Breakdown of the equivalence between gravitational mass and energy due to quantum effects

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We review our recent theoretical results about inequivalence between passive and active gravitational masses and energy in semiclassical variant of general relativity, where gravitational field is not quantized but matter is quantized. To this end, we consider the simplest quantum body with internal degrees of freedom - a hydrogen atom. We concentrate our attention on the following physical effects, related to electron mass. The first one is inequivalence between passive gravitational mass and energy at microscopic level. Indeed, quantum measurement of gravitational mass can give result, which is different from the expected, \( m \neq m_e + \frac{E_n}{P} \), where electron is initially in its ground state; \( m_e \) is the bare electron mass. The second effect is that the expectation values of both passive and active gravitational masses of stationary quantum states are equivalent to the expectation value of energy. The most spectacular effects are inequivalence of passive and active gravitational masses and energy at macroscopic level for ensemble of coherent superpositions of stationary quantum states. We show that, for such superpositions, the expectation values of passive and active gravitational masses are not related to the expectation value of energy by the famous Einstein’s equation, \( m \neq \frac{E}{P} \). In this review, we also improve several drawbacks of the original pioneering works.

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1. INTRODUCTION

Equivalence principle (EP) between gravitational and inertial masses in a combination with the local Lorentz invariance of spacetime is known to be a keystone of the classical general relativity [1,2]. In the current scientific literature, there exists a wide discussion if it can survive in the possible quantum theory of gravity (see, for example, the recent Refs.[3-5]). Since the quantum gravitation theory has not been elaborated yet, the EP is often studied in framework of the so-called semiclassical approach to quantum gravity, where gravitational field is not quantized, but the matter is quantized [3-5]. Note that the EP for a composite body is not a trivial notion even in general relativity in the absence of quantum effects. Indeed, as shown in Refs.[6-8], external gravitational field is coupled not directly with energy of a composite body but with the combination \( R + 3K + 2P \), where \( R, K, \) and \( P \) are rest, kinetic, and potential energies, respectively. As mentioned in Ref.[8], and considered in detail in Ref.[9], the above mentioned combination can be changed into expected total energy, if we choose the proper local coordinates, where the interval has the Minkowski’s form. Therefore, in classical general relativity passive gravitational mass is equivalent to inertial one for a composite body [8,9], as expected. On the other hand, as shown in Ref.[7], active gravitational mass of a composite classical body is equivalent to its energy only after averaging the gravitational mass over time. Semiclassical analysis [5] of the Einstein’s field equation has demonstrated that the expectation values of active gravitational mass and energy are equivalent only for stationary quantum states of a composite quantum body. Situation is different for ensemble of coherent quantum superpositions of stationary quantum states, where the expectation values of active gravitational mass can oscillate in time [5] even for superpositions with the constant expectation values of energy. The results of Ref.[5] are against the equivalence of active gravitational mass and energy even at macroscopic level in quantum gravity, which has to modify the EP. Note that quantum effects also change the status of the EP for passive gravitational mass of a quantum body with internal degrees of freedom. As discussed in [10], quantum effects break the equivalence of passive gravitational mass and energy at a microscopic level. Let us consider this phenomenon in more details. Suppose that there is a hydrogen atom in ground state, \( E_1 \), and we switch on gravitational field to measure electron mass, which is expected to be equal to \( m = m_e + \frac{E_n}{P} \), where \( m \) is electron passive gravitational mass, \( m_e \) is the bare electron mass. Contrary to the above mentioned common expectation, it has been shown in Ref.[10] that quantum measurements of the mass can give the following results \( m = m_e + \frac{E}{P} \), where \( E_n \) is energy of electron \( nS \) orbital in a hydrogen atom, although the corresponding probabilities are very small. Note that influence of quantum effects on the EP is even more dramatic and, as shown in Ref.[11], the equivalence between passive
gravitational mass and energy is broken even at a macroscopic level. Indeed, the above mentioned equivalence exists for the expectation values of the mass and energy only for stationary quantum states. In accordance with results of Ref.[11], the equivalence between the expectation values of passive gravitational mass and energy is broken for ensemble of coherent quantum superpositions of stationary quantum states in a hydrogen atom. Of course, all these statements are not restricted by the atom but are common properties of any quantum body with internal degrees of freedom.

2. GOAL

In Sec. 3, we discuss that there is no equivalence between passive gravitational mass and energy of electron in a hydrogen atom at a microscopic level, using the local Lorentz invariance, which defines electron wave functions in a gravitational field (we call it method-1). We start from quantum state with a definite electron energy in the absence of external gravitational field, $E_1$, and show that quantum measurement of the mass in the field can give values different from expected one, $m \neq m_e + \frac{\Delta E}{c^2}$, although the corresponding probabilities are small. In Sec. 4, we discuss the same results, using corrections to the Schrödinger equation for electron in a hydrogen atom, which contain the so-called virial term in gravitational field. We stress importance of the virial term for the breakdown of the above mentioned equivalence (and, thus, for the breakdown of the EP) at a microscopic level (we call this method-2). In Sec. 5, we discuss the breakdown of the equivalence between passive gravitational mass and energy (and, as a result - breakdown of the EP) at macroscopic level. We show that this equivalence survives for macroscopic ensemble of stationary quantum electron states in a hydrogen atom due to the so-called quantum virial theorem. On the other hand, it is also shown that for coherent ensembles of superpositions of stationary quantum states the above mentioned equivalence is not survived due to the quantum virial term. During our calculations in Secs. 2-5, we use property of the local Lorentz invariance of a spacetime in general relativity as well as consider passive gravitational mass to be a quantity proportional to weight of a composite body whose center of mass is fixed in gravitational field by some forces of non-gravitational origin. Finally, in Sec. 6, we discuss the EP between active gravitational mass and energy for electron in a hydrogen atom at macroscopic level. Our results are similar to that for passive gravitational mass. Indeed, we show that the equivalence (and, thus, the EP) survives for macroscopic ensembles of stationary quantum states, whereas for macroscopic coherent ensembles of superpositions of quantum states the equivalence is broken. In Sec. 7, we come to the conclusion that the EP has to be seriously reformulated in the presence of quantum effects in general relativity.

3. INEQUIVALENCE OF PASSIVE GRAVITATIONAL MASS AND ENERGY AT MICROSCOPIC LEVEL (METHOD-1)

3.1. Electron wave function in a hydrogen atom with a definite energy in the absence of gravitational field

Suppose that, at $t < 0$, there is no gravitational field and electron is in a ground state of a hydrogen atom, characterizing by the following wave function:

$$\Psi_1(r, t) = \exp\left(-\frac{i m_e c^2 t}{\hbar}\right) \exp\left(-\frac{i E_1 t}{\hbar}\right) \Psi_1(r),$$

which is solution of the corresponding Schrödinger equation:

$$i\hbar \frac{\partial \Psi_1(r, t)}{\partial t} = \left[ m_e c^2 - \frac{\hbar^2}{2m_e} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - \frac{e^2}{r} \right] \Psi_1(r, t).$$

[Here $E_1$ is electron ground state energy, $r$ is distance between electron with coordinates $(x, y, z)$ and proton; $\hbar$ is the Planck constant, $c$ is the velocity of light.]

