Learning Collaborative Policies to Solve NP-hard Routing Problems

Minsu Kim†  Jinkyoo Park‡  Joungho Kim†
Korea Advanced Institute of Science and Technology (KAIST)
† School of Electrical Engineering, ‡ Dept. Industrial & Systems Engineering
{min-su, jinkyoo.park, joungho}@kaist.ac.kr

Abstract

Recently, deep reinforcement learning (DRL) frameworks have shown potential for solving NP-hard routing problems such as the traveling salesman problem (TSP) without problem-specific expert knowledge. Although DRL can be used to solve complex problems, DRL frameworks still struggle to compete with state-of-the-art heuristics showing a substantial performance gap. This paper proposes a novel hierarchical problem-solving strategy, termed learning collaborative policies (LCP), which can effectively find the near-optimum solution using two iterative DRL policies: the seeder and reviser. The seeder generates as diversified candidate solutions as possible (seeds) while being dedicated to exploring over the full combinatorial action space (i.e., sequence of assignment action). To this end, we train the seeder’s policy using a simple yet effective entropy regularization reward to encourage the seeder to find diverse solutions. On the other hand, the reviser modifies each candidate solution generated by the seeder; it partitions the full trajectory into sub-tours and simultaneously revises each sub-tour to minimize its traveling distance. Thus, the reviser is trained to improve the candidate solution’s quality, focusing on the reduced solution space (which is beneficial for exploitation). Extensive experiments demonstrate that the proposed two-policies collaboration scheme improves over single-policy DRL framework on various NP-hard routing problems, including TSP, prize collecting TSP (PCTSP), and capacitated vehicle routing problem (CVRP).

1 Introduction

Routing is a combinatorial optimization problem, one of the prominent fields in discrete mathematics and computational theory. Among routing problems, the traveling salesman problem (TSP) is a canonical example. TSP can be applied to real-world problems in various engineering fields, such as robot routing, biology, and electrical design automation (EDA) [1, 2, 3, 4, 5] by expanding constraints and objectives to real-world settings: coined TSP variants are expanded version of TSP. However, TSP and its variants are NP-hard, making it challenging to design an exact solver [6].

Due to NP-hardness, solvers of TSP-like problems rely on mixed-integer linear programming (MILP) solvers [7] and handcrafted heuristics [8, 9]. Although they often provide a remarkable performance on target problems, the conventional approaches have several limitations. Firstly, in the case of MILP solvers, the objective functions and constraints must be formulated into linear forms, but many real-world routing applications, including biology and EDA, have a non-linear objective. Secondly, handcrafted heuristics rely on expert knowledge on target problems, thus hard to solve other problems. That is, whenever the target problem changes, the algorithm must also be re-designed.

Deep reinforcement learning (DRL)-routing frameworks [10, 11, 12] is proposed to tackle the limitation of conventional approaches. One of the benefits of DRL is that reward of DRL can be any
value, even from a black-box simulator; therefore, DRL can overcome the limitations of MILP on real-world applications. Moreover, DRL frameworks can automatically design solvers relying less on a handcrafted manner.

We note that the main objective of our research is not outperforming problem-specific solvers like the Concorde [9], a TSP solver. Our problem-solving strategy based on DRL, however, ultimately focuses on practical applications1 including intelligent transportation [13], biological sequence design [14], routing on electrical device [15] and device placement [16, 17]. Therefore, this paper evaluates the performance of DRL frameworks on TSP-like problems as a benchmark for potential applicability to practical applications, including speed, optimality, scalability, and expand-ability to other problems. TSP-like problems are excellent benchmarks as they have various baselines to compare with and can easily be modeled and evaluated.

Contribution. This paper presents a novel DRL scheme, coined learning collaborative policies (LCP), a hierarchical solving protocol with two policies: seeder and reviser. The seeder generates various candidate solutions (seeds), each of which will be iteratively revised by the reviser to generate fine-tuned solutions.

Having diversified candidate solutions is important, as it gives a better chance to find the best solution among them. Thus, the seeder is dedicated to exploring the full combinatorial action space (i.e., sequence of assignment action) so that it can provide as diversified candidate solutions as possible. It is important to explore over the full combinatorial action space because the solution quality highly fluctuates depending on its composition; however, exploring over the combinatorial action space is inherently difficult due to its inevitably many possible solutions. Therefore, this study provides an effective exploration strategy applying an entropy maximization scheme.

The reviser modifies each candidate solution generated by the seeder. The reviser is dedicated to exploiting the policy (i.e., derived knowledge about the problem) to improve the quality of the candidate solution. The reviser partitions the full trajectory into sub-tours and revises each sub-tour to minimize its traveling distance in a parallel manner. This scheme provides two advantages: (a) searching over the restricted solution space can be more effective because the reward signal corresponding to the sub-tour is less variable than that of the full trajectory when using reinforcement learning to derive a policy, and (b) searching over sub-tours of seeds can be parallelized to expedite the revising process.

The most significant advantage of our method is that the reviser can re-evaluate diversified but underrated candidates from the seeder without dropping it out early. Since the seeder explores the full trajectory, there may be a mistake in the local sub-trajectory. Thus, it is essential to correct such mistakes locally to improve the solution quality. The proposed revising scheme parallelizes revising process by decomposing the full solution and locally updating the decomposed solution. Thus it allows the revisers to search over larger solution space in a single inference than conventional local search (i.e., number of iteration of the reviser is smaller than that of conventional local search 2-opt [18], or DRL-based 2-opt [19]), consequently reducing computing costs. Therefore, we can keep the candidates without eliminating them early because of computing costs.

The proposed method is an architecture-agnostic method, which can be applied to various neural architectures. The seeder and reviser can be parameterized with any neural architecture; this research utilizes AM [12], the representative DRL model on combinatorial optimization, to parameterize the seeder and the reviser. According to the experimental results, the LCP improves the target neural architecture AM [12], and outperforms competitive DRL frameworks on TSP, PCTSP, and CVRP (\(N = 20, 50, 100, 500\), \(N\) : number of nodes) and real-world problems in TSPLIB [20]. Moreover, by conducting extensive ablation studies, we show proposed techniques, including entropy regularization scheme and revision scheme, clearly contribute to the performance improvement.

2 Related Works

There have been continuous advances in DRL frameworks for solving various routing problems. DRL framework can generate solvers that do not rely on the ground-truth label of target problems: it can be applied to un-explored problems. DRL-based approaches can be categorized into two parts;
constructive heuristics and improvement heuristics. We survey these two categories and current emerging hybrid approaches of machine learning (ML) with conventional solvers.

2.1 DRL-based Constructive Heuristics

Bello et al. [10] introduced an actor-critic algorithm with a policy parameterized by the pointer network [21]. They proposed a constructive Markov decision process (MDP), where the action is defined as choosing one of the un-served nodes to visit, given a partial solution; the policy is trained to add a node to provide a complete solution sequentially. Later, DRL-based constructive heuristics were developed to design the architecture of neural networks while preserving the constructive MDP [10]. Khalil et al. [11] proposed a DRL framework with a graph embedding structure. Nazari et al. [22], Duedon et al. [23] and Kool et al. [12] redesigned the pointer network [21] using the transformer [24] and trained it with a policy gradient method [25]. The AM by Kool et al. [12] reports substantial results on various NP-hard routing problems, including TSP, PCTSP, CVRP, and orienteering problem (OP) in high-speed computation.

AM-variants. After the meaningful success of the AM, many studies are expanded from the AM. Many engineering fields and industries apply AM into their domain. For example, Liao et al. [4] proposed a routing algorithm for the circuit using AM.

