Remarks on the Method of Modulus of Continuity and the Modified Dissipative Porous Media Equation

Yamazaki Kazuo
Stanford University
ky2168@stanford.edu

Abstract

In this paper we study the method of Modulus of Continuity to show the global well-posedness of some differential equations. In particular we prove the global well-posedness of the modified Porous Media Equation.

Keywords: Porous Media Equation; Quasi-geostrophic Equation, Criticality, Besov Space

1 Introduction

Porous Media Equation (PM) in $\mathbb{R}^3, t > 0$ is defined as follows:

$$\begin{cases}
\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \nu \Lambda^\alpha \theta = 0 \\
u = -\kappa (\nabla p + g\gamma \theta) \\
\text{div} u = 0 \\
\theta(x, 0) = \theta_0(x)
\end{cases}\quad (1)$$

where $\theta$ represents liquid temperature, $\nu > 0$ the dissipative coefficient, $\kappa$ the matrix medium permeability divided by viscosity in different directions respectively, $g$ the acceleration due to gravity, vector $\gamma$ the last canonical

12000MSC : 35Mxx, 35Qxx, 35Sxx
2The corresponding author can be reached at the following: ky2168@stanford.edu
vector $e_3$ and $\Lambda = (-\triangle)^{1/2}$. Moreover, $p$ is the liquid pressure and $u$ represents the liquid discharge by Darcy’s law. For simplicity we set $\kappa = g = 1$.

In this paper we study the Modified Porous Media Equation (MPM) defined as follows:

$$
\begin{aligned}
\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \nu \Lambda^\alpha \theta &= 0, \ x \in \mathbb{R}^3, \ t > 0 \\
u &= \Lambda^{\alpha-1} \{-\nabla p + \gamma \theta\} \\
\text{div} u &= 0 \\
\theta(x,0) &= \theta_0(x)
\end{aligned}
$$

(2)

for $\alpha \in (0,1)$. Our main result is below:

**Theorem 1.1**

Let $\nu > 0$, $0 < \alpha < 1$ and $\theta_0(x) \in H^m$, $m \in \mathbb{Z}^+$, $m > 5/2$. Then there exists a unique global solution $\theta$ to the MPM (2) such that

$$
\theta \in \mathcal{C}(\mathbb{R}^+; H^m) \bigcap L^2_{\text{loc}}(\mathbb{R}^+; H^{m+\alpha/2}).
$$

Moreover, $\forall \ \gamma \in \mathbb{R}^+$, we have $\partial^\gamma \theta \in L^\infty(\mathbb{R}^+, H^{m+\gamma\alpha})$.

We note that an analogous version for Quasi-geostrophic Equation (QG), which we define below (3), was done by [17]. Moreover, while we will employ the method of Modulus of Continuity (MOC), initiated by [12] in a periodic setting, the smoothing effects stated above, i.e. the spatial decay of the solution, allows us to circumvent the difficulty in a non-periodic setting. In this regard, we cite [1] and [9] in which the authors proved the global well-posedness of QG with initial data belonging to the critical space $\dot{B}_{\infty,1}^0$ and $H^1$ using the same technique. The $H^m, m > 5/2$, space to which the initial data $\theta_0$ belongs is not critical space, as that requires $H^{3/2}$.

A similar result to the Theorem 1.1 showing global regularity of MPM (2) is also possible through the method introduced by Caffarelli and Vasseur in [5] following the work in [4], [6] and [7]. A similar method following the work in [13] is also possible.

We stress that at first sight, modifying PM (1) by having $\Lambda^{\alpha-1}$ act on the $u$ term and finding its MOC based on the previous work on PM (1) in [23] seems somewhat difficult. As we will see, the $u$ term of PM (1) can be
decomposed to a linear combination of an identity and a singular integral operator acting on $\theta$ which we will denote by $P(\theta)$. The problematic term is the Riesz potential, namely $\Lambda^{\alpha-1}\theta$. We obviate from this issue by making a simple observation; see Proposition 3.3.

The outline of the rest of the paper is as follows:

1. Introduction
2. Local Results
3. Global Results
4. Appendix A: Besov Space, Mollifiers
5. Appendix B: Proofs of Local Results and More

Let us introduce some MOC of relevance. By definition, a MOC is a continuous, increasing and concave function $\omega: [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$. We say some function $\theta: \mathbb{R}^n \to \mathbb{R}^m$ has MOC $\omega$ if $|\theta(x) - \theta(y)| \leq \omega(|x - y|)$ holds for all $x, y \in \mathbb{R}^n$.

The idea of MOC has caught much attention since the paper [12], in which the authors proved the global regularity of the solution to the 2-D critical QG defined as follows:

$$\partial_t \theta + (u \cdot \nabla)\theta = -\kappa \Lambda^{\alpha} \theta$$

where $u = (u_1, u_2) = (-R_2 \theta, R_1 \theta)$, $R_i$ is Riesz Transform in $\mathbb{R}^2$, $i = 1, 2$, and $\kappa$ diffusivity constant. The variable $u$ represents velocity and $\theta$ potential temperature. In particular, we have the following result from [12]:

**Proposition 1.2**

If the function $\theta: \mathbb{R}^2 \to \mathbb{R}$ has MOC $\omega$, then $u$ of (3) has MOC as follows:

$$\Omega_1(\xi) = A(\int_0^\xi \frac{\omega(\eta)}{\eta} d\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^2} d\eta)$$

Their initiative motivated others to follow. Consider a pseudo-differential operator, or a modified Riesz Transform, $\tilde{R}_{\alpha,j}$ defined as follows:

$$\tilde{R}_{\alpha,j} f(x) = |D|^{\alpha-1} R_j f(x) = c_{\alpha,n} \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+\alpha}} f(x - y) dy.$$
1 ≤ j ≤ n, for f ∈ \( S(\mathbb{R}^n) \) where \( \mathcal{R} \) is Riesz Transform, \( \mathcal{S} \) Schwartz space, 0 < \( \alpha \) < 1 and \( c_{\alpha,n} \) the normalization constant. We have the following result due to [17]:

**Proposition 1.3**

If \( \theta \), as defined in Proposition 1.2, has MOC \( \omega \), then \( u = (-\tilde{\mathcal{R}}_{\alpha,2}\theta, \tilde{\mathcal{R}}_{\alpha,1}\theta) \) has the MOC of

\[
\Omega_2(\xi) = A_\alpha \left( \int_0^\xi \frac{\omega(\eta)}{\eta^{\alpha}} d\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^{\alpha+1}} d\eta \right)
\]

with some absolute constant \( A_\alpha > 0 \) depending only on \( \alpha \).

Next, consider in \( \mathbb{R}^3 \) a singular integral operator \( \mathcal{P}(\theta) \) by the convolution with a kernel:

\[
K(x) = \left( \frac{3x_1x_3}{|x|^5}, \frac{3x_2x_3}{|x|^5}, \frac{2x_2^2-x_1^2-x_2^2}{|x|^5} \right),
\]

essentially a double Riesz transform of \( \theta \). We elaborate on this kernel in more detail below. First, let us construct a MOC of such modified analogously to the Riesz Transform above:

**Proposition 1.4**

In \( \mathbb{R}^3 \) define a modified double Riesz Transform \( \tilde{\mathcal{R}}_{\alpha,j}\mathcal{R}_i(\theta) \) where \( \tilde{\mathcal{R}}_{\alpha,j} \) is that of Proposition 1.3 and its kernel is in the form of \( \sum_{i,j} c_{ij} \frac{x_i x_j}{|x|^\alpha} \) \( i,j \in \{1,2,3\} \) on each component.

