Survival probability for a class of multitype subcritical branching processes in random environment

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Abstract

We study the asymptotic behaviour of the survival probability of a multi-type branching processes in random environment. The class of processes we consider corresponds, in the one-dimensional situation, to the intermediately subcritical case. We show under rather general assumptions on the form of the offspring generating functions of particles that the probability of survival up to generation $n$ of the process initiated at moment zero by a single particle of any type is of order $\lambda^n n^{-1/2}$ for large $n$, where $\lambda \in (0, 1)$ is a constant specified by the Lyapunov exponent of the mean matrices of the process.

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1 Introduction and main results

Branching processes in random environment with one type of particles have been intensively investigated during the last two decades and their properties are well understood (see, for example, the survey [19] and the recent book by Kersting and Vatutin [14]). The multi-type case is much less studied and many basic problems such as the asymptotic behavior of the survival probability, limit theorems for the number of particles in the process and others are solved under rather heavy conditions, for example, for the cases when the mean matrices of the reproduction laws of particles in different generations have a common nonrandom left or right eigenvector corresponding to their Perron roots, or for some other relatively narrow classes of mean matrices (see [5] – [12], [18]).
This paper supplements some recent results (see [7], [15], [17], [20]) describing the asymptotic behavior of the survival probabilities of the critical and subcritical multitype branching processes evolving in random environment.

To formulate our main result we need some notation for $p$-dimensional vectors and $p \times p$ matrices. We usually make no difference in notation for row and column vectors. As we hope it will be clear from the context which form is selected in each case. Besides we write $e_j, j = 1, \ldots, p,$ for a vector whose $j$-th component is equal to 1 and the others are zeros; $0 = (0, \ldots, 0), 1 = (1, \ldots, 1)$ for zero and unit $p$-dimensional vectors.

The norm and scalar product of vectors $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_p)$ are denoted as

$$|x| = \sum_{i=1}^{p} |x_i|, \quad (x, y) = \sum_{i=1}^{p} x_i y_i.$$ 

We also use the notation $x^y = \prod_{i=1}^{p} x_i^y$ and define the norm of a matrix $m = (m(i, j))_{i,j=1}^{p}$ as

$$|m| = \sum_{i=1}^{p} \sum_{j=1}^{p} |m(i, j)|.$$ 

Let $\mathcal{P}(\mathbb{N}_0^p)$ be the space of all probability measures on the set $\mathbb{N}_0^p$ of $p$-dimensional vectors with nonnegative integer-valued components. For a measure $f \in \mathcal{P}(\mathbb{N}_0^p)$ we denote by $f[z]$ the mass assigning by the measure to the point $z = (z_1, \ldots, z_p) \in \mathbb{N}_0^p$. The function $f(s) := \sum_{z \in \mathbb{N}_0^p} f[z] s^z$, $s = (s_1, \ldots, s_p) \in [0,1]^p$, is the generating function for the distribution (measure) $f$. It will be convenient to denote (by taking some liberty) the distribution (measure) and the corresponding generating function by one and the same symbol $f$. We also need $p$-dimensional vectors

$$f = (f^{(1)}, \ldots, f^{(p)}) \in \mathcal{P}(\mathbb{N}_0^p) \times \cdots \times \mathcal{P}(\mathbb{N}_0^p) =: \mathcal{P}^p(\mathbb{N}_0^p),$$

whose components are probability measures $f^{(i)} \in \mathcal{P}(\mathbb{N}_0^p), i = 1, \ldots, p$. In what follows it will be sometimes convenient to call vectors $f \in \mathcal{P}^p(\mathbb{N}_0^p)$ simply as probability measures and the corresponding vectors $f(s)$ of generating functions as generating functions.

**Definition 1** A sequence $v = (f_1, f_2, \ldots)$ of probability measures on $(\mathbb{N}_0^p)^p$ is called a varying environment.

**Definition 2** Let $v = (f_n, n \geq 1)$ be a varying environment. A stochastic process $\{Z_n = (Z_n(1), \ldots, Z_n(p)), n \geq 0\}$ with values in the space $\mathbb{N}_0^p$ is called a branching process in the environment $v$, if, for any $z \in \mathbb{N}_0^p$ and $n \geq 1$

$$P(Z_n = z \mid Z_0, \ldots, Z_{n-1}) = f_n^z [z].$$
In the sequel the symbol $P_{z,v}(\cdot)$ will correspond to the distribution of the process in the varying environment $v$ under the initial value $Z_0 = z$.

We now introduce the notion of a multitype branching process in random environment specified on the corresponding probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define on the set $\mathcal{P}^p(\mathbb{N}_0^p)$ of probability measures the metric of total variation $d_{TV}$ by the formula

$$d_{TV}(f, g) = \frac{1}{2p} \sum_{z \in \mathbb{N}_0^p} |f[z] - g[z]|, \quad f, g \in \mathcal{P}^p(\mathbb{N}_0^p),$$

and supply $\mathcal{P}^p(\mathbb{N}_0^p)$ with the Borel $\sigma$-algebra generated by $d_{TV}$.

