Algorithms with Logarithmic or Sublinear Regret for Constrained Contextual Bandits

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Abstract

We study contextual bandits with budget and time constraints under discrete contexts, referred to as constrained contextual bandits. The budget and time constraints significantly increase the complexity of exploration-exploitation tradeoff because they introduce coupling among contexts. Such coupling effects make it difficult to obtain oracle solutions that assume known statistics of bandits. To gain insight, we first study unit-cost systems, where the costs of all actions under all contexts are identical. We develop near-optimal approximations of the oracle, which are then combined with the upper-confidence-bound (UCB) method in the general case where the expected rewards are unknown a priori. We show that our proposed algorithms, named UCB-PB and UCB-ALP, achieve logarithmic regret except in certain boundary cases. Last, we discuss the extension of the proposed algorithms into general-cost systems.

1. Introduction

Contextual multi-armed bandit (MAB) (Langford & Zhang, 2007; Lu et al., 2010) is a general model for exploration-exploitation tradeoff in the presence of side information. Specifically, in contextual bandits, an agent takes an action after observing a set of features, referred to as context, in each round. Then the agent receives a random reward where its expectation is a function of both context and action. Since only the reward of the action taken by the agent is revealed, the agent needs to balance between taking the best action based on the historical performance and exploring the potentially better alternative actions under a given context. Motivating examples for this model include online advertising (Tang et al., 2013), article recommendation (Li et al., 2010), online hiring in crowdsourcing (ul Hassan & Curry, 2014), etc. Much work has been done to achieve sublinear regret in contextual bandits under different context-reward models (Langford & Zhang, 2007; Slivkins, 2011), and more recent work (Agarwal et al., 2014) focuses on computationally efficient algorithms with minimum regret.

However, traditional contextual bandit models do not capture an important characteristic of real systems: the budget and time constraints. In practice, there is usually a cost associated with the resource consumed by each action. The system has a limited amount of resources, i.e., its budget, within a finite time horizon. Taking crowdsourcing as an example, an employer needs to pay the crowdsourced workers, and the budget for a given set of tasks is limited. Although budget constraints have been considered in recent work, e.g., total-budget constrained MAB (Tran-Thanh et al., 2010; 2012; Babaioff et al., 2012; Badanidiyuru et al., 2013) and individual-arm budget constrained MAB (Jiang & Srikant, 2013; Slivkins, 2013), the results are inapplicable in the case with observable contexts.

In this paper, we study contextual bandit problems with budget and time constraints, referred to as constrained contextual bandits, where the agent is given a budget $B$ and a time horizon $T$. In addition to a reward, a cost is incurred whenever an action is taken under a context. The bandit process ends when the agent runs out of either budget or time. The objective of the agent is to maximize the
expected total reward subject to the budget and time constraints.

The above constrained contextual bandit problem is a special case of Resourceful Contextual Bandits (RCB) (Badanidiyuru et al., 2014). In (Badanidiyuru et al., 2014), RCB is studied under more general settings with possibly infinite contexts, random costs, and multiple budget constraints. A Mixture Elimination algorithm is proposed and shown to achieve $O(\sqrt{T})$ regret. However, the benchmark for the definition of regret in (Badanidiyuru et al., 2014) is restricted to a finite policy set, and the Mixture Elimination algorithm is computationally inefficient. Recently, (Agrawal & Devanur, 2014) proposes computationally efficient algorithms for bandits with concave rewards and convex resource constraints, where the proposed algorithms are also extended to linear contextual bandits. However, (Agrawal & Devanur, 2014) focuses on static contexts where the contexts are bounded to actions, and it is unclear how to extend the results to the case with randomly arrival contexts.

To address these challenges, we consider a more restricted yet practical model with finite discrete contexts, fixed costs, and a single budget constraint. We also assume that the distribution of contexts is known to the agent as in (Badanidiyuru et al., 2014). These simplifications allow us to design easily-implementable algorithms that achieve $O(\log T)$ or $O(\sqrt{T})$ regret, which is defined more naturally as the performance gap from the oracle algorithm, i.e., the optimal algorithm with known statistics.

Even with simplified assumptions considered in this paper, the constrained contextual bandits are still challenging due to the budget and time constraints. Its main challenge lies in the difficulty of design and analysis of the oracle algorithm. In fact, with known statistics of bandits, the problem becomes a finite-horizon Markov decision process (MDP), where the system state can be described by the current context, the remaining time, and the remaining budget. Theoretically, such a MDP can be solved by dynamic programming (DP). However, using DP in our scenario is challenging: first, the implementation of DP is computationally complex due to the curse of dimensionality; second, it is difficult to obtain the benchmark for regret analysis, since the DP algorithm is implemented in a recursive manner and the explicit representation for its expected total reward is hard to obtain; third, it is difficult to extend the DP algorithm to the case without known statistics, because it is difficult to evaluate the impact of estimation errors on the performance of DP-type algorithms.

To address these difficulties, we start with unit-cost systems, where the costs for all actions under all contexts are identical and normalized as one. We first study the oracle algorithm and its approximations. With known statistics, the agent only needs to consider the best action under a given context, i.e., the action with highest expected reward under that context. Thus, the quality of a context is captured by its highest expected reward, i.e., the expected reward of its corresponding best action. As a special case, when there are only two contexts, skipping the worse context has no opportunity cost if the remaining budget is less than the remaining time. Thus, it is optimal for the agent to Procrastinate-for-the-Better (PB), i.e., to wait for the better context unless the amount of its remaining budget is no less than the remaining time.

When there are more than two contexts, however, significant complexity incurs when making a decisions under those contexts other than the best and worst contexts. Under these “medium-quality” contexts, the agent needs to balance between the instantaneous expected reward and the future expected reward, which is very difficult due to the coupling effect introduced by the budget and time constraints. Thus, we resort to approximations by relaxing the hard budget constraint to an average budget constraint. Specifically, we first obtain an upper bound on the expected total reward by solving a linear programming (LP) problem that maximizes the expected reward with average budget constraint $B/T$. This upper bound provides a benchmark for the regret analysis later. However, the policy from this static LP problem is suboptimal in practice as it does not take the remaining time $\tau$ and remaining budget $b_\tau$ into account. Hence, we propose an Adaptive Linear Programming (ALP) method that replaces the average budget constraint $B/T$ by the average remaining budget, i.e., $b_\tau/\tau$. Although the intuition behind ALP is natural, its performance analysis is non-trivial. Using concentration inequalities, we show that the average remaining budget $b_\tau/\tau$ under ALP concentrates near the average budget $B/T$ with high probability, and thus the ALP algorithm achieves an expected total reward within a constant from the optimum, except for certain boundary cases.

The insight from the known-statistics case allows us to study algorithms for the case where the expected rewards are unknown. We note that both the PB and ALP algorithms only require the ordering of the expected rewards rather than their actual values. This property allows us to combine PB and ALP with estimation methods that can correctly rank the expected rewards with high probability in a short period. In this paper, we combine with the upper-confidence-bound (UCB) algorithm (Auer et al., 2002) and propose a UCB-PB algorithm for two-context systems and a UCB-ALP algorithm for general multi-context systems. We show that for two-context systems, the UCB-PB algorithm achieves logarithmic regret under any given setting. For general multi-context systems, the UCB-ALP algorithm achieves $O(\log T)$ regret except for certain boundary cases, where its regret is $O(\sqrt{T})$.

Then, we relax the unit-cost assumption and discuss the extension to systems with heterogeneous costs. In this case, the agent needs to consider all actions even with known
Algorithms with Logarithmic or Sublinear Regret for Constrained Contextual Bandits

statistics, and a coupling effect occurs among actions under a context. However, the ALP algorithm can be extended to heterogeneous-cost systems with known statistics and is shown to achieve similar performance as in unit-cost systems. In the case without knowledge of expected rewards, the UCB-ALP algorithm is still applicable when the costs of all actions are identical under the same context.

In summary, the main contributions of this paper are:

- Starting with a unit-cost assumption, we first study algorithms in systems with known statistics. We propose an optimal PB algorithm for two-context systems and a near optimal ALP algorithm for general multi-context systems.
- Combining the insight from the known-statistics systems with the UCB method, we propose computational-efficient algorithms, UCB-PB and UCB-ALP, for systems without prior knowledge of the expected rewards. We show that UCB-PB achieves logarithmic regret in two-context systems, and UCB-ALP achieves $O(\log T)$ regret in general multi-context systems except in certain boundary cases (where it achieves $O(\sqrt{T})$ regret).
- We extend the ALP and UCB-ALP algorithms to heterogenous-cost systems. We show that ALP achieves similar performance as in unit-cost systems and UCB-ALP can be applied when all actions have identical costs under the same context.

2. System Model

We consider a contextual bandit problem with a context set $X = \{1, 2, \ldots, J\}$ and an action set $A = \{1, 2, \ldots, K\}$. At each round $t$, a context $X_t$ arrives according to independent and identical distributions with $P\{X_t = j\} = \pi_j$, $j \in X$. We assume that the context distribution vector $\pi = (\pi_1, \pi_2, \ldots, \pi_J)$ is known to the agent as in (Badanidiyuru et al., 2014). Each action $k \in A$ generates a non-negative reward $Y_{k,t}$, which is independent across actions conditioned on $X_t = j$. Under a given context $X_t = j$, the random reward $Y_{k,t}$ follows a distribution corresponding to $j$ and the expectation $E[Y_{k,t} | X_t = j] = u_{j,k}$ is unknown to the agent. Moreover, a cost is incurred if action $k$ is taken under context $j$. To gain insight into constrained contextual bandits, we consider fixed and known costs in this paper, where the cost is $c_{j,k} > 0$ when action $k$ is taken under context $j$. Similar to traditional contextual bandits, the context $X_t$ is observable at the beginning of round $t$, while only the reward of the action taken by the agent is revealed at the end of round $t$.

At the beginning of round $t$, the agent observes the context $X_t$ and takes an action $A_t$ from $\{0\} \cup A$, where “0” represents a dummy action that the agent skips the current context. Let $Y_t$ and $Z_t$ be the reward and cost for the agent in round $t$, respectively. Then, if the agent takes an action $A_t = k > 0$ in round $t$, the reward is $Y_t = Y_{k,t}$ and the cost is $Z_t = c_{X_t,k}$. Otherwise, when the agent takes the dummy action $A_t = 0$, neither reward nor cost is incurred, i.e., $Y_t = 0$ and $Z_t = 0$. In this paper, we focus on contextual bandits with a known time-horizon $T$ and limited budget $B$. The bandit process ends when the agent runs out of the budget or at the end of time $T$.

A contextual bandit algorithm $\Gamma$ is a function that maps the historical observations $H_{t-1} = (X_1, A_1, Y_1; X_2, A_2, Y_2; \ldots; X_{t-1}, A_{t-1}, Y_{t-1})$ and the current context $X_t$ to an action $A_t \in \{0\} \cup A$. Let $U_\Gamma(T, B)$ be the total reward obtained by algorithm $\Gamma$. The objective of the algorithm is to maximize the expected total reward for a given time-horizon $T$ and a budget $B$, i.e.,

$$\text{maximize } U_\Gamma(T, B) = \mathbb{E}_\Gamma \left[ \sum_{t=1}^{T} Y_t \right]$$

subject to $\sum_{t=1}^{T} Z_t \leq B$,

where the expectation is taken over all possible realizations of context and action rewards under $\Gamma$. Note that we consider a “hard” budget constraint, i.e., the total costs should not be greater than $B$ under any realization.

We measure the performance of the algorithm $\Gamma$ by comparing it with the oracle algorithm, which is the optimal algorithm with known statistics, including the knowledge of $\pi_j$’s, $u_{j,k}$’s, and $c_{j,k}$’s. Let $U^*(T, B)$ be the expected total reward obtained by the oracle algorithm. Then, the regret of the algorithm $\Gamma$ is defined as

$$R_\Gamma(T, B) = U^*(T, B) - U_\Gamma(T, B).$$

We are interested in the asymptotic regime where the time horizon $T$ and the budget $B$ grow to infinity in proportion, i.e., with a fixed ratio $\rho = B/T$.

As a starting point, we focus on unit-cost systems in Sections 3 and 4, where the costs are identical and normalized as $c_{j,k} = 1$ for all $j$’s and $k$’s, and the rewards are bounded and appropriately scaled such that $Y_{k,t} \in [0, 1]$ for all $k$’s and $t$’s. We will relax this assumption in Section 5.

In unit-cost systems, the quality of an action $k$ under context $j$ is fully captured by the expected reward $u_{j,k}$. Let $u_{j}^*$ be the highest expected reward under context $j$, and $k_j^*$ be the best action for context $j$, i.e.,

$$u_{j}^* = \max_{k \in A} u_{j,k},$$

$$k_j^* = \arg \max_{k \in A} u_{j,k}.$$

For ease of exposition, we assume that the best action under each context is unique, i.e., $u_{j,k} < u_{j,k'}^*$ for all $j$ and $k \neq k_j^*$. Similarly, we also assume that $u_{1}^* > u_{2}^* > \ldots > u_{J}^*$. 
For contexts $j$ and $j'$, and an action $k$, let $\Delta_{j,k}^{(j)}$ be the difference between the expected reward for action $k$ under context $j$ and the highest expected reward under context $j'$, i.e.,

$$\Delta_{j,k}^{(j')} = u_{j'}^* - u_{j,k}.$$

When $j' = j$, $\Delta_{j,k}^{(j)}$ is the difference of expected reward between the suboptimal action $k$ and the best action under context $j$.

### 3. The Oracle and Its Approximation

In this section, we consider unit-cost systems and focus on the oracle, i.e., the optimal algorithm where the statistics of bandits are known to the agent. We first identify the oracle for two-context systems. Then for general multi-context systems, we present an upper bound on the oracle and develop its near-optimal approximation.

Recall that by unit-cost, we assume that the cost is $Z_{k,t} = 1$ for all $1 \leq k \leq K$ and $t \geq 1$. Therefore, the quality of each context-action pair $(j, k)$ is captured by its expected reward $u_{j,k}$. With the knowledge of $u_{j,k}$’s, the oracle knows the best action $k_j^* = \arg\max_{1 \leq k \leq K} u_{j,k}$ and its expected reward $u_j^* = \max_{1 \leq k \leq K} u_{j,k}$ for any context $j$. Thus, the oracle needs to decide whether to take the best action under the current context or to skip it depending on the context $X_t$, the remaining time $\tau = T - t + 1$, and the remaining budget $b_\tau$.

When there are only two contexts, the oracle algorithm is trivial. Under the unit-cost assumption, skipping the worse context does not waste any opportunities if $b_\tau < \tau$. Thus, the agent can reserve budget for the better context, unless there is sufficient budget; i.e., we have the following algorithm:

**Procrastinate-for-the-Better (PB):** If $X_t = 1$ and $b_\tau > 0$, or if $b_\tau \geq \tau$, take action $A_t = k_{X_t}^*$; otherwise, $A_t = 0$.

We can verify that the above PB algorithm achieves the highest expected reward for any realization of the context arrival process. Thus, the PB algorithm is optimal in two-context systems.

When considering more general cases with $J > 2$, however, it is computationally intractable to obtain the oracle solution due to the coupling effect of the budget constraint under time horizon $T$. We resort to approximations based on constrained linear programming (LP).

