ON GAGLIARDO-NIRENBERG INEQUALITIES WITH VANISHING SYMBOLS

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Abstract. We prove interpolation inequalities of Gagliardo-Nirenberg type involving Fourier symbols that vanish on hypersurfaces in $\mathbb{R}^d$.

1. Introduction

In a recent paper by Fernández, Jeanjean, Mariš and the author the following inequality of Gagliardo-Nirenberg type was proved
\begin{equation}
\|u\|_q \lesssim \|(|D|^s - 1)u\|_2^{1-\kappa} \|u\|_2^\kappa \quad (u \in \mathcal{S}(\mathbb{R}^d)).
\end{equation}
Here, $(|D|^s - 1)u = \mathcal{F}^{-1}((|\cdot|^s - 1)\hat{u})$, the symbol $\lesssim$ stands for $\leq C$ for some positive number $C$ independent of $u$ and the parameters are supposed to satisfy
\begin{equation}
2 > s \geq 0, \quad \kappa \geq 1, \quad 2 \leq q < \infty, \quad d \in \mathbb{N}, \quad d \geq 2 \quad \text{and} \quad \frac{2(1 - \kappa)}{d + 1} \leq 1 - \frac{1}{q} \leq \frac{(1 - \kappa)s}{d},
\end{equation}
see \cite{14} Theorem 2.6. In this paper we investigate such inequalities in greater generality both by extending the analysis to a larger class of exponents, but also by allowing for more general Fourier symbols. We expect applications in the context of normalized solutions of elliptic PDEs and orbital stability \cite{11,29} or long-time behaviour \cite{36} of time-dependent PDEs just as in the case of the classical Gagliardo-Nirenberg Inequality \cite{28}. In \cite{13} and \cite{25} applications of \cite{11} to variational existence results and symmetry breaking phenomena for biharmonic nonlinear Schrödinger equations are given. For the existence and qualitative properties of maximizers in classical Gagliardo-Nirenberg inequalities we refer to \cite{2,12,24,30,35,37}. Interpolation inequalities in different spaces like Lorentz spaces, Besov spaces, BMO or weighted Lebesgue spaces can be found in \cite{5,7,11,18,27}.

We shall be concerned with inequalities of the form
\begin{equation}
\|u\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa} \|P_2(D)u\|_{r_2}^\kappa \quad (u \in \mathcal{S}(\mathbb{R}^d)),
\end{equation}
where $q,r_1,r_2 \in [1,\infty], \kappa \in [0,1]$ and $P_1,P_2 : \mathbb{R}^d \to \mathbb{R}$ are Fourier symbols that may vanish on a given smooth compact hypersurface $S \subset \mathbb{R}^d$, $d \geq 2$ with at least $k \in \{1,\ldots,d-1\}$ non-vanishing principal curvatures in each point. In the case $d = 1$ the symbols are allowed to have a finite set of zeros $S \subset \mathbb{R}$. We will assume that $P_1$ vanishes of order $\alpha_1$ on $S$ and behaves like $|\cdot|^s$ at infinity, see Assumption (A1),(A2) below for a precise statement. This covers \cite{11} as a special case where $d \geq 2$, $(\alpha_1,\alpha_2,s_1,s_2) = (1,0,s,0)$ and $S$ is the unit
sphere in $\mathbb{R}^d$, so $k = d - 1$. As an application of our results for (3) we obtain the following generalization of [14, Theorem 2.6].

**Theorem 1.** Assume $d \in \mathbb{N}, d \geq 2, \kappa \in [0, 1], s > 0$. Then
\[
\|u\|_{q} \lesssim \|(|D|^s - 1)u\|^1-r \|u\|^r \quad (u \in \mathcal{S}(\mathbb{R}^d))
\]
holds provided that the exponents $r \in [1, 2], q \in [2, \infty]$ satisfy
\[
\frac{2(1 - \kappa)}{d + 1} \leq \frac{1}{r - 1} - \frac{(1 - \kappa)s}{d} \quad \text{and} \quad \min \left\{ \frac{1}{r'}, \frac{1}{q'} \right\} \begin{cases} \geq \frac{d + 1 - 2s}{2d} & \text{if } \kappa > 0, \\ \geq \frac{d + 1}{2d} & \text{if } \kappa = 0. \end{cases}
\]

So our result from [14] is recovered as (2) is nothing but the special case $r = 2$ in the above theorem. We even obtain sufficient conditions for general $q, r_1, r_2 \in [1, \infty]$. In the one-dimensional case we obtain the following generalization of [14, Theorem 2.3].

**Theorem 2.** Assume $\kappa \in [0, 1], s > 0$. Then
\[
\|u\|_{q} \lesssim \|(|D|^s - 1)u\|^1-r \|u\|^r \quad (u \in \mathcal{S}(\mathbb{R}))
\]
holds provided that $q, r_1, r_2 \in [1, \infty]$ satisfy $1 - \kappa \leq \frac{1}{r_1} + \frac{s}{r_2} - \frac{1}{q} \leq (1 - \kappa)s$.

Both our main results arise as special cases of Theorem 3 and Theorem 4 where interpolation inequalities of the form (3) are proved for symbols $P_1, P_2 : \mathbb{R}^d \to \mathbb{R}$ that satisfy the following abstract conditions:

(A1) There is a compact hypersurface $S = \{ \xi \in \mathbb{R}^d : F(\xi) = 0 \}$ with $F \in C^\infty(\mathbb{R}^d)$, $|\nabla F| \neq 0$ on $S$ and at least $k \in \{1, \ldots, d - 1\}$ non-vanishing principal curvatures at each point such that $\{ \xi \in \mathbb{R}^d : P_i(\xi) = 0 \} \subset S$. For $\xi$ near $S$ we have $P_i(\xi) = a_{i+}(\xi)F(\xi)^{a_i} + a_{i-}(\xi)F(\xi)^{a_i}$ for smooth non-vanishing functions $a_{i+}, a_{i-}$ and $\alpha_i > -1$.

In the case $\alpha_i = 1$ additionally assume $a_{i-} = -a_{i+}$ and in the case $\alpha_i = 0$ additionally assume $a_{i-} = a_{i+}$.

(A2) There are $s_1, s_2 \in \mathbb{R}, \delta > 0$ such that for $\text{dist}(\xi, S) \geq \delta > 0$ the functions $Q_i(\xi) := (\xi)^{s_1}/P_i(\xi)$ satisfy for some $\varepsilon > 0$
\[
|\partial^\gamma Q_i(\xi)| \lesssim (\xi)^{-\varepsilon - |\gamma|} \quad \text{if } \gamma \in \mathbb{N}_0^d, \ 0 \leq |\gamma| \leq |d/2|,
\]
\[
|\partial^\gamma Q_i(\xi)| \lesssim (\xi)^{-|\gamma|} \quad \text{if } \gamma \in \mathbb{N}_0^d, \ |\gamma| = |d/2| + 1.
\]

Here and in the following we set $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ and $|\gamma| := (|\gamma_1|, \ldots, |\gamma_d|) := \gamma_1 + \ldots + \gamma_d$ for multi-indices $\gamma \in \mathbb{N}_0^d$, $F(\xi)_+ := \max\{F(\xi), 0\}$ and $F(\xi)_- := -\min\{F(\xi), 0\}$. In the case $d = 1$ assumption (A1) is supposed to mean $S = \{ \xi \in \mathbb{R} : F(\xi) = 0 \} = \{ \xi_1^*, \ldots, \xi_L^* \}$ with $F, P_i, a_{i+}, a_{i-}$ as above. Given the importance of the fractional Laplacian $(-\Delta)^{s/2} = |D|^s$ we mention that one may generalize this further by allowing the symbols $P_1, P_2$ to vanish at some finite set of points in $\mathbb{R}^d \setminus S$, see Remark 2. The choice $P_1 = P_2$ or $\kappa \in \{0, 1\}$ leads to Sobolev inequalities. In the elliptic case $-\Delta = |D|^2 - 1$ such results are due to Kenig, Ruiz, Sogge [21, Theorem 2.3], Gutiérrez [17, Theorem 6] and Evequoz [13]. Our most general result from Theorem 3 contains these results as a special case $(k, s_1, a_1, \kappa) = (d - 1, 2, 1, 0)$. Sharp results for special non-elliptic symbols with unbounded characteristic set $S$ are due to Kenig, Ruiz, Sogge [21, Theorem 2.1], Koch, Tataru [22] and Jeong, Kwon, Lee [20, Theorem 1.1].
Remark 1.

(a) In the case $S = \emptyset$ the main results of this paper hold without any assumption on $\alpha_1, \alpha_2$. Similarly, if the Fourier support of the given functions is contained in a fixed compact subset of $\mathbb{R}^d$, then all conditions involving $s_1, s_2$ can be ignored.

(b) Theorem 1 and 2 equally hold for symbols $P_i(|D|)$ where $P_i$ are polynomials of degree $s$ with simple zeros only or no zeros at all.

(c) Our analysis may be extended to vectorial differential operators with constant coefficients $P_1(D), P_2(D)$ where, according to Cramer's rule, the characteristic set $S$ is then supposed to satisfy $\{\det(P_i(\xi)) = 0\} \subset S$ for $i = 1, 2$. Such a situation occurs in the context of Maxwell's equations, Dirac equations or Lamé equations with constant coefficients.

(d) The Gagliardo-Nirenberg inequalities from this paper hold for functions with Fourier support in bounded smooth pieces of more general sets $S \subset \mathbb{R}^d$. In this way, unbounded characteristic sets $S$ or characteristic sets with singularities as in [26, Section 3] may be partially analyzed, but a full analysis remains to be done. In the special case of the wave and Schrödinger operator one may nevertheless implement the strategy from [14] to get such inequalities at least for $r = 2$, see Section 7.

(e) The admissible set of exponents for Gagliardo-Nirenberg inequalities may become larger by imposing symmetries. For instance, the Stein-Tomas Theorem for $O(d-k) \times O(k)$-symmetric functions from [30] may substitute the classical Stein-Tomas Theorem in Lemma 4 to prove better dyadic estimates. The latter yield larger values for $A_{r}(p, q)$ in (17), which allows to deduce Gagliardo-Nirenberg inequalities for a wider range of exponents.

Our strategy is as follows. We decompose the pseudo-differential operators $P_1(D), P_2(D)$ dyadically, both for frequencies close to the critical surface $S$ and at infinity. Assumption (A1) allows to analyze the first-mentioned part with the aid of Bochner-Riesz estimates from [10][26]. Here, only the parameters $\alpha_1, \alpha_2$ will play a role. Assumption (A2) will be used to estimate the second-mentioned part that only involves $s_1, s_2$. Interpolating the bounds for the dyadic operators in both frequency regimes then allows to conclude. We stress that the proof from [14] does not carry over from the $L^2(\mathbb{R}^d)$-setting since Plancherel’s Theorem does not have a counterpart in $L^r(\mathbb{R}^d)$ with $r \neq 2$.

2. Preliminaries

In the following we decompose a given Schwartz function $u \in \mathcal{S}(\mathbb{R}^d)$ in frequency space. We start by separating the frequencies close to the critical surface from the others by defining

\[ u_1 := \mathcal{F}^{-1}(\tau \hat{u}), \quad u_2 := \mathcal{F}^{-1}((1 - \tau)\hat{u}) \quad \text{where} \quad \tau \in C_0^\infty(\mathbb{R}^d), \quad \tau = 1 \text{ near } S. \]

More precisely, $\tau$ is chosen in such a way that $S$ admits local parametrizations in Euclidean coordinates within $\text{supp}(\tau)$, that $a_{+-}, a_{-+}$ from (A1) are uniformly positive near $S$ and that the functions $Q_i$ from (A2) behave as required for $\xi \in \mathbb{R}^d \setminus \text{supp}(\tau)$. The function $\tau$ is considered as fixed from now on. For both $u_1$ and $u_2$ we will introduce a dyadic decomposition into infinitely
many annular regions in order to prove our estimates mostly via Bourgain’s summation argument \cite{4}. We will need the following abstract version of this result from \cite{8} p.604.

**Lemma 1.** Let $\beta_1, \beta_2 \in \mathbb{R}, \theta \in (0, 1)$, let $(X_1, X_2)$ and $(Y_1, Y_2)$ be real interpolation pairs of Banach spaces. For $j \in \mathbb{N}$ let $T_j$ be linear operators satisfying

$$
\|T_j f\|_{Y_1} \leq M_1 2^{\beta_1 j} \|f\|_{X_1}, \quad \|T_j f\|_{Y_2} \leq M_2 2^{\beta_2 j} \|f\|_{X_2}.
$$

Then we have

$$
\| \sum_{j \in \mathbb{N}} T_j f \|_{(Y_1,Y_2)_{\theta,\infty}} \leq C(\beta_1, \beta_2) M_1^{1-\theta} M_2^\theta \|f\|_{(X_1,X_2)_{\theta,1}}
$$

provided that $(1-\theta)\beta_1 + \theta \beta_2 = 0$ with $\beta_1, \beta_2 \neq 0$. In the case $(1-\theta)\beta_1 + \theta \beta_2 < 0$ we have for all $r \in [1, \infty]$

$$
\| \sum_{j \in \mathbb{N}} T_j f \|_{(Y_1,Y_2)_{\theta,r}} \leq C M_1^{1-\theta} M_2^\theta \|f\|_{(X_1,X_2)_{\theta,r}}.
$$

The whole point of this result is \ref{5}; the estimate \ref{6} is a rather trivial consequence of the summability of the interpolated bounds

$$
\|T_j f\|_{(Y_1,Y_2)_{\theta,r}} \lesssim 2^{j((1-\theta)\beta_1+\theta \beta_2)} \|f\|_{(X_1,X_2)_{\theta,r}} \quad \text{for all } r \in [1, \infty].
$$

Here, $(Y_1, Y_2)_{\theta, r}, (X_1, X_2)_{\theta, r}$ denote real interpolation spaces \cite{3}. The choice $Y_1 = L^{q_1}(\mathbb{R}^d), Y_2 = L^{q_2}(\mathbb{R}^d)$ with $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, q_1 \neq q_2$ yields the Lorentz space $(Y_1, Y_2)_{\theta, r} = L^{q,r}(\mathbb{R}^d)$ whereas $q_1 = q_2 = q$ leads to $(Y_1, Y_2)_{\theta, r} = L^q(\mathbb{R}^d)$. In our context, the spaces $X_i$ are defined as the completion of $\{u \in \mathcal{S}(\mathbb{R}^d) : P_i(D)u \in L^r(\mathbb{R}^d)\}$ with respect to the norm $\|u\|_{X_i} := \|P_i(D)u\|_r$.

Exploiting assumption (A1),(A2) we find that for any given $u \in \mathcal{S}(\mathbb{R}^d)$ the function $P_i(D)u$ is a priori well-defined as a function in $L^\infty(\mathbb{R}^d)$ because $\xi \mapsto P_i(\xi)\hat{u}(\xi)$ is integrable due to $\alpha_i > -1$. (Choosing the completion of a smaller set one may extend the analysis to $\alpha_i \leq -1$.) The link to Gagliardo-Nirenberg-type inequalities is provided by the general interpolation property \cite{3} Theorem 3.1.2, namely

$$
\|f\|_{(X_1,X_2)_{\kappa,r}} \leq \|f\|_{X_1}^{1-\kappa} \|f\|_{X_2}^\kappa \quad (0 < \kappa < 1, 1 \leq r \leq \infty).
$$

In fact, choosing $X_1, X_2$ as above we obtain for $u \in \mathcal{S}(\mathbb{R}^d)$

$$
\|u\|_{(X_1,X_2)_{\kappa,r}} \leq \|P_1(D)u\|_{X_1}^{1-\kappa} \|P_2(D)u\|_{X_2}^\kappa \quad (0 < \kappa < 1, 1 \leq r \leq \infty).
$$

The same estimate holds for $(X_1, X_2)_{\kappa,r}$ replaced by the complex interpolation space $[X_1, X_2]_{\kappa}$. This can be deduced from \ref{7} and $[X_1, X_2]_{\kappa} \subset (X_1, X_2)_{\kappa,\infty}$, see \cite{3} Theorem 4.7.1.

