Provable Guarantees for Gradient-Based Meta-Learning

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Abstract

We study the problem of meta-learning through the lens of online convex optimization, developing a meta-algorithm bridging the gap between popular gradient-based meta-learning and classical regularization-based multi-task transfer methods. Our method is the first to simultaneously satisfy good sample efficiency guarantees in the convex setting, with generalization bounds that improve with task-similarity, while also being computationally scalable to modern deep learning architectures and the many-task setting. Despite its simplicity, the algorithm matches, up to a constant factor, a lower bound on the performance of any such parameter-transfer method under natural task similarity assumptions. We use experiments in both convex and deep learning settings to verify and demonstrate the applicability of our theory.

1. Introduction

The goal of meta-learning can be broadly defined as using the data of existing tasks to learn algorithms or representations that enable better or faster performance on unseen tasks. As the modern iteration of learning-to-learn (LTL) (Thrun & Pratt, 1998), research on meta-learning has been largely focused on developing new tools that can exploit the power of the latest neural architectures. Examples include the control of stochastic gradient descent (SGD) itself using a recurrent neural network (Ravi & Larochelle, 2017) and learning deep embeddings that allow simple classification methods to work well (Snell et al., 2017). A particularly simple but successful approach has been parameter-transfer via gradient-based meta-learning, which learns a meta-initialization \( \phi \) for a class of parametrized functions \( f_\theta : \mathcal{X} \rightarrow \mathcal{Y} \) such that one or a few stochastic gradient steps on a few samples from a new task suffice to learn good task-specific model parameters \( \hat{\theta} \). For example, when presented with examples \((x_i, y_i) \in \mathcal{X} \times \mathcal{Y}\) for an unseen task, the popular MAML algorithm (Finn et al., 2017) outputs

\[
\hat{\theta} = \phi - \eta \sum_i \nabla L(f_\theta(x_i), y_i) \quad (1)
\]

for loss function \( L : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}_+ \) and learning rate \( \eta > 0 \); \( \hat{\theta} \) is then used for inference on the task. Despite its simplicity, gradient-based meta-learning is a leading approach for LTL in numerous domains including vision (Li et al., 2017; Nichol et al., 2018; Kim et al., 2018), robotics (Al-Shedivat et al., 2018), and federated learning (Chen et al., 2018).

While meta-initialization is a more recent approach, methods for parameter-transfer have long been studied in the multi-task, transfer, and lifelong learning communities (Evgeniou & Pontil, 2004; Kuzborskij & Orabona, 2013; Pentina & Lampert, 2014). A common classical alternative to (1), which in modern parlance may be called meta-regularization, is to learn a good bias \( \phi \) for the following regularized empirical risk minimization (ERM) problem:

\[
\hat{\theta} = \arg \min_\theta \frac{\|\theta - \phi\|^2}{2\eta} + \sum_i L(f_\theta(x_i), y_i) \quad (2)
\]

Although there exist statistical guarantees and poly-time algorithms for learning a meta-regularization for simple models (Pentina & Lampert, 2014; Denevi et al., 2018b), such methods are impractical and do not scale to modern settings with deep neural architectures and many tasks. On the other hand, while the theoretically less-studied meta-initialization approach is often compared to meta-regularization (Finn et al., 2017), their connection is not rigorously understood.

In this work, we formalize this connection using the theory of online convex optimization (OCO) (Zinkevich, 2003), in which an intimate connection between initialization and regularization is well-understood due to the equivalence of online gradient descent (OGD) and follow-the-regularized-leader (FTRL) (Shalev-Shwartz, 2011; Hazan, 2015). In the lifelong setting of an agent solving a sequence of OCO tasks, we use this connection to analyze an algorithm that learns a \( \phi \), which can be a meta-initialization for OGD or a meta-regularization for FTRL, that the within-task regret of these algorithms improves with the similarity of the online tasks; here the similarity is measured by the distance between the optimal actions \( \theta^* \) of each task and is not known beforehand. This algorithm, which we call Follow-the-Meta-Regularized-Leader (FMRL or \texttt{Ephemeral})

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both computation and memory requirements, and in fact generalizes the gradient-based meta-learning algorithm Reptile (Nichol et al., 2018), thus providing a convex-case theoretical justification for a leading method in practice.

More specifically, we make the following contributions:

• Our first result assumes a sequence of OCO tasks $t = 1, \ldots, T$ whose optimal actions $\theta^*_t$ are inside a small subset $\Theta^*$ of the set of all possible actions. We show how Ephemeral can use these $\theta^*_t$ to make the average regret decrease in the diameter of $\Theta^*$ and do no worse on dissimilar tasks. Furthermore, we extend a lower bound of Abernethy et al. (2008) to the multi-task setting to show that Ephemeral is provably better than single-task learning and that one can do no more than a small constant-factor better sans stronger assumptions.

• Under a realistic assumption on the loss functions, we show that Ephemeral also has low-regret guarantees in the practical setting where the optimal actions $\theta^*_t$ are difficult or impossible to compute and the algorithm only has access to a statistical or numerical approximation. In particular, we show high probability regret bounds in the case when the approximation uses the gradients observed during within-task training, as is done in practice by Reptile (Nichol et al., 2018).

• We prove an online-to-batch conversion showing that task-specific parameters output by a meta-algorithm with low task-averaged regret have low generalization error, connecting our regret guarantees to statistical LTL (Baxter, 2000; Maurer, 2005).

• We verify several assumptions and implications of our theory using a new meta-learning dataset we introduce consisting of text-classification tasks solvable using convex methods. We further study the empirical suggestions of our theory in the deep learning setting.

1.1. Related Work

Gradient-Based Meta-Learning: The model-agnostic meta-learning (MAML) algorithm of Finn et al. (2017) pioneered this recent approach to LTL. A great deal of empirical work has studied and extended this approach (Li et al., 2017; Grant et al., 2018; Nichol et al., 2018; Jerfel et al., 2018); in particular, Nichol et al. (2018) develop Reptile, a simple yet equally effective first-order simplification of MAML, for which our analysis shows provable guarantees as a subcase. Theoretically, Franceschi et al. (2018) provide computational convergence guarantees for gradient-based meta-learning for strongly-convex functions, while Finn & Levine (2018) show that with infinite data MAML can approximate any function of task samples assuming a specific neural architecture as the model. In contrast to both results, we show finite-sample learning-theoretic guarantees for convex functions under a natural task-similarity assumption.

Online LTL: Learning-to-learn and multi-task learning (MTL) have both been extensively studied in the online setting, although our setting differs significantly from the one usually studied in online MTL (Abernethy et al., 2007; Dekel et al., 2007; Cavallanti et al., 2010). There, in each round an agent is told which of a fixed set of tasks the current loss belongs to, whereas our analysis is in the lifelong setting, in which tasks arrive one at a time. Here there are many theoretical results for learning useful data representations (Ruvolo & Eaton, 2013; Pentina & Lampert, 2014; Balcan et al., 2015; Alquier et al., 2017); the PAC-Bayesian result of Pentina & Lampert (2014) can also be used for regularization-based parameter transfer, which we also consider. Such methods are provable variants of practical shared-representation approaches, e.g. ProtoNets (Snell et al., 2017), but unlike our algorithms they do not scale to deep neural networks. Our work is especially related to Alquier et al. (2017), who also consider a dynamic, many-task notion of regret. We achieve similar bounds with a significantly more practical meta-algorithm, although within-task their results hold for any low-regret method whereas ours only hold for OCO.

Statistical LTL: While we focus on the online setting, our online-to-batch conversion results also imply generalization bounds for distributional meta-learning. The standard assumption of a distribution over tasks is due to Baxter (2000); Maurer (2005) further extended the hypothesis-space-learning framework to algorithm-learning. Recently, Amit & Meir (2018) showed PAC-Bayesian generalization bounds for this setting, although without implying an efficient algorithm. Closely related to our work are the regularization-based approaches of Denevi et al. (2018a;b), which provide statistical learning guarantees for Ridge regression with a meta-learned kernel or bias. Denevi et al. (2018b) is especially similar in spirit to our work in that it focuses on the usefulness of meta-learning compared to single-task learning, showing that their method is better than the $\ell_2$-regularized ERM baseline. In contrast to our work, neither work provides algorithms that scale to more complex models or addresses the connection between loss-regularization and gradient-descent-initialization.

2. Meta-Initialization & Meta-Regularization

In this paper we study simple methods of the form shown in Algorithm 1, in which we run a within-task online algorithm on each new task and then update the initialization or regularization of this algorithm using a meta-update online algorithm. Alquier et al. (2017) study a method of this form in which the meta-update is conducted using exponentially-weighted averaging. Our use of OCO for the meta-update makes this class of algorithms much more practical; for example, in the case of OGD for both the inner and outer loop we recover the Reptile algorithm of Nichol et al. (2018).
Algorithm 1: The general online-within-online algorithm we study. First-order gradient-based meta-learning uses OGD in both the inner and outer loop.

Pick a first meta-initialization \( \phi_1 \).

for task \( t \in [T] \) do
  Run a within-task online algorithm (e.g. OGD) on the
  losses of task \( t \) using initialization \( \phi_t \).
  Compute (exactly or approximately) the best fixed
  action in hindsight \( \theta_t^* \) for task \( t \).
  Update \( \phi_t \) using a meta-update online algorithm (e.g.
  OGD) on the meta-loss \( \ell_t(\phi) = \|\theta_t^* - \phi\|^2 \).
end for

In order to analyze this type of algorithm, we first discuss the OCO methods that make up both its inner and outer loop and the inherent connection they provide between initialization and regularization. We then make this connection explicit by formalizing the notion of learning a meta-initialization or meta-regularization as learning a parameterized Bregman regularizer. We conclude this section by proving convex-cone upper and lower bounds on the task-averaged regret.

### 2.1. Online Convex Optimization

In the online learning setting, at each time \( t = 1, \ldots, T \) an agent chooses action \( \theta_t \in \Theta \) and suffers loss \( \ell_t(\theta_t) \) for some adversarially chosen function \( \ell_t : \Theta \mapsto \mathbb{R} \) that subsumes the loss, model, and data in \( L(f_{\theta}(x), y) \) into one function of \( \theta \). The goal is to minimize regret – the difference between the total loss and that of the optimal fixed action:

\[
R_T = \sum_{t=1}^T \ell_t(\theta_t) - \min_{\theta \in \Theta} \sum_{t=1}^T \ell_t(\theta)
\]

When \( R_T = o(T) \) then as \( T \to \infty \) the average loss of the agent will approach that of an optimal fixed action.

For OCO, \( \ell_t \) is assumed convex and Lipschitz for all \( t \). This setting provides many practically useful algorithms such as online gradient descent (OGD). Parameterized by a starting point \( \phi \in \Theta \) and learning rate \( \eta > 0 \), OGD plays

\[
\theta_t = \text{Proj}_{\Theta}(\phi - \eta \sum_{s < t} \nabla \ell_s(\theta_s)) \tag{3}
\]

and achieves sublinear regret \( O(D \sqrt{T}) \) when \( \eta \propto \frac{D}{\sqrt{T}} \), where \( D \) is the diameter of the action space \( \Theta \).

Note the similarity between OGD and the meta-initialization update in Equation 1. In fact another fundamental OCO algorithm, follow-the-regularized-leader (FTRL), is a direct analog for the meta-regularization algorithm in Equation 2, with its action at each time being the output of \( \ell_2 \)-regularized ERM over the previous data:

\[
\theta_t = \arg\min_{\theta \in \Theta} \frac{1}{2\eta} \|\theta - \phi\|^2 + \sum_{s < t} \ell_s(\theta) \tag{4}
\]

Note that most definitions set \( \phi = 0 \). A crucial connection here is that on linear functions \( \ell_t(\cdot) = \langle \nabla \ell_t, \cdot \rangle \), OGD initialized at \( \phi = 0 \) plays the same actions \( \theta_t \in \Theta \forall t \in [T] \) as FTRL. Since linear losses are the hardest losses, in that low regret for them implies low regret for convex functions (Zinkevich, 2003), in the online setting this equivalence suggests that meta-initialization is a reasonable surrogate for meta-regularization because it is solving the hardest version of the problem. The OGD-FTRL equivalence can be extended to other geometries by replacing the squared-norm in (4) by a strongly-convex function \( R : \Theta \mapsto \mathbb{R}_+ \):

\[
\theta_t = \arg\min_{\theta \in \Theta} \frac{1}{\eta} R(\theta) + \sum_{s < t} \ell_s(\theta)
\]

In the case of linear losses this is the online mirror descent (OMD) generalization of OGD. For \( G \)-Lipschitz losses, OMD and FTRL have the following well-known regret guarantee \( \forall \theta^* \in \Theta \) (Shalev-Shwartz, 2011, Theorem 2.11):

\[
R_T \leq \frac{1}{\eta} R(\theta^*) + \eta G^2 T \tag{5}
\]

### 2.2. Task-Averaged Regret and Task Similarity

In this paper we consider the lifelong extension of the online learning setting, where \( t = 1, \ldots, T \) now index a sequence of online learning problems, in each of which the agent must sequentially choose \( m_t \) actions \( \theta_{t,i} \in \Theta \) and suffer loss \( \ell_{t,i} : \Theta \mapsto \mathbb{R} \). Since in meta-learning we are interested in doing well on individual tasks, we will aim to minimize a dynamic notion of regret in which the comparator changes with each task:

**Definition 2.1.** The **task-averaged regret** (TAR) of an online algorithm after \( T \) tasks with \( M = \{m_t\}_{t=1}^T \) steps is

\[
\bar{R}_M = \frac{1}{T} \sum_{t=1}^T \left( \frac{m_t}{\sum_{i=1}^{m_t} \ell_{t,i}(\theta_{t,i})} - \min_{\theta \in \Theta} \sum_{i=1}^{m_t} \ell_{t,i}(\theta_t) \right)
\]

As the comparator in this regret is dynamic, without very strong assumptions one cannot hope to achieve TAR decreasing in \( T \). A seeming remedy for this issue in our parameter-transfer setting is to subtract from TAR a “meta-comparator” that uses the optimal meta-initialization or meta-regularization in hindsight but with the same within-task algorithm. However, to prove regret sublinear in \( T \) using this approach, one has to use low-regret algorithms with tight constants on their upper and lower bounds, as otherwise the agent will always suffer an \( \Omega(\sqrt{m}) \)-worse loss on each task. Such tight bounds are known for very few algorithms (Abernethy et al., 2008). Our study of TAR is thus motivated by an interest in understanding average-case regret, as well as our derivation of an online-to-batch conversion for generalization bounds on distributional LTL. Note also that TAR is similar to the compound regret studied by Alquier et al. (2017), although they also compete with the best representation in hindsight.
Figure 1. Random projection of ERM parameters of 1-shot (left) and 32-shot (right) Mini-Wikipedia tasks, described in Section 4.

