Iterative Splitting Methods: Almost Asymptotic Symplectic Integrator for Stochastic Nonlinear Schrödinger Equation

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Abstract

In this paper we present splitting methods which are based on iterative schemes and applied to stochastic nonlinear Schrödinger equation. We will design stochastic integrators which almost conserve the symplectic structure.

The idea is based on rewriting an iterative splitting approach as a successive approximation method based on a contraction mapping principle and that we have an almost symplectic scheme, see [12] and [9].

We apply a stochastic differential equation, that we can decouple into a deterministic and stochastic part, while each part can be solved analytically. Such decompositions allow accelerating the methods and preserving, under suitable conditions, the symplecticity of the schemes.

A numerical analysis and application to the stochastic Schrödinger equation are presented.

Keywords: splitting methods, stochastic differential equations, iterative splitting schemes, stochastic Schrödinger equation.

AMS subject classifications. 35K25, 35K20, 74S10, 70G65.

1 Introduction

The motivation is to develop fast solver schemes to solve stochastic Hamiltonians in solitary waves and collisions.

The idea is based on almost asymptotic symplecticity for stochastic Hamiltonian partial differential equations, such underlying algorithms are applied to develop stochastic symplectic methods for solving a stochastic Schroedinger equations, see [12].

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It is shown that the noval schemes preserve the symplectic structure in an asymptotic regime, which means it is $O(\delta^{n+1})$ away from a symplectic scheme with $\delta \in (0, 1)$.

**Definition 1.1.** We consider a Hamiltonian system, while $u = (p, q)$ and we write:

$$
\frac{\partial u}{\partial t} = J\nabla_u H(u(t)),
$$

where $(p, q) \in \mathbb{R}^d \times \mathbb{R}^d$ and $J = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}$ and $I_d$ is the $d$-dimensional identity matrix, $\nabla_u$ is the gradient with respect to $u$.

We assume that $\phi_\tau$ is the solution operator with $u(t^{n+1}) = \phi_\tau(u^n)$, where $\tau$ is the time step and we have the following definition about the symplecticity:

- $\phi_\tau$ preserves the symplecticness of the system [1], if:

  $$
  \left. \left( \frac{\partial \phi_\tau}{\partial z(t)} \right)^T \right|_{t=t^n} J \left. \frac{\partial \phi_\tau}{\partial z(t)} \right|_{t=t^n} - J = 0,
  $$

- $\phi_\tau$ preserves the almost (or asymptotic) symplecticness of the system [1], if:

  $$
  \| \left. \left( \frac{\partial \phi_\tau}{\partial z(t)} \right)^T \right|_{t=t^n} J \left. \frac{\partial \phi_\tau}{\partial z(t)} \right|_{t=t^n} - J \| \leq C\delta^{n+1},
  $$

where $C$ is a constant with $\tau = \tilde{\tau}(\delta)$ and $\tilde{\tau}$ is a function of $\delta$, which is given from the solution method.

**Remark 1.1.** The idea of almost symplecticity has the origin of modifying the definition of symplecticity. For example, if one assume that $J$ depends on $u$, then one can proof, that we have an almost poisson structure and we preserve the poisson structure up to the second order, see [2]. Such ideas are also used in the development in pseudo-symplectic methods, see [1].

In the following, we deal with the stochastic nonlinear Schrödinger equation with multiplicative noise, which is given by

$$
i \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2|u|^{2\sigma} u + \epsilon u \circ \frac{dW}{dt}, \quad t > 0, \epsilon > 0, x \in \mathbb{R},
$$

where $u = u(x, t)$ is the complex-valued solution and $\circ$ denotes $\frac{dW}{dt}$ is defined as a real-valued white noise which is delta correlated in time and either smooth or delta correlated in space.

The deterministic nonlinear Schrödinger equation is given by

$$
i \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2|u|^{2\sigma} u,
$$

which is well-known in the literature [11].
# Iterative Splitting as a Successive Approximation Method

We can rewrite this to a Hamiltonian system by

\[ u = p + iq, \] where \( p \) and \( q \) are real-valued functions and we can separate it into the following form and we obtain a multi-symplectic system:

\[
\begin{align*}
\left( \begin{array}{c}
dP \\
dQ 
\end{array} \right) &= \left( \begin{array}{c}
-H_q(P,Q)dt - G_q(P,Q) \circ dW(t) \\
H_p(P,Q)dt + G_p(P,Q) \circ dW(t)
\end{array} \right) \\
\end{align*}
\]

where we have the symplectic structure \( dP \wedge dQ = dp \wedge dq \).