### 3. LARGE FREQUENCY ANALYSIS

We start with our analysis for large frequencies or, more precisely, for those frequencies with uniformly positive distance to the critical surface $S$ given by our assumption (A1). To this end we first choose a function $\eta$ such that

$$
\eta \in C_0^\infty(\mathbb{R}), \quad \text{supp}(\eta) \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2], \quad \sum_{j \in \mathbb{Z}} \eta(2^j \cdot) = 1 \text{ almost everywhere on } \mathbb{R},
$$

We recall that for a non-negative weight $\omega \in L^1_{\text{loc}}(\mathbb{R})$ and $1 \leq p < \infty$

$$
\|f\|_p(\mathbb{R}) \leq C_{p,\omega} \int_{\mathbb{R}} \omega^{1/p}(\xi) \|\hat{f}(\xi)^p\|_1 d\xi
$$

holds, where $C_{p,\omega}$ is a constant depending on $p$ and $\omega$. Let $\omega_0 = \sum_{j \in \mathbb{Z}} \eta(2^j \cdot)$, i.e. $\omega_0 \in L^1_{\text{loc}}(\mathbb{R})$.

For a non-negative weight $\omega_0 \in L^1_{\text{loc}}(\mathbb{R})$ and $1 \leq p < \infty$

$$
\|f\|_p(\mathbb{R}) \leq C_{p,\omega_0} \int_{\mathbb{R}} \omega_0^{1/p}(\xi) \|\hat{f}(\xi)^p\|_1 d\xi
$$

holds, where $C_{p,\omega_0}$ is a constant depending on $p$ and $\omega_0$. Let $\omega = \sum_{j \in \mathbb{Z}} \eta(2^j \cdot)$, i.e. $\omega \in L^1_{\text{loc}}(\mathbb{R})$.
Hence, shown, the result follows from Young’s Convolution Inequality because of

\[ T_j f := \mathcal{F}^{-1} \left( \eta(2^j|\xi - \xi_0|) \hat{f} \right) = K_j * f \quad \text{where} \]

\[ K_j(x) := \mathcal{F}^{-1} \left( \eta(2^j|\xi - \xi_0|) \right)(x) = 2^{-jd} \mathcal{F}^{-1} (\eta(|\cdot|)) (2^{-j}x) e^{ix\xi_0}. \]

Later on, we will choose \( \xi_0 \in S \) in order to have \( T_j u_2 = 0 \) for \( j \geq j_0 \) where \( j_0 \in \mathbb{Z} \) only depends on \( \xi_0 \) and \( \tau \). Indeed, (4) implies that \( \hat{u}_2(\xi) = (1 - \tau(\xi)\hat{u}(\xi) \) vanishes for frequencies \( \xi \) close to \( S \). As a consequence, only the bounds for \( j \searrow -\infty \) will be of importance.

**Lemma 2.** Assume \( d \in \mathbb{N} \) and let \( \eta \in C_0^\infty(\mathbb{R}) \), \( \xi_0 \in \mathbb{R}^d \). Then we have for \( j \in \mathbb{Z} \)

\[ \|T_j\|_{p \to q} \leq 2^{-jd(\frac{1}{p} - \frac{1}{q})} \quad \text{for } 1 \leq p \leq q \leq \infty. \]

**Proof.** For all \( r \in [1, \infty] \) we have \( \|K_j\|_r = 2^{-jd}\|\mathcal{F}^{-1}(\eta(|\cdot|))(2^{-j} \cdot)\|_r \lesssim 2^{-jd} \). Hence, for \( 1 \leq p \leq q \leq \infty \) and \( \frac{1}{p} := 1 + \frac{1}{q} - \frac{1}{r} \) we get from Young’s Convolution Inequality

\[ \|T_j f\|_q \lesssim \|K_j\|_r \|f\|_p \lesssim 2^{-jd} \|f\|_p \lesssim 2^{-jd(\frac{1}{p} - \frac{1}{q})} \|f\|_p. \]

\( \square \)

In the following, we will need a multiplier theorem in \( L^\mu(\mathbb{R}^d) \) for arbitrary \( \mu \in [1, \infty) \). The natural candidate - Mikhlin’s multiplier theorem [3] Theorem 6.1.6 - is only available for \( \mu \in (1, \infty) \). In order to avoid tiresome separate discussions we first provide a simple sufficient condition for a given function \( m : \mathbb{R}^d \to \mathbb{R} \) to be a \( L^\mu \)-multiplier for all \( \mu \in [1, \infty) \). The following result essentially says that a function \( m \) serves our purpose provided that its derivatives grow a bit slower near zero and decay a bit faster near infinity compared to the requirements of Mikhlin’s multiplier theorem.

**Proposition 1.** Let \( d \in \mathbb{N}, k := \lfloor d/2 \rfloor + 1 \) and \( m \in C^k(\mathbb{R}^d \setminus \{0\}) \). Then \( m \) is an \( L^\mu \) multiplier for all \( \mu \in [1, \infty) \) provided that there is \( \varepsilon > 0 \) such that

\[ |\partial^\alpha m(\xi)| \lesssim \langle \xi \rangle^{-2\varepsilon} |\xi|^{-k+\varepsilon} \quad \text{for all } \alpha \in \mathbb{N}_0^d \text{ such that } |\alpha| = k. \]

**Proof.** We show that the assumptions imply that \( \rho := \mathcal{F}^{-1} m \) is integrable. Once this is shown, the result follows from Young’s Convolution Inequality because of

\[ \|\mathcal{F}^{-1}(m \hat{f})\|_{\mu} = \|\rho \ast f\|_{\mu} \leq \|\rho\|_1 \|f\|_\mu. \]

We may w.l.o.g. assume \( 0 < \varepsilon \leq 2k - d \). For all \( \alpha \in \mathbb{N}_0^d, |\alpha| = k \) we have

\[ |\mathcal{F}((-ix)^\alpha \rho)(\xi)| = |\partial^\alpha \hat{\rho}(\xi)| = |\partial^\alpha m(\xi)| \lesssim \langle \xi \rangle^{-2\varepsilon} |\xi|^{-k+\varepsilon}. \]

Hence, \( \mathcal{F}(x^\alpha \rho) \) belongs to the space \( L^{\sigma_1}(\mathbb{R}^d) \cap L^{\sigma_2}(\mathbb{R}^d) \) where \( \sigma_1 := \frac{d}{k+\varepsilon/2}, \sigma_2 := \frac{d}{k-\varepsilon/2} \). Our choice for \( \varepsilon \) implies \( 1 \leq \sigma_1 \leq \sigma_2 \leq 2 \), so the Hausdorff-Young Inequality gives

\[ |x|^k \rho \in L^{\sigma_1}(\mathbb{R}^d) \cap L^{\sigma_2}(\mathbb{R}^d). \]

To conclude \( \rho \in L^1(\mathbb{R}^d) \) with Hölder’s Inequality it remains to check

\[ |x|^{-k} \in L^{\sigma_1}(\mathbb{R}^d) + L^{\sigma_2}(\mathbb{R}^d). \]

But this follows from \( |x|^{-k} 1_{|x| \leq 1} \in L^{\sigma_1}(\mathbb{R}^d) \) and \( |x|^{-k} 1_{|x| > 1} \in L^{\sigma_2}(\mathbb{R}^d) \) due to \( k\sigma_1 < d < k\sigma_2 \), which finishes the proof.

\( \square \)
Next we provide our estimates in the large frequency regime. To this end we analyze the mapping properties of \( T_j u := T_j (u_2) \) where \( T_j \) and \( u_2 = \mathcal{F}^{-1}((1 - \tau) \hat{u}) \) were defined in (8), (4), respectively.

**Proposition 2.** Assume \( d \in \mathbb{N} \) and (A2) with \( s_1, s_2 \in \mathbb{R} \). Then, for \( i = 1, 2 \),
\[
\|T_j u\|_q \lesssim 2^{j(s_i - d(\frac{1}{p} - \frac{1}{q}))} \|P_i(D)u\|_p, \quad \text{for } 1 \leq p \leq q \leq \infty, \quad j \in \mathbb{Z}.
\]

**Proof.** In order to use Lemma 2 for \( \xi_0 \in S \) we set \( \eta_i(z) := \eta(z)|z|^{-s_i} \) for \( z \in \mathbb{R} \). Then \( \eta \in C_0^\infty(\mathbb{R}), 0 \notin \text{supp}(\eta) \) implies \( \eta_i \in C_0^\infty(\mathbb{R}) \) for \( i = 1, 2 \). Moreover, we have for \( i = 1, 2 \) and \( j \in \mathbb{Z} \)
\[
T_j u = \mathcal{F}^{-1} \left( \eta(2^j|\xi - \xi_0|) \hat{u}_2(\xi) \right) = \mathcal{F}^{-1} \left( \eta_i(2^j|\xi - \xi_0|) (2^j|\xi - \xi_0|)^{s_i} \hat{u}_2(\xi) \right) = 2^{js_i} \mathcal{F}^{-1} \left( \eta_i(2^j|\xi - \xi_0|) m_i(\xi) P_i(\xi) \hat{u}(\xi) \right)
\]
where \( m_i(\xi) := (1 - \tau(\xi))|\xi - \xi_0|^{s_i}/P_i(\xi) \). Since \( \tau \) is smooth and identically 1 near \( \xi_0 \in S \), a calculation shows that our assumptions on \( P_i \) from (A2) imply that \( m_i \) satisfies the assumptions of Proposition 1. In fact, for \( |\alpha| = k := [d/2] + 1 \) and \( Q_i, \epsilon > 0 \) as in assumption (A2),
\[
|\partial^\alpha m_i(\xi)| \lesssim \sum_{0 \leq \gamma < \alpha} \left( \frac{\alpha}{\gamma} \right) |\partial^{\alpha - \gamma} ((1 - \tau(\xi))|\xi - \xi_0|^{s_i}(\xi)^{-s_i})| |\partial^\gamma Q_i(\xi)|
\]
\[
\lesssim 1 \cdot |\partial^\alpha Q_i(\xi)| + \sum_{0 \leq \gamma < \alpha} \langle \xi \rangle^{-|\alpha - \gamma| - 1} |\partial^\gamma Q_i(\xi)|
\]
\[
\lesssim \langle \xi \rangle^{-\epsilon - |\gamma|} + \langle \xi \rangle^{-|\alpha - \gamma| - 1} \langle \xi \rangle^{-|\gamma|}
\]
\[
\lesssim \langle \xi \rangle^{-\min(1, \epsilon) - |\alpha|}.
\]

Here we used the Leibniz rule. So, by Proposition 1 \( m_i \) is an \( L^\mu \)-multiplier for all \( \mu \in [1, \infty] \). Hence, Lemma 2 yields for all \( q \in [p, \infty] \)
\[
\|T_j u\|_q \lesssim 2^{js_i} \|\mathcal{F}^{-1}(\eta_i(2^j|\xi - \xi_0|) m_i(\xi) P_i(D)u(\xi))\|_q
\]
\[
\lesssim 2^{js_i - d(\frac{1}{p} - \frac{1}{q})} \|\mathcal{F}^{-1}(m_i(\xi) P_i(D)u(\xi))\|_p
\]
\[
\lesssim 2^{js_i - d(\frac{1}{p} - \frac{1}{q})} \|P_i(D)u\|_p.
\]

\( \square \)

Next we use these dyadic estimates to prove estimates of Gagliardo-Nirenberg type. We deduce our results from a detailed analysis of the special case \( P_i(D) = \langle D \rangle^{s_i} \) for \( s_1, s_2 \in \mathbb{R} \). This is possible due to
\[
\|\langle D \rangle^{s_i} u_2\|_p \lesssim \|P_i(D)u\|_p \quad (1 \leq p \leq \infty)
\]
for symbols \( P_1, P_2 \) as in (A2) thanks to Proposition 1. So we collect some mapping properties of the Bessel potential operators \( \langle D \rangle^{-s} \) where \( s > 0 \).

**Proposition 3.** Assume \( d \in \mathbb{N}, s > 0 \) and \( p, q, r \in [1, \infty], u \in \mathcal{S}(\mathbb{R}^d) \).
(i) If \( 0 \leq \frac{1}{p} - \frac{1}{q} < \frac{2}{d} \) then \( \|u\|_q \lesssim \|(D)^s u\|_p \).

(ii) If \( 0 \leq \frac{1}{p} - \frac{1}{q} = \frac{s}{d} \) and \( 1 < p, q < \infty \) then \( \|u\|_{q,r} \lesssim \|(D)^s u\|_{p,r} \) and \( \|u\|_q \lesssim \|(D)^s u\|_p \).

(iii) If \( 0 \leq \frac{1}{p} - \frac{1}{q} = \frac{s}{d} \) and \( s = d = 1 \) then \( \|u\|_\infty \lesssim \|(D) u\|_1 \).

(iv) If \( 0 \leq \frac{1}{p} - \frac{1}{q} = \frac{s}{d} \) and \( 1 < p < q < \infty \) then \( \|u\|_{q,\infty} \lesssim \|(D)^s u\|_1 \).

Proof. The parts (i),(iv) and the second part of (ii) are given in \([15, \text{Corollary 1.2.6}]\); the Lorentz space mapping properties from (ii) follow from real interpolation. The estimate (iii) follows from

\[
\|u\|_\infty \lesssim \|u\|_1 = \|m(D)(\langle D \rangle u)\|_1 \lesssim \|(D) u\|_1 \quad (u \in \mathcal{S}(\mathbb{R})).
\]

Here we used that \( m(\xi) := \xi(1 + |\xi|^2)^{-1/2} \) satisfies the assumptions of Proposition \([1] \). \( \square \)

We finally use these estimates to prove Gagliardo-Nirenberg inequalities for large frequencies.

**Proposition 4.** Assume \( d \in \mathbb{N}, \kappa \in [0, 1) \) and \((A2)\) for \( s_1, s_2 \in \mathbb{R} \). Then

\[
\|u_2\|_q \lesssim \|P_1(D)(\langle D \rangle u)\|_{r_2} \lesssim \|P_1(D)u\|_{r_1} \quad (u \in \mathcal{S}(\mathbb{R}^d))
\]

holds provided that the exponents \( q, r_1, r_2 \in [1, \infty] \) satisfy \( 0 \leq \frac{1}{r_1} - \frac{1}{r_2} \leq \frac{2}{d} \) as well as the following conditions in the endpoint case \( \frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{q} \):

(i) if \( q = \infty \) then \( \frac{1}{r_1} - \frac{2}{d} \neq 0 \) or \( r_1 = r_2 = \infty \), \( s_1 = s_2 = 0 \) or \( d = 1 \), \((r_1, r_2) = \left(\frac{1}{s_1}, \frac{1}{s_2}\right), s_1, s_2 \in \{0, 1\}\),

(ii) if \( 1 < q < \infty \) and \( \frac{1}{r_1} - \frac{2}{d} = \frac{1}{q} = \frac{1}{r_2} - \frac{2}{d} \) and if \( r_1 = 1, \kappa < 1 \)

\( 1 < r_2 < q, \kappa \geq \frac{2}{q} \) or \( r_2 = \infty, \frac{1}{q} \leq \kappa \leq \frac{1}{q} \),

(iii) if \( 1 < q < \infty \) and \( \frac{1}{r_1} - \frac{2}{d} = \frac{1}{q} = \frac{1}{r_2} - \frac{2}{d} \) if \( r_2 = 1, \kappa > 0 \)

\( 1 < r_1 < q, 1 - \kappa \geq \frac{2}{q} \) or \( r_1 = \infty, \frac{1}{q} \leq 1 - \kappa \leq \frac{1}{d} \).

