Cosmic String Current Stability

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Abstract

The stability of fermionic charge carriers on cosmic strings is considered. We show that neutral fermion currents in cosmic strings are always chiral or time-like, in contrast to the case of bosonic currents. The spectrum of bound states on an abelian SO(10) string is determined both before and after the electroweak phase transition. We determine the mass acquired by the zero mode at this transition. A range of charge carrier scattering processes are considered and corresponding decay rates calculated. Couplings between the carriers and the electroweak sector generate scattering from the plasma which can thermalise some currents. If the zero mode is isolated from the electroweak sector, then it survives. Current–current scattering is considered, but found to be unimportant in realistic settings where the string density is low.

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I. INTRODUCTION

Although much of the evolution of the Universe is well understood, there are still many cosmological phenomena for which a completely satisfactory explanation has yet to be found. Topological defects, such as cosmic strings, could provide mechanisms for structure formation, CMB anisotropy, and high energy cosmic rays \[1\]. Such defects form in many realistic unified theories.

In the past it has been difficult to evaluate the usefulness of such ideas due to a lack of data. This is now changing, and predictions of CMB anisotropies from simple cosmic string models have been made \[2\]. While these predictions show poor agreement with the observations, they do not take into account the full physics of strings models. Indeed, recent analysis which includes the effect of particle production as an energy loss mechanism from the string network show much improved agreement with data \[3\]. One possibility is that the strings carry conserved currents \[4\]. These currents will alter the evolution of a string network, which could lead to better agreement with observation. Indeed, an analytic analysis showed that a much denser string network results for electromagnetically coupled strings \[3\]. However, one significant implication of conserved currents is that they can stabilise loops of string. If persistent, these stabilised loops or ‘vortons’ \[6,7\], can easily dominate the energy density of the Universe, placing stringent constraints on the parameters of the model. An analysis has been made of the implications of this for particle physics models predicting current carrying strings \[8\].

Fermions are a natural choice for the charge carriers of such currents. Fermion zero modes exist in a wide class of cosmic string models. The fermions can be excited and move along the string, thus resulting in a current. Consequently, fermion conductivity occurs naturally in many supersymmetric and grand unified theories, such as SO(10). Most attention has been given to massless, chiral currents, as these are most likely to be stable. In this paper space- and time-like currents are also considered. These naturally occur in the bosonic models of superconducting strings \[9\]. In section II we show that space-like fermion bound states do not exist in the string core in any cosmic string model. We contrast this with the situation for bose current carriers. In section III the spectrum of time-like currents is investigated for an abelian string model. We determine the complete spectrum of fermion modes, including both zero modes and massive bound states in our analysis. Whilst traditionally, zero modes are considered for current carriers, low-lying bound states can also carry currents on strings. This analysis is applied to the SO(10) model. The effect of the electroweak phase transition on fermion currents is investigated in section IV.

One criticism of fermion currents is that, unlike scalar boson currents, they are not topologically stable. It is possible that they could decay, either directly into particles off the string, or through interactions with the surrounding plasma. If the decay rate is too high, currents will not last long enough to have any significant effect. On the other hand if the decay rate is too low the Universe could become vorton dominated. Such decays are examined in section V. We first examine the effect of plasma scattering on the string current. We show that this process can remove current carriers close to the phase transition, but not otherwise. We also consider the effect of current–current scattering and apply our analysis to a network of strings shortly after the phase transition. We show that this also has a negligible effect on the current stability. The results are summarised in section VI.
II. POSITIVITY CONDITION

In this section we consider the possible fermionic currents that can be carried by a string. Consider the following fermionic Lagrangian for a set of fermions $\psi_i$:

$$\mathcal{L} = \overline{\psi}_i i \not{D} \psi_i + [\overline{\psi}_i \phi_j m^D_{ijk} \psi_k + (\text{h. c.})] + [\overline{\psi}_i \phi_j m^M_{ijk} \psi_k + (\text{h. c.})] .$$  \hspace{1cm} (2.1)

Here, $\psi^c_i = \overline{C} \psi_i^T$, and $C$ is the charge conjugation matrix.

For shorthand we write, $M^D_{ik} = \phi_j m^D_{ijk}$ and $M^M_{ik} = \phi_j m^M_{ijk}$. The Dirac equation is then,

$$\gamma^0 i \not{D} \psi_i + [\gamma^0 M^D_{ik} + M^D_{ik} \gamma^0] \psi_k + [\gamma^0 M^M_{ik} - \gamma^0 C^T M^M C^{-1}] \psi^c_k = 0 .$$  \hspace{1cm} (2.2)

For concreteness we work with the Dirac representation of the gamma matrices. In this basis $\gamma^0$, $\gamma^3$ and $C = i \gamma^2 \gamma^0$ are real. Combining the Dirac equation and its charge conjugate into a single matrix equation we find,

$$[\hat{H}_0 + \gamma^0 \gamma^3 \hat{P}_3 + \hat{H}_m] \begin{pmatrix} \psi_i \\ \psi^c_i \end{pmatrix} = 0 ,$$  \hspace{1cm} (2.3)

where

$$\hat{H}_0 = \begin{pmatrix} iD_0 & 0 \\ 0 & iD_0^* \end{pmatrix} , \hspace{1cm} \hat{P}_3 = \begin{pmatrix} iD_3 & 0 \\ 0 & iD_3^* \end{pmatrix} ,$$

$$\hat{H}_m = \begin{pmatrix} i\gamma^0 \gamma^a D_a + \gamma^0 m^D_{ik} + M^D_{ik} \gamma^0 & \gamma^0 m^M_{ik} + \gamma^0 C^T M^M C^{-1} \\ M^M_{ik} \gamma^0 + C M^D_{ik} C \gamma^0 & i\gamma^0 \gamma^a D_a^* + \gamma^0 C^T M^D C + C M^M C \gamma^0 \end{pmatrix} .$$  \hspace{1cm} (2.4)

The index $a$ runs over the values 1 and 2. The operators $\hat{H}_0$ and $\hat{P}_3$ are clearly hermitian. If we restrict the mass matrices so that the Lagrangian contains only Lorentz scalar or pseudoscalar terms then $\hat{H}_m$ is also hermitian, and $\{\gamma^0 \gamma^3, \hat{H}_m\} = 0$.

Let a state $|\psi\rangle$ now represent the vector $(\psi_i, \psi^c_i)^T$. For a simultaneous eigenstate of $\hat{H}_0$ and $\hat{P}_3$ we have,

$$\langle \psi | \hat{H}_0 \hat{H}_0 | \psi \rangle = \langle \psi | (\gamma^0 \gamma^3 \hat{P}_3 + \hat{H}_m)(\gamma^0 \gamma^3 \hat{P}_3 + \hat{H}_m) | \psi \rangle ,$$  \hspace{1cm} (2.5)

or

$$w^2 = k_3^2 + \langle \psi | \{\gamma^0 \gamma^3 \hat{P}_3, \hat{H}_m\} | \psi \rangle + \langle \psi | \hat{H}_m \hat{H}_m | \psi \rangle .$$  \hspace{1cm} (2.6)

Now

$$\{\gamma^0 \gamma^3 \hat{P}_3, \hat{H}_m\} = [\hat{P}_3, \gamma^0 \gamma^3 \hat{H}_m] + \{\gamma^0 \gamma^3, \hat{H}_m\} \hat{P}_3 .$$  \hspace{1cm} (2.7)

For the mass matrices we are considering, the second term vanishes. Since $|\psi\rangle$ is an eigenstate of $\hat{P}_3$,

$$\langle \psi | [\hat{P}_3, \gamma^0 \gamma^3 \hat{H}_m] | \psi \rangle = \langle \psi | [k_3, \gamma^0 \gamma^3 \hat{H}_m] | \psi \rangle = 0 .$$  \hspace{1cm} (2.8)
Thus the anticommutator term vanishes and we have
\[ w^2 = k_3^2 + \langle \psi | \hat{H}_m \hat{H}_m | \psi \rangle . \] (2.9)

As we have seen, \( \hat{H}_m \) is hermitian, thus the expectation value is a weighted sum of the squares of real eigenvalues, i.e. it is positive definite:
\[ w^2 \geq k_3^2 . \] (2.10)

Thus for scalar and pseudoscalar mass terms, we see that there are no space-like fermions on the string and the total energy-momentum of the charge carriers is null or time-like. We can interpret this result physically as guaranteeing the stability of the string against the spontaneous formation of a fermionic condensate. This contrasts sharply with the bosonic case [9], where the bare string is unstable to the spontaneous formation of a bosonic condensate. This not only allows the formation of a uniform, static condensate, but also space-like excitations, i.e. static condensates whose phases wind along the string. The presence of such states relies on two features of the bosonic model which are not present in the fermionic case; the non-linearity of the equations of motion and the effective scalar mass squared being negative in the string core.

Of course if there are oppositely charged fermion carriers moving in different directions along the string, the charge current can be space-like, but the energy-momentum is still null or time-like.

III. BOUND STATES IN THE ABELIAN STRING MODEL

Having shown that only light- and time-like fermions exist on cosmic strings, we will now determine the complete spectrum of currents in a simple U(1) string model. A model with a suitable symmetry breaking has the Lagrangian
\[ \mathcal{L} = (D_\mu \Phi_S)^\dagger D^\mu \Phi_S - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{4} (|\Phi_S|^2 - \eta^2)^2 \] (3.1)
where \( D_\mu \Phi_S = (\partial_\mu - ieA_\mu) \Phi_S \) and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). This is just the abelian Higgs model. In the usual vacuum solution the mass of \( \Phi_S \) is \( M_S = \sqrt{\lambda \eta} \), and the gauge field \( A_\mu \) has mass \( M_V = \sqrt{2e\eta} \). The vacuum is topologically non-trivial, allowing cosmic string solutions to exist. To simplify the analysis, we consider strings with winding number 1. In this case \( \langle A_\mu \rangle = \delta_\mu^0 a(\rho)/er \) and \( \langle \Phi_S \rangle = \eta f(\rho)e^{i\theta} \), where \( \rho = M_S r \). The boundary conditions of the two functions are \( f(0) = a(0) = 0 \) and \( f(\infty) = a(\infty) = 1 \). The radius of the string is of order \( M_S^{-1} \).

