BERGMAN SPACES INDUCED BY DOUBLE WEIGHTS IN THE UNIT BALL OF $\mathbb{C}^n$

JUNTAO DU, SONGXIAO LI†, XIAOSONG LIU AND YECHENG SHI

ABSTRACT. In this paper, we introduce and study the weighted Bergman space $A^p_\omega$ with double weight $\omega$ in the unit ball of $\mathbb{C}^n$. Carleson measure and Volterra integral operators on $A^p_\omega$ are studied in this paper.

Keywords: Weighted Bergman space, Carleson measure, Volterra integral operator, double weight.

1. INTRODUCTION

Let $\mathbb{B}$ be the open unit ball of $\mathbb{C}^n$ and $\mathbb{S}$ the boundary of $\mathbb{B}$. When $n = 1$, then $\mathbb{B}$ is the open unit disk in complex plane $\mathbb{C}$ and always denoted by $\mathbb{D}$. Let $H(\mathbb{B})$ denote the space of all holomorphic functions on $\mathbb{B}$. For any two points $z = (z_1, z_2, \cdots, z_n)$ and $w = (w_1, w_2, \cdots, w_n)$ in $\mathbb{C}^n$, we define $\langle z, w \rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}$ and $|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$.

Let $d\sigma$ and $dv$ be the normalized surface and volume measures on $\mathbb{S}$ and $\mathbb{B}$, respectively. For $0 < p \leq \infty$, the Hardy space $H^p(\mathbb{B})$ (or $H^p$) is the space consisting of all functions $f \in H(\mathbb{B})$ such that

$$\|f\|_{H^p} := \sup_{0 < r < 1} M_p(r, f),$$

where

$$M_p(r, f) = \left( \int_\mathbb{S} |f(r\xi)|^p d\sigma(\xi) \right)^{\frac{1}{p}} < \infty, \text{ when } 0 < p < \infty,$$

and

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

For $-1 < \alpha < \infty$ and $0 < p < \infty$, the weighted Bergman space $A^p_\omega(\mathbb{B})$ (or $A^p_\omega$) consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{A^p_\omega} := \int_\mathbb{B} |f(z)|^p dv_\alpha(z) = c_\alpha \int_\mathbb{B} |f(z)|^p (1 - |z|^2)^\alpha dv(z) < \infty,$$

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where \( c_\alpha = \Gamma(n + \alpha + 1)/(\Gamma(n + 1)\Gamma(\alpha + 1)) \). When \( \alpha = 0 \), \( A^p_\omega(\mathbb{B}) = A^p(\mathbb{B}) \) is the standard Bergman space. It is known that \( f \in A^p_\omega \) if and only if \( (1 - |z|^2)\Re f(z) \in L^p(\mathbb{B}, dv_\omega) \). Moreover
\[
\|f\|_{A^p_\omega}^p = |f(0)|^p + \int_\mathbb{B} |\Re f(z)|^p(1 - |z|^2)dv_\omega(z).
\]

Here \( \Re f \) is the radial derivative of \( f \), i.e., \( \Re f(z) = \sum^n_{j=1} \frac{\partial f}{\partial \alpha_j} \). See [18, 19] for the theory of \( H^p \) and \( A^p_\omega \).

Suppose \( \omega \) is a radial weight (i.e., a positive and integrable function in \( B \) such that \( \omega(z) = \omega(|z|) \)). Let \( \hat{\omega}(r) = \int_1^r \omega(t)dt \). \( \omega \) is called a regular weight, denoted by \( \omega \in \mathcal{R} \), if there is a constant \( C > 0 \) such that
\[
\hat{\omega}(r) < C\omega(\frac{1 + r}{2}), \quad \text{when } 0 \leq r < 1.
\]
\( \omega \) is called a regular weight, denote by \( \omega \in \mathcal{R} \), if there is a constant \( C > 0 \) determined by \( \omega \), such that
\[
\frac{1}{C} < \frac{\hat{\omega}(r)}{(1 - r)\omega(r)} < C, \quad \text{when } 0 \leq r < 1.
\]
\( \omega \) is called a rapidly increasing weight, denote by \( \omega \in I \), if
\[
\lim_{r \to 1} \frac{\hat{\omega}(r)}{(1 - r)\omega(r)} = \infty.
\]

After a calculation, we see that \( I \cup \mathcal{R} \subset \hat{D} \). See [10, 11] for more details about \( I, \mathcal{R}, \hat{D} \).

In [11], J. Peláez and J. Rättyä introduced a new class function space \( A^p_\omega(\mathbb{D}) \), the weighted Bergman space induced by rapidly increasing weights in \( \mathbb{D} \). In [11], they investigated some basic properties of \( \omega \) with \( \omega \in \mathcal{R} \cup I \), described the \( q \)-Carleson measure for \( A^p_\omega(\mathbb{D}) \), gave equivalent characterizations of \( A^p_\omega(\mathbb{D}) \), characterized the boundedness, compactness and Schatten classes of Volterra integral operator \( J_{\gamma} \) on \( A^p_\omega(\mathbb{D}) \). In [10], J. Peláez extended many results from \( \omega \in \mathcal{R} \cup I \) to \( \omega \in \hat{D} \). See [10–16] for some results on \( A^p_\omega(\mathbb{D}) \) with \( \omega \in \hat{D} \).

Motivated by [11], we extend the Bergman space \( A^p_\omega(\mathbb{D}) \) with \( \omega \in \hat{D} \) to the unit ball. Let \( \omega \in \hat{D} \) and \( 0 < p < \infty \). The weighted Bergman space \( A^p_\omega(\mathbb{B}) = A^p_\omega \) is the space of all \( f \in H(\mathbb{B}) \) for which
\[
\|f\|_{A^p_\omega}^p := \int_\mathbb{B} |f(z)|^p\omega(z)dv(z) < \infty.
\]
It is easy to check that \( A^p_\omega \) is a Banach space when \( p \geq 1 \) and a complete metric space with the distance \( \rho(f, g) = \|f - g\|_{A^p_\omega}^p \) when \( 0 < p < 1 \). When \( \omega(z) = (1 - |z|^2)^{\alpha}(\alpha > -1) \), the space \( A^p_\omega \) becomes the classical weighted Bergman space \( A^p_\omega \).

Suppose that \( g \in H(\mathbb{D}) \). The integral operator \( J_{\gamma} \), called the Volterra type integral operator, is defined by
\[
J_{\gamma}f(z) = \int_0^\infty f(\xi)g'(\xi)d\xi, \quad f \in H(\mathbb{D}), \ z \in \mathbb{D}.
\]
The operator $J_g$ was first introduced by Pommerenke in [17]. He showed that $J_g$ is a bounded operator on the Hardy space $H^2(D)$ if and only if $g \in BMOA(D)$. See [1-5] for the study of the boundedness, compactness and the spectrum of $J_g$ in $H^p(D)$ and $A^p_0(D)$.

Let $g \in H(B)$. Define

$$T_g f(z) = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t}, \quad f \in H(B), \ z \in B.$$  

This operator is also called the Volterra type integral operator (or the Riemann-Stieltjes operator, or the Extended Cesàro operator). The operator $T_g$ was introduced by Z. Hu in [6] and studied, for example in [6-9]. In particular, J. Pau completely described the boundedness and compactness of $T_g$ between different Hardy spaces in the unit ball of $C^n$ in [9].

In this paper, we will investigate some properties of $A^p_\omega$ and study the boundedness and compactness of $T_g : A^p_\omega \to A^q_\omega$. $A^p_\omega$ is a ball in $S$ for all $\xi \in S$ and $r \in (0, 1)$. More information about $d(\cdot, \cdot)$ and $Q(\xi, r)$ can be found in [18, 19]. Lemma 4.6 in [19] is very useful in this paper, and we express it as follows.

**Lemma 1.** There exist positive constants $A_1$ and $A_2$ (depending on $n$ only) such that

$$A_1 \leq \frac{\sigma(Q(\xi, r))}{r^{2n}} \leq A_2$$

for all $\xi \in \mathbb{S}^n$ and $r \in (0, \sqrt{2})$. 

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2. Preliminary results

For any $\xi, \tau \in \overline{B}$, let $d(\xi, \tau) = |1 - \langle \xi, \tau \rangle|$. Then $d(\cdot, \cdot)$ is a nonisotropic metric. For $r > 0$ and $\xi \in \mathbb{S}$, let

$$Q(\xi, r) = \{ \eta \in \mathbb{S} : |1 - \langle \xi, \eta \rangle| \leq r^2 \}.$$ 

$Q(\xi, r)$ is a ball in $\mathbb{S}$ for all $\xi \in \mathbb{S}^n$ and $r \in (0, 1)$. More information about $d(\cdot, \cdot)$ and $Q(\xi, r)$ can be found in [18, 19]. Lemma 4.6 in [19] is very useful in this paper, and we express it as follows.

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For any \( a \in \mathbb{B} \setminus \{0\} \), let \( Q_a = Q(a/|a|, \sqrt{1-|a|}) \), and define
\[
S_a = S(Q_a) = \left\{ z \in \mathbb{B} : \frac{z}{|z|} \in Q_a, |z| < 1 \right\}.
\]

For convince, if \( a = 0 \), let \( Q_a = S \) and \( S_a = \mathbb{B} \). We call \( S_a \) the Carleson block. Now we give an estimate for the volume of \( S_a \). As usual, for a measurable set \( E \subset \mathbb{B} \),
\[
\omega(E) = \int_E \omega(z)dv(z).
\]

**Lemma 2.** Assume that \( \omega \in \mathcal{D}, r \in [0, 1] \) and \( \omega^*(r) = \int_0^1 \omega(s) \log \frac{1}{s}ds \). Then the following statements hold.

(i) \( \omega^* \in \mathcal{R} \) and \( \omega^*(r) \approx (1-r) \int_0^1 \omega(t)dt \) when \( r \in (\frac{1}{2}, 1) \);
(ii) There are \( 0 < a < b < +\infty \) and \( \delta \in [0, 1) \), such that
\[
\frac{\omega^*(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r \to 1} \frac{\omega^*(r)}{(1-r)^a} = 0;
\]
\[
\frac{\omega^*(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r \to 1} \frac{\omega^*(r)}{(1-r)^b} = \infty;
\]
(iii) \( \omega^*(r) \) is decreasing on \( [\delta, 1) \) and \( \lim_{r \to 1} \omega^*(r) = 0 \).
(iv) \( \omega(S_a) \approx (1-|a|)^n \int_0^1 \omega(r)dr \).

**Proof.** (i) – (iii) can be found in [11][12]. From [19], Lemma 1.8, we see that
\[
\int_\mathbb{B} f(z)dv(z) = 2n \int_0^1 r^{2n-1}dr \int_S f(r\xi)d\sigma(\xi).
\]
Then by Lemma[11] we have
\[
\omega(S_a) = 2n \int_{|a|}^1 r^{2n-1}\omega(r)dr \int_{Q(\frac{a}{|a|}, \sqrt{1-|a|})} d\sigma(\xi) \approx (1-|a|)^n \omega(|a|).
\]
The proof is complete. □

**Lemma 3.** There exists \( q = q(n) > 1 \), such that for all \( r \in \left(0, \frac{1}{\sqrt{n}}\right) \) and \( \xi \in \mathbb{S}^n \),
\[
\sigma(Q(\xi, qr)\setminus Q(\xi, r)) \approx r^{2n}.
\]

**Proof.** By Lemma[11] there exist \( A_2 > A_1 > 0 \), such that
\[
A_1 r^{2n} \leq \sigma(Q(\xi, r)) \leq A_2 r^{2n}, \text{ for all } r \in \left(0, \sqrt{2} \right) \text{ and } \xi \in \mathbb{S}^n.
\]
Fix a \( q > 1 \) such that \( A_1 q^{2n} > A_2 \). Then we have
\[
\sigma(Q(\xi, qr)\setminus Q(\xi, r)) \geq (A_1 q^{2n} - A_2)r^{2n} \approx r^{2n},
\]
and
\[
\sigma(Q(\xi, qr)\setminus Q(\xi, r)) \leq (A_2 q^{2n} - A_1)r^{2n} \leq r^{2n}.
\]
The proof is complete. □

To study the compactness of a linear operator, we need the following lemma which can be obtained in a standard way.
Lemma 4. Suppose $0 < p, q < \infty, \omega \in \hat{D}$ and $\mu$ is a positive Borel measure on $\mathbb{D}$. If $T : A^p_\omega \to A^q_\omega$ is linear and bounded, then $T$ is compact if and only if whenever $\{f_k\}$ is bounded in $A^p_\omega$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, \[ \lim_{k \to \infty} ||T f_k||^q_\omega = 0. \]

Lemma 5. Suppose $\omega \in \hat{D}, 0 < \alpha < \infty$. Then there exists a constant $C = C(\alpha, \omega, n)$ such that
\[ |f(z)|^\alpha \leq C M_\omega(|f|)z(\omega), \]
for all $f \in H(\mathbb{D})$. Here and henceforth,
\[ M_\omega(\varphi)(z) = \sup_{z \in S_a} \frac{1}{\omega(S_a)} \int_{S_a} |\varphi(\xi)| \omega(\xi) dv(\xi). \]

Proof. Fix $q = q(n)$ such that Lemma 4 holds. Let $r_0 = \max\left(\frac{1}{2}, 1 - \frac{1}{q}\right)$. First we suppose that $r_0 < |z| < 1$.

When $\frac{1+\beta}{2} < \rho < 1$, let $N$ be the largest nature number such that $q^N (1 - |z|) < 1$ and
\[ Q_k := \left\{ \xi \in \mathbb{S} : \left| 1 - \langle \xi, \frac{z}{|z|} \rangle \right| < q^k \left(1 - \frac{|z|}{\rho}\right) \right\}, \]
for $k = 0, 1, 2, \ldots, N$. Then,
\[ Q_0 \subset Q_1 \subset \cdots \subset Q_N \subset Q_{N+1} := \mathbb{S}. \]

When $\xi \in Q_{k+1} \setminus Q_k$ with $k = 1, 2, \ldots, N$, we have
\[ \left| 1 - \langle \frac{1}{\rho} z, \xi \rangle \right| = \left| 1 - \langle \xi, \frac{z}{|z|} \rangle + \langle \xi, \frac{z}{|z|} \rangle - \langle \xi, \frac{1}{\rho} z \rangle \right| \geq (q^k - 1) \left(1 - \frac{|z|}{\rho}\right). \]

Then for all $\xi \in Q_{k+1} \setminus Q_k$ with $k = -1, 0, 1, \ldots, N$, we have
\[ \left| 1 - \langle \frac{1}{\rho} z, \xi \rangle \right| \geq q^k \left(1 - \frac{|z|}{\rho}\right) \]
by the assumption $Q_{-1} = \emptyset$.