Some researches focus on increasing the performances of AM on classic routing problems like TSP by simple techniques. Kwon et al. [26] proposed the POMO, effective reinforcement learning method for AM. They proposed a new RL baseline that can reduce the training variance of AM using the problem-specific property of TSP and CVRP. In addition, they presented an effective post-processing algorithm for TSP and CVRP. However, their proposed method has a limitation in that it is problem-specific because it uses the domain properties of TSP and CVRP (e.g., their method is limited to be applied to PCTSP).

Xin et al. [27] proposed AM-style DRL-model, MDAM, for NP-hard routing problems. Their method learns multiple AM decoders and derives various solutions through the multiple decoders. The goal of increasing the solution diversity is similar to our research. However, our study is different where it increases the entropy of a single decoder and improves the mistakes of various solutions through a reviser.

2.2 DRL-based Improvement Heuristics

Unlike the constructive MDP, DRL-based improvement heuristics are designed to improve the completed solution iteratively. Most researches on DRL-based improvement heuristics are inspired by classical local search algorithms such as 2-opt [18] and the large neighborhood search (LNS) [28]. Chen et al. [29] proposed a DRL-based local search framework, termed NeuRewriter, that shows a promising performance on CVRP and job scheduling problems. Wu et al. [30], and Costa et al. [31] proposed a DRL-based TSP solver by learning the 2-opt. Their method improves the randomly generated solutions, unlike the method of Chen et al. [29] rewrites a solution given by a conventional heuristic solver. Hottung & Tierney [32] proposed a novel search method of VRP that destroys and repairs a solution repeatably inspired LNS. Their method gives promising performances on CVRP.

Improvement heuristic approaches generally show better performance than constructive heuristics but are usually slower than constructive heuristics. In the case of TSP, the number of neural network’s inferences of constructive heuristics is the same as the number of cities to visit. However, the number of inferences of the improvement heuristics is generally much larger.

2.3 Hybrid Approaches with Conventional Solvers

There are several studies on hybrid approaches with conventional solvers having promising performance recently. Lu et al. [33] proposed a hybrid method, where the policy is learned to control improvement operators (handcrafted heuristic). Significantly, they outperform the LKH3, which is widely considered as mountain to climb in machine learning (ML) communities. Joshi et al. [34] combined graph neural network (GNN) model with the beam search algorithm. They trained the GNN with supervised learning for generating a hit map of candidate nodes. Then trained GNN reduces a searching space for improvement heuristics. Similarly, Fu et al. [35] combined supervised
GNN model with Monte Carlo tree search (MCTS) and Kool et al. [36] combined supervised GNN model with dynamic programming. Their method achieves significant performances, showing ML method can effectively collaborate with conventional operational research (OR) methods.

The research scope of hybrid approaches and DRL-based methods is different. Hybrid approaches can overcome classical solvers in target tasks by collaborating with the classical solvers. However, hybrid approaches have inherited limitations from classical solvers that are poor expandability to other tasks. The DRL-based method can be applied to various real-world tasks without a classic solver; we can also utilize DRL-based to unexplored-ed tasks. This paper investigates the DRL-based NP-hard routing method without the help of classical solvers.

3 Formulation of Routing Problems

This section explains the Markov decision process (MDP) formulation for the given 2D Euclidean TSP as a representative example. The formulation of MDP for other problems is described in Appendix A.1.

The main objective of TSP is to find the shortest path of the Hamiltonian cycle. The TSP graph can be represented as a sequence of \( N \) nodes in 2D Euclidean space, \( s = \{ x_i \}_{i=1}^{N} \), where \( x_i \in \mathbb{R}^2 \). Then, the solution of TSP can be represented as the permutation \( \pi \) of input sequences:

\[
\pi = \bigcup_{t=1}^{N} \{ \pi_t \}, \quad \pi_t \in \{1, \ldots, N\}, \quad \pi_{t_1} \neq \pi_{t_2} \quad \text{if} \quad t_1 \neq t_2
\]

The objective is minimizing the tour length \( L(\pi | s) = \sum_{t=1}^{N-1} ||x_{\pi_{t+1}} - x_{\pi_t}||_2 + ||x_{\pi_N} - x_{\pi_1}||_2 \).

Then, we formulate the constructive Markov decision process (MDP) of TSP.

State. State of MDP is represented as a partial solution of TSP or a sequence of previously selected actions: \( \pi_{1:t-1} \).

Action. Action is defined as selecting one of un-served tasks. Therefore, action is represented as \( \pi_t \) where the \( \pi_t \in \{1, \ldots, N\} \setminus \{\pi_{1:t-1}\} \).

Cumulative Reward. We define cumulative reward for solution (a sequence of assignments) from problem instance \( s \) as negative of tourlength: \( -L(\pi | s) \).

Constructive Policy. Finally we define constructive policy \( p(\pi | s) \) that generates a solution \( \pi \) from TSP graph \( s \). The constructive policy \( p(\pi | s) \) is decomposed as:

\[
p(\pi | s) = \prod_{t=1}^{N} p_t(\pi_t | \pi_{1:t-1}, s)
\]

Where \( p_t(\pi_t | \pi_{1:t-1}, s) \) is a single-step assignment policy parameterized by parameter \( \theta \).

4 Learning Collaborative Policies

This section describes a novel hierarchical problem-solving strategy, termed learning collaborative policies (LCP), which can effectively find the near-optimum solution using two hierarchical steps,
4.1 Seeding Process

The seeder generates as diversified candidate solutions as possible while being dedicated to exploring the full combinatorial action space. To this end, the seeder is trained to solve the following problems.

**Solution space.** Solution space of seeder is a set of full trajectory solutions: \( \{\pi(1), ..., \pi(M)\} \). The \( M \) is the number of candidate solutions from the seeder: termed sample width.

**Policy structure.** Seeder is a constructive policy, as defined in section 3 as follows:

\[
p^S(\pi|s) = \prod_{t=1}^{t=N} p_{\theta_S}(\pi_t|\pi_{1:t-1}, s)
\]

The segment policy \( p_{\theta_S}(\pi_t|\pi_{1:t-1}, s) \), parameterized by \( \theta^S \), is derived form AM [12].

**Entropy Reward.** To force the seeder policy \( p^S \) to sample diverse solutions, we trained \( p^S \) such that the entropy \( H \) of \( p^S \) to be maximized. To this end, we use the reward \( R^S \) defined as:

\[
R^S = H \left( \pi \sim \prod_{t=1}^{t=N} p_{\theta_S}(\pi_t|\pi_{1:t-1}, s) \right) \approx \sum_{t=1}^{N} w_t H \left( \pi_t \sim p_{\theta_S}(\pi_t|\pi_{1:t-1}, s) \right)
\]

The entropy of constructive policy is appropriate for measuring solution diversity. However, computing the entropy of constructive policy is intractable because search space is too large: \( N! \). Therefore, we approximate it as a weighted sum of the entropy of segment policies \( p_{\theta_S}(\pi_t|\pi_{1:t-1}, s) \) evaluated at different time step.

We use a linear scheduler (time-varying weights) \( w_t = \frac{N - t}{Nw} \) to boost exploration at the earlier stage of composing a solution: higher randomness imposed by the higher weight \( w_t \) at the early stage tends to generate more diversified full trajectories later. The \( N_w \) is the normalizing factor, which is a hyperparameter.