### 3.1. Upper Bound: Static Linear Programming

In this section, we propose an upper bound for $U^*(T, B)$ by relaxing the hard constraint to an average constraint and solving the corresponding constrained LP problem.

Specifically, let $p_j \in [0, 1]$ be the probability that the agent takes action $k_j^*$ for context $j$, and $1 - p_j$ be the probability that the agent skips context $j$ (i.e., taking action $A_t = 0$). Denote the probability vector as $p = (p_1, p_2, \ldots, p_J)$. For a time horizon $T$ and budget $B$, consider the following LP:

$$\begin{align*}
(LP_{T,B}) \text{ maximize} & \quad \sum_{j=1}^J p_j \pi_j u_j^*, \\
\text{subject to} & \quad \sum_{j=1}^J p_j \pi_j \leq B/T, \\
& \quad p \in [0,1]^J.
\end{align*}$$

Define a threshold corresponding to the average budget $\rho = B/T$:

$$\tilde{j}(\rho) = \max\{j : \sum_{j'=1}^j \pi_{j'} \leq \rho\}. \tag{3}$$

Note that $\tilde{j}(\rho) = 0$ if $\pi_1 > \rho$. We can verify that the following solution is optimal for $LP_{T,B}$:

$$p_j(\rho) = \begin{cases} 
1, & \text{if } 1 \leq j \leq \tilde{j}(\rho), \\
\frac{\rho - \sum_{j'=1}^{\tilde{j}(\rho)} \pi_{j'}}{\pi_{\tilde{j}(\rho) + 1}}, & \text{if } j = \tilde{j}(\rho) + 1, \\
0, & \text{if } j > \tilde{j}(\rho) + 1.
\end{cases} \tag{4}$$

Thus, the optimal value of $LP_{T,B}$ is

$$v(\rho) = \sum_{j=1}^{\tilde{j}(\rho)} \pi_j u_j^* + p_{\tilde{j}(\rho) + 1}(\rho) \pi_{\tilde{j}(\rho) + 1} u_{\tilde{j}(\rho) + 1}^*. \tag{5}$$

This optimal value $v(\rho)$ can be viewed as the maximal expected reward in a single round with average budget $\rho$. When considering the entire horizon, the total expected reward becomes $\hat{U}(T, B) = TV(\rho)$. The average constraint in $LP_{T,B}$ relaxes the hard constraint in the original problem, while the oracle algorithm satisfies the hard budget constraint for any realization. Therefore, we have that $\hat{U}(T, B)$ is an upper bound on the expected total reward for the original problem.

**Lemma 1.** For a unit-cost system with known statistics, if the time horizon is $T$ and the budget is $B$, then $\hat{U}(T, B) \geq U^*(T, B)$.

With Lemma 1, we can bound the regret of any algorithm by comparing the performance of the algorithm with the upper bound $\hat{U}(T, B)$ rather than the expected total reward obtained by the oracle algorithm, i.e., $U^*(T, B)$. Since $\hat{U}(T, B)$ has a simple representation, as we will see later, it will significantly reduce the complexity of regret analysis.

### 3.2. Adaptive Linear Programming

We note that although the solution (4) provides an upper bound on the expected reward, using such a fixed algorithm
Adaptive Linear Programming: an ALP algorithm as follows: consider an LP problem \( LP_{\tau,b} \) which is the same as \( LP_{T,B} \) except that \( B/T \) in Eq. (2) is replaced with \( b/\tau \). Then, the optimal solution for \( LP_{\tau,b} \) can be obtained by replacing \( \rho \) in Eqs. (3), (4), and (5) with \( b/\tau \). We propose an ALP algorithm as follows:

**Adaptive Linear Programming:** At each round \( t \), when the remaining budget is \( b_\tau = b \), take an action \( A_t = k^*_X \), with probability \( p_{X_t}(b/\tau) \), and \( A_t = 0 \) with probability \( 1 - p_{X_t}(b/\tau) \).

Therefore, the expected total reward obtained by ALP is given by

\[
U_{ALP}(T, B) = \mathbb{E} \left[ \sum_{\tau=1}^{T} v(b_\tau/\tau) \right],
\]

where \( v(\cdot) \) is defined in (5) and the expectation is taken over the remaining budget \( b_\tau \).

Next, we study the evolution of the remaining budget \( b_\tau \) to evaluate the total expected reward under ALP. From Eq. (4), we can verify that when the remaining time is \( \tau \) and remaining budget is \( b_\tau = b \), the system consumes one unit of budget with probability \( b/\tau \), and consumes nothing with probability \( 1 - b/\tau \). Thus, when focusing on remaining budget, we can view the ALP algorithm as a “sampling problem without replacement” problem as follows.

**Mapping ALP to Sampling without Replacement:** Consider \( T \) balls in an urn, including \( B \) black balls and \( T - B \) white balls. Running ALP is equivalent to randomly drawing a ball without replacement. Taking an action \( A_t > 0 \) is equivalent to drawing a black ball and taking the dummy action \( A_t = 0 \) is equivalent to drawing a white ball. The event that \( b_\tau = b \) is equivalent to the event that the agent draws \( T - \tau \) balls, and the number of drawn black balls is \( B - b \).

Therefore, the amount of budget consumed by the agent, i.e., \( B - b_\tau \), follows the hypergeometric distribution and so does the remaining budget \( b_\tau \) due to the symmetry of the hypergeometric distribution (Dubhashi & Panconesi, 2009). Then, properties of the hypergeometric distribution can be used to characterize the evolution of remaining budget.

**Lemma 2.** Under the ALP algorithm, the remaining budget \( b_\tau \) has the following properties:

- \( b_\tau \) follows the hypergeometric distribution, i.e., for any \( 0 \leq b \leq \min\{\tau, b\} \),

\[
\mathbb{P}\{b_\tau = b\} = \binom{B}{b} \binom{T-B}{\tau-b} \binom{T}{\tau},
\]

- The expectation of \( b_\tau \) is \( \mathbb{E}[b_\tau] = \rho \tau \), and the variance of \( b_\tau \) is \( \text{Var}(b_\tau) = \frac{T-\tau}{\tau} \rho (1-\rho) \).

- For any positive number \( \delta \) satisfying \( 0 < \delta < \min\{\rho, 1-\rho\} \), the tail distribution of \( b_\tau \) satisfies

\[
\begin{align*}
\mathbb{P}\{b_\tau < (\rho-\delta)\tau\} & \leq e^{-2\delta^2 \tau}, \\
\mathbb{P}\{b_\tau > (\rho+\delta)\tau\} & \leq e^{-2\delta^2 \tau}.
\end{align*}
\]

Now, we are ready to investigate the performance of the ALP algorithm. We bound the regret of ALP by comparing its performance with the upper bound provided by \( LP_{T,B} \). Intuitively, from Lemma 2, the expectation of the average remaining budget, i.e., \( \mathbb{E}[b_\tau/\tau] \), remains unchanged over time. Thus, if the threshold stays the same, i.e., \( j(b_\tau/\tau) = j(\rho) \) for all possible \( b_\tau \)’s, then the expected single-step value of ALP will be the same as the optimal value of the static LP problem \( LP_{T,B} \), i.e., \( \mathbb{E}[v(b_\tau/\tau)] = v(\rho) \). The difference in the expected total reward results from the changing of thresholds. Lemma 2 also states that the average remaining budget \( b_\tau/\tau \) stays in a neighborhood of the initial average budget \( \rho \) with high probability. Hence, if the initial average budget \( \rho \) is not on boundaries, i.e., the critical values under which the threshold \( j(\rho) \) changes, then the probability of threshold changing is bounded. Therefore, we can show that the ALP algorithm achieves a very good performance within a constant distance from the optimum, except for certain boundary cases. Specifically, for \( 1 \leq j \leq J \), let \( q_j \) be the cumulative distribution function, i.e., \( q_j = \sum_{j'=1}^{j} \pi_{j'} \), and w.l.o.g., let \( q_0 = 0 \). The following theorem states the approximate optimality of ALP for the cases where \( \rho \neq q_j \) (\( j = 1, 2, \ldots, J - 1 \)). We note that \( j = 0 \) and \( j = J \) are trivial cases where ALP is optimal.

**Theorem 1.** Given any fixed \( \rho \in (0, 1) \) satisfying \( \rho \neq q_j \), \( j = 1, 2, \ldots, J - 1 \), the ALP algorithm achieves an \( O(1) \) regret. Specifically,

\[
U^*(T, B) - U_{ALP}(T, B) \leq \frac{u^*_i - u^*_j}{1 - e^{-2\delta^2 \tau}},
\]

where \( \delta = \min\{\rho - q_j(\rho), q_j(\rho) + 1 - \rho\} \).

**Proof.** The proof of this theorem uses the following two facts derived from Lemma 2: \( \mathbb{E}[v(b_\tau/\tau)] = v(\rho) \) if the threshold \( j(b_\tau/\tau) = j(\rho) \) for all possible \( b_\tau \)’s, and the probability that \( j(b_\tau/\tau) \neq j(\rho) \) decays exponentially. Please referred to Appendix B.1 of the supplementary material for details.

When considering the boundary cases, we can show similarly that the ALP achieves \( O(\sqrt{T}) \) regret.
4. Upper-Confidence-Bound Algorithms for Constrained Contextual Bandits

In this section, we come back to the original constrained contextual bandits, where the agent does not have information of the expected rewards. We still focus on unit-cost systems and assume the agent knows the context distribution.

As we can see in Section 3, the PB and ALP algorithms only require the ordering of the expected rewards. This property allows us to combine estimation policies that can provide correct ranking with high probability in a short period. Here, combining with the upper-confidence-bound (UCB) method (Auer et al., 2002), we propose UCB-PB and UCB-ALP algorithms for constrained contextual bandits.

4.1. UCB: Notations and Property

Let $C_{j,k}(t)$ be the number of times that action $k$ ($k \in \mathcal{A}$) has been taken under context $j$ up to round $t$. For $C_{j,k}(t) \geq 0$, let $\tilde{u}_{j,k}(t)$ be the empirical reward of action $k$ under context $j$ up to round $t$, i.e.,

$$\tilde{u}_{j,k}(t) = \frac{\sum_{t' = 1}^{t} 1(Y_{t'} = j, A_{t'} = k)}{C_{j,k}(t)}, \quad (8)$$

where $1(\cdot)$ is the indicator function. Furthermore, we define the UCB of the expected reward for the context-action pair $(j,k)$ as follows:

$$\hat{u}_{j,k}(t) = \tilde{u}_{j,k}(t) + \sqrt{\frac{\log t}{2 C_{j,k}(t)}}, \quad (9)$$

and define the UCB of the maximum expected reward under context $j$ as

$$\hat{u}_{j}^*(t) = \max_{k \in \mathcal{A}} \hat{u}_{j,k}(t). \quad (10)$$

As suggested in (Garivier & Cappé, 2011), we use a smaller coefficient in the exploration term $\sqrt{\frac{\log t}{2 C_{j,k}(t)}}$ than the traditional UCB algorithm (Auer et al., 2002) to achieve better performance.

We present the following property of UCB that is important in regret analysis.

**Theorem 2.** Given any fixed $\rho = q_{j}$, $j = 1, 2, \ldots, J - 1$, the ALP algorithm achieves $O(\sqrt{T})$ regret. Specifically,

$$U^*(T, B) - U_{\text{ALP}}(T, B) \leq \Theta(\rho)\sqrt{T} + \frac{u_1^* - u_J^*}{1 - e^{-2(\rho)^2}},$$

where $\Theta(\rho) = 2(u_1^* - u_J^*)\sqrt{|\rho(1 - \rho)|}$ and $\delta' = \min\{\rho - q_{j}(\rho) - 1, q_{j}(\rho) + 1 - \rho\}$.

**Proof.** See Appendix B.2 of the supplementary material.

4.2. UCB-PB for Two-Context Bandits

In this section, we propose the UCB-based Procrastinate-for-the-Better (UCB-PB) algorithm for solving the constrained contextual bandit problem with two contexts, i.e., $J = 2$.

**Algorithm 1 UCB-PB**

**Input:** Time horizon $T$, budget $B$;

**Init:** Remaining time $\tau = T$, remaining budget $b = B$;

$C_{j,k}(0) = 0$, $\hat{u}_{j,k}(0) = 1$, $\hat{u}_{j,k}(0) = 1$, for all $j \in \mathcal{A}$ and $k \in \mathcal{A}$; $\hat{u}_{j}^*(0) = 1$ for all $j \in \mathcal{A}$;

for $t = 1$ to $T$ do

$k_{j}^*(t) \leftarrow \arg \max_{k} \hat{u}_{j,k}(t)$, $\forall j$;

$\hat{u}_{j}^*(t) \leftarrow \hat{u}_{j,k}^*(t)$, $\forall j$;

$j_{j}^*(t) \leftarrow \arg \max_{j} \hat{u}_{j,k}(t)$;

if $b > \tau$ or ($0 < b < \tau$ and $X_t = j^*(t)$) then

Take action $k_{X_t}(t)$;

end if

Update $\tau, b, C_{j,k}(t), \hat{u}_{j,k}(t)$, and $\hat{u}_{j,k}(t)$;

end for

As shown in Algorithm 1, the agent maintains UCB estimates $\hat{u}_{j,k}(t)$’s for the expected rewards of all context-action pairs. In each round, the agent implements the PB algorithm defined in Section 3.

Next, we study the regret of the UCB-PB algorithm. Although the PB and UCB-PB algorithms seem trivial, the regret analysis of UCB-PB is not. From Lemma 3, we can see that if a suboptimal context-action pair has been executed enough times, then the probability of making a ranking error will be small. Unlike non-contextual bandits however, the context-action pair with the highest UCB in a round might not be executable, as the context of that round could be different. Fortunately, we can show that if a context-action pair has been observed to have the highest UCB among all context-action pairs for many times, then it
will be executed many times with high probability. Specifically, let \( \{X_t, A_t\} \) be the context-action pair that has the highest UCB in round \( t \), and let \( \hat{C}_{j,k}(t) = \sum_{t'=1}^t \mathbb{I}(X_{t'} = j, A_{t'} = k) \). Then, we have the following lemma, and its proof can be found in Appendix C.1 of the supplementary material.

**Lemma 4.** Under UCB-PB, for a positive number \( \epsilon \in (0, 1) \), we have
\[
\mathbb{P}\{ C_{j,k}(t) < \pi_j(1-\epsilon)n, \hat{C}_{j,k}(t) \geq n, b_\tau > 0 \} \leq e^{-2c^2 n}.
\]

(12)

Based on Lemmas 3 and 4, we show that the UCB-PB algorithm achieves logarithmic regret.

**Theorem 3.** For a constrained contextual bandit with unit-cost and two contexts, the UCB-PB algorithm achieves logarithmic regret as \( T \) goes to infinity, i.e.,
\[
\limsup_{T \to \infty} \frac{R_{\text{UCB-PB}}(T, B)}{\log T} \leq \sum_{k=1}^K \left[ \frac{27}{2 \pi^2 \Delta_{j,k}^{(1)}} + 2 \Delta_{j,k}^{(2)} \right] + \sum_{j=1}^2 \sum_{k \neq k'} \left[ \frac{2}{\Delta_{j,k}^{(1)}} + 2 \Delta_{j,k}^{(2)} \right].
\]

**Proof.** The above theorem is proved by partitioning the regret into two parts, where the first part results from the ranking errors among contexts, and the latter part results from the ranking errors among actions under each context. Then, using Lemmas 3 and 4, we can bound them, respectively. Details can be found in Appendix C.2 of the supplementary material.

