Hamiltonian Perturbations at the Second-Order Approximation

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Abstract. Integrability condition of Hamiltonian perturbations of integrable Hamiltonian PDEs of hydrodynamic type up to the second-order approximation is considered. Under a nondegeneracy assumption, we show that the Hamiltonian perturbation at the first-order approximation is integrable if and only if it is trivial, and that under a further assumption, the Hamiltonian perturbation at the second-order approximation is integrable if and only if it is quasi-trivial.

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1. Introduction and the Statements of the Results

Let $M$ be an $n$-dimensional complex manifold. Consider the following system of Hamiltonian PDEs of hydrodynamic type:

$$\partial_t(v^\alpha) = \eta^{\alpha\beta} \partial_x \left( \frac{\delta H_0}{\delta v^\beta(x)} \right), \quad v = (v^1, \ldots, v^n) \in M, \ x \in S^1, \ t \in \mathbb{R}, \ (1.1)$$

where $(\eta^{\alpha\beta})$ is a given symmetric invertible constant matrix, $H_0 := \int_{S^1} h_0(v) \, dx$ is a given local functional (called the Hamiltonian), and $\delta/\delta v^\beta(x)$ denotes the variational derivative. Here and below, free Greek indices take the integer values $1, \ldots, n$, and the Einstein summation convention is assumed for repeated Greek indices with one-up and one-down; the matrix $(\eta^{\alpha\beta})$ and its inverse $(\eta_{\alpha\beta})$ are used to raise and lower Greek indices, e.g., $v_\alpha := \eta_{\alpha\beta} v^\beta$. The Hamiltonian density $h_0(v)$ is assumed to be a holomorphic function of $v$. More explicitly, Eq. (1.1) have the form:

$$\partial_t(v^\alpha) = A^\alpha_{\gamma}(v) v^\gamma_x, \quad \text{where} \ A^\alpha_{\gamma}(v) := \eta^{\alpha\beta} \frac{\partial^2 h_0(v)}{\partial v^\beta \partial v^\gamma}. \quad \text{Basic assumption:} \ (A^\alpha_{\gamma}(v)) \text{ has pairwise distinct eigenvalues } \lambda_1(v), \ldots, \lambda_n(v) \text{ on an open dense subset } U \text{ of } M.
Let us perform a change of variables \((v^1, \ldots, v^n) \rightarrow (R_1, \ldots, R_n)\) with non-degenerate Jacobian locally on \(U\). We call \(R_1, \ldots, R_n\) a complete set of Riemann invariants, if evolutions along \(R_1, \ldots, R_n\) are all diagonal, namely,

\[
\partial_t(R_i) = V_i(R) \partial_x(R_i), \quad i = 1, \ldots, n, \tag{1.2}
\]

where \(V_i's\) are some functions of \(R = (R_1, \ldots, R_n)\). Below, free Latin indices take the integer values 1, \ldots, \(n\) unless otherwise indicated. Clearly, Eq. (1.2) imply that the gradients of Riemann invariants are eigenvectors of \(A^\beta_\alpha\), namely,

\[
A^\alpha_\beta R_{i,\alpha} = \lambda_i R_{i,\beta}, \quad V_i = \lambda_i \tag{1.3}
\]

with \(R_{i,\alpha} := \partial_\alpha(R_i)\). Similar notations like \(R_{i,j} := \partial_j(R_i), R_{i,j,k} := \partial_j\partial_k(R_i), \ldots\) will also be used. Here and below, \(\partial_\alpha := \partial_{v^\alpha}, \partial_i := \partial_{R_i}\).

It was proven by Tsarev [23] that the integrability of Eq. (1.1) is equivalent to the existence of complete Riemann invariants. Here, “integrability” means existence of sufficiently many conservation laws/infinitesimal symmetries (See Definition 2.2). It was shown by B. Dubrovin [10, 11] that existence of a complete set of Riemann invariants is equivalent to vanishing of the following Haantjes tensor:

\[
H_{\alpha\beta\gamma} := \left( A_{\alpha\rho\sigma} A_{\beta\phi} A_{\gamma\psi} + A_{\beta\rho\sigma} A_{\gamma\phi} A_{\alpha\psi} + A_{\gamma\rho\sigma} A_{\alpha\phi} A_{\beta\psi} \right) A^\rho_\nu A^\delta_\sigma A^\psi_\phi, \tag{1.4}
\]

where \(A_{\alpha\beta\gamma} := \partial_\alpha \partial_\beta \partial_\gamma(h_0)\) and \(\delta_{\alpha\beta\gamma} := \eta^\alpha_\epsilon \eta^\beta_\delta \eta^\gamma_\phi - \eta^\alpha_\epsilon \eta^\beta_\epsilon \eta^\gamma_\delta\). Note that \(H_{\alpha\beta\gamma}\) automatically vanishes if the signature \(\varepsilon(\alpha, \beta, \gamma) = 0;\) for \(n = 1\) or for \(n = 2\), the system (1.1) is always integrable.

We proceed to the study of Hamiltonian perturbations [4, 5, 9–11, 16, 18] of (1.1)

\[
\partial_t(v^\alpha) = \eta^{\alpha\beta} \partial_x \left( \frac{\delta H}{\delta v^\beta(x)} \right), \quad x \in S^1, \ t \in \mathbb{R}, \ v = (v^1, \ldots, v^n) \in M. \tag{1.5}
\]

Here, \(H := \int_{S^1} h \, dx = \sum_{j=0}^\infty \epsilon^j H_j\) with \(H_j := \int_{S^1} h_j(v, v_1, v_2, \ldots, v_j) \, dx\) is the Hamiltonian, and \(h_j\) are differential polynomials of \(v\) satisfying the following homogeneity condition:

\[
\sum_{\ell=1}^{\ell} \ell v^\alpha_\ell \frac{\partial h_j}{\partial v^\alpha_\ell} = j \, h_j, \quad j \geq 0. \tag{1.6}
\]

We recall that the variational derivative reads

\[
\frac{\delta H}{\delta v^\beta(x)} = \sum_{\ell=0}^\infty (-\partial_x)^\ell \left( \frac{\partial h}{\partial v^\beta_\ell} \right). \]

In the above formulae, \(v^\alpha_\ell := \partial_x^\ell(v^\alpha), \ \ell \geq 0,\) and we recall that a differential polynomial of \(v\) is a polynomial of \(v_1, v_2, \ldots\) whose coefficients are holomorphic functions of \(v\). The ring of differential polynomials of \(v\) is denoted by \(A_v\). We remark that according to [4, 14–16, 18] the Hamiltonian system (1.5) that we are considering is general. Note that the Hamiltonian operator \(\eta^{\alpha\beta} \partial_x\) defines
a Poisson bracket \( \{ , \} \) on the space of local functionals \( \mathcal{F} := \{ \int_{S^1} f \, dx \mid f \in \mathcal{A}_v[\epsilon] \} \), \( \{ , \} : \mathcal{F} \times \mathcal{F} \to \mathcal{F} \), by
\[
\{ F, G \} := \int_{S^1} \frac{\delta F}{\delta v^\alpha(x)} \eta^{\alpha\beta} \partial_x \left( \frac{\delta G}{\delta v^\beta(x)} \right) \, dx, \quad \forall \ F, G \in \mathcal{F}.
\]
(1.7)

It is helpful to view \( v^\alpha(x) \) as a “local functional” \( v^\alpha(x) = \int_{S^1} v^\alpha(y) \, \delta(y-x) \, dy \), called the coordinate functional. Then, one can write Eq. (1.5) in the form
\[
\partial_t(v^\alpha) = \{ v^\alpha(x), H \}.
\]

Clearly, a system of Hamiltonian PDEs of hydrodynamic type (1.1) can be obtained from (1.5) simply by taking the dispersionless limit: \( \epsilon \to 0 \).