**Training scheme.** To train the seeder, we use the REINFORCE [25] algorithm with rollout baseline \( b \) introduced by Kool et al. [12]. Then the gradient of each objective function is expressed as follows:

\[
\nabla J(\theta^S|s) = E_{\pi \sim p^S} \left[ \left( L(\pi|s) - \alpha R^S(p_{1:N}^S, \pi) - b(s) \right) \nabla \log(p^S) \right]
\]
We modify the context embedding vector $h$ (see red-points in Figure 3). The segment policy $\pi$ assigns the segment including $k$ as $\pi_{k+1:l}$. However, since the destination of reviser is $N$ and solving TSP. Hence, $\pi_{k+1:l}$ is a high dimensional embedding vector from the transformer-based encoder, not the first node $\pi_1$, which is designed for solving TSP. Hence, $h$ is a high dimensional embedding vector from the transformer-based encoder, and $N$ is the number of multi-head attention layers. $h^{(N)}_{\pi_{k-1:l}}$ is the embedding of previously selected nodes, and $h_{\pi_1}$ is the embedding of the first node. However, since the destination of reviser is $k_{k+l}$, not the first node $\pi_1$, we change the embedding of the first node $h_{\pi_1}$ to be the embedding of the last node $h_{\pi_{k+l}}$ for the context embedding as $h_{\pi_1} = [h^{(N)}_{\pi_{k-1:l}}, h^{(N)}_{\pi_{k+l-1}}, h^{(N)}_{\pi_{k+l}}]$. We use the ADAM [37] optimizer to obtain the optimal parameter $\theta^*$ that minimizes the objective function.

4.2 Revision Process

In the revision process, given a candidate solution, the reviser decomposes the candidate solution into $K$ segments and simultaneously finds the optimum routing sequence for each segment of each candidate solution. The reviser repeats this revising process $I$ times to find the best-updated candidate solution. To be specific, the reviser sequentially updates candidate solutions ($I$ times) by repeatedly decomposing the full trajectories computed from the previous iteration into segments and revising the segments to produce $M$ updated full trajectory solutions. To sum up, reviser solves $M \times K$ segments in parallel ($M$: number of candidate solutions, $K$: number of the segment in each candidate solution), $I$ times in series.

The proposed scheme has advantages over conventional local search methods or DRL-based improvement heuristics. It searches larger solution spaces in a single inference; therefore, it reduces iteration $I$. For example, 2-opt and DRL-2opt [19] search $O(N^2)$ solution space (if it is parallelizable, $O(MN^2)$), while the reviser searches $O(MK \times !)$ which is much larger (when the number of nodes of the segment $l$ is big enough) in a single inference. Hence we can reduce the number of iterations $I$ significantly compared to 2-opt, or DRL-2opt [19], thus expediting the speed of the solution search (see Appendix E).

**Solution space.** Solution space of reviser is a partial segment of full trajectory solution represented as $\pi_{k+1:l}$. The $k$ is starting index, and $l$ is the number of nodes of the segment. For details of assigning segment including $k$ and $l$, see Appendix A.3.

**Policy structure.** Reviser is a constructive policy as follows:

$$p^R(\pi_{k+1:l} | s) = \prod_{t=1}^{l-1} p_{\theta^R}(\pi_{k:t} | \pi_{k:t-1}, \pi_{k:t+1}, s)$$

The segment policy $p_{\theta^R}$, parameterized by $\theta^R$, is in the similar form with that of AM [12]. Each $\pi_k$ and $\pi_{k+1:l}$ indicate the starting point and the destination point of the partial segment, respectively (see red-points in Figure 3).

We modify the context embedding vector $h_{(c)}^{(N)} = [\tilde{h}^{(N)}, h^{(N)}_{\pi_{k-1:l}}, h^{(N)}_{\pi_1}]$ of AM, which is designed for solving TSP. Hence, $\tilde{h}$ is a high dimensional embedding vector from the transformer-based encoder, and $N$ is the number of multi-head attention layers. $\tilde{h}^{(N)}$ is the mean of the entire embedding, $h^{(N)}_{\pi_{k-1:l}}$ is the embedding of previously selected nodes, and $h^{(N)}_{\pi_1}$ is the embedding of the first node. However, since the destination of reviser is $\pi_{k+1:l}$, not the first node $\pi_1$, we change the embedding of the first node $h^{(N)}_{\pi_1}$ to be the embedding of the last node $h^{(N)}_{\pi_{k+l}}$ for the context embedding as $h_{(c)}^{(N)} = [\tilde{h}^{(N)}, h^{(N)}_{\pi_{k+l-1}}, h^{(N)}_{\pi_{k+l}}]$.

**Revision Reward:** negative of partial tour length $L^R(\pi_{k+1:l} | s) = \sum_{t=1}^{l+1} ||x_{\pi_{k+t}} - x_{\pi_{k+t-1}}||_2$. 

Figure 3: Training progress of the reviser. From randomly generated TSP, sub-problems are decomposed. Then reviser infers to make constructive action referring starting and destination node and previous selected actions. The reward is a negative length of the partial tour generated from the reviser.
**Training scheme.** The training process is mostly the same as described in section 4.1, except that we have modified the length term $L$ to $L^R$, and set $\alpha = 0$ to remove entropy reward $R^S$ for training the reviser. Note that the seeder and reviser are trained separately.

5 Experiments

This section reports the experimental results of the LCP scheme on TSP, PCTSP, and CVRP ($N = 20, 50, 100, 500$, $N$: number of nodes). Also, we report several ablation studies in section 5.3 and Appendix B-F. We evaluate performance on real-world TSPs in the TSPLIB in Appendix G.

**Training Hyperparamters.** Throughout the entire training process of the seeder and reviser, we have exactly the same hyperparameters as Kool et al. [12], except that the training batch size of our seeder is 1024. To train the seeder’s policy, we set $\alpha = 0.5$ (2) and $N_w = \sum_{i=1}^{N} i$ for linear weight $w_t = \frac{N-t}{N_w}$ for entropy scheduling.

Details in the experimental setting, including hyperparameters, dataset configuration, and run time evaluation, are described in Appendix A.5.

5.1 Target Problems and Baselines

We evaluate the performance of LCP in solving the three routing problems: TSP, PCTSP, and CVRP. We provide a brief explanation of them. The detailed descriptions for these problems are in Appendix A.1.

**Travelling salesman problem (TSP).** TSP is a problem to find the shortest Hamiltonian cycle given node sequences.

**Price collecting travelling salesman problem (PCTSP).** PCTSP [38] is a problem, where each node has a prize and a penalty. The goal is to collect the nodes with at least a minimum total prize (constraint) and minimize tour length added with unvisited nodes’ penalties.

**Capacitated vehicle routing problem (CVRP).** CVRP [39] is a problem where each node has a demand, while a vehicle must terminate the tour when the total demand limit is exceeded (constraint). The objective is to minimize the tour length.

For the baseline algorithms, we use two types of algorithms: conventional heuristics and DRL-based solvers. For the conventional heuristics, we use Gurobi [7] (the commercial optimization solver), and the OR Tools [40] (the commercial optimization solver) for all three problems. In Table 1, Gurobi ($t$) indicates time-limited Gurobi whose running time is restricted below $t$. In addition, OR Tools ($t$) is the OR Tools that allows additional local search over a duration of $t$. For problem-specific heuristics, we use Concorde [9] for TSP, the iterative local search (ILS) [12] for PCTSP, and LKH3 [41] for CVRP.

For the baselines using DRL-based solvers, we concentrated on the ability of the LCP scheme, which is improved performance over AM. Validating that the two-policies collaboration scheme outperforms the single-policy scheme (i.e., AM) is a crucial part of this research; thus, the most important metric for performance evaluation is improvement between vanilla AM the AM + LCP. Also, we reproduced other competitive DRL frameworks: current emerging improvement heuristics. We exclude recently proposed AM-style constructive heuristics, including the POMO [26] and MDAM [27] because they can be candidate collaborators with LCP, not competitors (e.g., POMO + LCP is possible). The detailed method for evaluation baselines in Table 1 is described as follows:

**TSP.** We follow baseline setting of Kool et al. [12] and Costa et al. [19]. We set DRL baselines including the S2V-DQN [11], EAN [23], GAT-T [30], DRL-2opt [19], and AM [12]. We show the results of S2V-DQN and EAN reported by Kool et al. [12], and the results of GAT-T reported by Costa et al. [19]. Then we directly reproduce the two most competitive DRL frameworks among baselines, the AM and DRL-2opt, in our machine to make a fair comparison of the speed.

