A finite difference approximation of a two dimensional time fractional advection-dispersion problem

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Abstract

Time fractional advection-dispersion equations arise as generalizations of classical integer order advection-dispersion equations and are increasingly used to model fluid flow problems through porous media. In this paper we develop an implicit finite difference method to solve a two-dimensional initial boundary value problem for the linear time fractional advection-dispersion equation with variable coefficients on a bounded domain. Consistency, stability and convergence of the method are proved in detail and the numerical experiments offer a good insight into the quality of the obtained approximations.

keywords: Caputo fractional derivative, two dimensional time fractional advection-dispersion problem, finite difference approximation, stability, convergence.

1 Introduction

Fractional derivatives were discussed as early as 1695, according to well established references on the subject like [4, 6, 10, 11] and, in recent years, many mathematical models based on the tools from fractional calculus have arisen. Our main interest is the modeling of transport phenomena in porous media by time fractional advection-dispersion equations. Many authors have considered the topic in recent years and we mention only a few: In the monograph category, we mention the survey of fractional calculus methods in hydrology by Benson et al. [1] and the book by Guo et al. [5] which is a numerical analysis book dedicated to the subject. In the research papers category, the more important reference is [14], whose subject and methods inspired us. The subject is a convergent implicit finite difference approximation for the numerical solution of a linear 2-D time fractional diffusion equation with variable coefficients. Other papers of interest are:

1. Mirza and Vieru (2017) [9]: They introduce fundamental solutions for a linear 2-D time fractional advection-dispersion equation with constant coefficients.

2. Cui (2015) [3]: The paper introduces high-order compact exponential schemes for 1-D time fractional convection-diffusion reaction equations with variable coefficients and 2-D time fractional convection-diffusion equations with variable convection coefficients depending on \((x, t)\) and \((y, t)\) respectively and constant diffusion coefficients.
3. Mejía and Piedrahita (2017): A predecessor of this work, that deals with an inverse problem for a linear 1-D time fractional advection-dispersion equation.

In this paper our goal is the numerical solution of a linear two dimensional time fractional advection-dispersion equation with variable coefficients. We may consider this problem as a model for the environmental task of tracking a contaminant in groundwater. Following [14], we introduce an implicit finite difference scheme and prove consistency, unconditional stability and convergence.

The paper is organized as follows. Section 2 introduces the initial-boundary value problem to be considered and the finite difference method proposed for its approximation. Section 3 deals with the consistency, stability and convergence of the scheme and section 4 is dedicated to illustrative numerical examples.

2 The setting

The prediction of the environmental consequences of groundwater contamination is an important goal for researchers. Our interest is to help in this prediction through a numerical approximation of a mathematical model based on a partial differential equation known as an advection-dispersion equation. Following some research works like [7] and [3], our equation has variable coefficients and a time fractional derivative rather than the classical time derivative. Other features of our model are: It considers the contaminant transport through a two dimensional porous medium with variable advection and dispersion function coefficients given by two components each. Moreover, the diffusion terms are in nondivergence form. For this matter we follow references [14, 12, 13, 2]. All of them show that nondivergence diffusion terms are worth and with Caputo time-fractional derivatives provide useful models of anomalous diffusion.

2.1 The initial-boundary value problem

We consider the two-dimensional initial-boundary value problem

\[ u_t^{(\alpha)}(x,y,t) + a(x,y,t)u_x(x,y,t) + b(x,y,t)u_y(x,y,t) = c(x,y,t)u_{xx}(x,y,t) + d(x,y,t)u_{yy}(x,y,t) + f(x,y,t) \]  

with initial condition

\[ u(x,y,0) = \psi(x,y), \quad (x,y) \in \Omega := (x_L, x_R) \times (y_L, y_R) \subset \mathbb{R}^2, \] 

and Dirichlet boundary condition

\[ u(x,y,t) = 0, \quad (x,y) \in \partial \Omega \times (0,T], \] 

where:

1. \( u(x,y,t) \) is the contaminant concentration.
2. \( c \) is the longitudinal dispersion variable coefficient.
3. \( d \) is the transversal dispersion variable coefficient.
4. \( a \) and \( b \) are the longitudinal and transversal advection coefficients respectively. They are basically the seepage or average pore water velocity and if one of the directions is predominant, only one of the advection function coefficients is nonzero.
5. \( f \) is a known source or sink term.
6. $u_i^{(\alpha)}$ is the Caputo fractional derivative of order $\alpha$ given by

$$u_i^{(\alpha)}(x, y, t) := \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_i(x, y, \xi)}{(t-\xi)\alpha} d\xi, & 0 < \alpha < 1, \\ u_i(x, y, t), & \alpha = 1, \end{cases}$$

7. The variable coefficient functions $a, b, c, d$ satisfy the following uniform bounds: There are two positive constants $A$ and $D$ so that

$$0 \leq a(x, y, t) \leq A, \quad 0 < D \leq c(x, y, t), \quad 0 < D \leq d(x, y, t).$$

The next subsection deals with the proposed finite difference approximation.

