BI-SPATIAL RANDOM ATTRACTOR, ERGODICITY AND A RANDOM
LIOUVILLE TYPE THEOREM FOR STOCHASTIC NAVIER-STOKES
EQUATIONS ON THE WHOLE SPACE

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ABSTRACT. This article concerns the random dynamics and asymptotic analysis of the well
known mathematical model,
\[
\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + \nabla p = f, \quad \nabla \cdot v = 0,
\]
the Navier-Stokes equations. We consider the two-dimensional stochastic Navier-Stokes
equations (SNSE) driven by a linear multiplicative white noise of Itô type on the whole space \( \mathbb{R}^2 \). Firstly, we prove that non-autonomous 2D SNSE generates a bi-spatial (\( L^2(\mathbb{R}^2), H^1(\mathbb{R}^2) \))-continuous random cocycle. Due to the bi-spatial continuity property of the random cocycle
associated with SNSE, we show that if the initial data is in \( L^2(\mathbb{R}^2) \), then there exists a unique
bi-spatial (\( L^2(\mathbb{R}^2), H^1(\mathbb{R}^2) \))-pullback random attractor for non-autonomous SNSE which is
compact and attracting not only in \( L^2 \)-norm but also in \( H^1 \)-norm. Next, we discuss the
existence of an invariant measure for the random cocycle associated with autonomous SNSE
which is a consequence of the existence of random attractors. We prove the uniqueness of
invariant measures for \( f = 0 \) and for any \( \nu > 0 \) by using the linear multiplicative structure
of the noise coefficient and exponential stability of solutions. Finally, we prove the existence
of a family of invariant sample measures for 2D autonomous SNSE which satisfies a random
Liouville type theorem. The above results for SNSE defined on \( \mathbb{R}^2 \) are totally new and we
observe that in contrast to Stratonovich noise, which is used widely in the literature to study
the random dynamics of SNSE, Itô noise is more adequate in the case of whole space. This
work settles down several open problems regarding random attractors, invariant measures
and ergodicity for 2D SNSE on whole space.

1. Introduction

1.1. Literature survey and motivations. It is well known that the explicit solutions of or-
dinary/partial differential equations (ODE/PDE) are very difficult to find (cf. [32, 54] and
references therein). Therefore, one requires a qualitative theory to understand the asym-
ptotic behavior of their solutions. The theory of attractors plays an important role to capture
the long time behavior of the solutions of ODE/PDE. The theory of global attractors for
the deterministic infinite-dimensional dynamical systems (DS) is well-investigated in [50, 55],
etc. The theory of global attractors (for deterministic DS) had been extended to random

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attractors (for compact random DS, cf. [1]) in the works [8, 9] etc., and it has been used for several physically relevant stochastic models (cf. [15, 33, 36, 66] etc. and references therein). The author in [58] introduced a sufficient and necessary criteria for the existence of a unique pullback random attractor for non-compact non-autonomous random DS and applied it to stochastic reaction-diffusion equations driven by additive noise. Later, it has been applied to several stochastic models, cf. [25, 35, 59, 60, 62], etc., and references therein.

The current work is mainly focused on the random dynamics of autonomous and non-autonomous 2D stochastic Navier-Stokes equations (SNSE) driven by a linear multiplicative white noise defined on the whole space. Given \( \tau \in \mathbb{R} \), we consider the following 2D SNSE on \( \mathbb{R}^2 \):

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p &= \mathbf{f} + \sigma \mathbf{v} \frac{dW}{dt}, & \text{in } \mathbb{R}^2 \times (\tau, \infty), \\
\nabla \cdot \mathbf{v} &= 0, & \text{in } \mathbb{R}^2 \times (\tau, \infty), \\
\mathbf{v}|_{t=\tau} &= \mathbf{v}_\tau, & x \in \mathbb{R}^2 \text{ and } \tau \in \mathbb{R}, \\
\mathbf{v}(x) &\to 0 & \text{as } |x| \to \infty,
\end{align*}
\]

where \( \mathbf{v}(x,t) \in \mathbb{R}^2 \) stands for the velocity field and \( p(x,t) \in \mathbb{R} \) denotes the pressure field, for all \( (x, t) \in \mathbb{R}^2 \times (\tau, +\infty) \). The external forcing \( \mathbf{f} \) is either autonomous (\( \mathbf{f}(x) \in \mathbb{R}^2 \), for \( x \in \mathbb{R}^2 \)) or non-autonomous (\( \mathbf{f}(x,t) \in \mathbb{R}^2 \), for \( (x, t) \in \mathbb{R}^2 \times (\tau, +\infty) \)). The coefficient \( \nu \) is positive and it represents the kinematic viscosity of the fluid and \( \sigma > 0 \) is known as the noise intensity. Here, the stochastic integral should be understood in the sense of Itô and \( W = W(t, \omega) \) is a one-dimensional two-sided Wiener process defined on some filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P}) \) (see Section 2).

The solvability of 2D deterministic/stochastic Navier-Stokes equations (NSE) on bounded or unbounded domains is well studied in the literature (existence as well as uniqueness), cf. [18, 21, 23, 45, 53, 55] etc., and references therein. In three-dimensions, the existence of at least one weak solution of 3D NSE is known due to Leary and Hopf (cf. [41, 29]). But the uniqueness is still a mathematically challenging open problem. Recently, in [4], it has been shown by a convex integration technique that the weak solutions of 3D Navier-Stokes equations are not unique in the class of weak solutions with finite kinetic energy. Therefore, in this work, our focus is on two-dimensions only.

The dynamics of 2D deterministic/stochastic NSE including the existence of unique global/pullback/random attractors also have a plenty of literature, cf. [2, 5, 22, 25, 51, 52, 55, 60] etc., and references therein. It has been observed that the existence of global/pullback/random attractors for 2D deterministic/stochastic NSE is well-studied either on periodic (cf. [10, 51, 55] etc.), bounded ([8, 25, 36] etc.), and unbounded Poincaré ([2, 5, 26, 52] etc.) domains, and not on the whole space. The existence of random attractors is based on a transformation which converts a stochastic system into an equivalent pathwise deterministic system. Such types of transformations are available in the literature when the noise is either additive or linear multiplicative. Moreover, it has been noticed in the literature that the SNSE driven by a linear multiplicative noise has been considered in the sense of Stratonovich. We consider the SNSE driven by a linear multiplicative noise in the Itô sense which is one of the motivations to do this work on the whole space \( \mathbb{R}^2 \). The existence of global/pullback attractors for deterministic NSE and random/pullback random attractors for SNSE driven by additive noise on the whole space is still an open problem.
The first aim of this work is to show the existence of a unique bi-spatial pullback random attractor for the non-autonomous SNSE (1.1) on the whole space $\mathbb{R}^2$. Stochastic PDE such as 2D SNSE (cf. [45]), stochastic convective Brinkman-Forchheimer equations (cf. [47]), stochastic fractional power dissipative equation (cf. [67]), stochastic semilinear Laplacian equations (cf. [42]) etc., have regularizing solutions. In other words, when initial data is in a Banach space (an initial space), the corresponding solution may belong to a more regular Banach space (a terminate space). To represent the better dynamics for such equations, the authors in [42] introduced the concept of bi-spatial random attractor which is an invariant and compact random set attracting the subsets of the initial space in the topology of the terminate space. Later, the authors in [11] generalized the work of [42] for non-autonomous random DS also, which has been applied to several physically relevant models, cf. [61, 63], etc., and references therein. Recently, the author in [67] introduced an abstract theory of bi-spacial pullback attractors for the bi-spacial continuous cocycle and successfully applied it to the stochastic fractional power dissipative equation. To the best of our knowledge, there is no result available in the literature on the existence of a unique bi-spatial pullback random attractor for 2D SNSE either on unbounded Poincaré domains or on the whole space $\mathbb{R}^2$.

The second goal of this article is to demonstrate the ergodic properties (via proving the existence of a unique invariant measure) for the 2D autonomous SNSE (1.1) which is one of the most considerable qualitative properties of a dynamical system. The ergodic properties of the randomly forced NSE on torus or bounded domains have been extensively studied by several researchers during the past two decades. The first work on the ergodicity for 2D SNSE on bounded domains (for any $\nu > 0$) driven by additive noise (see [19], for the existence of invariant measures) was done in [20], where the uniqueness of invariant measures was proved under some extra assumptions on the noise (cf. [20] for more details). But, in [16], the author proved the same results as in [20] under weaker assumptions on the noise. Later, instead of taking extra assumptions on the noise, the author in [43] proved the ergodicity for 2D SNSE driven by additive noise for sufficiently large $\nu$. The papers [17, 27, 28, 37, 39, 64, 65] etc., discussed various results on the ergodic behavior of different types of partial differential equations covering 2D SNSE and the works [13, 44], etc. are good expository articles on this subject, and references therein. The authors in [24, 39], etc. demonstrated the ergodicity property for 2D SNSE using the asymptotic coupling method, where the proof of uniqueness of invariant measures does not require any restriction on $\nu > 0$. In the literature, most of the works regarding the existence and uniqueness of invariant measures as well as ergodic behavior of 2D SNSE have been considered either on a torus or on bounded domains. In some of the works such as [2, 3] etc., the authors established the existence of invariant measures on unbounded Poincaré domains but they have not discussed the uniqueness. Recently, in [49], the author proved the ergodicity results for 2D SNSE on unbounded Poincaré domains, where the author used a random force of additive type having a special structure. It appears to us that the current work is the first one in which the existence and uniqueness of ergodic and strongly mixing invariant measures for 2D autonomous SNSE driven by a linear multiplicative noise in the whole space $\mathbb{R}^2$ is discussed for any $\nu > 0$ and the results are new in the context of multiplicative noise as well as for 2D SNSE on the whole space $\mathbb{R}^2$.

The final objective of this paper is to prove the existence of a family of invariant sample measures (see Definition 5.1 below) which satisfies a random Liouville type theorem (cf. [69]) for the 2D autonomous SNSE (1.1) on the whole space $\mathbb{R}^2$. This result is obtained with the aid of the existence of random attractors and an abstract theory available in [69, Theorem...
2.1. The authors in [69] established a sufficient criteria which guarantees the existence of a family of invariant sample measures satisfying a random Liouville type theorem. If the random statistical equilibrium has been reached by the 2D SNSE (1.1), then the statistical informations do not change with time, which is the random version of the Liouville Theorem in Statistical Mechanics (see [69]). If the shape of the random attractor \(A\) changes randomly with respect to time along with the sample points \(\omega \in \Omega\) but the measures of \(A\) on each time is the same, then the random Liouville theorem is satisfied by the family of measures. Since we are not proving the same as discussed above (see Theorem 5.7 below), therefore we call it as a random Liouville type theorem.

1.2. Difficulties and approaches. If one considers a damped 2D deterministic/stochastic NSE with linear damping, that is,

\[
\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + \alpha v + \nabla p = f + \sigma v \frac{dW}{dt}, \quad \nabla \cdot v = 0,
\]

where \(\alpha > 0\) and \(\sigma \geq 0\), then the existence of global/pullback/random attractors can be proved on the whole space (cf. [30, 31] etc.). Even though we do not have any linear damping term in (1.1), we obtain an equivalent pathwise deterministic system which contains a linear term (it is the advantage of considering the stochastic system (1.1) in the sense of Itô), see the system (2.8) below. This linear term helps us to prove the results on the whole space.

In order to prove the bi-spatial \((L^2(\mathbb{R}^2), H^1(\mathbb{R}^2))\)-pullback random attractors for the system (1.1), we need to estimate the nonlinear term \(\int_{\mathbb{R}^2} (u(x) \cdot \nabla) u(x) \cdot \Delta u(x) dx\) (see (3.9) below) in an appropriate way. We observe that one cannot use the estimates obtained in [56, Chapter 2, section 2.3] to estimate the above nonlinear term. In view of Hölder’s, Ladyzhenskaya’s ([40, Lemma 1, Chapter I]), Sobolev’s ([46, Theorem 7, pp. 190]) and interpolation ([46, Theorem 6, pp. 200]) inequalities, we estimate the above mentioned nonlinear term (see Remark 2.1 below). Note that for the periodic case, this integral is zero [56, Lemma 3.1], and for the bounded or unbounded Poincaré domains case, one can use the estimates form [56, Chapter 2, section 2.3] to obtain the results similar to this work. As the existence of global attractors for 2D deterministic NSE ((1.1) with \(\sigma = 0\)) is still an open problem, the upper semicontinuity of the obtained random attractors is also an open problem (Remark 3.13).

The existence of random attractors helps us to obtain the existence of invariant measures (see [9, Corollary 4.6] and Definition 4.3 below), and a family of invariant sample measures ([69, Theorem 2.1] and Definition 5.1 below) which satisfies the random Liouville type theorem for the system (1.1) (cf. [9, 69] etc.). The main difficulties arise in the uniqueness of invariant measures part. We consider the deterministic forcing term \(f = 0\) and obtain the uniqueness of invariant measure for any \(\nu > 0\), where the linear multiplicative structure of the white noise coefficient and exponential stability of solutions (cf. [7]) play a crucial role. The uniqueness of invariant measure for \(f \neq 0\) is still an open problem for 2D SNSE in \(\mathbb{R}^2\).

1.3. Novelties. Most of the results regarding the random dynamics and asymptotic analysis of 2D SNSE available in the literature are either on bounded domains (cf. [16, 24, 27, 36, 60], etc.) or on unbounded Poincaré domains (cf. [2, 3, 49], etc.). This work settles down several open problems concerning random dynamics of 2D SNSE defined on the whole space and this work appears to be the first one in this direction. Moreover, we emphasize here that all the results obtained in this paper hold true even in bounded as well as unbounded Poincaré domains. The major aims and novelties of this work are:
(i) For the 2D non-autonomous SNSE (1.1), we prove the existence of unique bi-spatial \((L^2(R^2), H^1(R^2))\)-pullback random attractors (Theorem 3.12).

(ii) For the 2D autonomous SNSE (1.1), for any \(f \in L^2(R^2)\), we establish the existence of an invariant measure in \(L^2(R^2)\) as well as in \(H^1(R^2)\) (Theorem 4.8 and Remark 4.9) and its uniqueness for \(f = 0\) (Theorem 4.11).

(iii) For the 2D autonomous SNSE (1.1), we show the existence of a family of invariant sample measures on \(L^2(R^2)\) which satisfies a random Liouville type theorem (Theorems 5.5 and 5.7).

The linear multiplicative structure of the Itô type noise coefficient helps us to resolve the above mentioned problems in \(R^2\). Moreover, we provide some remarks on the existence of \((L^2(R^2), L^2(R^2))\)-pullback random attractors, the upper semicontinuity of the random attractors with respect to domains, and the asymptotic autonomy of the obtained random attractors as well (Remarks 3.5, 3.14 and 3.15).

1.4. Outline. In the next section, we provide the necessary function spaces needed for the further analysis, and linear and bilinear operators to obtain an abstract formulation of the system (1.1). Furthermore, we have furnished an abstract formulation of the system (1.1), a pathwise deterministic system (2.8) which is equivalent to the system (1.1), random cocycle (2.9), and the universe of tempered sets, in the same section. In section 3, we first provide an abstract result for the existence of a unique bi-spatial pullback random attractor which is adapted from the work [67, Theorem 2.10] (Theorem 3.1). In order to apply the abstract result of Theorem 3.1, we prove that the random cocycle generated by the system (1.1) is a bi-spatial \((L^2(R^2), H^1(R^2))\)-continuous cocycle (Lemma 3.7). Next, we prove that the random cocycle has a random absorbing set (Lemma 3.9), the random cocycle is \((L^2(R^2), L^2(R^2))\)-pullback asymptotically compact (Lemma 3.10) and continuous with respect to the sample points (Lemma 3.11). Finally, using an abstract theory (Theorem 3.1), we establish the main result of this section (Theorem 3.12). We start with some basic definitions for invariant measures (adapted from [1]) in section 4. Next, we show that there exists an invariant measure for the system (1.1) in \(L^2(R^2)\) as well as in \(H^1(R^2)\) (Theorem 4.8 and Remark 4.9) for any \(f \in L^2(R^2)\) and the uniqueness of invariant measure for \(f = 0\) (Theorem 4.11). In the final section, we prove the existence of a family of invariant sample measures which satisfies the random Liouville type theorem for the system (1.1) (Theorems 5.5 and 5.7).

