Perelomov problem and inversion of the Segal-Bargmann transform

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We reconstruct a function by values of its Segal-Bargmann transform at points of a lattice.

1. Formulation of the result. Fix $\tau > 0$. For a function $f \in L^2(\mathbb{R})$, we define the coefficients

$$\gamma_{m,k} = \int_{-\infty}^{\infty} e^{-ikx - \tau mx} f(x) e^{-x^2/4} dx$$

where $m, k$ range in $\mathbb{Z}$. We intend to reconstruct $f$ by $\gamma_{m,k}$. As Perelomov showed, this is impossible for $\tau > \pi$; for $\tau \leq \pi$, the problem is overdetermined (see [7], [8], [2], more recent results in [6], [3]). There are many ways for reconstruction of $f$. We propose a formula (for $\tau < \pi$) that seems relatively simple and relatively closed.

Denote $q := e^{-2\pi \tau}$. Define the coefficients

$$E_m(\tau) = \frac{(-1)^m q^{m(m-1)/2}}{\prod_{l=1}^{\infty} (1 - q^l)^3} \sum_{j \geq 0} (-1)^j q^{j(j+2m+1)/2}$$

Then

$$f(x) = \frac{1}{2\pi} e^{x^2/4} \sum_m \left\{ E_m(\tau) e^{m\tau x} \sum_k \gamma_{m,k} e^{ikx} \right\}$$

The interior sum is an $L^2$-sum of a Fourier series, the exterior sum is a.s. convergent series.

Remark. 1. Our problem also is known in the theory of recognition and separation of waves (i.e., sound or electromagnetic oscillations). A.J.E.M. Janssen [4] proposed several non-equivalent formulae for reconstruction of $f$: the formula (1) easily follows from his considerations. Hence, this note clarifies Janssen’s results and present them in a final nice form.

2. Two-step way of reconstruction of $f$ was proposed by Yu. Lyubarskii, he uses the Lagrange formula to interpolate the Segal–Bargman transform of a function by its values at points of the lattice. Then we apply the inversion formula for the Segal–Bargmann transform. I do not know, is it possible to produce a nice one-step formula from this algorithm. Our calculation is based on the same idea but it gives another final result.

2. Preliminaries on $\theta$-functions. Let $0 < q < 1$. Denote

$$\Theta(z; q) := (1-z) \prod_{n=1}^{\infty} \frac{1 - q^n (1 - zq^n) (1 - z^{-1} q^n)}{1 - q^n} = \sum_{n=\infty}^\infty (-1)^n z^n q^{n(n-1)/2}$$

(this is the Jacobi triple identity, see, for instance, [1]). Obviously,

$$\Theta(qz; q) = -z^{-1} \Theta(z; q)$$

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Iterating this identity, we obtain
\[ \Theta(q^n z; q) = (-z)^{-n} q^{-n(n-1)/2} \Theta(z; q) \] (2)

The function
\[ \eta(z) = \exp \left\{ -\frac{1}{2 \ln q} \ln^2 |z| + \frac{1}{2} \ln q \ln |z| \right\} \] (3)
satisfies the recurrence equation \( \eta(qz) = |z|^{-1} \eta(z) \). Hence \( |\Theta(z; q)| \) can be represented in the form
\[ |\Theta(z; q)| = \eta(z) \psi(z); \quad \text{where } \psi(qz) = \psi(z) \] (4)

Obviously, \[ \Theta'(1; q) = \left. \frac{d}{dx} \Theta(x; q) \right|_{x=1} = -\prod (1 - q^n)^3 \]

Differentiating (2) and substituting \( z = 1 \), we obtain
\[ \Theta'(q^n; q) = (-1)^{n+1} q^{-n(n-1)/2} \Theta'(1; q) \] (5)

3. Interpolation problem. Denote \( g(x) = \frac{1}{2\pi} f(x) e^{-x^2/4} \). Applying the Poisson summation formula to the function \( g(x) e^{-\tau mx} \), we obtain
\[ e^{m \tau x} \sum_{k=-\infty}^{\infty} \gamma_{m,k} e^{ikx} = \sum_{j=-\infty}^{\infty} g(x + 2\pi j) e^{-2\pi \tau mj} \]

Denote the right-hand side of this identity by \( A_m \). Consider the function
\[ G_x(z) := \sum_{j=-\infty}^{\infty} g(x + 2\pi j) z^j \]
defined in the domain \( \mathbb{C} \setminus 0 \),
\[ G_x(q^m) = A_m \]

We obtain an interpolation problem for holomorphic functions, and solve it in a standard way (see [5]).

Denote
\[ \tilde{G}_x(z) = \sum_{n=-\infty}^{\infty} A_n \frac{\Theta(z; q)}{(z - q^n) \Theta(q^n; q)} = \sum_{n=-\infty}^{\infty} A_n \frac{(-1)^{n+1} q^{n(n-1)/2} \Theta(z; q)}{\prod (1 - q^j)^3 (z - q^n)} \] (6)

Obviously,
\[ G_x(q^n) = \tilde{G}_x(q^n) \] (7)

Hence,
\[ G_x(z) = \tilde{G}_x(z) + \Theta(z; q) \alpha(z) \] (8)
for certain function \( \alpha(z) \) holomorphic in \( \mathbb{C} \setminus 0 \).
LEMMA. $G_x(z) = \tilde{G}_x(z)$, i.e., $\alpha(z) = 0$.

