On the Brezis–Nirenberg problem for the \((p, q)\)-Laplacian

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Abstract
We prove some existence and nonexistence results for a class of critical \((p, q)\)-Laplacian problems in a bounded domain. Our results extend and complement those in the literature for model cases.

Keywords \((p, q)\)-Laplacian · Critical Sobolev exponent · Existence · Nonexistence

Mathematics Subject Classification Primary 35J92 · Secondary 35B33

1 Introduction
Consider the critical \((p, q)\)-Laplacian problem

\[
\begin{cases}
-\Delta_p u - \Delta_q u = b |u|^{s-2} u + |u|^{p^*-2} u \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), \(N \geq 2\), \(1 < q < p < N\), \(p^* = Np/(N - p)\) is the critical Sobolev exponent, \(1 < s < p^*\), and \(b > 0\). It was shown in Li and Zhang [1] that this problem has infinitely many solutions when \(1 < s < q\) and \(b > 0\) is sufficiently small. On the other hand, it was shown in Yin and Yang [2] that it has a nontrivial solution when \(p < s < p^*\) and \(b > 0\) is sufficiently large. Sufficient conditions for the existence of a nontrivial solution when \(s = p\) and \(b > 0\) is either small or large were given in Candito et
al. [3]. A rescaling of a result in Ho and Sim [4] shows that the related problem

\[
\begin{aligned}
\begin{cases}
-\Delta_p u - v \Delta_q u &= b |u|^{s-2} u + |u|^{p^*-2} u \quad \text{in } \Omega \\
u &= 0
\end{cases}
\end{aligned}
\tag{1.2}
\]

has a nontrivial solution when \( q < s < p \) and \( v, b > 0 \) are sufficiently small. The borderline case \( s = q \) does not seem to have been studied in the literature.

In the present paper we prove some existence results for a more general class of critical \((p, q)\)-Laplacian problems that, in particular, give a nontrivial solution of problem (1.1) for all \( b > 0 \) and a nontrivial solution of problem (1.2) for sufficiently small \( v > 0 \) and all \( b > 0 \). More specifically, our main results for the model problems (1.1) and (1.2) are the following:

**Theorem 1.1** Problem (1.1) has a nontrivial weak solution for all \( b > 0 \) in each of the following cases:

(i) \( 1 < q < N(p - 1)/(N - 1) \) and \( N^2(p - 1)/(N - 1)(N - p) < s < p^* \),

(ii) \( N(p - 1)/(N - 1) \leq q < p \) and \( Nq/(N - p) < s < p^* \).

In particular, problem (1.1) has a nontrivial weak solution for all \( b > 0 \) when \( N^2 - p(p + 1)N + p^2 \geq 0 \), \( q \leq (N - p)p/N \), and \( p < s < p^* \), and when \( N^2 - p(p + 1)N + p^2 > 0 \), \( q < (N - p)p/N \), and \( s = p \).

**Theorem 1.2** There exists \( v_0 > 0 \) such that problem (1.2) has a nontrivial weak solution for all \( v \in (0, v_0) \) and \( b > 0 \) in each of the following cases:

(i) \( N \geq p^2 \) and \( q < s < p^* \),

(ii) \( N < p^2 \) and either \( q < s < p \) or \( (Np - 2N + p)p/(N - p)(p - 1) < s < p^* \).

In particular, problem (1.2) has a nontrivial weak solution for all \( v \in (0, v_0) \) and \( b > 0 \) when \( q < s < p \), and when \( N \geq p^2 \) and \( s = p \).

In the borderline case \( s = q \) we show that problem (1.1) has no nontrivial weak solution for all sufficiently small \( b > 0 \) when \( \Omega \) is a star-shaped domain with \( C^1 \)-boundary (see Theorem 2.6). The proof of this nonexistence result will be based on a new Pohožaev type identity for the \((p, q)\)-Laplacian (see Theorem 2.8), which is of independent interest.

We refer the reader to Marano and Mosconi [5] for a survey of recent existence and multiplicity results for subcritical and critical \((p, q)\)-Laplacian problems in bounded domains.

## 2 Statement of results

We consider the critical \((p, q)\)-Laplacian problem

\[
\begin{aligned}
\begin{cases}
-\Delta_p u - \Delta_q u &= f(x, u) + |u|^{p^*-2} u \quad \text{in } \Omega \\
u &= 0
\end{cases}
\end{aligned}
\tag{2.1}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( N \geq 2 \), \( 1 < q < p < N \), \( p^* = Np/(N - p) \) is the critical Sobolev exponent, and \( f \) is a Carathéodory function on \( \Omega \times \mathbb{R} \) satisfying

\[
f(x, 0) = 0 \quad \text{for a.a. } x \in \Omega
\tag{2.2}
\]

and the subcritical growth condition

\[
|f(x, t)| \leq a_1 |t|^{r-1} + a_2 \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}
\tag{2.3}
\]
for some constants $a_1, a_2 > 0$ and $r \in (p, p^*)$. A weak solution of this problem is a function $u \in W_{0}^{1, p}(\Omega)$ satisfying

$$
\int_{\Omega} \left( |\nabla u|^{p - 2} \nabla u \cdot \nabla v + |\nabla u|^{q - 2} \nabla u \cdot \nabla v - f(x, u) v - |u|^{p - 2} u v \right) dx = 0
$$

