A Steenrod square on Khovanov homology

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Abstract

In a previous paper, we defined a space-level version $X_{Kh}(L)$ of Khovanov homology. This induces an action of the Steenrod algebra on Khovanov homology. In this paper, we describe the first interesting operation, $\text{Sq}^2: \text{Kh}^{i,j}(L) \to \text{Kh}^{i+2,j}(L)$. We compute this operation for all links up to 11 crossings; this, in turn, determines the stable homotopy type of $X_{Kh}(L)$ for all such links.

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1. Introduction

Khovanov homology, a categorification of the Jones polynomial, associates a bigraded abelian group $\text{Kh}^{i,j}(L)$ to each link $L \subset S^3$ (see [5]). In [8], we gave a space-level version of Khovanov homology. That is, to each link $L$ we associated stable spaces (finite suspension spectra) $X_{Kh}(L)$, well defined up to stable homotopy equivalence, so that the reduced cohomology $\tilde{H}^i(X_{Kh}(L))$ of these spaces is the Khovanov homology $\text{Kh}^{i,j}(L)$ of $L$. Another construction of such spaces has been given by Hu-Kriz-Kriz [4].

The space $X_{Kh}(L)$ gives algebraic structures on Khovanov homology which are not (yet) apparent from other perspectives. Specifically, while the cohomology of a spectrum does not have a cup product, it does carry stable cohomology operations. The bulk of this paper is devoted to giving an explicit description of the Steenrod square

$$\text{Sq}^2: \text{Kh}^{i,j}_{Z}(L) \to \text{Kh}^{i+2,j}_{Z}(L)$$

induced by the spectrum $X_{Kh}^{j}(L)$. First, we give a combinatorial definition of this operation $\text{Sq}^2$ in Section 2 and then prove that it agrees with the Steenrod square coming from $X_{Kh}(L)$ in Section 3.

The description is suitable for computer computation, and we have implemented it in Sage. The results for links with 11 or fewer crossings are given in Section 5. In particular, the operation $\text{Sq}^2$ is non-trivial for many links, such as the torus knot $T_{3,4}$. This implies a non-triviality result for the Khovanov space:

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Theorem 1.1. The Khovanov homotopy type $X_{\Kh}^{14}(T_{3,4})$ is not a wedge sum of Moore spaces.

Even simpler than $Sq^2$ is the operation $Sq^1: KH^i(L) \to KH^{i+1,j}(L)$. Let $\beta: KH^i(L) \to KH^{i+1,j}(L)$ be the Bockstein homomorphism, and $r: KH^i(L) \to KH^{i,j}(L)$ be the reduction mod 2. Then $Sq^1 = r\beta$, and is thus determined by the integral Khovanov homology; see also Subsection 2.5. As we will discuss in Section 4, the operations $Sq^1$ and $Sq^2$ together determine the Khovanov homotopy type $X_{\Kh}(L)$ whenever the Khovanov homology of $L$ has a sufficiently simple form. In particular, they determine $X_{\Kh}(L)$ for any link $L$ of 11 or fewer crossings; these homotopy types are listed in Table S1 (part of the supplementary material provided with the online version of this article).

The subalgebra of the Steenrod algebra generated by $Sq^1$ and $Sq^2$ is

$$A(1) = \frac{F_2\{Sq^1, Sq^2\}}{(Sq^1)^2, (Sq^2)^2 + Sq^1 Sq^2 Sq^1}$$

where $F_2\{Sq^1, Sq^2\}$ is the non-commuting extension of $F_2$ by the variables $Sq^1$ and $Sq^2$. By the Adem relations, the next Steenrod square $Sq^3$ is determined by $Sq^1$ and $Sq^2$, viz. $Sq^3 = Sq^1 Sq^2$. Therefore, the next interesting Steenrod square to compute would be $Sq^4: KH^{i}(L) \to KH^{i+4,j}(L)$.

The Bockstein $\beta$ and the operation $Sq^2$ are sometimes enough to compute Khovanov $K$-theory: in the Atiyah–Hirzebruch spectral sequence for $K$-theory, the $d_2$ differential is zero and the $d_3$ differential is the integral lift $\beta Sq^2 r$ of $Sq^3$ (see, for instance, [11, Proposition 16.6] or [7]). For grading reasons, this operation vanishes for links with 11 or fewer crossings; indeed, the Atiyah–Hirzebruch spectral sequence degenerates in these cases, and the Khovanov $K$-theory is just the tensor product of the Khovanov homology and $K^*(pt)$. In principle, however, the techniques of this paper could be used to compute Khovanov $K$-theory in some interesting cases. Similarly, in certain situations, the Adams spectral sequence may be used to compute the real connective Khovanov $KO$-theory using merely the module structure of Khovanov homology $Kh_{\F_2}(L)$ over $A(1)$.

2. The answer

2.1. Sign and frame assignments on the cube

Consider the $n$-dimensional cube $C(n) = [0,1]^n$, equipped with the natural CW complex structure. For a vertex $v = (v_1, \ldots, v_n) \in \{0,1\}^n$, let $|v| = \sum_i v_i$ denote the Manhattan norm of $v$. For vertices $u$ and $v$, declare $v \leq u$ if for all $i$, $v_i \leq u_i$; if $v \leq u$ and $|u - v| = k$, we write $v \leq_k u$.

For a pair of vertices $v \leq_k u$, let $C_{u,v} = \{ x \in [0,1]^n \mid \forall i: v_i \leq x_i \leq u_i \}$ denote the corresponding $k$-cell of $C(n)$. Let $C^*(C(n), F_2)$ denote the cellular cochain complex of $C(n)$ over $F_2$. Let $1_k \in C^k(C(n), F_2)$ denote the $k$-cocycle that sends all $k$-cells to 1.

The standard sign assignment $s \in C^1(C(n), F_2)$ (denoted $s_0$ in [8, Definition 4.5]) is the following 1-cochain. If $u = (\epsilon_1, \epsilon_{i-1}, 1, \epsilon_{i+1}, \ldots, \epsilon_n)$ and $v = (\epsilon_1, \ldots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \ldots, \epsilon_n)$, then

$s(C_{u,v}) = (\epsilon_1 + \cdots + \epsilon_{i-1}) \pmod{2} \in F_2$.

It is easy to see that $\delta s = 1_2$.

The standard frame assignment $f \in C^2(C(n), F_2)$ is the following 2-cochain. If $u = (\epsilon_1, \ldots, \epsilon_{i-1}, 1, \epsilon_{i+1}, \ldots, \epsilon_j-1, 1, \epsilon_{j+1}, \ldots, \epsilon_n)$ and $v = (\epsilon_1, \ldots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \ldots, \epsilon_j-1, 0$, then
that is, we add arcs at the 0-resolutions to record the crossings. The set of circles and arcs that

In this subsection, we recall the definition of the Khovanov chain complex associated to an

The set of all Khovanov generators in bigrading \((i,j)\) will always have a fixed

Proof. Let \(u = (\epsilon_1, \ldots, \epsilon_{i-1}, 1, \epsilon_{i+1}, \ldots, \epsilon_j, 1, \epsilon_{j+1}, \ldots, \epsilon_n)\) and \(v = (\epsilon_1, \ldots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \ldots, \epsilon_{j-1}, 0, \epsilon_{j+1}, \ldots, \epsilon_n)\). Then,

On the other hand,

thus completing the proof. \(\square\)

2.2. The Khovanov setup

In this subsection, we recall the definition of the Khovanov chain complex associated to an oriented link diagram \(L\). Assume \(L\) has \(n\) crossings that have been ordered, and let \(n_-\) denote the number of negative crossings in \(L\). In what follows, we will usually work over \(\mathbb{F}_2\), and we will always have a fixed \(n\)-crossing link diagram \(L\) in the background. Hence, we will typically drop both \(\mathbb{F}_2\) and \(L\) from the notation, writing \(KC = KC_{\mathbb{F}_2}(L)\) for the Khovanov complex of \(L\) with \(\mathbb{F}_2\)-coefficients and \(KC_{\mathbb{Z}}\) for the Khovanov complex of \(L\) with \(\mathbb{Z}\)-coefficients.

Given a vertex \(u \in \{0,1\}^n\), let \(D(u)\) be the corresponding complete resolution of the link diagram \(L\), where we take the 0 resolution at \(i\)th crossing if \(u_i = 0\), and the 1-resolution otherwise. We usually view \(D(u)\) as a resolution configuration in the sense of [8, Definition 2.1]; that is, we add arcs at the 0-resolutions to record the crossings. The set of circles and arcs that appear in \(D(u)\) are denoted \(Z(D(u))\) and \(A(D(u))\), respectively.

The Khovanov generators are of the form \(x = (D(u), x)\), where \(x\) is a labeling of the circles in \(Z(D(u))\) by elements of \(\{x_+, x_-\}\). Each Khovanov generator carries a bigrading \((\text{gr}_h, \text{gr}_q)\); \(\text{gr}_h\) is called the homological grading and \(\text{gr}_q\) is called the quantum grading. The bigrading is defined by

The set of all Khovanov generators in bigrading \((i,j)\) is denoted \(KG^{i,j}\). There is an obvious map \(\mathcal{F}: KG \to \{0,1\}^n\) that sends \((D(u), x)\) to \(u\). It is clear that if \(x \in KG^{i,j}\), then \(|\mathcal{F}(x)| = n_- + i\).
The Khovanov chain group in bigrading \((i,j)\), \(KC^{i,j}\), is the \(\mathbb{F}_2\) vector space with basis \(KG^{i,j}\); for \(x \in KG^{i,j}\), and \(c \in KC^{i,j}\), we say \(x \in c\) if the coefficient of \(x\) in \(c\) is 1, and \(x \notin c\) otherwise.

The Khovanov differential \(\delta\) maps \(KC^{i,j} \rightarrow KC^{i+1,j}\), and is defined as follows. If \(y = (D(v), y) \in KG^{i,j}\) and \(x = (D(u), x) \in KG^{i+1,j}\), then \(x \in \delta y\) if the following hold.

1. \(v \leq u\), that is, \(D(u)\) is obtained from \(D(v)\) by performing an embedded 1-surgery along some arc \(A_1 \in A(D(v))\). In particular, either,
   - (i) the endpoints of \(A_1\) lie on the same circle, say \(Z_1 \in D(v)\), which corresponds to two circles, say \(Z_2, Z_3 \in D(u)\); or,
   - (ii) the endpoints of \(A_1\) lie on two different circles, say \(Z_1, Z_2 \in D(v)\), which correspond to a single circle, say \(Z_3 \in D(u)\).

2. In Case ((1)(i)), \(x\) and \(y\) induce the same labeling on \(D(u) \setminus \{Z_2, Z_3\} = D(v) \setminus \{Z_1\}\); in Case ((1)(ii)), \(x\) and \(y\) induce the same labeling on \(D(u) \setminus \{Z_3\} = D(v) \setminus \{Z_1, Z_2\}\).

3. In Case ((1)(i)), either \(y(Z_1) = x(Z_2) = x(Z_3) = x_+\) or \(y(Z_1) = x_+\) and \(\{x(Z_2), x(Z_3)\} = \{x_+, x_-\}\); in Case ((1)(ii)), either \(y(Z_1) = y(Z_2) = x(Z_3) = x_+\) or \(\{y(Z_1), y(Z_2)\} = \{x_+, x_-\}\) and \(x(Z_3) = x_-\).

It is clear that if \(x \in \delta y\), then \(\mathcal{F}(y) \leq_1 \mathcal{F}(x)\). The Khovanov homology is the homology of \((KC, \delta)\); the Khovanov homology in bigrading \((i,j)\) is denoted \(Kh^{i,j}\). For a cycle \(c \in KC^{i,j}\), let \([c] \in Kh^{i,j}\) denote the corresponding homology element.

2.3. A first look at the Khovanov space

The Khovanov chain complex is actually defined over \(\mathbb{Z}\), and the \(\mathbb{F}_2\) version is its mod 2 reduction. The Khovanov chain group over \(\mathbb{Z}\) in bigrading \((i,j)\), \(KC\), is the free \(\mathbb{Z}\)-module with basis \(KG^{i,j}\). The differential \(\delta_2 : KC^{i,j}_\mathbb{Z} \rightarrow KC^{i+1,j}_\mathbb{Z}\) is defined by

\[
\delta_2 y = \sum_{x \in \delta y} (-1)^{(\mathcal{C}_\mathcal{F}(x), \mathcal{F}(y))} x. \tag{2.1}
\]

In [8, Theorem 1.1], we construct Khovanov spectra \(\mathcal{X}_\mathcal{K}\) satisfying \(\tilde{H}^i(\mathcal{X}_\mathcal{K}^j) = Kh^{i,j}_\mathbb{Z}\). Moreover, the spectrum \(\mathcal{X}_\mathcal{K}^j = \bigvee_j \mathcal{X}_\mathcal{K}^j\) is defined as the suspension spectrum of a CW complex \(\mathcal{C}_\mathcal{K}\), formally desuspended a few times [8, Definition 5.5] (this space is denoted \(Y = \bigvee_j Y_j\) in Section 3). Furthermore, there is a bijection between the cells (except the basepoint) of \(\mathcal{C}_\mathcal{K}\) and the Khovanov generators in \(KG\), which induces an isomorphism between \(\tilde{C}^*(\mathcal{C}_\mathcal{K})\), the reduced cellular cochain complex, and \((KC, \delta_2)\).

This allows us to associate homotopy invariants to Khovanov homology. Let \(\mathcal{A}\) be the (graded) Steenrod algebra over \(\mathbb{F}_2\), and let \(\mathcal{A}(1)\) be the subalgebra generated by \(Sq^1\) and \(Sq^2\). The Steenrod algebra \(\mathcal{A}\) acts on the Khovanov homology \(Kh\), viewed as the (reduced) cohomology of the spectrum \(\mathcal{X}_\mathcal{K}\). The (stable) homotopy type of \(\mathcal{X}_\mathcal{K}\) is a knot invariant, and therefore, the action of \(\mathcal{A}\) on \(Kh\) is a knot invariant as well.

2.4. The ladybug matching

Let \(x \in KG^{i+j}\) and \(y \in KG^{i,j}\) be Khovanov generators. Consider the set of Khovanov generators between \(x\) and \(y\):

\[\mathcal{G}_{x,y} = \{z \in KG^{i+1,j} \mid x \in \delta z, z \in \delta y\}\.]
Since $\delta$ is a differential, for all $x, y$, there are an even number of elements in $G_{x, y}$. It is well known that this even number is 0, 2 or 4. Indeed:

**Lemma 2.2** ([8, Lemma 5.7]). Let $x = (D(u), x)$ and $y = (D(v), y)$. The set $G_{x, y}$ has four elements if and only if the following hold.

(i) The vertices satisfy $v \leq 2u$, that is, $D(u)$ is obtained from $D(v)$ by doing embedded 1-surgeries along two arcs, say $A_1, A_2 \in A(D(v))$.

(ii) The endpoints of $A_1$ and $A_2$ all lie on the same circle, say $Z_1 \in Z(D(v))$. Furthermore, their endpoints are linked on $Z_1$, so $Z_1$ gives rise to a single circle, say $Z_2$, in $Z(D(u))$.

(iii) The labelings $x$ and $y$ agree on $Z(D(v)) \setminus \{Z_2\} = Z(D(u)) \setminus \{Z_1\}$.

