All-order Finiteness of the Higgs Boson Mass in the Dynamical Gauge-Higgs Unification

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Abstract

In the dynamical gauge-Higgs unification, it is shown that the mass of the Higgs boson (4D scalar field) in $U(1)$ gauge theory in $M^4 \times T^n$ ($n = 1, 2, 3, \ldots$) is finite to all order in perturbation theory as a consequence of the large gauge invariance. It is conjectured that the Higgs boson mass is finite in non-Abelian gauge theory in $M^4 \times S^1$, $M^4 \times (S^1/Z_2)$ and the Randall-Sundrum warped spacetime to all order in the rearranged perturbation theory where the large gauge invariance is maintained.

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In the standard model of electroweak interactions the Higgs scalar field plays a crucial role in inducing the electroweak symmetry breaking, giving finite masses to the weak bosons, quarks and leptons. Fundamental scalar fields are necessary in the minimal standard supersymmetric model (MSSM) and grand unified theories (GUT) as well. Interactions associated with the Higgs fields, however, are largely unconstrained.

The Higgs boson in the electroweak interactions is expected to be discovered at LHC (Large Hadron Collider) in the near future. We are facing the time when the Higgs sector in elementary particles is disclosed where the key structure in the symmetry breaking hides. On the theoretical side the interaction of the Higgs field poses a challenging problem concerning the stability of the Higgs boson mass against radiative corrections. It has been often argued that the Higgs boson mass suffers from quadratically divergent radiative corrections unless protected by symmetry. Supersymmetry provides desired protection, providing a leading candidate for physics beyond the standard model. MSSM, in particular, predicts a light Higgs boson with a mass $m_H < 130$ GeV. The experimental lower bound from the direct search for the Higgs boson is 114 GeV.\footnote{1}

Recently an alternative scenario has attracted attention from the viewpoint of unification and stabilization of the Higgs field. The Higgs field in four dimensions is unified with gauge fields within the framework of higher dimensional gauge theory. Low energy modes of extra-dimensional components of gauge potentials serve as 4D Higgs scalar fields. Many years ago Fairlie and Manton proposed such unification scheme in six dimensions compactified on $S^2$ by ad hoc symmetry ansatz.\footnote{2, 3} Justification of the ansatz by quantum dynamics was also discussed.\footnote{4} A few years later it was found that more natural scheme justified by dynamics is provided when the extra-dimensional space is non-simply connected.\footnote{5, 6} There appear Yang-Mills Aharonov-Bohm phases, $\theta_H$, associated with the gauge field holonomy, or the phases of Wilson line integrals along noncontractible loops. Classical vacua are degenerate with respect to values of $\theta_H$. The degeneracy is lifted by quantum effects, thus the quantum vacuum being dynamically determined by the location of the global minimum of the effective potential $V_{\text{eff}}(\theta_H)$. In non-Abelian gauge theory the gauge symmetry can be dynamically broken, depending on the value of $\theta_H$. This is called the Hosotani mechanism. Fluctuations of the Yang-Mills AB phases $\theta_H$ in four dimensions correspond to 4D Higgs fields. Higgs fields are unified with gauge field and the gauge symmetry is dynamically broken. The scheme is called the dynamical gauge-Higgs unification.
It was shown that the $\theta_H$-dependent part of the effective potential $V_{\text{eff}}(\theta_H)$ is finite at the one-loop level, irrespective of a regularization method employed. This is highly nontrivial, given the fact that gauge theory in higher dimensions is non-renormalizable. As the curvature of the effective potential at the global minimum is related to the mass of the 4D Higgs boson, $m_H$, it implies that a finite $m_H$ is generated radiatively with its value independent of the cutoff scale in non-renormalizable theory. Although higher dimensional, non-renormalizable gauge theory is employed, predictions obtained there have sound meaning which does not depend on the details of dynamics at the cutoff scale.

