Entanglement Spectrum as a Generalization of Entanglement Entropy: Identification of Topological Order in Non-Abelian Fractional Quantum Hall Effect States

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We study the “entanglement spectrum” (a presentation of the Schmidt decomposition analogous to a set of “energy levels”) of a many-body state, and compare the Moore-Read model wavefunction for the \( \nu = 5/2 \) fractional quantum Hall state with a generic 5/2 state obtained by finite-size diagonalization of the second-Landau-level-projected Coulomb interactions. Their spectra share a common “gapless” structure, related to conformal field theory. In the model state, these are the only levels, while in the “generic” case, they are separated from the rest of the spectrum by a clear “entanglement gap”, which appears to remain finite in the thermodynamic limit. We propose that the low-lying entanglement spectrum can be used as a “fingerprint” to identify topological order.

There has been increasing recent interest in using quantum entanglement as a probe to detect topological properties of many-body quantum states [1–4, 11], in particular the states exhibiting the fractional quantum Hall effect (FQHE) [5, 6, 7, 8, 9, 10, 11, 12, 13]. Among various measures of quantum entanglement, the entanglement entropy has by far been the favorite [12, 13]. Among the various measures of quantum entanglement, the entanglement spectrum is also “non-generic”, as it contains far fewer levels than the number of “energies”. The MR entanglement spectrum is also essentially a “fingerprint” which allows the associated CFT (which characterizes the topological order) to be identified.

If Landau-level mixing is ignored, the \( \nu = 5/2 \) FQHE system is equivalent to a \( \nu = 1/2 \) lowest Landau level with interaction pseudo-potentials [11] corresponding to a simple Coulomb interaction projected into the second Landau level (we did not refine this to include the quantum well form-factor, which may be important for quantitative modeling). In spherical geometry [11], the number of electrons (\( N_e \)) and the total number of Landau level orbitals (\( N_{\text{orb}} \)) are related by \( N_{\text{orb}} = 2N_e - 2 \). In second-quantization, a many-body state can be written in the basis of orbital occupations; we used the \( L_e \) eigenstate basis to divide the spherical surface at a line of latitude into two regions, so the \( N_{\text{orb}} \) orbitals are partitioned into \( N_{\text{orb}}^A \) around the north pole, and \( N_{\text{orb}}^B \) around the south pole (this is spatially the sharpest cut consistent with projection into a Landau level), which is a partition of the Fock space \( \mathcal{H} \) into two parts \( \mathcal{H}_A \otimes \mathcal{H}_B \).

A Schmidt decomposition (equivalent to the singular value decomposition of a matrix) of a many-body state \( |\psi\rangle \) gives

\[
|\psi\rangle = \sum_i e^{-\frac{1}{2}\xi_i} |\psi_A^i\rangle \otimes |\psi_B^i\rangle
\]

(1)

where \( \exp(-\frac{1}{2}\xi_i) \geq 0, \langle \psi_A^i | \in \mathcal{H}_A, |\psi_B^i\rangle \in \mathcal{H}_B \), and \( \langle \psi_A^i | \psi_B^j \rangle = \delta_{ij} \) giving the singular values and \( |\psi_A^i\rangle \) and \( |\psi_B^j\rangle \) the singular vectors. If the state is normalized, \( \sum_i \exp(-\xi_i) = 1 \), but it is not necessary to impose the normalization condition.

The \( \xi_i \)‘s are “energy levels” of a system with thermodynamic entropy at “temperature” \( T = 1 \) equivalent to the entanglement entropy, \( S = \sum_i \xi_i \exp(-\xi_i) \), which
TABLE I: The numbers in the parenthesis are values of \((N_{\text{orb}}, N_e)\), respectively for each system and partitioning as specified.

| \(N_e\) | \(P[0|0]\) | \(P[0|1]\) | \(P[1|1]\) |
|--------|--------|--------|--------|
| 10     | (7, 4) | (8, 4) | (9, 5) |
| 12 or 14 | (11, 6) | (12, 6) | (13, 7) |
| 16     | (15, 8) | (16, 8) | (17, 9) |
versa. However, the electron density anywhere on the
sphere must remain constant, which can be achieved if
the quasihole excitations in A and B are correlated (en-
tangled). This gives the empirical rules of counting the
levels. Take the spectrum in Figure 1(a) as an example.

