A Bell inequality which can be used to test locality more simply than Clauser-Horne inequality and which is violated by a larger magnitude of violation than Clauser-Horne-Shimony-Holt inequality

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Abstract

A correlation inequality is derived from local realism and a supplementary assumption. Unlike Clauser-Horne (CH) inequality [or Clauser-Horne-Shimony-Holt (CHSH) inequality] which is violated by quantum mechanics by a factor of $\sqrt{2}$, this inequality is violated by a factor of 1.5. Thus the magnitude of violation of this inequality is approximately 20.7% larger than the magnitude of violation of previous inequalities. Moreover, unlike CH (or CHSH) inequality which requires the measurement of five detection probabilities, the present inequality requires the measurement of only two detection probabilities. This inequality can therefore be used to test locality more simply than CH or CHSH inequality.

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Local realism is a philosophical view which holds that external reality exists and has local properties. Quantum mechanics vehemently denies that such a world view has any meaning for physical systems because local realism assigns simultaneous values to non-commuting observables. In 1965 Bell \cite{1} showed that the assumption of local realism, as postulated by Einstein, Podolsky, and Rosen (EPR) \cite{2}, leads to some constraints on the statistics of two spatially separated particles. These constraints, which are collectively known as Bell inequalities, are sometimes grossly violated by quantum mechanics. The violation of Bell inequalities therefore indicate that local realism is not only philosophically but also numerically incompatible with quantum mechanics. Bell’s theorem is of paramount importance for understanding the foundations of quantum mechanics because it rigorously formulates EPR’s assumption of locality and shows that all realistic interpretations of quantum mechanics must be nonlocal.

Bell’s original argument, however, can not be experimentally tested because it relies on perfect correlation of the spin of the two particles \cite{3}. Faced with this problem, Clauser-Horne-Shimony-Holt (CHSH) \cite{4}, Freedman-Clauser (FC) \cite{5}, and Clauser-Horne (CH) \cite{6} derived correlation inequalities for systems which do not achieve 100% correlation, but which do achieve a necessary minimum correlation. Quantum mechanics violates these inequalities by as much as $\sqrt{2}$. An experiment based on CHSH, or FC, or CH inequality utilizes one-channel polarizers in which the dichotomic choice is between the detection of the photon and its lack of detection. A better experiment is one in which a truly binary choice is made between the ordinary and the extraordinary rays \cite{7-10}. In this letter, we derive a correlation inequality for two-channel polarizer systems and we show that quantum mechanics vi-
ulates this inequality by a factor of 1.5. Thus the magnitude of violation of the inequality derived in this paper is approximately 20.7% larger than the magnitude of violation of previous inequalities of [4-10]. Moreover, we show that unlike CH (or CHSH) inequality which requires the measurements of five detection probabilities, the present inequality requires the measurement of only two detection probabilities. This inequality can therefore be used to test locality more simply than CH (or CHSH) inequality. This result can be particularly important for the experimental test of local realism.

We start by considering Bohm’s [11] version of EPR experiment in which an unstable source emits pairs of photons in a singlet state $|\Phi\rangle$. The source is viewed by two apparatuses. The first (second) apparatus consists of a polarizer $P_1$ ($P_2$) set at angle $a$ ($b$), and two detectors $D_1^\pm$ ($D_2^\pm$) put along the ordinary and the extraordinary beams. During a period of time $T$, the source emits, say, $N$ pairs of photons. Let $N^\pm \pm(a, b)$ be the number of simultaneous counts from detectors $D_1^\pm$ and $D_2^\pm$, $N^\pm(a)$ the number of counts from detectors $D_1^\pm$, and $N^\pm(b)$ the number of counts from detectors $D_2^\pm$. If the time $T$ is sufficiently long, then the ensemble probabilities $p^\pm \pm(a, b)$ are defined as

$$p^\pm \pm(a, b) = \frac{N^\pm \pm(a, b)}{N},$$

$$p^\pm(a) = \frac{N^\pm(a)}{N},$$

$$p^\pm(b) = \frac{N^\pm(b)}{N}. \quad (1)$$

We consider a particular pair of photons and specify its state with a parameter $\lambda$. Following Bell, we do not impose any restriction on the complexity of $\lambda$. It is a matter of indifference in the following whether $\lambda$ denotes a single variable or a set, or even a set of functions, and whether the variables are
discrete or continuous[4].”