2.2 The numerical scheme

Let the mesh points $x_i = x_L + i \Delta x, 0 \leq i \leq N_x, y_j = y_L + j \Delta y, 0 \leq j \leq N_y$ and $t_k = k \Delta t, 0 \leq k \leq N_t$, where $\Delta x = (x_R-x_L)/N_x$ and $\Delta y = (y_R-y_L)/N_y$ are the spatial grid sizes in the $x$- and $y$-direction, respectively, and $\Delta t = T/N_t$ is the time step size. The values of the functions $u, a, b, c, d$ and $f$ at the grid points are denoted by $u_{i,j}^k = u(x_i, y_j, t_k), a_{i,j}^k = a(x_i, y_j, t_k), b_{i,j}^k = b(x_i, y_j, t_k), c_{i,j}^k = c(x_i, y_j, t_k), d_{i,j}^k = d(x_i, y_j, t_k)$ and $f_{i,j}^k = f(x_i, y_j, t_k)$, respectively.

The initial condition is set as $u_{i,j}^0 = \psi(x_i, y_j)$. The Dirichlet boundary condition at $x = x_L$ is set as $u_{i,0}^k = 0$ and similarly on the other three sides of the boundary.

The Caputo fractional derivative at time $t_{k+1}$ is approximated by

$$u_i^{(\alpha)}(x_i, y_j, t_{k+1}) = \sigma_{\alpha, \Delta t} \sum_{s=0}^{N_t} \omega_s^{(\alpha)} (u_{i,j}^{k+1} - u_{i,j}^{k-s}) + O(\Delta t),$$

for $k = 0, \ldots, N_t - 1$, where $\sigma_{\alpha, \Delta t} = (\Delta t)^{-\alpha}/\Gamma(2-\alpha)$ and $\omega_s^{(\alpha)} = (s+1)^{1-\alpha} - s^{1-\alpha}$ for $s = 0, \ldots, N_t$, as described in [14].

Following [14], derivatives with respect to space coordinates are approximated by central difference formulae. Let $v_{i,j}^k$ be the numerical approximation to $u_{i,j}^k$. The discrete version of (1) is the implicit finite difference approximation (IFDA) given by

$$\sigma_{\alpha, \Delta t} \sum_{s=0}^{N_t} \omega_s^{(\alpha)} (v_{i,j}^{k+s+1} - v_{i,j}^{k-s}) + a_{i,j}^{k+1} \frac{v_{i+1,j}^{k+1} - v_{i-1,j}^{k+1}}{2\Delta x} + b_{i,j}^{k+1} \frac{v_{i+1,j+1}^{k+1} - v_{i+1,j-1}^{k+1}}{2\Delta y}$$

$$- c_{i,j}^{k+1} \frac{v_{i+1,j+1}^{k+1} - 2v_{i,j}^{k+1} + v_{i-1,j-1}^{k+1}}{(\Delta x)^2} + d_{i,j}^{k+1} \frac{v_{i,j+1}^{k+1} - v_{i,j-1}^{k+1}}{(\Delta y)^2} + f_{i,j}^{k+1},$$

for $i = 1, \ldots, N_x - 1, j = 1, \ldots, N_y - 1$ and $k = 0, \ldots, N_t - 1$.

2.3 Consistency

In order to prove consistency of scheme (IFDA), it is convenient to denote (1) by

$$S(u) = S(\partial_x, \partial_y, \partial_{xx}, \partial_{yy}) u = f(x, y, t),$$

where

$$S(u) = u_i^{(\alpha)}(x, y, t) + a(x, y, t)u_x(x, y, t) + b(x, y, t)u_y(x, y, t)$$

$$- c(x, y, t)u_{xx}(x, y, t) - d(x, y, t)u_{yy}(x, y, t)$$
Likewise, we establish the following alternative notation for scheme (IFDA)

\[ S_\Delta(v) = S_{\Delta t, \Delta x, \Delta y}(v_{i,j}^{k+1}), \]

where

\[
S_\Delta(v) = \sigma_{\alpha, \Delta t} \sum_{s=0}^{k} \omega_s^{(\alpha)} (v_{i,j}^{k-s+1} - v_{i,j}^{k-s}) + \alpha_{i,j} v_{i,j}^{k+1} \frac{v_{i+1,j}^{k+1} - v_{i-1,j}^{k+1}}{2\Delta x} + \beta_{i,j} \frac{v_{i,j+1}^{k+1} - v_{i,j-1}^{k+1}}{2\Delta y} - \frac{v_{i-1,j}^{k+1} - 2v_{i,j}^{k+1} + v_{i+1,j}^{k+1}}{(\Delta x)^2} - \frac{v_{i,j+1}^{k+1} - 2v_{i,j}^{k+1} + v_{i,j-1}^{k+1}}{(\Delta y)^2}.
\]

for \( i = 1, \ldots, N_x - 1, j = 1, \ldots, N_y - 1 \) and \( k = 0, \ldots, N_t - 1 \).

