Remarks on Black Hole Degrees of Freedom in String Theory

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Abstract

The Bekenstein-Hawking black hole area entropy law suggests that the quantum degrees of freedom of black holes may be realized as projections of quantum states unto the event horizon of the black hole. In this paper, we provide further evidence for this interpretation in the context of string theory. In particular, we argue that increase in the quantum entropy due to the capture of infalling fundamental strings appears in the form of horizon degrees of freedom.

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1. Introduction

The Bekenstein-Hawking area entropy law \[1\] implies that the quantum degrees of freedom of black holes may be realized as projections unto the event horizon of the black hole. While these degrees of freedom probably ultimately reside in the interior of the black hole, the holographic principle \[2\] suggests a natural realization of these degrees of freedom as quantum-gravitational states on the surface.

String - black hole correspondence \[3\] also has profound implications for the nature of the underlying quantum degrees of freedom of black holes. In particular, the fundamental states underlying black hole entropy, and ultimately quantum gravity from string theory, arise from string states, even in the strong coupling limit. While the precise nature of these states in the black hole limit is far from understood, the fact that not only the degeneracy but also the precise combinatorics underlying the counting of constituents are identical in both the quantum string states picture and the semiclassical black hole entropy calculation implies that there is perhaps far more to string-black hole correspondence than what one might reasonably have expected.

In recent work \[4\], the implications of correspondence were considered in two different physical pictures for black hole formation from strings. The first case concerns the transition, via an adiabatic increase in the string coupling, from the random walk picture of free strings \[5,6\] to a Schwarzschild black hole at the critical coupling, \(g_c\) \[7\]. The main finding was that the initial, random walk degrees of freedom transform into holographic, horizon degrees of freedom. In particular, in both pictures, the entropy of the system is given, up to a factor of order unity, by an integer representing the number of random walk steps (or number of string “bits” in a polymer string picture), which is also the number of area “pixels” on the corresponding black hole horizon. This latter configuration then represents a degeneracy of quantum gravitational states projected unto the horizon.

In the second case, string - black hole correspondence was applied to the constituent picture for Reissner-Nordstrom (RN) black hole solutions in string theory. Here it was
pointed out that the actual combinatorics of counting either string or black hole constituents was identical in both pictures. This “constituent correspondence”, in combination with the random walk correspondence, leads to a particularly simple interpretation of entropy enhancement in black hole dynamics in correspondence with analogous processes for string BPS states.

In this paper, we present two calculations in support of correspondence that show that the underlying degrees of freedom behind the Bekenstein/Hawking area entropy law \cite{1} are stringlike. In the first case, we consider a canonical ensemble of infalling point-like states in a Schwarzschild black hole background. Using a mean field theory approximation, we show that the increase in the entropy is accounted for by the number of steps in the stringy random walk picture. In the second case, we consider the extremal Reissner-Nordstrom black hole in the low-velocity approximation. The increase in entropy is again related in the quantum picture to string degrees of freedom, again in agreement with the black hole area entropy law.

2. Fundamental String Ensemble in Schwarzschild Background

It was argued in \cite{7,8} that the entropy of a Schwarzschild black hole in string theory is proportional to \( n = \sqrt{N_s} \), where \( N_s \) is the level number of a long-excited fundamental string which collapses into the Schwarzschild black hole at the critical value of the string coupling \( g_c \sim N_s^{-1/4} = n^{-1/2} \). The number \( n \) corresponds to the number of steps (or string “bits” \cite{3} in the random walk \cite{5} description of the string at zero coupling. Each step has length \( l_s \), the string scale, with each string “bit” having mass \( m = m_s = 1/l_s \). The idea is that if we start with this string at zero coupling and adiabatically increase its string coupling, then at \( g_c \), the string makes a transition into a black hole, such that the entropies of the string and black hole match up to a factor of order unity. In the black hole picture, the same number \( n \) represents the number of area “pixels” in the event horizon. In this view, the information in the quantum string states are projected out into the horizon.
Since the arguments behind this picture are rather general, it would be interesting to see whether this result can be supported by calculations of the entropy change in a black hole in string theory as the result of the capture of string states.

