Matrix equations and trilinear commutation relations

Sergey Klishevich∗

Institute for High Energy Physics, Protvino, Russia

Abstract

In this paper we discuss a general algebraic approach to treating static equations of matrix models with a mass-like term. In this approach the equations of motions are considered as consequence of parafermi-like trilinear commutation relations. In this context we consider several solutions, including construction of noncommutative spheres. The equivalence of fuzzy spheres and parafermions is underlined.

*E-mail: klishevich@ihep.ru
1 Introduction

The BFSS and IKKT matrix models [1, 2] are widely used in the context of string theories since it is believed that matrix models nonperturbatively describe collective degrees of freedom — branes (see also Ref. [3]). It was conjectured that superstring theories with brane degrees of freedom correspond to perturbative regimes of M-theory, which pretends to be theory of everything. In the light-cone frame it was conjectured to be described by an Yang-Mills type matrix mechanics (BFSS matrix model) [1]. The IKKT matrix model was offered as an effective theory for the large-N reduced model of super Yang-Mills theory while solutions of the model were interpreted as infinitely long static D-string configurations [2, 4].

Static equations of matrix models are very important since their solutions are used as an background (an vacuum configuration) when quantizing system with specific configuration (e.g. see Ref. [5, 6]). Therefore for applications it is important to have so many such solutions as possible. In this paper we offer a general algebraic approach to treating static equations of motions of matrix models and discuss various solutions.

The paper is organized as follows. In the next section we offer an algebraic approach to treating static matrix equations with a mass-like term which can be interpreted as an interaction with Ricci tensor in the case of curved space. In this approach the matrix equations of motions are considered as constraints on parafermi-like trilinear commutation relations. In the section 3 we discuss several solutions corresponding to the linear case, including construction of fuzzy spheres. Besides we have demonstrated equivalence of parafermions and noncommutative spheres. Brief discussion of results is presented in section 4.

2 An algebraic approach to static matrix equations

Let us consider the matrix model given by the action

\[
S = \int dt \, \text{Tr} \left( \dot{X}_\mu \dot{X}^\mu - \frac{1}{2} [X_\mu, X_\nu] [X^\mu, X^\nu] - R_{\mu\nu} X^\mu X^\nu \right). \tag{1}
\]

Here coordinates \(X^\mu\) are \(N \times N\) Hermitian matrices, all the components \(R_{\mu\nu}\) are numbers while Greek indices run form 1 to \(p\). Repeated upper and lower indices are implicitly summed over while for raising and lowering of indices the metric \(g_{\mu\nu}\) is used. The last term can be interpreted as a generally non-diagonal mass term or as a term representing the interaction of the vector field \(X_\mu\) with an external symmetric field \(R_{\mu\nu}\) (e.g. Ricci tensor in a space with nontrivial curvature). Advantage of such matrix models is that they admit stable vacuum solutions, which can be interpreted as compact branes (e.g. spherical branes). In the matrix model picture, \(N\) represents the number of the quantums on the backgrounds (or the number of D-instantons or D-particles). Without the mass term the model (1) is the bosonic part of the BSFF matrix model [1] however in principle the tensor \(R_{\mu\nu}\) can depend on \(N\) and vanish in large \(N\) limit. Later on we will discuss classical solutions only, therefore, we discard the fermionic degrees of freedom.

The action (1) leads to the following equations:

\[
[X^\nu, [X_\nu, X_\mu]] - \ddot{X}_\mu = R_{\mu\nu} X^\nu. \tag{2}
\]
The equations of motions for static solutions have the algebraic form of trilinear commutation relations:

\[ [X^\nu, [X_\nu, X_\mu]] = R_{\mu\nu} X^\nu. \]  

(3)

Solutions to these equations should correspond to even-dimensional D-p-branes.

If we treat the metric as pseudo-Euclidean then the relations (3) correspond to equations of motions of IKKT matrix model with a mass like term. The IKKT matrix model with a mass term was considered in Ref. [5].

It is worth noting that the nontrivial right-hand side in (3) make the algebra generated by coordinates \( X^\mu \) to be different from the homogeneous case [7].