Proof. As mentioned before, it is sufficient to prove the estimates in the prototypical case \( P_1(D) = (\langle D \rangle)^{s_1} \). So the case \( \kappa \in \{0, 1\} \) is covered by Proposition \([3, \text{(i),(ii),(iii)}]\). So we may concentrate on \( \kappa \in (0, 1) \) in the following. We combine Proposition \([2] \) and Lemma \([1] \) for the Bessel potential spaces \( X_i := P_i(D)^{-1}L^r(\mathbb{R}^d) = (\langle D \rangle)^{s_i}L^r(\mathbb{R}^d) \) and \( i = 1, 2 \). Here we use the identity

\[
u_2 = \sum_{j=-\infty}^{j_0} \mathcal{T}_j u \quad \text{where} \quad \|\mathcal{T}_j u\|_{q_i} \lesssim 2^{j(s_i - d(\frac{1}{r_i} - \frac{1}{q_i}))} \|u\|_{X_i} \quad (j \in \mathbb{Z}, 1 \leq r_i \leq q_i \leq \infty),
\]

see Proposition \([2] \). Our strategy is as follows. We first prove apply Lemma \([1] \) to get strong bounds. This will cover all non-endpoint cases \( \frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{1 - \kappa} \) as well as the endpoint cases involving \( q \in \{1, \infty\} \). The remaining discussion for \( 1 < q < \infty \) and \( 1 < r_1, r_2 < \infty \) can be taken from the literature, but the analysis for \( \{r_1, r_2\} \cap \{r_1, r_2\} \neq \emptyset \) is more delicate. We will first address the case \( \frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{1 - \kappa} \) where we prove our claim using complex and real interpolation theory. Finally, in the case \( \frac{1}{r_1} - \frac{1}{r_2} \neq \frac{2}{1 - \kappa} \) we will first deduce restricted weak-type bounds from Lemma \([1] \) and upgrade them to strong bounds by
interpolating the restricted weak-type bounds with each other. We will need in the following that our assumptions imply \( \sigma \geq 0 \).

**Step 1:** We start the interpolation procedure with (non-endpoint) exponents satisfying
\begin{equation}
0 \leq \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} < \frac{\sigma}{d}.
\end{equation}
In that case the interpolation estimate \( (6) \) with \( (Y_1, Y_2, \theta, r) := (L^{q_1}(\mathbb{R}^d), L^{q_2}(\mathbb{R}^d), \kappa, q) \) gives the bound
\begin{equation}
\|u_2\|_q = \|\sum_{j=-\infty}^{\infty} T_j u\|_q \lesssim \|u\|_{(X_1,X_2),s,q} \lesssim \|\langle D \rangle^{s_1} u\|_{r_1}^{1-\kappa} \|\langle D \rangle^{s_2} u\|_{r_2}^\kappa.
\end{equation}

Here, \( (6) \) applies because \( (11) \) allows to find \( q \in [r_1, \infty] \) such that
\[ (1 - \kappa) \left( s_1 - d \frac{1}{r_1} - \frac{1}{q_1} \right) + \kappa \left( s_2 - d \frac{1}{r_2} - \frac{1}{q_2} \right) > 0, \quad 1 \frac{1}{q} = \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2}.
\]
So the claim is proved for all non-endpoint exponents given by \( (11) \).

It remains to discuss the endpoint case \( 0 \leq \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\sigma}{d} \). Using \( (5) \) for \( Y_1 = Y_2 = L^q(\mathbb{R}^d) \) we get the claim for all exponents satisfying
\begin{equation}
0 \leq \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\sigma}{d} \quad \text{and} \quad q \geq \max\{r_1, r_2\}, \quad \frac{1}{r_1} - \frac{s_1}{d} \neq \frac{1}{r_2} - \frac{s_2}{d}.
\end{equation}

Here the latter two inequalities correspond to \( \beta_1, \beta_2 \neq 0 \) in Lemma \( \dagger \). From this we infer that the claimed endpoint estimates hold for \( q \in \{1, \infty\} \) via the following case distinction:
- **Case** \( q = 1 \): \( r_1 = r_2 = 1, s_1 = s_2 = 0 \) is trivial,
- **Case** \( q = 1 \): \( r_1 = r_2 = 1, s_1 = 0, s_1 \neq 0 \neq s_2 \) is covered by \( (12) \),
- **Case** \( q = \infty \): \( r_1 = r_2 = \infty, s_1 = s_2 = 0 \) is trivial,
- **Case** \( q = \infty \): \( \frac{1}{r_1} - \frac{s_1}{d} \neq 0 \neq \frac{1}{r_2} - \frac{s_2}{d} \) is covered by \( (12) \),
- **Case** \( q = \infty \): \( (d, r_1, r_2) = (1, \frac{1}{s_1}, \frac{1}{s_2}), s_1, s_2 \in \{0, 1\} \) is covered by Proposition \( \dagger \) (iii).

These are all cases involving \( q \in \{1, \infty\} \) and in particular claim (i) is proved. So we are left with those endpoint estimates for \( 1 < q < \infty \) that are not covered by \( (12) \).

**Step 2:** The claim holds for \( 1 < r_1, r_2 < \infty \) due to
\begin{equation}
\|u\|_q \lesssim \|\langle D \rangle u\|_\sigma \lesssim \|\langle D \rangle^{s_1} u\|_{r_1}^{1-\kappa} \|\langle D \rangle^{s_2} u\|_{r_2}^\kappa,
\end{equation}
where \( \frac{1}{\sigma} := \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} \). This is a consequence of Sobolev’s Embedding Theorem \( \dagger \) Theorem 6.5.1 and the complex interpolation result from \( \dagger \) Theorem 6.4.5 (7)]. So we may in the following assume \( \{r_1, r_2\} \cap \{1, \infty\} \neq \emptyset \). As announced earlier, we first deal with \( \frac{1}{r_1} - \frac{1}{r_2} = \frac{s_1 - s_2}{d} \).

**Step 3:** So assume we are in the endpoint case with \( 1 < q < \infty, \frac{1}{r_1} - \frac{1}{r_2} = \frac{s_1 - s_2}{d}, r_1 \leq r_2 \) (w.l.o.g.) and \( \{r_1, r_2\} \cap \{1, \infty\} \neq \emptyset \). Then \( \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\sigma}{d} \) implies \( \frac{1}{r_1} - \frac{s_1}{d} = \frac{1}{q} = \frac{1}{r_2} - \frac{s_2}{d} \). We distinguish the following cases:
- **Case** \( r_1 = 1, r_2 = 1 \): This case is excluded, so there is nothing to prove.
Case $r_1 = 1, 1 < r_2 < q$: By Proposition \(3\) (ii),(iv) we have $\|u\|_{q,\infty} \lesssim \|\langle D \rangle^{s_1} u\|_1$ as well as $\|u\|_{q,r_2} \lesssim \|\langle D \rangle^{s_2} u\|_{r_2}$. Applying the interpolation identity \(3\) Theorem 5.3.1]

$$L^q(\mathbb{R}^d) = (L^{q,\infty}(\mathbb{R}^d), L^{q,q}(\mathbb{R}^d))_{\kappa,q}, \quad \kappa \in (0, 1],$$

we infer for all $\kappa \in [\frac{1}{q}, 1]$,

$$\|u\|_q \lesssim \|u\|_{q,\infty}^{1-\kappa} \|u\|_{q,q}^{\kappa} \lesssim \|u\|_{q,\infty}^{1-\kappa} \|u\|_{q,r_2}^{\kappa} \lesssim \|\langle D \rangle^{s_1} u\|_1^{1-\kappa} \|\langle D \rangle^{s_2} u\|_{r_2}^{\kappa}.$$  

Case $r_1 = 1, r_2 = \infty$: We have to prove (10) for $\frac{1}{q} \leq \kappa \leq \frac{1}{q}$. It is sufficient to prove the claim first for $\kappa = \frac{1}{q}$ and then for $\kappa = \frac{1}{q}$. We use $\|u\|_{q,\infty} \lesssim \|\langle D \rangle^{s_1} u\|_1$ and

$$\|u\|_{q,2}^2 \lesssim \|\langle D \rangle^\frac{d}{q} u\|_2^2 = \int_{\mathbb{R}^d} \langle D \rangle^\frac{d}{q} u \cdot \langle D \rangle^\frac{d}{q} u \, dx \leq \|\langle D \rangle^{s_1} u\|_1 \|\langle D \rangle^{s_2} u\|_\infty.$$  

In (12) we subsequently used Proposition \(3\) (ii), the $L^2$-isometry property of the Fourier transform as well as $s_1 = \frac{d}{q}$, $s_2 = -\frac{d}{q}$. Real interpolation of these two estimates and $L^q(\mathbb{R}^d) = (L^{q,\infty}(\mathbb{R}^d), L^{q,2})_{2/q,q}$, which is (13) for $\kappa = \frac{2}{q}$, gives

$$\|u\|_q \lesssim \|u\|_{q,\infty}^{1-\frac{2}{q}} \|u\|_{q,2}^{\frac{2}{q}} \lesssim \|\langle D \rangle^{s_1} u\|_1^{1-\frac{2}{q}} \|\langle D \rangle^{s_2} u\|_\infty^\frac{2}{q}.$$  

So the claim holds for $\kappa = \frac{1}{q}$ and we now consider $\kappa = \frac{1}{q}$. Here we use Stein’s Interpolation Theorem \(31\) in a more general setting \(35\) Theorem 2.1] for the family of linear operators $T^s u := e^{s^2 \langle D \rangle^{s/2} u} u$ with $s \in \mathbb{C}, 0 \leq \text{Re}(s) \leq 1$. We have

$$\|T^{it} u\|_{\text{BMO}(\mathbb{R}^d)} = e^{-t^2} \|\langle D \rangle^{it} (\langle D \rangle^{-\frac{d}{q}} u)\|_{\text{BMO}(\mathbb{R}^d)} \lesssim \|\langle D \rangle^{-\frac{d}{q}} u\|_\infty,$$

$$\|T^{1+it} u\|_2 = e^{1-t^2} \|\langle D \rangle^{\frac{d}{q} - \frac{d}{q} u} u\|_2 \lesssim \|\langle D \rangle^{\frac{d}{q} u} u\|_2 \|\langle D \rangle^{-\frac{d}{q}} u\|_\infty.$$  

Here we used the validity of Mikhslin’s Multiplier Theorem in BMO(\mathbb{R}^d) to deduce that the operator norm $\langle D \rangle^{it} : L^\infty(\mathbb{R}^d) \to \text{BMO}(\mathbb{R}^d)$ is polynomially bounded with respect to $t$ and thus compensated by the mitigating factor $e^{-t^2}$ as $|t| \to \infty$. We refer to Proposition 3.4, Theorem 4.4 and the comments on page 20-21 in Tao’s Lecture notes \(34\) where such an application in the context of Stein’s interpolation theorem is explicitly mentioned. In view of \([\text{BMO}(\mathbb{R}^d), L^2(\mathbb{R}^d)]_\theta = L^{2/\theta}(\mathbb{R}^d)\) for $0 < \theta \leq 1$ we may plug in $\theta = \frac{2}{q}$ and get in view of $s_1 = \frac{d}{q}, s_2 = -\frac{d}{q}$

$$\|u\|_q = \|\langle D \rangle^{\frac{d}{q}} u\|_q \lesssim \|\langle D \rangle^{\frac{d}{q}} u\|_1^{1-\theta} \left(\|\langle D \rangle^{\frac{d}{q}} u\|_1 \|\langle D \rangle^{-\frac{d}{q}} u\|_\infty\right)^\theta = \|\langle D \rangle^{s_1} u\|_1^{\frac{2}{q}} \|\langle D \rangle^{s_2} u\|_\infty^{\frac{2}{q}}.$$  

Case 1 < $r_1 < r_2 = \infty$: We have to prove (10) for $1 < q < r_1, \kappa \geq \frac{q}{q}$. We consider $T^s u := e^{s^2 \langle D \rangle^{s_1} (s_1 - s_2) u}$ and obtain as before

$$\|T^{it} u\|_{\text{BMO}(\mathbb{R}^d)} \lesssim \|\langle D \rangle^{s_2} u\|_\infty, \quad \|T^{1+it} u\|_{r_1} \lesssim \|\langle D \rangle^{s_1} u\|_{r_1}.$$  

So we conclude for $\kappa := \frac{q}{q} = \frac{s_2}{s_2 - s_1}$

$$\|u\|_q = \|T^\kappa u\|_q \lesssim \|\langle D \rangle^{s_2} u\|_1^{1-\kappa} \|\langle D \rangle^{s_1} u\|_{r_1}^\kappa.$$
This proves the claim for $\kappa = \frac{1}{q}$. Since the desired bound for $\kappa = 1$ follows from Proposition 3 (ii), we get the claim for $\kappa \in \left[\frac{1}{q}, 1\right]$.

- Case 1 $< r_1 = r_2 = \infty$: This case does not occur because $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = -\frac{1}{q} < 0$.

**Step 4:** To prove the remaining estimates we first prove restricted weak-type estimates $\|u_2\|_{q,\infty} \lesssim \|u\|_{(X_1,X_2)_{\sigma,1}}$ for all exponents satisfying

$$0 \leq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\tau}{d} \quad \text{and} \quad 1 < q < \infty \quad \text{and} \quad \frac{1}{r_1} - \frac{1}{r_2} \neq \frac{s_1 - s_2}{d}.$$

For $s_1 = s_2 = 0$ this is implied by H"older’s Inequality, so we may assume $\tau > 0$ or $\tau = 0, (s_1, s_2) \neq (0, 0)$. For $\tau = 0, (s_1, s_2) \neq (0, 0), q = r_1 = r_2$ this is implied by the strong estimates in the case $\tau = 0$, so we may even assume $\tau > 0, (s_1, s_2) \neq (0, 0), (r_1, r_2) \neq (q, q)$. For the remaining exponents the weak estimate is a consequence of (6) because one can find $q_i \in [r_i, \infty)$ such that

$$(1-\kappa)\left(s_1 - d\left(\frac{1}{r_1} - \frac{1}{q_1}\right)\right) + \kappa\left(s_2 - d\left(\frac{1}{r_2} - \frac{1}{q_2}\right)\right) = 0,$$

$$\frac{1}{q} = \frac{1-\kappa}{q_1} + \frac{\kappa}{q_2}, \quad s_i - d\left(\frac{1}{r_i} - \frac{1}{q_i}\right) \neq 0, \quad q_1 \neq q_2.$$ 

Indeed, this condition is equivalent to $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\tau}{d}$ and finding $q_2$ such that

$$\frac{1}{q} - \frac{1-\kappa}{r_1} \leq \frac{\kappa}{q_2}, \quad q_2 \neq q_i, \quad \frac{1}{q} - (1-\kappa)\left(\frac{1}{r_1} - \frac{s_1}{d}\right) \neq \frac{\kappa}{q_2} \neq \frac{\kappa}{r_2} - \frac{s_2}{d},$$

and such a choice is possible due to our assumptions. (In the case $\tau = 0, (s_1, s_2) \neq (0, 0), (r_1, r_2) \neq (q, q)$ choose $q_2 = r_2, q_1 = r_1$.) In this way we obtain $\|u_2\|_{q,\infty} \lesssim \|u\|_{(X_1,X_2)_{\sigma,1}}$ for all exponents satisfying (10). We finally interpolate these restricted weak-type estimates with each other to prove strong estimates for exponents as in (10). To this end let $\delta > 0$ be sufficiently small (but fixed) and $\varepsilon := \delta\left(d\frac{s_1 - s_2}{d} - \frac{1}{r_1} + \frac{1}{r_2}\right) \neq 0$ and define $\hat{q}, q^*, \hat{\kappa}, \kappa^*$ via

$$\frac{1}{\hat{q}} - \varepsilon = \frac{1}{q} = \frac{1}{q^*} + \varepsilon \quad \text{and} \quad \hat{\kappa} - \delta = \kappa = \kappa^* + \delta.$$ 