The model can be extended by adding a Weyl fermion field, \( \Psi \). The extra fermionic part of the Lagrangian is then
\[ \mathcal{L}_{\text{fermions}} = \Psi^\dagger i\sigma^\mu D_\mu \Psi - \frac{1}{2} ig_\psi \Psi^\dagger \Phi_S \Psi + \frac{1}{2} ig^*_\psi \Psi^c i\Phi_S^* \Psi , \] (3.2)
where \( \sigma^\mu = (I, \sigma^i) \), \( \Psi^c = i\sigma^2 \Psi^* \) and \( D_\mu \Psi = (\partial_\mu - \frac{1}{2}ieA_\mu)\Psi \).
The variation of $\Phi_S$ means that the cosmic string acts as a potential well for the fermions. As well as the usual continuum states, there will be additional fermion states which exist only on the string. These are the fermion zero modes and massive bound states. The fermion field $\hat{\Psi}$ can be expressed in terms of momentum eigenstates. The bound states have only $z$- and angular momentum ($k_z, n$) since they are restricted to the string core. Apart from the massive bound states there are also massless chiral solutions. These have $n = 0$ and, as a consequence of their chirality, only travel in one direction along the string.

$$\hat{\Psi} = \sum_{k_z, n} \left( \hat{c}_{ik} U_{ik} e^{-iwt} + \hat{c}^\dagger_{ik} U^*_{ik} e^{iwt} \right) + \sum_{k_z > 0} \left( \hat{c}_{0k} U_{0k} e^{-iwt} + \hat{c}^\dagger_{0k} U^*_{0k} e^{iwt} \right) + \sum_k \text{(continuum states)} . \quad (3.3)$$

The index $i$ runs over the different masses of the bound states. It can be seen that the massless chiral states ($U_{0k}$) are real. The field equations for the spinors $U$ and $\tilde{U}$ can be found by varying (3.2) with respect to $\Psi^\dagger$. The wavefunctions are then found to have the form,

$$U_{ik} = \frac{\tilde{M}}{\sqrt{2\pi L}} \left( \frac{\sqrt{w + k_z} \chi_1(\rho)}{i \sqrt{w - k_z} \chi_2(\rho) e^{i\theta}} \right) e^{i(k_z + n \theta)}$$

$$\tilde{U}_{ik} = \frac{M}{\sqrt{2\pi L}} \left( \frac{\sqrt{w + k_z} \chi_3(\rho)}{-i \sqrt{w - k_z} \chi_4(\rho) e^{i\theta}} \right) e^{-i(k_z + n \theta)} \quad (3.4)$$

$$U_{0k} \propto \frac{\tilde{M}}{\sqrt{\pi L}} \left( \frac{1}{0} \right) e^{-\int \frac{M_f + \frac{a^2}{2 \rho} d\rho}{M_S} e^{ik_z z}} , \quad (3.5)$$

where $L$ is the length of the string, $\tilde{M}^{-1}$ is the effective radius of the wavefunctions and $M = |g_\psi| \eta \sim M_S$ is the fermion mass away from the string. The spinors are normalised so that $\int d^3x (|U_{ik}|^2 + |	ilde{U}_{ik}|^2) = 1$ and $\int d^3x |U_{0k}|^2 = 1/2$.

The allowed masses of the bound states ($M_B$) can be determined by finding normalisable solutions for the equations satisfied by the functions $\chi_i$. In a massive bound state’s rest frame these are

$$M_S \left( \partial_\rho - \frac{n}{\rho} + \frac{a}{2 \rho} \right) \chi_1 + M_B \chi_2 + M_f \chi_3 = 0 \quad (3.6)$$

$$M_S \left( \partial_\rho + \frac{n + 1}{\rho} - \frac{a}{2 \rho} \right) \chi_2 - M_B \chi_1 - M_f \chi_4 = 0 \quad (3.7)$$

$$M_S \left( \partial_\rho + \frac{n}{\rho} + \frac{a}{2 \rho} \right) \chi_3 + M_B \chi_4 + M_f \chi_1 = 0 \quad (3.8)$$

$$M_S \left( \partial_\rho - \frac{n - 1}{\rho} - \frac{a}{2 \rho} \right) \chi_4 - M_B \chi_3 - M_f \chi_2 = 0 \quad (3.9)$$

Of course $n$ must be an integer for the solution to be single valued and hence physical. Examining the above equations we see that given one solution, another can be obtained by putting $n \to -n$, $\chi_1 \leftrightarrow \chi_3$ and $\chi_2 \leftrightarrow \chi_4$. Thus the solutions will occur in pairs.
FIG. 1. Spectrum of $n = 0, \pm 1$ massive fermion bound states.

We will use a variation of the shooting method to determine the allowed values of $M_B$. At large $\rho$ the solutions of (3.6)–(3.9) have exponential behaviour. Two of them decay and so are acceptable. In the case of the small $\rho$ solutions, only two of them give a normalisable state.

Each of these 4 solutions can be numerically extended to some intermediate value of $\rho$ (of order 1). It can then be seen if any non-trivial combinations of the large and small $\rho$ solutions match up there. We find that the values of $M_B$ for which normalisable solutions exist satisfy

$$M^2 - M_B^2 = (M + \alpha_{ni} M_S)^2 > 0 \quad (3.10)$$

where the $\alpha_{ni}$ are functions of $M/M_S$. Figure 1 shows the variation of $\alpha_{ni}$ and the number of bound states with respect to the fermion to Higgs mass ratio ($M/M_S$). Each line corresponds to two bound states. For simplicity we have taken the Higgs and gauge field masses ($M_S$ and $M_V$) to be equal. With this choice of parameters all solutions have either $\chi_2$ or $\chi_4$ identically zero. Equations (3.6)–(3.9) can then be reduced to a second order problem. The solutions with $\chi_4 = 0$ are shown in the figures. It can be seen from figure 1 that the number of bound states increases with the size of the off-string fermion mass. For small values of $M$ there are just two massive bound states in addition to the chiral zero mode.

We find that in the region of parameter space in which $\alpha_{ni}$ is roughly constant, $\tilde{M}^2 \approx M_S \sqrt{M^2 - M_B^2}/2$. For the massless states, when $M_S = M_V$, this expression for $\tilde{M}$ is exact and can be proved analytically. In this case the two sides of (3.3) are actually equal.
Plots of the solutions for $M/M_S = 2$ are shown in figure 2. As expected they decay outside the string. We have thus found the full spectrum of fermion bound states for this model.

IV. CURRENTS AFTER THE ELECTROWEAK PHASE TRANSITION

The toy model of the previous section can be embedded in a phenomenologically credible grand unified theory, such as SO(10). Two suitable symmetry breakings which can give rise to cosmic strings are

$$\text{SO}(10) \rightarrow \Phi_S \rightarrow \text{SU}(5) \times \text{Z}_2 \rightarrow \mathcal{G}_{SM} \times \text{Z}_2 \rightarrow \text{SU}(3)_c \times \text{U}(1)_Q \times \text{Z}_2 . \quad (4.1)$$

$$\text{SO}(10) \rightarrow \text{SU}(5) \times \text{U}(1) \rightarrow \mathcal{G}_{SM} \times \text{U}(1) \rightarrow \Phi_S \rightarrow \mathcal{G}_{SM} \times \text{Z}_2 \rightarrow \text{SU}(3)_c \times \text{U}(1)_Q \times \text{Z}_2 \quad (4.2)$$

with $\mathcal{G}_{SM} = \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y$. The discrete $\text{Z}_2$ symmetry allows the formation of topologically stable cosmic strings. In this case the string gauge field is a neutral GUT boson, and $\Phi_S$ transforms under the 126 representation of SO(10). The electroweak Higgs field $\Phi$ transforms under the 10 representation.

The fermion sector of the SO(10) GUT contains all the Standard Model fermions, and an extra right-handed neutrino ($\nu^c$) for each family. Only right-handed neutrinos couple to $\langle \Phi_S \rangle$, while neutrinos of either helicity couple to $\langle \Phi \rangle$.

The abelian string’s gauge field has a non-trivial effect on the electroweak Higgs field. The components of $\Phi$ have charges $\pm 1/5$ with respect to the generator of the GUT string gauge field, so $\langle \Phi \rangle$ will not be constant in the presence of a string. A non-zero $Z$ field is also required to give a finite energy solution $[10]$.

The $\tau$-neutrino mass terms in a string background are

$$\left( \begin{array}{c} \nu^c \\ \nu \end{array} \right) \left( \begin{array}{cc} M_f(e^{i\theta}) & m_{\text{ew}} h(\rho) \\ m_{\text{ew}} h(\rho) & 0 \end{array} \right) \left( \begin{array}{c} \nu^c \\ \nu \end{array} \right)^c . \quad (4.3)$$
\( m_{\text{ew}} \sim 10^2 \text{GeV} \) is the electroweak energy scale and \( M \sim 10^{16} \text{GeV} \) is the GUT energy scale. For simplicity we will neglect mixing between different families. The function \( h \) gives the radial dependence of the component of \( \langle \Phi \rangle \) which couples to the neutrinos. The electroweak gauge field has \( \langle Z_\rho \rangle = -b(\rho)/(\sqrt{40}\epsilon c) \). The boundary conditions of \( h \) and \( b \) are \( b(0) = 0 \) and \( h(\infty) = b(\infty) = 1 \). Inside the string \( h \approx (m_{\text{ew}}/M)c = \text{constant}, \) with \( c \sim 10^4 \) \[11\]. While the GUT fields take their vacuum expectation values outside the string (whose radius is of order \( M^{-1} \)), the electroweak fields vary over a far larger region with radius of order \( m_{\text{ew}}^{-1} \).

Since \( \epsilon = m_{\text{ew}}/M \ll 1 \) the neutrino mass eigenvalues outside the string (or in a vacuum) are approximately \( M \) and \( m_{\text{ew}} = m^2_{\text{ew}}/M \). The corresponding mass eigenstates are then approximately \( \nu^c + \epsilon \nu \) and \( \nu - \epsilon \nu^c \). This illustrates the seesaw mechanism \[12\]. Although the \( \tau \)-neutrinos have the same couplings to the electroweak Higgs field as the top quark, the GUT Higgs ensures that \( \nu^c \) is superheavy and \( \nu \) is very light, as is required to agree with observation. Recent measurements have suggested that \( \nu \) does indeed have a small mass \[13\].