(iv) The labelings satisfy $y(Z_1) = x_+ \text{ and } x(Z_2) = x_-$.

In the construction of the Khovanov space, we made a global choice. This choice furnishes us with a **ladybug matching** $l$, which is a collection $\{l_{x, y}\}$, for $x, y \in KG$ with $|\mathcal{F}(x)| = |\mathcal{F}(y)| + 2$, of fixed point free involutions $l_{x, y} : G_{x, y} \to G_{x, y}$. The ladybug matching is defined as follows.

Fix $x = (D(u), x)$ and $y = (D(v), y)$ in $KG$ with $|u| = |v| + 2$; we will describe a fixed point free involution $l_{x, y}$ of $G_{x, y}$. The only case of interest is when $G_{x, y}$ has four elements; hence assume that we are in the case described in Lemma 2.2. Do an isotopy in $S^2$ so that $D(v)$ looks like Figure 2.1(a). (In the figure, we have not shown the circles in $Z(D(v)) \setminus \{Z_1\}$ and the arcs in $A(D(v)) \setminus \{A_1, A_2\}$.) Figure 2.1(b) shows the four generators in $G_{x, y}$ and the ladybug matching $l_{x, y}$. (Once again, we have not shown the extra circles and arcs.) It is easy to check (cf. [8, Lemma 5.8]) that this matching is well defined, that is, it is independent of the choice of isotopy and the numbering of the two arcs in $A(D(v)) \setminus A(D(u))$ as $\{A_1, A_2\}$.

**Figure 2.1.** The ladybug matching. We have shown the case when $G_{x, y}$ has four elements. (a) The configuration from Lemma 2.2. (b) The ladybug matching $l_{x, y}$ on $G_{x, y}$. 


Lemma 2.3. Let $x, y, z$ be Khovanov generators with $z \in G_{x,y}$. Let $z' = t_{x,y}(z)$. Then
\[ s(C_{x,y}(z)) + s(C_{x,y}(z')) = 1. \]

Proof. Let $u = \mathcal{F}(x)$, $v = \mathcal{F}(y)$, $w = \mathcal{F}(z)$ and $w' = \mathcal{F}(z')$. We have $v \leq u, w' \leq u, w$. It follows from the definition of ladybug matching (Figure 2.1(b)) that $w \neq w'$. Therefore, $u, v, w$ and $w'$ are precisely the four vertices that appear in the 2-cell $C_{u,v}$. Since $\delta s = 1$, $\delta s(C_{u,v}) = s(C_{w,v}) + s(C_{w,v}) + s(C_{w',v}) = 1$, as desired.

2.5. The operation $\text{Sq}^1$

Let $c \in KC^{i,j}$ be a cycle in the Khovanov chain complex. For $x \in KG^{i+1,j}$, let $G_c(x) = \{y \in KG^{i,j} \mid x \in \delta y, y \in c\}$.

Definition 2.4. A boundary matching $m$ for $c$ is a collection of pairs $(b_x, s_x)$, one for each $x \in KG^{i+1,j}$, where:

(i) $b_x$ is a fixed point free involution of $G_c(x)$, and
(ii) $s_x$ is a map $G_c(x) \to \mathbb{F}_2$, such that for all $y \in G_c(x)$,
\[ \{s_x(y), s_x(b_x(y))\} = \begin{cases} \{0, 1\} & \text{if } s(C_{x,y}, x(y)) = s(C_{x,y}, x(b_x(y))) \\ \{0\} & \text{otherwise.} \end{cases} \]

Since $c$ is a cycle, for any $x$ there are an even number of elements in $G_c(x)$. Hence, there exists a boundary matching $m$ for $c$.

Definition 2.5. Let $c \in KC^{i,j}$ be a cycle. For any boundary matching $m = \{(b_x, s_x)\}$ for $c$, define the chain $s_{m}^1(c) \in KG^{i+1,j}$ as
\[ s_{m}^1(c) = \sum_{x \in KG^{i+1,j}} \left( \sum_{y \in G_c(x)} s_x(y) \right) x. \]

Proposition 2.6. For any cycle $c \in KC^{i,j}$ and any boundary matching $m$ for $c$, $s_{m}^1(c)$ is a cycle. Furthermore,
\[ [s_{m}^1(c)] = \text{Sq}^1([c]). \]

Proof. The first Steenrod square $\text{Sq}^1$ is the Bockstein associated to the short exact sequence
\[ 0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0. \]
Since the differential in $KC_Z$ is given by Equation (2.1), a chain representative for $\text{Sq}^1([c])$ is the following:
\[ b = \sum_{x \in KG^{i+1,j}} \left( \frac{\#\{y \in G_c(x) \mid s(C_{x,y}, x(y)) = 0\} - \#\{y \in G_c(x) \mid s(C_{x,y}, x(y)) = 1\}}{2} \right) x. \]
It is easy to see that
\[
\sum_{y \in G_c(x)} s_x(y) = \left( \frac{\#\{y \mid s(C,\mathcal{F}(x),\mathcal{F}(y)) = 0\} - \#\{y \mid s(C,\mathcal{F}(x),\mathcal{F}(y)) = 1\}}{2} \right) \pmod{2},
\]
and hence \( b = sq^1_m(c) \).

2.6. The operation \( \text{Sq}^2 \)

Let \( c \in KC^{i,j} \) be a cycle. Choose a boundary matching \( m = \{(b_z,s_z)\} \) for \( c \). For \( x \in KG^{i+2,j} \), define
\[
G_c(x) = \{ (z,y) \in KG^{i+1,j} \times KG^{i,j} \mid x \in \delta z, z \in \delta y, y \in c \}.
\]

Consider the edge-labeled graph \( \mathfrak{G}_c(x) \), whose vertices are the elements of \( G_c(x) \) and whose edges are the following.

(e-1) There is an unoriented edge between \((z,y)\) and \((z',y)\), if the ladybug matching \( l_{x,y} \) matches \( z \) and \( z' \). This edge is labeled by \( f(C,\mathcal{F}(x),\mathcal{F}(y)) \in \mathbb{F}_2 \), where \( f \) denotes the standard frame assignment (Subsection 2.1).

(e-2) There is an edge between \((z,y)\) and \((z,y')\) if the matching \( b_z \) matches \( y \) with \( y' \). This edge is labeled by 0. Furthermore, if \( s_z(y) = 0 \) and \( s_z(y') = 1 \), then this edge is oriented from \((z,y)\) to \((z,y')\); if \( s_z(y) = 1 \) and \( s_z(y') = 0 \), then this edge is oriented from \((z,y')\) to \((z,y)\); and if \( s_z(y) = s_z(y') \), then the edge is unoriented.

Definition 2.7. Let \( f(\mathfrak{G}_c(x)) \in \mathbb{F}_2 \) be the sum of all the edge-labels (of the Type (e-1) edges) in the graph \( \mathfrak{G}_c(x) \).

Lemma 2.8. Each component of \( \mathfrak{G}_c(x) \) is an even cycle. Furthermore, in each component, the number of oriented edges is even.

Proof. Each vertex \((z,y)\) of \( \mathfrak{G}_c(x) \) belongs to exactly two edges: the Type (e-1) edge joining \((z,y)\) and \((l_{x,y}(z),y)\); and the Type (e-2) edge joining \((z,y)\) and \((z,b_z(y))\). This implies that each component of \( \mathfrak{G}_c(x) \) is an even cycle.

In order to prove the second part, vertex-label the graph as follows: To a vertex \((z,y)\), assign the number \( s(C,\mathcal{F}(x),\mathcal{F}(z)) + s(C,\mathcal{F}(x),\mathcal{F}(y)) \in \mathbb{F}_2 \). Lemma 2.3 implies that the Type (e-1) edges join vertices carrying opposite labels; and among the Type (e-2) edges, it is clear from the definition of boundary matching (Subsection 2.5) that the oriented edges join vertices carrying the same label, and the unoriented edges join vertices carrying opposite labels. Therefore, each cycle must contain an even number of unoriented edges; since there are an even number of vertices in each cycle, we are done.

This observation allows us to associate the following number \( g(\mathfrak{G}_c(x)) \in \mathbb{F}_2 \) to the graph.

Definition 2.9. Partition the oriented edges of \( \mathfrak{G}_c(x) \) into two sets, such that if two edges from the same cycle are in the same set, they are oriented in the same direction. Let \( g(\mathfrak{G}_c(x)) \) be the number modulo 2 of the elements in either set.
**Definition 2.10.** Let \( c \in KC^{i,j} \) be a cycle. For any boundary matching \( m \) for \( c \), define the chain \( sq_m^2(c) \in KC^{i+1,j} \) as

\[
sq_m^2(c) = \sum_{x \in KG^{i+2,j}} \left( \# |G_c(x)| + f(G_c(x)) + g(G_c(x)) \right) x.
\]

(Here, \( \# |G_c(x)| \) is the number of components of the graph, \( f(G_c(x)) \) is defined in Definition 2.7, and \( g(G_c(x)) \) is defined in Definition 2.9.)

We devote Section 3 to proving the following.

**Theorem 2.11.** For any cycle \( c \in KC^{i,j} \) and any boundary matching \( m \) for \( c \), \( sq_m^2(c) \) is a cycle. Furthermore,

\[
[sq_m^2(c)] = Sq^2([c]).
\]

**Corollary 2.12.** The operations \( sq_m^1 \) and \( sq_m^2 \) induce well-defined maps

\[
Sq^1: Kh^{i,j} \to Kh^{i+1,j} \quad \text{and} \quad Sq^2: Kh^{i,j} \to Kh^{i+2,j}
\]

that are independent of the choices of boundary matchings \( m \). Furthermore, these maps are link invariants, in the following sense: given any two diagrams \( L \) and \( L' \) representing the same link, there are isomorphisms \( \phi_{i,j}: Kh^{i,j}(L) \to Kh^{i,j}(L') \) making the following diagrams commute:

![Diagram](image)

**Proof.** This is immediate from Proposition 2.6, Theorem 2.11 and invariance of the Khovanov spectrum [8, Theorem 1.1]. \( \square \)

Indeed, we show in [9, Theorem 4] that we can choose the isomorphisms in Corollary 2.12 to be the canonical ones induced from an isotopy from \( L \) to \( L' \).

### 2.7. An example

For the reader’s convenience, we present an artificial example to illustrate Definitions 2.5 and 2.10.

**Example 2.1.** Assume \( KG^{i,j} = \{y_1, \ldots, y_5\} \), \( KG^{i+1,j} = \{z_1, \ldots, z_6\} \) and \( KG^{i+2,j} = \{x_1, x_2\} \), and the Khovanov differential \( \delta \) has the following form:
Finally, assume that the ladybug matching $I_{x_1, y_1}$ matches $z_1$ with $z_4$ and $z_2$ with $z_3$.

Let us start with the cycle $c \in K_{C^{++}}$ given by $c = \sum_{i=1}^{5} y_i$. In order to compute $Sq^1(c)$ and $Sq^2(c)$, we need to choose a boundary matching $m = \{ (b_{z_j}, s_{z_j}) \}$ for $c$. Let us choose the following boundary matching.

| $j$ | $b_{z_j}$ | $s_{z_j}$ |
|-----|----------|----------|
| 1   | $y_1 \leftrightarrow y_2$ | $y_1 \rightarrow 0, y_2 \rightarrow 1$ |
| 2   | $y_1 \leftrightarrow y_3$ | $y_1, y_3 \rightarrow 0$ |
| 3   | $y_1 \leftrightarrow y_2$ | $y_1 \rightarrow 0, y_2 \rightarrow 1$ |
| 4   | $y_1 \leftrightarrow y_3$ | $y_1, y_3 \rightarrow 0$ |
| 5   | $y_2 \leftrightarrow y_3, y_4 \leftrightarrow y_5$ | $y_2, y_4 \rightarrow 0, y_3, y_5 \rightarrow 1$ |
| 6   | $y_4 \leftrightarrow y_5$ | $y_4 \rightarrow 0, y_5 \rightarrow 1$ |

Then, the cycle $sq^1_{im}(c)$ is given by

$$sq^1_{im}(c) = \sum_{j=1}^{6} \left( \sum_{y \in G(z_j)} s_{z_j}(y) \right) z_j$$

$$= (s_{z_1}(y_1) + s_{z_2}(y_2))z_1 + (s_{z_2}(y_1) + s_{z_2}(y_3))z_2 + (s_{z_2}(y_1) + s_{z_3}(y_2))z_3$$

$$+ (s_{z_3}(y_1) + s_{z_3}(y_3))z_4 + (s_{z_3}(y_2) + s_{z_3}(y_3) + s_{z_3}(y_4) + s_{z_3}(y_5))z_5$$

$$+ (s_{z_3}(y_4) + s_{z_3}(y_5))z_6$$

$$= z_1 + z_3 + z_6.$$  

In order to compute $sq^2_{im}(c)$, we need to study the graphs $G_c(x_1)$ and $G_c(x_2)$, which are the following:
The Type (e-1) edges are represented by the dotted lines; they are unoriented and are labeled by elements of $F_2$. The Type (e-2) edges are represented by the solid lines; they are labeled by 0 and are sometimes oriented. Therefore, the cycle $sq^2_m(c)$ is given by

$$sq^2_m(c) = \sum_{j=1}^{2} (\#|G_c(x_j)| + f(G_c(x_j)) + g(G_c(x_j)))x_j$$

$$= (1 + 1 + 1)x_1 + (0 + 1 + 1)x_2$$

$$= x_1.$$

3. Where the answer comes from

This section is devoted to proving Theorem 2.11. The operation $Sq^2$ on a CW complex $Y$ is determined by the sub-quotients $Y^{(m+2)}/Y^{(m-1)}$. (In Subsection 3.1, we review an explicit description in these terms, due to Steenrod.) The space $K_{Kh}(L)$ is a formal de-suspension of a CW complex $Y = Y(L)$. So, most of the work is in understanding combinatorially how the $m$-, $(m+1)$- and $(m+2)$-cells of $Y(L)$ are glued together.

The description of $Y(L)$ from [8] is in terms of a Pontryagin–Thom-type construction. To understand just $Y^{(m+2)}/Y^{(m-1)}$ involves studying certain framed points in $\mathbb{R}^m$ and framed paths in $\mathbb{R} \times \mathbb{R}^m$. We will draw these framings from a particular set of choices, described in Subsection 3.2. Subsection 3.3 explains exactly how we assign framings from this set, and shows that these framings are consistent with the construction in [8]. Finally, Subsection 3.4 discusses how to go from these choices to $Y^{(m+2)}/Y^{(m-1)}$, and why the resulting operation $Sq^2$ agrees with the operation from Definition 2.10.

3.1. $Sq^2$ for a CW complex

We start by recalling a definition of $Sq^2$. The discussion in this section is heavily inspired by Steenrod [11, Section 12].

Let $K_m = K(\mathbb{Z}/2, m)$ denote the $m$th Eilenberg–MacLane space for the group $\mathbb{Z}/2$, so $\pi_m(K_m) = \mathbb{Z}/2$ and $\pi_i(K_m) = 0$ for $i \neq m$. Assume that $m$ is sufficiently large, say $m \geq 3$.

We start by discussing a CW structure for $K_m$. Since $\pi_i(K_m) = 0$ for $i < m$, we can choose the $m$-skeleton $K^{(m)}_m$ to be a single $m$-cell $e^m$ with the entire boundary $\partial e^m$ attached to the basepoint. To arrange that $\pi_m(K_m) = \mathbb{Z}/2$, it suffices to attach a single $(m+1)$-cell via a degree 2 map $\partial e^{m+1} \to K^{(m)}_m = S^m$.

