The Complexity of Binary Matrix Completion Under Diameter Constraints

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Abstract

We thoroughly study a novel but basic combinatorial matrix completion problem: Given a binary incomplete matrix, fill in the missing entries so that every pair of rows in the resulting matrix has a Hamming distance within a specified range. We obtain an almost complete picture of the complexity landscape regarding the distance constraints and the maximum number of missing entries in any row. We develop polynomial-time algorithms for maximum diameter three based on Deza’s theorem [Discret. Math. 1973] from extremal set theory. We also prove NP-hardness for diameter at least four. For the number of missing entries per row, we show polynomial-time solvability when there is only one and NP-hardness when there can be at least two. In many of our algorithms, we heavily rely on Deza’s theorem to identify sunflower structures. This paves the way towards polynomial-time algorithms which are based on finding graph factors and solving 2-SAT instances.

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1 Introduction

In combinatorial matrix completion problems, given an incomplete matrix over a fixed alphabet with some missing entries, the goal is to fill in the missing entries such that the resulting “completed matrix” (over the same alphabet) fulfills a desired property. Performing a parameterized complexity analysis, Ganian et al. [21, 22] and Eiben et al. [14, 15] recently contributed to this growing field by studying various desirable properties. More specifically, Ganian et al. [22] studied the two properties of minimizing the rank or of minimizing the number of distinct rows of the completed matrix. Ganian et al. [21] analyzed the complexity of completing an incomplete matrix so that it fulfills certain constraints and can be partitioned into subspaces of small rank. Eiben et al. [14] investigated clustering problems where one wants to partition the rows of the completed matrix into a given number of clusters of small radius or of small diameter. Finally, Koana et al. [30] studied two cases of completing...
the matrix into one which has small (local) radius. The latter two papers \cite{14, 30} rely on Hamming distance as a distance measure; in general, all considered matrix completion problems are NP-hard and thus the above papers \cite{14, 15, 21, 22, 30} mostly focused on parameterized complexity studies. In this work, we focus on a desirable property closely related to small radius, namely diameter bounds. Doing so, we further focus on the case of binary alphabet. For a matrix $T \in \{0, 1\}^{n \times \ell}$, let $\gamma(T) := \min_{i \neq i' \in [n]} d(T[i], T[i'])$ and $\delta(T) := \max_{i \neq i' \in [n]} d(T[i], T[i'])$, where $d$ denotes the Hamming distance and $T[i]$ denotes the $i$-th row of $T$. We use the special symbol $\square$ to represent a missing entry. Specifically, we study the following problem.

**Diameter Matrix Completion (DMC)**

**Input:** An incomplete matrix $S \in \{0, 1, \square\}^{n \times \ell}$ and $\alpha \leq \beta \in \mathbb{N}$.

**Question:** Is there a completion $T \in \{0, 1\}^{n \times \ell}$ of $S$ with $\alpha \leq \gamma(T)$ and $\delta(T) \leq \beta$?

We believe that DMC is a natural combinatorial matrix problem which may appear in the following contexts:

- In coding theory, one may want to “design” (by filling in the missing entries) codewords that are pairwise neither too close (parameter $\alpha$ in DMC) nor too far (parameter $\beta$ in DMC) from each other. One prime example is the completion into a Hadamard matrix \cite{28}. This is a special case of DMC with $n = \ell$ and $\alpha = \beta = n/2$.
- In computational biology, one may want to minimize the maximum distance of sequences in order to determine their degree of relatedness (thus minimizing $\beta$); missing entries refer to missing data points.\footnote{Somewhat surprisingly, although simple to define and well-motivated, in the literature there seems to be no systematic study of DMC and its computational complexity.}
- In data science, each row may represent an entity with its attributes, and solving the DMC problem may fulfill some constraints with respect to the pairwise (dis)similarity of the completed entities.
- In stringology, DMC seems to constitute a new and natural problem, closely related to several intensively studied consensus problems (many of which are NP-hard for binary alphabets) \cite{5, 2, 8, 9, 10, 26, 27, 33, 32, 36, 37}.

Somewhat surprisingly, although simple to define and well-motivated, in the literature there seems to be no systematic study of DMC and its computational complexity. We perform a fine-grained complexity study in terms of diameter bounds $\alpha, \beta$ and the maximum number $k$ of missing entries in any row. Note that in bioinformatics applications matrix rows may represent sequences with few corrupted data points, thus resulting in small values for $k$. In fact, the computational complexity with respect to this kind of parameters has been studied in the context of computational biology \cite{2, 5, 27}. We identify polynomial-time cases as well as NP-hard cases, taking significant steps towards a computational complexity dichotomy (polynomial-time solvable versus NP-hard), leaving fairly few cases open. While the focus of the previous works \cite{14, 15, 30} is on parameterized complexity studies, in this work we settle more basic algorithmic questions on the DMC problem, relying on several combinatorial insights, including results from (extremal) combinatorics (most prominently, Deza’s theorem \cite{12}). Indeed, we believe that exploiting sunflowers based on Deza’s theorem in combination with the corresponding use of algorithms for 2-SAT and graph factors is our most interesting technical contribution. In this context, we also observe the phenomenon that the running time bounds that we can prove for odd values of $\alpha$ (the “lower bound for

\footnote{Here, it would be particularly natural to also study the case of non-binary alphabets; however, most of our positive results probably only hold for binary alphabet.}
Complexity of DMC with respect to combinations of constant values for $\alpha$ and $\beta$.}

\[ \beta - \alpha \]

Figure 1 Overview of our results. Green (lighter) denotes polynomial-time solvability and red (darker) denotes NP-hardness. White cells indicate open cases.

Dissimilarity") are significantly better than the ones for even values of $\alpha$—indeed, for even values of $\alpha$ the running time exponentially depends on $\alpha$ while it is independent of $\alpha$ for odd values of $\alpha$. We survey our results in Figure 1 which also depicts remaining open cases.

1.0.0.1 Related work

The closest studies are the work of Hermelin and Rozenberg [27], Koana et al. [30], and Eiben et al. [14, 15]. Hermelin and Rozenberg [27] and Koana et al. [30] studied Constraint Radius Matrix Completion (ConRMC):

**Constraint Radius Matrix Completion (ConRMC)**

**Input:** An incomplete matrix $S \in (\Sigma \cup \Box)^{n \times \ell}$ and $r \in \mathbb{N}$.

**Question:** Is there a completion $T \in \Sigma^{n \times \ell}$ of $S$ and a row vector $v \in \Sigma^{\ell}$ such that $d(v, T[i]) \leq r[i]$ for all $i \in [n]$?

Note that ConRMC is defined for arbitrary alphabets. A more important difference between DMC and ConRMC is that in DMC we basically have to compare all rows against each other, but in ConRMC we have to compare one “center row” against all others. Indeed, this makes these two similarly defined problems quite different in many computational complexity aspects. See the example in Figure 2 that also illustrates significant differences between radius minimization and diameter minimization (the latter referring to $\delta(T) \leq \beta$ above). Recall that $k$ is the maximum number of missing entries in any row. The special case $k = 0$ is a generalization of the well-known CLOSEST STRING problem, which is NP-hard [18].

Hermelin and Rozenberg [27] proved that ConRMC is NP-hard even if $\max_{i \in [n]} r[i] = 2$ while it is polynomial-time solvable for $\max_{i \in [n]} r[i] = 1$. Koana et al. [30] provided a linear-time algorithm for $\max_{i \in [n]} r[i] = 1$. Moreover, Koana et al. [30] established fixed-parameter tractability with respect to $\max_{i \in [n]} r[i] + k$.

Eiben et al. [14] studied the following problem among others:

**Diameter Clustering Completion (DCC)**

**Input:** An incomplete matrix $S \in \{0, 1, \Box\}^{n \times \ell}$ and $r, c \in \mathbb{N}$.

**Question:** Is there a completion $T \in \{0, 1\}^{n \times \ell}$ of $S$ and a partition $(I_1, \ldots, I_c)$ of $[n]$ such that $\delta(T[I_i]) \leq r$ for every $i \in [n]$?
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![Figure 2](image_url) An illustration of matrix completion problems with the input matrix (left). Missing entries (and their completions) are framed by thick lines. The middle matrix is a completion of diameter four and the right matrix is a completion of radius three with the center vector below. Note that missing entries in the same column might be filled with different values to meet the diameter constraint, whereas this is never necessary for the radius constraint.

Here, for $I \subseteq [n]$, $T[I]$ is the submatrix comprising the rows with index $i \in I$. DMC and DCC are closely related. In fact, DMC with $\alpha = 0$ is equivalent to DCC with $c = 1$. However, the problems are incomparable: While DMC also models the aspect of achieving a minimum pairwise distance (not only a bounded diameter), DCC focuses on clustering. Eiben et al. \cite{14} showed NP-hardness for $c \geq 3$. They also showed that DCC on complete matrices is NP-hard for $r = 6$. Furthermore, using the classical sunflower lemma \cite{16}, they proved fixed-parameter tractability with respect to $r + c + \text{cover}$, where \text{cover} is the minimum number of rows and columns whose deletion results in a matrix without any missing entries. We remark that this parameter \text{cover} is not comparable to the maximum number $k$ of missing entries in any row. To see this, consider the following two square matrices $M_1, M_2 \in \{0,1,\square\}^{n \times n}$, where an entry is missing in $M_1$ if and only if it is on the main diagonal and an entry is missing in $M_2$ if and only if it is on the last row. Observe that $k = 1$ and \text{cover} = $n$ for $M_1$ and that $k = n$ and \text{cover} = 1 for $M_2$.

There are also numerous work on non-combinatorial matrix completion problems in the context of clustering \cite{23,24,31,34,13} such as $k$-center clustering and $k$-means clustering. These clustering problems are NP-hard even if the matrix has no missing entry. Often, the focus here is on developing approximation algorithms for the corresponding matrix completion problems.

Organization

We provide preliminaries in Section \ref{sec:preliminaries}. In Section \ref{sec:dmc}, we study DMC with fixed constants $\alpha \leq \beta$ (see Figure \ref{fig:prelim}). We start with polynomial-time solvability: Section \ref{sec:polytime} and Section \ref{sec:polytime2} cover the case $\alpha = 0$ and $\beta \leq 3$ and the case $\alpha > 0$ and $\beta \in \{\alpha, \alpha + 1\}$, respectively. We prove NP-hardness for $\beta \geq 2[\alpha/2] + 4$ in Section \ref{sec:nphardness}. In Section \ref{sec:parameters} we consider DMC with unbounded $\alpha$ and $\beta$ (see Figure \ref{fig:parameters}). We prove polynomial-time solvability in Section \ref{sec:parameters1} and NP-hardness in Section \ref{sec:parameters2}.

\section{Preliminaries} \label{sec:preliminaries}

For $m \leq n \in \mathbb{N}$, let $[m,n] := \{m, \ldots, n\}$ and let $[n] := [1,n]$.

For a matrix $T \in \{0,1\}^{n \times \ell}$, we denote by $T[i,j]$ the entry in the $i$-th row and $j$-th column ($i \in [n]$ and $j \in [\ell]$) of $T$. We use $T[i,:]$ (or $T[i]$ in short) to denote the row vector $(T[i,1], \ldots, T[i,\ell])$ and $T[:j]$ to denote the column vector $(T[1,j], \ldots, T[n,j])^T$. For subsets $I \subseteq [n]$ and $J \subseteq [\ell]$, we write $T[I,J]$ to denote the submatrix containing only the rows in
I and the columns in J. We abbreviate $T[I, \ell]$ and $T[n, J]$ as $T[I, :]$ (or $T[I]$ for short) and $T[\cdot, J]$, respectively. We use the special character $\square$ for a missing entry. A matrix $S \in \{0, 1, \square\}^{n \times \ell}$ is called incomplete if it contains a missing entry, and it is called complete otherwise. We say that $T \in \{0, 1\}^{n \times \ell}$ is a completion of $S \in \{0, 1, \square\}^{n \times \ell}$ if either $S[i, j] = \square$ or $S[i, j] = T[i, j]$ holds for all $i \in [n]$ and $j \in [\ell].$

Let $u, w \in \{0, 1, \square\}^\ell$ be row vectors. For $j \in [\ell]$ and $J \subseteq [\ell]$, let $u[j]$ denote the $j$-th entry of $u$ and let $u[J]$ denote the vector only containing entries in $J$. Let $D(u, w) := \{j \in [\ell] \mid u[j] \neq w[j] \land u[j] \neq \square \lor w[j] \neq \square\}$ be the set of column indices where $u$ and $v$ disagree (not considering positions with missing entries). The Hamming distance between $u$ and $w$ is $d(u, w) := |D(u, w)|$. Note that the Hamming distance obeys the triangle inequality $d(u, w) \leq d(u, v) + d(v, w)$ for a vector $v \in \{0, 1\}^\ell$. For a subset $J \subseteq [\ell]$, we also define $d_J(u, w) := d(u[J], w[J])$. Let $u', v', w' \in \{0, 1\}^\ell$ be complete row vectors. Then, it holds that $d(u', w') = |D(u', v') \Delta D(v', w')| = |D(u', v')| + |D(v', w')| - 2|D(u', v') \cap D(v', w')|.$ The binary operation $u \leftarrow v$ replaces the missing entries of $u$ with the corresponding entries in $v$ for $v \in \{0, 1\}^\ell$. We sometimes use string notation to represent row vectors, such as 001 for $(0, 0, 1)$.

## 3 Constant Diameter Bounds $\alpha$ and $\beta$

In this section, we consider the special case $(\alpha, \beta)$-DMC of DMC, where $\alpha \leq \beta$ are some fixed constants. We prove the results depicted in Figure 1A. To start with, we identify the following simple linear-time solvable case which will subsequently be used several times.

**Lemma 1.** DMC can be solved in linear time for a constant number $\ell$ of columns.

**Proof.** If $\alpha > 0$ and $\alpha > 2^{\ell}$, then there is no completion $T$ of $S$ with $\gamma(T) \geq \alpha > 0$. Thus, we can assume that the input matrix comprises at most $n\ell < 2^\ell \cdot \ell$ (that is, constantly many) entries for the case $\alpha > 0$. Suppose that $\alpha = 0$. Consider a set $V \subseteq [0, 1]^\ell$ in which the pairwise Hamming distances are at most $\beta$. We simply check whether each row vector in the input matrix can be completed to some row vector in $V$ in $O(n \cdot 2^\ell) = O(n)$ time. Since there are at most $2^{2^\ell}$ choices for $V$, this procedure can be done in linear time. ▶

### 3.1 Polynomial time for $\alpha = 0$ and $\beta \leq 3$

As an entry point, we show that $(0, 1)$-DMC is easily solvable. To this end, we call a column vector dirty if it contains both 0 and 1. Clearly, for $\alpha = 0$, we can ignore columns that are not dirty since they can always be completed without increasing the Hamming distances between rows. Hence, throughout this subsection, we assume that the input matrix contains only dirty columns. Now, any $(0, 1)$-DMC instance is a Yes-instance if and only if there is at most one dirty column in the input matrix:

**Lemma 2.** A matrix $S \in \{0, 1, \square\}^{n \times \ell}$ admits a completion $T \in \{0, 1\}^{n \times \ell}$ with $\delta(T) \leq 1$ if and only if $S$ contains at most one dirty column.

**Proof.** Suppose that $S$ contains two dirty columns $S[:, j_0]$ and $S[:, j_1]$ for $j_0 \neq j_1 \in [\ell]$. We claim that $\delta(T) \geq 2$ holds for any completion $T$ of $S$. Let $i \in [n]$. Then, there exist $i_0, i_1 \in [n]$ with $T[i, j_0] \neq T[i_0, j_0]$ and $T[i, j_1] \neq T[i_1, j_1]$. If $\delta(T) \leq 1$, then we obtain $T[i_0, j_0] = T[i, j_0]$ and $T[i_1, j_0] = T[i, j_0]$. Now we have $d(T[i_0], T[i_1]) \geq 2$ because $T[i_0, j_0] \neq T[i_1, j_0]$ and $T[i_0, j_1] \neq T[i_1, j_1]$. The reverse direction follows easily. ▶
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Lemma 2 implies that one can solve (0,1)-DMC in linear time. In the following, we extend this to a linear-time algorithm for (0,2)-DMC (Theorem 12) and a polynomial-time algorithm for (0,3)-DMC (Theorem 13).