Then $(\hat{q}, r_1, r_2, \hat{\kappa}), (q^*, r_1, r_2, \kappa^*)$ satisfies (10) and the reiteration property of real interpolation [3] Theorem 3.5.3] gives

$$\|u_1\|_q \lesssim \|u_1\|_{(L^\kappa(L^d), L^\delta)}^{1/\hat{q}} \lesssim \|u_1\|_{(X_1,X_2)_{\hat{\kappa},(X_1,X_2)_{\kappa,1}}}^{1/\hat{q}} \lesssim \|u_1\|_{(X_1,X_2)_{\kappa,1}} \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa} \|P_2(D)u\|_{r_2}^\kappa$$

Here the first bound uses $\frac{1}{\hat{q}} = \frac{1}{2}\left(\frac{1}{q^*} + \frac{1}{q}\right)$ and the third uses $\kappa = \frac{1}{2}(\hat{\kappa} + \kappa^*)$. This finishes the proof. \hfill \Box

We have thus proved that the Gagliardo-Nirenberg inequality (3) holds for non-critical frequencies whenever the exponents belong to the set

$$\mathcal{B}(\kappa) := \{(q, r_1, r_2) \in [1, \infty]^3 : (q, r_1, r_2) \text{ as in Proposition 4}\}.$$
Remark 2.
(a) The original Gagliardo-Nirenberg inequality \( \| \nabla^j v \|_q \lesssim \| \nabla^m v \|_1^{1-\kappa} \| v \|_2^{\kappa} \) from [28, p. 125] holds for \( j, m \in \mathbb{N} \) provided that \( \frac{1}{q} - \frac{j}{d} = (1 - \kappa)\left(\frac{1}{r_1} - \frac{m}{d}\right) + \frac{\kappa}{r_2} \) and \( \frac{1}{m} \leq 1 - \kappa < 1 \). Our result shows that “in most cases” the large frequency part of this estimate holds provided that \( \frac{1}{m} \leq 1 - \kappa < 1 \) holds and \( \frac{1}{q} - \frac{j + 1}{d} \geq (1 - \kappa)\left(\frac{1}{r_1} - \frac{m}{d}\right) + \frac{\kappa}{r_2} \). The exceptions are due to the fact that, in \( L^1(\mathbb{R}^d) \) or \( L^\infty(\mathbb{R}^d) \), the term \( \langle D \rangle^j u \) does not control \( D^j u \), i.e., not every single partial derivative of order \( j \). This is a consequence of the unboundedness of the Riesz transform on these spaces.
(b) Our proof indicates which function spaces to choose in order to get some endpoint estimates in the exceptional cases as well. Roughly speaking, one may replace \( L^q(\mathbb{R}^d) \) by \( L^{q,r}(\mathbb{R}^d) \) for suitable \( r > q \) and \( L^\infty(\mathbb{R}^d) \) by \( \text{BMO}(\mathbb{R}^d) \) on the left hand side. On the right hand side the Hardy space \( \mathcal{H}^1(\mathbb{R}^d) \) may replace \( L^1(\mathbb{R}^d) \).
(c) One may as well consider symbols \( P_i(D) \) that vanish at some finite set of points in \( \mathbb{R}^d \setminus S \). If for instance one has \( P_i(\xi) = b_i(\xi)\left|\xi - \xi^*\right|^4 \) near \( \xi^* \in \mathbb{R}^d \setminus S \) for \( t_1, t_2 > -d \) and non-vanishing \( b_i \in C^\infty(\mathbb{R}^d) \), then one finds as in Proposition 4 that the interpolation estimate holds in this frequency regime whenever \( \frac{1}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} > \frac{r}{d} \) where \( r := (1 - \kappa)t_1 + \kappa t_2 \). Under suitable extra conditions similar to the ones above, this may be extended to the endpoint case \( \frac{1}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{r}{d} \).
(d) The proof in the important special case \( 1 < r_1, r_2, q < \infty \) is much shorter than the complete analysis, see the beginning of Step 2.

4. Critical frequency analysis

We introduce a real number \( A_\varepsilon(p, q) \) such that \( \| \tilde{T}_j \|_{p \to q} \lesssim 2^{-j A_\varepsilon(p, q)} \) holds for suitably defined dyadic operators \( \tilde{T}_j \) that play the role of the \( T_j \) in the previous section. Unfortunately, the definition of \( A_\varepsilon(p, q) \) is rather complicated for \( d \geq 2 \). It involves the number

\[
A(p, q) := \min \{ A_0, A_1, A_2, A_2', A_3, A_3', A_4, A_4' \}
\]

where \( A_i = A_i(p, q) \) and \( A'_i = A_i(q', p') \) are given by

\[
A_0 = 1, \quad A_1 = \frac{k + 2}{2} \left( \frac{1}{p} - \frac{1}{q} \right), \quad A_2 = \frac{k + 2}{2} - \frac{k + 1}{q}
\]

as well as

\[
A_3 = \frac{2d - k}{2} - \frac{2d - k - 1}{q}, \quad A_4 = \frac{k + 2}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{2d - k - 2}{2} - \frac{2d - k - 2}{q}.
\]

The values \( A_0, A_1, A_1', A_2, A_2' \) will be important for \( 1 \leq p \leq 2 \leq q \leq \infty \) whereas all other exponents satisfying \( 1 \leq p \leq q \leq \infty \) come with \( A_3, A_3', A_4, A_4' \). Then we define for \( \varepsilon > 0 \)

\[
A_\varepsilon(p, q) := \frac{1}{p} - \frac{1}{q} \quad \text{if } d = 1, \quad A_\varepsilon(p, q) := A(p, q) - \varepsilon \cdot 1_{(p, q) \in \varepsilon} \quad \text{if } d \geq 2.
\]
Here, $E$ denotes a set of exceptional points where we do not have strong bounds, but only weak bounds or restricted weak-type bounds. It is given by

$$E := \left\{ (p, q) \in [1, \infty]^2 : \begin{array}{l}
\frac{1}{p} = \frac{k+2}{2(k+1)}, \quad \frac{1}{q} \leq \frac{k^2}{2(k+1)(k+2)} \quad \text{or} \\
\frac{1}{q} = \frac{k}{2(k+1)}, \quad \frac{1}{p} \geq \frac{k^2 + 6k + 4}{2(k+1)(k+2)} \end{array} \right\}$$

and coincides with the red points in Figure 1.

![Figure 1. Riesz diagram with the bounds for the mapping constant of $\tilde{T}_j$ from Lemma 4. The exceptional points from $E$ are coloured in red.](image)

We first prove dyadic estimates in the frequency regime close to the critical surface $S$. The latter can be locally parametrized as a graph $\xi_d = \psi(\xi')$ after some permutation of coordinates, where $\xi = (\xi', \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \simeq \mathbb{R}^d$. In view of (A1) we study operators of the
To prove (20) it thus suffices to show Lemma 3.

\[
\tilde{T}_j f := F^{-1} \left( \eta \left( 2^j (\xi - \psi(\xi')) \right) \chi(\xi') \hat{f}(\xi) \right) = \tilde{K}_j * f
\]
\[
\text{where } \tilde{K}_j := F^{-1} \left( \eta \left( 2^j (\xi - \psi(\xi')) \right) \chi(\xi') \right)
\]
and
\[
\psi \in C^\infty(\mathbb{R}^{d-1}), \ \chi \in C_0^\infty(\mathbb{R}^{d-1}) \text{ and at least } k \in \{1, \ldots, d-1\}
\]
eigenvalues of the Hessian $D^2 \psi$ are non-zero on $\text{supp}(\chi)$.

In the degenerate case $d = 1$ we interpret $\eta(2^j (\xi - \psi(\xi'))) \chi(\xi')$ as $\eta(2^j (\xi - c))$ for some constant $c \in \mathbb{R}$. Our analysis of the mapping properties of $\tilde{T}_j$ follows [26, Section 4]. Contrary to the situation for $T_j$, only the bounds for $j \not\to +\infty$ will be of importance. Repeating the proof of Lemma 2 gives the following result in the one-dimensional case.

**Lemma 3.** Assume $d = 1$ and $\eta \in C_0^\infty(\mathbb{R})$. Then we have
\[
\|\tilde{T}_j\|_{p \to \infty} \leq 2^{-j(\frac{d}{p} - \frac{1}{2})} \quad \text{for } 1 \leq p \leq q \leq \infty, \ j \in \mathbb{Z}.
\]
The bounds in higher dimensions are more complicated and depend on the number $k \in \{1, \ldots, d-1\}$ of non-vanishing principal curvatures of $S$. We first analyze the kernel function $\tilde{K}_j$ following [26, Lemma 4.4].

**Proposition 5.** Assume $d \in \mathbb{N}, d \geq 2$, let $\chi, \psi, k$ be as in (19) and $\eta \in C_0^\infty(\mathbb{R})$. Then the kernel function $\tilde{K}_j$ satisfies for $j \in \mathbb{Z}, j \geq j_0$
\[
\|\tilde{K}_j\|_r \lesssim 2^{-j\left(\frac{d-k}{2} - \frac{2d-k-1}{r+1}\right)} \text{ if } 1 \leq r \leq 2, \quad \|\tilde{K}_j\|_\infty \lesssim 2^{-j}.
\]

**Proof.** The bound $\|\tilde{K}_j\|_2 \lesssim 2^{-j/2}$ follows from Plancherel’s identity and (18). Indeed,
\[
\|\tilde{K}_j\|_2^2 = \int_{\mathbb{R}^d} \eta(2^j (\xi - \psi(\xi'))) \chi(\xi')^2 d(\xi', \xi_d)
\]
\[
= \int_{\mathbb{R}^{d-1}} \chi(\xi')^2 \left( \int_{\mathbb{R}} \eta(2^j t)^2 dt \right) d\xi'
\]
\[
= 2^{-j} \|\chi\|_2^2 \|\eta\|_2^2.
\]
To prove (20) it thus suffices to show $\|\tilde{K}_j\|_1 \lesssim 2^{-j(\frac{k+2}{2} - d)}$ as well as $\|\tilde{K}_j\|_\infty \lesssim 2^{-j}$ and to apply the Riesz-Thorin interpolation theorem. These two norm bounds for the kernel function are consequences of the pointwise bounds for arbitrary $N, M \in \mathbb{N}_0$
\[
|\tilde{K}_j(x)| \lesssim_{N,M} 2^{-j(1 + 2^{-j} |x_d|^{-M}(1 + |x'|)^{-N}} \text{ if } |x'| \geq c|x_d|,
\]
\[
|\tilde{K}_j(x)| \lesssim_{N,M} 2^{-j(1 + 2^{-j} |x_d|^{-M}(1 + |x|)^{-N}} \text{ if } |x'| \leq c|x_d|,
\]
where $c > 0$ is suitably chosen. Indeed, choosing $M, N$ sufficiently large we get
\[
\|\tilde{K}_j\|_1 \lesssim_{N,M} \int_{\mathbb{R}} \left( \int_{|x'| \leq c|x_d|} 2^{-j(1 + 2^{-j} |x_d|^{-M}(1 + |x'|)^{-N}} dx' \right) dx_d
\]
\[
+ \int_{\mathbb{R}} \left( \int_{|x'| \geq c|x_d|} 2^{-j(1 + 2^{-j} |x_d|^{-M}(1 + |x'|)^{-N}} dx' \right) dx_d
\]
\[ \lesssim_{M,N} 2^{-j} \int_{\mathbb{R}} (1 + 2^{-j} |x_d|)^{-M} |x_d|^{d-1} (1 + |x_d|)^{-\frac{k}{2}} dx_d \\
+ 2^{-j} \int_{\mathbb{R}} (1 + 2^{-j} |x_d|)^{-M} (1 + |x_d|)^{d-N} dx_d \]
\[ \lesssim_{M,N} 2^{-j} \int_{0}^{2^j} |x_d|^{d-1} (1 + |x_d|)^{-\frac{k}{2}} dx_d + 2^{j(M-1)} \int_{2^j}^{\infty} |x_d|^{d-\frac{k}{2}-1-M} dx_d \]
\[ \lesssim_{M,N} 2^{-j(\frac{k+2}{2}-d)}. \]

Here we used \( 2^j \geq 2^{j_0} > 0 \). So it remains to prove the pointwise bounds by adapting the arguments from [20]. We have

\[ \tilde{K}_j(x) = c_d 2^{-j}(F^{-1}\eta)(2^{-j}x_d) \int_{\mathbb{R}^{d-1}} e^{i(x' \cdot \xi' + x_d \psi(\xi'))} \chi(\xi') d\xi' \]

for some dimensional constant \( c_d > 0 \). We choose \( c > 0 \) so large that the smooth phase function \( \Phi(\xi') = x' \cdot \xi' + x_d \psi(\xi') \) satisfies \( |\nabla \Phi(\xi')| \geq c^{-1} |x'| \) for all \( \xi' \in \mathbb{R}^{d-1} \) whenever \( |x'| \geq c|x_d| \). In view of \( \chi \in C_0^\infty(\mathbb{R}^{d-1}) \) the method of non-stationary phase gives

\[ |\tilde{K}_j(x)| \lesssim_{N} 2^{-j}|(F^{-1}\eta)(2^{-j}x_d)|(1 + |x'|)^{-N} \]
\[ \lesssim_{N,M} 2^{-j}(1 + 2^{-j}|x_d|)^{-M}(1 + |x'|)^{-N} \text{ for } |x'| \geq c|x_d|. \]

In the second estimate we used that \( F^{-1}\eta \) is a Schwartz function. On the other hand, the theory of oscillatory integrals gives (see [32, p.361])

\[ |\tilde{K}_j(x)| \lesssim_{M} 2^{-j}(1 + 2^{-j}|x_d|)^{-M}(1 + |x_d|)^{-\frac{k}{2}} \text{ for } |x'| \leq c|x_d|. \]

\[ \square \]

Next we use Proposition 3 to find upper bounds for the operator norms of \( \tilde{T}_j \) as maps from \( L^p(\mathbb{R}^d) \) to \( L^q(\mathbb{R}^d) \) where \( 1 \leq p \leq q \leq \infty \). The latter condition is mandatory since \( \tilde{T}_j \) is a translation-invariant operator covered by Hörmander’s result from [19, Theorem 1.1].