From the algebraic point of view we can treat the matrix equations of motion (3) as constraints on a priori nonlinear algebra defined by the trilinear relations

\[ [X_\rho, [X_\mu, X_\nu]] = R_{\rho\mu\nu}(X, \mathcal{A}), \]  

(4)

where \( \mathcal{A} \) denotes a set of operators independent of the coordinates \( X_\mu \). The r.h.s. has the obvious index symmetries, which follow from the properties of the double commutator on the l.h.s. Such trilinear relations can be thought as a generalization of the trilinear commutation relations for parafermi systems [8].

To link the commutation relations (4) with matrix equations of motion (3) tensor operator \( R_{\nu\rho\mu}(X, \mathcal{A}) \) has to obey the condition

\[ g^{\rho\sigma} R_{\nu\rho\mu}(X, \mathcal{A}) \sim R_{\mu\nu} X^\nu \]  

(5)

So, the whole algebra is generated by the coordinates \( X_\mu \) and the set \( \mathcal{A} \) governed by the trilinear relations (4) and the constraints (5). Since we assume that the operators from the set \( \mathcal{A} \) do not appear in the equations (3) they can be treated as objects representing “topological” degrees of freedom (e.g. charges corresponding to D-branes) of the matrix model.

It is necessary to note that the offered approach is general because any solution to the equations (3) obeys the relations (4) and (5) for some tensor operator \( R_{\nu\rho\mu}(X, \mathcal{A}) \) since the equations (4) can be taken as its definition.

The relations (4) are too general and we restrict our self to the case when the set \( \mathcal{A} \) is trivial (proportional to identity or empty). When it is natural to consider \( R_{\nu\rho\mu}(X) \) as a regular function in \( X_\mu \), i.e. it can be represented as a series

\[ R_{\rho\mu\nu}(X) = R_{\rho\mu\nu}^{(0)} + R^\lambda_{\rho\mu\nu} X_\lambda + \ldots, \]

where in this case the constant tensor \( R_{\rho\mu\nu}^{(0)} \) is related to central charges of an algebra. The nonlinear dependence on the coordinates \( X_\mu \) in general leads to nonlinear algebras but in this paper we discuss the linear case only.

3 Linear case

Later on we will discuss the partial case, when the r.h.s. (4) is linear in the coordinates \( X_\mu \):

\[ [X_\rho, [X_\mu, X_\nu]] = R^\lambda_{\rho\mu\nu} X_\lambda, \]  

(6)
where $R^\lambda_{\rho(\mu\nu)}$ commutes with the all coordinates and will be treated as a constant tensor related to structure constants of the Lie algebra generated by the coordinates $X_\mu$. Besides, here we suppose that the coordinates generate all operators in the space.

The tensor $R^\lambda_{\rho\mu\nu}$ has the usual index symmetry of a Riemann tensor

$$R^\lambda_\rho(\mu\nu) = 0, \quad R^\lambda_{\rho\mu\nu} + R^\lambda_{\nu\rho\mu} + R^\lambda_{\lambda\rho\mu} = 0.$$  

We will see that in some cases this tensor indeed is a Riemann tensor of a symmetric space with the corresponding algebra defined by the trilinear commutations relations (6). Therefore, we will adopt this terminology for it. It is interesting to note that according to this terminology the massless solutions correspond to Ricci flat spaces.

Let us consider the following three cases:

1. Let the coordinates $X_\mu$ form a basis of some simple algebra $a$. Then the Riemann tensor has the form

$$R^\lambda_{\nu\mu\rho} = C^\lambda_{\nu\sigma} C^\sigma_{\mu\rho},$$

where $C^\lambda_{\nu\sigma}$ are structure constants of the simple algebra. This solution can be identified with a D-brane. If one takes a semi-simple algebra, then such a solution corresponds to a collection of D-branes.

