Equivalences among the chi measure, Hoffman constant, and Renegar’s distance to ill-posedness

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Abstract

We show the equivalence among the following three condition measures of a full column rank matrix $A$: the chi measure, the signed Hoffman constant, and the signed distance to ill-posedness. The latter two measures are constructed via suitable collections of matrices obtained by flipping the signs of some rows of $A$. Our results provide a procedure to estimate $\chi(A)$ thereby opening an avenue to identify classes of linear programs solvable in polynomial time in the real model of computation.

1 Introduction

We establish new equivalences among three types of condition measures of a matrix that play central roles in numerical linear algebra and in convex optimization: the chi measure [3, 7, 9, 31, 32], the Hoffman constant [15, 17, 19, 37], and Renegar’s distance to ill-posedness [29, 30]. We recall the definitions of these quantities in Section 2 below.

Let $A \in \mathbb{R}^{m \times n}$ be a full column rank matrix. The chi measure $\chi(A)$ arises naturally in weighted least-squares problems of the form $\min \| D^{1/2} (Ax - b) \|^2$, see, e.g., [4, 9, 10, 18]. The chi measure $\chi(A)$ is also a key component in the analysis of Vavasis and Ye’s interior-point algorithm for linear programming [23, 36]. A remarkable feature of Vavasis and Ye’s algorithm is its sole dependence on the matrix $A$ defining the primal and dual constraints.

The Hoffman constant $H(A)$ is associated to Hoffman’s Lemma [15, 17], a fundamental error bound for systems of linear constraints of the form $Ax \leq b$. The Hoffman constant and other similar error bounds are used to establish the convergence rate of a wide variety of optimization algorithms [2, 14, 16, 20, 22, 24, 26, 37, 37]. Renegar’s distance to ill-posedness $R(A)$ is a pillar for the concept of condition number in optimization introduced by Renegar in the seminal articles [29, 30] and subsequently extended in a number of articles [1, 5, 8, 11–13].

Our work is inspired by several relationships among $\chi(\cdot)$, $H(\cdot)$, and $R(\cdot)$ previously established in [6, 8, 27, 34, 35, 39]. In particular, it is known that if $A \in \mathbb{R}^{m \times n}$ is full column

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rank, then $\chi(A) \geq H(A)$ and if $Ax < 0$ is feasible then $H(A) = 1/R(A)$. However, $\chi(A)$ can be arbitrarily larger than $H(A)$ (see, e.g., [27]). Also, the equivalence between $\chi(A)$ and $1/R(A)$ breaks down when $Ax < 0$ is infeasible. Our main result (Theorem 1) shows that the lack of equivalence among these quantities can be rectified by considering signed versions of $H(\cdot)$ and $R(\cdot)$. In hindsight our equivalences are somewhat natural because $\chi(A)$ does not change when the signs of some rows of $A$ are flipped whereas both $H(A)$ and $R(A)$ evidently do. We show that $\chi(A)$ is exactly the largest $H(\tilde{A})$ over the collection of matrices $\tilde{A}$ obtained by flipping the signs of some rows of $A$. We also show that when all rows of $A$ are non-zero, $1/\chi(A)$ is the same as the smallest $R(A)$ over the collection of all matrices $\tilde{A}$ obtained by flipping the signs of some rows of $A$ so that $Ax < 0$ is feasible. Furthermore, we show that $\chi(A)$ is the same as $H(A)$ for the matrix $A$ obtained by stacking the rows of $A$ and $-A$. The latter equivalence together with the algorithmic machinery recently developed in [27] provides a procedure to compute or estimate $\chi(A)$. That computational ability in turn offers the potential to identify classes of linear programs that are solvable in polynomial time in the real model of computation via Vavasis-Ye’s interior-point algorithm [23, 36], since the number of arithmetic operations of Vavasis-Ye’s algorithm is polynomial on the dimensions of $A$ and on $\log(\tilde{\chi}(A)))$ for a variant $\tilde{\chi}(A)$ of $\chi(A)$.

