Bethe ansatz for the Temperley–Lieb spin chain with integrable open boundaries

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Abstract. In this paper we study the spectrum of the spin-1 Temperley–Lieb spin chain with integrable open boundary conditions. We obtain the eigenvalue expressions as well as its associated Bethe ansatz equations by means of the coordinate Bethe ansatz. These equations provide the complete description of the spectrum of the model.

Keywords: integrable spin chains (vertex models), quantum integrability (Bethe ansatz)
1. Introduction

Quantum integrable models and their associated classical vertex models have been widely investigated over the years [1, 2]. Many of these models have been studied by Bethe ansatz techniques with different boundary conditions and at zero or finite temperature and magnetic field.

Nevertheless there are still problems that cannot be treated by the standard techniques. One of these problems is the biquadratic spin-1 model [3]–[5]. The biquadratic model was shown to be invariant by the Temperley–Lieb algebra [6]. This property has made possible the discovery of new integrable quantum spin chains, which was achieved by exploiting the representation theory of the Temperley–Lieb algebra [7, 8].

Moreover the biquadratic model and its generalizations (Temperley–Lieb spin chains) were solved by the coordinate Bethe ansatz for periodic boundary conditions and free ends [3], [9]–[11]. Since then, the spectrum of the transfer matrices has been discussed by means of functional methods [12]. However, there is still no algebraic Bethe ansatz formulation for these models.

Recently the concept of Temperley–Lieb equivalence [1] was used in order to obtain the spectral properties of quantum spin chains of Temperley–Lieb type for periodic boundary conditions and free ends [13]. The study of the spectral multiplicities allowed the computation of the thermodynamic properties at finite temperature [13].

Besides that, the solution of the reflection equation, associated with the problem of Temperley–Lieb spin chains with integrable open boundaries was recently obtained [14, 15]. However, the computation of the spectra of these spin chains is still an open problem.

In order to fill this gap, in this paper we are interested in the spectra of the $U_q[sl(2)]$ Temperley–Lieb spin chain with integrable open boundary conditions [14]. We
use a suitable generalization of the coordinate Bethe ansatz [10] in order to obtain the
eigenvalues of the spin-1 Temperley–Lieb model with diagonal open boundaries.

The outline of the article is as follows. In section 2 we introduce the Temperley–Lieb
spin chain with integrable open boundaries. In section 3 we discuss the application of
the coordinate Bethe ansatz and obtain the eigenvalues of the Temperley–Lieb spin chain.
Our conclusions are given in section 4. In the appendix we discuss the completeness of
the spectrum for $L = 4$.

2. Temperley–Lieb spin chain

The cornerstone of the theory of quantum integrable models in one-dimension is given by
the Yang–Baxter equation,

$$ R_{12}(\lambda - \mu)R_{23}(\lambda)R_{12}(\mu) = R_{23}(\mu)R_{12}(\lambda)R_{23}(\lambda - \mu). \quad (1) $$

This equation provides the commutativity property of the transfer matrix $T(\lambda) =
\text{Tr}_A[T_A(\lambda)]$, where $T_A(\lambda) = R_{AL}(\lambda) \cdots R_{A1}(\lambda)$, $R_{12}(\lambda) = P_{12}R_{12}(\lambda)$ and $P_{12}$ is the
permutation operator. Quantum integrable spin chains with periodic boundary conditions
are obtained by means of the logarithmic derivative of the transfer matrix $T(\lambda)$.

The Temperley–Lieb invariant solutions of the Yang–Baxter equation (1) are well
known [16]. The spin-1 $U_q[sl(2)]$ solution can be written as

$$ \tilde{R}_{ij}(\lambda) = \frac{\sinh(\gamma - \lambda)}{\sinh \gamma} I_{ij} + \frac{\sinh \lambda}{\sinh \gamma} U_{ij}, \quad (2) $$

where $2 \cosh \gamma = q^2 + 1 + q^{-2}$ and the Temperley–Lieb operator is given by

$$ U_{12} = \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & q^{-2} & 0 & -q^{-1} & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & -q & 0 & q^2 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (3) $$

which is the projector onto the two-sites spin zero singlet written in the basis
$\{|+,0,-\rangle\}$.

The notion of integrability was extended to tackle open boundary problems [17]. On
the one hand, the $R$-matrix describes the bulk dynamics, and on the other hand, a new
set of matrices, the $K$-matrices, represent the interactions at the left and right ends of
the open spin chain. This is a consequence of the reflection equation, which reads

$$ R_{12}(\lambda - \mu)K_1(\lambda)R_{21}(\lambda + \mu)K_2(\mu) = K_2(\mu)R_{12}(\lambda + \mu)K_1(\lambda)R_{21}(\lambda - \mu). \quad (4) $$

In the case of open boundary conditions, the transfer matrix can be written as

$$ t(\lambda) = \text{Tr}_A[K_A^{(+)}(\lambda)T_A(\lambda)K_A^{(-)}(\lambda)[T_A(-\lambda)]^{-1}], \quad (5) $$

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where $K_A^{(+)}(\lambda)$ can be chosen as one of the solutions of the reflection equation (4). The other boundary matrix $K_A^{(+)}(\lambda)$ is obtained from the previous one by means of the isomorphism \cite{18},

\begin{equation}
K_A^{(+)}(\lambda) = K_A^{(-)}(-\lambda - \rho)^tV^tV,
\end{equation}

where $t$ means transposition, the crossing parameter is $\rho = -\gamma$ and the crossing matrix is given by

\begin{equation}
V = \begin{pmatrix} 0 & 0 & q \\ 0 & -1 & 0 \\ q^{-1} & 0 & 0 \end{pmatrix}.
\end{equation}

The integrable open spin chain is obtained by means of the logarithmic derivative of the transfer matrix (5), such that

\begin{equation}
H = \frac{\sinh \gamma}{2} \frac{d}{d\lambda} \ln t(\lambda) \bigg|_{\lambda=0} + \text{const},
\end{equation}

\begin{equation}
= \sum_{k=1}^{L-1} U_{k,k+1} + \frac{\sinh \gamma}{2} \frac{dK_A^{(-)}(\lambda)}{d\lambda} \bigg|_{\lambda=0} + \frac{\text{Tr}_A[K_A^{(+)}(0)U_{L,A}]}{\text{Tr}_A[K_A^{(+)}(0)]}.
\end{equation}

The solutions of the reflection equation (4) associated with the $R$-matrix (2) were recently obtained \cite{14}. Here we list only the diagonal ones, which we will use throughout this work.

