On harmonic quasiconformal quasi-isometries

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Abstract. The purpose of this paper is to explore conditions which guarantee Lipschitz-continuity of harmonic maps w.r.t. quasihyperbolic metrics. For instance, we prove that harmonic quasiconformal maps are Lipschitz w.r.t. quasihyperbolic metrics.

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1 Introduction

Let $G \subset \mathbb{R}^2$ be a domain and let $f : G \to \mathbb{R}^2, f = (f_1, f_2)$, be a harmonic mapping. This means that $f$ is a map from $G$ into $\mathbb{R}^2$ and both $f_1$ and $f_2$ are harmonic functions, i.e. solutions of the two-dimensional Laplace equation

$$\Delta u = 0.$$ (1.1)

The Cauchy-Riemann equations, which characterize analytic functions, no longer hold for harmonic mappings and therefore these mappings are not analytic. Intensive studies during the past two decades show that much of the classical function theory can be generalized to harmonic mappings (see the recent book of Duren [D] and the survey of Bshouty and Hengartner [BH]). The purpose of this paper is to continue the study of the subclass of quasiconformal and harmonic mappings, introduced by Martio in [Ma] and further studied in [M1, M2, M3, MK, PS]. The above definition of a harmonic mapping extends in a natural way to the case of vector-valued mappings $f : G \to \mathbb{R}^n, f = (f_1, \ldots, f_n)$, defined on a domain $G \subset \mathbb{R}^n, n \geq 2$.

We first recall the classical Schwarz lemma for the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$:

1.2. Lemma. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function with $f(0) = 0$. Then $|f(z)| \leq |z|, z \in \mathbb{D}$.

For the case of harmonic mappings this lemma has the following counterpart.

1.3. Lemma. ([He], [D, p. 77]) Let $f : \mathbb{D} \to \mathbb{D}$ be a univalent harmonic mapping with $f(0) = 0$. Then $|f(z)| \leq \frac{4}{\pi} \tan^{-1}|z|$ and this inequality is sharp for each point $z \in \mathbb{D}$. 

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The classical Schwarz lemma is one of the cornerstones of geometric function theory and it also has a counterpart for quasiconformal maps (\cite{A, LV, Vu2}). Both for analytic functions and for quasiconformal mappings it has a form that is conformally invariant under conformal automorphisms of \( \mathbb{D} \).

In the case of harmonic mappings this invariance is no longer true. In general, if \( \varphi : \mathbb{D} \to \mathbb{D} \) is a conformal automorphism and \( f : \mathbb{D} \to \mathbb{D} \) is harmonic, then \( \varphi \circ f \) is harmonic only in rare exceptional cases. Therefore one expects that harmonic mappings from the disk into a strip domain behave quite differently from harmonic mappings from the disk into a half-plane and that new phenomena will be discovered in the study of harmonic maps. For instance, it follows from Lemma 1.2 that holomorphic functions in plane do not increase hyperbolic distances. In general, planar harmonic mappings do not enjoy this property. On the other hand, we shall give here an additional hypothesis under which the situation will change, in the plane as well as in higher dimensions. It turns out that the property of the local uniform boundedness, which we are going to define, has an important role in our study.

For a domain \( G \subset \mathbb{R}^n, n \geq 2, x, y \in G \), let

\[
r_G(x,y) = \frac{|x - y|}{\min\{d(x),d(y)\}} \text{ where } d(x) = d(x, \partial G) = \inf\{|z - x| : z \in \partial G\}.
\]

If the domain \( G \) is understood from the context, we write \( r \) instead of \( r_G \). This quantity is used, for instance, in the study of quasiconformal and quasiregular mappings, cf. \cite{Vu2}. It is a basic fact that \cite[Theorem 18.1]{Va} for all \( n \geq 2, K \geq 1, c_2 > 0 \) there exists \( c_1 \in (0,1) \) such that whenever \( f : G \to fG \) is a quasiconformal mapping with \( G, fG \subset \mathbb{R}^n \) then \( x, y \in G \) and \( r_G(x,y) \leq c_1 \) imply \( r_{fG}(f(x),f(y)) \leq c_2 \). We call this property the local uniform boundedness of \( f \) with respect to \( r_G \).

