Controlled Generation of Dark Solitons with Phase Imprinting

Biao Wu, Jie Liu, and Qian Niu
Department of Physics, The University of Texas, Austin, Texas 78712-1081
(Dated: March 22, 2022)

The generation of dark solitons in Bose-Einstein condensates with phase imprinting is studied by mapping it into the classic problem of a damped driven pendulum. We provide simple but powerful schemes of designing the phase imprint for various desired outcomes. We derive a formula for the number of dark solitons generated by a given phase step, and also obtain results which explain experimental observations.

PACS numbers: 05.45.Yv, 03.75.Fi, 42.65.Tg

Solitons have been discovered in various classical nonlinear media, such as fluid, magnetic and optical systems, and have fascinated physicists for decades for their particle-like properties. Recently, dark solitons were observed in Bose-Einstein condensates (BECs) of dilute atomic gases, which are described by a macroscopic wave function. Dark solitons are produced by engineering the phase of this wave function with a technique known as phase imprinting, which was originally proposed and used to create vortices. Phase imprinting is to shine an off-resonance laser on a BEC thus create phase steps between different parts of the BEC. As a new tool of manipulating a wave function, its power and ability have not been studied in a systematic manner.

In this Letter we present a thorough analysis of the generation of dark solitons with phase imprinting in BECs. Our study is facilitated by a novel approach, which maps the soliton generation problem into a damped driven pendulum problem. This method makes it easier to find the dark solitons generated by a given phase step. More importantly, it changes our perspective on the problem of soliton generation. With this method we can show how to design and control the phase steps for various desired outcomes, such as a specific dark soliton and a specified number of solitons. We derive a formula, which relates the winding number of the pendulum motion to the number of dark solitons generated by a given phase step. In addition, we study the interesting phenomenon that counter-propagating dark solitons may be generated by one phase step, as observed in Ref. 3, and the physics behind it. Although our study is done in the context of a BEC, it can easily be applied to fiber optics, where dark solitons have potential applications in communication.

We study the phase imprinting on a quasi-one dimensional BEC, which is realizable experimentally and was indeed used to produce dark solitons in BECs. On the other hand, although the magnetic trap is important for the dynamics of BEC dark solitons, it has negligible effect on the generation of dark solitons, which has a much shorter time scale than the subsequent dynamics. Therefore, it is sufficient to neglect the trap and use the one dimensional nonlinear Schrödinger equation

\[ i \frac{\partial \psi(x,t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi(x,t)}{\partial x^2} + u_0^2 |\psi(x,t)|^2 \psi(x,t), \]  

where \( x \) is measured in units of \( \xi = 1 \mu m \), a typical length unit in this type of experiments, \( t \) in units of \( \frac{mc^2}{\hbar} \) (\( m \) is the atomic mass), \( \psi \) in units of the square root of \( n_0 \), the average density the condensate, and the speed of sound is \( u_0 = \sqrt{4\pi n_0 a_s \hbar^2} \), with \( a_s > 0 \) being the interatomic scattering length. For the experiment with rubidium, we have \( u_0 \approx 5.5 \) and the time of evolution around 10; for sodium experiment, we have \( u_0 \approx 1 \) and the time of evolution around 30.

A dark soliton is characterized by a local density minimum moving with constant speed against a uniform background. It has three characteristics, the depth of its density minimum, the phase step over its density notch, and its velocity. However, all the three are related to each other and can be specified by its velocity. The nonlinear Schrödinger equation is exactly solvable with the inverse scattering method, according to which the generation of dark solitons is determined by the Zakharov-Shabat eigenvalue equations

\[ i \frac{\partial U_1(x)}{\partial x} + u_0 \psi(x,0) U_2(x) = \lambda U_1(x), \]  
\[ i \frac{\partial U_2(x)}{\partial x} - u_0 \psi^*(x,0) U_1(x) = -\lambda U_2(x), \]

