Locally Hölder Continuity of the Solution Map to a Boundary Control Problem with Finite Mixed Control-State Constraints

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ABSTRACT
The local stability of the solution map to a parametric boundary control problem governed by semilinear elliptic equations with finite mixed pointwise constraints is considered in this paper. We prove that the solution map is locally Hölder continuous in $L^\infty$-norm of control variable when the strictly nonnegative second-order optimality conditions are satisfied for the unperturbed problem.

1. Introduction
Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with the boundary $\Gamma$ of class $C^{1,1}$. We consider the following parametric boundary control problem. For each parameter $\lambda \in L^\infty(\Gamma)$, determine a control function $u \in L^2(\Gamma)$, and a corresponding state function $y \in W^{1,r}(\Omega), 2 < r < 4$, which

$$
\text{minimize } J(y, u, \lambda) = \int_\Omega L(x, y(x)) \, dx + \int_\Gamma \left( \ell(x, y(x), \lambda(x)) + \alpha(\lambda(x)) u(x) + \frac{1}{2} \beta(\lambda(x)) u^2(x) \right) \, ds,
$$

subject to

$$
\begin{align*}
Ay + h(x, y) &= 0 \quad \text{in } \Omega, \\
\partial_n y &= u + \lambda \quad \text{on } \Gamma, \\
g_i(x, y(x), \lambda(x)) + u(x) &\leq 0 \text{ a.e. } x \in \Gamma, \; i = 1, 2, \ldots, m,
\end{align*}
$$

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where the maps \( L, h : \Omega \times \mathbb{R} \to \mathbb{R}, \ell, g_i : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are Carathéodory functions fulfilling \( L(x, \cdot), h(x, \cdot), \ell(x', \cdot, \cdot) \) and \( g_i(x', \cdot, \cdot) \) with \( i = 1, 2, \ldots, m \) are of class \( C^2 \) for a.e. \( x \in \Omega, x' \in \Gamma \); the functions \( \alpha, \beta : \mathbb{R} \to \mathbb{R} \) are of class \( C^2 \) and \( m \geq 2 \) is a positive integer. Moreover, \( A \) denotes a second-order elliptic operator of the form

\[
Ay(x) = -\sum_{i,j=1}^{2} D_j(a_{ij}(x)D_iy(x)) + a_0(x)y(x),
\]

where coefficients \( a_{ij} \in C^{0,1}(\overline{\Omega}) \) satisfy \( a_{ij}(x) = a_{ji}(x), a_0 \in L^\infty(\Omega), a_0(x) \geq 0 \) for a.e. \( x \in \Omega, a_0 \not\equiv 0 \), and there exists \( C_0 > 0 \) such that

\[
C_0 \| \xi \|^2 \leq \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \quad \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \quad \text{for a.e.} \quad x \in \Omega,
\]

in which \( \nu(x) = (\nu_1(x), \nu_2(x)) \) stands for the unit outward normal to \( \Gamma \) at the point \( x \). Here, the measure on the boundary \( \Gamma \) is the usual 1-dimensional measure induced by the parametrization (see [16]).

Throughout the paper, for each \( \lambda \in L^\infty(\Gamma) \) let us denote by \( (P(\lambda)) \) the problem (1) – (3), by \( \mathcal{F}(\lambda) \) the feasible set of \( (P(\lambda)) \) and by \( \mathcal{S}(\lambda) \) the solution set of \( (P(\lambda)) \). Given a fixed parameter \( \bar{\lambda} \in L^\infty(\Gamma) \), we call \( (P(\bar{\lambda})) \) the unperturbed problem and assume that \( \bar{z} = (\bar{y}, \bar{u}) \in \mathcal{F}(\bar{\lambda}). \) The goal of this paper is to study the behavior of \( \mathcal{S}(\lambda) \) when \( \lambda \) varies around \( \bar{\lambda} \) and to estimate the error \( \| \hat{y}_\lambda - \bar{y} \| + \| \hat{u}_\lambda - \bar{u} \| \) for \( (\hat{y}_\lambda, \hat{u}_\lambda) \in \mathcal{S}(\lambda). \)

The solution stability for nonlinear optimal control problems plays a vital role in parameter estimation problems and in numerical methods of finding optimal solutions (see [5, 6, 10]). The solution stability ensures that the solution set of perturbed problems is not very far away from the solution set of the unperturbed problems; in other words, the approximate solutions converge to the original solution.

The stability of the solution map to optimal control problems governed by elliptic equations has been studied by several authors recently. A family of linear-quadratic optimal control problems with pointwise mixed state-control constraints is considered in [1, 4]. The authors proved that the solution maps are single-valued and Lipschitz continuous in parameters in \( L^2 \)-norm (see [4]) and \( L^\infty \)-norm (see [1]) under some certain conditions. In general, the solution maps to nonlinear control problems may not be singleton (see [8, Example 4.1] or [13, Example 4.2]). Then several tools of set-valued analysis, variational analysis can be applied for researching these problems. For instance, in [8, 13, 15] the authors showed that the solution maps of some nonlinear control problems with one
pointwise control-state constraint are not singleton but they are lower semi-
continuous, upper semicontinuous and continuous in parameters by using first-
order necessary optimality conditions. However, by a general implicit function
theorem and some additional second-order conditions, Lipschitz continuity and
singleton of the solution maps to semilinear problems can be obtained. In [11],
Malanowski et al. showed that under coercivity condition the locally solutions
to the parameter dependent elliptic optimal control problems with nonlinear
boundary control are unique and Lipschitz continuous in parameter. Griesse
et al. [5] used the strongly second-order optimality conditions to prove that the
solution maps are Lipschitz continuous in parameters in $L^\infty$-norm of control
variable when they considered the semilinear elliptic optimal boundary con-
trol problems with finite nonlinear pointwise mixed control-state constraints.
Recently, Kien et al. [9] used the no-gap second-order optimality conditions and
the metric projections in Hilbert spaces to obtain the locally Hölder continuity
of the solution map at the reference point not only in $L^2$-norm but also in
$L^\infty$-norm of control variables. By the same method and using a larger critical
cone, Tuan [17] showed that the solution map is locally Lipschitz continuous
in parameter. Notice that the authors in [9] and [17] considered the distributed
control problem in which the number of constraints is two.

In this paper, we are going to develop the method in [9] to obtain the same
result for the boundary control problem (1)–(3). Namely, if $(\hat{y}, \hat{u}),\ (\hat{y}_\lambda, \hat{u}_\lambda)$ are
locally optimal solutions to the problems $P(\lambda)$ and $P(\lambda)$, respectively, then $\|\hat{y}_\lambda - 
\hat{y}\|_{W^{1,2}(\Omega)} + \|\hat{u}_\lambda - \hat{u}\|_{L^\infty(\Gamma)} \leq C\|\lambda - \lambda\|_{L^\infty(\Gamma)}^{1/2}$ with a constant $C > 0$ when $\lambda$
varies around $\bar{\lambda}$.

We emphasize that the number of constraints in our problem is any positive
integer $m$ and so, it may be greater than two. The estimation of $\|\hat{u}_\lambda - \hat{u}\|_{L^\infty(\Gamma)}$, as we see in Section 3, depends on the number of Lagrange multipliers which
equals the number of constraints. Therefore, the more constraints the problem
has, the more difficult it is to derive some estimations. Consequently, although
we follow the method in [9], some significant improvement in the proof tech-
nique will be required. In particular, we have to establish the extra assumption
(H5) which seems to generalize the assumption (H4) in [9].

The plan of this paper is as follows. In the next section, we set up assumptions
and state our main result. Section 3 is devoted to the proof of the main result.

2. Assumptions and the main result

Hereafter given a Banach space $X$, $x_0 \in X$ and $r > 0$, let us denote by $X^*$ the
dual space of $X$, and by $B_X(x_0, r), \overline{B_X(x_0, r)}$ the open ball and the closed ball with
center $x_0$ and radius $r$, respectively. We will write $B(x_0, r)$ and $\overline{B(x_0, r)}$ when no
confusion. Also, $B_X$ and $\overline{B_X}$ denote the open unit ball and the closed unit ball,
respectively.
Let $\bar{z} = (\bar{y}, \bar{u}) \in W^{1,r}(\Omega) \times L^2(\Gamma)$. For a number $R > 0$, we define

$$
\mathcal{F}_R(\lambda) = \mathcal{F}(\lambda) \cap B(\bar{z}, R),
$$

$$
\mathcal{S}_R(\lambda) = \left\{ (y_\lambda, u_\lambda) \in \mathcal{F}_R(\lambda) \mid J(y_\lambda, u_\lambda, \lambda) = \inf_{(y, u) \in \mathcal{F}_R(\lambda)} J(y, u, \lambda) \right\}.
$$

In this section, $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function which stands for $L, h$, and $\psi : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function which stands for $\ell, g_i$ with $i = 1, 2, \ldots, m$. Given a couple $(\bar{y}, \bar{u}) \in \mathcal{F}(\bar{n})$, symbols $\varphi[x], \varphi_y[x], \varphi[,], \varphi_{y[,]}$, $\varphi[x], \varphi_y[x], \varphi[,], \varphi_{y[\cdot]}$. etc, stand for $\varphi(x, \bar{y}(x)), \varphi_y(x, \bar{y}(x)), \varphi[,](\cdot, \bar{y}), \varphi_{y}(\cdot, \bar{y}), \varphi(x, \bar{y}(x), \bar{n}(x)), \psi_y(x, \bar{y}(x), \bar{n}(x)), \psi(\cdot, \bar{y}(\cdot), \bar{n}(\cdot)), \psi_y(\cdot, \bar{y}(\cdot), \bar{n}(\cdot))$, etc, respectively.

We now impose the following assumptions on $\varphi, \psi, \alpha$ and $\beta$.

\textbf{(H1)} $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class $C^2$ with respect to the second variable and for each $M > 0$, there exists a positive number $k_{\varphi,M}$ such that

$$
|\varphi(x, y_1) - \varphi(x, y_2)| + |\varphi_y(x, y_1) - \varphi_y(x, y_2)| + |\varphi_{yy}(x, y_1) - \varphi_{yy}(x, y_2)| \\
\leq k_{\varphi,M}|y_1 - y_2|
$$

for a.e. $x \in \Omega$ and for all $y_i$ satisfying $|y_i| \leq M$ with $i = 1, 2$. Furthermore, $\varphi(\cdot, 0)$ and $\varphi_y(\cdot, 0)$ belong to $L^\infty(\Omega)$.

\textbf{(H2)} $\psi : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class $C^2$ with respect to the second and the third variable, and for each $M > 0$, there exists a positive number $k_{\psi,M}$ such that one has

$$
|\psi(x, y_1, \lambda_1) - \psi(x, y_2, \lambda_2)| + |\psi_y(x, y_1, \lambda_1) - \psi_y(x, y_2, \lambda_2)| \\
+ |\psi_{yy}(x, y_1, \lambda_1) - \psi_{yy}(x, y_2, \lambda_2)| + |\psi_{\lambda\lambda}(x, y_1, \lambda_1) - \psi_{\lambda\lambda}(x, y_2, \lambda_2)| \\
\leq k_{\psi,M}|y_1 - y_2| + |\lambda_1 - \lambda_2|
$$

for a.e. $x \in \Gamma$ and for all $y_i, \lambda_i \in \mathbb{R}$ satisfying $|y_i| \leq M, |\lambda_i| \leq M$ with $i = 1, 2$. Furthermore, $\psi(\cdot, 0, 0)$ and $\psi_y(\cdot, 0, 0)$ belong to $L^\infty(\Gamma)$.