We now formalize our similarity assumption on the tasks \( t \in [T] \): their optimal actions \( \theta^*_t \) lie within a small subset \( \Theta^* \) of the action space. This is natural for studying gradient-based meta-learning, as the notion that there exists a meta-parameter \( \phi \) from which a good parameter for any individual task is reachable with only a few steps implies that they are all close together. We develop algorithms whose TAR scales with the diameter \( D^* \) of \( \Theta^* \); notably, this means they will not do much worse if \( \Theta^* = \Theta \), i.e. if the tasks are not related in this way, but will do well if \( D^* \ll D \). Importantly, our methods will not require knowledge of \( \Theta^* \).

**Assumption 2.1.** Assume each task \( t \in [T] \) consists of \( m_t \) convex \( G \)-Lipschitz loss functions \( \ell_{t,i} : \Theta \mapsto \mathbb{R} \) and let \( \theta^*_t \in \arg \min_{\theta \in \Theta} \sum_{i=1}^{m_t} \ell_{t,i}(\theta) \) be the minimum-norm optimal action in hindsight for task \( t \). Define \( \Theta^* \subset \Theta \) to be the minimal subset containing all \( \theta^*_t \forall t \in [T] \).

Note \( \theta^*_t \) is unique as the minimum of \( \| \cdot \|^2 \), a strongly convex function, over minima of a convex function. The algorithms in Section 2.4 assume an efficient oracle computing \( \theta^*_t \).

### 2.3. Parameterizing Bregman Regularizers

Following the main idea of gradient-based meta-learning, our goal is to learn a \( \phi \in \Theta \) such that an online algorithm such as OGD starting from \( \phi \) will have low regret. We thus treat regret as our objective and observe that in the regret of FTRL (5), the regularizer \( R \) effectively encodes a distance from the initialization to \( \phi^* \). This is clear in the Euclidean geometry for \( R(\theta) = \frac{1}{2} \| \theta - \phi \|^2 \), but can be extended via the Bregman divergence (Bregman, 1967), defined for \( f : S \mapsto \mathbb{R} \) everywhere-sub-differentiable and convex as

\[
B_f(x||y) = f(x) - f(y) - \langle \nabla f(y), x - y \rangle
\]

The Bregman divergence has many useful properties (Banerjee et al., 2005) that allow us to use it almost directly as a parameterized regularization function. However, in order to use OCO for the meta-update we also require it to be strictly convex in the second argument, a property that holds for the Bregman divergence of both the \( \ell_2 \) regularizer and the entropic regularizer \( R(\theta) = \langle \theta, \log \theta \rangle \) used for online learning over the probability simplex, e.g. with expert advice.

**Definition 2.2.** Let \( R : S \mapsto \mathbb{R} \) be 1-strongly-convex w.r.t. norm \( \| \cdot \| \) on convex \( S \subset \mathbb{R}^d \). Then we call the Bregman divergence \( B_R(x||y) : S \times S \mapsto \mathbb{R}^+ \) a Bregman regularizer if \( B_R(x||\cdot) \) is strictly convex for any fixed \( x \in S \).

Within each task, the regularizer is parameterized by the second argument and acts on the first. More specifically, for \( R = \frac{1}{2} \| \cdot \|^2 \) we have \( B_R(\theta||\phi) = \frac{1}{2} \| \theta - \phi \|^2 \), and so in the case of FTRL and OGD, \( \phi \) is a parameterization of the regularizer and the initialization, respectively. In the case of the entropic regularizer, the associated Bregman regularizer is the KL-divergence from \( \phi \) to \( \theta \) and thus meta-learning \( \phi \) can very explicitly be seen as learning a prior.

Finally, we use Bregman regularizers to formally define our parameterized learning algorithms:

**Definition 2.3.** FTRL\( _{\eta,\phi} \), for \( \eta \in \mathbb{R}^+ \), \( \phi \in \Theta \), where \( \Theta \) is some bounded convex subset \( \Theta \subset \mathbb{R}^d \), plays

\[
\theta_t = \arg \min_{\theta \in \Theta} B_R(\theta||\phi) + \eta \sum_{s<t} \ell_s(\theta)
\]

for Bregman regularizer \( B_R \). Similarly, OMD\( _{\eta,\phi} \) plays

\[
\theta_t = \arg \min_{\theta \in \Theta} B_R(\theta||\phi) + \eta \sum_{s<t} \langle \nabla \ell_s, \theta \rangle
\]

Here FTRL and OMD correspond to the meta-regularization (2) and meta-initialization (1) approaches, respectively. As \( B_R(\cdot||\phi) \) is strongly-convex, both algorithms have the same regret bound (5), allowing us to analyze them jointly.

### 2.4. Follow-the-Meta-Regularized-Leader

We now specify the first variant of our main algorithm, Follow-the-Meta-Regularized-Leader (Ephemeral ). In the case where the diameter \( D^* \) of \( \Theta^* \), as measured by the square root of the maximum Bregman divergence between any two points, is known. Starting with \( \phi_1 \in \Theta \), run FTRL\( _{\eta,\phi} \) or OMD\( _{\eta,\phi} \), with \( \eta \propto \frac{L^2}{\sqrt{m}} \), on the losses in each task. After each task, compute \( \phi_{t+1} \) using an OCO meta-update algorithm operating on the Bregman divergences \( B_R(\theta^*_t||\cdot) \). For \( D^* \) unknown, make an underestimate \( \bar{D} \) and multiply it by a factor \( \gamma > 1 \) each time \( B_R(\theta^*_t||\phi_t) > \bar{D}^2 \).

The following is a regret bound for this algorithm when the meta-update is either Follow-the-Leader (FTL), which plays the minimizer of all past losses, or OGD with adaptive step size. We call this Ephemeral variant Follow-the-Average-Leader (FAL) because in the case of FTL the algorithm uses the mean of the previous optimal parameters in hindsight as the initialization. Pseudo-code for this and other variants is given in Algorithm 2. For brevity, we state results for \( m_t = m \forall t \); detailed statements are in the supplement.

**Theorem 2.1.** Under Assumption 2.1, the FAL variant of Algorithm 2 with task similarity guess \( \bar{D} < D^* \), tuning parameter \( \gamma \), and Bregman regularizer \( B_R \) that is Lipschitz on \( \Theta^* \) achieves task-averaged regret

\[
\bar{R} \leq \left( G \left( \gamma D^* + \bar{D} \right) + \mathcal{O} \left( \frac{\log T}{T} \right) \right) \sqrt{m}
\]

where \( D^* = \max_{\theta,\phi \in \Theta^*} \sqrt{B_R(\theta||\phi)} \) the diameter of \( \Theta^* \) and \( \bar{D} = \min_{\phi \in \Theta} \frac{1}{D^* \gamma T} \sum_{t=1}^{T} B_R(\theta^*_t||\phi) \leq D^* \).
Proof Sketch. We give a proof for $R(\cdot) = \frac{1}{2} \| \cdot \|_2^2$ and known task similarity, i.e. $\tilde{D} = D^\ast, \gamma = 1$. A full proof is in the supplement. Use $\Delta_t(\phi) = B_R(\theta^*_t || \phi) = \frac{1}{2} \| \theta^*_t - \phi \|_2^2$ to denote the divergence to $\theta^*_t$ and let $\phi^* = \frac{1}{T} \sum_{t=1}^T \theta^*_t$. Note $\Delta_t$ is strongly-convex $\forall t \in [T]$ and $\phi^*$ is the minimizer of their sum, with average distance $\bar{D} = \frac{1}{1+T} \sum_{t=1}^T \Delta_t(\phi^*)$.

We can then expand Definition 2.1 for task-averaged regret:

$$
\bar{R} = \frac{1}{T} \sum_{t=1}^T \left( \sum_{i=1}^m (\hat{\ell}_{t,i}(\theta_{t,i}) - \min_{\theta_t \in \Theta} \sum_{i=1}^m \ell_{t,i}(\theta_t)) \right)
$$

$$
\leq \frac{1}{T} \sum_{t=1}^T \Delta_t(\phi_t) + nG^2m
$$

$$
= \frac{1}{T} \sum_{t=1}^T \Delta_t(\phi^*) + \frac{1}{T} \sum_{t=1}^T \Delta_t(\phi^*) + \frac{1}{T} \eta G^2m
$$

The first two lines just substitute the regret bound (5) of FTRL and OMD. The key step is the last one, where the regret is split into the left-hand loss of the meta-update algorithm and the right-hand loss of the loss incurred if we had always initialized at the mean $\phi^*$ of the optimal actions $\theta^*_t$.

Since $\Delta_1, \ldots, \Delta_T$ is a sequence of strongly-convex functions with minimizer $\phi^*$, and since each $\phi_t$ is determined by playing FTL or OGD on these same functions, the left-hand term is exactly the regret of these algorithms on strongly-convex functions, which is known to be $O(\log T)$ (Bartlett et al., 2008; Kakade & Shalev-Shwartz, 2008). Substituting $\eta = \frac{\sqrt{D^*}}{\sqrt{m}}$ and the definition of $\phi^*$ sets the right-hand side to

$$
\bar{R} \leq \frac{1}{T} \sum_{t=1}^T \Delta_t(\phi^*) + G \sqrt{D^*} \sqrt{m} + G \sqrt{D^*} \sqrt{m}
$$

Remark 2.1. Note that if we know the standard deviation $\tilde{D} = \frac{1}{T} \sum_{t=1}^T B_R(\theta^*_t || \phi)$ of the task parameters from their mean $\phi$, setting the learning rate $\eta_t = \frac{\tilde{D}}{\sqrt{m}}$ in Algorithm 2 and following the same analysis as above will give task-averaged regret $R \leq 2G \tilde{D} \sqrt{m}$, which is at least as good as the bound above since $\tilde{D} \leq D^*$ and is less sensitive to possible outlier tasks.

Theorem 2.1 shows that, so long as the similarity guess $\tilde{D}$ is not too large, the task-averaged regret of Ephemerol will scale with the task similarity $D^*$. The $\tilde{D}$ component shows that this bound improves if the $\theta^*_t$ are close on average; in the $\ell_2$-case we have $\tilde{D} \leq \frac{1}{2} D^*$. Furthermore, if $\Theta^* = \Theta$, i.e. if the tasks are not similar, then the algorithm will only do a constant factor worse than FTRL or OMD; this is similar to other “optimistic” methods that work well under regularity (Rakhlin & Sridharan, 2013; Jadabave et al., 2015). These results show that gradient-based meta-learning is useful in convex settings: under a simple notion of task similarity, using multiple tasks leads to better performance than the $O(\sqrt{m})$ regret of running the same algorithm in a single-
task setting. Furthermore, the algorithm scales well in terms of computation and memory requirements, and in the $\ell_2$ setting is very similar to Reptile (Nichol et al., 2018).

However, it is easy to see that an even simpler “strawman” algorithm achieves regret only a constant factor worse than Ephemeral: at time $t + 1$, simply initialize FTRL or OMD using the optimal parameter $\theta^*_t$ of task $t$. Of course, since such algorithms are often used in the few-shot setting of small $m$, a reduction in the average regret is practically significant; we observe this empirically in Figure 3. Indeed, in the proof of Theorem 2.1 the regret converges to the regret bound obtained by always playing the mean of the optimal actions if we somehow knew it beforehand, which will not occur when playing the strawman algorithm. Furthermore, the following lower-bound on the task-averaged regret, a multi-task extension of Abernethy et al. (2008, Theorem 4.2), shows that such constant factor reductions are the best we can achieve under our task similarity assumption:

**Corollary 2.1.** Assume $d \geq 3$ and that for each $t \in [T]$ an adversary must play a sequence of $m$ convex $G$-Lipschitz functions $\ell_{t,i} : \Theta \to \mathbb{R}$ whose optimal actions in hindsight $\arg\min_{\theta \in \Theta} \sum_{i=1}^m \ell_{t,i}(\theta)$ are contained in some fixed $\ell_2$-ball $\Theta^* \subset \Theta$ with center $\phi^*$ and diameter $D^*$. Then the adversary can force the agent to have TAR at least $\frac{GD^*}{\sqrt{m}}$.

More broadly, this lower bound shows that the learning-theoretic benefits of gradient-based meta-learning are inherently limited without stronger assumptions on the tasks. Nevertheless, Ephemeral-style algorithms are very attractive from a practical perspective, as their memory and computation requirements per iteration scale linearly in the dimension and not all at the number of tasks.

