ON SPECIAL CALIBRATED ALMOST COMPLEX STRUCTURES
AND MODULI SPACE

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Abstract. An \( \omega \)-admissible almost complex structure on a \( 2n \)-dimensional
symplectic manifold \((M, \omega)\) is a \( \omega \)-calibrated almost complex structure \( J \) ad-
mittting a nowhere vanishing \( \partial_J \)-closed \( (n, 0) \)-form \( \psi \). After giving some exam-
ple we consider the moduli space of admissible almost complex structures and
we study infinitesimal deformations. As special case, we write down explicit
computations for the complex torus.

1. Introduction

The interplay between symplectic and complex structures has been studied quite
extensively in the last years (see e.g. [1], [2], [9], [10], [14] and the refer-
ence therein). It is well known that on an a symplectic manifold \((M, \omega)\) there exist many almost
complex structures \( J \) such that

\[
\omega[x] (v, w) = \omega[x](J_x v, J_x w), \quad \omega[x](u, J_x u) > 0,
\]

for any \( x \in M, v, w, u \in T_x M, u \neq 0 \) (see e.g. [2], [9]). An almost complex
structure \( J \) on a symplectic manifold \((M, \omega)\) is said to be \( \omega \)-calibrated if it satisfies
condition (1). Furthermore the existence of special holomorphic structures on a
symplectic manifold (e.g. Kähler structures, Calabi-Yau structures) imposes strong
conditions on the topology of the manifold. Hence, it is natural to consider the non-
integrable case in order to have more flexible structures. In this context, the notion
of symplectic Calabi-Yau manifold has been considered in [4], [5] and it appears as
a natural generalization of Calabi-Yau manifold\(^1\). This generalization is different
from that one given by Hitchin in [12] in the context of generalized geometry.

Namely, a symplectic Calabi-Yau manifold is a symplectic manifold \((M, \omega)\)
endowed with an \( \omega \)-calibrated almost complex structure \( J \) and a complex volume
form \( \psi \) covariantly constant with respect to the Chern connection of \((M, J, \omega)\).
This is equivalent to have an almost Kähler manifold endowed with a \( \partial_J \)-closed
complex volume form. In dimension 6 this definition can be improved by requiring
that the real part of the complex volume form be closed. Such structures are called
symplectic Half-flat.

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\(^1\)In [5] symplectic Calabi-Yau manifolds were called \textit{generalized Calabi-Yau manifolds}. Here we change the terminology to avoid confusion with the case considered by Hitchin in [12].
In the present paper we study the moduli space of $\omega$-calibrated almost complex structures admitting a symplectic Calabi-Yau structure. Such almost complex structures will be called $\omega$-admissible.

First of all, we give an example of a non-admissible almost complex structure on a 6-dimensional compact symplectic manifold. This manifold provides an example of an almost Kähler manifold whose Chern connection has holonomy non-contained in $\text{SU}(3)$. Then, we study the infinitesimal deformations of the space of $\omega$-admissible almost complex structures and we compute the virtual tangent space to the moduli space. In the last part we apply our results to the torus showing that the standard complex structures is not rigid.

In section 2 we recall some preliminary results on complex manifolds and Hermitian geometry and give the basic definitions. In section 3 we give the example described above and an example of a compact almost complex 6-manifold which admits symplectic Calabi-Yau structures, but has no symplectic half-flat structure.

In section 4 we study the infinitesimal deformations of the space of $\omega$-admissible almost complex structures $\mathcal{AC}_\omega(M)$. We introduce the deformation form $\theta_L$ (see subsection 4.2 for the precise definition), which is a 1-form depending on the choice of an endomorphism of $TM$ and on a complex $\bar{\partial}$-closed volume form, and we prove the following

**Theorem.** Let $(M,\omega)$ be a symplectic manifold. Fix $J \in \mathcal{AC}_\omega(M)$ and consider a smooth curve $J_t$ in $\mathcal{AC}_\omega(M)$ close to $J$ and satisfying $J_0 = 0$. Then the derivative $\dot{J}_0$ of $J_t$ at $0$ is given by

$$\dot{J}_0 = 2JL,$$

where $L \in \text{End}(TM)$ satisfies the following conditions

$$L = \bar{\partial}J, \quad LJ = - JL$$

and for any nowhere vanishing $\bar{\partial}_J$-closed $\psi \in \Lambda^{\cdot,0}_J(M)$ the deformation form $\theta_L(\psi)$ is $\bar{\partial}_J$-exact.

A key tool to prove this theorem is proposition 4.2, which describes the relationship between the $\bar{\partial}$-operators of two close almost complex structures (see section 4).

In the compact case the previous theorem allows to define the virtual tangent space to the moduli space of admissible almost complex structures

$$\mathfrak{M}(\mathcal{AC}_\omega(M)) = \mathcal{AC}_\omega(M)/\text{Sp}_\omega(M).$$

In particular we can introduce the concepts of unobstructed and rigid $\omega$-admissible almost complex structures. See proposition 4.8 and definition 4.9.

In section 5 we apply the results of section 4 to the complex torus, showing that its standard complex structure is not rigid.

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2. Preliminaries

Let \((M, J)\) be an almost complex manifold manifold. Then the bundle of complex valued \(r\)-forms \(\Lambda^r_C(M)\) decomposes as

\[
\Lambda^r_C(M) = \bigoplus_{p+q=r} \Lambda^p_J(M),
\]

where \(\Lambda^p_J(M)\) is the bundle of \((p, q)\)-forms on \((M, J)\). According with the above decomposition, the exterior derivative \(d\) splits as

\[
d: \Lambda^p_J(M) \to \Lambda^{p+1}_J(M)
\]

splits as

\[
d: \Lambda^p_J(M) \to \Lambda^{p+2}_J(M) \oplus \Lambda^{p+1}_J(M) \oplus \Lambda^{p-1}_J(M) \oplus \Lambda^{p+1}_J(M) \\
\]

\[
d = A_J + \partial_J + \overline{\partial}_J.
\]

Let \(\omega\) be an almost symplectic structure on \(M\), i.e. \(\omega\) is a non-degenerate 2-form on \(M\). An almost complex structure \(J\) is said to be \(\omega\)-tamed if

\[
\omega[x](v, J_x v) > 0
\]

for any \(x \in M, v \in T_x M, v \neq 0\).

If \(J\) satisfies the extra condition

\[
\omega[x](J_x v, J_x w) = \omega[x](v, w),
\]

for any \(x \in M, v, w \in T_x M\), then \(J\) is said to be \(\omega\)-calibrated. In this case

\[
g_J[x](v, w) := \omega[x](v, J_x w),
\]

defines an almost Hermitian metric on \(M\) (i.e. a \(J\)-invariant Riemannian metric).

We denote by \(\mathcal{T}_\omega(M)\) and by \(\mathcal{C}_\omega(M)\) the space of \(\omega\)-tamed and \(\omega\)-calibrated almost complex structures on \(M\) respectively. It is well known that \(\mathcal{T}_\omega(M)\) is a contractible space (see e.g. [2]). In particular, the first Chern class of \((M, J)\) does not depend on the choice of \(J \in \mathcal{T}_\omega(M)\). Hence it is well defined the first Chern class of \((M, \omega)\).