It is known that if \( u \) is a smooth function, then at interior points of its domain the following equalities hold:

\[
\begin{align*}
\sigma^{(\alpha)}(x, y, t_k) - \sigma_{\alpha, \Delta t} \sum_{s=0}^{k} \omega_s^{(\alpha)} (u_{i,j}^{k-s+1} - u_{i,j}^{k-s}) &= O(\Delta t), \\
u_x(x, y, t_k) - \frac{u_{i+1,j}^{k+1} - u_{i-1,j}^{k+1}}{2\Delta x} &= O((\Delta x)^2), \\
u_y(x, y, t_k) - \frac{u_{i,j+1}^{k+1} - u_{i,j-1}^{k+1}}{2\Delta y} &= O((\Delta y)^2), \\
u_{xx}(x, y, t_k) - \frac{u_{i+1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i-1,j}^{k+1}}{(\Delta x)^2} &= O((\Delta x)^2), \\
u_{yy}(x, y, t_k) - \frac{u_{i,j+1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-1}^{k+1}}{(\Delta y)^2} &= O((\Delta y)^2).
\end{align*}
\]

Let us denote the addition of all right hand sides in (8)-(12) by

\[ O(\Delta) = O(\Delta t, (\Delta x)^2, (\Delta y)^2). \]

Thus,

\[ S(u) - S_\Delta(u) = O(\Delta t, (\Delta x)^2, (\Delta y)^2) \]

and we have proved

**Lemma 1.** The finite difference scheme (IFDA) is consistent with the partial differential equation (1).}

Other way to write (14) is

\[ f_{i,j}^{k+1} - S_\Delta(u) = O(\Delta t, (\Delta x)^2, (\Delta y)^2). \]

By setting

\[ \mu_1 = \frac{(\Delta t)^a}{2\Delta x}, \quad \mu_2 = \frac{(\Delta t)^a}{2\Delta y}, \quad \mu_3 = \frac{(\Delta t)^a}{(\Delta x)^2}, \quad \mu_4 = \frac{(\Delta t)^a}{(\Delta y)^2}, \quad \tau = \frac{1}{\sigma_{\alpha, \Delta t}} = (\Delta t)^a \Gamma(2 - \alpha), \]

and

\[
\begin{align*}
p_{i,j}^k &= \Gamma(2 - \alpha) [\mu_3 c_{i,j}^k - \mu_1 a_{i,j}^k], \quad q_{i,j}^k = \Gamma(2 - \alpha) [\mu_3 c_{i,j}^k + \mu_1 a_{i,j}^k], \\
r_{i,j}^k &= \Gamma(2 - \alpha) [\mu_4 d_{i,j}^k - \mu_2 b_{i,j}^k], \quad h_{i,j}^k = \Gamma(2 - \alpha) [\mu_4 d_{i,j}^k + \mu_2 b_{i,j}^k], \\
c_{i,j}^k &= 1 + p_{i,j}^k + r_{i,j}^k + q_{i,j}^k, \quad d_{i,j}^k = 1 + 2\Gamma(2 - \alpha) [\mu_3 c_{i,j}^k + \mu_4 d_{i,j}^k],
\end{align*}
\]

we split scheme (IFDA) in two stages:
1. For \( k = 0 \), it is
\[
- (p_{i,j}^1 v_{i+1,j}^1 + q_{i,j}^1 v_{i-1,j}^1) + e_{i,j}^1 v_{i,j}^1 - (r_{i,j}^1 v_{i,j+1}^1 + h_{i,j}^1 v_{i,j-1}^1) = \psi_{i,j} + \tau f_{i,j}^1
\]
for \( i = 1, \ldots, N_x - 1 \) and \( j = 1, \ldots, N_y - 1 \)

2. For \( k = 1, \ldots, N_t - 1 \), the scheme is
\[
- (p_{i,j}^{k+1} v_{i+1,j}^{k+1} + q_{i,j}^{k+1} v_{i-1,j}^{k+1}) + e_{i,j}^{k+1} v_{i,j}^{k+1} - (r_{i,j}^{k+1} v_{i,j+1}^{k+1} + h_{i,j}^{k+1} v_{i,j-1}^{k+1}) = v_{i,j}^k - \sum_{s=1}^{k} \omega_s^{(a)} (v_{i,j}^{k-s+1} - v_{i,j}^{k-s}) + \tau f_{i,j}^{k+1}
\]

where \( i = 1, \ldots, N_x - 1 \) and \( j = 1, \ldots, N_y - 1 \).

The next lemma provides the main features of the quadrature weights \( \omega_s^{(a)} \).

**Lemma 2.** The quadrature weights \( \omega_s^{(a)} \) are positive and \( \omega_s^{(a)} > \omega_{s+1}^{(a)} \) for all \( s = 0, 1, \ldots \)

In the following lemma the assumption on the size of the discretization parameters \( \Delta x \) and \( \Delta y \) is based on the uniform bounds for the coefficient functions and is independent of the time stepsize.