\forall v \in W_{0}^{1, p}(\Omega),

where $W_{0}^{1, p}(\Omega)$ is the usual Sobolev space with the norm $\|u\| = \left( \int_{\Omega} |\nabla u|^{p} dx \right)^{1/p}$. Weak solutions coincide with critical points of the $C^1$-functional

$$
E(u) = \int_{\Omega} \left( \frac{1}{p} |\nabla u|^{p} + \frac{1}{q} |\nabla u|^{q} - F(x, u) - \frac{1}{p^*} |u|^{p^*} \right) dx, \quad u \in W_{0}^{1, p}(\Omega),
$$

where $F(x, t) = \int_{0}^{t} f(x, \tau) d\tau$ is the primitive of $f$. Recall that a sequence $(u_j) \subset W_{0}^{1, p}(\Omega)$ such that $E(u_j) \to c$ and $E'(u_j) \to 0$ is called a $(PS)_c$ sequence. Let

$$
c^* = \frac{1}{N} S^{N/p}, \quad (2.4)
$$

where

$$
S = \inf_{u \in W_{0}^{1, p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^{p} dx}{\left( \int_{\Omega} |u|^{p^*} dx \right)^{p/p^*}}, \quad (2.5)
$$

is the best Sobolev constant. If $0 < c < c^*$, then every $(PS)_c$ sequence has a subsequence that converges weakly to a nontrivial critical point of $E$ (see Proposition 3.1).

Let

$$
\lambda_1 = \inf_{u \in W_{0}^{1, p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^{p} dx}{\int_{\Omega} |u|^{p} dx}, \quad \mu_1 = \inf_{u \in W_{0}^{1, q}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^{q} dx}{\int_{\Omega} |u|^{q} dx}, \quad (2.6)
$$

be the first Dirichlet eigenvalues of the $p$-Laplacian and the $q$-Laplacian, respectively. Assume that

$$
F(x, t) \leq \frac{\lambda}{p} |t|^p + \frac{\mu_1}{q} |t|^q \quad \text{for a.a.} \ x \in \Omega \text{ and } |t| < \delta \quad (2.7)
$$

for some $\lambda \in (0, \lambda_1)$ and $\delta > 0$. It follows from this and (2.3) that

$$
F(x, t) \leq \frac{\lambda}{p} |t|^p + \frac{\mu_1}{q} |t|^q + \alpha_3 |t|^r \quad \text{for a.a.} \ x \in \Omega \text{ and all } t \in \mathbb{R}
$$

for some constant $\alpha_3 > 0$, so

$$
E(u) \geq \int_{\Omega} \left[ \frac{1}{p} \left( 1 - \frac{\lambda}{\lambda_1} \right) |\nabla u|^{p} - \alpha_3 |u|^r - \frac{1}{p^*} |u|^{p^*} \right] dx.
$$

Since $p < r < p^*$, it follows that the origin is a strict local minimizer of $E$. On the other hand, it also follows from (2.3) that $E(tu) \to -\infty$ as $t \to +\infty$ for any $u \in W_{0}^{1, p}(\Omega) \setminus \{0\}$. So $E$ has the mountain pass geometry. Let

$$
\Gamma = \left\{ \gamma \in C([0, 1], W_{0}^{1, p}(\Omega)) : \gamma(0) = 0, \ E(\gamma(1)) < 0 \right\}.
$$
be the class of paths in $W_0^{1,p}(\Omega)$ joining the origin to the set $\{u \in W_0^{1,p}(\Omega) : E(u) < 0\}$, and set

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} E(u). \tag{2.8}$$

Since the origin is a strict local minimizer of $E$, $c > 0$. A standard deformation argument then shows that $E$ has a $(PS)_c$ sequence. The purpose of this paper is to give lower bounds on $F$ to guarantee that $c < c^*$ holds and hence this $(PS)_c$ sequence has a subsequence that converges weakly to a nontrivial solution of problem (2.1).

We assume that there is a ball $B_p(x_0) \subset \Omega$ such that

$$F(x, t) \geq bt^s \text{ for a.a. } x \in B_p(x_0) \text{ and all } t \geq 0 \tag{2.9}$$

for some constants $b > 0$ and $s \in (q, p^*)$.

**Theorem 2.1** Let $1 < q < p < N$ and assume (2.2), (2.3), (2.7), and (2.9). Then problem (2.1) has a nontrivial weak solution in each of the following cases:

(i) $q < N(p - 1)/(N - 1)$ and $s > N^2(p - 1)/(N - 1)(N - p)$,

(ii) $q \geq N(p - 1)/(N - 1)$ and $s > Nq/(N - p)$.

**Remark 2.2** We note that the two cases in Theorem 2.1 can be combined as

$$s > \max \left\{ \frac{N^2(p - 1)}{(N - 1)(N - p)}, \frac{Nq}{N - p} \right\}.$$  

In particular, we have the following corollary for the model problem

$$\begin{cases}
-\Delta_p u - \Delta_q u = b |u|^{s-2} u + |u|^{p^*-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases} \tag{2.10}$$

where $1 < p < N$.

**Corollary 2.3** Problem (2.10) has a nontrivial weak solution for all $b > 0$ in each of the following cases:

(i) $1 < q < N(p - 1)/(N - 1)$ and $N^2(p - 1)/(N - 1)(N - p) < s < p^*$,

(ii) $N(p - 1)/(N - 1) \leq q < p$ and $Nq/(N - p) < s < p^*$.

**Remark 2.4** It was shown in Yin and Yang [2] that problem (2.10) has a nontrivial solution when $p < s < p^*$ and $b > 0$ is sufficiently large. In contrast, Corollary 2.3 allows $s \leq p$ and gives a nontrivial solution for all $b > 0$. It also gives a nontrivial solution for all $s \in (p, p^*)$ and $b > 0$ when $N^2 - p(p + 1)N + p^2 \geq 0$ and $q \leq (N - p)\frac{p}{N}$, and for $s = p$ and all $b > 0$ when $N^2 - p(p + 1)N + p^2 > 0$ and $q < (N - p)\frac{p}{N}$.

When $p \leq 2 - 1/N$, case (i) in Corollary 2.3 cannot hold and the first inequality in case (ii) holds for $q > 1$, so we have the following corollary.

**Corollary 2.5** If $1 < q < p \leq 2 - 1/N$ and $Nq/(N - p) < s < Np/(N - p)$, then problem (2.10) has a nontrivial weak solution for all $b > 0$.

