Conformal, Subconformal and Spectral Universality in Incommensurate Spin Chains

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A renormalization scheme is developed to study an anisotropic quantum XY spin chain in a quasiperiodic transverse field. The critical phase of the quasi-particle excitations of the model with fractal wave functions exists in a finite parameter interval and is sandwiched between the extended and localized phases. The scaling properties of the critical phase fall into four distinct universality classes referred as spectral, subconformal, conformal and Harper. The spectral and conformal classes respectively describe the onsets of extended to critical and critical to localized transitions while the subconformal class describes the part of the phase diagram sandwiched between the conformal and spectral transitions. The Harper universality class describes the isotropic limit of the XY model. A decimation scheme is developed to compute the infinite sets of universal scaling ratios characterizing the wave functions in the four universality classes. The renormalization flow equations exhibit a limit cycle at the band center and at the band edges providing a new method for determining these energies with extremely high precision.

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I. INTRODUCTION

In recent years, renormalization group (RG) techniques have been the dominant theme in describing the scaling properties at the onset of phase transitions. These studies include the thermodynamical phase transitions, driven by thermal fluctuations, as the temperature is varied as well as the transitions which occur when a control parameter is varied. The later include the transitions to chaos in nonlinear dynamical systems and the phase transitions in quantum systems at zero temperature such as the metal-insulator transitions, driven by quantum fluctuations.

One-dimensional quantum spin chains provide very simple and interesting models that exhibit phase transitions at zero temperature as a function of anisotropy or magnetic field. In some cases, the \( T = 0 \) phase transitions of the quantum spin chain map onto the phase transitions of a 2D classical model in which the variable parameter is the temperature. These systems are also of experimental interest as various physical realizations of quantum spin chains have been found in quasi-1D magnetic systems. The Ising and the anisotropic XY model in a constant transverse field provide the simplest class of quantum models (that can be solved exactly) that exhibit the second order magnetic phase transition at a critical value of the magnetic field. The anisotropic model belongs to the universality class of the Ising model, i.e. the critical exponents at the onset of transition are the same as those of the Ising model. Therefore, such anisotropic models with a uniaxial anisotropy are usually referred as Ising like.

Motivated by the discovery of a quasi-crystal and the fabrication of magnetic superlattices, quantum spin chains in a quasiperiodic (QP) transverse field have been investigated extensively. Initial studies were confined to the case where the transverse magnetic field was inhomogenous with binary Fibonacci distribution. The quasiparticle excitation spectrum turned out to be a fractal for all parameter values. These studies were followed by other ones in which the magnetic field was oscillatory with the period incommensurate with respect to the period of the spin chain. Detailed numerical work showed that the quasiparticle excitations of the anisotropic quantum spin chain could be extended (E), localized (L) or critical (C) resulting in E-C and C-L transitions as the strength of the magnetic field was varied. These three different phases were respectively characterized by continuous, point like and singular continuous spectrum. It should be noted that in the isotropic limit the eigenvalue equation for the quasiparticle excitations of the model reduce to the the famous Harper equation exhibiting both extended and localized phases and hence the metal-insulator transition. The existence of a self-similar butterfly spectrum at the onset of transition aroused a great deal of interest in this model. The Harper equation has also been of interest due to the fact that it describes the two-dimensional electron gas in a magnetic field. Furthermore, this model has attracted the attention of mathematicians due to the small denominator problems.
The anisotropic XY model describes a perturbed Harper model and therefore provides a new class of quasiperiodic models which splits the single E-L transition of the Harper model into two transitions, namely E-C and C-L transitions. The existence of a fat C phase sandwiched between E and L phases provides a new paradigm for the metal-insulator transition in one-dimensional incommensurate systems. Analogous to the Harper model, in the anisotropic model both the E-C and C-L transitions are global in the sense that all quantum states are E, C or L simultaneously. Fig. 1 shows the spectral phase diagram of the model. The line in the phase diagram corresponding to the onset of localization also corresponds to the onset of magnetic phase transition from ordered to disordered phase providing an interesting example of a spectral and magnetic interplay. This line will be referred as the conformal line as along this line, the gap in the excitation spectrum vanishes and the model is conformally invariant. [10] The E-C transition in the quasiparticle wave functions does not affect the magnetic properties and hence the corresponding line will be referred as the spectral line. The fat C region sandwiched between the the spectral and conformal line will be called subconformal. The conformal and spectral lines intersect at the Harper limit.

In our previous short paper [11], we described a new decimation scheme to study nearest-neighbor (n.n.) tight binding models (TBM) and applied it to QP Ising chains exhibiting the C-L transition. The proposed scheme confirmed the existence of a fat C phase in the Ising limit. It also demonstrated the fact that the fat C phase was described by two distinct fixed points which respectively described the C-L transition at the conformally invariant point and the subconformal C phase. Since the decimation scheme was valid only for n.n. TBM’s, it described only the Ising limit of the general anisotropic model containing also a next-nearest-neighbor (n.n.n.) term. Therefore, these studies did not provide a full characterization of the C phase. In particular, the universality class for the E-C transition and for the conformal line (see Fig. 1) remained an open question.

In this paper we generalize our method to study the anisotropic model which includes Ising as a special case. The scalar decimation equations of our previous studies are replaced by vector equations. We show that the C phase is described by four distinct universality classes.

In Section 2, we describe the model and the corresponding tight binding form. In Section 3, we discuss the decimation scheme based on the Fibonacci lattice sites. Sections 4 and 5 respectively deal with the qualitative as well as quantitative characterization of universal features of the phase diagram. In section 6, we show how the decimation scheme can be generalized for an arbitrary path in the Farey tree. In section 7, we discuss the RG equations. Section 8 summarizes the various universality classes, and we end with some conclusions in Section 9.

II. SPIN HAMILTONIAN AND THE TIGHT BINDING EQUATION

The anisotropic XY model in a transverse field is given by

\[ H = -\sum_i \left[ \frac{1}{2} (J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y) + h_i \sigma_i^z \right] \]  

(1)

Here \( \sigma \) are the Pauli matrices describing the spin 1/2. \( J_x \) and \( J_y \) describe ferromagnetic exchange interactions. In our discussion, unless otherwise stated, \( h_i \) is an inhomogenous modulating field of periodicity \( \sigma \) which is incommensurate with the periodicity of the spin chain:

\[ h_i = \lambda \cos(2\pi(i \sigma + \phi)) \]  

(2)

We choose the parameter \( \sigma \) to be equal to the inverse golden mean \( \sigma = (\sqrt{5} - 1)/2 \). \( \phi \) is a constant phase factor.