Since $\omega \in \hat{D}$, by Lemma 2.1(ii) in [10], there esixt $C_0 = C_0(\omega) \geq 1$ and $\beta = \beta(\omega) > 0$, such that
\[ \hat{\omega}(r) \leq C_0 \left(\frac{1-r}{1-t}\right)^\beta \hat{\omega}(t), \text{ for all } 0 \leq r \leq t < 1. \]
Write $\alpha = x\gamma$, where $\gamma > 1 + \frac{2}{n} > 1$. Suppose $\frac{1}{r} + \frac{1}{s} = 1$. Then by Poisson Transform, H"older inequality and Theorem 1.12 in [19], we have

$$|f(z)|^s \leq \int_S \left( \frac{1 - \frac{1}{r^s}|z|^2}{1 - \langle \frac{1}{r} z, \xi \rangle} \right)^{\frac{n}{r^s}} |f(\rho \xi)|^s d\sigma(\xi)$$

$$\leq \left( \int_S \left( \frac{1 - \frac{1}{r^s}|z|^2}{1 - \langle \frac{1}{r} z, \xi \rangle} \right)^{\frac{n}{r^s}} |f(\rho \xi)|^s d\sigma(\xi) \right)^{\frac{1}{s}} \left( \int_S \left( \frac{1 - \frac{1}{r^s}|z|^2}{1 - \langle \frac{1}{r} z, \xi \rangle} \right)^{n\gamma - n} d\sigma(\xi) \right)^{\frac{1}{\gamma}}.$$
By Lemma 2, we have
\[
\frac{1}{(1 - |z|^n) \int_0^1 r^{2n-2} \omega(r) \, dr} \approx \frac{1}{(1 - |z|)^n \omega(z)} \leq \frac{(1 - |a|)^{n+\beta}}{(1 - |z|)^{n+\beta} \omega(S_k)} \leq \frac{Q^{(n+\beta)k}}{\omega(S_k)}.
\]

Then,
\[
|f(z)|^p \leq \sum_{k=0}^{N+1} \frac{1}{q^{(n-\beta)k}} \int_{S_k} |f(\xi)|^p \omega(\xi) \, d\xi \leq M_\omega(|f|^p)(z).
\]

Next we suppose that $|z| \leq r_0$. For all $a \in B$ such that $z \in S_a$, we have $|a| < |z| \leq r_0$. By Lemma 2, $\omega(S_a) \approx 1$. Then,
\[
\sup_{z \in S_a} \frac{1}{\omega(S_a)} \int_{S_a} |f(\xi)|^p \omega(\xi) \, d\xi \approx \sup_{z \in S_a} \int_{S_a} |f(\xi)|^p \omega(\xi) \, d\xi = \int_{B} |f(\xi)|^p \omega(\xi) \, d\xi.
\]

Using Cauchy formula, we have
\[
|f(z)|^p \leq \int_{B} |f(\xi)|^p \omega(\xi) \, d\xi \approx \sup_{z \in S_a} \frac{1}{\omega(S_a)} \int_{S_a} |f(\xi)|^p \omega(\xi) \, d\xi.
\]

The proof is complete. \qed

Here and henceforth, for all $a \in B$ and $0 < p < \infty$, set
\[
F_{a,p} = \left( \frac{1 - |a|^2}{1 - \langle z, a \rangle} \right)^{\frac{2n}{p} - 1}.
\]

We obtain the following lemma.

**Lemma 6.** Suppose $\omega \in \mathcal{D}$, $0 < p < \infty$ and $\gamma$ is large enough. For all $a \in B$,
\[
|F_{a,p}(z)| \approx 1, \ z \in S_a,
\]

and
\[
\|F_{a,p}\|_{A_\gamma^p} \approx \omega(S_a).
\]

**Proof.** For all $z \in S_a$, we have
\[
\frac{1 - |a|}{1 - \langle z, a \rangle} \leq \frac{1 - |a|}{1 - |z|} \leq \frac{1 - |a|}{1 - |a||z|} \leq 1
\]
and
\[
\frac{1 - |a|}{1 - \langle z, a \rangle} \geq \frac{1 - |a|}{1 - \langle \frac{z}{|z|}, a \rangle} \geq \frac{1 - |a|}{1 - |a| + 1 - |a|^2} \geq 1.
\]

It follows that (4) holds.

By Lemmas 1 and 2 we have
\[
\sigma(Q_a) \approx (1 - |a|)^n, \ \text{and} \ \omega(S_a) \approx (1 - |a|)^n \int_{|a|}^1 \omega(t) \, dt.
\]
By Theorem 1.12 in [19] and Lemma 2.1(iii) in [10], if $\gamma$ is large enough, we have
\[
\|F_{a,p}\|_{\mathcal{A}_b^p}^p = 2n(1 - |a|)^{\gamma + n} \int_0^1 \omega(r)r^{2n-1} \int_B \frac{1}{|1 - \langle r\xi, a\rangle|^{\gamma + n}} d\sigma(\xi)dr
\]
\[
\approx 2n(1 - |a|)^{\gamma + n} \int_0^1 r^{2n-1} \omega(r) \frac{1}{(1 - r|a|)^\gamma} dr
\]
\[
\leq 2n(1 - |a|)^{\gamma + n} \left( \int_{|a|}^1 \omega(r) \frac{1}{(1 - r)^\gamma} dr + \int_{|a|}^1 \omega(r) \frac{1}{(1 - |a|)^\gamma} dr \right)
\]
\[
\leq (1 - |a|)^{\gamma} \int_{|a|}^1 \omega(r)dr \approx \omega(S_a).
\]
By (4), $\|F_{a,p}\|_{\mathcal{A}_b^p} \geq \omega(S_a)$ is obvious. The proof is complete. \hfill \Box

In the rest this paper, we always assume $F_{a,p}$ satisfies the condition of Lemma 6.

In the last of this section, we define a $\alpha$–Carleson block $S_{a,\alpha}$ for all $a \in \mathbb{B}\setminus\{0\}$ and any fixed $\alpha \geq 0$. That is
\[
S_{a,\alpha} = \left\{ z \in \mathbb{B}: |a| < |z| < 1, \left| 1 - \left( \frac{z}{|z|} \cdot \frac{a}{|a|} \right) \right| \leq (\alpha + 1)(1 - |a|) \right\}.
\]
When $a = 0$, define $S_{a,\alpha} = \mathbb{B}^n$. Obviously, for all $a \in \mathbb{B}$, we have $S_{a,0} = S_a$ and $S_a \subset S_{a,\alpha}(\alpha \geq 0)$. The following proposition is useful in this paper.

**Proposition 1.** For any fixed $\alpha \geq 0$, there exist $N \in \mathbb{N}$, such that, for all $a \in \mathbb{B}$, there are $a_1, a_2, \ldots, a_N$ satisfying the following condition:
1. $k \leq N$ and $|a_1| = |a_2| = \cdots = |a_k| = |a|$;
2. $S_{a,\alpha} \subset \bigcup_{i=1}^k S_{a_i}$.

**Proof.** Suppose $a \in \mathbb{B}\setminus\{0\}$ is fixed. For any $\tau \in \mathbb{S}$, define
\[
E_\tau = Q(\tau, \frac{1}{2} \sqrt{1 - |a|}), \quad \text{and} \quad E_\tau^a = Q\left( \frac{a}{|a|}, \frac{1}{2} + \sqrt{1 + \alpha} \right) \sqrt{1 - |a|}.
\]
Since $\frac{\sigma(E_{\tau^a})}{\sigma(E_{\tau})} < \infty$, there are at most $M := M(a)$ elements $\xi_1, \xi_2, \ldots, \xi_M$ in $\mathbb{S}$ such that
(a) $E_{\xi_i} \cap E_{\xi_j} = \emptyset$ for all $1 \leq i < j \leq M$;
(b) $E_{\xi_i} \subset E_{\xi_j}^a$ for all $1 \leq i \leq M$.

Then we have
\[
Q\left( \frac{a}{|a|}, \sqrt{(1 + \alpha)(1 - |a|)} \right) \subset \bigcup_{i=1}^M Q(\xi_i, \sqrt{1 - |a|}).
\]
Otherwise, there is a $\xi \in Q\left( \frac{a}{|a|}, \sqrt{(1 + \alpha)(1 - |a|)} \right)$ but $\xi \notin \bigcup_{i=1}^M Q(\xi_i, \sqrt{1 - |a|})$. Then for any $\eta \in E_{\xi}$, we have
\[
d(\eta, \xi_i) \geq d(\xi, \xi_i) - d(\eta, \xi) > \sqrt{1 - |a|} - \frac{1}{2} \sqrt{1 - |a|} = \frac{1}{2} \sqrt{1 - |a|},
\]
and
\[
d(\eta, \frac{a}{|a|}) \leq d(\eta, \xi) + d(\xi, \frac{a}{|a|}) \leq \frac{1}{2} \sqrt{1 - |a|} + \sqrt{(1 + \alpha)(1 - |a|)}.
\]
That is a contraction with $M$ is the maximum number. By Lemma 1, we have

$$M \leq \frac{\sigma(E')}{\sigma(E_r)} \leq 1.$$ 

Then by letting $a_i = |a_l|$, we finish the proof. \hfill \Box

Remark 1. By Lemma 2 for any fixed $\alpha > 0$, $\omega(S_{a,\alpha}) \approx \omega(S_{a,0})$. Hence, many results described by Carleson block also hold for $\alpha$-Carleson block.

Remark 2. In the definition of $S_{a,\alpha}$, in order to get $\mathbb{B}\setminus\{0\} \subset \bigcup_{a \in \mathbb{B}\setminus\{0\}} S_{a,\alpha}$, we restrict $\alpha \geq 0$. When $-1 < \alpha < 0$, if we define $S_{a,0}$ the same way for $a \neq 0$, we have $\bigcup_{a \in \mathbb{B}\setminus\{0\}} S_{a,\alpha} = \mathbb{B}\setminus(1+\alpha)\mathbb{B}$. It is well known to the experts in this field, for a function in $H(\mathbb{B})$, the properties on $(1+\alpha)\mathbb{B}(1 < \alpha < 0)$ is not very important. So we can extend the field of $\alpha$ to $\alpha > -1$.

For $\xi \in \mathbb{S}$ and $r > 0$, a Carleson tube $S^*(\xi, r)$ can be define as

$$S^*(\xi, r) = \{z \in \mathbb{B} : |1 - \langle z, \xi \rangle| < r\}.$$ 

As we know, Carleson tube is very useful in the study of the function space on the unit ball of $\mathbb{C}$. For the convenience, we often restrict $0 < r < \delta$ for some $\delta > 0$. Here, we will compare Carleson tube with Carleson block.

Proposition 2. The following assertions hold.

(i) For any $\xi \in \mathbb{S}$ and $0 < r < 1$, there exists $a \in \mathbb{B}$ such that $S^*(\xi, r) \subset S_{a,2}$.

(ii) For any $a \in \mathbb{B}$ with $|a| > \frac{1}{2}$, there exist $\xi \in \mathbb{S}$ and $r > 0$ such that $S_a \subset S^*(\xi, r)$.

Proof. (i). For any $z \in S^*(\xi, r)$ with $0 < r < \frac{1}{2}$, by letting $a = (1-r)\xi$, we have $|z| > |a|$ and

$$|1 - \langle z, \xi \rangle| \leq |1 - \langle z, \xi \rangle| + \left|\langle z, \xi \rangle - \langle \frac{z}{|z|}, \xi \rangle\right| \leq r + 1 - |z| < 2(1 - |a|).$$

Then $S^*(\xi, r) \subset S_{a,2}$.

(ii). Suppose $a \neq 0$. Let $\xi = \frac{a}{|a|}$ and $r > 2(1 - |a|)$. For any $z \in S_a$, we have

$$|1 - \langle z, \xi \rangle| \leq \left|1 - \langle z, \frac{a}{|a|} \rangle\right| + \left|\langle \frac{z}{|z|}, \frac{a}{|a|} \rangle - \langle z, \xi \rangle\right| \leq 1 - |a| + 1 - |z| \leq 2(1 - |a|) < r.$$ 

Then $S_a \subset S^*(\xi, r)$. \hfill \Box

Remark 3. By Propositions 1 and 2, if a statement is described by Carleson tube, it also can be described by Carleson block. And the converse is also true. For the convenience to deal with radial weights, we use Carleson block in this paper.
3. The $q$-Carleson Measure for $A^p_\omega$

In this section, we give some description of $q$-Carleson measure for $A^p_\omega$ when $0 < p \leq q < \infty$. For a given Banach space (or a complete metric space) $X$ of analytic functions on $\mathbb{B}$, a positive Borel measure $\mu$ on $\mathbb{B}$ is called a $q$-Carleson measure for $X$ if the identity operator $Id : X \rightarrow L^q(\mu)$ is bounded. Moreover, if $Id : X \rightarrow L^q(\mu)$ is compact, then we say that $\mu$ is a vanishing $q$-Carleson measure for $X$.

**Theorem 1.** Let $0 < p \leq q < \infty$, $\omega \in \hat{D}$, and $\mu$ be a positive Borel measure on $\mathbb{B}$. Then the following statements hold:

(i) $\mu$ is a $q$-Carleson measure for $A^p_\omega$ if and only if
\[
\sup_{a \in \mathbb{B}} \frac{\mu(S_a)}{(\omega(S_a))^{\frac{q}{p}}} < \infty.
\]

Moreover, if $\mu$ is a $q$-Carleson measure for $A^p_\omega$, then the identity operator $Id : A^p_\omega \rightarrow L^q(\mu)$ satisfies
\[
\|Id\|_{A^p_\omega \rightarrow L^q(\mu)} \approx \sup_{a \in \mathbb{B}} \frac{\mu(S_a)}{(\omega(S_a))^{\frac{q}{p}}}.
\]

(ii) $\mu$ is a vanishing $q$-Carleson measure for $A^p_\omega$ if and only if
\[
\lim_{|a| \rightarrow 1} \frac{\mu(S_a)}{(\omega(S_a))^{\frac{q}{p}}} = 0.
\]

**Proof.** First assume that $\mu$ is a $q$-Carleson measure for $A^p_\omega$. By Lemma 6 we have
\[
\mu(S_a) \approx \int_{S_a} |F_{a,p}|^q d\mu(z) \leq \|F_{a,p}\|_{L^q(\mu)}^q = \|Id\|_{A^p_\omega \rightarrow L^q(\mu)}^q \left(\frac{\mu(S_a)}{(\omega(S_a))^{\frac{q}{p}}}\right).
\]

So,
\[
\sup_{a \in \mathbb{B}} \frac{\mu(S_a)}{(\omega(S_a))^{\frac{q}{p}}} \leq \|Id\|_{A^p_\omega \rightarrow L^q(\mu)}^q.
\]

Conversely, suppose $M := \sup_{a \in \mathbb{B}} \frac{\mu(S_a)}{(\omega(S_a))^{\frac{q}{p}}} < \infty$. We begin with proving that there exists a constant $K = K(p, q, \omega)$ such that
\[
\mu(E_s) \leq KM s^{-\frac{q}{p}} \|\varphi\|_{L^q(\omega)}^q,
\]

is valid for all $\varphi \in L^1_{\omega}$ and $0 < s < \infty$. Here $E_s = \{z \in \mathbb{B} : M_\omega(\varphi)(z) > s\}$.