The system is given by

\[
\begin{align*}
\left( \begin{array}{c}
 dp \\
 dq 
\end{array} \right) &= \left( \begin{array}{cc}
 0 & A_1 + A_2(p,q) \\
 -A_1 + A_2(p,q) & 0
\end{array} \right) \left( \begin{array}{c}
p \\
 q 
\end{array} \right) dt \\
&+ \varepsilon \left( \begin{array}{cc}
 0 & I \\
 -I & 0
\end{array} \right) \left( \begin{array}{c}
p \\
 q 
\end{array} \right) \circ dW,
\end{align*}
\]

where the matrices are given by the semi-discretization of the original system.

**Theorem 2.1.** The iterative splitting scheme is almost symplectic.

**Proof.** For the Hamiltonian system

\[ dy = F(y)dt + G(y)dW \]

we apply the successive approximation method:

\[ y^{n+1,i+1} = K(y^n, y^{n+1,i}, y^{n+1,i}) = y^n + F(y^{n+1,i+1})\Delta t + G(y^{n+1,i})\Delta W, \]

where we apply the linearised scheme:

\[ y^{n+1,i+1} = \tilde{K}(y^n, y^{n+1,i+1}, y^{n+1,i}) = y^n + \tilde{F}y^{n+1,i+1}\Delta t + \tilde{G}y^{n+1,i}\Delta W, \]

further, the contraction mapping is given by

\[
\begin{align*}
||\tilde{K}(y^n, y^{n+1,i+1}, y^{n+1,i}) - \tilde{K}(x^n, x^{n+1,i+1}, x^{n+1,i})|| \\
&\leq \tilde{\rho}||y^{n+1,i+1} - x^{n+1,i+1}||,
\end{align*}
\]

where \( \tilde{\rho} = \rho_1 + \rho_2 \) and \( \rho_1 = \Delta t||\tilde{F}|| \) and \( \rho_2 = \Delta W||\tilde{G}|| \).

## Almost Symplectic Scheme

In the following, we discuss the linearised equation in the algorithm.

We have the fixed-splitting discretisation step-size \( \tau \), on the time-interval \([t^n, t^{n+1}]\), and the stochastic time step \( \Delta W = W_{t^{n+1}} - W_{t^n} = \Delta tX \) (Wiener
process), where $X$ is a Gaussian distributed random variable with $E(X) = 0$ and $\text{Var}(X) = 1$, see [10].

We solve the following sub-problems consecutively for $i = 1, 2, \ldots, m + 1$.

\begin{equation}
\frac{dy_i(t)}{dt} = Ay_i(t)dt + By_{i-1}dW_i(t), \text{ with } y_i(t^n) = y^n \quad (12)
\end{equation}

and $y_0(t^n) = y^n, y_{-1} = 0.0$

where $y^n$ is the known successive approximation at the time-level $t = t^n$. The split approximation at the time-level $t = t^{n+1}$ is defined by $y^{n+1} = y_{m+1}(t^{n+1})$.

We can rewrite this into the following ODE form:

\begin{equation}
\frac{\partial y_i(t)}{\partial t} = Ay_i(t) + By_{i-1}\dot{W}_t, \text{ with } y_i(t^n) = y^n \quad (13)
\end{equation}

and $y_0(t^n) = y^n, y_{-1} = 0.0$.

where $\dot{W}_t = \frac{dW_t}{dt}$.

**Theorem 3.1.** We are given $A, B \in L(\Omega)$ linear bounded operators (e.g., due to the linearisation) and we consider the abstract Cauchy problem

\begin{equation}
\frac{dy(t)}{dt} = Aydt + BydW_t, \quad 0 < t \leq T \\
y(0) = y_0.
\end{equation}

(14)

Then the problem [12] has a unique solution; the iterations [13] over $i = 1, 2, \ldots, m + 1$ are convergent with order $O(\sqrt{\Delta t}^{n+1})$.

**Proof.** The problem [12] has a unique solution $c(t) = \exp((A \Delta t + B \Delta W)c_0$.

For the local error function $e_i(t) = y(t) - y_i(t)$, we have the relations

\begin{equation}
\frac{de_i(t)}{dt} = Ae_i(t) + Be_{i-1}\dot{W}_t, \quad t \in (t^n, t^{n+1}],
\end{equation}

$e_i(t^n) = 0$. (15)

Applying the method of variation of constants, the solution of the abstract Cauchy problem can be written as

\begin{equation}
e_i(t) = \int_{t^n}^{t} \exp(A(t-s))Be_i(s)dW_s, \quad t \in [t^n, t^{n+1}].
\end{equation}

(16)

Furthermore, we have

\begin{equation}
\|e_i\|_t \leq \|B\|\|e_{i-1}\| \int_{t^n}^{t} \|\exp(A(t-s))\|dW_s, \quad t \in [t^n, t^{n+1}].
\end{equation}

(17)

Based on our assumption that $A$ is bounded, we have

\begin{equation}
\|e_i\|_t \leq K\|B\|\sqrt{\Delta t}\|e_{i-1}\|, \quad t \in [t^n, t^{n+1}].
\end{equation}

(18)

where $\|\exp(A\Delta t)\| \leq K$, $t > 0$.