\textbf{(H3)} The functions $\alpha$ and $\beta$ are of class $C^2$ and for each $M > 0$, there exist numbers $k_{\alpha,M}, k_{\beta,M} > 0$ such that

$$
|\alpha(\lambda_1) - \alpha(\lambda_2)| \leq k_{\alpha}|\lambda_1 - \lambda_2|,
$$

$$
|\beta(\lambda_1) - \beta(\lambda_2)| \leq k_{\beta}|\lambda_1 - \lambda_2|
$$

for all $\lambda_i \in \mathbb{R}$ satisfying $|\lambda_i| \leq M$ with $i = 1, 2$. Moreover, there exists $\gamma > 0$ such that $\beta(\bar{\lambda}(x)) \geq \gamma$ for a.e. $x \in \Omega$.

\textbf{(H4)} $h(x, 0) = 0$ and $h_y(x, y) \geq 0, g_y[x'] \geq 0$ for a.e. $x \in \Omega, x' \in \Gamma$ and for all $y \in \mathbb{R}, i = 1, 2, \ldots, m$.

\textbf{(H5)} There exist measurable subsets $\Gamma_i$ of $\Gamma, i = 1, 2, \ldots, m$ such that

\begin{enumerate}
  \item $\Gamma = \bigcup_{i=1}^{m}\Gamma_i$,
  \item $\max \text{ess sup}_{x \in \Gamma_i} (g_{k,x'} - g_{i,x}) < 0$.
\end{enumerate}
It is noted that assumptions \((H1) - (H3)\) make sure that \(f, h\) and \(g_i, 1 \leq i \leq m\) are of class \(C^2\) around the referent point, while assumption \((H4)\) guarantees existence and uniqueness of solution to the state equation (2), and that the Robinson constraint qualification condition is valid. Finally, as we will see in Section 3, assumption \((H5)\) is essential for estimations in \(L^\infty\)-norm of control variable. Moreover, the condition \((ii)\) in \((H5)\) implies that the subsets \(\Gamma_k\) are disjoint.

Let \(s\) be the adjoint number of \(r\), that is, \(\frac{1}{r} + \frac{1}{s} = 1\) and \(\tau : W^{1,s}(\Omega) \rightarrow W^{1,s}(\Gamma)\) be the trace operator. Recall that for each given \(u \in W^{-1,r} (\Gamma)\), a function \(y \in W^{1,r}(\Omega)\) is said to be a (weak or variational) solution of (2) if

\[
\int_\Omega \sum_{i,j=1}^2 a_{ij}(x) D_i y(x) D_j v(x) dx + \int_\Omega a_0(x) y(x) v(x) dx + \int_\Omega h(x, y(x)) v(x) dx = \int_\Gamma (u(x) + \lambda(x)) \tau v(x) ds
\]

for all \(v \in W^{1,s}(\Omega)\). It is known from [14, Theorem 3.1] that under the assumption \((H4)\), for each \(u \in L^2(\Gamma)\) and \(\lambda \in L^\infty(\Gamma)\), the Equation (2) has a unique solution \(y \in W^{1,r}(\Omega)\) and there exists a constant \(C_1 > 0\) such that

\[
\|y\|_{W^{1,r}(\Omega)} \leq C_1 (\|u\|_{L^2(\Gamma)} + \|\lambda\|_{L^\infty(\Gamma)}).
\]

(5)

From \((H1) - (H3)\), it follows that for each fixed parameter \(\lambda \in B_{L^\infty(\Gamma)}(\lambda, \epsilon_1)\) and for any \((\hat{y}, \hat{u}) \in \mathcal{F}(\lambda)\) and \((y, u) \in W^{1,r}(\Omega) \times L^2(\Gamma)\), we have

\[
\langle \nabla_z J(\hat{y}, \hat{u}, \lambda), (y, u) \rangle = \int_\Omega L_y(x, \hat{y}(x)) y(x) dx + \int_\Gamma (\ell_y(x, \hat{y}(x), \lambda(x)) y(x)
\]

\[
+ \alpha(\lambda(x)) u(x) + \beta(\lambda(x)) \hat{u}(x) u(x) ) ds.
\]

**Definition 2.1.** We say that a couple \((y, u) \in W^{1,r}(\Omega) \times L^2(\Gamma)\) is a critical direction for \((P(\lambda))\) at \((\bar{y}, \bar{u})\) if the following conditions are fulfilled:

(i) \(\langle \nabla_z J(\bar{y}, \bar{u}, \lambda), (y, u) \rangle \leq 0\),

(ii) \[
\begin{align*}
Ay + h_y \cdot \lambda &= 0 & \text{in } \Omega, \\
\partial_\nu y &= u & \text{on } \Gamma,
\end{align*}
\]

(iii) \(g_i y(x) + u(x) \leq 0\) whenever \(g_i x + \bar{u}(x) = 0\), \(i = 1, 2, \ldots, m\).

Denote by \(\mathcal{C}(\bar{y}, \bar{u}, \lambda)\) the set of critical directions of \((P(\lambda))\) at \(\bar{z} = (\bar{y}, \bar{u})\). It is clear that \(\mathcal{C}(\bar{y}, \bar{u}, \lambda)\) is a closed and convex cone.

**Definition 2.2.** We say that the functions \(\vartheta_\lambda \in W^{1,s}(\Omega)\) and \(e_{\lambda 1}, e_{\lambda 2}, \ldots, e_{\lambda m} \in L^2(\Gamma)\) are Lagrange multipliers of \((P(\lambda))\) at \((\hat{y}_\lambda, \hat{u}_\lambda)\) if they satisfy the following conditions:
(i) The adjoint equation:
\[
\begin{cases}
A \vartheta_\lambda + h_\gamma(\cdot, \hat{y}_\lambda) \vartheta_\lambda = -L_y(\cdot, \hat{y}_\lambda) & \text{in } \Omega, \\
\vartheta_\lambda(\cdot, \hat{y}_\lambda) = -\ell_y(\cdot, \hat{y}_\lambda) - \sum_{i=1}^m g_{iy}(\cdot, \hat{y}_\lambda, \lambda) e_{\lambda,i} & \text{on } \Gamma;
\end{cases}
\]  

(6)

(ii) The stationary condition in \(u\):
\[-\vartheta_\lambda + \alpha(\lambda) + \beta(\lambda) \hat{u}_\lambda + \sum_{i=1}^m e_{\lambda,i} = 0;\]  

(7)

(iii) The complementary conditions:
\[e_{\lambda,i}(x) \geq 0, \quad e_{\lambda,i}(x)(g_i(x, \hat{y}_\lambda(x), \lambda(x)) + \hat{u}_\lambda(x)) = 0 \]  

for each \(i \in \{1, 2, \ldots, m\}\) and for a.e. \(x \in \Gamma\).

Let us denote by \(\Lambda[(\hat{y}_\lambda, \hat{u}_\lambda), \lambda]\) and \(\Lambda[(\bar{y}, \bar{u}), \bar{\lambda}]\) the sets of Lagrange multipliers of \((P(\lambda))\) at \((\hat{y}_\lambda, \hat{u}_\lambda)\) and \((P(\bar{\lambda}))\) at \((\bar{y}, \bar{u})\), respectively.

**Definition 2.3.** We say that \(\bar{z} = (\bar{y}, \bar{u})\) is a locally strongly optimal solution of \((P(\bar{\lambda}))\) if there exist numbers \(\delta > 0\) and \(\sigma > 0\) such that
\[J(z, \lambda) \geq J(\bar{z}, \bar{\lambda}) + \sigma \|z - \bar{z}\|^2 \quad \forall z \in B(\bar{z}, \delta) \cap \mathcal{F}(\bar{\lambda}).\]

We now state the main result of the paper.

**Theorem 2.1.** Suppose that the assumptions (H1) – (H4) are fulfilled and \((\vartheta, e_1, e_2, \ldots, e_m) \in \Lambda[(\bar{y}, \bar{u}), \bar{\lambda}]\) such that
\[
\int_\Omega (L_{yy}[x]y^2(x) + \vartheta h_{yy}[x]y^2(x))dx \\
+ \int_\Gamma (\ell_{yy}[x]y^2(x) + \beta(\bar{\lambda}(x))u^2(x) + \sum_{i=1}^m e_i(x)g_{y[y][x]y^2(x)}ds > 0
\]  

(9)

for all \((y, u) \in C[(\bar{y}, \bar{u}), \bar{\lambda}] \setminus \{(0,0)\}\). Then \((\bar{y}, \bar{u})\) is a locally strongly optimal solution of \((P(\bar{\lambda}))\) and there exist positive numbers \(R_0, S_0\) and \(l_0\) such that for all \(\lambda \in B_{L^\infty(\Gamma)}(\bar{\lambda}, S_0)\), every couple \((\hat{y}_\lambda, \hat{u}_\lambda) \in \mathcal{S}_{R_0}(\lambda)\) is a locally optimal solution of \((P(\lambda)), \hat{u}_\lambda \in L^\infty(\Gamma)\) and
\[\|\hat{y}_\lambda - \bar{y}\|_{W^{1,r}(\Omega)} + \|\hat{u}_\lambda - \bar{u}\|_{L^2(\Gamma)} \leq l_0\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2}.\]  

(10)

In addition, if assumption (H5) is satisfied then there exist positive numbers \(s_1\) and \(l_1\) such that for all \(\lambda \in B_{L^\infty(\Gamma)}(\bar{\lambda}, s_1)\) and \((\hat{y}_\lambda, \hat{u}_\lambda) \in \mathcal{S}_{R_0}(\lambda)\) one has
\[\|\hat{y}_\lambda - \bar{y}\|_{W^{1,r}(\Omega)} + \|\hat{u}_\lambda - \bar{u}\|_{L^\infty(\Gamma)} \leq l_1\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2}.\]  

(11)

3. **Proof of the main result**

In this section, we present the proof of Theorem 2.1. The proof will be divided into a sequence of lemmas.
We begin by proving the solution stability in $L^2$-norm of control variable. To this end, we shall transfer problem $P(\lambda)$ to a mathematical programming problem and apply [9, Proposition 3.1]. For this, we put
\[
Y := W^{1,r}(\Omega), \quad U := L^2(\Gamma), \quad Z := Y \times U, \quad \Pi = L^\infty(\Gamma),
\]
\[
E_0 := (W^{1,s}(\Omega))^*, \quad E := (L^2(\Gamma))^m, \quad K = \prod_{i=1}^m K_i
\]
where
\[
K_i = \{ v \in L^2(\Gamma) \mid v(x) \leq 0 \quad \text{a.e.} \quad x \in \Gamma \}, \quad i = 1, 2, \ldots, m.
\]
These spaces are separable and reflexive Banach space and $K$ is a unbounded, convex and closed subset in $E$. Let us define mappings
\[
H : Y \times U \times \Pi \to E_0, \quad H(y, u, \lambda) = \sum_{i,j=1}^2 a_{ij}D_i y D_j + a_0 y + h(\cdot, y) - \tau^*(u + \lambda),
\]
\[
G : Y \times U \times \Pi \to E, \quad G(y, u, \lambda) = (G_1(y, u, \lambda), G_2(y, u, \lambda), \ldots, G_m(y, u, \lambda)),
\]
where $\tau^*$ is the adjoint operator of the trace operator $\tau$, and
\[
(H(y, u, \lambda), v) := \int_\Omega \left( \sum_{i,j=1}^2 a_{ij}D_i y D_j v + a_0 y v + h(x, y)v \right) dx
\]
\[- \int_\Gamma (u + \lambda) v ds \quad \forall v \in W^{1,s}(\Omega),
\]
and $G_i(y, u, \lambda) := g_i(\cdot, y, \lambda) + u, i = 1, 2, \ldots, m$. Also, we define the set
\[
D(\lambda) = \{ z = (y, u) \in Y \times U \mid H(y, u, \lambda) = 0 \}.
\]
Since $r > 2$, $Y \hookrightarrow C(\overline{\Omega})$. Moreover, analysis similar to that in [12] shows that the mappings $F$ and $G$ are well-defined.

