Automorphism groups of finite-dimensional special odd Hamiltonian superalgebras in prime characteristic

Liming TANG¹,  Wende LIU¹,²

¹ Department of Mathematics, Harbin Institute of Technology, Harbin 150006, China
² School of Mathematical Sciences, Harbin Normal University, Harbin 150025, China

Abstract This paper is devoted to a study of the automorphism groups of three series of finite-dimensional special odd Hamiltonian superalgebras \( g \) over a field of prime characteristic. Our aim is to characterize the connections between the automorphism groups of \( g \) and the automorphism groups of the corresponding underlying superalgebras. Precisely speaking, we embed the former into the later. Moreover, we determine the images of the normal series of the automorphism groups and homogeneous automorphism groups of \( g \) under the embedded mapping.

Keywords Special odd Hamiltonian superalgebras, automorphism group

MSC 17B50, 17B40

1 Introduction

As is well known, the classification problem is still open for the finite-dimensional simple modular Lie superalgebras (see [2,9]) and the simple graded Lie superalgebras of Cartan type are quite useful in the classification problem (see, e.g., [3,5–7]). The automorphism groups of Lie superalgebras of Cartan type play a vital role in the future studies on structures and representations of Lie superalgebras of Cartan type (see, e.g., [6,7]). In [7], the automorphism groups were determined for finite-dimensional restricted modular Lie superalgebras \( W \), \( S \), \( H \), and \( K \). In [6], the automorphism groups were determined for the finite-dimensional restricted modular Lie superalgebras \( HO \).

The purpose of this paper is to discuss the automorphism groups of restricted special odd Hamiltonian superalgebras in prime characteristic. Our
methods are modeled on those used in [7] and meanwhile somewhat different from those papers mentioned above: we deal with three series of finite-
dimensional special odd Hamiltonian superalgebras containing non-simple ones. 
In addition, we also mention that some conclusions in this paper are general, 
which provide much information for investigating other problems of Lie super-

2 Basics 

In general, we adopt the conventions of [7]. In the following, $\mathbb{F}$ is a field of 
characteristic $p > 3$. Fix two $m$-tuples of positive integers

$$\ell := (t_1, t_2, \ldots, t_m), \quad \pi := (\pi_1, \pi_2, \ldots, \pi_m),$$

where $\pi_i := p^{\ell_i} - 1$. Let $\mathcal{O}(m; \ell)$ be the divided power algebra over $\mathbb{F}$ with a 
basis $\{x^{(\alpha)} \mid \alpha \in \mathbb{A}\}$, where $\mathbb{A} := \{\alpha \in \mathbb{N}^m \mid \alpha_i \leq \pi_i\}$. Let $\Lambda(m)$ be the exterior 
superalgebra over $\mathbb{F}$ of $m$ variables $x_{m+1}, x_{m+2}, \ldots, x_{2m}$. Let

$$\mathbb{B} = \{(i_1, i_2, \ldots, i_k) \mid m + 1 \leq i_1 < i_2 < \cdots < i_k \leq 2m, 1 \leq k \leq m\}.$$ 

For $u := (i_1, i_2, \ldots, i_k) \in \mathbb{B}$, set $|u| = k$ and write $x^u = x_{i_1} x_{i_2} \cdots x_{i_k}$. Write

$$\omega = (m + 1, \ldots, 2m).$$ 

The tensor product

$$\mathcal{O}(m, m; \ell) := \mathcal{O}(m; \ell) \otimes \Lambda(m)$$

is an associative superalgebra. Note that $\mathcal{O}(m, m; \ell)$ has a standard $\mathbb{F}$-basis

$$\{x^{(\alpha)} x^u \mid (\alpha, u) \in \mathbb{A} \times \mathbb{B}\}.$$ 

For convenience, put

$$I_0 = \{1, \ldots, m\}, \quad I_1 = \{m + 1, \ldots, 2m\}, \quad I = I_0 \cup I_1.$$ 

Let $D_r$ be the superderivation of $\mathcal{O}(m, m; \ell)$ such that

$$D_r(x^{(\alpha)}) = x^{(\alpha - \varepsilon_r)}, \quad r \in I_0; \quad D_r(x_s) = \delta_{rs}, \quad r, s \in I.$$ 

The generalized Witt superalgebra $W(m, m; \ell)$ is a free $\mathcal{O}(m, m; \ell)$-module with 
a basis $\{D_r \mid r \in I\}$. In particular, $W(m, m; \ell)$ has a standard $\mathbb{F}$-basis

$$\{x^{(\alpha)} x^u D_r \mid (\alpha, u, r) \in \mathbb{A} \times \mathbb{B} \times I\}.$$ 

Note that $\mathcal{O}(m, m; \ell)$ possesses a standard $\mathbb{Z}$-grading structure

$$\mathcal{O}(m, m; \ell) = \bigoplus_{r=0}^{\xi} \mathcal{O}(m, m; \ell)_{[r]}$$
by letting

$$\mathcal{O}(m, m; t)_r = \text{span}_F \{ x^{(\alpha)} x^u \mid |\alpha| + |u| = r \}, \quad \xi = |\pi| + m = \sum_{i \in I_0} p^i.$$  

