Abstract. We describe a canonical sequence of modular blowups of the Artin stack of stable weighted genus-two curves that diagonalize certain canonical derived objects of the original Artin stack. This provides a resolution of the primary component of the moduli space of genus two stable maps to projective space, and a partial desingularization of the same moduli space. Our approach should extend to higher genera.

1. Introduction

The moduli spaces $\overline{M}_{g,m}(\mathbb{P}^n, d)$ of stable maps into projective space has been one of the central objects in algebraic geometry since its introduction by Kontsevich in early 90’s ([5], [1], [2]). We are interested in these spaces because $\overline{M}_{g,m}(\mathbb{P}^n, d)$ carries certain tautological derived objects and bundle stacks whose topological invariants when integrated over the virtual fundamental class of $\overline{M}_{g,m}(\mathbb{P}^n, d)$ gives us the GW invariants of complete intersections in the projective space.

To this end, we like to find a modular (partial) desingularization of moduli space so that these objects and bundle stacks can be defined and evaluated. Our solution is to work on certain direct images sheaves (complex), study their degeneracy loci and construct modular blowing-up based on these loci to make the direct image complex diagonalizable. Once this is achieved, the partial desingularization of the moduli space $\overline{M}_g(\mathbb{P}^n, d)$ follows by standard argument.

In this paper, we will work out the $g = 2$ case in details. The results proved will allow us to prove the “hyperplane” property of the GW invariants of quintic Calabi-Yau threefolds conjectured in [6]. We hope this work will also shed light to constructing modular blowing up of higher genus moduli space $\overline{M}_g(\mathbb{P}^n, d)$.

We let $\mathcal{M}_2^{rt}$ be the Artin stack of prestable weighted curves of genus 2 (i.e. nodal curves whose irreducible components are decorated by weights $\in \mathbb{Z}_{\geq 0}$, cf. [3]); let

$$ \varsigma_k : \overline{M}_2(\mathbb{P}^n, d) \rightarrow \mathcal{M}_2^{rt}; \quad [u, C] \rightarrow (C, c_1(u^* \mathcal{O}_{\mathbb{P}^n}(k))), \tag{1.1} $$

be the induced representable morphism. In this paper, we construct a sequence of modular blowups of $\mathcal{M}_2^{rt}$ along smooth centers: $\mathcal{M}_2^{rt} \rightarrow \widetilde{\mathcal{M}}_2^{rt}$. We set

$$ \widetilde{\mathcal{M}}_2^{rt}(\mathbb{P}^n, d) = \overline{M}_2(\mathbb{P}^n, d) \times_{\varsigma_k, \mathcal{M}_2^{rt}} \mathcal{M}_2^{rt}, \quad k \geq 2. $$

Theorem 1. Let $(\pi, f, C)$ be the universal family on $\overline{M}_2(\mathbb{P}^n, d)$. Suppose $k \geq 2$, then the pullback to $\overline{M}_2(\mathbb{P}^n, d)$ of derived object $R\pi_*f^* \mathcal{O}_{\mathbb{P}^n}(k)$ is locally diagonalizable.

For the definition of locally diagonalizable object, see [2]
Corollary 2. Let the situation be as in Theorem 1, and let $N$ be any irreducible component of $\overline{M}_g^1(\mathbb{P}^n, d)$ with $(\pi_N^k, f_N^k)$ the pullback to $N$ of $(\pi, f)$. Then the direct image sheaf $(\pi_N^k)_\ast(f_N^k)_\ast\mathcal{O}_{\mathbb{P}^n}(k)$ is locally free.

To obtain a partial desingularization of $\overline{M}_2(\mathbb{P}^n, d)$, we need the analogue of Theorem 1 for $k = 1$. For this we need one more round of modular-blowing ups along smooth centers in a refinement of $\mathcal{M}_{g}^{wt}$. We let $\mathcal{P}_2$ be the relative Picard stack over $\mathcal{M}_2$; its objects are $(C, L)$, where $C$ is a prestable genus-2 curve and $L$ is a line bundle over $C$. We have representable morphisms

$$\varrho_k : \overline{M}_2(\mathbb{P}^n, d) \to \mathcal{P}_2; \quad [u, C] \mapsto (C, u^*\mathcal{O}_{\mathbb{P}^n}(1)).$$

Theorem 3. We have a sequence of blow-ups $\overline{\mathcal{P}}_2 \to \mathcal{P}_2$ so that if we let

$$\tilde{M}_2(\mathbb{P}^n, d) = \overline{M}_2(\mathbb{P}^n, d) \times_{\mathcal{P}_2} \overline{\mathcal{P}}_2,$$

then the pullback to $\tilde{M}_2(\mathbb{P}^n, d)$ of the derived object $R\pi_*i^*\mathcal{O}_{\mathbb{P}^n}(1)$ is locally diagonalizable. Further, for $d > 2$ the stack $\tilde{M}_2(\mathbb{P}^n, d)$ has normal crossing singularities.

To diagonalize $R\pi_*i^*\mathcal{O}_{\mathbb{P}^n}(k)$, we perform three sequences of blow-ups of $\mathcal{M}_{g}^{wt}$ to obtain the smooth stack $\mathcal{M}_{g}^{wt}$; in the case $k = 1$, we perform one more sequence of blow-ups of $\mathcal{P}_2 \times_{\mathcal{M}_{g}^{wt}} \mathcal{M}_{g}^{wt}$ to obtain the smooth stack $\mathcal{P}_2$. We describe briefly the blowing-up centers of these sequences of blowing-ups.

Our innovative approach improves the technique of [3] and combines with the idea of the derived modular blowups in [4]. To explain this, we introduce the Artin stack $\mathcal{M}_{g}^{div}$ of pairs $(C, D)$ where $C$ is a prestable genus-2 curve and $D$ is a simple divisor supported on the smooth locus of $C$. The stack $\mathcal{M}_{g}^{div}$ comes equipped with a universal curve $\rho : C \to \mathcal{M}_{g}^{div}$ and a universal divisor $\mathcal{D}$; there is a canonical morphism $\mathcal{M}_{g}^{div} \to \mathcal{M}_{g}^{wt}$, $(C, D) \to (C, c_1(\mathcal{O}_C(D)))$; via this morphism, we study the parallel diagonalization problem of the derived object $\rho_*\mathcal{O}_C(\mathcal{D})$. Locally over a smooth chart $\mathcal{V}$, the object $\rho_*\mathcal{O}_C(\mathcal{D})|_{\mathcal{V}}$ can be represented by a two term structural homomorphism $\varphi : \mathcal{O}_{\mathcal{V}}^{\otimes m+1} \to \mathcal{O}_{\mathcal{V}}^{\otimes 2}$ where $m = \deg \mathcal{D}$. We provide a detail analysis of this homomorphism in terms of local modular parameters. This analysis guides to the following blowups.

The first round blowups. Let $\Pi_A = [0, d] \cap \mathbb{Z}$. We introduce a (first order) depth function $\ell : \mathcal{M}_{g}^{wt} \to \Pi_A$; for each point $x \in \mathcal{M}_{g}^{wt}$, the invariant $\ell(x)$ quantitatively measures how far the homomorphism $\varphi : \mathcal{O}_{\mathcal{V}}^{\otimes m+1} \to \mathcal{O}_{\mathcal{V}}^{\otimes 2}$ from being of rank 1. For any $k \geq 1$, we let

$$\Theta^1_k = \{x \in \mathcal{M}_{g}^{wt} \mid \ell(x) = k\}.$$

We then blow up $\mathcal{M}_{g}^{wt}$ along (the proper transforms of) $\Theta^1_1, \Theta^1_2, \ldots, \Theta^1_d$. We let $\mathcal{N}_1$ be the final blowup stack of the first round.

The second round blowups. Let $\Pi_B = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq j < i \leq d\}$. We provide a total ordering of $\Pi_B$ and list its elements by its order as $\Pi_B = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ where $r = \frac{d(d+1)}{2}$. We then construct a (second order) depth function $\ell' : \mathcal{N}_1 \to \Pi_B$; for each point $x \in \mathcal{N}_2$, the invariant $\ell'(x)$ measures how far (the pullback of) the homomorphism $\varphi : \mathcal{O}_{\mathcal{V}}^{\otimes m+1} \to \mathcal{O}_{\mathcal{V}}^{\otimes 2}$ from being of rank 2. For any $k \geq 1$, we let

$$\Theta^2_k = \{x \in \mathcal{N}_2 \mid \ell'(x) = \alpha_k\}.$$
We then blow up $\mathcal{H}_1$ along (the proper transforms of) $\Theta^2_1, \Theta^2_2, \ldots, \Theta^2_2$. We let $\mathcal{H}_2$ be the final blowup stack of this round.

After the second round, for any point $x = (C, w) \in \mathfrak{M}^{\text{wt}}_2$, its neighborhood has been modified so that the pullback of the structural homomorphism $\varphi$ becomes diagonalizable, possibly except (1) when the smallest genus-2 subcurve $F$ of $C$, called the core of $C$, has rational tails attached at conjugate points (or Weierstrass points) of $F$; (2) when the core $F$ has exactly weight 2.

The third round blowups. For the first possibility (1), we construct a (third order) depth function $\ell'' : \mathfrak{H}_2 \to \Pi_C$ where $\Pi_C = [0, d] \cap \mathbb{Z}$. The rest goes almost the same as above. This round produces our stack $\widetilde{\mathfrak{M}}^{\text{wt}}_2$. We leave the second possibility (2) to the last round of blowing-ups.

We point out here that the $\widetilde{\mathfrak{M}}^{\text{wt}}_2$ diagonalizes the derived objects $R\pi_* f^* O_{\mathfrak{P}^n}(k)$ for all $k \geq 2$ except $R\pi_* f^* O_{\mathfrak{P}^n}(1)$. To resolve $R\pi_* f^* O_{\mathfrak{P}^n}(1)$ and hence a (partial) desingularization $\mathcal{M}_2(\mathbb{P}^n, d)$ ($d > 2$), we need one more round of blowups to resolve the locus of $\mathcal{M}_2(\mathbb{P}^n, d)$ whose general points are stable maps such that their restrictions to the cores are double covers of smooth rational curves.

The fourth round blowups. The procedure of this round is analogous to the third round (i.e., relying on a third order depth function). We let $\mathcal{H}_k$ be the closed substack of $\mathfrak{P}_2$ whose general points are pairs $(C, L)$ with $F$ the core of $C$ such that $\deg L|_F = 2$, $h^0(L|_F) = 2$ and the core $F$ have $k$ many rational tails. We then blow up the stack $\mathfrak{P}_2 \times \mathfrak{M}^{\text{wt}}_2 \mathcal{M}_2$ along (the proper transforms of ) $\mathcal{H}_1 \times \mathfrak{M}^{\text{wt}}_2, \mathcal{M}_2, \ldots, \mathcal{H}_m \times \mathfrak{M}^{\text{wt}}_2, \ldots$. We denote the resulting stack by $\mathfrak{P}_2$.

Once the Theorem 3 is established, applying the technique in [8], we obtain the local structure of the total transforms $\mathcal{M}_2(\mathbb{P}^n, d)$. We will provide the details of this proof in Section [7].

Throughout the paper, we work over an algebraically closed field of characteristic zero.

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2. Locally Diagonalizable Derived Objects

We begin with introducing diagonalizable derived objects, a key notion used throughout the article.

2.1. Diagonalizable homomorphism. Diagonalizable homomorphism is introduced in [4].

**Definition 2.1.1.** Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two locally free sheaves of $\mathcal{O}_U$-modules on a DM stack $U$. We say a homomorphism $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ is diagonalizable if there are integers $r, l_1$ and $l_2 \in \mathbb{Z}_{\geq 0}$, $p_i \in \Gamma(\mathcal{O}_U)$ such that the ideals $(p_i) \subset (p_{i+1})$, and isomorphisms $\mathcal{E}_i \cong \mathcal{O}_U^{\oplus r} \oplus \mathcal{O}_U^{\oplus l_1}$ such that

$$\varphi = \text{diag}[p_1, \ldots, p_r] \oplus 0 : \mathcal{O}_U^{\oplus r} \oplus \mathcal{O}_U^{\oplus l_1} \longrightarrow \mathcal{O}_U^{\oplus r} \oplus \mathcal{O}_U^{\oplus l_2},$$
where \( \text{diag}[p_1, \cdots, p_r] : \mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus r} \) is the diagonal homomorphism; \( 0 : \mathcal{O}_U^{\oplus 1} \to \mathcal{O}_U^{\oplus 2} \) is the zero homomorphism. We say \( \varphi \) is locally diagonalizable if there is an étale cover \( U_\alpha \) of \( U \) such that \( \varphi \) is diagonalizable over \( U_\alpha \).

It is direct to check that in case \( \varphi : \mathcal{E} \to \mathcal{F} \) is locally diagonalizable homomorphism of sheaves of \( \mathcal{O}_X \)-modules, then for every integral closed \( Z \subseteq X \), \( \ker(\varphi|_Z) \) is locally free.

2.2. Diagonalizable derived object.

**Definition 2.2.1.** Let \( \mathbf{E} \) be a two-term perfect derived object on a DM stack \( X \). We say \( \mathbf{E} \) is locally diagonalizable if there is an étale cover \( \{U\} \) of \( X \) such that \( \mathbf{E}|_U \) can be represented by a diagonalizable homomorphism \( \varphi : \mathcal{E}_1 \to \mathcal{E}_2 \) where \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are two locally free sheaves of \( \mathcal{O}_U \)-modules.

Using Proposition 3.2, [4] (the universality of diagonalization), one checks directly that the definition does not depend on the local representation of the object \( \mathbf{E} \). Here it is worth to point out that in Definition 2.1.1, if the requirement \( (p_{i+1}) \subset (p_i) \) is removed, then the “diagonalization” thus defined is not a property of the derived object but merely a property of a presentation of the object \( \mathbf{E} \) (cf. Example ??).

3. The Structures of the Tautological Derived Objects

We will set up the notation and derive the local description of the homomorphism that will be the basis of our modular blowing-ups mentioned in the introduction.

In studying \( \overline{\mathcal{M}}_2(\mathbb{P}^n, d) \) together with \( \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k) \), we will cover it by étale opens \( U \to \overline{\mathcal{M}}_2(\mathbb{P}^n, d) \) and pick degree \( k \) hypersurfaces \( H \subset \mathbb{P}^n \) in general position so that for each \( [u, C] \in U \), we have \( u^{-1}(H) \subseteq C \) is a simple divisor lie in the smooth locus of \( C \). We let \( \mathcal{M}_2^{\text{div}} \) be the Artin stack of prestable pairs \((C, D)\) where \( C \) are nodal curves and \( D \) are effective divisors on \( C \). Thus, the assignment \( [u, C] \to (C, u^{-1}(H)) \) define morphism

\[
(3.1) \quad \overline{\mathcal{M}}_2(\mathbb{P}^n, d) \supset U \longrightarrow \mathcal{M}_2^{\text{div}}.
\]

We pick an affine étale \( V \to \mathcal{M}_2^{\text{div}} \) so that the \( U \to \mathcal{M}_2^{\text{div}} \) factor through \( U \to V \).

3.1. Basic setup and the sheaf \( \mathcal{L} \). Using the (local) morphism (3.1), it suffices to study a parallel problem of \( \mathcal{M}_2^{\text{reg}} \) (for \( \mathcal{O}_{\mathbb{P}^n}(k) \)) and later for \( \mathcal{M}_2^{\text{reg}} \).

For a fixed \((C, D) \in \mathcal{M}_2^{\text{reg}}\) lying over \((C, w) \in \mathcal{M}_2^{\text{reg}}\), we pick an affine smooth chart \( V \to \mathcal{M}_2^{\text{reg}} \) containing \((C, D)\); let \( \rho : C \to V \) and \( D \subset C \) be the universal family on \( V \). We find disjoint sections \( A_1, A_2, B \subset C \) of \( C/V \), disjoint from \( D \). We list this package as

\[
(3.2) \quad \rho : C \longrightarrow V \to \mathcal{M}_2^{\text{reg}} \quad \text{and} \quad D \subset C; \quad \text{plus} \quad A = A_1 + A_2, B \subset C.
\]

Again, \( D \subset C \) is the universal divisor and \( A, B \subset C \) are auxiliary divisors.

We let

\[
\mathcal{L} = \mathcal{O}_C(D), \quad \mathcal{M} = \mathcal{O}_C(A - B), \quad \text{and} \quad \mathcal{O}_A(A) = \mathcal{O}_V(A) \otimes_{\mathcal{O}_C} \mathcal{O}_A,
\]

and consider the two term complex

\[
(3.3) \quad \varphi : \rho_* \mathcal{L}(A) \longrightarrow \rho_* \mathcal{O}_A(A)
\]

via the evaluation homomorphism.
Basic Assumptions. We assume throughout this section that
A. \( a_t = A_t \cap C \) and \( b = B \cap C \) lie in the core curve \( F \) of \( C \); in case \( F \) is separable, then \( a_1 \) and \( a_2 \) lie on different genus-one subcurves of \( F \); we assume that \( a_1, a_2 \) and \( b \) are in general position in \( F \).

B. In case \( D \) is simple, by an étale base change, we can assume that \( D = D_1 + \cdots + D_d \) is a union of \( d \) disjoint sections \( D_i \) of \( C \to \mathcal{V} \).

C. By shrinking \( \mathcal{V} \) if necessary, \( A \) and \( B \) holds for all fibers of \( C \to \mathcal{V} \).

Under the assumptions, we have \( R^1 \rho_* \mathcal{L}(A) = 0 \); \( \rho_* \mathcal{L}(A) \) is locally free, and \( R\rho_* \mathcal{L} \) is quasi-isomorphic to the complex \( (3.3) \).