We show that the resulting $(m+1)$-skeleton $K^{(m+1)}_m$ has $\pi_{m+1}(K^{(m+1)}_m) \cong \mathbb{Z}/2$. From the long exact sequence for the pair $(K^{(m+1)}_m, S^m)$,

$$\pi_{m+2}(K^{(m+1)}_m, S^m) \to \pi_{m+1}(S^m) \to \pi_{m+1}(K^{(m+1)}_m) \to \pi_{m+1}(K^{(m+1)}_m, S^m) \to \pi_m(S^m).$$

By excision (since $m$ is large), $\pi_{m+1}(K^{(m+1)}_m, S^m) \cong \pi_{m+1}(K^{(m+1)}_m/\mathbb{Z}) = \pi_{m+1}(\mathbb{Z}^{m+1}) = \mathbb{Z}$ and $\pi_{m+2}(K^{(m+1)}_m, S^m) \cong \pi_{m+2}(S^{m+1}) = \mathbb{Z}/2$. The maps $\pi_{m+1}(K^{(m+1)}_m, S^m) = \pi_{m+1}(S^{m+1}) \to \pi_i(S^m)$ are twice the Freudenthal isomorphisms. So, this sequence becomes

$$\mathbb{Z}/2 \to \mathbb{Z}/2 \to \pi_{m+1}(K^{(m+1)}_m) \to \mathbb{Z} \to \mathbb{Z}.$$
Therefore, there are fundamental cohomology classes \( \iota \in H^m(K_m; \mathbb{F}_2) \) and \( \text{Sq}^2(\iota) \in H^{m+2}(K_m; \mathbb{F}_2) \). It turns out that the element \( \text{Sq}^2(\iota) \) survives to \( H^{m+2}(K_m; \mathbb{F}_2) \).

Now, consider a CW complex \( Y \) and a cohomology class \( c \in H^m(Y; \mathbb{F}_2) \). The element \( c \) is classified by a map \( c : Y \to K_m \), so that \( c^* \iota = c \). We can arrange that the map \( c \) is cellular. So, we have an element \( \text{Sq}^2(c) = c^* \text{Sq}^2(\iota) \in H^{m+2}(Y; \mathbb{F}_2) \).

The element \( \text{Sq}^2(c) \) is determined by its restriction to \( H^{m+2}(Y^{(m+2)}; \mathbb{F}_2) \). So, to compute \( \text{Sq}^2(c) \) it suffices to give a cellular map \( Y^{(m+2)} \to K_m^{(m+2)} \) so that \( c \) pulls back to \( c \). Then, \( \text{Sq}^2(c) \) is the cochain which sends an \((m+2)\)-cell \( f^{m+2} \) of \( Y \) to the degree of the map \( f^{m+2}/\partial f^{m+2} \to e^{m+2}/\partial e^{m+2} \). Equivalently, \( \text{Sq}^2(c) \) sends \( f^{m+2} \) to the element \( c|_{\partial f^{m+2}} \in \pi_{m+1}(K_m^{(m+1)}) = \pi_{m+1}(S^m) = \mathbb{Z}/2 \). (In other words, \( \text{Sq}^2(c) \) is the obstruction to homotoping \( c \) so that it sends the \((m+2)\)-skeleton of \( Y \) to the \((m+1)\)-skeleton of \( K_m \).) Since \( K_m^{(m+2)} \) has no cells of dimension between 0 and \( m \), the map \( Y^{(m+2)} \to K_m^{(m+2)} \) factors through \( Y^{(m+2)}/Y^{(m-1)} \).

To understand the operation \( \text{Sq}^2 \) on Khovanov homology induced by the Khovanov homotopy type \( Y \), it remains to explicitly give the map \( c \) on \( Y^{(m+2)}/Y^{(m-1)} \). This will be done in Subsection 3.4, after we develop tools to understand the attaching maps for the \((m+2)\)-cells.

### 3.2. Frames in \( \mathbb{R}^3 \)

As discussed in Subsection 3.4, the sub-quotients \( Y^{(m+2)}/Y^{(m-1)} \) of the Khovanov space \( Y = Y(L) \) are defined in terms of framed points in \( \{0\} \times \mathbb{R}^m \) and framed paths in \( \mathbb{R} \times \mathbb{R}^m \) connecting these points.

A framing of a path \( \gamma : [0, 1] \to \mathbb{R}^{m+1} \) is a tuple \([v_1(t), \ldots, v_m(t)] \in (\mathbb{R}^{m+1})^m \) of orthonormal vector fields along \( \gamma \), normal to \( \gamma \). A collection of \( m \) orthonormal vectors \( v_1, \ldots, v_m \) in \( \mathbb{R}^{m+1} \) specifies a matrix in \( \text{SO}(m+1) \), whose last \( m \) columns are \( v_1, \ldots, v_m \) and whose first column is the cross product of \( v_1, \ldots, v_m \).

Now, suppose that \( p, q \in \{0\} \times \mathbb{R}^m \) and that we are given trivializations \( \varphi_p, \varphi_q \) of the normal bundles in \( \{0\} \times \mathbb{R}^m \) to \( p, q \) (that is, framings of \( p \) and \( q \)). On the one hand, we can consider the set of isotopy classes of framed paths from \( (p, \varphi_p) \) to \( (q, \varphi_q) \). On the other hand, we can consider the homotopy classes of paths in \( \text{SO}(m+1) \) from \( \varphi_p \) to \( \varphi_q \). There is an obvious map from isotopy classes of framed paths in \( \mathbb{R}^{m+1} \) to homotopy classes of paths in \( \text{SO}(m+1) \), by considering only the framing. This map is a surjection if \( m \geq 2 \) and a bijection if \( m \geq 3 \). In the case that \( m \geq 3 \), both sets have two elements.

The upshot is that if we want to specify isotopy classes of framed paths with given endpoints, and \( m \geq 3 \), then it suffices to specify a homotopy class of paths in \( \text{SO}(m+1) \).

The framings on both the endpoints and the paths relevant to constructing \( Y^{(m+2)}/Y^{(m-1)} \) will have a special form: they will be stabilizations of the \( m = 2 \) case. Specifically, we will write \( m = m_1 + m_2 \) with \( m_i \geq 1 \). Let \([e_1, e_{11}, \ldots, e_{1m_1}, e_{21}, \ldots, e_{2m_2}]\) denote the standard basis for \( \mathbb{R} \times \mathbb{R}^m \). Then all the points will have framings of the form:

\[
[v_1, e_{12}, \ldots, e_{1m_1}, v_2, e_{22}, \ldots, e_{2m_2}] \in (\{0\} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2})^m,
\]

for some \( v_1 \in \{\pm e_{11}\} \), \( v_2 \in \{\pm e_{21}\} \). So, to describe isotopy classes of framed paths connecting these points it suffices to describe paths of the form

\[
[v_1(t), e_{12}, \ldots, e_{1m_1}, v_2(t), e_{22}, \ldots, e_{2m_2}] \in (\mathbb{R} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2})^m.
\]

Therefore, to describe such paths, it suffices to work in \( \mathbb{R}^3 \) (that is, the case \( m = 2 \)). Denote the standard basis for \( \mathbb{R}^3 \) by \([e_1, e_2, e_3]\). We will work with the four distinguished frames in \( \{0\} \times \mathbb{R}^2 \), \([e_1, e_2], [-e_1, e_2], [e_1, -e_2], [-e_1, -e_2]\), which we denote by the symbols \( \pm, \mp, \pm, \mp \), respectively. By a coherent system of paths joining \( \pm, \mp, \pm, \mp \), we mean a choice of a path \( \varphi_1 \varphi_2 \) in \( \text{SO}(3) \) from \( \varphi_1 \) to \( \varphi_2 \) for each pair of frames \( \varphi_1, \varphi_2 \in \{\pm, \mp, \pm, \mp\} \), satisfying the following cocycle conditions.
Figure 3.1. Null-homotopy of the loop $\begin{pmatrix} + & + & - \end{pmatrix}$ in $\text{SO}(3)$. Viewing the arm as 2-dimensional, spanned by the tangent vector to the radius and the vector from the radius to the ulna, it traces out an extension of the map $S^1 \to \text{SO}(3)$ to a map $\mathbb{D}^2 \to \text{SO}(3)$.

(1) For all $\varphi \in \{+,-,-,+$, $+,-,+,-,$ $+,-,+,-,$ $-,-,-,-\}$, the loop $\varphi \cdot \varphi$ is nullhomotopic; and
(2) For all $\varphi_1, \varphi_2, \varphi_3 \in \{+,-,-,+,$ $+,-,+,-,$ $+,-,+,-,$ $-,-,-,-\}$, the path $\varphi_1 \varphi_2 \cdot \varphi_2 \varphi_3$ is homotopic (relative endpoints) to the path $\varphi_1 \varphi_3$.

We make a particular choice of a coherent system of paths, as follows.

(i) $\begin{pmatrix} + & - & + \\ - & + & - \end{pmatrix}$: Rotate $180^\circ$ around the $e_2$-axis, such that the first vector equals $\bar{e}$ halfway through.
(ii) $\begin{pmatrix} + & + & + \\ + & - & - \end{pmatrix}$: Rotate $180^\circ$ around the $e_1$-axis, such that the second vector equals $\bar{e}$ halfway through.
(iii) $\begin{pmatrix} - & - & - \\ - & + & + \end{pmatrix}$: Rotate $180^\circ$ around the $e_1$-axis, such that the second vector equals $-\bar{e}$ halfway through.
(iv) $\begin{pmatrix} + & + & - \\ + & + & + \\ - & + & - \end{pmatrix}$: Rotate $180^\circ$ around the $\bar{e}$-axis, such that the second vector equals $-e_1$ halfway through.

Lemma 3.1. The above choice describes a coherent system of paths.

Proof. We only need to check that each of the loops $\begin{pmatrix} + & - & + \\ - & + & - \end{pmatrix}$, $\begin{pmatrix} + & - & - \\ + & + & + \end{pmatrix}$ and $\begin{pmatrix} - & - & - \\ - & + & + \end{pmatrix}$ are null-homotopic. This is best checked with hand motions, as we have illustrated for the first loop in Figure 3.1(a).

Extending this slightly:

Definition 3.2. Fix $m_1$ and $m_2$, and let $m = m_1 + m_2$. By the four standard frames for $\mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, we mean the frames

$$\left[\pm e_{11}, e_{12}, \ldots, e_{1m_1}, \pm e_{21}, e_{22}, \ldots, e_{2m_2}\right] \in \left(\{0\} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}\right)^m.$$

Up to homotopy, there are exactly two paths between any pair of frames. By the standard frame paths in $\mathbb{R} \times \mathbb{R}^m$, we mean the one-parameter families of frames obtained by extending the coherent system of paths for $\text{SO}(3)$ specified above by the identity on $\mathbb{R}^{m-2} = \mathbb{R}^{m_1-1} \times \mathbb{R}^{m_2-1}$. Abusing terminology, we will sometimes say that any frame path homotopic (relative endpoints) to a standard frame path is itself a standard frame path. By a non-standard frame path, we mean a frame path which is not homotopic (relative endpoints) to one of the standard frame paths.

Define

$$\begin{align*}
\tau : \mathbb{R}^m &\rightarrow \mathbb{R}^m \\
\tau(x_1, \ldots, x_m) &= (-x_1, x_2, \ldots, x_m), \\
\delta : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} &\rightarrow \mathbb{R}^m \\
\delta(x_1, \ldots, x_m) &= (x_1, \ldots, x_{m_1}, -x_{m_1+1}, x_{m_1+2}, \ldots, x_m).
\end{align*}$$
Figure 3.2. The hexagon $\mathcal{M}_{\mathcal{C}(3)}(111,000)$. Each face corresponds to a product of lower-dimensional moduli spaces, as indicated.

Lemma 3.3. Suppose $\varphi_1$ and $\varphi_2$ are oppositely oriented standard frames. Then $\tau(\varphi_1 \varphi_2)$ is the non-standard frame path between $\tau(\varphi_1)$ and $\tau(\varphi_2)$. That is, $\tau$ takes standard frame paths between oppositely oriented frames to non-standard frame paths.

The map $s$ satisfies

$$
\begin{array}{ccc}
++ & \leftrightarrow & +-
\end{array}
\quad
\begin{array}{ccc}
+- & \leftrightarrow & -+
\end{array}
$$

In other words, $s$ takes the standard frame path $\overline{++}$ to the standard frame path $\overline{+-}$ for either $* \in \{+, -\}$.

Proof. This is a straightforward verification from the definitions.

3.3. The framed cube flow category

In this subsection, we describe certain aspects of the flow category $\mathcal{C}(n)$ associated to the cube $[0,1]^n$. For a more complete account of the story, see [8, Section 4]. The features of $\mathcal{C}(n)$ in which we are interested are the following.

(F-1) To a pair of vertices $u, v \in \{0,1\}^n$ with $v \leq k u$, $\mathcal{C}(n)$ associates a $(k-1)$-dimensional manifold with corners\footnote{It is also a $\langle k-1 \rangle$-manifold in the sense of [6].} called the moduli space $\mathcal{M}_{\mathcal{C}(n)}(u,v)$. We drop the subscript if it is clear from the context.

(F-2) For vertices $v < w < u$ in $\{0,1\}^n$, $\mathcal{M}(w,v) \times \mathcal{M}(u,w)$ is identified with a subspace of $\partial \mathcal{M}(u,v)$.

(F-3) Fix vertices $v \leq k u$ in $\{0,1\}^n$; let $\vec{0}, \vec{1} \in \{0,1\}^k$ be the minimum and the maximum vertex, respectively. Then $\mathcal{M}_{\mathcal{C}(n)}(u,v)$ can be identified with $\mathcal{M}_{\mathcal{C}(k)}(\vec{1}, \vec{0})$.

(F-4) $\mathcal{M}_{\mathcal{C}(n)}(\vec{1}, \vec{0})$ is a point, an interval and a hexagon, for $n = 1, 2, 3$, respectively, cf. Figure 3.2.

The cube flow category is also framed. In order to define framings, one needs to embed the moduli spaces into Euclidean spaces; one does so by neat embeddings (see [6, Definition 2.1.4] or [8, Definition 3.9]). Fix $d$ sufficiently large; for each $v \leq k u$, $\mathcal{M}(u,v)$ is neatly embedded in $\mathbb{R}_+^{k-1} \times \mathbb{R}^{kd}$. These embeddings are coherent in the sense that for each $v \leq k w \leq l u$, $\mathcal{M}(w,v) \times \mathcal{M}(u,w) \subset \partial \mathcal{M}(u,v)$ is embedded by the product embedding into $\mathbb{R}_+^{k-1} \times \mathbb{R}^{kd} \times \mathbb{R}_-^{l-1} \times \mathbb{R}^d = \mathbb{R}_+^{k-1} \times \{0\} \times \mathbb{R}_-^{l-1} \times \mathbb{R}^{kd+ld} \subset \partial(\mathbb{R}_+^{k+l-1} \times \mathbb{R}^{(k+l)d})$. The normal bundle to each
of these moduli spaces is framed. These framings are also coherent in the sense that the product framing on $\mathcal{M}(w, v) \times \mathcal{M}(u, w)$ agrees with the framing induced from $\mathcal{M}(u, v)$.