**PCTSP.** We follow baseline setting of Kool et al. [12]. We reproduce AM [12] for DRL baseline.

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2See source code in https://github.com/alstn12088/LCP
Table 1: Performance evaluation results of the LCP scheme compared with baseline heuristics and the DRL frameworks on TSP, PCTSP, and CVRP. The best costs (objective) among DRL frameworks are marked in bold. The H is heuristic and Solver is exact algorithm. The “LCP {640,2}” means that the sampling width $M$ is 640, and the number of iterations $I$ (of the reviser) is 2. We measure the performance in a limited time budget, which is 10 seconds per instance. The OB means “out of budget” and IF means “infeasible”, where the solver cannot generate solutions satisfying constraints of target problems in a limited time.

| Method | $N = 20$      |          |       |          |       |       |          |       |       |       |
|--------|--------------|----------|------|----------|------|------|----------|------|------|------|
|        | Cost Gap Time | Cost Gap Time | Cost Gap Time |
| Gurobi (Solver) | 3.84 0.00% 0.01s | 5.70 0.00% 0.01s | 7.76 0.00% 0.02s |
| OR Tools (H) | 3.85 0.37% | 5.80 1.76% | 8.12 4.53% |
| Concorde (H) | 3.84 0.00% 0.01s | 5.70 0.00% 0.01s | 7.76 0.00% 0.02s |
| S2V-DQN | 3.89 1.42% 5.99 5.16% 8.31 7.03% |
| EAN {M: 1280} | 3.84 0.11% 5.77 1.28% 8.75 12.7% |
| EAN+2OPT {M: 1280} | 3.84 0.09% 5.75 1.00% 8.12 4.64% |
| GAT-T {I: 5000} | 3.84 0.00% 5.71 0.20% 7.87 1.42% |
| Drl-2opt {I: 2000} | 3.84 0.00% 3.58s 5.70 0.12% 4.88s 7.83 0.87% 7.15s |
| AM {M: 1280} | 3.84 0.00% 0.18s 5.70 0.13% 0.30s 7.86 1.25% 0.57s |
| AM {M: 2560} | 3.84 0.00% 0.03s 5.70 0.10% 0.45s 7.85 1.13% 0.90s |
| AM {M: 7500} | 3.84 0.00% 0.06s 5.72 0.39% 0.29s 7.93 2.13% 1.21s |
| AM + LCP {640,10} | 3.84 0.00% 0.18s 5.70 0.13% 0.30s 7.86 1.25% 0.57s |
| AM + LCP {1280,10} | - 5.70 0.10% 0.45s 7.85 1.13% 0.90s |
| AM + LCP* {1280,45} | - 5.70 0.10% 0.45s 7.85 1.13% 0.90s |

| Method | $N = 50$      |          |       |          |       |       |          |       |       |       |
|--------|--------------|----------|------|----------|------|------|----------|------|------|------|
|        | Cost Gap Time | Cost Gap Time | Cost Gap Time |
| Gurobi (Solver) | 3.13 0.00% 0.01s | OB | OB |
| OR Tools (H) | 3.14 0.07% 0.01s | IF | IF |
| Gurobi (10s) (H) | 3.13 0.00% 0.01s | 5.17 15.6% 0.19s |
| OR Tools (10s) (H) | 3.14 0.05% 0.03s | 4.51 0.70% 0.31s | 6.35 6.21% 0.31s |
| OR Tools (60s) (H) | 3.13 0.01% 0.03s | 4.48 0.00% 1.80s | 6.08 1.56% 1.80s |
| ILS (H) | 3.16 0.77% 0.10s | 4.50 0.67% 0.72s | 5.98 0.00% 4.32s |
| AM {M: 1280} | 3.18 0.39% 0.03s | 4.52 0.74% 0.07s | 6.08 1.67% 0.17s |
| AM {M: 2560} | 3.15 0.41% 0.03s | 4.51 0.72% 0.10s | 6.07 1.57% 0.28s |
| AM + LCP {640,1} | 3.14 0.17% 0.04s | 4.50 0.51% 0.07s | 6.06 1.42% 0.15s |
| AM + LCP {1280,5} | 3.14 0.08% 0.10s | 4.49 0.32% 0.20s | 6.04 1.00% 0.39s |

| Method | $N = 100$     |          |       |          |       |       |          |       |       |       |
|--------|--------------|----------|------|----------|------|------|----------|------|------|------|
|        | Cost Gap Time | Cost Gap Time | Cost Gap Time |
| Gurobi (Solver) | 6.10 0.00% 0.01s | OB | OB |
| OR Tools (H) | 6.43 5.41% | 11.31 9.01% | 17.16 9.67% |
| LKH3 (H) | 6.14 0.58% 0.72s | 10.38 0.00% 2.52s | 15.65 0.00% 4.68s |
| RL {M: 10} | 6.40 4.92% | 11.15 7.46% | 16.96 8.39% |
| NLNS {I: 2000} | 6.19 1.47% 1.00s | 10.54 1.54% 1.63s | 16.00 2.17% 2.18s |
| AM {M: 1280} | 6.25 2.49% 0.05s | 10.62 2.40% 0.14s | 16.23 3.72% 0.34s |
| AM {M: 2560} | 6.25 2.39% 0.06s | 10.61 2.24% 0.31s | 16.17 3.34% 0.75s |
| AM {M: 7500} | 6.24 2.24% 0.09s | 10.59 2.06% 0.36s | 16.14 3.11% 1.42s |
| AM + LCP {640,1} | 6.17 1.15% 0.07s | 10.56 1.74% 0.15s | 16.05 2.58% 0.30s |
| AM + LCP {1280,1} | 6.16 0.92% 0.09s | 10.54 1.54% 0.20s | 16.03 2.43% 0.45s |
| AM + LCP {2560,1} | 6.15 0.84% 0.14s | 10.52 1.38% 0.31s | 16.00 2.24% 0.77s |
| AM + LCP {6500,1} | - - | 15.98 2.11% 1.73s |

**CVRP.** We follow baseline setting of Houttung & Tierney [32]. We report result of RL [22] based on Houttung & Tierney [32] and we reproduce AM [12] and NLNS [32].

### 5.2 Performance Evaluation

In this section, we report the performance of LCP on small-scale problems ($N = 20, 50, 100$) in Table 1. Then we provide a time-performance trade-off analysis including large-scale problems ($N = 500$). We note that time-performance analysis is significant because any method can find an optimal solution when given an infinite time budget. From the analysis, we can identify a specific time region, called winner region, where LCP performs the best in terms of both speed and performance.

**Performance evaluation on $N = 20, 50, 100$.** Our method outperforms all the DRL baselines and OR-tools in TSP, PCTSP, and CVRP, as clearly shown in Table 1.
Figure 4: Details of time-performance analysis. The left (fast) and lower (lower cost) trend indicates a better performance. In (e), the ILS cannot generate solution in time budget ($t < 1000$s).