To this end, we consider in this section the capture of \( n \) fundamental “test” strings, each of mass \( m_s = 1/l_s \), by a black hole of mass \( M = Nm_s \), with \( 1 << n << N \). This black hole has the same mass as a long, excited fundamental string of level \( N^2 \) at zero coupling. This latter string is equivalent to a random walk with \( N \) steps each of length \( l_s \).

This ensemble of fundamental strings in \( D = 4 \) may be thought of as arising, for example, from an ensemble of positively and negatively electrically charged BPS string states in \( D = 5 \) with total charge zero. If this latter ensemble collapses to a black hole, in \( D = 4 \) it will appear neutrally charged, i.e. a Schwarzschild solution. This is because the effect of the antisymmetric tensor, \( B_{MN} \) and any gauge fields arising from it due to compactification, are averaged to zero. A similar analysis to that below was done in [9]. The difference between the two calculations is that in [9], D0-branes of the same charge are considered, with the consequence of a zero-static force and even an \( O(v^2) \) zero force due to cancellation of long-range exchange forces. The leading order velocity-dependent forces are of order \( v^4 \), but the full Lagrangian is required to obtain the correct result. In our scenario, we consider a purely gravitational collapse, as the gauge forces average to zero in the mean-field limit. In any case, the calculation below for the Schwarzschild solution is of interest in its own right, whether or not it arises from a string compactification. Our viewpoint, however, is that the particle states, each of mass \( m \), arise as pointlike limits of string-like objects with length equal to a single string scale, \( l_s \), corresponding to a single string “bit” or single step in the random walk picture. The stringy character of the infalling states, whether point-like or string-like, is reflected in the random walk picture, with the corresponding number of extra “steps” acquired resulting from the capture of the same number of string bits.

In order to sensibly consider the strings as falling into a black hole, we must also
assume $g^2N \geq 1$, but $g^2n << 1$, which is clearly consistent with the above. Assume further that the coupling is such that the black hole background is not far removed from an adiabatic formation from a string state: $g > g_c(N)$, so that the black hole has formed, but $g$ is still of order $N^{-1/2}$.

The worldsheet action of the test string in a purely gravitational black hole background is given by

$$L_2 = -\frac{m}{2}\sqrt{-\gamma} = -\frac{m}{2}\sqrt{-\det \gamma_{ij}}, \quad (2.1)$$

where the worldsheet metric $\gamma_{ij} = \partial_i X^M \partial_j X^N g_{MN}$, where $g_{MN}$ is the background spacetime metric of the black hole. We may equivalently consider the Lagrangian $L_2 = -(m/2)(-\gamma)$, since the variation of the two actions leads to the the same equations of motion. In the low-velocity limit, this latter Lagrangian describes the dynamics of point particles with the same mass as the string bits. The equivalence of the string and particle dynamics makes sense provided the Schwarzschild radius $R_s >> \ell_s$, which follows from our assumption $n << N$. This assumption also allows us to neglect the bit-bit interactions.

We now wish to consider the capture of the $n$ particles by the black hole. For simplicity, assume the entire system is enclosed in a spherical volume $V = \frac{(4\pi/3)}{R_0^3}$. The canonical ensemble then describes the motion of the particles from $r = R_0$ to the Schwarzschild radius $R_s = 2GM$, where we also assume that $R_s << R_0$ and where $G$ is the four-dimensional Newton’s constant. Since we would like to mimic an adiabatic process, in which the total entropy does not undergo a violent change in the process of the capture, we restrict ourselves to the low-velocity limit.

