Rotation number of a unimodular cycle: an elementary approach

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Abstract

We give an elementary proof of a formula expressing the rotation number of a cyclic unimodular sequence \( L = u_1 u_2 \ldots u_d \) of lattice vectors \( u_i \in \mathbb{Z}^2 \) in terms of arithmetically defined local quantities. The formula has been originally derived by A. Higashitani and M. Masuda (arXiv:1204.0088v2 [math.CO]) with the aid of the Riemann-Roch formula applied in the context of toric topology. These authors also demonstrated that a generalized version of the ‘Twelve-point theorem’ and a generalized Pick’s formula are among the consequences or relatives of their result. Our approach emphasizes the role of ‘discrete curvature invariants’ \( \mu(a, b, c) \), where \( \{a, b\} \) and \( \{b, c\} \) are bases of \( \mathbb{Z}^2 \), as fundamental discrete invariants of modular lattice geometry.

1 Introduction

The following theorem of A. Higashitani and M. Masuda, proved in [H-M], is a close relative of the remarkable ‘Twelve-point theorem’ [PRV], [CRS], [F], [H-S], [K-N]. Like its predecessors, the ‘Twelve-point theorem’ and Pick’s formula, it is an intriguing and easily formulated statement about a sequence of lattice vectors, their rotation (winding) number and the associated, arithmetically defined local quantities.
Theorem 1. The rotation number $\text{Rot}(L)$ of a cyclic unimodular sequence $L = u_1u_2 \ldots u_d$ can be calculated as the weighted sum

$$\text{Rot}(L) = \frac{1}{12} \mu(L) + \frac{1}{4} \nu(L) = \frac{1}{12} \sum_{i=1}^{d} \mu(u_{i-1}, u_i, u_{i+1}) + \frac{1}{4} \sum_{i=1}^{d} \nu(u_i, u_{i+1}) \quad (1)$$

of locally defined quantities $\mu(L)$ and $\nu(L)$ where $\nu(u_i, u_{i+1}) := \det(u_i, u_{i+1}) \in \{-1, +1\}$ and $\mu(u_{i-1}, u_i, u_{i+1}) = a_i \in \mathbb{Z}$ is the integer determined by the equation

$$\det(u_{i-1}, u_i)u_i - \det(u_i, u_{i+1})u_{i+1} + a_i u_i = 0.$$

Theorem 1 may appear at first sight as quite elementary and not difficult to comprehend, however it has a deeper meaning and significance. Like its relative (and a consequence) the ‘Twelve-point theorem’, it is situated at the crossroads of several mathematical areas, illuminating and offering new perspectives on ‘well understood’ mathematical concepts.

The first proof [H-M] (see also [Ma, Section 5]) of the formula (1) was based on a Riemann-Roch type theorem (Noether formula) where the integers $a_i$ appeared as the self-intersection numbers of the corresponding homology classes of the associated ‘topological toric manifold’.

Our first objective in this paper is to give a conceptual and elementary proof of Theorem 1 which is based on a systematic analysis of the invariant $\mu(a, b, c)$. The second objective is to pave the way for the hypothetical higher dimensional analogues of Theorem 1. For this reason the exposition emphasizes the study of invariants $\mu(a, b, c)$ and their higher dimensional versions $\mu_j(a, b, a')$ (Section 6) as ‘discrete curvature invariants’ situated within unimodular lattice geometry. This point of view is similar to the approach of O. Karpenkov to ‘lattice trigonometry’, as part of his investigation of ‘lattice geometry invariants’ [K-1, K-2].

A different, short and elementary computational proof of Theorem 1 was subsequently included in the new version of the paper [H-M]. Their Lemma 1.3. fits in nicely in our approach so we took the opportunity to shorten the original proof ([Z], arXiv:1209.4981v1[math.CO]), retaining its transparency and conceptuality.

We observe in passing (Section 5.1) that the proof of Poonen and Rodriguez-Villegas of the ‘Twelve-point theorem’ [PRV], based on the properties of the holomorphic function (modular form) $\Delta(z)$, can also be used as the basis...
of a proof of Theorem \[\text{II}\] The variety of proofs and methods applied are perhaps an indication that this result deserves a further exploration so the paper ends with some open questions and conjectures.

2 Introductory definitions and remarks

2.1 Unimodular sequences

A sequence \(L = u_1 u_2 \ldots u_d\) of lattice vectors is called unimodular if \(\{u_i, u_{i+1}\}\) is a basis of the lattice \(\mathbb{Z}^2\) for each \(i\) or equivalently if \(\det(u_i, u_{i+1}) \in \{-1, +1\}\) for each \(i = 1, \ldots, d - 1\). Geometrically this condition means that for each \(i\) the triangle \(Ou_iu_{i+1}\) does not contain lattice points aside from the vertices.