Therefore, the problem $(P(\lambda))$ can be written in the following form
\[
(P(\lambda)) \begin{cases} 
J(y, u, \lambda) \to \min, \\
H(y, u, \lambda) = 0, \\
G(y, u, \lambda) \in K,
\end{cases}
\]
or in the simpler form
\[
(P(\lambda)) \begin{cases} 
J(z, \lambda) \to \min, \\
z \in \mathcal{F}(\lambda),
\end{cases}
\]
where $\mathcal{F}(\lambda) := D(\lambda) \cap G^{-1}(K)$ is the feasible set of problem $P(\lambda)$.

Recall that for given a closed set $D$ in $Z$ and a point $z \in D$, the adjacent tangent cone and the contingent cone to $D$ at $z$ are defined by
\[
T^0(D, z) = \{ h \in Z \mid \forall t_n \to 0^+, \exists h_n \to h, z + t_n h_n \in D \},
\]
\[ T(D, z) = \{ h \in \mathbb{Z} | \exists t_n \rightarrow 0^+, \exists h_n \rightarrow h, z + t_n h_n \in D \}, \]

respectively. These cones are closed and \( T^0(D, z) \subseteq T(D, z) \). It is well known that when \( D \) is convex, then
\[ T^0(D, z) = T(D, z) = \text{cone}(D - z) \]

and the normal cone to \( D \) at \( z \) is given by
\[ N(D, z) = \{ z^* \in \mathbb{Z}^* | \langle z^*, d - z \rangle \leq 0 \ \forall d \in D \}, \]

where \( \text{cone}(D - z) : = \{ k(d - z) : d \in D, k > 0 \} \) is the cone generated by \( (D - z) \).

**Lemma 3.1.** If the assumptions (H1)–(H4) are fulfilled then the following conditions hold:

(i) There exist positive numbers \( r_1, r_1', r_1'' \) such that for any \( \lambda \in B_\Pi(\bar{\lambda}, r_1') \), the mappings \( J(\cdot, \cdot, \lambda), H(\cdot, \cdot, \lambda) \) and \( G(\cdot, \cdot, \lambda) \) are twice Fréchet differentiable on \( B_Y(\bar{y}, r_1) \times B_U(\bar{u}, r_1') \). The mapping \( H(\cdot, \cdot, \cdot) \) is continuously Fréchet differentiable on \( B_Y(\bar{y}, r_1) \times B_U(\bar{u}, r_1') \times B_\Pi(\bar{\lambda}, r_1'') \).

(ii) There exist constants \( k_G > 0 \) and \( k_J > 0 \) such that
\[
\| G(z_1, \lambda_1) - G(z_2, \lambda_2) \| + \| \nabla G(z_1, \lambda_1) - \nabla G(z_2, \lambda_2) \| \\
\leq k_G(\| z_1 - z_2 \| + \| \lambda_1 - \lambda_2 \|),
\]

\[
\| J(z_1, \lambda_1) - J(z_2, \lambda_2) \| + \| \nabla J(z_1, \lambda_1) - \nabla J(z_2, \lambda_2) \| \\
\leq k_J(\| z_1 - z_2 \| + \| \lambda_1 - \lambda_2 \|)
\]

for all \( z_1, z_2 \in B_Y(\bar{y}, r_1) \times B_U(\bar{u}, r_1') \) and \( \lambda_1, \lambda_2 \in B_\Pi(\bar{\lambda}, r_1'') \).

(iii) The mapping \( H_y(\bar{z}, \bar{\lambda}) \) is bijective.

(iv) \( E = \nabla_z G(\bar{z}, \bar{\lambda})(T(D(\bar{\lambda}), \bar{z})) - \text{cone}(K - G(\bar{z}, \bar{\lambda})) \).

**Remark 3.1.** The condition (iv) in Lemma 3.1 means that the unperturbed problem \( P(\bar{\lambda}) \) satisfies the Robinson constraint qualification. According to [7, Theorem 2.5], this condition is equivalent to that there exists a number \( \rho_0 > 0 \) such that
\[ B_E(0, \rho_0) \subset \nabla_z G(\bar{z}, \bar{\lambda})(T(D(\bar{\lambda}), \bar{z}) \cap B_Z) - (K - G(\bar{z}, \bar{\lambda})) \cap B_E. \]

**Proof** Obviously, the assumptions (H1)–(H3) imply the conditions (i) and (ii). Moreover, for all \( \lambda \in B_Z \Pi(\bar{\lambda}, r_1') \) and \((\hat{y}, \hat{u}) \in \mathcal{F}(\lambda)\), an easy computation shows that
\[
\langle \nabla_z J(\hat{y}, \hat{u}, w), (y, u) \rangle = \int_{\Omega} L_y(x, \hat{y}(x)) y(x) dx + \int_{\Gamma} \left( \ell_y(x, \hat{y}(x), \lambda(x)) y(x) + \alpha(\lambda(x)) u(x) + \beta(\lambda(x)) \hat{u}(x) u(x) \right) ds,
\]

\[
\langle \nabla_z J(\bar{\lambda}, \bar{u}, w), (y, u) \rangle = \int_{\Omega} L_y(x, \bar{y}(x)) y(x) dx + \int_{\Gamma} \left( \ell_y(x, \bar{y}(x), \bar{\lambda}(x)) y(x) + \alpha(\bar{\lambda}(x)) u(x) + \beta(\bar{\lambda}(x)) \bar{u}(x) u(x) \right) dx,
\]

respectively.
\[ \nabla_y H(\bar{z}, \bar{\lambda}) y = \sum_{i,j=1}^{2} a_{ij} D_i y D_j + a_0 y + h_y[\cdot] y, \]
\[ \nabla_u H(\bar{z}, \bar{\lambda}) u = -\tau^* u, \]
\[ \nabla_z G(\bar{z}, \bar{\lambda}) = (\nabla_z G_1(\bar{z}, \bar{\lambda}), \nabla_z G_2(\bar{z}, \bar{\lambda}), \ldots, \nabla_z G_m(\bar{z}, \bar{\lambda})), \]
\[ \nabla_z G_i(\bar{z}, \bar{\lambda}) = (g_{iy}[\cdot], I_U) \quad \forall i = 1, 2, \ldots, m, \]
where \( I_U \) is the identity mapping of \( U \).

In order to show (iii), we take any \( f_0 \in E_0 \) and consider the equation
\[ H_y(y, u, \bar{\lambda}) y = f_0 \]
which is equivalent to
\[ \begin{aligned}
   Ay + h_y[\cdot] y &= f_0 \quad \text{in } \Omega, \\
   \partial_v y &= 0 \quad \text{on } \Gamma. 
\end{aligned} \quad (12) \]

By (H4), we have \( h_y[x] \geq 0 \) for a.e. \( x \in \Omega \). Therefore, the Equation (12) has a unique solution \( y \in W^{1,1}(\Omega) \) and so (iii) is valid.

It remains to prove (iv). It follows from (iii) that \( \nabla H(\bar{z}, \bar{\lambda}) \) is surjective. By [7, Lemma 2.2], we have
\[ T(D(\bar{\lambda}), \bar{z}) = \{(y, u) \in Z \mid H_y(y, \bar{\lambda}) y + H_u(\bar{z}, \bar{\lambda}) u = 0 \} \]
\[ = \{(y, u) \in Z \mid \begin{array}{l}
   Ay + h_y[\cdot] y = 0 \quad \text{in } \Omega, \\
   \partial_v y = u \quad \text{on } \Gamma. 
\end{array} \} \]

Therefore (iv) is equivalent to saying that, for any \((f_1, f_2, \ldots, f_m) \in (L^2(\Gamma))^m\), there exist \((y, u) \in Y \times U \) and \((v_1, v_2, \ldots, v_m) \in \text{cone}(K - G(\bar{z}, \bar{\lambda})) \) such that
\[ \begin{aligned}
   Ay + h_y[\cdot] y &= 0 \quad \text{in } \Omega, \\
   \partial_v y &= u \quad \text{on } \Gamma, \\
   f_i &= g_{iy}[\cdot] y + u - v_i, \quad i = 1, 2, \ldots, m. 
\end{aligned} \quad (13) \]

Write \( f_i = f_{i1} - f_{i2} \), where \( f_{i1}, f_{i2} \in K_i, i = 1, 2, \ldots, m \). Set \( u_i = f_{i1} \) with \( i = 1, 2, \ldots, m \) and \( u = \sum_{i=1}^{m} u_i \). Since \( f_{i1}(x) \leq 0 \) for all \( i = 1, 2, \ldots, m \), one has \( u(x) \leq 0 \) a.e. \( x \in \Gamma \).

Now, we consider the following equation
\[ \begin{aligned}
   Ay + h_y[\cdot] y &= 0 \quad \text{in } \Omega, \\
   \partial_v y &= u \quad \text{on } \Gamma. 
\end{aligned} \quad (14) \]

By the assumption (H4), the Equation (14) has a unique solution \( y_u \in Y \), that is
\[ \int_{\Omega} \sum_{i,j=1}^{2} a_{ij}(x) D_i y_u(x) D_j v(x) dx + \int_{\Omega} a_0(x) y_u(x) v(x) dx \\
+ \int_{\Omega} h_y[x] y_u(x) v(x) dx = \int_{\Gamma} u(x) \tau v(x) ds \quad (15) \]
for all $v \in W^{1,s}(\Omega)$. Setting $y_u^+ = \max\{0, y_u\}$, we have $y_u^+ \geq 0$ for a.e. $x \in \Omega$ and $y_u^+ \in W^{1,2}(\Omega) \hookrightarrow W^{1,s}(\Omega)$. Taking $v = y_u^+$ in the Equation (15), we obtain

$$
\int_{\Omega} \sum_{i,j=1}^{2} a_{ij}(x) D_i y_u(x) D_j y_u^+(x) dx + \int_{\Omega} a_0(x) y_u(x) y_u^+(x) dx
+ \int_{\Omega} h_j[x] y_u(x) y_u^+(x) dx = \int_{\Gamma} u(x) \tau y_u^+(x) ds,
$$

which is equivalent to

$$
\int_{\Omega} \sum_{i,j=1}^{2} a_{ij}(x) D_i y_u^+(x) D_j y_u^+(x) dx + \int_{\Omega} a_0(x) y_u^+(x) y_u^+(x) dx
+ \int_{\Omega} h_j[x] y_u^+(x) y_u^+(x) dx = \int_{\Gamma} u(x) \tau y_u^+(x) ds.
$$

Combining this with (4) and the assumption (H4) yields

$$
C_0 \|y_u^+\|_{H^1(\Omega)}^2 \leq \int_{\Gamma} u(x) \tau y_u^+(x) ds \leq 0
$$

for a constant $C_0 > 0$. This implies that $\|y_u^+\|_{H^1(\Omega)} = 0$ and so $\|y_u^+\|_{L^2(\Omega)} = 0$. Hence $y_u^+ = 0$ for a.e. $x \in \Omega$ which leads to $y_u \leq 0$ for a.e. $x \in \Omega$. Since $y_u \in C(\overline{\Omega})$, we get $y_u \leq 0$ for all $x \in \overline{\Omega}$. Combining this with the assumption (H4) yields $g_y[\cdot] y_u \leq 0$. It follows that

$$
g_y[\cdot] y_u + u - f_{i1} = g_y[\cdot] y_u + \sum_{j \neq i} u_j \in K_i.
$$

Define

$$
d_i = g_i[\cdot] + \overline{u} + g_y[\cdot] y_u + u - f_i, \quad i = 1, 2, \ldots, m.
$$

Then $d_i \in K_i$, and

$$
f_i = g_y[\cdot] y_u + u - (d_i - g_i[\cdot] - \overline{u}),
$$

for all $i = 1, 2, \ldots, m$, and $(y_u, u) \in T(D(\overline{\lambda}), \overline{\lambda})$. Hence, (13) is valid with $\nu_i = d_i - g_i[\cdot] - \overline{u}$ for $i = 1, 2, \ldots, m$. Consequently, (iv) is satisfied. The lemma is proved.

