Bloch-Siegert shift for multiphoton resonances

Peter Hagelstein and Irfan Chaudhary

Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, MA 02139, USA

Abstract

Recently there has been theoretical and experimental interest in Bloch-Siegert shifts in an intense photon field. A perturbative treatment becomes difficult in this multiphoton regime. We present a unitary transform and rotated model, which allows us to get accurate results away from the level anticrossings. A simple variational energy estimate leads to a new expression for the dressed two-level system energy which is accurate, and useful over a wide range of the dimensionless coupling constant.

PACS numbers: 32.60.+i,32.80.Bx,32.80.Rm,32.80.Wr
I. INTRODUCTION

The basic problem of a two-level system coupled to an oscillatory perturbation arose in the early on in quantum mechanics in association with studies of spin dynamics in a magnetic field. The closely related problem of a two-level coupled to a simple harmonic oscillator has been of interest for more than 30 years, at least since the work of Cohen-Tannoudji et al. The Hamiltonian for this problem can be written as

\[ \hat{H} = \Delta E \frac{\hat{s}_z}{\hbar} + \hbar \omega_0 \hat{a}^\dagger \hat{a} + U (\hat{a}^\dagger + \hat{a}) \frac{2\hat{s}_x}{\hbar} \]  

where we can write the spin operators \( \hat{s}_i \) in terms of the Pauli matrices as

\[ \hat{s}_i = \frac{\hbar}{2} \sigma_i \]

This model has been of interest recently in problems in which atoms interact with strong electromagnetic fields. We have become interested in this problem since it can exhibit coherent energy coupling between systems with characteristic energies that are very different. The multiphoton regime in which \( \Delta E \gg \hbar \omega_0 \) in this regard is of interest to us. Additionally, our focus has been on the large \( n \) limit.

It is well known that the dipolar interaction between photons and atoms causes the frequency of the atom to get shifted. The associated shift has been termed the Bloch-Siegert shift in the literature. A general analytic expression for the \((2k + 1)\)th resonance has been known for some time. Measurements of this shift have been reported recently up to \( k = 11 \). Various theoretical methods have been used to understand these results.

We have recently found a unitary transformation that separates the problem into three parts, each of which is relatively complicated. In the multiphoton regime we view the largest of these as an “unperturbed” dressed Hamiltonian, which we have found give energy eigenvalues that are close to those of the original problem away from the level anticrossings. In the present work, our attention is drawn to this aspect of the problem. Using this approach, we are able to develop a reasonably good estimate of the Bloch-Siegert shift using a simple variational approximation, one which we can compare with results obtained using other methods.

II. ROTATED HAMILTONIAN

As mentioned above, we have found it helpful to work in a rotated frame. Instead of considering the Hamiltonian in Equation (1), we consider the unitary equivalent Hamiltonian

\[ \hat{H}' = e^{-i\hat{\lambda}\hat{\sigma}_y} \hat{H} e^{i\hat{\lambda}\hat{\sigma}_y} \]  

where \( \hat{\lambda} \) is
The operator \( \hat{\lambda} \) depends explicitly on \( \hat{a} + \hat{a}^\dagger \).

This rotation can be carried out explicitly, leading to a complicated expression. It is useful to write the rotated Hamiltonian in the form

\[
\hat{H}' = \hat{H}_0 + \hat{V} + \hat{W}
\]

where we have defined \( \hat{H}_0, \hat{V}, \) and \( \hat{W} \) according to

\[
\hat{H}_0 = \sqrt{\Delta E^2 + 4U^2(\hat{a} + \hat{a}^\dagger)^2} \left( \frac{\hat{s}_z}{\hbar} \right) + \hbar \omega_0 \hat{a}^\dagger \hat{a}
\]