If $E_s = \emptyset$, (8) holds. If $E_s \neq \emptyset$, define $A_\varepsilon^s$ and $B_\varepsilon^s$ for each $\varepsilon > 0$ as follows.

\[
A_\varepsilon^s = \left\{ z \in \mathbb{B} : \int_{S_z} |\varphi(\xi)| \omega(\xi) d\nu(\xi) > s(\varepsilon + \omega(S_z)) \right\},
\]

and
\[
B_\varepsilon^s = \{ z \in \mathbb{B} : S_z \subset S_u \text{ for some } u \in A_\varepsilon^s \}.
\]
Then we have
\[ E_\varepsilon = \bigcup_{\varepsilon > 0} B^\varepsilon_s, \quad \text{and} \quad \mu(E_\varepsilon) = \lim_{\varepsilon \to 0^+} \mu(B^\varepsilon_s). \quad (9) \]

Let \( E \subset A^\varepsilon_s \) such that for all \( \xi, \eta \in E \) we have either \( \xi = \eta \) or \( Q_\xi \cap Q_\eta = \emptyset \). Since
\[
s \sum_{\xi \in E} (\varepsilon + \omega(S_\xi)) \leq \sup_{\xi \in E} \int_{S_\xi} |\varphi(z)| \omega(z) dv(z) \leq \|\varphi\|_{L^1}^\varepsilon, \quad (10)
\]
there are only finite elements in \( E \). By Lemma 5.6 in [19], there are \( \{z_1, z_2, \cdots, z_m\} \subset A^\varepsilon_s \) such that \( Q_{z_j} (1 \leq j \leq m) \) are disjoint and
\[
A^\varepsilon_s \subset \bigcup_{k=1}^m \{ z \in \mathbb{B} : Q_z \subset Q'_{z_k} \}, \quad (11)
\]
where
\[
Q'_{z_k} = Q\left(\frac{z_k}{|z_k|}, 5 \sqrt{1-|z_k|}\right). \]

For any \( z \in B^\varepsilon_s \), there is a \( u \in A^\varepsilon_s \) such that \( S_z \subset S_u \). So, \( Q_z \subset Q_u \). By (11), we have
\[
B^\varepsilon_s \subset \bigcup_{k=1}^m \{ z \in \mathbb{B} : Q_z \subset Q'_{z_k} \}. \quad (12)
\]

Let \( r_k = 1 - 25(1-|z_k|) \). If \( r_k > 0 \), let \( z'_k = \frac{r_k z_k}{|z_k|} \), and otherwise, let \( z'_k = 0 \). Then we have \( Q'_{z_k} \subset Q'_{z'_k} \). Therefore,
\[
\mu\left(\{ z \in \mathbb{B} : Q_z \subset Q'_{z'_k} \}\right) \leq \mu\left(\{ z \in \mathbb{B} : Q_z \subset Q'_{z_k} \}\right)
= \mu\left(\{ z \in \mathbb{B} : S_z \subset S'_{z_k} \}\right)
\leq \mu\left( S_{z'_k} \right) \leq M (\omega(S_{z'_k}))^{\\frac{q}{p}} \approx M (\omega(S_{z_k}))^{\frac{q}{p}}. \]

Here, the last equivalent relation can be get by Lemma [2] and \( \omega \in \mathcal{D} \). Then, by (10) and (12), we have
\[
\mu(B^\varepsilon_s) \leq \sum_{k=1}^m \mu\left(\{ z \in \mathbb{B} : Q_z \subset Q'_{z_k} \}\right)
\leq M \left( \sum_{k=1}^m \omega(S(Q_{z_k})) \right)^{\frac{q}{p}} \leq M s^{-\frac{p}{q}} \|\varphi\|_{L^1}^\varepsilon. \]

Let \( \varepsilon \to 0^+ \), we have \( K = K(p, q, \omega) \) such that
\[
\mu(E_\varepsilon) \leq KM s^{-\frac{p}{q}} \|\varphi\|_{L^1}^\varepsilon. \]

Then we obtain (8).

Next, we will show that \( \mu \) is a \( q \)-Carleson measure for \( A^p_\omega \). To do this, fix \( \alpha > \frac{1}{p} \) and let \( f \in A^p_\omega \). For \( s > 0 \), let
\[
|f|^{\frac{q}{p}} = \psi_{\frac{1}{p}, s} + \chi_{\frac{1}{s}, s},
\]
where
\[
\psi_{\frac{1}{p}, s}(z) = \begin{cases} |f(z)|^{\frac{q}{p}} & \text{if } |f(z)|^{\frac{q}{p}} > \frac{s}{2K} \\ 0 & \text{otherwise} \end{cases}
\]

and
\[
\chi_{\frac{1}{s}, s}(z) = \begin{cases} 0 & \text{if } |f(z)|^{\frac{q}{p}} > \frac{s}{2K} \\ \frac{s}{2K} & \text{otherwise} \end{cases}
\]
and $K$ is the constant in (8) such that $K \geq 1$. Since $p > \frac{1}{\alpha}$, the function $\psi_{\frac{1}{s}, s}$ belongs to $L^1_\omega$ for all $s > 0$, and
\[
M_\omega(|f|^\frac{1}{s}) \leq M_\omega(\psi_{\frac{1}{s}, s}) + M_\omega(\chi_{\frac{1}{s}, s}) \leq M_\omega(\psi_{\frac{1}{s}, s}) + \frac{s}{2K}.
\]
Then,
\[
\left\{ z \in \mathbb{B} : M_\omega(|f|^\frac{1}{s})(z) > s \right\} \subset \left\{ z \in \mathbb{B} : M_\omega(\psi_{\frac{1}{s}, s})(z) > \frac{s}{2} \right\}.
\]
By Lemma 5, (13), (8) and Minkowski's inequality (Fubini's Theorem in the case $p = q$) in order, we have
\[
\int_\mathbb{B} |f(z)|^p d\mu(z) \leq \int_\mathbb{B} \left( M_\omega(|f|^\frac{1}{s})(z) \right)^q d\mu(z)
\]
\[
= qa \int_0^\infty s^{q-1} \mu \left( \left\{ z \in \mathbb{B} : M_\omega(|f|^\frac{1}{s})(z) > s \right\} \right) ds
\]
\[
\leq qa \int_0^\infty s^{q-1} \mu \left( \left\{ z \in \mathbb{B} : M_\omega(\psi_{\frac{1}{s}, s})(z) > \frac{s}{2} \right\} \right) ds
\]
\[
\leq M \int_0^\infty s^{q-1} \frac{s}{K} \left\| \psi_{\frac{1}{s}, s} \right\|_{L_\omega}^q ds
\]
\[
= M \int_0^\infty s^{q-1} \left( \int_{\mathbb{B} \cap |f(z)|^\frac{1}{s} > \frac{s}{2K}} |f(z)|^\frac{1}{s} \omega(z) dv(z) \right)^q ds
\]
\[
\leq M \left( \int_\mathbb{B} |f(z)|^\frac{1}{s} \omega(z) \left( \int_0^{2K|f(z)|^\frac{1}{s}} s^{q-1} ds \right)^\frac{q}{2} dv(z) \right)^\frac{q}{2}
\]
\[
\leq M \left( \int_\mathbb{B} |f(z)|^p \omega(z) dv(z) \right)^\frac{q}{2}.
\]
So, we get $\|Id\|_{A^p_\alpha \rightarrow A^p_\alpha} \leq M$. Then we have proved the statement (i).

Suppose that $\mu$ is a vanishing $q$-Carleson measure for $A^p_\alpha$. Let
\[
f_{a,p}(z) = \left( \frac{1 - |a|^2}{1 - \langle z, a \rangle} \right)^\frac{q}{p} \frac{1}{(\omega(S_a))^\frac{q}{p}}
\]
for some $\gamma$ is large enough. By Lemmas 2 and 6, $f_{a,p}$ is bounded in $A^p_\alpha$ and converges to 0 uniformly on compact subset of $\mathbb{B}$ as $|a| \rightarrow 1$. By Lemma 6, we have
\[
\lim_{|a| \rightarrow 1} \|f_{a,p}\|_{L_\omega^q} = 0.
\]
Since
\[
\|f_{a,p}\|^q_{L_\omega^q} \geq \int_{S_a} |f(z)|^q d\mu(z) \approx \frac{\mu(S_a)}{(\omega(S_a))^\frac{q}{p}}
\]
we have
\[
\lim_{|a| \rightarrow 1} \frac{\mu(S_a)}{(\omega(S_a))^\frac{q}{p}} = 0.
\]
Suppose $\lim_{|a| \rightarrow 1} \frac{\mu(S_a)}{(\omega(S_a))^\frac{q}{p}} = 0$. We also exist $r = r(\epsilon) \in (0, 1)$ such that when $|a| > r$, $\frac{\mu(S_a)}{(\omega(S_a))^\frac{q}{p}} < \epsilon$. Let $d\mu_r(z) = \chi_{r \leq |z| < 1} d\mu(z)$. 

If \(|a| \geq r\), \(\mu_r(S_a) = \mu(S_a)\). Then suppose \(0 < |a| < r\). Since \(\sigma(\mathbb{S}) < \infty\) and
\[
\sigma(Q(\xi, \sqrt{1 - r}/2)) \approx (1 - r)^n > 0
\]
for all \(\xi \in \mathbb{S}\), there are at most \(N\) elements \(\xi_1, \xi_2, \ldots, \xi_N\) in \(Q_a\) such that
(a) \(Q(\xi_i, \sqrt{1 - r}/2) \cap Q(\xi_j, \sqrt{1 - r}/2) = \emptyset\) for \(1 \leq i < j \leq N\);
(b) \(Q(\xi_i, \sqrt{1 - r}/2) \subset Q(|a|/2, \sqrt{1 - |a|} + \sqrt{1 - r}/2)\) for all \(1 \leq i \leq N\).

Then we have
\[
Q_a \subset \bigcup_{i=1}^N Q(\xi_i, \sqrt{1 - r}). \tag{16}
\]
Otherwise, there is a \(\eta \in Q_a\) but \(\eta \notin \bigcup_{i=1}^N Q(\xi_i, \sqrt{1 - r})\). For any \(\tau \in Q(\eta, \sqrt{1 - r}/2)\), we have
\[
d(\tau, \xi_i) \geq d(\eta, \xi_i) - d(\eta, \tau) > \frac{\sqrt{1 - r}}{2},
\]
and
\[
d(\tau, \frac{a}{|a|}) \leq d(\tau, \eta) + d(\eta, \frac{a}{|a|}) < \frac{\sqrt{1 - r}}{2} + \sqrt{1 - |a|}.
\]
This is a contradiction with \(N\) is the maximum number. By Lemma 1, we have
\[
N \lesssim \left( \frac{\sqrt{1 - |a|} + \sqrt{1 - r}/2}{\sqrt{1 - r}/2} \right)^{2n} \approx \left( \frac{1 - |a|}{1 - r} \right)^n.
\]
By (16), we have
\[
E_a := \{ z \in S_a : r < |z| < 1 \} \subset \bigcup_{k=1}^N \left\{ z \in \mathbb{B} : r < |z| < 1, \frac{z}{|z|} \in Q(\xi_i, \sqrt{1 - r}) \right\}.
\]
Since
\[
\left\{ z \in \mathbb{B} : r < |z| < 1, \frac{z}{|z|} \in Q(\xi_i, \sqrt{1 - r}) \right\} = S_{\tau \xi_i},
\]
by Lemma 2, we have
\[
\mu_r(S_a) = \mu(E_a) \leq \sum_{i=1}^N \mu(S_{\tau \xi_i}) \leq \varepsilon \sum_{i=1}^N \left( \omega(S_{\tau \xi_i}) \right)^{\frac{q}{p}}
\approx N \varepsilon (1 - r)^{\frac{nq}{p}} \left( \int_r^1 \omega(t) \frac{dt}{t} \right)^{\frac{q}{p}}
\]
and
\[
\frac{\mu_r(S_a)}{(\omega(S_a))^{\frac{q}{p}}} \leq \varepsilon \left( \frac{1 - r}{1 - |a|} \right)^{\frac{nq}{p}} \left( \int_r^1 \omega(t) \frac{dt}{\omega(t)} \right)^{\frac{q}{p}} \leq \varepsilon. \tag{17}
\]
Then, \(\|Id\|_{A^p_{\omega} \rightarrow A^p_{\omega}}^{q} \lesssim \varepsilon\).
By Theorem 1 and Lemma 5, we have

\[ \limsup_{k \to \infty} ||f_k||_{A^q_\mu}^q = \limsup_{k \to \infty} \left( \int_B |f_k(z)|^q d\mu(z) + \int_B |f_k(z)|^q d\mu_\nu(z) \right) \]

\[ = \limsup_{k \to \infty} ||f_k||_{A^q_\mu}^q \leq \varepsilon \limsup_{k \to \infty} ||f_k||_{A^q_\mu}^q. \]

Since \( \varepsilon \) is arbitrary and \( \sup_{k \to \infty} ||f_k||_{A^q_\mu} < \infty \), \( \lim_{k \to \infty} ||f_k||_{L^q_\mu} = 0 \). So, \( \mu \) is a vanishing \( q \)-Carleson measure for \( A^q_\mu \). The proof is complete.

As a by-product of the proof of Theorem 1, we have the following result which is of independent interest.

**Corollary 1.** Let \( 0 < p \leq q < \infty \) and \( 0 < \alpha < \infty \) such that \( p\alpha > 1 \). Let \( \mu \) be a positive Borel measure on \( \mathbb{B} \) and \( \omega \in \mathcal{D} \). Then \( [M_\omega(\cdot)^\alpha] : L^p_\mu \to L^q_\mu \) is bounded if and only if \( (5) \) holds. Moreover,

\[ \|[M_\omega(\cdot)^\alpha]_q\|_{L^p_\mu \to L^q_\mu} \approx \sup_{a \in \mathbb{B}} \frac{\mu(S_a)}{(\omega(S_a))^\frac{1}{\alpha}}. \]

**Proof.** By \( (14) \) and \( (15) \), we obtain

\[ \|[M_\omega(\cdot)^\alpha]_q\|_{L^p_\mu \to L^q_\mu} \leq \sup_{a \in \mathbb{B}} \frac{\mu(S_a)}{(\omega(S_a))^\frac{1}{\alpha}}. \]

By Theorem 1 and Lemma 5, we have

\[ \sup_{a \in \mathbb{B}} \frac{\mu(S_a)}{(\omega(S_a))^\frac{1}{\alpha}} \approx \sup_{f \in A^q_\mu} \frac{||f||_{L^q_\mu}}{||f||_{A^q_\mu}} \approx \sup_{f \in A^q_\mu} \frac{||[M_\omega(\cdot)^\alpha]_q\|_{L^p_\mu \to L^q_\mu}}{||f||_{L^p_\mu}} \]

\[ \leq \|[M_\omega(\cdot)^\alpha]_q\|_{L^p_\mu \to L^q_\mu}. \]

The proof is complete.

## 4. Equivalent norms for \( A^p_\mu \) space

In this section, we give some equivalent norms for the space \( A^p_\mu \) on the unit ball. These norms are inherited from different equivalent \( H^p \) norms. First, we give some notations.