According to [2] and [12, Lemma 12], we have an exact formula for computing the cones $T(K, G(\overline{\lambda}, \overline{\lambda}))$ and $N(K, G(\overline{\lambda}, \overline{\lambda}))$ which will be used in the proof of next lemma:

$$
T(K, G(\overline{\lambda}, \overline{\lambda})) = \prod_{i=1}^{m} T(K_i, G_i(\overline{\lambda}, \overline{\lambda})),
$$

$$
N(K, G(\overline{\lambda}, \overline{\lambda})) = \prod_{i=1}^{m} N(K_i, G_i(\overline{\lambda}, \overline{\lambda})).
$$
where
\[ T(K_i, G_i(z, \lambda)) = \{ w \in L^2(\Gamma) \mid w(x) \in T((-\infty, 0], G_i(z, \lambda)(x)) \text{ a.e. } x \in \Gamma \} \]
\[ = \{ w \in L^2(\Gamma) \mid w(x) \leq 0 \text{ whenever } g_i(x) + \bar{u}(x) = 0 \} , \]
and
\[ N(K_i, G_i(z, \lambda)) = \{ w^* \in L^2(\Gamma) \mid w^*(x) \in N((-\infty, 0], G_i(z, \lambda)(x)) \text{ a.e. } x \in \Gamma \} \]
\[ = \left\{ w^* \in L^2(\Gamma) \mid \begin{array}{ll}
    w^*(x) \geq 0 & \text{whenever } g_i(x) + \bar{u}(x) = 0 \\
    w^*(x) = 0, & \text{otherwise}
\end{array} \right\} . \]

The solution stability in \( L^2 \)-norm is established in the following lemma. The proof of this lemma is followed by the same technique as in [9, Lemma 4.1]. For the convenience of the reader, we provide a proof here.

**Lemma 3.2.** Under the assumptions of Theorem 2.1, \((\bar{y}, \bar{u})\) is a locally strongly optimal solution of \((P(\bar{\lambda}))\) and there exist positive numbers \(R_0, s_0\) and \(l_0\) such that for all \(\lambda \in B_{L_{\infty}(\Gamma)}(\bar{\lambda}, s_0)\), every couple \((\hat{y}_\lambda, \hat{u}_\lambda) \in S_{R_0}(\lambda)\) is a locally optimal solution of \((P(\lambda))\) and
\[
\| \hat{y}_\lambda - \bar{y} \|_{L^\infty(\Omega)} + \| \hat{u}_\lambda - \bar{u} \|_{L^2(\Gamma)} \leq l_0 \| \lambda - \bar{\lambda} \|_{L^\infty(\Gamma)}^{1/2},
\]
(16)

**Proof** First, we show that \(\bar{z} = (\bar{y}, \bar{u})\) is a locally strongly optimal solution of \((P(\bar{\lambda}))\). By contradiction, suppose that this assertion is false. Then there is a sequence \(\{z_k = (y_k, u_k)\} \subset \mathcal{F}(\bar{\lambda}), z_k \to \bar{z}\) such that
\[
J(z_k, \bar{\lambda}) < J(\bar{z}, \bar{\lambda}) + o(t_k^2),
\]
(17)
where \(t_k := \|z_k - \bar{z}\|_Z \to 0\) as \(k \to \infty\). Putting \(\hat{z}_k = (\hat{y}_k, \hat{u}_k), \hat{y}_k = \frac{y_k - \bar{y}}{t_k}, \hat{u}_k = \frac{u_k - \bar{u}}{t_k}\), we have \(\|\hat{z}_k\|_Z = \|\hat{y}_k\|_Y + \|\hat{u}_k\|_U = 1\). Since \(Z\) is reflexive, we may assume that \(\hat{z}_k \rightharpoonup \hat{z} = (\hat{y}, \hat{u})\). Moreover, since the embedding \(Y = W^{1,r}(\Omega) \hookrightarrow C(\bar{\Omega})\) is compact, \(\hat{y}_k \to \hat{y}\) in \(C(\bar{\Omega})\).

**Claim 1.** \(\hat{z} \in \mathcal{C}(\bar{z}, \bar{\lambda})\).

Writing \(z_k = \bar{z} + t_k \hat{z}_k\), it follows from a first-order Taylor expansion and (17) that
\[
\nabla_z J(\bar{z}, \bar{\lambda}) \hat{z}_k + o(t_k) \leq o(t_k^2).
\]
Passing to the limit as \(k \to \infty\), we obtain \(\nabla_z J(\bar{z}, \bar{\lambda}) \hat{z} \leq 0\).

Since \(z_k \in \mathcal{F}(\bar{\lambda}), H(z_k, \bar{\lambda}) = H(\bar{z}, \bar{\lambda}) = 0\) for all \(k \geq 1\). By a similar argument, we get \(\nabla_z H(\bar{z}, \bar{\lambda}) \hat{z} = 0\), which is equivalent to
\[
\begin{cases}
    A\hat{y} + h(y) \hat{y} = 0 & \text{in } \Omega \\
    \partial_{\nu}(\hat{y}) = \hat{u} & \text{on } \Gamma.
\end{cases}
\]

Also, since \(G(z_k, \bar{\lambda}) - G(\bar{z}, \bar{\lambda}) \in K - G(\bar{z}, \bar{\lambda})\), we have
\[
\nabla_z G(\bar{z}, \bar{\lambda}) \hat{z}_k \in \frac{1}{t_k} (K - G(\bar{z}, \bar{\lambda})) \subset T(K, G(\bar{z}, \bar{\lambda})).
\]
Since \( T(K, G(\bar{z}, \bar{\lambda})) \) is weakly closed, we obtain \( \nabla_z G(\bar{z}, \bar{\lambda})\hat{z} \in T(K, G(\bar{z}, \bar{\lambda})) \). This is equivalent to
\[
g_{iy}[x]\hat{y}(x) + \hat{u}(x) \leq 0 \text{ whenever } g_i[x] + \bar{u}(x) = 0, \quad i = 1, 2, \ldots, m.
\]
Hence \( \hat{y} \) and \( \hat{u} \) satisfy the conditions (i)–(iii) of Definition 2.1, and so \((\hat{y}, \hat{u}) \in \mathcal{C}[\bar{z}, \bar{\lambda}]\).

**Claim 2.** \((\vartheta, e_1, e_2, \ldots, e_m) \in \Lambda[(\bar{y}, \bar{u}), \bar{\lambda}]\) is equivalent to the following conditions:
\[
\nabla_z \mathcal{L}(\bar{z}, \vartheta, e) = 0, \quad e \in N(K, G(\bar{z}, \bar{\lambda})), \quad e = (e_1, e_2, \ldots, e_m),
\]
where Lagrangian \( \mathcal{L}(z, \vartheta, e) \) is given by
\[
\mathcal{L}(z, \vartheta, e) = J(z, \bar{\lambda}) + \langle \vartheta, H(z, \bar{\lambda}) \rangle + \langle e, G(z, \bar{\lambda}) \rangle.
\]
In fact, the condition \( \nabla_z \mathcal{L}(\bar{z}, \vartheta, e) = 0 \) is rewritten as follows
\[
\begin{align*}
-A\vartheta + h_y(\bar{y})\vartheta + L_y(\cdot, \bar{y}) &= 0 & \text{in } \Omega, \\
\partial_i\vartheta + \ell_y[\cdot] + \sum_{i=1}^m g_y[\cdot]e_i &= 0 & \text{on } \Gamma,
\end{align*}
\]
\[
\alpha(\bar{\lambda}) + \beta(\bar{\lambda})\bar{u} - \vartheta + \sum_{i=1}^m e_{\lambda i} = 0,
\]
which is equivalent to the conditions (6) and (7); while the condition \( e \in N(K, G(\bar{z}, \bar{\lambda})) \) means \( e_i(x) \in N((-\infty, 0], g_i[x] + u(x)) \) for a.e. \( x \in \Gamma \), or
\[
e_i(x) \begin{cases} 
\geq 0 & \text{whenever } g_i[x] + \bar{u}(x) = 0 \\
= 0, & \text{otherwise}
\end{cases}
\]
for all \( i = 1, 2, \ldots, m \). By a simple argument, the above condition is equivalent to
\[
e_i(x) \geq 0, \quad e_i(x)(g_i[x] + u(x)) = 0 \text{ a.e. } x \in \Gamma, \quad i = 1, 2, \ldots, m
\]
which is the condition (8). Hence, the claim is justified.

**Claim 3.** \( \hat{z} = 0 \).

By a second-order Taylor expansion for \( \mathcal{L} \), one has
\[
\mathcal{L}(z_k, \vartheta, e) - \mathcal{L}(\bar{z}, \vartheta, e) = t_k \nabla_z \mathcal{L}(\bar{z}, \vartheta, e)\hat{z}_k + \frac{t_k^2}{2} \nabla_{zz}^2 \mathcal{L}(\bar{z}, \vartheta, e)(\hat{z}_k, \hat{z}_k) + o(t_k^2)
\]
\[
= 0 + \frac{t_k^2}{2} \nabla_{zz}^2 \mathcal{L}(\bar{z}, \vartheta, e)(\hat{z}_k, \hat{z}_k) + o(t_k^2).
\]
On the other hand, we get
\[
\mathcal{L}(z_k, \vartheta, e) - \mathcal{L}(\bar{z}, \vartheta, e)
\]
\[
= J(z_k, \bar{\lambda}) - J(\bar{z}, \bar{\lambda}) + \langle \vartheta, H(z_k, \bar{\lambda}) - H(\bar{z}, \bar{\lambda}) \rangle + \langle e, G(z_k, \bar{\lambda}) - G(\bar{z}, \bar{\lambda}) \rangle
\]
\[
= J(z_k, \bar{\lambda}) - J(\bar{z}, \bar{\lambda}) + \langle e, G(z_k, \bar{\lambda}) - G(\bar{z}, \bar{\lambda}) \rangle
\]
\[
\leq J(z_k, \bar{\lambda}) - J(\bar{z}, \bar{\lambda}) \leq o(t_k^2).
\]
Here we used the fact that \( e \in N(K, G(\bar{z}, \bar{\lambda})) \), \( H(z_k, \bar{\lambda}) = H(\bar{z}, \bar{\lambda}) = 0 \) and (17).
Therefore, we have

\[
\nabla^2_{zz} \mathcal{L}(\bar{z}, \vartheta, e)(\hat{z}_k, \hat{z}_k) \leq \frac{o(t_k^2)}{t_k^2}.
\]

(18)

By letting \( k \to \infty \), we have

\[
\nabla^2_{zz} \mathcal{L}(\bar{z}, \vartheta, e)(\hat{z}, \hat{z}) \leq 0.
\]

This is equivalent to

\[
\int_{\Omega} (L_{yy}[x] \hat{y}^2(x) + \vartheta h_{yy}[x] \hat{y}^2(x)) \, dx
\]

\[
+ \int_{\Gamma} (\ell_{yy}[x] \hat{y}^2(x) + \beta(\bar{\lambda}(x)) \hat{u}^2(x) + \sum_{i=1}^{m} e_i(x) g_{iyy}[x] \hat{y}^2(x)) \, ds \leq 0.
\]

Combining this with (9), we obtain \( \hat{z} = 0 \).

