MASSLESS ELEMENTARY PARTICLES IN A QUANTUM THEORY OVER A GALOIS FIELD

F. M. Lev

(E-mail: felixlev@hotmail.com)

We consider massless elementary particles in a quantum theory based on a Galois field (GFQT). We previously showed that the theory has a new symmetry between particles and antiparticles, which has no analogue in the standard approach. We now prove that the symmetry is compatible with all operators describing massless particles. Consequently, massless elementary particles can have only the half-integer spin (in conventional units), and the existence of massless neutral elementary particles is incompatible with the spin-statistics theorem. In particular, this implies that the photon and the graviton in the GFQT can only be composite particles.

Keywords: Galois fields, massless particles, modular representations

1 Introduction

At high energies any particle can be created and annihilated by other particles in different reactions. For this reason the property of a particle to be elementary or composite has no clear experimental meaning. However, in theory this property is well defined. By definition, a particle is called elementary if the full set of its wave functions forms a space of irreducible representation (IR) for the symmetry group or algebra in the given theory. Such an approach has been first proposed by Wigner in Ref. [1] where unitary IRs of the Poincare group have been constructed.

In the standard approach to quantum theory each elementary particle either has or does not have the corresponding antiparticle with
the same mass and spin. The latter case can also be treated in such a way that the particle and its antiparticle are the same. Elementary particles with such a property are called neutral.

Let us briefly discuss how the standard theory explains the existence of antiparticles. Consider, for example, the electron and the positron which are the antiparticles for each other. The explanation is based on the fact that the Dirac equation has solutions with both positive and negative energies. As noted by Dirac (see e.g. his Nobel lecture [2]), the existence of the negative energy solutions represents a difficulty which should be resolved. In the standard approach the solution is given in the framework of second quantization such that the creation and annihilation operators for the positron have the usual meaning but they enter the quantum Lagrangian with the coefficients representing the negative energy solutions. This is an implementation of the idea that the creation or annihilation of an antiparticle can be treated respectively, as the annihilation or creation of the corresponding particle with the negative energy. However, since negative energies have no direct physical meaning in the standard theory, this idea is implemented implicitly rather than explicitly. Note also that the electron and the positron are described by unitary IRs with positive energies, but these representations are fully independent. At the same time, IRs with negative energies are not used at all.

In papers [3] we have proposed an approach to quantum theory where the wave functions of the system under consideration are described by elements of a linear space over a Galois field, and the operators of physical quantities - by linear operators in this space. A detailed discussion of this approach has been given in a recent paper [4]. In particular, it has been shown that at some conditions such a description gives the same predictions as the standard approach. It has also been argued that the description of quantum systems in terms of Galois fields is more natural than the standard description in terms of complex numbers.

The first obvious conclusion about quantum theory based on a Galois field (GFQT) is as follows: since any Galois field has only a
finite number of elements, in the GFQT divergencies cannot exist in principle, and all operators are automatically well defined. It is also natural to expect that, since arithmetic of Galois field differs from the standard one, the GFQT has some properties which have no analog in the standard theory.

In particular, as shown in Ref. [4], in contrast to the standard approach, where a particle and its antiparticle are described by independent IRs of the symmetry group, in the GFQT a particle and its antiparticle are described by the same IR of the symmetry algebra. This automatically explains the existence of antiparticles and shows that a particle and its antiparticle represent different states of the same object. As a consequence, the GFQT possesses a new symmetry between particles and antiparticles, which has no analog in the standard quantum theory. Also the problem arises of whether neutral particles can be elementary or only composite.

The problem of existence of neutral elementary particles is of greatest interest for massless particles, e.g. for the photon and the graviton. For this reason, in the present paper (see Sects. 2 and 3) we consider the massless case. In Sect. 4 the new symmetry is described in detail, and in Sect. 5 it is shown that the vacuum condition is consistent only for particles with the half-integer spin (in conventional units). In Sect. 6 we prove that the symmetry is compatible with all the representation operators for massless particles. In Sect. 7 it is shown that, as a consequence, in the massless case the existence of massless neutral elementary particles in the GFQT is incompatible with the standard relation between spin and statistics.

Although the notion of the Galois field is extremely simple and elegant, the majority of physicists is not familiar with this notion. For this reason, in Ref. [4] an attempt has been made to explain the basic facts about Galois fields by using arguments which, hopefully, can be accepted by physicists. The readers who are not familiar with Galois fields can also obtain basic knowledge from the standard textbooks (see e.g. Refs. 5).
2 Representation operators of the anti de Sitter algebra

If a conventional quantum theory has a symmetry group (or algebra), then there exists a unitary representation of the group (or a representation of the algebra by Hermitian operators) in the Hilbert space describing the quantum system under consideration. In the present paper we assume that the symmetry algebra is the Galois field analog of the anti de Sitter (AdS) algebra so(2,3), and quantum systems are described by representations of this algebra in spaces over a Galois field (see Ref. [4] for details). The standard AdS group is ten-parametric, as well as the Poincare group. However, in contrast to the Poincare group, all the representation generators are angular momenta. In Ref. [4] we explained the reason why for our purposes it is convenient to work with the units $\hbar/2 = c = 1$. Then the representation generators are dimensionless, and the commutation relations for them can be written in the form

$$[M^{ab}, M^{cd}] = -2i(g^{ac}M^{bd} + g^{bd}M^{cd} - g^{ad}M^{bc} - g^{bc}M^{ad})$$

where $a, b, c, d$ take the values 0,1,2,3,5, and the operators $M^{ab}$ are antisymmetric. The diagonal metric tensor has the components $g^{00} = g^{55} = -g^{11} = -g^{22} = -g^{33} = 1$. In these units the spin of fermions is odd, and the spin of bosons is even. If $s$ is the particle spin then the corresponding IR of the su(2) algebra has the dimension $s + 1$. Note that if $s$ is interpreted in such a way then it does not depend on the choice of units (in contrast to the maximum eigenvalue of the $z$ projection of the spin operator).

For analyzing IRs implementing Eq. (1), it is convenient to work with another set of ten operators. Let $(a_j', a_j'', h_j)$ ($j = 1,2$) be two independent sets of operators satisfying the commutation relations

$$[h_j, a_j'] = -2a_j' \quad [h_j, a_j''] = 2a_j'' \quad [a_j', a_j''] = h_j$$

The sets are independent in the sense that for different $j$ they mutually commute with each other. We denote additional four operators
as $b', b'', L_+, L_-$. The meaning of $L_+, L_-$ is as follows. The operators $L_3 = h_1 - h_2, L_+, L_-$ satisfy the commutation relations of the su(2) algebra

$$ [L_3, L_+] = 2L_+ \quad [L_3, L_-] = -2L_- \quad [L_+, L_-] = L_3 $$

while the other commutation relations are as follows

$$ [a'_1, b'] = [a'_2, b'] = [a''_1, b''] = [a''_2, b''] = 0 $$
$$ [a'_1, L_-] = [a''_1, L_+] = [a'_2, L_+] = [a''_2, L_-] = 0 $$
$$ [h_j, b'] = -b' \quad [h_j, b''] = b'' \quad [h_1, L_+] = \pm L_\pm, \quad [h_2, L_\pm] = \mp L_\pm $$
$$ [b', L_-] = 2a'_1 \quad [b', L_+] = 2a'_2 \quad [b'', L_-] = -2a''_2 $$
$$ [b'', L_+] = -2a''_1, \quad [a'_1, b''] = [b', a''_1] = L_- $$
$$ [a'_2, b''] = [b', a''_1] = L_+, \quad [a'_1, L_+] = [a'_2, L_-] = b' $$
$$ [a''_2, L_+] = [a''_1, L_-] = -b'' $$

At first glance, these relations might seem rather chaotic, but they are in fact very natural in the Weyl basis of the so(1,4) algebra.

The relation between the above sets of ten operators is as follows

$$ M_{10} = i(a''_1 - a'_1 - a''_2 + a'_2) \quad M_{15} = a''_2 + a'_2 - a''_1 - a'_1 $$
$$ M_{20} = a''_1 + a''_2 + a'_1 + a'_2 \quad M_{25} = i(a''_1 + a'_2 - a''_1 - a'_2) $$
$$ M_{12} = L_3 \quad M_{23} = L_+ + L_- \quad M_{31} = -i(L_+ - L_-) $$
$$ M_{05} = h_1 + h_2 \quad M_{35} = b' + b'' \quad M_{30} = -i(b'' - b') $$

In addition, if $^*$ is used to denote the Hermitian conjugation, $L_+^* = L_-$, $a'_j^* = a''_j$, $b'^* = b''$ and $h_j^* = h_j$ then the operators $M^{ab}$ are Hermitian (we do not discuss the difference between selfadjoined and Hermitian operators).