The ensemble probabilities in Eq. (1) are defined as

\[
\begin{align*}
    p^{\pm \pm}(a, b) &= \int p(\lambda) p^{\pm}(a | \lambda) p^{\pm}(b | \lambda, a), \\
    p^{\pm}(a) &= \int p(\lambda) p^{\pm}(a | \lambda), \\
    p^{\pm}(b) &= \int p(\lambda) p^{\pm}(b | \lambda).
\end{align*}
\]  

Equations (2) may be stated in physical terms: The ensemble probability for detection of photons by detectors \(D_1^{\pm}\) and \(D_2^{\pm}\) [that is \(p^{\pm \pm}(a, b)\)] is equal to the sum or integral of the probability that the emission is in the state \(\lambda\) [that is \(p(\lambda)\)], times the conditional probability that if the emission is in the state \(\lambda\), then a count is triggered by the first detector \(D_1^{\pm}\) [that is \(p^{\pm}(a | \lambda)\)], times the conditional probability that if the emission is in the state \(\lambda\) and if the first polarizer is set along axis \(a\), then a count is triggered from the second detector \(D_2^{\pm}\) [that is \(p^{\pm}(b | \lambda, a)\)]. Similarly the ensemble probability for detection of photons by detector \(D_1^{\pm}\left(D_2^{\pm}\right)\) [that is \(p^{\pm}(a) \left[p^{\pm}(b)\right]\)] is equal to the sum or integral of the probability that the photon is in the state \(\lambda\) [that is \(p(\lambda)\)], times the conditional probability that if the photon is in the state \(\lambda\), then a count is triggered by detector \(D_1^{\pm}\left(D_2^{\pm}\right)\) [that is \(p^{\pm}(a | \lambda) \left[p^{\pm}(b | \lambda)\right]\)]. Note that Eqs. (1) and (2) are quite general and follow from the standard rules of probability theory. No assumption has yet been made that is not satisfied by quantum mechanics.

Hereafter, we focus our attention only on those theories that satisfy EPR criterion of locality: “Since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to first system[4].” EPR’s criterion of
locality can be translated into the following mathematical equation:

\[ p^\pm(b \mid \lambda, a) = p^\pm(b \mid \lambda). \quad (3) \]

Equation (3) is the hallmark of local realism. It is the most general form of locality that accounts for correlations subject only to the requirement that a count triggered by the second detector does not depend on the orientation of the first polarizer. The assumption of locality, i.e., Eq. (3), is quite natural since the two photons are spatially separated so that the orientation of the first polarizer should not influence the measurement carried out on the second photon.

In the following we show that equation (3) leads to validity of an equality that is sometimes grossly violated by the quantum mechanical predictions in the case of real experiments. First we need to prove the following algebraic

\[ p^+ (a, b, \lambda) = p^+ (a, \lambda) p^+ (b, \lambda). \]

Apparently by \( p^+ (a, b, \lambda) \), they mean the conditional probability that if the emission is in state \( \lambda \), then simultaneous counts are triggered by detectors \( D^+_1 \) and \( D^+_2 \). However, what they call \( p^+ (a, b, \lambda) \) in probability theory is usually called \( p^+ (a, b \mid \lambda) \) [note that \( p(x, y, z) \) is the joint probability of \( x, y \) and \( z \), whereas \( p(x, y \mid z) \) is the conditional probability that if \( z \) then \( x \) and \( y \)]. Similarly by \( p^+ (a, \lambda) [p^+ (b, \lambda)] \), CH mean the conditional probability that if the emission is in state \( \lambda \), then a count is triggered from the detector \( D^+_1 \) \( (D^+_2) \). Again what they call \( p^+ (a, \lambda) [p^+ (b, \lambda)] \) in probability theory is usually written as \( p^+ (a \mid \lambda) [p^+ (b \mid \lambda)] \) (again note that \( p(x, z) \) is the joint probability of \( x \) and \( z \), whereas \( p(x \mid z) \) is the conditional probability that if \( z \) then \( x \)). Thus according to standard notation of probability theory, CH criterion of locality may be written as

