Accelerated Stochastic ADMM with Variance Reduction

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Abstract

Alternating Direction Method of Multipliers (ADMM) is a popular method in solving Machine Learning problems. Stochastic ADMM was firstly proposed in order to reduce the per iteration computational complexity, which is more suitable for big data problems. Recently, variance reduction techniques have been integrated with stochastic ADMM in order to get a fast convergence rate, such as SAG-ADMM and SVRG-ADMM, but the convergence is still suboptimal w.r.t the smoothness constant. In this paper, we propose a new accelerated stochastic ADMM algorithm with variance reduction, which enjoys a faster convergence rate than all the other stochastic ADMM algorithms. We theoretically analyze its convergence rate and show its dependence on the smoothness constant is optimal. We also empirically validate its effectiveness and show its priority over other stochastic ADMM algorithms.

Introduction

The Alternating Direction Method of Multipliers (ADMM), firstly proposed by [Gabay and Mercier 1976, Glowinski and Marroco 1975], is an efficient and versatile tool, which can always guarantee good performance when handling large-scale data-distributed or big-data related problems, due to its ability of dealing with the objective functions separately and synchronously. Recent studies have shown that ADMM has a convergence rate of \(O(1/N)\) (where \(N\) is the number of iterations) for general convex problems [Monteiro and Svaiter 2013, He and Yuan 2012b, He and Yuan 2012a].

Therefore, ADMM has been widely applied to solve machine learning problems in real world, such as lasso, SVM [Boyd et al. 2011], group lasso [Meier, Van De Geer, and Bühlmann 2008], graph guided SVM [Ouyang et al. 2013], and top ranking [Kadkhodaie et al. 2015]. All these can be easily casted into an Empirical Risk Minimization (ERM) framework. That is

\[
\min_{x,y} \frac{1}{n} \sum_{i=1}^{n} f_i(x) + g(y), \quad \text{s.t.} \quad Ax + By = z, \tag{1}
\]

where \(\frac{1}{n} \sum_{i=1}^{n} f_i(x)\) is a loss function, each \(f_i(x) : \mathbb{R}^d \to \mathbb{R}\) is a convex component, and \(g(y) : \mathbb{R}^m \to \mathbb{R}\) is a convex regularizer. For example, given data samples \(\{(w_i,b_i)\}_{i=1}^{n}\) where \(w_i \in \mathbb{R}^d\) and \(b_i \in \mathbb{R}\), the group lasso problem can be reformulated as

\[
\min_{x,y} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} (w_i^T x - b_i)^2 + \nu \| y \|_1, \quad \text{s.t.} \quad Gx = y,
\]

where \(G\) is the matrix encoding the group information. However, in the ERM problem, when the size of the training dataset is large, the minimization procedure in each iteration can be computationally expensive, since it needs to access the whole training set. Many researchers resorted to the increment learning techniques to address such issue. Stochastic ADMM algorithm were proposed [Wang and Banerjee 2012, Ouyang et al. 2013, Suzuki 2013], though they only have a suboptimal convergence rate \(O(1/\sqrt{N})\). Recently, variance reduction techniques have been integrated with stochastic ADMM in order to get a faster convergence rate, such as SAG-ADMM [Zhong and Kwok 2014] and SVRG-ADMM [Zheng and Kwok 2016]. They all achieve a \(O(1/N)\) convergence rate when \(f(x)\) is both convex and smooth, and \(g(x)\) is convex.

At the same time, a few researchers tried to accelerate the traditional ADMM method by adding a momentum to the original solvers [Goldfarb, Ma, and Scheinberg 2013, Kadkhodaie et al. 2015]. An important work is the AL-ADMM algorithm proposed by [Ouyang et al. 2015], which improves the rate of convergence from \(O(1/N)\) to \(O(1/N^2)\) in terms of its dependence on the smoothness constant \(L_f\) of \(f(x)\).

Inspired by these two lines of research, we propose an accelerated stochastic ADMM algorithm called Accelerated SVRG-ADMM (ASVRG-ADMM), which incorporates both the variance reduction and the acceleration technique together. By utilizing a set of auxiliary
points, we speed up the original SVRG-ADMM algorithm without introducing much extra computation. We theoretically analyze the convergence rate of ASVRG-ADMM and show it is $O(1/N^2)$ for the smoothness constant $\parallel h \parallel$. Experimental results also show the proposed algorithm outperforms other stochastic ADMM algorithms in the big data settings.

Notation and Preliminaries

For a vector $x$, $\parallel x \parallel$ denotes the $l_2$-norm of $x$, and $\parallel x \parallel_1$ denotes the $l_1$-norm. For a matrix $X$, $\parallel X \parallel_2$ denotes its spectral norm. For a random variable $a$, we use $\mathbb{E}(a)$ to denote its expectation and $\mathbb{V}(a) = \mathbb{E}(\parallel a \parallel^2) - \mathbb{E}(\parallel a \parallel)^2$ to denote its variance. For a function $h(x)$, we denote its gradient and subgradient as $\nabla h$ and $\partial h(x)$ respectively.

A function $h(x)$ is $L$-smooth if it is differentiable and $\parallel h'(x) - h'(y) \parallel \leq L \parallel x - y \parallel$, or equivalently,

$$h(x) \leq h(y) + \langle \nabla h(y), x - y \rangle + \frac{L}{2} \parallel x - y \parallel^2.$$  

We call $L$ the smoothness constant.

For the convenience of notation, we denote $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$, $u = (x, y)$, and $w = (x, y, \lambda)$. We assume that the effective domain $\mathcal{X}$ of $x$ and $\mathcal{Y}$ of $y$ are bounded, that is

$$D_{x^*} = \sup_{x \in \mathcal{X}} \parallel x - x^* \parallel,$$

$$D_{y^*,B} = \sup_{y \in \mathcal{Y}} \parallel B(y - y^*) \parallel,$$  

$$D_{X,A} = \sup_{x_a, x_b \in \mathcal{X}} \parallel A(x_a - x_b) \parallel$$

exists and not equal to infinity, where $A$ and $B$ are matrices in the constraint of (1). We also assume that the optimal solution $u^* = (x^*, y^*)$ of problem (1) and the optimal of each $f_i$ exists.

Similar to those in [Ouyang et al. 2015], we denote the gap function as follows.

Definition (Gap Function). For any $w = (x, y, \lambda)$ and $\bar{w} = (\bar{x}, \bar{y}, \bar{\lambda})$, we define

$$Q(\bar{x}, \bar{y}, \bar{\lambda}; x, y, \lambda) = [f(x) + g(y) + \langle \bar{\lambda}, Ax + By - c \rangle - [f(\bar{x}) + g(\bar{y}) + \langle \bar{\lambda}, A\bar{x} + B\bar{y} - c \rangle].$$

For the simplicity of notation, we also write $Q(\bar{w}; w) = Q(\bar{x}, \bar{y}, \bar{\lambda}; x, y, \lambda)$.

Related Work

Here we focus on the constraint optimization problem

$$\min_{x,y} f(x) + g(y), \text{ s.t. } Ax + By = z,$$  

where $f(x)$ is both convex and $L_f$-smooth, and $g(y)$ is convex. Besides, we also assume that each $f_i$ is $L_i$-smooth.

