Double EPW-sextics with actions of $A_7$ and irrational GM threefolds

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Abstract

We construct two examples of projective hyper-Kähler fourfolds of $K3^{[2]}$-type with an action of the alternating group $A_7$, making them some of the most symmetric hyper-Kähler fourfolds. They are realized as so-called double EPW sextics and this allows us to construct an explicit family of irrational Gushel-Mukai threefolds.

Introduction

This article is primarily inspired by [DM21], where many ideas have been taken from and adjusted to our case. However, the search for symmetric hyper-Kähler manifolds has been of interest in recent years. One can find attempts at classification or construction of examples in [BH19, BS21] for K3 surface case or [HM19, Son21, Waw22, D-BvGKKW] in the case of hyper-Kähler fourfolds of $K3^{[2]}$-type. So our results should also be seen as continuation of these efforts.

We construct two explicit examples of hyper-Kähler fourfolds of $K3^{[2]}$-type with symplectic actions of the alternating group $A_7$. Which, according to the classification in [HM19], is one of the maximal finite groups one can obtain as a subgroup of symplectic automorphisms of a manifold of this type.

We describe the manifolds as so called double EPW-sextics. The rough idea is simple: given a Lagrangian subspace of $\bigwedge^3 \mathbb{C}^6$, there is a canonical way to associate an EPW-sextic to it and there exists a canonical double cover of the sextic under suitable generality assumptions. If the Lagrangian is preserved by the group action induced by an action of the six-dimensional space, then the sextic and its double cover inherits it. The fact that the cover is a hyper-Kähler manifold, for a general Lagrangian follows from [O’G06]. As an application, we exhibit two families of irrational GM-threefolds.

Our two hyper-Kähler fourfolds of $K3^{[2]}$-type are in a sense dual to each other and non-isomorphic. For each of them, the group of symplectic automorphisms $A_7$ fixes an ample divisor. The group of all polarized morphisms with respect to this divisor is $\mathbb{Z}/2\mathbb{Z} \times A_7$.

The structure of the article is as follows: in the first section we recall known facts about hyper-Kähler manifolds with associated lattices, EPW-sextics and
GM varieties, in the second section we outline the construction of the hyper-Kähler fourfolds and in the third we prove that for any of the two sextics we construct, each member of a family of GM threefolds associated with it is irrational. The appendix includes the codes that we run with GAP [GAP21] and Macaulay2 [M2].

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1 Preliminaries

This section is merely a collection of known facts about hyper-Kähler manifolds and associated lattices, EPW-sextics with their automorphisms and double covers, and GM varieties.

1.1 Hyper-Kähler manifolds and their lattices

A hyper-Kähler manifold is a compact, simply connected complex Kähler manifold such that $H^{2,0}(X) \cong H^0(X, \Lambda^2 \Omega_X) \cong \mathbb{C} \sigma_X$ where $\sigma_X$ is a nowhere degenerate holomorphic 2-form.

Let us briefly recall some theory on lattices of hyper-Kähler manifolds (see [Deb18, Huy99] for more details). By lattice we will mean a free $\mathbb{Z}$-module $L$ together with a bilinear form $L \times L \rightarrow \mathbb{Z}$ $(v, w) \mapsto v \cdot w$.

For $v \in L$, we put $v^2 = v \cdot v$. A submodule of a lattice is a sublattice with the induced bilinear form. If $M \subset L$ is a sublattice, its orthogonal complement in $L$ is $L^1 = \{ v \in L : v \cdot w = 0 \text{ for all } w \in M \}$. For a hyper-Kähler manifold $X$, the Beauville-Bogomolov form endows the second cohomology group $H^2(X, \mathbb{Z})$ with a lattice structure of index $(3, b_2(X) - 3)$ where $b_2$ is the second Betti number of the manifold. The Néron-Severi group (or lattice) of $X$ is defined as

$$NS(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X) \subset H^2(X, \mathbb{C}).$$

The transcendental lattice $T(X)$ is the orthogonal complement of $NS(X)$ in $H^2(X, \mathbb{Z})$. In the projective case, the Néron-Severi group is isomorphic with
the Picard group \( \text{Pic}(X) \). In this case, the induced bilinear form is hyperbolic (that is, of index \((1, k)\) for some \(k \in \mathbb{N}\)). Moreover, \(h^2 > 0\) for any ample class \(h \in \text{Pic}(X) = \text{NS}(X)\).

**Proposition 1.1.** For a projective hyper-Kähler manifold, the map

\[
\Psi: \text{Aut}(X) \to O(\text{Pic}(X))
\]

\[f \mapsto f^*\]

has a finite kernel.

**Proof.** [Deb18, Proposition 4.1]

From this, we have the following corollary.

**Corollary 1.2.** Let \(X\) be a projective hyper-Kähler manifold. A group of automorphisms \(G \subset \text{Aut}(X)\) is finite if and only if it fixes an ample class on \(X\).

**Proof.** If \(G\) is finite, then let \(h \in \text{Pic}(X) = \text{NS}(X)\) be any ample class. Then the class

\[
\eta = \sum_{g \in G} g^*h
\]

is invariant under \(G\). For any \(g \in G\), \(g^*h\) is necessarily Kähler because \(g\) is biholomorphic and it is therefore ample as \(g^*\) maps \(\text{Pic}(x)\) into itself. So also \(\eta\) is ample as Kähler classes form a cone.

