NONEXISTENCE RESULTS FOR A HIGHER-ORDER EVOLUTION EQUATION WITH AN INHOMOGENEOUS TERM DEPENDING ON TIME AND SPACE

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Abstract. We consider a higher-order evolution equation with an inhomogeneous term depending on time and space. We first derive a general criterion for the nonexistence of weak solutions. Next, we study the particular case when the inhomogeneity depends only on space. In that case, we obtain the first critical exponent in the sense of Fujita, as well as the second critical exponent in the sense of Lee and Ni.

1. Introduction

In this paper, we investigate the questions of existence and nonexistence of weak solutions to the inhomogeneous problem

\( \Box_k u = |u|^p + |\partial_t^{k-1} u|^q + w(t, x) \quad \text{in} \quad (0, \infty) \times \mathbb{R}^N. \)

Here \( \Box_k := \partial_t^k - \Delta \) \((k \geq 2)\), \( \partial_t^k := \frac{\partial^k}{\partial t^k} \), \( p, q > 1 \), \( N \geq 1 \) and \( w \geq 0 \) is a nontrivial \( L^1_{\text{loc}} \) function.

We mention below some motivations for studying problems of type (1.1).

In the case \( k = 2 \) and \( w \equiv 0 \), problem (1.1) reduces to

\( \Box_2 u = |u|^p + |\partial_t u|^q \quad \text{in} \quad (0, \infty) \times \mathbb{R}^N. \)

This problem can be considered as a natural combination of the problem

\( \Box_2 u = |u|^p \quad \text{in} \quad (0, \infty) \times \mathbb{R}^N \)

and the problem

\( \Box_2 u = |\partial_t u|^q \quad \text{in} \quad (0, \infty) \times \mathbb{R}^N. \)

John [12] proved that problem (1.3) in \( \mathbb{R}^3 \) admits as critical exponent \( p_c(3) := 1 + \sqrt{2} \), in the sense that: when \( 1 < p < p_c(3) \) the solution blows up in a finite time, while for \( p > p_c(3) \), there exists global solution. Strauss [24] conjectured that for all \( N \geq 2 \), problem (1.3) admits as critical exponent the real number \( p_c(N) \), which is the positive root of the equation

\( (N - 1)p^2 - (N + 1)p - 2 = 0, \)

that is,

\( p_c(N) = \frac{(N + 1) + \sqrt{N^2 + 10N - 7}}{2(N - 1)}. \)

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Later, several mathematicians have contributed to solve this conjecture, see e.g. [4, 5, 6, 11, 13, 16, 19, 20, 23, 25, 27, 29, 31]. For problem (1.4), it was conjectured (Glassey conjecture) that the critical exponent is given by
\[ q_c(N) := 1 + \frac{2}{N-1}. \]
In the case \( q \leq q_c(N) \), the nonexistence of global small solutions for problem (1.4) was investigated by several authors, see e.g. [1, 13, 18, 21, 30]. In the case \( q > q_c(N) \), the existence of global small solutions has been established in [8, 22, 26] for \( N \in \{2, 3\} \), and in [9] for \( N \geq 4 \), under the radial assumption of the initial data. In [7], Han and Zhou investigated the blow-up phenomenon for problem (1.2). Namely, they obtained blow-up results when \( p > p_c(N) \), \( q > q_c(N) \) and \( (p-1)((N-1)q-2) < 4 \). Moreover, in certain cases, they obtained an upper bound of the lifespan. In [10], Hidano, Wang and Yokoyama investigated problem (1.2) in the case \( N \in \{2, 3\} \). Namely, they determined the full region of \((p, q)\) for which there is global existence of small solutions. Moreover, a sharp lower bound of the lifespan was obtained for many \((p, q)\) when there is no global existence.

In [28], Zhang studied the inhomogeneous semilinear wave equation
\[ \Box u = |u|^p + w(x) \quad \text{in} \quad (0, \infty) \times \mathbb{R}^N, \]
where \( w \geq 0 \) is a nontrivial \( L^1_{\text{loc}} \) function. He proved that, if one assumes that stationary solutions are global solutions, then the critical exponent for problem (1.5) is \( p^* = \frac{N}{N-2} \), \( N \geq 3 \). Namely, he showed that, when \( 1 < p < p^* \), then problem (1.5) possesses no global solutions for any initial values; when \( p > p^* \), then problem (1.5) has global solutions (more precisely, stationary solutions) for some \( w > 0 \) and suitable initial values. Note that, in the blow-up case, no assumptions on the sign or decay of the initial values were supposed.

