CLASSIFICATION OF EXTREMAL VERTEX OPERATOR ALGEBRAS
WITH TWO SIMPLE MODULES

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Abstract. In recent work, Wang and the second author defined a class of ‘extremal’ vertex operator algebras, consisting of those with at least two simple modules and conformal dimensions as large as possible for the central charge. In this article we completely classify the potential character vectors of extremal VOAs with two simple modules. We find 15 candidates, all but one of which is known to be realized by a VOA. We discuss the remaining potential character vector, corresponding to a VOA with central charge 33, along with its connection to a new holomorphic VOA with central charge 32. The primary tool is the theory of vector-valued modular forms, in particular the work of Bantay and Gannon.

1. Introduction

In 1988, a foundational article of Mathur, Mukhi, and Sen [MMS88] pioneered an approach for the classification of characters of rational conformal field theories using what are now called modular linear differential equations (MLDEs). The study of this classification problem, via MLDEs and other methods, has continued over the last two decades, and in recent years there has been significant activity surrounding classification of characters of chiral CFTs with two characters.

We organize this classification problem as follows. We take vertex operator algebras as a mathematical model for chiral CFTs, and for a sufficiently nice (‘strongly rational’) VOA $V$ of central charge $c$, we consider its representation category $\mathcal{C} = \text{Rep}(V)$, which is a modular tensor category [Hua08]. One can recover the equivalence class of the central charge $c \mod 8$ from $\mathcal{C}$, and motivated by this we define an admissible genus to be a pair $(\mathcal{C}, c)$ consisting of a modular tensor category and a number $c$ in the appropriate class mod 8 (c.f. [Höh03]). It is natural to approach the problem of classification of VOAs and their characters by restricting to genera where both $\mathcal{C}$ and $c$ are sufficiently small in an appropriate sense.

We will take the rank of an MTC (i.e. the number of simple objects) as a measure of its size, although other options are possible. The smallest MTC is the trivial one, Vec, and a genus $(\text{Vec}, c)$ is admissible when $c \equiv 0 \mod 8$. There are a total of three VOAs in the genera $(\text{Vec}, 8)$ and $(\text{Vec}, 16)$, all coming from lattices. The problem becomes interesting at $(\text{Vec}, 24)$, where the classification of characters of VOAs can be read off from Schellekens’ famous list [Sch93]. The classification of VOAs in $(\text{Vec}, 24)$ is almost complete, but the uniqueness of a VOA with the same character as the Moonshine VOA has not been established. At $(\text{Vec}, 32)$ the explicit classification problem of characters is already intractable, even just for the simplest examples coming from even unimodular lattices.

This difficulty propagates to the study of genera $(\mathcal{C}, c)$ with $c$ large, as for any fixed $V \in (\mathcal{C}, c)$ one obtains from every $W \in (\text{Vec}, 32)$ a new VOA $V \otimes W \in (\mathcal{C}, c + 32)$. Thus if one wishes to consider classification problems for higher central charge and rank$(\mathcal{C}) > 1$,
it is necessary to restrict to a class of VOAs which excludes VOAs like $V \otimes W$. There is a natural notion of ‘primeness’ that one could consider in this context, but we will consider something slightly different.

The twist of a simple module $M \in \text{Rep}(V)$ is given by $\theta_M = e^{2\pi i h}$, where $h$ is the lowest conformal dimension of states in $M$. Thus for any (hypothetical) VOA $V \in (\mathcal{C}, c)$, we can recover the conformal dimensions of simple objects mod 1. Moreover, there is a priori bound $[\text{MMS88, Mas07}]$ on the conformal dimensions:

\begin{equation}
\ell := \frac{n}{2} + \frac{nc}{4} - 6 \sum_{j=0}^{n-1} h_j \geq 0
\end{equation}

where $V = M_0, \ldots, M_{n-1}$ are a complete list of simple $V$ modules and $h_j$ is the lowest conformal dimension of $M_j$. Moreover, $\ell$ is an integer. A strongly rational VOA $V \in (\mathcal{C}, c)$ with $\text{rank}(\mathcal{C}) > 1$ is called extremal $[\text{TW17}]$ if $\sum h_j$ is as large as possible for $c$ in light of (1.1). This is analogous to the extremality condition introduced by Höhn for holomorphic VOAs (i.e. VOAs $V$ with $\text{Rep}(V) = \text{Vec}$) $[\text{Höhn95}]$. Since the $h_j$ are determined mod 1 by $\mathcal{C}$, extremality is equivalent to the condition $\ell < 6$.

The classification of (characters of) extremal non-holomorphic VOAs appears to be a tractable piece of the unrestricted general classification problem. In $[\text{TW17}]$, it was demonstrated that when $\text{rank}(\mathcal{C})$ is 2 or 3, then the characters of a VOA are determined by its genus, and a list of potential character vectors was obtained up to central charge 48. In this article we give a complete classification of characters of extremal VOAs $V$ with $\text{rank}(\text{Rep}(V)) = 2$, with no restriction on the central charge.

**Main Theorem.** There are 15 potential characters of strongly rational extremal (i.e. $\ell < 6$) VOAs $V$ with exactly two simple modules. These characters are listed in Table A.1.

The theorem appears in the main body of the text as Theorem 3.13. Of the 15 potential characters in Table A.1, 14 of them have been realized by VOAs. The remaining case corresponds to the genus (Semion, 33), and we strongly believe that there is an extremal VOA in this genus. In Section 3.6 we describe a strategy for constructing this VOA which was described to us by Lam and Yamauchi, but at present the question remains open. The construction would depend on a certain interesting $c = 32$ holomorphic VOA with no weight-one states. The study of holomorphic VOAs with no weight-one states is a natural extension of Monstrous Moonshine and constitutes an important frontier in the theory of VOAs. The experimentally derived $c = 32$ candidate could serve as a first test case for attempts to construct such VOAs, and such a construction would be all the more satisfying for closing the last remaining case in the classification presented in our Main Theorem.

This article fits into a recent cluster of activity regarding classification of VOAs with two simple modules (or, more generally, two characters). Just recently, Mason, Nagatomo, and Sakai $[\text{MNS18}]$ used MLDEs to establish a classification result for VOAs with two simple modules satisfying certain additional properties, in the $\ell = 0$ regime. The VOAs covered by their classification are four affine VOAs and the Lee-Yang model; our $c = 33$ candidate has $\ell = 4$.

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1We are assuming that the characters of $V$ are linearly independent, which will always hold in the situation under consideration in this article, when $V$ has exactly two simple modules
In contrast, our paper uses techniques of Bantay and Gannon [BG07, Gan14] to compute fundamental matrices for spaces of vector-valued modular forms. Our approach is computational, in that we derive an explicit recurrence between potential character vectors in the genus $(C, c)$ and those in $(C, c \pm 24)$. By studying the long-term behavior of this recurrence, we are able to obtain effective bounds on the possible central charges of extremal VOAs.

This article is an adaptation of the undergraduate thesis [Gra18] of the first author, which obtained a classification of characters for extremal VOAs with two simple modules, and which focused on the case $c, h \geq 0$. Not long after the thesis was published online, an article [CM18] in the physics literature used MLDEs to obtain a classification similar to the one presented here, without having been aware of [Gra18].

The article is organized as follows. In Section 2, we review the classification of rank two modular tensor categories and modular data from the perspective of VOAs. In Section 3.1, we review the tools from [BG07] which we will use to describe character vectors of VOAs, particularly the fundamental and characteristic matrices associated to a suitable representation of $\text{PSL}(2, \mathbb{Z})$ and a bijective exponent matrix. In Section 3.2, we derive a recurrence relation which describes how characteristic matrices change under the transformation $c \mapsto c \pm 24$. In Sections 3.3 and 3.4, we study the long-term behavior of this recurrence in the positive $c$ and negative $c$ situations, respectively, and in Section 3.5 we put these tools together to obtain our main theorem. In Section 3.6, we describe in more detail our interesting $c = 33$ candidate characters. Finally, in Appendix A we give tables of numerical data used in the proof of the main theorem, as well as all 15 extremal characters in rank two.

### 2. Rank two modular tensor categories

In this article we will consider VOAs which are simple, of CFT type, self-dual, and regular (or equivalently, rational and $C_2$-cofinite [ABD04]). For brevity, we will use the term strongly rational to describe such VOAs. We refer the reader to [DLM97, ABD04] for background on the adjectives under consideration, but we will explain here the consequences which are relevant for our work.