Let \( \alpha > 1 \). The admissible approach region \( \Gamma_{\zeta,\alpha} \) (or \( \Gamma_\zeta \) for simply) for some \( \zeta \in \overline{\mathbb{B}} \setminus \{0\} \) can be defined as

\[ \Gamma_\zeta = \Gamma_{\zeta,\alpha} = \left\{ z \in \mathbb{B} : \left| 1 - \frac{\langle z, \zeta \rangle}{|\zeta|^2} \right| < \frac{\alpha}{2} \left( 1 - \frac{|z|^2}{|\zeta|^2} \right) \right\}. \]

When \( \zeta = 0 \), let \( \Gamma_\zeta = \{0\} \). Obviously, if \( r > 0 \) and \( r\zeta, \zeta \in \overline{\mathbb{B}} \), \( z \in \Gamma_\zeta \) if and only if \( rz \in \Gamma_{r\zeta} \). We define

\[ T_\zeta = T_{\zeta,\alpha} = \{ \zeta \in \overline{\mathbb{B}} : z \in \Gamma_\zeta \}. \]

We note that, in this paper, \( T_\zeta = T_{\zeta,\alpha} \) for some \( \alpha > 1 \) and \( S_\zeta = S_{\zeta,0} \).
It follows from Fubini’s Theorem, for a positive function \( \varphi \) and a finite positive measure \( \mu \), one has
\[
\int_B \varphi(z) d\mu(z) \approx \int_S \left( \int_{\Gamma_\eta} \varphi(z) \frac{d\mu(z)}{(1 - |z|^2)^\rho} \right) d\sigma(\eta).
\] (18)
See [9], for example. This fact will be used frequently in this paper.

**Proposition 3.** Suppose \( \alpha > 1 \) is fixed and \( \omega \in \hat{D} \). Then

(i) \( T_z \subset S_{z, \alpha} \).

(ii) If \( \alpha > 2 \), there exist a \( r = r(\alpha) \) and \( \beta > -1 \), such that \( S_{\frac{1+|z|}{2}, \beta} \subset T_z \) for all \( |z| > r \). If we restrict \( \alpha > 3 \), we can get such a \( \beta > 0 \).

(iii) If \( \alpha > 2 \), \( \omega(T_z) \approx \omega(S_z) \).

**Proof.** Suppose \( \zeta \in T_z \), that is,
\[
\left| 1 - \langle z, \frac{\zeta}{|\zeta|^2} \rangle \right| < \frac{\alpha}{2} \left( 1 - \frac{|z|^2}{|\zeta|^2} \right).
\]
So, we have \( |\zeta| \geq |z| \) and
\[
\left| 1 - \langle \frac{\zeta}{|\zeta|^2}, \frac{z}{|z|^2} \rangle \right| \leq \left| 1 - \langle z, \frac{\zeta}{|\zeta|^2} \rangle \right| + \left| \langle z, \frac{\zeta}{|\zeta|^2} \rangle - \langle z, \frac{\zeta}{|\zeta|^2} \rangle \right|
\]
\[
\leq (\alpha + 1) (1 - |z|).
\]
Therefore, \( \zeta \in S_{z, \alpha} \), i.e. \( T_z \subset S_{z, \alpha} \).

Suppose \( \zeta \in S_{\frac{1+|z|}{2}, \beta} \). Then we have \( |\zeta| > \frac{1+|z|}{2} \) and
\[
\left| 1 - \langle z, \frac{\zeta}{|\zeta|^2} \rangle \right| \leq \left| 1 - \langle \frac{z}{|z|^2}, \frac{\zeta}{|\zeta|^2} \rangle \right| + \left| \langle \frac{z}{|z|^2}, \frac{\zeta}{|\zeta|^2} \rangle - \langle z, \frac{\zeta}{|\zeta|^2} \rangle \right|
\]
\[
\leq \frac{\beta + 1}{2} (1 - |z|) + \left( 1 - \frac{|z|^2}{|\zeta|^2} \right)
\]
\[
\leq \left( \frac{\beta + 1 + |z|}{2} + 1 \right) \left( 1 - \frac{|z|^2}{|\zeta|^2} \right).
\]
Since \( 1 - \frac{|z|^2}{|\zeta|^2} > \frac{(1-|z|^2)(3|z|^2+1)}{(1+|z|^2)^2} \) and \( \lim_{|r| \to 1} \frac{(1+|z|^2)}{3|z|^2+1} = 1 \), (ii) holds.

By (i), (ii), Lemma[2] and Proposition[1] (iii) holds. The proof is complete. \( \square \)

In the following of this paper, for all \( z \in \mathbb{B} \) and \( f \in H(\mathbb{B}) \), if \( \omega \in \hat{D} \), let
\[
\omega^n(z) = \int_{|r|} r^{2n-1} \log \frac{r}{|z|} \omega(r) dr, \quad \text{and} \quad \Re f(z) = \langle \nabla f(z), \overline{z} \rangle,
\]
where \( \nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z) \right) \). \( \Re f \) and \( \nabla f \) are called the radial derivative and the complex gradient of \( f \), respectively.

The main result in this section is the following theorem.
Theorem 2. Let $0 < p < \infty$ and $\omega$ be a radial weight. Then

$$
\|f - f(0)\|_{A^p_{\omega}^0}^p = p^2 \int_{B} |\mathcal{R} f(z)|^2 \frac{|f(z) - f(0)|^{p-2}}{|z|^{2n}} \omega^n(z) dv(z)
$$

(19)

$$
\approx \int_{B} \left(\int_{F_u} |\mathcal{R} f(\xi)|^2 \left(1 - \frac{1}{|u|} \right)^{1-n} d\xi\right)^{\frac{p}{2}} \omega(u) dv(u).
$$

(20)

Moreover, if $p \geq 2$ and $\omega \in \mathcal{D}$,

$$
\|f - f(0)\|_{A^p_{\omega}^0}^p \approx \int_{B} |\mathcal{R} f(z)|^2 |f(z) - f(0)|^{p-2} \omega^*(z) dv(z).
$$

(21)

Proof. For $0 < r < 1$, let $f_r(z) = f(rz)$. By Theorem 4.22 in [19], we have

$$
\|f - f(0)\|_{H^p}^p = \frac{p}{2n} \int_{B} |\mathcal{R} f(rz)|^2 |f(z) - f(0)|^{p-2} |z|^{-2n} \log \frac{1}{|z|} dv(z).
$$

In the following, we always suppose $f(0) = 0$. Then Fubini's Theorem yields

$$
\|f\|_{A^p_{\omega}^0}^p = 2n \int_0^1 \|f\|_{H^p}^p \omega(r) r^{2n-1} dr
$$

$$
= p^2 \int_0^1 \left(\int_{B} |\mathcal{R} f(rz)|^2 |f(rz)|^{p-2} |z|^{-2n} \log \frac{1}{|z|} dv(z)\right) \omega(r) r^{2n-1} dr
$$

$$
= 2np^2 \int_0^1 \left(\int_{B} |\mathcal{R} f(rz)|^2 |f(rz)|^{p-2} |s|^{-2n} \log \frac{1}{s} ds\right) \omega(r) r^{2n-1} dr
$$

$$
= 2np^2 \int_0^1 \left(\int_{B} |\mathcal{R} f(tz)|^2 |f(tz)|^{p-2} t^{-1} \log \frac{t}{r} dt\right) \omega(r) r^{2n-1} dr
$$

$$
= 2np^2 \int_0^1 \left(\int_{B} |\mathcal{R} f(tz)|^2 |f(tz)|^{p-2} t^{-1} d\sigma(z) dt\right) \omega(r) r^{2n-1} dr
$$

$$
= p^2 \int_{B} \frac{|\mathcal{R} f(z)|^2 |f(z)|^{p-2}}{|z|^{2n}} \omega^n(z) dv(z)
$$

$$
\approx p^2 \int_{B} |\mathcal{R} f(z)|^2 |f(z)|^{p-2} \omega^*(z) dv(z).
$$

Hence (19) holds.

Suppose $p \geq 2$. By Theorem 4.17 in [19], we have

$$
|f(z)| \leq \frac{\|f\|_{H^p}}{(1 - |z|^2)^{\frac{p}{2}}}, \text{ for all } p > 0.
$$

So, for all $|z| < \frac{3}{4}$, we have

$$
|f(z)|^p = \left| f_z \left( \frac{5}{4} z \right) \right|^p \leq \left| f_z \right|_{H^p}^p \leq \frac{\int_{\frac{1}{5}}^1 \|f_z\|_{H^p}^p |r|^{2n-1} \omega(r) dr}{\int_{\frac{1}{5}}^1 |r|^{2n-1} \omega(r) dr} \leq \|f\|_{A^p_{\omega}^0}^p.
$$

Let $z = (z_1, z_2, \cdots, z_n) \in B$. By Cauchy Formula, we obtain

$$
\left| \frac{\partial f}{\partial z_i}(z) \right| \leq \|f\|_{A^p_{\omega}^0}.
$$
when $|z| < \frac{1}{2}$ and $i = 1, 2, \cdots , n$. So, we have

$$|\Re f(z)| = |\langle \nabla f(z), \overline{z} \rangle| \leq |z|\|f\|_{A^p_\omega^*}, \text{ when } |z| \leq \frac{1}{2}.$$ 

For all $\tau < \frac{1}{2}$, we have

$$\|f\|_{A^p_\omega^*} \lesssim \int_{\frac{1}{2}}^1 \|f\|_{H_p}^{2n-1} \omega(r)dr$$

$$\approx \int_{\frac{1}{2}}^1 \left( \int_{0}^{\tau} + \int_{\tau}^{\frac{1}{2}} \right) \left( \int_{S} |\Re f(rsz)|^2 |f(rsz)|^{p-2} s^{-1} \log \frac{1}{s} d\sigma(z)ds \right) \omega(r)r^{2n-1}dr$$

$$\lesssim \int_{\frac{1}{2}}^1 \left( \int_{\tau}^{\frac{1}{2}} \int_{S} |\Re f(rsz)|^2 |f(rsz)|^{p-2} s^{-1} \log \frac{1}{s} d\sigma(z)ds \right) \omega(r)r^{2n-1}dr$$

$$+ \|f\|_{A^p_\omega^*} \int_{0}^{\tau} s \log \frac{1}{s} ds.$$ 

Since $\lim_{\tau \to 0} \int_{\tau}^{\frac{1}{2}} s \log \frac{1}{s} ds = 0$, we can choose a fixed $0 < \tau < \frac{1}{2}$ such that

$$\|f\|_{A^p_\omega^*} \lesssim \int_{\frac{1}{2}}^1 \left( \int_{\tau}^{\frac{1}{2}} \int_{S} |\Re f(rsz)|^2 |f(rsz)|^{p-2} s^{-1} \log \frac{1}{s} d\sigma(z)ds \right) \omega(r)r^{2n-1}dr.$$ 

By Fubini’s Theorem, we have

$$\|f\|_{A^p_\omega^*} \lesssim \int_{\frac{1}{2}}^1 \left( \int_{\tau}^{\frac{1}{2}} \int_{S} |\Re f(tz)|^2 |f(tz)|^{p-2} t^{-1} \log \frac{r}{t} d\sigma(z)dt \right) \omega(r)r^{2n-1}dr$$

$$\lesssim \int_{\frac{1}{2}}^1 \int_{\tau}^{\frac{1}{2}} \left( \int_{S} r^{2n-1} \log \frac{r}{t} \omega(r)dr \right) \left| \Re f(tz) \right|^2 |f(tz)|^{p-2} t^{-1} d\sigma(z)dt$$

$$\approx \int_{\mathbb{C} \setminus \frac{\mathbb{C}}{\|z\|^{2n}}} \frac{|\Re f(z)|^2 |f(z)|^{p-2} |z|^{2n}}{\omega^* (z)} dv(z)$$

$$\lesssim \int_{\mathbb{C}} |\Re f(z)|^2 |f(z)|^{p-2} \omega^* (z) dv(z).$$

So, we get (21).

By Theorem B in [9], if $f(0) = 0$, we have

$$\|f\|_{H_p}^{p} \approx \int_{S} \left( \int_{\mathbb{C} \setminus \|z\|} |\Re f(z)|^2 (1 - |z|^2)^{-n} dv(z) \right)^{\frac{p}{2}} d\sigma(\zeta). \quad (22)$$
By Fubini’s Theorem, we have
\[
\|f\|_{A^p_ω}^p = 2n \int_0^1 \|f_i\|_{H_p, ω}^p r^{2n-1} dr
\]
\[
\leq 2n \int_0^1 \|f_i\|_{H_p, ω}^p r^{2n-1} dr.
\]
\[
\approx \int_{B \setminus \frac{1}{2}B} \left( \int_{Γ_u} |K f(ξ)|^2 \left( 1 - \frac{|ξ|^2}{|u|^2} \right)^{1-n} dν(ξ) \right)^\frac{p}{2} ω(u)|u|^{-np} dν(u)
\]
\[
\leq \int_{B \setminus \frac{1}{2}B} \left( \int_{Γ_u} |K f(ξ)|^2 \left( 1 - \frac{|ξ|^2}{|u|^2} \right)^{1-n} dν(ξ) \right)^\frac{p}{2} ω(u) dν(u).
\]
Similarly, by using the monotonicity of \(\|f_i\|_{H_p}\) and
\[
\int_0^\frac{1}{2} r^{2n-1} ω(r) dr \approx \int_0^1 r^{2n-1} ω(r) dr,
\]
we have
\[
\|f\|_{A^p_ω}^p = 2n \int_0^1 \|f_i\|_{H_p, ω}^p r^{2n-1} dr
\]
\[
\leq 2n \int_0^1 \|f_i\|_{H_p, ω}^p r^{2n-1} dr.
\]
\[
\approx \int_{B \setminus \frac{1}{2}B} \left( \int_{Γ_u} |K f(ξ)|^2 \left( 1 - \frac{|ξ|^2}{|u|^2} \right)^{1-n} dν(ξ) \right)^\frac{p}{2} ω(u)|u|^{-np} dν(u)
\]
\[
\leq \int_{B \setminus \frac{1}{2}B} \left( \int_{Γ_u} |K f(ξ)|^2 \left( 1 - \frac{|ξ|^2}{|u|^2} \right)^{1-n} dν(ξ) \right)^\frac{p}{2} ω(u) dν(u).
\]
Then, (20) holds. The proof is complete. \(\square\)

Suppose \(α > 1\), for any \(f \in H(\mathbb{B})\), let
\[
N(f)(u) = \sup_{z \in Γ_u} \{|f(z)|, u \in \mathbb{B} \setminus \{0\}\}.
\]
Then we have the following theorem.