\begin{equation}
K_{(1,0,0)}^{(-)}(\lambda) = \begin{pmatrix} k_{11}^{[1]}(\lambda) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K_{(0,1,0)}^{(-)}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k_{22}^{[1]}(\lambda) & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{equation}

\begin{equation}
K_{(0,0,1)}^{(-)}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k_{33}^{[1]}(\lambda) \end{pmatrix},
\end{equation}

where

\begin{equation}
k_{11}^{[1]}(\lambda) = \frac{\beta x_2(\lambda) [(1 + q^2)x_2(\lambda) + x_1(\lambda)] + 2 [x_1(\lambda)x_2'(\lambda) - x_1'(\lambda)x_2(\lambda)]}{-\beta x_2(\lambda) [q^{-2}x_2(\lambda) + x_1(\lambda)] + 2 [x_1(\lambda)x_2'(\lambda) - x_1'(\lambda)x_2(\lambda)]},
\end{equation}

\begin{equation}
k_{22}^{[1]}(\lambda) = \frac{\beta x_2(\lambda) [(q^{-2} + q^2)x_2(\lambda) + x_1(\lambda)] + 2 [x_1(\lambda)x_2'(\lambda) - x_1'(\lambda)x_2(\lambda)]}{-\beta x_2(\lambda) [x_2(\lambda) + x_1(\lambda)] + 2 [x_1(\lambda)x_2'(\lambda) - x_1'(\lambda)x_2(\lambda)]},
\end{equation}

\begin{equation}
k_{33}^{[1]}(\lambda) = \frac{\beta x_2(\lambda) [(q^{-2} + 1)x_2(\lambda) + x_1(\lambda)] + 2 [x_1(\lambda)x_2'(\lambda) - x_1'(\lambda)x_2(\lambda)]}{-\beta x_2(\lambda) [q^2x_2(\lambda) + x_1(\lambda)] + 2 [x_1(\lambda)x_2'(\lambda) - x_1'(\lambda)x_2(\lambda)]},
\end{equation}

and

\begin{equation}
x_1(\lambda) = \frac{\sinh(\gamma - \lambda)}{\sinh \gamma}, \quad x_2(\lambda) = \frac{\sinh \lambda}{\sinh \gamma}.
\end{equation}
There is an additional set of diagonal solutions of different form, given by

\[
K^{-\pm}_{(0,1,1)}(\lambda) = \begin{pmatrix}
1 & 0 & 0 \\
0 & k^{-\pm}_{1,1,1}(\lambda) & 0 \\
0 & 0 & k^{-\pm}_{1,1,1}(\lambda)
\end{pmatrix}, \quad K^{-\pm}_{(1,0,1)}(\lambda) = \begin{pmatrix}
k^{-\pm}_{2,2,2}(\lambda) & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & k^{-\pm}_{2,2,2}(\lambda)
\end{pmatrix},
\]

(12)

where

\[
k^{-\pm}_{1,1,1}(\lambda) = \frac{\beta x_2(\lambda) [q^{-2} x_2(\lambda) + x_1(\lambda)] + 2 [x_1(\lambda) x_2(\lambda) - x_1'(\lambda) x_2(\lambda)]}{-\beta x_2(\lambda) [(1 + q^2) x_2(\lambda) + x_1(\lambda)] + 2 [x_1(\lambda) x_2'(\lambda) - x_1'(\lambda) x_2(\lambda)]},
\]

\[
k^{-\pm}_{2,2,2}(\lambda) = \frac{\beta x_2(\lambda) [x_2(\lambda) + x_1(\lambda)] + 2 [x_1(\lambda) x_2'(\lambda) - x_1'(\lambda) x_2(\lambda)]}{-\beta x_2(\lambda) [(1 + q^2) x_2(\lambda) + x_1(\lambda)] + 2 [x_1(\lambda) x_2'(\lambda) - x_1'(\lambda) x_2(\lambda)]},
\]

\[
k^{-\pm}_{3,3,3}(\lambda) = \frac{\beta x_2(\lambda) [q^2 x_2(\lambda) + x_1(\lambda)] + 2 [x_1(\lambda) x_2'(\lambda) - x_1'(\lambda) x_2(\lambda)]}{-\beta x_2(\lambda) [(1 + q^2) x_2(\lambda) + x_1(\lambda)] + 2 [x_1(\lambda) x_2'(\lambda) - x_1'(\lambda) x_2(\lambda)]}.
\]

The boundary terms of Hamiltonian (8) are directly obtained from the above $K^{(\pm)}$ matrices. In particular, the left boundary acting non-trivially in the site 1 has the form

\[
B_1 = \frac{\sinh \gamma}{2} \frac{dK_1^{(-)}(\lambda)}{d\lambda} \bigg|_{\lambda=0} = \begin{pmatrix}
l_{11} & 0 & 0 \\
0 & l_{22} & 0 \\
0 & 0 & l_{33}
\end{pmatrix},
\]

(14)

while the right boundary acting in the last site $L$ is given by

\[
B_L = \frac{\text{Tr}_A[K_A^{(\pm)}(0) U_{L,A}]}{\text{Tr}_A[K_A^{(\pm)}(0)]} = \begin{pmatrix}
r_{11} & 0 & 0 \\
0 & r_{22} & 0 \\
0 & 0 & r_{33}
\end{pmatrix}_L,
\]

(15)

where

\[
r_{11} = \frac{q^{-2} k_{33}^+(0)}{k_{11}^+(0) + k_{22}^+(0) + k_{33}^+(0)}, \quad r_{22} = \frac{k_{22}^+(0)}{k_{11}^+(0) + k_{22}^+(0) + k_{33}^+(0)}, \quad r_{33} = \frac{k_{33}^+(0)}{k_{11}^+(0) + k_{22}^+(0) + k_{33}^+(0)}.
\]

(16)

Therefore, we can compute the left boundary terms (14) as

\[
B_{1}^{(1,[\lambda])} = \begin{pmatrix}
\frac{\beta \sinh \gamma}{2} & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad B_{1}^{(2,[\lambda])} = \begin{pmatrix}
0 & 0 & 0 \\
0 & \beta \sinh \gamma & 0 \\
0 & 2 & 0
\end{pmatrix}, \quad B_{1}^{(3,[\lambda])} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \frac{\beta \sinh \gamma}{2} \\
0 & 0 & 0
\end{pmatrix},
\]

(17)

corresponding to the left $K$-matrices $K^{(-)}_{(1,0,0)}, K^{(-)}_{(0,1,0)}$ and $K^{(-)}_{(0,0,1)}$, respectively.
Similarly, we have three more left boundaries corresponding to the $K$-matrices $K^{(-)}_{(0,1,1)}$, $K^{(-)}_{(1,0,1)}$ and $K^{(-)}_{(1,1,0)}$

\[
B^{(1,[II])}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\beta \sinh \gamma}{2} & 0 \\ 0 & 0 & \frac{\beta \sinh \gamma}{2} \end{pmatrix}, \quad B^{(2,[II])}_1 = \begin{pmatrix} \frac{\beta \sinh \gamma}{2} & 0 & 0 \\ 0 & \frac{\beta \sinh \gamma}{2} & 0 \\ 0 & 0 & \frac{\beta \sinh \gamma}{2} \end{pmatrix}_L
\]

\[
B^{(3,[II])}_1 = \begin{pmatrix} \frac{\beta \sinh \gamma}{2} & 0 & 0 \\ 0 & \frac{\beta \sinh \gamma}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}_L
\]

where $\beta$ is the left boundary free parameter.