**Claim 4.** There exists \( \sigma_0 > 0 \) such that \( \|\hat{u}_k\|_U > \sigma_0 \) for \( k \) large enough.

Since \( H(\bar{z}, \bar{\lambda}) = 0 \) and \( \nabla_y \mathcal{L}(\bar{y}, \bar{\mu}, \bar{\lambda}) \) is bijective, the Implicit Function Theorem implies that there exist balls \( B_{\bar{Y}}(\bar{y}, \epsilon_1), B_{U}(\bar{u}, \epsilon_2) \) and a mapping \( \phi : B_{U}(\bar{u}, \epsilon_2) \to B_{\bar{Y}}(\bar{y}, \epsilon_1) \) of class \( C^1 \) such that \( H(\phi(u), u, \bar{\lambda}) = 0 \) for all \( u \in B_{U}(\bar{u}, \epsilon_2) \). In particular, \( \phi \) is Lipschitz continuous on \( B_{U}(\bar{u}, \epsilon_2) \) with constant \( k_\phi > 0 \). Since \( z_k \to \bar{z} \), there exists a positive integer \( k_0 \) such that \( y_k = \phi(u_k) \) for \( k > k_0 \). By definition of \( (\hat{y}_k, \hat{u}_k) \), we have

\[
\|\hat{y}_k\|_Y = \frac{\|\phi(u_k) - \phi(\bar{u})\|_Y}{t_k} \leq \frac{k_\phi \|u_k - \bar{u}\|_U}{t_k} = k_\phi \|\hat{u}_k\|_U.
\]

It follows that

\[
1 = \|\hat{y}_k\|_Y + \|\hat{u}_k\|_U \leq (1 + k_\phi) \|\hat{u}_k\|_U
\]

and so \( \|\hat{u}_k\|_U \geq \sigma_0 \) for all \( k > k_0 \), where \( \sigma_0 := \frac{1}{1 + k_\phi} \).

Combining this with (18) and (H4) yields

\[
\frac{o(t_k^2)}{t_k^2} \geq \int_{\Omega} (L_{yy}[x] \hat{y}_k^2(x) + \vartheta h_{yy}[x] \hat{y}_k^2(x)) \, dx
\]

\[
+ \int_{\Gamma} (\ell_{yy}[x] \hat{y}_k^2(x) + \beta(\bar{\lambda}(x)) \hat{u}_k^2(x) + \sum_{i=1}^{m} e_i(x) g_{iyy}[x] \hat{y}_k^2(x)) \, ds
\]

\[
\geq \int_{\Omega} (L_{yy}[x] \hat{y}_k^2(x) + \vartheta h_{yy}[x] \hat{y}_k^2(x)) \, dx
\]

\[
+ \int_{\Gamma} (\ell_{yy}[x] \hat{y}_k^2(x) + \beta(\bar{\lambda}(x)) \hat{u}_k^2(x) + \sum_{i=1}^{m} e_i(x) g_{iyy}[x] y_k^2(x)) \, ds \quad (19)
\]

Notice that \( \hat{y}_k \to 0 \) in \( C(\Omega) \). By letting \( k \to \infty \), it follows from (19) that \( 0 \geq \gamma \sigma_0^2 \), which is impossible. Therefore, \( \bar{z} \) is a locally strongly optimal solution of \( (P(\bar{\lambda})) \).
Now we will use Proposition 3.1 in [9]. From Lemma 3.1 and the fact \( \bar{z} \) is a locally strongly optimal solution of \((P(\lambda))\), it follows that the assumptions of Proposition 3.1 in [9] are fulfilled. Therefore, there exist positive numbers \( R_0, s_0 \) and \( l_0 \) such that for all \( \lambda \in B(\bar{\lambda}, s_0) \), every couple \((\hat{y}_\lambda, \hat{u}_\lambda) \in S_{R_0}(\lambda)\) is a locally optimal solution of \((P(\lambda))\), and

\[
S_{R_0}(\lambda) \subset (\bar{y}, \bar{u}) + l_0 \| \lambda - \lambda_0 \|^1/2 \bar{B},
\]

which is equivalent to (16). The proof of the lemma is complete. \( \square \)

Notice that if the space of controls is \( U = L^q(\Gamma) \) then to obtain (19), we claim \( \int_\Gamma \hat{u}_k^2(x) ds = \| \hat{u}_k \|_{L^2(\Gamma)}^2 \geq \| \hat{u}_k \|_U^2 \geq \sigma_0^2 \). Hence \( q \leq 2 \). Besides the condition \( q > N - 1 \) is necessary to the state equation (2) has a solution \( y \in C(\Omega) \) where \( N \) is the dimension of the domain. Consequently, \( 2 > N - 1 \) and so \( N = 2 \).

According Lemma 3.1 in [9], there exist numbers \( \beta_0 > 0 \) and \( M_0 > 0 \) such that

\[
\|(e_{\lambda,1}, \ldots, e_{\lambda,m})\|_{(L^2(\Gamma))^m} + \| \vartheta_\lambda \|_{W^{1,\infty}(\Omega)} \leq M_0 \tag{20}
\]

for all \( \lambda \in B(\bar{\lambda}, \beta_0), (e_{\lambda,1}, \ldots, e_{\lambda,m}, \vartheta_\lambda) \in \Lambda(\hat{y}_\lambda, \hat{u}_\lambda, \lambda) \). Besides, by assumption (H3), there is a number \( \beta_1 > 0 \) such that for a.e. \( x \in \Gamma \), one has

\[
\beta(\lambda(x)) > \frac{\nu}{2}, \forall \lambda \in B(\bar{\lambda}, \beta_1). \tag{21}
\]

Without loss of generality we can assume that (10), (20), and (21) are valid for all \( \lambda \in B(\bar{\lambda}, s_0) \)

**Lemma 3.3.** For each \( \lambda \in B(\bar{\lambda}, s_0) \), if \((\hat{y}_\lambda, \hat{u}_\lambda) \in S_{R_0}(\lambda)\) then \( \hat{u}_\lambda \in L^\infty(\Gamma) \).

**Proof** Fix \( \lambda \in B(\bar{\lambda}, s_0) \). Since \((\hat{y}_\lambda, \hat{u}_\lambda) \in S_{R_0}(\lambda)\) is a locally optimal solution of \((P(\lambda))\), there exist Lagrange multipliers \( \vartheta_\lambda \in W^{1,\infty}(\Omega) \), \( e_{\lambda,1}, e_{\lambda,2}, \ldots, e_{\lambda,m} \in L^2(\Gamma) \) satisfying conditions (6)–(8). It follows from conditions (6) and (7) that for a.e. \( x \in \Gamma \),

\[
\alpha(\lambda(x)) + \beta(\lambda(x)) \hat{u}_\lambda(x) - \vartheta_\lambda(x) + \sum_{i=1}^m e_{\lambda,i}(x) = 0
\]
\[
e_{\lambda,i}(x) \in N((-\infty, 0], g_i(x, \hat{y}_\lambda(x), \lambda(x)) + \hat{u}_\lambda(x)) \quad \forall i = 1, 2, \ldots, m.
\]

Hence, setting \( \xi(x) := \sum_{i=1}^m e_{\lambda,i}(x) \), for a.e. \( x \in \Gamma \) one has

\[
\xi(x) = -\alpha(\lambda(x)) - \beta(\lambda(x)) \hat{u}_\lambda(x) + \vartheta_\lambda(x);
\]
\[
\xi(x) - \sum_{j \neq i} e_{\lambda,j}(x) \in N((-\infty, 0], g_i(x, \hat{y}_\lambda(x), \lambda(x)) + \hat{u}_\lambda(x)), \quad \forall i = 1, 2, \ldots, m.
\]

Let \( g(x, \hat{y}_\lambda(x), \lambda(x)) = \max_{1 \leq i \leq m} g_i(x, \hat{y}_\lambda(x), \lambda(x)) \) and \( G_i[x, \lambda] = g_i(x, \hat{y}_\lambda(x), \lambda(x)) + \hat{u}_\lambda(x) \) for all \( x \in \Gamma \). We claim that
\[ g(x, \hat{\lambda}(x), \lambda(x)) + \hat{u}_\lambda(x) \]

\[ = P_{(-\infty,0]} \left( \frac{1}{\beta(\lambda(x))} (\vartheta_\lambda(x) - \alpha(\lambda(x))) + g(x, \hat{\lambda}(x), \lambda(x)) \right) \]

for a.e. \( x \in \Gamma \), where \( P_{(-\infty,0]}(a) \) is the metric projection of \( a \) onto \( (-\infty,0] \).

For any \( i \in \{1,2,\ldots,m\} \) and \( \eta \in (-\infty,0] \), we have

\[ (\xi(x) - \sum_{j \neq i} e_{\lambda j}(x)) (\eta - G_i[x,\lambda]) \leq 0, \]

which implies that

\[ \xi(x)(\eta - G_i[x,\lambda]) \leq \sum_{j \neq i} e_{\lambda j}(x)(\eta - G_i[x,\lambda]) \]

\[ \leq \sum_{j \neq i} e_{\lambda j}(x)(\eta - G_j[x,\lambda]) + \sum_{j \neq i} e_{\lambda j}(G_j[x,\lambda] - G_i[x,\lambda]). \]

Notice that \( e_{\lambda i}(x) \geq 0 \ \forall i = 1,2,\ldots,m \). Therefore, if \( x \in \{x' \in \Gamma : g_i(x',\hat{\lambda}(x'),\lambda(x')) = g(x',\hat{\lambda}(x'),\lambda(x')) \} \) then one has

\[ \xi(x)(\eta - G_i[x,\lambda]) \leq 0, \]

which entails that

\[ \xi(x) \in N((-\infty;0],G_i[x,\lambda]), \]

and so,

\[ \vartheta_\lambda(x) - \alpha(\lambda(x)) - \beta(\lambda(x)) \hat{u}_\lambda(x) \in N((-\infty,0],g_i(x,\hat{\lambda}(x),\lambda(x)) + \hat{u}_\lambda(x)) \]

or

\[ \vartheta_\lambda(x) - \alpha(\lambda(x)) - \beta(\lambda(x)) \hat{u}_\lambda(x) \in N((-\infty,0],g(x,\hat{\lambda}(x),\lambda(x)) + \hat{u}_\lambda(x)). \]

(22)

Since \( \Gamma = \bigcup_{1 \leq i \leq m} \{x \in \Gamma : g_i(x,\hat{\lambda}(x),\lambda(x)) = g(x,\hat{\lambda}(x),\lambda(x)) \} \), the formula (22) is valid for all \( x \in \Gamma \). Hence, we conclude that

\[ \frac{1}{\beta(\lambda(x))} (\vartheta_\lambda(x) - \alpha(\lambda(x))) - \hat{u}_\lambda(x) \in N((-\infty,0],g(x,\hat{\lambda}(x),\lambda(x)) + \hat{u}_\lambda(x)) \]

or equivalently

\[ \frac{1}{\beta(\lambda(x))} (\vartheta_\lambda(x) - \alpha(\lambda(x))) + g(x,\hat{\lambda}(x),\lambda(x)) - (g(x,\hat{\lambda}(x),\lambda(x)) + \hat{u}_\lambda(x)) \]

\[ \in N((-\infty,0],g(x,\hat{\lambda}(x),\lambda(x)) + \hat{u}_\lambda(x)) \quad \text{a.e. } x \in \Gamma. \]

By [3, Theorem 5.2], for a.e. \( x \in \Gamma \) we obtain

\[ g(x,\hat{\lambda}(x),\lambda(x)) + \hat{u}_\lambda(x) \]

\[ = P_{(-\infty,0]} \left( \frac{1}{\beta(\lambda(x))} (\vartheta_\lambda(x) - \alpha(\lambda(x))) + g(x,\hat{\lambda}(x),\lambda(x)) \right). \]
Besides, from (10) and the embedding $W^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$, it follows that
\[ \|\hat{y}_\lambda - \bar{y}\|_{L^\infty(\Omega)} \leq C \|\hat{y}_\lambda - \bar{y}\|_{W^{1,r}(\Omega)} \leq Cl_0\|\lambda - \bar{\lambda}\|^{1/2} \leq Cl_0s_0^{1/2}. \]

Hence
\[ \|\hat{y}_\lambda\|_{L^\infty(\Omega)} \leq \|\bar{y}\|_{L^\infty(\Omega)} + Cl_0s_0^{1/2}. \]