The dynamical gauge-Higgs unification has been applied to both GUT and electroweak interactions. In particular, the dynamical gauge-Higgs unification of electroweak interactions in the Randall-Sundrum warped spacetime gives many interesting predictions in the Higgs field and gauge field phenomenology. The Higgs mass is predicted in the range between 140 GeV and 280 GeV, and the Kaluza-Klein mass scale is predicted in the range 1.5 TeV and 3.5 TeV. The universality of the weak gauge interactions is slightly broken, and the Yukawa couplings are significantly suppressed. These features can be measured at LHC and the future linear collider.

It becomes an important issue whether or not the mass of the 4D Higgs boson remains finite against higher order radiative corrections. If it does, the dynamical gauge-Higgs unification scenario gives robust predictions concerning the Higgs and gauge field phenomenology, justifying the use of non-renormalizable gauge theory in constructing a unified theory. It would give an alternative to supersymmetric theories to describe physics beyond the standard model. An important step in this direction has been taken by the author a few years ago in outlining a proof for the all-order finiteness of the effective potential $V_{\text{eff}}(\theta_H)$. As we will see below, the argument in ref. is valid in QED in arbitrary dimensions, whereas the case of non-Abelian gauge theory requires further elaboration.

Recently Maru and Yamashita performed detailed two-loop computations of the 4D Higgs boson mass in QED on $M^4 \times S^1$. Their result supports the all-order result for the effective potential in ref. In this paper we give a proof for the all-order finiteness of the effective potential $V_{\text{eff}}(\theta_H)$ in QED in $M^4 \times T^n$, where $T^n$ is an n-torus. We argue that the finiteness of the Higgs boson mass remains valid in non-Abelian gauge theory on $M^4 \times S^1$, $M^4 \times (S^1/Z_2)$, and the Randall-Sundrum (RS) warped spacetime as well. The most crucial ingredient in the proof is the large gauge invariance associated with $\theta_H$. 

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1. Yang-Mills AB phases $\theta_H$

In gauge theory defined on a non-simply connected space, a configuration of vanishing field strengths $F_{MN} = 0$ does not necessarily imply trivial. Consider an $SU(N)$ gauge theory on $\mathbb{M}^4 \times \mathbb{S}^1$ with coordinates $(x^\mu, y)$. Boundary conditions are given by

$$A_M(x, y + 2\pi R) = U A_M(x, y) U^\dagger,$$
$$\psi(x, y + 2\pi R) = e^{i\beta} T[U] \psi(x, y),$$

(1)

where $U \in SU(N)$. $T[U] \psi = U \psi$ or $U \psi U^\dagger$ for $\psi$ in the fundamental or adjoint representation, respectively. The boundary condition (1) guarantees that the physics is the same at $(x, y)$ and $(x, y + 2\pi R)$.

Under a gauge transformation

$$A'_M = \Omega A_M \Omega^\dagger - \frac{i}{g} \Omega \partial_M \Omega^\dagger,$$

(2)

$A'_M$ obeys a new set of boundary conditions

$$A'_M(x, y + 2\pi R) = U' A'_M(x, y) U'^\dagger,$$
$$U' = \Omega(x, y + 2\pi R) U \Omega(x, y)^\dagger,$$

(3)

provided $\partial_M U' = 0$. Gauge transformations $\Omega(x, y)$ which preserve the boundary conditions (so that $U' = U$) represent the residual gauge invariance. A set of eigenvalues $\{e^{i\theta_1}, \ldots, e^{i\theta_N}\}$ $(\sum_{j=1}^N \theta_j = 0)$ of $P \exp \{ig \int_0^{2\pi R} dy A_y\} \cdot U$ is invariant under residual gauge transformations. These phases $\theta_j$’s are Yang-Mills AB phases (holonomy phases) associated with a non-contractible loop in non-simply connected space which are collectively denoted as $\theta_H$. On flat space, constant configurations $A_y$ give nontrivial $\theta_H$. Although $\theta_H$ gives vanishing field strengths at the classical level, it affects physics at the quantum level.

