Holography, Cosmology  
and the  
Second Law of Thermodynamics

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Abstract  

We propose that in time dependent backgrounds the holographic principle should be replaced by the generalized second law of thermodynamics. For isotropic open and flat universes with a fixed equation of state, the generalized second law agrees with the cosmological holographic principle proposed by Fischler and Susskind. However, in more complicated spacetimes the two proposals disagree. A modified form of the holographic bound that applies to a post-inflationary universe follows from the generalized second law. However, in a spatially closed universe, or inside a black hole event horizon, there is no simple relationship that connects the area of a region to the maximum entropy it can contain.

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1 Introduction

The holographic principle proposes that the maximum number of degrees of freedom in a volume is proportional to the surface area \([1, 2]\). This principle is based on earlier studies by Bekenstein \([3]\) of maximum entropy bounds within a given volume. One argument used to motivate the holographic principle is as follows. Consider a region of space with volume \(V\), bounded by an area \(A\), which contains an entropy, \(S\), and assume that this entropy is larger than that of a black hole with the same surface area. Now throw additional energy into this region to form a black hole. Assuming that the Bekenstein-Hawking formula, \(S = A/4\), actually gives the entropy of the black hole, we conclude that the generalized second law of thermodynamics \([4]\) has been violated. To avoid this contradiction, the holographic principle proposes that the entropy inside a given region must satisfy \(S/A < 1\). However, this line of reasoning implicitly assumes that the black hole forms in an otherwise static background.

In what follows, we examine how this argument changes in more general, time-dependent, spacetimes, such as those encountered in cosmology. We argue the holographic bound is replaced by the simple requirement that physics respects the generalized second law of thermodynamics \([4]\). For static backgrounds this reduces to the holographic bound but, in general, the maximum entropy permitted inside a region is not related to its area by a simple formula.

Fischler and Susskind \([5]\) have proposed a generalization of the holographic principle to certain cosmological backgrounds. This proposal has been studied further in \([6, 8]\). For flat and open universes with time independent equations of state, we find that their bound is in accord with the generalized second law, and we propose a refinement of the Fischler-Susskind bound that applies to an inflationary universe after reheating.

Fischler and Susskind found that closed universes violate their cosmological holographic bound, and speculated that such backgrounds were either
inconsistent, or that new behavior sets in as the bound is violated. We argue that the evolution of closed universes does not violate the generalized second law, and that such backgrounds are thus self-consistent.

A related problem is the application of the holographic principle to a volume inside the event horizon of a black hole. The naïve holographic bound can easily be violated in such a region. Although this evolution respects the generalized second law, it appears that the price an observer pays for violating the holographic bound is to eventually encounter a curvature singularity. However it is possible for this fate to be delayed for cosmologically long time scales.

2 The Story So Far

Fischler and Susskind realized that while the holographic bound, \( S/A < 1 \), applies to an arbitrary region for the static case, its application to cosmological spacetimes is more subtle. Specifically, the homogeneous energy density, \( \rho \), of simple cosmological models implies a homogeneous entropy density, \( s \). Inside a (spatial) volume \( V \sim R^3 \), the total entropy is \( S = sV \), but \( S/A \sim sR \). Consequently, for a fixed \( s \) it is always possible to choose a volume large enough to violate the holographic bound. Fischler and Susskind resolve this problem by stipulating that the holographic bound only applies to regions smaller than the cosmological (particle) horizon \( \mathcal{H} \), which corresponds to the forward light-cone of an event occurring at (or infinitesimally after) the initial singularity. The comoving distance to the horizon, \( r_H \), is

\[
r_h = \int_0^{t_0} \frac{1}{a(t')} \, dt'
\]

while the corresponding physical distance is

\[
d_h = a(t)r_h = a(t) \int_0^{t_0} \frac{1}{a(t')} \, dt'.
\]
Here $a(t)$ is the scale factor of the Robertson-Walker metric, and obeys the evolution equations

\[
\left( \frac{\dot{a}}{a} \right)^2 = H^2 = \frac{\rho}{3} - \frac{k}{a^2} \tag{3}
\]

\[
\frac{\ddot{a}}{a} = -\frac{(\rho + 3p)}{6} \tag{4}
\]

where $k$ takes the values $\pm 1$ and 0, for solutions with positive, negative and zero spatial curvature.