3.2 Electron wave functions in a hydrogen atom in the presence of gravitational field

At $t = 0$, we perform the following Gedanken experiment. We switch on a weak centrosymmetric (e.g. the Earth’s) gravitational field, with position of center of mass of the atom (i.e., proton) being fixed in the field by some non-gravitational forces. It is known that, in a weak field approximation, curved spacetime is characterized by the following
of the order of $\phi E$

Note that it is easy to show that the above disregarded tidal terms have relative order of $\frac{\phi}{c^2}$. We stress that it is very important that the wave function (1) is not a solution of the Schrödinger equation (6) anymore if we don’t differentiate the potential $\phi(R)$, then new wave functions, written in the proper local coordinates (4) (with fixed proton’s position), satisfy at, $t, \tilde{t} > 0$, the similar Schrödinger equation:

$$i\hbar \frac{\partial \tilde{\Psi}(\tilde{r}, \tilde{t})}{\partial \tilde{t}} = \left[ m_e c^2 - \frac{\hbar^2}{2m_e} \left( \frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial^2}{\partial \tilde{y}^2} + \frac{\partial^2}{\partial \tilde{z}^2} \right) - \frac{e^2}{\tilde{r}} \right] \tilde{\Psi}(\tilde{r}, \tilde{t}).$$

(6)

On Earth’s surface they are very small and are of the relative order of $\frac{\phi}{c^2} \sim 10^{-17}$. We stress that it is very important that the wave function (1) is not a solution of the Schrödinger equation (6) anymore and, thus, is not characterized by definite energy and weight in the gravitational field (3). Moreover, a general solution of Eq.(6) can be written in the proper local coordinates in the following way:

$$\tilde{\Psi}(\tilde{r}, \tilde{t}) = \exp \left( \frac{-im_e c^2 \tilde{t}}{\hbar} \right) \sum_{n=1}^{\infty} \tilde{a}_n \Psi_n(\tilde{r}) \exp \left( \frac{-iE_n \tilde{t}}{\hbar} \right),$$

(7)

where the wave functions $\Psi_n(\tilde{r})$ are solutions for the so-called $nS$ atomic orbitals of a hydrogen atom with energies $E_n$ and are normalized in the proper local space,

$$\int \Psi_n^2(\tilde{r}) \, d^3\tilde{r} = 1.$$  

(8)

[Here we stress that, as possible to show, only $1S \rightarrow nS$ quantum transitions amplitudes are non-zero in a hydrogen atom in the gravitational field (3), which corresponds only to real wave functions. Therefore, we keep in Eq.(7) only $nS$ atomic orbitals and everywhere below disregard difference between $\Psi_n(r)$ and $\Psi_n^*(r) = \Psi_n(r).$]

Note that the normalized wave function (1) can be rewritten in the proper local spacetime coordinates (4) in the following way:

$$\Psi_1(\tilde{r}, \tilde{t}) = \exp \left( \frac{-im_e c^2 (1 - \frac{\phi}{c^2}) \tilde{t}}{\hbar} \right) \exp \left( \frac{-iE_1 (1 - \frac{\phi}{c^2}) \tilde{t}}{\hbar} \right) \times \left( 1 + \frac{\phi}{c^2} \right)^{3/2} \Psi_1 \left[ \left( 1 + \frac{\phi}{c^2} \right) \tilde{r} \right].$$

(9)

It is important that the gravitational field (3) can be considered as a sudden perturbation to the Hamiltonian (2), therefore, at $t = \tilde{t} = 0$, the wave functions (7) and (9) have to be equal to each other:

$$\left( 1 + \frac{\phi}{c^2} \right)^{3/2} \Psi_1 \left[ \left( 1 + \frac{\phi}{c^2} \right) \tilde{r} \right] = \sum_{n=1}^{\infty} \tilde{a}_n \Psi_n(\tilde{r}).$$

(10)
From Eq.(10), it directly follows that
\[
\tilde{a}_1 = \left(1 + \frac{\phi}{c^2}\right)^{3/2} \int_0^\infty \Psi_1 \left[\left(1 + \frac{\phi}{c^2}\right)\tilde{r}\right] \Psi_1(\tilde{r}) \ d^3\tilde{r} \tag{11}
\]
and
\[
\tilde{a}_n = \left(1 + \frac{\phi}{c^2}\right)^{3/2} \int_0^\infty \Psi_1 \left[\left(1 + \frac{\phi}{c^2}\right)\tilde{r}\right] \Psi_n(\tilde{r}) \ d^3\tilde{r}, \ n > 1. \tag{12}
\]

### 3.3. Probabilities and amplitudes

Below, we calculate quantum mechanical amplitudes (11) and (12) in a linear approximation with respect to the gravitational potential,
\[
\tilde{a}_1 = 1 + O\left(\frac{\phi^2}{c^4}\right), \tag{13}
\]
and
\[
\tilde{a}_n = \left(\frac{\phi}{c^2}\right) \int_0^\infty \frac{d\Psi_1(\tilde{r})}{d\tilde{r}} \tilde{r} \Psi_n(\tilde{r}) d^3\tilde{r}, \ n > 1. \tag{14}
\]

We stress that the wave function (7) is a series of wave functions, which have definite weights in the gravitational field (3). This means that they are characterized by the following definite electron passive gravitational masses,
\[
m_n = m_e + \frac{E_n}{c^2}. \tag{15}
\]

In accordance with the most general properties of quantum mechanics, this means that, if we do a measurement of gravitational mass for wave function (1) and (9), we obtain quantum values (15) with the probabilities: \(\tilde{P}_n = |\tilde{a}_n|^2\), where \(\tilde{a}_n\) are given by Eqs.(13) and (14).

Let us show that
\[
\int_0^\infty \frac{d\Psi_1(\tilde{r})}{d\tilde{r}} \tilde{r} \Psi_n(\tilde{r}) d^3\tilde{r} = \frac{V_{1,n}}{E_n - E_1}, \ n > 1, \tag{16}
\]
where \(\tilde{V}(\tilde{r})\) is the so-called quantum virial operator [12]:
\[
\tilde{V}(r) = -2\frac{\hbar^2}{2m_e} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{e^2}{r}, \tag{17}
\]
and
\[
V_{1,n} = \int_0^\infty \Psi_1(\tilde{r})\tilde{V}(\tilde{r})\Psi_n(\tilde{r})d^3\tilde{r}. \tag{18}
\]

To this end, we rewrite the Schrödinger equation in gravitational field (6) in terms of the initial coordinates \((x, y, z)\):
\[
(m_e c^2 + E_1)\Psi_1 \left[\left(1 - \frac{\phi}{c^2}\right) r\right] = \left[m_e c^2 - \frac{1}{(1 - \phi/c^2)^2} \frac{\hbar^2}{2m_e} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \right. \\
\left. - \frac{1}{(1 - \phi/c^2)} \frac{e^2}{r}\right] \Psi_1 \left[\left(1 - \frac{\phi}{c^2}\right) r\right]. \tag{19}
\]

Then, keeping as usual only terms of the first order with respect to the small parameter \(|\phi/c^2| \ll 1\), we obtain:
\[
E_1 \Psi_1(r) - \frac{\phi}{c^2} E_1 r \frac{d\Psi_1(r)}{dr} = -\frac{\hbar^2}{2m_e} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{e^2}{r} + \frac{\phi}{c^2} \tilde{V}(r)
\times
\left[\Psi_1(r) - \frac{\phi}{c^2} r \left(\frac{d\Psi_1(r)}{dr}\right) \right], \tag{20}
\]
and as a result
\[-E_1 r \left[ \frac{d\Psi_1(r)}{dr} \right] = \left[ -\frac{\hbar^2}{2m_e} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + \frac{e^2}{r} \right] \times \left[ -r \frac{d\Psi_1(r)}{dr} \right] + \hat{V}(r)\Psi_1(r). \tag{21} \]