\[ p^+ (a, b \mid \lambda) = p^+ (a \mid \lambda) p^+ (b \mid \lambda). \]

Now according to Bayes’ theorem,

\[ p^+ (a, b \mid \lambda) = p^+ (a \mid \lambda) p^+ (b \mid \lambda, a). \]

Substituting the above equation in CH’s criterion of locality, we obtain

\[ p^+ (b \mid \lambda, a) = p^+ (b \mid \lambda), \]

which for the ordinary equation is the same as Eq. (3).
the following inequality always holds:

**Theorem:** Given ten non-negative real numbers \( x_1^+, x_1^-, x_2^+, x_2^-, y_1^+, y_1^-, y_2^+, y_2^- \), \( U \) and \( V \) such that \( x_1^+, x_1^-, x_2^+, x_2^- \leq U \), and \( y_1^+, y_1^-, y_2^+, y_2^- \leq V \), then the following inequality always holds:

\[
Z = x_1^+ y_1^+ + x_1^- y_1^- - x_1^+ y_1^- - x_1^- y_1^+ + y_2^+ x_1^- + y_2^- x_1^-
- y_2^- x_1^- - y^-_2 x_1^+ + y_1^+ x_2^- + y_1^- x_2^+ - y_1^+ x_2^- - y_1^- x_2^+ - 2 x_2^+ y_2^-
- 2 x_2^- y_2^- + V x_2^+ + V x_2^- + U y_2^+ + U y_2^- + U V \geq 0. \tag{4}
\]

**Proof:** Calling \( A = y_1^+ - y_1^- \), we write the function \( Z \) as

\[
Z = x_2^+ \left( -2 y_2^+ + A + V \right) + x_2^- \left( -2 y_2^- - A + V \right)
+ \left( x_1^+ - x_1^- \right) \left( A + y_2^+ - y_2^- \right) + U y_2^+ + U y_2^- + U V. \tag{5}
\]

We consider the following eight cases:

1. First assume

\[
\begin{align*}
-2 y_2^+ + A + V &\geq 0, \\
-2 y_2^- - A + V &\geq 0, \\
A + y_2^+ - y_2^- &\geq 0.
\end{align*}
\]

The function \( Z \) is minimized if \( x_2^+ = 0, x_2^- = 0, \) and \( x_1^+ - x_1^- = -U \). Thus

\[
Z \geq -U \left( A + y_2^+ - y_2^- \right) + U y_2^+ + U y_2^- + U V
= U \left( -A + 2 y_2^- + V \right). \tag{6}
\]

Since \( V \geq A \) and \( y_2^- \geq 0, \) \( Z \geq 0. \)

2. Next assume

\[
\begin{align*}
-2 y_2^+ + A + V &< 0, \\
-2 y_2^- - A + V &\geq 0, \\
A + y_2^+ - y_2^- &\geq 0.
\end{align*}
\]

The function \( Z \) is minimized if \( x_2^+ = U, x_2^- = 0, \) and \( x_1^+ - x_1^- = -U \). Thus

\[
Z \geq U \left( -2 y_2^+ + A + V \right) - U \left( A + y_2^+ - y_2^- \right) + U y_2^+ + U y_2^- + U V
\]

\[
\]
\[ Z = 2U \left( V + y_2^- - y_2^+ \right). \tag{7} \]

Since \( V \geq y_2^+ \), and \( y_2^- \geq 0 \), \( Z \geq 0 \).

