Holographic like models as a five-dimensional rewriting of large-$N_c$ QCD

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Abstract

The AdS/QCD models are known to be tightly related with the QCD sum rules in the large-$N_c$ (called also planar) limit. Rewriting the theory of infinite tower of free stable mesons expected in the large-$N_c$ QCD as a five-dimensional theory we scrutinize to what extent the bottom-up holographic models may be viewed as an alternative language expressing the phenomenology of planar QCD sum rules. It is found that many features of AdS/QCD models can be thereby obtained without invoking prescriptions from the original AdS/CFT correspondence. Under some assumptions, all possibilities leading to simple Regge trajectories are classified and it is argued that the most phenomenologically consistent model is the one called ”soft wall model” in the holographic approach, with a preference to the positive-sign dilaton background.
1 Motivation and formulation of problem

In the last several years, numerous attempts to apply the ideas of AdS/CFT correspondence \[1,2\] to QCD have formed a rapidly developing line of researches in the modern field theory and phenomenology. The main efforts are concentrated on description of the hadron and glueball spectra, the chiral symmetry breaking, and the related phenomenology \[3–42\], although the range of applications is now much broader. The holographic methods are interesting as a language that allows to discuss within a uniform framework various approaches to modeling the interactions and spectral characteristics of light hadrons, heavy-light systems, hadron form factors, QCD phase diagram, and other phenomenological aspects which were previously the subjects of investigations for different communities. This unifying property brings up a natural question — why the AdS/QCD models are rather successful at reproducing the non-perturbative physics of strong interactions and how the holographic language is interrelated with other known phenomenological methods? At present there are plenty of papers devoted to the applications of holographic ideas but there is a lack of works which would try to answer this question\[1\]. The given research is intended to shed light on the problem by observing that rewriting the large-$N_c$ QCD in a fixed channel as a free five-dimensional theory, one is able to obtain models which are very similar to the holographic models of the bottom-up approach.

The AdS/QCD models are essentially based on the identification of 4D Kaluza-Klein (KK) harmonics of a 5D field with an infinite tower of colorless free meson states that is expected in the large-$N_c$ (called also planar) limit of QCD \[43,44\]. Such an identification can represent a convenient mathematical trick: Any field $\varphi(\vec{x}, z)$ living in a compact 5D space can be decomposed in 4D harmonics, $\varphi(\vec{x}, z) = \sum_{n=0}^{\infty} \varphi_n(z) \psi_n(\vec{x})$, which are 4D fields with masses determined by topology of the 5D space. For this reason we may rewrite an infinite tower of 4D fields $\psi_n(\vec{x})$ with equal quantum numbers and with a given spectrum $m_n$ as a single 5D field $\varphi(\vec{x}, z)$ propagating in a suitable background. Decomposing this field in 4D harmonics with normalizable coefficient functions $\varphi_n(z)$, the dependence on the fifth coordinate $z$ can be integrated out and one comes back to the original action for the tower of 4D fields $\psi_n(\vec{x})$. The AdS/CFT conjecture, however, has much richer dynamical content than just KK-rewriting — it asserts an exact correspondence between the generating functional of correlators in the 4D theory and the

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\[1\] A likely exception is the paper \[42\] where it was demonstrated numerically that the phenomenology of spin-1 mesons within the hard wall holographic models depends slightly on the choice of metric, the AdS metric, however, minimizes deviations from the experimental data.
effective action of the 5D theory in which the ultraviolet boundary values of the 5D fields are identified with the sources in 4D theory [2]. The question is whether we may interpret the KK-rewriting in such a way that it mimicked the correspondence above? The answer seems to be positive as is demonstrated in Section 2. The idea is that after integration over $z$ one obtains also boundary terms which may be identified with sources coupled to the fields $\psi_n(\vec{x})$. This identification yields the same results for the two-point correlators as within the holographic models.

In Section 3, under some assumptions we analyze all possibilities for the 5D backgrounds which lead to simple Regge like spectra and then, in Section 4, we select the most reasonable 5D rewriting of large-$N_c$ QCD basing on requirements to reproduce the analytic structure of two-point correlators and the expected phenomenological form of Regge spectrum, this rewriting turns out to be nothing but the so-called soft wall model [14].

The description of the chiral symmetry breaking is among the most discussed issues in the holographic approach [8,20,21,34]. We propose in Section 5 a modification of method of Refs. [8,9] that seems to allow for embedding this phenomenon into our approach.

In Section 6, we consider interrelation between holographic bottom-up approach and the phenomenology of QCD sum rules in the large-$N_c$ limit.

In the concluding Section 7, we discuss briefly the relation of the 5D formalism to some traditional phenomenological approaches.

## 2 Derivation of holographic like models via Kaluza-Klein reduction

With respect to the number of colors $N_c$, the meson masses behave as $m \sim N_c^0$ and their full decay widths do as $\Gamma \sim N_c^{-1}$. Since the meson masses vary slightly with $N_c$, it is thus often useful in the problems of finding the meson spectrum to take the limit of large $N_c$. In the extreme case $N_c \to \infty$, the mesons are infinitely narrow and non-interacting, in addition, their appear in infinite tower [43,44] at fixed quantum numbers. Such an infinite tower of resonance poles saturates completely the two-point correlation function of quark currents with quantum numbers corresponding to the given tower (see, e.g., the representation (106) in the vector channel). With the help of functional integral formalism, this situation can be formally described by the following action,

$$ I_{[O,J]} = (-1)^J \int d^4x \sum_{n=0}^{\infty} \left( \partial_\mu \phi_J^{(n)} \partial^\mu \phi_J^{(n)} - m^2_{n,J} \phi_J^{(n)} \phi_J^{(n)} + \phi_J^{(n)} O_J^{(n)} \right), \quad (1) $$
where we have included the couplings to external sources which appear in the corresponding generating functional of the connected correlators. Here \( \phi_{\mu_1 \mu_2 ... \mu_J} \), \( \mu_i = 0, 1, 2, 3 \) with the signature \((+ - - -)\), corresponds to a meson field of spin \( J \) whose quantum numbers \( I, G, P, C \) are not specified, and \( \mathcal{O}^{(n)}_J \) is a source which can be represented as

\[
\mathcal{O}^{(n)}_J = F^{(J)}_n \mathcal{O}_J,
\]

where \( F^{(J)}_n \) are the decay constants defined by

\[
\langle 0 | \mathcal{O}^{(n)}_J \phi^{(n)}_J \rangle = F^{(J)}_n \varepsilon_J
\]

for a meson \( \phi^{(n)}_J \) with ”polarization” \( \varepsilon_J \) and \( \mathcal{O}_J \) is a common source to which the states \( \phi^{(n)}_J \) are coupled with a ”coupling” \( F^{(J)}_n \). In case of high spin fields, \( J > 1 \), under \( \partial_\mu \) we mean the (general coordinate) covariant derivative \( \nabla_\mu \).

Another subtlety is that for \( J > 1 \) mesons many additional quadratic terms appear in the action \( (1) \), however, the auxiliary conditions can be chosen such that they do not contribute (to be discussed below). The representation \( (2) \) follows from the requirement to obtain finally the standard form for two-point correlators, namely integrating formally over the field \( \phi_J \) in the generating functional

\[
Z[\mathcal{O}_J] = \int D\phi_J e^{\mathcal{L}[\mathcal{O}_J]},
\]

and differentiating twice with respect to \( \mathcal{O}_J \) at \( \mathcal{O}_J = 0 \) one arrives at the sum over meson poles,

\[
\langle \mathcal{O}_J(q) \mathcal{O}_J(-q) \rangle \sim \sum_{n=0}^{\infty} \frac{(F^{(J)}_n)^2}{q^2 - m^2_{n,J} + i\varepsilon},
\]

for the two-point correlators in the momentum representation. As is usual in description of higher spin massive bosons, the tensor \( \phi_J \) is symmetric, traceless, \( \phi_{\mu\mu...} = 0 \), to provide the irreducible \( (\frac{1}{2}J, \frac{1}{2}J) \) representation of the homogeneous Lorentz group (or, equivalently, the positivity of energy), and satisfy the auxiliary condition \( \partial^\mu \phi_\mu... = 0 \) to give the required \( 2J+1 \) physical degrees of freedom. The tensor structure in the r.h.s. of representation \( (5) \) depends on the structure of terms with derivatives in the action \( (1) \), it will be not of interest for us.