Let $p$ be a prime number, and $F_{p^2}$ be a Galois field containing $p^2$ elements. This field has only one nontrivial automorphism $a \rightarrow \bar{a}$ (see e.g. Refs. [5, 4]) which is the analog of complex conjugation in the field of complex numbers. The automorphism can be defined as
$a \rightarrow \bar{a} = a^p$ [5]. Our goal is to find IRs, implementing the commutation relations (2-4) in spaces over $F_{p^2}$. Representations in spaces over fields of nonzero characteristics are called modular representations. A review of the theory of modular IRs can be found e.g. in Ref. [6]. In the present paper we do not need a general theory since modular IRs in question can be constructed explicitly. A modular analog of the Hilbert space is a linear space $V$ over $F_{p^2}$ supplied by a scalar product $(\cdot,\cdot)$ such that for any $x, y \in V$ and $a \in F_{p^2}$, $(x, y) \in F_{p^2}$ and the following properties are satisfied:

$$(x, y) = (y, x), \quad (ax, y) = \bar{a}(x, y), \quad (x, ay) = a(x, y)$$

(6)

In the modular case * is used to denote the Hermitian conjugation, such that $(Ax, y) = (x, A^*y)$.

Eq. (2) defines the commutation relations for representations of the $sp(2)$ algebra. These representations play an important role in constructing modular IRs of the $so(2,3)$ algebra. For this reason, following Refs. [3, 4], we describe below modular IRs of the $sp(2)$ algebra such that the representation generators are denoted as $a', a'', h$.

The Casimir operator of the second order for the algebra (2) has the form

$$K = h^2 - 2h - 4a'' = h^2 + 2h - 4a'a''$$

(7)

We will consider representations with the vector $e_0$, such that

$$a'e_0 = 0, \quad he_0 = q_0 e_0, \quad (e_0, e_0) = 1$$

(8)

One can easily prove [3, 4] that $q_0$ is "real", i.e. $q_0 \in F_p$ where $F_p$ is the residue field modulo $p$: $F_p = Z/\mathbb{Z}p$ where $Z$ is the ring of integers. The field $F_p$ consists of $p$ elements and represents the simplest possible Galois field.

Denote $e_n = (a'')^n e_0$. Then it follows from Eqs. (7) and (8), that for any $n = 0, 1, 2, ...$

$$he_n = (q_0 + 2n)e_n, \quad Ke_n = q_0(q_0 - 2)e_n,$$

(9)
\[ a'a''e_n = (n + 1)(q_0 + n)e_n \]  \hspace{1cm} (10)

\[ (e_{n+1}, e_{n+1}) = (n + 1)(q_0 + n)(e_n, e_n) \]  \hspace{1cm} (11)

The case \( q_0 = 0 \) is trivial and corresponds to zero representation, so we assume that \( q_0 \neq 0 \). Then we have the case when ordinary and modular representations considerably differ each other. Consider first the ordinary case when \( q_0 \) is any real positive number. Then IR is infinite-dimensional, \( e_0 \) is a vector with a minimum eigenvalue of the operator \( h \) (minimum weight) and there are no vectors with the maximum weight. This is in agreement with the well known fact that unitary IRs of noncompact groups are infinite dimensional. However in the modular case \( q_0 \) is one of the numbers \( 1, \ldots, p-1 \). The set \( (e_0, e_1, \ldots e_N) \) will be a basis of IR if \( a''e_i \neq 0 \) for \( i < N \) and \( a''e_N = 0 \). These conditions must be compatible with \( a'a''e_N = 0 \). Therefore, as follows from Eq. (11), \( N \) is defined by the condition \( q_0 + N = 0 \) in \( F_p \). As a result, \( N = p - q_0 \) and the dimension of IR is equal to \( p - q_0 + 1 \).

One might say that \( e_0 \) is the vector with the minimum weight while \( e_N \) is the vector with the maximum weight. However the notions of ”less than” or ”greater than” have only a limited sense in \( F_p \), as well as the notion of positive and negative numbers in \( F_p \). If \( q_0 \) is positive in this sense (see Ref. \cite{4} for details), then Eqs. (8) and (9) indicate that the modular IR under consideration can be treated as the modular analog of IR with ”positive energies”. However it is easy to see that \( e_N \) is the eigenvector of the operator \( h \) with the eigenvalue \(-q_0 \) in \( F_p \), and the same IRs can be treated as the modular analog of IRs with ”negative energies” (see Ref. \cite{4} for details).

3 Massless modular representations of the AdS algebra

There exists a vast literature on ordinary IRs of the so(2,3) algebra in Hilbert spaces. The representations relevant for elementary particles in
the AdS space have been constructed for the first time in Refs. [7, 8], while modular representations of algebra (2-4) have been investigated for the first time by Braden [9]. In Refs. [3, 4] we have reformulated his investigation in such a way that the correspondence between modular and ordinary IRs are straightforward. Our construction is described below.

We use the basis in which the operators \((h_j, K_j) \ (j = 1, 2)\) are diagonal. Here \(K_j\) is the Casimir operator (7) for algebra \((a'_j, a''_j, h_j)\). By analogy with Refs. [8, 9] we introduce the operators

\[
B^{++} = b'' - a_1''L_-(h_1 - 1)^{-1} - a_2''L_+(h_2 - 1)^{-1} + a_1''a_2''b'[(h_1 - 1)(h_2 - 1)]^{-1} \\
B^{+-} = L_+ - a_1''b'(h_1 - 1)^{-1} \\
B^{-+} = L_- - a_2''b'(h_2 - 1)^{-1} \\
B^{--} = b'
\]

and consider their action only on the space of "minimal" \(sp(2) \times sp(2)\) vectors, i.e. such vectors \(x\) that \(a'_jx = 0\) for \(j = 1, 2\), and \(x\) is the eigenvector of the operators \(h_j\).

It is easy to see that if \(x\) is a minimal vector such that \(h_jx = \alpha_jx\) then \(B^{++}x\) is the minimal eigenvector of the operators \(h_j\) with the eigenvalues \(\alpha_j + 1\), \(B^{+-}x\) - with the eigenvalues \((\alpha_1 + 1, \alpha_2 - 1)\), \(B^{-+}x\) - with the eigenvalues \((\alpha_1 - 1, \alpha_2 + 1)\), and \(B^{--}x\) - with the eigenvalues \(\alpha_j - 1\).

By analogy with the construction of ordinary representations with positive energy [7, 8], we require the existence of the vector \(e_0\) satisfying the conditions

\[
a'_je_0 = b'e_0 = L_+e_0 = 0 \quad h_je_0 = q_je_0 \\
(e_0, e_0) = 1 \quad (j = 1, 2)
\]

In the ordinary case the massless IRs are characterized by the condition \(q_2 = 1\). In the modular case we have the same condition but now \(q_2 \in F_p\).

It is well known that \(M^{05} = h_1 + h_2\) is the AdS analog of the energy operator, since \(M^{05}/2R\) becomes the usual energy when the AdS group is contracted to the Poincare one (here \(R\) is the radius of the
AdS space). As follows from Eqs. (2) and (4), the operators \((a'_1, a'_2, b')\) reduce the AdS energy by two units. Therefore in the conventional theory \(e_0\) is the state with the minimum energy. In this theory the spin in our units is equal to the maximum value of the operator \(L_3 = h_1 - h_2\) in the ”rest state”. For this reason we use \(s\) to denote \(q_1 - q_2\). In our units \(s = 2\) for the photon, and \(s = 4\) for the graviton. Note that in contrast to the Poincare invariant theories, massless particles in the AdS case do have states which can be treated as rest ones (see below).

The problem arises how to define the action of the operators \(B^{++}\) and \(B^{-+}\) on \(e_0\) which is the eigenvector of the operator \(h_2\) with the eigenvalue \(q_2 = 1\). A possible way to resolve ambiguities \(0/0\) in matrix elements is to write \(q_2\) in the form \(q_2 = 1 + \epsilon\) and take the limit \(\epsilon \to 0\) at the final stage of computations. This confirms a well known fact that analytical methods can be very useful in problems involving only integers. At the same time, one can justify the results by using only integers (or rather elements of the Galois field in question), but we will not go into details.