We consider random probability measures $F = (F(1), \ldots, F(p))$ being random vectors with values in the space $\mathcal{P}^p(\mathbb{N}_0^p)$, whose components are specified by the probability generating functions in $p$ variables:

$$F(i)(s) := \sum_{z \in \mathbb{N}_0^p} F(i)[z] s^z, \quad i = 1, \ldots, p.$$

**Definition 3** A sequence $\mathcal{V} = \{F_1, F_2, \ldots\}$ of random measures is called a random environment.

We say that the random environment $\mathcal{V}$ is generated by a sequence of independent identically distributed random variables if the random measures $F_1, F_2, \ldots$ are independent copies of a random probability measure $F$ with values in $\mathcal{P}^p(\mathbb{N}_0^p)$. In this paper we deal with such an environment only.

In what follows the symbols $\mathbb{P}$ and $\mathbb{E}$ denote probability and expectation for a branching process in a random environment in contrast to the symbols $\mathbb{P}$ and $\mathbb{E}$ applied in the case of a branching process in a varying environment.

**Definition 4** Let $\mathcal{V}$ be a random environment. A stochastic process

$$Z = \{Z_n = (Z_n(1), \ldots, Z_n(p)), n \geq 0\}$$

with values in $\mathbb{N}_0^p$ is called a $p$-type branching process in the random environment $\mathcal{V}$, if, for all $z, z_1, \ldots, z_k \in \mathbb{N}_0^p$ and any fixed environment $v$

$$\mathbb{P}(Z_1 = z_1, \ldots, Z_k = z_k \mid Z_0 = z; \mathcal{V} = v) = \mathbb{P}_{z,v}(Z_1 = z_1, \ldots, Z_k = z_k) \quad \mathbb{P}\text{-a.s.}$$

We use below the uppercase letters to denote variables or functions if we deal with a random environment, and the lowercase letters to denote the corresponding variables or functions if we deal with a fixed environment. For instance, the (random) distribution law of particles of the $(n-1)$th generation will be specified by a tuple $F_n = (F_n^{(1)}, \ldots, F_n^{(p)})$ of (random) probability generating functions in $p$ variables. Similarly, we denote by

$$M_n := (M_n(i,j))_{i,j=1}^p = \left( \frac{\partial F_n^{(i)}}{\partial s_j}(1) \right)_{i,j=1}^p$$
the mean matrix corresponding to the probability generating function $F_n$, and so on. Clearly, the random matrices $M_n$, $n \geq 1$, as well as the matrix

$$M = (M(i, j))_{i,j=1}^p := \left( \frac{\partial F(i)}{\partial s_j} (1) \right)_{i,j=1}^p$$

are independent and identically distributed under our conditions.

We define the cone

$$C := \{ x = (x_1, \ldots, x_p) \in \mathbb{R}^p : x_i \geq 0 \text{ for all } i = 1, \ldots, p \},$$

the sphere

$$S^{p-1} := \{ x : x \in \mathbb{R}^p, |x| = 1 \},$$

and the space $X := C \cap S^{p-1}$. In the sequel we need to consider the linear semi-group $S^+$ of $p \times p$ matrices with nonnegative elements each whose row and column includes at least one positive element. For a vector $x \in X$ and a matrix $m \in S^+$ we specify the projective actions

$$x \cdot m := \frac{x m}{|x m|}, \quad m \cdot x := \frac{m x}{|m x|}$$

and define a function $\rho$ on the product space $X \times S^+ = \{(x, m)\}$ by setting

$$\rho(x, m) := \log |x m|.$$ 

The function meets the so-called cocycle property meaning that for a vector $x \in X$ and matrices $m_1, m_2 \in S^+$

$$\rho(x, m_1 m_2) = \rho(x \cdot m_1, m_2) + \rho(x, m_1).$$

The measure $\mathbb{P}$, generated by a branching process in random environment (BPRE) with $p$ types of particles, specifies the corresponding probability measure on the Borel $\sigma$-algebra of the semi-group $S^+$. We agree to denote this measure as $\mathbb{P}$ as well, i.e., for a Borel subset $A \subseteq S^+$ we set

$$\mathbb{P}(M \in A) := \mathbb{P}(f : M = M(f) \in A).$$

Keeping in mind this agreement we introduce a number of assumptions to be valid throughout the paper. These assumptions are simplified versions of the conditions introduced in [21] and concern only properties of the restriction of $\mathbb{P}$ to the semi-group $S^+$.

- **Condition H1.** The set $\Theta := \{ \theta : 0 : \mathbb{E} \left[ |M|^{\theta} \right] < \infty \}$ is nonempty.