For the borderline case $s = q$ of problem (2.10) we prove a Pohožaev type nonexistence result. Recall that the corresponding nonexistence result for the $p$-Laplacian states that the problem

$$\begin{cases}
-\Delta_p u = \lambda |u|^{p^*-2} u + |u|^{p^*-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$
has no nontrivial weak solution in $W^{1,p}_0(\Omega)$ for $\lambda \leq 0$ when $\Omega$ is a star-shaped domain with $C^1$-boundary (see Guedda and Véron [6, Corollaries 1.2 & 1.3]). In contrast, we will show that the problem
\[
\begin{cases}
-\Delta_p u - \Delta_q u = \mu |u|^{q-2} u + |u|^{p^*-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]  
(2.11)
has no nontrivial weak solution even for small positive $\mu$.

**Theorem 2.6** Let $1 < q < p < N$. If $\Omega$ is a star-shaped domain with $C^1$-boundary and
\[
\mu \leq \frac{N(p-q)}{N(p-q)+pq} \mu_1,
\]  
(2.12)
then problem (2.11) has no nontrivial weak solution in $W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega)$.

**Remark 2.7** It was shown in Li and Zhang [1] that problem (2.10) has infinitely many solutions when $1 < s < q$ and $b > 0$ is sufficiently small. Theorem 2.6 shows that such a result cannot hold in general in the borderline case $s = q$.

To prove Theorem 2.6 we will first derive a Pohožaev type identity for the $(p,q)$-Laplacian operator that is of independent interest. Consider the problem
\[
\begin{cases}
-\Delta_p u - \Delta_q u = g(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]  
(2.13)
where $1 < q < p < N$ and $g$ is a continuous function on $\mathbb{R}$. Let $G(t) = \int_0^t g(\tau) \, d\tau$ be the primitive of $g$.

**Theorem 2.8** If $\Omega$ has $C^1$-boundary and $u \in W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega)$ is a weak solution of problem (2.13), then
\[
\left(\frac{1}{q} - \frac{1}{p}\right) \int_\Omega |\nabla u|^q \, dx - \int_\Omega \left[ G(u) - \frac{1}{p^*} u \, g(u) \right] \, dx
\]  
\[+ \frac{1}{N} \int_{\partial \Omega} \left[ \left( 1 - \frac{1}{p} \right) \left| \frac{\partial u}{\partial \nu} \right|^p + \left( 1 - \frac{1}{q} \right) \left| \frac{\partial u}{\partial \nu} \right|^q \right] (x \cdot \nu) \, d\sigma = 0,
\]  
(2.14)
where $\nu$ is the exterior unit normal to $\partial \Omega$.

Finally we prove a stronger existence result for the related problem
\[
\begin{cases}
-\Delta_p u - \nu \Delta_q u = f(x,u) + |u|^{p^*-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]  
(2.15)
when the parameter $\nu > 0$ is sufficiently small.

**Theorem 2.9** Let $1 < q < p < N$ and assume (2.2), (2.3), (2.9), and
\[
F(x,t) \leq \frac{\lambda}{p} |t|^p + b_0 |t|^\tilde{s} \quad \text{for a.a. } x \in \Omega \text{ and } |t| < \delta
\]  
(2.16)
for some $\lambda \in (0,\lambda_1)$, $b_0 > 0$, $\tilde{s} \in (q,s]$, and $\delta > 0$. Then there exists $\nu_0 > 0$ such that problem (2.15) has a nontrivial weak solution for all $\nu \in (0,\nu_0)$ in each of the following cases:

(i) $N \geq p^2$ and $q < s < p^*$,
(ii) $N < p^2$ and either $q < s < p$ or $(Np - 2N + p) \frac{p}{(N - p)(p - 1)} < s < p^*.$

**Remark 2.10** We note that $p < (Np - 2N + p) \frac{p}{(N - p)(p - 1)}$ when $N < p^2.$

In particular, we have the following corollary for the model problem

$$
\left\{ \begin{array}{ll}
-\Delta_p u - v \Delta_q u = b |u|^{q-2} u + |u|^{p^*-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{array} \right.
$$

(2.17)

where $1 < q < p < N$.

**Corollary 2.11** There exists $\nu_0 > 0$ such that problem (2.17) has a nontrivial weak solution for all $\nu \in (0, \nu_0)$ and $b > 0$ in each of the following cases:

(i) $N \geq p^2$ and $q < s < p^*$,

(ii) $N < p^2$ and either $q < s < p$ or $(Np - 2N + p) \frac{p}{(N - p)(p - 1)} < s < p^*$.

When $q < s < p$, we have the following corollary.

**Corollary 2.12** If $q < s < p$, then there exists $\nu_0 > 0$ such that problem (2.17) has a nontrivial weak solution for all $\nu \in (0, \nu_0)$ and $b > 0$.

**Remark 2.13** A rescaling of a result in Ho and Sim [4] shows that problem (2.17) has a nontrivial solution when $q < s < p$ and $\nu, b > 0$ are sufficiently small. In contrast, Corollary 2.12 gives a nontrivial solution for all $b > 0$.

### 3 Preliminaries

For $\nu \geq 0$, set

$$
E_\nu(u) = \int_{\Omega} \left( \frac{1}{p} |\nabla u|^p + \frac{v}{q} |\nabla u|^q - F(x, u) - \frac{1}{p^*} |u|^{p^*} \right) dx, \quad u \in W^{1, p}_0(\Omega).
$$

#### 3.1 A compactness result

Our existence results will be based on the following proposition, which extends Gazzola and Ruf [7, Lemma 1] and Arioli and Gazzola [8, Lemma 1] to the $(p, q)$-Laplacian.