Boundary conditions on an orbifold $\mathbb{M}^4 \times (\mathbb{S}^1/\mathbb{Z}_2)$ are given by

$$\begin{pmatrix} A^\mu \\ A_y \end{pmatrix}(x, z_j - y) = P_j \begin{pmatrix} A^\mu \\ -A_y \end{pmatrix}(x, z_j + y) P_j^\dagger,$$
$$\psi(x, z_j - y) = \pm T[P_j] \gamma^5 \psi(x, z_j + y).$$

(4)

Here $(z_0, z_1) = (0, \pi R)$, $P_j \in SU(N)$ and $P_j^2 = 1$. It follows that $A_M(x + 2\pi R) = U A_M(x, y) U^\dagger$ where $U = P_1 P_0$. The residual gauge invariance is given by $\Omega(x, y)$ satisfying

$$P_j = \Omega(x, z_j - y) P_j \Omega(x, z_j + y)^\dagger,$$
\[ U = \Omega(x, y + 2\pi R) U \Omega(x, y) \dagger. \] (5)

As \( A_y \) has an opposite parity to \( A_\mu \), there may or may not exist \( \theta_H \), depending on \( P_j \). We are interested in cases where \( \theta_H \) exists so that its fluctuation mode is identified as a 4D Higgs field. For instance, in the \( SU(3) \) model with \( P_j = \text{diag} (-1, -1, 1) \), the constant part of \( (A_{13}^y, A_{23}^y) \) forms \( \theta_H \). Its four-dimensional fluctuations correspond to the \( SU(2)_L \) doublet Higgs field. In the \( SO(5) \times U(1)_{B-L} \) model with \( P_j = \text{diag} (-1, -1, -1, -1, 1) \), \( A_{5j}^y (j = 1, \cdots, 4) \) form the \( SU(2)_L \) doublet Higgs field.

The Randall-Sundrum (RS) warped spacetime is given by

\[ ds^2 = e^{-\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \] (6)

where \( \eta_{\mu\nu} = \text{diag} (-1, -1, -1, 1) \), \( \sigma(y) = \sigma(y + 2\pi R) \) and \( \sigma(y) = k|y| \) for \( |y| \leq \pi R \). In the \( k \to 0 \) limit the RS spacetime becomes \( M^4 \times (S^1/Z_2) \). Gauge fields in the RS spacetime obey the same boundary conditions as in (4). In the \( SU(3) \) model with \( P_j = \text{diag} (-1, -1, 1) \), for instance, the zero mode of \( A_y \) has non-trivial \( y \)-dependence. It is related to \( \theta_H \) by

\[ gA_y = \frac{ke^{2ky}}{e^{2\pi kR} - 1} \theta_H \lambda^7 \equiv gA_y^c, \quad \lambda^7 = \begin{pmatrix} -i \\ i \end{pmatrix} \] (7)

In the following discussions we concentrate on one particular component of \( A_y \) and \( \theta_H \) as in (7), which corresponds to the neutral Higgs field in four dimensions. The argument can be generalized to theories with multiple Higgs fields, which arise, for instance, in theories on \( M^4 \times T^n \). In such cases one needs to consider a set of mutually independent \( \theta_H \)'s so that \( F_{MN} = 0 \) even with nonvanishing \( \theta_H \)'s.

2. Large gauge invariance

A gauge transformation in the RS spacetime given by

\[ \Omega(y) = \exp \left( i n \pi \frac{e^{2ky} - 1}{e^{2\pi kR} - 1} \lambda^7 \right), \quad (n: \text{an integer}) \] (8)

preserves the boundary conditions (4) but shifts \( \theta_H \) by \( 2\pi n \);

\[ \theta_H \to \theta_H' = \theta_H + 2\pi n. \] (9)

The transformation is called a large gauge transformation. It follows from the large gauge invariance that physical quantities such as the mass \( m_H \) of the 4D Higgs boson are periodic
functions of $\theta_H$ with a period $2\pi$. In $M^4 \times (S^1/Z_2)$ a large gauge transformation is given by $\Omega = \exp \left( iny/R \cdot \lambda^7 \right)$, whereas in $U(1)$ theory in $M^4 \times S^1$ it is given by $\Omega = \exp \left( iny/R \right)$.