By using the above prescription, we require that

\[
B^{++}e_0 = [b'' - a_1''L_-(h_1 - 1)^{-1}]e_0 \quad B^{-+}e_0 = L_-e_0
\]

if \(s \neq 0\) (and thus \(h_1 \neq 1\)), and

\[
B^{++}e_0 = b''e_0 \quad B^{-+}e_0 = B^{-+}e_0 = 0
\]

if \(s = 0\). As follows from the previous remarks, so defined operators transform minimal vectors to minimal ones, and therefore the element

\[
e_{nk} = (B^{++})^n(B^{-+})^ke_0
\]

is the minimal \(\text{sp}(2) \times \text{sp}(2)\) vector with the eigenvalues of the operators \(h_1\) and \(h_2\) equal to \(q_1 + n - k\) and \(q_2 + n + k\), respectively.

One can directly verify that, as follows from Eqs. (2-4)

\[
B^{-+}B^{++}(h_1 - 1) = B^{++}B^{-+}(h_1 - 2)
\]

\[
B^{+-}B^{++}(h_2 - 1) = B^{++}B^{+-}(h_2 - 2),
\]

(17)
and, in addition, as follows from Eq. (13) (see Ref. [4] for details)

\[
\begin{align*}
B^- e_{nk} &= a(n, k)e_{n-1,k} \\
B^+ e_{nk} &= b(n, k)e_{n,k-1}
\end{align*}
\]  \hspace{1cm} (18)

where

\[
\begin{align*}
a(n, k) &= \frac{n(q_1+q_2+n-3)(q_1+n-1)(q_2+n-2)}{(q_1+n-k-2)(q_2+n+k-2)} \\
b(n, k) &= \frac{k(s+1-k)(q_2+k-2)}{q_2+n+k-2}
\end{align*}
\]  \hspace{1cm} (19)

As follows from these expressions, the elements \(e_{nk}\) form a basis in the space of minimal \(\text{sp}(2) \times \text{sp}(2)\) vectors, and our next goal is to determine the range of the numbers \(n\) and \(k\).

Consider first the quantity \(b(0, k) = k(s+1-k)\) and let \(k_{\text{max}}\) be the maximum value of \(k\). For consistency we should require that if \(k_{\text{max}} \neq 0\) then \(k = k_{\text{max}}\) is the greatest value of \(k\) such that \(b(0, k) \neq 0\) for \(k = 1, \ldots k_{\text{max}}\). We conclude that \(k\) can take only the values of \(0, 1, \ldots s\).

Let now \(n_{\text{max}}(k)\) be the maximum value of \(n\) at a given \(k\). For consistency we should require that if \(n_{\text{max}}(k) \neq 0\) then \(n_{\text{max}}(k)\) is the greatest value of \(n\) such that \(a(n, k) \neq 0\) for \(n = 1, \ldots n_{\text{max}}(k)\). As follows from Eq. (19), in the massless case (when \(q_2 = 1\)) \(a(1, k) = 0\) for \(k = 1, \ldots s - 1\) if such values of \(k\) exist (i.e. when \(s \geq 2\)), and \(a(n, k) = n(s+n)\) if \(k = 0\) or \(k = s\). We conclude that at \(k = 1, \ldots s - 1\), the quantity \(n\) can take only the value \(n = 0\) while at \(k = 0\) or \(k = s\), the possible values of \(n\) are \(0, 1, \ldots n_{\text{max}}\) where \(n_{\text{max}} = p - s - 1\) (in contrast to the standard theory where \(n = 0, 1, \ldots \infty\)).

The full basis of the representation space can be chosen in the form

\[
e(n_1n_2nk) = (a_1^\alpha)^{n_1}(a_2^\alpha)^{n_2}e_{nk}
\]  \hspace{1cm} (20)

where, as follows from the results of the preceding section,

\[
\begin{align*}
n_1 &= 0, 1, \ldots N_1(n, k) \\
n_2 &= 0, 1, \ldots N_2(n, k) \\
N_1(n, k) &= p-q_1-n+k \\
N_2(n, k) &= p-q_2-n-k
\end{align*}
\]  \hspace{1cm} (21)

We conclude that, in contrast with the standard theory, where IRs of Lie algebras by Hermitian operators are necessarily infinite-
dimensional, massless modular IRs are finite-dimensional and even finite since the field $F_{p^2}$ is finite. This is in agreement with a general statement proved by Zassenhaus [10] that any modular IR is finite-dimensional.

Let us now discuss why IRs in question can be treated as massless. It is easy to see that

$$h_1 e(n_1 n_2 n k) = (q_1 + n - k + 2 n_1) e(n_1 n_2 n k)$$
$$h_2 e(n_1 n_2 n k) = (q_2 + n + k + 2 n_2) e(n_1 n_2 n k)$$
$$M^{05} e(n_1 n_2 n k) = (q_1 + q_2 + 2 n + 2 n_1 + 2 n_2) e(n_1 n_2 n k)$$ (22)

Therefore in the standard AdS theory the corresponding IR is characterized by the minimum AdS energy equal to $q_1 + q_2 = 2 q_2 + s$. Since in the usual case the mass is treated as the minimum energy, and the conventional energy is equal to $M^{05}/2R$, the conventional mass becomes zero when $q_2 = 1$ and $R \to \infty$. However this observation is still insufficient to conclude that $q_2 = 1$ is distinguished among other values of $q_2$ since $(2 q_2 + s)/2R \to 0$ when $R \to \infty$ if $q_2$ is any finite number. Let us recall that massless particles in conventional theory do not have ”rest states”, and for this reason the value of $s$ does not characterize the number of states in the corresponding IR of the su(2) algebra. Instead, massless particles are characterized by helicity which can have only two values: $s$ or $-s$. The AdS analog of this situation is that at $q_2 = 1$ and $n > 0$ there exist only the elements $e_{nk}$ with $k = 0$ and $k = s$. Only if $n = 0$, there exist the elements $e_{nk}$ with $k = 0, 1, \ldots s$. When the AdS algebra is contracted to the Poincare one (the meaning of contraction is well known [11]), the discrete spectrum becomes the continuous one, and the probability for a particle to have zero energy is negligible.

Taking into consideration the above remarks, in the literature the massless case is often characterized not only by the condition $q_2 = 1$, but also by the condition $s \geq 2$, since only in that case $1 \leq s - 1$. We assume only that $q_2 = 1$ while the spin can be arbitrary.
The above results can be summarized in the expressions

\[ B^{++}e_{nk} = e_{n+1,k} \quad (k = 0, s; \ n = 0, 1...n_{\text{max}} - 1) \]
\[ B^{--}e_{nk} = n(s + n)e_{n-1,k} \quad (k = 0, s; \ n = 1, ...n_{\text{max}}) \]
\[ B^{+-}e_{0k} = k(s + 1 - k)e_{0,k-1} \quad (k = 1, ...s) \]
\[ B^{-+}e_{0k} = e_{0,k+1} \quad (k = 0, 1, ...s - 1) \]  \hspace{1cm} (23)

while at other values of \( n \) and \( k \) the action of these operators on \( e_{nk} \) is equal to zero.

Our next task is to compute the quantities

\[ \text{Norm}(n_{1}n_{2}nk) = (e(n_{1}n_{2}nk), e(n_{1}n_{2}nk)). \]

By using Eqs. (11) and (23) one can show that

\[ \text{Norm}(n_{1}n_{2}nk) = F(n_{1}n_{2}nk)G(k), \quad \text{where} \]
\[ F(n_{1}n_{2}nk) = n_{1}!n_{2}!(n_{1} + n + s - k)!(n + n_{2} + k)! \]
\[ G(k) = s!/[(s - k)!]^2 \]  \hspace{1cm} (24)

In standard Poincare and AdS theories there also exist IRs with negative energies (as noted in Sect. 1, they are not used in the standard approach). They can be constructed by analogy with positive energy IRs. Instead of Eq. (13) one can require the existence of the vector \( e_{0}' \) such that

\[ a_{j}''e_{0}' = b''e_{0}' = L_{-}e_{0}' = 0 \quad h_{j}e_{0}' = -q_{j}e_{0}' \]
\[ (e_{0}', e_{0}') \neq 0 \quad (j = 1, 2) \]  \hspace{1cm} (25)

where the quantities \( q_{1}, q_{2} \) are the same as for positive energy IRs. It is obvious that positive and negative energy IRs are fully independent since the spectrum of the operator \( M^{05} \) for such IRs is positive and negative, respectively. At the same time, as shown in Ref. [4], \( the \ modular \ analog \ of \ a \ positive \ energy \ IR \ characterized \ by \ q_{1}, q_{2} \ in \ Eq. \ (13), \ and \ the \ modular \ analog \ of \ a \ negative \ energy \ IR \ characterized \ by \ the \ same \ values \ of \ q_{1}, q_{2} \ in \ Eq. \ (25) \ represent \ the \ same \ modular \ IR. \) Since this is the crucial difference between the standard quantum
theory and the GFQT, we give below the proof (which slightly differs from that in Ref. [4]).