A strongly rational VOA $V$ possesses finitely many simple modules $V = M_0, M_1, \ldots, M_n$. We denote the category of $V$-modules by $\text{Rep}(V)$, and write $\text{rank}(\text{Rep}(V))$ for the number of simple modules $n + 1$. We will assume throughout that every module $M_j$ is self-dual, as it simplifies the exposition and is satisfied in the rank two case.

We are primarily interested in the characters of $V$,

$$
\text{ch}_j(\tau) = q^{-c/24} \sum_{n=0}^{\infty} \dim M_j(n + h_j) q^{n + h_j}
$$

where as usual $q = e^{2\pi i \tau}$, $c$ is the central charge of $V$, $h_j$ is the smallest conformal dimension occurring in $M_j$, and $M_j(n + h_j)$ is the space of states of conformal dimension $n + h_i$. The foundational work of Zhu [Zhu96] demonstrated that the characters $\text{ch}_j$ define holomorphic functions on the upper half-plane, and that their span is invariant under the action of the modular group. Thus if we set

$$
\text{ch}(\tau) = \begin{pmatrix} 
\text{ch}_0 \\
\vdots \\
\text{ch}_n
\end{pmatrix}
$$

1
there exists a representation $\rho_V : \text{PSL}(2, \mathbb{Z}) \to \text{GL}(n + 1, \mathbb{C})$ such that

$$
(2.1) \quad \text{ch}(\gamma \cdot \tau) = \rho_V(\gamma) \text{ch}(\tau)
$$

for all $\gamma \in \text{PSL}(2, \mathbb{Z})$ (recall that we assumed each $M_i$ to be self-dual). Here $\gamma \cdot \tau$ denotes the natural action of $\text{PSL}(2, \mathbb{Z})$ on the upper half-plane.

By the work of Huang ([Hua08], see also [Hua05]), $\text{Rep}(V)$ is naturally a modular tensor category, and based on Huang’s work Dong-Lin-Ng [DLN15] showed that Zhu’s modular invariance is encoded by the $S$ and $T$ matrices of $\text{Rep}(V)$ (see [EGNO15] for more detail on the $S$ and $T$ matrices of a modular tensor category). Recall that the normalization of $S$ is only canonical up to a sign, and that for each choice of $S$ the normalization of $T$ is only canonical up to a third root of unitary. By [DLN15, Thm. 3.10] (based on [Hua08]), we have

$$
(2.2) \quad \rho_V \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = e^{-2\pi i c/24} \delta_{j,k} e^{2\pi i h_j}.
$$

We now consider strongly rational VOAs $V$ such that $\text{rank}(\text{Rep}(V)) = 2$. We will sometimes write $h$ instead of $h_1$ for the non-trivial lowest conformal dimension, and similarly we will sometimes write $M$ instead of $M_1$. A complete list of normalized $S$ matrices for modular tensor categories of rank 2 is given by [RSW09, Thm. 3.1]:

$$
(2.3) \quad \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & -1 \end{pmatrix} \quad \text{and} \quad \pm \frac{1}{\sqrt{2 + \alpha}} \begin{pmatrix} 1 & \alpha \\ \alpha & -1 \end{pmatrix}
$$

where $\varepsilon^2 = 1$ and $\alpha^2 = 1 + \alpha$.

Observe that

$$
\varepsilon^{2\pi i c/24} \text{ch}(i) = \sum_{n=0}^{\infty} \text{dim} M_j(n + h_j) e^{-2\pi i (n+h_j)} > 0
$$

and thus the phase of $\text{ch}(i)$ is independent of $j$. By (2.1), $\rho \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \text{ch}(i) = \text{ch}(i)$, and thus $\rho \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$ fixes a vector all of whose entries have the same phase. This observation allows us to refine (2.3) and conclude that if $\text{rank}(\text{Rep}(V)) = 2$ then $\rho \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$ must be one of:

$$
(2.4) \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2 + \varphi}} \begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}, \quad \frac{1}{\sqrt{3 - \varphi}} \begin{pmatrix} -1 & \varphi \\ \varphi & -1 \end{pmatrix}
$$

where we use positive square roots and $\varphi = \frac{1 + \sqrt{5}}{2}$ is the golden ratio.

By the classification of [RSW09], there are exactly two modular tensor categories realizing each of 2.4 as a normalization of its $S$ matrix, and these two are related by a reversal of the braiding. Fix one of these 8 modular tensor categories $\mathcal{C}$ and its normalized $S$-matrix from (2.4). We wish to see how much information about a hypothetical $V$ with $\text{Rep}(V) = \mathcal{C}$ we may recover. By definition, the non-normalized $T$ matrix of $\mathcal{C}$ is the diagonal matrix $e^{2\pi i h_j} \delta_{j,k}$, and thus the equivalence class of $h \mod 1$ is determined by $\mathcal{C}$. Observe that if $(S, T)$ are generators of a representation of $\text{PSL}(2, \mathbb{Z})$ then $(S, \zeta T)$ again generate a representation only if $\zeta^3 = 1$, and thus from (2.2) we can see that $c \mod 8$ is determined by $\mathcal{C}$ as well.

We summarize the 8 cases in Table 2.1. Each row corresponds to a modular tensor category, giving its normalized $S$ matrix from (2.4), the equivalence classes of central charge

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\textsuperscript{2}The authors thank Terry Gannon for pointing out that such a refinement is possible.
and minimal conformal weight of a hypothetical VOA realization, as well as a familiar name for the category and a VOA realizing the category, where appropriate/known.

| # | $S$ | $c \mod 8$ | $h \mod 1$ | Name | Extremal realization |
|---|---|---|---|---|---|
| 1 | $\frac{1}{\sqrt{2}} (\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix})$ | 1 | $\frac{1}{4}$ | Semion | $SU(2)_1$ at $c = 1$ |
| 2 | $\frac{1}{\sqrt{2}} (\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix})$ | 7 | $\frac{1}{4}$ | Semion | $E_{7,1}$ at $c = 7$ |
| 3 | $\frac{1}{\sqrt{2}} (\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix})$ | -3 | $-\frac{3}{4}$ | Semion$^\dagger$ | None |
| 4 | $\frac{1}{\sqrt{2}} (\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix})$ | -5 | $-\frac{1}{4}$ | Semion$^\dagger$ | None |
| 5 | $\frac{1}{\sqrt{2} + \varphi} (\begin{smallmatrix} 1 & \varphi \\ \varphi & -1 \end{smallmatrix})$ | $\frac{14}{5}$ | $\frac{2}{5}$ | Fib | $G_{2,1}$ at $c = \frac{14}{5}$ |
| 6 | $\frac{1}{\sqrt{2} + \varphi} (\begin{smallmatrix} \varphi & -1 \\ 1 & \varphi \end{smallmatrix})$ | $\frac{26}{5}$ | $\frac{3}{5}$ | Fib | $F_{4,1}$ at $c = \frac{26}{5}$ |
| 7 | $\frac{1}{\sqrt{3} - \varphi} (\begin{smallmatrix} -1 & \varphi-1 \\ \varphi-1 & 1 \end{smallmatrix})$ | $-\frac{22}{5}$ | $-\frac{1}{5}$ | Lee-Yang | Lee-Yang at $c = -\frac{22}{5}$ |
| 8 | $\frac{1}{\sqrt{3} - \varphi} (\begin{smallmatrix} -1 & \varphi-1 \\ \varphi-1 & 1 \end{smallmatrix})$ | $-\frac{18}{5}$ | $-\frac{4}{5}$ | Lee-Yang | None |

Table 2.1. The 8 rank two modular tensor categories from the perspective of VOAs, and an extremal realization where applicable

The genus of a strongly rational VOA $V$ is the pair $(\text{Rep}(V), c)$. In [TW17], the second author and Zhenghan Wang defined an extremal (non-holomorphic) VOA to be one with $\text{rank}(\text{Rep}(V)) > 1$ and such that the minimal conformal weights $h_j$ were as large as possible in light of a certain a priori bound [MMS88, Mas07]. See [TW17, §2.2] for more detail. When $\text{rank}(\text{Rep}(V)) = 2$, then $V$ is extremal when

$$0 \leq 1 + \frac{c}{2} - 6h < 6,$$

The quantity $\ell := 1 + \frac{c}{2} - 6h$ is always a non-negative integer, and has been used frequently in the study of VOAs (e.g. [MMS88, MNS18, GHM16] among many others).

The purpose of this article is to provide a list of all possible characters of extremal VOAs with $\text{rank}(\text{Rep}(V)) = 2$. Given a rank two modular tensor category $\mathcal{C}$ and central charge $c$ in the appropriate class mod 8 (as in Table 2.1), there is a unique rational number $h_{\text{ext}}$ in the appropriate class mod 1 satisfying (2.5). When $\mathcal{C}$ is fixed we will write $h_{\text{ext}}(c)$ to emphasize the dependence on $c$.