Let us multiply Eq.(21) on \( \Psi_1(r) \) and integrate over space,
\[-E_1 \int_0^\infty \Psi_n(r) \left[ \frac{d\Psi_1(r)}{dr} \right] d^3r = \int_0^\infty \Psi_n(r) \left[ -\frac{\hbar^2}{2m_e} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + \frac{e^2}{r} \right] \times \left[ -r \frac{d\Psi_1(r)}{dr} \right] d^3r + \int_0^\infty \Psi_n(r) \hat{V}(r)\Psi_1(r) d^3r. \tag{22} \]

Taking into account that the Hamiltonian operator is the Hermitian one, we rewrite Eq.(22) as
\[ E_1 \int_0^\infty \Psi_n(r) \left[ \frac{d\Psi_1(r)}{dr} \right] d^3r = E_n \int_0^\infty \Psi_n(r) r \left[ \frac{d\Psi_1(r)}{dr} \right] d^3r - \int_0^\infty \Psi_1(r) \hat{V}(r)\Psi_n(r) d^3r. \tag{23} \]

Then, Eqs.(16)-(18) directly follow from Eq.(23).

As a result, the calculated amplitudes (14) and the corresponding probabilities for \( n \neq 1 \) can be rewritten as functions of matrix elements (18) of the virial operator (17),
\[ \tilde{a}_n = \left( \frac{\phi}{c^2} \right) \frac{V_{1,n}}{E_n - E_1} \tag{24} \]
and
\[ \tilde{P}_n = |\tilde{a}_n|^2 = \left( \frac{\phi}{c^2} \right)^2 \left( \frac{V_{1,n}}{E_n - E_1} \right)^2. \tag{25} \]

Note that near the Earth’s surface, where \( \frac{\phi^2}{c^2} \approx 0.49 \times 10^{-18} \), the probability for \( n = 2 \) in a hydrogen atom can be calculated as
\[ \tilde{P}_2 = |\tilde{a}_2|^2 = 1.5 \times 10^{-19}, \tag{26} \]
where
\[ \frac{V_{1,2}}{E_2 - E_1} = 0.56. \tag{27} \]

It is important that non-zero matrix elements (18) of the virial operator (17) for \( n \neq 1 \) are also responsible for breakdown of the equivalence between active gravitational mass and energy for a quantum body with internal degrees of freedom [3].

4. INEQUIVALENCE OF PASSIVE GRAVITATIONAL MASS AND ENERGY AT MICROSCOPIC LEVEL (METHOD-2)

4.1. Schrödinger equation with a definite energy in the absence of gravitational field

As in the previous Section, at \( t < 0 \), gravitational field is zero and electron occupies ground state in a hydrogen atom, characterizing by the wave function (1). As we have already discussed, the wave function (1) corresponds to the 1S electron orbital and is known to be a ground state solution of Eq.(2).
4.2. Schrödinger equation in the presence of gravitational field

Let us consider the same Gedanken experiment as in Sec. 3. We switch on the weak gravitational field (3) and obtain Eq.(6) for the wave functions in the proper local spacetime coordinates (4). But, in this Section, we rewrite Eq.(6) in the initial spacetime coordinates, \( (t, x, y, z) \),

\[
\frac{i\hbar}{\partial t}\frac{\partial \Psi(r, t)}{\partial t} = \left\{ \left[ m_e c^2 - \frac{\hbar^2}{2m_e} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - \frac{e^2}{r} \right] + \left( \frac{\phi}{c^2} \right) \left[ m_e c^2 - \frac{\hbar^2}{2m_e} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - \frac{e^2}{r} + \hat{V}(r) \right] \right\} \Psi(r, t),
\]

(28)

where the virial operator \( \hat{V}(r) \), is equal to (17). From Eq.(28), it directly follows that the external gravitational field (3) is coupled not only to Hamiltonian (2) but also to the virial operator (17). It is important that the virial term (17) does not commute with the Hamiltonian (2), therefore, it breaks the equivalence of the passive gravitational mass and energy for electron in a hydrogen atom.

4.2.1. More general Lagrangian

Here, we derive Hamiltonian (28) from more general Lagrangian. Let us consider the Lagrangian of a three body system: a hydrogen atom and the Earth in inertial coordinate system, treating gravitation (3) as a small perturbation in the Minkowski’s spacetime. In this case, we can make use of the results of Ref.[7], where the corresponding n-body Lagrangian is calculated as a sum of the following four terms:

\[
L = L_{kin} + L_{em} + L_G + L_{e,G},
\]

(29)

where \( L_{kin}, L_{em}, L_G, \) and \( L_{e,G} \) are kinetic, electromagnetic, gravitational and electric-gravitational parts of the Lagrangian, respectively. We recall that, in our approximation, we keep in the Lagrangian and Hamiltonian only terms of the order of \( (v/c)^2 \) and \( |\phi|^2/c^2 \) as well as keep only classical kinetic and the Coulomb electrostatic potential energies couplings with external gravitational field. It is possible to show that, in our case, different contributions to the Lagrangian (29) can be simplified:

\[
L_{kin} + L_{em} = -Mc^2 - mp c^2 - me c^2 + \frac{v^2}{2} + \frac{e^2}{r},
\]

(30)

\[
L_G = G \frac{mp M}{R} + G \frac{me M}{R} + \frac{3}{2} G \frac{me M}{R} \frac{v^2}{c^2},
\]

(31)

\[
L_{e,G} = -2G \frac{M}{Re^2} \frac{e^2}{r},
\]

(32)

where, as usual, we use the inequality \( mp \gg me \), with \( mp \) being the bare proton mass.

If we keep only those terms in the Lagrangian, which are related to electron motion (as usual, proton is supposed to be supported by some non-gravitational forces in the gravitational field), then we can write the Lagrangian (30)-(32) in the following form:

\[
L = -me c^2 + \frac{v^2}{2} + \frac{e^2}{r} - \frac{\phi(R)}{c^2} \left[ me + 3me \frac{v^2}{2} - 2 \frac{e^2}{r} \right], \quad \phi(R) = -G \frac{M}{R},
\]

(33)

It is easy to show that the corresponding electron Hamiltonian is

\[
\hat{H} = \left\{ \left[ me c^2 - \frac{\hbar^2}{2me} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - \frac{e^2}{r} \right] + \left( \frac{\phi}{c^2} \right) \left[ me c^2 - \frac{\hbar^2}{2me} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - \frac{e^2}{r} + \hat{V}(r) \right] \right\}.
\]

(34)

Note that Eq.(34) exactly coincides with electron Hamiltonian (28), obtained by us in the previous Subsection.
4.2.2. More general Hamiltonian