3.2. Separating nodes and conjugate points. We call a node \( q \) (resp. an irreducible component) of \( C \) separating if \( C - q \) (resp. \( C - \Sigma \)) is disconnected. An inseparable curve is a curve with no separating node; an inseparable component of \( C \) is a smallest (via inclusion) inseparable subcurve of \( C \).

Two smooth points \( p \) and \( q \) of an inseparable genus-2 curve \( C \) are conjugate to each other if \( \mathcal{O}_C(p + q) \cong \mathcal{O}_C \); when \( p = q \), we call it a Weierstrass point of \( C \).

Any nodal genus-2 curve \( C \) can be obtained by attaching disjoint connected trees of rational curves at disjoint smooth points of its core \( F \). We call these connected trees tails. We call the unique irreducible component of a tail that meets \( F \) the pivotal component of the tail. A rational tail may contain several chains of rational curves; they share the same pivotal component.

Definition 3.2.1. We say that a rational tail \( T \) (resp. a maximal rational chain \( R \)) is of Weierstrass if its pivotal component intersects the core \( F \) at its Weierstrass point. We say that two rational tails (resp. two maximal rational chains) are conjugate if their pivotal components intersect \( F \) at two conjugate points of \( F \).

3.3. The first reduction. We let \( \rho : (C, D) \to \mathcal{V}, \ A,B \subset C \) and \( (C, D) \times_Y 0 = (C, D) \) be as in \( (3.2) \), satisfying the basic assumptions.

First, the standard inclusion \( \mathcal{O}_C \subset \mathcal{O}_C(D) \) provides us the obvious section \( 1 \in \Gamma(\rho_* \mathcal{L}) \). For other sections, we let \( \mathcal{M} = \mathcal{O}_C(A - B) \) and consider the inclusion

\[
\mathcal{M}(D_i) = \mathcal{O}_C(D_i + A - B) \xrightarrow{\sim} \mathcal{M}(D) = \mathcal{O}_C(D + A - B),
\]

and the induced inclusions

\[
\eta_i : \rho_* \mathcal{M}(D_i) \xrightarrow{\subset} \rho_* \mathcal{M}(D).
\]

Because of the basic setups, both are locally free. By Riemann-Roch, \( \rho_* \mathcal{M}(D_i) \) is invertible and \( \rho_* \mathcal{M}(D) \) has rank \( d \). We let

\[
(\varphi) : \rho_* \mathcal{M}(D) \to \rho_* \mathcal{O}_A(A) \quad \text{and} \quad (\varphi_i) : \rho_* \mathcal{M}(D_i) \to \rho_* \mathcal{O}_A(A)
\]

be the evaluation homomorphisms; then \( \varphi_i = \varphi \circ \eta_i \). We denote their sum by

\[
(\eta) : \oplus_{i=1}^d \rho_* \mathcal{M}(D_i) \to \rho_* \mathcal{M}(D), \quad \text{and} \quad (\varphi_i) : \oplus_{i=1}^d \rho_* \mathcal{M}(D_i) \to \rho_* \mathcal{O}_A(A).
\]

Lemma 3.3.1. We have (1). \( \rho_* \mathcal{L} \cong \mathcal{O}_\mathcal{V} \oplus \rho_* \mathcal{L}(-B) \); (2). \( \rho_* \mathcal{L}(-B) \cong \text{ker} \varphi \); (3). \( \eta \) is an isomorphism, and (4) \( (\varphi_i) = \varphi \circ (\eta) \). Consequently,

\[
\rho_* \mathcal{L} \cong \mathcal{O}_\mathcal{V} \oplus \text{ker}\{\varphi_i\}.
\]

Proof: The proofs of (1) and (2) are the same as for the genus-1 case, Lemma 4.10 of \( [3] \); we omit the details. We now prove (3). Since both \( \rho_* \mathcal{M}(D_i) \) and \( \rho_* \mathcal{M}(D) \) are locally free, we only need to show that for any closed \( z \in \mathcal{V}, \oplus_{i=1}^d d \rho_* \mathcal{M}(D_i) |_z \to \rho_* \mathcal{M}(D) |_z \)
$\rho_*, \mathcal{M}(D)|_x$ is an isomorphism. By base change, it suffices to show that the tautological homomorphism

$$
\bigoplus_{i=1}^d \eta_i(z) : \bigoplus_{i=1}^d H^0(\mathcal{O}_{C_i}(D_i + A - B)) \longrightarrow H^0(\mathcal{O}_{C_x}(D + A - B))
$$

is an isomorphism. Because both sides are of equal dimensions, it suffices to show that it is injective. The injectivity follows from that $D_1 \cap C_2, \ldots, D_d \cap C_2$ are distinct in $C_2$. The item (4) is a tautology. This proves the lemma. \qed

This proves that the homomorphism $\varphi$ in (3.4) is equivalent to the homomorphism $(\varphi_i)$, via the isomorphism $(\eta_i)$.

We will call either $\varphi$ or $(\varphi_i)$ a structural homomorphism. Note that each $\varphi_i$ has the form

$$(3.5) \quad \varphi_i = \varphi_{i,1} \oplus \varphi_{i,2} : \rho_* \mathcal{M}(D_i) \longrightarrow \rho_* \mathcal{O}_{A_i}(A_1) \oplus \rho_* \mathcal{O}_{A_i}(A_2).$$

These homomorphisms will be our focus in this section.

3.4. **The structural homomorphism.** We begin with introducing regular functions associated to node-smoothing. By the deformation theory of nodal curves, for each separating node $q \in C$ there is a regular function $\zeta_q \in \Gamma(\mathcal{O}_V)$ so that $\Sigma_q = \{\zeta_q = 0\}$ is the locus where the node $q$ is not smoothed; the divisor $\Sigma_q$ is a smooth divisor. We will refer $\zeta_q$ as a modular parameter.

To proceed, we introduce the notation of a node lies between two points (or components). We say a (separating) node $q$ separates (or lies between) two smooth points $x$ and $y$ in $C$ if and $x$ and $y$ lie in different connected components of $C - q$; in case we replace $y$ by a connected subcurve $B \subset C$, we say $q$ lies between $x$ and $B$ if $q$ is not a node of $B$, and $x$ and $B - q$ lie in different connected components of $C - q$. We denote by $N_{[x,y]}$ (resp. $N_{[x,B]}$) to be the set of nodes lying between $x$ and $y$ (resp. $B$).

Now back to the family $C \rightarrow V$ with $C = C \times_V 0$ the central fiber mentioned before. For $1 \leq i \leq m$ and $1 \leq i \leq 2$, we denote

$$\delta_i = D_i \cap C \quad \text{and} \quad a_s = A_s \cap C.$$ 

We form the products of modular parameters via

$$(3.6) \quad \zeta_{[\delta_i,a_s]} = \prod_{q \in N_{[\delta_i,a_s]}} \zeta_q, \quad \text{and} \quad \zeta_{\delta_i} = \prod_{q \in N_{[\delta_i,B]}} \zeta_q.$$

(Here as always, $F \subset C$ is its core curve.) In case $N_{[\delta_i,a_s]} = \emptyset$, we set $\zeta_{[\delta_i,a_s]} = 1$.

We fix trivializations

$$
(3.7) \quad \rho_* \mathcal{O}_{A_s}(A_s) \cong \mathcal{O}_V, \quad s = 1, 2; \quad \text{and the induced} \quad \rho_* \mathcal{O}_A(A) \cong \mathcal{O}_V^{\oplus 2}
$$

throughout this section.

**Proposition 3.4.1.** For $1 \leq i \leq m$ and $1 \leq s \leq 2$, after fixing trivializations $\rho_* \mathcal{M}(D_i) \cong \mathcal{O}_V$ and \cite[(3.7)], the evaluation homomorphism

$$(3.8) \quad \varphi_i^s : \rho_* \mathcal{M}(D_i) \longrightarrow \rho_* \mathcal{O}_{A_s}(A_s)$$

is given by the multiplication

$$(\varphi_i^s =) \ c_i^s \cdot \zeta_{[\delta_i,a_s]}^* : \mathcal{O}_V \longrightarrow \mathcal{O}_V, \quad c_i^s \in \Gamma(\mathcal{O}_V^*) \equiv \Gamma(\mathcal{O}_V^{\oplus 2}).$$

**Proof.** The proof is parallel to that of \cite[Prop. 4.13]{3}; thus will be omitted. \qed
3.5. The case of positive inseparable core. In this Subsection, we assume $F$ is inseparable and $\deg F \cap D > 0$. Since $\deg F \cap D = 1$ is covered in Proposition 3.4.1, we assume $\deg F \cap D \geq 2$.

We order the set $D \cap F = \{\delta_1, \cdots, \delta_s\}$ so that when $\deg D \cap F \geq 3$, then $\delta_1$ and $\delta_2$ are not conjugate to each other. As always, then $D_i$ are so ordered that $\delta_i = C \cap D_i$. We denote

$$\varphi^F = \bigoplus_{\delta_i \in D \cap F} \varphi_i : \bigoplus_{\delta_i \in D \cap F} \rho_*\mathcal{M}(D_i) \rightarrow \rho_*\mathcal{O}_A(A).$$

Proposition 3.5.1. Suppose $F$ of $C$ is inseparable, and $\deg D \cap F \geq 2$. Then there is a trivialization $\oplus_{i=1}^r \rho_*\mathcal{M}(D_i) \cong \Omega^2_{\mathcal{O}^r}$, together with (3.7), such that $\varphi^F$ takes the form

$$\varphi^F = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \end{bmatrix}$$

with $c_{ij} \in \Gamma(\Omega^2_{\mathcal{O}^r})$ (for all $i, j$) such that $\det(c_{ij})_{1 \leq i, j \leq 2}$ vanishes at 0 if and only if $\delta_1$ and $\delta_2$ are conjugate to each other.

Proof. The matrix form (3.9) follows directly from Proposition 3.4.1 using that $\zeta_{[\delta_i, \delta_j]} = 1$, for all $1 \leq i \leq r$ and $s = 1, 2$.

Next, $\det(a_{ij})_{1 \leq i, j \leq 2}$ vanishes at 0 if and only if $h^0(\mathcal{O}_F(\delta_1 + \delta_2 - b)) = 1$ and only if $h^0(\mathcal{O}_F(\delta_1 + \delta_2)) = h^0(\mathcal{O}_F(\delta_1 + \delta_2 - b)) + 1 = 2$ (because the point $b$ is in general position). The latter is equivalent to that $\delta_1$ and $\delta_2$ are conjugate to each other. \qed

For later use, we remark that when $\deg D \cap F \geq 3$, we can re-arrange (if necessary) such that $\delta_1$ and $\delta_2$ are not conjugate to each other. Thus, $\varphi^F$ is diagonalizable in this case.

3.6. A technical proposition on rational curves. We now focus on a maximal chain $R \subset C$ of ismooth rational curves $R = R_1 \cup \cdots \cup R_n$ of $C$ attached to a smooth point of the core $F \subset C$.

We fix some notations. First, we index the $R_j$ so that $R_1$ is attached to $F$, and $R_{j+1}$ is attached to $R_j$. We let $q_1 = F \cap R_1$, and $q_{j+1} = R_j \cap R_{j+1}$. Let $\zeta_{d_j}$ be the modular function associated with $q_j$. We list by the ordering

$$D \cap R = \{\delta_1, \delta_2, \cdots, \delta_r\}$$

such that if we define $e(i)$ be such that $\delta_i \in R_{e(i)}$, then $e$ is nondecreasing. We let

$$\varphi^R : \bigoplus_{\delta_i \in D \cap R} \rho_*\mathcal{M}(D_i) \rightarrow \rho_*\mathcal{O}_A(A)$$

be the restriction homomorphism.

Proposition 3.6.1. Let $R$ be a maximal chain of rational curves in $(C, D)$ as stated. Then after choosing suitable trivialization $\oplus_{\delta_i \in D \cap R} \rho_*\mathcal{M}(D_i) \cong \Omega^2_{\mathcal{O}^r}$ and (3.7), the homomorphism $\varphi^R$ takes the form

$$\varphi^R = \begin{bmatrix} c_{11}\zeta_{[\delta_1, a_1]} & c_{12}\zeta_{[\delta_1, a_1]}\zeta_{[\delta_2]} & c_{13}\zeta_{[\delta_1, a_1]}\zeta_{[\delta_2]}\zeta_{[\delta_3]} & \cdots \\ c_{21}\zeta_{[\delta_1, a_2]} & c_{22}\zeta_{[\delta_1, a_2]}\zeta_{[\delta_2]} & c_{23}\zeta_{[\delta_1, a_2]}\zeta_{[\delta_2]}\zeta_{[\delta_3]} & \cdots \end{bmatrix}.$$
with \(c_{11}, c_{21} \in \Gamma(\mathcal{O}_X^*)\) and \(c_{ij} \in \Gamma(\mathcal{O}_Y^*)\) for all \(j > 1\). Further, when \(F\) is inseparable, rank \(\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} (0) = 2\); rank \(\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} (0) = 2\) if and only if \(q_1\) is not a Weierstrass point of \(F\).

**Proof.** We focus on the maximal chain \(R \subset C\) of smooth rational curves \(R = R_1 \cup \cdots \cup R_h\) of \(C\) attached to a smooth point of the core \(F \subset C\).

Let \(\zeta_{q_j}\) be the modular function associated with \(q_j\). Since \(q_j\) is separating,

\[
\mathcal{C} \times \mathcal{V} (\zeta_{q_j} = 0) = \mathcal{G}_j \cup \mathcal{R}_j
\]

is a union of two families of nodal curves over \((\zeta_{q_j} = 0)\) intersecting (i.e. glued along) a section of nodes \(\Sigma_j = \mathcal{G}_j \cap \mathcal{R}_j\), such that \(\Sigma_j \cap \mathcal{C} = q_j\) and \(\mathcal{R}_j \cap \mathcal{C} = R_{j+1} \cup \cdots \cup R_h\).

We let \(\mathcal{M}_{2,ss}\) be the Artin stack of semi-stable genus two nodal curves. (A nodal curve is semi-stable if its smooth rational subcurves contain at least two nodes of the nodal curve.) By (successively) contracting rational subcurves that contain only one nodes of the ambient curves, we obtain a semi-stabilization \(\mathcal{M}_2 \to \mathcal{M}_{2,ss}\). We define

\[
\mathcal{M}_{2,ss}^\text{div} = \mathcal{M}_{2,ss} \times_{\mathcal{M}_2^\text{div}} \mathcal{M}_2^\text{div}.
\]

Possibly after shrinking and base changing \(\mathcal{V}\), we can assume that there is a smooth chart \(\mathcal{V} \to \mathcal{M}_{2,ss}\) with \(\mathcal{X} \to \mathcal{V}\) its universal family so that the composite \(\mathcal{V} \to \mathcal{M}_{2,ss}^\text{div} \to \mathcal{M}_{2,ss}\) lifts to \(g\) together with a semi-stable contraction \(\tilde{g}\) as shown

\[
\begin{array}{ccc}
\mathcal{V} & \to & \mathcal{M}_{2,ss}^\text{div} \\
\downarrow & & \downarrow \\
\mathcal{V} & \to & \mathcal{M}_{2,ss}
\end{array}
\]

(3.13) \(\tilde{g}\) contracts all rational tails of the curves in the family \(\mathcal{C} \to \mathcal{V}\).) We let \(\mathcal{D}_i = \tilde{g}(\mathcal{D}_i)\), which are family of smooth divisors of \(\mathcal{C} \to \mathcal{V}\), and let

\[
\mathcal{D}_R = \sum_{i=1}^{r} \mathcal{D}_i, \quad \mathcal{D}_R = \sum_{i=1}^{r} \mathcal{D}_i.
\]

Let \(\tilde{\rho} : \mathcal{C} \to \mathcal{V}\) be the second projection of (3.13). Because of our construction, the divisors \(\mathcal{R}_1, \cdots, \mathcal{R}_h\) are contracted under the morphism \(\tilde{g}\), and

\[
\tilde{g}^{-1}(\mathcal{D}_i) = \mathcal{R}_1 + \cdots + \mathcal{R}_{e(i)} + \mathcal{D}_i
\]

(3.14) where \(e\) is the index function introduced in (3.10).