We will not consider the baryons because their masses behave as \( m \sim N_c \) and their are not narrow at large-\( N_c \). The baryons emerge likely as solitonic objects in the large-\( N_c \) limit \[44\], our further discussions are definitely not applicable to such a case.
Our task is to rewrite the action (1) as an action of some free 5D theory that is not necessarily covariant with respect to the fifth coordinate \( z \), i.e. in the form

\[ S_{5D} = (-1)^J \int d^4x \, dz \, f_1(z) \left( \partial_M \varphi_J \partial^M \varphi^J - m_J^2 f_2(z) \varphi_J \varphi^J \right), \]

where \( M = 0, 1, 2, 3, 4 \), \( \varphi_J = \varphi_J(x, z) \), and \( f_1(z) \), \( f_2(z) \) are yet unknown functions of the fifth (space-like) coordinate \( z \). We require that the action (6) must respect the invariance under the 5D general coordinate transformations at vanishing \( |z| \). This implies two principal consequences. First,

\[ f_1(z) = e^{\Phi(z)} \sqrt{\det G_{MN}} = e^{\Phi(z)} a^5(z), \]

where \( G_{MN} \) is the metric tensor defined from the 5D metric,

\[ ds^2 = a^2(z)(dx^\mu dx^\mu - dz^2), \]

and possible deviations from the 5D covariance at large enough \( |z| \) are parametrized by means of a general factor \( e^{\Phi(z)} \), such a parametrization is convenient to draw parallels with the soft wall AdS/QCD models. Second, one has to respect the rule for contraction of indices, for instance,

\[ \partial_M \varphi_J \partial^M \varphi^J = \partial_M \varphi_{M_1...M_J} \partial_{M_1'} \varphi_{M_1'...M_{J-1}} G^{M_1M_1'} G^{M_{J-1}M_{J-1}} ... G^{M_{J-1}M_J}, \]

where

\[ G^{MN} = G^{-1}_{MN} = a^{-2}(z) \eta^{MN}. \]

We remind the reader again that \( \partial_M \) denotes the covariant derivative \( \nabla_M \) for high spin fields. Finally, the action is (up to a general normalization factor which is set to 1)

\[ S_{5D} = (-1)^J \int d^4x \, dz e^{\Phi(z)} a^{-2J+3}(z) \left\{ (\partial_\mu \varphi_J)^2 - (\partial_z \varphi_J)^2 - m_J^2 a^2(z) \varphi_J^2 \right\}. \]

We will regard the 5D fields with \( J > 0 \) as the gauge ones and use this gauge freedom to go to such a gauge that no additional quadratic terms appear in description of free higher-spin mesons. For instance, in the case of AdS space (or at least the AdS space in the UV limit), this is the axial gauge \([14][45]\),

\[ \varphi_{z...} = 0, \]

which will be implied in what follows. Together with the traceless condition this gauge fixes components of fields \( \varphi_J \).
Consider now the following Sturm-Liouville (SL) problem
\[
- \partial_z [g_J(z) \partial_z \varphi_n^{(J)}(z)] + g_J(z) a^2(z) n_J^2 \varphi_n^{(J)}(z) = m_{n,J}^2 g_J(z) \varphi_n^{(J)}(z).
\] (13)
with
\[
g_J(z) = e^{\Phi(z)} a^{-2J+3}(z).
\] (14)
We remind the reader the general results of SL theory in Appendix A. The equation (13) is the equation of motion for the action (11) in the gauge (12) if we admit
\[
\varphi(x, z) = e^{iqn x} \varphi_n(z), \quad q_n^2 = m_{n,J}^2.
\] (15)
The relation (15) is usually interpreted as the string excitation corresponding to 4D particle with physical momentum \( q_n \). We will not use such an interpretation and instead of this assumption consider a general mathematical problem: Given a spectrum \( m_{n,J}^2 \) in the action (1), find the functions \( g_J(z) \) and \( a(z) \) which being inserted in (13) would give \( m_{n,J}^2 \) as eigenvalues of some SL problem. This constitutes the main intermediate step in rewriting the action (1) as some free 5D field theory.

Under the conditions formulated in Appendix A, the SL problem (13) has the solutions \( \varphi_n^{(J)}(z) \) which are normalized as follows
\[
\int_{z_{\text{min}}}^{z_{\text{max}}} g_J(z) \varphi_m^{(J)}(z) \varphi_n^{(J)}(z) dz = \delta_{mn},
\] (16)
and form a complete set of functions, hence, the function \( \varphi_J(x, z) \) in the action (11) can be expanded in the 4D harmonics,
\[
\varphi_J(x, z) = \sum_{n=0}^{\infty} \phi_n^{(J)}(x) \varphi_n^{(J)}(z).
\] (17)

Let us substitute the expansion (17) in the action (11),
\[
S_{5D} = (-1)^J \int d^4x \, dz g_J(z) \sum_{m,n=0}^{\infty} \left\{ \varphi_m^{(J)} \varphi_n^{(J)} \partial_\mu \phi_m^{(n)} \partial_\nu \phi_n^{(n)} - \phi_J^{(m)} \phi_J^{(n)} \partial_\nu \varphi_m^{(n)} \partial_\mu \varphi_n^{(n)} - m_J^2 a^2(z) \varphi_m^{(J)} \varphi_n^{(J)} \phi_J^{(m)} \phi_J^{(n)} \right\},
\] (18)
where the notation (14) has been used. Integrating by parts and making use of Eq. (13), the second term in the action (15) can be rewritten as (omitting the general factor \(-\phi_J^{(m)} \phi_J^{(n)}\))
\[
\int_{z_{\text{min}}}^{z_{\text{max}}} dz g_J(z) \partial_z \varphi_m^{(J)} \partial_z \varphi_n^{(J)} = \varphi_m^{(J)} g_J(z) \partial_z \varphi_m^{(J)} \bigg|_{z_{\text{min}}}^{z_{\text{max}}} + \int_{z_{\text{min}}}^{z_{\text{max}}} dz \varphi_m^{(J)} g_J(z) \left( m_{n,J}^2 - a^2(z) m_J^2 \right) \phi_J^{(m)} \phi_J^{(n)}. \] (19)
Now we can integrate over $z$ in the action (18) with the help of (16) and (19), the result is

$$S_{5D} = (-1)^J \int d^4x \sum_{n=0}^{\infty} \left\{ \left( \partial^\mu \phi_{(n)}^J \right)^2 - m_{n,J}^2 \left( \phi_{(n)}^J \right)^2 \right. $$

$$- \phi_{(n)}^J g_J(z) \partial_z \varphi_{n}^{(J)}(x,z) \bigg|_{z_{\min}}^{z_{\max}} \right\}, \quad (20)$$

with $\varphi_J(x,z)$ given by (17). The action (20) has the form of the action (11) if we identify

$$O_{n}^{(n)} = -g_J(z) \partial_z \varphi_{n}^{(J)}(x,z) \bigg|_{z_{\min}}^{z_{\max}}. \quad (21)$$

If $O_{n}^{(n)} = 0$ then we have just rewritten the 4D action through the 5D one as in the Kaluza-Klein reduction. But we want to have an exact correspondence between the 4D action (11) and the 5D action (11), i.e. $O_{n}^{(n)} \neq 0$. By choosing $\beta = 0$ or $\beta = \pi/2$ in the boundary condition (A.3) (see Appendix A) we can nullify the term at $z = z_{\max}$ in the Eq. (21). Note in passing that for the vector mesons the first possibility is actually realized in the soft wall holographic models [14], where $z_{\max} \to \infty$, while the second one — in the hard wall models [8–10], where $z_{\max}$ is the infrared cutoff. Thus,

$$O_{n}^{(n)} = \lim_{z \to z_{\min}^+} g_J(z) \partial_z \varphi_{n}^{(J)}(z) \varphi_J(x,z). \quad (22)$$

Now we should identify the general source $O_J$ such that the decay constants $F_n^{(J)}$ in the representation (2) were non-zero finite numbers. In general, this identification depends on the choice of metric (the function $a(z)$) and of five-dimensional mass $m_J^2$. If we fix the AdS metric at $z \to z_{\min}$, $a(z) \sim 1/z$, $z \geq 0$, and $m_J^2$ in the form (72) then we should identify

$$O_J = zg_J \varphi_J/c \sim z^{2(J-1)} \varphi_J/c, \quad (23)$$

$$F_n^{(J)} = \lim_{z \to z_{\min}^+} c \frac{\partial_z \varphi_{n}^{(J)}(z)}{z}, \quad (24)$$

since the function $\varphi_{n}^{(J)}(z)$ in the realistic models (Eqs. (18) and (51)) behave as $z^2$ at small $z$. The factor $c$ appears because there is always the freedom to redefine $F_n \to cF_n$, $O \to O/c$ where the parameter $c$ can be fixed by matching to the high energy asymptotics of the corresponding QCD correlators [46]. The expression (24) reproduces the result of Ref. [8] for the vector mesons (up to a general factor depending on normalization of fields).