Also, one has
\[ \|\lambda\|_{L^\infty(\Gamma)} \leq \|\bar{\lambda}\|_{L^\infty(\Gamma)} + \|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)} \leq \|\bar{\lambda}\|_{L^\infty(\Gamma)} + s_0. \]

Putting $M = Cl_0s_0^{1/2} + \|\bar{y}\|_{L^\infty(\Omega)} + s_0 + \|\bar{\lambda}\|_{\infty}$, we have
\[ \|\bar{y}\|_{L^\infty(\Omega)} + \|\bar{\lambda}\|_{L^\infty(\Gamma)} \leq M, \quad \|\hat{y}_\lambda\|_{L^\infty(\Omega)} + \|\lambda\|_{L^\infty(\Gamma)} \leq M. \tag{24} \]

It follows from assumption (H2) and (24) that there exists $k_{g,M}$ such that
\[ |g_i(x, \hat{y}_\lambda(x), \lambda(x))| \leq |g_i(x, \hat{y}_\lambda(x), \lambda(x) - g_i(x, 0, 0)| + |g_i(x, 0, 0)| \leq k_{g,M}|\hat{y}_\lambda(x)| + |g_i(x, 0, 0)| \leq k_{g,M}|g_i(x, 0, 0)| \]
for all $i = 1, 2, \ldots, m$. Putting $k_{g,M} = \max_{1 \leq i \leq m} k_{g,M}$, we have
\[ |g_i(x, \hat{y}_\lambda(x), \lambda(x))| \leq k_{g,M}|g_i(x, 0, 0)| \]
for all $i = 1, 2, \ldots, m$. By a similar argument, the assumptions (H1), (H3), and (24) implies that there exist $k_{L,M}, k_{\ell,M}, k_{g,M}, k_{\alpha,M}$ such that
\[ |L_j(x, \hat{y}_\lambda(x))| \leq k_{L,M}|g_i(x, 0, 0)|, \]
\[ |\ell(x', \hat{y}_\lambda(x'), \lambda)| \leq k_{\ell,M}|g_i(x', 0, 0)|, \]
\[ |g_{ij}(x', \hat{y}_\lambda(x'), \lambda(x'))| \leq k_{g,M}|g_i(x', 0, 0)|, \]
\[ |\alpha(\lambda(x'))| \leq k_{\alpha,M}|\lambda(x')| \]
for a.e. $x \in \Omega, x' \in \Gamma$, for all $i = 1, 2, \ldots, m$. Hence $L_j(\cdot, \hat{y}_\lambda) \in L^\infty(\Omega)$ and $\ell(\cdot, \hat{y}_\lambda, \lambda), g_i(\cdot, \hat{y}_\lambda, \lambda), g_{ij}(\cdot, \hat{y}_\lambda, \lambda), \alpha(\lambda) \in L^\infty(\Gamma)$ and so, $g(\cdot, \hat{y}_\lambda, \lambda) \in L^\infty(\Gamma)$ and $\sum_{i=1}^m g_{ij}(\cdot, \hat{y}_\lambda, \lambda)e_{\lambda i} \in L^2(\Gamma)$. Combining this with (6) and (H4) yields $\vartheta_\lambda \in W^{1,r}(\Omega)$ and so, $\vartheta_\lambda \in C(\overline{\Omega})$.

Using the non-expansive property of metric projection (see [3, Proposition 5.3]) and the fact $P_{(-\infty, 0)}(0) = 0$, it follows from (23) that
\[ |\hat{u}_\lambda(x)| \leq |g(x, \hat{y}_\lambda(x), \lambda(x))| + \frac{1}{\beta(\lambda(x))}|(\vartheta_\lambda(x) - \alpha(\lambda(x))) + g(x, \hat{y}_\lambda(x), \lambda(x))| \]
\[ \leq |g(x, \hat{y}_\lambda(x), \lambda(x))| + \frac{1}{\gamma}|\vartheta_\lambda(x)| + |\alpha(\lambda(x))| + |g(x, \hat{y}_\lambda(x), \lambda(x))|. \]

Hence,
\[ |\hat{u}_\lambda(x)| \leq 2|g(x, \hat{y}_\lambda(x), \lambda(x))| + \frac{1}{\gamma}|\vartheta_\lambda(x)| + |\alpha(\lambda(x))| \]
for a.e. $x \in \Gamma$. Since $g(\cdot, \hat{y}_\lambda, \lambda), \tau \vartheta_\lambda, \alpha(\lambda) \in L^\infty(\Gamma)$, we obtain $\hat{u}_\lambda \in L^\infty(\Gamma)$. The lemma is proved.
It remains to prove (11) of Theorem 2.1. For this we have the following lemma.

**Lemma 3.4.** If the assumption (H5) is valid then there exist positive numbers $l_1$ and $s_1$ such that for all $\lambda \in B_{L^\infty(\Gamma)}(\bar{\lambda}, s_0)$, and any $(\hat{y}_\lambda, \hat{u}_\lambda) \in \mathcal{S}_0(\lambda)$ one has

$$\|\hat{y}_\lambda - \bar{y}\|_{W^{1,r}(\Omega)} + \|\hat{u}_\lambda - \bar{u}\|_\infty \leq l_1\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2}.$$ 

**Proof** Let $\lambda \in B_{L^\infty(\Gamma)}(\bar{\lambda}, s_0)$, and $(\hat{y}_\lambda, \hat{u}_\lambda) \in \mathcal{S}_0(\lambda)$, and $M = C_0s_0^{1/2} + \|\bar{y}\|_{L^\infty(\Omega)} + s_0 + \|\bar{\lambda}\|_\infty$. Analysis similar to that in the proof of Lemma 3.3 shows that

$$\|\bar{y}\|_{L^\infty(\Omega)} + \|\bar{\lambda}\|_{L^\infty(\Gamma)} \leq M, \quad \|\hat{y}_\lambda\|_{L^\infty(\Omega)} + \|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)} \leq M$$

and $g_i(\cdot, \hat{y}_\lambda, \lambda) \in L^\infty(\Gamma)$ for all $i = 1, 2, \ldots, m$. In particular, $g_i[\cdot] \in L^\infty(\Gamma)$ and so, $\sup_{x \in \Gamma_i} (g_i[x] - g_i[x]) > -\infty$ for all $1 \leq i, k \leq m$, $k \neq i$. Consequently, the assumption (H5) implies that there exists a number $\sigma_1 > 0$ such that

$$g_k[x] - g_i[x] \leq -\sigma_1 \text{ a.e. } x \in \Gamma_i,$$

which gives

$$G_k(\bar{y}, \bar{\lambda}) \leq G_i(\bar{y}, \bar{\lambda}) - \sigma_1 \leq -\sigma_1 < 0 \text{ a.e. } x \in \Gamma_i$$

for all $1 \leq i, k \leq m$, $k \neq i$, where $G_j(y, u, \lambda) = g_j(\cdot, y, \lambda) + u$ with $j = 1, 2, \ldots, m$.

It follows from (25) that there exist Lipschitz constants $k_{LM}, k_{LM}, k_{LM}, k_{LM}$, and $k_{LM}$ under which (H1), (H2), and (H3) hold. From the assumption (H2), we have

$$|g_i[x] - g_i(x, \hat{y}_\lambda(x), \lambda(x))| \leq k_{LM}(\hat{y}_\lambda(x) - \bar{y}(x)) + |\lambda(x) - \bar{\lambda}(x)|$$

$$\leq k_{LM}(C_0\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2} + \|\bar{\lambda}\|_{L^\infty(\Gamma)})$$

$$\leq k_{LM}(C_0\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2} + s_0^{1/2}\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2})$$

$$\leq k_{LM}(C_0 + s_0^{1/2})\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2} < \frac{\sigma_1}{4}$$

for a.e. $x \in \Omega$ and for all $i = 1, 2, \ldots, m$, whenever $\lambda \in B(\bar{\lambda}, s_1)$, where

$$s_1 := \min \left\{ \frac{\sigma_1^2}{16k_{LM}^2(C_0 + s_0^{1/2})^2}, s_0 \right\}.$$ 

Therefore, for all $i, k \in \{1, 2, \ldots, m\}$, $k \neq i$, $\lambda \in B(\bar{\lambda}, s_1)$, we have

$$g_k(x, \hat{y}_\lambda(x), \lambda(x)) - g_i(x, \hat{y}_\lambda(x), \lambda(x))$$

$$= (g_k(x, \hat{y}_\lambda(x), \lambda(x)) - g_k[x]) + (g_k[x] - g_i[x]) + (g_i[x] - g_i(x, \hat{y}_\lambda(x), \lambda(x)))$$

$$\leq |g_k(x, \hat{y}_\lambda(x), \lambda(x)) - g_k[x]| + |g_k[x] - g_i[x]| + |g_i[x] - g_i(x, \hat{y}_\lambda(x), \lambda(x))|$$

$$\leq \frac{\sigma_1}{4} - \sigma_1 + \frac{\sigma_1}{4} = -\frac{\sigma_1}{2} \text{ a.e. } x \in \Gamma_i.$$
This implies that
\[ G_k(\hat{u}, \hat{u}, \lambda) \leq \frac{\sigma_1^2}{2} \leq -\frac{\sigma_1}{2} < 0 \quad \text{a.e. in } \Gamma_i \]  
for all \( 1 \leq i, k \leq m, k \neq i, \lambda \in B(\lambda, s_1) \).

Fix \( \lambda \in B(\lambda, s_1) \) and \( (\hat{y}, \hat{u}) \in S_{R_0}(\lambda) \). Since \( (\hat{y}, \hat{u}) \) and \( (\bar{y}, \bar{u}) \) are locally optimal solutions of \( (P(\lambda)) \) and \( (P(\bar{\lambda})) \), respectively, there exist the Lagrange multipliers \( (\theta, e_1, e_2, \ldots, e_m) \in \Lambda[(\hat{y}, \hat{u}), \lambda] \) and \( (\theta, e_1, e_2, \ldots, e_m) \in \Lambda[(\bar{y}, \bar{u}), \bar{\lambda}] \) satisfying

\[
\begin{align*}
A\theta + h_y(\cdot, \hat{y}, \lambda) \theta &= -L_y(\cdot, \hat{y}, \lambda) & \text{in } \Omega \\
\partial_y \theta + \sum_{i=1}^m g_{iy}(\cdot, \hat{y}, \lambda) e_i &= 0 & \text{on } \Gamma,
\end{align*}
\]

(28)

\[
\begin{align*}
A\theta + h_y[\cdot] \theta &= -L_y[\cdot] & \text{in } \Omega \\
\partial_y \theta &= -\ell_y[\cdot] - \sum_{i=1}^m g_{iy}[\cdot] e_i & \text{on } \Gamma,
\end{align*}
\]

(29)

\[
-\theta + \alpha(\lambda) + \beta(\lambda) \hat{u} + \sum_{i=1}^m e_i = 0,
\]

(30)

\[
-\theta + \alpha(\bar{\lambda}) + \beta(\bar{\lambda}) \bar{u} + \sum_{i=1}^m e_i = 0,
\]

(31)

\[
e_{2,i}(x) (g_i(x, \hat{y}(x), \lambda(x)) + \hat{u}(x)) = 0, \quad e_{2,i}(x) \geq 0,
\]

(32)

\[
e_{1}(x) (g_i[x] + \bar{u}(x)) = 0, \quad e_{1}(x) \geq 0,
\]

(33)

\[
g_i(x, \hat{y}(x), \lambda(x)) + \hat{u}(x) \leq 0,
\]

(34)

\[
g_i[x] + \bar{u}(x) \leq 0
\]

(35)

for a.e. \( x \in \Gamma \), for all \( i = 1, 2, \ldots, m \).

