SQUEEZED COHERENT STATES AS THERMAL-LIKE ONES
(ON THE PROBLEM OF INCORPORATING THERMODYNAMICS INTO QUANTUM THEORY)

A.D. Sukhanov 1, O.N. Golubeva 2, and V.G. Bar’yakhtar 3

1 Theoretical Physics Laboratory, Joint Institute for Nuclear Research, Dubna, Russia
ogol@oldi.ru
2 People’s Friendship University of Russia, Moscow, Russia. ogol@mail.ru
3 Institute of Magnetism, National Academy of Sciences of Ukraine, Kiev, Ukraine. victor.baryakhtar@gmail.com

Abstract

In our paper [1], we proposed an original approach to the incorporation of stochastic thermodynamics into quantum theory. It is based on the concept of consistent inclusion of the holistic stochastic environmental influence modeled by arbitrary Bogoliubov vacua. In our study, we implement a possibility of describing the states of the system "an object + arbitrary vacuum" based on the wave function. The fact that the quantum language is required becomes especially appreciable in the cases where thermal and quantum fluctuations occur simultaneously. This allows analyzing different types of quantum states and estimating the degree of their applicability to the description of thermal effects. The main result of our study reduces to the fact that squeezed coherent state (SCS) cannot be used to consistently explain quantum-thermal effects under conditions of object equilibrium with the stochastic environment at any temperatures.

1. Generalization of the concept of thermal equilibrium to quantum phenomena

As is well known, in the standard thermodynamics, the thermal equilibrium state is fixed by the special parameter, namely, the Kelvin temperature, which does not reach the zero value in principle. As this takes place, this state satisfies the zeroth law of thermodynamics, i.e., the assertion that the object temperature $T$ and the environment temperature $T_0$ are equal (if only on the average). In this case, it is meant that the model of a classical thermostat is brought in correspondence with the external macro-environment. As a result, only thermal fluctuations that are significant only at high temperatures are considered within the framework of the standard thermodynamics. In addition, it is tacitly taken that the unavoidable quantum stochastic environmental influence is not taken into account.

At the same time, it is very significant at sufficiently low temperatures, where it manifests itself through quantum fluctuations. On the contrary, the concept of thermal equilibrium is not used in quantum mechanics (i.e., at zero Kelvin temperature), because the presence of any thermal contact with the environment is not assumed. Thus, as if the standard thermodynamics and quantum mechanics are at different poles of the temperature scale. Nevertheless, it turns out that the concept of thermal equilibrium can be joined to that of the apparatus of quantum mechanics. The fact that the thermal equilibrium as a fundamental concept of equilibrium thermodynamics has the stability property, which is also inherent in certain quantum states, must be used in this case. It also must be kept in mind that a peculiar uncertainty relation between the temperature $T$ and the entropy $S$ [2] in the form

$$\Delta T \cdot \Delta S \geq k_B$$

(1)
is valid in thermal equilibrium. We note that, at the same time, the uncertainty relation is traditionally related to the most important statements of quantum mechanics. In the most general case, it has the form of the Schrödinger uncertainty relation (SUR)

$$\Delta p \cdot \Delta q \geq \left| \langle \psi | \delta \hat{p} \cdot \delta \hat{q} | \psi \rangle \right| \equiv |\tilde{R}_{p,q}|^2,$$

(2)

where the notation $\Delta p \equiv \sqrt{\langle \Delta p \rangle^2}$ and $\Delta q \equiv \sqrt{\langle \Delta q \rangle^2}$ is used in the left-hand side of the inequality, and the variances of the momentum $\langle \Delta p \rangle^2$ and the coordinate $\langle \Delta q \rangle^2$ in the state $|\psi\rangle$ are calculated in accordance with the definitions.

The complex quantity in the right-hand side of this expression

$$\tilde{R}_{p,q} = \langle \Delta p | \Delta q \rangle = \langle |\Delta \hat{p} \Delta \hat{q}| \rangle$$

(3)

has a double sense. On the one hand, it is the amplitude of the transition from the state $|\Delta q\rangle$ to the state $|\Delta p\rangle$. However, on the other hand, it can be treated as a mean or the quantum correlator with respect to the arbitrary state $|\rangle$ of the operator $\hat{S}_R \equiv \Delta \hat{p} \Delta \hat{q}$, which we called previously [1] the operator of stochastic influence – Schrödingerian. In this case, the fact that this quantity is nonzero is an underlying attribute of a nonclassical theory in which the stochastic environmental influence on the object plays an important role.

Thus, a certain crossing between the conceptual apparatuses of these two theories has already been outlined. To extend it, we study the possibility of bringing the set of quantum states described by wave functions in correspondence with equilibrium thermal states. To do this, we proceed from the following considerations.