By definition, an almost complex structure \(J\) is said to be integrable (or a complex structure) if the Nijenhuis tensor

\[
N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]
\]

vanishes.

In the sequel we will use the following fact (see e.g. [13]):

let \((M, J)\) be a compact almost complex manifold and \(f: M \to \mathbb{C}\) be a \(J\)-holomorphic function, then \(f\) is constant.

Now we are going to recall the definition of symplectic Calabi-Yau structure. Let \((M, \omega, J)\) be an almost Kähler manifold and let \(\nabla^{LC}\) be the Levi Civita connection of the metric \(g_J\) associated with \((\omega, J)\). Then the Chern connection is defined as

\[
\nabla = \nabla^{LC} - \frac{1}{2} J \nabla^{LC} J
\]

and it satisfies the following properties

\[
\nabla g = 0, \quad \nabla J = 0, \quad T^\nabla = \frac{1}{2} N_J,
\]

\(T^\nabla\) being the torsion of \(\nabla\).
Remark 2.1. We recall that $\nabla^{0,1} = \overline{\partial}_J$ (see e.g. [8]), where $\nabla^{0,1}$ denotes the $(0,1)$-component of $\nabla$.

We have the following definition (see [5])

**Definition 2.2.** A symplectic Calabi-Yau manifold consists of $(M, \omega, J, \psi)$, where $(M, \omega)$ is a $2n$-dimensional symplectic manifold, $J$ is an $\omega$-calibrated almost complex structure on $M$ and $\psi$ is a nowhere vanishing $(n,0)$-form on $M$ satisfying

$$\nabla \psi = 0,$$

where $\nabla$ is the Chern connection of $(\omega, J)$.

**Remark 2.3.** Note that a Calabi-Yau manifold is in particular a symplectic Calabi-Yau manifold. Indeed in this case the Chern connection is the Levi Civita one, since $J$ is integrable.

In the 6-dimensional case we can improve the previous definition by requiring that the real part of $\psi$ is $d$-closed; namely

**Definition 2.4.** A symplectic half-flat manifold is the datum of $(M, \omega, J, \psi)$, where $(M, \omega)$ is a 6-dimensional symplectic manifold, $J$ is an $\omega$-calibrated almost complex structure on $M$, $\psi$ is a nowhere vanishing $(3,0)$-form such that

$$\nabla \psi = 0, \quad \psi \wedge \overline{\psi} = -\frac{4}{3} i \omega^3, \quad d \text{Re} \psi = 0.$$

These structures are just the intersection between symplectic and half-flat ones. The latter have been introduced and studied by Hitchin and Chiossi-Salamon (see [11] and [3]). In this situation is possible to perform (special) Lagrangian geometry by considering Lagrangian submanifolds calibrated by $\text{Re} \psi$ (see [4]).

**Remark 2.5.** Note that the condition $\nabla \psi = 0$ is redundant since it can be showed that, given a nowhere vanishing $\psi \in \Lambda^{3,0}_J(M)$, then the following facts are equivalent (see [3])

$$\begin{cases}
\psi \wedge \overline{\psi} = i \lambda \omega^3 \\
d \text{Re} \psi = 0
\end{cases} \iff \nabla \psi = 0, \quad (A_J + \overline{A}_J) \psi = 0,$$

where $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $A_J: \Lambda^{p,q}_J(M) \to \Lambda^{p+2,q-1}_J(M)$ denotes the $(p + 2, q - 1)$-component of the exterior derivative of a $(p,q)$-form.

We introduce now the definition of admissible almost complex structures. Namely, if $(M, \omega)$ is a symplectic manifold, an $\omega$-calibrated almost complex structure $J$ on $M$ will be called admissible if there exists $\psi \in \Lambda^{n,0}_J(M)$ such that the triple $(\omega, J, \psi)$ is a symplectic Calabi-Yau structure on $M$. More precisely we state the following

**Definition 2.6.** Let $(M, \omega)$ be a symplectic manifold. An almost complex structure $J$ on $M$ is said to be $\omega$-admissible if

1. $J$ is $\omega$-calibrated;
2. there exists a nowhere vanishing $\psi \in \Lambda^{n,0}_J(M)$ such that $\overline{\partial}_J \psi = 0$.

In the next sections we will study some properties of admissible almost complex structures and we will introduce the Moduli space of such structures.
3. Examples

We start by giving an example of an almost complex structure which admits a symplectic Calabi-Yau structure, but it has no symplectic half-flat structures.

Example 3.1. Let

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}$$

be the complex Heisenberg group and let $\Gamma \subset G$ be the subgroup with integral entries. Then $M = G/\Gamma$ is the Iwasawa manifold. It is known that $M$ is symplectic, but it has no Kähler structures (see [6]).

Let $z_r = x_r + ix_{r+3}$, $r = 1, 2, 3$, and set

$$\alpha_1 = dx_1, \quad \alpha_2 = dx_3 - x_1 dx_2 + x_4 dx_5, \quad \alpha_3 = dx_5,$$

$$\alpha_4 = dx_4, \quad \alpha_5 = dx_2, \quad \alpha_6 = dx_6 - x_4 dx_2 - x_1 dx_5,$$

then $\{\alpha_1, \ldots, \alpha_6\}$ are invariant 1-forms on $G$, so that $\{\alpha_1, \ldots, \alpha_6\}$ is a global coframe on $M$. We immediately get

$$\begin{align*}
\{d\alpha_1 = d\alpha_3 = d\alpha_4 = d\alpha_5 = 0 \\
d\alpha_2 = -\alpha_1 \wedge \alpha_5 - \alpha_3 \wedge \alpha_4 \\
d\alpha_6 = -\alpha_4 \wedge \alpha_5 - \alpha_1 \wedge \alpha_3.
\end{align*}$$

Let $\{\xi_1, \ldots, \xi_6\}$ be the dual frame of $\{\alpha_1, \ldots, \alpha_6\}$; then

$$\begin{align*}
J(\xi_r) = \xi_{r+3} & \quad r = 1, 2, 3 \\
J(\xi_{4+r}) = -\xi_r & \quad r = 1, 2, 3,
\end{align*}$$

defines an almost complex structure on $M$ calibrated by the symplectic form

$$\omega = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6.$$

Let $\psi = (\alpha_1 + i \alpha_4) \wedge (\alpha_2 + i \alpha_5) \wedge (\alpha_3 + i \alpha_6)$, then a direct computation gives

$$\begin{align*}
\psi \wedge \overline{\psi} = -\frac{4}{3} \omega^3 \\
\overline{\nabla}_J \psi = 0.
\end{align*}$$

Hence the conditions above and remark 2.1 imply

$$\nabla \psi = 0.$$

Hence $(\omega, J, \psi)$ is a symplectic Calabi-Yau structure on $M$.

Now we prove that there are no nowhere vanishing $(3,0)$-forms $\eta$ on $M$ such that

$$d\Re \eta = 0.$$ 

In particular $(M, J)$ does not admit any symplectic half-flat structure.