From the isomorphism (6), we have the $K^{(+)}(\lambda)$-matrices evaluated at $\lambda = 0$, given as

\[
K^{(+)}_{(1,0,0)}(0) = \begin{pmatrix} q^{-2}j_{[II]}^{[1]} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{pmatrix}, \quad K^{(+)}_{(0,1,0)}(0) = \begin{pmatrix} q^{-2} & 0 & 0 \\ 0 & j_{[II]}^{[22]} & 0 \\ 0 & 0 & q^2 \end{pmatrix},
\]

\[
K^{(+)}_{(0,0,1)}(0) = \begin{pmatrix} q^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2j_{[II]}^{[33]} \end{pmatrix},
\]

where

\[
j_{[II]}^{[1]} = \frac{2 + (1 + q^2)\alpha \sinh \gamma}{2 - q^{-2}\alpha \sinh \gamma}, \quad j_{[II]}^{[2]} = \frac{2 + (q^{-2} + q^2)\alpha \sinh \gamma}{2 - \alpha \sinh \gamma},
\]

\[
j_{[II]}^{[33]} = \frac{2 + (1 + q^{-2})\alpha \sinh \gamma}{2 - q^2\alpha \sinh \gamma},
\]

and

\[
K^{(+)}_{(0,1,1)}(0) = \begin{pmatrix} q^{-2} & 0 & 0 \\ 0 & j_{[II]}^{[1]} & 0 \\ 0 & 0 & q^2j_{[II]}^{[1]} \end{pmatrix}, \quad K^{(+)}_{(1,0,1)}(0) = \begin{pmatrix} q^{-2}j_{[II]}^{[1]} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2j_{[II]}^{[22]} \end{pmatrix},
\]

\[
K^{(+)}_{(1,1,0)}(0) = \begin{pmatrix} q^{-2}j_{[II]}^{[33]} & 0 & 0 \\ 0 & j_{[II]}^{[33]} & 0 \\ 0 & 0 & q^2 \end{pmatrix},
\]

where

\[
j_{[II]}^{[1]} = \frac{2 + q^{-2}\alpha \sinh \gamma}{2 - (1 + q^2)\alpha \sinh \gamma}, \quad j_{[II]}^{[2]} = \frac{2 + \alpha \sinh \gamma}{2 - (q^2 + q^{-2})\alpha \sinh \gamma},
\]

\[
j_{[II]}^{[33]} = \frac{2 + q^2\alpha \sinh \gamma}{2 - (1 + q^{-2})\alpha \sinh \gamma},
\]

and $\alpha$ is the right boundary free parameter.

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Therefore, using (16) we can write the corresponding right boundary terms

\[
B^{(1,[I])}_L = \begin{pmatrix}
\frac{2 - q^{-2} \alpha \sinh \gamma}{4 \cosh \gamma} & 0 & 0 \\
0 & \frac{2 - q^{-2} \alpha \sinh \gamma}{4 \cosh \gamma} & 0 \\
0 & 0 & \frac{2 + (1 + q^2) \alpha \sinh \gamma}{4 \cosh \gamma}
\end{pmatrix}_L,
\]

\[
B^{(2,[I])}_L = \begin{pmatrix}
\frac{2 - \alpha \sinh \gamma}{4 \cosh \gamma} & 0 & 0 \\
0 & \frac{2 + (q^2 + q^{-2}) \alpha \sinh \gamma}{4 \cosh \gamma} & 0 \\
0 & 0 & \frac{2 - \alpha \sinh \gamma}{4 \cosh \gamma}
\end{pmatrix}_L,
\]

\[
B^{(3,[I])}_L = \begin{pmatrix}
\frac{2 + (1 + q^{-2}) \alpha \sinh \gamma}{4 \cosh \gamma} & 0 & 0 \\
0 & \frac{2 - q^2 \alpha \sinh \gamma}{4 \cosh \gamma} & 0 \\
0 & 0 & \frac{2 - q^2 \alpha \sinh \gamma}{4 \cosh \gamma}
\end{pmatrix}_L,
\]

\[
B^{(1,[II])}_L = \begin{pmatrix}
\frac{2 + q^{-2} \alpha \sinh \gamma}{4 \cosh \gamma} & 0 & 0 \\
0 & \frac{2 + q^{-2} \alpha \sinh \gamma}{4 \cosh \gamma} & 0 \\
0 & 0 & \frac{2 - (1 + q^2) \alpha \sinh \gamma}{4 \cosh \gamma}
\end{pmatrix}_L,
\]

\[
B^{(2,[II])}_L = \begin{pmatrix}
\frac{2 + \alpha \sinh \gamma}{4 \cosh \gamma} & 0 & 0 \\
0 & \frac{2 - (q^2 + q^{-2}) \alpha \sinh \gamma}{4 \cosh \gamma} & 0 \\
0 & 0 & \frac{2 + \alpha \sinh \gamma}{4 \cosh \gamma}
\end{pmatrix}_L,
\]

\[
B^{(3,[II])}_L = \begin{pmatrix}
\frac{2 - (1 + q^{-2}) \alpha \sinh \gamma}{4 \cosh \gamma} & 0 & 0 \\
0 & \frac{2 + q^2 \alpha \sinh \gamma}{4 \cosh \gamma} & 0 \\
0 & 0 & \frac{2 + q^2 \alpha \sinh \gamma}{4 \cosh \gamma}
\end{pmatrix}_L.
\]

Here we have 6 different integrable boundaries related by the isomorphism (6). However, it is worth noting that other combinations of the boundaries are allowed \( B^{(i,[a,b])}_{L,j} = B^{(1,[a])}_L + B^{(j,[b])}_L \) with \( i, j \in \{1, 2, 3\}, a, b \in \{I, II\} \) resulting in 36 integrable boundaries for the spin-1 \( U_q[sl(2)] \) Temperley–Lieb Hamiltonian.

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The isomorphism (see e.g. table 1).

\[
E_{11} = \begin{pmatrix}
|+ + + +\rangle \\
|+0 + +\rangle \\
|+ + 0 +\rangle
\end{pmatrix}
\]
\[
E_{12} = \begin{pmatrix}
|+ + 0 +\rangle \\
|+0 + 0\rangle \\
|+0 0 +\rangle
\end{pmatrix}
\]
\[
E_{13} = \begin{pmatrix}
|+ 0 + +\rangle \\
|+0 0 +\rangle \\
|+0 + 0\rangle
\end{pmatrix}
\]
\[
E_{21} = \begin{pmatrix}
|0 + + +\rangle \\
|0 + 0 +\rangle \\
|0 + 0 0\rangle
\end{pmatrix}
\]
\[
E_{22} = \begin{pmatrix}
|0 + 0 +\rangle \\
|0 0 + 0\rangle \\
|0 0 0 +\rangle
\end{pmatrix}
\]
\[
E_{23} = \begin{pmatrix}
|0 0 + +\rangle \\
|0 0 0 +\rangle \\
|0 + 0 0\rangle
\end{pmatrix}
\]
\[
E_{31} = \begin{pmatrix}
|0 + + +\rangle \\
|0 0 + +\rangle \\
|0 + 0 +\rangle
\end{pmatrix}
\]
\[
E_{32} = \begin{pmatrix}
|0 + 0 +\rangle \\
|0 0 0 +\rangle \\
|0 + 0 0\rangle
\end{pmatrix}
\]
\[
E_{33} = \begin{pmatrix}
|0 0 + +\rangle \\
|0 0 0 +\rangle \\
|0 + 0 0\rangle
\end{pmatrix}
\]

The action of the boundary terms on the Hilbert space is given by

\[
B_{1,L}^{(i,j|[a,b])} \frac{1}{\sigma_1 \ldots \sigma_L} = E_{\sigma \tau}^{(i,j|[a,b])} \frac{1}{\sigma_1 \ldots \sigma_L}
\]

where \( E_{\sigma \tau}^{(i,j|[a,b])} = l_{\sigma \sigma}^{(i,[a])} + r_{\tau \tau}^{(j,[b])} \) and the sites are indexed by \( \sigma, \tau \in \{1, 2, 3\} = \{+, 0, -\} \).