2. Mathematical Formulation

This section is devoted for providing the necessary function spaces and operators needed to obtain the main results of this work. Further, we provide an abstract formulation of the system (1.1), a pathwise deterministic system equivalent to the system (1.1) (which helps us to define the non-autonomous random dynamical system) and the universe of tempered sets.

2.1. Function spaces. We define the space \(V := \{v \in C^\infty_0(R^2; R^2) : \nabla \cdot v = 0\}\), where \(C^\infty_0(R^2; R^2)\) denotes the space of all \(R^2\)-valued infinite times differentiable functions with compact support in \(R^2\). Let \(H\) and \(V\) denote the completion of \(V\) in \(L^2(R^2) := L^2(R^2; R^2)\) and \(H^1(R^2) := H^1(R^2; R^2)\) norms, respectively. The spaces \(H\) and \(V\) are endowed with the norms \(\|v\|_H := \int_{R^2} |v(x)|^2 dx\) and \(\|v\|_V^2 := \int_{R^2} |v(x)|^2 dx + \int_{R^2} |\nabla v(x)|^2 dx\), respectively. The inner product in the Hilbert space \(H\) is represented by \(\langle \cdot, \cdot \rangle\). The duality pairing between the spaces \(V\) and \(V^*\) is denoted by \(\langle \cdot, \cdot \rangle\). The \(m\)-th order Sobolev spaces are denoted by
\[ \mathbb{H}^m(\mathbb{R}^2) := H^m(\mathbb{R}^2; \mathbb{R}^2) \] with the norm \[ \|v\|_{\mathbb{H}^m(\mathbb{R}^2)}^2 := \sum_{|\alpha| \leq m} \int_{\mathbb{R}^2} |D^\alpha v|^2 \, dx, \] where \( k = (k_1, k_2) \) is the multiindex.

2.2. **Linear operator.** Let \( P : L^2(\mathbb{R}^2) \to \mathbb{H} \) denote the Helmholtz-Hodge orthogonal projection (cf. [40]). The projection operator \( P : L^2(\mathbb{R}^2) \to \mathbb{H} \) can be expressed in terms of the Riesz transform (cf. [48]), and the operators \( P \) and \( \Delta \) commute, that is, \( P\Delta = \Delta P \). Let us define the Stokes operator

\[ Av := -P\Delta v = -\Delta v, \quad v \in D(A) := \mathcal{V} \cap \mathbb{H}^2(\mathbb{R}^2). \]

The operator \( A : \mathcal{V} \to \mathcal{V}^* \) is linear and continuous. Moreover, the usual norm of \( \mathbb{H}^m(\mathbb{R}^2) \) is equivalent to (cf. [46, Proposition 1, pp. 169])

\[ \|v\|_{\mathbb{H}^m(\mathbb{R}^2)} = \left( \int_{\mathbb{R}^2} (1 + |\xi|^2)^m |\hat{\varphi}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} = \| (I - \Delta)^{\frac{m}{2}} v \|_{\mathbb{H}} = \| (I + A)^{\frac{m}{2}} v \|_{L^2(\mathbb{R}^2)}, \quad (2.1) \]

where \( \hat{\varphi}(\cdot) \) is the Fourier transform of \( \varphi(\cdot) \).

2.3. **Bilinear operator.** Let us define the trilinear form \( b(\cdot, \cdot, \cdot) : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R} \) by

\[ b(u, v, w) = \int_{\mathbb{R}^2} (u(x) \cdot \nabla) v(x) \cdot w(x) \, dx = \sum_{i,j=1}^2 \int_{\mathbb{R}^2} u_i(x) \frac{\partial v_i(x)}{\partial x_i} w_j(x) \, dx. \]

An integration by parts gives

\[ \begin{cases} b(u, v, u) = 0, & \text{for all } u, v \in \mathcal{V}, \\ b(u, v, w) = -b(u, w, v), & \text{for all } u, v, w \in \mathcal{V}. \end{cases} \quad (2.2) \]

If \( u, v \) are such that the linear map \( b(u, v, \cdot) \) is continuous on \( \mathcal{V} \), the corresponding element of \( \mathcal{V}^* \) is denoted by \( B(u, v) \). We also denote \( B(v) = B(v, v) = P[(v \cdot \nabla)v] \).

**Remark 2.1.** The following estimate on \( b(\cdot, \cdot, \cdot) \) plays a crucial role in the sequel. Applying Hölder’s, Ladyzhenskaya’s ([40, Lemma 1, Chapter I]), Sobolev’s ([46, Theorem 7, pp. 190]) and interpolation ([46, Theorem 6, pp. 200]) inequalities, and (2.1), respectively, we get

\[ \begin{align*}
|b(u, v, w)| & \leq \|u\|_{L^1(\mathbb{R}^2)} \|\nabla v\|_{L^1(\mathbb{R}^2)} \|w\|_{\mathbb{H}} \\
& \leq C \|u\|_{\mathbb{H}}^{1/2} \|\nabla u\|_{\mathbb{H}}^{1/2} \|v\|_{\mathbb{H}} \|w\|_{\mathbb{H}} \\
& \leq C \|u\|_{\mathbb{H}}^{1/2} \|\nabla u\|_{\mathbb{H}}^{1/2} \|v\|_{\mathbb{H}} \|v\|_{L^2(\mathbb{R}^2)} \|w\|_{\mathbb{H}} \\
& \leq C \|u\|_{\mathbb{H}}^{1/2} \|\nabla u\|_{\mathbb{H}}^{1/2} \left( \left\|v\right\|_{\mathbb{H}}^{1/2} + \|\nabla v\|_{\mathbb{H}} \right) \|v\|_{\mathbb{H}} \|v\|_{\mathbb{H}}^{1/2} \|w\|_{\mathbb{H}} \\
& \leq C \|u\|_{\mathbb{H}}^{1/2} \|\nabla u\|_{\mathbb{H}}^{1/2} \|v\|_{\mathbb{H}}^{1/2} \left( \left\|v\right\|_{\mathbb{H}}^{1/2} + \|A v\|_{\mathbb{H}} \right) \|w\|_{\mathbb{H}} \\
& \quad + C \|u\|_{\mathbb{H}}^{1/2} \|\nabla u\|_{\mathbb{H}}^{1/2} \|v\|_{\mathbb{H}}^{1/2} \left( \left\|v\right\|_{\mathbb{H}}^{1/2} + \|A v\|_{\mathbb{H}} \right) \|w\|_{\mathbb{H}}. 
\end{align*} \quad (2.3) \]

**Remark 2.2.** Note that \( \langle B(u, u - v), u - v \rangle = 0 \), which implies that

\[ \langle B(u) - B(v), u - v \rangle = \langle B(u - v, v), u - v \rangle = -\langle B(u, v - u), v \rangle. \quad (2.4) \]
2.4. Abstract formulation. Taking the projection $\mathcal{P}$ on SNSE (1.1), we obtain for $t \geq \tau$, $\tau \in \mathbb{R}$

$$\begin{align*}
\begin{cases}
\frac{dv}{dt} + \nu Au + B(v) = f + \sigma v \frac{dW}{dt}, & \text{in } \mathbb{R}^2 \times (\tau, \infty), \\
v|_{t=\tau} = v_\tau, & x \in \mathbb{R}^2,
\end{cases}
\end{align*}$$

(2.5)

where $\sigma > 0$ and the stochastic integral is understood in the Itô sense. Here, $W(t, \omega)$ is the standard scalar Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega = \{\omega \in C(\mathbb{R}; \mathbb{R}) : \omega(0) = 0\},$$

endowed with the compact-open topology given by the complete metric

$$d_\Omega(\omega, \omega') := \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\|\omega - \omega'\|_m}{1 + \|\omega - \omega'\|_m},$$

where $\|\omega - \omega'\|_m := \sup_{-m \leq t \leq m} |\omega(t) - \omega'(t)|$,

and $\mathcal{F}$ is the Borel sigma-algebra induced by the compact-open topology of $(\Omega, d_\Omega)$. $\mathbb{P}$ is the two-sided Wiener measure on $(\Omega, \mathcal{F})$. Also, define $\{\theta_t\}_{t \in \mathbb{R}}$ by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R} \text{ and } \omega \in \Omega.$$

Hence, $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system (MDS). Moreover, there exists a $\theta_\tau$-invariant set $\tilde{\Omega} \subseteq \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that for each $\omega \in \tilde{\Omega}$,

$$\frac{|\omega(t)|}{t} \to 0 \text{ as } |t| \to +\infty. \quad (2.6)$$

Throughout this work, we will not distinguish between $\tilde{\Omega}$ and $\Omega$.

Next, for a given $t \in \mathbb{R}$ and $\omega \in \tilde{\Omega}$, let $z(t, \omega) = e^{-\sigma \omega(t)}$. Then, $z$ satisfies the equation

$$dz = \frac{\sigma^2}{2} zdt - \sigma z dW. \quad (2.7)$$

Let $u$ be a new variable given by

$$u(t; \tau, \omega, u_\tau) = z(t, \omega)v(t; \tau, \omega, v_\tau) \quad \text{with} \quad u_\tau = z(\tau, \omega)v_\tau,$$

where $v(t; \tau, \omega, v_\tau)$ and $z(t, \omega)$ are the solutions of (2.5) and (2.7), respectively. Then $u(\cdot; \tau, \omega, u_\tau)$ satisfies the following:

$$\begin{align*}
\begin{cases}
\frac{du}{dt} + \nu Au + \alpha u + \frac{1}{z(t, \omega)} B(u) = z(t, \omega)f, & t > \tau, \\
|u|_{t=\tau} = u_\tau, & x \in \mathbb{R}^2,
\end{cases}
\end{align*}$$

(2.8)

in $V^*$, where $\alpha = \frac{\sigma^2}{2} > 0$. An application of the standard Faedo-Galerkin approximation method ensures that for all $t > \tau$, $\tau \in \mathbb{R}$, for every $u_\tau \in H$ and $f \in L^2_{\text{loc}}(\mathbb{R}; V^*)$, (2.8) has a unique weak solution $u \in C([\tau, +\infty); H) \cap L^2_{\text{loc}}(\tau, +\infty; V)$ (cf. [55]). Furthermore, if $f \in L^2_{\text{loc}}(\mathbb{R}; H)$, then one can show that $u \in C((\tau, +\infty); V)$ (cf. [51, Corollary 4], see Lemma 3.6 below). For $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $v_\tau \in H$, define a map $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H$ given by

$$\Phi(t, \tau, \omega, v_\tau) = v(t + \tau; \tau, \theta_{-\tau} \omega, v_\tau) = \frac{u(t + \tau; \tau, \theta_{-\tau} \omega, u_\tau)}{z(t + \tau, \theta_{-\tau} \omega)}, \quad (2.9)$$
with \( \mathbf{u}_\tau = z(\tau, \theta_{-\tau}\omega)\mathbf{v}_\tau \). Note that \( \Phi \) is a random cocycle (cf. [59]). Assume that \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is a family of non-empty subsets of \( \mathbb{H} \) satisfying, for every \( c > 0, \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\lim_{t \to -\infty} e^{ct} \| D(\tau + t, \theta_t\omega) \|_{\mathbb{H}}^2 = 0.
\] (2.10)

Let us define the \textit{universe} of tempered subsets of \( \mathbb{H} \) as

\[
\mathcal{D} := \{ D = \{ D(\tau, \omega) : \tau \in \mathbb{R} \text{ and } \omega \in \Omega \} : D \text{ satisfying (2.10)} \}.
\]

3. Bi-spatial Pullback Random Attractor for SNSE

This section is devoted to establish the existence of unique (\( \mathbb{H}, \mathcal{V} \))-pullback random attractors (belongs to class \( \mathcal{D} \)) for non-autonomous SNSE on the whole space \( \mathbb{R}^2 \). In order to prove the existence of unique bi-spatial pullback random attractors, we use an abstract result established in the work [67]. The following theorem is adapted from the work [67]. For the basic definition of the terms that are used in the following theorem, the readers are referred to see [67, Section 2].

**Theorem 3.1** (Theorem 2.10, [67]). Let \( (X, d_X) \) and \( (Y, d_Y) \) be two complete metric spaces. Let \( \Phi \) be a random cocycle on \( X \) (over MDS \( (\Omega, \mathcal{F}, \mathcal{P}, \{ \theta_t \}_{t \in \mathbb{R}}) \)) which is \( (X, Y) \)-continuous and \( \mathcal{D} \) be inclusion closed universe in \( X \). Suppose that

(i) \( \Phi \) has a closed pullback \( \mathcal{D} \)-random absorbing set \( K = \{ K(\tau, \omega) : \tau \in \mathbb{R} \text{ and } \omega \in \Omega \} \in \mathcal{D} \) in \( X \);

(ii) \( \Phi \) is \( (X, X) \)-pullback asymptotically compact in \( X \);

(iii) For every fixed \( t > 0, \tau \in \mathbb{R} \) and \( x \in X \), the mapping \( \Phi(t, \tau, \cdot, x) : \Omega \to Y \) is \( (\mathcal{F}, \mathcal{B}(Y)) \)-measurable.

Then the random cocycle \( \Phi \) possesses a unique \( (X, Y) \)-pullback random attractor

\[
\mathcal{A} = \{ A(\tau, \omega) : \tau \in \mathbb{R} \text{ and } \omega \in \Omega \} \in \mathcal{D},
\]

where

\[
\mathcal{A}(\tau, \omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \Phi(t, \tau - t, \theta_{-t}\omega, K(\tau - t, \theta_{-t}\omega)) \chi_X.
\] (3.1)

Moreover, it can also be structured by the \( Y \)-metric, that is,

\[
\mathcal{A}(\tau, \omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \Phi(t, \tau - t, \theta_{-t}\omega, K(\tau - t, \theta_{-t}\omega)) \chi_Y.
\] (3.2)

The following assumption on the external forcing term \( f \) is needed to prove the results of this section.

**Hypothesis 3.2.** For the external forcing term \( f \in L^2_{\text{loc}}(\mathbb{R}; \mathbb{H}) \), there exists a number \( \delta \in [0, \alpha) \) such that for every \( c > 0 \),

\[
\lim_{s \to -\infty} e^{cs} \int_{-\infty}^{0} e^{\delta \zeta} \| f(\cdot, \zeta + s) \|_{\mathbb{H}}^2 d\zeta = 0.
\] (3.3)

A direct consequence of the above Hypothesis 3.2 is as follows.
Proposition 3.3 (Proposition 4.2, [35]). Assume that Hypothesis 3.2 holds. Then
\[ \int_{-\infty}^{\tau} e^{\delta \zeta} \| f(\cdot, \zeta) \|_{H^2}^2 d\zeta < \infty, \quad \text{for all } \tau \in \mathbb{R}, \] (3.4)
where \( \delta \) is the same as in (3.3).

Example 3.4. Take \( f(\cdot, t) = t^p f_1 \), for any \( p \geq 0 \) and \( f_1 \in \mathbb{H} \). Note that the conditions (3.4)-(3.3) do not need \( f \) to be bounded in \( \mathbb{H} \) at \( \pm \infty \).