Now assume \(G\) fixes an ample class \(h\). Let \(\Psi\) be as in Proposition 1.1, and put \(\Psi_G = \Psi|_G\), so \(\ker \Psi_G\) is finite. Now define the quotient \(\tilde{G} = G/ \ker \Psi_G\) and note that \(\tilde{G}\) acts faithfully on \(\text{NS}(X)\). Take \(N\) to be the orthogonal complement of \(h\) in \(\text{NS}(X)\). Since \(\text{NS}(X)\) is of index \((1, k)\) for some integer \(k\) and \(h^2 > 0\) as \(h\) is ample, then \(N\) is negative definite. Therefore \(N\) has finitely many isometries (as for any integer \(z\), there are only finitely many vectors \(v \in N\) such that \(v^2 = z\)). But any isometry \(f \in O(\text{NS}(X))\) which fixes \(h\) is uniquely determined by its restriction to \(N\). In conclusion \(|\tilde{G}| \leq |O(N)|\) which means that \(\tilde{G}\) is finite. This implies that also \(G\) is finite as \(|G| = |\ker \Psi_G| \cdot |\tilde{G}|\).

### 1.2 EPW-sextics, their double covers and automorphisms

EPW-sextics are singular sextic hypersurfaces in \(\mathbb{P}^5\), firstly constructed by Eisenbud, Popescu and Walter ([EPW01](#)). They come provided with a natural double cover, and O’Grady showed that the cover of a generic EPW is a hyper-Kähler which is deformation equivalent to the Hilbert square of a K3 surface ([O’G06](#)).

We briefly recall the construction.

Let us fix once and for all a 6-dimensional \(\mathbb{C}\)-vector space \(V_6\) with a volume form \(\text{vol}: \bigwedge^6 V_6 \to \mathbb{C}\). This induces a symplectic form \((\alpha, \beta) := \text{vol}(\alpha \wedge \beta)\) on
\[ \Lambda^3 V_6 \cong \mathbb{C}^{20}. \] Consider the sub vector-bundle \( F \subset \Lambda^3 V_6 \otimes \mathcal{O}_{\mathbb{P}(V_6)} \) whose fiber over \([v] \in \mathbb{P}(V_6)\) is given by:

\[
F_v = \{ \alpha \in \Lambda^3 V_6; \alpha \wedge v = 0 \}.
\]

Since the symplectic form is zero on \( F \) and \( 2 \dim(F_v) = 20 = \dim(\Lambda^3 V_6) \), the sub vector-bundle is Lagrangian for the symplectic form induced by the one on \( \Lambda^3 V_6 \). Let \( LG(\Lambda^3 V_6) \) be the Lagrangian Grassmannian parametrizing the Lagrangian subspaces, and fix an element \( A \in LG(\Lambda^3 V_6) \). There is a map

\[
\lambda_A : F \to \Lambda^3 (V_6/A) \otimes \mathcal{O}_{\mathbb{P}(V_6)}
\]

which is obtained by taking the quotient and let

\[
Y_A = V(\det \lambda_A).
\]

One can compute \( \det F \cong \mathcal{O}_{\mathbb{P}(V_6)}(-6) \), hence \( \det \lambda_A \in H^0(\mathbb{P}(V_6), \mathcal{O}_{\mathbb{P}(V_6)}(6)) \) and if that is not the zero section then \( Y_A \) is a sextic hypersurface. It is known that for a generic \( A \in LG(\Lambda^3 V_6) \) one has \( Y_A \not\cong \mathbb{P}(V_6) \), even though it is not the case for any \( A \) (e.g. \( A = F_{v_0} \) for some \( v_0 \)). An EPW-sextic is a sextic hypersurface of \( \mathbb{P}^5 \) which is projectively equivalent to \( Y_A \) for some \( A \in LG(\Lambda^3 V_6) \).

Put

\[
Y_A[k] = \{ [v] \in \mathbb{P}(V_6) | \dim(A \cap F_v) \geq k \}
\]

and note that \( Y_A[1] = Y_A \).

We say that \( A \) does not contain decomposable vectors if it has no nonzero vectors of the form \( x \wedge y \wedge z \) (i.e. \( \mathbb{P}(A) \cap \text{Gr}(3, V_6) = \emptyset \) in \( \mathbb{P}(\Lambda^3 V_6) \)). In this section we will always assume that \( A \) has no decomposable vectors.

An important fact about automorphisms of \( Y_A \) is that

\[
\text{Aut}(Y_A) = \{ g \in \text{PGL}(V_6) | (\Lambda^3 g)(A) = A \}
\]

and this is a finite group by ([DK18, Proposition B.9]).

