Distributed coordination for seeking the optimal Nash equilibrium of aggregative games

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Abstract—This paper aims to design a distributed coordination algorithm for solving a multi-agent decision problem with a hierarchical structure. The primary goal is to search the Nash equilibrium of a noncooperative game such that each player has no incentive to deviate from the equilibrium under its private objective. Meanwhile, the agents can coordinate to optimize the social cost within the set of Nash equilibria of the underlying game. Such an optimal Nash equilibrium problem can be modeled as a distributed optimization problem with variational inequality constraints. We consider the scenario where the objective functions of both the underlying game and social cost optimization problem have a special aggregation structure. Since each player only has access to its local objectives while cannot know all players’ decisions, a distributed algorithm is highly desirable. By utilizing the Tikhonov regularization and dynamical averaging tracking technique, we propose a distributed coordination algorithm by introducing an incentive term in addition to the gradient-based Nash equilibrium seeking, so as to intervene players’ decisions to improve the system efficiency. We prove its convergence to the optimal Nash equilibrium of a monotone aggregative game with simulation studies.

Index Terms—aggregative games, optimal Nash equilibrium, hierarchical decision problem, distributed coordination

I. INTRODUCTION

Consider a group of players $\mathcal{N} = \{1, \ldots, N\}$, where each player $i \in \mathcal{N}$ decides its strategy $x_i \in \mathbb{R}^m$. The $i$th player is characterized by two cost functions $f_i(x_i, \bar{x}) : \mathbb{R}^{2m} \rightarrow \mathbb{R}$ and $g_i(x_i, \bar{x}) : \mathbb{R}^{2m} \rightarrow \mathbb{R}$, which depend on $x$ and the aggregate $\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$. The primary goal of the players is to seek a Nash equilibrium (NE) in an aggregative game $P_e$, i.e., player $i$ solves the following optimization problem given other players’ decisions,

$$\min_{x_i \in \mathbb{R}^m} f_i(x_i, \frac{1}{N} \sum_{i=1}^{N} x_i + \frac{1}{N} \sum_{j \neq i} x_j). \quad (P_i)$$

A profile $\bar{x} = [\tilde{x}_1^T, \ldots, \tilde{x}_N^T]^T$ is a Nash equilibrium when

$$\tilde{x}_i \in \arg\min_{x_i \in \mathbb{R}^m} f_i(x_i, \frac{1}{N} \sum_{i=1}^{N} x_i + \frac{1}{N} \sum_{j \neq i} x_j), \quad \forall i \in \mathcal{N}$$

Denote by $X^* \subseteq \mathbb{R}^{mN}$ the set of Nash equilibria to $(P_e)$. The secondary goal is to cooperatively solve the distributed constrained aggregative optimization problem:

$$\min_{x = [x_1^T, \ldots, x_N^T]^T} g(x) = \sum_{i=1}^{N} g_i(x_i, \bar{x}), \text{ s.t. } x \in X^*. \quad (P)$$

The ultimate goal of this work is to design a distributed coordination algorithm to seek a $x^* \in \arg\min_{x \in X^*} \{g(x)\}$ of $(P)$. We consider the aforementioned problem since game theory is increasingly applied to control and decision problems over networks such as network load balancing [1], smart grids [2], and intelligent transportation systems [3]. Meanwhile, the class of aggregative games in $(P_e)$ has attracted significant attention recently due to its wide applications, such as charging management plug-in electric vehicles [4, 5], demand-response [6, 7], spectrum sharing [8] and public goods problem [3]. Hence, seeking NE of aggregative games is also of interest, and we list [10,13] for a few.

Even though NE represents the incentive compatibility state of a game, it can be non-unique for a variety of problems. For example, there exist two equilibria in the stag hunt game, and [14] proves that in a congestion game, the number of Nash equilibria is exponential in the number of users and routes. The efficiency of Nash equilibria has drawn much attention for decades, since Nash equilibria do not always optimize the overall system performance or lead to the lowest social cost (like [7]), taking the Prisoner’s Dilemma as a famous example. To measure the efficiency of different Nash equilibria, [14] proposes to use the ratio between the social cost of the worst Nash equilibrium and the global optimum, called “Price of Anarchy” (PoA). Within this avenue, [15] studies a noncooperative spectrum sharing game with two players and provides the upper bound of PoA at a high signal to noise ratio. [16] considers aggregative charging games in smart grids, and provides the upper bounds of PoA for three types of charging price functions. In addition to considering the efficiency under the worst Nash equilibrium, how to obtain a best Nash equilibrium that optimizes the social welfare/cost is also very meaningful, and is referred as the optimal NE problem [17]. However, we cannot expect that the existing NE seeking algorithms (like in [10, 12], and [13]) can naturally converge to the optimal NE. As such, certain intervention and control must be imposed. There are a few works studying how to intervening in the players to obtain an optimal NE. For example, [18] proposes a completely uncoupled learning rule to select an efficient NE in a Pareto optimal sense. Based on [18], [19] further proposes and proves learning procedures that can lead to a Pareto optimal action profile for repeated games, but the converged profile is not necessarily an NE. The authors of [20] propose to affect players’ strategies by imposing an external incentive target. This method is able to converge to an optimal NE, which however necessitates a central planner and needs to change the players’ objectives. Recently, [21] studies an optimization problem with variational inequality constraints, and demonstrates that the optimal NE problem can be reformulated as it. Based on the Tikhonov trajectory, it also develops a converging algorithm, but needs either a center or requires full information of all players.