We also consider a weaker form of this property and say that \( f : G \to fG \) with \( G, fG \subset \mathbb{R}^n \) satisfies the weak property of local uniform boundedness on \( G \) (with respect to \( r_G \)) if there is a constant \( c > 0 \) such that \( r_G(x,y) \leq 1/2 \) implies \( r_{fG}(f(x),f(y)) \leq c \).

Univalent harmonic mappings fail to satisfy the weak local uniform boundedness as a rule, see Example 2.1 below. Note that quasiconformal mappings satisfy the local uniform boundedness and so do quasiregular mappings under appropriate conditions.

We show that if \( f : G \to fG \) is harmonic then \( f \) is Lipschitz w.r.t. quasihyperbolic metrics on \( G \) and \( fG \) if and only if it satisfies the weak property of local uniform boundedness. The proof is based on a higher dimensional version of the Schwarz lemma: harmonic maps satisfy the inequality (2.9) below. An inspection of the proof shows that this property is crucial and leads to generalizations of the result.

One of the main results of this paper deals with the class of quasiconformal harmonic mappings, in particular when the domain of definition is the upper half space \( \mathbb{H}^n \). For example, we show that if \( h \) is a quasiconformal harmonic mapping of the upper half space \( \mathbb{H}^n \) onto itself and \( h(\infty) = \infty \), then \( h \) is quasi-isometry with respect to both the Euclidean and the Poincaré distance.
2 Uniform continuity

We first show that univalent harmonic mappings fail to satisfy the local uniform boundedness in general.

2.1. Example. The univalent harmonic mapping \( f : \mathbb{H}^2 \to f(\mathbb{H}^2) \), \( f(z) = \pi - \arg(z + i \text{Im}z) \), fails to satisfy the local uniform boundedness property with respect to \( r_{\mathbb{H}^2} \).

Let \( z_1 = \rho e^{i\pi/4} \), \( z_2 = \rho e^{i3\pi/4} \), \( w_1 = f(z_1) \) and \( w_2 = f(z_2) \). Then \( r_{\mathbb{H}^2}(z_1, z_2) = 2 \) and \( r_{f(\mathbb{H}^2)}(w_1, w_2) = \frac{\pi}{\sqrt{2}\rho} \) if \( \rho \) is small enough and we see that the local uniform boundedness fails.

2.2. Example. The univalent harmonic mapping \( f : \mathbb{H}^2 \to \mathbb{H}^2 \), \( f(z) = \text{Re} z \text{Im} z + i \text{Im}z \), fails to satisfy the local uniform boundedness property with respect to \( r_{\mathbb{H}^2} \).

For a harmonic mapping \( f(z) = h(z) + g(z) \), we introduce the following notation
\[
l_f(z) = |h'(z)| - |g'(z)|, \quad L_f(z) = |h'(z)| + |g'(z)|.\
\]

2.3. Example. Let \( f(z) = \ln(|z|^2) + 2i\text{y} \). Then \( f(z) = h(z) + \overline{g(z)} \), where \( h(z) = \ln z + z \) and \( g(z) = \ln z - z \). Since \( h'(z) = 1 + 1/z \) and \( g'(z) = -1 + 1/z \), we have \( \nu(z) = g'(z)/f'(z) \) and \( |\nu(z)| < 1 \) for \( z \in \Pi^+ = \{ z : \text{Re} z > 0 \} \).

Moreover, \( f \) is quasiconformal on every compact subset \( D \subset \Pi^+ \) and \( l_h, L_h \) are bounded from above and below on \( D \). Therefore \( f \) is quasi-isometry on \( D \) and by Lemma 2.14 below, \( f \) satisfies the property of local uniform boundedness on \( D \).

From now on we consider the restriction of \( f \) to \( V = \{ z : x > 1, 0 < y < 1 \} \).

Then \( fV = \{ w = (u, v) : u > \ln(1 + v^2/4), 0 < v < 2 \} \).

We are going to show that:

- \( f \) satisfies the property of local uniform boundedness, but \( f \) is not quasiconformal on \( V \).