In order to show this let $\eta \in \Lambda_{3,0}^M(M)$; then there exists $f \in C^\infty(M, \mathbb{C})$ such that

$$\eta = f \psi.$$

Let $f = u + iv$ and set

$$du = \sum_{i=1}^n u_i \alpha_i, \quad dv = \sum_{i=1}^n v_i \alpha_i.$$
A direct computation shows that
\[
d\Re \eta = (u_6 + v_3) \alpha_{3456} + (v_2 + u_5) \alpha_{2456} + (-u_1 + v_4) \alpha_{1345} + \\
+ (-u_6 - v_3) \alpha_{1236}(-u_3 + v_6) \alpha_{2346} + (-u_5 - v_2) \alpha_{1235} + \\
+ (u_1 - v_4) \alpha_{1246} + (u_3 - v_6) \alpha_{1356} + v \alpha_{1245} + \\
-v \alpha_{1346} + (u + u_4 + v_1) \alpha_{1456} + (u - u_4 - v_1) \alpha_{1234} + \\
+ (-u_2 + v_5) \alpha_{2345} + (u_2 - v_5) \alpha_{1256},
\]
where \(\alpha_{ijk} = \alpha_i \wedge \alpha_j \wedge \alpha_k\). Hence \(d\Re \eta = 0\) if and only if \(u = v = 0\). \(\square\)

The next nilmanifold provides an example of a compact almost complex manifold with vanishing first Chern class and such that there are no Hermitian metrics whose Chern connection has holonomy contained in SU(3). This is in contrast with the integrable case, in view of Calabi-Yau theorem. For other examples involving the Bismut connection see [7].

**Example 3.2.** Let
\[
H(3) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R} \right\}
\]
be the 3-dimensional Heisenberg group and let \(\Gamma \subset H(3)\) be the cocompact lattice of matrices with integral entries. Then \(X = S^1 \times H(3)/\Gamma\) is called the *Kodaira-Thurston* manifold.

Let \(M = \mathbb{T}^2 \times X\), where \(\mathbb{T}^2\) is the 2-dimensional standard torus. Then \(M\) can be viewed as \(\mathbb{R}^6/\sim\), where
\[
[(x_1, x_2, x_3, x_4, x_5, x_6)] = [(x_1 + m_1, x_2 + m_2, x_3 + m_3, x_4 + m_4, x_5 + m_5, x_6 + m_4 x_5 + m_6)],
\]
for any \((m_1, m_2, m_3, m_4, m_5, m_6) \in \mathbb{Z}^6\). The 1-forms
\[
\alpha_1 = dx_1, \quad \alpha_2 = dx_2, \quad \alpha_3 = dx_3, \\
\alpha_4 = dx_4, \quad \alpha_5 = dx_5, \quad \alpha_6 = dx_6 - x_4 dx_5,
\]
define a global coframe on \(M\). We have
\[
d\alpha_i = 0, \text{ for } i = 1, \ldots, 5, \\
d\alpha_6 = -\alpha_4 \wedge \alpha_5.
\]
The 2-form
\[
\omega = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4 + \alpha_5 \wedge \alpha_6,
\]
is a symplectic structure on \(M\). Let \(J\) be the almost complex structure defined on the dual frame of \(\{\alpha_1, \ldots, \alpha_6\}\) by the relations
\[
J(X_1) := X_2, \quad J(X_3) := X_4, \quad J(X_5) := X_6, \\
J(X_2) := -X_1, \quad J(X_4) := -X_3, \quad J(X_6) := -X_5.
\]

\(J\) is an \(\omega\)-calibrated almost complex structure on \(M\). The form
\[
\psi := (\alpha_1 + i\alpha_2) \wedge (\alpha_3 + i\alpha_4) \wedge (\alpha_5 + i\alpha_6)
\]
is a nowhere vanishing section of \(\Lambda^3_{\mathbb{R}}(M)\). We have
\[
\overline{\partial} J \psi = -(\alpha_3 - i\alpha_4) \wedge \psi.
\]
Now we prove that there are no nowhere vanishing $(3,0)$-forms $\eta$ on $M$ such that $\overline{\partial}_J \eta = 0$. Set

$$Z_j = \frac{1}{2}(X_j - iJX_j), \quad j = 1, 2, 3$$

and

$$\zeta_j = \alpha_j + iJ\alpha_j, \quad j = 1, 2, 3.$$  

Any $\eta \in \Lambda^{3,0}_J(M)$ is a multiple of $\psi$, i.e. there exists a function $f = u + iv \in C^\infty(M, \mathbb{C})$ such that $\psi = f \eta$. Then

$$\overline{\partial}_J \eta = \overline{\partial}_J (f \psi) = \overline{\partial}_J f \wedge \psi + f \overline{\partial}_J \psi = (\sum_{j=1}^{3} \overline{\partial}_j(f) \zeta_j - f \overline{\partial}_2) \wedge \psi.$$  

Therefore $\overline{\partial}_J \eta = 0$ if and only if the following systems of PDE’s are satisfied:

a. $\begin{cases} \partial_{x_1} u - \partial_{x_2} v = 0 \\ \partial_{x_2} u + \partial_{x_1} v = 0, \end{cases}$

b. $\begin{cases} \partial_{x_3} u - \partial_{x_4} v - u = 0 \\ \partial_{x_4} u + \partial_{x_3} v - v = 0, \end{cases}$

c. $\begin{cases} \partial_{x_5} v - \partial_{x_6} u - x_4 \partial_{x_6} u = 0 \\ \partial_{x_6} u + \partial_{x_5} v + x_4 \partial_{x_5} v = 0, \end{cases}$

where the unknowns $u, v$ are functions on $\mathbb{R}^6$ satisfying

$u(x) = u(x_1 + m_1, x_2 + m_2, x_3 + m_3, x_4 + m_4, x_5 + m_5, x_6 + m_4x_5 + m_6),$

$v(x) = v(x_1 + m_1, x_2 + m_2, x_3 + m_3, x_4 + m_4, x_5 + m_5, x_6 + m_4x_5 + m_6)$

for any $m_1, \ldots, m_6 \in \mathbb{Z}$.

Equations a. imply that $f = f(x_3, x_4, x_5, x_6)$. Since $f$ is a function on $M$, then $f$ is $\mathbb{Z}$-periodic in $x_3, x_5, x_6$. Therefore we can take the Fourier expansion of $u$ and $v$. Set

$$u(x_3, x_4, x_5, x_6) = \sum_{N \in \mathbb{Z}^3} u_N(x_4) e^{2\pi i (n_3x_3 + n_5x_5 + n_6x_6)},$$

$$v(x_3, x_4, x_5, x_6) = \sum_{N \in \mathbb{Z}^3} v_N(x_4) e^{2\pi i (n_3x_3 + n_5x_5 + n_6x_6)},$$

where $N = (n_3, n_5, n_6)$. Then

$$\partial_{x_5} u = \sum_{N \in \mathbb{Z}^3} 2\pi i n_5 u_N(x_4) e^{2\pi i (n_3x_3 + n_5x_5 + n_6x_6)},$$

and the same relation holds for $\partial_{x_6} u, \partial_{x_4} v, \partial_{x_5} v$. Hence, by plugging (2) and the other expressions for the derivatives of $u, v$ into equations c., we get

$$\begin{cases} (n_5 + x_4 n_6) u_N(x_4) - n_6 v_N(x_4) = 0 \\ n_6 u_N(x_4) + (n_5 + x_4 n_6) v_N(x_4) = 0, \end{cases}$$

for any $N = (n_3, n_5, n_6) \in \mathbb{Z}^3$. If $(n_5 + x_4 n_6)^2 + n_6^2 \neq 0$ then $u_N(x_4) = v_N(x_4) = 0$. Therefore if $f$ satisfies equations a. and c. then $f = f(x_3, x_4)$. In particular $f$
must be $\mathbb{Z}^2$-periodic. By equations b. we immediately get $f \equiv 0$. Hence $J$ is not admissible.

