THE KNOT INVARIANT ASSOCIATED TO TWO-PARAMETER QUANTUM ALGEBRAS

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Abstract. Using the skew-Hopf pairing, we obtain $\mathcal{R}$-matrix for the two-parameter quantum algebra $U_{v,t}$. We further construct a strict monoidal functor $\mathcal{F}$ from the tangle category $(\text{OTa}, \otimes, \emptyset)$ to the category $(\text{Mod}, \otimes, \mathbb{Q}(v, t))$ of $U_{v,t}$-modules. As a consequence, the quantum knot invariant of the tangle $L$ of type $(n, n)$ is obtained by the action of $\mathcal{F}$ on the closure $\tilde{L}$ of $L$.

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1. Introduction

Knot theory is the mathematical discipline with the unusually diverse applications, such as statistical mechanics [K11], symbolic logic and set theory [K16], quantum field theory [W], quantum computing [NSS], etc. Reshetikhin and Turaev [RT] related quantum algebras to knot invariants, often referred to as quantum invariants. They generalized the Jones polynomial of links and the related Jones-Conway and Kauffman polynomials to the case of graphs in $\mathbb{R}^3$. Inspired by the result, a large number of researchers began to pay attention to quantum invariants, such as Zhang [ZGB], Kauffman [KM] and Clark [C].

In [FL], the first author and Li provided a novel presentation of the two-parameter quantum algebra $U_{v,t}(g)$ by a geometric approach, where both parameters $v$ and $t$ have geometric meaning. Moreover, this presentation unifies the various quantum algebras in literature. By various specialization, one can obtain one-parameter quantum algebras [L], two-parameter quantum algebras [BW], quantum superalgebras [CFYW] and multi-parameter quantum algebras [HPR]. It is a natural question that whether this two-parameter quantum algebra provide a new knot invariant. This is what we will explore in this paper. In [C], Clark gave a partial answer to this

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question by proving that the quantum knot invariants for $\mathfrak{osp}(1|2n)$ and $\mathfrak{so}(2n+1)$ are essentially the same.

The solution of the Yang-Baxter equation is called $\cal R$-matrix which connects quantum algebras to the knot invariant theory. In the construction of quantum knot invariants, invariance under the Reidemeister III move holds naturally by attaching a copy of the $\cal R$-matrix to each crossing. The construction of $\cal R$-matrices for various quantum groups is very meaningful and interesting. In [L, Chapter 32], Lusztig provided a framework to construct the $\cal R$-matrix of one-parameter quantum algebras via the quasi-$\cal R$-matrix. In [FX], we defined a skew-Hopf pairing on the deformed quantum algebra which unifies various quantum algebras in literatures, such as one-parameter quantum algebras [L], two-parameter quantum algebras [BW], quantum superalgebras [CFYW] and multi-parameter quantum algebras [HPR]. This provided a tool to construct $\cal R$-matrix for quantum algebras other than one-parameter one. For instance, in [BW], Benkart and Witherspoon constructed the $\cal R$-matrix for two-parameter quantum algebra $U_{r,s}(\mathfrak{sl}_n)$ by the Hopf pairing which can be viewed as a special case of that in [FX].

In this paper, we construct the $\cal R$-matrix for the two-parameter quantum algebra $U_{v,t}$ by using skew-Hopf pairing. This recovers the constructions of $\cal R$-matrix in [L] and [BW] under certain assumptions. We further provide the functor $\mathcal{T} : (\mathcal{OTa}, \otimes, \emptyset) \rightarrow (\mathcal{Mod}, \otimes, \mathbb{Q}(v,t))$, where $\mathcal{OTa}$ and $\mathcal{Mod}$ are the categories of tangles and $U_{v,t}$-modules, respectively. This produces the machinery and correspondence for the construction of quantum invariants via representations of two-parameter quantum algebra $U_{v,t}$. Furthermore, given a tangle $L$ of type $(n,n)$, we can get an endomorphism $\mathcal{T}(\tilde{L})$ of the ground field $\mathbb{Q}(v,t)$, where $\tilde{L}$ is the closure of $L$.

This paper is organized as follows. In Section 2, we recall the definition of two-parameter quantum algebra $U_{v,t}$ from [FL] and formulate the quasi-$\cal R$-matrix $\Theta$ of $U_{v,t}$. Part of the results are new for two-parameter quantum groups. In Section 3, we construct the $\cal R$-matrix of two-parameter quantum algebras $U_{v,t}$. In Section 4, we construct the functor between the categories of tangles and the categories of $U_{v,t}$-modules.

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2. The two-parameter quantum algebra $U_{v,t}$

We briefly review the definition of the two-parameter quantum algebra $U_{v,t}$ in [FL]. Given a Cartan datum $(I, \cdot)$, let $\Omega = (\Omega_{ij})_{i,j \in I}$ be an integer matrix satisfying that

(a) $\Omega_{ii} \in \mathbb{Z}_{>0}$, $\Omega_{ij} \in \mathbb{Z}_{\leq 0}$ for all $i \neq j \in I$;
(b) $\frac{\Omega_{ii} + \Omega_{jj}}{\Omega_{ij}} \in \mathbb{Z}_{\leq 0}$ for all $i \neq j \in I$;
(c) the greatest common divisor of all $\Omega_{ii}$ is equal to 1.
To $\Omega$, we associate the following three bilinear forms on $\mathbb{Z}[I]$.

$$
\langle i, j \rangle = \Omega_{ij}, \quad \forall i, j \in I.
$$

$$
[i, j] = 2\delta_{ij}\Omega_{ii} - \Omega_{jj}, \quad \forall i, j \in I.
$$

$$
i \cdot j = \langle i, j \rangle + \langle j, i \rangle, \quad \forall i, j \in I.
$$

2.1. The free algebra $\mathcal{J}$. For indeterminates $v$ and $t$, we set $v_i = v^{i/2}$ and $t_i = t^{i/2}$. Denoted by $v_\nu = \prod_i v_i^{\nu_i}$, $t_\nu = \prod_i t_i^{\nu_i}$ and $\text{tr}(\nu) = \sum_{i \in I} \nu_i \in \mathbb{N}$, for any $\nu = \sum \nu_i i \in \mathbb{N}[I]$.

Let $\mathcal{J}$ be the free unital associative algebra over $\mathbb{Q}(v, t)$ generated by the symbols $\theta_i$, $\forall i \in I$. By setting the degree of the generator $\theta_i$ to be $i$, the algebra $\mathcal{J}$ becomes an $\mathbb{N}[I]$-graded algebra. For any $\nu \in \mathbb{N}[I]$, we denote by $\mathcal{J}_\nu$ the subspace of all homogenous elements of degree $\nu$. We have $\mathcal{J} = \oplus_{\nu \in \mathbb{N}[I]} \mathcal{J}_\nu$ and denote by $|x|$ the degree of a homogenous element $x \in \mathcal{J}$.

2.1.1. The tensor product $\mathcal{J} \otimes \mathcal{J}$. On the tensor product $\mathcal{J} \otimes \mathcal{J}$, we define an associative $\mathbb{Q}(v, t)$-algebra structure by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = v_1^{\langle y_1 \rangle - \langle x_2 \rangle} (\langle y_1 \rangle - \langle x_2 \rangle \langle y_1 \rangle - \langle y_1 \rangle) x_1 y_1 \otimes x_2 y_2,$$

for homogeneous elements $x_1, x_2, y_1$ and $y_2$ in $\mathcal{J}$. It is associative since the forms $\langle \cdot, \cdot \rangle$ and “$\cdot$” are bilinear.

Let $r : \mathcal{J} \to \mathcal{J} \otimes \mathcal{J}$ be the $\mathbb{Q}(v, t)$-algebra homomorphism such that

$$
r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i, \quad \text{for all } i \in I.
$$

Proposition 2.1. [FL Proposition 13] There is a unique symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{J}$ with values in $\mathbb{Q}(v, t)$ such that

(a) $\langle 1, 1 \rangle = 1$;

(b) $\langle \theta_i, \theta_j \rangle = \delta_{ij} \frac{1}{1-v_i} \bar{r}_i$, for all $i, j \in I$;

(c) $\langle x, y' y'' \rangle = (r(x), y' \otimes y'')$, for all $x, y', y'' \in \mathcal{J}$;

(d) $\langle x' x'', y \rangle = (x' \otimes x'', r(y))$, for all $x', x'', y \in \mathcal{J}$.

Here the bilinear form on $\mathcal{J} \otimes \mathcal{J}$ is defined by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = t^{2\langle x_1 \rangle \langle x_2 \rangle} (x_1, y_1) (x_2, y_2).$$

2.1.2. Let $\sigma : \mathcal{J} \to \mathcal{J}^{op}$ be a twisted anti-involution such that

$$
\sigma(\theta_i) = \theta_i, \quad \text{and} \quad \sigma(xy) = t^{\langle y \rangle - \langle x \rangle} \sigma(y) \sigma(x)
$$

for any homogenous elements $x, y \in \mathcal{J}$.

Let $\rho : \mathcal{J} \otimes \mathcal{J} \to \mathcal{J} \otimes \mathcal{J}$ be a linear map defined by

$$
\rho(x \otimes y) = t^{\langle y \rangle - \langle x \rangle} y \otimes x, \quad \forall x, y \in \mathcal{J}.
$$

We set $\mathcal{J} r = \rho \circ r$.

Lemma 2.2. We have $r(\sigma(x)) = (\sigma \otimes \sigma) \mathcal{J} r(x)$, for all $x \in \mathcal{J}$. 

Proof. We show this lemma by induction on $|x|$. If $x = \theta_i$, it follows the definition of $\sigma$ and $t^r$.

Assume that it holds for any homogenous elements $x'$ and $x''$. We shall show that it holds for $x = x'x''$. Let's write $r(x') = \sum x'_1 \otimes x'_2$ and $r(x'') = \sum x''_1 \otimes x''_2$, with all factors being homogeneous. Then we have

$$ r(x'x'') = r(x')r(x'') = \sum v|\sigma| |\sigma| t(|\sigma| |\sigma|) x'_1 x''_1 \otimes x'_2 x''_2. $$

By the definition of $r$, $(\sigma \otimes \sigma)^t r(x'x'')$ is equal to

$$ r(\sigma(x')) = \sum t(|\sigma| |\sigma|) \sigma(x'_2) \otimes \sigma(x'_1) \text{ and } r(\sigma(x'')) = \sum t(|\sigma| |\sigma|) \sigma(x''_2) \otimes \sigma(x''_1). $$

Thus,

$$ r(\sigma(x') \sigma(x'')) = t(|\sigma| |\sigma|) \sigma(x''_2) \otimes \sigma(x''_1) \sigma(x'_2) \otimes \sigma(x'_1). $$

By hypothesis, we have $r(\sigma(x')) = \sigma(x'_2) \otimes \sigma(x'_1)$ and $r(\sigma(x'')) = \sigma(x''_2) \otimes \sigma(x''_1)$.