Motivated by the appealing properties of distributed algorithms for large-scale networks, we propose solving the problem $(P)$ over a network. With the distributed algorithm, each player preserves its local objective functions $f_i$ and $g_i$, and needs no center to collect and broadcast the aggregation variable. By applying the gradient tracking technique, we propose a distributed and single-timescale algorithm for this problem with provable convergence guarantee.

The rest of the paper is organized as follows. We list the standing assumptions of problems $(P_e)$ and $(P)$ in Section II, and then design a distributed algorithm along with the main results in Section III. We validate the proposed algorithm through numerical simulations in Section IV, and conclude the paper in Section V.

Notations. Let $x = [x_1^T, \ldots, x_N^T]^T$ stack the vectors $x_1, \ldots, x_N$, and $x_{-i} = [x_1^T, \ldots, x_i^T, \ldots, x_N^T]^T$. Denote $\nabla x f_i(x_i, \bar{x})$ by $\nabla x f_i(x_i, \bar{x})$ and $\nabla x f_i(x_i, \bar{x})$ by $\nabla x f_i(x_i, \cdot)$. Let $\|x\|$ denote the Euclidean two-norm. An $N$ dimension vector with all entries being...
one is denoted by $1_N$, and the $m \times m$ identity matrix is denoted by $I_m$. Denote a graph by $G = (N, E)$, where $N$ is the set of nodes, and $E \subseteq N \times N$ denotes the set of ordered pairs, called edges. A graph $G$ is called undirected when for any edge $(i, j) \in E$, there exists an edge $(j, i) \in E$. An undirected graph $G$ is called connected if there exists a path between any two nodes. Define the weighted adjacency matrix $W = [w_{ij}]_{i,j=1}^N$ as $w_{ij} > 0$ if $(i, j) \in E$ else $w_{ij} = 0$.

II. STANDING ASSUMPTIONS

We make the following assumptions on the problem (1).

Assumption 1. For each player $i \in N$,

(a) $f_i(x_i, x + \frac{1}{N} \sum_{j \neq i} x_j)$ is convex in $x_i \in \mathbb{R}^m$ for every fixed $x_{-i}$;

(b) $f_i(\cdot, \beta)$ is continuously differentiable in $x_i \in \mathbb{R}^m$ for every fixed $\beta \in \mathbb{R}^m$, and $f_i(x, \beta)$ is continuously differentiable in $\beta \in \mathbb{R}^m$ for every fixed $x_i \in \mathbb{R}^m$.

Let $V \in \mathbb{R}^{mN}$, and $x$ is an NE if and only if it is a solution to the variational inequality $VI(V, F)$,

$$
F(x)^T(x - \bar{x}) \geq 0 \text{ for all } x \in \mathbb{R}^{mN}.
$$

The solution set of (2) is denoted as $SOL(V, F)$.

Proposition 1. Under Assumption 1, a vector $\bar{x}$ solves $VI(V, F)$ if and only if $F(\bar{x}) = 0$.

We require the set of Nash equilibria to be nonempty, and impose some Lipschitz continuity on the mappings defined in (1).

Assumption 2. (a) The map $F(x)$ is $L_F$-Lipschitz continuous and monotone in $x \in \mathbb{R}^{mN}$ and $\nabla f_i(x, y) + \frac{1}{N} \nabla_2 f_j(x, y)$ is also $L_F$-Lipschitz continuous in $(x, y) \in \mathbb{R}^{mN} \times \mathbb{R}^{mN}$, i.e., for any $(x, y), (x', y') \in \mathbb{R}^{mN} \times \mathbb{R}^{mN}$,

$$
||F(x) - F(x')|| \leq L_F ||x - x'||,
$$

$$(F(x) - F(x'))^T(x - x') \geq 0,
$$

$$
||\nabla_1 f_i(x, y) + \frac{1}{N} \nabla_2 f_j(x, y) - \nabla_1 f_i(x', y') - \frac{1}{N} \nabla_2 f_j(x', y')|| \leq L_F (||x - x'|| + ||y - y'||).
$$

(b) $x^*$ is a nonempty set.

For problem (2), let

$$
\nabla_1 g_i(x) = \nabla_1 g_i(x_i, \bar{x}) + \frac{1}{N} \sum_{j=1}^N \nabla_2 g_i(x_j, \bar{x}), \quad i \in N,
$$

$$
\nabla g(x, y) = [\nabla g_1(x_1, y_1)^T, \ldots, \nabla g_N(x_N, y_N)^T]^T,
$$

$$
\nabla_2 g(x, y) = [\nabla_2 g_1(x_1, y_1)^T, \ldots, \nabla_2 g_N(x_N, y_N)^T]^T,
$$

$$
\nabla g(x) = \nabla_1 g(x, 1_N \otimes \bar{x}) + \frac{1}{N} 1_N^T \otimes I_m \nabla_2 g(x, 1_N \otimes \bar{x}).
$$

We impose the following assumptions on $g(x)$.