Let us now compare the Kaluza-Klein like calculations above with the technique used in the standard holographic approach. Within the latter, one
evaluates the action (11) on the solution of Eq. (13) that would give in our notations
\[ S_{5D} = (-1)^J \int d^4x g_J(z) \partial_z \varphi^J \varphi_J \big|_{z=z_{\text{min}}} \] (25)

Supposing that \( \varphi^J_0 \) is the Fourier transform of the source of QCD operator \( O^J \) at the UV boundary and accepting \( \varphi^J(q, z) = \varphi(q, z) \varphi^J_0(q) \) with the boundary condition \( z^{J-1} \varphi(q, z) \big|_{z \to +0} = 1 \) (we generalize the matter to the case of arbitrary spin \( J \)) one differentiates the Eq. (25) twice with respect to the source and arrives at the same sum over meson poles as in the expression (5) using the spectral expansion of the Green’s function for the Eq. (13),
\[ G^{(J)}(z, z', q) = \sum_{n=0}^{\infty} \frac{\varphi_n^{(J)}(z') \varphi_n^{(J)}(z)}{q^2 - m_{n,J}^2 + i\varepsilon}. \] (26)

This very coincidence enables the assumed holographic duality [2]
\[ Z_{4D}[O_J] = S_{5D}[\varphi^J(x, z_{\text{min}})] \text{ at } z_{\text{min}}^{J-1} \varphi^J(x, z_{\text{min}}) = O_J \] (27)
at the level of two-point correlation functions in the approximation of classical field theory.

It should be noted finally that we did not need the interpretation of the fifth coordinate \( z \) as inverse energy scale that is used in the standard holographic approach.

3 Regge spectrum

On the theoretical and phenomenological [17] grounds, the spectrum of light mesons is expected to be of the Regge form,
\[ m_{n,J}^2 \simeq An + BJ + C, \] (28)
where \( A, B, C \) are parameters, \( J \) is the spin, and \( n \) is the ”radial” quantum number that enumerates the states with identical quantum numbers lying on the higher daughter trajectories. In addition, one expects \( A \approx B \), i.e. the behavior (see, e.g., [48, 49] for the recent discussions)
\[ m_{n,J}^2 \sim n + J. \] (29)

In the given Section, we examine systematically at which conditions the Regge spectrum can be obtained.
For further analysis it is convenient to cast the Eq. (13) into the form of the Schrödinger equation via the substitution (we omit below the index \((J)\) and the notation of dependence on \(z\))

\[ \phi_n = e^{-\Phi/2}a^{J-3/2}\psi_n. \]  

(30)

The result is

\[ -\psi''_n + U\psi_n = m^2_n\psi_n, \]  

(31)

\[ U = \frac{\Phi''}{2} + \left(\frac{\Phi'}{2}\right)^2 + \left(\frac{3}{2} - J\right)\frac{\Phi'\alpha' + a'' + (\frac{1}{2} - J)\frac{(\alpha')^2}{\alpha} + \alpha^2m^2_J}{}, \]  

(32)

where the prime denotes the derivative with respect to \(z\). As is known from Quantum Mechanics, the linear dependence \(m^2_n \sim n\) takes place if \(U \sim z^2\) at least at large \(z\), i.e. if one has the oscillator type of potential. On the other hand, the Regge behavior \(m^2_{n,J} \sim J\) holds if there is a shift of energy levels (= masses square) \(m^2_n \sim n + C(J)\) such that \(C(J) \sim J\). With the help of formulae in Appendix B, the both conditions can be achieved simultaneously at many choices of the functions \(\Phi(z)\) and \(a(z)\) in the potential (32). This freedom can be restricted significantly if we make the following simplifying assumptions:

(a) The functions \(\Phi\) and \(a\), i.e. the shape of dilaton and the metric in the AdS/CFT language, do not depend on spin \(J\);

(b) The functions \(e^\Phi\) and \(a\) are continuous and differentiable on the interval \((0, \infty)\) or \((-\infty, \infty)\) (depending on a model);

(c) Subleading in \(J\) and \(n\) corrections to the Regge spectrum are absent.

Under these assumptions we are able to classify all possible models yielding the Regge like spectrum.

The most important class of models that we will refer to as models \(I\) corresponds to the choice,

\[ I: \Phi = \pm\lambda^2z^2; a = (R/z)^k, \text{ where } 0 < z < \infty. \]

The parameters \(\lambda\) and \(R\) have dimension of mass and inverse mass correspondingly. Without loss of generality, we set \(\lambda = 1\) and \(R = 1\). The dependence on \(\lambda\) and \(R\) can be restored in the final expressions by replacing

\[ 2^2\text{Although there are speculations in the literature on that point [27].} \]

\[ 3^3\text{In reality, this seems to be not the case. The form of these corrections, however, is not known even approximately, it depends strongly on a model and is highly speculative in the phenomenology. We would not like to boil down into such speculations.} \]
\[ z \rightarrow \lambda z, \; m_J \rightarrow m_J R^k, \; m_n \rightarrow m_n/\lambda. \] The equation (31)-(32) reads then as follows

\[ -\psi''_n + \left\{ z^2 \pm [k (2 J - 3) + 1] + \frac{[k (J - \frac{3}{2}) - \frac{3}{4}]^2 - \frac{1}{4}}{z^2} + \frac{m_J^2}{z^2} \right\} \psi_n = m_n^2 \psi_n. \] (33)

The normalizable solutions of Eq. (33) under different subcases are given below (see Appendix B).

IA: \( \Phi = z^2; \; a = z^{-k}, \; k > 1; \; m_J = 0. \)

The spectrum is

\[ m_n^2 = 4(n + 1) + 2(|\xi_{k,J}| + \xi_{k,J}), \quad n = 0, 1, 2, \ldots, \] (34)

where

\[ \xi_{k,J} = k \left( J - \frac{3}{2} \right) - \frac{1}{2}. \] (35)

The eigenfunctions (see the notation (30))

\[ \varphi_n = \sqrt{\frac{2n!}{(|\xi_{k,J}| + n)!}} e^{-z^2} z^{\xi_{k,J} - \xi_{k,J}} L_{n}^{\xi_{k,J}}(z^2). \] (36)

The spectrum is absent for

\[ \frac{3}{2} < J < \frac{1}{k} + \frac{3}{2}. \] (37)

The restriction (37) means that at \( 1 < k < 2 \) there is no finite discrete spectrum for \( J = 2 \) states. The case \( k = 2 \) is special, the equation (33) is then

\[ -\psi''_n + (z^2 + 3) \psi_n = m_n^2 \psi_n, \] (38)

that yields formally the spectrum of masses and eigenfunctions given by Eqs. (B.4) and (B.5). We must remember, however, that, first, due to the pole in the metric the problem is defined in \( 0 < z < \infty \) while the eigenfunctions (B.5) are normalized in \( -\infty < z < \infty \), second, we should have \( \varphi_n(0) = 0 \), i.e. only odd \( n \) in (B.5) should be selected. Keep this in mind, we arrive at the spectrum

\[ J, k = 2: \quad m_{2l+1}^2 = 4(l + 3), \quad l = 0, 1, 2, \ldots, \] (39)

and normalized eigenfunctions

\[ J, k = 2: \quad \varphi_{2l+1} = \frac{\pi^{-1/4}}{2^{l+1}} \frac{1}{(2l+1)!} e^{-z^2} z^{-1} H_{2l+1}(z). \] (40)
IA⁻: \( \Phi = -z^2; a = z^{-k}, k > 1; m_J = 0 \).
The equation (33) yields the spectrum
\[
m_n^2 = 4(n + 1) + 2(|\xi_{k,J} - \xi_{k,J}|), \quad n = 0, 1, 2, \ldots,
\]
and the eigenfunctions
\[
\varphi_n = \sqrt{\frac{2n!}{(|\xi_{k,J}| + n)!}} e^{z|\xi_{k,J}| - \xi_{k,J}} L_{n}^{(|\xi_{k,J}|)}(z^2).
\]

As before, the spectrum of \( J = 2 \) mesons is absent for \( 1 < k < 2 \) and for \( k = 2 \) is given by
\[
J, k = 2: \quad m_{2l+1}^2 = 4l, \quad l = 0, 1, 2, \ldots,
\]
\[
J, k = 2: \quad \varphi_{2l+1} = \frac{\pi^{-1/4}}{2^{l+1}} \sqrt{\frac{1}{(2l + 1)!}} z^{-1} H_{2l+1}(z).
\]

IB⁺: \( \Phi = z^2; a = 1/z \).
The equation (33) is
\[
- \psi''_n + \left\{ z^2 + 2(J - 1) + \frac{(J - 2)^2 + m_J^2 - 1/4}{z^2} \right\} \psi_n = m_n^2 \psi_n.
\]
The spectrum
\[
m_n^2 = 4n + 2(\xi_J + J), \quad n = 0, 1, 2, \ldots,
\]
where
\[
\xi_J = \sqrt{(J - 2)^2 + m_J^2}.
\]
The eigenfunctions
\[
\varphi_n = \sqrt{\frac{2n!}{(\xi_J + n)!}} e^{-z^2} z^{\xi_J + 2 - J} L_{n}^{\xi_J}(z^2).
\]
The spectrum is absent for \( 0 \leq \xi_J < 1/2 \) and represents a special case for \( \xi_J = 1/2 \) that can be analyzed as in the model IA for any concrete value of \( m_J^2 \).