\[
\hat{V} = \frac{i\hbar \omega_0}{2} \left\{ \frac{U}{\Delta E} \left[ 1 + \frac{2U(\hat{a} + \hat{a}^\dagger)}{\Delta E} \right] \right\} (\hat{a} - \hat{a}^\dagger) + (\hat{a} - \hat{a}^\dagger) \left\{ \frac{U}{\Delta E} \left[ 1 + \frac{2U(\hat{a} + \hat{a}^\dagger)}{\Delta E} \right] \right\} \left( \frac{2\hat{s}_y}{\hbar} \right)
\]

\[
\hat{W} = \hbar \omega_0 \left\{ \frac{U}{\Delta E} \left[ 1 + \frac{2U(\hat{a} + \hat{a}^\dagger)}{\Delta E} \right]^2 \right\}
\]

Even though this rotated Hamiltonian is much more complicated than the one we started out with, we have found that it has useful properties that are of interest. For example, from numerical calculations and also from the WKB approximation, we have found that the “unperturbed” part of the Hamiltonian \( \hat{H}_0 \) is a good approximation to the initial Hamiltonian \( \hat{H} \) away from level anticrossings. The “perturbation” \( \hat{V} \) contains the part of the Hamiltonian that is most responsible for level splitting at the anti-crossing (as we will discuss elsewhere). Finally, the remaining term \( \hat{W} \) gives rise to a minor correction to \( \hat{H}_0 \).

III. BLOCH-SIEGERT SHIFT

To illustrate the usefulness of this rotation, we focus on one issue in particular: the development of an interesting approximation for the energy eigenvalues. For this, we need focus only on \( \hat{H}_0 \). Consider the time-independent Schrödinger equation for the “unperturbed” part of the rotated problem

\[
E \psi = \hat{H}_0 \psi
\]
Because $\hat{H}_0$ is somewhat complicated, it is not straightforward to develop exact analytic solutions. We have used numerical solutions to develop some understanding of the eigenfunctions and associated eigenvalues. From these numerical studies, it has become clear that there exists a useful analytic approximation.

We adopt an approximate solution of the form

$$
\psi_t = |n\rangle|s,m\rangle
$$

where $|n\rangle$ is a pure SHO eigenstate, and $|s,m\rangle$ is a spin function. The variational estimate for the energy eigenvalue in this approximation is

$$
E_t = \langle \psi_t | \hat{H}_0 | \psi_t \rangle
$$

This approximate energy can be evaluated explicitly to give

$$
E_t = \langle n | \sqrt{\Delta E^2 + 4U^2(\hat{a} + \hat{a}^\dagger)^2} | n \rangle m + n\hbar\omega_0
$$

This result is interesting in that it leads directly to a simple interpretation of the system in terms of a dressed two-level system and unperturbed oscillator, which is reasonably good away from the level anticrossings. Moreover, the dressed two-level system energy is involved in the calculation of the Bloch-Siegert shift in the literature.

Let us write the variational energy as

$$
E_t = \Delta E(g)m + \hbar\omega_0 n
$$

where we define $g$ according to

$$
g = \frac{U\sqrt{n}}{\Delta E}
$$

and $\Delta E(g)$ as

$$
\Delta E(g) = \Delta E\left<n \left| \sqrt{1 + \frac{4g^2(\hat{a} + \hat{a}^\dagger)^2}{n}} \right| n \right>
$$

This approximation is very good, as can be seen in Figure 1 where the approximation is plotted along with the exact numerical results. The Bloch-Siegert resonance condition can be written in terms of $\Delta E(g)$ as

$$
\Delta E(g) = (2k + 1)\hbar\omega_0
$$

It is possible to develop a simpler approximation for this approximate dressed energy based on the WKB approximation. We may write
FIG. 1: Plot of $\Delta E(g)$ computed from direct numerical solutions for $\Delta E = 11\hbar \omega_0$ and $n = 10^8$, and also the approximate $\Delta E(g)$ based on the approximation of Equation (12). The two curves are so close that they cannot be distinguished.

\[
\Delta E(g) = \frac{\Delta E}{\pi} \int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} \sqrt{\frac{1 + 8g^2y^2/n}{\epsilon - y^2}} dy
\]  

with $\epsilon = 2n + 1$. Results from this approximation can also not be distinguished from the exact results if plotted as done previously in Figure 1.