Let $e_0$ be a vector satisfying Eq. (13). Denote $N_1 = p - q_1$ and $N_2 = p - q_2$. We will prove that the vector $x = (a_1'')^{N_1}(a_2'')^{N_2}e_0$ satisfies the conditions (25), i.e. $x$ can be identified with $e'_0$.

As follows from Eq. (9), the definition of $N_1, N_2$ and the results of the preceding section, the vector $x$ is the eigenvector of the operators $h_1$ and $h_2$ with the eigenvalues $-q_1$ and $-q_2$, respectively, and, in addition, it satisfies the conditions $a_1''x = a_2''x = 0$.

Let us now prove that $b''x = 0$. Since $b''$ commutes with the $a_j''$, we can write $b''x$ in the form

$$b''x = (a_1'')^{N_1}(a_2'')^{N_2}b''e_0$$

As follows from Eqs. (4) and (13), $a_2'b''e_0 = L_+e_0 = 0$ and $b''e_0$ is the eigenvector of the operator $h_2$ with the eigenvalue $q_2 + 1$. Therefore, $b''e_0$ is the minimal vector of the sp(2) representation which has the dimension $p - q_2 = N_2$. Therefore $(a_2'')^{N_2}b''e_0 = 0$ and $b''x = 0$.

The next stage of the proof is to show that $L_-x = 0$. As follows from Eq. (4) and the definition of $x$,

$$L_-x = (a_1'')^{N_1}(a_2'')^{N_2}L_-e_0 - N_1(a_1'')^{N_1-1}(a_2'')^{N_2}b''e_0$$

We have already shown that $(a_2'')^{N_2}b''e_0 = 0$, and therefore it suffice to prove that the first term in the r.h.s. of Eq. (27) is equal to zero. As follows from Eqs. (4) and (13), $a_2'L_-e_0 = b'e_0 = 0$, and $L_-e_0$ is the eigenvector of the operator $h_2$ with the eigenvalue $q_2 + 1$. Therefore $(a_2'')^{N_2}L_-e_0 = 0$ and we have proved that $L_-x = 0$.

The fact that $(x, x) \neq 0$ immediately follows from the definition of the vector $x$ and the results of the preceding section. Therefore the vector $x$ can be indeed identified with $e'_0$ and the above statement is proved.

The matrix elements of the operator $M^{ab}$ are defined as

$$M^{ab}e(n_1n_2nk) = \sum_{n_1'n_2'n'k'} M^{ab}(n_1'n_2'n'k', n_1n_2nk)e(n_1'n_2'n'k')$$

(28)
In the modular case the trace of each operator $M^{ab}$ is equal to zero. For the operators $(a'_j, a''_j, L_\pm, b'_j, b''_j)$ this is clear immediately: since they do not contain nonzero diagonal elements at all, they necessarily change one of the quantum numbers $(n_1 n_2 n k)$. The proof for the diagonal operators $h_1$ and $h_2$ is as follows. For each IR of the $sp(2)$ algebra with the minimal weight $q_0$ and the dimension $N + 1$, the eigenvalues of the operator $h$ are $(q_0, q_0 + 2, ... q_0 + 2N)$. The sum of these eigenvalues is equal to zero in $F_p$ since $q_0 + N = 0$ in $F_p$ (see the preceding section). Therefore we conclude that

$$
\sum_{n_1 n_2 n k} M^{ab}(n_1 n_2 n k, n_1 n_2 n k) = 0 \quad (29)
$$

This property is very important for investigating a new symmetry between particles and antiparticles in the GFQT (see Sect. [4]).

4 New symmetry between particles and antiparticles in GFQT

Since $(n_1 n_2 n k)$ is the complete set of quantum numbers for the elementary particle in question, we can define operators describing annihilation and creation of the particle in the states with such quantum numbers. Let $a(n_1 n_2 n k)$ be the operator of particle annihilation in the state described by the vector $e(n_1 n_2 n k)$. Then the adjoint operator $a(n_1 n_2 n k)^*$ has the meaning of particle creation in that state. Since we do not normalize the states $e(n_1 n_2 n k)$ to one (see the discussion in Ref. [4]), we require that the operators $a(n_1 n_2 n k)$ and $a(n_1 n_2 n k)^*$ should satisfy either the anticommutation relations

$$\{a(n_1 n_2 n k), a(n'_1 n'_2 n' k')^*\} =
Norm(n_1 n_2 n k)\delta_{n_1, n'_1} \delta_{n_2, n'_2} \delta_{n, n'} \delta_{k, k'} \quad (30)$$

or the commutation relation

$$[a(n_1 n_2 n k), a(n'_1 n'_2 n' k')^*] =
Norm(n_1 n_2 n k)\delta_{n_1, n'_1} \delta_{n_2, n'_2} \delta_{n, n'} \delta_{k, k'} \quad (31)$$
Then, taking into account the fact that the matrix elements satisfy the proper commutation relations, it is easy to demonstrate that the operators $M^{ab}$ in the secondly quantized form

$$M^{ab} = \sum \left\{ M^{ab}(n'_1n'_2n'k',n_1n_2nk) \right\}$$

satisfy the commutation relations in the form (1) or (2) if the $(a, a^*)$ operators satisfy either Eq. (30) or Eq. (31). Here and henceforth we use a convention that summation over repeated indices is implied.

In the standard theory, where the particle and its antiparticle are described by independent IRs, Eq. (32) describes either the quantized field for particles or antiparticles. To be precise, let us assume that the operators $a(n_1n_2nk)$ and $a(n_1n_2nk)^*$ are related to particles while the operators $b(n_1n_2nk)$ and $b(n_1n_2nk)^*$ satisfy the analogous commutation relations and describe the annihilation and creation of antiparticles. Then in the standard theory the operators of the quantized particle-antiparticle field are given by

$$M^{ab}_{\text{standard}} = \sum \left\{ M^{ab}_{\text{particle}}(n'_1n'_2n'k',n_1n_2nk) \right\}$$

$$a(n'_1n'_2n'k')^*a(n_1n_2nk)/\text{Norm}(n_1n_2nk) \right\} +$$

$$\sum \left\{ M^{ab}_{\text{antiparticle}}(n'_1n'_2n'k',n_1n_2nk) \right\}$$

$$b(n'_1n'_2n'k')^*b(n_1n_2nk)/\text{Norm}(n_1n_2nk) \right\}$$

where the quantum numbers $(n_1n_2nk)$ in each sum take the values allowable for the corresponding IR.

In contrast to the standard theory, Eq. (32) describes the quantized field for particles and antiparticles simultaneously. When the values of $(n_1n_2nk)$ are much less than $p$, the contribution of such values correctly describes particles (see Ref. [4] for details). The problem arises whether this expression correctly describes the contribution of antiparticles in the GFQT. Indeed, when the AdS energy is negative, the operator $a(n_1n_2nk)$ cannot be treated as the annihilation operator and $a(n_1n_2nk)^*$ cannot be treated as the creation operator.

Let us recall (see Sect. 3) that at any fixed values of $n$ and $k$, the quantities $n_1$ and $n_2$ can take only the values 0, 1...$N_1(n, k)$ and
0, 1...\(N_2(n, k)\), respectively (see Eq. (21)). We use \(Q_1(n, k)\) and \(Q_2(n, k)\) to denote \(q_1+n-k\) and \(q_2+n+k\), respectively. Then, as follows from Eq. (22), the element \(e(n_1n_2nk)\) is the eigenvector of the operators \(h_1\) and \(h_2\) with the eigenvalues \(Q_1(n, k) + 2n_1\) and \(Q_2(n, k) + 2n_2\), respectively. As follows from the results of Sect. 2, the first IR of the \(\text{sp}(2)\) algebra has the dimension \(N_1(n, k) + 1\) and the second IR has the dimension \(N_2(n, k) + 1\). If \(n_1 = N_1(n, k)\) then it follows from Eq. (22) that the first eigenvalue is equal to \(-Q_1(n, k)\) in \(F_p\), and if \(n_2 = N_2(n, k)\) then the second eigenvalue is equal to \(-Q_2(n, k)\) in \(F_p\). We use \(\tilde{n}_1\) to denote \(N_1(n, k) - n_1\) and \(\tilde{n}_2\) to denote \(N_2(n, k) - n_2\). Then it follows from Eq. (22) that \(e(\tilde{n}_1\tilde{n}_2nk)\) is the eigenvector of the operator \(h_1\) with the eigenvalue \(-(Q_1(n, k) + 2n_1)\) and the eigenvector of the operator \(h_2\) with the eigenvalue \(-(Q_2(n, k) + 2n_2)\).