**Proposition 3.1** Let $1 < q < p < N$ and assume (2.3). If $0 < c < c^*$, then every $(PS)_c$ sequence has a subsequence that converges weakly to a nontrivial critical point of $E_\nu$.

**Proof** Let $(u_j) \subset W^{1, p}_0(\Omega)$ be a $(PS)_c$ sequence, i.e.,

$$
E_\nu(u_j) = \int_{\Omega} \left( \frac{1}{p} |\nabla u_j|^p + \frac{v}{q} |\nabla u_j|^q - F(x, u_j) - \frac{1}{p^*} |u_j|^{p^*} \right) dx = c + o(1) \quad (3.1)
$$

and

$$
(E'_\nu(u_j), v) = \int_{\Omega} \left( |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v + \frac{v}{q} |\nabla u_j|^{q-2} \nabla u_j \cdot \nabla v - f(x, u_j) v 
- |u_j|^{p^*-2} u_j v \right) dx = o(\|v\|) \quad \forall v \in W^{1, p}_0(\Omega). \quad (3.2)
$$
Taking \( v = u_j \) in (3.2) gives
\[
\int_\Omega \left( |\nabla u_j|^p + v |\nabla u_j|^q - f(x, u_j) u_j - |u_j|^{p^*} \right) \, dx = o(\|u_j\|). \tag{3.3}
\]
Fix \( \sigma \in (p, p^*) \). Dividing (3.3) by \( \sigma \) and subtracting from (3.1) gives
\[
\left( \frac{1}{p} - \frac{1}{\sigma} \right) \int_\Omega |\nabla u_j|^p \, dx + \left( \frac{1}{q} - \frac{1}{\sigma} \right) v \int_\Omega |\nabla u_j|^q \, dx + \int_\Omega \left[ \frac{1}{\sigma} - \frac{1}{p^*} \right] |u_j|^{p^*} - F(x, u_j) + \frac{1}{\sigma} f(x, u_j) u_j \right] \, dx = c + o(1) + o(\|u_j\|). \tag{3.4}
\]
Since \( q < p < \sigma < p^* \), it follows from this and (2.3) that \( (u_j) \) is bounded in \( W^{1, p}(\Omega) \). So a renamed subsequence converges to some \( u \) weakly in \( W^{1, p}(\Omega) \), strongly in \( L^r(\Omega) \), and a.e. in \( \Omega \). Then \( u \) is a critical point of \( E_v \) by the weak continuity of \( E_v' \) (see Li and Zhang [1, Lemma 2.3]).

Suppose \( u = 0 \). Then (3.1) and (3.3) reduce to
\[
\int_\Omega \left( \frac{1}{p} |\nabla u_j|^p + \frac{v}{q} |\nabla u_j|^q - \frac{1}{p^*} |u_j|^{p^*} \right) \, dx = c + o(1) \tag{3.5}
\]
respectively. Equation (3.5) together with (2.5) gives
\[
\|u_j\|^p \leq \frac{\|u_j\|^{p^*}}{S^{p^*/p}} + o(1). \tag{3.6}
\]
If \( \|u_j\| \to 0 \) for a renamed subsequence, then (3.4) gives \( c = 0 \), contrary to our assumption that \( c > 0 \). So \( \|u_j\| \) is bounded away from zero and hence (3.6) implies that
\[
\|u_j\|^p \geq S^{N/p} + o(1).
\]
Now dividing (3.5) by \( p^* \) and subtracting from (3.4) gives
\[
c = \int_\Omega \left[ \frac{1}{p} - \frac{1}{p^*} \right] |\nabla u_j|^p + \left( \frac{1}{q} - \frac{1}{p^*} \right) v |\nabla u_j|^q \right] \, dx + o(1) \geq \frac{1}{N} S^{N/p} + o(1),
\]
so \( c \geq c^* \), contrary to assumption. □

### 3.2 Some estimates

Let \( \rho > 0 \) be as in (2.9), take a cut-off function \( \psi \in C_0^\infty(B_{\rho}(0)) \) such that \( 0 \leq \psi \leq 1 \) and \( \psi = 1 \) on \( B_{\rho/2}(0) \), and set
\[
u_\varepsilon(x) = \psi(x) \left. \psi(x) \right|_{(\varepsilon^p/(p-1))^p} + \left. |x|^{p/(p-1)} (N-p)/p \right|_{\varepsilon^p/(p-1)}^{p^*} \quad \text{for } \varepsilon > 0,
\]
where \( |\cdot|_{p^*} \) denotes the norm in \( L^{p^*}(\Omega) \). Then \( |v_\varepsilon|_{p^*} = 1 \). Recall that
\[
f(\varepsilon) = \Theta(g(\varepsilon))
\]
as \( \varepsilon \to 0 \) if there exist constants \( c, C > 0 \) such that
\[
|g(\varepsilon)| \leq |f(\varepsilon)| \leq C |g(\varepsilon)|
\]
for all sufficiently small \( \varepsilon > 0 \). We have the estimates
\[
\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^p \, dx = S + \Theta(\varepsilon^{(N-p)/(p-1)}),
\]
where \( S \) is as in (2.5),
\[
\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^q \, dx = \begin{cases} 
\Theta(\varepsilon^{N(p-q)/p}), & q > \frac{N(p-1)}{N-1}, \\
\Theta(\varepsilon^{(N-p)/(N-1)} \log \varepsilon), & q = \frac{N(p-1)}{N-1}, \\
\Theta(\varepsilon^{(N-p)q/(p-1)}), & q < \frac{N(p-1)}{N-1},
\end{cases}
\]
and
\[
\int_{\mathbb{R}^N} v_\varepsilon^s \, dx = \begin{cases} 
\Theta(\varepsilon^{(Np-(N-p)s)/p}), & s > \frac{N(p-1)}{N-p}, \\
\Theta(\varepsilon^{N/p} \log \varepsilon), & s = \frac{N(p-1)}{N-p}, \\
\Theta(\varepsilon^{(N-p)s/(p-1)}), & s < \frac{N(p-1)}{N-p},
\end{cases}
\]
as \( \varepsilon \to 0 \) (see Drábek and Huang [9]).