**Theorem 3.** Let \(0 < p < \infty\) and \(ω\) be a radial weight. Then for all \(f \in H(\mathbb{B})\),
\[
\|f\|_{A^p_ω}^p \leq \|N(f)\|_{L^p(\mathbb{S})}^p \leq \|f\|_{A^p_ω}^p.
\]

**Proof.** By Theorem A in [9], we have \(\|N(f)\|_{L^p(\mathbb{S})} \leq \|f\|_{H_p}^p\). For any \(u \in \mathbb{B} \setminus \{0\}\), let \(u = rξ\) for \(r = |u|\) and \(ξ \in \mathbb{S}\). Then
\[
Γ_u = \left\{ z \in \mathbb{B}^n : 1 - \langle z, \frac{u}{|u|^2} \rangle \leq \frac{α}{2} \left( 1 - \frac{|ξ|^2}{|u|^2} \right) \right\}
\]
\[
= \left\{ \frac{z}{r} \in \mathbb{B}^n : 1 - \langle \frac{z}{r}, ξ \rangle \leq \frac{α}{2} \left( 1 - \frac{|ξ|^2}{r^2} \right) \right\},
\]
and
\[ N(f)(u) = \sup_{r \in \mathcal{T}} \left\{|f(r\frac{z}{r})|\right\} = N(f)(\xi). \]

Therefore,
\[ \|N(f)\|_{L_p^q}^p = 2n \int_0^1 \|N(f_r)\|_{L_p^q}^p r^{2n-1} \omega(r) dr \leq 2n \int_0^1 \|f_r\|_{L_p^q}^p r^{2n-1} \omega(r) dr = \|\omega\|_{A_q^p}. \]

The fact that \( \|f\|_{A_q^p} \leq \|N(f)\|_{L_p^q}^p \) is obvious. The proof is complete. \( \Box \)

5. VOLterra INTEGRAL OPERATOR FROM \( A_{\omega}^p \) TO \( A_{\omega}^q \)

In this section, we will describe the boundedness and compactness of \( T_g : A_{\omega}^p \to A_{\omega}^q \). For this purpose, we first introduce some new function spaces.

Let \( 0 < p \leq q < \infty \), \( g \in H(\mathbb{B}) \) and \( \omega \in \dot{D} \). We say that \( g \) belongs to \( C_{q,p}(\omega^*) \), if the measure \(|\mathcal{R}g(z)|^2 \omega^*(z)dv(z)\) is a \( q \)-Carleson measure for \( A_{a}^p \). \( g \in C_{0}^{q,p}(\omega^*) \) if \(|\mathcal{R}g(z)|^2 \omega^*(z)dv(z)\) is a vanishing \( q \)-Carleson measure for \( A_{a}^p \).

If \( 0 < p \leq q < \infty \), Theorem 1 shows that \( C_{q,p}(\omega^*) \) depends only on \( \frac{q}{p} \). Consequently, for \( 0 < p \leq q < \infty \), we will write \( C_{\kappa}(\omega^*) \) instead of \( C_{q,p}(\omega^*) \) where \( \kappa = \frac{q}{p} \). Similarly, we can define \( C_{0}^{\kappa}(\omega^*) \). Thus, if \( \kappa \geq 1 \), \( C_{\kappa}(\omega^*) \) consists of those \( g \in H(\mathbb{B}) \) such that
\[ \|g\|_{C^{\kappa}(\omega^*)} = |g(0)| + \sup_{a \in \mathbb{B}} \int_{S_a} |\mathcal{R}g(z)|^2 \omega^*(z)dv(z) (\omega(S_a))^\kappa < \infty. \]

Before state and prove the main results in this section, we state some lemmas which will be used.

**Lemma 7.** Let \( 0 < p, q < \infty \), \( g \in H(\mathbb{B}) \) and \( \omega \in \dot{D} \).

(i) If \( T_g : A_{\omega}^p \to A_{\omega}^q \) is bounded, then
\[ M_{\omega}(r, \mathcal{R}g) \leq \frac{\omega_{p}^{-\frac{1}{q}}(S_r)}{1 - r}, \quad 0 < r < 1. \]

(ii) If \( T_g : A_{\omega}^p \to A_{\omega}^q \) is compact, then
\[ M_{\omega}(r, \mathcal{R}g) = o \left( \frac{\omega_{p}^{-\frac{1}{q}}(S_r)}{1 - r} \right), \quad r \to 1. \]

Here, \( S_r \) means any Carleson block \( S_a \) with \( |a| = r \).

**Proof.** Assume \( T_g : A_{\omega}^p \to A_{\omega}^q \) is bounded. Let
\[ f_{a,p}(z) = \frac{F_{a,p}(z)}{(\omega(S_a))^{\frac{1}{q}}}, \quad (1 - |a|^2)^{\frac{n+\gamma}{p}} \frac{1}{(\omega(S_a))^{\frac{n}{2}}} \]
for some $\gamma$ which is large enough such that Lemma 6 holds. For all $\frac{1}{2} < r < 1$ and $h \in A^{q}_\omega$, we have

$$\|h\|_{A^{q}_\omega}^q \geq \int_{B_\gamma \cap B} |h(z)|^q |\omega(z)|dv(z)$$

$$\geq 2nM^q_\omega(r,h) \int_r^1 r^{2n-1} |\omega(r)|dr \approx \hat{\omega}(r)M^q_\omega(r,h).$$

Then, when $\frac{1}{2} < r < 1$, for all $a \in B$, by Lemma 6, we have

$$M^q_\omega(r,T_gf_{a,p}) \leq \frac{\|T_gf_{a,p}\|_{A^{q}_\omega}}{\hat{\omega}(r)} \leq \frac{\|T_g\|_{A^{q}_\omega \to A^{q}_\omega} \|f_{a,p}\|_{A^{q}_\omega}}{\hat{\omega}(r)} \leq \frac{1}{\hat{\omega}(r)}. \quad (23)$$

By Theorem 4.17 in [19], we have

$$|f(z)| \leq \frac{\|f\|_{H^q}}{(1 - |z|^2)^{\frac{q}{2}}}. \tag{24}$$

Letting $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in B$ and $i = 1, 2, \cdots, n$, by Cauchy Formula, we obtain

$$\left| \frac{\partial f}{\partial \xi_i}(\xi) \right| = \frac{1}{2\pi} \left| \int_{|\eta - \xi| = \frac{1}{|Z|}} \frac{f(z(\xi, i, \eta))}{(\eta - \xi_i)^2} d\eta \right| \leq \frac{\|f\|_{H^q}}{(1 - |\xi|^2)^{\frac{q}{2} + 1}},$$

here, $z(\xi, i, \eta) = (\xi_1, \cdots, \xi_{i-1}, \xi_i, \xi_{i+1}, \cdots, \xi_n)$. So, we have

$$|\Re f(z)| \leq \frac{\|f\|_{H^q}}{(1 - |z|^2)^{\frac{q}{2} + 1}}.$$

Letting $f_r(z) = f(rz)$ for all $0 < r < 1$, when $|a| > \frac{1}{2}$, by (23), we have

$$\left| \frac{\Re g(a)}{(\omega(S_\omega))^\frac{1}{2}} \right| = \left| \Re (T_gf_{a,p})(a) \right| = \left| \Re ((T_gf_{a,p})_{\omega^{\frac{1}{2}}})(\frac{2a}{1 + |a|}) \right| \leq \frac{\|((T_gf_{a,p})_{\omega^{\frac{1}{2}}})_{H^q}\|_{H^q}}{(1 - |a|)^{\frac{q}{2} + 1}} \approx \frac{M^q_{\omega^{\frac{1}{2}}}(T_gf_{a,p})}{(1 - |a|)^{\frac{q}{2} + 1} \hat{\omega}(\frac{1}{2})} \approx \frac{1}{(1 - |a|)^{\frac{q}{2} + 1} \hat{\omega}^{\frac{1}{2}}(a)}. \tag{25}$$

By Lemma 4, we have

$$|\Re g(a)| \leq \frac{\omega^{\frac{q}{2} - 1}(S_\omega)}{1 - |a|},$$

which implies the desired result.

(ii) Assume that $T_g : A^{q}_\omega \to A^{q}_\omega$ is compact. By Lemma 6, $\{f_{a,p}\}$ is bounded and converges to 0 uniformly on compact subset of $B$. By Lemma 4,

$$\lim_{|a| \to 1} \|T_gf_{a,p}\|_{A^{q}_\omega} = 0.$$
By (23), for any given \( \varepsilon > 0 \), there exists a \( r_{0} \in (0, 1) \), such that when \( |a| > r_{0} \),

\[
M_{y}^{2}(r, T_{g}f_{a,p}) \leq \frac{\varepsilon}{\omega(r)}.
\]

Then by repeating the proof of (i), we can prove (ii). The proof is complete. \( \square \)

**Lemma 8.** Let \( 0 < \kappa < \infty, \omega \in \mathcal{D} \) and \( g \in H(\mathbb{B}) \). Then the following statements hold.

(i) \( g \in C^{2k+1}(\omega^{*}) \) if and only if

\[
M_{\infty}(r, Rg) \lesssim \frac{\omega^{k}(S_{\omega})}{1 - r}, \quad 0 < |a| = r < 1.
\]

(ii) \( g \in C_{0}^{2k+1}(\omega^{*}) \) if and only if

\[
M_{\infty}(r, Rg) = o \left( \frac{\omega^{k}(S_{\omega})}{1 - r} \right), \quad r \to 1.
\]

**Proof.** Let \( r_{0} > 0 \) be fixed and \( D(a, r_{0}) \) be the Bergman metric ball at \( a \) with radius \( r_{0} \). By Lemma 2.20 in [19], there exists \( B = B(r_{0}) > 1 \) such that, for all \( z \in D(a, r_{0}) \),

\[
B^{-1} < 1 - |a| \quad \text{B^{-1}} < \frac{1 - |a|}{1 - |a, z|} < B.
\]

When \( |a| > \max \{ \frac{B - 1}{B}, \frac{2B}{2B + 1} \} = \frac{2B}{2B + 1} \), let

\[
\beta_{3}(a) = \frac{a - (2B + 1)(1 - |a|)a}{|a|} \in \mathbb{B}.
\]

Then, \( 1 - |\beta_{3}(a)| = (2B + 1)(1 - |a|) \). For all \( z \in D(a, r_{0}) \), we have

\[
|z| > 1 + B|a| - B > |\beta_{3}(a)|
\]

and

\[
\left| 1 - \left( \frac{\beta_{3}(a)}{\beta_{3}(a)} \right), \frac{z}{|z|} \right| \leq |1 - \langle a, z \rangle| + |\langle a, z \rangle - \langle a, \frac{z}{|z|} \rangle| + |\langle a, \frac{z}{|z|} \rangle - \langle a, \frac{z}{|z|} \rangle| \\
\leq B(1 - |a|) + |a|(1 - |z|) + (1 - |a|) \\
\leq (2B + 1)(1 - |a|) = 1 - |\beta_{3}(a)|.
\]

Therefore, \( D(a, r_{0}) \subset S_{\beta_{3}(a)} \) for all \( |a| > \max \{ \frac{B - 1}{B}, \frac{2B}{2B + 1} \} \).

Assume that \( g \in C^{2k+1}(\omega^{*}) \). It is enough to prove (25) holds for \( |a| > \frac{2B}{2B + 1} \). By Lemma [2] we have \( \omega^{*} \in \mathcal{R} \) and

\[
\omega^{*}(t) \approx \omega^{*}(s), \quad \text{if} \quad 1 - t \approx 1 - s \quad \text{and} \quad s, t \in (0, 1).
\]
By Lemmas 1, 2 and Lemma 1.7 in [11], there is a $C$ such that

\[(1-|a|^2)^{\gamma+1} \omega^*(a) |\Re g(a)|^2 \leq C(r_0) \omega^*(a) \int_{D(a,r_0)} |\Re g(z)|^2 dv(z) \]

\[\leq C \int_{D(a,r_0)} |\Re g(z)|^2 \omega^*(z) dv(z) \]

\[\leq C \int_{S_{\beta_j(a)}} |\Re g(z)|^2 \omega^*(z) dv(z) \]

\[\leq C \omega^{2\kappa+1}(S_{\beta_j(a)}) \leq C \omega^{2\kappa+1}(S_a). \]

Hence, there exists $C = C(k, r_0, \omega)$, such that

\[|\Re g(a)| \leq C\|g\|_{C^{2\kappa+1}(\omega^*)} \frac{\omega^*(S_a)}{1-|a|}, \quad \text{when } |z| \geq \frac{2B}{2B+1}. \]

Therefore, (25) holds.

Conversely, suppose that

\[M := \sup_{0<|d|<r<1} \frac{(1-r)M_\alpha(r, \Re g)}{\omega^*(S_a)} < \infty. \]

By Lemmas 1, 2 and Lemma 1.7 in [11], we have

\[\int_{S_a} |\Re g(z)|^2 \omega^*(z) dv(z) = 2n \int_{|d|} \int_{Q_d} |\Re g(r\xi)|^2 \omega^*(r) r^{n-1} d\sigma(\xi) dr \]

\[\leq M \int_{|d|} \int_{Q_d} \frac{\omega^{2\kappa}(S_r)}{(1-r)^2} \omega^*(r) r^{n-1} d\sigma(\xi) dr \]

\[\approx M(1-|a|)^n \int_{|d|} \frac{\omega^{2\kappa}(S_r)}{(1-r)^2} \omega^*(r) dr \]

\[\approx M \omega^{2\kappa+1}(S_a). \]

It follows that $g \in C^{2\kappa+1}(\omega^*)$.

The assertion (ii) can be proved by modifying the above proof in a standard way and we omit the details. The proof is complete. \qed

**Theorem 4.** Let $0 < p \leq q < \infty$, $\omega \in \hat{D}$, $\kappa = \frac{1}{p} - \frac{1}{q}$ and $g \in H(\mathbb{B})$.

(i) If $nk \geq 1$, then $T_g : A^p_\omega \to A^q_\omega$ is bounded if and only if $g$ is constant.

(ii) If $nk < 1$, then the following conditions are equivalent:

\[ T_g : A^p_\omega \to A^q_\omega \text{ is bounded; } \]

\[ M_\alpha(r, \Re g) \leq \frac{\omega^*(S_r)}{1-r} \]

\[ g \in C^{2\kappa+1}(\omega^*). \]

(iii) The following conditions are equivalent.

\[ T_g : A^p_\omega \to A^q_\omega \text{ is bounded; } \]

\[ g \in C^1(\omega^*). \]

**Proof.** By Lemmas 2, 7 and 8 we see that (i) holds, and (iiia) $\Rightarrow$ (iib) $\Leftrightarrow$ (iic). Let $d\mu_\omega(z) = |\Re g(z)|^2 \omega^*(z) dv(z)$. First, we prove the statement (ii).
Suppose that (iiic) holds and \( q = 2 \). Then \( d\mu_\kappa \) is a 2-Carleson measure for \( A^p_\omega \). By using (21) and Theorem 1, we have

\[
||T_\kappa f||_{A^p_\omega}^2 \approx \int_{\mathbb{B}} |f(z)|^2 |\Re g(z)|^2 \omega^*(z) dv(z) \lesssim ||f||^2_{A^p_\omega}.
\]

So, \( T_\kappa : A^p_\omega \to A^2_\omega \) is bounded.

Assume \( f \in H^\infty \). By Lemmas 2 and 7, \( f \in H^\infty \) and \( g \in H(\mathbb{B}) \) we get

\[
\sup_{0 < \frac{1}{4} < t \in \mathbb{B}} \frac{|\Re g(tz)|}{t} < \infty, \quad \int_0^\frac{1}{4} f(tz) \Re g(tz) \frac{dt}{t} \leq ||f||_{H^\infty} \int_0^\frac{1}{4} |\Re g(tz)| \frac{dt}{t} < \infty, \tag{26}
\]

and

\[
\left| \int_0^\frac{1}{4} f(tz) \Re g(tz) \frac{dt}{t} \right| \leq ||f||_{H^\infty} \int_0^\frac{1}{4} (1 - |t|^2)^{\alpha - 1} \Re^* (t|z|) dt < \infty. \tag{27}
\]

That is to say \( T_\kappa f \in H^\infty \).

