Stability of bound states in the light-front Yukawa model

V.A. Karmanov*, J. Carbonell† and M. Mangin-Brinet†

*Lebedev Physical Institute, Leninsky Pr. 53, 119991 Moscow, Russia
†Institut des Sciences Nucléaires, 53, Av. des Martyrs, 38026 Grenoble, France

Abstract. We show that in the system of two fermions interacting by scalar exchange, the solutions for $J^P=0^+$ bound states are stable without any cutoff regularization, for values of the coupling constant $\alpha$ below a critical value $\alpha_c$. This latter is calculated from an eigenvalue equation.

INTRODUCTION

The Yukawa model, describing a system of two fermions interacting by scalar exchange $(L_{\text{int}} = g\bar{\psi}\psi\phi(s))$, is instructive for studying the relativistic bound states and for developing the renormalization methods. It is also a main ingredient in building the $NN$ interaction, which contains an important contribution of scalar meson exchanges. The bound state problem and the renormalization in the Yukawa model were studied [1] in the framework of standard light-front dynamics [2]. The relativistic two-nucleon wave functions have been also calculated perturbatively [3] in the explicitly covariant version of light-front dynamics [4], where the state vector is defined on the invariant plane $\omega \cdot x = 0$ with $\omega^2 = 0$ (see for review [5]). In this work, the Bonn $NN$ model was used with the corresponding form factors [6] and the problem of cut-off dependence was not analyzed.

In reference [7], we investigated the stability of the bound states relative to the high momentum contributions of the kernel, when the cutoff tends to infinity. Below we present the results of our study and compare them with those obtained in [1].

THE CUTOFF DEPENDENCE OF THE BINDING ENERGY

We consider the two fermion wave function with total angular momentum $J = 0^+$. In the fermion spin indices, it is a $2 \times 2$ matrix determined, due to the parity conservation, by two independent elements – the spin components $f_{i=1,2}$. Since the wave function is defined on the light-front plane $\omega \cdot x = 0$, components $f_i(k, \theta)$ depend not only on the relative momentum $k$, but also on the angle $\theta$ between $\vec{k}$ and $\vec{n} = \vec{\omega}/|\vec{\omega}|$. The system of equations for $f_i$ contains a $2 \times 2$ matrix kernel $K_{ij}$, calculated by using the explicitly covariant light-front graph techniques [4, 5]. These equations and the analytical expressions of kernels are given in [7]. Though their form differs from the ones used in [1], we have shown that they are strictly equivalent.
Let us consider the equations on the finite interval $0 \leq k \leq k_{\text{max}}$. The dependence of their solutions on the cutoff $k_{\text{max}}$ in the limit $k_{\text{max}} \to \infty$ is determined by the kernels asymptotics. The kernel $K_{22}$ is repulsive and cannot generate a collapse, whereas $K_{11}$ is attractive. Therefore, to investigate the stability, we can consider the one channel problem for the component $f_1$, which satisfies the equation:

$$
[M^2 - 4(k^2 + m^2)] f_1(k, \theta) = \frac{m^2}{2 \pi^2} \int K_{11}(k, \theta; k', \theta') f_1(k', \theta') \frac{d^3 k'}{\varepsilon_{k'}},
$$

(1)

where $\varepsilon_{k'} = \sqrt{m^2 + k'^2}$.

Our analysis uses the fact that at $k \to \infty$ the integral in the r.h.s. of (1) is dominated by the region $k' \propto k$, i.e. $k' \to \infty$ with a fixed ratio $k'/k = \gamma$ [8]. One can therefore replace in (1) both wave function and kernel by their asymptotics, which have the form [7]:

$$
f_1(k, z) = \frac{h_1(z)}{k^2 + \beta}, \quad K_{11} = -\frac{\pi \alpha}{m^2} \begin{cases} \sqrt{\pi} A_{11}(z, z', \gamma), & \text{if } \gamma < 1 \\ A_{11}(z, z', 1/\gamma)/\sqrt{\gamma}, & \text{if } \gamma > 1 \end{cases}
$$

(2)

with $0 < \beta < 1$ and

$$
A_{11}(z, z', \gamma) = \int_0^{2\pi} \frac{d\phi}{2\pi \sqrt{\gamma}} \frac{2\gamma(1 - z z') - (1 + \gamma^2) \sqrt{1 - z^2} \sqrt{1 - z'^2} \cos \phi}{(1 + \gamma^2)(1 + |z - z'| - z z') - 2\gamma \sqrt{1 - z^2} \sqrt{1 - z'^2} \cos \phi},
$$

(3)

where $\alpha = g^2/4\pi$ and $z = \cos \theta$. By setting $A_{11}(z, z', \gamma) \equiv 1$ and $\alpha = 2\pi m \alpha'$ in (2), the asymptotics of $K_{11}$ becomes identical to asymptotics of the momentum space kernel corresponding to the non-relativistic potential $-\alpha'/r^2$. As it is well known [9], there exists for this potential a critical coupling constant $\alpha_c' = 1/(4m)$, that corresponds to $\alpha_c = \pi/2$. The inspection of (3) shows that in the Yukawa model the function $A_{11}$ is smaller than for the $-\alpha'/r^2$ potential: $0 \leq A_{11}(z, z', \gamma) \leq 1$. Therefore in this model, one can expect a larger critical coupling constant i.e. $\alpha_c > \pi/2$, what is confirmed by numerical calculations.