Once \( \Phi \) gains a non-zero expectation value, the \( \nu^c \) currents are no longer solutions of the field equations. We will start by considering the fate of the massless \( \nu^c \) currents. It seems likely that they mix with a \( \nu \) state to give a low mass bound state. We denote the resulting mass of the bound state by \( \mu M \), where the dimensionless parameter \( \mu \) is expected to be very small. Using the ansatz

\[
\nu^c = \begin{pmatrix} \chi_1(\rho) \cos(\mu Mt) \\ \chi_2(\rho) e^{i\theta} \sin(\mu Mt) \end{pmatrix}, \quad \nu = \begin{pmatrix} \zeta_1(\rho) e^{-i\theta} \cos(\mu Mt) \\ \zeta_2(\rho) \sin(\mu Mt) \end{pmatrix}
\]

we can look for solutions which reduce to the zero mode solutions of (3.6)–(3.9) when \( \langle \Phi \rangle = 0 \). Putting \( M_S = M \) for simplicity, the field equations reduce to

\[
\left( \partial_\rho + \frac{a}{2\rho} + f \right) \chi_1 + \mu \chi_2 + \epsilon h \zeta_1 = 0 \quad (4.5)
\]

\[
\left( \partial_\rho + \frac{2 - a}{2\rho} - f \right) \chi_2 - \mu \chi_1 - \epsilon h \zeta_2 = 0 \quad (4.6)
\]

\[
\left( \partial_\rho + \frac{10 - 2b - 3a}{10\rho} \right) \zeta_1 + \mu \zeta_2 + \epsilon h \chi_1 = 0 \quad (4.7)
\]

\[
\left( \partial_\rho + \frac{2b + 3a}{10\rho} \right) \zeta_2 - \mu \zeta_1 - \epsilon h \chi_2 = 0 \quad (4.8)
\]

As with the abelian string, the existence of bound states can be investigated by examining the large and small \( \rho \) solutions of (4.5)–(4.8), and then trying to match them at intermediate \( \rho \). There are two well behaved solutions at small \( \rho \) and two at large \( \rho \) if \( \mu < \epsilon^2 \).

We will attempt to do this by finding approximate solutions in a simple ‘top hat’ approximation of the string background. This has \( f(\rho) = a(\rho) = \Theta(\rho), b(\rho) = \Theta(\epsilon \rho) \) and \( h(\rho) = \epsilon c + (1 - \epsilon c)\Theta(\epsilon \rho) \), where \( \Theta \) is the Heaviside step function. It is then possible to find analytic solutions for \( \rho > \epsilon^{-1} \). For smaller \( \rho \) the solutions can be expanded in terms of the tiny parameter \( \epsilon \).

Requiring the approximate solutions to match up at \( \rho = 1 \) and \( \rho = \epsilon^{-1} \) will give an expression for \( \mu \). It is satisfied by \( \mu = \sqrt{2} \gamma c \epsilon^{16/5} \) with \( \gamma = e^{\Gamma(6/5,1)} - 1/2 \approx 1.6 \). \( \Gamma \) is the...
incomplete gamma function. To leading order in $\epsilon$, the corresponding approximate solution is

\[
\begin{align*}
\chi_1 &= 1 \\
\chi_2 &= \frac{c}{\sqrt{2}}(\gamma - 1)c^{16/5}/\rho \\
\zeta_1 &= -\frac{c^2}{2}\epsilon^2/\rho \\
\zeta_2 &= -\sqrt{2}\epsilon^{6/5}
\end{align*}
\]

for $\rho < 1$,

\[
\begin{align*}
\chi_1 &= \frac{e^{1-\rho}}{\sqrt{\rho}} + \epsilon^4 c\gamma \left(\frac{5}{4} iF_1\left(\frac{4}{5}, \frac{9}{5}; \rho\right) e^{-\rho/\rho^{3/10}} - 2\epsilon^{6/5} e^{-1/\epsilon}/\sqrt{\rho}\right) \\
\chi_2 &= \sqrt{2}c e^{\rho/\sqrt{\rho}} \left(\sqrt{\rho} - \frac{\gamma e^{1-\rho}}{2}\epsilon^{6/5} + \sqrt{2}\epsilon^2 e^{-1/\epsilon}/\sqrt{\rho}\right) \\
\zeta_1 &= \epsilon^2 \left(e^\Gamma\left(\frac{6}{5}, \rho\right) - \gamma\right) \rho^{-7/10} \\
\zeta_2 &= -2\epsilon^{6/5} \rho^{-3/10}
\end{align*}
\]

for $1 < \rho < \epsilon^{-1}$, and

\[
\begin{align*}
\left(\zeta_1 - \epsilon\chi_1\right) \left(\zeta_2 - \epsilon\chi_2\right) &= -\left(\frac{\gamma c\epsilon^{6/5}}{\sqrt{2}}\right) \epsilon^{6/5} e^{-c^2/\rho} / \sqrt{\rho} \\
\left(\chi_1 + \epsilon\zeta_1\right) \left(\chi_2 + \epsilon\zeta_2\right) &= O(\epsilon^{-1/5}) \frac{e^{-\rho+1/\epsilon}}{\sqrt{\rho}}
\end{align*}
\]

for $\rho > \epsilon^{-1}$. $iF_1$ is a confluent hypergeometric function.

The mass of the new bound state is $M_\mu \sim 10^{-16}$ eV. Unlike the original zero mode, it is a mixture of left- and right-handed neutrinos. Similarly, the massless $\nu^c_\mu$ and $\nu^c_\tau$ currents will also gain tiny masses at the electroweak phase transition. Since these states now couple to light off-string states, they can escape from curved strings [6]. Applying the arguments in ref. [6] the rate of decay of states with energy of order $M_S$ is calculated to give a lifetime of order $10^{-40}$ s. This would prevent the Universe becoming vorton dominated. In addition the electroweak bound states are spread over a far greater region than the original $\nu^c$ current, so interactions with other fields will also be increased. If any massive bound states survive the phase transition they will also rapidly decay into left-handed neutrinos, although it seems more likely that they will simply be changed into free states by the transition.

The above mechanism will work in other theories where zero modes mix with free massless particles. If there is mixing between two massive species of right and left moving currents, zero modes can mix to give a low mass bound state. However, as there is no low mass off-string state these cannot escape. Such theories always feature zero modes moving in opposite directions. These can interact with each other, which reduces the current. It is still possible for a net current to persist, and so fermion currents may provide constraints on these models after all.

If a zero mode is isolated from the electroweak sector, then it persists. Consequently the resulting currents also persist. In particular, zero modes in grand unified theories or
FIG. 3. Feynman diagrams contributing to plasma scattering (a,b), massive bound state decay (c) and vorton decay (d).

supersymmetric theories where there is no coupling between the zero mode and the electroweak sector, result in currents which are not destroyed. Generally, such currents give rise to stable vortons, which constrain the underlying particle physics theory [8]. This is likely to be particularly true in D-type supersymmetric theories.

V. DECAY RATES OF CHARGE CARRIERS

Having established the spectrum of fermion states in an abelian string background, we now turn our attention to the stability of charge carriers on these strings. In the absence of other particles, currents carried by zero modes on isolated, straight strings are stable on grounds of energy and momentum conservation. However, in a realistic setting there are many processes which can depopulate zero modes. Strings are not isolated and in the early Universe they and their bound states will interact with the hot plasma. Also, bound states on different strings, or different parts of a single curved string, can scatter from one another. If there are couplings within the theory that allow a heavy neutrino zero mode to scatter from a light plasma particle to produce a light fermion-antifermion pair via an intermediate electroweak Higgs boson, we can have charge carrier decay from the Feynman diagrams in figures 3a and 3b. Also of interest is the decay of massive bound modes into light fermions (see figure 3c). In theories which allow such interactions, the charge carriers can decay. It is also possible for heavy neutrino currents on different strings to decay by exchanging a Higgs particle, as in figure 3d. The aim of this section is to calculate these decay rates.

The details of the calculations are given in the appendices, here we highlight the important differences between the calculation in the background of the string and the corresponding trivial background case. We initially consider our system to be restricted to some box of finite volume, $V$. Following the canonical procedure we decompose the field operators into a sum over orthonormal wavefunctions and creation/annihilation operators. Throughout we normalise our wavefunctions according to,

$$
\int d^3x \phi_k^\dagger(x)\phi_{k'}(x) = \delta_{kk'} .
$$

We take the following decomposition for scalar fields,

$$
\hat{\Phi}(t, x) = \sum_k \frac{1}{\sqrt{2w}}(\hat{a}_k \phi_k(x)e^{-iwt} + \hat{b}_k^\dagger \phi_{-k}(x)e^{iwt}) ,
$$
where $w^2 = k \cdot k + m_{\phi,k}^2$. The necessary commutation relations are, $[\hat{a}_k, \hat{a}_k^\dagger] = [\hat{b}_k, \hat{b}_k^\dagger] = \delta_{kk'}$.

The corresponding decomposition of fermionic fields is,

$$\hat{\Psi}(t, \mathbf{x}) = \sum_k (\hat{c}_k U_k(\mathbf{x}) e^{-iwt} + \hat{d}_k^\dagger V_k(\mathbf{x}) e^{iwt}) ,$$

where the states $U_k$ and $V_k$ are spinor valued and the sum is over momentum states. In this case we impose anticommutation relations $\{\hat{c}_k, \hat{c}_k^\dagger\} = \{\hat{d}_k, \hat{d}_k^\dagger\} = \delta_{kk'}$.

