HEEGAARD FLOER HOMOLOGY OF SURGERIES ON TWO-BRIDGE LINKS

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Abstract. We give an $O(p^2)$ time algorithm to compute the generalized Heegaard Floer complexes $A_{s_1,s_2}(\mathcal{L})$'s for a two-bridge link $\mathcal{L} = b(p,q)$ by using nice diagrams. Using the link surgery formula of Manolescu-Ozsváth, we also show that $\text{HF}^-$ and their $d$-invariants of all integer surgeries on two-bridge links are determined by $A_{s_1,s_2}(\mathcal{L})$'s. We obtain a polynomial time algorithm to compute $\text{HF}^-$ of all the surgeries on two-bridge links, with $\mathbb{Z}/2\mathbb{Z}$ coefficients. In addition, we calculate some examples explicitly: $\text{HF}^-$ and the $d$-invariants of all integer surgeries on a family of hyperbolic two-bridge links including the Whitehead link.

1. Introduction

1.1. Background and motivation. Heegaard Floer homology is a package of invariants of 3-manifolds invented by Ozsváth and Szabó, using holomorphic disks and Heegaard splittings of the 3-manifold [13, 12]. It detects the Thurston norm and fiberedness of a 3-manifold [3, 11, 10]. Furthermore, it fits into a kind of 3+1 dimensional topological quantum field theory, which is important in the study of smooth structures on 4-manifolds. Unlike other Floer homological invariants, Heegaard Floer homology is combinatorially computable, and there are several algorithms for computing various versions of it. Manolescu, Ozsváth and Sarkar described knot Floer homology combinatorially using grid diagrams in [7]. Sarkar and Wang in [18] found an algorithm for computing $\hat{HF}(M^3)$ over $\mathbb{Z}/2\mathbb{Z}$ by using nice Heegaard diagrams. Lipshitz, Ozsváth and Thurston used bordered Floer homology to give another algorithm for computing $\hat{HF}(M^3)$ in [5]. In [8], Manolescu, Ozsváth and Thurston showed that the plus and minus versions of Heegaard Floer homology (over $\mathbb{Z}/2\mathbb{Z}[U]$) can also be described combinatorially, by using link surgery and grid diagrams. (Admittedly, the MOT algorithm has a high time complexity.) Improving these algorithms and developing new methods for computations are still important and interesting questions.

1.2. The basic idea. This paper is aimed at studying the Heegaard Floer homology of surgeries on two-component links by using the link surgery formula due to Manolescu-Ozsváth [6]. When $\mathcal{L}$ is a two-bridge link $b(p,q)$, we find a fast algorithm for computing the Floer homology of surgeries on $L$, $\text{HF}^{-}(S^3_{\Lambda}(\mathcal{L}))$ over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, where $\Lambda$ is the framing matrix of a surgery. (Here, $\text{HF}^{-}(S^3_{\Lambda}(\mathcal{L}))$ is the $U$-completion of $HF^-$.) See [6] Section 2.) This algorithm uses genus-0 nice diagrams and algebraic arguments to simplify the Manolescu-Ozsváth surgery formula. Its worst-case time complexity is a polynomial of $p$ and $\det(\Lambda)$.

Let us mention some related work. In [17], Rasmussen studied Heegaard Floer homology of surgeries on two-bridge knots. In [16], Ozsváth and Szabó developed a formula for the Heegaard Floer homology of surgeries on knots. The paper [6] presents a generalization of this formula to the case of links. Two sets of data are needed in the surgery formula in [6]: the generalized Floer complexes $A_{s}(\mathcal{L})$'s and the maps in the surgery formula, namely the maps $I_s^{L'}, D_s^{L'}$ connecting the complexes associated to oriented sublinks. In general, the Heegaard Floer homology of link surgeries is more difficult to compute, due to more involved algebraic structures. However, in some cases, computations using this surgery formula can be simplified.

The main complexity in the link surgery formula is the counting of the holomorphic domains on the Heegaard surface, which corresponds to holomorphic bigons and polygons in the symmetric
product. For the special case of two-bridge links, we directly find a formula for the counts of holomorphic bigons. Furthermore, the general link surgery formula involves counting holomorphic polygons in the symmetric product for computing some cobordism maps, and this is of considerably high time complexity. Here we notice that, for two-bridge links, all these maps can be determined algebraically.

Note that this paper provides new examples of hyperbolic 3-manifolds for which we can compute their Heegaard Floer homology.

1.3. Main results and organization. In Section 2, we review some preliminaries for the link surgery formula, including the generalized Floer complexes, polygon maps and nice diagrams.

In Section 3, using the Schubert normal form of two-bridge links we get a class of nice Heegaard diagrams called Schubert Heegaard diagrams, in which every region is either a bigon or a square. We can explicitly describe all the composite bigons on a Schubert Heegaard diagram, and hence the Floer differentials. Further, we get a formula for the Alexander gradings of all intersection points, thus giving a formula for the multi-variable Alexander polynomial of a two-bridge link \( b(p, q) \) in terms of \( p, q \). See Theorem 3.18 and Proposition 3.13 below for the precise statements. This implies that \( A^+ (\tilde{L}) \) can be directly computed from this diagram. For a two-bridge link \( \tilde{L} = b(p, q) \), we get an \( O(p^2) \) time algorithm for computing \( A^+ (\tilde{L}) \). We also found different two-bridge links (modulo \( \mathbb{Z}/2\mathbb{Z} \) coefficients) sharing the same multi-variable Alexander polynomial, signature, and linking number.

In Section 4, we review the link surgery formula from [6] for two-component links \( \tilde{L} = L_1 \cup L_2 \) with basic diagrams. First, we review some algebraic tools, hyperboxes of chain complexes. In order to see the algebraic structure of the link surgery formula, we define a twisted gluing of squares of chain complexes. Then the link surgery formula is a twisted gluing of certain squares of chain complexes derived from \( L \). These squares are constructed in [6] by means of complete system of hyperboxes, which is a set of compatible Heegaard diagrams for the sublinks. For any two-component link, we define a type of complete system of hyperboxes which generalizes the basic systems used in [6], called a primitive system of hyperboxes. We also show that any basic diagram of \( \tilde{L} = L_1 \cup L_2 \) produces a primitive system.

In Section 5, we use algebraic arguments to show some rigidity results of the destabilization maps \( D^1 \)'s, \( M \subset L \) up to chain homotopy, for two-bridge links. Further, if we perturb the destabilization maps \( D^1 \)'s by chain homotopy, i.e. replace \( D^1_{s+L_2} \) by \( D^1_{s+L_2} \cong D^1_{s+L_1} \), we can construct a new square of chain complexes called the perturbed surgery complex. Using the rigidity results, we show that the perturbed surgery complex is isomorphic to the original complex in the link surgery formula. Based on the perturbed surgery complex, we give the algorithm for computing \( \text{HF}^- (S^3_A (\tilde{L})) \) mentioned before.

Throughout this paper, we use \( \mathbb{F} = \mathbb{Z}/2\mathbb{Z} \) coefficients. The main result we obtain is the following:

**Theorem 1.1.** Suppose \( \tilde{L} \) is an oriented two-bridge link with framing \( \Lambda \). Let \( \mathcal{H}^L \) be a basic Heegaard diagram of \( \tilde{L} \) and let \( \mathcal{H} \) be a primitive system induced by \( \mathcal{H}^L \). After we determine the \( \mathbb{F}[[U_1, U_2]] \)-modules \( A^- (\tilde{L}) \)'s sitting at the vertices of the square in the link surgery formula, any choices of

- \( \mathbb{F}[[U_1]] \)-linear chain homotopy equivalences \( \tilde{D}_{s_1}^{-L_1} \) for the edge maps,
- \( \mathbb{F}[[U_1]] \)-linear chain homotopies for the diagonal maps

yield a perturbed surgery complex \( (\tilde{\mathcal{C}}^- (\mathcal{H}^L, \Lambda), \tilde{D}^-) \) which is isomorphic to the original surgery complex in [6] as an \( \mathbb{F}[[U_1]] \)-module. By imposing the \( U_2 \)-action to be the same as the \( U_1 \)-action, the \( \mathbb{F}[[U_1, U_2]] \)-module \( H_* (\tilde{\mathcal{C}}^- (\mathcal{H}, \Lambda), \tilde{D}^-) \) becomes isomorphic to the homology \( \text{HF}^- (S^3_A (\tilde{L})) \). This isomorphism preserves the absolute grading.

See Theorem 5.13 below for a more precise statement and the proof.
Corollary 1.2. For a two-bridge link $\hat{L}$, knowledge of the $A_{\infty}^L(\hat{L})$ determines $HF^-$ of all the surgeries on $L$.

In Section 6, we compute some examples explicitly: the surgeries on $b(4n, 2n+1), n \in \mathbb{N}$, which are two sequences of hyperbolic two-bridge links generalizing the Whitehead link and the torus link $T(2, 4)$. See Proposition 6.9 and Theorem 6.10. Actually in the course of the computation, we also show that the Whitehead link is an L-space link, which means all of its large surgeries are L-spaces, i.e. $A_{s_1, s_2}^L(\text{Wh})$’s all have homology $F[[U]]$. This provides examples of hyperbolic L-spaces.

To compute these examples, we study the filtered homotopy type of $CFL^-(L)$. In Section 6.2, we prove that when $L = b(4n, 2n+1)$, the filtered chain homotopy type of $CFL^-(L)$ is determined by the filtered chain homotopy type of $\hat{CFL}(L)$. See Proposition 6.5 and Proposition 6.6 for the precise statements. Basically, this is based on an observation that the Alexander polytope is simple and there are several symmetries on $CFL^-(L)$, which give constraints for the differentials in $CFL^-(L)$. From $CFL^-(L)$ we derive all the $A_{\infty}^L(L)$’s and the inclusion maps. Finally, using the perturbed surgery complex, we compute the Floer homology of their surgeries and the associated $d$-invariants in Section 6.3.

Since $CFL^-(L)$ is the same as $\hat{A}_{\infty}^L(L)$ viewed as a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex (with the Alexander filtration), Corollary 1.2 means the filtered homotopy type of $CFL^-(L)$ contains all the information about the Floer homology of the surgeries on $L$, when $L$ is a two-bridge link. In [15], it is shown that, for an alternating two-component link $L$, the filtered chain homotopy type of $\hat{CFL}(L)$ is determined by the set of data:

- the multi-variable Alexander polynomial $\Delta_L(x, y)$,
- the signature $\sigma(L)$,
- the linking number $\text{lk}(L)$,
- the filtered homotopy type of $\hat{CFK}(L_i)$ of each component.

However, it is hard to determine the filtered homotopy type of $CFL^-(L)$ in general. For two-bridge links, the Schubert Heegaard diagrams show that the $U_1, U_2$-differentials in $CFL^-(L)$ are quite simple, since the bigons always contain exactly one basepoint. In addition, every component of a two-bridge link is the unknot. Thus, for two-bridge links, we raise the following question:

Question 1.3. Given an oriented two-bridge link $L$, is the filtered homotopy type of $CFL^-(L)$ determined by the set of data $\{\Delta_L(x, y), \sigma(L), \text{lk}(L)\}$?

We note that $HF^-$ of surgeries on two-bridge links may also be computed by other methods. For example, as long as one of the framing coefficients is not 0, one can view one component as a knot in a lens space and compute using the grid diagram methods in [11]. Another method is to consider these surgeries as surgeries on $(1,1)$-knots in lens spaces and use the method of [4]. Nevertheless, the method in this paper is more conceptual. Some of the arguments here could be potentially used for other classes of links. In fact, Theorem 1.1 and Corollary 1.2 can be directly generalized to the two-component links with every component being an L-space knot.

1.4. Acknowledgments. This project was written under the supervision of my adviser Ciprian Manolescu. I am greatly indebted to him for his continuous guidance and support, and for his numerous valuable suggestions and encouragements.

2. Heegaard diagrams and generalized Floer complexes

In this section, we give the precise definitions of what we need in the link surgery formula, including Heegaard diagrams, generalized Floer complexes, polygon maps, and nice diagrams. Here, the Heegaard diagram is adapted for a link inside a 3-manifold with multiple basepoints; the generalized Floer complexes of a link $L$ are derived from the filtered complex $CFL^-(L)$, and they govern the large surgeries; the polygon maps are used in constructing cobordism maps and certain
maps in the link surgery formula; knowledge of nice diagrams are also introduced to deal with two-bridge links.

2.1. Heegaard diagrams of links. We give the most general definition of Heegaard diagrams for an oriented link $\overrightarrow{L}$ inside a 3-manifold $M^3$. When the link $\overrightarrow{L} = \emptyset$, the Heegaard diagram is simply for $M^3$.

**Definition 2.1** (Heegaard diagram of links). A *multi-pointed Heegaard diagram* for the oriented link $\overrightarrow{L}$ in $M^3$ is the data of the form $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$, where:

- $\Sigma$ is a closed, oriented surface of genus $g$;
- $\alpha = \{\alpha_1, \ldots, \alpha_{g+k-1}\}$ is a collection of disjoint, simple closed curves on $\Sigma$ which span a $g$-dimensional lattice of $H_1(\Sigma; \mathbb{Z})$, hence specify a handlebody $U_\alpha$; the same goes for $\beta = \{\beta_1, \ldots, \beta_{g+k-1}\}$ specify a handlebody $U_\beta$.
- $w = \{w_1, \ldots, w_k\}$ and $z = \{z_1, \ldots, z_m\}$ (with $k \geq m$) are collections of points on $\Sigma$ with the following property. Let $\{A_i\}_{i=1}^k$ be the connected components of $\Sigma - \alpha_1 - \cdots - \alpha_{g+k-1}$ and $\{B_i\}_{i=1}^k$ be the connected components of $\Sigma - \beta_1 - \cdots - \beta_{g+k-1}$. Then there is a permutation $\sigma$ of $\{1, \ldots, m\}$ such that $w_i \in A_i \cap B_i$ for $i = 1, \ldots, k$, and $z_i \in A_i \cap B_{\sigma(i)}$ for $i = 1, \ldots, m$, such that connecting $w_i$ to $z_i$ inside $A_i$ and connecting $z_i$ to $w_{\sigma(i)}$ inside $B_i$ will give rise to the link $\overrightarrow{L}$.

**Definition 2.2** (Admissible diagrams). A *periodic domain* is a two-chain $\phi$ on $\Sigma$ which is a linear combination of components of $\Sigma - \alpha \cup \beta$ with integral coefficients, such that the local multiplicity of $\phi$ at every $w_i \in w$ is 0 and the boundary of $\phi$ is a integral combination of $\alpha$- and $\beta$-curves. A multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$ is called admissible if every non-trivial periodic domain has some positive local multiplicities and some negative local multiplicities.

**Definition 2.3** (Basic diagrams of links). An admissible Heegaard diagram of $\overrightarrow{L}$ is called basic, if $l = k = m$, meaning there are exactly two basepoints $w_i, z_i$ for every component $L_i$ and no free basepoints.

**Remark 2.4.** (1) The definitions of pointed Heegaard moves are systematically formulated in [6] section 4.

(2) In order to avoid the issue of naturality, we fix the Heegaard surface $\Sigma$ as an embedded surface in the underlying 3-manifold $M^3$. Thus, a Heegaard diagram is equivalent to a self-indexed Morse function.

(3) In this paper we will only consider maximally colored diagrams in the sense of [6].

2.2. Generalized Floer complexes. Here we define some chain complexes of a Heegaard diagram for an oriented link in $S^3$, which govern the large surgeries on this link. Suppose $\overrightarrow{L} = \overrightarrow{L}_1 \cup \overrightarrow{L}_2 \cup \cdots \cup \overrightarrow{L}_l$, and $\overrightarrow{M}$ is an oriented sublink of $\overrightarrow{L}$, where $\overrightarrow{M}$ may not have the induced orientation from $\overrightarrow{L}$ on each component. By $\overrightarrow{L} - \overrightarrow{M}$, we denote the oriented link obtained by deleting all the components of $\overrightarrow{M}$ from $\overrightarrow{L}$.

The identity $H_1(S^3 - \overrightarrow{L}) \cong \mathbb{Z}^l$ provides a way to record the Spin$^c$ structures over $S^3$ relative to $L$ as an affine lattice over $\mathbb{Z}^l$.

**Definition 2.5** ($\mathbb{H}(L)$ and reduction maps). Define the affine lattice $\mathbb{H}(\overrightarrow{L})$ over $H_1(S^3 - \overrightarrow{L})$ as follows:

$$\mathbb{H}(\overrightarrow{L})_i = \frac{lk(\overrightarrow{L}_i, \overrightarrow{L} - \overrightarrow{L}_i)}{2} + \mathbb{Z} \subset \mathbb{Q}, \mathbb{H}(\overrightarrow{L}) = \bigoplus_i \mathbb{H}(\overrightarrow{L})_i,$$

together with its completion

$$\widehat{\mathbb{H}(\overrightarrow{L})}_i = \mathbb{H}(\overrightarrow{L})_i \cup \{-\infty, +\infty\}, \widehat{\mathbb{H}(\overrightarrow{L})} = \bigoplus_i \widehat{\mathbb{H}(\overrightarrow{L})}_i.$$
The map $\psi^M : \mathbb{H}(\overrightarrow{L}) \to \mathbb{H}(\overrightarrow{L} - M)$ is defined by $\psi_i^M(s) = s - \overrightarrow{M}/2$. More precisely, let $M = L_{j_1} \cup \ldots \cup L_{j_m}$. Then for all $i$ not in $\{j_1, \ldots, j_m\}$, let $L_i = (L - M)_{k_i}$, set

$$\psi_i^M : \mathbb{H}(\overrightarrow{L})_i \to \mathbb{H}(\overrightarrow{L} - M)_{k_i}, s_i \to s_i - \frac{\text{lk}(\overrightarrow{L}_i, \overrightarrow{M})}{2}. \quad (2.1)$$

The map $\psi^M_i$ is defined to be the direct sum of the maps $\psi_i^M$, for those $i$’s with $L_i$ not in $M$.

The reduction maps $\psi^M_i$ are used in the definition of the destabilization maps in Section 4.3.2 and also in the statement of link surgery formula in Section 4.2.

For convenience to define the generalized Floer complexes, here we focus on Heegaard diagrams with only one pair of basepoints $w_i, z_i$ on each component and allow free basepoints. Given an admissible multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$ for $\overrightarrow{L}$ with exactly two basepoints $z_i$ and $w_i$ for each link component $L_i$, we consider the Lagrangian pair $\mathcal{T}_\alpha, \mathcal{T}_\beta$ in $\text{Sym}^{g+k-1}(\Sigma)$ and the Floer complex $CF(\mathcal{T}_\alpha, \mathcal{T}_\beta)$. There is an Alexander multi-grading $\Lambda : \mathcal{T}_\alpha \cap \mathcal{T}_\beta \to \mathbb{H}(L)$ characterized by the property

$$A_i(x) - A_i(y) = n_{z_i}(\phi) - n_{w_i}(\phi), \forall \phi \in \pi_2(x, y)$$

and a normalization condition on the Alexander polynomial. The Alexander grading induces a filtration on $CF^-(\mathcal{T}_\alpha, \mathcal{T}_\beta)$. Given a Spin$^c$ structure on $S^3 - L$, i.e. an element $s \in \mathbb{H}(L)$, we associate a chain complex $\mathfrak{F}^-(\mathcal{H}, s)$ called the generalized Heegaard Floer complex using the Alexander filtration. We introduce variables $U_i$ with $1 \leq i \leq l$ for each link component $L_i$, and $U_i$ with $l + 1 \leq i \leq k$ for each free basepoint $w_i$.

**Definition 2.6 (Generalized Floer complex).** For $s \in \mathbb{H}(L)$, the generalized Floer complex $\mathfrak{F}^-(\mathcal{H}, s)$ is the free module over $\mathcal{R} = \mathbb{F}[U_1, \ldots, U_l]$ generated by $\mathcal{T}_\alpha \cap \mathcal{T}_\beta \in \text{Sym}^{g+k-1}(\Sigma)$, and equipped with the differential:

$$\partial_s^x = \sum_{y \in \mathcal{T}(\alpha) \cap \mathcal{T}(\beta)} \sum_{\phi \in \pi_2(x, y)} \#(\mathcal{M}(\phi)/\mathcal{R}) \cdot U_1^{E_{s_1}^1(\phi)} \cdots U_l^{E_{s_l}^1(\phi)} \cdot U_{l+1}^{n_{w_{l+1}}(\phi)} \cdots U_k^{n_{w_k}(\phi)} \cdot y,$$  \quad (2.2)

where $E_{s_i}^i(\phi)$ is defined by

$$E_{s_i}^i(\phi) = \max\{s - A_i(x), 0\} - \max\{s - A_i(y), 0\} + n_{z_i}(\phi) = \max\{A_i(x) - s, 0\} - \max\{A_i(y) - s, 0\} + n_{w_i}(\phi).$$

For simplicity, we also write

$$U E_s(\phi) = \prod_{i=1}^l U_i^{E_{s_i}^i(\phi)} \prod_{i=l+1}^k U_i^{n_{w_i}(\phi)}.$$

When the Heegaard diagram in the context is unique, we simply denote $\mathfrak{F}^-(\mathcal{H}, s)$ by $A_{s_1, s_2}^-(L)$ or $A_{s_1, s_2}^-(L)$, where $s = (s_1, s_2)$. The direct product of all the generalized Floer complexes forms the first input of the surgery formula.

**Remark 2.7.** Let us explain the relation between $A_s^-(L)$ and $CFL^-(L)$. First, the filtered chain complex $CFL^-(L)$ defined in [15] is the chain complex $CF^-(S^3)$ with the Alexander filtration. Second, the subcomplexes forming the Alexander filtration are isomorphic to the $A_s^-(L)$’s. The Equation (2.2) is an explicit formulation of those differentials in $A_s^-(L)$. 
2.3. Polygon maps and homotopy equivalences between Floer complexes. In the Fukaya category of a symplectic manifold \((X, \omega)\) (when it is well-defined), the product of morphisms

\[ \mu^2 : \text{Hom}(L_1, L_2) \otimes \text{Hom}(L_0, L_1) \to \text{Hom}(L_0, L_1) \]

is defined by counting holomorphic triangles. In general, higher products are defined by means of holomorphic polygons. In Heegaard Floer theory, the polygon maps are defined similarly. However, the technical issue is the compactness of moduli spaces of holomorphic polygons in the symmetric product of the Heegaard surface, i.e. whether the polygon counts are finite. This problem breaks down to periodic domains on the Heegaard surface. Admissibility of Heegaard multi-diagrams solves this problem. For more details, one can see Section 4.4 in [6].

**Definition 2.8** (Strongly equivalent Heegaard diagrams). (1) If two Heegaard diagrams \(H\) and \(H'\) have the same underlying Heegaard surface \(\Sigma\), and their collections of curves \(\beta\) and \(\beta'\) are related by isotopies and handleslides only (supported away from the basepoints), we say that \(\beta\) and \(\beta'\) are strongly equivalent.

(2) Two multi-pointed Heegaard diagrams \(H = (\Sigma, \alpha, \beta, w, z, \tau)\), \(H' = (\Sigma', \alpha', \beta', w', z', \tau')\) are called strongly equivalent, if \(\Sigma = \Sigma', w = w', z = z', \tau = \tau'\), the curve collections \(\alpha\) and \(\alpha'\) are strongly equivalent, and \(\beta\) and \(\beta'\) are strongly equivalent as well.

(3) We say that two Heegaard diagrams \(H\) and \(H'\) differ by a surface isotopy if there is a self-diffeomorphism \(\phi : \Sigma \to \Sigma\) isotopic to the identity and supported away from the link \(L\), such that \(\Sigma = \Sigma'\) and \(\phi\) takes all the attaching curves and basepoints on \(\Sigma\) to the corresponding one on \(\Sigma'\). If \(H\) and \(H'\) are surface isotopic, we write \(H \cong H'\).

**Definition 2.9** (Triangle maps). Let \((\Sigma, \alpha, \beta, \gamma, w, z)\) be a generic, admissible Heegaard triple-diagram, where \(\beta\) and \(\gamma\) are strongly equivalent, such that \(H = (\Sigma, \alpha, \beta, w, z)\), \(H' = (\Sigma, \alpha, \gamma, w, z)\) are both Heegaard diagrams of the link \(L\) and \(H' = (\Sigma, \beta, \gamma, w, z)\) is the Heegaard diagram of the unlink in \#(\S^1 \times \S^2). Then we can define the triangle map

\[ f_{\alpha, \gamma} : A^- (T_\alpha, T_\beta, s) \otimes A^- (T_\beta, T_\gamma, s') \to A^- (T_\alpha, T_\gamma, s+s') \]

\[ f_{\alpha, \gamma} (x \otimes y) = \sum_{z \in \mathcal{T}_\alpha T_\beta} \sum_{\phi \in \pi_2(x,y,z) | \mu(\phi) = 0} \#(M(\phi)) \cdot U_{E_{s,s'}(\phi)} z, \]

where

\[ E_{s,s'}(\phi) = U_{E_{s_1,s_1}(\phi)} \ldots U_{E_{s_l,s_l}(\phi)} U_{n_{s_1}n_{s_2}(\phi)} \ldots U_{n_{s_k}(\phi)} \]

\[ E_{s,s'}(\phi) = \max\{A_i(x) - s, 0\} + \max\{A_i(y) - s', 0\} - \max\{A_i(z) - s - s', 0\} + n_{w_i}(\phi). \]

**Definition 2.10** (Quadrilateral maps). Let \((\Sigma, \eta^0, \eta^1, \eta^2, w, z)\) be a generic, admissible multi-diagram, such that there are two equivalence classes of strongly equivalent attaching curves among \(\{\eta^i\}\), and \(\eta^0, \eta^3\) are in different equivalent classes so that \((\Sigma, \eta^0, \eta^3, w, z)\) is a Heegaard diagram for the link \(L\). Now we can define the quadrilateral maps

\[ f_{\eta^0, \eta^1}(x_1 \otimes x_2 \otimes x_3) = \sum_{y \in \mathcal{T}_{\eta^0} \cap \mathcal{T}_{\eta^3}} \sum_{\phi \in \pi_2(x_1,x_2,x_3,y) | \mu(\phi) = -1} \#(M(\phi)) \cdot U_{E_{s_1,s_2,s_3}(\phi)} y, \]
where
\[ U_{E_{s_1,s_2,s_3}} = U_{E_{1}^{1},E_{2}^{1},E_{3}^{1}} \cdots U_{E_{1}^{k},E_{2}^{k},E_{3}^{k}} \cdot U_{1}^{n_{w_1}}, \cdots U_{k}^{n_{w_k}}, s_i = (s_1^i, \ldots, s_k^i), i = 1, 2, 3, \]
\[ E_{s_1,s_2,s_3} = \max \{ A_i(x_1) - s_1, 0 \} + \max \{ A_i(x_2) - s_2, 0 \} + \max \{ A_i(x_3) - s_3, 0 \} - \max \{ A_i(y) - s_1 - s_2 - s_3, 0 \} + n_{w_i} \phi. \]

One can define higher polygon counts \( f_{\eta_1, \ldots, \eta_l} \) similarly, although the case \( l > 3 \) will not be needed in this paper. For simplicity, we ignore the subscripts of \( f_{\eta_1, \ldots, \eta_l} \). An important property of polygon maps is the so-called quadratic \( A_{\infty} \)-associativity equation
\[ \sum_{0 \leq i < j \leq l} f(x_1, \ldots, x_i, f(x_{i+1}, \ldots, x_j), x_{j+1}, \ldots, x_l) = 0. \]

2.4. Nice diagrams. In [15, Sarkar and Wang] use nice Heegaard diagrams to combinatorially compute the \( \widehat{HF}(M) \). This algorithm is based on a fact: in a nice diagram \( \mathcal{H} = (\Sigma, \alpha, \beta, w) \), the index-1 pseudo-holomorphic disks in \( \text{Sym}^{g+k-1}(\Sigma) \) with \( n_{w_i} = 0 \) have simple domains on \( \Sigma \) and can be combinatorially counted.

Definition 2.11 (Nice diagrams). A Heegaard diagram \( \mathcal{H} = (\Sigma, \alpha, \beta, w) \) is called nice, if any region (i.e. connect component of \( \Sigma - \alpha - \beta \)) without any basepoint \( w_i \in w \) is either a bigon or a square. For \( x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \), a domain \( \phi \in \pi_2(x, y) \) is called an empty embedded \( 2n \)-gon, if it is an embedded disk with \( 2n \) vertices on its boundary, such that for each vertex \( v \), \( \mu_v(\phi) = 1 \), and it does not contain any \( x_i \) or \( y_i \) in its interior. An empty embedded \( 4 \)-gon is also called empty embedded square.

Remark 2.12. The notation \( \pi_2(x, y) \) in [15] denotes the sets of domains, namely \( 2 \)-chains \( \phi \) on \( \Sigma \) such that \( \partial(\partial\phi|_\alpha) = y - x \), whereas in this paper \( \pi_2(x, y) \) denotes the homology classes of Whitney disks in \( \text{Sym}^{g+k-1}(\Sigma) \) from \( x \) to \( y \).

Theorem 2.13 ([15]). Let \( \phi \in \pi_2(x, y) \) be a domain on a nice diagram such that \( \mu(\phi) = 1 \) and \( n_{w_i}(\phi) = 0, \forall i \). If \( \phi \) has a holomorphic representative, then \( \phi \) is either an empty embedded bigon or an empty embedded square. Conversely, if \( \phi \in \pi_2(x, y) \) with \( n_{w_i}(\phi) = 0, \forall i \) is an empty embedded bigon or an empty embedded square, then the product complex structure on \( \Sigma \times D^2 \) achieves transversality for \( \phi \) under a generic perturbation of the \( \alpha \) and the \( \beta \) curves, and \( \mu(\phi) = 1 \) as well as \( \#M(\phi)/\mathbb{R} = 1 \) (mod 2).

The above theorem enables us to combinatorially count differentials in \( \widehat{CF}(\Sigma) \) on a nice diagram, by counting empty embedded bigons and squares. Following the same lines of the proof, we can obtain the following adaption.

Proposition 2.14. Suppose \( \mathcal{H} = (\Sigma, \alpha, \beta, w, z) \) is a Heegaard diagram such that any region of \( \Sigma \) is either a bigon or a square. Then, there is a 1-1 correspondence between the differentials in \( \mathfrak{M}^{-}(\mathcal{H}, \infty) \) and the set of empty embedded bigons and empty embedded squares. Thus, the complex \( \mathfrak{M}^{-}(\mathcal{H}, s) \) can be described combinatorially.

3. Generalized Floer complexes of two-bridge links

In this section, we combinatorially compute the generalized Floer complexes \( A^{-}_{s_1,s_2}(\mathcal{L}) \) for all two-bridge links \( \mathcal{L} \) by using nice diagrams.