where \( \psi(x,0) \) is the initial condition. The ZS equations can have discrete eigenvalues \( \lambda_i \) with magnitude smaller than \( u_0 \). Corresponding to each \( \lambda_i \), a dark soliton with velocity \( -\lambda_i \) is generated. For phase imprinting, we have \( \psi(x,0) = e^{iS(x)} \), where \( S(x) \) is the imprinted phase. In this letter, for simplicity, we will concentrate on the right phase step, which increases monotonically from the left to the right, and stays constant at the boundaries,

\[ \frac{dS(x)}{dx} \geq 0, \quad \frac{dS(x)}{dx} = 0 \quad \text{as} \quad |x| \rightarrow \infty. \]
It is straightforward to generalize our method and results to the left phase step and more general phase imprinting, or even to the density engineering.

We solve the ZS equations by mapping them into a simple pendulum problem, which is physically more intuitive and mathematically much simpler. Note that, for discrete eigenvalues $\lambda_i$, the ZS wave functions $U_1 = \sqrt{\rho_1} e^{i\varphi_1}$ and $U_2 = \sqrt{\rho_2} e^{i\varphi_2}$ have the boundary conditions

$$\rho_1, \rho_2 \to 0, \quad \varphi_1, \varphi_2 \to \text{constants}, \quad |x| \to \infty.$$  

As is well known, the quantity $|U_1|^2 - |U_2|^2$ is conserved and independent of $x$, which, combined with Eq.(4), leads to the conclusion that $|U_1|^2 - |U_2|^2 = \rho$ for a discrete eigenvalue $\lambda_i$. In light of this, we make the following transformation

$$U_1 = i \sqrt{\rho} e^{i(\theta - \varphi + S)/2}, \quad U_2 = \sqrt{\rho} e^{i(\theta + \varphi - S)/2},$$  

which turns the ZS equations into a pair of very simple equations

$$\dot{\varphi} = \frac{d\varphi}{dx} = 2\lambda + \dot{S} - 2u_0 \sin \varphi, \quad (7)$$  

$$\ddot{\rho} = \frac{d\rho}{dx} = 2u_0 \rho \cos \varphi, \quad (8)$$

with $\dot{\theta} = 0$, where the overhead dot denotes the spatial derivative. Remarkably, equation (7) involves only $\varphi$, and can be viewed as a damped massless pendulum driven by the force $2\lambda + \dot{S}$ if we regard $x$ as time. This pendulum has two fixed points, $P_s$ and $P_u$, when $\dot{S} = 0$ and $|\lambda| < u_0$. The point $P_s$ is at $\varphi_0 = \sin^{-1} \frac{\lambda}{u_0}$ and is stable; the other one $P_u$ is at $\pi - \varphi_0$ and is unstable, as shown in Fig.1B.

The solutions of Eq.(7) transformed from the ZS eigenfunctions for a discrete eigenvalue always start at $P_s$ and end at $P_u$. This can be checked by noticing that the boundary conditions (4) become $\rho \to 0$ and $\dot{\varphi} \to 0$, and the discrete eigenvalues have magnitude smaller than $u_0$.

The correspondence between the pendulum solutions going from $P_s$ to $P_u$ and dark solitons can be appreciated in context of the pendulum problem (8) itself without referencing to the ZS equations. With a given phase step, the pendulum equation (8) has a solution starting at the stable fixed point $P_s$ for each $|\lambda| < u_0$, as shown in Fig.1B. For future convenience, we name this kind of pendulum solutions proto-soliton solutions. Because of the asymptotic behavior, $\dot{S} = 0$ at $|x| \to \infty$, the proto-soliton solution can only end up at either $P_s$ or $P_u$. Since $P_s$ is the stable fixed point, for most values of $\lambda$ the pendulum comes back to $P_s$ after several rounds of rotation (Fig.1B). Only for finite number of $\lambda$’s the pendulum will end up at the unstable fixed point $P_u$ (Fig.1A). When this happens, we say the proto-soliton solutions become the soliton solutions. This set of special $\lambda$’s are just the discrete eigenvalues $\lambda_i$ of the ZS equations while the soliton solutions correspond to the ZS eigenfunctions.