This induces naturally a $\mathbb{Z}$-grading structure

$$W(m, m; t) = \bigoplus_{i=-1}^{\xi-1} W(m, m; t)_{[i]}$$

by letting

$$W(m, m; t)_{[i]} = \text{span}_F \{ f D_r \mid f \in \mathcal{O}(m, m; t)_{[i+1]}, \ r \in I \}. $$

For a vector superspace $V = V_0 \oplus V_1$, we write $p(x) = \theta$ for the parity of a homogeneous element $x \in V_\theta$, $\theta \in \mathbb{Z}_2$. The symbol $p(x)$ implies that $x$ is $\mathbb{Z}_2$-homogeneous. Obviously,

$$p(D_j) = \mu(j) := \begin{cases} 0, & j \in I_0, \\ 1, & j \in I_1. \end{cases}$$

Let

$$i' = \begin{cases} i + m, & i \in I_0, \\ i - m, & i \in I_1. \end{cases}$$

For $a \in \mathcal{O}(m, m; t)$, write

$$T_H(a) = \sum_{i \in I} (-1)^{p(D_i)p(a)} D_i(a) D_i'.$$ 

Then $T_H$ is a linear operator from $\mathcal{O}(m, m; t)$ into $W(m, m; t)$. We have

$$[T_H(a), T_H(b)] = T_H(T_H(a)(b)), \quad \forall a, b \in \mathcal{O}(m, m; t).$$

Then

$$HO(m, m; t) := \{ T_H(a) \mid a \in \mathcal{O} \}$$

is a finite-dimensional simple Lie superalgebra, which is called the odd Hamiltonian superalgebra (see [6]). Let

$$\text{div} : W(m, m; t) \to \mathcal{O}(m, m; t)$$

be the divergence, which is a linear mapping, such that

$$\text{div}(f D_r) = (-1)^{p(D_r)p(f)} D_r(f), \quad f \in \mathcal{O}(m, m; t), \ r \in I.$$ 

Note that $\text{div}$ is an even superderivation of $W(m, m; t)$ into $W(m, m; t)$-module $\mathcal{O}(m, m; t)$. Put

$$S'(m, m; t) = \{ D \in W(m, m; t) \mid \text{div}(D) = 0 \}.$$
Then $S'(m, m; \mathfrak{l})$ is a $\mathbb{Z}$-graded subalgebra of $W(m, m; \mathfrak{l})$. Let

$$SHO'(m, m; \mathfrak{l}) = S'(m, m; \mathfrak{l}) \cap HO(m, m; \mathfrak{l}),$$

$$\overline{SHO}(m, m; \mathfrak{l}) = [SHO'(m, m; \mathfrak{l}), SHO'(m, m; \mathfrak{l})],$$

$$SHO(m, m; \mathfrak{l}) = [\overline{SHO}(m, m; \mathfrak{l}), \overline{SHO}(m, m; \mathfrak{l})].$$

We call these algebras the special odd Hamiltonian superalgebras (see [5]). Among these Lie superalgebras, only $SHO(m, m; \mathfrak{l})$ is simple. In the below, we usually omit the parameters $(m, m; \mathfrak{l})$ and write $\mathfrak{g}$ for $SHO'$, $\overline{SHO}$, or $SHO$.

3 Automorphism groups

Let $\mathcal{A}$ be a finite-dimensional superalgebra and $\mathcal{Q}$ a sub Lie superalgebra of the full superderivation superalgebra $\text{Der}\,\mathcal{A}$. As in [8] (see also [7]), put

$$\text{Aut}(\mathcal{A} : \mathcal{Q}) = \{\sigma \in \text{Aut}\,\mathcal{A} | \overline{\sigma}(\mathcal{Q}) \subset \mathcal{Q}\},$$

where

$$\overline{\sigma}(D) := \sigma D \sigma^{-1}, \quad D \in \mathcal{Q}. $$

Then $\text{Aut}(\mathcal{A} : \mathcal{Q})$ is a subgroup of $\text{Aut}\,\mathcal{A}$, which is referred to as the admissible automorphism group of $\mathcal{A}$ (to $\mathcal{Q}$). Suppose that in addition $\mathcal{A}$ is a $\mathbb{Z}$-graded superalgebra $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}[i]$. The subgroup of $\text{Aut}\,\mathcal{A}$,

$$\text{Aut}^*\mathcal{A} := \{\sigma \in \text{Aut}\,\mathcal{A} | \sigma(\mathcal{A}[j]) \subset \mathcal{A}[j], \forall j \in \mathbb{Z}\},$$

is called the homogeneous automorphism group of $\mathcal{A}$. Correspondingly, the subgroup

$$\text{Aut}^*(\mathcal{A} : \mathcal{Q}) = \text{Aut}^*\mathcal{A} \cap \text{Aut}(\mathcal{A} : \mathcal{Q})$$

is called the homogeneous admissible automorphism group of $\mathcal{A}$.