To solve problem (2), ADMM starts with the augmented Lagrangian of the original problem:

$$L_\beta(x, y, \lambda) := f(x) + g(y) - \langle \lambda, Ax + By - z \rangle + \frac{\beta}{2} \parallel Ax + By - z \parallel^2,$$

where $\beta > 0$ is a constant, and $\lambda$ is the vector of Lagrangian multipliers. In each round, ADMM minimizes $L_\beta$ with respect to the variables $x$ and $y$ alternatively given the other fixed, followed by an update of the vector $\lambda$. Specifically, it updates $x, y$, and $\lambda$ as follows.

$$x_t = \arg\min_{x} L_\beta(x, y_{t-1}, \lambda_{t-1}),$$

$$y_t = \arg\min_{y} L_\beta(x_t, y, \lambda_{t-1}),$$

$$\lambda_t = \lambda_{t-1} - \beta(Ax_t + By_t - c).$$

In order to avoid the access to the whole dataset in the $x$ update and further reduce the per-iteration computation complexity, stochastic ADMM algorithm(SADMM) was proposed by [Ouyang et al. 2013]. However, their method can only achieve a suboptimal convergence rate $O(\frac{\log(1/\epsilon)}{\sqrt{N}} + \frac{1}{N})$, where $c$ is a constant that depends on $D_{x^*}, D_{y^*,B}$ and $D_{X,A}$, and $\sigma$ is the upper bound of the variance of stochastic gradients. Typically, $\sigma$ and $L_f D_s^2$ can be rather large and $c$ is small compared to them. Based on [Ouyang et al. 2013], [Azadi and Sra 2014] proposed an accelerated stochastic ADMM algorithm(ASADMM) to reduce the dependence of $L_f$, which has a $O(\frac{\log(1/\epsilon)}{N^{3/4}} + \frac{c}{\sqrt{N}} + \frac{D_{Y,A}}{\sqrt{N}})$ convergence rate.

The $O(\frac{L_f D_s^2}{N})$ is optimal w.r.t. the smoothness constant $L_f$, but due to the high variance of stochastic gradients, their method is far away from optimal.

Borrowed idea from variance reduction techniques used in stochastic gradient descent literature, SAG-ADMM [Zhong and Kwok 2014] and SVRG-ADMM [Zheng and Kwok 2016] are recently proposed. These methods enjoy similar convergence rate $O(\frac{c_1 D_s^2}{N} + \frac{\epsilon}{N} + \frac{D_{Y,A}}{\sqrt{N}})$, where $c_1$ is a constant which depends on $D_{x^*}, D_{y^*,B}$, $D_{X,A}$, and $\parallel B \parallel_2$, and can be rather small compared to $L_f D_s^2$. (different algorithms may have different $c_1$).

Recently, acceleration technique has been integrated with variance reduction in stochastic gradient descent literature [Hien et al. 2016] [Allen-Zhu 2016], which improves the convergence rate from $O(1/N)$ to $O(1/N^2)$, but it is still not clear how to combine this two techniques in stochastic ADMM literature to get a more effective method.

ASVRG-ADMM Algorithm

In this section, we first propose an general ASVRG-ADMM framework for solving (3), and then analyze relation between the convergence rate and the parameter settings. Finally, we conclude our Accelerated SVRG-ADMM algorithm by specifying the particular parameter setting. The proposed framework is presented in Algorithm 1.

\footnote{The overall convergence is $O(\frac{aL_f}{N} + \frac{c_1}{\sqrt{N}})$, where $aL_f$ will dominate $b$ in most ERM problems, since big dataset means big smoothness constant.}
Algorithm 1: ASVRG-ADMM Framework

1. Initialization: $\tilde{x}_0 = x_{m,0}, \tilde{y}_0 = y_{m,0}, \tilde{\lambda}_0 = \lambda_{m,0}$ such that $A\tilde{x}_0 + B\tilde{y}_0 - c = 0$
2. For $s = 1, 2, \ldots, N$ do
   3. Update $\alpha_{ts}, \alpha_{ts} g, \alpha_{ts}, \theta_s$ and $\rho_s$;
   4. $x_{0,s} = x_{m,s-1}, y_{0,s} = y_{m,s-1}, \lambda_0 = \lambda_{m,s-1}$;
   5. $x_{1,s} = \arg\min_{x_{1,s}} f(x_{1,s}) + \langle A x_{1,s} - B y_{1,s} - c \rangle$;
   6. $\tilde{v}_s = \frac{1}{m} \sum_{i=1}^{m} (f_i(x_{1,s}) - f_i(x_{0,s}));$
   7. For $t = 1, 2, \ldots, m$ do
      8. $x_{t,s} = \alpha_{ts} x_{t-1,s} + \alpha_{ts} x_{t-1,s} + \alpha_{ts} x_{t-1,s};$
      9. Sample $i_t$ uniformly from $\{1, 2, \ldots, n\}$;
      10. $f_{i,s} = \nabla f_{i,s}(x_{t,s})$;
      11. $x_{t,s} = \arg\min_{x_{t,s}} L_{i,s}(x, y_{t,s}) + \langle \lambda_{t,s}, x_{t,s} \rangle$;
      12. $f_{i,s} = \alpha_{i,k} x_{t,s} - \alpha_{i,k} x_{t,s} + \alpha_{i,k} x_{t,s} - \alpha_{i,k} x_{t,s};$
      13. $v_{i,s} = \arg\min_{v_{i,s}} g(y_{i,s}) - \langle \lambda_{i,s}, A x_{t,s} + B y_{i,s} - c \rangle;$
      14. $y_{t,s} = \alpha_{i,s} y_{t,s} \alpha_{i,s} y_{t,s} + \alpha_{i,s} y_{t,s} + \alpha_{i,s} y_{t,s};$
      15. $\lambda_{t,s} = \lambda_{t-1,s} + \rho_s \alpha_{i,k} x_{t,s} + B y_{i,s} - c;$
      16. $x_{t,s} = \alpha_{i,k} \lambda_{t,s} - \alpha_{i,k} \lambda_{t,s} + \alpha_{i,k} \lambda_{t,s} - \alpha_{i,k} \lambda_{t,s};$
   end
   17. $\tilde{x}_s = \frac{1}{m} \sum_{t=1}^{m} x_{t,s};$
   18. $\tilde{y}_S = \frac{1}{m} \sum_{t=1}^{m} y_{t,s};$
   19. $\tilde{\lambda}_s = \frac{1}{m} \sum_{t=1}^{m} \lambda_{t,s};$
20. end

Output: $\hat{w}_N = \frac{1}{1 + \alpha_{i,k, N+1} m} w_{m,N} + \frac{\alpha_{i,k, N+1} m}{1 + \alpha_{i,k, N+1} m} \tilde{w}_N$

In this framework, the updates of $x$ and $y$ are actually alternatingly minimizing the weighted linearized augmented Lagrangian:

$$L_{i,s}(x, y, \lambda_{t,s}, \chi) = f(x_{t,s}) + \langle \chi_{t,s}, x_{t,s} \rangle + \langle \lambda_{t,s}, A x_{t,s} + B y_{t,s} - c \rangle + \frac{\eta_s}{2} \|A x_{t,s} + B y_{t,s} - c\|^2$$

where $f(x)$ in the original augmented Lagrangian $L(x, y, \lambda)$ is linearized as $f(x_{t,s}) + \langle \chi_{t,s}, x_{t,s} \rangle + \frac{\eta_s}{2} \|x_{t,s} - x_{t-1,s}\|^2$ and a weight parameter $\theta_s$ is added to the constraint measure $\|A x_{t,s} + B y_{t,s} - c\|^2$. However, we must notice that $\nabla f(x_{t,s})$ is an unbiased estimation of $\nabla f(x_{t,s})$ instead of $\nabla f(x_{t,s})$. Here, $\chi$ is an indicator variable that is either 0 or 1. If $\chi = 0$, augmented term $\|A x + B y_{t,s} - c\|^2$ is preserved, and the update in line (10) of Algorithm is

$$x_{t,s} = (\eta_s I + \theta_s A^T A)^{-1} (\eta_s x_{t,s} - v_{t,s} - A^T \chi_{t,s} - \theta_s A^T (B y_{t,s} - c))$$

