Casimir Effect In Krein Space Quantization

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Abstract

An explicit calculation of Casimir effect through an alternative approach of field quantization [1, 2], has been presented in this paper. In this method, the auxiliary negative norm states have been utilized, the modes of which do not interact with the physical states or real physical world. Naturally these modes cannot be affected by the physical boundary conditions. Presence of negative norm states play the rule of an automatic renormalization device for the theory.

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1 Introduction

Due to appearance of infrared divergence in the two point function for the minimally coupled scalar field in de Sitter space, a new method of field quantization has been presented [1, 3]. The minimally coupled scalar field plays an important role in the inflationary model as well as in the linear quantum gravity [4, 5]. Infrared divergence however appears in the linear gravity and the minimally coupled scalar field in de Sitter space with a great deal of similarity [4]. In the case of the linear gravity, this divergence does not manifest itself in the quadratic part of the effective action in the one-loop approximation. This means that the pathological behavior of the graviton propagator is gauge dependent and so should not appear in an effective way as a physical quantity [6].

A covariant quantization of minimally coupled scalar field cannot be constructed by positive norm states alone [7]. To cope with this problem the following method of quantization has been utilized to achieve a naturally renormalized theory. It has been proven that the use of the two sets of solutions (positive and negative norms states) are an unavoidable feature for preservation of (1) causality (locality), (2) covariance, and (3) elimination of the infrared divergence for the

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minimally coupled scalar field in de Sitter space [1]. Preserving the covariance principle and ignoring the positivity condition, similar to Gupta-Bleuler quantization of the electrodynam ic equations in Minkowski space, we have performed the field quantization in the Krein space, in combined Hilbert and anti-Hilbert space [8]. Most interesting result of this construction is the convergence of the Green’s function at large distances, which means that the infrared divergence is gauge dependent [1, 9]. The ultraviolet divergence in the stress tensor disappears as well, in other words the quantum free scalar field in this method is automatically renormalized. The effect of “unphysical” states (negative norm states) appears in the above theory as a natural renormalization element.

It is important to note that by the use of this method, a natural renormalization of the following problems, have been already attained:

- the massive free field in de Sitter space [1],
- the graviton two point function in de Sitter space [10],
- the one-loop effective action for scalar field in a general curved space-time [11],
- tree level scattering amplitude of scalar field with one graviton exchange in de Sitter space [12].

Following above achievements and through the same new approach, the Casimir force between two parallel plates in Minkowski space has been calculated in this paper. In section 2 we briefly recall the Casimir force in the Hilbert space quantization. Section 3 is devoted to the calculation of the Casimir force in the Krein space quantization in 2 and 4 dimensional space time. Brief conclusion and outlook are given in final section.

## 2 Casimir Force

The Casimir effect is a small attractive force, which acts between two parallel uncharged conducting plates. It is due to quantum vacuum fluctuation of the field operator between two parallel plates. For simplicity, we consider the two dimensional space time and one component field in this section. We start with the quantization of the scalar field $\phi(t, x)$. In two dimensions, the scalar field equation, *i.e.* the Klein-Gordon equation, is ($c = \hbar = 1$):

$$\frac{\partial^2 \phi(t, x)}{\partial t^2} - \frac{\partial^2 \phi(t, x)}{\partial x^2} + m^2 \phi(t, x) = 0.$$  \hfill (1)

Inner products, which defined the norms, is defined by [13]

$$\langle \phi_1, \phi_2 \rangle = -i \int_{t=\text{cons.}} \phi_1(t, x) \frac{\partial}{\partial t} \phi_2^*(t, x) dx.$$ \hfill (2)

Two sets of solutions of (1) are given by:

$$u_p(k, x, t) = \frac{e^{ikx-iwt}}{\sqrt{(2\pi)^2w}} = \frac{e^{-ikx}}{\sqrt{(2\pi)^2w}}, \quad u_n(k, x, t) = \frac{e^{-ikx+iwt}}{\sqrt{(2\pi)^2w}} = \frac{e^{ikx}}{\sqrt{(2\pi)^2w}},$$ \hfill (3)
where \( w(k) = k^0 = (k^2 + m^2)^{1/2} \geq 0 \). These \( u(k, x, t) \) modes are orthonormalized by the following relations:

\[
(u_p(k, x, t), u_p(k', x, t)) = \delta(k - k'),

(u_n(k, x, t), u_n(k', x, t)) = -\delta(k - k'),

(u_p(k, x, t), u_n(k', x, t)) = 0.
\]

(4)

\( u_p \) modes are positive norm states and the \( u_n \)'s are negative norm states. By imposing the physical boundary condition on the positive norm states,

\[
u_p(k, 0, t) = u_p(k, a, t) = 0,
\]

(5)

we obtain

\[
u_p(k_N, x, t) = \left(\frac{1}{a \omega_N}\right)^{1/2} e^{-i \omega_N t} \sin k_N x,
\]