Using the Jordan-Wigner transformation, the model can be transformed into that of a free spinless fermion Hamiltonian

\[ H = -\sum_i \left[ \frac{1}{2} (J_x + J_y) c_i^\dagger c_{i+1} + \frac{1}{2} (J_x - J_y) c_i^\dagger c_{i+1} + h_i c_i^\dagger c_i + h.c. \right] \]  

(3)

We see that the lack of the O(2) symmetry in the spin Hamiltonian results in a lack of the \( U(1) \) symmetry \( (c_i^\dagger c_i U) \) in the corresponding fermion Hamiltonian. The system can be diagonalized using the methods described by Lieb et al. [3]. The resulting eigenvalue equations for the quasi-particle excitations are

\[ J_y \psi_{i+1} + J_x \psi_{i-1} + 2h_i \psi_i = E \eta_i \]  

(4)

\[ J_x \eta_{i+1} + J_y \eta_{i-1} + 2h_i \eta_i = E \psi_i \]  

(5)

The above two coupled equations with n.n. terms can be combined into a single equation with n.n.n. interaction:

\[ E^2 \psi_i = J_x J_y (\psi_{i-2} + \psi_{i+2}) + 2(J_y h_{i-1} + J_x h_i) \psi_{i-1} + 2(J_y h_i + J_x h_{i+1}) \psi_{i+1} + (J_x^2 + J_y^2 + 4h_i^2) \psi_i \]  

(6)

Alternatively, the coupled equations can be rewritten as a single vector equation involving only a n.n. interaction:

\[ J(y,x) \Psi_{i+1} + J(x,y) \Psi_{i-1} + 2h_i \Psi_i = E \Psi_i \]  

(7)

where \( J(x,y) \) is a 2 \times 2 matrix whose nondiagonal entries are zero while its diagonal elements are \( J_y \) and \( J_x \) (see Eqs. 4 and 5) and \( \Psi \) is a two-dimensional vector.

In the isotropic limit, the coupled equations become degenerate and the two components of \( \Psi \) become identical. The resulting equation is the famous Harper equation which is a paradigm for quasiperiodic models exhibiting the metal-insulator transition. The anisotropic
The fact that the anisotropy doubles the size of the Hilbert space can be seen also in the semiclassical limit. The anisotropic model in the semiclassical limit describes the motion of a particle with the Hamiltonian \[ H = \frac{1}{2}(J_x + J_y)\cos(p) + \frac{i}{2}(J_x - J_y)\sin(p) + \lambda \cos(x), \] (9)

where \( p = \frac{h}{\tau} \frac{d}{dx} \) is the momentum operator. We see that the anisotropy perturbs the system along a complex direction. This explains why the Hilbert space has doubled its dimension in the anisotropic case.

In the Ising limit \( J_y = 0 \), the matrices \( J(x,y) \) of the vector equation are noninvertible. In this case, we will always work with the TBM which contains only n.n. terms:

\[ 2J_x h_{i+1} \psi_{i+1} + 2J_x h_i \psi_{i-1} + (J_x^2 + 4h_i^2) \psi_i = E^2 \psi_i. \] (10)

An interesting limit of the Ising model is the case \( J_x \to \infty \) and \( \lambda \to 0 \) where the TBM contains no diagonal disorder:

\[ 2h_i \psi_{i+1} + 2h_i \psi_{i-1} = E \psi_i \] (11)

It is easy to show that the model is always self-dual and has been numerically shown to have only critical states. Therefore we refer to it as the critical model. It should be noted that this TBM also describes the isotropic XY model with oscillatory exchange interaction \( J_x(i) = h_i \) and a constant field. This should be contrasted with the Harper model which describes the isotropic XY model with constant exchange and oscillatory field.

The numerical phase diagram \[ \text{Fig. 1.} \] for the anisotropic model describing a perturbed Harper model is shown in Fig. 1. Whenever \( \lambda \) and one of the exchange interactions become equal either the E-C or C-L transition takes place. The E-C transition corresponds always to a smaller absolute value of \( \lambda \), i.e. to the exchange interaction whose absolute value is smaller. Whenever \( J_x \) and \( J_y \) differ a fat \( C \) phase is observed in the phase diagram. Therefore, the fattening of the critical point is due to the breaking of the \( U(1) \) symmetry which is a consequence of the broken \( O(2) \) symmetry in spin space. An interesting consequence of the spectral and magnetic interplay is the fact that the onset of the C-L transition is coincident with the onset of the magnetic transition to the long range order (LRO) where two-point long range spin-spin correlations vanish. These numerical observations regarding the onset of E-C and C-L transitions and the fact that C-L transition coincides with magnetic transition still remain unexplained.

The energy spectrum of the anisotropic model is characterized by a gap in the spectrum which vanishes along the conformal line. Therefore, unlike the isotropic limit, \( E = 0 \) is not an eigenvalue of the system in most part of the phase diagram. As seen below, this introduces another degree of complexity in analysing the scaling properties of the model.

III. FIBONACCI DECIMATION

A. Formulation

One of the challenging questions posed by the fat C phase is how the scaling properties vary within the phase. A priori there are two main possibilities: either the scaling can be described by only few different universality classes or the scaling properties vary continuously through the phase. The former case would correspond to a few distinct RG fixed points whereas the latter case could be described by a line of RG fixed points in function space.

In recent years, various RG schemes have been proposed to study the scaling properties of the critical point of the Harper equation. \[ \text{[14], [15], [16], [17]} \] Our methodology is somewhat similar to that of Ostlund et al. \[ \text{[14]} \] as it describes the scaling properties of the wave functions for specific values of the energy. Furthermore, in analogy with their scheme, our decimation scheme has been applied to study the breakup of KAM tori in the circle and standard maps. \[ \text{[18], [19]} \] However, the method of Ostlund and Pandit \[ \text{[14]} \], which was based on transfer matrices, had limited success due to the fact that at the localization threshold infinite products of transfer matrices diverged for almost all values of the phase \( \phi \). We propose a new decimation scheme where instead of multiplying transfer matrices, the TBM itself is decimated. The main advantage is the reduction in the number of functions needed to carry out the renormalization. The cost we have to pay is that our recursion relations will be slightly more complicated. However, it turns out that with fewer functions we are able to eliminate directions which lead to divergences. This not only helps in approaching the RG problem but also provides practical means of calculating various essential quantities like the localization threshold and the minimum \( (E_{\text{min}}) \) and maximum \( (E_{\text{max}}) \) energy eigenvalues in general TBM’s. The knowledge of machine precision \( E_{\text{min}} \) and \( E_{\text{max}} \) comes from the study of the states with Bloch index \( k = 0 \) and \( k = 1/2 \) in addition to \( k = 1/4 \) which respectively correspond to \( E_{\text{min}} \),
$E_{\text{max}}$ and $E = 0$ quantum states. The previous studies of Ostlund and Pandit [14] and those of Dominguez, Wiecko and Jose [17] were restricted to the Bloch index $k = 1/4$ corresponding to $E = 0$ part of the spectrum.

We will first describe the decimation scheme based on the Fibonacci lattice sites. This procedure describes only three critical scaling ratios out of infinity that are required to fully characterize the wave function. However, the Fibonacci decimation scheme is quite sufficient to confirm the fat C phase and to calculate critical parameters and energies. A decimation scheme based on an arbitrary path in the Farey tree (see sections 6 and 7) will determine the infinite set of universal scaling ratios needed to describe the universality completely. As discussed later in the paper, the dominant peaks of the wave functions are described by $\sigma^3$ decimation.