4. Moduli spaces of admissible almost complex structures

Let $(M, \omega)$ be a symplectic manifold with vanishing first Chern class. By using notation of section 2 let

$$C_\omega(M) = \{ J \in \text{End}(TM) | J^2 = -I, \omega(\cdot, \cdot) = \omega(J\cdot, J\cdot), \omega(\cdot, J\cdot) > 0 \}$$

be the space of $\omega$-calibrated almost complex structures on $M$. Let $J$ be an $\omega$-calibrated almost complex structure; then we say that $\tilde{J} \in C_\omega(M)$ is close to $J$ if $\det(I - \tilde{JJ}) \neq 0$. It is known that the space of $\omega$-calibrated almost complex structures close to a fixed $J$ is parametrized by the tangent bundle symmetric endomorphisms anti-commuting with $J$ and having norm less than 1: namely $\tilde{J}$ is close to $J$ if and only if there exists a unique $L \in \text{End}(TM)$ such that

$$\begin{cases} 
\tilde{J} = RJR^{-1}, \\
^tL = L, \\
LJ = -JL, \\
||L|| < 1,
\end{cases}$$

where $R = I + L$ and the norm $|| \cdot ||$ is taken with respect to $g_J$ (see [2]). Denote by

$$\mathcal{AC}_\omega(M) = \{ J \in C_\omega(M) | J \text{ is } \omega - \text{admissible} \}.$$ 

Then the symplectic group

$$\text{Sp}_\omega(M) = \{ \phi \in \text{Diff}(M) | \phi^*\omega = \omega \}$$

acts on $\mathcal{AC}_\omega(M)$ by conjugation

$$(\phi, J) \mapsto \phi^{-1} \circ J \circ \phi.$$ 

Let

$$\mathcal{M}(\mathcal{AC}_\omega(M)) = \mathcal{AC}_\omega(M)/\text{Sp}_\omega(M)$$

be the relative Moduli Space.

4.1. Deformation of $\partial J$. In order to give a description of $\mathcal{M}(\mathcal{AC}_\omega(M))$, we have to describe the behavior of the $\partial$ operator for an almost complex structure $\tilde{J}$ close to a fixed $J$. We start considering the following

Proposition 4.1. Let $R = I + L$ be an arbitrary isomorphism of $TM$. Then

$$(3) \quad RdR^{-1} \gamma = d\gamma + [\tau_L, d]\gamma + \sigma_L \gamma$$

for any exterior form $\alpha$ in $M$ of positive degree, where:

- $\tau_L$ is the zero order derivation defined on the $r$-forms by $\tau_L \gamma(X_1, X_2, \ldots, X_r) = \gamma(LX_1, X_2, \ldots, X_n) + \gamma(X_1, LX_2, \ldots, X_n) + \cdots + \gamma(X_1, X_2, \ldots, LX_n)$;

- $[\tau_L, d] = \tau_L d - d\tau_L$;
• $\sigma_L$ is the operator defined on the 1-form as

$$\sigma_L \alpha(X,Y) := \alpha(R^{-1}(N_L(X,Y))),$$

being $N_L(X,Y) := [LX,LY] - L[LX,Y] - L[X,LY] + L^2[X,Y]$, and extended on the forms of arbitrary degree by the Leibniz rule.

**Proof.** Let $\alpha \in \Lambda^1(M)$ and let $X, Y \in TM$. We have

$$RdR^{-1} \alpha(X,Y) = dR^{-1} \alpha(RX,RY) = RX\alpha(Y) - RY\alpha(X) + \alpha(R^{-1}[RX,RY])$$

$$= X\alpha(Y) - Y\alpha(X) + \alpha([X,Y]) + LX\alpha(Y) - LY\alpha(X)$$

$$+ \alpha(R^{-1}[RX,RY] - [X,Y])$$

$$= da(X,Y) + LX\alpha(Y) - LY\alpha(X) + \alpha(R^{-1}[RX,RY] - [X,Y]).$$

Moreover

$$\tau_L da(X,Y) = da(LX,LY) + da(X,LY)$$

$$= LX\alpha(Y) - Y\alpha(X) + \alpha([X,Y]) + X\alpha(LY) - LY\alpha(X) + \alpha([X,LY])$$

and

$$dr_L \alpha(X,Y) = X\alpha(LY) - Y\alpha(LX) + \alpha(L[X,Y]).$$

Therefore we obtain

$$(RdR^{-1} - [\tau_L,d])\alpha(X,Y) = da(X,Y) + \alpha(R^{-1}[RX,RY] - [X,Y]) + \alpha(L[X,Y] - [X,Y]).$$

Now we have

$$R(R^{-1}[RX,RY] - [LX,Y] - [X,LY] + L[X,Y] - [X,Y]) =$$

$$= [RX,RY] - R[LX,Y] - R[X,LY] + RL[X,Y] - R[X,Y] =$$

$$= [LX,LY] + [LX,Y] + [X,LY] + [X,Y] - [LX,Y] - L[LX,Y]$$

$$= [X,LY] - L[X,LY] + L[X,Y] + L^2[X,Y] - [X,Y] =$$

$$= N_L(X,Y),$$

i.e.

$$R^{-1}[RX,RY] - [LX,Y] - [X,LY] + L[X,Y] - [X,Y] = R^{-1}(N_L(X,Y)).$$

Hence we have

$$(RdR^{-1} - [\tau_L,d])\alpha(X,Y) = da(X,Y) + \alpha(R^{-1}(N_L(X,Y)))$$

which proves the proposition when $\alpha$ is a 1-form. Since the operators on the two sides of formula (3) satisfy Leibnitz rule, the proof is complete. \hfill \Box

Now we are ready to give the following

**Proposition 4.2.** Let $J, \tilde{J}$ be close almost complex structures in $\mathcal{C}_\omega(M)$. Let $\overline{\partial}_J$, $\overline{\partial}_{\tilde{J}}$ be the $\overline{\partial}$-operators with respect to $J, \tilde{J}$ respectively. Then

1. $R\overline{\partial}_{\tilde{J}} f = \overline{\partial}_J f + L\partial_J f,$