3.1. Schubert normal form. A two-bridge link/knot can be obtained by closing a rational tangle. For the definition of rational tangles, one can see the reference [9] chapter 9 and [2] chapter 7E, 12D. Let us adopt the notations in [2]. By \( b(p, q) \) where \( \gcd(p, q) = 1 \), we denote the two-bridge link/knot according to the rational tangle of slope \( \frac{q}{p} \).
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\[ a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_{p-1} \quad a_{2p-1} \quad a_{2p-2} \quad a_{2p-3} \quad a_{2p-4} \]

\[ a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_{p-1} \quad a_{2p-1} \quad a_{2p-2} \quad a_{2p-3} \quad a_{2p-4} \]

\[ a_i \rightarrow a'_{q-i} \]

**Figure 3.1.** The Schubert form: the neighborhoods of the two over-bridges.

\[ O_1 \quad O_2 \quad U_1 \quad U_2 \]

**Figure 3.2.** The Schubert normal form of the Whitehead link.

**Definition 3.1** (Schubert normal form). For a two-bridge link/knot \( L = b(p,q) \), the *Schubert normal form* is a canonical projection of \( L \) with two over-bridges and two under-bridges, where we regard the projection plane as a sphere \( S \) in \( S^3 \). The two over-bridges \( O_1, O_2 \) are straight segments on the projection plane, and each component of the other two under-bridges \( U_1, U_2 \) crosses \( O_1, O_2 \) alternatively. Together with the lower half space (which is a ball in \( S^3 \)), the under-bridges \( U_1, U_2 \) form a rational tangle of slope \( \frac{q}{p} \). Moreover, if \( L \) has two components, we arrange the notation such that \( L_i = O_i \cup U_i \). We denote the Schubert form by \( (S, O_1, O_2, U_1, U_2) \).

Concretely, the Schubert normal form can be obtained by gluing two disks \( D^p_1, D^p_2 \) shown in Figure 3.1. The endpoint \( a_i \) is glued to \( a'_{q-i} \), where all the subscripts are modulo \( 2p \). When \( L \) has two components, \( L \) can be endowed with a canonical orientation induced by the orientation of \( \overrightarrow{O_1} = a_0a'_p, \overrightarrow{O_2} = a'_0a'_p \), which is also shown in Figure 3.1.

**Example 3.2.** In Figure 3.2, we show the Schubert normal form of the Whitehead link \( b(8,3) \).

**Remark 3.3.** Here in the definition of Schubert normal form, we set the straight arcs to be over-bridges, while in other places the straight arcs are set to be under-bridges. However, given \( p, q \), these two links are the same up to taking the mirror of each other.

**Fact 3.4.** Let \( b(p,q) \) denote the two-bridge link defined as above, where \( p, q \in \mathbb{Z}, \gcd(p,q) = 1, p > 0 \). Then
When $p$ is odd, the link $b(p,q)$ is a knot; and when $p$ is even, $b(p,q)$ has two components.

(2) (Schubert, [2], Theorem 12.6.) As an oriented link/knot, $b(p,q)$ is equivalent to $b(p',q')$ if and only if $p' = p, q' \equiv q^\pm 1 \pmod{2p}$; as an unoriented link/knot, $b(p,q)$ is equivalent to $b(p',q')$ if and only if $p' = p, q' \equiv q^\mp 1 \pmod{p}$.

(3) (12.8 in [2]) When $b(p,q)$ has two components, the link $b(p,-q)$ is the mirror of $b(p,q)$, and the link $b(p,q+p)$ can be obtained by changing the orientation on one component of $b(p,q)$.

(4) (Remark 12.7, [2]) The linking number can be computed by the formula:

$$\text{lk}(b(p,q)) = - \sum_{i=1}^{\frac{p}{2}} (-1)^{\lfloor \frac{2i-1}{p} \rfloor}.$$ 

(5) (Theorem 9.3.6, [3]) The signature can be computed by the formula:

$$\sigma(b(p,q)) = \sum_{i=1}^{p-1} (-1)^{\lfloor \frac{iq}{p} \rfloor}.$$ 

### 3.2. Heegaard diagrams of two-bridge links

In this section, we construct nice Heegaard diagrams of two-bridge links by using their Schubert forms.

**Definition 3.5.** A bridge presentation of a link $L$ is a topological pair $(L,S)$ inside $S^3$, such that

- $S$ is an embedded sphere transversely intersecting $L$,
- $S^3 - S \cong B_1 \cup B_2$, where $B_1, B_2$ are homeomorphic to the unit ball $B^3$,
- each pair $(L \cap B_i, B_i)$ with $i = 1, 2$ is homeomorphic to the pair $\left(\{P_j\}_{j=1}^k \times I, D^2 \times I\right)$, where $\{P_j\}_{j=1}^k$ is a set of points in the interior of the unit disk $D^2$.

The minimum over all possible $k$ is called the bridge number of the link, denoted by $\text{br}(L)$.

Every bridge presentation of $L$ gives rise to a genus-0 multi-pointed Heegaard diagram for $L$. Let the sphere $S$ be the Heegaard surface. In each ball $B_i$, choose $k-1$ disjoint proper disks to divide $B_i$ into $k$ chambers, such that every component of the $k$ bridges is in a distinct chamber. The boundaries of these disks are the alpha, beta curves. The basepoints $w_i, z_i$ are the intersection points of $L$ and $S$. In other words, by pushing all the bridges onto the sphere $S$, we obtain a projection of $L$ consisting of $2k$ arcs $a_1, \cdots, a_k, b_1, \cdots, b_k$, such that the arcs $\{a_i\}$ are disjoint, the arcs $\{b_j\}$ are disjoint, and the arcs $\{a_i\}$ are always over the arcs $\{b_j\}$. Then the boundaries of tubular neighborhoods of the arcs $\{a_i\}_{i=1}^{k-1}, \{b_j\}_{j=1}^{k-1}$ are the alpha, beta curves.

**Definition 3.6** (Schubert Heegaard diagrams). Let $\mathbf{L} = \mathbf{L}_1 \cup \mathbf{L}_2$ be a two-bridge link, and let $(S,O_1, O_2, U_1, U_2)$ be its Schubert form. The Schubert Heegaard diagram of $L$ is the Heegaard diagram $\mathcal{H} = (S^2, \{\alpha\}, \{\beta\}, \{z_1, z_2\}, \{w_1, w_2\})$, where

- $\alpha = \partial N(O_1), \beta = \partial N(U_1)$ with $N(O_1), N(U_1)$ being disjoint tubular neighborhoods of $O_1, U_1$ on $S$,
- $\{z_1, w_1\} = \{L_1 \cap S\}$ and $\{z_2, w_2\} = \{L_2 \cap S\}$.

Concretely, regarding the Schubert form as the gluing of two disks $D^2_1, D^2_2$ in Figure 3.1, we can take $\alpha = \partial D^2_1$ and $\beta = \partial N(U_1)$. The basepoint $z_1$ can be any point in $D^2_1$ near $a_0$, and the basepoint $w_1$ can be any point in $D^2_1$ near $a_p$; whereas the basepoint $z_2$ can be any point in $D^2_2$ near $a_0'$, and the basepoint $w_2$ can be any point in $D^2_2$ near $a_p'$.

**Example 3.7.** The two-bridge link $b(8,3)$ is the Whitehead link $Wh$ (or its mirror due to the convention). The Schubert Heegaard diagram of $Wh$ is in Figure 3.3.

**Notations 3.8.** Since we will repeatedly discuss the Schubert Heegaard diagram, it is convenient to make a notational convention for all the intersection points and regions as follows.
The components of $S - \beta$ are both disks, denoted by $D_i^\alpha, D_i^\beta$ such that the disk $D_i^\beta$ is a neighborhood of $U_i$.

There is a total of four bigons among the components of $S - (\alpha \cup \beta)$, and each of them contains a distinct basepoint in $\{z_1, z_2, w_1, w_2\}$. All the other components are squares.

We label all the $p + 1$ components of $D_i^\alpha - \beta$ by $X_0, X_1, ..., X_p$, and label all the $p + 1$ components of $D_i^\beta - \beta$ by $Y_0, Y_1, ..., Y_p$, such that $X_0, X_p, Y_0, Y_p$ are bigons and $w_1 \in X_0, w_2 \in Y_0, z_1 \in X_p, z_2 \in Y_p$.

There is a total of $2p$ intersection points of $\alpha$ and $\beta$. We label them by $b_0, b_1, ..., b_{2p-1}$ clockwise, such that $b_0, b_{2p-1}$ are vertices of $X_0$, and $b_{2p-1}, b_p$ are vertices of $X_p$. All the subscripts are modulo $2p$.

The above properties and conventions are illustrated in Figure 3.4.

**Lemma 3.9.** The Schubert Heegaard diagram of the two-bridge link $b(p, q)$ is a nice diagram. By Proposition 2.14, the generalized Floer complexes of the Schubert Heegaard diagram are combinatorial.

From the property of Schubert normal form, it follows a direct description of the Schubert Heegaard diagram.

**Lemma 3.10.** In the Schubert Heegaard diagram of $b(p, q)$,

1. in the disk $D_i^\alpha$, the points $b_i$ and $b_{2p-1-i}$ are connected by a $\beta$-arc,
(2) in the disk $D^2_2$, the points $b_i$ and $b_j$ are connected by a $\beta$-arc if and only if $i + j \equiv 2q - 1 \pmod{2p}$.

3.3. The multi-variable Alexander polynomial of two-bridge links. With the help of link Floer homology, we can directly calculate the multi-variable Alexander polynomial of knots and links. In [15], there is a formula of the Euler characteristic of $HF\hat{L}(L)$:

$$\sum_{h \in H(L)} \chi(HF\hat{L}_d(L, h)) \cdot e^h = \prod_{i=1}^l (T_i^2 - T_i^{-2}) \Delta_L.$$  

Definition 3.11 (Thin complex and $E_2$-collapsed complex). Suppose $(C, \partial)$ is a $\mathbb{Z}^2$-filtered chain complex of $\mathbb{F}$-vector spaces. Let $(i, j)$ denote the filtration, and let $g$ denote the internal grading. The complex $(C, \partial)$ is called \textit{thin}, if $i + j - g$ is a constant for all elements in $C$. The chain complex $C$ is called $E_2$-\textit{collapsed}, if the differential can be decomposed as $\partial = \partial_1 + \partial_2$, such that $F(\partial_1(x)) = F(x) - (1, 0)$ and $F(\partial_2(x)) = F(x) - (0, 1)$, where $F(x)$ is the $\mathbb{Z}^2$-filtration of $x$.

Remark 3.12. A thin complex is $E_2$-collapsed. The classification of $E_2$-collapsed complexes of $\mathbb{F}$-vector spaces is shown in [15] Section 12.1.

Proposition 3.13. Let $L = b(p, q)$ be a two-bridge link, where $p$ is even and $-p < q < p$. Let $A(b_i)$ be the Alexander grading, and let $q^{-1}$ be the number theoretical reciprocal of $q$ modulo $2p$. Then

$$A(b_i) - A(b_{i-1}) = \begin{cases} (-1)^{\lfloor q^{-1}i/p \rfloor}, & i \text{ is even,} \\ (0, (-1)^{\lfloor q^{-1}i/p \rfloor}), & i \text{ is odd.} \end{cases}$$

Furthermore, we have $A_1(b_i) + A_2(b_i) - M(b_i)$ is a constant, i.e. not dependent on $i$, where $M(b_i)$ is the Maslov grading. In other words, the chain complex $A_{-\infty, +\infty}(L)$ is thin.

Proof. We use Notations 3.8. There is a set of bigons of Maslov index 1 connecting $b_i$ and $b_{i+1}$, for $i = 0, 1, \ldots, 2p - 2$. Each of these bigons is a part of one of the disks $D^\beta_1$ and $D^\beta_2$. In fact, these bigons can be obtained by chasing the under-bridges $U_1$ and $U_2$.

The under-bridge $U_1$ starts from $z_1 = a_p = a'_{q-p}$ and passes the disks $D^\alpha_2$ and $D^\alpha_1$ alternately. For $i$ even, at the point $a_i$, if the under-bridge $U_1$ is pointing out of $D^\alpha_2$, then $i \equiv p - 2kq \pmod{2p}$ for some $k$ with $0 < k < \frac{p}{2}$, which is equivalent to $\lfloor \frac{i - q^{-1}}{p} \rfloor$ is even. In this case, there is a bigon $\phi$ from $b_i$ to $b_{i-1}$ with a single basepoint $z_1$ on it, and thereby

$$A(b_i) - A(b_{i-1}) = ((-1)^{\lfloor q^{-1}i/p \rfloor}, 0).$$

If the under-bridge $U_1$ is pointing into $D^\alpha_1$, then $i \equiv 2kq - p \pmod{2p}$ for some $k$ with $0 < k < \frac{p}{2}$, which is equivalent to $\lfloor i - q^{-1} \rfloor$ is odd. In this case, there is a bigon $\phi$ from $b_{i-1}$ to $b_i$ with a single basepoint $z_1$ on it, and still

$$A(b_i) - A(b_{i-1}) = ((-1)^{\lfloor q^{-1}i/p \rfloor}, 0).$$

Similarly, by keeping track of $U_2$, we can prove the other cases. For $i$ odd, at the point $a_i$, if the under-bridge $U_2$ is pointing off $D^\alpha_1$, then $i \equiv p - q - 2kq \pmod{2p}$ for some $k$ with $0 < k < \frac{p}{2}$, which is equivalent to $\lfloor 1 + i - q^{-1} \rfloor$ is even. In this case, there is a bigon $\phi$ of index 1 from $b_i$ to $b_{i-1}$ with a single basepoint $z_2$ on it. Thus, we have $A(b_i) - A(b_{i-1}) = (0, (-1)^{\lfloor q^{-1}i/p \rfloor})$ for $i$ odd.

From Lipshitz’s formula $\mu(\phi) = e(\phi) + \mu_{b_i}(\phi) + \mu_{b_{i-1}}(\phi)$, it follows $\mu(\phi) = 1$, and thereby for all $i$,

$$A_1(b_i) + A_2(b_i) - M(b_i) = A_1(b_{i-1}) + A_2(b_{i-1}) - M(b_{i-1}).$$

\qed
Lemma 3.15. Then $f$ regions in $-\rightarrow$ link $k$gaard diagram of the arc $f$.

Proof. Definition 3.14. In Schubert Heegaard diagram, these bigons are always in a similar form of Figure 3.5, where the Floer complexes for two-bridge links.

The pattern of empty embedded bigons of Maslov index $\Delta b(126,47) = \Delta b(126,55)$ that share the same Alexander polynomial, signature, and linking number, but are not the same or mirror to each other.

$\Delta b(126,47)(x,y) = \Delta b(126,55)(x,y) = -15 + \frac{8}{x} + 8x + \frac{y}{y} + 8y - \frac{4}{xy} - 4xy - \frac{4x}{y} - \frac{4y}{x}$,

$\sigma(b(126,47)) = \sigma(b(126,55)) = 3$,

$\text{lk}(126,47) = \text{lk}(126,55) = 1$.

3.4. The Floer complexes for two-bridge links. Let $\mathcal{H} = (S, \alpha, \beta, w,z)$ be the Schubert Heegaard diagram of $b(p,q)$. By Lemma 3.9, the generalized Floer complex $A^-(\mathcal{H}, s)$ is combinatorial. It consists of counting the empty embedded bigons of Maslov index 1 on $S$, since here $g+k-1 = 1$.

The pattern of empty embedded bigons of Maslov index 1 is illustrated in Figure 3.5 as in [18]. In Schubert Heegaard diagram, these bigons are always in a similar form of Figure 3.5 where the function $f_p$ is defined as follows.

**Definition 3.14.** For all $n, m, k \in \mathbb{Z}$, let $\text{Mod}(n, m, k)$ be the residue of $n$ modulo $m$ starting from $k$, that is,

$$\text{Mod}(n, m, k) \equiv n \pmod{m} \quad \text{and} \quad k \leq \text{Mod}(n, m, k) \leq k + m - 1.$$ 

Then $f_p$ is defined by

$$f_p(n) = |\text{Mod}(n, 2p, -p + 1)|.$$

**Lemma 3.15.** In the Schubert Heegaard diagram $(S, \{\alpha\}, \{\beta\}, \{w_1, w_2\}, \{z_1, z_2\})$ of the two-bridge link $L = b(p,q)$, the regions in $D_1^{\beta}$ are $X_0, Y_q, X_{f_p(2q)}, Y_{f_p(3q)}, \ldots, X_{f_p(pq)} = X_p$ consecutively, and the regions in $D_2^{\beta}$ are $Y_0, X_q, Y_{f_p(2q)}, X_{f_p(3q)}, \ldots, Y_{f_p(pq)} = Y_p$ consecutively.

**Proof.** Note that $D_1^{\beta}$ is the regular neighborhood of the under-bridge $U_i$. The region $X_i$ contains the arc $a_{i-1} \subset L$, and the region $Y_j$ contains arc $a_j' \subset L$. Conversely, the point $a_j$ is contained in $X_{f_p(i)}$, and the point $a_j'$ is contained in $Y_{f_p(j)}$. Thus since $a_i, a_{q-i}$ are glued together and $a_{i-1}, a_{q+i}$
are glued together, $X_{f_p(i)}$ is adjacent to $Y_{f_p(q-i)}$ and $Y_{f_p(q+i)}$. Since $Y_0$ is in $D^3_2$ and it is adjacent to $X_q$, the region $X_q$ is adjacent to $Y_{f_p(2q)}$. Inductively, we can show in $D^3_2$, $Y_{f_p(kq)}$ is adjacent to $X_{f_p((k-1)q)}$ and $X_{f_p([k+1]q)}$. A similar argument applies to $D^3_1$.

**Definition 3.16.** In the bigon $\phi$, denote the number of $\alpha$ arcs in $\phi$ by $n_\alpha(\phi)$, and denote the number $\beta$ arcs in $\phi$ by $n_\beta(\phi)$.

Every bigon $\phi$ is uniquely determined by $n_\alpha(\phi), n_\beta(\phi)$ and the basepoint on it.

**Lemma 3.17 (Patterns of bigons).** In the Schubert Heegaard diagram of $b(p,q)$, suppose $\phi$ is an empty embedded bigon of index 1 in $\pi_2(b_1, b_j)$. Then

$$(i,j) = ((1-n_\alpha)q+n_\beta-1, (1-n_\alpha)q-n_\beta), \quad \text{if } w_1 \in \phi, n_\alpha \text{ is odd,}$$

$$(i,j) = (n_\alpha q-n_\beta, n_\alpha q+n_\beta-1), \quad \text{if } w_1 \in \phi, n_\alpha \text{ is even,}$$

$$(i,j) = (n_\alpha q-n_\beta, n_\alpha q+n_\beta-1), \quad \text{if } w_2 \in \phi, n_\alpha \text{ is odd,}$$

$$(i,j) = ((1-n_\alpha)q+n_\beta-1, (1-n_\alpha)q-n_\beta), \quad \text{if } w_2 \in \phi, n_\alpha \text{ is even.}$$

Furthermore, given $m, n \in \mathbb{Z}$ and a basepoint $p \in \{w_1, w_2, z_1, z_2\}$, there exists at most one empty embedded bigon $\phi$ with $n_\alpha(\phi) = m$, $n_\beta(\phi) = n$, and $n_{w_1}(\phi) = 1$ if and only if the condition $P_1(m,n)$ holds.

The condition $P_1(m,n)$ is as follows:

1. either $m = 1$, or if $m > 1$, then the set of intervals: $[0, n-1]$ and all intervals $[f_p(2iq) - n+1, f_p(2iq) + n-1]$ with $1 \leq 2i \leq m - 1$ are pairwise disjoint intervals in $[0, p]$;
2. either $m = 1$, or if $m > 1$, then the set of intervals: all intervals $[f_p((2i+1)q) - n + 1, f_p((2i+1)q) + n-1]$ with $1 \leq 2i + 1 \leq m - 1$ are also pairwise disjoint intervals in $[1, p-1]$.

Similarly, there exists an empty embedded bigon $\phi$ of index 1 with $n_\alpha(\phi) = m$, $n_\beta(\phi) = n$, and $n_{w_1}(\phi) = 1$ if and only if the condition $P_2(m,n)$ holds.

The condition $P_2(m,n)$ is as follows:

1. either $n = 1$, or $n > 1$ and the set of intervals: $[0, m-1]$ and all intervals $[f_p(2iq) - m+1, f_p(2iq) + m-1]$ with $1 \leq 2i \leq n-1$ are pairwise disjoint intervals in $[0, p]$;
2. either $n = 1$, or $n > 1$ and the set of intervals: all intervals $[f_p((2i+1)q) - m + 1, f_p((2i+1)q) + m-1]$ with $1 \leq 2i + 1 \leq n-1$ are also pairwise disjoint intervals in $[1, p-1]$.

In addition, for $i = 1, 2$, there is a one-to-one correspondence between the set of all the empty embedded bigons with $n_{w_i} = 1$ and the set of empty embedded bigons with $n_{z_i} = 1$, where the bigon $\phi \in \pi_2(b_1, b_{j'})$ with $n_\alpha = m, n_\beta = n, n_{w_i} = 1$ is sent to the bigon $\phi' \in \pi_2(b_i + p, b_{j+1} + p)$ with $n_\alpha = m, n_\beta = n, n_{z_i} = 1$.

**Proof.** Suppose $\phi$ is a bigon of index 1 in $\pi_2(b_1, b_{j'})$ with $n_{w_i}(\phi) = 1$. Combining Lemma 3.10 and Lemma 3.15 we can get the formula of $(i,j)$ out of Figure 3.5 by induction on $n_\alpha, n_\beta$. The initial step is $(i,j) = (q-1, q)$, for $n_\alpha = n_\beta = 1$. Similarly, we can show the other case where $n_{w_1}(\phi) = 1$.

For the second part, the sufficient and necessary condition of when there exists an empty embedded bigon is that all the regions in the bigon are not overlapped. By Lemma 3.15, it is not hard to get the formulas by induction.

Finally, notice that there is a symmetry of the Heegaard Schubert diagram which sends $b_k$ to $b_{k-p}$ and exchanges $w_i$ to $z_i$ for all $1 \leq k \leq 2p, i = 1, 2$. This symmetry directly gives the one-to-one correspondence between the bigons with $w_i$ and the ones with $z_i$.

Consequently, we get an algorithm for computing $A_s^-(b(p,q))$ as follows.
Theorem 3.18. Let \( L = b(p,q) \) be a two-bridge link and \( \mathcal{H} \) be the Schubert Heegaard diagram. Define functions \( F_i : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}/2p\mathbb{Z} \times \mathbb{Z}/2p\mathbb{Z}, i = 1, 2, \) by

\[
F_1(m, n) = \begin{cases} ((1 - m)q + n - 1, (1 - m)q - n), & \text{if } m \text{ is odd}, \\ (mq - n, mq + n - 1), & \text{if } m \text{ is even}. \end{cases}
\]

\[
F_2(m, n) = \begin{cases} ((1 - m)q + n - 1, (1 - m)q - n), & \text{if } m \text{ is even}, \\ (mq - n, mq + n - 1), & \text{if } m \text{ is odd}. \end{cases}
\]

The conditions \( P_i(m, n), i = 1, 2 \) are as in Lemma 3.17. Then, the complex \( \mathcal{A}^{-}(\mathcal{H}, +\infty, +\infty) \) is a free \( \mathbb{F}[\{U_1, U_2\}] \)-module generated by \( g_0, \ldots, g_{2p-1} \) with differentials

\[
\partial g_i = \sum_{j=0}^{2p-1} (\lambda_{i,j} + \mu_{i,j} + \mu_{i,j} + p)g_j + \sum_{j=0}^{2p-1} \lambda_{i,j} U_1 g_j + \sum_{j=0}^{2p-1} \mu_{i,j} U_2 g_j,
\]

where the coefficients \( \lambda_{i,j}, \mu_{i,j} \in \mathbb{Z}/2\mathbb{Z} \) are determined by the following equations

\[
\lambda_{i,j} = \#\{(m, n) \in \mathbb{N} \times \mathbb{N} | 1 \leq m, n \leq p, F_1(m, n) = (i, j), P_1(m, n) \text{ is true} \} \quad (\text{mod } 2),
\]

\[
\mu_{i,j} = \#\{(m, n) \in \mathbb{N} \times \mathbb{N} | 1 \leq m, n \leq p, F_2(m, n) = (i, j), P_2(m, n) \text{ is true} \} \quad (\text{mod } 2).
\]

Remark 3.19. One can get an algorithm of \( O(p^2) \) time complexity for computing \( A^{-}_s(b(p,q)) \). To compute \( A^{-}_s(b(p,q)) \), we only need to know \( A^{-}_{+\infty, +\infty}(b(p,q)) \), which is determined by all the counting of bigons, i.e. those \( \lambda_{i,j} \)'s and \( \mu_{i,j} \)'s. Computing \( \lambda_{i,j} \)'s and computing \( \mu_{i,j} \)'s are similar. In order to get all the \( \lambda_{i,j} \)'s, one can nest two loops. The outer loop is indexed by \( n \geq 1 \), and the inner loop is indexed by \( m \geq 1 \) with a test condition \( P_1(m,n) \). When \( P_1(m,n) \) is true, we change the value of \( \lambda_{F_1(m,n)} \) by \( 1 \) (mod 2) and keep running the inner loop; when \( P_1(m,n) \) is false, we stop the inner loop and go back to the outer loop.

Let us estimate the time complexity. Switching the \( \alpha \) and \( \beta \) roles converts the Heegaard diagram of \( b(p,q) \) to its mirror \( b(-p,-q) \). Thus, we assume \( 0 < q < p \). First, when \( P_1(m,n) \) is true and \( n > q, m \) must be 1, as otherwise the second part of \( P_1(m,n) \) would imply \( f_p(q) - n + 1 = q - n + 1 \geq 1 \). Thus we force the outer loop stop when \( n = q + 1 \). Computing the other \( \lambda_{F_1(m,n)} \)'s with \( (m,n) = (1,n) \), \( n > p \) can be done within \( O(p) \) operations. Second, if \( P_1(m,n) \) is true, then \( m \leq (p + 1)/n \). This is because the first part of \( P_1(m,n) \) implies that there are \( m \) pieces of open intervals \( (f_p(2i) - n + \frac{1}{2}, f_p(2i) + n - \frac{1}{2}) \) pairwise disjoint in \( (-\frac{1}{2}, p + \frac{1}{2}) \). Thus, at the \( n \)th step of the outer loop, the inner loop stops within \( [(p+1)/n] \) steps. Finally, testing \( P_1(m,n) \) can be done within \( 2m \) steps. In fact, when \( m \) is even, we can check if the new interval \( [f_p(mq) - n + 1, f_p(mq) + n - 1] \) is disjoint from the other \( \frac{m}{2} - 1 \) intervals in the first part of \( P_1(m,n) \) which are already ordered in the previous step. If it is disjoint from the other intervals, we put it in the correct position in the order. This is done within \( 2m \) operations. It is similar when \( m \) is odd. Thus the time complexity is of the order

\[
T(p,q) = \left( \sum_{n=1}^{q} \frac{\lfloor (p+1)/n \rfloor}{n} \right) + O(p) \leq \left( \sum_{n=1}^{q} \left( \frac{p+1}{n} \right)^2 + \frac{p+1}{n} \right) + O(p).
\]

Since \( n \leq q \leq p, (p+1)/n \geq 1 \), thus

\[
T(p,q) \leq 2 \sum_{n=1}^{q} \left( \frac{(p+1)^2}{n^2} \right) + O(p) = O(p^2).
\]

4. Link surgery formula

In this section, we review the link surgery formula of Manolescu-Ozsváth for two-component links with basic diagrams. In Section 4.1, we review some algebra on hyperboxes of chain complexes and introduce twisted gluing of squares of chain complexes. In Section 4.2, we express the link surgery
formulas for a two-component link as a twisted gluing of certain squares of chain complexes derived from the link. These squares are elaborated in Section 4.3, by using primitive systems of hyperboxes. The primitive systems of hyperboxes are generalizations of the basic systems of hyperboxes used in \[6\]. One can consult \[9\] for the full generality of link surgery formula with general Heegaard diagrams. We assume that the reader is familiar with Heegaard Floer homology \[13, 12, 14, 15\].

Throughout, \(L = L_1 \cup L_2\) will be an oriented link in \(S^3\), and \(\hat{M}\) will denote an oriented sublink of \(\hat{L}\) which may not have the induced orientation from \(\hat{L}\) on each component.

4.1. **Hyperboxes of chain complexes.**

4.1.1. **Hyperboxes of chain complexes.**

**Definition 4.1** (Hyperbox). An \(n\)-dimensional hyperbox of size \(d = (d_1, ..., d_n) \in \mathbb{Z}_{\geq 0}^n\) is the subset 
\[ E(d) = \{(\varepsilon_1, ..., \varepsilon_n) \in \mathbb{Z}_{\geq 0}^n : 0 \leq \varepsilon_i \leq d_i\}. \]

If \(E(d) = \{0, 1\}^n\), then \(E(d)\) is called a hypercube, denoted by \(E_n\).

**Definition 4.2** (Hyperbox of chain complexes). Let \(R\) be an \(\mathbb{F}\)-algebra. An \(n\)-dimensional hyperbox of chain complexes of size \(d \in \mathbb{Z}_{\geq 0}^n\) is a collection of \(\mathbb{Z}\)-graded \(R\)-modules 
\[ (C_\varepsilon)_{\varepsilon \in E(d)}; C_\varepsilon = \bigoplus_{\ast \in \mathbb{Z}} C_{\ast \varepsilon}, \]

together with a collection of \(R\)-linear maps 
\[ D^\varepsilon_{\varepsilon'} : C_{\varepsilon'} \to C_{\varepsilon' + \varepsilon}, \]

one map for each \(\varepsilon' \in E(d)\) and \(\varepsilon \in E_n\) such that \(\varepsilon' + \varepsilon \in E(d)\). The maps are required to satisfy the relations 
\[ \sum_{\varepsilon' \leq \varepsilon} D_{\varepsilon' + \varepsilon'}^\varepsilon \circ D_{\varepsilon'}^\varepsilon = 0, \]

for all \(\varepsilon' \in E(d), \varepsilon \in E_n\) such that \(\varepsilon' + \varepsilon \in E(d)\).

By abuse of notation, we sometimes let \(D^\varepsilon\) stand for any of its maps \(D^\varepsilon_{\varepsilon'}\). Note that a hypercube of chain complexes \(H\) gives rise to a total complex of the hypercube \(\text{Tot}(H)\).

**Example 4.3** (1-dimensional hyperboxes). A 1-dimensional hyperbox of chain complexes is a sequence of chain complexes \(C_n\), together with a sequence of chain maps \(f_n : C^{(n-1)} \to C^{(n)}\).

\[ C^{(0)} \xrightarrow{f_1} C^{(1)} \xrightarrow{f_2} C^{(2)} \xrightarrow{f_3} \ldots \xrightarrow{f_{n-1}} C^{(n-1)} \xrightarrow{f_n} C^{(n)}. \]

The total complex of a 1-dimensional hypercube of chain complexes can be regarded as a mapping cone. Therefore, we also call a 1-dimensional hyperbox of chain complexes a sequence of chain complexes.