A unimodular sequence \(L = u_1 u_2 \ldots u_d\) is called cyclic if \(\det(u_d, u_1) \in \{-1, +1\}\). A cyclic unimodular sequence \(L\) (of length \(d\)) naturally defines a \(d\)-periodic unimodular sequence \(W = \ldots LLL \ldots = \ldots u_{-1}u_0u_1u_2 \ldots u_d u_{d+1} \ldots\) where \(u_i = u_j\) if \(i \equiv j \mod d\).

2.2 Local and global \(\mu\)-invariants

The invariants \(\mu(L)\) and \(\nu(L)\) of a cyclic unimodular sequence \(L = u_1 \ldots u_d\) are already introduced in Theorem \[\text{II}\] as the sums

\[
\mu(L) = \sum_{i=1}^{d} \mu(u_{i-1}, u_i, u_{i+1}) \quad \nu(L) = \sum_{i=1}^{d} \nu(u_i, u_{i+1}).
\] (2)

The \(\mu\)-invariant of a unimodular sequence \((u, v, w)\) is described as the unique integer \(a = \mu(u, v, w)\) determined by the equation

\[
\det(u, v)u + \det(v, w)w + aw = 0.
\] (3)

Together with the associated \(\nu\)-invariant \(\nu(u, v) := \det(u, v)\) this is a basic discrete angle invariant of (planar) modular lattice geometry. Higher dimensional analogues of these invariants are introduced and discussed in Section \[\text{VI}\] (see Definition \[\text{I}\]).

A possible ambiguity arises if \(L = u_1 u_2 u_3\) is a cyclic unimodular sequence. For this reason the term ‘\(\mu\)-invariant’ is reserved for the number \(\mu(u_1, u_2, u_3)\) (local \(\mu\)-invariant) while \(\mu(L) = \mu(u_1, u_2, u_3) + \mu(u_2, u_3, u_1) + \mu(u_3, u_1, u_2)\) is the corresponding global \(\mu\)-invariant.
2.3 Rotation number

Let \( P = P(a_1, \ldots, a_d) \) be a closed, oriented, polygonal curve in the plane with points \( a_i \) as vertices and \( \overline{a_i a_{i+1}} = [a_i, a_{i+1}] \) as oriented edges \( (a_{d+1} := a_1) \). If the origin \( O \) is not on \( P \) it has a rotation number (or winding number) defined by,

\[
\text{Rot}(P) = \frac{1}{2\pi} \sum_{i=1}^{d} \nu(a_i, a_{i+1}) \angle(a_i O a_{i+1})
\]

(4)

where \( \nu(a_i, a_{i+1}) \) is the sign of the determinant \( \det(a_i, a_{i+1}) \) and \( \angle(a_i O a_{i+1}) \) is the measure of the angle \( a_i O a_{i+1} \).

Given a cyclic, unimodular sequence \( L = u_1 u_2 \ldots u_d \) the associated closed unimodular polygon is \( P_L = P(u_1, u_2, \ldots, u_d) \). The rotation number \( \text{Rot}(L) \) of \( L \) is by definition the rotation number of the polygonal curve \( P_L \).

It is well known that \( \text{Rot}(P) \) can be defined with the aid of elementary homology theory. We do not need this definition here but we shall occasionally use the term unimodular cycle \( [L] \) to describe the collection \( \{\overline{u_i u_{i+1}}\}_{i=1}^{d} \) of oriented edges of \( P_L \) which may be written also as a formal sum,

\[
[L] := \overrightarrow{u_1 u_2} + \overrightarrow{u_2 u_3} + \ldots + \overrightarrow{u_{d-1} u_d} + \overrightarrow{u_d u_1}.
\]

In this context the decomposition \( [L] = [L_1] + [L_2] \), that appears in Section 4, simply indicates that \( [L] = [L_1] \cup [L_2] \) is a union of sets with signed elements (multisets) where the elements with different sign, i.e. the edges with different orientation, are supposed to cancel out. The following proposition is, in light of the equation (4), an immediate consequence.

**Proposition 2.** If \( L, L_1 \) and \( L_2 \) are cyclic, unimodular sequences such that \( [L] = [L_1] + [L_2] \) then \( \text{Rot}(L) = \text{Rot}(L_1) + \text{Rot}(L_2) \).