Suppose that (iiic) holds and \( q > 2 \). Let \( \beta = \frac{(2q+1)\omega}{2q+2} \) and \( \beta' = \frac{(2q+1)\omega}{q-2} \). By (21) and Hölder inequality, we have

\[
||T_\kappa f||_{A^p_\omega}^q \approx \int_{\mathbb{B}} |f(z)|^{2q} |T_\kappa f(z)|^{q-2} |\Re g(z)|^2 \omega^*(z) dv(z)
\]

\[
\leq \left( \int_{\mathbb{B}} |f(z)|^{2q} d\mu_\kappa(z) \right)^{\frac{q}{2q-2}} \left( \int_{\mathbb{B}} |T_\kappa f(z)|^{q-2} d\mu_\kappa(z) \right)^{\frac{1}{q-2}}
\]

\[
\lesssim ||f||_{A^p_\omega}^2 ||T_\kappa f||^{q-2}_{A^{q-2}_\omega} = ||f||_{A^p_\omega}^2 ||T_\kappa f||_{A^q_\omega}^{q-2}.
\]

Therefore, when \( q > 2 \),

\[
||T_\kappa f||_{A^q_\omega} \leq ||f||_{A^p_\omega}, \quad \text{for all } f \in H^\infty. \tag{28}
\]

For all \( f \in A^p_\omega \), we have \( \lim_{r \to 1} ||f - f_r||_{A^p_\omega} = 0 \). For any fixed \( z \in \mathbb{B} \), by (26) and (27), there exists a \( C = C(g, \omega) \) such that

\[
\lim_{r \to 1} ||T_\kappa f(z) - T_\kappa f_r(z)|| \leq C \lim_{r \to 1} ||f - f_r||_{H^\infty} = 0.
\]

Then \( \lim_{r \to 1} T_\kappa f_r(z) = T_\kappa f(z) \) for all \( z \in \mathbb{B} \). At the same time, by (28), there is a \( F \in A^q_\omega \), such that \( \lim_{r \to 1} ||T_\kappa f_r - F||_{A^q_\omega} = 0 \). Therefore, \( F = T_\kappa f \) and

\[
||T_\kappa f||_{A^q_\omega} = \lim_{r \to 1} ||T_\kappa f_r||_{A^q_\omega} \leq \lim_{r \to 1} ||f||_{A^q_\omega} = ||f||_{A^q_\omega}.
\]

So, (iiic) deduce (iiia) when \( q > 2 \).
Suppose that (iiic) holds and \( q < 2 \). Let \( \tau = \frac{(2-q)p}{q} \). By (20), Hölder inequality and Theorem 3, we have

\[
\|T_g f\|_{A_{\omega}^p}^q \approx \int_B \left( \int_{\Gamma_u} |f(\xi)|\Re g(\xi)|^2 \left( 1 - \frac{|\xi|^2}{|u|^2} \right)^{1-n} \, d\nu(\xi) \right)^{\frac{q}{2}} \omega(u) \, dv(u)
\]

\[
\leq \int_B |N(f)(u)|^{\frac{q}{2}} \left( \int_{\Gamma_u} |f(\xi)|^{2-\tau} |\Re g(\xi)|^2 \left( 1 - \frac{|\xi|^2}{|u|^2} \right)^{1-n} \, d\nu(\xi) \right)^{\frac{q}{2}} \omega(u) \, dv(u)
\]

\[
\leq \|N(f)|^{\frac{2-q}{2}}_{A_{\omega}^p} J_1^{\frac{q}{2}} \leq \|f\|_{X_{\omega}^p}^{\frac{2-q}{2}} J_1^{\frac{q}{2}},
\]

where

\[
J_1 = \int_B \int_{\Gamma_u} |f(\xi)|^{2-\tau} |\Re g(\xi)|^2 \left( 1 - \frac{|\xi|^2}{|u|^2} \right)^{1-n} \, d\nu(\xi) \omega(u) \, dv(u).
\]

By (18), Fubini’s Theorem and \( \omega^*(t) \approx \int_0^t r^{2n-1} \omega(r)(1 - \frac{t}{r}) \, dr \), we have

\[
J_1 = 2n \int_0^1 r^{2n-1} \omega(r) \, dr \int_d \, d\sigma(\eta) \int_{\Gamma_u} |f(\xi)|^{2-\tau} |\Re g(\xi)|^2 \left( 1 - \frac{|\xi|^2}{|r|^2} \right)^{1-n} \, d\nu(\xi)
\]

\[
= 2n \int_0^1 r^{4n-1} \omega(r) \, dr \int_d \, d\sigma(\eta) \int_{\Gamma_u} |f(rz)|^{2-\tau} |\Re g(rz)|^2 (1 - |z|^2)^{-n} \, dv(z)
\]

\[
\approx \int_0^1 r^{4n-1} \omega(r) \, dr \int_B |f(rz)|^{2-\tau} |\Re g(rz)|^2 (1 - |z|^2) \, dv(z)
\]

\[
\approx \int_0^1 r^{4n-1} \omega(r) \, dr \int_0^{s^{2n-1}} \, ds \int_B |f(rsn\eta)|^{2-\tau} |\Re g(rsn\eta)|^2 (1 - s^2) \, d\sigma(\eta)
\]

\[
\approx \int_0^1 r^{2n-1} \omega(r) \, dr \int_0^{\tau} \, dt \int_B |f(t\eta)|^{2-\tau} |\Re g(t\eta)|^2 (1 - \frac{t}{\tau}) ^{-n} \, d\sigma(\eta)
\]

\[
\approx \int_0^1 t^{2n-1} \, dt \int_B |f(t\eta)|^{2-\tau} |\Re g(t\eta)|^2 \, d\sigma(\eta) \int_0^1 r^{2n-1} \omega(r)(1 - \frac{t}{\tau}) \, dr
\]

\[
\approx \int_B |f(\xi)|^{2-\tau} |\Re g(\xi)|^2 \omega^*(\xi) \, dv(\xi)
\]

(31)

\[
\leq \|f\|_{X_{\omega}^p}^{2-\tau} = \|f\|^2_{X_{\omega}^p}.
\]

So, \( \|T_g f\|_{A_{\omega}^p} \leq \|f\|_{X_{\omega}^p} \). Then we finish the proof of assertion (ii).

Next we prove the assertion (iii). When \( p = 2 \), (iiiia) \( \iff \) (iiib) is obvious. When \( p < 2 \), by the the proof of (iiic) \( \iff \) (iiia) when \( q < 2 \), we obtain (iiib) \( \iff \) (iiiia).

Suppose \( 2 < p \leq 4 \) and (iiib) holds. For all \( \|f\|_{X^{\omega}} < \infty \), by (21), we have

\[
\|T_g f\|^p_{A_{\omega}^p} \approx \int_B |T_g f(z)|^{p-2} |f(z)|^2 |\Re g(z)|^2 \omega^*(z) \, dv(z)
\]

\[
\leq \|f\|^2_{X^{\omega}} \int_B |T_g f(z)|^{p-2} |\Re g(z)|^2 \omega^*(z) \, dv(z).
\]
By \( g \in C^1(\omega^*) \), we have \( T_{\omega} \) is bounded on \( A_{\omega}^{p-2} \). Since \( f \in H^\infty \subset A_{\omega}^{p-2} \), \( T_{\omega}f \in A_{\omega}^{p-2} \). By Theorem 1, we have

\[
\int_{\mathbb{B}} |T_{\omega}g(z)|^2 \omega(z)dv(z) \leq \|T_{\omega}g\|_{A_{\omega}^{p-2}} \leq \|f\|_{A_{\omega}^{p-2}}^2.
\]

Then following the proof of (iic)\( \Rightarrow (iia) \) when \( q > 2 \), we obtain (iiib)\( \Rightarrow (iii a) \) when \( 2 < p \leq 4 \). Using mathematical induction, we have (iiib)\( \Rightarrow (iii a) \) when \( p > 2 \).

So, it remains to show that (iii a)\( \Rightarrow (iiib) \) when \( p \neq 2 \).

Suppose \( p > 2 \) and (iii a) holds. By the calculations from (30) to (31), Hölder inequality, Theorem 3 and (20), we have

\[
\begin{align*}
\int_{\mathbb{B}} |f(z)|^p |\Re g(z)|^2 \omega^*(z)dv(z) & \approx \int_{\mathbb{B}} \int_{\Gamma_u} |f(\xi)|^p |\Re g(\xi)|^2 \left(1 - \frac{|\xi|^2}{|u|^2}\right)^{1-n} dv(\xi)\omega(u)dv(u) \\
& \leq \int_{\mathbb{B}} |N(f)(u)|^{p-2} \int_{\Gamma_u} |f(\xi)|^2 |\Re g(\xi)|^2 \left(1 - \frac{|\xi|^2}{|u|^2}\right)^{1-n} dv(\xi)\omega(u)dv(u) \\
& \leq \|N(f)\|^2_{L_p^\mathbb{B}} \left( \int_{\mathbb{B}} \left( \int_{\Gamma_u} |f(\xi)|^2 |\Re g(\xi)|^2 \left(1 - \frac{|\xi|^2}{|u|^2}\right)^{1-n} dv(\xi) \right)^{\frac{2}{p}} \omega(u)dv(u) \right)^{\frac{p}{2}} \\
& \approx \|f\|_{A_{\omega}^{p-2}}^2 \|T_{\omega}f\|_{A_{\omega}^{p-2}}^2 \leq \|f\|_{A_{\omega}^p}. \tag{32}
\end{align*}
\]

So, \( |\Re g(z)|^2 \omega^*(z)dv(z) \) is a \( p \)-Carleson measure for \( A_{\omega}^p \), and thus \( g \in C^1(\omega^*) \).

Suppose \( 0 < p < 2 \) and (iii a) holds. Recall that \( d\mu_{\omega}(z) = |\Re g(z)|^2 \omega^*(z)dv(z) \).

Then by Lemma 7 and its proof we get

\[
g \in \mathcal{B} \quad \text{and} \quad \|g\|_{\mathcal{B}} \leq \|T_{\omega}\|. \tag{33}
\]

Here, \( \mathcal{B} \) is the Bloch space on the unit ball and \( \|T_{\omega}\| \) is \( \|T_{\omega}\|_{A_{\omega}^\infty \rightarrow A_{\omega}^\infty} \). Let \( F_{a,p} \) be defined as (3) for some \( \gamma \) large enough. Let \( 1 < \tau_1, \tau_2 < \infty \) such that \( \frac{\tau_2}{\tau_1} = \frac{p}{2} < 1 \), and let \( \tau'_1, \tau'_2 \) be the conjugate indexes of \( \tau_1, \tau_2 \).

By Lemma 2, Proposition 3, Hölder inequality and (20), for any \( a \in \mathcal{B} \) with \( |a| \geq \frac{1}{2} \), we have

\[
\begin{align*}
\mu_{\gamma}(S_a) & \approx \int_{S_a} |F_{a,p}(z)|^2 |\Re g(z)|^2 \left(1 - |z|^2\right)^{1-n} dv(z) \\
& \approx \int_{\mathcal{B}} \left( \int_{S_a \cap \Gamma_u} |F_{a,p}(z)|^2 |\Re g(z)|^2 \left(1 - |z|^2\right)^{1-n} dv(z) \right)^{\frac{1}{\tau'_1}} \left( \int_{\mathcal{B}} \omega(u)dv(u) \right)^{\frac{1}{\tau'_2}} \omega(u)dv(u)
\end{align*}
\]
Since Fubini’s Theorem, Lemma 6, Propositions 1 and 3, Remark 1, Lemma 2

Then we have

where

\[ C \]

\[ \leq \left( \int_\mathbb{B} \left( \int_{S_a \cap \Gamma_u} |F_{a,p}(z)|^2 |\mathcal{R}g(z)|^2 (1 - |z|^2)^{-n} dv(z) \right)^{\frac{r_3}{r_1}} \omega(u) dv(u) \right) \]

\[ \leq \left( \int_\mathbb{B} \left( \int_{S_a \cap \Gamma_u} |F_{a,p}(z)|^2 |\mathcal{R}g(z)|^2 (1 - |z|^2)^{-n} dv(z) \right)^{\frac{r_3}{r_1}} \omega(u) dv(u) \right) \]

\[ \leq \|T_g F_{a,p}\|_{L^p_{\omega, \tau_1}} \|J_2\|_{L^1_{\omega, \tau_1}} \]

where

\[ J_2(u) = \int_{S_a \cap \Gamma_u} |F_{a,p}(z)|^2 |\mathcal{R}g(z)|^2 (1 - |z|^2)^{-n} dv(z). \]

Since \( \frac{r_3}{r_1} > 1 \), we have \( \left( \frac{r_3}{r_1} \right)^r = \frac{r_3 (r_1 - 1)}{r_1 - r_3} > 1 \). Let \( \tau_3 = \frac{r_3 (r_1 - 1)}{r_1 - r_3} \). We have

\[ ||J_3||_{L^1_{\omega, \tau_1}} = \sup_{\|h\|_{L^1} \leq 1} \left| \int_\mathbb{B} h(u) J_2(u) \omega(u) dv(u) \right|. \]

By using Fubini’s Theorem, Lemma 6, Propositions 1 and 3, Remark 1, Lemma 2, and Corollary 1, in order, we have

\[ \left| \int_\mathbb{B} h(u) J_2(u) \omega(u) dv(u) \right| \leq \int_\mathbb{B} |h(u)| J_2(u) \omega(u) dv(u) \]

\[ \leq \int_{S_a} |\mathcal{R}g(z)|^2 (1 - |z|^2)^{-n} \int_{S_a} |h(u)| \omega(u) dv(u) dv(z) \]

\[ \leq \int_{S_a} |\mathcal{R}g(z)|^2 M_\omega(|h|)(z) \omega^*(z) dv(z) = \int_{S_a} M_\omega(|h|)(z) d\mu_g(z) \]

\[ \leq \left( \mu_g(S_a) \right)^{\frac{r_3}{r_1}} \left( \int_{S_a} (M_\omega(|h|)(z))^{r_3} d\mu_g(z) \right)^{\frac{1}{r_3}} \]

\[ \leq \left( \mu_g(S_a) \right)^{\frac{r_3}{r_1}} \left( \sup_{\omega \in S_a} \frac{\mu_g(S_a)}{\omega(S_a)} \right)^{\frac{1}{r_3}} \|h\|_{L^{r_3}}. \]

Then we have

\[ \mu_g(S_a) \leq \|T_g F_{a,p}\|_{L^p_{\omega, \tau_1}} \left( \mu_g(S_a) \right)^{\frac{r_3}{r_1}} \left( \sup_{\omega \in S_a} \frac{\mu_g(S_a)}{\omega(S_a)} \right)^{\frac{1}{r_3}}. \]

By the process of obtaining (34), if we replace \( g(z) \) by \( g_r(z) = g(rz) \), we have

\[ \mu_{g_r}(S_a) \leq \|T_g F_{a,p}\|_{L^p_{\omega, \tau_1}} \left( \mu_{g_r}(S_a) \right)^{\frac{r_3}{r_1}} \left( \sup_{\omega \in S_a} \frac{\mu_{g_r}(S_a)}{\omega(S_a)} \right)^{\frac{1}{r_3}}. \]

We now claim that there exists a constant \( C = C(\omega) > 0 \) such that

\[ \sup_{\frac{1}{2} < r < 1} ||T_g F_{a,p}\|_{L^p_{\omega, \tau_1}}^p \leq C ||T_g\|_{L^1_{\omega, \tau_1}}^p \omega(S_a), \quad \frac{1}{2} \leq |a| < 1. \]
Taking this for granted for a moment, by (35) and (36) we have

$$\sup_{|a| \geq \frac{1}{2}} \frac{\mu_{g}(S_{a})}{\omega(S_{a})} \leq ||T_{g}||^{2}, \quad \text{for all} \quad \frac{1}{2} < r < 1.$$ 

By Fatou’s Lemma,

$$\sup_{|a| \geq \frac{1}{2}} \frac{\mu_{g}(S_{a})}{\omega(S_{a})} \leq ||T_{g}||^{2}.$$ 

Using the skill we proved Proposition 1, there exist $|a_{1}| = |a_{2}| = \cdots = |a_{N}| = \frac{1}{2}$ such that $B = \bigcup_{i=1}^{N} S_{a_{i}} \cup \frac{1}{2}B$. Then $\mu(B) < \infty$. So $g \in C^{1}(\omega^{*})$.