The pair $(\mathcal{C}, c)$ of a modular tensor category and appropriate choice of $c$ is called an admissible genus [Höh03]. For every admissible genus $(\mathcal{C}, c)$ described by Table 2.1 there is a representation $\rho_c : PSL(2, \mathbb{Z}) \to U(2, \mathbb{C})$ whose $S$ matrix is given by the entry of the table, and whose $T$ matrix is obtained by rescaling the categorical $T$ matrix by $e^{-2\pi i c/24}$. These representations are simply a choice of normalization of the categorical $S$ and $T$ matrices, and their existence do not depend in any way on vertex operator algebras. However, they are defined in such a way that if there is a strongly rational VOA $V$ with central charge $c$ and $\text{Rep}(V) = \mathcal{C}$, then $\rho_V = \rho_c$.

3. Characters of VOAs with two simple modules

3.1. Characters and vector-valued modular forms. The work of Bantay and Gannon [BG07, Gan14] on vector-valued modular forms provides powerful tools for studying the characters of vertex operator algebras. We briefly recall the points which will be most
important for us, and refer the reader to these references, especially [BG07, §2], for more detail.

Let $\rho : \text{PSL}(2, \mathbb{Z}) \to \text{GL}(d, \mathbb{C})$ be an irreducible representation of the modular group, and assume that $\rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is diagonal with finite order. Let $X : \mathbb{H} \to \mathbb{C}$ be a holomorphic function on the upper half-plane which satisfies

$$X(\gamma \cdot \tau) = \rho(\gamma)X(\tau)$$

for all $\gamma \in \text{PSL}(2, \mathbb{Z})$ and $\tau \in \mathbb{H}$. Choose a diagonal matrix $\Lambda$ such that $\rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = e^{2\pi i \Lambda}$, called an exponent matrix. For any choice of exponent matrix, we may Fourier expand

$$q^{-\Lambda}X(q) = \sum_{n \in \mathbb{Z}} X[n]q^n$$

for coefficients $X[n] \in \mathbb{C}^d$. Let $M(\rho)$ denote the space of functions $X$ satisfying (3.1) such that $X[n] = 0$ for $n$ sufficiently negative (observe that this does not depend on the choice of $\Lambda$).

Given a choice of exponent $\Lambda$, we define the principal part map

$$\mathcal{P}_\Lambda : M(\rho) \to \text{span}\{vq^{-n} : n > 0, v \in \mathbb{C}^d\}$$

by

$$\mathcal{P}_\Lambda X = \sum_{n < 0} X[n]q^n$$

where $X[n]$ are as in (3.2).

An exponent matrix is called bijective if $\mathcal{P}_\Lambda$ is an isomorphism. For $\xi \in \{1, \ldots, d\}$ let $e_\xi \in \mathbb{C}^d$ be the corresponding standard basis vector. Given a choice of bijective exponent matrix, let $X^{(\xi)} \in M(\rho)$ be the function with $\mathcal{P}_\Lambda X^{(\xi)} = q^{-1}e_\xi$. In this case, $X^{(1)}, \ldots, X^{(d)}$ form a basis for $M(\rho)$ as a free $\mathbb{C}[[J]]$-module, where

$$J = q^{-1} + 196884q + \cdots$$

is the $J$-invariant. The fundamental matrix $\Xi$ is given by

$$\Xi = [X^{(1)} | \cdots | X^{(d)}].$$

The characteristic matrix $\chi$ is given by the constant terms of $\Xi$ taken in the $q$-expansion (shifted by $\Lambda$ as in (3.2)). That is,

$$\chi = [X^{(1)}[0] | \cdots | X^{(d)}[0]].$$

Now fix as in Section 2 a modular tensor category $\mathcal{C}$ of rank two, and a choice of real number $c$ in the appropriate class mod 8. From this data we specified a representation $\rho_c$ of $\text{PSL}(2, \mathbb{Z})$ with the property that if there exists a VOA $V$ with central charge $c$ and $\text{Rep}(V) = \mathcal{C}$, then its character vector $(\text{ch}_j)$ satisfies $(\text{ch}_j) \in M(\rho_c)$. The key observation of [TW17] is that

$$\Lambda(c) = \begin{pmatrix} 1 - \frac{c}{24} & 0 \\ 0 & h_{\text{ext}}(c) - \frac{c}{24} \end{pmatrix} = \begin{pmatrix} \lambda_0(c) & 0 \\ 0 & \lambda_1(c) \end{pmatrix}$$

is a bijective exponent for $\rho_c$, where $h_{\text{ext}}$ is the real number lying in the appropriate class mod 1 which satisfies (2.5). Thus by the definition of fundamental matrix we have:
**Theorem 3.1** ([TW17, Thm. 3.1]). Let $\mathcal{C}$ be a modular tensor category of rank two, and let $c$ be a real number in the appropriate class mod 8. If $V$ is an extremal VOA with central charge $c$ and rank(Rep($V$)) = $\mathcal{C}$, then its character appears as the first column of the fundamental matrix corresponding to the bijective exponent $\Lambda(c)$.

Let $\chi(c) = (\chi(c)_{ij})_{i,j=0}^{1}$ be the characteristic matrix taken with respect to $\Lambda(c)$. Thus if $V$ is an extremal VOA with central charge $c$ (and rank(Rep($V$)) = 2) we have $\chi(c)_{00} = \dim V(1)$. We will determine the possible values of $c$ for which there exists an extremal VOA by showing that for $|c|$ sufficiently large, one of $\chi(c)_{00}$ or $\chi(c)_{10}$ is not a non-negative integer.

### 3.2. General recurrence.

They key idea [Gra18] is to derive a recurrence relating the pair $(\chi(c + 24), h_{\text{ext}}(c + 24))$ to $(\chi(c), h_{\text{ext}}(c))$, and then study the long-term behavior of this recurrence. In fact, to handle the case $c \to +\infty$, one may derive a simple recurrence involving only the diagonal entries of $\chi$ [Gra18, Lem. 6.4]. To handle the case $c \to -\infty$ we will use all of the entries of $\chi$, and the relation will be slightly more complicated as a result.

Let $M_{2 \times 2}$ be the set of $2 \times 2$ complex matrices whose bottom-left entry is non-zero, and let $M_{2 \times 2}^\pm$ be the set of matrices whose top-right entry is non-zero. Define functions

$$f_\pm : M_{2 \times 2}^\pm \times (\mathbb{R} \setminus \mathbb{Z}) \to M_{2 \times 2}^\pm \times (\mathbb{R} \setminus \mathbb{Z})$$

by

$$f_+ \left[ \begin{pmatrix} x & y \\ z & w \end{pmatrix}, h \right] = \begin{pmatrix} w+h(x-240) \\ (h+1)(h+2) \\ x+h(w+240) \end{pmatrix}$$

and

$$f_- \left[ \begin{pmatrix} x & y \\ z & w \end{pmatrix}, h \right] = \begin{pmatrix} -w+(h-2)(x+240) \\ h(h-3)(h-4) \\ -x+(h-2)(w-240) \end{pmatrix}.$$

By direct computation, one may check that these functions are invertible and $f_+^{-1} = f_-$. We will show that $f_\pm$ take characteristic matrices to characteristic matrices, but first we must check:

**Lemma 3.2.** Let $\chi$ be the fundamental matrix corresponding to a $2 \times 2$ bijective exponent $\Lambda$. Then $\chi \in M_{2 \times 2}^+ \cap M_{2 \times 2}^-$.  

**Proof.** We show $\chi \in M_{2 \times 2}^+$ and the other step is similar. Let $X = X^{(1)}$ be the first column of the fundamental matrix $\Xi$ corresponding to $\Lambda = (\lambda_0 \ 0 \ \lambda_1)$. If the bottom-left entry of $\chi$ were 0, this would imply that $X$ was of the form

$$X = \begin{pmatrix} q^{\lambda_0} (q^{-1} + \cdots) \\ q^{\lambda_1} (z_1 q + \cdots) \end{pmatrix}$$

for some $z_1 \in \mathbb{C}$. By [Gan14, Thm. 4.1], $\Lambda' = (\lambda_0 - 1 \ 0 \ \lambda_1 + 1)$ is again a bijective exponent. But examining (3.3) we see that $\mathcal{P}_{\Lambda'} X = 0$, which is a contradiction. \qed

**Lemma 3.3.** Let $(\mathcal{C}, c)$ be an admissible genus from Table 2.1, and let $\chi(c)$ denote the characteristic matrix of the representation $\rho$, taken with respect to the bijective exponent $\Lambda(c)$. Then $[\chi(c + 24), h_{\text{ext}}(c + 24)] = f_\pm [\chi(c), h_{\text{ext}}(c)]$. 