2.4 Modular vs. distance geometry

The formula (1) exhibits interesting similarities and differences when compared with more conventional formulas for the rotation number of a planar curve \( \Gamma \), say the formula (4) or its smooth analogue,

\[
\text{Rot}(\Gamma) = \frac{1}{2\pi} \int_{\Gamma} xdy - ydx.
\]

(5)
All these formulas are local, expressing the rotation number as a sum (integral) of quantities defined locally on the curve. All of them apply to closed curves, considering that each cyclic unimodular sequence $L$ is associated a closed unimodular polygonal line $P_L$. On closer inspection they even display an interesting analogy between the set $S^1$ of unit vectors in $\mathbb{R}^2$ and the set $\text{Prim}(\mathbb{Z}^2)$ of primitive lattice vectors. Indeed, the local quantity integrated in (5) is the (infinitesimal) angle $d\theta = d\arctan(y/x)$ where $\theta : \mathbb{R}^2 \setminus \{0\} \to S^1$ is the function $\theta(v) = \|v\|^{-1}v$, whereas a similar role of “discrete angle functions” is in formula (1) played by the functions $\mu$ and $\nu$.

It is fairly certain that formulas (1) and (5) (or (1) and (4)) cannot be directly related, however it is likely that both can be derived from more general, unifying principles. Discovering these principles appears to be an interesting research problem.

### 3 Properties of the invariant $\mu$

**Proposition 3.** Suppose that $(u, v, w)$ is an ordered triple of lattice vectors such that $(u, v)$ and $(v, w)$ are ordered bases of $\mathbb{Z}^2$. Then,

$$\mu(u, v, w) = \det(u, v)\det(v, w)\det(w, u).$$  \hspace{1cm} (6)

**Proof:** By definition $\mu(u, v, w) = a$ is the integer determined by the equation

$$\det(u, v)u + \det(v, w)w + av = 0.$$

A multiplication of both sides of this equality by $\det(u, v)\det(v, w)$ and a comparison with the equation (7) in the following lemma (Lemma 4) yields the desired formula. \hfill \Box

**Lemma 4.** Suppose that $u, v, w \in \mathbb{R}^2$. Then,

$$\det(u, v)w + \det(v, w)u + \det(w, u)v = 0 \hspace{1cm} (7)$$

Moreover, if $\text{Span}(u, v, w) = \mathbb{R}^2$ then (7) is up to a scalar factor the only linear dependence between vectors $u, v, w$.

**Corollary 5.** If $u, v, w$ is a unimodular sequence then $\mu(w, v, u) = -\mu(u, v, w)$ and $\mu(-u, -v, -w) = \mu(u, v, w)$. More generally $\mu(O(u), O(v), O(w)) = \det(O)\mu(u, v, w)$ for each $O \in GL(\mathbb{Z}, 2)$. 

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As an illustration let us check Theorem 1 for the case of cyclic unimodular sequences of length 3 and special sequences of length 4.

**Example 6.** Assume that \( L = (u, v, w) \) is a cyclic unimodular sequence. Let \( \alpha := \det(v, w), \beta := \det(w, u) \) and \( \gamma := \det(u, v) \) (Figure 1). Then \( \mu(L) = 3\alpha\beta\gamma \) (by Proposition 3) and \( \nu(L) = \alpha + \beta + \gamma \). Assume initially that \( 0 \in \conv\{u, v, w\} \) and that the sequence is positively oriented (Figure 1 (a)) which means that \( \alpha = \beta = \gamma = 1 \). Then \( \Rot(u, v, w) = 1 \) while \( \mu(L) = \nu(L) = 3 \). Similarly if \( -\alpha = \beta = \gamma = 1 \) (Figure 1 (b)) then \( \Rot(L) = 0, \mu(L) = -3 \) and \( \nu(L) = 1 \). This is sufficient to establish the formula,

\[
\Rot(L) = (1/12)\mu(L) + (1/4)\nu(L) = (1/4)(\alpha\beta\gamma + \alpha + \beta + \gamma)
\]

since both sides of the second equality are symmetric with respect to \( \alpha, \beta, \gamma \) and both change the sign if the orientation of \( L \) is reversed.

**Example 7.** Let \( L = \{a, x, b, -x\} \) be a special cyclic, unimodular sequence of length 4 (Figure 2). Observe that \( \mu(L) = 0 \) since for example \( \mu(b, -x, a) = -\mu(a, x, b) \). If \( \alpha = \det[a, x] \) and \( \beta = \det[x, b] \) then \( \nu(L) = 2(\alpha + \beta) \) and the equality \( \Rot(L) = (1/2)(\alpha + \beta) = (1/4)\nu(L) \) follows by inspection of Figure 2.

