton mass is related to intrinsic spin. This makes some sense since, firstly, it implies identical intrinsic spin in all three massive leptons and, secondly, its absence in the components in the lepton masses without radiative corrections would then imply that they are spinless as required of the Higgs field. Whether the Higgs can also be considered responsible for the $R(4! - 2)$ part of the lepton mass is an open question. This piece of mass, identical to the electron rest mass, we might attribute it to the 4! piece we ‘throw’ away when we decompose a large scalar factorial unit. Clearly we need the Higgs for correct dimensions in the Lagrangian:

$$\mathcal{L} = g\psi_R\psi_L$$

of any fermion.

This prompts a search for the pattern of Higgs components appropriate for the massive quarks and leptons. For consistency we will use the same basic structure that we employed for generating the Higgs components for the vector gauge bosons since the same Higgs generates the masses for both sets of particles. From the known spectrum of masses clearly the symmetry will be broken. Using the components already listed for the tau and the muon the symmetry breaks up as follows (using a $2.S_{10}$ to couple to the fermions in comparison to the $3.S_{10}$ we used to couple to the adjoint representation of the vector gauge bosons):
The known quark masses have been used as a guide to the decomposition. The first

generation quarks and leptons have been left massless (and consequently an amount of mass
equal to $5.5!+8.4!$ unaccounted for!). The pion mass expressions however imply that, under
some circumstances, the up acquiring a contribution of $5!$ and the down of $4!$. Detailed study
of meson ground state masses (unpublished \[13\]) appears to indicate that these missing
components - the $5.5!+8.4!$ - do not appear in any of the second or third generation quarks.
It is possible they contribute to the $\omega$ terms that appear in the hadron mass expressions or
more likely they in fact manifest as the baseline tetrahedral units in both quarks and leptons
(for example the ($4!-2$) part of the electron mass). As mentioned, such a contribution from
the Higgs is needed to get the dimensions of the Lagrangian right for all Dirac particles.
From the table the Higgs contribution to the quark masses are; top $\approx 159.9$GeV, bottom
$\approx 4.65$GeV, charm $\approx 863$MeV and strange $\approx 24.9$MeV. It is interesting to note that the
fifth flavour line appears to be breaking up in SU(5) multiplets and the third quark flavour
line in SU(3) multiplets. The lepton terms are dominated by singlets.

| Fermion | 9.9! | 8.8! | 7.7! | 6.6! | 5.5! | 4.4! |
|---------|------|------|------|------|------|------|
| top     | 2    | 10/8 | 0    | 0    | 0    | 0    |
| bottom  | 0    | 5/8  | 0    | 0    | 0    | 0    |
| charm   | 0    | 0    | 1    | 3/6  | 0    | 0    |
| strange | 0    | 0    | 0    | 1/6  | 3/5  | 0    |
| up      | 0    | 0    | 0    | 0    | 0    | 0    |
| down    | 0    | 0    | 0    | 0    | 0    | 0    |
| $\nu_\tau$ | 0  | 0    | 0    | 0    | 0    | 0    |
| $\tau$  | 0    | 1/8  | 1    | 2/6  | 0    | 0    |
| $\nu_\mu$ | 0  | 0    | 0    | 0    | 0    | 0    |
| $\mu$   | 0    | 0    | 0    | 1    | 2/5  | 0    |
| $\nu_e$ | 0    | 0    | 0    | 0    | 0    | 0    |
| $e$     | 0    | 0    | 0    | 0    | 0    | 0    |
With these postulates in mind can we make a guess at the Higgs mass? Obviously it must be related to a factorial number if these mass expressions have any real physical meaning. The minimum choice would, from the above discussion, be a multiple of 10! which translates into an integral multiple of about 83GeV. However this is based on the (T-2)! bosonic geometry not on a scalar geometry. Could we propose a scalar geometry such as T! (which would be scalar under the assumption of no massless generators)? Unfortunately this translates into a mass of the order of 11.05TeV which appears to be an order of magnitude too high for a prospective Higgs scalar which is expected to be of the order of 1TeV or less.

XVIII. NEUTRINOS

The theory of neutrinos has been left to last in this paper because this is probably the most difficult area of the theory to analyse.

Let us first consider what the geometry of a neutrino should be from first principles and see where the problem lies. The massive leptons in this theory are particles which define a tetrahedral 3-space continuum. The different masses of the e, \( \mu \) and \( \tau \) derive solely from the different Higgs components listed in the previous table (Section XV).