Our final formula is a corollary of this lemma. Indeed, $g(x)$ is the Laurent coefficient of $G_x(z)$ in $z^0$; it remains to evaluate the Laurent expansion of

$$(z - q^n)^{-1}\Theta(z; q) = (z - q^n)^{-1} \sum_{l=-\infty}^{\infty} (-1)^l z^l q^{((l-1)/2)}$$

Assuming $|z| > q^n$, we obtain

$$(z^{-1} + z^{-2} q^n + z^{-3} q^{2n} + \ldots) \cdot \sum_{l=-\infty}^{\infty} (-1)^l z^l q^{((l-1)/2)}$$

and we obtain (11) as a coefficient in the front of $z^0$.

4. Proof of Lemma. We represent the identity (8) in the form

$$G_x(z)/\Theta(z; q) = \tilde{G}_x(z)/\Theta(z; q) + \alpha(z) \quad (9)$$

For a function $\Phi(z)$ we denote

$$M_k[\Phi] := \max_{|z| = q^{k+1}/2} |\Phi(z)|$$

We intend to analyze the behavior of these maxima for summands of (9) as $k \to \pm \infty$.

A) First,

$$\infty > \int_\mathbb{R} |f(x)|^2 dx \geq \int_0^{2\pi} \left( \sum_{j=-\infty}^{\infty} |f(x + 2\pi j)|^2 \right) dx$$

Hence (by the Fubini theorem) the value

$$V_x := \sum_{j=-\infty}^{\infty} |f(x + 2\pi j)|^2$$

is finite for almost all $x$.

B) By the Schwartz inequality,

$$|G_x(z)| = \left| \sum f(x + 2\pi j) e^{-(x+2\pi j)^2/4z^j} \right| \leq \left( \sum |f(x + 2\pi j)|^2 \right)^{1/2} \left( \sum e^{-(x+2\pi j)^2/2|z|^2} \right)^{1/2} = V_x^{1/2} \left[ e^{-x^2 \Theta(-|z|^2 e^{-2\pi x e^{-2\pi^2 z^2}; e^{-4\pi^2}})} \right]^{1/2}$$

Applying (3)-(4), we obtain for $|G_x(z)|$ an upper estimate of the form

$$|G_x(z)| \leq \exp \left\{ \frac{1}{4\pi^2} \ln^2 |z| + O(\ln|z|) + O(1) \right\} \quad (10)$$
In particular,

\[ |A_m| = |G_x(q^m)| \leq \exp\left( \frac{\ln^2 q}{4\pi^2} m^2 + O(m) + O(1) \right) \]

By (5),

\[ \Theta'(q^m; q) = \exp\left\{ -m^2 \ln q/2 + O(m) + O(1) \right\} \]

Since \((-\ln q) = 2\pi \tau < 2\pi^2\), we obtain the following estimate

\[ |A_m/\Theta'(q^m; q)| \leq \exp\left\{ -\varepsilon m^2 \right\} \]

C) Consider the summand \( \tilde{G}_x(z)/\Theta(z; q) \) in (9)

\[ M_k \left[ \tilde{G}_x(z)/\Theta(z; q) \right] = \sum_m A_m \Theta(q^m; q) \cdot \frac{1}{z - q^m} \leq \sum_m \frac{e^{-\varepsilon m^2}}{|q^{k+1/2} - q^m|} \]

Next,

\[ |q^{k+1/2} - q^m| = q^m |1 - q^{-m+k+1/2}| \geq q^m (1 - q^{1/2}) \]

This implies the boundedness of the sequence \( M_k[\cdot] \).

Secondly,

\[ |q^{k+1/2} - q^m| \geq q^{k+1} (1 - q^{1/2}) \]

Hence, \( M_k[\cdot] \) tends to 0 as \( k \to -\infty \).

D) By (3), (10)

\[ M_k \left[ \Theta(z)^{-1} \right] \sim \eta(z)^{-1} \bigg|_{z = q^{k+1/2}} \]

By (5), (14)

\[ M_k \left[ G_x(z)/\Theta(z; q) \right] \to 0 \quad \text{as } k \to \pm \infty \]

E) We have

\[ M_k [\alpha(z)] \leq M_k \left[ G_x(z)/\Theta(z; q) \right] + M_k \left[ \tilde{G}_x(z)/\Theta(z; q) \right] \]

Thus \( M_k[\alpha(z)] \) tends to 0 as \( k \to -\infty \); and remains bounded as \( k \to +\infty \). Since \( \alpha(z) \) is holomorphic in \( \mathbb{C} \setminus 0 \), we have \( \alpha(z) = 0 \).

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