**Remark.** Suppose \( a, x, b \) is a unimodular sequence. Then \( a \pm x, x, b, a, x, b \pm x \) and \( a, -x, b \) are also unimodular and,

\[
\begin{align*}
\mu(a \pm x, x, b) &= \mu(a, x, b) \mp \det(x, b) \\
\mu(a, x, b \pm x) &= \mu(a, x, b) \mp \det(a, x)
\end{align*}
\]
3.1 Exchange lemma and the Jacobi identity

**Proposition 8.** (Exchange Lemma) Suppose that $a, x, b$ and $u, x, v$ are (not necessarily cyclic) unimodular sequences. Then the sequences $a, x, v$ and $u, x, b$ are also unimodular and

$$\mu(a, x, b) + \mu(u, x, v) = \mu(a, x, v) + \mu(u, x, b).$$

**(8)**

**Proof:** Let us write the defining equations of the corresponding $\mu$-invariants.

\[
\begin{align*}
det(a, x)a + det(x, b)b + \mu(a, x, b)x &= 0 \\
det(u, x)u + det(x, v)v + \mu(u, x, v)x &= 0 \\
det(a, x)a + det(x, v)v + \mu(a, x, v)x &= 0 \\
det(u, x)u + det(x, b)b + \mu(u, x, b)x &= 0
\end{align*}
\]

The equality (8) is obtained by subtracting the third and fourth equation from the sum of the first two equations. \qed

**Proposition 9.** (Jacobi identity) Suppose that $(a, x), (b, x), (c, x)$ are bases of the lattice $\mathbb{Z}^2$ which implies the unimodularity of $C = (a, x, b), A = (b, x, c)$ and $B = (c, x, a)$. Then,

$$\mu(a, x, b) + \mu(b, x, c) + \mu(c, x, a) = 0$$

**(9)**

$$\mu(a, x, b) = \mu(a, x, c) + \mu(c, x, b)$$

**(10)**

**Proof:** Since $\mu(z, y, x) = -\mu(x, y, z)$ (Corollary 5) the equalities (9) and (10) are equivalent so it is sufficient to prove only one of them.
The Plücker relation associated to the $2 \times 4$ matrix $M = [a \ b \ c \ x]$ is
\[
det(a, b)det(c, x) - det(a, c)det(b, x) + det(a, x)det(b, c) = 0.
\]
On multiplying both sides of this equality by $det(a, x)det(b, x)det(c, x)$ we obtain on the left-hand side the expression
\[
det(a, b)det(a, x)det(b, x) - det(a, c)det(a, x)det(c, x) + det(b, c)det(b, x)det(c, x)
\]
which by Proposition 3 immediately leads to the equation (9).

\[\Box\]

4 The additivity of \(\mu\) and \(\nu\)

In this section we describe a reduction procedure for cyclic unimodular sequences which serves as a basis for an inductive proof of the formula (1).

The general idea is to express a given sequence \(L\), or rather the associated cycle \([L]\) (Section 2.3), as a sum \(L = L_1 + L_2\) of simpler sequences such that \(\mu(L) = \mu(L_1) + \mu(L_2)\) and \(\nu(L_1) + \nu(L_2)\). Since the rotation number has a similar behavior \(Rot(L) = Rot(L_1) + Rot(L_2)\) (Proposition 2), this eventually reduces (1) to the case of cyclic, unimodular sequences of length 2 and 3 where the formula is easily verified (Example 6).

4.1 Removing self-intersections

Suppose that \(L = \ldots u_{-1}xu_+1 \ldots v_{-1}xv_+1 \ldots\) is a cyclic unimodular sequence which has a self-intersection in the sense that a primitive vector \(x\) appears twice in \(L\), Figure 3 (a). Then \(L\) allows a “shortcut decomposition” \(L = L_1 + L_2\) (more precisely \([L] = [L_1] + [L_2]\)) where \(L_1 = \ldots u_{-1}xv_+1 \ldots\) and \(L_2 = \ldots v_{-1}xu_+1 \ldots\) are subsequences of \(L\), Figure 3 (a).

Proposition 10. Suppose that a cyclic unimodular sequence \(L\) has a self-intersection (Figure 3 (a)). If \(L = L_1 + L_2\) is the associated shortcut decomposition then \(\mu(L) = \mu(L_1) + \mu(L_2)\) and \(\nu(L) = \nu(L_1) + \nu(L_2)\).