For these algorithms, we make use of a concept from extremal set theory, known as $\Delta$-systems [29]. We therefore consider matrices as certain set systems.

\begin{itemize}
  \item \textbf{Definition 3.} For a matrix $T \in \{0,1\}^{n \times \ell}$, let $T$ denote the set system $\{D(T[i], T[n]) \mid i \in [n-1]\}$. Moreover, for $x \in \mathbb{N}$, let $T_x$ denote the set system $\{D(T[i], T[n]) \mid i \in [n-1], d(T[i], T[n]) = x\}$.
\end{itemize}

The set system $T$ contains the subsets (without duplicates) of column indices corresponding to the columns where the row vectors $T[1], \ldots, T[n-1]$ differ from $T[n]$. For given $T[n]$, all the rows of $T$ can be determined from $T$, as we have binary alphabet.

The concept of $\Delta$-systems has previously been used to obtain efficient algorithms [14, 17, 19]. They are defined as follows (see also Figure 3):

\begin{itemize}
  \item \textbf{Definition 4 (Weak $\Delta$-system).} A set family $F = \{S_1, \ldots, S_m\}$ is a weak $\Delta$-system if there exists an integer $\lambda \in \mathbb{N}$ such that $|S_i \cap S_j| = \lambda$ for any pair of distinct sets $S_i, S_j \in F$. The integer $\lambda$ is called the intersection size of $F$.
  
  \item \textbf{Definition 5 (Strong $\Delta$-system, Sunflower).} A set family $F = \{S_1, \ldots, S_m\}$ is a strong $\Delta$-system (or sunflower) if there exists a subset $C \subseteq S_1 \cup \cdots \cup S_m$ such that $S_i \cap S_j = C$ for any pair of distinct sets $S_i, S_j \in F$. We call the set $C$ the core and the sets $P_i = S_i \setminus C$ the petals of $F$.
\end{itemize}

Clearly, every strong $\Delta$-system is a weak $\Delta$-system.

Our algorithms employ the combinatorial property that under certain conditions the set system $T$ of a matrix $T$ with bounded diameter forms a strong $\Delta$-system (which can be algorithmically exploited). We say that a family $F$ of sets is $h$-uniform if $|S| = h$ holds for each $S \in F$. Deza [12] showed that an $h$-uniform weak $\Delta$-system is a strong $\Delta$-system if its cardinality is sufficiently large (more precisely, if $|F| \geq h^2 - h + 2$). Moreover, Deza [11] also proved a stronger lower bound for uniform weak $\Delta$-systems in which the intersection size is exactly half of the cardinality of each set. (We remark that our algorithm could rely on the weaker bound of Deza but using the stronger bound yields a faster algorithm.)

\begin{itemize}
  \item \textbf{Lemma 6 ([11 Théorème 1.1])}. Let $F$ be a $(2\mu)$-uniform weak $\Delta$-system with intersection size $\mu$. If $|F| \geq \mu^2 + \mu + 2$, then $F$ is a strong $\Delta$-system.
\end{itemize}

We extend this result to the case in which the set size is odd.

\begin{itemize}
  \item \textbf{Lemma 7.} Let $F$ be a $(2\mu + 1)$-uniform weak $\Delta$-system.
\end{itemize}

(i) If the intersection size of $F$ is $\mu + 1$ and $|F| \geq \mu^2 + \mu + 3$, then $F$ is a strong $\Delta$-system.

\begin{figure}[h]
  \centering
  \begin{tabular}{|c|c|c|c|}
    \hline
    1 & 2 & 3 & \{2,3\} \\
    \hline
    \{1,3\} \\
    \hline
    \{1,2\} \\
    \hline
  \end{tabular}
  \quad
  \begin{tabular}{|c|c|c|c|c|}
    \hline
    1 & 2 & 3 & 4 & 5 \\
    \hline
    \{1,2\} \\
    \hline
    \{1,3\} \\
    \hline
    \{1,4\} \\
    \hline
    \{1,5\} \\
    \hline
  \end{tabular}
  \caption{Illustration of a weak $\Delta$-system with intersection size one (left) and a strong $\Delta$-system with core $\{1\}$ (right).}
\end{figure}
(ii) If the intersection size of $F$ is $\mu$ and $|F| \geq (\mu + 1)^2 + \mu + 3$, then $F$ is a strong $\Delta$-system.

**Proof.** Let $S \in F$ and let $F' = \{T \triangle S \mid T \in F \setminus \{S\}\}$. Here $T \triangle S$ denotes the symmetric difference $(T \setminus S) \cup (S \setminus T)$. Note that $F'$ is a $(2\mu)$-uniform weak $\Delta$-system with intersection size $\mu$:

- For each $T \in F \setminus \{S\}$, we have $|T \triangle S| = |S \setminus T| + |T \setminus S| = 2\mu$.
- We show that $|(T \triangle S) \cap (U \triangle S)| = \mu$ for each distinct $T, U \in F \setminus \{S\}$. We rewrite

$$
|\left((T \setminus U) \cup (S \setminus T)\right) \cap (U \setminus S) \cup (S \setminus U)| = |(T \setminus U) \cup (S \setminus T)\cap (U \setminus S) \cup (S \setminus U)| = |\left((T \setminus U) \cup (S \setminus T)\right) \cap (S \setminus (T \cup U))| = |((T \setminus U) \setminus S)| + |(S \setminus (T \cup U))|.
$$

Here the third equality follows from $(T \setminus S) \cap (S \setminus U) = (S \setminus T) \cap (U \setminus S) = \emptyset$. Let $\kappa = |S \cap T \cap U|$. Since $|S \cap T| = |S \cap U| = \mu + 1$, it follows that $|(S \cap T) \setminus U| = |(S \cap U) \setminus T| = \mu - \kappa + 1$. Thus, we obtain

$$
|S \setminus (T \cup U)| = |S| - |S \cap T \setminus U| - |S \cap U \setminus T| - |S \cap T \cap U| = (2\mu + 1) - (\mu - \kappa + 1) - (\mu - \kappa + 1) - \kappa = \kappa - 1.
$$

Moreover, we obtain

$$
|\left((T \setminus U) \setminus S\right)| = |T \setminus U| - |S \cap T \cap U| = \mu - \kappa + 1.
$$

Now we have $|(T \triangle S) \cap (U \triangle S)| = |(S \setminus (T \cup U))| + |((T \cap U) \setminus S)| = \mu.$

Now, Lemma 9 implies that $F'$ is a strong $\Delta$-system. Let $C'$ be the core of $F'$. Note that $|(T \triangle S) \cap S| = |S \setminus T| = \mu$ for each $T \in F \setminus \{S\}$ or equivalently $|T \cap S| = \mu$ for each $T' \in F'$. We claim that $T' \cap S = C'$ for each $T' \in F'$. Suppose not. Then, we have two cases: $C' \setminus S \neq \emptyset$ or $(T' \cap S \setminus S) \neq \emptyset$. We show that $C' \setminus S \neq \emptyset$ holds for the latter case as well. Since $|S \cap C'| = |T' \cap S| - |(T' \cap S) \cap C|$, we have $|S \cap C'| < |T' \cap S| = \mu$. This gives us $|C' \setminus S| = |C'| - |S \cap C'| > 0$. We thus have $C' \setminus S \neq \emptyset$. It follows that there exists an element $x \in (T' \setminus C') \cap S$ for each $T' \in F'$. Since the set family $\{T' \setminus C' \mid T' \in F'\}$ is pairwise disjoint, it gives us $|S| \geq \mu^2 + \mu + 2 > 2\mu + 1$, a contradiction. Thus, $F$ is a sunflower with its core being $S \setminus C'$.

**Observation 8.** Let $T \in \{0, 1\}^{n \times \ell}$ be a matrix with $\delta(T) \leq 2$. For each $T_1 \in T_1$ and $T_2, T_2' \in T_2$, it holds that $T_1 \subseteq T_2$ and that $|T_2 \cap T_2'| \geq 1$ (otherwise there exists a pair of rows with Hamming distance at least three).
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Theorem 12. \( (0, 2) \)-DMC can be solved in \( O(n\ell) \) time.

Figure 4 Illustration of Lemma 11 with \( n = 6 \). A black cell denotes a value different from row \( T[6] \). In (b) the set system \( T_2 \) forms a sunflower with core \{2\}. In both cases the radius is one.

The next lemma states that \( |T_2| \) restricts the number of columns.

Lemma 9. Let \( T \in \{0, 1\}^{n \times \ell} \) be a matrix consisting of only dirty columns with \( \delta(T) \leq 2 \). If \( T_2 \neq \emptyset \), then \( \ell \leq |T_2| + 1 \).

Proof. First, observe that \( \ell = |\bigcup_{T_1 \in F_1} T_1 \cup \bigcup_{T_2 \in F_2} T_2| \) because each column of \( T \) is dirty. Thus, it follows from Observation 8 that \( \ell = |\bigcup_{T_2 \in T_2} T_2| \). We prove the lemma by induction on \( |T_2| \). Clearly, we have at most two columns if \( |T_2| = 1 \). Suppose that \( |T_2| \geq 2 \). For \( T_2 \in T_2 \), we claim that

\[
|T_2| = \left| \bigcup_{T_2' \in T_2} T_2' \right| = \left| \bigcup_{T_2' \in T_2 \setminus \{T_2\}} T_2' \right| + \left| T_2 \setminus \bigcup_{T_2' \in T_2 \setminus \{T_2\}} T_2' \right| \leq |T_2| + 1.
\]

The induction hypothesis gives us that \( |\bigcup_{T_2' \in T_2 \setminus \{T_2\}} T_2'| \leq |T_2| \). For the other term, observe that \( |T_2 - \bigcup_{T_2' \in T_2 \setminus \{T_2\}} T_2'| \leq |T_2 - T_2| = |T_2| - |T_2 \cap T_2'| \) for \( T_2' \in T_2 \setminus \{T_2\} \). Hence, it follows from Observation 8 that the second term is at most 1.

Next, we show that a matrix with diameter at most two has radius at most one as long as it has at least five columns. Thus, we can solve DMC by solving CONRC with radius one, which can be done in linear time via a reduction to 2-SAT \cite{29}. We use the following lemma concerning certain intersections of a set with elements of a sunflower.

Lemma 10 (\cite{12}, Lemma 8). Let \( \lambda \in \mathbb{N} \), let \( F \) be a sunflower with core \( C \), and let \( X \) be a set such that \( |X \cap S| \geq \lambda \) for all \( S \in F \). If \( |F| > |X| \), then \( \lambda \leq |C| \) and \( |X \cap C| \geq \lambda \).

Lemma 11. Let \( T \in \{0, 1\}^{n \times \ell} \) be a matrix with \( \delta(T) \leq 2 \). If \( \ell \geq 5 \), then there exists a vector \( v \in \{0, 1\}^\ell \) such that \( d(v, T[i]) \leq 1 \) for all \( i \in [n] \).

Proof. If \( T_2 = \emptyset \), then we are immediately done by definition, because \( d(T[n], T[i]) \leq 1 \) for all \( i \in [n] \) (see Figure 1a for an illustration). Since \( \ell \geq 5 \), Lemma 9 implies \( |T_2| \geq 4 \).

It follows from Observation 8 that \( T_2 \) is a 2-uniform weak \( \Delta \)-system with intersection size one (see Figure 1b). Thus, \( T_2 \) is a sunflower by Lemma 6. Let \( \{j_{\text{core}}\} \) denote the core of \( T_2 \). Note that \( |T_1 \cap T_2| \geq 1 \) holds for each \( T_1 \in T_1 \) and \( T_2 \in T_2 \) by Observation 8. Now we can infer from Lemma 10 (let \( X = T_1 \), \( \lambda = 1 \), and \( F = T_2 \)) that \( \mathcal{T} \subseteq \{T_1\} \), where \( T_1 = \{j_{\text{core}}\} \).

Hence, it holds that \( d(v, T[i]) \leq 1 \) for all \( i \in [n] \), where \( v \in \{0, 1\}^\ell \) is a row vector such that \( v[j_{\text{core}}] = 1 - T[n, j_{\text{core}}] \) and \( v[j] = T[n, j] \) for each \( j \in [\ell] \setminus \{j_{\text{core}}\} \).
Proof. Let \( S \in \{0, 1, \square\}^{n \times \ell} \) be an input matrix of \((0, 2)\)-DMC. If \( \ell \leq 4 \), then we use the linear-time algorithm of Lemma 1. Henceforth, we assume that \( \ell \geq 5 \).

We claim that \( S \) is a Yes-instance if and only if the CONRMC instance \( I = (S, 1^n) \) is a Yes-instance.

\((\Rightarrow)\) Let \( T \) be a completion of \( S \) with \( \delta(T) \leq 2 \). Since \( \ell \geq 5 \), there exists a vector \( v \) such that \( d(v, T[i]) \leq 1 \) for all \( i \in [n] \) by Lemma 11. It follows that \( I \) is a Yes-instance.

\((\Leftarrow)\) Let \( v \) be a solution of \( I \). Let \( T \) be the matrix such that for each \( i \in [n] \), \( T[i] = S[i] \leftarrow v \) (recall that \( u \leftarrow v \) denotes the vector obtained from \( u \) by replacing all missing entries of \( u \) with the entries of \( v \) in the corresponding positions). Then, we have \( d(v, T[i]) \leq 1 \) for each \( i \in [n] \). By the triangle inequality, we obtain \( d(T[i], T[i']) \leq d(v, T[i]) + d(v, T[i']) \leq 2 \) for each \( i, i' \in [n] \).

Since ConRMC can be solved in linear time when \( \max_{i \in [n]} r[i] = 1 \) [30, Theorem 1], it follows that \((0, 2)\)-DMC can be solved in linear time.

We next show polynomial-time solvability of \((0, 3)\)-DMC (Theorem 18). The overall idea is, albeit technically more involved, similar to \((0, 2)\)-DMC. We first show that the set family \( T \) of a matrix \( T \) with \( \delta(T) = 3 \) contains a sunflower by Lemma 7. We then show that such a matrix has a certain structure which again allows us to reduce the problem to the linear-time solvable special case of ConRMC with radius one.

We start with an observation on a matrix whose diameter is at most three.

\begin{itemize}
  \item \textbf{Observation 13.} Let \( T \in \{0, 1\}^{n \times \ell} \) be a matrix with \( \delta(T) \leq 3 \). It holds for each \( T_1 \in T_1, T_2 \in T_2, \) and \( T_3, T_3' \in T_3 \) that \( T_1 \subseteq T_3, T_2 \cap T_3 \neq \emptyset, \) and \( |T_3 \cap T_3'| \geq 2 \) (otherwise there exists a pair of rows with Hamming distance four).
\end{itemize}

From Observation 13 we obtain (by induction) the following lemma analogously to Lemma 9.

\begin{itemize}
  \item \textbf{Lemma 14.} Let \( T \in \{0, 1\}^{n \times \ell} \) be a matrix consisting of dirty columns with \( \delta(T) \leq 3 \). If \( T_3 \neq \emptyset, \) then \( \ell \leq |T_2| + |T_3| + 2 \).
\end{itemize}

\textbf{Proof.} First, observe that \( \ell = |\bigcup_{T_1 \in T_1} T_1 \cup \bigcup_{T_2 \in T_2} T_2 \cup \bigcup_{T_3 \in T_3} T_3| \) because each column of \( T \) is dirty. Thus, it follows from Observation 8 that \( \ell = |\bigcup_{T_2 \in T_2} T_2 \cup \bigcup_{T_3 \in T_3} T_3| \). We prove the lemma by induction on \( |T_2| + |T_3| \). We have at most three columns if \( |T_2| + |T_3| = 1 \). Suppose that \( |T_2| + |T_3| \geq 2 \). For \( T \in T_2 \cup T_3 \) of minimize size, we claim that

\[
\ell = \left| \bigcup_{T'' \in T_2 \cup T_3} T'' \right| = \left| \bigcup_{T' \in T_2 \cup T_3 \setminus \{T\}} T' \right| + \left| T \setminus \bigcup_{T' \in T_2 \cup T_3 \setminus \{T\}} T' \right| \leq |T_2| + |T_3| + 2.
\]

The induction hypothesis gives us that \( |\bigcup_{T'' \in T_2 \cup T_3 \setminus \{T\}} T''| \leq |T_2| + |T_3| + 1 \). For the other term, since we can assume that \( T_2 \cup T_3 \setminus \{T\} \) has a set \( T_3 \) from \( T_3 \), we have \( |T \setminus \bigcup_{T'' \in T_2 \cup T_3 \setminus \{T\}} T''| \leq |T \setminus T_3| = |T| - |T \cap T_3|. \) Hence, it follows from Observation 8 that the second term is at most 1.

