D2-brane RR-charge on $SU(2)$

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Abstract

We compute RR charges of D2-branes on a background with $H$-field which belongs to a nontrivial cohomology class. We discover that the RR charge depends on the configuration of the background ‘electric’ RR field. This result explains the ambiguity in the definition of the RR charge previously observed in the SU(2) WZW model.
1 Introduction

Computation of RR charges of D2 branes in the SU(2) WZW model attracted considerable attention since this is the simplest example where we deal with D-branes on a curved background. Moreover, the field strength $H = dB$ of the $B$-field in this model is given by the volume form on the 3-sphere, and, hence, belongs to a non-trivial cohomology class. The first calculation by Bachas, Douglas and Schweigert [1] produced the set of charges with irrational ratios contradicting the standard idea of charge quantization. A resolution of this paradox was proposed by Taylor [2] by taking into account the contribution of bulk fields into the RR charge. In the case when $H$ belongs to a nontrivial cohomology class the Taylor’s mechanism produces an integral but ambiguous answer [3],[4]. For instance, it is argued that in the $SU(2)$ WZW model at level $k$ which describes string propagation on a 3-sphere, the charge is defined modulo $k$ [4] or modulo $(k + 2)$ [3]. This ambiguity finds a mathematical interpretation in terms of twisted K-theory (see [5],[6],[7],[8]). In this paper we reconsider the Taylor’s argument in the case of nontrivial $H$-field and discover that the RR charge depends on the background configuration of the electric components of the RR field strength. On a compact manifold, such as a 3-sphere, the RR field equations possess nontrivial solutions even in the absence of external sources. Adding such a free solution to the background configuration changes the RR charge of the brane. Hence, neglecting this extra contribution leads to the charge ambiguity.

We start by exploring the analogy between D-brane physics and Maxwell’s electromagnetism outlined by Taylor. We first analyse a simple model of an expanding dipole on the circle which then provides an intuition for computing RR charges of D2 branes on a 3-sphere.

2 Electromagnetic analogy

Recall the dimensional reduction in electromagnetism. We start with the Maxwell’s theory in the space-time dimension $D = m + 1$. The fields in our theory are the gauge field $A = A_\mu dx^\mu$ and the (background) metric $g_{\mu\nu}$. The electromagnetic field interacts with the external current represented by an $m$-form $J = - \ast \tilde{J}$, such that $\tilde{J} = - \rho dt + j \cdot dx$ with $\rho$ density of the electric charge and $j$ density of the electric current. The action is given by formula,

$$ S = \int \left[ \frac{1}{2} \ast F \wedge F + A \wedge J \right] , $$

where $F = dA$ is the field strength of $A$.

Now assume that the $m$th spacial direction is compactified on a circle of length $l_m$, and that the field configuration is always independent of the $m$:th coordinate $x_m$. We can then dimensionally reduce the system to $D' = (m - 1) + 1$ dimensions. The $D = (m+1)$ metric decomposes into the $D' = (m-1)+1$ metric, the 1-form $g_m \sim g_{\mu\nu}$ and the 0-form $g_{mm}$. Similarly, the fields $A$, $F$ and $J$ decompose as follows,
The 0-form $A_m$
The 1-form $dA_m = F_m$
The 1-form $A$
The 2-form field strength $F = dA$
The (m-2)-form dual field strength $\ast F$
The (m-1)-form current $J$

In terms of dimensionally reduced variables the action reads,

$$S = l_m \int \left[ \frac{1}{2} * F \wedge F + A \wedge J + g_m \wedge (F_m \wedge * F + A_m J) + g_{nm} (F_m \wedge * F_m) \right]$$  \hspace{1cm} (2)

Since the energy momentum tensor $T$ is defined as $T^{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}}$ we find that the momentum in the $m$:th direction is given by formula,

$$P_m = \int_V T_{m0} = l_m \int_V [F_m \wedge * F + A_m J]$$  \hspace{1cm} (3)

with $V$ implying integration over all of space. It is convenient to split $P_m$ into two parts,

$$p_m = l_m \int_V A_m J, \quad p'_m = l_m \int_V F_m \wedge * F,$$

where, roughly speaking, $p_m$ is the momentum carried by the electric current $J$, and $p'_m$ is the momentum contained in the field configuration.