The perturbed system (1.5) is called integrable if its dispersionless limit is integrable and each conservation law of (1.1) can be extended to a conservation law of (1.5). In this paper, we start with a system of integrable Hamiltonian PDEs of hydrodynamic type, and study the conditions such that the perturbation (1.5) is integrable up to the second-order approximation.

**Theorem 1.1.** Assume that the matrix \( (A^\alpha_\beta) \) associated with (1.1) has distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \) on an open dense subset \( U \subset M \). Assume that (1.1) is integrable and denote by \( R = (R_1, \ldots, R_n) \) the associated complete Riemann invariants. A Hamiltonian perturbation of (1.1) of the form \( H = H_0 + \epsilon H_1 + \mathcal{O}(\epsilon^2) \) with \( H_0 = \int_{S^1} h(v) \, dx \), \( H_1 = \int_{S^1} \sum_{i=1}^n p_i(R) R_{ix} \, dx \) is integrable at the first-order approximation iff either of the following is true:

(i) it is trivial;

(ii) the following equations hold true for \( p_i \):
\[
\omega_{ij,k} - \omega_{ik,j} = a_{ij} \omega_{ik} + a_{ji} \omega_{jk} - a_{ik} \omega_{ij} - a_{ki} \omega_{kj}, \quad \forall \ \epsilon(i, j, k) = \pm 1.
\]
(1.8)

Here, \( a_{ij} \) and \( \omega_{ij} \) are defined by
\[
a_{ij} := \frac{\lambda_{ij}}{\lambda_i - \lambda_j}, \quad \omega_{ij} := \frac{p_{i,j} - p_{j,i}}{\lambda_i - \lambda_j}, \quad \forall \ i \neq j.
\]
(1.9)

In the above statement, we recall that a Hamiltonian perturbation is called trivial if it is Miura equivalent to its dispersionless limit; for more details about triviality, see Sect. 2. Due to Theorem 1.1, to study the integrable Hamiltonian perturbation (1.5) of an integrable PDE of hydrodynamic type (1.1) up to the second-order approximation, it suffices to consider the case with vanishing \( H_1 \). Here, it should also be noted that the basic assumption proposed in the beginning of the paper has been assumed as it is written again in the statement.

**Theorem 1.2.** Assume that the matrix \( (A^\alpha_\beta) \) associated with (1.1) has distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \) on an open dense subset \( U \subset M \) and that \( \lambda_{i,i}(v) \neq 0 \) for \( v \in U \). Assume that (1.1) is integrable and denote by \( R = (R_1, \ldots, R_n) \) the associated complete Riemann invariants. A Hamiltonian perturbation of (1.1) of the form
\[
H = H_0 + \epsilon^2 H_2 + \mathcal{O}(\epsilon^3)
\]
(1.10)
with $H_0 = \int_{S^1} h_0(u) \, dx$, $H_2 = \int_{S^1} \sum_{i,j=1}^{n} d_{ij}(R) R_{ix} R_{jx} \, dx$ ($d_{ij} = d_{ji}$) is $O(\epsilon^2)$-integrable iff either of the followings is true:

(i) it is quasi-trivial;
(ii) there exist functions $C_i(R_i)$, $i = 1, \ldots, n$ such that

$$d_{ii} = -C_i(R_i) \lambda_{i,i},$$

$$\left( \frac{d_{ij}}{\lambda_i - \lambda_j} \right)_{,k} + \left( \frac{d_{jk}}{\lambda_j - \lambda_k} \right)_{,i} + \left( \frac{d_{ki}}{\lambda_k - \lambda_i} \right)_{,j} = 0, \quad \forall \varepsilon(i,j,k) = \pm 1.$$  

For the meaning of quasi-triviality, see Sects. 2 and 3. Note that an equivalent description of (1.11)–(1.12) is that the density $h_2$ can be written in the form

$$h_2 = -\sum_{i=1}^{n} C_i(R_i) \lambda_{i,i} R_{ix}^2 + \frac{1}{2} \sum_{i \neq j} (\lambda_i - \lambda_j) s_{ij} R_{ix} R_{jx},$$

where $s_{ij} := \phi_{i,j} - \phi_{j,i}$ for some functions $\phi_i(R)$.

For the cases $n = 1, 2$, Theorems 1.1 and 1.2 agree with the results of [20] and [9].

The paper is organized as follows. In Sect. 2, we review some terminologies about Hamiltonian PDEs. In Sect. 3, we study integrability of (1.5) up to the second-order approximation. An example of non-integrable perturbation is given in Sect. 4.

2. Preliminaries

In this section, we will recall several terminologies in the theory of Hamiltonian perturbations; more terminologies can be found in, e.g., [6–8,10,12,16,22,23].

**Definition 2.1.** A local functional $F_0 = \int_{S^1} f_0(v) \, dx$ is called a conserved quantity of (1.1) if

$$\frac{dF_0}{dt} = 0.$$  

Here, the density $f_0(v)$ is a given holomorphic function of $v$.

We also often call a conserved quantity a conservation law. Note that for simplicity we will exclude the degenerate ones with $f_0(v) \equiv \text{const}$ from conservation laws.

Since (1.1) is a Hamiltonian system, Eq. (2.1) can be written equivalently as

$$\{H_0, F_0\} = 0,$$

where $\{ , \}$ denotes the Poisson bracket defined in (1.7). (This is straightforward to verify.) According to Noether’s theorem, (2.1) is also equivalent to the statement that the following Hamiltonian flow generated by $F_0$

$$v^\alpha_s := \{v^\alpha(x), F_0\}$$
Definition 2.2. The PDE system (1.1) is called integrable if it possesses an infinite family of conserved quantities parametrized by \( n \) arbitrary functions of one variable.

A necessary and sufficient condition for integrability of (1.1) is the vanishing of the Haantjes tensor \( H_{\alpha\beta\gamma} \) as recalled already in the introduction.