In the GFQT the operators \(b(n_1n_2nk)\) and \(b(n_1n_2nk)^*\) cannot be independent of \(a(n_1n_2nk)\) and \(a(n_1n_2nk)^*\). The meaning of the operators \(b(n_1n_2nk)\) and \(b(n_1n_2nk)^*\) should be such that if the values of \((n_1n_2n)\) are much less than \(p\), these operators can be interpreted as those describing the annihilation and creation of antiparticles. Therefore it is reasonable to think that the operator \(b(n_1n_2nk)\) should be defined in such a way that it is proportional to \(a(\tilde{n}_1, \tilde{n}_2, n, k)^*\) and \(b(n_1n_2nk)^*\) should be defined in such a way that it is proportional to \(a(\tilde{n}_1, \tilde{n}_2, n, k)\). In this way we can directly implement the idea that the creation of the antiparticle with the positive energy can be described as the annihilation of the particle with the negative energy, and the annihilation of the antiparticle with the positive energy can be described as the creation of the particle with the negative energy. As noted in Sect. 1 in the standard theory this idea is implemented implicitly.

As follows from the well known Wilson theorem \((p - 1)! = -1\) in \(F_p\) (see e.g. [5]) and Eq. (24)

\[
F(n_1n_2nk)F(\tilde{n}_1\tilde{n}_2nk) = (-1)^s
\]  

(34)

We now define the \(b\)-operators as follows.

\[
a(n_1n_2nk)^* = \eta(n_1n_2nk)b(\tilde{n}_1\tilde{n}_2nk)/F(\tilde{n}_1\tilde{n}_2nk)
\]  

(35)
where $\eta(n_1n_2nk)$ is some function. Note that in the standard theory the CPT-transformation in Schwinger’s formulation transforms $a^*$ to $b$ (see e.g. Refs. [12, 13]), but in that case the both operators refer only to positive energies, in contrast to Eq. (35). In contrast to the standard CPT-transformation, where the sets $(a, a^*)$ and $(b, b^*)$ are independent, Eq. (35) represents not a transformation but a definition.

As a consequence of this definition,

$$a(n_1n_2nk) = \bar{\eta}(n_1n_2nk)b(\tilde{n}_1\tilde{n}_2nk)^*/F(\tilde{n}_1\tilde{n}_2nk)$$

$$b(n_1n_2nk)^* = a(\tilde{n}_1\tilde{n}_2nk)F(n_1n_2nk)/\bar{\eta}(\tilde{n}_1\tilde{n}_2nk)$$

$$b(n_1n_2nk) = a(\tilde{n}_1\tilde{n}_2nk)^*F(n_1n_2nk)/\eta(\tilde{n}_1\tilde{n}_2nk)$$  \hspace{1cm} (36)

Eqs. (35) and (36) define a possible symmetry when the set $(a, a^*)$ is replaced by the set $(b, b^*)$. Let us call it the AB symmetry. To understand whether it is indeed a new symmetry, we should investigate when so defined $(b, b^*)$ operators satisfy the same commutation or anticommutation relations as the $(a, a^*)$ operators, and whether the operators $M^{ab}$ written in terms of $(b, b^*)$ have the same form as in terms of $(a, a^*)$.

As follows from Eqs. (30) and (31), the $b$-operators should satisfy either

$$\{b(n_1n_2nk), b(n'_1n'_2n'k')^*\} =$$

$$\text{Norm}(n_1n_2nk)\delta_{n_1n'_1}\delta_{n_2n'_2}\delta_{nkn'}\delta_{kk'}$$

\hspace{1cm} (37)

in the case of anticommutators or

$$[b(n_1n_2nk), b(n'_1n'_2n'k')^*] =$$

$$\text{Norm}(n_1n_2nk)\delta_{n_1n'_1}\delta_{n_2n'_2}\delta_{nkn'}\delta_{kk'}$$

\hspace{1cm} (38)

in the case of commutators.

Now, as follows from Eqs. (24), (30), (34-36), Eq. (37) is satisfied if

$$\eta(n_1n_2nk)\bar{\eta}(n_1, n_2, nk) = (-1)^s$$  \hspace{1cm} (39)

At the same time, in the case of commutators it follows from Eqs. (24), (31) and (34,36) that Eq. (38) is satisfied if

$$\eta(n_1n_2nk)\bar{\eta}(n_1, n_2, nk) = (-1)^{s+1}$$  \hspace{1cm} (40)
We now represent $\eta(n_1n_2nk)$ in the form

$$\eta(n_1n_2nk) = \alpha f(n_1n_2nk)$$ (41)

where $f(n_1n_2nk)$ should satisfy the condition

$$f(n_1n_2nk)\bar{f}(n_1, n_2, nk) = 1$$ (42)

Then $\alpha$ should be such that

$$\alpha\bar{\alpha} = \pm(-1)^s$$ (43)

where the plus sign refers to anticommutators and the minus sign to commutators, respectively. If the spin-statistics theorem is satisfied, i.e. we have anticommutators for odd values of $s$ and commutators for even ones (this is the well known Pauli theorem in local quantum field theory [14]) then the r.h.s. of Eq. (43) is equal to -1.

Eq. (43) is a consequence of the fact that our basis is not normalized to one (see Ref. [4] for discussion). In the standard theory such a relation is impossible but if $\alpha \in F_p^2$, a solution of Eq. (43) exists. Indeed, we can use the fact that any Galois field is cyclic with respect to multiplication [5]. Let $r$ be a primitive root of $F_p^2$. This means that any element of $F_p^2$ can be represented as a power of $r$. As mentioned in Sect. 2, $F_p^2$ has only one nontrivial automorphism which is defined as $\alpha \rightarrow \bar{\alpha} = \alpha^p$. Therefore if $\alpha = r^k$ then $\alpha\bar{\alpha} = r^{(p+1)k}$. On the other hand, since $r^{(p^2-1)} = 1$, we conclude that $r^{(p^2-1)/2} = -1$. Therefore there exists at least a solution with $k = (p - 1)/2$.

5 Vacuum condition

Although we have called the sets $(a, a^*)$ and $(b, b^*)$ annihilation and creation operators for particles and antiparticles, respectively, it is not clear yet whether these operators indeed can be treated in such a way.

In the standard approach, this can be ensured by using the following procedure. One requires the existence of the vacuum vector $\Phi_0$ such that

$$a(n_1n_2nk)\Phi_0 = b(n_1n_2nk)\Phi_0 = 0 \quad \forall \ n_1, n_2, n, k$$ (44)
Then the elements
\[ \Phi_+(n_1n_2nk) = a(n_1n_2nk)^*\Phi_0 \quad \Phi_-(n_1n_2nk) = b(n_1n_2nk)^*\Phi_0 \quad (45) \]
have the meaning of one-particle states for particles and antiparticles, respectively.

However, if one requires the condition (44) in the GFQT then it is obvious from Eqs. (35) and Eq. (36), that the elements defined by Eq. (45) are null vectors. Note that in the standard approach the AdS energy is always greater than the mass while in the GFQT the AdS energy is not positive definite. We can therefore try to modify Eq. (44) as follows. Let us first break the set of elements \((n_1n_2nk)\) into two equal nonintersecting parts (defined later), \(S_+\) and \(S_-\), such that if \((n_1n_2nk) \in S_+\) then \((\tilde{n}_1\tilde{n}_2nk) \in S_-\). Then, instead of the condition (44) we require
\[ a(n_1n_2nk)\Phi_0 = b(n_1n_2nk)\Phi_0 = 0 \quad \forall (n_1, n_2, n, k) \in S_+ \quad (46) \]
In that case the elements defined by Eq. (45) will indeed have the meaning of one-particle states for \((n_1n_2nk) \in S_+\).