(3) Next assume \[
\begin{cases}
-2y_2^+ + A + V \geq 0, \\
-2y_2^- - A + V < 0, \\
A + y_2^+ - y_2^- \geq 0.
\end{cases}
\]

The function \( Z \) is minimized if \( x_2^+ = 0, x_2^- = U \), and \( x_1^+ - x_1^- = -U \). Thus

\[ Z \geq U \left( -2y_- - A + V \right) - U \left( A + y_2^+ - y_2^- \right) + Uy_2^+ + Uy_2^- + UV \]
\[ = 2U \left( V - A \right). \tag{8} \]

Since \( V \geq A \), \( Z \geq 0 \).

(4) Next assume \[
\begin{cases}
-2y_2^+ + A + V \geq 0, \\
-2y_2^- - A + V \geq 0, \\
A + y_2^+ - y_2^- < 0.
\end{cases}
\]

The function \( Z \) is minimized if \( x_2^+ = 0, x_2^- = 0 \), and \( x_1^+ - x_1^- = U \). Thus

\[ Z \geq U \left( A + y_2^+ - y_2^- \right) + Uy_2^+ + Uy_2^- + UV \]
\[ = U \left( A + 2y_2^+ + V \right). \tag{9} \]

Since \( V \geq A \) and \( y_2^+ \geq 0 \), \( Z \geq 0 \).

(5) Next assume \[
\begin{cases}
-2y_2^+ + A + V < 0, \\
-2y_2^- - A + V < 0, \\
A + y_2^+ - y_2^- \geq 0.
\end{cases}
\]

The function \( Z \) is minimized if \( x_2^+ = U, x_2^- = U \), and \( x_1^+ - x_1^- = -U \). Thus

\[ Z \geq U \left( -2y_2^+ + A + V \right) + U \left( -2y_2^- - A + V \right) - U \left( A + y_2^+ - y_2^- \right) \]
\[ + Uy_2^+ + Uy_2^- + UV \]
\[ = U \left( -2y_2^+ - A + 3V \right). \tag{10} \]
Since $V \geq A$ and $V \geq y_2^+$, $Z \geq 0$.

(6) Next assume
\[
\begin{cases}
-2y_2^+ + A + V < 0, \\
-2y_2^+ - A + V \geq 0, \\
A + y_2^+ - y_2^- < 0.
\end{cases}
\]

The function $Z$ is minimized if $x_2^+ = U$, $x_2^- = 0$, and $x_1^+ - x_1^- = U$. Thus
\[
Z \geq U \left( -2y_2^+ + A + V \right) + U \left( A + y_2^+ - y_2^- \right) + Uy_2^+ + Uy_2^- + UV
= \ 2U \left( A + V \right) .
\] (11)

Since $V \geq A$, $Z \geq 0$.

(7) Next assume
\[
\begin{cases}
-2y_2^- + A + V \geq 0, \\
-2y_2^- - A + V < 0, \\
A + y_2^+ - y_2^- < 0.
\end{cases}
\]

The function $Z$ is minimized if $x_2^+ = 0$, $x_2^- = U$, and $x_1^+ - x_1^- = U$. Thus
\[
Z \geq U \left( -2y_2^- - A + V \right) + U \left( A + y_2^+ - y_2^- \right) + Uy_2^+ + Uy_2^- + UV
= \ 2U \left( y_2^- + y_2^+ + V \right) .
\] (12)

Since $V \geq y_2^-$ and $y_2^+ \geq 0$, $Z \geq 0$.