2. $R\overline{\partial}_J R^{-1} \alpha = \overline{\partial}_J \alpha + [\tau_L, d]^{p,q+1} \alpha + \sigma_L^{p,q+1} \alpha,$

where $f \in C^\infty(M, \mathbb{C})$, $\alpha \in \Lambda^p,q(M)$, $\tilde{J} = RJ^{-1} R^{-1}$, $R = I + L$, $[\tau_L, d]^{p,q+1}$, $(\sigma_L)^{p,q+1}$ denote the projection of the bracket $[\tau_L, d] = \tau_L d - d\tau_L$ and of the operator $\sigma_L$ on the space $\Lambda^{p,q+1}_J(M)$, respectively.
Proof. 1. Let $f \in C^\infty(M, \mathbb{C})$. We have
\[
R\bar{\partial}_jf = R(df)^{\bar{0},1} = (Rdf)^{\bar{0},1} = (df + Ldf)^{\bar{0},1} = \bar{\partial}_jf + L\partial_j f^{\bar{0},1},
\]
where the subscript $0, 1$ denotes the projection onto $\Lambda_{\bar{j}}^{0,1}(M)$.

2. Let $\alpha \in \Lambda_{\bar{j}}^{p,q}(M)$. Then we have
\[
R\bar{\partial}_j R^{-1}_j \alpha = R(dR^{-1}_j \alpha)^{\bar{0}, q+1} = (RdR^{-1}_j \alpha)^{\bar{0}, q+1} = (d\alpha)^{\bar{0}, q+1} + [\tau_L, d]^{\bar{0}, q+1} + \sigma_L^{q, p+1} \alpha,
\]
where the subscript $p, q + 1$ denotes the projection onto $\Lambda_{\bar{j}}^{p, q+1}(M)$. \qed

4.2. The deformation form. In this subsection we introduce a $(0,1)$-form which will be a useful tool to study infinitesimal deformations of admissible almost complex structures.

Let $(M, \omega)$ be a symplectic manifold, $J \in \mathcal{C}_\omega(M)$ and $\psi \in \Lambda_{\bar{j}}^{n,0}(M)$ nowhere vanishing. For any endomorphism $L$ of $TM$ commuting with $J$ there exist unique forms $\mu_L(\psi), \tau_L(\psi) \in \Lambda^{0,1}(M)$ satisfying the following relations
\[
(\tau_L \overline{A}_J \psi)^{n,1} = \mu_L(\psi) \wedge \psi, \quad \partial_j \tau_L \psi = \gamma_L(\psi) \wedge \psi.
\]

Definition 4.3. The $(0,1)$-form
\[
\theta_L(\psi) := \mu_L(\psi) - \gamma_L(\psi)
\]
is called the deformation form of $L$.

Note that if $J$ is integrable, then $\theta_L(\psi) = -\gamma_L(\psi)$, since $\overline{A}_J = 0$.

The following lemma gives the behavior of $\theta_L(\psi)$ when the complex volume form $\psi$ changes:

Lemma 4.4. Let $\psi, \psi' \in \Lambda_{\bar{j}}^{n,0}(M)$ be $\overline{\partial}_j$-closed. Let $\{Z_1, \ldots, Z_n\}$ be a local $(1,0)$-frame and $\{\zeta_1, \ldots, \zeta_n\}$ be the dual frame.

Then
1. $\mu_L(\psi') = \mu_L(\psi)$,
2. $\gamma_L(\psi') = \gamma_L(\psi) + \eta(f)$,

where $\psi' = f \psi$ and $\eta(f)$ is the $(0,1)$-form defined locally as
\[
\eta(f) = -\frac{1}{f} \sum_{k,r=1}^n Z_k(f) L_{kr} \overline{\zeta}_r,
\]

where
\[
L(Z_i) = \sum_{k=1}^n L_{ik} Z_k.
\]

Proof. By definition we have
\[
\mu_L(\psi') \wedge \psi' = (\tau_L \overline{A}_J \psi')^{n,1} = (\tau_L \overline{A}_J f \psi)^{n,1} = f(\tau_L \overline{A}_J \psi)^{n,1}
= f \mu_L(\psi) \wedge \psi = \mu_L(\psi) \wedge \psi'.
\]
Therefore 1. is proved.
We have
\[
\gamma_{L}(\psi')\wedge\psi' = \partial_{J}f\psi' = \partial_{J}\tau_{L}f\psi \\
= \partial_{J}f\wedge\tau_{L}\psi + f\partial_{J}\tau_{L}\psi \\
= \partial_{J}f\wedge\tau_{L}\psi + f\gamma_{L}(\psi)\wedge\psi \\
= \partial_{J}f\wedge\tau_{L}\psi + \gamma_{L}(\psi)\wedge\psi'.
\]
Now we express \(\partial_{J}f\wedge\tau_{L}\psi\) in terms of \(\psi\'). With respect to the local \((1,0)\)-frame \(\{Z_{1},\ldots,Z_{n}\}\) we get
\[
\psi = h\zeta_{1} \cdots \zeta_{n},
\]
where \(h\) is a local nowhere vanishing smooth function and
\[
\partial_{J}f = \sum_{k=1}^{n}Z_{k}(f)\zeta_{k}.
\]
Now we have
\[
\tau_{L}\psi = hL(\zeta_{1}) \cdots \zeta_{n} + \cdots + h\zeta_{1} \cdots L(\zeta_{n})
\]
\[
= h\sum_{r=1}^{n}\{L_{1r}\zeta_{r} \cdots \zeta_{n}\} + \cdots + h\sum_{r=1}^{n}\{(-1)^{n-1}L_{n\tau}\zeta_{r} \cdots \zeta_{n-1}\}.
\]
Therefore
\[
\partial_{J}f\wedge\tau_{L}\psi = -\sum_{k,r=1}^{n}Z_{k}(f)L_{k\tau}\zeta_{r} \wedge \psi
\]
\[
= -\frac{1}{f}\sum_{k,r=1}^{n'}Z_{k}(f)L_{k\tau}\zeta_{r} \wedge \psi'.
\]
Hence
\[
\gamma_{L}(\psi') = \gamma_{L}(\psi) + \eta(f),
\]
i.e. 2. is proved.
\[
\square
\]
**Corollary 4.5.** Assume that \((M,\omega)\) is compact. Then the deformation form \(\theta_{L}\) does not depend on the choice of the \(\partial_{J}\)-closed complex volume form on \(M\).

**Proof.** Since \(M\) is compact if \(\psi,\psi'\) are two complex volume form on \(M\) satisfying \(\partial_{J}\psi = \partial_{J}\psi' = 0\), then \(\psi = c\psi'\) for some constant \(c\) on \(M\). Hence by lemma 4.4
\[
\theta_{L}(\psi) = \theta_{L}(\psi')
\]
for any \(L \in \text{End}(TM)\) anticommuting with \(J\).

\[
\square
\]

**4.3. Infinitesimal deformations of admissible complex structures.** In this subsection we compute the infinitesimal deformations of admissible almost complex structures on a symplectic manifold. We start with the following

**Theorem 4.6.** Let \((M,\omega)\) be a symplectic manifold. Fix \(J \in \mathcal{AC}(M)\) and consider a smooth curve \(J_{t}\) in \(\mathcal{AC}(M)\) of almost complex structures close to \(J\) satisfying \(J_{0} = 0\). Then the derivative \(J_{0}\) of \(J_{t}\) at 0 is given by
\[
\dot{J}_{0} = 2JL,
\]
where \(L \in \text{End}(TM)\) satisfies the following conditions
\[
L = \nabla_{L}, \quad LJ = -JL
\]
and for any nowhere vanishing $\overline{\partial}_J$-closed $\psi \in \Lambda^n_j(M)$ the $(0,1)$-form $\theta_L(\psi)$ is $\overline{\partial}_J$-exact.