For \( \varepsilon > 0 \) and \( 0 < \delta \leq 1 \), set
\[
u_{\varepsilon,\delta}(x) = \frac{v_{\varepsilon,\delta}(x)}{|u_{\varepsilon,\delta}|_{p^*}},
\]
and hence
\[
|u_{\varepsilon,\delta}|_{p^*} = \delta^{-(N-p)/(p-1)}|u_{\varepsilon,\delta}|_{p^*}.
\]

It follows from (3.10) and (3.11) that
\[
v_{\varepsilon,\delta}(x) = \delta^{-(N-p)/p} \nu_{\varepsilon,\delta}(x/\delta).
\]

Moreover,
\[
\nabla v_{\varepsilon,\delta}(x) = \delta^{-N/p} \nabla \nu_{\varepsilon,\delta}(x/\delta)
\]
and hence
\[
\int_{\mathbb{R}^N} \left| \nabla v_{\varepsilon,\delta} \right|^q \, dx = \delta^{-Nq/p} \int_{\mathbb{R}^N} \left| \nabla \nu_{\varepsilon,\delta}(x/\delta) \right|^q \, dx = \delta^{N(p-q)/p} \int_{\mathbb{R}^N} \left| \nabla \nu_{\varepsilon,\delta} \right|^q \, dx.
\]

Combining (3.12) and (3.13) with (3.7)–(3.9) gives us the following estimates.
Lemma 3.2 As $\varepsilon \to 0$ and $\varepsilon/\delta \to 0$,

\[
\int_{\mathbb{R}^N} |\nabla v_{\varepsilon,\delta}|^p \, dx = S + \Theta((\varepsilon/\delta)^{(N-p)/(p-1)}),
\]

(3.14)

and

\[
\int_{\mathbb{R}^N} |\nabla v_{\varepsilon,\delta}|^q \, dx = \begin{cases} 
\Theta(\varepsilon^{N(p-q)/p}), & q > \frac{N(p-1)}{N-1} \\
\Theta(\varepsilon^{N(N-1)p}/(N-1)^p \log (\varepsilon/\delta)), & q = \frac{N(p-1)}{N-1} \\
\Theta((\varepsilon(N-p)/p-1) \delta^{N(p-1)-(N-1)q}/(p-1)), & q < \frac{N(p-1)}{N-1},
\end{cases}
\]

(3.15)

Next we prove the following proposition. Note that the first limit in (3.17) holds trivially when $\nu = 0$ and hence the second limit is sufficient for (3.18) to hold for $E_0$.

Proposition 3.3 If $(\varepsilon_j), (\delta_j)$ are sequences such that $\varepsilon_j \to 0$, $0 < \delta_j \leq 1$, $\varepsilon_j/\delta_j \to 0$,

\[
\frac{\nu}{j} \int_{\mathbb{R}^N} |\nabla \nu_{\varepsilon_j,\delta_j}|^q \, dx \to 0, \quad \frac{\varepsilon_j/\delta_j}{}(N-p)/(p-1) \int_{\mathbb{R}^N} v_{\varepsilon_j,\delta_j}^s \, dx \to 0,
\]

(3.17)

then

\[
\max_{t \geq 0} E_\nu(t \nu_{\varepsilon_j,\delta_j}(x - x_0)) < c^*
\]

(3.18)

for all sufficiently large $j$.

Proof Write $v_j(x) = \nu_{\varepsilon_j,\delta_j}(x - x_0)$. Since $v_j(x) = 0$ for all $x \in \Omega \setminus B_{\rho}(x_0)$,

\[
F(x, tv_j(x)) \geq bt^s v_j(x)^s
\]

for a.a. $x \in \Omega$ and all $t \geq 0$

by (2.9), so

\[
E_\nu(tv_j) \leq \frac{t^p}{p} \int_{\Omega} |\nabla v_j|^p \, dx + \frac{t^{q’}}{q} \int_{\Omega} |\nabla v_j|^q \, dx - bt^s \int_{\Omega} v_j^s \, dx - \frac{t^p}{p^*} =: \varphi(t).
\]

Suppose that the conclusion of the lemma is false. Then there are renamed subsequences $(\varepsilon_j), (\delta_j)$ and $t_j > 0$ such that

\[
\varphi(t_j) = \frac{t^p}{p} \int_{\Omega} |\nabla v_j|^p \, dx + \frac{t^{q’}}{q} \int_{\Omega} |\nabla v_j|^q \, dx - bt_j^s \int_{\Omega} v_j^s \, dx - \frac{t^p}{p^*} \geq c^* \quad (3.19)
\]

and

\[
t_j \varphi’(t_j) = \frac{t^{p’}}{p} \int_{\Omega} |\nabla v_j|^p \, dx + t_j^{q’} \int_{\Omega} |\nabla v_j|^q \, dx - sbt_j^s \int_{\Omega} v_j^s \, dx - t_j^{p^*} = 0. \quad (3.20)
\]

By Lemma 3.2,

\[
\int_{\Omega} |\nabla v_j|^p \, dx \to S, \quad \int_{\Omega} |\nabla v_j|^q \, dx \to 0, \quad \int_{\Omega} v_j^s \, dx \to 0.
\]
So (3.19) implies that the sequence \( \{t_j\} \) is bounded and hence converges to some \( t_0 > 0 \) for a subsequence. Passing to the limit in (3.20) gives
\[
S^p t_0^p - t_0^p = 0, \tag{3.21}
\]
so \( t_0 = S^{(N-p)/p^2} \).