By analogy with the standard approach, we can try to define the set \(S_+\) such that for the corresponding values of \((n_1n_2nk)\) the AdS energy \(E = s + 2(n + n_1 + n_2 + 1)\) is positive. However, as already noted, the meaning of positive and negative is not quite clear in \(F_p\). We can treat the AdS energy as positive if all of the quantities \((nn_1n_2)\) are much less than \(p\) but in other cases such a treatment would be problematic. We believe that in modern physics there still exists a lack of understanding, to what extent the positivity of energy is important. For this reason our goal will be restricted to that of constructing the set \(S_+\) in a mathematically consistent way.

We will say that the AdS energy \(E\) is positive if \(E\) is one of the values \(1, 2, \ldots, (p-1)/2\) and negative if it is one of the values \(-1, -2, \ldots, -(p-1)/2\). If \(E\) is positive then we require that the corresponding element \((n_1n_2nk)\) belongs to \(S_+\), and if \(E\) is negative then the corresponding element \((n_1n_2nk)\) belongs to \(S_-\). The problem arises with such elements that the corresponding value of \(E\) is equal to zero in \(F_p\). Let us recall
that $E$ is the eigenvalue of $M^{05} = h_1 + h_2$, the eigenvalue of $h_1$ is equal to $E^{(1)} = 1 + s + n - k + 2n_1$ and the eigenvalue of $h_2$ is equal to $E^{(2)} = 1 + n + k + 2n_2$. The value of $E$ can be equal to zero in three cases: $E^{(1)}$ is positive and $E^{(2)}$ is negative; $E^{(1)}$ is negative and $E^{(2)}$ is positive; $E^{(1)} = E^{(2)} = 0$. We can require that in the first case the corresponding element $(n_1 n_2 n k)$ belongs to $S_+$ and in the second case — to $S_-$. However, the third case is still problematic.

As follows from the results of Sects. 2 and 3, the case $E^{(1)} = 0$ can occur only if $\tilde{n}_1 = n_1$ where $\tilde{n}_1 = N_1(n, k) - n_1$ and $N_1(n, k)$ is given by Eq. (21). Analogously the case $E^{(2)} = 0$ can occur only if $\tilde{n}_2 = n_2$ where $\tilde{n}_2 = N_2(n, k) - n_2$. Therefore the case $E^{(1)} = 0$ can occur only if $N_1(n, k)$ is even and $n_1 = N_1(n, k)/2$. Analogously, the case $E^{(2)} = 0$ can occur only if $N_2(n, k)$ is even and $n_2 = N_2(n, k)/2$.

It is now clear that if the third case can occur then the whole construction becomes inconsistent. Indeed, since $b(n_1 n_2 n k)$ is proportional to $a(\tilde{n}_1 \tilde{n}_2 n k)^*$ then, if $\tilde{n}_1 = n_1$, $\tilde{n}_2 = n_2$ and $\Phi_0$ is annihilated by both $a(n_1 n_2 n k)$ and $b(n_1 n_2 n k)$, it is also annihilated by both $a(n_1 n_2 n k)$ and $a(n_1 n_2 n k)^*$. However this contradicts Eqs. (30) and (31).

Since $q_1 = 1 + s$ and $q_2 = 1$ in the massless case, it follows from Eq. (21) that if $s$ is even then $N_1(n, k)$ and $N_2(n, k)$ are either both even or both odd. Therefore in that case we will necessarily have a situation when for some values of $(n k)$, $N_1(n, k)$ and $N_2(n, k)$ are both even. In that case $E^{(1)} = E^{(2)} = 0$ necessarily takes place for $n_1 = N_1(n, k)/2$ and $n_1 = N_1(n, k)/2$. Moreover, since for each $(n k)$ the number of all possible values of $(n_1 n_2 n k)$ is equal to $(N_1(n, k) + 1)(N_2(n, k) + 1$, this number is odd (therefore one cannot divide the set of all possible values into the equal nonintersecting parts $S_+$ and $S_-$).

On the other hand, if $s$ is odd then for all the values of $(n k)$ we will necessarily have a situation when either $N_1(n, k)$ is even and $N_2(n, k)$ is odd or $N_1(n, k)$ is odd and $N_2(n, k)$ is even. Therefore for each value of $(n k)$ the case $E^{(1)} = E^{(2)} = 0$ is impossible, and the number of all possible values of $(n_1 n_2 n k)$ is even.

We conclude that the condition (46) is mathematically consistent only if $s$ is odd, or in other words, if the particle spin in usual
units is half-integer.

Although the interpretation of each operator from the set $(a, a^*, b, b^*)$ as creation or annihilation one depends on the way of breaking the elements $(n_1 n_2 n k)$ into $S_+$ and $S_-$, the consistency requirement, that the case $E^{(1)} = E^{(2)} = 0$ should be excluded, does not depend on the choice of $S_+$ and $S_-$. For this reason we believe, that the results of this section give a strong indication that in the massless case only particles with the half-integer spin can be elementary. Then as follows from the spin-statistics theorem [14], massless elementary particles can be described only by anticommutation relations, i.e. they can be only fermions. However, since the spin-statistics theorem has not been proved in the GFQT yet, in the subsequent sections we consider both anticommutators and commutators.

6 Compatibility of the AB symmetry with representation operators

Let us consider the operators (32) and use the fact that in the modular case the trace of the operators $M^{ab}$ is equal to zero (see Eq. (29)). Therefore, as follows from Eqs. (30) and (31), we can rewrite Eq. (32) as

\[
M^{ab} = \mp \sum \left\{ M^{ab}(n'_1 n'_2 n' k', n_1 n_2 n k) a(n_1 n_2 n k) a(n'_1 n'_2 n' k')^* / \text{Norm}(n_1 n_2 n k) \right\} \tag{47}
\]

where the minus sign refers to anticommutators and the plus sign - to commutators. Using Eqs. (34,36) and (41,43), we then obtain

\[
M^{ab} = - \sum \left\{ M^{ab}(n'_1 n'_2 n' k', n_1 n_2 n k) f(n'_1 n'_2 n' k') \bar{f}(n_1 n_2 n k) b(n_1 n_2 n k)^* b(n'_1 n'_2 n' k') / [F(n'_1 n'_2 n' k') G(k)] \right\} = - \sum \left\{ M^{ab}(n_1 n_2 n k, n'_1 n'_2 n' k') f(n_1 n_2 n k) \bar{f}(n'_1 n'_2 n' k') b(n'_1 n'_2 n' k')^* b(n_1 n_2 n k) / [F(n_1 n_2 n k) G(k')] \right\} \tag{48}
\]

in both cases.
We first consider the AdS energy operator which is diagonal. As follows from Eq. (22), in the massless case the matrix elements of the $M^{05}$ operator are given by

$$M^{05}(n'_1n'_2n'_k'n_1n_2nk) = (2 + s + 2n + 2n_1 + 2n_2)\delta_{n_1n'_1}\delta_{n_2n'_2}\delta_{nn'}\delta_{kk'} \quad (49)$$

Therefore the operator (32) in this case can be written as

$$M^{05} = \sum_{n_1n_2nk} [(s + 2(n + n_1 + n_2 + 1)]a(n_1n_2nk)^* \times a(n_1n_2nk)/\text{Norm}(n_1n_2nk) \quad (50)$$

At the same time, as follows from Eqs. (42), (48), (49) and the definition of the transformation $n_1 \rightarrow \tilde{n}_1$, $n_2 \rightarrow \tilde{n}_2$ (see Sect. 4)

$$M^{05} = \sum_{n_1n_2nk} [(s + 2(n + n_1 + n_2 + 1)]b(n_1n_2nk)^* \times b(n_1n_2nk)/\text{Norm}(n_1n_2nk) \quad (51)$$

In Eqs. (50) and (51), the sum is taken over all the values of $(n_1n_2nk)$ relevant to the particle modular IR. At the same time, for the correspondence with the standard case, we should consider only the values of the $(n_1n_2n)$ which are much less than $p$ (see Refs. [3, 4]). The derivation of Eq. (51) demonstrates that the contribution of those $(n_1n_2n)$ originates from such a contribution of $(n_1, n_2)$ to Eq. (50) that $(\tilde{n}_1, \tilde{n}_2)$ are much less than $p$. In this case the $(n_1, n_2)$ are comparable to $p$. Therefore, if we consider only such states that the $(n_1n_2n)$ in the $a$ and $b$ operators are much less than $p$ then the AdS Hamiltonian can be written in the form

$$M^{05} = \sum'_{n_1n_2nk} [(s + 2(n + n_1 + n_2 + 1)]a(n_1n_2nk)^* \times a(n_1n_2nk) + b(n_1n_2nk)^*b(n_1n_2nk)]/\text{Norm}(n_1n_2nk) \quad (52)$$

where $\sum'_{n_1n_2nk}$ means that the sum is taken only over the values of the $(n_1n_2nk)$ which are much less than $p$. In this expression the contributions of particles and antiparticles are written down explicitly and the corresponding standard AdS Hamiltonian is positive definite.