This novel approach has tremendous advantages over the existing methods for the study of ZS equations [9]. With the above analysis it is clear that we can discard Eq.(8) and focus only on the pendulum equation (8), which is much simpler than the ZS equations. As an example, we solve Eq.(7) for the case of the “sudden” limit in which the phase imprinted is a step function, $S(x) = \alpha \Theta(x)$. In this case, the force is a $\delta$ function, $\dot{S} = \alpha \delta(x)$. Integrating Eq.(7), we have $\varphi(0_+) - \varphi(0_-) = (\pi - \varphi_0) - \varphi_0 = \alpha$ which gives us $\pi - 2\varphi_0 = \alpha$. So only one soliton is generated, and its speed is $\lambda = u_0 \sin \varphi_0 = u_0 \cos(\alpha/2)$. This recovers the result in Ref.[1].

More importantly, this method changes our perspective on the problem of generating dark solitons. It allows
us ask and answer the inverse question, “what phase step is needed to produce a specified dark soliton?” This is achieved with the following steps:

1) pick $\lambda$, the soliton that one wants to create;

2) choose a curve, $\varphi = f(\varphi)$, connecting the pair of fixed points $P_s$ and $P_u$ corresponding to $\lambda$ in Fig. 4;

3) substitute $\dot{\varphi} = f(\varphi)$ into Eq. (9), solve it for $S(x)$.

The obtained phase step $S(x)$ creates the soliton $\lambda$. Note that any curve lying entirely in the upper half of the plane correspond to right phase steps, $\dot{S} > 0$.

Obviously, there are infinite number of paths connecting a pair of $P_s$ and $P_u$, thus there are infinite number of phase steps that generate a certain dark soliton. We seek one best step in terms of expenditure of energy. Imprinting a phase step $S(x)$ injects energy into the system,

$$E = \int_{-\infty}^{\infty} dx \frac{S^2}{2} = \int_{0}^{\pi} d\varphi \frac{\dot{S}^2}{4\lambda - 4u_0 \sin \varphi + 2\dot{S}}. \quad (9)$$

The minimum value of this energy corresponds to the smallest disturbance to the system by phase imprinting or also the smallest amount of noise in the output. Viewing $E$ as a functional of $\dot{S}$, using variational analysis, we have the phase step of the least disturbance,

$$S(x) = 4 \tan^{-1} \left( \frac{u_0 - \lambda}{u_0^2 - \lambda^2} \tanh(x \sqrt{u_0^2 - \lambda^2}) \right). \quad (10)$$

This phase step generates a soliton of velocity $-\lambda$, and possibly other solitons. Curve 1 in Fig. 2 is one example of this kind.

One interesting aspect of the phase step (10) is that the step height increases as $\lambda$ gets smaller when $\lambda$ is positive. This agrees with the experimental results that the soliton speed decreases when the step height increases \[\frac{3}{2}.\]

More interesting is when $\lambda$ becomes negative. For $\lambda < 0$, our numerical study shows that the step (10) usually generates more than one solitons with one traveling to the right with speed $|\lambda|$ and the rest to the left. That implies that counter-propagating solitons can be generated by one phase step, as observed experimentally \[\frac{3}{2}.\] This phenomenon is certainly not special to the phase step \[\frac{3}{2},\] it happens to any phase step obtained from a path connecting a pair of the fixed points of a negative $\lambda$ in the upper half plane of Fig. 2. However, this phenomenon is rather mysterious and against physical intuition.