Let us derive the Hamiltonian (28), (34) from more general arguments. The so-called gravitational Stark effect (i.e., the mixing effect between even and odd wave functions in a hydrogen atom in gravitational field) was studied in Ref.[13] in the weak external gravitational field (3). Note that the corresponding Hamiltonian was derived in $1/c^2$ approximation and a possibility of center of mass of the atom motion was taken into account. The main peculiarity of the calculations in the above-mentioned paper was the fact that not only terms of the order of $\phi/c^2$ were calculated, as in our case, but also terms of the order of $\phi'/c^2$. Here, we use a symbolic notation $\phi'$ for the first derivatives of gravitational potential. In accordance with the existing tradition, we refer to the latter terms as to the tidal ones. Note that the Hamiltonian (3.24) was obtained in Ref.[13] directly from the Dirac equation in a curved spacetime of general relativity. As shown in Ref.[13], it can be rewritten for the corresponding Schrödinger equation as a sum of the four terms:

$$H(\hat{P}, \hat{p}, \hat{R}, r) = H_0(\hat{P}, \hat{p}, r) + H_1(\hat{P}, \hat{p}, \hat{R}, r) + H_2(\hat{p}, r) + H_3(\hat{P}, \hat{p}, \hat{R}, r),$$

(35)

$$H_0(\hat{P}, \hat{p}, r) = m_e c^2 + m_p c^2 + \left[\frac{\hat{P}^2}{2(m_e + m_p)} + \frac{\hat{p}^2}{2\mu}\right] - \frac{e^2}{r},$$

(36)

$$H_1(\hat{P}, \hat{p}, \hat{R}, r) = \left\{ m_e c^2 + m_p c^2 + \left[3\frac{\hat{P}^2}{2(m_e + m_p)} + 3\frac{\hat{p}^2}{2\mu} - 2\frac{e^2}{r}\right]\right\} \left(\frac{\phi - g\hat{R}}{c^2}\right),$$

(37)

$$H_2(\hat{p}, r) = \frac{1}{c^2} \left(\frac{\hat{p}}{m_e} - \frac{\hat{p}}{m_p}\right) [- (\mathbf{g}\cdot\hat{r}) \hat{p}^2 + i\hbar \hat{p}]$$

$$+ \frac{1}{c^2} \mathbf{g} \left( \frac{s_e}{m_e} - \frac{s_p}{m_p} \right) \times \hat{p} + \frac{e^2 (m_p - m_e) g \hat{r}}{2(m_e + m_p) c^2} \cdot \hat{r},$$

(38)

$$H_3(\hat{P}, \hat{p}, \hat{R}, r) = \frac{3}{2} \frac{i\hbar \mathbf{g} \mathbf{P}}{2(m_e + m_p) c^2} + \frac{3}{2} \frac{g(s_e + s_p) \times \mathbf{P}}{(m_e + m_p) c^2}$$

$$- \frac{(\mathbf{g}\cdot\hat{P})\hat{P} + (\mathbf{P}\cdot\mathbf{g})\hat{P}}{m_e + m_p} - \frac{i\hbar \mathbf{g} \mathbf{P}}{m_e + m_p},$$

(39)

where $\mathbf{g} = -G\frac{\mu}{R^2}$. Note that we use the following notations in Eqs. (35)-(39): $\hat{R}$ and $\mathbf{P}$ stand for coordinate and momentum of a hydrogen atom center of mass, respectively; whereas, $\mathbf{r}$ and $\mathbf{p}$ stand for relative electron coordinate and momentum in center of mass coordinate system; $\mu = m_e m_p / (m_e + m_p)$ is the so-called reduced electron mass. We point out that $H_0(\hat{P}, \hat{p}, r)$ is the Hamiltonian of a hydrogen atom in the absence of the field. It is important that the Hamiltonian $H_1(\hat{P}, \hat{p}, \hat{R}, r)$ describes couplings not only of the bare electron and proton masses with the gravitational field (3) but also couplings of electron kinetic and potential energies with the field. And finally, the Hamiltonians $H_2(\hat{p}, r)$ and $H_3(\hat{P}, \hat{p}, \hat{R}, r)$ describe only the tidal effects.

Let us strictly derive the Hamiltonian (28), (34), which has already been semi-quantitatively derived, from the more general Hamiltonian (35)-(39). As was already mentioned, we use the approximation, where $m_p \gg m_e$, and, therefore, $\mu = m_e$. In particular, this allows us to consider proton as a heavy classical particle. We recall that we need to derive the Hamiltonian of the atom, whose center of mass is at rest with respect to the Earth. Thus, we can omit center of mass kinetic energy and center of mass momentum. As a result, the first two contributions to electron part of the total Hamiltonian (35)-(39) can be written in the following way:

$$\hat{H}_0(\hat{p}, r) = m_e c^2 + \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r}$$

(40)

and

$$\hat{H}_1(\hat{p}, r) = \left\{ m_e c^2 + \left[3\frac{\hat{p}^2}{2m_e} - 2\frac{e^2}{r}\right]\right\} \left(\frac{\phi}{c^2}\right),$$

(41)
where we place center of mass of the atom at point \( \tilde{R} = 0 \). Now, let us study the first tidal term (38) in the total Hamiltonian (35). At first, we pay attention that \( |g| \simeq |\phi|/R_0 \). Then, as well known, in a hydrogen atom \(|r| \sim h/|p| \sim r_B \) and \( p^2/(2m_e) \sim e^2/r_B \). These values allow us to evaluate the first tidal term (38) in the Hamiltonian (35) as \( H_2 \sim (r_B / R_0) (|\phi|/c^2)(e^2/r_B) \sim 10^{-17} (|\phi|/c^2)(e^2/r_B) \). Note that this value is \( 10^{-17} \) times smaller than \( H_1 \sim (|\phi|/c^2)(e^2/r_B) \) and \( 10^{-8} \) times smaller than the second correction with respect to the small parameter \( |\phi|/c^2 \). Therefore, we can disregard the contribution (38) to the total Hamiltonian (35). As to the second tidal term (39) in the total Hamiltonian, we pay attention that it is exactly zero in the case, where \( P = 0 \), considered in this review. Therefore, we can conclude that the Hamiltonian (40),(41), derived in this Subsubsection, exactly coincides with that, semi-quantitatively derived by us earlier [see Eqs.(28),(34)].

### 4.3. Gravitational field as a perturbation to the Hamiltonian

It is important that the gravitational field (3), under the condition of our Gedanken experiment, can be considered as the following sudden perturbation, \( \tilde{U}_1(r, t) \), to the Hamiltonian (2) in the absence of gravitational field:

\[
\tilde{U}_1(r, t) = \left( \frac{\phi}{c^2} \right) \left[ m_e c^2 - \frac{\hbar^2}{2m_e} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right] - \frac{e^2}{r} + \tilde{V}(r) \Theta(t),
\]

where \( \Theta(t) \) is the so-called step function. Then, a general solution of Eq.(28) can be written in the following way:

\[
\Psi(r, t) = \exp \left( \frac{-i\tilde{m}_e c^2 t}{\hbar} \right) \Psi_1^0(r) \exp \left( \frac{-iE_1 t}{\hbar} \right) + \exp \left( \frac{-i\tilde{m}_e c^2 t}{\hbar} \right) \sum_{n>1} \tilde{a}_n \Psi_n(r) \exp \left( \frac{-iE_n t}{\hbar} \right),
\]

where the wave functions \( \Psi_n(r) \) are solutions for the \( nS \) orbitals in a hydrogen atom and are normalized,

\[
\int |\Psi_1^0(r)|^2 \, d^3r = 1, \quad \int |\tilde{\Psi}_n(r)|^2 \, d^3r = 1, \quad n > 1.
\]

[It is easy to show that perturbation (42) can results only in non-zero quantum transitions between \( 1S \) and \( nS \) electron orbitals, therefore, we keep in Eq.(43) only \( \Psi_1^0(r) \) wave functions, which are real.]