(6)

where:

\[
\omega_N = (m^2 + k_N^2)^{1/2}, \quad k_N = \frac{N \pi}{a}, \quad N = 1, 2, 3, \ldots
\]

(7)

This is a typical case where the Casimir effect arises. In this case, the scalar product is:

\[
(u_p(k_N, x, t), u_p(k_N', x, t)) = \delta_{N,N'}.
\]

(8)

The standard quantization of the field is performed by means of the expansion

\[
\phi(t, x) = \sum_N [a_N u_p(k_N, x, t) + a_N^\dagger u_p^*(k_N, x, t)],
\]

(9)

where \( a_N, a_N^\dagger \) are the annihilation and creation operators obeying the commutation relations

\[
[a_N, a_N^\dagger] = \delta_{N,N'}, [a_N, a_{N'}] = [a_N^\dagger, a_{N'}^\dagger] = 0.
\]

(10)

The vacuum state in the presence of boundary condition is defined by

\[
a_N |0\rangle = 0.
\]

(11)

The vacuum energy of this state is calculated at this stage. The operator of the energy density is given by the 00-component of the energy-momentum tensor of the scalar field in the two-dimensional space-time

\[
T_{00} = \frac{1}{2} \left\{ [\partial_t \phi(t, x)]^2 + [\partial_x \phi(t, x)]^2 \right\}.
\]

(12)

Substituting Eq. (9) into Eq. (12) accounting (10) and (11) one easily obtains

\[
\langle 0 | T_{00} | 0 \rangle = \frac{1}{2a} \sum_{N=1}^{\infty} \omega_N - \frac{m^2}{2a} \sum_{N=1}^{\infty} \frac{\cos 2k_N x}{\omega_N}.
\]

(13)

The total vacuum energy of the interval \((0, a)\) is obtained by the integration of (13) as follow:

\[
E_0(a) = \int_0^a \langle 0 | T_{00} | 0 \rangle dx = \frac{1}{2} \sum_{N=1}^{\infty} \omega_N.
\]

(14)
The second, oscillating term in the right-hand side of (13) does not contribute to this result.

The expression (14) for the vacuum state energy of the quantized field (between above boundaries) is generally the standard starting point in the theory of the Casimir effect. Evidently the quantity \( E_0(a) \) is infinite. There are many regularization procedures. Here we use one of the simplest methods, i.e., we introduce an exponentially damping function \( \exp(-\delta \omega_N) \).

In the limit \( \delta \to 0 \) the regularization is removed. For simplicity let us consider the regularized vacuum energy of the above interval for a massless field \((m = 0)\). In this case

\[
E_0(a, \delta) = \frac{1}{2} \sum_{N=1}^{\infty} \frac{\pi N}{a} \exp(-\frac{\delta \pi N}{a}) = \frac{\pi}{8a} \sinh^2 \frac{\delta \pi}{2a}. \tag{15}
\]

In the limit of small \( \delta \) one obtains;

\[
E_0(a, \delta) = \frac{a}{2\pi \delta^2} + E(a) + O(\delta^2), E(a) = -\frac{\pi}{24a}, \tag{16}
\]

i.e., the vacuum energy is represented as a sum of a singular term and a finite contribution.

Let us compare this result with the corresponding result for the unbounded axis. Here instead of (6) we have the positive frequency solutions in the form of travelling waves (3). The sum in the field operator (9) is interpreted now as an integral with the measure \( \frac{dk}{2\pi} \), and the commutation relations contain delta functions \( \delta(k - k') \), instead of the Kronecker symbols. Let us call the vacuum state defined by:

\[
a_k |0_M\rangle = 0, \tag{17}
\]

the Minkowski vacuum to underline the fact that it is defined in free space without any boundary conditions. Repeating exactly the same simple calculation which was performed for the above interval, we obtain the divergent expression for the vacuum energy density in Minkowski vacuum;

\[
\langle 0_M|T_{00}|0_M\rangle = \frac{1}{2\pi} \int_0^\infty \omega dk, \tag{18}
\]

and for the total vacuum energy on the axis

\[
E_{0M}(-\infty, +\infty) = \frac{1}{2\pi} \int_0^\infty \omega dk L, \tag{19}
\]

where \( L \to \infty \) is the normalization length. Let us separate the interval \((0, a)\) of the whole axis whose energy should be compared with (14):

\[
E_{0M}(a) = \frac{E_{0M}(-\infty, +\infty)}{L} = \frac{a}{2\pi} \int_0^\infty \omega dk. \tag{20}
\]

To calculate (20) we use the same regularization method as the above case, i.e., we introduce the exponentially damping function under the integral;

\[
E_{0M}(a) = \frac{a}{2\pi} \int_0^\infty k e^{-\delta k} dk = \frac{a}{2\pi \delta^2}. \tag{21}
\]

The obtained result coincides with the first term in the right-hand side of (16). Consequently, the renormalized vacuum energy of the interval \((0, a)\) in the presence of boundary conditions can be defined as[14]:

\[
E_0^{\text{Ren}}(a) = \lim_{\delta \to 0} [E_0(a, \delta) - E_{0M}(a, \delta)] = -\frac{\pi}{24a}. \tag{22}
\]
This leads us to calculation of following attractive force between the above boundaries:

$$F(a) = -\frac{\pi}{24a^2}$$

(23)

In this case, the renormalization corresponds to removing the vacuum energy of the unbounded space from the total energies of corresponding bounded case. The renormalized energy $E(a)$ monotonically decreases as the boundary points approach each other. This points to the presence of an attractive force between the conducting planes.