We will assume that (1.1) is integrable and study its perturbations. Recall that vanishing of the Haantjes tensor ensures the existence of a complete set of Riemann invariants \( \{R_1, \ldots, R_n\} \). We have

\[
A_{\beta}^{\alpha} R_{i,\alpha} = \lambda_i R_{i,\beta},
\]

\[
M_{\beta}^{\alpha} R_{i,\alpha} = \mu_i R_{i,\beta}.
\]

Here, \( \mu_i \) are eigenvalues of \( (M_{\beta}^{\alpha}) \). For a generic conserved quantity \( F_0 \), the eigenvalues \( \mu_1, \ldots, \mu_n \) on the \( U \) are also pairwise distinct. In terms of \( \lambda_i, \mu_i \), the flow commutativity is equivalent to

\[
a_{ij} = b_{ij}, \quad \forall i \neq j,
\]

where

\[
a_{ij} := \frac{\lambda_{i,j}}{\lambda_i - \lambda_j}, \quad b_{ij} := \frac{\mu_{i,j}}{\mu_i - \mu_j}.
\]

The compatibility condition

\[
\mu_{i,jk} = \mu_{i,kj}, \quad \forall \varepsilon(i,j,k) = \pm 1
\]

for Eq. (2.6) reads as follows

\[
(\mu_i - \mu_k)(a_{ij,k} - a_{ik,j}) - (\mu_j - \mu_k)(a_{ij,k} + a_{ij}a_{jk} + a_{ik}a_{kj} - a_{ij}a_{ik}) = 0.
\]

Definition 2.2 requires that equation (2.8) is true for infinitely many \( F_0 \) parametrized by \( n \) arbitrary functions of one variable. So the coefficients of \( \mu_i - \mu_k \) and of \( \mu_j - \mu_k \) must vanish:

\[
a_{ij,k} - a_{ik,j} = 0, \quad \forall \varepsilon(i,j,k) = \pm 1,
\]

\[
a_{ij,k} + a_{ij}a_{jk} + a_{ik}a_{kj} - a_{ij}a_{ik} = 0, \quad \forall \varepsilon(i,j,k) = \pm 1.
\]

Note that (2.10) is implied by Eqs. (2.9) and (2.7).

Definition 2.3. A local functional \( F := \sum_{j=0}^{\infty} \epsilon^j F_j \) is called a conserved quantity of (1.5), if

\[
\frac{dF}{dt} = 0.
\]

Here, \( F_j = \int_{S_1} f_j(v, v_1, \ldots, v_j) \, dx, \ j \geq 0 \) with \( f_j \) being differential polynomials of \( v \) homogeneous of degree \( j \).
Conserved quantities (or say conservation laws) considered in this paper are always of the form as in Definition 2.3.

Equation (2.11) can be equivalently written as

$$\{H, F\} = 0,$$

which is recast into an infinite sequence of equations

$$\{H_0, F_0\} = 0,$$
$$\{H_0, F_1\} + \{H_1, F_0\} = 0,$$
$$\{H_0, F_2\} + \{H_1, F_1\} + \{H_2, F_0\} = 0,$$

etc.

**Definition 2.4.** A Hamiltonian perturbation (1.5) is called integrable if its dispersionless limit (1.1) is integrable and generic conservation laws of (1.1) can be extended to those of (1.5). For $N \geq 1$, (1.5) is called $O(\epsilon^N)$-integrable if its dispersionless limit (1.1) is integrable and every generic conservation law $F_0$ of (1.1) can be extended to a local functional $F$, s.t.

$$\{H, F\} = O(\epsilon^{N+1}).$$  \hfill (2.12)

One important tool of studying Hamiltonian perturbations is to use Miura-type and quasi-Miura transformations [16]. Recall that a Miura-type transformation near identity is given by an invertible map of the form

$$v \mapsto w, \quad w^\alpha := \sum_{j=0}^\infty \epsilon^j W_j^\alpha(v, v_1, \ldots, v_\ell), \quad W_0^\alpha = v^\alpha,$$  \hfill (2.13)

where $W_j^\alpha$, $j \geq 0$ are differential polynomials of $v$ homogeneous of degree $j$ with respect to the degree assignments $\deg v_\ell^\alpha = \ell$, $\ell \geq 1$. A Miura-type transformation is called canonical if there exists a local functional $K$, such that

$$w^\alpha = v^\alpha + \epsilon \left\{ v^\alpha(x), K \right\} + \frac{\epsilon^2}{2!} \left\{ \left\{ v^\alpha(x), K \right\}, K \right\} + \cdots$$  \hfill (2.14)

where $K = \sum_{j=0}^\infty \epsilon^j K_j$. Two Hamiltonian perturbations of the same form (1.5) are called equivalent if they are related via a canonical Miura-type transformation. A Hamiltonian perturbation (1.5) is called trivial if it is equivalent to (1.1).

A map of the form (2.13) is called a quasi-Miura transformation, if $W_\ell^\alpha$, $\ell \geq 1$ are allowed to have rational and logarithmic dependence in $v_x$. The Hamiltonian perturbation (1.5) is called quasi-trivial or possessing quasi-triviality, if it is related via a canonical quasi-Miura transformation to (1.1). We recall that many interesting nonlinear PDE systems possess quasi-triviality; for example, it was shown in [12] that if (1.5) is bihamiltonian then it is quasi-trivial. The precise definition used in this paper for quasi-Miura transformation will be given in the next section.
3. Proofs of Theorems 1.1 and 1.2

In this section, we study integrability of the Hamiltonian system \((1.5)\) up to the second-order approximation, and prove Theorems 1.1 and 1.2.

Assume that \((1.1)\) is integrable.

We start with the first-order approximation. Let us first look at the integrability condition of the \(O(\epsilon^1)\)-approximation. Denote
\[
H = H_0 + \epsilon H_1 + O(\epsilon^2) \tag{3.1}
\]
with \(H_1 = \int_{S^1} \tilde{p}_\alpha(u) u^\alpha dx = \sum_{i=1}^n \int_{S^1} p_i(R) R_{ix} dx.\) Here, the functions \(p_\alpha\) and \(p_i\) are assumed to satisfy \(\tilde{p}_\alpha = \sum_{i=1}^n p_i R_{i,\alpha}\).