Our goal is to use Lemma 4 to derive that \( T_3 \) forms a sunflower, that is, we need that \( |T_3| \geq 5 \). The next lemma shows that this holds when \( T \) has at least 14 dirty columns.

\begin{itemize}
  \item \textbf{Lemma 15.} Let \( T \in \{0, 1\}^{n \times \ell} \) be a matrix consisting of \( \ell \geq 14 \) dirty columns with \( \delta(T) = 3 \). Then, \( |T_3| \geq 5 \) (for some reordering of rows).
\end{itemize}

\textbf{Proof.} Assume that the rows are reordered such that \( |T_3| \) is maximized. If \( |T_3| \leq 4 \), then we have \( |T_2| \geq 8 \) by Lemma 14. Let \( T_3 \in T_3 \). By Observation 13, \( T_2 \cap T_3 \neq \emptyset \) holds for
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Figure 5 Illustration (for smaller ℓ) of Lemma 10 (a) (left) and (b) (right). A black cell indicates that the entry differs from the last row vector in the corresponding column.

each \( T_2 \in \mathcal{T}_2 \). There are at most three sets \( T_2 \in \mathcal{T}_2 \) with \( T_2 \subseteq T_3 \). Thus, there are at least five sets \( T_2 \in \mathcal{T}_2 \) such that \( |T_2 \cap T_3| = 1 \). For each of these five sets, it holds that \( |T_2 \Delta T_3| = |T_2| + |T_3| - 2|T_2 \cap T_3| = 3 \). This contradicts the choice of the row reordering.

With Lemma 15 at hand, we are ready to reveal the structure of a diameter-three matrix (see Figure 5 for an illustration).

Lemma 16. Let \( T \in \{0, 1\}^{n \times \ell} \) be a matrix with \( \delta(T) \leq 3 \) and \( |T_3| \geq 5 \). Then, there exist \( j_1 \neq j_2 \in \{\ell\} \) such that the following hold:

- \( T_1 \subseteq \{j_1, j_2\} \) for each \( T_1 \in \mathcal{T}_1 \).
- \( T_2 \cap \{j_1, j_2\} \neq \emptyset \) for each \( T_2 \in \mathcal{T}_2 \).
- \( T_3 \supseteq \{j_1, j_2\} \) for each \( T_3 \in \mathcal{T}_3 \).

Moreover, exactly one of the following holds for \( T_2' = \{T_2 \in \mathcal{T}_2 \mid j_1 \in T_2 \land j_2 \notin T_2\} \) and \( T_2'' = \{T_2 \in \mathcal{T}_2 \mid j_1 \notin T_2 \land j_2 \in T_2\} \):

(a) \( T_2' = \emptyset \) or \( T_2'' = \emptyset \).
(b) \( T_2' = \{\{j_1, j_3\}\} \) and \( T_2'' = \{\{j_2, j_3\}\} \) for some \( j_3 \in \{\ell\} \).

Proof. Note that \( T_3 \) is 3-uniform by definition and note also that it is a weak \( \Delta \)-system with intersection size two by Observation 13. Hence, \( T_3 \) is a strong \( \Delta \)-system by Lemma 7. Let \( \{j_1, j_2\} \) be its core. Then, we have \( \{j_1, j_2\} \subseteq T_3 \) for each \( T_3 \in \mathcal{T}_3 \). It follows from Observation 13 and Lemma 10 that \( T_1 \subseteq \{j_1, j_2\} \) for each \( T_1 \in \mathcal{T}_1 \) and \( T_2 \cap \{j_1, j_2\} \neq \emptyset \) for each \( T_2 \in \mathcal{T}_2 \).

Now we show that either (a) or (b) holds. Suppose that \( |T_2'| \geq 2 \) and \( |T_2''| \geq 1 \), and let \( T_3 \neq T_2' \in \mathcal{T}_2' \) and \( T_3 \neq T_2'' \in \mathcal{T}_2'' \). Then, either \( T_2' \cap T_2'' = \emptyset \) or \( T_2' \cap T_2'' = \emptyset \) must hold. However, this is a contradiction because the corresponding row vectors have Hamming distance four. Thus, we have that \( |T_2'| \leq 1 \) or \( T_2'' = \emptyset \). Analogously, we obtain \( T_2' = \emptyset \) or \( |T_2''| \leq 1 \). If \( T_2' = \emptyset \) or \( T_2'' = \emptyset \), then (a) is satisfied. Otherwise, we have \( |T_2'| = |T_2''| = 1 \). Since \( |T_2' \Delta T_2''| = |T_2'| + |T_2''| - 2|T_2' \cap T_2''| \leq \delta(T) \leq 3 \), we obtain \( T_2' \cap T_2'' \neq \emptyset \) for each \( T_2' \in \mathcal{T}_2' \) and \( T_2'' \in \mathcal{T}_2'' \). Hence, (b) holds.

The following lemma establishes a connection to ConRMC. For \( v, v' \in \{0, 1\}^\ell \) and \( J \subseteq \{\ell\} \), we write \( d_J(v, v') \) to denote \( d(v[J], v'[J]) \).

Lemma 17. Let \( T \in \{0, 1\}^{n \times \ell} \) be a matrix consisting of dirty columns with \( \delta(T) = 3 \). If \( \ell \geq 14 \), then there exists \( v \in \{0, 1\}^\ell \) such that at least one of the following holds:
(a) There exists $j \in [\ell]$ such that $d_{\ell \setminus \{j\}}(v, T[i]) \leq 1$ for all $i \in [n]$.
(b) There exist three column indices $J = \{j_1, j_2, j_3\} \subseteq [\ell]$ such that all of the following hold for each $i \in [n]$:

- $d(v_{j_1}, t_{i,j_1}) \leq 2$.
- If $d(v_{j_1}, t_{i,j_1}) \geq 1$, then $d_{\ell \setminus J}(v, T[i]) = 0$.
- If $d(v_{j_1}, t_{i,j_1}) = 0$, then $d_{\ell \setminus J}(v, T[i]) \leq 1$.

Here $v_j = (v[j_1], v[j_2], v[j_3])$ and $t_{i,j} = (T[i,j_1], T[i,j_2], T[i,j_3])$ for each $i \in [n]$.

Proof. From Lemma 15 we can assume that $|T_3| \geq 5$. Hence, Lemma 16 applies. Let $j_1$ and $j_2$ be the according column indices. Let $v \in \{0,1\}^\ell$ be the row vector with

$$v[j] = \begin{cases} 1 - T[i,j] & \text{if } j \in \{j_1, j_2\}, \\ T[i,n] & \text{otherwise,} \end{cases}$$

for each $j \in [\ell]$. We claim that [a] corresponds to Lemma 16[a] and [b] corresponds to Lemma 16[b].

Suppose that Lemma 16[a] holds with $T_3' = \emptyset$ (the case $T_3' = \emptyset$ is completely analogous). We prove that $d_{\ell \setminus \{j_1\}}(v, T[i]) \leq 1$ for all $i \in [n]$. Since $d_{\ell \setminus \{j_2\}}(v, T[i]) = |\{j_1\} \triangle (D(T[i], T[i]) \setminus \{j_2\})|$, it suffices to show that $|\{j_1\} \triangle (T \setminus \{j_2\})| \leq 1$ holds for all $T \in T$. Due to Lemma 16 we have

- $|\{j_1\} \triangle (T \setminus \{j_2\})| \leq |\{j_1\} \triangle (T \setminus \{j_2\})| \leq 1$ for each $T \in T \cup T_3$ since $j_1 \in T$.

Hence, [a] is true.

Now, assume that Lemma 16[b] holds and let $J = \{j_1, j_2, j_3\}$. If there exists $i \in [n]$ with $d(v_{j_1}, t_{i,j_1}) = 3$, then this implies $\{j_3\} \in T_1$ which yields the contradiction $\{j_3\} \subseteq \{j_1, j_2\}$.

Further, for each $T \in T_1 \cup T_2$, we have $T \setminus J = \emptyset$. Hence, for the row vector $T[i]$ corresponding to $T$, we have $d_{\ell \setminus J}(v, T[i]) = 0$. Now, let $T_3 \in T_3$ with corresponding row vector $T[i]$. If $T_3 = \{j_1, j_2, j_3\}$, then $d(v_{j_1}, t_{i,j_1}) = 1$ and $d_{\ell \setminus J}(v, T[i]) = 0$. Otherwise, we have $T_3 = \{j_1, j_2, j\}$ for some $j \in [\ell] \setminus J$. Hence, $d(v_{j_1}, t_{i,j_1}) = 0$ and $d_{\ell \setminus J}(v, T[i]) = |T_3 \setminus \{j_1, j_2\}| = 1$. Hence, [b] is true. ▶

Based on the connection to ConRMC, we obtain a polynomial-time algorithm.

**Theorem 18.** $(0,3)$-DMC can be solved in $O(n^{\ell^2})$ time.

Proof. We first apply Theorem 12 to determine whether there exists a completion $T \in \{0,1\}^{n \times \ell}$ of $S \in \{0,1,\emptyset\}^{n \times \ell}$ such that $\delta(T) \leq 2$. If not, then it remains to determine whether there exists a completion $T$ with $\delta(T) = 3$. We can assume that $\ell \geq 14$ by Lemma 12.

We solve the problem by solving several instances of ConRMC based on Lemma 17.

For $j \in [\ell]$, let $I_j = (S[i, j] \setminus \{j\}, 1^n)$ be a ConRMC instance and let $I_1 = \{I_j \mid j \in [\ell]\}$. These instances correspond to Lemma 17[a].

Now, we describe the instances corresponding to Lemma 17[b]. Let $j_1, j_2, j_3 \in [\ell]$ be three distinct column indices and let $v_1, v_2, v_3 \in \{0,1\}$. We define an instance $I_{v_1,v_2,v_3} = (S_{j_1,j_2,j_3}, r)$ of ConRMC as follows:

- $S_{j_1,j_2,j_3} = S[i, \ell] \setminus \{j_1, j_2, j_3\}$.
- For each $i \in [n]$, let

$$r[i] = \begin{cases} 0 & \text{if } (S[i,j_1] = 1 - v_1) \lor (S[i,j_2] = 1 - v_2) \lor (S[i,j_3] = 1 - v_3), \\ 1 & \text{otherwise}. \end{cases}$$
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Let \( I_2 \) contain those instances \( I_{v_1,v_2,v_3} \) in which for each \( i \in [n] \) at least one of \( S[i,j_1] \neq 1 - v_1 \), \( S[i,j_2] \neq 1 - v_2 \), or \( S[i,j_3] \neq 1 - v_3 \) holds. We claim that \( I_1 \) is a Yes-instance and only if at least one instance in \( I_1 \) or \( I_2 \) is a Yes-instance.

- If \( I_j \in I_1 \) is a Yes-instance, then there exists \( v \in \{0,1\}^{|\ell|} \) such that \( d(v', S[i, [\ell] \setminus \{j\}]) \leq 1 \) for each \( i \in [n] \). Let \( T \) be the completion of \( S \) in which \( T[i, [\ell] \setminus \{j\}] = S[i, [\ell] \setminus \{j\}] \leftrightarrow v \) for each \( i \in [n] \). Then, we have
  \[
  d(T[i], T[i']) \leq d(T[i, [\ell] \setminus \{j\}], T[i', [\ell] \setminus \{j\}]) + 1 \\
  \leq d(v, T[i, [\ell] \setminus \{j\}]) + d(v, T[i', [\ell] \setminus \{j\}]) + 1 \leq 3
  \]
  for each \( i, i' \in [n] \).

- If \( I_{v_1,v_2,v_3} = (S_{j_1,j_2,j_3}, r) \in I_2 \) is a Yes-instance with solution \( v' \in \{0,1\}^{|\ell| - 3} \), then let \( v \in \{0,1\}^{|\ell|} \) be the row vector obtained from \( v' \) by inserting \( v_1, v_2, \) and \( v_3 \) in the \( j_1 \)-th, \( j_2 \)-th, and \( j_3 \)-th column, respectively, and let \( T \) be the completion of \( S \) in which \( T[i] = S[i] \leftrightarrow v \) or each \( i \in [n] \). We prove that \( \delta(T) \leq 3 \). Let \( R_x \subseteq [n] \) be the set of row indices \( i \) with \( r[i] = x \) for \( x \in \{0,1\} \). Then, we have that
  \[
  d(v, T[i]) = d_{(j_1,j_2,j_3)}(v, T[i]) + d_{(\ell \setminus \{j_1,j_2,j_3\})}(v, T[i]) \leq 2 + 0 = 2 \text{ for each } i \in R_0 \\
  d(v, T[i]) = d_{(j_1,j_2,j_3)}(v, T[i]) + d_{(\ell \setminus \{j_1,j_2,j_3\})}(v, T[i]) \leq 0 + 1 = 1 \text{ for each } i \in R_1.
  \]
  By the triangle inequality, we obtain \( d(T[i], T[i']) \leq 3 \) for each \( i, i' \in [n] \) with \( i \in R_1 \) or \( i' \in R_1 \). Thus, it suffices to show \( \delta(T[R_0]) \leq 3 \). Since \( T[i, [\ell] \setminus \{j_1,j_2,j_3\}] = T[i', [\ell] \setminus \{j_1,j_2,j_3\}] = v' \) for each \( i, i' \in R_0 \), this clearly holds.

The reverse direction is easily verified using Lemma 17.

Overall, we construct \( O(\ell^3) \) ConRMC instances, each of which can be solved in \( O(n\ell^4) \) time [39] Theorem 1]. Hence, \( (0,3) \)-DMC can be solved in \( O(n\ell^4) \) time.

Our algorithms work via reductions to CONRMC. Although CONRMC imposes an upper bound on the diameter implicitly by the triangle inequality, it is seemingly difficult to enforce any lower bounds (that is, \( \alpha > 0 \)). In the next subsection, we will see polynomial-time algorithms for \( \alpha > 0 \), based on reductions to the graph factorization problem.

### 3.2 Polynomial time for \( \beta = \alpha + 1 \)

We now give polynomial-time algorithms for \((\alpha, \beta)\)-DMC with constant \( \alpha > 0 \) given that \( \beta \leq \alpha + 1 \). As in Section 3.1, our algorithms exploit combinatorial structures revealed by Deza’s theorem (Lemmas 6 and 7). Recall that \( \mathcal{T} \) denotes a set system obtained from a complete matrix \( T \) (Definition 3). We show that \( \mathcal{T} \) essentially is a sunflower when \( \gamma(T) \geq \alpha \) and \( \delta(T) \leq \alpha + 1 \). For the completion into such a sunflower, it suffices to solve the following matrix completion problem, which we call SUNFLOWER MATRIX COMPLETION.