### 4.3. UCB-ALP for Multi-Context Bandits

We now study the general multi-context bandit problem with \( J \geq 2 \). For \( J \geq 2 \), we propose a UCB-based adaptive linear programming (UCB-ALP) algorithm, as shown in Algorithm 2. As indicated by the name, the UCB-ALP algorithm maintains UCB estimates of expected rewards for all context-action pairs and then implements the ALP algorithm. Note that the UCB estimates \( \hat{u}_{\cdot,k}^* \)'s may be non-decreasing in \( j \). Thus, the solution of \( \mathcal{LP}_{\tau,b} \) based on \( \hat{u}_{\cdot,k}^* \) depends on the actual order of \( \hat{u}_{\cdot,k}^* \)'s and may be different from Eq. (4). We use \( \hat{p}_j(\cdot) \) rather than \( p_j(\cdot) \) to indicate this difference.

Next, we study the regret of UCB-ALP. Recall that \( q_j = \sum_{j'=1}^J \pi_{j'} (1 \leq j' \leq J) \) as defined in Section 3. We first study the regret of UCB-ALP for the non-boundary cases, i.e., \( \rho \neq q_j (1 \leq j \leq J-1) \), and discuss boundary cases later.

As mentioned earlier, in Algorithm 2, the solution of \( \mathcal{LP}_{\tau,b} \) mainly depends on the ordering of UCBs and its estimation error is one source of regret. Recall that \( \hat{j}(\rho) = \max\{j : \sum_{j'=1}^j \pi_{j'} \leq \rho \} \) is the threshold for the static LP problem \( \mathcal{LP}_{\tau,b} \). We define the following events that capture all possible ordering results based on UCBs:
\[
\mathcal{E}_{\text{rank},0}(t) = \{ \forall j \leq \hat{j}(\rho), \hat{u}_{\cdot,k}^* > \hat{u}_{\cdot,k}^* + 1 \};
\]
\[
\mathcal{E}_{\text{rank},1}(t) = \{ \exists j > \hat{j}(\rho), \hat{u}_{\cdot,k}^* > \hat{u}_{\cdot,k}^* + 1 \};
\]
\[
\mathcal{E}_{\text{rank},2}(t) = \{ \exists j > \hat{j}(\rho) + 1, \hat{u}_{\cdot,k}^* > \hat{u}_{\cdot,k}^* + 1 \}.
\]

The event \( \mathcal{E}_{\text{rank},0}(t) \) indicates a roughly correct context ranking because the UCB-ALP will obtain a correct solution for \( \mathcal{LP}_{\tau,b} \), if the average remaining budget \( b_\tau / \tau \) satisfies \( b_\tau / \tau \in [q_{\hat{j}(\rho)}, q_{\hat{j}(\rho)+1}] \). The last two events \( \mathcal{E}_{\text{rank},s}(t), s = 1, 2 \), represent two types of context ranking errors. Let \( T(s) \) be the number of times that ranking event \( \mathcal{E}_{s}(t) \) occurs up to round \( T \), i.e.,
\[
T(s) = \sum_{t=1}^T \mathbb{I}(\mathcal{E}_{s}(t)), 0 \leq s \leq 2.
\]

The following lemma shows that under UCB-ALP, the number of ranking errors is bounded by \( O(\log T) \). Details of the proof can be found in Appendix D.1 of the supplementary material.

**Lemma 5.** Given \( \pi_j \)'s, \( u_{j,k} \)'s and a fixed \( \rho \in [0,1] \), \( \rho \neq q_j (1 \leq j \leq J-1) \), under the UCB-ALP algorithm, we have
\[
\limsup_{T \to \infty} \frac{\mathbb{E}[T(s)]}{\log T} \leq \sum_{j=1}^J \sum_{k=1}^K 2 \hat{q}_j(\rho)(1+\Delta_{j,k}^{(1)})(\Delta_{j,k}^{(1)}+1)^{2} + K\hat{\gamma}(\rho).
\]
where \( g_j = \min \left\{ \pi_j, \frac{1}{2} (\rho - q_j(\rho)), \frac{1}{2} (q_j(\rho) + 1 - \rho) \right\} \).

Under the assumption that the agent knows the context distribution, we can verify that Lemma 2 holds under UCB-ALP. Thus, the average remaining budget \( b_{\tau} / \tau \) stays in a neighborhood of \( \rho \) with high probability. Combining with Lemma 5, we can show that UCB-ALP achieves logarithmic regret except in boundary cases.

**Theorem 4.** Given \( \pi_j \)'s, \( u_{j,k} \)'s and a fixed \( \rho \in [0,1], \rho \neq q_j \) (1 ≤ \( j \leq J - 1 \)), the UCB-ALP algorithm achieves logarithmic regret as \( T \) goes to infinity. Specifically,

\[
\limsup_{T \to \infty} \frac{R_{\text{UCB-ALP}}(T, B)}{\log T} \leq \Theta^{(c)}_{nb} + \Theta^{(a)},
\]

where

\[
\Theta^{(c)}_{nb} = \left[ \bar{u}^* + v(\rho) \right] \left\{ \sum_{j=1}^{j(\rho)-1} \sum_{k=1}^{K} \frac{27}{2g_j(\rho) + 1 + [\Delta_j(\rho)]^2} \right\} + \sum_{j=j(\rho)+1}^{J} \sum_{k=1}^{K} \frac{27}{2g_j(\rho) [\Delta_j(\rho)]^2} + K J,
\]

\[
\Theta^{(a)} = \sum_{j=1}^{J} \sum_{k \neq k^*_j} \left( \frac{2}{\Delta_j(k)} + 2\Delta_j(k) \right),
\]

\( \bar{u}^* = \sum_{j=1}^{J} \pi_j u^*_j \), and \( v(\rho) \) is defined in (5).

**Proof.** Similar to Theorem 3, we divide the regret \( R_{\text{UCB-ALP}}(T, B) \) into two parts, the regret from context ranking errors and the regret from action ranking errors, and bound them respectively. The difference is that when bounding the regret from context ranking errors, we need to consider both the context ranking errors and the fluctuation of the remaining budget. Details can be found in Appendix E.2 of the supplementary material.

Using similar methods in the proof of Theorems 2 and 4, we obtain the upper bound for the regret of UCB-ALP on boundaries, i.e., \( \rho = q_j \).

**Theorem 5.** Given \( \pi_j \)'s, \( u_{j,k} \)'s and a fixed \( \rho = q_j \) (\( j = 1, 2, \ldots, J - 1 \)), the UCB-ALP algorithm achieves \( O(\sqrt{T}) \) regret. Specifically,

\[
R_{\text{UCB-ALP}}(T, B) \leq \Theta^{(c)} \sqrt{T} + [\Theta^{(c)}_{b} + \Theta^{(a)}] \log T + O(1),
\]

where \( \Theta^{(c)} = 2(u^*_1 - u^*_j) \sqrt{\rho(1-\rho)} \),

\[
\Theta^{(c)}_{b} = \left[ \bar{u}^* + v(\rho) \right] \left\{ \sum_{j=1}^{j(\rho)-1} \sum_{k=1}^{K} \frac{27}{2g_j(\rho) + 1 + [\Delta_j(\rho)]^2} \right\} + \sum_{j=j(\rho)+1}^{J} \sum_{k=1}^{K} \frac{27}{2g_j(\rho) [\Delta_j(\rho)]^2} + K J,
\]

\[ g'_j = \min \left\{ \pi_j, \frac{1}{2} (\rho - q'_j(\rho)) \right\}, \]

where \( \bar{u}^* \) and \( v(\rho) \) are defined in Theorem 4.

We keep both the \( \sqrt{T} \) and \( \log T \) terms in Theorem 5 because the constant in the \( \log T \) term is much larger than that in the \( \sqrt{T} \) term. Therefore, the \( \log T \) term may dominate the regret particularly when the number of context-action pairs is large.

### 5. Constrained Contextual Bandits with Heterogeneous Costs

In this section, we discuss how to use the insight from unit-cost systems for the design of algorithms in heterogeneous-cost systems where cost \( c_{j,k} \) depends on \( j \) and \( k \). We only discuss the main results here due to space limitations. Details can be found in Appendix E.3 of the supplementary material.

When the statistics of bandits are known to the agent, the ALP algorithm can be generalized to heterogeneous-cost systems. With heterogeneous costs, the quality of an action \( k \) under a context \( j \) is roughly captured by its *normalized expected reward*, defined as \( \eta_{j,k} = u_{j,k}/c_{j,k} \). However, the agent cannot only focus on the “best” action, i.e., \( k^*_j = \arg \max_{k \in A} \eta_{j,k} \), for context \( j \). This is because there may exist another action \( k' \) such that \( \eta_{j,k'} < \eta_{j,k^*_j} \), but \( u_{j,k'} > u_{j,k^*_j} \) (and of course, \( c_{j,k'} > c_{j,k^*_j} \)). If the budget allocated to context \( j \) is sufficient, then the agent may take action \( k' \) to maximize the expected reward. Therefore, the ALP algorithm in this case needs to decide the probability to take action \( k \) under context \( j \), by solving an LP problem with an additional constraint that only one action can be taken under each context. We can show that ALP achieves \( O(1) \) regret in non-boundary cases, and \( O(\sqrt{T}) \) regret in boundary cases. We note that the regret analysis of ALP in this case is much more difficult due to the additional constraint that couples all actions under each context.

When the expected rewards are unknown, it is difficult in general to combine ALP with the UCB method since the ALP algorithm in this case not only requires the ordering of \( \eta_{j,k} \)'s, but also the ordering of \( u_{j,k} \)'s and the ratios \( \frac{u_{j,k_1} - u_{j,k_2}}{c_{j,k_1} - c_{j,k_2}} \). As a special case, when all actions have the same cost under a given context, i.e., \( c_{j,k} = c_j \) for all \( k \) and \( j \), the UCB-ALP can be extended by defining the UCBs for the normalized expected rewards \( \eta_{j,k} \) with appropriate
scaling such that \( Y_{k,t}/c_j \in [0,1] \) if \( X_t = j \). However, it is still an open problem to design algorithms for general heterogeneous-cost systems.

6. Conclusion and Future Work

We study algorithms for constrained contextual bandits with a budget \( B \) and a time horizon \( T \). We first study algorithms for unit-cost systems. Starting with the case where the statistics of bandits are known, we identify an optimal algorithm PB for two-context systems and propose a near-optimal algorithm ALP for general multi-context systems. Then, combining with the UCB method, we present UCB-PB and UCB-ALP algorithms for the case where the expected rewards are unknown. We show that UCB-ALP achieves \( O(\log T) \) regret except in boundary cases, where it achieves \( O(\sqrt{T}) \) regret. The insight from the unit-cost systems can be used in the design and analysis of constrained contextual bandits with heterogeneous costs. It is our future work to further study systems with unknown context distribution and/or general costs.

References

Agarwal, Alekh, Hsu, Daniel, Kale, Satyen, Langford, John, Li, Lihong, and Schapire, Robert E. Taming the monster: A fast and simple algorithm for contextual bandits. In *International Conference on Machine Learning (ICML)*, 2014.

Agrawal, Shipra and Devanur, Nikhil R. Bandits with concave rewards and convex knapsacks. In *ACM Conference on Economics and Computation*, pp. 989–1006. ACM, 2014.

Auer, Peter, Cesa-Bianchi, Nicolo, and Fischer, Paul. Finite-time analysis of the multiarmed bandit problem. *Machine learning*, 47(2-3):235–256, 2002.

Babaioff, Moshe, Dughmi, Shaddin, Kleinberg, Robert, and Slivkins, Aleksandrs. Dynamic pricing with limited supply. In *ACM Conference on Electronic Commerce (EC)*, pp. 74–91, 2012.

Badanidiyuru, Ashwinkumar, Kleinberg, Robert, and Slivkins, Aleksandrs. Bandits with knapsacks. In *IEEE 54th Annual Symposium on Foundations of Computer Science (FOCS)*, pp. 207–216, 2013.

Badanidiyuru, Ashwinkumar, Langford, John, and Slivkins, Aleksandrs. Resourceful contextual bandits. In *Conference on Learning Theory (COLT)*, 2014.

Dubhashi, Devdatt P and Panconesi, Alessandro. *Concentration of measure for the analysis of randomized algorithms*. Cambridge University Press, 2009.

Garivier, Aurélien and Cappé, Olivier. The KL-UCB algorithm for bounded stochastic bandits and beyond. In *Conference on Learning Theory (COLT)*, pp. 359–376, 2011.

Golovin, Daniel and Krause, Andreas. Dealing with partial feedback, 2009. URL http://courses.cms.caltech.edu/cs253/slides/cs253-07-ucb1.pdf.

Jiang, Chong and Srikant, R. Bandits with budgets. In *IEEE 52nd Annual Conference on Decision and Control (CDC)*, pp. 5345–5350. IEEE, 2013.

Langford, John and Zhang, Tong. The epoch-greedy algorithm for contextual multi-armed bandits. In *Advances in Neural Information Processing Systems (NIPS)*, pp. 817–824, 2007.

Li, Lihong, Chu, Wei, Langford, John, and Schapire, Robert E. A contextual-bandit approach to personalized news article recommendation. In *ACM International Conference on World Wide Web (WWW)*, pp. 661–670. ACM, 2010.

Lu, Tyler, Pál, Dávid, and Pál, Martin. Contextual multi-armed bandits. In *International Conference on Artificial Intelligence and Statistics*, pp. 485–492, 2010.

Slivkins, Aleksandrs. Contextual bandits with similarity information. In *Conference on Learning Theory (COLT)*, pp. 679–701, 2011.

Slivkins, Aleksandrs. Dynamic ad allocation: Bandits with budgets. *arXiv preprint arXiv:1306.0155*, 2013.

Tang, Liang, Rosales, Romer, Singh, Ajit, and Agarwal, Deepak. Automatic ad format selection via contextual bandits. In *ACM International Conference on Information & Knowledge Management*, pp. 1587–1594. ACM, 2013.

Tran-Thanh, Long, Chapman, Archie, Munoz De Cote Flores Luna, Jose Enrique, Rogers, Alex, and Jennings, Nicholas R. \( \epsilon \)-First policies for budget-limited multi-armed bandits. In *AAAI Conference on Artificial Intelligence*, 2010.