In what follows, $\Omega$ will be the region of space-time contained between two hyperplanes $x^0 = 0$ and $x^0 = T$, and we assume that the fields decay sufficiently fast at the spatial infinity so as we can integrate by parts. Over the time $T$, the momenta $p_m$ and $p'_m$ evolve as follows,

$$\delta p_m = l_m \int_\Omega d (A_m J) = l_m \int_\Omega F_m \wedge J$$  \hspace{1cm} (4)

$$\delta p'_m = l_m \int_\Omega d (F_m \wedge * F).$$  \hspace{1cm} (5)

Momentum conservation follows from the Maxwell’s equation, $d \ast F = J$,

$$F_m \wedge J + d (F_m \wedge * F) = F_m \wedge (J - d \ast F) = 0$$  \hspace{1cm} (6)

implying $\delta P_m = \delta p_m + \delta p'_m = 0$.

Following Taylor we consider the following special situation: let $J$ be produced by a slowly expanding dipole consisting of charges $\pm q$. One first separates the charges from each other, and then very slowly moves the charge $+q$ along some path $C$. The corresponding current $J$ is given by a $\delta$-function supported on $C$, $J = -q \delta_C$. Then,

$$\delta p_m = l_m \int_\Omega dA_m \wedge J = -q l_m \int_C dA_m = -q l_m (A^f_m - A^i_m),$$
where $A^{i,f}_m$ are the values of $A_m$ at the end-points of $C$. Of course, $\delta p'_m = -\delta p_m$ to ensure momentum conservation.

The situation becomes somewhat more interesting when in addition to $x^m$ one more direction is compactified. For instance, consider a $2 + 1$-dimensional configuration where both spatial directions are compactified. The dimensional reduction on the $m$th (now $m = 2$) direction yields a $1 + 1$-dimensional system where the spacial coordinate is compactified on a circle of length $l_1$.

Again, we consider an expanding dipole, but now after the charge $+q$ makes a full turn around the circle, it can annihilate again with the charge $-q$. As before, this gives

$$\delta p_m = -ql_m(A^f_m - A^i_m).$$

Note that

$$l_m(A^f_m - A^i_m) = l_m \int_{S_1} d^{1}x (\partial_1 A_m) = l_m \int_{S_1} F_m = \Phi;$$

where $S_1$ is the circle spanned by the 1st direction and $\Phi$ is the flux of the field $F$ through the torus spanned by both spatial directions. We conclude that $\delta p_m = -q\Phi$.

Seemingly, we arrive at a paradox since in general $\delta p_m \neq 0$ while the initial and final charge configurations are the same. The latter, however, does not imply that the field configurations at $x^0 = 0$ and $x^0 = T$ are the same. Indeed, while expanding the dipole we create an electric field between the charges. By Maxwell’s equation,

$$*F|_0^T = \int_0^T dx^0 \partial_0 *F = \int_0^T J_0 = \int_0^T j = \delta q,$$

where $j$ is the electric current and $\delta q$ is the charge that passed through the given point of the circle during the period from $x^0 = 0$ to $x^0 = T$. This extra electric field changes the momentum stored in the field configuration,

$$\delta p'_m = l_m \int_{S_1} F_m \wedge *F|_0^T = l_m \int_{S_1} F_m(*F(T) - *F(0)) = ql_m \int_{S_1} F_m = q\Phi. \tag{8}$$

We conclude that the field configuration (the electric field) contains the information about the history of our system. Moving a charge $n$ times around the circle produces a uniform electric field proportional to $n$. This electric field (together with the background magnetic field) carries momentum, and, since the space is compact, the configuration with uniform electric and magnetic fields does not decay.