We see that \( f \) is not quasiconformal on \( V \), because \( |\nu(z)| \to 1 \) as \( z \to \infty \), \( z \in V \).

For \( s > 1 \), define \( V_s = \{ z : 1 < x < s, 0 < y < 1 \} \). Note that \( f \) is qc on \( V_s \) and therefore \( f \) satisfies the property of local uniform boundedness on \( V_s \) for every \( s > 1 \).

We consider separately two cases.

Case A. \( z \in V_4 \). If \( r > 1 \) is big enough, then \( d(z, \partial V_r) = d(z, \partial V) \) and \( d(f(z), \partial f(V_r)) = d(f(z), \partial f(V)) \) and therefore \( f \) satisfies the property of local uniform boundedness on \( V_4 \) with respect to \( r_V \).

Case B. It remains to prove that \( f \) satisfies the property of local uniform boundedness on \( V \setminus V_4 \) with respect to \( r_V \).

Observe first that for \( z, z_1 \in V \) and \( |z_1| \geq |z| \geq 1 \), we have the estimate
\[
\ln \left( \frac{|z_1|}{|z|} \right) \leq \frac{|z_1|}{|z|} - 1 \leq |z_1 - z|,
\]
and therefore for \( z, z_1 \in V \)
\[
|f(z_1) - f(z)| \leq 4|z_1 - z|. \quad (2.4)
\]
We write
\[ \partial V = [1, 1 + i] \cup A \cup B; A = \{(x, 0) : x \geq 1\}, B = \{(x, 1) : x \geq 1\}. \]

Then
\[ \partial (fV) = f(\partial V) \subset f[1, 1 + i] \cup (fA) \cup (fB) \]
and by the definition of \( f \) we see that
\[ fA = \{(x, 0) : x \geq 0\}, \quad fB = \{(x, 2) : x \geq \ln 2\}, \quad f[1, 1 + i] \subset [0, \ln 2] \times [0, 2]. \]

Clearly for \( w \in fV \)
\[ d(w) = \min\{d(w, fA), d(w, fB), d(w, f[1, 1 + i])\}, \]
and for \( \text{Re}w > 1 + \ln 2 \) and \( w \in fV \), we find
\[ d(w) = \min\{d(w, fA), d(w, fB)\}. \quad (2.5) \]

For \( z \in V \setminus V_4 \) we have \( \text{Re}f(z) \geq \ln(16) > 1 + \ln 2 \) and therefore, in view of the definition of \( f \), (2.5) yields \( d(f(z)) = 2d(z) \). This together with (2.4) shows that \( f \) satisfies the property of local uniform boundedness on \( V \setminus V_4 \).

Let \( B^n(x, r) = \{z \in \mathbb{R}^n : |z - x| < r\} \), \( S^{n-1}(x, r) = \partial B^n(x, r) \) and let \( \mathbb{B}^n, S^{n-1} \) stand for the unit ball and the unit sphere in \( \mathbb{R}^n \), respectively. Sometimes we write \( \mathbb{D} \), instead of \( \mathbb{B}^2 \). For a vector-valued function \( f : G \to \mathbb{R}^n \), where \( G \subset \mathbb{R}^n \), is a domain, we define
\[ |f'(x)| = \max_{|h|=1} |f'(x)h| \quad \text{and} \quad l(f'(x)) = \min_{|h|=1} |f'(x)h|, \]
when \( f \) is differentiable at \( x \in G \). We adopt the standard terminology and notation for \( K \)-quasiconformal (\( K \)-qc) and \( K \)-quasiregular (\( K \)-qr) mappings from \([LV],[Va],[Vu2]\).

For a domain \( G \subset \mathbb{R}^n \) let \( \rho : G \to (0, \infty) \) be a function. We say that \( \rho \) is a weight function or a metric density if for every locally rectifiable curve \( \gamma \) in \( G \), the integral
\[ l_\rho(\gamma) = \int_\gamma \rho(x)ds \]
is well-defined. In this case we call \( l_\rho(\gamma) \) the \( \rho \)-length of \( \gamma \). A metric density defines a metric \( d_\rho : G \times G \to (0, \infty) \) as follows. For \( a, b \in G \), let
\[ d_\rho(a, b) = \inf_\gamma l_\rho(\gamma) \]
where the infimum is taken over all locally rectifiable curves in \( G \) joining \( a \) and \( b \). It is an easy exercise to check that \( d_\rho \) satisfies the axioms of a metric. For instance, the hyperbolic (or Poincaré) metric of \( \mathbb{D} \) is defined in terms of the density \( \rho(x) = c/(1 - |x|^2) \) where \( c > 0 \) is a constant.

