On a theorem of Goussarov

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Abstract

In this paper, the easier methods of my thesis are applied to give a simple proof of a theorem of Goussarov. This theorem relates two possible notions of finite type equivalence of knots, links, or string links, showing that the resulting filtrations are the same up to a degree shift by a factor of two. This is then applied to the situation of rooted clasps to show that rooted clasper surgeries of sufficiently high degree must preserve type $k$ invariants. As a consequence, grope cobordisms of sufficiently high class must preserve type $k$ invariants. This result is applied in [CT] to show Theorem 2 of that paper.

1 Introduction

The heart of this paper is a simple proof of a theorem of Goussarov, using an elementary tool developed in my thesis. His theorem relates two possible notions of finite type invariants, the standard one which he co-invented with Vassiliev, and a more subtle one based on what he calls “interdependent” modifications of links. He proves that these invariants coincide up to a degree shift by a factor of two.

The reason I wrote this paper is Corollary 4, that a grope cobordism preserves Vassiliev invariants up to roughly half the degree. This corollary is needed to prove Theorem 2 of [CT], which states that grope cobordism of class $c$ is precisely the same as surgery on simple claspers (see [3] or [CT]) with grope degree $c$ or more. It is much easier to see that grope cobordism coincides with rooted clasper surgeries, for which Goussarov’s interdependent modifications are well-suited.

In my thesis I considered a problem similar to Corollary 4, the difference being that I considered knots bounding gropes, which is a much stronger notion than cobounding a grope with the unknot. In that case, I obtained a sharp answer, namely a knot bounding a grope of class $k$ is $\lceil k/2 \rceil$-trivial. The current paper implies $\lfloor (k-1)/2 \rfloor$, which is not very far away. Achieving the sharp result is a lot of work for not much gain.

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Goussarov’s main idea was to define filtrations using alternating sums over “disjoint variations.” For instance, one may take the variations to be crossing changes of a knot, as he did in [G1], defining the usual finite type filtration independently of Vassiliev. In [G2] he considered a more subtle notion of variation, an “interdependent modification,” which involves replacing an arc of a circle with an arbitrary other arc in the link complement with the same endpoints. Evidently he was trying to find more subtle invariants than those which are of standard finite type. But his discovery, which is the main subject of this paper, was that they are the same up to a degree change. Most recently, he invented and developed a notion of 3-manifold finite type invariants, where the variation is a surgery along a “Y.” (Equivalently, a Matveev move on a genus 3 handlebody.) This notion appears in [GGP] for example. In [GR] Garoufalidis and Rozansky use a variation of this notion to study pairs (Homology 3-sphere, Knot), where the leaves of the “Y” must link the knot trivially. If it weren’t for his untimely death, Goussarov surely would have produced more exotic and beautiful notions.

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2 Two filtrations

Let $X$ be the set of isotopy classes of links or string links with a fixed number of components. The Vassiliev filtration of $\mathbb{Z}[X]$ is a descending filtration $\mathbb{Z}[X] = F^0 \supseteq F^1 \supseteq F^2 \supseteq \cdots$. Each piece $F^k$ is spanned by alternating sums of the following form. Let $x \in X$ be a knot, link, or string link. Choose $k+1$ framed arcs from $x$ to itself, which guide homotopies of $x$ supported in neighborhoods of these arcs, which take the little piece of $x$ at one end of the arc and push this across the other end. Such a move is called a finger move. (Such finger moves can always be realized as crossing changes in some projection of $x$.) Let the set of these finger moves be called $S$. Then define $[x; S] = \sum_{\sigma \in S} (-1)^{|\sigma|} x_\sigma$, where $x_\sigma$ is $x$ modified by the finger moves in $\sigma$. Now, by definition $F^k$ is additively generated by elements of the form $[x; S]$, where $S$ has $k+1$ finger moves in it.

One can replace single finger moves by groups of finger moves in the above definition, and this is well-known to give the same filtration. See [G1] for a proof.

An alternative filtration can be defined in the same way using moves more general than finger moves. One way to think of these moves is that they are guided by circles which are attached to $x$ along a subarc. The move is to replace the part of $x$ running across the circle with the other arc of the circle. $k+1$ disjoint circles lead to generators of this alternative filtration, which I denote by $F^alt_k$. Goussarov calls these interdependent moves. The reason is that such

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1 I use the term group to indicate that a collection of moves is to be viewed as a single variation.
moves cannot necessarily be realized by independent, i.e. disjoint, groups of finger moves.

As before, one can replace single moves along circles by groups of moves along circles and the same filtration is achieved, the proof being modified *mutatis mutandis*.

The main result of [G2], which is stated there in the dual setting, is that $\mathcal{F}_k^v = \mathcal{F}_{2k}^{alt} = \mathcal{F}_{2k+1}^{alt}$. It is proved in two steps.