For a perfect fluid, in a flat ($k = 0$) universe, whose pressure and density satisfy $\rho = \omega p$, the solution of equations (3) and (4) is straightforward

\[
a(t) = a_0 \left( \frac{t}{t_0} \right)^q, \quad q = \frac{2}{3} \left( 1 + \frac{1}{1 + \omega} \right). \tag{5}
\]

In particular, if $\omega = 0$ we recover the equation of state for dust, while $\omega = 1/3$ is the appropriate value for a hot (relativistic) gas or radiation. In general,

\[
dH = \frac{t}{1 - q}. \tag{6}
\]

The comoving entropy density is constant, so with $k = 0$ it follows that when measured over the horizon volume,

\[
\frac{S}{A} \propto t^{1 - 3q}. \tag{7}
\]

If $q < 1/3$ ($\omega > 1$) the holographic bound is violated at late times but, as Fischler and Susskind explain, such a cosmological model is not viable since a perfect fluid with $\omega > 1$ has a speed of sound greater than the speed of light.

In realistic cosmological models the equation of state is far from that of a perfect fluid with constant $\omega$. Even simple models of the big bang combine dust and radiation and make a transition between $\omega = 1/3$ and $\omega = 0$, since the energy density of radiation drops faster than the density of dust as the universe expands. More importantly, during an inflationary epoch in the
primordial universe $\dot{a}$ is, by definition, positive; so the pressure and $\omega$ must be negative.

One of the original motivations for inflation was that it endows the primordial universe with a substantial entropy density. Inflationary models generate entropy after inflation has finished, when energy is transferred from the scalar field which drove the inflationary expansion to radiation and ultra-relativistic particles. This process is referred to as reheating, and the equation of state usually changes from $\omega < -1/3$, to that of a radiation dominated universe whose subsequent evolution is described by the “standard” model of the hot big bang. The comoving entropy density is not constant, and $S/A$ is thus a more complicated function of time than it is in models with constant $\omega$.

The maximum temperature, $T$, attained after inflation is model dependent, and the resulting entropy density is proportional to $T^3$ only if we assume a relativistic gas. Inflation can make the cosmological horizon arbitrarily large; for instance in a class of realistic models it may be $10^{1000}$ times greater than in the absence of inflation. Applying the original Fischler-Susskind formulation of the holographic principle leads to a value of $S/A$ massively greater than unity for almost any realistic inflationary model. This difficulty is noted by Rama and Sarkar [8], and they propose various smaller volumes over which to measure the entropy. In general, their formulation is not consistent with the one we propose in the next section.

3 Holography and the Generalized Second Law

One of the initial motivations for the holographic principle was based on the generalized second law of thermodynamics. The generalized second law states

$$\delta S_{\text{mat}} + \delta S_{\text{bh}} \geq 0 ,$$

(8)
where $S_{\text{mat}}$ is the entropy of matter outside black holes, and $S_{\text{bh}}$ is the Bekenstein-Hawking entropy of the black holes. This law has not been proven, but is expected to follow from most of the current approaches to quantum gravity and has been tested in many non-trivial situations [3]. Assuming that this law is correct, the holographic principle follows from it if we consider a region of space embedded in an approximately static background (such as Minkowski space, or anti-de Sitter space), as discussed in the introduction.