**Proof of Theorem 1.1.** Denote by \(\tilde{\theta}_{\alpha\beta}\) the exterior differential of the 1-form \(\tilde{p}_\alpha du^\alpha\)
\[
\tilde{\theta}_{\alpha\beta} = \tilde{p}_{\alpha,\beta} - \tilde{p}_{\beta,\alpha}. \tag{3.2}
\]
In the coordinate chart of the Riemann invariants \(R_1, \ldots, R_n\), we have
\[
\theta_{ij} = \partial_i u^\alpha \tilde{\theta}_{\alpha\beta} \partial_j u^\beta = p_{i,j} - p_{j,i}. \tag{3.3}
\]
The \(O(\epsilon^1)\)-integrability says any local functional \(F_0 = \int_{S^1} f(u) dx\) satisfying
\[
\{H_0, F_0\} = 0
\]
can be extended to a local functional
\[
F = F_0 + \epsilon F_1 + O(\epsilon^2),
\]
such that
\[
\{H, F\} = O(\epsilon^2). \tag{3.4}
\]
Here, the local function \(F_1\) is of the form
\[
F_1 = \int_{S^1} \tilde{q}_\alpha(u) u^\alpha dx = \sum_{i=1}^n \int_{S^1} q_i(R) R_{ix} dx. \tag{3.5}
\]
Eq. (3.3) reads as follows
\[
\{H_0, F_1\} + \{H_1, F_0\} = 0,
\]
which is equivalent to
\[
\tilde{\theta}_{\alpha\gamma} M_{\beta}^\gamma + \tilde{\theta}_{\beta\gamma} M_{\alpha}^\gamma = \tilde{\Theta}_{\alpha\gamma} A_{\beta}^\gamma + \tilde{\Theta}_{\beta\gamma} A_{\alpha}^\gamma \tag{3.6}
\]
or, in the coordinate system of the Riemann invariants, to
\[
\frac{\theta_{ij}}{\lambda_i - \lambda_j} = \frac{\Theta_{ij}}{\mu_i - \mu_j}, \quad \forall \ i \neq j. \tag{3.7}
\]
Here, \(\tilde{\Theta}_{\alpha\beta} := \tilde{q}_{\alpha,\beta} - \tilde{q}_{\beta,\alpha}, \Theta_{ij} := q_{i,j} - q_{j,i}.\) The compatibility condition of (3.6) is given by
\[
\Theta_{ij,k} + \Theta_{jk,i} + \Theta_{ki,j} = 0, \quad \forall \ v(i, j, k) = \pm 1. \tag{3.8}
\]
Introduce the notations
\[
\omega_{ij} = \frac{\theta_{ij}}{\lambda_i - \lambda_j}, \quad i \neq j. \tag{3.9}
\]
Then, Eq. (3.7) imply
\[ \partial_i [\omega_{ij} (\mu_j - \mu_i)] + \partial_j [\omega_{jk} (\mu_j - \mu_k)] + \partial_k [\omega_{ki} (\mu_k - \mu_i)] = 0, \]
i.e.,
\[ \omega_{ij,k} (\mu_i - \mu_j) + \omega_{ij} (\mu_i,k - \mu_j,k) + \text{cyclic} = 0, \quad \forall \varepsilon(i, j, k) = \pm 1. \] (3.9)
Substituting Eqs. (2.6), (2.7) in Eq. (3.9), we obtain
\[ \omega_{ij,k} (\mu_i - \mu_j) + \omega_{ij} (a_{ik} (\mu_i - \mu_k) - a_{jk} (\mu_j - \mu_k)) + \text{cyclic} = 0, \]
(3.10)
from which we obtain that for any pairwise distinct \( i, j, k \),
\[ (\mu_i - \mu_k) (\omega_{ij,k} + \omega_{ij} a_{ik} - \omega_{jk} a_{ji}) + \text{cyclic} = 0. \] (3.11)
As a result, we conclude that
\[ \omega_{ij,k} + \omega_{ij} a_{ik} - \omega_{jk} a_{ji} + \text{cyclic} = 0, \quad \forall \varepsilon(i, j, k) = \pm 1, \] (3.12)
\[ -\omega_{ij,k} - \omega_{ij} a_{jk} + \omega_{jk,i} + \omega_{jk} a_{ji} + \text{cyclic} = 0, \quad \forall \varepsilon(i, j, k) = \pm 1. \] (3.13)
This arguments above can be reversed to get (3.7). We therefore conclude that integrability at the first order of approximation is equivalent to (1.8).

Let us now consider the condition of (quasi-)triviality at the first order of approximation. The Hamiltonian perturbation (3.1) is quasi-trivial at the first-order approximation, if there exists a local functional
\[ K_0 = \int_{S^1} k_0(v) \, dx \]
such that
\[ \{H_0, K_0\} = H_1. \] (3.14)
Clearly, quasi-triviality at the first-order approximation is the same as triviality at the first-order approximation. Equation (3.14) is equivalent to the existence of a function \( \psi \) satisfying
\[ \tilde{p}_\alpha = \frac{\partial k_0}{\partial u^\gamma} A_\alpha^\gamma + \frac{\partial \psi}{\partial u^\alpha}. \] (3.15)
Eliminating \( \psi \) in the above equation we find the following equivalent equation to (3.14):
\[ \tilde{\theta}_{\alpha\beta} = \frac{\partial^2 k_0}{\partial u^\gamma \partial u^\gamma} A_\alpha^\gamma - \frac{\partial^2 k_0}{\partial u^\alpha \partial u^\beta} A_\beta^\gamma. \] (3.16)
In the coordinate chart of Riemann invariants, Eqs. (3.15) and (3.16) become
\[ p_i = \lambda_i k_{0,i} + \psi_{,i}, \] (3.17)
\[ \frac{\theta_{ij}}{\lambda_i - \lambda_j} = k_{0,ij} + a_{ij} k_{0,i} + a_{ji} k_{0,j}, \quad i \neq j. \] (3.18)
The compatibility condition of Eq. (3.18) is given by

$$\partial k_{0,ij} = \partial_j k_{0,ik}, \quad \forall \varepsilon(i, j, k) = \pm 1,$$

which yields

$$\partial \left( \frac{\theta_{ij}}{\lambda_i - \lambda_j} - a_{ij} k_{0,i} - a_{ji} k_{0,j} \right) = \partial_j \left( \frac{\theta_{ik}}{\lambda_i - \lambda_k} - a_{ik} k_{0,i} - a_{ki} k_{0,k} \right).$$  \tag{3.19}$$

Substituting Eq. (3.18) into (3.19), we find

$$\omega_{ij,k} - a_{ij} \omega_{ik} - a_{ji} \omega_{jk} - k_{0,i}(a_{ji} a_{jk} - a_{ij,k}) + k_{0,k}(a_{ki} a_{kj} - a_{ki,j})$$
$$+ k_{0,j}(a_{ji} a_{ik} + a_{jk} a_{ki}) = \omega_{ik,j} - a_{ik} \omega_{ij} - a_{ki} \omega_{kj} - k_{0,k}(a_{ki} a_{kj} - a_{ki,j}) + k_{0,j}(a_{ji} a_{ik} + a_{jk} a_{ki}).$$  \tag{3.20}$$

Finally substituting Eqs. (2.9) and (2.10) into (3.20), we have

$$\omega_{ij,k} - a_{ij} \omega_{ik} - a_{ji} \omega_{jk} = \omega_{ik,j} - a_{ik} \omega_{ij} - a_{ki} \omega_{kj}, \quad \forall \varepsilon(i, j, k) = \pm 1.$$  \tag{3.21}$$

The procedure can again be reversed. So we proved the equivalence between (1.8) and triviality at the first-order approximation. The theorem is proved. \quad \Box

We proceed with the second-order approximation. Let

$$H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \mathcal{O}(\epsilon^3)$$  \tag{3.22}$$

be a Hamiltonian perturbation of (1.1) with

$$H_2 = \int_{S^1} \tilde{\alpha}_{\alpha\beta}(v) v_x^\alpha v_x^\beta \, dx = \int_{S^1} \sum_{i,j=1}^n d_{ij} R_{ix} R_{jx} \, dx$$  \tag{3.23}$$

and

$$\tilde{\alpha}_{\alpha\beta} = \tilde{\alpha}_{\beta\alpha}, \quad d_{ij} = d_{ji} := \tilde{\alpha}_{\alpha\beta} v_x^\alpha v_x^\beta.$$  \tag{3.24}$$

Assume as always that (1.1) is integrable, and assume that (3.22) is $\mathcal{O}(\epsilon^1)$-integrable. According to Theorem 1.1, there exists a canonical Miura-type transformation reducing $H_1$ to the zero functional. So the assumption that $H_1 = 0$ used in (1.10) in the statement of Theorem 1.1 does not lose generality as we already pointed it out in the Introduction.

**Proof of Theorem 1.2.** The proof will be given with the following order: firstly, we show that $\mathcal{O}(\epsilon^2)$-integrability implies (1.11)–(1.12); secondly, we show that (1.11)–(1.12) is equivalent to quasi-triviality at the second-order approximation; thirdly, we show that quasi-triviality implies $\mathcal{O}(\epsilon^2)$-integrability.