Assumption 3. (a) The global objective function $g$ is differentiable and $g_{\beta}$-strongly convex in $x \in \mathbb{R}^{mN}$. Also, $\nabla g$ and $\nabla_1 g(x, y) + \frac{1}{N} 1_N^T \otimes I_m \nabla_2 g(x, y)$ are $L_2$ Lipschitz continuous in $(x, y) \in \mathbb{R}^{mN} \times \mathbb{R}^{mN}$, i.e., for any $(x, y), (x', y') \in \mathbb{R}^{mN} \times \mathbb{R}^{mN}$,

$$
||\nabla g(x) - \nabla g(x')|| \leq L_1 ||x - x'||,
$$

$$(\nabla g(x) - \nabla g(x'))^T(x - x') \geq \mu_g ||x - x'||^2,
$$

$$
||\nabla_1 g(x, y) + \frac{1}{N} 1_N^T \otimes I_m \nabla_2 g(x, y) - \nabla_1 g(x', y') - \frac{1}{N} 1_N^T \otimes I_m \nabla_2 g(x', y')|| \leq L_1 (||x - x'|| + ||y - y'||).
$$

(b) $\nabla_2 g(x, y)$ is $L_2$ Lipschitz continuous in $(x, y) \in \mathbb{R}^{mN} \times \mathbb{R}^{mN}$, i.e., for any $(x, y), (x', y') \in \mathbb{R}^{mN} \times \mathbb{R}^{mN}$,

$$
||\nabla_2 g(x, y) - \nabla_2 g(x', y')|| \leq L_2 (||x - x'|| + ||y - y'||).
$$

Note that we only require the global function $g(x)$ to be strongly convex, while each local function $g_i(x_i, \bar{x})$ is not necessarily strongly convex or convex. Since the objective function $g$ is strongly convex, (1) has a unique optimal solution, denoted as $x^*$.

Let the interaction among the players be described by an undirected graph $G = (N, E)$ with the adjacency matrix denoted by $W$. We impose the following conditions on $G$.

Assumption 4. The communication graph $G$ is undirected and connected. Besides, the adjacency matrix $W = [w_{ij}]_{i,j=1}^N$ is nonnegative and doubly stochastic, i.e., $W 1_N = 1_N$ and $1_N^T W = 1_N^T$.

III. DISTRIBUTED ALGORITHM AND MAIN RESULTS

We propose a distributed Tikhonov regularized algorithm to compute the optimal NE as follows.

Algorithm 1: Distributed seeking for the optimal NE

Initialization: For each $i \in N$, let $x_i^0 \in \mathbb{R}^m$ and set $v_i^0 = x_i^0$, $y_i^0 = \nabla_2 g_i(x_i^0, v_i^0)$;

Iteration: for all $k \geq 0$,

$$
x_i^{k+1} = x_i^k - \gamma_k \left( \nabla_1 f_i(x_i^k, v_i^k) + \frac{1}{N} \nabla_2 f_i(x_i^k, v_i^k) + \eta_k (\nabla_1 g_i(x_i^k, v_i^k) + y_i^k) \right),
$$

$$
v_i^{k+1} = \sum_{i=1}^N w_{ij} v_i^k + x_i^{k+1} - x_i^k,
$$

$$
y_i^{k+1} = \sum_{i=1}^N w_{ij} y_i^k + \nabla_2 g_i(x_i^{k+1}, v_i^{k+1}) - \nabla_2 g_i(x_i^k, v_i^k),
$$

where $\gamma_k, \eta_k$ are positive step-sizes.

In the algorithm, the local variables $v_i$ and $y_i$ are used to track the dynamical aggregate $\bar{x}$ and $\frac{1}{N} \sum_{i=1}^N \nabla_2 g_i(x_i, \bar{x})$, respectively. Then each player $i$ uses $\nabla_1 f_i(x_i^k, v_i^k) + \frac{1}{N} \nabla_2 f_i(x_i^k, v_i^k)$ as an estimate of the gradient $\nabla_1 f_i(x_i^k, \bar{x}^k)$ and adds an incentive term $\nabla_1 g_i(x_i^k, v_i^k)$ as an estimate of the gradient $\nabla_1 g_i(x_i^k)$ to optimize the social cost.

Algorithm 1 can be represented compactly as follows:

$$
x_i^{k+1} = x_i^k - \gamma_k \left( \nabla_1 f_i(x_i^k, v_i^k) + \frac{1}{N} \nabla_2 f_i(x_i^k, v_i^k) + \eta_k (\nabla_1 g_i(x_i^k, v_i^k) + y_i^k) \right),
$$

$$
v_i^{k+1} = W \otimes I_m v_i^{k+1} - x_i^k,
$$

$$
y_i^{k+1} = W \otimes I_m y_i^k + \nabla_2 g_i(x_i^{k+1}, v_i^{k+1}) - \nabla_2 g_i(x_i^k, v_i^k).
$$

We impose the following conditions on the step-sizes $\gamma_k$ and $\eta_k$. 
Assumption 5. (Step-size update rules) Set $\gamma_k = \frac{\eta_k}{\mu_k^{(k+1)}} > 0$ and $\eta_k = \frac{\eta_0}{\mu_k^{(k+1)}} > 0$ for $k \geq 0$, where $a, b$ satisfy $0 < b < a < 1$ and $a + b < 1$.

