Classifying spaces with virtually cyclic stabilizers for linear groups

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Abstract

We show that every discrete subgroup of $GL(n, \mathbb{R})$ admits a finite dimensional classifying space with virtually cyclic stabilizers. Applying our methods to $SL(3, \mathbb{Z})$, we obtain a four dimensional classifying space with virtually cyclic stabilizers and a decomposition of the algebraic $K$-theory of its group ring.

1 Introduction

A classifying space of a discrete group $\Gamma$ for a family of subgroups $\mathcal{F}$ is a $\Gamma$-CW complex $X$ with stabilizers in $\mathcal{F}$ such that $X^n_H$ is contractible for every $H \in \mathcal{F}$. Such a space is also called a model for $E_{\mathcal{F}} \Gamma$. A model for $E_{\mathcal{F}} \Gamma$ always exists for any given discrete group $\Gamma$ and a family of subgroups $\mathcal{F}$, but it need not be of finite type or finite dimensional (see [17]). The smallest possible dimension of a model for $E_{\mathcal{F}} \Gamma$ is the geometric dimension of $\Gamma$ for the family $\mathcal{F}$, denoted by $gd_{\mathcal{F}}(\Gamma)$. When $\mathcal{F}$ is the family of finite, respectively, virtually cyclic subgroups of $\Gamma$, $E_{\mathcal{F}} \Gamma (gd_{\mathcal{F}}(\Gamma))$ is denoted by $E_{\mathcal{F}} \Gamma (gd(\Gamma))$, respectively, $E_{\mathcal{F}} \Gamma (gd(\Gamma))$.

For any group $\Gamma$, one always has $gd(\Gamma) \leq gd(\Gamma) + 1$ (see [20]). In all examples known so far, a group $\Gamma$ admits a finite dimensional model for $E_{\mathcal{F}} \Gamma$ if it admits a finite dimensional model for $E_{\mathcal{F}} \Gamma$. However, it is still an open problem whether this is always the case. It is known that the invariant $gd(\Gamma)$ can be arbitrarily larger than $gd(\Gamma)$ (see [8]).

Questions concerning finiteness properties of $E_{\mathcal{F}} \Gamma$ and $E_{\mathcal{F}} \Gamma$ have been especially motivated by the Farrell–Jones isomorphism conjecture in $K$- and $L$-theory (see below and [6, 13, 19]). Finding models for $E_{\mathcal{F}} \Gamma$ with good finiteness properties has been proven to be much more difficult than for $E_{\mathcal{F}} \Gamma$. So far, such models have been found for polycyclic-by-finite groups [20], word-hyperbolic groups [11], relatively hyperbolic groups [14], countable elementary amenable group of finite Hirsch length [7, 8] and groups acting isometrically with discrete orbits on separable complete $CAT(0)$-spaces, such as mapping class groups and finitely generated linear groups over fields of positive characteristic [9, 18].

Here, we will show that certain subgroups of $GL(n, \mathbb{C})$ admit a finite dimensional classifying space with virtually cyclic stabilizers.

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Theorem A. Let $\Gamma$ be a discrete subgroup of $\text{GL}(n, \mathbb{R})$ such that the Zariski closure of $\Gamma$ in $\text{GL}(n, \mathbb{R})$ has dimension $m$. Then $\Gamma$ admits a model for $E\Gamma$ of dimension $m + 1$.

Recall that a subgroup $\Gamma$ of $\text{GL}(n, \mathbb{C})$ is said to be of integral characteristic if the coefficients of the characteristic polynomial of every element of $\Gamma$ are algebraic integers. It follows that $\Gamma$ has integral characteristic if and only if the characteristic roots of every element of $\Gamma$ are algebraic integers (see [22] §2). The standard embedding $\Gamma \hookrightarrow \text{GL}(n, \mathbb{C}) \hookrightarrow \text{SL}(n+1, \mathbb{C})$ allows one to consider $\Gamma$ as a subgroup of integral characteristic of $\text{SL}(n+1, \mathbb{C})$.

Theorem B. Let $\Gamma$ be a finitely generated subgroup of $\text{GL}(n, \mathbb{C})$ of integral characteristic such that there is an upper bound on the Hirsch lengths of its finitely generated unipotent subgroups. Then $\Gamma$ admits a finite dimensional model for $E\Gamma$.

Corollary. Let $\mathbb{F}$ be an algebraic number field and suppose $\Gamma$ is a subgroup of $\text{GL}(n, \mathbb{F})$ of integral characteristic. Then $\Gamma$ admits a finite dimensional model for $E\Gamma$.

Theorems A and B and the Corollary will be proven in Section 4.

The $K$-theoretical Farrell–Jones conjecture (e.g. see [6,19]) predicts that for a group $\Gamma$ and a ring $R$, the assembly map
\[ \mathcal{H}_n^L(E\Gamma; K_R) \to \mathcal{H}_n^L(\{\ast\}; K) = K_n(R[\Gamma]) \]
is an isomorphism for every $n \in \mathbb{Z}$. Here $K_n(R[\Gamma])$ is the algebraic $K$-theory of the group ring $R[\Gamma]$ and $\mathcal{H}_n^L(\{\ast\}; K_R)$ is a generalized equivariant homology theory defined using the $K$-theory spectrum $K_R$. This conjecture has been proven for many important classes of groups (and rings), including $\text{SL}(n, \mathbb{Z})$ when $R$ is finitely generated as an abelian group (see [4]).