By comparing the exponents of $v$ and $t$ in (2.2) and (2.3) with $|x'| = |x'_1| + |x'_2|$ and $|x''| = |x''_1| + |x''_2|$, we have

$$ r(\sigma(x'x'')) = (\sigma \otimes \sigma)^t r(x'x''). $$

This finishes the proof. \hfill \Box

2.1.3. Let $\ast : \mathbb{Q}(v, t) \to \mathbb{Q}(v, t)$ be the unique $\mathbb{Q}$-algebra involution such that

$$ \mathfrak{v} = v^{-1} \text{ and } \mathfrak{t} = t. $$

Let $\mathfrak{v} : \mathfrak{f} \to \mathfrak{f}$ be the unique $\mathbb{Q}$-algebra involution such that

$$ \mathfrak{p} \theta_i = \mathfrak{p} \theta_i, \quad \forall p \in \mathbb{Q}(v, t), \ i \in I. $$

It's clear that $|\mathfrak{v}| = |x|$ for any homogeneous element $x \in \mathfrak{f}$.

Let $\mathfrak{f} \otimes \mathfrak{f}$ be the $\mathbb{Q}(v, t)$-vector space $\mathfrak{f} \otimes \mathfrak{f}$ with the associative $\mathbb{Q}(v, t)$-algebra structure given by

$$ (x_1 \otimes x_2)(y_1 \otimes y_2) = v^{-|y_1||x_2|} t(|y_1||x_2| - |x_2||y_1|) x_1 y_1 \otimes x_2 y_2. $$

Then $\mathfrak{f} \otimes \mathfrak{f}$ is the $\mathbb{Q}(t)$-algebra isomorphism.

Let $\mathfrak{f} : \mathfrak{f} \to \mathfrak{f} \otimes \mathfrak{f}$ be the $\mathbb{Q}(t)$-algebra homomorphism defined by

$$ \mathfrak{v}(x) = \mathfrak{r}(x), \quad \forall x \in \mathfrak{f}. $$

Then we have $\mathfrak{v}(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$. 
Lemma 2.3. For any \( x \in \mathfrak{g} \), by setting \( r(x) = \sum x_1 \otimes x_2 \), we have
\[
\mathfrak{r}(x) = \sum v^{-|x_1|\cdot|x_2|} t^{(|x_2|,|x_1|)} - (|x_1|,|x_2|) x_2 \otimes x_1.
\]

Proof. By the definition of \( \mathfrak{r} \), we shall show that
\[
r(\mathfrak{r}) = \sum v^{||x_1|\cdot|x_2|} t^{(|x_2|,|x_1|)-(|x_1|,|x_2|)} \mathfrak{r}_2 \otimes \mathfrak{r}_1.
\]

The proof for Lemma 5 in [FL] works through if we replace \( v \) by \( v^{-1} \).

We note that the coassociativity property of \( r \) implies the coassociativity property of \( \mathfrak{r} \), i.e., \((\mathfrak{r} \otimes 1)\mathfrak{r} = (1 \otimes \mathfrak{r})\mathfrak{r}\).

2.1.4. The maps \( r_i \) and \( \mathfrak{r}_i \). For any \( i \in I \), let \( r_i \) (resp. \( \mathfrak{r}_i \)) be the unique linear map satisfying the following properties.
\[
r_i(1) = 0, \quad r_i(\theta_j) = \delta_{ij}, \quad \forall j \in I \quad \text{and} \quad r_i(xy) = v^{i\cdot|y|} t^{(|y|,i)-(i,|y|)} r_i(x)y + x r_i(y); \\
\mathfrak{r}_i(1) = 0, \quad \mathfrak{r}_i(\theta_j) = \delta_{ij}, \quad \forall j \in I \quad \text{and} \quad \mathfrak{r}_i(xy) = \mathfrak{r}_i(x)y + v^{i\cdot|x|} t^{(|x|,i)-(i,|y|)} \mathfrak{r}_i(y).
\]

By an induction on \(|x|\), we can show that \( r(x) = r_i(x) \otimes \theta_i \) (resp. \( \mathfrak{r}(x) = \mathfrak{r}_i(x) \otimes \mathfrak{r}(x) \)) plus other terms.

2.1.5. Quantum serre relations. Let \( \mathfrak{J} \) be the radical of the bilinear form \((-, -)\). It is clear that \( \mathfrak{J} \) is a two-sided ideal of \( \mathfrak{g} \). Denote the quotient algebra of \( \mathfrak{g} \) by
\[
\mathfrak{g} = \mathfrak{g}/\mathfrak{J}.
\]

Recall the quantum integers from [FL]. For any \( n \in \mathbb{N} \), we have
\[
[n]_{v,t} = \frac{(vt)^n - (vt^{-1})^{-n}}{vt - (vt^{-1})^{-1}}, \quad [n]_{v,t}^! = \prod_{k=1}^{n} [k]_{v,t}.
\]

Denote by
\[
\theta_i^{(n)} = \frac{\theta_i^n}{[n]_{v,t}^!}.
\]

Proposition 2.4. [FL, Proposition 14] The generators \( \theta_i \) of \( \mathfrak{g} \) satisfy the following identities.
\[
\sum_{p+p' = 1-2\frac{i+j}{i+j}} (-1)^p t_i^{-p'(p'-2|j|_i + 2|j|_i)} \theta_i^{(p)} \theta_j^{(p')} = 0, \quad \forall i \neq j \in I.
\]

2.2. The presentation of the two-parameter quantum algebra \( U_{v,t} \). By Drinfeld double construction, we get the following presentation of the entire two-parameter quantum algebra \( U_{v,t} \), generated by symbols \( E_i, F_i, K_i^{\pm 1}, K_i'^{\pm 1}, \forall i \in I \), and subjects to the following relations.

\begin{align*}
(R1) \quad & K_i^{\pm 1} K_i'^{\mp 1} = K_i'^{\pm 1} K_i^{\mp 1} = 1, \\
(R2) \quad & K_i E_j K_i'^{-1} = v^{i(j) - (i,j)} E_j, \quad K_i F_j K_i'^{-1} = v^{-i(j) - (i,j)} E_j, \\
& K_i' F_j K_i^{-1} = v^{j(i) - (j,i)} F_j, \quad K_i' F_j K_i^{-1} = v^{-j(i) - (j,i)} F_j.
\end{align*}
\[(R3) \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_j'}{v_i - v_j^{-1}}.\]

\[(R4) \quad \sum_{p+p' = 1-2 \frac{i+j}{r}} (-1)^{p + p'} \frac{p(p'-2, (i+j)+2)}{v_i - v_j^{-1}} E_i(p) E_j(p') = 0, \quad \text{if} \ i \neq j,\]

\[
\sum_{p+p' = 1-2 \frac{i+1+j}{r}} (-1)^{p + p'} \frac{p(p'-2, (i+j)+2)}{v_i - v_j^{-1}} F_i(p') F_j(p) = 0, \quad \text{if} \ i \neq j.
\]

The algebra \(U_{v,t}\) has a Hopf algebra structure with the comultiplication \(\Delta\), the counit \(\varepsilon\) and the antipode \(S\) given as follows.

\[
\Delta(K_i^{\pm1}) = K_i^{\pm1} \otimes K_i^{\pm1}, \quad \Delta(K_i'^{\pm1}) = K_i'^{\pm1} \otimes K_i'^{\pm1},
\]

\[
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i',
\]

\[
\varepsilon(K_i^{\pm1}) = \varepsilon(K_i'^{\pm1}) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad S(K_i^{\pm1}) = K_i^{-1},
\]

\[
S(K_i'^{\pm1}) = K_i'^{-1}, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i'^{-1}.
\]

Let \(U_{v,t}^+\) (resp. \(U_{v,t}^-\)) be a subalgebra of \(U_{v,t}\) generated by \(E_i\) (resp. \(F_i\)). From the Drinfeld double construction, we know that there are two well-defined algebra homomorphisms \(\iota^+ : f \to U_{v,t}^+\ (x \to x^+)\) and \(\iota^- : f \to U_{v,t}^-\ (x \to x^-, t \to t^-)\) such that \(E_i = \theta_i^+, F_i = \theta_i^-\) for all \(i \in I\).

Let \(\sigma^+ : U^+ \to U^+\) (resp. \(\sigma^- : U^- \to U^-\)) be an anti-involution such that \(\sigma^+(x^+) = \sigma(x)^+\) (resp. \(\sigma^-(x^-) = \sigma(x)^-\)) for all \(x \in f\).

**Lemma 2.5.** For all \(x \in f\), we have

(i) \(S(x^+) = (-1)^{tr[x]v_{|x|}} v_{|x|}^{-1} K_{|x|}^{-1} \sigma(x)^+\),

(ii) \(S(x^-) = (-1)^{tr[x]v_{|x|}} v_{|x|}^{-1} K_{|x|}^{-1} \sigma(x)^-\).

**Proof.** The proofs of (i) and (ii) are similar. We shall only show (i) and leave (ii) to readers. If \(x = \theta_i\), (i) is straightforward by the definition of \(S(E_i)\).