According with the standard time-dependent perturbation theory [12], the corrected wave-function of ground state, \( \Psi_1^0(r) \), as well as the corrections to mass and energy of ground state in Eq.(43) can be written as:

\[
\Psi_1^0(r) = \Psi_1(r) + \left( \frac{\phi}{c^2} \right) \sum_{n>1} \frac{V_{n,1}}{E_1 - E_n} \Psi_n(r),
\]

\[
\tilde{m}_e = \left( 1 + \frac{\phi}{c^2} \right) m_e, \quad \tilde{E}_1 = \left( 1 + \frac{\phi}{c^2} \right) E_1,
\]

where \( V_{n,1} \) is matrix element of the virial operator (17):

\[
V_{n,1} = \int \Psi_n(r) \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right] \Psi_1(r) \, d^3r.
\]

Note that the very last term in Eq.(45) corresponds to the so-called red shift in gravitational field. It is due to the expected contribution to passive gravitational mass from electron binding energy in the atom. As to the coefficients \( a_n \) with \( n \neq 1 \) in Eq.(43), they can be also written in terms of the virial operator matrix elements,

\[
a_n = - \left( \frac{\phi}{c^2} \right) \left( \frac{V_{n,1}}{E_1 - E_n} \right),
\]

and coincides with Eq.(24). Note that the wave function (43)-(47), which corresponds to electron ground energy level in the presence of the gravitational field (3) (i.e., \( t > 0 \)), is a series of eigenfunctions of electron energy operator, taken in the absence of the field. Therefore, if we measure energy, in electron quantum state (43)-(47), we obtain the following quantized values for electron gravitational mass:

\[
m_n = m_e + \frac{E_n}{c^2},
\]
where we omit the red shift effect. From Eqs.(43)-(48), we can state that the expected Einstein’s equation, \( m = m_e + \frac{E_n}{c^2} \), survives in our case with probability close to 1, whereas with the following small probabilities,

\[
P_n = |a_n|^2 = \left( \frac{\phi}{c^2} \right)^2 \frac{V_{n,1}^2}{(E_n - E_1)^2}, \quad n \neq 1,
\]

it is broken. The reason for this breakdown is that, the virial term (17) does not commute with the Hamiltonian (2) in the absence of gravitational field. As a result, electron wave functions with definite passive gravitational masses are not characterized by definite energies in the absence of gravitational field. It is important that our current results coincide with that obtained in Sec.3 by different method.

### 4.4. Experimental aspects

Here, let us describe another Gedanken experiment, where gravitational field is adiabatically switched on. To this end, we consider wave function (1) to be valid at \( t \rightarrow -\infty \) and apply the following perturbation, due to the gravitational field (3), for the Hamiltonian (2):

\[
\tilde{U}_2(r, t) = \left( \frac{\phi}{c^2} \right) \left[ m_e c^2 - \frac{\hbar^2}{2m_e} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right] - \frac{e^2}{r} + \hat{V}(r) \exp(\lambda t), \quad \lambda \rightarrow 0.
\]

Then, at \( t \approx 0 \) (i.e., in the presence of the field), the electron wave function can be written as

\[
\Psi(r, t) = \exp \left( \frac{-i\tilde{m}_e c^2 t}{\hbar} \right) \Psi_1(r) \exp \left( \frac{-i\tilde{E}_1 t}{\hbar} \right) + \exp \left( \frac{-i\tilde{m}_e c^2 t}{\hbar} \right) \sum_{n>1} a_n \Psi_n(r) \exp \left( \frac{-iE_n t}{\hbar} \right).
\]

Application of the standard time-dependent perturbation theory \[12\] in the case of adiabatic switching on gravitational field results in:

\[
\Psi_1^1(r) = \Psi_1(r) + \left( \frac{\phi}{c^2} \right) \sum_{n>1} \frac{V_{n,1}}{E_1 - E_n} \Psi_n(r),
\]

\[
\tilde{m}_e = \left( 1 + \frac{\phi}{c^2} \right) m_e, \quad \tilde{E}_1 = \left( 1 + \frac{\phi}{c^2} \right) E_1,
\]

and

\[
a_n = 0, \quad P_n = 0.
\]

Thus, in adiabatic limit, the phenomenon of quantization of passive gravitational mass (15),(48) disappears. This means that the possible experimental observation of the above mentioned phenomenon has to be done in quickly changing gravitational field. It is important that step-like function, \( \Theta(t) \), which was used to derive Eq.(48), does not mean motion of a source of gravity with velocity higher than the speed of light. We can use step-like function if significant change of gravitational field happens quicker than the characteristic period of quasiclassical rotation of electron in a hydrogen atom. In the case under consideration, we need the time about \( \delta t \leq t_0 = \frac{2\pi \hbar}{E_2 - E_1} \sim 10^{-15} \text{s} \).

Of course, there exist much more convenient quantum systems with higher values of the parameter \( t_0 \), where the above discussed phenomenon could be observed. We recall that all excited energy levels are quasistationary and, thus, decay with time by emitting photons. Therefore, it is much more efficient to detect emitted photons than to directly measure a weight. As to the relatively small probabilities (24) of the mass quantization, it is not too small and can be compensated by large value of the Avogadro number, \( N_A \approx 6 \times 10^{23} \). In other words, for macroscopic number of the atoms, we may have large number of emitted photons. For instance, the number of excited electrons (i.e., emitted photons) for 1000 moles of the atoms is estimated as

\[
N_n = 2.95 \times 10^8 \times \left( \frac{V_{n,1}}{E_n - E_1} \right)^2, \quad N_2 = 0.9 \times 10^8.
\]
5. INEQUIVALENCE BETWEEN PASSIVE GRAVITATIONAL MASS AND ENERGY AT MACROSCOPIC LEVEL

In Sec. 5, we perform our Gedanken experiment, where we switch on the gravitational field (3) for \( t > 0 \), using the gravitational field as a sudden perturbation (42). We consider two different cases: macroscopic ensemble of stationary quantum states and macroscopic ensemble of coherent superpositions of stationary quantum states. In this section we disregard small probabilities of the order of \( \frac{\phi}{c^2} \) [see Eqs.(25) and (49)] and, thus, ignore mass quantization phenomenon.