Remark 3.5. One can prove the existence of unique pullback random attractors in \( \mathbb{H} \) by replacing \( \mathbb{H} \) with \( V^* \) in Hypothesis 3.2 (cf. [59]). But, in order to prove the existence of pullback random attractor in a more regular space \( V \), one has to consider Hypothesis 3.2.

The following Lemma helps us to prove the \((\mathbb{H}, V)\)-continuity of random cocycle \( \Phi \) as well as the \((\mathcal{F}, \mathcal{B}(V))\)-measurability of \( \Phi(t, \tau, \cdot, x) : \Omega \to V \), for every fixed \( t > 0 \), \( \tau \in \mathbb{R} \) and \( x \in \mathbb{H} \).

Lemma 3.6. For \( u_\tau \in \mathbb{H} \) and \( f \in L^p_{loc}(\mathbb{R}; \mathbb{H}) \), there exists random variables \( \rho(t, \tau, \omega, u_\tau, f) \), \( \tilde{\rho}(t, \tau, \omega, u_\tau, f) \) and \( \tilde{\rho}(t, \tau, \omega, u_\tau, f) \) such that
\[ \| u(t) \|_{\mathbb{H}}^2 + \| u(\zeta) \|_{\mathbb{H}}^2 d\zeta + 2\nu \int_\tau^t \| \nabla u(\zeta) \|_{\mathbb{H}}^2 d\zeta \leq \rho(t, \tau, \omega, u_\tau, f), \quad \text{for all } t \geq \tau, \] (3.5)
and
\[ \begin{aligned}
&\int t \| \nabla u(\zeta) \|_{\mathbb{H}}^2 d\zeta \leq \tilde{\rho}(t, \tau, \omega, u_\tau, f) \quad \text{and} \\
&\int t \| \nabla u(\zeta) \|_{\mathbb{H}}^2 d\zeta \leq \tilde{\rho}(t, \tau, \omega, u_\tau, f), \quad \text{for all } t > \tau.
\end{aligned} \] (3.6)

Proof. From the first equation of the system (2.8), (2.2) and Young’s inequality, we obtain
\[ \frac{d}{dt} \| u(t) \|_{\mathbb{H}}^2 + \alpha \| u(t) \|_{\mathbb{H}}^2 + 2\nu \| \nabla u(t) \|_{\mathbb{H}}^2 \leq \frac{z^2(t, \omega)}{\alpha} \| f(t) \|_{\mathbb{H}}^2, \] (3.7)
for a.e. \( t \geq \tau \), which gives
\[ \| u(t) \|_{\mathbb{H}}^2 + \alpha \int_\tau^t \| u(\zeta) \|_{\mathbb{H}}^2 d\zeta + 2\nu \int_\tau^t \| \nabla u(\zeta) \|_{\mathbb{H}}^2 d\zeta \leq \| u_\tau \|_{\mathbb{H}}^2 + \frac{1}{\alpha} \int_\tau^t z^2(\zeta, \omega) \| f(\zeta) \|_{\mathbb{H}}^2 d\zeta \\
\quad := \rho(t, \tau, \omega, u_\tau, f), \] (3.8)
for all \( t \geq \tau \). Taking the inner product of the first equation in (2.8) with \( \nabla u(\cdot) \), using (2.3) and Young’s inequality, we find
\[ \begin{aligned}
\frac{1}{2} \frac{d}{dt} \| \nabla u(t) \|_{\mathbb{H}}^2 + \nu \| \nabla u(t) \|_{\mathbb{H}}^2 + & \frac{\alpha}{2} \| u(t) \|_{\mathbb{H}}^2 + \frac{C}{z^4(t, \omega)} \| u(t) \|_{\mathbb{H}}^2 \| \nabla u(t) \|_{\mathbb{H}}^2 + \frac{C}{z^4(t, \omega)} \| u(t) \|_{\mathbb{H}}^2 \| \nabla u(t) \|_{\mathbb{H}}^2 \\
\leq & \frac{\nu}{2} \| \nabla u(t) \|_{\mathbb{H}}^2 + \frac{\alpha}{2} \| u(t) \|_{\mathbb{H}}^2 + \frac{C}{z^4(t, \omega)} \| u(t) \|_{\mathbb{H}}^2 \| \nabla u(t) \|_{\mathbb{H}}^2 \\
&+ \frac{C}{z^4(t, \omega)} \| u(t) \|_{\mathbb{H}}^2 \| \nabla u(t) \|_{\mathbb{H}}^2 + C z^2(t, \omega) \| f(t) \|_{\mathbb{H}}^2, \quad \text{(3.9)}
\end{aligned} \]
for a.e. $t \geq \tau$. From (3.9), we get
\[
\frac{d}{dt} \| \nabla u(t) \|_{H}^{2} \leq \frac{C}{z^{4}(t, \omega)} \| u(t) \|_{H}^{2} \| \nabla u(t) \|_{H}^{4} + \frac{C}{z^{4}(t, \omega)} \| u(t) \|_{H}^{6} + \frac{C}{z^{4}(t, \omega)} \| u(t) \|_{H}^{4} \| \nabla u(t) \|_{H}^{2} + \frac{C}{z^{2}(t, \omega)} \| u(t) \|_{H}^{3} \| \nabla u(t) \|_{H}^{2} + Cz^{2}(t, \omega) \| f(t) \|_{H}^{2},
\]
(3.10)
for a.e. $t \geq \tau$. For $t > \tau$, using (3.8), we have
\[
\int_{\tau}^{t} \| \nabla u(\zeta) \|_{H}^{2} d\zeta \leq \rho(t, \tau, \omega, u_{\tau}, f),
\]
(3.11)
\[
C \int_{\tau}^{t} \frac{\| u(\zeta) \|_{H}^{2} \| \nabla u(\zeta) \|_{H}^{2}}{z^{4}(\zeta, \omega)} d\zeta \leq C \sup_{\zeta \in [\tau, t]} \left\{ \frac{\| u(\zeta) \|_{H}^{2}}{z^{4}(\zeta, \omega)} \right\} \int_{\tau}^{t} \| \nabla u(\zeta) \|_{H}^{2} d\zeta
\]
\[
\leq C \sup_{\zeta \in [\tau, t]} \left\{ \frac{1}{z^{4}(\zeta, \omega)} \right\} \left[ \rho(t, \tau, \omega, u_{\tau}, f) \right]^{2},
\]
(3.12)
\[
:= \rho_{1}(t, \tau, \omega, u_{\tau}, f),
\]
\[
C \int_{\tau}^{t} \frac{\| u(\zeta) \|_{H}^{6}}{z^{4}(\zeta, \omega)} d\zeta + C \int_{\tau}^{t} \frac{\| u(\zeta) \|_{H}^{4} \| \nabla u(\zeta) \|_{H}^{2}}{z^{4}(\zeta, \omega)} d\zeta + C \int_{\tau}^{t} \frac{\| u(\zeta) \|_{H}^{2} \| \nabla u(\zeta) \|_{H}^{2}}{z^{2}(\zeta, \omega)} d\zeta + C \int_{\tau}^{t} \frac{\| u(\zeta) \|_{H}^{2}}{z^{2}(\zeta, \omega)} d\zeta \leq C \sup_{\zeta \in [\tau, t]} \left[ \rho(t, \tau, \omega, u_{\tau}, f) \right]^{2} + C \sup_{\zeta \in [\tau, t]} \left[ \rho(t, \tau, \omega, u_{\tau}, f) \right]^{2} + C \sup_{\zeta \in [\tau, t]} \left[ \rho(t, \tau, \omega, u_{\tau}, f) \right]^{2}
\]
(3.13)
\[
:= \rho_{2}(t, \tau, \omega, u_{\tau}, f).
\]
In view of uniform Gronwall’s lemma (cf. [55, Lemma 1.1, pp. 91]) along with (3.11)-(3.13), we get
\[
\| \nabla u(t) \|_{H}^{2} \leq \left[ \frac{\rho(t, \tau, \omega, u_{\tau}, f)}{t - \tau} + \rho_{2}(t, \tau, \omega, u_{\tau}, f) \right] \exp \{ \rho_{1}(t, \tau, \omega, u_{\tau}, f) \}
\]
(3.14)
\[
:= \tilde{\rho}(t, \tau, \omega, u_{\tau}, f), \text{ for all } t > \tau.
\]
Moreover, (3.9) gives
\[
\int_{\tau}^{t} \| A u(\zeta) \|_{H}^{2} d\zeta \leq \tilde{\rho} \left( \frac{t + \tau}{2}, \tau, \omega, u_{\tau}, f \right) + C \{ \rho(t, \tau, \omega, u_{\tau}, f) \}^{2} \sup_{\zeta \in \left( \frac{t - \tau}{2}, \frac{t + \tau}{2} \right]} \left[ \tilde{\rho}(\zeta, \tau, \omega, u_{\tau}, f) \right] \frac{z^{4}(\zeta, \omega)}{z^{4}(\zeta, \omega)}
\]
\[
+ \rho_{2}(t, \tau, \omega, u_{\tau}, f)
\]
(3.15)
\[
:= \tilde{\rho}(t, \tau, \omega, u_{\tau}, f), \text{ for all } t > \tau,
\]
which completes the proof. \(\square\)

In order to apply the abstract result stated in Theorem 3.1, \(\Phi\) should be \((H, V)\)-continuous. The following Lemma shows that our random cocycle is \((H, V)\)-continuous.

**Lemma 3.7.** Assume that \(f \in L^{2}_{\text{loc}}(\mathbb{R}; H)\). Then, the solution of (2.8) is continuous in \(V\) with respect to the initial data in \(H\).
Proof. Let $u_1(t) := u_1(t; \tau, \omega, u_{1,\tau})$ and $u_2(t) := u_2(t; \tau, \omega, u_{2,\tau})$ be two solutions of the system (2.8). Then $\mathcal{W}(\cdot) = u_1(\cdot) - u_2(\cdot)$ with $\mathcal{W}(\tau) = u_{1,\tau} - u_{2,\tau}$ satisfies
\[
\frac{d\mathcal{W}(t)}{dt} + \nu A\mathcal{W}(t) + \alpha \mathcal{W}(t) = -\frac{1}{z(t, \omega)} \{ B(u_1(t)) - B(u_2(t)) \},
\] (3.14)
for a.e. $t \geq \tau$ in $\mathbb{V}^*$. Multiplying (3.14) with $\mathcal{W}(\cdot)$ and then integrating over $\mathbb{R}^2$, we get
\[
\frac{1}{2} \frac{d}{dt} \| \mathcal{W}(t) \|^2_H + \nu \| \nabla \mathcal{W}(t) \|^2_H + \alpha \| \mathcal{W}(t) \|^2_H = -\frac{1}{z(t, \omega)} \langle B(u_1(t)) - B(u_2(t)), \mathcal{W}(t) \rangle,
\] (3.15)
for a.e. $t \geq \tau$. Using (2.4), Hölder’s, Ladyzhenskaya’s and Young’s inequalities, we obtain
\[
\left| \frac{1}{z(t, \omega)} \langle B(u_1) - B(u_2), \mathcal{W} \rangle \right| = \left| \frac{1}{z(t, \omega)} \langle B(\mathcal{W}), u_2 \rangle \right| \leq \frac{\nu}{2} \| \nabla \mathcal{W} \|^2_H + \frac{C}{z^4(t, \omega)} \| u_2 \|^2_H \| \nabla u_2 \|^2_H \| \mathcal{W} \|^2_H.
\] (3.16)
Making use of (3.16) in (3.15), we get
\[
\frac{d}{dt} \| \mathcal{W}(t) \|^2_H + \nu \| \nabla \mathcal{W}(t) \|^2_H \leq \frac{C}{z^4(t, \omega)} \| u_2(t) \|^2_H \| \nabla u_2(t) \|^2_H \| \mathcal{W}(t) \|^2_H,
\] (3.17)
for a.e. $t \geq \tau$ and an application of Gronwall’s inequality implies
\[
\| \mathcal{W}(t) \|^2_H \leq \exp \left\{ C \int_\tau^t \frac{1}{z^4(\zeta, \omega)} \| u_2(\zeta) \|^2_H \| \nabla u_2(\zeta) \|^2_H d\zeta \right\} \| \mathcal{W}(\tau) \|^2_H
\leq \exp \left\{ C \sup_{\zeta \in [\tau, t]} \left[ \frac{1}{z^4(\zeta, \omega)} \| u_2(\zeta) \|^2_H \right] \int_\tau^t \| \nabla u_2(\zeta) \|^2_H d\zeta \right\} \| \mathcal{W}(\tau) \|^2_H
\leq \exp \left\{ \frac{C[\rho(t, \tau, \omega, u_{2,\tau}, f)]^2}{\sup_{\zeta \in [\tau, t]} z^4(\zeta, \omega)} \right\} \| \mathcal{W}(\tau) \|^2_H := \rho_5(t, \tau, \omega, u_{2,\tau}, f) \| \mathcal{W}(\tau) \|^2_H,
\] (3.18)
for all $t \geq \tau$. Furthermore, integrating (3.17) over $[\tau, t]$ with $t > \tau$, we get
\[
\int_\tau^t \| \nabla \mathcal{W}(\zeta) \|^2_H d\zeta \leq \| \mathcal{W}(\tau) \|^2_H + C \int_\tau^t \frac{1}{z^4(\zeta, \omega)} \| u_2(\zeta) \|^2_H \| \nabla u_2(\zeta) \|^2_H \| \mathcal{W}(\zeta) \|^2_H d\zeta
\leq \left[ 1 + \frac{C[\rho(t, \tau, \omega, u_{2,\tau}, f)]^2}{\sup_{\zeta \in [\tau, t]} z^4(\zeta, \omega)} \right] \| \mathcal{W}(\tau) \|^2_H
\] := \rho_6(t, \tau, \omega, u_{2,\tau}, f) \| \mathcal{W}(\tau) \|^2_H,
(3.19)
for all $t > \tau$. Taking the inner product of (3.14) with $A\mathcal{W}(\cdot)$, and using (2.3) and Young’s inequality, we find
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \mathcal{W}(t) \|^2_H + \nu \| A\mathcal{W}(t) \|^2_H + \alpha \| \nabla \mathcal{W}(t) \|^2_H
\leq -\frac{1}{z(t, \omega)} \langle B(u_1(t)) - B(u_2(t)), A\mathcal{W}(t) \rangle
\].
\[
\frac{1}{z(t, \omega)} \left[ b(b(t, u_1(t), A \mathcal{U}(t)) + b(u_2(t), \mathcal{U}(t), A \mathcal{U}(t)) \right] \\
\leq \frac{\nu}{2} \| A \mathcal{U}(t) \|_{H}^2 + C \left[ \| u_1(t) \|_{H}^2 + \| A u_1(t) \|_{H}^2 + \| \nabla u_1(t) \|_{H}^2 + \frac{\| u_2(t) \|_{H}^2}{z^2(t, \omega)} + \frac{\| \nabla u_2(t) \|_{H}^2}{z^2(t, \omega)} \right] \| \mathcal{U}(t) \|_{H}^2 + C \frac{z^4(t, \omega)}{z^4(t, \omega)} \| \mathcal{U}(t) \|_{H}^2 + C \left[ \| u_1(t) \|_{H}^2 + \| \nabla u_1(t) \|_{H}^2 \right] \| \mathcal{U}(t) \|_{H}^2 \frac{z^4(t, \omega)}{z^4(t, \omega)} \| \mathcal{U}(t) \|_{\tilde{H}}^2, \tag{3.20}
\]

for a.e. \( t \geq \tau \). Replacing \( t \) by \( \zeta \) in (3.20) and multiplying by \( (\zeta - \frac{\tau + t}{2}) \) with \( \zeta \in [\frac{\tau + t}{2}, t] \), we obtain

\[
\left[ \zeta - \frac{\tau + t}{2} \right] \frac{d}{d\zeta} \| \nabla \mathcal{U}(\zeta) \|_{H}^2 \\
\leq C \frac{z^4(\zeta, \omega)}{z^4(\zeta, \omega)} \left[ \| u_1(\zeta) \|_{H}^2 + \| \nabla u_1(\zeta) \|_{H}^2 + \| u_2(\zeta) \|_{H}^2 + \| \nabla u_2(\zeta) \|_{H}^2 \right] \| \nabla \mathcal{U}(\zeta) \|_{H}^2 \\
+ C \left[ \| u_1(\zeta) \|_{H}^2 + \| \nabla u_1(\zeta) \|_{H}^2 \right] \| \mathcal{U}(\zeta) \|_{H}^2 \\
+ C \left[ \| u_2(\zeta) \|_{H}^2 + \| \nabla u_2(\zeta) \|_{H}^2 \right] \| \mathcal{U}(\zeta) \|_{\tilde{H}}^2. \tag{3.21}
\]