Our main interest is to study the formulation of holographic style bounds in a general background. In time dependent situations, we propose that the general principle which replaces the holographic principle is simply: *that the generalized second law of thermodynamics holds.*

In the examples considered below, we assume that changes are quasi-static, so that at all times the entropy is maximized to an arbitrarily good approximation, subject to constraints. In these situations we can make the stronger statement: *for all time, the entropy is maximized subject to the constraints.*

For volumes embedded in certain backgrounds we may use these principles to deduce holographic style bounds on the entropy, but this does not appear to be possible in general. We will now illustrate these observations with a number of examples.

### 3.1 Flat Universe

Let us consider isotropic, homogeneous and spatially flat cosmologies. The comoving entropy density in these models is constant.\(^1\) In order to formulate

\(^1\)This discussion, like that of Fischler and Susskind, assumes that a flat or open universe necessarily expands indefinitely, while a closed universe recollapses in a finite time. However, a spatially flat or open universe with a negative vacuum energy density (cosmological constant) can recollapse, just as a positive vacuum energy can cause a closed universe to expand indefinitely. Our discussion can easily be generalized to these cases.
a holographic bound, we need to introduce a length scale that defines the size of the spatial region under consideration. We argue that the relevant length scale is the Hubble length $1/H$. Physically, $1/H$ is the distance at which a point appears to recede at the speed of light, due to the overall expansion of the universe.

To see that this is the relevant length scale consider a small gravitational perturbation of this background. Small perturbations to a spatially flat, homogeneous and isotropic universe with wavelengths larger than $1/H$ do not grow with time [9], provided the equation of state remains constant. If perturbations do not grow, black holes cannot form, $\delta S_{\text{bh}} = 0$, and the generalized second law reduces to $\delta S_{\text{mat}} \geq 0$, which is satisfied by any physically reasonable equation of state.

Perturbations with wavelengths shorter than the Hubble length will tend to collapse and form black holes via the Jeans instability. Thus consistency with the generalized second law suggests a holographic bound may hold for regions smaller than the Hubble volume.

If $\omega$ is constant the particle horizon, $d_H$, and $1/H$ are related to one another by a factor of order unity, and we recover the Fischler-Susskind formulation. However, if inflation has taken place the particle horizon is much larger than $1/H$, which depends only on the instantaneous expansion rate, and not on the integrated history of the universe. As we will see later, the Hubble volume is the relevant region to consider in this case.

In principle, any initial value of $S/A$ is consistent with the generalized second law. However, if $S/A < 1$ initially, this condition is satisfied at all later times, provided $\omega$ is fixed and less than unity [5]. With some additional assumptions, we can also bound $S/A$ at early times.

As an example, consider the energy density and entropy density of a relativistic gas at temperature $T$:

$$\rho = \frac{\pi^2}{30} n_s T^4,$$  \hspace{1cm} (9)
\[ s = \frac{2\pi^2}{45} n_\ast T^3, \]  

(10)

where \( n_\ast \) is the number of bosonic degrees of freedom plus \( 7/8 \) times the number of fermionic degrees of freedom. Using equation (3) to relate \( \rho \) and \( H \), we find

\[ \frac{S}{A} \leq \sqrt{n_\ast T}, \]

(11)

up to constant factors. Since the density must be less than unity for quantum gravitational corrections to be safely ignored, the maximum temperature is proportional to \( n_\ast^{-1/4} \), and the maximum value of \( S/A \) inside a Hubble volume is proportional to \( n_\ast^{1/4} \). Violating \( S/A < 1 \) significantly at a sub-Planckian energy requires an enormous value of \( n_\ast \). Thus, in the absence of fine tuning the holographic bound originally proposed by Fischler and Susskind is satisfied at all post-Planckian times.

### 3.2 Closed Universe

For an isotropic closed universe with fixed equation of state, Fischler and Susskind found [5] that even if \( S/A < 1 \) initially on particle horizon sized regions, it could be violated at later times. This violation is possible even while the universe is still in its expansion phase.