The quasihyperbolic metric \( k = k_G \) of \( G \) is a particular case of the metric \( d_\rho \) when \( \rho(x) = \frac{1}{d(x, \partial G)} \) (see [Vu2]).
2.1 Higher dimensional version of Schwarz lemma

Before giving a proof of the higher dimensional version of the Schwarz lemma we first establish notation.

Suppose that \( h : \overline{B}^n(a, r) \to \mathbb{R}^n \) is a continuous vector-valued function, harmonic on \( B^n(a, r) \), and let

\[
M_a = \omega_n^* \{ h y - h a : y \in S^{n-1}(a, r) \}.
\]

Let \( h = (h^1, h^2, \ldots, h^n) \). A modification of the estimate in \( [GT, \text{Equation (2.31)}] \) gives

\[
r |\nabla h^k(a)| \leq n M_a, \quad k = 1, \ldots, n.
\]

We next extend this result to the case of vector valued functions. See also \( [Bu] \) and \( [ABR, \text{Theorem 6.16}] \).

2.6. Lemma. Suppose that \( h : \overline{B}^n(a, r) \to \mathbb{R}^n \) is a continuous mapping, harmonic in \( B^n(a, r) \). Then

\[
r |h'(a)| \leq n M_a.
\] (2.7)

Proof. Without loss of generality, we may suppose that \( a = 0 \) and \( h(0) = 0 \). Let

\[
K(x, y) = K_y(x) = \frac{r^2 - |x|^2}{n \omega_n r |x - y|^n},
\]

where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \).

Then

\[
h(x) = \int_{S^{n-1}(0, r)} K(x, t) h(t) ds, \quad x \in B^n(0, r),
\]

where \( ds \) is the \((n-1)\)-dimensional surface measure on \( S^{n-1}(0, r) \).

A simple calculation yields

\[
\frac{\partial}{\partial x_j} K(x, \xi) = \frac{1}{n \omega_n r} \left( \frac{-2 x_j}{|x - \xi|^n} - n(r^2 - |x|^2) \frac{x_j - \xi_j}{|x - \xi|^{n+2}} \right).
\]

Hence, for \( 1 \leq j \leq n \), we have

\[
\frac{\partial}{\partial x_j} K(0, \xi) = \frac{\xi_j}{\omega_n r^{n+1}}.
\]

Let \( \eta \in S^{n-1} \) be a unit vector and \( |\xi| = r \). For given \( \xi \), it is convenient to write \( K_\xi(x) = K(x, \xi) \) and consider \( K_\xi \) as function of \( x \).

Then

\[
K'_\xi(0) \eta = \frac{1}{\omega_n r^{n+1}} (\xi, \eta).
\]

Since \( |(\xi, \eta)| \leq |\xi||\eta| = r \), we see that

\[
|K'_\xi(0) \eta| \leq \frac{1}{\omega_n r^n}, \quad \text{and therefore} \quad |\nabla K_\xi(0)| \leq \frac{1}{\omega_n r^n}.
\]

This last inequality yields

\[
|h'(0)(\eta)| \leq \int_{S^{n-1}(a, r)} |\nabla K^y(0)||h(y)| ds_y \leq \frac{M_0 \omega_n r^{n-1}}{\omega_n r^n} = \frac{M_0 n}{r}.
\]
and the proof is complete. □

Let $G \subset \mathbb{R}^n$, be a domain, let $h : G \to \mathbb{R}^n$ be continuous. For $x \in G$ let $B_x = B^n(x, \frac{1}{4}d(x))$ and

$$M_x = \omega_h(x) = \sup \{|hy - hx| : y \in B_x\}. \tag{2.8}$$

If $h$ is a harmonic mapping, then the inequality (2.7) yields

$$\frac{1}{4}d(x)|h'(x)| \leq n \omega_h(x), \quad x \in G. \tag{2.9}$$

We also refer to (2.9) as the inner gradient estimate.