Some of our equivalences are reminiscent of results previously developed by Tunçel [34] and by Todd, Tunçel, and Ye [33] to compare a variant $\tilde{\chi}(A)$ of $\chi(A)$ and Ye’s condition measure [35] for polyhedra of the form $\{A^Ty : y \geq 0, \|y\|_1 = 1\}$.

## 2 Definition of $\chi(\cdot)$, $H(\cdot)$, and $R(\cdot)$

Let $A \in \mathbb{R}^{m \times n}$ have full column rank. The chi measure of $A$ is defined as

$$\chi(A) = \sup \{\|(A^T \text{Diag}(d)A)^{-1} A^T \text{Diag}(d)\| : d \in \mathbb{R}_+^m\}.$$ 

In this expression and throughout the paper, $\text{Diag}(d) \in \mathbb{R}^{m \times m}$ denotes the diagonal matrix whose vector of diagonal entries is $d \in \mathbb{R}^m$. Also, we write $\| \cdot \|$ to denote the canonical Euclidean norms in $\mathbb{R}^m$ and $\mathbb{R}^n$, and the corresponding induced operator norm (or equivalently the spectral norm) in $\mathbb{R}^{m \times n}$. The underlying space will always be clear from the context. Several authors [3, 7, 31, 32] independently showed that $\chi(A)$ is finite as long as $A$ is full column rank. See [9] for a detailed discussion.

Let $A \in \mathbb{R}^{m \times n}$. The Hoffman constant $H(A)$ of $A$ is defined as

$$H(A) = \sup \left\{ \frac{\text{dist}(u, P_A(b))}{\|(Au - b)\|} : b \in A(\mathbb{R}^n) + \mathbb{R}_+^m \text{ and } u \not\in P_A(b) \right\},$$

where $P_A(b) := \{ x \in \mathbb{R}^n : Ax \leq b \}$ and $\text{dist}(u, P_A(b)) = \min\{\|u - x\| : x \in P_A(b)\}$. Hoffman [17] showed that $H(A)$ is always finite. Other proofs of this fundamental result can be found in [15, 27, 37].

Let $A \in \mathbb{R}^{m \times n}$ be such that $Ax < 0$ is feasible. Renegar’s distance to ill-posedness of $A$ is defined as

$$R(A) := \inf \{\|\Delta A\| : (A + \Delta A)x < 0 \text{ is infeasible}\}.$$ 

Renegar introduced the distance to ill-posedness as a main building block to develop the concept of condition number for optimization problems [29, 30].
The following proposition, which recalls properties previously established in [19, 27, 28, 39], is our starting point.

**Proposition 1.** Let $A \in \mathbb{R}^{m \times n}$. If $A$ has full column rank then

$$\chi(A) \geq H(A).$$

(1)

On the other hand, if $Ax < 0$ is feasible then

$$H(A) = \frac{1}{R(A)}.$$  

(2)

### 3 Equivalences among $\chi(\cdot), H(\cdot), \text{and } R(\cdot)$

Let $A \in \mathbb{R}^{m \times n}$. The following two collections $\mathbb{S}(A)$ and $\mathbb{D}(A)$ of signed matrices associated to $A$ play a central role in our main developments. Let

$$\mathbb{S}(A) := \{\text{Diag}(d)A : d \in \{-1, 1\}^m\},$$

and

$$\mathbb{D}(A) := \{\hat{A} \in \mathbb{S}(A) : \hat{A}x < 0 \text{ is feasible}\}.$$

We are now ready to state our main result.

**Theorem 1.** Let $A \in \mathbb{R}^{m \times n}$ have full column rank. Then

$$\chi(A) = \max_{\hat{A} \in \mathbb{S}(A)} H(\hat{A}) = H(A),$$

(3)

where $A \in \mathbb{R}^{2m \times n}$ is the matrix obtained by stacking $A$ and $-A$, that is, $A = \begin{bmatrix} A \\ -A \end{bmatrix}$.