The estimations [13] result in

\begin{equation}
\|e_{m+1}\| = K\sqrt{\Delta t}^{m+1}\|e_0\| + O(\sqrt{\Delta t}^{m+2}),
\end{equation}

(19)

which proves our statement.
Furthermore, the almost asymptotic symplecticity of the scheme is given as:

**Theorem 3.2.** Consider the algorithm and let be the solver step of the algorithm. Then for any , there exists where and the time-step and we have

\[
\left( \frac{\partial \phi_{\Delta t}^{n}}{\partial y_{0}} \right)^{t} J \left( \frac{\partial \phi_{\Delta t}^{n}}{\partial y_{0}} \right) - J \| \leq C \delta^{m+1}, \forall y_{0} \in \Omega,
\]

where is a constant.

**Proof.** The algorithm has the following solution:

\[
\phi_{\Delta t}^{n} = \exp(A \Delta t) y_{n} + \int_{t_{n}}^{t_{n+1}} \exp(A(t_{n+1} - s)) B y_{n-1}(s) \, dW_{s}. \tag{21}
\]

Furthermore, we have

\[
\| \frac{\partial \phi_{\Delta t}^{n}}{\partial y_{0}} \| \leq \| \exp(A \Delta t) B \Delta W_{t} \|,
\]

\[
\leq C \|B\| \sqrt{\Delta t}, \tag{22}
\]

and the recursion is given by

\[
\left( \frac{\partial \phi_{\Delta t}^{n}}{\partial y_{0}} \right)^{t} J \left( \frac{\partial \phi_{\Delta t}^{n}}{\partial y_{0}} \right) - J \| \leq \tilde{C} \sqrt{\Delta t}^{t+1}, \tag{23}
\]

when the estimations result in

\[
\left( \frac{\partial \phi_{\Delta t}^{n}}{\partial y_{0}} \right)^{t} J \left( \frac{\partial \phi_{\Delta t}^{n}}{\partial y_{0}} \right) - J \| \leq C \delta^{m+1}, \forall y_{0} \in \Omega, \tag{24}
\]

and \( \sqrt{\tau} \leq \frac{\delta}{K_{1}} \), which proves our statement.

\( \square \)

4 Numerical Methods

In the following, we treat the different numerical methods.

The underlying equation is given as

\[
i \frac{\partial u}{\partial t} = \lambda \frac{1}{2} \frac{\partial^{2} u}{\partial x^{2}} + 2 |u|^{2\sigma} u + \epsilon u \circ \frac{dW}{dt}, \quad t > 0, \epsilon > 0, x \in \mathbb{R}, \tag{25}\]

where the initial values are given as \( u_{t_{0}} = u_{0}, \lambda \in \mathbb{R} \) and \( W \) is a Wiener process.

We apply a semi-discretisation via finite difference schemes and obtain the ODE problem

\[
i \frac{\partial u}{\partial t} = Au + B(u)u + C u \circ \frac{dW}{dt}, \quad t > 0, \epsilon > 0, x \in \mathbb{R}, \tag{26}\]

where the operators are given by

\[
A = \lambda \frac{1}{\Delta x^{2}} [1 - 2 1], \tag{27}\]

\[
B(u) = 2 |u|^{2\sigma} \tag{28}\]

\[
C = \epsilon u, \tag{29}\]

where we apply the different splitting schemes.
4.1 Linearised stochastic Schrödinger equation

We consider the following linearised stochastic Schrödinger equation:

\[
i\frac{\partial u}{\partial t} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} u + V(x, t)u + \psi |u|^2 u + \epsilon u \circ dW dt, \quad (x, t) \in [0, 1] \times [0, 1],
\]

where \( u_0(x) = \exp(\sin(2x)) \).

We assume periodic boundary conditions

\[
u(x, 0) = u_0(x), \quad x \in [0, 1],
\]

\[
u(0, t) = u(1, t), \quad t \in [0, 1],
\]

where \( \Omega = [x_L, x_R] \), e.g. \( x_L = 0, x_R = 1.0 \) and \( \epsilon \) is small.