For our purpose it is convenient to have the following lemma.

2.10. Lemma. Let $G$ and $G'$ be two domains in $\mathbb{R}^n$, and let $\sigma$ and $\rho$ be two continuous metric densities on $G$ and $G'$, respectively, which define the elements of length $ds = \sigma(z)|dz|$ and $ds = \rho(w)|dw|$, respectively; and suppose that $f : G \to G'$, is a $C^1$-mapping. If $\rho(f(z))|f'(z)| \leq c \sigma(z)$, $z \in G$, then $d_{\rho}(f(z_2), f(z_1)) \leq c d_{\sigma}(z_2, z_1)$, $z_1, z_2 \in G$.

The proof of this result is straightforward and it is left to the reader as an exercise.

Let $f$ be a map from a metric space $(M, d_M)$ into another metric space $(N, d_N)$.

- We say that $f$ is a pseudo-isometry if there exist two positive constants $a$ and $b$ such that for all $x, y \in M$,

$$a^{-1}d_M(x, y) - b \leq d_N(f(x), f(y)) \leq ad_M(x, y).$$

- We say that $f$ is a quasi-isometry or a bi-Lipschitz mapping if there exists a positive constant $a \geq 1$ such that for all $x, y \in M$,

$$a^{-1}d_M(x, y) \leq d_N(f(x), f(y)) \leq ad_M(x, y).$$

Let $G$ be a proper subdomain of $\mathbb{R}^n$. Define $j_G(x, y) = \ln(1 + r_G(x, y))$ for $x, y \in G$; it is clear that

$$j_G(x, y) = \ln(1 + r_G(x, y)) \leq r_G(x, y). \tag{2.11}$$

2.11. Lemma. ([GP], [Vu2] (3.4), Lemma 3.7) Let $G$ be a proper subdomain of $\mathbb{R}^n$.

(a) If $x, y \in G$ and $|y - x| \leq d(x)/2$, then $k_G(x, y) \leq 2j_G(x, y)$.

(b) For $x, y \in G$ we have $k_G(x, y) \geq j_G(x, y) \geq \ln \left(1 + \frac{|y-x|}{d(x)}\right)$.

For the convenience of the reader we begin our discussion for the unit disk case.

2.12. Theorem. Suppose that $h : \mathbb{D} \to \mathbb{R}^2$ is harmonic and satisfies the weak property of local uniform boundedness.
(c) Then \( h : (\mathbb{D}, k_\mathbb{D}) \to (h(\mathbb{D}), k_{h(\mathbb{D})}) \) is Lipschitz.

(d) If, in addition, \( h \) is a qc mapping, then \( h : (\mathbb{D}, k_\mathbb{D}) \to (f(\mathbb{D}), k_{f(\mathbb{D})}) \) is a quasi-isometry.

Proof. The part (d) is proved in [M2].

For the proof of part (c) fix \( x \in \mathbb{D} \) and \( y \in B_x = B^n(x, \frac{1}{2}d(x)) \). Then \( d(y) \geq \frac{3}{4}d(x) \) and therefore \( r(x, y) < 1/2 \). By the hypotheses \( |hy - hx| \leq c d(h(x)) \). The Schwarz lemma, applied to \( B_x \), yields in view of (2.8)

\[
\frac{1}{4}d(x)|h'(x)| \leq 2M_x \leq 2c d(h(x))
\]

The proof of part (c) follows from Lemma 2.11. □

A similar proof applies for higher dimensions; the following result is a generalization of the part (c) of Theorem 2.12.

2.13. Theorem. Suppose that \( h : G \to \mathbb{R}^n \) is a harmonic mapping. Then the following conditions are equivalent

1. \( h \) satisfies the weak property of local uniform boundedness.
2. \( h : (G, k_G) \to (h(G), k_{h(G)}) \) is Lipschitz.

By Theorem 2.20 below, (1) implies (2). We have only to prove that (2) implies (1). This implication follows from the next lemma.

