Strings in the Einstein’s paradigm of matter

Vladimir Dzhunushaliev *

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Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195, Berlin, Germany

and

Dept. Phys. and Microel. Engineer., Kyrgyz-Russian Slavic University
Bishkek, Kievskaya Str. 44, 720000, Kyrgyz Republic

Abstract

Spherically symmetric solutions with a flux of electric and magnetic fields in Kaluza-Klein gravity are considered. It is shown that under the condition (electric charge \(q\) \(\approx\) (magnetic charge \(Q\)) \((q > Q)\) these solutions are like a flux tube stretched between two Universes. The longitudinal size of this tube depends on the value of \(\delta = 1 - Q/q\). The cross section of the tube can be chosen \(\approx l_{Pl}\). In this case this flux tube looks like a 1-dimensional object (thread) stretched between two Universes. The propagation of gravitational waves on the thread is considered. The corresponding equations are very close to the classical string equations. This result allows us to say that the thread between two Universes is similar to the string attached to two \(D\)-branes.

1 Introduction

In string theory there is a problem which can not be resolved inside of the theory: Is there an inner structure of the string? If the string is some kind of matter then string theory can not be a fundamental theory of Everything. We need to have a theory for the string matter. If such matter has an elementary structure then we must have a theory for this structure and so on up to infinity. In fact this is analogous to the well known old problem of the inner structure of electron. Probably the most radical resolution of this problem was suggested by Einstein. As usually his approach is very revolutionary: the electron does not have any material structure, it is a bridge between two asymptotically flat spaces (a wormhole in the modern language). One can say that this is Einstein’s paradigm for the fundamental structure of matter.

*E-mail: dzhun@hotmail.kg
Surprisingly, this point of view is more adaptable to strings then to point particles. The reason is that it is very difficult to find vacuum wormhole solutions of Einstein equations, but string-like solutions arise as very natural objects in vacuum 5D Kaluza-Klein theory. In this paper we would like to present such solutions and interpret them as string-like objects.

2 String-like solutions

In common case the 5D metric $G_{AB}$ ($A, B = 0, 1, 2, 3, 5$) has the following form

$$G_{AB} = \left( \begin{array}{cc} g_{\mu\nu} - \phi^2 A_\mu A_\nu & -\phi^2 A_\mu \\ -\phi^2 A_\mu & -\phi^2 \end{array} \right) \quad (1)$$

$\mu, \nu = 0, 1, 2, 3$ are the 4D indices. According to the Kaluza-Klein point of view $g_{\mu\nu}$ is the 4D metric; $A_\mu$ is the usual electromagnetic potential and $\phi$ is some scalar field. The 5D Einstein vacuum equations are

$$R_{AB} - \frac{1}{2} G_{AB} R = 0, \quad (2)$$

where $R_{AB}$ is the 5D Ricci tensor and after the dimensional reduction we have the following 4D equations for the (electrogravity + scalar field) theory (for the reference see, for example, 1)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{\phi^2}{2} T_{\mu\nu} - \frac{1}{\phi} \left[ \nabla_\mu (\partial_\nu \phi) - g_{\mu\nu} \Box \phi \right], \quad (3)$$

$$\nabla_\nu F^{\mu\nu} = -\frac{3}{\phi} \partial_\nu \phi F^{\mu\nu}, \quad (4)$$

$$\Box \phi = -\frac{\phi^4}{4} F^{\alpha\beta} F_{\alpha\beta}. \quad (5)$$

where $R_{\mu\nu}$ is the 4D Ricci tensor; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the 4D Maxwell tensor and $T_{\mu\nu}$ is the energy-momentum tensor for the electromagnetic field.