An integration by parts leads to

\[
\int_{\frac{\tau + t}{2}}^{t} \left[ \zeta - \frac{\tau + t}{2} \right] \frac{d}{d\zeta} \| \nabla \mathcal{U}(\zeta) \|_{H}^2 d\zeta = \frac{t - \tau}{2} \| \nabla \mathcal{U}(t) \|_{H}^2 - \int_{\frac{\tau + t}{2}}^{t} \| \nabla \mathcal{U}(\zeta) \|_{H}^2 d\zeta. \tag{3.22}
\]

Integrating (3.21) over \( [\frac{\tau + t}{2}, t] \) and using (3.22), we arrive at

\[
\frac{t - \tau}{2} \| \nabla \mathcal{U}(t) \|_{H}^2 \\
\leq \int_{\frac{\tau + t}{2}}^{t} \| \nabla \mathcal{U}(\zeta) \|_{H}^2 d\zeta + C \left( t - \tau \right) \sup_{\zeta \in [\frac{\tau + t}{2}, t]} \left[ \frac{1}{z^4(\zeta, \omega)} \left( \| u_1(\zeta) \|_{H}^2 + \| \nabla u_1(\zeta) \|_{H}^2 \right) \\
+ \| u_2(\zeta) \|_{H}^2 \right] d\zeta + C \left( t - \tau \right) \sup_{\zeta \in [\frac{\tau + t}{2}, t]} \| \mathcal{U}(\zeta) \|_{H}^2 \\
\times \int_{\frac{\tau + t}{2}}^{t} \| \nabla \mathcal{U}(\zeta) \|_{H}^2 d\zeta + C \left( t - \tau \right) \sup_{\zeta \in [\frac{\tau + t}{2}, t]} \| \nabla u_1(\zeta) \|_{H}^2 \\
+ \| u_2(\zeta) \|_{H}^2 \right] \frac{z^4(\zeta, \omega)}{z^4(\zeta, \omega)} \| \mathcal{U}(\zeta) \|_{\tilde{H}}^2 d\zeta \\
\leq \rho_0(t, \tau, \omega, u_{2, \tau}, f) \| \mathcal{U}(t) \|_{H}^2 + C \left( t - \tau \right) \sup_{\zeta \in [\frac{\tau + t}{2}, t]} \left[ \frac{1}{z^4(\zeta, \omega)} \left( \rho(\zeta, \tau, \omega, u_{1, \tau}, f) \\
+ \rho(\zeta, \tau, \omega, u_{1, \tau}, f) + \rho(t, \tau, \omega, u_{2, \tau}, f) \tilde{\rho}(\zeta, \tau, \omega, u_{2, \tau}, f) \right) \right] \rho_0(t, \tau, \omega, u_{2, \tau}, f) \| \mathcal{U}(t) \|_{H}^2
\]
3.6 holds. Then, for every 
exists \delta > 0 such that there exists 
R < \parallel u_{\tau,\omega} \parallel_{\Omega} \} \}
\rho_5(t, \tau, \omega, u_{\tau,\omega}) \parallel \mathcal{W}(\tau) \parallel_{\Omega}^2 + \frac{C}{2} (t - \tau)^2 \left[ \rho(t, \tau, \omega, u_{1,\tau}, f) + \right.
\frac{\{ \rho(t, \tau, \omega, u_{2,\tau}, f) \}^2}{\sup_{z \in [\tau,\omega]} \{ z(\zeta, \omega) \}^2} \right] \rho_5(t, \tau, \omega, u_{2,\tau}, f) \parallel \mathcal{W}(\tau) \parallel_{\Omega}^2,
(3.23)
for all t > \tau, where we have used (3.8), (3.6) and (3.19)-(3.20). It implies from (3.23) that there exist a positive random variable \rho_7(t, \tau, \omega, u_{\tau,\omega}, f) such that for all t > \tau,
\parallel \nabla \mathcal{W}(t) \parallel_{\Omega}^2 \leq \rho_7(t, \tau, \omega, u_{1,\tau}, u_{2,\tau}, f) \parallel \mathcal{W}(\tau) \parallel_{\Omega}^2,
(3.24)
which completes the proof.

The following Lemma helps us to prove the existence of pullback \Omega\text{-}random absorbing set.

Lemma 3.8. Assume that Hypothesis 3.2 holds. Then, for every \((\tau, \omega, D) \in \mathbb{R} \times \Omega \times \Omega\), there exists \(\mathcal{F} = \mathcal{F}(\tau, \omega, D) > 0\) such that for all \(t \geq \mathcal{F}\) and \(s \geq \tau - t\), the solution \(u(\cdot)\) of the system (2.8) satisfies (with \(\omega\) replaced by \(\theta_{-\tau}\omega\))
\[\parallel u(s; \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) \parallel_{\Omega}^2 + 2\nu \int_{\tau-t}^{s} e^{\alpha(\zeta-s)} \parallel \nabla u(\zeta; \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) \parallel_{\Omega}^2 \, d\zeta \leq e^{\alpha(t-s)} \frac{e^{-\alpha t}}{\alpha} \int_{-\infty}^{s} e^{\alpha \zeta [z(\zeta, \theta_{-\tau}\omega)]^2} \parallel f(\cdot, \zeta) \parallel_{\Omega}^2 \, d\zeta.\]
(3.25)
where \(u_{\tau-t} \in D(\tau - t, \theta_{-\tau}\omega)\).

Proof. Applying the variation of constant formula to (3.7) and replacing \(\omega\) by \(\theta_{-\tau}\omega\) in the above inequality, we obtain
\[\parallel u(s; \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) \parallel_{\Omega}^2 + 2\nu \int_{\tau-t}^{s} e^{\alpha(\zeta-s)} \parallel \nabla u(\zeta; \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) \parallel_{\Omega}^2 \, d\zeta \leq e^{\alpha(t-s)} e^{-\alpha t} \parallel u_{\tau-t} \parallel_{\Omega}^2 + \frac{e^{-\alpha s}}{\alpha} \int_{\tau-t}^{s} e^{\alpha \zeta [z(\zeta, \theta_{-\tau}\omega)]^2} \parallel f(\cdot, \zeta) \parallel_{\Omega}^2 \, d\zeta.\]
(3.26)
Since \(u_{\tau-t} \in D(\tau - t, \theta_{-\tau}\omega)\), we have
\[e^{-\alpha t} \parallel u_{\tau-t} \parallel_{\Omega}^2 \leq e^{-\alpha t} \parallel D(\tau - t, \theta_{-\tau}\omega) \parallel_{\Omega}^2 \to 0 \text{ as } t \to \infty.\]
(3.27)
Therefore, there exists \(\mathcal{F} = \mathcal{F}(\tau, \omega, D) > 0\) such that \(e^{-\alpha t} \parallel u_{\tau-t} \parallel_{\Omega}^2 \leq 1\) for all \(t \geq \mathcal{F}\). Thus
\[e^{\alpha(t-s)} e^{-\alpha t} \parallel u_{\tau-t} \parallel_{\Omega}^2 \leq e^{\alpha(t-s)}, \text{ for all } t \geq \mathcal{F}.\]
(3.28)
Now, it is only left to estimate the final term of (3.26). Let \(\tilde{\omega} = \theta_{-\tau}\omega\). Then by (2.6), we have that there exists \(R < 0\) such that for all \(\zeta \leq R\),
\[-2\sigma \tilde{\omega}(\zeta) \leq -((\alpha - \delta)\zeta),\]
where \(\delta\) is the positive constant appearing in (3.4). Therefore,
\[\parallel z(\zeta, \tilde{\omega}) \parallel_{\Omega}^2 = e^{-2\sigma \tilde{\omega}(\zeta)} \leq e^{-(\alpha - \delta)\zeta}\]
and we have for all \(\zeta \leq R\),
\[e^{\alpha \zeta [z(\zeta, \theta_{-\tau}\omega)]^2} \parallel f(\cdot, \zeta) \parallel_{\Omega}^2 \leq e^{(\alpha - \delta)\zeta} e^{\alpha \zeta [z(\zeta, \theta_{-\tau}\omega)]^2} \parallel f(\cdot, \zeta) \parallel_{\Omega}^2 \leq e^{\alpha \zeta} \parallel f(\cdot, \zeta) \parallel_{\Omega}^2.\]
Therefore, it follows from Hypothesis 3.2 that for every $s \in \tau - t, \tau \in \mathbb{R}$ and $\omega \in \Omega$,
\[
\int_{-\infty}^{s} e^{\alpha \zeta} [z(\zeta, \theta_{-r} \omega)]^2 \|f(\cdot, \zeta)\|^2_{H^2} d\zeta < \infty. \tag{3.29}
\]
Hence, from (3.26), (3.28) and (3.29), (3.25) follows. \hfill \Box

Next Lemma shows the existence of a pullback $\mathcal{D}$-random absorbing set. The proofs of following two Lemmas are analogous to the proofs of [59, Lemmas 5.2 and 5.3] with some minor modifications, hence we are omitting it here.

**Lemma 3.9.** Assume that Hypothesis 3.2 holds. Then the continuous cocycle $\Phi$ associated with the system (2.5) possesses a closed measurable pullback $\mathcal{D}$-random absorbing set $\mathcal{K} = \{ \mathcal{K}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}$, where $\mathcal{K}(\tau, \omega)$ is defined by
\[
\mathcal{K}(\tau, \omega) = \{ v \in \mathbb{H} : \|v\|^2_{H} \leq \mathcal{M}(\tau, \omega) \}, \tag{3.30}
\]
and $\mathcal{M}(\tau, \omega)$ is given by
\[
\mathcal{M}(\tau, \omega) = [z(\tau, \theta_{-r} \omega)]^{-2} + \frac{[z(\tau, \theta_{-r} \omega)]^{-2}}{\alpha} \int_{-\infty}^{\tau} e^{\alpha (\zeta - \tau)} [z(\zeta, \theta_{-r} \omega)]^2 \|f(\cdot, \zeta)\|^2_{H^2} d\zeta.
\]

Next Lemma demonstrates the pullback $\mathcal{D}$-asymptotic compactness of non-autonomous random DS $\Phi$.

**Lemma 3.10.** Assume that Hypothesis 3.2 holds. Then the continuous cocycle $\Phi$ associated with system (2.5) is $(\mathbb{H}, \mathbb{H})$-pullback $\mathcal{D}$-asymptotically compact.

Since, $\omega(\cdot)$ has sub-exponential growth (cf. [6, Lemma 11]), $\Omega$ can be written as $\Omega = \bigcup_{N \in \mathbb{N}} \Omega_N$, where
\[
\Omega_N := \{ \omega \in \Omega : |\omega(t)| \leq Ne^{|t|}, \text{ for all } t \in \mathbb{R} \}, \text{ for every } N \in \mathbb{N}.
\]
The following lemma plays a crucial role in proving the $(\mathcal{F}, \mathcal{B}(\mathcal{V}))$-measurability of $\Phi(t, \tau, .., x) : \Omega \rightarrow \mathcal{V}$, for every fixed $t > 0$, $\tau \in \mathbb{R}$ and $x \in \mathcal{H}$.

**Proposition 3.11.** Suppose that $\tau \in \mathbb{R}$, $t > \tau$, $f \in L^{2}_{\text{loc}}(\mathbb{R}; \mathbb{H})$ and $u_{\tau} \in \mathbb{H}$. For each $N \in \mathbb{N}$, the mapping $\omega \mapsto u(t; \tau, \omega, u_{\tau})$ (solution of (2.8)) is continuous from $(\Omega_N, d_{\Omega_N})$ to $\mathcal{V}$.

**Proof.** Assume that $\omega_k, \omega_0 \in \Omega_N$, $N \in \mathbb{N}$ such that $d_{\Omega_N}(\omega_k, \omega_0) \rightarrow 0$ as $k \rightarrow +\infty$. Let $\mathcal{U}^k := u^k - u^0$, where $u^k = u(t; \tau, \omega_k, u_\tau)$ and $u^0 = u(t; \tau, \omega_0, u_\tau)$ for $t \geq \tau$. Then, $\mathcal{U}^k$ satisfies:
\[
\frac{d\mathcal{U}^k}{dt} = -\nu A \mathcal{U}^k + \alpha \mathcal{U}^k - \frac{1}{z(t, \omega_k)} B(u^k) + \frac{1}{z(t, \omega_0)} B(u^0) + [z(t, \omega_k) - z(t, \omega_0)] f, \tag{3.31}
\]
in $\mathcal{V}^*$ (in the weak sense). Taking the inner product with $\mathcal{U}^k(\cdot)$ in (3.31), and using (2.2) and (2.4), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\mathcal{U}^k\|_{H}^2 = -\nu \|\nabla \mathcal{U}^k\|_{H}^2 - \alpha \|\mathcal{U}^k\|_{H}^2 - \frac{1}{z(t, \omega_k)} b(\mathcal{U}^k, u^0, \mathcal{U}^k) + \frac{1}{z(t, \omega_0)} b(u^0, \mathcal{U}^k, u^0) + \left[ z(t, \omega_k) - z(t, \omega_0) \right] (f, \mathcal{U}^k). \tag{3.32}
\]
Using Hölder’s and Young’s inequalities, we obtain
\[
|z(t, \omega_k) - z(t, \omega_0)| (f, \mathcal{U}^k) \leq C |z(t, \omega_k) - z(t, \omega_0)|^2 \|f\|^2_{H} + \frac{\alpha}{2} \|\mathcal{U}^k\|_{H}^2. \tag{3.33}
\]
Applying Hölder’s, Ladyzhenskaya’s and Young’s inequalities, we estimate
\[ \left| \frac{1}{z(t, \omega_k)} b(\mathcal{U}^k, u^0, \mathcal{U}^k) \right| \leq \frac{C}{z^2(t, \omega_k)} \| \nabla u^0 \|_{\mathbb{H}}^2 \| \mathcal{U}^k \|_{\mathbb{H}}^2 + \frac{\nu}{4} \| \nabla \mathcal{U}^k \|_{\mathbb{H}}^2 \]  
(3.34)
and
\[ \left| \left[ \frac{1}{z(t, \omega_k)} - \frac{1}{z(t, \omega_0)} \right] b(u^0, \mathcal{U}^k, u^0) \right| \leq C \left[ \frac{1}{z(t, \omega_k)} - \frac{1}{z(t, \omega_0)} \right] ^2 \| u^0 \|_{\mathbb{H}}^2 \| \nabla u^0 \|_{\mathbb{H}}^2 + \frac{\nu}{4} \| \nabla \mathcal{U}^k \|_{\mathbb{H}}^2 . \]
(3.35)