\section{Perturbation theory}

In developing perturbation theory in the path integral formalism, we separate $A_M$ into the classical part $A_M^c$ and the quantum part $A_M^q$;

$$A_M = A_M^c + A_M^q .$$

In the $SU(3)$ model in the RS spacetime, $A_M^c$ is defined in (7) and $A_M^c = 0$. The quantum part $A_M^q$ and fermion fields $\psi$ are integrated in the path integral. The background field gauge is specified with a gauge-fixing term $\text{Tr} f_{g.f.}(A_M)^2$ where

$$f_{g.f.} = e^{2\sigma} \eta^{\mu\nu} D^c_\mu A^q_\nu + e^{2\sigma} D^c_\mu \left( e^{-2\sigma} A^q_\mu \right) ,$$

$$D^c_M A^q_N \equiv \partial_M A^q_N + ig [A^c_M, A^q_N] .$$

$\sigma = 0$ in the flat space. The quadratic part of the effective Lagrangian which includes the gauge-fixing term and associated ghost term is simplified in the background field gauge.

Under a large gauge transformation (8), $A_M^c(\theta_H)$ is transformed to $A_M^c(\theta_H) = A_M^c(\theta_H + 2\pi)$, while $A_M^q$ to $A_M^q = \Omega A_M^q \Omega^\dagger$. It follows that $f_{g.f.}(A_M^c) = \Omega f_{g.f.}(A_M) \Omega^\dagger$ so that the gauge-fixing term is invariant under the large gauge transformation. The invariance implies that the effective potential obtained in the new gauge $V_{\text{eff}}(\theta_H + 2\pi)$ is the same as that in the old gauge $V_{\text{eff}}(\theta_H)$;

$$V_{\text{eff}}(\theta_H + 2\pi) = V_{\text{eff}}(\theta_H) .$$

We stress that the periodicity in $\theta_H$ of $V_{\text{eff}}$ is a consequence of the large gauge invariance.

The perturbation theory is developed with respect to $A_M^q$ and $\psi$. The total Lagrangian is decomposed as

$$\mathcal{L} = \mathcal{L}^{(2)}(A_M^q, c, \bar{c}, \psi; \theta_H) + \mathcal{L}^{(3)}_a(A_M^q, c, \bar{c}, \psi; g)$$

$$+ \mathcal{L}^{(3)}_b(A_M^q; g\theta_H) + \mathcal{L}^{(4)}(A_M^q; g)$$

where $\mathcal{L}^{(2)}$ is bilinear in $A_M^q$, $\psi$, and ghost fields $c, \bar{c}$. $\mathcal{L}^{(2)}$ depends on $\theta_H$. $\mathcal{L}^{(3)}_a$ and $\mathcal{L}^{(4)}$ are cubic and quadratic interactions, respectively, which are present with vanishing $\theta_H$. 

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\[ \mathcal{L}_a^{(3)} = O(g) \] and \[ \mathcal{L}^{(4)} = O(g^2) \]. The additional cubic interaction \[ \mathcal{L}_b^{(3)} \] arises from the term \[ \frac{1}{2} g^2 \text{Tr} [A_M, A_N] [A^M, A^N] \]. With (17) it becomes

\[ \mathcal{L}_b^{(3)} = 2g\theta_H \frac{k e^{2ky}}{e^{2\pi k R} - 1} \text{Tr} [A_\mu, A_\nu] [A^{\mu}, A^{\nu}, \lambda^7] \, . \] (14)

It depends on \( g \) and \( \theta_H \) in the combination of \( g\theta_H \). In five dimensions the gauge coupling constant \( g \) has a mass dimension \(-\frac{1}{2}\); \( [g] = M^{-1/2} \). \( \mathcal{L}_a^{(3)} \) and \( \mathcal{L}^{(4)} \) give non-renormalizable interactions, while \( \mathcal{L}_b^{(3)} \) gives a super-renormalizable interaction.