Proof. Let $\psi \in \Lambda^n_j(M)$ be a nowhere vanishing complex form satisfying $\overline{\partial}_J \psi = 0$. Fix an $\omega$-calibrated almost complex structure $\bar{J}$ close to $J$. Then

$$\bar{J} = RJR^{-1},$$

where

$$R = I + L, \quad L = tL, \quad LJ + JL = 0, \quad \|L\| < 1.$$  

The form $R^{-1} \psi \in \Lambda^n_j(M)$ is nowhere vanishing. Any other section $\psi'$ which trivializes $\Lambda^n_j(M)$ is a multiple of $\psi$, namely

$$\psi' = f R^{-1} \psi,$$

with $f \in C^\infty(M,\mathbb{C})$, $f(p) \neq 0$ for any $p \in M$. Hence the almost complex structure $\bar{J}$ is $\omega$-admissible if and only if there exists $f \in C^\infty(M,\mathbb{C})$ such that

$$\overline{\partial}_J(f R^{-1} \psi) = 0,$$

where $f \neq 0$.

By formulæ of proposition [1,2] we have

$$R \overline{\partial}_J(f R^{-1} \psi) = R(\overline{\partial}_J f \wedge R^{-1} \psi + f \overline{\partial}_J R^{-1} \psi)$$

$$= R(\overline{\partial}_J f) \wedge \psi + fR \overline{\partial}_J R^{-1} \psi$$

$$= \overline{\partial}_J f \wedge \psi + L \partial_J f \wedge \psi + f(\overline{\partial}_J \psi + [\tau_L, d]^{n,1} \psi + (\sigma_L)^{n,1} \psi)$$

$$= \overline{\partial}_J f \wedge \psi + L \partial_J f \wedge \psi + f([\tau_L, d]^{n,1} \psi + (\sigma_L)^{n,1} \psi),$$

i.e.

$$R \overline{\partial}_J(f R^{-1} \psi) = \overline{\partial}_J f \wedge \psi + L \partial_J f \wedge \psi + f([\tau_L, d]^{n,1} \psi + (\sigma_L)^{n,1} \psi).$$

(4) $$R \overline{\partial}_J(f R^{-1} \psi) = \overline{\partial}_J f \wedge \psi + L \partial_J f \wedge \psi + f([\tau_L, d]^{n,1} \psi + (\sigma_L)^{n,1} \psi).$$

Let us consider a smooth curve of $\omega$-admissible complex structures $J_t$ close to $J$, such that $J_0 = J$. For any $t$ there exists $L_t \in \text{End}(TM)$ such that if $R_t = I + L_t$, then

$$J_t = R_t J R_t^{-1},$$

for $L_t J + JL_t = 0, \|L_t\| < 1$. We may assume $L_0 = 0$. We set $L_0 = L$. Then

$$J_0 = JL - LJ = 2JL.$$  

For any $t$ there exists $f_t: M \to \mathbb{C}$, $f_t \neq 0$, such that

$$\overline{\partial}_{J_t}(f_t R_t^{-1} \psi) = 0.$$  

Hence by formula (4) $J_t$ is $\omega$-admissible if and only exists $f_t: M \to \mathbb{C}$, such that $f_t \neq 0$ and

$$\overline{\partial}_{J_t} f_t \wedge \psi + L_t \partial_J f_t \wedge \psi + f_t([\tau_{L_t}, d]^{n,1} \psi + f_t(\sigma_{L_t})^{n,1} \psi = 0.$$  

(5) $$\overline{\partial}_{J_t} f_t \wedge \psi + L_t \partial_J f_t \wedge \psi + f_t([\tau_{L_t}, d]^{n,1} \psi + f_t(\sigma_{L_t})^{n,1} \psi = 0.$$  

We may assume without loss of generality that

$$f_0 = 1.$$  

Since $\frac{d}{dt} \sigma_{L_t}|_{t=0} = 0$, by taking the derivative of (5) at $t = 0$ we get

$$\overline{\partial}_J f_0 \wedge \psi + [\tau_L, d]^{n,1} \psi = 0.$$  

(6) $$\overline{\partial}_J f_0 \wedge \psi + [\tau_L, d]^{n,1} \psi = 0.$$
Let us compute $[\tau_L, d]^{n,1} \psi$. Since $d = A_J + \partial_J + \overline{\partial}_J + \overline{A}_J$, we have

$$(\tau_L d \psi)^{n,1} = (\tau_L A_J \psi)^{n,1},$$

$$(d \tau_L \psi)^{n,1} = \partial_J \tau_L \psi.$$ 

By the definition of $\mu_L(\psi)$, $\gamma_L(\psi)$ we have

$$(\tau_L \overline{A}_J \psi)^{n,1} = \mu_L(\psi) \wedge \psi, \quad \partial_J \tau_L \psi = \gamma_L(\psi) \wedge \psi.$$ 

Hence (6) reduces to

$$\overline{\partial}_J \tilde{\psi} + \mu_L(\psi) \wedge \psi - \gamma_L(\psi) \wedge \psi = 0,$$

so that (6) is equivalent to

$$\overline{\partial}_J \tilde{\psi} + \mu_L(\psi) - \gamma_L(\psi) = 0,$$

i.e. $\theta_L(\psi)$ is $\overline{\partial}_J$-exact. □

We give the following

**Definition 4.7.** Let $(M, \omega)$ be a symplectic manifold and $J \in \mathcal{AC}_\omega(M)$. The tangential space to the moduli space $\mathfrak{M}(\mathcal{AC}_\omega(M))$ at $[J]$ is the vector space

$$T_{[J]} \mathfrak{M}(\mathcal{AC}_\omega(M)) := \{ J, \gamma : (-\epsilon, \epsilon) \rightarrow \mathcal{AC}_\omega(M) \text{ is a smooth curve s.t. } \gamma(0) = J \},$$

where $\mathcal{O}_J(M)$ denotes the orbit of $J$ under the action of $\text{Sp}_\omega(M)$.

