Quantum detection theory and optimum strategy in quantum radar system

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Abstract: As new concept radar combing quantum technology with radar technology, quantum radar is an extension of traditional radar theory and has the potential to break through the limit of conventional radar detection performance. We study quantum radar optimum decision strategies for various signal and background forms and compare them with traditional photon-counting strategies. Quantum optimum strategy gives better detection probability with fixed false-alarm probability, thus reducing needed signal strength to detect the target, but its validity strongly depends on exact knowledge of signal forms. We briefly discuss the influence of deviating signal amplitudes and phases, and find that in the condition that the false-alarm probability is fixed, quantum optimum strategy is still useful when the signal form varies in a small region.

1 Introduction

The need to detect signals in noise and clutters in radar called forth a theory of signal detection. In traditional detection theory, we use probability distribution functions to describe signals and backgrounds at the receiver, and optimum strategies are derived with some specific criterion. In recent years, the successes of quantum technology lead us to the consideration of quantum radar. New quantum radar systems such as quantum illumination radar \([1, 2]\) and quantum-enhanced radar \([3, 4]\) with squeezed-vacuum injection and phase-sensitive amplification are suggested. Besides, in quantum radar, the quantum detection theory is essential and quantum optimum strategies can in principle give better results compared with traditional ones. So far a large number of researches on quantum optimum strategies as well as their realisations have been done \([5–10]\), but most of them are about quantum communications. In communication systems signal forms are already known, but in target detecting radar systems situations are different. We are not to choose from limited kinds of signal forms, but to judge if echoes contain any reflecting waves from targets or just noise and clutters. The targets could be anywhere, so signals may have arbitrary amplitudes and phases. In this paper, we will study quantum optimum strategy in quantum radar system, and find out its applicability and advantage over traditional strategies in target detection.

The paper is organised as follows. In Section 2, we study the quantum optimum strategies for binary decisions between two coherent states. The effects of squeezing are also considered. In Section 3, we study the detection of a coherent state in thermal noise, and the suboptimum threshold detection is compared with traditional photon-counting. In Section 4, we briefly discuss the influence of signal forms on quantum optimum strategy performance. In Section 5, we conclude.

2 Quantum binary decisions with the neyman-Pearson criterion

In quantum mechanics, quantum systems are described by density operators, and a quantum measurement is implemented by a device associated with a probability-operator measure (p.o.m.). If there are \(M\) hypotheses \(H_i\) \((i = 1, 2, \ldots, M)\) about the state of a quantum system \(S\), of which the \(i\)th is the proposition that its density operator is \(\rho_i\), then the p.o.m. needs only \(M\) components \(\Pi_i\) \((i = 1, 2, \ldots, M)\), which we call detection operators. These operators must be non-negative-definite Hermitian operators, summing to the identity,

\[
\sum_{i=1}^{M} \Pi_i = 1, \quad (1)
\]

and according to quantum mechanics

\[
\Pr \{j|k\} = \text{Tr}(\rho_j \Pi_k), \quad (j, k) = 1, 2, \ldots, M \quad (2)
\]

is the probability of choosing hypotheses \(H_j\) when \(H_k\) is true. As to binary decisions, there are only two hypotheses between which to choose, which are customarily labelled \(H_0\) and \(H_1\). The system \(S\) has density operator \(\rho_0\) and \(\rho_1\) under each hypothesis, and their prior probabilities are \(\zeta_0\) and \(\zeta_1\), with \(\zeta_0 + \zeta_1 = 1\). The detection operators can be termed \(\Pi_1 = \Pi_0\) and \(\Pi_0 = \Pi_1\) for simplicity. Our goal is to find the device, or strategy, that gives best performance in hypotheses testing. With the Bayes criterion the optimum strategy should have least average cost

\[
\hat{C} = \sum_{i=1}^{M} \sum_{j=1}^{M} \zeta_j C_{ij} \text{Tr}(\rho_i \Pi_i), \quad (3)
\]

where \(C_{ij}\) is the cost of choosing \(H_i\) when \(H_j\) is true. Specifically, when \(C_{ij} = 1 - \delta_{ij}\), optimum strategy gives the least probability of error

\[
P_e = 1 - \sum_{j=1}^{M} \zeta_j \Pr \{j|\bar{j}\} \quad (4)
\]

This is often used in quantum communications.

When dealing with target detection problems, prior probabilities are generally unknown, so we prefer the Neyman-Pearson criterion, which amounts to fixing the false-alarm probability and maximising the detection probability. As previous work had already shown \([5]\), finding optimum strategy with the Neyman-Pearson criterion is equivalent to finding eigenvalues and corresponding eigenvectors of the operator \(\rho_i - \rho_0\rho_i\rho_0\),

\[
(\rho_i - \rho_0 \rho_i |\rho_0\rangle \langle \rho_0|) = \eta_i |\rho_i\rangle \langle \rho_i|, \quad (5)
\]
where $\chi$ is a parameter linked with predetermined false-alarm probability. Under most circumstances this is hard to achieve, and in this section we will at first consider the simplest example that can be formalised analytically, i.e. binary decisions between two pure states.