- **Condition H2.** There exists a positive number $\Delta > 1$ such that

$$1 \leq \frac{\max_{i,j} M(i,j)}{\min_{i,j} M(i,j)} \leq \Delta.$$
• **Condition H3.** There exists $\delta > 0$ such that
\[
\inf_{x \in X} \mathbb{P} (M : \log |Mx| > \delta) > 0.
\]

Along with random matrices $M_n$ and $M$ we introduce the random Hessian matrices
\[
B^{(i)} := \left( \frac{\partial^2 F^{(i)}}{\partial s_k \partial s_l}(1) \right)_{k,l=1}^p, \quad B_n^{(i)} := \left( \frac{\partial^2 F_n^{(i)}}{\partial s_k \partial s_l}(1) \right)_{k,l=1}^p,
\]
and set
\[
\mathcal{B} := \sum_{i=1}^p |B^{(i)}|, \quad \mathcal{T} := \frac{\mathcal{B}}{|M|^2}, \quad \mathcal{B}_n := \sum_{i=1}^p |B_n^{(i)}|, \quad \mathcal{T}_n := \frac{\mathcal{B}_n}{|M_n|^2}.
\]

Thus, $\mathcal{T}_n$ are independent probabilistic copies of $\mathcal{T}$. We shall impose, along with Conditions H1 – H3 the following restriction on the distribution of $\mathcal{T}$.

• **Condition H4.** There exists an $\varepsilon > 0$ such that
\[
\mathbb{E} \left[ |M| \log |\mathcal{T}|^{1+\varepsilon} \right] < \infty.
\]

Using the standard subadditivity arguments, one can easily infer that for every $\theta \in \Theta$ the limit
\[
\lambda (\theta) := \lim_{n \to \infty} \left( \mathbb{E} \left[ |M_n \cdots M_1|^\theta \right] \right)^{1/n} < \infty
\]
is well defined. This function is an analog of the moment generating function for the associated random walk in the case of single-type BPRE’s.

Set
\[
\Lambda (\theta) := \log \lambda (\theta), \quad \theta \in \Theta.
\]

Here is our main result.

**Theorem 5** Assume that Conditions H1 – H4 are valid, the point $\theta = 1$ belongs to the interior of the set $\Theta$ and, in addition, $\Lambda'(0) < 0$ and $\Lambda'(1) = 0$. Then there exist positive constants $C^-$ and $C^+$ such that, for all $i = 1, ..., p$ and all $n \geq 1$
\[
\frac{C^-}{\sqrt{n}} \lambda^n(1) \leq \mathbb{P} \left( |Z_n| > 0 \left| Z_0 = e_i \right. \right) \leq \frac{C^+}{\sqrt{n}} \lambda^n(1).
\]

Dyakonova [7] has proved a statement more precise than (1) under stronger restrictions. Namely, she has shown that if all possible realisations of $M$ have a common deterministic left eigen-vector $v$ corresponding to the Perron root $\chi(M)$ of $M$ and some other technical conditions are valid then there exists a vector $C = (C_1, ..., C_p)$ with strictly positive components such that,
\[
\mathbb{P} \left( |Z_n| > 0 \left| Z_0 = e_i \right. \right) \sim \frac{C_i}{\sqrt{n}} \lambda^n(1), \quad n \to \infty.
\]
Note that the assumption $\Lambda'(1) = 0$ reduces in this special case to the condition $\mathbb{E}[\chi(M) \log \chi(M)] = 0$. In the single-type case the last condition corresponds to the so-called intermediately subcritical BPRE’s (see, for instance, [2] or [14], chapter 8).

2 Auxiliary results

Denote by $C(X)$ the set of all continuous functions on $X$. For $\theta \in \Theta$, $g \in C(X)$, and $x \in X$ define the transition operators

$$P_\theta g(x) := \mathbb{E}[|Mx|^{\theta} g(M \cdot x)]$$

and

$$P^*_\theta g(x) := \mathbb{E}[|M^T x|^{\theta} g(M^T \cdot x)],$$

where $M^T$ is the matrix transposed to $M$.

If Conditions $H_1$ – $H_3$ hold, then, according to Proposition 3.1 in [3], $\lambda(\theta)$ is the spectral radius of $P_\theta$ and $P^*_\theta$ and there exist unique strictly positive functions $r_\theta, r^*_\theta \in C(X)$ and unique probability measures $l_\theta$ and $l^*_\theta$ subject to the scalings

$$\int_X r_\theta(x) dl_\theta(x) = 1, \quad \int_X r^*_\theta(x) dl^*_\theta(x) = 1$$

and possessing the properties

$$l_\theta P_\theta = \lambda(\theta) l_\theta, \quad P_\theta r_\theta = \lambda(\theta) r_\theta.$$ (2)

$$l^*_\theta P^*_\theta = \lambda(\theta) l^*_\theta, \quad P^*_\theta r^*_\theta = \lambda(\theta) r^*_\theta.$$ (2)

Following [4], we introduce the functions

$$p^\theta_n(x, m) := \frac{|mx|^{\theta} r_\theta(m \cdot x)}{\lambda^n(\theta) r_\theta(x)}, \quad x \in X.$$ (3)

It is easy to see that, for $n \geq 1$, $x \in X$ and $m \in S^+$

$$\mathbb{E} \left[ p^\theta_{n+1}(x, Mm) \right] = p^\theta_n(x, m)$$ (4)

and, in view of (2)

$$\mathbb{E} \left[ p^\theta_n(x, L_{n,1}) \right] = 1.$$ (5)