Using the universal property of classifying spaces for families, one can construct a $\Gamma$-equivariant inclusion of $E\Gamma$ into $F\Gamma$. By a result of Bartels (see [3] Th. 1.3.1) this inclusion induces a split injection $\mathcal{H}_n^L(E\Gamma; K_R) \to \mathcal{H}_n^L(F\Gamma; K_R)$. Hence, there is an isomorphism
\[ \mathcal{H}_n^L(E\Gamma; K_R) \cong \mathcal{H}_n^L(E\Gamma; K_R) \oplus \mathcal{H}_n^L(F\Gamma, E\Gamma; K_R). \]

If $\Gamma = \text{SL}(3, \mathbb{Z})$, the term $\mathcal{H}_n^L(E\Gamma; K_R)$ can be computed using a 3-dimensional cocompact model for $F\Gamma$ constructed by Soulé (see [24]). In Theorem 5.4 we describe the term $\mathcal{H}_n^L(E\Gamma, E\Gamma; K_R)$ using a 4-dimensional model for $E\Gamma$ we construct in Section 5.

2 A push-out construction

A general method to obtain a model for $E\Gamma$ from a model for $F\Gamma$ is given by Lück and Weiermann in [20] §2. We will briefly recall this method.

Let $\Gamma$ be a discrete group and consider the set $\mathcal{S}$ of all infinite virtually cyclic subgroups of $\Gamma$. Two infinite virtually cyclic subgroup of $\Gamma$ are said to be equivalent if they have infinite intersection in $\Gamma$. One easily verifies that this defines an equivalence relation on $\mathcal{S}$. If $H \in \mathcal{S}$, then its equivalence class will be denoted by $[H]$.

The set of all equivalence classes of elements of $\mathcal{S}$ will be denoted by $[\mathcal{S}]$. Note
that the conjugation action of $\Gamma$ on $\mathcal{H}$ passes to $[\mathcal{H}]$. The stabilizer of an $[H] \in \mathcal{H}$ under this action is the subgroup

$$N_\Gamma[H] = \{ g \in \Gamma \mid |H \cap H^g| = \infty \}$$

of $\Gamma$. By definition, $N_\Gamma[H]$ only depends on the equivalence class of $[H]$ of $H$. We may therefore always assume that $[H]$ is represented by an infinite cyclic group $H = \langle t \rangle$. Hence, one can write

$$N_\Gamma[H] = \{ g \in \Gamma \mid \exists n, m \in \mathbb{Z} \setminus \{ 0 \} : g^{-1} t^n g = t^m \}.$$

This group is called the commensurator of $H$ in $\Gamma$. Some references, e.g. [8], actually denote this group by $\text{Comm}_\Gamma[H]$ instead of $N_\Gamma[H]$. Note that $N_\Gamma[H]$ always contains $H$ as a subgroup.

Let $\mathcal{I}$ be a complete set of representatives $[H]$ of the orbits of the conjugation action of $\Gamma$ on $\mathcal{H}$. For each $[H] \in \mathcal{I}$, let $\mathcal{F}[H]$ be the family of subgroups of $N_\Gamma[H]$ containing all finite subgroup of $N_\Gamma[H]$ and all infinite virtually cyclic subgroup of $N_\Gamma[H]$ that are equivalent to $H$.

**Theorem 2.1 ([20, Theorem 2.3]).** Let

$$\bigsqcup_{[H] \in \mathcal{I}} \Gamma \times N_\Gamma[H] E_{\mathcal{F}[H]} N_\Gamma[H] \xrightarrow{i} E \Gamma \xrightarrow{\pi} \bigsqcup_{[H] \in \mathcal{I}} \Gamma \times N_\Gamma[H] E_{\mathcal{F}[H]} N_\Gamma[H] \xrightarrow{\downarrow \downarrow} Y$$

be a $\Gamma$-equivariant push-out diagram of $\Gamma$-CW-complexes such that for each $[H] \in \mathcal{I}$, the map $f_{[H]}$ is cellular and $N_\Gamma[H]$-equivariant and $i$ is a cellular inclusion of $\Gamma$-CW-complexes. Then the push-out $Y$ is a model for $E \Gamma$.

Using ([20, Remark 2.5], one arrives at the following corollary.

**Corollary 2.2 ([20, Remark 2.5]).** If there exists a natural number $d$ such that for each $[H] \in \mathcal{I}$,

- $\text{gd}(N_\Gamma[H]) \leq d - 1$,
- $\text{gd}_{\mathcal{F}[H]}(N_\Gamma[H]) \leq d$,

and such that $\text{gd}(\Gamma) \leq d$, then $\text{gd}(\Gamma) \leq d$.

### 3 On the structure of $N_\Gamma[H]$ in linear groups

We recall that a real algebraic group is the set of real points of a linear algebraic group defined over $\mathbb{R}$. Throughout this section, we will use some basic facts about (real) algebraic groups for which we refer to [5]. For any subgroup $K$ of $\text{GL}(n, \mathbb{R})$, we will denote the Zariski closure of $K$ in $\text{GL}(n, \mathbb{R})$ by $\overline{K}$. The notions “connected” and “discrete” will refer to the Hausdorff topology and not to the Zariski topology.