Now, suppose that the associated filtration $(\mathcal{A}_m)_{m \in \mathbb{Z}}$ of $\mathcal{A}$, where

$$\mathcal{A}_m := \oplus_{i \geq m} \mathcal{A}[i], \quad \forall i \in \mathbb{Z},$$

is invariant under automorphisms of $\mathcal{A}$. Put

$$\text{Aut}_i\mathcal{A} = \{\sigma \in \text{Aut}\,\mathcal{A} | (\sigma - 1)(\mathcal{A}_j) \subset \mathcal{A}_{i+j}, \forall j \in \mathbb{Z}\}, \quad i \geq 0.$$ 

Then $\text{Aut}_i\mathcal{A}$ is a normal subgroup of $\text{Aut}\,\mathcal{A}$ for each $i \geq 0$. We call

$$\text{Aut}\,\mathcal{A} = \text{Aut}_0\mathcal{A} > \text{Aut}_1\mathcal{A} > \cdots$$
the standard normal series of $\text{Aut}\mathcal{A}$. Put
\[
\text{Aut}_i(\mathcal{A} : \mathcal{D}) = \text{Aut}_i\mathcal{A} \cap \text{Aut}(\mathcal{A} : \mathcal{D}), \quad i \geq 0.
\]
We call the normal series
\[
\text{Aut}(\mathcal{A} : \mathcal{D}) = \text{Aut}_0(\mathcal{A} : \mathcal{D}) > \text{Aut}_1(\mathcal{A} : \mathcal{D}) > \cdots
\]
the standard admissible normal series of $\text{Aut}(\mathcal{A} : \mathcal{D})$.

Let us consider the homomorphism of groups:
\[
\Phi: \text{Aut}(\mathcal{O} : \mathfrak{g}) \to \text{Aut}\mathfrak{g},
\]
\[
\sigma \mapsto \tilde{\sigma}|_{\mathfrak{g}},
\]
\[
(1)
\]
where
\[
\tilde{\sigma}(D) = \sigma D \sigma^{-1}, \quad D \in \mathfrak{g}.
\]
In the following, we are going to study the automorphism groups of the restricted Lie superalgebras under consideration via the homomorphism of groups.

As in [5], we adopt the convention that $x^{(\alpha)} = 0$ for $\alpha \notin \mathbb{A}$. For $(\alpha, u) \in \mathbb{A} \times \mathbb{R}$, put
\[
I(\alpha, u) = \{i \in I_0 \mid \partial_i \partial_{x^i}(x^{(\alpha)}x^u) \neq 0\}, \quad \tilde{I}(\alpha, u) = \{i \in I_0 \mid x^{(\alpha+\epsilon_i)}x^ix^u \neq 0\}.
\]

From [5], we have the following result.

**Lemma 1** The following properties hold:
\[
\text{SHO}' = \overline{\text{SHO}} \oplus \text{span}_F\{T_H(x^{(\pi)}x^u) \mid I(\alpha, u) = \tilde{I}(\alpha, u) = \emptyset\}, \quad (2)
\]
\[
\overline{\text{SHO}} = \text{SHO} \oplus \text{span}_F\{[T_H(x^{(\pi)}), T_H(x^\omega)]\}, \quad (3)
\]
\[
\overline{\text{SHO}} = \bigoplus_{i=-1}^{\xi-1} \overline{\text{SHO}}[i], \quad \text{SHO} = \bigoplus_{i=-1}^{\xi-1} \text{SHO}[i], \quad \xi = \sum_i p^i, \quad (4)
\]
\[
\overline{\text{SHO}}[i] = \text{SHO}[i], \quad \overline{\text{SHO}}[i] = [\overline{\text{SHO}}[-1], \overline{\text{SHO}}[i+1]], \quad (5)
\]
\[
\overline{\text{SHO}}[\xi-4] = \text{SHO}'[\xi-4] = \text{span}_F\{[T_H(x^{(\pi)}), T_H(x^\omega)]\}. \quad (6)
\]

**Lemma 2** The associated filtration of $X$ is invariant under automorphisms of $X$, where $X = \mathcal{O}$ or $\mathfrak{g}$.

**Proof** For $X = \text{SHO}'$ or $\mathcal{O}$, the conclusion follows from [4] or [7]. For $X = \overline{\text{SHO}}$ or $\text{SHO}$, the proof is similar to that of [4, Theorem 2].

**Lemma 3** Let $(G, [p])$ be a restricted Lie superalgebra. Assume that $L$ is a subalgebra of $G$, which is a sum of subsuperspaces:
\[
L = [L, L] + V.
\]
If $[L, L]$ is a restricted subalgebra of $G$ and $V_\mathbf{1}^{[p]}$ is contained in $L_\mathbf{1}$, then $(L, [p])$ is restricted.