**Theorem 1** $\mathcal{F}_k^v \subset \mathcal{F}_{2k+1}^{alt}$.

**Theorem 2** $\mathcal{F}_{2k}^{alt} \subset \mathcal{F}_k^v$.

**Theorem 1** is the easier of the two. I provide a repackaged version of Goussarov’s proof for completeness. **Theorem 2** is harder and this is the one for which I provide a new simplified proof.

### 3 Proofs of Theorems 1 and 2

*Proof of Theorem 1*

Each finger move has two interdependent moves associated to it as follows. A symmetric way to view a finger move along an arc $\eta$ is to push the strands of $x$ at both endpoints of $\eta$ along $\eta$ so that they crash through each other at some point in the middle of $\eta$. For each of the two modified strands of $x$, there is a disk cobounding that strand with the strand before the finger move, the disk having been swept out by the isotopy. These two disks intersect in a single clasp. The boundaries of these two disks are the circles guiding the two interdependent moves associated to the finger move. Now I claim the alternating sum over a set of finger moves is, up to a sign, the same as the alternating sum over the associated interdependent moves. This is essentially because if one only does one of a given pair associated to a finger move, this is just an isotopy. Let $S = \{s_i\}$ be the set of $k+1$ finger moves, and $T = \{a_i, b_i\}$ the associated set of $2k+2$ interdependent moves. Inductively I show that

$$\sum_{\sigma \subset T} (-1)^{|\sigma|} x_\sigma = (-1)^k \sum_{\mu \subset S} (-1)^{|\mu|} x_\mu,$$

which will be sufficient to prove the theorem. This left hand sum breaks up into

$$\sum_{\tau \subset \{a_2, \ldots, b_{k+1}\}} (-1)^{|\tau|} (x_\tau - x_\tau \cup \{a_1\} - x_\tau \cup \{b_1\} + x_\tau \cup \{a_1, b_1\}),$$

and since the first three terms are equal, we get

$$- \sum_{\tau \subset \{a_2, \ldots, b_{k+1}\}} (-1)^{|\tau|} x_\tau + \sum_{\tau \subset \{a_2, \ldots , b_{k+1}\}} (-1)^{|\tau|} (x_{a_1})_\tau,$$
and by induction this is just

\[ (-1)^k \sum_{\mu \subseteq \{s_2, \ldots, s_{k+1}\}} ((-1)^{|\mu|}x_{\mu} - (-1)^{|\mu|}x_{(s_1) \cup \mu}) \]

which equals \((-1)^k \sum_{\mu \subseteq \mathcal{S}} (-1)^{|\mu|}x_{\mu}\) as desired. \(\square\)

**Proof of Theorem 2**

Given a set of \(2k + 1\) interdependent moves, \(\mathcal{S} \) on \(x\), we wish to show that \([x; \mathcal{S}] \in \mathcal{F}_k^v\). The strategy is to show that \([x; \mathcal{S}]\) is congruent modulo \(\mathcal{F}_k^v\) to sums of simpler alternating sums \([x; \mathcal{S}']\), where the complexity of such alternating sums is recorded by a graph I will define in a moment. After iteration, we reduce the problem to alternating sums for which a direct argument is possible to show that they are congruent to 0.

Fix a projection of \(x\) and the circles in \(\mathcal{S}\). (That is, think of it as a planar picture with over and under crossing data.) The projection is assumed to be such that \(x\) does not cross itself where circles are attached. In order to keep track of the complexity of the interdependence of the moves, I now define a graph associated to the picture. It has \(2k + 1\) vertices corresponding to the \(2k + 1\) circles that guide the interdependent moves. Fix an ordering on these vertices, say they are called \(v_1, \ldots, v_{2k+1}\). Draw an edge between \(v_i\) and \(v_j\) if \(i < j\), but the circle corresponding to \(v_i\) crosses over the one corresponding to \(v_j\). Draw an edge from a vertex to itself if the corresponding circle is knotted. Finally, label a vertex with a star if \(x\) crosses over the corresponding circle.

For each edge of the graph, one can do an obvious group of crossing changes of \(x\) union the circles, which eliminates that edge. That is, to eliminate an edge between \(v_i\) and \(v_j\), where \(i < j\), do those crossing changes which make \(v_i\) always pass under \(v_j\) at each crossing. Similarly, to eliminate a star, there is a group of crossing changes that always make \(x\) pass under a given circle. Finally there is a group of crossing changes eliminating a self-edge, i.e. by unknotting the corresponding circle.