5.1. Equivalence between passive gravitational mass and energy of stationary quantum states

Suppose that, at \( t < 0 \), there is no gravitational field and we have macroscopic ensemble of hydrogen atoms with electrons being in their ground states (1). At \( t > 0 \), we perform our Gedanken experiment: we switch on gravitational field, which is treated as the perturbation (42) in inertial system. Let us for the moment consider one atom. At \( t > 0 \), general solution for electron wave function is

\[
\Psi(r, t) = \exp\left(-i\frac{\tilde{m}_e c^2 t}{\hbar}\right) \Psi_1(r) \exp\left(-i\frac{\tilde{E}_1 t}{\hbar}\right) + \exp\left(-i\frac{m_e c^2 t}{\hbar}\right) \sum_{n>1} a_n \Psi_n(r) \exp\left(-i\frac{E_n t}{\hbar}\right),
\]

If we disregard small probabilities \( a_n \) for \( n > 1 \), which were considered in Secs. 3 and 4, we can rewrite the Eq.(55) as

\[
\Psi(r, t) = \exp\left(-i\frac{\tilde{m}_e c^2 t}{\hbar}\right) \Psi_1(r) \exp\left(-i\frac{\tilde{E}_1 t}{\hbar}\right).
\]

In accordance with the quantum perturbation theory [12], first order correction to energy of wave function (56) can be written as:

\[
\tilde{m}_e = \left(1 + \frac{\phi}{c^2}\right)m_e, \quad \tilde{E}_1 = \left(1 + \frac{\phi}{c^2}\right)E_1,
\]

which is well known red shift [1]. It is important that, in Eq.(57), there is no correction due to the quantum virial term (42),(46). The virial term correction is zero due the so-called quantum virial theorem [12], which claims that for any value of \( n \), including \( n = 1 \):

\[
V_{n,n} = \int \Psi_n(r) \tilde{V}(r) \Psi_n(r) d^3r = 0.
\]

Eq.(57) directly demonstrates the equivalence of gravitational mass and energy at macroscopic level.

5.2. Inequivalence between passive gravitational mass and energy for macroscopic coherent ensemble of superpositions of stationary quantum states

Suppose that, in the absence of gravitational field (i.e., at \( t < 0 \)), we have macroscopic ensemble of coherent superpositions of two wave functions, corresponding to ground state wave function, \( \Psi_1(r) \), and first excited energy level wave function, \( \Psi_2(r) \), in a hydrogen atom:

\[
\Psi(r, t) = \frac{1}{\sqrt{2}} \exp\left(-i\frac{m_e c^2 t}{\hbar}\right) \left[ \exp\left(-i\frac{E_1 t}{\hbar}\right) \Psi_1(r) + \exp\left(-i\frac{E_2 t}{\hbar}\right) \Psi_2(r) \right].
\]

Coherent ensemble of such wave functions, where the difference between phases of functions \( \Psi_1(r) \) and \( \Psi_2(r) \) is fixed, is possible to create by using lasers. We perform the same Gedanken experiment and, therefore, suddenly switch on the gravitational field (3) at \( t > 0 \) [see the corresponding perturbation (42) to the Hamiltonian (2)]:

\[
U_1(r, t) = \frac{\phi}{c^2} [m_e c^2 + \tilde{H}_0(r) + \tilde{V}(r)] \Theta(t).
\]
Then, if we disregard small probabilities of the order of \( \frac{c^2}{|c|^2} \) [see Eq.(25)] and, thus, don’t take into account the mass quantization phenomenon (15), we can consider wave function (59) as wave function of two level system and can use the corresponding variant of the time-dependent perturbation theory. In accordance with this theory [12], the wave functions in the gravitational field (3) can be written as:

\[
\Psi_1(r, t) = \exp\left(-\frac{im_e^2 c^2 t}{\hbar}\right) \left[ \exp\left(-\frac{i(E_2 - E_1)t}{\hbar}\right) a_1(t) \Psi_1(r) + \exp\left(-\frac{iE_2 t}{\hbar}\right) a_2(t) \Psi_2(r) \right].
\]

Using the results of the time-dependent perturbation theory, it is possible to find equations to determine the functions \( a_1(t) \) and \( a_2(t) \):

\[
\frac{d a_1(t)}{dt} = -i U_{11}(t) a_1(t) - i U_{12}(t) \exp\left(-\frac{i(E_2 - E_1)t}{\hbar}\right) a_2(t),
\]

\[
\frac{d a_2(t)}{dt} = -i U_{22}(t) a_2(t) - i U_{21}(t) \exp\left(-\frac{i(E_2 - E_1)t}{\hbar}\right) a_1(t),
\]

where

\[
U_{11}(t) = \Theta(t) \frac{\phi}{c^2} \int \Psi_1^*(r) [m_e c^2 + \hat{H}(r) + \hat{V}(r)] \Psi_1(r) d^3 r = \Theta(t) \frac{\phi}{c^2} (m_e c^2 + E_1),
\]

\[
U_{12}(t) = \Theta(t) \frac{\phi}{c^2} \int \Psi_1^*(r) [m_e c^2 + \hat{H}(r) + \hat{V}(r)] \Psi_2(r) d^3 r = \Theta(t) \frac{\phi}{c^2} V_{12},
\]

\[
U_{22}(t) = \Theta(t) \frac{\phi}{c^2} \int \Psi_2^*(r) [m_e c^2 + \hat{H}(r) + \hat{V}(r)] \Psi_2(r) d^3 r = \Theta(t) \frac{\phi}{c^2} (m_e c^2 + E_2),
\]

\[
U_{21}(t) = \Theta(t) \frac{\phi}{c^2} \int \Psi_2^*(r) [m_e c^2 + \hat{H}(r) + \hat{V}(r)] \Psi_1(r) d^3 r = \Theta(t) \frac{\phi}{c^2} V_{21},
\]

where \( V_{ij} \) are matrix elements of the virial operator (17). After substitution of Eqs.(63) in Eqs.(62), it is possible to find that the function (59) is

\[
\Psi_1(r, t) = \exp\left(-\frac{im_e^2 c^2 t}{\hbar}\right) \left[ \Psi_1^1(r, t) + \Psi_2^1(r, t) \right],
\]

where

\[
\Psi_1^1(r, t) = \frac{1}{\sqrt{2}} \exp\left[-i \frac{(m_e c^2 + E_1) \phi t}{c^2 \hbar}\right] \exp\left(-i \frac{E_1 t}{\hbar}\right) \left[ 1 - \frac{\phi V_{12}}{c^2(E_2 - E_1)} \right] \Psi_1(r) + \frac{1}{\sqrt{2}} \exp\left(-i \frac{E_2 t}{\hbar}\right) \frac{\phi V_{12}}{c^2(E_2 - E_1)} \Psi_1(r)
\]

and

\[
\Psi_2^1(r, t) = \frac{1}{\sqrt{2}} \exp\left[-i \frac{(m_e c^2 + E_2) \phi t}{c^2 \hbar}\right] \exp\left(-i \frac{E_2 t}{\hbar}\right) \left[ 1 - \frac{\phi V_{21}}{c^2(E_1 - E_2)} \right] \Psi_2(r) + \frac{1}{\sqrt{2}} \exp\left(-i \frac{E_1 t}{\hbar}\right) \frac{\phi V_{21}}{c^2(E_1 - E_2)} \Psi_2(r).
\]

It is possible to demonstrate that with accuracy to the first order of the small parameter, \( \frac{\phi}{c^2} \ll 1 \), the wave function (64)-(66) can be written as

\[
\Psi_1(r, t) = \frac{1}{\sqrt{2}} \exp\left[-i \frac{(m_e c^2 + E_1)(1 + \phi)t}{c^2 \hbar}\right] \left[ 1 - \frac{\phi V_{12}}{c^2(E_2 - E_1)} \right] \Psi_1(r) + \frac{\phi V_{21}}{c^2(E_1 - E_2)} \Psi_2(r)
\]

and

\[
\Psi_2(r, t) = \frac{1}{\sqrt{2}} \exp\left[-i \frac{(m_e c^2 + E_2)(1 + \phi)t}{c^2 \hbar}\right] \left[ 1 - \frac{\phi V_{21}}{c^2(E_1 - E_2)} \right] \Psi_2(r) + \frac{\phi V_{12}}{c^2(E_2 - E_1)} \Psi_1(r).
\]
where the wave function is normalized with the same accuracy:

\[ \int |\Psi_1(r,t)|^2 \Psi_1(r,t) d^3r = 1 + O\left(\frac{\phi^2}{c^4}\right). \]  

(68)

Taking into account that we consider macroscopic coherent ensemble of superposition of quantum states (59), (67), it is easy to calculate the expectation value of energy per one electron in gravitational field for wave function (67):

\[ <E> = m_e c^2 \left(1 + \frac{\phi}{c^2}\right) + \frac{E_1 + E_2}{2c^2} \left(1 + \frac{\phi}{c^2}\right) + V_{12} \frac{\phi}{c^2}. \]  

(69)

Note that the first term and the second one are expected. On the other hand, the last term contains contribution to the weight of macroscopic coherent ensemble from the virial term (17) and breaks the equivalence of passive gravitational mass and energy for quantum superposition of stationary state.