Proof It is enough to prove that $x^{[p]} \in L_\mathbf{1}$ for arbitrary $x \in L_\mathbf{1}$. Since $L = [L, L] + V$, we have $x = y + z$ for some $y \in [L, L_\mathbf{1}], z \in V_\mathbf{1}$. Because

$$x^{[p]} = y^{[p]} + z^{[p]} + \sum_{i=1}^{p-1} S_i(y, z),$$

where

$$\sum_{i=1}^{p-1} S_i(y, z) \in [L, L_\mathbf{1}], \quad z^{[p]} \in L_\mathbf{1},$$

we can conclude that $x^{[p]} \in L_\mathbf{1}$ for all $x \in L_\mathbf{1}$. □

The following conclusion is analogous to those of other Lie superalgebras of Cartan type (see [6,7]). For a $\mathbb{Z}$-graded vector space and $x \in V$, write $zd(x)$ for the $\mathbb{Z}$-degree of $x$.

**Proposition 1** $g(m, m; \mathbf{t})$ is restricted if and only if $\mathbf{t} = 1$.

Proof Suppose that $g(m, m; \mathbf{t})$ is restricted. Then $(adD_i)^p$ are inner derivations of $g$ for all $i \in I_0$. Hence,

$$zd(adD_i)^p \geq -1.$$ 

On the other hand, noticing that $zd(adD_i) = -1$, we obtain that

$$zd(adD_i)^p = -p.$$ 

As a consequence,

$$(adD_i)^p = 0, \quad \forall i \in I_0.$$ 

Assert that $\mathbf{t} = 1$. Otherwise, $\mathbf{t} > 1$, applying $(adD_i)^p$ to $T_H(x^{(p+1)c_i}) \in g$, we get

$$(adD_i)^p \neq 0, \quad i \in I_0.$$ 

But this contradicts that

$$(adD_i)^p = 0, \quad \forall i \in I_0.$$ 

If $\mathbf{t} = 1$, then $W(m, m; 1)$ is a restricted Lie superalgebra with respect to the usual $p$-mapping (see [9]). For $g = SHO$, the conclusion follows directly from [1]. For $g = SHO$, in view of (3), for arbitrary $\mathbf{y} \in SHO_\mathbf{1}$, we have

$$\mathbf{y} = y + \lambda [T_H(x^{(\pi)}), T_H(x^{(\omega)})],$$

where $y \in SHO_\mathbf{1}, \lambda \in \mathbb{F}$, and $[T_H(x^{(\pi)}), T_H(x^{(\omega)})]$ is even. Thanks to Lemma 3, it remains to show that $[T_H(x^{(\pi)}), T_H(x^{(\omega)})]^p$ does lie in $SHO_\mathbf{1}$. In fact, a direct computation shows

$$[T_H(x^{(\pi)}), T_H(x^{(\omega)})]^p = 0.$$
For \( g = \text{SHO}' \), according to (2) and Lemma 3, it remains to show that \( T_H(x^\alpha x^u)^p \in \text{SHO}'_0 \), where \( T_H(x^\alpha x^u) \) is even and

\[
I(\alpha, u) = \tilde{I}(\alpha, u) = \emptyset.
\]

Note that

\[
T_H(x^\alpha x^u)^p = \begin{cases} 
T_H(x^\alpha x^u), & \alpha = \varepsilon_i, \ u = \langle i' \rangle \text{ for some } i \in I_0, \\
0, & \text{otherwise.}
\end{cases}
\]

The proof is complete.

For the rest of this section, we shall restrict our attention to the restrictedness case. Suppose that \( g \) is restricted, that is, \( g = (m, m; \underline{1}) \), and correspondingly, \( X := X(m, m; \underline{1}) \), where \( X = \mathcal{O} \) or \( W \). In the following, let \( M_{2m}(\mathcal{O}) \) be the \( \mathbb{F} \)-algebra of all \( 2m \times 2m \) matrices over \( \mathcal{O} \). Denote by \( \text{pr}_{[0]} \) and \( \text{pr}_1 \) the projections of \( \mathcal{O} \) onto \( \mathcal{O}_{[0]} = \mathbb{F} \) and \( \mathcal{O}_{1} \), respectively. For \( A = (a_{ij}) \in M_{2m}(\mathcal{O}) \), put

\[
\text{pr}_{[0]}(A) = (\text{pr}_{[0]}(a_{ij})), \quad \text{pr}_1(A) = (\text{pr}_1(a_{ij})).
\]

For a \( \mathbb{Z} \)-graded Lie superalgebra \( L \), the grading \( (L_i)_{i \in \mathbb{Z}} \) is called transitive if

\[
\{A \in L_i \mid [A, L_{-1}] = 0\} = \{0\}, \quad \forall \ i \geq 0.
\]

**Lemma 4**  The following statements hold:

(i) (see [7]) Suppose that \( \{E_1, \ldots, E_{2m}\} \) is an \( \mathcal{O} \)-basis of \( W \). Then

\[
\{\text{pr}_{[-1]}(E_1), \ldots, \text{pr}_{[-1]}(E_{2m})\}
\]

is an \( \mathbb{F} \)-basis of \( W_{[-1]} \), where \( \text{pr}_{[-1]} \) is the projection of \( W \) onto \( W_{[-1]} \).