(8) Finally assume
\[
\begin{cases}
-2y_2^+ + A + V < 0, \\
-2y_2^- + A + V < 0, \\
A + y_2^+ - y_2^- < 0.
\end{cases}
\]

The function $Z$ is minimized if $x_2^+ = U$, $x_2^- = 0$, and $x_1^+ - x_1^- = U$. Thus
\[
Z \geq U \left( -2y_2^+ + A + V \right) + U \left( -2y_2^- - A + V \right) + U \left( A + y_2^+ - y_2^- \right) \\
+ \ Uy_2^+ + Uy_2^- + UV
= \ U \left( -2y_2^- + A + 3V \right) .
\] (13)

Since $V \geq A$ and $V \geq y_2^-$, $Z \geq 0$, and the theorem is proved.
Now let \( a \) (\( b \)) and \( a' \) (\( b' \)) be two arbitrary orientation of the first (second) polarizer, and let

\[
x^\pm_1 = p^\pm(a | \lambda), \quad x^\pm_2 = p^\pm(a' | \lambda),
\]
\[
y^\pm_1 = p^\pm(b | \lambda), \quad y^\pm_2 = p^\pm(b' | \lambda).
\]  

(14)

Obviously for each value of \( \lambda \), we have

\[
p^\pm(a | \lambda) \leq 1, \quad p^\pm(a' | \lambda) \leq 1,
\]
\[
p^\pm(b | \lambda) \leq 1, \quad p^\pm(b' | \lambda) \leq 1.
\]  

(15)

Inequalities (4) and (15) yield

\[
p^+(a | \lambda)p^+(b | \lambda) + p^-(a | \lambda)p^-(b | \lambda) - p^+(a | \lambda)p^-(b | \lambda)
\]
\[
- p^-(a | \lambda)p^+(b | \lambda) + p^+(a' | \lambda)p^+(a | \lambda) + p^+(b' | \lambda)p^+(a | \lambda)
\]
\[
- p^+(b' | \lambda)p^-(a | \lambda) - p^+(b' | \lambda)p^+(a | \lambda) + p^+(b | \lambda)p^+(a' | \lambda)
\]
\[
+ p^-(b | \lambda)p^-(a' | \lambda) - p^+(b | \lambda)p^-(a' | \lambda) - p^-(b | \lambda)p^+(a' | \lambda)
\]
\[
- 2p^+(a' | \lambda)p^+(b' | \lambda) - 2p^-(a' | \lambda)p^-(b' | \lambda) + p^+(a' | \lambda)
\]
\[
+ p^-(a' | \lambda) + p^+(b' | \lambda) + p^-(b' | \lambda) \geq -1.
\]  

(16)

Multiplying both sides of (16) by \( p(\lambda) \), integrating over \( \lambda \) and using Eqs. (2), we obtain

\[
p^{++}(a, b) + p^{--}(a, b) - p^{+-}(a, b) - p^{-+}(a, b) + p^{++}(b', a) +
\]
\[
p^{-+}(b', a) - p^{+-}(b', a) - p^{-+}(b', a) + p^{++}(b, a') +
\]
\[
p^{+-}(b, a') - p^{+-}(b, a') - p^{-+}(b, a') - 2p^{++}(a', b') -
\]
\[
2p^{--}(a', b') + p^{++}(a') + p^{--}(a') + p^{+-}(b') + p^{--}(b') \geq -1.
\]  

(17)

All local realistic theories must satisfy inequality (17).
In the atomic cascade experiments, an atom emits two photons in a cascade from state $J = 1$ to $J = 0$. Since the pair of photons have zero angular momentum, they propagate in the form of spherical wave. Thus the probability $p(d_1, d_2)$ of both photons being simultaneously detected by two detectors in the directions $d_1$ and $d_2$ is \cite{5,3}

$$p(d_1, d_2) = \eta^2 \left( \frac{\Omega}{4\pi} \right)^2 g(\theta, \phi),$$

where $\eta$ is the quantum efficiency of the detectors, $\Omega$ is the solid angle of the detector, $\cos \theta = d_1 \cdot d_1$, and angle $\phi$ is related to $\Omega$ by

$$\Omega = 2\pi (1 - \cos \phi).$$

Finally the function $g(\theta, \phi)$ is the angular correlation function and in the special case is given by \cite{1}

$$g(\pi, \phi) = 1 + \frac{1}{8} \cos^2 \phi (1 + \cos \phi)^2.$$ (20)