For each $n \geq 1$ let $\mathcal{F}_n$ be the $\sigma$-algebra generated by random elements $Z_1, Z_2, \ldots, Z_n$ and $F_1, F_2, \ldots, F_n$. It follows from (5) that

$$\mathbb{P}^\theta_n(A) := \mathbb{E} \left[ p^\theta_n(x, L_{n,1}) 1_A \right]$$

is a probability measure on $\mathcal{F}_n$ (here $1_A$ is the indicator of the event $A$). Furthermore, (1) implies that the sequence of measures $\{\mathbb{P}^\theta_n, n \geq 1\}$ is consistent and
can be extended to a probability measure $\mathbb{P}^\theta$ on our original probability space $(\Omega, \mathcal{F})$. Denote by $\mathbb{E}^\theta[\cdot]$ the expectation taken with respect to this measure.

Now we take $\theta = 1$ and introduce a homogeneous Markov chain $\{X_n, n \geq 0\}$ with values in $\mathbb{X}$, where

$$X_0 := x \in \mathbb{X} \text{ and } X_n := x \cdot (L_{n,1})^T, \ n \geq 1.$$ 

Observe that $|x \cdot (L_{n,1})^T| > 0$ by Condition $\text{H2}$. Since the matrices $M_n$ are i.i.d. with respect to the measure $\mathbb{P}^1$, the transition probabilities of the chain are specified, for any vector $x \in \mathbb{X}$ and any Borel function $\phi : \mathbb{X} \to \mathbb{R}$ by the relation

$$Q\phi(x) := \int_{S^+} \phi(x \cdot m^T) \mathbb{P}^1(d m).$$ 

We fix a vector $x \in \mathbb{X}$, a number $a < 0$ and introduce a sequence $\{S_n, n \geq 0\}$ by the equalities

$$S_0 = a, \ S_n = S_0 + \log \left| x M_1^T M_2^T \cdots M_n^T \right|, \ n \geq 1.$$ 

Denote $\mathbb{P}^1_{x,a}$ the conditional measure, generated by the measure $\mathbb{P}^1$, and $\mathbb{E}^1_{x,a}[\cdot]$ the corresponding conditional expectation given the event $\{X_0 = x, S_0 = a\}$.

Let

$$\mu := \min \{n \geq 1 : S_n \geq 0\}$$

be the first moment when the sequence $\{S_n, n \geq 1\}$ enters the set $[0, \infty)$.

Modifying in a natural way the arguments used in [21] or in Appendix to [15] one can conclude that given the conditions of Theorem 5 the function $h : \mathbb{X} \times (-\infty, 0) \to [0, \infty)$, specified by the relation

$$h(x, a) := \lim_{n \to \infty} \mathbb{E}_{x,a}[-S_n; \mu > n],$$

possesses the property

$$\mathbb{E}_{x,a}[h(X_1, S_1); \mu > 1] = h(x, a). \quad (6)$$

We need the following upper and lower estimates for $h(x, a)$ which are reformulations to our setting the respective results from [21].

**Lemma 6** (compare with Theorem 1.1. in [21]) Under Conditions $\text{H1} - \text{H3}$, there exist constants $R > 0$ and $0 < C < \infty$ such that, for all $(x, a) \in \mathbb{X} \times (-\infty, 0)$

$$\max\{C^{-1}, |a| - R\} < h(x, a) \leq C(1 + |a|) \quad (7)$$

and

$$1 + |a| \leq (R + 1)(1 + h(x, a)). \quad (8)$$

The next result is a restatement of a part of Theorem 1.2 from [21].
Lemma 7 Let Conditions H1 − H3 be valid. Then, for any pair \((x,a) \in \mathbb{X} \times (-\infty,0)\) as \(n \to \infty\)

\[
P_{x,a}(\mu > n) \sim \frac{2}{\sigma \sqrt{2\pi n}} h(x,a),
\]

where \(\sigma \in (0,\infty)\) is a constant. Moreover, there exists a constant \(C > 0\) such that, for any pair \((x,a) \in \mathbb{X} \times (-\infty,0)\)

\[
P_{x,a}(\mu > n) \leq \frac{C (1+|a|)}{\sqrt{n}}
\]  

(9)

for all \(n \geq 1\).

Recall that \(\mathcal{F}_n, n \geq 1\), is the \(\sigma\)-algebra generated by the random variables \(Z_0, Z_1, ..., Z_n, F_1, F_2, ..., F_n\).

We introduce a new measure \(\hat{\mathbb{P}}_1\) on the flow of \(\sigma\)-algebras \(\{\mathcal{F}_n, n \geq 1\}\) by setting

\[
\hat{\mathbb{P}}_{x,a}^1[Y_n] := \frac{1}{h(x,a)} P_{x,a}(Y_n h(X_n, S_n) ; \mu > n)
\]

for any \((x,a) \in \mathbb{X} \times (-\infty,0)\) and nonnegative random variable \(Y_n\) measurable with respect to the \(\sigma\)-algebra \(\mathcal{F}_n\).

It follows from (6) and the Markov property that the respective measure \(\hat{\mathbb{P}}_{x,a}^1\) is well defined (compare with the similar definition in [21]).