Let us denote

\[
LG(\Lambda^3 V_6)^0 = \{ A \in LG(\Lambda^3 V_6) \mid \mathbb{P}(A) \cap \text{Gr}(3, V_6) = \emptyset, Y_A[3] = \emptyset \} =
\]

\[
= \{ A \in LG(\Lambda^3 V_6) \mid \text{Sing} Y_A = Y_A[2], \text{Sing} Y_A[2] = Y_A[3] = \emptyset \}.
\]

For \( A \) from the above subset (which is open in the Lagrangian Grassmanian), there is a canonical double cover \( \pi_A : \tilde{Y}_A \to Y_A \) branched along the surface \( Y_A[2] \), which is of great interest for us since for a generic \( A \) it is a hyper-Kähler deformation equivalent to the Hilbert square of a K3 surface ([O’G06]). \( \tilde{Y}_A \) carries a canonical polarization \( H = \pi_A^* O_{Y_A}(1) \) and the image of the morphism \( \tilde{Y}_A \to \mathbb{P}(H^0(\tilde{Y}_A, H^\vee)) \) is isomorphic to \( Y_A \).
Every automorphism of $Y_A$ induces an automorphism of $\tilde{Y}_A$ that fixes the class $H$ (proof of [DK18 Proposition B.8(b)]), conversely any automorphism of $\tilde{Y}_A$ that fixes $H$ induces an isomorphism $\mathbb{P}(H^0(\tilde{Y}_A, H)^\vee) \cong \mathbb{P}(V_6)$ hence descends to an automorphism of $Y_A$. Denote by $\text{Aut}_H(\tilde{Y}_A)$ the group of automorphisms that fix the class $H$ and by $\iota$ the covering involution of $\pi_A$. The discussion above gives a central extension

$$1 \to \langle \iota \rangle \to \text{Aut}_H(\tilde{Y}_A) \to \text{Aut}(Y_A) \to 1,$$

moreover denoting by $\text{Aut}^s_H(\tilde{Y}_A)$ the subgroup of $\text{Aut}_H(\tilde{Y}_A)$ consisting of symplectic automorphisms, one gets an extension

$$1 \to \text{Aut}^s_H(\tilde{Y}_A) \to \text{Aut}_H(\tilde{Y}_A) \to \mu_r \to 1$$

with $\mu_r$ a finite group of order $r$. Note that the image of $\iota$ in $\mu_r$ is given by $-1$. Consider the embedding $\text{Aut}(Y_A) \hookrightarrow \text{PGL}(V_6)$ and let $G$ be the inverse image of $\text{Aut}(Y_A)$ via the canonical map $\text{SL}(V_6) \to \text{PGL}(V_6)$. $G$ is an extension of $\text{Aut}(Y_A)$ by the cyclic group $\langle \gamma \rangle$ of order 6, so we have an induced representation of $G$ on $\bigwedge^3 V_6$ and this factors through a representation of $\tilde{\text{Aut}}(Y_A) := G / \langle \gamma^2 \rangle$. Since $A$ is preserved by this action, we have a morphism of central extensions

$$1 \longrightarrow \langle \gamma^3 \rangle \longrightarrow \tilde{\text{Aut}}(Y_A) \longrightarrow \text{Aut}(Y_A) \longrightarrow 1$$

and by [DM21 Lemma A.1] the vertical maps are injective.

**Proposition 1.3 (Kuznetsov).** Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors. Then the extensions (2) and (3) are trivial and $r = 2$. In particular there is an isomorphism

$$\text{Aut}_H(\tilde{Y}_A) \cong \text{Aut}(Y_A) \times \langle \iota \rangle$$

which splits (2) and the factor $\text{Aut}(Y_A)$ corresponds to the subgroup $\text{Aut}_H^s(\tilde{Y}_A)$.

**Proof.** [DM21 Proposition A.2]

Since $A$ has no decomposable vectors there is a canonical connected double covering

$$\tilde{Y}_A[2] \to Y_A[2]$$

by [DK19 Theorem 5.2(2)].

Clearly there is a morphism $\text{Aut}(Y_A) \to \text{Aut}(Y_A[2])$ and as $Y_A$ is not contained in any hyperplane, the morphism is injective. As [DM21 Proposition A.6 (Kuznetsov)] shows, the group of lifts of automorphisms of $Y_A$ to automorphisms of $Y_A[2]$ is isomorphic to $\tilde{\text{Aut}}(Y_A)$, hence there is an injection $\tilde{\text{Aut}}(Y_A) \hookrightarrow \text{Aut}(\tilde{Y}_A[2])$.
Recall that the analytic representation of a finite group $G$ acting on an Abelian variety $X$ is the composition

$$G \to \text{End}_\mathbb{Q}(X) \to \text{End}_\mathbb{C}(T_{X,0}). \quad (5)$$

We recall the useful Proposition 1.4. Suppose the surface $Y_A[2]$ is smooth. The restriction of the analytic representation of $\text{Aut}(Y_A[2])$ on $\text{Alb}(Y_A[2])$ to the subgroup $\text{Aut}(Y_A)$ is the injective middle vertical map in the diagram (4).

Proof. [DM21] Proposition A.7 \qed

1.3 GM varieties and their automorphisms

Let $V_5$ be a 5-dimensional complex vector space. A GM variety (shorthand for Gushel-Mukai) of dimension $n = 3, 4, 5$ is the smooth complete intersection of the Grassmannian $\text{Gr}(2, V_5) \subset \mathbb{P}(\bigwedge^2 V_5)$ with a linear space $\mathbb{P}^{n+4}$ and a quadric. These varieties are Fano varieties with Picard number 1, index $n - 2$ and degree 10.