Since every update in $\chi$ involves a matrix inverse, which can be quite complicated when $A$ is large, we can set $\chi = 1$ to linearize this part to simplify the computation. When $\chi = 1$, the augmented term $\|A x + B y - c\|^2$ in $L_{i,s}(x, y, \lambda_{t,s})$ is linearized as $\langle A x_{t,s} + B y_{t,s} - c, A x_{t,s} \rangle$ the $t$-th iteration, and the update in line (10) of Algorithm is simplified as

$$x_{t,s} = x_{t-1,s} - \frac{1}{\eta_s} (A^T \chi_{t,s} + v_{t,s} + \theta_s A^T (A x_{t-1,s} + B y_{t-1,s} - c)).$$

Similar to other accelerated methods, we construct a set of auxiliary sequences $\{x^{ag}\}$, $\{y^{ag}\}$ and $\{\lambda^{ag}\}$. Here the superscript "ag" stands for "aggregate" and "md" stands for "middle". We can see that all $\{x^{ag}\}$ and $\{x^{ag}\}$ are actually weighted sums of all the previous $\{x_{i,k}\}$ and all $\{y^{ag}\}$ and $\{\lambda^{ag}\}$ are weighted sums of previous $\{y_{i,k}\}$ and $\{\lambda_{i,k}\}$ respectively. We require all the weight parameters $\alpha_{1,s}, \alpha_{2,s}, \alpha_{3,s}$ belong to (0, 1) and their sums equal 1 to make all the auxiliary points to be a convex combination of three previous point. If the weights $\alpha_{3,s} = 1$ for all $s$, then $x^{ag} = x_{t,s}$, and the aggregate points $x^{ag} = x_{t,s} + y^{ag} = y_{t,s}$ and $\lambda^{ag} = \lambda_{t,s}$. In this situation, if we set all $\theta_s = \theta_s = \beta$, the ASVRG-ADMM becomes SVRG-ADMM. However, with carefully chosen parameters we can significantly improve the rate of convergence.

Convergence Analysis

In this subsection, we will give the convergence analysis of the proposed framework, and conclude our ASVRG-ADMM algorithm.

Before we address the main theorem on the convergence rate, we will firstly try to bound the variance of the stochastic gradient. In order to control the variance of the stochastic gradient, we first take a snapshot $\tilde{x}_{s-1}$ at the beginning of each outer iteration and calculate its full gradient $\tilde{v}_s$, and then randomly sample one $f_{i_k}(x)$ and update $v_{i_k} = \nabla f_{i_k}(x_{t,s}) - \nabla f_{i_k}(\tilde{x}_{s-1}) + \tilde{v}_s$ in each inner loop. Similar to [Allen-Zhu et al. 2018] and [Hien et al. 2019], we can use a different upper bound from the analysis of SVRG-ADMM.

Lemma 1. In each of the inner iteration, the variance of $v_{i_k}$ is bounded by

$$E\|v_{i_k} - \nabla f(x_{t,s})\|^2 \leq 2LQ(f(\tilde{x}_{s-1}) - f(x_{t,s}))$$

where $LQ = \max_i \{L_i\}$.

Proof.

$$E\|v_{i_k} - \nabla f(x_{t,s})\|^2 = E\|\nabla f_i(x_{t,s}) - \nabla f_i(\tilde{x}_{s-1}) + \nabla f(\tilde{x}_{s-1}) - \nabla f(x_{t,s})\|^2$$

$$\leq E\|\nabla f_i(x_{t,s}) - \nabla f_i(\tilde{x}_{s-1})\|^2$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} 2LQ(f_i(\tilde{x}_{s-1}) - f_i(x_{t,s} - x_{t,s} - x_{t,s}))$$

$$\leq 2LQ(f(\tilde{x}_{s-1}) - f(x_{t,s}) - \nabla f(x_{t,s}) - \nabla f(x_{t,s} - x_{t,s})).$$
The first equality is according to the definition of \( v_{t,s} \).
In the first inequality, we use \( \mathbb{E}[a - \mathbb{E}(a)]^2 \leq \mathbb{E}[a]^2 \).
The second inequality is because of the smoothness of each \( f_i(x) \), and the last one is just due to the definition of \( L_Q \).

Utilizing the above lemma, we are able to obtain an upper bound of the progress of the gap function \( Q \) in each inner loop.

**Lemma 2.** In Algorithm 1, if we choose \( \theta_s \geq \rho_s \) and \( \eta_s \geq L_s \alpha_{2,s} + \chi \theta_s \| A \|_2^2 \), we have

\[
\begin{align*}
\mathbb{E}(x_t^*, y^*, \lambda; w_{t-1,s}) - & \alpha_{3,s} Q(x^*, y^*, \lambda; w_{t-1,s}) \\
\leq & \frac{\eta_s}{2} \| x_{t,s} - x^* \|_2^2 - \| x_{t,s} - x^* \|_2^2 \\
& + \frac{1}{2 \rho_s} (\| \lambda_{t-1,s} - \lambda \|_2^2 - \| \lambda_{t,s} - \lambda \|_2^2) \\
& + \frac{\chi \theta_s}{2} \| A (x_{t-1,s} - x^*) \|_2^2 \\
& + \frac{\rho_s}{2} \| A x_s + B y_{t,s} - c \|_2^2 \\
& - \frac{\rho_s}{2} \| A x_s + B y_{t,s} - c \|_2^2,
\end{align*}
\]

where \( L_s \geq L_Q/\alpha_{3,s} + L_f \) and the expectation is w.r.t. \( v_{t,s} \).

The proof of this inequality is lengthy and tedious, so we leave it in the supplement. We can see that this bound is not only related to \( x_{t,s} \) and \( x_{t-1,s} \), but also related to \( \tilde{x}_{t-1,s} \). It is very common in snapshot based algorithms. Lemma 2 proves crucial for doing the induction to obtain the next step towards our final conclusion.

**Lemma 3.** If in the \((s+1)\)-th outer iteration, we choose

\[
\frac{1 - \alpha_{2,s+1}}{\alpha_{2,s}^2} = \frac{1}{\alpha_{2,s}^2} \quad \text{and} \quad \frac{\alpha_{3,s+1}}{\alpha_{2,s}^2} = \frac{1 - \alpha_{2,s+1}}{\alpha_{2,s}^2}, \quad \text{and make} \quad \theta_s = \beta_1 \alpha_{2,s}, \rho_s = \frac{\beta_3}{\alpha_{2,s}}, \eta_s = (\bar{L} + \chi \beta_1 \| A \|_2^2) \alpha_{2,s}.
\]

Then in the \( N \)-th iteration, we have

\[
\mathbb{E}[f(\hat{x}_N) + g(\tilde{y}_N) - f(x^*) - g(y^*) + \gamma \| A \hat{x}_N + B \tilde{y}_N - c \|_2^2] \\
\leq \frac{\alpha_{2,N+1}^2}{1 + m \alpha_{3,N+1}^2} \left( f(\hat{x}_0) + g(\tilde{y}_0) - f(x^*) - g(y^*) \right) \\
+ \bar{L} + \chi \beta_1 \| A \|_2^2 \frac{1}{2} D_{2,s}^2 + \frac{1}{2 \rho_s} \gamma^2 + \chi \beta_1 \frac{1}{2} D_{A,X}^2 + \frac{\beta_3}{2} D_{y^*,b}^2,
\]

where we need choose \( \beta_1 \) and \( \beta_2 \) to satisfy \( \theta_s \geq \rho_s \), and \( L \geq \max_s \{ L_Q/\alpha_{3,s} + L_f \} \).