Lemma 8 Let Conditions H1 − H3 be valid and \(Y_k\) be a random variable measurable with respect to the \(\sigma\)-algebra \(\mathcal{F}_k, k \geq 1\). Then, for any pair \((x,a) \in \mathbb{X} \times (-\infty,0)\)

\[
\lim_{n \to \infty} E_{x,a}^1(\mu > n) = \hat{\mathbb{P}}_{x,a}^1[Y_k].
\]  

(10)

Moreover, if \(Y_1, Y_2, ...\) is a sequence of uniformly bounded random variables adopted to the filtration \(\{\mathcal{F}_n, n \geq 1\}\) and converging \(\hat{\mathbb{P}}_{x,a}^1\) a.s. as \(n \to \infty\) to a random variable \(Y_\infty\), then

\[
\lim_{n \to \infty} E_{x,a}^1(\mu > n) = \hat{\mathbb{P}}_{x,a}^1[Y_\infty].
\]  

(11)

Proof. We follow with minor changes the line of proving lemma 2.5 in [1] (see also Lemma 5.2 in [14]). Let

\[
m_{x,a}(n) := P_{x,a}(\mu > n).
\]

Clearly,

\[
E_{x,a}^1(\mu > n) = \frac{1}{P_{x,a}(\mu > n)} E_{x,a}^1(Y_k; \mu > n)
\]

\[
= \frac{1}{P_{x,a}(\mu > n)} E_{x,a}^1(Y_k m_{X_k, S_k} (n-k); \mu > k).
\]

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In view of Lemma 7
\[
\lim_{n \to \infty} \frac{\mathbb{E}_{X_n, \alpha} (n - k)}{\mathbb{E}_{X_n, \alpha} (\mu > n)} = \frac{h(X_k, S_k)}{h(x, a)} \quad \mathbb{P}_{X_n, \alpha} \text{-a.s.}
\]
and there exists a constant \( C > 0 \) such that
\[
\frac{\mathbb{E}_{X_n, \alpha} (n - k)}{\mathbb{E}_{X_n, \alpha} (\mu > n)} \leq C \frac{h(X_k, S_k)}{h(x, a)}.
\]
The estimate
\[
\mathbb{E}_{X_n, \alpha} \left[ Y_k \frac{h(X_k, S_k)}{h(x, a)} ; \mu > k \right] = \mathbb{E}_{X_n, \alpha} [Y_k] < \infty
\]
allows us to apply the dominated convergence theorem to get
\[
\lim_{n \to \infty} \mathbb{E}_{X_n, \alpha} [Y_k | \mu > n] = \mathbb{E}_{X_n, \alpha} \left[ Y_k \lim_{n \to \infty} \frac{\mathbb{E}_{X_n, \alpha} (n - k)}{\mathbb{E}_{X_n, \alpha} (\mu > n)} ; \mu > k \right] = \frac{1}{h(x, a)} \mathbb{E}_{X_n, \alpha} [Y_k h(X_k, S_k) ; \mu > k] = \mathbb{E}_{X_n, \alpha} [Y_k],
\]
proving (11).

To check the validity of (11) fix \( \gamma > 1 \), assume for simplicity that \( |Y_n| \leq 1 \) for all \( n \geq 1 \), and observe that in view of Lemma 7
\[
\lim_{n \to \infty} \sup_{\gamma} \lim_{n \to \infty} \frac{\mathbb{E}_{X_n, \alpha} [Y_n ; \mu > n, \mu \leq \gamma n]}{\mathbb{P}_{X_n, \alpha} (\mu > n)} \leq \lim_{n \to \infty} \frac{\mathbb{P}_{X_n, \alpha} (\mu > n, \mu \leq \gamma n)}{\mathbb{P}_{X_n, \alpha} (\mu > n)} \leq \lim_{n \to \infty} \left( 1 - \gamma^{-1/2} \right) = 0.
\]
Further we write for sufficiently large \( n \)
\[
\Delta_{k,n} (\gamma) := \mathbb{E}_{X_n, \alpha} [Y_n - Y_k | \mu > \gamma n] = \mathbb{E}_{X_n, \alpha} \left[ Y_n - Y_k \frac{\mathbb{E}_{X_n, \alpha} ((\gamma - 1) n)}{\mathbb{P}_{X_n, \alpha} (\mu > n)} ; \mu > n \right] \leq 2C \sqrt{\gamma} \frac{1}{h(x, a)} \mathbb{E}_{X_n, \alpha} [Y_n - Y_k ; h(X_n, S_n) ; \mu > n] = 2C \sqrt{\frac{\gamma}{\gamma - 1}} \mathbb{E}_{X_n, \alpha} [Y_n - Y_k].
\]
Since \( Y_m \to Y_{\infty} \) \( \mathbb{P}_{X_n, \alpha} \text{-a.s.} \) as \( m \to \infty \) by the conditions of the lemma, letting first \( n \) to inifinity and than \( k \) to inifinity \( \Delta_{k,n} (\gamma) \) vanishes for any fixed \( \gamma > 1 \).