If we insert polarizers in front of the detectors, then the quantum mechanical predictions for joint detection probabilities are \cite{4,3}

$$p^+ (a) = p^- (a) = \eta \left( \frac{\Omega}{8\pi} \right), \quad p^+ (b) = p^- (b) = \eta \left( \frac{\Omega}{8\pi} \right),$$

$$p^{++} (a, b) = p^{--} (a, b) = \eta^2 \left( \frac{\Omega}{8\pi} \right)^2 g(\theta, \phi) [1 + \cos 2(a - b)],$$

$$p^{+-} (a, b) = p^{-+} (a, b) = \eta^2 \left( \frac{\Omega}{8\pi} \right)^2 g(\theta, \phi) [1 - \cos 2(a - b)].$$ (21)

In experiments which are feasible with present technology \cite{5,12}, because $\Omega \ll 4\pi$, only a very small fraction of photons are detected Thus inequality (17) can not be used to test the violation of Bell’s inequality. We now state a supplementary assumption, and we show that this assumption is sufficient
to make these experiments (where $\Omega \ll 4\pi$) applicable as a test of local theories (it is important to emphasize that a supplementary assumption is required primarily because the solid angle covered by the aperture of the apparatus, $\Omega$, is much less than $4\pi$ and not because the efficiency of the detectors, $\eta$, is much smaller than 1. In fact in the previous experiments (Ref. 12), the efficiency of detectors were larger than 90%. However, because $\Omega \ll 4\pi$, all previous experiments needed supplementary assumptions to test locality). The supplementary assumption is: For every emission $\lambda$, the detection probability by detector $D^+$ (or $D^-$) is less than or equal to the sum of detection probabilities by detectors $D^+$ and $D^-$ when the polarizer is set along any arbitrary axis. If we let $r$ be an arbitrary direction of the first or second polarizer, then the above supplementary assumption may be translated into the following inequalities

$$
\begin{align*}
 p^+(a | \lambda) &\leq p^+(r | \lambda) + p^-(r | \lambda), & p^-(a | \lambda) &\leq p^+(r | \lambda) + p^-(r | \lambda), \\
p^+(a' | \lambda) &\leq p^+(r | \lambda) + p^-(r | \lambda), & p^-(a' | \lambda) &\leq p^+(r | \lambda) + p^-(r | \lambda), \\
p^+(b | \lambda) &\leq p^+(r | \lambda) + p^-(r | \lambda), & p^-(b | \lambda) &\leq p^+(r | \lambda) + p^-(r | \lambda), \\
p^+(b' | \lambda) &\leq p^+(r | \lambda) + p^-(r | \lambda), & p^-(b' | \lambda) &\leq p^+(r | \lambda) + p^-(r | \lambda).
\end{align*}
$$

Now using relations (4), (14) and (22), and applying the same argument that led to inequality (17), we obtain the following inequality

$$
\begin{align*}
 &\left[ p^{++}(a, b) + p^{--}(a, b) - p^{++}(a, b) - p^{--}(a, b) - p^{++}(a', b') + p^{--}(a', b') ight. \\
&\left. - p^{--}(a', b') \right. - p^{++}(b', a) + p^{++}(b, a') + p^{--}(b, a') - p^{++}(b, a') \\
&\left. - p^{--}(b, a') - 2p^{++}(a', b') - 2p^{--}(a', b') + p^{++}(a', r) + p^{--}(a', r) \\
&\left. + p^{++}(a', r) + p^{--}(a', r) + p^{++}(r, b') + p^{--}(r, b') + p^{++}(r, b') \right]
\end{align*}
$$
Note that in the above inequality the number of emissions \( N \) from the source (something which can not be measured experimentally, see Eq. (1)) is eliminated from the ratio. Inequality (23) contains only double-detection probabilities. Quantum mechanics violates this inequality in case of real experiments where the solid angle covered by the aperture of the apparatus, \( \Omega \), is much less than \( 4\pi \).