We employ the following transformation and change of variables:

\[
\begin{pmatrix}
\dot{\eta} \\
\dot{\xi}
\end{pmatrix} = \begin{pmatrix}
0 & A(t, x, \eta, \xi) \\
-A(t, x, \eta, \xi) & 0
\end{pmatrix} \begin{pmatrix}
\eta \\
\xi
\end{pmatrix}
\]

We apply a finite difference discretisation and the matrices are given as

\[
A(t, x, \eta, \xi) = A_1(t, x) + A_2(t, x, \eta, \xi) + A_3(t, x),
\]

\[
A_1(t, x) = -\frac{1}{2} \frac{\Delta x^2}{\Delta x^2}[1 - 2 1],
\]

\[
A_2(t, x, \eta, \xi) = \left( V(x) + 2(\eta^2 + \xi^2) \right) [0 1 0],
\]

\[
A_3(t, x) = \epsilon \Delta W [0 1 0].
\]

\[
A_1(t, x) = -\frac{1}{2} \frac{1}{\Delta x^2} \begin{pmatrix}
-2 & 1 & 0 & \ldots & 0 & 1 \\
1 & -2 & 1 & 0 & \ldots & 0 \\
0 & 1 & -2 & 1 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 1 & -2
\end{pmatrix},
\]

\[
A_2(t, x, \eta, \xi) = \begin{pmatrix}
\tilde{V}(x_1) & 0 & 0 & \ldots & 0 & 0 \\
0 & \tilde{V}(x_2) & 0 & \ldots & 0 & 0 \\
0 & 0 & \tilde{V}(x_3) & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & \tilde{V}(x_M)
\end{pmatrix},
\]

\[
\tilde{V}(x_i) = V(x_i) + 2(\eta^2(x_i, t^n-1) + \xi^2(x_i, t^n-1)),
\]

\[
A_3(t, x) = \begin{pmatrix}
\epsilon & 0 & 0 & \ldots & 0 & 0 \\
0 & \epsilon & 0 & 0 & \ldots & 0 \\
0 & 0 & \epsilon & 0 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & \epsilon
\end{pmatrix},
\]
where we have $V(x,t) = 1.0$, $\epsilon = 1$, $\Delta x = 0.1, 0.01, 0.001$.

We apply the operator splitting schemes:

$$
\begin{pmatrix}
\eta_{n+1} \\
\xi_{n+1}
\end{pmatrix} = \exp(\Delta t \tilde{A}_1) \exp(\Delta t \tilde{A}_2(\eta^n, \xi^n)) \cdot \exp\left(-\frac{1}{2} \Delta t (\tilde{A}_2^T \tilde{A}_3 + \tilde{A}_3 \Delta W_i) \begin{pmatrix}
\eta^n \\
\xi^n
\end{pmatrix}\right), 1 \leq n \leq N,
$$

$$
\eta^0 = (\exp(\sin(2x_1)), \ldots, \exp(\sin(2x_M)))^t, \xi^0 = (0, \ldots, 0)^t
$$

where $\Delta t = t_{n+1} - t_n$, the random variable $W_i$ is based on a Wiener process with $\Delta W_i = W_{i+1} - W_i = \sqrt{\Delta t} X$, and $X$ is a Gaussian distributed random variable with $E(X) = 0$ and $Var(X) = 1$. This means we have $\Delta W_i = rand\sqrt{\Delta t}$.

The splitting operators are

$$
\tilde{A}_1 = \begin{pmatrix}
0 & A_1(t, x, \eta, \xi) \\
-A_1^T(t, x, \eta, \xi) & 0
\end{pmatrix} \in \mathbb{R}^{2m \times 2m},
$$

$$
\tilde{A}_2 = \begin{pmatrix}
0 & A_2(t, x, \eta, \xi) \\
-A_2^T(t, x, \eta, \xi) & 0
\end{pmatrix} \in \mathbb{R}^{2m \times 2m},
$$

$$
\tilde{A}_3 = \begin{pmatrix}
0 & A_3(t, x, \eta, \xi) \\
-A_3^T(t, x, \eta, \xi) & 0
\end{pmatrix} \in \mathbb{R}^{2m \times 2m}
$$

We present the different convergent time-steps results for $|u| = \sqrt{\eta^2 + \xi^2}$.

The analytical solution is

$$
\begin{pmatrix}
\eta_{n+1} \\
\xi_{n+1}
\end{pmatrix} = \exp((\tilde{A}_1 + \tilde{A}_2(\eta^n, \xi^n) - \frac{1}{2} (\tilde{A}_2^T \tilde{A}_3 + \tilde{A}_3 \Delta W_i)) \Delta t + \tilde{A}_3 \Delta W_i) \begin{pmatrix}
\eta^n \\
\xi^n
\end{pmatrix}, 1 \leq n \leq N,
$$

$$
\eta^0 = (\exp(\sin(2x_1)), \ldots, \exp(\sin(2x_M)))^t, \xi^0 = (0, \ldots, 0)^t
$$

where $\Delta t = t_{n+1} - t_n$, the random variable $W_i$ is based on a Wiener process with $\Delta W_i = W_{i+1} - W_i = \sqrt{\Delta t} X$, and $X$ is a Gaussian distributed random variable with $E(X) = 0$ and $Var(X) = 1$. This means we have $\Delta W_i = rand\sqrt{\Delta t}$.