Assume that (3.22) with $H_1 = 0$ is $\mathcal{O}(\epsilon^2)$-integrable. This means that, for a generic conservation law $F_0$ of (1.1), there exists a local functional of the form

$$F_2 = \int_{S^1} \tilde{D}_{\alpha\beta}(u) v_x^\alpha u_x^\beta \, dx = \sum_{i,j=1}^n \int_{S^1} D_{ij}(R) R_{ix} R_{jx} \, dx$$  \tag{3.25}$$

such that

$$\{H_0, F_2\} + \{H_2, F_0\} = 0.$$  \tag{3.26}$$
Note that equation (3.26) implies
\begin{align}
M_\rho^\alpha \tilde{d}_\rho^\beta - M_\beta^\alpha \tilde{d}_\rho^\sigma = A_\rho^\alpha \tilde{D}_\rho^\beta - A_\beta^\alpha \tilde{D}_\rho^\sigma, \\
M_\gamma^\alpha \tilde{d}_\rho^\beta,\gamma + M_\rho^\alpha \tilde{d}_\rho^\beta,\gamma - M_\sigma^\alpha \tilde{d}_\rho^\gamma,\sigma - M_\rho^\alpha \tilde{d}_\rho^\gamma,\sigma = -M_\rho^\alpha \tilde{d}_\rho^\beta - M_\sigma^\alpha \tilde{d}_\rho^\gamma - M_\beta^\alpha \tilde{d}_\rho^\sigma \\
= (M \leftrightarrow A, d \leftrightarrow D). 
\end{align}
(3.27)

In the coordinate system of the complete Riemann invariants, (3.27) and (3.28) become
\begin{align}
\frac{D_{ij}}{\mu_i - \mu_j} = \frac{d_{ij}}{\lambda_i - \lambda_j}, \quad \forall \ i \neq j, \\
\lambda_{i,i}D_{ij} + \lambda_{j,i}D_{ji} + \lambda_{i,j}D_{il} + (\lambda_i - \lambda_l)D_{li,j} \\
+ (\lambda_j - \lambda_l)D_{li,j} + (\lambda_i - \lambda_l)D_{ij,l} \\
= \mu_{i,i}d_{ij} + \mu_{j,i}d_{ji} + \mu_{i,j}d_{il} + (\mu_i - \mu_l)d_{li,j} + (\mu_j - \mu_l)d_{li,j} \\
+ (\mu_i - \mu_j)d_{ij,l}, \quad \forall \ i, j, l. 
\end{align}
(3.29)

Here, in the derivation of (3.30), we have used (3.29).

Taking \( j = l = i \) in (3.30), we obtain
\begin{align}
\lambda_{i,i}D_{ii} = \mu_{i,i}d_{ii}. 
\end{align}
(3.31)

By assumption, in the subset \( U \) of \( M \), \( \lambda_i \) satisfy \( \lambda_i \neq 0 \). Thus, there exist functions \( C_i(R) \) such that
\begin{align}
D_{ii} = -C_i(R)\mu_{i,i}, \quad d_{ii} = -C_i(R)\lambda_{i,i}. 
\end{align}
(3.32)

Taking \( l = j \) and \( i \neq j \) in (3.30), we find
\begin{align}
\lambda_{j,j}D_{jj} + (\lambda_i - \lambda_j)D_{jj,i} = \mu_{j,j}d_{jj} + (\mu_j - \mu_i)d_{jj,i}, \quad \forall \ j \neq i. 
\end{align}
(3.33)

Substituting (3.32) into (3.33) and using (2.9) we obtain
\begin{align}
C_{j,i}((\lambda_i - \lambda_j)\mu_{i,j} - (\mu_i - \mu_j)\lambda_{j,i}) = 0, \quad \forall \ j \neq i, 
\end{align}
(3.34)

which implies
\begin{align}
C_{j,i} = 0, \quad \forall \ j \neq i, 
\end{align}
i.e.,
\begin{align}
C_j(R) = C_j(R_j). 
\end{align}

Taking \( l = i \) and \( j \neq i \) in (3.30) and using (3.31), (3.33), we find
\begin{align}
\lambda_{i,i}D_{ij} + (\lambda_i - \lambda_j)D_{ij,i} = \mu_{i,i}d_{ij} + (\mu_i - \mu_j)d_{ij,i}. 
\end{align}
(3.35)

Taking \( j = i \) and \( l \neq i \) in (3.30) and using (3.33), we find
\begin{align}
\lambda_{i,i}D_{ii} + (\lambda_i - \lambda_l)D_{ii,i} = \mu_{i,i}d_{ii} + (\mu_i - \mu_l)d_{ii,i}, 
\end{align}
(3.36)

which coincides with (3.35). It is straightforward to check that (3.29) and (2.9) imply (3.35). So (3.35) does not give new constraints to \( d_{ij}, i \neq j \).

Now we use (3.30) with \( \varepsilon(i, j, l) = \pm 1 \). First, by (3.29) it is convenient to write
\begin{align}
D_{ij} = s_{ij}(\mu_i - \mu_j), \quad d_{ij} = s_{ij}(\lambda_i - \lambda_j), \quad i \neq j, 
\end{align}
(3.37)
where \(s_{ij}\) are some anti-symmetric fields. Substituting (3.37) in (3.30) and using (2.9), we obtain
\[
(s_{lj,i} + s_{ji,l} + s_{il,j})(\lambda_i - \lambda_l)(\mu_j - \mu_l) - (\lambda_j - \lambda_l)(\mu_i - \mu_l) = 0, \quad \forall \varepsilon(i, j, l) = \pm 1.
\]
(3.38)

Hence,
\[
s_{lj,i} + s_{ji,l} + s_{il,j} = 0, \quad \forall \varepsilon(i, j, l) = \pm 1.
\]
(3.39)

This proves (1.11)–(1.12).

We now consider the condition of quasi-triviality for (3.22) with \(H_1 = 0\). Such a perturbation is called \emph{quasi-trivial} if there exists a local functional \(K\) of the form
\[
K = \epsilon K_1 + O(\epsilon^2), \quad K_1 = \int_{S^1} k_1(u; u_x) \, dx,
\]
(3.40)
such that
\[
H_0 + \epsilon \{H_0, K\} = H.
\]
(3.41)

Here, \(k_1\) is also required to satisfy the following homogeneity condition:
\[
\sum_{r \geq 1} r u_\beta^\alpha \frac{\partial}{\partial u_\rho^\alpha} \left( \frac{\partial k_1}{\partial u_\beta} - \partial_x \left( \frac{\partial k_1}{\partial u_x^\beta} \right) \right) = \frac{\partial k_1}{\partial u_\beta} - \partial_x \left( \frac{\partial k_1}{\partial u_x^\beta} \right).
\]
(3.42)

(The above (3.40)–(3.42) is the precise definition used in this paper for quasi-triviality at the second-order approximation.)