### 3. Provable Guarantees for Practical Gradient-Based Meta-Learning

In the previous section we showed that an algorithm with access to the best actions in hindsight of each task could learn a good meta-initialization or meta-regularization. In practice we may wish to be more computationally efficient and use a simpler-to-compute quantity for the meta-update. In addition, in the i.i.d. case few-shot ERM may not be a good task representation and a task similarity assumption on the true risk minimizers may be more relevant. In this section we first show how two simple variants of Ephemeral handle these settings. Finally, we also provide an online-to-batch conversion result for task-averaged regret that implies good generalization guarantees when any of the variants of Ephemeral are run in the distributional LTL setting.

#### 3.1. Simple-to-Compute Meta-Updates

The FAL variant of Ephemeral uses each task’s minimum-norm optimal action in hindsight $\theta^*_t$ to perform a meta-update. While $\theta^*_t$ is efficiently computable in some cases, in most cases it is more efficient and practical to use an estimate instead. This is especially true when applying these methods in the deep learning setting; for example, Nichol et al. (2018) find that taking the average within-task gradient works well. Furthermore, in the batch setting, when each task $t$ consists of $m_t$ i.i.d. samples drawn from an adversarially chosen distribution, a more natural notion of task similarity would depend on the true risk minimizer of each task, of which $\theta^*_t$ is just an estimate. We thus extend the results of Section 2.4 to handle these considerations by proving regret bounds for two variants of Ephemeral: one for the adversarial setting which uses the final action on task $t$ as the meta-update, and one for the stochastic setting which uses the average iterate. We call these methods FLI-Online and FLI-Batch, respectively, where FLI stands for Follow-the-Last-Iterate.

However, to achieve these guarantees we need to make some assumptions on the within-task loss functions. This is unavoidable because we need estimates of the optimal actions of different tasks to be nearby; in general, for some $\theta \in \Theta$ a convex function $f : \Theta \to \mathbb{R}$ can have small $f(\theta) - f(\theta^*)$ but large $\|\theta - \theta^*\|$ if $f$ does not increase quickly away from the minimum. This makes it impossible to use guarantees on the loss of an estimate of $\theta^*_t$ to bound its distance from $\theta^*_t$. We therefore make assumptions that some aggregate loss, e.g. the expectation or sum of the within-task losses, satisfies the following growth condition:

**Definition 3.1.** A function $f : \Theta \to \mathbb{R}$ has $\alpha$-quadratic-growth $(\alpha\text{-QG})$ w.r.t. $\|\|_{\theta \in \Theta}$ for $\alpha > 0$ if for any $\theta \in \Theta$ and $\theta^*$ its closest minimum of $f$ we have

$$\frac{\alpha}{2} \|\theta - \theta^*\|^2 \leq f(\theta) - f(\theta^*)$$

QG has recently been used to provide fast rates for both offline and online GD that hold for practical problems such as LASSO and logistic regression under data-dependent assumptions (Karimi et al., 2016; Garber, 2019). It can be shown to hold for $f(\theta) = g(A\theta)$ for $g$ strongly-convex and some $A \in \mathbb{R}^{m \times d}$; in this case $\alpha \geq \sigma_{\min}(A)$ (Karimi et al., 2016). Note that $\alpha$-QG will also be satisfied when $f$ itself is strongly-convex, making the former a weaker condition.

We start with FLI-Online; as shown in Algorithm 2, this variant is the same as FAL except that the meta-update is performed using the last action of FTRL, i.e. the regularized empirical risk minimizer. To provide regret guarantees in this setting, we stipulate that the average loss is $\alpha$-growing and strengthen the task similarity notion slightly:

**Assumption 3.1.** Let each task $t = 1, \ldots, T$ consist of $m_t$ convex $G$-Lipschitz loss functions $\ell_{t,i} : \Theta \to \mathbb{R}$ s.t. the total loss $L_t(\theta) = \sum_{i=1}^{m_t} \ell_{t,i}(\theta)$ is $\alpha\text{-QG}$ w.r.t. $\|\|$. Define $\Theta^* \subset \Theta$ s.t. $\Theta^* \supset \arg\min_{\theta \in \Theta} L_t(\theta) \forall t \in [T]$.

In contrast to Assumption 2.1, we require $\Theta^*$ to contain all
optimal actions and not only the one with minimal norm. Furthermore, we require that the growth factor is $\Omega(m)$. While this is a stronger requirement than usually assumed, in Figure 2 we show that it holds in certain real and synthetic settings. Note that this growth factor will always hold in the case of the losses themselves being $\alpha$-strongly-convex.

Under such data-dependent assumptions, and if the within-task algorithm is FTRL, we have the following bound:

**Theorem 3.1.** Under Assumption 3.1, the FLI-Online variant of Algorithm 2 with task similarity guess $\tilde{D} < D^*$, tuning parameter $\gamma$, and within-task algorithm FTRL with Lipschitz Bregman regularizer $B_R$ for $R$ strongly-smooth w.r.t. $\| \cdot \|$ achieves task-averaged regret

$$\bar{R} \leq \left( G (\gamma D^* + \tilde{D} + o_m(1)) + O \left( \frac{\log T}{T} \right) \right) \sqrt{m}$$

with $D^*, \tilde{D}$ as in Theorem 2.1 and $o_m(1) = O \left( \frac{\sqrt{\frac{1}{\pi m}} \log \frac{T}{\pi}} \right)$.

Note that above regret is very similar to that in Theorem 2.1 apart from a per-task error term decreasing in $m$ that is due to the use of an estimate of $\theta^*_t$.

We now turn to the FLI-Batch algorithm, which uses each task’s average action for the meta-update. To give a regret bound here, we assume that at each task $t$, an adversary picks a distribution over loss functions, from which $m_t$ samples are drawn i.i.d. This follows the batch-within-online setting of Alquier et al. (2017). We thus use the distance between the true-optimizers for the task similarity assumption:

**Assumption 3.2.** Let each task $t = 1, \ldots, T$ consist of $m_t$ convex $G$-Lipschitz loss functions $\ell_{t,i} : \Theta \rightarrow [0, 1]$ sampled i.i.d. from distribution $P_t$ s.t. the expected total loss $E_{P_t^{m_t}} L_t(\theta) = \sum_{i=1}^{m_t} E_{\ell_{t,i}} \ell(\theta)$ is $\alpha$-QG w.r.t. $\| \cdot \|$. Define $\Theta^* \subset \Theta$ s.t. $\star^* \Rightarrow \min_{\theta \in \Theta} \mathbb{E}_{P_t^{m_t}} L_t(\theta) \forall t \in [T]$.

We can show a high probability bound on the task-averaged regret assuming strongly-smooth regularization:

**Theorem 3.2.** Under Assumption 3.2, the FLI-Batch variant of Algorithm 2 with task similarity guess $\tilde{D} < D^*$, tuning parameter $\gamma$, and Lipschitz Bregman regularizer $B_R$ for $R$ strongly-smooth w.r.t. $\| \cdot \|$ achieves TAR

$$\bar{R} \leq \left( G (\gamma D^* + \tilde{D} + o_m(1)) + O \left( \frac{\log T}{T} \right) \right) \sqrt{m}$$

w.p. $1 - \delta$, with $D^*, \tilde{D}$ as in Theorem 2.1 and the error term $o_m(1) = O \left( \frac{\sqrt{\frac{1}{\pi m}} \log \frac{T}{\pi}} \right)$.

### 3.2. Distributional Learning-to-Learn

While gradient-based LTL algorithms are largely online, their goals are often statistical. Here we review distributional LTL and prove an online-to-batch conversion showing that low TAR implies low risk for within-task learning.

As formulated by Baxter (2000), distributional LTL assumes a distribution $Q$ over task distributions $P$ over $\mathcal{X} \times \mathcal{Y}$. Given $m$ i.i.d. data samples $(x_{t,i}, y_{t,i}) \sim P_t$ from $T$ i.i.d. task samples $P_t \sim Q$, we seek to do well in expectation when new samples $(x_{t,i}, y_{t,i}) \sim P$ are drawn from a new distribution $P \sim Q$ and we must learn how to predict $y$ given $x$ for $(x, y) \sim P$. This models a setting with not enough data to learn $P$ on its own, i.e. $m$ is small, but where tasks are somehow related and thus we can use samples from $Q$ to reduce the number of samples needed from $P$. Parameter-transfer LTL lies within the algorithm-learning framework of Maurer (2005), where tasks samples are used to learn a learning algorithm $A_\phi : (\mathcal{X} \times \mathcal{Y})^m \rightarrow \Theta$ parameterized by $\phi \in \Phi$ that takes $m$ data points and returns a prediction algorithm $f_\theta : \mathcal{X} \rightarrow \mathcal{Y}$ parameterized by $\theta \in \Theta$. Theorem 3.3 bounds the within-task expected risk under task-averaged regret guarantees for any task sample from the same distribution $Q$. For Ephemeral, the procedure picks a task $t \in [T]$ uniformly at random, runs FTRL$_{\eta,\phi_t}$, or OMD$_{\eta,\phi_t}$ on samples from $P \sim Q$, and outputs the average iterate as the learned parameter. Note that guarantees on randomly or mean iterates are standard, although in practice we use $t = T$ and the last iterate as the learned parameter.

**Theorem 3.3.** Suppose convex losses $\ell_{t,i} : \Theta \rightarrow [0, 1]$ are generated by sampling i.i.d. $P_t \sim Q, \{\ell_{t,i} \}_{t,i} \sim P_{t}^{m}$ for some distribution $Q$ over task distributions $P_t$. Let $\hat{A}$ be the state before some task $t \in [T]$ picked uniformly at random of algorithm $\hat{A}$ with task-averaged regret $\hat{R}$. Then if $m$ new loss functions $\{\ell_{t,i} \}_{t,i} \sim P_{t}^{m}$ are sampled from a new task distribution $P \sim Q$, running $\hat{A}$ on these losses will generate $\theta_1, \ldots, \theta_m \in \Theta$ s.t. w.p. $1 - \delta$ their mean $\bar{\theta}$ satisfies

$$\mathbb{E}_{\mathbb{P}_{t}^{P \sim Q}} \mathbb{E}_{\mathbb{P}_{t}^{P \sim Q}} \ell(\bar{\theta}) = \hat{O} \left( \mathbb{E} + \frac{1}{m} \log \frac{1}{\delta} + \sqrt{\frac{\mathbb{E}}{m} \log \frac{1}{\delta}} \right)$$

where the outer expectation is over sampling $\hat{A}$ and

$$\mathbb{E} = \left( \mathbb{E}_{\mathbb{P}_{t}^{P \sim Q}} \min_{\theta \in \Theta} \frac{1}{m} \sum_{i=1}^{m} \ell_i(\theta) \right) + \frac{\hat{R}}{m} + \sqrt{\frac{1}{T} \log \frac{1}{\delta}}$$

![Plot of the smallest $L(\theta) - L(\theta^*)$ as $\|\theta - \theta^*\|_2$ increases for logistic regression over a mixture of four 50-dimensional Gaussians (left) and over a four-class text classification task over 50-dimensional CBOW (right). For both the $\alpha$ factor of the quadratic-growth condition scales linearly with the number of samples $m$.](image)
Figure 3. TAR of Ephemeral and the strawman method for FTRL (left) and of variants of Ephemeral for OGD (right). Ephemeral is much better than the strawman at low $m$, showing the significance of Theorem 2.1 in the few-shot case. As predicted by Theorems 3.1 and 3.2, FLI regret converges to that of FAL as $m$ increases.

The result follows by nesting two different single-task online-to-batch conversions (Cesa-Bianchi et al., 2004; Cesa-Bianchi & Gentile, 2005) via Jensen’s inequality. Note that the first term is the expected empirical risk of the ERM, which is small in many practical settings, such as for linear models over non-atomic distributions with $m < d$. Thus, apart from the fast-decaying $\frac{1}{m} \log \frac{1}{\delta}$ term, proving a low task-averaged regret multiplicatively improves the bound as $T \to \infty$ on the risk of a new task sampled from $Q$.

4. Empirical Results

A major benefit of Ephemeral is its practicality. In particular, the FLI-Batch variant is scalable without modification to high-dimensional, non-convex models. Here its practical effectiveness is evidenced by the success of first-order MAML and similar algorithms, as our method is a generalization of Reptile (Nichol et al., 2018), which performs slightly worse than MAML on the Omniglot benchmark (Lake et al., 2017) but better on the harder Mini-ImageNet benchmark (Ravi & Larochelle, 2017). With this evidence, empirically our main goal is to validate our theory in the convex setting, although we also examine implications for deep meta-learning.

4.1. Convex Setting

We introduce a new dataset of 812 classification tasks, each consisting of sentences from one of four Wikipedia pages which we use as labels. We call this dataset Mini-Wikipedia. Our use of text classification to examine the convex setting is motivated by the well-known effectiveness of linear models over simple representations (Wang & Manning, 2012; Arora et al., 2018). We use logistic regression over continuous-bag-of-words (CBOW) vectors built using 50-dimensional GloVe embeddings (Pennington et al., 2014). The similarity of these tasks is verified by seeing if their optimal parameters are close together. As shown before in Figure 1, we find when $\Theta$ is the unit ball that even in the 1-shot setting the tasks have non-vacuous similarity; for 32-shots the parameters are contained in a set of radius 0.32.

We next compare Ephemeral to the “strawman” algorithm from Section 2, which simply uses the previous optimal action as the initialization. For both algorithms we use task similarity guess $\hat{D} = 0.1$ and tuning parameter $\gamma = 1.1$. As expected, we see in Figure 3 that the improvement of Ephemeral over the strawman is especially prominent for few-shot learning, showing that the theoretical task-similarity-based improvement we achieve is practically significant when individual tasks have very few samples. We also see that FLI-Batch, which uses an estimate of the best parameter for the meta-update, approaches the performance of FAL as the number of spaces increases and thus its estimate improves.