IB⁻: \( \Phi = -z^2; a = 1/z \).
The equation (33) is
\[
- \psi''_n + \left\{ z^2 - 2(J - 1) + \frac{(J - 2)^2 + m_J^2 - 1/4}{z^2} \right\} \psi_n = m_n^2 \psi_n.
\]
The spectrum
\[ m_n^2 = 4(n + 1) + 2(\xi_J - J), \quad n = 0, 1, 2, \ldots, \quad (50) \]
\[ \varphi_n = \sqrt{\frac{2n!}{(\xi_J + n)!}} z^{\xi_J + 2 - J} L_n^{\xi_J}(z^2). \quad (51) \]
As before, the spectrum is absent for \( 0 \leq \xi_J < 1/2 \) and is special for \( \xi_J = 1/2 \).

**IC±:** \( \Phi = \pm z^2; \ a = z^{-k}, \ 0 < k < 1. \)

For \( m_J = 0 \), this model is identical to the IA± one but with one important distinction: The interval of forbidden spins (37) is now larger, in the limit \( k \to +0 \) only the scalar and vector modes can be described. In the case \( m_J \neq 0 \), we are not aware of analytical solutions, presumably they violate our assumption (c). It is clear, however, that at small enough and large enough \( z \) the solutions are approximately given by the \( m_J = 0 \) case.

**ID±:** \( \Phi = \pm z^2; \ a = z^{-k}, \ k > 1; \ m_J \neq 0. \)

The spectrum is given by
\[ m_n^2 = 4n \pm k(2J - 3) + 1 + \ldots, \quad n = 0, 1, 2, \ldots, \quad (52) \]
where the corrections presumably violate our assumption (c). The eigenfunctions are not known, at large enough \( z \) they are approximately given by the model IA±.

The second class of models that we will denote the models II corresponds formally to the limit \( k = 0 \) in the models I, namely

**II:** \( \Phi = \pm \lambda z^2; \ a = \text{const}, \ -\infty < z < \infty. \)

Without loss of generality, we may set \( \lambda = 1 \) and \( a = 1 \). Note that the absolute value of \( m_J^2 \) in the potential (32) is not fixed because of freedom in choosing \( a \). The model II has two different variants which are given below.

**II±:** \( \Phi = z^2. \)

The spectrum is
\[ m_n^2 = 2(n + 1) + m_J^2, \quad n = 0, 1, 2, \ldots, \quad (53) \]
\[ \psi_n = \sqrt{\frac{1}{2^n n! \pi^{-1/4}}} e^{-z^2} H_n(z). \quad (54) \]

**II−:** \( \Phi = -z^2. \)

The spectrum is
\[ m_n^2 = 2n + m_J^2, \quad n = 0, 1, 2, \ldots, \quad (55) \]
\[ \psi_n = \sqrt{\frac{1}{2^n n! \pi^{-1/4}}} H_n(z). \quad (56) \]
There is the third, rather exotic class of models, it would correspond to the choice \( k = -1 \) in the model I but with a broader possibility for the choice of function \( \Phi \),

**III**: \( a = z/R \).

The models III can be divided into the following subcases.

**IIIA**: \( \Phi = \pm \lambda^2 z^2; \; \lambda^4 + m_J^2/R^2 > 0 \).

The equation (51) is

\[
- \psi''_n + \left\{ \left( \lambda^4 + \frac{m_J^2}{R^2} \right) z^2 \pm 2 \lambda^2 (2 - J) + \frac{(J - 1)^2 - 1/4}{z^2} \right\} \psi_n = m_n^2 \psi_n. \tag{57}
\]

The slope of Regge trajectories includes the 5D mass \( m_J \), hence, this quantity should not depend on \( J \). Without loss of generality, we may set \( \lambda^4 + m_J^2/R^2 = 1 \) and \( R = 1 \). The equation (57) does not describe the vector mesons, for \( J \neq 1 \) the spectrum is

\[
m_n^2 = 4n + 2|J - 1| + 2 \pm 2 \lambda^2 (2 - J), \quad n = 0, 1, 2, \ldots, \tag{58}
\]

\[
\varphi_n = \sqrt{\frac{2n!}{(|J - 1| + n)!}} e^{-\frac{1}{2} z^2} z^{J-1+|J-1|} L_n^{J-1}(z^2). \tag{59}
\]

An interesting particular case here is the choice \( \lambda = 0 \) that corresponds to the absence of dilaton background in the holographic models, the slope of Regge trajectories is then determined by the universal 5D mass square \( m_J^2 \).

**IIIB**: \( \Phi = b \log (\lambda z); \; (m_J/R)^2 > 0 \).

Choosing the units \( \lambda = 1 \), the equation (31) reads

\[
- \psi''_n + \left\{ \left( \frac{m_J}{R} \right)^2 z^2 + \frac{(J - 1 - b)^2 + b - 1/4}{z^2} \right\} \psi_n = m_n^2 \psi_n. \tag{60}
\]

Here the slope is always determined by \( m_J^2 \). Setting for simplicity \( (m_J/R)^2 = 1 \), the spectrum is

\[
m_n^2 = 4n + 2 \xi_{J,b} + 2, \quad n = 0, 1, 2, \ldots, \tag{61}
\]

where

\[
\xi_{J,b} = \sqrt{(J - 1 - b)^2 + b}, \tag{62}
\]

and the eigenfunctions are given by

\[
\varphi_n = \sqrt{\frac{2n!}{(\xi_{J,b} + n)!}} e^{-z^2/2} z^{J-1+\xi_{J,b}-b/2} L_n^{\xi_{J,b}}(z^2). \tag{63}
\]

This model describes the mesons of all spins if \( b \geq (\sqrt{2} - 1)/2 \); at \( b = (\sqrt{2} - 1)/2 \), the spectrum of \( J = 1 \) mesons is given by Eqs. (B.4) and (B.5) with the same reservation as for the model IA.
4 Choosing the most viable model

After having classified all 5D models leading to simple Regge trajectories we should impose some criteria which would select the most viable model. Our first criterion is that a model should describe the mesons of all spins in a uniform way, i.e. without ”pathologies” at certain values of \( J \). On this ground, we reject the models of type IIIA and of type IC.

The models in question can be matched to QCD by means of the quark-hadron duality principle that is considered in Section 6. The requirement of such a duality for the vector mesons constitutes our second selection criterion. This kind of matching for the present models cannot be fulfilled for the \( J > 1 \) mesons (at least we could not find the relevant examples), this is not, however, a serious drawback since the issue of quark-hadron duality for the \( J \neq 1 \) mesons is questionable by itself\(^4\). Let us show that the models of type II do not pass this criterium.

Within the models under consideration, the condition of duality \((111)\) translates into

\[
\lim_{z \to 0} \left[ \partial_z \varphi_n(z) \right]^2 \sim m_n^2 \partial_n m_n^2, \quad n \gg 1, \tag{64}
\]

where the l.h.s. is the residue in the case of type II models and \( \varphi_n(z) \sim H_n(z) \) (see Eqs. \((54)\) and \((56)\)). First of all, in order to have non-zero residues, we must define the type II models on the interval \( 0 < z < \infty \), this changes slightly the normalization of wave functions \((54)\) and \((56)\). Due to the identity

\[
\partial_z H_n(z) = 2nH_{n-1}(z), \tag{65}
\]

only the states with uneven \( n \) survive: \( n = 2l + 1, \ l = 0, 1, 2, \ldots \). Making use of the property

\[
H_{2l}(0) = (-1)^l 2^l (2l - 1)!!, \tag{66}
\]

and the Regge form of the spectrum, \( m_n^2 \sim n \), we arrive at the expression

\[
\left[ \partial_z \varphi_{2l+1}(0) \right]^2 \sim m_{2l+1}^2 h_l, \quad h_l = \frac{[(2l - 1)!!]^2}{(2l)!}. \tag{67}
\]

The quantity \( h_l \) is a degreasing function of index \( l \) while it must be a constant (at least for large \( l \)) for the Regge like spectrum. Thus, the condition of duality \((64)\) is not fulfilled.