(ii) Let \( L = \oplus_{i \geq -1} L_i \) be a finite-dimensional \( \mathbb{Z} \)-graded subalgebra of \( W \) and \( L_{[-1]} = W_{[-1]} \). Suppose that \( \phi \in \text{AutL} \) preserves the associated filtration invariant. If \( \{G_i \mid i \in I\} \subset L \) is an \( \mathcal{O} \)-basis of \( W \), then so is \( \{\phi(G_i) \mid i \in I\} \).

(iii) Let \( L = \oplus_{i \geq -1} L_i \) be a finite-dimensional transitive \( \mathbb{Z} \)-graded subalgebra of \( W \) and \( L_{[-1]} = W_{[-1]} \). Suppose that \( \sigma, \tau \in \text{AutL} \) preserve the associated filtration of \( L \) invariant. If \( \sigma|_{L_{[-1]}} = \tau|_{L_{[-1]}} \), then \( \sigma = \tau \).

**Proof** (ii) By the assumption, \( \phi \) induces an \( \mathbb{F} \)-isomorphism of the quotient space:

\[
\phi : L/L_0 \to L/L_0, \quad \dim L/L_0 = 2m.
\]

Denote by \( G_i \) the image \( G_i \) under the canonical map \( L \to L/L_0 \). Then \( \{G_i \mid i \in I\} \) is an \( \mathbb{F} \)-basis of \( L/L_0 \). Suppose

\[
(\phi(G_1), \ldots, \phi(G_{2m}))^T = A(D_1, \ldots, D_{2m})^T,
\]

Automorphism groups of finite-dimensional special odd Hamiltonian superalgebras
where \( A \in M_{2m}(\mathcal{O}) \). Then
\[
(\phi(G_1), \ldots, \phi(G_{2m}))^T = \text{pr}_{[0]}(A)(D_1, \ldots, D_{2m})^T + \text{pr}_1(A)(D_1, \ldots, D_{2m})^T.
\]
Since \( W_{[-1]} = L_{[-1]} \), we have
\[
(\overline{\phi(G_1)}, \ldots, \overline{\phi(G_{2m})})^T = (\overline{\phi(G_1)}, \ldots, \overline{\phi(G_{2m})})^T = \text{pr}_{[0]}(A)(\overline{D_1}, \ldots, \overline{D_{2m}})^T.
\]
This implies that \( \text{pr}_{[0]}(A) \in \text{GL}(m, m) \). From [7, Lemma 2], \( A \) is invertible, and therefore, \( \{\phi(G_i) \mid i \in I\} \) is an \( \mathcal{O} \)-basis of \( W \).

(iii) Use induction on \( k \). For \( k = -1 \), the conclusion holds. For any \( E \in L_{[k]} \), \( D_i \in L_{[-1]} \), we have
\[
[\sigma(E), \sigma(D_i)] = [\tau(E), \tau(D_i)] \in L_{[k-1]},
\]
that is,
\[
[(\sigma - \tau)(E), \sigma(D_i)] = 0, \quad \forall i \in I.
\]
Since \( L \) is transitive and the associated filtration of \( L \) is invariant under automorphism of \( L \), we have \( (\sigma - \tau)(E) = 0 \), namely, \( \sigma = \tau \). \( \square \)

**Lemma 5** Suppose \( \phi \in \text{Aut} \mathfrak{g} \). Then there exist \( y_j \in \mathcal{O}_1 \) with \( p(y_j) = \mu(j) \) such that
\[
\phi(D_i)(y_j) = \delta_{ij}, \quad i, j \in I.
\]
Furthermore, the matrix \( (\phi(D_i)(y_j))_{i,j} \) is invertible.