3 RR-charge of D2-branes on $SU(2)$

Our main interest in this paper is the RR-charge of D2-branes. Following the reasoning outlined in the previous section we will deduce the D0 RR-charge from the type IIA supergravity action and show that just as the momentum of the electromagnetic system couldn’t be determined by the charge configuration alone D0-charge can only be unambiguously defined when taking into account non-trivial free field solutions that are stable even in the absence of sources.
D2-branes are found in type IIA supergravity but their origin can be traced back to M-theory. From this perspective the D2-branes are membranes charged with respect to the 3-form $C$ and the RR charge is just one particular component of 11-dimensional momentum. The low-energy limit of M-theory is 11-dimensional supergravity which can be dimensionally reduced to type IIA supergravity by compactifying a spatial direction $x^m$ on a circle. While doing so the 3-form $C$ decomposes into the RR 3-form $C^{(3)}$ and the NS-NS 2-form $B$. Momentum in the $m$-direction is interpreted as D0-brane charge since D0-branes carry charge under the R-R 1-form $C^{(1)}_\mu \sim g_{\mu m}$. From these fields one obtains the 3-form $H = dB$ and the 4-form $G^{(4)} = dC^{(3)} - C^{(1)} \wedge H$. The fields interact with external sources (D2-branes) through the 7-form $J_{D^2}$ defined so that for any 3-form $A^{(3)}$

$$\int_\Omega J_{D^2} \wedge A^{(3)} = \mu_2 \int_{\Omega(\Sigma)} A^{(3)}$$

where $\Omega(\Sigma)$ is the world-volume of the brane $\Sigma$ and $\mu_2$ is a unit D2-brane charge. With kinetic terms canonically normalized (setting the prefactor $1/2\kappa_{10}$ to unity) the IIA supergravity action is (where $\ldots$ denotes terms not involving the field $C^{(1)}$):

$$S_{IIA} = \int_\Omega \left\{ \frac{1}{2} G^{(4)} \wedge *G^{(4)} + \ldots \right\}$$

$$= \int_\Omega \left\{ -C^{(1)} \wedge H \wedge *G^{(4)} + \ldots \right\}$$

The D2-brane world-volume action is a sum of a Born-Infeld action and a WZW action. Since the Born-Infeld action does not involve the field $C^{(1)}$ we ignore it in our considerations and focus on the WZW part which is:

$$S^{D2}_{WZW} = \mu_2 \int_{\Omega(\Sigma)} \left\{ C^{(3)} + C^{(1)} \wedge \mathcal{F} \right\} = \int_\Omega \left\{ J_{D^2} \wedge C^{(3)} + J_{D^2} \wedge C^{(1)} \wedge \mathcal{F} \right\}$$

with $\mathcal{F} = B + 2\pi \alpha' F$. $F$ is the 2-form field strength of the $U(1)$ gauge field on the brane. Equations of motion give $d \star G^{(4)} = J_{D^2} + H \wedge G^{(4)}$ i.e D2-branes are sources of the ‘electrical’ $\star G^{(4)} - B \wedge G^{(4)}$ field. \[[\ref{1}], \[\ref{9}]\] With no D4-branes we have $dG^{(4)} = 0$.

Collecting terms that couple to $C^{(1)}$ we find the total D0-charge to be

$$Q_{D0} = -\int_V \left\{ \star G^{(4)} \wedge H + J_{D^2} \wedge \mathcal{F} \right\}.$$ 

Again, it’s convenient to split $Q_{D0}$ into two parts; one associated with $J_{D^2}$ and the other with the field configuration:

$$Q^{D2}_{D0} = -\int_V J_{D^2} \wedge \mathcal{F}, \quad Q^{\text{field}}_{D0} = -\int_V \star G^{(4)} \wedge H.$$ 

The D0-charge evolves in time as follows:

$$\delta Q^{D2}_{D0} = -\int_\Omega d(J_{D^2} \wedge \mathcal{F})$$
\[ \delta Q_{D0}^{\text{field}} = - \int_{\Omega} d(*G^{(4)} \wedge H) \] (14)