2. If the noncommutative coordinates $X_\mu$ form only a part of basis of some simple algebra then this algebra has a $\mathbb{Z}_2$-grading defined by the relation $\text{gr} X_\mu = 1$. Therefore, such an algebra can be decomposed as $p \oplus m$, where $p = \text{span}\{X_\mu\}$ and $m = \text{span}\{[X_\mu, X_\nu]\}$, with the structural relations

$$[m, m] \subset m, \quad [m, p] \subset p, \quad [p, p] \subset m.$$  

(7)

The subspace $p$ can be identified with a tangent space of the corresponding symmetric space. On the other hand one can represent the commutators of the coordinates in the form

$$[X_\mu, X_\nu] = \frac{1}{2} R^\lambda_{\rho\mu\nu} M_\lambda^\rho,$$

where $M_\lambda^\rho \in \mathfrak{so}(p) \supset m$, therefore, the noncommutative coordinates can be associated with the covariant derivatives on the symmetric space.

3. In general, it is not forbidden for the algebra generated by the operators $X_\mu$ subjected to (6) to be infinite-dimensional. Later we will briefly discuss an explicit example to this case.

Some comments are in order. The coordinates $X_\mu$ can be represented by finite $N \times N$ matrices iff the algebras in the first two cases are compact. Since a Lie group is a symmetric space as well, hence, the first case can be included into the second. Therefore, we will not discuss this case here. In the cases corresponding to symmetric spaces the metric should be proportional to the Ricci tensor.
3.1 Parafermions and noncommutative spheres

Let us start with the case of a constant curvature space, which is the most simple case among symmetric spaces. The Riemann tensor has the form

$$R^\lambda_{\mu\nu} = \frac{\mathcal{R}}{p(p-1)} (\delta^\lambda_\mu g_{\nu\rho} - \delta^\lambda_\nu g_{\mu\rho}),$$

(8)

where $\mathcal{R}$ is a scalar curvature. We look for the solution for the coordinates in the form of the rescaled operators

$$X_\mu = \frac{1}{2} \sqrt{\frac{\mathcal{R}}{p(p-1)}} G_\mu,$$

where Hermitian operators $G_\mu$ obey the commutation relations

$$[G_\rho, [G_\mu, G_\nu]] = 4 (g_{\nu\rho} G_\mu - g_{\mu\rho} G_\nu).$$

(9)

The operators $G_\mu$ generate the algebra $\mathfrak{so}(p+1), p \in \mathbb{N}$:

$$[G_\mu, G_\nu] = 4i M_{\mu\nu},$$

$$[G_\mu, M_{\rho\nu}] = i (g_{\mu\nu} G_\rho - g_{\mu\rho} G_\nu),$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i (g_{\mu\rho} M_{\sigma\nu} - g_{\mu\sigma} M_{\rho\nu} + g_{\nu\rho} M_{\mu\sigma} - g_{\nu\sigma} M_{\mu\rho}).$$

(10)

**Even dimensional case.** Let us consider the case of even $p$. Then the Hermitian operators $G_\mu$ can be represented in the following form:

$$G_{2k-1} = a_k^\dagger + a_k, \quad G_{2k} = i(a_k^\dagger - a_k),$$

where the mutually conjugate operators $a_k, a_k^\dagger$ obey the basic parafermi commutation relations

$$[a_k, [a_l^\dagger, a_m]] = 2g_{kl} a_m, \quad [a_k, [a_l^\dagger, a_m^\dagger]] = 2g_{kl} a_m^\dagger - 2g_{km} a_l^\dagger, \quad [a_k, [a_l, a_m]] = 0.$$  

(11)

So, this case corresponds to a parafermi system. For parafermi systems the most important representation is the Fock representation defined by the relations:

$$a_k |0\rangle = 0, \quad a_k a_l^\dagger |0\rangle = n \delta_{kl} |0\rangle, \quad n \in \mathbb{N}.$$  

(12)

This representation is labeled by the natural number $n$ called the order of the parafermi quantization. This representation can also be given in terms of the Green ansatz:

$$G_\mu = \sum_{\alpha=1}^{n} G_{\mu}^{(\alpha)},$$

(13)

where the summands obey the relations

$$\{ G_{\mu}^{(\alpha)}, G_{\nu}^{(\alpha)} \} = 2g_{\mu\nu} \cdot \mathbb{I}, \quad [G_{\mu}^{(\alpha)}, G_{\nu}^{(\beta)}] = 0, \quad \alpha \neq \beta.$$  