2.14. Lemma. Let \( G \) be a proper subdomain of \( \mathbb{R}^n \). If \( f : (G, k_G) \to (f(G), k_{f(G)}) \) is Lipschitz with the multiplicative constant \( c_2 \), then \( f \) satisfies the weak property of local uniform boundedness.

Proof. Fix \( x, y \in G \) with \( r_G(x, y) \leq 1/2 \). Then \( |y - x| \leq d(x)/2 \) and therefore by Lemma 2.11

\[
k_G(x, y) \leq 2j_G(x, y) \leq 2r_G(x, y) \leq 1.
\]

Hence \( k_{G'}(fx, fy) \leq c_2 \). Since \( j_{G'}(fx, fy) \leq k_{G'}(fx, fy) \leq c_2 \), we find \( j_{G'}(fx, fy) = \ln(1 + r_{G'}(fx, fy)) \leq c_2 \) and therefore \( r_{G'}(fx, fy) \leq e^{c_2} - 1 \). □

2.15. Corollary. Suppose that \( G \subset \mathbb{R}^n \), \( h : G \to hG \) is harmonic and K- qc. Then \( h : (G, k_G) \to (h(G), k_{h(G)}) \) is a pseudo-isometry.

In [Vu2, Example 11.4] (see also [Vu1, Example 3.10]) an example was given to show that the analytic function \( f : \mathbb{D} \to G, G = \mathbb{D}\setminus\{0\} \), \( f(z) = \exp((z+1)/(z-1)) \), fails to be uniformly continuous as a map

\[
f : (\mathbb{D}, k_\mathbb{D}) \to (G, k_G).
\]

Therefore bounded analytic functions fail to satisfy the weak local uniform boundedness property. The situation will be different for instance if the boundary of the image domain is a continuum containing at least two points. Note that if \( k_G \) is
replaced by the hyperbolic metric $\lambda G$ of $G$, then $f : (\mathbb{D}, k_\mathbb{D}) \rightarrow (G, \lambda G)$ is Lipschitz.

2.16. Theorem. Suppose that $G \subset \mathbb{R}^n$, $f : G \rightarrow \mathbb{R}^n$ is $K$-qr and $G' = f(G)$. Let $\partial G'$ be a continuum containing at least two distinct points. If $f$ is a harmonic mapping, then $f : (G, k_G) \rightarrow (G', k_{G'})$ is Lipschitz.

Proof. Fix $x \in G$ and let $B_x = B^n(x, d(x)/4)$. If $|y - x| \leq d(x)/4$, then $d(y) \geq 3d(x)/4$ and therefore,

$$r_G(y, x) \leq \frac{4|y - x|}{3d(x)}.$$

Because $j_G(x, y) = \ln(1 + r_G(x, y)) \leq r_G(x, y)$, using Lemma 2.11(a), we find

$$k_G(y, x) \leq 2j_G(y, x) \leq 2/3 < 1.$$

By [Vu2, Theorem 12.21] there exists a constant $c_2 > 0$ depending only on $n$ and $K$ such that

$$k_{G'}(f y, f x) \leq c_2 \max\{k_G(y, x)^\alpha, k_G(y, x)\}, \alpha = K^{1/(1-n)},$$

and hence, using Lemma 2.11(b) and $k_G(y, x) \leq 1$, we see that

$$|f y - f x| \leq e^{c_2}d(f x), \quad \text{i.e.} \quad M_x = \omega_f(x) \leq e^{c_2}d(f x). \quad (2.17)$$

By (2.9) applied to $B_x = B^n(x, d(x)/4)$, we have

$$\frac{1}{4}d(x)|f'(x)| \leq 2M_x$$

and therefore using the inequality (2.17), we have

$$\frac{1}{4}d(x)|f'(x)| \leq 2c d(f(x)),$$

where $c = e^{c_2}$; and the proof follows from Lemma 2.10. □

2.2 Pseudo-isometry

In this subsection, we give a sufficient condition for a qc mapping $f : G \rightarrow f(G)$ to be a pseudo-isometry w.r.t. quasihyperbolic metrics on $G$ and $f(G)$. First we adopt the following notation.