There is a bijection between the set of isomorphism classes of GM varieties of dimension $n$ and isomorphism classes of triples $(V_6, V_5, A)$, where $A \subset \bigwedge^3 V_6$ is Lagrangian with no decomposable vectors and $V_5 \subset V_6$ is a hyperplane that satisfies

$$\dim(A \cap \bigwedge^3 V_5) = 5 - n.$$ 

The correspondence is outlined in [DK18] Theorem 3.10 and Proposition 3.13(c)]. If $X$ is a GM associated with $(V_6, V_5, A)$, then we have

$$\text{Aut}(X) \cong \{g \in \text{Aut}(A)|g(V_5) = V_5\} \quad (6)$$

by [DK18] Theorem 3.10 and Corollary 3.11]. So for a GM variety $X$, we can define an associated EPW sextic as $Y_A$ where $(V_6, V_5, A)$ is the class associated with $X$ by the above correspondence. The other way around, if $A$ is a Lagrangian space of $\bigwedge^3 V_6$, we define a family of GM varieties associated with $A$ as consisting of all the GM varieties $X$ such that $Y_A$ is an EPW sextic associated with $X$.

2 EPW-sextics with an action of $\mathcal{A}_7$

The general idea is to find a Lagrangian which is invariant under the action of the group $\mathcal{A}_7$, to get an invariant EPW-sextic. Our first attempt, the most na"{i}ve way to proceed, is to consider the natural representation of $\mathcal{A}_7$ on a 7-dimensional space and quotient out by the trivial subrepresentation (the one generated by the sum of basis vectors). This leads to an irreducible 6-dimensional representation and with exactly the same construction as it follows, one gets an invariant EPW-sextic. This is in fact a reduced quartic so we had to discard it and look for a non-reduced sextic.
According to [WCN85], there exists a group (going by the notation from the atlas) $3.A_7$ such that $3.A_7/\langle \omega \rangle \cong A_7$ with $\omega$ an element of order 3. This group has a unique irreducible representation $\rho : 3.A_7 \rightarrow \mathbb{C}^6$ that we call $\mathcal{V}_6$ and is generated by the elements (ATLAS)

$$\alpha = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \xi_3^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-1 + \xi_3 & 0 & 1 - \xi_3 & -\xi_3 & \xi_3 & 1 \\
2 & 0 & -1 & -1 & 0 & -1
\end{bmatrix},$$

$$\beta = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -\xi_3 & 0 & \xi_3 & 1
\end{bmatrix},$$

where $\xi_3$ is a primitive third root of unity. Note that this induces a representation on $\bigwedge^3 \mathcal{V}_6$. Moreover, since $\omega$ has order 3 the representation $\mathcal{V}_6$ induces an action of the quotient $A_7$ on $\bigwedge^3 \mathcal{V}_6$, we denote this (faithful) representation of $A_7$ by $W$.

The irreducible complex representations of $A_7$ of dimension smaller or equal than 20 have dimensions 1, 6, 10, 10, 14, 14 and 15 and can be read in Table 1.

**Lemma 2.1.** The representation $W$ decomposes as the direct sum of the only two irreducible 10-dimensional representations $R_1, R_2$ of the group $A_7$, moreover the underlying vector spaces $A_1, A_2 \subset \bigwedge^3 \mathcal{V}_6$ of those representations are Lagrangian.

**Proof.** The fact that $W$ has the mentioned decomposition is just a computation of characters (we used GAP), the subrepresentations being Lagrangian is easily checked with computer algebra. See Appendix A.1 and B. \hfill $\Box$

As a consequence of Lemma 1 setting $A = A_1, A_2$ leads to an EPW-sextic $Y_A \subset \mathbb{P}^5$ which is invariant under the action of $A_7$. $Y_{A_1}$ and $Y_{A_2}$ are dual (cf. [O’G06] Section 3]). From now on $A$ will denote one of the two specific Lagrangian spaces.

**Proposition 2.2.** The Lagrangian space $A$ has no decomposable vectors, the degeneracy locus $Y_A[3]$ is empty, so $A \in LG(\bigwedge^3 \mathcal{V}_6)^0$, and in consequence the EPW-sextic $Y_A$ is singular along the degree 40 smooth surface $Y_A[2]$. Hence the double cover $\overline{Y}_A \rightarrow Y_A$ is a smooth hyper-Kähler fourfold.

**Proof.** Following [O’G12] we need to prove that $A$ does not belong to $\Sigma$ and $\Delta$. According to [Fer09] Proposition 1.5 the singular locus is given by the union of the 40-degree surface $Y_A[2]$ with planes $\mathbb{P}(U)$ where $U$ is a three-dimensional subspace of $W$ such that $\bigwedge^3 U \subset A$. 


Our computation with Macaulay2 shows that the singular locus has degree 40 (see Appendix A.2.1) so it must coincide with $Y_\Lambda[2]$ (cf. [O’G12, Corollary 1.10]), thus there are no decomposable vectors in $\Lambda$.