**Example 4.4** (2-dimensional hyperboxes). A square of chain complexes is a 2-dimensional hypercube of chain complexes:

\[ C^{(0,0)} \xrightarrow{D^{(0,0)}_{(1,0)}} C^{(1,0)} \]

Here \(D^{(1,0)}\) and \(D^{(0,1)}\) are chain maps, and \(D^{(1,1)}\) is a chain homotopy between \(D^{(0,1)} \circ D^{(1,0)}\) and \(D^{(1,0)} \circ D^{(0,1)}\). We can regard the total complex of this square as the mapping cone of.
A rectangle of chain complexes is a 2-dimensional hyperbox of chain complexes. It consists of squares of chain complexes. A rectangle of chain complexes of size $(m, 1)$ can also be regarded as a sequence of mapping cones, i.e. a size $(m)$ 1-dimensional hyperbox of mapping cones.

Let $R = (C, D)$ be a hyperbox of chain complexes of size $(d_1, d_2, \ldots, d_n)$. Fixing $1 \leq i \leq n$, for any integer $0 \leq l \leq d_i$, we have a hyperbox $R_{\varepsilon_i = l} = (C_{\varepsilon_i = l}, D_{\varepsilon_i = l})$ of size $(d_1, \ldots, d_{i-1}, 0, d_{i+1}, \ldots, d_n)$, which consists of the chain complexes $C^{(\varepsilon_1, \ldots, \varepsilon_n)}$ with $\varepsilon_i = l$. The differentials $D_{\varepsilon_i = l}$ consist of all the differentials $D_{\varepsilon_0}$ of $(C, D)$ inside $R_{\varepsilon_i = l}$.

**Remark 4.5.** In general, a hyperbox of chain complexes is not a chain complex. But a hypercube is a chain complex considered as the total complex, and it can also be regarded as a mapping cone in many ways.

**4.1.2. Compression.** From a hyperbox of chain complexes $H = ((C^\varepsilon)_{\varepsilon \in \mathbb{E}(d)}, (D^\varepsilon)_{\varepsilon \in \mathbb{E}_n})$, we can obtain a hypercube of chain complexes $\hat{H} = (\hat{C}^\varepsilon, \hat{D}^\varepsilon)_{\varepsilon \in \mathbb{E}_n}$, thus generating a total complex $\text{Tot}(\hat{H})$. The process of turning $H$ into $\hat{H}$ is called compression.

**Example 4.6** (Compression of 1-dimensional hyperboxes). Let $R$ be a hyperbox of dimension 1, see Example 4.3. The compression $\hat{R}$ is the mapping cone of the composition of the maps $f_1, \ldots, f_n$

$$C^{(0)} \xrightarrow{f_n \circ \cdots \circ f_1} C^{(n)}.$$

**Example 4.7** (Compression of 2-dimensional hyperboxes). Consider a rectangle of chain complexes $R$ of size $(n, 1)$:

As in 4.3, we can regard this rectangle as a 1-dimensional hyperbox of mapping cones $\text{cone}(k_i), i = 0, 1, \ldots, n$. The compression of 1-dimensional hyperboxes induces the compression of rectangles of chain complexes as follows

$$C^{(0, 0)} \xrightarrow{f_n \circ \cdots \circ f_1} C^{(n, 0)}$$

$$k_0 \xrightarrow{\hat{H}} k_n$$

$$C^{(0, 1)} \xrightarrow{g_n \circ \cdots \circ g_1} C^{(n, 1)},$$

where

$$\hat{H} = \sum_{i=1}^{n} f_1 \circ \cdots \circ f_{i-1} \circ H_i \circ g_{i+1} \circ \cdots \circ g_n.$$
For higher dimensional hyperbox, the compression is defined similarly by induction, once we fix an order of the coordinate axes. Let us describe this procedure using the language of composing chain maps of hyperboxes. One can check that it is the same as the compression by means of the algebra of songs introduced in [6].

Let \(0 H = \left(\left(1 C^e\right)_{\varepsilon \in \mathbb{E}(d)}, \left(1 D^e\right)_{\varepsilon \in \mathbb{E}(n)}\right), 1 H = \left(\left(1 C^e\right)_{\varepsilon \in \mathbb{E}(d)}, \left(1 D^e\right)_{\varepsilon \in \mathbb{E}(n)}\right)\) be two hyperboxes of chain complexes, having the same size \(d \in \mathbb{Z}_{\geq 0}\). Let \((d, 1) \in \mathbb{Z}_{n+1}^{\geq 0}\) be the sequence obtained from \(d\) by adding 1 at the end.

**Definition 4.8 (Chain maps of hyperboxes).** A chain map \(F: 0 H \rightarrow 1 H\) is a collection of linear maps

\[
F_{\varepsilon^0}^\varepsilon : 0 C_{\varepsilon^0} \rightarrow 1 C_{\varepsilon^0 + \varepsilon}^\varepsilon \]

satisfying

\[
\sum_{e' \leq \varepsilon} \left(D_{\varepsilon^0 + e'}^0 \circ F_{\varepsilon^0}^{e'} + F_{\varepsilon^0}^{e-e'} \circ D_{\varepsilon^0}^{e'}\right) = 0,
\]

for all \(\varepsilon^0 \in \mathbb{E}(d), \varepsilon \in \mathbb{E}(n)\) such that \(\varepsilon^0 + \varepsilon \in \mathbb{E}(d)\).

In other words, a chain map between the hyperboxes \(0 H\) and \(1 H\) is an \((n + 1)\)-dimensional hyperbox of chain complexes, of size \((d, 1)\), such that the sub-hyperbox corresponding to \(\varepsilon_{n+1} = 0\) is \(0 H\) and the one corresponding to \(\varepsilon_{n+1} = 1\) is \(1 H\). The maps \(F\) are those maps \(D\) in the new hyperbox that increase \(\varepsilon_{n+1}\) by 1. Direct computations show the associativity \((F \circ G) \circ H = F \circ (G \circ H)\).

For a \(n\)-dimensional hyperbox \(H\) of size \(d = (d_1, ..., d_n)\), we fix an order of the axes, say, the increasing order \(1, 2, ..., n\). The hyperbox \(H\) can be decomposed into \(d_n\) pieces of hyperboxes of size \((d_1, ..., d_{n-1}, 1)\), which is a chain map \(F_i : H_{\varepsilon_{n+1} + 1} \rightarrow H_{\varepsilon_{n+1} = i}^e\). Thus the composition \(F_{d_n} \circ \cdots \circ F_1\) is a hyperbox of size \((d_1, ..., d_{n-1}, 1)\), and we call it the compression along the \(i^{th}\) axis \(\text{Comp}_i(H)\). If we keep doing compressions for the other axes, then we get the compression \(\hat{H} = \text{Comp}_1 \circ \cdots \circ \text{Comp}_n(H)\).

4.1.3. **Gluing of squares.** In the link surgery formula, an algebraic operation occurs, which we could call a twisted gluing of hypercubes. It consists in repeatedly gluing mapping cones \(A \xrightarrow{f} B, A \xrightarrow{g} B\) to get a new mapping cone \(A \xrightarrow{f+g} B\). In this section we describe this operation in detail for the case of two-component links. We call it the twisted gluing of framed product squares.

**Remark 4.9.** In Section 4.1.1, a hypercube of chain complexes requires a \(\mathbb{Z}\)-grading on it. However, after gluing of hypercubes, it does not always admit a \(\mathbb{Z}\)-grading, but admits a \(\mathbb{Z}/2\mathbb{Z}\)-grading. Now we only require a \(\mathbb{Z}/2\mathbb{Z}\)-grading on each chain complex sitting at a vertex in the hypercube.

**Definition 4.10 (Gluing of squares).** Suppose there are four squares of chain complexes \(R_{i,j} = (C_{i,j}^e, D_{i,j}^e), i, j = 0, 1\) as listed below,

\[
\begin{array}{cccc}
R_{i,j} : & C_{i,j}^{(0,0)} & \xrightarrow{D_{i,j}^{(1,0)}} & C_{i,j}^{(1,0)} \\
& & D_{i,j}^{(0,1)} & \xrightarrow{D_{i,j}^{(1,1)}} & D_{i,j}^{(1,1)} & \xrightarrow{D_{i,j}^{(1,1)}} & C_{i,j}^{(1,1)} \\
& C_{i,j}^{(0,1)} & \xrightarrow{D_{i,j}^{(1,0)}} & C_{i,j}^{(1,0)} \\
\end{array}
\]

The squares \(\{R_{i,j}\}_{i,j}\) are called gluable, if \(C_{i,0}^{e} = C_{0,1}^{e} = C_{1,0}^{e} = C_{1,1}^{e}\) for all \(e \in \mathbb{E}_2\) and \(D_{i,0}^{(1,0)} = D_{i,1}^{(1,0)} = D_{i,0}^{(0,1)} = D_{i,1}^{(0,1)} = D_{i,j}^{(0,1)} = D_{i,j}^{(1,1)} = D_{i,j}^{(1,0)} = D_{i,j}^{(0,0)}\) for all \(i, j = 0, 1\). Then we can define \(R = (C^e, D^e)\) to be the gluing of \(R_{i,j}\)’s as below, where we suppress the subscripts \(i, j, e\) of \(C_{i,j}^e\). One can check that \(R\)
is a square of chain complexes.

\[
R := \begin{array}{c}
\begin{array}{c}
C^{(0,0)} \\
D_0^{(0,0)} + D_1^{(0,0)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
D_0^{(1,0)} + D_1^{(1,0)} \\
\sum_{i,j} D_{i,j}^{(1,1)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
C^{(1,0)} \\
D_0^{(0,0)} + D_1^{(0,0)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
C^{(1,1)} \\
D_0^{(0,0)} + D_1^{(0,0)}
\end{array}
\end{array}
\end{array}
\]

**Definition 4.11** (Framed product square). A \( \mathbb{Z}^2 \)-product square of chain complexes is a direct product of squares of chain complexes

\[
R = \prod_{s \in \mathbb{Z}^2} R_s,
\]

where \( R_s \) is a square of chain complexes for all \( s \in \mathbb{Z}^2 \). We call \( s \) the coordinate of any element \( x \in R_s \). The function

\[
F : R \to \mathbb{Z}^2, \quad F(x) = s, \forall s \in R_s,
\]

is called the framing of \( R \). In order to denote the framing, we write a framed product square as a pair

\[
(R, F) = \prod_{s \in \mathbb{Z}^2} (C^e_s, D^e_s).
\]

We can shift the framing \( F \) by a set of vectors in \( \mathbb{Z}^2 \), \( V = \{v^e\}_{e \in \mathbb{E}_2} \), to get a new framing \( F^V \), such that

\[
\forall x \in C^e_s, \quad F^V(x) = F(x) + v^e.
\]

We call the new framed product square \((R, F^V)\) the shifted square of \( R \) by \( V \), and simply denote it by \( R[V] \). Thus, we can write \( R[V] = \prod_{s \in \mathbb{Z}^2} (\tilde{C}^e_s, \tilde{D}^e_s) \), where \( \tilde{C}^e_{s+v^e} = C^e_s, \forall e \in \mathbb{E}_2, \forall s \in \mathbb{Z}^2 \).

**Definition 4.12** (Framed gluable). Let \((R_{i,j}, F_{i,j})\) with \( i, j = 0, 1 \) be a set of framed product squares of chain complexes. The set of four squares \( \{R_{i,j}\}_{i,j} \) is called framed gluable, if \( \{R_{i,j}\}_{i,j} \) are gluable as squares of chain complexes and all the framings \( F_{i,j}, \forall i, j \) are the same. Then, the result is called the framed gluing of \((R_{i,j}, F_{i,j})\)’s.

**Definition 4.13** (Twisted gluing). Let \((R_{i,j}, F_{i,j})\) with \( i, j = 0, 1 \) be a set of framed product squares of chain complexes. For any matrix \( \Lambda = (\Lambda_1, \Lambda_2) \in \mathbb{Z}^{2 \times 2} \), let

\[
V_{i,j}(\Lambda) = \{v^e = \Lambda \cdot (i \varepsilon_1, j \varepsilon_2)\}_{\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{E}_2}, \quad \forall i, j = 0, 1.
\]

Then there are four shifted squares \( R_{i,j}[V_{i,j}(\Lambda)] \), with \( i, j = 0, 1 \). As long as these four shifted squares \( \{R_{i,j}[V_{i,j}(\Lambda)]\}_{i,j = 0, 1} \) are framed gluable, we define the \( \Lambda \)-twisted gluing of \( \{R_{i,j}\}_{i,j} \) to be the framed gluing of \( \{R_{i,j}[V_{i,j}(\Lambda)]\}_{i,j} \), denoted by \( R^\Lambda \). See Figure 4.1 for an example of twisted gluing.

**Example 4.14** (Twisted gluing in link surgery formula). Suppose for any \((i, j) \in \mathbb{E}_2\),

\[
R_{i,j} = \prod_{s \in \mathbb{Z}^2} R_{s, i,j}
\]

with \( R_{s, i,j} = (C^e_{s, i,j}, D^e_{s, i,j}) \) is a framed product square of chain complexes with the natural framing \( F_{i,j}(x) = s, \forall x \in C_{s, i,j} \). Let \( \mathcal{L} \) be a two-component link and \( \text{lk} \) be the linking number. Given any surgery framing matrix

\[
\Lambda = \begin{pmatrix}
\lambda_1 & \text{lk} \\
\text{lk} & \lambda_2
\end{pmatrix},
\]

...
as long as the following identities hold for all $s \in \mathbb{Z}^2$,

$$
C_{s,0,0}^{(0,0)} = C_{s,1,0}^{(0,0)} = C_{s,1,1}^{(0,0)} = C_{s,1,1},
$$

$$
C_{s,0,0}^{(1,0)} = C_{s,0,1}^{(1,0)} = C_{s-\Lambda_1,1,0}^{(1,0)} = C_{s-\Lambda_1,1,1}^{(1,0)},
$$

$$
C_{s,0,0}^{(0,1)} = C_{s-\Lambda_2,0,1}^{(0,1)} = C_{s-\Lambda_2,0,1}^{(0,1)} = C_{s-\Lambda_2,0,1},
$$

$$
C_{s,0,0}^{(1,1)} = C_{s-\Lambda_1,1,0}^{(1,1)} = C_{s-\Lambda_1,1,0}^{(1,1)} = C_{s-\Lambda_1-\Lambda_2,1,1},
$$

the shifted squares $\{R_{i,j}[V_{i,j}(\Lambda)]\}_{i,j}$ are framed gluable. Thus we define the twisted gluing of squares $R^\Lambda$. 

**Figure 4.1. An example of twisted gluing.** This is an example of $\left(\begin{array}{cc} -1 & 1 \\ 1 & 2 \end{array}\right)$-twisted gluing of four squares $\{R_{s,i,j} = \prod_{s \in \mathbb{Z}^2} R_{s,i,j}\}_{i,j=0,1}$, where $R_{s,i,j} = (C_{s,i,j}^s, D_{s,i,j})$. Since $C_{s,i,j}$ is identified with some $C_{s,i,j}^s$, we omit the subscripts $i,j$ in the picture. Every shaded circle encloses a factor $R_{s,0,0}$ of the $\mathbb{Z}^2$-product square $R_{0,0}$ with some $s \in \mathbb{Z}^2$. The yellow parallelogram indicates the $D$-maps of $R_{0,1}$, which is shifted by the vector $\Lambda_2$; whereas the red parallelogram indicates the $D$-maps of $R_{1,0}$, which is shifted by the vector $\Lambda_1$. The gray parallelogram indicates all the maps of the square $R_{1,1}$, which is shifted by using both $\Lambda_1$ and $\Lambda_2$. 


Remark 4.15. The twisted glued square $R^4$ no longer decomposes as a $\mathbb{Z}^2$ direct product. However, it decomposes as a direct sum $\bigoplus_{u \in \mathbb{Z}^2/\Lambda} R^4(u)$, where $\Lambda$ is viewed as a lattice spanned by $\Lambda_1, \Lambda_2$. The equivalence classes $\mathbb{Z}^2/\Lambda$ correspond to the Spin$^c$ structures over the surgery manifold on a link in $S^3$.

4.2. Link surgery formula for a two-component link $\overrightarrow{L} = L_1 \cup L_2$. In order to denote the orientations of the sublinks, we use $\pm$ signs to denote the positive and negative orientations, where the positive orientation is the induced orientation from $\overrightarrow{L}$ and the negative orientation is the opposite orientation. Let $L = +L_1 \cup +L_2$, that is, $L$ has the same orientation as $\overrightarrow{L}$.

The link surgery formula is the total complex of a square of chain complexes: the chain complexes at the vertices are the generalized Floer complexes described in Section 2.2, and the maps in the square are defined by means of complete systems of hyperboxes (see section 4.3 for the definition). From a complete system of hyperboxes of $L$, we get four sets of squares of chain complexes $R_{s,i,j}$, where $s \in \mathbb{H}(L), i, j \in \{0, 1\}$:

\begin{align}
(4.2) \quad R_{s,0,0} : \quad & \mathfrak{A}^- (\mathcal{H}^L, s) \xrightarrow{\Phi_{s}^{+L_1}} \mathfrak{A}^- (\mathcal{H}^{L_2}, \psi^{+L_1}(s)) \\
& \xrightarrow{\Phi_{s}^{+L_2}} \mathfrak{A}^- (\mathcal{H}^{L_1}, \psi^{+L_2}(s)) \xrightarrow{\Phi_{s}^{+L_1 \cup +L_2}} \mathfrak{A}^- (\mathcal{H}^{\theta}, \psi^{+L_1 \cup +L_2}(s));
\end{align}

\begin{align}
(4.3) \quad R_{s,1,0} : \quad & \mathfrak{A}^- (\mathcal{H}^L, s) \xrightarrow{\Phi_{s}^{-L_1}} \mathfrak{A}^- (\mathcal{H}^{L_2}, \psi^{-L_1}(s)) \\
& \xrightarrow{\Phi_{s}^{-L_2}} \mathfrak{A}^- (\mathcal{H}^{L_1}, \psi^{-L_2}(s)) \xrightarrow{\Phi_{s}^{-L_1 \cup -L_2}} \mathfrak{A}^- (\mathcal{H}^{\theta}, \psi^{-L_1 \cup -L_2}(s));
\end{align}

\begin{align}
(4.4) \quad R_{s,0,1} : \quad & \mathfrak{A}^- (\mathcal{H}^L, s) \xrightarrow{\Phi_{s}^{+L_1}} \mathfrak{A}^- (\mathcal{H}^{L_2}, \psi^{+L_1}(s)) \\
& \xrightarrow{\Phi_{s}^{+L_2}} \mathfrak{A}^- (\mathcal{H}^{L_1}, \psi^{+L_2}(s)) \xrightarrow{\Phi_{s}^{+L_1 \cup -L_2}} \mathfrak{A}^- (\mathcal{H}^{\theta}, \psi^{+L_1 \cup -L_2}(s));
\end{align}

\begin{align}
(4.5) \quad R_{s,1,1} : \quad & \mathfrak{A}^- (\mathcal{H}^L, s) \xrightarrow{\Phi_{s}^{-L_1}} \mathfrak{A}^- (\mathcal{H}^{L_2}, \psi^{-L_1}(s)) \\
& \xrightarrow{\Phi_{s}^{-L_2}} \mathfrak{A}^- (\mathcal{H}^{L_1}, \psi^{-L_2}(s)) \xrightarrow{\Phi_{s}^{-L_1 \cup +L_2}} \mathfrak{A}^- (\mathcal{H}^{\theta}, \psi^{-L_1 \cup +L_2}(s)),
\end{align}

where $\psi^M$ is defined in Equation (2.1). Thus, we have four framed product squares with the natural framings

$$R_{i,j} = \prod_{s \in \mathbb{H}(L)} R_{s,i,j}.$$
Definition 4.16 (Link surgery formula). For any surgery framing matrix $\Lambda$, the shifted squares \( \{R_{i,j}[V_{i,j}(\Lambda)]\}_{i,j} \) are framed gluable according to Equations (4.1). The link surgery formula for the framed link \((L, \Lambda)\) is the total complex of the $\Lambda$-twisted gluing of \( \{R_{i,j}\}_{i,j} \) as follows,

\[
(C^-(\mathcal{H}, \Lambda), D^-) := \prod_{s \in \mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^L, s) \xrightarrow{\Phi^+L_1+\Phi^-L_1} \prod_{s \in \mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{L_2}, \psi^+L_1(s)) \xrightarrow{\Phi^+L_2+\Phi^-L_2} \prod_{s \in \mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{L_1}, \psi^+L_2(s)) \xrightarrow{\Phi^+L_1+\Phi^-L_1} \prod_{s \in \mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^\emptyset, \psi^+L_1(s)),
\]

where $\Phi^0 = \prod_{s \in \mathbb{H}(L)} \Phi^0_s$ with $\circ = \pm L_1, \pm L_2, \pm L_1 \cup \pm L_2$.

The map

\[
\tilde{\Phi}^M_s : \mathfrak{A}^-(\mathcal{H}^L, s) \rightarrow \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(s))
\]

is defined by

\[\tilde{\Phi}^M_s = D^M_{\mathfrak{p}^M(s)} \circ T^M_s.\]

We will spell out the constructions of $D^M_{\mathfrak{p}^M(s)}$ and $T^M_s$ in the next sections by using primitive systems of hyperboxes.

For the $\Lambda$-twisted gluing of squares, there is a direct sum splitting of the complex

\[
C^-(\mathcal{H}, \Lambda) = \bigoplus_{u \in \mathbb{H}(L)/H(L, \Lambda)} C^-(\mathcal{H}, \Lambda, u),
\]

where we identify $\mathbb{H}(L)/H(L, \Lambda)$ with $\text{Spin}^c(S^3(L))$.

**Theorem 4.17** (Manolescu-Ozsváth Link Surgery Theorem, Theorem 7.7 of [6] for two-component links). Fix a primitive system of hyperboxes $\mathcal{H}$ for an oriented two-component link $\mathcal{L}$ in $S^3$, and fix a framing $\Lambda$ of $L$. Then for any $u \in \text{Spin}^c(S^3(L)) \cong \mathbb{H}(L)/H(L, \Lambda)$, there is an isomorphism of $\mathbb{F}[[U]]$-modules

\[
H_*C^-(\mathcal{H}, \Lambda, u, D^-) \cong HF^-(S^3(L), u),
\]

where $\mathbb{F}[[U]] = \mathbb{F}[[U_1, U_2]]/(U_1 - U_2)$.

Here, we let $HF^-$ denote the completion of $HF^-$ with respect to the maximal ideal $(U)$ in the ring $\mathbb{F}[U]$. Since completion is an exact functor, $HF^-$ can be regarded as the homology of the complex $CF^- = CF^\infty \otimes_{\mathbb{F}[[U]]} \mathbb{F}[[U]]$, where $\mathbb{F}[[U]]$ is the completion of $\mathbb{F}[U]$. When $s$ is a torsion $\text{Spin}^c$ structure of a 3-manifold $M$, if

\[
HF^-(M, s) = \mathbb{F}[[U]] \oplus T
\]

with $T$ a torsion $\mathbb{F}[[U]]$-module, then

\[
HF^-(M, s) = \mathbb{F}[U] \oplus T.
\]

For more details, see Section 2 in [6].

**Remark 4.18.** The link surgery theorem states that all the $U_i$-actions are the same in the homology of the surgery complex.

**Remark 4.19.** Although all the squares in Equations (4.2) to (4.5) posses $\mathbb{Z}$-gradings, the surgery complex $C^-(\mathcal{H}, \Lambda)$ does not always have a $\mathbb{Z}$-grading after the twisted gluing. In [6], an absolute grading was also given in Section 7.4, which is the same as the absolute grading on Heegaard Floer homology of the surgery manifold.
4.3. Inclusion maps and destabilization maps.

4.3.1. Inclusion maps. In the link surgery formula, we need a set of chain maps $I^\overline{M}$ in (4.6) which are called inclusion maps. Here, we define the inclusion maps for all links with arbitrary number of components. In the knot case, the inclusion maps correspond to the maps $h_s$ and $v_s$ from $[16].$

**Definition 4.20.** Let $\overline{M}$ be an oriented sublink of $\overline{L}$. Define 

\[
I_+(\overline{L}, \overline{M}) = \{i : \overline{L} \text{ and } \overline{M} \text{ share the same orientation on } L_i\};
\]

\[
I_-(\overline{L}, \overline{M}) = \{i : \overline{L} \text{ and } \overline{M} \text{ have different orientations on } L_i\}.
\]

A projection map $p^\overline{M} : \mathbb{H}(L) \to \mathbb{H}(L)$ is defined component-wisely as follows:

\[
p_i^\overline{M}(s) = \begin{cases} 
+\infty & \text{if } i \in I_+(\overline{L}, \overline{M}) \\
-\infty & \text{if } i \in I_-(\overline{L}, \overline{M}) \\
n & \text{otherwise}
\end{cases}
\]

**Definition 4.21** (Inclusion maps). Suppose $\overline{M} \subset \overline{L}$ is an oriented sublink, and $s = (s_1, s_2) \in \mathbb{H}(L)$ satisfies $s_i \neq \mp \infty$ for those $i \in I_\pm(\overline{L}, \overline{M})$. Let $\mathcal{H}$ be a Heegaard diagram of $L$. The inclusion map $I^\overline{M}_s : \mathfrak{A}^-(\mathcal{H}^L, s) \to \mathfrak{A}^-(\mathcal{H}^L, p^\overline{M}(s))$ is defined by the formula

\[
I^\overline{M}_s(x) = \prod_{i \in I_+(\overline{L}, \overline{M})} U^\max(A_i(x) - s_i, 0) \prod_{i \in I_-(\overline{L}, \overline{M})} U^\max(s_i - A_i(x), 0)x.
\]

One can verify this is a chain map.

**Example 4.22.** Suppose $\overline{L}$ is a two-component link and $\mathcal{H}$ is a basic Heegaard diagram of $L$. We have the following inclusion maps

- $I^{+L_1}_{s_1, s_2} : A^-_{s_1, s_2} \to A^-_{s_1, s_2} + \infty, s_2$.
- $I^{-L_1}_{s_1, s_2} : A^-_{s_1, s_2} \to A^-_{s_1, s_2} - \infty, s_2$.
- $I^{+L_2}_{s_1, s_2} : A^-_{s_1, s_2} \to A^-_{s_1, s_2} + \infty, s_2$.
- $I^{-L_2}_{s_1, s_2} : A^-_{s_1, s_2} \to A^-_{s_1, s_2} - \infty, s_2$.
- $I^{+L_1 \cup L_2}_{s_1, s_2} : A^-_{s_1, s_2} \to A^-_{s_1, s_2} + \infty, + \infty$.
- $I^{-L_1 \cup L_2}_{s_1, s_2} : A^-_{s_1, s_2} \to A^-_{s_1, s_2} - \infty, - \infty$.
- $I^{+L_1 \cup L_2}_{s_1, s_2} : A^-_{s_1, s_2} \to A^-_{s_1, s_2} + \infty, - \infty$.
- $I^{-L_1 \cup L_2}_{s_1, s_2} : A^-_{s_1, s_2} \to A^-_{s_1, s_2} - \infty, + \infty$.

4.3.2. Destabilization maps. Let $\overline{L} = L_1 \cup L_2$. We set

- $J(\overline{L}_i) = \{(s_1, s_2) \in \mathbb{H}(L) | s_i = +\infty\}$, $J(\overline{L}_i) = \{(s_1, s_2) \in \mathbb{H}(L) | s_i = -\infty\}$.

Now let $\overline{M} \subset L$ be an oriented sublink, and let $\overline{M}_i$ be all the oriented components of $\overline{M}$. Define

\[
J(\overline{M}) = \bigcap_i J(\overline{M}_i).
\]

For $s \in J(\overline{M})$, there is a destabilization map

\[
D^\overline{M}_s : \mathfrak{A}^-(\mathcal{H}^L, s) \to \mathfrak{A}^-(\mathcal{H}^{L - \overline{M}}, \psi^\overline{M}(s)),
\]

which gives rise to the map $D^\overline{M}_s(p^\overline{M}(s))$ in (4.6). Note that $p^\overline{M}(s) \in J(\overline{M})$ for any $s \in \mathbb{H}(L)$. In the knot case, the destabilization map corresponds to the map identifying $C\{i > 0\}$ and $C\{j > 0\}$. We will give the definition in the next section.

**Example 4.23.** Let $s = (s_1, s_2) \in \mathbb{H}(L)$ and $\overline{M} = \pm L_1$, then $p^{\pm L_1}(s) = (\pm \infty, s_2)$. The destabilization map

\[
D^{\pm L_1}_{\pm \infty, s_2} : \mathfrak{A}^-(\mathcal{H}^L, \pm \infty, s_2) \to \mathfrak{A}^-(\mathcal{H}^{L_2}, s_2 - \frac{\text{lkn}(L_1, \pm L_2)}{2})
\]

is a chain homotopy equivalence.
If we consider sublinks $\tilde{M} = \pm L_1 \cup \pm L_2$, then we will get destabilization maps from $\mathfrak{A}^-(H^L, \pm \infty, \pm \infty)$ to $\mathfrak{A}^-(H^0, 0)$, namely,

$$
D_{+L_1 \cup L_2}^{\pm L_1 \cup L_2} : \mathfrak{A}^-(H^L, +\infty, +\infty) \to \mathfrak{A}^-(H^0, 0),
$$

$$
D_{-\infty \cup +\infty}^{L_1 \cup L_2} : \mathfrak{A}^-(H^L, -\infty, +\infty) \to \mathfrak{A}^-(H^0, 0),
$$

$$
D_{+\infty \cup -\infty}^{L_1 \cup L_2} : \mathfrak{A}^-(H^L, +\infty, -\infty) \to \mathfrak{A}^-(H^0, 0),
$$

$$
D_{-\infty \cup -\infty}^{L_1 \cup L_2} : \mathfrak{A}^-(H^L, -\infty, -\infty) \to \mathfrak{A}^-(H^0, 0).
$$

4.3.3. Primitive system of hyperboxes. In \cite{6}, complete system of hyperboxes is defined in order to define the destabilization maps.

Definition 4.24 (Complete pre-system of hyperboxes). A complete pre-system of hyperboxes $\mathcal{H}$ representing the link $\tilde{L}$ consists of a collection of hyperboxes of Heegaard diagrams, subject to certain compatibility conditions as follows. For each pair of subsets $M \subseteq L' \subseteq L$, and each orientation $\tilde{M}$ on $M$, the complete pre-system assigns a hyperbox $H^{\tilde{M}, \tilde{M}}$ for the pair $(\tilde{L}, \tilde{M})$, where $\tilde{L}$ has the induced orientation from $\tilde{L}$. Moreover, the hyperbox $H^{\tilde{M}, \tilde{M}}$ is required to be compatible with both $H^{\tilde{M}, \tilde{M}}$ and $H^{\tilde{M}, \tilde{M}}$.