Tran-Thanh, Long, Chapman, Archie C, Rogers, Alex, and Jennings, Nicholas R. Knapsack based optimal policies for budget-limited multi-armed bandits. In *AAAI Conference on Artificial Intelligence*, 2012.

ul Hassan, Umair and Curry, Edward. A multi-armed bandit approach to online spatial task assignment. In *IEEE International Conference on Ubiquitous Intelligence and Computing*. IEEE, 2014.
Appendices

A. Numerical Experiments

In this section, we evaluate the regret of the proposed algorithms through numerical simulations. We study the performance of the proposed algorithms for two-context and general multi-context in unit-cost systems, respectively. In the case with known statistics, we compare the proposed PB (two-context case) and ALP algorithms with Fixed LP (FLP) algorithm that uses a fixed average budget constraint \( B/T \). Then, the UCB-based FLP, i.e., UCB-FLP, is evaluated in the case without knowledge of expected rewards. We also evaluate algorithms for the case without knowledge of context distribution. When the context distribution is unknown to the agent, we propose an algorithm, called Empirical ALP (EALP), that uses the empirical distribution (histogram) of context for making decisions, in the case with known expected rewards. Then, the UCB-based EALP is proposed for the case without knowledge of expected rewards. The results are averaged from 5,000 independent runs of the simulations.

A.1. Two-Context Systems

We first consider a two-context scenario with \( K = 3 \) arms and Bernoulli rewards: the context distribution vector is \( \pi = [0.4, 0.6] \), the expected rewards are \( u_1 = 0.8 \times [1/3, 2/3, 1] \) for context 1, and \( u_2 = 0.4 \times [1/3, 2/3, 1] \) for context 2. The boundary is \( q_1 = \pi_1 = 0.4 \) and we study the cases with normalized budget \( \rho = 0.39, 0.4, \) and 0.41, respectively.

Figure 1 shows the regret of different algorithms in the case with known expected rewards. In the non-boundary cases (i.e., \( \rho = 0.39, 0.41 \)), the ALP algorithm achieves near optimal performance. Even without the knowledge of context distribution, the EALP algorithm performs much better than FLP. In the boundary case, i.e., \( \rho = 0.4 \), the regret of ALP increases with \( T \) but is still lower than that of FLP. The EALP algorithm achieves higher regret than ALP and FLP due to the empirical distribution errors.

Figure 2 shows the regret of different algorithms in the case without knowledge of expected rewards. We can see that in the non-boundary cases, UCB-ALP and UCB-EALP achieves regret that is very close to UCB-PB and outperforms UCB-FLP. Interestingly, we can even see that UCB-ALP achieves slightly lower regret than UCB-PB in the case with \( \rho = 0.41 \). This is because under UCB-PB, the better context may be skipped and wasted if it does not have the highest UCB. In contrast, the UCB-ALP algorithm may allocate certain resource to the better context, even when it does not have the highest UCB. On the boundary case, the regrets of UCB-ALP and UCB-EALP become larger than that of UCB-PB, but are still sublinear in \( T \).

A.2. Multi-Context Systems

Next, we study a multi-context scenario with \( J = 10 \) contexts, \( K = 5 \) arms, and Bernoulli rewards. Specifically, the context distribution vector is \( \pi = [0.025, 0.05, 0.075, 0.15, 0.2, 0.2, 0.15, 0.075, 0.05, 0.025] \). The expected reward of action \( k \) under context \( j \) is \( u_{j,k} = \frac{j}{J} \). One boundary in this system is \( q_0 = 0.5 \). We study the cases with average budget \( \rho = 0.49, 0.5, \) and 0.51, respectively. In this case, it is difficult to calculate the expected total reward obtained by the oracle solution. Thus, we calculate the regret by comparing with the upper bound, i.e., \( \hat{U}(T,B) = Tv(\rho) \).

Figure 3 shows the regret of different algorithms in the case with known expected rewards. In the non-boundary cases, both the ALP and EALP algorithm achieve similar performance as in the two-context case. The regret of EALP is even lower than FLP in the boundary case, since the ratio of contexts that are executed with correct probability is higher than that in the two-context systems.

Figure 4 shows the regret of different algorithms in the case without knowledge of expected rewards. We can see that all algorithms achieve sublinear regret, but the difference between the non-boundary cases and the boundary case is small. As we can see from Theorem 5, this is because when the number of contexts and the number of actions are large, the constant in the \( \log T \) term is much larger than that in the \( \sqrt{T} \) term. Hence, the \( \log T \) term dominates the regret and the impact of the \( \sqrt{T} \) term could be small.

B. Near Optimality of ALP

B.1. Theorem 1: Non-Boundary Cases

According to Lemma 1, \( \hat{U}(T,B) \) is an upper bound on the total expected reward. Thus,
\[
\begin{align*}
U^*(T,B) - U_{ALP}(T,B) & \leq \hat{U}(T,B) - U_{ALP}(T,B) \\
& = \sum_{\tau=1}^{T} \left\{ v(\rho) - \mathbb{E}[v(b_{\tau}/\tau)] \right\}.
\end{align*}
\]

To evaluate the gap between the single-round values, we define an auxiliary function \( \tilde{v}(b/\tau) \) for a given \( \rho \) as follows:
\[
\tilde{v}(b/\tau) = \sum_{j=1}^{\tilde{j}(\rho)} \pi_j u_j^* + \pi_{\tilde{j}(\rho)+1} \tilde{p}_{\tilde{j}(\rho)+1}(b/\tau) u_{\tilde{j}(\rho)+1}^* \tag{15}
\]
where
\[
\tilde{p}_{\tilde{j}(\rho)+1}(b/\tau) = \frac{b/\tau - \sum_{j=1}^{\tilde{j}(\rho)} \pi_j}{\pi_{\tilde{j}(\rho)+1}}.
\]

This auxiliary function bridges the gap of single-round values, \( v(\rho) \) and \( \mathbb{E}[v(b_{\tau}/\tau)] \), as follows:
Figure 1. Comparison of algorithms for the two-context systems with perfect knowledge ($\pi_1 = 0.4, \pi_2 = 0.6$), (a) $\rho = 0.39$, (b) $\rho = 0.4$, (c) $\rho = 0.41$.

Figure 2. Comparison of algorithms for the two-context systems without perfect knowledge ($\pi_1 = 0.4, \pi_2 = 0.6$), (a) $\rho = 0.39$, (b) $\rho = 0.4$, (c) $\rho = 0.41$.

Figure 3. Comparison of algorithms for the multi-context systems with perfect knowledge ($Q_5 = 0.5$), (a) $\rho = 0.49$, (b) $\rho = 0.5$, (c) $\rho = 0.51$.

Figure 4. Comparison of algorithms for the multi-context systems without perfect knowledge ($Q_5 = 0.5$), (a) $\rho = 0.49$, (b) $\rho = 0.5$, (c) $\rho = 0.51$. 
First, we note that $\tilde{v}(b/\tau)$ uses the same threshold $\tilde{j}(\rho)$ as in $v(\rho)$. The only difference between $\tilde{v}(b/\tau)$ and $v(\rho)$ is that $\tilde{v}(b/\tau)$ uses the instantaneous average budget $b/\tau$ instead of the fixed average budget $\rho$. Considering all possible $b$’s and according to Lemma 2, we have

$$v(\rho) - \mathbb{E}[\tilde{v}(b/\tau)] = \{\rho - \mathbb{E}[b/\tau]\} u_{j(\rho)+1}^* = 0. \quad (16)$$

Second, compared with $v(b/\tau)$, the difference of the auxiliary function $\tilde{v}(b/\tau)$ comes from the event of $j(b/\tau) \neq \tilde{j}(\rho)$, which occurs when $b/\tau < q_{j(\rho)}$ or $b/\tau > q_{j(\rho)+1}$. Because $\rho \neq q_j$, $1 \leq j \leq J - 1$, there exists a positive number $\delta = \min\{\rho - q_j, q_{j+1} - \rho\}$ such that for all $\rho - \delta < \rho < \rho + \delta$,

$$\tilde{j}(\rho') = \tilde{j}(\rho).$$

Therefore, for all $b$ satisfying $\rho - \delta \leq b/\tau \leq \rho + \delta$, $v(b/\tau) = \tilde{v}(b/\tau)$.

If $b/\tau < \rho - \delta$, then

$$\tilde{v}(b/\tau) - v(b/\tau) \leq (u_1^* - u_j^*) \sum_{j=j(b/\tau)+1}^{\tilde{j}(\rho)} \pi_j.$$  \quad (17)

Similarly, if $b/\tau > \rho + \delta$, then

$$\tilde{v}(b/\tau) - v(b/\tau) \leq (u_1^* - u_j^*) \sum_{j=j(b/\tau)+1}^{\tilde{j}(\rho)} \pi_j.$$  \quad (18)

Summing all the above three cases ($\rho - \delta \leq b/\tau \leq \rho + \delta$), $b/\tau < \rho - \delta$, and $b/\tau > \rho + \delta$ and using Eq. (16), we have

$$v(\rho) - \mathbb{E}[\tilde{v}(b/\tau)] = v(\rho) - \mathbb{E}[\tilde{v}(b/\tau)] + \mathbb{E}[\tilde{v}(b/\tau) - v(b/\tau)]$$

$$\leq \frac{1}{2} \sum_{b/\tau < \rho - \delta} \mathbb{P}(b/\tau) + \frac{1}{2} \sum_{b/\tau > \rho + \delta} \mathbb{P}(b/\tau)$$

$$\leq q_{j(\rho)}(u_1^* - u_j^*) \mathbb{P}(b/\tau < \rho - \delta)$$

$$+ (1 - q_{j(\rho)}) (u_1^* - u_j^*) \mathbb{P}(b/\tau > \rho + \delta)$$

$$\leq (u_1^* - u_j^*) e^{-2\delta^2 \tau}. \quad (19)$$

The conclusion of Theorem 1 then follows by substituting Eq. (19) into Eq. (14).

**B.2. Theorem 2: Boundary Cases**

The proof of Theorem 2 is similar to that of Theorem 1, except for the case of $\rho - \delta \leq b/\tau < \rho$. Specifically, when $\rho = q_j(\rho)$, let $\delta = \min\{\rho - q_j, q_{j+1} - \rho\}$. From the proof of Theorem 1, we know that

$$v(\rho) - \mathbb{E}[\tilde{v}(b/\tau)] = 0. \quad (20)$$

In addition, if $\rho \leq b/\tau \leq \rho + \delta$, then $\tilde{j}(b/\tau) = \tilde{j}(\rho)$ and $\tilde{v}(b/\tau) = \tilde{v}(b/\tau)$.

If $\rho - \delta \leq b/\tau < \rho$, we have $\tilde{j}(b/\tau) = \tilde{j}(\rho) - 1$, and

$$\tilde{v}(b/\tau) - v(b/\tau) \leq (u_1^* - u_j^*) \sum_{j=j(b/\tau)+1}^{\tilde{j}(\rho)} \pi_j.$$  \quad (21)

Moreover, we still have (17) if $b/\tau < \rho - \delta$, and (18) if $b/\tau > \rho + \delta$.

Compared with the proof of Theorem 1, we know that the only difference relies on the case of $\rho - \delta \leq b/\tau < \rho$. Thus, summing all the above cases and using the results in the analysis of Theorem 1, we have

$$\mathbb{E}[\tilde{v}(b/\tau) - \tilde{v}(b/\tau)]$$

$$\leq (u_1^* - u_j^*) \left\{ \mathbb{E}[b/\tau - \rho] + e^{-2\delta^2 \tau} \right\}$$

$$\leq (u_1^* - u_j^*) \left[ \frac{\text{Var}(b/\tau)}{\tau^2} + e^{-2\delta^2 \tau} \right].$$
Consequently,

\[
\begin{align*}
U^*(T, B) - U_{ALP}(T, B) & \leq \hat{U}(T, B) - U_{ALP}(T, B) \\
& = \sum_{\tau=1}^{T} \{ v(\rho) - \mathbb{E}[v(b_{\tau}/\tau)] \} \\
& = \sum_{\tau=1}^{T} \{ v(\rho) - \mathbb{E}[\hat{v}(b_{\tau}/\tau)] \} + \sum_{\tau=1}^{T} \mathbb{E}[\hat{v}(b_{\tau}/\tau) - v(b_{\tau}/\tau)] \\
& \leq (u^*_1 - u^*_j) \sum_{\tau=1}^{T} \left[ \sqrt{\frac{\text{Var}(b_{\tau})}{\tau^2}} + e^{-2\beta^2 \tau} \right] \\
& = (u^*_1 - u^*_j) \sum_{\tau=1}^{T} \left[ \frac{(T - \tau)\rho(1 - \rho)}{(T - 1)\tau} + e^{-2\beta^2 \tau} \right] \\
& \leq (u^*_1 - u^*_j) \sqrt{\rho(1 - \rho)} \sum_{\tau=1}^{T} \left[ \frac{1}{\tau^2} + \frac{u^*_1 - u^*_j}{1 - e^{-2\beta^2 \tau}} \right] \\
& \leq 2\sqrt{\rho(1 - \rho)}(u^*_1 - u^*_j) \sqrt{T} + \frac{u^*_1 - u^*_j}{1 - e^{-2\beta^2 \tau}}.
\end{align*}
\]

C. Regret of UCB-PB

C.1. Lemma 4: Occurrence Bound

Recall that \((\hat{X}_t, \hat{A}_t)\) is the context-action pair that has the highest UCB in round \(t\), and the remaining time \(\tau = T - t + 1\). Let \(\hat{X}_t = (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_t)\) and \(A_t = (\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_t)\). Furthermore, let \(\hat{x}_t\) and \(\hat{a}_t\) be the realizations of \(\hat{X}_t\) and \(A_t\), respectively. Let \(\hat{C}_{j, k}(\hat{a}, \hat{a}) (t) = \sum_{t'=1}^{T} 1(\hat{x}_{t'} = j, \hat{a}_{t'} = a)\). Then,

\[
\mathbb{P}\{C_{j, k}(t) < \pi_j(1 - \epsilon)n, \hat{C}_{j, k}(t) \geq n, b_r > 0 \} = \sum_{\hat{C}_{j, k}(\hat{a}, \hat{a}) (t) \geq n} \mathbb{P}\{C_{j, k}(t) < \pi_j(1 - \epsilon)n, (\hat{X}_t, \hat{A}_t) = (\hat{x}_t, \hat{a}_t), b_r > 0 \}.
\]

For any possible realization \((\hat{x}_t, \hat{a}_t)\), we evaluate the probability \(\mathbb{P}\{C_{j, k}(t) < \pi_j(1 - \epsilon)n, (\hat{X}_t, \hat{A}_t) = (\hat{x}_t, \hat{a}_t), b_r > 0 \}\) from round \(t\) to round \(1\). We note that the context arrives independently with \(\mathbb{P}\{X_t = j\} = \pi_j\) and under UCB-PB, the action \(A_t\) will be taken under context \(j\) if \(\hat{X}_t = j, \hat{A}_t = k\), and the remaining budget \(b_r > 0\). In other words, \(\mathbb{P}\{X_t = j, A_t = k|\hat{X}_t = j, \hat{A}_t = k, b_r > 0\} = \pi_j\). Thus, if \((\hat{x}_t, \hat{a}_t) = (j, k)\), we introduce a Bernoulli distributed random variable \(W_t\) as \(\text{Bern}(\pi_j)\) and \(\mathbb{P}\{W_t = 0\} = 1 - \pi_j\) (independent of \(\hat{X}_t\) and \(\hat{A}_t\)). Then, we have:

\[
\begin{align*}
\mathbb{P}\{C_{j, k}(t) < \pi_j(1 - \epsilon)n, (\hat{X}_t, \hat{A}_t) = (\hat{x}_t, \hat{a}_t), b_r > 0 \} & \leq \mathbb{P}\{C_{j, k}(t - 1) + W_t < \pi_j(1 - \epsilon)n, \\
(\hat{X}_t, \hat{A}_t) = (\hat{x}_t, \hat{a}_t), b_r > 0 \}.
\end{align*}
\]