We let \(r_i\) be the number of points in \(R_i \cap \mathcal{D}\); thus \(\sum_{i=1}^{h} r_i = r\). We denote \(\mathcal{R} = \sum_{i=1}^{h} (r_i + \cdots + r_h) \mathcal{R}_i\); then

\[
\tilde{g}^* \mathcal{O}_\mathcal{C}(\mathcal{D}_R + \mathcal{A} - \mathcal{B}) = \mathcal{O}_\mathcal{C}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}).
\]

Then we have

\[
\rho_* \mathcal{O}_\mathcal{C}(\mathcal{D}_R + \mathcal{A} - \mathcal{B}) \subseteq \rho_* \mathcal{O}_\mathcal{C}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}) = \tilde{\rho}_* \mathcal{O}_\mathcal{C}(\mathcal{D}_R + \mathcal{A} - \mathcal{B}).
\]

Here the identity follows from combining (3.15) and the identities

\[
\rho_* \mathcal{O}_\mathcal{C}(\mathcal{D}_R + \mathcal{A} - \mathcal{B}) = \tilde{\rho}_* \tilde{g}_* \mathcal{O}_\mathcal{C}(\mathcal{D}_R + \mathcal{A} - \mathcal{B}) = \tilde{\rho}_* \mathcal{O}_\mathcal{C}(\mathcal{D}_R + \mathcal{A} - \mathcal{B}),
\]

where the last equality holds since both \(\mathcal{C}\) and \(\mathcal{D}\) are smooth, \(\tilde{g} : \mathcal{C} \to \mathcal{D}\) is a proper divisorial contraction.
As before, we denote \( a_i = F \cap A_i, \) \( a = a_1 + a_2, \) and \( b = F \cap B. \) Since they are in general position, we have \( H^1(\mathcal{O}_F(rp_1 + a - b)) = 0. \) Thus by our Basis Sets up on \( V, \) we have that \( \bar{\rho}_* \mathcal{O}_C(\overline{D}_R + \overline{A} - \overline{B}) \) is a free \( \Gamma(\mathcal{O}_V) \)-module. For convenience, we denote

\[
M = \Gamma(\bar{\rho}_* \mathcal{O}_C(\overline{D}_R + A - B)) \quad \text{and} \quad \overline{M} = \Gamma(\bar{\rho}_* \mathcal{O}_C(\overline{D}_R + \overline{A} - \overline{B})).
\]

We now pick a basis of \( \overline{M}. \) Since \( a \) and \( b \) are in general positions, by a simple vanishing argument, we conclude that \( H^j(\mathcal{O}_F(a - b)) = 0 \) for all \( j. \) Hence, the restriction homomorphism

\[
H^0(\mathcal{O}_F(rp_1 + a - b)) \rightarrow H^0(\mathcal{O}_F(rp_1)|_{rp_1})
\]

is an isomorphism. Therefore, we can pick a basis \( s_1, \ldots, s_r \) of \( H^0(\mathcal{O}_F(rp_1 + a - b)) \) so that \( s_i \) has vanishing order \( (r - i) \) at \( p_1. \)

Next, we let \( H_i = (\zeta_i = 0) \subseteq V, \) and let \( H = \bigcup_{i=1}^r H_i. \) We let

\[
\bar{\rho}_H : \mathcal{T}_H := \mathcal{T} \times_Y H \rightarrow H
\]

be the projection; let \( \bar{\Sigma}_i = \bar{g}(\Sigma_i), \) where \( \Sigma_i = G_i \cap R_i. \) For the same reason, \( \overline{M}_R := (\bar{\rho}_H)^*(\mathcal{O}_C(\overline{D}_R + \overline{A} - \overline{B})) \) is a free \( \Gamma(\mathcal{O}_\mathcal{C}) \)-module, and has a basis \( \bar{S}_{H,i}, \cdots \bar{S}_{H,r} \) so that \( \bar{S}_{H,i} \) lifts to \( S_{H,i} \in \Gamma(\mathcal{O}_C(\overline{D}_R + \overline{A} - \overline{B})(-(r - i)\Sigma_i)) \) such that \( S_{H,i} \) are non-vanishing along \( \Sigma_i. \)

By shrinking \( V \) if necessary, we can extend \( \bar{S}_{H,i} \) to \( \bar{S}_i \in \overline{M} \) so the later (as a free \( \Gamma(\mathcal{O}_V) \)-module) is generated freely by \( \bar{S}_1, \cdots, \bar{S}_r. \) Thus \( \bar{g}^* \bar{S}_1, \cdots, \bar{g}^* \bar{S}_r \) freely generates the free \( \Gamma(\mathcal{O}_V) \)-module \( \Gamma(\mathcal{O}_C(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B})). \)

**Sublemma.** Recall \( \zeta_{[q_j]} = \prod_{i=1}^j \zeta_{[q_i]}, \) and let \( S_i = \zeta_{[q_i]} \cdot \bar{g}^* \bar{S}_i. \) Then the submodule \( M \subseteq \overline{M} \) is a free rank \( r \) \( \Gamma(\mathcal{O}_V) \)-module generated by \( S_1, \cdots, S_r. \)

**Proof of Sublemma.** We begin with introduction useful convention. We let \( U_i \subseteq C \) be an affine open that contains the generic point \( \xi_i \) of \( R_i. \) We let \( u_i \in \Gamma(\mathcal{O}_{U_i}) \) be so that \( R_i \cap U_i = (u_i = 0) \subseteq U_i. \) Since \( R_i \to H_i \) is a family of rational curves and has general fibers isomorphic to \( \mathbb{P}^1, \) we can find a rational function \( v_i \) on \( U_i \cap R_i \) so that \( v_i \) restricts to general fibers of \( R_i \cap U_i \to H_i \) are birational maps to \( k^1. \) We next let \( \xi'_i \) be the generic point of \( \Sigma_i. \) Since \( \bar{g}(\xi_i) = \xi'_i, \) the field \( K_i = \mathcal{O}_{\xi_i} \) contains \( K'_i = \mathcal{O}_{\xi'_i} \) as its subfield. Because of our choice of \( v_i, \) we have \( K_i = K'_i(v_i). \)

We let \( \hat{\mathcal{O}}_{U_i} \) be the formal completion of \( \mathcal{O}_{U_i} \) along \( \xi_i. \) Then \( \hat{\mathcal{O}}_{U_i} = K_i[[\hat{u}_i]], \) where \( \hat{u}_i \) is the image of \( u_i \) in \( \hat{\mathcal{O}}_{U_i}. \) (For any \( f \in \Gamma(\mathcal{O}_V), \) we denote by \([f] \) its image in \( \hat{\mathcal{O}}_{U_i} \) via the pullback \( \mathcal{O}_V \to \mathcal{O}_{U_i} \) and the completion map.)

We next pick an open \( U'_i \subseteq C \) that contains the generic point \( \xi'_i \) of \( \Sigma_i. \) By shrinking \( U_i \) if necessary, we can assume \( \bar{g}(U_i) \subseteq U'_i. \) We then fix a trivialization \( \mathcal{O}_{U'_i}(\overline{D}_R + \overline{A} - \overline{B}) \cong \mathcal{O}_{U'_i}. \) And form the induced trivialization

\[
\mathcal{O}_{U_i}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}) \cong \bar{g}^* \mathcal{O}_{U'_i}(\overline{D}_R + \overline{A} - \overline{B}) \cong \mathcal{O}_{U_i}.
\]

Using this trivialization, we can identify any section in \( \Gamma(\mathcal{O}_{U_i}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B})) \) with a regular function in \( \Gamma(\mathcal{O}_{U_i}), \) thus obtaining its image in \( \hat{\mathcal{O}}_{U_i}. \) We denote this process by

\[
\gamma \in \Gamma(\mathcal{O}_{U_i}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B})) \mapsto [\gamma] \in \mathcal{O}_{U_i}(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}) \otimes \mathcal{O}_{U_i} \hat{\mathcal{O}}_{U_i} \cong \hat{\mathcal{O}}_{U_i}.
\]
Note that by our choice of $\bar{S}_j$, (i.e. its vanishing along $\Sigma_{j, i}$) $\tilde{g}^*\bar{S}_j|U_i$ has vanishing order $r - j$ along $R_i$; thus $[\tilde{g}^*\bar{S}_j] = e_j(\bar{u}_i)^{r - j}$, where $e_j \neq 0 \in \tilde{O}_{\Sigma_i} = K_i[\bar{u}_i]$ such that $\bar{u}_i$ does not divide $e_j$.

We now prove the Sublemma. Let $F' \in \Gamma(\rho_*\mathcal{O}_C(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}))$. By (3.16), it lies in $\mathfrak{M}$ if and only if

$$F'|_{\mathfrak{R}} = 0 \in \Gamma(\mathfrak{O}_R(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B})).$$

Let $\tilde{r}_i = r_i + \cdots + r_v$; it is the multiplicity of $R_i$ in $\mathfrak{R}$. Since $\tilde{g}^*\bar{S}_j$ has vanishing order $r - j$ along $R_i$, and since $\zeta_{[\alpha]}$ has vanishing order max$(0, j - (r - \tilde{r}_i))$ along $R_i$, $S_j = \zeta_{[\alpha]} \cdot \tilde{g}^*\bar{S}_j$ has vanishing order $(r - j) + (j - r + \tilde{r}_i) = \tilde{r}_i$ along $R_i$. Thus $S_j$ lies in $\mathfrak{M}$.

We now prove that $\mathfrak{M}$ is generated by $S_j$’s. Let $F \in \mathfrak{M}$. Using (3.10), we denote its image in $\Gamma(\rho_*\mathcal{O}_C(\mathcal{R} + \mathcal{D}_R + \mathcal{A} - \mathcal{B}))$ by $F'$. Using the isomorphism below (3.10), and that $\mathfrak{M}$ is a free $\Gamma(\mathfrak{O}_V)$-module generated freely by $\bar{S}_i$, $F'$ can be uniquely expressed as

$$F' = f_1 \cdot \tilde{g}^*\bar{S}_1 + \cdots + f_r \cdot \tilde{g}^*\bar{S}_r, \quad f_i \in \Gamma(\mathfrak{O}_V).$$

By (3.11), $F'|_{\mathfrak{R}} = 0$. In particular, $(u_i)^{\tilde{r}_i}$ divides $F'|_{U_i}$.

We claim that for any positive integer $\alpha$, $u_i^\alpha$ divides $F'|_{U_i}$ if and only if $f_j|_{U_i}$ has vanishing order max$(0, \alpha - r - j)$ along $R_i$, for all $1 \leq j \leq r$. As the if direction is trivial, we prove the only if part. Taking their respective images in $\tilde{O}_{U_i}$, $u_i^\alpha$ dividing $F'|_{U_i}$ implies that

$$(\bar{u}_i)^\alpha \text{ divides } [F'] = \tilde{f}_1 \cdot [\tilde{g}^*\bar{S}_1] + \cdots + \tilde{f}_r \cdot [\tilde{g}^*\bar{S}_r] \in K_i[[\bar{u}_i]].$$

By replacing $v_j$ by $v_j^{-1}$ if necessary, we can assume that $\zeta_{[\alpha]} = c_i \bar{u}_i v_i$, where $c_i \neq 0 \in K_i'$. Let $\alpha_j$ be the order of which $f_j|_{U_i}$ is divisible by $\zeta_{[\alpha]}$, then $\tilde{f}_j = c_j(\zeta_{[\alpha]}\hat{\zeta}_{[\alpha]})^{\alpha_j}$, where $c_j \neq 0 \in K_i'$. (The case $f_j = 0$ is trivially true and is ignored.)

Adding that $[\tilde{g}^*\bar{S}_j] = e_j(\bar{u}_i)^{r - j}$ mentioned earlier, (3.19) translates to that $(\bar{u}_i)^\alpha$ divides

$$\sum_{j=1}^r \tilde{f}_j \cdot [\tilde{g}^*\bar{S}_j] = \sum_{j=1}^r e_j(c_i \bar{u}_i v_i)^{\alpha_j} \cdot (\bar{u}_i)^{r - j} = \sum_{k \geq 0} \sum_{\alpha_j} (e_j^c c_j^k e_j)(\bar{u}_i)^{k+r-j}$$

in $K_i[[\bar{u}_i]] = K_i'(v_i)[[u_i]]$. As $\bar{u}_i$ does not divide $c_j^k e_j$, this divisibility holds if $(\bar{u}_i)^\alpha$ divides each individual $c_i^k(\bar{u}_i)^{k+r-j}$, which holds if, $\alpha \leq \alpha_j + r - j$ for each $j$ where $f_j \neq 0$. This proves the claim.

Applying the claim to $\alpha = \tilde{r}_i$, we conclude that $(u_i)^{\tilde{r}_i}$ dividing $F'|_{U_i}$ implies that $\alpha_j \geq \tilde{r}_i + r - j$. This proves that $f_j/\zeta_{[\alpha_j]}$, which is a meromorphic function on $\mathcal{V}$, is regular over an open subset of $\mathcal{V}$ containing the generic point of $\mathcal{H}_i$, for each $1 \leq i \leq r$. Since $f_j/\zeta_{[\alpha_j]}$ is regular over $\mathcal{V} - \cup_i \mathcal{H}_i$, by Hartogs Lemma, it is regular over $\mathcal{V}$.

This proves that $F$ lies in the $\Gamma(\mathfrak{O}_V)$ span of $S_1, \cdots, S_r$. Since $\mathfrak{M}$ is a rank $r$ free $\Gamma(\mathfrak{O}_V)$-module, it is freely generated by $S_1, \cdots, S_r$. This proves the Sublemma. \hfill $\square$

We continue the proof of the Proposition. Let $b_{ij} = S_j|_{A_i} \in \Gamma(\rho_*\mathcal{O}_{A_i}(A_i) \cong \Gamma(\mathfrak{O}_{V_i}))$. Then under $\{S_1, \cdots, S_r\}$ and (3.7), $\varphi^R$ takes form

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots \\ b_{21} & b_{22} & b_{23} & \cdots \\ b_{31} & b_{32} & b_{33} & \cdots \end{bmatrix}.$$
By Proposition 3.4.1 under a basis change of $M$, $\varphi^R$ also takes form

\[(3.21) \quad A = \begin{bmatrix}
\alpha_{11}\zeta_{[\delta_1, a_1]} & \alpha_{12}\zeta_{[\delta_2, a_1]} & \alpha_{13}\zeta_{[\delta_3, a_1]} & \cdots \\
\alpha_{21}\zeta_{[\delta_1, a_2]} & \alpha_{22}\zeta_{[\delta_2, a_2]} & \alpha_{23}\zeta_{[\delta_3, a_2]} & \cdots 
\end{bmatrix},\]

where $a_{ij} \in \Gamma(\mathcal{O}_V)$. Therefore, $B = A(\mu_{ij})$ for a $(\mu_{ij}) \in GL(r, \mathcal{O}_V)$. Since $\zeta_{[q_1, a_1]}$ divides the first row of (3.21), it divides the first row of (3.20) as well. Because $\zeta_{[q_1, a_1]}$ are coprime with $\zeta_{[\delta_j]}$ ($1 \leq j \leq r$), we have $\zeta_{[q_1, a_1]}$ divides $b_{1j}$. Hence we can write

\[b_{1j} = c_{1j}\zeta_{[q_1, a_1]} .\]

Let $(\nu_{ij}) = (\mu_{ij})^{-1}$. Then

\[a_{11}\zeta_{[\delta_1, a_1]} = \nu_{11}\zeta_{[\delta_1, a_1]} + \nu_{21}\zeta_{[\delta_2, a_1]} + \cdots .\]

Hence, $a_{11} = \nu_{11}\zeta_{[\delta_1, a_1]} + \nu_{21}\zeta_{[\delta_2, a_1]} + \cdots$. Thus $0 \neq a_{11}(0) = \nu_{11}(0)c_{11}(0)$. Shrinking $V$ if necessary, we can assume $c_{11} \in \Gamma(\mathcal{O}_V)$. This brings the first row of (3.20) to the desired form. Similar arguments can be applied to the second row. This proves the form in (3.12).

We prove the last statement of the proposition. Since $s_1, s_2 \in H^0(\mathcal{O}_F(rp_1 + a - b))$ has vanishing order $r - 1$ and $r - 2$ (respectively) at $p_1$, one verifies directly that the image of $H^0(\mathcal{O}_F(2p_1 + a - b))$ in $H^0(\mathcal{O}_F(rp_1 + a - b))$ (via the inclusion) is the subspace spanned by $s_1, s_2$. By identifying the image of $H^0(\mathcal{O}_F(2p_1 + a - b))$ with this subspace, we obtain a homomorphism

\[H^0(\mathcal{O}_F(2p_1 + a - b)) \to \mathbf{k}^2,\]

where $s_{12} = \begin{bmatrix}
s_1(a_1) & s_1(a_2) \\
s_2(a_1) & s_2(a_2)
\end{bmatrix} = \begin{bmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{bmatrix}(0)$. Thus, rank $\begin{bmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{bmatrix}(0) = 1$ if and only if $h^0(\mathcal{O}_F(2p_1 - b)) = 1$. Since $b$ is a general point, the evaluation homomorphism at the point $b$, $H^0(\mathcal{O}_F(2p_1)) \to \mathbf{k}$, is surjective. Hence the exact sequence

\[0 \to H^0(\mathcal{O}_F(2p_1 - b)) \to H^0(\mathcal{O}_F(2p_1)) \to \mathbf{k} \to 0\]

implies that $h^0(\mathcal{O}_F(2p_1 - b)) = 1$ is equivalent to $h^0(\mathcal{O}_F(2p_1)) = 2$, which is equivalent to that $p_1$ is a Weierstrass point.

Likewise, the natural image of $H^0(\mathcal{O}_F(3p_1 + a - b))$ in $H^0(\mathcal{O}_F(rp_1 + a - b))$ is the subspace spanned by $s_1, s_2$ and $s_3$. Assume rank $\begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
c_{21} & c_{22} & c_{23}
\end{bmatrix}(0) \leq 1$. By similar arguments, the $\leq 1$ implies this $h^0(\mathcal{O}_F(3p_1 - b)) \geq 2$, and consequently $h^0(\mathcal{O}_F(3p_1)) \geq 3$, which is impossible by Riemann-Roch Theorem. This completes the proof of the proposition. $\Box$

3.7. Relation between two chains of rational curves. Suppose that the curve $C$ has an inseparable core $F$ and contains two disjoint rational chains $R_a$ and $R_b$ that are attached to $F$ at two distinct smooth points $q_a$ and $q_b$. Let $\delta_a$ (resp. $\delta_b$) be the first marked point on $R_a$ (resp. $R_b$); we let $D_a$ and $D_b$ be two divisors such that $\delta_i = D_i \cap R_i$, $i = a, b$. Consider the homomorphism

\[\varphi_{ab} : \rho_*\mathcal{M}(D_a) \oplus \rho_*\mathcal{M}(D_b) \to \rho_*\mathcal{O}_A(A) .\]

Apply (3.12), Proposition 3.6.1 to the chain $R_a$ and $R_b$ simultaneously, we find that under trivializations

\[\rho_*\mathcal{M}(D_a) \cong \mathcal{O}_V , \quad \rho_*\mathcal{M}(D_b) \cong \mathcal{O}_V \quad \text{and} \quad \rho_*\mathcal{O}_A(A_i) .\]
\(\varphi_{ab}\) can be represented by a matrix \(
abla_{a_{11}} \delta_{a_{11}} \nabla_{b_{11}} \delta_{b_{11}} \nabla_{a_{21}} \delta_{a_{21}} \nabla_{b_{21}} \delta_{b_{21}}\) where \(a_{ij}\) and \(b_{ij}\) are invertible functions over \(V\). Recall that the maximal rational chains \(R_a\) and \(R_b\) are conjugate if \(q_a\) and \(q_b\) are conjugate points of \(F\).