We also compute the singular locus $Y_\Lambda[3]$ of $Y_\Lambda[2]$ (see Appendix A.2.2), and it turns out to be empty. Which finishes the prove.

The double cover and its features were already described by O’Grady in [O’G06].

**Corollary 2.3.** The fourfold $\tilde{Y}_\Lambda$ has a symplectic action of the group $A_7$ and the action fixes the polarization $H$ (i.e. $A_7 \hookrightarrow \text{Aut}^s_H(\tilde{Y}_\Lambda)$).

**Proof.** Use Proposition 1.3 and (1).

**Lemma 2.4.** The group $\text{Aut}^s_H(\tilde{Y}_\Lambda)$ is finite.

**Proof.** From Corollary 1.2 as (by definition) it fixes $H$. We can also prove it in a somehow more direct way. We know from Proposition 1.3 that $\text{Aut}^s_H(\tilde{Y}_\Lambda) \cong \text{Aut}(Y_\Lambda)$. Proposition 2.2 ensures that $\Lambda$ has no decomposable vectors and [DK18, Proposition B.9] guarantees that the last group is finite.

**Proposition 2.5.** There is an isomorphism $\text{Aut}^s_H(\tilde{Y}_\Lambda) \cong A_7$.

**Proof.** Using the fact that $\text{Aut}^s_H(\tilde{Y}_\Lambda)$ is finite combined with [HM19, Theorem A and Table 6], one concludes that the inclusion $A_7 \hookrightarrow \text{Aut}^s_H(\tilde{Y}_\Lambda)$ is in fact an isomorphism.

Now we are ready to show that the two examples we found $\tilde{Y}_{A_1}$ and $\tilde{Y}_{A_2}$ are not isomorphic as polarized manifolds. We will need the following lemma.

**Lemma 2.6.** There are no $f \in \text{GL}(V_6)$ such that $\wedge^3 f(A_1) = A_2$.

**Proof.** Set $h = \wedge^3 f$ and denote the representations $R_i = (A_i, \rho_i)$ for $i = 1, 2$. Notice that $h : A_1 \to A_2$ defines an isomorphism of representations, and so a faithful representation

$$(A_2, h \circ \rho_1 \circ h^{-1}) \cong R_1$$

which is then not isomorphic to $R_2$. This means that one has the inclusions

$$A_7 \subset \langle (h \circ \rho_1 \circ h^{-1})(g), \rho_2(g) | g \in A_7 \rangle \subset \text{Aut}(Y_{A_2})$$

where the first one is strict and the second follows from (1) since all the automorphisms of the middle group are expressed by third wedges of automorphisms of $V_6$ which preserve the Lagrangian $A_2$. We conclude again using the isomorphisms

$$\text{Aut}(Y_{A_2}) \cong \text{Aut}^s_H(\tilde{Y}_{A_2}) \cong A_7$$

from Propositions 1.3 and 2.5 to get a contradiction.
Proposition 2.7. The manifolds $(\tilde{Y}_{A_1}, H_1)$ and $(\tilde{Y}_{A_2}, H_2)$ are not isomorphic as polarized manifolds where $H_i = \pi_i^* \mathcal{O}_{Y_{A_i}}(1)$ for $i = 1, 2$.

Proof. By [O’G15, page 486], if $A_1, A_2 \in \mathcal{L}(\Lambda^3 V_6)^0$ are not in the same orbit of $\text{PGL}(\Lambda^3 V_6)$, then $\tilde{Y}_{A_1}$ and $\tilde{Y}_{A_2}$ have different periods, so they cannot be isomorphic. Lemma 2.6 finishes the proof.

We can note that as $Y_{A_1}$ and $Y_{A_2}$ are dual to each other, that makes $\tilde{Y}_{A_1}$ and $\tilde{Y}_{A_2}$ dual in the sense of the involution of the moduli space of double EPW sextics described in [O’G06, Section 6]. We also obtain the following information on the constructed manifolds.

Proposition 2.8. The transcendental lattice

$$T(Y_{\mathbb{A}}) \cong \begin{pmatrix} 6 & 0 \\ 0 & 70 \end{pmatrix},$$

i.e. there exists \{v_1, v_2\} a basis of $T(Y_{\mathbb{A}})$ such that $v_1^2 = 6$, $v_2^2 = 70$, and $v_1 \cdot v_2 = 0$.

Proof. Based on [Waw22, Table 1], a projective hyper-Kähler fourfold of type $K3^{[2]}$ admitting an action of a nontrivial overgroup of $\mathcal{A}_7$ and fixing an ample vector $h$ such that $h^2 = 2$ in $\text{NS}(X) \subset H^2(X, \mathbb{Z})$ must have the transcendental lattice from the statement. But by [O’G06], the polarization $H$ discussed earlier satisfies $H^2 = 2$.

3 Irrational GM threefolds

Let $X_{\mathbb{A}}$ be any (smooth) GM variety of dimension 3 or 5 associated with $\mathbb{A}$ (as defined as the end of Section 1) and let $\text{Jac}(X_{\mathbb{A}})$ be its intermediate Jacobian.