**Lemma 3** (Grid size assumptions). *If the variable coefficients \( a, b, c \) and \( d \) satisfy condition \( [2] \) of subsection \([2,7]\) and \( \max \{\Delta x, \Delta y\} \leq 2D/A \), then \( p_{i,j}^k \geq 0 \) and \( r_{i,j}^k \geq 0 \) for each \( i = 1, \ldots, N_x - 1 \), \( j = 1, \ldots, N_y - 1 \) and \( k = 1, \ldots, N_t - 1 \).*

**Proof.** The proof consists on the following straightforward computations:
\[
p_{i,j}^k = \Gamma (2 - \alpha) (\Delta t)^\alpha (\Delta x)^2 \left[ c_{i,j}^k - \frac{\Delta x}{2} a_{i,j}^k \right] \geq - \frac{\tau}{(\Delta x)^2} \left[ D - \frac{\Delta x}{2} A \right] \geq 0
\]

and
\[
r_{i,j}^k = \frac{\tau}{(\Delta y)^2} \left[ d_{i,j}^k - \frac{\Delta y}{2} b_{i,j}^k \right] \geq \frac{\tau}{(\Delta y)^2} \left[ D - \frac{\Delta y}{2} A \right] \geq 0.
\]

A closing remark is useful at this point: Assuming the hypothesis of the grid size assumption lemma \([3]\) the variable coefficients \( p, q, r \) and \( h \) of scheme \([18,19]\) are nonnegative functions. This will help in proving the main theoretical results in the next section.

**2.4 The linear system**

Let
\[
v^k = [v_{s,1}^k \ v_{s,2}^k \ \cdots \ v_{s,N_y-1}^k]^T
\]
\( \text{for each } 0 \leq k < N_t, \) where \( A^{(k)} \) is the \((N - 1)^2 \times (N - 1)^2\) matrix of coefficients resulting from the system of difference equations at the gridpoints at level \( t = t_k \),
\[
v^k = [v_{s,1}^k \ v_{s,2}^k \ \cdots \ v_{s,N_y-1}^k]^T
\]
with $v_{*,j}^k = [v_{1,j}^k \ v_{2,j}^k \ \cdots \ \cdot \ \cdot \ \cdot \ v_{N-1,j}^k]^T$, and $y^k = [y_{*,1}^k \ \ y_{*,2}^k \ \ \cdots \ \ y_{*,N-1}^k]^T$ with

$$y_{*,j}^k = \begin{cases} 
\psi_{*,j} + \tau f_{*,j}^1, & k = 0, \\
\gamma \psi_{*,j} + \tau f_{*,j}^2, & k = 1, \\
\gamma \psi_{*,j} + \sum_{s=1}^{k-1} (\omega_s^*(\alpha) - \omega_{s+1}^*(\alpha)) v_{*,j}^{k-s} + \omega_k^*(\alpha) \psi_{*,j} + \tau f_{*,j}^{k+1}, & 1 < k < N_t,
\end{cases}$$

where $\psi_{*,j} = [\psi_{1,j} \ \psi_{2,j} \ \cdots \ \psi_{N-1,j}]^T$, $f_{*,j}^k = [f_{1,j}^k \ f_{2,j}^k \ \cdots \ f_{N-1,j}^k]^T$ and $\gamma = (2 - 2^{1-\alpha})$.

Eq. (21) requires, at each time step, to solve a linear system where the right-hand side $y^k$ utilizes all the history of the computed solution up to that time, and $A^{(k)}$ is a band matrix with a block structure. Each block is a $(N-1) \times (N-1)$ matrix and together they give $A^{(k)}$ the following form

$$A^{(k)} = \begin{bmatrix}
T_1^k & D_1^k & 0 & \cdots & 0 \\
\tilde{D}_1^k & T_2^k & D_2^k & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \tilde{D}_{N-2}^k & T_{N-1}^k \\
\end{bmatrix}.$$

In this expression each $T_\ell^k$ is a tridiagonal matrix given by

$$T_\ell^{(k)} = \begin{bmatrix}
e_{1,\ell}^k & -p_{1,\ell}^k & 0 & \cdots & 0 \\
-q_{2,\ell}^k & e_{2,\ell}^k & -p_{2,\ell}^k & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -q_{N-2,\ell}^k & e_{N-2,\ell}^k & -p_{N-2,\ell}^k \\
0 & \cdots & 0 & -q_{N-1,\ell}^k & e_{N-1,\ell}^k & \end{bmatrix},$$

while $D_\ell^k = [d_{1,\ell}^k]$ and $\tilde{D}_\ell^k = [\tilde{d}_{1,\ell}^k]$ are diagonal matrices defined by $d_{i,\ell}^k = -r_{i,\ell}^k$ and $\tilde{d}_{i,\ell}^k = -\tilde{h}_{i,\ell}^k$, for $i = 1, \ldots, N-1$.

**Remark 1.** Note that for each $1 \leq \ell \leq (N-1)^2$, there exists exactly one $1 \leq \ell_i \leq N - 1$ such that the resulting diagonal entry $A_{i,i}^{(k)}$ of (22) is determined by

$$A_{i,i}^{(k)} := e_{i,\ell_i}^k = 1 + p_{i,\ell_i}^k + q_{i,\ell_i}^k + \tilde{r}_{i,\ell_i}^k + \tilde{h}_{i,\ell_i}^k.$$  

The off-diagonal entries $A_{i,j}^{(k)}$ with $i \neq j$, can be determined in the same way.