The Higgs components are considered to be ‘internalised’ in the tetrahedral geometry; that is, they are located inside the boundary of the geometry in a time-less space. The number of actual vertices in the geometry depends on the leading term of the Higgs mass components. Thus, the tau lepton, which has components 8! + 7.7! +2.6! is an 8-vertex geometry with 4 vertices internalised inside the tetrahedron (the 8! mass component relates to a \( S_8 \) permutation symmetry of the vertices). The boundary of the geometry which embeds in the surrounding space-time is identical to the electron and has a tetrahedral morphology. The \( S_6 \) and \( S_7 \) symmetries are sub-groups of the 8-vertex geometry. Similarly we can see, from the Higgs components, that the muon must be a 6-vertex geometry with two internalised vertices.

Let us hypothesise that the corresponding neutrinos conserve vertex number; this is
geometrically the most intuitively obvious choice. This means that the $\nu_e$ is an 8 vertex (but two-dimensional) geometry, the $\nu_\mu$ is a 6-vertex (two-dimensional) geometry and the $\nu_\tau$ a 4-vertex geometry flat ‘square’. The proposed geometric forms are all two-dimensional and are as follows;

The absence of any Higgs components for mass generation in the neutrinos means that they cannot be Dirac particles in this theory. Let us see why.

It is instructive to consider first the case of the electron neutrino. The electron has a basic mass unit formed from the tetrahedral symmetry group because it forms a 3-volume in space-time. This mass is given by a tetrahedral $S_4$ symmetry. The corresponding neutrino must be a 4-vertex geometry but morphologically distinct to the electron. Because in affine-set geometry all tetrahedrons are equal, the unique choice would appear to be a flat 4-vertex square. But this must be massless in this theory as has been previously explained. Worse still, such a geometry represents a boson! (Unlike with the case of the tetrahedron, no spinor group can be associated with a two-dimensional square symmetry as the geometry generator acts as a vector).

There appears to be only one way around this impasse and that is to assume that the structure of space-time is more complex than has been listed in the hypotheses so far. We add the following;

Hypothesis 11; That the time dimension for spinors is retarded with respect to bosons.
This is analogous to admitting a violation of special relativity into the theory but with special restrictions. What it means is that the neutrino can be a flat square geometry with respect to time as defined for an electron but remains a tetrahedral geometry with respect to a photon and time as defined for a macroscopic observer; so it will be a spinor with respect to any boson interacting with it. For this to be achieved we need two copies of the Lorentz group; one for spinors and one for bosons (surprisingly CPT symmetry between particles and anti-particles will be maintained in such a scheme because the symmetry is broken between bosons and fermions; not between two fermions so that particles and anti-particles would still have the same mass and lifetime etc).

In terms of cardinality this means admitting the existence of a cardinality \( \aleph_1 \) such that

\[ \aleph_0 < \aleph_1 < c \]

as permitted by the formal undecidability of the continuum hypothesis. This means that, in the absence of the weak interaction, the continuum hypothesis applies to the structure of the universe but the addition of the weak interaction, and specifically the Higgs field, is equivalent to adding an extra order of cardinality to space-time. The propagating neutrino sweeps out a space of cardinality \( \aleph_1 \) which is less than that of the continuum. Its’ weak radiative corrections must then be responsible for the elevation of the cardinality of the contained space to that of the continuum and in so doing generate a mass for the neutrino.

A good geometric way to think of this is as follows. In this theory gravitation is the gauge field of individual intervals in the boundaries of geometries - one dimensional ‘phase’- and electro-magnetism is the gauge field of triangles in the boundaries of geometries - a two-dimensional gauge field (a U(1) phase in this case). The weak interaction is like a ‘volume’ gauge field; a three-dimensional phase of the geometry and, because this dimensionality is associated with mass in this theory, the corresponding bosons must be massive.

In the case of the electron, the contained 3-volume within the tetrahedral boundary in the absence of the weak interaction is a space of cardinality \( \aleph_1 \) which weak radiative corrections elevate to cardinality \( c \). The weak interaction pushes up the cardinality of the contained
space within the tetrahedron. The mass gap generated for the electron is simply the weak radiative correction to its’ mass which was previously calculated at about 7eV. We expect a similar mass gap between the electron neutrino and the photon but this value seems to be ruled out by experiment. Nevertheless, in the absence of any other physics modifying things, this value is a definite prediction for the $\nu_e$ mass in this theory; a little under 7eV ($\approx 6.95eV$).