Proof. It is clear from (2.5), which characterizes $h_{\text{ext}}$, that $h_{\text{ext}}(c + 24) = h_{\text{ext}}(c) \pm 2$, and by examining Table 2.1 we see that $h_{\text{ext}}(c)$ is never an integer. Since $f_{\pm}^{-1} = f_{\mp}$ it suffices to show that $[\chi(c + 24), h_{\text{ext}}(c + 24)] = f_{+}[\chi(c), h_{\text{ext}}(c)]$.

Let $h = h_{\text{ext}}(c)$. By definition we have

\[
\Lambda := \Lambda(c) = \begin{pmatrix} 1 - \frac{c}{24} & 0 \\ 0 & h - \frac{c}{24} \end{pmatrix}
\]

and

\[
\Lambda_{+} := \Lambda(c + 24) = \begin{pmatrix} -\frac{c}{24} & 0 \\ h + 1 - \frac{c}{24} & 0 \end{pmatrix} = \Lambda(c) + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let $\Xi := \Xi(c)$ and $\Xi_{+} := \Xi(c + 24)$ be the fundamental matrices corresponding to the bijective exponents $\Lambda$ and $\Lambda_{+}$, respectively, for the representation $\rho_{c} = \rho_{c+24}$. We may expand

\[
\Xi = \begin{pmatrix} \mathbf{X}^{(1)} & \mathbf{X}^{(2)} \end{pmatrix} = q^{\Lambda} \left( q^{-1} + \sum_{n \geq 0} x_{n} q^{n} \sum_{n \geq 0} y_{n} q^{n} \right)
\]

and

\[
\Xi_{+} = \begin{pmatrix} \mathbf{X}^{(1)}_{+} & \mathbf{X}^{(2)}_{+} \end{pmatrix} = q^{\Lambda_{+}} \left( q^{-1} + \sum_{n \geq 0} x_{n}^{+} q^{n} \sum_{n \geq 0} y_{n}^{+} q^{n} \right).
\]

Our goal is to show that

\[
f_{+} \left[ \begin{pmatrix} x_{0} & y_{0} \\ z_{0} & w_{0} \end{pmatrix}, h \right] = \begin{pmatrix} x_{0}^{+} & y_{0}^{+} \\ z_{0}^{+} & w_{0}^{+} \end{pmatrix}, h + 2 \right].
\]

To do this, we must obtain formulas for $x_{0}^{+}, y_{0}^{+}, z_{0}^{+}$ and $w_{0}^{+}$ in terms of $x_{0}, y_{0}, z_{0},$ and $w_{0}$.

Using (3.4), we have

\[
\Xi_{+} = q^{\Lambda} \left( q^{-2} + \frac{q^{-1} + \cdots}{z_{0}^{+} q^{n}} \right).
\]

Thus

\[
\mathcal{P}_{\Lambda} \mathbf{X}^{(1)}_{+} = \left( q^{-2} + x_{0}^{+} q^{-1} \right) \quad \text{and} \quad \mathcal{P}_{\Lambda} \mathbf{X}^{(2)}_{+} = \left( y_{0}^{+} q^{-1} \right).
\]

Since $\mathcal{P}_{\Lambda}$ is injective, we have

\[
\mathbf{X}^{(2)}_{+} = y_{0}^{+} \mathbf{X}^{(1)}.
\]

By (3.6), this identity reads

\[
\left( \begin{array}{c} y_{0}^{+} q^{-1} + \cdots \\ 1 + \cdots \end{array} \right) = \left( \begin{array}{c} y_{0}^{+} q^{-1} + \cdots \\ y_{0}^{+} z_{0} + \cdots \end{array} \right)
\]

and thus

\[
y_{0}^{+} = \frac{1}{z_{0}}.
\]

Note that $z_{0} \neq 0$ by Lemma 3.2. This gives the formula for $y_{0}^{+}$ which corresponds to (3.5).

Substituting (3.8) into (3.7) yields $\mathbf{X}^{(2)}_{+} = \frac{1}{z_{0}} \mathbf{X}^{(1)}$, from which we conclude

\[
w_{0}^{+} = \frac{z_{1}}{z_{0}}.
\]
by considering the second \( q \)-coefficient in the second entry of \( X_+^{(2)} \). While this gives an expression for \( w_0^+ \) in terms of \( z \) variables, it is not the expression we are looking for due to the presence of the higher order coefficient \( z_1 \). We will derive an expression for \( z_1 \) in terms of lower order coefficients later in the proof (3.15).

For now, we continue on and find expressions for \( x_0^+ \) and \( z_0^+ \). By direct calculation,

\[
P_\Lambda ((J + x_0^+ - x_0)X^{(1)} - z_0X^{(2)}) = \left( \frac{q^{-2} + x_0^+q^{-1}}{0} \right) = P_\Lambda X^{(1)}_+
\]

where \( J(q) = q^{-1} + 196884q + \cdots \). Since \( P_\Lambda \) is injective, we have

(3.10) \((J + x_0^+ - x_0)X^{(1)} - z_0X^{(2)} = X^{(1)}_+ \).

We now multiply both sides of (3.10) by \( q^{-\Lambda} \), expanding out the left-hand side and substituting the expression from (3.6) for the right-hand side, to obtain

\[
\left( \frac{q^{-2} + x_0^+q^{-1} + \cdots}{\gamma_0 + \gamma_1q + \cdots} \right) = \left( \frac{q^{-2} + x_0^+q^{-1} + \cdots}{0 + z_0^+q + \cdots} \right)
\]

where

\[
\gamma_0 = (x_0^+ - w_0 - x_0)z_0 + z_1, \quad \text{and} \quad \gamma_1 = 196884z_0 - w_1z_0 - x_0z_1 + x_0^+z_1 + z_2.
\]

Thus \( \gamma_0 = 0 \) and \( \gamma_1 = z_0^+ \). The former yields

(3.11) \( x_0^+ = x_0 + w_0 - \frac{z_1}{z_0} \).

Substituting (3.11) into the equation \( z_0^+ = \gamma_1 \) and simplifying yields

(3.12) \( z_0^+ = -\frac{z_1^2}{z_0} + z_1w_0 + z_0(196884 - w_1) + z_2 \).

Our aim now is to replace the higher order coefficients \( w_1, z_1, \) and \( z_2 \) appearing in (3.9), (3.11) and (3.12) with expressions in terms of \( x_0, y_0, z_0, \) and \( w_0 \). To do this, we use the differential equation derived by Bantay-Gannon [BG07, Eqn. (2.14)]

(3.13) \[
\frac{1}{2\pi\iota} \frac{dE}{d\tau} - \Xi(\tau)D(\tau) = 0
\]

where

\[
D(\tau) = \frac{1}{E(\tau)} [(J(\tau) - 240)(\Lambda - 1) + \chi + [\Lambda, \chi]]
\]

for \( E(\tau) = q^{-1} - 240 - 141444q - \cdots \). Examining the coefficient of \( q \) in the bottom-right entry of (3.13) yields

(3.14) \[
w_1 = \frac{1}{2}(w_0(w_0 + 240) - (h - 2)y_0z_0 + 338328(h - 1 - \frac{c}{24})).
\]

Similarly, examining the coefficients of \( q \) and \( q^2 \) in the bottom-left entry of (3.13) yields

(3.15) \[
z_1 = \frac{x_0 + h(w_0 + 240)}{h + 1}z_0
\]

and

(3.16) \[
z_2 = \frac{x_0(z_1 + 240z_0) + z_0(hw_1 + 240hw_0 + 199044h - 338328\frac{c}{24})}{h + 2}
\]
respectively.

We have now obtained expressions (3.14), (3.15), and (3.16) for \( w_1, z_1, \) and \( z_2 \) in terms of lower order coefficients. We may substitute these formulas into our expressions (3.9), (3.11), and (3.12) for \( w_0^+, x_0^+, \) and \( z_0^+ \) to obtain the formulas of (3.5). Combining with our earlier expression (3.8) for \( y_0^+ \) completes the proof.

\[ \square \]

3.3. Recurrence for large positive \( c \). We will show that for \( n \) sufficiently large, \( \chi(c + 24n)_{00} < 0 \), and moreover we will obtain an effective bound on such an \( n \). We will do this by iterating \( f_+ \), although in fact a simpler function will suffice.