The following result was kindly communicated to us by Herbert Abels.

**Proposition 3.1** (H. Abels). Let $G$ be a real algebraic group and suppose $R$ is its algebraic radical. Suppose $\Gamma$ is a discrete subgroup of $G$ such that the $\pi(\Gamma)$ is Zariski dense in $G/R$, where $\pi : G \to G/R$ is the natural quotient map. Then $\pi(\Gamma)$ is discrete.
Proof. Denote by $C$ the identity component of the closure of the group $\pi(\Gamma)$ in $G/R$ in the Hausdorff topology. We need to show that $C$ is trivial. By Corollary 1.3 of [1], it is solvable. Since $C$ is normalised by $\pi(\Gamma)$, it is also normalised by its Zariski closure $G/R$. We obtain that $C$ is a (Hausdorff and hence Zariski) connected solvable normal subgroup of the semisimple group $G/R$ and hence it is trivial. \hfill $\Box$

Now, let us assume that $\Gamma$ is a discrete subgroup of $GL(n,\mathbb{R})$ and let $[H]$ be an equivalence class of infinite virtually cyclic subgroups of $\Gamma$.

**Lemma 3.2.** There is a representative $H \in [H]$ such that $\overline{H}$ is a Zariski connected abelian normal subgroup of $N^c[H] \leq GL(n,\mathbb{R})$ and $N^c[H] = N^c(H)$, the normaliser of $H$ in $\Gamma$.

**Proof.** Let $H \in [H]$. An algebraic group has only finitely many Zariski connected components. Up to passing to a finite-index subgroup of $H$ we may therefore assume that $\overline{H}$ is a Zariski connected algebraic group. Moreover, since $H$ is abelian, so is $\overline{H}$. Let $x \in N^c[H]$. By definition, $H^x \cap H$ is a finite index subgroup of $H$. This implies that $\overline{H}^x \cap \overline{H}$ is an algebraic subgroup of $\overline{H}$ of the same dimension. Because $\overline{H}$ is Zariski connected, we conclude that $\overline{H}^x = \overline{H}$. Since $N^c[H]$ normalizes $\overline{H}$, it follows that $\overline{H}$ is normal in $N^c[H]$. From this we deduce that $\overline{H} \cap N^c[H]$ is a normal abelian subgroup of $N^c[H]$. Since every discrete subgroup of a finite dimensional abelian Lie group is finitely generated (e.g. see [23] Proposition 3.8), the structure theorem of finitely generated abelian groups implies that up to passing to a finite-index subgroup, $H$ is contained in an finite rank free abelian subgroup $A$ of $\overline{H} \cap N^c[H]$ that is normal in $N^c[H]$. But this implies that $H$ is also normal in $N^c[H]$. Indeed, take $g \in N^c[H]$. Since $A$ is normal in $N^c[H]$, conjugation by $g$ induces an automorphism $\varphi$ of $A$. Note that $H$ has a finite index infinite cyclic overgroup $H'$ in $A$ that has a primitive generator $h$, meaning that $h$ is not a proper power of any other element in $A \cong \mathbb{Z}'$. Because $g \in N^c[H] = N^c[H']$, there exists $s, t \in \mathbb{Z} \setminus \{0\}$ such that $s \varphi(h) = th$. Since $\varphi(h)$ is also a primitive element, it now follows that $s$ must divide $t$ and vice versa. Hence $s = \pm t$. It follows that $H'$ is normal in $N^c[H]$ and so is $H$. \hfill $\Box$

We continue assuming that $\overline{H}$ is a Zariski connected abelian normal subgroup of $N^c[H]$ and $N^c[H] = N^c(H)$. Let $m$ be the dimension of $\Gamma$.

Recall also that the notion of Hirsch length $h(S) \in \mathbb{Z}_{\geq 0}$ is defined for all virtually solvable groups $S$. The Hirsch length is stable under passing to finite index subgroups. It behaves additively with respect to group extensions of virtually solvable groups and satisfies $h(\mathbb{Z}) = 1$. It also satisfies the relation $h(S) = \sup\{h(S') | S' \}$ is a finitely generated subgroup of $S$.

**Proposition 3.3.** There exists a short exact sequence

$$1 \rightarrow N \rightarrow N^c[H]/H \rightarrow Q \rightarrow 1$$

where $Q$ is a discrete subgroup of a $k$-dimensional semisimple algebraic group and $N$ is a finitely generated solvable group of Hirsch length $h(N) \leq m - k - 1$.