Proof: The relation for \(\nu\) is obvious while the additivity of \(\mu\) follows from the Exchange Lemma (Proposition 8). Indeed, the difference \(\mu(L) - \mu(L_1) - \mu(L_2)\) reduces to,
\[
\mu(u_{-1}, x, u_+1) + \mu(v_{-1}, x, v_+1) - \mu(u_{-1}, x, v_+1) - \mu(v_{-1}, x, u_+1) = 0.
\]
Proposition 11. Suppose that \( L = \ldots u_{-1} x_0 x_1 \ldots x_k \ldots x_1 x_0 u_{+1} \ldots \) is a cyclic unimodular sequence which has a fragment \( X \) where it goes back and forth (Figure 3 (b)). Let \( L_1 = \ldots u_{-1} x_0 u_{+1} \ldots \) be the cyclic unimodular sequence obtained from \( L \) by removing the segment \( X = x_0 x_1 \ldots x_k \ldots x_1 x_0 \). Then,

\[
\mu(L) = \mu(L_1) \quad \text{and} \quad \nu(L) = \nu(L_1).
\]

Proof: The second equality is obvious while the first follows from the anti-symmetry of the \( \mu \)-invariant (Corollary 5) and the Jacobi identity (Proposition 9). Indeed,

\[
\mu(L) - \mu(L_1) = \mu(u_{-1}, x_0, x_1) + \mu(x_1, x_0, u_{+1}) - \mu(u_{-1}, x_0, u_{+1}) = 0
\]

by an application of identity (10). \( \square \)

4.2 Removing triangles

Suppose that a cyclic unimodular sequence \( L = \ldots v_{-1} v_0 v_{+1} \ldots \) has a built-in unimodular triangle \( L_1 = v_{-1} v_0 v_{+1} \); this situation arises if \( \det(v_{-1}, v_{+1}) \in \{-1, +1\} \). Then the sequence \( L_0 \) obtained from \( L \) by deleting the vector \( v_0 \) is also a cyclic unimodular sequence and there is a homological decomposition \( L = L_0 + L_1 \) (meaning that \( [L] = [L_0] + [L_1] \)). The following proposition shows that the invariants \( \mu \) and \( \nu \) behave in the expected way.
Proposition 12. Let us assume that a cyclic unimodular sequence $L$ has three consecutive vectors $v_{-1}, v_0, v_{+1}$ such that $v_{-1}, v_{+1}$ is a basis of the lattice $\mathbb{Z}^2$. Then $L_0$ obtained from $L$ by deleting the vector $v_0$ and the three-term sequence $L_1 = (v_{-1}, v_0, v_{+1})$ are both cyclic unimodular sequences and,

$$
\mu(L) = \mu(L_0) + \mu(L_1) \quad \nu(L) = \nu(L_0) + \nu(L_1)
$$

(11)

Proof: As before the relation $\nu(L) = \nu(L_0) + \nu(L_1)$ is straightforward. The difference $\mu(L) - \mu(L_0) - \mu(L_1)$ is equal to $A - B - C$ where

$$
A = \mu(v_{-2}, v_{-1}, v_0) + \mu(v_{-1}, v_0, v_{+1}) + \mu(v_0, v_{+1}, v_{+2})
$$

$$
B = \mu(v_{-2}, v_{-1}, v_{+1}) + \mu(v_{-1}, v_{+1}, v_{+2})
$$

$$
C = \mu(v_{-1}, v_0, v_{+1}) + \mu(v_0, v_{+1}, v_{-1}) + \mu(v_{+1}, v_{-1}, v_0).
$$

All terms in $A - B - C$ cancel out as a consequence of the Jacobi identity (Proposition 9 equation (10)) applied on vertices $v_{-1}$ and $v_{+1}$ of the triangle $L_1 = v_{-1}v_0v_{+1}$ (Figure 4). \hfill \Box

4.3 Splitting cycles

Proposition 12 is a special case of a general “unimodular cycle splitting principle” which is also a consequence of the Jacobi identity and the Exchange Lemma.
Proposition 13. Suppose that a cyclic unimodular sequence

\[ L = \ldots u_{-1}x_0u_1 \ldots v_{-1}x_1v_1 \ldots \]

(as depicted in Figure 3 (c)) admits a unimodular shortcut \( X = x_0 \ldots x_1 \) giving rise to two new cyclic unimodular sequences \( L_1 = \ldots u_{-1}Xv_1 \ldots \) and \( L_2 = \ldots v_{-1}X' u_1 \ldots \) (where \( X' := x_1 \ldots x_0 \)). Then,

\[ \mu(L) = \mu(L_1) + \mu(L_2) \quad \text{and} \quad \nu(L) = \nu(L_1) + \nu(L_2). \]

Proof: The second part of the proposition (referring to the invariant \( \nu \)) is again obvious. The first equality can be proved directly, along the lines of the proof of Proposition 12, or deduced as a consequence of Propositions 10 and 11.