Before we conclude our main theorem of the convergence rate of ASVRG-ADMM, we need to have a look at all the parameter constraints in Lemma 3.

\[
\begin{align*}
\alpha_{2,s+1} = \frac{1 - \alpha_{2,s+1}}{\alpha_{2,s+1}}, & \quad \alpha_{3,s+1} = \frac{1 - \alpha_{2,s+1}}{\alpha_{2,s+1}}, \\
\alpha_{1,s} + \alpha_{2,s} + \alpha_{3,s} = & \frac{\alpha_{2,s+1}}{\alpha_{2,s}}, \\
\theta_{s} = & \frac{\beta_1}{\alpha_{2,s}}, \beta_{t,s} = \frac{\beta_3}{\alpha_{2,s}}, \theta_s \geq \rho_s, \\
\eta_{t,s} = & L \alpha_{2,s} + \chi \| A \|_2^2 \theta_s, \quad \bar{L} \geq \max_s \{ L_Q/\alpha_{3,s} + L_f \}.
\end{align*}
\]

To satisfy the constraints on weight parameters \( \alpha_{1,s}, \alpha_{2,s}, \alpha_{3,s} \), we can actually calculate the update rule of each

\[
\alpha_{1,s+1} = \alpha_{1,s} \left( 1 - \alpha_{2,s+1} \right), \\
\alpha_{2,s+1} = \frac{\sqrt{\alpha_{2,s}^2 + 4 \alpha_{3,s}^2} - \alpha_{2,s}}{2}, \\
\alpha_{3,s+1} = (1 - \alpha_{1,s}) \left( 1 - \alpha_{2,s+1} \right).
\]

The following lemma depicts the property of this sequence.

**Lemma 4.** If \( \alpha_{1,s} \in (0, 1), \alpha_{2,s} \in \left(0, \frac{2}{\bar{L} \gamma + 1} \right), \alpha_{3,s} \in (0, 1), \) and their sum equals to 1, then \( \alpha_{1,s+1} \in (0, \alpha_{1,s}), \alpha_{2,s+1} \in (0, \min\{\alpha_{2,s}, \frac{2}{\bar{L} \gamma + 1}\}) \), \( \alpha_{3,s+1} \in (\alpha_{3,s}, 1) \) and their sum equals to 1, too.

Proof. Denote \( h(a) = \sqrt{a^2 + 4 \alpha_{3,s}^2} - a \) as a function of \( a \), we have \( \nabla h(a) > 0 \) for \( a \in (0, 1) \). Then \( \alpha_{1,s+1} \leq h(\frac{2}{\bar{L} \gamma + 1}) \).

Since \( h(\frac{2}{\bar{L} \gamma + 1}) < \frac{2}{\bar{L} \gamma + 1} \), then \( \alpha_{1,s+1} < \frac{2}{\bar{L} \gamma + 1} \). Besides, it is easy to verify \( \alpha_{2,s+1} - \alpha_{2,s} \leq 0 \) and \( 0 < \alpha_{2,s+1} < 1 \). then we have \( \alpha_{2,s+1} \in (0, \min\{\alpha_{2,s}, \frac{2}{\bar{L} \gamma + 1}\}) \). Since \( \alpha_{1,s+1} = \alpha_{1,s} \left( 1 - \alpha_{2,s+1} \right) \), then \( \alpha_{1,s+1} \in (0, \alpha_{1,s}) \).

As \( \alpha_{3,s+1} - \alpha_{3,s} = \alpha_{1,s} + \alpha_{2,s} - \alpha_{1,s+1} - \alpha_{2,s+1} \), then \( \alpha_{3,s+1} \in (\alpha_{3,s}, 1) \).

This means \( \{\alpha_{1,s}\} \) and \( \{\alpha_{2,s}\} \) are decreasing and \( \{\alpha_{3,s}\} \) is increasing. Besides, we also have all \( \alpha_{2,s} \leq \frac{2}{\bar{L} \gamma + 1} \) if \( \alpha_{1,s} \leq \frac{2}{\bar{L} \gamma + 1} \). Actually, we can verify that if \( \alpha_{2,s} = 2/3 \), then \( \alpha_{2,s} \rightarrow \frac{2}{\bar{L} \gamma + 1} \) as \( s \rightarrow \infty \), and \( \beta_1 = N, \beta_2 = \frac{1}{N} \) will ensure \( \theta_s \geq \rho_s \). Since \( \alpha_{3,s} \) is increasing, \( \bar{L} \geq L_Q/\alpha_{3,s} + L_f \) will satisfy the constraint. Based on all these, we are now ready to present our main theorem of convergence rate.

**Theorem 1.** If we initialize \( \alpha_{2,1} = \frac{2}{3}, \alpha_{3,1} \in (0, \frac{1}{N}) \), \( \alpha_{1,1} = 1 - \alpha_{2,1} - \alpha_{3,1} \), then with proper parameter setting the same as in lemma 4 and \( \beta_1 = N, \beta_2 = \frac{1}{N}, \bar{L} = L_Q/\alpha_{3,1} + L_f \), we have

\[
\mathbb{E}[f(\hat{x}_N) + g(\tilde{y}_N) - f(x^*) - g(y^*) + \gamma \| A \hat{x}_N + B \tilde{y}_N - c \|_2^2] \\
= \mathcal{O}\left( \frac{1}{N^2} \left( f(\hat{x}_0) + g(\tilde{y}_0) - f(x^*) - g(y^*) \right) + \frac{L_Q + L_f D_{2,s}}{m N^2} \right) \\
+ \frac{1}{m N^2} \left( \chi \| A \|_2^2 \frac{1}{2} D_{2,s}^2 + \gamma^2 + \chi D_{A,X}^2 + D_{y^*,b}^2 \right).
\]

Proof. Since \( \alpha_{2,N+1} \leq \frac{2}{\bar{L} \gamma + 1} \) and \( \frac{1}{1 + m \alpha_{3,N+1}} \leq \frac{1}{1 + m \alpha_{3,1}} \),
then according to \ref{thm:convergence}, we have
\[
\begin{align*}
& f(\hat{x}_N) + g(\hat{y}_N) - f(x^*) - g(y^*) + \gamma \| Ax + B y - c \|^2 \\
& \leq \frac{4}{\alpha^2_{s,1}(N+3)^2} f(\hat{x}_0) + g(\hat{y}_0) - f(x^*) - g(y^*) \\
& + \frac{2LQ}{(N+3)^2} \left[ \| A \|^2 D_x^2 + \gamma^2 + \chi D_{A,X}^2 + D_{y^*,B}^2 \right] \\
& + \frac{2(LQ + Lf)}{(N+3)^2(1 + m\alpha_{s,1,3})} D_x^2 \\
& = O\left( \frac{1}{N^2} (f(\hat{x}_0) + g(\hat{y}_0) - f(x^*) - g(y^*)) + \frac{LQ + Lf}{mN^2} D_x^2 \\
& + \frac{1}{mN} \| A \|^2 D_x^2 + \gamma^2 + \chi D_{A,X}^2 + D_{y^*,B}^2 \right)
\end{align*}
\]

The convergence result in \textbf{Theorem 1} mainly consists of the convergence of three parts:

- \( f(\hat{x}_0) + g(\hat{y}_0) - f(x^*) - g(y^*) \): This part measures the influence of the initial objective value on the convergence rate. Actually in most ERM problems such as (group/fused) lasso, \( l_1/l_2 \) penalized regression/logistic regression, \( f(\hat{x}_0) + g(\hat{y}_0) - f(x^*) - g(y^*) \) can be quite small if we simply initialized \( \hat{x}_0 \) to 0 and \( \hat{y}_0 \) to 0. For example, in two label classification, for any commonly used regularizer \( g(y)(l_1/l_2 \) norm/group/fused lasso penalty), if \( f(x) \) is the logistic loss then \( f(0) + g(0) = I_n(2) \); if \( f(x) \) is the square loss then \( f(0) + g(0) = \frac{x}{2} \). Since \( f(x_0) + g(y_0) \) here must be greater than 0, then this part can be really small compared to the other two parts.

- \( (LQ + Lf)D_x^2 \): This is the dominant part of the convergence since \( LQ \) and \( Lf \) can be rather large for ERM problems with a big dataset, and we assume \( LQ = O(Lf) \) without lost of generality.