Assuming that we are to search some airspace using quantum radar, the receiver’s measurements give us information about whether a target exists or not. Traditional radar detections are based on electromagnetic waves’ amplitudes and phases, while quantum radar can further exploit quantum resources. Quantum optimum strategies are in general better than traditional ones. Consider the case that if a target exists, the signal contains a coherent state $\{\beta, \xi\}$ and the background is still a coherent state $\{\beta_0\}$. Their inner-product is [11]

$$\langle \beta_0 | \beta, \xi \rangle = \sqrt{\text{sech}\text{ch}} \exp\left\{-\frac{1}{2} |\beta|_F^2 \right\} |\beta| \sqrt{\text{sech}} \text{r} \right\}.$$  

and the photon number distribution function $p(n)$ of $|\beta, \xi\rangle$ is

$$p(n) = \frac{\left(\tan hr\right)^n}{2^n \cos hr} \exp\left\{-\frac{1}{2} |\beta|_F^2 \right\} \left[1 + e^{2\beta_0^* i_{\beta_0}} \tan hr\right] .$$

The results are shown in Fig. 2, together with results of a coherent state $\{\beta_0\}$ and the background is still a coherent state $|\beta_0\rangle$. Their inner-product is [11]

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and the photon number distribution function $p(n)$ of $|\beta, \xi\rangle$ is

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where $N_b$ is the average number of photons supplied by the Plank-distributed background light,

$$N_b = \frac{1}{e^{\pi f/\hbar} - 1}. \quad (16)$$

$h$ is reduced Planck’s constant, $f$ is the frequency, $K$ is Boltzmann constant, and $T$ is effective absolute temperature. Under hypothesis $H_1$ the density operator has displaced Gaussian form

$$\rho = \frac{1}{\pi N_b} \int \exp \left( -\frac{a^2 + a'^2}{2N_b} \right) |a\rangle \langle a'| d^2 a, \quad (17)$$

with an amplitude $\mu = N_b^{1/2}$. In general, to derive the optimum strategy we must find solutions of (5), which is extremely difficult in this case.

As optimum strategy seems unpractical, we turn to suboptimal strategies, one of which is the threshold detector. Its detection operator $\Pi_\theta$ is given by

$$\Pi_\theta = \frac{1}{N_b + 1/2} (a + a^*). \quad (18)$$

The outcome $y$ of measuring the operator $\Pi_\theta$ is a Gaussian random variable because the density operators have Gaussian $P$-representations. Its mean value and variance under both hypotheses are [5]

$$\langle y \rangle_0 = 0, \quad \langle y \rangle_1 = \frac{4\mu}{2N_b + 1}, \quad \sigma_{y0}^2 = \sigma_{y1}^2 = \frac{4}{2N_b + 1}. \quad (19)$$

So the false-alarm and detection probabilities for the threshold detector are

$$P_{fa} = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2} dx, \quad P_d = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-x^2/2} dx, \quad (20)$$

where

$$x = \sqrt{2N_b + 1} y_0, \quad D_q = \frac{4\mu}{\sqrt{2N_b + 1}}. \quad (21)$$

and $y_0$ is pre-assigned decision level.

For comparison we also consider photon-counting. The photon number distribution functions for both density operators are

$$p_{\theta}(n) \approx \frac{N_b^n}{(N_b + 1)^{n+1}} \exp \left( -\frac{N_b}{N_b + 1} \right),$$

$$p_d(n) = \frac{N_b^n}{(N_b + 1)^{n+1}} \exp \left[ -\frac{N_b}{N_b + 1} + \frac{N_b}{N_b + 1} \right]. \quad (22)$$

The results are shown in Fig. 3. We can see that if $N_b$ is small (about <0.3), photon-counting strategies are better than threshold detections. This is because the threshold operator has least variance $\sigma_{min} = 2$, and small false-alarm probability thus demands a large decision level and hence some misdetection probability if the signal does not contain sufficient photons, even if the number of noise photons is close to zero.

To better see when threshold detection works, we illustrate average number of noise photons $N_b$ with regard to the frequency $f$ in Fig. 4. The effective absolute temperature is fixed at $T = 290K$. If quantum radar works at optical frequencies ($f \sim 10^5 GHz$), little noise photons exist, and the threshold strategy is of no advantage to detection. At microwave frequencies ($f \sim 1 GHz$) where the number of noise photons exceeds 10, threshold detection can be helpful.

### 4 Optimum strategy dependence on signal forms

The results in above sections show that quantum optimum/suboptimum detection strategies have better performance than traditional ones in certain conditions. However, the derivation of optimum strategies strongly depends on signal/background forms. In quantum communication system, the receiver makes decisions between several fixed signal forms (noise may exist as well), while in radar systems echoes can have arbitrary amplitudes and phases as the target may exist at any distance. In Section 2, we have shown that when we want to make decisions between two coherent states, quantum optimum strategies are helpful, as can be seen from Fig. 1. However, we should notice that one optimum strategy corresponds to one specific pair of coherent states. This means that if we want to achieve optimum detection for targets at arbitrary distance, infinite strategies are needed, which is impossible. So, the question is that if an optimum strategy can be utilised to a series of signal forms while detection probabilities are still better than traditional photon-counting strategies.