**Lemma 3.4.** Let \( g : \mathbb{C}^2 \times (\mathbb{R} \setminus \mathbb{Z}) \to \mathbb{C}^2 \times (\mathbb{R} \setminus \mathbb{Z}) \) be the function

\[
g[x, w, h] = \left[ \frac{w + h(x - 240)}{h + 1}, \frac{x + h(w + 240)}{h + 1}, h + 2 \right],
\]

and let \( g^n \) denote its \( n \)-fold iterate. Then

\[
g^n[x, w, h] = \left[ \frac{nw + (h + n - 1)(x - 240n)}{h + 2n - 1}, \frac{nx + (h + n - 1)(w + 240n)}{h + 2n - 1}, h + 2n \right].
\]

**Proof.** This follows by a straightforward induction. \( \square \)

**Lemma 3.5.** Let \((\mathbb{C}, c)\) be an admissible genus from Table 2.1, and let \( \chi(c) \) denote the characteristic matrix of the representation \( \rho \) with respect to the bijective exponent \( \Lambda(c) \). Suppose that \( h_{\text{ext}}(c) > 0 \). Then \( \chi(c + 24n)_{00} < 0 \) when

\[
n > |M| + \sqrt{M^2 + 960 |(h_{\text{ext}}(c) - 1)\chi(c)_{00}|}
\]

where \( M = \chi(c)_{00} + \chi(c)_{11} - 240(h_{\text{ext}}(c) - 1) \).

**Proof.** Set \( a = \chi(c)_{00}, d = \chi(c)_{11}, \) and \( h = h_{\text{ext}}(c) \). By Lemma 3.3 and Lemma 3.4, we have

\[
\chi(c + 24n)_{00} = \frac{nd + (h + n - 1)(a - 240n)}{h + 2n - 1}.
\]

Since we assume \( h > 0 \), when \( n \geq 0 \) we have \( h + 2n - 1 > 0 \). Thus \( \chi(c + 24n)_{00} < 0 \) if and only if

\[
0 > nd + (h + n - 1)(a - 240n) = -240n^2 + Mn + (h - 1)\chi_{00}.
\]

The right-hand side of (3.17) is a quadratic polynomial in \( n \) which is concave down. Thus (3.17) holds when \( n \) exceeds the largest real root of that quadratic (and it holds trivially if the quadratic has no real roots). The conclusion of the lemma now follows immediately from the quadratic formula. \( \square \)

The purpose of Lemma 3.5 is to reduce the question of classifying extremal VOAs to a finite one. We apply it 24 times to obtain the following.
Theorem 3.6. For every rank two modular tensor category \( C \), there is an explicitly computable \( c_{\max} \) such that there are no extremal VOAs in the genus \((C,c)\) when \( c > c_{\max} \). The values are given in the following table, and the numbering of categories is the same as Table 2.1.

\[
\begin{array}{|c|c|c|}
\hline
\# & C & c_{\max} \\
\hline
1 & \text{Semion} & 57 \\
2 & \text{Semion} & 39 \\
3 & \text{Semion}^\dagger & 67 \\
4 & \text{Semion}^\dagger & 37 \\
5 & \text{Fib} & 174 \\
6 & \text{Fib} & 186 \\
7 & \text{Lee-Yang} & \frac{338}{5} \\
8 & \text{Lee-Yang} & \frac{222}{5} \\
\hline
\end{array}
\]

(3.18)

Proof. Let us first take \( C \) to be the Semion MTC. In this case, \( c \equiv 1 \mod 8 \). We consider first the case \( c \equiv 1 \mod 24 \). For \( c = 1 \), we can compute the characteristic matrix \( \chi(1) = \begin{pmatrix} 3 & 26752 \\ 2 & -247 \end{pmatrix} \), for example using the method of [TW17] (based on [BG07]). We can compute \( h_{\text{ext}}(1) = 14 \) from the definition of \( h_{\text{ext}} \) and the fact that \( h \equiv 14 \mod 1 \). Applying Lemma 3.5 with this data, we see that \( \chi(1 + 24n) < 0 \) when \( n > 0 \).

Thus if \( n_{\max} = 0 \), we have \( \chi(1 + 24n) < 0 \) when \( n > n_{\max} \). By Theorem 3.1, there are no extremal VOAs in the genera \((C,1 + 24n)\) when \( n > n_{\max} \).

We can repeat the above exercise for the values \( c = 9 \) and \( c = 17 \), and three times again for each row of Table 2.1. The resulting characteristic matrices, \( h_{\text{ext}} \), and \( n_{\max} \) are given in Table A.2. For each category \( C \), the value \( c_{\max} \) in the table (3.18) is the maximum of the three values of \( c + 24n_{\max} \), corresponding to the three possible classes of \( c \mod 24 \). \( \square \)

3.4. Recurrence for very negative \( c \). We will show that for \( n \) sufficiently large, we have \( |\chi(c - 24n)_{10}| < 1 \). Since \( \chi(c - 24n)_{10} \neq 0 \) by Lemma 3.2, this will guarantee that \( \chi(c - 24n)_{10} \) is not an integer. As with the case of very positive \( c \), we will avoid finding an explicit expression for \( f_{\pm}[\chi(c), h_{\text{ext}}(c)] \). Instead, we extract the following pieces of the data which will be easier to work with. Let \( \alpha(c) = \chi(c)_{10} - \chi(c)_{11} \) and \( \beta(c) = \chi(c)_{10} \chi(c)_{01} \).

The utility of studying \( \beta(c) \) is the following.

Lemma 3.7. Let \((C,c)\) be an admissible genus from Table 2.1, and suppose that \( |\chi(c)_{10}| \leq 1 \) and \( |\beta(c - 24n)| > 1 \) for all \( n \geq 1 \). Then \( |\chi(c - 24n)_{10}| < 1 \) for all \( n \geq 0 \).

Proof. By Lemma 3.3 we have \( \chi(c - 24)_{10} = \chi(c)_{01}^{-1} \). Thus

\[
\beta(c) = \chi(c)_{10} \chi(c)_{01} = \frac{\chi(c)_{10}}{\chi(c - 24)_{10}}.
\]

Thus if we know that \( |\chi(c)_{10}| \leq 1 \) and \( |\beta(c - 24)| > 1 \), we can conclude that \( |\chi(c - 24)_{10}| < 1 \). We repeat this argument \( n \) times to complete the proof. \( \square \)
The see how \( \beta(c) \) depends on \( c \), we introduce the function

\[
k : \mathbb{C}^2 \times (\mathbb{R} \setminus \mathbb{Z}) \to \mathbb{C}^2 \times (\mathbb{R} \setminus \mathbb{Z})
\]
given by

\[
k[\alpha, \beta, h] = \left[ \frac{\alpha(h-1) + 480(h-2)}{h-3}, \frac{(h-3)^2(h\beta - 746496) + (\alpha + 120(h-1))^2}{(h-4)(h-3)^2}, h-2 \right].
\]

This function was chosen so that:

**Lemma 3.8.** Let \((\mathbb{C}, c)\) be an admissible genus from Table 2.1. Then

\[\alpha(c - 24), \beta(c - 24), h_{\text{ext}}(c - 24) = k[\alpha(c), \beta(c), h_{\text{ext}}(c)].\]

**Proof.** This follows by direct algebraic manipulation applied to Lemma 3.3 and the formula for \( f_- \).

It is now an algebra exercise to determine the long-term behavior of \( \alpha(c - 24n) \) and \( \beta(c - 24n) \).

**Lemma 3.9.** The \( n \)-fold iterate of \( k \) is given by

\[
k^n[\alpha, \beta, h] = \left[ \frac{\alpha(h-1) + 480n(h - n - 1)}{h-2n - 1}, \frac{n(h - n - 1)(\alpha + 120(h-1))^2}{(h-2n)(h-2n-1)^2(h-2n-2)} + \frac{h(h-2)\beta - 746496n(h-n-1)}{(h-2n)(h-2n-2)}, h-2n \right].
\]

**Proof.** The formula may be verified by a straightforward induction using the definition of \( k \).

We carefully examine the expression obtained in Lemma 3.9 to obtain a criterion to bound \( |\beta(c + 24n)| > 1 \).

**Lemma 3.10.** Let \((\mathbb{C}, c)\) be an admissible genus from Table 2.1, and suppose that \( h_{\text{ext}}(c) < 0 \) and \( \beta(c) > 1 \). Then \( |\beta(c - 24n)| > 1 \) whenever

\[
n > \frac{|\alpha(c) - 120(1 - h_{\text{ext}}(c))| (1 - h_{\text{ext}}(c))}{860}.
\]

**Proof.** Let \( \beta_n = \beta(c - 24n) \), which by Lemma 3.8 and Lemma 3.9 is given by the formula

\[
\beta_n = \frac{n(h - n - 1)(\alpha + 120(h-1))^2}{(h-2n)(h-2n-1)^2(h-2n-2)} + \frac{h(h-2)\beta - 746496n(h-n-1)}{(h-2n)(h-2n-2)}.
\]

where \( \alpha = \alpha(c) \) and \( h = h_{\text{ext}}(c) < 0 \).