Now we would like to present the spherically symmetric solutions with nonzero flux of electric and/or magnetic fields 2. For our spherically symmetric 5D metric we take

$$ds^2 = \frac{dt^2}{\Delta(r)} - \frac{dr^2}{\Delta(r)} e^{2\psi(r)} [d\chi + \omega(r) dt + Q \cos \theta d\varphi]^2 - dr^2 - a(r)(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (6)$$

where $\chi$ is the 5-th extra coordinate; $r, \theta, \varphi$ are 3D spherical-polar coordinates; $n$ is constant; $r \in \{-r_H, +r_H\}$ ($r_H$ may be equal to $\infty$), $t_0$ is some constant; $Q$ is the magnetic charge as $(\theta\varphi)$ -component of the Maxwell tensor $F_{23} = -\sin \theta$ and consequently the radial magnetic field is $H_r = Q/a$.

In our case the metric 3 give us the 4D metric

$$ds^2 = \frac{dt^2}{\Delta(r)} - dr^2 - a(r)(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (7)$$
and the following components of electromagnetic potential \( A_\mu \),

\[
A_0 = \omega(r) \quad \text{and} \quad A_\varphi = Q \cos \theta \tag{8}
\]

that gives the Maxwell tensor

\[
F_{10} = \omega(r)' \quad \text{and} \quad F_{23} = -Q \sin \theta. \tag{9}
\]

This means that we have radial Kaluza-Klein electric \( E_r \propto F_{01} \) and magnetic \( H_r \propto F_{23} \) fields.

Substituting this ansatz into the 5D Einstein vacuum equations \((\chi t)\) gives us

\[
\frac{\Delta''}{\Delta} - \frac{\Delta'^2}{\Delta^2} + \frac{\Delta'\omega'}{\Delta a} + \frac{\Delta'\psi'}{\Delta} + \frac{q^2}{a^2 \Delta^2} e^{-4\psi} = 0, \tag{10}
\]

\[
\frac{a''}{a} + \frac{a' \psi'}{a} - \frac{2}{a} + \frac{Q^2}{a^2 \Delta} e^{2\psi} = 0, \tag{11}
\]

\[
\psi'' + \psi'^2 + \frac{a' \psi'}{a} - \frac{Q^2}{2a^2 \Delta} e^{2\psi} = 0, \tag{12}
\]

\[
-\frac{\Delta'^2}{\Delta^2} + \frac{a'^2}{a^2} - 2 \frac{\Delta'\psi'}{\Delta} - \frac{4}{a} + \frac{a' \psi'}{a} + \frac{q^2}{a^2 \Delta^2} e^{-4\psi} + \frac{Q^2}{a^2 \Delta} e^{2\psi} = 0 \tag{13}
\]

\( q \) is some constant which physical sense will be discussed below; these equations are derived after substitution the expression \((14)\) for the electric field in the initial Einstein’s equations \((\chi t)\). The 5D \((\chi t)\)-Einstein’s equation (4D Maxwell equation) is taken as having the following solution

\[
\omega' = \frac{q}{l_0 a \Delta^2} e^{-3\psi}. \tag{14}
\]

For the determination of the physical sense of the constant \( q \) let us write the \((\chi t)\)-Einstein’s equation in the following way:

\[
(l_0 \omega' \Delta^2 e^{3\psi} 4\pi a)' = 0. \tag{15}
\]

The 5D Kaluza-Klein gravity after the dimensional reduction says us that the Maxwell tensor is

\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{16}
\]

That allows us to write in our case the electric field as \( E_r = \omega' \). Eq.\((15)\) with the electric field defined by \((16)\) can be compared with the Maxwell’s equations in a continuous medium

\[
\text{div} \vec{D} = 0 \tag{17}
\]

where \( \vec{D} = \epsilon \vec{E} \) is an electric displacement and \( \epsilon \) is a dielectric permeability. Comparing Eq.\((15)\) with Eq.\((17)\) we can say that the magnitude \( q/a = \omega' \Delta^2 e^{3\psi} \) is like to the electric displacement and in this case the dielectric permeability
is $\epsilon = \Delta^2 e^{3\psi}$. It means that $q$ can be taken as the Kaluza-Klein electric charge because the flux of electric field is $\Phi = 4\pi a D = 4\pi q$.