Suppose a graph has at least \(k + 1\) edges plus stars. Let \(T\) denote a set of \(k + 1\) groups of crossing changes on \(x\) union the circles, each of which removes an edge or star from the graph. Each such group of crossing changes induces a group of crossing changes on each summand \(x_\sigma\) of \([x; \mathcal{S}]\). Thus each \(x_\sigma\) is congruent modulo \(\mathcal{F}_k^v\) to \(\sum_{\emptyset \neq \tau \subset T} \pm x_{\sigma \tau}\). Then

\[ [x; \mathcal{S}] = \sum_{\sigma \subset \mathcal{S}} (-1)^{|\sigma|}x_\sigma \]
\[ = \sum_{\sigma \subset \mathcal{S}} (-1)^{|\sigma|} \sum_{\emptyset \neq \tau \subset T} \pm (x_\sigma)_\tau \]
\[ = \sum_{\emptyset \neq \tau \subset T} \pm \sum_{\sigma \subset \mathcal{S}_\tau} (-1)^{|\sigma|}x_\sigma \]
\[ = \sum_{\emptyset \neq \tau \subset T} \pm [x, \mathcal{S}_\tau], \]
where \( S_\tau \) is the set of interdependent moves guided by the circles modified by \( \tau \). Thus it suffices to show that \( [x; S_\tau] \in F_k^v \) for \( \tau \neq \emptyset \). Each of the resulting sets \( S_\tau, \tau \neq \emptyset \), of interdependent moves has fewer edges plus stars in the resulting graph. We can always iterate this simplification unless the number of edges plus stars is less than or equal to \( k \). It is hence sufficient to consider this case.

On such a graph, I claim that there are \( k + 1 \) unstarred vertices, no pair of which has a connecting edge. This follows from the claim that there are at least \( k + 1 \) connected components without any stars on them. To see this, let \( st \) denote the number of stars. Then \( E + st \leq k \) implies \( E \leq k - st \). The Euler characteristic can be computed in two different ways: \( V - E = b_0 - b_1 \), so that \( b_0 \geq b_0 - b_1 = 2K + 1 - E \geq (k + 1) + st \). Thus \( b_0 - st \geq k + 1 \).

The fact that the vertices are unstarred means they each bound a disk, and the lack of edges implies the disks are disjoint. There are now \( k + 1 \) groups of finger moves of \( x \) union the circles which push everything out of the \( k + 1 \) disks. Thus, as above, modulo \( F_k^v \), we can assume that at least one of the circles guiding a move bounds an embedded disk with no intersections with anything in its interior. This means that the move is an isotopy. However \( [x; S] \) is obviously 0 if any of the moves in \( S \) are isotopies.

\[ \square \]

\section{Application to rooted claspers and grope cobordism}

For the reader’s convenience in following the proof of Theorem 3, I remind him that a rooted clasper is a clasper in the sense of Habiro (see \cite{H}), embedded in the complement of a knot or link, such that there is one root leaf, which is a zero framed leaf linking the knot as a little meridian, whereas the other leaves can be embedded arbitrarily.

\begin{proof}
\end{proof}

Surgering along the clasper, the original knot is modified inside a regular neighborhood of the clasper union the disk bounding its root leaf to get \( K_2 \). We will find \( 2k + 1 \) groups of interdependent moves as follows. For each non-root leaf of the clasper there is an interdependent move of a subarc of the leaf so that the result is a tiny leaf bounding a disk avoiding intersections with anything. Each of these moves on the clasper descend to a group of interdependent moves which take strands of the knot running through the neighborhood of the leaf and move them to a position corresponding to the modified clasper. Thus we have found a set \( S \) of \( 2k + 1 \) groups of interdependent moves on \( K_2 \). Thus the surgered knot \( K_2 \), is congruent modulo \( F_{2k}^{alt} \) to the sum \( -\sum_{\emptyset \neq \sigma \subset S} (-1)^{\sigma}(K_2)_\sigma \), but for each nonempty \( \sigma \), \( (K_2)_\sigma \) is by construction just \( K_1 \) modified by a clasper with at least one trivial leaf. Such clasper surgeries do not change the knot, hence \( K_2 \) is congruent to \( -\sum_{\emptyset \neq \sigma \subset S} (-1)^{\sigma} K_1 = K_1 \).

\[ \square \]
Corollary 4 Suppose two knots $K_1$ and $K_2$ are related by a grope cobordism of class $2k + 1$. Then $K_1 - K_2 \in \mathcal{F}_k$.

[Proof]
By Proposition 6 of [CT], we may assume that the grope cobordism has all of its stages of genus one. By Theorem 9 of the same paper, this is just a rooted clasper surgery with $2k + 1$ non-root leaves.

Remark: Even though this Corollary is used in [CT] to prove Theorem 2 of that paper, Theorem 2 is not used to prove Theorem 9 or Proposition 6, so there is no logical circuitry. Corollary 4 could have been proven directly in the same way as Theorem 3, using techniques of my thesis [C], but as there is no logical necessity I have avoided the added complication.

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