Combining (3.32)-(3.35), we arrive at (replacing \( t \) by \( \zeta \))
\[ \frac{d}{d\zeta} \| \mathcal{U}^k(\zeta) \|_{\mathbb{H}}^2 + \alpha \| \mathcal{U}^k(\zeta) \|_{\mathbb{H}}^2 + \nu \| \nabla \mathcal{U}^k(\zeta) \|_{\mathbb{H}}^2 \leq P_1(\zeta) \| \mathcal{U}^k(\zeta) \|_{\mathbb{H}}^2 + Q_1(\zeta), \]
for a.e. \( t \geq \tau \), where
\[ P_1^k = \frac{C}{z^2(\zeta, \omega_k)} \| \nabla u^0 \|_{\mathbb{H}}^2 , \]
\[ Q_1^k = C |z(\zeta, \omega_k) - z(\zeta, \omega_0)|^2 \| f \|_{\mathbb{H}}^2 + C \frac{1}{z(\zeta, \omega_k)} \frac{1}{z(\zeta, \omega_0)} ^2 \| u^0 \|_{\mathbb{H}}^2 \| \nabla u^0 \|_{\mathbb{H}}^2 . \]

Now, from the fact that \( f \in L^2_{\text{loc}}(\mathbb{R}; \mathbb{H}) \) and \( u^0 \in C([\tau, +\infty); \mathbb{H}) \cap L^2_{\text{loc}}(\tau, +\infty; \mathbb{V}) \), we conclude that for all \( t \geq \tau \)
\[ \lim_{k \to +\infty} \int_{\tau}^{t} P_1^k(\zeta) d\zeta \leq C(t, \tau, \omega_0) \quad \text{and} \quad \lim_{k \to +\infty} \int_{\tau}^{t} Q_1^k(\zeta) d\zeta = 0. \]
(3.37)

Making use of Gronwall’s inequality in (3.36), we get for all \( t \geq \tau \)
\[ \| \mathcal{U}^k(t) \|_{\mathbb{H}}^2 \leq e^{\int_{\tau}^{t} P_1^k(\zeta) d\zeta} \left[ \int_{\tau}^{t} Q_1^k(\zeta) d\zeta \right] . \]
(3.38)

In view of (3.37)-(3.38), we find for all \( t \geq \tau \)
\[ \| \mathcal{U}^k(t) \|_{\mathbb{H}}^2 \to 0 \quad \text{as} \quad k \to +\infty. \]
(3.39)

Moreover, (3.36) along with (3.37)-(3.38) imply
\[ \int_{\tau}^{t} \| \mathcal{U}^k(\zeta) \|_{\mathbb{H}}^2 d\zeta \to 0 \quad \text{as} \quad k \to +\infty, \quad \text{for all} \quad t \geq \tau. \]
(3.40)

Taking the inner product with \( A \mathcal{U}^k(\cdot) \) in (3.31), we get
\[ \frac{1}{2} \frac{d}{dt} \| \nabla \mathcal{U}^k \|_{\mathbb{H}}^2 = -\nu \| A \mathcal{U}^k \|_{\mathbb{H}}^2 - \alpha \| \nabla \mathcal{U}^k \|_{\mathbb{H}}^2 - \frac{1}{z(t, \omega_k)} b(\mathcal{U}^k, \mathcal{U}^k, A \mathcal{U}^k) \]
\[ - \frac{1}{z(t, \omega_k)} b(u^0, \mathcal{U}^k, A \mathcal{U}^k) - \frac{1}{z(t, \omega_k)} b(\mathcal{U}^k, u^0, A \mathcal{U}^k) \]
\[ - \left( \frac{1}{z(t, \omega_k)} - \frac{1}{z(t, \omega_0)} \right) b(u^0, u^0, A \mathcal{U}^k) + [z(t, \omega_k) - z(t, \omega_0)](f, A \mathcal{U}^k) . \]
(3.41)

Using Hölder’s and Young’s inequalities, we obtain
\[ \| [z(t, \omega_k) - z(t, \omega_0)](f, A \mathcal{U}^k) \| \leq C \| z(t, \omega_k) - z(t, \omega_0) \| \| f \|_{\mathbb{H}}^2 + \frac{\nu}{8} \| A \mathcal{U}^k \|_{\mathbb{H}}^2 . \]
(3.42)
Using (2.3) and Young’s inequality, we estimate
\[
\left| \frac{1}{z(t, \omega_k)} b(\mathcal{U}^k, \mathcal{U}^k, A\mathcal{U}^k) \right| \leq \frac{\nu}{8} \|A\mathcal{U}^k\|_H^2 + C \|\mathcal{U}^k\|_H^2 + \frac{C}{z^4(t, \omega_k)} \|\mathcal{U}^k\|_H^4 \|\nabla \mathcal{U}^k\|_H^2
\]
\[
+ \frac{C}{z^2(t, \omega_k)} \|\mathcal{U}^k\|_H^2 \|\nabla \mathcal{U}^k\|_H^2 + \frac{C}{z^4(t, \omega_k)} \|\mathcal{U}^k\|_H^2 \|\nabla \mathcal{U}^k\|_H^4.
\]
(3.43)

\[
\left| \frac{1}{z(t, \omega_k)} b(u^0, \mathcal{U}^k, A\mathcal{U}^k) + \frac{1}{z(t, \omega_k)} b(\mathcal{U}^k, u^0, A\mathcal{U}^k) \right|
\]
\[
\leq \frac{\nu}{8} \|A\mathcal{U}^k\|_H^2 + C \left[ \|u^0\|_H^2 + \|Au^0\|_H^2 + \|\nabla u^0\|_H^2 + \frac{\|u^0\|_H^2}{z^2(t, \omega_k)} + \frac{\|\nabla u^0\|_H^2}{z^2(t, \omega_k)} + \frac{\|\nabla u^0\|_H^2}{z^4(t, \omega_k)} \right]
\]
\[
\times \|\mathcal{U}^k\|_H^2 + \frac{C}{z^4(t, \omega_k)} \left[ \|u^0\|_H^2 + \|\nabla u^0\|_H^2 + \|u^0\|_H^2 \|\nabla u^0\|_H^2 \right] \|\nabla \mathcal{U}^k\|_H^2,
\]
(3.44)
\[
\text{and}
\]
\[
\left| \left[ \frac{1}{z(t, \omega_k)} - \frac{1}{z(t, \omega_0)} \right] b(u^0, u^0, A\mathcal{U}^k) \right|
\]
\[
\leq C \left[ \frac{1}{z(t, \omega_k)} - \frac{1}{z(t, \omega_0)} \right]^2 \left[ \|\nabla u^0\|_H^4 + \|u^0\|_H^2 \|\nabla u^0\|_H^2 + \|u^0\|_H^2 \|Au^0\|_H^2 + \|\nabla u^0\|_H^2 \|Au^0\|_H^2 \right]
\]
\[
+ \frac{\nu}{8} \|A\mathcal{U}^k\|_H^2.
\]
(3.45)

Combining (3.41)-(3.45), we reach at (replacing t by \(\zeta\))
\[
\frac{d}{d\zeta} \|\nabla \mathcal{U}^k(\zeta)\|_H^2 \leq \hat{P}_1^k(\zeta) \|\nabla \mathcal{U}^k(\zeta)\|_H^2 + \hat{Q}_1^k(\zeta),
\]
(3.46)

for a.e. \(t \geq \tau\), where
\[
\hat{P}_1^k = \frac{C}{z^2(\zeta, \omega_k)} \|\mathcal{U}^k\|_H^2 + \frac{C}{z^4(\zeta, \omega_k)} \left[ \|\mathcal{U}^k\|_H^4 + \|\mathcal{U}^k\|_H^2 \|\nabla \mathcal{U}^k\|_H^2 + \|u^0\|_H^2 + \|\nabla u^0\|_H^2 \right]
\]
\[
+ \|u^0\|_H^2 \|\nabla u^0\|_H^2,
\]
\[
\hat{Q}_1^k = C \left[ 1 + \|u^0\|_H^2 + \|Au^0\|_H^2 + \|\nabla u^0\|_H^2 + \frac{\|u^0\|_H^2}{z^2(\zeta, \omega_k)} + \frac{\|\nabla u^0\|_H^2}{z^2(\zeta, \omega_k)} + \frac{\|u^0\|_H^2}{z^4(\zeta, \omega_k)} \right] \|\mathcal{U}^k\|_H^2
\]
\[
+ C \left[ \frac{1}{z(\zeta, \omega_k)} - \frac{1}{z(\zeta, \omega_0)} \right]^2 \left[ \|\nabla u^0\|_H^4 + \|u^0\|_H^2 \|\nabla u^0\|_H^2 + \|u^0\|_H^2 \|Au^0\|_H^2 \right]
\]
\[
+ \|\nabla u^0\|_H^2 \|Au^0\|_H^2 \right] + C |z(\zeta, \omega_k) - z(\zeta, \omega_0)|^2 \|f\|_H^2.
\]

Multiplying (3.46) by \((\zeta - \frac{\tau}{2})\) with \(\zeta \in [\frac{\tau}{2}, t]\) and making use of integration by parts as same as in (3.22), we obtain
\[
\frac{t - \tau}{2} \|\nabla \mathcal{U}^k(t)\|_H^2 \leq \left[ 1 + \sup_{\zeta \in [\frac{\tau}{2}, t]} \hat{P}_1^k(\zeta) \right] \int_{\frac{\tau}{2}}^t \|\nabla \mathcal{U}^k(\zeta)\|_H^2 d\zeta + \int_{\frac{\tau}{2}}^t \hat{Q}_1^k(\zeta) d\zeta.
\]
(3.47)
Using the fact $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ and $d_{\Omega_N}(\omega_k, \omega_0) \to 0$, the estimates established in (3.8) and (3.6), and convergences obtained in (3.39)-(3.40), we conclude from (3.47) that $\|\nabla U^k(t)\|^2_H \to 0$ as $k \to +\infty$, for all $t > \tau$, which completes the proof along with (3.39).

Now, we are in a position to state and prove our main result of this work.

**Theorem 3.12.** Suppose that Hypothesis 3.2 holds. Then the non-autonomous random DS $\Phi$ has a unique $(H, V)$-pullback $D$-random attractor $A = \{A(\tau, \omega) : \tau \in \mathbb{R} \text{ and } \omega \in \Omega\} \in D$ given by

$$A(\tau, \omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \Phi(t, \tau - t, \theta_\tau \omega, K(\tau - t, \theta_\tau \omega)),$$

where $K(\tau, \omega)$ is the absorbing set obtained in Lemma 3.9.

**Proof.** Since $\Phi$ is $(H, V)$-continuous random cocycle (see Lemma 3.7) and $D$ is an inclusion closed universe, we apply the abstract result from the work \cite{67} (see Theorem 3.1 above) to prove this theorem. Lemma 3.9 shows that $\Phi$ has closed pullback $D$-random absorbing set, Lemma 3.10 reveals that $\Phi$ is $(H, H)$-pullback asymptotically compact and Lemma 3.11 proves that $\Phi(t, \tau, \cdot, v_\tau) : \Omega \to V$ is $(\mathcal{F}, \mathcal{B}(V))$-measurable for every fixed $t > 0$, $\tau \in \mathbb{R}$ and $v_\tau \in H$. We conclude that all the three conditions stated in Theorem 3.1 for $(H, V)$-continuous random cocycle $\Phi$ are satisfied. Hence, an application of Theorem 3.1 completes the proof. \qed

**Remark 3.13.** We observe that the method introduced in \cite{57} for the upper semicontinuity is not suitable for proving the upper semicontinuity of random attractors obtained in Theorem 3.12 as $\sigma \to 0$. Because the right hand side of (3.25) will tend to $\infty$ as $\frac{\sigma^2}{2} = \alpha \to 0$ and we will not able to find a random set which contains $\bigcup_{0 < \sigma \leq 1} K_\sigma(\tau, \omega)$, where $K_\sigma(\tau, \omega)$ represents the absorbing set corresponding to each $\sigma$. Therefore the existence of pullback attractors for 2D deterministic NSE as well as the upper semicontinuity of random attractors for 2D SNSE on the whole space are still challenging open problems.

**Remark 3.14.** In \cite{34}, the upper semicontinuity of the random attractors with respect to domains, that is, when domain changes from bounded to unbounded (Poincaré) domain is proved for stochastic convective Brinkman-Forchheimer equations (cf. \cite{34}, Theorem 6.10). The authors have also discussed the upper semicontinuity of the random attractors with respect to domains for SNSE driven by additive noise as a remark (cf. \cite{34}, Remark 6.11). Using the similar arguments as in the work \cite{34}, one can establish the upper semicontinuity of the random attractors with respect to domains for the 2D non-autonomous SNSE (2.5) on the whole space.

**Remark 3.15.** The authors in \cite{60} prove the asymptotic autonomy of pullback random attractors for 2D SNSE on unbounded Poincaré domains. They consider the additive as well as linear multiplicative white noise where the stochastic integration has been taken in the sense of Stratonovich. Using the similar ideas as in the work \cite{60}, one can obtain the asymptotic autonomy robustness of random attractors for the 2D non-autonomous SNSE (2.5). It means
that if $A_{\infty}(\omega)$ is the unique random attractor for the 2D autonomous SNSE (2.5), then one can prove that
\[ \lim_{\tau \to -\infty} \text{dist}_H(A(\tau, \omega), A_{\infty}(\omega)) = 0, \ P\text{-a.s.} \ \omega \in \Omega, \] (3.48)
where $\text{dist}_H(\cdot, \cdot)$ denotes the Hausdorff semi-distance between two non-empty subsets of $\mathbb{H}$ (cf. [60] for the detailed proof).

**Remark 3.16.** One can use another transformation also to change the system (2.5) into an equivalent pathwise deterministic system. Let $y(\cdot)$ be a solution of the one-dimensional Ornstein-Uhlenbeck equation:
\[ dy(t, \omega) + y(t, \omega)dt = dW(t). \] (3.49)
Let us define
\[ u(t; \tau, \omega, u_{\tau}) = e^{-\sigma y(t, \omega)} v(t; \tau, \omega, v_{\tau}), \text{ with } u_{\tau} = e^{-\sigma y(\theta_{\tau}, \omega)} v_{\tau}, \]
where $y$ satisfies (3.49), and $v(\cdot)$ is the solution of (2.5). Then $u(\cdot)$ satisfies the following system:
\[ \frac{du}{dt} + \nu Au + \frac{\sigma^2}{2} u + e^{\sigma y(\theta_{\tau}, \omega)} B(u) = f e^{-\sigma y(\theta_{\tau}, \omega)} + \sigma y(\theta_{\tau}, \omega) u, \ t \geq \tau, \] (3.50)
which is a pathwise deterministic system and is equivalent to the system (2.5). It can be easily seen that the system (3.50) is similar to the system (2.8), and one can use the same ideas for the above system to get the results of this work.

4. Invariant Measures and Ergodicity

This section is devoted for the existence and uniqueness of invariant measures and ergodicity for random DS $\Phi$ associated with the system (2.5). Since, in this section and the next section, we are applying the abstract theory established in the works [9] and [69], respectively, we restrict our deterministic forcing term to be independent of $t$ in both the sections. From now onward, $f$ is time-independent and $f \in \mathbb{H}$ (autonomous case).

The authors in the work [9] proved that if a random DS $\Phi$ has a compact invariant random set, then there exists an invariant measure for $\Phi$ (cf. [9, Corollaries 4.4 and 4.6]). Hence, the existence of invariant measures for the 2D autonomous SNSE (2.5) is a direct consequence of [9, Corollaries 4.4 and 4.6] and Theorem 3.12, because the random attractor itself is a compact invariant random set. Let us recall some definitions and results from [1] and [9]. Let $\mathbb{X}$ be a Polish space and $\Phi$ be a random DS over $\theta$.