The results of [4] imply that the black hole entropy, whether seen in the string limit or in the black hole limit, is given by the total number of random walk steps. So the increase in the entropy of the black hole as a result of the capture of the $n$ particles is simply $\Delta S \sim n$. This result should be reproduced from the canonical ensemble calculation of the entropy of the particles in the black hole background, at least to within a factor of
the order unity.

The Lagrangian for a single particle of mass $m$ in the Schwarzschild background in the low-velocity limit is given by

$$\mathcal{L} = -\frac{m}{2} \left( \Omega - \Omega^{-1} \dot{r}^2 - r^2 \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) \right), \quad (2.2)$$

where $(r, \theta, \phi)$ are spherical coordinates in three dimensions and $\Omega = 1 - 2GM/r$, where $M$ is the mass of the black hole.

In terms of the conjugate momenta, $p_r = \partial \mathcal{L} / \partial \dot{r}$, $p_\theta = \partial \mathcal{L} / \partial \dot{\theta}$ and $p_\phi = \partial \mathcal{L} / \partial \dot{\phi}$, the Hamiltonian for this particle $H = \dot{r} p_r + \dot{\theta} p_\theta + \dot{\phi} p_\phi - \mathcal{L}$ is given by

$$H = \frac{m\Omega}{2} + \frac{\Omega p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2 \theta}. \quad (2.3)$$

The single particle partition function is then

$$W = \int d^3x \int d^3p \exp(-\beta H) = 4\pi \left( \frac{2\pi m}{\beta} \right)^{3/2} e^{-\frac{\beta m}{2}} \int_{2GM}^{R_0} dr r^2 e^{\frac{\beta GmM}{r}} \sqrt{\Omega}, \quad (2.4)$$

where $\beta = 1/T_H \sim GM$. Note that the domain of the configuration space is limited by the horizon of the already formed black hole. Note also that in our approximation, $\beta m \sim l_p^2 M m \sim g^2 l_s^2 M m \sim g^2 l_s M \sim g^2 N \sim 1$. The integral $I_1$ in the right hand side of (2.4) may be rewritten as

$$I_1 = 16(GM)^3 e^{-\frac{\beta m}{2}} \int_0^{u_0} e^{-\beta m u^2/2} \frac{du}{(1 - u^2)^4}, \quad (2.5)$$

where $u(r) = \sqrt{1 - 2GM/r}$ and $u_0 = u(R_0)$. The integral $I_2$ in the right hand side of (2.3) can be estimated as follows:

$$e^{-\beta m/2} I_3 < I_2 < I_3, \quad (2.6)$$

where $I_3 = \int_0^{u_0} \frac{du}{(1 - u^2)^4} \sim (R_0/R_s)^3$, for $R_s << R_0$ (see, e.g., (11)). It follows from $\beta m \sim 1$ that $I_2 \sim I_3$ and

$$W \sim \left( \frac{m}{\beta} \right)^{3/2} (GM)^3 \frac{V}{(GM)^3} = \left( \frac{m}{\beta} \right)^{3/2} V. \quad (2.7)$$
For \( n \) identical (but distinguishable) masses, the partition function is

\[
Z = W^n = \left( \frac{m}{\beta} \right)^{3n/2} V^n.
\] (2.8)

The energy of the \( n \) masses as they approach the horizon is then given by

\[
\delta E = -\frac{\partial \ln Z}{\partial \beta} = \frac{3n}{2\beta},
\] (2.9)

which is consistent with equipartition of degrees of freedom, each having energy \((1/2)k_B T\)
(where \( k_B \), Boltzmann’s constant, has been set to 1 throughout above). The entropy is given by

\[
\delta S = \beta E + \ln Z \sim n.
\] (2.10)

This is consistent with the Bekenstein-Hawking formula, in which

\[
\delta S_{BH} \sim GM\delta M \sim GMnm \sim n\beta m \sim n,
\] (2.11)

where we have used \( \delta E \sim \delta M \sim nm \sim n/\beta \).