As a corollary of theorem 4.6 we have the following

**Proposition 4.8.** Let $(M, \omega)$ be a compact symplectic manifold of dimension $2n$ and let $J \in \mathcal{AC}_\omega(M)$. Then the tangent space to the Moduli space $\mathfrak{M}(\mathcal{AC}_\omega(M))$ at the point $[J]$ satisfies

$$T_{[J]} \mathfrak{M}(\mathcal{AC}_\omega(M)) \subseteq \{ JL \in \text{End}(TM) \mid L = ^tL, LJ = -JL, \theta_L \text{ is } \overline{\partial}_J \text{-exact} \} \cap \{ L_X J \in TM \mid X \in TM \text{ and } L_X \omega = 0 \}.$$

**Proof.** By corollary 4.5 the $(0, 1)$-form $\theta_L$ does not depend on the choice of the volume form $\psi$. Therefore, by theorem 4.6 if $J_t$ is a smooth curve in $\mathcal{AC}_\omega(M)$ satisfying $J_0 = J$, then $J_t = 2tJL$, where $L$ is a symmetric endomorphism of $TM$ anticommuting with $J$ and such that $\theta_L$ is $\overline{\partial}_J$-exact. A standard argument shows that

$$T_{[J]} \mathcal{O}_J(M) = \{ L_X J \in TM \mid X \in TM \text{ and } L_X \omega = 0 \}$$

and this completes the proof. □

In analogy to the classical case, it is natural to consider the following

**Definition 4.9.** The vector space

$$TV_{[J]} \mathfrak{M}(\mathcal{AC}_\omega(M)) := \{ JL \in \text{End}(TM) \mid L = ^tL, LJ = -JL, \theta_L \text{ is } \overline{\partial}_J \text{-exact} \} \cap \{ L_X J \in TM \mid X \in TM \text{ and } L_X \omega = 0 \}$$

will be called the virtual tangent space to the moduli space $\mathfrak{M}(\mathcal{AC}_\omega(M))$ at $[J]$.

An $\omega$-admissible almost complex structure will be called

- non-obstructed if $T_{[J]} \mathfrak{M}(\mathcal{AC}_\omega(M)) = TV_{[J]} \mathfrak{M}(\mathcal{AC}_\omega(M))$,

- rigid if $TV_{[J]} \mathfrak{M}(\mathcal{AC}_\omega(M)) = \{ 0 \}$. 

We end this section with the following

**Remark 4.10.** We observe the following:

i) In contrast to the case of deformation of complex structures (see e.g. [15]), in this context, as far as we know, the fact that $J$ in non-obstructed cannot be interpreted as a cohomological condition, since the $\overline{\partial}_J$-operator does not give rise to a cohomology on the manifold;

ii) All the considerations made in this section can be done by considering almost symplectic manifolds instead of symplectic manifolds.

5. Admissible complex structures on the Torus

In this section we apply the results of section 4 to the torus, computing explicitly the virtual tangent space to $\mathcal{M}(\mathcal{AC}_\omega(T^{2n}))$.

Let $T^{2n} = \mathbb{C}^n/\mathbb{Z}^{2n}$ be the standard complex torus of real dimension $2n$ and let \{z_1, \ldots, z_n\} be coordinates on $\mathbb{C}^n$, $z_\alpha = x_\alpha + ix_{\alpha+n}$ for $n = 1, \ldots, n$. Then

$$\omega_n = \frac{i}{2} \sum_{\alpha=1}^n dz_\alpha \wedge d\overline{z}_\alpha,$$

$$\psi_n = dz_1 \wedge \cdots \wedge dz_n$$

define a Calabi-Yau structure on $T^{2n}$. Therefore the standard complex structure $J_n$ is a $\omega_n$-admissible complex structure on $T^{2n}$.

Now we want to deform $J_n$, computing the virtual tangent space $TV_{J_n} \mathcal{M}(\mathcal{AC}_\omega(M))$ to the Moduli space $\mathcal{M}(\mathcal{AC}_\omega(M))$. According to the previous section, given a symmetric $L \in \text{End}(TM)$ that anticommutes with $J_n$, we have to write down the $(0,1)$-form $\gamma_L = -\theta_L$ defined by

$$\overline{\partial}_J(\tau_L \psi_n) = \gamma_L \wedge \psi_n.$$

Let

$$L = \sum_{s,r=1}^n \{ L_{rs} dz_s \otimes \frac{\partial}{\partial \overline{z}_r} + \overline{L}_{rs} \overline{dz}_s \otimes \frac{\partial}{\partial z_r} \},$$

where \{L_{rs}\} are $\mathbb{Z}^{2n}$-periodic functions. Then we get

$$\tau_L \psi_n = L(dz_1) \wedge \cdots \wedge dz_n + \cdots + dz_1 \wedge \cdots \wedge L(dz_n)$$

$$= \sum_{r=1}^n \{ L_{1r} d\overline{z}_r \wedge \cdots \wedge dz_n \} + \cdots + \sum_{r=1}^n \{ (-1)^{n-1} L_{nr} d\overline{z}_r \wedge \cdots \wedge dz_{n-1} \}$$

$$= \sum_{r,s=1}^n (-1)^{r+1} L_{rs} d\overline{z}_r \wedge dz_1 \wedge \cdots \wedge \overline{dz}_s \wedge \cdots \wedge dz_n,$$

where $\wedge$ means that the corresponding term is omitted. Therefore we obtain

$$\partial_J(\tau_L \psi) = - \sum_{r,s=1}^n \frac{\partial}{\partial z_s} L_{rs} d\overline{z}_r \wedge \psi,$$

i.e.

$$\gamma_L = - \sum_{r,s=1}^n \frac{\partial}{\partial z_s} L_{rs} d\overline{z}_r.$$
Then if $J_t \in AC_\omega(M)$ is a smooth curve satisfying $J_0 = J_n$, then $J_0 = 2J_n L$, where

$$L = L', \quad J_n L + LJ_n = 0$$

and the $(0,1)$-form

$$\gamma_L = - \sum_{r,s=1}^{n} \frac{\partial}{\partial z_r} L_{sr} \, dz_r$$

is $\overline{\partial} J_n$-exact. In order to compute $TV_{J_n}(\mathcal{M}(\mathcal{C}_\omega(T^{2n})))$, we have to write down the Lie derivative $L_X J_n$, for $X \in \text{End}(TM)$, such that $L_X \omega_n = 0$. Let $X = \sum_{r=1}^{2n} a_r \frac{\partial}{\partial x_r}$ be a real vector field on $T^{2n}$, then a direct computation gives

$$L_X \omega_n = 0 \iff \begin{cases} \frac{\partial}{\partial x_r} a_{n+s} - \frac{\partial}{\partial x_s} a_{n+r} = 0 \\ \frac{\partial}{\partial x_r} a_s + \frac{\partial}{\partial x_{n+s}} a_{n+r} = 0 \\ \frac{\partial}{\partial x_r} a_{n+s} - \frac{\partial}{\partial x_s} a_{n+r} = 0 \end{cases}$$

for $r, s = 1, \ldots, n$. Furthermore

$$L_X (J_n) \left( \frac{\partial}{\partial z_r} \right) = \sum_{r=1}^{2n} -i \frac{\partial a_r}{\partial z_s} \frac{\partial}{\partial x_r} + \frac{\partial a_r}{\partial z_s} J_n \left( \frac{\partial}{\partial x_r} \right)$$

$$= \sum_{r=1}^{2n} -i \frac{\partial a_r}{\partial z_s} \left( \frac{\partial}{\partial x_r} - i J_n \frac{\partial}{\partial x_r} \right)$$

$$= \sum_{r=1}^{n} -i \frac{\partial a_r}{\partial z_s} \left( \frac{\partial}{\partial x_r} - i \frac{\partial}{\partial x_{r+n}} \right) - i \frac{\partial a_{r+n}}{\partial z_s} \left( \frac{\partial}{\partial x_{r+n}} - i \frac{\partial}{\partial x_r} \right)$$

$$= - \sum_{r=1}^{n} \frac{\partial}{\partial z_s} (ia_r + a_{r+n}) \left( \frac{\partial}{\partial x_r} + i \frac{\partial}{\partial x_{r+n}} \right)$$

$$= -2 \sum_{r=1}^{n} \frac{\partial}{\partial z_s} (ia_r + a_{r+n}) \frac{\partial}{\partial z_r},$$

i.e.