With the initial state $v_1 = x_0, y_0 = \nabla g_1(x_0, v_0)$, similarly to [4], Lemma 2, we can easily obtain the following lemma.

Lemma 1. Define $v_k = \frac{1}{\gamma} \sum_{i=1}^N v_i^k$ and $y_k = \frac{1}{\gamma} \sum_{i=1}^N y_i^k$. Under Assumption 2, we have that

$$\theta^k = \bar{x}^k, \quad y^k = \frac{1}{N} \sum_{i=1}^N \nabla g(x_i^k, v_i^k), \quad \forall k \geq 0.$$  

From Assumptions 2 and 3 we can easily obtain some properties of the ergodic map $F + \eta_k \nabla g$.

Lemma 2. Let Assumptions 2 and 3 hold. Then $F + \eta_k \nabla g$ is $(\eta_k \mu_k)$-strongly monotone and $(L_F + \eta_k L_1)$ Lipschitz continuous, and $|F(x_i^k, y_i^k) - \frac{1}{\gamma} \sum_{i=1}^N \nabla g(x_i^k, y_i^k)| = (L_F + \eta_k L_1)$ Lipschitz continuous.

Since $F + \eta_k \nabla g$ is strongly monotone and Lipschitz continuous, there exists an unique solution $x_{\infty}^k$ to $F(R_{n\times m} \setminus F + \eta_k \nabla g)$. The sequence $\{x_{\infty}^k\}$ is called the Tikhonov trajectory of the problem $P_k$.

In the following, we show that the trajectory asymptotically converges to the optimal solution to Problem $P_k$ and provide the upper bound on the error of $\|x_{\infty}^k - x_{\infty}^{k-1}\|$. The proof is similar to that of Lemma 4.5, so we give the proof in Appendix for completeness. The established properties of the Tikhonov trajectory will be used in the convergence analysis of Algorithm 1.

Lemma 3. (Properties of Tikhonov trajectory) Let Assumptions 2 and 3 hold. Then we have

(a) the sequence $\{x_{\infty}^k\}$ converges to $x^*$;

(b) there exists a scalar $\bar{C} > 0$ such that $\|x_{\infty}^k - x_{\infty}^{k-1}\| \leq \frac{\bar{C}}{\eta_k} \nabla g(x_{\infty}^k, y_{\infty}^k)$ for all $k \geq 1$, where $\Gamma_{k-1} = 1 - \eta_k \mu_k$.

We use $\|x_i^k - x_{\infty}^{k-1}\|, \|v_i^k - 1 - N \otimes \bar{v}\|, \|y_i^k - 1 - N \otimes \bar{y}\|$ to show our convergence, of which the first term represents the distance between the iterate and the Tikhonov trajectory, the second term measures the consensus violation for the aggregate and the third term measures the consensus violation for the gradient of the global cost function. Define $\Delta_k = \left[\|x_i^k - x_{\infty}^{k-1}\|, \|v_i^k - 1 - N \otimes \bar{v}\|, \|y_i^k - 1 - N \otimes \bar{y}\| \right]^T$. The following result provides a recursive relation of the three metrics.

Proposition 2. Consider Algorithm 1 under Assumptions 2 and 3. If $0 < \gamma_0 = \frac{1}{(L_F + \eta_0 L_1)}$, we have $\Delta_{k+1} \leq H_{k+1} \Delta_k + h_{k+1}$ for all $k \geq 1$, where $H_k = [h_k,i,j]_{i,j=1}^3, h_k = [h_k,i]_{i=1}^3$ are given by

$$H_{k,11} = 1 - \gamma_k \eta_k \mu_k, \quad H_{k,12} = \gamma_k (L_F + \eta_k L_1), \quad H_{k,13} = \frac{C^2(\eta_0 L_1)}{\eta_k} \Gamma_{k-1},$$

$$H_{k,22} = \rho + (L_F + \eta_k L_1), \quad H_{k,23} = 4L_2 \gamma_k (L_F + \eta_k L_1), \quad H_{k,32} = L_2 \frac{W - L_1}{L_1} + 2L_2 \gamma_k (L_F + \eta_k L_1),$$

$$H_{k,33} = \rho + 2L_2 \gamma_k \eta_k, \quad h_{k,1} = \frac{C^2(\eta_0 L_1)}{\eta_k} \Gamma_{k-1}, \quad h_{k,2} = 2 \gamma_k \frac{C^2(\eta_0 L_1)}{\mu_k \eta_k} \Gamma_{k-1},$$

$$h_{k,3} = \frac{4L_2 \gamma_k C^2(\eta_0 L_1)}{\mu_k \eta_k} \Gamma_{k-1}.$$

Proof. First we show $\Delta_{k+1} \leq \sum_{i=1}^3 H_{k,i} \Delta_{k,i} + h_{k,i}$. From the definition of $x_{\infty}^k$ and Proposition 1 we know that $F(x_{\infty}^k) + \eta_k \nabla g(x_{\infty}^k) = 0$. By (3),