## 3 The approximation

In this section we prove the unconditional stability and the convergence of scheme (18)-(19). Both results are inspired by [14]. Let $v^k$ be given by (20) for $k = 0, \ldots, N_t$. 


3.1 Stability

Theorem 4. If the hypotheses of lemma 3 hold, scheme (18)-(19) for the homogeneous \( f \equiv 0 \) initial-boundary value problem (1)-(2)-(3) is unconditionally stable.

Proof.

1. Scheme (18): Let \( \| v^1 \|_\infty = |v^1_{l,m}| = \max_{i,j} |v^1_{i,j}| \). We show that \( \| v^1 \|_\infty \leq \| v^0 \|_\infty \)

\[
|v^1_{l,m}| = (1 + p^1_{l,m} + q^1_{l,m} + r^1_{l,m} + h^1_{l,m}) |v^1_{l,m}|
- (p^1_{l,m} |v^1_{l,m}| + q^1_{l,m} |v^1_{l,m}|) - (r^1_{l,m} |v^1_{l,m}| + h^1_{l,m} |v^1_{l,m}|)
\leq (1 + p^1_{l,m} + q^1_{l,m} + r^1_{l,m} + h^1_{l,m}) |v^1_{l,m}|
- (p^1_{l,m} |v^1_{l,m}| + q^1_{l,m} |v^1_{l,m}| - (r^1_{l,m} |v^1_{l,m+1}| + h^1_{l,m} |v^1_{l,m-1}|))
\leq |(1 + p^1_{l,m} + q^1_{l,m} + r^1_{l,m} + h^1_{l,m}) |v^1_{l,m}|
- (p^1_{l,m} |v^1_{l,m+1}| + q^1_{l,m} |v^1_{l,m-1}| - (r^1_{l,m} |v^1_{l,m+1}| + h^1_{l,m} |v^1_{l,m-1}|))
\leq |v^1_{l,m}| = |v^0_{l,m}| = \| v^0 \|_\infty.
\]

2. Scheme (19): The proof is by induction over \( k \). Suppose \( \| v^n \|_\infty \leq \| v^0 \|_\infty \) for \( n = 1, \ldots, k \).
We prove the inequality \( \| v^{k+1} \|_\infty \leq \| v^0 \|_\infty \). From now on, we will write \( \omega_s \) rather than \( \omega_s^{(n)} \).
Notice that the right hand side in scheme (19) is

\[
v^k_{i,j} - \sum_{s=1}^{k} \omega_s (v^{k-s}_{i,j+1} - v^{k-s}_{i,j}) = \omega_0 v^k_{i,j} - \omega_1 v^{k-1}_{i,j} + \omega_2 v^{k-2}_{i,j} - \cdots - \omega_k v^0_{i,j}
= (\omega_0 - \omega_1) v^k_{i,j} + (\omega_1 - \omega_2) v^{k-1}_{i,j} + (\omega_2 - \omega_3) v^{k-2}_{i,j} + \cdots + (\omega_{k-1} - \omega_k) v^0_{i,j}.
\]

Now we look at the left hand side. Let \( \| v^{k+1} \|_\infty = \| v^{k+1}_{l,m} \| = \max_{i,j} |v^{k+1}_{i,j}| \).

\[
|v^{k+1}_{l,m}| = (1 + p^{k+1}_{l,m} + q^{k+1}_{l,m} + r^{k+1}_{l,m} + h^{k+1}_{l,m}) |v^{k+1}_{l,m}|
- (p^{k+1}_{l,m} |v^{k+1}_{l,m}| + q^{k+1}_{l,m} |v^{k+1}_{l,m}|) - (r^{k+1}_{l,m} |v^{k+1}_{l,m}| + h^{k+1}_{l,m} |v^{k+1}_{l,m}|)
\leq (1 + p^{k+1}_{l,m} + q^{k+1}_{l,m} + r^{k+1}_{l,m} + h^{k+1}_{l,m}) |v^{k+1}_{l,m}|
- (p^{k+1}_{l,m} |v^{k+1}_{l,m}| + q^{k+1}_{l,m} |v^{k+1}_{l,m}| - (r^{k+1}_{l,m} |v^{k+1}_{l,m+1}| + h^{k+1}_{l,m} |v^{k+1}_{l,m-1}|))
\leq |(1 + p^{k+1}_{l,m} + q^{k+1}_{l,m} + r^{k+1}_{l,m} + h^{k+1}_{l,m}) |v^{k+1}_{l,m}|
- (p^{k+1}_{l,m} |v^{k+1}_{l,m+1}| + q^{k+1}_{l,m} |v^{k+1}_{l,m-1}| - (r^{k+1}_{l,m} |v^{k+1}_{l,m+1}| + h^{k+1}_{l,m} |v^{k+1}_{l,m-1}|))
\leq (\omega_0 - \omega_1) v^k_{l,m} + \cdots + (\omega_{k-1} - \omega_k) v^0_{l,m} + \omega_k v^0_{l,m} = \| v^0 \|_\infty.
\]