Assume that (i) holds for \(x_1\) and \(x_2\). We shall show that it holds for \(x = x_1 x_2\). Let’s write \(S(x^+) = (-1)^{tr[x_1]v_{|x_1|}} v_{|x_1|}^{-1} K_{|x_1|}^{-1} \sigma(x_1)^+ + S(x_2^+) = (-1)^{tr[x_2]v_{|x_2|}} v_{|x_2|}^{-1} K_{|x_2|}^{-1} \sigma(x_2)^+\). Then, we have

\[
S((x_1 x_2)^+) = S(x_2^+) S(x_1^+) = (-1)^{tr[|x_1|+|x_2|]} v_{|x_1|+|x_2|}^{-1} K_{|x_1|+|x_2|}^{-1} \sigma(x_1)^+ K_{|x_1|}^{-1} \sigma(x_1)^+.
\]

This finishes the proof. \(\square\)

**Lemma 2.6.** For all \(x, x', x'' \in f\), let \(r_i^+(x^+) = \iota^+ \circ r_i^+(x)\) and \(r_i^+(x^+) = \iota^+ \circ r_i^+(x)\). Then we have

(i) \(r_i^+(x x'') = v^{i,|x'|} (|x''|)^{i,|x''|} r_i^+(x') x'' + x' r_i^+(x'')\),
Lemma 2.7. For all \( r(x) \) and \( r(y) \),

\[(ii) \quad i^{r+}((x')x'') = i^{r+}(x')x'' + v^{|x'|}t^{|i|x'|} - \langle x'|i \rangle x' + i^{r+}(x''),\]

\[(iii) \quad x^+F_i - F_i x^+ = \frac{r_i^+(x^+)K_i - K_i'^r(r^+(x^+))}{v_i - v_i^{-1}},\]

\[(iv) \quad \Delta(x^+) = x^+ \otimes 1 + \sum_i r_i^+(x^+)K_i \otimes E_i + \cdots = K_{[x]} \otimes x^+ + \sum_i E_i K_{[x]-i} \otimes r^+(x^+) + \cdots.\]

\[
\text{Proof.} \quad \text{Statement (i) (resp. (ii)) directly follows the definition of } r^+ \text{ and } r_i \text{ (resp. } i^r). \]

We now show (iii) by induction on \(|x|\). The case that \( x = \theta_i \) is trivial. Assume that (iii) holds for \( x' \) and \( x'' \). Then we have

\[
\begin{align*}
(x' x'')^+ F_i - F_i (x' x'')^+ & = x'^+ (F_i x'^+ + \frac{r_i^+(x'^+)K_i - K_i'^r(r^+(x'^+))}{v_i - v_i^{-1}}) - F_i x'^+ x''^+ \\
& = \frac{r_i^+(x'^+)K_i - K_i'^r(r^+(x'^+))}{v_i - v_i^{-1}} + x'^+ \frac{r_i^+(x'^+)K_i - K_i'^r(r^+(x'^+))}{v_i - v_i^{-1}} \\
& = \frac{r_i^+(x'^+)K_i x'^+ + x'^+ r_i^+(x'^+)K_i}{v_i - v_i^{-1}} - \frac{K_i'^r(r^+(x'^+))x'^+ + x'^+ K_i'^r(r^+(x'^+))}{v_i - v_i^{-1}} \\
& = \frac{r_i^+((x' x'')^+)K_i + K_i'^r((x' x'')^+)}{v_i - v_i^{-1}}.
\end{align*}
\]

This proves (iii).

By (i) and (ii), statement (iv) can be shown by induction on \(|x|\). \(\square\)

Lemma 2.7. For all \( y, y', y'' \in f \), let \( r_i^-(y^-) = i^- \circ r_i(x) \) and \( r^-(y^-) = i^- \circ i^r(x) \).

Then we have

\[(i) \quad r_i^-((y y')^-) = v^{|y'|}t^{|i^i|y'|} - (|y'|, i) r_i^-(y^-) y''^- + y^- r_i^- (y''^-),\]

\[(ii) \quad i^r^-((y y')^-) = i^r^-(y^-) y''^- + v^{|y'|}t^{|(i|y'|) - (i, |y'|)} y^- i^r (y''^-),\]

\[(iii) \quad E_i y^- - y^- E_i = \frac{K_i(r^- (y^-) - r_i^- (y^-))K_i'}{v_i - v_i^{-1}},\]

\[(iv) \quad \Delta(y^-) = y^- \otimes K_{[y]} + \sum_i r_i^-(y^-) \otimes F_i K_{[y]-i} + \cdots = 1 \otimes y^- + \sum_i E_i \otimes r_i^- (y^-) K_i' + \cdots.\]

The proof of this lemma is similar to that of Lemma 2.6.

2.3. The quasi-\( R \)-matrix \( \Theta \). In this section, we shall simply write \( U \) instead of \( U_{v,t} \).

We define a bar involution \( \overline{\gamma} : U \to U \) such that

\[
\overline{E_i} = E_i, \quad \overline{F_i} = F_i, \quad \overline{K_i} = K_i', \quad \overline{K_i'} = K_i; \quad \overline{\rho \tau} = \overline{\rho} \cdot \overline{\tau}, \quad \forall \rho, \tau \in \mathbb{Q}(v, t), x \in U.
\]

Let \( \overline{\gamma} : U \otimes U \to U \otimes U \) be the \( \mathbb{Q}(t) \)-algebra homomorphism given by \( \overline{\gamma} \otimes \overline{\gamma} \) and \( \overline{\Delta} : U \to U \otimes U \) the \( \mathbb{Q}(t) \)-algebra homomorphism given by \( \overline{\Delta}(x) = \overline{\Delta(x)} \). Thus, we have

\[
\begin{align*}
\overline{\Delta}(E_i) & = E_i \otimes 1 + K_i' \otimes E_i, \quad \overline{\Delta}(K_i) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \overline{\Delta}(K_i') = K_i'^{\pm 1} \otimes K_i'^{\pm 1}, \\
\overline{\Delta}(F_i) & = 1 \otimes F_i + F_i \otimes K_i, \quad \overline{\Delta}(K_i'^{\pm 1}) = K_i'^{\pm 1} \otimes K_i'^{\pm 1}.
\end{align*}
\]
Let \((U \otimes U)^{\wedge}\) be the completion of the vector space \(U \otimes U\) with respect to the descending sequence of vector spaces

\[ \mathcal{H}_N = (U^{+}U^0 \left( \sum_{\nu \geq N} U^\nu \right)) \otimes U + U \otimes (U^{-}\mathbf{U}^0 \left( \sum_{\nu \geq N} U^\nu \right)) \]

for \(N = 1, 2, \ldots\).

We set

\[ \{i, j\} = v^{i, j} (j, i) - (i, j), \forall i, j \in I, \]

which is a multiplicative bilinear form on \(\mathbb{Z}[I] \times \mathbb{Z}[I]\).

**Lemma 2.8.** [FX Proposition 4] Let \(U_{\geq 0}\) (resp. \(U_{\leq 0}\)) be the subalgebra of \(U\) generated by \(E_i\) and \(K_i\) (resp. \(F_i\) and \(K'_i\)) for all \(i\) in \(I\). We denote \(K_{-\mu}\) (resp. \(K'_{-\mu}\)) by \(K^{-1}_{\mu}\) (resp. \(K'^{-1}_{\mu}\)) for all \(\mu \in \mathbb{N}[I]\). There is a skew-Hopf pairing \((, \phi\) : \(U_{\geq 0} \times U_{\leq 0} \rightarrow \mathbb{Q}(v, t)\) such that

(i) \((1, 1)_{\phi} = 1\),

(ii) \((E_i, F_j)_{\phi} = \delta_{ij}(v_i^{-1} - v_i)^{-1}, \forall i, j \in I\),

(iii) \((K_\mu x, K'_\nu y)_{\phi} = \{\mu, \nu\}\{\mu, [y]\} x, \forall \mu, \nu \in \mathbb{Z}[I], x \in U^+, y \in U^-\).

For any homogenous elements \(x \in U^+, y \in U^-\), we have

\[(xK_\mu, yK'_\nu)_{\phi} = (x, y)_{\phi}(K_\mu, K'_\nu)_{\phi}, \forall \mu, \nu \in \mathbb{Z}[I]. \tag{2.5} \]

By Lemma 2.6(iv) and Lemma 2.7(iv), we have

\[(xK_\mu, yK'_\nu)_{\phi} = (\Delta(x)\Delta(K_\mu), y \otimes K'_\nu)_{\phi} = (xK_\mu, y)_{\phi}(K_\mu, K'_\nu)_{\phi} = (x \otimes K_\mu, \Delta^\mu(y))_{\phi}(K_\mu, K'_\nu)_{\phi} = (x, y)_{\phi}(K_\mu, K'_\nu)_{\phi}. \]

This proves \((2.5)\).

**Lemma 2.9.** For all \(x \in U^+\) and \(y \in U^-\), we have

(i) \((x, F_iy)_{\phi} = (v_i^{-1} - v_i)^{-1}(i^{+}x, y)_{\phi},\)

(ii) \((x, yF_i)_{\phi} = (v_i^{-1} - v_i)^{-1}(r_i^{+}x, y)_{\phi},\)

(iii) \((E_i x, y)_{\phi} = (v_i^{-1} - v_i)^{-1}(x, i^{+}y)_{\phi},\)

(iv) \((xE_i, y)_{\phi} = (v_i^{-1} - v_i)^{-1}(x, r_i^{-}y)_{\phi}.\)

**Proof.** The proofs of the four equations are similar. We shall only show (i) and leave others to readers.

By the definition of skew-Hopf pairing \((, \phi\) in [X97 Section 2.2], Lemma 2.6(iv) and Lemma 2.8(iii), we have

\[(x, F_i y)_{\phi} = (\Delta(x), F_i \otimes y)_{\phi} = (E_i K_{[x]^{-i}} \otimes i^{+}x, F_i \otimes y)_{\phi} = (E_i K_{[x]^{-i}} F_i)(i^{+}x, y)_{\phi} = (E_i, F_i)(i^{+}x, y)_{\phi} = (v_i^{-1} - v_i)^{-1}(i^{+}x, y)_{\phi}.\]
Lemma 2.10. For all \( x, y \in \mathfrak{f} \), we have \((x^+, y^-)_\phi = (\sigma^+(x^+), \sigma^-(y^-))_\phi\).

Proof. It is straightforward to check it when \( x = \theta_i \) and \( y = \theta_j \) for some \( i, j \in I \). Assume that the lemma holds for \( x_1 \) and \( x_2 \). We shall show that it holds for \( x = x_1x_2 \).