**Proof.** Let $R$ be the algebraic radical of the Zariski closure $\overline{N^c[H]} \leq GL(n,\mathbb{R})$. There is a short exact sequence

$$1 \rightarrow R \rightarrow \overline{N^c[H]} \xrightarrow{\pi} S \rightarrow 1$$
where \( S = \mathbb{N}_c[H]/R \) is semisimple. Since \( \mathbb{N}_c[H] \) is Zariski dense in \( \mathbb{N}_c[H] \) we conclude that \( Q = \pi(\mathbb{N}_c[H]) \) is Zariski dense in \( S \). Hence, by Proposition 2.1 it follows that \( \pi(\mathbb{N}_c[H]) \) is a discrete subgroup of the semisimple real algebraic group \( S \). Since the Zariski connected abelian normal subgroup \( H \) of \( \mathbb{N}_c[H] \) has finitely many Hausdorff connected components, up to passing to a finite-index subgroup, we may assume that \( H \) is contained in \( R \). Denoting \( N = (R \cap \mathbb{N}_c[H])/H \), we obtain a short exact sequence

\[
1 \rightarrow N \rightarrow \mathbb{N}_c[H]/H \rightarrow Q \rightarrow 1.
\]

Since every discrete subgroup of a connected solvable Lie group is finitely generated (see [23, Proposition 3.8]) and \( R \), being an algebraic group, has finitely many connected components, it follows that \( N \) is a finitely generated solvable group. Suppose \( S \) had dimension \( k \). Then \( R \) has dimension \( m - k \). The dimension of \( R \) is an upper bound for \( \text{gd}(R \cap \mathbb{N}_c[H]) \) (e.g. see \([17, \text{Theorem } 4.4]\)). Moreover, \( \text{gd}(R \cap \mathbb{N}_c[H]) \) is bounded from below by the Hirsch length of \( R \cap \mathbb{N}_c[H] \), since the Hirsch length of a solvable group coincides with its rational homological dimension (see \([26]\)). It follows that \( h(R \cap \mathbb{N}_c[H]) \leq m - k \). Hence, the Hirsch length of \( N \) is at most \( m - k - 1 \).

4 \hspace{1em} The proofs of the main theorems

We are now ready to prove Theorems A and B and their Corollary.

Proof of Theorem A. Since the dimension of the Zariski closure of \( \Gamma \) in \( GL(n, \mathbb{R}) \) is an upper bound for \( \text{gd}(\Gamma) \) (e.g. see \([17, \text{Theorem } 4.4]\)), we have \( \text{gd}(\mathbb{N}_c[H]) \leq \text{gd}(\Gamma) \leq m \) for every infinite cyclic subgroup \( H \) of \( \Gamma \). Now fix \([H] \in \mathcal{S}\) and consider the exact sequence

\[
1 \rightarrow N \rightarrow \mathbb{N}_c[H]/H \rightarrow Q \rightarrow 1
\]

resulting from Proposition 3.3. By \([8, \text{Lemma } 4.2]\), we have \( \text{gd}_{\mathcal{S}[H]}(\mathbb{N}_c[H]) = \text{gd}(\mathbb{N}_c[H]/H) \) for every \([H] \in \mathcal{S}\). If the Hirsch length of \( N \) is at most 1, then since \( N \) is finitely generated it follows that \( N \) is virtually cyclic. So, every finite extension \( T \) of \( N \) must be virtually cyclic as well. In this case one has \( \text{gd}(T) \leq 1 \). If the Hirsch length of \( N \) is a least 2, then it follows from \([10, \text{Corollary } 4]\) that every finite extension \( T \) of \( N \) has \( \text{gd}(T) \leq h + 1 \). Since \( \text{gd}(S) \leq k \) by \([17, \text{Theorem } 4.4]\), it follows from \([8, \text{Corollary } 2.3]\) that \( \text{gd}_{\mathcal{S}[H]}(\mathbb{N}_c[H]) \leq (m - k - 1) + 1 + k = m \) for every \([H] \in \mathcal{S}\). The theorem now follows from Corollary 2.2.

Proof of Theorem B. We may assume that \( \Gamma \) is a subgroup of \( SL(n, \mathbb{C}) \) of integral characteristic. Let \( A \) be the finitely generated unital subring of \( \mathbb{C} \) generated by the matrix entries of a finite set of generators of \( \Gamma \) and their inverses. Then \( \Gamma \) is a subgroup of \( SL(n, A) \).

Let \( \mathcal{F} \) denote the quotient field of \( A \). Proceeding as in the proof of Theorem 3.3 of \([23]\), we obtain an epimorphism \( \rho : \Gamma \rightarrow H_1 \times \cdots \times H_r \) such that the kernel \( U \) of \( \rho \) is a unipotent subgroup of \( H \) and for each \( 1 \leq i \leq r \), \( H_i \) is a subgroup of some \( GL(n, \mathcal{F}) \) of integral characteristic where the canonical action of \( H_i \) on \( \mathcal{F}^n \) is irreducible. Following the proof of Proposition 2.3 of \([24]\), we have that for each subgroup \( H_i \), there exists a finite field extension \( L_i \) of \( \mathbb{Q} \) such that \( H_i \) is isomorphic to a subgroup \( H_i' \) of some \( GL(n_i, L_i) \), which is absolutely irreducible and of integral characteristic. Now, according to the proof of Proposition 2.1 of
[2], each $H_i'$ embeds as a discrete subgroup of $\text{GL}(m_i, \mathbb{R})^{s_i} \times \text{GL}(m_i, \mathbb{C})^{h_i}$ for some nonnegative integers $r_i$ and $s_i$. So, by Theorem A, we have that $gd(H_1 \times \cdots \times H_r) < \infty$. Applying Corollary 6.1 of [28], we obtain that

$$gd(\Gamma) \leq gd(H_1 \times \cdots \times H_r) + h + 3$$

where $h$ is the Hirsch length of $U$.