Finally, we evaluate the performance of Ephemeral and (first-order) MAML in the distributional setting on this NLP task. On each task we standardize data using the mean and deviation of the training features. For Ephemeral we use the FAL variant with OGD as the within-task algorithm, with learning rate set using the average deviation of the optimal task parameters from the mean optimal parameter, as suggested in Remark 2.1. For MAML, we use a hyperparameter sweep to determine the within-task and meta-update learning rates; for our algorithm, we simply use the root average squared distance of all tasks in hindsight, which from Theorem 2.1 can be seen to be minimizing the upper bound on the within-task regret. As shown in Figure 4, even though Ephemeral does not require any learning-rate tuning, unlike the MAML procedure, the algorithm performs comparably – slightly better for $m \geq 8$ and slightly worse for $m < 4$.

4.2. Deep Learning

While our algorithm generalizes Reptile, an already-effective gradient-based meta-learning algorithm (Nichol et al., 2018), we can still see if improvements suggested by our theory help for neural network LTL. To this end we study controlled modifications to the settings used in the Reptile experiments on 5-way and 20-way Omniglot (Lake et al., 2017) and 5-way Mini-ImageNet classification (Ravi & Larochelle, 2017). For both datasets, we use the same
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Figure 5. Performance of Reptile (the FLI variant of Ephemeral using OGD within-task) on 5-shot 5-way Mini-ImageNet (left), 1-shot 5-way Omniglot (center), and 5-shot 20-way Omniglot (right) while varying the number of training samples. Increasing the number of samples per training task improves performance even when using the same number of samples at meta-test time.

Figure 6. Performance of Reptile (the FLI variant of Ephemeral using OGD within-task) on 5-shot 5-way Mini-ImageNet (left), 1-shot 5-way Omniglot (center), and 5-shot 20-way Omniglot (right) while varying the number of training iterations. The benefit of more iterations is not clear for Mini-ImageNet, but an improvement is seen on Omniglot. The number of iterations at meta-test time is 50.

Our theoretical results point to the importance of accurately computing the within-task parameter before the meta-update; Theorem 2.1 assumes access to the optimal parameter in hindsight, whereas Theorems 3.1 and 3.2 allow computational and stochastic approximations that result in an additional error term decaying with $m$, the number of within-task examples. This becomes relevant in the non-convex setting with many thousands of tasks, where it can be infeasible to find even a local optimum.

The theory thus suggests that using a better estimate of the within-task parameter for the meta-update may lead to lower regret, and thus lower generalization error. We can attain a better estimate by using more samples on each task, to reduce stochastic noise, or by running more gradient steps on each task, to reduce approximation error. It is not obvious that these changes will improve performance – it may be better to learn a few-shot learning algorithm using the same settings at meta-train and meta-test time. However, in practice the Reptile authors use more task samples – 10 for Omniglot and 15 for Mini-ImageNet – at meta-train time than the number of shots – at most 5 – used for evaluation. On the other hand, they use far fewer within-task gradient steps – 5 for Omniglot and 8 for Mini-ImageNet – at meta-train time than the 50 iterations used for evaluation.

We study how varying these two settings – the number of task samples and the number of within-task iterations – changes performance at meta-test time. In Figure 5, we see that more within-task samples provides a significant improvement in performance for Mini-ImageNet and Omniglot, with many fewer meta-iterations needed to reach good test performance. Reducing the number of meta-iterations is important in practice as it corresponds to fewer tasks needed for training, although for a better stochastic approximation each task needs more samples. On the other hand, increasing the number of training iterations does not need more samples, and we see in Figure 6 that raising this value can also lead to better performance, especially on 20-way Omniglot, although the effect is less clear for Mini-ImageNet, with the use of more than 8 training iterations reducing performance. The latter result is likely due to over-fitting on specific tasks, with task similarity in this stochastic setting likely holding for the true rather than empirical risk minimizers, as in Assumption 3.2. The broad patterns shown above also hold for several other parameter settings, which we depict in greater detail in the supplement.
5. Conclusion
In this paper we undertook a study of a broad class of gradient-based meta-learning methods using the theory of online convex optimization. Our results show the usefulness of running such methods compared to single-task learning under the assumption that individual task parameters are close together. The fully online guarantees of our meta-algorithm, Ephemeral, can be extended to practically relevant approximate meta-updates, the batch-within-online setting, and distributional LTL.

Apart from these results, the simplicity of Ephemeral makes it extensible to various settings of practical interest in meta-learning, such as for federated learning and differential privacy. In the theoretical direction, future work can consider more sophisticated notions of task-similarity, such as for multi-modal or continuously-evolving settings. While we have studied only the parameter-transfer setting, deriving statistical or low-regret guarantees for practical and scalable representation-learning remains an important research goal.

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A. Background and Results for Online Convex Optimization

Throughout the appendix we assume all subsets are convex and in \( \mathbb{R}^d \) unless explicitly stated. Let \( \| \cdot \| \) be the dual norm of \( \| \cdot \|_* \), which we assume to be any norm on \( \mathbb{R}^d \), and note that the dual norm of \( \| \cdot \|_2 \) is itself. For sequences of scalars \( \sigma_1, \ldots, \sigma_T \in \mathbb{R} \) we will use the notation \( \sigma_{1:t} \) to refer to the sum of the first \( t \) of them. In the online learning setting, we will use the shorthand \( \nabla_i \) to denote the subgradient of \( \ell_i : \Theta \rightarrow \mathbb{R} \) evaluated at action \( \theta_i \in \Theta \). We will use \( \text{Conv}(S) \) to refer to the convex hull of a set of points \( S \).

A.1. Convex Functions

We first state the related definitions of strong convexity and strong smoothness:

**Definition A.1.** An everywhere sub-differentiable function \( f : S \mapsto \mathbb{R} \) is \( \alpha \)-strongly-convex w.r.t. norm \( \| \cdot \| \) if

\[
f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\alpha}{2} \| y - x \|^2 \quad \forall \ x, y \in S
\]

**Definition A.2.** An everywhere sub-differentiable function \( f : S \mapsto \mathbb{R} \) is \( \beta \)-strongly-smooth w.r.t. norm \( \| \cdot \| \) if

\[
f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\beta}{2} \| y - x \|^2 \quad \forall \ x, y \in S
\]

We now turn to the Bregman divergence and a discussion of several useful properties (Bregman, 1967; Banerjee et al., 2005):

**Definition A.3.** Let \( f : S \mapsto \mathbb{R} \) be an everywhere sub-differentiable strictly convex function. Its Bregman divergence is defined as

\[B_f(x||y) = f(x) - f(y) - \langle \nabla f(y), x - y \rangle\]

The definition directly implies that \( B_f(x||y) \) preserves the (strong or strict) convexity of \( f \) for any fixed \( y \in S \). Strict convexity further implies \( B_f(x||y) \geq 0 \quad \forall \ x, y \in S \), with equality iff \( x = y \). Finally, if \( f \) is \( \alpha \)-strongly-convex, or \( \beta \)-strongly-smooth, w.r.t. \( \| \cdot \| \) then Definition A.1 implies \( B_f(x||y) \geq \frac{\alpha}{2} \| x - y \|^2 \), or \( B_f(x||y) \leq \frac{\beta}{2} \| x - y \|^2 \), respectively.

**Claim A.1.** Let \( f : S \mapsto \mathbb{R} \) be a strictly convex function on \( S \), \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) be a sequence satisfying \( \alpha_{1:n} > 0 \), and \( x_1, \ldots, x_n \in S \). Then

\[
\bar{x} = \frac{1}{\alpha_{1:n}} \sum_{i=1}^{n} \alpha_i x_i = \arg \min_{y \in S} \sum_{i=1}^{n} \alpha_i B_f(x_i||y)
\]

**Proof.** \( \forall \ y \in S \) we have

\[
\sum_{i=1}^{n} \alpha_i (B_f(x_i||y) - B_f(x_i||\bar{x})) = \sum_{i=1}^{n} \alpha_i (f(x_i) - f(y) - \langle \nabla f(y), x_i - y \rangle - f(x_i) + f(\bar{x}) + \langle \nabla f(\bar{x}), x_i - \bar{x} \rangle)
\]

\[
= (f(\bar{x}) - f(y) + \langle \nabla f(y), y \rangle) \alpha_{1:n} + \sum_{i=1}^{n} \alpha_i (-\langle \nabla f(\bar{x}), \bar{x} \rangle + \langle \nabla f(\bar{x}) - \nabla f(y), x_i \rangle)
\]

\[
= (f(\bar{x}) - f(y) - \langle \nabla f(y), \bar{x} - y \rangle) \alpha_{1:n}
\]

\[
= \alpha_{1:n} B_f(\bar{x}||y)
\]

By Definition A.3 the last expression has a unique minimum at \( y = \bar{x} \). □
A.2. Online Algorithms

Here we provide a review of the online algorithms we use within each task. Our focus is on two closely related meta-algorithms, Follow-the-Regularized-Leader (FTRL) and (linearized lazy) Online Mirror Descent (OMD). For a given Bregman regularizer \( B_R \), starting point \( \phi \in \Theta \), and fixed learning rate \( \eta > 0 \), the algorithms are as follows:

- FTRL plays \( \theta_{t+1} = \arg \min_{\theta \in \Theta} B_R(\theta \mid \phi) + \eta \sum_{s \leq t} \ell_s(\theta) \).
- OMD plays \( \theta_{t+1} = \arg \min_{\theta \in \Theta} B_R(\theta \mid \phi) + \eta \sum_{s \leq t} \langle \nabla \ell_s, \theta \rangle \).

This formulation makes the connection between the two algorithms – that they are equivalent in the linear case \( \ell_s(\cdot) = \langle \nabla \ell_s, \cdot \rangle \) – very explicit. There exists a more standard formulation of OMD that is used to highlight its generalization of OGD – the case of \( B_R(\theta \mid \phi) = \frac{1}{2} \| \theta - \phi \|^2 \) and the fact that the update is carried out in the dual space induced by the regularizer \( R \) (Hazan, 2015, Section 5.3). However, we will only need the following regret bound for FTRL, which since it holds for all \( G \)-Lipschitz convex functions also holds for OMD when \( \| \nabla \ell_t \|_* \leq G \) \( \forall t \in [T] \) (Shalev-Shwartz, 2011, Theorem 2.11):

\[
R_T \leq \frac{B_R(\theta^* \mid \phi)}{\eta} + \eta G^2 T
\]

(6)

We next review the online algorithms we use for the meta-update. The main requirement here is logarithmic regret guarantees for the case of strongly convex loss functions. Two well known algorithms do so with the following guarantee on a sequence of functions indexed by \( t = 1, \ldots, T \) that are \( \alpha_t \)-strongly-convex w.r.t. \( \| \cdot \| \) and \( G_t \)-Lipschitz w.r.t. \( \| \cdot \|_* \):

\[
R_T \leq \frac{1}{2} \sum_{t=1}^{T} \frac{G_t}{\alpha_{1:t}}
\]

(7)

The algorithms and sources for this regret guarantee are the following:

- Follow-the-Leader (FTL), which plays \( \theta_{t+1} = \arg \min_{\phi} \sum_{s \leq t} \ell_s(\phi) \) (Kakade & Shalev-Shwartz, 2008, Theorem 2).
- Adaptive Online Gradient Descent (OGD), which plays \( \theta_{t+1} = \phi_t - \frac{1}{\alpha_{1:t}} \nabla \ell_t(\phi_t) \) (Bartlett et al., 2008, Theorem 2.1).

Of course, these are not the only algorithms achieving logarithmic regret on strongly convex functions. For example, the popular AdaGrad algorithm also does so (Duchi et al., 2010, Theorem 13). However, the proof of the main result requires that the meta algorithm only play points in the convex hull of the points seen thus far. This is because we must stay in the smaller meta-learned subset that we assume contains all the optimal parameters. Since we do not know this subset, we cannot use the projections most online methods use to remain feasible. We can easily show in the following claim that FTL and OGD satisfy these requirements but leave the extension to different meta-update algorithms, either of this or of the main proof, to future work.

**Claim A.2.** Let \( B_R \) be a Bregman regularizer on \( S \) and consider any \( \theta_1, \ldots, \theta_T \in S^* \) for some convex subset \( S^* \subset S \). Then for loss sequence \( \alpha_1 B_R(\theta_1 \mid \cdot), \ldots, \alpha_T B_R(\theta_T \mid \cdot) \) for any positive scalars \( \alpha_1, \ldots, \alpha_T \in \mathbb{R}_+ \), if we assume \( \phi_t \in S^* \) then FTL will play \( \phi_t \in S^* \) \( \forall t \) and OGD will as well if we further assume \( R(\cdot) = \frac{1}{2} \| \cdot \|^2 \).

**Proof.** The proof for FTL follows directly from Claim A.1 and the fact that the weighted average of a set of points is in their convex hull. For OGD we proceed by induction on \( t \). The base case \( t = 1 \) holds by the assumption \( \phi_t \in S^* \). In the inductive case, note that \( B_R(\theta_t \mid \phi_t) = \frac{1}{2} \| \theta_t - \phi_t \|^2 \) so the gradient update is \( \phi_{t+1} = \phi_t + \frac{\alpha_t}{\alpha_{1:t}} (\theta_t - \phi_t) \), which is on the line segment between \( \phi_t \) and \( \theta_t \), so the proof is complete by the convexity of \( S^* \ni \phi_t, \theta_t \). \( \square \)
A.3. Online-to-Batch Conversion

Finally, since we are also interested in distributional meta-learning, we discuss standard techniques for converting regret guarantees into generalization bounds, which are usually named online-to-batch conversions. In particular, for OCO we have the following bound on the risk of the average over the actions taking by an online algorithm, a result of applying Jensen’s inequality to Proposition 2 in Cesa-Bianchi & Gentile (2005):

**Proposition A.1.** Let $\theta_1, \ldots, \theta_T$ be the actions of an online algorithm and let $\ell_1, \ldots, \ell_T : \Theta \mapsto [0, 1]$ be convex loss functions drawn i.i.d. from some distribution $D$. Then w.p. $1 - \delta$ we have

$$
\mathbb{E}_{\ell \sim D} \ell(\bar{\theta}) \leq L_T + \frac{36}{T} \log \frac{T L_T + 3}{\delta} + 2 \sqrt{\frac{L_T}{T} \log \frac{T L_T + 3}{\delta}}
$$

where $\bar{\theta} = \frac{1}{T} \theta_{1:T}$ and $L_T = \frac{1}{T} \sum_{t=1}^{T} \ell_t(\theta_t)$ is the average loss suffered by the agent.