It remains to prove (36). For any fixed $r \in (\frac{1}{2}, 1)$, when $\frac{1}{2} < |a| \leq \frac{1}{2-r}$, by triangle inequality, we have

$$|1 - \langle z, a \rangle| \leq |1 - \langle \frac{z}{r}, a \rangle| + \frac{1-r}{2-r} 2 \left|1 - \langle \frac{z}{r}, a \rangle\right|, \quad |z| \leq r.$$ 

When $\frac{1}{2} < |a| \leq \frac{1}{2-r}$, by (20), we have

$$||T_{g}F_{a,p}||_{A_{p}^{\infty}} \approx \int_{B} \left(\int_{\Gamma_{u}} |F_{a,p}(\xi)|^{2} |R_{g}(\xi)|^{2} \left(1 - \frac{|\xi|^{2}}{|ru|^{2}}\right)^{1-n} \frac{dv(\xi)}{r^{n}}\right)^{\frac{2}{q}} \omega(u)dv(u)$$

$$= \int_{B} \left(\int_{\Gamma_{u}} |F_{a,p}(\frac{\xi}{r})|^{2} |R_{g}(\xi)|^{2} \left(1 - \frac{|\xi|^{2}}{|ru|^{2}}\right)^{1-n} \frac{dv(\xi)}{r^{n}}\right)^{\frac{2}{q}} \omega(u)dv(u)$$

$$\leq \int_{B} \left(\int_{\Gamma_{u}} |F_{a,p}(\xi)|^{2} |R_{g}(\xi)|^{2} \left(1 - \frac{|\xi|^{2}}{|ru|^{2}}\right)^{1-n} \frac{dv(\xi)}{r^{n}}\right)^{\frac{2}{q}} \omega(u)dv(u)$$

$$\approx \int_{B} \left(\int_{\Gamma_{u}} |F_{a,p}(\xi)|^{2} |R_{g}(\xi)|^{2} \left(1 - \frac{|\xi|^{2}}{|ru|^{2}}\right)^{1-n} \frac{dv(\xi)}{r^{n}}\right)^{\frac{2}{q}} \omega(u)dv(u)$$

$$\approx \int_{0}^{\infty} \left(\int_{\Gamma_{u}} |F_{a,p}(\xi)|^{2} |R_{g}(\xi)|^{2} \left(1 - \frac{|\xi|^{2}}{r^{2}}\right)^{1-n} \frac{dv(\xi)}{r^{n}}\right)^{\frac{2}{q}} d\sigma(\eta) \omega(\frac{t}{r})dt.$$ 

By (22) and the similar calculation, we have

$$||T_{g}F_{a,p}||_{H^{p}} \approx \int_{\Gamma_{u}} \left(\int_{\Gamma_{u}} |F_{a,p}(\xi)|^{2} |R_{g}(\xi)|^{2} \left(1 - \frac{|\xi|^{2}}{r^{2}}\right)^{1-n} \frac{dv(\xi)}{r^{n}}\right)^{\frac{2}{q}} d\sigma(\eta).$$

Then

$$||T_{g}F_{a,p}||_{A_{p}^{\infty}}^{p} \leq \int_{0}^{\infty} ||(T_{g}F_{a,p})_{t}||_{H^{p}}^{p} \omega(t)dt$$

$$\leq \int_{0}^{\infty} ||(T_{g}F_{a,p})_{t}||_{H^{p}}^{p} \omega(\frac{t}{r})dt$$

$$\leq ||T_{g}F_{a,p}||_{A_{p}^{\infty}}^{p}$$

$$\leq ||T_{g}||^{p} \omega(S_{a}).$$ 

(37)
Proof. By Lemmas 7 and 8, we have (35), Lemma 7 (33) and γ is large enough, we have

\[ \|T_g, F_{a,p}\|_{A^0_p}^p \approx \int_B \left( \int_{\Gamma_u} |F_{a,p}(\xi)|^2 |\Re g_r(\xi)|^2 \left( 1 - \frac{\|\xi\|^2}{|u|^2} \right)^{1-n} dv(\xi) \right)^{\frac{p}{2}} \omega(u)dv(u) \]

\[ \lesssim M^p_\omega(r, \Re g) \int_B \left( \int_{\Gamma_u} |F_{a,p}(\xi)|^2 |\Re g_r(\xi)|^2 \left( 1 - \frac{\|\xi\|^2}{|u|^2} \right)^{1-n} dv(\xi) \right)^{\frac{p}{2}} \omega(u)dv(u) \]

\[ \lesssim \|T_g\|^p \left( \frac{1 - |a|^2}{1 - \langle z, a \rangle} \right)^{\frac{mn}{p} - 1} \|g\|^p_{A^0_p} \approx \|T_g\|^p \omega(S_a). \]

When \(|a| \geq \frac{1}{2r}\) and \(|\langle z, a \rangle| > r\), by (20), we have

\[ \|T_g, F_{a,p}\|_{A^0_p}^p \approx \int_B \left( \int_{\Gamma_u} |F_{a,p}(\xi)|^2 |\Re g_r(\xi)|^2 \left( 1 - \frac{\|\xi\|^2}{|u|^2} \right)^{1-n} dv(\xi) \right)^{\frac{p}{2}} \omega(u)dv(u) \]

\[ \lesssim (1 - |a|)^{\gamma + n} \int_B \left( \int_{\Gamma_u} |\Re g_r(\xi)|^2 \left( 1 - \frac{\|\xi\|^2}{|u|^2} \right)^{1-n} dv(\xi) \right)^{\frac{p}{2}} \omega(u)dv(u) \]

\[ \approx (1 - |a|)^{\gamma + n}\|g\|^p_{A^0_p} \leq \|T_g\|^p_{A^0_p} \omega(S_a) \leq \|T_g\|^p \omega(S_a). \]

It follows that (36) holds. The proof is complete.

**Theorem 5.** Let \(0 < p \leq q < \infty\), \(\omega \in \tilde{\mathcal{D}}\), \(\kappa = \frac{1}{p} - \frac{1}{q}\) and \(g \in H(\mathbb{B})\).

(i) If \(\kappa < 1\), then the following conditions are equivalent:

(iia) \(T_g : A^0_p \to A^0_q\) is compact;

(iib) \(M_\omega(r, \Re g) = o \left( \frac{\omega'(S_a)}{1-r} \right)\);

(iic) \(g \in C^{2\kappa+1} (\omega^*)\).

(ii) The following conditions are equivalent.

(iia) \(T_g : A^0_p \to A^0_p\) is compact;

(iib) \(g \in C^1 (\omega^*)\).

**Proof.** By Lemmas 7 and 8 we have (ia) \(\Rightarrow (ib) \Rightarrow (ic)\). Let \(d\mu_g(z) = |\Re g(z)|^2 \omega^*(z)dv(z)\). First, we prove (i).

Suppose that (ic) holds and \(q = 2\). Then \(\mu_g\) is a vanishing \(2\)-Carleson measure for \(A^0_p\). Using (21), we have

\[ \|T_g f\|_{A^0_p}^2 \approx \int_B |f(z)|^2 |\Re g(z)|^2 \omega^*(z)dv(z). \]

So, \(T_g : A^0_p \to A^0_p\) is compact by Theorem 1.
Suppose that (ic) holds. By Theorem 4 \( T_g : A^p_\omega \to A^q_\omega \) is bounded. For every \( \varepsilon > 0 \), there is a \( r \in (0, 1) \), such that
\[
\sup_{|a| \geq r} \frac{\mu_g(S_a)}{(\omega(S_a))^{2\alpha+1}} < \varepsilon.
\]
For any measurable subset \( E \) of \( \mathbb{B} \), define \( \mu_{g,r}(E) = \mu_g(E \cap (\mathbb{B} \setminus r\mathbb{B})) \). By (17),
\[
\sup_{a \in \mathbb{B}} \frac{\mu_{g,r}(S_a)}{(\omega(S_a))^{2\alpha+1}} \leq \varepsilon.
\]
When \( q > 2 \), let \( \beta = \frac{(2\alpha+1)q}{2q+2} \) and \( \beta' = \frac{(2\alpha+1)q}{q-2} \). For any \( 0 < r < 1 \), by (21), Hölder inequality and Theorem 1 we have
\[
\|T_g f\|^q_{A^0_\omega} \approx \left( \int_{r\mathbb{B}} + \int_{\mathbb{B} \setminus r\mathbb{B}} \right) |f(z)|^2 |T_g f(z)|^{q-2} |\Re g(z)|^2 \omega^*(z) dv(z)
\]
\[
\leq \left( \int_{r\mathbb{B}} |f(z)|^{2\beta} d\mu_g(z) \right)^\frac{1}{\beta} \left( \int_{r\mathbb{B}} |T_g f(z)|^{(q-2)\gamma} d\mu_g(z) \right)^\frac{1}{\gamma'}
\]
\[
+ \left( \int_{\mathbb{B} \setminus r\mathbb{B}} |f(z)|^{2\beta} d\mu_g(z) \right)^\frac{1}{\beta} \left( \int_{\mathbb{B} \setminus r\mathbb{B}} |T_g f(z)|^{(q-2)\gamma} d\mu_g(z) \right)^\frac{1}{\gamma'}
\]
\[
\leq \sup_{|z| \leq r} |f(z)|^2 \left( \int_{r\mathbb{B}} |T_g f(z)|^{(q-2)\gamma} d\mu_g(z) \right)^\frac{1}{\gamma'}
\]
\[
+ \sup_{|z| \leq r} |f(z)|^2 \cdot \|T_g f\|^{q-2}_{A^0_\omega} + \varepsilon^\frac{1}{\gamma} \|f\|_{A^0_\omega}^{2\beta} \|T_g f\|^{(q-2)\gamma}_{A^0_\omega}
\]
\[
= \sup_{|z| \leq r} |f(z)|^2 \cdot \|T_g f\|^{q-2}_{A^0_\omega} + \varepsilon^\frac{1}{\gamma} \|f\|_{A^0_\omega}^{2\beta} \|T_g f\|^{q-2}_{A^0_\omega}.
\]
Then by Lemma 4 (ia) holds.
When \( 0 < q < 2 \), by (29) and (31), we have
\[
\|T_g f\|^q_{A^0_\omega} \leq \|f\|^{\frac{2-q}{q}}_{A^0_\omega} \left( \int_{\mathbb{B}} |f(z)|^2-\tau |\Re g(z)|^2 \omega^*(z) dv(z) \right)^\frac{q}{2}
\]
\[
\leq \|f\|^{\frac{2-q}{q}}_{A^0_\omega} \left( \sup_{|z| \leq r} |f(z)|^{2-\tau} + \int_{\mathbb{B}} |f(z)|^{2-\tau} d\mu_{g,r}(z) \right)^\frac{q}{2}
\]
\[
\leq \|f\|^{\frac{2-q}{q}}_{A^0_\omega} \left( \sup_{|z| \leq r} |f(z)|^{2-\tau} + \varepsilon \|f\|_{A^0_\omega}^{2-\tau} \right)^\frac{q}{2}
\]
\[
= \|f\|^{\frac{2-q}{q}}_{A^0_\omega} \left( \sup_{|z| \leq r} |f(z)|^{2-\tau} + \varepsilon \|f\|_{A^0_\omega}^{2-\tau} \right)^\frac{q}{2}.
\]
Here \( \tau = \frac{(2-q)\beta}{q} \). By Lemma 4 \( T_g : A^p_\omega \to A^q_\omega \) is compact. So, we finish the proof of (ic)⇒(ia).
When \( p = 2 \), (iia)⇒(iib) is obvious. By the proof of (ic)⇒(ia), we get (iib)⇒(iia).
Suppose \( p > 2 \) and (iia) holds. By (32), we have
\[
\int_{\mathbb{B}} |f(z)|^p |\Re g(z)|^2 \omega^\prime(z) dv(z) \leq \|f\|_{A^p_{\omega}}^p \|T_g f\|_{A^p_{\omega}}^2.
\]
Let \( f_{a,p}(z) = \frac{f_{a,p}(z)}{\|f_{a,p}\|_{A^p_{\omega}}} \) for some \( \gamma \) is large enough. Then we have
\[
\frac{\mu_g(S_a)}{\omega(S_a)} = \int_{S_a} |f_{a,p}(z)|^p d\mu_g(z) \leq \int_{\mathbb{B}} |f_{a,p}(z)|^p |\Re g(z)|^2 \omega^\prime(z) dv(z) \leq \|T_g f_{a,p}\|_{A^p_{\omega}}^2.
\]
By Lemma 4, (iib) holds.

Suppose \( 0 < p < 2 \) and (iia) holds. Let \( f_{a,p}(z) = \frac{f_{a,p}(z)}{\|f_{a,p}\|_{A^p_{\omega}}} \) for some \( \gamma \) is large enough. Then \( \sup_{a \in \mathbb{B}} \frac{\mu_g(S_a)}{\omega(S_a)} < \infty \). By (34), we have
\[
\frac{\mu_g(S_a)}{\omega(S_a)} \leq \|T_g f_{a,p}\|_{A^p_{\omega}} \left( \sup_{a \in \mathbb{B}} \frac{\mu_g(S_a)}{\omega(S_a)} \right)^{\frac{2}{p^*}}.
\]
By Lemma 4, (iib) holds. The proof is complete. \( \square \)

6. INCLUSION RELATIONS ABOUT \( C^1(\omega^\prime)(C^1_0(\omega^\prime)) \)

In this section, we discuss the inclusion relationship between \( C^1(\omega^\prime)(C^1_0(\omega^\prime)) \) and some other function spaces, such as \( \mathcal{B}(\mathcal{B}_0) \) and \( BMOA(VMOA) \).