In [17], Mitidieri and Pohozaev considered the inhomogeneous exterior problem
\[ \left\{ \begin{array}{ll}
\Box u & \geq |u|^p + w(x) \quad \text{in} \quad (0, \infty) \times \Omega, \\
u & \geq 0 \quad \text{on} \quad (0, \infty) \times \partial \Omega, \\
\partial_t^{k-1} u(0, x) & \geq 0 \quad \text{in} \quad \Omega,
\end{array} \right. \]
where \( \Omega \) is the exterior of a ball of center 0 and radius \( R > 0 \), and \( w \geq 0 \) is a nontrivial \( L^1_{\text{loc}} \) function. Using the test function method, they proved that, if \( 1 < p < p^* \), then problem (1.6) admits no weak solutions.

As far as we know, problems of type (1.1) were not considered previously in the literature. Before stating the main results related to problem (1.1), let us mention in which sense solutions are considered. Just before, let us fix some notations. Denote
\[ Q = (0, \infty) \times \mathbb{R}^N. \]
By \( C^2_c(Q) \), we mean the space of \( C^2 \) real valued functions compactly supported in \( Q \). Let
\[ p^*(N) = \begin{cases} \infty & \text{if} \ N \in \{1, 2\}, \\
\frac{N}{N-2} & \text{if} \ N \geq 3. \end{cases} \]

**Definition 1.1.** We say that \((u, \partial_t^{k-1} u) \in L^p_{\text{loc}}(Q) \times L^q_{\text{loc}}(Q)\) is a weak solution to problem (1.1), if for any \( \varphi \in C^2_c(Q) \), there holds
\[ \int_Q (|u|^p \varphi + |\partial_t^{k-1} u|^q + w(t, x)) \varphi \, dx \, dt = - \int_Q \partial_t^{k-1} u \partial_t \varphi \, dx \, dt - \int_Q u \Delta \varphi \, dx \, dt. \]
Our first main theorem provides a general nonexistence result for problem (1.1).

**Theorem 1.2.** Let \( w \in L^1_{\text{loc}}(Q) \), \( w \geq 0 \). Suppose that there exist \( 0 < c_1 < c_2 < 1 \) such that

\[
\limsup_{T \to \infty} T^{-\frac{q}{q-1} \left[ 1 - \frac{N(p-1)}{2p} \right] - 1} \int_{c_1 T}^{c_2 T} \int_{0 < |x| < T^{\frac{2p}{q(p-1)}}} w(t, x) \, dx \, dt = +\infty.
\]

Then problem (1.1) admits no weak solutions.

**Remark 1.3.** (i) Note that no conditions on the sign or decay of the initial values \( \partial_t u(0, x) \), \( i = 0, 1, \ldots, k - 1 \), are imposed in Theorem 1.2.

(ii) Observe that condition (1.9) is independent of \( k \).

In the particular case \( w = f(t)g(x) \), one deduces from Theorem 1.2 the following nonexistence result.

**Corollary 1.4.** Let \( w = f(t)g(x) \), where \( f \in L^1_{\text{loc}}((0, \infty)), g \in L^1_{\text{loc}}(\mathbb{R}^N) \), \( f, g \geq 0 \) and \( g \neq 0 \). Suppose that there exist \( 0 < c_1 < c_2 < 1 \) such that

\[
\limsup_{T \to \infty} T^{-\frac{q}{q-1} \left[ 1 - \frac{N(p-1)}{2p} \right] - 1} \int_{c_1 T}^{c_2 T} f(t) \, dt = +\infty.
\]

Then problem (1.1) admits no weak solutions.

**Example 1.5.** Let \( f \in L^1_{\text{loc}}((0, \infty)), f \geq 0 \), be a function satisfying, for some \( C > 0 \) and \(-1 + \frac{1}{q} < \sigma < -1 + \frac{N}{2}\),

\[
f(t) \geq C t^{\frac{q\sigma}{q-1}},
\]

for \( t \) sufficiently large. Let \( w = f(t)g(x) \), where \( g \in L^1_{\text{loc}}(\mathbb{R}^N) \), \( g \geq 0 \) and \( g \neq 0 \). For \( T \) sufficiently large, one has

\[
\int_T^T f(t) \, dt \geq C(\sigma, q) T^{1 + \frac{q\sigma}{q-1}},
\]

where \( C(\sigma, q) > 0 \) is a constant that depends only of \( \sigma \) and \( q \). Hence, one obtains

\[
T^{-\frac{q}{q-1} \left[ 1 - \frac{N(p-1)}{2p} \right] - 1} \int_T^T f(t) \, dt \geq C(\sigma, q) T^{\frac{q}{q-1} \left[ \sigma + 1 - \frac{N(p-1)}{2p} \right]}. \]