4. Finiteness in Abelian gauge theory

With the property (12), \( V_{\text{eff}}(\theta_H) \) is expanded in a Fourier series;

\[ V_{\text{eff}}(\theta_H) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta_H} \, . \] (15)

We are going to show that \( V_{\text{eff}}(\theta_H) - a_0 \) is finite to all order in perturbation theory except at a discrete set of values of \( \theta_H \) in Abelian gauge theory, where \( \mathcal{L}_b^{(3)} \) and \( \mathcal{L}^{(4)} \) are absent. The argument given in ref. [30] remains intact in this case. Expand \( V_{\text{eff}}(\theta_H) \) in \( g \);

\[ V_{\text{eff}}(\theta_H) = \sum_{\ell=0}^{\infty} g^{2\ell} V_{\text{eff}}^{(2\ell)}(\theta_H) \, . \] (16)

To each order in \( g \) there are a finite number of bubble diagrams contributing to \( V_{\text{eff}}^{(2\ell)}(\theta_H) \) in \( U(1) \) theory. In each diagram \( \theta_H \) appears in fermion propagators \( S_F \). In flat space \( S_F \) behaves as \( [p_\mu, \gamma^\mu + (n - a - \theta_H/2\pi) R^{-1} \gamma^5 - m_F]^{-1} \) where \( a \) is a constant. The periodicity in \( \theta_H \) is recovered after summing over internal momentum indices \( n \) in the fifth dimension. Here the translational invariance on \( S^1 \) is important. One can expand \( V_{\text{eff}}^{(2\ell)} \) in a Fourier series.

\[ V_{\text{eff}}^{(2\ell)}(\theta_H) = \sum_{n=-\infty}^{\infty} a_n^{(2\ell)} e^{in\theta_H} \, . \] (17)

We proceed to show that \( V_{\text{eff}}^{(2\ell)}(\theta_H) - a_0^{(2\ell)} \) is finite.

Each diagram may be UV (ultraviolet)-divergent. Now differentiate \( V_{\text{eff}}^{(2\ell)} \) with respect to \( \theta_H \) sufficiently many times, say, \( q \) times. The divergence degree is lowered by \( q \). Since there are only a finite number of diagrams in \( V_{\text{eff}}^{(2\ell)} \), \( d^q V_{\text{eff}}^{(2\ell)}/d\theta_H^q \) becomes finite for sufficiently large \( q \). Hence \( n^q a_n^{(2\ell)} \) becomes finite. It follows that \( V_{\text{eff}}^{(2\ell)} - a_0^{(2\ell)} \) is finite.

As shown in ref. [30], the argument can break down at a discrete set of values of \( \theta_H \) when the fermion mass \( m_F \) vanishes. Differentiation with respect to \( \theta_H \) decreases the degree of
UV divergence, but increases the degree of IR (infrared) divergence when \( m_F = 0 \) and \( a + (\theta_H/2\pi) = \) an integer. In even (odd) dimensions it induces a singularity of the type of \( \delta_{2\pi}(\theta_H) (\ln \sin \theta_H) \) or its derivatives. We note that the argument remains valid in arbitrary dimensions, in which case there are a multiple number of \( \theta_H \)'s. Thus we have proven a theorem.

**Theorem** In \( U(1) \) gauge theory defined in \( M^4 \times T^n \) (\( n = 1, 2, 3, \cdots \)) the \( \theta_H \)-dependent part of \( V_{\text{eff}}(\theta_H) \) is finite, except at a discrete set of values of \( \theta_H \), in each order in perturbation theory.