**SUNFLOWER MATRIX COMPLETION (SMC)**

**Input:** An incomplete matrix \( S \in \{0, 1, \Box\}^{n \times \ell} \) and \( s, m \in \mathbb{N} \).

**Question:** Is there a completion \( T \in \{0, 1\}^{n \times \ell} \) of \( S \) such that \( D(T[1], T[n]), \ldots, D(T[n-1], T[n]) \) are pairwise disjoint sets each of size at most \( s \) and \( \sum_{i \in [n-1]} |D(T[i], T[n])| \geq m \). Intuitively speaking, the problem asks for a completion into a sunflower with empty core and bounded petal sizes. All algorithms presented in this subsection are via reductions to SMC. First, we show that SMC is indeed polynomial-time solvable. We prove this using a well-known polynomial-time algorithm for the graph problem \((g, f)\)-FACTOR [20].
(g, f)-Factor

Input: A graph \( G = (V, E) \), functions \( g, f : V \rightarrow \mathbb{N} \), and \( m' \in \mathbb{N} \).

Question: Does \( G \) contain a subgraph \( G' = (V, E') \) such that \( |E'| \geq m' \) and \( g(v) \leq \deg_{G'}(v) \leq f(v) \) for all \( v \in V \)?

\[ \alpha, \alpha \]

Lemma 19. For constant \( s > 0 \), SMC can be solved in \( O(n^s \sqrt{n + \ell}) \) time.

Proof. Let \((S, s, m)\) be an SMC instance. Let \( a^x_j \) be the number of occurrences of \( x \in \{0, 1\} \) in \( S[i,j] \) for each \( j \in [\ell] \). We can assume that \( a^0_j \geq a^1_j \) for each \( j \in [\ell] \) (otherwise swap the occurrences of 0’s and 1’s in the column). If \( a^0_j \geq 2 \) and \( S[m, j] = 1 \) for some \( j \in [\ell] \), then we can return No since there will be two intersecting sets. Also, if \( a^0_j \geq 2 \), then we return No.

We construct an instance of \((g, f)\)-Factor as follows. We introduce a vertex \( u_i \) for each \( i \in [n-1] \) and a vertex \( v_j \) for each \( j \in [\ell] \). The resulting graph \( G \) will be a bipartite graph with one vertex subset \( \{u_1, \ldots, u_{n-1}\} \) representing rows and the other \( \{v_1, \ldots, v_{\ell}\} \) representing columns. Essentially, we add an edge between \( u_i \) and \( v_j \) if the column \( S[i,j] \) can be completed such that the \( i \)-th entry differs from all other entries on \( S[i,j] \) (see Figure 5 for an illustration). Intuitively, such an edge encodes the information that column index \( j \) can be contained in a petal of the sought sunflower. Formally, there is an edge \( \{u_i, v_j\} \) if and only if there is a completion \( t_j \in \{0, 1\}^n \) of \( S[i,j] \) in which \( t_j[h] = 1 - t_j[h] \) for all \( h \in [n-1] \setminus \{i\} \). We set \( g(u_i) := 0 \) and \( f(u_i) := s \) for each \( i \in [n-1] \), \( g(v_j) := a^0_j \) and \( f(v_j) := 1 \) for each \( j \in [\ell] \), and \( m' := m \). This construction can be done in \( O(n^s \ell) \) time. To see this, note that the existence of an edge \( \{u_i, v_j\} \) only depends on \( a^0_j, a^1_j \), and \( S[i,j] \).

- If \( a^0_j \leq 1 \) and \( a^1_j = 0 \), then add the edge \( \{u_i, v_j\} \). The corresponding completion \( t_j \) can be seen as follows:
  - If \( S[h,j] \neq \square \) for all \( h \in [n-1] \), then let \( t_j[i] := 1 \) and let \( t_j[h] := 0 \) for all \( h \in [n] \setminus \{i\} \).
  - If \( S'[h,j] = 0 \) for some \( h \in [n-1] \), then let \( t_j[i] := 1 \) and let \( t_j[h] := 0 \) for all \( h \in [n] \setminus \{i\} \). Otherwise, let \( t_j[h] := 1 \) for all \( h \in [n] \setminus \{i\} \).
- If \( a^0_j = 1 \) and \( a^1_j = 1 \), then add the edge \( \{u_i, v_j\} \) if \( S[i,j] \neq \square \).
- If \( a^0_j \geq 2 \) and \( a^1_j = 0 \), then add the edge \( \{u_i, v_j\} \) if \( S[i,j] = \square \).
- If \( a^0_j \geq 2 \) and \( a^1_j = 1 \), then add the edge \( \{u_i, v_j\} \) if \( S[i,j] = 1 \) (because \( S[n,j] \) must be completed with 0).

The correctness of the reduction easily follows from the definition of an edge: If \( T \) is a solution for \((S, s, m)\), then the corresponding subgraph of \( G \) contains the edge \( \{u_i, v_j\} \) for each \( i \in [n-1] \) and each \( j \in D(T[i], T[n]) \). Conversely, a completion of \( S \) is obtained from a subgraph \( G' \) by taking for each edge \( \{u_i, v_j\} \) the corresponding completion \( t_j \) as the \( j \)-th column. Note that no vertex \( v_j \) can have two incident edges since \( f(v_j) = 1 \). Moreover, if \( v_j \) has no incident edges, then this implies that \( g(v_j) = a^1_j = 0 \). Hence, we can complete all missing entries in column \( j \) by 0.

Regarding the running time, note that the constructed graph \( G \) has at most \( n^s \ell \) edges and \( \sum_{i \in [n-1]} f(u_i) \in O(n) \) and \( \sum_{j \in [\ell]} f(v_j) \in O(\ell) \). Since \((g, f)\)-Factor can be solved in \( O(|E| \sqrt{f(V)}) \) time \( [20] \) for \( f(V) = \sum_{v \in V} f(v) \), SMC can be solved in \( O(n^s \sqrt{n + \ell}) \) time.

Using Lemma 19, we first show that \((\alpha, \alpha)\)-DMC can be solved in polynomial time.

Theorem 20. \((\alpha, \alpha)\)-DMC can be solved in \( O(n^s \sqrt{n + \ell}) \) time.
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\begin{figure}[h]
\centering
\begin{tabular}{cccccc}
  & $v_1$ & $v_2$ & $v_3$ & $v_4$ & $v_5$ \\
\hline
$u_1$ & 1 & 1 & 0 & 0 & 0 \\
$u_2$ & 0 & 0 & 1 & 0 & 0 \\
$u_3$ & 1 & 0 & 0 & 0 & 1 \\
$u_4$ & 1 & 0 & 0 & 1 & 0 \\
\end{tabular}
\caption{A completion of a $5 \times 5$ incomplete matrix (left). The known entries are highlighted in gray. A bipartite graph as constructed in the reduction (right). Note that the entries framed by thick lines (which differ from all others in the same column) correspond to the subgraph represented by the thick lines.}
\end{figure}

**Proof.** We first show that $(\alpha, \alpha)$-DMC can easily be solved if $\alpha$ is odd. Consider row vectors $u, v, w \in \{0, 1\}^\ell$ and let $U := D(u, v)$ and $W := D(v, w)$. Then, $d(u, v) + d(v, w) = |U| + |W| + |([U] + [W] - 2[U \cap W] = 2(|U| + |W| - |U \cap W|)$ and hence $d(u, v) + d(v, w) + d(w, u)$ is even. Thus, we can immediately answer **No** if $n \geq 3$. It is also easy to see that DMC can be solved in linear time if $n \leq 2$.

We henceforth assume that $\alpha$ is even. Eiben et al. [13, Theorem 34] provided a linear-time algorithm for $(0, \alpha)$-DMC with constant $n$ (and arbitrary $\alpha$) using reductions to integer linear programming (ILP). To ensure that each pairwise distance is at most $\alpha$, we express this property as a linear constraint. By simply adding an analogous constraint enforcing that each pairwise distance is at least $\alpha$, it follows that $(\alpha, \alpha)$-DMC can also be solved in linear time for constant $n$. So we can assume that $n \geq (\alpha/2)^2 + (\alpha/2) + 3$ (otherwise $(\alpha, \alpha)$-DMC can be solved in linear time). We claim that there is a completion $T$ of $S$ with $\gamma(T) = \delta(T) = \alpha$ if and only if the SMC instance $(S', \alpha/2, \alpha n/2)$ is a **Yes**-instance for the matrix $S' \in \{0, 1, \square\}^{(n+1) \times \ell}$ obtained from $S$ with an additional row vector $\square^\ell$.

\(\Rightarrow\) Let $T$ be a completion of $S$ with $\gamma(T) = \delta(T) = \alpha$. Then, $T$ is a weak $\Delta$-system with intersection size $\alpha/2$. For any two sets $U, W \in T$, we have $|U \cap W| = (|U| + |W| - |U \triangle W|)/2 = \alpha/2$. Since $|T| \geq (\alpha/2)^2 + (\alpha/2) + 2$, Lemma 6 tells us that $T$ is a sunflower. Let $C$ be the core of $T$. Consider the completion $T'$ of $S'$ such that

- $T'[n, :] = T$,
- $T'[n + 1, j] = 1 - T[n, j]$ for each $j \in C$, and
- $T'[n + 1, j] = T[n, j]$ for each $j \notin C$.

Note that $D(T'[i], T'[n + 1]) = D(T[i], T[n]) \setminus C$ for each $i \in [n - 1]$. Note also that $D(T'[n], T'[n + 1]) = C$. Hence, $D(T'[1], T'[n + 1]), \ldots, D(T'[n], T'[n + 1])$ are pairwise disjoint sets of size $\alpha/2$.

\(\Leftarrow\) Let $T'$ be a completion of $S'$ such that $D(T'[1], T'[n + 1]), \ldots, D(T'[n], T'[n + 1])$ are pairwise disjoint sets of size $\alpha/2$. For the completion $T = T'[n, :]$ of $S$, it holds that $d(T[i], T[i']) = |D(T[i], T'[n + 1])| \leq |D(T'[i], T'[n + 1])| + |D(T'[i'], T'[n + 1])| = \alpha$ for each $i, i' \in [n]$.

Now we proceed to develop polynomial-time algorithms for the case $\alpha + 1 = \beta$. We will make use of the following observation made by Froese et al. [19, Proof of Theorem 9].

**Observation 21.** Let $T \in \{0, 1\}^{n \times \ell}$ with $\gamma(T) \geq \alpha$ and $\delta(T) \leq \beta = \alpha + 1$. For $T_\alpha \neq T_\beta \in T_\alpha$ and $T_\beta \neq T_\alpha \in T_\beta$, it holds that $|T_\alpha \cap T_\beta| = [\alpha/2]$, $|T_\beta \cap T_\alpha| = [\beta/2]$, and $|T_\alpha \cap T_\beta| = [\alpha/2] = [\beta/2]$. 

\[\Box\]
Theorem 20. Let $T, T' \in \mathcal{T}_\alpha \cup \mathcal{T}_\beta$, we have $|T| + |T' - 2|T \cap T'| \in \{\alpha, \beta\}$. If $T, T' \in \mathcal{T}_\alpha$ or

$T, T' \in \mathcal{T}_\beta$, then $|T| + |T' - 2|T \cap T'|$ is even, and thus $|T| + |T' - 2|T \cap T'| \in 2\{\alpha/2 = 2\beta/2\}$. It follows that $|T_\alpha \cap T_\beta| = \lfloor \alpha/2 \rfloor$ and $|T_\beta \cap T'_\beta| = \lceil \beta/2 \rceil$. For the last expression, $|T_\alpha \cap T'_\beta| = \alpha + 2\beta - 2|T_\alpha \cap T'_\beta| \in \{\alpha, \beta\}$. Hence, $|T_\alpha \cap T'_\beta| = \alpha/2 = \beta/2$.

Surprisingly, an odd value of $\alpha$ seems to allow for significantly more efficient algorithms than an even value.

\begin{itemize}
  \item \textbf{Theorem 22.} $(\alpha, \beta)$-DMC with $\beta = \alpha + 1$ can be solved in
  \begin{enumerate}
    \item $O(ns\sqrt{n+1})$ time for odd $\alpha$, and
    \item $(nt)^{O(n^2)}$ time for even $\alpha$.
  \end{enumerate}
\end{itemize}

Proof. (i) We can assume that $n \geq \beta^2/2 + \beta + 7$ holds since otherwise the problem is linear-time solvable via a reduction to ILP as in the proof of Theorem 20. Suppose that $\mathbf{S}$ admits a completion $\mathbf{T}$ with $\gamma(\mathbf{T}) \geq \alpha$ and $\delta(\mathbf{T}) \leq \beta$. Since $\mathbf{T} = \mathcal{T}_\alpha \cup \mathcal{T}_\beta$ and $|T| \geq \beta^2/2 + \beta + 6$, it follows that $\max\{|T_\alpha|, |T_\beta|\} \geq c := \omega(\beta/2)^2 + (\beta/2)^2 + 3$. We consider two cases depending on the size of $T_\alpha$ and $T_\beta$.

1. Suppose that $|T_\alpha| \geq c$. Since $\mathcal{T}_\alpha$ is a weak $\Delta$-system with intersection size $(\alpha - 1)/2$, $\mathcal{T}_\alpha$ is a sunflower with a core of size $(\alpha - 1)/2$ and petals of size $(\alpha + 1)/2$ by Lemma 10. We claim that $\mathcal{T}_\beta = \emptyset$. Suppose not and let $T_\beta \in \mathcal{T}_\beta$. We then obtain $|T_\alpha \cap T_\beta| = (\alpha + 1)/2$ for all $T_\alpha \in \mathcal{T}_\alpha$ by Observation 21, which contradicts Lemma 10.

2. Suppose that $|T_\beta| \geq c$. By Lemma 10, $T_\beta$ is a sunflower whose core $C$ has size $\beta/2$. By Observation 21 and Lemma 10, $T_\alpha \supseteq C$ holds for each $T_\alpha \in \mathcal{T}_\alpha$. Now suppose that there exist $T_\alpha \neq T_\beta \in \mathcal{T}_\alpha$. Since $C \subset T_\alpha$ and $C \subset T_\beta$, it follows that $|T_\alpha \cap T_\beta| \geq \beta/2$, thereby contradicting Observation 21. Hence, we have $|T_\alpha| \leq 1$.

We construct an instance $I$ of SMC covering both cases above, as in Theorem 20. We use the matrix $\mathbf{S}'$ obtained from $\mathbf{S}$ by appending a row vector $\mathbf{0}$, and we set $s := \beta/2$ and $m := ns - s$. Basically, we allow at most one “petal” to have size $s - 1$. We return \textbf{Yes} if and only if $I$ is a \textbf{Yes}-instance. The correctness can be shown analogously to the proof of Theorem 20.

(i) Suppose that there is a completion $\mathbf{T}$ of $\mathbf{S}$ with $\gamma(\mathbf{T}) \geq \alpha$ and $\delta(\mathbf{T}) \leq \beta$. Again, we can assume that $n > 2c$ for $c := (\beta/2)^2 + (\beta/2)^2 + 4$, and consider a case distinction regarding the size of $T_\alpha$ and $T_\beta$.

1. Suppose that $|T_\alpha| \geq c$ and $|T_\beta| \geq c$. It follows from Observation 21 and Lemmas 6 and 7 that $T_\alpha$ and $T_\beta$ are sunflowers. Let $C_\alpha$ and $C_\beta$ be the cores of $T_\alpha$ and $T_\beta$, respectively. Note that $|C_\alpha| = \alpha/2$ and $|C_\beta| = \alpha/2 + 1$, and hence $C_\alpha \subseteq C_\beta$ holds by Observation 21 and Lemma 10. Let $j \in [\ell]$ be such that $C_\alpha \cup \{j\} = C_\beta$ and let $\mathbf{T}' := \mathbf{T}[;[\ell] \setminus \{j\}]$. Then, the set family $\mathbf{T}'$ is a sunflower with a core of size $\alpha/2$ and petals of size $\alpha/2$. Hence, there exists $j \in [\ell]$ such that the $(\alpha, \alpha)$-DMC instance $\mathbf{S}[;[\ell] \setminus \{j\}]$ is a \textbf{Yes}-instance. On the other hand, if there is a completion $\mathbf{T}'$ of $\mathbf{S}[;[\ell] \setminus \{j\}]$ with $\gamma(\mathbf{T}') = \delta(\mathbf{T}') = \alpha$, then $\gamma(\mathbf{T}) \geq \alpha$ and $\delta(\mathbf{T}) \leq \alpha + 1$ hold for any completion $\mathbf{T}$ of $\mathbf{S}$ with $\mathbf{T}[;[\ell] \setminus \{j\}] = \mathbf{T}'$.