The partitioning $P[0|0]$ results in the root configuration
$110011001100101010010$. The counting for the levels at small $\Delta L$ for $P[0|0]$ and $P[1|1]$ can be obtained similarly.

For an infinite system in the thermodynamic limit, the
above idea gives an empirical counting rule of the number
of levels at any $\Delta L$, i.e., it is the number of independent
quasihole excitations upon the semi-infinite root config-
uration uniquely defined by the partitioning. For a finite
system, this rule explains the counting for small $\Delta L$; for large $\Delta L$, the finite size limits the maximal an-
gular momentum that can be carried by an individual
quasihole. Therefore the number of levels at large $\Delta L$
in a finite system will be smaller than the number ex-
pected in an infinite system. Not only is this empirical
rule consistent with all our numerical calculation, but
it also explains why $P[0|0]$ and $P[0|1]$ have essentially
identical low-lying structures. This is because the (semi-
infinite) configuration $\ldots 110010101010101010010$ is essentially equiva-
lent to $\ldots 1100110010100110010001010010$ (with an extra “0” attached to the right). We expect that $P[0|0]$ and $P[0|1]$ become exactly identical in the thermodynamic limit.

For completeness, we list the root configurations asso-
ciated with the first few low-lying levels in Figure 1(c).

$$
\begin{align*}
\Delta L &= 0 : & 110011001100101010010 \\
\Delta L &= 1 : & 110011001100100110010 \\
\Delta L &= 2 : & 110011001100100110010 \\
\Delta L &= 3 : & 110011001100100110010 \\
\end{align*}
$$

Figure 2 shows the spectra of the system of the same
size as in Figure 1, i.e., $N_e = 16$ and $N_{orb} = 30$, but for
the ground state of the Coulomb interaction projected
into the second Landau level, obtained by direct diag-
onalization. Interestingly, the low-lying levels have the
same counting structure as the corresponding Moore-
Read case. We identify these low-lying levels as the
“CFT” part of the spectrum, in contrast to the other
generic, non-CFT levels that are expected for generic
many-body states. At relatively small $\Delta L$ (up to a limit
which grows with the size of the system), the CFT lev-
els are separated from the generic levels by a clear gap,
which we define as the distance from the average of the
CFT levels to the bottom of the generic levels.
Figure 3 shows the value of this gap (at $\Delta L = 0, 1, 2$ respectively) as a function of the size of the system, based on which we speculate that the gap between the CFT and non-CFT levels remains finite in the thermodynamic limit for all $\Delta L$. The observed fact that the structure of the low-lying spectrum is essentially identical to that of the Moore-Read state, as well as the existence of the “entanglement gap”, serve as evidence that the “realistic” $\nu = 5/2$ FQH states is indeed modeled by the Moore-Read state.

Assuming that the gap does remain finite in the thermodynamic limit, characterization of the entanglement spectrum is a reliable way to identify a topologically ordered state. While finite-size numerical studies often show impressive (e.g., 99%) overlaps between model wave-functions (Laughlin, Moore-Read, etc.) and “realistic” states at intermediate system sizes, this cannot persist in the thermodynamic limit. Furthermore, the entanglement spectrum is a property of the ground state wave-function itself, as oppose to the physical excitations of a system with boundaries, so allows direct comparison between model states and physical ones.

The asymptotic behavior of the characters of a CFT (the count of independent states at each Virasoro level, in each sector) defines both the effective conformal anomaly $\tilde{c}$ of the CFT, and the quantum dimension of each sector. To the extent that there is a clear separation of the “gapless”, low-lying CFT-like modes and the generic (but “gapped”) modes, one can count the number of “gapless” modes as a function of momentum parallel to the boundary separating the two regions. For a finite-size system, these will match the CFT characters up to some limit that grows with system size. These numbers are integers, so are not subject to numerical error, and in principle, both $\tilde{c}$ and the quantum dimensions can be extracted from their behavior as the system size grows.

As a critical point is approached, the “entanglement gap” may still be finite but well below the “temperature” $T = 1$ at which the von Neumann entropy is evaluated. We suggest that the direct study of the low-lying entanglement spectrum is a far more meaningful way to characterize bipartite entanglement. Equivalently, the $T \to 0$ “low-temperature” entropy of the modified family of density matrices $\tilde{\rho}^{(1/T)}$ may prove useful, as this corresponds to the thermodynamic entropy of the entanglement spectrum at temperature $T$.

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