This theorem allows us to prove an additional stability bound. Let \( v^0_{l,m} \) and \( \tilde{v}^0_{l,m} \) be the initial discrete values corresponding to two initial conditions \( \psi_{ij} \) and \( \tilde{\psi}_{ij} \). We may think of two different
measurements of the initial concentration. Furthermore, let \( \tilde{v}_{ij}^k \) and \( \tilde{v}_{ij}^k \) be the corresponding discrete approximations obtained by the numerical schemes (18) and (19). Let \( \varepsilon_{ij}^k = v_{ij}^k - \tilde{v}_{ij}^k \) and

\[
E^k = \left[ \varepsilon_{*,1}^k \ v_{*,2}^k \ \cdots \ \varepsilon_{*,N_y-1}^k \right]^T
\]

where \( \varepsilon_{*,j}^k = [\varepsilon_{1,j}^k \ \varepsilon_{2,j}^k \ \cdots \ \varepsilon_{N_x-1,j}^k]^T \), \( j = 1, \ldots, N_y - 1 \).

**Corollary 1.** If the hypotheses of lemma 3 are satisfied, the numerical errors induced by initial-value conditions in scheme (18), (19) for the inhomogeneous initial-boundary value problem do not propagate. More precisely, they satisfy the bound

\[
\|E^k\|_\infty \leq \|E^0\|_\infty , \quad k = 1, 2, \ldots
\]

### 3.2 Convergence

Stability and convergence proofs follow similar patterns. Let \( \varepsilon_{ij}^k = u_{ij}^k - \tilde{v}_{ij}^k \) and

\[
E^k = \left[ \varepsilon_{*,1}^k \ v_{*,2}^k \ \cdots \ \varepsilon_{*,N_y-1}^k \right]^T
\]

where \( \varepsilon_{*,j}^k = [\varepsilon_{1,j}^k \ \varepsilon_{2,j}^k \ \cdots \ \varepsilon_{N_x-1,j}^k]^T \), \( j = 1, \ldots, N_y - 1 \). The convergence of the scheme is given by the following theorem.

**Theorem 5.** If the hypotheses of lemma 3 hold, then

\[
\|E^k\|_\infty \leq \|E^0\|_\infty + (\Delta t)^\alpha O(\Delta), \quad k = 1, 2, \ldots
\]

where \( O(\Delta) \) is defined by (13).

**Proof.** The proof is by induction on \( k \).

Case \( k = 1 \). We show that

\[
\|E^1\|_\infty \leq \|E^0\|_\infty + (\Delta t)^\alpha O(\Delta).
\]

Let \( \|E^1\|_\infty = \|\varepsilon_{1,m}\| = \max_{i,j} \|\varepsilon_{i,j}\| \). In this case the scheme under consideration is (18).
Now suppose
\[ \| E^s \|_\infty \leq \| E^0 \|_\infty + (\Delta t)^\alpha O(\Delta) \]
holds for \( s = 1, 2, \ldots, k \). We prove the result for \( s = k + 1 \). Let \( \| E^{k+1} \|_\infty = |\varepsilon_{lm}^{k+1}| = \max_{i,j} |\varepsilon_{ij}^{k+1}| \). By the same argument as before,
\[
|\varepsilon_{lm}^{k+1}| \leq \left| \left( 1 + p_{l,m}^{k+1} + q_{l,m}^{k+1} + r_{l,m}^{k+1} + s_{l,m}^{k+1} \right) u_{l,m}^{k+1} - \left( p_{l,m}^{k+1} u_{l+1,m}^{k+1} + q_{l,m}^{k+1} u_{l-1,m}^{k+1} \right) \right|
\]
\[
- \left( r_{l,m}^{k+1} u_{l+1,m}^{k+1} + s_{l,m}^{k+1} u_{l-1,m}^{k+1} \right) \right|
\]
\[
- \left( 1 + p_{l,m}^{k+1} + q_{l,m}^{k+1} + r_{l,m}^{k+1} + s_{l,m}^{k+1} \right) v_{l,m}^{k+1} + \left( p_{l,m}^{k+1} v_{l+1,m}^{k+1} + q_{l,m}^{k+1} v_{l-1,m}^{k+1} \right) \right|
\]
\[
+ \left( r_{l,m}^{k+1} v_{l+1,m}^{k+1} + s_{l,m}^{k+1} v_{l-1,m}^{k+1} \right) \right| .
\]