2. Suppose that $|T_\alpha| \geq c$ and $|T_\beta| < c$. The same argument as above shows that $T_\alpha \cap T_\beta \cap C$ holds for each $T_\alpha \in \mathcal{T}_\alpha$ and $T_\beta \in \mathcal{T}_\beta$, where $C$ is the size-$\alpha/2$ core of sunflower $T_\alpha$. Let $I_\beta = \{i \in [n - 1] | d(T[i], T[n]) = \beta\}$ be the row indices that induce the sets in $T_\beta$ and let $J_\alpha = \bigcup_{T_\beta \in T_\beta} T_\beta$. Consider $\mathbf{T}' = \mathbf{S}[n \setminus I_\beta, [\ell] \setminus (C \cup J_\alpha)]$ and note that the family $\mathbf{T}'$ consists of pairwise disjoint sets, each of size $\alpha/2$. We use this observation to obtain a reduction to SMC. The idea is to test all possible choices for $\mathbf{T}'$, that is, we simply try out all possibilities to choose the following sets:
\[ C \subseteq \ell \text{ of size exactly } \alpha/2. \]
\[ I^c \subseteq [n-1] \text{ of size at most } c. \]
\[ J^c \subseteq [\ell] \setminus C \text{ of size at most } \beta \cdot c \text{ such that } d(\ell \setminus (C \cup J^c), (S[i^c], S[n])) = 0 \text{ for all } i^c \in I^c. \]

For each possible choice, we check whether it allows for a valid completion. Formally, it is necessary that the following exist:

- A completion \( t_{C} \) of \( S[n, C] \) such that \( S[i, j] \neq t_{C}[j] \) for all \( i \in [n-1] \) and \( j \in C \).
- A completion \( t_{J^c} \) of \( S[n, J^c] \) such that \( d(t_{J^c}, S[i, J^c]) = 0 \) for all \( i \in [n-1] \setminus I^c \).
- A completion \( t_{i^c} \) of \( S[i^c, J^c] \) for each \( i^c \in I^c \) such that \( d(t_{i^c}, t_{J^c}) = \alpha/2 + 1 \) for each \( i^c \in I^c \) and \( d(t_{i^c}, t_{J^c}) = \alpha \) for each \( i^c \neq i' \) \( i^c \in I^c \).

The existence of the above completions can be checked in \( O(n) \) time. We then construct an SMC instance \( (S', \alpha/2, (n - |I^c| - 1) \cdot \alpha/2) \), where \( S' \) is an incomplete matrix with \( n' = n - |I^c| \) rows and \( \ell - |C| - |J^c| \) columns defined as follows:

- \( S'[n' - 1] = S[n - 1 \setminus I^c, \ell \setminus (C \cup J^c)]. \)
- \( S'[n', j] = \square \) for each \( j \in \ell \setminus (C \cup J^c) \) such that \( S[i^c, j] = \square \) for all \( i^c \in I^c \cup \{n\} \).
- \( S'[n', j] = S[i, j] \) for each \( j \in \ell \setminus (C \cup J^c) \) such that \( S[i, j] \neq \square \) for some \( i^c \in I^c \cup \{n\} \).

Overall, we solve at most \( (n\ell)O(\alpha^3) \) SMC instances and hence it requires \( (n\ell)O(\alpha^3) \) time.

3. Suppose that \( |\mathcal{T}_1| < c \) and \( |\mathcal{T}_3| \geq c \). Let \( i \in [n-1] \) be such that \( d(T[i], T[n]) = \beta \). Then, \( d(T[i], T[i']) = \alpha \) holds for each \( i' \in [n-1] \setminus \{i\} \) with \( d(T[i'], T[n]) = \beta \). Since there are at least \( c - 1 = (\beta/2)^2 + (\beta/2) + 3 \) such row indices, it follows that this case is essentially equivalent to the previous case (by considering row \( i \) as the last row).

\[ \square \]

A natural question is whether one can extend our approach above to the case \( \beta = \alpha + 2 \) (particularly \( \alpha = 1 \) and \( \beta = 3 \)). The problem is that the petals of the sunflowers \( \mathcal{T}_2 \) and \( \mathcal{T}_3 \) may have nonempty intersections. Thus, reducing to SMC to obtain a polynomial-time algorithm is probably hopeless.

### 3.3 NP-hardness

Hermelin and Rozenberg [27] Theorem 5] proved that CONRMC (under the name CLOSEST STRING WITH WILDCARDS) is NP-hard even if \( r[i] = 2 \) for all \( i \in [n] \). Using this result, we prove the following.

\[ \blacktriangleleft \textbf{Theorem 23.} \ (\alpha, \beta)-DMC \text{ is NP-hard if } \beta \geq 2[\alpha/2] + 4. \]

\[ \textbf{Proof.} \] We give a polynomial-time reduction from CONRMC. Let \( (S \in \{0, 1, \square\}^{n \times \ell}, r) \) be a CONRMC instance with \( r[i] = 2 \) for all \( i \in [n] \).

Let \( C \in \{0, 1\}^{(n+1) \times m} \) be the binary matrix with \( m := (n - 1) \cdot [\alpha/2] + \beta - 2 \) columns obtained by horizontally stacking

- the \((n+1) \times (n+1)\) identity matrix \([\alpha/2]\) times and
- the column vector \((0^{n+1})^T \beta - 2[\alpha/2] - 2\) times.

Since the pairwise row Hamming distances in the identity matrix are all two, we have that:

\[ d(C[i], C[i']) = 2[\alpha/2] \text{ for each } i \neq i' \in [n] \text{ and} \]
\[ d(C[i], C[n+1]) = \beta - 2 \text{ for each } i \in [n]. \]
Consider the matrix $S' \in \{0, 1, \square\}^{(n+1)\times(\ell+m)}$ obtained from $S$ by adding the row $\square^\ell$ and then horizontally appending $C$. We show that there exists a vector $v \in \{0, 1\}^\ell$ with $d(v, S[i]) \leq 2$ for all $i \in [n]$ if and only if $S'$ admits a completion $T'$ with $\gamma(T') \geq \alpha$ and $\delta(T') \leq \beta$.

$(\Rightarrow)$ Let $T'$ be the completion of $S'$ such that for each $i \in [n+1]$, $T'[i, [\ell]] = S'[i] \leftarrow v$. Then, we have the following:

1. $\gamma(T') \geq \gamma(C) = 2[\alpha/2] \geq \alpha$.
2. $d(T'[i], T'[n+1]) = d(S[i], v) + \beta - 2 \leq \beta$ for each $i \in [n]$.
3. By the triangle inequality, $d(T'[i], T'[i']) \leq d(v, S[i]) + d(v, S[i']) + d(C[i], C[i']) \leq 2 + 2 + 2[\alpha/2] \leq \beta$ holds for each $i, i' \in [n]$.

$(\Leftarrow)$ Let $v = T'[n+1, [\ell]]$. It is easy to see that $d(S[i], v) = d(T'[i], T'[n+1]) - d(C[i], C[n+1]) \leq \beta - (\beta - 2) \leq 2$ holds for each $i \in [n]$. □

It remains open whether NP-hardness also holds for $(\alpha, \alpha + 3)$-DMC with $\alpha \geq 1$ (recall that $(0, 3)$-DMC is polynomial-time solvable). In Section 4.1 however, we show NP-hardness for $\beta = \alpha + 3$ when $\alpha$ and $\beta$ are part of the input.

4 Bounded Number $k$ of Missing Entries per Row

In this section, we consider DMC with $\alpha$ and $\beta$ being part of the input, hence not necessarily being constants. We consider the maximum number $k$ of missing entries in any row as a parameter (DMC is clearly trivial for $k = 0$). We obtain two polynomial-time algorithms and two NP-hardness results. Our polynomial-time algorithms are based on reductions to 2-SAT.

4.1 Polynomial-time algorithms

We show that DMC can be solved in polynomial time when $k = 1$, via a reduction to 2-SAT. For a Boolean variable $x$, we use $(x = 1)$ and $(x \neq 0)$ to denote the positive literal $x$. Similarly, we use $(x = 0)$ and $(x \neq 1)$ for the negative literal $\neg x$.

**Theorem 24.** DMC can be solved in $O(n^2 \ell)$ time

(i) for $k = 1$, and

(ii) for $k = 2$ and $\alpha = \beta$.

**Proof.** [1] We construct a 2-CNF formula $\phi$ of polynomial size such that $\phi$ is satisfiable if and only if the input matrix $S$ admits a completion $T$ with $\gamma(T) \geq \alpha$ and $\delta(T) \leq \beta$.

First, we compute the distances $d(S[i], S[i'])$ for each $i, i' \in [n]$ in $O(n^2 \ell)$ time. Clearly, if there exists a pair with distance less than $\alpha - 2$ or larger than $\beta$, then we have a **No**-instance. Let $I \subseteq [n]$ be the set of row indices corresponding to row vectors with a missing entry and let $j_i \in [\ell]$ be such that $S[i, j_i] = \square$.

We introduce a variable $x_i$ for each $i \in I$, where $x_i$ is set to true if $S[i, j_i]$ is completed with a 1. We construct the formula $\phi$ as follows:

- For each $i < i' \in [n]$ with $d(S[i], S[i']) = \alpha - 2$:
  - If $i \not\in I$ or $i' \not\in I$, or $j_i = j_i'$, then return **No**. Otherwise, add the clauses $(x_i = 1 - S[i', j_i])$ and $(x_i = 1 - S[i, j_i'])$.
- For each $i < i' \in [n]$ with $d(S[i], S[i']) = \alpha - 1$:
  - If $i \not\in I$ and $i' \not\in I$, then return **No**.
Thus, \( \phi \) we add the following clauses:

- For each \( i < i' \in [n] \) with \( d(S[i], S[i']) = \beta \):
  - If \( i \in I \) and \( i' \not\in I \), then add the clause \( (x_i = S[i', j_i]) \).
  - If \( i \not\in I \) and \( i' \in I \), then add the clause \( (x_{i'} = S[i, j_{i'}]) \).
  - If \( i \in I \) and \( i' \in I \) and \( j_i = j_{i'} \), then add the clauses \( (x_i \lor \neg x_{i'}) \) and \( (\neg x_i \lor x_{i'}) \).
  - If \( i \in I \) and \( i' \in I \) and \( j_i \neq j_{i'} \), then add the clause \( (x_i \neq S[i', j_i] \lor x_{i'} \neq S[i, j_{i'}]) \).

It is easy to see that these clauses ensure that \( \gamma(T) \geq \alpha \). Similarly, to ensure that \( \delta(T) \leq \beta \), we add the following clauses:

- For each \( i < i' \in [n] \) with \( d(S[i], S[i']) = \beta - 1 \):
  - If \( i \in I \) and \( i' \not\in I \), then add the clause \( (x_i = S[i', j_i]) \).
  - If \( i \not\in I \) and \( i' \in I \), then add the clause \( (x_{i'} = S[i, j_{i'}]) \).
  - If \( i \in I \) and \( i' \in I \) and \( j_i = j_{i'} \), then add the clauses \( (x_i \lor \neg x_{i'}) \) and \( (\neg x_i \lor x_{i'}) \).
  - If \( i \in I \) and \( i' \in I \) and \( j_i \neq j_{i'} \), then add the clause \( (x_i = S[i', j_i]) \) and \( (x_{i'} = S[i, j_{i'}]) \).

Thus, \( \phi \) is of size \( O(n^2) \) and can be solved in \( O(n^2) \) time \([1]\). The correctness follows directly from the construction.

\[ \text{(ii)} \] Since \( k = 2 \), there are at most four possible ways to complete the row vector \( S[n] \). So we can afford to try all possible completions of \( S[n] \). Without loss of generality, assume that \( S[n] = 0^n \). We show that DMC can be solved in \( O(n^2) \) time.

First, we check whether \( \alpha - 2 \leq d(S[i], 0^n) \leq \alpha \) holds for each \( i \in [n-1] \) (otherwise, we return \( \text{No} \)). We do the following for each \( i \in [n-1] \):

- \( S[i] \) contains exactly one missing entry: If \( d(S[i], 0^n) = \alpha - 1 \), then fill the missing entry by 1. If \( d(S[i], 0^n) = \alpha \), then fill the missing entry by 0. If \( d(S[i], 0^n) = \alpha - 2 \), then return \( \text{No} \).
- \( S[i] \) contains exactly two missing entries: If \( d(S[i], 0^n) = \alpha - 2 \), then fill both missing entries by 1. If \( d(S[i], 0^n) = \alpha \), then fill both missing entries by 0.

Now, each row vector either contains no missing entry or exactly two missing entries. Let \( I_0 \subseteq [n] \) be the set of row indices corresponding to row vectors without any missing entry and let \( I_2 = [n] \setminus I_0 \). We check in \( O(n^2) \) time whether all pairwise Hamming distances in \( S[I_0] \) are \( \alpha \). If not, then we return \( \text{No} \). Note that we have \( d(S[i], 0^n) = \alpha - 1 \) for each \( i \in I_2 \), and thus there are exactly two ways to complete \( S[i] \): One missing entry filled by 1 and the other by 0, or vice versa. For each \( i \in I_2 \), let \( j^1_i < j^2_i \in [n] \) be such that \( S[i, j^1_i] = S[i, j^2_i] = \emptyset \).

We verify whether the following necessary conditions hold:

- \( d(S[i_0], S[i]) = \alpha - 1 \) for each \( i_0 \in I_0 \) and \( i \in I_2 \) with \( S[i_0, j^1_i] = S[i_0, j^2_i] \).
- \( d(S[i_0], S[i]) = \alpha - 2, \alpha \) for each \( i_0 \in I_0 \) and \( i \in I_2 \) with \( S[i_0, j^1_i] \neq S[i_0, j^2_i] \).
- Let \( i \neq i' \in I_2 \) be such that \( \{j^1_i, j^2_i\} \cap \{j^1_{i'}, j^2_{i'}\} = \emptyset \). Observe that if \( S[i', j^1_{i'}] = S[i', j^2_{i'}] \), then the completion of \( S[i] \) increases the distance between \( S[i] \) and \( S[i'] \) by exactly one. Otherwise, the distance either stays the same or increases by exactly two. It is analogous for the completion of \( S[i'] \). Thus, the following must hold for each \( i \neq i' \in I_2 \) with \( \{j^1_i, j^2_i\} \cap \{j^1_{i'}, j^2_{i'}\} = \emptyset \):
  - \( d(S[i], S[i']) = \alpha - 2 \) if \( S[i, j^1_i] = S[i, j^2_i] \) and \( S[i', j^1_{i'}] = S[i', j^2_{i'}] \).
  - \( d(S[i], S[i']) = \alpha - 3, \alpha - 1 \) if either \( S[i, j^1_i] = S[i, j^2_i] \) and \( S[i', j^1_{i'}] = S[i', j^2_{i'}] \), or \( S[i, j^1_i] \neq S[i, j^2_i] \) and \( S[i', j^1_{i'}] = S[i', j^2_{i'}] \).
  - \( d(S[i], S[i']) = \alpha - 4, \alpha - 2, \alpha \) if \( S[i, j^1_i] \neq S[i, j^2_i] \) and \( S[i', j^1_{i'}] \neq S[i', j^2_{i'}] \).
Let \( i \neq i' \in I_2 \) be such that \( j^2_i = j^1_{i'} \). Note that \( S[i, j^2_i] \) and \( S[i', j^1_{i'}] \) are completed by the same value, if and only if \( S[i, j^1_{i'}] \) and \( S[i', j^2_i] \) are completed by the same value. Hence, the following must hold for each \( i \neq i' \in I_2 \) with \( j^2_i = j^1_{i'} \):

- \( d(S[i], S[i']) \in \{ \alpha - 2, \alpha \} \) if \( S[i, j^2_i] = S[i', j^1_{i'}] \).
- \( d(S[i], S[i']) \in \{ \alpha - 3, \alpha - 1 \} \) if \( S[i, j^2_i] \neq S[i', j^1_{i'}] \).