**Proof of Corollary.** As in the proof of Theorem B, $\Gamma$ fits into an extension

$$1 \to U \to \Gamma \to H_1 \times \cdots \times H_r \to 1$$

where $U$ is a unipotent subgroup and for each $1 \leq i \leq r$, $H_i$ is a subgroup of integral characteristic of some $\text{GL}(n_i, \mathbb{F})$ such that the canonical action of $H_i$ on $\mathbb{F}^n$ is irreducible. Following the proof of Proposition 2.1 of [2], we obtain that each $H_i$ is isomorphic to a discrete subgroup of $\text{GL}(m_i, \mathbb{R})^{s_i} \times \text{GL}(m_i, \mathbb{C})^{h_i}$ for some positive integers $r_i$ and $s_i$. By considering the upper central series of the subgroup of strictly upper triangular matrices $\text{Tr}(n, \mathbb{F})$ of $\text{GL}(n, \mathbb{F})$ and noticing that the additive group of $\mathbb{F}$ has finite Hirsch length because it is isomorphic to a finite direct product of copies of $(\mathbb{Q}, +)$, it follows that $\text{Tr}(n, \mathbb{F})$ has finite Hirsch length. Since the subgroup $U$ of $\Gamma$ is conjugate in $\text{GL}(n, \mathbb{C})$ to a subgroup of $\text{Tr}(n, \mathbb{F})$, it also has finite Hirsch length. Just as in the proof of Theorem B, we now conclude that $gd(\Gamma) < \infty$. □

5 **The case of SL(3, Z)**

Consider the group $\text{SL}(3, \mathbb{Z})$ and let $\mathcal{J}$ be a set of representatives of the orbits of the conjugation action of $\text{SL}(3, \mathbb{Z})$ on the set of equivalence classes of infinite virtually cyclic subgroups of $\text{SL}(3, \mathbb{Z})$ (see Section 2). We note that the infinite virtually cyclic subgroups of $\text{SL}(3, \mathbb{Z})$ are listed, up to isomorphism, in [25] and [27]. From this classification it follows that every infinite virtually cyclic subgroup $V$ of $\Gamma$ that is not isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$ fits into a short exact sequence

$$1 \to \mathbb{Z} \oplus \mathbb{Z}_2 \to V \to \mathbb{Z}_2 \to 1.$$

**Definition 5.1.** We define the following subsets of $\mathcal{J}$.

(a) The set $\mathcal{J}_1$ contains all $[H] \in \mathcal{J}$ such that $[H]$ has a representative whose generator has two complex conjugate eigenvalues and one real eigenvalue different from 1;

(b) The set $\mathcal{J}_2$ contains all $[H] \in \mathcal{J}$ such that $[H]$ has a representative whose generator has exactly one eigenvalue that is a root of unity.

(c) The set $\mathcal{J}_3$ contains all $[H] \in \mathcal{J}$ such that $[H]$ has a representative with a generator all of whose eigenvalues are real and not equal to ±1;

(d) The set $\mathcal{J}_4$ contains all $[H] \in \mathcal{J}$ such that $[H]$ has a representative with a generator all of whose eigenvalues equal 1 and which cannot be conjugated into the center of the strictly upper triangular matrices in $\text{SL}(3, \mathbb{Z})$.

(e) The set $\mathcal{J}_5$ contains all $[H]$ such that $[H]$ has a representative with a generator that can be conjugated into the center of the strictly upper triangular matrices in $\text{SL}(3, \mathbb{Z})$.  

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Lemma 5.2. One can write \( \mathcal{I} \) as a disjoint union
\[
\mathcal{I} = \mathcal{I}_1 \sqcup \mathcal{I}_2 \sqcup \mathcal{I}_3 \sqcup \mathcal{I}_4
\]
and the set \( \mathcal{I}_3 \) contains exactly one element.

Proof. This is left as an easy exercise to the reader. \( \square \)

The group \( \Gamma = \text{SL}(3, \mathbb{Z}) \) is a discrete subgroup of \( \text{GL}(3, \mathbb{R}) \). Hence, we know from Lemma 3.2 that for every equivalence class \( [H] \) of infinite virtually cyclic subgroups of \( \Gamma \), there exists an representative \( H \) such that \( N_\Gamma[H] = N_\Gamma(H) \). Using this fact, we will now determine for each \( [H] \in \mathcal{I} \) the structure of the group \( N_\Gamma[H] \).

Lemma 5.3. For each \( [H] \in \mathcal{I} \), the following holds.

(a) If \( [H] \in \mathcal{I}_1 \), then \( N_\Gamma[H] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

(b) If \( [H] \in \mathcal{I}_2 \), then \( N_\Gamma[H] \) has a subgroup of index at most two isomorphic to \( \mathbb{Z}_2 \oplus \mathbb{Z} \).

(c) If \( [H] \in \mathcal{I}_3 \), then \( N_\Gamma[H] \cong \mathbb{Z}_2 \oplus \mathbb{Z}^2 \).