The above results show that as far as the operator $M^{05}$ is concerned, Eq. (35) indeed defines a new symmetry since $M^{05}$ has the
same form in terms of \((a, a^*)\) and \((b, b^*)\) (compare Eqs. (50) and (51)). Note that we did not assume that the theory was C-invariant (in the standard theory C-invariance can be defined as the transformation
\[
a(n_1n_2nk) \leftrightarrow b(n_1n_2nk).
\]
It is well known that C-invariance is not a fundamental symmetry. In the standard theory only CPT-invariance is fundamental since, according to the famous CPT-theorem [15], any local Poincare invariant theory is automatically CPT-invariant. Our assumption is that Eq. (35) defines a fundamental symmetry in the GFQT. To understand its properties one has to investigate not only \(M^{05}\) but other representation generators as well.

By analogy with the case of the operator \(M^{05}\), it is easy to show that at the same conditions, the operators \(h_1\) and \(h_2\) have the same form in terms of \((a, a^*)\) and \((b, b^*)\).

Consider now the operator \(a_1''\) (see Sect. 2). As follows from its definition, its matrix elements are given by
\[
a_1''(n'_1n'_2n'_k'n_1n_2nk) = \delta_{n_1,n'_1-1}\delta_{n_2n'_2}\delta_{nn'}\delta_{kk'} \tag{53}
\]
and therefore, as follows from Eq. (52), the secondly quantized form of \(a_1''\) is
\[
a_1'' = \sum_{n_1=0}^{N_1-1} \sum_{n_2nk} \{a(n_1+1,n_2nk)^*a(n_1n_2nk)/\text{Norm}(n_1n_2nk)\} \tag{54}
\]
We have to prove that in terms of \((b, b^*)\) this operator has the same form, i.e.
\[
a_1'' = \sum_{n_1=0}^{N_1-1} \sum_{n_2nk} \{b(n_1+1,n_2nk)^*b(n_1n_2nk)/\text{Norm}(n_1n_2nk)\} \tag{55}
\]
As follows from Eqs. (48) and (53), Eq. (55) is indeed valid if
\[
f(n_1n_2nk)f(n_1-1,n_2nk) = -1 \tag{56}
\]
Since the action of the operator \( a'_1 \) can be written as
\[
a'_1 e(n_1 n_2 n k) = a'_1 a'' e(n_1 - 1, n_2 n k)
\]
then, as follows from Eq. (11), the matrix elements of the operator \( a'_1 \) are given by
\[
a'_1(n'_1 n'_2 n' k' n_1 n_2 n k) = n_1(Q_1(n, k) + n_1 - 1)\delta_{n_1,n'_1+1} \delta_{n_2 n'_2} \delta_{n n'} \delta_{k k'}
\] (57)

Therefore, as follows from Eq. (32), the secondly quantized form of this operator is
\[
a'_1 = \sum_{n_1=1}^{N_1} \sum_{n_2 n k} \{n_1(Q_1(n, k) + n_1 - 1)a(n_1 - 1, n_2 n k)^* \over a(n_1 n_2 n k)/\text{Norm}(n_1 n_2 n k)\}
\] (58)

By analogy with the proof of Eq. (55), one can prove that in terms of \((b, b^*)\) this operator has the same form, i.e.
\[
a'_1 = \sum_{n_1=1}^{N_1} \sum_{n_2 n k} \{n_1(Q_1(n, k) + n_1 - 1)b(n_1 - 1, n_2 n k)^* \over b(n_1 n_2 n k)/\text{Norm}(n_1 n_2 n k)\}
\] (59)

if
\[
f(n_1 n_2 n k)\bar{f}(n_1 + 1, n_2 n k) = -1
\] (60)

Analogously we can prove that the secondly quantized operators \( a''_2 \) and \( a'_2 \) also have the same form in terms of \((a, a^*)\) and \((b, b^*)\) if
\[
f(n_1 n_2 n k)\bar{f}(n_1, n_2 + 1, n k) = -1
\]
\[
f(n_1 n_2 n k)\bar{f}(n_1, n_2 - 1, n k) = -1
\] (61)

As follows from Eqs. (52), (56) (60) and (61), the function \( f(n_1 n_2 n k) \) necessarily has the form
\[
f(n_1 n_2 n k) = (-1)^{n_1+n_2} f(n, k)
\] (62)

where the function \( f(n, k) \) should satisfy the condition
\[
f(n, k)\bar{f}(n, k) = 1
\] (63)
The next step is to investigate whether the remaining operators \((b', b'', L_+, L_-)\) have the same form in terms of \((a, a^*)\) and \((b, b^*)\). We discuss the operator \(b'\) since computations with the other operators are analogous (and simpler).

As follows from Eqs. (41) and (20),

\[
b' e(n_1 n_2 n k) = b'(a_1^\prime)^n_1 (a_2^\prime)^n_2 e_{nk} = (a_1^\prime)^n_1 b'(a_2^\prime)^n_2 e_{nk} + \\
n_1 (a_1^\prime)^{n_1 - 1} L_+(a_2^\prime)^n_2 e_{nk} = (a_1^\prime)^n_1 (a_2^\prime)^n_2 b' e_{nk} + \\
n_2 (a_1^\prime)^n_1 (a_2^\prime)^{n_2 - 1} L_- e_{nk} + n_1 (a_1^\prime)^{n_1 - 1} (a_2^\prime)^{n_2} L_+ e_{nk} + \\
n_1 n_2 (a_1^\prime)^{n_1 - 1} (a_2^\prime)^{n_2 - 1} b'' e_{nk}
\] (64)

By using Eq. (12) we can express the action of the \((b', b'', L_+, L_-)\) operators on the minimal vectors in terms of the \(B\) operators:

\[
b'' = B^{++} + a_1^\prime B^{+-}(h_1 - 1)^{-1} \pm a_2^\prime B^{--}(h_2 - 1)^{-1} + \\
a_1^\prime a_2^\prime B^{--}[(h_1 - 1)(h_2 - 1)]^{-1} \quad L_+ = B^{+-} + a_1^\prime B^{++}(h_1 - 1)^{-1} \\
L_- = B^{--} + a_2^\prime b'(h_2 - 1)^{-1} \quad b' = B^{--}
\] (65)

and then use Eq. (23).

In such a way we can explicitly compute \(b' e(n_1 n_2 n k)\) in Eq. (64), find the matrix elements of \(b'\) by using Eq. (28) and write the operator \(b'\) in the secondly quantized form by using Eq. (32). The result is

\[
b' = \sum_{n=0}^{n_{\text{max}}-1} \sum_{k=0}^{s} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \{(n_1 n_2) \\
a(n_1 - 1, n_2 - 1, n + 1, k) a(n_1 n_2 n k) / \text{Norm}(n_1 n_2 n k)\} + \\
\sum_{n=1}^{n_{\text{max}}} \sum_{k=0}^{s} \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \{(n + s - k + n_1)(n + k + n_2) \\
a(n_1 n_2, n - 1, k) a(n_1 n_2 n k) / \text{Norm}(n_1 n_2 n k)\} + \\
\sum_{k=0}^{s-1} \sum_{n_1=0}^{N_1} \sum_{n_2=1}^{N_2} \left\{[n_2(s - k + n_1)/(s - k)] \\
a(n_1, n_2 - 1, 0, k + 1) a(n_1 n_2 0 k) / \text{Norm}(n_1 n_2 0 k)\} + \\
\sum_{k=1}^{s} \sum_{n_1=1}^{N_1} \sum_{n_2=0}^{N_2} \{n_1(n_2 + k)(s + 1 - k) \\
a(n_1 - 1, n_2, 0, k - 1) a(n_1 n_2 0 k) / \text{Norm}(n_1 n_2 0 k)\}
\] (66)

where (see Sect. 3) \(n_{\text{max}} = p - 1 - s\), \(N_1 = N_1(n, k)\) and \(N_2 = N_2(n, k)\).
The next step is to express the \((a, a^*)\) operators in terms of the \((b, b^*)\) operators by using Eqs. (35) and (36), and use Eqs. (34), (41), (43) and (62). The result is as follows. Eq. (66) has the same form in terms of \((a, a^*)\) and \((b, b^*)\) only if

\[
f(n, k) = c(-1)^n	ag{67}\]

where \(c\) is any constant such that \(c\bar{c} = 1\).