We return No if at least one of the above fails. Clearly this requires \( O(n^2 \ell) \) time.

Now, we construct a 2-CNF formula which is satisfiable if and only if our DMC instance is a Yes-instance. We introduce a variable \( x_i \) for each \( i \in I_2 \), which basically encodes the completion of \( S[i] \). Intuitively speaking, \((S[i, j^1_{i}], S[i, j^2_i])\) are filled by \((1,0)\) if \( x_i \) is true and by \((0,1)\) if \( x_i \) is false. We add clauses as follows:

- For each \( i_0 \in I_0 \) and \( i \in I_2 \) with \( d(S[i_0], S[i]) = \alpha - 2 \), add a singleton clause \((x_i)\) if \( (S[i_0, j^1_{i_0}], S[i_0, j^2_{i_0}]) = (0,1) \) and \((\neg x_i)\) otherwise.

- For each \( i \neq i' \in I_2 \) with \( \{j^1_i, j^2_i\} \cap \{j^1_{i'}, j^2_{i'}\} = \emptyset \), \( S[i, j^1_i] = S[i', j^2_{i'}] \), and \( S[i, j^2_i] \neq S[i', j^1_{i'}] \), add a clause \((x_i)\) if either \( (S[i, j^1_i], S[i, j^2_i]) = (0,1) \) and \( d(S[i], S[i']) = \alpha - 3 \), or \( (S[i, j^1_i], S[i, j^2_i]) = (1,0) \) and \( d(S[i], S[i']) = \alpha - 1 \). Analogously, we add a clause \((\neg x_i)\) or \((\neg x_{i'})\) for the case \( \{j^1_i, j^2_i\} \cap \{j^1_{i'}, j^2_{i'}\} = \emptyset \), \( S[i, j^1_i] = S[i', j^2_{i'}] \), and \( S[i, j^2_i] \neq S[i', j^1_{i'}] \).

- For each \( i \neq i' \in I_2 \) with \( \{j^1_i, j^2_i\} \cap \{j^1_{i'}, j^2_{i'}\} = \emptyset \), \( S[i, j^1_i] \neq S[i', j^2_{i'}] \), and \( S[i, j^2_i] \neq S[i', j^1_{i'}] \).

- Suppose that \( d(S[i], S[i']) = \alpha - 4 \). Add a clause \((x_i)\) if \( (S[i', j^1_i], S[i', j^2_{i'}]) = (0,1) \) and \((\neg x_i)\) otherwise. Moreover, add a clause \((x_{i'})\) if \( (S[i, j^1_i], S[i, j^2_{i'}]) = (0,1) \) and \((\neg x_{i'})\) otherwise.

- Suppose that \( d(S[i], S[i']) = \alpha - 2 \). If \( (S[i, j^1_i], S[i, j^2_{i'}]) = (S[i', j^1_{i'}], S[i', j^2_i]) \), then add clauses \((x_i \lor x_{i'})\) and \((\neg x_i \lor \neg x_{i'})\). Note that these clauses are satisfied if and only if \( x_i = x_{i'} \). Otherwise, we add clauses \((x_i \lor \neg x_{i'})\) and \((\neg x_i \lor x_{i'})\), which are satisfied if and only if \( x_i = x_{i'} \).

- Suppose that \( d(S[i], S[i']) = \alpha \). Add a clause \((x_i)\) if \( (S[i', j^1_i], S[i', j^2_{i'}]) = (1,0) \) and \((\neg x_i)\) otherwise. Moreover, add a clause \((x_{i'})\) if \( (S[i, j^1_i], S[i, j^2_{i'}]) = (1,0) \) and \((\neg x_{i'})\) otherwise.

- For each \( i \neq i' \in I_2 \) with \( j^2_i = j^1_{i'} \).

- If \( d(S[i], S[i']) = \alpha - 2 \) and \( S[i, j^2_i] = S[i', j^1_{i'}] = 0 \), then add a clause \((x_i \lor \neg x_{i'})\).

- If \( d(S[i], S[i']) = \alpha \) and \( S[i, j^2_i] = S[i', j^1_{i'}] = 0 \), then add clauses \((\neg x_i)\) and \((x_{i'})\).

- If \( d(S[i], S[i']) = \alpha - 2 \) and \( S[i, j^2_i] = S[i', j^1_{i'}] = 1 \), then add a clause \((\neg x_i \lor x_{i'})\).

- If \( d(S[i], S[i']) = \alpha \) and \( S[i, j^2_i] = S[i', j^1_{i'}] = 1 \), then add clauses \((x_i)\) and \((\neg x_{i'})\).

- If \( d(S[i], S[i']) = \alpha - 3 \) and \( (S[i, j^1_i], S[i, j^2_{i'}]) = (1,0) \), then add clauses \((x_i)\) and \((x_{i'})\).

- If \( d(S[i], S[i']) = \alpha - 1 \) and \( (S[i, j^1_i], S[i', j^2_{i'}]) = (1,0) \), then add a clause \((\neg x_i \lor \neg x_{i'})\).

- If \( d(S[i], S[i']) = \alpha - 3 \) and \( (S[i, j^1_i], S[i', j^2_{i'}]) = (0,1) \), then add clauses \((\neg x_i)\) and \((\neg x_{i'})\).

- If \( d(S[i], S[i']) = \alpha - 1 \) and \( (S[i, j^1_i], S[i', j^2_{i'}]) = (0,1) \), then add a clause \((x_i \lor x_{i'})\).

- For each \( i \neq i' \in I_2 \) with \( j^1_i = j^1_{i'} \) and \( j^2_i = j^2_{i'} \), add clauses \((x_i \lor \neg x_{i'})\) and \((\neg x_i \lor x_{i'})\) if \( d(S[i], S[i']) = \alpha - 2 \) and add clauses \((x_i \lor x_{i'})\) and \((\neg x_i \lor \neg x_{i'})\) if \( d(S[i], S[i']) = \alpha \).

It is easy to check that the constructed formula is correct. The formula contains \( O(n^2) \) clauses and can thus be solved in \( O(n^2) \) time [1].
The Complexity of Binary Matrix Completion Under Diameter Constraints

\[ T = \begin{bmatrix} 001 & 111 & 001 & 00000000 \\ 111 & 111 & 001 & 00000000 \\ 111 & 111 & 010 & 11111111 \\ 111 & 111 & 110 & 11111111 \end{bmatrix} \]

**Figure 7** An illustration of the reduction from ORTHOGONAL VECTORS, where \( \mathcal{U} = \{010, 110\} \) and \( \mathcal{V} = \{110, 101\} \).

Next, we show that the quadratic dependency on \( n \) in the running time of Theorem 24 is inevitable under the Orthogonal Vectors Conjecture (OVC), which states that ORTHOGONAL VECTORS cannot be solved in \( O(n^{2-\epsilon} \cdot \ell^c) \) time for any \( c, \epsilon > 0 \) (it is known that the Strong Exponential Time Hypothesis implies the OVC \( \text{[38]} \)).

**ORTHOGONAL VECTORS**

**Input:** Sets \( \mathcal{U}, \mathcal{V} \) of row vectors in \( \{0, 1\}^\ell \) with \( |\mathcal{U}| = |\mathcal{V}| = n \).

**Question:** Are there row vectors \( u \in \mathcal{U} \) and \( v \in \mathcal{V} \) such that \( u[j] \cdot v[j] = 0 \) holds for all \( j \in [\ell] \)?

**Theorem 25.** DMC cannot be solved in \( O(n^{2-\epsilon} \cdot \ell^c) \) time for any \( c, \epsilon > 0 \), unless OVC breaks.

**Proof.** We reduce from ORTHOGONAL VECTORS. Let \( u_1, \ldots, u_n, v_1, \ldots, v_n \in \{0, 1\}^\ell \) be row vectors. Consider the matrix \( T \in \{0, 1\}^{2n \times 6\ell} \) where

\[
T[i, [3j - 2, 3j]] = \begin{cases} 001 & \text{if } u_i[j] = 0, \\ 111 & \text{if } u_i[j] = 1, \\ \end{cases} \quad T[i, [3\ell + 3j - 2, 3\ell + 3j]] = 000,
\]

\[
T[n + i, [3j - 2, 3j]] = \begin{cases} 010 & \text{if } v_i[j] = 0, \\ 111 & \text{if } v_i[j] = 1, \\ \end{cases} \quad T[n + i, [3\ell + 3j - 2, 3\ell + 3j]] = 111,
\]

for each \( i \in [n] \) and \( j \in [\ell] \) (see Figure 7 for an illustration). We show that \( \delta(T) = 5\ell \) if and only if there are \( i, i' \in [n] \) such that \( u_i \) and \( v_{i'} \) are orthogonal. By construction, we have

\[
d(T[i, [3j - 2, 3j]], T[n + i', [3j - 2, 3j]]) = \begin{cases} 2 & \text{if } u_i[j] = 0 \text{ or } v_{i'}[j] = 0, \\ 0 & \text{otherwise,} \end{cases}
\]

for any \( i, i' \in [n] \) and \( j \in [\ell] \). Thus, it holds for any orthogonal vectors \( u_i \) and \( v_{i'} \) that \( d(T[i], T[n + i']) = 5\ell \). Conversely, suppose that there exist \( i < i' \in [2n] \) such that \( d(T[i], T[i']) = 5\ell \). It is easy to see that \( i \in [n] \) and \( i' \in [n + 1, 2n] \) hold (since otherwise \( d(T[i], T[i']) \leq 3\ell \)). Then, the vectors \( u_i \) and \( v_{i'-n} \) are orthogonal. \( \blacksquare \)

### 4.2 NP-hardness

Next, we prove the following two NP-hardness results. In particular, it turns out that DMC is NP-hard even for \( \alpha = \beta \) when \( \alpha \) and \( \beta \) are unbounded. This is in contrast to Theorem 20 where we showed that DMC is polynomial-time solvable when \( \alpha = \beta \) is fixed.

**Theorem 26.** DMC is NP-hard

(i) for \( k = 2 \) and \( \beta - \alpha \geq 3 \), and
(ii) for \( k = 3 \) and \( \alpha = \beta \).

The proof for Theorem 26 is based on reductions from NP-hard variants of 3-SAT. The most challenging technical aspect of the reductions is to ensure the upper and lower bounds on the pairwise row Hamming distances. To overcome this challenge, we adjust pairwise row distances by making heavy use of a specific matrix, in which one pair of rows has distance exactly two greater than any other:

**Lemma 27.** For each \( n \geq 3 \) and \( i < i' \in [n] \), one can construct in \( n^{O(1)} \) time, a matrix \( B^{n}_{i,i'} \in \{0,1\}^{n \times \ell} \) with \( n \) rows and \( \ell := \binom{n}{2} - (2n-1) \) columns such that for all \( h \neq h' \in [n] \),

\[
d(B^{n}_{i,i'}[h], B^{n}_{i,i'}[h']) = \begin{cases} 
\gamma(B^{n}_{i,i'}) + 2 & \text{if } (h, h') = (i, i'), \\
\gamma(B^{n}_{i,i'}) & \text{otherwise}.
\end{cases}
\]

**Proof.** First, we define a binary matrix \( A^{n} \in \{0,1\}^{(n-2) \times (n-2)} \) as follows:

\[
A^{n} := \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\
\vdots & & & & & & & & \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1
\end{bmatrix},
\]

where \( I \) is the \( (n-2) \times (n-2) \) identity matrix. Note that \( d(A^{n}[1], A^{n}[2]) = 2 \) and \( d(A^{n}[h], A^{n}[h']) = 4 \) for all \( h < h' \in [n] \) with \( (h, h') \neq (1,2) \). We also define the matrix \( A^{n}_{h,h'} \) obtained from \( A^{n} \) by swapping the row vectors \( A^{n}[1] \) (and \( A^{n}[2] \)) with \( A^{n}[h] \) (and \( A^{n}[h'] \)), respectively for each \( h < h' \in [n] \). The matrix \( A^{n}_{h,h'} \) is a matrix in which the distance between the \( h \)-th and \( h' \)-th row vectors are exactly two smaller than all other pairs. Now we use the matrix \( A^{n}_{h,h'} \) to obtain a binary matrix in which the distance of a certain pair of rows is exactly two greater than all others. We define \( B^{n}_{i,i'} \in \{0,1\}^{n \times \ell} \) as the matrix obtained by horizontally stacking \( \binom{n}{2} - 1 \) matrices \( A^{n}_{h,h'} \) for all \( h < h' \in [n] \) with \( (h, h') \neq (i, i') \):

\[
B^{n}_{i,i'} := [A^{n}_{1,1} \cdots A^{n}_{i,n} \cdots A^{n}_{i,i+1} \cdots A^{n}_{i,i'-1} A^{n}_{i,i'+1} \cdots A^{n}_{i,n} \cdots A^{n}_{h-1,n}].
\]

Observe that \( d(B^{n}_{i,i'}[h], B^{n}_{i,i'}[i']) = 4 \cdot (\binom{n}{2} - 1) = 2n(n-1) - 4 \), since \( d(A^{n}[h], A^{n}[h']) = 4 \) for all \( h < h' \in [n] \) with \( (h, h') \neq (i, i') \). Note also that for each \( h < h' \in [n] \) with \( (h, h') \neq (i, i') \), we have \( d(B^{n}_{i,i'}[h], B^{n}_{i,i'}[h']) = 2n(n-1) - 6 \) because the distance between \( A^{n}_{i,i'}[h] \) and \( A^{n}_{i,i'}[h'] \) is four for every \( h < h' \in [n] \) except that it is smaller by two for the pair \( A^{n}_{i,i'}[i] \) and \( A^{n}_{i,i'}[i'] \). It is easy to see that the matrix \( B^{n}_{i,i'} \) can be constructed in polynomial time.

**Theorem 26** \[(1)\] We reduce from the following NP-hard variant of 3-SAT known as (3, B2)-SAT \[4]:

(3, B2)-SAT

**Input:** A Boolean formula in conjunctive normal form, in which each literal occurs exactly twice and each clause contains exactly three literals of distinct variables.

**Question:** Is there a satisfying truth assignment?
We divide our proof into two parts as follows. We first provide a set $\mathcal{C}$ of incomplete matrices and describe certain completion rules such that the given formula of $(3, B_2)$-SAT is satisfiable if and only if the matrices $\mathcal{C}$ can be completed under those rules. We then show that one can construct in polynomial time a single incomplete matrix $S$ containing each matrix in $\mathcal{C}$ as a submatrix, such that $S$ admits a solution if and only if the submatrices in $\mathcal{C}$ can be completed according to the rules. We are going to exploit the matrix $\mathbf{B}_{i,i'}^{n}$ of Lemma [27] for this construction.