Let \( y \in \mathfrak{f} \) and \( r(y) = \sum y_1 \otimes y_2 \) with \( y_1, y_2 \) homogeneous. Then we have

\[
\Delta(y^-) = \sum v^{-|y_1|\cdot|y_2|} t^{\langle |y_1|, |y_2| \rangle} y_2^{-} \otimes K'_{|y_2|} y_1^{-}.
\]

By Lemma 2.2

\[
(\sigma(y))_- = \sum v^{-|y_1|\cdot|y_2|} \sigma(y_2) \otimes \sigma(y_1). \quad (2.6)
\]

By (2.1), (2.6) and Lemma 2.8(iii), we have

\[
(\sigma^+(x^+_1 x^+_2), \sigma^-(y^-))_\phi
\]

\[
-\sum v^{-|y_1|\cdot|y_2|} t^{\langle |y_1|, |y_2| \rangle} (\sigma^+(x^+_2) \otimes \sigma^+(x^+_1), \sigma^-(y^-))_\phi
\]

\[
= \sum v^{-\langle |y_1|, |y_2| \rangle} \sigma^+(x^+_2) \otimes \sigma^+(x^+_1) - \sum v^{-|y_1|\cdot|y_2|} K'_{|y_1|} \sigma^-(y^-) \otimes \sigma^-(y^-)_\phi
\]

\[
= \sum \sigma^+(x^+_2) \sigma^-(y^-)_\phi(\sigma^+(x^+_1), \sigma^-(y^-))_\phi.
\]

Similarly, we have

\[
(x^+_1 x^+_2, y^-)_\phi = (x^+_1 \otimes x^+_2, \Delta^\text{op}(y^-))_\phi
\]

\[
= \sum v^{-|y_1|\cdot|y_2|} t^{\langle |y_1|, |y_2| \rangle} (x^+_1 \otimes x^+_2, K'_{|y_2|} y_1^{-} \otimes y_2^-)_\phi
\]

\[
= \sum v^{-|y_1|\cdot|y_2|} t^{\langle |y_1|, |y_2| \rangle} (x^+_1, y_1^{-})_\phi x^+_2 \otimes y_2^-)
\]

By (2.7) and (2.8), we have

\[
(x^+_1 x^+_2, y^-)_\phi = (\sigma^+(x^+_1 x^+_2), \sigma^-(y^-))_\phi.
\]

This finishes the proof. ∎

By the relation \((R2)\) in Section 2.2, the subalgebra \( U^+ \) has the following decomposition

\[
U^+ = \bigoplus_{\mu \in \mathbb{N}[I]} U^+_\mu,
\]

where

\[
U^+_\mu = \{ u \in U^+ | K_i u = v^{i\cdot|u|} t^{\langle |u|, i \rangle - \langle i, |u| \rangle} u K_i, \ K'_i u = v^{-i\cdot|u|} t^{\langle |u|, i \rangle - \langle i, |u| \rangle} u K'_i, \forall i \in I \}.
\]

The weight space \( U^+_\mu \) is spanned by all the monomials \( E_{i_1} \cdots E_{i_t} \) with grading \( \mu \).

Similarly, the subalgebra \( U^- \) has a decomposition \( U^- = \bigoplus_{\mu \in \mathbb{N}[I]} U^-_{-\mu} \) and the spaces \( U^+_\mu \) and \( U^-_{-\mu} \) are nondegenerately paired under the skew-Hopf pairing \((\cdot, \cdot)_\phi \).
Then we may select a basis $B$ of $U^-$ such that $B_\mu = B \cap U^-$. Let $\{b^* | b \in B_\mu\}$ be the basis of $U^+\mu$ dual to $B_\mu$ under $(,)_{\phi}$.

**Lemma 2.11.** Let $x \in U^\lambda_+$ and $y \in U^-\lambda$ for any $\lambda \in \mathbb{N}[I]$. Then,

(i) $\Delta(x) = \sum_{0 \leq \mu \leq \lambda} \sum_{b \in B_\mu} (x, b'_{\phi})_{\phi} b'^* K_\mu \otimes b^*$,

(ii) $\Delta(y) = \sum_{0 \leq \mu \leq \lambda} \sum_{b \in B_\mu} (b^* b^*, y)_{\phi} b \otimes b'_{K_\mu}$.

**Proof.** The proofs of (i) and (ii) are similar. We shall only show (i).

As $x \in U^\lambda_+$, we have $\Delta(x) = \sum_{0 \leq \mu \leq \lambda} U^\lambda_\mu K_\mu \otimes U^\mu_\mu$. Let $h_{b,b'}^{\mu} \in \mathbb{Q}(v,t)$ be such that

$$\Delta(x) = \sum_{0 \leq \mu \leq \lambda} \sum_{b \in B_\mu} h_{b,b'}^{\mu} b'^* K_\mu \otimes b^*.$$

Then for all $b_1 \in B_{\lambda,\mu}$, $b_2 \in B_\mu$ and $\mu$, we have

$$(x, b_1 b_2)_{\phi} = (\Delta(x), b_1 \otimes b_2)_{\phi} = \sum_{0 \leq \mu \leq \lambda} \sum_{b \in B_\mu, b' \in B_{\lambda,\mu}} h_{b,b'}^{\mu} (b'^* K_\mu \otimes b^*, b_1 \otimes b_2)_{\phi} = \sum_{0 \leq \mu \leq \lambda} \sum_{b \in B_\mu, b' \in B_{\lambda,\mu}} h_{b,b'}^{\mu} (b'^* K_\mu, b_1)_{\phi} (b^*, b_2)_{\phi} = h_{b_2, b_1}^{\mu}.$$

This finishes the proof. \[\square\]

For each $x \in U^\mu_\mu$ and $y \in U^-\mu$, we have

$$x = \sum_{b \in B_\mu} (x, b)_{\phi} b^*, \quad y = \sum_{b \in B_\mu} (b^*, y)_{\phi} b. \quad (2.9)$$

For $\mu \in \mathbb{N}[I]$, we define

$$\Theta_\mu = \sum_{b \in B_\mu} b \otimes b^* \in U^-\mu \otimes U^\mu_\mu.$$

Set $\Theta_\mu = 0$ if $\mu \notin \mathbb{N}[I]$.

**Lemma 2.12.** For all $i \in I$, $\mu \in \mathbb{N}[I]$, we have

(i) $(K_i \otimes K_i) \Theta_\mu = \Theta_\mu (K_i \otimes K_i)$,

(ii) $(K'_i \otimes K'_i) \Theta_\mu = \Theta_\mu (K'_i \otimes K'_i)$,

(iii) $(E_i \otimes 1) \Theta_\mu + (K_i \otimes E_i) \Theta_{\mu-i} = \Theta_\mu (E_i \otimes 1) + \Theta_{\mu-i} (K'_i \otimes E_i)$,

(iv) $(1 \otimes F_i) \Theta_\mu + (F_i \otimes K'_i) \Theta_{\mu-i} = \Theta_\mu (1 \otimes F_i) + \Theta_{\mu-i} (F_i \otimes K'_i)$.

**Proof.** The first two are easy to check. We shall show (iii) and (iv) can be shown similarly. The proof of (iii) goes in a similar way as that for Lemma 4.10 in [BW]. For the convenience of the readers, we present it here. By Lemma 2.7(iii), Lemma
Lemma 2.15. It satisfies that (2.9(iii)-(iv) and (2.9), we have
\[(E_i \otimes 1)\Theta = (\Theta \otimes 1)(E_i \otimes 1)\]
\[= \sum_{b \in B_\mu} (E_ib - bE_i) \otimes b^*\]
\[= (v_i - v_i^{-1})^{-1} \sum_{b \in B_\mu} (K_i(\iota r^-(b)) - r_i^{-1}(b)K_i') \otimes b^*\]
\[= (v_i - v_i^{-1})^{-1} \sum_{b \in B_\mu} (\sum_{b' \in B_{\mu^{-1}}} (b', \iota r^-(b))_\phi b' \otimes b^* - \sum_{b' \in B_{\mu^{-1}}} (b', r_i^{-1}(b))_\phi b'K_i' \otimes b^*)\]
\[= - \sum_{b \in B_\mu} K_i \sum_{b' \in B_{\mu^{-1}}} (Eib', b)_\phi b' \otimes b^* + \sum_{b' \in B_{\mu^{-1}}} (b'E_i, b)_\phi b'K_i' \otimes b^*\]
\[= \sum_{b' \in B_{\mu^{-1}}} b'K_i' \otimes \sum_{b \in B_\mu} (b'E_i, b)_\phi b^* - \sum_{b' \in B_{\mu^{-1}}} K_i b' \otimes \sum_{b \in B_\mu} (Eib', b)_\phi b^*\]
\[= \Theta_{\mu^{-1}}(K_i' \otimes E_i) - (K_i \otimes E_i)\Theta_{\mu^{-1}}.\]

This finishes the proof. \(\Box\)

Proposition 2.13. Let \(\Theta_0 = 1 \otimes 1\) and \(\Theta = \sum_{\nu \in [I]} \Theta_\nu \in (U \otimes U)^\wedge\). Then we have
\[\Delta(u)\Theta = \Theta\Delta(u)\] for all \(u \in U\) (where this identity is in \((U \otimes U)^\wedge\)).

This proposition follows from Lemma 2.12. The element \(\Theta\) defined in this proposition is called the quasi-\(R\)-matrix.

Corollary 2.14. We have \(\Theta\wedge = \Theta\otimes = 1 \otimes 1\) (equality in \((U \otimes U)^\wedge\)).

The proof is similar to those for Corollary 4.1.3 in [L].

We define \((,)_\wedge : U^+ \times U^- \to \mathbb{Q}(v, t)\) by
\[(x, y)_\wedge = \langle \phi (x), \overline{y} \rangle, \quad \forall x \in U^+, y \in U^-\] (2.10)

It satisfies that \((1, 1)_\wedge = 1\) and \((E_i, F_j)_\wedge = \delta_{ij} (v_i - v_i^{-1})^{-1}\).

Lemma 2.15. \((x^+, y^-)_\wedge = (-1)^{tr[x^+]v^-|y^-|/2}v_-(x^+, \sigma^-(y^-))_\wedge, \forall x, y \in \mathcal{I}\).

Proof. It is straightforward to check it when \(x = \theta_i\) and \(y = \theta_j\) for some \(i, j \in I\).

Let \(x \in \mathcal{I}\) and \(r(x) = \sum x_1 \otimes x_2\) with \(x_1, x_2\) homogeneous. Assume that the lemma holds for \(y_1\) and \(y_2\). We shall show that it holds for \(y = y_1y_2\).