With these normalisations we have a simple interpretation of the amplitude,

$$\mathcal{A} = \langle 1, 2 | S | 3, 4 \rangle .$$

The probability of the interaction, characterised by $S$, converting the initial state, $|1, 2\rangle$, into the final state, $|3, 4\rangle$, is simply,

$$P = |\mathcal{A}|^2 .$$

We can now consider a simple, second order tree diagram in $\Phi \Psi^\dagger \Psi^c$ theory, where the interaction is given by,

$$\mathcal{H}_{\text{int}} = ig \Phi \Psi_1^\dagger i\sigma^2 \Psi_2^c + ig^* \Phi^* \Psi_3^\dagger i\sigma^2 \Psi_4^c + (\text{h. c.}) .$$

The amplitude is given by,

$$\mathcal{A} = \langle \text{in} | T |g|^2 i^2 \int d^4 x \Psi_1^\dagger(x) i\sigma^2 \Psi_2^c(x) \Phi_I(x) \int d^4 y \Psi_3^\dagger(y) i\sigma^2 \Psi_4^c(y) \Phi_I^*(y) |\text{out}\rangle ,$$

where incoming and outgoing states have the form,

$$\langle \text{in} | = \langle 0 | \hat{c}_1 \hat{c}_2 , \quad |\text{out}\rangle = \hat{c}_4^\dagger \hat{c}_3^\dagger |0\rangle .$$

Expanding the field operators in the usual way we obtain,

$$\mathcal{A} = |g|^2 \sum_k \int d^4 x \int d^4 x' U_1^\dagger(\mathbf{x}) i\sigma^2 U_2^c(\mathbf{x}) e^{i(w_1 + w_2)t}$$

$$\times \langle 0 | T \Phi_I(x) \Phi_I(y) |0\rangle U_3^T(\mathbf{x'}) i\sigma^2 U_4(\mathbf{x'}) e^{-i(w_3 + w_4)t'} .$$

Expressing the Green’s function as a sum over a complete set of states, we have,

$$\mathcal{A} = |g|^2 \sum_k \int d^4 x \int d^4 x' U_1^\dagger(\mathbf{x}) i\sigma^2 U_2^c(\mathbf{x}) e^{i(w_1 + w_2 - w_I)t}$$

$$\times \frac{\phi_k(\mathbf{x}) \phi_k^*(\mathbf{x'})}{T[w_I^2 - k_I^2 - m_{\phi,k}^2]} U_3^T(\mathbf{x'}) i\sigma^2 U_4(\mathbf{x'}) e^{-i(w_3 + w_4 - w_I)t'} ,$$

where $T$ is the temporal extent of the region we are considering and $m_{\phi,k}$ is the (constant) effective mass of the scalar mode.

It is useful to consider briefly how this calculation would proceed in a trivial background. In a space-time box of volume $V T$, we have,
\[ \sum_k \rightarrow \frac{V T}{(2\pi)^4} \int d^4k_I . \] (5.11)

Each wavefunction introduces a factor of $V^{-1/2}$ to the amplitude. The spatial integrations yield finite volume approximations to energy and momentum conserving delta functions which we denote $\delta^4_{T V}$, where $\int e^{ik \cdot x} d^4x = (2\pi)^4 \delta^4_{T V}(k)$ and $\delta^4_{T V}(0) = TV(2\pi)^{-4}$.

The value of the propagator is fixed by energy/momentum conservation and, up to dimensionless factors, we have,

\[ A \sim \frac{|g|^2}{V^2} \frac{1}{k^2_I - m^2} \delta^4_{T V}(k_1 + k_2 - k_3 - k_4) , \] (5.12)

where $k_I = k_1 + k_2 = k_3 + k_4$.

The interaction cross section is given by,

\[ \sigma = \sum_{\text{final states}} \frac{V|A|^2}{T v_{\text{rel}}} . \] (5.13)

Replacing this sum by integrals over the final state momenta and including all the spinor factors, we find,

\[ \sigma = \frac{|g|^4}{16\pi} \frac{k^2_I}{(k^2_I - m^2)^2} . \] (5.14)

Now we can consider a situation of interest. If the incoming particles are a zero mass particle of energy $w_{zm}$ and a light plasma particle of energy of order the temperature, then $k^2_I \sim w_{zm}T$ and,

\[ \sigma \sim \frac{|g|^4 w_{zm}T}{(w_{zm}T - m^2)^2} . \] (5.15)

Having discussed the calculation in a trivial background, we can now consider the string background. The major difference arises in the spatial integration associated with the position of the initial vertex. Let particle 1 be the bound state,

\[ \psi_1(x) \sim e^{i w_{t-1} x} e^{-i n_{1} t} \frac{\tilde{M}}{\sqrt{L}} e^{-\tilde{M}r} , \] (5.16)

where $\tilde{M}^{-1}$ is the radius of the bound state. As this state is localised close to the string, it will only overlap significantly with the lowest angular mode components of the incoming scattering wave and so the integration over the position of the initial vertex will be dominated by these modes (and for the same reason the lowest modes of the propagator).

We consider the incoming particle to be a plane wave asymptotically and do a mode by mode matching of this incoming plane wave onto the states in the string background. As we only have square integrable wavefunctions in the string background, in general we must also include an outgoing scattered wave: this is the source of Aharonov-Bohm scattering.
The general features of both the incoming particle and intermediate particle wavefunctions are a growth region for $k_T r \lesssim 1$ and an oscillatory region for $k_T r \gtrsim 1$, where $k_T$ is the momentum of the particle transverse to the string $[14]$. The transverse momentum of the incoming particle will be much less than $\hat{M}$, so we expect some power law type behaviour in the region of overlap with the bound state. Similarly, if the transverse momentum of the intermediate particle, $k_{IT}$ is smaller than $\hat{M}$, its wavefunction will also be roughly a power law in the important overlap region. The integration over the initial vertex position then produces a number that is suppressed due to the limited extent of the bound state wavefunction, but is only mildly dependent on $k_{IT}$. Only when $k_{IT}$ exceeds $\hat{M}$ do we find a wavefunction oscillating in the overlap region. Thus only for $k_{IT} \gtrsim \hat{M}$ does the integral decrease. Instead of a momentum conserving delta function, this spatial integration yields something more like a step function, which permits transverse momentum non-conservation up to the mass scale $\hat{M}$. This behaviour is natural as the string breaks transverse translation invariance and the object bound to the string has an effective radius $\hat{M}^{-1}$.

The violation of momentum conservation in the string background opens up significant areas of phase space that are forbidden in the trivial background. Of particular importance is the possibility of resonant scattering. We consider first the decay of a charge carrier into a fermion and a Higgs boson. In the string background transverse momentum can be acquired from the string to place the Higgs particle on shell in a much larger region of phase space. This introduces a factor of $\hat{M}^2 M_B$. The small size of the region in which the wavefunctions overlap gives an extra factor of $\hat{M}^{-2}$. The full decay into three fermions can then be considered to be an initial decay into a light fermion and physical electroweak Higgs boson, followed by Higgs decay. In appendix $[\square]$ it is shown that the massive bound state lifetime is

$$\tau \sim (|g_\nu|^2 M_B)^{-1},$$

where $g_\nu$ is the Yukawa coupling in the neutrino electroweak mass term. Thus (5.17) results in a small lifetime.

In arriving at this lifetime we have made certain assumptions, in particular we have neglected the possibility that the incoming and outgoing fermionic wavefunctions may be amplified near the string $[14]$. We have also neglected back-reaction on the string. As a significant contribution to the amplitude comes from large transverse momentum non-conservation, the use of a static string background might not be a good approximation. However, if the string recoils it will absorb some of the energy of the interaction. This will reduce the amount of momentum non-conservation required to make the outgoing particles on-shell. Thus these approximations are not likely to increase the lifetime, and (5.17) should therefore be taken as an overestimate.

The fate of massless bound states is also complicated by momentum non-conservation. Of particular interest are the bound states that stabilise vortons. Contraction of the vorton will ensure that the states at the Fermi surface will have GUT scale momenta. The lifetime of these high momentum states is critical, if they decay the vorton will contract and promote low momentum states to high momentum. Energy and $z$-momentum conservation prevents the massless states decaying spontaneously, in contrast to the massive bound states. It is however possible for them to decay by interaction with plasma particles or other zero
If $M_S > M$ it is also possible for high energy massless currents on a curved string to decay by tunneling to free heavy neutrinos. However the rate will not be significant unless $M_S \gg M$ \cite{1}.

If we consider one of these states scattering from a typical plasma particle, we have a centre of mass energy of order $\sqrt{M_S T}$, well above the mass of the intermediate particle. Transverse momentum non-conservation again allows for resonant scattering. Including amplification of the incoming plasma particle wavefunction, the lifetime of these high momentum zero modes is found to be,

$$\tau \sim \frac{\tilde{M}^2}{|g_\nu|^2 T^3} \left( \frac{T}{M} \right)^{2Q},$$

where $Q$ is the charge of the plasma particle under the string gauge field. We have taken the plasma particles to be massless and ignored any temperature dependent corrections to their mass. Providing $\sqrt{w_{zm} T} > m_\phi \sim T$, resonant scattering is possible and the cross-section is largely independent of the zero mode energy, $w_{zm}$. The unamplified lifetime of the modes thus scales with the plasma density. Conversely, the amplification factor decreases with increasing temperature as the ratio of the GUT scale to the typical thermal energy grows.

In the radiation dominated era we can take $t = \alpha T^{-2}$ and the lifetime becomes,

$$\tau \sim \tilde{M}^{2-2Q} |g_\nu|^{-2} T^{2Q-3} = \tilde{M}^{2-2Q} |g_\nu|^{-2} \alpha^{Q-3/2} t_i^{Q-1/2 - Q}. \tag{5.19}$$

The probability of a zero mode state scattering in time interval $dt$ is $dt/\tau$, thus the probability of a zero mode state scattering after some time $t_i$ is,

$$P(\text{decay after } t_i) = 1 - e^{-\int_{t_i}^{\infty} \frac{dt}{\tau}} = 1 - \exp \left\{ -\tilde{M}^{2Q-2} |g_\nu|^{-2} \alpha^{3/2-Q} t_i^{Q-1/2} \right\}. \tag{5.20}$$

If the magnitude of the exponent is small there is a small probability, thus zero modes are stable if,

$$t_i > O \left( |\tilde{M}|^{2Q-2} |g_\nu|^2 \alpha^{3/2-Q} \right)^{2/(1-2Q)). \tag{5.21}$$

Now, $\alpha \sim M_{Pl}/10$ and in the case of SO(10), $Q = 3/10$, leading to the condition,

$$t_i > O \left( |\tilde{M}|^{-7} |g_\nu|^{10} \alpha^6 \right) > O \left( \frac{M_{Pl}}{10 M} \right)^6 |g_\nu|^{10} \tilde{M}^{-1}. \tag{5.22}$$

As the lifetime varies only slightly faster than $T^{-2}$, this result for $t_i$ is very sensitive to the Yukawa coupling. For $\tilde{M} \sim 10^{18}$GeV, if $|g_\nu| = 1$ zero mode states populated after $t_i \sim 10^{15} t_{GUT}$ will be stable, while if $|g_\nu| \lesssim 0.03$, this scattering is never significant. In the SO(10) model $g_\nu$ is also the Yukawa coupling for the corresponding quarks, thus there is an epoch when $\nu^c$ zero modes will scatter from the string, but $\nu^c$ and $\nu^\mu$ zero modes will never scatter by this process. Thus the interaction with plasma particles can not significantly remove zero modes from the string. Note that it is also possible to create currents using the above interactions in reverse. Hence, if thermal equilibrium is reached the number density of zero modes will be of order $T$. 