Equation (3.42) is equivalent to the following linear PDE system:
\[
u_\alpha^\beta \nu_\beta^\alpha u_\rho^\gamma + k_1 u_\alpha^\beta u_\rho^\gamma = 0,
\]
(3.43)
\[
u_\alpha^\beta k_1 u_\alpha^\beta u_\rho^\gamma - \nu_\alpha^\beta k_1 u_\rho^\gamma u_\alpha^\beta - k_1 u_\rho^\gamma = 0.
\]
(3.44)

From Eq. (3.41), we obtain
\[
\{H_0, K_1\} = H_2,
\]
which is equivalent to
\[
\delta \frac{\delta}{\delta u^\rho(x)} \left( H_2 + \int_{S^1} \frac{\delta k_1}{\delta u^\rho(x)} A_\gamma^\alpha u_\gamma^x \, dx \right) = 0.
\]
(3.45)

Eq. (3.45) read more explicitly as follows:
\[
\sum_{j=0}^{2} (-1)^j \frac{\partial}{\partial u^\rho} \left[ \delta_{\alpha\beta} u_\alpha^\beta u_\rho^\gamma + A_\gamma^\alpha u_\gamma^x \left( \frac{\partial k_1}{\partial u^\alpha} - \partial_x \left( \frac{\partial k_1}{\partial u_x^\alpha} \right) \right) \right] = 0.
\]
(3.46)

Comparing the coefficients of \(u_\sigma^{xxx}\) of both sides of Eq. (3.46) gives
\[
A_\rho^\alpha k_1 u_\rho^\gamma u_\alpha^\beta = A_\rho^\alpha k_1 u_\rho^\gamma u_\alpha^\beta.
\]
(3.47)

In terms of the Riemann invariants, Eq. (3.47) read
\[
\sum_{i \neq j} k_1 R_{i,x} R_{j,x} R_{i,\sigma} R_{j,\rho} (\lambda_j - \lambda_i) = 0,
\]
which imply
\[
k_1 R_{i,x} R_{j,x} = 0, \quad \forall i \neq j.
\]
(3.48)
Lemma 3.1. Up to a total $x$-derivative, $k_1$ must have the form

$$k_1 = \sum_{i=1}^{n} C_i(R_1, \ldots, R_n)R_{ix} \log R_{ix} - C_i(R_1, \ldots, R_n)R_{ix} + \phi_i(R_1, \ldots, R_n)R_{ix}$$

(3.49)

for some $C_i, \phi_i$. Moreover, if $k_1$ has the form (3.49) then it satisfies (3.43), (3.44), (3.47).

Proof. Eq. (3.48) imply that $k_1$ must have the variable separation form

$$k_1 = \sum_{i=1}^{n} B_i(R_1, \ldots, R_n; R_{ix}).$$

(3.50)

Noting that

$$k_{1,u_x} = \sum_{i=1}^{n} k_{1,R_{ix}R_i,\alpha},$$

$$k_{1,u_x^\alpha u_x^\beta} = \sum_{i,j=1}^{n} k_{1,R_{ix}R_{jx}R_i,\alpha R_j,\beta},$$

$$k_{1,u_x^\alpha u_x^\beta u_x^\gamma} = \sum_{i,j,k=1}^{n} k_{1,R_{ix}R_{jx}R_{kx}R_i,\alpha R_j,\beta R_k,\gamma}$$

and substituting Eq. (3.50) into Eq. (3.43), we obtain

$$R_{ix} B_{i,R_{ix}R_{ix}R_{ix}} + 2B_{i,R_{ix}R_{ix}} = 0.$$ 

If follows that

$$B_i = E_i(R) + \phi_i(R)R_{ix} + C_i(R)R_{ix} \log R_{ix} - C_i(R)R_{ix}$$

(3.51)

for some functions $C_i, \phi_i, E_i$. Finally, noticing that

$$k_{1,u^\alpha} = \sum_{i=1}^{n} \left( k_{1,R_i R_{ix}} + k_{1,R_{ix} R_i,\beta} u_x^\sigma \right),$$

$$k_{1,u_x^\alpha u_x^\beta} = \sum_{i,j=1}^{n} \left( k_{1,R_{ix} R_j R_i,\beta} + k_{1,R_{ix} R_j x R_i,\beta} u_x^\sigma \right) R_i,\alpha + \sum_{i=1}^{n} k_{1,R_{ix} R_i,\alpha \beta},$$

$$k_{1,u_x^\alpha u_x^\beta u_x^\gamma} = \sum_{i,j,k=1}^{n} \left( k_{1,R_{ix} R_j x R_k,\gamma} R_{k,\gamma} + k_{1,R_{ix} R_j x R_{kx} R_{k,\gamma} u_x^\sigma} + k_{1,R_{ix} R_j x R_{kx} R_{k,\gamma} u_x^\sigma} \right) R_i,\alpha R_j,\beta$$

and substituting (3.50), (3.51) into (3.44), we obtain

$$\partial_\beta \left( \sum_{i=1}^{n} E_i(R) \right) = 0,$$

(3.52)

which finishes the proof. \qed
Now collect the terms of (3.46) containing $u^\beta_x u^\sigma_x$:

$$u^\beta_x u^\sigma_x \left( A^\alpha_\rho \frac{\partial^3 k_1}{\partial u_x^\rho \partial u_x^\beta \partial u_x^\sigma} + A^\alpha_\beta \frac{\partial^3 k_1}{\partial u_x^\gamma \partial u_x^\beta \partial u_x^\sigma} - 2 A^\alpha_\sigma \frac{\partial^3 k_1}{\partial u_x^\gamma \partial u_x^\rho \partial u_x^\sigma} \right) = 0. \tag{3.53}$$

**Lemma 3.2.** If $k_1$ satisfies (3.48), then it automatically satisfies (3.53).

**Proof.** We have

LHS of (3.53)

$$= u^\beta_x u^\sigma_x \sum_{i,j,l=1}^n k_{1,R_{ix}} R_{i,x} R_{i,\alpha} (A^\alpha_\rho R_{i,\beta R_{j,\sigma}} + A^\alpha_\beta R_{i,\rho R_{j,\sigma}} - 2 A^\alpha_\sigma R_{i,\beta R_{j,\rho}})$$

$$= u^\beta_x u^\sigma_x \sum_{i=1}^n k_{1,R_{ix}} R_{i,x} R_{i,\alpha} (A^\alpha_\rho R_{i,\beta R_{i,\sigma}} + A^\alpha_\beta R_{i,\rho R_{i,\sigma}} - 2 A^\alpha_\sigma R_{i,\beta R_{i,\rho}})$$

$$= u^\beta_x u^\sigma_x \sum_{i=1}^n k_{1,R_{ix}} R_{i,x} \lambda_i (R_{i,\rho R_{i,\beta R_{i,\sigma}} + R_{i,\beta R_{i,\rho R_{i,\sigma}} - 2 R_{i,\sigma R_{i,\beta R_{i,\rho}}}) = 0. $$

The lemma is proved. \qed

Comparing the coefficients of $u^\beta_x$ of the both sides of (3.46) yields

$$2 A^\alpha_\rho k_{1,u^\alpha_x u^\rho_x u^\gamma_x} u^\gamma_x - A^\alpha_\beta k_{1,u^\alpha_x u^\beta_x u^\gamma_x} u^\gamma_x - 3 A^\alpha_\gamma k_{1,u^\alpha_x u^\gamma_x} u^\gamma_x - A^\alpha_\epsilon k_{1,u^\alpha_x u^\beta_x u^\epsilon_x u^\gamma_x}$$

$$+ A^\alpha_\beta (k_{1,u^\alpha_x} u^\rho_x - k_{1,u^\alpha_x} u^\alpha_x) + A^\alpha_\rho (k_{1,u^\alpha_x} u^\beta_x - k_{1,u^\alpha_x} u^\beta_x) - 2 \tilde{d}_{\rho \beta} = 0. \tag{3.54}$$

Substituting (3.49) into (3.54), we obtain the following lemma.