Here we recall that \( U_{\sigma \sigma}^{(i,[a])} \) and \( r_{\tau \tau}^{(j,[b])} \) are the matrix elements of the boundary matrices \( B_{1}^{(i,[a])} \) and \( B_{L}^{(j,[b])} \) respectively.

In section 3, we will restrict ourselves to the case of integrable boundaries related by the isomorphism \( B_{1,L}^{(i,[a])} = B_{1,L}^{(i,[a])} = B_{1}^{(i,[a])} + B_{L}^{(i,[a])} \),

\[
E_{\sigma \tau}^{(i,[a])} = E_{\sigma \tau}^{(i,[a])} = l_{\sigma \sigma}^{(i,[a])} + r_{\tau \tau}^{(i,[a])}
\]

and we shall use the coordinate Bethe ansatz in order to obtain the eigenvalues of the Hamiltonian (8).

3. Coordinate Bethe ansatz

In most of the cases where the Bethe ansatz is successfully applied, one can build up all the eigenstates from just one reference state, and usually there exist only a few such reference states available. In the case of Temperley–Lieb spin chains, in contrast, we have exponentially degenerated ground states, which implies that we have a very large number of reference states. In fact we have \( 3 \times 2^{L-1} \) natural eigenstates which can be used as reference states. This explains the difficulties in constructing all the eigenstates from just one reference state [9]–[11].

The reason for such differences is that the bulk part of the Hamiltonian (the Temperley–Lieb operator \( U_{k,k+1} \)) is the projector operator onto the two-site spin zero singlet. This implies that there exist \( 3 \times 2^{L-1} \) states which are eigenstates of the bulk Hamiltonian with zero eigenvalues. Therefore, these states are eigenstates of the Hamiltonian and of its boundary part \( B_{1,L}^{(i,[a])} \) with eigenvalues \( E_{\sigma \tau}^{(i,[a])} \) (see e.g. table 1).

Moreover, apart from the natural degeneracy of the boundary eigenvalues \( E_{\sigma \tau}^{(i,[a])} \), one can see from the structure of the boundary matrix \( K^{(\pm)} \) that not all \( E_{\sigma \tau}^{(i,[a])} \) are independent.
Table 2. The relation among the boundary (and Hamiltonian) eigenvalues for different solutions of the reflection equation. These relations hold true for any $a \in \{I, II\}$.

| $\mathcal{E}^{(1),[a]}_{11}$ | $= \mathcal{E}^{(1),[a]}_{12}$ | $= \mathcal{E}^{(1),[a]}_{13}$ |
|---------------------------|-----------------------------|-----------------------------|
| $\mathcal{E}^{(1),[a]}_{21}$ | $= \mathcal{E}^{(1),[a]}_{22}$ | $= \mathcal{E}^{(1),[a]}_{23}$ |
| $\mathcal{E}^{(2),[a]}_{11}$ | $= \mathcal{E}^{(2),[a]}_{13}$ | $= \mathcal{E}^{(2),[a]}_{12}$ |
| $\mathcal{E}^{(2),[a]}_{21}$ | $= \mathcal{E}^{(2),[a]}_{23}$ | $= \mathcal{E}^{(2),[a]}_{22}$ |
| $\mathcal{E}^{(3),[a]}_{11}$ | $= \mathcal{E}^{(3),[a]}_{13}$ | $= \mathcal{E}^{(3),[a]}_{12}$ |
| $\mathcal{E}^{(3),[a]}_{12}$ | $= \mathcal{E}^{(3),[a]}_{21}$ | $= \mathcal{E}^{(3),[a]}_{23}$ |
| $\mathcal{E}^{(3),[a]}_{22}$ | $= \mathcal{E}^{(3),[a]}_{23}$ | $= \mathcal{E}^{(3),[a]}_{32}$ |
| $\mathcal{E}^{(3),[a]}_{33}$ | $= \mathcal{E}^{(3),[a]}_{32}$ | $= \mathcal{E}^{(3),[a]}_{31}$ |

In fact they are also degenerate and can be grouped in four blocks for each integrable boundary related by the isomorphism (see table 2).

In face of the large number of reference states, the standard construction of all the eigenstates seems to be impracticable, though it is possible. However, in order to obtain the eigenvalues of the Hamiltonian it is enough to work with a few reference states. In fact, we can take one reference state from each block of eigenvalues $\mathcal{E}_{\sigma\tau}^{(i,[a])}$. From now on, we drop the label for different solutions of the reflection equation from the boundary eigenvalues, such that $\mathcal{E}_{\sigma\tau}^{(i,[a])} = \mathcal{E}_{\sigma\tau}$.

### 3.1. Ferromagnetic reference state

We shall start by considering the pseudo-particle as a singlet over the standard ferromagnetic state. Therefore, it is convenient to start our ansatz with the following linear combination of the basis states [11],

$$|\Omega(k)\rangle = q^{-2}|++\cdots++\cdots+\cdots k k 0 0\cdots+\cdots++\cdots+\cdots-\cdots-\cdots-\cdots-\cdots+\cdots+\cdots+\cdots+\cdots\rangle,$$

which is an eigenstate of $U_{k,k+1}$ such that

$$U_{k,k+1}|\Omega(k)\rangle = Q|\Omega(k)\rangle, \quad U_{k+1,k+2}|\Omega(k)\rangle = |\Omega(k + 1)\rangle,$$

$$U_{k,k+1}|\Omega(k \pm 1)\rangle = |\Omega(k)\rangle, \quad U_{k-1,k}|\Omega(k)\rangle = |\Omega(k - 1)\rangle,$$

where $Q = (q^2 + 1 + q^{-2})$. Therefore, the action of the Hamiltonian $H = \sum_{k=1}^{L-1} U_{k,k+1} + B_{1,L}$ over this state results in

$$H|\Omega(k)\rangle = Q|\Omega(k)\rangle + \mathcal{E}_{11}|\Omega(k)\rangle + |\Omega(k - 1)\rangle + |\Omega(k + 1)\rangle, \quad 1 < k < L - 1$$

$$H|\Omega(1)\rangle = Q|\Omega(1)\rangle + \mathcal{E}_{11}|\Omega(1)\rangle + |\Omega(0)\rangle + |\Omega(2)\rangle$$

$$H|\Omega(L - 1)\rangle = Q|\Omega(L - 1)\rangle + \mathcal{E}_{11}|\Omega(L - 1)\rangle + |\Omega(L - 2)\rangle + |\Omega(L)\rangle,$$

where $B_{1,L}|\Omega(k)\rangle = \mathcal{E}_{11}|\Omega(k)\rangle, 1 < k < L - 1$, $|\Omega(0)\rangle = (B_{1,L} - \mathcal{E}_{11})|\Omega(1)\rangle$ and $|\Omega(L)\rangle = (B_{1,L} - \mathcal{E}_{11})|\Omega(L - 1)\rangle$. In addition to the previous relation, we have a set of closing relations

$$H|\Omega(0)\rangle = \Delta_{1}^{(1)}|\Omega(1)\rangle + \mathcal{E}_{v,1,1}|\Omega(0)\rangle,$$