This is because:

\[
\begin{align*}
\mathbb{P}\{C_{j, k}(t) < \pi_j(1 - \epsilon)n, (\hat{X}_t, \hat{A}_t) = (\hat{x}_t, \hat{a}_t)\} & = \mathbb{P}\{C_{j, k}(t) < \pi_j(1 - \epsilon)n, (\hat{X}_{t-1}, \hat{A}_{t-1}) = (\hat{x}_{t-1}, \hat{a}_{t-1}) \} \\
|\{X_t = j, A_t = k, b_r > 0\} & \mathbb{P}\{X_t = j, A_t = k, b_r > 0\} \\
\leq \mathbb{P}\{C_{j, k}(t - 1) + W_t < \pi_j(1 - \epsilon)n, \\
(\hat{X}_{t-1}, \hat{A}_{t-1}) = (\hat{x}_{t-1}, \hat{a}_{t-1}), b_r > 0 \} \\
\times \mathbb{P}\{X_t = j, A_t = k, b_r > 0\} \\
= \mathbb{P}\{C_{j, k}(t - 1) + W_t < \pi_j(1 - \epsilon)n, \\
(\hat{X}_t, \hat{A}_t) = (\hat{x}_t, \hat{a}_t), b_r > 0 \}.
\end{align*}
\]

Similarly, if \((\hat{x}_t, \hat{a}_t) \neq (j, k)\), we have

\[
\begin{align*}
\mathbb{P}\{C_{j, k}(t) < \pi_j(1 - \epsilon)n, (\hat{X}_t, \hat{A}_t) = (\hat{x}_t, \hat{a}_t)\} & \leq \mathbb{P}\{C_{j, k}(t - 1) < \pi_j(1 - \epsilon)n, \\
(\hat{X}_t, \hat{A}_t) = (\hat{x}_t, \hat{a}_t), b_r > 0 \}.
\end{align*}
\]

Using the similar technique to rounds \(t - 1, t - 2, \ldots, 1\), we have

\[
\begin{align*}
\mathbb{P}\{C_{j, k}(t) < \pi_j(1 - \epsilon)n, (\hat{X}_t, \hat{A}_t) = (\hat{x}_t, \hat{a}_t)\} & \leq \mathbb{P}\{\sum_{t'=\hat{x}_{t'}}^{t} W_{t'} < \pi_j(1 - \epsilon)n, \\
(\hat{X}_t, \hat{A}_t) = (\hat{x}_t, \hat{a}_t), b_r > 0 \} \\
& \leq \mathbb{P}\{\sum_{t'=\hat{x}_{t'}}^{t} W_{t'} < \pi_j(1 - \epsilon)n, \\
& \times \mathbb{P}\{X_t = j, A_t = k, b_r > 0\} \} \quad (22)
\end{align*}
\]

The last equality in the above equation is obtained from the independency of \(W_{t'}\) and \((\hat{X}_t, \hat{A}_t)\). When \(\hat{C}_{j, k}(\hat{a}, \hat{a}) (t) \geq n\), \(\sum_{t'=\hat{x}_{t'}}^{t} W_{t'}\) is the sum of \(\hat{C}_{j, k}(\hat{a}, \hat{a}) (t)\) (which is no less than \(n\)) i.i.d. Bernoulli distributed random variables with success probability \(\pi_j\). Using Hoeffding-Chebyshev bound (Dubhashi & Panconesi, 2009) and summing over all possible realizations \((\hat{x}_t, \hat{a}_t)\) satisfying \(\hat{C}_{j, k}(\hat{a}, \hat{a}) (t) \geq n\), we obtain the conclusion of the lemma.

C.2. Theorem 3: Regret of UCB-PB

Note that the regret is defined as the difference between the expected total rewards achieved by the UCB-PB algorithm and the oracle algorithm. For the oracle algorithm, let \(C_j(t) = \sum_{t'=1}^{t} 1(\hat{X}_{t'} = j, A_{t'} = k_j^\star)\) be the number of times that the context-action pair \((j, k_j^\star)\) has been executed up to round \(t\). For the UCB-PB algorithm, recall that \(C_{j, k}(t) = \sum_{t'=1}^{t} 1(\hat{X}_{t'} = j, A_{t'} = k)\) is the number of times that the context-action pair \((j, k)\) has been executed up to round \(t\), and let \(C_j(t) = \sum_{k=1}^{K} C_{j, k}(t)\). Then the
The regret of UCB-PB can be expressed as
\[
R_{\text{UCB-PB}}(T, B) = R_{\text{UCB-PB}}^c(T, B) + R_{\text{UCB-PB}}^a(T, B),
\]
where
\[
R_{\text{UCB-PB}}^c(T, B) = \sum_{j=1}^J u_j^* \E[C_j^*(T)] - \sum_{k=1}^K C_j(k(T))
\]
and
\[
R_{\text{UCB-PB}}^a(T, B) = \sum_{j=1}^J \sum_{k \neq k^*} \Delta(j,k) \E[C_j(k(T))].
\]

The expression of \( R_{\text{UCB-PB}}^c(T, B) \) uses the fact that both the oracle algorithm and UCB-PB will exhaust their entire budget, i.e., \( \sum_{j=1}^J C_j^*(T) = \sum_{j=1}^J C_j(T) = B \).

Eq. (23) clearly shows that the regret of UCB-PB roots from the following two reasons: the first part \( R_{\text{UCB-PB}}^c(T, B) \) results from taking (unnecessary) actions under the suboptimal context, the other part \( R_{\text{UCB-PB}}^a(T, B) \) results from taking suboptimal actions under a given context. For the latter part, we show in Lemma 6 that \( R_{\text{UCB-PB}}^a(T, B) = \Theta(\log T) \) using similar techniques for traditional UCB methods (Golovin & Krause, 2009).

**Lemma 3.** Under UCB-PB, the regret due to the action ranking errors within context \( j \) satisfies
\[
R_{\text{UCB-PB}}^a(T, B) \leq \sum_{j=1}^J \sum_{k \neq k^*} \left[ \frac{2}{\Delta(j,k)} + 2 \Delta(j,k) \log T + 2 \Delta(j,k) \right].
\]

**Proof.** For \( k \neq k^* \), let \( \ell_{j,k} = \frac{2 \log T}{\Delta(j,k)^2} \). According to

Next, we show that the first part \( R_{\text{UCB-PB}}^c(T, B) \) is also of order \( O(\log T) \). Recall that \( (\hat{X}_t, \hat{A}_t) \) is the context-action pair that has the highest UCB in round \( t \). Moreover, let \( \hat{C}_j(t) \) be the number of events that context \( j \) has the maximal index up to round \( t \), i.e., \( \hat{C}_j(t) = \sum_{t=1}^T \I(\hat{X}_t = j) \), and \( C_j(k(t), t) \) be the number of events that the context-action pair \( (j, k) \) has the highest UCB up to round \( t \), i.e., \( C_j(k(t), t) = \sum_{t=1}^T \I(\hat{X}_t = j, \hat{A}_t = k) \). We show that the UCB-PB algorithm mistakes the suboptimal context as the optimal context for at most \( O(\log T) \) times, i.e., \( \E[C_2^*(T) - C_2(T)] \leq \E[C_2(T)] = O(\log T) \).

Specifically, consider the suboptimal context \( j = 2 \). For \( 1 \leq k \leq K \), we have
\[
\E[C_2^*(T)] \leq \ell_{2,k} + \sum_{t=1}^T \P\{ \hat{X}_t = 2, \hat{A}_t = k \},
\]
where \( \ell_{2,k} = \frac{2 \log T}{\tau^2(1-\epsilon)^2} \), and \( \epsilon \in (0, 1) \).

Based on Lemma 4, we have
\[
\P\{ C_2^*(T) - C_2(k(t), t) \leq \ell_{2,k} \} \leq e^{-2\epsilon^2 \ell_{2,k}^2} T^{-4}.
\]

Thus,
\[
\P\{ \hat{X}_t = 2, \hat{A}_t = k, \hat{C}_2(k(t), t) \leq \ell_{2,k} \} \leq \P\{ \hat{X}_t = 2, \hat{A}_t = k, \hat{C}_2(k(t), t) \leq \ell_{2,k} \} \leq \P\{ C_2(k(t), t) < \pi_2(1-\epsilon) \} \leq e^{-2\epsilon^2 \ell_{2,k}^2} T^{-4}.
\]

**Lemma 6.** Under UCB-PB, the regret due to the action ranking errors within context \( j \) satisfies
\[
\hat{C}_{2,k}(t-1) \geq \frac{1}{2} C_{2,k}(b_r > 0) \\
\leq \mathbb{P}\{\tilde{u}_{2,k}(t) > \tilde{u}_{1,k}(t) | C_{2,k}(t-1) \geq \tau_2(1 - \epsilon)\} + \mathbb{P}\{C_{2,k}(t-1) < \tau_2(1 - \epsilon)\} \\
\leq 2t^{-1} + T^{-4},
\]
where the last inequality results from Lemma 3 (note that for \( j = 2 \), \( \tilde{u}_{2,k}(t) < \tilde{u}_{1,k}(t) \)) and Eq. (25).

Summing over all actions, we have
\[
\mathbb{E}[\hat{C}_2(T)] = \sum_{k=1}^{K} \mathbb{E}[\hat{C}_{2,k}(T)] \\
\leq \sum_{k=1}^{K} \frac{1}{2} (2t^{-1} + T^{-4}) \\
= \sum_{k=1}^{K} \left[ \frac{2}{\tau_2(1 - \epsilon) \epsilon^2(u_1^* - u_2,k)^2} + 2 \right] \log T + O(1).
\]

According to Lemma 2, we have
\[
\mathbb{P}\{\mathcal{E}_{\text{budget},0}(t)\} = \mathbb{P}\{b_r < (\rho - \delta)\tau\} + \mathbb{P}\{b_r > (\rho + \delta)\tau\} \\
\leq 2e^{-2\delta^2(T-t+1)}.
\]

Back to the ranking event \( \mathcal{E}_{\text{rank},1}(t) \), we have
\[
\mathbb{P}(\mathcal{E}_{\text{rank},1}(t)) \leq \mathbb{P}(\mathcal{E}_{\text{budget},0}(t)) + \mathbb{P}(\mathcal{E}_{\text{rank},1}(t) \cap \mathcal{E}_{\text{budget},0}(t)).
\]

Note that the event \( \mathcal{E}_{\text{rank},1}(t) \) can be divided as following:
\[
\mathcal{E}_{\text{rank},1}(t) \subseteq \bigcup_{1 \leq j \leq \hat{j}(\rho), 1 \leq k \leq K} \mathcal{E}_{\text{rank},1}(j,k),
\]
where for \( 1 \leq j \leq \hat{j}(\rho) \) and \( 1 \leq k \leq K \),
\[
\mathcal{E}_{\text{rank},1}(j,k) = \{ \forall j' > j, \tilde{u}_{j',k}(t) < \tilde{u}_{j,k}(t) \}, \quad \hat{u}_{j,k}(t) \leq \tilde{u}_{j,k}(t), k_j^*(t) = k \}.
\]

The last equality is obtained by letting \( \epsilon = 2/3 \). Substituting Eqs. (24) and (26) into (23) and using the fact that \( u_1^* - u_2,k \) for all \( k \), we can obtain the conclusion of Theorem 3.

D. Regret of UCB-ALP

D.1. Lemma 5: Bounds of Context Ranking Errors

We only prove the conclusion for the case of \( s = 1 \) as the other case can be analyzed similarly. From Algorithm 2, we can see that the evolution of the remaining budget also affects the execution of the UCB-ALP algorithm. Under the assumption of known context distribution, it can be verified that Lemma 2 holds under UCB-ALP, i.e., the remaining budget \( b_r \) follows the hypergeometric distribution and has the properties described in Lemma 2. We define an event \( \mathcal{E}_{\text{budget},0}(t) \) as follows,
\[
\mathcal{E}_{\text{budget},0}(t) = \{(\rho - \delta)\tau \leq b_r \leq (\rho + \delta)\tau\},
\]
where \( \delta \) is given by
\[
\delta = \frac{1}{2} \min\{\rho - q_{j,\rho}, q_{j,\rho}+1 - \rho\}.
\]
where 
\[ C_{j}(\rho)+1,k(t) = \sum_{t'=1}^{t} 1(X_{t'} = \hat{j}(\rho) + 1, A_{t'} = k) \]

is the number that the context-action pair \((\hat{j}(\rho) + 1, k)\) has been executed up to round \(t\).

For the first term, we note that the event \(C_{\text{rank,1}}^{(j,k)}(t), \mathcal{E}_{\text{budget},0}(t)\) implies that \(\hat{u}_{j,k}^{*} \leq \hat{u}_{\hat{j}(\rho) + 1,k}^{*}\). Because \(u_{j,k}^{*} > u_{\hat{j}(\rho) + 1,k}^{*}\) for all \(j \leq \hat{j}(\rho)\) and \(k\), according to Lemma 3, we have
\[
P\{C_{\text{rank,1}}^{(j,k)}(t), \mathcal{E}_{\text{budget},0}(t) \mid C_{\text{rank,1}}^{(j,k)}(t) \}
\leq P\{\hat{u}_{j,k}^{*} \leq \hat{u}_{\hat{j}(\rho) + 1,k}^{*}\}
\leq 2t^{-1}. \tag{29}\]

For the second term, we note that since context \(\hat{j}(\rho) + 1\) arrives with probability \(\pi_{\hat{j}(\rho) + 1}\) independent of the observations, we have
\[
P\{X_{t} = \hat{j}(\rho) + 1, A_{t} = k \mid C_{\text{rank,1}}^{(j,k)}(t), \mathcal{E}_{\text{budget},0}(t)\}
= \min\{\delta, \pi_{\hat{j}(\rho) + 1}\} = g_{\hat{j}(\rho) + 1}^{*}.\]

Similar to Lemma 4, we have
\[
P\{C_{\text{rank,1}}^{(j,k)}(t) \leq \hat{u}_{\hat{j}(\rho) + 1,k}^{*}\}
\leq e^{-2\hat{j}(\rho) + 1,k} \leq T^{-4}. \tag{30}\]

Substituting Eqs. (29) and (30) into Eq. (28), we have
\[
E[N_{j,k}(T)] \leq \hat{\hat{j}}(\hat{j}(\rho) + 1,k) + \sum_{t=1}^{T} (2t^{-1} + T^{-4})
\leq \hat{\hat{j}}(\hat{j}(\rho) + 1,k) + 2\log T + 2 + T^{-3}. \tag{31}\]

Substituting Eq. (31) to Eq. (27) and letting \(\epsilon = 2/3\) in \(\hat{j}(\hat{j}(\rho) - 1,k)\), we have
\[
E[T^{(1)}] \leq \frac{2e^{-2\hat{j}(\hat{j}(\rho) + 1,k)}}{1 - e^{-2\hat{j}(\hat{j}(\rho) + 1,k)}},
\leq 2\hat{\hat{j}}(\hat{j}(\rho) + 1,k) \log T + O(1) \leq \frac{2\hat{\hat{j}}(\hat{j}(\rho) + 1,k)}{1 - e^{-2\hat{j}(\hat{j}(\rho) + 1,k)}} + 2K\hat{j}(\rho) \log T + O(1). \tag{32}\]

The above analyses use the following two fundamental facts: one is that the event \(C_{\text{rank,1}}^{(j,k)}(t) \cap \mathcal{E}_{\text{budget},0}(t)\) indicates a context ranking error, which will happen with small probability if the suboptimal context-action pair has been executed sufficiently many times; the other is that when \(E_{\text{rank,1}}^{(j,k)}(t) \cap \mathcal{E}_{\text{budget},0}(t)\) occurs, the suboptimal context-action pair will be executed with a positive probability.