Therefore, by Corollary 1.4, one deduces that, if

\[
1 < p < \frac{N}{N - 2(\sigma + 1)} \quad \text{and} \quad q > \max\left\{ 1, \frac{2}{N} \right\},
\]

then problem (1.1) admits no weak solutions.

In the case \( w = g(x) \), the next theorem provides the critical exponent for problem (1.1) in the sense of Fujita [3].

**Theorem 1.6** (First critical exponent). (I) Let \( w = g(x), g \in L^1_{\text{loc}}(\mathbb{R}^N), g \geq 0, g \neq 0, q > 1 \) and \( 1 < p < p^*(N) \), where \( p^*(N) \) is given by (1.7). Then problem (1.1) admits no weak solutions.

(II) If \( N \geq 3 \) and \( p > p^*(N) \), then for all \( q > 1 \), problem (1.1) has global positive solutions for some \( w = g(x) > 0 \) and suitable initial values.
Remark 1.7. (i) In this paper, we are concerned essentially with blow-up results. Comparing with the delicate contributions on global existence on the homogeneous case, the existence result (II) in Theorem 1.6 (same remark for the existence result (II) in Theorem 1.8) is just a consequence of elliptic results. We hope this can be improved in a future work.
(ii) If one excludes stationary solutions as global solutions, then it is not certain that the exponent $p^*(N)$ is still critical.
(iii) We do not know whether the exponent $p^*(N)$ belongs to the blow-up case or not.

Further, for $a < N$, we define the sets
\[
\mathbb{I}_a = \{ g \in C(\mathbb{R}^N) | g(x) \geq 0, g(x) \geq C|x|^{-a} \text{ for } |x| \text{ large} \}
\]
and
\[
\mathbb{J}_a = \{ g \in C(\mathbb{R}^N) | g(x) > 0, g(x) \leq C|x|^{-a} \text{ for } |x| \text{ large} \},
\]
where $C > 0$ is a constant (independent of $x$).

Denote
\[
a^* = \frac{2p}{p-1}.
\]

The next result provides the second critical exponent for problem (1.1) in the sense of Lee and Ni [15].

**Theorem 1.8** (Second critical exponent). Let $N \geq 3$, $q > 1$ and $p > p^*(N)$.

(I) If $a < a^*$ and $w = g(x) \in \mathbb{I}_a$, then problem (1.1) admits no weak solutions.

(II) If $a^* \leq a < N$, then problem (1.1) has positive global solutions for some $w = g(x) \in \mathbb{J}_a$ and suitable initial values.

The rest of the paper is organized as follows. In Section 2, we provide some preliminary estimates that will be used later in the proofs of our main results. In Section 3, we prove Theorem 1.2 and Corollary 1.4. In Section 4, we prove Theorems 1.6 and 1.8.

2. Preliminary estimates

Given $0 < c_1 < c_2 < 1$, let $\eta, \xi \in C^\infty((0, \infty))$ be two functions satisfying
\[
\eta \geq 0, \quad \text{supp}(\eta) \subset (0, 1), \quad \eta(t) = 1, \quad c_1 \leq t \leq c_2
\]
and
\[
0 \leq \xi \leq 1, \quad \xi(\sigma) = \begin{cases} 
1 & \text{if } 0 \leq \sigma \leq 1, \\
0 & \text{if } \sigma \geq 2.
\end{cases}
\]

For $T > 0$, let
\[
\phi_T(t, x) = \eta \left( \frac{t}{T} \right)^{\ell} \xi \left( \frac{|x|^2}{T^{2\theta}} \right)^{\ell} = \lambda_T(t) \mu_T(x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^N,
\]
where $\ell, \theta > 0$ are to be chosen.

**Lemma 2.1.** Let $m > 1$ and $\ell \geq \frac{m}{m-1}$. There holds
\[
\int_Q \phi_T^{-1} \left| \partial_t \phi_T \right|^{m-1} \frac{m}{m-1} \, dx \, dt \leq CT^{1+N\theta-\frac{m}{m-1}}, \quad T > 0.
\]
Here and below, \( C \) is a positive constant (independent of \( T \)), whose value may change from line to line.