We can write \( \beta_n = \frac{p(n)}{q(n)} \) for \( q(n) = (h - 2n)(h - 2n-1)^2(h-2n-2) \) and \( p \) a certain polynomial of \( n \). To show \( |\beta_n| > 1 \), it suffices to show that \( |p(n)| > |q(n)| \). Since \( q(n) > 0 \) by inspection when \( n \geq 1 \), it suffices to show that \( p(n) > q(n) \), or that \( r(n) := p(n) - q(n) > 0 \).
Through straightforward manipulation of the formula for $\beta_n$, we have $r(n) = r_1(n) + r_2(n)$ where

$$r_1(n) = 2985968n^4 + 5971936(1 - h)n^3 + (3732456(1 - h)^2 + 4)n^2 + (746488(1 - h)^3 + 4(1 - h))n + 4(-h)n(2 - h)(1 - h + n) + (-h)(2 - h)(\beta - 1)(1 - h + 2n)^2$$

$$r_2(n) = -(1 - h + n)n(\alpha - 120(1 - h))^2$$

Since $h < 0$ and $\beta > 1$, every term of $r_1(n)$ is positive. Thus to show $r(n) > 0$ it suffices to find a term of $r_1(n)$ which controls $r_2(n)$.

To show

$$2985968n^4 + r_2(n) > 0,$$

it suffices to show

$$2985968n^4 > (1 - h + n)^2(\alpha - 120(1 - h))^2.$$ 

This will follow from the simple estimate Lemma 3.11 below with $A = 2985968$, $B = (\alpha - 120(1 - h))^2$, and $C = 1 - h$, provided

$$2985968n^2 > 2(\alpha - 120(1 - h))^2(1 + (1 - h)^2).$$

This would follow from

$$2985968n^2 > 4(\alpha - 120(1 - h))^2(1 - h)^2,$$

or equivalently

$$(2985968)^{1/2}n > 2|\alpha - 120(1 - h)|(1 - h).$$

This is an immediate consequence of our assumption (3.19). □

We used the following simple observation in the proof of Lemma 3.10.

**Lemma 3.11.** Let $A, B, C,$ and $n$ be positive real numbers with $n \geq 1$. Then if

$$An^2 > 2B(1 + C^2)$$

it follows that

$$An^4 > B(n + C)^2.$$ 

**Proof.** It suffices to show $An^4 > 2B(n^2 + C^2)$, or equivalently $(An^2 - 2B)n^2 > 2BC^2$. Instead, we may show $An^2 - 2B > 2BC^2$ since $n \geq 1$ and $An^2 > 2B$. This follows immediately from our hypothesis. □

We now apply Lemma 3.10 in 24 cases to obtain a lower bound on the central charge of extremal VOAs.
Theorem 3.12. For every rank two modular tensor category $\mathcal{C}$, there is an explicitly computable $c_{\text{min}}$ such that there are no extremal VOAs in the genus $(\mathcal{C}, c)$ when $c < c_{\text{min}}$. The values are given in the following table. The numbering of categories is the same as Table 2.1.

| # | $\mathcal{C}$       | $c_{\text{min}}$ |
|---|---------------------|-------------------|
| 1 | Semion             | $-23$             |
| 2 | Semion             | $-17$             |
| 3 | Semion$^\dagger$   | $-13$             |
| 4 | Semion$^\dagger$   | $-19$             |
| 5 | Fib                | $-\frac{196}{5}$ |
| 6 | Fib                | $-\frac{94}{5}$  |
| 7 | Lee-Yang           | $-\frac{62}{5}$  |
| 8 | Lee-Yang           | $-\frac{98}{5}$  |

(3.20)

Proof. As in the proof of Theorem 3.6, we will work through the necessary computation when $\mathcal{C} = \text{Semion}$ and obtain a bound which holds for $c \equiv 1 \mod 24$. Since $h_{\text{ext}}(1) > 0$, we must instead consider $c = -23$ in order to apply Lemma 3.10. We compute $h_{\text{ext}}(-23) = -\frac{7}{4}$ using (2.5), and we compute

$$
\chi(-23) = \left( \frac{713}{11}, \frac{57264144384}{29762}, \frac{3397}{11} \right)
$$

and from there $\alpha(-23) = \frac{4110}{11}$ and $\beta(-23) = \frac{23346112}{121}$. Thus by Lemma 3.10 we have $|\beta(-23 - 24n)| > 1$ when $n > 0.13\ldots$. Taking $n_{\text{max}} = 0$, we have $|\beta(-23 - 24n)| > 1$ when $n > n_{\text{max}}$. As $\left|\chi(-23 - 24n_{\text{max}})_{10}\right| < 1$, we conclude that $\left|\chi(-23 - 24n)_{10}\right| < 1$ for all $n > n_{\text{max}}$, and thus by Theorem 3.1 there cannot be an extremal VOA in the genus $(\mathcal{C}, c)$ when $c < -23$ and $c \equiv 1 \mod 24$. We repeat this argument for the other two equivalence classes of $c \mod 24$, and the value $c_{\text{min}}$ from (3.20) is the minimum of the allowed values.

We apply the above procedure to each of the 8 modular categories appearing in Table 2.1. The data from each of the cases is given in Table A.3. □
3.5. **Main result.** Combining Theorem 3.6 and Theorem 3.12, we obtain for every rank two modular tensor category \( \mathcal{C} \) a pair of numbers \( c_{\text{min}} \) and \( c_{\text{max}} \) such that if \( V \) is an extremal VOA in the genus \( (\mathcal{C}, c) \), then \( c_{\text{min}} \leq c \leq c_{\text{max}} \). We can now compute the characteristic matrix of every remaining pair \( (\mathcal{C}, c) \) (e.g. by Lemma 3.3), and throw away any for which the first column does not consist of positive integers.

**Theorem 3.13.** Let \( V \) be a strongly rational extremal VOA with two simple modules. Then it lies in one of the following genera (and its character vector is given in Table A.1).

| \( \mathcal{C} \) | \( c \) | Realization? | \( h_{\text{ext}} \) | \( \ell \) |
|---------------------|---------|--------------|----------------|------|
| Semion 1            | 1       | \( SU(2)_1 \) | \( \frac{1}{4} \) | 0    |
| Semion 9            | \( SU(2)_1 \otimes E_{8,1} \) | \( \frac{5}{3} \) | 2    |
| Semion 17           | ?       | \( SU(2)_1 \otimes E_{8,1} \) | \( \frac{5}{3} \) | 4    |
| Semion 33           | ?       | \( SU(2)_1 \otimes E_{8,1} \) | \( \frac{5}{3} \) | 4    |
| Semion 7            | \( E_{7,1} \) | \( \frac{5}{3} \) | 0    |
| Semion 15           | \( E_{7,1} \otimes E_{8,1} \) | \( \frac{5}{3} \) | 4    |
| Semion 23           | [GHM16] | \( \frac{5}{3} \) | 2    |

| \( \mathcal{C} \) | \( c \) | Realization? | \( h_{\text{ext}} \) | \( \ell \) |
|---------------------|---------|--------------|----------------|------|
| Fib 14              | \( \frac{14}{5} \) | \( SU(2)_1 \) | \( \frac{2}{5} \) | 0    |
| Fib 54              | \( \frac{54}{5} \) | \( SU(2)_1 \otimes E_{8,1} \) | \( \frac{2}{5} \) | 4    |
| Fib 94              | \( \frac{94}{5} \) | [GHM16] | \( \frac{7}{5} \) | 2    |
| Fib 26              | \( \frac{26}{5} \) | \( F_{4,1} \) | \( \frac{3}{5} \) | 0    |
| Fib 66              | \( \frac{66}{5} \) | \( F_{4,1} \otimes E_{8,1} \) | \( \frac{3}{5} \) | 4    |
| Fib 106             | \( \frac{106}{5} \) | [GHM16] | \( \frac{8}{5} \) | 2    |
| Lee-Yang 22         | \( \frac{22}{5} \) | \( SU(2)_1 \otimes E_{8,1} \) | \( \frac{1}{5} \) | 0    |
| Lee-Yang 18         | \( \frac{18}{5} \) | \( SU(2)_1 \otimes E_{8,1} \) | \( \frac{1}{5} \) | 4    |