Table 2: Ablation study of LCP components on TSP, PCTSP, and CVRP ($N = 100$). The optimal gap is measured by comparing it with state-of-the-art solvers. The Entropy Regularization indicates training the seeder with $R^S$, while the default is the uniform scheduling. The best performances are marked in bold.

| Component of the LCP | TSP          | PCTSP        | CVRP         |
|----------------------|--------------|--------------|--------------|
| Entropy Regularization | TSP          | PCTSP        | CVRP         |
| Weight Scheduling    | Reviser      | cost         | gap          | cost         | gap          | cost         | gap          |
| ✓                    | ✓            | 7.96         | 2.65%        | 6.08         | 1.64%        | 16.29        | 3.43%        |
| ✓                    | ✓            | 7.94         | 2.45%        | 6.07         | 1.62%        | 16.20        | 2.86%        |
| ✓                    | ✓            | 7.86         | 1.32%        | 6.04         | 1.13%        | 16.20        | 2.86%        |
| ✓                    | ✓            | 7.82         | 0.88%        | 6.04         | 1.02%        | 16.12        | 2.37%        |

Note that for TSP ($N = 100$), we applied two types of revisers, each of which is denoted LCP and LCP*, respectively. The details are described in Appendix A.4 with pseudo-code. Our LCP and LCP* outperform DRL-2opt, the current state-of-the-art DRL-based improvement heuristic in $N = 20, 50, 100$, surpass 0.33% in $N = 100$.

In PCTSP, LCP outperforms AM with less time. Our method (AM + LCP (640,1)) outperforms the OR-Tools (10s), with 4× and 2× faster speed in $N = 50, 100$ respectively. Compared to the ILS, our method (AM + LCP (1280,5)) underperforms by 1.0%, but has 11× faster speed for $N = 100$.

For CVRP, our method outperforms competitive DRL frameworks.

**Time-performance analysis on $N = 100, 500$.** In Figure 4, we describe the time-performance analysis. We cannot control the speed of the Concorde, ILS, and LKH3. We can control the speed of DRIL solvers by adjusting sample width $M$ or the number of iterations $I$. For PCTSP, we can change the speed of OR-tools by managing the time for additional local searches.

Our scheme clearly outperforms DRL-solvers in terms of both speed and performance. For PCTSP ($N = 100, 500$) and CVRP ($N = 500$), our method achieves the winner region of $t < 10$, which is best performed in a specific time region among all kind of baseline solvers (for CVRP ($N = 100$), our method achieves the winner region of $t < 5$).

**Performance on TSPLIB [20] data:** see Appendix G.
5.3 Ablation Study

In this section, we conduct an ablation study on LCP components. We leave further ablation studies to Appendix B-F.

Ablation study of collaborative policies. In Table 2, we ablate three significant components of LCP and show the experimental results for every case. In the case of vanilla AM, having none of LCP components, the performance is the poorest. On the other hand, collaboration of seeder trained with linearly scheduled-entropy and the reviser shows the best performance. Therefore, the experimental results empirically validate our proposal of hierarchically collaborating two policies and also demonstrate the effectiveness of using a linearly scheduled-entropy term shown in section 4.1 and Figure 2.

Ablation study of entropy regularization: see Appendix B.

Ablation study of SoftMax temperature: see Appendix C.

Ablation study of application of LCP to pointer network [10, 21]: see Appendix D.

Comparison with reviser and other improvement heuristics: see Appendix E.

Training convergence of seeder and reviser in different PyTorch seeds: see Appendix F.

6 Discussion

In this paper, we proposed a novel DRL scheme, learning collaborative policies (LCP). The extensive experiments demonstrate that our two-policies collaboration algorithm (i.e., LCP) outperforms conventional single-policy DRL frameworks, including AM [12], on various NP-hard routing problems, such as TSP, PCTSP, and CVRP.

We highlight that LCP is a reusable scheme, can solve various problems. The neural architecture of the seeder and reviser proposed in this paper is derived from AM [12]. It can be substituted by other architectures, such as the pointer network [10, 21] and AM-style architectures including POMO [26] and MDAM [27]. If further studies on neural architecture for combinatorial optimization are carried out, the seeder and reviser can be improved further.

Also, LCP can be directly applied to other combinatorial optimization tasks, including TSP with time windows (TSPTW), orienteering problem (OP), multiple TSP (mTSP), variations of the vehicle routing problem (VRP), and other practical applications.

Further Works. We made an important first step: two-policies collaboration where each policy specializing in exploration or exploitation can improve conventional single-policy systems on combinatorial optimization tasks. The important direction of further research is introducing more sophisticated strategies to explore or exploit combinatorial solution space. New exploration strategies for overcoming the proposed approximated entropy maximization scheme are needed. Also, it is necessary to investigate more effective exploitation strategies beyond the proposed revision scheme.

Acknowledgements and Disclosure of Funding

This research is supported in part by the KAIST undergraduates research program (URP), 2019. We thank Hankook Lee and Prof. Jinwoo Shin for building part of this project in the URP. We thank Joonsang Park, Keeyoung Son, Hyunwook Park, Haeyeon Rachel Kim, and our anonymous reviewers for feedback and discussions.
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A Details of Experiments

A.1 Detailed Explanation of Target Problems.

This paper solves three NP-hard routing problems, traveling salesman problem (TSP), prize collecting
TSP (PCTSP), and capacitated vehicle routing problem (CVRP). This section provides detailed
descriptions of PCTSP and CVRP (for TSP, see section 3).

Prize Collecting Travelling Salesman Problem (PCTSP). The PCTSP is similar to TSP, while
there are differences in that we do not have to visit all the nodes and that the destination is not the
first node but the depot node, i.e., a tour is not a cycle. Let \( N \) be the number of nodes. The problem
instance of PCTSP is \( s = \{(x_i, \lambda_i, \mu_i)\}_{i=1}^{N+1} \), where the \( x_i \in \mathbb{R}^2 \) is in 2D euclidean coordinates, \( \lambda_i \in \mathbb{R} \) is the penalty of unvisited node, and \( \mu_i \in \mathbb{R} \) is the prize of visited node.

\[
L((\pi|s) = \sum_{i=1}^{k} ||x_{\pi_i} - x_{\pi_{i-1}}||_2 + ||x_{\pi_1} - x_{n+1}||_2 + ||x_{\pi_k} - x_{n+1}||_2
\]

\[
\lambda((\pi|s) = \sum_{i \notin \pi} \lambda_i
\]

The \( L((\pi|s) \) is the tour length, and \( \lambda((\pi|s) \) is the total penalty of the unvisited nodes. The \( k = |\pi| \), \( k \leq N \) because the entire tour does not contain all of the nodes. There is a constraint of minimum
prize \( \mu(c) \) as follows:

\[
\sum_{i \in \pi} \mu_i \geq \mu(c)
\]

Capacitated Vehicle Routing Problem (CVRP). In CVRP, the vehicle can no longer visit nodes,
when it exceeds the maximum demand \( v(c) \). Thus, the vehicle has to go back to the depot node,
and start another tour. The vehicle can make \( k \) number of tours, \( \pi = \{\pi(i)\}_{i=1}^{k} \), where the first
and last element of each sub-tour (sub-permutation) \( \pi(i) \) is the depot node. Let \( N \) be the number
of nodes. Then the instance of CVRP expressed as \( s = \{(x_i, v_i)\}_{i=1}^{N+1} \), where the depot node is
\( s_{n+1} = (x_{n+1}, 0) \). The objective of CVRP is minimizing \( f \):

\[
f(\pi|s) = \sum_{i=1}^{k} \sum_{j=1}^{l_i} ||x_{\pi(i)_j} - x_{\pi(i)_{j-1}}||_2
\]

The \( l_i = |\pi(i)| \). For every tour \( \pi(i) \), their is constraint on maximum demand \( v(c) \):

\[
\sum_{j \in \pi(i)} v_j \leq v(c)
\]

The MDP formulation is mostly same as TSP. There is action restriction rule based on constraint
(4), where every action except selecting depot node is restricted when it may exceeds the maximum
demand (i.e. when it make violation of (4) when selecting node other than depot node).