**Lemma 4.** Assume \( d \in \mathbb{N}, d \geq 2 \) and let \( \chi, \psi, k \) are as in [19] and \( \eta \in C_0^\infty(\mathbb{R}) \). Then, for any fixed \( \varepsilon > 0 \),

\[ \| \tilde{T}_j \|_{p \rightarrow q} \lesssim 2^{-j A_\varepsilon(p,q)} \text{ for } 1 \leq p \leq q \leq \infty, \ j \in \mathbb{Z}, j \geq j_0. \]

**Proof.** We first analyze the range \( 1 \leq p \leq 2 \leq q \leq \infty \). Plancherel’s Theorem gives

\[ \| \tilde{T}_j f \|_2 = \| \eta (2^j(\xi_d - \psi(\xi'))) \chi(\xi') \tilde{f} \|_2 \lesssim \| \tilde{f} \|_2 = \| f \|_2 \]

due to \( \eta, \chi \in L^\infty(\mathbb{R}^d) \). The Stein-Tomas Theorem for surfaces with \( k \) non-vanishing principal curvatures [32, p.365] yields as in [20, Lemma 4.3]

\[ \| \tilde{T}_j f \|_q \lesssim 2^{-\frac{k}{2}} \| f \|_2, \quad \| \tilde{T}_j f \|_2 \lesssim 2^{-\frac{k}{2}} \| f \|_q \quad \text{if } \frac{1}{q} \leq \frac{k}{2(k+2)}. \]

The Restriction-Extension operator \( f \mapsto F^{-1}(\hat{f} \sigma_M) \) for compact pieces \( M \) of hypersurfaces with \( k \) non-vanishing principal curvatures has the mapping properties from [20, Corollary 5.1],
so it is bounded for \((p, q)\) belonging to the pentagonal region

\[
\frac{1}{p} > \frac{k + 2}{2(k + 1)}, \quad \frac{1}{q} < \frac{k}{2(k + 1)}, \quad \frac{1}{p} - \frac{1}{q} \geq \frac{2}{k + 2}.
\]

So for these exponents and \(M_t := \{\xi = (\xi', \xi_d) \in \text{supp}(\chi) \times \mathbb{R} : \xi_d - \psi(\xi') = t\}\) with induced surface measure \(d\sigma_{M_t} = (1 + |\nabla \psi(\xi')|^2)^{1/2} d\xi'\) we have for \(\hat{g}(\xi) := \chi(\xi')\hat{f}(\xi)(1 + |\nabla \psi(\xi')|^2)^{-1/2}\)

\[
\|\hat{T}_j f\|_q \lesssim \int_{\mathbb{R}} |\eta(2^j t)||\mathcal{F}^{-1}(\hat{g} d\sigma_{M_t})|_q dt \lesssim \int_{\mathbb{R}} |\eta(2^j t)||g|_p dt \lesssim 2^{-j\|f\|_p}.
\]

Moreover, [26] Corollary 5.1 yields restricted weak-type bounds from \(L^{p,1}(\mathbb{R}^d)\) to \(L^{q,\infty}(\mathbb{R}^d)\) for all \((p, q)\) belonging to the closure of the above-mentioned pentagon, which implies \(\|\hat{T}_j f\|_{q,\infty} \lesssim 2^{-j\|f\|_{p,1}}\) in the same manner. Interpolating all these bounds gives

\[
\|\hat{T}_j\|_{p \to q} \lesssim 2^{-j(\min\{A_0, A_1, A_2, A'_2\} - \varepsilon - 1_{(p, q) \in \varepsilon})} = 2^{-j A_2(p, q)} \quad \text{for } 1 \leq p \leq 2 \leq q \leq \infty, \varepsilon > 0.
\]

This finishes the analysis in the case \(1 \leq p \leq 2 \leq q \leq \infty\). For \(2 \leq p \leq q \leq \infty\) or \(1 \leq p \leq q \leq 2\) we get from Proposition 5

\[
\|\hat{T}_j\|_{1 \to 1} + \|\hat{T}_j\|_{\infty \to \infty} \lesssim \|\hat{K}_j\|_1 \lesssim 2^{-j(d+2)-d}.
\]

Interpolating the estimates for \((p, q) = (\infty, \infty)\) with the ones for \(p = 2, q \geq 2\) from above yields the estimates in the region \(A'_3, A'_4\); the dual ones follow analogously. So we get

\[
\|\hat{T}_j\|_{p \to q} \lesssim 2^{-j \min\{A_3, A'_4\}} = 2^{-j A_4(p, q)},
\]

which proves the claim. \(\square\)

The optimality of our constants is open. It would be interesting to see whether recent results and techniques for oscillatory integral operators by Guth, Hickman, Iliopoulou [16] or Kwon, Lee [23] (Proposition 2.4, Proposition 2.5) can be adapted to prove better bounds, especially in the range \(1 \leq p \leq q < 2\) or \(2 \leq p \leq q \leq \infty\). Any theorem leading to a larger value of \(A_4(p, q)\) will automatically provide a larger range of exponents \(q, r_1, r_2\) for which our Gagliardo-Nirenberg inequalities hold. Candidates for such values \(\geq A_4(p, q)\) are given in [10] Lemma 2.2 and [26] Lemma 4.4), but it seems nontrivial to make use of those in our setting. Next we use the estimates for \(\hat{T}_j\) to discuss the relevant operators at distance \(2^{-j}\) from the critical surface where \(j \nearrow +\infty\).

**Proposition 6.** Assume \(d \in \mathbb{N}\) and \((A1)\) with \(\alpha_1, \alpha_2 > -1\). Then there are bounded linear operators \(T_j : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)\) and \(j_0 \in \mathbb{Z}\) with \(\sum_{j = j_0}^{\infty} T_j u = u_1\) such that, for \(i = 1, 2\) and any given \(\varepsilon > 0\), we have for all \(u \in \mathcal{S}(\mathbb{R}^d),\)

\[
\|T_j u\|_q \lesssim 2^{j(\alpha_i - A_i(p, q))}\|P_i(D)u\|_p \quad \text{for } 1 \leq p \leq q \leq \infty, j \in \mathbb{Z}, j \geq j_0.
\]

**Proof.** Recall \(u_1 = \mathcal{F}^{-1}(\tau u)\) where \(\tau\) was chosen in [4]; we first consider the case \(d \geq 2\). According to Assumption \((A1)\) there are \(\tau_1, \ldots, \tau_L \in C_0^\infty(\mathbb{R}^d)\) such that \(\tau_1 + \ldots + \tau_L = \tau\) holds and \(S \cap \text{supp}(\tau) = \{\xi \in \text{supp}(\tau) : \xi_d = \psi_l(\xi')\text{ where } \xi = \Pi_l \xi\}\). Here, \(\Pi_l\) denotes some
permutation of coordinates in \( \mathbb{R}^d \). Since \( P \) vanishes of order \( \alpha \) near the surface in the sense of Assumption (A1), we may write

\[
P(\xi)^{-1} \tau_i(\xi) = \left[ \tau_{i+}(\xi) (\tilde{\xi}_d - \psi_l(\tilde{\xi}))^{-\alpha} + \tau_{i-}(\xi) (\tilde{\xi}_d - \psi_l(\tilde{\xi}))^{-\alpha} \right] \chi_l(\tilde{\xi})
\]

with \( \tau_{i+}, \tau_{i-} \in C_0^\infty(\mathbb{R}^d), \chi_l \in C_0^\infty(\mathbb{R}^d-1), \tilde{\xi} := \Pi_l \xi \).

for suitable functions \( \chi_l, \psi_l \) that satisfy (19). In view of this we define

\[
\mathcal{T}_j := \sum_{l=1}^L T_j^l \quad \text{where} \quad T_j^l u := F^{-1} \left( \eta_l(\xi) \tilde{u}(\xi) \eta(2^j (\tilde{\xi}_d - \psi_l(\tilde{\xi}))) \chi_l(\tilde{\xi}) \right) \quad (\tilde{\xi} = \Pi_l \xi).
\]

Since 0 does not belong to the support of \( \eta \), there is \( j_0 \in \mathbb{Z} \) such that \( u_1 = \sum_{j=j_0}^\infty \mathcal{T}_j u \) in the sense of distributions. We introduce the smooth function \( \eta_h(z) := \eta(z)|z|^{-\alpha_1} \). Then Lemma 4 yields

\[
\| \mathcal{T}_j u \|_q \lesssim \sum_{l=1}^L \| T_j^l u \|_q \\
= \sum_{l=1}^L \| F^{-1} \left( \eta(2^j (\tilde{\xi}_d - \psi_l(\tilde{\xi}))) \chi_l(\tilde{\xi}) \tau_l(\xi) \tilde{u}(\xi) \right) \|_q \\
= \sum_{l=1}^L \| F^{-1} \left( \eta(2^j (\tilde{\xi}_d - \psi_l(\tilde{\xi}))) \chi_l(\tilde{\xi}) \tau_l(\xi) P_l(\widetilde{D} u(\xi)) \right) \|_q \\
\geq \sum_{l=1}^L 2^{j \alpha_1} \| F^{-1} \left( \eta(2^j (\tilde{\xi}_d - \psi_l(\tilde{\xi}))) \chi_l(\tilde{\xi}) (\tau_{l+}(\xi) + \tau_{l-}(\xi)) P_l(\widetilde{D} u(\xi)) \right) \|_q \\
\lesssim \sum_{l=1}^L 2^{j(\alpha_1 - A_1(p,q))} \| F^{-1} \left( \tau_{l+}(\xi) + \tau_{l-}(\xi) \right) P_l(\widetilde{D} u(\xi)) \|_p \\
\lesssim 2^{j(\alpha_1 - A_1(p,q))} \| P_l(\widetilde{D} u(\xi)) \|_p
\]

In the last inequality we used that \( \tau_{l+}, \tau_{l-} \) are \( L^p \)-multipliers since their Fourier transforms are integrable. \( \square \)

In the forthcoming analysis we shall need the following auxiliary result. The proof mainly follows Stein’s analysis of oscillatory integrals on \( [32, \text{p.380-386}] \).

**Proposition 7.** Assume \( 0 \leq \alpha < \frac{1}{2} \) and that \( \chi, \psi \) are as in (19), \( \tau \in C_0^\infty(\mathbb{R}^d) \), set

\[
L_\alpha u := F^{-1} \left( (\xi_d - \psi(\xi))^\alpha \chi(\xi) \tau(\xi) u \right).
\]

Then \( L_\alpha : L^2(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d) \) is a bounded linear operator for \( q := \frac{2(k+2)}{k+2-4\alpha} \).

**Proof.** Define the family of distributions \( \gamma_s \) as in \( [32, \text{p.381}] \) (called \( \alpha_s \) in this book) via

\[
\gamma_s(y) = \frac{e^{s^2}}{\Gamma(s)} y^{s-1} \zeta(y) 1_{y > 0} \quad \text{if} \ \Re(s) > 0.
\]
where $\zeta$ is smooth with compact support and $\zeta(y) = 1$ for $|y| \leq y_0$ where $y_0$ is chosen so large that $\zeta(\xi_d - \psi(\xi')) = 1$ holds whenever $\chi(\xi')\tau(\xi) \neq 0$. The family $(\gamma_s)$ is extended to all $s \in \mathbb{C}$ via analytic continuation. Then introduce the family of linear operators

$$M_s f := \mathcal{F}^{-1} \left( \chi(\xi')^2 \gamma_s (\xi_d - \psi(\xi')) \hat{f} \right).$$

Plancherel’s Identity gives

$$\| M_s f \|_2 \lesssim \| f \|_2 \quad \text{if } \Re(s) = 1.$$

On the other hand

$$M_s f = \Phi * f, \quad \Phi(z) := \hat{\gamma}_s(-z_d) \cdot \int_{\mathbb{R}^{d-1}} \chi(\xi')^2 e^{iz \cdot (\xi', \psi(\xi'))} \, d\xi'$$

From eq. (15) in [32] and eq. (32) in [26] we infer

$$|\Phi(z)| \lesssim (1 + |z_d|)^{-\Re(s)} (1 + |z_d|)^{-\frac{k}{2}} \lesssim 1 \quad \text{if } \Re(s) = -\frac{k}{2}.$$

We conclude

$$\| M_s f \|_\infty \lesssim \| f \|_1 \quad \text{if } \Re(s) = -\frac{k}{2}.$$

Furthermore, for any given Schwartz functions $f, g$ the function $s \mapsto \int_{\mathbb{R}^d} (M_s f)g$ is holomorphic in the open strip $-\frac{k}{2} < \Re(s) < 1$ with continuous extension to the boundary. So the family $M_s$ is admissible for Stein’s Interpolation Theorem [31, Theorem 1] and we obtain

$$\| M_{1-2\alpha} f \|_q \lesssim \| f \|_q' \quad \text{if } \theta \in [0, 1], \quad 1 - 2\alpha = (1 - \theta) \cdot (\frac{k}{2}) + \theta \cdot 1, \quad \frac{1}{q} = \frac{1 - \theta}{\infty} + \frac{\theta}{2}.$$  

This leads to $\theta = \frac{2(2k+2-4\alpha)}{2(k+1)}$ and $q = \frac{2(k+2)}{k+2-4\alpha}$. In view of $0 < 2\alpha < 1$ this implies

$$\| \mathcal{F}^{-1} \left( \chi(\xi')^2 (\xi_d - \psi(\xi')) \tau^{2\alpha} \zeta(\xi_d - \psi(\xi')) \hat{f} \right) \|_q \lesssim \| f \|_q'.$$

Now we consider functions $\hat{f} = \tau^{2\alpha} \hat{g}$. By choice of $\zeta$ and of $y_0$ we then have

$$\| \mathcal{F}^{-1} \left( \chi(\xi')^2 (\xi_d - \psi(\xi')) \tau^{2\alpha} \tau(\xi') \hat{g} \right) \|_q \lesssim \| \mathcal{F}^{-1}(\tau^{2\alpha} \hat{g}) \|_q' \lesssim \| \hat{g} \|_q'.$$

This implies the claim given that this operator coincides with $L_\alpha L^*_\alpha$. □

We now use the dyadic estimates from Proposition 3 to prove Gagliardo-Nirenberg inequalities in the special case $P_1(D) = P_2(D)$ where the exponents satisfy $A_\alpha(p, q) = \alpha \in [0, 1]$. This result plays the same role in the critical frequency regime as Proposition 3 does in the non-critical regime. For $d \geq 2$ we concentrate on exponents with $1 \leq p \leq 2 \leq q \leq \infty$.

**Lemma 5.** Assume $d \in \mathbb{N}$ and let $P := P_1 = P_2$ satisfy (A1) for $\alpha := \alpha_1 = \alpha_2 \in [0, 1]$. Then

$$\| u_1 \|_q \lesssim \| P(D) u \|_p$$

holds for all $u \in \mathcal{S}(\mathbb{R}^d)$ provided that

(i) $d = 1$ and $1 \leq p, q \leq \infty$ satisfy $\frac{1}{p} - \frac{1}{q} = \alpha$ and, if $0 < \alpha < 1$, $(p, q) \notin \{(1, \frac{1}{1-\alpha}), (\frac{1}{\alpha}, \infty)\}$,

(ii) $d \geq 2$ and $1 \leq p \leq 2 \leq q \leq \infty$ satisfy $\frac{1}{p} - \frac{1}{q} = \frac{2\alpha}{k+2}$ and $\min\{\frac{1}{p}, \frac{1}{q}\} > \frac{k+2\alpha}{2(k+1)}$.

The estimate $\| u_1 \|_{q, \infty} \lesssim \| P(D) u \|_p$ holds for exponents as in (i),(ii) or

(iii) $d = 1, p = 1, q = \frac{1}{1-\alpha}$ if $\alpha \in (0, 1)$,
\[ (iv) \quad d \geq 2, 1 \leq p < \frac{2(k+1)}{k+2-2\alpha}, q = \frac{2(k+1)}{k+2-2\alpha} \quad \text{if} \quad \alpha \in (\frac{1}{2}, 1]. \]

**Proof.** With the same notations as before we have

\[
P(\xi)^{-1} \tau_l(\xi) = \left[ \tau_{l+}(\xi)(\tilde{\xi} - \psi_l(\tilde{\xi}))^{-\alpha} + \tau_{l-}(\xi)(\tilde{\xi} - \psi_l(\tilde{\xi}))^{-\alpha} \right] \chi_l(\tilde{\xi})
\]

with \( \tau_{l+}, \tau_{l-} \in C_0^\infty(\mathbb{R}^d), \chi_l \in C_0^\infty(\mathbb{R}^{d-1}), \tilde{\xi} := \Pi_l \xi. \)

for functions \( \chi_l, \psi_l \) that satisfy \([13]\). So

\[ u_1 = \sum_{j=j_0}^\infty T_j u. \]

Assuming \( 1 \leq p \leq 2 \leq q \leq \infty \) are chosen as above we obtain (ii),(iv) as follows:

- **Case** \( d \geq 2, \alpha = 0. \)
  Our assumptions give that \( A_\epsilon(p, q) = \alpha = 0 \) only occurs for \( p = q = 2 \). Here the estimate \( \|u_1\|_2 \lesssim \|P(D)u\|_2 \) follows from Plancherel’s Theorem.

- **Case** \( d \geq 2, \alpha \in (0,1). \)
  We first consider the case \( \alpha < \frac{1}{2} \). By assumption, \((\frac{1}{p}, \frac{1}{q})\) lies on the green diagonal line in Figure 2. By Proposition 7 the claimed inequality holds for the endpoints of that line given by \( p = 2, q = \frac{2(k+2)}{k+2-4\alpha} \) and its dual \( p = \frac{2(k+2)}{k+2+4\alpha}, q = 2 \). Interpolating these two estimates with each other provides the desired inequality for all tuples on the green line in Figure 2 and thus proves the claim for \( \alpha < \frac{1}{2} \).
  