In deriving quantum optimum detection strategies for binary decisions between two coherent states in Section 2, we solve (5), and $\lambda$ is chosen that the false-alarm probability equals $10^{-6}$. In fact $\lambda$ is the solution of the equation

$$P_{fa} = \frac{1 - \lambda - 2h}{\sqrt{(1 - \lambda^2) + 4\hbar^2 + \lambda^2}}. \quad (23)$$

We illustrate the value of $\lambda$ for $h$ in Fig. 5. In our case where $P_{fa} = 10^{-6}$, $\lambda$ is very large unless $h$ is close to 0 or 1. Furthermore, from (10) and the fact that the false-alarm probability is extremely small, we have $P_d = h$ if $h$ is not too small (about larger than $10^{-3}$). So, $\lambda$
specific optimum strategy derived from a known background and an estimated signal can still be useful if actual $N_s$ is close to our estimation. So far we have omitted the phases of coherent states, but actually

$$|\psi\rangle = \sqrt{N_e^0}\psi_0^0.$$  

Now we briefly illustrate the effects of phases. Still we assume $N_b = 10$, and if the signal and background coherent states satisfy $\alpha_0 = 10^{1/2}$, $\alpha_1 = 24^{1/2}$ and $\theta_0 = \theta_1 = 0$, optimum strategy gives $P_{fa} = 10^{-6}$ and $P_d = 0.95$. We let $\theta_1$ vary from 0 to $2\pi$ while keeping $\theta_0 = 0$, and calculate $h$. The results are shown in Fig. 7. The parameter $h$ quickly goes close to 1 as signal phase deviates from 0, so an optimum strategy can only work in a small phase range.

5 Conclusion

In this paper, we have studied quantum optimum strategies for different signal and background forms. Our results include binary decisions between two coherent states, between a coherent state and a squeezed state, and of a coherent state in thermal noise. Compared with traditional photon-counting strategies, quantum optimum strategies need less signal photons to achieve the detection probability $P_d = 0.95$ with fixed false-alarm probability $P_{fa} = 10^{-6}$.

The derivation of quantum optimum strategy depends on signal and background forms. Unlike quantum communication systems, in quantum radar the signal can have arbitrary amplitudes and phases because the target can be at any distance. This nature of target detection thus demands a robust detection strategy that can be used for a wide range of signal amplitudes and phases. We have briefly discussed the influence of signal deviations and find that for binary decisions between two coherent states a quantum optimum strategy is only valid when actual average number of signal photons and phase lie within a small region.

6 References

[1] Lloyd, S.: ‘Enhanced sensitivity of photodetection via quantum illumination’, *Science*, 2008, 321, (5895), p. 1463
[2] Barzanjeh, S., Guha, S., Weedbrook, S., et al.: ‘Microwave quantum illumination’, *Phys.Rev.Lett.*., 2015, 114, p. 080503
[3] Caves, C.M.: ‘Quantum limits on noise in linear amplifiers’, *Phys. Rev.*, 1982, D26, pp. 1817–1839
[4] Ou, Z.Y., Pereira, S.F., Kimble, H.J.: ‘Quantum noise reduction in optical amplification’, *Phys. Rev. Lett.*, 1993, 70, (21), pp. 3239–3242
[5] Helstrom, C.W.: ‘Quantum detection and estimation theory’ (Academic Press, New York, NY, USA, 1976)
[6] Kennedy, R.S.: ‘A near-optimum receiver for the binary coherent state quantum channel’, MIT Research Laboratory of Electronics Quarterly Progress Report, 111, Cambridge, MA, USA, 1973
[7] Dolinar, Jr., S.J.: ‘An optimum receiver for the binary coherent state quantum channel’, MIT Research Laboratory of Electronics Quarterly Progress Report, 111, Cambridge, MA, USA, 1973
[8] Guha, S., Habif, J.L., Takeoka, M.: ‘Appraising helstrom limits to optical pulse-position demodulation using single photon detection and optical feedback’, *J. Mod. Opt.*, 2011, 58, (3–4), pp. 257–265
[9] Izumi, S., Takeoka, M., Enu, K., et al.: ‘Quantum receivers with squeezing and photon-number-resolving detectors’, *Phys. Rev.*, 2013, A87, p. 042328
[10] Müller, C.R., Marquardt, C.: ‘A robust quantum receiver for phase shift keyed signals’, *New J. Phys.*, 2015, 17, (3), p. 032003
[11] Scully, M.G., Zubairy, M.S.: ‘Quantum optics’ (Cambridge University Press, Cambridge, UK, 1997)