By Lemma 2.3 we have \(\Delta(x^+) = \sum_{v} v|x_1^1| x_2 f(|x_2|, |x_1|, (|x_2|, |x_1|) x_2^1 K|x_1| \otimes x_2 = 1|x_2^1 K|x_1| \otimes x_1^1.\) Then,
\[(x^+, y_1^1 y_2^-)_\wedge = \langle \Delta(x^+), y_1^1 \otimes y_2^- \rangle_\wedge\]
\[= \sum_{v} v|x_1^1| x_2 f(|x_2|, |x_1|, (|x_2|, |x_1|) x_2^1 K|x_1|, y_1^1 \rangle_\wedge (x_2^1, y_2^-)_\wedge (x_1^1, y_2^-)_\wedge\]
\[= \sum_{v} v|x_1^1| x_2 f(|x_2|, |x_1|, (|x_2|, |x_1|) x_2^1 K|x_1|, y_1^1 \rangle_\wedge (x_2^1, y_1^1) \wedge (x_1^1, y_2^-)_\wedge.\]
By (2.10), we have
\[
(x^+, y_1 y_2^-) = \sum v^{-|x_1||x_2|} \nu^{-|x_1| |x_2|} (x^+_1, y_1^-) \bar{\nu}(x^+_1, y_2^-) \bar{\nu}.
\]
\[
= \sum (-1)^{tr(|x_1| + |x_2|)} v^{-|x_1| |x_2| + |y_1| |y_2|} v^{-|x_1| + |x_2|} v^{-|y_1| + |y_2|} (x^+_1, \sigma^-(y_2^-)) \phi(x^+_2, \sigma^-(y_1^-)) \phi.
\]
(2.11)

On the other hand,
\[
(x^+, \sigma^-(y_1^- y_2^-)) = t^{\nu} \nu^{-|y_1| |y_2|} (\Delta(x^+), \sigma^-(y_2^-) \sigma^-(y_1^-)) \phi.
\]
\[
= t^{\nu} \nu^{-|y_1| |y_2|} \sum (x^+_1 K_{|x_2|} \otimes x^+_2, \sigma^-(y_2^-) \otimes \sigma^-(y_1^-)) \phi
\]
\[
= t^{\nu} \nu^{-|y_1| |y_2|} \sum (x^+_1, \sigma^-(y_2^-)) \phi \sum (x^+_2, \sigma^-(y_1^-)) \phi.
\]
(2.12)

This lemma follows from (2.11) and (2.12) with \(|x_1| = |y_2|\) and \(|x_2| = |y_1|\). □

While \(\bar{\Theta}\) can be evaluated easily, it will be more convenient to have the following alternate description of \(\bar{\Theta}\) using the property of \((,)_\bar{\nu}\).

**Lemma 2.16.** With the same notations as in Proposition 2.13, \(\bar{\Theta} = \sum_{\nu} \bar{\Theta}_\nu\) is given by
\[
\bar{\Theta}_\nu = (-1)^{tr \nu} \nu^{-|x_1||x_2|} \sum b \otimes \sigma^+(b^*) \in U_{|-\nu} \otimes U_{\nu}^+.
\]

**Proof.** Since \(\Theta\) is independent of the choice of basis, we let
\[
\Theta_\nu = \sum_{b \in B_\nu} \bar{b} \otimes \bar{b}^*, \quad \forall \nu \in \mathbb{N}[I],
\]
where \(\bar{b} \in \mathbb{B} = \{\bar{b}| b \in B\}\). Then
\[
\bar{\Theta}_\nu = \sum_{b \in B_\nu} \bar{b} \otimes \overline{\bar{b}^*}, \quad \forall \nu \in \mathbb{N}[I].
\]
(2.13)

Note that \(\overline{\bar{b}} = b\).

We show the relation between \(\overline{\bar{b}}\) and \(\sigma^+(b^*)\). There exists an element \(\overline{\bar{b}}' \in B_\nu\) such that \((\bar{b}' , \overline{\bar{b}}')_\phi = \delta_{b', b}\). By (2.10) and Lemma 2.15, we have
\[
\delta_{b, b'} = (\bar{b}' , \overline{\bar{b}}')_\phi = (\overline{\bar{b}} , \bar{b}')_\phi = (-1)^{tr \nu} v^{-\nu/2} v^{-\nu}_{-\nu} (\overline{\bar{b}} , \sigma^-(b'))_\phi
\]
Therefore, we have
\[
\overline{\bar{b}}' = (-1)^{tr \nu} v^\nu_{-\nu} v^{-\nu/2} v_{-\nu} (\overline{\bar{b}} , \sigma^-(b'))_\phi.
\]
(2.14)

By Lemma 2.10 we have
\[
\sigma^-(b)^* = \sigma^+(b^*).
\]
(2.15)
The Lemma follows from (2.13), (2.14) and (2.15). □
3. The $R$-matrix for two-parameter quantum algebras

3.1. The module of $U_{v,t}$. A $U_{v,t}$-module $M$ is called a weight module if it admits a decomposition $M = \bigoplus_{\lambda \in \mathbb{N}[I]} M_\lambda$ of vector spaces such that

$$M_\lambda = \{ m \in M | K_i \cdot m = v^{i,\lambda} c_{i,\lambda} m, \ K'_i \cdot m = v^{-i,\lambda} c_{i,\lambda} m, \ \forall i \in I \},$$

where

$$c_{i,\lambda} = t^{\langle \lambda, i \rangle - \langle i, \lambda \rangle}.$$

For any $m \in M_\lambda$, we denote by $|m| = \lambda$.

For all $m \in \mathbb{Q}(v,t)$, we define

$$u \cdot m = \varepsilon(u) m, \ \forall u \in U_{v,t},$$

where $\varepsilon$ is the counit of $U_{v,t}$. Then, $\mathbb{Q}(v,t)$ is a trivial module of $U_{v,t}$.

Let $M$ be $U_{v,t}$-module and $M^* = \text{Hom}_{\mathbb{Q}(v,t)}(M, \mathbb{Q}(v,t))$. We define $u \cdot n^* \in M^*$ by

$$u \cdot n^*(m) = n^*(S(u) \cdot m), \ \forall u \in U_{v,t}, n^* \in M^*, m \in M,$$

then $M^*$ is also a $U_{v,t}$-module.

For any $U_{v,t}$-modules $M$ and $N$, we can construct the $U_{v,t}$-module $M \otimes N = M \otimes_{\mathbb{Q}(v,t)} N$ via the coproduct. In particular, we have $U_{v,t}$-modules $M^* \otimes M$ and $M \otimes M^*$.

For any $i \in I, \lambda \in \mathbb{N}[I]$, we denote by

$$v_{-\lambda} = v^{-1}_\lambda, \ \text{and} \ \ c_{i,-\lambda} = c^{-1}_{i,\lambda}.$$

Lemma 3.1. Fix a $U_{v,t}$-module $M$.

1. Let $\text{ev} : M^* \otimes M \to \mathbb{Q}(v,t)$ be the $\mathbb{Q}(v,t)$-linear map defined by

$$m^* \otimes n \mapsto m^*(n), \ \forall m^* \in M^*, n \in M.$$

Then $\text{ev}$ is a $U_{v,t}$-module epimorphism.

2. Let $\text{qtr} : M \otimes M^* \to \mathbb{Q}(v,t)$ be the $\mathbb{Q}(v,t)$-linear map defined by

$$m \otimes n^* \mapsto v_2^{|m|} n^*(m), \ \forall m \in M, n^* \in M^*.$$

Then $\text{qtr}$ is a $U_{v,t}$-module epimorphism.

3. Let $\text{coev} : \mathbb{Q}(v,t) \to M^* \otimes M$ be the $\mathbb{Q}(v,t)$-linear map defined by

$$1 \mapsto \sum_{w \in B} v_2^{|w|} w^* \otimes w$$

for some homogeneous $\mathbb{Q}(v,t)$-basis $B$ of $M$. Then $\text{coev}$ is a $U_{v,t}$-module monomorphism.

4. Let $\text{coqtr} : \mathbb{Q}(v,t) \to M \otimes M^*$ be the $\mathbb{Q}(v,t)$-linear map defined by

$$1 \mapsto \sum_{w \in B} w \otimes w^*$$

for some homogeneous $\mathbb{Q}(v,t)$-basis $B$ of $M$. Then $\text{coqtr}$ is a $U_{v,t}$-module monomorphism.
Proof. The surjectivity and injectivity of the maps are clear. We shall show that these maps preserve the action of generators of $U_{e,t}$. It is straightforward to verify these maps holds for $K_1$ and $K'_1$. It’s enough to check that these maps preserve the action of $E_i$ and $F_i$ for all $i \in I$. Show that

\[ \text{ev}(\Delta(E_i)m^* \otimes n) = \text{ev}(\Delta(F_i)m^* \otimes n) = 0, \quad \forall m^* \in M^*, n \in M, \quad (3.2) \]

\[ \text{qtr}(\Delta(E_i)m \otimes n^*) = \text{qtr}(\Delta(F_i)m \otimes n^*) = 0, \quad \forall m \in M, n^* \in M^*, \quad (3.3) \]

\[ \Delta(E_i) \sum_{w \in B} v^2_{|w|} w^* \otimes w = \Delta(F_i) \sum_{w \in B} v^2_{|w|} w^* \otimes w = 0, \quad (3.4) \]

\[ \Delta(E_i) \sum_{w \in B} w \otimes w^* = \Delta(F_i) \sum_{w \in B} w \otimes w^* = 0. \quad (3.5) \]

We shall prove (3.3) and (3.4) for the action of $E_i$, the remaining cases can be shown similarly.