3.6. **The next monster?** There is exactly one set of potential characters for an extremal VOA with two simple modules which has not yet been realized, which corresponds to the \( (\text{Semion}, c = 33) \) row in Theorem 3.13:

\[
q^{-33/24} \left( \frac{1 + 3q + 86004q^2 + \cdots}{q^{\frac{3}{2}}(565760 + 192053760q + \cdots)} \right).
\]

We strongly believe that a VOA \( V \) with this character vector exists. Such a \( V \) would satisfy \( \dim V(1) = 3 \), and we expect that the corresponding affine subVOA \( W \subset V \) is \( SU(2)_1 \). The coset \( W^c \) has central charge 32, and \( \text{Rep}(W^c) \) would have four simple objects \( W^c = M_0, M_1, M_2, \) and \( M_3 \) with \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) fusion rules. Applying the Bantay-Gannon techniques, one can compute the character vector

\[
(ch_{M_j})_{j=0}^3 = q^{-32/24} \left( \frac{1 + 0q + 69616q^2 + 34668544q^3 + \cdots}{q^{9/4}(426192 + 121366368q + \cdots)q^{7/4}(10245 + 11330970q + \cdots)q^{2}(69888 + 34664448q + \cdots)} \right).
\]

We would have \( V = W \otimes W^c \oplus N \otimes M_2 \), where \( N \) is the non-trivial simple \( W \)-module.

We now describe a strategy for constructing \( W^c \) which was described to us by Ching-Hung Lam and Hiroshi Yamauchi. In order to find the \( c = 32 \) VOA \( W^c \), it may be easier to look for its holomorphic extension \( \tilde{V} = W^c \oplus M_3 \). This VOA would have character vector

\[
\text{ch}_{\tilde{V}} = q^{-32/24}(1 + 0q + 139504q^2 + 69332992q^3 + \cdots).
\]

This is akin to Evans and Gannon’s suggestion that a (not yet realized) Haagerup VOA might find a natural home inside the Moonshine VOA with \( c = 24 \) [EG11, §5.2.2]. Lam and
Yamauchi propose to construct such a $\tilde{V}$ by first finding another $c = 32$ holomorphic VOA $\tilde{V}'$ with $\dim \tilde{V}'(1) = 0$, along with an involution $\theta \in \text{Aut}(\tilde{V}')$ such that the unique $\theta$-twisted $\tilde{V}'$-module has lowest conformal dimension $7/4$. The associated simple current extension of the $\theta$-invariant vectors is then $\tilde{V}$. It appears to be possible that $\tilde{V}'$ can be constructed by a twisted orbifold of the rank 32 Barnes-Wall lattice, taking advantage of the theory of framed VOAs [LY08], but that is unclear at this point.

One difficulty in this approach is the fact that $\dim \tilde{V}(1) = 0$. The most powerful tools for the theory of existence and uniqueness of VOAs rely on a large weight-one space, so that the VOA can be controlled by its affine (‘classical’) part. This is exemplified by the fact that the only remaining case in the classification of $c = 24$ holomorphic VOAs is to establish that there is a unique realization of the Monster VOA character.

Thus we find ourselves in a position analogous to the early days of Monstrous Moonshine, with the benefit of modern context. We have the character vector of a proposed holomorphic VOA $\tilde{V}$ with $\dim \tilde{V}(1) = 0$, now with $c = 32$. We also have the suggestion that it can be built (indirectly) out of an interesting unimodular lattice. It is an important challenge to develop the theory of holomorphic VOAs with no weight-one states, and we believe that this example provides a small, but interesting, candidate to use as a test case.
As in [TW17] we can compute potential characters vectors for each of the allowed genera of Theorem 3.13. The realizations labelled [GHM16] arise as cosets of affine VOAs. 

| C    | c  | Realization? | Character |
|------|----|--------------|-----------|
| Semion | 1  | $SU(2)_1$    | $q^{-1/24} \left( 1 + 3q + 4q^2 + \cdots \right)$  |
| Semion | 9  | $SU(2)_1 \otimes E_{8,1}$ | $q^{-9/24} \left( 1 + 251q + 4872q^2 + \cdots \right)$  |
| Semion | 17 | [GHM16]       | $q^{-17/24} \left( 1 + 323q + 60860q^2 + \cdots \right)$  |
| Semion | 33 | ?             | $q^{-33/24} \left( 1 + 3q + 86004q^2 + \cdots \right)$  |
| Semion | 7  | $E_{7,1}$    | $q^{-7/24} \left( 1 + 133q + 1673q^2 + \cdots \right)$  |
| Semion | 15 | $E_{7,1} \otimes E_{8,1}$ | $q^{-15/24} \left( 1 + 381q + 38781q^2 + \cdots \right)$  |
| Semion | 23 | [GHM16]       | $q^{-23/24} \left( 1 + 69q + 131905q^2 + \cdots \right)$  |
| Fib  | $\frac{14}{5}$ | $G_{2,1}$   | $q^{-7/60} \left( 1 + 14q + 42q^2 + \cdots \right)$  |
| Fib  | $\frac{54}{5}$ | $G_{2,1} \otimes E_{8,1}$ | $q^{-27/60} \left( 1 + 262q + 7638q^2 + \cdots \right)$  |
| Fib  | $\frac{94}{5}$ | [GHM16]   | $q^{-47/60} \left( 1 + 188q + 62087q^2 + \cdots \right)$  |
| Fib  | $\frac{26}{5}$ | $F_{4,1}$   | $q^{-13/60} \left( 1 + 52q + 377q^2 + \cdots \right)$  |
| Fib  | $\frac{66}{5}$ | $F_{4,1} \otimes E_{8,1}$ | $q^{-33/60} \left( 1 + 300q + 17397q^2 + \cdots \right)$  |
| Fib  | $\frac{106}{5}$ | [GHM16]   | $q^{-33/60} \left( 1 + 106q + 84429q^2 + \cdots \right)$  |
| Lee-Yang | $-\frac{22}{5}$ | Lee-Yang | $q^{11/60} \left( 1 + 0q + q^2 + \cdots \right)$  |
| Lee-Yang | $\frac{18}{5}$ | L-Y $\otimes E_{8,1}$ | $q^{-3/20} \left( 1 + 248q + 4125q^2 + \cdots \right)$  |

**Table A.1.** Characters of strongly rational extremal VOAs with two simple modules
The following table (which was used in the proof of Theorem 3.6) has 24 rows, in groups of 3. Each group corresponds to a modular tensor category from Table 2.1, and each row within the group selects a representative of an equivalence class of admissible $c \mod 24$. The table provides the characteristic matrix (computed as in [TW17]), the extremal conformal dimension $h_{\text{ext}}(c)$ computed from the definition (2.5), and a value $n_{\text{max}}$ such that $\chi(c + 24n)_{00} < 0$ when $n > n_{\text{max}}$, as computed using Lemma 3.5.