Once again, analysis similar to that in the proof of Lemma 3.3 shows that \( \theta \in W^{1,r}(\Omega) \). Moreover, by (5), we have

\[
\| \theta \|_{W^{1,r}(\Omega)} \leq C_1 \left( \| L_y(\cdot, \hat{y}, \lambda) \|_{L^2(\Omega)} + \| \ell_y(\cdot, \hat{y}, \lambda) + \sum_{i=1}^m g_{iy}(\cdot, \hat{y}, \lambda) e_i \|_{L^2(\Gamma)} \right)
\]

\[
\leq C_1 \left( \left( k_{LM}M + \| L(\cdot, 0) \|_{L^\infty(\Omega)} \right) |\Omega|^{1/2} + \left( k_{LM}M + \| \ell(\cdot, 0, 0) \|_{L^\infty(\Gamma)} \right) |\Gamma|^{1/2} + \left( m k_{g0}M + \sum_{i=1}^m \| g_{iy}(\cdot, 0, 0) \|_{L^\infty(\Gamma)} \right) M_0 \right) := M_1.
\]

In particular, \( \theta \in W^{1,r}(\Omega) \) and \( \| \theta \| \leq M_1 \). Notice that \( M_1 \) do not depend on \( \lambda \). From this and the embedding \( W^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega}) \), it follows that there exists a positive number \( M_2 \) independent of \( \lambda \) such that

\[
\| \theta \|_{C(\overline{\Omega})} + \| \theta \|_{C(\overline{\Omega})} + \| \theta \|_{L^2(\Omega)} + \| \tau \theta \|_{L^\infty(\Gamma)} + \| \tau \theta \|_{L^\infty(\Gamma)} < M_2.
\]
Besides, it follows from (26), (27), (32), and (33) that \( e_k = e_{\lambda k} = 0 \) a.e. in \( \Gamma_i \) for all \( 1 \leq i, k \leq m, k \neq i \). Combining this with (30) and (31) we obtain

\[
e_i = \vartheta - \alpha(\lambda) - \beta(\lambda)\bar{u} \quad \text{and} \quad e_{\lambda i} = \vartheta_{\lambda} - \alpha(\lambda) - \beta(\lambda)\bar{u}_{\lambda} \text{ a.e. in } \Gamma_i.
\]

Hence

\[
e_i = (\vartheta - \alpha(\lambda) - \beta(\lambda)\bar{u})\chi_i \quad \text{and} \quad e_{\lambda i} = (\vartheta_{\lambda} - \alpha(\lambda) - \beta(\lambda)\bar{u}_{\lambda})\chi_i \text{ a.e. in } \Gamma_i,
\]

where \( \chi_i \) is the characteristic function of the set \( \Gamma_i, i = 1, 2, \ldots, m \), that is

\[
\chi_i(x) = \begin{cases} 
1 & \text{if } x \in \Gamma_i, \\
0 & \text{if } x \notin \Gamma_i.
\end{cases}
\]

Substituting (36) into (28) and (29), we see that

\[
\begin{align*}
A\vartheta_{\lambda} + h_y(\cdot, \hat{y}_{\lambda})\vartheta_{\lambda} &= -L_y(\cdot, \hat{y}_{\lambda}) \text{ in } \Omega, \\
\vartheta_{\lambda} + \sum_{i=1}^{m} g_{i\lambda}(\cdot, \hat{y}_{\lambda}, \lambda)\chi_i\vartheta_{\lambda} &= -\ell_y(\cdot, \hat{y}_{\lambda}, \lambda) + \sum_{i=1}^{m} g_{\lambda i}(\cdot, \hat{y}_{\lambda}, \lambda)(\alpha(\lambda) + \beta(\lambda)\bar{u}_{\lambda}) \chi_i \text{ on } \Gamma,
\end{align*}
\]

By subtracting the above equations, we get

\[
\begin{align*}
A(\vartheta - \vartheta_{\lambda}) + h_y(\cdot, \hat{y}_{\lambda})(\vartheta - \vartheta_{\lambda}) &= L_y(\cdot, \hat{y}_{\lambda}) - L_y[\cdot] + (h_y(\cdot, \hat{y}_{\lambda}) - h_y(\cdot, \hat{y}_{\lambda}))\vartheta_{\lambda} \text{ in } \Omega, \\
\vartheta_{\lambda} + \sum_{i=1}^{m} g_{i\lambda}(\cdot, \hat{y}_{\lambda}, \lambda)\chi_i(\vartheta - \vartheta_{\lambda}) &= \ell_y(\cdot, \hat{y}_{\lambda}, \lambda) - \ell_y[\cdot] + \sum_{i=1}^{m} g_{\lambda i}(\cdot, \hat{y}_{\lambda}, \lambda)(\alpha(\lambda) + \beta(\lambda)\bar{u}_{\lambda}) \chi_i \text{ on } \Gamma.
\end{align*}
\]

By (H4) and (5), we get

\[
\|\vartheta - \vartheta_{\lambda}\|_{C(\overline{\Omega})} \leq C \|\vartheta - \vartheta_{\lambda}\|_{W^{1, r}(\Omega)} \leq CC_1 \left( \|I_{\Omega}\|_{L^2(\Omega)} + \|I_{\Gamma}\|_{L^2(\Gamma)} \right),
\]

where

\[
I_{\Omega} := L_y(\cdot, \hat{y}_{\lambda}) - L_y[\cdot] + (h_y(\cdot, \hat{y}_{\lambda}) - h_y(\cdot, \hat{y}_{\lambda}))\vartheta_{\lambda},
\]

\[
I_{\Gamma} := \ell_y(\cdot, \hat{y}_{\lambda}, \lambda) - \ell_y[\cdot] + \sum_{i=1}^{m} (g_{i\lambda}(\cdot, \hat{y}_{\lambda}, \lambda) - g_{\lambda i}(\cdot, \hat{y}_{\lambda}, \lambda))\chi_i\vartheta_{\lambda}
\]

\[
+ \sum_{i=1}^{m} g_{\lambda i}(\cdot, \hat{y}_{\lambda}, \lambda)(\alpha(\lambda) + \beta(\lambda)\bar{u}_{\lambda}) \chi_i - \sum_{i=1}^{m} g_{i\lambda}(\cdot, \hat{y}_{\lambda}, \lambda)(\alpha(\lambda) + \beta(\lambda)\bar{u}_{\lambda}) \chi_i.
\]
We have
\[
\|I_{\Omega}\|_{L^2(\Omega)} \leq \|(h_y(\cdot, \hat{\lambda}) - h_y[\cdot])\hat{\vartheta}_{\lambda}\|_{L^2(\Omega)} + \|L_y(\cdot, \hat{\lambda}) - L_y[\cdot]\|_{L^2(\Omega)} \\
\leq \|h_y[\cdot] - h_y(\cdot, \hat{\lambda})\|_{L^\infty(\Omega)} \|\vartheta_{\lambda}\|_{L^2(\Omega)} + \|L_y(\cdot, \hat{\lambda}) - L_y[\cdot]\|_{L^\infty(\Omega)}|\Omega|^{1/2} \\
\leq k_{hM}\|\hat{\lambda} - \bar{\lambda}\|_{L^\infty(\Omega)}M_2 + |\Omega|^{1/2}k_{LM}\|\hat{\lambda} - \bar{\lambda}\|_{L^\infty(\Omega)} \\
\leq (k_{hM}M_2 + |\Omega|^{1/2}k_{LM})\|\hat{\lambda} - \bar{\lambda}\|_{L^\infty(\Omega)} \\
\leq (k_{hM}M_2 + |\Omega|^{1/2}k_{LM})C_0\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2}.
\]
Hence
\[
\|I_{\Omega}\|_{L^2(\Omega)} \leq k_1\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2},
\] (38)
where \(k_1 := (k_{hM}M_2 + |\Omega|^{1/2}k_{LM})C_0\). It remains to estimate \(\|I_{\Gamma}\|_{L^2(\Gamma)}\). For this, we have
\[
\|I_{\Gamma}\|_{L^2(\Gamma)} \leq \|\ell_y(\cdot, \hat{\lambda}, \lambda) - \ell_y[\cdot]\|_{L^2(\Gamma)} + \sum_{i=1}^m (g_{iy}(\cdot, \hat{\lambda}, \lambda) - g_{iy}[\cdot])\varpi_i\vartheta_{\lambda} \|_{L^2(\Gamma)} \\
+ \left| \sum_{i=1}^m g_{iy}[\cdot]\alpha(\lambda) + \beta(\lambda)\bar{\varpi}_i \right| - \sum_{i=1}^m g_{iy}(\cdot, \hat{\lambda}, \lambda)\alpha(\lambda) + \beta(\lambda)\hat{\varpi}_i \right| \|_{L^2(\Gamma)}.
\] (39)
We now give estimates for terms on the right hand side of (39). For the first term, using (10), we have
\[
\|\ell_y[\cdot] - \ell_y(\cdot, \hat{\lambda}, \lambda)\|_{L^2(\Gamma)} \leq \|\ell_y[\cdot] - \ell_y(\cdot, \hat{\lambda}, \lambda)\|_{L^\infty(\Gamma)}|\Gamma|^{1/2} \\
\leq k_{\ellM}(\|\bar{\varpi} - \hat{\varpi}\|_{L^\infty(\Gamma)} + \|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)})|\Gamma|^{1/2} \\
\leq k_{\ellM}(C_0\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2} + s_1^{1/2}\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2})|\Gamma|^{1/2} \\
\leq k_2\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2},
\] (40)
where \(k_2 := k_{\ellM}(C_0 + s_1^{1/2})|\Gamma|^{1/2}\). Similarly, for the second term, one has
\[
\sum_{i=1}^m (g_{iy}(\cdot, \hat{\lambda}, \lambda) - g_{iy}[\cdot])\varpi_i\vartheta_{\lambda} \|_{L^2(\Gamma)} \\
\leq \sum_{i=1}^m \|g_{iy}(\cdot, \hat{\lambda}, \lambda) - g_{iy}[\cdot]\|_{L^\infty(\Gamma)}\|\varpi_i\vartheta_{\lambda}\|_{L^2(\Gamma)} \\
\leq \sum_{i=1}^m k_{gM}(\|\hat{\lambda} - \bar{\lambda}\|_{L^\infty(\Omega)} + \|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)})\|\varpi_i\vartheta_{\lambda}\|_{L^2(\Gamma)} \\
\leq k_{gM}(\|\hat{\lambda} - \bar{\lambda}\|_{L^\infty(\Omega)} + \|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}\|\vartheta_{\lambda}\|_{L^2(\Gamma)} \\
\leq k_{gM}(C_0\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2} + s_1^{1/2}\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2})M_2 \\
\leq k_{gM}M_2(C_0 + s_1^{1/2})\|\lambda - \bar{\lambda}\|_{L^\infty(\Gamma)}^{1/2}
\] (41)
\begin{equation}
\leq k_3 \| \lambda - \lambda^* \|_{L^\infty(\Gamma)}^{1/2},
\end{equation}

where \( k_3 := k_{gM} M_2 \left( C_0 + s_1^{1/2} \right) \).