So the increase of the black hole entropy can essentially be accounted for by the addition of the bit degrees of freedom, arising from a stringy, random-walk picture. This type of calculation can be repeated for string-like falling states, with the same result: the increase in the black hole’s entropy is essentially due to the number of bits acquired.

3. Charged Extremal Black Holes and Constituent Correspondence

Now we consider the implications of correspondence for extremal black holes. Here we make use of the “constituent correspondence” between string states and singly charged black hole constituents. For example, a generalization of the extremal Reissner-Nordstrom black hole solution in four dimensions has metric \([12]\)

\[
ds^2 = -\prod_{i=1}^{4} \left( 1 + \frac{Q_i}{r} \right)^{-1/2} dt^2 + \prod_{i=1}^{4} \left( 1 + \frac{Q_i}{r} \right)^{1/2} \left( dr^2 + r^2 d\Omega_2^2 \right).
\] (3.1)
Following the conventions of [13], this solution arises from a compactification of string theory in ten dimensions so that \( Q_1 = (4G_4 R_9 / \alpha') N_1 = N_1 q_{1,0}, \) \( Q_2 = (\alpha' / 2 R_4) N_2 = N_2 q_{2,0}, \) \( Q_3 = (4G_4 / R_9) N_3 = N_3 q_{3,0} \) and \( Q_4 = (R_4 / 2) N_4 = N_4 q_{4,0} \), where \( R_4 \) and \( R_9 \) are compactification radii for the fourth and ninth spatial dimensions in \( D = 10 \), respectively. The \( N_i \gg 1 \) represent eigenvalues of number operators in the string picture (assumed all to be, say, right movers), with the \( q_{i,0} \) representing unit charges for each of the four species, which therefore have mass \( m_{i,0} = q_{i,0} / l_p^2 \).

The area of the black hole is \( A = 4 \pi R^2 = 4 \pi \sqrt{Q_1 Q_2 Q_3 Q_4} \), leading to a black hole entropy \( S_{BH} = A / 4G_4 = 2 \pi \sqrt{N_1 N_2 N_3 N_4} = S_0 \). One of the major recent successes of string theory has been the recovery of this entropy as a quantum mechanical entropy due to the degeneracy of the charged BPS states corresponding to the \( N_i \) [14]. This entropy can also be interpreted as the number of “pixels” on the horizon of the black hole [4].

For nonextremal black holes, the entropy formula gets generalized to include number operators for oppositely charged states

\[
S = 2 \pi (\sqrt{N_1} + \sqrt{\bar{N}_1})(\sqrt{N_2} + \sqrt{\bar{N}_2})(\sqrt{N_3} + \sqrt{\bar{N}_3})(\sqrt{N_4} + \sqrt{\bar{N}_4}).
\]  

(3.2)

As in the above discussion of the Schwarzschild case, we wish to consider the capture of incoming quanta by the extremal black hole and attempt to interpret the corresponding increase in entropy in terms of the quantum degrees of freedom of the system. This necessarily turns the extremal black hole into a non-extremal one, even if the incoming charges are all of the right sign, if only because of the acquired kinetic energy, which results in a departure from the extremality condition on the total mass (or energy). For simplicity, consider the capture of \( n_1 \) constituents/quanta of species “1”, (i.e. each state has charge \( q_{1,0} \)), with \( 1 \ll n_1 \ll N_i \). In the limit of zero kinetic energy (e.g. an adiabatic approach of the quanta), the black hole simply acquires the extra \( n_1 \) charges and the extremal entropy formula is generalized by simply replacing \( N_1 \) with \( N_1 + n_1 \) so that \( S_1 = S_0 \sqrt{N_1 + n_1} / \sqrt{N_1} \approx S_0 (1 + n_1 / N_1) \), to leading order.
We wish to consider the case in which the kinetic energy of the incoming quanta, while nonzero, is small. In this case, assume an average KE of \((1/2)m_{1,0}\bar{v}^2\), with \(\bar{v} << 1\). Assume the black hole absorbs these quanta and settles down into a stationary state. By conservation of energy, the black hole eventually becomes non-extremal with \(N_1 + n_1 + \bar{n}_1\) right movers and \(\bar{n}_1\) left movers, such that \(\bar{n}_1 = n_1 \bar{v}^2/4 << n_1\). Here the presence of \(\bar{n}_1\) in both left and right movers is a result of the kinetic energy being transformed into pairs of right and left movers. The entropy is now given by