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Within the SO(10) model there is also the possibility of mediating these processes by GUT mass Higgs fields with zero VEV. In this case the Yukawa coupling need not be small, but the centre of mass energy of the interaction is only of order the intermediate particle mass for $T \sim T_{\text{GUT}}$. Thus below the GUT temperature the reaction rates for these processes are rapidly suppressed by powers of $T/T_{\text{GUT}}$.

None of these plasma scattering processes can remove $\nu_e$ and $\nu_\mu$ zero modes, and are only significant for $\nu_\tau$ immediately after the phase transition. Thus, they are unable to prevent the vorton density from dominating the energy density of the Universe.

The plasma scattering processes considered above failed to remove zero modes due to the decreasing plasma density at late times. A distinct category of process is the scattering of a zero mode on one string by a second zero mode on another string. This is particularly relevant for vortons as they form small loops with a typical radius only one or two orders of magnitude larger than the string width. We thus have zero modes moving in opposite directions on opposite sides of the vorton. For simplicity the decay rate can be calculated by considering two straight, anti-parallel strings (appendix D). Physically one expects a suppression due to the finite separation of the initial and final vertices. As the initial and final vertices are confined to different pieces of string, the amplitude contains a factor of $e^{ik_T R}$, where $k_T$ is the transverse momentum of the intermediate particle and $2R$ is the separation of the strings. When the integration over the intermediate momenta is performed, this factor can cause the integrand to oscillate, leading to a suppression of the amplitude. This occurs if $k_T R$ is large where the standard propagator factor is significant.

If the intermediate particle is an electroweak Higgs boson, then the standard propagator factor peaks for $k_T$ around the electroweak scale, $k_T R$ is tiny in the important region, there is no oscillation and no suppression. The cross-section in this case is found to be,

$$\sigma \sim \frac{|g_\nu|^4}{(M_{\text{GUT}} R)^4}.$$  (5.23)

This cross-section is dimensionless as the scattering is effectively in one spatial dimension. In this case resonant scattering is not possible, thus the Higgs width does not enter and the rate therefore contains a factor of $g_\nu^4$.

Conversely, for a GUT scale mass as the intermediate particle, the standard propagator factor peaks for $k_T \sim M_{\text{GUT}}$, giving $k_T R \sim 10 - 100$. In this regime the reaction rate is found to display the exponential suppression expected on physical grounds,

$$\sigma \sim \frac{|g_\nu|^4}{(M_{\text{GUT}} R)^4} e^{-4M_{\text{GUT}} R}.$$  (5.24)

The exponential suppression in (5.24) makes such processes irrelevant in all physical situations.

Taken at face value, if electroweak particles mediate current–current scattering on different segments of string then (5.23) gives a short lifetime for charge carriers on a vorton. However, there are complications in directly applying this result derived for straight strings to curved vortons. The presence of charge carriers on the vorton loops results in the loop carrying angular momentum, which must be conserved in any physical process under investigation. The above calculation does not take into account conservation of angular
momentum, as the system considered has no rotational symmetry. However, the angular momentum gives rise to a centrifugal barrier, suppressing the above decay channel. Combined with energy conservation, this would prevent massless modes on the string scattering into massive modes. However, the fermionic spectrum has not been calculated for a circular loop and there is no reason to expect the zero modes to remain massless and thus, angular momentum conservation can only be considered in a consistent rotationally invariant calculation. The calculation above works consistently with straight strings, thus while (5.23) may not be directly applicable to vorton decay, it is relevant for interactions on non-circular loops and scattering of currents on a string network.

At very early times in the friction dominated regime, the string correlation length is small. The interaction of (5.23) could be operative. However the string density drops too quickly for it to be significant. As the density of current carriers builds up, the inter-string separation increases, so the cross-section decreases significantly. Consequently, the current-current interaction is unable to reduce the number of current carriers on the string and prevent vorton formation.

VI. SUMMARY

In this paper the existence and form of fermion bound states and their corresponding currents on cosmic strings were investigated. Only time-like and light-like fermions can exist on the string, in contrast to the bosonic case. Using numerical methods, the discrete spectrum of states for an abelian string model with one fermion field was determined. We found that the number of bound states increased as the Yukawa coupling grew. For very low values there are just two massive bound states and a zero mode.

Bound mode states will always occur. The occupancy of these states depends on the decay modes of the carriers. Since the cosmic string breaks Lorentz invariance, transverse momentum is not conserved in interactions in a string background. This leads to an enlarged phase space and increased cross sections. On the other hand, the fact that the bound states are confined to the string reduces the overlap of the particle wavefunctions, tending to reduce the cross sections.

Unless the massive carriers are isolated from the electroweak sector, they will decay and the states will empty on a time scale of $10^{-33}$ s. If the carriers are stable, they will persist on the string and carry angular momentum, contributing to vorton formation.

In order for the massless current carriers to decay they must interact with other particles, such as carriers on other strings or plasma particles. The most significant decays involve electroweak Higgs intermediate states. Non-conservation of momentum means that plasma scattering is usually resonant. However the small size of the Yukawa couplings and the plasma density mean that the rate of this decay is too small to be cosmologically significant (at least for $\nu_{eR}$ and $\nu_{\mu R}$ current carriers). Decays involving GUT boson intermediates are also possible, although the momentum non-conservation is not large enough to make these decays resonant. The interaction rate in this case is also small.

Massless current carriers can also decay by scattering with currents on other strings. The cross section for this interaction is tiny unless the two strings are very close together. For GUT mass intermediate states appropriate densities are never physically realised, but for
electroweak intermediate states suitable densities arise immediately after string formation. A situation similar to this occurs when the current is on a string loop. In this case the current will decay rapidly. Unfortunately this result cannot be applied directly to circular loops, where angular momentum conservation must be taken into account. Consequently, it cannot be applied directly to the case of cosmic vortons. This situation requires more detailed analysis [15].

In the SO(10) model, after the electroweak phase transition, the situation changes dramatically. The right-handed neutrino zero modes mix with left-handed neutrinos allowing currents on curved strings to decay by tunneling into free left-handed neutrinos. This averts any cosmological disaster.

However, if the zero mode is isolated from the electroweak sector, then the decay processes considered in this paper are not operative. As a consequence, the zero mode survives. For GUT scale strings, this results in a cosmological disaster as discussed in [8].

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APPENDIX A: BASICS: PROPAGATORS, INTERACTIONS AND WAVEFUNCTIONS

1. The Scalar Propagator

We will begin by finding the propagator, $G(x, y)$, which satisfies,

$$\hat{O}G(x, y) = (\partial_t^2 - \nabla^2 + V(r)^2)G(x, y) = -i\delta^4(x - y),$$

(A1)

where $\hat{O}\Phi = 0$ is the equation of motion. $V(r)^2$ represents the position dependent potential that the scalar field experiences in the string background. The standard connection with the time ordered product holds here too;

$$\hat{O} \langle 0| T\Phi(x)\Phi^*(x') |0\rangle = -i\delta^4(x - y).$$

(A2)

We now have to find a suitable representation of this Green’s function. Consider the quantity,

$$G(x - x') = i \sum_k \frac{e^{-i\omega\tau}}{\tau[n^2 - k^2 - m_{\phi,k}^2]} \phi_k(x)\phi_k^*(x'),$$

(A3)

where $\tau = t - t'$ and the sum extends over states of all masses, i.e. we include off-shell states in the sum. We will work in a space-time box and so impose boundary conditions at $t = -\tau/2$ and $t = \tau/2$. The allowed energies are then quantised in just the same way as the momenta. Normalising these states to unity introduces a factor of $1/\sqrt{\tau}$ into
each wavefunction. We will display this factor explicitly so that only the $1/\sqrt{V}$ factors are implicit in any wavefunction.

The full wavefunctions, $\phi_k(x) = e^{-iwt} \phi_k(x)$, satisfy,

$$\hat{O}\phi_k(x) = [-w^2 + \nabla^2 + V(r)^2]e^{-iwt} \phi_k(x) = [-w^2 + k^2 + m_{\phi,k}^2]e^{-iwt} \phi_k(x),$$

(A4)

where $m_{\phi,k}$ is the (constant) effective mass of the scalar mode of momentum $k$.

To see that we have a Green’s function, act with the operator on $G$ and use the fact that the off shell states are a complete set of solutions of the 4-dimensional eigenvalue problem. We tame the singularity in the usual way with an $i\epsilon$ to produce the Feynman propagator.

2. The Interaction

Now consider the simple, second order tree diagram in a theory with

$$\mathcal{H}_{\text{int}} = ig\phi \Psi_1^\dagger i\sigma^2 \Psi_2^2 + ig^* \Phi^* \Psi_3^\dagger i\sigma^2 \Psi_4^4 + (\text{h. c.}) .$$

(A5)

We will start by considering incoming and outgoing states of the form

$$(\text{in}) = \langle \Psi \phi \rangle, \quad (\text{out}) = \phi \Psi \Psi$$

(A6)

Substituting in the expressions for the external fields (5.3) we arrive at the VEV of the time ordered product in the usual way,

$$A = |g|^2 \sum_k \int d^4x \int d^4x' U_1^\dagger(x)U_2^\dagger(x)e^{i(w_1+w_2-w_I)t} \times \frac{\phi_k(x)\phi_k^*(x')}{T[k_f^2 - m_{\phi,k}^2]} U_3^\dagger(x')U_4^\dagger(x')e^{-i(w_3+w_4-w_I)t'}.$$  

(A7)

3. The Amplitude

First we will evaluate the amplitude (A7) in a trivial background, using cartesian coordinates. We will be mainly interested in 2-component left-handed massless Weyl fermions, with Lagrangian $L = \Psi^\dagger i\sigma^\mu D^\mu \Psi$. The wavefunctions in (5.2) and (5.3) are then

$$\phi_k(x) = \frac{e^{ik\cdot x}}{\sqrt{V}}$$

$$U_k(x) = \frac{e^{ik\cdot x}}{\sqrt{V}} u_k \quad V_k(x) = \frac{e^{-ik\cdot x}}{\sqrt{V}} e^{-i\beta} u_k$$

with $u_k = \frac{1}{\sqrt{2w}} \left( \frac{\sqrt{w+k_z}}{\sqrt{w-k_z}e^{i\beta}} \right),$  

(A8)

where $k_x + ik_y = k_{Te^{i\beta}}$. Since we are considering massless fermions $w = |k|$. The sum over intermediate states can be approximated by an integral,
\[ \sum_k \rightarrow \frac{V T}{(2\pi)^4} \int d^4 k_1 . \]  

The amplitude then becomes,

\[ A = |g|^2 \frac{1}{(2\pi)^4 V^2} \int d^4 k d^4 x d^4 x' u^\dagger_{k_1} i \sigma^2 u^*_{k_2} e^{ik_1 \cdot k_2 - k_3 - k_4} \delta_T (k_1 + k_2 - k_3 - k_4) . \]  

We will define \((2\pi)^4 \delta^4_T (k) = \int e^{ik \cdot x} d^4 x\) to be the equivalents of the usual delta functions at finite time and volume, with \(\delta^4_T (0) = TV (2\pi)^{-4}\). The large time and volume limits are implied.