Inequality (23) may be considerably simplified if we invoke some of the symmetries that are exhibited in atomic-cascade photon experiments. For a pair of particles in a singlet state, the quantum mechanical detection probabilities \( p_{QM}^{\pm\pm} \) and expected value \( E_{QM} \) exhibit the following symmetry

\[
p_{QM}^{\pm\pm} (a, b) = p_{QM}^{\pm\pm} (|a - b|), \quad E_{QM} (a, b) = E_{QM} (|a - b|).
\]

We assume that the local theories also exhibit the same symmetry

\[
p^{\pm\pm} (a, b) = p^{\pm\pm} (|a - b|), \quad E (a, b) = E (|a - b|),
\]

where \( E (|a - b|) \) is the expected value of detection probabilities in local realistic theories and is defined as

\[
E (|a - b|) = p^{++} (|a - b|) - p^{+-} (|a - b|) - p^{-+} (|a - b|) + p^{--} (|a - b|).
\]

Note that there is no harm in assuming Eqs. (25) since they are subject to experimental test (CHSH [1], FC [4], and CH [6] made the same assumptions). Using the above symmetry, inequality (23) is simplified to

\[
\left[ E (|a - b|) + E (|b - a'|) + E (|b' - a|) - 2p^{++} (|a' - b'|) - 2p^{--} (|a' - b'|) \right] \geq -1.
\]
\[ +p^+ (| a' - r |) + p^- (| a' - r |) + p^+ (| a' - r |) + p^- (| a' - r |) \\
+ p^+ (| r - b' |) + p^- (| r - b' |) + p^+ (| r - b' |) + p^- (| r - b' |) \]
\[
\left[ p^+ (0°) + p^- (0°) + p^+ (0°) + p^- (0°) \right] \geq -1.
\]  

(27)

We now take \(a'\) and \(b'\) to be along direction \(r\), and we take \(a, b,\) and \(a'\) to be three coplanar axes, each making 120° with the other two, that is we choose the the following orientations, \(| a - b | = | b' - a | = | b - a' | = 120°\) and \(| a' - b' | = | a' - r | = | r - b' | = 0°\). Furthermore if we define \(K\) as

\[ K = p^+ (0°) + p^- (0°) + p^+ (0°) + p^- (0°) \]

(28)

then the above inequality is simplified to

\[ \frac{3E (120°) - 2p^+ (0°) - 2p^- (0°)}{K} \geq -1. \]

(29)

Using the quantum mechanical probabilities [i.e., Eqs. (21)] inequality (29) becomes \(-1.5 \geq -1\), which is certainly impossible. Quantum mechanics therefore violates inequality (29) by a factor of 1.5, whereas it violates CH (or CHSH) inequality by by a factor of \(\sqrt{2}\). Thus the magnitude of violation of inequality (29) is approximately 20.7% larger than the magnitude of violation of the previous inequalities [4-10].

Moreover, inequality (29) can be used to test locality considerably more simply than CH or CHSH inequality. CH inequality may be written as

\[ \frac{3p(\phi) - p(3\phi) - p(a', \infty) - p(\infty, b)}{p(\infty, \infty)} \leq 0. \]

(30)

The above inequality requires the measurements of five detection probabilities:

(1) The measurement of detection probability with both polarizers set along the 22.5° axis [that is \(p(22.5°)\)].
(2) The measurement of detection probability with both polarizers set along
the $67.5^\circ$ axis [that is $p(67.5^\circ)$].

(3) The measurement of detection probability with the first polarizer set
along $a'$ axis and the second polarizer being removed [that is $p(a',\infty)$].

(4) The measurement of detection probability with the first polarizer removed
and the second polarizer set along $b$ axis [that is $p(\infty,b)$].

(5) The measurement of detection probability with both polarizers removed
[that is $p(\infty,\infty)$].