Then \( \tilde{\mathcal{R}}_{\alpha,j}\mathcal{R}_i(\theta) \) has a MOC as follows:

\[
\Omega(\xi) = C_\alpha \xi^{1-\alpha} \int_0^\xi \frac{\omega(\eta)}{\eta} d\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^{\alpha+1}} d\eta
\]

for some \( C > 0 \).

**Proof of Proposition 1.4**

The idea of this proof is borrowed from [23]. We simply substitute \( \Omega_1 \) from Proposition 1.2 for \( \omega \) in \( \Omega_2 \) of Proposition 1.3. Direct computation gives the following:
\[ \Omega(\xi) = A_\alpha \left( \int_0^\xi \frac{\Omega_1(\eta)}{\eta^\alpha} d\eta + \xi \int_\xi^\infty \frac{\Omega_1(\eta)}{\eta^{1+\alpha}} d\eta \right) \]

\[ \leq C(\xi^{1-\alpha} \int_0^\xi \frac{\omega(\eta)}{\eta} d\eta + \xi^{2-\alpha} \int_\xi^\infty \frac{\omega(\eta)}{\eta^2} d\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^{\alpha+1}} d\eta) \]

\[ \leq C(\xi^{1-\alpha} \int_0^\xi \frac{\omega(\eta)}{\eta} d\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^{\alpha+1}} d\eta) \]

Q.E.D.

Let us explain the motivation behind the kernels introduced before Proposition 1.4. We derive a relation between \( u \) and \( \theta \) of PM (1) following the method in [3]; for its generalization, see Lemma 4.6 in the Appendix A. As a vector identity we have -\( \text{curl} \ (\text{curl} \ u) + \nabla \ \text{div} \ u = \triangle \ u \) which is reduced to -\( \text{curl} (\text{curl} \ u) = \triangle \ u \) by divergence free property. Hence, we obtain

\[ \triangle u = \left( \frac{\partial^2 \theta}{\partial x_1 \partial x_3}, \frac{\partial^2 \theta}{\partial x_2 \partial x_3}, -\frac{\partial^2 \theta}{\partial x_1^2} - \frac{\partial^2 \theta}{\partial x_2^2} \right). \]

Taking the inverse of the Laplacian,

\[ u = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \left( \frac{\partial^2 \theta}{\partial x_1 \partial x_3}, \frac{\partial^2 \theta}{\partial x_2 \partial x_3}, -\frac{\partial^2 \theta}{\partial x_1^2} - \frac{\partial^2 \theta}{\partial x_2^2} \right) dy \]

from which standard Integration By Parts (IBP) gives

\[ u(x, t) = -\frac{2}{3}(0, 0, \theta(x, t)) + \frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^3} K(x - y) \theta(y, t) dy = C\theta + \mathcal{P}(\theta) \]

for \( x \in \mathbb{R}^3 \) where \( K(x) = \left( \frac{3x_1x_3}{|x|^4}, \frac{3x_2x_3}{|x|^4}, \frac{2x_3^2 - x_1^2 - x_2^2}{|x|^6} \right) \), the kernel introduced before Proposition 1.4, and \( C \) a constant. Throughout this paper we let this \( C \) be one.

**Remark**

*In fact, it can be shown that the double Riesz transform of \( \theta \), \( \mathcal{P}(\theta) \) has the identical MOC as that of Proposition 1.2 and the MOC of the additional term \( C\theta \) can be absorbed into the first integral of \( \Omega_1 \) as well. If done so, the MOC \( \Omega_2 \) of Proposition 1.3 will suffice for our need and we do not even need*
to construct our own MOC but simply use that in [17]. We choose to rely on Proposition 1.4 instead as the integrand of \( \Omega \) being similar to that of \( \Omega_1 \) of Proposition 1.2 allows us to construct our own MOC (4) and (5) closely related to that introduced in [12], consequently making the computation in Section 3 simpler.

The scaling invariance of the PM is \( \lambda^{\alpha-1} \theta(\lambda x, \lambda^\alpha t) \) for \( \lambda > 0 \), same as the case of QG. This makes \( \alpha = 1 \) the threshold of sub- and super-criticality. Recall that the method of MOC in the first place was introduced in order to prove the global regularity of the critical 2-D QG. Observe that if \( \theta(x, t) \) solves QG (3), then so does \( \theta(\lambda x, \lambda t) \). Not so, when \( \alpha \neq 1 \) and its cost is usually the initial condition (cf. [20]).

However, one may modify the QG (3) as below so that for any \( \alpha \in (0, 1) \), its scaling invariance may be similar to that of critical (cf. [4]):

\[
\partial_t \theta + (u \cdot \nabla) \theta + \kappa \Lambda^\alpha \theta = 0
\]

with \( u = \Lambda^{\alpha-1}(-R_2 \theta, R_1 \theta) \). Observe that this PDE enjoys the rescaling of \( \theta(x, t) \to \theta(\lambda x, \lambda^\alpha t) \). For this reason, as we will see in (4) and (5), we will construct a MOC that is unbounded so that finding one MOC \( \omega \) which is globally preserved in time implies that all the MOC \( \omega_\lambda(\xi) = \omega(\lambda \xi) \) will also be globally preserved.

It takes only a glance at MPM (2) to realize that it was defined in the same spirit.

2 Local Results

The purpose of this section is to introduce local results. We note that double Riesz Transform remains bounded in any space in which an ordinary Riesz Transform is bounded and that the results for the latter case has been obtained by [17]. The method of proof is similar to that introduced in [14] through regularizing (2) and relying on Picard’s Theorem. Let us first set some notations; additional information can be found in Appendix A.

Denote by \( S' \) the space of tempered distributions, \( S'(\mathbb{R}^n)/P(\mathbb{R}^n) \) the quotient space of tempered distributions modulo polynomials, \( \mathcal{F}(\xi) = \hat{f}(\xi) \) the Fourier Transform, and \( \| \cdot \|_X \) the norm of Banach space \( X \), e.g.
\[ H^m = \{ f \in L^2(\mathbb{R}^n) : \| f \|^2_{H^m} = \sum_{0 \leq |\beta| \leq m} \| D^\beta f \|^2_{L^2} < \infty \}. \]

Next we take the usual dyadic unity partition of Littlewood-Paley decomposition. Let us denote two nonnegative radial functions \( \chi, \phi \in C^\infty(\mathbb{R}^n) \) supported in \( \{ \xi \in \mathbb{R}^n : |\xi| \leq \frac{3}{4} \} \) and \( \{ \xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{5}{4} \} \) respectively such that
\[
\chi(\xi) + \sum_{j \geq 0} \phi(2^{-j} \xi) = 1, \forall \xi \in \mathbb{R}^n; \quad \sum_{j \in \mathbb{Z}} \phi(2^{-j} \xi) = 1, \forall \xi \neq 0
\]
and \( \chi(\xi) = \hat{\Phi}(\xi), \phi(\xi) = \hat{\Psi}(\xi) \).