$$\|x_{k+1}^k - x_{\infty}^k\| = \|x^k - \gamma_k (\nabla f(x_i^k, v_i^k) + \frac{1}{N} \nabla f(x_i^k, v_i^k) + \eta_k \nabla g(x_i^k, v_i^k)) - x_{\infty}^k + \gamma_k (F(x_{\infty}^k) + \eta_k \nabla g(x_{\infty}^k))\| \leq \|x^k - x_{\infty}^k\| (F(x_i^k) + \eta_k \nabla g(x_i^k)) + \gamma_k (F(x_{\infty}^k) + \eta_k \nabla g(x_{\infty}^k))$$

$$+ \gamma_k (F(x_{\infty}^k) + \eta_k \nabla g(x_{\infty}^k)) \geq \|\nabla f(x_i^k, v_i^k) + \frac{1}{N} \nabla f(x_i^k, v_i^k) + \eta_k \nabla g(x_i^k, v_i^k)\| \geq \gamma_k \|\nabla f(x_i^k, v_i^k) + \frac{1}{N} \nabla f(x_i^k, v_i^k) + \eta_k \nabla g(x_i^k, v_i^k)\|.$$  

By Lemma 1 we have

$$\|x^k - x_{\infty}^k\| + \gamma_k (F(x_{\infty}^k) + \eta_k \nabla g(x_{\infty}^k)) \leq \|x^k - x_{\infty}^k\| + \gamma_k (F(x_{\infty}^k) + \eta_k \nabla g(x_{\infty}^k)) \leq (1 - \gamma_k \eta_k \mu_k)\|x^k - x_{\infty}^k\|.$$  

This combined with (6) and (7) produces $\|x^k - x_{\infty}^k\| \leq (1 - \gamma_k \eta_k \mu_k)\|x^k - x_{\infty}^k\|.$
where $\rho$ is the spectral radius of $W - 1_N 1_N$ and $0 < \rho < 1$ by Lemma 1. For the second term above, from (3) and $F(x_{n_k}^*) + \eta_k \nabla g(x_{n_k}^*) = 0$, we obtain

$$\|x^{k+1} - x_k\| = \gamma_k \|\nabla f(x_k, v_k) + \frac{1}{N} \nabla_f f(x_k, v_k) + \eta_k (\nabla g(x_k, v_k) + \gamma_k \bar{v})\|
\leq \gamma_k \|\nabla f(x_k, v_k) + \frac{1}{N} \nabla_f f(x_k, v_k) + \eta_k (\nabla g(x_k, v_k) + \gamma_k \bar{v})\|
+ \eta_k \gamma_k \|y_k - 1_N \otimes \bar{v}\|
\leq \gamma_k (L_F + \eta_k L_1)(\|x_k - x_{n_k}^*\| + \|v_k - 1_N \otimes \bar{v}\|)
+ \eta_k \gamma_k \|y_k - 1_N \otimes \bar{v}\|
\leq \gamma_k (L_F + \eta_k L_1)(\|x_k - x_{n_k}^*\| + \|v_k - 1_N \otimes \bar{v}\|)
+ \gamma_k \eta_k \|y_k - 1_N \otimes \bar{v}\|$$

(11)

where the second inequality is derived similar to that of (8). Note by Lemma 1 that

$$\|1_N \otimes v_k - 1_N \otimes \bar{v}_{n_k}\| = N \|x_k - x_{n_k}^*\|^2
= N \left( \sum_{i=1}^{N} \left( x_k^i - x_{n_k,i}^* \right)^2 \right)
\leq \frac{1}{N} \left( \sum_{i=1}^{N} \left( x_k^i - x_{n_k,i}^* \right)^2 \right)
\leq \left( \sum_{i=1}^{N} x_k^i - x_{n_k,i}^* \right)^2
\leq \|x_k - x_{n_k}^*\|^2,$$

(12)

where the second inequality we use the fact that $(\sum_{i=1}^{N} a_i)^2 \leq \sum_{i=1}^{N} a_i^2$. This together with (11) implies that

$$\|x^{k+1} - x_k\| \leq 2\gamma_k (L_F + \eta_k L_1)(\|x_k - x_{n_k}^*\| + \|v_k - 1_N \otimes \bar{v}\|)
+ \gamma_k (L_F + \eta_k L_1)(\|x_k - x_{n_k}^*\| + \|v_k - 1_N \otimes \bar{v}\|)
+ \gamma_k \eta_k \|y_k - 1_N \otimes \bar{v}\|$$

(13)

Substituting (12) into (10), we have

$$\|v^{k+1} - 1_N \otimes \bar{v}\| \leq (\rho + \gamma_k (L_F + \eta_k L_1))(\|v_k - 1_N \otimes \bar{v}\|)
+ 2\gamma_k (L_F + \eta_k L_1)(\|x_k - x_{n_k}^*\| + \gamma_k \eta_k \|y_k - 1_N \otimes \bar{v}\|).$$

Then the desired inequality is obtained by adding and subtracting $x_{n_k-1}$ in the second term.