**Proof.** Let \( T > 0 \). By the definition of the function \( \phi_T \), one has

\[
\int_Q \phi_T^{-\frac{1}{m-1}} |\partial_t \phi_T|^{\frac{m}{m-1}} \, dx \, dt = \left( \int_0^T \lambda_T^{-\frac{1}{m-1}} |\lambda_T'|^{\frac{m}{m-1}} \, dt \right) \left( \int_{\mathbb{R}^N} \mu_T(x) \, dx \right).
\]

Using the change of variable \( x = T^\theta y \), one obtains

\[
\int_{\mathbb{R}^N} \mu_T(x) \, dx = T^{N\theta} \int_{0 \leq |y|^2 \leq 2} \xi(|y|^2) \ell \, dy = CT^{N\theta}.
\]

By the change of variable \( t = T s \), one gets

\[
\int_0^T \lambda_T^{-\frac{1}{m-1}} |\lambda_T'|^{\frac{m}{m-1}} \, dt = CT^{1-\frac{m}{m-1}} \int_0^1 \eta(s) \ell^{-\frac{m}{m-1}} \eta'(s) \, ds,
\]

hence

\[
\int_0^T \lambda_T^{-\frac{1}{m-1}} |\lambda_T'|^{\frac{m}{m-1}} \, dt = CT^{1-\frac{m}{m-1}}.
\]

Finally, \((2.1)-(2.3)\) yield the desired estimate.

**Lemma 2.2.** Let \( m > 1 \) and \( \ell \geq \frac{2m}{m-1} \). There holds

\[
\int_Q \phi_T^{-\frac{1}{m-1}} |\Delta \phi_T|^{\frac{m}{m-1}} \, dx \, dt \leq CT^{1+N\theta-\frac{2m\theta}{m-1}}, \quad T > 0.
\]

**Proof.** Let \( T > 0 \). There holds

\[
\int_Q \phi_T^{-\frac{1}{m-1}} |\Delta \phi_T|^{\frac{m}{m-1}} \, dx \, dt = \left( \int_0^T \lambda_T \, dt \right) \left( \int_{\mathbb{R}^N} \mu_T^{-\frac{1}{m-1}} |\Delta \mu_T|^{\frac{m}{m-1}} \, dx \right).
\]

Using the change of variable \( t = T s \), one obtains

\[
\int_0^T \lambda_T \, dt = T \int_0^1 \eta(s) \ell \, ds = CT.
\]

By the change of variable \( x = T^\theta y \), one has

\[
\int_{\mathbb{R}^N} \mu_T^{-\frac{1}{m-1}} |\Delta \mu_T|^{\frac{m}{m-1}} \, dx \leq CT^{N\theta-\frac{2m\theta}{m-1}} \left( \int_{1 \leq |y|^2 \leq 2} \xi(|y|^2) \ell^{-\frac{m}{m-1}} |y|^{2m} \, dy + \int_{1 \leq |y|^2 \leq 2} \xi(|y|^2) \ell^{-\frac{m}{m-1}} |y|^{2m} \, dy \right),
\]

hence

\[
\int_{\mathbb{R}^N} \mu_T^{-\frac{1}{m-1}} |\Delta \mu_T|^{\frac{m}{m-1}} \, dx \leq CT^{N\theta-\frac{2m\theta}{m-1}}.
\]

Finally, \((2.4)-(2.6)\) yield the desired estimate.
3. A GENERAL NONEXISTENCE RESULT

In this section, we prove the general nonexistence result given by Theorem 1.2, as well as Corollary 1.4.

Proof of Theorem 1.2. Suppose that \((u, \partial_t^{k-1}u) \in L^p_{\text{loc}}(Q) \times L^q_{\text{loc}}(Q)\) is a weak solution to problem (1.1). For \(T > 0\), taking \(\varphi = \phi_T\) in (1.8), one obtains

\[
\int_Q |u|^p \phi_T \, dx \, dt + \int_Q |\partial_t^{k-1} u|^q \phi_T \, dx \, dt + \int_Q w(t, x) \phi_T \, dx \, dt
\]

\[
\leq \int_Q |\partial_t^{k-1} u| \, |\partial_t \phi_T| \, dx \, dt + \int_Q |u| \Delta \phi_T \, dx \, dt.
\]