\[
S_2 = 2\pi(\sqrt{N_1 + n_1 + \bar{n}_1} + \sqrt{\bar{n}_1})(\sqrt{N_2N_3N_4}) = S_1 + 2\pi\sqrt{\bar{n}_1 N_2N_3N_4},
\]

to leading order. The last term represents the gain in entropy due to the kinetic energy of the quanta. We argue below that this result is consistent with considerations of correspondence and heuristic arguments on the quantum degrees of freedom underlying this entropy.

Unlike the Schwarzschild case, we are unable to use the canonical ensemble, simply because the temperature of the extremal black hole is zero. Nevertheless, it is possible from heuristic, semiclassical considerations to connect the above result to the possible quantum degrees of freedom of black holes.

The incoming constituents, if stationary, experience zero force due to the extremal black hole background. This is due to the cancellation of attractive gravitational and dilatational forces against repulsive gauge forces as the result of supersymmetry and the saturation of the BPS bound. When moving in the BH background in the low-velocity limit, the constituents experience velocity-dependent forces which, however, lead to velocity independent paths, governed by the metric on moduli space. In other words, in this limit, the path of the quasi-static solutions are an excellent approximation to the actual paths, being tangent to them. This is a crucial point in our analysis: the incoming constituents may arrive in different times, but essentially follow the same paths in the low-velocity limit. The total energy can then be divided amongst the different constituents via a simple partition function.
For the particular case we are considering, the most general metric on moduli space of this sort was computed in [13], based on the work of [15,16]. Arguments similar to those of [17] can also be made to confirm that, by calculating four-point amplitudes of corresponding string states, it can be shown that the interaction Lagrangian is velocity-dependent, with the same form as that of the metric-on-moduli calculation. This identity is simply the result of supersymmetry.

Without rederiving the results of [13], it is sufficient to note that for quanta of species “1” of mass $m_1$, the leading order term is a four-body interaction between the four species leading to the Lagrangian

$$\mathcal{L}_{int} = -\frac{m_1}{2} \left[ 1 - F(r) \left( \dot{r}^2 + r^2 \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) \right], \quad (3.4)$$

where $F(r) = 1 + \frac{C_{234}}{r^3}$ and where $C_{234} \sim Q_1 Q_2 Q_3$. In this scenario, the horizon is initially at $r = 0$, but after the capture expands to a radial position $\delta R$. It is easy to show that $\delta R \sim (Q_1 Q_2 Q_3 Q_4)^{1/4} \sqrt{n_1/N_1} \ll C_{234}^{1/3}$ provided we assume that the $N_i$ are of the same order and $n_1 \ll N_1$. So in the classical picture, the incoming quanta fall all the way to $r = 0$, but a back-reaction calculation should correct this to $r \sim \delta R$. In the absence of such a calculation, we assume, to leading order, that the black hole horizon remains at $r = 0$. So the classical trajectory for the particles in the black hole background must pass through $r = 0$. The quantum (or at least semiclassical) entropy increase generated by the particles arises from quantum fluctuations about the classical trajectories.