Eq. (13) gives us the following relationship between the Kaluza-Klein electric and magnetic charges

$$1 = \frac{q^2 + Q^2}{4a(0)}$$

where $a(0) = a(r = 0)$. From Eq. (18) it is seen that the charges can be parameterized as $q = 2\sqrt{a(0)} \sin \alpha$ and $Q = 2\sqrt{a(0)} \cos \alpha$.

As the relative strengths of the Kaluza-Klein fields are varied it was found that the solutions to the metric in Eq. (6) evolve in the following way:

1. $0 \leq Q < q$. The solution is a regular flux tube. The throat between the surfaces at $\pm r_H$ is filled with both electric and magnetic fields. The longitudinal distance between the $\pm r_H$ surfaces increases, and the cross-sectional size does not increase as rapidly as $r \to r_H$ with $q \to Q$. Essentially, as the magnetic charge is increased one can think that the $\pm r_H$ surfaces are taken to $\pm \infty$ and the cross section becomes constant. The radius $r = r_H$ is defined by the following way

$$ds^2(r = \pm r_H) = 0.$$

2. $Q = q$. In this case the solution is an infinite flux tube filled with constant electric and magnetic fields. The cross-sectional size of this solution is constant ($a = \text{const.}$). A similar object was derived within the context of 4D dilaton gravity in Ref. [5].

3. $0 \leq q < Q$. In this case we have a singular flux tube located between two $(\pm)$ electric and magnetic charges located at $\pm r_{\text{sing}}$. Thus the longitudinal size of this object is again finite, but now the cross-sectional size decreases as $r \to r_{\text{sing}}$. At $r = \pm r_{\text{sing}}$ this solution has real singularities which we interpret as the locations of the charges.

The evolution of the solution from a regular flux tube, to an infinite flux tube, to a singular flux tube, as the relative magnitude of the charges is varied, is presented in Fig. (1).

Most important for us here is the case with $q \approx Q$ (but $q > Q$). From Eq. (18) we have $q = 2a_0 \sin(\pi/4 + \delta) \approx a_0 \sqrt{2}(1 + \delta)$ and $Q \approx a_0 \sqrt{2}(1 - \delta)$.

### 3 Long flux tube solutions

Now we would like to consider the case with $q \equiv Q$ ($q > Q$) ($|r| \leq r$). As we see above the case $q = Q$ is the infinite flux tube with (electric field) = (magnetic field). Our basic aim is to show that every such object with $2\delta = 1 - Q/q \ll 1$ and cross section $\approx l_P^2$ can be considered as a string-like object.

To do this we will numerically investigate solutions of Eq’s (10)-(13) for different $\delta$’s. The numerical investigations of our system of equations is complicated as it is very insensitive to $\delta \ll 1$. In order to avoid this problem we
introduce the following new functions

\[ \Delta(x) = \frac{f(x)}{\cosh^2(x)}, \]  
\[ \psi(x) = \phi(x) + \ln \cosh(x) \]  

where \( x = r/a(0)^{1/2} \) is a dimensionless radius. It is necessary to note that when \( f(x) = 1 \) and \( \phi(x) = 0 \) we have the above-mentioned infinite flux tube. We have the following equations for these functions

\[ f'' - 4f' \tanh(x) - 2f (1 - 3 \tanh^2(x)) - \left( \frac{f' - 2f \tanh(x)}{f} \right)^2 + \]
\[ \frac{a'}{a} (f' - 2f \tanh(x)) + (f' - 2f \tanh(x)) (\phi' + \tanh(x)) + \frac{q^2}{a^2} e^{-4\phi} = 0, \]  
\[ a'' - 2 + a' (\phi' + \tanh(x)) + \frac{Q^2 f}{a} e^{2\phi} = 0, \]  
\[ \phi'' + \frac{1}{\cosh^2(x)} (\phi' + \tanh(x))^2 + \frac{a'}{a} (\phi' + \tanh(x)) - \frac{Q^2 f}{2a^2} e^{2\phi} = 0 \]  

The corresponding solutions of these equations with the different \( \delta \) are presented in Fig’s (2)-(4). We see that (at least in the investigated area of \( \delta \)) at the point \( r = \pm r_H \) the function \( a(r_H) \approx 2a_0 \) (here \( r_H \) is the point where \( \Delta(r_H) = 0 \) and \( a_0 = a(r = 0) \)). The most interesting aspect is that \( a(r_H) \) does not grow as \( r \to r_H \) and consequently the flux tube with \( \delta \ll 1 \) (in the region \( |r| \leq r_H \)) can be considered as a string-like object attached to two Universes. For brevity we will call such objects “threads.”