First, we show $\text{qtr}(\Delta(E_i)m \otimes n^*) = 0$. For homogenous elements $m \in M, n^* \in M^*$, we have

\[ \text{qtr}(\Delta(E_i)m \otimes n^*) = \text{qtr}(E_i \cdot m \otimes n^* + K_i \cdot m \otimes E_i \cdot n^*) \]

\[ = v^2_{[m]}n^*(E_i \cdot m) + v^2_{[m]}(E_i \cdot n^*)(K_i \cdot m) \]

\[ = v^2_{[m]}n^*(E_i \cdot m) + v^2_{[m]}n^*(-K_i^{-1}E_iK_i \cdot m) \]

\[ = v^2_{[m]}n^*(E_i \cdot m) - v^2_{[m]}n^*(E_i \cdot m) = 0. \]

Next, we show that $\Delta(E_i) \sum_{w \in B} v^2_{|w|} w^* \otimes w = 0$. Observe that $x = \sum m^* \otimes n = 0$ if and only if $x(m') := \sum m^*(m')n = 0$ for all $m' \in M$. Let $x = \Delta(E_i) \sum_{w \in B} v^2_{|w|} w^* \otimes w$. Then we have

\[ x(m) = \sum_{w \in B} v^2_{|w|}(E_i \cdot w^*(m)w + (K_i \cdot w^*)(m)E_i \cdot w), \quad \forall m \in M. \]

For any $w_0 \in B$, we have

\[ x(w_0) = \sum_{w \in B} v^2_{|w|}(-K_i^{-1}E_i \cdot w_0)w + w^*(K_i^{-1} \cdot w_0)E_i \cdot w) \]

\[ = -\sum_{w \in B} v^2_{|w|}v^{-i(|w|+i)}c_i{|w|+i}^{-1}w^*(E_i \cdot w_0)w \]

\[ + \sum_{w \in B} v^2_{|w|}v^{-i|w|}c_i{|w|}^{-1}w^*(w_0)E_i \cdot w \]

\[ = -v^2_{|w_0|+i}v^{-i(|w_0|+i)}c_i{|w_0|}^{-1}E_i \cdot w_0 + v^2_{|w_0|}v^{-i|w_0|}c_i{|w_0|}^{-1}E_i \cdot w_0 \]

\[ = 0. \]

Hence, $x = 0$.

This finishes the proof. \qed
3.2. The $\mathcal{R}$-matrix of $U_{v,t}$-module. We shall construct a $U_{v,t}$-module isomorphism $\mathcal{R}_{M,M'} : M \otimes M' \to M' \otimes M$ for any finite dimensional weight modules $M$ and $M'$, by the method used by Jantzen [Jan, Chap. 7] for the quantum algebras $U_q(\mathfrak{g})$.

The map $\mathcal{R}_{M,M'}$ is the composite of three linear transformations $P, \tilde{f}, \Theta$ defined as follows.

Let $P : M \otimes M' \to M' \otimes M$ be the $\mathbb{Q}(v,t)$-linear bijection defined by

$$P(m \otimes m') = m' \otimes m, \quad \forall m \in M, m' \in M'.$$

Recall that $(, \phi)$ is a skew-Hopf pairing defined in Lemma 2.8. For any $\lambda, \mu \in \mathbb{Z}[I]$, we define the map $f : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Q}(v,t)^\times$ by

$$f(\lambda, \mu) = (K_\lambda, K'_\mu)_\phi^{-1}. \quad (3.6)$$

Then we have

$$
\begin{align*}
    f(\lambda + \mu, \nu) &= f(\lambda, \nu)f(\mu, \nu) \\
    f(\lambda, \mu + \nu) &= f(\lambda, \mu)f(\lambda, \nu) \\
    f(\lambda, \mu) &= f(-\lambda, -\mu) \\
    f(i, \mu) &= v^{-i}\mu_{ci,i} \\
    f(\lambda, i) &= v^{-i}\lambda_{ci,i},
\end{align*}

(3.7)
$$

where $\nu$ is also in $\mathbb{Z}[I]$.

We define a bijective linear map $\tilde{f} : M \otimes M' \to M \otimes M'$ by

$$\tilde{f}(m \otimes m') = f(\lambda, \mu)m \otimes m', \quad \forall m \in M_\lambda, m' \in M'_\mu, \lambda, \mu \in \mathbb{Z}[I]. \quad (3.8)$$

Recall the definition of $\Theta$ from Proposition 2.13. The linear transformation $\Theta = \Theta_{M,M'} : M \otimes M' \to M \otimes M'$ is well-defined.

**Proposition 3.2.** For any finite dimensional weight module $M, M'$ of $U_{v,t}$, we define $\mathcal{R} = \mathcal{R}_{M,M'} : M \otimes M' \to M' \otimes M$ by $\mathcal{R} = \Theta \circ \tilde{f} \circ P$. Then $\mathcal{R}$ is a $U_{v,t}$-module isomorphism.

**Proof.** By Corollary 2.14, $\mathcal{R}$ is invertible transformation. Then, we shall show that $\mathcal{R}$ is a $U_{v,t}$-module homomorphism, i.e.,

$$\Delta(u)\mathcal{R}(m \otimes m') = \mathcal{R}(\Delta(u)m \otimes m'), \quad \forall u \in U_{v,t}, m \in M, m' \in M'.$$

By Proposition 2.13 we have

$$\Delta(u)\mathcal{R}(m \otimes m') = \Theta\Delta(u)\tilde{f} \circ P(m \otimes m') = \Theta(f(|m'|, |m|)\Delta(u)(m \otimes m)).$$

So it is suffices to show

$$\tilde{f} \circ P(\Delta(u)m \otimes m')) = f(|m'|, |m|)\Delta(u)(m' \otimes m)$$

for all $u \in U_{v,t}$. Hence it is enough to show that this equality holds all generators of $U_{v,t}$. For $u = K_\nu, K'_\nu$, this is straightforward. The cases $u = E_i$ and $u = F_i$ are similar, so we shall prove the first case.
By (2.4) and (3.7), we have
\[
\tilde{f} \circ P(\Delta(E_i) m \otimes m')
\]
\[= f(|m'|, i + |m|) m' \otimes E_i m + f(i + |m'|, |m|) E_i m' \otimes K_i m
\]
\[= f(|m'|, |m|) e^{-i |m'| c_i |m'} \otimes E_i m + f(|m'|, |m|) E_i m' \otimes m
\]
\[= f(|m'|, |m|) (K_i m' \otimes E_i m + E_i m' \otimes m)
\]
\[= f(|m'|, |m|) \sum (E_i)(m' \otimes m).
\]
This finishes the proof. \qed

For any finite dimension weight $U_{v,t}$-module $M_1, M_2, M_3$, we have maps $\mathcal{R}_{12}, \mathcal{R}_{23} : M_1 \otimes M_2 \otimes M_3 \rightarrow M_2 \otimes M_1$ defined as $\mathcal{R} \otimes Id$ and $Id \otimes \mathcal{R}$, respectively. We shall now verify that $\mathcal{R}$ satisfy the quantum Yang-Baxter equation
\[\mathcal{R}_{12} \circ \mathcal{R}_{23} \circ \mathcal{R}_{12} = \mathcal{R}_{23} \circ \mathcal{R}_{12} \circ \mathcal{R}_{23}.
\]

We will need the following lemma.

For $1 \leq s, l \leq 3$, we define $\tilde{f}_{sl}$ on $M_1 \otimes M_2 \otimes M_3$ via $\tilde{f}_{sl}(m_1 \otimes m_2 \otimes m_3) = f(|m_s|, |m_l|) m_1 \otimes m_2 \otimes m_3$. Let $\Theta^{op} = \sum_{\mu} \sum_{b \in B_{\mu}} b^* \otimes b, \Theta_{12} = \sum_{\mu} \sum_{b \in B_{\mu}} b \otimes b^* \otimes 1$, and we define the other expressions in a similar way. Letting $\Theta_{sl}^f = \Theta_{sl} \circ \tilde{f}_{sl}$, we have the following identities for operators on $M_1 \otimes M_2 \otimes M_3$.

**Lemma 3.3.** (i) $(\Delta \otimes 1)(\Theta^{op}) \circ \tilde{f}_{31} \circ \tilde{f}_{32} = \Theta_{31}^f \circ \Theta_{32}^f$.
(ii) $\tilde{f}_{31} \circ \tilde{f}_{32} \circ \Theta_{12} = \Theta_{12} \circ \tilde{f}_{31} \circ \tilde{f}_{32}$.

**Proof.** We shall give a detailed proof of (i). For any $m_1 \in M_1, m_2 \in M_2, m_3 \in M_3$, by Lemma 2.11 we have
\[
(\Delta \otimes 1)(\Theta^{op}) \circ \tilde{f}_{31} \circ \tilde{f}_{32}(m_1 \otimes m_2 \otimes m_3)
\]
\[= f(|m_3|, |m_1|) f(|m_3|, |m_2|) (\Delta \otimes 1)(\sum_{\mu} \sum_{b \in B_{\mu}} b^* \otimes b)(m_1 \otimes m_2 \otimes m_3)
\]
\[= f(|m_3|, |m_1|) f(|m_3|, |m_2|) (\sum_{\mu, b_1 \in B_{\mu}} \sum_{0 \leq \lambda \leq \mu, b \in B_{\lambda}} (b^*_{\mu} b_{\mu} b_2 K_{\lambda} \otimes b_1) (m_1 \otimes m_2 \otimes m_3)
\]
\[= f(|m_3|, |m_1|) f(|m_3|, |m_2|) (\sum_{\mu} \sum_{0 \leq \lambda \leq \mu, b \in B_{\lambda}} b_2 K_{\lambda} m_1 \otimes b_1 m_2 \otimes b_2 m_3.
\]

On the other hand,
\[
\Theta_{31}^f \circ \Theta_{32}^f(m_1 \otimes m_2 \otimes m_3)
\]
\[= f(|m_3|, |m_2|) \Theta_{31}^f (\sum_{\mu, b' \in B_{\mu}} m_1 \otimes b^* m_2 \otimes b' m_3)
\]
\[= f(|m_3|, |m_2|) \sum_{\nu, b' \in B_{\nu}} \sum_{\zeta, b'' \in B_{\zeta}} f(|m_3| - \nu, |m_1|) b'' m_1 \otimes b^* m_2 \otimes b'' b' m_3.
\]

By Lemma 2.8 and (3.6), we have
\[K_{\nu} \cdot m_1 = f(-\nu, |m_1|) m_1.
\]
We get the second expression by replacing the variables $\lambda$ and $\mu - \lambda$ in the first expression with $\nu$ and $\zeta$, respectively. This proves (i).

Identity (ii) follows directly from (3.7).

We thus obtain the following crucial property of $R$.

**Proposition 3.4.** For any finite dimension weight $U_{v,t}$-modules $M_1$, $M_2$, and $M_3$, we have

$$R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23} : M_1 \otimes M_2 \otimes M_3 \rightarrow M_3 \otimes M_2 \otimes M_1.$$ 

**Proof.** This proposition follows from Lemma 3.3, Proposition 3.2 and the following identities.

$$P_{12} \circ \Theta_{12}^f = \Theta_{12}^f \circ P_{12}, \quad P_{23} \circ \Theta_{23}^f = \Theta_{23}^f \circ P_{23}, \quad P_{23} \circ \Theta_{12}^f = \Theta_{13}^f \circ P_{23}, \quad P_{12} \circ \Theta_{13}^f = \Theta_{24}^f \circ P_{12}.$$ 

□

**Remark.** By specialization of $R$-matrix, we recover the one for various quantum algebras in literatures.

(1) By setting $t = 1$, the $R$-matrix in Proposition 3.2 degenerates into the one for one-parameter quantum algebras [L, Chapter 32].