**Proposition 3.7.1.** Under the above notations, \(\det \begin{bmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{bmatrix}(0) = 0\) if and only if \(R_a\) and \(R_b\) are conjugate.

**Proof.** The proof is elementary. By semi-stable reduction, one can reduce it to the proof as in Proposition 3.5.1. We omit further details. \(\square\)

4. **Dual graphs and Modular vocabularies**

We will use the dual graphs of stable weighted curves decorated by vocabularies and weights to describe the blowing-up loci inductively.

**4.1. Weighted graph and a partition of \(\mathcal{M}_2\).** We introduce a modified version of dual graph.

**Definition 4.1.1.** Given a weighted curve \(x = (C, w) \in \mathcal{M}_2^w\), we define its dual graph \(\tau_x = \tau\) to be the graph whose vertices \(V_\tau\) are the inseparable components of \(C\); an edge \(e \in E_\tau\) connecting vertices \(v_1, v_2 \in V_\tau\) corresponds to a node connecting \(C_{v_1}\) with \(C_{v_2}\). We call a vertex \(o \in \tau\) a root if \(C_o\) is contained in the core \(F\) of \(C\); we label each vertex \(v \in V_\tau\) by the pair of the genus of \(C_v\) and the (total) weight of \(C_v\). We let \(V_{\tau}^{\ast}\) be all non-root vertices of \(\tau\). We call a non-root vertex a tail vertex.

The dual graph \(\tau\) of \((C, w)\) is stable if for any weight-0 genus-0 vertex \(v\), its valence (the number of edges connecting to \(v\)) is at least three.

We denote by \(\Lambda\) the set of dual graphs of weighted curves in \(\mathcal{M}_2^w\). Note that each \(\tau \in \Lambda\) either has a single root or its roots form a chain. We let \(o(\tau)\) be the subgraph of \(\tau\) consisting of roots and edges connecting them; we call \(o(\tau)\) the root graph or simply the root. The weight \(w(o(\tau))\) is the total weight of the roots of \(\tau\).

We give \(\Lambda\) a topology: we say \(\tau'\) is in the closure of \(\tau\) if there is a one parameter family of curves in \(\mathcal{M}_2^w\) so that the dual graph of its general fiber (resp. special fiber) is \(\tau\) (resp. \(\tau'\)). For \(\tau \in \Gamma\), we denote by \(\overline{\tau}\) the closure of \(\tau\) in \(\Gamma\).

For any \(\tau \in \Lambda\), we denote by \(\mathcal{M}_2^w(\tau) \subset \mathcal{M}_2^w\) be the locally closed substack (with reduced structure) of \(\mathcal{M}_2^w\) whose dual graph is \(\tau\); we define \(\mathcal{M}_2^w(\tau) = \bigcup \{\mathcal{M}_2^w(\tau') \mid \tau' \in \overline{\tau}\}\).

**Proposition 4.1.2.** For any \(\tau \in \Lambda\), \(\mathcal{M}_2^w(\overline{\tau})\) is the closure of \(\mathcal{M}_2^w(\tau);\) further, \(\mathcal{M}_2^w(\tau)\) and \(\mathcal{M}_2^w(\overline{\tau})\) are smooth.

This defines a partition \(\mathcal{M}_2^w = \coprod_{\tau \in \Lambda} \mathcal{M}_2^w(\tau)\).

**4.2. Words and vocabularies.** We introduce words and vocabularies.

**Definition 4.2.1.** Given a finite set \(Q\), a word \(w\) in \(Q\) is an array of elements in \(Q\), repetition allowed. When \(Q\) is understood, we call elements in \(Q\) alphabets. The array of empty alphabet is also a word, called the trivial word, usually denoted by \(\phi\). The first alphabet of a nonempty word is called the pivot of the word. The trivial word has no pivot. A vocabulary is an (unordered) finite collection of words.
Definition 4.2.2. We call \( p \) a prefix of a word \( w \) if \( p \) is a word and there is another word \( w' \) so that \( w = pw' \). The common prefix of a vocabulary \( s \) is the maximal word \( p \) that is a prefix of every word in the vocabulary \( s \). The words in \( s \) are called co-prime if their common prefix is trivial.

Definition 4.2.3. Let \( A \) be a finite set and \( s = \{ w_a \}_{a \in A} \) ve a vocabulary of words in \( Q \) with common prefix \( p \). We write \( w_a = pw_{a}^{\tau} \), and call \( s^{\tau} = \{ w_{a}^{\tau} \}_{a \in A} \) the residual vocabulary of \( s \) and \( p \cdot s^{\tau} \) the factored form of \( s \). In case \( w_a = p \) for some \( a \), we say \( w_a \) is an excellent word in \( s \). In particular, \( \phi \), if appears in \( s \), is an excellent word.

Note that the words in the residual vocabulary \( s^{\tau} \) are co-prime.

Given a word \( w \) in \( Q \), we define its multiplicity to be the function \( \langle w \rangle : Q \to \mathbb{N} \) so that \( \langle w \rangle(q) \) is the number of appearances of \( q \) in \( w \). Given two \( i \) and \( j \in \mathbb{N}^{Q} \), we say \( i \preceq j \) if \( i(q) \preceq j(q) \) for all \( q \in Q \). For two words \( w_a \) and \( w_b \) in \( Q \), we say \( w_a \preceq w_b \) if \( \langle w_a \rangle \preceq \langle w_b \rangle \). This provides a partial ordering among words in \( Q \).

Given a vocabulary \( s \), we let \( s_{\min} \subset s \) be the subset of minimal words in \( s \).

Definition 4.2.4. A word \( w \) is linear if \( \langle w \rangle \) takes values in \( \{0, 1\} \).

We remark that the trivial word \( \phi \) is minimal and linear.

Definition 4.2.5. A vocabulary \( s = \{ w_a \}_{a \in A} \) is perfect if there are \( a \neq b \) such that \( \langle w_a \rangle \preceq \langle w_b \rangle \preceq \langle w_a \rangle \) for all \( c \in A - \{ a \} \).

Definition 4.2.6. If \( s \) contains a unique excellent word \( w_a \), and \( s \setminus w_a \) also contains an excellent word \( w_b \), we call \( w_b \) a secondary excellent word.

It is easy to check that a vocabulary with a secondary excellent word is perfect.

We list the useful properties of the vocabularies that will arise through our modular blowing-up of \( \mathcal{M}_2^{\text{wt}} \).

Let \( x \in \mathcal{M}_2^{\text{wt}} \) be a closed point, we will see that its alphabets \( Q_x \) are identified with tail vertexes of \( V_x^{\tau} \) and edges of \( E_0(\tau_x) \). We call the alphabets identified with \( V_x^{\tau} \) the tail alphabets.

Let \( \mathcal{M} \to \mathcal{M}_2^{\text{wt}} \) be an intermediate modular blowing-up mentioned in the introduction. For \( x \in \mathcal{M} \) a closed point and \( x \in \mathcal{M}_2^{\text{wt}} \) its image in \( \mathcal{M}_2^{\text{wt}} \), we will see that the alphabets \( Q_x \) of \( x \) is a union of \( Q_{\lambda} \) with the alphabets associated to the exceptional divisors of \( \mathcal{M} \to \mathcal{M}_2^{\text{wt}} \) and \( \lambda \) (for the third round) and \( \theta \) for the fourth round; we call alphabets in \( Q_{\lambda} \) (resp. associated with the exceptional divisors) the original alphabets (resp. exceptional alphabets). A word all of its alphabets are tail (resp., original or exceptional) alphabets is called a tail (resp., original or exceptional) word.

Definition 4.2.7. Let \( x \in \mathcal{M} \) over \( \bar{x} \in \mathcal{M}_2^{\text{wt}} \) be as stated. We list the following properties of which the vocabulary \( s_x = \{ w_a \}_{a \in A} \) in \( Q_x \) might have:

(P1). all minimal original words in \( s_x \) are linear;

(P2). any two minimal original words in \( s_x \) with distinct initials have no common alphabets;

(P3). any minimal word that is not a tail word is of the form \( vw \) such that \( v \) is a tail word and \( v \) contains no tail alphabets;

\[ \text{Warning: this is not the lexicographic ordering; we do not order the alphabets.} \]
4.3. Nodal divisors and divisorial labelings. We introduce more notations.

Definition 4.3.1. A nodal divisor of $\mathfrak{M}^{wt}_2$ consists of a smooth Artin stack $\mathcal{S}$, a proper local immersion $\mathcal{S} \to \mathfrak{M}^{wt}_2$ such that its image is a divisor, that the pullback family $\mathcal{C} = C \times_{\mathfrak{M}^{wt}_2} \mathcal{S}$ contains two subfamilies of proper nodal curves $F_1$ and $F_2 \subset \mathcal{C}$, of which $\mathcal{C} = F_1 \cup F_2$ and $F_1 \cap F_2$ is a section of $\mathcal{C} \to \mathcal{S}$.

Let $\Sigma$ be the set of nodal divisors of $\mathfrak{M}^{wt}_2$. We divide $\Sigma$ into two parts. Let $[\mathcal{S}] \in \Sigma$, and let $x \in \mathcal{S}$ be a closed point. Then the splitting $\mathcal{C} = F_1 \cup F_2$ shows that the curve $C = C_x$ has its associated splitting node $q = C_1 \cap C_2$, where $C_1 = C \cap F_1$. Thus, $q$ associates to an edge of the graph $\tau_x$ of $x$. We define $[\mathcal{S}] \in \Sigma^o$ (resp. $[\mathcal{S}] \in \Sigma^s$) if this edge lies in (resp. not in) the root subgraph $o(\tau_x) \subset \tau_x$. It is direct to check that this definition does not depend on the choice of $x \in \mathcal{S}$.

For any point $x = (C, w) \in \mathfrak{M}^{wt}_2$, we describe an injective map

$$\iota^*_x : V^*_x \to \Sigma^*,$$

that is the (reduced) divisorial labeling of $x$. Taking any vertex $v \in V^*_x$, from its construction, there is a unique edge $e$ emanating from $v$ such that when the edge $e$ is removed, $v$ and the root $o(\tau_v)$ lie in different components; by definition $e$ corresponds to a separating node $q$ of $C$; we let $\mathcal{S}_v$ be the nodal divisor associated to $q$ and define $\iota^*_x(v) = [\mathcal{S}_v]$. This defines the labeling $\iota^*_x$.

We give $\Sigma^*$ a partial ordering: we say $\mathcal{S}$ and $\mathcal{S}' \in \Sigma^*$ have $\mathcal{S} < \mathcal{S}'$ if there is a $[C] \in \mathcal{S} \cap \mathcal{S}'$ such that after letting $q$ and $q' \in C$ be the nodes associated with $\mathcal{S}$ and $\mathcal{S}'$, respectively, $F - q$ and $q'$ lie in different connected components of $C - q$. It is direct to check that this partial ordering is well-defined.

4.4. Modular and derived vocabularies. In this subsection, we set up all the foundational materials necessary for the constructions of modular blowups.

To each point $x = (C, w) \in \mathfrak{M}^{wt}_2$, we first construct a vocabulary $s_x$, called the reduced modular vocabulary of $x$, which consists of $|w|$ many words with (tail) alphabets in $V^*_x$ where $\tau = \tau_x$ and $|w| = \sum_{v \in V_x} w(v)$ is the total weight of $(C, w)$. When the core $F$ of $C$ contains more than one inseparable components, we additionally construct a pair of vocabularies $s^\pm_x$ with $|w|$ many words in $V_x$. These vocabularies are constructed to simulate the matrix presentation of the structural homomorphism in Proposition 3.6.1.

Let $v \in V^*_x$: it is contained in a unique chain of vertices $\omega v_{1} \cdots v_{r-1} v_{r}$, $\omega \in o(\tau)$, $v_{j} \geq 1 \in V^*_x$ and $v_{r} = v$. Letting $w_{1} = w(v_{1})$, we introduce $s_{v_{1}} = \{v_{1}, \cdots, v_{r}^{w_{1}}\}$. Suppose the words associated to $v_{1}, \cdots, v_{r-1}$ have been constructed, we let $w$ be the longest word in $s_{v_{1}} \cup \cdots \cup s_{v_{r-1}}$; let $s_{v_{r}}$ be the word $v_{1} \cdots v_{r-1} v_{r}$; let $w_{1} = w(v_{1})$, and let $s_{w_{1}} = (w_{1}^{w_{1}}s_{v_{1}}^{w_{1}})$. (Our convention is when $v = abc$, then $v^{3} = abcaabca$, etc.) For any $v \in o(\tau)$, we let $s_{v} = \{\omega, \cdots, \omega\}$ where the trivial word $\omega$ occurs $w(v)$ many times. We set

$$(\text{4.1})\quad s_{x} = \cup_{v \in V_x} s_{v}.$$ 

This is a vocabulary with alphabets in $V^*_x$; it contains $|w|$ many words; every word of $s_{x}$ is associated to a positively weighted inseparable component of $C$. We call this the reduced vocabulary of the point $x \in \mathfrak{M}^{wt}_2$. To emphasize the dependence on the
point $x$ of the index set of $s_x$, in what follows, we sometimes write $s_x = \{w_a\}_{a \in A_x}$. We note here that each $a \in A_x$ is uniquely associated to a positive vertex $v_a$ of $\tau_x$; for a positive vertex $v \in \tau_x$, there are exactly $w(v)$ many elements of $A_x$ that are associated to $v$.

By construction, every word in $s_x$ (including the trivial ones) has its parent root in $o(\tau)$; if we write $s_x = \{w_a\}_{a \in A_x}$, we denote by $o(a) = o(w_a)$ the parent root of $w_a$. It is useful to observe that $s_x$ contains $w(o(\tau_x))$ many excellent words (recall that a trivial word is an excellent word); as we will see, our modular blowing-up centers will rely on the number of the excellent words in $s_x$.

To simulate the two components of the structural homomorphism (Proposition 3.6.1), we introduce a pair $s_x^\pm$ of vocabularies for every point $x = (C, w) \in \mathcal{M}_w^\text{wt}$. We distinguish three cases.

1. $\#(V_{(\tau_x)}) = 1$. We let $s_x^\pm = s_x$;

2. $\#(V_{(\tau_x)}) = 2$. Then we can write $o(\tau) = o_--o_+$. We let $q = C_{o_--} \cap C_{o_+}$; it corresponds to the unique edge of the root graph $o(\tau)$. We write $s_x = \{w_a\}_{a \in A_x}$. We then decompose $s_x = \{w_a\}_{a \in A^-_x} \cup \{w_b\}_{b \in A^+_x}$ such that the parent root $o(w_a) = o_-$ for any $a \in A^-_x$ and $o(w_b) = o_+$ for any $b \in A^+_x$. Then we let

$$s_x^- = \{w_a\}_{a \in A^-_x} \cup \{w_b\}_{b \in A^+_x} \quad \text{and} \quad s_x^+ = \{w_a\}_{a \in A^-_x} \cup \{w_b\}_{b \in A^+_x}.$$  

Note that $A_x = A^-_x \cup A^+_x$ (this is the only case where this identity holds); $\{w_a\}_{a \in A^\pm_x}$ is a proper sub-vocabulary of tails words in $s_x^\pm$.

3. $\#(V_{(\tau_x)}) > 2$. We assume $o(\tau) = o_-a_1 \cdots o_l o_+(l \geq 1)$. As earlier, we write $s_x = \{w_a\}_{a \in A_x}$; we let $o_{a_i}$ be the parent root of $w_a$ for every $a \in A_x$. For any $1 \leq i \leq l + 1$, we let $q_i = C_{o_-} \cap C_{o_{a_i}}$ where we set $o_0 = o_-$ and $o_{l+1} = o_+$. Note that there is a canonical one-to-one correspondence between the set $\{q_1, \cdots, q_{l+1}\}$ and the edge set $E_{(\tau_x)}$ (by indexing the edges from $o_-$ to $o_+$). Letting

$$q_i^- = q_1 \cdots q_i \quad \text{and} \quad q_i^+ = q_{i+1} \cdots q_{l+1},$$

we introduce

$$s_x^- = \{q_i^- w_a\}_{a \in A_x} \quad \text{and} \quad s_x^+ = \{q_i^+ w_a\}_{a \in A_x}.$$  

(For instance, when $w_a = \emptyset$, then $w(o_{a_i}) > 0$; in this case, $w_a = \emptyset$ gives rise to one $q_i^\pm$ in $s_x^\pm$.) We let $A^\pm_x = \{a \in A_x \mid o(w_a) = o_a\pm\}; A^-_x \cup A^+_x \neq A_x$ in this case; $\{w_a\}_{a \in A^\pm_x}$ is a proper sub-vocabulary of tails words in $s_x^\pm$.