The condition \((64)\) is also very restrictive for the models of type I. First of all, it is easy to see that the residues are finite and non-zero constants only\(^4\) in the sense that the partonic logarithm given by the free quark loop does not yield a clear-cut main contribution to the two-point functions.
if the 5D vector field is massless. In this case (see Eq. (65) and expressions for the relevant wave functions)

\[
\lim_{z \to 0} g(z) \partial_z \varphi_n(z) \sim \sqrt{\frac{n!}{(n + \frac{k+1}{2})!}} L^{k+1}_{n+\frac{k+1}{2}}(0) \sim \sqrt{\frac{n!}{(n + k+1)!}} \left( \frac{n + \frac{k+1}{2}}{n} \right) \sim \sqrt{\frac{(n + \frac{k+1}{2})!}{n!}}. \quad (68)
\]

Substituting this expression in the duality condition (64) we conclude that the latter holds for \( k = 1 \) only, i.e. for the AdS background in the limit \( z \to +0 \). This situation corresponds to the models of type IB with \( m^2_J = 0 \).

Our third selection criterion is the requirement to have in a natural way the spectrum (29), i.e. the slopes of spin and radial trajectories must coincide. We should emphasize a remarkable fact that the third and the second criteria have a significant overlap. For instance, the form of spectrum (29) selects immediately the models IB among the models of type I. Also it excludes the model IIIB. Thus, the phenomenological spectrum (29) and the quark-hadron duality principle strongly suggest that the AdS background in the limit \( z \to +0 \) is the only possible 5D realization of large-\( N_c \) QCD.

We note further that the spectrum depends on spin \( J \) not only by construction (due to the covariant way for contraction of Lorentz indices) but also through the dependence on \( J \) of 5D mass \( m_J \). The latter dependence is \textit{ad hoc} and requires invoking some speculative assumptions. It would be natural therefore to have a minimal impact of this dependence on the final results. In the models of type II, the spectrum depends on \( J \) completely through \( m^2_J \), such a realization of Regge trajectories looks unfortunate and for this reason it does not worth serious consideration.

The third criterion might be suggestive in the choice between the models IB\(^+\) and IB\(^-\). In order to have a simple Regge-like spectrum (our assumption (c))

\[
m^2_{n,J} = 4(n + J + C), \quad (69)
\]

where \( C \) is a constant and the value of the slope was determined by the choice \( \lambda = 1 \) in the dilaton background \( e^{\pm \lambda^2 z^2} \), we must impose the following dependence of 5D mass on spin \( J \) (see Eqs. (46), (49), and (50))

\[
\text{IB}^+ : \quad m^2_J = 4J(C + 1) + 4(C^2 - 1), \quad (70)
\]

\[
\text{IB}^- : \quad m^2_J = 8J^2 + 4J(3C - 2) + 4C(C - 2). \quad (71)
\]

The condition \( m^2_{J=1} = 0 \) obtained above yields for the both cases one acceptable solution \( C = 0 \) that is, however, not valid for the \( J = 0 \) case in the
model IB\(^{-}\). The five dimensional mass is then

\begin{align*}
\text{IB}^+ : & \quad m^2_J = 4(J - 1), \\
\text{IB}^- : & \quad m^2_J = 8J(J - 1), \quad J > 0.
\end{align*}

(72)  
(73)

The both expression result in the spectrum

\[ m^2_{n,J} = 4(n + J), \quad J > 0. \]  
(74)

The spectrum (74) turns out to be not valid for the scalars of the model IB\(^+\) as well because the equation (45) does not have finite discrete spectrum in this case, i.e. when \(m^2_{J=0} = -4\) according to Eq. (72). Thus the scalar mesons represent a special case in the models IB and should be considered separately. It seems to be natural to supplement the relations (72) and (73) by the condition

\[ m^2_{J=0} = 0, \]  
(75)

The requirement (75) entails a change of prescription (24) for the residues, namely the latter must be replaced for the scalar case by

\[ F_n = \lim_{z \to z_{\text{min}} + 0} c \frac{\partial_x \varphi_n(z)}{z^3}. \]  
(76)

The condition (75) leads to the following spectrum for the scalar mesons,

\begin{align*}
\text{IB}^+ : & \quad m^2_{n,0} = 4(n + 1), \\
\text{IB}^- : & \quad m^2_{n,0} = 4(n + 2),
\end{align*}

(77)  
(78)

i.e. the scalars are degenerate with the vector mesons in the model IB\(^+\) and with the \(J = 2\) tensor mesons in the model IB\(^-\).

Consider now the impact of 5D mass \(m_J\) on the final results of models IB. For this purpose let us just set \(m_J = 0\), the spectra then are

\begin{align*}
\text{IB}^+ : & \quad m^2_{n,J<2} = 4(n + 1), \\
& \quad m^2_{n,J\geq2} = 4(n + J - 1); \\
\text{IB}^- : & \quad m^2_{n,J<2} = 4(n + 2 - J), \\
& \quad m^2_{n,J\geq2} = 4n.
\end{align*}

(79)  
(80)  
(81)  
(82)

We see that the impact of \(m_J\) is rather slight in the model IB\(^+\) and is decisive in the model IB\(^-\). According to our third criterion, the model IB\(^+\) looks
therefore more attractive phenomenologically than the model IB$^\pm$. This conclusion, however, may turn out to be misleading if we attempt to match the present phenomenological approach to the theory of free higher spin fields propagating in AdS$_5$ background where the mass coefficient behaves as $J^2 - J - 4$ (see, e.g., \cite{55,50} and references therein), the related discussions are beyond the scope of the present paper.

Let us compare our results with the assumptions used in the holographic models. According to the principle of AdS/CFT correspondence, each operator $O_J(x)$ in the 4D field theory corresponds to a field $\varphi_J(x,z)$ in the 5D bulk theory that has a mass $m_J$ determined via the relation \cite{2}

$$m_J^2 = (\Delta_J - J)(\Delta_J + J - 4), \quad (83)$$

where $\Delta_J$ is the canonical dimension of the operator $O_J(x)$. One can construct two types of minimal interpolating operators which describe the mesons in QCD,

$$O_J^{t=2} = \bar{q}(\gamma_5)\gamma_\mu D_\mu \cdots D_\mu J q, \quad (84)$$

$$O_J^{t=3} = \bar{q}(\gamma_5)D_\mu \gamma_\mu \cdots D_\mu J q, \quad (85)$$

where $t$ denotes the twist, $t = \Delta - J$. If we restrict ourselves by the operators with minimal twist (a custom first approximation in QCD) then $\Delta_J = J + 2$, hence, according to Eq. (83), we arrive exactly at the relation (72)\footnote{Neglecting the axial anomaly in the pseudoscalar isoscalar channel that is suppressed in the large-$N_c$ limit.}! The operators of the minimal twist, however, cannot interpolate the scalar mesons, in this case one uses the interpolator (85) that predicts $\Delta_0 = 3$, hence, $m_{J=0}^2 = -3$. This does not contradict to our analysis where the scalar case was also special. We would then obtain for the model IB$^+$

$$m_{J=0}^2 = -3 : \quad m_{n,0}^2 = 4(n + 1/2). \quad (86)$$

The phenomenology should tell us which spectrum of scalars is preferable, (77) or (86). If the spectrum (77) is definitely better phenomenologically then, within the holographic approach, one should seemingly use the interpolator (85) for the scalars also, i.e. make contraction of indices and interpolate the scalars by the twist-four operator $O_J^{t=4} = \bar{q}(\gamma_5)\gamma_\mu D_\mu q$, thus $\Delta_0 = 4$ and $m_{J=0}^2 = 0$ according to Eq. (83). The matter is however complicated by the fact that, firstly, this interpolator is equivalent to $O_J^{t=4} = m_q \bar{q}(\gamma_5)q$ due to the QCD equations of motion and thus it can be neglected in the chiral limit, $m_q = 0$, secondly, one can construct a purely gluonic twist-four interpolator $O_J^{t=4} = \alpha_s G_{\mu\nu}$ that makes difficult to distinguish between the
quark-antiquark states and glueballs in the scalar isoscalar channel within the holographic approach.

The assumption of the AdS background in the limit \( z \to 0 \) that is exploited in the AdS/QCD models matches perfectly our results as was remarked above.

The spectrum (74) was first obtained in the holographic model of Ref. [14]. We note that the sign of dilaton background was opposite in that model (corresponding to the model IB\(^-\) in our classification), this would lead to a different spectrum in our approach which would coincide with the Eq. (74) for the vector mesons only (compare the Eqs. (16) and (50)). The reason of this disagreement lies in a different way of introduction of higher spin fields\(^7\) in [14]. According to the arguments of recent papers [37, 38] based on realization of confinement in the string picture, the dilaton background should have positive sign in the soft wall AdS/QCD models, i.e. the most consistent models are those of type IB\(^+\) in our classification. We have arrived at the same conclusion without using any string arguments.

5 Chiral symmetry breaking

The incorporation of the Chiral Symmetry Breaking (CSB) into 5D approaches is often a questionable subject. Nevertheless we will try to outline a possible description of the CSB phenomenon that is compatible with our scheme of 5D holographic like rewriting.