In this case the scheme is (19) and as before, we write \( \omega_s \) instead of \( \omega_s^{(\alpha)} \). The last expression becomes
\[
\left| u_{l,m}^{k+1} + \tau \left( S\Delta \left( u_{l,m}^{k+1} \right) - \sigma_{l,m} \sum_{s=0}^{k} \omega_s \left( u_{l,m}^{k-s+1} - u_{l,m}^{k-s} \right) \right) \right|
\]
\[
- \left| v_{l,m}^{k} + \sum_{s=1}^{k} \omega_s \left( v_{l,m}^{k-s+1} - v_{l,m}^{k-s} \right) - \tau f_{l,m}^{k+1} \right|
\]
\[
= \left| u_{l,m}^{k+1} + \tau S \left( u_{l,m}^{k+1} \right) + \tau O(\Delta) - \sum_{s=0}^{k} \omega_s \left( u_{l,m}^{k-s+1} - u_{l,m}^{k-s} \right) \right|
\]
\[
- \left| v_{l,m}^{k} + \sum_{s=1}^{k} \omega_s \left( v_{l,m}^{k-s+1} - v_{l,m}^{k-s} \right) - \tau f_{l,m}^{k+1} \right|
\]
\[
= \left| \sum_{s=1}^{k} \omega_s \left( v_{l,m}^{k-s+1} - v_{l,m}^{k-s} \right) \right|
\]
\[
\left| \sum_{s=1}^{k} \omega_s \left( v_{l,m}^{k-s+1} - v_{l,m}^{k-s} \right) \right|
\]
\[
\leq \sum_{s=0}^{k-1} \left( \omega_s - \omega_{s+1} \right) \left| \varepsilon_{l,m}^{k-s} \right| + \omega_k \varepsilon_{l,m}^{0} + \tau O(\Delta)
\]
\[
\leq \sum_{s=0}^{k-1} \left( \omega_s - \omega_{s+1} \right) \left| \varepsilon_{l,m}^{k-s} \right| + \omega_k \left| E^0 \right| + \tau O(\Delta)
\]
\[
= \| E^0 \|_\infty + (\Delta t)^\alpha O(\Delta).
\]
### Table 1: Absolute errors and order of convergence at $t = 1$ for Example 4.1.

| $\Delta t$ | $\Delta x = \Delta y$ | Max. error | Order |
|------------|------------------------|------------|-------|
| $\alpha = 0.1$ |                        |            |       |
| $1/16$     | $1/4$                  | 1.440e-01  | -     |
| $1/32$     | $1/8$                  | 4.070e-02  | 1.823 |
| $1/64$     | $1/16$                 | 1.043e-02  | 1.964 |
| $1/128$    | $1/32$                 | 2.607e-03  | 2.001 |
| $1/256$    | $1/64$                 | 6.530e-04  | 1.997 |
| $\alpha = 0.5$ |                        |            |       |
| $1/16$     | $1/4$                  | 1.415e-01  | -     |
| $1/32$     | $1/8$                  | 4.055e-02  | 1.803 |
| $1/64$     | $1/16$                 | 1.045e-02  | 1.957 |
| $1/128$    | $1/32$                 | 2.625e-03  | 1.993 |
| $1/256$    | $1/64$                 | 6.627e-04  | 1.986 |
| $\alpha = 0.9$ |                        |            |       |
| $1/16$     | $1/4$                  | 1.588e-01  | -     |
| $1/32$     | $1/8$                  | 4.434e-02  | 1.841 |
| $1/64$     | $1/16$                 | 1.189e-02  | 1.899 |
| $1/128$    | $1/32$                 | 3.365e-03  | 1.821 |
| $1/256$    | $1/64$                 | 1.053e-03  | 1.676 |

### 4 Numerical Examples

In order to demonstrate the reliability of our numerical method, three examples are presented. The absolute errors in the approximation $v$ of $u$ at time $t = t_k$ are measured by the maximum norm

$$
\|v^k - u(t_k)\|_\infty := \max_{i,j} |v^k_{i,j} - u^k_{i,j}|
$$

**Example 4.1.** We consider the time fractional advection-dispersion equation

$$
u^{(\alpha)}_k(x, y, t) + a(x, y, t)u_x(x, y, t) + b(x, y, t)u_y(x, y, t)
= c(x, y, t)u_{xx}(x, y, t) + d(x, y, t)u_{yy}(x, y, t) + f(x, y, t)
$$
on a finite square domain $\Omega = (0, 1) \times (0, 1)$ for $0 \leq t \leq 1$, with the initial condition

$$u(x, y, 0) = \sin \pi x \sin \pi y, \quad (x, y) \in \Omega,$$

and the boundary condition

$$u(x, y, t) = 0, \quad (x, y) \in \partial \Omega \times (0, 1].$$

The advection and dispersion coefficients are given by

$$a(x, y, t) = \frac{1}{\sin \pi y}, \quad b(x, y, t) = \frac{1}{\sin \pi x},$$
Table 2: Absolute errors and order of convergence at \( t = 1 \) for Example 4.2.