Note that so far we have considered macroscopic coherent ensemble of superposition of stationary wave functions, which is characterized by constant difference of phases, \( \alpha = 0 \), between the first and the second quantum states. If we introduce more general macroscopic coherent ensemble,

\[ \Psi(r,t) = \frac{1}{\sqrt{2}} \exp\left(-\frac{im_e c^2 t}{\hbar}\right) \left[ \exp\left(-\frac{iE_1 t}{\hbar}\right) \Psi_1(r) + \exp(i\alpha) \exp\left(-\frac{iE_2 t}{\hbar}\right) \Psi_2(r) \right], \]  

(70)

the expectation value of energy in the gravitational field (3) is changed:

\[ <E> = m_e c^2 \left(1 + \frac{\phi}{c^2}\right) + \frac{E_1 + E_2}{2c^2} \left(1 + \frac{\phi}{c^2}\right) + V_{12} \cos \alpha \frac{\phi}{c^2}. \]  

(71)

It is important that Eqs.(69), (71) directly demonstrate the breakdown of the equivalence between gravitational mass and energy at macroscopic level for coherent ensemble of superposition of stationary quantum state. On the other hand, for the non-coherent ensembles, phase \( \alpha \) is not fixed in Eq.(71) and, thus, the last virial terms quickly averages to zero.

6. INEQUIVALENCE BETWEEN ACTIVE GRAVITATIONAL MASS AND ENERGY AT MACROSCOPIC LEVEL

In this section, we review our results \[5, 10\], where we showed that active gravitational mass and energy were inequivalent to each other at macroscopic level for coherent ensembles of quantum superpositions of stationary states.

6.1. Active gravitational mass in classical physics

Here, we determine electron active gravitational mass in a classical model of a hydrogen atom, which takes into account electron kinetic and potential energies \[7\]. More specifically, we consider a particle with small bare mass \( m_e \), moving in the Coulomb electrostatic field of a heavy particle with bare mass \( m_p \gg m_e \). Our task is to find gravitational potential at large distance from the atom, \( R \gg r_B \), where \( r_B \) is the the so-called Bohr radius (i.e., effective "size" of a hydrogen atom). Below, we use the so-called weak field gravitational theory \[1, 7\], where the post-Newtonian gravitational potential can be represented as \[8, 10\]

\[ \phi(R,t) = -G \frac{m_p + m_e}{R} - G \int \frac{\Delta T_{\alpha\beta}^{\text{kin}}(t,r) + \Delta T_{\alpha\beta}^{\text{pot}}(t,r)}{c^2 R} d^3r, \]  

(72)

where \( \Delta T_{\alpha\beta}^{\text{kin}}(t,r) \) and \( \Delta T_{\alpha\beta}^{\text{pot}}(t,r) \) are contributions to stress-energy tensor density, \( T_{\alpha\beta}(t,r) \), due to kinetic and the Coulomb potential energies, respectively. We point out that, in Eq.(72), we disregard all retardation effects. Thus, in the above-discussed approximation, electron active gravitational mass is equal to

\[ m_e^a = m_e + \frac{1}{c^2} \int \left[ \Delta T_{\alpha\beta}^{\text{kin}}(t,r) + \Delta T_{\alpha\beta}^{\text{pot}}(t,r) \right] d^3r. \]  

(73)
Let us calculate $\Delta T_{\alpha\alpha}^{\text{kin}}(t, r)$, using the standard expression for stress-energy tensor density of a moving relativistic point mass \cite{1,2}:

$$T_{\alpha\beta}^{\text{kin}}(r, t) = \frac{m_e v^\alpha(t) v^\beta(t)}{\sqrt{1 - v^2(t)/c^2}} \delta^3[r - r_e(t)], \quad (74)$$

where $v^\alpha$ is a four-velocity, $\delta^3(...)$ is the three dimensional Dirac $\delta$-function, and $r_e(t)$ is a three dimensional electron trajectory.

From Eqs.(73),(74), it directly follows that

$$\Delta T_{\alpha\alpha}^{\text{kin}}(t) = \int \Delta T_{\alpha\alpha}^{\text{kin}}(t, r) d^3 r = \frac{m_e [c^2 + v^2(t)]}{\sqrt{1 - v^2(t)/c^2}} - m_e c^2. \quad (75)$$

Note that, although calculations of the contribution from potential energy to stress energy tensor are more complicated, they are straightforward and can be done by using the standard formula for stress energy tensor of electromagnetic field \cite{2},

$$T^{\mu\nu}_{\text{em}} = \frac{1}{4\pi} \left[ F^{\mu\alpha} F^\nu_\alpha - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right], \quad (76)$$

where $\eta_{\alpha\beta}$ is the Minkowski metric tensor, $F^{\alpha\beta}$ is the so-called tensor of electromagnetic field \cite{2}. In this review, we use approximation, where we do not take into account magnetic field and keep only the Coulomb electrostatic field. In this approximation, we can simplify Eq.(76) and obtain from it the following expression:

$$\Delta T_{\alpha\alpha}^{\text{pot}}(t) = \int \Delta T_{\alpha\alpha}^{\text{pot}}(t, r) d^3 r = -2 \frac{e^2}{r(t)}, \quad (77)$$

where $e$ is the electron charge. As directly follows from Eqs.(75),(77), electron active gravitational mass can be represented in the following way:

$$m_e^a = \left[ \frac{m_e c^2}{(1 - v^2/c^2)^{1/2}} - \frac{e^2}{r} \right] / c^2 + \left[ \frac{m_e v^2}{(1 - v^2/c^2)^{1/2}} - \frac{e^2}{r} \right] / c^2. \quad (78)$$

We note that the first term in Eq.(78) is the expected one. Indeed, it is the total energy contribution to the mass, whereas the second term is the so-called relativistic virial one \cite{12}. It is important that it depends on time. Therefore, in classical physics, active gravitational mass of a composite body depends on time too. Nevertheless, in this situation, it is possible to introduce averaged over time electron active gravitational mass. This procedure results in the expected equivalence between averaged over time active gravitational mass and energy \cite{3}:

$$< m_e^a > t = \left( \frac{m_e c^2}{(1 - v^2/c^2)^{1/2}} - \frac{e^2}{r} \right) / c^2 + \left( \frac{m_e v^2}{(1 - v^2/c^2)^{1/2}} - \frac{e^2}{r} \right) / c^2 = m_e + E/c^2. \quad (79)$$