**Proof** In view of Lemma 4 (ii), since \( \{D_1, \ldots, D_{2m}\} \) is an \( \mathcal{O} \)-basis of \( W \), so is \( \{\phi(D_1), \ldots, \phi(D_{2m})\} \). For \( j \in I \setminus \{1'\} \), we have \( T_H(x_1x_j) \in \mathfrak{g} \). Assume
\[
\phi(T_H(x_1x_j)) = \sum_{l=1}^{2m} a_{jl}\phi(D_l), \quad a_{jl} \in \mathcal{O}.
\]
In particular, \( \phi \) preserves the associated filtration invariant. By Lemma 1, this forces \( a_{jl} \in \mathcal{O}_1 \). We can obtain
\[
\phi([D_l, T_H(x_1x_j)]) = \sum_{l=1}^{2m} (\phi(D_l)(a_{jl}))\phi(D_l). \tag{7}
\]
On the other hand, since
\[
[D_l, T_H(f)] = T_H(D_l(f)),
\]
we have
\[
\phi([D_l, T_H(x_1x_j)]) = \delta_{ij}\phi(D_{1'}) + (-1)^{\mu(j)}\delta_{i1}\phi(D_{j'}). \tag{8}
\]
Comparing (7) with (8), we have
\[
\phi(D_i)(a_{jl}) = \delta_{ij}.
\]
Put $y_j = a_{j'}$ for all $j \in \mathcal{I}\setminus\{1'\}$. For $j \in \mathcal{I}\setminus\{1'\}$, we know that
\[
\phi(D_i)(y_j) = \delta_{ij}, \quad y_j \in \mathcal{O}_1,
\]
\[
p(y_j) = p(a_{j'}) = \mu(j') + \mu(1') = \mu(j).
\]
For $j = 1'$, the element $T_H(x_1 x_1' - x_2 x_2')$ lies in $\mathfrak{g}$. Suppose
\[
\phi(T_H(x_1 x_1' - x_2 x_2')) = 2m \sum_{l=1}^{2m} a_l \phi(D_l).
\]
Then
\[
\phi([D_i, T_H(x_1 x_1' - x_2 x_2')]) = 2m \sum_{l=1}^{2m} (\phi(D_i)(a_l)) \phi(D_l), \quad a_l \in \mathcal{O}_1.
\] (9)

On the other hand,
\[
\phi([D_i, T_H(x_1 x_1' - x_2 x_2')]) = \delta_{11'} \phi(D_{1'}) - \delta_{11} \phi(D_1) - \delta_{12} \phi(D_2) + \delta_{12} \phi(D_2).
\] (10)

Comparing (9) with (10), we have
\[
\phi(D_i)(a_{1'}) = \delta_{11'}.
\]

Putting $y_{1'} = a_{1'}$, we have $p(a_{1'}) = \mu(1')$. The proof is complete. □

Our main intent is to describe the connections between $\text{Aut} \mathfrak{g}$ and $\text{Aut}(\mathcal{O} : \mathfrak{g})$. Recall (1), the homomorphism of groups. We have the following result.

**Theorem 1**  \( \Phi \) is an isomorphism of groups.

**Proof**  We shall merely prove that $\Phi$ is bijective. Let us first show that $\Phi$ is injective, that is, $\ker(\Phi) = \{1_\mathcal{O}\}$. To that aim, let $\sigma \in \text{Aut}(\mathcal{O} : \mathfrak{g})$ such that $\tilde{\sigma}|_{\mathfrak{g}} = 1_{\mathfrak{g}}$. For $j \in \mathcal{I}$, since $\tilde{\sigma}|_{\mathfrak{g}} = 1_{\mathfrak{g}}$ and
\[
T_H(x_j)(x_k) = \sigma(T_H(x_j)(x_k)) = T_H(x_j)(\sigma(x_k)),
\]
we conclude that
\[
x_k - \sigma(x_k) \in \mathbb{F}, \quad k \in \mathcal{I}.
\]
On the other hand, Lemma 2 ensures that $\sigma(x_k) \in \mathcal{O}_1$. It follows that
\[
\sigma(x_k) - x_k \in \mathcal{O}_1 \cap \mathbb{F},
\]
and therefore,
\[
\sigma(x_k) = x_k, \quad \forall k \in \mathcal{I}.
\]
Since $\mathcal{O}$ is generated by $x_r$, $r \in \mathcal{I}$, it follows that $\sigma = 1_\mathcal{O}$.

The remaining task is to show that $\Phi$ is surjective. Let $\phi \in \text{Aut} \mathfrak{g}$. By Lemma 5, there exist $y_j \in \mathcal{O}$ with $p(y_j) = \mu(j)$ such that
\[
\phi(D_i)(y_j) = \delta_{ij}, \quad j \in \mathcal{I}.
\]
Suppose
\[ \phi(D_i) = \sum_{j=1}^{2m} a_{ij} D_j, \quad a_{ij} \in \mathcal{O}. \]

Then the matrix \((\phi(D_i)(y_j))\) is equal to \((a_{ij})(D_i y_j)\), and therefore,
\[ (\delta_{ij}) = (\phi(D_i)(y_j)) = \text{pr}_{[0]}(a_{ij})(D_i y_j). \]

It follows that \(\text{pr}_{[0]}(D_i y_j)\) is invertible. By [7, Lemma 2.5], there exists an endomorphism of \(\mathcal{O}\) such that
\[ \sigma(x_i) = y_j, \quad i, j \in I. \]  \hspace{1cm} (11)

Then \(\sigma\) is even. We assert that \(\sigma \in \text{Aut}\mathcal{O}\). In fact, from (11), it is easy to see that \(\sigma\) preserves the associated filtration of \(\mathcal{O}\) invariant, that is,
\[ \sigma(\mathcal{O}_i) \subset \mathcal{O}_i, \quad \forall i \geq 0. \]