The solution is given by $|u|$ and the errors are

$$
||u_{\text{refer}}(x,t) - u_{i,j}(t)||_{L^2(0,T)} = \Delta x \sum_{n=1}^{N} (u_{\text{refer}}(x_i,t) - u(x_i,t))^2,
$$

$$
E(||u_{\text{refer}}(x,t) - u_{i,j}(x,t)||) = \frac{1}{N} \sum_{n=1}^{N} |u_{\text{refer}}(x_i,t) - u(x_i,t)|,
$$

In the following figures, we present the results for the error of the iterative splitting schemes, see Fig. 1.
Figure 1: The $L_2$-errors of the iterative splitting scheme

Figure 2: The results of the A–B splitting with $\Delta t = 0.005, \Delta x = 0.005$.

In the following figures, we present the results for the different splitting schemes, see Fig. 2.

Remark 4.1. With more iterative steps, we see an improvement in the numerical results. With two to three iterative steps, we obtain nearly the analytical solution. Here, we could see the almost asymptotic behaviour of the scheme.

4.2 Deterministic Schroedinger equation: Perturbations

We consider the following equation:

\[
    i\hbar \frac{\partial u}{\partial t} = Hu,
\]  

\[
    Hu = -\frac{1}{2} \frac{\partial^2}{\partial x^2} u + \left( \frac{1}{1 + \sin^2(x)} + \lambda |u|^2 \right) u,
\]

where $\lambda = 30$, $u_0 = \exp(\sin(2x))$.

We employ the following transformation and change of variables:

$u = \eta + i\xi$
\[ \begin{pmatrix} \dot{\eta} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} 0 & A(t, x, \eta, \xi) \\ -A(t, x, \eta, \xi) & 0 \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix} \]  
(52)

\[ A(t, x, \eta, \xi) = A_1(t, t) + A_2(t, x, \eta, \xi), \]  
(53)

\[ A_1(t, x) = -\frac{1}{2} \frac{1}{\Delta x^2} [1 - 21], \]  
(54)

\[ A_2(t, x, \eta, \xi) = \left( \frac{1}{1 + \sin^2(x)} + \lambda(\eta^2 + \xi^2) \right) [0 1 0]. \]  
(55)

The underlying discretised matrices for the splitting schemes are given as:

\[ \tilde{A}_1 = \begin{pmatrix} 0 & A_1(t, x) \\ -A_1(t, x) & 0 \end{pmatrix}, \]  
(56)

\[ \tilde{A}_2(t, x, \eta(t^n-1), \xi(t^n-1), x) = \begin{pmatrix} 0 & A_2(t, x, \eta(t^n-1), \xi(t^n-1), x) \\ -A_2(t, x, \eta(t^n-1), \xi(t^n-1), x) & 0 \end{pmatrix}. \]  
(57)

In the next list of schemes we discuss different splitting scheme. The first splitting scheme is known as an A-B splitting or Lie-Trotter splitting scheme, see [13], while we apply multiplicative the different separated operators. The second splitting scheme is known as an iterative splitting scheme, see [4]. Such a scheme apply iteratively the separated operators based on a fix-point approximation, see [7].

We will employ the following splitting schemes:

- **A–B splitting**

  \[ u_n = \exp(t\tilde{A}_1) \exp(t\tilde{A}_2)u_{n-1}, 1 \leq n \leq N, \]  
(58)

- **Strang splitting scheme**

  \[ u_n = \exp(t/2\tilde{A}_1) \exp(t/2\tilde{A}_2)u_{n-1}, 1 \leq n \leq N, \]  
(59)

- **Weighted Iterative Splitting 1:** We define a relaxed iterative splitting method based on the critical value \( \lambda \):

  \[ \dot{u}_i = (\hat{A}_1 + (1-\omega)\hat{A}_2)u_i + \omega\hat{A}_2u_{i-1}, \]  
(60)

  \[ = \hat{A}_1u_i + \hat{A}_2u_{i-1}, \]  
(61)

and \( \hat{A}_1 = \hat{A}_1 + (1-\omega)\hat{A}_2, \hat{A}_2 = \omega\hat{A}_2 \) and \( \omega = \frac{1}{\lambda} \).