Indeed, if \( X = x_0Yx_1 \) (and \( X' = x_1Y'x_0 \)) then there exist decompositions \( L \cong Ax_0Bx_1, L_1 \cong AX, L_2 \cong BX' \), where \( \cong \) expresses equality of cyclic unimodular sequences ‘up to a cyclic permutation’. Then by Proposition 11 \( \mu(L) = \mu(Ax_0Yx_1Y'x_0Bx_1) \) and by Proposition 10 \( \mu(Ax_0Yx_1Y'x_0Bx_1) = \mu(L_1) + \mu(L_2) \).

\[ \square \]

5 The proof of Theorem 1

Before commencing the proof of Theorem 1 we formulate a lemma which was used in [H-M] for a similar purpose, see also [E, Section 2.5] (Exercises) where a lemma of this kind was used as a combinatorial basis for classification of smooth toric surfaces.

Lemma 14. ([H-M, Lemma 1.3]) If \( L = u_1u_2 \ldots u_d \) is a cyclic, unimodular sequence of length \( d \geq 3 \) then there exist three consecutive vectors \( u_{j-1}, u_j, u_{j+1} \) such that

\[ \mu(u_{j-1}, u_j, u_{j+1}) \in \{0, 1, -1\}. \]

Proof of Theorem 1: The proof is by induction on the length \( d \) of the cyclic, unimodular sequence \( L = u_1u_2 \ldots u_d \). In the rather trivial case \( d = 2 \), \( \mu(L) = \nu(L) = \text{Rot}(L) = 0 \). The case \( d = 3 \) is established by Example 6 so let us suppose that \( d \geq 4 \). By Lemma 14 we know that there exists an index \( j \) such that \( \mu(u_{j-1}, u_j, u_{j+1}) = a_j \in \{-1, 0, +1\} \).
Assume that $a_j \in \{-1, +1\}$. Then \(\det(u_{j-1}, u_{j+1}) \in \{-1, +1\}\) as well (see (6) in Proposition 3) and we are allowed to use Proposition 12. More explicitly, \(L\) has a decomposition \(L = L_1 + L_2\) (in the sense of Section 4) where \(L_1 = u_1 \ldots u_{j-1}u_{j+1} \ldots u_d\) and \(L_2\) is a triangular, cyclic, unimodular sequence \(L_2 = u_{j-1}u_ju_{j+1}\). By the induction hypothesis the formula (1) holds for both \(L_1\) and \(L_2\) hence, in light of the additivity of the invariants \(\mu, \nu\) and \(\text{Rot}\) (Propositions 2 and 12), it holds also for \(L\).

In the case \(a_j = 0\) we again have a (one-step) unimodular shortcut, this time from \(u_{j-1}\) to \(u_{j+2}\) (Figure 5). We again have a decomposition \(L = L_1 + L_2\) where \(L_1 = u_1 \ldots u_{j-1}u_{j+2} \ldots u_d\) and \(L_2\) is a special cyclic, unimodular sequence \(L_2 = u_{j-1}u_ju_{j+1}u_{j+2}\) of length 4. We know by Example 7 that formula (11) holds for \(L_2\) hence, again by additivity (Propositions 2 and 13) and the induction hypothesis, it holds also for \(L\).

5.1 Other proofs of Theorem 1

Here we briefly outline the history of Theorem 1 and some of its immediate predecessors. We record some of the main ideas used for its proof, illustrating the diversity of methods and the relevance of this result for different mathematical disciplines.

The original proof of A. Higashitani and M. Masuda [H-M, v2], see also [Ma, Section 5], relied on toric topology (Noether formula). This elegant proof nicely illustrates the ongoing project of modifying or replacing some of the methods of algebraic geometry (toric varieties) by the ideas of equivariant algebraic topology (topological toric manifolds).
Probably the earliest appearance of a result that can be directly linked to Theorem [1] is [F, Section 2.5]. Fulton does not formulate it as a separate statement about winding numbers, however in one of the exercises he states that the equality  

\[ a_1 + \ldots + a_d = 3d - 12 \]

follows from the assumption that \( L = u_1 u_2 \ldots u_d \) is (in our notation) a cyclic unimodular sequence such that \( \nu(u_i, u_{i+1}) = +1 \) and \( \mu(u_{i-1}, u_i, u_{i+1}) = a_i \) for each \( i \). His proof is inductive and uses a ‘topological constraint’ on \( L \) similar to Lemma [1].