Analogous computations for the operators \((b^L L_+ L_-)\) show that if Eq. (67) is satisfied then they have the same form in terms of \((a, a^*)\) and \((b, b^*)\). Therefore, as follows from Eqs. (41) and (62), the final solution for \(\eta(n_1n_2nk)\) is

\[
\eta(n_1n_2nk) = \alpha f(n_1n_2nk) \quad f(n_1n_2nk) = (-1)^{n_1+n_2+n} \tag{68}\]

where \(\alpha\) satisfies Eq. (43).

We have proved that the AB symmetry defined by Eq. (35) is indeed a fundamental symmetry in the GFQT (at least for massless elementary particles).

7 Problem of existence of neutral elementary particles

Suppose now that the particle in question is neutral, i.e. the particle coincides with its antiparticle. On the language of the operators \((a, a^*)\) and \((b, b^*)\) this means that these sets are the same, i.e. \(a(n_1n_2nk) = b(n_1n_2nk)\) and \(a(n_1n_2nk)^* = b(n_1n_2nk)^*\). As a consequence, Eq. (35) has now the form

\[
a(n_1n_2nk)^* = \eta(n_1n_2nk)a(\tilde{n}_1\tilde{n}_2nk)/F(\tilde{n}_1\tilde{n}_2nk) \tag{69}\]

and therefore

\[
a(n_1n_2nk) = \tilde{\eta}(n_1n_2nk)a(\tilde{n}_1\tilde{n}_2nk)^*/F(\tilde{n}_1\tilde{n}_2nk) \tag{70}\]

As follows from Eqs. (41) and (43), these expressions are compatible with each other only if

\[
f(n_1n_2nk)\bar{f}(\tilde{n}_1, \tilde{n}_2, nk) = \pm 1 \tag{71}\]
where the plus sign refers to anticommutators and the minus sign to commutators, respectively. Therefore the problem arises whether Eqs. (68) and (71) are compatible with each other. As follows from Eq. (21), (62), (63) and the definition of the transformations $n_j \rightarrow \tilde{n}_j$ (see Sect. 4)

\[ f(n_1n_2nk)\tilde{f}(\tilde{n}_1\tilde{n}_2nk) = (-1)^s \]  

(72)

By comparing Eqs. (71) and (72) we conclude that they are incompatible with each other if the spin-statistics theorem is satisfied. Therefore massless particles in the GFQT cannot be elementary but only composite.

8 Discussion

In the present paper we have considered massless IRs in a quantum theory based on a Galois field (GFQT). One of the crucial differences between the GFQT and the standard theory is that in the GFQT a particle and its antiparticle represent different states of the same object. As a consequence, the annihilation and creation operators for a particle and its antiparticle can be directly expressed in terms of each other. This imposes additional restrictions on the structure of the theory. In particular, Eq. (35) defines a new symmetry which has no analog in the standard theory. We have shown in Sect. 6 that this is indeed a symmetry in the massless case since the representation operators have the same form in terms of annihilation and creation operators for particles and antiparticles. It has been also shown in Sect. 5 that the new symmetry is compatible with the vacuum condition only for particles with the half-integer spin (in conventional units). As a consequence, as shown in Sect. 7, the existence of massless neutral elementary particles in the GFQT is incompatible with the spin-statistics theorem. It will be shown in a separate paper that these results can be extended to the massive case as well.

Is it natural that the requirement about the normal connection between spin and statistics excludes the existence of neutral elementary particles? If there is no restriction imposed by the spin-statistics
theorem then we cannot exclude the existence of neutral elementary particles in the GFQT. However, such an existence seems to be rather unnatural. Indeed, since one modular IR simultaneously describes a particle and its antiparticle, the AdS energy operator necessarily contains the contribution of the both parts of the spectrum, corresponding to the particle and its antiparticle (see Eq. (52)). If a particle were the same as its antiparticle then Eq. (52) would contain two equal contributions and thus the value of the AdS energy would be twice as big as necessary.

Although the conclusion about the nonexistence of neutral elementary particles has been made for both bosons and fermions, it is obvious that the case of bosons is of greater importance. A possibility that the photon is composite has been already discussed in the literature. For example, in Ref. [16] a model where the photon is composed of two Dirac singletons [17] has been investigated. However, in the framework of the standard theory, the compositeness of the photon is only a possible (and attractive) scenario while in the GFQT this is inevitable.

It is well known that the standard local quantum field theory (LQFT) has achieved very impressive success in comparing theory and experiment. In particular, quantum electrodynamics and the electroweak theory are based on the assumption that the photon is the elementary particle. For this reason one might doubt whether our conclusion has any relevance to physics. At the same time the LQFT has several well known drawbacks and inconsistencies. The majority of physicists believes that [13] the LQFT should be ‘taken as is’, but at the same time it is a ‘low energy approximation to a deeper theory that may not even be a field theory, but something different like a string theory’ [13].

We argued in [3, 4] that the future quantum physics will be based on a Galois field. In that case the theory does not contain actual infinity, all operators are well defined, divergencies cannot exist in principle etc. We believe however that not only this makes the GFQT very attractive.
For centuries, scientists and philosophers have been trying to understand why mathematics is so successful in explaining physical phenomena (see e.g. Ref. [18]). However, such a branch of mathematics as number theory and, in particular, Galois fields, have practically no implications in physics. Historically, every new physical theory usually involved more complicated mathematics. The standard mathematical tools in modern quantum theory are differential and integral equations, distributions, analytical functions, representations of Lie algebras in Hilbert spaces etc. At the same time, very impressive results of number theory about properties of natural numbers (e.g. the Wilson theorem) and even the notion of primes are not used at all! The reader can easily notice that the GFQT involves only arithmetic of Galois fields (which are even simpler than the set of natural numbers). The very possibility that the future quantum theory could be formulated in such a way, is of indubitable interest.

References

[1] E.P. Wigner, Ann. Math. 40, 149 (1939).

[2] P.A.M. Dirac, in The World Treasury of Physics, Astronomy and Mathematics, p. 80, Timothy Ferris ed., (Little Brown and Company, Boston-New York-London, 1991).

[3] F.M. Lev, Yad. Fiz. 48, 903 (1988); J. Math. Phys. 30, 1985 (1989); J. Math. Phys. 34, 490 (1993).

[4] F.M. Lev, hep-th/0206078.

[5] B.L. Van der Waerden, Algebra I, (Springer-Verlag, Berlin Heidelberg New York, 1967); K. Ireland and M. Rosen, A Classical Introduction to Modern Number Theory, Graduate Texts in Mathematics-87, (New York - Heidelberg - Berlin: Springer, 1987); H. Davenport, The Higher Arithmetic, (Cambridge University Press, 1999).
[6] E.M. Friedlander and B.J. Parshall, Bull. Am. Math. Soc. 17, 129 (1987).

[7] C. Fronsdal, Rev. Mod. Phys. 37, 221 (1965).

[8] N.T. Evans, J. Math. Phys. 8, 170 (1967).

[9] B. Braden, Bull. Amer. Math. Soc. 73, 482 (1967); Thesis, Univ. of Oregon, (Eugene, OR, 1966).

[10] H. Zassenhaus, Proc. Glasgow Math. Assoc. 2, 1 (1954).

[11] E. Inonu and E.P. Wigner, Nuovo Cimento, IX, 705 (1952).

[12] Yu.V. Novozhilov, An Introduction to Elementary Particle Theory (Nauka, Moscow, 1972).

[13] S. Weinberg, Quantum Theory of Fields, (Cambridge, Cambridge University Press, 1995).

[14] W. Pauli, Phys. Rev. 58, 116 (1940).

[15] W. Pauli, in N. Bohr and the Development of Physics, (Pergamon Press, London, 1955); G. Gravert, G. Luders and G. Rollnik, Fortschr. Phys. 7, 291 (1959).

[16] M. Flato and C. Fronsdal, Lett. Math. Phys. 2, 421 (1978); L. Castell and W. Heidenreich, Phys. Rev. D24, 371 (1981); C. Fronsdal, Phys. Rev. D26, 1888 (1982).

[17] P.A.M. Dirac, J. Math. Phys. 4, 901 (1963).

[18] E.P. Wigner, in The World Treasury of Physics, Astronomy and Mathematics, p. 526, Timothy Ferris ed., (Little Brown and Company, Boston-New York-London, 1991).