We define $Q_x = V_{(\tau_x)}^* \cup \{q_1, \cdots, q_{l+1}\}$; this can be identified with $V_{(\tau_x)}^* \cup E_{(\tau_x)}$. Then $s_x^\pm$ are vocabularies with alphabets in $Q_x$. It is obvious that the injective map $\iota^*_{(\tau_x)} : V_{(\tau_x)}^* \to \Sigma^*$ uniquely extend to an injective map

$$\iota_x : Q_x \to \Sigma, \quad v \to \mathcal{S}_v,$$

called the divisorsial labeling associated to the point $x$.

An instrumental notion to be used throughout the inductive modular blowups is derived vocabulary. Take any point $x = (C, w) \in \mathcal{M}_w^\text{wt}$, we let its derived vocabulary be

$$t_x = s_x^- \cup s_x^+.$$  

In any case, observe that $t_x$ contains an excellent word (necessarily a trivial word) if and only if there is a root $o \in o(\tau_x)$ such that $q(o) \cdot w(o) > 0$. It is easy to check that $t_x$ coincides with its residual vocabulary $t_x'$ for any $x \in \mathcal{M}_w^\text{wt}$. 


We make a useful remark. By tracing the indexes of the words, we see that \((\tau_x, s_{\pm}^x)\) and \((\tau_x, t_x)\) uniquely determines each other. Also useful is the introduction of the depth function

\[
\ell : \mathcal{M}_{\text{wt}}^2 \rightarrow [0, d] \cap \mathbb{Z}, \quad x \rightarrow \#(\text{ini}(t_x^e)).
\]

When \(t_x^e\) contains a trivial word, we let \(\ell(x) = 0\). (Depth functions will be extended to all intermediate modular blowups.)

![Figure 1. Two trees](image)

**Example 4.4.1.** \((#(V_{o(\tau)}) = 1.)\) Consider a point \(x \in \mathcal{M}_{\text{wt}}^2\) such that its weighted graph is pictured in the left of Figure 1. We assume that \(C_o\) is a smooth genus-2 curve of weight 0; \(w(a) = 2, w(b) = 0, w(c) = 3, w(d) = 2\). In the notation of \([3]\), \(\tau_x = o(0)[a(2), b[c(3), d(2)]]\). In this case, its associated modular vocabularies \(s_{\pm}^x = t_x^e = [a, a; bc, bcbc; bd, bd]\).

**Example 4.4.2.** \((#(V_{o(\tau)}) = 2.)\) Let \(x = (C, w) \in \mathcal{M}_{\text{wt}}^2\) be such that its weighted graph is pictured in the middle of Figure 1. We assume that \(C_{o \pm}\) are smooth elliptic curves of weight 0 and \(C_o, C_a, C_b\) are smooth rational curves of weight 2. We have

\[
\begin{bmatrix}
  s_{-}^x \\
  s_{+}^x
\end{bmatrix} = \begin{bmatrix}
  a & a^2q & bq & b^2q \\
  aq & a^2q & b & b^2
\end{bmatrix}, \quad t_x = [a, a^2, bq, b^2q, aq, a^2q, b, b^2].
\]

**Example 4.4.3.** \((#(V_{o(\tau)}) = 3.)\) Let \(x = (C, w) \in \mathcal{M}_{\text{wt}}^2\) be such that its weighted graph is pictured in the right of Figure 1. We assume that \(C_{o \pm}\) are smooth elliptic curves of weight 0 and \(C_o, C_{a_1}, C_{b_1}\) are smooth rational curves of weight 1. We have

\[
\begin{bmatrix}
  s_{-}^x \\
  s_{+}^x
\end{bmatrix} = \begin{bmatrix}
  a_1 & a_2 & q_1 & q_1q_2b \\
  q_2q_1a_1 & q_2q_1a_2 & q_1 & q_2
\end{bmatrix}, \quad t_x = [a_1, a_2, q_1, q_1q_2b; q_2q_1a_1, q_2q_1a_2, q_2, b].
\]

It is easy to check

**Proposition 4.4.4.** For any point \(x \in \mathcal{M}_{\text{wt}}^2\), its derived vocabulary \(t_x\) satisfies the properties specified in Definition 4.2.7.

**Definition 4.4.5.** Let \(x = (C, w) \in \mathcal{M}_{\text{wt}}^2\); we define its decorated graph to be \((\tau_x, s_{\pm}^x)\); it comes with a derived vocabulary \(t_x\) and a divisorial labeling \(\iota_x : V_{\tau_x} \rightarrow \Sigma\).

**Definition 4.4.6.** In general, a decorated graph \((\tau, s_{\pm})\) consists of a weighted graph \(\tau\), a set of alphabets \(Q\) and a pair of vocabulary \(s_{\pm}\) in \(Q\); it comes with a derived
vocabulary \( t = s^- \cup s^+ \). The pair \((\tau, t)\) is called admissible if \(\tau\) is stable and \(t\) satisfies the properties specified in Definition 1.2.7.

By definition, \((\tau, s^\pm)\) and \((\tau, t)\) uniquely determine each other.

5. Modular Blowups

We use the constructed decorated graphs to construct successively the modular blowing-ups we aim. The modular blowing-ups consists of four rounds, each consisting of finite blowing-ups along transversal smooth centers. Each round is constructed by induction.

5.1. First round of modular blowing-ups. The initial stack \(\mathcal{M}_0\) of the first round blowups is \(\mathfrak{M}^\text{st}_2\). The blowing up centers of this round lie over the subset

\[ \mathcal{P}_0 = \{ x \in \mathfrak{M}^\text{st}_2 \mid \text{the derived vocabulary } t_x \text{ contains no excellent word} \} \]

We let \( \Pi_A = [0, d] \cap \mathbb{Z} \). The depth function \( \ell : \mathfrak{M}^\text{st}_2 \rightarrow \Pi_A \) will be extended to any intermediate blowup stack; it measures the progress of blowing-ups.

**Inductive Assumptions A.** For each \( 0 \leq k \leq d \), we have a smooth \(\mathcal{M}_k\), a set \(\Sigma_k\) of smooth divisors of \(\mathcal{M}_k\) such that

(a1). each closed point \( x \in \mathcal{M}_k \) has its set \( Q_x \) of alphabets together with an injective map \( \iota_x : Q_x \rightarrow \Sigma_k \), called the divisorial labeling; any finite subsets of \(\Sigma_k\) intersect transversally in \(\mathcal{M}_k\);

(a2). each closed point \( x \in \mathcal{M}_k \) has a decorated weighted graph \((\tau_x, s^\pm_x)\) with \(\tau_x = \tau_x^k\) where \( x \in \mathfrak{M}^\text{st}_2 \) is the image of \(x\) and the vocabulary \(s^\pm_x\) of words in \(Q_x\); it has a derived vocabulary \(t_x\); if introduce the (depth) function,

\[ \ell_k : \mathcal{M}_k \rightarrow \Pi_A, \ x \rightarrow \#(\text{ini}(t_x)) \]

then either \(\ell_k(x) = 0\) or \(\ell_k(x) \geq k + 1\); further, \((\tau_x, t_x)\) is admissible;

(a3). if we let \(\mathcal{P}_k = \{ x \in \mathcal{M}_k \mid t_x \text{ contains no excellent word} \} \), then, the subset \(\mathcal{B}_{k+1} = \{ x \in \mathcal{P}_k \mid \ell_k(x) = k + 1 \}\) is a codimension \(k + 1\) smooth closed substack of \(\mathcal{M}_k\); \(\mathcal{M}_{k+1}\) is the blowing-up of \(\mathcal{M}_k\) along \(\mathcal{B}_{k+1}\);

(a4). we have injective \(\Sigma_k \rightarrow \Sigma_{k+1}\) that send \([D] \in \Sigma_k\) to \([D'] \in \Sigma_{k+1}\), where \(D' \subset \mathcal{M}_{k+1}\) is the proper transform of \(D\).

As always, we let \(\mathcal{E}_v\) denote the divisor of \(\mathcal{M}_k\) corresponding to \(\iota_x(v)\) for any \(v \in Q_x\).

We now begin with the initial package. The initial \(\mathcal{M}_0\) is \(\mathfrak{M}^\text{st}_2\). For the other data, we let \(\Sigma_0\) be the set of nodal divisors introduced in the previous section; for \(x \in \mathcal{M}_0\), we let \(Q_x\), the pair of vocabularies \(s^\pm_x\), the derived vocabulary \(t_x\), and \(\iota_x\) be as defined in the previous section. It is easy to check that \(\mathcal{M}_0\) satisfies the properties a1)-a4). As \(\mathcal{B}_1\) is a smooth divisor, blowing up along it does nothing. So, \(\mathcal{M}_1 = \mathcal{M}_0\); we retain all the associated data. As a convention, we rename \(\mathcal{B}_1 \subset \mathcal{M}_0\) as \(\mathcal{E}_1 \subset \mathcal{M}_1\) and call \(\mathcal{E}_1\) the exceptional divisor of \(\mathcal{M}_1 \rightarrow \mathcal{M}_0\).

Suppose we have constructed the intermediate modular blowing-ups \(\mathcal{M}_0, \cdots, \mathcal{M}_k\) together with their associated data satisfying the properties a1)-a4).

By the inductive assumption, \(\mathcal{M}_{k+1}\) is smooth. We now describe its associated data. We denote by \(\mathcal{E}_1 \subset \mathcal{M}_{k+1}\) the proper transform of the exceptional divisor of \(\mathcal{M}_1 \rightarrow \mathcal{M}_{k+1}\); we define \(\Sigma_{k+1} = \Sigma_1 \cup \{ [\mathcal{E}_1], \cdots, [\mathcal{E}_{k+1}] \}\); we introduce exceptional alphabets \(\varepsilon_1, \cdots, \varepsilon_{k+1}\) and for \(x \in \mathcal{M}_{k+1}\) define \(Q_x = Q_x^k \cup \{ \varepsilon_1, \cdots, \varepsilon_{k+1} \}\), where
$x \in \mathcal{M}_{k+1}^\omega$ is the image of $x \in \mathcal{M}_{k+1}$; we define $t_x : Q_x \to \Sigma_{k+1}$ be the extension of $t_x$ via $t_x(\varepsilon) = \left[\varepsilon\right] \in \Sigma_{k+1}$. We let $\tau_x = t_x$.

We next construct the vocabularies $s^\perp_x$ and its derived vocabulary $t_x$. Since $(\tau_x, t_x)$ determines $(\tau, s^\perp)$, it suffices to construct $t_x$.

**Construction 5.1.1.** (Constructing new vocabulary\(^2\)) If $x \not\in \mathcal{E}_{k+1}$, we let $t_x = t_{x'}$ where $x' \in \mathcal{M}_k$ is the image of $x \in \mathcal{M}_{k+1}$. In the case $x \in \mathcal{E}_{k+1}$, we write $t_x = p_x \cdot t_{x'}$ in factored form. Since $x' \in B_{k+1}$, $t_k(x') = k + 1$, that is, $\#\text{ini}(t_{x'}) = k + 1$. Take any $p \in \text{ini}(t_{x'})$. For each $w \in t_{x'}$, we define $w(x)$ to be $w$ with all $p$ appearing in $w$ replaced by $\varepsilon_{k+1}$ (resp. by $\varepsilon_{k+1}p$) if $x \not\in \mathcal{E}'_{p}$ (resp. if $x \in \mathcal{E}'_{p}$) where $\mathcal{E}'_{p}$ is the proper-transform of $\mathcal{E}_p$. We define $t_x = \{w(x)\}_{w \in t_{x'}}$. This constructs $t_x$ and hence $s^\perp_x$.

**Lemma 5.1.2.** The stack $\mathcal{M}_{k+1}$ satisfies the Inductive Assumptions $A$.

**Proof.** (a1) follows from the standard fact that blowing up a smooth stack along smooth closed substack preserves transversality.

(a2). Let $x \in \mathcal{M}_{k+1}$ be any closed point lying over $x' \in \mathcal{M}_k$. In the case $x \not\in \mathcal{E}_{k+1}$, then $x' \not\in B_{k+1}$, hence by (a2) of $\mathcal{M}_k$, we have that $t_x = t_{x'}$ either contains an excellent word $(t_{k+1}(x) = 0)$ or its residual vocabulary $t_{x'}$ has more than $k + 2$ initial alphabets $(t_{k+1}(x) \geq k + 2)$. In the case $x \in \mathcal{E}_{k+1}$, by (a3), there is at least one $p \in \text{ini}(t_{x'})$ such that $x \not\in \mathcal{E}_p$. Next, for any $w = pv \cdots \in t_{x'}$, with $p$ its initial (we allow $v \cdots = o$), we have $w(x) = \varepsilon_{k+1}w(x)'$ such that $w(x)'$ has initial $p$ if $x \in \mathcal{E}'_p$ and $w(x)'$ has initial $v$ if $x \not\in \mathcal{E}'_p$, unless $w(x)' = o$ which happens exactly when $w = p$ and $x \not\in \mathcal{E}'_p$. In any case, if we factor $t_{x'} = p_x t_{x'}'$, one sees that $t_x = p_x t_{x'}$; we define $t_x = \{w(x)'\}_{w \in t_{x'}}$, with its prefix $p$. Assume that none of words in $w(x)'$ equals $o$ $(t_{k+1}(x) \neq 0)$. By (P1) and (P2) of Proposition 4.2.7, one sees that $\#\text{ini}(t_{x'}) \geq k + 1$. Because none of $w(x)'$ equals $o$, there is $w_a = p v_a \cdots \in t_{x'}$, with $v_a \cdots \neq o$ and $x \not\in \mathcal{E}'_p$. Then, by (P4) of Proposition 4.2.7, $\#\text{ini}(t_{x'}) \geq k + 2$ $(t_{k+1}(x) \geq k + 2)$. It remains to check that $t_{x'}$ is admissible, but, this is a direct check using Definition 4.2.7.

(a3). Consider any points $x \in B_{k+2}$, i.e., $\#\text{ini}(t_{x'}) \geq k + 2$. We let $\text{ini}(t_{x'}) = \{v_1, \cdots, v_{k+2}\}$. Using induction, it is straightforward to check the following: in a neighborhood $x \in U \subset \mathcal{M}_{k+1}$ we have $B_{k+2} \cap U = \bigcap_{i=1}^{k+2} \mathcal{E}_{v_i} \cap U$. It follows that $B_{k+2}$ is smooth of codimension $k + 2$.

(a4) comes with the construction.

Combining all the above, we see that the first round blowups can be carried out and process terminates at $\mathcal{M}_d$. We rename this final stack by $\mathcal{R}_d$.

It is useful to summarize.

**Proposition 5.1.3.** Every point $x \in \mathcal{R}_d$ has a decorated graph $(\tau_x, s^\perp_x)$ such that its derived vocabulary $t_x$ contains at least one excellent word.

**Proof.** This follows from direct checks (cf. Examples 4.4.2 and 4.4.3). \[\square\]

As we will see, this implies that one of the two components of the structural homomorphism $\varphi$ has a minimal entry (in terms of divisibility) corresponding to an excellent word; using it, we can eliminate the remainder entries in that component, making it into a desired form.

\(^2\)The same construction will be reused in all the subsequent modular blowups.
We introduce secondary derived vocabularies; they are defined only for special points in the intermediate blown-ups.

**Definition 5.1.4.** Let \( x \in \mathcal{M}_{k+1} \); the secondary derived vocabulary \( t'_x \) of \( x \) is defined only in the following three situations.

1. When \( \#(V_{o(a)}(x)) = 1 \) and when \( t_x \) contains a unique excellent word \( p_x \); in this case we define \( t'_x = s_x \setminus p_x \).

2. When \( \#(V_{o(a)}(x)) > 1 \) and when the perfect words of \( t_x \) belong to exactly one of \( s^+_x \), say \( s^+_x \); in this case we define \( t'_x = s^+_x \setminus w^+_a \), where \( s^+_x = \{ w^+_a \}_{a \in A_x} \), and \( i \) is the smallest integer so that there is a perfect word \( w^+_a = p_x \) with \( o(w^+_a) = a_i \);

3. When \( \#(V_{o(a)}(x)) > 1 \) and when \( t_x \) has exactly two perfect words, one in each of \( s^+_x \), such that they are indexed by the same element \( a_0 \in A_x \); in this case, we let \( t'_x = t_x \setminus \{ w^+_a, w^+_a \} \), where \( s^+_x = \{ w^+_a \}_{a \in A_x} \).

One checks from the construction that \( t'_x \) does not depend on the choice of \( a_0 \), other than the indexing.

**Example 5.1.5.** We first continue with Example 4.4.1. For the point \( x \in \mathcal{M}_2 \), we have
\[
\mathfrak{g}_x = \{ a, a, b, bc, bc, bc, b(bdb) \}.
\]
For \( x \in \mathcal{M}_1 \) lying over \( x \in \mathcal{M}_2 \), either \( t_x \) contains more than one excellent words, or we have the following four cases where \( t_x \) has a unique perfect word.

1. \( t_x = \varepsilon_2 \{ a, a, a, b, bc, bc, bc, b, bdb \} \) (this is achieved in \( \mathcal{M}_2 \));
2. \( t_x = \varepsilon_2 \{ a, a, a, b, bc, bc, bc, b, bdb \} \) for some \( x' \in \mathcal{M}_2 \) lying over \( x \);
3. \( t_x = \varepsilon_2 \{ a, a, a, a, b, b, b, bdb \} \);
4. \( t_x = \varepsilon_2 \{ a, a, a, a, b, b, b, bdb \} \).

Observe that \( t_x \) contains at most one minimal exceptional word.