Similarly to PCTSP, our reward is \( R = R_f + \alpha R_S \), where \( R_f = -f \). Policy structure and training
scheme are the same as TSP.
A.2 Detailed Implementation of Seeder in Inference Phase

This section provides implementation details of the seeder for the experiments. Our seeder $p^S$ is parameterized by the AM [12], which is the transformer [24] based encoder-decoder model. For the details of the architecture of the AM, see Kool et al. [12].

**SoftMax Temperature.** The output of the AM architecture is the compatibility of the query of all nodes $u_{(c)}$ (see (7) in Kool et al. [12]). Then the probability of selecting nodes can be expressed with the SoftMax function as follows:

$$p_{q^S}(\pi_t = i|\pi_{1:t-1}, s) = \text{SoftMax}(u_{(c)}, i, T)$$

$$\text{SoftMax}(u_{(c)}, i, T) = \frac{e^{u_{(c)i} \theta}}{\sum_j e^{u_{(c)j} \theta}}$$

The SoftMax temperature $T$ is an important hyperparameter of the sampling of the seeder. Note that if $T \approx 0$ then $p_{q^S}(\pi_t = i|\pi_{1:t-1}, s)$ will select greedy samples: i.e. $\pi_t = \text{argmax}(p_{q^S}(\pi_t = i|\pi_{1:t-1}, s))$. If $T \approx \infty$, it will be the same as random search. In the training phase, we set $T = 1$. The details of setting $T$ in the inference phase (i.e. in experiments) is described in Appendix A.5.

A.3 Detailed Implementation of Reviser

This section describes the detailed implementation of the reviser for each target problem.

**Travelling Salesman Problem (TSP).** Let the reviser($L$) is trained on TSP with $L$ nodes. During the training phase, starting node and end node is restricted to be selected; thus $L - 2$ nodes are in action space.

During the inference phase, the length of segment (the number of nodes in the action space) $l = L - 2$ (the -2 is because starting and end node are restricted), number of segment $K = N/L$ ($N$: number of nodes of seed from seeder). The starting point of each segment is represented as $\{k, k + L, k + 2L, ...\}$, where the every segment is disjoint each other.

The reviser repeats revising process by re-assigning $1 \leq k \leq L$. In the experiments, we simply assigned $k_i = (k_{i-1} + 1)$ iteratively, $1 \leq i \leq I$. See Algorithm 1 for details.

Let $I$ be number of iteration of revision process. For LCP* in Table 1, we used reviser(10). For LCP* in Table 1, and TSPLIB experiment in Appendix G and large-scale TSP ($N = 500$), we used reviser(20) ($I = 25$) and reviser(10) ($I = 20$) see Algorithm 2 for details.

For CVRP and PCTSP, there is depot node, we do not need additional process process to make reviser unlike TSP’s depot node itself being the starting node. Therefore we can use seeder (without entropy reward) as reviser. We note that reviser implementation of other problems of VRP variants, TSP variants are straightforward with a slight change in seeder. Also, most of problems can be also revised by TSP reviser as well.

**Prize Collecting Travelling Salesman Problem (PCTSP).** For the small-scale problems ($N = 20, 50, 100$), after the seeder generates seeds (intermediate solutions), the reviser does not select unvisited nodes or drop visited nodes. That is, the selection of visited nodes is fixed and the reviser only tunes the order of the visited nodes. Thus, we use reviser(10) for small scale experiments. The decomposition rule is same as TSP.

For the large-scale problems ($N = 500$), we set $K = 5, L = 100$. We used the seeder trained in PCTSP ($N = 100$) as reviser on large-scale tasks.

**Capacitated Vehicle Routing Problem (CVRP).** For small-scale problems ($N = 20, 50, 100$) we used reviser(10) with $K = 10$. For large-scale problems ($N = 500$) we used seeder trained on CVRP ($N = 100$) as reviser. We set $K = 2$ and $N = 250$.

Solution of CVRP has sub-tours $\pi_{(i)}$; the starting point of each segment is the same as starting point of the sub-tour. For parallelization (make segment length same), we make padding nodes (depot nodes) when the number of nodes in the sub-tour smaller than the assigned segment length $L$, end of sub-tours.
A.4 Algorithmic Details of LCP.

This section provides a pseudo-code-based explanation of section 4 and Appendix A.3 for clear understanding. The Algorithm 1 is for single reviser, Algorithm 2 is for double reviser (reviser1, reviser2). These algorithms mainly target TSP; application to PCTSP and CVRP is mostly similar (some difference in decomposition method) as described in Appendix A.3.

Algorithm 1 LCP ($M$: sample width, $K$: # of segment, $l$: length of segment, $I$: # of iteration)

1: $k = 1, L = l + 2$
2: $\{\pi(1), ..., \pi(M)\} \sim p^S$
3: for $i = 1 : I$ do
   4: $B = \{\pi_{k:k+L}, ..., \pi_{k:k+KL}\}, ..., \{\pi_{k:k+L}, ..., \pi_{k:k+KL}\}$: Decompose to make $B$.
   5: $R = \text{argmax}(p^R(B))$: Make revised segment $R$ by $p^R$.
   6: $\{\pi(1), ..., \pi(M)\} = \text{Composite}(R)$: Composition to full trajectory solutions.
   7: $k = k + 1$: Re-assign segment.
8: end for
9: $\pi(*) = \text{Best}(\{\pi(1), ..., \pi(M)\})$: Selecting best solution among candidates.

Algorithm 2 LCP* ($M$: sample width, $K_1, K_2, l_1, l_2, I_1, I_2$)

1: $k = 1, L_1 = l_1 + 2$: Initialization for Reviser 1 (R1)
2: $\{\pi(1), ..., \pi(M)\} \sim p^S$
3: for $i = 1 : I_1$ do
   4: $B = \{\pi_{k:k+L_1}, ..., \pi_{k:k+K_1L_1}\}, ..., \{\pi_{k:k+L_1}, ..., \pi_{k:k+K_1L_1}\}$: Decompose to make $B$.
   5: $R = \text{argmax}(p^{R1}(B))$: Make revised segment $R$ by $p^{R1}$.
   6: $\{\pi(1), ..., \pi(M)\} = \text{Composite}(R)$: Composition to full trajectory solutions.
   7: $k = k + 1$: Re-assign segment.
8: end for
9: $k = 1, L_2 = l_2 + 2$: Initialization for Reviser 2 (R2)
11: for $i = 1 : I_2$ do
   12: $B = \{\pi_{k:k+L_2}, ..., \pi_{k:k+K_2L_2}\}, ..., \{\pi_{k:k+L_2}, ..., \pi_{k:k+K_2L_2}\}$: Decompose to make $B$.
   13: $R = \text{argmax}(p^{R2}(B))$: Make revised segment $R$ by $p^{R2}$.
   14: $\{\pi(1), ..., \pi(M)\} = \text{Composite}(R)$: Composition to full trajectory solutions.
   15: $k = k + 1$: Re-assign segment.
16: end for
17: $\pi(*) = \text{Best}(\{\pi(1), ..., \pi(M)\})$: Selecting best solution among candidates.

A.5 Details of Experimental Setting

Dataset. We follow the method introduced in Kool et al. [12], random generation of the datasets of TSP, PCTSP, and CVRP based on the provided code\(^3\).

Runtime comparison. A single GPU (NVIDIA RTX 2080 Ti) and a single CPU (Intel i7-9700K) are used for all the experiments, with few exceptions. The speed of heuristic solvers in Table 1 is from Kool et al. [12], where it was performed using two CPUs (2 × Xeon E5-2630).