The algorithm is

\[ \dot{u}_1 = \hat{A}_1u_1, \]  
(62)

\[ \dot{u}_2 = \hat{A}_1u_2 + \hat{A}_2u_1, \]  
(63)

\[ \dot{u}_3 = \hat{A}_1u_3 + \hat{A}_2u_2, \]  
(64)

\[ \dot{u}_4 = ... \]  
(65)
and is solved as:

\[c_1(t) = \exp(\hat{A}_1 t) c(t^n),\]  
(66)

\[c_2(t) = c_1(t) + c_1(t) \int_0^t \left[ \hat{A}_2, \exp(s \hat{A}_1) \right] ds ,\]  
(67)

\[c_3(t) = c_2(t) + c_1(t) \int_0^t \left[ \hat{A}_2, \exp(s \hat{A}_1) \right][\hat{A}_2, \phi_1(t \hat{A}_1)] ds ,\]  
(68)

\[c_3(t) = c_2(t) + c_1(t) \left( [\hat{A}_2, \exp(t \hat{A}_1)][\hat{A}_2, \phi_2(t \hat{A}_1)] + [\hat{A}_2, \hat{A}_1 \exp(t \hat{A}_1)][\hat{A}_2, \phi_3(t \hat{A}_1)] \right) + O(t^3) ,\]  

where the given \(\phi_i\) is defined as:

\[\phi_0(\hat{A}_1 t) = \exp(\hat{A}_1 t),\]  
(69)

\[\phi_i(\hat{A}_1 t) = \int_0^t \phi_{i-1}(\hat{A}_1 s) ds,\]  
(70)

\[\phi_i(\hat{A}_1 t) = \frac{\phi_{i-1}(\hat{A}_1 t) - I \frac{t^{i-1}}{(i-1)!}}{\hat{A}_1}.\]  
(71)

- **Weighted Iterative Splitting 2:** We define a relaxed iterative splitting method based on the critical value \(\lambda\):

\[u_i = \hat{A}_1 u_i + \omega \hat{A}_2 u_{i-1},\]  
(72)

with \(u_i(t^n) = u^n,\)  
(73)

and \(u_0(t^n) = u^n, u_{-1} = 0,\)  
(74)

where \(u^n\) is the known split approximation at the time level \(t = t^n\). The split approximation at the time level \(t = t^{n+1}\) is defined as \(u^{n+1} = u_{2m+1}(t^{n+1})\). The parameter \(\omega \in [0, 1]\). For \(\omega = 0\), we have the sequential splitting method, and for \(\omega = 1\) we have the iterative splitting method.

The following figures present the results for the different splitting schemes, see Fig. 3.

**Remark 4.2.** Here, we have compared the standard splitting scheme with our iterative splitting approach. Based on the resolution of the analytical solution, we obtain the same results as for the standard schemes.
4.3 Deterministic nonlinear Schrödinger equation

We consider the equation

\[
 i \frac{\partial u}{\partial t} = Hu + \epsilon u \circ \frac{dW}{dt}, \quad t > 0, \epsilon > 0, x \in \mathbb{R},
\]

(75)

\[
 Hu = \left( \frac{\partial^2}{\partial x^2} + 2|u|^{2\sigma} \right) u,
\]

(76)

with \( \sigma = 1.0 \) and \( \epsilon = 0.0 \).

We choose the initial condition:

\[
 u|_{t=0} = \frac{1}{\sqrt{2}} \sec\left( \frac{1}{\sqrt{2}}(x - 25) \right) \exp\left(-i \frac{x}{20}\right).
\]

(77)

Then the exact single-soliton solution is

\[
 u(x, t) = \frac{1}{\sqrt{2}} \sec\left( \frac{1}{\sqrt{2}}(x - \frac{t}{10} - 25) \right) \exp\left(-i \frac{x}{20} + \frac{199}{400} t\right).
\]

(78)

We employ the following transformation and change of variables:

\[
 u = \eta + i \xi
\]

We have

\[
 \begin{pmatrix}
 \dot{\eta} \\
 \dot{\xi}
 \end{pmatrix} =
 \begin{pmatrix}
 0 & A(t, x, \eta, \xi) \\
 -A(t, x, \eta, \xi) & 0
 \end{pmatrix}
 \begin{pmatrix}
 \eta \\
 \xi
 \end{pmatrix}
\]

(79)

\[
 A(t, x, \eta, \xi) = A_1(t, x) + A_2(t, x, \eta, \xi),
\]

(80)

\[
 A_1(t, x) = \frac{1}{\Delta x^2}[1 - 2 \mathbf{1}],
\]

(81)

\[
 A_2(t, x, \eta, \xi) = \left( 2(\eta^2 + \xi^2)^\sigma \right) \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.
\]

(82)
The underlying discretised matrices for the splitting schemes are
\[
\tilde{A}_1 = \begin{pmatrix}
0 & A_1(t, x) \\
-A_1(t, x) & 0
\end{pmatrix},
\]
\[
\tilde{A}_2(t, x, \eta(t^{n-1}, x), \xi(t^{n-1}, x)) = \begin{pmatrix}
0 & A_2(t, x, \eta(t^{n-1}, x), \xi(t^{n-1}, x)) \\
-A_2(t, x, \eta(t^{n-1}, x), \xi(t^{n-1}, x)) & 0
\end{pmatrix}.
\]

We consider the following splitting schemes:

- **A–B splitting**
  \[
  u_n = \exp(t \tilde{A}_1) \exp(t \tilde{A}_2) u_{n-1}, 1 \leq n \leq N,
  \]

- **Strang splitting scheme**
  \[
  u_n = \exp(t/2 \tilde{A}_1) \exp(t \tilde{A}_2) \exp(t/2 \tilde{A}_1) u_{n-1}, 1 \leq n \leq N,
  \]

- **Weighted Iterative Splitting 1**: We define a relaxed iterative splitting method based on the critical value \( \lambda \):
  \[
  \dot{u}_i = (\tilde{A}_1 + (1 - \omega)\tilde{A}_2)u_i + \omega \tilde{A}_2 u_{i-1},
  \]
  \[
  = \tilde{A}_1 u_i + \tilde{A}_2 u_{i-1},
  \]
  and \( \tilde{A}_1 = \tilde{A}_1 + (1 - \omega)\tilde{A}_2, \tilde{A}_2 = \omega \tilde{A}_2 \) and \( \omega = \frac{1}{\lambda} \).

The algorithm is

\[
\dot{u}_1 = \tilde{A}_1 u_1,
\]
\[
\dot{u}_2 = \tilde{A}_1 u_2 + \tilde{A}_2 u_1,
\]
\[
\dot{u}_3 = \tilde{A}_1 u_3 + \tilde{A}_2 u_2,
\]
\[
\dot{u}_4 = \ldots
\]

and is solved as

\[
c_1(t) = \exp(\tilde{A}_1 t) c(t^n),
\]
\[
c_2(t) = c_1(t) + c_1(t) \int_0^t [\tilde{A}_2, \exp(s \tilde{A}_1)] ds,
\]
\[
c_2(t) \approx c_1(t) + c_1(t) t [\tilde{A}_2, \exp(t \tilde{A}_1)],
\]

The following figures present the results for the different splitting schemes, see Fig. 4.

We apply \(|u|\) for each solution and obtain the following errors:

\[
\|u_{\text{refer}}(x, t) - u_{i,j}(t)\|_{L^2(0,T)} = \Delta x \sum_{n=1}^N (u_{\text{refer}}(x_i, t) - u(x_i, t))^2,
\]
\[
E(\|u_{\text{refer}}(x, t) - u_{i,j}(x, t)\|) = \frac{1}{N} \sum_{n=1}^N |u_{\text{refer}}(x_i, t) - u(x_i, t)|,
\]
Figure 4: Results of the iterative splitting approach.

Figure 5: The $L_2$-errors of the different splitting schemes, where in the left figure, we have $\Delta t = 0.002$ and $\Delta x = 0.01$ and in the right figure, we have $\Delta t = 0.002$ and $\Delta x = 0.02$.

The following figures present the results for the errors of the iterative splitting schemes, see Fig. 5.

**Remark 4.3.** In both resolution in time and space the iterative splitting method is more accurate than the standard A–B and Strang splitting schemes. Here, we see an improvement based on the successive approximation idea and obtain a more accurate linearisation than for the standard schemes.

### 4.4 Stochastic nonlinear Schrödinger equation

We consider the equation

$$i\frac{\partial u}{\partial t} = Hu + \epsilon u \circ dW, \quad t > 0, \epsilon > 0, x \in \mathbb{R},$$

$$Hu = \left( \frac{\partial^2}{\partial x^2} + 2|u|^{2\sigma} \right) u,$$

with $\sigma = 1.0$ and $\epsilon = 0.0$. 

We choose the initial condition
\[ u|_{t=0} = \frac{1}{\sqrt{2}} \sec\left( \frac{1}{\sqrt{2}} (x - 25) \right) \exp\left( -i \frac{x}{20} \right). \]  
(99)

For the reference solution, we apply a fine resolution Strang splitting. We employ the following transformation and change of variables:
\[ u = \eta + i \xi \]
\[ \begin{pmatrix} \dot{\eta} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} 0 & A(t, x, \eta, \xi) \\ -A(t, x, \eta, \xi) & 0 \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix} \]
(100)
\[ A(t, x, \eta, \xi) = A_1(t, x) + A_2(t, x, \eta, \xi) + A_3(t, x), \]
(101)
\[ A_1(t, x) = \frac{1}{\Delta x^2} \left[ 1 - 2 \right], \]
(102)
\[ A_2(t, x, \eta, \xi) = \left( \frac{2(\eta^2 + \xi^2)\sigma}{\sigma} \right) \left[ 0 \ 1 \ 0 \right], \]
(103)
\[ A_3(t, x) = \epsilon \Delta W \left[ 0 \ 1 \ 0 \right], \]
(104)
\[ \epsilon \Delta W \]
(105)