We define \( \forall f \in S'(\mathbb{R}^n) \) the nonhomogeneous Littlewood-Paley operators:
\[
\triangle_{-1} f := \Phi * f, \quad \triangle_j f := \Psi_{2^{-j}} * f, \forall j \in \mathbb{Z}^+ \cup \{0\}
\]
where \( \hat{\Psi}_{2^{-j}}(\xi) = \phi(2^{-j} \xi) \) and the homogeneous defined as
\[
\hat{\Delta}_j f = \Psi_{2^{-j}} * f, \forall j \in \mathbb{Z}.
\]

With these Littlewood-Paley operators, we define Besov spaces for \( p, q \in [1, \infty], s \in \mathbb{R} \), the nonhomogeneous and homogeneous respectively:
\[
B^s_{p,r} := \{ f \in S'(\mathbb{R}^n) : \| f \|_{B^s_{p,r}} := \| \triangle_{-1} f \|_{L^p} + (\sum_{j \geq 0} 2^{jsr} \| \triangle_j f \|_{L^p})^{1/r} < \infty \}
\]
\[
\dot{B}^s_{p,r} := \{ f \in S'(\mathbb{R}^n/)P(\mathbb{R}^n) : \| f \|_{\dot{B}^s_{p,r}} := (\sum_{j \in \mathbb{Z}} 2^{jsr} \| \hat{\Delta}_j f \|_{L^p})^{1/r} < \infty \}.
\]

Now let us state the main results of this section:

**Proposition 2.1**
Let \( \nu > 0, \ 0 < \alpha < 1 \) and \( \theta_0 \in H^m, \ m \in \mathbb{Z}^+, \ m > 5/2 \). Then there exists a unique solution \( \theta \in C([0, T], H^m) \cap L^2([0, T], H^{m+\frac{5}{2}}) \) to the MPM (2) where \( T = T(\alpha, \| \theta_0 \|_m) > 0 \). Moreover, we have \( t^\alpha \theta \in L^\infty T H^{m+\gamma} \forall \gamma \geq 1 \).

**Proposition 2.2**
Let \( T^* \) be the maximal local existence time of \( \theta \in C([0, T^*], H^m) \cap L^2([0, T^*], H^{m+\alpha/2}) \). If \( T^* < \infty \), then we have \( \int_0^{T^*} \| \nabla \theta(t, \cdot) \|_{L^\infty} \, dt = \infty \).

The proofs are found in the Appendix B.
3 Global Results

We extend our results to the global case. Below we use $\xi = |x - y|$ interchangeably and omit the subscript $\alpha$ but use instead $i = 1, 2, 3, \ldots$ to indicate different constants; when no confusion arises. By the Blow-up Criterion Proposition 2.2, we only have to show that $\int_0^T \|\nabla \theta\|_{L^\infty} < \infty$ for $\theta$ a local solution of MPM (2) up to time $T > 0$. We utilize the following two observations made in [12]:

Proposition 3.1

If $\omega$ is a MOC for $\theta(x, t) : \mathbb{R}^n \to \mathbb{R}$ for all $t > 0$, then $|\nabla \theta|(x) \leq \omega'(0)$ for all $x \in \mathbb{R}^n$

For this reason, we shall construct a MOC $\omega$ such that $\omega'(0) < \infty$. Next,

Proposition 3.2

Assume $\theta$ has a strict MOC for all $t < T$; i.e. $\forall x, y \in \mathbb{R}^n, |\theta(x, t) - \theta(y, t)| < \omega(|x - y|)$, but not for $t > T$. Then, there exists $x, y \in \mathbb{R}^n, x \neq y$ such that $\theta(x, T) - \theta(y, T) = \omega(|x - y|)$ for $\omega$ satisfying

$\omega''(0+) = -\infty$

Consequently, the only scenario in which a MOC $\omega$ is lost is if there exists a moment $T > 0$ such that $\theta$ has the MOC $\omega$ for all $t \in [0, T]$ and two distinct points $x$ and $y$ such that $\theta(x, T) - \theta(y, T) = \omega(|x - y|)$. Below we rule out this possibility by showing that in such case, $\frac{\partial}{\partial t}[\theta(x, t) - \theta(y, t)]_{t=T} < 0$. For this purpose, let us write

$$
\frac{\partial}{\partial t}[\theta(x, T) - \theta(y, T)] = -[(u \cdot \nabla \theta)(x, T) - (u \cdot \nabla \theta)(y, T)] - [(\Lambda^\alpha \theta)(x, T) - (\Lambda^\alpha \theta)(y, T)]
$$

We call the first bracket Convection term and the second Dissipation term. Our agenda now is to first estimate the Convection and Dissipation terms, to be specific find upper bounds that depend on $\omega$. Then we will construct the MOC $\omega$ explicitly that assures us that the sum of the two terms is negative to reach the desired result.

Estimates on the Convection and Dissipation Terms

We propose the following estimate for our Convection Term:

Proposition 3.3
If $\omega$ is a MOC for $\theta(x,t): \mathbb{R}^3 \to \mathbb{R}$, a local solution to MPM (2) for all $t \leq T$, then

$$(u \cdot \nabla \theta)(x, T) - (u \cdot \nabla \theta)(y, T) \leq C\Omega(\xi)\omega'(\xi)$$

where $u = \Lambda^{-1}\{\theta + \mathcal{P}(\theta)\}$ and $\Omega(\xi)$ is that of Proposition 1.4.

The proof relies on the following observation due to [15] and [19]. From the expression of $\nabla u = (-\partial_{x_1}\partial_{x_3}\theta, -\partial_{x_2}\partial_{x_3}\theta, \partial_{x_1}^2\theta + \partial_{x_2}^2\theta)$ in the case of PM (1) derived in Section 1, the Fourier multiplier of such operator is clear; each component is a linear combination of terms like $\frac{\xi_i\xi_j}{|\xi|^2}$, $i, j = 1, 2, 3$ which belongs to $C^\infty(\mathbb{R}^3 \setminus \{0\})$ and homogeneous of degree zero. Hence, it is clear that for the MPM (2) we can express $u$ as

$$\sum_{i,j} c_{ij} \Lambda^{-1}(-\Delta)^{-1}\partial_i\partial_j \theta = \sum_{i,j} \tilde{R}_{\alpha,j} R_i \theta$$

where $\tilde{R}_{\alpha,j}$ is that of Proposition 1.2. Therefore, the MOC for $u = \Lambda^{-1}\{\theta + \mathcal{P}(\theta)\}$ of MPM (2) is that of Proposition 1.4.

**Remark**

We note that in [23], the authors considered the MOC of the Convection term of PM (1) separately; i.e. $\omega(\xi)$ for $\theta$ term and another for the $\mathcal{P}(\theta)$. In our case, the same strategy would lead to having to deal with $\Lambda^{-1}\theta$ term.

Now we only need to compute

$$u \cdot \nabla \theta(x) - u \cdot \nabla \theta(y)$$

$$= \frac{d}{dh} \left[ \theta(x + hu(x)) - \theta(y + hu(y)) \right]$$

$$= \lim_{h \to 0} \left[ \frac{\theta(x + hu(x)) - \theta(y + hu(y))}{h} \right] - \frac{\theta(x) - \theta(y)}{h}$$

$$= \lim_{h \to 0} \left[ \frac{\theta(x + hu(x)) - \theta(y + hu(y))}{h} - \omega(\xi) \right]$$

$$\leq \lim_{h \to 0} \left[ \frac{\omega(\xi + h\Omega(\xi)) - \omega(\xi)}{h} \right]$$

$$= \Omega(\xi)\omega'(\xi)$$
For the estimate on Dissipation term, we borrow below from [12], note their result is general in dimension:

\[
C_2 \left[ \int_0^{\xi/2} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^{1+\alpha}} d\eta + \int_{\xi/2}^{\infty} \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^{1+\alpha}} d\eta \right]
\]

The Explicit Construction of the Modulus of Continuity

The MOC (4) and (5) that we construct below is naturally a hybrid of those introduced in [12] and [23]. With \( \alpha \in (0, 1), r \in (1, 1 + \alpha), 0 < \gamma < \delta < 1 \), define

\[
\omega(\xi) = \xi - \xi^r \quad \text{when } 0 \leq \xi \leq \delta \quad (4)
\]

\[
\omega'(\xi) = \frac{\gamma}{\xi \left( \frac{2\alpha^2 + \alpha + 1}{\alpha} \right) + \log(\xi)} \quad \text{when } \delta < \xi \quad (5)
\]