Using similar analyses for the cases of \(s = 2\), we can show \(E[T^{(2)}] = O(\log T)\) with slight modifications. Specifically, let \(j^{*}(t)\) the context with the largest UCB among all contexts worse than \(\hat{j}(\rho) + 1\), i.e., \(j^{*}(t) = \arg \max_{j \geq \hat{j}(\rho) + 1} u_{j}^{*}\). Then \(E_{\text{rank,2}}^{(j,k)}(t) \cap \mathcal{E}_{\text{budget},0}(t)\) indicates that \(\hat{u}_{j^{*}(t),k^{*}(t)}(t) \leq \hat{u}_{\hat{j}(\rho) + 1,k}^{*}\) and the action \(k^{*}(t)\) will be taken under context \(j^{*}(t)\) with a positive probability at least \(g_{j}(t)\). The conclusion then follows by considering all possible \(j^{*}(t)\)’s and \(k^{*}(t)\)’s.

**D.2. Theorem 4: Regret of UCB-ALP for Non-boundary Cases**

Note that the total reward of the oracle solution \(U^{*}(T,B) \leq U(T,B)\). Thus, we can bound the regret of UCB-ALP by comparing its total expected reward \(U_{\text{UCB-ALP}}(T,B)\) with \(U(T,B)\), i.e.,
\[
R_{\text{UCB-ALP}}(T,B) = U^{*}(T,B) - U_{\text{UCB-ALP}}(T,B)
\leq U(T,B) - U_{\text{UCB-ALP}}(T,B)
= TV(p) - \sum_{j=1}^{K} u_{j,k}^{*}E[C_{j,k}(T)]. \tag{33}\]

where \(C_{j}(T) = \sum_{k=1}^{K} C_{j,k}(T)\) is the total number that actions have been taken under context \(j\) up to round \(T\). The total expected reward of UCB-ALP can be further divided as
\[
U_{\text{UCB-ALP}}(T,B) = \sum_{j=1}^{J} u_{j}^{*}E[C_{j}(T)] - \sum_{j=1}^{J} \sum_{k=1}^{K} \Delta_{j,k}^{(j)}E[C_{j,k}(T)]
= \sum_{j=1}^{J} u_{j}^{*}E[C_{j}(T)] - \sum_{j=1}^{J} \sum_{k=1}^{K} \Delta_{j,k}^{(j)}E[C_{j,k}(T)]. \tag{34}\]

Consequently, the regret of UCB-ALP can be bounded as
\[
R_{\text{UCB-ALP}}(T,B) \leq R_{\text{UCB-ALP}}(T,B) + R_{\text{UCB-ALP}}^{(a)}(T,B), \tag{34}\]

where
\[
R_{\text{UCB-ALP}}^{(c)}(T,B) = \sum_{j=1}^{J} \sum_{k=1}^{K} \Delta_{j,k}^{(j)}E[C_{j,k}(T)].
\]

and
\[
R_{\text{UCB-ALP}}^{(a)}(T,B) = \sum_{j=1}^{J} \sum_{k=1}^{K} \Delta_{j,k}^{(j)}E[C_{j,k}(T)].
\]
Eq. (34) clearly shows that the regret of the UCB-ALP algorithm can be divided into two parts: the first part $R_{\text{UCB-ALP}}^b(T, B)$ is from the deviation of remaining budget $b_\tau$ and context-ranking errors, the second part $R_{\text{ALP}}(T, B)$ is from taking suboptimal actions under a given context. For the second part, the results of Lemma 6 can be extended to UCB-ALP, i.e.,

$$R_{\text{ALP}}^b(T, B) = \sum_{j=1}^{J} \sum_{k \neq k'} \left[ \frac{2 \Delta_{j,k}}{2} \log T + 2 \Delta_{j,k} \right] \tag{35}$$

Next, we show that the first part $R_{\text{ALP}}^b(T, B)$ is also in the order of $O(\log T)$. Let $\nu_{\text{ALP}}(\tau, b_\tau)$ be the best single round value obtained by UCB-ALP in round $T - \tau + 1$, i.e.,

$$\nu_{\text{ALP}}(\tau, b_\tau) = \hat{\nu}_j(\tau, b_\tau)\pi_j u_j^*.$$  

Note that we have separately considered the regret due to action ranking errors in $R_{\text{ALP}}^b(T, B)$ and we only need to consider the best action in $\nu_{\text{ALP}}(\tau, b_\tau)$ for each context. Let $\Delta_{\tau}$ be the single-round difference between UCB-ALP and the upper bound, i.e.,

$$\Delta_{\tau} = \nu(\rho) - \nu_{\text{ALP}}(\tau, b_\tau).$$

Then $R_{\text{ALP}}^b(T, B) = \sum_{\tau=1}^{T} \mathbb{E}[\Delta_{\tau}]$. We study the expectation $\mathbb{E}[\Delta_{\tau}]$ under all possible situations. For a random variable $X$ and event $\mathcal{E}$, let $\mathbb{E}[X, \mathcal{E}] = \mathbb{E}[X 1(\mathcal{E})]$. Then, the expectation $\mathbb{E}[X] = \mathbb{E}[X, \mathcal{E}] + \mathbb{E}[X, \overline{\mathcal{E}}]$. Therefore,

$$\mathbb{E}[\Delta_{\tau}] = \sum_{s=0}^{2} \mathbb{E}[\Delta_{\tau}, \mathcal{E}_{\text{rank}, s}(T - \tau + 1)]. \tag{36}$$

We first consider the case of $s = 0$ and convert the expectation value into other two cases. Considering all possible value of $b_\tau$, we have

$$\mathbb{E}[\Delta_{\tau}, \mathcal{E}_{\text{rank}, 0}(T - \tau + 1)] = \sum_{b=0}^{B} \mathbb{E}[\Delta_{\tau} | b_\tau = b, \mathcal{E}_{\text{rank}, 0}(T - \tau + 1)] \times \mathbb{P}(b_\tau = b, \mathcal{E}_{\text{rank}, 0}(T - \tau + 1)). \tag{37}$$

For the probability, we have

$$\mathbb{P}(b_\tau = b, \mathcal{E}_{\text{rank}, 0}(T - \tau + 1)) = \mathbb{P}(b_\tau = b) - \sum_{s=1}^{3} \mathbb{P}(b_\tau = b, \mathcal{E}_{\text{rank}, s}(T - \tau + 1)). \tag{38}$$

For the conditioned expectation, we note that $\mathcal{E}_{\text{rank}, 0}(T - \tau + 1)$ provides a roughly context rank in the sense that if $b_\tau/\tau$ is close to $\rho$, then $v_{\text{UCB-ALP}}(\tau, b_\tau) = v(b_\tau/\tau)$, where $v(b_\tau/\tau)$ is the single round value with the correct context rank. Specifically, letting $\delta = \frac{1}{2} \min\{\rho - q_j(\rho), q_j(\rho) + 1 - \rho\}$. If $b \in [\rho - \delta, \rho + \delta]$, then $v_{\text{UCB-ALP}}(\tau, b) = v(b_\tau/\tau)$, and thus,

$$\mathbb{E}[\Delta_{\tau} | b_\tau = b, \mathcal{E}_{\text{rank}, 0}(T - \tau + 1)] = v(\rho) - v(b_\tau/\tau). \tag{39}$$

Combining Eqs. (37) to (39) and using the facts that $v(\rho) \geq 0$ and $v_{\text{UCB-ALP}}(\tau, b) \geq 0$, we have

$$\mathbb{E}[\Delta_{\tau}, \mathcal{E}_{\text{rank}, 0}(T - \tau + 1)] \leq v(\rho) - \sum_{b=0}^{B} v(b_\tau/\tau) \mathbb{P}(b_\tau = b)$$

$$+ \sum_{b=0}^{B} v(b_\tau/\tau) \mathbb{P}(b_\tau = b, \mathcal{E}_{\text{rank}, s}(T - \tau + 1)) \leq 2 \bar{\nu}^* e^{-2\delta^2}, \tag{40}$$

Recall that under UCB-ALP, the remaining budget $b_\tau$ follows hypergeometric distribution. Using the same method as the analysis of Eq. (19), we have

$$v(\rho) - \sum_{b=0}^{B} \bar{\nu}(\tau, b) \mathbb{P}(b_\tau = b) \leq (u_1^* - u_j^*) e^{-2\delta^2}. \tag{41}$$

In addition,

$$\sum_{b=0}^{B} \bar{\nu}(\tau, b) \mathbb{P}(b_\tau = b) \leq 2 \bar{\nu}^* e^{-2\delta^2}, \tag{42}$$

where $\bar{\nu}^* = \sum_{j=1}^{J} \pi_j u_j^*$ is the expected reward without budget constraint.

Moreover,

$$\sum_{b=0}^{B} v(b_\tau/\tau) \mathbb{P}(b_\tau = b, \mathcal{E}_{\text{rank}, s}(T - \tau + 1)) \leq \bar{\nu}^* \mathbb{P}(\mathcal{E}_{\text{rank}, s}(T - \tau + 1)), \tag{43}$$

Substituting Eqs. (41) and (43) into Eq. (40), we have

$$\mathbb{E}[\Delta_{\tau}, \mathcal{E}_{\text{rank}, 0}(T - \tau + 1)] \leq (u_1^* - u_j^*) e^{-2\delta^2} + \bar{\nu}^* \sum_{s=1}^{3} \mathbb{P}(\mathcal{E}_{\text{rank}, s}(T - \tau + 1)). \tag{44}$$
When the rank is wrong, i.e., \(1 \leq s \leq 2\), since \(\Delta v_{\tau} \leq v(\rho)\) under any possible ranking results, we have
\[
E[\Delta v_{\tau}, \mathcal{E}_{\text{rank},s}(T - \tau + 1)] \\ \leq v(\rho)P\{\mathcal{E}_{\text{rank},s}(T - \tau + 1)\}.
\] (45)
Substituting Eqs. (44) and (45) into Eq. (36), we have
\[
E[\Delta v_{\tau}] = [u_1^* - u_j^* + \bar{u}^*]e^{-2\delta^2 T} + [\bar{u}^* + v(\rho)]\sum_{s=1}^{2} P\{\mathcal{E}_{\text{rank},s}(T - \tau + 1)\}.
\]
Note that \(R^{(c)}_{\text{UCB-ALP}}(T, B) = \sum_{\tau=T}^{T} E[\Delta v_{\tau}]\). Thus
\[
R^{(c)}_{\text{UCB-ALP}}(T, B) \leq [u_1^* - u_j^* + \bar{u}^*]e^{-2\delta^2} + \left[\bar{u}^* + v(\rho)\right]\sum_{s=1}^{2} E[T^{(s)}],
\]
where \(T^{(s)} = \sum_{t=1}^{T} \mathcal{E}_{\text{rank},s}(t)\) (\(s = 1, 2\)) is the number of type-\(s\) ranking errors. The conclusion of the theorem follows by using the conclusion of Lemma 5 and Eq. (35).

### D.3. Theorem 5: Regret of UCB-ALP for Boundary Cases

The proof of Theorem 5 is similar to Theorem 4 with slight modification on the threshold.

First, we note that the regret of UCB-ALP \(R^{(a)}_{\text{UCB-ALP}}(T, B)\) can still be partitioned as (34) and the second term \(R^{(a)}_{\text{UCB-ALP}}(T, B)\) can still be bounded as (35). We only need to analyze the first term \(R^{(c)}_{\text{UCB-ALP}}(T, B)\).

We note that fundamentally, the context \(\hat{j}(\rho) + 1\) for \(\rho \neq q_j\) and the context \(\hat{j}(\rho)\) for \(\rho = q_j\) are both the minimum context with positive probability in the static LP problem. Thus, we can define the context ranking events \(\mathcal{E}_{\text{rank},s}(t)\) (\(0 \leq s \leq 2\)) similar to the analysis of Theorem 4, with \(\hat{j}(\rho) + 1\) replaced by \(\hat{j}(\rho)\). We can verify that Lemma 5 can be extended to include the boundary cases:

\[
\limsup_{T \to \infty} \frac{E[T^{(1)}]}{\log T} \leq \sum_{j > j(\rho)} K \sum_{k=1}^{27} \frac{2\delta_{j(\rho)}[\Delta_{j(\rho),k}]^2}{2g_{j(\rho)}[\Delta_{j(\rho),k}]^2} + \frac{K[\hat{j}(\rho) - 1]}{2\delta_{j(\rho)}[\Delta_{j(\rho),k}]^2} + \frac{K[J - \hat{j}(\rho)]}{2g_{j(\rho)}[\Delta_{j(\rho),k}]^2},
\]
\[
\limsup_{T \to \infty} \frac{E[T^{(2)}]}{\log T} \leq \sum_{j \geq j(\rho) + 1} K \sum_{k=1}^{27} \frac{2\delta_{j(\rho)}[\Delta_{j(\rho),k}]^2}{2g_{j(\rho)}[\Delta_{j(\rho),k}]^2} + \frac{K[J - \hat{j}(\rho)]}{2g_{j(\rho)}[\Delta_{j(\rho),k}]^2},
\]
where
\[
g_{j} = \min \left\{ \pi_j, \frac{1}{2}(\rho - q_{j(\rho) - 1}), \frac{1}{2}(q_{j(\rho) + 1} - \rho) \right\}.
\]
Then,
\[
R^{(c)}_{\text{UCB-ALP}}(T, B) = \sum_{\tau=T}^{T} E[\Delta v_{\tau}],
\]
where
\[
E[\Delta v_{\tau}] = \sum_{s=0}^{2} E[\Delta v_{\tau}, \mathcal{E}_{\text{rank},s}(T - \tau + 1)].
\]
For the case of \(s = 0\),
\[
E[\Delta v_{\tau}, \mathcal{E}_{\text{rank},0}(T - \tau + 1)] = \sum_{b=0}^{B} E[\Delta v_{\tau} | b_{\tau} = b, \mathcal{E}_{\text{rank},0}(T - \tau + 1)] \times P\{b_{\tau} = b\}
\]
When \(b/\tau \in [\rho - \delta, \rho + \delta]\) and \(\mathcal{E}_{\text{rank},0}(T - \tau + 1)\) occurs, we have \(\Delta v_{\tau} \leq (u_1^* - u_j^*)|\rho - b/\tau|\). Moreover, \(\Delta v_{\tau} \leq v(\rho)\) under any condition. Thus,
\[
E[\Delta v_{\tau}, \mathcal{E}_{\text{rank},0}(T - \tau + 1)] \leq u_1^* \E[|b_{\tau}/\tau - \rho|] + v(\rho) \sum_{b \notin [\rho - \delta, \rho + \delta]} P\{b_{\tau} = b\}
\]
\[
\leq u_1^* \sqrt{\text{Var}(b_{\tau})/\tau^2} + 2v(\rho)e^{-2\delta^2 \tau}.
\]
For the other cases of \(s = 1, 2\), we have
\[
E[\Delta v_{\tau}, \mathcal{E}_{\text{rank},s}(T - \tau + 1)] \leq v(\rho)P\{\mathcal{E}_{\text{rank},s}(T - \tau + 1)\}.
\]
Then we can bound \(R^{(c)}_{\text{UCB-ALP}}(T, B)\) by summing over the entire horizon. The conclusion then follows by adding the bound of \(R^{(c)}_{\text{UCB-ALP}}(T, B)\) and \(R^{(a)}_{\text{UCB-ALP}}(T, B)\).