In the above definition, there is some compatibility condition we have not spelled out. A complete system of hyperboxes is a complete pre-system with some additional conditions regarding the surface isotopies connecting those hyperboxes. In a complete system of hyperboxes, every hyperbox of Heegaard diagrams induces a hyperbox of generalized Floer complexes. Instead of explaining these compatibility conditions, we give a special complete system of hyperboxes for two-component links satisfying these conditions, which illustrates the main idea. They are called primitive system of hyperboxes.

When the sublink $\tilde{L}'$ has the induced orientation from $\tilde{L}$, we simply denote it by $L'$. Thus, we use notation $H^{\tilde{M}, \tilde{M}} = H^{\tilde{M}, \tilde{M}}$. In a complete pre-system $\mathcal{H}$, we have four zero dimensional hyperboxes of Heegaard diagrams, $H_{L, \mathcal{H}}, H_{L, \mathcal{H}}, H_{L, \mathcal{H}}, H_{L, \mathcal{H}}$, where $H_{L, \mathcal{H}}$ is a Heegaard diagram of $L$, $H_{L, \mathcal{H}}$ is a Heegaard diagram of $L_1$, and $H_{L, \mathcal{H}}$ is a Heegaard diagram of $S^4$. We denote the four Heegaard diagrams simply by $H_L, H_{L_1}, H_{L_2}, H_0$.

Given a basic Heegaard diagram $H_L = (\Sigma, \alpha, \beta, \{w_1, w_2\}, \{z_1, z_2\})$ of $L$, from Equation (2.2), we see $\mathfrak{A}^-(H_L, (+\infty, s_2))$ is counting $n_{w_2}(\phi)$ without using $z_1$, thus as the same as deleting $z_1$. Moreover $H_L = (\Sigma, \alpha, \beta, \{w_1, w_2\}, \{z_2\})$ is a Heegaard diagram of $L_2$ with one free basepoint $w_1$. We call this diagram the reduction of $H_L$ at $+L_1$, denoted by $r+L_1(H_L)$; see \cite{6} Definition 4.17. Hence, we have an identification between $\mathfrak{A}^-(H_L, (+\infty, s_2))$ and $\mathfrak{A}^-(r+L_1(H_L), s_2 - \frac{lk(L_1, L_2)}{2})$. Similarly, we define $r-L_1(H_L)$ to be the diagram obtained from $H_L$ by deleting $w_1$ and relabeling $z_1$ as $w_1$. We have an identification between $\mathfrak{A}^-(H_L, (-\infty, s_2))$ and $\mathfrak{A}^-(r-L_1(H_L), s_2 - \frac{lk(L_1, L_2)}{2})$, since $\mathfrak{A}^-(H_L, (-\infty, s_2))$ uses basepoints $\{z_1, w_2\} \subset H_L$.

Moreover, the diagrams $r-L_1(H_L)$ and $r+L_1(H_L)$ are related by Heegaard moves, for they represent the same knot $L_2$. By definition, there is an arc $c$ in $\Sigma - \alpha$ connecting $w_1$ and $z_1$, so we can move $z_1$ along $c$ to $w_1$, by a sequence of Heegaard moves. Moving a basepoint to cross some $\beta$-curve can be done by a sequence of handleslides and isotopies of $\beta$-curves, stabilizations, and destabilization followed by a surface isotopy. However, if we need stabilizations/destabilizations, we can modify the original Heegaard diagram $H_L$ by these stabilizations in the beginning. Thus, we can always get a diagram $\tilde{H}_L$, such that there is a sequence of Heegaard moves only of handleslides and isotopies of $\beta$-curves together with some surface isotopy from $r-L_1(H_L)$ to $r+L_1(H_L)$. In sum, there is an surface isotopy $h : \Sigma \to \Sigma$ supported in a small neighborhood of $c$ (so $h$ fixes other basepoints and all the $\alpha$-curves), such that $h(w_1) = z_1$ and $h(r+L_1(H_L))$ is strongly equivalent to $r-L_1(H_L)$ via handleslides and isotopies of $\beta$-curves.
Definition 4.25 (Primitive Heegaard diagrams). For any basic Heegaard diagram $\mathcal{H}$ of an oriented link $\widetilde{L} = \widetilde{L}_1 \cup \widetilde{L}_2$, there are surface isotopies $h_i^r : \Sigma \to \Sigma$ supported in a small neighborhood of the arc $c_i$ connecting $w_i$ and $z_i$ in $\Sigma - \alpha$, such that $h_i^r(w_i) = z_i$ and $h_i^r$ preserves the $\alpha$-curves and the other basepoints. They are unique up to isotopy. The basic Heegaard diagram is called primitive, if it is admissible and $r_{-L_i}(\mathcal{H})$ is strongly equivalent to $h_i^r(r_{+L_i}(\mathcal{H}))$ for both $i = 1, 2$.

From the above discussion, we can get the following lemma.

Lemma 4.26. Let $L$ be an oriented two-component link, and let $\mathcal{H}$ be a basic admissible Heegaard diagram of $L$. Then there is an index one/two stabilization turning $\mathcal{H}$ into a primitive Heegaard diagram $\mathcal{H}$.

Fixing a primitive Heegaard diagram $\mathcal{H}^L$ for an oriented two-component link $L$, we can get two sequences of strongly equivalent Heegaard diagrams $\mathcal{H}^{L_i}$:

\begin{align*}
(4.7) \quad & \mathcal{H}^{L,-L_1}_1 : r_{-L_1}(\mathcal{H}^L) \to \mathcal{H}^{L,-L_1}(L_1) = h_1^r(r_{+L_1}(\mathcal{H}^L)), \\
(4.8) \quad & \mathcal{H}^{L,-L_2}_1 : r_{-L_2}(\mathcal{H}^L) = \mathcal{H}^{L,-L_2}(\emptyset) \to \mathcal{H}^{L,-L_2}(L_2) = h_2^r(r_{+L_2}(\mathcal{H}^L)).
\end{align*}

These induce another two sequences of strongly equivalent Heegaard diagrams $\mathcal{H}^{L_i}$:

\begin{align*}
(4.9) \quad & \mathcal{H}^{L_1,-L_1} : r_{-L_1}(\mathcal{H}^L) = \mathcal{H}^{L_1,-L_1}(\emptyset) \to \mathcal{H}^{L_1,-L_1}(L_1) = h_1^r(r_{+L_1}(\mathcal{H}^L)), \\
(4.10) \quad & \mathcal{H}^{L_2,-L_2} : r_{-L_2}(\mathcal{H}^L) = \mathcal{H}^{L_2,-L_2}(\emptyset) \to \mathcal{H}^{L_2,-L_2}(L_2) = h_2^r(r_{+L_2}(\mathcal{H}^L)),
\end{align*}

together with a square of strongly equivalent Heegaard diagrams $\mathcal{H}^{L_i}$:

\begin{align*}
(4.11) \quad & r_{-L_1 \cup L_2}(\mathcal{H}) = \mathcal{H}^{L_1 \cup L_2,-L_1 \cup L_2}(\emptyset) \to \mathcal{H}^{L_1 \cup L_2,-L_1 \cup L_2}(L_1) = h_1^r((r_{+L_1 \cup L_2}(\mathcal{H}) \to h_1^r((r_{+L_1 \cup L_2}(\mathcal{H}) \to h_2^r(r_{+L_2}(\mathcal{H}))).
\end{align*}

These almost produce a complete system of hyperbox $\mathcal{H}$ except for the admissibility of $\mathcal{H}$. We call this system a primitive almost complete system of hyperbox.

Definition 4.27 (Primitive almost complete system of hyperbox). Given a primitive Heegaard diagram $\mathcal{H}^L = (\Sigma, \alpha, \beta, \{w_1, w_2\}, \{z_1, z_2\})$ of $\widetilde{L} = \widetilde{L}_1 \cup \widetilde{L}_2$, there exists a primitive almost complete system of hyperbox $\mathcal{H}$ associated to $\mathcal{H}^L$ consisting of

- four 0-dimensional hyperboxes of Heegaard diagrams:
  \[
  \mathcal{H}_{L}^L = \mathcal{H}, \quad \mathcal{H}_{L_1}^1 = r_{+L_1}(\mathcal{H}), \quad \mathcal{H}_{L_2}^2 = r_{+L_2}(\mathcal{H}), \quad \mathcal{H}_0 = r_{L_1 \cup L_2}(\mathcal{H});
  \]

- eight 1-dimensional hyperboxes of Heegaard diagrams:
  \[
  \mathcal{H}_{L_1 \pm L_1}^L, \quad \mathcal{H}_{L_1 \pm L_i}^L, \forall i = 1, 2,
  \]

where $\mathcal{H}_{L_1 \pm L_1}^L, \mathcal{H}_{L_1 \pm L_i}^L$ are trivial hyperboxes, i.e. just a Heegaard diagram, and $\mathcal{H}_{L_i \pm L_i}^L$ are described in Equations from (4.7) to (4.10);

- four 2-dimensional hyperboxes of Heegaard diagrams: one trivial hyperbox $\mathcal{H}_{L_1 \cup L_2}^L$, two degenerate hyperboxes:
  \[
  \mathcal{H}_{L_1 \cup L_2}^L \pm L_2 = \mathcal{H}_{L_2}^L, \quad \mathcal{H}_{L_1 \cup L_2}^L \pm L_1 = \mathcal{H}_{L_1}^L,
  \]

and a square of strongly equivalent Heegaard diagrams $\mathcal{H}^{L_i}$, which is described in Equation (4.11).
Definition 4.28 (Primitive system of hyperboxes). Given a primitive diagram $H^L$ and the induced primitive almost complete systems of hyperboxes $H$, if the admissibility of $H$ is not satisfied, we can enlarge the hyperbox in $H$ to achieve admissibility, thus getting a complete system of hyperboxes. We call the result a primitive system of hyperboxes $H$.

Indeed, if $H_{L_1}^{-1}$ is not admissible, i.e. $(\Sigma, \alpha, \beta, \beta', w, z)$ is not admissible, then we can insert an isotopy of $\beta$, namely $\beta''$, such that both $(\Sigma, \alpha, \beta, \beta''', w, z)$ and $(\Sigma, \alpha, \beta'', \beta', w, z)$ are admissible. Suppose $\{D_1, \ldots, D_m\}$ is a basis of the $\mathbb{Q}$-vector space of the periodic domains in $(\Sigma, \alpha, \beta, \beta', w, z)$ with only positive multiplicities. Let $D_1^c$ be the union of all the regions which are not in $D_1$. Then $D_1^c \neq \emptyset$, since $w_1D_1 = 0$. As $(\Sigma, \alpha, \beta, w, z)$ and $(\Sigma, \alpha, \beta', w, z)$ are both admissible, the boundary of $D_1$ must contain a $\beta$-vector and a $\beta'$-vector. Thus there is a $\beta$-arc $b$ and a $\beta'$-arc $b'$ on $D_1 \cap D_1^c$. So we can find a path $\gamma$ in $D_1$ connecting $b$ to $b'$, and then do a finger move of the $\beta$-curve containing $b$ along $\gamma$ to get negative multiplicities for $D_1$ (see [13] for the definition of finger move). Similarly we deal with the other $\beta_i$'s. Finally, the new $\beta$ in the above process is chosen to be $\beta''$. Similar arguments work for the case of the square $H_{L_1}^{-1} \cup -L_2$.

Therefore to achieve admissibility, we can enlarge the square of Heegaard diagram $H_{L_1}^{-1} \cup -L_2$ to $H_{L_1}^{-1} \cup -L_2$:

$$(\Sigma, \alpha, \beta_{11}, w, z) \rightarrow (\Sigma, \alpha, \beta_{13}, w, z)$$

$$(\Sigma, \alpha, \beta_{31}, w, z) \rightarrow (\Sigma, \alpha, \beta_{33}, w, z)$$

$$H_{L_1}^{-1} : (\Sigma, \alpha, \beta_1, w, z) \rightarrow (\Sigma, \alpha, \beta_2, w, z) \rightarrow (\Sigma, \alpha, \beta_3, w, z).$$

In order to send every hyperbox of Heegaard diagrams $H_{L_1}^{-1}$ to a hyperbox of chain complexes $\mathfrak{A}^{-}(H_{L_1}^{-1}, s)$, we need a set of $\Theta$ chain elements. We call the choice of these $\Theta$-elements a filling of the hyperboxes Heegaard diagrams. Let us explain $\Theta$-elements case by case.

For $H_{L_1}^{-1}$, we have a sequence of strongly equivalent Heegaard diagrams of $L_1 - L_1$:

$$(\Sigma, \alpha, \beta_1, w, z) \rightarrow (\Sigma, \alpha, \beta_2, w, z) \rightarrow (\Sigma, \alpha, \beta_3, w, z).$$

We choose a cycle element $\Theta_{\beta_1, \beta_2}$ representing the maximal degree element in the homology of $\mathfrak{A}^{-}(\mathbb{T}_{\beta_1}, \mathbb{T}_{\beta_2}, 0)$. Then we define a chain homotopy equivalence $D_{\beta_1, \beta_2} : \mathfrak{A}^{-}(\mathbb{T}_{\beta_1}, \mathbb{T}_{\beta_2}, s) \rightarrow \mathfrak{A}^{-}(\mathbb{T}_{\beta_1}, \mathbb{T}_{\beta_2}, s)$ by using triangle maps $D_{\beta_1, \beta_2}(x) = f_{\alpha, \beta_1, \beta_2} \otimes \Theta_{\beta_1, \beta_2}$. Similarly, we get a chain homotopy equivalence $D_{\beta_2, \beta_3} : \mathfrak{A}^{-}(\mathbb{T}_{\beta_2}, \mathbb{T}_{\beta_3}, s) \rightarrow \mathfrak{A}^{-}(\mathbb{T}_{\beta_2}, \mathbb{T}_{\beta_3}, s)$ by choosing $\Theta_{\beta_2, \beta_3} \in \mathfrak{A}^{-}(\mathbb{T}_{\beta_2}, \mathbb{T}_{\beta_3}, 0)$. Thus, $D^{-1}_{L_1} = D_{\beta_2, \beta_3} \circ D_{\beta_1, \beta_2}$ is also a chain homotopy equivalence. Let us put a subscript on $D^{-1}_{L_1}$ for labeling the $\text{Spin}^c$ structure. Since $\mathfrak{A}^{-}(H_{L_1}, (+\infty, s_2)) = \mathfrak{A}(r_+L_1(H_{L_1}), s_2 - \frac{\text{lk}(+L_1, +L_2)}{2})$, $\mathfrak{A}^{-}(H_{L_1}, (-\infty, s_2)) = \mathfrak{A}(r_-L_1(H_{L_1}), s_2 - \frac{\text{lk}(-L_1, -L_2)}{2})$, we write

$$D^{-1}_{L_1}_{-\infty, s_2} : \mathfrak{A}^{-}(H_{L_1}, (-\infty, s_2)) \rightarrow \mathfrak{A}^{-}(H_{L_1}, (+\infty, s_2 + \text{lk}(+L_1, +L_2))).$$

or simply

$$D^{-1}_{L_1}_{-\infty, s_2} : A^{-}_{-\infty, s_2} \rightarrow A^{-}_{+\infty, s_2 + \text{lk}}.$$

Similarly we define $D^{-1}_{L_2}_{-\infty} : A^{-}_{s_1, -\infty} \rightarrow A^{-}_{s_1, +\text{lk}, +\infty}$. For the 2-dimensional hyperbox of Heegaard diagrams $H_{L_1}^{-1} \cup -L_2$, we can get a square of chain complexes. Let us first look at the upper left quarter of $H_{L_1}^{-1} \cup -L_2$.
In the above diagram, the $\Theta$-elements on the edges are arbitrary cycles representing the maximal degree elements in their homology groups. Let $c = f_{\beta_1\beta_2\beta_{22}}(\Theta_{\beta_1\beta_2\beta_2\beta_{22}}) + f_{\beta_1\beta_{21}\beta_{22}}(\Theta_{\beta_1\beta_{21}\beta_{22}})$. The equation

$$
\partial(f_{\beta_1\beta_2\beta_{22}}(\Theta_{\beta_1\beta_2\beta_{22}}) + f_{\beta_1\beta_{21}\beta_{22}}(\Theta_{\beta_1\beta_{21}\beta_{22}})) = f_{\beta_1\beta_2\beta_{22}}((\partial\Theta_{\beta_1\beta_2\beta_{22}}) + f_{\beta_1\beta_{21}\beta_{22}}(\Theta_{\beta_1\beta_{21}\beta_{22}}(\partial\Theta_{\beta_2\beta_{22}}))) = 0
$$

shows that $c$ is a cycle in $\mathfrak{A}_\mu(T_{\beta_11}, T_{\beta_{22}}, 0)$, where $\mu$ equals to the maximal degree of the homology of $\mathfrak{A}(T_{\beta_11}, T_{\beta_{22}}, 0)$. Since the curves $\beta_{\star\star}$ are all strongly equivalent, up to chain homotopy equivalences, we can only consider the case when they are all small Hamiltonian isotopies of each other. By Lemma 9.7 in [13], we can see that $f_{\beta_1\beta_2\beta_{22}}(\Theta_{\beta_1\beta_2\beta_{22}})$ and $f_{\beta_1\beta_{21}\beta_{22}}(\Theta_{\beta_1\beta_{21}\beta_{22}})$ both represent the maximal degree element in the homology of $CF(T_{\beta_11}, T_{\beta_{22}})$. Thus, $c$ is $0$ in the homology. So there is a $\Theta_{\beta_1\beta_{22}}$ such that $\partial\Theta_{\beta_1\beta_{22}} = c$, where $\partial = f_{\beta_1\beta_{22}}$. In sum,

$$f_{\beta_1\beta_2\beta_{22}}(\Theta_{\beta_1\beta_2\beta_{22}}) + f_{\beta_1\beta_{21}\beta_{22}}(\Theta_{\beta_1\beta_{21}\beta_{22}}) = f_{\beta_1\beta_{22}}(\Theta_{\beta_1\beta_{22}}).$$

From the quadratic $A_\infty$ associativity Equation (2.3), we have a square of chain complexes

$$
\mathfrak{A}(T_{\alpha}, T_{\beta_{22}}, s) \xrightarrow{D_{\beta_1\beta_{21}}} \mathfrak{A}(T_{\alpha}, T_{\beta_{22}}, s) \xrightarrow{D_{\beta_1\beta_{22}}} \mathfrak{A}(T_{\alpha}, T_{\beta_{22}}, s),
$$

where $D_{\beta_1\beta_{22}}(x) = f_{\alpha\beta_1\beta_2\beta_{22}}(x \otimes \Theta_{\beta_1\beta_2\beta_{22}}) + f_{\alpha\beta_1\beta_{21}\beta_{22}}(x \otimes \Theta_{\beta_1\beta_{21}\beta_{22}}) + f_{\alpha\beta_1\beta_{22}}(x \otimes \Theta_{\beta_2\beta_{22}})$, and

$$
D_{\beta_1\beta_{12}}(x) = f_{\alpha\beta_1\beta_{12}}(x \otimes \Theta_{\beta_1\beta_{12}}),
D_{\beta_1\beta_{22}}(x) = f_{\alpha\beta_1\beta_{22}}(x \otimes \Theta_{\beta_2\beta_{22}}),
D_{\beta_2\beta_{22}}(x) = f_{\alpha\beta_2\beta_{22}}(x \otimes \Theta_{\beta_2\beta_{22}}).
$$

Similarly, we can choose other $\Theta$-elements on $H_{L_1+L_2}$, and get a rectangle of chain complexes of size $(2, 2)$. We denote it by $\mathfrak{A}(H_{L_1+L_2}, s)$.

**Definition 4.29.** We define the destabilization map $D^{-L_1\cup-L_2}_\alpha$ to be the diagonal map in the compression of $\mathfrak{A}(H_{L_1+L_2}, s)$:

$$
\mathfrak{A}(r_{-L_1+L_2}, s) \xrightarrow{D^{-L_1\cup-L_2}_\alpha} \mathfrak{A}(r_{-L_1+L_2}, s)
$$

Since $\mathfrak{A}(r_{-L_1+L_2}, s) = \mathfrak{A}(H_{L_1}, s)$, we denote it by $D^{-L_1\cup-L_2}_\alpha$.

As all the other hyperboxes of Heegaard diagrams are trivial, the following identities hold

$$D^{+L_1}_{\infty, s_2} = id, \; D^{+L_2}_{s_1, \infty} = id, \; D^{-L_1+L_2}_{\infty, \infty} = 0, \; D^{+L_1\cup-L_2}_{\infty, \infty} = 0, \; D^{+L_1\cup-L_2}_{\infty, \infty} = 0.$$
Now we can build all the rectangles of chain complexes as follows, where \( \text{lk} = \text{lk}(L_1, L_2) \).

\[
\begin{align*}
\mathcal{R}_{s,1,1} := & \quad A_{s_1,s_2}^{-} \xrightarrow{r_{s_1,s_2}^{-L_2}} A_{s_1,-\infty}^{-} \xrightarrow{D_{s_1,-\infty}} A_{s_1,+\text{lk}+\infty}^{-} \\
& \quad A_{-\infty,s_2}^{-} \xrightarrow{r_{-\infty,s_2}^{-L_2}} A_{-\infty,-\infty}^{-} \xrightarrow{D_{-\infty,-\infty}} A_{-\infty,+\text{lk}+\infty}^{-}, \\
& \quad A_{+\infty,s_2+\text{lk}}^{-} \xrightarrow{r_{+\infty,s_2+\text{lk}}^{-L_2}} A_{+\infty,+\infty}^{-}.
\end{align*}
\]

\[
\begin{align*}
\mathcal{R}_{u,0,1} := & \quad A_{s_1,s_2}^{-} \xrightarrow{r_{s_1,s_2}^{-L_2}} A_{s_1,-\infty}^{-} \xrightarrow{D_{s_1,-\infty}} A_{s_1,+\text{lk}+\infty}^{-} \\
& \quad A_{-\infty,s_2}^{-} \xrightarrow{r_{-\infty,s_2}^{-L_2}} A_{-\infty,-\infty}^{-} \xrightarrow{D_{-\infty,-\infty}} A_{-\infty,+\text{lk}+\infty}^{-}, \\
& \quad A_{+\infty,s_2+\text{lk}}^{-} \xrightarrow{r_{+\infty,s_2+\text{lk}}^{-L_2}} A_{+\infty,+\infty}^{-}.
\end{align*}
\]

The squares \( R_{s,i,j} \)'s used in Equations (4.2) to (4.5) are defined to be the compressions of \( \mathcal{R}_{s,i,j} \)'s.

5. Applying the surgery formula to two-bridge links

In this section, we show some algebraic rigidity results for the chain maps between certain chain complexes up to chain homotopy. This provides a way to determine the destabilization maps in the surgery complex of two-bridge links up to chain homotopy. Using these maps to replace the original maps in the surgery complex, we construct a perturbed surgery complex. We further show that it has the same homology as the original one. Based on the perturbed surgery formula, we give an algorithm for computing the homology of surgeries on two-bridge links.

5.1. Algebraic rigidity results. There is a short exact sequence in the Exercise 3.6.1 in [19] as follows. Suppose \( P_s, Q^s \) are (co)chain complexes of \( R \)-modules, and \( P, d(P) = \text{Im}(d) \) are both projective \( R \)-modules. Then there is an exact sequence

\[
0 \to \prod_{p+q=n-1} \text{Ext}_R^1(H_p(P_s), H^q(Q^s)) \to H_n(\text{Hom}(P_s, Q^s)) \to \prod_{p+q=n} \text{Hom}_R(H_p(P_s), H^q(Q^s)) \to 0.
\]

For completeness, we give a proof here adapted to the setting of \( \mathbb{Z}/2\mathbb{Z} \)-graded chain complexes.

**Lemma 5.1.** Let \( P_s, Q^s \) be \( \mathbb{Z}/2\mathbb{Z} \)-graded chain complexes of \( R \)-modules. If \( P \) and \( d(P) = \text{Im}(d) \) are projective modules, then there is a short exact sequence for any \( n, p, q \in \mathbb{Z}/2\mathbb{Z} \),

\[
(5.1) \quad 0 \to \bigoplus_{p+q=n+1} \text{Ext}_R^1(H_p(P), H_q(Q)) \to H_n(\text{Hom}(P, Q)) \to \bigoplus_{p+q=n} \text{Hom}_R(H_p(P), H_q(Q)) \to 0.
\]

**Proof.** First, all the indices \( n, p, q, i, j \) are in \( \mathbb{Z}/2\mathbb{Z} \). Since \( d(P) \) is projective, the short exact sequence \( 0 \to Z^P \to P \to d(P) \to 0 \) splits, thus giving that \( P = d(P) \oplus Z^P \). Thereby, \( Z^P \) is projective...
and thus $\text{Ext}^1_R(Z^P, M) = 0$ for all $R$-module $M$. Also, by $\text{Ext}^1_R(d(P), M) = 0, \forall M$ we get an exact sequence

$$0 \to \text{Hom}_R(d(P_p), Q) \to \text{Hom}_R(P, Q) \to \text{Hom}_R(Z^P_p, Q) \to 0.$$  

These assemble to a short exact sequence of chain complexes

$$(5.2) \quad 0 \to \bigoplus_{p+q=n} \text{Hom}_R(d(P_p), Q) \to (\text{Hom}_R(P, Q))_n \to \bigoplus_{p+q=n} \text{Hom}_R(Z^P_p, Q) \to 0.$$  

Actually, it is not hard to check the following commuting diagram

$$
\begin{array}{ccccccc}
0 & \to & \bigoplus_{p+q=n} \text{Hom}_R(d(P_p), Q) & \to & (\text{Hom}_R(P, Q))_n & \to & \bigoplus_{p+q=n} \text{Hom}_R(Z^P_p, Q) & \to & 0 \\
& & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \to & \bigoplus_{p+q=n+1} \text{Hom}_R(d(P_p), Q) & \to & (\text{Hom}_R(P, Q))_{n+1} & \to & \bigoplus_{p+q=n+1} \text{Hom}_R(Z^P_p, Q) & \to & 0.
\end{array}
$$  

Since $dP$ is projective, the short exact sequence $0 \to d(Q_{j-1}) \to Z^Q_j \to H_j(Q) \to 0$ gives a short exact sequence

$$0 \to \text{Hom}_R(dP_i, d(Q_{j-1})) \to \text{Hom}_R(dP_i, Z^Q_j) \to \text{Hom}_R(dP_i, H_j(Q)) \to 0.$$  

Furthermore, the differential in $\text{Hom}_R(dP_i, Q)$ is $dQ$, so from the above exact sequence it follows that $H_n(\text{Hom}_R(d(P, Q)) = \bigoplus_{p+q=n} \text{Hom}_R(d(P_p), H_q(Q)), p, q, n \in \mathbb{Z}/2\mathbb{Z}$. Since $Z^P$ is projective, similarly we have

$$H_n(\text{Hom}_R(Z^P, Q)) = \bigoplus_{p+q=n} \text{Hom}_R(Z^P_p, H_q(Q)).$$  

Thus the long exact sequence of homology from Equation (5.2) is

$$(5.3) \quad \cdots \to \bigoplus_{p+q=n} \text{Hom}_R(Z^P_{p+1}, H_q(Q)) \xrightarrow{\partial_{p+1}} \bigoplus_{p+q=n} \text{Hom}_R(d(P_p), H_q(Q)) \to H_n(\text{Hom}_R(P, Q)) \to \cdots.$$  

A diagram chasing shows that the connecting morphism $\partial : \text{Hom}(Z^P, H_q(Q)) \to \text{Hom}(d(P), H_q(Q))$ is the restriction.

Hence, the short exact sequence $0 \to dP_{i+1} \to Z^P_i \to H_i(P) \to 0$ can produce the exact sequence

$$0 \to \text{Hom}_R(H_p(P), H_q(Q)) \to \text{Hom}_R(Z^P_p, H_q(Q)) \xrightarrow{\partial_{p+q}} \text{Hom}_R(dP_{p+1}, H_q(Q))$$  

$$\to \text{Ext}^1_R(H_p(P), H_q(Q)) \to \text{Ext}^2_R(Z^P_p, H_q(Q)) = 0,$$

thus $\text{Ker}(\partial_{p+q}) \cong \text{Hom}_R(H_p(P), H_q(Q))$ and $\text{Coker}(\partial_{p+q}) \cong \text{Ext}^1_R(H_p(P), H_q(Q))$. Finally, the exact sequence in Equation (5.1) comes from Equation (5.3). \hfill \square

Let $(C_\ast, \partial_\ast)$ be a chain complex of $\mathbb{F}$-vector spaces, with $U_1, U_2$-actions which drop the $\mathbb{Z}$-grading by 2. Consider $C$ as a $\mathbb{F}[[U_1, U_2]]$-module. Even though the $U_1, U_2$-actions do not preserve the $\mathbb{Z}$-grading, we will still call $C$ a complex of $\mathbb{F}[[U_1, U_2]]$-modules.

**Proposition 5.2.** Let $A, B$ be complexes of $\mathbb{F}[[U_1, U_2]]$-modules with $U_1, U_2$-actions dropping grading by 2, and $A, dA$ are both free $\mathbb{F}[[U_1, U_2]]$-modules. Suppose $H_\ast(A) \cong H_\ast(B) \cong \mathbb{F}[[U_1, U_2]]/(U_1-U_2)$, precisely, $H_{2k}(A) \cong H_{2k}(B) \cong \mathbb{F}$ for all $k \leq 0$ and $H_i(A) = H_i(B) = 0$ otherwise, where $U_1 \cdot H_{2k}(A) = H_{2k+2}(A), U_1 \cdot H_{2k}(B) = H_{2k+2}(B)$ for both $i = 1, 2$. If $F, G : A \to B$ are both quasi-isomorphisms as $\mathbb{F}[[U_1, U_2]]$-modules, then $F$ and $G$ are homotopic as $\mathbb{F}[[U_1, U_2]]$-modules.
Proof. First, the \( \mathbb{Z} \)-grading of \( A, B \) induces a \( \mathbb{Z}/2\mathbb{Z} \)-grading on both \( A \) and \( B \), and \( U_1, U_2 \)-action preserves the induced \( \mathbb{Z}/2\mathbb{Z} \)-grading, thus we regard \( A, B \) as \( \mathbb{Z}/2\mathbb{Z} \)-graded chain complexes of \( \mathbb{F}[[U_1, U_2]] \)-modules. In order to distinguish these two gradings, we put brackets on the numbers to represent \( \mathbb{Z}/2\mathbb{Z} \)-gradings. Hence we have \( H_0(A) = H_0(B) = \mathbb{F}[[U_1, U_2]]/(U_1 - U_2) \), \( H_1(A) = H_1(B) = 0 \).