Since $\mathcal{L}_{int}$ is independent of $\theta$, angular momentum is conserved $(1 + C_{234}/r^3)r^2\dot{\theta} = \text{const.} = L/m$. The radial equation can then be written as

$$\ddot{r} F + \frac{r^2 F'}{2} = \frac{L^2}{m^2 r^2 F} \left( \frac{F'}{2F} + \frac{1}{r} \right). \quad (3.5)$$

This equation can also be obtained from conservation of total energy

$$E = \frac{m}{2} F(r)(\dot{r}^2 + \frac{L^2}{m^2 r^2 F^2}). \quad (3.6)$$
Note that from either (3.5) or (3.6), as \( r \to 0 \), \( \dot{r} \to 0 \), so that the capture does indeed represent a transformation of the initial kinetic energy of the particles into additional mass for the black hole. In the limit \( r \to 0 \), the radial equation of motion takes the form

\[
\ddot{r} - \frac{3\dot{r}^2}{2r} = -\left(\frac{L^2}{2m^2C_{23}^2}\right)r^3
\]

(3.7)

with solution \( r = \dot{r} = \ddot{r} = 0 \). Clearly, a more precise analysis would yield a solution with \( r = a \), with \( 0 < a << C_{23}^{1/3} \) for the above assumptions, as the horizon must expand with the added mass. Without performing such a calculation we can still gain some information on the quantum fluctuations about such a final position for the horizon from the above equations since, to leading order, they describe the dynamics of the incoming quanta. Furthermore, the horizon expansion is a far smaller distance than the length scale of the equations, so that departures from the correct equations are at most of order \( a \). So now we assume a solution to the corrected classical equations of the form \( r = a \), with \( \dot{r} = \ddot{r} = 0 \), let us examine the quantum fluctuations about this solution. Setting \( r = a + y \), with \( y << a \), and to leading order in \( y \), (3.7) yields

\[
\ddot{y} = -\frac{L^2 a^3}{2m^2C_{23}^2}(1 + \frac{3y}{a}).
\]

(3.8)

In terms of \( z = y + a/3 \), this equation has the harmonic oscillator form

\[
\ddot{z} = -\frac{3L^2 a^2}{2m^2C_{23}^2}z.
\]

(3.9)

Note that the shift to the \( z \)-variable show that the classical equations are not reliable to within the order \( a \). Nevertheless, the quantum fluctuations can be understood from (3.9). This is because the negative sign in the right hand side of (3.7), coming from (3.5), holds to higher length scales (or lower order) than \( a \), so that whatever the correct equations including back-reaction may be, the perturbations about the final horizon position (or capture position of the particles) will be oscillatory. In other words, we would expect that a full calculation including the back-reaction would not affect the quantum oscillations, and hence the quantum entropy. The significance of this will be discussed below.
Now consider the oscillating modes in (3.9). These contribute an energy \( \delta E = (n + 1/2)\hbar \omega \) at level \( n \), where \( \omega = \sqrt{3L^2a^2/2m^2C_{234}^2} \) is the angular frequency for the oscillations. Setting \( \hbar = 1 \) and for large numbers, \( \delta E \sim n\omega \sim nLa/mC_{234} \). However, \( \delta E = 2m_{1,0}\bar{n}_1 \). Using both expressions for \( \delta E \) leads to \( n \sim \bar{n}_1c_{234} \sim \bar{n}_1N_2N_3N_4 \). As noted above, the trajectories of the particles in the low-velocity limit are independent of the actual velocity, so that an energy level \( n \) can be achieved by any partition of the integer \( n \). So the degeneracy \( d(n) \) of the quantum states is given by the partition function \( P(n) \) of the integer \( n \). This implies that the entropy generated by these modes \( \delta S = \ln d(n) \sim \ln P(n) \sim \sqrt{n} \sim \sqrt{\bar{n}_1N_2N_3N_4} \), in agreement with (3.3) and the area law black hole entropy above. This argument can easily be generalized for the case of arbitrary incoming constituents from the four different species: \( n_1, n_2, n_3 \) and \( n_4 \). One again finds that the entropy increase \( \delta S \) matches the result expected from the area law. Finally, since the entropy dependence on the constituents matches in both pictures, one can interpret the area law entropy in terms of the quantum degrees of freedom available to the constituents in the process of black hole formation.