B. Poonen and F. Rodriguez-Villegas in their very interesting paper [PRV] use the properties of the modular form \( \Delta(z) \) to construct a homomorphism \( \Phi : \widetilde{SL}_2(\mathbb{Z}) \to \mathbb{Z} \), where \( \widetilde{SL}_2(\mathbb{Z}) \) is an extension of the group \( SL_2(\mathbb{Z}) \). They formulate a generalization of the ‘Twelve-point theorem’ to so called ‘legal loops’ (cyclic unimodular sequences) but they stop short of formulating Theorem [1]. However, in the course of the proof they demonstrate how \( \Phi \) can be used to evaluate the associated rotation number. They also observe (Section 10) that the \( \mu \)-invariant can be interpreted as a ‘combinatorial analogue of an exterior angle’ and discuss a possible connection with the Gauss-Bonnet theorem.

The second proof of A. Higashitani and M. Masuda [H-M, v3] is closer in spirit to our proof in Section 5. This proof is also inductive, short and elementary, and may be a useful alternative for those interested in a direct, computational proof of Theorem [1].

6 Generalizations and questions

It is quite natural to ask for higher dimensional analogues of the formula (1) and other results from previous sections. As a step in this direction we introduce the concept of a unimodular map and discuss higher dimensional generalizations of invariants \( \mu \) and \( \nu \) from Sections 2 and 3.

**Definition 15.** Suppose that \( M^d \) is an oriented, triangulated, closed manifold. A map \( \phi : M^d \to \mathbb{R}^{d+1} \) is called unimodular if for each \( d \)-simplex \( (a_0, a_1, \ldots, a_d) \) in \( M^d \) the set \( \{\phi(a_0), \phi(a_1), \ldots, \phi(a_d)\} \) is a basis of the lattice \( \mathbb{Z}^{d+1} \subset \mathbb{R}^{d+1} \).

**Problem 1:** Suppose that \( M^d \) is an oriented, triangulated, closed manifold. Let \( \phi : M^d \to \mathbb{R}^{d+1} \) be a unimodular map. If \( [M] \in H_d(M^d; \mathbb{Z}) \) is the fundamental class of \( M \) then

\[ \phi_*([M]) \in H_d(\mathbb{R}^{d+1} \setminus \{0\}; \mathbb{Z}) \cong \mathbb{Z} \]
and $\phi_\ast([M]) = k\iota$ for some integer $k$ where $\iota$ is a generator of $H_d(\mathbb{R}^{d+1} \setminus \{0\}; \mathbb{Z})$. The problem is to find a formula expressing the integer $k$ in terms of locally defined quantities, analogous to (1).

There are natural candidates for the ‘locally defined quantities’ which are relatives and higher dimensional analogues of the invariant $\mu$ introduced in Section 3. Suppose that $A = \{a_1, a_2, b_1, \ldots, b_n\}$ is a collection of integer vectors such that both $A \setminus \{a_1\}$ and $A \setminus \{a_2\}$ are bases of $\mathbb{Z}^n$. Let

$$\nu_1 = [a_1; b] := \det(a_1, b_1, \ldots, b_n), \nu_2 = [a_2; b] := \det(a_2, b_1, \ldots, b_n).$$

(12)

It is not difficult to show (generalizing the formula (3) from Section 2.2) that there exists a unique sequence of integers $\mu_1, \ldots, \mu_n$ such that,

$$[a_1; b]a_1 - [a_2; b]a_2 + \mu_1 b_1 + \ldots + \mu_n b_n = 0.$$  

(13)

![Figure 6: Higher order $\mu$-invariants $\mu_j(a_1, b, a_2)$.](image)

Definition 16. The numbers $\mu_i = \mu_i(a_1, b, a_2)$, that appear in the equation (13), are referred to as the $\mu$-invariants associated to the collection $A = \{a_1, a_2, b_1, \ldots, b_n\}$.

The following proposition records some of the first, most basic properties of the $\mu$-invariants $\mu_i(a_1, b, a_2)$.

Proposition 17. Given a sequence $b = (b_1, \ldots, b_n)$ and a permutation $\pi$ of integers $1, 2, \ldots, n$ let $b^\pi$ be the sequence $(b_{\pi(1)}, \ldots, b_{\pi(n)})$. Suppose that both $(a_1, b)$ and $(a_2, b)$ are ordered bases of the lattice $\mathbb{Z}^{n+1}$. Then,

(a) $\mu_i(a_2, b, a_1) = -\mu_i(a_1, b, a_2)$;

(b) $\mu_i(a_1, b^\pi, a_2) = \text{sgn}(\pi)\mu_i(a_1, b, a_2)$.
Proof: Both equalities follow from the uniqueness of the representation (13).