The underlying discretised matrices for the splitting schemes are
\[ \tilde{A}_1 = \begin{pmatrix} 0 & A_1(t, x) \\ -A_1(t, x) & 0 \end{pmatrix}, \]
(106)
\[ \tilde{A}_2(t, x, \eta(t^{n-1}, x), \xi(t^{n-1}, x)) = \begin{pmatrix} 0 & A_2(t, x, \eta(t^{n-1}, x), \xi(t^{n-1}, x)) \\ -A_2(t, x, \eta(t^{n-1}, x), \xi(t^{n-1}, x)) & 0 \end{pmatrix}, \]
(107)
\[ \tilde{A}_3 = \begin{pmatrix} 0 & A_3(t, x) \\ -A_3(t, x) & 0 \end{pmatrix}, \]
(108)
and
\[ \tilde{A}_4 = \tilde{A}_1 + \tilde{A}_2. \]
(109)

- A–B splitting

We apply the operator splitting schemes as:
\[ \begin{pmatrix} \eta^{n+1} \\ \xi^{n+1} \end{pmatrix} = \exp(\Delta t \tilde{A}_1) \exp(\Delta t \tilde{A}_2(\eta^n, \xi^n)) \exp\left( -\frac{1}{2} \Delta t \left( \tilde{A}_3^2 \tilde{A}_3 + \tilde{A}_3 \Delta W_i \right) \right) \begin{pmatrix} \eta^n \\ \xi^n \end{pmatrix}, 1 \leq n \leq N, \]
\[ \eta^0(x_i) = \frac{1}{\sqrt{2}} \sec\left( \frac{1}{\sqrt{2}} (x_i - 25) \right) \cos\left( -\frac{x_i}{20} \right), i = 1, \ldots, M, \]
(111)
\[ x^0_i = \frac{1}{\sqrt{2}} \sec\left( \frac{1}{\sqrt{2}} (x_i - 25) \right) \sin\left( -\frac{x_i}{20} \right), i = 1, \ldots, M, \]
(112)
where $\Delta t = t^{n+1} - t^n$, the random variable $W_t$ is based on a Wiener process with $\Delta W_t = W_{t^{n+1}} - W_{t^n} = \sqrt{\Delta t} X$, and $X$ is a Gaussian distributed random variable with $E(X) = 0$ and $Var(X) = 1$. This means we have $\Delta W_t = \text{rand} \sqrt{\Delta t}$.

- Iterative splitting scheme:
  First iterative step
  
  \[ X_{1,n}(t) = \left( \frac{\eta_n^{n+1}}{\xi_n^{n+1}} \right) = \exp(\Delta t \tilde{A}_4) \left( \frac{\eta_n^n}{\xi_n^n} \right), \]

Second iterative step

\[ X_{2,n}(t) = X_{1,n}(t) + X_{1,n}(t) \left[ \tilde{A}_3, \int_0^t \exp(\tilde{A}_4 s) dW_s \right], \quad t \in (t^n, t^{n+1}], \]
\[ X_{2,n}(t) = X_{1,n}(t) + X_{1,n}(t) \left[ \tilde{A}_3, C_1(t) \right], \quad t \in (t^n, t^{n+1}], \]

The stochastic integral is computed as a Stratonovich integral:

\[ C_1(t) = \int_0^t \exp(A s) dW_s = \sum_{j=0}^{N-1} \exp(A \frac{t_j + t_{j+1}}{2}) (W(t_{j+1}) - W(t_j)), \]
\[ \Delta t = t/N, t_j = \Delta t + t_{j-1}, t_0 = 0. \]

We apply $|u|$ for each solution and deal with the following errors:

\[ \| u_{\text{refer}}(x,t) - u_{i,j}(t) \|_{L_2(0,T)} = \Delta x \sum_{n=1}^{N} (u_{\text{refer}}(x, t) - u(x, t))^2, (116) \]
\[ E(\| u_{\text{refer}}(x,t) - u_{i,j}(x,t) \|) = \frac{1}{N} \sum_{n=1}^{N} |u_{\text{refer}}(x, t) - u(x, t)|. \]

The following figures present the results for the error of the iterative splitting schemes, see Fig. 6.

Remark 4.4. In both resolution in time and space the iterative splitting method is more accurate than the standard $A-B$ and Strang splitting schemes. Here, we obtain an improvement based on the successive approximation scheme.

5 Conclusion

We discuss the problems of using novel iterative splitting schemes to solve stochastic nonlinear Schrödinger equations. We could prove the almost asymptotic symplectic behaviour of the novel scheme. The improvement with more
iterative steps allows resolving the nonlinearity and obtaining an improved symplectic scheme. While standard splitting schemes have drawbacks as regards linearisation and symplecticity, we could derive a combination of both higher accuracy and conservation of the symplecticity. In the future, we will take into account larger equation systems for a realistic application.

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