First of all, since we have no quarks, no microscopic model of CSB is possible, the best we may do is to describe the consequences of CSB on the hadron level. The first such consequence is the phenomenological fact that the masses of parity partners which seemingly belong to the same chiral multiplet are quite different. To reflect this effect one should introduce a mechanism for this mass splitting. Within the 5D framework, the chiral symmetry does not exist at all because there is no analogue for the matrix \( \gamma_5 \) in five dimensions, the CSB can be therefore simulated only indirectly by means of somewhat different description of states with equal spin but opposite parity. We will restrict ourselves by the vector and scalar sectors only.

The second consequence of CSB consists in the appearence of massless (in the chiral limit) pseudoscalar meson due to the Goldstone theorem. The

\(^7\)We have introduced the higher spin fields in a direct way as in paper [36]. In the latter paper (and analogous ones), however, the hadron spin is subsequently devided into the orbital momentum of quark-antiquark pair and the proper quark spin while we do not use such a model assumption.
both consequences should be described with the help of a mechanism that explains, e.g., why the relation \( m_\pi = 0 \) is naturally related with \( m_\rho^2 \gtrsim 2m_\rho^2 \) (instead of \( m_\rho = m_\rho \) as naively expected from the linear chiral symmetry).

The simplest such a mechanism that is realized in the AdS/QCD models and that we will consider as well is borrowed from the low-energy effective field theories: One introduces a scalar field \( X \) that acquires a non-zero vacuum expectation value (v.e.v.) \( X_0(z) \) and is coupled to the axial-vector field \( A_\mu \) through the covariant derivative [8, 9],

\[
S_{CSB} = \int d^4x \int dz \sqrt{|G_{MN}|} \epsilon^{\mu}(z) \left( |D_M X|^2 - m_{X}^2 |X|^2 \right), \tag{87}
\]

where

\[
D_M X = \partial_M X - ig_5 A_M X. \tag{88}
\]

As usual, it is enough to restrict ourselves by the quadratic in fields part in the action (87) because the equations of motion can provide a non-zero v.e.v. already in this case if the bulk space is curved. This simplification turns out especially convenient for us since it is much easier to integrate over \( z \) in order to see the equivalent 4D effective theory. The equation of motion determining the v.e.v. \( X_0 \) is nothing but the SL equation (13) for the massless scalar particle. According to a recipe based on the AdS/CFT correspondence [51], it must behave at \( z = 0 \) as

\[
X_0(z) \big|_{z \to 0} = C_1 z + C_2 z^3, \tag{89}
\]

where \( C_1 \) is associated with the current quark mass, \( C_1 \sim m_q \), and \( C_2 \) with the quark condensate, \( C_2 \sim \langle \bar{q}q \rangle \). This interpretation implies a somehow well established correspondence of the model to QCD. As we do not have such a correspondence, it is more honest to say that the incorporation of the 4D massless scalar particle can be related to the spontaneous appearance of two order parameters with mass dimension one and three and this property may be exploited to mimic the CSB.

It is easy to establish a general self-consistency condition for such a description of the CSB. Substituting \( \varphi = X = z^h \) into the Eq. (13) with \( a(z) = z^{-k} \) and zero r.h.s. \( (k > 0, h > 0) \) and retaining the leading terms at \( z \to 0 \), one has

\[
m_{X}^2 z^{h-5k} = h(h - 1 - 3k)z^{h-3k-2}, \tag{90}
\]

that yields immediately \( k = 1 \), i.e. this design of CSB is possible at \( m_{X}^2 \neq 0 \) only in the AdS background at \( z \to 0 \). The second condition is \( m_{X}^2 = h(h - 4) \) that both for \( h = 1 \) and for \( h = 3 \) gives \( m_{X}^2 = -3 \) in agreement with the Eq. (83) at \( \Delta_0 = 3 \). The case \( m_{X}^2 = 0 \) we will briefly analyze later.
The fact that the pion does not belong to the corresponding linear pseudoscalar trajectory and should be considered separately is in agreement with the phenomenology. The solution \( X_0(z) \), however, turns out to be non-normalizable in the known bottom-up holographic models. This is a serious problem for our approach because we cannot integrate over \( z \) and see the underlying 4D image of the description. A way out may be the following. The equation of motion for \( X \) represents a second order linear differential equation that has two independent solutions,

\[
X_0(z) = C_1 X_1(z) + C_2 X_2(z). \tag{91}
\]

The solution \( X_1(z) \) spoils the normalizability at \( z \to 0 \) while (in the soft wall models) \( X_2(z) \) does at \( z \to \infty \). We do not see any reasons why \( X_0(z) \) must have everywhere a continuous derivative. Taking this into account, we can construct the following normalizable solution,

\[
X_0(z) = C_2 X_2(z)|_{z \leq z_0} + C_1 X_1(z)|_{z > z_0} \tag{92}
\]

with the continuity condition

\[
C_1 X_1(z_0) = C_2 X_2(z_0). \tag{93}
\]

Another restriction on the input parameters comes from the normalization \[106\],

\[
C_2^2 \int_{z_{\text{min}}}^{z_0} g_0(z) X_2^2(z) \, dz + C_1^2 \int_{z_0}^{z_{\text{max}}} g_0(z) X_1^2(z) \, dz = 1. \tag{94}
\]

The system of equations \((93), (94)\) allows to exclude two of three parameters \( C_1, C_2, z_0 \).

If \( z_0 \) is associated with the inverse energy scale then it is natural to interpret \( z_0^{-1} \) as the CSB scale, \( \Lambda_{\text{CSB}} \approx 4\pi f_\pi \approx 1 \div 1.2 \text{ GeV} \). The physics of strong interactions is known to be substantially different below \( \Lambda_{\text{CSB}} \) and above \( \Lambda_{\text{CSB}} \), the effective emergence of this scale represents in fact the third consequence of the CSB in QCD. This consequence of the CSB seems to be not incorporated into the existing AdS/QCD models.

Let us consider a concrete example — the model IB\(^+\) that has been selected above as the most viable phenomenologically. The corresponding solution for \( X_0(z) \) is known \[38\],

\[
X_1(z) = z e^{-z^2} U(-1/2, 0; z^2), \tag{95}
\]

\[
X_2(z) = z^3 e^{-z^2} M(1/2, 2; z^2), \tag{96}
\]
where $U$ and $M$ stay for the Kummer function $U$ and the Kummer function $M$ correspondingly. They have the asymptotics

$$X_1(z) \xrightarrow{z \to 0} z/\sqrt{\pi}, \quad X_1(z) \xrightarrow{z \to \infty} z^2 e^{-z^2},$$

and

$$X_2(z) \xrightarrow{z \to 0} z^3, \quad X_2(z) \xrightarrow{z \to \infty} 1/\sqrt{\pi}.$$  

The continuity condition (93) takes the form

$$C_1 U(-1/2, 0; z_0^2) = C_2 z_0^2 M(1/2, 2; z_0^2),$$  

while the normalization (94) yields

$$C_2^2 \int_0^{z_0} z^3 e^{-z^2} M^2(1/2, 2; z^2) dz + C_1^2 \int_{z_0}^{\infty} z^{-1} e^{-z^2} U^2(-1/2, 0; z^2) dz = 1.$$  

The solution of sum rules (120) gives a remarkable relation for the slope of meson trajectories, $a \simeq \Lambda_{\text{CSB}}^2$. Identification of $z_0$ with $\Lambda_{\text{CSB}}^{-1}$ means then in our units for $z$ that $z_0 = 1/2$. The system of equations (99), (100) yields numerically $C_1 \approx \pm 1.2$, $C_2 \approx \pm 3.5$.

The substitution of solution (92) back into the action (87) gives the vacuum energy. As follows from the asymptotics (98), the vacuum energy density $\varepsilon_{\text{vac}}$ will be divergent logarithmically at $z = 0$,

$$\varepsilon_{\text{vac}} = 9 C_2^2 \log z |_{z \to 0} + \text{const.}$$  

By subtracting the divergence in Eq. (101), we obtain the renormalized vacuum energy density. In the case of numerical example above it is

$$\varepsilon_{\text{vac}}^{\text{(ren)}} \approx -29.3 |_{z \leq 0.5} + 3.2 |_{z > 0.5} = -26.1.$$  

The negative value for $\varepsilon_{\text{vac}}^{\text{(ren)}}$ is in accord with expectations from the gluodynamics [52].

Comparing the prescription (89) with the solution (92) and the asymptotics (98), we would say in AdS/QCD language that the CSB is mimicked in the chiral limit (zero current quark mass). This is self-consistent as long as we have assumed initially the existence of exactly massless scalar particle.