Interestingly, if we were to perform the analogous calculation with the absence of at least one or more of the four species of constituents, we would obtain equations with no oscillatory modes at all. For example, if in the above we set \( Q_4 = 0 \), then we would replace \( F(r) \) with \( G(r) = 1+C_{23}/r^2 \), where \( C_{23} \sim Q_2Q_3 \). Then the negative sign on the right hand side of the radial equation is replaced with a positive sign, with the consequence that the equation for quantum fluctuations yields only decaying modes, instead of oscillatory ones. This leads to no increase in the entropy, and ultimately to the result that such black holes possess zero entropy. This is entirely consistent with the Bekenstein-Hawking expression, since in this case the “horizon” has zero area and so the black hole entropy is zero.

4. Discussion

The above arguments are highly heuristic and require more precise calculations for
confirmation. Ultimately, a kind of “matching” between semiclassical results from general relativity and quantum degrees of freedom in string theory will lead us to a firmer understanding of quantum gravity in string theory. For example, a correct back-reaction calculation already requires an understanding of this matching, or equivalently, the phase transition in going from the string to the black hole picture.

The main intent of the above calculations is to further support the notion of string-black hole correspondence, namely that stringy degrees of freedom are the basis for the quantum properties of black holes. It would be interesting to see whether similar findings can be obtained for charged, non-extremal black holes, in which combinations of arguments of the previous two sections can be made. Another possibility is to investigate the case of rotating black holes and strings with angular momentum.
References

[1] J. Bekenstein, Lett. Nuov. Cimento 4 (1972) 737; Phys. Rev. D7 (1973) 2333; Phys. Rev. D9 (1974) 3292; S. W. Hawking, Nature 248 (1974) 30; Comm. Math. Phys. 43 (1975) 199.

[2] G. ’t Hooft, gr-qc/9310026; L. Susskind, L. Thorlacius and J. Uglum, Phys. Rev. D48 (1993) 3743.

[3] L. Susskind, hep-th/9309143.

[4] R. R. Khuri, Nucl. Phys. B588 (2000) 253.

[5] P. Salomonson and B. S. Skagerstam, Nucl. Phys. B268 (1986) 349; Physica A158 (1989) 499; D. Mitchell and N. Turok, Phys. Rev. Lett. 58 (1987) 1577; Nucl. Phys. B294 (1987) 1138.

[6] See C. B. Thorn, hep-th/9607204 and references therein; see also O. Bergman and C. B. Thorn, Nucl. Phys. B502 (1997) 309.

[7] G. T. Horowitz and J. Polchinski, Phys. Rev. D57 (1998) 2557.

[8] R. R. Khuri, Phys. Lett. B470 (1999) 73.

[9] H. Liu and A. A. Tseytlin, JHEP 9801 (1998) 010.

[10] See M. J. Duff, R. R. Khuri and J. X. Lu, Phys. Rep. B259 (1995) 213.

[11] I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series and Products, (1965) Academic Press (London).

[12] M. Cvetic and D. Youm, Phys. Rev. D53 (1996) 584; M. Cvetic and A. A. Tseytlin, Phys. Rev. D53 (1996) 5619.

[13] D. M. Kaplan and J. Michelson, Phys. Lett. B410 (1997) 125; J. Michelson, Phys. Rev. D57 (1998) 1092.

[14] A. Strominger and C. Vafa, Phys. Lett. B379 (1996) 99.

[15] R. C. Ferrell and D. M. Eardley, Phys. Rev. Lett. 59 (1987) 1617.

[16] K. Shiraishi, Nucl. Phys. B402 (1993) 399.

[17] R. R. Khuri and R. C. Myers, Phys. Rev. D52 (1995) 6988.