The function \( \omega \) is clearly continuous and \( \omega(0) = 0 \). The first derivative of (4) is

\[
\omega'(\xi) = 1 - r\xi^{r-1} \geq 0 \quad (6)
\]

for \( \delta \) sufficiently small and hence increasing. Moreover, \( \omega'(0) = 1 < \infty \). The positivity of (5) is clear, and hence the unboundedness. The second derivative of (4) is

\[
\omega''(\xi) = -r(r - 1)\xi^{r-2} \quad (7)
\]

which is clearly negative; notice also

\[
\lim_{\xi \to 0^+} \omega''(\xi) = -\infty \quad (8)
\]

Finally, for small enough \( \delta \) we have

\[
\omega'(\delta^-) = 1 - r\delta^{r-1} \approx 1 \quad (9)
\]

while

\[
\omega'(\delta^+) = \frac{\gamma\alpha^2}{\delta(2\alpha^2 + \alpha + 1)} < \frac{\gamma}{2\delta} < \frac{1}{2} \quad (10)
\]

Therefore, the concavity is achieved. Now we consider two cases

**The case** \( 0 \leq \xi \leq \delta \)

10
In this case we have
\[ \frac{\omega(\eta)}{\eta} \leq 1 \ \forall \ \eta \geq 0, \]  
(11)
with which we immediately obtain
\[ \int_0^\xi \frac{\omega(\eta)}{\eta} d\eta \leq \xi \]  
(12)
and
\[ \int_\xi^\delta \frac{\omega(\eta)}{\eta^{\alpha+1}} d\eta = \int_\xi^\delta \frac{\eta - \eta^\alpha}{\eta^{\alpha+1}} d\eta \leq \int_\xi^\delta \eta^{-\alpha} d\eta \leq \frac{\delta^{1-\alpha}}{1-\alpha} \]  
(13)
Moreover,
\[ \int_\delta^\infty \frac{\omega(\eta)}{\eta^{\alpha+1}} d\eta = \frac{\omega(\delta)\delta^{-\alpha}}{\alpha} + \frac{1}{\alpha} \int_\delta^\infty \omega'(\eta)\eta^{-\alpha} d\eta \]  
(14)
\[ \leq \frac{\delta^{1-\alpha}}{\alpha} + \frac{\gamma}{\alpha} \int_\delta^\infty \frac{1}{\eta^{1+\alpha}} \left[ \frac{2\alpha^2 + \alpha + 1}{\alpha^2} \right] + \log\left( \frac{\eta}{\delta} \right) d\eta \]  
\[ \leq \frac{\delta^{1-\alpha}}{\alpha} + \frac{\gamma\delta^{-\alpha}}{2\alpha} \int_1^\infty \frac{1}{z^{1+\alpha}} dz \]  
\[ = \frac{\delta^{1-\alpha}}{\alpha} + \frac{\gamma\delta^{-\alpha}}{2\alpha^2} \]
where the first inequality is by (11). Thus, we have
\[ \int_\delta^\infty \frac{\omega(\eta)}{\eta^{\alpha+1}} d\eta \leq \frac{2\alpha + 1}{2\alpha^2} \delta^{1-\alpha} \]  
(15)
because \( \gamma < \delta \). Since \( \omega'(\xi) \leq \omega'(0) = 1 \), the contribution from the positive side is limited to
\[ C_1[\xi^{1-\alpha}\xi + \xi\left\{ \frac{\delta^{1-\alpha}}{1-\alpha} + \left( \frac{2\alpha + 1}{2\alpha^2} \right)\delta^{1-\alpha} \right\}] \]  
(16)
The work from [12] shows that the first integrand of the dissipation term gives
\[ \int_0^{\xi/2} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^{1+\alpha}} d\eta \leq -C_2\xi^{r-\alpha} \]  
(17)
Therefore, adding (16) and (17) gives
\[ \xi[C_1 \xi^{1-\alpha} + \frac{1}{1-\alpha} + \frac{2\alpha+1}{2\alpha^2}]\xi^{1-\alpha} - C_2 \xi^{r-\alpha-1} < 0 \quad (18) \]

is achieved as \( \delta \to 0 \) forcing \( \xi \to 0 \); note \( r - \alpha - 1 < 0 \).

**The Case** \( \delta < \xi \)

In this case we only have

\[ \frac{\omega(\eta)}{\eta} \leq 1 \quad (19) \]

for \( \eta \in [0, \delta] \). Using \( \omega(\eta) \leq \omega(\xi) \forall \delta \leq \eta \leq \xi \), we obtain

\[ \int_0^\xi \frac{\omega(\eta)}{\eta} d\eta = \int_0^\delta \frac{\omega(\eta)}{\eta} d\eta + \int_\delta^\xi \frac{\omega(\eta)}{\eta} d\eta \leq \delta + \omega(\xi) \log(\frac{\xi}{\delta}) \quad (20) \]

Observing that

\[ \omega(\xi) \geq \omega(\delta) = \delta - \delta^r \geq \frac{\delta}{2} \quad (21) \]

if \( \delta \) is small enough, we reach from (20)

\[ \int_0^\xi \frac{\omega(\eta)}{\eta} d\eta \leq \omega(\xi)(2 + \log(\frac{\xi}{\delta})) \quad (22) \]

Next,

\[ \int_\xi^{\infty} \frac{\omega(\eta)}{\eta^{\alpha+1}} d\eta = \frac{\omega(\xi)\xi^{-\alpha}}{\alpha} + \frac{1}{\alpha} \int_\xi^{\infty} \omega'(\eta)\eta^{-\alpha} d\eta \quad (23) \]

\[ \leq \frac{\omega(\xi)\xi^{-\alpha}}{\alpha} + \frac{\gamma}{\alpha} \int_\xi^{\infty} \frac{1}{\eta^{1+\alpha}} d\eta \]

\[ = \frac{\omega(\xi)\xi^{-\alpha}}{\alpha} + \frac{\gamma \xi^{-\alpha}}{\alpha^2} \]

and hence

\[ \int_\xi^{\infty} \frac{\omega(\eta)}{\eta^{\alpha+1}} d\eta \leq \omega(\xi)\xi^{-\alpha}(\frac{\alpha+1}{\alpha^2}) \quad (24) \]

if we take \( \gamma \) small enough such that

\[ 2\log(2)\gamma < \frac{\delta}{2} \leq \omega(\xi) \quad (25) \]
which follows from (21); the need for the multiplication by $2\log(2)$ on $\gamma$ becomes clear below. Therefore, the contribution from the positive side is limited to

$$C_1 \Omega(\xi) \omega'(\xi) \leq C_1 [\xi^{1-\alpha} \omega(\xi)(2 + \log(\frac{\xi}{\delta})) + \omega(\xi) \xi^{-\alpha}\left(\frac{\alpha + 1}{\alpha^2}\right)\omega'(\xi)$$

(26)

$$= C_1 \omega(\xi) \xi^{-\alpha} \left[\xi(2 + \log(\frac{\xi}{\delta})) + \left(\frac{\alpha + 1}{\alpha^2}\right)\right] \frac{\gamma}{\xi\left(\frac{(2\alpha^2 + \alpha + 1)}{\alpha^2} + \log(\frac{\xi}{\delta})\right)}$$

$$= C_1 \omega(\xi) \xi^{-\alpha} \gamma$$

On the other hand, we have the following estimate from [12]:

$$\int_{\xi/2}^{\infty} \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^{1+\alpha}} d\eta \leq -C_2 \omega(\xi) \xi^{-\alpha}$$

(27)

which is due to

$$\omega(2\eta + \xi) - \omega(2\eta - \xi) \leq \omega(2\xi) = \omega(\xi) + \gamma \int_{\xi}^{2\xi} \frac{1}{\eta\left(\frac{(2\alpha^2 + \alpha + 1)}{\alpha^2} + \log(\frac{\eta}{\delta})\right)} d\eta$$

$$\leq \omega(\xi) + \gamma \log(2) \leq \frac{3}{2} \omega(\xi)$$

by concavity and (25). In sum, we obtain from (26) and (27)

$$C_1 \omega(\xi) \xi^{-\alpha} \gamma - C_2 \omega(\xi) \xi^{-\alpha}$$

(28)

$$\leq \omega(\xi) \xi^{-\alpha} [C_1 \gamma - C_2] < 0$$

where the last inequality is attainable for $\gamma$ sufficiently small. Q.E.D.