The framed cube flow category $\mathcal{C}_C(n)$ is needed in the construction of the Khovanov homotopy type. The cube flow category can be framed in multiple ways. However, all such framings lead to the same Khovanov homotopy type [8, Proposition 6.1]; hence it is enough to consider a specific framing. Consider the following partial framing.

**Definition 3.4.** Let $s \in C^1(C(n), \mathbb{F}_2)$ and $f \in C^2(C(n), \mathbb{F}_2)$ be the standard sign assignment and the standard frame assignment from Subsection 2.1. Fix $d$ sufficiently large.

(i) Consider $v \leq_1 u$ in $\{0, 1\}^n$. Embed the point $\mathcal{M}(u, v)$ in $\mathbb{R}^d$; let $[e_1, \ldots, e_d]$ be the standard basis in $\mathbb{R}^d$. For framing the point $\mathcal{M}(u, v)$, choose the frame $[e_1, e_2, \ldots, e_d]$ if $s(C_{u,v}) = 0$, and choose the frame $[-e_1, e_2, \ldots, e_d]$ if $s(C_{u,v}) = 1$.

(ii) Consider $v \leq_2 u$ in $\{0, 1\}^n$; let $w_1$ and $w_2$ be the two other vertices in $C_{u,v}$. Choose a proper embedding of the interval $\mathcal{M}(u, v)$ in $\mathbb{R}^d \times \mathbb{R}^{2d}$; let $[\bar{e}, e_{11}, \ldots, e_{1d}, e_{21}, \ldots, e_{2d}]$ be the standard basis for $\mathbb{R} \times \mathbb{R}^{2d}$. The two endpoints $\mathcal{M}(w_1, v) \times \mathcal{M}(u, w_1)$ of the interval $\mathcal{M}(u, v)$ are already framed in $\{0\} \times \mathbb{R}^{2d}$ by the product framings, say $\varphi_1$. Since $s$ is a sign assignment, the framings of the two endpoints, $\varphi_1$ and $\varphi_2$, are opposite, and hence can be extended to a framing on the interval. Any such extension can be treated as a path joining $\varphi_1$ and $\varphi_2$ in $SO(2d + 1)$, cf. Subsection 3.2. If $f(C_{u,v}) = 0$, choose an extension so that the path is a standard frame path; if $f(C_{u,v}) = 1$, choose an extension so that the path is a non-standard frame path.

As we will see in Subsection 3.4, in order to study the $Sq^2$ action, one only needs to understand the framings of the 0-dimensional and the 1-dimensional moduli spaces. Therefore, the information encoded in Definition 3.4 is all we need in order to study the $Sq^2$ action. However, before we proceed onto the next subsection, we need to check the following.

**Lemma 3.5.** The partial framing from Definition 3.4 can be extended to a framing of the entire cube flow category $\mathcal{C}_C(n)$.

**Proof.** We frame the cube flow category in [8, Proposition 4.12] inductively: We start with coherent framings of all moduli spaces of dimension less than $k$; after changing the framings in the interior of the $(k - 1)$-dimensional moduli spaces if necessary, we extend this to a framing of all $k$-dimensional moduli spaces.

Therefore, in order to prove this lemma, we merely need to check that the framings of the zero- and one-dimensional moduli spaces from Definition 3.4 can be extended to a framing of the two-dimensional moduli spaces.

Fix $v \leq_3 u$, and fix a neat embedding of the hexagon $\mathcal{M}(u, v)$ in $\mathbb{R}^2_+ \times \mathbb{R}^{3d}$. Let $[e_1, e_2, e_{11}, \ldots, e_{1d}, e_{21}, \ldots, e_{2d}, e_{31}, \ldots, e_{3d}]$ be the standard basis for $\mathbb{R}^2_+ \times \mathbb{R}^{3d}$. The boundary $K$ is a framed 6-gon embedded in $\partial(\mathbb{R}^2_+ \times \mathbb{R}^{3d})$. Let us flatten the corner in $\partial(\mathbb{R}^2_+ \times \mathbb{R}^{3d})$ so that $[\bar{e}_1 = -\bar{e}_2, e_{11}, \ldots, e_{3d}]$ is the standard basis in the flattened $\mathbb{R} \times \mathbb{R}^{3d}$. After this flattening operation, we can treat $K$ as a framed 1-manifold in $\mathbb{R} \times \mathbb{R}^{3d}$ (see Figure 3.3) which in turn represents some element $\eta_{u,v} \in \pi_4(S^3) = \mathbb{Z}/2$ by the Pontryagin–Thom correspondence. We want to show that $K$ is null-concordant, that is, that $\eta_{u,v} = 0$.

As in Subsection 3.2, $K$ can also be treated as a loop in $SO(3d + 1)$, and thus represents some element $h_{u,v} \in H_1(SO(3d + 1); \mathbb{Z}) = \mathbb{F}_2$. The element $h_{u,v}$ is non-zero if and only if $\eta_{u,v}$ is zero; therefore, we want to show $h_{u,v} = 1$.

Let $t_1, t_2, t_3, w_1, w_2$ and $w_3$ be the six vertices between $u$ and $v$ in the cube, with $w_1 \leq_1 t_1, t_2$ and $w_2 \leq_1 t_1, t_3$ and $w_3 \leq_1 t_2, t_3$. For $i \in \{1, 2, 3\}$, let $s(C_{u,t_i}) = a_i$, $s(C_{u,v}) =$
Figure 3.3. The embedding of $\partial \mathcal{M}(u,v)$. Left: the embedding in $\partial (\mathbb{R}^2 \times \mathbb{R}^3)$. Right: the corresponding embedding in $\mathbb{R} \times \mathbb{R}^3$ obtained by flattening the corner.

Figure 3.4. Framing on the hexagon. Left: notation for the framings at the vertices and edges of the hexagon. Right: the graph $G$.

c_i, f(\mathcal{C}_{u,w_i}) = f_i$ and $f(\mathcal{C}_{t,v}) = g_i$. Finally, let $s(\mathcal{C}_{t_1,w_1}) = b_1$, $s(\mathcal{C}_{t_2,w_2}) = b_2$, $s(\mathcal{C}_{t_3,w_3}) = b_3$, $s(\mathcal{C}_{t_2,w_3}) = b_4$, $s(\mathcal{C}_{t_3,w_2}) = b_5$ and $s(\mathcal{C}_{t_3,w_3}) = b_6$. This information is encoded in the first part of Figure 3.4.

Consider the planar graph $G$ in the second part of Figure 3.4. Some edges are solid, some are dashed, and some are dotted. The vertices represent frames in $\{0\} \times \mathbb{R}^3$ as follows: if a vertex is labeled $cba$, then it represents the frame 

$$[(-1)^e e_1, e_2, \ldots, e_{1d}, (-1)^e e_{21}, e_{22}, \ldots, e_{2d}, (-1)^e e_{31}, e_{32}, \ldots, e_{3d}].$$

Each edge represents a frame path joining the frames at its endpoints as follows.

(i) If the edge is solid, that is, if it is at the boundary of the hexagon, then it is one of the edges of the framed 6-cycle $K$. 

(ii) If the edge is dashed, then it represents the image under flattening of the following path in \( \{0\} \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \); it is constant on the first \( \mathbb{R}^d \) and is a standard frame path on the remaining \( \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \).

(iii) If the edge is dotted, then it represents the image under flattening of the following path in \( \mathbb{R}^+ \times \{0\} \times \mathbb{R}^d \times \mathbb{R}^d \); it is constant on the last \( \mathbb{R}^d \) and is a standard frame path on the remaining \( \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \).

The element \( h_{u,v} \in H_1(\text{SO}(3d + 1)) \) is represented by the solid 6-cycle in \( G \). In order to compute \( h_{u,v} \), we will compute the homology classes of some other cycles in \( G \).

Consider the solid-dashed 5-cycle joining \( c_1 b_1 a_1, c_1 0 a_1, c_1 0 0, c_1 0 a_2 \) and \( c_1 b_3 a_2 \). Modulo extending by the constant map on \( \mathbb{R}^d \), the four dashed edges represent standard frame paths in \( \mathbb{R} \times \mathbb{R}^d \) and the solid edge is standard if and only if \( f_1 = 0 \). Therefore, this cycle represents the element \( f_1 \in H_1(\text{SO}(3d + 1)) \). We denote this by writing \( f_1 \) in the pentagonal region bounded by this 5-cycle in \( G \). The homology classes represented by the other 5-cycles are shown in Figure 3.4.

Next consider the dotted-dashed 4-cycle connecting \( c_1 b_1 a_1, 0 0 a_1, 0 0 0 \) and \( c_1 0 0 \). If \( a_1 = 0 \), then the dashed edges represent the constant paths, and the two dotted edges represent the same path; therefore, the cycle is null-homologous. Similarly, if \( c_1 = 0 \), the cycle is null-homologous as well. Finally, if \( a_1 = c_1 = 1 \), it is easy to check from the definition of standard paths (Subsection 3.2) that the cycle represents the generator of \( H_1(\text{SO}(3d + 1)) \). Therefore, the cycle represents the element \( a_1 c_1 \). The contributions from such 4-cycles are also shown in Figure 3.4.

Finally, consider the dotted-dashed 2-cycle connecting \( c_1 b_1 a_1 \) and \( c_1 0 a_1 \). If \( b_1 = 0 \), then both edges represent constant paths, and hence the cycle is null-homologous. So, let us concentrate on the case when \( b_1 = 1 \). Let

\[
\varphi_1 = [(-1)^{c_1} e_{11}, \ldots, e_{1d}, -e_{21}, \ldots, e_{2d}, (-1)^{a_1} e_{31}, \ldots, e_{3d}]
\]

and

\[
\varphi_2 = [(-1)^{c_1} e_{11}, \ldots, e_{1d}, e_{21}, \ldots, e_{2d}, (-1)^{a_1} e_{31}, \ldots, e_{3d}]
\]

be the two frames in \( \{0\} \times \mathbb{R}^{2d} \). The dashed edge represents the path from \( \varphi_1 \) to \( \varphi_2 \), where the \((d+2)^{\text{th}}\) vector rotates \( 180^\circ \) in the \((\bar{e}_2, e_{21})\)-plane and equals \( e_2 = -\bar{e}_1 \) halfway through. The dotted edge also represents a path where the \((d+2)^{\text{th}}\) vector rotates \( 180^\circ \) in the \((\bar{e}_1, e_{21})\)-plane. However, halfway through, it equals \( \bar{e}_1 \) if \( c_1 = 0 \), and it equals \( -\bar{e}_1 \) if \( c_1 = 1 \). Hence, when \( b_1 = 0 \), the dotted-dashed 2-cycle is null-homologous if and only if \( c_1 = 1 \). Therefore, the cycle represents the element \( b_1 (c_1 + 1) = b_1 c_1 + b_1 \). These contributions are also shown in Figure 3.4.

Now,

\[
h_{u,v} = (f_1 + f_2 + f_3 + g_1 + g_2 + g_3) + (b_1 + b_2 + b_3 + b_4 + b_5 + b_6)
\]

\[
+ (b_1 c_1 + b_3 c_1 + b_4 c_3 + b_6 c_3 + b_5 c_2 + b_2 c_2)
\]

\[
+ (a_1 c_1 + a_2 c_1 + a_3 c_3 + a_3 c_2 + a_1 c_2)
\]

\[
= (c_1 + c_2 + c_3) + (b_1 + b_2 + b_3 + b_4 + b_5 + b_6) \quad \text{(by Lemma 2.1)}
\]

\[
+ (c_1 (a_1 + a_2 + b_1 + b_3) + c_2 (a_1 + a_3 + b_2 + b_5) + c_3 (a_2 + a_3 + b_4 + b_6))
\]

\[
= (c_1 + c_2 + c_3) + (b_1 + b_2 + b_3 + b_4 + b_5 + b_6)
\]

\[
+ (c_1 + c_2 + c_3) \quad \text{(since } s \text{ is a sign assignment)}
\]

\[
= (b_1 + b_2 + c_1 + c_2) + (b_3 + b_4 + c_1 + c_3) + (b_5 + b_6 + c_2 + c_3)
\]

\[
= 1 + 1 + 1 \quad (s \text{ is still a sign assignment})
\]

\[
= 1,
\]

as desired. \( \square \)
3.4. \( Sq^2 \) for the Khovanov homotopy type

Fix a link diagram \( L \) and an integer \( \ell \), and let \( \mathcal{X}^\ell_{\text{Kh}}(L) \) denote the Khovanov homotopy type constructed in [8]. We want to study the Steenrod square

\[
\text{Sq}^2 : \tilde{H}^\infty(\mathcal{X}^\ell_{\text{Kh}}(L)) \longrightarrow \tilde{H}^{\infty+2}(\mathcal{X}^\ell_{\text{Kh}}(L)).
\]

The spectrum \( \mathcal{X}^\ell_{\text{Kh}}(L) \) is a formal de-suspension \( \Sigma^{-N}Y_\ell \) of a CW complex \( Y_\ell \) for some sufficiently large \( N \). Therefore, we want to understand the Steenrod square

\[
\text{Sq}^2 : H^{N+\kappa}(Y_\ell; \mathbb{F}_2) \longrightarrow H^{N+\kappa+2}(Y_\ell; \mathbb{F}_2).
\]

Before we get started, we give names to a few maps which will make regular appearances. Fix \( m_1, m_2 \geq 2 \) and let \( m = m_1 + m_2 \). First, recall that we have maps \( r, s : \mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^m \) given by Formulas (3.1) and (3.2), respectively. Next, let

\[
\Pi : \mathbb{D}^m \longrightarrow K_{m}^{(m+1)}
\]

be the composition of the projection map \( \mathbb{D}^m \to \mathbb{D}^m / \partial \mathbb{D}^m = S^m \) and the inclusion \( S^m \to K_{m}^{(m+1)} \). Let

\[
\Xi : [0, 1] \times \mathbb{D}^m \longrightarrow K_{m}^{(m+1)}
\]

be the map induced by the identification of \([0, 1] \times \mathbb{D}^m \) with the \((m + 1)\)-cell \( e^{m+1} \) of \( K_{m}^{(m+1)} \); the map \( \Xi \) collapses \([0, 1] \times \partial \mathbb{D}^m \) to the basepoint and maps each of \([0] \times \mathbb{D}^m \) and \([1] \times \mathbb{D}^m \) to the \( m \)-skeleton \( S^m \subset K_{m}^{(m+1)} \) by \( \Pi \) and \( \Pi \circ r \), respectively. The map \( \Xi \) factors through a map

\[
\tilde{\Xi} : [0, 1] \times S^m \longrightarrow K_{m}^{(m+1)}.
\]

We construct a CW complex \( X \) as follows. Choose real numbers \( \epsilon \) and \( R \) with \( 0 < \epsilon \ll R \). Then:

**Step 1:** Start with a unique 0-cell \( e^0 \).

**Step 2:** For each Khovanov generator \( x_i \in KG^{\kappa, \ell} \), \( X \) has a corresponding cell

\[
f_i^m = \{0\} \times \{0\} \times [-\epsilon, \epsilon]^{m_1} \times [-\epsilon, \epsilon]^{m_2}.
\]

The boundary of \( f_i^m \) is glued to \( e^0 \).