Charge conservation properties follows from

\[
\begin{align*}
\delta Q_{D0} &= - \int_{\Omega} \left\{ d(J_{D2} \wedge F) + d(*G^{(4)} \wedge H) \right\} \\
&= - \int_{\Omega} \left\{ d \left( J_{D2} \wedge (B + 2\pi \alpha' F) + *G^{(4)} \wedge H \right) - H \wedge G^{(4)} \wedge H \right\} \\
&= - \int_{\Omega} \left\{ d \left( J_{D2} \wedge B + H \wedge G^{(4)} \wedge B + *G^{(4)} \wedge H \right) - 2\pi \alpha' J_{D2} \wedge dF \right\} \\
&= - \int_{\Omega} \left\{ dd(*G^{(4)} \wedge B) - 2\pi \alpha' J_{D2} \wedge dF \right\} \\
&= 2\pi \alpha' \mu_2 \int_{\Omega(\Sigma)} dF \\
\end{align*}
\] (15)

where we have used the property \( H \wedge H = 0 \). Note that while \( dF = 0 \) on any specific brane, we wish to consider a process where we expand a brane and in this context \( dF \) should be understood as the change of \( F \) as we pass through different brane configurations. Evidently, in this framework, charge is in general only conserved as long as we don’t expand or shrink the brane. However, from momentum conservation in M-theory we know that the charge has to be conserved, so if we wish to expand a brane we have to add charge from external sources. Using \([10]\)

\[ \mu_p = (2\pi)^{-p+1} e^{-\frac{2\pi}{k}} \] (16)

we find that the amount of charge we add when expanding a brane is

\[ \delta Q_{D0} = \mu_0 \int_{S^3} \frac{dF}{2\pi} \] (17)

Now we turn to the special case of a \( D2 \)-brane on the \( SU(2) \) group manifold. If we parametrize the manifold with \((\psi, \theta, \phi)\) where \( \psi \) describes the ”latitude” on the 3-sphere \((\psi = \pi \text{ and } \psi = 0 \text{ corresponding to } -e \text{ and } e \text{ respectively})\) and \((\theta, \phi)\) parametrizes the 2-spheres corresponding to fixed \( \psi \), then the brane is a 2-sphere sitting at fixed \( \psi = \frac{\pi n}{k} \) with \( n, k \) integers satisfying \( 0 < n < k \) \([11], [12]\). In the semi-classical limit \( k \to \infty \) we have a continuum of allowed brane configurations.

Imagine taking a \( D2 \)-brane from \( e \) and expanding it all the way to \( -e \). At the pole it becomes singular and decouples from the \( B \)-field so that we can transport it back to \( e \) without any loss or gain in momentum. (Note that since we work within the semi-classical limit \( k \to \infty \) we disregard quantum corrections that would change this picture.) This is the analog of taking the full circle in the dipole case. The net change in \( Q_{D0} \) of one such winding around \( S^3 \) is

\[ \delta Q_{D0} = \mu_0 \int_{S^3} \frac{dF}{2\pi} \] (18)

In the WZW level \( k \) model on \( SU(2) \) \([11], [13], [14]\) the \( H \)-field is given by the volume 3-form to be \([1]\),

\[ H = 2k\alpha' \sin^2 \psi \sin \theta d\psi d\theta d\phi \] (19)
Solving $H = dB$ gives

$$B = k\alpha' \left( \psi - \frac{\sin 2\psi}{2} \right) \sin \theta d\theta d\phi$$  \hspace{1cm} (20)

which is a well defined choice except at the point $\psi = \pi$. Another choice, $B'(\psi) = -B(\pi - \psi)$ is well defined everywhere except at the point $\psi = 0$. The two choices are related by a gauge transformation. Gauge invariance of $F = B + 2\pi \alpha' F$ yields

$$F = -\frac{k\psi}{2\pi} \sin \theta d\theta d\phi \text{ or } F' = -F(\pi - \psi).$$ \hspace{1cm} (21)

Evaluating the integral we find

$$\delta Q_{D0} = -\mu_0 k$$ \hspace{1cm} (22)

This is just the $mod$ $k$ ambiguity of RR-charge of $D2$-branes on $SU(2)$. Now we consider a general scenario where we first expand a membrane from pole-to-pole (as we did above) $\lambda$ times and then make an extra expansion under which the membrane traces out a 3-volume $\Gamma$ and ends up as the membrane $\Sigma$. Then, from (17):

$$\delta Q_{D0} = \mu_0 \left( \int_{\Sigma} \frac{F(\Gamma)}{2\pi} - \lambda k \right)$$ \hspace{1cm} (23)

with $F(\Gamma)$ being $F$ as defined on $\Gamma$. Since we started out with no charge, $\delta Q_{D0}$ should equal our expression (12) for the total charge $Q_{D0} = Q_{D2}^{D0} + Q_{field}^{D0}$. Using the expression following the first equality in (11) (rather then the second as we did to arrive at (12)) we find:

$$Q_{D2}^{D0} = \mu_2 \int_{\Sigma} F$$ \hspace{1cm} (24)