In this scheme, all sites except those labelled by the Fibonacci numbers $F_n$ ($n = 0, 1, 2, ...$) are decimated. This results in a TBM connecting a function $\Psi$ at two neighboring Fibonacci sites:

$$\Psi(i + F_{n+1}) = c_n(i)\Psi(i + F_n) + d_n(i)\Psi(i) \quad (12)$$

In the Harper and the Ising limits, the above equation is a scalar equation with scalar "decimation" functions $c_n(i)$ and $d_n(i)$. For the general anisotropic case, $\Psi(i)$ is taken to be the vector $(\psi_i, \eta_i)$ of the coupled eigenvalue equation and the multiplying factors $c_n$ and $d_n$ are $2 \times 2$ "decimation" matrices. In this case, the name "decimation function" refers to the entries of these matrices, which are functions of the lattice site $i$. The index $n$ labels the "level" of the decimation. As the fractional part of $F_n\sigma$ decays to zero as $n \to \infty$, we expect the scaling of the wave function for consecutive Fibonacci sites to show some regularities which should also come up in the decimation functions. Using the defining property of the Fibonacci numbers, $F_{n+1} = F_n + F_{n-1}$, the following recursion relations are obtained for $c_n$ and $d_n$ [13]:

$$c_{n+1}(i) = c_n(i + F_n)c_{n-1}(i + F_n) - d_n^{-1}(i)d_{n+1}(i) \quad (13)$$
$$d_{n+1}(i) = -d_n(i)d_n(i + F_n) + c_n(i + F_n)d_{n-1}(i + F_n)c_n^{-1}(i). \quad (14)$$

As seen from the above equations, the decimated matrices describe a flow in Fibonacci space provided their inverse exist.

In order to iterate these equations, we need the initial conditions for the matrices at the levels $n = 1$ and $n = 2$, say. For the standard Fibonacci numbers with $F_1 = F_2 = 1$, it is possible to take $c_1$ to be the unity matrix and let $d_1$ vanish. The non-trivial initial conditions at $n = 2$ are obtained directly from the defining equations (4-5) for the TBM. The decimation could be carried out also with other choices for $F_1$ and $F_2$ but this would require more elaborate initial conditions.

Comparing our decimation scheme with that of Ostlund et al. [14], it should be noticed that our approach based on the special TBM form requires half the number of functions needed for the method based on transfer matrices. As discussed below, our new approach eliminates most of the divergences encountered in the previous approach.

III.B Results and applications

The recursion relations for the decimation functions were iterated numerically for extremely large size systems (upto sizes 500,000) required to study lattice of $F_{24}$ spins. The renormalized $2 \times 2$ decimation matrices were then diagonalized.

Numerical iterations of the scalar decimation equations for the Harper and the Ising models result in well-defined asymptotic solutions for the functions $c_n$ and $d_n$ as $n \to \infty$ in both E and C phases. For the vector decimation of the general anisotropic model, asymptotic decimation matrices and their inverse were found to exist in the $E$ phase as well as along the $E - C$ transition line in the phase diagram. However, along the conformal line and within the fat C phase for the general anisotropic case (except the Ising limit), the $c$- or $d$-matrices were non-invertible for some lattice sites and hence our decimation equations could not be used there. We used alternative means to study this part of the phase diagram as described in the next section.

The asymptotic behaviour of the decimation functions as $n \to \infty$ is different in the E, C, and L phases and also depends sensitively on the energy $E$ and the phase $\phi$. In the L phase, $c_n$ always diverges and $d_n$ tends to zero as one would expect by looking at Eq. (12). In the E and C phases, the decimation functions remain bounded for all $n$ (except for some cases in the E phase, see Table I). Furthermore, at the band edges corresponding to the minimum and maximum eigenvalue $E_{\text{min}}$ and $E_{\text{max}}$ (and also at the band center for Harper), the decimation functions converge on well-defined limit cycles provided the phase factor $\phi$ is tuned to some special values. For arbitrary $\phi$ and also for other eigenvalues, the decimation functions oscillate in a rather irregular way with increasing $n$ converging perhaps on a strange attractor.

Table I shows the trivial limit cycles for the E phase. The asymptotic solutions are independent of the chosen value of $\phi$ and hence do not depend upon the site index $i$. This provides a rather interesting example of the dimensional reduction for a system with infinite degrees of freedom. It should be noted that the limit cycle is almost independent of the anisotropy $g$.

For the Harper model and also along the conformal line in the phase diagram the existence of a six-cycle can be seen for the zero-energy state. For most of the phase diagram, the model has no zero-energy eigenstate and hence the determination of a limit cycle requires eigenenergy to an extremely high precision. The values of the other parameters determines how sensitive the solution is to an error in the energy. Typically, in order to see the limit cycle with three digit precision, energies are required to
have machine precision. In the worst case of a delayed crossover, where the parameter values are very close but not exactly on a transition line, it may happen that machine precision energy may give an asymptotic limit cycle with single digit precision. Obtaining machine precision energies is almost an impossible task even for the tridiagonal matrices. (See Table III) Table II illustrates how the fixed point found in the E phase can be used to numerically determine \( E_{\text{max}} \) for an arbitrary value of \( \lambda \). A good initial estimate of \( E_{\text{max}} \) can be obtained via a direct diagonalization of the TBM with periodic boundary conditions. This initial estimate can then be improved by imposing the fixed point property. The resulting energy eigenvalues are much more accurate than the ones obtained by the diagonalization. [20]

Unlike the trivial limit cycle of the E phase, the C phase is characterized by non-trivial asymptotic 6-cycles observed at the band edges for carefully chosen values of \( \phi \). Furthermore, the decimation functions depend upon \( i \) (see Figs. 5 and 6). Our numerical results show that \( \phi = 1/4 \), for which the QP potential has the symmetry \( h_{-i} = -h_i \), always gives a desired six-cycle throughout the fat C phase. The value \( \phi = 0 \) leads also to a reflection symmetry (due to various symmetries in the anisotropic model the phase can be defined mod 1/2). The phase \( \phi = 1/4 \) is however special because for the part of the phase diagram where zero energy state is an eigenstate (Harper model and the anisotropic model along the conformal line), this value of \( \phi \) causes the main peak of the wave function to lie at \( i = 0 \) and the resulting wave function is bounded. (See Section IV)

In the cases where \( E = 0 \) is not an eigenstate, \( \phi = 1/4 \) was found to shift the main peak in a very orderly fashion (related to number theory) for finite lattices. For the Fibonacci lattices of size \( F_n \), the main peak is located at \( p_k \), where \( p_k \) satisfies the recursion \( p_{k+1} - p_k = 4(p_k - p_{k-1}) \). Therefore, the shifts in the peak positions \( p_k - p_{k-1} \) give the rational approximants of \( \sigma^2 \). For the odd Fibonacci lattice sizes the shifts \( p_k - p_{k-1} \), are given by the even Fibonacci sequence 8, 34, 144,... while that for the lattices of size 8, 34, 144,..., the shifts are 21, 89, 377,... The shift in the main peak between two neighboring odd-even cases is 4, 17, 72, 305,... This systematic shift in the main peak implies that the continuously varying critical \( \phi \) for locating the main peak at \( i = 0 \) can be found up to arbitrary precision by the formula \( \phi = \lim_{k \to \infty} < \sigma p_k + 1/4 > \), where \( < > \) denotes the fractional part. For the fat C phase, this phase factor is found to vary continuously as \( p_0, p_1, p_2 \) vary inside the fat C phase. The knowledge of \( \phi \) is essential in confirming the universality as discussed in the next two sections. [21]

Detailed numerical iterations of the decimation equations reveal that the fat C phase of the model is described by four distinct limit cycles:

1. Spectral limit cycle along the E-C transition line.
2. Conformal limit cycle along the conformal line (C-L line).
3. Subconformal limit cycle, for the fat C regime bounded by E-C and C-L transitions.
4. Harper limit cycle corresponding to the isotropic limit of the model.