It shows us that the necessary conditions for the consideration of our solution as a string-like object is definitely satisfied in the region \( |r| \leq r_H \) and outside of
Figure 2: The functions $a(x)$ with the different magnitudes of $\delta$. The curves 1,2,3,4 correspond to the following magnitudes of $\delta = 10^{-3}, 10^{-6}, 10^{-10}, 10^{-13}$.

Figure 3: The functions $f(x)$ with the different magnitudes of $\delta$. The curves 1,2,3,4 correspond to the following magnitudes of $\delta = 10^{-3}, 10^{-6}, 10^{-10}, 10^{-13}$.

Figure 4: The functions $\phi(x)$ with the different magnitudes of $\delta$. The curves 1,2,3,4 correspond to the following magnitudes of $\delta = 10^{-3}, 10^{-6}, 10^{-10}, 10^{-13}$.
this region the cross section will increase and consequently such approach will not be correct. For example, it is easy to see for the special case when \( Q = 0 \) \[2\]. In this case there is the exact solution

\[
\begin{align*}
\Delta &= r_0^2 + r^2, \\
\psi &= 0
\end{align*}
\]

And immediately we see that \( a \approx r^2 \) (at \( a \to \infty \)) and \( g_{tt} \approx -1 + O(1/r^2) \), the same is for the others cases with \( Q < q \). It is interesting to note that here we have the change of the 4D metric signature \[7\].

We can also insert this part of solution between two Reissner - Nordst"om black holes \[3\] and join the flux tube with each black hole on the event horizon. For both cases for us is interesting only the part \( |r| \leq r_H \) of the solution because here the transversal linear sizes are much less the longitudinal characteristic length.

Here we would like to investigate the behavior of all functions close to \( r = r_H \).

We propose the following approximate form of the metric

\[
\begin{align*}
\Delta (r) &= \Delta_1 (r_H - r) + \cdots, \\
a (r) &= a_1 + a_2 (r_H - r) + \cdots, \\
\psi (r) &= \psi_1 + \psi_2 (r_H - r) + \cdots
\end{align*}
\]

From Eq. \[10\] we see that

\[
a_1 = \frac{q}{\Delta_1} e^{-2\psi_1}
\]

Now we would like to show that at this point we have \( ds^2 = 0 \) (where \( d\chi = dr = d\theta = d\varphi = 0 \) and \( G_{55} (r = r_H) = 0 \)). The \((tt)\) component of the metric is

\[
G_{tt} (r) = \frac{1}{\Delta (r)} - R_1^2 \Delta (r) e^{2\psi (r)} \omega^2 (r)
\]

According to Eq’s \[14\], \[28\]-\[30\] we see that

\[
\omega' (r) \approx \frac{q e^{-3\psi_1}}{R_1 a_1 \Delta_1^2 (r - r_H)^2}
\]

consequently

\[
\omega (r) \approx \frac{q}{R a_1 (r - r_H)} e^{-3\psi_1}
\]
\[ G_{tt}(r) \approx \frac{1}{\Delta_1 (r - r_H)^2} \left[ 1 - \frac{q^2 e^{-4\psi_1}}{a_1^2 \Delta_1^2} \right] \]  

(35)

from Eq. (31) we see that \( G_{tt}(r_H) = 0 \) and consequently \( ds^2 = 0 \). The same is true for \( r = -r_H \).