4 Appendix A: Besov Space and Mollifiers

Besov Space

We introduce the two types of coupled space-time Besov spaces. They are $L^{p}([0, T], B^{s}_{p,r})$, abbreviated by $L^{p}_{T}B^{s}_{p,r}$, defined by the norm

$$\|f\|_{L^{p}_{T}B^{s}_{p,r}} = \|||2^{js}||\Delta_{j}f\||_{L^{p}}\|_{L^{r}[0,T]}$$

and $\tilde{L}^{p}([0, T], B^{s}_{p,r})$, abbreviated by $\tilde{L}^{p}_{T}B^{s}_{p,r}$, called the Chemin-Lerner’s space-time space which are the set of tempered distribution $f$ satisfying the norm
We list useful results below:

**Lemma 4.1 Bernstein’s Inequality**

Let \( f \in L^p(\mathbb{R}^n) \) with \( 1 \leq p \leq q \leq \infty \) and \( 0 < r < R \). Then \( \forall k \in \mathbb{Z}^+ \cup \{0\} \), and \( \lambda > 0 \), \( \exists \) a constant \( C_k > 0 \) such that

\[
(\text{a}) \quad \sup_{|\alpha|=k} \| \partial^\alpha f \|_{L^q} \leq C \lambda^{k+n(1/p-1/q)} \| f \|_{L^p} \quad \text{if supp} \mathcal{F} f \subset \{ \xi : |\xi| \leq \lambda r \},
\]

\[
(\text{b}) \quad C_k^{-1} \lambda^k \| f \|_{L^p} \leq \sup_{|\alpha|=k} \| \partial^\alpha f \|_{L^p} \leq C_k \lambda^k \| f \|_{L^p} \quad \text{if supp} \mathcal{F} f \subset \{ \xi : \lambda r \leq |\xi| \leq \lambda R \}.
\]

and if we replace derivative \( \partial^\alpha \) by the fractional derivative, the inequalities remain valid only with trivial modifications.

**Lemma 4.2 Besov Embedding (cf. [18])**

Assume \( s \in \mathbb{R} \) and \( p, q \in [1, \infty] \).

(a) If \( 1 \leq q_1 \leq q_2 \leq \infty \), then \( \dot{B}_{p,q_1}^s(\mathbb{R}^n) \subset \dot{B}_{p,q_2}^s(\mathbb{R}^n) \).

(b) If \( 1 \leq p_1 \leq p_2 \leq \infty \) and \( s_1 = s_2 + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \), then \( \dot{B}_{p_1,q_1}^{s_1}(\mathbb{R}^n) \subset \dot{B}_{p_2,q_2}^{s_2}(\mathbb{R}^n) \).

**Lemma 4.3 (cf. [21])**

(a) For \( f \in S' \) with \( \text{supp} \mathcal{F} f \subset \{ \xi : |\xi| \leq r \} \), \( \exists C = C(n) \) such that for \( 1 \leq p \leq q \leq \infty \),

\[
\| f \|_q \leq C r^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \| f \|_p
\]

(b) Analogously, if \( \text{supp} \mathcal{F} f \subset \{ \xi : |\xi| \simeq r \} \), then

\[
\| f \|_q \simeq r^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \| f \|_p
\]

(c) Denoting Riesz transform by \( \mathcal{R} \), for \( s > n/p, 1 < p < \infty, 1 \leq r \leq \infty \),

\[
\| \mathcal{R} f \|_{B^s_{p,r}} \leq C \| f \|_{B^s_{p,r}}
\]

(d) Analogously, for \( 1 \leq p \leq \infty \) and \( 1 \leq r \leq \infty \), we have

\[
\| \mathcal{R} f \|_{\dot{B}^s_{p,r}} \leq C \| f \|_{\dot{B}^s_{p,r}}
\]
Next, we define the transport-diffusion equation, $0 < \alpha < 1$

$$(TD)_\alpha \begin{cases} \partial_t \theta + u \cdot \nabla \theta + \nu |D|^{\alpha} \theta &= f \\ \text{div} u &= 0 \\ \theta|_{t=0} &= \theta_0 \end{cases}$$

**Proposition 4.4** *(cf. [16], [17])*

Let $-1 < s < 1, 1 \leq \rho_1 \leq \rho \leq \infty, p, r \in [1, \infty], f \in L^1_{\text{loc}}(\mathbb{R}^+, B^{s+\frac{\alpha}{\rho_1} - \alpha}_{p,r})$ and $u$ a divergence-free vector field in $L^1_{\text{loc}}(\mathbb{R}^+; \text{Lip}(\mathbb{R}^n))$. Suppose $\theta$ is a $C^\infty$ solution of $(TD)_\alpha$. Then $\exists C = C(n, s, \alpha)$ s.t. $\forall t \in \mathbb{R}^+$,

$$\nu^\frac{1}{r} \| \theta \|_{L^r B^{s+\alpha/r}_{p,r}} \leq C e^{CV(t)} (\| \theta_0 \|_{B^{s+\alpha/r}_{p,r}} + \nu^\frac{1}{r_1} \| f \|_{L^r B^{s+\alpha/r_1}_{p,r}}),$$

where $V(t) := \int_0^t \| \nabla u(\tau) \|_{L^\infty} \, d\tau$.

Moreover, if $u = |D|^{\alpha-1}(R^\perp(\theta))$, the inequality above still holds with $V(t) := \int_0^t (\| \nabla u(\tau) \|_{L^\infty} + \| |D|^{\alpha}\theta(\tau) \|_{L^\infty}) \, d\tau$. Taking $p = r = 2, \rho = \rho_1 = \infty$ gives

$$\| \theta \|_{L^\infty B^{s}_{H^s}} \leq C e^{C\tilde{V}(t)} (\| \theta_0 \|_{H^s} + \nu^{-1} \| f \|_{L^\infty B^{s-\alpha}_{H^s}}).$$

An outline of the proof of Proposition 4.4 which consists of using paradifferential calculus and Lagrangian coordinate method combined with commutator estimates as well as results from [8], [10] and [11], can be found in [16].

For our case with $u = \Lambda^{\alpha-1}(\theta + \mathcal{P}(\theta))$, it suffices to note that the statement of the Proposition applies for $u = R^\perp(\theta)$ and $u = \theta + \mathcal{P}(\theta)$ identically and the modification by $|D|^{\alpha-1}$ is same in the case of the modified QG and MPM. Therefore, $\tilde{V}$ above applies for MPM. Furthermore, for our purpose, we note that $\tilde{V}(T) \leq CT \| \theta \|_{L^\infty B^m}$ by Sobolev embedding. The complete proof can be found in [22].