Therefore, it induces a linear transformation \(\sigma_i\) of \(\mathcal{O}_i/\mathcal{O}_{i+1}\). Note that the matrix of \(\sigma_1\) relative to \(\mathbb{F}\)-basis \(\{x_1+\mathcal{O}_2, \ldots, x_{2n}+\mathcal{O}_2\}\) is just \((\text{pr}_{[0]}(D_i y_j))\). This implies that \(\sigma_1\) is bijective. Proceeding by induction on \(i \geq 1\), we get \(\sigma_i\) is bijective. Hence, \(\sigma\) is bijective. Note that
\[ \tilde{\sigma}(D_i)(y_j) = (\sigma D_i \sigma^{-1})(y_j) = \sigma(D_i x_j) = \delta_{ij} = \phi(D_i)(y_j), \quad \forall i, j \in I. \]

Since \(\{y_j \mid j \in I\}\) generates \(\mathcal{O}\), we conclude that
\[ \tilde{\sigma}(D_i) = \phi(D_i), \quad i \in I. \]

By Lemma 4 (iii), this makes it clear that \(\tilde{\sigma}|_{\mathcal{O}} = \phi\). The theorem follows. \(\square\)

In the next theorem, we give a further discussion about the relations between \(\text{Aut}(\mathcal{O} : \mathcal{g})\) and \(\text{Aut} \mathcal{g}\) and prove that \(\Phi\) preserves the standard normal series and the homogeneous automorphism groups.

**Theorem 2** The following identities hold:

(i) \(\Phi(\text{Aut}_i(\mathcal{O} : \mathcal{g})) = \text{Aut}_i \mathcal{g}\) for \(i \geq 0\),

(ii) \(\Phi(\text{Aut}^*(\mathcal{O} : \mathcal{g})) = \text{Aut}^* \mathcal{g}\).

**Proof** (i) First, we show the inclusion ‘\(\subset\)’. Let \(\sigma \in \text{Aut}_i(\mathcal{O} : \mathcal{g})\). Then \(\sigma^{-1} \in \text{Aut}_i(\mathcal{O} : \mathcal{g})\). For \(k \in \mathbb{N}\) and \(f \in \mathcal{O}_k\), we may assume
\[ \sigma^{-1}(f) = f + f', \quad f' \in \mathcal{O}_{i+k}, \]
\[ \sigma(D_j(f)) = D_j(f) + f'', \quad f'' \in \mathcal{O}_{i+k-1}. \]

According to Lemma 2, we have
\[ \sigma(D_j(f')) \in \mathcal{O}_{i+k-1}. \]
Thus, from (12) and (13), we obtain
\[ \tilde{\sigma}(D_j)(f) = D_j(f) + f'' + \sigma(D_j(f')). \]
We obtain that
\[ \tilde{\sigma}(D_j)(f) \equiv D_j(f) \pmod{\mathcal{O}_{i+k-1}}. \]
This implies that
\[ \tilde{\sigma}(D_j) \equiv D_j \pmod{W_{i-1}}, \quad \forall \ j \in I. \]
A straightforward calculation shows that
\[ \tilde{\sigma}(fD_j) = \sigma(f)\tilde{\sigma}(D_j), \quad \forall \ j \in I. \]
Then it is easy to see that
\[ \tilde{\sigma}(fD_j) \equiv fD_j \pmod{W_{k-1+i}}. \]
Therefore,
\[ \tilde{\sigma} \in \text{Aut}_i W \cap \text{Aut}_i \mathfrak{g} \subset \text{Aut}_i \mathfrak{g}. \]
Hence,
\[ \Phi(\text{Aut}_i(\mathfrak{g} : \mathfrak{g})) \subset \text{Aut}_i \mathfrak{g}. \]
Next, we show the inclusion ‘⊇’. Let \( \tilde{\sigma} \in \text{Aut}_i \mathfrak{g} \) with \( i \geq 0 \) and \( \sigma = \Phi^{-1}(\tilde{\sigma}) \).
Given \( j \in I \), pick \( k \in I \setminus \{j\} \). We have
\[ T_H(x_k'x_j) = (-1)^{\mu(k') + \mu(k')\mu(j)} x_j D_k + (-1)^{\mu(j)} x_k' D_{j'}. \]
Then
\[ (-1)^{\mu(k') + \mu(k')\mu(j)} \sigma(x_j)(\tilde{\sigma}(D_k)) + (-1)^{\mu(j)} \sigma(x_k')(\tilde{\sigma}(D_{j'})) \]
\[ \equiv (-1)^{\mu(k') + \mu(k')\mu(j)} x_j D_k + (-1)^{\mu(j)} x_k' D_{j'} \pmod{\mathfrak{g}_i}. \] (12)
Notice that \( \tilde{\sigma} \in \text{Aut}_i \mathfrak{g} \) and \( W_{[-1]} = \mathfrak{g}_{[-1]} \). We have
\[ \tilde{\sigma}(D_k) = D_k + E_1, \quad \tilde{\sigma}(D_{j'}) = D_{j'} + E_2, \] (13)
where \( E_1, E_2 \in \mathfrak{g}_{i-1} \). By Lemma 2, it is easily seen that \( \sigma(x_j)E_1, \sigma(x_k')E_2 \in W_i \). Thus, from (12) and (13), we obtain
\[ (-1)^{\mu(k') + \mu(k')\mu(j)}(\sigma(x_j) - x_j)D_k + (-1)^{\mu(j)}(\sigma(x_k') - x_k')D_{j'} \equiv 0 \pmod{W_i}. \]
Since \( k' \neq j \), we obtain \( \sigma(x_j) \equiv x_j \pmod{\mathcal{O}_{i+1}} \). Now, using induction on \( |\alpha| + |\mu| \), one may prove that
\[ \sigma(x^{(\alpha)}x^u) \equiv x^{(\alpha)}x^u \pmod{\mathcal{O}_{|\alpha| + |\mu|+1}}. \]
This implies that \( \sigma \in \text{Aut}_i \mathfrak{g}, \) and therefore, \( \sigma \in \text{Aut}_i(\mathfrak{g} : \mathfrak{g}) \). Hence,
\[ \Phi(\text{Aut}_i(\mathfrak{g} : \mathfrak{g})) \supset \text{Aut}_i \mathfrak{g}. \]
(ii) The proof is similar to that of (i), we omit it here. □