4 Gravitational waves on the thread.

In this section we would like to consider the propagation of small perturbations (gravitational waves) of the 5D metric along the thread. It is not too hard to show [6] that the equations for these perturbations are

\[
\begin{align*}
&h_{A;B;C}^C + h_{B;A;C}^C - h_{AB;C}^C - h_{;A;B} = 0, \\
&h_{AB} = G_{AB} - G_{AB}^{(0)} \quad \text{and} \quad h = h_A^A
\end{align*}
\]  

(36)

(37)

where \( G_{AB}^{(0)} \) is the background metric [11] and \( G_{AB} \) is the metric with the perturbations. Eq. (36) can be simplified in the case of waves with large frequency

\[ h_{AB;C}^C = 0 \]  

(38)

with gauging

\[ \left( h_A^B - \frac{1}{2} \delta_A^B h \right) ;_B = 0. \]  

(39)

Let us introduce coordinates \( \mathcal{X}^A = h_{AB}, \ A = (A,B) = 0,1,2,\cdots,15 \). As mentioned above our thread approximately looks like a 1-dimensional object with a cross section of order of Planck scale \( l_{Pl}^2 \) and a classical longitudinal length. This leads to the possibility that the gravitational waves depend only on the time \( t \) and the longitudinal coordinate \( r \), i.e. \( h_{AB} = h_{AB}(t,r) \). Thus Eq. (38) has the following form

\[ (\mathcal{X}^A)^{;C}_{;C} = 0 \]  

(40)

or

\[ (\mathcal{X}^A)^{;m}_{;m} + (\mathcal{X}^A)^{;m}_{;m} = 0 \]  

(41)

where \( a = t,r \) and \( m = \chi,\theta,\phi \). We see that the second term does not contain derivatives.

It is interesting to compare these equations for the thread with the classical string equations

\[ \Box X^\mu = \left( \frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) X^\mu = 0 \]  

(42)

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The long flux tube solution of the 5D Kaluza–Klein theory with 
(electric charge) $\approx$ (magnetic charge) can be considered as a thread with a
Planckian cross section and classical longitudinal size attached to two Universes.
Such a picture is very close to Guendelman’s idea about the possibility of
connecting a region where some dimensions are compact to another region where
compactification does not exist.

where $X^n$ are the coordinates of the string and $\sigma, \tau$ are the parameters on the
sheet. We see that the difference is connected with the second term $(X^A)_{m}^{m}$.
This difference between the thread and the string equations has the origin in the
inner structure: the thread has a cross section $(\approx l_{Pl}^2)$ but the string is a pure
1-dimensional object. Nevertheless, the gravitational objects on the thread can
have propagating solutions.

## 5 Discussion

Thus 5D Kaluza-Klein theory has a very interesting solutions which can be
considered as a thread between two Universes (see, Fig. (5)). The cross section
of such objects can be very small $\approx l_{Pl}$, but the longitudinal length can take any
value. This remark gives us grounds to say that we have an unique object: in
the transverse direction it is a quantum object, but in the longitudinal direction
a classical one. It gives us the unique possibility to investigate quantum gravity
only in one dimension. For example, a 1-dimensional spacetime foam on the
thread and so on.

Our thread presented on Fig. (5) looks like a string attached to two $D$-branes
with the ends acting as sources of electric/magnetic charges in the external
Universes. Nevertheless there is the difference: strings are located in some
external spacetime but our threads do not need such space. Nevertheless, it is
well known that any space with Lorentzian metric can be embedded into some
multidimensional spacetime.
Our investigation of the small perturbations propagating on the thread show us that the corresponding equations for the gravitational waves are sufficiently close to the classical string equations. Physically this difference appears because the thread has an inner transversal structure whereas the string is a pure 1-dimensional object.

Finally, we see in Kaluza - Klein gravity can exist 1-dimensional objects and the corresponding equations for the gravitational waves propagating on these objects are similar to the corresponding equations. Our investigation shows us that these 1-dimensional objects (threads in our notations) have an inner structure in the spirit of Einstein’s idea that matter should be constructed from “Nothing”.

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