**Step 3:** For each Khovanov generator \( y_j \in KG^{\kappa+1, \ell} \), \( X \) has a corresponding cell

\[
f_j^{m+1} = \{0, R\} \times \{0\} \times [-R, R]^{m_1} \times [-\epsilon, \epsilon]^{m_2}.
\]

The boundary of \( f_j^{m+1} \) is attached to \( X^{(m)} \) as follows. If \( y_j \) occurs in \( \delta_2 x_i \) with sign \( \epsilon_{i,j} \in \{\pm 1\} \), then we embed \( f_i^m \) in \( \partial f_j^{m+1} \) by a map of the form

\[
\begin{align*}
(0, 0, x_1, x_2, \ldots, x_{m_1}, y_1, \ldots, y_{m_2})
\quad \longmapsto \quad
(0, 0, \epsilon_{i,j} x_1 + a_1, x_2 + a_2, \ldots, x_{m_1} + a_{m_1}, y_1, \ldots, y_{m_2})
\end{align*}
\]

for some vector \( (a_1, \ldots, a_{m_1}) \). Call the image of this embedding \( C_i(j) \). Then \( C_i(j) \) is mapped to \( f_i^m \) by the specified identification, and \( (\partial f_j^{m+1}) \setminus \bigcup_i C_i(j) \) is mapped to the basepoint \( e^0 \).

We choose the vectors \( (a_1, \ldots, a_{m_1}) \) so that the different images \( C_i(j) \) are disjoint. Write \( p_{i,j} = (0, a_1, \ldots, a_{m_1}, 0, \ldots, 0) \in f_j^{m+1} \).

**Step 4:** For each Khovanov generator \( z_k \in KG^{\kappa+2, \ell} \), \( X \) has a corresponding cell

\[
f_k^{m+2} = \{0, R\} \times \{0, R\} \times [-R, R]^{m_1} \times [-R, R]^{m_2}.
\]
The boundary of $f^m_k$ is attached to $X^{(m+1)}$ as follows. First, we choose some auxiliary data.

(i) If $z_k$ occurs in $\delta z y_j$ with sign $\epsilon_{j,k} \in \{\pm 1\}$, then we embed $f^{m+1}_j$ in $\partial f^{m+2}_k$ by a map of the form

$$(x_0, 0, x_1, \ldots, x_m, y_1, \ldots, y_m) \mapsto (x_0, 0, x_1, 2, \ldots, x_m, 0, \epsilon_{j,k} y_1 + b_1, \ldots, y_m + b_{m_2}) \quad (3.4)$$

for some vector $(b_1, \ldots, b_{m_2})$. Call the image of this embedding $C_j(k)$. Once again, we choose the vectors $(b_1, \ldots, b_{m_2})$ so that the different $C_j(k)$'s are disjoint.

(ii) Let $x_i \in K G^{w, \ell}$. If the set of generators $\mathcal{G}_{x_i,y_i}$ between $z_k$ and $x_i$ is non-empty, then $G_{z_k,x_i}$ consists of 2 or 4 points, and these points are identified in pairs (via the ladybug matching of Subsection 2.4). Write $G_{z_k,x_i} = \{ y_{j',\ell} \}$. For each $j',\ell$, the cell $C_{i}(j',\ell)$ can be viewed as lying in the boundary of $C_{j}(k)$. Consider the point $p_{i,j',\ell}$ in the interior of $C_{i}(j',\ell) \subset \partial C_{j}(k)$. Each of the points $p_{i,j',\ell}$ inherits a framing, that is, a trivialization of the normal bundle to $p_{i,j',\ell}$ in $\partial C_{j}(k)$, from the map $f^m_i \to \partial f^{m+2}_k$.

$$(0, 0, x_1, \ldots, x_m, y_1, \ldots, y_m) \mapsto (0, 0, \epsilon_{j',\ell} x_1 + a_1, x_2 + a_2, \ldots, x_m + a_{m_1}, \epsilon_{j',\ell} y_1 + b_1, \ldots, y_m + b_{m_2})$$

Note that the framing of $p_{i,j',\ell}$ is one of the standard frames for $R^m = R^{m_1} \times R^{m_2}$ (Definition 3.2).

The pair of generators $(x_i, z_k)$ specifies a 2-dimensional face of the hypercube $C(n)$. Let $C_{k,i}$ denote this face. The standard frame assignment $f$ of Subsection 2.1 assigns an element $f(C_{k,i}) \in F_2$ to the face $C_{k,i}$.

The matching of elements of $G_{z_k,x_i}$ matches the points $p_{i,j',\ell}$ in pairs. Moreover, it follows from the definition of the sign assignment that matched pairs of points have opposite framings. For each matched pair of points choose a properly embedded arc connecting the pair of points. The endpoints of $\zeta$ are framed. Extend this to a framing of the normal bundle to $\zeta$ in $\partial f^{m+2}_k$. If $f(C_{k,i}) = 0$, then choose this framing to be isotopic relative boundary to a standard frame path for $\{0\} \times R \times R^m$; if $f(C_{k,i}) = 1$, then choose this framing to be isotopic relative boundary to a non-standard frame path for $\{0\} \times R \times R^m$.

We call these arcs $\zeta$ Pontryagin–Thom arcs, and denote the set of them by $\{ \zeta_{i,1,1}, \ldots, \zeta_{i,A,n} \}$, where the arc $\zeta_{i,a}$ comes from the generator $x_i \in K G^{w,\ell}$.

The choice of these auxiliary data is illustrated in Figure 3.5. Now, the attaching map on $\partial f^{m+2}_k$ is given as follows.

(i) The interior of $C_j(k)$ is mapped to $f^{m+1}_j$ by (the inverse of) the identification in Formula (3.4).

(ii) A tubular neighborhood of each Pontryagin–Thom arc $\text{nbd}(\zeta_{i,a})$ is mapped to $f^m_i$ as follows. The framing identifies

$$\text{nbd}(\zeta_{i,a}) \cong \zeta_{i,a} \times [-\epsilon, \epsilon]^{m_1+m_2} \cong \zeta_{i,a} \times f^m_{i,a}$$

With respect to this identification, the map is the obvious projection to $f^m_{i,a}$.

(iii) The rest of $\partial f^{m+2}_k$ is mapped to the basepoint $e^0$.

**Proposition 3.6.** Let $X$ denote the space constructed above and $Y_\ell = Y_\ell(L)$ the CW complex from [8] associated to $L$ in quantum grading $\ell$. Then

$$\Sigma^{N+\kappa-m} X = Y_\ell^{(N+\kappa+2)} / Y_\ell^{(N+\kappa-1)}.$$
Figure 3.5. The attaching map corresponding to a generator $z_7$. The boundary matching arcs are also shown; $\eta_{4,1}$ is boundary-coherent while $\tilde{\eta}_{5,1}$ and $\tilde{\eta}_{6,1}$ are boundary-incoherent.

Here, $m_1 = m_2 = 1$ and we have identified $([0,R] \times \{0\} \times [-R,R] \times [-R,R]) \cup (\{0\} \times [0,R] \times [-R,R] \times [-R,R]) \cong [-R,R] \times [-R,R] \times [-R,R] \times [-R,R]$ via a reflection on the first summand and the identity map on the second summand.

Proof. The construction of $X$ above differs from the construction of $Y_\ell$ in [8] as follows.

(i) We have collapsed all cells of dimension less than $m$ to the basepoint, and ignored all cells of dimension bigger than $m + 2$.

(ii) In the construction above, we have suppressed the 0-dimensional framed moduli spaces, instead speaking directly about the embeddings of cells that they induce. The 0-dimensional moduli spaces correspond to the points $(a_1, \ldots, a_{m_1})$ and $(b_1, \ldots, b_{m_2})$ in Formulas (3.3) and (3.4), respectively. Their framings are induced by the maps in Formulas (3.3) and (3.4).

(iii) In the construction of [8, Definition 3.23], each of the cells above were multiplied by $[0,R]^{p_1} \times [-R,R]^{p_2} \times [-\epsilon,\epsilon]^{p_3}$ for some $p_1, p_2, p_3 \in \mathbb{N}$ with $p_1 + p_2 + p_3 = N + \kappa - m$; and the various multiplicands were ordered differently. This has the effect of suspending the space $X$ by $(N + \kappa - m)$-many times.

(iv) The framings in [8] were given by an obstruction-theory argument [8, Proposition 4.12], while the framings here are given explicitly by the standard sign assignment and standard frame assignment. This is justified by Lemma 3.5.

Thus, up to stabilizing, the two constructions give the same space. \qed

Therefore, it is enough to study the Steenrod square $Sq^2: H^m(X;\mathbb{F}_2) \to H^{m+2}(X;\mathbb{F}_2)$. Fix a cohomology class $[c] \in H^m(X;\mathbb{F}_2)$. Let

$$c = \sum_{f_i^m} c_i (f_i^m)^*$$

be a cocycle representing $[c]$. Here, the $c_i$ are elements of $\{0, 1\}$. 
We want to understand the map \( c : X \to K^{(m+2)}_m \) corresponding to \( c \). We start with \( c^{(m)} : X^{(m)} \to K^{(m)}_m \) on \( f^m_j \), this map is defined as follows:

(i) the projection \( \Pi \) composed with the identification \([−\epsilon, \epsilon]^m = \mathbb{D}^m\) if \( c_j = 1\);
(ii) the constant map to the basepoint of \( K^{(m)}_m \) if \( c_j = 0\).

To extend \( c \) to \( X^{(m+1)} \), we need to make one more auxiliary choice:

**Definition 3.7.** A topological boundary matching for \( c \) consists of the following data for each \((m+1)\)-cell \( f^{m+1}_j \): a collection of disjoint, embedded, framed arcs \( \eta_{j,i} \) in \( f^{m+1}_j \) connecting the points \( \coprod_{i \mid c_i = 1} p_{i,j} \subset \partial f^{m+1}_j \) in pairs, together with framings of the normal bundles to the \( \eta_{j,i} \).

The normal bundle in \( f^{m+1}_j \) to each of the points \( p_{i,j} \) inherits a framing from Formula (3.3).

(i) Trivialize \( T f^{m+1}_j \) using the following inclusion:

\[
f^{m+1}_j = [0, R] \times \{0\} \times [-R, R]^{m1} \times [-\epsilon, \epsilon]^{m2} \to \mathbb{R}^{m+1} \\
(\ell, 0, x_1, \ldots, x_{m1}, y_1, \ldots, y_{m2}) \to (\ell, x_1, \ldots, x_{m1}, y_1, \ldots, y_{m2}).
\]

We require the framing of \( \eta_{j,i} \) to be isotopic relative boundary to one of the standard frame paths for \( \mathbb{R} \times \mathbb{R}^m \).

(ii) If \( \eta_{j,i} \) is boundary-coherent, then the framing of \( \eta_{j,i} \) is compatible with the framing of its boundary.

(iii) If \( \eta_{j,i} \) is boundary-incoherent, then the framing of one end of \( \eta_{j,i} \) agrees with the framing of the corresponding \( p_{i1,j} \) while the framing of the other end of \( \eta_{j,i} \) differs from the framing of \( p_{i2,j} \) by the reflection \( r : \mathbb{R}^m \to \mathbb{R}^m \).

Each boundary-incoherent arc in a topological boundary matching inherits an orientation: it is oriented from the endpoint \( p_{i,j} \) at which the framings agree, to the endpoint at which the framings disagree.

**Lemma 3.8.** A topological boundary matching for \( c \) exists.

**Proof.** Since \( c \) is a cocycle, for each \((m+1)\)-cell \( f^{m+1}_j \), we have

\[
\sum_i \epsilon_{i,j} c_i = c \left( \sum_i \epsilon_{i,j} f^m_i \right) = c(\partial f^{m+1}_j) \equiv 0 \pmod{2}.
\]

Together with our condition on \( m \), this ensures that a topological boundary matching for \( c \) exists.

The map \( c^{(m+1)} : X^{(m+1)} \to K^{(m+1)}_m \) is defined using the topological boundary matching as follows. On \( f^{m+1}_j \):

(i) The map \( c^{(m+1)} \) sends the complement of a neighborhood of the \( \eta_{j,i} \) to the basepoint.
(ii) If $\eta_{j,j}$ is boundary-coherent, then the framing of $\eta_{j,j}$ identifies a neighborhood of the arc $\eta_{j,j}$ with $\eta_{j,j} \times \mathbb{D}^m$. With respect to this identification, the map $c^{(m+1)}$ is projection $\eta_{j,j} \times \mathbb{D}^m \to \mathbb{D}^m \overset{\Pi}{\longrightarrow} K_m^{(m+1)}$.

Note that $c^{(m)}$ induces a map $(\partial \eta_{j,j}) \times \mathbb{D}^m$, and that the compatibility condition of the framing of $\eta_{j,j}$ with the framing of $\partial \eta_{j,j}$ implies that the map $c^{(m+1)}$ extends the map $c^{(m)}$.

(iii) If $\eta_{j,j}$ is boundary-incoherent, then the orientation and framing identify a neighborhood of $\eta_{j,j}$ with $[0, 1] \times \mathbb{D}^m$. With respect to this identification, $c^{(m+1)}$ is given by the map $\Xi$.

Again, $c^{(m)}$ induces a map $\{0, 1\} \times \mathbb{D}^m$, and the compatibility condition of the framing of $\partial \eta_{j,j}$ implies that the map $c^{(m+1)}$ extends the map $c^{(m)}$.

Now, fix an $(m + 2)$-cell $f^{m+2}$. We want to compute the element

$$c_{|\partial f^{m+2}} \in \pi_{m+1}(K_m^{(m+1)}) = \mathbb{Z}/2.$$

As described above,

$$\partial f^{m+2} = ([0] \times [0, R] \times [-R, R]^m) \cup ([0] \times [0, R] \times [-R, R]^m)$$

$$\cup ([0, 1] \times [0, R] \times [-R, R]^m) \cup ([0, 1] \times [0, R] \times [-R, R]^m)$$

has corners. The map $c_{|\partial f^{m+2}}$ will send

$$([0, 1] \times [0, R] \times [-R, R]^m) \cup ([0, R] \times [0, R] \times [-R, R]^m) \cup ([0, R] \times [0, 1] \times \partial([-R, R]^m))$$

to the basepoint. We straighten the corner between the other two parts of $\partial f^{m+2}$ via the map

$$([0] \times [0, R] \times [-R, R]^m) \cup ([0, R] \times [0, R] \times [-R, R]^m) \rightarrow [-R, R] \times [-R, R]^m$$

$$(0, t, x_1, \ldots, x_m, y_1, \ldots, y_m) \mapsto (t, x_1, \ldots, x_m, y_1, \ldots, y_m)$$

$$(t, 0, x_1, \ldots, x_m, y_1, \ldots, y_m) \mapsto (-t, x_1, \ldots, x_m, y_1, \ldots, y_m).$$

We will suppress this straightening from the notation in the rest of the section.

Let $\zeta_1, \ldots, \zeta_k \subset S^{m+1} = \partial f^{m+2}$ be the Pontryagin–Thom arcs corresponding to $f$. Let $\tilde{\eta}_{j,j}$ be the preimages in $S^{m+1} = \partial f^{m+2}$ of the topological boundary matching. The union

$$\bigcup_{j,j} \tilde{\eta}_{j,j} \cup \bigcup_{i} \zeta_i$$

is a one-manifold in $S^{m+1}$. Each of the arcs $\zeta_i \subset \partial f^{m+2}$ comes with a framing. Each of the arcs $\tilde{\eta}_{j,j} \subset \partial f^{m+2}$ also inherits a framing: the pushforward of the framing of $\eta_{j,j}$ under the map of Formula (3.4). The map $c_{|\partial f^{m+2}}: S^{m+1} \rightarrow K_m^{(m+1)}$ is induced from these framed arcs as follows.