**Definition 4.1 (Skew-product, [1]).** Given a random DS $\Phi$, the mapping
\[ \Theta_t : \Omega \times \mathbb{X} \ni (\omega, x) \mapsto (\theta_t(\omega, \Phi(t, \omega, x))) = \Theta_t(\omega, x) \in \Omega \times \mathbb{X}, \ t \in \mathbb{R}^+, \] (4.1)
is a measurable DS on $(\Omega \times \mathbb{X}, \mathcal{F} \otimes \mathcal{B}(\mathbb{X}))$ which is called the skew product of metric DS $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ and random DS $\Phi(t, \omega)$ on $\mathbb{X}$.

**Definition 4.2 (Invariant measure for $\Theta$, [1]).** A probability measure $\rho$ on $(\Omega \times \mathbb{X}, \mathcal{F} \otimes \mathcal{B}(\mathbb{X}))$ is called invariant for $\Theta$ corresponding to $\Phi$, if $\Theta_t \rho = \rho$ for all $t \in \mathbb{R}^+$.

**Definition 4.3 (Invariant measure for $\Phi$, [1]).** A probability measure $\rho$ on $(\Omega \times \mathbb{X}, \mathcal{F} \otimes \mathcal{B}(\mathbb{X}))$ is called invariant for $\Phi$ ($\Phi$-invariant), if
(i) $\Theta_t \rho = \rho$ for all $t \in \mathbb{R}^+$,
(ii) $\pi_t \rho = \mathbb{P}$, that is, the first marginal of $\rho$ is $\mathbb{P}$, where $\pi_t : \Omega \times \mathbb{X} \ni (\omega, x) \mapsto \omega \in \Omega$.

Define

$$\mathcal{P}_\mathbb{P}(\Omega \times \mathbb{X}) := \{ \rho : \rho \text{ is probability measure on } (\Omega \times \mathbb{X}, \mathcal{F} \otimes \mathcal{B}(\mathbb{X})) \text{ with marginal } \mathbb{P} \text{ on } (\Omega, \mathcal{F}) \}.$$  

**Definition 4.4** (Sample measure, [1]). Let $\rho \in \mathcal{P}_\mathbb{P}(\Omega \times \mathbb{X})$. A mapping $\rho(\cdot) : \Omega \times \mathcal{B}(\mathbb{X}) \to [0, 1]$ is called a sample measure (or disintegration) of $\rho$ with respect to $\mathbb{P}$ if

(i) for all $E \in \mathcal{B}(\mathbb{X})$, $\omega \mapsto \rho_\omega(E)$ is $\mathcal{F}$-measurable,
(ii) for $\mathbb{P}$-a.s. $\omega \in \Omega$, $E \mapsto \rho_\omega(E)$ is a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$,
(iii) for all $G \in \mathcal{F} \times \mathcal{B}(\mathbb{X})$,

$$\rho(G) = \int_{\Omega} \int_{\mathbb{X}} 1_G(\omega, x) \rho_\omega(dx) \mathbb{P}(d\omega). \quad (4.2)$$

Define two $\sigma$-algebras corresponding to the past and future, respectively, by

$$\mathcal{F}^- = \sigma\{ \omega \mapsto \Phi(t, \theta_{-t}\omega) : 0 \leq t \leq \tau \} \quad \text{and} \quad \mathcal{F}^+ = \sigma\{ \omega \mapsto \Phi(t, \theta_{+t}\omega) : t, \tau \geq 0 \}.$$  

**Theorem 4.5** (Corollary 4.6, [9]). Let $\omega \mapsto \mathcal{A}(\omega)$ be a $\Phi$-invariant compact set which is measurable with respect to the past $\mathcal{F}^-$ for a random DS $\Phi$ and $\Phi$ is a random DS whose one-point motions form a Markov family, and such that $\mathcal{F}^+$ and $\mathcal{F}^-$ are independent. Then there exists an invariant measure $\mu$ for the associated Markov semigroup. Furthermore, the limit

$$\rho_\omega = \lim_{t \to \infty} \Phi(t, \theta_{-t}\omega)\mu$$

exists $\mathbb{P}$-a.s., $\mu = \int_{\Omega} \rho_\omega d\mathbb{P}(\omega) = \mathbb{E}[\rho]$, and $\rho$ is a Markov measure.

4.1. **Existence.** Since the deterministic forcing term in the system (2.5) is time independent, the non-autonomous random DS will reduce to autonomous random DS. Therefore, the random DS $\Phi : \mathbb{R}^+ \times \Omega \times \mathbb{H} \to \mathbb{H}$, for $t \geq \tau$, $\omega \in \Omega$ and $\mathbf{v}_0 \in \mathbb{H}$, is defined by

$$\Phi(t - \tau, \theta_{\tau}\omega, \mathbf{v}_\tau) = \mathbf{v}(t; \tau, \omega, \mathbf{v}_\tau) = \frac{\mathbf{u}(t; \tau, \omega, \mathbf{u}_\tau)}{z(t, \omega)} \text{ with } \mathbf{u}_\tau = z(t, \omega)\mathbf{v}_\tau. \quad (4.3)$$

For a Banach space $\mathbb{X}$, let $\mathcal{B}_b(\mathbb{X})$ be the space of all bounded and Borel measurable functions on $\mathbb{X}$, and $C_b(\mathbb{X})$ be the space of all bounded and continuous functions on $\mathbb{X}$. Let us define the transition operator $\{T_t\}_{t \geq 0}$ by

$$T_t g(\mathbf{x}) = \int_{\Omega} g(\Phi(\omega, t, \mathbf{x})) d\mathbb{P}(\omega) = \mathbb{E}[g(\Phi(t, \mathbf{x}))], \quad (4.4)$$

for all $g \in \mathcal{B}_b(\mathbb{H})$, where $\Phi$ is the random DS corresponding to the 2D SNSE (2.5) defined by (4.3). Since $\Phi$ is continuous (Lemma 3.7), the following result holds due to [2, Proposition 3.8].

**Lemma 4.6.** The family $\{T_t\}_{t \geq 0}$ is Feller, that is, $T_t g \in C_b(\mathbb{H})$ if $g \in C_b(\mathbb{H})$. Moreover, for any $g \in C_b(\mathbb{H})$, $T_t g(\mathbf{x}) \to g(\mathbf{x})$ as $t \downarrow 0$.

**Definition 4.7.** A Borel probability measure $\mu$ on $\mathbb{H}$ is called an invariant measure for a Markov semigroup $\{T_t\}_{t \geq 0}$ of Feller operators on $C_b(\mathbb{H})$ if and only if

$$T_t^* \mu = \mu, \quad t \geq 0,$$
where \( \left( T^*_t \mu \right)(\Gamma) = \int_{\Omega} P_t(y, \Gamma) \mu(dy) \), for \( \Gamma \in \mathcal{B}(\mathbb{H}) \) and \( T^*_t(y, \cdot) \) is the transition probability, \( T^*_t(y, \Gamma) = T^*_t(\chi_t(y)), \ y \in \mathbb{H} \).

One can establish that \( \Phi \) is a Markov random DS (cf. [9, Theorem 5.6]), that is, \( T_{s_1+s_2} = T_{s_1}T_{s_2} \), for all \( s_1, s_2 \geq 0 \). It is known by Theorem 4.5 that there exists a Feller invariant probability measure \( \mu \) for a Markov random DS \( \Phi \) which has an invariant compact random set on a Polish space. Hence we have the following result due to Theorems 3.12 and 4.5.

**Theorem 4.8.** Assume that \( f \in \mathbb{H} \). Then, the Markov semigroup \( \{T_t\}_{t \geq 0} \) induced by the flow \( \Phi \) on \( \mathbb{H} \) has an invariant measure \( \mu \). The associated flow-invariant Markov measure \( \rho \) on \( \Omega \times \mathbb{H} \) has the property that its disintegration \( \omega \mapsto \rho_\omega \) is supported by the attractor \( \mathcal{A}(\omega) \).

**Remark 4.9.** Theorem 4.8 establishes the existence of invariant measures in \( \mathbb{H} \). Since we have proved the existence of a unique \( (\mathbb{H}, \mathcal{V}) \)-pullback \( \mathfrak{D} \)-random attractor, it can be shown in a similar way (same as in the case of \( \mathbb{H} \)) that there exists an invariant measure for the 2D SNSE (2.5) in \( \mathcal{V} \) also.

### 4.2. Uniqueness

In this section, we prove the uniqueness of invariant measures for the system (2.5) by using the linear structure of multiplicative noise and the exponential stability of solutions (cf. [7]). For this purpose, we consider the external forcing \( f = 0 \) in the system (2.5).

**Lemma 4.10.** Assume that \( f = 0 \). Then, there exists \( T(\omega) > 0 \) such that the solution of the system (2.5) satisfies the following exponential estimate:

\[
\|v(t)\|_\mathbb{H}^2 \leq e^{-\sigma^2 t}\|v(0)\|_\mathbb{H}^2, \quad \text{for all} \quad t \geq T(\omega). \tag{4.5}
\]

**Proof.** Applying Itô’s formula to the process \( \|v(\cdot)\|_\mathbb{H}^2 \), we get

\[
\|v(t)\|_\mathbb{H}^2 = \|v(0)\|_\mathbb{H}^2 - 2\nu \int_0^t \|\nabla v(\zeta)\|_\mathbb{H}^2 d\zeta + \sigma^2 \int_0^t \|v(\zeta)\|_\mathbb{H}^2 d\zeta + 2\sigma \int_0^t \|v(\zeta)\|_\mathbb{H}^2 dW(\zeta).
\]

Again applying the Itô formula to the process \( \log \|v(\cdot)\|_\mathbb{H}^2 \), we deduce

\[
\log \|v(t)\|_\mathbb{H}^2 = \log \|v(0)\|_\mathbb{H}^2 - 2\nu \int_0^t \frac{\|\nabla v(\zeta)\|_\mathbb{H}^2}{\|v(\zeta)\|_\mathbb{H}^2} d\zeta + \sigma^2 \int_0^t \frac{\|v(\zeta)\|_\mathbb{H}^2}{\|v(\zeta)\|_\mathbb{H}^2} d\zeta \\
+ 2\sigma \int_0^t \frac{\|v(\zeta)\|_\mathbb{H}^2}{\|v(\zeta)\|_\mathbb{H}^2} dW(\zeta) - \frac{4\sigma^2}{2} \int_0^t \frac{\|v(\zeta)\|_\mathbb{H}^2}{\|v(\zeta)\|_\mathbb{H}^2} d\zeta
\leq \log \|v(0)\|_\mathbb{H}^2 - \sigma^2 t + 2\sigma W(t). \tag{4.6}
\]

Since \( \lim_{t \to \infty} \frac{W(t)}{t} = 0 \), \( \mathbb{P} \)-a.s., we can find a set \( \Omega_0 \subset \Omega \) with \( \mathbb{P}(\Omega_0) = 0 \), such that for every \( \omega \in \Omega \setminus \Omega_0 \), there exists \( T(\omega) > 0 \) such that for all \( t \geq T(\omega) \), we have

\[
\frac{2\sigma W(t)}{t} \leq \frac{\sigma^2}{2},
\]

which completes the proof along with (4.6).

**Theorem 4.11.** Assume that \( f = 0 \) and \( v_0 \in \mathbb{H} \) be given. Then, there is a unique invariant measure to the system (2.5) which is ergodic and strongly mixing.
Proof. For \( \psi \in \text{Lip}(\mathbb{H}) \) (Lipschitz \( \psi \)) and an invariant measure \( \mu \), we have for all \( t \geq T(\omega) \),
\[
\left| T_t \psi(v_0) - \int_{\mathbb{H}} \psi(u_0) \mu(du_0) \right| \\
= \left| \mathbb{E}[\psi(v(t, v_0))] - \int_{\mathbb{H}} T_t \psi(u_0) \mu(du_0) \right| \\
= \left| \int_{\mathbb{H}} \mathbb{E}[\psi(v(t, v_0)) - \psi(v(t, u_0))] \mu(du_0) \right| \\
\leq L_\psi \int_{\mathbb{H}} \mathbb{E}[\|v(t, v_0) - v(t, u_0)\|_H] \mu(du_0) \\
\leq L_\psi \int_{\mathbb{H}} \mathbb{E}[\|v(t, v_0)\|_H] \mu(du_0) + L_\psi \int_{\mathbb{H}} \mathbb{E}[\|v(t, u_0)\|_H] \mu(du_0) \\
\leq L_\psi \mathbb{E}\left[e^{-\frac{\omega^2}{2}t}\left(\|v_0\|_H + \int_{\mathbb{H}} \|u_0\|_H \mu(du_0)\right)\right] \to 0 \text{ as } t \to \infty,
\]
since \( \int_{\mathbb{H}} \|u_0\|_H \mu(du_0) < +\infty \). Hence, we conclude
\[
\lim_{t \to \infty} T_t \psi(v_0) = \int_{\mathbb{H}} \psi(u_0) d\mu(u_0), \text{ } \mu\text{-a.s., for all } v_0 \in \mathbb{H} \text{ and } \psi \in \text{C}_b(\mathbb{H}), \tag{4.7}
\]
by the density of \( \text{Lip}(\mathbb{H}) \) in \( \text{C}_b(\mathbb{H}) \). Since we have a stronger result that \( T_t \psi(v_0) \) converges exponentially fast to the equilibrium, this property is known as the exponential mixing property. Now suppose that \( \tilde{\mu} \) is another invariant measure, then we have for all \( t \geq T(\omega) \),
\[
\left| \int_{\mathbb{H}} \psi(v_0) \mu(dv_0) - \int_{\mathbb{H}} \psi(u_0) \tilde{\mu}(du_0) \right| \\
= \left| \int_{\mathbb{H}} T_t \psi(v_0) \mu(dv_0) - \int_{\mathbb{H}} T_t \psi(u_0) \tilde{\mu}(du_0) \right| \\
= \left| \int_{\mathbb{H}} \int_{\mathbb{H}} [T_t \psi(v_0) - T_t \psi(u_0)] \mu(dv_0) \tilde{\mu}(du_0) \right| \\
\leq L_\psi \int_{\mathbb{H}} \int_{\mathbb{H}} \mathbb{E}[\|v(t, v_0) - v(t, u_0)\|_H] \mu(dv_0) \tilde{\mu}(du_0) \\
\leq L_\psi \int_{\mathbb{H}} \mathbb{E}[\|v(t, v_0)\|_H] \mu(dv_0) + L_\psi \int_{\mathbb{H}} \mathbb{E}[\|v(t, u_0)\|_H] \tilde{\mu}(du_0) \\
\leq L_\psi \mathbb{E}\left[e^{-\frac{\omega^2}{2}t}\left(\|v_0\|_H \mu(dv_0) + \int_{\mathbb{H}} \|u_0\|_H \tilde{\mu}(du_0)\right)\right] \to 0 \text{ as } t \to \infty, \tag{4.8}
\]
since \( \int_{\mathbb{H}} \|v_0\|_H \mu(dv_0) < +\infty \) and \( \int_{\mathbb{H}} \|v_0\|_H \mu(dv_0) < +\infty \). Since \( \mu \) is the unique invariant measure for \( (T_t)_{t \geq 0} \), it follows from \cite[Theorem 3.2.6]{12} that \( \mu \) is ergodic also. \[ \square \]

**Remark 4.12.** Since, for \( f = 0 \), Theorem 4.11 demonstrates the unique invariant measure in \( \mathbb{H} \). As \( \mathbb{V} \subset \mathbb{H} \) is a closed subspace, the uniqueness of invariant measure in \( \mathbb{V} \) also follows from Theorem 4.11.
5. Invariant Sample Measures and Random Liouville Type Theorem

In this section, we demonstrate the existence of a family of invariant sample measures which satisfies a random Liouville type theorem (cf. [69]) for the 2D autonomous SNSE (2.5) on the whole space $\mathbb{R}^2$. As discussed in the previous section, the deterministic forcing term $f$ is time-independent and $f \in H$.