  Now consider the case \( \alpha \geq \frac{1}{2} \). Our assumptions imply that \((\frac{1}{p}, \frac{1}{q})\) lies on the blue line in Figure 2 with endpoints excluded. In particular, \((\frac{1}{p}, \frac{1}{q})\) is in the interior of the \( A_1 \)-region, so \( A(\tilde{p}, \tilde{q}) = \frac{k+2}{2} (\frac{1}{p} - \frac{1}{q}) \) for all \((\tilde{p}, \tilde{q})\) close to \((p, q)\). For small \( \delta > 0 \) we choose \( \frac{1}{q_1} = \frac{1}{q} + \frac{\delta}{2}, \frac{1}{q_2} = \frac{1}{q} - \frac{\delta}{2} \). Interpolating the estimates for \((p, q_1)\) and \((p, q_2)\) with interpolation parameter \( \theta = \frac{1}{2} \) gives, due to \((1-\theta)A_\epsilon(p, q_1) + \theta A_\epsilon(p, q_2) = \alpha \), the weak estimate \( \|u\|_{q, \infty} \lesssim \|P(D)u\|_p \). Here we used \( u_1 = \sum_{j=j_0}^\infty T_j u \), the dyadic estimates from Proposition 8 and the Interpolation Lemma 11. These weak estimates hold for all \((\frac{1}{p}, \frac{1}{q})\) on the blue line with endpoints excluded. Interpolating these inequalities with each other gives \( \|u\|_q \lesssim \|P(D)u\|_p \) for the same set of exponents, which proves (ii) for \( \alpha \in (0,1) \).
  
  The prove the weak estimate from (iv) assume \( \alpha \in (\frac{1}{2}, 1) \). For any given \((\frac{1}{p}, \frac{1}{q})\) on the dashed horizontal blue line in Figure 2 with left endpoint excluded we can choose \( q_1, q_2 \) as above and the same argument gives \( \|u\|_{q, \infty} \lesssim \|P(D)u\|_p \). Since these exponents are given by \( 1 \leq p < \frac{2(k+1)}{k+2-2\alpha} \) and \( q = \frac{2(k+1)}{k+2+2\alpha} \), we are done.

- **Case** \( d \geq 2, \alpha = 1. \)
  It was shown in \([26], \text{Section 5}\) that the linear operators \( (P(D) + i\delta)^{-1} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d) \) are uniformly bounded with respect to small \( |\delta| > 0 \) given that our additional regularity assumptions on \( P \) from (A1) imply that \( S = \{ \xi \in \mathbb{R}^d : P(\xi) = 0 \} \) is a smooth compact manifold with \( |\nabla P| \neq 0 \) on \( S \). This implies \( \|u_1\|_q \lesssim \|P(D)u\|_p \) and analogous arguments yield the weak bounds claimed in (iv).
Figure 2. Riesz diagram showing the exponents $1 \leq p \leq 2 \leq q \leq \infty$ satisfying $A_\varepsilon(p, q) = \alpha$ in the case $\alpha = \alpha_1 \in (\frac{1}{2}, 1)$ (blue) and for $\alpha = \alpha_2 \in (0, \frac{1}{2})$ (green). For the green resp. non-dashed blue exponent pairs Lemma 5 (i),(ii) gives $\|u\|_q \leq \|P(\mathcal{D})u\|_p$. In the case $\alpha = \alpha_2$ the corresponding estimates from [26, Theorem 1.4 (ii)] only hold for exponents on the magenta line. The picture was produced with parameter values $(d, k, \alpha_1, \alpha_2) = (4, 2, \frac{3}{4}, \frac{1}{4})$.

Next we turn to the one-dimensional case $d = 1$. The representation formula then reads

$$u_1 = \sum_{l=1}^{L} \mathcal{F}^{-1} \left( [\tau_+ (\xi) (\xi - \xi^*_l)^{\frac{\alpha}{\tau_+}} + \tau_- (\xi) (\xi - \xi^*_l)^{\frac{\alpha}{\tau_-}}] \hat{P}(\mathcal{D})u \right)$$

where $\{P(\xi) = 0\} = \{\xi_1^*, \ldots, \xi_L^*\}$. Using our assumption $\frac{1}{p} - \frac{1}{q} = \alpha$ we obtain the claims (i),(iii) from the following arguments:

- Case $d = 1, \alpha = 0$.

We then have $p = q$ and we first analyze $1 < p = q < \infty$. In this case the Hilbert transform $f \mapsto \mathcal{F}^{-1}(\text{sign}(\xi) \hat{f})$ is bounded on $L^p(\mathbb{R})$, and so is $f \mapsto \mathcal{F}^{-1}(\text{sign}(\xi - \xi^*_l) \hat{f})$. 

(23) $u_1 = \sum_{l=1}^{L} \mathcal{F}^{-1} \left( [\tau_+ (\xi) (\xi - \xi^*_l)^{\frac{\alpha}{\tau_+}} + \tau_- (\xi) (\xi - \xi^*_l)^{\frac{\alpha}{\tau_-}}] \hat{P}(\mathcal{D})u \right)$
for \( l = 1, \ldots, L \). So the representation formula \((23)\) implies \( \|u_1\|_p \lesssim \|P(D)u\|_p \).

In the case \( p = q \in \{1, \infty\} \) we make use of our additional regularity assumption \( \tau_l := \tau_+ = \tau_- \) from \((A1)\), so

\[
\|u_1\|_p \leq \sum_{l=1}^L \|F^{-1}(\tau_l P(D)u)\|_p \lesssim \sum_{l=1}^L \|F^{-1}(\tau_l) * (P(D)u)\|_p \lesssim \|P(D)u\|_p.
\]

Here we used that \( F^{-1}(\tau_l) \) is a Schwartz function for \( l = 1, \ldots, L \).

- **Case** \( d = 1, \alpha \in (0, 1) \)
  
  If \( 1 < p < q < \infty \) we deduce the claimed estimate from the boundedness of the Hilbert transform on \( L^q(\mathbb{R}) \) and the Riesz potential estimate \( \|F^{-1}(|\cdot|^{-\alpha} \hat{f})\|_q \lesssim \|f\|_p \).

  For \( p = 1, 0 < \alpha < 1 \) we have a weak estimate \( \|F^{-1}(|\cdot|^{-\alpha} \hat{f})\|_{q, \infty} \lesssim \|f\|_1 \), see [15, Theorem 1.2.3]. Note that the Hilbert transform is bounded on \( L^{p, \infty}(\mathbb{R}) \) as well by real interpolation.

- **Case** \( d = 1, \alpha = 1 \).
  
  We now have \( \frac{1}{p} - \frac{1}{q} = 1 \), so \( p = 1, q = \infty \). We exploit the additional smoothness assumption \( \tau_+ = -\tau_- \) from \((A1)\). Then \( P \in C^\infty(\mathbb{R}) \) is a smooth function with simple zeros \( \xi^*_1, \ldots, \xi^*_L \). To prove the claimed inequality we start with the trivial estimate \( \|v\|_\infty \lesssim \|v\|_1 = \|F^{-1}(i\xi \hat{v})\|_1 \) for all \( v \in S(\mathbb{R}) \). Translation in Fourier space gives \( \|v\|_\infty \lesssim \|F^{-1}(i(\xi - \xi^*) \hat{v})\|_1 \) for all \( u \in S(\mathbb{R}), l = 1, \ldots, L \). So \((23)\) implies as above

\[
\|u_1\|_\infty \lesssim \sum_{l=1}^L \|F^{-1}((\xi - \xi^*)^{-1} \tau_l P(D)u)\|_\infty \\
\lesssim \sum_{l=1}^L \|F^{-1}(\tau_l P(D)u)\|_1 \\
\lesssim \|P(D)u\|_1.
\]

As remarked in Figure [2], claim (ii) of the previous lemma improves upon the corresponding bounds from [26, Theorem 1.4] in the case \( 0 < \alpha < \frac{1}{2} \). We finally combine all these estimates to prove Gagliardo-Nirenberg inequalities in the critical frequency regime. Given the rather complicated definition of \( A_\alpha(p, q) \), an explicit characterization of the admissible exponents is possible in principle, but extremely laborious. We prefer to avoid most of the computations. Instead, we describe the set of admissible exponents in an abstract way and provide the required computations in the reasonably simple special case \( 1 \leq p \leq 2 \leq q \leq \infty \) that allows to prove our main results. Proceeding in this way it becomes clear, how eventual improvements of Lemma 4 affect the final range of exponents. Once more we exploit Bourgain’s summation argument, which allows us to argue almost as in the large frequency regime. On a formal level, comparing Lemma 2 (large frequencies) with Lemma 4 (critical frequencies), we essentially have to replace \( s_j - d(\frac{1}{\tau_j} - \frac{1}{q}) \) by \( A_\alpha(\rho_j, q) - \alpha \), because the summation index now ranges from some \( j = j_0 \) to \( +\infty \) and not from \( j = j_0 \) to \( -\infty \). It will be convenient to formulate our sufficient conditions in terms of \( \overline{\alpha} := (1 - \kappa)\alpha_1 + \kappa\alpha_2 \).
We provide a definition of the set $\mathcal{A}(\kappa)$ of exponents $(q, r_1, r_2)$ that are admissible for
\begin{equation}
\|u_1\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^\kappa \quad (u \in \mathcal{S}(\mathbb{R}^d)).
\end{equation}
Lemma 5 provides the definition for $\kappa \in \{0, 1\}$, namely
\begin{equation}
\mathcal{A}(0) := \{(q, r_1, r_2) \in [1, \infty]^3 : (q, r_1, \alpha_1) \text{ as in Lemma 5 (i),(ii)}\},
\end{equation}
\begin{equation}
\mathcal{A}(1) := \{(q, r_1, r_2) \in [1, \infty]^3 : (q, r_2, \alpha_2) \text{ as in Lemma 5 (i),(ii)}\}.
\end{equation}
In the case $0 < \kappa < 1$ the definition is more involved and relies on the Interpolation Lemma 1 and the dyadic estimates for critical frequencies from Proposition 6. Combining the latter with (6) we obtain
\begin{equation}
\|u\|_q \lesssim \|u\|_{(x_1, x_2)_{n,q}} \text{ and deduce (24) for exponents } (q, r_1, r_2) \text{ belonging to the set}
\end{equation}
\begin{equation}
\mathcal{A}_1(\kappa) := \{(q, r_1, r_2) \in [1, \infty]^3 : \text{ There are } \varepsilon > 0, q_1 \in [r_1, \infty], q_2 \in [r_2, \infty], \text{ such that}
\end{equation}
\begin{equation}
\frac{1}{q} = \frac{1-\kappa}{q_1} + \frac{\kappa}{q_2} \text{ and } (1-\kappa)A_\varepsilon(r_1, q_1) + \kappa A_\varepsilon(r_2, q_2) > \frac{\varepsilon}{\tau} \}
\end{equation}
This result covers all non-endpoint cases in our considerations further below. Using (5) with $Y_1 = Y_2 = L^q(\mathbb{R}^d)$ we obtain $\|u_1\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^\kappa$ for exponents
\begin{equation}
\mathcal{A}_2(\kappa) := \{(q, r_1, r_2) \in [1, \infty]^3 : q \geq \max\{r_1, r_2\} \text{ and there is } \varepsilon > 0 \text{ such that}
\end{equation}
\begin{equation}
(1-\kappa)A_\varepsilon(r_1, q) + \kappa A_\varepsilon(r_2, q) = \tau, \ A_\varepsilon(r_1, q) \neq \alpha_i (i = 1, 2) \}
\end{equation}
Next we use $\|u\|_q = \|u\|_q^{1-\kappa}\|u\|_q^\kappa$ to deduce further estimates from Lemma 5 for exponents in
\begin{equation}
\mathcal{A}_3(\kappa) := \{(q, r_1, r_2) \in [1, \infty]^3 : (q, r_1, \alpha_1), (q, r_2, \alpha_2) \text{ as in Lemma 5 (i),(ii)}\}.
\end{equation}
Using (5) with $Y_1 = L^q(\mathbb{R}^d), Y_2 = L^{q_2}(\mathbb{R}^d)$ we get the weak bound $\|u_1\|_{q,\infty} \lesssim \|u\|_{(x_1, x_2)_{n,1}}$ for exponents belonging to
\begin{equation}
\mathcal{A}_3^\mu(\kappa) := \{(q, r_1, r_2) \in [1, \infty]^3 : \text{ There are } \varepsilon > 0, q_1 \in [r_1, \infty], q_2 \in [r_2, \infty] \text{ such that}
\end{equation}
\begin{equation}
(1-\kappa)A_\varepsilon(r_1, q_1) + \kappa A_\varepsilon(r_2, q_2) = \tau, \ \frac{1}{q} = \frac{1-\kappa}{q_1} + \frac{\kappa}{q_2}, \ \alpha_i \neq A_\varepsilon(r_i, q_i), q_1 \neq q_2 \}
\end{equation}
Interpolating the (weak or strong) endpoint estimates for $\mathcal{A}_2(\kappa) \cup \mathcal{A}_3(\kappa) \cup \mathcal{A}_3^\mu(\kappa)$ with each other exactly as in the final step of the proof of of Proposition 4 we deduce $\|u_1\|_q \lesssim \|u\|_{(x_1, x_2)_{n,q}} \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^\kappa$ for exponents from
\begin{equation}
\mathcal{A}_4(\kappa) := \{(q, r_1, r_2) \in [1, \infty]^3 : \text{ There are } \varepsilon \neq 0, \delta > 0, \bar{q}, \bar{q}^* \in [1, \infty], \bar{\kappa}, \bar{\kappa}^* \in (0, 1) \text{ with}
\end{equation}
\begin{equation}
\frac{1}{\bar{q}} - \varepsilon = \frac{1}{\bar{q}^*} = \frac{1}{q} = \frac{1-\kappa}{q_1} + \frac{\kappa}{q_2}, \ \bar{\kappa} - \delta = \kappa = \kappa^* + \delta
\end{equation}
\begin{equation}
(\bar{q}, r_1, r_2) \in \mathcal{A}_4^\mu(\bar{\kappa}) \cup \mathcal{A}_3(\bar{\kappa}) \cup \mathcal{A}_2(\bar{\kappa}), \ (q^*, r_1, r_2) \in \mathcal{A}_4^\mu(\kappa^*) \cup \mathcal{A}_3(\kappa^*) \cup \mathcal{A}_2(\kappa^*) \}
\end{equation}
Summarizing these interpolation results we obtain the following interpolation inequality in the critical frequency regime.

**Proposition 8.** Assume $d \in \mathbb{N}$, $\kappa \in [0,1]$ and (A1) for $\alpha_1, \alpha_2 > -1$. Then

$$
\|u\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa} \|P_2(D)u\|_{r_2}^\kappa \quad (u \in \mathcal{S}(\mathbb{R}^d))
$$

holds provided that $(q,r_1,r_2) \in \mathcal{A}(\kappa) := \mathcal{A}_1(\kappa) \cup \mathcal{A}_2(\kappa) \cup \mathcal{A}_3(\kappa) \cup \mathcal{A}_4(\kappa)$.