Subtracting (3.21) from (3.20) and using (3.14) gives
\[
S(t_j^p - t_0^p) + v_j^q \int_\Omega |\nabla v_j|^q \, dx - sbt_j^p \int_\Omega v_j^p \, dx - (t_j^p - t_0^p) = \Theta((\varepsilon_j/\delta_j)^{(N-p)/(p-1)}).
\]

Then
\[
(p \sigma_j^{p-1} - p^* \tau_j^{p*-1}) (t_j - t_0) = sbt_j^p \int_\Omega v_j^p \, dx - vt_j^q \int_\Omega |\nabla v_j|^q \, dx + \Theta((\varepsilon_j/\delta_j)^{(N-p)/(p-1)}) \tag{3.22}
\]
for some \( \sigma_j \) and \( \tau_j \) between \( t_0 \) and \( t_j \) by the mean value theorem. Since \( t_j \to t_0, \sigma_j, \tau_j \to t_0 \) and hence
\[
p \sigma_j^{p-1} - p^* \tau_j^{p*-1} \to p \sigma_0^{p-1} - p^* \tau_0^{p*-1} = -(p^* - p) t_0^{p*-1}
\]
by (3.21). So (3.22) together with (3.17) gives
\[
t_j = t_0 - \left( \frac{sb t_0^{(p^*-s-1)}}{p^* - p} + o(1) \right) \int_\Omega v_j^p \, dx < t_0
\]
for all sufficiently large \( j \).

Dividing (3.20) by \( p^* \), subtracting from (3.19), using (3.14), and writing \( c^* \) in terms of \( t_0 \) gives
\[
\frac{1}{N} S^p t_j^p + \left( \frac{1}{q} - \frac{1}{p^*} \right) v_j^q \int_\Omega |\nabla v_j|^q \, dx - b \left( 1 - \frac{s}{p^*} \right) t_j^s \times \int_\Omega v_j^s \, dx \geq \frac{1}{N} S^p t_0^p + \Theta((\varepsilon_j/\delta_j)^{(N-p)/(p-1)}).
\]

This together with \( t_j < t_0 \) and (3.17) gives
\[
b \left( 1 - \frac{s}{p^*} \right) t_0^s \leq 0,
\]
a contradiction since \( s < p^* \) and \( t_0 > 0 \).

\[\square\]

## 4 Proofs

### 4.1 Proof of Theorem 2.1

Lemma 3.2 gives the following estimates for the quotients in (3.17).

**Lemma 4.1** If \( s > N(p - 1)/(N - p) \), then
\[
\begin{align*}
\int_{\mathbb{R}^N} |\nabla v_{\varepsilon_j, \delta_j}|^q \, dx & = \begin{cases} 
\Theta((N-p)s-Nq/p), & q > \frac{N(p-1)}{N-1} \\
\Theta(\varepsilon_j^{(N-p)s-Nq/p} |log(\varepsilon_j/\delta_j)|), & q = \frac{N(p-1)}{N-1} \\
\Theta(\varepsilon_j^{(N-p)(s+q/(p-1) - Np)/p} \delta_j^{(N(p-1) - (N-1)q)/(p-1)}), & q < \frac{N(p-1)}{N-1}
\end{cases} 
\end{align*}
\]
and
\[ \frac{(\varepsilon_j/\delta_j)^{(N-p)/(p-1)}}{\int_{\mathbb{R}^N} v_{\varepsilon_j, \delta_j}^s \, dx} = \Theta(\varepsilon_j^{[(N-p)(p-1)s-(Np-2N+p)p]/(p-1)} \delta_j^{-(N-p)/(p-1)}) \]

We are now ready to prove Theorem 2.1.

**Proof of Theorem 2.1** As we have noted in the introduction, it suffices to show that the mountain pass level \( c \) defined in (2.8) is below the threshold level \( c^* \) in (2.4). For any \( u \in W_0^1, p(\Omega) \setminus \{0\} \), \( E(tu) \to -\infty \) as \( t \to +\infty \) and hence \( \exists t_u > 0 \) such that \( E(t_u u) < 0 \). Then the line segment \( \{tu : 0 \leq t \leq t_u\} \) belongs to \( \Gamma \) and hence
\[ c \leq \max_{0 \leq t \leq t_u} E(tu) \leq \max_{t \geq 0} E(tu). \quad (4.1) \]

In each of the two cases in the theorem, we will construct sequences \( (\varepsilon_j) \), \( (\delta_j) \) such that \( \varepsilon_j \to 0 \), \( 0 < \delta_j \leq 1 \), \( \varepsilon_j/\delta_j \to 0 \), and (3.17) with \( \nu = 1 \) holds, and conclude from Proposition 3.3 and (4.1) that \( c < c^* \).

(i) Let \( q < N(p-1)/(N-1) \) and \( s > N^2(p-1)/(N-1)(N-p) \). We take a sequence \( \varepsilon_j \to 0 \) and set \( \delta_j = \varepsilon_j^2 \), where \( \kappa \in [0, 1) \) is to be determined. Since
\[ s > \frac{N^2(p-1)}{(N-1)(N-p)} > \frac{N(p-1)}{N-p}, \]
Lemma 4.1 gives
\[
\frac{1}{\int_{\mathbb{R}^N} |\nabla v_{\varepsilon_j, \delta_j}|^q \, dx} = \Theta(\varepsilon_j^{[(N-p)q/(p-1)]-\kappa/[N(p-1)-(N-1)q]/(p-1)})
\]
where
\[ \kappa = \frac{Np(p-1) - (N-p)(p-1)s - (N-p)q}{[N(p-1) - (N-1)q]p}, \]
and
\[ (\varepsilon_j/\delta_j)^{(N-p)/(p-1)} \int_{\mathbb{R}^N} v_{\varepsilon_j, \delta_j}^s \, dx = \Theta(\varepsilon_j^{[(N-p)(p-1)s-(Np-2N+p)p]/(p-1)-\kappa(N-p)/(p-1)}) \]
where
\[ \overline{\kappa} = \frac{(N-p)(p-1)s - (Np - 2N + p)p}{(N-p)p}. \]

We want to choose \( \kappa \in [0, 1) \) so that \( \kappa > \underline{\kappa} \) and \( \kappa < \overline{\kappa} \). This is possible if and only if \( \underline{\kappa} < \overline{\kappa} \), \( \underline{\kappa} < 1 \), and \( \overline{\kappa} > 0 \). Tedious calculations show that these inequalities are equivalent to
\[ s > \frac{N^2(p-1)}{(N-1)(N-p)}, \]
\[ s > \frac{Nq}{N-p}. \]
and

\[ s > \frac{N^2(p-1)}{(N-1)(N-p)} - \frac{N-p}{(N-1)(p-1)}, \]

respectively, all of which hold under our assumptions on \( q \) and \( s \).