As stated earlier, all the four limit cycles were found for \( E_{\text{min}} \) as well as for \( E_{\text{max}} \) (and also for \( \lambda = 0 \) for Harper).

It is interesting that the spectral limit cycle does not depend upon the actual value of the \( E_{\text{min}} \) and \( E_{\text{max}} \) which varies continuously along the E-C transition line. For the subconformal limit cycle, both the energy and the strength of the field \( \lambda \) were irrelevant parameters. Therefore, the subconformal limit cycle is identical to the limit cycle observed for the critical model (Eq. 11).

Table III compares the eigenenergies obtained by diagonalizing tri-diagonal matrices of varying sizes with those obtained by the Newton method based on the existence of the six-cycle. This table clearly shows the superiority of the decimation method compared to the exact diagonalization even for tri-diagonal matrices. As stated earlier, a very precise value of \( E_{\text{min}} \) is essential to confirm the C phase at subconformal points. Near the conformally invariant point we observe that \( E_{\text{min}} \) behaves like \( J_x^2 c(1 - \lambda)^2 \) with \( c \approx 0.3708641926 \) and \( z \approx 1.38897 \). This gives a rather good estimate for \( E_{\text{min}} \) through the whole fat C phase. Close to the limit \( \lambda = 0 \), \( E_{\text{min}} \) behaves like \( J_x + E_{\text{min}}^c \lambda/2 + ... \), where \( E_{\text{min}}^c \) is the band edge of the critical model. This fact together with Eq. (11) implies a continuous connection between the band edge of the critical model and \( E_{\text{min}} \) of the Ising model.

In order to confirm the universality of the observed limit cycles, the decimation procedure was repeated for the multiharmonic QP potentials

\[
h(i) = \frac{\lambda}{\sqrt{1 + \alpha^2}} (\cos(2\pi i \sigma + \phi)) + \cos(6\pi i \sigma + \phi))
\]

The observed limit cycles for this generic potential were identical to those observed for a single harmonic field in all the universality classes discussed above. Therefore, we conclude that the fat C phase persists for generic potentials.

Numerics to demonstrate the limit cycle in the decimation functions for multi-harmonic field is complicated by the fact that the phase boundaries have to be known to a very high precision. We were again assisted by the existence of limit cycles which provided a very efficient Newton method to locate the phase boundaries. In Table IV, we display the flow of the decimation functions to an asymptotic 6-cycle for the Ising model at the onset of the C-L transition where the model is conformally invariant.

The results of this table are used in determining the C-L phase boundary as shown in Table V.
The region of the phase diagram where the vector decimation cannot be applied due to noninvertibility of the decimation matrices, will be studied in the later part of the next section. Based on those studies, we conjecture that these regimes do not introduce any new limit cycles.

IV. SYMMETRIC AND ASYMMETRIC WAVE FUNCTIONS

The existence of a $p$-cycle for the decimation functions often implies that the self-similarity in the wave function is described by the equation, \[ \Psi(i) \approx \Psi([\sigma^p i + 1/2]) \] (16)

([ ] denotes the integer part). This equality with $p = 6$ (or $p = 3$ if the absolute values are considered) was originally found for the Harper model at its critical point. Our studies generalize the validity of this result for the whole class of TBM’s discussed here. In case where both the decimation functions and the wave function are bounded, a limit cycle for the decimations functions necessitates the above-form self-similarity for the wave function and vice versa. \[ \text{[22]} \]

For a given eigenstate with known eigenvalue $E$, the TBM provides an iterative scheme to obtain the wave function at all sites in terms of $\Psi(0)$ and $\Psi(1)$. We choose $\psi_0 = 1$ and adjust the phase factor $\phi$ so that the maximum of the absolute value of the wave function $\psi$ is at $i = 0$. This method always gives a bounded solution for the wave function. The unknowns $\eta_0$, $\psi_1$, and $\eta_1$ are then determined requiring that the above equation becomes exact as $i$ tends to infinity. Eq. (12) implies \[ \Psi(F_{n+p}) = C_{n,p} \Psi(F_n) + D_{n,p} \Psi(0) \] (17)

where \[ C_{n,p} = c_{n+p-1}(0)c_{n+p-2}(0)\ldots c_n(0) \] (18)

\[ D_{n,p} = \sum_{i=0}^{p-1} C_{n+i+1,p-i-1}d_{n+i}(0) \quad (C_{n,p,0} = \text{unity}) \] (19)

If there is an asymptotic $p$-cycle for the decimation functions, $C_{n,p}$ and $D_{n,p}$ approach a fixed point in the limit $n \to \infty$. Assuming that $\Psi(F_{n+p}) = \Psi(F_n) = \zeta \Psi(0)$ we can write Eq. (17) as an eigenvalue equation and solve for the eigenvalue $\zeta$ and the eigenvector $\Psi(0)$ (i.e. $\eta_0$). An estimate for $\Psi(1)$ is then obtained by applying the recursion relation (12) backwards $n + p - 2$ times. This section describes wave functions obtained in this way.

Figs. 2 and 3 show the wave functions corresponding to different parts of the phase diagram. It is interesting to note that for Harper, the wave function is symmetric about $i = 0$ wherease for the Ising case at the conformally invariant point, it is completely asymmetrical.

The symmetry in Harper is obvious from the TBM. The asymmetry of the $E = 0$ eigenstate in the Ising case is a consequence of the fact that in this limit the wave function is given by the simple recursion $J_x \psi_{i-1} + 2h_i \psi_i = 0$. Because $h_0 = 0$ with $\phi = 1/4$, we obtain $\psi_i = 0$ for all $i < 0$. A more physical picture of the wave function could be obtained by choosing another phase factor for which $h_i$ would be nonvanishing for all finite $i$. However, even in this case there would always exist an $i$ so that $h_i \approx 0$ and the wave function would essentially vanish on a finite left-hand-side neighborhood of that lattice site. The asymmetry can be understood also from the semiclassical analysis of the TBM. \[ \text{[22]} \]

In the Ising case, the eigenvalue equation in the semiclassical limit is (from Eq. (9))

\[ \exp(ip)\psi(x) + \lambda \cos(x)\psi(x) = E \eta(x) \] (20)

\[ \exp(ip)\eta(x) + \lambda \cos(x)\eta(x) = E \psi(x) \] (21)

For $E = 0$, if $\cos(x_0) = 0$ for some $x_0$, then $\exp(ip)\psi(x_0) = \exp(ip)\eta(x_0) = 0$. Since $p$ is the generator of space translation, the above equation implies that $\psi(x)$ and $\eta(x)$ respectively vanish to the left and right of $x_0$.