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\[ H|\Omega(L)\rangle = \Delta_i^{(1)}|\Omega(L-1)\rangle + \mathcal{E}_{1u_1}|\Omega(L)\rangle, \]  

where \( \Delta_i^{(1)} = (\mathcal{E}_{21} - \mathcal{E}_{11}) + q^2(\mathcal{E}_{31} - \mathcal{E}_{11}), \Delta_i^{(1)} = (\mathcal{E}_{12} - \mathcal{E}_{11}) + q^{-2}(\mathcal{E}_{13} - \mathcal{E}_{11}). \) In order to cover all the solutions of the reflection equation, we introduce the following notation, \( \bar{v}_1 = (2,2,3) \) and \( \bar{u}_1 = (3,2,2), \) whose elements \( i \in \{1,2,3\} \) represent different solutions of the reflection equation (for any \( a \in \{I,II\} \)). In the above relations, we exploited the fact that

\[
U_{1,2}|\Omega(0)\rangle = \Delta_i^{(1)}|\Omega(1)\rangle, \\
B_{1,L}|\Omega(0)\rangle = \mathcal{E}_{v_{1,1}}|\Omega(0)\rangle, \\
U_{L-1,L}|\Omega(L)\rangle = \Delta_i^{(1)}|\Omega(L-1)\rangle, \\
B_{1,L}|\Omega(L)\rangle = \mathcal{E}_{1u_1}|\Omega(L)\rangle.
\]

3.1.1. One-particle state. In the first non-trivial sector, we assume the following ansatz for the eigenstates

\[ |\Psi_1\rangle = \sum_{k=1}^{L-1} A(k)|\Omega(k)\rangle. \]  

Imposing that the eigenvalue equation \( H|\Psi_1\rangle = E_1|\Psi_1\rangle \) is fulfilled, we obtain a set of equations for the function \( A(k) \),

\[
(Q + \mathcal{E}_{11} - E_1)A(k) + A(k-1) + A(k+1) = 0, \quad 1 < k < L - 1 \tag{39}
\]

\[
(\mathcal{E}_{v_{1,1}} - E_1)A(0) + \Delta_i^{(1)}A(1) = 0, \tag{40}
\]

\[
(\mathcal{E}_{1u_1} - E_1)A(L) + \Delta_i^{(1)}A(L-1) = 0. \tag{41}
\]

Taking the ansatz for the amplitude

\[ A(k) = a(\theta)\xi^k - a(-\theta)\xi^{-k}, \]  

and substituting in equation (39) provides the following expression for the energy eigenvalues

\[ E_1 = \mathcal{E}_{11} + Q + \xi + \xi^{-1}. \]  

(43)

The parameter \( \xi \) and the ratio of the amplitudes \( a(\theta)/a(-\theta) \) are fixed by equations (40) and (41), which results in the Bethe ansatz equation

\[ \xi^{2L} = \left( \frac{Q + \mathcal{E}_{11} - \mathcal{E}_{v_{1,1}} + \xi^{-1} + \xi(1 - \Delta_i^{(1)})}{Q + \mathcal{E}_{11} - \mathcal{E}_{v_{1,1}} + \xi + \xi^{-1}(1 - \Delta_i^{(1)})} \right) \left( \frac{Q + \mathcal{E}_{11} - \mathcal{E}_{1u_1} + \xi^{-1} + \xi(1 - \Delta_r^{(1)})}{Q + \mathcal{E}_{11} - \mathcal{E}_{1u_1} + \xi + \xi^{-1}(1 - \Delta_r^{(1)})} \right). \]  

(44)

3.1.2. Two-particle state. In the next particle sector, we have two interacting pseudo-particles, which can be represented as a product of two pseudo-particle eigenstates, as given by

\[ |\Psi_2\rangle = \sum_{k_1+1<k_2} A(k_1, k_2)|\Omega(k_1, k_2)\rangle, \]  

(45)

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where

$$|\Omega(k_1, k_2)\rangle = \sum_{i,j=-1}^1 (-1)^{i+j} q^{i+j-2} |k_1(-i, i); k_2(-j, j)\rangle. \quad (46)$$

We can split the action of the Hamiltonian on the state $|\Omega(k_1, k_2)\rangle$ into four cases:

(i) The case where two pseudo-particles are separated in the bulk,

$$H|\Omega(k_1, k_2)\rangle = (2Q + \mathcal{E}_{11})|\Omega(k_1, k_2)\rangle + |\Omega(k_1 - 1, k_2)\rangle + |\Omega(k_1 + 1, k_2)\rangle$$

$$+ |\Omega(k_1, k_2 - 1)\rangle + |\Omega(k_1, k_2 + 1)\rangle, \quad 1 < k_1 + 2 < k_2 < L - 1. \quad (47)$$

(ii) The case where the pseudo-particles are separated but one of them or both are at the boundaries

$$H|\Omega(1, k_2)\rangle = (2Q + \mathcal{E}_{11})|\Omega(1, k_2)\rangle + |\Omega(0, k_2)\rangle + |\Omega(2, k_2)\rangle$$

$$+ |\Omega(1, k_2 - 1)\rangle + |\Omega(1, k_2 + 1)\rangle, \quad 5 < k_2 < L - 1 \quad (48)$$

$$H|\Omega(k_1, L - 1)\rangle = (2Q + \mathcal{E}_{11})|\Omega(k_1, L - 1)\rangle + |\Omega(k_1 - 1, L - 1)\rangle + |\Omega(k_1 + 1, L - 1)\rangle$$

$$+ |\Omega(k_1, L - 1)\rangle + |\Omega(k_1, L)\rangle, \quad 1 < k_1 < L - 3 \quad (49)$$

$$H|\Omega(1, L - 1)\rangle = (2Q + \mathcal{E}_{11})|\Omega(1, L - 1)\rangle + |\Omega(0, L - 1)\rangle + |\Omega(2, L - 1)\rangle$$

$$+ |\Omega(1, L - 1)\rangle + |\Omega(1, L)\rangle. \quad (50)$$

(iii) The case where the particles are neighbours in the bulk

$$H|\Omega(k, k + 2)\rangle = (2Q + \mathcal{E}_{11})|\Omega(k, k + 2)\rangle + |\Omega(k - 1, k + 2)\rangle + |\Omega(k, k + 3)\rangle$$

$$+ |\Omega(k + 1, k + 2)\rangle + |\Omega(k, k + 1)\rangle, \quad 1 < k < L - 3. \quad (51)$$

(iv) The case where the particles are neighbours at the boundaries

$$H|\Omega(1, 3)\rangle = (2Q + \mathcal{E}_{11})|\Omega(1, 3)\rangle + |\Omega(0, 3)\rangle + |\Omega(2, 3)\rangle$$

$$+ |\Omega(1, 2)\rangle + |\Omega(1, 4)\rangle. \quad (52)$$

$$H|\Omega(L - 3, L - 1)\rangle = (2Q + \mathcal{E}_{11})|\Omega(L - 3, L - 1)\rangle + |\Omega(L - 4, L - 1)\rangle$$

$$+ |\Omega(L - 2, L - 1)\rangle + |\Omega(L - 3, L - 2)\rangle + |\Omega(L - 3, L)\rangle. \quad (53)$$