**Lemma 3.3.** The functions $C_i$ must satisfy

$$C_{i,j} = 0, \quad \forall \ i \neq j. \tag{3.55}$$

**Proof.** Noting that

$$k_{1,R_{ix}} = C_i \log R_{i,x} + \phi_i, $$

$$k_{1,R_{ix}} R_{j,x} = C_{i,j} \log R_{i,x} + \phi_{i,j},$$

$$k_{1,R_{ix}} R_{j,x} = C_{1} \delta_{ij} R_{i,x}^{-1},$$

we find that the only possible terms containing $\log R_{i,x}$ in Eq. (3.54) are

$$A^\alpha_\rho \left( k_{1,u^\alpha_x} u^\beta_x - k_{1,u^\alpha_x} u^\beta_x \right), \quad A^\alpha_\beta \left( k_{1,u^\alpha_x} u^\rho_x - k_{1,u^\alpha_x} u^\beta_x \right).$$

If follows that $\sum_{i,j=1}^n C_{i,j} (\lambda_i - \lambda_j) (R_{i,\beta R_{j,\rho}} + R_{i,\rho R_{j,\beta}}) \log R_{i,x} = 0$, which yields

$$\sum_{j \neq i} C_{i,j} (\lambda_i - \lambda_j) (R_{i,\beta R_{j,\rho}} + R_{i,\rho R_{j,\beta}}) = 0.$$

This gives (3.55). The lemma is proved. \qed
Lemma 3.4. The $\tilde{d}_{\alpha\beta}$ must have the form

$$
\tilde{d}_{\alpha\beta} = -\frac{1}{2} \sum_{i=1}^{n} C_i(R_i)(\lambda_{i,\alpha}R_{i,\beta} + \lambda_{i,\beta}R_{i,\alpha}) + \frac{1}{2} \sum_{i \neq j} s_{ij} (\lambda_i - \lambda_j) R_{i,\alpha}R_{j,\beta},
$$

where $s_{ij} = \phi_{i,j} - \phi_{j,i}$ for some functions $\phi_i$.

Proof. Using Eq. (3.54), we obtain

$$
2 \tilde{d}_{\alpha\beta} u_{\alpha}^{\alpha} u_{\beta}^{\beta} = -2 \sum_{i=1}^{n} C_i(R_i) \lambda_{i,x} R_{i,x} + \sum_{i,j=1}^{n} s_{ij} (\lambda_i - \lambda_j) R_{i,x} R_{j,x}. 
$$

(3.57)

The lemma is proved. □

Let us further show that the expression (3.56) is equivalent to the expression (1.13) (therefore is also equivalent to (1.11)–(1.12)). Indeed, in the coordinate chart of the complete Riemann invariants, (3.56) becomes

$$
d_{ij} = -\frac{1}{2} \left( C_i(R_i) \lambda_{i,j} + C_j(R_j) \lambda_{j,i} \right) + \frac{1}{2} \sum_{i \neq j} s_{ij} (\lambda_i - \lambda_j),
$$

(3.58)

where $s_{ij} = \phi_{i,j} - \phi_{j,i}$ for some functions $\phi_i$. It then suffices to show that $-\frac{1}{2} \sum_{i=1}^{n} C_i(R_i) \lambda_{i,j} + C_j(R_j) \lambda_{j,i}, \forall i \neq j$ can be absorbed into the term $\frac{1}{2} \sum_{i \neq j} s_{ij} (\lambda_i - \lambda_j)$. This is true because

$$
\partial_k \left( C_i(R_i) \lambda_{i,j} + C_j(R_j) \lambda_{j,i} \frac{\lambda_j - \lambda_k}{\lambda_k - \lambda_i} \right) + \partial_l \left( C_j(R_j) \lambda_{j,k} + C_k(R_k) \lambda_{k,j} \frac{\lambda_j - \lambda_k}{\lambda_k - \lambda_l} \right) + \partial_j \left( C_k(R_k) \lambda_{k,i} + C_i(R_i) \lambda_{i,k} \frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_k} \right) = 0, \quad \forall \varepsilon(i,j,k) = \pm 1. 
$$

(3.59)

Finally, let us check that equalities (3.46) hold true if $\tilde{d}_{\alpha\beta}$ and $k_1$ are given by (3.56) and (3.49). Collecting the rest terms of both sides of (3.46), we find that it suffices to show

$$
- (\tilde{d}_{\alpha\beta,\rho} u_{\rho}^{\alpha} u_{\rho}^{\beta} - 2 \tilde{d}_{\rho\beta,\gamma} u_{\rho}^{\alpha} u_{\gamma}^{\beta})
= A_{\alpha}^{\gamma} u_{\gamma}^{\gamma} \left( k_{1,u^\alpha u^\rho} - u_{\alpha}^{\rho} k_{1,u^\gamma u^\alpha} \right) - A_{\rho}^{\alpha} u_{\rho}^{\gamma} \left( k_{1,u^\gamma u^\alpha} - u_{\rho}^{\gamma} k_{1,u^\alpha u^\gamma} \right)
- A_{\gamma\beta}^{\rho} u_{\rho}^{\gamma} u_{\rho}^{\beta} \left( k_{1,u^\gamma u^\rho} - u_{\gamma}^{\rho} k_{1,u^\rho u^\gamma} \right) + A_{\gamma\rho}^{\alpha} u_{\rho}^{\gamma} u_{\alpha}^{\rho} \left( k_{1,u^\gamma u^\rho} - u_{\gamma}^{\rho} k_{1,u^\rho u^\gamma} \right),
$$

(3.60)

where $A_{\alpha\beta}^{\rho} := \eta^{\alpha\beta} \partial_{\rho} \partial_{\beta} (h)$. Indeed, the contribution of $\phi_i$-terms is just a result of canonical Miura-type transformation and note that Eq. (3.46) depend on $k_1$ linearly, so we can assume $\phi_i = 0$, $i = 1, \ldots, n$. Then, by straightforward calculations, we find that the both sides of Eq. (3.60) are equal to

$$
- \sum_{i=1}^{n} C_i(R_i) \left( \lambda_{i,\beta}R_{i,\rho} + \lambda_{i,\rho}R_{i,\beta} \right) u_{\rho}^{\beta}.
$$

Hence, we have proved that the Hamiltonian perturbation (3.22) is quasi-trivial at the second-order approximation if $\tilde{d}_{\alpha\beta}$ has the form (1.13).

We proceed with proving that quasi-triviality at the second-order approximation implies $O(\epsilon^2)$-integrability. We have shown that there exist functions
$C_i(R_i)$ and $\phi_i(R)$ such that Eqs. (3.56) hold true. And the quasi-triviality is generated by $\epsilon K_1 + \mathcal{O}(\epsilon^2)$:

$$K_1 = \int_{S^1} \sum_{i=1}^{n} C_i(R_i) R_{ix} \log R_{ix} - C_i(R_i) R_{ix} + \phi_i(R_1, \ldots, R_n) R_{ix} \, dx. \quad (3.61)$$

For a generic conservation law $F_0 = \int_{S^1} f_0(v) \, dx$ of (1.1), denote by $\mu_1, \ldots, \mu_n$ the distinct eigenvalues of the Hessian $(M^\alpha_\beta)$ of $f_0$. The calculations above can be applied to $F_0$, which give

$$F_2 := \{F_0, K_1\} = \int_{S^1} \left( -\sum_{i=1}^{n} C_i(R_i) \mu_i x R_{ix} + \frac{1}{2} \sum_{i \neq j} (\mu_i - \mu_j) s_{ij} R_{ix} R_{jx} \right) \, dx. \quad (3.62)$$

Then, using the Jacobi identity, we obtain $\{H_0, F_2\} + \{H_2, F_0\} = 0$. Hence, we have proved the $\mathcal{O}(\epsilon^2)$-integrability.