(i) A tubular neighborhood of each Pontryagin–Thom arc $\zeta_i$ is mapped via

$$\text{nbd}(\zeta_i) \cong \zeta_i \times \mathbb{D}^m \rightarrow \mathbb{D}^m \overset{\Pi}{\longrightarrow} S^m,$$

where the first isomorphism is induced by the framing.

(ii) A tubular neighborhood of each boundary-coherent $\tilde{\eta}_{j,j}$ is mapped via

$$\text{nbd}(\tilde{\eta}_{j,j}) \cong \tilde{\eta}_{j,j} \times \mathbb{D}^m \rightarrow \mathbb{D}^m \overset{\Pi}{\longrightarrow} S^m,$$

where the first isomorphism is induced by the framing.

(iii) A tubular neighborhood of each boundary-incoherent $\tilde{\eta}_{j,j}$ is mapped via

$$\text{nbd}(\tilde{\eta}_{j,j}) \cong [0, 1] \times \mathbb{D}^m \overset{\Xi}{\longrightarrow} K_m^{(m+1)},$$

where the first isomorphism is induced by the orientation and framing.

(iv) The map $c$ takes the rest of $S^{m+1}$ to the basepoint of $K_m^{(m+1)}$. 

Let $K$ be a component of $\bigcup_j \zeta_i \cup \bigcup_{j,j'} \tilde{\eta}_{j,j'}$. Relabeling, let $p_1, \ldots, p_{2k}$ be the points $p_{i,j}$ on $K$, $\tilde{\eta}_1, \ldots, \tilde{\eta}_k$ the sub-arcs of $K$ coming from the topological boundary matching and $\zeta_1, \ldots, \zeta_k$ the sub-arcs of $K$ coming from the Pontryagin–Thom data. Order these so that $\partial \zeta_i = \{p_{2i-1}, p_{2i}\}$ and $\partial \tilde{\eta}_i = \{p_{2i}, p_{2i+1}\}$.

We define an isomorphism $\Phi: \text{nbd}(K) \to K \times D^m$ as follows. First, the framing of $\zeta_1$ induces an identification of the normal bundle $N_{p_1} K$ with $D^m$. Second, the framing of each arc $\gamma \in \{\zeta_1, \tilde{\eta}_1\}$ induces a trivialization of the normal bundle $N\gamma$. Suppose that the framing of $K$ has already been defined at the endpoint $p_i$ of $\gamma$. Then the trivialization of $N\gamma$ allows us to transport the framing of $p_i$ along $\gamma$. This transported framing is the framing of $K$ along $\gamma$.

Note that the framing of $K$ along $\gamma$ may not agree with the original framing of $\gamma$; but the two either agree or differ by the map

$$K \times D^m \to K \times D^m \quad (x, v) \mapsto (x, r(v)),$$

depending on the parity of the number of boundary-incoherent arcs traversed from $p_1$. In particular, it is not a priori obvious that the framing we have defined is continuous at $p_1$; but this follows from the following lemma:

**Lemma 3.9.** An even number of the arcs in $K$ are boundary-incoherent.

**Proof.** The proof is essentially the same as the proof of the second half of Lemma 2.8, and is left to the reader. \qed

Call an arc $\gamma$ in $K$ $r$-colored if the original framing of $\gamma$ disagrees with the framing of $K$, and Id-colored if the original framing of $\gamma$ agrees with the framing of $K$. Write $\Psi = c \circ \Phi^{-1}: K \times D^m \to K^{(m+1)}$. Explicitly, the map $\Psi$ is given as follows.

(i) If $\gamma_i$ is one of the Pontryagin–Thom arcs or is a boundary-coherent topological boundary matching arc then a neighborhood of $\gamma_i$ in $K \times D^m$ is mapped to $S^m \subset K^{(m+1)}$ by the map

$$\gamma_i \times D^m \ni (x, v) \mapsto \Pi(v) \in K^{(m+1)}_m \quad \text{if } \gamma_i \text{ is Id-colored}$$

$$\gamma_i \times D^m \ni (x, v) \mapsto \Pi(r(v)) \in K^{(m+1)}_m \quad \text{if } \gamma_i \text{ is } r\text{-colored}.$$

(ii) If $\tilde{\eta}_i$ is boundary-incoherent, then the framing of $K$ and the orientation of $\tilde{\eta}_i$ induce an identification $\text{nbd}(\tilde{\eta}_i) \cong [0, 1] \times D^m$. With respect to this identification, $\text{nbd}(\tilde{\eta}_i)$ is mapped to $K^{(m+1)}_m$ by the map

$$(t, v) \mapsto \Xi(t, v) \quad \text{if } \tilde{\eta}_i \text{ is Id-colored},$$

$$(t, v) \mapsto \Xi(t, r(v)) \quad \text{if } \tilde{\eta}_i \text{ is } r\text{-colored}.$$

Let $\Psi'$ be the projection $K \times D^m \to D^m \overset{\Pi}{\longrightarrow} S^m$. These maps are summarized in the following diagram:

\[
\begin{array}{ccc}
\text{nbd}(K) & \xrightarrow{\Phi} & K \times D^m \\
\downarrow & & \downarrow \\
& \xrightarrow{\Psi'} & K^{(m+1)}_m \\
\end{array}
\]

\[\Xi\]

It is immediate from the definitions that the top triangle commutes. Our next goal is to show that the other triangle commutes up to homotopy.
**Proposition 3.10.** The map $\Psi$ is homotopic relative $(K \times \partial \mathbb{D}^m) \cup \{p_1\} \times \mathbb{D}^m$ to $\iota \circ \Psi'$, that is, the bottom triangle of Diagram (3.5) commutes up to homotopy relative $(K \times \partial \mathbb{D}^m) \cup \{p_1\} \times \mathbb{D}^m$.

The proof of Proposition 3.10 uses a model computation. Consider the map $\Xi: [0, 1] \times S^m \to K_{m+1}$. Concatenation in $[0, 1]$ endows Hom$([0, 1] \times S^m, K_{m+1})$ with a multiplication, which we denote by $\ast$. Let $t: [0, 1] \to [0, 1]$ be the reflection $f(t, x) = f(1 - t, x)$. Using $t$, we obtain a map $\Xi \circ (t \times Id): [0, 1] \times S^m \to K_{m+1}$. Finally, using the map $t$ we obtain a map $\Xi \circ (Id \times t): [0, 1] \times S^m \to K_{m+1}$.

**Lemma 3.11.** Assume that $m \geq 3$. Then both $\Xi \ast [\Xi \circ (t \times Id)]$ and $\Xi \ast [\Xi \circ (Id \times t)]$ are homotopic (relative boundary) to the map $[0, 1] \times S^m \to S^m \subset K_m$ given by $(t, x) \mapsto x$ (that is, the constant path in $O(m + 1)$ with value $Id$).

**Proof.** The statement about $\Xi \ast [\Xi \circ (t \times Id)]$ is obvious. For the statement about $\Xi \ast [\Xi \circ (Id \times t)]$, let $H_m = \Xi \ast [\Xi \circ (Id \times t)]: [0, 1] \times S^m \to K_{m+1}$. We can view $H_m$ as an element of $\pi_1(\Omega^m K_{m+1}) \cong \pi_{m+1}(K_{m+1})$. Moreover, the map $H_m$ is the $(m - 1)$-fold suspension of the map $H_1: [0, 1] \times K^1 \to K^1_{m+1}) = \mathbb{R}P^2$. But the suspension map

$$\Sigma^i: \pi_2(\mathbb{R}P^2) \to \pi_{i+2}(\Sigma^i \mathbb{R}P^2)$$

is nullhomotopic for $i \geq 2$; see, for instance, [14, Proposition 6.5 and discussion before Proposition 6.11]. So, it follows from our assumption on $m$ that $H_1 \in \pi_1(\Omega^m K_{m+1})$ is homotopically trivial. Bearing in mind that our loops are based at the identity map $S^m \to S^m \subset K_m$, this proves the result.

**Proof of Proposition 3.10.** As in the proof of Lemma 3.11, we can view $\Psi$ as an element of $\pi_1(\Omega^m K_{m+1}, \iota)$, that is, a loop of maps $S^m \to K_{m+1}$ based at the map $\iota: S^m \to K_{m+1}$. From its definition, $\Psi$ decomposes as a product of paths,

$$\Psi = \Psi_{\gamma_1} \ast \cdots \ast \Psi_{\gamma_k},$$

one for each arc $\gamma_i$ in $K$. Here, $\Psi_{\gamma_i}$ is an element of the fundamental groupoid of $\Omega^m K_{m+1}$, with endpoints in $\{\iota, \iota \circ t\}$. The path $\Psi_{\gamma_i}$ is:

(i) The constant path based at either $\iota$ or $\iota \circ t$ if $\gamma_i$ is one of the Pontryagin–Thom arcs or is a boundary-coherent topological boundary matching arc.

(ii) One of the paths $\Xi, \Xi \circ (Id \times t), \Xi \circ (t \times Id)$ or $\Xi \circ (t \times Id) \circ (Id \times t)$ if $\gamma_i$ is a boundary-incoherent topological boundary matching arc.

Contracting the constant paths, $\Psi$ can be expressed as

$$\Psi = \Psi_{\eta_{i_1}} \ast \Psi_{\eta_{i_2}} \ast \cdots \ast \Psi_{\eta_{i_A}},$$

where the $\eta_{\alpha}$ are boundary-incoherent. By Lemma 3.9, $A$ is even. Moreover, $\Psi_{\eta_{\alpha}}$ is either $\Xi$ or $\Xi \circ (t \times Id) \circ (Id \times t)$ if $\alpha$ is odd, and is either $\Xi \circ (t \times Id)$ or $\Xi \circ (Id \times t)$ if $\alpha$ is even. So, by Lemma 3.11, the concatenation $\Psi_{\eta_{2\alpha-1}} \ast \Psi_{\eta_{2\alpha}}$ is homotopic to the constant path $\iota$. The result follows. \qed

The pair $(K, \Phi)$ specifies a framed cobordism class $[K, \Phi] \in \Omega^m \mathbb{Z} = \mathbb{Z}/2$. 

Proposition 3.12. The element $[K, \Phi] \in \mathbb{Z}/2$ is given by the sum of the following.

(i) The number 1.
(ii) The number of Pontryagin–Thom arcs in $K$ with the non-standard framing.
(iii) The number of arrows on $K$ which point in a given direction.

Proof. First, exchanging the standard and non-standard framings on an arc changes the overall framing of $K$ by 1. So, it suffices to prove the proposition in the case that all of the Pontryagin–Thom arcs in $K$ have the standard framing.

Second, the framing on each boundary-matching arc is standard if the corresponding $(m+1)$-cell occurs positively in $\partial f^{m+2}$, and differs from the standard framing by the map $s$ of Subsection 3.2 if the $(m+1)$-cell occurs negatively in $\partial f^{m+2}$. In the notation of Subsection 3.2, the framings of the boundary matching arcs are among $\{\pm, \pm, \pm, \pm\}$. So, by Lemma 3.3, $s$ takes the standard frame path on a boundary-matching arc to a standard frame path. In sum, each of the arcs $\tilde{\eta}_i$ is framed by a standard frame path.

So, by Lemma 3.3, the framing of $K$ at each arc $\gamma_i$ is standard if $\gamma_i$ is Id-colored, and non-standard if $\gamma_i$ is $r$-colored. Thus, it suffices to show that the number of $r$-colored arcs agrees modulo 2 with the number of arrows on $K$ which point in a given direction.

Let $\tilde{\eta}_{i_1}, \ldots, \tilde{\eta}_{i_A}$ be the boundary-incoherent boundary matching arcs in $K$. Then we have the following.

(1) There are an odd number of arcs strictly between $\tilde{\eta}_{i_\alpha}$ and $\tilde{\eta}_{i_{\alpha+1}}$. Moreover:
   (i) these arcs are all $r$-colored if $\alpha$ is odd;
   (ii) these arcs are all Id-colored if $\alpha$ is even.

(2) If $\tilde{\eta}_{i_\alpha}$ and $\tilde{\eta}_{i_{\alpha+1}}$ are oriented in the same direction, then exactly one of $\tilde{\eta}_{i_\alpha}$ and $\tilde{\eta}_{i_{\alpha+1}}$ is $r$-colored.

(3) If $\tilde{\eta}_{i_\alpha}$ and $\tilde{\eta}_{i_{\alpha+1}}$ are oriented in opposite directions, then either both of $\tilde{\eta}_{i_\alpha}$ and $\tilde{\eta}_{i_{\alpha+1}}$ are $r$-colored or both of $\tilde{\eta}_{i_\alpha}$ and $\tilde{\eta}_{i_{\alpha+1}}$ are Id-colored.

It follows that there are an even (respectively odd) number of $r$-colored arcs in the interval $[\tilde{\eta}_{i_{2\alpha+1}}, \tilde{\eta}_{i_{2\alpha}}]$ if $\tilde{\eta}_{i_{2\alpha+1}}$ and $\tilde{\eta}_{i_{2\alpha}}$ are oriented in the same (respectively opposite) directions; and all of the arcs in the interval $(\tilde{\eta}_{i_{2\alpha+1}}, \tilde{\eta}_{i_{2\alpha+1}})$ are Id-colored. So, the number of $r$-colored arcs agrees with the number of arcs which point in a given direction.

Finally, the contribution 1 comes from the fact that the constant loop in $\text{SO}(m)$ corresponds to the non-trivial element of $\Omega^1\mathbb{F}$.

Proof of Theorem 2.11. Let $\Phi_K$ and $\Psi_K$ be the maps associated above to each component $K$ of $\bigcup_i \zeta_i \cup \bigcup_{j,j} \tilde{\eta}_{j,j}$. Then $\Psi_K \circ \Phi_K$ induces an element $[\Psi_K \circ \Phi_K]$ of $\pi_{m+1}(K^{m+1}) \cong \mathbb{Z}/2$ (by collapsing everything outside a neighborhood of $K$ to the basepoint). Let $z$ be the generator of the Khovanov complex corresponding to the cell $f^{m+2}$ and $c$ the element of the Khovanov chain group corresponding to the cocycle $c$. Then it suffices to show that the sum

$$
\sum_K [\Psi_K \circ \Phi_K]
$$

agrees with the expression

$$
(\#\{c(z)\} + f(c(z)) + g(c(z)))z,
$$

from Formula (2.3).
By Proposition 3.10,

\[ [K, \Phi] \rightarrow \rightarrow [\Psi_K \circ \Phi_K] \]

\[ \Omega_1^{fr} = \pi_{m+1}(S^m) \xrightarrow{\iota_*} \pi_{m+1}(K_{m+1}). \]

The element \([K, \Phi_K] \in \mathbb{Z}/2\) is computed in Proposition 3.12, and it remains to match the terms in that proposition with the terms in Formula (3.6).