By Lemma [5.1] we have

\[
0 \to \bigoplus_{[p] \in \mathbb{Z}/2\mathbb{Z}} \text{Ext}^1_{\mathbb{F}[[U_1, U_2]]}(H_{[p]+1}(A_*), H_{[p]}(B_*)) \to H_0(\text{Hom}(A_*, B_*)) \to 0
\]

thus \( H_0(\text{Hom}(A_*, B_*)) = \text{Hom}_{\mathbb{F}[[U_1, U_2]]}(\mathbb{F}[[U_1, U_2]]/(U_1 - U_2), \mathbb{F}[[U_1, U_2]]/(U_1 - U_2)) = \mathbb{F}[[U_1, U_2]]/(U_1 - U_2) \). Since \( H_0(\text{Hom}(A_*, B_*)) \) is the group of chain homotopy equivalence classes of chain maps from \( A \) to \( B \), this means the chain maps from \( A \) to \( B \) are classified by their action on homology. Since \( F \) and \( G \) are both quasi-isomorphisms, they are homotopic as \( \mathbb{F}[[U_1, U_2]] \)-modules.

Let \( H : A \to B \) be any homotopy such that \( F - G = H \partial + \partial H, H \cdot U_1 = U_1 \cdot H \). Then \( H \) shifts the \( \mathbb{Z}/2\mathbb{Z} \)-grading by 1, thus shifting the \( \mathbb{Z} \)-grading by odd numbers. Thus, let \( H = \sum_{i \in \mathbb{Z}} H_{2i+1} \), where \( H_{2i+1} : A_* \to B_{*+1} \). Since

\[
F - G = H \partial + \partial H = \sum_{i \in \mathbb{Z}} (H_{2i+1} \partial + \partial H_{2i+1})
\]

preserves the original \( \mathbb{Z} \)-grading, we have \( H_{2i+1} \partial + \partial H_{2i+1} = 0, \forall i \neq 0 \). So we can replace the homotopy \( H \) by \( H_1 : A_* \to B_{*+1} \), thus being a chain homotopy of the original \( \mathbb{Z} \)-graded chain complexes.

\[\square\]

Remark 5.3. The prototype of the complexes in the previous Proposition is the simplest Heegaard Floer chain complex of the unknot in \( S^3 \). Let \( C^u \) be the chain complex of \( \mathbb{F}[[U_1, U_2]] \)-modules generated by \( x, y \) with differential \( \partial x = (U_1 - U_2)y \), where \( y, x \) are of gradings 0, -1 respectively.

Corollary 5.4. Let \( A, B \) be complexes of \( \mathbb{F}[[U_1, U_2]] \)-modules with \( U_1, U_2 \)-actions dropping grading by 2. Suppose \( A \) is chain homotopy equivalent to the complex \( C^u \) as \( \mathbb{F}[[U_1, U_2]] \)-modules, and \( H_*(B) \cong \mathbb{F}[[U_1, U_2]]/(U_1 - U_2) \) as in Proposition 5.2. Then for any quasi-isomorphisms \( F, G : A \to B \) of \( \mathbb{F}[[U_1, U_2]] \)-modules, \( F \) and \( G \) are chain homotopic as \( \mathbb{F}[[U_1, U_2]] \)-modules.

Proof. Let \( h_1 : A \to C^u, h_2 : C^u \to A \) be the chain homotopy equivalences, such that \( h_1 \circ h_2 \cong id_{C^u}, h_2 \circ h_1 \cong id_A \). Then by Proposition 5.2, \( F \circ h_2 \) is homotopic to \( G \circ h_2 \) as \( \mathbb{F}[[U_1, U_2]] \)-modules. Hence, \( F \circ h_2 \circ h_1, G \circ h_2 \circ h_1 \) are homotopic as \( \mathbb{F}[[U_1, U_2]] \)-modules, and thus so are \( F \) and \( G \). \[\square\]

Proposition 5.5. Let \( A_*, B_* \) be complexes of \( \mathbb{F}[[U]] \)-modules with \( U \)-action dropping grading by 2, and \( A, B \) are both free \( \mathbb{F}[[U]] \)-modules. Suppose \( H_*(A) = H_*(B) = \mathbb{F}[[U]] \), precisely, \( H_{2k}(A) \cong H_{2k}(B) \cong \mathbb{F} \) for all \( k \leq 0 \) and \( H_1(A) = H_1(B) = 0 \) otherwise, where \( U \cdot H_{2k}(A) = H_{2k-2}(A), U \cdot H_{2k}(B) = H_{2k-2}(B) \). If \( F, G : A \to B \) are both quasi-isomorphisms of \( \mathbb{F}[[U]] \)-modules, then \( F, G \) are chain homotopic as maps of \( \mathbb{F}[[U]] \)-modules.

Moreover, if \( H, K \) are both chain homotopies as homomorphisms of \( \mathbb{F}[[U]] \)-modules between any two chain maps \( f, g : A \to B \), i.e. \( H \partial + \partial H = K \partial + \partial K = f - g \), then \( H - K = \partial T + T \partial \), for some \( \mathbb{F}[[U]] \)-module homomorphism \( T : A_* \to B_{*+2} \).

Proof. First, we regard \( A, B \) as \( \mathbb{Z}/2\mathbb{Z} \)-graded complexes of \( \mathbb{F}[[U]] \)-modules. Since a P.I.D. is hereditary, every submodule of a free module over a P.I.D. is a projective module. See [11] Definition 4.2.10 and Exercise 4.2.6. Thus, \( d(A), d(B) \) are both projective \( \mathbb{F}[[U]] \)-modules. Applying Lemma
we can compute
\[
H_0(\text{Hom}(A, B)) = \text{Hom}(F[[U]], F[[U]]) = F[[U]],
\]
\[
H_1(\text{Hom}(A, B)) = \text{Ext}_F(F[[U]], F[[U]]) = 0.
\]

The first identity implies that the quasi-isomorphisms \( F, G \) are chain homotopic as \( \mathbb{Z}/2\mathbb{Z} \)-graded complexes via \( H \). In order to get a homotopy between \( F \) and \( G \) preserving the \( \mathbb{Z} \)-grading, we decompose \( H = \sum_{i \in \mathbb{Z}} H_{2i+1} \), where \( H_{2i+1} : A_i \to B_{i+2} \). Then similarly to Proposition 5.2 the map \( H_1 \) is also a chain homotopy between \( F, G \).

Since \( \partial(H - K) + (H - K)\partial = 0 \), the second identity implies that \( H - K \in Z_1(\text{Hom}(A, B)) = B_1(\text{Hom}(A, B)) \). This means there is a homomorphism of \( F[[U]] \)-modules \( T : A \to B \) preserving the \( \mathbb{Z}/2\mathbb{Z} \)-grading, such that \( H - K = \partial T + T \partial \). Thus, the map \( T \) can be decomposed as \( T = \sum_{i \in \mathbb{Z}} T_{2i} \), where \( T_{2i} : A_i \to B_{i+2} \). From the fact that \( H - K = \sum_{i \in \mathbb{Z}} \partial T_{2i} + T_{2i} \partial \) maps \( A_i \) into \( B_{i+1} \), it follows that \( \partial T_{2i} + T_{2i} \partial : A_i \to B_{i+2} \) vanish for all \( i \neq 1 \). Thus \( T = T_2 : A_2 \to B_{2+2} \).

Corollary 5.6. Suppose the complexes \( A_2 \) and \( B_2 \) are as in Proposition 5.5. Then, \( A_2 \) and \( B_2 \) are chain homotopy equivalent as \( F[[U]] \)-modules.

Proof. From the proof of Proposition 5.5 we see \( H_0(\text{Hom}(A_2, B_2)) = \text{Hom}(H_0(A), H_0(B)) = F[[U]] \), which implies that there exists a quasi-isomorphism \( h : A \to B \) as \( \mathbb{Z}/2\mathbb{Z} \)-graded chain complex of \( F[[U]] \)-modules. Decompose \( h \) as \( h = h_0 + h_1 \) such that for all \( a \in A_n \), \( h_0(a) \in B_n \), \( h_1(a) \in \bigoplus_{i \neq 0} B_{n+2i} \). Then \( h \partial A = \partial B h \) implies that \( h_0 \partial A = \partial B h_0 \), \( h_1 \partial A = \partial B h_1 \), so \( h_0 \) is also a chain map preserving the \( \mathbb{Z} \)-grading. Since \( U h = h U \) and the \( U \)-action drops the \( \mathbb{Z} \)-grading by 2, we have \( h_0 U + U h_0 = 0 \). In addition, \( h_0 \) is also a quasi-isomorphism. Hence, on the homology level, \( h_0(1) = 1 \in F[[U]] \).

Similarly, we have another quasi-isomorphism \( g_0 : B_2 \to A_2 \) preserving the \( \mathbb{Z} \)-grading, such that \( g_0 \partial B = \partial A g_0 \). Then, \( g_0 h_0 : A_2 \to A_2 \) is a quasi-isomorphism. From Proposition 5.5 it follows that \( g_0 h_0 - id_{A_2} = \partial H + H \partial \), where \( H \) is chain homotopy of \( \mathbb{Z} \)-graded complexes commuting with the \( U \)-action. Similarly, we have that \( h_0 g_0 \) is homotopic to \( id_{B_2} \).

5.2. Destabilization maps. In the link surgery formula, the part of polygon counts for defining the destabilization maps is difficult to read off from the Heegaard diagram. However, in the case of two-braid links we can use the algebraic rigidity result to avoid the difficulty.

In a primitive system of hyperboxes, all the destabilization maps we need are listed below:

\[
D_{-L_1}^{-L_2} : \mathfrak{A}(\mathcal{H}^L, -s_1, -s_2) \to \mathfrak{A}(\mathcal{H}^L, +s_2, +s_1 + \text{lk}(L_1, L_2)),
\]
\[
D_s^{L_2} : \mathfrak{A}(\mathcal{H}^L, -s_1, -s_2) \to \mathfrak{A}(\mathcal{H}^L, s_1 + \text{lk}(L_1, L_2), +s_2),
\]
\[
D_{-L_1}^{L_2} : \mathfrak{A}(\mathcal{H}^L, -s_1, -s_2) \to \mathfrak{A}(\mathcal{H}^L, +s_1, +s_2).
\]

Second, notice that all the domains and targets of these maps have homology \( F[[U_1, U_2]]/U_1 - U_2 \), which is isomorphic to \( F[[U_1]] \) as an \( F[[U_1]] \)-module. By Proposition 5.5 we can substitute \( D_{-L_1}^{L_2} \) by any \( F[[U_1]] \)-linear homotopy equivalence, since they are all homotopic as homomorphisms of \( F[[U_1]] \)-modules. We can also substitute the diagonal maps \( D_{-L_1}^{L_2} \) by any \( F[[U_1]] \)-linear homotopy shifting grading by 1, since they are homotopic up to higher \( F[[U_1]] \)-linear homotopy. We will show an invariance theorem of the surgery square under perturbations of the edge maps and the diagonal maps in the next sections.

5.3. Perturbed surgery complex for two-bridge links. The rigidity results in Section 5.1 allow us to perturb the edge and diagonal maps up to homotopies, in the surgery square for two-bridge links. However, in order to obtain a square of chain complexes, we still need some more modifications of the square.
First, suppose we have a hypercube \((C^\varepsilon, D^\varepsilon)\). If we change \(D^\varepsilon_{\varepsilon_0}\) to \(D^\varepsilon_{\varepsilon_0+\varepsilon} = D^\varepsilon_{\varepsilon_0} + \Delta D^\varepsilon_{\varepsilon_0}\) for all \(\varepsilon\) with \(\|\varepsilon\| > 0\), then in order to have a hypercube again, we need to have
\[
\sum_{\varepsilon' \leq \varepsilon} (D^\varepsilon_{\varepsilon_0+\varepsilon'} + \Delta D^\varepsilon_{\varepsilon_0+\varepsilon'}) \circ (D^\varepsilon_{\varepsilon_0} + \Delta D^\varepsilon_{\varepsilon_0}) = 0,
\]
\[
\sum_{\varepsilon' \leq \varepsilon} \Delta D^\varepsilon_{\varepsilon_0+\varepsilon'} \circ D^\varepsilon_{\varepsilon_0} + D^\varepsilon_{\varepsilon_0+\varepsilon'} \circ \Delta D^\varepsilon_{\varepsilon_0} + \Delta D^\varepsilon_{\varepsilon_0+\varepsilon'} \circ \Delta D^\varepsilon_{\varepsilon_0} = 0.
\]

This formula provides a necessary condition to inductively perturb the maps from edges to the longest diagonal. Based on the above principles, we get the following procedures to construct the perturbed surgery square.

Suppose \(\mathcal{H}\) be a primitive system of hyperboxes of a two-bridge link \(L\) and consider Equation (4.12). Now we choose an arbitrary \(F[\{U_i\}]\)-linear quasi-isomorphism \(D_{s_1}^{-L_i}\) for substituting \(D_{s_1}^{-L_i}\).

By Proposition 5.5 \(D_{s_1, s_2}^{-L_i}\) and \(D_{s_1}^{-L_i}\) are homotopic by a \(F[\{U_i\}]\)-linear homotopy \(H_{s_1}^{-L_i}\):
\[
\tilde{D}_{s}^{-L_i} = D_{s}^{-L_i} + H_{s}^{-L_i} \partial_{s}^{-} + \partial_{s}^{-} \mu_{s}(s) H_{s}^{-L_i}.
\]

Then, we choose any \(F[\{U_i\}]\)-linear maps \(\tilde{F}_{s_1, -\infty}^{-L_i}, \tilde{F}_{-\infty, s_2}^{-L_i}, \tilde{D}_{-\infty, -\infty}^{-L_i}\) which are homotopies in each square of Equation (4.12), such that the following rectangles are hyperboxes of chain complexes:

\[
\begin{array}{c}
A_{s_1, s_2}^{-}\quad I_{s_1, s_2}^{-L_i} \quad A_{s_1, -\infty}^{-}\quad I_{s_1, s_2}^{-L_i} \quad A_{s_1, -\infty}^{-1}\quad I_{s_1, s_2}^{-L_i} \quad A_{s_1, +\infty}^{-}\quad I_{s_1, s_2}^{-L_i} \quad A_{s_1, +\infty}^{-} \quad I_{s_1, s_2}^{-L_i} \quad A_{s_1, +\infty}^{-}\quad I_{s_1, s_2}^{-L_i} \quad A_{s_1, +\infty}^{-} \quad I_{s_1, s_2}^{-L_i} \quad A_{s_1, +\infty}^{-}\quad I_{s_1, s_2}^{-L_i} \quad A_{s_1, +\infty}^{-} \quad I_{s_1, s_2}^{-L_i} \quad A_{s_1, +\infty}^{-}\quad I_{s_1, s_2}^{-L_i} \quad A_{s_1, +\infty}^{-} \quad I_{s_1, s_2}^{-L_i} \quad A_{s_1, +\infty}^{-}\quad I_{s_1, s_2}^{-L_i} \quad A_{s_1, +\infty}^{-}
\end{array}
\]

**Definition 5.7** (Perturbed surgery square). The above rectangles in Equation (5.4) are called perturbed surgery rectangles for the two-bridge link \(L\). After compressing them, we get four sets of
Remark 5.8. In the definition, a perturbed surgery square depends on the choices of the maps \(\tilde{D}_{s_1,s_2}, \bar{D}_{s_1,s_2}, \bar{D}_{s_1,s_2}, D_{s_1,s_2}, D_{s_1,s_2} \). However, we will show it is isomorphic to the original square as \(\mathbb{F}[[U_1]]\)-module.

5.4. Invariance of the perturbed surgery complex. Now we establish the invariance of the perturbed surgery complex for two-bridge links under the change of edge maps and some diagonal maps up to chain homotopies.

**Proposition 5.9.** Let \(R\) be a \(\mathbb{F}\)-algebra. Suppose \(f, g : A \to B\) be two chain maps between two chain complexes of \(R\)-modules. If \(f, g\) are homotopic to each other by \(f \sim g\), then the mapping cones \(\text{cone}(f), \text{cone}(g)\) are isomorphic.

**Proof.** We directly construct the isomorphism between the mapping cones \(\text{cone}(f)\) and \(\text{cone}(g)\).

\[
K_1|_A = id_A \oplus H, \quad K_1|_B = id_B,
\]

\[
K_2|_A = id_A \oplus H, \quad K_2|_B = id_B.
\]

In fact, \(K_1\) is a chain map, since \(\forall a \in A, b \in B\),

\[
K_1 \partial f(a) + \partial g K_1(a) = K_1(\partial A(a) + f(a)) + \partial g(a + H(a)) = \partial(a) + H \partial A(a) + f(a) + \partial A(a) + g(a) + \partial_B H(a) = 0,
\]

\[
K_1 \partial f(b) + \partial g K_1(b) = K_1 \partial B(b) + \partial g(b) = \partial_B(b) + \partial_B(b) = 0.
\]

Moreover, \(K_2 K_1\) is \(id_{\text{cone}(f)}\), since

\[
K_2 K_1(a) = K_2(a + H(a)) = a + H(a) + H(a) = a,
\]

\[
K_2 K_1(b) = K_2(b) = b, \forall a \in A, b \in B.
\]

\[
\begin{array}{c}
A \xrightarrow{id_A} A \xrightarrow{id_A} A \\
B \xrightarrow{id_B} B \xrightarrow{id_B} B
\end{array}
\]

\[
\begin{array}{c}
A \xrightarrow{f} B \xrightarrow{id_B} B \xrightarrow{H} A \xrightarrow{id_A} A
\end{array}
\]

\[
\begin{array}{c}
A \xrightarrow{g} B \xrightarrow{id_B} B \xrightarrow{H} A \xrightarrow{id_A} A
\end{array}
\]

There is a hyperbox version of Proposition [5.9].
Definition 5.10. A hyperbox of chain complexes $R$ is said to be isomorphic to another hyperbox $R'$, if there are chain maps of hyperboxes $F : R \to R'$, $G : R' \to R$, such that $F \circ G = \text{id}_{R'}$, $G \circ F = \text{id}_R$.

Proposition 5.11. Let $R = \left((C^\varepsilon)_{\varepsilon \in \mathbb{E}(d,1)}, (D^\varepsilon)_{\varepsilon \in \mathbb{E}(n+1)}\right)$ be a hyperbox of chain complexes of size $(d,1) \in \mathbb{Z}_{\geq 1}^{n+1}$. If all the edge maps $D^{(0,1)} = \text{id}$, where $0 = (0, ..., 0) \in \mathbb{Z}^n$, then $R$ induces an isomorphism from the subhyperbox $R^{\varepsilon_{n+1}=0}$ to the subhyperbox $R^{\varepsilon_{n+1}=1}$.

Proof. We first show the case of hypercubes by induction, i.e., $d = (1, ..., 1) \in \mathbb{Z}^n$.

When $n = 1$, this is exactly Proposition 5.8. When $n > 1$, let us make some notations at first. There is a $(n-1)$-dimensional subhypercube corresponding to $\varepsilon_n = \varepsilon_{n+1} = 0$, denoted by $R^{00}$, and there is also a $(n-1)$-dimensional subhypercube corresponding to $\varepsilon_n = 0, \varepsilon_{n+1} = 1$, denoted by $R^{01}$. Similarly, the subhypercube corresponding to $\varepsilon_n = 1, \varepsilon_{n+1} = 0$ is denoted by $R^{10}$, and the hypercube corresponding to $\varepsilon_n = \varepsilon_{n+1} = 1$ is denoted by $R^{11}$. Then we can view the hypercube $R$ as the following square of hypercubes:

$$
\begin{array}{ccc}
R^{00} & f & R^{10} \\
\downarrow h_1 & & \downarrow h_2 \\
R^{01} & f' & R^{11}
\end{array}
$$

Notice that $f, f', h_1, h_2$ are chain maps of hypercubes, and $H$ is a chain homotopy of hypercubes between the chain maps. In other words, we have $h_1 \circ D|_{R^{00}} = D|_{R^{10}} \circ h_1, h_2 \circ D|_{R^{01}} = D|_{R^{11}} \circ h_2, H \circ D|_{R^{00}} + D|_{R^{11}} \circ H = h_2 \circ f + f' \circ h_1$.

By induction, the $(n-1)$-dimensional subhypercube corresponding to $\varepsilon_n = 0$ induces the isomorphism $h_1$. Thus, we have a chain map of hypercubes $h_1^{-1} : R^{01} \to R^{00}$, such that $h_1 h_1^{-1} = \text{id}_{R^{01}}$ and $h_1^{-1} h_1 = \text{id}_{R^{00}}$. Similarly, we have $h_2^{-1} : R^{11} \to R^{10}$ as the inverse of $h_2$. The hypercube $R$ induces a chain map $h_1 + H + h_2$ from the subhypercube $R^{\varepsilon_{n+1}=0}$ to the subhypercube $R^{\varepsilon_{n+1}=1}$. We show that the chain map of hyperboxes $h_1 + H + h_2 : C^{\varepsilon_{n+1}=0} \to C^{\varepsilon_{n+1}=1}$ is an isomorphism, by directly constructing the inverse $h_1^{-1} + h_2^{-1} + H h_1^{-1} + h_2^{-1} : R^{\varepsilon_{n+1}=1} \to R^{\varepsilon_{n+1}=0}$, which is induced by the following hypercube:

$$
K = \begin{array}{ccc}
R^{01} & f' & R^{11} \\
\downarrow h_2^{-1} \circ H \circ h_1^{-1} & & \downarrow h_2^{-1} \\
R^{00} & f & R^{10}
\end{array}
$$

Here, the map $h_2^{-1} \circ H \circ h_1^{-1}$ is the composition of maps $h_2^{-1}, H, h_1^{-1}$.

The following two rectangles of hypercubes show that $h_1^{-1} + h_2^{-1} H h_1^{-1} + h_2^{-1}$ is the inverse of $h_1 + H + h_2$.
Next, to prove the general case for hyperboxes, we do induction on the size of $H$, while fixing $n$. We claim that for any $k$ with $1 \leq k \leq n - 1$, if the proposition is true for all hyperboxes $R$ of size $(\mathbf{d}, 1)$ where $\mathbf{d} = (d_1, \ldots, d_k, 0, 0, \ldots, 0) \in \mathbb{Z}^n_{\geq 0}$, then the proposition is also true for all hyperboxes $S$ of size $(\mathbf{d}', 1)$ where $\mathbf{d}' = (d_1', \ldots, d_{k+1}', 0, \ldots, 0) \in \mathbb{Z}^n_{\geq 0}$.

Let $S_{i,j} = S_{i+k+1,j}$ with $i \in \{0, 1, \ldots, d_{k+1}\}$, $j = \{0, 1\}$ be the subhyperbox corresponding to those complexes with $\varepsilon_{k+1} = i$, $\varepsilon_{n+1} = j$. Thereby, the subhyperbox $S_{i,j}$ is of size $\mathbf{d}'_k = (d_1', \ldots, d_k', 0, \ldots, 0) \in \mathbb{Z}^n_{\geq 0}$. So we can regard $S_{i,j}$ as a $k$-dimensional hyperbox of size $\mathbf{d}'_k = (d_1', \ldots, d_k')$. For all $\varepsilon \in \mathbb{E}(\mathbf{d}'_k)$, we denote the chain complex of $S$ sitting at $(\varepsilon, i, 0, 0, \ldots, 0, j)$ by $(S_{i,j})_{\varepsilon}$.

We can decompose the hyperbox $S$ as a rectangle of hyperboxes as the following diagram:

\[
\begin{array}{cccccccccc}
S_{0,0} & \xrightarrow{f_1} & S_{1,0} & \xrightarrow{f_2} & S_{2,0} & \cdots & \xrightarrow{f_{d_{k+1}}'} & S_{d_{k+1},0} \\
S_{0,1} & \xrightarrow{f_1'} & S_{1,1} & \xrightarrow{f_2'} & S_{2,1} & \cdots & \xrightarrow{f_{d_{k+1}}'} & S_{d_{k+1},1},
\end{array}
\]

where $f_1, \ldots, f_{d_{k+1}}', f_1', \ldots, f_{d_{k+1}}', h_0, \ldots, h_{d_{k+1}}$ are chain maps of hyperboxes and $H_1, \ldots, H_{d_{k+1}}$ are chain homotopies of hyperboxes.

By the induction hypothesis, the subhyperbox $S_{i+k+1,j} = S_{i,0}$ is of size $(d_1', \ldots, d_k', 0, \ldots, 0, 1)$, and thereby induces the isomorphism of hyperboxes $h_j : S_{i,0} \to S_{i,1}$. Let the inverse of $h_j$ be $h_j^{-1} : S_{i,0} \to S_{i,1}$. We define a set of homotopies of hyperboxes $h_j^{-1} \circ H_j \circ h_j^{-1} : S^{j,1} \to S^{j,0}$, for any $j \in \{1, 2, \ldots, d_{k+1}\}$ by the following equations, for all $\varepsilon^0 \in \mathbb{E}(\mathbf{d}'_k), \varepsilon \in \mathbb{E}(k)$ such that $\varepsilon^0 + \varepsilon \in \mathbb{E}(\mathbf{d}'_k)$,

\[
(h_j^{-1} \circ H_j \circ h_j^{-1})_{\varepsilon^0} = \sum_{\varepsilon', \varepsilon'' \in \mathbb{E}(k)} (h_j^{-1})_{\varepsilon^0 + \varepsilon''} \circ (H_j)_{\varepsilon'' + \varepsilon'} \circ (h_j^{-1})_{\varepsilon^0}.
\]

We simply denote $h_j^{-1} \circ H_j \circ h_j^{-1}$ by $h_j^{-1} H_j h_j^{-1}$. From the definition of $h_j^{-1} H_j h_j^{-1}$, we can show the associativity of compositions of maps of hyperboxes. Thus, $H_j D|_{S_j=0} + D|_{S_j=1} H_j = h_j f_j + f_j h_j^{-1}$ and $h_j D|_{S_j=0} = D|_{S_j=1} h_j$ implies that

\[
h_j^{-1} \circ H_j \circ h_j^{-1} \circ D|_{S_j=1} + D|_{S_j=0} \circ h_j^{-1} \circ H_j \circ h_j^{-1} = f_j \circ h_j^{-1} + h_j^{-1} \circ f_j.
\]

Therefore, we can construct the following the hypercube $T$

\[
\begin{array}{cccccccccc}
S_{0,0} & \xrightarrow{f_1} & S_{1,0} & \xrightarrow{f_2} & S_{2,0} & \cdots & \xrightarrow{f_{d_{k+1}}'} & S_{d_{k+1},0} \\
S_{0,1} & \xrightarrow{f_1'} & S_{1,1} & \xrightarrow{f_2'} & S_{2,1} & \cdots & \xrightarrow{f_{d_{k+1}}'} & S_{d_{k+1},1},
\end{array}
\]

which induces a chain map from the subhyperbox $S_{i+k+1,0} = 0$ to the subhyperbox $S_{i+k+1,0}$.

**Corollary 5.12.** On a rectangle of chain complexes $R = (C^e, D^e)$, if we change the diagonal maps $D^e_{(1)}$ by a higher homotopy, i.e. $D^e_{(1)} = D^e_{(1)} + H^e_{(1)} D^e + D^e_{(1)} H^e_{(1)}$ and $H^e_{(1)}$ with $H^e_{(1)} : C^e_{(1)} \to C^e_{(1)}$, then the new rectangle $R' = (C^e, D^e)$ is isomorphic to $R$. 


Theorem 5.13. Suppose $L$ is an oriented two-bridge link with the framing $\Lambda$. For any $\mathbb{F}[[U]]$-linear quasi-isomorphisms $\tilde{D}^{-L_{t}}_{s_{1},s_{2}}$ and $\tilde{F}^{-L_{1} \cup L_{2}}_{s_{1},s_{2}}$, $\tilde{F}^{\pm L_{1} \cup \pm L_{2}}_{s_{1},s_{2}}$, $\tilde{D}^{-L_{1} \cup \pm L_{2}}_{-\infty,-\infty}$, the perturbed surgery complex $(\tilde{C}^{-}(\mathcal{H}^{L}, \Lambda), \tilde{D}^{-})$ is isomorphic to the original surgery complex in $[0]$ as $\mathbb{F}[[U]]$-module. By imposing the $U_{2}$-action to be the same as $U_{1}$-action, the $\mathbb{F}[[U_{1}, U_{2}]]$-module $H_{\ast}(\tilde{C}^{-}(\mathcal{H}^{L}, \Lambda), \tilde{D}^{-})$ is isomorphic to the homology $HF^{-}(S^{3}_{\Lambda}(L))$. Furthermore, this isomorphism preserves the absolute grading.

Proof. First, we restrict our scalars to $\mathbb{F}[[U]]$. By Proposition 5.11, the following cubes show that the top square in each cube is isomorphic to the bottom one.

This means when an edge map is changed up to a $\mathbb{F}[[U]]$-linear chain homotopy in a square $R$, we are able to change the diagonal maps correspondingly to guarantee the new square is isomorphic to the original one as an $\mathbb{F}[[U]]$-module. By inductions on the edges of rectangles $R'_{s,i,j}$ in Equation (4.12), we can show that after changing the edge maps $D^{-L_{t}}_{s_{1},s_{2}}$ by $\tilde{D}^{-L_{t}}_{s_{1},s_{2}}$ and changing some diagonal maps accordingly, the result rectangle $R'_{s,i,j}$ is isomorphic to $R_{s,i,j}$ as $\mathbb{F}[[U]]$-modules. In fact, we only have changed diagonal maps among the positions of $\tilde{F}^{\pm L_{1} \cup \pm L_{2}}_{s_{1},s_{2}}$, $\tilde{F}^{\pm L_{1} \cup \pm L_{2}}_{s_{1},s_{2}}$, $\tilde{D}^{\pm L_{1} \cup \pm L_{2}}_{-\infty,-\infty}$ in (5.4), where we can keep applying the rigidity results in Proposition 5.5. Thereby, Corollary 5.12 implies the perturbed rectangles in Equation (5.4) are isomorphic to those rectangles $R'_{s,i,j}$’s, and thus isomorphic to the original $R_{s,i,j}$’s in Equation (4.12) as $\mathbb{F}[[U]]$-modules. After compressing these rectangles and gluing them together, the perturbed surgery complex is isomorphic to the original surgery complex as an $\mathbb{F}[[U]]$-module.

From Theorem 4.17 it follows that the $U_{1}, U_{2}$ actions in $H_{\ast}(\tilde{C}^{-}(\mathcal{H}^{L}, \Lambda), \tilde{D}^{-})$ are the same. Thus by imposing the $U_{2}$-action as the same as the $U_{1}$-action on the $\mathbb{F}[[U_{1}]]$-module $H_{\ast}(\tilde{C}^{-}(\mathcal{H}^{L}, \Lambda), \tilde{D}^{-})$, we get an isomorphism as $\mathbb{F}[[U_{1}, U_{2}]]$-module between $H_{\ast}(\tilde{C}^{-}(\mathcal{H}^{L}, \Lambda), \tilde{D}^{-})$ and $H_{\ast}(\tilde{C}^{-}(\mathcal{H}^{L}, \Lambda), \tilde{D}^{-})$. As all the rigidity results respect the gradings, the above isomorphism also preserves the grading.