### 3 Casimir force in Krein space quantization

In this section we calculate zero point energy and casimir force in Krein space quantization thoroughly. In the previous paper [2], we present the free field operator in the Krein space quantization

$$\phi(t, x) = \phi_p(t, x) + \phi_n(t, x),$$

(24)

where

$$\phi_p(t, x) = \int dk[a(k)u_p(k, x, t) + a^\dagger(k)u_p^*(k, x, t)],$$

$$\phi_n(t, x) = \int dk[b(k)u_n(k, x, t) + b^\dagger(k)u_n^*(k, x, t)],$$

and $a(k)$ and $b(k)$ are two independent operators. Creation and annihilation operators are constrained to obey the following commutation rules

$$[a(k), a(k')] = 0, \quad [a^\dagger(k), a^\dagger(k')] = 0, \quad [a(k), a^\dagger(k')] = \delta(k - k'),$$

(25)

$$[b(k), b(k')] = 0, \quad [b^\dagger(k), b^\dagger(k')] = 0, \quad [b(k), b^\dagger(k')] = -\delta(k - k'),$$

(26)

$$[a(k), b(k')] = 0, \quad [a^\dagger(k), b^\dagger(k')] = 0, \quad [a^\dagger(k), b(k')] = 0.$$

(27)

The vacuum state $|\Omega>$ is then defined by

$$a^\dagger(k) |\Omega >= |1_k >; \quad a(k) |\Omega >= 0,$$

(28)

$$b^\dagger(k) |\Omega >= |\bar{1}_k >; \quad b(k) |\Omega >= 0,$$

(29)

$$b(k) |1_k >= 0; \quad a(k) |\bar{1}_k >= 0,$$

(30)

where $|1_k>$ is called a one particle state and $|\bar{1}_k>$ is called a one “unparticle state”.

By imposing the physical boundary condition on the field operator, only the positive norm states are affected. The negative modes do not interact with the physical states or real physical world, thus they can not be affected by the physical boundary conditions as well. In this case, the field operator is defined as:

$$\phi(t, x) = \sum_N [a_Nu_p(k_N, x, t) + a_N^\dagger u_p^*(k_N, x, t)] + \int dk[b(k)u_n(k, x, t) + b^\dagger(k)u_n^*(k, x, t)].$$

(31)

Substituting the above field operator in (12) and using Eqs (28) and (29), one easily obtains:

$$\langle\Omega | T_{00}^{Kre} |\Omega \rangle = \frac{1}{2a} \sum_{N=1}^\infty \omega_N - \frac{m^2}{2a} \sum_{N=1}^\infty \frac{\cos 2k_N x}{\omega_N} - \frac{1}{2\pi} \int_0^\infty \omega dk.$$  

(32)
The total vacuum energy for the interval \((0, a)\) is obtained by the integration of (32) as follow:

\[
E_{0}^{Kre}(a) = \int_{0}^{a} \langle \Omega | T_{00}^{Kre} | \Omega \rangle dx = -\frac{\pi}{24a},
\]  

(33)

which is exactly the previous result. Then the attractive Casimir force between the conducting planes in the Krein space quantization is then

\[
F(a) = -\frac{\partial(E_{0}^{Kre}(a))}{\partial a} = -\frac{\pi}{24a^{2}}.
\]  

(34)

The Casimir force for the Electromagnetic field in 4-dimensional space time, \(i.e.\) a real physical case, in the Krein space quantization has been calculated as well. It resulted in the following Casimir force [15]

\[
F(a) = -\frac{\partial(E_{0}^{Kre}(a))}{\partial a} = -\frac{\pi^{2}}{240a^{4}}.
\]  

(35)

Due to its similarity with the above 2-dimensional case it was not necessary to present the explicit calculation. It is important to note that once again the natural renormalization of the theory has been established.

## 4 Conclusion

The negative frequency solutions of the field equation are needed for the covariant quantization in the minimally coupled scalar field in de Sitter space. Contrary to the Minkowski space, the elimination of de Sitter negative norms in this case breaks the de Sitter invariance. In other words for restoring of the de Sitter invariance, one needs to take into account the negative norm states \(i.e.\) the Krein space quantization. This provides a natural tool for renormalization of the theory [1]. In the present paper, Casimir force in Minkowski space-time, has been calculated through the Krein space quantization. Once again it is found that the theory is automatically renormalized.

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