In fact, looking at the lepton mass expressions given in Section XV it is easily seen that the masses of the corresponding neutrinos are degenerate all with mass 6.95eV in the absence of any physical properties associated with the ‘triangle’ ‘wings’ on the $\nu_\mu$ and $\nu_\tau$ in the previous figure (these are expected to be scalar, not spin-1, and so should not imply fractional EM charge - the only other possibility is that the triangles are paired with opposite spins in which case both the $\nu_\mu$ and the $\nu_\tau$, but not the $\nu_e$, will have non-zero magnetic moments).

A possible source of non-degeneracy of mass of the neutrinos may be derived from radiative corrections to the Higgs components in this theory which have been assumed to be zero (the only multiplier to the factors 6.6!+2.5! - the assumed ‘Higgs’ components of the muon for example - is the number 1; no electro-magnetic or weak radiative correction appears). Using the available data it is possible to put theoretical constraints on mass differences arising from radiative corrections to the Higgs field (which would imply non-standard physics). One readily obtains from the above table and the standard deviations of the empirical values;

$$\text{Mass } \nu_\tau - \text{Mass } \nu_\mu < 0.7MeV$$

$$\text{Mass } \nu_\mu - \text{Mass } \nu_e < 9eV$$

to 95% CL. As the charged lepton masses become known more precisely these constraints can be further restricted.

Can we have sterile neutrinos?

If the triangular ‘wings’ on the $\nu_\mu$ and $\nu_\tau$ are spin 0, and if there is a mass difference
between states to provide some phase space, then it should be possible to produce the following weird geometries through decay mediated by a virtual Higgs particle;

\[ \nu_s \nu_s \]

both of which have zero lepton number and in turn would decay to the electron neutrino. If the triangular wings are pairs of spin-1 these states cannot exist but instead a variety of anomalous non-standard interactions between neutrinos and matter must be possible causing interconversions of one neutrino form to another; always complicated by the fact that, in this model, special relativity will also break down in the neutrino sector.

**XIX. CONCLUSION**

In summary mass expressions for all the quarks, all the leptons and the gauge bosons (under the assumption of massless gluons and photons) have been presented which are concise and simple and all related to a permutation symmetry embedded in affine-set geometry. In particular, precise mass expressions have been presented for the leptons, the nucleons and the vector gauge bosons all of which are testable in precision experiments. Theoretical restrictions have been placed on the possible values the Higgs particle mass can be in this schema; if it exists at all as a free particle (and in this theory there is reason to believe that it may not!) it should appear as an integer multiple of about 83.67GeV. The most likely candidate is a multiplier of 5 (to account for the ‘3’ of the vector gauge bosons and the ‘2’ of the fermions). However, the most logical value for the Higgs is in fact the true tetrahedral
scalar boson 12! but this is more than 10 TeV and so appears to be ruled out.

The best precision test of the theory is the tau lepton mass which, in the absence of radiative corrections to the Higgs field, can be calculated precisely using the known values of $M_e$ and $M_\mu$. The current prediction is $M_\tau = 1776.95\text{MeV}$.

[1] J Balmer, Verh. Naturf. Ges. Basel 7 548 (1885)
[2] J.J.Sakurai. Advanced Quantum Mechanics. Addison-Wesley. p271.
[3] See for example; Kaku; quantum field theory.
[4] M Singer, P Sutcliffe. Symmetric Instantons and Skyrme Fields. \texttt{hep-th/9901075}
[5] M.R.Douglas, G.Moore. D Branes, Quivers and ALE Instantons. \texttt{hep-th/9603167}
[6] C.V.Johnson, R.C.Myers. Aspects of Type IIB theory on ALE Spaces. Phys.Rev. D55 (1997) 6382.
[7] A.Hanany and Y He. Non-abelian Finite Gauge Field Theory. \texttt{hep-th/9811183}
[8] B.R. Greene, C I Lazaroiu and M Raugas. D branes on Non-abelian Threefold Quotient Singularities. \texttt{hep-th/9811201}
[9] T.Muto. Non-abelian sub-groups of SU(3) \texttt{hep-th/9811258}
[10] A.Klemm, M.G.Schmidt. Phys.Lett. B245, 53(1990).
[11] P.Bantay. Permutation Orbifolds. \texttt{hep-th/9603176}
[12] G.R. Filewood in Proc. Fifth Marcel Grossman meeting. Wld. Scientific. 1979.
[13] G.R. Filewood. Unpublished.