IV. COMPARISON WITH OTHER RESULTS

As the results available in the literature of interest here are given for small $g$, we require a power series expansion of Equation (12):

\[
\left\langle n \left| \sqrt{\Delta E^2 + 4U^2(\hat{a} + \hat{a}^\dagger)^2} \right| n \right\rangle \rightarrow \Delta E \left[ 1 + 4g^2 - 12g^4 + \cdots \right]
\]  

To compare to the exact expression obtained by Ahmad and Bullough, we write the resonance condition using their notation as

\[
\frac{\omega_0^2}{\omega^2} = (2k + 1)^2 \left[ 1 - \frac{2}{n(n + 1)} \frac{b^2}{\omega^2} + \frac{n^2 + n - 1}{2n^3(n + 1)^3} \frac{b^4}{\omega^4} + \cdots \right]
\]  

We translate this into our notation to obtain

\[
\frac{\Delta E^2}{\omega_0^2} = (2k + 1)^2 \left[ 1 - \frac{2}{k(k + 1)} \frac{U^2n}{\omega_0^2} + \frac{k^2 + k - 1}{2k^3(k + 1)^3} \frac{U^4n^2}{\omega_0^4} + \cdots \right]
\]  

Keeping in mind the resonance condition [Equation (13)], this is consistent with
\[ \Delta E(g) = \Delta E \left[ 1 + a(k)g^2 + b(k)g^4 + \cdots \right] \tag{18} \]

where
\[ a(k) = 4 + \frac{1}{k(k+1)} \]
\[ b(k) = -12 - \frac{8k^4 + 16k^3 + 3k^2 - 5k - 1}{4k^3(k+1)^3} \]

We can now see that in the large \( k \) limit, Equation (18) reduces to our result. It should also be noted that the deviations for finite \( k \) are \( O(1/k^2) \).

Another approximate multiphoton result is given by Ostrovsky and Horsdal-Pedersen. This result (written in their notation) for the Bloch-Siegert shift is
\[ N\omega = \omega_{ba} \left( 1 + \frac{1}{4}q^2 - \frac{3}{64}q^4 + O(q^6) \right) \tag{19} \]

This result is equivalent to our result [Equation (15)] to fourth order in \( g \).

V. CONCLUSION

The unitary equivalent Hamiltonian provides a simple way to understand the Bloch-Siegert shift. We have found in our work that the unperturbed part of the rotated Hamiltonian (\( \hat{H}_0 \)) produces a reasonably accurate estimate of the dressed two-level system energy in the large \( n \) limit away from the level anticrossings. In the discussion presented here, we have shown that a simple variational estimate for the energy associated with \( \hat{H}_0 \) in the rotated frame produces good results away from the level anticrossings over a wide range of the dimensionless coupling strength \( g \). The approximation is found to be in agreement with previous work in the limit that \( g \) is small.

* Electronic address: plh@mit.edu
† Electronic address: irfanc@mit.edu

1 F. Bloch and A. Siegert, Phys. Rev. 57 522 (1940).
2 J. Shirley, Phys. Rev. 138, B979 (1965).
3 C. Cohen-Tannoudji, J. Dupont-Roc, and C. Fabre, J. Phys. B: At. Mol. Phys. 6 L214 (1973).
4 F. Ahmad and R. K. Bullough J. Phys. B: At. Mol. Phys. 7 L275 (1974).
5 D. Fregenal et al., Phys. Rev. A 69 031401(R) (2004).
6 V. N. Ostrovsky and E. Horsdal-Pedersen Phys. Rev. A 70 033413 (2004).
7 M. Førre Phys. Rev. A 70 013406 (2004).