**Example 5.1.6.** Next, we examine Example 4.4.2. We have
\[
\begin{bmatrix}
    \mathfrak{g}_x^- \\
    \mathfrak{g}_x^+
\end{bmatrix} = \begin{bmatrix}
    a & a^2 & b & b^2; \\
    a & a^2 & b & b^2
\end{bmatrix}, \quad \mathfrak{t}_x = [a, a^2, bq, b^2; aq, a^2q, b, b^2].
\]
At \( x \in \mathcal{M}_1 \) lying over \( x \in \mathcal{M}_2 \), either \( t_x \) contains more than one excellent words, or we have one case (modulo switching the roles of \( s^+_x \)) where \( t_x \) has a unique perfect word.
\[
t_x = \varepsilon_2 \{ a, a, bq, b, bbd \} \text{ having \( t'_x = \{ \varepsilon_2q, b, bbd \} \).}
\]
Observe that \( t_x \) contains at most one minimal word of the form \( \varepsilon q b \) where \( \varepsilon \) is an exceptional word; this occurs for the same reason as the previous example.

**Example 5.1.7.** Consider now Example 4.4.3. We have
\[
\begin{bmatrix}
    \mathfrak{g}_x^- \\
    \mathfrak{g}_x^+
\end{bmatrix} = \begin{bmatrix}
    a_1 & a_2 & q_1 & q_1q_2^b; \\
    q_2q_1a_1 & q_2q_1a_2 & q_2 & b
\end{bmatrix}, \quad \mathfrak{t}_x = [a_1, a_2, q_1, q_1q_2^b, q_2q_1a_1, q_2q_1a_2, q_2, b].
\]
There are five initials in \( \mathfrak{t}_x \), so we move to \( \mathcal{M}_5 \) directly. At \( x \in \mathcal{M}_1 \) lying over \( x \in \mathcal{M}_2 \), either \( t_x \) contains more than one excellent words, or we have the following four cases (modulo switching the roles of \( a_1 \) and \( a_2 \)) where \( t_x \) has a unique perfect word.

1. \( t_x = \varepsilon_5 \{ a, a_2, q_1, q_1q_2^b, q_2^5q_1a_1, q_2^5q_1a_2, q_2, b \} \); 
\( t'_x = \varepsilon_5 \{ q_2^5q_1a_1, q_2, b \} \);
(2) \( t_x = \varepsilon[a_1, a_2, \varnothing, \varepsilon q_2 \varepsilon q_5 b; q_2 \varepsilon q_5 \varepsilon a_1, q_2 \varepsilon q_5 \varepsilon a_2, q_2, b] \);
\( t'_x = \varepsilon_5[q_2 \varepsilon q_5 \varepsilon a_1, q_2 \varepsilon q_5 \varepsilon a_2, b] \);

(3) \( t_x = \varepsilon_5[a_1, a_2, q_1, q_1 \varepsilon q_2 \varepsilon q_5; q_2 \varepsilon q_5 \varepsilon a_1, q_2 \varepsilon q_5 \varepsilon a_2, q_2, \varnothing] \);
\( t'_x = \varepsilon_5[a_1, a_2, \varnothing; q_1 \varepsilon q_2 \varepsilon q_5] \);

(4) \( t_x = \varepsilon_5[a_1, a_2, q_1, q_1 \varepsilon q_2 \varepsilon q_5; \varepsilon q_5 q_1 \varepsilon a_1, \varepsilon q_5 q_1 \varepsilon a_2, \varnothing, b] \);
\( t'_x = \varepsilon_5[a_1, a_2, q_1 \varepsilon q_5 \varepsilon b] \);

Observe that in (2), there are two minimal words of the form \( \text{ww} \) where \( \text{w} \) is a tail word and \( \text{w} \) contains no tail alphabets.

Before we proceed to the next round, we record, for the later use, the following properties about the stack \( \mathcal{M}_{k+1} \) in the first round. We follow the previous notations.

**Lemma 5.1.8.** Fix \( k \geq 0 \). Let \( x \in \mathcal{M}_{k+1} \) be such that \( \#(V_{\alpha(x)}) = 1 \). Suppose \( i \) is the largest such that \( x \in \mathcal{E}_{i+1} \subset \mathcal{M}_{k+1} \) and \( t'_x \) is defined. Then exactly one of the following holds for \( t'_x \):

(a) \( (t'_x)_{\text{min}} \) contains no exceptional words and \( \#\text{ini}(t'_x) = i \);

(b) \( \varepsilon_2 \cdots \varepsilon_{i+1} \) is a unique minimal exceptional word of \( (t'_x)_{\text{min}} \) and \( \#\text{ini}(t'_x) = i+1 \).

Further, in either case, \( t'_x \) satisfies the properties specified in Definition 4.2.7.

**Proof.** We prove by induction. When \( k = 0 \), we only need to consider \( x \in \mathcal{E}_1 \) (\( i = 0 \)). In this case, (a) holds; (b) is vacuous.

Assume the statements hold for \( \mathcal{M}_k \). Now consider \( \mathcal{M}_{k+1} \) and we suppose \( i \) is the largest such that \( x \in \mathcal{E}_{i+1} \subset \mathcal{M}_{k+1} \) and \( t_x \) contains a unique excellent word . If \( i < k \), then \( x \notin \mathcal{E}_{k+1} \), by the construction, \( t_x = t_{x'} \) where \( x' \in \mathcal{M}_k \) is the image of \( x \). By inductive assumption, exactly one of (a) and (b) holds for \( t_{x'} \); thus the same for \( t'_x \). Now suppose \( x \in \mathcal{E}_{k+1} \). By Construction 4.1.4, if we write \( t_{x'} = p_{x'} \{ w \} \) with its residual vocabulary \( \mathcal{V}_{x'} = \{ w \} \) (where \( x' \in \mathcal{B}_{k+1} \subset \mathcal{M}_k \) is the image of \( x \)), then \( t_x = p_x \{ w(x) \} \in \mathcal{E}_{i+1} \); we can write \( w(x) = \varepsilon_{i+1} w(x') \) so that \( p_x = p_{x'} \varepsilon_{i+1} \) is the prefix of \( t_x \) and \( t'_x = \{ w(x') \} \in \mathcal{E}_{i+1} \). Since \( t_x \) contains a unique excellent word, \( t'_x \) contains exactly one trivial word; we may suppose that \( w(x') = \varnothing \) for a unique \( w \in t'_x \); that is, \( w = p \), with \( p \) an initial of \( t_{x'} \), and \( x \notin \mathcal{E}_{i+1} \).

From the previous section, we have \( Q_x = V_{\tau_x} \cup \{ q_1, \ldots, q_{i+1} \} \). If \( p = q_i \) for some \( i \), then one checks directly that only (a) holds. Otherwise, the alphabet \( p \in V_{\tau_x} \); as a vertex of \( \tau_x \), it has two cases: (1) \( w(p) = 1 \); in this case, no other words of \( g_{x'} \) contains \( p \). Then, one sees that \( (t'_x)_{\text{min}} \) contains no exceptional words and \( \#\text{ini}(t'_x)_{\text{min}} = k \); (2) \( w(p) > 1 \); in this case, one shows directly (e.g., by induction) that following the word \( w = p \), \( t'_x \) contains another word \( w' = p \varepsilon_{i+2} \cdots \varepsilon_{k+1} p \) (which is not a minimal word of \( t'_x \)). Then, one sees that \( \varepsilon_2 \cdots \varepsilon_{k+1} \) is a minimal word of \( t'_x \), \( (t'_x)_{\text{min}} \setminus \{ \varepsilon_2 \cdots \varepsilon_{k+1} \} \) contains no exceptional words, and \( \#\text{ini}(t'_x) = k + 1 \). In all cases, \( t'_x \) inherits the properties specified in Definition 4.2.7.

**Lemma 5.1.9.** Fix \( k \geq 0 \). Let \( x \in \mathcal{M}_{k+1} \) such that \( \#(V_{\alpha(x)}) = 2 \). We let \( q = C_{a_2} \cap C_{a_3} \). Suppose \( i \) is the largest such that \( x \in \mathcal{E}_{i+1} \subset \mathcal{M}_{k+1} \) and \( t_x \) exists.

Then exactly one of the following holds for \( t_x \):

(a) \( (t_x)_{\text{min}} \) contains no exceptional words and \( \#\text{ini}(t_x) = i \);

(b) \( \varepsilon_2 \cdots \varepsilon_{i+q} \) is a unique minimal exceptional word of \( (t_x)_{\text{min}} \) and \( \#\text{ini}(t_x) = i + 1 \).

Further, in either case, \( t_x \) satisfies the properties specified in Definition 4.2.7.

**Proof.** The proof is (almost) identical to that Lemma 5.1.8 we omit details. □
Lemma 5.1.10. For $k \geq 2$. Let $x \in \mathfrak{M}_1$ be such that $\#(V_{\mathfrak{M}_1}(\tau_3)) > 2$. Suppose $i$ is the largest such that $x \in \mathcal{E}_{i+1} \subset \mathcal{M}_{k+1}$ and $\tau'_x$ is defined. Then the following holds for $\tau'_x$: any minimal word of $\tau'_x$ that is not a tail word is of the form $\mathbf{w} \mathbf{v}$ such that $\mathbf{v}$ is a word in $\{q_1, \ldots, q_{i+1}\} \cup \{\varepsilon_i\}$ and $\mathbf{w}$ is an original tail word of $\tau$ where $x \in \mathcal{M}^{\tau}$ is the image of $x \in \mathfrak{M}_1$. Further, $\tau'^{e}_s$ satisfies the properties specified in Definition 4.2.7.

Proof. The proof is also largely parallel to that of Lemma 5.1.8; it follows from direct checks; we omit further details. \qed

Definition 5.1.11. The non-tail minimal word as described in each of the above three lemmas is called a distinguished minimal word. The exceptional alphabets together with $q$ and $\{q_1, \ldots, q_{i+1}\}$ are called distinguished alphabets.

5.2. The second round blowups. The second round begins with the initial stack $\mathcal{M}_0 = \mathfrak{M}_1$; the blowing up centers of this round are built over the subset

$$\mathcal{P}_0 = \{x \in \mathcal{M}_0 \mid \tau'_x \text{ exists and contains no excellent word}\}^3$$

The stacks in the second round is indexed by the set

$$\Pi_B = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq j < i \leq d\}.$$

We give the set $\Pi_B$ the lexicographic ordering: $(i, j) < (i', j')$ if $i < i'$ or $i = i'$ and $j > j'$. We list $\Pi_B$ as $\{\alpha_0, \alpha_1, \cdots, \alpha_r\}$ by its ordering where $r = \#(\Pi_B) - 1 = \frac{d(d+1)}{2}$.

For any $x \in \mathcal{P}_0$, we let $\epsilon(x) = 0$ if $(\tau'_x)_{\text{min}}$ does not contain any distinguished minimal words (cf. Lemmas 5.1.8, 5.1.10); otherwise, we let $\epsilon(x)$ be the maximum of the numbers of distinguished alphabets in the distinguished minimal words. We introduce the (secondary depth) function

$$\ell' : \mathcal{P}_0 \rightarrow \Pi_B, \quad x \rightarrow (\#(\text{ini}(\tau'_x)), \epsilon(x)).$$

The secondary depth function will extend to any intermediate blowup stack; it is used to measure the progress of blowing-ups.

Inductive Assumptions B. For each $\alpha_k \in \Pi$, we have a smooth $\mathcal{M}_k$, a set $\Sigma_k$ of smooth divisors of $\mathcal{M}_k$ such that

(b1). each closed point $x \in \mathcal{M}_k$ has its set $Q_x$ of alphabets together with an injective map $\iota_x : Q_x \rightarrow \Sigma_k$, called the divisorial labeling; any finite subsets of $\Sigma_k$ intersect transversally in $\mathcal{M}_k$; each closed point $x \in \mathcal{M}_k$ has a decorated weighted graph $(\tau_x, s^+_x)$ with $\tau_x = \tau_2$ where $\tau \in \mathcal{M}^\tau$ is the image of $x$ and the vocabulary $s^+_x$ of words in $Q_x$; it comes with the derived vocabulary $\tau_x$;

(b2) we let $\mathcal{P}_k = \{x \in \mathcal{M}_k \mid \tau'_x \text{ exists}\}$ (cf. Definition 5.1.7); for each point $x \in \mathcal{P}_k$, $(\tau_x, \tau'_x)$ is admissible; further, we let $\epsilon(x)$ be the maximum of the numbers of distinguished alphabets in the distinguished minimal words $(\epsilon(x) = 0$ if $(\tau'_x)_{\text{min}}$ does not contain any distinguished minimal words);

(b3). if we introduce the (secondary depth) function,

$$\ell'_k : \mathcal{P}_k \rightarrow \Pi_B, \quad x \rightarrow (\#(\text{ini}(\tau'_x)), \epsilon(x)),$$

3We note here that each time we begin a new round of inductive blowups, we may recycle some notations from the previous, such as $\mathcal{P}_k$, $\Pi_k$, $\mathcal{M}_k$, etc..
then either \( t'_k(x) = \alpha_0 \) or \( t'_k(x) \geq \alpha_{k+1} \); further, the subset

\[
B_{k+1} = \{ x \in \mathcal{P}_k \mid \#\init(t'_k) = \alpha_{k+1} \}
\]

is a codimension \( i \) smooth closed substack of \( \mathcal{M}_k \) where \( \alpha_{k+1} = (i, j) \); \( \mathcal{M}_{k+1} \) is the blowing-up of \( \mathcal{M}_k \) along \( B_{k+1} \);

(b4). We have injective \( \Sigma_k \to \Sigma_{k+1} \) that send \([D] \in \Sigma_k \) to \([D'] \in \Sigma_{k+1} \), where \( D' \subset \mathcal{M}_{k+1} \) is the proper transform of \( D \).

(We note here that the inductive assumptions of this round is in (fairly) large amount parallel to the one used in the first round, but with its own distinguished characteristics. The main difference is the role of new depth function. It is possible to mingle the two into one to achieve conciseness (e.g., induction on \( \Pi A \cup \Pi B \)). But, we decide to separate for preciseness and clarity.)

We now begin to describe and check the initial package for this round. This is indexed by \( \alpha_0 = (0, 0) \in \Pi_B \). We let the initial stack \( \mathcal{M}_0 \) be the final stack \( \mathfrak{N}_1 \) of the first round; the divisors of \( \Sigma_0 \) are the proper transforms of nodal divisors and the exceptional divisors, \( \{ \mathcal{E}'_j \}_{j=2} \), created in the first round; any finite collection of these divisors intersect transversally because blowing up of smooth stacks along smooth centers preserve transversality. Each closed point \( x \in \mathcal{M}_0 \) lying over \( x \in \mathfrak{N}^0_2 \) has its alphabets \( Q_x = Q_x \cup \{ \varepsilon_j \mid x \in \mathcal{E}'_j \} \), together with the induced divisorial labeling \( t_x : Q_x \to \Sigma_0 \); the closed point \( x \in \mathcal{M}_0 \) has the decorated graph \((\tau_x, s_x)\) and its associated derived vocabulary \( t_x \). (By Proposition 5.1.3 each \( t_x \) contains (at least) one excellent word.) This checks (b1). By Definition 5.1.4 and Lemmas 5.1.8,5.1.10 (b2) follows readily. (b3) follows from a direct check. (b4) comes with the construction of \( \mathfrak{N}_1 \). This checks the case of \( \alpha_0 \). As \( B_1 \) is a smooth divisor, blowing up along it does nothing. So, \( \mathcal{M}_1 = \mathcal{M}_0 \); we retain all the associated data. As a convention, we rename \( B_1 \subset \mathcal{M}_0 \) as \( \mathcal{E}'_1 \subset \mathcal{M}_1 \) and call \( \mathcal{E}'_1 \) the exceptional divisor of \( \mathcal{M}_1 \to \mathcal{M}_0 \).

Suppose we have constructed the intermediate modular blowing-ups \( \mathcal{M}_0, \cdots, M_k \) together with their associated data satisfying the properties (b1)-(b4).

The next stack \( \mathcal{M}_{k+1} \) is the blowing up of \( \mathcal{M}_k \) along \( B_{k+1} \); by the inductive assumption, it is smooth. We now describe its associated data. We denote by \( \mathcal{E}'_1 \subset \mathcal{M}_{k+1} \) the proper transform of the exceptional divisor of \( \mathcal{M}_1 \to \mathcal{M}_{k-1} \); we define \( \Sigma_{k+1} = \Sigma_1 \cup \{ \mathcal{E}'_1, \cdots, \mathcal{E}'_{k+1} \} \); we introduce the secondary exceptional alphabets \( \varepsilon_1, \cdots, \varepsilon_{k+1} \) and for \( x \in \mathcal{M}_{k+1} \) define \( Q_x = Q_{x_1} \cup \{ \varepsilon'_1, \cdots, \varepsilon'_{k+1} \} \) where \( x_1 \in \mathfrak{N}_1 \) is the image of \( x \in \mathcal{M}_{k+1} \); we define \( \iota_x : Q_x \to \Sigma_{k+1} \) be the extension of \( \iota_{x_1} \) via \( \iota_x(\varepsilon'_i) = [\mathcal{E}'_i] \in \Sigma_{k+1} \). We let \( \tau_x = \tau_{x_1} \).

It remains to construct new vocabularies.

**Construction 5.2.1.** (Constructing new vocabularies)\(^4\) If \( x \notin \mathcal{E}'_{k+1} \), we let \( s^\pm_x = s^\pm_{x'} \) (and hence \( t_x = t_{x'} \)) where \( x' \in \mathcal{M}_k \) denotes the image of \( x \in \mathcal{M}_{k+1} \). In the case \( x \in \mathcal{E}'_{k+1} \), we write \( t'_x = p_x \cdot t_{x'} \) in factored form. Since \( x' \in \mathcal{B}_{k+1} \), \( t'_x(x') = \alpha_{k+1} \), that is, \( \#\init(t'_x) = i \) and \( \epsilon(x') = j \) where \( \alpha_{k+1} = (i, j) \). Take any \( p \in \init(t'_x) \).