(ii) Let \( q \geq N(p-1)/(N-1) \) and \( s > Nq/(N-p) \). We take a sequence \( \varepsilon_j \to 0 \) and set \( \delta_j = 1 \). Since

\[ s > \frac{Nq}{N-p} \geq \frac{N^2(p-1)}{(N-1)(N-p)} > \frac{N(p-1)}{N-p}, \]

Lemma 4.1 gives

\[
\int_{\mathbb{R}^N} |\nabla v_{\varepsilon_j, \delta_j}|^q \, dx \leq \begin{cases} \Theta \left( \varepsilon_j \frac{(N-p)s-Nq}{p} \right), & q > \frac{N(p-1)}{N-1} \\ \Theta \left( \varepsilon_j \frac{(N-p)s-Nq}{p} \log \varepsilon_j \right), & q = \frac{N(p-1)}{N-1} \end{cases}
\]

and

\[
\int_{\mathbb{R}^N} v_{\varepsilon_j, \delta_j}^q \, dx = \Theta \left( \varepsilon_j \frac{(N-p)(p-1)s-(Np-2N+p)p}{p(p-1)} \right).
\]

Since \( s > Nq/(N-p) \), the first limit in (3.17) holds. The second limit also holds since

\[
\frac{Nq}{N-p} \geq \frac{N^2(p-1)}{(N-1)(N-p)} > \frac{(Np-2N+p)p}{(N-p)(p-1)}.
\]

\[ \square \]

### 4.2 Proofs of Theorems 2.6 and 2.8

First we prove Theorem 2.8.

**Proof of Theorem 2.8** Integrating the easily verified identity

\[
[ \text{div} \left( |\nabla u|^{p-2} \nabla u + |\nabla u|^{q-2} \nabla u \right) + g(u)] (x \cdot \nabla u) = \left( \frac{N}{p} - 1 \right) |\nabla u|^p + \left( \frac{N}{q} - 1 \right) |\nabla u|^q - NG(u) + \text{div} \left( |\nabla u|^{p-2} \nabla u + |\nabla u|^{q-2} \nabla u \right) (x \cdot \nabla u) - x \left( \frac{|\nabla u|^p}{p} + \frac{|\nabla u|^q}{q} \right) + x G(u)
\]

over \( \Omega \) gives

\[
\left( \frac{N}{p} - 1 \right) \int_{\Omega} |\nabla u|^p \, dx + \left( \frac{N}{q} - 1 \right) \int_{\Omega} |\nabla u|^q \, dx - N \int_{\Omega} G(u) \, dx + \int_{\partial \Omega} \left[ (|\nabla u|^{p-2} \nabla u + |\nabla u|^{q-2} \nabla u) (x \cdot \nabla u) - x \left( \frac{|\nabla u|^p}{p} + \frac{|\nabla u|^q}{q} \right) \right] \cdot \nu \, d\sigma = 0
\]

(4.2)

since \( u \) is a weak solution of problem (2.13). On \( \partial \Omega \), tangential derivatives of \( u \) vanish since \( u = 0 \) and hence \( \nabla u \frac{\partial u}{\partial v} \). So \( (\nabla u \cdot \nu)(x \cdot \nabla u) = |\nabla u|^2 (x \cdot \nu) \) and \( |\nabla u| = \left| \frac{\partial u}{\partial v} \right| \) on \( \partial \Omega \).
and hence (4.2) reduces to
\[
\left(\frac{N}{p} - 1\right) \int_{\Omega} |\nabla u|^p \, dx + \left(\frac{N}{q} - 1\right) \int_{\Omega} |\nabla u|^q \, dx - N \int_{\Omega} G(u) \, dx \\
+ \int_{\partial \Omega} \left[ \left(1 - \frac{1}{p}\right) \left| \frac{\partial u}{\partial \nu} \right|^p + \left(1 - \frac{1}{q}\right) \left| \frac{\partial u}{\partial v} \right|^q \right] (x \cdot \nu) \, d\sigma = 0.
\]
(4.3)

On the other hand, testing problem (2.13) with \( u \) gives
\[
\int_{\Omega} |\nabla u|^p \, dx + \int_{\Omega} |\nabla u|^q \, dx - \int_{\Omega} u g(u) \, dx = 0.
\]
(4.4)

Multiplying (4.4) by \( \frac{N}{p} - 1 \) and subtracting from (4.3) gives (2.14).

Now we prove Theorem 2.6.