The figures suggest that the phase diagram for arbitrary $g$ can be classified as either symmetric or asymmetric. Along the conformal line in the phase diagram, the wave function is always asymmetric about the center. However, in the subconformal region of the C phase the system slowly recovers its symmetry asymptotically. For the critical model, the wave function fully restores its symmetry and in fact qualitatively resembles the Harper case. For the critical model, the phase $\phi = (3 - 2\sigma)/4$ setting the main peak at $i = 0$ is such that the potential has the symmetry $h_{-i} = -h_{i+1}$. This implies the reflection symmetry for the wave function. Furthermore, the E-C transition is also characterized by an asymptotically symmetric wave function. Fig. 4 shows the Hull function, obtained by plotting the wave function $|\psi_i|$ as a function of $<i\sigma>$, in the four universality classes.

In the region where the vector decimation could not be implemented, we have to use other methods. Along the conformal line for $E = 0$ state, the coupled equations (4-5) decouple. It turns out that either the forward or backward iteration of Eq. (4) is extremely stable beginning with an arbitrary initial condition. In other words, we can give $\psi_{\pm N}$ an arbitrary initial value and iterate in the stable direction to obtain $\psi_i$ for $i = \ldots, -M, -M+1, \ldots, M$. These $\psi$-values are very accurate if $N$ is much larger than $M$. In the subconformal region of the anisotropic model, the wave function is obtained by the diagonalization method and thus it is there not so accurate as in other parts of the phase diagram.

In the next section, we provide a quantitative characterization of the universality classes.
V. INFINITE SET OF UNIVERSAL SCALING RATIOS

Ostlund and Pandit [14] described the universality of the Harper critical point in terms of the scaling ratio

\[ \zeta = \lim_{n \to \infty} \frac{\left| \psi(F_{3n})/\psi(0) \right|}{|}, \]

(22)

This scaling ratio along with Eq. (16) clearly describes the wave function in the C phase by implying that it is neither extended nor localized. Eq. (22) describes the decay of wave functions at asymptotic Fibonacci sites with respect to the central peak. The existence of a limit cycle with period 3 in the decimation equation (considering the absolute values) gives three different values of \( \zeta \). Two of them are listed in Table VI. The interesting question is what happens to the similar scaling ratios at sites other than the Fibonacci’s. Does the wave function repeat itself in the same way as it does for the Fibonacci sites? Our detailed numerical studies suggest that it is true for all sites that a critical wave function repeats itself: i.e. for every given site, there exists a whole sequence of sites \( Q_k \) where the wave function approaches the same amplitude which is a universal fraction of the main peak. This sequence is given by the recursion relation

\[ Q_{k+1} = 4Q_k + Q_{k-1}. \]

(23)

The recursion is such that \( Q_k/Q_{k+1} \rightarrow \sigma^3 \) as \( k \rightarrow \infty \).

This implies that the scale invariance of the wave function is described in terms of an infinite set of scaling ratios \( \zeta \),

\[ \zeta(Q_0, Q_1) = \lim_{k \to \infty} \frac{|\psi(Q_k)/\psi(0)|}{}, \]

(24)

where \( Q_k \) is obtained by the above recursion relation. The set obtained by varying the two integers \( Q_0, Q_1 \) is complete and hence specifies the location and height of all the peaks of the wave functions. Stated differently, this result implies that the wave function at every \( Q_k \) site repeats itself at sites given by the above recursion relation. It is easy to see that this generalization includes the previous results [14] related to the Fibonacci sites as a special case. (See Table VI).

We next investigate the sequences that result in the dominant peaks in the wave function. Our detailed studies show that out of infinity of the scaling ratios, the \( \zeta \)'s for the dominant peaks are obtained from those values of \( Q_0 \) and \( Q_1 \) that are of the order of 4.

Furthermore, it turns out that the dominant peaks can be labeled in two ways. This also helps in regrouping the infinite set of \( \zeta \)'s. First, with every sequence determined by \( Q_0 \) and \( Q_1 \), we associate a set of n “harmonic” sequences (\( nQ_0, nQ_1, \ldots \)). Each corresponding to its own unique \( \zeta \). We notice that the dominant \( \zeta \)'s correspond to lower harmonics as shown in Table VII. It is interesting to note that the \( (0,2,8,34,\ldots) \) sequence associated with the Fibonacci numbers can be considered as the second harmonic of the sequence \( (0,1,4,17,72,\ldots) \) associated with the rational approximants of \( \sigma^2 \). It should be noted that unlike the other two sequences \( (1,3,13,55,\ldots) \) and \( (5,21,89,\ldots) \), the sequence consisting of even integers \( (0,2,8,34,\ldots) \) does not belong to the sequences associated with \( \sigma^3 \). However, the fact that it can be regarded as a higher harmonic of one of the \( \sigma^3 \) sequence suggests that the important peaks are determined by \( \sigma^3 \) periodicity and its harmonics, instead of \( \sigma \) periodicity.

Alternatively, the location of the dominant peaks in the wave functions can also be related to the number theoretical properties of \( \sigma^3 \) using the Farey tree [23]. Denoting the left and right branches of the Farey tree by 0 and 1, the rational approximants of \( \sigma^3 \) can be represented by the path 001111000011111... which we will denote as 0314014... . We conjecture that the dominant peaks in the wave functions occur on the sites which correspond to the denominators of the rational numbers on this main path and on the side paths that differ minimally from the main path.

VI. DECIMATION SCHEME BASED ON ARBITRARY FAREY PATHS

The infinite set of scaling ratios can be obtained by generalizing the Fibonacci decimation scheme. As described in the previous section, every site in the spin chain can be labelled by a symbol sequence corresponding to a definite path in the Farey tree. We develop a decimation scheme based on this. The basic idea underlying the decimation can be understood by considering three Farey levels denoted as \( f \) (father), \( m \) (mother), \( d \) (daughter). Our decimation equations are,

\[ \Psi(i + Q_d) = c_n(i)\Psi(i + Q_m) + d_n(i)\Psi(i) \]

(25)

\[ \Psi(i + Q_d) = a_n(i)\Psi(i + Q_f) + b_n(i)\Psi(i) \]

(26)

where \( Q_d = Q_m + Q_f \). In order to obtain the recursion relations for the decimation functions \( c_n, d_n, a_n, b_n \), we consider the next level denoted as \( g \) (granddaughter). Now, \( Q_g = Q_f + Q_d \) or \( Q_g = Q_m + Q_d \) depending upon whether the \( Q_g \) is obtained by taking a step in the same direction or the different direction from the one from \( Q_m \) to \( Q_d \). The above two routes to \( g \) level can be denoted as \( (01) \to (01) \) and \( (01) \to (10) \).

The recursion relations for \((01) \to (01)\) are

\[ a_{n+1}(i) = c_n(i + Q_f)\alpha_n(i) + d_n(i + Q_f) \]

(27)

\[ b_{n+1}(i) = c_n(i + Q_f)\beta_n(i) \]

(28)

\[ c_{n+1}(i) = c_n(i + Q_f) + d_n(i + Q_f)\alpha_n^{-1}(i) \]

(29)

\[ d_{n+1}(i) = -d_n(i + Q_f)\beta_n(i)\alpha_n^{-1}(i) \]

(30)

and for \((01) \to (10)\)
\begin{align*}
a_{n+1}(i) &= a_n(i + Q_n)c_n(i) + b_n(i + Q_m) \\
b_{n+1}(i) &= a_n(i + Q_m)d_n(i) \\
c_{n+1}(i) &= a_n(i + Q_m) + b_n(i + Q_m)c_n^{-1}(i) \\
d_{n+1}(i) &= -b_n(i + Q_m)d_n(i)c_n^{-1}(i)
\end{align*}

It should be noted that the Fibonacci decimation corresponds to the symbol sequence 101010..., in which case the recursion relations can be written in the form in which \(a_n\) and \(b_n\) become redundant.