We let \( \nu_{x'} \) (resp. \( \nu_x \)) for any of \( s^\pm_x \) (resp. \( s^\pm_{x'} \), to be constructed). For each \( \nu \in \nu_{x'} \), we define \( \nu(x) \) to be \( \nu \) with all \( p \) appearing in \( \nu \) replaced by \( \varepsilon'_{k+1} \) (resp. by \( \varepsilon'_{k+1} \)) if \( x \notin \mathcal{E}'_{p} \) (resp. if \( x \in \mathcal{E}'_{p} \)) where \( \mathcal{E}'_{p} \) is the proper-transform of \( \mathcal{E}'_{p} \).

We define \( \nu_x = \{ \nu(x) \}_{\nu \in \nu_{x'} \} \). This constructs \( s^\pm_x \) (hence also \( t_x \)).

---

\(^4\)This is largely parallel to Construction 5.2.1.
Lemma 5.2.2. Then stack $\mathcal{M}_{k+1}$ satisfies the Inductive Assumptions B.

Proof. The large portion of the proof is parallel to Lemma 5.1.2.

(b1) comes with the construction.
(b2) follows from Lemmas 5.1.8, 5.1.10.
(b3) follows by mimicking the proof of Lemma 5.1.2, (a2). We omit the details.
(b4) follows readily. $\square$

Thus, the induction can be carried out; the process terminates at $\mathcal{M}_r$ where
\[ r = \frac{d(d+1)}{2}. \]
We denote this final stack by $\mathcal{N}_2$.

We summarize.

Proposition 5.2.3. Every point $x \in \mathcal{N}_2$ has the decorated graph $(\tau_x, s^\pm_x)$ such that
its derived vocabulary $t_x$ contains (at least) one excellent word; if its secondary
derived vocabulary $t'_x$ exists, then $t'_x$ also contains (at least) one excellent word.

Definition 5.2.4. A point $x \in \mathcal{N}_2$ is a critical point if one of the following holds:
(1). the secondary derived vocabulary $t'_x$ does not exist and $t_x$ has exactly two perfect
words;
(2). its secondary derived vocabulary $t'_x$ exists, and both $t_x$ and $t'_x$ have unique
excellent words.$^5$

As will see, the above implies that at any non-critical point $x$, the pullback to $\mathcal{N}_2$ of structural homomorphism can be diagonalized, using any two excellent words of $t_x$; for the generic critical points, the pullback to $\mathcal{N}_2$ can also be diagonalized, using the two perfect words or the first excellent word of $t_x$ and the unique secondary excellent word of $t'_x$. It remains to treat special critical points.

5.3. The third round blowups. Next, we consider the special modular blowups.

Beginning with $\mathcal{N}_2$, the blowing up centers of this round lie over
\[ C_{\text{cri}} = \{ x \in \mathcal{N}_2 \mid x \text{ is a critical point} \}. \]

Let $\mathcal{M}_{2,2}$ be the Artin stack of prestable two pointed curves of genus 2. We introduce
\[ K^\circ = \{ (C, p_1, p_2) \in \mathcal{M}_{2,2} \mid C \text{ is smooth; } p_1, p_2 \text{ are conjugate points of } C \}. \]

Let $K$ be the closure of $K^\circ$.

Proposition 5.3.1. Let $(C, p_1, p_2) \in \mathcal{M}_{2,2}$ and $C$ is singular. Then $(C, p_1, p_2)$ lies in $K$ if
(1). when $C$ is inseparable, $h^0(\mathcal{O}_C(p_1 + p_2)) = 2$;
(2). when $C = E_1 \cup E_2$ where $E_i$ are smooth elliptic curves joined at one point $q$, then
both $p_1, p_2$ lie on one of $E_i$, say $E_1$, and $\mathcal{O}_{E_1}(p_1 + p_2) = \mathcal{O}_{E_1}(2q)$.
Consequently, $K$ is a smooth divisor in $\mathcal{M}_{2,2}$.

Proof. This follows from a direct check. $\square$

For any point $(C, p_1, p_2) \in S$, we say $p_1$ and $p_2$ are conjugate.

$^5$It would be interesting to interpret these critical points as the critical points of some energy functions.
Next, let $\mathcal{M}_{2,1}$ be the Artin stack of prestable one pointed curves of genus 2. We introduce
$$W^\circ = \{(C, p) \in \mathcal{M}_{2,1} \mid C \text{ is smooth; } p \text{ is a Weierstrass point of } C\}.$$ Let $W$ be the closure of $W^\circ$. Then, as a special case of Proposition 5.3.1, we have

**Proposition 5.3.2.** Let $(C, p) \in \mathcal{M}_{2,1}$ and $C$ is singular. Then $(C, p)$ lies in $W$ if
d (1) when $C$ is inseparable, $h^0(\mathcal{O}_C(2p)) = 2$;
(2) when $C = E_1 \cup E_2$ where $E_i$ are smooth elliptic curves joined at one point $q$, suppose that $p$ lies on $E_1$, then $\mathcal{O}_{E_1}(2p) = \mathcal{O}_{E_1}(2q)$.
Consequently, $W$ is a smooth divisor in $\mathcal{M}_{2,1}$.

Let $x \in C_{\text{cri}} \subset \mathcal{N}_2$ be any critical point. There are two cases. First, the
secondary derived vocabulary $\lambda'_x$ does not exist and $\tau_x$ has exactly two perfect
words; if $\#(V_{\alpha(x)}) = 1$, we write $s_x = \{w_a, a \in A_x\}$, and let $w_{a_1}$ and $w_{a_2}$ be the
two perfect words for some $a_1 \neq a_2 \in A_x$; if $\#(V_{\alpha(x)}) > 1$, we write $s_x^\pm = \{w_a^\pm, a \in A_x\}$
since $\lambda'_x$ does not exist, we may assume that the two perfect words can be written
as $w_{a_1} \in s_x^-$ and $w_{a_2} \in s_x^+$ such that $a_1 \neq a_2 \in A_x$. Second, $\lambda'_x$ exists, and $\tau_x$ (resp.
$\lambda'_x$) contains a unique perfect word $p_{x,1}$ (resp. $p_{x,2}$); write $s_x^+ = \{w_a^+, a \in A_x\}$; we may
assume (W.L.G) that $p_{x,1} = w_{a_1} \in s_x^-$ for some $a_1 \in A_x$. By construction, we can write
$\lambda'_x = \{w_a^+, a \in A_x \backslash \{a_1\}\};$ we assume $p_{x,2} = w_{a_2}$ for some $a_2 \in A_x \backslash a_1$. (Warning:
these $a_1$, $a_2$ are not the marked points that we chose on the core curve as in the
Basic Assumptions, 3.1.)

We now introduce a characteristic function $\chi : C_{\text{cri}} \to \{0,1\}$. (Recall that every
vertex index $a \in A_x$ is associated to a positive vertex $v_a \in V_{\tau_x}$.) In case $v_{a_i} \in V_{\alpha(x)}$
($i = 1, 2$), we define $\chi(x) = 0$. Otherwise, we have $v_{a_1}, v_{a_2} \in V_{\tau_x}$; we let $R_1 (R_2)$ be
the rational chain that contains $C_{v_{a_1}} (C_{v_{a_2}})$; we define $\chi(x) = 0$ (resp. $= 1$) when
$R_1$ and $R_2$ are not conjugate (resp. are conjugate). Here when $R_1 = R_2$, being
conjugate means being of Weierstrass. We may extend the characteristic function to $\chi : \mathcal{N}_2 \to \{0,1\}$ by letting $\chi(x) = 0$ for any point $x \notin C_{\text{cri}}$.

We next introduce a subset of $C_{\text{cri}}$
$$C'_{\text{cri}} = \{x \in \mathcal{N}_2 \mid \chi(x) = 1\}.$$ By Proposition 5.3.1, $C'_{\text{cri}}$ is a smooth closed substack of $\mathcal{M}_2$.

We now let $\mathcal{M}_w$ be the blowup of $\mathcal{M}_2$ along $C'_{\text{cri}}$ and call the exceptional divisor the
$\lambda$-divisor. Next, we introduce (new) decorated graphs for $\mathcal{M}_w$ and its associated
derived vocabulary of third order. First, associated to $\mathcal{M}_w$, we let $\Sigma_0$ be the first
union of the $\lambda$-divisor with the proper transforms of the associated divisors of $\mathcal{M}_w$.

**Definition 5.3.3.** Take any point $x \in \mathcal{M}_w$; we let $x' \in \mathcal{M}_2$ be its image. We let
$\tau_x = \tau_{x'}$. If $x' \notin C'_{\text{cri}}$, we let $Q_x = Q_{x'}$, $s_x^\pm = s_{x'}^\pm$, $\tau_x = \tau_{x'}$, and $\lambda'_x = \lambda'_{x'}$ (if exists); otherwise, there are two cases; (1). $\lambda'_x$ does not exist and $\tau_x$ contains exactly
two perfect words $w_{a_1} = w_{a_2} = p_x$ for some $a_1 \neq a_2 \in A_x$ where $p_x$ is the prefix
of $\tau_x$; in this case, we let $Q_x = Q_{x'} \cup \lambda$ and $\tau_x$ be obtained from $\tau_{x'}$ by substituting
each of the two perfect words by $p_{x,1}$; we let its third-order derived vocabulary $\lambda'_x$
be obtained from $\lambda'_{x'}$ by deleting the two perfect words and replacing them by $p_{x,1}$;
(2). $\lambda'_x$ exists, and $\tau_x$ (resp. $\lambda'_x$) contains a unique perfect word $p_{x,1}$ (resp. $p_{x,2}$); in
this case, we let $Q_x = Q_{x'} \cup \lambda$ and $\tau_x$ (resp. $\lambda'_x$) be obtained from $\tau_{x'}$ (resp. $\lambda'_{x'}$) by
substituting its (unique) secondary perfect word $p_{x,2}$ by $p_{x,3}$; we let its third-order
derived vocabulary $\lambda'_x = \tau_x \backslash p_{x,1}$ where $p_{x,1}$ is the unique perfect word of $\tau_{x'}$. Recall
that $(\tau_x, \lambda'_x)$ uniquely determines a decorated graph $(\tau_x, s_x^\pm)$. 

Let \( \Pi_C = [0, d] \cap \mathbb{Z} \). We let
\[
P_0 = \{ x \in \mathfrak{M}_w \mid t'_x \text{ exists and } \lambda \in \text{ini}(t''_x) \}.
\]
We introduce the (third order) depth function
\[
\ell'' : P_0 \rightarrow \Pi_C, \quad x \rightarrow \#(\text{ini}(t''_x)).
\]
(We may extend it to \( \ell'' : \mathfrak{M}_w \rightarrow \Pi_C \) by letting \( \ell''(x) = 0 \) for any \( x \notin P_0 \), although will not be used.)

**Inductive Assumptions C.** For each \( 0 \leq k \leq d \), we have a smooth \( \mathcal{M}_k \), a set \( \Sigma_k \) of smooth divisors of \( \mathcal{M}_k \) such that

(c1). each closed point \( x \in \mathcal{M}_k \) has its set \( Q_x \) of alphabets together with an injective map \( t_x : Q_x \rightarrow \Sigma_k \), called the divisorial labeling; any finite subsets of \( \Sigma_k \) intersect transversally in \( \mathcal{M}_k \);

(c2). each closed point \( x \in \mathcal{M}_k \) has a decorated weighted graph \( (\tau_x, s^x_\pm) \) with \( \tau_x = \tau_x \) where \( \tau_x \in \mathfrak{M}_w \) is the image of \( x \) and the vocabulary \( s^x_\pm \) of words in \( Q_x \); we let \( \tilde{P}_k = \{ x \in \mathcal{M}_k \mid t'_x \text{ exists and } \lambda \in \text{ini}(t''_x) \} \); if introduce the (depth) function
\[
\ell'_k : \tilde{P}_k \rightarrow \Pi_C, \quad x \rightarrow \#(\text{ini}(t''_x)),
\]
then \( \ell'_k(x) \geq k + 1 \); further, \( (\tau_x, t''_x) \) is admissible;

(c3). the subset \( B_{k+1} = \{ x \in \tilde{P}_k \mid \ell_k(x) = k + 1 \} \) is a codimension \( k + 1 \) smooth closed substack of \( \mathcal{M}_k \); \( \mathcal{M}_{k+1} \) is the blowing-up of \( \mathcal{M}_k \) along \( B_{k+1} \);

(c4). we have injective \( \Sigma_k \rightarrow \Sigma_{k+1} \) that send \( [D] \in \Sigma_k \rightarrow [D'] \in \Sigma_{k+1} \), where \( D' \subset M_{k+1} \) is the proper transform of \( D \).

For the proofs, one checks directly that \( B_1 = \emptyset \) because \( \text{ini}(t''_x) \) contains \( \lambda \) and at least another alphabet), hence \( \mathcal{M}_1 = \mathcal{M}_0 \). The remainder of this round is totally parallel to the first round; the arguments to carry it out is almost identical to those of the first round. It would be duplicating to provide the details, so we omit them.

We let the final blowup be \( \tilde{\mathfrak{M}}_2^\text{rt} \). We summarize its property.

**Proposition 5.3.4.** For each point \( x \in \tilde{\mathfrak{M}}_2^\text{rt} \), either \( \lambda \notin Q_x \) or \( \lambda \in Q_x \) but \( \lambda \notin \text{ini}(t''_x) \).

It is easy to see that when \( \lambda \in Q_x \) but \( \lambda \notin \text{ini}(t''_x) \), then \( t''_x \) contains a trivial word label by the same index of \( \lambda \) in \( t'_x \) where \( x \in \mathfrak{M}_w \) is the image of \( x \).

### 5.4. The fourth round.

Finally, in this round we treat the critical points (of first order). Recall from Definition 5.2.1 a point \( x \in \tilde{\mathfrak{M}}_2^\text{rt} \) is a critical point (of first order) if \( t_x \) has exactly two perfect words (necessarily trivial words). Note that a small neighborhood of any such a critical point has not been modified during the first three rounds.

Let \( \mathcal{M}_0 = \mathfrak{P}_2 \times \tilde{\mathfrak{M}}_2^\text{rt} \). For any \( x \in \mathcal{M}_0 \), \( x' \in \tilde{\mathfrak{M}}_2^\text{rt} \), where \( x' \) is the image of \( x \). Then for each \( x \in \mathcal{M}_0 \), we let we let the point \( x \) inherit all the data associated to \( x' \in \tilde{\mathfrak{M}}_2^\text{rt} \).

For any \( x \in \mathcal{M}_0 \), we let \( \tau_x = (C, L) \in \mathfrak{P}_2 \) be its image. We introduce the subset \( \mathcal{P}_0 \subset \mathcal{M}_0 \) consisting of points \( x \) such that

(1). \( x' \) is critical of first order where \( x' \in \tilde{\mathfrak{M}}_2^\text{rt} \) is the image of \( x \);

(2) \( \deg L|_F = 2, h^0(L|_F) = 2 \) where \( F \) is the core of \( C \).

For any \( x \in \mathcal{M}_0 \), we let \( \tau_x = \tau_{x'} \) where \( x' \in \tilde{\mathfrak{M}}_2^\text{rt} \) is the image of \( x \); we modify the (third order) derived vocabulary \( t'_x \) to \( t''_x \). If \( x \notin \mathcal{P}_0 \), we let \( Q_x = Q_{x'} \) and
6.1. Diagonalizations of the canonical derived objects. We let \( \mathcal{V} \to \mathcal{M}_2^{\text{div}} \) whose image contains the pair \((C, D)\) be as in (5.2): \( \mathcal{V}' = \mathcal{V} \times_{\mathcal{M}_2^{\text{wt}}} \mathcal{M}_2^{\text{wt}}, \quad \tilde{\mathcal{V}} = \mathcal{V} \times_{\mathfrak{q}_2} \tilde{\mathfrak{p}}_2 \). The technical part of this section is to prove the following propositions.

**Proposition 6.1.1.** Assume that the pair \((C, D)\) does not lie over \( \mathcal{H}_i \) (for any \( i \geq 1 \)). Then the pullback of \( R\rho_* \mathcal{L} \) to \( \mathcal{V}' \) is diagonalizable.

**Proposition 6.1.2.** The pullback of \( R\rho_* \mathcal{L} \) to \( \tilde{\mathcal{V}} \) is diagonalizable.

Granting the two Propositions, we have

**Theorem 6.1.3.** The pullback of the derived object \( R\pi_* \mathcal{O}_{\mathbb{P}^n}(k) \) to \( \tilde{\mathcal{M}}_2^k(\mathbb{P}^n, d) \) is locally diagonalizable for all \( k \geq 2 \).
Proof: Fix $k \geq 2$. As before we cover $\overline{\mathcal{M}}_2(\mathbb{P}^n, d)$ with charts $\{U/V\}$ (the charts depend on $k$) such that the image $V \to \mathcal{M}_2^{\text{div}}$ contains the pair $(C, D)$. If $\deg F \cap D \neq 0$, then $\deg F \cap D \geq 4$ since $k \geq 2$. Thus, the pair $(C, D)$ does not lie over $\mathcal{H}_i$ (for any $i \geq 1$). The theorem then follows immediately from Proposition 6.1.1. □

Theorem 6.1.4. The pullback of the derived object $R\pi_*\mathcal{O}_{\overline{\mathbb{P}}^n}(1)$ to $\overline{\mathcal{M}}_2(\mathbb{P}^n, d)$ is locally diagonalizable.

Proof. This is immediate from Proposition 6.1.2. □

6.2. The structural homomorphism in original modular vocabularies. In this subsection, for each $x = (C, w) \in \mathcal{M}_2^{\text{wt}}$, we set up the structural homomorphism in terms of the decorated graph $(\tau_x, s_x^\pm)$.