In the above relations, we have introduced new states whose definition are given by

$$|\Omega(0, k_2)\rangle = (B_{1,L} - \mathcal{E}_{11})|\Omega(1, k_2)\rangle, \quad (54)$$

$$|\Omega(k_1, L)\rangle = (B_{1,L} - \mathcal{E}_{11})|\Omega(k_1, L - 1)\rangle, \quad (55)$$

$$|\Omega(0, L - 1)\rangle + |\Omega(1, L)\rangle = (B_{1,L} - \mathcal{E}_{11})|\Omega(1, L - 1)\rangle, \quad (56)$$

$$|\Omega(k + 1, k + 2)\rangle + |\Omega(k, k + 1)\rangle = U_{k+1,k+2}[\Omega(k, k + 2)]. \quad (57)$$

The action of the Hamiltonian on these states can be written as follows

$$H|\Omega(0, k_2)\rangle = \Delta^{(1)}|\Omega(1, k_2)\rangle + \mathcal{E}_{v_{1,1}}|\Omega(0, k_2)\rangle + |\Omega(0, k_2 - 1)\rangle$$

$$+ Q|\Omega(0, k_2)\rangle + |\Omega(0, k_2 + 1)\rangle, \quad (58)$$

$$H|\Omega(k_1, L)\rangle = \Delta^{(1)}|\Omega(k_1, L - 1)\rangle + \mathcal{E}_{v_{1,1}}|\Omega(k_1, L)\rangle + |\Omega(k_1 - 1, L)\rangle$$

$$+ Q|\Omega(k_1, L)\rangle + |\Omega(k_1 + 1, L)\rangle, \quad (59)$$

$$H|\Omega(k, k + 1)\rangle = Q|\Omega(k, k + 1)\rangle + |\Omega(k - 1, k + 1)\rangle + |\Omega(k, k + 2)\rangle + \mathcal{E}_{11}|\Omega(k, k + 1)\rangle. \quad (60)$$

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In order to obtain the eigenvalues, we have to substitute the above relations in the eigenvalue equation \((H|\Psi_2) = E_2|\Psi_2\)). This will provide us with the following set of equations for the amplitude \(A(k_1, k_2)\),

\[
(2Q + \mathcal{E}_{11} - E_2)A(k_1, k_2) + A(k_1 - 1, k_2) + A(k_1 + 1, k_2) + A(k_1, k_2 - 1) + A(k_1, k_2 + 1) = 0,
\]

\[
(\mathcal{Q} + \mathcal{E}_{11} - E_2)A(k, k + 1) + A(k - 1, k + 1) + A(k, k + 2) = 0,
\]

\[
(\mathcal{Q} + \mathcal{E}_{v_{1,1}} - E_2)A(0, k_2) + A(0, k_2 - 1) + A(0, k_2 + 1) + \Delta^{(1)}_{v}A(1, k_2) = 0,
\]

\[
(\mathcal{Q} + \mathcal{E}_{1u,1} - E_2)A(k_1, L) + A(k_1 - 1, L) + A(k_1 + 1, L) + \Delta^{(1)}_{v}A(k_1, L - 1) = 0.
\]

One can obtain the eigenvalues from equation (61),

\[
E_2 = \mathcal{E}_{11} + 2Q + \xi_1 + \xi^{-1}_1 + \xi_2 + \xi^{-1}_2,
\]

provided that the following parametrization for the amplitudes is assumed

\[
A(k_1, k_2) = \sum_P \varepsilon_P a(\theta_1, \theta_2) \xi_1^{k_1} \xi_2^{k_2},
\]

where the sum extends over all permutations and negations of momenta \((\theta_i)\), such that \(\xi_i = e^{i\theta_i}\), and \(\varepsilon_P\) is the signature of permutations and negations. This structure already reflects the existence of the boundary.

On the other hand, equation (62) is the meeting condition for the two pseudo-particle states. Using the ansatz (66), we obtain the following phase shifts,

\[
a(\theta_2, \theta_1) = \left(\frac{s(\theta_2, \theta_1)}{s(\theta_1, \theta_2)}\right) a(\theta_1, \theta_2),
\]

\[
a(\theta_2, -\theta_1) = \left(\frac{s(\theta_2, -\theta_1)}{s(-\theta_1, \theta_2)}\right) a(-\theta_1, \theta_2),
\]

\[
a(-\theta_2, \theta_1) = \left(\frac{s(-\theta_2, \theta_1)}{s(\theta_1, -\theta_2)}\right) a(\theta_1, -\theta_2),
\]

\[
a(-\theta_2, -\theta_1) = \left(\frac{s(-\theta_2, -\theta_1)}{s(-\theta_1, -\theta_2)}\right) a(-\theta_1, -\theta_2),
\]

where

\[
s(\theta_1, \theta_2) = 1 + \xi_1 \xi_2 + \xi_1 Q.
\]

At this point, we still have two remaining equations (63) and (64), which introduce the boundary effects. One can introduce the expressions (65) and (66) in the equation for the left boundary (63), which results in

\[
a(-\theta_1, \theta_2) = F_l(\theta_1) a(\theta_1, \theta_2),
\]

\[
a(-\theta_2, \theta_1) = F_l(\theta_2) a(\theta_2, \theta_1),
\]

where

\[
F_l(\theta_1) = \left(\frac{Q + \mathcal{E}_{11} - \mathcal{E}_{v_{1,1}} + \xi_1^{-1} + \xi_1(1 - \Delta^{(1)}_{v})}{Q + \mathcal{E}_{11} - \mathcal{E}_{v_{1,1}} + \xi_1 + \xi^{-1}_1(1 - \Delta^{(1)}_{v})}\right).
\]

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Likewise, for right boundary, one obtains the following relations
\[
a(\theta_2, -\theta_1) = \xi_1^{2L} F_r(\theta_1) a(\theta_2, \theta_1), \tag{75}
\]
\[
a(\theta_1, -\theta_2) = \xi_2^{2L} F_r(\theta_2) a(\theta_1, \theta_2), \tag{76}
\]
where
\[
F_r(\theta_1) = \frac{(Q + \mathcal{E}_{11} - \mathcal{E}_{1u_s} + \xi_1 + \xi_1^{-1}(1 - \Delta^{(I)})}{Q + \mathcal{E}_{11} - \mathcal{E}_{1u_s} + \xi_1^{-1} + \xi_1(1 - \Delta^{(I)})}. \tag{77}
\]
Combining these relations with the phase shift relations (67)–(70), we obtain the Bethe ansatz equations
\[
\xi_1^{2L} = F_i(\theta_1) F_r(\theta_1)^{-1} \left( \frac{s(\theta_1, \theta_2)}{s(\theta_2, \theta_1)} \right) \left( \frac{s(\theta_2, -\theta_1)}{s(-\theta_1, \theta_2)} \right), \tag{78}
\]
\[
\xi_2^{2L} = F_i(\theta_2) F_r(\theta_2)^{-1} \left( \frac{s(\theta_2, \theta_1)}{s(\theta_1, \theta_2)} \right) \left( \frac{s(\theta_1, -\theta_2)}{s(-\theta_2, \theta_1)} \right). \tag{79}
\]