Using the \(\varepsilon\)-Young inequality with \(\varepsilon = 1\), one gets

\[
\int_Q |u| \Delta \phi_T \, dx \, dt \leq \int_Q |u|^p \phi_T \, dx \, dt + C \int_Q \phi_T^{\frac{1}{p-1}} |\Delta \phi_T|^{\frac{p}{p-1}} \, dx \, dt.
\]

Similarly, one has

\[
\int_Q |\partial_t^{k-1} u| \, |\partial_t \phi_T| \, dx \, dt \leq \int_Q |\partial_t^{k-1} u|^q \phi_T \, dx \, dt + C \int_Q \phi_T^{\frac{1}{q-1}} |\partial_t \phi_T|^{\frac{q}{q-1}} \, dx \, dt.
\]

It follows from (3.1)–(3.3) that

\[
\int_Q w(t, x) \phi_T \, dx \, dt \leq C \left( I_1(T) + I_2(T) \right),
\]

where

\[
I_1(T) = \int_Q \phi_T^{\frac{1}{p-1}} |\Delta \phi_T|^{\frac{p}{p-1}} \, dx \, dt
\]

and

\[
I_2(T) = \int_Q \phi_T^{\frac{1}{q-1}} |\partial_t \phi_T|^{\frac{q}{q-1}} \, dx \, dt.
\]

Taking \(\ell = \max \left\{ \frac{q}{q-1}, \frac{2p}{p-1} \right\}\) and using Lemma 2.2 with \(m = p\), one obtains

\[
I_1(T) \leq CT^{1+N\theta-\frac{2p\theta}{p-1}}.
\]

Using Lemma 2.1 with \(m = q\), one obtains

\[
I_2(T) \leq CT^{1+N\theta-\frac{q}{q-1}}.
\]

Hence, we deduce that

\[
\int_Q w(t, x) \phi_T \, dx \, dt \leq C \left( T^{1+N\theta-\frac{2p\theta}{p-1}} + T^{1+N\theta-\frac{q}{q-1}} \right).
\]

Taking \(\theta = \frac{(p-1)q}{2(q-1)p}\), one has

\[
1 + N\theta - \frac{2p\theta}{p-1} = 1 + N\theta - \frac{q}{q-1} = 1 + \frac{q}{q-1} \left[ \frac{N(p-1)}{2p} - 1 \right]
\]

and

\[
\int_Q w(t, x) \phi_T \, dx \, dt \leq CT^{1+\frac{q}{q-1} \left[ \frac{N(p-1)}{2p} - 1 \right]}.
\]
On the other hand, by the definition of the function $\phi_T$, since $w \geq 0$, one gets

$$\int_Q w(t, x)\phi_T \,dx \,dt \geq \int_{c_2 T} c_1 T \int_{0<T<T^\frac{p-1}{2p(q-1)}}^c w(t, x) \,dx \,dt.$$ \hspace{1cm} (3.5)

It follows from (3.4) and (3.5) that

$$T^\frac{q}{q-1} \left[1 - \frac{N(p-1)}{2p} \right] - 1 \int_{c_1 T}^{c_2 T} \int_{0<T<T^\frac{p-1}{2p(q-1)}}^c w(t, x) \,dx \,dt \leq C,$$

which contradicts (1.9). This proves Theorem 1.2. \hspace{1cm} \Box

**Proof of Corollary 1.4.** Let $T > 0$ be large enough. One has

$$\int_{c_1 T}^{c_2 T} \int_{0<T<T^\frac{p-1}{2p(q-1)}}^c w(t, x) \,dx \,dt = \left( \int_{c_1 T}^{c_2 T} f(t) \,dt \right) \left( \int_{0<T<T^\frac{p-1}{2p(q-1)}}^c g(x) \,dx \right) \geq \left( \int_{0<T<T^\frac{p-1}{2p(q-1)}}^c g(x) \,dx \right) \left( \int_{c_1 T}^{c_2 T} f(t) \,dt \right) = C \int_{c_1 T}^{c_2 T} f(t) \,dt.$$

Hence, using (1.10), one deduces that (1.9) is satisfied. Therefore, the result follows from Theorem 1.2. \hspace{1cm} \Box

### 4. First and second critical exponents

We first prove the Fujita-type result given by Theorem 1.6.