Thirdly, we show that $\Delta_{k+1,3} \subseteq \sum_{i=1}^{3} H_{k,i,3} \Delta_{k+h_{i,3}}$. By (5), we obtain that

$$\begin{align*}
\hat{y}^{k+1} &= \hat{y} + \frac{1}{\gamma_k} \otimes I_m(\nabla g(x^{k+1}, v^{k+1}) - \nabla g(x_k, v_k)).
\end{align*}$$

Since $W$ is doubly stochastic, by using $\|I_m - \frac{1}{1-N} \otimes I_m\| = 1$, we derive

$$\|y^{k+1} - 1_N \otimes \bar{v}\| = \|W \otimes I_m y_k - 1_N \otimes \bar{y} + \nabla g(x^{k+1}, v^{k+1}) - \nabla g(x_k, v_k) - \frac{1}{1-N} \otimes I_m(\nabla g(x^{k+1}, v^{k+1}) - \nabla g(x_k, v_k))\|
\leq \|W \otimes I_m y_k - 1_N \otimes \bar{y}\| + \|\nabla g(x^{k+1}, v^{k+1}) - \nabla g(x_k, v_k)\|
\leq \rho \|y^{k+1} - 1_N \otimes \bar{y}\| + L_2(\|x^{k+1} - x_k\| + \|v^{k+1} - v_k\|)).$$

(14)

where Assumption (3b) has been used in the last inequality. For the last term in (13), from (4) and noting that $W 1_N = 1_N$, we have

$$\|v^{k+1} - v_k\| = \|W \otimes I_m y_k + x^{k+1} - x_k\| - \|W - I_N\| \|v^{k+1} - 1_N \otimes \bar{v}\| + \|x^{k+1} - x_k\|$$

(15)

and

$$\|y^{k+1} - 1_N \otimes \bar{y}\| \leq \rho \|y^{k+1} - 1_N \otimes \bar{y}\| + L_2(\|x^{k+1} - x_k\| + \|v^{k+1} - v_k\|))$$

(16)

Substituting (12) and (14) into (13) gives rise to

$$\|y^{k+1} - 1_N \otimes \bar{y}\| \leq \rho \|y^{k+1} - 1_N \otimes \bar{y}\| + L_2(\|x^{k+1} - x_k\| + \|v^{k+1} - v_k\|))$$

(17)

and

$$\|v^{k+1} - v_k\| \leq \rho \|v^{k+1} - v_k\| + L_2(\|x^{k+1} - x_k\| + \|v^{k+1} - v_k\|))$$

(18)

Adding and subtracting $x_{n_k-1}$ in the second term, we obtain the desired inequality.

Now we introduce a lemma for our convergence proof.

Lemma 4. [24, Lemma 2.2.3] Let $\{v^k\}$, $\{e^k\}$, and $\{\eta_k\}$ be nonnegative sequences of real numbers such that

$$v^{k+1} \leq \eta_k v^k + e^k, \text{ for all } k \in \mathbb{N}.$$ 

Then lim$_{k \to \infty} v^k = 0$.

Let

$$c_1 = \eta_k^2 \mu_2 L_2(L_F + \eta_k L_1) + 8\eta_k L_2(L_F + \eta_k L_1)^2,$$

$$c_2 = 0.5\eta_k^2 \mu_2 L_2(1 - \rho)^2 + 2\eta_k L_2(L_F + \eta_k L_1)^2 + 2\rho \eta_k L_2(1 - \rho)^2(L_F + \eta_k L_1),$$

$$c_3 = 0.125\mu_2 (1 - \rho)^2.$$ 

The next theorem gives the convergence result of Algorithm 1.

Theorem 1. Consider Algorithm 1 under Assumptions 7.1 If $\gamma$ satisfies

$$0 < \gamma < \min\left\{ \frac{1}{\eta_k \mu_2 + 4L_2\eta_k}, \frac{1}{\eta_k \mu_2 + 2L_2\eta_k}, \frac{1}{\eta_k \mu_2 + 4L_2\eta_k}, \frac{1}{\eta_k \mu_2 + 2L_2\eta_k}, \frac{1}{\eta_k \mu_2 + 4L_2\eta_k} \right\},$$

then

(1) $\|\Delta_{k+1}\| \leq (1-0.5\gamma \eta_k \mu_2) \|\Delta_k\| + \Theta (\Omega - 1)$ for all $k \geq 1$, where $\Theta \leq \frac{1}{\max\{1, 2\gamma (L_F + \eta_k L_1), 4L_2\gamma (L_F + \eta_k L_1)\}} \sqrt{3}/\mu_2$;

(2) lim$_{k \to \infty} \|\Delta_k\| = 0$;