(2) By setting $v = (rs^{-1})^{\frac{1}{2}}$ and $t = (rs)^{-\frac{1}{2}}$, the $R$-matrix in Proposition 3.2 coincides with the one in [BW].

4. **Knot invariants associated to the two-parameter quantum algebra $U_{v,t}$**

4.1. **Tangles.** Recall the definition of the tangle from [Kas]. Let $[0]$ be the empty set and $[n] = \{1, 2, \cdots, n\}$ for any integer $n > 0$. We denote by $J$ the closed interval $[0, 1]$ and by $\mathbb{R}^2$ the real plane.

**Definition 4.1.** [Kas, Section X.5] Let $k$ and $l$ be nonnegative integers. A tangle $L$ of type $(k, l)$ is the union of a finite number of pairwise disjoint simple oriented polygonal arcs in $X = \mathbb{R}^2 \times J$ such that the boundary $\partial L$ of $L$ satisfies the condition

$$\partial L = L \cap (\mathbb{R}^2 \times \{0, 1\}) = ([k] \times \{0\} \times \{0\}) \cup ([l] \times \{0\} \times \{1\}).$$

For a tangle $L$ of type $(k, l)$, there exists two finite sequences $s(L)$ and $b(L)$ consisting of $+$ and $-$ signs. Let $s(L) = (\varepsilon_1, \cdots, \varepsilon_k)$ and $b(L) = (\eta_1, \cdots, \eta_l)$. For $1 \leq i \leq k$, let $\varepsilon_i = +$ (resp. $\varepsilon_i = -$) if the point $(i, 0, 0)$ is an origin (resp. an endpoint) of $L$. On the contrary, for $1 \leq i \leq l$, let $\eta_i = +$ (resp. $\eta_i = -$) if the point $(i, 0, 1)$ is an endpoint (resp. an origin) of $L$.

Figure 1
From left to right, we denote the oriented tangles shown in Figure 1 by the symbols $\uparrow, \downarrow, \bowtie, \bowtie, \bowtie, X_+$ and $X_-$, respectively.

**Example** (1) For the tangles $\uparrow$ and $\downarrow$, we have

$$s(\uparrow) = (+), b(\uparrow) = (+), s(\downarrow) = (-) \text{ and } b(\downarrow) = (-).$$

(2) For the tangles $\bowtie$, we have $s(\bowtie) = (-, +), b(\bowtie) = \emptyset$.

The product $\circ$ and the tensor product $\otimes$ on the set of all isotopy types of tangles are defined as follows:

$$L \circ T = \begin{array}{c} L \\ \hline T \end{array}, \quad L \otimes T = \begin{array}{c} L \\ \hline T \end{array},$$

where $L$ and $T$ are tangles. The product $L \circ T$ is well defined when $s(L) = b(T)$.

**4.2. The strict monoidal category $(OTa, \otimes, \emptyset)$ of tangles.** Let $OTa$ be a category. The objects of $OTa$ are all finite sequences consisting of $+$ and $-$ including the empty sequence. Given two finite sequences $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ and $\nu = (\nu_1, \ldots, \nu_l)$, the oriented $(k, l)$-tangle $L$ represents a morphism from $\varepsilon$ to $\nu$ such that $s(L) = \varepsilon$ and $b(L) = \nu$. The tensor product of the objects $\varepsilon$ and $\nu$ is the object $(\varepsilon, \nu)$. The composition (resp. tensor product) of morphisms is the product (resp. tensor product) of tangles. It is clear that $(OTa, \otimes, \emptyset)$ is a strict monoidal category [Kas, Proposition XII.2.1].

In what follows, we shall omit the $\otimes$ sign between morphisms if there is no dangers of confusion.

**Theorem 4.2.** [Tur90, Theorem 3.2] The category $OTa$ is generated by the morphisms $\bowtie, \bowtie, \bowtie, \bowtie, X_+$ and $X_-$, and is presented by them together with the relations:

(i) $(\bowtie \uparrow) \circ (\uparrow \bowtie) = \uparrow = (\uparrow \bowtie) \circ (\bowtie \uparrow)$;
(ii) $(\bowtie \downarrow) \circ (\downarrow \bowtie) = \downarrow = (\downarrow \bowtie) \circ (\bowtie \downarrow)$;
(iii) $(\downarrow \downarrow \bowtie) \circ (\downarrow \uparrow \downarrow \bowtie) \circ (\downarrow \downarrow X_\pm \downarrow \uparrow \downarrow) \circ (\downarrow \bowtie \uparrow \downarrow \downarrow) \circ (\bowtie \downarrow \downarrow)$
= $(\downarrow \downarrow \downarrow) \circ (\downarrow \bowtie \uparrow \downarrow \downarrow) \circ (\downarrow \downarrow X_\pm \downarrow \uparrow \downarrow) \circ (\downarrow \downarrow \uparrow \bowtie \downarrow \downarrow) \circ (\downarrow \downarrow \bowtie)$;
(iv) $X_+ \circ X_- = X_- \circ X_+ = \uparrow \uparrow$;
(v) $(X_+ \uparrow) \circ (\uparrow X_+) \circ (X_+ \uparrow) = (\uparrow X_+) \circ (X_+ \uparrow) \circ (\uparrow X_+)$;
(vi) $(\uparrow \bowtie) \circ (\uparrow X_+) \circ (\uparrow \bowtie) = (\uparrow X_+) \circ (X_+ \uparrow) \circ (\uparrow X_+)$;
(vii) $Y \circ T = \downarrow \uparrow$, $T \circ Y = \downarrow \uparrow$,

where $Y = (\downarrow \uparrow \bowtie) \circ (\downarrow X_+ \downarrow) \circ (\bowtie \uparrow \downarrow)$ and $T = (\bowtie \uparrow \downarrow) \circ (\downarrow X_- \downarrow) \circ (\downarrow \uparrow \bowtie)$.

**4.3. The strict monoidal category $(\text{Mod}, \otimes, \mathbb{Q}(v, t))$ of $U_{v,t}$-modules.** Let $\{M(i)\}_{i \in \{+, -\}}$ be a collection of $U_{v,t}$-modules where $M(\cdot)$ is a finite dimensional weight module of the two parameter quantum algebra $U_{v,t}$ with the highest weight $\lambda$ and $M(-)$ is its
dual module. To each finite sequence \( j = (i_1, \cdots, i_n) \) with \( i_1, \cdots, i_n \in \{+, -\} \) we associate the \( U_{v,t} \)-module

\[ M(j) = (\cdots ((M(i_1) \otimes_{U_{v,t}} M(i_2)) \otimes_{U_{v,t}} M(i_3)) \otimes \cdots) \otimes_{U_{v,t}} M(i_n)). \]

For the empty sequence \( \emptyset \), we set \( M(\emptyset) = \mathbb{Q}(v,t) \).

Consider the category Mod whose objects are the pairs \( (j, M(j)) \) for all finite sequences \( j \) of elements of \( \{+, -\} \). The morphisms \( (j, M(j)) \to (j', M(j')) \) consist of all \( U_{v,t} \)-linear homomorphisms \( M(j) \to M(j') \). Composition of morphisms is the usual composition of homomorphisms. For simplicity, we denote the object \( (j, M(j)) \) by \( M(j) \). The tensor product \( \otimes \) is defined by setting \( M(j) \otimes M(j') = M(j, j') \).

4.4. Knot invariants. Recall that \text{ev}, \text{qtr}, \text{coev}, and \text{coqtr} are the maps defined in Lemma 3.1. Likewise, let \( R \) be the map defined in Proposition 3.2.

We denote the maps \( \text{id}_+, \text{id}_-, \text{ev}, \text{qtr}, \text{coev}, \text{coqtr}, R \text{ and } R^{-1} \) by the symbols \( \uparrow, \downarrow, \leftarrow, \rightarrow, \leftrightarrow, X_+ \text{ and } X_- \), respectively. Then we have some substantial diagrammatic identities.

**Lemma 4.3.** We have four equalities of diagrams

\[ \begin{array}{ccc}
\uparrow \downarrow & = & \leftarrow \\
\downarrow \uparrow & = & \rightarrow
\end{array} \]

**Proof.** The proofs of the four equalities are similar. We show the first equality in detail. In terms of morphisms, we wish to show

\[ (\text{qtr} \otimes \text{id}_+) \circ (\text{id}_+ \otimes \text{coev}) = \text{id}_+. \]

Let \( B \) be a homogeneous basis of \( M(+) \) and \( B^* \) the dual basis of \( M(-) \). Then for any \( w_0 \in B \), we have

\[ (\text{qtr} \otimes \text{id}_+) \circ (\text{id}_+ \otimes \text{coev})(w_0) = \sum_{w \in B} v_{|w|}^2 (\text{qtr} \otimes \text{id}_+)(w \otimes w^* \otimes w) \]

\[ = \sum_{w \in B} v_{|w|}^2 v_{-|w|}^2 w^*(w_0)w = w_0. \]

To distinguish \( \leftrightarrow \) from \( \leftrightarrow \), we let

\[ \mathcal{R}_{+,+} = \leftrightarrow, \quad \mathcal{R}_{+-} = \leftarrow. \]

**Lemma 4.4.** We have four equalities of diagrams

\[ \begin{array}{ccc}
(1) & = & \\
(2) & = & \\
\left\{ \begin{array}{ccc}
\end{array} \right. \end{array} \]

\[ \begin{array}{ccc}
\left\{ \begin{array}{ccc}
\end{array} \right. \end{array} \]
Proof. The proofs of (1) – (4) are similar. We shall only show (1) in detail. It is equivalent to show that (1) holds for the following equality

\[ \varphi = \psi, \]

where

\[ \varphi = \text{qtr} \circ (\text{id}_+ \otimes \text{qtr} \otimes \text{id}_-) \circ (\text{id}_+ \otimes \text{id}_+ \otimes R_{-}), \]

\[ \psi = \text{qtr} \circ (\text{id}_+ \otimes \text{qtr} \otimes \text{id}_-) \circ (R_{+} \otimes \text{id}_- \otimes \text{id}_-). \]