We also have the following from [2]:

**Proposition 4.5**

Let $u$ be a $C^\infty$ divergence-free vector field and $f$ a $C^\infty$ function. Assume that $\theta$ is a solution of $(TD_\alpha)$. Then for $p \in [1, \infty]$, we have

$$\| \theta(t) \|_{L^p} \leq \| \theta_0 \|_{L^p} + \int_0^t \| f(\tau) \|_{L^p} \, d\tau$$
We introduce another result, relevant to our estimate of the Convection term:

Lemma 4.6 (cf. [19])
Let \( m \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) be a homogeneous function of degree 0, and \( T_m \) be the corresponding multiplier operator defined by \( (T_m f) = m \hat{f} \), then there exists \( a \in \mathcal{C} \) and \( \Omega \in C^\infty(S^{n-1}) \) with zero average such that for any Schwartz function \( f \),

\[
T_m f = af + PV \frac{\Omega(x')}{|x|^n} \ast f.
\]

**Mollifier**
Given an arbitrary radial function \( \rho(|x|) \in C^\infty_0(\mathbb{R}^3), \rho \geq 0, \int_{\mathbb{R}^3} \rho \, dx = 1 \), we define the mollifier operator \( T_\epsilon : L^p(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3), 1 \leq p \leq \infty, \epsilon > 0 \), by

\[
(T_\epsilon f)(x) = \epsilon^{-3} \int_{\mathbb{R}^3} \rho \left( \frac{x - y}{\epsilon} \right) f(y) \, dy \quad \forall \ f \in L^p(\mathbb{R}^3).
\]

**Lemma 4.8**
For \( m \in \mathbb{Z}^+ \cup \{0\} \), \( s \in \mathbb{R} \), \( k \in \mathbb{R}^+ \), Then

(i) \( \forall f \in C_0, T_\epsilon f \to f \) uniformly on a compact set \( U \) in \( \mathbb{R}^3; \|T_\epsilon f\|_{L^\infty} \leq \|f\|_{L^\infty} \).

(ii) \( \forall f \in H^m(\mathbb{R}^3), D^\beta(T_\epsilon f) = T_\epsilon(D^\beta f); \forall f \in H^s(\mathbb{R}^3), |D|^s(T_\epsilon f) = T_\epsilon(|D|^s f) \).

(iii) \( \forall f \in H^s(\mathbb{R}^3), \lim_{\epsilon \to 0} \|T_\epsilon f - f\|_{H^s} = 0 \) and \( \|T_\epsilon f - f\|_{H^{s+1}} \leq c\epsilon \|f\|_{H^s} \).

(iv) \( \forall f \in H^m(\mathbb{R}^3), \|T_\epsilon f\|_{H^{m+k}} \leq c \epsilon^{-k} \|f\|_{H^m} \) and \( \|T_\epsilon D^k f\|_{L^\infty} \leq c \epsilon^{-(1+k)} \|f\|_{L^2} \).

5 **Appendix B: Proofs of Local Results and More**

In this section we sketch out the proofs from Section 2 Local Results. Let us regularize MPM (2) writing \( \Lambda = |D|: \)

\[
\begin{cases}
\theta^\epsilon_t + T_\epsilon((T_\epsilon u^\epsilon) \cdot \nabla(T_\epsilon \theta^\epsilon)) + \nu T_\epsilon(|D|^\alpha \theta^\epsilon) = 0 \\
u^\epsilon = |D|^{\alpha - 1}(C(\theta^\epsilon) + P(\theta^\epsilon)) \\
\theta^\epsilon|_{t=0} = \theta_0(x)
\end{cases}
\tag{29}
\]

The author is grateful to Professor Liutang Xue for pointing out this fact.
Observe that (29) can be reduced to an ODE,
\[ \frac{d}{dt} \theta^\epsilon = F^\epsilon(\theta^\epsilon); \theta^\epsilon|_{t=0} = \theta_0 \in H^m \]  
(30)
where \( F^\epsilon(\theta^\epsilon) = -\nu T^\epsilon |D|^{\alpha/2} \theta^\epsilon - \mathcal{T}_\epsilon(\mathcal{T}_\epsilon \nabla(\mathcal{T}_\epsilon \theta^\epsilon)) = F_1^\epsilon(\theta^\epsilon) - F_2^\epsilon(\theta^\epsilon). \)

Naturally the following Proposition can be proven by Picard’s Theorem; we refer readers to [14] for proof.

**Proposition 5.1 Global Existence of Regularized Solutions**
Let \( \theta_0 \in H^m, m \in \mathbb{Z}^+ \cup \{0\}, m > 5/2. \) Then \( \forall \epsilon > 0, \) there exists a unique global solution \( \theta^\epsilon \in C^1([0, \infty), H^m) \) to the regularized MPM (29).

We may also assume the following result with identical proof found in [14]:

**Proposition 5.2**
The unique regularized solution \( \theta^\epsilon \in C^1([0, \infty), H^m) \) to (29) satisfies below:

\[ \frac{1}{2} \frac{d}{dt} \|\theta^\epsilon\|^2_{H^m} + \nu \mathcal{T}_\epsilon |D|^{\alpha/2} \theta^\epsilon \|_{H^m}^2 \leq C_{m,\alpha} (\|\nabla \mathcal{T}_\epsilon \theta^\epsilon\|_{L^\infty} + \|\mathcal{T}_\epsilon \theta^\epsilon\|_{L^3}) \|\theta^\epsilon\|^2_{H^m} \]  
(31)

and

\[ \sup_{0 \leq t \leq T} \|\theta^\epsilon\|_{L^2} \leq \|\theta_0\|_{L^2} \]

Below we show that if \( m > 5/2, \exists [0, T] \) and subsequence convergent to a limit function \( \theta \) that solves the MPM (2). The strategy is to first obtain the uniform bounds of \( H^m \) norm in the interval \([0, T]\) independent of \( \epsilon \), and show that in \([0, T]\), these approximate solutions are contracting in \( L^2 \) norm. By applying Interpolation Inequality, we will prove convergence as \( \epsilon \to 0 \) and pass the limit. Moreover, we outline the proof of the uniqueness and smoothing effects.

We first show below that \( (\theta^\epsilon) \) the family of solution is uniformly bounded in \( H^m \). We have

\[ \frac{d}{dt} \|\theta^\epsilon\|_{H^m} \leq C_{m,\alpha} \|\mathcal{T}_\epsilon \nabla \theta^\epsilon\|_{L^\infty} \|\theta^\epsilon\|_{H^m} \leq C_{m,\alpha} \|\theta^\epsilon\|_{H^m}^2 \]

by (31) and Sobolev Embedding as \( m > \frac{5}{2}. \) Thus, \( \forall \epsilon > 0, \) we have
\[ \sup_{0 \leq t \leq T} \| \theta^\epsilon \|_{H^m} \leq \frac{\| \theta_0 \|_{H^m}}{1 - c_{m,\alpha} T \| \theta_0 \|_{m}} \] (32)

which implies that for \( T < \frac{1}{c_{m,\alpha} \| \theta_0 \|_{m}} \), \( (\theta^\epsilon) \) is uniformly bounded in \( C([0, T], H^m) \). Next, by (31) and (32) after integrating in time \([0, T]\) we obtain

\[ \nu^{1/2} ||D|^{\alpha/2} \theta^\epsilon||_{L^2([0,T], H^m)} \leq C(\| \theta_0 \|_{H^m}, T) \] (33)

This with the \( L^2 \) energy inequality after (31) gives

\[ \nu^{1/2} ||\theta^\epsilon||_{L^2([0,T], H^{m+\delta})} \leq C(\| \theta_0 \|_{H^m}, T), \]

the desired uniform bound.