- \( \chi \| A \|^2 D_x^2 + \gamma^2 + \chi D_{A,X}^2 + D_{y^*,B}^2 \): Here \( D_x \), \( D_{A,X} \), \( D_{y^*,B} \) are constant denoting the boundary of the effective domain of \( x \) and \( y \). They are always assumed to be finite in the analysis of stochastic ADMM algorithms and are typically small compared to \( (LQ + Lf)D_x^2 \). It also shows the linearization of \( \| Ax + By - c \|^2 (\chi = 1) \) will not effect the convergence significantly.

Thus if we make the count of iter loop as \( m = n \), then the convergence rate of ASVRG-ADMM can be written as \( O\left( \frac{LQ}{mN^2} D_x^2 + \frac{\gamma}{mN} + \frac{\chi}{mN} \right) \), where \( c_2 \) and \( c_3 \) are rather small compared to \( LfD_x^2 \). If we use similar notation, the convergence rate of SAG-ADMM is \( O\left( \frac{LQ}{mN} D_x^2 + \frac{\gamma}{N} + \frac{\chi}{N} \right) \), where \( c_4, c_5, c_6 \) are all small compared to \( LfD_x^2 \) in most of the ERM problems. Typically, \( c_3, c_5 \) are of the same order, \( c_2, c_4, c_6 \) are of the same order and \( c_3, c_5 \) are much smaller than \( c_2, c_4, c_6 \). It is clear from the above analysis that our proposed accelerated stochastic ADMM algorithm(ASVRG-ADMM) are much better than other stochastic variance reduction ADMM without acceleration in dealing with ERM problem. Moreover, when the dataset is large and \( f(x) \) has a large smoothness constant, the second part will dominate the convergence and the convergence rate will be optimal \( O(1/N^2) \) w.r.t. the smoothness constant, while other methods only achieve \( O(1/N) \) convergence. As for the vanilla stochastic ADMM algorithms without variance reduction SADMM\cite{Ouyang2013} and ASADMM \cite{Azadi2014}, both of their analyses include the variance of the stochastic gradient \( \sigma \) explicitly, and the convergence rate is \( O\left( \frac{\sqrt{D_x}}{\sigma \sqrt{N}} \right) \) w.r.t. \( \sigma \), which is uncontrollable and can be extremely bad in practice.

**Discussion on the parameters**

Let’s have a deeper insight into the parameters in the ASVRG-ADMM algorithm. The weight parameters \( \alpha_1, \alpha_2, \alpha_3 \) are updated once in each outer iteration, and stay the same in the inner loop. In each inner loop, the update of \( x_{t,s}^{md} \) and \( x_{t,s}^{ag} \) are:

\[
\begin{align*}
x_{t,s}^{md} &= \alpha_1 x_{t-1,s}^{ag} + \alpha_2 x_{t-1,s} + \alpha_3 \hat{x}_{s-1} \\
x_{t,s}^{ag} &= \alpha_1 x_{t-1,s}^{ag} + \alpha_2 x_{t-1,s} + \alpha_3 \hat{x}_{s-1}.
\end{align*}
\]

It seems that both \( x_{t,s}^{md} \) and \( x_{t,s}^{ag} \) are somehow trapped at \( \hat{x}_{s-1} \), and we even gradually increase the weight on \( \hat{x}_{s-1} \). However, it is not incomprehensible since we are actually analyze the convergence of sequences \{\( \hat{x}_s \)\} (since \( \alpha_{3,N+1}m \gg 1 \), \( \hat{w}_s \approx \hat{w}_s \)). According to line 18 in Algorithm \ref{alg:asvg_admm},

\[
\hat{x}_s = \frac{\alpha_1}{m} \sum_{i=0}^{m-1} x_{i,s}^{ag} + \frac{\alpha_2}{m} \sum_{i=1}^{m} x_{i,s} + \alpha_3 \hat{x}_{s-1},
\]

and it is updated after we use one full gradient and n stochastic gradient. While in the deterministic accelerated ADMM algorithm proposed by \cite{Ouyang2013}, the update of the auxiliary points are

\[
\begin{align*}
x_s^{md} &= \alpha_1 x_{s-1}^{ag} + \alpha_2 x_{s-1} + \alpha_3 \hat{x}_{s-1} \\
x_s^{ag} &= \alpha_1 x_{s-1}^{ag} + \alpha_2 x_{s-1} + \alpha_3 \hat{x}_{s-1}.
\end{align*}
\]

Here \( x_s \) is updated based on the full gradient, and \( \alpha_2, \alpha_3 \) is decreased similar to \( \alpha_2, \alpha_3 \) in our algorithm. Since \( x_s^{ag} \) is updated based on one full gradient and the convergence of their algorithm is w.r.t. \( x_s^{ag} \), \( x_s^{ag} \) here is similar to \( \hat{x}_s \) instead of \( x_s^{ag} \) in our ASVRG-ADMM algorithm. As the weight of previous \( x_{s-1}^{ag} \) is also gradually increased, it is not counterintuitive for us to gradually increase the weight of \( \hat{x}_{s-1} \). Actually, when \( m = 1 \), ASVRG-ADMM is actually the same as the deterministic accelerated ADMM method proposed by \cite{Ouyang2013}.

Besides, we have to mention that the parameter update rules we use here are different from all other accelerated stochastic methods with similar procedure \cite{Allen-Zhu2016, Hu2016}, where they fix the weight parameter \( \alpha_3 \) of \( \hat{x}_{s-1} \) to be a constant, and only change \( \alpha_1, \alpha_2 \).
Since \( \eta_s = \bar{L}\alpha_{2,s} + \chi\|A\|^2\theta_s \) and \( \theta_s = \beta_1\alpha_{2,s} \), the matrix inverse in x update rule (3) is 
\[
(\eta_s I + \theta_s A^T A)^{-1} = \alpha_{2,s}^{-1}(\bar{L}I + \beta_1 A^T A)^{-1}.
\]

Then we can pre-compute \((\bar{L}I + \beta_1 A^T A)^{-1}\) and store it in order to decrease the computation. According to the update rules of \( \hat{\eta}, \eta_s, \rho_s \), we gradually decrease \( \theta_s \) and \( \eta_s \), and increase \( \rho_s \). This means we increase the step size of the x-update and the \( \lambda \)-update.

### Experimental Results

In this section, we give the experimental results of the proposed algorithm. We consider the generalized lasso\cite{Tibshirani2011}. More specifically, we solve the following optimization problem:

\[
\min_{x,y} \frac{1}{n} \sum_{i=1}^{n} f_i(x) + \nu\|Fx\|_1,
\]

where the penalty matrix \( F \) gives information about the underlying sparsity pattern of \( x \). While proximal methods have been used to solve this problem\cite{Liu2010,Barbero2011}, the existence of \( F \) makes the underlying proximal step difficult to solve. This problem can be rewritten as

\[
\min_{x,y} \frac{1}{n} \sum_{i=1}^{n} f_i(x) + \nu\|y\|_1, \quad \text{s.t.} \quad Fx - y = 0.
\]

It can be solved efficiently by ADMM methods by dealing with \( x \) and \( y \) separately. In our experiments, we focus on the graph guided fused lasso\cite{Kim2009}, where \( F = [G; I] \) is constructed based on a graph and \( G \) denotes the sparsity pattern of graph. We use SVRG-ADMM\cite{Zheng2016} and SAG-ADMM\cite{Zhong2014} as our baseline. Since we can linearize \( \|Fx - y\|_2^2 \) in the augmented lagrangian in all these three methods, here in our implementation we use this technique to reduce the computation complexity.