Let us consider the case of vector mesons. First of all we note that the trick above does not allow to introduce the massless vector particles as bound states — one can check that although non-normalizable massless solutions exist for the spin-one mesons, both of them are non-normalizable at $z \to 0$. Thus, we are save from the appearance of massless vectors.$^8$

$^8$We would note that there is a related logical uneasiness in some AdS/QCD models: If one introduces a non-normalizable scalar field, why is this forbidden in the vector channel?
The axial-vector sector is a traditional place for modelling the CSB physics. We will not boil down to the model building of this kind (the most popular way is the introduction of longitudinal component for $A_M$ and of the pion field $\pi$ through the parametrization $X = X_0 e^{i2\pi}$) since in the chiral limit it does not seem to bring new interesting results. An important consequence of the term (87) in the action is that the masses of axial-vector mesons get shifted due to coupling to the v.e.v. $X_0$ through the covariant derivative (88). Within the model IB$, the new mass spectrum can be in principle calculated via the substitution $m_j^2 \rightarrow m_j^2 + g_5^2 X_0^2(z)$ into the Eq. (45). Although the exact analytic solution is not known, it is clear that the masses will be enhanced because the new contribution to the "potential" of Shr"odinger equation (45) is always positive. In addition, from the solution (92) and the asymptotics (97), (98) follows that the holographic potential in the Eq. (45) will be (taking into account that $m_{J=1}^2 = 0$)

$$U|_{z > z_0} = z^2 \left( 1 + g_5^2 C_1^2 e^{-2z^2} \right) + \frac{3}{4z^2},$$

$$U|_{z < z_0} = z^2 \left( 1 + g_5^2 C_2^2 z^2 \right) + \frac{3}{4z^2}.$$  

(103)

(104)

This behavior demonstrates that the spectrum of axial-vector states approaches rapidly to the spectrum of vector mesons and the axial-vector residues are well-defined at least for high enough excitations. The rate of such a "Chiral Symmetry Restoration" (CSR) is exponential that is in agreement with the analysis [53,54] based on the QCD sum rules. This qualitative feature differs from the previous results obtained within the soft wall holographic models [34,38] in which the shift between the $V$ and $A$ masses square tends to a constant and the axial-vector residues cannot be determined from the expression (24). It must be noted that the described behavior does not contradict to the CS non-restoration scenario observed in the phenomenology [48,49]. The latter states that the mesons on the leading Regge trajectories do not have parity partners and the spectral degeneracy has the form

$$m_{n,J}^- \approx m_{n,J}^+ \quad (J \text{ uneven}); \quad m_{n+1,J}^+ \approx m_{n,J}^- \quad (J \text{ even}),$$

(105)

where $\pm$ refer to parity. The model above can be accommodated to such a scenario.
6 QCD sum rules in the large-$N_c$ limit vs. AdS/QCD

The momentum representation for the vector (V) and axial-vector (A) two-point functions in the large-$N_c$ limit of QCD can be written as (neglecting the nonpole terms)

\[\langle J^V_\mu(q) J^V_\nu(-q) \rangle = \Pi^{\perp}_{\mu\nu}(q) \sum_{n=0}^{\infty} \frac{F^2_{V,n}}{q^2 - m^2_{V,n} + i\varepsilon}, \quad (106)\]

\[\langle J^A_\mu(q) J^A_\nu(-q) \rangle = q_\mu q_\nu f^2_\pi + \Pi^{\perp}_{\mu\nu}(q) \sum_{n=0}^{\infty} \frac{F^2_{A,n}}{q^2 - m^2_{A,n} + i\varepsilon}, \quad (107)\]

where $F_n$ are the corresponding electromagnetic meson decay constants, $f_\pi$ denotes the pion weak decay constant, and

\[\Pi^{\perp}_{\mu\nu}(q) = -\eta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\]

is the transverse projector.

On the other hand, one can write the Operator Product Expansion (OPE) for those two-point functions at large Euclidean momentum, $Q^2 = -q^2$,

\[\langle J_\mu(q) J_\nu(-q) \rangle = \Pi^{\perp}_{\mu\nu}(q) Q^2 \left( C_0 \ln \frac{\Lambda^2}{Q^2} + \sum_{i=1}^{\infty} \frac{C_i}{Q^{2i}} \right). \quad (109)\]

Here $\Lambda$ is a renormalization scale and the coefficients $C_i$ depend weakly on $\Lambda$ and $Q^2$. It is easy to show that in order to obtain the analytic behavior (109) the residues in Eqs. (106) and (107) must satisfy the following condition (see, e.g., [53] for a detailed discussion)

\[F^2_n \sim m^2_n \partial_n m^2_n. \quad (110)\]

Thus, given the relation (110), the infinite number of meson poles is equivalent to the partonic logarithm in the Eq. (109) plus some corrections. This property we refer to as the quark-hadron duality in the large-$N_c$ limit (brief reviews can be found in [53, 55]). Matching the Eq. (109) to the representations (106) and (107) in the Euclidean region, one obtains the QCD sum rules which represent certain relations on the parameters of the meson spectrum [53–57].

Consider the simplest example — the linear spectrum

\[m^2_{V,A,n} = a(n + b_{V,A,n}), \quad (111)\]
where the slope $a$ is assumed to be universal. According to the condition (110), the residues must behave as (we neglect possible corrections to the relation (110) [53])

$$F_{V,A;n}^2 \sim m_{V,A;n}^2 F_{V,A}^2, \quad F_{V,A}^2 = \text{const}. \quad (112)$$

Performing the summation in $n$ in the expression (106), one arrives at [54,57]

$$\langle J_\mu^V(q) J_\nu^V(-q) \rangle = \Pi_{\mu\nu}^V(q) Q^2 \left[ \ln \frac{a}{Q^2} + \sum_{i=1}^{\infty} \frac{(-1)^i a^i B_i(b_V)}{i Q^{2i}} + \text{const} \right], \quad (113)$$

where $B_i(x)$ are the Bernoulli polynomials. The OPE for the two-point functions at $N_c \gg 1$ and in the chiral limit reads as follows [58]

$$\langle J_\mu^{V,A}(q) J_\nu^{V,A}(-q) \rangle = \Pi_{\mu\nu}^{V,A}(q) Q^2 \left[ \frac{N_c}{12 \pi^2} \left( 1 + \frac{\alpha_s}{\pi} \right) \ln \frac{A^2}{Q^2} + \frac{\alpha_s}{12 \pi} \frac{\langle G^2 \rangle}{Q^4} + \frac{4 \xi^{V,A}}{9 \pi \alpha_s} \frac{\langle \overline{q} q \rangle^2}{Q^6} + O \left( \frac{1}{Q^8} \right) \right]. \quad (114)$$

Here $\langle G^2 \rangle$ and $\langle \overline{q} q \rangle$ denote the gluon and quark condensates and $\xi^V = -7, \xi^A = 11$. Comparing the Eq. (114) with (113) and with analogous expression for the axial-vector two-point function, one obtains the following sum rules (we consider the leading order in $\alpha_s$),

$$F_V^2 = F_A^2 \equiv F^2 = \frac{N_c a}{24 \pi^2}, \quad (115)$$

$$0 = b_V - \frac{1}{2}, \quad (116)$$

$$\frac{f^2}{F^2} = b_A - \frac{1}{2}, \quad (117)$$

$$\frac{\alpha_s \langle G^2 \rangle}{12 \pi F^2 a} = b_{V,A}^2 - b_{V,A} + \frac{1}{6}, \quad (118)$$

$$- \frac{2 \xi^{V,A} \pi \alpha_s \langle \overline{q} q \rangle^2}{3 F^2 a^2} = b_{V,A} (b_{V,A} - 1/2) (b_{V,A} - 1). \quad (119)$$

The sum rules (115)-(119) cannot be satisfied by the simple linear spectrum (111), a natural solution of the problem is to consider nonlinear corrections to the ansatz (111) or/and not place some states on the linear trajectory. We note, however, that the sum rules (115)-(116), (117)-(118)- (119)… are progressively less and less reliable due to the asymptotic nature of OPE and...
growing anomalous dimensions of condensate terms. The simple linear spectrum is able to satisfy the most reliable relations (115)-(117). Identifying $F \equiv F_\rho$ and making use of the KSFR relation [59] $F_\rho^2 = 2f_\pi^2$ as in the original Weinberg sum rules [60], we have immediately the solution

$$b_V = 1/2, \quad b_A = 1, \quad a = 48\pi^2 f_\pi^2/N_c,$$

which is in a good agreement with the phenomenology (for instance, the Weinberg relation $m_{a_1}^2 = 2m_\rho^2$ follows automatically). This solution for the intercept $b_{V,A}$ yields the spectrum of the generalized Lovelace-Shapiro (LS) dual amplitude [61]. The difference $b_A - b_V$ emerges from the CSB dynamics that is directly seen in the Eq. (117) (namely the fact that $f_\pi \neq 0$). Within the LS amplitude, the same difference arises from taking into account the Adler self-consistency condition [62] which is a logical consequence of the Goldstone theorem.