Substituting (2) into equation (1), we obtain for $g_1(z)$ [8]:

$$
\int_{-1}^{+1} dz' H_{\beta}(z, z') h_1(z') = \lambda h_1(z)
$$

(4)

with $\lambda = 1/\alpha$ and

$$
H_{\beta}(z, z') = \int_0^1 \frac{d\gamma}{2\pi \sqrt{\gamma}} A_{11}(z, z', \gamma) \cosh (\beta \log \gamma).
$$

(5)

Equation (4) is an eigenvalue equation for $\lambda$, parametrized by $\beta$. It provides the relation between the coupling constant $\alpha$ and $\beta$, determining the power law (2) of the wave function asymptotics. The r.h.s. of equation (1) becomes divergent for $\beta \leq 0$. Hence, the equation $\beta(\alpha_c) = 0$ determines the critical coupling constant $\alpha_c$. 

RESULTS

In the numerical calculations, the constituent masses were taken equal to \(m=1\) and the mass of the exchanged scalar is \(\mu=0.25\). The numerical solution \(\alpha(\beta)\), found from equation (4) with the function \(A_{11}(\gamma, z, z')\) given by (3), is plotted in figure 1. The critical coupling constant is obtained for \(\beta = 0\) for which the eigenvalue is \(\lambda_c = 0.269\). It corresponds to \(\alpha_c = 1/\lambda_c = 3.72\), as shown in figure 1 at \(\beta = 0\).

In figure 2, we have plotted the mass square \(M^2\) of the two fermion system, found from (1), as a function of the cutoff \(k_{\text{max}}\) for two fixed values of the coupling constant below and above the critical value \(\alpha_c = 3.72\). One can see two dramatically different behaviors depending on the value of the coupling constant \(\alpha\). For \(\alpha = 3\), i.e. \(\alpha < \alpha_c\), the result is convergent. On the contrary, for \(\alpha = 4\), i.e. \(\alpha > \alpha_c\), the result is clearly divergent: \(M^2\) decreases logarithmically as a function of \(k_{\text{max}}\) and becomes even negative. Though

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**FIGURE 1.** Function \(\alpha(\beta)\) determined by eq. (4). The critical coupling constant is: \(\alpha_c = \alpha(\beta = 0) = 3.72\). The values discussed in the text: \(\alpha = 1.096, \beta = 0.819\) and \(\alpha = 2.480, \beta = 0.548\) are on the curve.

**FIGURE 2.** Cutoff dependence of the binding energy in the \(J = 0^+\) state, in the one-channel problem \((f_1)\), for two fixed values of the coupling constant below and above the critical value \(\alpha_c = 3.72\). One can see two dramatically different behaviors depending on the value of the coupling constant \(\alpha\). For \(\alpha = 3\), i.e. \(\alpha < \alpha_c\), the result is convergent. On the contrary, for \(\alpha = 4\), i.e. \(\alpha > \alpha_c\), the result is clearly divergent: \(M^2\) decreases logarithmically as a function of \(k_{\text{max}}\) and becomes even negative. Though
the negative values of $M^2$ are physically meaningless, they are formally allowed by equations. The first degree of $M$ does not enter neither in the equation nor in the kernel, and $M^2$ crosses zero without any singularity.

We have examined the asymptotical behavior of the wave function $f_1(k, z)$ and found that it very accurately follows the power law (2) with the power $\beta(\alpha)$ given in figure 1. For instance for a binding energy $B = 2m - M = 0.05$ ($\alpha = 1.096$) a direct measurement in the numerical solution plotted in figure 3 gives $\beta = 0.820 \pm 0.002$ whereas the solution of equation (4) for the corresponding $\alpha$ gives $\beta = 0.819$. The same kind of agreement

![Figure 3](image)

**FIGURE 3.** Asymptotic behavior of the $J = 0^+$ wave function components $f_{1,2}$ for $B=0.05$, $\alpha=1.096$, $\mu=0.25$. The slope coefficients are $\beta_1 = 0.820$ and $\beta_2 \approx 0$.

was found for $B = 0.5$ ($\alpha = 2.480$, $\beta = 0.548 \pm 0.002$).

We conclude that the $J = 0$ – or (1+, 2-) state in the classification [1] – is stable (i.e. convergent relative to the cutoff $k_{\text{max}} \to \infty$) for coupling constant $\alpha$ below the critical value $\alpha_c = 3.72$. In this point, our conclusion differs from the one settled in [1], where it was stated that the integrals in the r.h.s. of the equations diverge logarithmically with cutoff. Above the critical value the integrals indeed diverge and the system collapses.

In the $J = 1$ state the system is found to be always unstable, as pointed out in [1].

Thus, by an analytical method, confirmed by numerical calculations, we have shown that the Yukawa model is not cutoff dependent for coupling constant below a critical value. The results obtained should be taken into account for instance in the renormalization procedures.

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