**Proof of Theorem 1.6.** Part (I) follows immediately from Corollary 1.4 with $f \equiv 1$.

(II) It is well-known that the elliptic equation

$$-\Delta u = u^p + g(x), \quad x \in \mathbb{R}^N,$$

where $N \geq 3$ and $p > \frac{N}{N-2}$, admits positive solutions for some $g > 0$ (see e.g. [2]). Clearly, if $u$ is a positive solution to (4.1), then it is a global positive solution to (1.1) with $w = g(x)$ and suitable initial data. This completes the proof of Theorem 1.6. \hspace{1cm} \Box

Next, we prove Theorem 1.8 which provides the second critical exponent for problem (1.1).

**Proof of Theorem 1.8.** (I) Let $w = g(x) \in \mathbb{I}_a$. For $T$ large enough, one has

$$\int_{0<T<T^\frac{p-1}{2p(q-1)}}^c g(x) \,dx \geq \int_{0<T<T^\frac{p-1}{2p(q-1)}}^c g(x) \,dx \geq C \int_{0<T<T^\frac{p-1}{2p(q-1)}}^c g(x) \,dx \geq C T^{\frac{q}{q-1} \left[1 - \frac{N(p-1)}{2p} \right]}.$$

Hence, for any $0 < c_1 < c_2 < 1$, one obtains

$$T^\frac{q}{q-1} \left[1 - \frac{N(p-1)}{2p} \right] - 1 \int_{c_1 T}^{c_2 T} \int_{0<T<T^\frac{p-1}{2p(q-1)}}^c g(x) \,dx \,dt = C T^\frac{q}{q-1} \left[1 - \frac{N(p-1)}{2p} \right] \int_{0<T<T^\frac{p-1}{2p(q-1)}}^c g(x) \,dx \geq C T^\frac{q}{q-1} \left[1 - \frac{N(p-1)}{2p} \right].$$
Since \( a < a^* \), (1.9) is satisfied, which yields the desired result.

(II) Let \( a^* \leq a < N \). We take

\[
u(x) = \varepsilon(1 + |x|^2)^{-\frac{\delta}{2}}, \quad x \in \mathbb{R}^N,
\]

where

\[
a - 2 \leq \delta < N - 2 \quad \text{and} \quad 0 < \varepsilon < \left[\frac{\delta(N - \delta - 2)}{N - \delta} - 1\right]^{\frac{N - \delta}{2}}.
\]

One can show easily that

\[-\Delta u = u^p + g(x), \quad x \in \mathbb{R}^N,
\]

where

\[
g(x) = \varepsilon\delta \left(N + (N - \delta - 2)|x|^2\right) (1 + |x|^2)^{-\frac{\delta}{2}} - \varepsilon^p(1 + |x|^2)^{-\frac{\delta p}{2}}.
\]

Using (4.2), for all \( x \in \mathbb{R}^N \), one obtains

\[
g(x) \geq \varepsilon\delta \min\{N, N - \delta - 2\} (1 + |x|^2)^{-\frac{\delta}{2}} - \varepsilon^p(1 + |x|^2)^{-\frac{\delta p}{2}}
\]

\[
= \varepsilon\delta(N - \delta - 2)(1 + |x|^2)^{-\frac{\delta}{2}} - \varepsilon^p(1 + |x|^2)^{-\frac{\delta p}{2}}
\]

\[
= \varepsilon(1 + |x|^2)^{-\frac{\delta}{2}} \left(\delta(N - \delta - 2) - \varepsilon^p - (1 + |x|^2)^{-\frac{\delta p}{2}}\right)
\]

\[
\geq \varepsilon(1 + |x|^2)^{-\frac{\delta}{2}} \left(\delta(N - \delta - 2) - \varepsilon^p\right) > 0.
\]

On the other hand, using (4.2), for \(|x|\) large, one has

\[
g(x) \leq \varepsilon\delta \left(N + (N - \delta - 2)|x|^2\right) (1 + |x|^2)^{-\frac{\delta}{2}} - \varepsilon^p(1 + |x|^2)^{-\frac{\delta p}{2}}
\]

\[
\leq \varepsilon\delta \max\{N, N - \delta - 2\} (1 + |x|^2)^{-\frac{\delta}{2}} - 1
\]

\[
= \varepsilon\delta N(1 + |x|^2)^{-\frac{\delta}{2}} - 1
\]

\[
\leq \varepsilon\delta N(1 + |x|^2)^{-\frac{\delta}{2}}
\]

\[
\leq C|x|^{-a}.
\]

Hence, \( u \) is a global positive solution to problem (1.1) with \( w = g(x) \in J_a \) and suitable initial data. This completes the proof of Theorem 1.8. \( \square \)

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