(14)
The natural matrix representation of the Green ansatz is given on the vector space $V_a = V^\otimes n$:

$$G^{(1)} = \Gamma_\mu \otimes 1 \otimes \cdots \otimes 1, \quad G^{(2)} = 1 \otimes \Gamma_\mu \otimes \cdots \otimes 1, \quad \ldots, \quad G^{(n)} = 1 \otimes \cdots \otimes 1 \otimes \Gamma_\mu,$$

where $V$ is the irreducible representation space of the Clifford algebra

$$\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu} \cdot 1.$$  

The operators $G^{(\alpha)}$ can be represent as a linear combination of the fermi creation-annihilation operators:

$$G^{(\alpha)}_{2k-1} = a^{(\alpha)}_k \rightarrow a^{(\alpha)}_k e^{-i\omega t}, \quad a^{(\alpha)}_k \rightarrow a^{(\alpha)}_k e^{i\omega t}.$$

The vector space $V_a$ is the corresponding Fock space generated by polynomials of the operators $a^{(\alpha)}_k$ acting on the vacuum. The symmetrized traceless vector space $V_{st} = \text{Sym}_t V^\otimes n$ is a proper subspace of $V_a$, which contains the vacuum of the Fock representations on $V_a$ and $V_{st}$. Besides, this symmetrized space is invariant with respect to the action of the operators (13). The Fock representation of the parafermi system (11), (12) belongs to the space $V_{st}$ since the corresponding vacuum state does. It is known Ref. [9] that the irreducible Fock representation of the parafermi relations is a proper subspace of $V_a$, therefore, since the vacuum belong to $V_{st}$ this space coincides with the irreducible representation space of the parafermi system. This proofs the following theorem.

**Theorem:** The matrix representation (13)-(16) on the symmetrized traceless vector space $\text{Sym}_t V^\otimes n$ is the irreducible Fock representation of the parafermi commutation relations of the order $n$.

**Time dependent solution.** Using the one-parametric unitary transformation of the Fock representation

$$a_k \rightarrow a_k e^{-i\omega t}, \quad a_k \rightarrow a_k e^{i\omega t},$$

we can write down the time dependent solution of the matrix equations (2) in terms of the time dependent matrices

$$G_\mu = \sum_{k=1}^{p/2} \left( B^k_\mu a_k e^{-i\omega t} + (B^k_\mu)^* a_k e^{i\omega t} \right),$$

where

$$B^l_{2k-1} = \delta_{kl}, \quad B^l_{2k} = i\delta_{kl}.$$  

The coordinates can be represented in the form

$$X_\mu = \frac{r}{2\sqrt{p(p-1)}} G_\mu,$$

where from the (9) in follows that the parameter $r$ is defined by the relation

$$r^2 = R - p\omega^2.$$  

Since the parameter $r$ has to be positive real number the scalar curvature and frequency are restricted: $R > 0$ and $\omega^2 < R/p$. 


Besides the harmonic time dependence we can consider the coordinates with exponential dependence, \( \ddot{X}_\mu - \omega^2 X_\mu = 0 \). In this case there is no constraints on the parameter \( r \).

**Large N limit.** It is known that for the Fock representations of the parafermi system the rescaled operators \( a_k/\sqrt{n} \) and \( a^*_k/\sqrt{n} \) in the limit \( n \to \infty \) go to the Heisenberg algebra. Therefore, in this limit the coordinates (18) have the commutation relation \( [X_\mu, X_\nu] = \theta_{\mu\nu} \cdot \mathbb{I} \). In other words, the noncommutative manifold defined by the coordinates (18) in large \( N \) limit goes to the fuzzy plane \( \mathbb{R}^p_\theta \) representing a noncommutative D-p-brane. This solution is a BPS one [1].

In the paper [10] the representation (13)-(16) on the symmetrized vector space was used to realize the noncommutative 4-sphere. Later similar representations of higher even-dimensional fuzzy spheres were realized on the symmetrized traceless vector space \( \text{Sym}_t V \otimes^n \) [11, 12]. Thus as we have seen the irreducible Fock representations of parafermions are equivalent to noncommutative spheres.