We conduct experiments on four real-world datasets from web machine learning repositories. All the datasets were downloaded from the LIBSVM website, and their basic information is listed in Table 1. We repeat all the experiments for 10 times and report the average results. For the parameters in ASVRG-ADMM, we initialize \( \alpha_{2,1} = 2/3 \), \( \alpha_{3,1} = 1/10 \), and all other parameters are set as in Theorem 1. For parameters in SAG-ADMM and SVRG-ADMM, we all choose them as indicated in their papers, and SAG-ADMM is initialized by running SADMM for \( n \) iteration. All the methods are implemented in Matlab, and all the experiments are performed on a Windows server with two Intel Xeon E5-2690 CPU and 128GB memory.

If the algorithm uses one full gradient or \( n \) stochastic gradients, we call it uses one effective pass of data. We report the objective value over effective passes of data. We can see that in all the four datasets, our ASVRG-ADMM algorithm outperforms the other two algorithms. On the first three data set, the smoothness constant is small and all the three method converges after 20 effective passes of data, the ASVRG-ADMM method converges faster then the other two algorithms. When \( L_f \) is large (on the covtype dataset), all the three algorithms do not converge in 50 passes of data, but the objective of ASVRG-ADMM decreases much faster than the other two algorithms.

### Conclusion

In this paper, we combine the variance reduction technique and the acceleration technique in stochastic ADMM together, we devised a new stochastic method with faster convergence rate, especially in term of the smoothness constant of \( f(x) \). It is of great significance.
since big dataset often means a large smoothness constant. Our experimental results also validate the effectiveness of our algorithm.

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In the third inequality, we use the Cauchy-Schwartz inequality.

\[ f(x_{t,s}^a) \leq f(x_{t,s}^m) + \langle \nabla f(x_{t,s}^m), x_{t,s}^a - x_{t,s}^m \rangle + \frac{L_f}{2} \| x_{t,s}^a - x_{t,s}^m \|^2. \]

According to line 7 and line 11 in Algorithm \[\boxed{\text{Algorithm 1}}\] we have \[\| x_{t,s}^a - x_{t,s}^m \| = \alpha_2, (x_{t,s} - x_{t-1,s}).\] Substituting this into the above inequality, we have

\[ f(x_{t,s}^a) \leq f(x_{t,s}^m) + \langle \nabla f(x_{t,s}^m), x_{t,s}^a - x_{t,s}^m \rangle + \frac{L_f \alpha_3}{2} \| x_{t,s}^a - x_{t,s}^m \|^2. \]

By the optimality of Line 12 in Algorithm \[\boxed{\text{Algorithm 1}}\] we have

\[ v_{t,s} = \nabla f(x_{t,s}^m) - v_{t,s}, x_{t,s}^a - x_{t,s}^m \leq f(x_{t,s}^m) - f(x_{t,s}^a) \leq f(x_{t,s}^m) - f(x_{t,s}^a). \]

The second inequality is due to Lemma 1 and the convexity of \( f(x) \).

According to the optimality of Line 12 in Algorithm \[\boxed{\text{Algorithm 1}}\] and the convexity of \( g(y) \), we have

\[ g(y_{t,s}) - g(y) \leq \langle \nabla g(y_{t,s}), y_{t,s} - y \rangle = \langle \theta_s(A x_{t,s}^p + c) + \lambda_{t-1,s}, B(y - y_{t,s}) \rangle. \]
According to the definition of gap function, we have

\[
\mathbb{E}Q(w; w_{t-1}^{ag}) - \alpha_{1,s}Q(w; w_{t-1}^{ag}) - \alpha_{3,s}Q(w; \tilde{w}_{s-1}) = \begin{cases} 
[f(x_{t,s}^{ag}) + g(y_{t,s}^{ag}) + \langle \lambda, Ax_{t,s} + By_{t,s} - c \rangle - [f(x) + g(y) + \langle \lambda_{t,s}, Ax + By - c \rangle] \\
-\alpha_{1,s}([f(x_{t,s}^{ag}) + g(y_{t,s}^{ag}) + \langle \lambda, Ax_{t-1,s} + By_{t-1,s} - c \rangle - [f(x) + g(y) + \langle \lambda_{t-1,s}, Ax + By - c \rangle]) \\
-\alpha_{3,s}([f(\tilde{x}_{s-1}) + g(\tilde{y}_{s-1}) + \langle \lambda, A\tilde{x}_{s-1} + B\tilde{y}_{s-1} - c \rangle - [f(x) + g(y) + \langle \lambda_{s-1}, Ax + By - c \rangle]) \\
= [f(x_{t,s}^{ag}) - \alpha_{1,s}f(x_{t-1,s}^{ag}) - \alpha_{2,s}f(\tilde{x}_{s-1})] + [g(y_{t,s}^{ag}) - \alpha_{1,s}g(y_{t-1,s}^{ag}) - \alpha_{2,s}g(y)] \\
-\alpha_{3,s}(g(\tilde{y}_{s-1}) + \alpha_{2,s}(\lambda, Ax_{s,t} + By_{s,t} - c) - \alpha_{2,s}(\lambda_{s,t}, Ax + By - c) \\
\leq [f(x_{t,s}^{ag}) - \alpha_{1,s}f(x_{t-1,s}^{ag}) - \alpha_{2,s}f(\tilde{x}_{s-1}) + \alpha_{2,s}[g(y_{t,s}^{ag}) - g(y)] \\
+\alpha_{2,s}(\lambda, Ax_{s,t} + By_{s,t} - c) - \alpha_{2,s}(\lambda_{s,t}, Ax + By - c) \\
\end{cases}
\]

Combining (5), (6) and (7), we have

\[
\begin{aligned}
\mathbb{E}Q(w; w_{t-1}^{ag}) - \alpha_{1,s}Q(w; w_{t-1}^{ag}) - \alpha_{3,s}Q(w; \tilde{w}_{s-1}) &
\leq \alpha_{2,s}[(\theta_{s}\langle \lambda_{t,s} - \lambda_{t-1,s} \rangle + \lambda_{t-1,s} + \theta_{s}A(\tilde{x}_{t,s} - x_{t,s}) + \theta_{s}B(y_{t-1,s} - y_{t,s}), A(x_{t,s} - x)) \\
&+ \frac{\alpha_{2,s}L}{2}||x_{t,s} - x_{t-1,s}||^2_2 + (\theta_{s}(Ax_{t,s} + By_{t,s} - c) + \lambda_{t-1,s}, B(y - y_{t,s})) \\
&+ (\lambda, Ax_{s,t} + By_{s,t} - c) - (\lambda_{t,s}, Ax + By - c)] \\
&\leq \alpha_{2,s}[(\theta_{s}\langle \lambda_{t,s} - \lambda_{t-1,s} \rangle + \lambda_{t-1,s} + \theta_{s}A(\tilde{x}_{t,s} - x_{t,s}) + \theta_{s}B(y_{t-1,s} - y_{t,s}), A(x_{t,s} - x)) \\
&+ \frac{\alpha_{2,s}L}{2}||x_{t,s} - x_{t-1,s}||^2_2 + (\theta_{s}(Ax_{t,s} + By_{t,s} - c) + \lambda_{t-1,s}, B(y - y_{t,s})) \\
&+ (\lambda, Ax_{s,t} + By_{s,t} - c) - (\lambda_{t,s}, Ax_{t,s} - Ax) + (\lambda_{t,s}, By_{s,t} - By_{t,s})] \\
&\leq \alpha_{2,s}[(\theta_{s}\langle \lambda_{t,s} - \lambda_{t-1,s} \rangle + \lambda_{t-1,s} + \theta_{s}A(\tilde{x}_{t,s} - x_{t,s}) - \lambda_{t,s}, Ax + By - c)] \\
&+ \frac{\alpha_{2,s}L}{2}||x_{t,s} - x_{t-1,s}||^2_2 \\
\end{aligned}
\]