Q.E.D.

We now show that the solutions \( \theta^\epsilon \) to regularized MPM (29) form a contraction in the low norm \( C([0,T], L^2(\mathbb{R}^3)) \); i.e. \( \forall \epsilon, \tilde{\epsilon}, \exists C = C(\| \theta_0 \|_{H^m}, T) \) such that

\[ \sup_{0 \leq t \leq T} ||\theta^\epsilon - \tilde{\theta}^\tilde{\epsilon}||_{L^2} \leq C \max\{\epsilon, \tilde{\epsilon}\} \]

We take

\[ \theta^\epsilon - \tilde{\theta}^\tilde{\epsilon} = -\nu (T^2_\epsilon |D|^{\alpha} \theta^\epsilon - T^2_{\tilde{\epsilon}} |D|^{\alpha} \tilde{\theta}^\tilde{\epsilon}) - (T_\epsilon ((T_\epsilon u^\epsilon) \cdot \nabla (T_\epsilon \theta^\epsilon)) - (T_{\tilde{\epsilon}} ((T_{\tilde{\epsilon}} u_{\tilde{\epsilon}}) \cdot \nabla (T_{\tilde{\epsilon}} \tilde{\theta}^\tilde{\epsilon}))) \]

and multiply by \( \theta^\epsilon - \tilde{\theta}^\tilde{\epsilon} \) and integrate to get

\[ \begin{align*}
(\theta^\epsilon - \theta^\epsilon_0, \theta^\epsilon - \tilde{\theta}^\tilde{\epsilon}) &= -\nu (T^2_\epsilon |D|^{\alpha} \theta^\epsilon - T^2_{\tilde{\epsilon}} |D|^{\alpha} \tilde{\theta}^\tilde{\epsilon}) \theta^\epsilon - \theta^\epsilon) \\
&= \frac{1}{2} \left(I + \frac{1}{2} \right) \end{align*} \]

We bound I and II separately by standard method using the fact that Riesz potentials are bounded in \( L^p \) space; for details, see [17]. Thus, we have

\[ \sup_{0 \leq t \leq T} ||\theta^\epsilon - \tilde{\theta}^\tilde{\epsilon}||_{L^2} \leq e^{c(M)T} \max\{\epsilon, \tilde{\epsilon}\} + \| \theta^\epsilon_0 - \tilde{\theta}^\tilde{\epsilon}_0 \|_{L^2} \leq C(M, T) \max\{\epsilon, \tilde{\epsilon}\} \] (34)
where $M$ is an upper bound from (32). From this we deduce that \{\theta^\epsilon\} is Cauchy in $C([0, T], L^2(\mathbb{R}^3))$ and hence converges to $\theta \in C([0, T], L^2(\mathbb{R}^3))$. We apply the Interpolation Inequality to $\theta^\epsilon - \theta$, and using (32) and (34) we obtain

$$
sup_{0 \leq t \leq T} \|\theta^\epsilon - \theta\|_s \leq C_s sup_{0 \leq t \leq T}(\|\theta^\epsilon - \theta\|^{1 - \frac{s}{m}}_{L^2} \|\theta^\epsilon - \theta\|^{\frac{s}{m}}_{H^m}) \leq C(\|\theta_0\|_{H^m}, T, s) \epsilon^{1 - \frac{s}{m}}
$$

which gives $\theta \in C([0, T], H^s(\mathbb{R}^3)), 0 \leq s < m$

Also, from $\theta^\epsilon_t = -\nu T^\epsilon_\alpha [D^\alpha \theta^\epsilon - T_\epsilon((-T^\epsilon_\alpha \cdot \nabla T^\epsilon_{\theta^\epsilon}))$, we see that $\theta^\epsilon_t$ converges to $-\nu [D^\alpha \theta - u \cdot \nabla \theta]$ in $C([0, T], C(\mathbb{R}^3))$. As $\theta^\epsilon \to \theta$, the distribution limit of $\theta^\epsilon_t$ must be $\theta_t$; i.e. $\theta$ is a classical solution of MPM (2). From (32) and (33) we also have $\theta \in L^\infty([0, T], H^m(\mathbb{R}^3)) \cap L^2([0, T], H^{m + \frac{3}{2}}(\mathbb{R}^3))$.

Next, we show $\theta \in C([0, T], H^m(\mathbb{R}^3))$. Firstly, we have

$$
\|\theta(t) - \theta(t')\|_{H^m}^2 = C_0 \|\theta(t) - \theta(t')\|^2_{B_{2/1,2}^m}
\leq C_0 \sum_{-1 \leq j \leq J} 2^{2jm} \|\Delta_j \theta(t) - \Delta_j \theta(t')\|_{L^2}^2 + 2C_0 \sum_{j > J} 2^{2jm} \|\Delta_j \theta\|_{L^\infty}^2 L^2
$$

and

\[
\|\nabla u\|_{L^\infty} + \|\nabla [D^\alpha \theta]\|_{L^\infty} \\
\leq C(\|\nabla [D^\alpha \theta]\|_{B^{0,1}_{\infty,1}} + \|\theta\|_{H^m}) \\
= C(\sum_{j \leq -1} \|\hat{\Delta}_j \{\nabla [D^\alpha \theta] + \mathcal{P}(\theta)\}\|_{L^\infty} + \|\theta\|_{H^m}) \\
+ \sum_{j \geq 0} \|\hat{\Delta}_j \{\nabla [D^\alpha \theta] + \mathcal{P}(\theta)\}\|_{L^\infty} + \|\theta\|_{H^m}) \\
\leq C(\sum_{j \leq -1} 2^{j} \|\hat{\Delta}_j \{\theta + \mathcal{P}(\theta)\}\|_{L^\infty} + \|\theta\|_{H^m}) \\
+ \sum_{j \geq 0} 2^{j(\alpha - 1)} \|\hat{\Delta}_j \{\nabla [\theta] + \mathcal{P}(\theta)\}\|_{L^\infty} + \|\theta\|_{H^m}) \\
\leq C(\sum_{j \leq -1} 2^{j(\alpha + \frac{1}{2})} \|\hat{\Delta}_j \theta\|_{L^2} + \sum_{j \geq 0} 2^{j(\alpha - 1)} \|\hat{\Delta}_j \nabla \theta\|_{L^\infty} + \|\theta\|_{H^m} \leq C\|\theta\|_{H^m}
\]

Next, by Besov embedding and Prop. 4.4, we have
\[ \| \theta \|_{L^\infty_t B_{2,2}^m} \leq \| \theta \|_{L^\infty_t H^m} \leq C e^{e^T} \| \theta_0 \|_{H^m} \leq C e^{e^T} \| \theta \|_{L^\infty_t H^m} \| \theta_0 \|_{H^m} < \infty \]  

(35)

Hence, we know \( \exists J = J(T, \delta) \) such that
\[
\sum_{j > J} 2^{2jm} \| \Delta_j \theta \|_{L^2_t L^2}^2 \leq \frac{\delta^2}{4C_0}
\]

We apply Mean Value Theorem to get
\[
\sum_{-1 \leq j \leq J} 2^{2jm} \| \Delta_j \theta(t) - \Delta_j \theta(t') \|_{L^2_t L^2}^2 \leq |t - t'|^2 \sum_{-1 \leq j \leq J} 2^{2jm} \| \Delta_j (\partial_t \theta) \|_{L^\infty_t L^2}^2
\]

\[
\leq C |t - t'|^2 2^{2J} \| \partial_t \theta \|_{L^\infty_t H^{m-1}}^2
\]

On the last term, we have
\[
\| \partial_t \theta \|_{H^{m-1}} \leq \nu \| D^\alpha \theta \|_{H^{m-1}} + \| u \cdot \nabla \theta \|_{H^{m-1}}
\]
\[
\leq \nu \| \theta \|_{H^{m-1+\alpha}} + \| u \theta \|_{H^m} \leq C (\| \theta \|_{H^m} + \| u \theta \|_{H^m} \| \theta \|_{H^m})
\]
\[
\leq C (\| \theta \|_{H^m} + \| \theta \|_{H^{m-1}}^2) \leq C (\| \theta_0 \|_{H^m}, T)
\]

i.e. \( \partial_t \theta \in L^\infty([0, T], H^{m-1}) \); hence the desired continuity.