This fact and the first part of the lemma show that
\[
\lim_{n \to \infty} \mathbb{E}_{X_n, \alpha} [Y_n | \mu > \gamma n] = \lim_{k \to \infty} \lim_{n \to \infty} \mathbb{E}_{X_n, \alpha} [Y_k | \mu > \gamma n] = \mathbb{E}_{X_n, \alpha} [Y_{\infty}]
\]
for any \( \gamma > 1 \). Hence, writing for \( k < n \) and \( \gamma > 1 \)
\[
\mathbb{E}_{X_n, \alpha} [Y_n | \mu > n] = \mathbb{E}_{X_n, \alpha} [Y_n | \mu > \gamma n] \mathbb{P}_{X_n, \alpha} (\mu > n) + \mathbb{E}_{X_n, \alpha} [Y_n ; \mu > n, \mu \leq \gamma n] \mathbb{P}_{X_n, \alpha} (\mu > n),
\]
for any \( \gamma > 1 \).
observing that
\[
\frac{\mathbb{E}^1_{x,a}[Y_n; \mu > n, \mu \leq \gamma n]}{\mathbb{P}^1_{x,a}(\mu > n)} = O\left(\frac{\mathbb{E}^1_{x,a}(\mu > n, \mu \leq \gamma n)}{\mathbb{P}^1_{x,a}(\mu > n)}\right)
\]
and using Lemma 7 we conclude that, as \( n \to \infty \)
\[
\mathbb{E}^1_{x,a}[Y_n|\mu > n] = \left(\hat{\mathbb{E}}^1_{x,a}[Y_\infty] + o(1)\right) \left(\frac{1}{\sqrt{\gamma}} + o(1)\right) + O\left(1 - \frac{1}{\sqrt{\gamma}}\right).
\]
Letting now sequentially \( n \) to infinity and \( \gamma \) to 1 completes the proof of the lemma.

The next lemma is a generalization of Lemma 3.1 of [15] to our setting.

**Lemma 9** Under the conditions of Theorem 5 for any pair \((x, a)\) \(\in X \times (-\infty, 0)\)
\[
\hat{\mathbb{E}}^1_{x,a} \left[ \sum_{n=1}^{\infty} T_n e^{S_n} \right] < \infty.
\]
The proof of this lemma has practically no differences with the proof of Lemma 4 in [17] and we omit it.

### 3 Proof of Theorem 5

For every environmental sequence \(F_n\) and \(0 \leq k < n\) define
\[
F_{k,n}(s) = \left(F_{k,n}^{(1)}(s), \ldots, F_{k,n}^{(p)}(s)\right) := F_{k+1}(F_{k+2}(\ldots(F_n(s))\ldots)),
\]
\[
F_{n,k}(s) = \left(F_{n,k}^{(1)}(s), \ldots, F_{n,k}^{(p)}(s)\right) := F_n(F_{n-1}(\ldots(F_{k+1}(s))\ldots))
\]
and set \(F_{n,n}(s) := s\). It is immediate from the definition of the process \(Z\) that
\[
\mathbb{E}[s^Z_0|Z_0 = e_i] = \mathbb{E}[F_{0,n}^{(i)}(s)].
\]
Letting \(s = 0\) and using the independency of the environmental components we get
\[
\mathbb{P}\left(|Z_n| > 0|Z_0 = e_i\right) = 1 - \mathbb{E}[F_{0,n}^{(i)}(0)] = \mathbb{E}[1 - F_{n,0}^{(i)}(0)].
\]
Set
\[
L_{n,k} := M_nM_{n-1}\ldots M_k, \ 1 \leq k \leq n
\]
and let \(L_{n,n+1}\) be the \(p \times p\) identity matrix.
We take \(\theta = 1\) in (3) and apply the corresponding change of measure to the representation
\[
1 - F_{0,n}^{(i)}(0) \overset{d}{=} 1 - F_{n,0}^{(i)}(0) = (e_i, 1 - F_{n,0}(0)).
\]
From now on we agree to consider \( e_i \) as a row vector, and \( e_i^T \) as its transpose. Since \((e_i,1-F_{n,0}(0))\) is measurable with respect to \( F_n \), it follows that

\[
E[1 - F_{0,n}^{(i)}(0) = \lambda^n(1)r_1(e_i^T)E \left[p_n(e_i^T, L_n, 1) \left| \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|} \right| r_1(L_{n,1}e_i^T) \right].
\]

To prove the theorem we need to show that there exist positive constants \( C^- \) and \( C^+ \) such that

\[
\frac{C^-}{\sqrt{n}} \leq E_1 \left[ \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|} \right] \leq \frac{C^+}{\sqrt{n}}
\]

for all \( n \geq 1 \).

First observe that \( r_1(x) \) is a positive function on the compact \( X \). Hence

\[
0 < c_1 \leq r_1(x) \leq c_2 < \infty
\]

for some constants \( c_1 \) and \( c_2 \). Thus, to complete the proof of Theorem 5 it is sufficient to demonstrate that

\[
\frac{c_3}{\sqrt{n}} \leq E_1 \left[ \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|} \right] \leq \frac{c_4}{\sqrt{n}}
\]

for some positive constants \( c_3 \) and \( c_4 \).