By construction, the graph \(\mathcal{G}_e(z)\) is exactly \(\bigcup_i \zeta_i \cup \bigcup_{j,j'} \tilde{\eta}_{j,j'}\), and the orientations of the oriented edges of \(\mathcal{G}_e(z)\) match up with the orientations of the boundary-incoherent \(\tilde{\eta}_i\). So, the first term in Formula (3.6) corresponds to part (i) of Proposition 3.12, and the third term in Formula (3.6) corresponds to part (iii) of Proposition 3.12. Finally, since the framings of the Pontryagin–Thom arcs differ from the standard frame paths by the standard frame assignment \(f\), the second term of Formula (3.6) corresponds to part (ii) of Proposition 3.12. This completes the proof.

4. The Khovanov homotopy type of width three knots

It is immediate from Whitehead’s theorem that if \(\hat{H}_i(X)\) is trivial for all \(i \neq m\), then \(X\) is a Moore space; in particular, the homotopy type of \(X\) is determined by the homology of \(X\) in this case. This result can be extended to spaces with non-trivial homology in several gradings, if one also keeps track of the action of the Steenrod algebra. To determine the Khovanov homotopy types of links up to 11 crossings, we will use such an extension due to Whitehead [13] and Chang [3], which we review here. (For further discussion along these lines, as well as the next larger case, see [2, Section 11].)

**Proposition 4.1.** Consider quivers of the form:

\[
\begin{array}{ccc}
A & \xrightarrow{s} & B \\
\downarrow f & & \downarrow g \\
C
\end{array}
\]

where \(A, B\) and \(C\) are \(\mathbb{F}_2\)-vector spaces, and \(gf = 0\). Such a quiver uniquely decomposes as a direct sum of the following quivers:

\[
\begin{array}{cccccccc}
(S-1) & \mathbb{F}_2 & 0 & 0 & (S-2) & 0 & \mathbb{F}_2 & 0 & (S-3) & 0 & 0 & \mathbb{F}_2 \\
(P-1) & \mathbb{F}_2 & \xrightarrow{\text{Id}} \mathbb{F}_2 & 0 & (P-2) & 0 & \mathbb{F}_2 & \xrightarrow{\text{Id}} \mathbb{F}_2 & (X-1) & \mathbb{F}_2 & 0 & \mathbb{F}_2 \\
(X-2) & \mathbb{F}_2 & \xrightarrow{\text{Id}} \mathbb{F}_2 & \mathbb{F}_2 & (X-3) & \mathbb{F}_2 & \xrightarrow{\text{Id}} \mathbb{F}_2 & (X-4) & \mathbb{F}_2 & \xrightarrow{(\text{Id}_0)} \mathbb{F}_2 \oplus \mathbb{F}_2 & \xrightarrow{(0,\text{Id})} \mathbb{F}_2
\end{array}
\]

**Proof.** We start with uniqueness. In such a decomposition, let \(s_i\) be the number of \((S-i)\) summands, \(p_i\) be the number of \((P-i)\) summands and \(x_i\) be the number of \((X-i)\) summands. Consider the nine pieces of data.

(i) The dimensions of the \(\mathbb{F}_2\)-vector spaces \(A, B\) and \(C\), say \(d_1, d_2, d_3\), respectively.
(ii) The ranks of the maps \(f\) and \(g\), say \(r_f, r_g\), respectively.
(iii) The dimensions of the $F_2$-vector spaces $\text{im}(s)$, $\text{im}(s|_{\ker(f)})$, $\text{im}(g) \cap \text{im}(s)$ and $\text{im}(g) \cap \text{im}(s|_{\ker(f)})$, say $r_1, r_2, r_3, r_4$, respectively.

We have

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
s_1 \\
s_2 \\
s_3 \\
p_1 \\
p_2 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{pmatrix}
= \begin{pmatrix}
d_1 \\
d_2 \\
d_3 \\
d_4 \\
\end{pmatrix}
\]

and therefore, the numbers $s_i, p_i$ and $x_i$ are determined as follows:

\[
\begin{pmatrix}
s_1 \\
s_2 \\
s_3 \\
p_1 \\
p_2 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
d_1 \\
d_2 \\
d_3 \\
d_4 \\
\end{pmatrix}
\]

For existence of such a decomposition, we carry out a standard change-of-basis argument. Choose generators for $A$, $B$, and $C$, and construct the following graph. There are three types of vertices, $A$-vertices, $B$-vertices and $C$-vertices, corresponding to generators of $A$, $B$ and $C$, respectively. There are three types of edges, $f$-edges, $g$-edges and $s$-edges, corresponding to the maps $f$, $g$ and $s$ as follows: for a an $A$-vertex and $b$ a $B$-vertex, if $b$ appears in $f(a)$, then there is an $f$-edge joining $a$ and $b$; the $g$-edges and $s$-edges are defined similarly.

We will do a change of basis, which will change the graph, so that in the final graph, each vertex is incident to at most one edge of each type. This will produce the required decomposition of the quiver.

We carry out the change of basis in the following sequence of steps. Each step accomplishes a specific simplification of the graph; it can be checked that the later steps do not undo the earlier simplifications.

(i) We ensure that no two $f$-edges share a common vertex. Fix an $f$-edge joining an $A$-vertex $a$ to a $B$-vertex $b$. Let $\{a_i\}$ be the other $A$-vertices that are $f$-adjacent to $b$ and $\{b_j\}$ the other $B$-vertices that are $f$-adjacent to $a$. Then change basis by replacing each $a_i$ by $a_i + a$, and by replacing $b$ with $b + \sum_j b_j$.

(ii) By the same procedure as Step (i), we ensure that no two $g$-edges share a common vertex. Since $gf = 0$, this ensures that no $B$-vertex is adjacent to both an $f$-edge and a $g$-edge. Call an $A$-vertex an $A_1$-vertex (respectively $A_2$-vertex) if it is adjacent (respectively non-adjacent) to an $f$-edge; similarly, call a $C$-vertex a $C_1$-vertex (respectively $C_2$-vertex) if it is adjacent (respectively non-adjacent) to a $g$-vertex.

(iii) Next, we isolate the $s$-edges that connect $A_2$-vertices to $C_2$-vertices. Fix an $s$-edge joining an $A_2$-vertex $a$ to a $C_2$-vertex $c$. If $\{a_i\}$ (respectively $\{c_j\}$) are the other $A$-vertices (respectively $C$-vertices) that are $s$-adjacent to $c$ (respectively $a$), then change basis by replacing each $a_i$ by $a_i + a$ and by replacing $c$ with $c + \sum_j c_j$.

(iv) The next step is to isolate the $s$-edges that connect $A_1$-vertices to $C_2$-vertices. Once again, fix an $s$-edge joining an $A_1$-vertex $a$ to a $C_2$-vertex $c$. Let $\{a_i\}$ (respectively $\{c_j\}$) be the
other $A$-vertices (respectively $C$-vertices) that are $s$-adjacent to $c$ (respectively $a$). Let $b_i$ be the $B$-vertex that is $f$-adjacent to $a_i$ (observe, each $a_i$ is an $A_1$-vertex), and let $b$ be the $B$-vertex that is $f$-adjacent to $a$. Then change basis by replacing each $a_i$ by $a_i + a$, by replacing each $b_i$ by $b_i + b$ and by replacing $c$ with $c + \sum_j c_j$.

(v) Similarly, we can isolate the $s$-edges that connect $A_2$-vertices to $C_1$-vertices. As before, fix an $s$-edge joining an $A_2$-vertex $a$ to a $C_1$-vertex $c$. Let $\{a_i\}$ (respectively $\{c_j\}$) be the other $A$-vertices (respectively $C$-vertices) that are $s$-adjacent to $c$ (respectively $a$). Let $b_j$ be the $B$-vertex that is $g$-adjacent to $c_j$ and let $b$ be the $B$-vertex that is $g$-adjacent to $c$. Then change basis by replacing each $a_i$ by $a_i + a$, by replacing $b$ with $b + \sum_j b_j$ and by replacing $c$ with $c + \sum_j c_j$.

(vi) Finally, we have to isolate the $s$-edges that connect $A_1$-vertices to $C_1$-vertices. This can be accomplished by a combination of the previous two steps.

The graph now has the desired form, completing the proof.

We are interested in stable spaces $X$ satisfying the following conditions.

(i) The only torsion in $H^*(X; \mathbb{Z})$ is 2-torsion.
(ii) The cohomology groups satisfy $\tilde{H}^i(X; \mathbb{F}_2) = 0$ if $i \neq 0, 1, 2$.

Then the quiver

$$
\tilde{H}^0(X; \mathbb{F}_2) \xrightarrow{\text{Sq}^2} \tilde{H}^1(X; \mathbb{F}_2) \xrightarrow{\text{Sq}^1} \tilde{H}^2(X; \mathbb{F}_2)
$$

is of the form described in Proposition 4.1. In Examples 4.1–4.5, we will describe nine such spaces whose associated quivers are the nine irreducible ones of Proposition 4.1.

**Example 4.1.** The associated quivers of $S^0$, $S^1$ and $S^2$ are (S-1), (S-2) and (S-3), respectively. The associated quivers of $\Sigma^{-1}\mathbb{R}P^2$ and $\mathbb{R}P^2$ are (P-1) and (P-2), respectively.

**Example 4.2.** The space $\mathbb{C}P^2$ has cohomology

$$
\tilde{H}^4(\mathbb{C}P^2; \mathbb{Z}) \xrightarrow{\mathbb{Z}} \tilde{H}^4(\mathbb{C}P^2; \mathbb{F}_2) \xrightarrow{\mathbb{F}_2} \\
\tilde{H}^3(\mathbb{C}P^2; \mathbb{Z}) \xrightarrow{0} \tilde{H}^3(\mathbb{C}P^2; \mathbb{F}_2) \xrightarrow{0} \\
\tilde{H}^2(\mathbb{C}P^2; \mathbb{Z}) \xrightarrow{\mathbb{Z}} \tilde{H}^2(\mathbb{C}P^2; \mathbb{F}_2) \xrightarrow{\mathbb{F}_2}
$$

(The fact that $\text{Sq}^2$ has this form follows from the fact that for $x \in H^n$, $\text{Sq}^n(x) = x \cup x$.) Therefore, the stable space $X_1 := \Sigma^{-2}\mathbb{C}P^2$ has (X-1) as its associated quiver.
**Example 4.3.** The space $\mathbb{R}P^5/\mathbb{R}P^2$ has cohomology

\[
\begin{array}{ccc}
\tilde{H}^5(\mathbb{R}P^5/\mathbb{R}P^2; \mathbb{Z}) & Z & \tilde{H}^5(\mathbb{R}P^5/\mathbb{R}P^2; \mathbb{F}_2) \\
\tilde{H}^4(\mathbb{R}P^5/\mathbb{R}P^2; \mathbb{Z}) & \mathbb{F}_2 & \tilde{H}^4(\mathbb{R}P^5/\mathbb{R}P^2; \mathbb{F}_2) \\
\tilde{H}^3(\mathbb{R}P^5/\mathbb{R}P^2; \mathbb{Z}) & 0 & \tilde{H}^3(\mathbb{R}P^5/\mathbb{R}P^2; \mathbb{F}_2)
\end{array}
\]

To see that $\text{Sq}^2$ has the stated form, consider the inclusion map $\mathbb{R}P^5/\mathbb{R}P^2 \to \mathbb{R}P^6/\mathbb{R}P^2$. The map $\text{Sq}^3: H^3(\mathbb{R}P^6/\mathbb{R}P^2) \to H^6(\mathbb{R}P^6/\mathbb{R}P^2)$ is an isomorphism (since it is just the cup square). By the Adem relations, $\text{Sq}^3 = \text{Sq}^1 \text{Sq}^2$, so $\text{Sq}^2: H^3(\mathbb{R}P^6/\mathbb{R}P^2) \to H^5(\mathbb{R}P^6/\mathbb{R}P^2)$ is non-trivial. So, the corresponding statement for $\mathbb{R}P^5/\mathbb{R}P^2$ follows from naturality. Therefore, the stable space $X_2 := \Sigma^{-3}(\mathbb{R}P^5/\mathbb{R}P^2)$ has $(X-2)$ as its associated quiver.

**Example 4.4.** The space $\mathbb{R}P^4/\mathbb{R}P^1$ has cohomology

\[
\begin{array}{ccc}
\tilde{H}^4(\mathbb{R}P^4/\mathbb{R}P^1; \mathbb{Z}) & \mathbb{F}_2 & \tilde{H}^4(\mathbb{R}P^4/\mathbb{R}P^1; \mathbb{F}_2) \\
\tilde{H}^3(\mathbb{R}P^4/\mathbb{R}P^1; \mathbb{Z}) & 0 & \tilde{H}^3(\mathbb{R}P^4/\mathbb{R}P^1; \mathbb{F}_2) \\
\tilde{H}^2(\mathbb{R}P^4/\mathbb{R}P^1; \mathbb{Z}) & Z & \tilde{H}^2(\mathbb{R}P^4/\mathbb{R}P^1; \mathbb{F}_2)
\end{array}
\]

(The answer for $\text{Sq}^2$ again follows from the fact that it is the cup square.) Therefore, the stable space $X_3 := \Sigma^{-2}(\mathbb{R}P^4/\mathbb{R}P^1)$ has $(X-3)$ as its associated quiver.

**Example 4.5.** The space $\mathbb{R}P^2 \wedge \mathbb{R}P^2$ has cohomology

\[
\begin{array}{ccc}
\tilde{H}^4(\mathbb{R}P^2 \wedge \mathbb{R}P^2; \mathbb{Z}) & \mathbb{F}_2 & \tilde{H}^4(\mathbb{R}P^2 \wedge \mathbb{R}P^2; \mathbb{F}_2) \\
\tilde{H}^3(\mathbb{R}P^2 \wedge \mathbb{R}P^2; \mathbb{Z}) & \mathbb{F}_2 & \tilde{H}^3(\mathbb{R}P^2 \wedge \mathbb{R}P^2; \mathbb{F}_2) \\
\tilde{H}^2(\mathbb{R}P^2 \wedge \mathbb{R}P^2; \mathbb{Z}) & 0 & \tilde{H}^2(\mathbb{R}P^2 \wedge \mathbb{R}P^2; \mathbb{F}_2)
\end{array}
\]

(The answer for $\text{Sq}^2$ follows from the product formula: $\text{Sq}^2(a \wedge b) = a \wedge \text{Sq}^2(b) + \text{Sq}^1(a) \wedge \text{Sq}^1(b) + \text{Sq}^2(a) \wedge b$.) Therefore, the quiver associated to the stable space $X_4 := \Sigma^{-2}(\mathbb{R}P^2 \wedge \mathbb{R}P^2)$ is isomorphic to $(X-4)$. 
The following is a classification theorem from [2, Theorems 11.2 and 11.7].

**Proposition 4.2.** Let $X$ be a simply connected CW complex such that:

(i) the only torsion in the cohomology of $X$ is 2-torsion;

(ii) there exists $m$ sufficiently large so that the reduced cohomology $	ilde{H}^i(X;\mathbb{F}_2)$ is trivial for $i \neq m, m+1, m+2$.