Thus for distributions over bounded convex loss functions we can run a low-regret online algorithm and perform asymptotically as well as ERM in hindsight w.h.p. However, for our lifelong algorithms we are also considering online algorithms over the regret of within-task algorithms as a function of the initialization and learning rates. One can obtain a good action in expectation by picking one at random (Cesa-Bianchi et al., 2004, Proposition 1):

**Proposition A.2.** Let $\theta_1, \ldots, \theta_T$ be the actions of an online algorithm and let $\ell_1, \ldots, \ell_T : \Theta \mapsto [0, 1]$ be loss functions drawn i.i.d. from some distribution $D$. Then we have

$$
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{\ell \sim D} \ell(\theta_t) \leq \frac{1}{T} \sum_{t=1}^{T} \ell_t(\theta_t) + \sqrt{\frac{2}{T} \log \frac{1}{\delta}} \quad \text{w.p. } 1 - \delta
$$

$$
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{\ell \sim D} \ell(\theta_t) \geq \frac{1}{T} \sum_{t=1}^{T} \ell_t(\theta_t) - \sqrt{\frac{2}{T} \log \frac{1}{\delta}} \quad \text{w.p. } 1 - \delta
$$

Note that Cesa-Bianchi et al. (2004) only prove the first inequality; the second follows via the same argument but applying the symmetric version of the Azuma-Hoeffding inequality (Azuma, 1967). There is also a deterministic way to pick an action from $\theta_1, \ldots, \theta_T$ using a penalized ERM approach with a high probability bound (Cesa-Bianchi et al., 2004, Theorem 4); however, the algorithm, while computable in polynomial time, is practically not very efficient in our setting.
B. Proofs of Main Theoretical Results

B.1. Upper and Lower Bounds for Task-Averaged Regret (Theorem 2.1 and Corollary 2.1)

We start with some technical lemmas. The first lower-bounds the regret of FTL when the loss functions are quadratic.

Lemma B.1. For any \( \theta_1, \ldots, \theta_T \in S \) and positive scalars \( \alpha_1, \ldots, \alpha_T \in \mathbb{R}_+ \) define \( \phi_t = \frac{1}{\alpha_t} \sum_{s=1}^{t} \alpha_s \theta_s \) and let \( \phi_0 \) be any point in \( S \). Then

\[
\sum_{t=1}^{T} \alpha_t \|	heta_t - \phi_{t-1}\|_2^2 - \sum_{t=1}^{T} \alpha_t \|	heta_t - \phi_T\|_2^2 \geq 0
\]

Proof. We proceed by induction on \( T \). The base case \( T = 1 \) follows directly since \( \phi_1 = \theta_1 \) and so the second term is zero.
In the inductive case we have

\[
\sum_{t=1}^{T-1} \alpha_t \|	heta_t - \phi_{t-1}\|_2^2 - \sum_{t=1}^{T-1} \alpha_t \|	heta_t - \phi_{T-1}\|_2^2 \geq 0
\]

so it suffices to show

\[
\phi_{T-1} = \arg \min_{\theta_{T}} \sum_{t=1}^{T} \alpha_t \|	heta_t - \phi_{t-1}\|_2^2 - \sum_{t=1}^{T} \alpha_t \|	heta_t - \phi_T\|_2^2
\]

in which case \( \phi_T = \phi_{T-1} \) and both added terms are zero, preserving the inequality. The gradient and Hessian are

\[
2 \alpha_T (\theta_T - \phi_{T-1}) + \frac{2 \alpha_T}{\alpha_1:T} \sum_{t=1}^{T-1} \alpha_t (\theta_t - \phi_T) - 2 \alpha_T (\theta_T - \phi_T) \left( 1 - \frac{\alpha_T}{\alpha_1:T} \right)
\]

so the problem is strongly convex and thus has a unique global minimum. Setting the gradient to zero yields

\[
0 = \theta_T - \phi_{T-1} + \frac{1}{\alpha_1:T} \sum_{t=1}^{T-1} \alpha_t \theta_t - \frac{1}{\alpha_1:T} \sum_{t=1}^{T-1} \alpha_t \phi_T - \theta_T + \frac{\alpha_T}{\alpha_1:T} \theta_T + \phi_T - \frac{\alpha_T}{\alpha_1:T} \phi_T = \phi_T - \phi_{T-1} \implies \theta_T = \phi_{T-1}
\]

We use this to show logarithmic regret of FTL when the loss functions are Bregman regularizers with changing first arguments. Note that such functions are in general only strictly convex, so the above bounds (7) cannot be applied directly.

Lemma B.2. Let \( B_R \) be a Bregman regularizer on \( S \) w.r.t. \( \| \cdot \| \) and consider any \( \theta_1, \ldots, \theta_T \in S \). Then for loss sequence \( \alpha_1 B_R(\theta_1\| \cdot \|), \ldots, \alpha_T B_R(\theta_T\| \cdot \|) \) for any positive scalars \( \alpha_1, \ldots, \alpha_T \in \mathbb{R}_+ \) we have regret bound

\[
R_T \leq \frac{G_R^2 + 1}{2} \sum_{t=1}^{T} \alpha_t
\]

where \( G_R \) is the Lipschitz constant of the Bregman regularizer \( B_R(\theta_1\| \cdot \|) \) for any \( t \in [T] \) on \( S \) w.r.t. the Euclidean norm.

Proof. Defining \( \bar{\phi} = \frac{1}{\alpha_1:T} \sum_{t=1}^{T} \alpha_t \theta_t \), we apply Claim A.1 and Lemma B.1 to get

\[
R_T = \sum_{t=1}^{T} \alpha_t B_R(\theta_t\| \bar{\phi}) - \min_{\phi \in S} \sum_{t=1}^{T} \alpha_t B_R(\theta_t\| \phi)
\]

\[
\leq \sum_{t=1}^{T} \alpha_t B_R(\theta_t\| \bar{\phi}) - \sum_{t=1}^{T} \alpha_t B_R(\theta_t\| \bar{\phi}) + \frac{1}{2} \sum_{t=1}^{T} \alpha_t \|\theta_t - \bar{\phi}\|_2^2 - \frac{1}{2} \sum_{t=1}^{T} \alpha_t \|\theta_t - \bar{\phi}\|_2^2
\]

\[
= \sum_{t=1}^{T} \alpha_t B_R(\theta_t\| \bar{\phi}) + \frac{\alpha_T}{2} \|\theta_T - \bar{\phi}\|_2^2 - \min_{\phi \in S} \sum_{t=1}^{T} \alpha_t B_R(\theta_t\| \phi) + \frac{\alpha_T}{2} \|\theta_T - \phi\|_2^2
\]

Since Bregman regularizers are convex in the second argument, the above is the regret of playing FTL on a sequence of \( \alpha_t \)-strongly-convex losses. Applying Kakade & Shalev-Shwartz (2008, Theorem 2) yields the result.\( \square \)
We are now ready to prove the result for the Follow-the-Average-Leader (FAL) variant of our main algorithm, where meta-updates are performed using the optimal task parameters. The following is an expanded version of Theorem 2.1:

**Theorem B.1.** Make Assumption 2.1. Let $B_R$ be a Bregman regularizer w.r.t. $\| \cdot \|_R$ associated to some $l$-strongly-convex function $R : \Theta \rightarrow \mathbb{R}$ on action space $\Theta$ and let $D^* = \max_{\theta \in \Theta} \sqrt{B_R(\theta^* || \theta^*)}$ be the diameter of $\Theta^*$. Then the FAL variant of Algorithm 2 will have TAR

$$R_M^* \leq \frac{G}{T} \left( \sum_{t=1}^{T} \left( \frac{(G^2 + 1)\sqrt{m_t}}{2D^* \sqrt{m_{1:t}}} + \frac{B_R(\theta^*_t || \phi^*_t)}{D^*} + \hat{D}_t^* \right) \sqrt{m_t} + D \sqrt{m_t} + (D^* + 1) \left( \frac{\hat{D}_t^* - \hat{D}_t}{\gamma - 1} \right) \sqrt{m_t} \right)$$

for $D = \max_{\theta \in \Theta} \sqrt{B_R(\theta || \phi^*_t)}$, $\phi^* = \arg \min_{\phi \in \Theta} \sum_{t=1}^{T} B_R(\theta^*_t || \phi) \sqrt{m_t}$, $\hat{D}_t^* = \max\{\gamma D^*, \hat{D}\}$, $\hat{m}_t^* = \max_k m_{t_k}$ for times $t_k$, $k = 0, \ldots$ when the diameter guess is violated, and Lipschitz constant $G^*$ of the Bregman regularizer $B_R(\theta^*_t || \cdot)$ for any $t \in [T]$ w.r.t. the Euclidean norm.

**Proof.** We first expand the task-averaged regret as a sum over the regret on each task, which we can bound for the case of both FTRL and OMD by Equation 6:

$$\mathcal{R}_T = \sum_{t=1}^{T} \mathcal{R}_t^{\text{META}_{\eta_t}} \leq \sum_{t=1}^{T} \frac{B_R(\theta^*_t || \phi_t)}{\eta_t} + \eta_t G^2 m_t = G \sum_{t=1}^{T} \left( \frac{B_R(\theta^*_t || \phi_t)}{D_t} + D_t \right) \sqrt{m_t}$$

We analyze this regret by defining two “cheating” sequences: $\hat{\phi}_t = \phi_t$ on all $t$ except $t = 1$, when we set $\hat{\phi}_1 = \theta^*_1$; similarly, $D_t = D_1$ on all $t$ except $t = 1$ and any $t$ s.t. $B_R(\theta^*_t || \phi_t) > D_t^2$, when we set $D_t = D^*$. In order to do this we add outside of the summation the corresponding regret of the true sequences whenever on of them is not the same as its “cheating” sequence. From Algorithm 2 recall that for $t = 1$ the agent suffers regret (6) for $\eta_1$ set using the maximum distance $D$ from the initialization $\phi_1$. Note further that $B_R(\theta^*_t || \phi_t) > D_t^2$ corresponds exactly to the times that the violation count $k$ is incremented in Algorithm 2 and thus occurs at most $\log \gamma \frac{D^2}{D_t}$ times, as we multiply the diameter guess by $\gamma$ each time it happens, which together with Lemma A.2 ensures that $\phi_t$ remains within $D_t^*$ of all the optima $\theta^*_t$. We index these times by $k = 0, \ldots$ so that at each $k$ the agent suffers regret (6) for $\eta_k$ set using $g^{k+1} D_t$. These modifications lead to the regret bound being split into three components:

$$\mathcal{R}_T \leq \frac{B_R(\theta^*_1 || \phi_1)}{D} + D \sqrt{m_1} + \sum_{t=1}^{T} \left( \frac{B_R(\theta^*_t || \phi_t)}{D_t} + \hat{D}_t \right) \sqrt{m_t} + \sum_{k=0}^{\lfloor \log \gamma \frac{D^2}{D_t} \rfloor} \left( \frac{B_R(\theta^*_t || \phi_k)}{D_k} + g^{k+1} \hat{D}_t \right) \sqrt{m_k}$$

Since $B_R(\theta^*_1 || \phi_1) \leq D^2$ the first component is bounded by $D \sqrt{m_1}$. We then consider the second term. For any $t$ let $f_t(D) = \frac{B_R(\theta^*_t || \phi_t)}{D} + D$. Its derivative is $\partial_D f_t = 1 - \frac{B_R(\theta^*_t || \phi_t)}{D^2}$, which is nonnegative whenever $D^2 \geq B_R(\theta^*_t || \phi_t)$. Therefore when $D_t \leq D^*$ we have $f(D_t) \leq f(D^*)$, as by definition both are greater than $\sqrt{B_R(\theta^*_t || \phi_t)}$ and so $f$ is increasing on the interval between them. On the other hand, for $D_t \geq D^*$ either $D_t \leq \gamma D^* \forall t$ by the tuning rule or, if we initialized $D > D^*$, then $D_t = D^*$ for $t$, so either way we have $f(D_t) \leq \frac{B_R(\theta^*_t || \phi_t)}{D^2} + \hat{D}_t^*$. Since $\gamma \geq 1$ this bounds $f(D_t)$ in the previous case $D_t \leq D^*$ as well, so we have

$$\sum_{t=1}^{T} \left( \frac{B_R(\theta^*_t || \phi_t)}{D_t} + \hat{D}_t \right) \sqrt{m_t} \leq \sum_{t=1}^{T} \left( \frac{B_R(\theta^*_t || \phi_t)}{D^*} + \hat{D}_t^* \right) \sqrt{m_t}$$

where we define $\phi^* = \arg \min_{\phi \in \Theta} \sum_{t=1}^{T} \sqrt{m_t} B_R(\theta^*_t || \phi)$. Note that since $\hat{\phi}_1 = \theta^*_1 \Rightarrow B_R(\theta^*_1 || \phi_1) = 0$, the first summation is bounded by the regret of $\text{META}_{\eta_t}$ on a sequence of Bregman regularizers with multipliers $\sqrt{m_t}$ and gradients bounded by $G^*$. Thus using Lemma B.2 leads to a final bound on the second term of

$$\sum_{t=1}^{T} \left( \frac{B_R(\theta^*_t || \phi_t)}{D_t} + \hat{D}_t \right) \sqrt{m_t} \leq \sum_{t=1}^{T} \left( \frac{(G^2 + 1) \sqrt{m_t}}{2D^* \sqrt{m_{1:t}}} + \frac{B_R(\theta^*_t || \phi_t)}{D^*} + \hat{D}_t^* \right) \sqrt{m_t}$$
We now bound the last term. Assuming $\hat{D} < D^*$ and $\gamma > 1$, as otherwise this term is zero, we have

$$\sum_{k=0}^{\left\lfloor \frac{\log_\gamma \frac{D^*}{\hat{D}} \right\rfloor} \left( B_R(\theta_{t,k}^* \| \phi_{t,k}) + \gamma^k \hat{D} \right) \sqrt{m_{t,k}} \leq \sqrt{\hat{m}_{t,k}} \sum_{k=0}^{\left\lfloor \frac{\log_\gamma \frac{D^*}{\hat{D}} \right\rfloor} \frac{D^*}{\gamma^k \hat{D}} + \gamma^k \hat{D} \leq (D^* + 1) \sqrt{\hat{m}_{t,k}} \frac{\gamma D^* - \hat{D}}{\gamma - 1}$$

Combining the bounds in Equation (9) completes the proof. \(\square\)

The following lower bound, an expanded version of Corollary 2.1, which extends Theorem 4.2 of Abernethy et al. (2008) to the multi-task setting, shows that the above TAR guarantee is optimal up to a constant multiplicative factor.