From the fact that $Y_{\mathbb{A}}[2]$ is smooth, by [DK20, Theorem 1.1] there are a canonical principal polarization $\theta$ on the Albanese variety $\text{Alb}(\tilde{Y}_{\mathbb{A}}[2])$ and a canonical isomorphism

$$(\text{Jac}(X_{\mathbb{A}}), \theta_{X_{\mathbb{A}}}) \cong (\text{Alb}(\tilde{Y}_{\mathbb{A}}[2]), \theta)$$

(7)

between principally polarized Abelian varieties. Furthermore, the tangent spaces at the origin of these varieties are isomorphic to $\mathbb{A}$. Explicitly,

$$T_{\text{Alb}(\tilde{Y}_{\mathbb{A}}[2]), 0} \cong T_{\text{Jac}(X_{\mathbb{A}}), 0} \cong \mathbb{A}.$$  

(8)

One of the key features of the action of $\mathcal{A}_7$ we presented, is the following

Proposition 3.1. The principally polarized variety $(\text{Jac}(X_{\mathbb{A}}), \theta_{X_{\mathbb{A}}})$ is indecomposable.
Proof. Suppose it was isomorphic to a product of \( m \geq 2 \) nonzero indecomposable principally polarized abelian varieties.

Since \( \mathcal{A}_7 \cong \text{Aut}(Y_A) \), the diagram (4) reads:

\[
\begin{array}{cccccc}
1 & \longrightarrow & \langle \gamma^3 \rangle & \longrightarrow & \tilde{\mathcal{A}}_7 & \xrightarrow{\psi} & \mathcal{A}_7 & \longrightarrow & 1 \\
& & \downarrow \rho_a & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \text{GL}(\mathbb{A}) & \xrightarrow{\pi} & \text{PGL}(\mathbb{A}) & \longrightarrow & 1
\end{array}
\]

where \( \tilde{\mathcal{A}}_7 \) is an extension of \( \mathcal{A}_7 \) by the group of order two, \( \rho_a \) is the analytic representation \( \tilde{\mathcal{A}}_7 \rightarrow \text{GL}(T_{\text{Jac}(X_A)},0) \) by Proposition 1.4 and \( \rho \) is the irreducible representation \( \mathcal{A}_7 \). Now \( \rho_a \neq \rho \circ \psi \) since both representations are faithful, but the equality \( \pi \circ \rho_a = \pi \circ \rho \circ \psi \) holds by construction (using the commutativity of the diagram (1)). This means that the two actions on \( \mathbb{A} \) differ by a scalar multiplication, hence if one of the representations decomposes then the other must do so as well. We supposed that the Jacobian is a product and so the analytic representation decomposes in the sum of the tangent spaces of the components, but this is a contradiction since \( \mathcal{A}_7 \) was irreducible as \( \mathcal{A}_7 \)-representation.

\[\square\]

Combining this property with the group having a big cardinality, we get the result sought after:

**Theorem 3.2.** Any smooth GM threefold associated with the Lagrangian \( \mathbb{A} \) is irrational.

**Proof.** Same proof as [DM21, Theorem 5.2], but we include it for completeness. We want to use the Clemens-Griffiths criterion: if the Jacobian is not the product of Jacobians of curves, then the variety is irrational.

Since \( (\text{Jac}(X_A), \theta_{X_A}) \) is indecomposable, we can reduce to treat the case where \( (\text{Jac}(X_A), \theta_{X_A}) \cong (\text{Jac}(C), \theta_{2C}) \) for \( C \) a curve of genus 10. We have a faithful action of \( \mathcal{A}_7 \) on the Jacobian, by the Torelli theorem the group of automorphisms of that abelian variety is either \( \text{Aut}(C) \) or \( \text{Aut}(C) \times \mathbb{Z}/2\mathbb{Z} \). In conclusion \( \mathcal{A}_7 \) embeds in one of those two groups, but this is a contradiction since \( |\mathcal{A}_7| = 2520 \) and \( |\text{Aut}(C)| < 756 \) [Mir95, Theorem 3.7].

\[\square\]

### A Appendix

We provide codes for the computations we run using computer algebra. For the computations of characters we used \text{GAP} [GAP21], and \text{Macaulay2} [M2] for all the rest. For the sake of conciseness, we omit some technical details. The full code can be found in the auxiliary files attached on arxiv.
A.1 Finding the Lagrangians subspaces