| # | $\mathcal{C}$ | $c$ | $\chi(c)$ | $h_{\text{ext}}(c)$ | $n_{\text{max}}$ |
|---|---|---|---|---|---|
| 1 | Semion | 1 | $\begin{pmatrix} 3 & 26752 \\ 2 & -247 \end{pmatrix}$ | $\frac{1}{3}$ | 0 |
| 1 | Semion | 9 | $\begin{pmatrix} 251 & 26752 \\ 2 & 1 \end{pmatrix}$ | $\frac{1}{3}$ | 2 |
| 1 | Semion | 17 | $\begin{pmatrix} 323 & 88 \\ 1632 & -319 \end{pmatrix}$ | $\frac{5}{3}$ | 0 |
| 2 | Semion | 7 | $\begin{pmatrix} 133 & 1248 \\ 56 & -377 \end{pmatrix}$ | $\frac{3}{4}$ | 0 |
| 2 | Semion | 15 | $\begin{pmatrix} 381 & 1248 \\ 56 & -129 \end{pmatrix}$ | $\frac{3}{4}$ | 1 |
| 2 | Semion | 23 | $\begin{pmatrix} 69 & 10 \\ 32384 & -65 \end{pmatrix}$ | $\frac{7}{4}$ | 0 |
| 3 | Semion$^\dagger$ | 11 | $\begin{pmatrix} -319 & 1632 \\ 88 & 323 \end{pmatrix}$ | $\frac{3}{4}$ | 0 |
| 3 | Semion$^\dagger$ | 19 | $\begin{pmatrix} -247 & 2 \\ 26752 & 3 \end{pmatrix}$ | $\frac{7}{4}$ | 2 |
| 3 | Semion$^\dagger$ | 27 | $\begin{pmatrix} 1 & 2 \\ 26752 & 251 \end{pmatrix}$ | $\frac{7}{4}$ | 0 |
| 4 | Semion$^\dagger$ | 5 | $\begin{pmatrix} -65 & 32384 \\ 10 & 69 \end{pmatrix}$ | $\frac{1}{3}$ | 0 |
| 4 | Semion$^\dagger$ | 13 | $\begin{pmatrix} -377 & 56 \\ 1248 & 133 \end{pmatrix}$ | $\frac{5}{4}$ | 1 |
| 4 | Semion$^\dagger$ | 21 | $\begin{pmatrix} -129 & 56 \\ 1248 & 381 \end{pmatrix}$ | $\frac{5}{4}$ | 0 |
| 5 | Fib | $\frac{14}{5}$ | $\begin{pmatrix} 14 & 12857 \\ 7 & -258 \end{pmatrix}$ | $\frac{3}{5}$ | 0 |
| 5 | Fib | $\frac{54}{5}$ | $\begin{pmatrix} 262 & 12857 \\ 7 & -10 \end{pmatrix}$ | $\frac{3}{5}$ | 1 |
| 5 | Fib | $\frac{94}{5}$ | $\begin{pmatrix} 188 & 46 \\ 4794 & -184 \end{pmatrix}$ | $\frac{7}{5}$ | 0 |
| 6 | Fib | $\frac{26}{5}$ | $\begin{pmatrix} 52 & 3774 \\ 26 & -296 \end{pmatrix}$ | $\frac{3}{5}$ | 0 |
| 6 | Fib | $\frac{66}{5}$ | $\begin{pmatrix} 300 & 3774 \\ 26 & -48 \end{pmatrix}$ | $\frac{3}{5}$ | 1 |
| 6 | Fib | $\frac{106}{5}$ | $\begin{pmatrix} 106 & 17 \\ 15847 & -102 \end{pmatrix}$ | $\frac{8}{5}$ | 0 |
Table A.2. Values of $n_{\text{max}}$ computed from Lemma 3.5

| # | $\mathfrak{C}$ | $c$ | $\chi(c)$ | $h_{\text{ext}}(c)$ | $\alpha(c)$ | $\beta(c)$ | $n_{\text{max}}$ | $\chi_{10}(c)$ |
|---|---|---|---|---|---|---|---|---|
| 7 | Lee-Yang | $\frac{58}{5}$ | \[\begin{pmatrix} 902 \\ 87 \\ 410 \end{pmatrix} \] | $\frac{4}{5}$ | 0 |
| 7 | Lee-Yang | $\frac{98}{5}$ | \[\begin{pmatrix} 26999 \\ 1 \\ 1 \end{pmatrix} \] | $\frac{2}{5}$ | 2 |
| 7 | Lee-Yang | $\frac{138}{5}$ | \[\begin{pmatrix} 3 \\ 26999 \\ 249 \end{pmatrix} \] | $\frac{2}{5}$ | 0 |
| 8 | Lee-Yang | $\frac{22}{5}$ | \[\begin{pmatrix} 32509 \\ 59 \end{pmatrix} \] | $\frac{1}{5}$ | 1 |
| 8 | Lee-Yang | $\frac{62}{5}$ | \[\begin{pmatrix} 57 \\ 190 \end{pmatrix} \] | $\frac{6}{5}$ | 1 |
| 8 | Lee-Yang | $\frac{102}{5}$ | \[\begin{pmatrix} 57 \\ 438 \end{pmatrix} \] | $\frac{6}{5}$ | 1 |

The following table (which was used in the proof of Theorem 3.12) again has 24 rows, in the same groups of 3. Each group corresponds to a modular tensor category from Table 2.1, and each row within the group selects a representative of an equivalence class of admissible $c$ mod 24. The table provides the characteristic matrix (computed as in [TW17]), the extremal conformal dimension $h_{\text{ext}}(c)$ computed from the definition (2.5), $\beta(c)$ and $\alpha(c)$ computed directly from the characteristic matrix. Using this data, we apply Lemma 3.10 to obtain a value $n_{\text{max}}$ such that $|\beta(c - 24n)| > 1$ when $n > n_{\text{max}}$. In fact, $n_{\text{max}} = 0$ in all cases.
| #  | \(c\) | \(\chi(c)\) | \(h_{ext}(c)\) | \(\alpha(c)\) | \(\beta(c)\) | \(n_{max}\) | \(\chi_{10}(c)\) |
|----|------|------------|-------------|-------------|-------------|-------------|-------------|
| 3  | Semion\(^\dagger\) | -13 | \(\frac{299}{3} \quad \frac{1827924480}{3} \quad -287\) | \(-\frac{5}{4}\) | \(\frac{586}{3}\) | \(\frac{1521920}{9}\) | 0 | \(\frac{1}{1632}\) |
| 4  | Semion\(^\dagger\) | -3 | \(\frac{1857}{7} \quad \frac{83232768}{7} \quad -93\) | \(-\frac{3}{4}\) | \(\frac{1950}{7}\) | \(\frac{10404096}{49}\) | 0 | \(\frac{1}{56}\) |
| 4  | Semion\(^\dagger\) | -11 | \(\frac{121}{7} \quad \frac{827924480}{7} \quad -1829\) | \(-\frac{3}{4}\) | \(\frac{1950}{7}\) | \(\frac{10404096}{49}\) | 0 | \(\frac{1}{56}\) |
| 4  | Semion\(^\dagger\) | -19 | \(\frac{1501}{11} \quad \frac{62591041536}{11} \quad -1457\) | \(-\frac{7}{4}\) | \(\frac{2958}{11}\) | \(\frac{21260544}{121}\) | 0 | \(\frac{1}{32384}\) |
| 5  | Fib | -26 | \(\frac{91}{7} \quad \frac{13051833}{7} \quad -83\) | \(-\frac{8}{5}\) | 87 | \(\frac{567471}{4}\) | 0 | \(\frac{1}{46}\) |
| 5  | Fib | -66 | \(\frac{32712244109}{1} \quad \frac{32712244109}{1} \quad -690\) | \(-\frac{8}{5}\) | 4656 | \(\frac{33076081}{169}\) | 0 | \(\frac{1}{12857}\) |
| 5  | Fib | -106 | \(\frac{742}{13} \quad \frac{32712244109}{13} \quad -3914\) | \(-\frac{8}{5}\) | 4656 | \(\frac{33076081}{169}\) | 0 | \(\frac{1}{12857}\) |
| 6  | Fib | -14 | \(\frac{26}{17} \quad \frac{1951158}{17} \quad -22\) | \(-\frac{2}{5}\) | 48 | 114774 | 0 | \(\frac{1}{17}\) |
| 6  | Fib | -54 | \(\frac{295}{1} \quad \frac{745916226}{1} \quad -43\) | \(-\frac{7}{5}\) | 338 | \(\frac{3359983}{17}\) | 0 | \(\frac{1}{3774}\) |
| 6  | Fib | -94 | \(\frac{47}{1774} \quad \frac{745916226}{1774} \quad -291\) | \(-\frac{7}{5}\) | 338 | \(\frac{3359983}{17}\) | 0 | \(\frac{1}{3774}\) |
| 7  | Lee-Yang | 18 | \(\frac{248}{1} \quad \frac{310124}{1} \quad -4\) | \(-\frac{1}{5}\) | 244 | 310124 | 0 | 1 |
| 7  | Lee-Yang | -22 | \(\frac{0}{1} \quad \frac{310124}{1} \quad -244\) | \(-\frac{1}{5}\) | 244 | 310124 | 0 | 1 |
| 7  | Lee-Yang | -62 | \(\frac{1054}{11} \quad \frac{1667924403}{11} \quad -1010\) | \(-\frac{6}{5}\) | 2064 | \(\frac{40681083}{242}\) | 0 | \(\frac{1}{902}\) |
| 8  | Lee-Yang | -18 | \(\frac{802}{3} \quad \frac{35954954}{3} \quad -46\) | \(-\frac{4}{5}\) | 848 | \(\frac{1892366}{9}\) | 0 | \(\frac{1}{57}\) |
| 8  | Lee-Yang | -58 | \(\frac{58}{3} \quad \frac{35954954}{3} \quad -790\) | \(-\frac{4}{5}\) | 848 | \(\frac{1892366}{9}\) | 0 | \(\frac{1}{57}\) |
| 8  | Lee-Yang | -98 | \(\frac{140}{3} \quad \frac{5726299516}{3} \quad -136\) | \(-\frac{9}{5}\) | 276 | \(\frac{3346756}{19}\) | 0 | \(\frac{1}{32384}\) |

Table A.3. Values of \(n_{max}\) computed from Lemma 3.10
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