### E. Constrained Contextual Bandits with Heterogeneous Costs

In this section, we consider the case where the cost for each action \(k\) under context \(j\) is fixed at \(c_{j,k}\), which may be different for different \(j\) and \(k\). We discuss how to use the insight from unit-cost systems in designing algorithms for heterogeneous-cost systems.

#### E.1. Approximation of the Oracle Algorithm

Similar to unit-cost systems, we first study the case with known statistics. We generalize the upper bound and the ALP algorithm in Section 3 to general-cost systems.

#### E.1.1. Upper Bound

With known statistics, the agent knows the context distribution \(\pi_j\)’s, the costs \(c_{j,k}\)’s, and the expected rewards...
Theorem 8. For any given \( \rho_j \geq 0 \), there exists an optimal solution of \( SP_j \), i.e., \( p^*_j = (p_{j,1}, p_{j,2}, \ldots, p_{j,K}) \), satisfies:

1. For \( k_1 \), if there exists another action \( k_2 \), such that \( \eta_{j,k_1} \leq \eta_{j,k_2} \) and \( u_{j,k_1} \leq u_{j,k_2} \), then \( p^*_{j,k_1} = 0 \);

2. For \( k_1 \), if there exists two actions \( k_2 \) and \( k_3 \), such that \( \eta_{j,k_2} \leq \eta_{j,k_1} \leq \eta_{j,k_3} \), \( u_{j,k_2} \geq u_{j,k_1} \geq u_{j,k_3} \), and \( c_{j,k_1} - c_{j,k_3} \leq \frac{u_{j,k_1} - u_{j,k_2}}{c_{j,k_2} - c_{j,k_3}} \), then \( p^*_{j,k_1} = 0 \).

Intuitively, the first part of Lemma 7 shows that if an action has small normalized and original expected reward, then it can be removed. The second part of Lemma 7 shows that if an action has small normalized expected reward and medium original expected reward, but the increasing rate is smaller than another action with larger expected reward, then it can also be removed.

Proof. The key idea of this proof is that, if the conditions is satisfied, and there is a feasible solution \( p_j = (p_{j,1}, p_{j,2}, \ldots, p_{j,K}) \) such that \( p_{j,k_1} > 0 \), then we can construct another feasible solution \( p'_j \) such that \( p'_{j,k_1} = 0 \), without reducing the objective value \( v_j(p_j) \).

We first prove part (1). Under the conditions of part (1), if \( p_j \) is a feasible solution of \( SP_j \) with \( p_{j,k_1} > 0 \), then consider another solution \( p'_j \), where \( p'_{j,k} = p_{j,k} \) for \( k \neq k_1 \) or \( k \neq k_2 \), \( p'_{j,k_1} = 0 \), and \( p'_{j,k_2} = p_{j,k_1} + p_{j,k_2} \), \( k 
.pdf
With Lemma 7, the agent can ignore some actions that will obviously be allocated zero probability under a given context \( j \). We call the set of the remaining actions as candidate set for context \( j \), denoted as \( \mathcal{A}_j \). We propose an algorithm to construct the candidate action set for context \( j \), as shown in Algorithm 3.

**Algorithm 3** Find Candidate Set for Context \( j \)

**Input:** \( c_{j,k} \)'s, \( u_{j,k} \)'s, for all \( 1 \leq k \leq K \);

**Output:** \( \mathcal{A}_j \);

**Init:** \( \mathcal{A}_j = \{ 1, 2, \ldots , K \} \);

Calculate normalized rewards: \( \eta_{j,k} = u_{j,k}/c_{j,k} \);

Sort actions in descending order of their normalized rewards:

\[
\eta_{j,k_1} \geq \eta_{j,k_2} \geq \ldots \geq \eta_{j,k_K}.
\]

for \( a = 2 \) to \( K \) do

if \( \exists a' < a \) such that \( u_{j,k_a} \leq u_{j,k_a'} \) then

\( \mathcal{A}_j = \mathcal{A}_j \setminus \{ k_a \} \);

end if

end for

\( a = 1; \)

while \( a \leq K - 1 \) do

Find the action with highest increasing rate:

\[
a^* = \arg \max_{a' \neq a, k_a \in \mathcal{A}_j} \left( \frac{u_{j,k_{a'}} - u_{j,k_a}}{c_{j,k_{a'}} - c_{j,k_a}} \right).
\]

Remove the medium actions:

\( \mathcal{A}_j = \mathcal{A}_j \setminus \{ k_{a'}: a < a' < a^* \} \).

end while

For context \( j \), assume that the candidate set \( \mathcal{A}_j = \{ k_{j,1}, k_{j,2}, \ldots , k_{j,K_j} \} \) has been sorted in descending order of their normalized rewards, i.e., \( \eta_{j,k_{j,1}} \geq \eta_{j,k_{j,2}} \geq \ldots \geq \eta_{j,k_{j,K_j}} \). From Algorithm 3, we know that \( u_{j,k_{j,1}} < u_{j,k_{j,2}} < \ldots < u_{j,k_{j,K_j}} \), and \( c_{j,k_{j,1}} < c_{j,k_{j,2}} < \ldots < c_{j,k_{j,K_j}} \).

The agent now only needs to consider the actions in the candidate set \( \mathcal{A}_j \). To decouple the “intra-context” constraint (48), we introduce the following transformation:

\[
p_{j,k_{j,a}} = \begin{cases} 
\tilde{p}_{j,k_{j,a}} - \tilde{p}_{j,k_{j,a+1}}, & \text{if } 1 \leq a \leq K_j - 1, \\
\tilde{p}_{j,k_{j,K_j}}, & \text{if } a = K_j,
\end{cases}
\]

where \( \tilde{p}_{j,k_{j,a}} \in [0, 1] \), and \( \tilde{p}_{j,k_{j,a}} \geq \tilde{p}_{j,k_{j,a+1}} \) for \( 1 \leq a \leq K_j - 1 \). Substituting the transformations into \( (\mathcal{SP}_j) \) and reorganize it as

\[
(\mathcal{SP}_j) \text{ maximize } \sum_{a=1}^{K_j} \tilde{p}_{j,k_{j,a}} \tilde{u}_{j,k_{j,a}};
\]

subject to

\[
\tilde{p}_{j,k_{j,a}} \leq \rho_j, \\
\tilde{p}_{j,k_{j,a}} \geq \tilde{p}_{j,k_{j,a+1}}, \quad 1 \leq a \leq K_j - 1,
\]

\[
\tilde{p}_{j,k_{j,a}} \in [0, 1], \quad \forall a,
\]

where

\[
\tilde{u}_{j,k_{j,a}} = \begin{cases} 
u_{j,k_{j,1}}, & \text{if } a = 1, \\
u_{j,k_{j,a}} - \nu_{j,k_{j,a-1}}, & \text{if } 2 \leq a \leq K_j,
\end{cases}
\]

\[
\tilde{c}_{j,k_{j,a}} = \begin{cases} \tilde{c}_{j,k_{j,1}}, & \text{if } a = 1, \\
\tilde{c}_{j,k_{j,a}} - \tilde{c}_{j,k_{j,a-1}}, & \text{if } 2 \leq a \leq K_j.
\end{cases}
\]

Next, we show that the constraint (52) can indeed be removed. For each \( k_{j,a} \), we can view \( \tilde{c}_{j,k_{j,a}} \) and \( \tilde{u}_{j,k_{j,a}} \) as the cost and expected reward of a virtual action. Let \( \eta_{j,k_{j,a}} = \tilde{u}_{j,k_{j,a}}/\tilde{c}_{j,k_{j,a}} \) be the normalized expected reward of virtual action \( k_{j,a} \). For \( a = 1 \), using \( \eta_{j,k_{j,1}} \), we can show that \( \tilde{p}_{j,k_{j,1}} \geq \tilde{p}_{j,k_{j,2}} \). For \( 2 \leq K_j - 1 \), using \( \eta_{j,k_{j,a}} - \eta_{j,k_{j,a-1}} = \frac{\tilde{u}_{j,k_{j,a}} - \tilde{u}_{j,k_{j,a-1}}}{\tilde{c}_{j,k_{j,a}} - \tilde{c}_{j,k_{j,a-1}}} \), we can show that \( \eta_{j,k_{j,1}} \geq \eta_{j,k_{j,2}} \geq \eta_{j,k_{j,3}} \). In other words, we can verify that \( \tilde{p}_{j,k_{j,1}} \geq \tilde{p}_{j,k_{j,2}} \geq \ldots \geq \tilde{p}_{j,k_{j,K_j}} \). Thus, without constraint (52), there is still an optimal solution \( \tilde{p}_j = [\tilde{p}_{j,k_{j,1}}, \tilde{p}_{j,k_{j,2}}, \ldots , \tilde{p}_{j,k_{j,K_j}}] \), such that \( \tilde{p}_{j,k_{j,1}} \geq \tilde{p}_{j,k_{j,2}} \geq \ldots \geq \tilde{p}_{j,k_{j,K_j}} \). Hence, we can remove the constraint (52), and thus decouple the probability constraint under a context.

With the above transformations, we can thus rewrite the global LP problem

\[
(\mathcal{LP}'_{T,B}) \text{ maximize } \sum_{j=1}^{J} \sum_{a=1}^{K_j} \pi_j \tilde{p}_{j,k_{j,a}} \tilde{u}_{j,k_{j,a}},
\]

subject to

\[
\sum_{j=1}^{J} \sum_{a=1}^{K_j} \pi_j \tilde{p}_{j,k_{j,a}} \tilde{c}_{j,k_{j,a}} \leq B/T,
\]

\[
\tilde{p}_{j,k_{j,a}} \in [0, 1], \quad \forall j, \text{ and } 1 \leq a \leq K_j.
\]

The solution of \( (\mathcal{LP}'_{T,B}) \) follows a threshold structure. We sort all context-(virtual)-action pairs \( (j, k_a) \) in descending order of their normalized expected reward. Let \( j^{(i)}, k^{(i)} \) be the context index and action index of the \( i \)-th pair, respectively. Namely, \( \eta_{j^{(i)},k^{(i)}} \geq \eta_{j^{(i+1)},k^{(i+1)}} \geq \ldots \geq \eta_{j^{(M)},k^{(M)}} \), where \( M = \sum_{j=1}^{J} K_j \) is the total number of candidate actions for all contexts. Define a threshold corresponding to \( \rho = B/T \),

\[
\tilde{u}(\rho) = \max \left\{ i : \sum_{v'=1}^{i} \pi_{j^{(v')}} \tilde{c}_{j^{(v')},k^{(v')}} \leq \rho \right\},
\]
where \( \rho = B/T \) is the average budget. We can verify that the following solution is optimal for \( \widehat{LP}_{T,B}^i \):

\[
\tilde{p}_{j(i),k(i)}(\rho) = \begin{cases} 
1, & \text{if } 1 \leq i \leq \tilde{i}(\rho), \\
\frac{\rho - \sum_{i=1}^{\tilde{i}(\rho)} \pi_{j(i)} \epsilon_{j(i),k(i)}}{\pi_{\tilde{i}(\rho)+1} \epsilon_{\tilde{i}(\rho)+1}}, & \text{if } i = \tilde{i}(\rho) + 1, \\
0, & \text{if } i > \tilde{i}(\rho) + 1.
\end{cases}
\]

Then, the optimal solution of \( \widehat{LP}_{T,B}^i \) can be calculated using the reverse transformation from \( \tilde{p}_{j,k}(\rho) \)’s to \( p_{j,k}(\rho) \)’s.

### E.1.2. ALP Algorithm

Similar to unit-cost systems, the ALP algorithm replaces the average constraint \( B/T \) in \( \widehat{LP}_{T,B}^i \) with the remaining budget \( b \), and obtains probability \( p_{j,k}(b/\tau) \). Under context \( j \), the ALP algorithm take action \( k \) with probability \( p_{j,k}(b/\tau) \).

Unlike unit-cost systems, the remaining budget \( b \) does not follow any classic distribution in heterogeneous-cost systems. However, we can show that the concentration property still holds for this general case by using the method of averaged bounded differences (Dubhashi & Panconesi, 2009).

#### Lemma 8

For \( 0 < \delta < 1 \), there exists a positive number \( \kappa \), such that under the ALP algorithm, the remaining budget \( b_\tau \) satisfies

\[
\begin{align*}
\mathbb{P}\{b_\tau > (1 + \delta)\rho \tau\} & \leq e^{-\kappa \delta^2 \tau}, \\
\mathbb{P}\{b_\tau < (1 - \delta)\rho \tau\} & \leq e^{-\kappa \delta^2 \tau}.
\end{align*}
\]

**Proof.** We prove the lemma using the method of averaged bounded differences (Dubhashi & Panconesi, 2009). The process is similar to Section 7.1 in (Dubhashi & Panconesi, 2009), except that we consider the remaining budget and the successive differences of the remaining budget are bounded by \( c_{\text{max}} \).