Evaluating the integrals gives

\[ A = |g|^2 \frac{1}{(2\pi)^4 V^2} \int d^4 k d^4 x d^4 x' u^\dagger_{k_1} i \sigma^2 u^*_{k_2} e^{ik_1 \cdot k_2 - k_3 - k_4} \delta_T (k_1 + k_2 - k_3 - k_4) , \]  

with \(k_I = k_1 + k_2 = k_3 + k_4\).

The total cross section for the interaction is obtained by summing over the possible final states

\[ \sigma = \sum_{\text{final states}} \frac{V |A|^2}{T v_{\text{rel}}} . \]  

The sum over final states can be replaced by integrals over the final state momenta,

\[ \sigma = \frac{V}{T v_{\text{rel}}} \int \frac{d^3 k_3}{(2\pi)^3} \int \frac{d^3 k_4}{(2\pi)^3} \frac{|g|^4 (2\pi)^8}{V^4} \delta^4_T (0) \delta^4_T (k_1 + k_2 - k_3 - k_4) \frac{|u^T_{k_1} i \sigma^2 u_{k_2}|^2}{(k_I^2 - m_\phi^2)^2} \delta^4_T (k_3 + k_4) . \]  

The relative velocity of the incoming particles is \(v_{\text{rel}} = k_1^\mu k_2^\mu / (w_1 w_2)\). Using this and \(|u^T_{k} i \sigma^2 u_{k'}|^2 = k^\mu k'_\mu / (2w w') = (k + k')^2 / (4w w')\) gives

\[ \sigma = \frac{|g|^4}{8 (2\pi)^2} \frac{k_I^2}{(k_I^2 - m_\phi^2)^2} \int \frac{d^3 k_3}{w_3} \frac{d^3 k_4}{w_4} \delta^4_T (k_3 + k_4 - k_I) . \]  

The integral is Lorentz invariant, and can easily be evaluated in the centre of momentum frame to give

\[ \sigma = \frac{|g|^4}{16 \pi} \frac{k_I^2}{(k_I^2 - m_\phi^2)^2} . \]  

\[ \textbf{4. The Higgs Width} \]

The part of the electroweak Higgs which couples to neutrinos can decay into two quarks, \(\Phi \rightarrow q + q'\), where \(q = u, c, t\). This is a first order process with rate given by
\[ \Gamma_\phi = 3 \sum_{q=u,c,t} \int \frac{V^2 d^3k_1 d^3k_2}{T(2\pi)^6} \left| \frac{i g_q^* (2\pi)^4 u^T_{k_1} i \sigma^2 u_{k_2} \delta(m_\phi - w_1 - w_2) \delta^3(k_1 + k_2)}{\sqrt{2m_\phi} V^3} \right|^2 \]

\[ = 3 \sum_{q=u,c,t} \frac{|g_q|^2 m_\phi}{8(2\pi)^2} \int \frac{d^3k_1 d^3k_2}{w_1 w_2} \delta(m_\phi - w_1 - w_2) \delta^3(k_1 + k_2) \]

\[ = 3 \sum_{q=u,c,t} \frac{|g_q|^2 m_\phi}{16\pi} . \quad (A16) \]

5. Wavefunctions in a Cosmic String Background

Having established our normalisations and conventions in the familiar trivial background, we now turn to the calculation in the background of a cosmic string. We work in a cylindrical box of length \( L \) and volume \( V \). The normalised on-shell states are given in section \([\text{III}]\).

In the string background it is natural to perform an expansion in angular mode number, yet we still wish to consider an incoming plane wave. The two expansions can be matched by first expanding the plane wave in terms of Bessel functions;

\[ e^{ik(x \cos \beta + y \sin \beta)} = e^{i(k_x x + k_y y)} = \sum_{q=-\infty}^{\infty} i^q J_q(kr) e^{iq(\theta - \beta)} , \quad (A17) \]

and then asymptotically matching these Bessel functions mode by mode to the modes in the string background. As we only have square integrable wavefunctions in the string background, in general we must also include an outgoing scattered wave. This outgoing wave is the source of Aharanov-Bohm scattering. Using an angular mode decomposition of the intermediate particle, all the wavefunctions at the initial vertex are in the form of cylindrical modes.

We must now consider the participating wavefunctions. Let particle 1 be the bound state,

\[ \Psi_1(x) \sim e^{i w_1 t - i k_{1z} z - i n_1 \theta} \frac{\bar{M}}{\sqrt{L}} e^{-\bar{M}r} . \quad (A18) \]

As this state is bound to the string, it will only overlap significantly with the lowest angular mode components of the incoming scattering wave, and so the integration over the position of the initial vertex will be dominated by these components. Similarly, we need only consider the lowest angular modes of the propagator.

The modes of a generic fermion wavefunction in the string background have three important regions. If the transverse momentum is \( k_T \), these regions are \( k_T r \gtrsim 1, 1/k_T \gtrsim r \gtrsim 1/\bar{M} \) and \( r \ll 1/\bar{M} \). In the large \( r \) region, \( k_T r \gtrsim 1 \), we have some Bessel function at large argument. This region dominates the normalisation integral. In the intermediate region, \( 1/k_T \gtrsim r \gtrsim 1/\bar{M} \we have some Bessel function at small argument. The order of this Bessel function is shifted by the string gauge field \([14]\). In the lowest angular momentum modes there is a mixture of Bessel functions of the first and second type, leading to one component of the spinor varying like \( r^{-Q} \) in this region, where \( Q \) is the magnitude of the charge of
the fermion under the string gauge field. Finally, inside the string, \( r \lesssim 1/\tilde{M} \), the spinor components tend to constants as \( r \to 0 \). The effect of the intermediate region is thus to amplify the wavefunctions by \( O([\tilde{M}/kT]^Q) \). In the following appendices we calculate unamplified cross-sections. The amplification factors, which depend on the fermion charges, will be added at the end.

For the outgoing states a construction as above is less intuitive. We can think of the incoming wave undergoing classical scattering and some small part of it also partaking in the quantum scattering. The corresponding interpretation for the outgoing wave is that the quantum scattering excites states consisting of an outgoing plane wave and outgoing Aharanov-Bohm scattered waves. Whilst this does not lend itself to a clean interpretation in terms of two-to-two scattering, it is just another manifestation of the inappropriateness of the plane wave as the asymptotic state in the background of the string. The \( 1/r \) fall off of the gauge field of the string introduces long range interactions in the string background which we will neglect. This corresponds to assuming that the outgoing states are free.

The outgoing states we are interested in are light and have little interaction with the string. Thus we make the approximation that the outgoing states do not interact with the string fields and the plane wave expansion employed for a free plane wave can be used for the outgoing states.

**APPENDIX B: SCATTERING FROM A MASSLESS BOUND STATE ON A STRING**

We now consider the problem of real physical interest: a plasma fermion scattering from a bound fermion via an intermediate scalar to produce two light fermions. We consider theories with couplings of the form:

\[
H_{\text{int}} = ig_\nu \Phi \Psi\nu^\dagger i\sigma^2 \Psi^*\nu + ig_u \Phi \Psi^u i\sigma^2 \Psi^*u + (\text{h. c.}),
\]

(B1)

where \( \Psi_\nu \) is the heavy neutrino, \( \Phi \) is a light Higgs field, and the other \( \Psi \) are light fermions. In the SO(10) model the couplings \( g_\nu \) and \( g_u \) are the same.

For the SO(10) model the scattering of interest is \( \nu_\nu \text{str} + \nu \to u + u^c \), where \( \nu_\nu \text{str} \) is the massless current carrier. Figure 3a shows the corresponding Feynman diagram. The scattering amplitude is then

\[
A = \langle \nu_\nu \text{str}(k_1) \nu(k_2) | S | u(k_3) u^c(k_4) \rangle = g_\nu g_u^* \sum_k \int d^4x \int d^4y U_{0k_1}(x) i\sigma^2 U_{k_2}^*(x) e^{i(w_1 + w_2)t}
\]

\[
\times G(x - x') U_{k_3}(x') i\sigma^2 U_{k_4}(x') e^{-i(w_3 + w_4)t'}.
\]

(B2)

To keep things simple, we will use plane wave approximations of the Higgs and light fermion states [A8]. Using these and the expression for \( U_{0k} \) (B.3), the above amplitude can be expanded to give a similar expression to (A10). Most of the integrals can be done in the same way as before, leading to
\[ A = g_\nu g_u^* \frac{\tilde{M}}{\sqrt{2\pi}L^3} (2\pi)^2 \int d^2x \delta^{(t,z)}(k_1 + k_2 - k_3 - k_4) \]

\[ \times \frac{u_{k_3}^* i\sigma^2 u_{k_4} e^{i(k_1-k_2)x^{(2)}}}{k_1^2 - m_\phi^2 + im_\phi \Gamma_\phi} \sqrt{1 - \frac{k_{z2}}{w_2}} e^{-i\beta_2} e^{-\int^{\tilde{M}f+\frac{w}{2}\pi dr'}} , \]

(B3)

where \( k \cdot x^{(2)} = k_x x + k_y y \). The Higgs particle momentum satisfies \( k_I = k_3 + k_4 \) and \( (k_I)_{z,t} = (k_1 + k_2)_{z,t} \). The above \( d^2x \) integral is most easily evaluated in polar coordinates. Using a Bessel function expansion of a plane wave (A17) gives

\[ \int e^{i(k_x x + k_y y)} d\theta = 2\pi J_0(kr) \]

(B4)

Using a numerical solution for the \( r \) dependence of the zero mode, we find

\[ \int J_0(kr)e^{-\int^{\tilde{M}f+\frac{w}{2}\pi dr'} rdr} \approx \frac{c_0}{M^2} e^{-k^2/(2\tilde{M}^2)} . \]

(B5)

\( c_0 \approx 0.7 \) is a slowly varying function of \( M/M_S \). These two results allow the integrals in (B3) to be evaluated, giving

\[ A = g_\nu g_u^* \frac{(2\pi)^3}{\sqrt{2\pi}L^3} \frac{c_0}{\tilde{M}} \sqrt{1 - \frac{k_{z2}}{w_2}} e^{-i\beta_2} \delta^{(t,z)}(k_1 + k_2 - k_3 - k_4) \]

\[ \times \frac{u_{k_3}^* i\sigma^2 u_{k_4}}{k_1^2 - m_\phi^2 + im_\phi \Gamma_\phi} e^{-(k_1-k_2)^2/(2\tilde{M}^2)} . \]

(B6)

The corresponding expression for scattering in a trivial background has a delta function instead of the gaussian. The physical significance of this is that transverse momentum is not conserved at the first vertex. This is to be expected since the string breaks transverse Lorentz invariance. The extra momentum is absorbed or provided by the string, up to the scale \( \tilde{M} = \sqrt{M_S M/2} \). This is the effective energy scale of the string with respect to bound state interactions. There is also an extra \( \tilde{M}^{-1} \) factor, which is a combination of the bound state normalisation and the fact the interaction at the first vertex is confined to the string core, whose effective area is \( \tilde{M}^{-2} \).