Let \( m_1, m_2 \in M(+) \) and \( m_3^*, m_4^* \in M(-) \). On the left hand side, by Lemma 2.5, we have

\[
\varphi(m_1 \otimes m_2 \otimes m_3^* \otimes m_4^*) \\
= \sum_{\nu} \sum_{b \in B_{\nu}} v^2_{|m_1|-|m_2|} \left| m_4^* \right| f(|m_4^*|, |m_3^*|) (b \cdot m_4^*)(m_2)(b^* \cdot m_3^*)(m_1) \\
= \sum_{\nu} \sum_{b \in B_{\nu}} v^2_{|m_1|-|m_2|} \left| m_4^* \right| f(|m_4^*|, |m_3^*|) \nu(\left| m_2 \right| - \left| m_1 \right| - \nu) c_{-\nu, |m_1| + |m_2|} m_4^*(\sigma^-(b)m_2)m_3^*(\sigma^+(b^*)m_1).
\]

On the right hand side, by Lemma 2.10, we rewrite the presentation of \( \Theta \) in the basis \( \sigma^-(B) \) such as

\[ \Theta = \sum_{\nu} \sum_{b \in B_{\nu}} \sigma^-(b) \otimes \sigma^+(b^*). \]

Then, we have

\[
\psi(m_1 \otimes m_2 \otimes m_3^* \otimes m_4^*) \\
= \sum_{\nu} \sum_{b \in B_{\nu}} v^2_{|m_1|-|m_2|} \left| m_4^* \right| f(|m_4^*|, |m_3^*|) \nu(\left| m_2 \right| - \left| m_1 \right| - \nu) c_{-\nu, |m_1| + |m_2|} m_4^*(\sigma^-(b)m_2)m_3^*(\sigma^+(b^*)m_1).
\]

It is enough to show that

\[ f(|m_2|, |m_1|) = f(|m_4^*|, |m_3^*|) \nu(\left| m_2 \right| - \left| m_1 \right| - \nu) c_{-\nu, |m_1| + |m_2|}. \]

This equality holds for \( |m_1| = -|m_3^*| - \nu \) and \( |m_2| = -|m_4^*| + \nu. \)

By this lemma, we have the following corollary.

**Corollary 4.5.** We have four equalities of diagrams

\[ = = = = \]
Lemma 4.6. We have two equalities of diagrams

\[
(f^{-1}(\lambda, \lambda)v_\lambda^2)^{-1} = f^{-1}(\lambda, \lambda)v_\lambda^2,
\]

where \( \lambda \) is the highest weight of \( M(+) \).

Proof. We denote

\[
\varphi = (\text{id}_+ \otimes \text{qtr}) \circ (R_{+,+} \otimes \text{id}_+) \circ (\text{id}_+ \otimes \text{coqtr}),
\]

\[
\psi = (\text{id}_+ \otimes \text{qtr}) \circ (R_{+,+}^{-1} \otimes \text{id}_+) \circ (\text{id}_+ \otimes \text{coqtr}).
\]

Since \( \varphi \) and \( \psi \) are \( U_{v,e} \)-module homomorphisms from \( M(+) \) to \( M(+) \), both \( \varphi \) and \( \psi \) must be a multiple of the identity which is completely determined by the image of an extremal weight vector.

Let \( m_\lambda, m_{-\lambda} \in M(+) \) be nonzero highest-weight and lowest-weight vectors. We have

\[
\varphi(m_\lambda) = (\text{id}_+ \otimes \text{qtr}) \circ (R_{+,+} \otimes \text{id}_+) \sum_{w \in B} m_\lambda \otimes w \otimes w^* = f(\lambda, \lambda)v_\lambda^2 m_\lambda.
\]

Thus \( \text{id}_+ = f^{-1}(\lambda, \lambda)v_\lambda^2 \varphi. \)

By Corollary 2.14 we have

\[
R_{+,+}^{-1} = P \circ \tilde{f}^{-1} \circ \Theta.
\]

By Lemma 2.16 we compute

\[
\psi(m_{-\lambda}) = (\text{id}_+ \otimes \text{qtr}) \circ (R_{+,+}^{-1} \otimes \text{id}_+) \sum_{w \in B} m_{-\lambda} \otimes w \otimes w^* = f^{-1}(\lambda, \lambda)v_{\lambda}^2 m_{-\lambda}.
\]
Then we have id\(_+\) = (f\(^{-1}\)(\(\lambda, \lambda\))\(v_\lambda^2\))\(^{-1}\) \(\psi\) following (3.7).

This finishes the proof. \(\Box\)

**Lemma 4.7.** We have two equalities of diagrams

\[
\begin{align*}
\text{Diagram 1} & = \text{Diagram 2} \quad (1) \\
\text{Diagram 3} & = \text{Diagram 4} \quad (2)
\end{align*}
\]

**Proof.** The proofs of (1) and (2) are similar. We shall only show (1) in detail.

Denoting \(\cong\) by \(\mathcal{R}_{+,-}\), (1) is equivalent to

\[
\mathcal{R}_{+,-} = \varphi,
\]

where \(\varphi = (\text{id}_- \otimes \text{id}_+ \otimes \text{qrt}) \circ (\text{id}_- \otimes \mathcal{R}_{+,-}^{-1} \otimes \text{id}_-) \circ (\text{coev} \otimes \text{id}_+ \otimes \text{id}_-).\)

For any \(m_1 \in M(+)\) and \(m_2^* \in M(-)\), we have

\[
\mathcal{R}_{+,-}(m_1 \otimes m_2^*) = f([m_2^*, |m_1|]) \sum_{\nu} \sum_{b \in B_\nu} b m_2^* \otimes b^* m_1. \quad (4.1)
\]

By Lemma 2.10, we have the representation of \(\mathcal{C}\) in the basis \(\sigma(B)\) such as

\[
\mathcal{C} = \sum_{\nu} (-1)^{|v_\nu|^+} v_{-\nu} \sum_{b \in B_\nu} \sigma^{-}(b) \otimes b^*.
\]

Then we compute

\[
(id_- \otimes \mathcal{R}_{+,-}^{-1} \otimes id_-) \circ (\text{coev} \otimes \text{id}_+ \otimes \text{id}_-)(m_1 \otimes m_2^*)
= \sum_{\nu} (-1)^{|v_\nu|^+} v_{-\nu} \sum_{w \in B} \sum_{b \in B} v_\nu^2 f^{-1}(|w| - \nu, |m_1| + \nu) w^* \otimes b^* m_1 \otimes \sigma^{-}(b) w \otimes m_2^*. \quad (4.2)
\]

By Lemma 2.5, we have

\[
(id_- \otimes \text{id}_+ \otimes \text{qrt})(\sum_{w \in B} v_\nu^2 f^{-1}(|w| - \nu, |m_1| + \nu) w^* \otimes b^* m_1 \otimes \sigma^{-}(b) w \otimes m_2^*)
= \sum_{w \in B} v_\nu^2 f^{-1}(|w| - \nu, |m_1| + \nu) m_2^*(\sigma(\nu) w) w^* \otimes b^* m_1
= \sum_{w \in B} (-1)^{|v_\nu|^+} v_{-\nu} v^{-\nu}|w| c_{\nu,|w|} v_\nu^2 f^{-1}(|w| - \nu, |m_1| + \nu) (bm_2^*)(w) w^* \otimes b^* m_1
= (-1)^{|v_\nu|^+} v_{-\nu} v^{-\nu}(m_2^* - \nu) c_{\nu,|m_2^*|,|m_1| + \nu} bm_2^* \otimes b^* m_1. \quad (4.3)
\]
By (3.7), (4.2) and (4.3), we have
\[ \varphi(m_1 \otimes m_2^*) = f(|m_2^*|, |m_1|) \sum \sum_{\nu \in \mathcal{B}_\nu} b_{m_2^*} \otimes b^* m_1. \]  (4.4)

By (4.1) and (4.4), we have
\[ \mathcal{R}_{+,-}(m_1 \otimes m_2^*) = \varphi(m_1 \otimes m_2^*). \]

This finishes the proof. \( \square \)

By this lemma, we have the following corollary.

**Corollary 4.8.** We have two equalities of diagrams

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1} \\
= \\
\includegraphics[width=0.2\textwidth]{diagram2}
\end{array}
\]

**Theorem 4.9.** There exists a strict tensor functor \( \mathcal{T} \) from the tangle category \((\text{OTa}, \otimes, \emptyset)\) to the \(U_{v,t}\)-module category \((\text{Mod}, \otimes, \mathbb{Q}(v,t))\) such that \( \mathcal{T}((+)) = M(+) \), \( \mathcal{T}((-)) = M(-) \), and
\[
\begin{align*}
\mathcal{T}(X_+) & = (f(\lambda, \lambda)v^2_{-\lambda})^{-1} \mathcal{R}, & \mathcal{T}(\bowtie) & = \text{coev}, & \mathcal{T}(\bowtie) & = \text{coqtr}, \\
\mathcal{T}(X_-) & = f(\lambda, \lambda)v^2_{-\lambda} \mathcal{R}^{-1}, & \mathcal{T}(\bowtie) & = \text{qtr}, & \mathcal{T}(\bowtie) & = \text{ev},
\end{align*}
\]

where \( \lambda \) is the highest weight of \( M(+) \).

**Proof.** To prove the proposition, it is enough to show that the evaluations of \( \mathcal{T} \) at both sides of the relations of Theorem 4.2 coincide.

The relations (i), (ii), (iii), (vi) and (vii) follow from Lemma 4.3, Corollary 4.5, Lemma 4.6 and Corollary 4.8, respectively. The relations (iv) and (v) hold for the properties of \( \mathcal{R} \). \( \square \)

**Remark** Given a tangle \( L \) of type \((n, n)\) for any \( n \in \mathbb{N} \), we get the closure \( \tilde{L} \) of \( L \) by connecting the origin and the endpoint one by one with no intersection. For example, see Figure 2. Furthermore, the evaluation of the functor \( \mathcal{T} \) on \( \tilde{L} \) is an endomorphism of ground field \( \mathbb{Q}(v,t) \). Therefore, \( \mathcal{T}(\tilde{L})(1) \) is a binary polynomial with the parameters \( v \) and \( t \). That is called the quantum knot invariant. We expect this is a refinement of the knot invariant associated to the one-parameter case.

\[
\begin{array}{c}
L = \\
\tilde{L} =
\end{array}
\]

**Figure 2**
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