Since \(2\langle a, b \rangle = ||a + b||^2 - ||a||^2 - ||b||^2\), we have

\[
\langle x_{t-1,s} - x_{t,s}, x_{t,s} - x \rangle = \frac{1}{2}(||x_{t-1,s} - x||^2 - ||x_{t,s} - x||^2 - ||x_{t-1,s} - x_{t,s}||^2) \\
(\lambda - \lambda_{t,s} - \lambda_{t-1,s} - \lambda_{t,s} - \lambda_{t-1,s}) = \frac{1}{2}(||\lambda_{t,s} - \lambda||^2 - ||\lambda_{t,s} - \lambda||^2 - ||\lambda_{t,s} - \lambda_{t-1,s}||^2). \\
\]

Since \(B(y - y_{t,s}) = (Ax + By - c) - \frac{1}{\rho_{t,s}}(\lambda_{t,s} - \lambda_{t-1,s}) + A(x_{t,s} - x)\), we have

\[
\begin{aligned}
-\langle (\theta_{s}\langle \lambda_{t,s} - \lambda_{t-1,s} \rangle, A(x_{t,s} - x)) + (\theta_{s}\langle \lambda_{t,s} - \lambda_{t-1,s} \rangle, B(y - y_{t,s})) \\
&+ (\theta_{s}\langle \lambda_{t,s} - \lambda_{t-1,s} \rangle, Ax_{t,s} - x) + (\theta_{s}\langle \lambda_{t,s} - \lambda_{t-1,s} \rangle, Ax + By - c)] \\
&\leq \frac{\alpha_{2,s}L}{2}||x_{t,s} - x_{t-1,s}||^2_2 + (\theta_{s}\langle \lambda_{t,s} - \lambda_{t-1,s} \rangle, (Ax + By - c) \\
\end{aligned}
\]

(8)
We also have

\[-\theta_s \langle \hat{x}_{t,s} - x_{t,s}, A(x_{t,s} - x) \rangle + \theta_s \langle y_{t,s} - y_{t-1,s}, A(x_{t,s} - x) \rangle = \frac{\chi \theta_s}{2} \left( \|A(x_{t-1,s} - x_{t,s})\|^2 + \|A(x_{t-1,s} - x)\|^2 - \|A(x_{t,s} - x)\|^2 \right) + \frac{\theta_s}{2} \|Ax_{t,s} + By_{t,s} - c\|^2 - \|Ax_{t,s} + By_{t,s} - c\|^2 \]

Substituting (9), (10), (11) and (12) into (8), we have

\[-\theta_s \langle \hat{x}_{t,s} - x_{t,s}, A(x_{t,s} - x) \rangle + \theta_s \langle y_{t,s} - y_{t-1,s}, A(x_{t,s} - x) \rangle \leq \frac{\chi \theta_s}{2} \left( \|A(x_{t-1,s} - x_{t,s})\|^2 + \|A(x_{t-1,s} - x)\|^2 - \|A(x_{t,s} - x)\|^2 \right) + \frac{\theta_s}{2} \|Ax_{t,s} + By_{t,s} - c\|^2 - \|Ax_{t,s} + By_{t,s} - c\|^2 \]

Substituting (9), (10), (11) and (12) into (5), we have

\[EQ(w; w^0_{t,s}) - \alpha_{1,t,s}Q(w; w^0_{t-1,s}) - \alpha_{3,t,s}Q(w; \tilde{w}_{s-1}) \leq \alpha_{2,t}\left[ \frac{\eta_s}{2} \left( \|x_{t-1,s} - x\|^2 - \|x_{t,s} - x\|^2 \right) + \frac{1}{2 \rho_s} (\|\lambda_{t-1,s} - \lambda\|^2 - \|\lambda_{t,s} - \lambda\|^2) \right]

Since $Ax^* + By^* - c = 0$, if we choose $\theta_s \geq \rho_s$ and $\eta_s \geq \frac{\tilde{L}}{\alpha_{2,t}} + \chi \theta_s \|A\|_2^2$, we have

\[EQ(x^*, y^*, \lambda; w^0_{t,s}) - \alpha_{1,t}Q(x^*, y^*, \lambda; w^0_{t-1,s}) - \alpha_{3,t}Q(x^*, y^*, \lambda; \tilde{w}_{s-1}) \leq \alpha_{2,t}\left[ \frac{\eta_s}{2} \left( \|x_{t-1,s} - x^*\|^2 - \|x_{t,s} - x^*\|^2 \right) + \frac{1}{2 \rho_{t,s}} (\|\lambda_{t-1,s} - \lambda\|^2 - \|\lambda_{t,s} - \lambda\|^2) \right]

\[\frac{1}{\alpha_{2,t}} - \frac{\chi \beta_1}{\alpha_{2,t}} Q(x^*, y^*, \lambda; w^0_{t,s}) - \frac{\alpha_{3,t}}{\alpha_{2,t}} Q(x^*, y^*, \lambda; \tilde{w}_{s-1}) \leq \tilde{L} + \chi \beta_1 \|A\|_2^2 \left( \|x_{t-1,s} - x^*\|^2 - \|x_{t,s} - x^*\|^2 \right) + \frac{\beta_2}{2} (\|\lambda_{t-1,s} - \lambda\|^2 - \|\lambda_{t,s} - \lambda\|^2) \]

Proof of Lemma 3

Proof. In the $s$-th outer iteration, if we make $\theta_s = \beta_1 \alpha_{2,t}$, $\rho_s = \frac{\beta_2}{\alpha_{2,t}}$, $\eta_s = (\tilde{L} + \chi \beta_1 \|A\|_2^2) \alpha_{2,t}$, then according to Lemma 2 we have
Adding up $t$ from 1 to $m$ in the $s$–th outer iteration, we have

\[
\frac{1}{\alpha_{2,s}^2} \mathbb{E} Q(x^*, y^*, \lambda; w^g_{0,s}) + \sum_{t=1}^{m-1} \frac{1 - \alpha_{1,s}^1}{\alpha_{2,s}^2} \mathbb{E} Q(x^*, y^*, \lambda; w^g_{t,s}) \\
\leq \frac{1}{\alpha_{2,s}^2} \mathbb{E} Q(x^*, y^*, \lambda; w^g_{0,s}) + \frac{\alpha_{3,s}^m}{\alpha_{2,s}^2} \mathbb{E} Q(x^*, y^*, \lambda; \tilde{w}_{s-1}) \\
+ \frac{L + \chi \beta_1 \|A\|^2}{2} (\|x_{0,s} - x^*\|^2 - \|x_{m,s} - x^*\|^2) + \frac{1}{2\beta_2} (\|\lambda_{0,s} - \lambda\|^2 - \|\lambda_{m,s} - \lambda\|^2) \\
+ \frac{\chi \beta_1}{2} (\|A(x_{m,s} - x^*)\|^2 - \|A(x_{0,s} - x^*)\|^2) + \frac{\beta_1}{2} (\|Ax^* + By_{0,s} - c\|^2 - \|Ax^* + By_{m,s} - c\|^2) \\
\] (16)

If \( \frac{1}{\alpha_{2,s}} = \frac{1 - \alpha_{2,s+1}}{\alpha_{2,s+1}} \) and \( \frac{1 - \alpha_{1,s}}{\alpha_{2,s}} = \frac{\alpha_{3,s+1}}{\alpha_{2,s+1}} \), we have

\[
\frac{1 - \alpha_{2,s+1}}{\alpha_{2,s+1}} \mathbb{E} Q(x^*, y^*, \lambda; w^g_{0,s}) + \sum_{t=1}^{m-1} \frac{\alpha_{3,s+1}^m}{\alpha_{2,s+1}} \mathbb{E} Q(x^*, y^*, \lambda; w^g_{t,s}) \\
\leq \frac{1}{\alpha_{2,s}^2} \mathbb{E} Q(x^*, y^*, \lambda; w^g_{0,s}) + \frac{\alpha_{3,s}^m}{\alpha_{2,s}^2} \mathbb{E} Q(x^*, y^*, \lambda; \tilde{w}_{s-1}) \\
+ \frac{L + \chi \beta_1 \|A\|^2}{2} (\|x_{0,s} - x^*\|^2 - \|x_{m,s} - x^*\|^2) + \frac{1}{2\beta_2} (\|\lambda_{0,s} - \lambda\|^2 - \|\lambda_{m,s} - \lambda\|^2) \\
+ \frac{\chi \beta_1}{2} (\|A(x_{m,s} - x^*)\|^2 - \|A(x_{0,s} - x^*)\|^2) + \frac{\beta_1}{2} (\|Ax^* + By_{0,s} - c\|^2 - \|Ax^* + By_{m,s} - c\|^2) \\
\] (17)