5.1. Invariant sample measures. First of all, let us recall some basic definitions from the work [69].

**Definition 5.1** (Invariant sample measure, [69]). For $t \geq \xi$ and $\omega \in \Omega$, suppose that $\Phi(t - \xi, \theta_t \omega, \cdot)$ be a random DS over the measurable DS $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ with state space $X$. A family of Borel probability measures $\{\rho_{\theta_t \omega}\}_{t \in \mathbb{R}}$ on $X$ is called the invariant sample measure for $\Phi(t - \xi, \theta_t \omega, \cdot)$, if for $\mathbb{P}$-a.s. $\omega \in \Omega$ and for all $E \in \mathcal{B}(X)$,

$$
\rho_{\theta_t \omega}(E) = \rho_{\theta_t \omega}(\Phi(t - \xi, \theta_t \omega, E)), \ t, \xi \in \mathbb{R}, \ t \geq \xi.
$$

**Definition 5.2** (Generalized Banach limit, [69]). A generalized Banach limit is any linear functional, denoted by $\lim \limits_{t \to +\infty}$, defined on the space of all bounded real-valued functions on $[0, +\infty)$ and satisfying

(i) $\lim \limits_{t \to +\infty} \zeta(t) \geq 0$ for non-negative functions $\zeta(\cdot)$ on $[0, +\infty)$,

(ii) $\lim \limits_{t \to +\infty} \zeta(t) = \lim \limits_{t \to +\infty} \zeta(t)$ if the usual limit $\lim \limits_{t \to +\infty} \zeta(t)$ exists.

In order to apply the abstract results from the work [69], we require that $\Phi$ has a random attractor, and for given $v^* \in H$, $t \in \mathbb{R}$ and $\omega \in \Omega$, the $H$-valued mapping $\xi \mapsto \Phi(t - \xi, \theta_t \omega, v^*)$ is continuous and bounded on $(-\infty, t]$. The existence of random attractors is shown in Theorem 3.12. The following lemma is helpful to obtain the continuity and boundedness (discussed above) of the mapping $\Phi$.

**Lemma 5.3.** For given $v^* \in H$, $t \in \mathbb{R}$ and $\omega \in \Omega$, the $H$-valued mapping $\xi \mapsto \Phi(t - \xi, \theta_t \omega, v^*)$ is right continuous on $(-\infty, t]$.

**Proof.** Since $\Phi(t - \xi, \theta_t \omega, v^*) = v(t; \xi, \omega, v^*) = u(t; \xi, \omega, u^*)z^{-1}(t, \omega)$, it is enough to prove that $u(t; \xi, \omega, u^*)$ with $u^* = v^*z(\cdot, \omega)$ is right continuous on $(-\infty, t]$. Let us fix $\xi^* \in \mathbb{R}$ and $u^* \in H$, $\omega \in \Omega$. Now, we only need to prove that for any given $\varepsilon > 0$, we can find a positive real number $\varepsilon^* = \varepsilon^*(\varepsilon, \xi^*, \omega, u^*)$ such that

$$
\|u(\tau; \xi, \omega, u^*) - u^*\|_H < \varepsilon, \ \text{whenever} \ \xi \in (\xi^*, \xi^* + \varepsilon^*), \ \tau \in (\xi, \xi^* + \varepsilon^*),
$$

(5.1)

where $u(\tau; \xi, \omega, u^*)$ is the solution of the system (2.8) with the initial data $u^*$ and initial time $\xi$.

We have

$$
\|u(\tau; \xi, \omega, u^*) - u^*\|_H^2 = \|u(\tau; \xi, \omega, u^*)\|_H^2 - \|u^*\|_H^2 - 2(u(\tau; \xi, \omega, u^*) - u^*, u^*)
$$

$$
= \int_{\xi}^{\tau} \frac{d}{d\zeta}\|u(\xi, \omega, u^*)\|_H^2 d\zeta - 2(u(\tau; \xi, \omega, u^*) - u^*, u^*). \quad (5.2)
$$

From (3.7), we have

$$
\int_{\xi}^{\tau} \frac{d}{d\zeta}\|u(\xi, \omega, u^*)\|_H^2 d\zeta \leq \frac{\|f\|_H^2}{\alpha} \int_{\xi}^{\tau} z^2(\zeta, \omega) d\zeta < +\infty. \quad (5.3)
$$
From (3.8), we find
\[
\sup_{\xi \in [\xi^*-1,\xi^*+1]} \|u(\zeta; \xi, \omega, u^*)\|_{H^1}^2 \leq \sup_{\xi \in [\xi^*-1,\xi^*+1]} \left[ \|u^*\|_{H^1}^2 + \|f\|_{H^\alpha}^2 \int_\xi^\zeta z^2(r, \omega)dr \right]
\leq \|u^*\|_{H^1}^2 + \|f\|_{H^\alpha}^2 \int_{\xi^*-1}^{\xi^*} z^2(r, \omega)dr,
\]
which is finite and the right hand side is independent of \(\xi\). It implies from (5.3) that we can find a positive real number \(\varepsilon_1^* = \varepsilon_1^*(\varepsilon, \xi^*, \omega, u^*)\) such that
\[
\int_\xi^\tau \frac{d}{d\zeta} \|u(\zeta; \xi, \omega, u^*)\|_{H^1}^2 d\zeta \leq \frac{\varepsilon_1^*}{2}, \text{ whenever } \zeta \in (\xi, \xi^* + \varepsilon_1^*), \tau \in (\xi, \xi^* + \varepsilon_1^*). \tag{5.5}
\]
Now we estimate the final term on the right hand side of (5.2). From the fact that
\[
u(\cdot; \xi, \omega, u^*) \in C((\xi, +\infty); \mathbb{H}) \cap L^2_{loc}(\xi, +\infty; V),
\]
and (5.4), we can find a positive number \(K(\omega, \xi^*, u^*)\) which is independent of \(\xi\) such that
\[
\max_{\zeta \in [\xi^*-1,\xi^*+1]} \|u(\zeta; \xi, \omega, u^*)\|_{H^1}^2 \leq K(\omega, \xi^*, u^*), \text{ for all } \xi \in [\xi^*-1, \xi]. \tag{5.6}
\]
Since \(V\) is dense in \(\mathbb{H}\), for \(\varepsilon > 0\) same as in (5.1), we can find an element \(\tilde{u} \in V\) such that
\[
\|\tilde{u} - u^*\|_{H^1} \leq \frac{\varepsilon_1^*}{8(K(\omega, \xi^*, u^*) + \|u^*\|_{H^1}). \tag{5.7}
\]
Thus, for \(\xi \in (\xi^*, \xi^* + \varepsilon_1^*)\) and \(\tau \in (\xi, \xi^* + \varepsilon_1^*)\), in view of (5.6) and (5.7), we obtain
\[
|(u(\tau; \xi, \omega, u^*) - u^*, \tilde{u})| \leq |(u(\tau; \xi, \omega, u^*) - u^*, u^*)| + |(u(\tau; \xi, \omega, u^*) - u^*, \tilde{u} - u^*)|
\]
\[
\leq |(u(\tau; \xi, \omega, u^*) - u^*, \tilde{u})| + \frac{\varepsilon_1^*}{8}. \tag{5.8}
\]
Now, we consider
\[
|(u(\tau; \xi, \omega, u^*) - u^*, \tilde{u})| = \left| \int_\xi^\tau \frac{d}{d\zeta} u(\zeta; \xi, \omega, u^*)d\zeta, \tilde{u} \right|
\]
\[
\leq \|\tilde{u}\|_{V} \left( \int_\xi^\tau \left| \frac{d}{d\zeta} u(\zeta; \xi, \omega, u^*) \right|^2 d\zeta \right)^{1/2} (\tau - \xi)^{1/2}. \tag{5.9}
\]
From the first equation of the system (2.8), for any \(\tilde{u} \in V\), we have
\[
\left| \int \frac{d}{d\zeta} u(\zeta; \xi, \omega, u^*) \right| \leq \nu \|\nabla u(\zeta; \xi, \omega, u^*)\|_{H} \|\nabla \tilde{u}\|_{H} + \alpha \|u(\zeta; \xi, \omega, u^*)\|_{H} \|\tilde{u}\|_{H}
\]
\[
+ \frac{\sqrt{2}}{z(t, \omega)} \|u(\zeta; \xi, \omega, u^*)\|_{H} \|\nabla u(\zeta; \xi, \omega, u^*)\|_{H} \|\nabla \tilde{u}\|_{H} + z(t, \omega) \|f\|_{H} \|\tilde{u}\|_{H},
\]
\[
\leq C \left[ \|u(\zeta; \xi, \omega, u^*)\|_{H} + \|u(\zeta; \xi, \omega, u^*)\|_{H} \right]
\]
\[
+ \frac{1}{z(t, \omega)} \|u(\zeta; \xi, \omega, u^*)\|_{H} \|\nabla u(\zeta; \xi, \omega, u^*)\|_{H} + z(t, \omega) \|f\|_{H} \|\tilde{u}\|_{V},
\]
which gives

\[
\left\| \frac{d}{d\zeta} u(\zeta; \xi, \omega, u^*) \right\|_{V^*}^2 \leq C \left[ \| u(\zeta; \xi, \omega, u^*) \|_{V}^2 + \frac{1}{\varepsilon^2(t, \omega)} \| u(\zeta; \xi, \omega, u^*) \|_{H}^2 \right] + \varepsilon^2(t, \omega) \| f \|_{H}^2. 
\] (5.10)

Combining (5.9) and (5.10), we get

\[
\left\| [u(\tau; \xi, \omega, u^*) - u^*, \tilde{u}] \right\| \leq \left( \int_\xi^\tau \left[ \| u(\zeta; \xi, \omega, u^*) \|_{V}^2 + \frac{1}{\varepsilon^2(t, \omega)} \| u(\zeta; \xi, \omega, u^*) \|_{H}^2 \right] \, d\zeta \right)^{1/2} \| \tilde{u} \|_V (\tau - \xi)^{1/2}. 
\] (5.11)

From the fact that \( u(\cdot; \xi, \omega, u^*) \in C([\xi, +\infty); \mathbb{H}) \cap L^2_{\text{loc}}(\xi, +\infty; V) \), continuity of \( \varepsilon(\cdot, \omega) \), (5.4) and (5.11), it follows that for \( \varepsilon > 0 \) same as in (5.1), we can find a positive real number \( \varepsilon^*_2 = \varepsilon^*_2(\varepsilon, \xi^*, \omega, u^*) \) such that

\[
\left\| [u(\tau; \xi, \omega, u^*) - u^*, \tilde{u}] \right\| \leq \frac{\varepsilon^2}{8}, \quad \text{whenever} \quad \xi \in (\xi^* - \varepsilon^*_2), \quad \varepsilon \in (\xi^* - \varepsilon^*_2). 
\] (5.12)

Taking \( \varepsilon^* = \min\{\varepsilon^*_1, \varepsilon^*_2\} \), we obtain (5.1) by combining (5.2), (5.5), (5.8) and (5.12).

One can prove that, for given \( \nu^* \in \mathbb{H} \), \( t \in \mathbb{R} \) and \( \omega \in \Omega \), the \( \mathbb{H} \)-valued mapping \( \xi \mapsto \Phi(t - \xi, \theta_{\xi}\omega, \nu^*) \) is left continuous on \(( -\infty, t] \) following the analogous steps as in the proof of Lemma 5.3. In addition, in view of the attracting property of the random attractor, we have that the \( \mathbb{H} \)-valued map \( \xi \mapsto \Phi(t - \xi, \theta_{\xi}\omega, \nu^*) \) is bounded also on \(( -\infty, t] \). We provide the following result on the continuity and boundedness of the \( \mathbb{H} \)-valued mapping \( \xi \mapsto \Phi(t - \xi, \theta_{\xi}\omega, \nu^*) \) on \(( -\infty, t] \).

**Lemma 5.4.** For given \( \nu^* \in \mathbb{H} \), \( t \in \mathbb{R} \) and \( \omega \in \Omega \), the \( \mathbb{H} \)-valued mapping \( \xi \mapsto \Phi(t - \xi, \theta_{\xi}\omega, \nu^*) \) is continuous and bounded on \(( -\infty, t] \).

Now, in light of the abstract result on invariant sample measures which was established in [69, Theorem 2.1], we can say (using Theorem 3.12 and Lemma 5.4) that the 2D SNSE (2.5) has a family of invariant sample measures on the phase space \( \mathbb{H} \) which satisfies a random Liouville type theorem. Therefore we have the following result:

**Theorem 5.5.** Let \( f \in \mathbb{H} \) and \( \Phi \) be the random DS generated by the 2D SNSE (2.5) over the metric DS \(( \Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}}) \) with the state space \( \mathbb{H} \). Then for a given generalized Banach limit \( \text{LIM} \) and a given continuous function \( \varphi(\cdot) : \mathbb{R} \mapsto \mathbb{H} \), there exists a family of Borel probability measures \( \{\rho_{\theta_{\xi}\omega}\}_{t \in \mathbb{R}} \) on \( \mathbb{H} \) such that the support of \( \rho(\theta_{\xi}\omega) \) is contained in \( \mathcal{A}(\theta_{\xi}\omega) \) and

\[
\int_{\mathbb{H}} \varphi(\nu) \, d\rho_{\theta_{\xi}\omega}(\nu) = \int_{\mathcal{A}(\theta_{\xi}\omega)} \varphi(\nu) \, d\rho_{\theta_{\xi}\omega}(\nu)
= \text{LIM}_{t \to -\infty} \frac{1}{t - \tau} \int_{\tau}^{t} \varphi(\Phi(t - \xi, \theta_{\xi}\omega, \nu(\xi))) \, d\xi
= \text{LIM}_{t \to -\infty} \frac{1}{t - \tau} \int_{\tau}^{t} \varphi(\Phi(t - \xi, \theta_{\xi}\omega, \nu(\xi))) \, d\rho_{\theta_{\xi}\omega}(\nu) \, d\xi, 
\] (5.13)
for any non-negative, real valued continuous functional \( \varphi \) on \( \mathbb{H} \). In addition, for all \( \omega \in \Omega \), \( \rho_{\theta, \omega} \) is invariant with respect to the random DS \( \Phi \) in the sense that
\[
\int_{\mathcal{A}(\theta, \omega)} \varphi(v) \, d\rho_{\theta, \omega}(v) = \int_{\mathcal{A}(\theta, \omega)} \varphi(\Phi(t - \tau, \theta, \omega, v)) \, d\rho_{\theta, \omega}(v), \quad \text{for all } t \geq \tau. \tag{5.14}
\]

5.2. Random Liouville type theorem. Our next aim is to study a random Liouville type theorem for the 2D autonomous SNSE (2.5). For this purpose, we require the definition of the class of test functions. Let us rewrite the first equation of the system (2.5) as
\[
dv = L(v, t, \omega) = [-\nu \Lambda v - B(v) + f] \, dt + \sigma v dW. \tag{5.15}
\]

**Definition 5.6.** Let us define the class \( \mathcal{D} \) of test functions as the set of non-negative, real valued continuous functionals \( \Lambda = \Lambda(\omega) \) on \( \mathbb{H} \) which are bounded on bounded subset of \( \mathbb{H} \) and the following conditions hold:

(i) for any \( y \in \mathbb{V} \), the Fréchet derivative \( \Lambda'(y) \) exists: for each \( y \in \mathbb{V} \), there exists an element \( \Lambda'(y) \) such that
\[
\frac{|\Lambda(y + u) - \Lambda(y) - (\Lambda'(y), u)|}{\|u\|_V} \to 0 \quad \text{as } \|u\|_V \to 0, \quad u \in \mathbb{V};
\]

(ii) \( \Lambda'(y) \in \mathbb{V} \) for all \( y \in \mathbb{V} \), and the mapping \( y \mapsto \Lambda'(y) \) is continuous and bounded as a functional from \( \mathbb{V} \) to \( \mathbb{V} \);

(iii) for every global solution \( v(t; \cdot, \cdot, \cdot) \) of the system (2.5), for each \( \omega \in \Omega \),
\[
\frac{d}{dt} \Lambda(v) = \langle L(v, t, \omega), \Lambda'(v) \rangle, \tag{5.16}
\]

holds.