Recall that a function \( f \in H(\mathbb{B}) \) is said to belong to the Bloch space, denoted by \( \mathcal{B} = \mathcal{B}(\mathbb{B}) \), if
\[
\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re f(z)| < \infty.
\]
It is well known that \( \mathcal{B} \) is a Banach space with the above norm. Let \( \mathcal{B}_0 \), called the little Bloch space, denote the subspace of \( \mathcal{B} \) consisting of those \( f \in \mathcal{B} \) for which
\[
\lim_{|z| \to 1} (1 - |z|^2) |\Re f(z)| = 0.
\]
Let \( \nabla f \) denote the invariant gradient of \( f \), i.e.,
\[
(\nabla f)(z) = \nabla (f \circ \varphi_z)(0).
\]
A function \( f \in H(\mathbb{B}) \) is said to belong to the space \( BMOA \) if
\[
\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\nabla f(z)|^2 G(z, a) d\lambda(z) < \infty.
\]
Here \( d\lambda(z) = (1 - |z|^2)^{-n-1} dv(z), \ G(z, a) = g(\varphi_a(z)), \)
\[
g(z) = \frac{n+1}{2n} \int_1^{|z|} (1 - t^2)^{n-1} t^{-2n+1} dt.
\]
Let \( VMOA \) denote the subspace of \( BMOA \) for which
\[
\lim_{|z| \to 1} \int_{\mathbb{B}} |\nabla f(z)|^2 G(z, a) d\lambda(z) = 0.
\]

**Proposition 4.** (i) If \( \omega \in \dot{\mathcal{D}} \), then \( C^1(\omega^\prime) \subset A^p_{\omega} \) for all \( p > 0 \).
(ii) If \( \omega \in \dot{\mathcal{D}} \), then \( BMOA \subset C^1(\omega^\prime) \subset \mathcal{B} \) and \( VMOA \subset C^1_0(\omega^\prime) \subset \mathcal{B}_0 \).
Lemma 2, we have $V \subseteq BMOA = B_0$.

Proof. (i). Suppose $g \in C^1(\omega^*)$. By Theorem 2

\[
\|g - g(0)\|_{A^2_{\omega^*}}^2 \approx \int_{B} |R_g(z)|^2 \omega^*(z)dv(z) \leq \|1\|_{A^2_{\omega^*}}^2.
\] (38)

So, $g \in A^2_{\omega^*}$. Similarly, we have

\[
\|g - g(0)\|_{A^4_{\omega^*}}^4 \approx \int_{B} |g(z) - g(0)|^2 |R_g(z)| \omega^*(z)dv(z) \leq \|g - g(0)\|_{A^2_{\omega^*}}^2.
\]

By mathematical induction, for all $k \in \mathbb{N}$, $g - g(0) \in A^{2k}_{\omega^*}$. So, $C^1(\omega^*) \subseteq A^p_{\omega^*}$ for all $p > 0$.

(ii). By Theorems 41, 5 and Lemma 7, we have $C^1(\omega^*) \subseteq B$ and $C^1(\omega^*) \subseteq B_0$. Suppose $g \in BMOA$, let $d\mu^*_g(z) = (1 - |z|^2)|R_g(z)|^2dv(z)$ and

\[
S^*(\xi, r) = \{z \in B : |1 - \langle z, \xi \rangle| < r\}, \text{ for } \xi \in \mathbb{S}.
\]

By Theorem 5.14 in [19], we have

\[
M := \sup \left\{ \frac{\mu^*_S(S^*(\xi, r))}{r^n} : \xi \in \mathbb{S}, 0 < r < \delta \right\} < \infty,
\]

where $\delta$ is any fixed positive constant. By Proposition 2 and the proof of it, for any $a \in B$ with $|a| > \frac{3}{7}$, there are $\xi \in \mathbb{S}$ and $r = 3(1 - |a|)$, such that $S_a \subseteq S^*(\xi, r)$. By Lemma 2 we have

\[
\int_{S_a} |R_g(z)|^2 \omega^*(z)dv(z) \leq \frac{\omega(a) \int_{S_a} |R_g(z)|^2(1 - |z|)dv(z)}{(1 - |a|)^n \omega(a)} \leq \frac{\int_{S^*(\xi, r)} |R_g(z)|^2(1 - |z|^2)dv(z)}{r^n} \leq M.
\]

So, $g \in C^1(\omega^*)$. That is $BMOA \subseteq C^1(\omega^*)$. Similarly, by Theorem 5.19 in [19], we have $VMOA \subseteq C^1(\omega^*)$.

(iii). Suppose that $\{g_k\}$ is a Cauchy sequence in $C^1(\omega^*)$. By (38) and Theorem 1 $\{g_k\}$ is a Cauchy sequence in $A^2_{\omega^*}$. Then we have $g \in A^2_{\omega^*}$ such that $\lim_{k \to \infty} \|g_k - g\|_{A^2_{\omega^*}} = 0$. 

By Theorem 2 Fatou Lemma and Theorem 1 for any $f \in A^2_\omega$, we have
\[
\|T_\omega f\|^2_{A^2_\omega} \approx \int_B |f(z)|^2 |\mathcal{R} g(z)|^2 \omega^*(z) dv(z)
\]
\[
\approx \int_B |f(z)|^2 \liminf_{k \to \infty} |\mathcal{R} g_k(z)|^2 \omega^*(z) dv(z)
\]
\[
\leq \liminf_{k \to \infty} \int_B |f(z)|^2 |\mathcal{R} g_k(z)|^2 \omega^*(z) dv(z)
\]
\[
\leq \liminf_{k \to \infty} \|g_k\|_{C^1(\omega^*)}^2 \|f\|^2_{A^2_\omega}.
\]
So, $T_\omega : A^2_\omega \to A^2_\omega$ is bounded. Then $g \in C^1(\omega^*)$. Similarly, for all $f \in A^2_\omega$, we have
\[
\int_B |f(z)|^2 (|\mathcal{R} g - \mathcal{R} g_j(z)|^2 \omega^*(z) dA(z) \approx \|(T_\omega - T_\omega) f\|^2_{A^2_\omega}
\]
\[
\leq \liminf_{k \to \infty} \|g_j - g_k\|_{C^1(\omega^*)}^2 \|f\|^2_{A^2_\omega}.
\]
By Theorem 1 $\lim \|g - g_j\|_{C^1(\omega^*)} = 0$. So, $C^1(\omega^*)$ is a Banach space.

Suppose $\{g_k\}$ is a Cauchy sequence in $C^1_0(\omega^*)$. Then there exists $g \in C^1(\omega^*)$ such that $\lim_{k \to \infty} \|g_k - g\|_{C^1(\omega^*)} = 0$. Let $\{f_j\}$ be a bounded sequence in $A^2_\omega$ such that $\{f_j\}$ converges to 0 uniformly on compact subsets of $\mathbb{B}$. By Theorems 1 and 2 we have
\[
\|T_\omega f_j\|_{A^2_\omega} \leq \|T_\omega g_k f_j\|_{A^2_\omega} + \|T_\omega f_j\|_{A^2_\omega} \leq \|g - g_k\|_{C^1(\omega^*)}^2 \|f_j\|_{A^2_\omega} + \|T_\omega f_j\|_{A^2_\omega}.
\]
For any given $\varepsilon > 0$, we can choose a $k \in \mathbb{N}$ such that $\|g - g_k\|_{C^1(\omega^*)} < \varepsilon^2$. By Lemma 4
\[
\lim_{j \to \infty} \|T_\omega f_j\|_{A^2_\omega} \leq \varepsilon \sup_{j \geq 1} \{\|f_j\|_{A^2_\omega}\}.
\]
Then $T_\omega : A^2_\omega \to A^2_\omega$ is compact. So, $g \in C^1_0(\omega^*)$. That is, $C^1_0(\omega^*)$ is a closed subspace of $C^1(\omega^*)$.

(iv) Suppose $\omega \in \mathcal{R}$. By observation (v) after Lemma 1.1 in [11], there exists $\beta > -1$ and $\delta \in (0, 1)$, such that $\omega(\cdot)(1-\cdot)^\beta$ is decreasing on $[\delta, 1)$. Without loss of generality, let $\delta = 0$. Then for all $g \in \mathcal{B}$ and $a \in \mathbb{B}$, by Lemmas 1 and 2 we have
\[
\int_{S_a} |\mathcal{R} g(z)|^2 \omega^*(z) dv(z) \approx \int_{S_a} |\mathcal{R} g(z)|^2 (1-|z|)^{2+\beta} \frac{\omega(a)}{(1-|a|)^\beta} dv(z)
\]
\[
\leq \frac{\|g\|_{2, \omega(a)}}{(1-|a|)^\beta} \int_{S_a} (1-|z|)^{\beta} dv(z)
\]
\[
\approx \frac{\|g\|_{2, \omega(a)}}{(1-|a|)^\beta} \int_{S_a} (1-|z|)^{\beta} dv(z)
\]
\[
\approx \|f\|_{2, \omega(a)}.
\]
Then $\mathcal{B} \subset C^1(\omega^*)$. So, $\mathcal{B} = C^1(\omega^*)$. Similarly, $\mathcal{B}_0 = C^1_0(\omega^*)$.

(v) and (vi) have been proved in [11] when $n = 1$, so they also hold for $n > 1$. The proof is complete.

Proposition 5. Let $\omega \in \hat{\mathcal{D}}$ and $g \in C^1(\omega^*)$. The following statements are equivalent.
(i) \( g \in C_0^1(\omega^*) \);
(ii) \( \lim_{r \to 1} \|g - g_r\|_{C^1(\omega^*)} = 0 \), here \( g_r(z) = g(rz) \);
(iii) There is a sequence of polynomials \( \{p_k\} \) such that \( \lim_{r \to 1} \|g - p_k\|_{C^1(\omega^*)} = 0 \).

Proof. (i)\( \Rightarrow \) (ii). Suppose \( g \in C_0^1(\omega^*) \). Let \( \gamma \) be large enough and

\[
f_{a,2}(z) = \frac{F_{a,2}(z)}{(\omega(S_a))^2}.
\]

Then Lemma 4, Theorems 1 and 2 yield

\[
\lim_{|a| \to 1} \|T_g(f_{a,2})\|_{A_2^c}^2 \approx \lim_{|a| \to 1} \int_B |f_{a,2}(z)|^2 |\Re g(z)|^2 \omega^*(z)dv(z) = 0
\]  

(39)

and

\[
\int_B |(\Re g - \Re g_r)(z)|^2 \omega^*(z)dv(z) \leq \|T_{g-r,f_{a,2}}\|_{A_2^c}^2.
\]  

(40)

By (39) and (ii) in Proposition 4 for any \( \varepsilon > 0 \), there is a \( r_0 \in (\frac{1}{2}, 1) \) such that

\[
\|T_g(f_{a,2})\|_{A_2^c} < \varepsilon, \quad \text{and} \quad (1 - |a|)|\Re g(a)| < \varepsilon, \quad \text{when} \quad |a| > r_0.
\]  

(41)

By (37), if \( r_0 < |a| < \frac{1}{2}r \), we have

\[
\|T_{g-r,f_{a,2}}\|_{A_2^c}^2 \leq \|T_{g,f_{a,2}}\|_{A_2^c}^2 + \|T_{g,f_{a,2}}\|_{A_2^c}^2 \leq \|T_{g,f_{a,2}}\|_{A_2^c}^2 \leq \varepsilon^2.
\]  

(42)

If \( |a| > \max \{r_0, \frac{1}{2}r\} \), by (21), we have

\[
\|T_{g-r,f_{a,2}}\|_{A_2^c}^2 \leq \|T_{g,f_{a,2}}\|_{A_2^c}^2 + \|T_{g,f_{a,2}}\|_{A_2^c}^2
\]

\[
\leq \varepsilon^2 + \int_B |\Re g_r(z)|^2 |f_{a,p}(z)|^2 \omega^*(z)dv(z)
\]

\[
\leq \varepsilon^2 + M_\infty^2(r, \Re g) \int_B |f_{a,p}(z)|^2 \omega^*(z)dv(z).
\]

By Theorem 1.12 in [19] and Lemma 2 if \( \gamma \) is large enough, we have

\[
\int_B |f_{a,p}(z)|^2 \omega^*(z)dv(z) = \frac{(1 - |a|^2)^{\gamma+n}}{\omega(S_a)} \int_B |\omega^*(z)|dv(z)
\]

\[
\approx \frac{(1 - |a|^2)^{\gamma+n} \omega^*(a)}{(1 - |a|^2)^{\gamma-1} \omega(S_a)} \approx (1 - |a|^2)^2.
\]

So, when \( |a| > \max \{r_0, \frac{1}{2}r\} \), by (41),

\[
\|T_{g-r,f_{a,2}}\|_{A_2^c}^2 \leq \varepsilon^2 + (1 - |a|^2)^2 M_\infty^2(2 - \frac{1}{|a|}, \Re g) \leq \varepsilon^2.
\]  

(43)

By (40), (42) and (43), we obtain

\[
\sup_{|a| > r_0} \frac{\int_{S_a} |(\Re g - \Re g_r)(z)|^2 \omega^*(z)dv(z)}{\omega(S_a)} \leq \varepsilon^2.
\]
When \(|a| \leq r_0\), we have
\[
\frac{\int_{S_a} |(\Re g - \Re g_r)(z)|^2 \omega^*(z) dv(z)}{\omega(S_a)} \leq \frac{\|\Re g - \Re g_r\|^2_{A^{2,\omega^*}}}{\omega(S_{r_0})}.
\]
So, there is a \(r_1 \geq r_0\), such that
\[
\sup_{r > r_1} \frac{\|\Re g - \Re g_r\|^2_{A^{2,\omega^*}}}{\omega(S_{r_0})} \leq \varepsilon^2.
\]
Therefore,
\[
\sup_{a \in \mathbb{B}, \, r > r_1} \frac{\int_{S_a} |(\Re g - \Re g_r)(z)|^2 \omega^*(z) dv(z)}{\omega(S_a)} \leq \varepsilon^2.
\]
Then \((ii)\) holds.

\((iii)\) \(\Rightarrow\) \((i)\). For any polynomial \(p_n\), we have \(\|\Re p_n\|_{H^\omega} < \infty\). Then by Lemmas 1 and 2, for \(|a| > \frac{1}{2}\) we have
\[
\frac{\int_{S_a} |\Re p_n(z)|^2 \omega^*(z) dv(z)}{\omega(S_a)} \leq \|\Re p_n\|^2_{H^\omega} \frac{(1 - |a|) \hat{\omega}(a) \int_{S_a} dv(z)}{\omega(S_a)} \approx (1 - |a|)^2 \|\Re p_n\|^2_{H^\omega}.
\]
So, \(p_n \in C^1_0(\omega^*)\). Then by \((iii)\) of Proposition 4, \((i)\) holds.

\((ii)\) \(\Rightarrow\) \((iii)\). For all \(n \in \mathbb{N}\), there is a polynomial \(p_n\) such that
\[
\|\Re g_{1 - \frac{1}{n}} - \Re p_n\|_{H^\omega} < \frac{1}{n}.
\]
Since
\[
\|g - p_n\|_{C^1(\omega^*)} \leq \|g - g_{1 - \frac{1}{n}}\|_{C^1(\omega^*)} + \|g_{1 - \frac{1}{n}} - p_n\|_{C^1(\omega^*)} \leq \|g - g_{1 - \frac{1}{n}}\|_{C^1(\omega^*)} + \|\Re g_{1 - \frac{1}{n}} - \Re p_n\|_{H^\omega},
\]
we obtain \((iii)\). The proof is complete. \(\square\)

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