According to the convexity of $f(x)$ and $g(y)$ and the linearity of $\lambda(Ax + By - c)$ (w.r.t. $x$ and $y$), we have

\[
Q(x^*, y^*, \lambda; \tilde{w}_s) = f(\tilde{x}_s) - f(x^*) + g(\tilde{y}_s) - g(y^*) + (\lambda, Ax^* + By^* - c) \\
\leq \frac{1}{m} \sum_{t=1}^{m} (f(\tilde{x}_{t,s}^g) - f(x^*) + g(\tilde{y}_{t,s}^g) - g(y^*) + (\lambda, Ax^* + By^* - c)) \\
= \frac{1}{m} \sum_{t=1}^{m} Q(x^*, y^*, \lambda; w_{t,s}^g) \\
\]

Since $x_{0,s} = x_{m,s-1}$, $y_{0,s} = y_{m,s-1}$ and $\lambda_{0,s} = \lambda_{m,s-1}$, we have

\[
\frac{1}{\alpha_{2,s+1}^2} \mathbb{E} Q(x^*, y^*, \lambda; w^g_{0,s}) + \frac{\alpha_{3,s+1}^m}{\alpha_{2,s+1}^2} \mathbb{E} Q(x^*, y^*, \lambda; \tilde{w}_{s}) \\
\leq \frac{1}{\alpha_{2,s}^2} \mathbb{E} Q(x^*, y^*, \lambda; w^g_{0,s}) + \frac{\alpha_{3,s}^m}{\alpha_{2,s}^2} \mathbb{E} Q(x^*, y^*, \lambda; \tilde{w}_{s-1}) \\
+ \frac{L + \chi \beta_1 \|A\|^2}{2} (\|x_{m,s-1} - x^*\|^2 - \|x_{m,s} - x^*\|^2) + \frac{1}{2\beta_2} (\|\lambda_{m,s-1} - \lambda\|^2 - \|\lambda_{m,s} - \lambda\|^2) \\
+ \frac{\chi \beta_1}{2} (\|A(x_{m,s} - x^*)\|^2 - \|A(x_{m,s-1} - x^*)\|^2) + \frac{\beta_1}{2} (\|Ax^* + By_{m,s-1} - c\|^2 - \|Ax^* + By_{m,s} - c\|^2) \\
\] (18)
Summing $s$ from 1 to $N$, we have
\[
\frac{1}{\alpha_{2,N+1}} \mathbb{E} Q(x^*, y^*, \lambda; u_{m,N}^{ag}) + \frac{\alpha_{3,N+1}m}{\alpha_{2,N+1}} \mathbb{E} Q(x^*, y^*, \lambda; \tilde{w}_N)
\]
\[
\leq \frac{1}{\alpha_{2,1}} Q(x^*, y^*, \lambda; u_{m,0}^{ag}) + \frac{\alpha_{3,1}m}{\alpha_{2,1}} Q(x^*, y^*, \lambda; \tilde{w}_0)
\]
\[
+ \frac{L + \chi \beta_1 \|A\|_2^2}{2} (\|x_{m,0} - x^*\|^2 - \|x_{m,N} - x^*\|^2) + \frac{1}{2\beta_2} (\|\lambda_{m,0} - \lambda\|^2 - \|\lambda_{m,N} - \lambda\|^2)
\]
\[
+ \frac{\chi \beta_1}{2} (\|A(x_{m,N} - x^*)\|^2 - \|A(x_{m,0} - x^*)\|^2) + \frac{\beta_1}{2} (\|Ax^* + By_{m,0} - c\|^2 - \|Ax^* + B\lambda_{m,N} - c\|^2)
\]
\[
\tag{19}
\]
According to the convexity of $Q(x^*, y^*, \lambda; \cdot)$, if we set $\tilde{w}_N = \frac{1}{1 + \alpha_{3,N+1}m} u_{m,N}^{ag} + \frac{\alpha_{3,N+1}m}{1 + \alpha_{3,N+1}m} \tilde{w}_N$, we have
\[
\frac{1 + \alpha_{3,N+1}m}{\alpha_{2,N+1}} \mathbb{E} Q(x^*, y^*, \lambda; \tilde{w}_N)
\]
\[
\leq \frac{1}{\alpha_{2,1}} Q(x^*, y^*, \lambda; u_{m,0}^{ag}) + \frac{\alpha_{3,1}m}{\alpha_{2,1}} Q(x^*, y^*, \lambda; \tilde{w}_0)
\]
\[
+ \frac{1}{2\beta_2} \|\lambda_{m,0} - \lambda\|^2 + \frac{\chi \beta_1}{2} (\|A(x_{m,N} - x^*)\|^2 + \frac{1}{2} \|Ax^* + By_{m,0} - c\|^2)
\]
\[
\tag{20}
\]
The above inequality is true for all $\lambda$, hence it also holds in the ball $\mathbb{B}_0 = \{\lambda : \|\lambda\|_2 \leq \gamma\}$, it follows that
\[
\max_{\lambda \in \mathbb{B}_0} Q(x^*, y^*, \lambda; \tilde{x}, \tilde{y}, \tilde{\lambda}) = \max_{\lambda \in \mathbb{B}_0} \left\{ f(\tilde{x}) + g(\tilde{y}) - f(x^*) - g(y^*) + \langle \lambda, Ax + By - c \rangle \right\}
\]
\[
= f(\tilde{x}) + g(\tilde{y}) - f(x^*) - g(y^*) + \gamma \|Ax + By - c\|^2
\]
\[
\tag{21}
\]
If we make both side of inequality (20) the max of $\lambda \in \mathbb{B}_0$, then we have
\[
\mathbb{E} [f(\tilde{x}_N) + g(\tilde{y}_N) - f(x^*) - g(y^*) + \gamma \|Ax_N + By_N - c\|^2]
\]
\[
\leq \frac{\alpha_{2,N+1}^2}{1 + m\alpha_{3,N+1}} \left( \frac{1}{\alpha_{2,1}} (f(\tilde{x}_0) + g(\tilde{y}_0) - f(x^*) - g(y^*)) + \frac{L + \chi \beta_1 \|A\|_2^2}{2} \|x_{m,0} - x^*\|^2 \right.
\]
\[
+ \max_{\lambda \in \mathbb{B}_0} \frac{1}{2\beta_2} \|\lambda_{m,0} - \lambda\|^2 + \frac{\chi \beta_1}{2} \|A(x_{m,N} - x^*)\|^2 + \frac{\beta_1}{2} \|By^* + B\lambda_{m,0}\|^2 \\
\leq \frac{\alpha_{2,N+1}^2}{1 + m\alpha_{3,N+1}} \left( \frac{1 + m\alpha_{3,1}}{\alpha_{2,1}} f(\tilde{x}_0) + g(\tilde{y}_0) - f(x^*) - g(y^*)) + \frac{L + \chi \beta_1 \|A\|_2^2}{2} D_{x^*}
\]
\[
+ \frac{1}{2\beta_2} \|A\|_{2,A,X} + \frac{\beta_1}{2} \|D_{g^*,b}\|^2
\]
\[
\tag{22}
\]
, where the first inequality is due to the initialization $A\tilde{x}_0 + B\tilde{y}_0 - c = 0$ and $Ax^* + By^* - c = 0$. 