3.1.3. General sector. The generalization to any number \( n \) of pseudo-particles goes along the same lines as before, although the calculation becomes cumbersome. Therefore, we just present the final results. The energy eigenvalues are given by the sum of single pseudo-particle energies
\[
E_n = \mathcal{E}_{11} + \sum_{k=1}^{n} (Q + \xi_k + \xi_k^{-1}), \tag{80}
\]
where \( n \) ranges from 0 to \( L/2 \), and the corresponding Bethe ansatz equations depend on the phase shift of two pseudo-particles and on the boundary factors
\[
\xi_k^{2L} = F_i(\theta_k) F_r(\theta_k)^{-1} \prod_{j=1, j \neq k}^{n} \left( \frac{s(\theta_k, \theta_j)}{s(\theta_j, \theta_k)} \right) \left( \frac{s(\theta_j, -\theta_k)}{s(-\theta_k, \theta_j)} \right). \tag{81}
\]

3.2. Other reference states

In order to obtain the whole spectrum of the Hamiltonian, we have to consider additional reference states. This has to be done for each different boundary eigenvalue \( \mathcal{E}_{\sigma \tau} \) (26) where \( \sigma, \tau \in \{1, 2, 3\} \). As a result of that, we must have as many as reference states (and consequently Bethe ansatz equations) as boundary eigenvalues. Let us recall that the Hamiltonian has boundary terms \( B_{1i}^{(i,a)} \), where \( i \in \{1, 2, 3\} \) and \( a \in \{I, II\} \), whose eigenvalues are \( \mathcal{E}_{\sigma \tau}^{(i,a)} \), but the labels \( (i, [a]) \) are suppressed.

In principle, we have nine boundary eigenvalues \( \mathcal{E}_{\sigma \tau} \). If we choose one reference state for each boundary eigenvalue (e.g. the first state of each block of table 1 extended to \( L \)-sites) and proceed along the same lines as section 3.1.3, we obtain nine eigenvalue expressions
\[
E_n^{(\sigma, \tau)} = \mathcal{E}_{\sigma \tau} + \sum_{k=1}^{n} (Q + \xi_k + \xi_k^{-1}), \tag{82}
\]
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as well as their associated Bethe ansatz equations

\[ \xi^{2L}_k = F^{(\sigma, \tau)}_l(\theta_k) F^{(\sigma, \tau)}_r(\theta_k)^{-1} \prod_{j=1, j \neq k}^n \left( \frac{s(\theta_k, \theta_j)}{s(-\theta_k, \theta_j)} \right) \left( \frac{s(\theta_j, -\theta_k)}{s(-\theta_k, \theta_j)} \right) , \]  

where \( \xi_k = e^{i \theta_k} , s(\theta_j, \theta_k) = 1 + \xi_j \xi_k + \xi_j Q \) and

\[ F^{(\sigma, \tau)}_l(\theta_k) = \left( \frac{Q + \mathcal{E}_{\sigma\tau} - \mathcal{E}_{\nu_{l\sigma} \tau} + 1 - \Delta^{(\sigma)}_k}{Q + \mathcal{E}_{\sigma\tau} - \mathcal{E}_{\nu_{l\sigma} \tau} + 1} \right)^{1 - \Delta^{(\sigma)}_k} , \]
\[ F^{(\sigma, \tau)}_r(\theta_k) = \left( \frac{Q + \mathcal{E}_{\sigma\tau} - \mathcal{E}_{\nu_{r\sigma} \tau} + 1 - \Delta^{(\tau)}_k}{Q + \mathcal{E}_{\sigma\tau} - \mathcal{E}_{\nu_{r\sigma} \tau} + 1} \right)^{1 - \Delta^{(\tau)}_k} , \]

\[ \Delta^{(1)}_k = (\mathcal{E}_{21} - \mathcal{E}_{11}) + q^2 (\mathcal{E}_{31} - \mathcal{E}_{11}) , \]
\[ \Delta^{(2)}_k = (\mathcal{E}_{31} - \mathcal{E}_{21}) + q^{-2} (\mathcal{E}_{11} - \mathcal{E}_{21}) , \]
\[ \Delta^{(3)}_k = (\mathcal{E}_{11} - \mathcal{E}_{12}) + q^2 (\mathcal{E}_{12} - \mathcal{E}_{13}) , \]
\[ \Delta^{(3)}_k = (\mathcal{E}_{11} - \mathcal{E}_{12}) + q^{-2} (\mathcal{E}_{13} - \mathcal{E}_{12}) . \]

The remaining indices \( \nu_{l, \sigma} \) are defined by \( \vec{\nu}_1 = (2, 2, 3), \vec{\nu}_2 = (1, 1, 3), \vec{\nu}_3 = (1, 1, 2) \) and the \( u_{i, \tau} \) are given by \( \vec{u}_1 = (3, 2, 2), \vec{u}_2 = (3, 1, 1), \vec{u}_3 = (2, 1, 1) \).

However, we can see from table 2 that most of these equations degenerate into each other, resulting in four equations for each integrable boundary. We have verified numerically the completeness of the spectrum up to \( L = 6 \) sites. The completeness of the spectrum for \( L = 4 \) is discussed in the appendix.

### 4. Conclusion

In this paper we obtained the spectrum of the spin-1 \( U_q[sl(2)] \) Temperley–Lieb spin chain with diagonal open boundary conditions. We have identified that this model has a large number of possible reference states. By selecting a small subset of these states, we managed to obtain four eigenvalue expressions and their associated Bethe ansatz equations by means of a generalization of the coordinate Bethe ansatz. This provides the complete description of the spectrum of the model for any values of the boundary parameters. We verified that the Bethe ansatz results are in agreement with the direct diagonalization of the Hamiltonian up to six sites.

Apart from the new results, we believe that this work also brings a better understanding of the coordinate Bethe ansatz construction of the eigenstates. Although with this new perspective it is possible to construct all the eigenstates for finite system size, this seems to be rather impracticable. Therefore, we still leave the problem of counting the spectral multiplicities as an open question. We also hope that this work will shed some light on the algebraic Bethe ansatz construction for the Temperley–Lieb spin chains.
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Appendix. Numerical check for \( L = 4 \)

This appendix is concerned with the study of the completeness of the spectrum for \( L = 4 \). In order to give an explicit example, we chose the integrable boundary \( B^{(i=1,a=1)} \) and we set the parameters \( q = 0.244\,877\,13, \alpha = 0.334\,570\,53 \) and \( \beta = 0.336\,609\,233 \). The direct diagonalization of the Hamiltonian (8) provides us with 25 different eigenvalues.

The natural eigenvalues of the Hamiltonian are given in Table A.1.

In order to determine the remaining eigenvalues, one has to use the eigenvalue expressions and the Bethe ansatz equations (82) and (83).

One can start looking at pseudo-particles (spin-1 singlets) over the ferromagnetic reference state (27). Therefore we have to solve the Bethe ansatz equations (82) and (83) for \((\sigma = 1, \tau = 1)\),

\[
E_n^{(1,1)} = \mathcal{E}_{11} + \sum_{k=1}^{n} (Q + \xi_k + \xi_k^{-1}),
\]

and Bethe ansatz equations

\[
\xi_k^8 = \left( \frac{Q + \mathcal{E}_{11} - \mathcal{E}_{21} + \xi_k^{-1} + \xi_k (1 - \Delta^{(1)}_l)}{Q + \mathcal{E}_{11} - \mathcal{E}_{21} + \xi_k + \xi_k^{-1} (1 - \Delta^{(1)}_l)} \right) \left( \frac{Q + \mathcal{E}_{11} - \mathcal{E}_{13} + \xi_k^{-1} + \xi_k (1 - \Delta^{(1)}_r)}{Q + \mathcal{E}_{11} - \mathcal{E}_{13} + \xi_k + \xi_k^{-1} (1 - \Delta^{(1)}_r)} \right) \\
\times \prod_{j \neq k}^{n} \left( \frac{s(\theta_k, \theta_j)}{s(\theta_j, \theta_k)} \right) \left( \frac{s(\theta_j, -\theta_k)}{s(-\theta_k, \theta_j)} \right),
\]

for the case of one and two particles \((n = 1, 2)\). We obtain 11 additional eigenvalues of the Hamiltonian. These results are summarized in Table A.2.