5. Finiteness in non-Abelian gauge theory

We proceed to the non-Abelian case, limiting ourselves to gauge theory in five dimensions. In non-Abelian gauge theory propagators of fermions, gauge fields, and ghost fields depend on \( \theta_H \). Further \( \mathcal{L}_b^{(3)} \) and \( \mathcal{L}^{(4)} \) are present, the former of which explicitly depends on \( \theta_H \). The presence of \( \mathcal{L}_b^{(3)} \) complicates the argument for the finiteness of the \( \theta_H \)-dependent part of \( V_{\text{eff}}(\theta_H) \). Naively one might expand \( V_{\text{eff}}(\theta_H) \) in a power series of \( g \) as in (16). If the expansion made sense, the periodicity of \( V_{\text{eff}}^{(2\ell)}(\theta_H) \) would result and the argument presented in the Abelian case would apply. However this cannot be true as indicated by the following observation. Consider \( O(g^2) \) corrections in \( M^4 \times S^1 \). Among them there is a two-loop diagram generated by two vertices of \( \mathcal{L}_b^{(3)} \), which is proportional to \((g\theta_H)^2\). Gauge field propagators also depend on \( \theta_H \). After loop integrals and sums, it gives a contribution of the form \((g\theta_H)^2 h(\theta_H)\) where \( h(\theta_H) \) is periodic. It follows that \( V_{\text{eff}}^{(2)}(\theta_H) \) would not be periodic. This implies that \( V_{\text{eff}}(\theta_H, g) \) is singular at \( g = 0 \) in non-Abelian gauge theory. The expansion (16) would not be valid.

To distinguish contributions from \( \mathcal{L}_b^{(3)} \) and from \( \mathcal{L}_a^{(3)} \) and \( \mathcal{L}^{(4)} \), we denote the coupling constant in the former by \( \hat{g} \); \( \mathcal{L}_b^{(3)}(A_M; \hat{g} \theta_H) \). Let us develop perturbation theory in both parameters \( g \) and \( \hat{g} \), expanding \( V_{\text{eff}}(\theta_H) \) as

\[
V_{\text{eff}}(\theta_H) = \sum_{\ell=0}^{\infty} g^{2\ell} \left\{ V_a^{(2\ell)}(\theta_H) + V_b^{(2\ell)}(\theta_H, \hat{g} \theta_H) \bigg|_{\hat{g}=g} \right\}. \tag{18}
\]

\( V_a^{(2\ell)}(\theta_H) \) contains a finite number of diagrams generated by the vertices \( \mathcal{L}_a^{(3)} \) and \( \mathcal{L}^{(4)} \), whereas \( V_b^{(2\ell)}(\theta_H, \hat{g} \theta_H) \) is a sum of infinitely many diagrams which are \( O(g^{2\ell}) \) and nonvanishing powers of \( \hat{g} \) from \( \mathcal{L}_b^{(3)} \). \( V_b^{(2\ell)} \) depends on \( \theta_H \) through propagators and \( \mathcal{L}_b^{(3)} \). To have the periodicity in \( \theta_H \), it seems necessary to have contributions of all order in \( \hat{g} \) in \( V_b^{(2\ell)} \). However, it is not clear whether or not \( V_a^{(2\ell)} + V_b^{(2\ell)} \), for instance, is periodic in \( \theta_H \).
There exists special circumstance in five dimensions. \( \mathcal{L}_b^{(3)} \) gives super-renormalizable cubic interactions. The expansion parameter in \( M^4 \times S^1 \) or \( M^4 \times (S^1/Z_2) \) is \( \hat{g} \theta_H / R \), whereas \( \hat{g} \theta_H k \) in the RS warped spacetime. The expansion parameter has mass dimension \( +\frac{1}{2} \). This implies that the number of UV divergent diagrams in \( V_b^{(2\ell)} \) is finite. Each divergent diagram contains a power of \( g \theta_H \) from vertices and a product of propagators. The ultraviolet behavior of propagators is the same both in flat and RS spacetime. Hence \( dq V_b^{(2\ell)}/d\theta^4_H \) becomes UV-finite by taking sufficiently large \( q \).