### 3.2 About solutions with a nontrivial Weyl tensor

Above we discussed the cases related to the symmetric spaces with a trivial Weyl tensor. However, the non-triviality of the Weyl tensor is important for constructing “massless” solutions (with \( R_{\mu\nu} = 0 \)). Indeed, in the trilinear relations (19) one can change the Riemann tensor for the Weyl one

\[
[X_\rho, [X_\mu, X_\nu]] = W^\lambda_{\rho\mu\nu} X_\lambda
\]

while the metric is kept the same. Obviously, the new trilinear commutation relations (19) will provide solutions of the necessary type.

Let us consider the case of the symmetric space \( SU(n)/SO(n) \). The basis of the algebra \( \mathfrak{su}(n) \) can be represented by matrices \( s_{ab} + im_{ab} \), where \( s_{ab}^\dagger = s_{ab} \), \( s_{ab} = s_{ba} \) and \( m_{ab} \) span a Hermitian basis of the algebra \( \mathfrak{so}(n) \). We will identify the operators \( s_{ab} \) with the coordinates \( X_\mu \), i.e. in this case \( \mu \) is a multi-index. The commutation relations between \( s_{ab} \) and \( m_{cd} \) are

\[
[s_{ab}, s_{cd}] = -2i(\delta_{(a| m_{b|d} + \delta_{d(a m_{b|c})}),} \quad [s_{ab}, m_{cd}] = 2i(\delta_{c(a s_{b|d})} - \delta_{d(a s_{b|c})}).
\]

Therefore, the Riemann tensor in (19) has the form

\[
R^{(lm)}_{(ef)(ab)(cd)} \sim 4 \left( \delta_{(e|(a| \delta_{b)(e \delta_f)(l)m} - \delta_{(a|(c| \delta_d)(e \delta_f)(l)m}) \right).
\]

The Ricci tensor is proportional to the metric

\[
R_{(ab)(cd)} \sim g_{(ab)(cd)} = \delta_{a(c} \delta_{d)b} - \frac{1}{n} \delta_{ab} \delta_{cd}.
\]

Comparing the Riemann tensor (20) with the Ricci one it is possible to conclude that the corresponding Weyl tensor is nontrivial, therefore nontrivial trilinear commutation relations (19) providing a massless solution can be constructed.
The case of $A$-statistics. In this case the basic commutation relations are given in the form [14]

\[
\begin{align*}
[[a_k, a_l], a_m] &= \delta_{kl}a_m + \delta_{lm}a_k, \\
[[a_k, a_k], a_m] &= \delta_{kl}a_m + \delta_{lm}a_k, \\
[a_k, a_l] &= [a_k, a_l] = 0,
\end{align*}
\] (21)

where $k, l, m = 1, \ldots, M$.

In terms of the basis of the Hermitian operators $X_\mu$:

\[
X_k = a_k^\dagger + a_k, \quad X_{n+k} = i(a_k^\dagger - a_k)
\]

the commutation relations (21) can be represented in the form (6) with the Riemann tensor

\[
R^{\alpha\beta\mu\nu} = \delta^{\alpha\mu}g_{\beta\nu} - \delta^{\alpha\nu}g_{\beta\mu} + J^{\alpha}_{\mu}J^{\beta}_{\nu} - J^{\alpha}_{\nu}J^{\beta}_{\mu} + 2J^{\alpha}_{\beta}J^{\mu\nu},
\] (22)

where

\[
g_{\mu\nu} = \begin{pmatrix}
I_{M \times M} & 0 \\
0 & -I_{M \times M}
\end{pmatrix}, \quad J_{\mu\nu} = \begin{pmatrix}
0 & I_{M \times M} \\
I_{M \times M} & 0
\end{pmatrix}.
\]

The commutation relations (6) with this Riemann tensor coincide with those of covariant derivatives on the complex projective plane $\mathbb{C}P^M$. The tensor $J_{\mu\nu}$ is a complex structure tensor on this space. It is easily to see that the associated Weyl tensor is nontrivial in this case.

Consider a Fock representation for the A-statistics (21) given by polynomials in operators $a_k^\dagger$ over a ground state $|0\rangle$ formally defined by the same relations (12). In this case the natural parameter $n$ is called as an order of the $A$-statistics. This representation is finite-dimensional and in the large $N$ limit ($n \to \infty$) this representation goes to that of Heisenberg algebra (a noncommutative plane) like the parafermionic representation does.