The theorem is proved.

\[\square\]

4. Example

The two component irrotational water wave equations in $1 + 1$ dimensions [1, 25] are given by

$$\int_{-\infty}^{\infty} e^{-ikx} \, dx \left\{ i \eta \cosh [k(1 + \mu \eta)] - \frac{q_x}{\epsilon} \sinh [k(1 + \mu \eta)] \right\} = 0, \quad (4.1)$$

$$q_t + \eta + \frac{\mu}{2} q_x^2 = \frac{\mu \epsilon^2}{2} \frac{(\eta + \mu q_x \eta_x)^2}{1 + \mu^2 \epsilon^2 \eta_x^2} + \frac{\sigma \epsilon^2 \eta_x}{(1 + \mu^2 \epsilon^2 \eta_x^2)^{3/2}}. \quad (4.2)$$

Here, $\mu$ and $\sigma$ are constants. For simplicity, we will only consider the case $\sigma \equiv 0$. Denote $r = 1 + \mu \eta$, $v = \mu q_x$. Then, we can rewrite (4.1)–(4.2) as the perturbation of a system of Hamiltonian PDEs of hydrodynamic type:

$$r_t = (1 + Q)^{-1} \sum_{j=1}^{\infty} \frac{(-1)^j \epsilon^{2j-2}}{(2j-1)!} \partial_x^{2j-1} (r^{2j-1} v), \quad (4.3)$$

$$v_t = -r_x - vv_x + \frac{\epsilon^2}{2} \partial_x \left( v r_x + (1 + Q)^{-1} \sum_{j=1}^{\infty} \frac{(-1)^j \epsilon^{2j-2}}{(2j-1)!} \partial_x^{2j} r^{2j-1} \right) \left(1 + \epsilon^2 r_x^2 \right)^{-1}, \quad (4.4)$$

where $Q$ is an operator defined by $Q := \sum_{j=1}^{\infty} \frac{(-1)^j \epsilon^{2j}}{(2j)!} \partial_x^{2j} \circ r^{2j}$. The dispersionless limit of (4.3)–(4.4) was studied by Whitham [24] and is integrable. Now we look at the second-order approximation of (4.3)–(4.4):

$$r_t = -(rv)_x + \epsilon^2 \left( -r^2 r_x v_x - \frac{1}{3} r^3 v_{xx} \right)_x + \mathcal{O}(\epsilon^4), \quad (4.5)$$

$$v_t = -r_x - vv_x + \epsilon^2 \left( \frac{1}{2} r^2 v_x^2 \right)_x + \mathcal{O}(\epsilon^4). \quad (4.6)$$
This approximation has the Hamiltonian structure:

\[
(r_t, v_t)^T = \left( \begin{array}{cc} 0 & \partial_x \\ \partial_x & 0 \end{array} \right) \left( \frac{\delta H}{\delta r(x)}, \frac{\delta H}{\delta v(x)} \right)^T,
\]

\[H = H_0 + \epsilon^2 H_2 + O(\epsilon^3),\]  
(4.7)

\[H_0 = -\int_{S^1} \frac{1}{2} r v^2 + \frac{r^2}{2} \, dx, \quad H_2 = \int_{S^1} \frac{1}{6} r^3 v_x^2 \, dx.\]  
(4.8)

**Proposition 4.1.** The system (4.3)–(4.4) is not integrable in the sense of Definition 2.2.

**Proof.** The Riemann invariants are

\[R_1 = v + \sqrt{r}, \quad R_2 = v - \sqrt{r}.\]

And the eigenvalues are

\[\lambda_1 = -v - \sqrt{r} = -\frac{3}{2} R_1 - \frac{1}{2} R_2, \quad \lambda_2 = -v + \sqrt{r} = -\frac{1}{2} R_1 - \frac{3}{2} R_2.\]  
(4.10)

This gives \(\lambda_{1,1} = \lambda_{2,2} = -3/2\). According to Theorem 1.2, the perturbation (4.8) is quasi-trivial at the second-order approximation iff the following equation has a solution:

\[- \left((R_1 - R_2)(\phi_{1,2} - \phi_{2,1})\right) R_{1x} R_{2x} \]

\[+ \frac{3}{2} C_1(R_1) R_1^2 + \frac{3}{2} C_2(R_2) R_2^2 = \frac{(R_1 - R_2)^6}{384} (R_{1x} + R_{2x})^2.\]

However, the solution set to this equation is empty as the coefficients of \(R_1^2\) on the both sides already produce a contradiction. The proposition is proved. \(\square\)

Let us provide additional but more straightforward evidence supporting the already proved statement of Proposition 4.1. It is easy to verify that up to the second-order approximations, system (4.3)–(4.4) has four linearly independent conservation laws:

\[\int_{S^1} r \, dx, \quad \int_{S^1} v \, dx, \quad \int_{S^1} r v \, dx, \quad -H.\]

We will show these form all possible conservation laws in all-order for (4.3)–(4.4). (They are actually indeed conservation laws all-order, but we do not prove this in the present paper; instead we refer to [1,3,25].) The precise statement that we will now prove is that only the following four conservation laws of the dispersionless limit of (4.3)–(4.4)

\[\int_{S^1} r \, dx, \quad \int_{S^1} v \, dx, \quad \int_{S^1} r v \, dx, \quad \int_{S^1} \frac{1}{2} r v^2 + \frac{r^2}{2} \, dx\]  
(4.11)

can be extended to conservation laws at the second-order approximation for (4.3)–(4.4). To see this, denote \(u^1 = r, u^2 = v\), and let

\[F = F_0 + \epsilon^2 F_2 + O(\epsilon^3) = \int_{S^1} f(u) \, dx + \epsilon^2 \int_{S^1} D_{\alpha\beta}(u) u^\alpha_x u^\beta_x + O(\epsilon^3)\]
be a conserved quantity of (4.3)–(4.4) at the second-order approximation. Then, we have

\[ f_{vv} = r f_{rr}, \]  
\[ \mu_1 = f_{rv} - \sqrt{r} f_{rr}, \quad \mu_2 = f_{rv} + \sqrt{r} f_{rr}, \]  
\[ d_{11} = d_{22} = \frac{1}{384} (r_1 - r_2)^2, \]  
\[ D_{11} = -\frac{\partial R_1(\mu_1)}{576} (r_1 - r_2)^6, \quad D_{22} = -\frac{\partial R_2(\mu_2)}{576} (r_1 - r_2)^6. \]

Substituting these equations in (3.33) and using (4.12), we find \( f_{rrv} = 0 \). It yields five solutions:

\[ f = r, \quad f = v, \quad f = rv, \quad f = \frac{1}{2} r v^2 + \frac{1}{2} r^2, \quad f = \frac{v^2}{2} + r \log r. \]  

However, through one by one verifications, only the first four can be (and are indeed) extended to the second-order approximation.

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