(3) lim$_{k \to \infty} x_k = x^*.$

Proof. (1) Let $\alpha_k = 1 - 0.5\gamma \eta_k \mu_2$ and define $H_k = \lfloor h_{k,1,3} \rfloor$ and $\hat{h}_k = \lfloor h_{k,1,3} \rfloor$ as follows:

$$\begin{align*}
\hat{h}_{k,11} &= h_{k,11}, \quad \hat{h}_{k,12} = h_{k,12}, \\
\hat{h}_{k,23} &= h_{k,23}, \quad \hat{h}_{k,23} = h_{k,23}, \\
\hat{h}_{k,33} &= h_{k,33}, \quad \hat{h}_{k,33} = h_{k,33}.
\end{align*}$$

where $\Theta \leq \frac{1}{\max\{1, 2\gamma (L_F + \eta_k L_1), 4L_2\gamma (L_F + \eta_k L_1)\}} \sqrt{3}/\mu_2$.
Note that
\[ H_{k,22} - H_{k,22} = 1 - \frac{1}{2} - 0.5\gamma_k \eta_k \mu_k - \gamma_k (L_F + \eta_k L_1), \]
and by Assumption 5 that \( \{ \gamma_k \}^\infty_{k=0}, \{ \eta_k \}^\infty_{k=0} \) are decreasing strictly positive sequences, we have \( \gamma_k \leq 1 < \frac{1}{\eta_k \mu_k + 2(L_F + \eta_k L_1)} \leq \frac{1-\rho}{\eta_k \mu_k + 2(L_F + \eta_k L_1)} \). Thus \( H_{k,22} > H_{k,22} \) for all \( k \geq 1 \). Similarly, we have \( H_{k,33} > H_{k,33} \) for all \( k \geq 1 \) since \( \gamma_0 < \frac{1-\rho}{\eta_0 \mu_0 + 2(L_F + \eta_0 L_1)} \). Since \( \{ \eta_k \}^\infty_{k=0} \) is a decreasing strictly positive sequence, by invoking the definition of \( \Theta \), we have \( h_k \leq \tilde{h}_k \) and \( h_k \leq \tilde{h}_k \) for any \( k \geq 0 \). Thus, we have \( \| \Delta k+1 \| \leq \rho(\hat{h}_k) \Delta k+1 \| + \Theta \Gamma k-1 \), where \( \rho(\hat{h}_k) \) denotes the spectral norm of \( \tilde{h}_k \).

We proceed to show that \( \rho(\tilde{h}_k) < \alpha_k \) for all \( k \geq 1 \). By Lemma 5 it suffices to show that \( 0 \leq H_{k,ii} < \alpha_k \) for \( i = 1, 2, 3 \) and \( \det(\alpha_k I - \tilde{h}_k) > 0 \). Firstly, we show that \( 0 \leq H_{k,ii} < \alpha_k \) for \( i = 1, 2, 3 \). It can be seen that \( H_{k,ii} < \alpha_k \) for \( i = 1, 2, 3 \) from its definition. Since \( 0 < \rho < 1 \) and \( \gamma_0 < \frac{1-\rho}{\eta_0 \mu_0 + 2(L_F + \eta_0 L_1)} \) by (16), we have \( \gamma_k < \frac{\gamma_0}{\eta_k \mu_k} \) and thus \( H_{k,11} > 0 \). We can further obtain that \( \gamma_k < \frac{\gamma_0}{\eta_k \mu_k} \) by Lemma 5 implying that \( H_{k,22} = \tilde{H}_{k,33} > 0 \). Next, we show that \( \det(\alpha_k I - \tilde{h}_k) > 0 \) for all \( k \geq 1 \).

Since \( \gamma_k > 0 \), it suffices to show that \( c_1 \gamma_k^2 + c_2 \gamma_k < c_3 \gamma_k \). Note that \( c_1, c_2, c_3 \) are those introduced. By invoking the step-size update rule that \( \gamma_k < \frac{1}{\eta_k \mu_k} \), we have \( \gamma_k \gamma_0 < c_3 \gamma_0 \). And because \( 0 < \gamma_k < \frac{\gamma_0}{\eta_k \mu_k} \), we have \( c_1 \gamma_0^2 + c_2 \gamma_0^2 < c_3 \gamma_0 \) implying that \( \det(\alpha_k I - \tilde{h}_k) > 0 \). Thus, for all \( k \geq 1 \), we have \( \rho(\tilde{h}_k) < \alpha_k = 1 - 0.5\gamma_k \eta_k \mu_k \) provided that \( \gamma_0 \) satisfies (16). We can conclude that for all \( k \geq 1 \), we have \( \| \Delta k+1 \| \leq 0.5\gamma_k \eta_k \mu_k \| \Delta k+1 \| + \Theta \Gamma k-1 \).