In order to explain the infinite set of \(\zeta\)'s we first need the decimation functions along the "main" Farey route to \(\sigma^i\) which defines four universal scaling ratios (there is an 8-cycle in the decimation functions). Other universal \(\zeta\)'s are then obtained by determining the decimation functions along finite side paths from the main route. It should be noted that the most dominant peak in the wave functions (corresponding to largest \(\zeta\)) occurs at 1, 4, 17, 72, ... (see Figs. 2 and 3) and can be obtained only by \(\sigma^8\) decimation. In the previous RG scheme the scaling ratio \(\zeta\) corresponding to this sequence was not predicted.

VII. RENORMALIZATION GROUP EQUATION

The universal functions and the scaling ratios can in principle be obtained from RG equations. We have already derived recursion relations for the decimation functions or matrices which can be used as the starting point for the definition of a suitable renormalization operator. However, in order to get the final equations in a solvable form it is essential to get rid of the discrete lattice index \(i\). Because the QP potential has been constructed using a period-1 function, the decimation functions/matrices \(a_n(i), b_n(i), c_n(i), d_n(i)\) can be thought of being functions of the fractional part of \(i\sigma\), denoted by \(<i\sigma>\), only. This gives us a continuous variable. But we also need to consider the scaling. Assuming a \(p\)-cycle for the decimation functions at \(i = 0\), we observe that if \(a_n, b_n, c_n, d_n\) are defined as functions of the renormalized variable \(x = (-\sigma)^{-n} < i\sigma>\), any decimation function of the level \(n\) maps roughly onto the corresponding function of the level \(n+p\) for \(x \in [0, 1]\) (see Figs. 5 and 6). This indicates that \(x\) provides us with a proper continuous variable.

The next step is to rewrite the recursion equations for the decimation matrices/functions in a form which uses the above renormalized variable \(x\). For simplicity, we restrict ourselves to the Fibonacci decimation. By using the relation \(F_n\sigma = F_{n-1} - (\sigma)^n\), the nonlocal recursion relations (13-14) can be transformed into local RG equations

\begin{align*}
c_{n+1}(x) &= c_n(-\sigma x - 1)c_{n-1}(\sigma^2 x + \sigma) \\
&\quad -d_{n-1}^{-1}(-\sigma x)d_{n+1}(x)
\end{align*}

\(d_{n+1}(x) = -d_n(-\sigma x)|d_n(-\sigma x - 1) + c_n(-\sigma x - 1)d_{n-1}(\sigma^2 x + \sigma)|c_n^{-1}(-\sigma x)\)

This defines our renormalization operator which can be now applied to determining the universal cycles. We tried that in the case of the critical Harper model \((E = 0)\). Our detailed numerical studies showed that \(c_n(x), d_n(x)\) approached a six-cycle \((c_n^2, d_n^2) \rightarrow (c_n^3, d_n^3) \rightarrow (c_n^2, -d_n^2) \rightarrow (c_n^3, -d_n^3) \rightarrow (c_n^2, d_n^3) \rightarrow (c_n^3, d_n^2)\) where all the functions of the six-cycle appeared bounded and smooth for the full measure of \(x\) values. This suggested the functions \(c_n^*, d_n^*\) could be expanded in a power series and one could use the Newton method to solve for the coefficients. However, simple circular complex domains centered on the real axis did not give convergent results. We believe that the equations can be solved using more elaborate domains. It is probably necessary to let the centers of at least some of the complex disks to be located off the real axis so that the radius of convergence can be increased. This appears true especially for the expansion of the inverse of \(c_n\) for which this radius is only unity if the center of the corresponding domain lies at the origin of the complex plain. Therefore, the determination of scaling properties using eigenvalues of linearized RG equations about the limit cycle remains open.

As to the RG for an arbitrary Farey path, the main difficulty arises from the number-theoretical problem of generalizing the relation giving the fractional part of \(F_n\sigma\) to products of the type \(Q_f\sigma\) and \(Q_m\sigma\) where \(Q_f\) and \(Q_m\) are arbitrary integers appearing in the Farey tree. All we can prove at the moment is that with the recursion (23) the fractional part of \(Q_k\sigma\) (with arbitrary \(Q_0, Q_1\)) approaches either 0 or 1/2 and the rate of convergence is given by \((-\sigma)^3\). However, there are still a lot of details which need to be worked out in the general case.

VIII. SUMMARY OF THE UNIVERSALITY CLASSES

Based on all the infinite set of scaling ratios, we conjecture that the C phase of the model is governed by four different fixed points of the renormalization group defining four universality classes, which we refer as spectral, conformal, subconformal and Harper. We show that the anisotropy is a relevant parameter as an infinitesimal anisotropy takes the system to one of the three possible new universality classes, away from the Harper universality describing the isotropic limit of the model. Except for the conformal universality, the other three universality classes exhibit an asymptotic reflection symmetry of the wave function.

We checked in all the above four cases that the universality is rigid with respect to replacing the single harmonic \(h_i\) by an odd two-harmonic function. Both the in-
finite set of \( \zeta \)'s as well as the decimation functions were found to be universal.

We also computed the exponent \( \beta \) defined by Ostlund et al. \[14\] which measures the degree of localization around the site where the wave function is peaked.

\[
- \beta = \frac{1}{p \ln(\sigma^{-1})} \lim_{n \to \infty} \left[ \ln \left( \frac{L_n}{L_{n+p}} \right) \right]
\]

(37)

where

\[
L_n = \frac{1}{Q_n} \sum_{i=-Q_n}^{Q_n} \psi_i^2
\]

(38)

The \( p \) in the above equation is the period of the limit cycle. Unlike the scaling ratio \( \zeta \), no close form expression for \( \beta \) could be found in terms of the decimation functions. \( \beta \) was computed directly from the wave function.

The exponent \( \beta = 0 \) in the E phase, \( -1 \) in the L phase and varies between these two values in the C phase. Table VIII shows \( \beta \) in four different universality classes. As expected the absolute value of \( \beta \) is greater at the conformal fixed point compared to the subconformal case. Comparing \( \beta \) at the onset of localization in the Harper and Ising-like cases leads us to conclude that the LRO slightly suppresses the degree of localization. However, the degree of localization remains constant inside the C phase. Since the subconformal C phase existing in finite window in \( \lambda \) is described by a single fixed point this implies that \( \lambda \) is an irrelevant parameter in this phase.

All the other universality classes except the spectral one can be described by scalar decimation functions. This made the study of the spectral universality a lot more difficult. Additional complications came from the facts that \( E_{\min} \) was non-zero and the phase factor \( \phi \) was continuously changing. This explains why the wave function and the scaling ratios \( \zeta \) are known with less precision in that case than for the other three universality classes.