The total cross section is again obtained by squaring the modulus of the amplitude and integrating over the final state momenta. After squaring it is convenient to introduce two extra \( \delta \)-functions

\[ |A|^2 = |g_\nu g_u|^2 \frac{T}{V^3} \frac{(2\pi)^3 c_0^2}{4M^2} \left(1 - \frac{k_{z2}}{w_2}\right) \frac{k_I^2}{|k_I^2 - m_\phi^2 + im_\phi \Gamma_\phi|^2} e^{-(k_1-k_2)^2/(2\tilde{M}^2)} \]

\[ \times \int \frac{d^2k_I}{w_3 w_4} \delta^4(k_I - k_3 - k_4) . \]

(B7)

In this case \( v_{rel} = 1 - k_{z2}/w_2 \). The \( k_3 \) and \( k_4 \) integrals are Lorentz invariant and can be evaluated in the centre-of-final-momentum frame, as in (A14). Hence

\[ \sigma = \frac{V}{T v_{rel}} \int V d^3k_3 \frac{(2\pi)^3 k_3^2}{4M^2} |A|^2 \]

\[ = g_\nu g_u^* \frac{c_0^2}{16\pi^2 \tilde{M}^2} \int d^2k_I \frac{k_I^2 \Theta(k_I^2)}{|k_I^2 - m_\phi^2 + im_\phi \Gamma_\phi|^2} e^{-(k_1-k_2)^2/(2\tilde{M}^2)} . \]

(B8)
If $w_1^2 - k_{Tz}^2 < m_{\phi}^2$, the first term of (B9) does not contribute to (B8) and $\sigma \sim |g_{\nu} g_u|^2 / \tilde{M}^2$. If $w_1^2 - k_{Tz}^2 \ll m_{\phi}^2$ then $\sigma \sim |g_{\nu} g_u|^2 (w_1^2 - k_{Tz}^2)^2 / (\tilde{M}^2 m_{\phi}^4)$. Otherwise, to leading order, the $O(1)$ terms can dropped and

$$\sigma = |g_{\nu} g_u|^2 \frac{e_0^2}{16\pi M^2} \frac{m_{\phi}}{\Gamma_{\phi}} \frac{e^{-[(k_{IT}^2+k_{2T}^2)/\tilde{M}^2]}}{I_0(2k_{IT}k_{2T}/\tilde{M}^2)}$$

with $k_{IT}^2 = w_{in}^2 - k_{inz}^2 - m_{\phi}^2$. The angular integral is evaluated using $\int_0^{2\pi} d\theta e^{x\cos\theta} = 2\pi I_0(x)$. The plasma particles will have energies of order $T$, so if $w_1$ is less than the GUT scale then $k_{IT}, k_{2T} < \tilde{M}$ and the exponential terms can then be dropped.

Scatterings involving charm and top quarks will give similar contributions to (B10). Summing these and substituting (A16) gives

$$\sigma = |g_{\nu} g_u|^2 \frac{e_0^2}{M^2} \left( \Theta(w_{in}^2 - k_{inz}^2 - m_{\phi}^2) + O(|g|^2) + O(T w_1/\tilde{M}^2) \right).$$

The above calculation only involves the neutral component of the intermediate Higgs field. Similar scatterings involving the other components also occur, such as $\nu_{str} + e \rightarrow d + u^c$. In SO(10) there are a total of 5 such scatterings. They all have similar cross sections to (B11).

It is also possible for plasma particles to interact with the neutrino currents by exchanging a virtual Higgs particle, as in figure 3b. Conservation of momentum implies that the Higgs particle is always space-like, and so the scattering is never resonant. Thus the dominant contributions to the total cross section come from (B11) and similar interactions.

The rate at which the currents interact with the plasma is given by $\Gamma_{\text{int}} = n_{\text{plas}}^{\text{eq}} \langle \sigma v_{\text{rel}} \rangle$, where $n_{\text{plas}}^{\text{eq}}$ is the equilibrium number density of each species of plasma particle and $\langle \sigma v_{\text{rel}} \rangle$ is the thermally averaged cross section. $n_{\text{plas}} \sim T^3$ and $\sigma \sim |g_{\nu}|^2 / \tilde{M}^2$, so

$$\Gamma_{\text{int}} \sim \frac{|g_{\nu}|^2}{M^2 T^3}.$$
GUT scale. The mass of a massive bound state will then be some fraction of the GUT scale. Using the expressions (3.4), (A3), (A8), the amplitudes for the above decays in the rest frame of the bound state are

\[
\mathcal{A}_1 = \langle \nu^c_{\text{str}}(k_1) | S | \bar{\nu}(k_2) \Phi(k_3) \rangle \\
= ig_\nu \int d^4x U^\dagger_{k_1} \frac{i\sigma^2 V_{k_2}^*}{\sqrt{2w_3}} e^{i(k_2-k_3) \cdot x} e^{i(w_1-w_2-w_3)t} \\
= ig_\nu \int d^2x \frac{\tilde{M}(2\pi)^2}{\sqrt{8\pi w_3 L^2}} e^{i(k_2+k_3) \cdot x(\gamma)} \delta(t) \delta(t) (k_1 - k_2 - k_3) \\
\times \left( \sqrt{1 - \frac{k_{\perp}^2}{w_2}} e^{i\theta} + i \frac{k_{\perp}^2}{w_2} \chi_2 e^{i\beta_2} \right),
\]

(C1)

\[
\mathcal{A}_2 = \langle \nu^c_{\text{str}}(k_1) | S | \nu(k_2) \Phi(k_3) \rangle \\
= ig_\nu^* \int d^4x \bar{U}^T_{k_1} \frac{i\sigma^2 U_{k_2}}{\sqrt{2w_3}} e^{i(w_1-w_2-w_3)t} \\
= ig_\nu^* \int d^2x \frac{\tilde{M}(2\pi)^2}{\sqrt{8\pi w_3 L^2}} e^{i(k_2+k_3) \cdot x(\gamma)} \delta(t) (k_1 - k_2 - k_3) \sqrt{1 - \frac{k_{\perp}^2}{w_2}} \chi_3 e^{i(\beta_2+\theta)}.
\]

(C2)

As with the massless current scattering [B3], we will use a Bessel function expansion (A17) of \( e^{i(k_2+k_3) \cdot x(\gamma)} \), and approximations of the radial integrals.

\[
\int J_0(kr) \chi_2(Mr) r dr \approx \frac{c_2}{M^2} e^{-k^2/(2\tilde{M}^2)}, \quad \int J_1(kr) \chi_1,3(Mr) r dr \approx \frac{c_{1,3}k}{M^2 M_B} e^{-k^2/(2\tilde{M}^2)},
\]

(C3)

with \( c_2 \approx -0.7 \) and \( c_1 \approx c_3 \approx 0.3 \). Thus

\[
\mathcal{A}_1 \approx -\frac{g_\nu(2\pi)^3}{MV \sqrt{8\pi w_3 L}} \left( \sqrt{1 - \frac{k_{\perp}^2}{w_2}} \frac{k_{\perp}^2 + k_{\perp}^2}{M_B} + \sqrt{1 + \frac{k_{\perp}^2}{w_2}} \frac{k_{\perp}^2}{M_B} \right) \chi_2 \delta(t) (k_1 - k_2 - k_3) e^{-k^2/(2\tilde{M}^2)},
\]

(C4)

\[
\mathcal{A}_2 \approx -\frac{g_\nu^*(2\pi)^3}{MV \sqrt{8\pi w_3 L}} \sqrt{1 - \frac{k_{\perp}^2}{w_2}} \frac{k_{\perp}^2 + k_{\perp}^2}{M_B} \frac{k_{\perp}^2}{M_B} \delta(t) (k_1 - k_2 - k_3) e^{-k^2/(2\tilde{M}^2)},
\]

(C5)

where \((k_3)_{x,y} = (k_2 + k_3)_{x,y}\). As with the massless scattering in appendix B, transverse momentum is not conserved and there is a \( \tilde{M}^{-1} \) suppression due to the bound state being confined to the string core. The 2-component vector \( k_s \) gives the momentum contributed by the string. The two amplitudes correspond to different decay products so the total decay rate is obtained by squaring and then adding them, and finally summing over the outgoing state momenta. Before doing this, it is useful to introduce 2 delta functions to separate out the \( k_s \) dependence.

\[
\Gamma = \frac{1}{T} \int \frac{V^2 d^3k_2 d^3k_3}{(2\pi)^6} \left( |\mathcal{A}_1|^2 + |\mathcal{A}_2|^2 \right)

= \frac{|g_\nu|^2}{4M^2(2\pi)^3} \int d^2k_{\perp} \frac{d^3k_2 d^3k_3}{w_2 w_3} \delta^4(k_1 + k_s - k_2 - k_3) \chi_2 \delta(t) (k_1 - k_2 - k_3) \chi_2 \frac{k_s \cdot k_2}{M_B} \frac{k_s \cdot k_3}{M_B} \left( \frac{k_s^2}{M_B} \right) w_2 + 2c_1 c_2 \frac{k_s \cdot k_2}{M_B} + c_2^2 w_2 \right) e^{-k^2/(2\tilde{M}^2)}. \]