Part I

Let $\phi$ be an instance of $(3, B_2)$-SAT with clauses $C_0, \ldots, C_{m-1}$. We define the following matrix for each clause $C_i$

$$C_i := \begin{bmatrix}
    l_i^1 & 0 & 0 & 0 & 1 & 1 \\
    0 & 0 & 1 & 1 & 0 & 1 \\
    0 & 0 & 0 & 0 & l_i^3 & 0 & 1 \\
    1 & 0 & 1 & 0 & 1 & 0 & c_i
\end{bmatrix}.$$  

Here we use $l_i^1, l_i^2, l_i^3$, and $c_i$ to represent two missing entries for notational purposes. Note that the matrices $C_i$ are identical for all $i \in [0, m - 1]$. We will prove that $\phi$ is satisfiable if and only if it is possible to complete matrices $\mathcal{C} := \{C_i | i \in [0, m - 1]\}$ while satisfying the following constraints:

1. The missing entries $l_i^1$ are filled by 10 or 01 for each $i \in [0, m - 1]$ and $j \in [3]$.
2. The missing entries $c_i$ are filled by 00, 01, or 10 for each $i \in [0, m - 1]$.
3. If the missing entries $c_i$ are filled by 00 (01, 10), then $l_i^1$ ($l_i^2, l_i^3$, respectively) are filled by 10 for each $i \in [0, m - 1]$.
4. Let $Z$ be the set such that $(i, j, i', j') \in Z$ if and only if the $j$-th literal in $C_i$ and the $j'$-th literal in $C_{i'}$ correspond to the same variable and one is the negation of the other for each $i < i' \in [0, m - 1]$ and $j, j' \in [3]$. If $(i, j, i', j') \in Z$, then either $l_i^j$ or $l_{i'}^{j'}$ is filled by 01.

Note that there are three choices for filling in $c_i$ by Constraint [2]. The intuitive idea is that the choice of $c_i$ dictates which literal (in binary encoding) in the clause $C_i$ is satisfied. We then obtain a satisfying truth assignment for $\phi$, as we shall see in the following claim.

\textbf{Claim 28.} The formula $\phi$ is satisfiable if and only if the matrices $\mathcal{C}$ can be completed according to Constraints [1 to 4].

\textbf{Proof.} ($\Rightarrow$) If there exists a truth assignment $\tau$ satisfying $\phi$, then at least one literal in the clause $C_i$ evaluates to true for each $i \in [0, m - 1]$. We choose an arbitrary number $l_i \in [3]$ such that the $l_i$-th literal of $C_i$ is satisfied in $\tau$ for each $i \in [0, m - 1]$. For each $i \in [0, m - 1]$ we complete the matrix $C_i$ as follows:

- If $l_i = 1$, then the missing entries $c_i, l_i^1, l_i^2, l_i^3$ are filled by 00, 10, 01, 01, respectively.
- If $l_i = 2$, then the missing entries $c_i, l_i^1, l_i^2, l_i^3$ are filled by 01, 01, 10, 01, respectively.
- If $l_i = 3$, then the missing entries $c_i, l_i^1, l_i^2, l_i^3$ are filled by 10, 01, 01, 10, respectively.

It is easy to verify that Constraints [1 to 3] are satisfied. We claim that Constraint [4] is also satisfied. Suppose to the contrary that there exists an $(i, j, i', j') \in Z$ such that the missing entries $l_i^j$ and $l_{i'}^{j'}$ are both filled by 10. Then, we have $l_i = j$ and $l_{i'} = j'$, meaning that $\tau$ satisfies both $x$ and $\neg x$ (a contradiction).
We provided matrices $C_i$ as well as the constraints on the completion of $C$ in Part I. Now, we describe how to construct a matrix $S$ that admits a completion $T$ with $\gamma(T) \geq \alpha$ and $\delta(T) \leq \beta$ if and only if $C$ can be completed fulfilling Constraints [1],[2],[3],[4]. First, for each matrix $C_i \in C$, we introduce a matrix $C'_i \in \{0,1,\square\}^{11 \times 8}$ containing $C_i$ by adding row vectors as follows (see Figure 8). These additional rows will help to encode Constraints [1],[2],[3],[4] and [5].

- The first four row vectors of $C'_i$ are identical to the row vectors of $C_i$.
- The row vectors $C'_i[9]$, $C'_i[6]$, and $C'_i[7]$ are obtained by completing the missing entries in $C_i[1]$, $C_i[2]$, and $C_i[3]$, respectively, with 00.
- The row vectors $C'_i[8]$, $C'_i[9]$, and $C'_i[10]$ are obtained by completing the missing entries in $C_i[1]$, $C_i[2]$, and $C_i[3]$, respectively, with 11.
- The row vector $C'_i[11]$ is obtained by completing the missing entries in $C_i[4]$ with 00.

Next, we construct a matrix $C \in \{0,1,\square\}^{11m \times 8m}$ from the matrices $C'_i$ as follows (see also Figure 8): We start with an empty matrix of size $11m \times 8m$. We first place $C'_0,\ldots,C'_{m-1}$ on the diagonal. Then, we place 01 at the intersection of the row containing $l^T_i$ ($l^T_i$) and the columns containing $l^C_i$ ($l^C_i$) respectively, for each $(i,j,i',j') \in Z$. This essentially encodes Constraint [4]. Finally, let the remaining entries be all 0. The formal definition is given as follows:

- $C[[11i + 1, 11i + 11], [8i + 1, 8i + 8]] = l^T_i$ for each $i \in [0, m - 1]$.
- $C[[11i + j, 8i + 2j]] = 1$ and $C[[11i' + j', 8i + 2j']] = 1$ for each $(i,j,i',j') \in Z$.
- All other entries are 0.

\begin{figure}[h]
\begin{center}
\begin{tabular}{cccccccc}
  $l^T_1$ & 0 & 0 & 0 & 0 & 1 & 1 \\
  0 & $l^T_2$ & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & $l^T_3$ & 0 & 1 \\
  1 & 0 & 1 & 0 & 1 & $c_i$ & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & 1 \\
  0 & 0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 1 \\
  1 & 1 & 0 & 0 & 0 & 1 & 1 \\
  0 & 0 & 1 & 1 & 0 & 0 & 1 \\
  0 & 0 & 0 & 1 & 1 & 0 & 1 \\
  1 & 0 & 1 & 0 & 1 & 0 & 0 \\
\end{tabular}
\end{center}
\caption{The matrix $C'_i \in \{0,1,\square\}^{11 \times 8}$ (left). The rows $\{11i + j, 11i' + j'\}$ and the columns $\{8i + 2j - 1, 8i + 2j, 8i + 2j' - 1, 8i + 2j'\}$ of $C$ for $(i,j,i',j') \in Z$ (right).}
\end{figure}
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Let \( n = 11m \) be the number of rows in \( C \). Now we define seven "types" \( H_1, \ldots, H_7 \) of row index pairs. The first four types correspond to Constraints [1] to [4]. In Claim [27], we show how to enforce Constraints [1] to [4] by appending an appropriate number of matrices given in Lemma [27]. The other three types are defined based on the number of missing entries. For each \( h < h' \in [n] \),

\( (h, h') \in H_1 \) if \( h = 11i + j \) and \( h' = 11i + j' \) for some \( i \in [0, m - 1] \) and \( (j, j') \in \{(1, 5), (2, 6), (3, 7), (1, 8), (2, 9), (3, 10)\} \).

\( (h, h') \in H_2 \) if \( h = 11i + j \) and \( h' = 11i + j' \) for some \( i \in [0, m - 1] \) and \( (j, j') = (4, 11) \).

\( (h, h') \in H_3 \) if \( h = 11i + j \) and \( h' = 11i + j' \) for some \( i \in [0, m - 1] \) and \( (j, j') \in \{(1, 4), (2, 4), (3, 4)\} \).

\( (h, h') \in H_4 \) if \( h = 11i + j \) and \( h' = 11i' + j' \) for \( (i, j, i', j') \in \mathbb{Z} \).

For each \( h < h' \in [n] \) with \( (h, h') \notin H_1, \ldots, H_4 \),

\( (h, h') \in H_5 \) if both \( C[h] \) and \( C[h'] \) have missing entries.

\( (h, h') \in H_6 \) if exactly one of \( C[h] \) and \( C[h'] \) have missing entries.

\( (h, h') \in H_7 \) if neither of \( C[h] \) and \( C[h'] \) has missing entries.

For each type of row index pairs, we “adjust” the pairwise distances using Lemma [27] to encode Constraints [1] to [4]. Before doing so, we prove an auxiliary claim stating that \( \delta(C) \leq 8 \).

\( \triangleright \) Claim 29. \( d(C[h], C[h']) \leq 8 \) for each \( h < h' \in [n] \).

Proof. It suffices to show that \( d(C[h], 0^{8m}) \leq 4 \) for each \( h \in [n] \), because we then have \( d(C[h], C[h']) \leq d(C[h], 0^{8m}) + d(C[h'], 0^{8m}) \leq 8 \) by the triangle inequality. Suppose that \( h = 11i + j \) for \( i \in [0, m - 1] \) and \( j \in [3] \). Then, the row vector \( C[h] \) contains at most two 1’s in \( C_i \), and exactly two 1’s elsewhere because each literal appears in the formula \( \phi \) exactly twice. It follows that \( C[h] \) contains at most four 1’s. Hence, we can assume that \( h = 11i + j \) for \( i \in [0, m - 1] \) and \( j \in [4, 11] \). Then, the row vector \( C[h] \) contains at most four 1’s, because all 1’s appear in \( C_i \). This shows the claim.

\( \triangleright \) Claim 30. There exists a \( \beta \in \mathbb{N} \) and a complete matrix \( D \) over \( \{0, 1\} \) with \( n \) rows such that all of the following hold for \( S = [C \quad D] \):

\( d(S[h], S[h']) = \beta - 1 \) for each \( (h, h') \in H_1 \) (cf. Constraint [1]).

\( d(S[h], S[h']) = \beta - 1 \) for each \( (h, h') \in H_2 \) (cf. Constraint [2]).

\( d(S[h], S[h']) = \beta - 3 \) for each \( (h, h') \in H_3 \) (cf. Constraint [3]).

\( d(S[h], S[h']) \in \{\beta - 3, \beta - 2\} \) for each \( (h, h') \in H_4 \) (cf. Constraint [4]).

\( d(S[h], S[h']) \in \{\beta - 3, \beta - 2\} \) for each \( (h, h') \in H_5 \).

\( d(S[h], S[h']) \in \{\beta - 2, \beta - 1\} \) for each \( (h, h') \in H_6 \).

\( d(S[h], S[h']) \in \{\beta - 1, \beta\} \) for each \( (h, h') \in H_7 \).

Proof. We obtain the matrix \( D \) by horizontally stacking \( B_{h,h'}^{c_{h,h'}} \) of Lemma [27] \( c_{h,h'} \) times (where \( c_{h,h'} \in \mathbb{N} \) is to be defined) for each \( h < h' \in [n] \). Recall that \( d(B_{h,h'}^{i,j'}, B_{h,h'}^{i',j'}) \) equals \( \gamma(B_{h,h'}^{i,j'}) + 2 \) if \( (h, h') = (i, i') \) and \( \gamma(B_{h,h'}^{i,j'}) \) otherwise and let

\[ \beta = \sum_{h < h' \in [n]} c_{h,h'} \cdot \gamma(B_{h,h'}^{n}) + 11. \]
We define \( c_{h,h'} \) for each \((h,h') \in H_1 \cup H_2 \cup H_3 \) as follows.

- Let \( c_{h,h'} = 4 \) for each \((h,h') \in H_1 \). Then, we have \( d(S[h], S[h']) = 2 + 2 \cdot 4 - \beta - 11 = \beta - 1 \).
- Let \( c_{h,h'} = 5 \) for each \((h,h') \in H_2 \). Then, we have \( d(S[h], S[h']) = 0 + 2 \cdot 5 - \beta - 11 = \beta - 1 \).
- Let \( c_{h,h'} = 2 \) for each \((h,h') \in H_3 \). Then, we have \( d(S[h], S[h']) = 4 + 2 \cdot 2 - \beta - 11 = \beta - 3 \).

For the remainder (that is, \((h,h') \in H_4 \cup \cdots \cup H_7 \) ), it has to be shown that there exists \( c_{h,h'} \in \mathbb{N} \) such that \( d(S[h], S[h']) \in \{x, x+1\} \) holds for \( x \geq \beta - 3 \). Let \( c_{h,h'} = [(11+x-\beta-d(C[h], C[h']))/2] \). Clearly, \( c_{h,h'} \) is an integer and it holds that \( c_{h,h'} \geq 0 \) because \( x-\beta \geq -3 \) and \( d(C[h], C[h']) \leq 8 \) by Claim 29. Moreover, we have \( d(S[h], S[h']) = x \) if \( 11 - x + \beta - d(C[h], C[h']) \) is even and \( d(S[h], S[h']) = x + 1 \) otherwise.

Finally, we show that Constraints 1 to 4 are essentially the same as the pairwise row distance constraints on the matrix \( S \) of Claim 30.

\[ \text{Claim 31.} \quad \text{The matrices } C \text{ can be completed according to Constraints 1 to 4 if and only if } S \text{ admits a completion } T \text{ with } \beta - 3 \leq \delta(T) \leq \beta. \]

\[ \text{Proof.} \quad (\Rightarrow) \text{ Let } T \text{ be the matrix where the missing entries of } S \text{ are filled as in the completion of } C. \text{ First, note that } \gamma(T) = \gamma(S) \geq \beta - 3 \text{ by Claim 30. We show that } d(T[h], T[h']) \leq \beta \text{ holds for each } h < h' \in [n]. \]

- Suppose that \((h,h') \in H_1 \). Then, the missing entries in \( S[h] \) are filled by 10 or 01 by Constraint 1 and \( S[h'] \) (which has no missing entries) has 00 or 11 in the corresponding positions. Hence, \( d(T[h], T[h']) = d(S[h], S[h']) + 1 = \beta \).
- Suppose that \((h,h') \in H_2 \). Then, the missing entries in \( S[h] \) are filled by 00, 01, or 10 by Constraint 2 and \( S[h'] \) (which has no missing entries) has 00 or 11 in the corresponding positions. Hence, \( d(T[h], T[h']) = d(S[h], S[h']) + 1 = \beta \).
- Suppose that \((h,h') \in H_3 \). Note that \( S[h] \) has missing entries \( l'_i \) and \( S[h'] \) has missing entries \( c_i \) for \( i \in [0, m-1] \) and \( j \in [3] \). Let \( c_i^1, c_i^2 \) be the completion of \( c_i \) for \( c_i^1, c_i^2 \in \{0, 1\} \). If \( T[h] \) has \( 1 - c_i^1 \) and \( 1 - c_i^2 \) in the corresponding positions, then \( d(T[h], T[h']) = d(S[h], S[h']) + 2 = \beta - 1 \). Otherwise, \( T[h] \) matches in at least one position where \( T[h'] \) has missing entries \( c_i \). Therefore, \( d(T[h], T[h']) \leq d(S[h], S[h']) + 3 = \beta \).
- Suppose that \((h,h') \in H_4 \). Note that \( S[h] \) has missing entries \( l'_i \) and \( S[h'] \) has missing entries \( l'_j \). Also note that \( S[h] \) and \( S[h'] \) have 01 where the other row vector has missing entries. Since either \( l'_i \) or \( l'_j \) must be completed by 01 due to Constraint 4, we have \( d(T[h], T[h']) = d(S[h], S[h']) \leq \beta - 2 + 2 = \beta \).
- Suppose that \((h,h') \in H_5 \cup H_6 \cup H_7 \). Let \( x \in \{0, 1, 2\} \) be the number of row vectors with missing entries in \( \{S[h], S[h']\} \). Then, we have \( d(S[h], S[h']) \in \{\beta - x + 1, \beta - x\} \). If \( S[h] \) has missing entries, then \( S[h'] \) has 00 in the corresponding positions, and vice versa. Since the missing entries are filled by 00, 01, or 10 according to Constraints 1 to 3, we have \( d(T[h], T[h']) \leq d(S[h], S[h']) + x = \beta \).

(\(\Leftarrow\)) We complete the matrices in \( C \) in the same way as in the completion of \( S \). It is easy to verify all Constraints 1 to 4 are satisfied.
Note that $\gamma(T) \geq \gamma(S) \geq \beta - 3$ for any completion $T$ of $S$. Hence, it follows from Claims 28 and 31 that $\phi$ is satisfiable if and only if the DMC instance $(S, \alpha, \beta)$ is a \textbf{Yes}-instance, for any $\alpha \leq \beta - 3$. This concludes the proof of Theorem 26 (i).