It should be noted that the universality classes obtained here are for the fractal quasi-particle wavefunctions and do not tell us anything about the scaling properties of the fractal quasi-particle energy spectrum. Very recently, Chaves \[24\] showed that the energy level statistics follow inverse power law distribution with the exponent \( 3/2 \) throughout the fat C phase. This implies that the fractal dimension of the C phase is a constant throughout the phase and is equal to \( .5 \). \[25\]. However, the multifractal analysis of the spectrum \[ \beta \] showed that the \( f(\alpha) \) curve describes the scaling properties of the energy spectrum is different in the four universality classes. Therefore, we conjecture that even the multifractals describing the quasi-particle energy fall within the same four classes. We believe that the exact RG treatment describing the spectrum as a whole as proposed by Wilkinson \[16\] could be generalized to confirm this conjecture.

**IX. CONCLUSIONS**

The work described here confirms that the anisotropic spin model describing the perturbed Harper with broken \( U(1) \) symmetry fattens the critical point of the Harper model. In analogy with the mode-locked phase, this fat C phase existing in a finite window can be described as \( C - locked \) phase whose scaling properties fall within four different universality classes. The conformal and the spectral universality classes define the boundaries of the phase diagram which sandwich the subconformal class. In the isotropic limit, the three universality classes degenerate to the Harper universality. It is rather interesting that the subconformal and conformal parts of the phase diagram belong to different universality classes. This provides a novel example of spectral and magnetic interplay showing how a magnetic phase transition (which occurs also in a periodic model) affects the critical phase characteristics of aperiodic systems. The spectral transition which exists only in an aperiodic system (and does not affect the magnetic properties such as the long-range correlation function) defines its own universality class distinct from the conformal and the subconformal cases.

The E phase of the quasiperiodic models is the KAM phase where the wave functions are extended and hence the quasi-periodicity has minimal effects. As shown in the Fig. 7, in contrast with the C phase, the Hull function is smooth here. It is rather surprising that the universality in this phase is almost independent of the anisotropy \( g \). For the periodic system, the anisotropy is a relevant parameter as it provides a preferred axis for spin alignment. The fact that the universal functions of the E phase are insensitive to the anisotropy adds a new dimension to the KAM theory: the smooth Hull function of the integrable limit \( \lambda = 0 \) survives not only the \( \lambda \) perturbation in the E phase, but the anisotropy leaves the universality class unaltered. This is different from the C phase where an infinitesimal anisotropy alters the universal characteristics. Therefore, we conclude that the anisotropy responsible for the long range order does not affect the KAM phase and the quasi-periodicity responsible for the spectral transition does not affect the long range order.

It is interesting to compare our results with those obtained with a TBM related to the stability of the ground state in the classical Frenkel-Kontorova model at the onset of pinning-depinning transition. \[19\] This pinning-depinning transition corresponds to the breakup of KAM tori in area preserving maps. \[28\] In this case the analog of the fractal wave function in the C phase is the derivative of the Hull function describing a QP critical ground state (Fig. 8). The reason why here the wave function corresponds to the derivative of the Hull function rather than the Hull function itself is the fact that the TBM follows from a variation of the ground state. With the
initial conditions on the dominant symmetry line (this is analogous to our choice of the phase $\phi$), the Fibonacci decimation leads to an asymptotic universal fixed point.\[18\] It is interesting to speculate the existence of a C-locked critical Hull function in some generalized classical Frenkel-Kontorova models or the corresponding area preserving maps.\[24\]

Very recently, Faddeev et al.\[27\] have shown that the Harper equation describing isotropic limit of the XY model can be solved using generalized Bethe Ansatz. It will be interesting to know if the coupled equations (4-5) describing the anisotropic model are amendable to similar treatment. Since in the case of a periodic system ($\sigma = 0$), the anisotropy introduces an integrable perturbation on the isotropic model, it is possible that even with quasiperiodicity that may be the case. We hope that our studies will motivate rigorous analysis of the quantum spin models discussed here.

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[20] We find the same 3-cycle as for $E_{\text{min}}$ in the Harper model also just below $E_{\text{max}}$ at $E = 1.943387097623$ with $\lambda = .5$. The significance of this cycle is not clear to us.
[21] For reasons which we believe to be related to the asymptotic validity of the equation for determining $\phi$, this method of determining the phase factor was not always succesful.
[22] Our numerical studies suggest that it is possible that a) the wave function has the self-similarity but the asymptotic decimation functions are ill-defined, b) the decimation functions show a $p$-cycle but the wave function is unbounded (e.g. in subconformal region of the Ising model with $\phi = 1/4$).
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FIG. 1. Phase diagram for the quasi-particle excitations: With $J_x = 1 - g$ and $J_y = 1 + g$, the figure shows the E, C and L phases in $\lambda - g$ plane. In the E phase, wave functions are extended and hence resemble those of a periodic system. This phase is also called the KAM phase as the Hull functions are self-similar (see Figs. 2 and 3) exhibiting atmost algebraic localization while in the L phase, they are exponentially localized. The onsets of E-C and C-L transitions are respectively the spectral line (thin line) and the conformal line (thick line). The two lines meet in the isotropic (Harper) limit. The $g = -1$ describes the Ising limit with no E or KAM type phase.
FIG. 2. (a-e) show the absolute values of the wave functions, along the line in the phase diagram where the model is conformally invariant \( (E_{\text{min}} = 0) \). For fixed \( J_x = 1 > J_y \geq 0 \), the conformal line is obtained by setting \( \lambda = 1 \) and varying \( J_y \) (see ref. 5): (a) \( J_y = 0 \), (b) \( J_y = .99 \), (c) \( J_y = .999 \), (d) \( J_y = .9999 \), and (e) \( J_x = J_y = 1 \). The figures (a) and (e) correspond respectively to the Ising and isotropic limits while (b), (c) and (d) are Ising like. We see that the wave functions are highly asymmetric about the peaks in the Ising and Ising like cases. However, in the isotropic limit, the wave functions recover the symmetry.

FIG. 3. (a) shows the wave function below the conformally invariant point for the Ising model \( (J_y = 0) \) with \( \lambda = .95 \). (b) is the wave function of the critical model with no diagonal disorder. (c) shows the wave function at the onset of the spectral transition with \( J_x = 1 \), \( J_y = 1.5 \), and \( \lambda = 1 \). In all these cases, the wave function is always asymptotically symmetrical (about \( i = 0 \)). In the limit \( \lambda \to 0 \), but finite, the wave function is fully symmetric (b).

FIG. 4. (a-d) respectively show the Hull function \( |\psi(x)| \) vs \( x =< i \sigma > \) in the four universality classes, namely the Harper \( (E = 0) \), conformal \( (J_y = 0, J_x = 1, \lambda = 1, E = 0) \), subconformal \( (J_x = 1, J_y = 1.5, \lambda = 1, E_{\text{min}} = -1.420111920492815) \), and spectral \( (J_x = 1, J_y = 1.5, \lambda = 1, E_{\text{min}} = 0.17429863452742) \).

FIG. 5. (a-d) show the decimation function \( c_n(x) \) \((n = 6 - 9)\) for the critical Harper model with \( E = 0, \phi = 1/4 \). As a consequence of the 3-cycle for the absolute values of the decimation function, \( c_0(x) = \pm c_0(x) \).