(C6)
The $d^3k_2$ and $d^3k_3$ integrals can be evaluated using

$$\int \frac{d^3k_2}{w_2} \frac{d^3k_3}{w_3} k_2^\mu \delta^4(k - k_2 - k_3) = k_\mu \pi \left(1 - \frac{m_\phi^2}{k^2}\right)^2 \Theta(k^2 - m_\phi^2). \quad (C7)$$

This can most easily be shown in the centre of outgoing momentum frame, and then Lorentz transformed to a general frame. We expand the remaining $d^2k_s$ integral as a power series in $m_\phi/M_B$.

$$\Gamma = \frac{|g_\nu|^2 M_B}{16\pi M^2} \int_0^{\sqrt{M_B^2 - m_\phi^2}} d k_s \left[c_2^2 + (c_1^2 + c_3^2 + 2c_1c_2) \frac{k_s^2}{M_B^2} \right] \left(1 - \frac{m_\phi^2}{M_B^2 - k_s^2}\right)^2 e^{-k_s^2/\tilde{M}^2}$$

$$= M_B \frac{|g_\nu|^2}{32\pi} \left[c_2^2(1 - e^{-\lambda}) - (c_1^2 + c_3^2 + 2c_1c_2)[\lambda^{-1} + e^{-\lambda}(1 - \lambda^{-1})]\right] + O\left(\frac{m_\phi^2}{M_B^2}\right). \quad (C8)$$

where $\lambda = (M_B/\tilde{M})^2 \sim 1$.

The massive bound state lifetime is then

$$\tau \sim (|g_\nu|^2 M_B)^{-1}, \quad (C9)$$

which is small.

**APPENDIX D: ZERO MODE SCATTERING ON NEARBY STRINGS**

In addition to plasma interactions, it is also possible for massless fermion charge-carriers to decay by interacting with currents on other strings. This will be most significant when the separation of the strings is small.

We will consider two infinite straight strings with opposite windings, running parallel to each other. We define $2R$ to be the displacement of the second string, with $R$ orthogonal to the $z$-axis. This calculation will also be of relevance to string loops, $R$ is then the radius of the corresponding loop. For a typical vorton (a loop stabilised by a current) $R \sim 10 - 100 M_S^{-1}$.

Since the second string has opposite winding to those considered in section III, the solution (3.5) is not valid. Instead

$$\vec{U}_{0k} \propto \frac{\tilde{M}}{\sqrt{\pi L}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp \left( - \int \frac{M}{M_S} f + \frac{a}{2\rho_{2R}} d\rho_{2R} \right) e^{i k_z z} \quad (D1)$$

should be used, with $\rho_{2R}$ a radial coordinate centred on $2R$ rather than the origin. These states have $k_z < 0$, and opposite chirality to (3.3).

The cross section for direct interactions between currents on different strings will be proportional to the square of the overlap of their wavefunctions, which is of order $\exp(-4MR)$. Interactions involving the exchange of a virtual particle do not suffer from this suppression, and provide the dominant decay channel. We will consider the exchange of a virtual Higgs particle $\tilde{\nu}_{str} + \tilde{\nu}_{str} \rightarrow \nu + \nu$, as shown in figure 3d (the arrows indicate the direction of the bound state momenta along the strings).
\[ \mathcal{A} = \langle \bar{\nu}_e (k_1) \bar{\nu}_e (k_2) | S | \nu (k_3) \nu (k_4) \rangle \]
\[ = 2 |g_\nu|^2 \sum_k \int d^4x \int d^4y \left( \frac{\bar{U}_{0k_1}^T (x) i \sigma^2 U_{k_3} (x) e^{i(w_1 - w_3)t} \bar{U}_{0k_2}^T (x') i \sigma^2 V_{k_4}^* (x') e^{i(w_2 - w_4)t'}}{L \nu} \right. \]
\[ - \bar{U}_{0k_1}^T (x) i \sigma^2 V_{k_4}^* (x') e^{i(w_1 - w_3)t} \bar{U}_{0k_2}^T (x') i \sigma^2 U_{k_3} (x) e^{i(w_2 - w_4)t'} \) \( G(x - x') \). \] (D2)

Substituting (3.5), (A3), (A8) and (D1), with \( \bar{U}_{0k} = U_{0k} \), and evaluating the integrals gives
\[ \mathcal{A} = \frac{4\pi |g_\nu|^2 c^2_0}{LVM^2} \left\{ \sqrt{1 - \frac{k_{3z}}{w_3}} \sqrt{1 + \frac{k_{4z}}{w_4}} e^{i\beta_3 + i\beta_4} \mathcal{I}(k_{1z} - k_{3z}, w_1 - w_3) e^{2ik_4 \cdot R} \right. \]
\[ - \left. \sqrt{1 + \frac{k_{3z}}{w_3}} \sqrt{1 - \frac{k_{4z}}{w_4}} \mathcal{I}^*(k_{2z} - k_{3z}, w_2 - w_3) e^{2ik_4 \cdot R} \right\} \delta^{(t,z)}(k_1 + k_2 - k_3 - k_4) \] (D3)

where
\[ \mathcal{I}(k_z, w) = \int \frac{d^2k}{m_\phi^2 + k_z^2 - w^2 + k_T^2} \exp \left( -\frac{(k + k_3)^2_T}{2M^2} - \frac{(k - k_4)^2_T}{2M^2} - 2ik \cdot R \right). \] (D4)

Applying the convolution theorem gives
\[ \mathcal{I}(k_z, w) = 2\tilde{M}^2 \int d^2X K_0 \left( 2\sqrt{m_\phi^2 + k_z^2 - w^2} |R - X| \right) \]
\[ \times e^{-X^2\tilde{M}^2} e^{i(k_3 - k_4) \cdot X} e^{-(k_4 + k_3)^2_T/4\tilde{M}^2}. \] (D5)

The main contribution from the integrand occurs around \( X = 0 \). Since the modified Bessel function \( K_0 \) varies slowly in this region, (D3) can be approximated by putting \( X = 0 \) in the argument of the Bessel function. This gives
\[ \mathcal{I}(k_z, w) \approx 2\pi e^{-k_{3T}^2/2\tilde{M}^2} e^{-k_{4T}^2/2\tilde{M}^2} K_0 \left( 2\sqrt{m_\phi^2 + k_z^2 - w^2} R \right). \] (D6)

The exponential factors come from non-conservation of transverse momentum at each string and the Bessel function is a result of the limited range of the virtual Higgs particle.

Substituting (D6) into (D3), squaring and multiplying by \( L/ \mathcal{T} v_{\text{rel}} \) gives the cross section. \( v_{\text{rel}} = 2 \) and so,
\[ \sigma = \frac{2 |g_\nu|^4 c_0^4}{(2\pi)^4 M^4} \int dw_3 dw_4 d\Omega_3 d\Omega_4 \delta^{(t,z)}(k_1 + k_2 - k_3 - k_4) w_3 w_4 e^{-k_{3T}^2/\tilde{M}^2} e^{-k_{4T}^2/\tilde{M}^2} \]
\[ \left\{ (w_3 - k_{3z})(w_4 + k_{4z}) K_0^2 \left( 2\sqrt{m_\phi^2 + (k_{1z} - k_{3z})^2 - (w_1 - w_3)^2} R \right) + (k_3 \leftrightarrow k_4) \right. \]
\[ -2k_{3T} k_{4T} \text{Re} \left( e^{i(\beta_3 - 2k_3 \cdot R)} e^{i(\beta_4 + 2k_4 \cdot R)} \right) K_0 \left( 2\sqrt{m_\phi^2 + (k_{1z} - k_{3z})^2 - (w_1 - w_3)^2} R \right) \]
\[ K_0 \left( 2\sqrt{m_\phi^2 + (k_{1z} - k_{1z})^2 - (w_1 - w_4)^2} R \right) \}. \] (D7)

The cross section gives the transition rate for unit incident flux. The incident flux in this case is \( L/v_{\text{rel}} \), instead of \( V/v_{\text{rel}} \) as in (B11), because the incoming particles are confined to the one-dimensional strings. This gives a dimensionless cross section instead of the usual (length)\(^{-2} \).
The Bessel function means that the first term of (D7) is exponentially suppressed except when $m^2_\Phi + (k_{1z} - k_{3z})^2 - (w_1 - w_3)^2$ is small. We will use the approximation $K_0(2z) = -\log(z)$ for $z < 1$ and $K_0(2z) = 0$ elsewhere. In this region the transverse outgoing momenta are small, and the gaussian factors can be dropped to first order. If $m_\phi R \ll 1$, the Higgs mass can also be neglected.

The integral of the second term of (D7) is equal to the integral of the first. The third is of order $k^2_{3T} k^2_{1T}$, which is small in the region where the $K_0$ factors are not exponentially small. Thus the third term of (D7) can be neglected. Using the above approximations gives

$$\sigma \sim \frac{|g_\nu|^4}{(MR)^4} \min(1, (w_{\text{CoM}} R)^4),$$

(D8)

where $w_{\text{CoM}}$ is the incoming centre of mass energy.

If $m_\phi R$ is large the above approximations do not apply. This occurs if $\Phi$ is a GUT boson, or the electroweak Higgs field at high temperatures. In this case an asymptotic expansion of the $K_0$ factors can be used to estimate (D7), and

$$\sigma \sim \frac{|g_\nu|^4 m_\phi}{M^4 R^3} e^{-4m_\phi R}.$$  

(D9)

Taken at face value, (D8) gives a short lifetime for charge carriers on a vorton. However, there are complications in directly applying this result derived for straight strings to curved vortons. The above calculation does not take into account conservation of this angular momentum, as the system considered has no rotational symmetry. For a perfectly circular loop we would expect angular momentum to be conserved. Combined with energy conservation, this would prevent massless modes on the string scattering into massive modes. However, the fermionic spectrum has not been calculated for a circular loop and there is no reason to expect the zero modes to remain massless. Angular momentum conservation should only be considered in a consistent rotationally invariant calculation. The calculation above works consistently with straight strings, thus while (D8) may not be directly applicable to vorton decay, it is relevant for interactions on non-circular loops and scattering of currents on a string network.
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