Then the homotopy type of $X$ is determined by the isomorphism class of the quiver

\[
\begin{array}{ccc}
H^m(X;\mathbb{F}_2) & \xrightarrow{Sq^2} & H^{m+1}(X;\mathbb{F}_2) \\
& \xrightarrow{Sq^2} & H^{m+2}(X;\mathbb{F}_2)
\end{array}
\]

as follows: Decompose the quiver as in Proposition 4.1; let $s_i$ be the number of (S-i) summands, $1 \leq i \leq 3$; let $p_i$ be the number of (P-i) summands, $1 \leq i \leq 2$; and let $x_i$ be the number of (X-i) summands, $1 \leq i \leq 4$. Then $X$ is homotopy equivalent to

\[
Y := \left( \bigvee_{i=1}^3 \bigvee_{j=1}^{s_i} \Sigma^{m+i-1} \right) \vee \left( \bigvee_{i=1}^2 \bigvee_{j=1}^{p_i} \Sigma^{m+i-2} \mathbb{R}P^2 \right) \vee \left( \bigvee_{i=1}^4 \bigvee_{j=1}^{x_i} \Sigma^m X_i \right).
\]

In light of Corollary 4.4, the following seems a natural link invariant.

**Definition 4.3.** For any link $L$, the function $\text{St}(L) : \mathbb{Z}^2 \to \mathbb{N}^4$ is defined as follows: Fix $(i, j) \in \mathbb{Z}^2$; for $k \in \{i, i+1\}$, let $\text{Sq}^k_1$ denote the map $\text{Sq}^k_1 : \text{K}^{k+2-j}_L \to \text{K}^{k+3-j}_L$.

Let $r_1$ be the rank of the map $\text{Sq}_1^2 : \text{K}^{i,j}_L \to \text{K}^{i+2,j}_L$; let $r_2$ be the rank of the map $\text{Sq}_1^{3,i} \ker \text{Sq}_1^{i+1}$; let $r_3$ be the dimension of the $\mathbb{F}_2$-vector space $\text{im} \text{Sq}_1^{i+1} \cap \text{im} \text{Sq}_2^1$; and let $r_4$ be the dimension of the $\mathbb{F}_2$-vector space $\text{im} \text{Sq}_1^{i+1} \cap \text{im} \text{Sq}_2^3 \ker \text{Sq}_1^{i+1}$. Then,

\[
\text{St}(i, j) := (r_2 - r_4, r_1 - r_2 - r_3 + r_4, r_4, r_3 - r_4).
\]

**Corollary 4.4.** Suppose that the Khovanov homology $\text{Kh}_2(L)$ satisfies the following properties.

(i) The Khovanov homology $\text{Kh}_2^{i,j}(L)$ lies on three adjacent diagonals, say $2i - j = \sigma, \sigma + 2, \sigma + 4$.

(ii) There is no torsion other than 2-torsion.

(iii) There is no torsion on the diagonal $2i - j = \sigma$.

Then the homotopy types of the stable spaces $\mathcal{X}^{i,j}_{\text{Kh}}(L)$ are determined by $\text{Kh}_2(L)$ and $\text{St}(L)$ as follows: Fix $j$; let $i = (j + \sigma)/2$; let $\text{St}(i, j) = (x_1, x_2, x_3, x_4)$; then $\mathcal{X}^{i,j}_{\text{Kh}}(L)$ is stably homotopy equivalent to the wedge sum of

\[
\left( \bigvee_{i=1}^{x_1} \Sigma^{i-2} \mathbb{C}P^2 \right) \vee \left( \bigvee_{i=1}^{x_2} \Sigma^{i-3}(\mathbb{R}P^5/\mathbb{R}P^2) \right) \vee \left( \bigvee_{i=1}^{x_3} \Sigma^{i-2}(\mathbb{R}P^4/\mathbb{R}P^1) \right) \vee \left( \bigvee_{i=1}^{x_4} \Sigma^{i-2} \mathbb{R}P^2 \right).
\]
and a wedge of Moore spaces. In particular, $\mathcal{X}_{\text{Kh}}^j(L)$ is a wedge of Moore spaces if and only if $x_1 = x_2 = x_3 = x_4 = 0$.

Proof. The first part is immediate from Proposition 4.2. To wit, if one decomposes the quiver

\[
\begin{array}{c}
\text{K}_2^j \\
\xrightarrow{\text{Sq}^2} \\
\text{K}_2^{j+1} \\
\xrightarrow{\text{Sq}^2} \\
\text{K}_2^{j+2}
\end{array}
\]

as a direct sum of the nine quivers of Proposition 4.1, Equation (4.1) implies that the number of $(X-i)$ summands will be $x_i$. The 'if' direction of the second part follows from the first part. For the 'only if' direction, observe that the rank of $\text{Sq}^2: \text{Kh}^j_{i,j} \to \text{K}_2^{j+2}$ is $x_1 + x_2 + x_3 + x_4$; therefore, if $\mathcal{X}_{\text{Kh}}^j(L)$ is a wedge of Moore spaces, $\text{Sq}^2 = 0$, and hence $x_1 = x_2 = x_3 = x_4 = 0$. □

5. Computations

It can be checked from the databases [12] that all prime links up to 11 crossings satisfy the conditions of Corollary 4.4. Therefore, their homotopy types are determined by the Khovanov homology $\text{Kh}_Z$ and the function $\text{St}$ of Definition 4.3. In Table S1, as a part of the supplementary material provided with the online version of this article, we present the values of $\text{St}$. To save space, we only list the links $L$ for which the function $\text{St}(L)$ is not identically $(0, 0, 0, 0)$; and for such links, we only list tuples $(i, j)$ for which $\text{St}(i, j) \neq (0, 0, 0, 0)$. For the same reason, we do not mention $\text{Kh}_Z(L)$ in the table; the Khovanov homology data can easily be extracted from [12].

After collecting the data for the PD-presentations from [12], we used several Sage programs for carrying out this computation. (To get more information about Sage, visit http://www.sagemath.org/) In addition to being part of the supplementary material provided with the online version of this article, all the programs and computations are also available at https://github.com/sucharit/KhovanovSteenrod.

Remark 5.1. Let $m(L)$ denote the mirror of $L$. In [8, Conjecture 10.1] we conjecture that $\mathcal{X}_{\text{Kh}}^j(m(L))$ and $\mathcal{X}_{\text{Kh}}^j(L)$ are Spanier–Whitehead dual. In particular, the action of $\text{Sq}^i$ on $\text{K}_2^j(L)$ and $\text{K}_2^j(m(L))$ should be transposes of each other. This conjecture provides some justification for the fact that Table S1 does not list the results for both mirrors of chiral knots.

For disjoint unions, we conjecture in [8, Conjecture 10.2] that $\mathcal{X}_{\text{Kh}}(L_1 \# L_2)$ is the smash product of $\mathcal{X}_{\text{Kh}}(L_1)$ and $\mathcal{X}_{\text{Kh}}(L_2)$. So, Table S1 only lists non-split links.

The expected behavior of $\mathcal{X}_{\text{Kh}}$ under connected sums is more complicated: in [8, Conjecture 10.5] we conjecture that $\mathcal{X}_{\text{Kh}}(L_1 \# L_2) \simeq \mathcal{X}_{\text{Kh}}(L_1) \otimes_{\text{K}_2} \mathcal{X}_{\text{Kh}}(L_2)$, where $\otimes$ denotes the tensor product of module spectra. So, like for Khovanov homology itself, the Khovanov homotopy type of a connected sum of links is not determined by the Khovanov homotopy types of the individual links: the module structures are required. Nonetheless, we have restricted Table S1 to prime links.

Proof of Theorem 1.1. From Table S1, we see that for $T_{3,4} = 8_{19}$, $\text{St}(2, 11) = (0, 1, 0, 0)$. Therefore, by Corollary 4.4, $\mathcal{X}_{\text{Kh}}^{11}(T_{3,4})$ is not a wedge sum of Moore spaces. □
Example 5.2. Consider the knot $K = 10_{45}$. From [12], we know its Khovanov homology:

|   | $-9$ | $-8$ | $-7$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ |
|---|------|------|------|------|------|------|------|------|------|------|
| $-3$ | .    | .    | .    | .    | .    | .    | .    | Z    | .    | .    |
| $-5$ | .    | .    | .    | .    | .    | .    | .    | Z    | .    | .    |
| $-7$ | .    | .    | .    | .    | .    | Z    | Z    | .    | .    | .    |
| $-9$ | .    | .    | .    | .    | .    | Z    | Z    | .    | .    | .    |
| $-11$ | .    | .    | Z    | Z    | .    | .    | .    | .    | .    | .    |
| $-13$ | .    | .    | Z    | F₂  | Z    | .    | .    | .    | .    | .    |
| $-15$ | Z    | Z    | .    | .    | .    | .    | .    | .    | .    | .    |
| $-17$ | .    | F₂  | .    | .    | .    | .    | .    | .    | .    | .    |
| $-19$ | F₂  | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| $-21$ | Z    | .    | .    | .    | .    | .    | .    | .    | .    | .    |

and from Table S1, we know the function $\text{St}(K)$:

$\text{St}(-4, -9) = (0, 0, 0, 1),$
$\text{St}(-6, -13) = (0, 0, 1, 0),$
$\text{St}(-7, -15) = (0, 1, 0, 0).$

Therefore, via Corollary 4.4, we can compute Khovanov homotopy types:

$\mathcal{X}_{\text{Kh}}^{-3}(K) \sim S^0,$
$\mathcal{X}_{\text{Kh}}^{-5}(K) \sim S^0,$
$\mathcal{X}_{\text{Kh}}^{-7}(K) \sim \Sigma^{-3}(S^0 \vee S^1),$  
$\mathcal{X}_{\text{Kh}}^{-9}(K) \sim \Sigma^{-6}(\mathbb{RP}^2 \wedge \mathbb{RP}^2),$
$\mathcal{X}_{\text{Kh}}^{-11}(K) \sim \Sigma^{-5}(S^0 \vee S^1 \vee S^1 \vee S^2),$
$\mathcal{X}_{\text{Kh}}^{-13}(K) \sim \Sigma^{-8}(\mathbb{RP}^4/\mathbb{RP}^1 \vee \Sigma \mathbb{RP}^2),$
$\mathcal{X}_{\text{Kh}}^{-15}(K) \sim \Sigma^{-10}(\mathbb{RP}^5/\mathbb{RP}^2 \vee S^4),$
$\mathcal{X}_{\text{Kh}}^{-17}(K) \sim \Sigma^{-8}(S^0 \vee S^1),$
$\mathcal{X}_{\text{Kh}}^{-19}(K) \sim \Sigma^{-10}\mathbb{RP}^2,$
$\mathcal{X}_{\text{Kh}}^{-21}(K) \sim \Sigma^{-9}S^0.$

Example 5.3. The Kinoshita–Terasaka knot $K_1 = K11n42$ and its Conway mutant $K_2 = K11n34$ have identical Khovanov homology. From Table S1, we see that $\text{St}(K_1) = \text{St}(K_2)$. Therefore, by Corollary 4.4, they have the same Khovanov homotopy type.

The Kinoshita–Terasaka knot and its Conway mutant is an example of a pair of links that are not distinguished by their Khovanov homologies. In an earlier version of the paper, we asked the following question:

Question 5.1. Does there exist a pair of links $L_1$ and $L_2$ with $\text{Kh}_\mathbb{Z}(L_1) = \text{Kh}_\mathbb{Z}(L_2)$, but $\mathcal{X}_{\text{Kh}}(L_1) \not\sim \mathcal{X}_{\text{Kh}}(L_2)$?

We provided the following partial answer:

Example 5.4. The links $L_1 = L11n383$ and $L_2 = L11n393$ have isomorphic Khovanov homology in quantum grading $(-3)$: $\text{Kh}_\mathbb{Z}^{-2,-3} = \mathbb{Z}^3, \text{Kh}_\mathbb{Z}^{-1,-3} = \mathbb{Z}^3 \oplus \mathbb{F}_2, \text{Kh}_\mathbb{Z}^{-6,-3} = \mathbb{Z}^2, [12].$ However, $\text{St}(L_1)(-2, -3) = (0, 2, 0, 0)$ and $\text{St}(L_2)(-2, -3) = (0, 1, 0, 0)$ (Table S1); therefore, $\mathcal{X}_{\text{Kh}}^{-3}(L_1)$ is not stably homotopy equivalent to $\mathcal{X}_{\text{Kh}}^{-3}(L_2)$.

Since the first version of this paper, C. Seed has independently computed the $\text{Sq}^1$ and $\text{Sq}^2$ action on the Khovanov homology of knots, and has answered Question 5.1 in the affirmative in [10].

We conclude with an observation and a question. Since all prime links up to 11 crossings satisfy the conditions of Corollary 4.4, their Khovanov homotopy types are wedges of various
suspensions of $S^0$, $\mathbb{RP}^2$, $\mathbb{CP}^2$, $\mathbb{RP}^5/\mathbb{RP}^2$, $\mathbb{RP}^4/\mathbb{RP}^1$ and $\mathbb{RP}^2 \wedge \mathbb{RP}^2$; and this wedge sum decomposition is unique since it is determined by the Khovanov homology $KH_\mathbb{Z}$ and the function $St$. Example 5.2 already exhibits all but one of these summands; it does not have a $\mathbb{CP}^2$ summand. A careful look at Table S1 reveals that neither does any other link up to 11 crossings. The conspicuous absence of $\mathbb{CP}^2$ naturally leads to the following question.

**Question 5.2.** Does there exist a link $L$ for which $X_j^{KH}(L)$ contains $\Sigma^m \mathbb{CP}^2$ in some wedge sum decomposition, for some $j, m$?

**Supplementary data**

Supplementary material is available with the online version of this article.

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**References**

1. J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics (University of Chicago Press, Chicago, IL, 1995), Reprint of the 1974 original.
2. H. J. Baues, ‘Homotopy types’, Handbook of algebraic topology (North-Holland, Amsterdam, 1995) 1–72.
3. S. C. Chang, ‘On algebraic structures and homotopy invariants’, Bull. Acad. Polon. Sci. Cl. III 4 (1956) 797–800 (1957).
4. P. Hu, D. Kriz and I. Kriz, ‘Field theories, stable homotopy theory and Khovanov homology’, Preprint, 2012, arXiv:1203.4773.
5. M. Khovanov, ‘A categorification of the Jones polynomial’, Duke Math. J. 101 (2000) 359–426.
6. G. Laures, ‘On cobordism of manifolds with corners’, Trans. Amer. Math. Soc. 352 (2000) 5667–5688 (electronic).
7. T. Lawson (mathoverflow.net/users/360), ‘Third differential in Atiyah Hirzebruch spectral sequence’, MathOverflow, http://mathoverflow.net/questions/62644 (version: 2011-04-22).
8. R. Lipshitz and S. Sarkar, ‘A Khovanov stable homotopy type’, J. Amer. Math. Soc., Preprint, 2011, arXiv:1112.3932.
9. R. Lipshitz and S. Sarkar, ‘A refinement of Rasmussen’s s-invariant’, Duke Math. J., Preprint, 2012, arXiv:1206.3532.
10. C. Seed, ‘Computations of the Lipshitz–Sarkar Steenrod square on Khovanov homology’, Preprint, 2012, arXiv:1210.1882.
11. N. E. Steenrod, ‘Cohomology operations, and obstructions to extending continuous functions’, Adv. in Math. 8 (1972) 371–416.
12. The Knot Atlas, http://katlas.org/Data/ (version: 2012-02-02).
13. J. H. C. Whitehead, ‘A certain exact sequence’, Ann. of Math. (2) 52 (1950) 51–110.
14. J. Wu, ‘Homotopy theory of the suspensions of the projective plane’, Mem. Amer. Math. Soc. 162 (2003) x+130.

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