Remark 5.14. In the above theorem, the homology of the unknot is $\mathbb{F}[[U_{1}, U_{2}]]/(U_{1} - U_{2})$ as an $\mathbb{F}[[U_{1}, U_{2}]]$-module. There is no analogue of the Proposition 5.5 for homotopies over the ring $\mathbb{F}[[U_{1}, U_{2}]]$. This is why we restrict our scalars to $\mathbb{F}[[U]]$. This idea is due to Ciprian Manolescu.

5.5. Algorithm for computing $HF^{-}(S^{3}_{\Lambda}(L))$ for two-bridge links. Let $L$ be a two-bridge link.

First, we use the algorithm in Section 3.4 to compute all the $A_{\Lambda}(L)$’s.

Second, by solving linear equations, we find $\mathbb{F}[[U_{1}, U_{2}]]$-linear quasi-homomorphisms

$$
\tilde{D}^{-L_{1}}_{-\infty,s_{2}} : A^{-\infty,s_{2}} \rightarrow A^{-\infty,s_{2}+\text{lk}}, \quad \tilde{D}^{-L_{2}}_{s_{1},-\infty} : A_{s_{1},-\infty} \rightarrow A_{s_{1}+\text{lk},-\infty}.
$$

Finding chain maps is a problem of solving linear equations modulo 2, which has fast algorithm in the case of sparse matrices. In order to find a quasi-isomorphism without computing the homology, we adopt an area filtration on the complexes as follows. From the Schubert diagram $\mathcal{H}^{L}$, one
can see that the diagram $r_{-L_i}(H^L)$ is isotopic to the standard genus-0 diagram of the unknot with one free basepoint and two intersection points $x, y$ of attaching curves. Let $C^u$ be the chain complex of $\mathbb{F}[[U_1, U_2]]$-modules generated by $x, y$, with differential $\partial x = (U_1 - U_2)y$. Thus there is a chain homotopy equivalence by counting holomorphic triangles from $A^-_{-\infty, s_2}$ to $C^u$, denoted by $F: A^-_{-\infty, s_2} \to C^u$.

Consider the Heegaard diagram by removing $z_1$, then an area filtration argument shows this chain homotopy equivalence $F: A^-_{-\infty, s_2} \to C^u$ is in the form of

$$F(b_0) = x + \text{lower terms},$$
$$F(b_{-1}) = y + \text{lower terms},$$

where the lower terms are referred to the area filtration. In fact, as long as a chain map $G: C^u \to A^-_{-\infty, s_2}$ is in the form of

$$G(x) = b_0 + \text{lower terms},$$
$$G(y) = b_{-1} + \text{lower terms},$$

then it is a quasi-isomorphism. This is because $F \circ G: C^u \to C^u$ is in the form of

$$F \circ G(x) = x + \text{lower terms},$$
$$F \circ G(y) = y + \text{lower terms},$$

which is an isomorphism of groups by Lemma 9.10 in [13]. In order to find an area filtration, we can set every bigon and square to be of the same area to be of the same area 1 on the Schubert Heegaard diagram so that every periodic domain has area 0. We can also similarly determine $F$.

Third, we plug in all the maps $I^M$ and $\tilde{D}^{L_i}$ to $\text{(5.4)}$. By Corollary 5.4 these $\tilde{D}^{L_i}$’s are chain homotopic to $D^{L_i}$’s as maps of $\mathbb{F}[[U_1, U_2]]$-modules. Thus, following the same line in the proof of Theorem 5.13, we can find $\mathbb{F}[[U_1, U_2]]$-linear diagonal maps $\tilde{F}$’s and $\tilde{D}^{L_i}$ to make those rectangles to be hyperboxes of chain complexes. Finding such maps is also a problem of solving linear equations.

Finally, after compressing all the rectangles and doing $\Lambda$-twisted gluing of these squares, we obtain the perturbed surgery complex $(\tilde{C}^-(H^L, \Lambda), \tilde{D}^-)$. Then, we compute the homology over the polynomial ring $\mathbb{F}[[U_1, U_2]]$ and there are several algorithms of polynomial time. This $\mathbb{F}[[U_1, U_2]]$-module might not be isomorphic to $HF^-(S^3_\Lambda(L))$. However, by Theorem 5.13 as an $\mathbb{F}[[U_1]]$-module, it is isomorphic to $HF^-(S^3_\Lambda(L))$. So we impose the $U_2$-action on the homology of $(\tilde{C}^-(H^L, \Lambda), \tilde{D}^-)$ to be the same as the $U_1$-action. By Theorem 5.13 now it is isomorphic to $HF^-(S^3_\Lambda(L))$ as an $\mathbb{F}[[U_1, U_2]]$-module.

We note that the surgery complex is infinitely generated over $\mathbb{F}[[U_1, U_2]]$. Hence, before finding the perturbed surgery complex, we need to do truncations for a fixed framing matrix $\Lambda$, as described in [6] Section 8.3. The time complexity of doing truncations is a polynomial of $\det(\Lambda)$.

**Remark 5.15.** Indeed, in the second and third steps, we only need $\mathbb{F}[[U_1]]$-linear quasi-isomorphisms $\tilde{D}^{L_i}$’s and $\mathbb{F}[[U_1]]$-linear diagonal maps $\tilde{F}$’s and $\tilde{D}^{L_1 \cup L_2}$ to replace those $\mathbb{F}[[U_1, U_2]]$-linear maps. The reason we use $\mathbb{F}[[U_1, U_2]]$-linear maps is that over $\mathbb{F}[[U_1, U_2]]$ the module $A^-_s$ is finitely generated and thus easier to use in computer programs.

**Example 5.16.** Consider $(0, 0)$ surgery on the unlink $L = L_1 \cup L_2$ and look at the $(0, 0)$ $\text{Spin}^c$ structure $s_0$. The general Floer complexes of the unlink $A^-_s(L)$’s are all $C^u$, where $C^u$ is defined in Remark 5.3. Since the Alexander gradings $A(x) = A(y) = (0, 0)$, the inclusion maps $I^L_{s_0}$ are all the identities for $i = 1, 2$. It follows that $\Phi^L_{s_0}$ and $\Phi^-_{s_0}$ are chain homotopic by Proposition 5.2.
Hence, we can get the perturbed surgery complex for the Spin⁶ structure s₀ as follows

\[
\begin{array}{ccc}
C^u & \rightarrow & C^u \\
\downarrow & & \downarrow \\
0 & \leftarrow & 0 \\
\end{array}
\]

where \( F \) is an \( \mathbb{F}[[U_1, U_2]] \)-linear map shifting the gradings by 1 satisfying \( \partial F = F \partial \). Thus, \( F \) can either be 0 or the following map \( f : C^u \rightarrow C^u \), where \( f(x) = y, f(y) = 0 \).

**Case I:** For \( F = 0 \), the homology of the perturbed complex is \( (\mathbb{F}[[U_1, U_2]]/(U_1 - U_2))^\oplus 4 \).

**Case II:** For \( F = f \), the homology is \( (\mathbb{F}[[U_1, U_2]]/(U_1 - U_2))^\oplus 2 \oplus (\mathbb{F}[[U_1, U_2]]/(U_1 - U_2))^\oplus 4 \).

However, as \( \mathbb{F}[[U]] \)-modules, both homology groups are isomorphic to \( \mathbb{F}[[U]]^\oplus 4 \). Thus, by imposing the \( U_2 \)-action to be the same as \( U_1 \), we obtain the correct homology is \( (\mathbb{F}[[U_1, U_2]]/(U_1 - U_2))^\oplus 4 \).

5.6. Further discussions of the perturbed surgery complex. Besides replacing the maps in a hypercube of chain complexes up to homotopy, we can also replace the chain complexes sitting at the vertices in the hypercube up to chain homotopy equivalences. Sometimes this procedure allows us to simplify the computations of the homology of a hypercube.

**Lemma 5.17.** Let \( A, \tilde{A}, B, \tilde{B} \) be chain complexes. Suppose \( h_A : A \rightarrow \tilde{A}, h_{\tilde{A}} : \tilde{A} \rightarrow A \) are the chain homotopy equivalences, with \( h_A h_{\tilde{A}} \cong \text{id}_A, h_{\tilde{A}} h_A \cong \text{id}_A \). Similarly, we have the chain homotopy equivalences \( h_B, h_{\tilde{B}} \) and the chain homotopies \( K_B, K_{\tilde{B}} \) with \( h_B h_{\tilde{B}} \cong \text{id}_{\tilde{B}}, h_{\tilde{B}} h_B \cong \text{id}_B \). Let \( f : A \rightarrow B \) be any chain map. Then, the mapping cones \( \text{cone}(f) \) and \( \text{cone}(h_B f h_{\tilde{A}}) \) are chain homotopy equivalent via the chain maps \( H_1 : \text{cone}(f) \rightarrow \text{cone}(h_B f h_{\tilde{A}}), H_2 : \text{cone}(h_B f h_{\tilde{A}}) \rightarrow \text{cone}(f) \) induced by the following squares of chain complexes

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
\tilde{A} & \xrightarrow{h_B f h_{\tilde{A}}} & \tilde{B} \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
\tilde{A} & \xrightarrow{h_B f h_{\tilde{A}}} & \tilde{B} \\
\end{array}
\]

**Proof.** We directly compute the compositions \( H_1 \circ H_2 \) and \( H_2 \circ H_1 \) to check that they are both chain homotopic to the identities. The composition \( H_2 \circ H_1 \) is the compression of the juxtaposition of the two squares, which is the square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
\tilde{A} & \xrightarrow{h_B f h_{\tilde{A}}} & \tilde{B} \\
\end{array}
\]
where \( \bar{F} = K_B f h_{\tilde{A}} h_A + h_{\tilde{B}} h_B f K_A \). The following cube shows that \( H_2 \circ H_1 \) is homotopic to \( \text{id}_{\text{cone}(f)} \).

![Diagram]

where \( \bar{F} = K_B f h_{\tilde{A}} h_A + h_{\tilde{B}} h_B f K_A \). Direct computation verifies that the above cube is a hypercube chain complex. We only check the longest diagonal here:

\[
K_B f + f K_A + K_B f h_{\tilde{A}} h_A + h_{\tilde{B}} h_B f K_A = K_B f (\partial_A K_A + K_A \partial_A) + (\partial_B K_B + K_B \partial_B) f K_A \\
= K_B f K_A \partial A + \partial_B K_B f K_A.
\]

Similarly, \( H_1 \circ H_2 \) is chain homotopic to \( \text{id}_{\text{cone}(h_{\tilde{B}} f h_{\tilde{A}})} \).

Now we generalize this lemma to a hypercube version. The proof is similar, so we omit it.

**Proposition 5.18.** Suppose \( A \) and \( \tilde{A} \) are two chain homotopy equivalent \( n \)-dimensional hypercubes, and so do \( B \) and \( \tilde{B} \). Then the \( (n+1) \)-dimensional hypercube \( \text{cone}(A \xrightarrow{f} B) \) is chain homotopy equivalent to \( \text{cone}(\tilde{A} \xrightarrow{h_{\tilde{B}} f h_{\tilde{A}}^{-1}} \tilde{B}) \), where \( h_B: B \to \tilde{B}, h_{\tilde{A}}: \tilde{A} \to A \) are chain homotopy equivalences.

Iterating these conjugation constructions, we have the following proposition.

**Proposition 5.19.** Let \( H = (C^e, D^e) \) be a \( n \)-dimensional hypercube of chain complexes. Suppose we have that \( \tilde{C}^e \) is chain homotopy equivalent to \( C^e \) for all \( e \in \mathbb{E}_n \). Then there exists a hypercube \( H = (\tilde{C}^e, \tilde{D}^e) \) which is chain homotopy equivalent to \( H \).

For the purpose of this paper, we give an example of the 2-dimensional case.

**Example 5.20.** Let \( H \) be the following square of chain complexes

\[
H = \begin{array}{ccc}
C_1 & \xrightarrow{f_1} & C_2 \\
\downarrow{g_1} & & \downarrow{g_2} \\
C_3 & \xrightarrow{f_2} & C_4
\end{array}
\]

Then suppose we have a set of chain homotopy equivalences \( h_i: C_i \to \tilde{C}_i, \bar{h}_i: \tilde{C}_i \to C_i \), where \( h_i \circ \bar{h}_i \) is homotopic to \( \text{id}_{\tilde{C}_i} \) via \( K_i \) and \( \bar{h}_i \circ h_i \) is homotopic to \( \text{id}_{C_i} \) via \( K_i \). Then compressing the following rectangle, we obtain the desired square \( \tilde{H} \).

![Diagram]
Thus,

\[
\tilde{H} = \xymatrix{ & \hat{C}_1 \ar[dr]^{h_2 f_1 \hat{h}_1} & \hat{C}_2 \\
C_3 \ar[ur]_{h_3 g_1 \hat{h}_1} & \hat{F} & C_4 \\
& \hat{C}_3 \ar[ur]_{h_4 f_2 \hat{h}_2} & }
\]

where \( \hat{F} = h_4 g_2 K_3 g_1 \hat{h}_1 + h_4 F \hat{h}_1 + h_4 f_2 K_2 f_1 \hat{h}_1. \)

If we know the chain homotopy types of all the \( A_g \)'s, we can also replace the chain complexes in the perturbed surgery complex by the conjugation construction. We will still call it the perturbed surgery complex. This is used in simplifying the computations in the next section.

6. Examples

6.1. The complexes \( \hat{CF}(L) \) for two-bridge links \( L \). We recall from [15] that for a link \( L \), the filtered chain complex \( \hat{CF}(L) \) is a chain complex of \( S^3 \) with a filtration induced from \( L \). More precisely, fixing a Heegaard diagram \( H^L \) of \( L \subset S^3 \), we obtain a chain complex of \( \mathbb{F} \)-modules \( \hat{CF}(H^L) \), generated by the intersection points of \( T_\alpha \) and \( T_\beta \) in the symmetric product. There is an Alexander filtration on \( \hat{CF}(H^L) \). It is shown that given different Heegaard diagrams of \( L \), \( H^L_1 \) and \( H^L_2 \), there is a chain homotopy equivalence from \( \hat{CF}(H^L_1) \) to \( \hat{CF}(H^L_2) \), which preserves the Alexander filtration. Thus, the filtered chain homotopy equivalence class of these chain complexes is called the filtered chain homotopy type of \( \hat{CF}(L) \). By abuse of notation, we also let \( \hat{CF}(L) \) be some filtered complex in this equivalence class. Similarly, we define the filtered chain homotopy type of \( CF^{-}(L) \), by looking at the Alexander filtered chain complex \( CF^{-}(H^L) \).

We represent \( \mathbb{Z}^2 \)-filtered complexes graphically by dots and arrows on the \( x-y \) coordinate plane, with the dots representing generators, the arrows representing differentials, and the coordinates representing filtrations.

**Theorem 6.1** (Theorem 12.1 in [15]). Suppose \( \overline{L} = \overline{L}_1 \cup \overline{L}_2 \) is an oriented two-component alternating link. Then the filtered chain homotopy type of \( \hat{CF}(L) \) is determined by the following data:

(1) the multi-variable Alexander polynomial of \( L \), \( \Delta_L \);
(2) the signature of \( L \), \( \sigma(L) \), and the linking number of \( L \), \( \text{lk}(L) \);
(3) the filtered chain homotopy type of \( \hat{CF}(L_1) \) and \( \hat{CF}(L_2) \).

In fact, for alternating two-component links, \( \hat{CF}(L) \) is filtered chain homotopy equivalent to a simplified filtered chain complex \( \hat{CF}_{\text{OS}}(L) \). The simplified complex is a direct sum of five different types of \( \mathbb{Z} \oplus \mathbb{Z} \)-filtered chain complexes \( B_{(d)}[i,j], H^L_{(d)}[i,j], \mathbb{V}_{(d)}[i,j], X^L_{(d)}[i,j] \) and \( Y^L_{(d)}[i,j] \). These basic filtered complexes are described in Section 12.1 of [15]; the filtered complex \( B_{(d)}[i,j] \) looks like a box and the others look like zigzags. See Figure 6.1.

**Corollary 6.2.** If \( L \) is an oriented two-bridge link, then the filtered homotopy type of \( \hat{CF}(L) \) is determined by \( \sigma(L), \text{lk}(L) \) and the multi-variable Alexander polynomial \( \Delta_L(x,y) \). More concretely, let

\[
l = \text{lk}(\overline{L}) + \frac{\sigma(\overline{L}) - 1}{2},
\]

(1) if \( l \geq 0 \), let \( a = \frac{1-\sigma-\text{lk}}{2}, \ b = \frac{-1-\sigma-\text{lk}}{2} \), then we have that \( \hat{CF}(L) \) is filtered chain homotopic to

\[
Y^L_{(a)}[a,a] \oplus Y^L_{(-1)}[b,b] \oplus \bigoplus_k B_{(d_k)}[i_k,j_k],
\]
where those $d_k,i_k,j_k$’s are determined by the Alexander polynomial $\Delta_L$;

(2) if $l < 0$, then we have that the $\hat{CFL}(L)$ is filtered chain homotopic to

\[
X_{(0)}^{(l)}\left[\frac{lk}{2}, \frac{lk}{2}\right] \oplus X_{(-1)}^{(l)}\left[\frac{lk}{2}, \frac{lk}{2}\right] \oplus \bigoplus_k B_{(d_k)}[i_k,j_k],
\]

where those $d_k,i_k,j_k$’s are determined by the Alexander polynomial $\Delta_L$.

**Example 6.3.** Let $Wh$ denote the Whitehead link. Since $\text{lk}(Wh) = 0$, $\sigma(Wh) = -1$, we get $l = -1$. Notice that $\text{lk} = 0$ implies that the signature doesn’t depend on the orientations of the link. Thus the filtered chain homotopy type of $\hat{CFL}(Wh)$ is

\[
X_{(0)}^{1}[0,0] \oplus X_{(-1)}^{0}[0,0] \oplus \bigoplus_k B_{(d_k)}[i_k,j_k],
\]

where those $d_k,i_k,j_k$ are determined by the Alexander polynomial. If we consider the mirror of $Wh$, we have $\sigma(\overline{Wh}) = 1$. Similarly, the filtered chain homotopy type of $\hat{CFL}(\overline{Wh})$ is

\[
Y_{(0)}^{0}[0,0] \oplus Y_{(-1)}^{1}[1,1] \oplus \bigoplus_k B_{(d_k')}[i_k',j_k'].
\]

In the following diagram, $\hat{CFL}(Wh)$ and $\hat{CFL}(\overline{Wh})$ are illustrated, where each dot represents a generator and each arrow represents a differential.

\[
\text{(6.1)} \quad \sigma(Wh) = 1 : \quad \sigma(Wh) = -1 :
\]

We find that it is easier to work with $Wh$ with $\sigma = -1$, since all the $A_+^-(Wh)$ have homology $\mathbb{F}[[U_1,U_2]]/(U_1 - U_2)$, so that we can apply the rigidity results. (One can compare this to the case of the right-handed trefoil knot versus the left-handed trefoil knot.)

**Remark 6.4.** The way of decomposing the complex $\hat{CFL}(\overline{L})$ into direct sums of $B,H,V,X,Y$ is not canonical. We can do some base-changes to change the above direct sum decomposition of $\hat{CFL}$ such that the patterns of the arrows don’t change.
6.2. The filtered homotopy type of $CFL^-(L)$ for some two-bridge links. Given a two-bridge link $L$, we can use the Schubert Heegaard diagram to combinatorially find the filtered complex $CFL^-(L)$. However, this description is too cumbersome. Instead, here we use algebraic arguments to determine the filtered homotopy type of $CFL^-(L)$ in some special examples.

Consider the Schubert Heegaard diagram $H$ for $L$ and the $\mathbb{Z}_2$-filtered chain complex $\widehat{CF}(H)$. Then there is a filtered chain homotopy equivalence $F : \widehat{CF}(H) \to \widehat{CFL}_{OS}(L)$. Thus, $F$ induces an isomorphism on the homology of their associated graded, i.e. the link Floer homology. In fact, the homology of their associated graded are just the chain groups themselves, because $\widehat{CF}(H)$ and $\widehat{CFL}_{OS}(L)$ are both thin (with no differentials in their associated graded). So $F$ is an isomorphism.

In this section, we show that for the two-bridge links $b(4n, 2n + 1)$, the filtered homotopy type of $CFL^-(L)$ is determined by $\widehat{CFL}(L)$. Since $\widehat{CFL}(L)$ can be decomposed as direct sums of $B, X, Y$’s, our goal is to show that $CFL^-(L)$ can be viewed as a square of chain complexes of these $B, X, Y$’s.

Using continuous fractions, we can get the 4-plat presentations of $b(4n, 2n + 1)$, thus providing the diagram in Figure 6.2. In addition, there is a convention issue of signs of the signature. We adopt the convention compatible with Corollary 6.2, so that $\sigma(b(8k, 4k + 1)) = -1$.

Proposition 6.5. For the two-bridge link $L = b(8k, 4k+1)$, the filtered homotopy type of $CFL^-(L)$ is determined by the Alexander polynomial, signature and linking number of $L$, or equivalently by $\widehat{CFL}(L)$. Precisely, we have $CFL^-(L) = CFL^-(\text{Wh}) \oplus \bigoplus_{i=1}^{k-1} (N, \partial^-)$, where $CFL^-(\text{Wh})$ and $(N, \partial^-)$ are described in Figure 6.3.

Proof. By Theorem 6.1, the Ozsváth-Szabó simplified complex $\widehat{CFL}_{OS}(L)$ can be computed in terms of $\Delta_L(x, y) = k\Delta_{\text{Wh}}(x, y) = k\frac{(x-1)(y-1)}{\sqrt{xy}}$, $\text{lk} = 0$, and $\sigma(L) = -1$. Then, we compute that

$$\widehat{CFL}_{OS}(L) = A \oplus B \oplus C \oplus D = \bigoplus_{i=1}^{k} A^{(i)} \oplus \bigoplus_{i=1}^{k} B^{(i)} \oplus \bigoplus_{i=1}^{k} C^{(i)} \oplus \bigoplus_{i=1}^{k} D^{(i)}.$$  

"n right-handed half twists"
Figure 6.3. The filtered complex $CFL^-(Wh)$ and $(N, \partial^-)$. The horizontal red arrows and vertical red arrows have $U_1$ and $U_2$ coefficients respectively.

Figure 6.4. $\widehat{CFL}_{OS}(b(8k, 4k \pm 1))$. On the left side, the figure illustrates the Alexander grading of $A, B, C, D$ summands, where $k = 2$. On the right side, it indicates the filtered homotopy types of $(A^{(i)}, \hat{\partial}, (B^{(i)}, \hat{\partial}), (C^{(i)}, \hat{\partial}), (D^{(i)}, \hat{\partial})$, which all have the filtered homotopy types as boxes, except for $(D^{(1)}, \hat{\partial})$.

See Figure 6.4 for the filtered homotopy type of $A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}$, where we denote the generators in $A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}$ by $a_j^{(i)}, b_j^{(i)}, c_j^{(i)}, d_j^{(i)}$, $j = 1, 2, 3, 4$, respectively.

Given $\widehat{CFL}_{OS}(L)$, let us investigate the possibilities for $CFL^-(L)$. The differential $\partial^-$ in $CFL^-(L)$ decomposes into

$$\partial^- = \hat{\partial} + \partial_{U_1 U_2} = \partial_{A_1} + \partial_{A_2} + \partial_{U_1} + \partial_{U_2},$$

where $\partial_{U_1 U_2}(x) = \partial_{U_1}(x) + \partial_{U_2}(x)$ consists of the components in $\partial^-(x)$ with coefficients of $U_1, U_2$ powers, and $\hat{\partial}(x) = \partial_{A_1}(x) + \partial_{A_2}(x)$ is decomposed by the Alexander filtration. As stated before, here $\partial_{U_i}$ has the form of $\partial_{U_i}(x) = U_i y$ for $i = 1, 2$, i.e., the $\partial_{U_i}$-arrows are all of length 1. A close examination of $U_1, U_2$ powers and the Alexander filtrations in the coefficients of the following
identity provides that
\[ 0 = (\partial^-)^2 = (\partial + \partial_{U_1,U_2})^2 = (\partial_{A_1} + \partial_{A_2} + \partial_{U_1} + \partial_{U_2})^2 \implies [\partial, \partial_{U_1,U_2}] = 0, \quad \partial_{U_1,U_2}^2 = \partial^2 = 0 \implies [\partial_{A_1}, \partial_{U_1}] = [\partial_{A_2}, \partial_{U_1}] = [\partial_{A_2}, \partial_{U_2}] = [\partial_{A_1}, \partial_{U_2}] = 0, \quad \partial_{A_1} \partial_{U_2} = \partial_{A_2} \partial_{U_1} = \partial_{A_2} \partial_{U_2} = 0. \]

where \([f,g] = fg + gf\).

At this point, we first consider the Whitehead link. The \(\text{CFL}(Wh)\) is shown by the right term in Equation (6.1), and the bullets are labeled as in Figure 6.4. By looking at the vertical arrows only, the equations \(\partial_{U_2}^2 = \partial_{A_2} = [\partial_{A_2}, \partial_{U_2}] = 0\) give rise to the two possibilities of the rightmost column as follows, according to whether \(\partial_{U_2}(c_4^{(1)})\) is 0 or not.

Consider the Heegaard diagram of \(L_1\) obtained from \(\mathcal{H}\) by deleting \(w_2\), i.e. the reduction \(r_{-L_2}(\mathcal{H})\). The differentials in \(\hat{CFL}(r_{-L_2}(\mathcal{H}))\) count the bigons without basepoints \(w_1, z_1, z_2\) on \(\mathcal{H}\), which are the same as bigons with basepoint \(w_2\). Thus, the complex \(\hat{CFL}(r_{-L_2}(\mathcal{H}))\) can be obtained by ignoring the arrows \(\partial_{A_2}\) and setting \(U_2 = 1\). So the vertical homology of \(CFL^-(L)\) using only the \(\partial_{U_2}\) arrows is the knot Floer homology of the unknot \(L_1, \mathbb{F} \oplus \mathbb{F}\), supported in the filtration \(A_1 = \tau(L_1) + \frac{lk(L)}{2} = 0\). Thus, the right-hand side in the above diagram is ruled out. A similar argument applies to the leftmost column. In sum,
\[
\partial_{U_2}(b_1^{(1)}) = U_2a_1^{(1)}, \quad \partial_{U_2}(b_2^{(1)}) = U_2a_2^{(1)}, \quad \partial_{U_2}(c_4^{(1)}) = U_2d_4^{(1)}, \quad \partial_{U_2}(c_3^{(1)}) = U_2d_3^{(1)}.
\]
Together with \(\partial_{A_1}, \partial_{U_2} = \partial_{U_2} \partial_{A_1}\), we get
\[
\partial_{A_1} \partial_{U_2}(b_4^{(1)}) = \partial_{U_2} \partial_{A_1}(b_4^{(1)}) = \partial_{U_2} b_4^{(1)} = U_2a_4^{(1)} \implies \partial_{U_2}(b_4^{(1)}) = U_2a_4^{(1)} \text{ or } U_2(a_4^{(1)} + d_1^{(1)}).
\]

Thus, \(\partial_{A_2} \partial_{U_2} = \partial_{U_2} \partial_{A_2}\) implies that \(\partial_{U_2}(b_3^{(1)}) = a_3^{(1)}\) and \(\partial_{U_2}(c_1^{(1)}) = d_1^{(1)}\), \(\partial_{U_2}(c_2^{(1)}) = 0\).

Next, \(\partial_{U_2} \partial_{A_2} = \partial_{A_2} \partial_{U_2}\) implies that \(\partial_{U_2}(d_2^{(1)}) \in U_2 \cdot D\). To determine \(\partial_{U_2}(d_2^{(1)})\), we consider the complex \(CFL^-(L) \otimes_{\mathbb{F}} [[U_1, U_2]](\mathbb{F}[[U_1, U_2]]/U_1) = CFL^-(L)/(U_1 \cdot CFL^-(L))\), i.e. setting \(U_1 = 0\). The homology of this complex can be computed from the long exact sequence of homologies, and it is \(\mathbb{F}[[U_2]]/U_2\) as an \(\mathbb{F}[[U_2]]\)-module. Meanwhile, to compute this homology we can also use the \(A_1\)-filtration and kill acyclic subcomplexes and acyclic quotient complexes. Taking the vertical
homology of this complex with respect to $\partial A_2$ leaves only $d_2^{(1)}$ and $d_2^{(4)}$. Thus, from the homology constraint we have computed, it follows that $\partial U_2(d_2^{(1)}) = U_2 d_1^{(1)}$. Thus, we recover all the $U_2$-arrows. See the above figure, where the dashed arrow is undetermined. Similarly, we can get all the $U_1$-arrows. By changing basis, $a_2^{(i)} = a_2^{(i)} + 1$ and $b_2^{(i)} = b_2^{(i)} + 1$, we can get rid of the dashed arrows, which gives the picture of $CFL^-(\mathcal{W})$ in Figure 6.3.

When $k > 1$, we follow the same line of argument, together with doing more changes of basis to prune the arrows. First, consider the rightmost column, i.e. $R = \text{Span}_{\mathbb{F}[\mathbb{U}_2]}\{d_3^{(i)}, d_4^{(i)}, c_3^{(i)}, c_4^{(i)}\}_{i=1}^k$ with the differentials $\partial A_2 + \partial U_2$. Assume $\partial U_2(c_3^{(i)}) = U_2 \cdot \sum_{m=1}^{k} \lambda_{i,m} d_4^{(m)}$. Then $\partial U_2 \partial A_2 = \partial A_2 \partial U_2$ implies that $\partial U_2(c_3^{(i)}) = U_2 \cdot \sum_{m=1}^{k} \lambda_{i,m} d_3^{(m)}$. So the matrix $D = (\lambda_{i,m})$ represents the differential $\partial U_2$ in the upside down vertical complex $(R \otimes_{\mathbb{F}[\mathbb{U}_2]} \mathbb{F}[\mathbb{U}_2])/(U_2 - 1, \partial U_2)$. Since its homology is 0, the matrix $D$ is invertible. In other words, the $\partial U_2$-arrows form an isomorphism from $\text{Span}_{\mathbb{F}}\{c_3^{(i)}, c_4^{(i)}\}_{i=1}^k$ to $\text{Span}_{\mathbb{F}}\{d_3^{(i)}, d_4^{(i)}\}_{i=1}^k$. Thus, we can find a new basis of $C$, namely $\{\tilde{c}_3^{(i)}, \tilde{c}_4^{(i)}, \tilde{c}_3^{(i)}, \tilde{c}_4^{(i)}\}_{i=1}^k$, such that

$$\partial U_2(\tilde{c}_3^{(i)}) = U_2 \cdot d_3^{(i)}, \quad \partial U_2(\tilde{c}_4^{(i)}) = U_2 \cdot d_4^{(i)}, \quad \forall 1 \leq i \leq k,$$

while the pattern of the $\hat{\partial}$ is preserved. In addition, $[\partial U_2, \partial A_1] = 0$ implies that

$$\partial U_2(\tilde{c}_1^{(i)}) = U_2 \cdot d_1^{(i)}, \quad \partial U_2(\tilde{c}_2^{(i)}) = U_2 \cdot d_2^{(i)}, \quad \forall 2 \leq i \leq k,$$

$$\partial U_2(\tilde{c}_1^{(1)}) = U_2 \cdot d_1^{(1)}, \quad \partial U_2(\tilde{c}_2^{(1)}) = 0.$$

From the fact that the vertical homology of $CFL^-(L)$ with respect to the differential $\partial A_2 + \partial U_2$ is $\mathbb{F}[\mathbb{U}_2]/U_2$, it follows that $\partial U_2(d_2) = U_2 \cdot d_1$.