First of all, we assume that, in the general case, the existence of inequalities similar to (1) in thermodynamics and to (2) in quantum theory is provided by the nonzero value of the correlator forming their left-hand side. It is also necessary to take into account that the thermal equilibrium has the stability property. It can be assumed that the correlation between the corresponding quantities in thermodynamic inequality (1) also exists and plays a certain role in the preservation of the stability of the thermal equilibrium state. Therefore, it would be natural to begin by determining a stable quantum state that can be assumed to be adequate to the thermal equilibrium state if the stochastic thermal influence is taken into account additionally. Thus, in our opinion, the following conditions must be satisfied in the sought quantum state:

a) the correlator of form (3) is nonzero;
b) the Schrödinger uncertainty relation (SUR) of form (2) has the form of the equality, i.e., becomes saturated, providing the state stability.

Of course, condition b) is reached when the SUR left-hand side

$$\Delta p \cdot \Delta q \equiv \mathcal{U}P,$$

(4)

called the uncertainties product and regarded as a single holistic quantity takes a value that is as minimum as possible.

To find the general form of the wave function corresponding to the sought state, we use the Schwartz-von-Neumann theorem [3]. In accordance with the corollary from this theorem, the equality sign in SUR (2) is reached in the state $|\psi_\alpha\rangle$ when the vectors $|\delta p\rangle$ and $|\delta q\rangle$ are proportional to each other in the Hilbert space, i.e.,

$$\delta \hat{p} |\psi_\alpha\rangle = (i\gamma \cdot e^{i\alpha}) \delta \hat{q} |\psi_\alpha\rangle.$$  

(5)
Here, the dimensional parameter $\gamma > 0$ and the phase parameter $\alpha \neq 0$. To simplify our calculation, we assume that in this state, the average values of the momentum and the coordinate are zero so that $\delta p = \hat{p}$ and $\delta q = \hat{q}$ in (3) and below. From formula (5), the equation

$$(\hat{p} - i\gamma e^{i\alpha} \hat{q})|\psi_{\alpha}\rangle = 0$$

(6)
can be obtained. It is interesting that it resembles (up to the normalization factor) the result of the action of the annihilation operator

$$\hat{a} = \frac{\hat{p} - i\zeta \hat{q}}{\sqrt{2}\hbar\zeta}$$
on the state $|\psi_{\alpha}\rangle$. Expression (6) in the coordinate representation takes the form of the differential equation for the unknown function $\psi_{\alpha}(q)$:

$$\frac{\hbar}{i} \frac{d}{dq} \psi_{\alpha} - (i\gamma e^{i\alpha})q \cdot \psi_{\alpha} = 0.$$ (7)

Solving it and using the normalization condition, in the general case, we obtain the complex function

$$\psi_{\alpha}(q) = [2\pi(\Delta q_0)^2 \frac{1}{\cos \alpha}]^{-1/4} \exp \left\{ -\frac{q^2}{4(\Delta q_0)^2} e^{i\alpha} \right\}$$ (8)
as a universal wave function $\psi_{\alpha}(q)$. Here,

$$(\Delta q_0)^2 = \frac{\hbar}{2\gamma}, \quad \alpha \neq \frac{\pi}{2}.$$ (9)

The physical meaning of the function $\psi_{\alpha}(q)$ is clarified if the fact that for $\alpha = 0$, Eq. (8) for the function $\psi_0(q)$ is equivalent to the equation for the state $|0\rangle$ (which is adopted to be called the cold vacuum) is taken into account. Accordingly, we assume that for arbitrary value $\alpha \neq 0$, the state $\psi_{\alpha}$ describes an arbitrary vacuum. We note that unlike the approach proposed above, the standard way of obtaining the state of the arbitrary vacuum, which we used in [2], is related to the application of the Bogoliubov $(u, v)$-transformation. The coordinate variance $(\Delta q_{\alpha})^2$ in the arbitrary-vacuum state $|\psi_{\alpha}\rangle$, calculated using wave function (8) has the form

$$(\Delta q_{\alpha})^2 = \int_{-\infty}^{+\infty} \psi_{\alpha}^* q^2 \psi_{\alpha} dq = \frac{\hbar}{2\gamma} \cdot \frac{1}{\cos \alpha}.$$ (10)

The momentum variance $(\Delta p_{\alpha})^2$ and the “coordinate–momentum” correlator $|\langle \delta p_{\alpha} | \delta q_{\alpha} \rangle|$ are related by the proportional dependence with the coordinate variance $(\Delta q_{\alpha})^2$

$$(\Delta p_{\alpha})^2 = \gamma^2 (\Delta q_{\alpha})^2 = \frac{\hbar\gamma}{2} \cdot \frac{1}{\cos \alpha};$$ (11)

$$|\langle \delta p_{\alpha} | \delta q_{\alpha} \rangle| = |i\gamma e^{i\alpha}| \langle \delta q_{\alpha} | \delta q_{\alpha} \rangle = \gamma (\Delta q_{\alpha})^2 = \frac{\hbar}{2} \cdot \frac{1}{\cos \alpha}.$$ (12)