**Proof of Theorem 2.6** Suppose problem (2.11) has a nontrivial weak solution \( u \in W^{1,p}_{0}(\Omega) \cap W^{2,p}(\Omega) \). Taking \( g(t) = \mu |t|^{q-2} t + |t|^{p^* - 2} t \) in (2.14) and combining with (2.12) and (2.6) gives
\[
\frac{1}{N} \int_{\partial \Omega} \left[ \left(1 - \frac{1}{p}\right) \left| \frac{\partial u}{\partial \nu} \right|^p + \left(1 - \frac{1}{q}\right) \left| \frac{\partial u}{\partial v} \right|^q \right] (x \cdot \nu) \, d\sigma = \left(\frac{1}{q} - \frac{1}{p^*}\right) \mu \int_{\Omega} |u|^q \, dx \\
- \left(\frac{1}{q} - \frac{1}{p}\right) \int_{\Omega} |\nabla u|^q \, dx \leq \left(\frac{1}{q} - \frac{1}{p}\right) \left( \mu_{1} \int_{\Omega} |u|^q \, dx - \int_{\Omega} |\nabla u|^q \, dx \right) \leq 0.
\]
(4.5)

Without loss of generality, we may assume that \( \Omega \) is star-shaped with respect to the origin. Then \( x \cdot \nu > 0 \) on \( \partial \Omega \), so the first integral in (4.5) is nonnegative and hence equality holds throughout (4.5). This implies that \( u \) is an eigenfunction of the \( q \)-Laplacian associated with the eigenvalue \( \mu_{1} \) and \( \partial u/\partial v = 0 \) on \( \partial \Omega \), contradicting the Hopf lemma (see Vázquez [10, Theorem 5]).

\( \square \)

### 4.3 Proof of Theorem 2.9

We have
\[
E_{\nu}(u) = E_{0}(u) + \frac{\nu}{q} \int_{\Omega} |\nabla u|^q \, dx, \quad u \in W^{1,p}_{0}(\Omega).
\]

Taking \( \nu = 0 \) and \( \delta_{j} = 1 \) in Proposition 3.3 and noting that \( v_{\varepsilon,1} = v_{\varepsilon} \) gives the following proposition.

**Proposition 4.2** If
\[
\frac{\varepsilon^{(N-p)/(p-1)}}{\int_{\mathbb{R}^{N}} v_{\varepsilon} \, dx} \to 0 \text{ as } \varepsilon \to 0,
\]
(4.6)

then
\[
\max_{t \geq 0} E_{0}(tv_{\varepsilon}(x - x_{0})) < c^{*}
\]
for all sufficiently small \( \varepsilon > 0 \).

Equation (3.9) gives the following estimate for the quotient in (4.6).
Lemma 4.3 We have 
\[
\epsilon^{(N-p)/(p-1)} \int_{\mathbb{R}^N} v_\epsilon^s \, dx = \begin{cases} 
\Theta(\epsilon^{(N-p)(p-1)}s-(Np-2N+p)p/p(p-1)), & s > \frac{N(p-1)}{N-p} \\
\Theta(\epsilon^{(N-p^2)/p(p-1)}/\log \epsilon), & s = \frac{N(p-1)}{N-p} \\
\Theta(\epsilon^{(N-p)(p-s)/p(p-1)}), & s < \frac{N(p-1)}{N-p}.
\end{cases}
\]

We are now ready to prove Theorem 2.9.

Proof of Theorem 2.9 The proof is similar to that of Theorem 2.1, so we will be sketchy. Let 
\[
\Gamma_v = \left\{ \gamma \in C([0,1], W^{1,p}_0(\Omega)) : \gamma(0) = 0, \ E_v(\gamma(1)) < 0 \right\},
\]
set
\[
c_v := \inf_{\gamma \in \Gamma_v} \max_{u \in \gamma([0,1])} E_v(u),
\]
and note that \( c_v > 0 \) by (2.16). It suffices to show that \( c_v < c^* \) for sufficiently small \( v \). We will show that
\[
c_0 := \inf_{\gamma \in \Gamma_0} \max_{u \in \gamma([0,1])} E_0(u) < c^*. \tag{4.7}
\]
Then there is a path \( \gamma_0 \in \Gamma_0 \) such that 
\[
\max_{u \in \gamma_0([0,1])} E_0(u) < c^*.
\]
For all sufficiently small \( v > 0 \),
\[
E_v(\gamma_0(1)) = E_0(\gamma_0(1)) + \frac{v}{q} \int_{\Omega} |\nabla \gamma_0(1)|^q \, dx < 0
\]
and
\[
\max_{u \in \gamma_0([0,1])} E_v(u) \leq \max_{u \in \gamma_0([0,1])} E_0(u) + \frac{v}{q} \left( \max_{u \in \gamma_0([0,1])} \int_{\Omega} |\nabla u|^q \, dx \right) < c^*,
\]
so \( \gamma_0 \in \Gamma_v \) and
\[
c_v \leq \max_{u \in \gamma_0([0,1])} E_v(u) < c^*.
\]

To show that (4.7) holds, it suffices to show that
\[
\max_{t \geq 0} E_0(tu_0) < c^* \tag{4.8}
\]
for some \( u_0 \in W^{1,p}_0(\Omega) \setminus \{0\} \) as in the proof of Theorem 2.1. In each of the two cases in the theorem, we will show that (4.6) holds and conclude from Proposition 4.2 that (4.8) holds for \( u_0 = v_\epsilon(x - x_0) \) with \( \epsilon > 0 \) sufficiently small.

(i) Let \( N \geq p^2 \) and \( q < s < p^* \). If \( s > N(p-1)/(N-p) \), then
\[
(N-p)(p-1) - (Np-2N+p)p = N(p-1)^2 - (Np-2N+p)p = N - p^2,
\]
and if \( s < N(p-1)/(N-p) \), then
\[
(N-p)(p-s) > (N-p)p - N(p-1) = N - p^2.
\]
So (4.6) follows from Lemma 4.3.
(ii) Let \( N < p^2 \). Then
\[
P < \frac{N(p-1)}{N-p} < \frac{(Np - 2N + p)}{(N-p)(p-1)}.
\]
So if \( q < s < p \), then \( s < N(p-1)/(N-p) \), and if \( (Np - 2N + p)/(N-p)(p-1) < s < p^* \), then \( s > N(p-1)/(N-p) \). In either case, (4.6) follows from Lemma 4.3.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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