We refer readers to [68, p. 477] for such functions satisfying the conditions of Definition 5.6. Next, we provide a random Liouville type theorem for the 2D autonomous SNSE (2.5).

**Theorem 5.7.** Under the conditions of Theorem 5.5, for each \( \omega \in \Omega \), the following random Liouville type equation
\[
\int_{\mathcal{A}(\theta, \omega)} \Lambda(v) \, d\rho_{\theta, \omega}(v) - \int_{\mathcal{A}(\theta, \omega)} \Lambda(v) \, d\rho_{\theta, \omega}(v) = \int_{\tau}^{t} \int_{\mathbb{H}} \langle L(v, r, \omega), \Lambda'(v) \rangle \, d\rho_{\theta, \omega}(v) \, dr, \quad \text{for all } t \geq \tau, \tag{5.17}
\]
holds for all test functions \( \Lambda \in \mathcal{D} \).

**Proof.** Using Theorem 5.5 and the properties of test function (see Definition 5.6), one can obtain the proof by using similar arguments as in the proof of [69, Theorem 3.2], hence we omit it here. \( \square \)

**Remark 5.8.** The result of above Theorem 5.7 can be seen as a random Liouville theorem. If the random statistical equilibrium has been attained by 2D SNSE, then the statistical informations do not change with respect to time, that is, \( \Lambda'(v(\cdot, \cdot, \cdot, \omega, \cdot)) = 0 \). Therefore, it implies from (5.14) and (5.17) that for each \( \omega \in \Omega \),
\[
\int_{\mathcal{A}(\theta, \omega)} \varphi(v) \, d\rho_{\theta, \omega}(v) = \int_{\mathcal{A}(\theta, \omega)} \varphi(\Phi(t - \tau, \theta, \omega, v)) \, d\rho_{\theta, \omega}(v) = \int_{\mathcal{A}(\theta, \omega)} \varphi(v) \, d\rho_{\theta, \omega}(v), \tau \in \mathbb{R}. \tag{5.18}
\]
The above equality (5.18) says that the sample measures \( \{\rho_{\theta^i \omega}\}_{t \in \mathbb{R}} \) are invariant under the action of the RDS \( \Phi \). It declares that, for each \( \omega \in \Omega \), the shape of the random attractor \( \mathcal{A}(\theta^i \omega) \) could change randomly with the evolution of time from \( \tau \) to \( t \), along with the sample points \( \omega \in \Omega \), but the measures of \( \mathcal{A}(\theta^\tau \omega) \) and \( \mathcal{A}(\theta^t \omega) \) would be the same. This is the random version of the Liouville Theorem in Statistical Mechanics. Therefore, we say that the invariant sample measures \( \{\rho_{\theta^i \omega}\}_{t \in \mathbb{R}} \) of the 2D SNSE (2.5) satisfy a random Liouville type theorem.

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References

[1] L. Arnold, Random Dynamical Systems, Springer-Verlag, Berlin, Heidelberg, New York, 1998.
[2] Z. Brzéniak and Y. Li, Asymptotic compactness and absorbing sets for 2D stochastic Navier-Stokes equations in some unbounded domains, Trans. Amer. Math. Soc., 358(12)(2006), 5587-5629.
[3] Z. Brzeźniak, E. Motyl and M. Ondrejat, Invariant measure for the stochastic Navier-Stokes equations in unbounded 2D domains, Ann. Probab. 45(5) (2017), 3145–3201.
[4] T. Buckmaster and V. Vicol, Nonuniqueness of weak solutions to the Navier-Stokes equation, Annals of Mathematics, 189(1) (2019), 101-144.
[5] T. Caraballo, G. Lukaszewicz and J. Real, Pullback attractors for non-autonomous 2D-Navier-Stokes equations in some unbounded domains, C. R. Math. Acad. Sci. Paris, 342(4) (2006), 263-268.
[6] T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuss and J. Valero, Asymptotic behaviour of a stochastic semilinear dissipative functional equation without uniqueness of solutions, Discrete Contin. Dyn. Syst. Ser. B, 14 (2) (2010), 439-455.
[7] T. Caraballo, J.A. Langa and T. Taniguchi, The exponential behaviour and stabilizability of stochastic 2D-Navier-Stokes equations, J. Dynam. Differential Equations, 179(2) (2002), 714–737.
[8] H. Crauel, A. Debussche and F. Flandoli, Random attractors, J. Dynam. Differential Equations 9(2)(1995), 307-341.
[9] H. Crauel and F. Flandoli, Attractors for random dynamical systems, Probab. Theory Related Fields, 100 (1994), 365-393.
[10] H. Cui and P. E. Kloeden, Convergence rate of random attractors for 2D Navier-Stokes equation towards the deterministic singleton attractor, Chapter 10 in Contemporary Approaches and Methods in Fundamental Mathematics and Mechanics, Springer, 2021.
[11] H. Cui, Y. Li and J. Yin, Existence and upper semicontinuity of bi-spatial pullback attractors for smoothing cocycles, Nonlinear Anal., 128 (2015), 303-324.
[12] G. Da Prato and J. Zabczyk, Ergodicity for Infinite Dimensional Systems, London Mathematical Society Lecture Notes, 229, Cambridge University Press, 1996.
[13] A. Debussche, Ergodicity results for the stochastic Navier-Stokes equations: an introduction, In: Beirão da Veiga, H., Flandoli, F. (eds.) Topics in mathematical fluid mechanics, 2073, pp. 23–108. Springer, Heidelberg (2013).
[14] W. E, J.C. Mattingly, and Y. Sinai. Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation, Comm.Math. Phys., 224(1) (2001), 83–106.
[15] X. Feng and B. You, Random attractors for the two-dimensional stochastic g-Navier-Stokes equations, Stochastics, 92(4) (2020), 613-626.
[16] B. Ferrario, Ergodic results for stochastic Navier-Etokes equation, Stoch. Stoch. Rep., 60(3-4) (1997), 271–288.
[17] B. Ferrario, Stochastic Navier-Stokes equations: Analysis of the noise to have a unique invariant measure, Ann. Mat. Pura Appl. (4), CLXXVII (1999), 331–347.

[18] B. Fernando, S. S. Sritharan and M. Xu, A simple proof of global solvability for 2-D Navier-Stokes equations in unbounded domains, Differential Integral Equations, 23 (3-4), (2010), 223–235.

[19] F. Flandoli, Dissipativity and invariant measures for stochastic Navier-Stokes equations, NoDEA Non-linear Differential Equations Appl., 1(4) (1994), 403–423.

[20] F. Flandoli and B. Maslowski, Ergodicity of the 2D Navier-Stokes equations under random perturbations, Comm. Math. Phys. 172(1) (1995), 119–141.

[21] C. Foias, O. Manley, R. Rosa and R. Temam, Navier-Stokes Equations and Turbulence, Cambridge University Press, 2008.

[22] B. Gess, W. Liu and A. Schenke, Random attractors for locally monotone stochastic partial differential equations, J. Differential Equations, 269 (2020), 3414–3455.

[23] N. Glatt-Holtz and M. Ziane, Strong pathwise solutions of the stochastic Navier-Stokes system, Adv. Differential Equations, 14 (5/6) (2009), 567–600.

[24] N. Glatt-Holtz, J. C. Mattingly and G. Richards, On unique ergodicity in nonlinear stochastic partial differential equations, J. Stat. Phys. 166(3) (2017), 618-649.

[25] A. Gu, B. Guo and B. Wang, Long term behavior of random Navier-Stokes equations driven by colored noise, Discrete Contin. Dyn. Syst. Ser. B, 25(7) (2020), 2495-2532.

[26] A. Gu, K. Lu and B. Wang, Asymptotic behavior of random Navier-Stokes equations driven by Wong-Zakai approximations, Discrete Contin. Dyn. Syst. Ser. B, 39(1) (2019), 185-218.

[27] M. Hairer and J.C. Mattingly, Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing, Ann. of Math. 164(3) (2006), 993–1032.

[28] M. Hairer and J.C. Mattingly. Spectral gaps in Wasserstein distances and the 2D stochastic Navier-Stokes equations, J. Differential Equations, 269 (2020), 2050–2091.

[29] E. Hopf, ¨Uber die Anfangswertaufgabe f ¨ur die hydrodynamischen Grundgleichungen, Math. Nachr., 4 (1951), 213–231.

[30] A. Ilyin, K. Patni and S. Zelik, Upper bounds for the attractor dimension of damped Navier-Stokes equations in $\mathbb{R}^2$, Discrete Contin. Dyn. Syst., 36(4) (2016).

[31] J. Jiang and Y. Hou, The global attractor of g-Navier-Stokes equations with linear dampness on $\mathbb{R}^2$, Appl. Math. Comput., 215(3) (2009), 1068-1076.

[32] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Encyclopedia of Mathematics and its Applications, Series Number 54, 1995.

[33] K. Kinra and M. T. Mohan, Existence and upper semicontinuity of random pullback attractors for 2D and 3D non-autonomous stochastic convective Brinkman-Forchheimer equations on whole space, Submitted, https://arxiv.org/pdf/2205.02099.pdf.

[34] K. Kuksin and A. Shirikyan, Stochastic dissipative PDEs and Gibbs measures, Comm. Math. Phys. 213(2) (2000), 291–330.

[35] A. Kulik and M. Scheutzow, Generalized couplings and convergence of transition probabilities, Probab. Theory Relat. Fields, 171(1) (2018), 333-376.

[36] O. A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, New York, 1969.

[37] J. Leray, Sur le mouvement d’un liquide visqueux emplissant l’espace, Acta Math., 63(1) (1934), 193–248.
[42] Y. Li, A. Gu and J. Li, Existence and continuity of bi-spatial random attractors and application to stochastic semilinear Laplacian equations, *J. Differential Equations*, **258**(2) (2015), 504-534.

[43] J. C. Mattingly, Ergodicity of 2D Navier-Stokes equations with random forcing and large viscosity, *Comm. Math. Phys.*, **206**(2) (1999), 273–288.

[44] J.C. Mattingly, On recent progress for the stochastic Navier-Stokes equations, In *Journées “Équations aux Dérivées Partielles”*, pages Exp. No. XI, 52. Univ. Nantes, Nantes, 2003.

[45] J. L. Menaldi and S. S. Sritharan, Stochastic 2-D Navier-Stokes equation, *Appl. Math. Optim.*, **46** (2002), 31–53.

[46] D. Mitrovic and D. Zubrinic, *Fundamentals of Applied Functional Analysis: Distributions-Sobolev Spaces-Nonlinear Elliptic Equations*, Pitman Monographs and Surveys in Pure and Applied Mathematics 91, 1998.

[47] M. T. Mohan, Stochastic convective Brinkman-Forchheimer equations, *Submitted*, https://arxiv.org/abs/2007.09376.

[48] M. T. Mohan and S. S Sritharan, Stochastic Euler equations of fluid dynamics with Lévy noise, *Asymptot. Anal.*, **99** 1–2 (2016), 67–103.

[49] V. Nersesyan, Ergodicity for the randomly forced Navier-Stokes system in a two-dimensional unbounded domain, *Ann. Henri Poincaré*, **23**(6) (2022), 2277–2294.

[50] J. C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge University Press, 2001.

[51] J. C. Robinson, Attractors and Finite-Dimensional Behaviour in the 2D Navier-Stokes Equations, *International Scholarly Research Notices*, **2013**, Article ID 291823, 29 pages, 2013.

[52] R. Rosa, The global attractor for the 2D Navier-Stokes flow on some unbounded domains, *Nonlinear Anal.*, **32** (1998), 71-85.

[53] K. Sakthivel and S. S. Sritharan, Martingale solutions for stochastic Navier-Stokes equations driven by Lévy noise, *Evol. Equ. Control Theory*, **1**(2) (2012), 355–392, doi: 10.3934/eect.2012.1.355.

[54] L. Shi, L. Shi, D. Turaev and L. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics*, Part II, World Scientific, New Jersey, 2001.

[55] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, vol. 68, Applied Mathematical Sciences, Springer, 1988.

[56] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, Second Edition, CBMS-NSF Regional Conference Series in Applied Mathematics, 1995.

[57] B. Wang, Upper semicontinuity of random attractors for non-compact random dynamical systems, *Electron. J. Differential Equations*, **2009** (2009), 1–18.

[58] B. Wang, Suffcient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differential Equations*, **253** (5) (2012), 1544-1583.

[59] B. Wang, Periodic random attractors for stochastic Navier-Stokes equations on unbounded domain, *Electron. J. Differential Equations*, **2012** (59) (2012), 1-18.

[60] R. Wang, K. Kinra and M. T. Mohan, Asymptotically autonomous robustness in probability of random attractors for stochastic Navier-Stokes equations on unbounded Poincaré domains, *Submitted*, https://arxiv.org/pdf/2208.06808.pdf.

[61] R. Wang, Y. Li and B. Wang, Bi-spatial pullback attractors of fractional nonclassical diffusion equations on unbounded domains with (p,q)-growth nonlinearities, *Appl. Math. Optim.*, **84** (2021), 425–461.

[62] X. Wang, J. Shen, K. Lu and B. Wang, Wong-Zakai approximations and random attractors for non-autonomous stochastic lattice systems, *J. Differential Equations*, **280** (2021), 477-516.

[63] R. Wang and B. Wang, Random dynamics of non-autonomous fractional stochastic p-Laplacian equations on $\mathbb{R}^N$, *Banach J. Math. Anal.*, **15**, 19 (2021).

[64] E. Weinan, J.C. Mattingly and Y.G. Sinai, Gibrssian dynamics and ergodicity for the stochastically forced Navier-Stokes equation, *Comm. Math. Phys.*, **224** (2001), 83–106.

[65] E. Weinan and J. C. Mattingly, Ergodicity for the Navier-Stokes equation with degenerate random forcing: finite-dimensional approximation, *Comm. Pure Appl. Math.*, **54**(11) (2001), 1386–1402.

[66] J. Xu and T. Caraballo, Long time behavior of stochastic nonlocal partial differential equations and Wong-Zakai approximations, *SIAM J. Math. Anal.*, **54**(3) (2022), 2792-2844.
[67] W. Zhao, Pullback attractors for bi-spatial continuous random dynamical systems and application to stochastic fractional power dissipative equation on an unbounded domain, *Discrete Contin. Dyn. Syst. Ser. B*, **24**(7) (2019), 3395-3438.

[68] C. Zhao, Y. Li and T. Caraballo, Trajectory statistical solutions and Liouville type equations for evolution equations: abstract results and applications, *J. Differential Equations*, **269** (2020), pp. 467-494.

[69] C. Zhao, J. Wang and T. Caraballo, Invariant sample measures and random Liouville type theorem for the two-dimensional stochastic Navier-Stokes equations, *J. Differential Equations*, **317** (2022), 474-494.