If in addition all rows of $A$ are nonzero then

$$\chi(A) = \max_{\hat{A} \in \mathbb{D}(A)} H(\hat{A}) = \max_{\hat{A} \in \mathbb{D}(A)} \frac{1}{R(\hat{A})}.$$  

(4)

The identity (4) in Theorem 1 has the following natural extension when some rows of $A$ are zero. Given $A \in \mathbb{R}^{m \times n}$, let $A \in \mathbb{R}^{\tilde{m} \times n}$ denote the submatrix of $A$ obtained by dropping the zero rows from $A$. If $A \in \mathbb{R}^{m \times n}$ has full column rank then so does $\hat{A}$ and Theorem 1 implies that

$$\chi(A) = \chi(\hat{A}) = \max_{\hat{A} \in \mathbb{D}(\hat{A})} H(\hat{A}) = \max_{\hat{A} \in \mathbb{D}(\hat{A})} \frac{1}{R(\hat{A})}.$$  

(5)

The identity (5) in turn suggests an extension of $\chi(\cdot)$ to general (not necessarily full rank) matrices and general (not necessarily Euclidean) norms since both $H(\cdot)$ and $R(\cdot)$ are defined in full generality and satisfy (2).

The proof of Theorem 1 relies on the two key building blocks stated as Proposition 2 and Proposition 3 below. We will use the following convenient notation. For a positive integer
m, let \([m]\) denote \(\{1, \ldots, m\}\). For \(A \in \mathbb{R}^{m \times n}\) and \(J \subseteq [m]\), we let \(A_J \in \mathbb{R}^{J \times n}\) denote the submatrix of \(A\) defined by the rows indexed by \(J\).

The first key building block for the proof of Theorem 1 is the following characterization of \(\chi(\cdot)\) from [2]. The same characterization is also stated and proved in [39] by adapting a technique from [33].

**Proposition 2.** Let \(A \in \mathbb{R}^{m \times n}\) have full column rank. Then

\[
\chi(A) = \max_{J \subseteq [m], |J| = n} \max_{A_J \text{ non-singular}} \|v\|.
\]

The second building block for the proof of Theorem 1 is the following characterization of \(H(\cdot)\) discussed in [27] but that can be traced back to [19, 37, 39].

**Proposition 3.** Let \(A \in \mathbb{R}^{m \times n}\). Then

\[
H(A) = \max_{J \subseteq [m], |J| = n} \max_{A_J \text{ non-singular}} \max_{v \in \mathbb{R}_+^J, \|A_J v\| = 1} \|v\| = \max_{\hat{A} \in \mathcal{S}(A)} \max_{v \in \mathbb{R}_+^m, \|A_J v\| = 1} \|v\|,
\]

where \(\mathcal{J}(A) = \{J \subseteq [m] : A_J x < 0 \text{ is feasible}\}\) and \(\mathcal{J}(A) \subseteq \mathcal{J}(A)\) is the collection of maximal sets in \(\mathcal{J}(A)\).

**Proof of Theorem 1.** Let \(J\) and \(v\) be optimal for the characterization of \(\chi(A)\) in Proposition 2. Then for \(d = \text{sign}(v) \in \{-1, 1\}^m\) and \(\hat{A} := \text{Diag}(d)A \in \mathcal{S}(A)\) Proposition 3 implies that

\[
H(\hat{A}) \geq \|v\| = \chi(A).
\]

On the other hand, the construction of \(\chi(A)\) and Proposition 1 imply that for all \(\hat{A} \in \mathcal{S}(A)\)

\[
\chi(A) = \chi(\hat{A}) \geq H(\hat{A}).
\]

Thus the first identity in (3) follows. To prove the second identity in (3), notice that \(J \subseteq [2m]\) is such that \(|J| = n\) and \(A_J\) non-singular if and only if there exists \(I \subseteq [m]\) such that \(|I| = n\), \(A_I\) is non-singular, and \(J = I_+ \cup (m + I_-)\) for some partition \(I = I_+ \cup I_-\) of \(I\). If \(d \in \{-1, 1\}^m\) satisfies \(d_i = 1, i \in I_+\) and \(d_i = -1, i \in I_-\) then

\[
\max_{v \in \mathbb{R}_+^I, \|A_J v\| = 1} \|v\| = \max_{v \in \mathbb{R}_+^I, \|A_J v\| = 1} \|v\|,
\]