To prove that DMC is NP-hard for $\alpha = \beta$ and $k = 3$, we provide a polynomial-time reduction from another NP-hard variant of 3-SAT 35:

**Cubic Monotone 1-in-3 SAT**

**Input:** A Boolean formula in conjunctive normal form, in which each variable appears exactly three times and each clause contains exactly three distinct positive literals.

**Question:** Is there a truth assignment that satisfies exactly one literal in each clause?

Our reduction heavily depends on the fact that $\alpha = \beta$. This is contrary to the reduction in part 47 which in fact works for any $\alpha \leq \beta - 3$.

Let $\phi$ be an instance of \textsc{Cubic Monotone 1-in-3 SAT}. Our proof has two parts: First, we provide an incomplete matrix $C$ and we show that $\phi$ is a \textbf{Yes}-instance if and only if $C$ can be completed under certain constraints. Then, we obtain an instance $(S, \alpha, \alpha)$ of DMC by adjusting the pairwise row distances with the help of Lemma 27.

Suppose that $\phi$ contains variables $x_1, \ldots, x_m$ and clauses $C_1, \ldots, C_m$, where $C_i = (C_i^1 \vee C_i^2 \vee C_i^3)$ for each $i \in [m]$. First, we define matrices $C_1, C_3 \in \{0,1,\square\}^{m \times 2m}$ and $C_2, C_4 \in \{0,1,\square\}^{m \times 3m}$. The incomplete matrices $C_1$ and $C_4$ represent variables and clauses, respectively. The matrices $C_2$ and $C_3$ are complete. We use $a_i$ (and $b_i$) to represent two (three, respectively) missing entries in $C_1$ ($C_4$, respectively) for each $i \in [m]$. For each $i \in [m]$ and $j \in [m]$, let

$$C_1[i, \{2j - 1, 2j\}] = \begin{cases} a_i & \text{if } i = j, \\ 00 & \text{otherwise.} \end{cases}$$

$$C_2[i, \{3j - 2, 3j\}] = \begin{cases} 011 & \text{if } x_i = C_i^1, \\ 101 & \text{if } x_i = C_i^2, \\ 110 & \text{if } x_i = C_i^3, \\ 000 & \text{otherwise.} \end{cases}$$

$$C_3[i, \{2j - 1, 2j\}] = \begin{cases} 10 & \text{if } x_i \text{ is in } C_j, \\ 00 & \text{otherwise.} \end{cases}$$

$$C_4[i, \{3j - 2, 3j\}] = \begin{cases} b_i & \text{if } i = j, \\ 000 & \text{otherwise.} \end{cases}$$

We obtain an incomplete matrix $C \in \{0,1,\square\}^{(2m+1) \times (5m+1)}$ by appending a column vector $(0^{m+1})^T$ and a row vector $0^{5m+1}$ to the following matrix

$$\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}.$$ 

Refer to Figure 9 for an illustration.

Intuitively speaking, we will use the first $m$ rows to encode the variables and the following $m$ rows to encode the clauses.

\textasteriskcentered Claim 32. There is a truth assignment that satisfies exactly one literal in clause $C_i$ for each $i \in [m]$ if and only if there is a completion $C'$ of $C$ such that

1. $d(C'[i], C[2m+1]) = d(C[i], C[2m+1]) + 1$ for each $i \in [2m]$.
2. $d(C'[i], C'[m + i']) = d(C[i], C[m + i']) + 3$ for each $i, i' \in [m]$ such that $x_i$ is in $C_i'$.

Proof. ($\Rightarrow$) Let $\tau$ be a truth assignment satisfying exactly one literal in each clause of $\phi$. Consider the matrix $C'$ obtained by completing $C$ as follows for each $i \in [m]$:

\begin{itemize}
\item For $j \in [m]$ such that $x_j$ is in $C_i$, we have $C'[i] = C[i] + a_i = C[i] + 1$.
\item For $j \in [m]$ such that $x_j$ is in $C_i'$, we have $C'[i] = C[i] + a_i = C[i] + 1$.
\end{itemize}
We show that $B$ only if $\varphi$ has a satisfying assignment. To determine how many times we append the remaining two are 01. Thus, exactly one literal in $(101)$, the equality above holds. If $x_i$ is false in $\tau$, then $a_i'$ and $b_i'$ are 10 and 100, respectively. Thus, the equality above holds. If $x_i$ is false in $\tau$, then $a_i' = 01$ and $b_i' \in \{010,001\}$. Again the equality above holds.

$(\Leftarrow)$ Let $a_i'$ and $b_i'$ be the completion of $a_i$ and $b_i$ in $C'$ for each $i \in [m]$. Due to the first constraint, exactly one entry in $a_i'$ and $b_i'$ must be 1 for each $i \in [m]$. Now consider the truth assignment $\tau$ that assigns $x_i$ to true if $a_i' = 10$ and false if $a_i' = 01$ for each $i \in [m]$. We show that $\tau$ satisfies exactly one literal in each clause of $\varphi$. Consider $i \in [m]$ with $C_i = (x_i \lor x_{i_2} \lor x_{i_3})$. By the second constraint, we have

$$\sum_{j=1}^{3} d(C'[i_j], C'[m+i]) - d(C[i_j], C[m+i]) = 9.$$  

Rewriting the left-hand side in terms of $a_i', a_i', a_i', b_i'$, we obtain

$$d(011, b_i') + d(101, b_i') + d(110, b_i') + d(10, a_i') + d(10, a_i') + d(10, a_i') = 9.$$  

Since $b_i' \in \{010,010,001\}$, it follows that the first three terms sum up to exactly 5 and hence $d(10, a_i') + d(10, a_i') + d(10, a_i') = 4$. This means that exactly one of $a_i', a_i', a_i', b_i'$ is 10 and the remaining two are 01. Thus, exactly one literal in $C_i$ is satisfied. \hfill $\checkmark$

We will build an incomplete matrix $S$ with $2m+1$ rows from $C$ by horizontally appending matrices $B_{i,i'}^{2m+1}$ of Lemma 27 such that $S$ admits a completion $T$ with $\gamma(T) = \delta(T)$ if and only if $\varphi$ has a satisfying assignment. To determine how many times we append $B_{i,i'}^{2m+1}$, we observe the following about the pairwise distances in $C$ (see Figure 9):

- For each $i \in [m]$, $d(C[i], C[2m+1]) = 7$ and $d(C[m+i], C[2m+1]) = 3$. 

\begin{figure}[h]
\centering
\begin{tabular}{cccccccc|c}
\hline
$a_1$ & 00 & 00 & 00 & 011 & 011 & 011 & 000 & 1 \\
00 & $a_2$ & 00 & 00 & 101 & 101 & 000 & 011 & 1 \\
00 & 00 & $a_3$ & 00 & 110 & 000 & 101 & 101 & 1 \\
00 & 00 & 00 & $a_4$ & 000 & 110 & 110 & 110 & 1 \\
\hline
10 & 10 & 10 & 00 & $b_1$ & 000 & 000 & 000 & 0 \\
10 & 00 & 10 & 00 & 000 & $b_2$ & 000 & 000 & 0 \\
10 & 00 & 10 & 00 & 000 & 000 & $b_3$ & 000 & 0 \\
00 & 10 & 10 & 10 & 000 & 000 & 000 & $b_4$ & 0 \\
00 & 00 & 00 & 00 & 000 & 000 & 000 & 000 & 0 \\
\hline
\end{tabular}
\caption{An example of $C$ for $\varphi = (x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_2 \lor x_4) \land (x_1 \lor x_3 \lor x_4) \land (x_2 \lor x_3 \lor x_4)$.}
\end{figure}
For each \( i \neq i' \in [m] \), \( d(C[i], C[i']) = 12 - 2c_{i,i'} \) and \( d(C[m+i], C[m+i']) = 6 - 2c_{i,i'} \). Here \( c_{i,i'} \in \{0, 1, 2, 3\} \) is the number of clauses that contain both \( x_i \) and \( x_{i'} \), and \( c_{i,i'} \in \{0, 1, 2, 3\} \) is the number of variables that are both in \( C_i \) and \( C_{i'} \).

For each \( i, i' \in [m] \) with \( x_i \in C_i \), \( d(C[i], C[m+i]) = 7 \).

For each \( i, i' \in [m] \) with \( x_i \) not in \( C_i \), \( d(C[i], C[m+i]) = 10 \).

Now we construct \( S \) as follows. Recall that \( d(B_{i',i'}^{2m+1}[i], B_{i,i'}^{2m+1}[i']) = \gamma(B_{i',i'}^{2m+1}) + 2 \) and \( d(B_{i',i'}^{2m+1}[h], B_{i,i'}^{2m+1}[h']) = \gamma(B_{i',i'}^{2m+1}) \) for all \( h < h' \in [2m+1] \) with \( (h, h') \neq (i, i') \).

For each \( i \), \( \alpha \) with \( \alpha + 1 \) and we increase \( \alpha \) by \( \alpha \cdot (B_{i',i'}^{2m+1}) \) each time \( eB_{i,i'} \) is appended to \( C \). We horizontally append the following matrices:

- \( 3B_{i,2m+1} \) and \( 5B_{m+i,2m+1} \) for each \( i \in [m] \).
- \( c_{i,i'}B_{i,i'} \) and \( (c_{i,i'} + 3)B_{m+i,m+i} \) for each \( i < i' \in [m] \).
- \( 2B_{i,m+i} \) for each \( i, i' \in [m] \) with \( x_i \) in \( C_i \).
- \( 1B_{i,m+i} \) for each \( i, i' \in [m] \) with \( x_i \) not in \( C_i \).

Note that for each \( i, i' \in [2m+1] \), \( d(S[i], S[i']) = d(C[i], C[i']) + 2 \cdot n_{i,i'} + \alpha - 14 \), where \( n_{i,i'} \) is the number of appended \( B_{i,i'} \)’s. Thus, the pairwise row distances in \( S \) are given as follows:

- For each \( i \in [m] \), \( d(S[i], S[2m+1]) = 7 + 2 \cdot 3 + \alpha - 14 = \alpha - 1 \) and \( d(S[m+i], S[2m+1]) = 3 + 2 \cdot 5 + \alpha - 14 = \alpha - 1 \).
- For each \( i \neq i' \in [m] \), \( d(S[i], S[i']) = (12 - 2c_{i,i'}) + 2 \cdot c_{i,i'} + \alpha - 14 = \alpha - 2 \), and \( d(S[m+i], S[m+i']) = (6 - 2c_{i,i'}) + 2 \cdot (c_{i,i'} + 3) + \alpha - 14 = \alpha - 2 \).
- For each \( i, i' \in [m] \) with \( x_i \) in \( C_{i'} \), \( d(S[i], S[m+i']) = 7 + 2 \cdot 2 + \alpha - 14 = \alpha - 3 \).
- For each \( i, i' \in [m] \) with \( x_i \) not in \( C_{i'} \), \( d(S[i], S[m+i']) = 10 + 2 \cdot 1 + \alpha - 14 = \alpha - 2 \).

Finally, we prove that one can complete \( C \) as specified in Claim 32 if and only if one can complete \( S \) into a matrix \( T \) with \( \gamma(T) = \delta(T) = \alpha \).

Claim 33. There is a completion \( C' \) of \( C \) such that

1. \( d(C'[i], C[2m+1]) = d(C[i], C[2m+1]) + 1 \) for each \( i \in [2m] \).
2. \( d(C'[m+i], C[m+i']) = d(C[i], C[m+i']) + 3 \) for each \( i, i' \in [m] \) such that \( x_i \) is in \( C_{i'} \).

if and only if there is a completion \( T \) of \( S \) with \( d(T[i], T[i']) = \alpha \) for all \( i \neq i' \in [2m+1] \).

Proof. \((\Rightarrow)\) Consider the completion \( T \) of \( S \) in which each missing entry is completed as in \( C' \). Let \( a'_i \) and \( b'_i \) be the completion of \( a_i \) and \( b_i \) for each \( i \in [m] \). We have \( a'_i \in \{10, 01\} \) and \( b'_i \in \{100, 010, 001\} \) due to the first constraint. Now we examine each pairwise row distance.

- For each \( i \in [m] \), \( d(T[i], T[2m+1]) = d(S[i], S[2m+1]) + d(a'_i, 00) = (\alpha - 1) + 1 + \alpha = \alpha + 1 \) and \( d(T[m+i], T[2m+1]) = d(S[m+i], S[2m+1]) + d(b'_i, 00) = (\alpha - 1) + 1 + \alpha = \alpha + 1 \).
- For each \( i \neq i' \in [m] \), \( d(T[i], T[i']) = d(S[i], S[i']) + d(a'_i, 00) + d(b'_i, 00) = (\alpha - 2) + 1 + 1 + \alpha = \alpha + 1 \).
- For each \( i, i' \in [m] \) with \( x_i \) in \( C_{i'} \), \( d(T[i], T[m+i']) = d(S[i], S[m+i']) + d(C[i], C[m+i']) = (\alpha - 3) + 3 = \alpha \) because of the second constraint.
- For each \( i, i' \in [m] \) with \( x_i \) not in \( C_{i'} \), \( d(T[i], T[m+i']) = d(S[i], S[m+i']) + d(a'_i, 00) + d(b'_i, 00) = (\alpha - 2) + 1 + 1 + \alpha = \alpha + 1 \).
Theorem 23 we have

We conjecture that one can adapt the proof of Theorem 26 (ii) to show the NP-hardness of

we conducted a detailed complexity analysis in terms of polynomial-time solvable versus

Together with the recent work of Eiben et al. [14, 15], we are seemingly among the first in

This concludes the proof of the claim.

Combining Claim 32 and Claim 33, we have that φ is a Yes-instance if and only if the DMC instance (S, α, α) is a Yes-instance.

The problem of deciding whether an incomplete matrix S ∈ {1, −1, θ}n×n can be completed into a Hadamard matrix [28], is equivalent to DMC with n = ℓ and α = β = n/2.

We conjecture that one can adapt the proof of Theorem 25[11] to show the NP-hardness of this problem. We also conjecture that DMC with k = 3 is actually NP-hard for every value of β − α. Similar reductions might work here as well. By contrast, we believe the case k = 2 and β − α = 1 to be polynomial-time solvable, again by reducing to 2-SAT.

5 Conclusion

Together with the recent work of Eiben et al. [14] [15], we are seemingly among the first in the context of stringology that make extensive use of Deza’s theorem and sunflowers. While Eiben et al. [14] [15] achieved classification results in terms of parameterized (in)tractability, we conducted a detailed complexity analysis in terms of polynomial-time solvable versus NP-hard cases. Figure 1 provides a visual overview on our results for Diameter Matrix Completion (DMC), also spotting concrete open questions.

Going beyond open questions directly arising from Figure 1, we remark that it is known that the clustering variant of DMC can be solved in polynomial time when the number of clusters is two and the matrix is complete [25]. Hence, it is natural to ask whether our tractability results can be extended to this matrix completion clustering problem as well. Furthermore, we proved that there are polynomial-time algorithms solving DMC when β ≤ 3 and α = 0 (Theorems [12] and [18]). This leads to the question whether these algorithms can be extended to matrices with arbitrary alphabet size. Next, we are curious whether the phenomenon we observed in Theorem 22 concerning the exponential dependence of the running time for (α, α + 1)-DMC when α is even but independence of α when it is odd can be further substantiated or whether one can get rid of the “α-dependence” in the even case. In terms of standard parameterized complexity analysis, we wonder whether DMC is fixed-parameter tractable with respect to β + k (in our NP-hardness proof for the case β = 4 (Theorem 23), we have k ∈ Θ(ℓ)). Finally, performing a multivariate fine-grained complexity analysis in the same spirit as in recent work for Longest Common Subsequence [6] would be another natural next step.

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