The following is the scheme of the GAP code that computes the bases of two Lagrangian subspaces of $\bigwedge^3 V_6$ represented as $\mathbb{C}^{20}$ (more precisely $\mathbb{Q}[\xi]_{21}$ in the code) respectively invariant under two non-isomorphic 10-dimensional representations $R_1$ and $R_2$ of $A_7$. We compute the the invariant subspaces using the formula for a projection

$$P_i = \sum_{g \in A_7} \chi_{R_i}(g) \phi_W(g),$$

where $\chi_{R_i}$ is the character of the representation $R_i$ ($i = 1, 2$) and $\phi_W : A_7 \rightarrow \text{GL}(\mathbb{C}^{20})$ is the 20-dimensional representation. For more details on the characters, see Appendix B.

```
InducedMapOnWedge := function(Mat)
#computes the matrix of the linear map induced by the matrix Mat on
#the third Exterior Power of the underlying space in the basis of
#lexographically ordered simple vectors
#obtained through multiplication of the canonical basis
#
end;;

CheckLagrangianWedge3_6 := function(L)
#checks if a subspace of Wedge^3V_6 is a Lagrangian space
#
end;;

OrbitSpace := function(vec, Gr, F)
#returns the space spanned by the orbit of
#the vector vec by the group Gr over the field F
#
end;;

#http://brauer.maths.qmul.ac.uk/Atlas/alt/A7/gap0/3A7G1-Ar6B0.g
#6-dimensional representation of 3.A7
#defined in the ring Z extended by 21st root of unity
Gens_3A7_6 := [
#...
];;

#induced representation on Wedge^3C^6; it acts as A7 now
Gens_A7_20 := [InducedMapOnWedge(Gens_3A7_6[1]),
InducedMapOnWedge(Gens_3A7_6[2])]
];;

b := E(7)+E(7)^2+E(7)^4;;
B := -1-b;;
CC := ConjugacyClasses(A7_20);
ImportantCC := [1]; #only the classes of nonzero are important for computation
for C in CC do
    if Trace(Representative(C)) <> 0 then
        Add(ImportantCC, C);
    fi;
od;
```

#below we mark which classes have traces -b and -B in 10 dimensional representations
for ind in [1..Length(ImportantCC)] do
    if Trace(Representative(ImportantCC[ind])) = -1 then
        if flag then
            ind1 := ind;
            flag := false;
        else
            ind2 := ind;
        fi;
    else
        fi;
od;
e1 := CanonicalBasis(Rationals^20)[1];

P_1 := 0 * IdentityMat(20,20);;
for ind in [1..Length(ImportantCC)] do
    Paux := 0 * IdentityMat(20);
    for g in ImportantCC[ind] do
        Paux := Paux + g;
    od;
    if ind = ind1 then
        P_1 := P_1 + b * Paux;
    elif ind = ind2 then
        P_1 := P_1 + B * Paux;
    else
        P_1 := P_1 + Trace(Representative(ImportantCC[ind]))/2 * Paux;
    fi;
od;
A_1 := OrbitSpace(v_1, A7_20, CF(21));
CheckLagrangianWedge3_6(R_1);
#returns true
#the computation as above for A_2 follows

A.2  Supplement to the proof of Proposition 2.2

This section consists of two subsections, the aim is to justify computations about
the singular locus of $Y_A$, in particular conclude that it consists of a smooth
surface. In the first subsection we compute equations for the EPW-sextic and
its singular locus over an appropriate finite field, so that those objects can sit
as fibers in a flat family whose central fiber is the solution to the equation on
complex numbers. In the second subsection we compute the locus $Y_A[3]$ and
conclude that dimension and degree of the spaces over the complex numbers are
the same as the ones over the finite field, proving that $A \in LG(\Lambda^3 V_6)^o$.

A.2.1  Computation of the EPW and its singular locus with Macaulay2

The following Macaulay2 code computes equations over a finite field for the
EPW sextic associated to a Lagrangian space, check that the sextic is irreducible
and has the right degree. Lastly, we compute the singular locus of the EPW and
check it is a surface of degree 40. The two Lagrangian subrepresentations
we take into account are defined over $\mathbb{Q}[\xi_{21}]$.

We choose $p = 127$ so that the cyclotomic polynomial decomposes as

$$\Phi_{21}(v) \equiv_{127} \prod_{j=1}^{12} g_j(v),$$

in the polynomial ring $\mathbb{F}_{127}[v]$ in exactly as many factors as $[\mathbb{Q}[\xi_{21}]: \mathbb{Q}] = 12$ so that the decomposition of the ideal $(127)$ in the ring of integers $\mathbb{Z}[\xi_{21}]$ is given by

$$(127) = \prod_{j=1}^{12} q_j$$

where $q_j = (127, g_j(\xi_{21}))$ and hence the residue field of $\mathbb{Z}[\xi_{21}]$ at any prime $q$ in the decomposition is exactly $\mathbb{F}_{127}$. Set $D$ as the DVR obtained by localizing $\mathbb{Z}[\xi_{21}]$ at any such $q$. To be more explicit we can put $q = (127, \xi_{21} - 25)$ which is one of the factors in the decomposition.