Next proposition is a general conclusion for a finite-dimensional $\mathbb{Z}$-graded superalgebra $\mathcal{A} = \oplus_{i \in \mathbb{Z}} \mathcal{A}_i$ for which $\text{Aut}\mathcal{A}$ preserves the associated filtration invariant.

**Proposition 2** Suppose that $\mathcal{A} = \oplus_{i \in \mathbb{Z}} \mathcal{A}_i$ is a finite-dimensional $\mathbb{Z}$-graded superalgebra over $\mathbb{F}$ and the associated filtration of $\mathcal{A}$ is invariant under $\text{Aut}\mathcal{A}$. Then $\text{Aut}_i\mathcal{A}$ ($i \geq 1$) are solvable normal subgroups of $\text{Aut}\mathcal{A}$.

**Proof** Since $\text{Aut}\mathcal{A}$ preserves the associated filtration invariant, we have

$$[\text{Aut}_i\mathcal{A}, \text{Aut}_j\mathcal{A}] \subset \text{Aut}_{i+j}\mathcal{A}, \quad i, j \geq 0.$$  

Then the normal series

$$\text{Aut}_1\mathcal{A} > \text{Aut}_2\mathcal{A} > \cdots$$

is abelian (that is, $\text{Aut}_i\mathcal{A}/\text{Aut}_{i+1}\mathcal{A}$ are abelian groups for all $i \geq 1$) and reaches 0. This shows that $\text{Aut}_i\mathcal{A}$ is solvable. □

For any special odd Hamiltonian superalgebra $\mathfrak{g}$, by Proposition 2, $\text{Aut}_i\mathfrak{g}$ ($i \geq 1$) are solvable normal subgroups of $\text{Aut}\mathfrak{g}$.

**Acknowledgements** The authors thank the referees for careful reading and valuable suggestions. This work was supported by the Natural Science Foundation for Distinguished Young Scholars, Heilongjiang Province (JC201004) and the National Natural Science Foundation of China (Grant No. 11171055).

**References**

1. Bai W, Liu W D, Ni L. Derivations of the finite-dimensional special odd Hamiltonian Lie superalgebras. arXiv: 1007.1098math.RT
2. Bouarroudj S, Leites D. Simple Lie superalgebras and nonintegrable distributions in characteristic $p$. J Math Sci, 2007, 141: 1390–1398
3. Fu J Y, Zhang Q C, Jiang C P. The Cartan-type modular Lie superalgebra $\mathcal{K}O$. Commun Algebra, 2006, 34: 107–128
4. He Y H, Liu W D, Li B. Filtration structure of finite dimensional special odd Hamiltonian superalgebras in prime characteristic. J Beijing Inst Technol, 2009, 18: 488–491
5. Liu W D, He Y H. Finite dimensional special odd Hamiltonian superalgebras in prime characteristic. Commun Contemp Math, 2009, 11: 523–546
6. Liu W D, Zhang Y Z. Finite-dimensional odd Hamiltonian superalgebras over a field of prime characteristic. J Aust Math Soc, 2005, 79: 113–130
7. Liu W D, Zhang Y Z. Automorphism groups of restricted Cartan-type Lie superalgebras. Commun Algebra, 2006, 34: 3767–3784
8. Wilson R L. Automorphisms of graded Lie algebras of Cartan type. Commun Algebra, 1975, 3: 591–613
9. Zhang Y Z. Finite-dimensional Lie superalgebras of Cartan type over fields of prime characteristic. Chin Sci Bull, 1997, 42: 720–724