If $V$ is a subset of $\mathbb{R}^n$ and $u : V \rightarrow \mathbb{R}^m$, we define

$$\text{osc}_V u = \sup\{|u(x) - u(y)| : x, y \in V\}.$$

Suppose that $G \subset \mathbb{R}^n$ and $B_x = B(x, d(x)/2)$. Let $OC^1(G)$ denote the class of $f \in C^1(G)$ such that

$$d(x)|f'(x)| \leq c_1 \text{osc}_{B_x}f \quad (2.18)$$
for every $x \in G$. Similarly, let $SC^1(G)$ be the class of functions $f \in C^1(G)$ such that

$$|f'(x)| \leq ar^{-1}\omega_f(x, r) \quad \text{for all } B^n(x, r) \subset G,$$

(2.19)

where $\omega_f(x, r) = \sup\{|f(y) - f(x)| : y \in B^n(x, r)\}$.

2.20. Theorem. Suppose that $G \subset \mathbb{R}^n$, $f : G \to G'$, $f \in OC^1(G)$ and it satisfies the weak property of local uniform boundedness with a constant $c$ on $G$. Then

(e) $f : (G, k_G) \to (G', k_{G'})$ is Lipschitz.

(f) In addition, if $f$ is $K$-qc, then $f$ is pseudo-isometry w.r.t. quasihyperbolic metrics on $G$ and $f(G)$.

Proof. By the hypothesis

$$\text{osc}_{B_x} f \leq c_2 d(f(x))$$

(2.21)

for every $x \in G$. This inequality together with (2.18) gives $d(x)|f'(x)| \leq c_3 d(f(x))$. Now an application of Lemma 2.10 gives part (e). Since $f^{-1}$ is qc, an application of [GO] Theorem 3] on $f^{-1}$ gives part (f). □

In order to apply the above method we introduce subclasses of $OC^1(G)$ (see, for example, below (2.22)).

Let $f : G \to G'$ be a $C^2$ function and $B_x = B(x, d(x)/2)$. We denote by $OC^2(G)$ the class of functions which satisfy the following condition:

$$\sup_{B_x} d^2(x)|\Delta f(x)| \leq c \text{osc}_{B_x} f$$

(2.22)

for every $x \in G$.

If $f \in OC^2(G)$, then by Theorem 3.9 in [GT],

$$d(x)|f'(x)| \leq c_1 \text{osc}_{B_x} f$$

(2.23)

for every $x \in G$ and therefore $OC^2(G) \subset OC^1(G)$.

Now the following result follows from the previous theorem.

2.24. Corollary. Suppose that $G \subset \mathbb{R}^n$ is a proper subdomain, $f : G \to G'$ is $K$-qc and $f$ satisfies the condition (2.22). Then $f : (G, k_G) \to (G', k_{G'})$ is Lipschitz.

We will now give some examples of classes of functions to which Theorem 2.20 is applicable. Let $SC^2(G)$ denote the class of $f \in C^2(G)$ such that

$$|\Delta f(x)| \leq ar^{-1}\sup\{|f'(y)| : y \in B^n(x, r)\},$$

for all $B^n(x, r) \subset G$, where $a$ is a positive constant. Note that the class $SC^2(G)$ contains every function for which $d(x)|\Delta f(x)| \leq a|f'(x)|$, $x \in G$. It is clear that $SC^1(G) \subset OC^1(G)$ and by the mean value theorem, $OC^2(G) \subset SC^2(G)$. For example, in [P] it is proved that $SC^2(G) \subset SC^1(G)$ and that the class $SC^2(G)$ contains harmonic functions, eigenfunctions of the ordinary Laplacian if $G$ is bounded, eigenfunctions of the hyperbolic Laplacian if $G = \mathbb{R}^n$ and thus our results are applicable for instance to these classes.
Let \( \mathbb{H}^n \) denote the half-space in \( \mathbb{R}^n \). If \( D \) is a domain in \( \mathbb{R}^n \), by \( QCH(D) \) we denote the set of Euclidean harmonic quasi-conformal mappings of \( D \) onto itself.

In particular if \( x \in \mathbb{R}^3 \), we use notation \( x = (x_1, x_2, x_3) \) and we denote by \( \partial_{x_k} f = f_{x_k}' \) the partial derivative of \( f \) with respect to \( x_k \).