**Corollary B.1.** Suppose the action space $\Theta \subset \mathbb{R}^d$ for $d \geq 3$ and for each task $t \in [T]$ an adversary must play a a sequence of $m_t$ convex $G$-Lipschitz functions $\ell_{t,i}: \Theta \rightarrow \mathbb{R}$ whose optimal actions in hindsight $\arg \min_{\theta \in \Theta} \sum_{i=1}^{m_t} \ell_{t,i}(\theta)$ are contained in some fixed $\ell_2$-ball $\Theta^* \subset \Theta$ with center $\phi^*$ and diameter $D^*$. Then the adversary can force the agent to have task-averaged regret at least $\frac{G D^*}{2t} \sum_{i=1}^{T} \sqrt{m_t}$.  

**Proof.** Let $\{\theta_{t,i}\}_{i=1}^{m_t}$ be the sequence of actions of the agent on task $t$. Define $c(\theta) = \frac{G}{2} \max\{0, \|\theta - \phi^*\|_2 - D^*\}$, which is 0 on $\Theta^*$ and an upward-facing cone with vertex $\left(\phi^*, -\frac{G D^*}{2}\right)$ and slope $\frac{G}{2}$ on the complement. The strategy of the adversary at round $i$ of task $t$ will be to play $\ell_{t,i}(\theta) = (\nabla_{t,i} \theta, \theta - \phi^*) + c(\theta)$, where $\nabla_{t,i}$ satisfies $\|\nabla_{t,i}\|_2 = \frac{G}{2}$, $\langle \nabla_{t,i}, \theta_{t,i} - \phi^* \rangle = 0$, and $\langle \nabla_{t,i}, \sum_{j<i} \nabla_{t,j} \rangle = 0$. Such a $\nabla_{t,i}$ always exists for $d \geq 3$. Note that these conditions imply that along any direction from $\phi^*$ the total loss $\sum_{i=1}^{m_t} \ell_{t,i}(\theta)$ is increasing outside $\Theta^*$ and so is minimized inside $\Theta^*$, so we have

$$\min_{\theta \in \Theta^*} \sum_{i=1}^{m_t} \ell_{t,i}(\theta) = \min_{\theta \in \Theta^*} \sum_{i=1}^{m_t} \langle \nabla_{t,i} \theta, \theta - \phi^* \rangle = \min_{\|\theta - \phi^*\|_2 \leq \frac{D^*}{2}} \left\langle \theta - \phi^*, \sum_{i=1}^{m_t} \nabla_{t,i} \right\rangle = -\frac{D^*}{2} \left\| \sum_{i=1}^{m_t} \nabla_{t,i} \right\|_2$$

Note that the condition $\langle \nabla_{t,i}, \theta_{t,i} - \phi^* \rangle = 0$ and the nonnegativity of $c(\theta)$ implies that the loss of the agent is at least 0, and so the agent’s regret on task $t$ satisfies $R_{m_t} \geq \frac{D^*}{2} \left\| \sum_{i=1}^{m_t} \nabla_{t,i} \right\|_2$. By the condition $\langle \nabla_{t,i}, \sum_{j<i} \nabla_{t,j} \rangle = 0$ we have that

$$\left\| \sum_{j=1}^{i} \nabla_{t,j} \right\|_2^2 = \left\| \nabla_{t,i} + \sum_{j<i} \nabla_{t,j} \right\|_2^2 = \left\| \nabla_{t,i} \right\|_2^2 + \left\| \sum_{j<i} \nabla_{t,j} \right\|_2^2 = \frac{G^2}{4} + \left\| \sum_{j<i} \nabla_{t,j} \right\|_2^2$$

and so by induction on $i$ with base case $\|\nabla_{t,1}\|_2 = \frac{G}{2}$ we have $\|\sum_{i=1}^{m_t} \nabla_{t,i}\|_2 = \frac{G}{2} \sqrt{m_t} \Rightarrow R_{m_t} \geq \frac{G D^*}{4t} \sqrt{m_t}$. 

Substituting the regret on each task into $R = \frac{1}{T} \sum_{t=1}^{T} R_{m_t}$ completes the proof. \(\square\)
B.2. Approximate Meta-Updates under Quadratic Growth (Theorems 3.1 and 3.2)

For the case of Follow-the-Last-Iterate (FLI) we need a bound on the distance between the last iterate of FTRL/OMD and the best parameter in hindsight. This necessitates further assumptions on the loss functions besides convexity, as a task may otherwise have functions with very small losses, even far away from the optimal parameter, in which case the last iterate of FTRL/OMD will be far away if the initial point is far away from the optimum. Here we make use of the $\alpha$-QG assumptions on the average loss functions to obtain stability of the estimates w.r.t. the true loss. We first prove a general theorem assuming this bound; regret bounds for FLI-Online and FLI-Batch will then follow from brief arguments.

**Theorem B.2.** Let $B_R$ be a Bregman regularizer w.r.t. $\|\cdot\|$ associated to some 1-strongly-convex and $\beta$-strongly-smooth function $R : \Theta \rightarrow \mathbb{R}$ on action space $\Theta$, and let $\ell_1, \ldots, \ell_T$ be a sequence of convex losses of $T$ tasks with $m_t$ samples each that are $G$-Lipschitz w.r.t. $\|\cdot\|$. Assume that the meta-updates are run according one of the FLI variants of Algorithm 2 with meta-update action $\theta_t$. Then if

$$\frac{1}{2}||\hat{\theta}'_t - \hat{\theta}_t||^2 \leq \left(\frac{B_R(\hat{\theta}'_t||\phi_t)}{\eta_t} + \delta_t^2\right) \frac{1}{\alpha m_t}$$

for some reference actions $\hat{\theta}'_t \in \Theta^* \subset \Theta$ we have that

$$\bar{R}_M \leq G \left(\sum_{t=1}^{T} \left(\frac{\beta^2 (G + 1) \sqrt{m_t}}{2D^* \sqrt{m_1}} + \frac{\beta^2 B - R(\hat{\theta}'_t||\phi^*)}{D^*} + \hat{D}_t^* + \epsilon_t \right) \sqrt{m_t} + D \sqrt{m_1} + \frac{3\gamma D^2 \sqrt{m_1}}{\alpha_m} \right)$$

where we have $D = \max_{\theta \in \Theta} \sqrt{B_R(\theta||\phi_1)}$, $D^* = \max_{s,t \in [T]} \sqrt{B_R(\theta'_s||\theta'_t)}$, $\phi^* = \arg \min_{\phi \in \Theta} \sum_{t=1}^{T} B_R(\theta'_t||\phi) \sqrt{m_t}$, $\hat{D}_t^* = \max\{\gamma D^*, \hat{D}\}$, $m_t^* = \max_k m_{t_k}$ for times $t_k, k = 0, \ldots$ when the diameter guess is violated, $m_{\min} = \min_t \sqrt{m_t}$, Lipschitz constant $\hat{G}$ of the Bregman regularizer $B_R(\hat{\theta}||\cdot)$ for any $t \in [T]$ w.r.t. the Euclidean norm, and error terms

$$\epsilon_t = \gamma D \sqrt{\beta} + \frac{8\beta D^* \Delta_t^2}{\alpha \sqrt{m_1}} + \frac{\beta}{D} \left(\frac{\epsilon_t}{\sqrt{m_t}} + \Delta_t' + 2\Delta_{\max}\right) \left(\frac{\epsilon_t}{\sqrt{m_t}} + \Delta_t' \right)$$

$$\Delta_t' \leq \sqrt{\frac{GD}{\alpha \sqrt{m_1}}} + \frac{\delta_t}{\alpha \sqrt{m_1}}, \quad \Delta_t' \leq \left(D^* + \frac{\Delta}{2}\right) \left(\frac{\beta G}{\alpha D m_t} + \frac{\delta_t}{\alpha m_t}\right), \quad \Delta_{\max} \leq D^* + \hat{\Delta}$$

$$\hat{\Delta} = 2\Delta_t' + \max_{t>1} \frac{2\delta_t}{\sqrt{m_1}} + \left(2D^* + \max_{t} \frac{6\delta_t}{\sqrt{m_1}}\right) \left(\frac{\beta G}{\alpha D \sqrt{m_{\min}}} + \frac{6\beta GD}{\alpha D \sqrt{m_{\min}}}\right)$$
Proof. The analysis extends that of Theorem B.1. We similarly define a sequence \( \hat{\phi}_t = \phi_t \) \( \forall t > 1 \) and \( \hat{\theta}_1 = \hat{\theta}_1 \). Let \( D' = \max_{t,s} \{ \frac{1}{\sqrt{2}} \| \theta'_t - \theta'_s \| \} \) be the diameter guess of \( \Theta^* \) w.r.t. \( \frac{1}{2} \cdot \| \cdot \|_2 \), \( \Delta'_t = \frac{1}{\sqrt{2}} \| \hat{\theta}_t - \theta'_t \| \), and \( \Delta_t = \frac{1}{\sqrt{2}} \| \hat{\theta}_t - \phi_t \| \). We can bound the latter quantity as follows:

\[
\Delta_{\text{max}} \frac{\sqrt{2}}{2} = \max_{t > 1} \| \hat{\phi}_t - \hat{\theta}_t \| \\
\leq \max_{t > 1} \| \hat{\phi}_t - \hat{\phi}'_t \| + \| \hat{\phi}'_t - \hat{\theta}_t \| \\
\leq \max_{t > 1} \| \hat{\phi}_t - \hat{\phi}'_t \| + \max_{s < t} \| \hat{\phi}'_t - \hat{\phi}'_s \| + \| \hat{\phi}'_s - \hat{\theta}_s \| \\
\leq D' \sqrt{2} + \max_{t > 1} \| \hat{\phi}'_t - \hat{\theta}_t \| + \max_{s < t} \| \hat{\phi}'_s - \hat{\theta}_s \| \\
\leq D' \sqrt{2} + 2 \max_{t > 1} \| \hat{\phi}'_t - \hat{\theta}_t \| \\
\leq D' \sqrt{2} + 2 \| \phi'_t - \hat{\theta}_t \| + 2 \max_{t > 1} \| \phi'_t - \hat{\phi}_t \| \\
\leq D' \sqrt{2} + 2 \| \phi'_t - \hat{\phi}_t \| + \max_{t > 1} \sqrt{\frac{8}{\alpha m_t} \left( \frac{B_R(\phi'_t, \phi_t)}{\eta_t} + \delta_t^2 \right)} \\
\leq D' \sqrt{2} + \Delta'_1 \sqrt{8} + 2 \max_{t > 1} \sqrt{\beta \| \phi'_t - \phi_t \|_2 G \sqrt{m_t} + \delta_t \sqrt{\frac{8}{\alpha m_t}}} \\
\leq D' \sqrt{2} + \Delta'_1 \sqrt{8} + 2 \max_{t > 1} \| \phi'_t - \hat{\theta}_t \| + \| \theta'_t - \phi_t \|, \sqrt{\frac{\beta G}{\alpha D \sqrt{m_{\text{min}}}}} + \delta_t \sqrt{\frac{8}{\alpha m_t}} \\
\leq D' \sqrt{2} + \Delta'_1 \sqrt{8} + (D' \sqrt{8} + 6 \max_{t > 1} \| \theta'_t - \hat{\theta}_t \|) \sqrt{\frac{\beta G}{\alpha D \sqrt{m_{\text{min}}}}} + \max_{t > 1} \delta_t \sqrt{\frac{8}{\alpha m_t}} \\
\leq D' \sqrt{2} + \Delta'_1 \sqrt{8} + \max_{t > 1} \delta_t \sqrt{\frac{8}{\alpha m_t}} + \left( D' \sqrt{8} + 6 \max_{t} \sqrt{\frac{2}{\alpha m_t} \left( \frac{B_R(\theta'_t, \phi_t)}{\eta_t} + \delta_t^2 \right)} \right) \sqrt{\frac{\beta G}{\alpha D \sqrt{m_{\text{min}}}}} \\
\leq D' \sqrt{2} + \Delta'_1 \sqrt{8} + \max_{t > 1} \delta_t \sqrt{\frac{8}{\alpha m_t}} + \left( D' \sqrt{8} + 6 \max_{t} \sqrt{\frac{2}{\alpha m_t} \left( \frac{B_R(\theta'_t, \phi_t)}{\eta_t} + \delta_t^2 \right)} \right) \sqrt{\frac{\beta G}{\alpha D \sqrt{m_{\text{min}}}}} + \frac{6 \beta GD}{\alpha D} \frac{2}{m_{\text{min}}}
\]