However, in order to describe the whole spectrum, one needs to use a few additional reference states. One can use the spin-down ferromagnetic state associated with the eigenvalue \( \mathcal{E}_{33} \), which results in

\[
E_n^{(3,3)} = \mathcal{E}_{33} + \sum_{k=1}^{n} (Q + \xi_k + \xi_k^{-1}),
\]

Table A.1. The 4 energy eigenvalues associated with the natural eigenstates of the Hamiltonian.

| \( E \)   | Natural eigenvalues |
|----------|---------------------|
| 0.153\,456\,14 | \( \mathcal{E}_{11} = \mathcal{E}_{12} \) |
| 1.627\,514\,41 | \( \mathcal{E}_{13} \) |
| -1.329\,584\,29 | \( \mathcal{E}_{21} = \mathcal{E}_{22} = \mathcal{E}_{32} = \mathcal{E}_{31} \) |
| 0.144\,473\,97 | \( \mathcal{E}_{33} = \mathcal{E}_{23} \) |
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Table A.2. The 11 energy eigenvalues and the corresponding Bethe roots for $L = 4$ for one and two pseudo-particles over the spin-up ferromagnetic state $|++++]$.

| $E$       | BA roots                                                                 |
|-----------|---------------------------------------------------------------------------|
| 37.048 872 49 | $\{\xi_1, \xi_2\} = \{0.946 44 \div i0.322 88, -0.235 144 \pm i0.971 961\}$ |
| 19.892 901 25 | $\xi_1 = 0.946 423$                                                      |
| 19.370 407 95 | $\{\xi_1, \xi_2\} = \{-0.054 7786, 1.261 52\}$                         |
| 18.558 666 31 | $\{\xi_1, \xi_2\} = \{-0.051 6155, 1.803\}$                            |
| 18.427 951 02 | $\xi_1 = 0.269 041 \pm i0.963 129$                                      |
| 17.885 363 93 | $\{\xi_1, \xi_2\} = \{-0.052 5908, 0.663 208 \pm i0.748 435\}$         |
| 16.662 926 84 | $\xi_1 = -0.613 471 \pm i0.789 717$                                     |
| 15.623 048 65 | $\{\xi_1, \xi_2\} = \{-0.052 4364, -0.440 045 \pm i0.897 976\}$        |
| -1.247 215 62 | $\xi_1 = -0.052 3981$                                                    |
| -1.173 696 34 | $\{\xi_1, \xi_2\} = \{-0.052 3981, -17.6061\}$                        |
| 0.230 971 38  | $\xi_1 = -0.056 8115$                                                    |

Table A.3. The 5 energy eigenvalues and the corresponding Bethe roots for $L = 4$ for one particle over the spin-down ferromagnetic state $|--|--$.

| $E$       | BA roots                                                                 |
|-----------|---------------------------------------------------------------------------|
| 19.889 475 51 | $\xi_1 = 0.911 516$                                                      |
| 18.421 522 33 | $\xi_1 = 0.270 318 \pm i0.962 771$                                       |
| 16.654 841 81 | $\xi_1 = -0.613 022 \pm i0.790 066$                                      |
| -1.247 677 82 | $\xi_1 = -0.052 4215$                                                    |
| 0.222 426 53  | $\xi_1 = -0.056 8129$                                                    |

and Bethe ansatz equations

$$
\xi^8_k = \frac{(Q + \mathcal{E}_{33} - \mathcal{E}_{13} + \xi_k^{-1} + \xi_k(1 - \Delta_k^{(3)}))}{(Q + \mathcal{E}_{33} - \mathcal{E}_{13} + \xi_k + \xi_k^{-1}(1 - \Delta_k^{(3)}))} \frac{(Q + \mathcal{E}_{33} - \mathcal{E}_{32} + \xi_k^{-1} + \xi_k(1 - \Delta_k^{(3)}))}{(Q + \mathcal{E}_{33} - \mathcal{E}_{32} + \xi_k + \xi_k^{-1}(1 - \Delta_k^{(3)}))} \times \prod_{j \neq k}^{n} \frac{s(\theta_j, \theta_j)}{s(-\theta_k, \theta_j)}.
$$

(A.4)

We have 5 additional eigenvalues which result from the one-particle sector, as given in table A.3. The two-particle eigenvalues, by construction, are the same as the two-particle eigenvalues given already in table A.2.

Finally, we use one of the reference states from the lower block of the table 1, e.g. $|0 ++ +\rangle$. This provides us with the following equations,

$$
E^{(2,1)}_1 = \mathcal{E}_{21} + Q + \xi_1 + \xi_1^{-1},
$$

(A.5)

and

$$
\xi^8_1 = \frac{(Q + \mathcal{E}_{21} - \mathcal{E}_{11} + \xi_1^{-1} + \xi_1(1 - \Delta_1^{(2)}))}{(Q + \mathcal{E}_{21} - \mathcal{E}_{11} + \xi_1 + \xi_1^{-1}(1 - \Delta_1^{(2)}))} \frac{(Q + \mathcal{E}_{21} - \mathcal{E}_{23} + \xi_1^{-1} + \xi_1(1 - \Delta_1^{(1)}))}{(Q + \mathcal{E}_{21} - \mathcal{E}_{23} + \xi_1 + \xi_1^{-1}(1 - \Delta_1^{(1)})}. \quad (A.6)
$$

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Table A.4. The remaining 5 energies eigenvalues and the corresponding Bethe roots for \( L = 4 \) and one particle over the state \(|0++\rangle\).

| \( E \) | BA roots |
|-------|----------|
| 18.637 212 08 | \( \xi_1 = 0.621 579 \) |
| 17.803 453 41 | \( \xi_1 = 0.698 313 \pm i0.715 793 \) |
| 15.581 441 42 | \( \xi_1 = -0.412 693 \pm i0.910 87 \) |
| -1.256 025 88 | \( \xi_1 = -0.056 7986 \) |
| -1.247 667 48 | \( \xi_1 = -0.056 8257 \) |

The solution of the above equation provides us with the 5 remaining eigenvalues. These energies are given in table A.4.

It is worth noting that the Bethe ansatz equation associated with the eigenvalues \( E_n^{(1,3)} \), \( E_n^{(2,2)} \) and \( E_n^{(3,1)} \) hold for \( L \geq 5 \). This means that for the present example \( (B^{(1,1)}) \), one would also have to solve the Bethe ansatz equation for \( E_n^{(1,3)} \) for larger lattices \( (L \geq 5) \). The other equations associated with \( E_n^{(2,2)} \) and \( E_n^{(3,1)} \) are identical to the previous one.

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