(d) If \( [H] \in \mathcal{I}_4 \), then \( N_\Gamma[H] \) has a subgroup of index at most two isomorphic to \( \mathbb{Z}_2 \).

(e) If \( [H] \in \mathcal{I}_5 \), then \( N_\Gamma[H] \) is isomorphic to \( \text{Tr}(3, \mathbb{Z}) \rtimes \varphi \left( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right) \), where
\[
\text{Tr}(3, \mathbb{Z}) = \left\{ x, y, z \mid [x, y] = z, [x, z] = e, [y, z] = e \right\}
\]
is isomorphic to the group of strictly upper triangular integral matrices,
\[
\varphi((1, 0))(x) = x^{-1}, \varphi((1, 0))(y) = y^{-1}, \varphi((1, 0))(z) = z
\]
and
\[
\varphi((0, 1))(x) = x^{-1}, \varphi((0, 1))(y) = y, \varphi((0, 1))(z) = z^{-1}.
\]

Proof. Take \( [H] \in \mathcal{I} \) and let \( A \in \Gamma \) be an infinite order matrix such that \( N_\Gamma[H] = N_\Gamma(H) \), where \( (A) = H \). Note that we may replace \( A \) by a power of \( A \) in order to assume that \( A \) does not have any eigenvalues that are non-trivial roots of unity.

First assume that \( [H] \in \mathcal{I}_1 \sqcup \mathcal{I}_2 \). This means that all eigenvalues of \( A \) are different from 1. The characteristic polynomial \( p(x) \) of \( A \) is therefore irreducible over \( \mathbb{Q} \). Indeed, if \( p(x) \) was reducible over \( \mathbb{Q} \), then \( A \) would have a rational eigenvalue \( \mu \). But since \( A \in \text{SL}(3, \mathbb{Z}) \), it follows from the rational root theorem that \( \mu = \pm 1 \), which is a contradiction. Also note that the normalizer of \( H \) must equal the centralizer of \( H \). Indeed, an element of the normalizer of \( H \) that does not commute with \( A \) must send a eigenvector of \( A \) with eigenvalue \( \mu \) to an eigenvector of \( A \) with eigenvalue \( \mu^{-1} \), which would imply that \( A \) has an eigenvalue equal to 1. As illustrated for example in [12] Prop. 3.7 and [22] section 4], an application of the Dirichlet unit theorem shows that the centralizer \( C_\Gamma(H) \) of \( A \) in \( \Gamma \) equals \( \mathbb{Z}^{r+s-1} \oplus \mathbb{Z}_2 \), where \( r \) is the number of real roots of \( p(x) \) and \( 2s \) is the number of complex roots of \( p(x) \). Hence, if all eigenvalues of \( A \) are real then \( N_\Gamma(H) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2 \) and if \( A \) has two complex conjugate eigenvalues then \( N_\Gamma(A) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \). This proves (a) and (c).
Secondly, assume that \([H] \in \tilde{\mathcal{H}}\). Then \(A\) has exactly one eigenvalue equal to 1. Hence, \(A\) is conjugate in \(\text{SL}(3, \mathbb{Z})\) to a matrix of the form

\[
\begin{bmatrix}
1 & a & b \\
0 & 1 & 0 \\
0 & 0 & M
\end{bmatrix}.
\]

Since we are only interested in the structure of the normalizer of \(H\) up to isomorphism, we may as well assume that \(A\) is of this form. If a matrix \(B \in \text{SL}(3, \mathbb{Z})\) commutes with \(A\) it must preserve the 1-dimensional eigenspace of \(A\) with eigenvalue 1. Therefore, this is also an eigenspace of \(B\), with eigenvalue \(\pm 1\). We conclude that \(B\) must be of the form

\[
B = \begin{bmatrix}
\pm 1 & x & y \\
0 & 1 & 0 \\
0 & 0 & N
\end{bmatrix}.
\]

By elementary matrix computations, one checks that such a matrix \(B\) commutes with \(A\) if and only if it is of the form

\[
(M^t - \text{Id}) \begin{bmatrix} x \\ y \end{bmatrix} = (N^t - \text{Id}) \begin{bmatrix} a \\ b \end{bmatrix}.
\]

Since \((M^t - \text{Id})\) is an invertible matrix, \(B\) is completely determined by \(N\) and the fact that its commutes with \(N\). We therefore obtain an isomorphism \(C_{\Gamma}(A) = C_{\text{GL}(2, \mathbb{Z})}(M)\). By analyzing centralizers in \(\text{GL}(2, \mathbb{Z})\), for example using the Dirichlet unit theorem, it follows that the centralizer \(C_{\Gamma}(A)\) is \(\mathbb{Z}_2 \oplus \mathbb{Z}\). This proves (b).