Specifically, let \( \tilde{c}_{\tau} \), \( 1 \leq \tau' \leq T \) be the budget consumed under ALP, and let \( \tilde{c}_{\tau'} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_T) \). Then the remaining budget at round \( t \) (the remaining time \( \tau = T - t + 1 \)), i.e., \( b_{T-t+1} \) is a function of \( \tilde{c}_t \). We note that under ALP, the expectation of the ratio between the remaining budget and the remaining time does not change, i.e., for any \( b \leq \sum_{j=1}^{T} \pi_j c_{j,T} \), if \( b_{\tau} = b \), then \( \mathbb{E}\{b_{\tau-1}/(\tau-1)\} = b/\tau \). Thus, we can verify that for any \( 1 \leq \tau' \leq t \), we have

\[
\mathbb{E}\{b_{T-t+1}/\tilde{c}_{\tau'}\} = b_{T-t+1}/\tilde{c}_{\tau+1} = \frac{b_{T-t+1}}{T - t + 1}(t - \tau').
\]

Note that \( \Delta b = b_{T-t+2} - b_{T-t+1} \leq c_{\text{max}} \) and \( b_{T-t+2} \geq -c_{\text{max}} \), we have

\[
\mathbb{E}\{b_{T-t+1}/\tilde{c}_{\tau'}\} - \mathbb{E}\{b_{T-t+1}/\tilde{c}_{\tau'-1}\} \leq \frac{\max_{0 \leq \Delta b \leq c_{\text{max}}} \{(|\Delta b - b_{T-t'+2}| - b_{T-t'+1})/T - t' + 1 \}}{T - t' + 1} \leq 2c_{\text{max}}(T - t + 1)/(T - t' + 1).
\]

Moreover,

\[
\sum_{\tau'=1}^{t} \left[ 2c_{\text{max}}(T - t + 1)/(T - t' + 1) \right] ^2 \leq 4c^2_{\text{max}}(T - t + 1)^2 \sum_{\tau'=1}^{t} \frac{1}{(T - t' + 1)^2} \leq 4c^2_{\text{max}}(T - t + 1)^2 \sum_{\tau'=T-t+1}^{\tau} \frac{1}{(\tau')^2} \approx 4c^2_{\text{max}}(T - t + 1)^2 \int_{T-t+1}^{T} \frac{1}{(\tau')^2} \, d\tau' \leq 4c^2_{\text{max}}(T - t + 1)^2 \frac{1}{T}. \]

According to Theorem 5.3 in (Dubhashi & Panconesi, 2009), and noting \( \tau = T - t + 1, \mathbb{E}\{b_\tau\} = \rho \tau \), we have

\[
\mathbb{P}\{b_\tau > (1 + \delta)\rho \tau\} \leq e^{-\kappa \delta^2 \tau},
\]

and similarly,

\[
\mathbb{P}\{b_\tau < (1 - \delta)\rho \tau\} \leq e^{-\kappa \delta^2 \tau},
\]

Choosing \( \kappa = \frac{\delta^2}{c_{\text{max}}^2} \) concludes the proof. \( \square \)

Then, using similar methods in Section 3, we can show that the generalized ALP algorithm achieves \( O(1) \) regret in non-boundary cases, and \( O(\sqrt{T}) \) regret in boundary cases, where the boundaries are now defined as \( Q_i = \sum_{i'=1}^{i} \pi_{j(i')}\tilde{c}_{j(i')},k(i') \).

### E.2. \( \epsilon \)-First ALP Algorithm

When the expected rewards are unknown, it is difficult to combine UCB method with the proposed ALP for general systems. As a special case, when all actions have the same cost under a given context, i.e., \( c_{j,k} = c_j \) for all \( k \) and \( j \), the normalized expected reward \( \eta_{j,k} \) represents the quality of action \( k \) under context \( j \). In this case, the candidate set for each context only contains one action, which is the action
with the highest expected reward. Thus, the ALP algorithm for the known statistics case is simple. When the expected rewards are unknown, we can extend the UCB-ALP algorithm by managing the UCB for the normalized expected rewards.

When the costs for different actions under the same context are heterogeneous, it is difficult to combine ALP with the UCB method since the ALP algorithm in this case not only requires the ordering of $\eta_{j,k}$‘s, but also the ordering of $u_{j,k}$‘s and the ratios $\frac{\eta_{j,k} - u_{j,k}}{\eta_{j,k} - \bar{u}_{j,k}}$. We propose an $\epsilon$-First ALP Algorithm that explores and exploits separately: the agent takes actions under all contexts in the first $[\epsilon(T)]$ rounds to estimate the expected rewards, and runs ALP based on the estimates in the remaining $T - [\epsilon(T)]$ rounds.

**Algorithm 4 $\epsilon$-First ALP**

**Input:** Time horizon $T$, budget $B$, exploration budget ratio $\epsilon$, and $c_{j,k}$‘s, for all $j$ and $k$;

**Init:** Remaining budget $b = B$;

for $t = 1$ to $[\epsilon(T)]$ do

if $b > 0$ then

Obtain the probabilities $p_{j,k}(b/\tau)$’s by solving the problem $(LP'_{\epsilon,j,k})$ with $u_{j,k}$ replaced by $\bar{u}_{j,k}$;

Take action $A_t =$ arg min$_{k \in \mathcal{A}} C_{X_t,k}$ (with random tie-breaking);

Observe the reward $R_{A_t,t};$

Update the reward estimate:

$\bar{u}_{X_t,A_t} = \frac{(C_{X_t,A_t} - 1)\bar{u}_{X_t,A_t} + Y_{A_t,t}}{C_{X_t,A_t}}.$

end if

end for

for $t = [\epsilon(T)] + 1$ to $T$ do

Remaining time $\tau = T - t + 1$;

if $b > 0$ then

Obtain the probabilities $p_{j,k}(b/\tau)$’s by solving the problem $(LP'_{\epsilon,j,k})$ with $u_{j,k}$ replaced by $\bar{u}_{j,k}$;

Take action $A_t =$ arg min$_{k \in \mathcal{A}} C_{X_t,k}$ (with random tie-breaking);

Observe the reward $R_{A_t,t};$

Update the reward estimate:

$\bar{u}_{X_t,A_t} = \frac{(C_{X_t,A_t} - 1)\bar{u}_{X_t,A_t} + Y_{A_t,t}}{C_{X_t,A_t}}.$

end if

end for

Let $\xi_{j,k_1,k_2} = \frac{u_{j,k_1} - u_{j,k_2}}{\eta_{j,k_1} - \eta_{j,k_2}}$ for $j \in \mathcal{X}, k_1, k_2 \in \{0\} \cup \mathcal{A}$, and $k_1 \neq k_2$ (recall that $u_{j,0} = 0$ and $c_{j,0} = 0$ for the dummy action). Let $\Delta_{\min}$ be the minimal difference between any $\xi_{j_1,k_1,k_2}$ and $\xi_{j_2,k_2,k_2}$, i.e.,

$$\Delta_{\min} = \min_{(j_1,k_1,k_2) \in \mathcal{X}} \min_{(j_2,k_1,k_2) \in \{0\} \cup \mathcal{A}} |\xi_{j_1,k_1,k_2} - \xi_{j_2,k_1,k_2}|.$$

Moreover, let $\pi_{\min} = \min_{j \in \mathcal{X}} \pi_j$ and $\bar{\xi}_{j,k_1,k_2}$ be the estimate of $\xi_{j,k_1,k_2}$ at the end of the exploration stage, i.e.,

$$\bar{\xi}_{j,k_1,k_2} = \frac{u_{j,k_1} - \bar{u}_{j,k_2}}{c_{j,k_1} - c_{j,k_2}}.$$

Then, the following lemma states that under $\epsilon$-First ALP with a sufficiently large $\epsilon(T)$, the agent will obtain a correct ordering of $\xi_{j,k_1,k_2}$‘s with high probability at the end of the exploration stage.

**Lemma 9.** Under $\epsilon$-First ALP, if

$$\epsilon(T) \geq \frac{K}{(1 - \delta)\pi_{\min}} + \log T \max \left\{ \frac{1}{\delta^2}, \frac{16K}{(1 - \delta)\pi_{\min}\Delta_{\min}^2} \right\},$$

then for any contexts $j_1,j_2 \in \mathcal{X}$, and actions $k_{11},k_{12},k_{21},k_{22} \in \{0\} \cup \mathcal{A}$, if $\xi_{j_1,k_{11},k_{12}} < \xi_{j_2,k_{21},k_{22}}$, then at the end of the $[\epsilon(T)]$th round, we have

$$\mathbb{P}\left\{ \bar{\xi}_{j_1,k_{11},k_{12}} \geq \bar{\xi}_{j_2,k_{21},k_{22}} \right\} \leq (J + 4)T^{-2}.$$

Moreover, the agent ranks all the $\xi_{j,k_1,k_2}$‘s correctly with probability no less than $1 - (4K + 1)JT^{-2}$.

**Proof.** We first analyze the number of executions for each context-action pair $(j,k)$ in the exploration stage. Let $N_j = \sum_{t=1}^{[\epsilon(T)]} \mathbb{1}(X_t = j)$ be the number of occurrences of context $j$ up to round $[\epsilon(T)]$. Recall that the contexts $X_t$ arrive i.i.d. in each round. Thus, using Hoeffding-Chernoff Bound for each context $j$, we have

$$\mathbb{P}\left\{ \forall j \in \mathcal{X}, N_j \geq (1 - \delta)\pi_j[\epsilon(T)] \right\} \geq 1 - \sum_{j=1}^{J} \mathbb{P}\left\{ N_j < (1 - \delta)\pi_j[\epsilon(T)] \right\} \geq 1 - J e^{-2\pi_j[\epsilon(T)]} \geq 1 - J e^{-2 \log T} = 1 - JT^{-2} \quad (59)$$

On the other hand, the lower bound $(1 - \delta)\pi_j[\epsilon(T)] \geq K + \frac{16K \log T}{\Delta_{\min}^2}$ from the implementation of the exploration stage in Algorithm 4, we know that if $N_j \geq (1 - \delta)\pi_j[\epsilon(T)]$, then

$$C_{j,k} \geq \left[ 1 + \frac{16 \log T}{\Delta_{\min}^2} \right] \geq \frac{16 \log T}{\Delta_{\min}^2}, \quad \forall k \in \mathcal{A}. \quad (60)$$

Therefore,

$$\mathbb{P}\left\{ \forall j \in \mathcal{X}, \forall k \in \mathcal{A}, C_{j,k} \geq \frac{16 \log T}{\Delta_{\min}^2} \right\} \geq 1 - JT^{-2} \quad (61)$$

Next, we study the relationship between the estimates $\hat{\xi}_{j_1,k_{11},k_{12}}$ and $\hat{\xi}_{j_2,k_{21},k_{22}}$ at the end of the exploration stage.
We note that
\[
\xi_{j_1,k_1,k_1} \geq \xi_{j_2,k_2,k_2} \leftrightarrow \left( \xi_{j_1,k_1,k_1} - \xi_{j_1,k_1,k_1} - \xi_{j_2,k_2,k_2} + \xi_{j_1,k_1,k_1} \right) \geq 0
\]

\[
\xi_{j_2,k_2,k_2} - \xi_{j_1,k_1,k_1} + \frac{\xi_{j_1,k_1,k_1} - \xi_{j_1,k_1,k_1}}{2} \geq 0
\]

\[
\xi_{j_1,k_1,k_1} - \xi_{j_2,k_2,k_2} - \xi_{j_1,k_1,k_1} + \frac{\xi_{j_1,k_1,k_1} - \xi_{j_1,k_1,k_1}}{2} \geq 0
\]

Thus, for the event \( \xi_{j_1,k_1,k_1} \geq \xi_{j_2,k_2,k_2} \) to be true, we require that at least one term (with the sign) in the last inequality above is no less than zero. Conditioned on \( C_{j,k} \geq \frac{16 \log T}{(\Delta_{\min})^2} \), we can bound the probability of each term according to the Hoeffding-Chernoff bound. For example, for the first term, since \( |c_{j_1,k_1} - c_{j_1,k_1}| \geq 1 \) and \( \xi_{j_2,k_2,k_2} - \xi_{j_1,k_1,k_1} \geq \Delta_{\min} \), we have

\[
P\left\{ \frac{\bar{u}_{j_1,k_1} - u_{j_1,k_1}}{c_{j_1,k_1} - c_{j_1,k_1}} - \frac{\xi_{j_2,k_2,k_2} - \xi_{j_1,k_1,k_1}}{4} \geq 0 \right\}
\]

\[
\leq \frac{16 \log T}{(\Delta_{\min})^2} \leq e^{-2 \log T} = T^{-2}
\]

The conclusion then follows by considering the event \( \{ C_{j,k} \geq \frac{16 \log T}{(\Delta_{\min})^2}, \forall j \in X, \forall k \in X \} \) and its negation.

**Theorem 6.** Under \( \epsilon \)-First ALP, if

\[
\epsilon(T) \geq \frac{K}{(1 - \delta)^2} \log T \max\left\{ \frac{16K}{\delta^2}, \frac{16K}{(1 - \delta)\pi_{\min}(\Delta_{\min})^2} \right\}
\]

then the regret of \( \epsilon \)-First ALP satisfies:

1) if \( \rho = B/T \neq Q_i \), then \( R_{\epsilon-\text{FirstALP}}(T, B) = O(\log T) \);

2) if \( \rho = B/T = Q_i \), then \( R_{\epsilon-\text{FirstALP}}(T, B) = O(\sqrt{T}) \).

**Proof.** (Sketch) The key idea of proving this theorem is considering the event where the \( \xi_{j_1,k_1,k_2} \)’s are ranked correctly and its negation. When the \( \xi_{j_1,k_1,k_2} \)’s are ranked correctly, we can use the properties of the ALP algorithm with modification on the time horizon and budget (subtracting the time and budget in the exploration stage, which is \( O(\log T) \)); otherwise, if the agent obtains a wrong ranking, the regret is bounded as \( O(1) \) because the probability is \( O(T^{-2}) \) and the reward in each round is bounded.

**E.3. Deciding \( \epsilon(T) \) without Priori Information**

In Theorem 6, the agent requires the value of \( \Delta_{\min} \) to calculate \( \epsilon(T) \). This is usually impractical since the expected rewards are unknown a priori. Thus, without the knowledge of \( \Delta_{\min} \), we propose a Confidence Level Test (CLT) algorithm for deciding when to end the exploration stage.

Specifically, assume \( \Delta_{\min} > 0 \) and is unknown by the agent. In each round of the exploration stage, the agent tries to solve the problem \( (LP_{\tau,b}) \) with \( u_{j,k} \) replaced by \( \bar{u}_{j,k} \) using comparison, i.e., using Algorithm 3 and sorting the virtual actions. For each comparison, the agent tests the confidence level according to Algorithm 5. If all comparisons pass the test, i.e., \( \text{flagSucc} = \text{true} \) for all comparisons, then the agent ends the exploration stage and starts the exploitation stage.

**Algorithm 5 Confidence Level Test (CLT)**

**Input:** Time horizon \( T \), estimates \( \xi_{j_1,k_1,k_1}, \xi_{j_2,k_2,k_2} \); number of executions \( C_{j_1,k_1}, C_{j_2,k_2} \) and \( C_{j_2,k_2} \).

**Output:** \( \text{flagSucc} \);

**Init:** \( \text{flagSucc} = \text{false} \),

\[ \delta = \frac{\xi_{11,k_1,k_2} - \xi_{j_2,k_2,k_2}}{2} \]

if \( e^{-2 \Delta_{\min}} \leq T^{-2} \) and \( e^{-2 \Delta_{\min}} \leq T^{-2} \) then

\( \text{flagSucc} = \text{true}; \)

end if

return \( \text{flagSucc} \).

Next, we show that the \( \epsilon \)-First policy with CLT will achieve \( O(\log T) \) regret except for the boundary cases, where it achieves \( O(\sqrt{T}) \) regret. On one hand, according to Hoeffding-Chernoff bound, if all comparisons pass the confidence level test, then with probability at least \( 1 - J K^2 T^{-2} \), the algorithm obtains the correct rank and provide a right solution for the problem \( (LP_{\tau,b}) \). On the other hand, because \( \Delta_{\min} > 0 \), from the analysis in the previous section, we know that the exploration stage will end within \( O(\log T) \) rounds with high probability. Therefore, the expected regret is the same as that in the case with known \( \Delta_{\min} \).