Intuitively, once a right phase step (12) is imprinted on a BEC cloud, atoms in the imprinted areas will start moving to the right. Since the atoms outside of the imprinted area do not move, a dip with a bump to its right will appear as a result (see Fig. 3). Due to the stronger repulsive interaction from the bump, the dip will be pushed to the left. As dark solitons come into form in the dip, one would expect that the dark solitons generated by a right phase step would always move to the left. Our numerical simulation shows that this intuitive picture is only correct at the beginning of time evolution. But after a very short time, as shown in Fig. 3, one soliton in its early forming stage may change its direction and start moving to the right. More interestingly, the phase difference $\Delta S = S(+\infty) - S(-\infty)$ is reduced by $2\pi$ when this happens, as seen in Fig. 3C. This means that the initially imprinted the phase difference $\Delta S$ can not always be maintained in the course of time evolution [12].

We are also able to design a phase step which generates a specified number of solitons. This is based upon the formula for the number of solitons generated by a given phase step. To derive the formula, we need to examine the winding number $W(\lambda)$ of the proto-soliton solution, which is defined as the number of rounds of rotation that the pendulum makes in the solution. For a given phase step $S(x)$, we have $W(\lambda) \leq W(\lambda')$ if $\lambda < \lambda'$, and $W(\lambda') - W(\lambda) = 1$ when there is only one eigenvalue $\lambda_1$ between $\lambda$ and $\lambda'$. Therefore, the number of soliton generated by $S(x)$ is $N_s = W(u_0) - W(-u_0)$, and the number of solitons traveling to the right is $N_r = W(0) - W(-u_0)$. As an example we consider a special but still quite general case, a phase step which can be cut into pieces so that $\dot{S}$ can be considered as
constant within each piece. Finding the winding number for each piece and adding them together, we have

\[ N_s \approx \text{INT} \left( \int_{-\infty}^{\infty} \frac{dx}{\pi} \sqrt{\frac{S^2}{4} + u_0 S} \right) - \text{INT} \left( \int_{+\infty}^{\infty} \frac{dx}{\pi} \sqrt{\frac{S^2}{4} - u_0 S} \right) + 1, \]  

where the second integral is done over the intervals on which \( S > 4u_0 \), and INT is the integer function which keeps only the integer part of a real number.

To imprint a phase which generates exactly \( n \) solitons, we only need to find a phase step that has \( W(u_0) = n \) and \( W(-u_0) = 0 \). This is achieved by drawing a path connecting \( \varphi = \pi/2 \) and \( \varphi = 2n\pi + \pi/2 \), the fixed point of the pendulum driven by a constant force \( 2u_0 \). At the same time we make sure it lies under the curve, \( \dot{\varphi} = 6u_0 - 2u_0 \sin \varphi \), the darkened dashed line in Fig. 2. The phase step obtained from this path certainly has \( W(u_0) = n \). Also this phase step has \( W(-u_0) = 0 \). To see this, note that the phase step satisfies \( 0 \leq \dot{S} \leq 4u_0 \). Therefore, we always have \( \dot{\varphi} \leq 0 \) at \( \varphi = \pi/2 \) for the case \( \lambda = -u_0 \) thus the pendulum can never pass the position \( \pi/2 \) and make a full round of rotation. The simplest example is curve 3 in Fig. 2, which represent a linear phase step generating exactly two solitons.

For the linear step, Eq. (11) is exact, and it is used to compute the number of solitons generated for different step heights \( \alpha \) and different step widths \( a \), as shown in Fig. 3A. Also for the simple case, Eq. (11) can be solved analytically, and the eigenvalues \( \lambda_i \) are given by

\[ \lambda^2 - u_0^2 + \frac{\alpha}{2a} = \sqrt{\left(\lambda + \frac{\alpha}{2a}\right)^2 - u_0^2} \tan(a\sqrt{\left(\lambda + \frac{\alpha}{2a}\right)^2 - u_0^2}). \]

By solving this equation, we obtain the soliton velocities, and find the number of solitons traveling to the right, which is plotted in Fig. 3B. As a whole, Fig. 3 serves as a reference table, where experimentalists can find the right parameters, the step height and step width, to generate desired number of solitons.

We thank Roberto Diener for helpful discussions. This work is supported by the NSF, the Robert A. Welch Foundation, and the NSF of China.

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