We may as well keep using the notations $c_j^{(i)}$ for the new basis. Applying similar arguments for the leftmost column with respect to vertical arrows, we can change the basis of $A$ without changing the pattern of $\hat{\partial}$, such that

$$\partial U_2(b_j^{(i)}) = U_2 a_j^{(i)}, \quad \forall j = 1, 2, \forall i = 1, \ldots, k.$$

Then $\partial A_1 \partial U_2 = \partial U_2 \partial A_1$ implies

$$\partial U_2(b_3^{(i)}) = U_2 a_3^{(i)} + \sum_{m=1}^{k} \varepsilon_{i,m} U_2 d_1^{(m)}, \quad \forall i = 1, \ldots, k, \varepsilon_{i,m} \in \mathbb{F}.$$

Thus, $\partial U_2(b_3^{(i)}) = U_2 a_3^{(i)} + \sum_{m=2}^{k} \varepsilon_{i,m} U_2 d_2^{(m)}, \quad \forall i = 1, \ldots, k$. Do base-changes:

$$\tilde{a}_4^{(i)} = a_4^{(i)} + \sum_{m=1}^{k} \varepsilon_{i,m} d_1^{(m)}, \quad \tilde{a}_3^{(i)} = a_3^{(i)} + \sum_{m=2}^{k} \varepsilon_{i,m} d_2^{(m)}.$$

We can preserve the pattern of $\hat{\partial}$, such that under the new basis (where we keep using the notations $a_j^{(i)}$) all the vertical arrows are pruned as $\partial U_2(b_j^{(i)}) = U_2 a_j^{(i)}, \quad \forall j = 1, 2, 3, 4, \forall i = 1, \ldots, k$. Similarly, by changing the bases of $A$ and $B$ simultaneously, we can prune the horizontal arrows in the top row, while preserving the pattern of $\hat{\partial}, \partial U_2$ on $A$ and $B$, such that $\partial U_1(b_2^{(i)}) = U_1 c_2^{(i)}, \quad \forall j = 1, 4, \forall i = 1, \ldots, k$. Then $\partial A_2 \partial U_1 = \partial U_1 \partial A_2$ implies that $\partial U_1(b_2^{(i)}), \partial U_1(b_3^{(i)})$ are determined.

Similarly, all the horizontal arrows from $B$ can be pruned by changing the basis of $C$. Suppose

$$\partial U_1(b_4^{(i)}) = U_1 \left( \sum_{m=1}^{k} \lambda_{i,m} c_4^{(m)} + \sum_{m=1}^{k} \mu_{i,m} d_3^{(m)} \right).$$
Proposition 6.6. For the two-bridge link \( L = b(8k + 4, 4k + 3) \), the filtered homotopy type of \( \text{CFL}^-(L) \) is determined by the filtered homotopy type of \( \text{CFL}(L) \) (and hence by the Alexander polynomial, signature and linking number). Furthermore, \( \text{CFL}^-(L) = \text{CFL}^-(T(2, 4)) \oplus \bigoplus_{i=1}^{k-1} (N, \partial^-) \), where \((N, \partial^-)\) is as in Figure 6.3 and \( \text{CFL}^-(T(2, 4)) \) is as follows.

![Diagram](image-url)

6.3. Computations of surgeries on \( b(8k, 4k + 1) \). In this section, we compute the homology of surgeries on the two-bridge link \( b(8k, 4k + 1) \) and their \( d \)-invariants explicitly. Here, we make a convention of the \( d \)-invariants of \( \text{HF}^- \) different from [14]. We require that \( d(\text{HF}^-(S^3)) = 0 \). Thus, the \( d \)-invariants computed here are the same as the \( d \)-invariants for \( HF^+ \).

We will first compute for the Whitehead link, following three steps: computations of \( A_{s}^-(\text{Wh}) \), computations of the inclusion maps \( I_{s, \tau}^\text{Wh} \), and the computations of the homology of the surgeries on \( \text{Wh} \).

Lemma 6.7. \( H_s(A_{s}^-(\text{Wh})) = \mathbb{F}[[U_1, U_2]]/(U_1 - U_2) = \mathbb{F}[[U]] \) for all \( s \in \mathbb{H}^\text{Wh} = \mathbb{Z}^2 \).

Proof. By Proposition 6.5, we can decompose \( A_{-\infty, +\infty}^+(\text{Wh}) \) into a square of chain complexes, i.e.

\[
\begin{align*}
A &\longrightarrow D \\
\uparrow & \quad \uparrow \\
B &\longrightarrow C,
\end{align*}
\]

where the summands \( A, B, C, D \) are described in Proposition 6.5. Since \( A_{-\infty, +\infty}^- \) can be viewed as a mapping cone from \( A \oplus B \oplus C \) to \( D \), we get a short exact sequence of chain complexes

\[
0 \rightarrow D \rightarrow A_{-\infty, +\infty}^- \rightarrow A \oplus B \oplus C \rightarrow 0.
\]

From the fact that \( H_s(A_{-\infty, +\infty}^\text{Wh}) = \mathbb{F}[[U]] \) and \( [d_1] = [d_3] = 1 \in H_s(A_{-\infty, +\infty}^-) \).

All the other complexes \( A_{s_1, s_2}^-(\text{Wh}) \) can be actually obtained by taking various reflections on \( A_{-\infty, +\infty}^- \). Note that Equation (2.2) implies that the differentials in \( A_{s}^- \) are only changed by...
$U_1, U_2$ powers from $A^{-}_{+\infty,+\infty}$. In order to read off the correct powers of $U_1, U_2$-coefficients, we can change the $\mathbb{Z}^2$-filtration of $A^{-}_{+\infty,+\infty}$, such that the upward and rightward arrows in $A^{-}_{+\infty,+\infty}$ are with $U_1$ and $U_2$ coefficients respectively. For instance, when $s_1 > 0, s_2 = 0$, we can flip the summands $A$ and $D$ about the $A_1$-axis to obtain the complex $A^{-}_{s_1, 0}$. For convenience, we denote the vertical reflections of $A, B, C, D$ by $\overline{A}, \overline{B}, \overline{C}, \overline{D}$. Thus, the complex $A^{-}_{s_1, 0}$ is still a square of chain complexes as follows:

$$
\begin{array}{c}
\overline{A} \rightarrow \overline{D} \\
\uparrow \quad \uparrow \\
B \rightarrow C.
\end{array}
$$

Thus, the fact that $\overline{A}, \overline{B}, \overline{C}$ are acyclic implies that $H_*(A^{-}_{s_1, 0}) = H_*(\overline{D}) = \mathbb{F}[U]$. Similarly, we denote the horizontal reflections of $A, B, C, D$ by $|A, |B, |C, |D$ respectively. Thus, the complex $A^{-}_{0,s_1}$ with $s_1 > 0$ is the following square of chain complexes

$$
\begin{array}{c}
A \rightarrow |D \\
\uparrow \quad \uparrow \\
B \rightarrow |C.
\end{array}
$$

Following the same line, we list all the filtered homotopy types of $A^{-}_{s}$’s together with some generators of their homologies in Table 6.1. Since $\max A_1 = \max A_2 = 1$, $\min A_1 = \min A_2 = -1$, the notation $+\infty$ means a positive integer $s$, while $-\infty$ means a negative integer $s$.

| $A^{-}_{-\infty,+\infty}$ | $A^{-}_{0,+\infty}$ | $A^{-}_{+\infty,+\infty}$ |
|--------------------------|-------------------|--------------------------|
| $|A \rightarrow |D$       | $A \rightarrow |D$  | $A \rightarrow |D$       |
| $|B \rightarrow |C$,      | $|B \rightarrow |C$, | $|B \rightarrow |C$,      |
| $[d_1] = 1 \in H_*(A^{-}_{-\infty,+\infty})$; | $[d_1] = 1 \in H_*(A^{-}_{0,+\infty})$; | $[d_1] = [d_3] = 1 \in H_*(A^{-}_{+\infty,+\infty})$; |
| $[a_2] = 1 \in H_*(A^{-}_{-\infty,0})$;     | $[a_2] = 1 \in H_*(A^{-}_{0,0})$;     | $[d_3] = 1 \in H_*(A^{-}_{+\infty,0})$;     |
| $[a_2] = [c_2] = 1 \in H_*(A^{-}_{-\infty,-\infty})$; | $[c_2] = 1 \in H_*(A^{-}_{0,-\infty})$; | $[d_3] = [c_2] = 1 \in H_*(A^{-}_{+\infty,-\infty})$. |

Table 6.1. $A^{-}_{s}(Wh)$ and generators of their homology.

Note that $A, \overline{A}, |A, |B, |B, |C, |C$ are all acyclic, and $D, \overline{D}, |D, |D$ all have the same homology $\mathbb{F}[U]$. We can use the same argument for $A^{-}_{+\infty,+\infty}$ to show that $A^{-}_{-\infty,+\infty}, A^{-}_{0,+\infty}, A^{-}_{+\infty,+\infty}, A^{-}_{0,0}, A^{-}_{+\infty,0},$ and $A^{-}_{+\infty,-\infty}$ all have the same homology $\mathbb{F}[U]$. For those other $A^{-}_{s}$, we can use the conjugation symmetry, that is, $H_*(A^{-}_{s}(L)) = H_*(A^{-}_{-s}(L)), \forall s \in \mathbb{H}(L), \forall L$. This is because $A^{-}_{s}$’s are quasi-isomorphic to the Floer complexes of large surgeries on $L$.

Now we explain the generators of their homologies in Table 6.1. The chain complex $A^{-}_{0,-\infty}$ can be viewed as a mapping cone of a chain map from cone($B \rightarrow A$) to cone($\overline{C} \rightarrow |D$). Because
cone$(\mathcal{B} \to \mathcal{A})$ is acyclic, the generator of $H_*(\text{cone}(\mathcal{C} \to \mathcal{D}))$ is also a generator of $H_*(A_{\infty,-\infty})$. Since $H_*(\mathcal{C}) = \mathbb{F}[[U]]/U, H_*(\mathcal{D}) = \mathbb{F}[[U]]$, we derive a short exact sequence

$$0 \to \mathbb{F}[[U]] \to \mathbb{F}[[U]] \to \mathbb{F}[[U]]/U \to 0$$

from the long exact sequence of the homologies $\cdots \to H_*(\mathcal{D}) \to H_*(\text{cone}(\mathcal{C} \to \mathcal{D})) \to H_*(\mathcal{C}) \to \cdots$. Because $[c_2] = 1 \in H_*(\mathcal{C}) = \mathbb{F}[[U]]/U$ and $[c_2] \in \text{cone}(\mathcal{C} \to \mathcal{D})$ is mapped to $[c_2] \in H_*(\mathcal{C})$, the above short exact sequence implies that $[c_2] = 1 \in H_*(\text{cone}(\mathcal{C} \to \mathcal{D}))$, and thus $[c_2] = 1 \in H_*(A_{\infty,-\infty})$.

Similar arguments show that $[a_2] = [c_2] = 1 \in H_*(A_{-\infty,-\infty})$ and $[d_3] = 1 \in H_*(A_{-\infty,-\infty})$. Moreover, in the complex $A_{-\infty,-\infty}$, the equations $\partial^c c_1 = U c_2 + d_1, \partial^c d_2 = d_1 + U d_3$ imply that $[c_2] = [d_3] = 1 \in H_*(A_{-\infty,-\infty})$.

Taking the grading into account, we adopt the formula of the $\mathbb{Z}/2\mathbb{Z}$ grading defined on the surgery complex for a Spin$^c$ structures $u \in \mathbb{H}(L)/H(L, \Lambda)$ in [6] Section 7.4,

$$(6.2) \quad \mu(s, x) = \mu^M_s(x) + \nu(s) - ||M||, x \in \mathfrak{A}^-(H^L - \overline{M}, \psi^M(s)),$$

where $s \in u$ and $\mu^M_s = \mu_{\psi^M}(s)$ is a natural $\mathbb{Z}$-grading defined on $\mathfrak{A}^-(L - \overline{M}, \psi^M(s))$. In the torsion case, the quadratic function $\nu$ can be chosen as 0. The natural $\mathbb{Z}$-grading $\mu^0 = \mu_{s_1, s_2}$ on each $A_{s_1, s_2}$ is given by

$$\mu_{s_1, s_2}(x) = M(x) - 2 \sum_{i=1}^2 \max\{A_i(x) - s_i, 0\},$$

where $M(x)$ is the Maslov grading. When we use the Schubert Heegaard diagram, $A_1(x) + A_2(x) - M(x)$ is constant. Thus, up to a shift of a constant number, we can take $M(x) = A_1(x) + A_2(x)$ for all $x \in A_{s_1, s_2}$. In the primitive system we identify $\mathfrak{A}^-(L - \overline{M}, \psi^M(s))$ with some $A_{s_1, s_2}$ (where $s_1'$, $s_2'$ can evaluate $+\infty$), so the grading $\mu^M$ is actually $\mu^0$. We define some rules of $\infty$ as follows:

$$0 \cdot (+\infty) = +\infty; \quad s \cdot (+\infty) = +\infty, \forall s \in \mathbb{R}; \quad s \cdot (-\infty) = -\infty, \forall s \in \mathbb{R}; \quad (\pm 1) \cdot (+\infty) = \pm \infty.$$

Recall the notations in Example 4.14. The complexes $C^{(s_1, s_2)} = A_{s_1 + \epsilon_1 - \infty, s_2 + \epsilon_2 - \infty}$, $\epsilon_1 \in \{0, 1\}$ are setting at the position $(s_1, s_2)$ in the square and with the index $(s_1, s_2)$ in the product complex $C^{(s_1, s_2)} = \prod_{s_1, s_2} C^{(s_1, s_2)}$.

We define the grading $\mu^{s_1, s_2}$ on the complex $C^{(s_1, s_2)}$ by the formula:

$$\mu^{s_1, s_2}_{s_1, s_2}(x) = M(x) - 2 \sum_{i=1}^2 \max\{A_i(x) - s_i - \epsilon_i(\infty), 0\} - \epsilon_1 - \epsilon_2.$$
where \( c_1(s) = [2s] - \Lambda_1 - \Lambda_2 = (2s_1 - p_1, 2s_2 - p_2) \in \mathbb{Z}^2/\Lambda. \) In the perturbed surgery complex, since all the perturbed maps have the same degrees as the original, we can compute the gradings still using Equation (6.3).

Now we restrict our scalars to \( \mathbb{F}[[U]] \). By Proposition 5.5 and Lemma 6.7 up to \( \mathbb{F}[[U]] \)-linear chain homotopy, all the edge maps \( \Phi_{s,L}^\pm \) are classified by their actions on the homologies. The actions of \( \Phi_{s,L}^\pm \) on homologies are determined by the corresponding inclusion maps \( I_{s,L}^\pm \). We denote the induced maps on homologies by \( (I_{s,L}^\pm)_*: \mathbb{F}[[U]] \to \mathbb{F}[[U]] \).

**Lemma 6.8.** Regarding the inclusion maps, we have the following results for \( I_{s,L}^\pm \), where \( s = (s_1, s_2) \).

- If \( s_1 > 0 \), then \( (I_{s,L}^\pm)_{s_1} = \text{id} \).
- If \( s_1 = 0, s_2 \neq 0 \), then \( (I_{s,L}^\pm)_{s_2} = \text{id} \).
- If \( s_1 = s_2 = 0 \), then \( (I_{s,L}^\pm)_{s_1} = U \cdot \text{id} \).
- If \( s_1 < 0 \), then \( (I_{s,L}^\pm)_{s_1} = U^{-s_1} \cdot \text{id} \).
- If \( s_1 > 0 \), then \( (I_{s,L}^\pm)_{s_1} = U^{s_1} \cdot \text{id} \).
- If \( s_1 = 0, s_2 \neq 0 \), then \( (I_{s,L}^\pm)_{s_2} = \text{id} \).
- If \( s_1 = s_2 = 0 \), then \( (I_{s,L}^\pm)_{s_1} = U \cdot \text{id} \).
- If \( s_1 < 0 \), then \( (I_{s,L}^\pm)_{s_1} = \text{id} \).

**Proof.** In fact, when \( s_1 > 0 \), by definition \( I_{0,s_2}^{L_1} = \text{id} \). When \( s_1 = 0, s_2 > 0 \), we have \( I_{0,s_2}^{L_1}(d_1) = d_1 \). Therefore by Table 6.1, the inclusion map \( I_{0,s_2}^{L_1} \) acts on the homology as \( \text{id} : \mathbb{F}[[U]] \to \mathbb{F}[[U]] \). When \( s_1 = 0, s_2 < 0 \), we have \( I_{0,s_2}^{L_1}(c_2) = c_2 \). Therefore by Table 6.1, the inclusion map \( I_{0,s_2}^{L_1} \) acts on homology as the identity. When \( s_1 = s_2 = 0 \), we have \( I_{0,0}^{L_1}(d_3) = U_1 \cdot d_3 \). Therefore by Table 6.1, \( I_{0,0}^{L_1} \) acts on homology as \( U \cdot \text{id} \). When \( s_1 < 0, s_2 \leq 0 \), we have \( I_{s_1,s_2}^{L_1}(a_2) = U_1^{-s_1} \cdot a_2 \). In the complex \( A_{s_1,s_2}^\pm, s_2 \leq 0 \), the equation \( \partial^- a_3 = a_2 + U_1 d_3 \) implies that \( [a_2] = U_1[d_3] \in H_*(A_{s_1,s_2}^-) \). Thus, \( [a_2] = U \in \mathbb{F}[[U]] = H_*(A_{s_1,s_2}^-) \). Therefore, it follows that \( (I_{s_1,s_2}^{L_1})_* = U^{-s_1} \cdot \text{id} \), when \( s_1 < 0, s_2 \leq 0 \).

In the same way, we get the following results for \( I_{s,L}^\pm \), where \( s = (s_1, s_2) \).

Now we can compute the homology of surgeries on \( S \). In each case, we write down the \( d \)-invariants, which are the gradings of the top element in each \( \mathbb{F}[[U]] \) summand.

**Proposition 6.9.** Let Wh be the Whitehead link, \( \Lambda = \text{diag}(p_1, p_2) \) and \( Y \) be the surgery manifold \( S^3_\Lambda(\text{Wh}) \). Then \( \text{Spin}^c(Y) \) can be identified with \( \mathbb{Z}^2/\Lambda \cong \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z} \), so we use \( (t_1, t_2) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z} \) to denote the \( \text{Spin}^c \) structures over \( Y \). Then, the Floer homology of \( Y \) is as follows.

- If \( p_1 = p_2 = 0 \), then \( \text{HF}^-(Y, (t_1, t_2)) = \begin{cases} \mathbb{F}[[U]]^{\oplus 4}, & \text{if } (t_1, t_2) = (0, 0); \\ 0, & \text{otherwise}. \end{cases} \) with \( d = -1, -1, 0, 0 \).

- If \( p_1 > 0, p_2 = 0 \), then \( \text{HF}^-(Y, (t_1, t_2)) = \begin{cases} \mathbb{F}[[U]]^{\oplus 2}, & \text{if } (t_1, t_2) = (t_1, 0); \\ 0, & \text{otherwise}. \end{cases} \) Their \( d \)-invariants are \( d(Y, (0, 0)) = \frac{p_1}{4} - \frac{7}{4}, \frac{p_1}{4} - 3 \), and \( d(Y, (t_1, 0)) = \frac{(2s_1 + p_1)^2}{4p_1} + \frac{1}{4}, \frac{(2s_1 + p_1)^2}{4p_1} - \frac{3}{4} \), when \( t_1 \neq 0 \), where \( s_1 \) is an integer in the class \( t_1 \in \mathbb{Z}/p_1\mathbb{Z} \) such that \(-p_1 < s_1 \leq 0 \).

- If \( p_1 = 0, p_2 = 0 \), then \( \text{HF}^-(Y, (t_1, t_2)) = \begin{cases} \mathbb{F}[[U]]^{\oplus 2} \oplus (\mathbb{F}[[U]])/U, & \text{if } (t_1, t_2) = (0, 0); \\ \mathbb{F}[[U]]^{\oplus 2}, & t_1 \neq 0, t_2 = 0; \\ 0, & \text{otherwise}. \end{cases} \) Their \( d \)-invariants are \( d(Y, (t_1, 0)) = \frac{(2s_1 - p_1)^2}{4p_1} + \frac{3}{4}, \frac{(2s_1 - p_1)^2}{4p_1} - \frac{1}{4} \), when \( s_1 \) is an integer in the class \( t_1 \in \mathbb{Z}/p_1\mathbb{Z} \) such that \( p_1 < s_1 \leq 0 \).
• If $p_1 > 0, p_2 > 0$, then $\mathbf{HF}^{-}(Y, (t_1, t_2)) = \mathbb{F}[[U]]$, $\forall (t_1, t_2) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$. Their $d$-invariants are $d(Y, (0, 0)) = \frac{p_1 + p_2 - 10}{4}$, and $d(Y, (t_1, t_2)) = \frac{(2s_1 + p_1)^2}{4p_1} + \frac{(2s_2 + p_2)^2}{4p_2} - \frac{1}{2}$, when $(t_1, t_2) \neq (0, 0)$, where $s_i$ is an integer in the class $t_i \in \mathbb{Z}/p_i\mathbb{Z}$ such that $-p_i < s_i \leq 0$.

• If $p_1 > 0, p_2 < 0$, then $\mathbf{HF}^{-}(Y, (t_1, t_2)) = \left\{ \begin{array}{ll} \mathbb{F}[[U]], & (t_1, t_2) = (0, 0); \\ \mathbb{F}[[U]], & \text{otherwise.} \end{array} \right.$ Their $d$-invariants are $d(Y, (t_1, t_2)) = \frac{p_1 (2s_1 + p_1)^2}{4p_1} + \frac{(2s_2 - p_2)^2}{4p_2} + 1$, where $s_i$ is an integer in the class $t_i \in \mathbb{Z}/p_i\mathbb{Z}$ such that $-|p_i| < s_i \leq 0$.

• If $p_1, p_2 < 0$, then $\mathbf{HF}^{-}(Y, (t_1, t_2)) = \left\{ \begin{array}{ll} \mathbb{F}[[U]] \oplus (\mathbb{F}[[U]]/U), & (t_1, t_2) = (0, 0); \\ \mathbb{F}[[U]], & \text{otherwise.} \end{array} \right.$ Their $d$-invariants are $d(Y, (t_1, t_2)) = \frac{(2s_1 - p_1)^2}{4p_1} + \frac{2s_2 - p_2)^2}{4p_2}$, where $s_i$ is an integer in the class $t_i \in \mathbb{Z}/p_i\mathbb{Z}$ such that $-|p_i| < s_i \leq 0$.

Proof. First, let’s look at the $(0, 0)$-surgery on the Whitehead link. The surgery complex splits into a direct product of squares of chain complexes according to Spin$^c$ structures. See Figure 6.5. In the $(s_1, s_2)$ Spin$^c$ structure, the factor of the direct product is the following square of chain complexes:

$$
\begin{array}{c}
A^+_{s_1, s_2} \Phi^{L_1}_{s_1, s_2} + \Phi^{-L_2}_{s_1, s_2} A^+_{-\infty, s_2} \\
\sum_{s_1, s_2} \Phi^{L_1}_{s_1, s_2} + \Phi^{-L_2}_{s_1, s_2} A^+_{\infty, \infty, s_2} \\
A^-_{s_1, \infty} + \Phi^{L_1}_{s_1, \infty} + \Phi^{-L_2}_{s_1, \infty} A^+_{\infty, \infty, s_2} \\
A^+_{\infty, \infty, \infty} \Phi^{L_1}_{\infty, \infty} + \Phi^{-L_2}_{\infty, \infty} A^+_{\infty, \infty, \infty} \\
\end{array}
$$

For the torsion Spin$^c$ structure $(0, 0) \in \mathbb{Z}^2$, since $\Phi^{L_1}_{0, 0} \simeq \Phi^{-L_2}_{0, 0}$, $\Phi^{L_2}_{0, 0} \simeq \Phi^{-L_2}_{0, 0}$, $\Phi^{L_1}_{0, \infty} \simeq \Phi^{-L_1}_{0, \infty}$, $\Phi^{L_2}_{0, \infty} \simeq \Phi^{-L_2}_{0, \infty}$, the perturbed surgery complex is as follows:

$$
\begin{array}{c}
A^-_{0, 0} \xrightarrow{0} A^+_{-\infty, 0} \\
\xrightarrow{0} \xrightarrow{0} A^+_{\infty, \infty, \infty} \\
\end{array}
$$

Therefore, the homology is $\mathbb{F}[[U]]^{\oplus 4}$ generated by $d_1 \in C_{(0, 0)}^{(0, 0)}$, $d_3 \in C_{(0, 0)}^{(1, 0)}$, $d_1 \in C_{(0, 0)}^{(1, 0)}$, $d_3 \in C_{(0, 0)}^{(1, 1)}$. Since $c_1((0, 0)) = (0, 0)$, $\chi(W) = 2$, $\sigma(W) = 0$, from Equation (6.3), we also get their absolute gradings $\mu_{0, 0, 0}((d_1)) = -1$, $\mu_{0, 0, 0}((d_3)) = 0$, $\mu_{0, 0, 0}((d_1)) = 0$, $\mu_{0, 0, 0}((d_3)) = -1$.

For the non-torsion Spin$^c$ structure $(s_1, s_2) \in \mathbb{Z}^2$, $s_1 > 0$, since $\Phi^{L_1}_{s_1, s_2} + \Phi^{-L_2}_{s_1, s_2}$ acts on homology as $id + U^{s_1} \cdot id = (1 + U^{s_1}) \cdot id$, which is a quasi-isomorphism, it follows that the homology of this Spin$^c$ structure is 0. Indeed, one can consider the horizontal filtration for this square, whose associated graded is the direct sum of the two acyclic horizontal rows. A similar argument applies to all the other non-torsion Spin$^c$ structures.

Second, let’s look at the $(p_1, 0)$-surgery with $p_1 \neq 0$, which gives rise to a manifold with $b_1 = 1$. Suppose $p_1 > 0$. In order to compute the homology, we need some filtrations to kill acyclic subcomplexes and quotient complexes. Let $\mathcal{F}_1(C_{(s_1, s_2)}^{(\epsilon_1, \epsilon_2)}) = -s_1, \mathcal{F}_2(C_{(s_1, s_2)}^{(\epsilon_1, \epsilon_2)}) = s_1 - (\epsilon_1 - 1)p_1$. Without loss of generality, see Figure 6.6 for the illustration of the surgery complex and the truncation in the case of $\Lambda = (1, 0)$.

For any $(t_1, t_2) \in \text{Spin}^c(Y) = \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}$ with $t_2 \neq 0$, the Floer homology is 0. Indeed, we can consider the union of all these Spin$^c$ structures, which corresponds to the subcomplex

$$
\mathcal{R}_1 = \bigoplus_{s_2 \neq 0} C_{(s_1, s_2)}^{(0, 0)} \oplus C_{(s_1, s_2)}^{(1, 0)} \oplus C_{(s_1, s_2)}^{(0, 1)} \oplus C_{(s_1, s_2)}^{(1, 1)}.
$$
Since $\Phi^{+L_2}_{s_1,s_2} + \Phi^{-L_2}_{s_1,s_2} \neq 0$ acts on homology as $id + U^{[s_2]} \cdot id = (1 + U^{[s_2]}) \cdot id$, which is a quasi-isomorphism, the following square is acyclic:

$$
\begin{array}{ccc}
A^-_{s_1,s_2} & \xrightarrow{\Phi^{+L_1}_{s_1,-s_2}, \Phi^{-L_1}_{s_1,-s_2}} & A^-_{\infty,s_2} \\
\Phi^{+L_2}_{s_1,s_2} + \Phi^{-L_2}_{s_1,s_2} & \Phi^{+L_1 \cup +L_2}_{s_1,s_2} + \Phi^{-L_1 \cup -L_2}_{s_1,s_2} & \Phi^{+L_2}_{s_1,s_2} + \Phi^{-L_2}_{s_1,s_2}
\end{array}
$$

The associated graded complex of $F_1$ splits as a direct product of the above squares, so $R_1$ is acyclic.

For the Spin$^c$ structure $(t_1,0)$, we first kill the acyclic subcomplex

$$R_2 = \bigoplus_{s_1 > 0} C^{(s_1,s_2)}_{(s_1,0)}.$$

Since the inclusion map $I^{+L_1}_{s_1,0}$ is $id$ for all $s_1 > 0$, the associated graded complex of the filtration $F_1$ splits as a direct product of acyclic complexes in the form of

$$
\begin{array}{ccc}
C^{(0,0)}_{(s_1,0)} & \xrightarrow{\Phi^{+L_1}_{s_1,0}} & C^{(1,0)}_{(s_1,0)} \\
\Phi^{+L_2}_{s_1,0} + \Phi^{-L_2}_{s_1,0} & \Phi^{+L_1 \cup +L_2}_{s_1,s_2} + \Phi^{-L_1 \cup -L_2}_{s_1,s_2} & \Phi^{+L_2}_{s_1,s_2} + \Phi^{-L_2}_{s_1,s_2}
\end{array}
$$

Thus $R_2$ is acyclic.

On the other hand, we have another acyclic subcomplex

$$R_3 = \bigoplus_{F_2 \leq 0} C^{(s_1,s_2)}_{(s_1,0)}.$$
Figure 6.6. The surgery complex for $\Lambda = (1, 0)$. Every dot represents a complex $C^s_0$ which is a certain generalized Floer complex $A^s_\mathbb{M}(Wh)$, and in every shaded circle the complexes $C^s_0$’s have the same subscript $s$. Every arrow represents a $\Phi$-map according to the endpoints of the arrow, where we omit the subscripts. All the parallel arrows share the same type of $\Phi$, i.e., having the same superscript $\mathbb{M}$. The arrows with circled numbers $1, 2, 3, 4$ are $\Phi_{s_1,0}^{+L_2}, \Phi_{s_1,0}^{-L_2}, \Phi_{s_1,0}^{L_2∪L_2}, \Phi_{s_1,0}^{-L_2∪L_1}$, $\Phi_{s_1,0}^{+L_2}, \Phi_{s_1,0}^{-L_2}$, and $\Phi_{s_1,0}^{+L_2∪L_1} + \Phi_{s_1,0}^{-L_2∪L_1}$ respectively. The regions $R_1, R_2, R_3$ divided by the (thicker) lines are corresponded to the acyclic subcomplexes $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$. The shaded region $Q$ corresponds to the truncated complex $Q$.