$$L_X (J_n) = -2 \sum_{r,s=1}^{n} \left\{ \frac{\partial}{\partial z_s} (ia_r + a_{r+n}) \, dz_s \otimes \frac{\partial}{\partial z_r} + \frac{\partial}{\partial z_s} (-ia_r + a_{r+n}) \, dz_r \otimes \frac{\partial}{\partial z_r} \right\}.$$

Therefore $L = L_X J_n$, with $L_X \omega = 0$ if and only if

$$L_{\tau_r} = 2 \frac{\partial}{\partial z_s} (a_r - ia_{r+n}),$$

where $a_r, a_{r+n}$ are periodic functions on $\mathbb{R}^{2n}$ satisfying

$$\begin{cases} \frac{\partial}{\partial x_r} a_{n+s} - \frac{\partial}{\partial x_s} a_{n+r} = 0 \\ \frac{\partial}{\partial x_r} a_s + \frac{\partial}{\partial x_{n+s}} a_{n+r} = 0 \\ \frac{\partial}{\partial x_r} a_{n+s} - \frac{\partial}{\partial x_s} a_{n+r} = 0 \end{cases}.$$
Now we show that the standard complex structure $J_n$ on $T^{2n}$ is not rigid. In order to see this, we show that if $L$ belongs to tangent space to the orbit of $J_n$ at $J_n$ and it has constant coefficients, then $L$ is zero.

Let $J_n L \in \text{End}(T^2 \mathbb{T}^{2n})$, where $L$ is symmetric, anticommutes with $J_n$ and it is such that $L_{rs}$ are constant functions.

By equation (7) there exists $X \in TM$ such that $J_n L = L X (J_n)$ if and only if

$$
(J_n L)_{rs} = 2 \frac{\partial}{\partial z_s} (a_r - i a_{r+n}) - \frac{\partial}{\partial x_s} (a_r - i a_{r+n}) = \left( \frac{\partial a_r}{\partial x} - \frac{\partial a_{r+n}}{\partial x_{s+n}} - i \left( \frac{\partial a_{r+n}}{\partial x_s} + \frac{\partial a_r}{\partial x_{s+n}} \right) \right).
$$

Therefore

$$
\begin{align*}
\frac{\partial a_r}{\partial x} - \frac{\partial a_{r+n}}{\partial x_{s+n}} &= \text{constant} \\
\frac{\partial a_{r+n}}{\partial x_s} + \frac{\partial a_r}{\partial x_{s+n}} &= \text{constant},
\end{align*}
$$

that imply

$$\frac{\partial^2 a_r}{\partial x^2} + \frac{\partial^2 a_r}{\partial x_{s+n}^2} = 0$$

for any $r, s = 1, \ldots, n$.

It follows that the $\{a_r\}$ are harmonic functions on the standard torus $T^{2n}$ and then they are constant. Therefore any constant $0 \neq L \in \text{End}(T^2 \mathbb{T}^{2n})$ anticommuting with $J_n$ defines a non-trivial element of $T[J_n] \mathfrak{M}(\mathcal{AC}_\omega(M))$. Moreover any constant endomorphisms $L_1, L_2$ of such type give rise to different elements of $T[J_n] \mathfrak{M}(\mathcal{AT}_\omega(M))$. Hence $J_n$ is not rigid.

REFERENCES

[1] Apostolov V., Draghici T.: The curvature and the integrability of almost-Kähler manifolds: a survey, Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001), 25-53, Fields Inst. Commun., 35, Amer. Math. Soc., Providence, RI, 2003.

[2] Holomorphic curves in symplectic geometry. Edited by Michèle Audin and Jacques Lafontaine. Progress in Mathematics, 117. Birkhäuser Verlag, Basel, 1994. xii+328 pp.

[3] Chiossi S., Salamon S.: The intrinsic torsion of SU(3) and $G_2$ structures. Differential geometry, Valencia, 2001, 115–133, World Sci. Publishing, River Edge, NJ, 2002.

[4] de Bartolomeis P.: Geometric Structures on Moduli Spaces of Special Lagrangian Submanifolds, Ann. di Mat. Pura ed Applicata, IV, Vol. CLXXIX, (2001), pp. 307–342.

[5] de Bartolomeis P., Tomassini A.: On the Maslov Index of Lagrangian Submanifolds of Generalized Calabi-Yau Manifolds, to appear in Int. J. of Math.

[6] Cordero L. A., Fernández M., Gray A. Symplectic manifolds with no Kähler structure, Topology 25 (1986), no. 3, pp. 375–380.

[7] Fino A., Grantcharov G.: Properties of manifolds with skew-symmetric torsion and special holonomy, Adv. Math. 189 (2004), no. 2, pp. 439–450.

[8] Gauduchon P.: Hermitian connections and Dirac operators. Bull. Un. Mat. Ital. B (7) 11 (1997), no. 2, suppl., pp. 257–288.

[9] Gromov M.: Pseudoholomorphic curves in symplectic manifolds. Invent. Math. 82 (1985), no. 2, pp. 307–347.

[10] Gross M., Huybrechts D., Joyce D.: Calabi-Yau manifolds and related geometries, Lectures from the Summer School held in Nordfjordeid, June 2001. Universitext. Springer-Verlag, Berlin, 2003. viii+239 pp.

[11] Hitchin N.J.: Stable forms and special metrics, Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), 70–89, Contemp. Math., 288, Amer. Math. Soc., Providence, RI, 2001.
[12] Hitchin N.: Generalized Calabi-Yau manifolds, *Q. J. Math.* 54 (2003), no. 3, pp. 281–308.

[13] Hopf E.: Elementare Bemerkungen ueber die Losung parzieller Differentialgleichungen zweiter Ordnung von elliptischen Typus, *Sitzungber. Preuss. Akad. Wiss. phys. math. Kl.* 19 (1927), pp. 147–152.

[14] Joyce, Dominic D.: *Compact manifolds with special holonomy*, Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. xii+436 pp.

[15] Kodaira K.: *Complex manifolds and deformation of complex structures*, translated from the 1981 Japanese original by Kazuo Akao. Reprint of the 1986 English edition. Classics in Mathematics. Springer-Verlag, Berlin, 2005. x+465 pp.

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