While expanding the brane we also create a $*G^{(4)}$ field and the charge stored in this created field together with the $H$ field gives a contribution

$$Q_{field}^{D0} = -\int_{\Omega} d\left( *G^{(4)} \wedge H \right) = -\int_{\Omega} J_{D2} \wedge H$$

$$= -\lambda \mu_2 \int_{S^3} H - \mu_2 \int_{\Gamma} H = -\mu_2 \int_{\Gamma} H - \lambda \mu_0 k$$ \hspace{1cm} (25)

Let $B(\Gamma)$ and $F(\Gamma)$ be choices of $B$ and $F$ that are well defined everywhere on $\Gamma$, then

$$Q_{D0} = \int_{\Sigma} \mu_2 (B + 2\pi \alpha' F) - \mu_2 \int_{\Sigma} B(\Gamma) - \lambda \mu_0 k = \mu_0 \left( \int_{\Sigma} \frac{F(\Gamma)}{2\pi} - \lambda k \right).$$ \hspace{1cm} (26)

As expected this is identical to (23). In order for this formula to make sense we need to relate $\lambda$ and $\Gamma$ to physical properties of our configuration rather than referring to the history of it: Let $M$ be the (compact) spatial directions transverse to $S^3$ and assume that the above described process of expanding a membrane takes place.
between \( x^0 = 0 \) and \( x^0 = T \). Then, at \( x^0 = T \) the flux of \( *G(4) - B \wedge G(4) \) on \( M \) at any given point \( x \) on \( S^3 \) is

\[
\int_M (\ast G^{(4)}(x) - B(x) \wedge G^{(4)}(x)) = \int_{M \times [0,T]} dx^0 \partial_0 (\ast G^{(4)}(x) - B(x) \wedge G^{(4)}(x))
\]

\[
= \int_{M \times [0,T]} d (\ast G^{(4)}(x) - B(x) \wedge G^{(4)}(x)) = \int_{M \times [0,T]} J_{D2}(x) = \int_{\Omega} J_{D2} \wedge \delta^{(3)}_{S^3}(x)
\]

\[
= \lambda \mu_2 \int_{S^3} \delta^{(3)}_{S^3}(x) + \mu_2 \int_{\Gamma} \delta^{(3)}_{S^3}(x) = (\lambda + \delta_{\Gamma}(x)) \mu_2
\]

where we have used the equations of motion and compactness of \( M \). In the above \( \delta^{(3)}_{S^3}(x) \) is a deltafunction type 3-form on \( S^3 \) with support limited to the point \( x \) and \( \delta_{\Gamma}(x) \) satisfies

\[
\delta_{\Gamma}(x) = \begin{cases} 
1 & \text{if } x \in \Gamma \\
0 & \text{if } x \notin \Gamma 
\end{cases}
\]

The general expression (26) for \( D0 \) charge on \( SU(2) \) is our main result. Since \( \Gamma \) and \( \lambda \) are determined by \( \int_M (\ast G^{(4)} - B(x) \wedge G^{(4)}(x)) \) we find that \( Q_{D0} \) is unambiguous. Also, integrality of \( \int_{\Sigma} F_2 \pi \) and \( \lambda \) gives \( Q_{D0} \) integral in units of \( \mu_0 \).

4 Conclusions

We have shown that ambiguities in the RR-charge on D-branes in B-field backgrounds can be removed by taking into account nontrivial configurations of the RR ‘electric’ field. This holds whenever all transverse directions \( M \) are compact. If there are non-compact directions the electric field configuration is unstable since \( \int_M (\ast G^{(4)} - B \wedge G^{(4)}) \) is no longer preserved. This allows charge to be transported to infinity and probably explains the charge ambiguity in the CFT approach.

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