We point out that, in Eq.(79), the averaged over time virial term is zero due to the classical virial theorem. It is easy to show that for non-relativistic case our Eqs.(78),(79) can be simplified to

$$m_e^a = m_e + \left( \frac{m_e v^2}{2} - \frac{e^2}{r} \right) / c^2 + \left( 2 \frac{m_e v^2}{2} - \frac{e^2}{r} \right) / c^2 \quad (80)$$

and

$$< m_e^a > t = m_e + \left( \frac{m_e v^2}{2} - \frac{e^2}{r} \right) / c^2 + \left( 2 \frac{m_e v^2}{2} - \frac{e^2}{r} \right) / c^2 = m_e + E/c^2. \quad (81)$$

### 6.2. Active gravitational mass in quantum physics

In this Subsection, we consider the so-called semiclassical theory of gravity \cite{14}, where, in the Einstein’s field equation, gravitational field is not quantized but the matter is quantized:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} \langle \tilde{T}_{\mu\nu} \rangle. \quad (82)$$
Here, $< \hat{T}_{\mu\nu} >$ is the expectation value of quantum operator, corresponding to the stress-energy tensor. To make use of Eq.(82), we have to rewrite Eq.(80) for electron active gravitational mass using momentum, instead of velocity. Then, we can quantize the obtained result:

$$\hat{m}_e^a = m_e + \left( \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right) / c^2 + \left( \frac{2}{2m_e} \right) / c^2. \tag{83}$$

Note that Eq.(83) represents electron active gravitational mass operator. As directly follows from it, the expectation value of electron active gravitational mass can be written as

$$< \hat{m}_e^a > = m_e + \left( \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right) / c^2 + \left( \frac{2}{2m_e} \right) / c^2, \tag{84}$$

where third term is the virial one.

6.2.1. Equivalence of the expectation values of active gravitational mass and energy for stationary quantum states

Now, we consider a macroscopic ensemble of hydrogen atoms with each of them being in the n-th energy level. For such ensemble, the expectation value of the mass (83) is

$$< \hat{m}_e^a > = m_e + \frac{E_n}{c^2}. \tag{85}$$

In Eqs.(84),(85), we take into account that the expectation value of the virial term is equal to zero in stationary quantum states due to the quantum virial theorem [12]. Thus, we can make the following important conclusion: in stationary quantum states, active gravitational mass of a composite quantum body is equivalent to its energy at a macroscopic level [3, 10].

6.2.2. Inequivalence between active gravitational mass and energy for macroscopic coherent ensemble of quantum superpositions of stationary states

Below, we introduce the simplest macroscopic coherent ensemble of quantum superpositions of the following stationary states in a hydrogen atom,

$$\Psi(r, t) = \frac{1}{\sqrt{2}} \exp\left(-i\frac{m_e c^2 t}{\hbar}\right) \left[ \Psi_1(r) \exp\left(-i\frac{E_1 t}{\hbar}\right) + \exp(i\alpha) \Psi_2(r) \exp\left(-i\frac{E_2 t}{\hbar}\right) \right], \tag{86}$$

where $\Psi_1(r)$ and $\Psi_2(r)$ are the normalized wave functions of the ground state (1S) and first excited state (2S), respectively. We stress that it is possible to create the coherent superposition, where $\alpha = const$ for all macroscopic ensemble, by using lasers. It is easy to show that the superposition (86) corresponds to the following constant expectation value of energy in the absence of gravitational field,

$$< E > = m_e c^2 + \frac{E_1 + E_2}{2}. \tag{87}$$

Nevertheless, as seen from Eq.(84), the expectation value of electron active gravitational mass operator for the wave function (86) is not constant and exhibits time-dependent oscillations:

$$< \hat{m}_e^a > = m_e + \frac{E_1 + E_2}{2c^2} + \frac{V_{1,2}}{c^2} \cos \left[ \alpha + \frac{(E_1 - E_2)t}{\hbar} \right], \tag{88}$$

where $V_{1,2}$ is matrix element of the virial operator,

$$V_{1,2} = \int \Psi_1(r) \left( \frac{2}{2m_e} - \frac{e^2}{r} \right) \Psi_2(r) \ d^3r, \tag{89}$$

between the above-mentioned two stationary quantum states. It is important that the oscillations (88),(89) directly demonstrate breakdown of the equivalence between the expectation values of active gravitational mass and energy for coherent quantum superpositions of stationary states [3, 10]. We pay attention to the fact that such quantum time-dependent oscillations are very general and are not restricted by the case of a hydrogen atom. They are of a pure quantum origin and do not have classical analogs.
6.3. Experimental aspects

In this short Subsection, we suggest an idealized experiment, which allows to observe quantum time-dependent oscillations of the expectation values of active gravitational mass (88). In principle, it is possible to create a macroscopic ensemble of the coherent quantum superpositions of electron stationary states in some gas with high density. It is important that these superpositions have to be characterized by the feature that each atom (or molecule) has the same phase difference between two wave function components, $\Psi_1(r)$ and $\Psi_2(r)$. In this case, the macroscopic ensemble of the atoms (or molecules) generates gravitational field, which oscillates in time similar to Eq.(88), which, in principle, can be measured. It is important to use such geometrical distributions of the molecules and a test body, where oscillations (88) are "in phase" and, thus, do not cancel each other.

7. SUMMARY

In conclusion, in the review, we have discussed in detail breakdown of the equivalence between active and passive gravitational masses of an electron and its energy in a hydrogen atom. We stress that the considered phenomena are very general and are not restricted by atomic physics and the Earth’s gravitational field. In other words, the above discussed phenomena exist for any quantum system with internal degrees of freedom and at any gravitational field. In this review, we also have improved several drawbacks of the original pioneering works.

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[1] C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation (W.H. Freeman and Co, New York, USA, 1973).
[2] L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields, 4th edn. (Butterworth-Heinemann, Oxford, 2003).
[3] M. Zych and C. Brukner, Nature Phys. 14, (2018) 1027.
[4] G. Rosi, D’Amico, L.Cacciapuoti, F. Sorrentino, M.Prevedelli, M. Zych, C. Brukner, G.M. Tito, Nature Commun. 8, (2016) 15529.
[5] A.G. Lebed, J. Phys., Conf. Ser. 738, (2016) 012036.
[6] C.W. Misner and P. Putnam, Phys. Rev. 116, (1959) 1045.
[7] K. Nordtvedt, Class. Quantum Grav. 11, (1994) A119.
[8] S. Carlip, Am. J. Phys. 66, (1998) 409.
[9] M. Zycz, L. Rudnicki, and I. Pikovski, (2018) arXiv:1808.05831v1.
[10] A.G. Lebed, Int. J. Mod. Phys. 26, (2017) 1730022.
[11] A.G. Lebed, in Proc. 15th Marcel Grossmann Meeting on General Relativity, eds. Elia Baitstell, Robert T. Jantzen, and Remo Ruffini, Open access (World Scientific, Singapore, 2019).
[12] D. Park, Introduction to the Quantum Theory, 3rd edn. (Dover Publications, New York, USA, 2005).
[13] E. Fischbach, B.S. Freeman and W.K. Cheng, Phys. Rev. D 23, (1981) 2157.
[14] N.D. Birrell and P.C.W. Davies, Quantum Fields in Curved Space, 3rd edn. (Cambridge University Press, Cambridge, UK, 1982).