5. **Gagliardo-Nirenberg Inequalities, Proofs of Theorem 1 and Theorem 2**

We first discuss the one-dimensional case. As before, we use the notation

$$
\overline{\sigma} := (1 - \kappa)\alpha_1 + \kappa\alpha_2 \quad \text{and} \quad \overline{\sigma} := (1 - \kappa)s_1 + \kappa s_2.
$$

**Theorem 3.** Assume $d = 1, \kappa \in [0,1]$ and that (A1),(A2) hold for $s_1, s_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 > -1$ such that $0 < \overline{\sigma} \leq \overline{\sigma}$. Then

$$
\|u\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa} \|P_2(D)u\|_{r_2}^\kappa \quad (u \in \mathcal{S}(\mathbb{R}))
$$

holds provided that $q, r_1, r_2 \in [1, \infty]$ satisfy $\overline{\sigma} \leq \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} \leq \overline{\sigma}$ as well as the conditions (i),(ii),(iii) and (iv),(v),(vi) in the endpoint cases $\frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \overline{\sigma}$ and $\overline{\sigma} = \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q}$, respectively:

(i) if $q = \infty$ then $\frac{1}{r_1} - s_1 \neq 0 \neq \frac{1}{r_2} - s_2$ or $(r_1, r_2) = (\frac{1}{s_1}, \frac{1}{s_2})$, $s_1, s_2 \in \{0,1\}$,

(ii) if $1 < q < \infty$, $\frac{1}{r_1} - \frac{\alpha_1}{d} = \frac{1}{q} = \frac{1}{r_2} - \frac{\alpha_2}{d}$ and $r_1 = 1$ then

$1 < r_2 < q$, $1 - \kappa \geq \frac{\alpha_2}{d}$ or $r_2 = \infty, \frac{1}{q} \leq 1 - \kappa \leq \frac{1}{q}$,

(iii) if $1 < q < \infty$ and $\frac{1}{r_1} - \frac{\alpha_1}{d} = \frac{1}{q} = \frac{1}{r_2} - \frac{\alpha_2}{d}$ and $r_2 = 1$ then

$1 < r_1 < q$, $1 - \kappa \geq \frac{1 - \alpha_1}{d}$ or $r_1 = \infty, \frac{1}{q} \leq 1 - \kappa \leq \frac{1}{q}$,

(iv) if $q = \infty$ then $\frac{1}{r_1} - \alpha_1 \neq 0 \neq \frac{1}{r_2} - \alpha_2$ or $(r_1, r_2) = (\frac{1}{\alpha_1}, \frac{1}{\alpha_2})$, $\alpha_1, \alpha_2 \in \{0,1\}$,

(v) if $1 < q < \infty$, $\frac{1}{r_1} - \alpha_1 = \frac{1}{q} = \frac{1}{r_2} - \alpha_2$ then

$\alpha_1, \alpha_2 \in \{0,1\}$ and $r_1 = 1, \kappa < 1$ only if $1 < r_2 < q$, $\kappa \geq \frac{\alpha_2}{d}$,

(vi) if $1 < q < \infty$, $\frac{1}{r_1} - \alpha_1 = \frac{1}{q} = \frac{1}{r_2} - \alpha_2$ then

$\alpha_1, \alpha_2 \in \{0,1\}$ and $r_2 = 1, \kappa > 0$ only if $1 < r_1 < q$, $1 - \kappa \geq \frac{\alpha_1}{d}$.

**Proof.** Proposition 3 shows that the large frequency part of the inequality (involving $s_1, s_2$ and thus (i),(ii),(iii)) holds. In view of Proposition 3 it remains to show that all exponents satisfying $\overline{\sigma} \leq \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q}$ with (iv),(v),(vi) in the endpoint case $\overline{\sigma} = \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q}$ are covered by $\mathcal{A}(\kappa)$. In the case $\kappa = 0$ this holds by definition of $\mathcal{A}(0)$ from (25) because the requirement $(r_1, q) \notin \{1, \frac{1}{1 - \alpha}, \frac{1}{d}, \infty\}$ if $0 < \alpha < 1$ from Lemma 3(i) is met by (iv),(v),(vi). The discussion for $\kappa = 1$ is analogous. So from now on consider the case $0 < \kappa < 1$.

We now retrieve some information about $\mathcal{A}(\kappa)$ by exploiting the formula $A_\varepsilon(p,q) = \frac{1}{p} - \frac{1}{q}$ for $1 \leq p \leq q \leq \infty$, see (17). Going back to the definition of the sets $\mathcal{A}_i(\kappa)$ we find

$$
\mathcal{A}_1(\kappa) = \left\{ (q,r_1,r_2) \in [1,\infty]^3 : \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} > \overline{\sigma} \right\},
$$
\[ \mathcal{A}_2(\kappa) \supset \left\{ (q, r_1, r_2) \in [1, \infty]^3 : \frac{1 - \kappa}{r_1} + \frac{\kappa - \frac{1}{q}}{r_2} = \alpha_i, \ 0 \leq \frac{1}{r_i} - \frac{1}{q} \neq \alpha_i \text{ for } i = 1, 2 \right\} , \]

\[ \mathcal{A}_3(\kappa) \supset \left\{ (q, r_1, r_2) \in [1, \infty]^3 : \frac{1 - \kappa}{r_1} + \frac{\kappa - \frac{1}{q}}{r_2} = \alpha, \ \frac{1}{r_i} - \frac{1}{q} = \alpha_i \in [0, 1] \text{ and } (r_i, q) \not\in \left\{ \left( 1, \frac{1}{1 - \alpha_i} \right), \left( \frac{1}{\alpha_i}, \infty \right) \right\} \text{ if } \alpha_i \in (0, 1) \text{ for } i = 1, 2 \right\} . \]

Since the interpolation inequality holds for these exponents, our claim is proved in the following cases:

- \( \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} > \alpha \): see \( \mathcal{A}_1(\kappa) \).
- \( \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} = \alpha \) and \( q = 1 \): we necessarily have \( \alpha = 0, r_1 = r_2 = 1 \), which is covered by \( \mathcal{A}_2(\kappa) \) for \( \alpha_1, \alpha_2 \neq 0 \) or \( \mathcal{A}_3(\kappa) \) for \( \alpha_1 = \alpha_2 = 0 \), respectively.
- \( \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} = \alpha \) and \( q = \infty \): \( \frac{1}{r_1} - \alpha_1 \neq 0 \neq \frac{1}{r_2} - \alpha_2 \) is covered by \( \mathcal{A}_2(\kappa) \) and \( \frac{1}{r_1} - \alpha_1 = 0 = \frac{1}{r_2} - \alpha_2 \) with \( \alpha_1, \alpha_2 \in \{0, 1\} \) is covered by \( \mathcal{A}_3(\kappa) \).

So it remains to show the remaining endpoint estimates dealing with \( 1 < q < \infty \). By definition of \( \mathcal{A}_4(\kappa) \) we have restricted weak-type estimates for exponents from

\[ \mathcal{A}_4(\kappa) \supset \left\{ (q, r_1, r_2) \in [1, \infty]^3 : \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \alpha, \text{ and there are } q_1 \in [r_1, \infty), q_2 \in [r_2, \infty] \right\} . \]

such that \( q_1 \neq q_2, \frac{1}{r_i} - \frac{1}{q_i} \neq \alpha_i (i = 1, 2), \frac{1 - \kappa}{q_1} + \frac{\kappa}{q_2} = \frac{1}{q} \)

\[ \left\{ (q, r_1, r_2) \in [1, \infty]^3 : \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \alpha, \ 1 < q < \infty \right\} . \]

(Indeed, thanks to \( \alpha > 0 \) we may choose \( \frac{1}{q_1} := \frac{1}{r_1} - \varepsilon \) and \( \frac{\kappa}{q_2} := \frac{1}{q} - \frac{1 - \kappa}{q_1} \) for small \( \varepsilon > 0 \) provided that \( 1 \leq r_1 < \infty \), analogously for \( r_2 < \infty \).) This implies

\[ \mathcal{A}_4(\kappa) \supset \left\{ (q, r_1, r_2) \in [1, \infty]^3 : \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \alpha, \ 1 < q < \infty, \frac{1}{r_1} - \frac{1}{r_2} \neq \alpha_1 - \alpha_2 \right\} . \]

This yields the claim for the following exponents:

- \( \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} = \alpha, \ 1 < q < \infty \) and \( \frac{1}{r_1} - \frac{1}{r_2} \neq \alpha_1 - \alpha_2 \), which is covered by \( \mathcal{A}_4(\kappa) \),
- \( \frac{1 - \kappa}{r_1} + \frac{\kappa}{r_2} = \alpha, \ 1 < q < \infty \) and \( \frac{1}{r_1} - \frac{1}{q} = \alpha_i \in [0, 1] \) with \( (r_i, q) \neq (1, \frac{1}{1 - \alpha_i}) \) if \( \alpha_i \in (0, 1) \), which is covered by \( \mathcal{A}_3(\kappa) \).

So it remains to prove the claim for

\[ 1 < q < \infty, \frac{1}{r_1} - \alpha_1 = \frac{1}{q} = \frac{1}{r_2} - \alpha_2 \quad \text{ and } \quad \left[ r_1 = 1 < r_2 < q, \ 1 > \kappa \geq \frac{r_2}{q} \right. \text{ or } \left. r_2 = 1 < r_1 < q, \ 1 > \kappa \geq \frac{r_1}{q}. \right] \]

By symmetry we may concentrate on \( r_1 = 1 < r_2 < q, \ 1 > \kappa \geq \frac{r_2}{q} \) where the estimate follows from

\[ \| u \|_q \lesssim \| u \|_{q, \infty}^{1 - \kappa} \| u \|_{q, r_2}^{\kappa} \lesssim \| u \|_{q, \infty}^{1 - \kappa} \| u \|_{q, r_2}^{\kappa} \lesssim \| P_1(D)u \|_{1 - \kappa} \| P_2(D)u \|_{r_2}. \]
Here we used Proposition 3 (iv) and (ii) \( (r = r_2) \). This finishes the proof. \( \square \)

**Proof of Theorem 2:** We apply Theorem 3 to the symbols \( P_i(D) = |D|^s - 1, s > 0 \) and \( P_2(D) = I \) that satisfy the hypotheses of the Theorem for \((\alpha_1, \alpha_2, s_1, s_2) = (1, 0, s, 0)\). Then \( \overline{\sigma} = 1 - \kappa, \overline{\sigma} = (1 - \kappa)s \), so Theorem 3 implies that the Gagliardo-Nirenberg Inequality holds provided that \( 1 - \kappa \leq \frac{\kappa}{\alpha_1} + \frac{\kappa}{\alpha_2} - \frac{1}{q} \leq (1 - \kappa)s \). The latter restriction comes from Theorem 3 (i) and one checks that (ii)-(vi) are not restrictive for our choice of parameters \((\alpha_1, \alpha_2, s_1, s_2) = (1, 0, s, 0), s > 0\). \( \square \)

We continue with the higher-dimensional case where a computation of \( \mathcal{A}(\kappa) \cap \mathcal{B}(\kappa) \) is rather cumbersome. To simplify the discussion we concentrate on the special case \( r_1 = r_2 = r \in [1, 2] \) and \( q \in [2, \infty] \) and only consider the special ansatz \( q_1 = q_2 = q \) in the definition of the sets \( \mathcal{A}_i(\kappa) \).

**Theorem 4.** Assume \( d \in \mathbb{N}, d \geq 2, \kappa \in [0, 1] \) and that \((A1),(A2)\) hold for \( s_1, s_2 \in \mathbb{R} \) and \( \alpha_1, \alpha_2 > -1 \) such that \( 0 \leq \overline{\sigma} \leq 1 \). Then 
\[
\|u\|_q \lesssim \|P_1(D)u\|_r^{1 - \kappa} \|P_2(D)u\|_{r^*}^\kappa \quad (u \in \mathcal{S}(\mathbb{R}^d))
\]
holds provided that \( \overline{\sigma} < 1, \alpha_1 \neq \alpha_2, 0 < \kappa < 1 \) and the exponents \( r \in [1, 2], q \in [2, \infty] \) satisfy 
\[
\frac{2\pi}{k + 2} \leq \frac{1}{r} - \frac{1}{q} \leq \frac{\overline{\sigma}}{d} \quad \text{and} \quad \min \left\{ \frac{1}{r}, \frac{1}{q} \right\} \geq \frac{k + 2\pi}{2(k + 1)}
\]
as well as \((q, r) \neq (\infty, \frac{d}{2})\) if \( s_1 = s_2 = \overline{\sigma} \in (0, d] \). In the case \( \overline{\sigma} = 1 \) or \( \alpha_1 = \alpha_2 \) or \( \kappa \in \{0, 1\} \) the same is true provided that the last condition in (26) is replaced by \( \min \left\{ \frac{1}{r}, \frac{1}{q} \right\} > \frac{k + 2\pi}{2(k + 1)} \).

**Proof.** The conditions for large frequencies (involving \( s_1, s_2 \)) were shown to be sufficient in Proposition 4. So we concentrate on the critical frequency part involving \( \alpha_1, \alpha_2 \). The following computations are based on the formula \( A_\varepsilon(r, q) = A(r, q) - \varepsilon \cdot 1_{(p, q) \in \mathcal{E}} \) where 
\[
A(r, q) = \min \left\{ 1, \frac{k + 2}{2} \left( \frac{1}{r} - \frac{1}{q} \right), \frac{k + 2}{2} - \frac{k + 1}{q}, \frac{k + 1}{2} + \frac{k + 1 + 1}{r} \right\}
\]
for \( 1 \leq r \leq 2 \leq q \leq \infty \), see (17) and Figure 1. Our definitions of \( \mathcal{A}_1(\kappa), \mathcal{A}_2(\kappa), \mathcal{A}_3(\kappa) \) yield in the case \( 0 < \kappa < 1 \)
\[
\mathcal{A}_1(\kappa) \supset \{ (q, r, \varepsilon) \in [2, \infty] \times [1, 2]^2 : A_\varepsilon(r, q) > \overline{\sigma} \text{ for some } \varepsilon > 0 \},
\]
\[
\mathcal{A}_2(\kappa) \supset \{ (q, r, \varepsilon) \in [2, \infty] \times [1, 2]^2 : A_\varepsilon(r, q) = \overline{\sigma} \text{ for some } \varepsilon > 0, \alpha_1 \neq \overline{\sigma} \neq \alpha_2 \},
\]
\[
\mathcal{A}_3(\kappa) \supset \{ (q, r, \varepsilon) \in [2, \infty] \times [1, 2]^2 : A_\varepsilon(r, q) = \overline{\sigma} \text{ for some } \varepsilon > 0, \alpha_1 = \overline{\sigma} = \alpha_2 \in [0, 1] \}
\]
and \( \min \left\{ \frac{1}{r}, \frac{1}{q} \right\} > \frac{k + 2\pi}{2(k + 1)} \).

From \( \mathcal{A}(\kappa) \supset \mathcal{A}_1(\kappa) \cup \mathcal{A}_2(\kappa) \cup \mathcal{A}_3(\kappa) \) we thus get
\[
\mathcal{A}(\kappa) \supset \{ (q, r, \varepsilon) \in [2, \infty] \times [1, 2]^2 : A_\varepsilon(r, q) \geq \overline{\sigma} \text{ for some } \varepsilon > 0 \}
\]
if \( A_\varepsilon(r, q) = \overline{\sigma} = \alpha_1 = \alpha_2 \in [0, 1] \) then \( \min \left\{ \frac{1}{r}, \frac{1}{q} \right\} > \frac{k + 2\pi}{2(k + 1)} \).
Since $A_\varepsilon(r,q) \geq \kappa$ for some $\varepsilon > 0$ is equivalent to
\[
\frac{1}{r} - \frac{1}{q} \geq \frac{2\kappa}{k+2} \quad \text{and} \quad \min\left\{ \frac{1}{r}, \frac{1}{q} \right\} \geq \frac{\kappa}{k+1} \quad \text{if} \quad \kappa < 1 \quad \text{and} \quad \alpha_1 \neq \alpha_2 \quad \text{or} \quad \frac{\kappa}{k+1} \quad \text{if} \quad \kappa = 1 \quad \text{and} \quad \alpha_1 = \alpha_2.
\]
This proves the claim for $0 < \kappa < 1$. In the case $\kappa \in \{0,1\}$ the claim follows from (25) and Lemma 5(i),(ii).