3.3 The generalized Dolan-Grady relations

The interrelationship between the equations (3), one-mode parafermion and the Dolan-Grady relations was noted in Ref. [15]. Here we discuss application of the generalized Dolan-Grady relations [16].

The real form of the generalized Dolan-Grady relations can be represented as

\[
[G_\nu, [G_\mu, G_\nu]] = \epsilon_\nu G_\mu, \quad [G_\rho, G_\sigma] = 0, \quad \epsilon_\nu = \pm 1,
\] (23)

where $G_\mu$ are Hermitian operators, the indices run from 1 to $p$, there is no summation over the index $\nu$, the indices $\mu$ and $\nu$ are supposed to acquire adjacent values while the second identity is supposed for the case when the indices $\rho$ and $\sigma$ are not adjacent. The values 1 and natural $p$ are considered as adjacent.

However, it is worth noting that if one takes the metric in the form $g_{\mu\nu} = \delta_{\mu\nu}$ when for the coefficients $\epsilon_\nu$ obeying the relation

\[
\epsilon_i + \epsilon_{i+2} = 0,
\]
where we imply the periodic condition $\epsilon_{i+p} = \epsilon_i$, the generalized Dolan-Grady relations provide a stationary solution to the usual BFSS model (massless solution). Such a solution exists only for $p$ to be a number divisible by 4. So, this solution should correspond to even D-p-branes as it is required.

The generalized Dolan-Grady relations define $\mathfrak{sl}(p)$ Onsager algebra \[16]. Unlike the usual Onsager algebra, the theory of finite-dimensional representations in the general case of $\mathfrak{sl}(p)$ Onsager algebra is not elaborated. Till now, the only known such representations can be defined in terms of the operators

$$g_{2k-1} = \frac{1}{2} \sigma_k^x, \quad g_{2k} = \frac{1}{2} \sigma_k^x \sigma_{k+1}^y,$$

where $k = 1, 2, \ldots, L$. The case $G_{\mu} = g_{\mu}$ corresponds to the $\mathfrak{sl}(2L)$ Onsager algebra, while for $2L = mp$ ($m \in \mathbb{N}$) the operators of the form

$$G_{\mu} = \sum_{s=0}^{m-1} g_{\mu+ps},$$

where $\mu = 1, \ldots, p$, generate the $\mathfrak{sl}(p)$ Onsager algebra \[16]. The limit $L \to \infty$ (or more specifically $m \to \infty$ while $p$ is fixed) corresponds to the large $N$ limit.

One can consider more restrictive commutation relations than the generalized Dolan-Grady relations \[23\] taking the “Riemann tensor” of the form

$$R^\lambda_{\nu \rho \mu} \sim \epsilon_\nu \left( \delta_\mu^\lambda \delta_\rho^\nu (\delta_{\mu,\nu+1} + \delta_{\mu,\nu-1}) - \delta_\rho^\lambda \delta_\mu^\nu (\delta_{\rho,\nu+1} + \delta_{\rho,\nu-1}) \right).$$

The corresponding “Ricci tensor” is $R_{\nu \rho} \sim \epsilon_\nu \delta_{\nu \rho}$, therefore, in the case of the BFSS model it cannot be taken as a metric if $\epsilon_\nu$ have different signs.

The relations \[23\] can also be used to construct solutions of the IKKT model. In this case there is the same restriction on $p$.

### 4 Conclusion

In this paper we offered a general algebraic approach to treating static equations of motions of Yang-Mills type matrix models. We have considered several solutions, including discussed in the literature construction of higher dimensional noncommutative spheres. We underlined equivalence of fuzzy spheres and parafermions and demonstrated that this parafermionic solution can be extended to include harmonic time dependence but with restrictions on the scalar curvature and frequency.

Relying on the noted analogy between the equations of motion of Yang-Mills type matrix models and the parafermi-like commutation relations we hope that the numerous ideas and methods elaborated for parastatistics (e.g. the Green representation) will be useful in developments related to the matrix equations.

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