To estimate the third term, we first have from (25) that

\begin{equation}
|g_{iy}(\cdot, \hat{\lambda}, \lambda)|_{L^\infty(\Gamma)} \leq k_{gM} M + \sum_{1 \leq i \leq m} \|g_{iy}(\cdot, 0, 0)\|_{L^\infty(\Gamma)} := M_3,
\end{equation}

\begin{equation}
\|\beta(\lambda)\|_{L^\infty(\Gamma)} \leq M_4, \quad \|\beta(\lambda)\|_{L^\infty(\Gamma)} \leq M_4 \quad \text{with} \ M_4 := k_{\beta M} M + |\beta(0)|,
\end{equation}

\begin{equation}
\|\alpha(\lambda)\|_{L^\infty(\Gamma)} \leq M_5, \quad \|\alpha(\lambda)\|_{L^\infty(\Gamma)} \leq M_5 \quad \text{with} \ M_5 := k_{\alpha M} M + |\alpha(0)|,
\end{equation}

for all \( i = 1, 2, \ldots, m \). Using (43)–(45) and (10), we get

\begin{align*}
&\left\| \sum_{i=1}^m g_{iy}(\cdot, \hat{\lambda}, \lambda)(\alpha(\lambda) + \beta(\lambda) \hat{u}_\lambda) \chi_i - \sum_{i=1}^m g_{iy}(\cdot, \hat{\lambda}, \lambda)(\alpha(\lambda) + \beta(\lambda) \hat{u}_\lambda) \chi_i \right\|_{L^2(\Gamma)} \\
&= \left\| \sum_{i=1}^m g_{iy}(\cdot, \hat{\lambda}, \lambda)(\alpha(\lambda) + \beta(\lambda) \hat{u}_\lambda - \alpha(\lambda) - \beta(\lambda) \hat{u}) \right. \\
&\quad + (g_{iy}(\cdot, \hat{\lambda}, \lambda) - g_{iy}(\cdot, \hat{\lambda}, \lambda))(\alpha(\lambda) + \beta(\lambda) \hat{u}) \chi_i \right\|_{L^2(\Gamma)} \\
&\leq \sum_{i=1}^m \|g_{iy}(\cdot, \hat{\lambda}, \lambda)(\alpha(\lambda) - \alpha(\lambda)) \chi_i \|_{L^2(\Gamma)} \\
&\quad + \sum_{i=1}^m \|g_{iy}(\cdot, \hat{\lambda}, \lambda)(\beta(\lambda) \hat{u}_\lambda - \beta(\lambda) \hat{u}) \chi_i \|_{L^2(\Gamma)} \\
&\quad + \sum_{i=1}^m \|(g_{iy}(\cdot, \hat{\lambda}, \lambda) - g_{iy}(\cdot, \hat{\lambda}, \lambda))(\alpha(\lambda) + \beta(\lambda) \hat{u}) \chi_i \|_{L^2(\Gamma)} \\
&\leq \sum_{i=1}^m M_3 k_{\alpha M} \|\lambda - \lambda^*\|_{L^\infty(\Gamma)} \|\chi_i\|_{L^2(\Gamma)} + \sum_{i=1}^m M_3 \|\beta(\lambda) \hat{u}_\lambda - \beta(\lambda) \hat{u}\|_{L^2(\Gamma)} \\
&\quad + k_{gM} \left( \|\hat{\lambda} - \lambda\|_{L^\infty(\Omega)}^{1/2} + \|\lambda - \lambda^*\|_{L^\infty(\Gamma)} \right) \|\alpha(\lambda) + \beta(\lambda) \hat{u}\|_{L^2(\Gamma)} \\
&\leq M_3 k_{\alpha M} \|\Gamma\|_{L^\infty(\Gamma)} \|\lambda - \lambda^*\|_{L^\infty(\Gamma)} + \sum_{i=1}^m M_3 \|\beta(\lambda) \hat{u}_\lambda - \hat{u}\|_{L^2(\Gamma)} \\
&\quad + k_{gM} \left( C_0 \|\lambda - \lambda^*\|_{L^\infty(\Gamma)}^{1/2} + \|\lambda - \lambda^*\|_{L^\infty(\Gamma)} \right) \left( M_5 \|\Gamma\|_{L^2(\Gamma)}^{1/2} + M_4 \|\hat{u}\|_{L^2(\Gamma)} \right) \\
&\leq M_3 k_{\alpha M} \|\Gamma\|_{L^\infty(\Gamma)} \|\lambda - \lambda^*\|_{L^\infty(\Gamma)} + M_3 M_4 \|\hat{u}_\lambda - \hat{u}\|_{L^2(\Gamma)} \\
&\quad + 2M_3 k_{\beta M} \|\lambda - \lambda^*\|_{L^\infty(\Gamma)} \|\hat{u}\|_{L^2(\Gamma)}
+ k_{g\mathcal{M}} \left( \frac{1}{\beta(\lambda(x))} \left( \beta(\lambda(x)) - \alpha(\lambda(x)) \right) + \sum_{i=1}^{m} g_i(x, \hat{y}_\lambda(x), \lambda(x)) \chi_i(x) \right)

and

\sum_{i=1}^{m} g_i[x] \chi_i(x) + \overline{u}(x) = P_{(-\infty, 0]} \left( \frac{1}{\beta(\lambda(x))} \left( \beta(x) - \alpha(\lambda(x)) \right) + \sum_{i=1}^{m} g_i[x] \chi_i(x) \right)

for a.e. \( x \in \Gamma \). This implies that

\hat{u}_\lambda(x) - \overline{u}(x) = P_{(-\infty, 0]} \left( \frac{1}{\beta(\lambda(x))} \left( \beta(\lambda(x)) - \alpha(\lambda(x)) \right) + \sum_{i=1}^{m} g_i(x, \hat{y}_\lambda(x), \lambda(x)) \chi_i(x) \right)

- P_{(-\infty, 0]} \left( \frac{1}{\beta(\lambda(x))} \left( \beta(\lambda(x)) - \alpha(\lambda(x)) \right) + \sum_{i=1}^{m} g_i[x] \chi_i(x) \right)

+ \sum_{i=1}^{m} g_i[x] \chi_i(x) - \sum_{i=1}^{m} g_i(x, \hat{y}_\lambda(x), \lambda(x)) \chi_i(x) \quad \text{a.e. } x \in \Gamma.
Using the non-expansive property of metric projections, we have

\[ |\hat{u}_\lambda(x) - \overline{u}(x)| \leq \left( \frac{1}{\beta(\lambda(x))} (\vartheta_{\lambda}(x) - \alpha(\lambda(x))) + \sum_{i=1}^{m} g_i(x, \hat{y}_{\lambda}(x), \lambda(x)) \chi_i(x) \right) \]

\[ - \left( \frac{1}{\beta(\lambda(x))} (\vartheta(x) - \alpha(\lambda(x))) + \sum_{i=1}^{m} g_i[x] \chi_i(x) \right) \]

\[ + \left| \sum_{i=1}^{m} g_i[x] \chi_i(x) - \sum_{i=1}^{m} g_i(x, \hat{y}_{\lambda}(x), \lambda(x)) \chi_i(x) \right| \]

\[ \leq \frac{1}{\beta(\lambda(x))} (\vartheta_{\lambda}(x) - \alpha(\lambda(x))) - \frac{1}{\beta(\lambda(x))} (\vartheta(x) - \alpha(\lambda(x))) \]

\[ + 2 \left| \sum_{i=1}^{m} (g_i[x] - g_i(x, \hat{y}_{\lambda}(x), \lambda(x))) \chi_i(x) \right|. \] (47)

Let us estimate the terms on the right hand side of (47). Using \((H3), (21), (46)\) and embedding \(W^{1,r}(\Omega) \hookrightarrow C(\Omega)\), we have the following estimation of the first term:

\[ \left| \frac{1}{\beta(\lambda(x))} (\vartheta_{\lambda}(x) - \alpha(\lambda(x)) - \frac{1}{\beta(\lambda(x))} (\vartheta(x) - \alpha(\lambda(x))) \right| \]

\[ = \left| \frac{1}{\beta(\lambda(x))} (\vartheta_{\lambda}(x) - \alpha(\lambda(x)) - \vartheta(x) + \alpha(\lambda(x))) \right| + \]

\[ + \left| \left( \frac{1}{\beta(\lambda(x))} - \frac{1}{\beta(\lambda(x))} \right) (\vartheta(x) - \alpha(\lambda(x))) \right| \]

\[ \leq \frac{2}{\gamma} (|\vartheta_{\lambda}(x) - \vartheta(x)| + |\alpha(\lambda(x)) - \alpha(\lambda(x))|) + \]

\[ + \frac{4}{\gamma^2} |\beta(\lambda(x)) - \beta(\lambda(x))||\vartheta(x)| + |\alpha(\lambda(x))|) \]

\[ \leq \frac{2}{\gamma} \left( Ck_6 \|\lambda - \overline{\lambda}\|_{L_\infty}^{1/2} + k_{\alpha M} \|\lambda(x) - \overline{\lambda}(x)\| + \frac{2}{\gamma} k_{\beta M} \|\lambda(x) - \overline{\lambda}(x)\|_{(M_2 + M_5)} \right) \]

\[ \leq \frac{2}{\gamma} \left( Ck_6 \|\lambda - \overline{\lambda}\|_{L_\infty}^{1/2} + \left( k_{\alpha M} + \frac{2}{\gamma} k_{\beta M} (M_2 + M_5) \right) \|\lambda - \overline{\lambda}\|_{L_\infty(\Gamma)} \right) \]

\[ \leq \frac{2}{\gamma} \left( Ck_6 + \left( k_{\alpha M} + \frac{2}{\gamma} k_{\beta M} (M_2 + M_5) \right) s_1^{1/2} \right) \|\lambda - \overline{\lambda}\|_{L_\infty(\Gamma)}^{1/2} \]

\[ \leq k_7 \|\lambda - \overline{\lambda}\|_{L_\infty(\Gamma)}, \] (48)

where \(k_7 := \frac{2}{\gamma} \left( Ck_6 + \left( k_{\alpha M} + \frac{2}{\gamma} k_{\beta M} (M_2 + M_5) \right) s_1^{1/2} \right)\).
To estimate the second term, from (H2) and (10), we have

$$2 \left| \sum_{i=1}^{m} (g_i[x] - g_i(x,\hat{\lambda}_i(x),\lambda(x))) \chi_i(x) \right|$$

$$\leq 2k_gM(|\hat{\lambda}_i(x) - \bar{\lambda}(x)| + |\lambda(x) - \bar{\lambda}(x)|)$$

$$\leq 2k_gM(\|\hat{\lambda} - \bar{\lambda}\|_{L^{\infty}(\Omega)} + \|\lambda - \bar{\lambda}\|_{L^{\infty}(\Gamma)})$$

$$\leq 2k_gM(C\|\hat{\lambda} - \bar{\lambda}\|_{W^{1,r}(\Omega)} + \|\lambda - \bar{\lambda}\|_{L^{\infty}(\Gamma)})$$

$$\leq 2k_gM(Cl_0\|\lambda - \bar{\lambda}\|_{L^{\infty}(\Gamma)} + s_{1/2}^{1/2}\|\lambda - \bar{\lambda}\|_{L^{\infty}(\Gamma)})$$

$$\leq k_8\|\lambda - \bar{\lambda}\|_{L^{\infty}(\Gamma)}$$

(49)

for a.e. \( x \in \Gamma \), where \( k_8 = 2k_gM(Cl_0 + s_{1/2}^{1/2}) \).

Combining (47)–(49) we obtain

$$|\hat{u}_\lambda(x) - \bar{u}(x)| \leq (k_7 + k_8)\|\lambda - \bar{\lambda}\|_{L^{\infty}(\Gamma)}^{1/2} \quad \text{a.e. } x \in \Gamma.$$ 

Thus, setting \( k_9 := k_7 + k_8 \), we have

$$\|\hat{u}_\lambda - \bar{u}\|_{L^{\infty}(\Gamma)} \leq k_9\|\lambda - \bar{\lambda}\|_{L^{\infty}(\Gamma)}^{1/2}.$$ 

Combining this with (10) yields

$$\|\hat{y}_\lambda - \bar{y}\|_{W^{1,r}(\Omega)} + \|\hat{u}_\lambda - \bar{u}\|_{L^{\infty}(\Gamma)} \leq (l_0 + k_9)\|\lambda - \bar{\lambda}\|_{L^{\infty}(\Gamma)}^{1/2}.$$ 

Therefore, the lemma is proved with the choice \( l_1 = l_0 + k_9 \).

The proof of Theorem 2.1 is complete.

**Disclosure statement**

The authors report there are no competing interests to declare.

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