All the run time described in section 5 is the average time per instance. For the run time measurement in Table 1, we directly reproduce the AM and DRL-2opt with our machine. The implementation of DRL-2opt is based on code\(^4\) provided by the Costa et al. [31]. The results of the NLNS is from code\(^5\), and others are from Kool et al. [12].

For measuring speed, it is essential to set proper evaluation batch size $B$, which is the number of instances solving in parallel. However, it is difficult to make an absolutely fair setting of parallelization.

\(^3\)https://github.com/wouterkool/attention-learn-to-route
\(^4\)https://github.com/paulorocosta/learning-2opt-drl
\(^5\)https://github.com/ahottung/NSLNS
For example, heuristic methods are mainly performed in CPU, but DRL frameworks are performed in GPU. The parallelization ability of GPU is usually higher than CPU. Hence, DRL frameworks will show fast speed in high $B$.

Moreover, a fair comparison between DRL frameworks is also difficult. The DRL-2opt has merit with parallelization, but it has slow serial speed because they need iterative inferences (about 2000 iterations). In contrast, our LCP scheme has a fast serial speed because our seeder needs only one inference and reviser needs few iteration. Our revision process is parallelizable for a single instance because the reviser solves the decomposed seeds simultaneously.

For the experiments, we give a restriction of $B \leq 100$. Because, from a practical point of view, 10000 instances of the same scale rarely occur at once, we should restrict the evaluation batch size reasonably. Furthermore, many real-world problems require the sequential solving of routing problems: i.e., it is proper to measure speed in low $B$.

For the run time measurements in large scale experiment ($N = 500$), we set $B = 1$ for all of the baselines. The experiments are performed in the same CPU (Intel i7-9700K) and GPU (Nvidia RTX2080Ti), except for the DRL-2opt and NLNS. For DRL-2opt and NLNS, we set $B = 10$ (10% of total number of instances). Also, note that we reproduce the Concorde, Gurobi, OR Tools, ILS and LKH3 in large scale experiment ($N = 500$), based on code provided by Kool et al. [12].

**Temperature scaling.** In the experiments in small scale $N = 20, 50, 100$, we set the SoftMax temperature $T$ of the AM as 1, which is the default setting reported in Kool et al. [12]. For large scale experiment $N = 500$, we tuned $T$ for AM as 0.1. See Table 4 for the setting of $T$ for the LCP scheme.

| Table 3: Number of instances in each experiment. |
|-----------------------------------------------|
| Number of Instances | 10000 | 100 | 1000 | 1000 | 100 | 1000 | 10000 |

| Table 4: SoftMax temperature of LCP scheme in Table 1, Table 2, and Figure 4. |
|-----------------------------------------------|
| $n = 20$ | $n = 50$ | $n = 100$ | $n = 500$ |
| TSP | 2 | 2 | 2 | 0.1 |
| PCTSP | 3 | 2 | 1.5 | 0.2 |
| CVRP | 3 | 2 | 1 | 0.3 |
B Ablation Study of Scaled Entropy Regularization

This section provides a detailed ablation study of $\alpha$, a hyperparameter of scaled entropy reward $R^S$. We target TSP ($N = 100$) with randomly generated 100 instances. We use reviser(10) with $I = 10$ in the experiment. The SoftMax temperature $T$ of the seeder is fixed as 2. As shown in Table 5, when $\alpha = 0.2, 0.3$, the performance of LCP with uniform scaled entropy regularization exceeds that of linear scaling. However, when $\alpha = 0.4, 0.5$, LCP with linearly scaled entropy regularization outperforms uniform scaling, where they achieve best performances among all cases $\alpha = 0.2, 0.3, 0.4, 0.5$.

Table 5: Cost (lower is better) of LCP scheme with different training hyperparameter $\alpha$ and types of the scaled entropy regularization. The best performances are indicated in bold among the same $\alpha$. The * indicates best performances. The Seeding indicates cost after the seeding process. The Revision indicates cost after the revision process, which is finalized value.

| $\alpha$ | Uniform Scaling | Linear Scaling |
|----------|-----------------|----------------|
|          | Seeding | Revision | Seeding | Revision |
| 0.2      | 7.90    | **7.85** | 7.91    | 7.87     |
| 0.3      | 7.91    | **7.86** | 7.91    | 7.86     |
| 0.4      | 7.92    | 7.88     | 7.89    | **7.83** |
| 0.5      | 7.91    | 7.86     | 7.89    | **7.84** |
C Ablation to SoftMax Temperature

As shown in Figure 5, the experimental results demonstrate that our method is robust to variations on temperature scaling, whereas the AM [12] is vulnerable to high temperature. We remark that the optimal SoftMax temperature of our approach is higher than the AM. For example, in TSP ($N = 50$), the AM performs the best at $T = 2.0$, while our method gives the best performance at $2.0 < T < 3.5$. It demonstrates that while high temperature provides diverse solutions, the quality of the solution is insufficient because the fitness landscape of the AM is steeper than our scheme. In TSP ($N = 500$), both methods have a steep fitness landscape; thus, very low temperature $T = 0.1$ gives a reasonable solution. While our approach gives merit to temperature robustness on large scales with significant performance gain, a policy that has a wider fitness landscape is still needed.
D Applying the LCP to Other DRL Frameworks

Section 5 mentions that our LCP scheme can easily be applied to other on-policy DRL frameworks. This section presents empirical validation of LCP’s flexibility by using the LCP scheme to pointer network \([21, 10]\) in TSP \((N = 20)\). In the experiments, the seeder is parameterized by the pointer network, while the setting of the reviser is similar to previous experiments; we use reviser(10) with five iterations \((I = 5)\). The pointer network’s implementation is based on code provided by Kool et al. [12], the training hyperparameter is the same as them except we set the batch size as 1024, and \(\alpha = 0.5\). The SoftMax temperature \(T\) of the seeders is fixed as 1. The dataset of 1000 instances is randomly generated using the same method as previous experiments. As shown in Table 6, the sampling method (Pointer Network \{(1280)\}) with our LCP achieves the best performance among all settings, where it gives a significant performance increment compared to the vanilla pointer network.

Remarkably, the sampling method of the seeder itself reduces performance. The cost of the pointer network’s sampling method is 7.33, while the greedy method gives 3.95. However, with LCP, the cost of the sampling method drastically reduced, finally exceeding the greedy method even if LCP supports the greedy method. It demonstrates that the main idea of LCP of revising diverse seeds is promising to pointer network. Even if the sampled 1280 seeds are not reasonable solutions, seeds with diversity are revised in parallel. Eventually, the best solution among the revised seeds has outstanding performance.