On the other hand, the generalized second law is expected to hold over regions the size of the particle horizon in a closed universe. To be definite, suppose this violation occurs while the universe is still in its expansion phase. One certainly expects that a region with an excessive entropy density could begin to collapse via the Jeans instability and form a black hole. However, if the size of this region is similar to that of the particle horizon, its collapse will necessarily take at least a Hubble time. Consequently, a violation of the holographic bound of Fischler and Susskind remains consistent with the second law for cosmologically long time-scales.

Of course, in order to find \( S/A > 1 \) well before the closed universe reaches its final singularity, we must consider a volume that is a substantial fraction
of the overall universe. The collapse of this region into a single black hole is not a small perturbation of the background Friedmann universe. Thus, the evolution equations for the unperturbed universe are not expected to accurately describe the entropy density of the collapsing region.

### 3.3 Open Universe

The behavior of isotropic open (negative spatial curvature) universes is similar to that of isotropic flat universes. If $S/A < 1$ initially, it remains so at later times [5]. An argument that $S/A < 1$ remains valid at earlier times can likewise be made in a similar way to the flat case.

### 3.4 Inside a Black Hole

Another interesting time dependent background is the region inside the event horizon of a black hole. The generalized second law will apply to a spatial volume inside the event horizon if the volume is out of thermal contact with other regions. However it is straightforward to argue that the entropy can exceed the surface area in such a volume whose size is on the order of the horizon size. Consider a large ball of gravitationally collapsing dust. The entropy of the ball is approximately constant during the course of the collapse. However, the size of the ball can contract to zero at the singularity, giving rise to a violation of the holographic bound.

We see no reason why an observer inside such a region should not be able to actually measure a violation of the holographic bound. A direct measurement is difficult since the observer will typically hit the singularity within a light-crossing time of the black hole horizon. However, if the observer has the additional information that the entropy density is constant, s/he can infer a violation of holography via local measurements.
3.5 The Inflationary Universe

We can view an inflationary universe as a Friedmann universe with a time-dependent equation of state. During the reheating phase at the end of inflation there is a sharp change in the equation of state, as energy is transferred from the inflaton field to radiation (or ultra-relativistic particles). This raises the entropy of the universe in a homogeneous way. After this sudden increase in the entropy density it is possible to violate Fischler and Susskind’s bound when it is applied to regions the size of the particle horizon. Of course, a sharp homogeneous increase in the entropy density is permitted by the generalized second law.

The process of reheating is model dependent. To simplify the discussion assume that reheating takes place instantaneously. After reheating, the equation of state need not change significantly. The post-inflationary universe closely resembles a homogeneous and isotropic universe which never inflated, with the exception that the particle horizon of the universe after inflation is much larger than that of a universe which did pass through an inflationary phase. We may therefore adopt the results for Friedmann universes with a constant equation of state.

The post-inflationary universe is accurately approximated by an isotropic Friedmann spacetime, so if $S/A < 1$ when measured over a Hubble volume at the end of inflation, this inequality will continue to be satisfied at later times. Moreover, immediately after reheating the energy density is typically well below the Planck scale so $S/A \ll 1$ in the absence of extreme fine-tuning, as discussed above. This bound differs from that of Rama and Sarkar, and we obtain no specific constraints on inflationary models beyond the usual assumption that the energy density is sub-Planckian during and after inflation.
4 Conclusions

We have proposed that the holographic principle should be replaced by the generalized second law of thermodynamics [4] in time dependent backgrounds. In static backgrounds, the generalized second law reduces to the holographic principle of ’t Hooft and Susskind [1, 2]. In cosmological backgrounds corresponding to isotropic flat and open universes with a fixed equation of state the second law implies the entropy bound of Fischler and Susskind [3] over regions the size of the particle horizon. However for closed universes, and inside black hole event horizons, a useful holographic bound cannot be deduced from the second law. Finally, we proposed a modified version of the holographic bound which applies to spatial regions of the post-inflationary universe that are smaller than the Hubble volume.

Acknowledgments

We thank Robert Brandenberger for useful comments. This work was supported in part by DOE grant DE-FE0291ER40688-Task A.

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