To prove the finiteness by generalizing the argument presented in the \( U(1) \) case, it is necessary to express \( V_{\text{eff}}(\theta_H) \) as a sum of gauge-invariant subsets of diagrams such that each subset is periodic in \( \theta_H \). It is likely that a whole set of diagrams in \( V_b^{(2\ell)} \) is contained in one of those subsets and that each subset contains only a finite number of UV-divergent diagrams. The argument breaks down in six or higher dimensions, where \( \mathcal{L}_b^{(3)} \) becomes marginal or non-renormalizable.

Thus we conjecture the following.

[Conjecture] In non-Abelian gauge theory defined in \( M^4 \times S^1 \), \( M^4 \times (S^1/Z_2) \), and the Randall-Sundrum warped spacetime, the \( \theta_H \)-dependent part of \( V_{\text{eff}}(\theta_H) \) is finite in each gauge invariant subset of diagrams, except at a discrete set of values of \( \theta_H \).

6. Higgs boson mass

The Higgs boson corresponds to four-dimensional fluctuations of \( \theta_H \). Its mass \( m_H \) is related to the curvature of the four-dimensional \( V_{\text{eff}}(\theta_H) \) at the minimum.\cite{14, 19, 20, 28}

For instance, in the \( SU(3) \) model in the Randal-Sundrum warped spacetime

\[
m_H^2 = \frac{\pi g^2 R (e^{2\pi k R} - 1)}{2k} \frac{d^2 V_{\text{eff}}}{d\theta_H^2}.
\]

In \( U(1) \) gauge theory in \( M^4 \times T^n \) a similar formula is obtained for the mass of 4D scalar fields arising from zero modes of extra-dimensional components of gauge potentials; \( m_H^2 \sim g^2 R^2 (d^2 V_{\text{eff}}/d\theta_H^2) \). We note that \( V_{\text{eff}}(\theta_H) = M_{KK}^4 f(\theta_H) \) where \( M_{KK} \) is the Kaluza-Klein mass scale and \( f(\theta_H) \) is dimensionless.

Although higher derivatives of \( V_{\text{eff}}(\theta_H) \) can be afflicted with infrared divergence at a discrete set of values of \( \theta_H \) as explained above, the global minimum in all non-Abelian models investigated so far is located at a regular point when the symmetry is dynamically broken. In \( U(1) \) theory with periodic (anti-periodic) fermions the global minimum occurs at \( \theta_H = \pi \) (0), in either case of which \( V_{\text{eff}}(\theta_H) \) is regular at the minimum.
Thus the finiteness of the $\theta_H$-dependent part of $V_{\text{eff}}(\theta_H)$ implies the finiteness of
the Higgs boson mass. One concludes that in $U(1)$ gauge theory defined on $M^4 \times T^n$
($n = 1, 2, 3, \cdots$) the mass of the Higgs boson (4D scalar fields) is finite in each order in
perturbation theory. Radiative corrections are finite, being independent of the cutoff scale.
Recent two-loop analysis by Maru and Yamashita \cite{33} supports the result in the present
paper and the earlier argument in ref. \cite{30}. The large gauge invariance plays a crucial role
in the proof.

In non-Abelian gauge theory a proof for the finiteness of $V_{\text{eff}}(\theta_H)$ is incomplete. Based
on the argument leading to the conjecture stated above, we expect that the Higgs boson
mass in non-Abelian gauge theory in $M^4 \times S^1$, $M^4 \times (S^1/Z_2)$ and the Randall-Sundrum
warped spacetime is finite in each order in the rearranged perturbation theory where the
large gauge invariance is maintained. Although gauge theory in higher dimensions is non-
renormalizable in perturbation theory, the Higgs mass evaluated in the dynamical gauge-
Higgs unification has well-defined meaning free from the cutoff scale in the theory. We note
that it has been argued that non-Abelian gauge theory in five dimensions can be defined in
non-perturbative renormalization group approach \cite{31} and in the lattice formulation \cite{32}.
We shall come back to a more detailed analysis of the non-Abelian case separately.

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