Turning back to the holographic approach, we note that the spectrum (74) that often appears in the soft-wall AdS/QCD models, in fact, does not describe the $\rho$ and $\omega$ mesons as opposed to the usual belief, it corresponds rather to the axial-vector mesons. We are not aware of any natural way for accommodation of the intercept $b_V = 1/2$ within the holographic approach and/or the technique presented in the previous sections, unless some speculative extra assumptions are involved [27,37]. As follows from the discussions above, the correct vector spectrum should naturally stem from a successful implementation of the CSB physics in the holographic model.

As seen from the expansion (113), the spectrum of masses (at least in the vector channels) determines the gauge-invariant QCD condensates; if there are several such condensates of equal dimension it determines a certain linear combination of them. For this reason, any attempts of inclusion of QCD condensates into the soft wall holographic models face a serious problem of double counting. This remark, however, is not relevant for the hard wall models [8,10], where the masses are given by roots of Bessel function, $m_n \sim n$, and for this reason it turns out that the corrections to the partonic logarithm does not reproduce the analytic structure of OPE.

Our previous discussions incline us to conclude that the bottom-up holographic approach may represent just an alternative language for expressing the phenomenology known from the QCD sum rules in the large-$N_c$ limit with practically the same number of input parameters and ensuing accuracy.

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9The situation resembles the almost 40 years old discussions about a still unresolved problem: In contrast to the Veneziano dual amplitude that gave rise to the modern string theory, the LS amplitude does not ensue from any string model; the incorporation of the CSB effect into the string approach is a formidable task up to now.
This is directly seen from the bottom-up derivation of AdS/QCD like models in Section 2 and \textit{a posteriori} from the type of questions which are addressed within the holographic models. First of all, the problem of finding the fittest spectrum in the sum rules is reformulated within the AdS/QCD models as the problem of finding an appropriate 5D background and boundary conditions on the 5D fields which give the required spectrum as a result of solution of the corresponding SL problem while in the sum rules the spectrum is looked for empirically ignoring its model-dependent origin. The subsequent phenomenology, however, is very similar in the both approaches. For instance, both sum rules and most of holographic models incorporate the pion via the Partial Conservation of Axial Current hypothesis. As is known from the phenomenology of sum rules \cite{53}, it is difficult to accommodate the pion as the first state in the tower of pseudoscalar resonances, one should rather consider it separately as a special state, the same conclusion seems to follow from the AdS/QCD models.

7 Concluding remarks

The phenomenology expected in the large-$N_c$ limit of QCD can be compactly rewritten in terms of some phenomenological 5D theory, namely the infinite towers of narrow meson states coupled to external sources can be formally represented as some 5D single fields propagating in a suitable background. This allows to replace sometimes lengthy manipulations with infinite series of meson poles by compact operations with those 5D fields. Needless to say that such an introduction of 5D formalism represents a certain operational progress and one could even draw a parallel with the introduction of continual integral into the field theory in the sense that manipulations in the latter formalism are usually understood as a compact formal way to write lengthy operations with series of perturbation theory. The modern 5D model building, however, involves various speculative assumptions borrowed from the AdS/CFT correspondence which look often unfounded in their extrapolation to real QCD. In the present work, we have explicitly demonstrated that 5D models very similar to the so-called soft wall holographic models can be derived within the 5D formalism imposing some requirements of phenomenological consistency without use of AdS/CFT dictionary. These models represent just another language for expressing the Regge phenomenology within the framework of QCD sum rules.

The 5D formalism proposes new ways for description of the chiral symmetry breaking. The related attempts cannot be viewed as another language for expressing the matching of low-energy effective field theories for strong
interactions to QCD sum rules (see, e.g., [63] among others), but are similar in spirit though. The reason is that the effective models deal with the lowest mass (the first radial excitation at best) mesons while the 5D models are able to describe the impact of CSB on the whole tower of excited states. Another reason consists in the fact that the existing 5D descriptions of CSB are vague from the 4D point of view. We have tried to advance in "visualizing" a 5D description of CSB in 4D terms but could not provide a concrete example. The main problem to be solved is the derivation of GOR relation for the pion mass within the presented approach.

We did not discuss an explicit incorporation of quarks and QCD running coupling since without knowledge of the confinement mechanism, at least in the planar limit of QCD, any such attempt is doomed to be rather speculative.

It must be emphasized that the considered 5D approach is completely phenomenological and hardly can be directly related to any fundamental extra-dimensional theory. First of all, the 5D background of the soft wall model that we have also obtained cannot be a solution of 5D Einstein equations as it breaks explicitly the 5D general covariance. A possible way out could consist in constructing an alternative mechanism for mass generation. Such an attempt for flat metric (an "effective" holographic model) was undertaken in Ref. [64] (see also [65]) where the Higgs mechanism was used. Second, we have ignored all long-standing problems in description of higher spin fields. If we included a gravitational dynamics that would yield particular metrics in question we would have the problem of gravitational coupling for higher spin fields — such a theory does not exist. For this reason we believe that within the present approach it is meaningless to analyze the backreaction of metric due to condensation of scalar field when modeling the CSB physics as well as to address other questions of relation to consistent extra-dimensional theories until some fundamental problems in field theory are solved. Further discussions of relevant problems can be found in [66].

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Appendix A: Sturm-Liouville theorem

In this Appendix, we briefly remind the reader the main results of the Sturm-Liouville (SL) theory. The SL equation is

\[-\partial_z [p(z) \partial_z \varphi] + q(z) \varphi = \lambda \omega(z) \varphi. \tag{A.1}\]

Here the function \(p(z) > 0\) has a continuous derivative, the functions \(q(z) > 0\) and \(\omega(z) > 0\) are continuous on the finite closed interval \([z_{\text{min}}, z_{\text{max}}]\). Loosely speaking, the SL problem consists in finding the values of \(\lambda\) for which there exists a non-trivial solution of Eq. (A.1) satisfying certain boundary conditions. Under the assumptions that \(p(z)^{-1} q(z),\) and \(\omega(z)\) are real-valued integrable functions over the interval \([z_{\text{min}}, z_{\text{max}}]\), with the boundary conditions of the form

\[
\varphi(z_{\text{min}}) \cos \alpha - p(z_{\text{min}}) \varphi'(z_{\text{min}}) \sin \alpha = 0, \tag{A.2}
\]

\[
\varphi(z_{\text{max}}) \cos \beta - p(z_{\text{max}}) \varphi'(z_{\text{max}}) \sin \beta = 0, \tag{A.3}
\]

where \(\alpha, \beta \in [0, \pi]\) and prime denotes the derivative, the SL theorem states that

- There is infinite discrete set of real eigenvalues \(\lambda_n, n = 0, 1, 2, \ldots\).
- Up to a normalization constant, there is a unique eigenfunction \(\varphi_n(z)\) corresponding to each eigenvalue \(\lambda_n\) and this eigenfunction has exactly \(n - 1\) zeros in \([z_{\text{min}}, z_{\text{max}}]\).
- The normalized eigenfunctions form an orthonormal basis

\[
\int_{z_{\text{min}}}^{z_{\text{max}}} \varphi_m(z) \varphi_n(z) \omega(z) dz = \delta_{mn}. \tag{A.4}
\]

Thus the solutions of the SL problem form a complete set of functions in the interval \([z_{\text{min}}, z_{\text{max}}]\), i.e. they can be used for expansion of arbitrary functions in that interval.

Appendix B: Schrödinger equations appearing in the work

The spectrum is typically determined by the equation

\[-\psi''_n + \left[ z^2 + \frac{b^2}{z^2} - \frac{1}{4} / z^2 \right] + c \psi_n = m_n^2 \psi_n, \tag{B.1}\]
that is exactly solvable. The spectrum of normalizable modes is

\[ m_n^2 = 4n + 2|b| + 2 + c, \quad n = 0, 1, 2, \ldots, \] (B.2)

the corresponding normalized eigenfunctions are

\[ \psi_n = \sqrt{\frac{2n!}{(|b| + n)!}} e^{-z^2/2} z^{n+1/2} L_n^{|b|}(z^2), \] (B.3)

where \( L_n^{|b|} \) are associated Laguerre polynomials.

In the case \( |b| = 1/2 \), the solutions of Eq. (B.1) are different. The spectrum is

\[ m_n^2 = 2n + 1 + c, \quad n = 0, 1, 2, \ldots, \] (B.4)

and the corresponding normalized eigenfunctions are given by

\[ \psi_n = \sqrt{\frac{1}{2^{2n} n!}} \pi^{-1/4} e^{-z^2/2} H_n(z), \] (B.5)

where \( H_n \) are Hermite polynomials.

In the case \( |b| < 1/2 \), there is no finite discrete spectrum.

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