Finally, assume that all eigenvalues of \(A\) equal 1. In this case \(A\) can be conjugated inside \(\text{SL}(3, \mathbb{Z})\) to a strictly upper triangular matrix. Hence, we may again assume that \(A\) is a strictly upper triangular matrix. If \(A\) is of the form

\[
\begin{bmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{bmatrix}
\]

where \(a\) and \(b\) are both non-zero, then one may check by elementary matrix computations that a matrix \(B \in \text{SL}(3, \mathbb{Z})\) commutes with \(A\) if and only if it is of the form

\[
\begin{bmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{bmatrix}
\]

where \(ay - bx = 0\). This shows that in this case the centralizer of \(H\) in \(\Gamma\) is isomorphic to \(\mathbb{Z}^2\), and hence the normalizer \(N_{\Gamma}(H)\) has a subgroup of index at most two isomorphic to \(\mathbb{Z}^2\). If on the other hand, \(A\) is of the form (1) where \(ab = 0\) then \(A\) can be conjugated in \(\text{SL}(3, \mathbb{Z})\) to matrix of the form (1) where \(a\) and \(b\) are both zero. In this case the centralizer \(C_{\Gamma}(H)\) is isomorphic to the group

\[
\left\{ \begin{bmatrix}
\pm 1 & x & z \\
0 & 1 & y \\
0 & 0 & \pm 1
\end{bmatrix} \bigg| x, y, z \in \mathbb{Z} \right\}.
\]
One can now easily verify via explicit matrix computation that the normalizer \( N_\Gamma(H) \) is isomorphic to a semi-direct product of \( C_\Gamma(H) \) with

\[
\left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.
\]

It follows that \( N_\Gamma(H) \) is isomorphic to the semi-direct product

\[
\text{Tr}(3, \mathbb{Z}) \rtimes \left( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right).
\]

In \([24]\), a 3-dimensional model \( X \) for \( E\Gamma \) is constructed. This model has the property that the orbit-space \( \Gamma \setminus X = B\Gamma \) is contractible. Moreover, this model is of minimal dimension since \( \Gamma \) contains the strictly upper triangular matrices \( \text{Tr}(3, \mathbb{Z}) \), which has cohomological dimension 3. Since for each \( [H] \in \mathcal{H} \), \( N_\Gamma[H] \) is either virtually-\( \mathbb{Z} \), virtually-\( \mathbb{Z}^2 \) or virtually-\( \text{Tr}(3, \mathbb{Z}) \) by the lemma above, a model for \( E_\mathcal{H}[H] \) can be chosen to be either \( \mathbb{R} \), \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \), respectively (see, e.g., \([17]\) Ex. 5.26). Moreover, \( H \) can be chosen to be normal in \( N_\Gamma[H] \) in which case a model for \( E_\mathcal{H}[H] \) is given by a model for \( E_\mathcal{H}[H]/H \), where the action is obtained via the projection \( N_\Gamma[H] \to N_\Gamma[H]/H \). Hence, a model for \( E_\mathcal{H}[H] \) can be chosen to be either \( \{ \ast \} \), \( \mathbb{R} \) or \( \mathbb{R}^2 \), respectively. Using the universal property of classifying spaces for families, one obtains a cellular \( \Gamma \)-equivariant map

\[
f : \bigsqcup_{[H] \in \mathcal{H}} \Gamma \times N_\Gamma[H] E_\mathcal{H}[H] \to X
\]

Note that the mapping cylinder \( M_f \) of \( f \) is a 4-dimensional model for \( E\Gamma \), since \( E_\mathcal{H}[H] \) is at most 3-dimensional and \( M_f \) is \( \Gamma \)-homotopy equivalent to \( X \). We obtain an equivariant cellular inclusion

\[
i : \bigsqcup_{[H] \in \mathcal{H}} \Gamma \times N_\Gamma[H] E_\mathcal{H}[H] \to M_f.
\]

Using \( i \) and the models for \( E_\mathcal{H}[H] \) described above, one can construct a \( \Gamma \)-equivariant push-out diagram that by Theorem \([2.1]\) produces a 4-dimensional model \( Y \) for \( E\Gamma \). We claim that this model is of minimal dimension. Indeed, take \( \Gamma \)-orbits of the push-out diagram constructed above and consider the long exact Mayer–Vietoris cohomology sequence with \( \mathbb{Q} \)-coefficients obtained from the resulting push-out diagram. This leads to the exact sequence

\[
H^3(B\Gamma, \mathbb{Q}) \to H^3(BN_\Gamma[H], \mathbb{Q}) \to H^4(B\Gamma, \mathbb{Q}) \to 0,
\]

where \( [H] \in \mathcal{H} \). Since \( B\Gamma \) is contractible and \( N_\Gamma[H] \cong \text{Tr}(3, \mathbb{Z}) \rtimes \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) by the lemma above , we obtain an isomorphism

\[
H^3(\text{Tr}(3, \mathbb{Z}) \rtimes \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Q}) \cong H^4(B\Gamma, \mathbb{Q}).
\]