A fundamental solution in space \( \mathbb{R}^3 \) of the Laplace equation is \( \frac{1}{|x|} \). Let \( U_0 = \frac{1}{|x|+\varepsilon_3|} \), where \( \varepsilon_3 = (0, 0, 1) \). Define \( h(x) = (x_1+\varepsilon_1 U_0, x_2+\varepsilon_2 U_0, x_3) \). It is easy to verify that \( h \in QCH(\mathbb{H}^3) \) for small values of \( \varepsilon_1 \) and \( \varepsilon_2 \).

3.1. Theorem. If \( h \in QCH(\mathbb{H}^n) \) and \( h(\infty) = \infty \), then both \( h : (\mathbb{H}^n, | \cdot |) \rightarrow (\mathbb{H}^n, | \cdot |) \) and \( h : (\mathbb{H}^n, \rho_{\mathbb{E}^n}) \rightarrow (\mathbb{H}^n, \rho_{\mathbb{E}^n}) \) are bi-Lipschitz where \( \rho_{\mathbb{E}^n} \) is the Poincaré distance.

Proof. It suffices to deal with the case \( n = 3 \) as the proof for the general case is similar. Let \( h = (h_1, h_2, h_3) \).

Using the Herglotz representation of the positive harmonic function \( h_3 \) (see Theorem 7.24 and Corollary 6.36 \([ABR]\)), we get \( h_3(x) = a x_3 \), where \( a \) is a positive constant. Without loss of generality we can suppose that \( a = 1 \).

Since \( h_3(x) = x_3 \), we have \( \partial_{x_3} h_3(x) = 1 \), and therefore \( |h'_{x_3}(x)| \geq 1 \). In a similar way, \( |g'_{x_3}(x)| \geq 1 \), where \( g = h^{-1} \). Hence, there exists a constant \( c = c(K) \),

\[
|h'(x)| \leq c \quad \text{and} \quad 1/c \leq l(h'(x)) .
\]

Therefore partial derivatives of \( h \) and \( h^{-1} \) are bounded from above; and, in particular, \( h \) is Euclidean bi-Lipschitz.

Since \( h_3(x) = x_3 \),

\[
\frac{|h'(x)|}{h_3(x)} \leq \frac{c}{x_3};
\]

and hence, by Lemma 2.10 \( \rho(h(a), h(b)) \leq c \rho(a, b) \). \( \square \)

3.1 Open problems

In order to formulate a few open problems, we need to introduce some convenient notation.

For \( u : \mathbb{R}^2 \rightarrow \mathbb{R} \), we define

\[
P[u](x) = \frac{x_3}{2\pi} \int_{\mathbb{R}^2} \frac{u(y)}{|x-y|^3} dy ,
\]

where \( y = (y_1, y_2) \) and \( dy = dy_1 dy_2 \).

Question 1. What is the analog of Theorem 3.1 for \( n \)-harmonic functions?

Question 2. Let \( f = (f_1, f_2) \) be a homeomorphism of \( \mathbb{R}^2 \) onto itself with \( f(\infty) = \infty \) and let \( h(x) = (P[f_1](x), P[f_2](x), x_3) \) be a normalized solution of the corresponding Dirichlet problem in \( \mathbb{H}^3 \). Find a sufficient condition for \( f \) such that

(a) \( h \) is one-to-one,

(b) \( h \in QCH(\mathbb{H}^3) \).
Also, it is natural to ask whether a similar result holds for \( n \)-harmonic functions.

**Question 3.** What is the analog statement of Theorem 3.1 for the unit ball?

**Question 4.** Suppose that \( G \subset \mathbb{R}^n \) is a proper subdomain, \( f : G \to \mathbb{R}^n \) is harmonic \( K \)-qc and \( G' = f(G) \). Determine whether \( f \) is a quasi-isometry w.r.t. quasihyperbolic metrics on \( G \) and \( G' \).

This is true for \( n = 2 \). It seems that one can modify the proof of Proposition 4.6 in [TW] and show that this is true for the unit ball if \( n \geq 3 \) and \( K < 2^{n-1} \).

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