| \( \Delta t \) | \( \Delta x = \Delta y \) | Absolute error | Order of convergence |
|---------------|-----------------|----------------|---------------------|
| \( \alpha = 0.1 \) | \( \varepsilon = 1e-1 \) | \( \varepsilon = 1e-3 \) | \( \varepsilon = 1e-5 \) | \( \varepsilon = 1e-1 \) | \( \varepsilon = 1e-3 \) | \( \varepsilon = 1e-5 \) |
| 1/16 | 1/4 | 1.103e-01 | 1.431e-01 | 1.451e-01 | - - - | - - - | - - - | - - - |
| 1/32 | 1/8 | 2.982e-02 | 5.154e-02 | 5.622e-02 | 1.887 | 1.473 | 1.368 |
| 1/64 | 1/16 | 7.703e-03 | 1.448e-02 | 1.711e-02 | 1.953 | 1.831 | 1.716 |
| 1/128 | 1/32 | 1.927e-03 | 3.386e-03 | 4.484e-03 | 1.999 | 2.097 | 1.932 |
| \( \alpha = 0.5 \) | \( \varepsilon = 1e-1 \) | \( \varepsilon = 1e-3 \) | \( \varepsilon = 1e-5 \) | \( \varepsilon = 1e-1 \) | \( \varepsilon = 1e-3 \) | \( \varepsilon = 1e-5 \) |
| 1/16 | 1/4 | 1.108e-01 | 1.445e-01 | 1.468e-01 | - - - | - - - | - - - | - - - |
| 1/32 | 1/8 | 3.001e-02 | 5.079e-02 | 5.451e-02 | 1.884 | 1.508 | 1.430 |
| 1/64 | 1/16 | 7.817e-03 | 1.406e-02 | 1.617e-02 | 1.941 | 1.853 | 1.753 |
| 1/128 | 1/32 | 1.974e-03 | 3.169e-03 | 4.090e-03 | 1.986 | 2.150 | 1.983 |
| \( \alpha = 0.9 \) | \( \varepsilon = 1e-1 \) | \( \varepsilon = 1e-3 \) | \( \varepsilon = 1e-5 \) | \( \varepsilon = 1e-1 \) | \( \varepsilon = 1e-3 \) | \( \varepsilon = 1e-5 \) |
| 1/16 | 1/4 | 1.228e-01 | 1.451e-01 | 1.468e-01 | - - - | - - - | - - - |
| 1/32 | 1/8 | 3.377e-02 | 4.401e-02 | 4.523e-02 | 1.862 | 1.721 | 1.701 |
| 1/64 | 1/16 | 9.633e-03 | 1.219e-02 | 1.321e-02 | 1.810 | 1.852 | 1.776 |
| 1/128 | 1/32 | 2.899e-03 | 3.743e-03 | 3.981e-03 | 1.733 | 1.703 | 1.730 |

Table 2: Absolute errors and order of convergence at \( t = 1 \) for Example 4.2.

Numerical experiments for fractional derivatives of orders \( \alpha = 0.1, \alpha = 0.5 \) and \( \alpha = 0.9 \) are listed in Table 1. The columns for absolute errors and order of convergence are the main features of this table. This is a somewhat extreme example due to the fact that the advection coefficients \( a \) and \( b \) do not satisfy the bounds condition (5).

**Example 4.2.** As a second example we consider the time fractional advection-dispersion equation

\[
\begin{align*}
u_t^{(\alpha)}(x, y, t) + \frac{1}{1 + x} u_x(x, y, t) + \frac{1}{1 + y} u_y(x, y, t) &= \varepsilon \Delta u(x, y, t) + f(x, y, t), \quad \varepsilon > 0,
\end{align*}
\]

on \( \Omega = (0, 1) \times (0, 1) \) for \( 0 \leq t \leq 1 \), with initial condition

\[
u(x, y, 0) = \sin \pi x \sin \pi y, \quad (x, y) \in \Omega,
\]

boundary condition

\[
u(x, y, t) = 0, \quad (x, y) \in \partial \Omega \times (0, 1],
\]
and source or sink term
\[ f(x, y, t) = \left( \frac{2t^{2-\alpha}}{\Gamma(3 - \alpha)} + 2\varepsilon\pi^2(t^2 + 1) \right) \sin \pi x \sin \pi y + \pi(t^2 + 1) \left( \frac{\cos \pi x \sin \pi y}{x + 1} + \frac{\cos \pi \sin \pi y}{y + 1} \right). \]

The exact solution is
\[ u(x, y, t) = (t^2 + 1) \sin \pi x \sin \pi y. \]

This is a test for the behavior of the method in the presence of very small diffusion coefficients, an almost degenerate parabolic equation. Numerical results are provided in Table 2. As before, three different fractional derivative orders are taken into account and there are results for three different diffusion coefficients: \( \varepsilon = 10^{-1}, \varepsilon = 10^{-3} \) and \( \varepsilon = 10^{-5} \).

**Example 4.3.** Finally we solve the time fractional diffusion equation
\[ u_t^{(\alpha)}(x, y, t) + u_x(x, y, t) + u_y(x, y, t) = \Delta u(x, y, t), \]
on the finite square domain \( \Omega = (0, 1) \times (0, 1) \) for \( 0 \leq t \leq 1 \), with the initial condition
\[ u(x, y, 0) = \sin \pi x \sin \pi y, \quad \text{for} \ (x, y) \in \Omega, \]
and the boundary condition
\[ u(x, y, t) = 0, \quad (x, y) \in \partial \Omega \times (0, 1]. \]
Figure 1 illustrates the computed solutions for $t = 1$ for several values of $\alpha$. No exact solutions are known for this problem but the pictures illustrate the continuous dependence of the solutions on the fractional derivative order.

5 Final remarks

In this paper we introduce a new implicit finite difference approximation for the solution of an initial boundary value problem for a two dimensional time fractional advection-dispersion equation with variable coefficients. Proofs of consistency, stability and convergence are included and so are illustrative numerical experiments. Extensions of this method to different equations or to support the solution of inverse problems are the subject of current research.

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