The uniqueness is proven by standard way of using the difference of two different solutions, multiplication, integration and Gronwall’s inequality (cf. [17]). For the smoothing effects, take \( t^\gamma \theta, \gamma > 0 \) in (TD)_\alpha below:

\[
\partial_t (t^\gamma \theta) + u \cdot \nabla (t^\gamma \theta) + \nu |D|^\alpha (t^\gamma \theta) = \gamma t^{\gamma-1} \theta; (t^\gamma \theta)|_{t=0} = 0
\]

Assume \( T \geq 1 \) without loss of generality. We show the following:

\[
\| t^\gamma \theta(t) \|_{L^\infty_t L^2} + \| t^\gamma \theta(t) \|_{L^\infty_t H^{m+\gamma}} \leq C (T^{\gamma+1} + e^{C(\gamma+1)T \| \theta \|_{L^\infty_t H^m}}) \| \theta_0 \|_{H^m}
\]

which implies

\[
\| t^\gamma \theta(t) \|_{L^\infty_t H^{m+\gamma}} \leq C (T^{\gamma+1} + e^{C(\gamma+1)T \| \theta \|_{L^\infty_t H^m}}) \| \theta_0 \|_{H^m}
\]

The proof is done through induction on \( \gamma \) and interpolation to apply for all \( \gamma \in \mathbb{R}^+ \); the readers are referred to [17] for detail.
Finally, the blow up criterion is proven. In similar fashion to (31) we can obtain
\[
\frac{1}{2} \frac{d}{dt} \|\theta\|_{H^m}^2 + \nu \|D^\frac{\alpha}{2} \theta\|_{H^m}^2 \leq C_{m,\alpha} (\|\nabla \theta\|_{L^\infty} + \|\theta\|_{L^3}) \|\theta\|_{H^m}^2
\]

Now Gronwall’s inequality, the fact that \(\theta_0 \in H^m, m \in \mathbb{Z}^+\) and \(m > \frac{5}{2}\) and Sobolev inequality show that if the blow-up time \(T^* < \infty\), then
\[
\int_0^{T^*} \|\nabla \theta\|_{L^\infty} dt = \infty
\]
Q.E.D.
This completes the proofs of both Proposition 2.1 and 2.2.

6 Acknowledgment

The author expresses gratitude to Professor Gautam Iyer and Jiahong Wu for their teaching and Professor Miao Changxing and Liutang Xue and the referee for their helpful comments and suggestions.

References

[1] Abidi, H., Hmidi, T.: On the global wellposedness of the critical quasi-geostrophic equation. SIAM J. Math. Anal. 40, 167-185 (2008).
[2] Cordoba, A., Cordoba, D.: A Maximum Principle Applied to Quasi-geostrophic Equations. Commun. Math. Phys. 249, 511-528 (2004)
[3] Castro, A., Cordoba, D., Francisco, G., Orive, R.: Incompressible Flow in Porous Media with Fractional Diffusion. Arxiv. http://arxiv.org/abs/0806.1180 (2008). Accessed 08/03/09
[4] Constantin, P., Iyer, G., Wu, J.: Global Regularity for a Modified Critical Dissipative Quasi-geostrophic Equation. Indiana Univ. Math. J., Vol. 57, No. 6 (2008)
[5] Caffarelli, L., Vasseur, A.: Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. Arxiv. http://arxiv.org/abs/math/0608447 (2006). Accessed 09/01/09
[6] Constantin, P., Wu, J.: Holder continuity of solutions of supercritical dissipative hydrodynamic transport equations. Arxiv. http://arxiv.org/abs/math/0701594 (2007). Accessed 09/01/09

[7] Constantin, P., Wu, J.: Regularity of Holder continuous solutions of the supercritical quasi-geostrophic equation. Arxiv. http://arxiv.org/abs/math/0701592 (2007). Accessed 09/01/09

[8] Cannone, M., Miao, C., Wu G.: On the Inviscid Limit of the Two-Dimensional Navier-Stokes Equations with Fractional Diffusion. Adv. in Math. Sci. and App., Vol. 18, No. 2 (2008)

[9] Dong H., Du., D.: Global well-posedness and a decay estimate for the critical dissipative quasi-geostrophic equation in the whole space. Discrete Contin. Dyn. Syst. (2008)

[10] Danchin, R.: Uniform Estimates for Transport-Diffusion Equations J. of Hyperbolic Differ. Equ. Vol. 4, No. 1 1-17 World Scientific Publishing Company (2007)

[11] Hmidi, T., Keraani S.: On the Global Solutions of the Super-critical 2D Quasi-geostrophic Equation in Besov Spaces. Adv. Math. 214 pp. 618-638 (2007)

[12] Kiselev, A., Nazarov, F., Volberg, A.: Global Well-posedness for the Critical 2D Dissipative Quasi-geostrophic Equation. Invent. Math. 167, 445-453 (2007)

[13] Kiselev, A., Nazarov, F.: A variation on a theme of Caffarelli and Vasseur. Arxiv. http://arxiv.org/abs/0908.0923 (2009). Accessed 09/01/09

[14] Majda, A. J., Bertozzi, A. L.: Vorticity and Incompressible Flow. Cambridge Texts in Applied Mathematics, 2002.

[15] Miao C. ”reply from Miao” E-mail to author. 25 Jul. 2009.

[16] Miao C., Wu G.: Global Well-posedness of the Critical Burgers Equation in Critical Besov Spaces. J. of Differ. Equ. Vol. 247, Issue 6, 15 1673-1693(2009)

[17] Miao, C., Xue, L.: Global Wellposedness for a Modified Critical Dissipative Quasi-geostrophic Equation. Arxiv. http://arxiv.org/abs/0901.1368 (2009). Accessed 08/03/09
[18] Wu, J.: Lower Bounds for an Integral Involving Fractional Laplacians and the Generalized Navier-Stokes Equations in Besov Spaces. Commun. Math. Phys. 263, 803-831 (2005)

[19] Xue, L.: On the well-posedness of incompressible flow in porous media with supercritical diffusion. Applicable Analysis, 88: 4, 547-561 (2009)

[20] Xinwei, Y.: Remarks on the Global Regularity for the Super-critical 2D dissipative Quasi-geostrophic Equation. J. Math. Anal. and Appl. 339 359 -371 (2008)

[21] Xiaofent, L., Wang, M., Zhang, Z.: Local Well-posedness and Blowup Criterion of the Boussinesq Equation in Critical Besov Spaces. J. math. fluid. mech. 28 April (2009)

[22] Yamazaki, K.: Remarks on the Method of Modulus of Continuity and the Modified Dissipative Porous Media Equation Detailed Version. Homepage of Kazuo Yamazaki. Web. http://stanford.edu/~ky2168/main.htm (2009). Accessed 08/28/2009

[23] Yuan, B., Yuan, J.: Global Well-posedness of Incompressible Flow in Porous Media with Critical Diffusion in Besov Spaces. J. Differ. Equ. 246 4405-4422 (2009)