Take any point $x = (C, w) \in \mathcal{M}_2^{\text{wt}}$, we let $(C, D) \in \mathcal{M}_2^{\text{div}}$ be any fixed lift of $(C, w)$; the pair $(C, D)$ comes with a smooth chart $V$ as in the previous subsection.

Also, coming with the point $x = (C, w)$ are the alphabets $Q_x$ and the decorated graph $(\tau_x, s_x^\pm)$. We write $s_x^\pm = \{w^\pm_a\}_{a \in A_x}$. Every index $a \in A_x$ is associated with a positive vertex $v$ of $\tau_x$; for each positive $v \in \tau_x$ there are $w(v)$ many elements in $A_x$ that are associated to $v$. In comparison, for every positive $v \in \tau_x$, $C_v$ contains $w(v)$ many marked points $\delta_{v,1}, \cdots, \delta_{v,w(v)}$, ordered as (3.10) for $v \in V^*_x$ or ordered in any (fixed) way for $v \in V_{o(\tau_x)}$. Once the ordering of $D$ is fixed, from the constructions of the original vocabularies (1.3), it is easy to see that there is a unique one to one correspondence between $D$ and $A_x$.

\begin{equation}
(6.1)
D \rightarrow A_x, \quad \delta_j \rightarrow a_j.
\end{equation}

In the remainder of this subsection, we continue to freely use the notations from the previous sections. Then we have the structural homomorphism

\begin{equation}
(6.2) \quad \varphi : \bigoplus_{\delta_j \in D} \rho_* \mathcal{M}(D_i) \rightarrow \rho_* \mathcal{O}_A(A).
\end{equation}

Let $\zeta = (\zeta_v)_{v \in V^*_x \cup E_{o(\tau_x)}}$, where $\zeta_v$ is a local modular parameter for the nodal divisor $\mathcal{E}_v$ (cf. 3.4). Here, we can have a canonical identification $Q_x = V^*_x \cup E_{o(\tau_x)}$; hence, we will substitute $V^*_x \cup E_{o(\tau_x)}$ by $Q_x$. Now, for each word $w \in s_x^\pm$, we introduce

\begin{equation}
(6.3) \quad \zeta^{(m)} = \prod_{v \in Q_x} \zeta_v^{(m)(v)}.
\end{equation}

Now we write $s_x = \{w^\pm_a\}_{a \in A_x}$. This way, by combining Propositions 3.4.1 and 3.6.1, we can express the structural homomorphism $\varphi$ as $[\varphi^-, \varphi^+]^T$:

\begin{equation}
(6.4) \quad \varphi^\pm = \bigoplus_{\delta_j \in D} c_j \zeta^{(m^\pm_j)},
\end{equation}

where $c_j = (c^-_j, c^+_j)^T \in \Gamma(\mathcal{O}_V)^{\oplus 2}$ and $c^-_j, c^+_j$ are local functions as defined in Propositions 3.4.1 and 3.6.1.

Our remaining task of this section is to diagonalize the structural homomorphism. We begin to state the cases that $\varphi^\pm$ is already diagonalized.

Definition 6.2.1. For any pair $(C, D)$ with the core $F$, we say that it is nonspecial if every positive inseparable component $E$ of $F$ contains at least $q(E)$ many marked points; in case that $\deg D \cap F$ is exactly 2, then $D \cap F$ is not a pair of conjugate points.
By Corollary 3.5.1 we have

**Proposition 6.2.2.** Assume that \((C, D)\) is nonspecial. Then we can find a frame such that \(\varphi\) is diagonalized as \(\varphi = [I_2, 0, \ldots, 0]\), where \(I_2\) is the identity matrix.

The diagonalizations (in the sense of [3]) of the structural homomorphism in all the remainder cases require modular blowups. We give a (non-) example.

### 6.3. The structural homomorphism under modular blowups

We continue to follow the previous notations. Let \(\mathcal{M} \twoheadrightarrow \mathfrak{M}_2^{\text{rt}}\) be any intermediate modular blowup. For any point \(x \in \mathcal{M}\), we have an associated vocabulary \(s_x\) in \(Q_x\) and a divisorial labeling \(i_x: Q_x \to \Sigma\). For any \(v \in Q_x\), let \(\mathcal{G}_v\) be the corresponding divisor; we let \(\zeta_v\) be a local regular function such its vanishing locus (locally) defines \(\mathcal{G}_v\). Let \(\zeta_\xi\) be its image. So, there is a smooth chart \(V_\xi; \zeta_\xi\) where \(\zeta_\xi = \zeta_{\xi'}\) and \(\zeta_\xi\) is realized as the closed substack of \(V_\xi\) defined by \(\zeta_\xi\)’s associated derived vocabulary (of certain order); so we let \(\text{ini}(\zeta_\xi) = \{v_0, \ldots, v_k\}\) and let \(\zeta_\xi, i\) be the local parameter of the divisor \(\mathcal{G}_v\) (0 \leq i \leq k). The (local) blowup around the point \(x\) is performed along the locus \(\zeta_\xi; \xi' = \cdots = \xi_{k+1} = 0\). Consider \(V_{\xi'} \times \mathcal{M}_{k+1}\); it can be realized as the closed substack of \(V_{\xi'} \times \mathfrak{P}^k\) subject to the relation \(\zeta_\xi;\xi' = \cdots = \zeta_{\xi_{k+1}} = 0\) along \(\xi_{k+1}\) whose homogeneous coordinate of \(\mathfrak{P}^k\). Then \(x\) lies in the chart \(V_x = \{u_i \neq 0\}\) for some \(i\). Then by Constructions 5.1.1 and 5.2.1 it is routine to check that \(\varphi^\dagger_{V_x} = \bigoplus_{D \in D} c_j \zeta^{(w_j^\dagger)}\). We omit further details.

### 6.4. Proofs of Propositions 6.1.1 and 6.1.2

**Proof of Proposition 6.1.1** By the assumption, we only need consider \(\mathfrak{M}_2^{\text{rt}}\). Take any point \(x \in \mathfrak{M}_2^{\text{rt}}\). We let \(x'\) be its image in \(\mathfrak{M}_2\). We have \(\varphi^\dagger_{V_x'} = \bigoplus_{D \in D} c_j \zeta^{(w_j^\dagger)}\) where \(s_x^\dagger = (w_j^\dagger)_{a \in A_{x'}}\) (and \(A_{x'} = A_x\)). By Proposition 5.2.3 we have that the point point \(x' \in \mathfrak{M}_2\) has the decorated graph \((\tau_x, s_x^\dagger)\) such that its derived vocabulary \(t_{x'}\) either contains two excellent words or \(t_{x'}\) contains a unique excellent
Thus, we may assume that $s' \subseteq s$ for all $b$ by using invertible (shrinking the chart if necessary); then a direct computation, e.g., elimination by using the similar proof of Lemma 6.3.1, one checks that under the basis change, we have $\varphi_{V'}(\delta, \varepsilon) \in N_2$ (i.e., the bias toward $s$), we have $\zeta_{x', a}^{(m'_{x', a})} \mid \zeta_{x', a}^{(m'_{x', a})} \mid \zeta_{x', a}^{(m'_{x', a})}$ for all $b \neq a, a_j \in A_x'$. If $det(c_1, c_j)(0) \neq 0$, we then assume $det(c_i, c_j)(0)$ is invertible (shrinking the chart if necessary); then a direct computation, e.g., elimination by using $\zeta_{x', a}^{(m'_{x', a})}$ and $\zeta_{x'}^{(m_{x'} a)}$, shows that $\varphi_{V'}$, is diagonalizable, hence, so is $\varphi_{V'}$. If $det(c_1, c_j)(0) = 0$, using $\zeta_{x', a}^{(m'_{x', a})}$, we can eliminate the remainder terms $\zeta_{x', a}^{(m'_{x', a})}$ of $\varphi_{V'}$, for all $a \neq a_i$; further, $\varphi_{V'}^+ = \bigoplus_{h, \in D} \zeta_{x', a}^{(m'_{x', a})}$; these are the forms of $\varphi_{V'}$ after some basis changes (in the domain free sheaf only). Then using the similar proof of Lemma 6.3.1 one checks that under the basis change, we have $\varphi_{V'}^+ = \bigoplus_{h, \in D} \zeta_{x', a}^{(m'_{x', a})}$ where $\zeta_{x', a}^+ = \{m_{x', a}^+\}_{a \in A_x}$. By the construction of $\mathcal{N}_1$, there is $h \neq i$ such that $\zeta_{x', a}^{(m'_{x', a})} \mid \zeta_{x', a}^{(m'_{x', a})} \mid \zeta_{x', a}^{(m'_{x', a})}$ for all $h \neq a, a_j \in A_x'$. The remainder arguments are the same.

Proof of Proposition 6.1.2: The proof is parallel to the above proof, and will be omitted.

7. LOCAL DEFINING EQUATIONS OF $\mathcal{M}_2(\mathbb{P}^n, d)$ AND ITS BLOWUPS

7.1. The local open immersions. As before (cf. (3.2)), we cover $\mathcal{M}_2(\mathbb{P}^n, d)$ by charts $\{U/V\}$. We let $\mathcal{E}_V$ be the total space of the vector bundle $\rho_*\mathcal{L}(A)\otimes n$ and let

$$p : \mathcal{E}_V \to V$$

be the projection. We also set

$$\rho_*\mathcal{L}(A)\otimes n | A = \rho_*\mathcal{L}(A)\otimes n | A_1 \oplus \rho_*\mathcal{L}(A)\otimes n | A_2.$$ 

Then the tautological restriction homomorphism

$$\text{rest} : \rho_*\mathcal{L}(A)\otimes n \longrightarrow \rho_*\mathcal{L}(A)\otimes n | A$$

lifts to a section

$$\Phi \in \Gamma(\mathcal{E}_V, p^*\rho_*\mathcal{L}(A)\otimes n | A)).$$

\footnote{In what follows, we still use $s^+_{x'}$ (resp. $s^+_{x''}$), even though in this case $s^+_{x'} = s_{x'}$ (resp. $s^+_{x''} = s_x$); this is for the purpose of the uniform presentation so that we can readily quote the same proof for the case $\#(V_0(x_{1'), x_{2}')) > 1$ below.}
By choosing suitable trivialization, the homomorphism “rest” of (7.1) can be identified with the homomorphism
\[(0 \oplus \varphi)^{\oplus n} : (\mathcal{O}_V \oplus \rho_*\mathcal{M}(\mathcal{D}))^{\oplus n} \to (\rho_*\mathcal{O}_A(A))^{\oplus n}\]
where \(\varphi\) is the homomorphism as defined in (3.4). Then, the section \(\Phi\) of (7.2) is the lift of \((0 \oplus \varphi)^{\oplus n}\).

**Theorem 7.1.1.** Let \(\mathcal{U} = \mathcal{V} \times_{\mathcal{M}^{\text{div}}_2} \mathcal{V}\). Then there is a canonical open immersion
\[(7.4) \quad \mathcal{U} \to (\Phi = 0) \subset \mathcal{E}_V.\]

**Proof.** This theorem is the natural extension of Theorem 2.16 of [3]; the proofs are also parallel. 

**Corollary 7.1.2.** Assume that \(d > 2\) and \((C, D) \in \mathcal{M}^{\text{div}}_2\) is nonspecial. Then by shrinking \(\mathcal{V}\) if necessary, we have that \(\mathcal{U} = \mathcal{V} \times_{\mathcal{M}^{\text{div}}_2} \mathcal{V}\) is smooth. In particular, the primary component of \(\mathcal{M}^2(\mathbb{P}^n, d)\) is generically smooth of expected dimension.

**Proof.** We can trivialize \(\rho_*\mathcal{M}(\mathcal{D})\) and \(\rho_*\mathcal{O}_A(A)\) so that \(\mathcal{E}_V \cong \mathcal{V} \times (\mathbb{A}^{d+1})^n\).

We let \(w_i^j \in \mathbb{A}^d\) for all \(1 \leq i \leq n\) and \(0 \leq j \leq d\). Recall that the homomorphism “rest” of (7.1) can be identified with the homomorphism
\[ (0 \oplus \varphi)^{\oplus n} : (\mathcal{O}_V \oplus \rho_*\mathcal{M}(\mathcal{D}))^{\oplus n} \to (\rho_*\mathcal{O}_A(A))^{\oplus n}.\]

By Proposition 6.2.2, \(0 \oplus \varphi\) can be represented by
\[ 0 \oplus \varphi = [0, I_2, 0, \ldots, 0].\]

Hence, the equation \(\Phi = 0\) is equivalent to the following equations
\[(7.5) \quad w_1^i = w_2^i = 0, \quad \text{for all } 1 \leq i \leq n,\]
which obviously implies that \(\mathcal{U}\) is isomorphic to an open subset of \(\mathcal{V} \times (\mathbb{A}^{d-1})^n\).

Note that \(\dim(\mathcal{V} \times (\mathbb{A}^{d-1})^n) = 3 + d + (d - 1)n = d(n + 1) - n + 3\) which is the expected dimension of \(\mathcal{M}^2(\mathbb{P}^n, d)\). The assertion then follows. □

When \(d = 2\), the interior of the primary component is made of stable maps that are double covers of smooth rational curves. It is smooth but of wrong dimension unless \(n = 1\). The one-tail and two-tail loci are also smooth and of the wrong dimension.

### 7.2. Induced open immersions of the blowup

Let the notations be as before; we form the fiber product \(\mathcal{V} = \mathcal{V} \times_{\mathcal{M}^{\text{div}}_2} \mathcal{P}_2\). Then
\[ \mathcal{E}_{\mathcal{V}} = \mathcal{E}_V \times_{\mathcal{V}} \mathcal{V} \]
is the total space of the pull back bundle \(\beta^*\rho_*\mathcal{L}(A)^{\oplus n}\) where \(\beta : \mathcal{V} \to \mathcal{V}\) is the projection. The tautological restriction homomorphism “rest” of (7.1) pullbacks to give rise to
\[ (7.6) \quad \tilde{\text{rest}} : \beta^*\rho_*\mathcal{L}(A)^{\oplus n} \to \beta^*\rho_*\mathcal{L}(A)^{\oplus n}|_A;\]
the section \(\Phi\) of (7.2) pullbacks to give rise to
\[ (7.7) \quad \tilde{\Phi} \in \Gamma(\mathcal{E}_{\mathcal{V}}, \tilde{\beta}^*\beta^*\rho_*\mathcal{L}(A)^{\oplus n}|_A) \]
where \(\tilde{\beta} : \mathcal{E}_{\mathcal{V}} \to \mathcal{V}\) is the projection. Let \(\mathcal{U} = \mathcal{V} \times \mathcal{U}\). Then, the immersion \(\mathcal{U} \to \mathcal{E}_V\) of Theorem 7.1.1 naturally induces an immersion \(\mathcal{U} \to \mathcal{E}_{\mathcal{V}}\).
Theorem 7.2.1. There is a canonical induced open immersion

\[ \tilde{U} \to (\tilde{\Phi} = 0) \subset E_{\tilde{V}}. \]

Suppose that \( d > 2 \) and (by Proposition 6.1.2) the homomorphism

\[ \beta^* \varphi : \beta^* \rho_* M(D) \to \beta^* \rho_* O_A(A) \]

can be diagonalized as

\[
\begin{bmatrix}
z_1 & 0 & 0 & \cdots & 0 \\
0 & z_2 & 0 & \cdots & 0
\end{bmatrix}.
\]

Then \( \tilde{U} \) has normal crossing singularities; the primary component of \( \tilde{U} \) whose general points are defined by \( z_1 z_2 \neq 0 \) is smooth. Consequently, \( \tilde{M}_2(\mathbb{P}^n, d) \) has normal crossing singularities; the primary component of \( \tilde{M}_2(\mathbb{P}^n, d) \) is smooth and of expected dimension.

Proof. First, Theorem 7.1.1 induces an open immersion \( \tilde{U} \to (\tilde{\Phi} = 0) \subset E_{\tilde{V}}. \)

We can trivialize \( \beta^* \rho_* M(D) \) and \( \beta^* \rho_* O_A(A) \) such that

\[ E_{\tilde{V}} \cong \tilde{V} \times (\mathbb{A}^{d+1})^n. \]

We let \( w^i_j \in \mathbb{A}^1 \) for all \( 1 \leq i \leq n \) and \( 0 \leq j \leq d \). Then the equation \( \tilde{\Phi} = 0 \) is equivalent to the following equations

\[ z_1 w^i_1 = 0, \quad z_2 w^i_2 = 0, \quad \text{for all } 1 \leq i \leq n. \]

Since the primary component of \( \tilde{U} = \tilde{V} \times_{\tilde{V}} U \) consists of general points with non-vanishing \( z_1 z_2 \), we see that the primary component is defined by

\[ w^i_1 = w^i_2 = 0, \quad \text{for all } 1 \leq i \leq n. \]

Thus, the primary component of \( \tilde{U} \) is smooth.

It remains to check dimension. Note that \( \dim E_{\tilde{V}} = 3 + d + (d+1)n \). Hence the dimension of the primary component of \( \tilde{U} \) is

\[ 3 + d + (d+1)n - 2n = d(n+1) - n + 3 \]

which is the virtual dimension of \( \tilde{M}_2(\mathbb{P}^n, d) \). \( \Box \)

When \( d = 2 \), the main component of \( \tilde{M}_2(\mathbb{P}^n, 2) \) is smooth of dimension \( 2n + 4 \); this is of wrong dimension unless \( n = 1 \) since the virtual dimension is \( n + 5 \); the entire moduli can be treated by hand.

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