Note that \( \Delta'_1 \) is bounded as

\[
\Delta'_1 = \frac{1}{\sqrt{2}} \| \hat{\theta}_1 - \theta'_1 \| \leq \sqrt{\frac{\frac{1}{m} B_R(\theta'_1, \phi_1)}{\alpha m_1} + \delta_1^2} \leq \sqrt{\frac{GD}{\alpha \sqrt{m_1}}} + \frac{\delta_1}{\sqrt{\alpha m_1}}
\]

Define \( \Delta' \sqrt{2} \) to be the sum of all terms except the first term in (10), and note that from that argument we have

\[
\| \theta'_t - \hat{\phi}_t \| \leq \max_{s < t} \| \theta'_t - \hat{\phi}'_s \| \leq \max_{s < t} \| \theta'_t - \hat{\phi}'_s \| + \| \hat{\phi}'_s - \hat{\theta}_s \| \leq D' \sqrt{2} + \Delta' \sqrt{2}
\]

which we can use to bound \( \Delta'_t \) for \( t > 1 \):

\[
\Delta'_t = \frac{1}{\sqrt{2}} \| \hat{\theta}_t - \theta'_t \| \leq \sqrt{\frac{\frac{1}{m} B_R(\theta'_t, \phi_t)}{\alpha m_t} + \delta_t^2} \leq \| \theta'_t - \hat{\phi}_t \| \sqrt{\frac{\beta G}{2 \alpha D m_t}} + \frac{\delta_t}{\sqrt{\alpha m_t}} \leq \left( D' + \frac{\Delta'}{2} \right) \sqrt{\frac{\beta G}{\alpha D m_t}} + \frac{\delta_t}{\sqrt{\alpha m_t}}
\]

We now define a second sequence \( \tilde{D}_t = D_t \) on all \( t \) except whenever \( \frac{1}{m} B_R(\hat{\theta}_t, \phi_1) > D_t^2 \), when we set \( \tilde{D}_t = \Delta_{\text{max}} \sqrt{2} \). As before this corresponds to the times that the violation count \( k \) is incremented in Algorithm 2, which by the update rule occurs at most \( \lfloor \log \gamma \Delta_{\text{max}} \sqrt{2} \rfloor \) times. We then follow an analysis similar to that for Thm. B.1 to split the regret into three
components:
\[
\frac{\tilde{R}_M T}{G} = \frac{1}{G} \sum_{t=1}^{T} \mathbf{R}_{m_t}(\text{TASK}_{\eta_t, \phi_t}) \\
\leq \sum_{t=1}^{T} \frac{B_R(\theta^*_t||\phi_t)}{\eta_t G} + \eta G m_t \\
\leq \left( \frac{B_R(\theta^*_1||\phi_1)}{\eta_1 G} + \eta_1 G m_1 \right) + \sum_{t=1}^{T} \left( \frac{B_R(\theta^*_t||\phi_t)}{D_t} + D_t \right) \sqrt{m_t} + \sum_{k=0}^{[\log_2 \hat{\Delta}_{\max} \sqrt{\beta}]} \left( \frac{B_R(\theta^*_{k+1}||\phi_{k+1})}{\gamma^k D} + \gamma^k \hat{D} \right) \sqrt{m_k}
\]

As before, the first term is bounded by \( D \sqrt{m_1} \). We analyze the second term by applying \( \beta \)-strong-smoothness, the triangle inequality, substituting error terms \( \hat{\Delta}_{\max} \) and \( \Delta'_t \), and then following the argument in Theorem B.1 to replace the \( \hat{D}_t \) terms in the summation by \( \hat{\Delta}_{\max} \):
\[
\sum_{t=1}^{T} \left( \frac{B_R(\theta^*_t||\phi_t)}{D_t} + D_t \right) \sqrt{m_t} \leq \sum_{t=1}^{T} \left( \frac{\beta ||\theta^*_t - \hat{\theta}_t||^2 + 2||\theta^*_t - \hat{\theta}_t|| \cdot ||\hat{\theta}_t - \hat{\phi}_t|| + ||\hat{\theta}_t - \hat{\phi}_t||^2}{D_t} + D_t \right) \sqrt{m_t} \\
\leq \sum_{t=1}^{T} \left( \frac{\beta B_R(\tilde{\theta}_t||\phi_t)}{D_t} + D_t + \frac{\beta}{2D^*} \left( ||\hat{\theta}_t - \tilde{\theta}_t|| + \hat{\Delta}_{\max} \sqrt{\beta} \right) ||\hat{\theta}_t - \tilde{\theta}_t|| \right) \sqrt{m_t} \\
\leq \sum_{t=1}^{T} \left( \frac{B_R(\hat{\theta}_t||\tilde{\phi}_t)}{\hat{\Delta}_{\max}} + \max \{ \hat{D}, \gamma \hat{\Delta}_{\max} \} \right) \sqrt{\beta m_t} \\
+ \frac{\beta}{D^*} \sum_{t=1}^{T} \left( \frac{\varepsilon_t}{\sqrt{m_t}} + \Delta'_t + 2\hat{\Delta}_{\max} \right) \left( \frac{\varepsilon_t}{\sqrt{m_t}} + \Delta'_t \right) \sqrt{m_t} \\
\leq \beta \sum_{t=1}^{T} \left( \frac{B_R(\hat{\theta}_t||\tilde{\phi}_t) - B_R(\hat{\theta}_t||\hat{\phi}^*)}{D^*} + \max \{ \hat{D}, \gamma \hat{\Delta}_{\max} \} \sqrt{\beta} \right) \sqrt{m_t} \\
+ \sum_{t=1}^{T} \left( \frac{\beta B_R(\hat{\theta}_t||\hat{\phi}^*)}{\hat{D}^*} + \max \{ \hat{D}, \gamma \hat{\Delta}_{\max} \} \sqrt{\beta} \right) \sqrt{m_t} \\
+ \frac{\beta}{D^*} \sum_{t=1}^{T} \left( \frac{\varepsilon_t}{\sqrt{m_t}} + \Delta'_t + 2\hat{\Delta}_{\max} \right) \left( \frac{\varepsilon_t}{\sqrt{m_t}} + \Delta'_t \right) \sqrt{m_t}
\]

Here \( \hat{\phi}^* = \arg \min_{\phi \in \Theta} \sum_{t=1}^{T} B_R(\hat{\theta}_t||\phi) \sqrt{m_t} \). As in the proof of Theorem B.1, the first term is bounded by the regret of META\( \hat{\theta}_t \) on a sequence of Bregman regularizer with multipliers \( \sqrt{m_t} \) and gradients bounded by \( \hat{G} \), a bound that can be computed using Lemma B.2. In the second term, we have by \( \beta \)-strong-smoothness and the triangle inequality that
\[
B_R(\hat{\theta}_t||\phi^*) \leq \frac{\beta}{2} \left( ||\hat{\theta}_t - \theta'_t|| + ||\theta'_t - \phi^*|| + ||\phi^* - \hat{\phi}^*|| \right)^2 \leq \beta B_R(\theta'_t||\phi^*) + 8\beta \Delta^2_t
\]

Thus we have the bound
\[
\sum_{t=1}^{T} \left( \frac{B_R(\theta^*_t||\phi_t)}{D_t} + D_t \right) \sqrt{m_t} \leq \sum_{t=1}^{T} \left( \frac{\beta^2 (G^2 + 1) \sqrt{m_t}}{2D^* \sqrt{m_{t-1}}} + \frac{\beta^2 B_R(\theta'_t||\phi^*)}{D^*} + \hat{D}^*_t + \mathcal{E} \right) \sqrt{m_t}
\]

Assuming \( \hat{D} < \hat{\Delta}_{\max} \sqrt{\beta} \) and \( \gamma > 1 \), as otherwise the last term is zero, we have the following bound using a somewhat looser analysis than in Theorem B.1:
\[
\sum_{k=0}^{[\log_2 \hat{\Delta}_{\max} \sqrt{\beta}]} \left( \frac{B_R(\theta^*_{k+1}||\phi_{k+1})}{\gamma^k D} + \gamma^k \hat{D} \right) \sqrt{m_k} \leq \sum_{k=0}^{[\log_2 \hat{\Delta}_{\max} \sqrt{\beta}]} \left( \frac{D^2}{\gamma^k D} + \gamma^k \hat{D} \right) \leq 3\gamma D^2 \sqrt{m_\gamma} \]

Combining terms completes the proof.
To apply Theorem B.2 to FLI we need to provide the assumed bounds on \( \| \theta'_t - \hat{\theta} \| \) and \( \| \theta'_t - \theta^*_t \| \), for \( \hat{\theta} \) the meta-update parameter, \( \theta^*_t \) the optimal actions in hindsight, and \( \theta'_t \) the task-similarity assumption parameter. For FLI-Online, the task-similarity assumption is on the optimal actions in hindsight, so \( \theta'_t = \theta^*_t \). We thus only need to provide the first bound:

**Lemma B.3.** Let \( \ell_1, \ldots, \ell_T \) be a sequence of convex losses on \( S \) with sum \( L(\theta) = \sum_{t=1}^{T} \ell_t(\theta) \) being \( \alpha\text{-QG} \) w.r.t. \( \| \cdot \| \) and let \( \theta' \in \arg \min_{\theta \in S} L(\theta) \) be any of its optimal actions in hindsight. Then the last iterate \( \hat{\theta} = \arg \min_{\theta \in S} B_R(\theta) + \eta L(\theta) \) of running FTRL\(_{\eta,\phi} \) with Bregman regularizer \( B_R \) w.r.t. \( \| \cdot \| \) satisfies

\[
\frac{1}{2} \| \theta' - \hat{\theta} \|^2 \leq \frac{B_R(\theta')|\phi| - B_R(\hat{\theta})|\phi|}{\alpha \eta}
\]

**Proof.** Since \( B_R \) is 1-strongly-convex w.r.t. \( \| \cdot \| \) we have by definition of any \( \theta'_t \) and \( \hat{\theta}_t \) that

\[
B_R(\theta'_t|\phi_t) + \eta L(\theta'_t) \geq B_R(\hat{\theta}_t|\phi_t) + \eta L(\hat{\theta}_t) + \frac{1}{2} \| \theta'_t - \hat{\theta}_t \|^2 \geq B_R(\hat{\theta}_t|\phi_t) + \eta L(\hat{\theta}_t)
\]

On the other hand since the sum is \( \alpha\text{-QG} \) we have by definition

\[
L(\hat{\theta}_t) \geq L(\theta'_t) + \frac{\alpha}{2} \| \theta'_t - \hat{\theta}_t \|^2
\]

Multiplying the second inequality by \( \eta \) and adding it to the first yields the result. \( \square \)

The following then implies Theorem 3.1:

**Corollary B.2.** Make Assumption 3.1. Let \( B_R \) be a Bregman regularizer w.r.t. \( \| \cdot \| \) associated to some 1-strongly-convex, \( \beta \)-strongly-smooth function \( R : \Theta \rightarrow \mathbb{R} \) on action space \( \Theta \) and let \( D^* = \max_{\theta \in [T]} \sqrt{B_R(\theta^*_t)}|\theta_t^*| \) be the diameter of \( \Theta^* \). Then if we set the within-task algorithm TASK\(_{\eta,\phi} \) to be FTRL\(_{\eta,\phi} \) with meta-updates according to the FLI-Online variant of Algorithm 2 we will have

\[
\bar{R}_M \leq \frac{G}{T} \left( \sum_{t=1}^{T} \left( \frac{\beta^2 (\bar{G} + 1) \sqrt{\hat{m}_t}}{2D^* \sqrt{m_{1:t}}} + \frac{\beta^2 B_R(\theta^*_t|\phi^*_t)}{D^*} + \hat{D}^*_\gamma + \bar{E}^*_m \right) \right) \sqrt{m_{1:t} + D \sqrt{m_{1:t}}} + \frac{3 \gamma D^2 \sqrt{m_{\min}}}{(\gamma - 1)D}
\]

where \( D, \phi^*_t, \hat{D}^*_\gamma, \) and \( \hat{m}^*_\gamma \) are as in Theorem B.2 and \( \bar{E} = \text{Poly} \left( \sqrt{\frac{1}{\alpha + \epsilon_{\min}}} \right) \).