The $\mu$-invariants are quite natural. They implicitly appear in many constructions involving lattice polytopes and fans, see for example the ‘Oda’s criterion’ (Theorem 4.12 in [E, Chapter V]). Together with the $\nu$-invariant $\nu(a, b) = [a; b]$ they are the obvious choice for the ‘unimodular discrete curvature invariants’ relevant for the hypothetical generalizations of Theorem 1.

This may serve as a justification for a deeper study of $\mu$-invariants and as an example we prove a generalization of Proposition 9 from Section 3.1.

Proposition 18. (Jacobi identity) Suppose that $a_1, a_2, a_3 \in \mathbb{Z}^n$. Let $b = (b_1, b_2, \ldots, b_n)$ be a sequence of integer vectors such that all three collections $(a_1, b), (a_2, b), (a_3, b)$ are bases of $\mathbb{Z}^n$. Then for each $j \in \{1, \ldots, n\}$

$$\mu_j(a_1, b, a_2) + \mu_j(a_2, b, a_3) + \mu_j(a_3, b, a_1) = 0. \quad (14)$$

Proof: By adding up the three equations (13), associated to the pairs $(a_1, a_2), (a_2, a_3), (a_3, a_1)$, one deduces the equality

$$\sum_{j=1}^{n}(\mu_j(a_1, b, a_2) + \mu_j(a_2, b, a_3) + \mu_j(a_3, b, a_1))b_j = 0 \quad (15)$$

and the result follows from the linear independence of vectors $b_1, \ldots, b_n$. □

Problem 2: Explore the higher dimensional analogues (generalizations) of results from Sections 3 and 4.

It is not difficult to prove that a unimodular sequence $L = u_1u_2\ldots u_n$ is completely determined by the initial elements $u_1$ and $u_2$ and the ‘discrete curvature invariants’ $\nu(u_i, u_{i+1}), \mu(u_i, u_{i+1}, u_{i+2})$. Moreover, one can more or less freely prescribe in advance these numbers, as made precise by the following proposition.

Proposition 19. Let $(u_1, u_2)$ be an ordered basis of the lattice $\mathbb{Z}^2$. Suppose that

$$\nu_{12}, \nu_{23}, \ldots, \nu_{n-1,n} \quad \text{and} \quad \mu_1, \mu_2, \ldots, \mu_{n-1}$$

are sequences of integers such that $\nu_{12} = \det(u_1, u_2)$, $\nu_{i,i+1} \in \{-1, +1\}$ for each $i = 1, \ldots, n-1$ and $\mu_i \in \mathbb{Z}$ for each $i = 2, \ldots, n-1$. Then there exists a unique unimodular sequence $u_1u_2\ldots u_n$, which extends the basis $(u_1, u_2)$, such that $\nu_{i,i+1} = \nu(u_i, u_{i+1}) = \det(u_i, u_{i+1})$ for each $i = 1, \ldots, n-1$ and $\mu_j = \mu(u_{j-1}, u_j, u_{j+1})$ for each $j = 2, \ldots, n-1$.  

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**Proof:** Suppose that $u_1, \ldots, u_k$ are already constructed for some $k$, where $2 \leq k \leq n - 1$. The equation (3), that defines the $\mu$-invariant, rewritten as follows

$$\nu_{k-1,k}u_{k-1} + \nu_{k,k+1}u_{k+1} + \mu_ku_k = 0,$$  \hspace{1cm} (16)

uniquely determines the vector $u_{k+1}$.

This observation shows how the sequence $(u_i)_{i=1}^n$ can be recursively constructed and it remains to check that $\nu(u_k, u_{k+1}) = \nu_{k,k+1}$ for each $k = 1, \ldots, n - 1$ and $\mu(u_{k-1}, u_k, u_{k+1}) = \mu_k$ for each $k = 2, \ldots, n - 1$.

It follows from the equality (16) that

$$\nu_{k-1,k}\nu(u_{k-1}, u_k) = \nu_{k,k+1}\nu(u_k, u_{k+1})$$

and if we already know that $\nu_{k-1,k} = \nu(u_{k-1}, u_k)$ we obtain $\nu_{k,k+1} = \nu(u_k, u_{k+1})$ as an immediate consequence. From here and the defining equation (16) we also deduce the equality $\mu_k = \mu(u_{k-1}, u_k, u_{k+1})$. □

**Problem 3:** Explore the higher dimensional generalizations of Proposition 19.

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