Thus, we assume that the function $\psi_{\alpha}$ describes the equilibrium with the arbitrary vacuum. However, the intuitive assumption of the possibility of describing thermal states by the functions $\psi_{\alpha}$ requires a further substantiation. To do this, it is necessary (if it is possible) to relate the obtained expressions to the Kelvin temperature, which has no direct pre-image in quantum mechanics, and also to establish which values of the parameter $\alpha$ correspond to the thermal states.
2. Squeezed coherent states of the arbitrary vacuum as thermal-like states

At this stage, we analyze squeezed coherent state (SCSs) in order to establish how adequate to the some kind of thermal states they are. The cold-vacuum states are described by real wave functions and occur in the case where the parameter $\alpha$ is zero in formula (8). Accordingly, $\cos \alpha = 1$ in formulas (14) and (15) so that the coordinate and momentum variances have the forms

$$\overline{(\Delta p_0)^2} = \frac{\hbar \gamma}{2};$$  \hspace{1cm} (13)

$$\overline{(\Delta q_0)^2} = \frac{\hbar}{2\gamma}.$$  \hspace{1cm} (14)

In what follows, we establish that such wave functions describing the cold-vacuum states cannot be interpreted as truly thermal ones. Expressions (13) and (14) allows obtaining the average values of the kinetic $\langle K \rangle$ and potential $\langle \Pi \rangle$ energies of the system under consideration in the case where $\overline{(p_0)^2} = 0$ and $\overline{(q_0)^2} = 0$:

$$\langle K \rangle = \frac{1}{2} \overline{(\Delta p_0)^2} = \frac{\hbar \gamma}{4},$$  \hspace{1cm} (15)

$$\langle \Pi \rangle = \frac{1}{2} \overline{(\Delta q_0)^2} = \frac{\hbar \gamma}{4}.$$  \hspace{1cm} (16)

Thus, the total energy of the quantum oscillator

$$U = \langle K \rangle + \langle \Pi \rangle = \frac{\hbar \gamma}{2},$$  \hspace{1cm} (17)

which corresponds to the zero-oscillation energy $\frac{\hbar \omega}{2}$ if $\gamma$ is endowed with the meaning of the frequency: $\gamma \rightarrow \omega$. In other words, the function $\psi_0$ corresponds to the state at the zero Kelvin temperature. Of course, it occurs solely under conditions under which there is no uncontrolled thermal influence, so that it can be considered as thermal exclusively in the Pickwickian sense.

The problem whether the state $|\psi_0\rangle$ can nevertheless be interpreted as an equilibrium one deserves additional discussions. It is easy to see that on the one hand, it provides the SUR saturation. But, on the other hand, relation (2) in this case transforms into the saturated Heisenberg UR

$$\Delta p_0 \cdot \Delta q_0 = \overline{\langle \psi_0 | \frac{1}{2} [\hat{p}, \hat{q}] | \psi_0 \rangle} = \frac{\hbar}{2}.$$  \hspace{1cm} (18)

Here, $\frac{\hbar}{2} \equiv \mathcal{J}_0$ is the measure of the purely quantum environmental influence. Thus, at $\alpha = 0$, each of the quantities ($U_P$ and the correlator $\overline{\langle \psi_0 | \hat{p} \cdot \hat{q} | \psi_0 \rangle}$) have values as minimum as possible independently. In other words, the state $|\psi_0\rangle$ can indeed be regarded as a stable state that is similar to the equilibrium; but the cold vacuum plays the role of as “thermostat” in this case, because this state corresponds to the minimum vacuum energy $U_0 = \frac{\hbar \omega}{2}$.

Thus, it is improper to bring squeezed coherent states in correspondence with any equilibrium thermal states. Only the equilibrium with the cold vacuum, which is typical
of maximally isolated systems (i.e., systems experiencing a solely unavoidable quantum stochastic influence), can be discussed notably arbitrarily. Probably, it is no mere chance that considering squeezed states, Umedzawa [4] called them not truly thermal ones but thermal-like states. Further search for the possibility of bringing the states $|\psi_\alpha\rangle$ (with the arbitrary value $\alpha \neq 0$) in correspondence with the thermal states requires studying correlated coherent states.

References

[1] A.D. Sukhanov and O.N. Golubjeva. Arbitrary vacuum as a model of stochastic influence of environment: on the problem of incorporating thermodynamics into quantum theory. Physics of Particles and Nuclei Letters, 9(3):303. Pleiades Publishing, Ltd. (2012).

[2] A.D. Sukhanov and O.N. Golubjeva. Toward a quantum generalization of equilibrium statistical thermodynamics: $(\hbar, k)$ - dynamics. Theoretical and Mathematical Physics, 160(2):1177. Springerlink (2009)

[3.] J.von Neumann. Mathematical Foundations of Quantum Mechanics. Princeton University Press, 1996

[4] H.Umezawa. Advanced Field Theory. Micro-, macro-, and Thermal Physics. AIP, N.-Y. 1993.