In contrast, the inequality derived in this paper [i.e., inequality (29)] requires
the measurements of only two detection probabilities:

(1) The measurement of detection probability with both polarizers set along
the $0^\circ$ axis [that is $p(0^\circ)$].

(2) The measurement of detection probability with both polarizers set along
the $120^\circ$ axis [that is $p(120^\circ)$].

Inequality (29) is also experimentally simpler than FC inequality \( [\text{Eq. } 29] \) (it
should be noted that FC inequality is derived under the assumptions that
(i) $p(a',\infty)$ is independent of $a'$, (ii) $p(b',\infty)$ is independent of $b'$. These
assumptions, however, should be tested experimentally). FC inequality may
be written as

$$\frac{p(22.5^\circ) - p(67.5^\circ)}{p(\infty,\infty)} \leq 0.25. \quad (31)$$

The above inequality requires the measurement of at least three detection
probabilities:

(1) The measurement of detection probability with both polarizers set along
the $22.5^\circ$ axis [that is $p(22.5^\circ)$].

(2) The measurement of detection probability with both polarizers set along
the 67.5° axis [that is \( p(67.5°) \)].

(3) The measurement of detection probability with both polarizers removed [that is \( p(\infty, \infty) \)].

In contrast inequality (29) requires the measurements of only two detection probabilities.

A final comment is in order about the Bell inequality [inequality (29)] that was derived in this paper. The analysis that led to inequality (29) is not limited to atomic-cascade experiments and can easily be extended to experiments which use phase-momentum \[13\], or use high energy polarized protons or \( \gamma \) photons [14-15] to test Bell’s limit. For example in the experiment by Rarity and Tapster \[13\], instead of inequality (2) of their paper, the following inequality (i.e., inequality (29) using their notations) may be used to test locality:

\[
\frac{3E(120°) - 2C_{a3b4}(0°) - 2C_{a4b3}(0°)}{K} \geq -1
\]

where \( C_{ai, bj}(\phi_a, \phi_b) \) \((i = 3, 4; j = 3, 4)\) is the counting rate between detectors \( D_{ai} \) and \( D_{bj} \) with phase angles being set to \( \phi_a, \phi_b \) (See Fig. 1 of \[13\]). The following set of orientations \( (\phi_a, \phi_b) = (\phi_{\nu'}, \phi_a) = (\phi_b, \phi_{\nu'}) = 120° \), and \((\phi_{\nu'}, \phi_{\nu'}) = 0°\) leads to the largest violation of inequality (32). Using the optimum orientation of phase angles, the magnitude of violation of inequality (32) is approximately 20.7% larger than the magnitude of violation of inequality (2) of \[13\]. This result can be particularly important for experiments using phase-momentum to test locality. Similarly, in high-energy experiments, inequality (29) can lead to a larger magnitude of violation. For example, in spin correlation proton-proton scattering experiments \[14\], inequality (29) leads to a magnitude of violation of approximately 20.7%
larger than the results reported by Lamehirachti and Mittig [15].

In summary, we have demonstrated that the conjunction of Einstein’s locality [Eq. (3)] with a supplementary assumption [inequality (22)] leads to validity of inequality (29) that is sometimes grossly violated by quantum mechanics. Inequality (29), which may be called strong inequality [16], defines an experiment which can actually be performed with present technology and which does not require the number of emissions $N$. Quantum mechanics violates this inequality by a factor of 1.5, whereas it violates the previous strong inequalities (for example CHSH inequality of 1969 [4], or CH inequality of 1974 [5]) by a factor of $\sqrt{2}$. Thus the magnitude of violation of the inequality derived in this paper is approximately 20.7% larger than the magnitude of violation of CH (or CHSH) inequality. Moreover, inequality (29) requires the measurements of only two detection probabilities (at polarizere angles $0^\circ$ and $120^\circ$), whereas CH or CHSH inequality requires the measurements of five detection probabilities. This result can be of considerable significance for the experimental test of locality where the time during which the source emits particles is usually very limited and it is highly desirable to perform the least number of measurements.
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