**Proof of Theorem 1** We apply Theorem 4 to Lemma 5(i),(ii).

**Proof of Theorem 11** We apply Theorem 4 to $P_1(D) = |D|^{s} - 1$, $P_2(D) = I$. Again, the hypotheses of the Theorem hold for $(\alpha_1, \alpha_2, s_1, s_2, k) = (1, 0, s, 0, d-1)$ because $S$ is the unit sphere with $d-1$ non-vanishing principal curvatures.

6. Local Gagliardo-Nirenberg inequalities

In [14] it was shown that a “local” version of Gagliardo-Nirenberg inequalities is of interest, too. Here one looks for a larger set of exponents where (3) holds under the additional hypothesis $\|P_1(D)u\|_{r_1} \leq R\|P_2(D)u\|_{r_2}$ where $R > 0$ is fixed, see Corollary 2.10 in that paper. A simple consequence of our estimates above is the following.

**Corollary 1.** Assume $d \in \mathbb{N}$, $\kappa \in [0,1]$ and (A1),(A2) for $s_1, s_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 > -1$. Then the inequality
\[
\|u\|_{q} \lesssim (R^{\kappa - \kappa_1} + R^{\kappa - \kappa_2})\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa_1}
\]
holds for all $u \in \mathcal{S}(\mathbb{R}^d)$ and satisfying $\|P_1(D)u\|_{r_1} \leq R\|P_2(D)u\|_{r_2}$ provided that $(q, r_1, r_2) \in \mathcal{A}(\kappa_1) \cap \mathcal{B}(\kappa_2)$ holds for some $\kappa_1, \kappa_2 \in [0, \kappa]$.

*Proof.* Choose $\kappa_1, \kappa_2$ as required. Then Proposition 5 gives
\[
\|u_1\|_{q} \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa_1}
\]
\[
= (\|P_1(D)u\|_{r_1}\|P_2(D)u\|_{r_2})^{\kappa - \kappa_1}. \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa_1}
\]
\[
\lesssim R^{\kappa - \kappa_1}\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa_1}.
\]
Similarly, Proposition 4 implies
\[
\|u_2\|_{q} \lesssim R^{\kappa - \kappa_2}\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa_1}.
\]
Summing up these inequalities gives the claim.

In the context of our particular example $P_1(D) = |D|^{s} - 1$, $s > 0$ and $P_2(D) = I$ this gives the following generalization of [14, Corollary 2.10].

**Corollary 2.** Assume $d \in \mathbb{N}$, $d \geq 2$, $\kappa \in (0,1)$, $s > 0$. Then
\[
\|u\|_{q} \lesssim (R^{\kappa} + 1)(\|D|^{s} - 1)\|u\|_{r}^{1-\kappa}\|u\|_{r}^{\kappa}
\]
holds for all $u \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\|(|D|^{s} - 1)\|_{r} \leq R\|u\|_{r}$, provided that $(q, r) \neq (\infty, \frac{2}{s})$ if $0 < s \leq d$ and

(i) $d = 1$, $1 \leq r, q \leq \infty$ and $1 - \kappa \leq \frac{1}{r} - \frac{1}{q} \leq s$ or

(ii) $d \geq 2$, $1 \leq r \leq 2 \leq q \leq \infty$ and $\frac{2(1-\kappa)}{k+2} \leq \frac{1}{r} - \frac{1}{q} \leq \frac{s}{d}$, $\min\{\frac{1}{r'}, \frac{1}{q'}\} \geq \frac{k+2-2\kappa}{2(k+1)}$.
7. Gagliardo-Nirenberg inequalities with unbounded characteristic sets

In the previous sections we provided a systematic study of Gagliardo-Nirenberg Inequalities where the characteristic set $S$ of the symbols is smooth and compact. In the case of unbounded characteristic sets our analysis works for Schwartz functions whose Fourier transform is supported in some smooth and compact piece of $S$, but an argument for general Schwartz functions is lacking so far, even in the case of simple differentiable operators with suitable scaling behaviour like the wave operator or the Schrödinger operator. In the $L^2$-setting, a less technical approach based on Plancherel’s identity can be used. We follow the ideas presented in [14] to prove Gagliardo-Nirenberg inequalities of the form

\begin{align}
\|u\|_q &\lesssim \|\partial_t u - \Delta u\|_{L^r}^{1-\kappa}\|u\|_r^\kappa \quad (u \in S(\mathbb{R}^d)), \\
\|v\|_q &\lesssim \|i\partial_t v - \Delta v\|_{L^r}^{1-\kappa}\|v\|_r^\kappa \quad (v \in S(\mathbb{R}^d)).
\end{align}

where $r = 2$. We denote the space-time variable by $z = (x, t) \in \mathbb{R}^{d-1} \times \mathbb{R} = \mathbb{R}^d$.

**Theorem 5.** Let $d \in \mathbb{N}$. Then (27) holds provided that $r = 2$, $q = \frac{2d}{2d-4\kappa}$ where $\frac{1}{2} \leq \kappa \leq 1$ if $d \geq 3$ and $\frac{1}{2} < \kappa \leq 1$ if $d = 2$.

**Proof.** We first consider the case $d \geq 3$, define $C_t := \{\xi = (\xi', \xi_d) \in \mathbb{R}^d : \xi_d^2 - |\xi'|^2 = t\}$ and the induced surface measure $\sigma_t$. Then we have the representation formula

$$u(z) = c_d \int_{\mathbb{R}^d} \hat{u}(\xi) e^{i\xi \cdot z} d\xi = \frac{c_d}{2} \int_{\mathbb{R}} \int_{C_t} \hat{u}(\xi) |\xi|^{-1} e^{i\xi \cdot z} d\sigma_t(\xi) \, dt$$

where $c_d = (2\pi)^{-d/2}$. Strichartz’ inequality from [33] (Theorem I, case III (b)) implies that we have for $\frac{2(d+1)}{d-4\kappa} \leq q \leq \frac{2d}{d-2}$

$$\|u\|_q \lesssim \int_{\mathbb{R}} \left\| \mathcal{F}^{-1} \left( \hat{u} \cdot |^{-1} \right) d\sigma_t \right\|_q \, dt$$

$$\lesssim \int_{\mathbb{R}} |t|^{\frac{d-1}{2} - \frac{d}{q}} \|\hat{u} \cdot |^{-1}\|_{L^2(C_t, d\sigma_t)} \, dt$$

$$\lesssim \int_{\mathbb{R}} |t|^{\frac{2}{d} - \frac{d}{q}} \|\hat{u} \cdot |^{-1/2}\|_{L^2(C_t, d\sigma_t)} \, dt.$$

Here, the factor $|t|^{\frac{d-1}{2} - \frac{d}{q}}$ is obtained via scaling and in the last estimate we used $|\xi| \geq \sqrt{|t|}$ for $\xi \in C_t$. On the other hand, Plancherel’s Theorem gives

$$\|\partial_t u - \Delta u\|_2^2 = \int_{\mathbb{R}^d} |\xi_d^2 - |\xi'|^2| \hat{u}(\xi)|^2 d\xi$$

$$= \frac{1}{2} \int_{\mathbb{R}} \int_{C_t} |t|^2 |\hat{u}(\xi)|^2 |\xi|^{-1} d\sigma_t(\xi) \, dt$$
This leads to
\[
\varphi(t) := \|\hat{u} \cdot |\cdot|^{-1/2}\|_{L^2(C_t, d\sigma_t)}^2 \quad \text{for} \quad 2 \leq q \leq \frac{2(d+1)}{d-1}, \quad \frac{1}{q} = 1 - \frac{\theta}{d-1} \quad \text{with} \quad \theta = \frac{d-1}{d-3+4\kappa}.
\]

Writing \(\varphi(t) := \|\hat{u} \cdot |\cdot|^{-1/2}\|_{L^2(C_t, d\sigma_t)}^2\) it remains to prove that the quotient
\[
\frac{\int_{\mathbb{R}}|t|^{\frac{4q-2}{q}} |\varphi(t)| dt}{(\int_{\mathbb{R}}|t|^2 |\varphi(t)|^2 dt)^{1/2}}
\]
is bounded independently of \(\varphi\). According to [14, Lemma 2.1], with \(w(t) = |t|^{\frac{d-2}{d-1}}\), \(w_1(t) = 1\) and \(w_2(t) = t\), this is the case if and only if the following quantity is finite:
\[
\sup_{s > 0} \left\| \frac{w}{(w_1^2 + sw_2^2)^{1/2}} \right\|_{L^2(\mathbb{R})} = \sup_{s > 0} \left( \frac{\int_{\mathbb{R}}|t|^{\frac{d-2}{2}} |\cdot|^{\frac{d}{2}} dt}{1 + st^2} \right)^{1/2} \leq \frac{1}{\kappa} \left( \int_{\mathbb{R}}|\rho|^{\frac{d-2}{2}} d\rho \right)^{1/2}.
\]

This leads to \(q = \frac{2d}{d-4+4\kappa}\). In view of \(2(d+1) \leq q \leq \frac{2d}{d-2}\), this requires \(\frac{1}{2} \leq \kappa \leq \frac{d+2}{2(d+1)}\), but the upper bound for \(\kappa\) may be removed just as in [14, p.20-21] by combining the already established inequality for \(\frac{2(d+1)}{d-1}\) with
\[
\|u\|_q \leq \|u\|_{2^{\frac{d+1}{d-1}}}^{1-\theta} \|u\|_{\frac{2(d+1)}{d-1}}^{\theta} \quad \text{for} \quad 2 \leq q \leq \frac{2(d+1)}{d-1}, \quad \frac{1}{q} = 1 - \frac{\theta}{d-1} + \frac{\theta}{2(d+1)}.
\]

In the case \(d = 2\) the analogous reasoning based on Theorem I, case III (c) [33] shows that the above estimates are valid for \(6 = \frac{2(d+1)}{d-1} \leq q < \frac{2d}{d-2} = \infty\) and thus \(\frac{1}{2} < \kappa \leq \frac{d+2}{2(d+1)}\). The same interpolation trick then allows to extend this to the whole range \(\kappa > \frac{1}{2}\).

We now apply this method to the Schrödinger operator.

**Theorem 6.** Let \(d \in \mathbb{N}, d \geq 2\). Then (28) holds provided that \(r = 2, q = \frac{2(d+1)}{d-3+4\kappa}\) and \(\frac{1}{2} \leq \kappa \leq 1\).

**Proof.** Define \(\mathcal{P}_t := \{\xi = (\xi', \xi_d) \in \mathbb{R}^d : \xi_d - |\xi'|^2 = t\}\) and the induced surface measure \(d\sigma_t\). Plancherel’s identity gives
\[
\|u\|_2^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{v}(\xi', t + |\xi'|^2)|^2 d\xi' dt
\]
\[
= \int_{\mathbb{R}} |t|^{\frac{d-1}{2}} \int_{\mathbb{R}^{d-1}} |\hat{v}(\sqrt{t} \xi', t(1 + |\xi'|^2))|^2 d\xi' dt
\]
\[
= \int_{\mathbb{R}} |t|^{\frac{d-1}{2}} \int_{\mathbb{R}^{d-1}} |\hat{v}(\sqrt{1 + 4|\xi'|^2})|^2 d\xi' dt
\]
where \( \hat{t}_i(\xi) := \hat{\nu}(\sqrt{t}\xi', t\xi_d)(1 + 4|\xi'|^2)^{-1/4} \). Similarly,

\[
\|i\partial_t v - \Delta v\|^2 = \int_{\mathbb{R}} t^{2 + \frac{4}{d}} \|\hat{t}_i\|^2_{L^2(\mathcal{P}_1, d\sigma_1)} dt
\]

Strichartz’ inequality from \([33]\) (Theorem I, case I) implies for \( q = \frac{2(d+1)}{d-1} \)

\[
\|v\|_q = \left\| c_d \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \hat{\nu}(\xi', t + |\xi'|^2) e^{iz(\xi', t + |\xi'|^2)} d\xi' dt \right\|_q \\
\leq \int_{\mathbb{R}} \left\| \int_{\mathbb{R}^{d-1}} \hat{\nu}(\xi', t + |\xi'|^2) e^{iz(\sqrt{\xi', t}(1 + |\xi'|^2))} d\xi' \right\|_q dt \\
\leq \int_{\mathbb{R}} |t| \left\| \mathcal{F}^{-1} \left( \hat{\nu}(\sqrt{\xi', t}\xi_d)(1 + 4|\xi'|^2)^{-1/2} d\sigma_1 \right) \right\|_q dt \\
= \int_{\mathbb{R}} |t| \left\| \mathcal{F}^{-1} \left( \hat{\nu}(\sqrt{\xi', t}\xi_d)(1 + 4|\xi'|^2)^{-1/2} d\sigma_1 \right) \right\|_q dt \\
\leq \int_{\mathbb{R}} |t| \left\| \mathcal{F}^{-1} \left( \hat{\nu}(\sqrt{\xi', t}\xi_d)(1 + 4|\xi'|^2)^{-1/2} d\sigma_1 \right) \right\|_{L^2(\mathcal{P}_1, d\sigma_1)} dt \\
\leq \int_{\mathbb{R}} |t| \left\| \mathcal{F}^{-1} \left( \hat{\nu}(\sqrt{\xi', t}\xi_d)(1 + 4|\xi'|^2)^{-1/2} d\sigma_1 \right) \right\|_{L^2(\mathcal{P}_1, d\sigma_1)} dt.
\]

We set \( \varphi(t) := |t|^{\frac{d+1}{d-1}} \|\hat{t}_i\|_{L^2(\mathcal{P}_1, d\sigma_1)} \) and it remains to show that the quotient

\[
\frac{\int_{\mathbb{R}} |t|^{\frac{d+1}{d-1}} \varphi(t) dt}{(\int_{\mathbb{R}} t^2 \varphi(t)^2 dt)^{\frac{1}{2}}} (\int_{\mathbb{R}} \varphi(t)^2 dt)^{\frac{1}{2}}
\]

is bounded independently of \( \varphi \). We apply \([14], \text{Lemma 2.1}\) once more.

\[
\sup_{s>0} s^{\frac{1-d}{2}} \left( \int_{\mathbb{R}} |t|^{\frac{d+1}{d-1}} \frac{\varphi(t)}{1 + st^2} dt \right)^{\frac{1}{2}} = \sup_{s>0} s^{\frac{1-d}{2}} \left( \left( \frac{1}{\sqrt{s}} \right)^{\frac{d+1}{2}} \int_{\mathbb{R}} |\rho|^{\frac{d+1}{2}} - \frac{d+1}{2} \frac{d+1}{2} d\rho \right)^{\frac{1}{2}} \\
= \sup_{s>0} s^{\frac{1-d}{2}} \left( \frac{d+1}{2} \frac{d+1}{2} + \frac{d+1}{4} \right) \left( \int_{\mathbb{R}} |\rho|^{\frac{d+1}{2}} - \frac{d+1}{2} \frac{d+1}{2} d\rho \right)^{\frac{1}{2}}
\]

This term is indeed finite for \( q = \frac{2(d+1)}{d-1} \) and \( \kappa = \frac{1}{2} \), which proves the claim in this special case. The claim for general \( \kappa \geq \frac{1}{2} \) follows as above by interpolation. \( \square \)
We conjecture that at least for $1 < r \leq 2 \leq q < \infty$ and $0 < \kappa < 1$ the inequality (27) actually holds for exponents

$$\frac{1}{r} - \frac{1}{q} = \frac{2(1-\kappa)}{d}, \quad \min \left\{ \frac{1}{r}, \frac{1}{q'} \right\} \geq \frac{d-2\kappa}{2(d-1)}$$

whereas the corresponding inequality involving the Schrödinger operator holds whenever

$$\frac{1}{r} - \frac{1}{q} = \frac{2(1-\kappa)}{d+1}, \quad \min \left\{ \frac{1}{r}, \frac{1}{q'} \right\} \geq \frac{d+1-2\kappa}{2d}.$$

Note that the Sobolev inequalities [20, Theorem 1.1] then take the form of the endpoint estimate $\kappa = 0$ in (29).

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