Table 6: Ablation study of LCP components on TSP \((N = 20)\). The optimal gap is measured by comparing it with state-of-the-art solvers. As in Table 2, the Entropy Regularization indicates training the seeder with \(R^S\), while the default is the uniform scaling. The Linear Scaling means entropy regularization method with linear weight scheduling. The best performances are marked in bold. The Pointer Network (greedy) indicates Pointer Network with greedy selection, and the Pointer Network \{(1280)\} means the sampling method where the sample width is 1280.

| Component of the LCP | Pointer Network (greedy) | Pointer Network \{(1280)\} |
|----------------------|--------------------------|-----------------------------|
| Entropy Regularization | Weight Scheduling | Reviser | cost | gap | cost | gap |
| ✓ ✓ ✓ | ✓ ✓ ✓ | ✓ ✓ ✓ | 3.89 | 1.24% | 3.85 | 0.20% |
| ✓ ✓ ✓ | ✓ ✓ ✓ | ✓ ✓ ✓ | 3.89 | 1.18% | 3.85 | 0.24% |
| ✓ ✓ ✓ | ✓ ✓ ✓ | ✓ ✓ ✓ | 3.89 | 1.27% | 3.85 | 0.21% |
| ✓ ✓ ✓ | ✓ ✓ ✓ | ✓ ✓ ✓ | 3.95 | 2.62% | 7.32 | 90.29% |
| ✓ ✓ ✓ | ✓ ✓ ✓ | ✓ ✓ ✓ | 3.95 | 2.71% | 7.30 | 89.77% |
| ✓ ✓ ✓ | ✓ ✓ ✓ | ✓ ✓ ✓ | 3.95 | 2.63% | 7.33 | 90.75% |
| ✓ ✓ ✓ | ✓ ✓ ✓ | ✓ ✓ ✓ | 3.89 | 1.24% | 3.85 | 0.20% |
E Comparison with Reviser and Improvement Heuristics

Figure 6: The time-cost graph on TSP ($N = 100, 500$).

This experiment was conducted to test the reviser’s performance. Seeder+reviser outperforms Seeder+DRL-2opt in both time and performance. This experimental result demonstrates that the reviser performs well in a fast and accurate improvement heuristic role. Since reviser can be designed easily by modifying the seeder appropriately, when a new type of seeder (constructive heuristic) is proposed, we can create a high-performance improvement heuristic (i.e., reviser) accordingly.
Experiments of Training Convergence in Different PyTorch Seeds

Figure 7: Training graph of the seeder on TSP, PCTSP and CVRP ($N = 100$).

Figure 8: Training graph of the reviser.

This experiment verifies that the proposed seeder and reviser converge in various PyTorch seeds. DRL algorithms are sometimes unstable in convergence to the random seed; therefore, it is crucial to carry out experiments with random seeds. Through this experiment, we can see that both seeder and reviser converge steadily in 4 random seeds.
### G Details of Real World Experiments on TSP

This experiment evaluates the solver’s performance on 35 instances extracted from TSPLIB. This experiment considers the solver’s performance in real-world problems and tests how well the model trained on a fixed scale \( N = 100 \) fits in instances with various scales.

In this experiment, we experimented with reviser\((20)\) \( (I = 25) \), reviser\((10)\) \( (I = 20) \), and sample width \( M = 2560 \).

We set the sample width of AM as \( M = 40000 \) and the number of iterations of DRL-2opt as \( I = 2000 \).

We outperformed the baseline solver in 22 cases out of 35, and the average optimal gap outperformed DRL-2opt by 2.5%. It also showed an overwhelmingly faster speed compared to the DRL-2opt and AM.

Table 7: Performance comparison in real-world instances in TSPLIB. The (time) indicates time spending. We select the best performance among SoftMax temp \( T \in \{0.1, 0.5, 1, 2\} \) in both AM and ours. Therefore, we multiply 4 \( \times \) to spent time.

| Instance | Opt. | AM [12] | DRL-2opt [19] | AM + LCP (ours) |
|----------|------|---------|---------------|----------------|
|          |      | Cost    | Gap | Time | Cost | Gap | Time | Cost | Gap | Time |
| eil51    | 426  | 435     | 2.11%| 13s  | 427  | 0.23%| 460s | 429  | 0.73%| 13s  |
| berlin52 | 7.542| 8663    | 14.86%| 14s | 7974 | 5.73%| 460s | 7550 | 0.10%| 13s  |
| st70     | 675  | 690     | 2.18%| 23s | 680  | 0.74%| 540s | 680  | 0.74%| 13s  |
| eil76    | 538  | 555     | 3.18%| 27s | 552  | 2.60%| 540s | 547  | 1.64%| 18s  |
| pr76     | 180,159| 110,956| 2.59%| 14s | 111,085| 2.60%| 540s | 108,633| 0.44%| 18s  |
| rd99     | 1,211| 1,309   | 8.09%| 44s | 1,388| 14.62%| 680s | 1,292| 6.67%| 26s  |
| KroA100  | 21,282| 23,227| 9.14%| 46s | 23,751| 11.60%| 680s | 21,910| 2.95%| 26s  |
| KroB100  | 22,141| 23,227| 8.23%| 46s | 23,790| 7.45%| 680s | 22,476| 1.51%| 26s  |
| KroC100  | 20,749| 21,868| 5.40%| 46s | 22,672| 9.27%| 680s | 21,337| 2.84%| 26s  |
| KroD100  | 21,282| 22,984| 7.94%| 46s | 22,672| 9.27%| 680s | 21,337| 2.84%| 26s  |
| KroE100  | 22,068| 22,686| 2.80%| 46s | 23,334| 5.39%| 680s | 22,488| 1.90%| 26s  |
| eil101   | 629  | 654     | 4.03%| 46s | 635  | 0.95%| 680s | 645  | 2.59%| 26s  |
| lin105   | 14,379| 16,516| 14.87%| 49s | 16,156| 12.36%| 680s | 14,934| 3.86%| 26s  |
| pr124    | 59,030| 63,936| 8.30%| 72s | 61,073| 14.60%| 680s | 62,659| 3.86%| 37s  |
| bier127  | 118,282| 125,256| 5.90%| 72s | 121,122| 2.40%| 720s | 128,832| 9.27%| 26s  |
| ch130    | 6,110 | 6,279   | 2.76%| 77s | 6,175| 1.06%| 790s | 6,145| 0.57%| 38s  |
| pr136    | 96,772| 101,927| 5.33%| 84s | 98,453| 1.74%| 820s | 98,285| 5.67%| 38s  |
| pr144    | 58,537| 63,778| 8.95%| 93s | 61,207| 4.56%| 720s | 60,571| 3.47%| 43s  |
| kroA150  | 26,524| 28,658| 8.05%| 102s | 30,078| 13.40%| 900s | 27,501| 3.68%| 44s  |
| kroB150  | 26,130| 27,565| 5.49%| 102s | 28,169| 7.80%| 900s | 26,962| 3.18%| 44s  |
| pr152    | 73,682| 79,442| 7.82%| 101s | 75,301| 2.20%| 720s | 75,539| 2.52%| 44s  |
| u159     | 42,080| 50,656| 20.38%| 111s | 42,716| 1.51%| 840s | 46,640| 10.84%| 45s  |
| rat195   | 2,323 | 2,518   | 8.14%| 168s | 2,955| 27.21%| 1080s | 2,574| 10.81%| 57s  |
| kroA200  | 29,368| 33,313| 13.43%| 173s | 32,522| 10.74%| 1200s | 31,172| 6.14%| 86s  |
| ts225    | 126,643| 138,000| 8.97%| 223s | 127,731| 0.86%| 1100s | 134,827| 6.46%| 113s |
| tsp225   | 3,919 | 4,837   | 23.42%| 224s | 4,354| 11.10%| 1100s | 4,487| 14.50%| 113s |
| pr226    | 80,369| 90,390| 12.47%| 228s | 91,560| 13.92%| 940s | 85,262| 6.09%| 113s |
| kroB200  | 2,378 | 2,586   | 8.81%| 306s | 2,490| 4.71%| 1380s | 2,508| 5.49%| 34s  |
| gil126   | 42,029| 47,288| 12.51%| 397s | 46,065| 9.60%| 1470s | 46,540| 10.72%| 158s |
| lin318   | 15,281| 17,053| 11.59%| 458s | 16,159| 8.10%| 1870s | 16,519| 8.10%| 209s |
| pcb442   | 140,217| 160,594| 49.75%| 744s | 143,590| 33.92%| 1760s | 130,996| 22.18%| 228s |
| avg. gap | 0.00%| 9.90%   | 7.63%| 5.14%|