As we are working with \( \mathbb{Q} \)-coefficients, an application of the Lyndon–Hochschild–Serre spectral sequence tells us that

\[
H^3(\text{Tr}(3, \mathbb{Z}) \rtimes \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Q}) \cong H^3(\text{Tr}(3, \mathbb{Z}), \mathbb{Q})^\mathbb{Z}_2 \oplus \mathbb{Z}_2.
\]
Moreover, since \( \text{Tr}(3, \mathbb{Z}) \) fits into the central extension
\[
1 \to \mathbb{Z} \cong \langle z \rangle \to \text{Tr}(3, \mathbb{Z}) \to \mathbb{Z}^2 \cong \langle x, y \rangle \to 0,
\]
another application of the Lyndon–Hochschild–Serre spectral sequence yields
\[
H^3(\text{Tr}(3, \mathbb{Z}), \mathbb{Q}) \oplus \mathbb{Z}_2 \cong H^3((x, y), H^1((z), \mathbb{Q}))^{\mathbb{Z}_2}.
\]
Using the explicit description of the map \( \varphi : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \text{Aut}(\text{Tr}(3, \mathbb{Z})) \) in the lemma above, and the fact that
\[
H^3((x, y), H^1((z), \mathbb{Q})) = \text{Hom}(\Lambda^3((x, y)), \text{Hom}(\Lambda^1((z)), \mathbb{Q}) \cong \mathbb{Q},
\]
one checks that the action of \( H \) on \( H^3((x, y), H^1((z), \mathbb{Q})) \) is trivial. We conclude that \( H^3(\mathbb{B} \Gamma, \mathbb{Q}) \cong \mathbb{Q} \), proving that there cannot exists a model for \( \mathbb{E} \Gamma \) of dimension strictly smaller than 4.

As mentioned in the introduction, for \( \Gamma = \text{SL}(3, \mathbb{Z}) \), the Farrell–Jones conjecture implies that for any ring \( R \) that is finitely generated as an abelian group, one has
\[
K_n(R[\Gamma]) \cong \mathcal{K}_n(\mathbb{E} \Gamma; R) \oplus \mathcal{K}_n(\mathbb{E} \Gamma; K_0)
\]
for every \( n \in \mathbb{Z} \). Using the model \( Y \) for \( \mathbb{E} \Gamma \) constructed above and Lemma 5.3 we obtain a description of the term \( \mathcal{K}_n(\mathbb{E} \Gamma; R) \). We summarize this description in the following theorem. Note that given a \( \Gamma \)-map \( f : X \to Y \), the homology group \( \mathcal{H}_n^\Gamma(Y, X; K_0) \) is by definition the relative homology group \( \mathcal{H}_n^\Gamma(M_f, X; K_0) \), where \( M_f \) is the mapping cylinder of \( f \).

**Theorem 5.4.** Let \( \Gamma = \text{SL}(3, \mathbb{Z}) \) and let \( R \) be a ring that is finitely generated as an abelian group. Then,
\[
K_n(R[\Gamma]) \cong \mathcal{K}_n(\mathbb{E} \Gamma; K_0) \oplus \mathcal{K}_n(\mathbb{E} \Gamma; \mathbb{R}; K_0) \oplus \mathcal{K}_n(\mathbb{E} \Gamma; \mathbb{Z}; K_0)
\]
where,

(a) for \( [H] \in \mathcal{A}_1 \), \( N_{\Gamma}[H] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), \( E_{N_{\Gamma}}[H] = \{ \ast \} \), \( E_{N_{\Gamma}}[H] = \mathbb{R} \) and
\[
\mathcal{K}_n(\mathcal{A}_1) = \bigoplus_{[H] \in \mathcal{A}_1} \mathcal{K}_n^N([H] \{ \ast \}, \mathbb{R}; K_0),
\]

(b) for \( [H] \in \mathcal{A}_1 \), \( N_{\Gamma}[H] \) has a subgroup of index at most two isomorphic to \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), \( E_{N_{\Gamma}}[H] = \{ \ast \} \), \( E_{N_{\Gamma}}[H] = \mathbb{R} \) and
\[
\mathcal{K}_n(\mathcal{A}_1) = \bigoplus_{[H] \in \mathcal{A}_1} \mathcal{K}_n^{N_{\Gamma}}([H] \{ \ast \}, \mathbb{R}; K_0),
\]

(c) for \( [H] \in \mathcal{A}_2 \), \( N_{\Gamma}[H] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), \( E_{N_{\Gamma}}[H] = \mathbb{R} \), \( E_{N_{\Gamma}}[H] = \mathbb{R} \) and
\[
\mathcal{K}_n(\mathcal{A}_2) = \bigoplus_{[H] \in \mathcal{A}_2} \mathcal{K}_n^{N_{\Gamma}}([H], \mathbb{R}; K_0),
\]
(d) for $[H] \in \tilde{\mathcal{I}}_2$, $N_f[H]$ has a subgroup of index at most two isomorphic to $\mathbb{Z}^2$, $E_{\mathcal{F}[H]}N_f[H] = EN_f[H]/H = \mathbb{R}$, $EN_f[H] = \mathbb{R}^2$ and

$$\mathcal{M}_n(\tilde{\mathcal{I}}_2) = \bigoplus_{[H] \in \tilde{\mathcal{I}}_2} \mathcal{M}_n^{N_f[H]}(\mathbb{R}, \mathbb{R}^2; K_R),$$

(e) for $[H] \in \tilde{\mathcal{I}}_3$, $N_f[H] \cong \text{Tr}(3, \mathbb{Z}) \rtimes \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $E_{\mathcal{F}[H]}N_f[H] = EN_f[H]/H = \mathbb{R}^2$, $EN_f[H] = \mathbb{R}^3$ and

$$\mathcal{M}_n(\tilde{\mathcal{I}}_3) = \mathcal{M}_n^{N_f[H]}(\mathbb{R}^2, \mathbb{R}^3; K_R).$$

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