**Estimate in (12) from above.** We fix a pair \((x,a) \in X \times (-\infty, 0)\) and use the decomposition

\[
E_1 \left[ \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|} \right] = E_{x,a} \left[ \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|} \right] = E_{x,a} \left[ \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|} : \mu \leq n \right] + E_{x,a} \left[ \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|} : \mu > n \right].
\]

Write \( L_{n,k} = (l_{n,k}(q,r))^q_{q=r=1} \). Note that if Condition H3 is valid then, according to Lemma 2 in [13] for any \( n,k \) and any tuple \( 1 \leq h,g,q,r \leq p \)

\[
\Delta^{-2} \leq \frac{l_{n,k}(h,g)}{l_{n,k}(q,r)} \leq \Delta^2.
\]

Using for \( 0 \leq k \leq n-1 \) the inequality

\[
(e_i, 1 - F_{n,0}(0)) \leq (e_i, L_{n,k+1}(1 - F_{k,0}(0))) \leq (e_i, L_{n,k+1}) \leq p^2 \max_{q,r} l_{n,k+1}(q,r)
\]
and the estimate
\[ |L_{n,1}e_i^T| = |L_{n,k+1}L_{k,1}e_i^T| \geq \min_{q,r} l_{n,k+1}(q, r)|L_{k,1}e_i^T| \geq \Delta^{-2} \max_{q,r} l_{n,k+1}(q, r)|L_{k,1}e_i^T| \]
we see that
\[ \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|} \leq p^2 \frac{\max_{q,r} l_{n,k+1}(q, r)}{\min_{q,r} l_{n,k+1}(q, r)} \frac{1}{|L_{k,1}e_i^T|} \leq \frac{\Delta^2 p^2}{|L_{k,1}e_i^T|}. \]

Hence we deduce
\[ \mathbb{E}_{e_i,0}^1 \left( \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|}; \mu \leq n \right) \leq \Delta^2 p^2 \mathbb{E}_{e_i,0}^1 \left( \frac{1}{|L_{k,1}e_i^T|}; \mu \leq n \right) \]

or, in view of \( |L_{k,1}e_i^T| = |e_i (L_{k,1})^T| \)
\[ \mathbb{E}_{e_i,0}^1 \left( \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|}; \mu \leq n \right) \leq \Delta^2 p^2 \mathbb{E}_{e_i,0}^1 \left( \min_{0 \leq k \leq n} \frac{1}{|e_i (L_{k,1})^T|}; \max \log |e_i (L_{k,1})^T| \geq -a \right). \]

To evaluate the right-hand side of this inequality we use the estimates
\[ \mathbb{E}_{e_i,0}^1 \left[ e^{-\max_{0 \leq k \leq n} \log |e_i (L_{k,1})^T|}; \max \log |e_i (L_{k,1})^T| \geq -a \right] \leq \sum_{j=-a}^{\infty} e^{-j} \mathbb{P}_{e_i,0}^1 \left( j < \max \log |e_i (L_{k,1})^T| \leq j + 1 \right) \leq \sum_{j=-a}^{\infty} e^{-j} \mathbb{P}_{e_i,j} \left( \mu > n \right) \leq \frac{C}{\sqrt{n}} \sum_{j=-a}^{\infty} e^{-j(j+1)}, \]
where the last inequality is justified by (9). Whence, for the first term at the right-hand side of (13) we obtain
\[ \mathbb{E}_{e_i,0}^1 \left( \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|}; \mu \leq n \right) \leq \frac{\Delta^2 p^2 C}{\sqrt{n}} \sum_{j=-a}^{\infty} e^{-j(j+1)}. \]

For the second term in (13) we apply (9) once again to conclude that
\[ \mathbb{E}_{e_i,0}^1 \left( \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|}; \mu > n \right) \leq \mathbb{E}_{e_i,0}^1 \left( \frac{(e_i, L_{n,1}1)}{|L_{n,1}e_i^T|}; \mu > n \right) \leq \frac{\Delta^2 p^2 \mathbb{E}_{e_i,0}^1 (\mu > n)}{\sqrt{n}}. \]

Thus,
\[ \sqrt{n} \mathbb{E}_{e_i,0}^1 \left( \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1}e_i^T|} \right) \leq C \Delta^2 p^2 (1 + |a|) \left( 1 + \sum_{j=-a}^{\infty} e^{-j(j+1)} \right) \]
which leads to the desired estimate from above in \((12)\).