FIG. 6. (a-d) show the decimation function \( c_n(x) \) \((n = 6 - 9)\) for the Ising model with \( E = 0, \phi = 1/4 \) at the conformally invariant point. As a consequence of the 3-cycle for the absolute values of the decimation function, \( c_0(x) = \pm c_0(x) \).

FIG. 7. The wave function (a) and the smooth Hull function (b) in the E phase at \( \lambda = .5 \), \( J_x = 1 \), \( J_y = 1.5 \), and \( E = E_{\text{max}} = 2.620638044392 \). Unlike the C phase, the Hull function is smooth.

FIG. 8. The derivative of the Hull function at the onset of breaking of analyticity in the Frenkel-Kontorova model.

TABLE I. The limiting subcritical \( (E_{\text{phase}}) \) decimation functions for the general anisotropic model \( (g \neq 0) \) and the Harper limit \( (g = 0) \). As the functions do not depend on the phase \( \phi \), there cannot be any dependence on the lattice site \( i \) either. For the general anisotropic case, the cycles are seen only after diagonalizing the decimation matrices. For the Harper limit \( E_{\text{min}} = -E_{\text{max}} \) while for the anisotropic model \( E_{\text{min}} \) is positive. Note that \( E_{\text{min}} \) of the anisotropic case goes into \( E = 0 \) of the Harper model as \( g \) goes to zero.

| \( E \)       | \( c^* \) | \( d^* \) | cycle |
|--------------|----------|----------|-------|
| \( g \neq 0 \):         |          |          |       |
| \( E_{\text{min}} \)   | 1, 0, \infty | 0, 1, \infty | 3     |
| \( E_{\text{max}} \)   | \( \sigma^{-1} \) | -\( \sigma \) | 1     |
| \( g = 0 \):          |          |          |       |
| 0   | 1, 0, \(-\infty\), \(-1\), \(-\infty\) | 0, \(-1\), \(-\infty\), 0, 1, \infty | 6     |
| \( E_{\text{min}} \)   | \( \sigma^{-1}, -\sigma^{-1}, -\sigma^{-1} \) | \( \sigma, -\sigma, \sigma \) | 3     |
| \( E_{\text{max}} \)   | \( \sigma^{-1} \) | -\( \sigma \) | 1     |

TABLE II. Estimates for \( E_{\text{max}} \) of the Harper model with \( \lambda = .5 \) obtained from the condition \( c_n(0) = \sigma^{-1} \). The \( n \) here refers to the \( n^{th} \) order Fibonacci number with \( F_{25} = 46368 \).

| \( n \) | \( E \)         |
|--------|----------------|
| 10     | 2.144122230954376 |
| 13     | 2.144103497103577 |
| 16     | 2.144103742043093 |
| 19     | 2.144103738794523 |
| 22     | 2.144103738837256 |
| 25     | 2.144103738836694 |
TABLE III. $E_{\text{min}}$ for the Ising model obtained by the diagonalization (size $N$ finite) and the decimation scheme ($N = \infty$).

| $\lambda$ | $N$     | $E_{\text{min}}$              |
|---------|---------|------------------------------|
| 0.95   | 1597    | 1.51567E-2                   |
|        | 2584    | 1.51484E-2                   |
|        | 4181    | 1.51481E-2                   |
| $\infty$ | 1.514806760988E-2 $\pm$ 1.E-14 |
| 0.995  | 4181    | 6.1929E-4                    |
|        | 6765    | 6.1862E-4                    |
|        | 10946   | 6.1853E-4                    |
| $\infty$ | 6.18420285E-4 $\pm$ 1.E-12   |

TABLE IV. The decimation functions at $i = 0$ for the Ising model with $\lambda = 1$, $E = 0$, $\phi = 1/4$. The table clearly shows the asymptotic six-cycle.

| $n$ | $c_n(0)$ | $d_n(0)$ |
|-----|----------|----------|
| 3   | -2.245709795011781 | -0.6002603574022150 |
| 4   | 8.109008292790231   | -1.272168729380112  |
| 5   | 1.010919961015582    | -0.3666408658531275  |
| 6   | 3.86722946686237      | -0.7949593051332248  |
| 7   | -7.553298439810634    | 1.18294059162462     |
| 8   | 1.100178678647771     | 0.376420348828012    |
| 9   | -3.710289833700136     | -0.7701476581532440  |
| 10  | 7.758396265078099      | -1.19467615297624    |
| 11  | 1.085391362405107      | -0.37380576364522718 |
| 12  | 3.746636404387838      | -0.7734760239491446  |
| 13  | -7.718184336265150     | 1.192015326586235    |
| 14  | 1.089002996132289      | 0.3742871557654810   |
| 15  | -3.7389868728922117    | -0.7728088313940935  |
| 16  | 7.727825304953185      | -1.192558926144134   |
| 17  | 1.088713364711348      | -0.374175202327470   |
| 18  | -3.740771704903936     | -0.7729562819664260  |
| 19  | 7.725577268994126      | 1.19245496911469     |
| 20  | 1.088369395725458      | 0.3742014064535257   |
| 21  | -3.740355675780264     | -0.7729194653703071  |
| 22  | 7.726105073162105      | -1.19248586259755    |
| 23  | 1.088319151696401      | -0.3741951424798995  |
| 24  | 3.740453282920909      | -0.7729293725799885  |
| 25  | -7.726061715596284     | 1.19249956626364     |

TABLE V. Estimates for the conformally invariant point of the Ising model where the third harmonic with $\alpha = 0.2$ has been added to $h(i)$. The estimates were obtained by imposing the condition $c_n(0) = 1.08832$ for $\phi = 1/4$ (see the first column corresponding to $n = 23$ in Table IV).

| $n$ | $\lambda$ |
|-----|------------|
| 11  | 1.367474785 |
| 14  | 1.366202286 |
| 17  | 1.366282249 |
| 20  | 1.366277923 |
| 23  | 1.366278173 |
| 26  | 1.366278174 |

TABLE VI. $\zeta$'s corresponding to some of the dominant peaks in the Harper, Ising (both conformal and subconformal), and the spectral universality classes.

| sequence Harper conformal subconformal spectral |
|-----------------|-----------------|-----------------|-----------------|
| 0,1,4,17,2,...  | .516            | .744            | .676            | .72             |
| 1,1.5,21,89,... | .315            | .622            | .590            | .54             |
| 1,2,9,38,...   | .256            | .297            | .337            | .58             |
| 1,3,13,55,...  | .143            | .496            | .231            | .37             |

TABLE VII. $\zeta$'s corresponding to a sequence and its harmonics for the Harper model.

| harmonic sequence | $\zeta$ |
|------------------|---------|
| 1                | 3, 13, 55, 233,... | 0.143 |
| 2                | 6, 26, 110, 466,... | 0.1456 |
| 3                | 9, 39, 165, 699,... | 0.1485 |
| 4                | 12, 52, 220, 932,... | 0.0183 |
| 5                | 15, 65, 275, 1165,... | 0.025 |
TABLE VIII. The exponent $\beta$ for the four universality classes.

| Universality Class | $\beta$     |
|--------------------|-------------|
| Harper             | -.639 ± .005|
| conformal          | -.602 ± .001|
| subconformal       | -.423 ± .001|
| spectral           | -.175 ± .005|