Hence Proposition 3 implies that

\[
H(A) = \max_{J \subseteq [2m], |J| = n} \max_{A_J \text{ non-singular}} \max_{v \in \mathbb{R}_+^J, \|A_J v\| = 1} \|v\| = \max_{A \in \mathcal{S}(A)} \max_{J \subseteq [m], |J| = n} \max_{A_J \text{ non-singular}} \max_{v \in \mathbb{R}_+^J, \|A_J v\| = 1} \|v\| = \max_{A \in \mathcal{S}(A)} H(\hat{A}).
\]

The second identity in (3) thus follows.

The crux of the proof of (3) is the following one-to-one correspondence between \(\mathcal{J}(A)\) and \(\mathbb{D}(A)\).
Claim. Suppose all rows of $A$ are nonzero. Then $J \in \mathcal{J}(A)$ if and only if $J = ([m] \setminus I) \cup (m + I)$ for some $I \subseteq [m]$ such that $\hat{A} \in \mathcal{D}(A)$ where $\hat{A}$ is the matrix obtained by flipping the signs of the rows of $A$ indexed by $I$.

This claim, Proposition 3 and Proposition 1 imply that

$$H(A) = \max_{J \in \mathcal{J}(A)} \max_{v \in \mathbb{R}^m} \frac{\|v\|}{\|A_J^Tv\|} = \max_{\hat{A} \in \mathcal{D}(A)} H(\hat{A}) = \max_{\hat{A} \in \mathcal{D}(A)} \frac{1}{\mathcal{R}(\hat{A})}. \quad (6)$$

The third step follows from Proposition 3 and the fact that $\mathcal{J}(\hat{A}) = \{[m]\}$ if $\hat{A}x < 0$ is feasible. Identity (4) follows from (6) and (3).

To finish, here is a proof of the above claim. For $u \in \mathbb{R}^n$ let $J_u := \{j : A_ju < 0\}$. Observe that $J \in \mathcal{J}(A)$ if and only if $J \subseteq J_u$ for some $u \in \mathbb{R}^n$. Since all rows of $A$ are nonzero, it follows that $J \in \mathcal{J}(A)$ if and only if $J = J_u$ for some $u \in \mathbb{R}^n$ such that all entries of $Au$ are non-zero. When the latter holds, we have $J_u = ([m] \setminus I_u) \cup (m + I_u)$ for $I_u = \{i : A_iu > 0\}$, and $A_{[m] \setminus I_u}u < 0$, $A_{I_u}u > 0$ which is equivalent to $\hat{A} \in \mathcal{D}(A)$ where $\hat{A}$ is the matrix obtained by flipping the signs of the rows of $A$ indexed by $I_u$.

\begin{proof}

\end{proof}

4 Conclusion

We showed that if $A \in \mathbb{R}^{m \times n}$ has full column rank and nonzero rows then

$$\chi(A) = \max_{\hat{A} \in \mathcal{S}(A)} H(\hat{A}) = \max_{\hat{A} \in \mathcal{D}(A)} H(\hat{A}) = \max_{\hat{A} \in \mathcal{D}(A)} \frac{1}{\mathcal{R}(\hat{A})} = H(A), \quad (7)$$

where $A \in \mathbb{R}^{2m \times n}$ is the matrix obtained by stacking the rows of $A$ and $-A$. The first expression in (7) takes the maximum over the collection of matrices $\mathcal{S}(A)$ which has exponential size in $m$. The second and third expressions in (7) take the maximum over the smaller but harder to describe collection of matrices $\mathcal{D}(A)$. By contrast, the last expression in (7) is the Hoffman constant of the single matrix $A \in \mathbb{R}^{2m \times n}$. The identity $\chi(A) = H(A)$ and the machinery developed in [27] provide a novel algorithmic procedure to compute or estimate $\chi(A)$. This computational capability in turn creates an avenue to identify families of linear programs that are solvable in polynomial time in the real model of computation via Vavasis-Ye’s interior-point algorithm [23, 36].

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