Finally, from part (1) we have \( 0 < \alpha_k < 1 \). Secondly, by using the update rule of \( \gamma_k \) and \( \eta_k \) and \( 0 < a + b < 1 \), it follows that
\[
\sum_{k=0}^\infty (1 - \alpha_k) = \sum_{k=0}^\infty 0.5\gamma_k \eta_k \mu_k = 0.5\gamma_0 \eta_0 \mu_0 \sum_{k=0}^\infty \frac{1}{(k+1)a+b} \geq 0.5\gamma_0 \eta_0 \mu_0 \sum_{k=0}^\infty \frac{1}{k+1} = \infty.
\]
Thirdly, because \( x^k < x \) if \( x > 1 \) and \( 0 < b < 1 \), we can obtain
\[
\Gamma k-1 = \frac{\eta k-1}{\eta_k} - 1 = (1 + \frac{1}{b})^b - 1 \leq 1 + \frac{1}{k} - 1 = \frac{1}{k}.
\]
Then we have
\[
\Theta \Gamma k-1 \leq \Theta (k+1)^{1+b} \leq \frac{\Theta (k+1)^{1+b}}{0.5\gamma_0 \eta_0 \mu_0}.
\]

Since \( a + b < 1 \), we obtain that \( \lim_{k\to\infty} \Theta (k+1)^{1+b} = 0 \). Therefore, by Lemma 4 we have \( \lim_{k\to\infty} \| \Delta_k \| = 0 \).

\[ (3) \lim_{k\to\infty} \| x_k - x_{k-1} \| = 0 \] combined with Lemma 4 proves that \( \lim_{k\to\infty} x_k = x^* \).

IV. NUMERICAL STUDIES

We considered a monotone aggregative game, meanwhile the players coordinate to search a Nash equilibrium that minimizes the social cost. Based on a dynamical averaging tracking and Tikhonov regularization, we develop a distributed and single-timescale framework for seeking the optimal Nash equilibrium. Future works can be

V. CONCLUSION
Fig. 1: The infeasibility measured by $\ln \| F(x^k) \|$.

Fig. 2: The optimality gap measured by $\ln \frac{\| x^k - x^* \|}{\| x^0 - x^* \|}$.

Fig. 3: The decision trajectories of players generated by the algorithm. Each hyperplanes $S_i$ is a cut of the NE set $X^*$, while the star point corresponds with the optimal NE $x^*$.

placed on various aspects, such as Nash games, complicated strategy sets, unbalanced directed graphs, and the relaxation of the strongly convex social cost to a convex or non-convex objective.

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the limit points of \(\{\eta_k\}\) that imply \(\nabla g(\eta_k)\) by Assumption 5, we obtain that \(F(\eta_k)\) is bounded. Further by the coercivity of \(g\), we have
\[
\eta_k \nabla g(\eta_k) = 0.
\]

Let \(x_1 = x_\eta_k\), \(x_2 = x^*\), and add (21) and (22) together, we obtain
\[
(\hat{x} - x_\eta_k) \geq (F(\hat{x}) - F(x_\eta_k))^T (x^* - x_\eta_k).
\]

Recalling by Assumption 2 that the map \(F(x)\) is monotone, we have
\[
\nabla g(x_\eta_k) = 0.
\]

From the strong convexity of \(g\), we obtain that for all \(k \geq 0\),
\[
g(x^*) \geq g(x_\eta_k) + \nabla g(x_\eta_k)^T (x^* - x_\eta_k) + \frac{\mu}{2}\|x^* - x_\eta_k\|^2
\]
\[
\geq g(x^*),
\]
where we neglect the term \(\frac{\mu}{2}\|x^* - x_\eta_k\|^2\). By Assumptions 2, 3, and Lemma 2 both \(x^*\) and \(x_\eta_k\) exist and are unique. Therefore, \(g(x_\eta_k)\) is bounded. Further by the coercivity of \(g\) (due to its strong convexity), the sequence \(\{x_\eta_k\}\) is bounded implying that it must have at least one limit point. Let \(\{x_{\eta_k}\}_{k \in \mathbb{K}}\) be an arbitrary subsequence such that
\[
\lim_{k \to \infty} x_{\eta_k} = \hat{x}.
\]

We now show that \(\hat{x} \in \text{SOL} (\mathbb{R}^{m_N}, F)\). Taking the limit on both sides of (22) with respect to the aforementioned subsequence and using the continuity of \(F\) and \(\nabla g\), we obtain that for all \(y \in \mathbb{R}^{m_N}\),
\[
(F(\hat{x}) + \lim_{k \to \infty} \eta_k \nabla g(\hat{x}))^T (y - \hat{x}) \geq 0.
\]

Because the sequence \(\{x_{\eta_k}\}\) is bounded, there exists a compact ball \(\mathcal{X} \subseteq \mathbb{R}^{m_N}\) such that \(\bigcup_{k \in \mathbb{K}} x_{\eta_k} \subseteq \mathcal{X}\). Combining with the continuity of \(\nabla g\), we know that there exists a constant \(C > 0\) such that
\[
\|\nabla g(\eta_k)\| \leq C \text{ for all } k \geq 0.
\]

Recalling that \(\lim_{k \to \infty} \eta_k = 0\) by Assumption 5, we obtain that \(F(\hat{x})^T (y - \hat{x}) \geq 0\) for all \(y \in \mathbb{R}^{m_N}\), implying \(\hat{x}\) is a feasible solution to Problem (7). Next we show that \(\hat{x}\) is the optimal solution. From (23) we know that
\[
g(\hat{x}^*) \geq g(\lim_{k \to \infty} \eta_k \nabla g(\hat{x})) = g(\hat{x}).
\]

Hence, by the uniqueness of \(x^*\), all the limit points of \(\{x_{\eta_k}\}\) converge to \(x^*\) and this proof is completed.

(2) See the proof of [17, Lemma 4.5(b)].