**Estimate in \((12)\) from below.** For a generating function \(f\), the corresponding mean matrix

\[
m = \left( \frac{\partial f(i)}{\partial s^j} (1) \right)_{i,j=1}^p,
\]

and a matrix \(a\) with nonnegative elements define

\[
\psi_{f,a}(s) := \frac{|a|}{|a(1 - f(s))|} - \frac{|a|}{|a m (1 - s)|}, \quad s \in [0, 1]^p \setminus \{1\}.
\]

Let \(a_i = (a(j, k))^p_{j,k=1}\) be the matrix with \(a(i, i) = 1\) and \(a(k, l) = 0\) for all \((k, l) \neq (i, i)\). Then, clearly,

\[
1 - F_{n,0}^{(i)}(s) = |a_i (1 - F_{n,0}(s))|.
\]

Using the definition of \(\psi\), we write

\[
\frac{1}{1 - F_{n,0}^{(i)}(s)} = \frac{|a_i|}{a_i (1 - F_{n,0}(s))} = \frac{1}{|a_i M_n (1 - F_{n-1,0}(s))|} + \psi_{F_{n-1,0}(s)},
\]

\[
= \frac{1}{a_i L_{n,n} |a_i L_{n,n} (1 - F_{n-1,0}(s))|} + \frac{1}{a_i L_{n,n+1}} \psi_{F_{n-1,0} L_{n,n+1}(F_{n-1,0}(s))} + \frac{1}{a_i L_{n,n-1}} |1 - F_{n-2,0}(s)|
\]

\[
+ \frac{1}{a_i L_{n,n}} \psi_{n-1,a L_{n,n}} (F_{n-2,0}(s)) + \frac{1}{a_i L_{n,n+1}} \psi_{F_{n-1,0} a L_{n,n+1}(F_{n-1,0}(s))}.
\]

Iterating this procedure, we obtain

\[
\frac{1}{1 - F_{n,0}^{(i)}(s)} = \frac{1}{|a_i L_{n,1}(1 - s)|} + \sum_{k=1}^n \frac{1}{a_i L_{n,k+1}} \psi_{F_{k-1,0} a L_{n,k+1}(F_{k-1,0}(s))}. \tag{15}
\]

In view of \((15)\) we have

\[
\mathbb{E}_{e_{i,a}} \left[ \frac{(e_i, 1 - F_{n,0}(0))}{|L_{n,1} e_i^T|}; \mu > n \right] = \mathbb{E}_{e_{i,a}} \left[ \frac{|e_i L_{n,1}|}{|L_{n,1} e_i^T|}; \mu > n \right],
\]

where

\[
\Xi_{n} := \left( 1 + \sum_{k=1}^n \frac{|e_i L_{n,1}|}{|a_i L_{n,k+1}|} \psi_{F_{k-1,0} a L_{n,k+1}(F_{k-1,0}(0))} \right)^{-1}.
\]

Using \((14)\) we conclude that

\[
\frac{|e_i L_{n,1}|}{|L_{n,1} e_i^T|} \geq \frac{\min_{q,r} l_{n,1}(q, r)}{p \max_{q,r} l_{n,1}(q, r)} \geq \frac{1}{p \Delta^2}.
\]
Further, it is known (see Lemma 5 in [17]), that, for all \( s \in [0,1)^p \setminus \{1\} \)

\[
0 \leq \psi_{\mathbf{F}_k, \mathbf{a}_n, \mathbf{L}_{n,k+1}}(s) \leq \Delta p^2 T_k
\]

and, evidently,

\[
\frac{|e_{\mathbf{L}_{n,1}}|}{|a_{\mathbf{L}_{n,k+1}}|} = \frac{|e_{\mathbf{L}_{n,k+1}}|}{|a_{\mathbf{L}_{n,k+1}}|} \leq \frac{|e_{\mathbf{L}_{n,k+1}}|}{|a_{\mathbf{L}_{n,k+1}}|} = |L_{k,1}|.
\]

Thus, there exists a positive constant \( c_6 \) such that

\[
\Xi_n \geq Y_n := c_6 \left(1 + \sum_{k=1}^{n} |L_{k,1}| T_k\right)^{-1}
\]

\[
\geq c_6 \left(1 + \sum_{k=1}^{\infty} |L_{k,1}| T_k\right)^{-1} =: Y_{\infty} > 0 \quad \mathbb{P}_1^{1,x,a} \text{ a.s.}
\]

(the last in view of Lemma 9). Thus,

\[
\mathbb{P}_1^{1,x,a} \left[ \left\{ \frac{|e_{\mathbf{L}_{n,1}}|}{|L_{n,1} e_1^T|} \Xi_n ; \mu > n \right\} \geq \frac{1}{p \Delta^2} \mathbb{P}_1^{1,x,a} [Y_n ; \mu > n] \right]
\]

\[
= \frac{1}{p \Delta^2} \mathbb{P}_1^{1,x,a} [Y_n ; \mu > n] \mathbb{P}_1^{1,x,a} (\mu > n).
\]

Hence, using Lemma 7 Lemma 8 for the sequence \( Y_n \to Y_{\infty} \mathbb{P}_1^{1,x,a} \text{ a.s. as } n \to \infty \) and (7) we deduce that

\[
\liminf_{n \to \infty} \sqrt{n} \mathbb{P}_1^{1,x,a} \left[ \left\{ \frac{|e_{\mathbf{L}_{n,1}}|}{|L_{n,1} e_1^T|} \Xi_n ; \mu > n \right\} \right] > 0,
\]

proving the estimate from below in (12). This completes the proof of Theorem 5.

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