In fact, since the inclusion maps $I_{s_1,0}^{-L_1}$ and $I_{s_1,0}^{-L_1}$ are both $id$ when $s_1 < 0$, the associated graded complex of the filtration $\mathcal{F}_2$ splits as a direct product of acyclic complexes in the form of

$$
\begin{array}{c}
\text{C}_{s_1,0}^{(0,0)} & \xrightarrow{\Phi_{s_1,0}^{L_1}} & \text{C}_{s_1,0}^{(1,0)} \\
\Phi_{s_1,0}^{+L_2} & \Phi_{s_1,0}^{-L_2} & \Phi_{s_1,0}^{L_2∪L_2} & \Phi_{s_1,0}^{-L_2∪L_1} \\
\text{C}_{s_1,0}^{(0,1)} & \xrightarrow{\Phi_{s_1,0}^{-L_1}} & \text{C}_{s_1,0}^{(1,1)}
\end{array}
$$

Thus $\mathcal{R}_3$ is acyclic. So the quotient complex $Q = \mathcal{C}^- / \mathcal{R}_1 ∪ \mathcal{R}_2 ∪ \mathcal{R}_3$ is a direct product of

$$
\begin{array}{c}
\text{C}_{s_1,0}^{(0,0)} & \xrightarrow{\Phi_{s_1,0}^{L_1}} & \text{C}_{s_1,0}^{(1,0)} \\
\Phi_{s_1,0}^{+L_2} & \Phi_{s_1,0}^{-L_2} & \Phi_{s_1,0}^{L_2∪L_2} & \Phi_{s_1,0}^{-L_2∪L_1} \\
\text{C}_{s_1,0}^{(0,1)} & \xrightarrow{\Phi_{s_1,0}^{-L_1}} & \text{C}_{s_1,0}^{(1,1)}
\end{array}
$$

where $-p_1 + 1 \leq s_1 \leq 0$. From the computations of the inclusion maps, we know that $\Phi_{s_1,0}^{L_2} \sim \Phi_{s_1,0}^{-L_2}$. Thus the homology of each $\text{Spin}^c$ structure $(t_1, 0) ∈ \mathbb{Z}/p_1\mathbb{Z} ⊕ \mathbb{Z}$ is $\mathbb{F}[(U)]^{\mathbb{Z}/2}$. Note that $\chi(W) = 2, \sigma(W) = 1$. When $-p_1 + 1 \leq s_1 < 0$, the complex $C_{s_1,0}^{(0,0)}$ has $d_2$ as a generator of its homology of grading $\mu_{s_1,0}^{0,0}(d_2) = s_1 + p_2 / 4 + 1 / 4$, and the complex $C_{s_1,0}^{(0,0)}$ has $d_1$ as a generator of its homology with grading $\mu_{s_1,0}^{0,0}(d_1) = p_1 / 4 - 3 / 4$. While for the $(0, 0)$ $\text{Spin}^c$ structure, $C_{s_1,0}^{(0,0)}$ has $d_1$ as a generator of its homology with grading $\mu_{s_1,0}^{0,0}(d_1) = p_1 / 4 - 3 / 4$. 
The case of \( p_1 < 0 \) is similar. We first kill the acyclic subcomplex
\[
\mathcal{R}_1 = \bigoplus_{s_2 \neq 0} (C_{(s_1, s_2)}^{(0, 0)} \oplus C_{(s_1, s_2)}^{(1, 0)} \oplus C_{(s_1, s_2)}^{(0, 1)} + C_{(s_1, s_2)}^{(1, 1)}).
\]
Thus, the homology for the \( \text{Spin}^c \) structure \((t_1, t_2)\) with \( t_2 \neq 0 \) is 0. Next, we kill the acyclic quotient complexes
\[
\mathcal{R}_2 = \bigoplus_{s_1 > 0} C_{(s_1, 0)}^{(\varepsilon_1, \varepsilon_2)}, \quad \mathcal{R}_3 = \bigoplus_{s_1 - \varepsilon_1 p_1 < 0} C_{(s_1, 0)}^{(\varepsilon_1, \varepsilon_2)}.
\]
In the \((0, 0)\) \( \text{Spin}^c \) structure, the remaining complexes are as follows
\[
\begin{array}{cccc}
C_{(p_1, 0)}^{(1, 0)} & \Phi_{0, 0}^{L_1} & C_{(p_1, 0)}^{(0, 0)} & \Phi_{0, 0}^{L_1} & C_{(p_1, 0)}^{(1, 0)} \\
\Phi_{+\infty, 0}^{L_2} + \Phi_{-\infty, 0}^{L_2} & \Phi_{0, 0}^{L_1 + L_2} + \Phi_{-0, 0}^{L_2} + \Phi_{0, 0}^{L_1 + L_2} & \Phi_{0, 0}^{L_1 + L_2 + \Phi_{0, 0}^{L_2}} + \Phi_{0, 0}^{L_1 + L_2} & \Phi_{0, 0}^{L_1 + L_2} + \Phi_{0, 0}^{L_1 + L_2} \\
C_{(p_1, 0)}^{(1, 1)} & \Phi_{0, +\infty}^{L_1} & C_{(p_1, 0)}^{(0, 1)} & \Phi_{0, +\infty}^{L_1} & C_{(p_1, 0)}^{(1, 1)}
\end{array}
\]
Since \( \Phi_{+\infty, 0}^{L_2}, \Phi_{-\infty, 0}^{L_2} \) are chain homotopic, we can replace \( \Phi_{+\infty, 0}^{L_2} + \Phi_{-\infty, 0}^{L_2} \) by 0 in the perturbed surgery complex. Therefore, we can also replace the diagonal maps by 0. Thus, we have two split complexes,
\[
\begin{array}{cccc}
C_{(p_1, 0)}^{(1, 0)} & \Phi_{0, 0}^{L_1} & C_{(p_1, 0)}^{(0, 0)} & \Phi_{0, 0}^{L_1} & C_{(p_1, 0)}^{(1, 0)} \\
C_{(p_1, 0)}^{(1, 1)} & \Phi_{0, +\infty}^{L_1} & C_{(p_1, 0)}^{(0, 1)} & \Phi_{0, +\infty}^{L_1} & C_{(p_1, 0)}^{(1, 1)}
\end{array}
\]
Since \( C_{(p_1, 0)}^{(1, 0)} = C_{(p_1, 0)}^{(0, 0)} \) and \( \Phi_{0, 0}^{L_1} \simeq \Phi_{-0, 0}^{L_1} \), we can replace \( \Phi_{0, 0}^{L_1} \) by \( \Phi_{0, 0}^{L_1} \) in the perturbed complex. By changing basis, we can split the first row as cone(\( \Phi_{0, 0}^{L_1} \)) \( \oplus \) \( C_{(p_1, 0)}^{(1, 0)} \) with homology \( \mathbb{F}[U] \oplus (\mathbb{F}[U]/U) \). Similarly, from that \( \Phi_{0, +\infty}^{L_1} \) are quasi-isomorphisms, it follows that the second row is quasi-isomorphic to \( C_{(0, 0)}^{(1, 1)} \) by changing basis. Thus in the \((0, 0)\) \( \text{Spin}^c \) structure, the homology is \( \mathbb{F}[U] \oplus 2 \oplus (\mathbb{F}[U]/U) \).

For the other \( \text{Spin}^c \) structures \((t_1, 0)\), \( t_1 \neq 0 \), the remaining complexes are as follows
\[
\begin{array}{cccc}
C_{(s_1, 0)}^{(1, 0)} & \Phi_{0, +\infty}^{L_1} + \Phi_{-0, +\infty}^{L_2} & C_{(s_1, 0)}^{(1, 1)} \\
\end{array}
\]
where the integer \( s_1 \) is in the residue class \( t_1 \in \mathbb{Z}/p_1 \mathbb{Z} \) such that \( p_1 < s_1 < 0 \). Similarly, we replace \( \Phi_{+\infty, 0}^{L_2} + \Phi_{-\infty, 0}^{L_2} \) by 0, and get that the homology is \( \mathbb{F}[U] \oplus 2 \). The correction terms can be computed similarly with \( \chi(W) = 2, \sigma(W) = -1 \).

Finally, let’s look at the \((p_1, p_2)\)-surgery, where \( p_1 p_2 \neq 0 \). This breaks down to three cases: \( p_1, p_2 > 0 \), \( p_1, p_2 < 0 \), and \( p_1 p_2 < 0 \). We apply the truncation tricks shown in [10] Section 8.3.

(1) When \( p_1, p_2 > 0 \), the \((p_1, p_2)\)-surgery is actually a large surgery, so its homology can be derived from \( A_s^- \) directly. However, we still compute them by elementary methods. We construct two filtrations,
\[
\mathcal{F}_{00}(C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}) = -s_1 - s_2, \quad \mathcal{F}_{11}(C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}) = s_1 - (\varepsilon_1 - 1)p_1 + s_2 - (\varepsilon_2 - 1)p_2.
\]
Without loss of generality, see Figure 6.7 for the illustration of the surgery complex and the truncation in the case of \( \Lambda = (1, 1) \).
We first consider an acyclic subcomplex
\[ \mathcal{R}_1 = \bigoplus_{\max\{s_1, s_2\} > 0} C^{(\varepsilon_1, \varepsilon_2)}_{(s_1, s_2)}. \]

In fact, since the inclusion maps \( I_{s_1, s_2}^+ \), \( s_1 > 0 \) and \( I_{s_1, s_2}^-, s_2 > 0 \) are both id’s, the associated graded complex of the filtration \( \mathcal{F}_{00} \) splits as a direct product of acyclic squares \( R_{s, 0, 0} \) in Equation 4.2:

Thus \( \mathcal{R}_1 \) is acyclic.

There is another acyclic subcomplex
\[ \mathcal{R}_2 = \bigoplus_{\max\{s_1 - (\varepsilon_1 - 1)p_1, s_2 - (\varepsilon_2 - 1)p_2\} \leq 0} C^{(\varepsilon_1, \varepsilon_2)}_{(s_1, s_2)}. \]

One can directly check \( \mathcal{R}_2 \) is a subcomplex by computation. Because the inclusion maps \( I_{s_1, s_2}^-, s_1 < 0 \) and \( I_{s_1, s_2}^-, s_2 < 0 \) are both id’s, the associated graded complex of \( \mathcal{F}_{11} \) splits as a product of acyclic squares \( R_{s, 1, 1} \):

where \( s_1 + p_1 \leq 0, s_2 + p_2 \leq 0. \) Thus \( \mathcal{R}_2 \) is acyclic.

Let \( C_1 = \mathcal{C}/(\mathcal{R}_1 + \mathcal{R}_2) \). Inside \( C_1 \), there are two acyclic subcomplexes
\[ \mathcal{R}_3 = \bigoplus_{s_1 - (\varepsilon_1 - 1)p_1 \leq 0, p_1 \leq 1, s_2 \leq 0} C^{(\varepsilon_1, \varepsilon_2)}_{(s_1, s_2)} \cap C_1 = \bigoplus_{s_1 - (\varepsilon_1 - 1)p_1 \leq 0, p_1 \leq 1, s_2 \leq 0} (C^{(0,0)}_{(s_1, s_2)} \oplus C^{(1,0)}_{(s_1 + p_1, s_2)}), \]
\[ \mathcal{R}_4 = \bigoplus_{s_2 - (\varepsilon_2 - 1)p_2 \leq 0, p_1 \leq 1, s_1 \leq 0} C^{(\varepsilon_1, \varepsilon_2)}_{(s_1, s_2)} \cap C_1 = \bigoplus_{s_2 - (\varepsilon_2 - 1)p_2 \leq 0, p_1 \leq 1, s_1 \leq 0} (C^{(0,0)}_{(s_1, s_2)} \oplus C^{(1,1)}_{(s_1, s_2 + p_2)}). \]

In fact, the associated graded complex of \( \mathcal{F}_{11} \) on \( \mathcal{R}_3 \) splits as a direct product of acyclic complexes
\[ C^{(0,0)}_{(s_1, s_2)} \xrightarrow{\Phi_{s_1, s_2}^{+L_1}} C^{(1,0)}_{(s_1 + p_1, s_2)} \] because the inclusion map \( I_{s_1, s_2}^-, s_1 < 0 \) is id. Thus \( \mathcal{R}_3 \) is acyclic. Similar argument applies to \( \mathcal{R}_4 \).

At last, we look at the quotient complex
\[ Q = C_1/(\mathcal{R}_3 + \mathcal{R}_4) = \bigoplus_{-p_1 < s_1 \leq 0, -p_2 < s_2 \leq 0} C^{(0,0)}_{(s_1, s_2)}, \]
where \( C^{(0,0)}_{(s_1, s_2)} = A_{s_1, s_2}^- \). There is only one \( A_{s_1, s_2}^- \) left in each Spin\(^c\) structure \( Y \) with homology \( \mathbb{F}[\left[U\right]] \). For \( (s_1, s_2) = (0, 0) \), the complex \( C^{(0,0)}_{(0,0)} = A_{0,0}^- \) has \( d_1 \) as a generator of its homology with grading \( \mu_{0,0}^1(d_1) = \frac{p_1 + p_2 - 10}{4} \). For \( -p_1 < s_1 < 0 \), the complex \( C^{(0,0)}_{(s_1, s_2)} = A_{s_1, s_2}^- \) has \( a_2 \) as a generator of its homology with grading \( \mu_{s_1, s_2}^1(a_2) = \frac{s_1^2}{p_1} + \frac{s_2^2}{p_2} + s_1 + s_2 + \frac{p_1 + p_2 - 2}{4} \). Similarly, we have the same formula for \( -p_2 < s_2 < 0, -p_1 < s_1 \leq 0 \).
We have an acyclic quotient complex \( R \) for \( \Phi \). In fact, since

\[
\Phi + L_1 \cup L_2, \Phi + L_1 \cup -L_2, \Phi - L_1 \cup L_2, \Phi - L_1 \cup -L_2
\]

respectively. The regions \( R_1, R_2, R_3, R_4 \) divided by the (thicker) lines are corresponded to the acyclic subcomplexes \( R_1, R_2, R_3, R_4 \). The shaded region \( Q \) corresponds to the truncated complex \( Q \).

(2) When \( p_1 p_2 < 0 \), we might as well suppose \( p_1 > 0, p_2 < 0 \) due to the symmetry of the two components. We construct four filtrations

\[
\mathcal{F}_{00}(C_{(s_1,s_2)}) = -s_1 + s_2, \quad \mathcal{F}_{01}(C_{(s_1,s_2)}) = -s_1 - s_2 + (\varepsilon_2 - 1)p_2,
\]

\[
\mathcal{F}_{10}(C_{(s_1,s_2)}) = s_1 - (\varepsilon_1 - 1)p_1 + s_2, \quad \mathcal{F}_{11}(C_{(s_1,s_2)}) = s_1 - (\varepsilon_1 - 1)p_1 - s_2 + (\varepsilon_2 - 1)p_2.
\]

Without loss of generality, see Figure 6.8 for the illustration of the surgery complex and the truncation in the case of \( \Lambda = (1, -1) \). We first kill an acyclic subcomplex \( R_1 \) composed of \( C_{(s_1,s_2)}^{(\varepsilon_1,\varepsilon_2)} \) with \( s_1 > 0 \). Indeed, the associated graded complex of \( \mathcal{F}_{00} \) on \( R_1 \) splits as a direct product of acyclic squares, since the inclusion map \( I_{s_1,s_2}^{+L_1} \), \( s_1 > 0 \) is id.

We have another acyclic subcomplex

\[
R_2 = \bigoplus_{s_1 - (\varepsilon_1 - 1)p_1 \leq 0} C_{(s_1,s_2)}^{(\varepsilon_1,\varepsilon_2)}.
\]

In fact, since \( \Phi_{-L_1}^{s_1,s_2} \) are quasi-isomorphisms when \( s_1 < 0 \), the associated graded of the filtration \( \mathcal{F}_{10} \) for \( R_2 \) splits as a direct product of acyclic squares \( R_{s_1,0} \). Thus \( R_2 \) is acyclic.

Thus, \( C \) is quasi-isomorphic to the quotient complex

\[
C_1 = C/(R_1 + R_2) = \bigoplus_{-p_1 < s_1 \leq 0} (C_{(s_1,s_2)}^{(0,0)} \oplus C_{(s_1,s_2)}^{(0,1)}).
\]

We have an acyclic quotient complex \( R_3 \) of \( C_1 \)

\[
R_3 = \bigoplus_{-p_1 < s_1 \leq 0, s_2 > 0} (C_{(s_1,s_2)}^{(0,0)} \oplus C_{(s_1,s_2)}^{(0,1)}),
\]

since the inclusion maps \( I_{s_1,s_2}^{+L_2} \), \( s_2 > 0 \) are all the identities. Furthermore, we have another acyclic quotient complex \( R_4 \) of \( C_1 \)

\[
R_4 = \bigoplus_{-p_1 < s_1 \leq 0, s_2 < 0} (C_{(s_1,s_2)}^{(0,0)} \oplus C_{(s_1,s_2)}^{(0,1)}).
\]
Thus, the homology of the surgery complex for $\Lambda = (1, -1)$. The arrows with circled numbers 1, 2, 3, 4 are $\Phi^{+L_1 \cup L_2}$, $\Phi^{+L_1 \cup -L_2}$, $\Phi^{-L_1 \cup L_2}$, and $\Phi^{-L_1 \cup -L_2}$ respectively. The regions $R_1, R_2, R_3, R_4$ divided by the (thicker) lines are corresponded to the acyclic subcomplexes $R_1, R_2, R_3, R_4$. The shaded region $Q$ corresponds to the truncated complex $Q$.

Thus $C$ is quasi-isomorphic to
\[
Q = C_1((R_3 \cup R_4) = \bigoplus_{-p_1 < s_1 \leq 0} (C_{(s_1, 0)}^{(0, 0)} \oplus C_{(s_1, 0)}^{(0, 1)} \oplus C_{(s_1, p_2)}^{(0, 1)}) \oplus \bigoplus_{-p_1 < s_1 \leq 0, p_2 < s_2 < 0} C_{(s_1, s_2)}^{(0, 1)}.
\]

In the Spin$^c$ structure $(t_1, 0) \in \mathbb{Z}/p_1 \mathbb{Z} \oplus \mathbb{Z}/p_2 \mathbb{Z}$, we have the complex as follows,
\[
C_{(s_1, 0)}^{(0, 0)} = A_{s_1, 0}^{-} \xrightarrow{\Phi_{s_1, 0}^{+L_2}} A_{s_1, +\infty}^{-} = C_{(s_1, 0)}^{(0, 1)}
\]
\[
A_{s_1, 0}^{-} \xrightarrow{\Phi_{s_1, 0}^{-L_2}} A_{s_1, +\infty}^{-} = C_{(s_1, p_2)}^{(0, 1)},
\]
where $s_1$ is an integer such that $-p_1 < s_1 \leq 0$ and $s_1 \equiv t_1 (\text{mod } p_1)$. Since the inclusion maps $I_{0,0}^{+L_2}$ induce the same action on homology, $\Phi_{0,0}^{\pm L_2}$ are chain homotopic to each other. By Corollary 5.6, we can replace $A_{0,0}^{-}, A_{0, +\infty}$ by the complex $\mathbb{F}[[U_1, U_2]] \xrightarrow{U_1-U_2} \mathbb{F}[[U_1, U_2]]$, where the generators are $g_1, g_2$. Then, we can replace the chain maps $I_{0,0}^{\pm L_2}$ by the same chain map $\tilde{I}$, where $\tilde{I}(g_i) = U_1 g_i$. Thus, the homology of the $(0, 0)$ Spin$^c$ structure can be computed by this perturbed complex, which is $\mathbb{F}[[U]] \oplus \mathbb{F}[[U]]/U$. From above computation, the generator corresponding to $\mathbb{F}[[U]]$ is actually the generator of $H_* (C_{(0,0)}^{(0, 1)})$, which is $d_1 \in A_{s_1, +\infty}$ with grading $\mu_{0,1}^{(0,1)}(d_1) = \frac{p_1 + p_2}{4}$ by Equation (6.3).

On the other hand, since the inclusion maps $I_{s_1,0}^{-L_2}$, $s_1 < 0$ are all quasi-isomorphisms, we can kill the acyclic quotient complex $A_{s_1, 0}^{-} \xrightarrow{\Phi_{s_1, 0}^{-L_2}} A_{s_1, +\infty}^{-}$. Thus, the homology for the Spin$^c$ structure $(t_1, 0) \in \mathbb{Z}/p_1 \mathbb{Z} \oplus \mathbb{Z}/p_2 \mathbb{Z}$ with $t_1 \neq 0$ is $\mathbb{F}[[U]]$ generated by $d_1 \in A_{s_1, +\infty}$ of grading $\mu_{s_1, 0}^{(0,1)}(d_1) = \frac{s_1^2}{p_1} + s_1 + \frac{p_1 + p_2}{4}$, where $s_1$ is an integer with $-p_1 < s_1 < 0$ in the class $t_1$ modulo $p_1$.

In every other Spin$^c$ structure in the complex $Q$, there is only one complex $C_{(s_1, s_2)}^{(0, 1)} = A_{s_1, +\infty}^{-}, -p_1 < s_1 \leq 0$ of homology $\mathbb{F}[[U]]$ of grading $\frac{s_1^2}{p_1} + \frac{s_2^2}{p_2} + s_1 - s_2 + \frac{p_1 + p_2}{4}$.
(3) The last case is when $p_1, p_2$ are both negative integers. We use two filtrations

$$F_{00}(C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}) = s_1 + s_2, \quad F_{11}(C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}) = -s_1 + (\varepsilon_1 - 1)p_1 - s_2 + (\varepsilon_2 - 1)p_2.$$ 

Without loss of generality, see Figure 6.9 for the illustration of the surgery complex and the truncation in the case of $\Lambda = (-1, -1)$.

We first kill an acyclic quotient complex

$$R_1 = \bigoplus_{\max\{s_1, s_2\} > 0} C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}.$$ 

By considering the filtration $F_{00}$, we can see that $R_1$ is acyclic. We also have another acyclic quotient complex

$$R_2 = \bigoplus_{\min\{s_1 - \varepsilon_1, p_2 - \varepsilon_2, p_2\} < 0} C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}.$$ 

In fact, the inclusion maps $I_{s_1, s_2}^{L_i}, s_i < 0$ are quasi-isomorphisms. Thus the associated graded complex of the filtration $F_{11}$ splits as a direct product of acyclic complexes

$$R_{s_1, s_2} \cap (\mathcal{C} \setminus R_1),$$

where $\min\{s_1, s_2\} < 0$ and $R_{s_1, s_2}$ is in Equation (4.5). Therefore $R_2$ is acyclic.

Hence, the subcomplex $Q = \mathcal{C} \setminus (R_1 \cup R_2)$ is quasi-isomorphic to $\mathcal{C}$, where

$$Q = \bigoplus_{\max\{s_1, s_2\} \leq 0, \min\{s_1 - \varepsilon_1, p_1 - 1, s_2 - \varepsilon_2, p_2\} \geq 0} C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}.$$ 

In the Spin$^c$ structure $(t_1, t_2), t_1 \neq 0, t_2 \neq 0$, there is only one complex $C_{(s_1, s_2)}^{(1, 1)}$ in $Q$, thus having the homology $\mathbb{F}[[U]]$ with grading $(\frac{2s_1 - p_1}{4p_1} + \frac{2s_2 - p_2}{4p_2} + \frac{1}{2})$, where $s_1, s_2$ are negative integers in the residue classes $t_1, t_2$ such that $s_i \geq p_i + 1, i = 1, 2$. In the Spin$^c$ structure $(0, t_2), t_2 \neq 0$, there are three complexes $C_{(0, s_2)}^{(0, 1)}, C_{(0, s_2)}^{(1, 1)}, C_{(p_2, s_2)}^{(1, 1)}$ in $Q$, where $s_2$ is an integer in the residue class $t_2$ such that $p_2 < s_2 < 0$. Since the inclusion map $I_{s_1, s_2}^{L_1}, s_2 \neq 0$ are quasi-isomorphisms, we can replace $\Phi_{s_2}^{L_1}$ in the perturbed complex and thus split it as a direct sum, $\text{cone}(\Phi_{s_2}^{L_1}) C_{(0, s_2)}^{(1, 1)}$, by changing the basis. Thus the homology is the same as the homology of $C_{(0, s_2)}^{(1, 1)} = A_{-\infty, +\infty}$, which is $\mathbb{F}[[U]]$ generated by $[d_1]$ with grading $\frac{p_1}{4} + \frac{2s_2 - p_2}{4p_2} + \frac{1}{2}$. It is similar for $(t_1, 0) \in \text{Spin}^c(Y), t_1 \neq 0$.

The most interesting Spin$^c$ structure is $(0, 0)$. It consists of nine complexes, which are also illustrated in Figure 6.9. By Corollary 5.6 and the discussion in Section 5.6, in the perturbed complex we can replace all the $A_+^{-\infty}$ by the complex $\mathbb{F}[[U]]$ with 0 differential and replace the edge maps by the corresponding maps on homology. Finally, the perturbed complex is the following chain complex

$$\begin{array}{c}
\mathbb{F}[[U]] \xrightarrow{1} \mathbb{F}[[U]] \xrightarrow{1} \mathbb{F}[[U]] \\
\mathbb{F}[[U]] \xrightarrow{U} \mathbb{F}[[U]] \xrightarrow{U} \mathbb{F}[[U]] \\
\mathbb{F}[[U]] \xrightarrow{1} \mathbb{F}[[U]] \xrightarrow{1} \mathbb{F}[[U]].
\end{array}$$

Direct computation shows that

$$\text{HF}^{-}(S^3_\Lambda (Wh), (0, 0)) = \mathbb{F}[[U]] \oplus (\mathbb{F}[[U]]/U),$$

when $\Lambda = \text{diag}(p_1, p_2)$ with $p_1, p_2 < 0$. Thereby, $[d_1] = 1 \in H_*(C_{(p_1, p_2)}^{(1, 1)})$ is a generator of the $\mathbb{F}[[U]]$ summand with the absolute grading $\frac{p_1 + p_2 + 2}{4}$.\qed
THEOREM 6.10. Let $\widetilde{L}$ be the two-bridge link $b(8k, 4k + 1)$, $k \in \mathbb{N}$ and $\Lambda = \text{diag}(p_1, p_2), p_1, p_2 \in \mathbb{Z}$ be the framing matrix of an integer surgery on $L$. As in Proposition 6.9, we use $(t_1, t_2) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$ to denote the $\text{Spin}^c$ structures over $S^3(\widetilde{L})$. Then, we have the Floer homology

$$\text{HF}^{-}(S^3(\widetilde{L}), (t_1, t_2)) = \begin{cases} \text{HF}^{-}(S^3(\text{Wh}), (0, 0)) \oplus \mathbb{F}^{2(k-1)}, & (t_1, t_2) = (0, 0), \\ \text{HF}^{-}(S^3(\text{Wh}), (t_1, t_2)), & \text{otherwise.} \end{cases}$$

The correction terms of the elements in the $\text{HF}^{-}(S^3(\text{Wh}))-\text{summand}$ are the same as in $\text{HF}^{-}(S^3(\text{Wh}))$.

Proof. By Proposition 6.5, $\text{CFL}^{-}(L) = \text{CFL}^{-}(\text{Wh}) \oplus \bigoplus_{i=1}^{k-1}(N, \partial^i)$. Let $\mathcal{N} = \bigoplus_{i=1}^{k-1}(N, \partial^i)$. We define $\mathcal{N}'$ similarly as $\mathcal{N}_s$ in (2.2). Concretely, suppose $G$ be a set of homogeneous generators of $\mathcal{N}$ as a $\mathbb{F}[[U_1, U_2]]$-module, and for $x \in G$,

$$\partial x = \sum_{y \in G} k_{xy} y,$$

where $k_{xy} \in \mathbb{F}[[U_1, U_2]]$. Let $A(x) = (A_1(x), A_2(x))$ denote the Alexander filtration of $x \in G$. Define $\mathcal{N}_s$ by

$$\partial x = \sum_{y \in G} k_{xy} \cdot U_1^{\max\{A_1(x) - s_1, 0\} - \max\{A_1(y) - s_1, 0\}} U_2^{\max\{A_2(x) - s_2, 0\} - \max\{A_2(y) - s_2, 0\}} \cdot y.$$

Thus $A_s^{-}(L) = A_{s}^{-}(\text{Wh}) \oplus \mathcal{N}'$. Thus all the inclusion maps $I^{\pm L_i}, i = 1, 2$ preserve this direct sum decomposition. Since the complexes $\mathcal{N}_{s_1, \pm \infty}$ are acyclic complexes, we can choose $\tilde{D}^{\pm L_2}_{s_1, -\infty} : A^{-}_{s_1, -\infty}(L) \rightarrow A^{-}_{s_1, +\infty}(L)$ to be

$$\tilde{D}^{\pm L_2}_{s_1, -\infty} = D(\text{Wh})^{\pm L_2}_{s_1, -\infty} \oplus 0,$$

where $D(\text{Wh})^{\pm L_2}_{s_1, -\infty}$ is the destabilization map for $\text{Wh}$. Therefore $\tilde{\Phi}^{\pm L_i}_s = \Phi(\text{Wh})^{\pm L_i}_s \oplus \Phi^{\pm L_i}_{\mathcal{N}, s}$, where $\Phi^{\pm L_i}_{\mathcal{N}, s} = 0 : \mathcal{N}_s \rightarrow \mathcal{N}_{s}^{\pm L_i(s)}$. 
Thus the perturbed surgery complex $(\tilde{C}^{-}(\tilde{L},\Lambda),\tilde{D}^{-})$ is a direct sum of two twisted gluing of squares
\[(\tilde{C}^{-}(\tilde{L},\Lambda),\tilde{D}^{-}) = (C^{-}(Wh,\Lambda),D^{-}) \oplus \prod_{s=(s_1,s_2) \in \mathbb{Z}^2} \left( N_s \oplus N_{s_1,+\infty} \oplus N_{s_2,+\infty} \oplus N_{+\infty,+\infty},\tilde{D}^{-}\right).\]

From the fact that any $N_s$ with $s \neq (0,0)$ is acyclic, it follows that $H_{s}(\tilde{C}^{-}(\tilde{L}),\tilde{D}^{-}) = H_{s}(C^{-}(Wh),D^{-}) \oplus H_{s}(N_{0,0})$. For that $N_{0,0}$ belongs to the $(0,0)$ Spin$^c$ structure and $H_{s}(N_{0,0}) = F[[U]]/U$, we have the equations (6.4). The absolute gradings are inherited from $H_{s}(C^{-}(Wh),D^{-})$. □

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