**Proof.** By Lemma B.3, the assumptions in Theorem B.2 hold with \( \delta_t = 0, \epsilon_t = 0 \ \forall \ t \in [T] \). \( \square \)

For FLI-Batch, we use arguments similar to those used for online-to-batch conversion to provide the bounds:

**Lemma B.4.** Let \( \ell_1, \ldots, \ell_T : S \rightarrow [0, 1] \) be a sequence of convex losses on \( S \) drawn i.i.d. from some distribution \( D \) with expected sum \( \mathbb{E}_D L(\theta) = \sum_{t=1}^{T} \mathbb{E}_{t \sim D} \ell(\theta) \) \( \alpha\text{-QG} \) w.r.t. \( \| \cdot \| \) and let \( \theta' = \arg \min_{\theta \in S} \mathbb{E}_{t \sim D} \ell(\theta) \) be any of its true risk minimizers. Then the average iterate \( \hat{\theta} = \frac{1}{T} \theta_{1:T} \) of running FTRL\(_{\eta,\phi} \) or OMD\(_{\eta,\phi} \) on \( \ell_1, \ldots, \ell_T \) satisfies w.p. \( 1 - \delta \)

\[
\frac{1}{2} \| \theta' - \hat{\theta} \|^2 \leq \frac{1}{2T} B_R(\theta'|\phi) + \eta G^2 T + \frac{8 T \log \frac{2}{\delta}}{\alpha}
\]

**Proof.** By definition of \( \hat{\theta} \) and \( \theta' \) we have w.p. \( 1 - \delta \) that

\[
\frac{\alpha}{2T} \| \theta' - \hat{\theta} \|^2 \leq \frac{1}{T} \mathbb{E}_{(\ell_t)_{t \sim D^*}} \sum_{t=1}^{T} \ell_t(\theta_t) - \frac{1}{T} \mathbb{E}_{(\ell_t)_{t \sim D^*}} \sum_{t=1}^{T} \ell_t(\theta_t) \quad \text{(apply \( \alpha\text{-QG} \) and Jensen’s inequality)}
\]

\[
\leq \frac{1}{T} \sum_{t=1}^{T} \ell_t(\theta_t) - \frac{1}{T} \sum_{t=1}^{T} \ell_t(\theta') + \sqrt{\frac{8 T \log \frac{2}{\delta}}{\alpha}} \quad \text{(apply Proposition A.2 twice)}
\]

\[
\leq \frac{1}{T} B_R(\theta'|\phi) + \eta G^2 T + \sqrt{\frac{8 T \log \frac{2}{\delta}}{\alpha}} \quad \text{(substitute the regret of FTRL/OMD (5))}
\]

\( \square \)
Lemma B.5. Let $\ell_1, \ldots, \ell_T : S \mapsto [0, 1]$ be a sequence of convex losses on $S$ drawn i.i.d. from some distribution $D$ with expected sum $\mathbb{E}_D L(\theta) = \sum_{t=1}^{T} \mathbb{E}_{\ell_t \sim D} \ell(\theta)$ w.r.t. $\|\cdot\|$ and let $\theta' = \arg \min_{\theta \in \Theta} \mathbb{E}_{\ell_t \sim D} \ell(\theta)$ be any of its true risk minimizers. Then every optimal action in hindsight $\theta^* \in \arg \min_{\theta \in S} L(\theta)$ satisfies w.p. $1 - \delta$.

$$\frac{1}{2} \|\theta^* - \theta'\|^2 \leq \frac{\sqrt{8T \log \frac{2}{\delta}}}{\alpha}$$

Proof. By definition of $\theta^*$ and $\theta'$ we have w.p. $1 - \delta$ that

$$\frac{\alpha}{2T} \|\theta^* - \theta'\|^2 \leq \frac{1}{T} \mathbb{E}_{(\ell_t)_{t \sim D^T}} \sum_{t=1}^{T} \ell_t(\theta^*) - \frac{1}{T} \mathbb{E}_{(\ell_t)_{t \sim D^T}} \sum_{t=1}^{T} \ell_t(\theta') \quad \text{(apply } \alpha\text{-QG)}$$

$$\leq \frac{1}{T} \sum_{t=1}^{T} \ell_t(\theta^*) - \frac{1}{T} \sum_{t=1}^{T} \ell_t(\theta') + \sqrt{\frac{8}{T} \log \frac{2}{\delta}} \quad \text{(apply Proposition A.2 twice)}$$

$$\leq \sqrt{\frac{8}{T} \log \frac{2}{\delta}} \quad \text{(definition of regret)}$$



The following then implies Theorem 3.2:

Corollary B.3. Make Assumption 3.2. Let $B_R$ be a Bregman regularizer w.r.t. $\|\cdot\|$ associated to some $1$-strongly-convex, $\beta$-strongly-smooth function $R : \Theta \mapsto \mathbb{R}$ on action space $\Theta$ and let $D^* = \max_{s,t \in [T]} \sqrt{B_R(\theta_s^* || \theta_t^*)}$ be the diameter of $\Theta^*$. Then the FLI-Batch variant of Algorithm 2 will have w.p. $1 - \delta$ a TAR of

$$\hat{R}_M \leq \frac{G}{T} \left( \sum_{t=1}^{T} \left( \frac{\beta^2 (\hat{G} + 1) \sqrt{m_t}}{2D^* \sqrt{m_{1:t}}} + \frac{\beta^2 B_R(\theta_s^* || \theta_t^*)}{D^*} + \hat{D}_t^* + \mathcal{E}_m \right) \right) \sqrt{m_t} + D \sqrt{m_1} + \frac{3\gamma D^2 \sqrt{m_\gamma^*}}{(\gamma - 1) \hat{D}}$$

where $D, \phi^*, \hat{D}_t^*$, and $\hat{m}_t^*$ are as in Theorem B.2 and $\mathcal{E}_M = \text{Poly} \left( \sqrt{\frac{1}{\alpha_m \log \frac{4T}{\delta}}} \right)$.

Proof. By Lemma B.4 the first assumption in Theorem B.2 holds w.p. $1 - \frac{\delta}{2}$ for $\delta_t^2 = \eta_t G^2 m_t + \sqrt{8m_t \log \frac{4T}{\delta}}$, where note that $\eta_t \leq \frac{D}{c \sqrt{m_{\min}}} \forall t \in [T]$. By Lemma B.5 the second assumption holds w.p. $1 - \frac{\delta}{2}$ for $\epsilon_t^2 = \frac{8m_t \log \frac{4T}{\delta}}{4}$. $\square$
B.3. Online-to-Batch Conversion for Task-Averaged Regret (Theorem 3.3)

Theorem B.3. Consider a sequence of $T m$ convex loss functions $\ell_{t,i} : \Theta \mapsto [0, 1]$ generated by sampling $T$ i.i.d. distributions $P_t \sim Q, t \in [T]$ and then sampling $m$ i.i.d. loss functions $\ell_{t,i} \sim P_t, i \in [m]$ from each. Let $A$ be an online algorithm with TAR $\bar{R}_M(A)$ whose instantiations before each task $t$ are denoted by $A_t$. If we pick $\hat{A}$ uniformly at random from these instantiations and use it to sequentially pick actions $\theta_1, \ldots, \theta_m$ for $m$ new loss functions $\ell_1, \ldots, \ell_m \sim P$ sampled i.i.d. from a new distribution $P \sim Q$ then w.p. $1 - \delta$ for $\bar{\theta} = \frac{1}{m} \sum_{i=1}^{m} \theta_i$, we will have

$$E \ E_{P \sim Q} \ell_\bar{\theta} \leq E \left( L_m + \frac{36}{m} \log \frac{2mL_m + 6}{\delta} \right) + 2 \sqrt{\frac{E}{m} \log \frac{2mL_m + 6}{\delta}}$$

where the outer expectation is over the randomness of sampling $\hat{A} \sim U\{A_1, \ldots, A_T\}$ and

$$E = \min_{\theta \in \Theta} \left( \frac{1}{m} \sum_{i=1}^{m} \ell_i(\theta) \right) + \frac{2 \bar{R}_M(A)}{m} + \sqrt{\frac{2}{T} \log \frac{2}{\delta}}$$

Proof. By Proposition A.1 and Jensen’s inequality we have w.p. $1 - \frac{\delta}{2}$

$$E \ E_{P \sim Q} \ell(\bar{\theta}) \leq E \ E_{P \sim Q} \left( L_m + \frac{36}{m} \log \frac{2mL_m + 6}{\delta} \right) + 2 \sqrt{\frac{E}{m} \log \frac{2mL_m + 6}{\delta}}$$

for $L_m = \frac{1}{m} \sum_{i=1}^{m} \ell_i(\theta_i)$. Substituting the definition of regret and applying Proposition A.2 over the sequence of regrets $\frac{1}{2m} R_m(\cdot) + \frac{1}{2}$, which map to $[0, 1]$ since the losses are in $[0, 1]$, yields

$$E \ E_{P \sim Q} L_m = E \ E_{P \sim Q} \min_{\theta \in \Theta} \left( \frac{1}{m} \sum_{i=1}^{m} \ell_i(\theta) + \frac{R_m(\hat{A})}{m} \right)$$

$$= \ E_{\ell_i \sim P, \theta \in \Theta} \left( \min_{\theta \in \Theta} \frac{1}{m} \sum_{i=1}^{m} \ell_i(\theta) \right) + \frac{1}{T} \sum_{t=1}^{T} \frac{R_m(A_t)}{m}$$

$$\leq \ E_{\ell_i \sim P, \theta \in \Theta} \left( \min_{\theta \in \Theta} \frac{1}{m} \sum_{i=1}^{m} \ell_i(\theta) \right) + \frac{2}{T} \sum_{t=1}^{T} \frac{R_m(A_t)}{m} + \sqrt{\frac{2}{T} \log \frac{2}{\delta}}$$

w.p. $1 - \frac{\delta}{2}$. Combining these two bounds yields the result.
C. Computing the Quadratic Growth Factor

For our analysis of the FLI variants of Algorithm 2 we consider a class of functions related to strongly convex functions that satisfy the quadratic growth (QG) condition:

$$\frac{\alpha}{2} \|\theta - \theta^*\|^2 \leq f(\theta) - f(\theta^*) \quad (11)$$

By Theorem 2 of Karimi et al. (2016), in the convex case QG is equivalent, up to multiplicative constants, with the Polyak-Łojaciewicz (PL) inequality (Polyak, 1963). Using the latter condition, Karimi et al. (2016) further show that functions of form $f(A\theta)$ for $f$ strongly-convex satisfy the PL inequality, and thus also QG, with constant $\alpha = \Omega(\sigma_{\min}(A))$. This provides data-dependent guarantees for a variety of practical problems, including least-squares and logistic regression. Garber (2019) shows a similar result for expectations of such functions with the QG constant depending now on $\lambda_{\min}(E A^T A)$; in order to do so they assume the constraint set is a polytope, e.g. an $\ell_1$ or $\ell_\infty$ ball.

For our results we require a stronger condition, namely that if $L$ is a sum of $m$ convex losses then $L$ satisfies $\alpha m$-QG. While this additive property holds directly if the losses are strongly-convex, in the general case it does not. Furthermore, the spectral lower bound on $\alpha$ studied by Karimi et al. (2016) and Garber (2019) is an underestimate; for example, in the strongly-convex case, where $A^T A$ is the identity, the lower bound will be 1 even though their sum is $m$-QG.

Here we derive an alternative approach for verifying $\alpha$-QG for a convex Lipschitz function $f$ constrained to a ball of radius $B$. Note that since the functions are Lipschitz, we can focus on computing the minimal difference between $f(\theta)$ and $f(\theta^*)$ over all $\theta$ located some fixed distance $\delta$ away from any minimizer $\theta^*$ of $f$ over the ball:

$$\varepsilon_{\delta} = \min \quad f(\theta) - f(\theta^*)$$
$$\text{s.t.} \quad \|\theta - \theta^*\|_2^2 \geq \delta^2$$
$$\|\theta\|_2 \leq B$$

Then if $f$ is $\alpha$-QG, Equation 11 implies that $\alpha_{\delta} = \frac{2\varepsilon_{\delta}}{\delta^2}$ should be a constant, or equivalently that $\varepsilon_{\delta} = \Omega(\delta^2)$. While the above problem is non-convex due to the first constraint, note that

$$\delta^2 \leq \|\theta - \theta^*\|_2^2 = \|\theta\|_2^2 - 2\langle \theta, \theta^* \rangle + \|\theta^*\|_2^2 \leq B^2 - 2\langle \theta, \theta^* \rangle + \|\theta^*\|_2^2$$

which is a linear constraint since $\theta^*$ is constant. Therefore we have

$$\varepsilon_{\delta} \geq \min \quad f(\theta) - f(\theta^*)$$
$$\text{s.t.} \quad 2\langle \theta^*, \theta \rangle \leq B^2 - \delta^2 + \|\theta^*\|_2^2$$
$$\|\theta\|_2 \leq B$$

which is a convex program amenable to standard solvers; we employ the Frank-Wolfe method (Frank & Wolfe, 1956).
D. Experimental Details

D.1. Constructing Mini-Wikipedia

We briefly describe the construction of Mini-Wikipedia. First we take Wikipedia pages whose titles correspond to lemmas in the WordNet corpus (Fellbaum, 1998). We then use the hypernymy structure in this corpus to separate the pages into four semantically meaningful meta-classes; this is necessary when using linear classification as the task similarity only depends on the classifier and not the representation. Finally, we take the longest sentences from each page to construct $m$-shot tasks of $4m$ samples each, for $m = 1, 2, 4, \ldots, 32$.

D.2. Complete Deep Learning Results

Below are plots for all evaluations on Omniglot and Mini-ImageNet. As our algorithm generalizes the Reptile method of Nichol et al. (2018), we use code they make available at https://github.com/openai/supervised-reptile and vary the parameters train-shots and inner-itors.

Figure 7. Performance of the FLI variant of Ephemeral with OGD within-task (Reptile) on 5-way Mini-ImageNet when varying the number of task samples and the number of iterations per training task. In the left-hand plots we use 1-shot at meta-test time; in the right-hand plots we use 5-shots. 50 iterations are used at meta-test time in both cases.

Figure 8. Performance of the FLI variant of Ephemeral with OGD within-task (Reptile) on 5-way Omniglot when varying the number of task samples and the number of iterations per training task. In the left-hand plots we use 1-shot at meta-test time; in the right-hand plots we use 5-shots. 50 iterations are used at meta-test time in both cases.

Figure 9. Performance of the FLI variant of Ephemeral with OGD within-task (Reptile) on 20-way Omniglot when varying the number of task samples and the number of iterations per training task. In the left-hand plots we use 1-shot at meta-test time; in the right-hand plots we use 5-shots. 50 iterations are used at meta-test time in both cases.