In the following code we read the roots of unity in $\mathbb{F}_{127}$ via the left-down association in the following commutative diagram where we chose $q = (127, \xi_{21} - 25)$

\[
\begin{array}{c}
\mathbb{Z}[x] \longrightarrow \mathbb{Z}[x]/(\Phi_{21}(x)) \longrightarrow \mathbb{Z}[\xi_{21}] \\
\downarrow \quad \downarrow \\
\mathbb{F}_{127}[x] \longrightarrow \mathbb{F}_{127}[x]/(x - 25) \longrightarrow \mathbb{F}_{127}
\end{array}
\]

which is in fact the same procedure as taking the residue field at the prime $q$ of $D$.

```plaintext
p = 127;
F = ZZ/p;
R = F[v];
I = ideal (v^12-v^11+v^9-v^8+v^6-v^4+v^3-v+1); --21st cyclotomic polynomial
J = decompose(I);
length J --returns 12
K = toField(R/J_0);
P = K[x,y,z,t,u,w];
--Matrix of coordinates
M = matrix {
  { 0,0,0,0,0,0,0,u,-t,z},
  { 0,0,0,0,-u,t,0,0,-y},
  { 0,0,0,0,-z,0,0,y,0},
  { 0,0,0,-t,z,0,-y,0,0},
  { 0,0,0,0,0,0,0,0,0},
  { 0,0,0,0,0,0,0,0,-x,0},
  { 0,0,0,0,0,0,0,x,0,0},
  { 0,0,0,0,0,0,0,0,0},
  { 0,0,0,0,0,0,0,0,0},
}
MM = M + transpose ( M ); --Mat1 is the symmetric matrix associated to the basis of the lagrangian A1
Lambda1 = Mat1-MM; --The EPW is given by
```
d1 = det(Lambda1); --polynomial of degree 6, the equation for the EPW
I1 = ideal d1;

degree(I1) --returns 6

s1 = ideal singularLocus I1; --singular locus of the EPW over the finite field

dim(s1) --returns 3, so the projective dim is 2

degree(s1) --returns 40

The code for computing the equations for EPW-sextics is based on the Appendix of [KKM22].

A.2.2 The singular locus is smooth

Consider the proper map \( \mathbb{P}^5_D \rightarrow \text{Spec } D \), this induces a map \( V(I) \rightarrow \text{Spec } D \) where \( V(I) \) is the zero locus of our homogeneous equation in \( D[x_0, \ldots, x_5] \). In this setting, the fiber over the ideal \( (0) \) is the solution to the equation with coefficients in \( \mathbb{Q}[\xi_{21}] \) and the fiber over the ideal \( q \) is the solution to the equation with coefficients in the residue field \( \mathbb{F}_{127} \). Since the map is proper, the image must be either the closed point \( q \) or the entire scheme \( \text{Spec } D \), in particular if the fiber \( X_q \) is empty then the fiber \( X_{(0)} \) must be empty as well.

We compute the locus \( Y_{A\{3\}} \) as the zero locus of the ideal generated by the \( 8 \times 8 \) minors of the symmetric matrix whose zero locus is the EPW. The outcome of the computation is that the locus \( Y_{A\{3\}} \) is empty over the finite field and this implies that it is also empty over the complex numbers, proving that \( A \not\in \Delta \).

The solutions over the two fields lie in a flat family and then the singular locus of \( Y_{A\{2\}} \) has dimension 2 and degree 40 over the complex numbers. In conclusion the singular locus of the EPW coincides with the smooth surface \( Y_{A\{2\}} \), since \( Y_{A\{2\}} \) has already degree 40 [O'G12, Corollary 1.10] and is already contained in the singular locus. This gives \( A \not\in \Sigma \).

\hspace{1cm} --The ideal generated by the 8x8 minors describes the locus Y_A{1[3]}
J1 = minors(8,Lambda1);

\hspace{1cm} -- Dimension of Y_A{1[3]} (affine chart)
dim(J1)

\hspace{1cm} --Homogeneous ideal whose associated variety is Y_A{1[3]}
J1h = saturate homogenize(J1,x);

\hspace{1cm} --Projective variety Y_A{1[3]}
Z1 = Proj(P/J1h);
dim(Z1) --returns -2

Variables that are not defined here are the same as above.

B Appendix

Below is the character for \( A_7 \) including the 20-dimensional one reducible representation we obtained on the exterior power space.
Table 1: Character table of $A_7$

| Conj. class | id  | $[ab^{-1}ab]$ | $[a]$ | $[a^{-1}bab]$ | $[a^{-1}bab^2]$ | $[b]$ | $[abab^2]$ | $[ab]$ | $[a^{-1}b]$ |
|-------------|-----|----------------|-------|---------------|-----------------|------|-------------|-------|-------------|
| $V_0$       | 1   | 1             | 1     | 1             | 1               | 1    | 1           | 1     | 1           |
| $V_6$       | 6   | 2             | 3     | 0             | 0               | 1    | -1          | -1    | -1          |
| $V_{10}$    | 10  | -2            | 1     | 1             | 0               | 0    | 1           | $\frac{1}{2}(1-i\sqrt{7})$ | $\frac{1}{2}(1+i\sqrt{7})$ |
| $V_{12}$    | 10  | -2            | 1     | 1             | 0               | 0    | 1           | $\frac{1}{2}(1+i\sqrt{7})$ | $\frac{1}{2}(1-i\sqrt{7})$ |
| $V_{14}$    | 14  | 2             | -1    | 2             | 0               | -1   | 2           | 0     | 0           |
| $V_{14}'$   | 14  | 2             | -1    | 2             | 0               | -1   | -1          | 0     | 0           |
| $V_{15}$    | 15  | -1            | 3     | 0             | -1              | 0    | -1          | 1     | 1           |
| $W$         | 20  | -4            | 2     | 2             | 0               | 0    | 2           | -1    | -1          |

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