On the winding number for particle trajectories in a disk-like vortex patch of the Euler equations

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Abstract

We consider vortex patch solutions of the incompressible Euler equation in the plane. When the origin is fixed by the patch velocity, we show that the winding number around the origin for most particles in the patch grows linearly when the initial patch is close to a disk enough.

1 Introduction

We consider the incompressible Euler equation in vorticity form in $\mathbb{R}^2$:

$$\partial_t \omega + u \cdot \nabla \omega = 0 \quad \text{for } x \in \mathbb{R}^2 \text{ and for } t > 0,$$

$$\omega|_{t=0} = \omega_0 \quad \text{for } x \in \mathbb{R}^2,$$

where the Biot-Savart law is given by $u = (K * \omega)$ with

$$K(x) := \frac{1}{2\pi} \frac{x^\perp}{|x|^2} = \frac{1}{2\pi} \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right).$$

When $\omega_0$ lies on $L^1 \cap L^\infty$, the existence and uniqueness of a global-in-time weak solution is due to Yudovich [10]. We are concerned with vortex patch solutions; when $\omega_0 = 1_{\Omega_0}$ for some bounded open set $\Omega_0 \subset \mathbb{R}^2$, the corresponding Yudovich solution $\omega$ has the form of $\omega(t) = 1_{\Omega_t}$ where $\Omega_t = \{ \phi_x(t) \in \mathbb{R}^2 | x \in \Omega_0 \}$. Here $\phi_x(\cdot)$ is the unique solution to the system

$$\frac{d}{dt} \phi_x(t) = u(t, \phi_x(t)) \quad \text{for } t > 0 \quad \text{and} \quad \phi_x(0) = x,$$

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which is well-defined thanks to the estimate
\[ \|u(t)\|_{L^\infty} \leq C\|\omega_0\|_{L^1 \cap L^\infty}, \quad t \geq 0 \tag{1.2} \]
and the log-Lipschitz estimate
\[ |u(t, x) - u(t, x')| \leq C\|\omega_0\|_{L^1 \cap L^\infty}|x - x'| \ln\left(\frac{1}{|x - x'|}\right), \quad |x - x'| \leq \frac{1}{2}, \quad t \geq 0 \tag{1.3} \]
(e.g. see Section 8.2.3 in [11]).

In a recent related work [5], the travel distance for fluid particles
\[ d_x(t) := \int_0^t |u(s, \phi_x(s))|ds \]
was considered, and it was established that for most particles on the initial vortex patch, this quantity grows linearly in time when the initial patch is disk-like. Here we say that a patch on \( \Omega \) is disk-like if the measure of the symmetric difference
\[ \Omega \triangle B_r := (\Omega \setminus B_r) \cup (B_r \setminus \Omega) \]
is small enough for some \( r > 0 \) where we denote \( B_r := \{x \in \mathbb{R}^2 \mid |x| < r\} \).

The goal of this paper is to produce an estimate on the winding number around the origin for fluid particle trajectories, rather than the travel distance. To be precise, we define the notion of winding number, under the assumption that \( u \) satisfies \( u(t, 0) = 0 \) for all time. Then the origin is fixed for all time, and the winding number of a particle \( x \in \mathbb{R}^2 \setminus \{0\} \) up to time \( t > 0 \) is defined by the integral
\[ N_x(t) := \frac{1}{2\pi} \int_0^t \frac{u_{\tan}(s, \phi_x(s))}{|\phi_x(s)|} ds, \]
where \( u_{\tan} \) is the speed in the angular direction with respect to the origin (see the definition (1.3)). This is well-defined since \( \phi_x(s) \neq 0 \) for all \( s \geq 0 \) as long as \( x \neq 0 \). It is not difficult to check that this definition coincides with the usual notion of winding number (for curves in \( \mathbb{R}^2 \) not intersecting the origin) applied to the curve \( \gamma_x : [0, t] \mapsto \mathbb{R}\setminus\{0\} \) defined by \( \gamma_x(s) = \phi_x(s) \). For example, we observe that for the circular vortex patch supported on the unit disc \( D \), we have, for any \( t > 0 \),
\[ \frac{N_x(t)}{t} = \frac{1}{4\pi}, \quad \forall x \in D \setminus \{0\}, \tag{1.4} \]

simply because \( u_{\tan}(t, x) = \frac{1}{2}|x| \) for \( t \geq 0 \) and \( x \in D \setminus \{0\} \).

Compared with the task of estimating the travel distance, there are some additional difficulties in treating the winding number. First, this quantity may decrease in time (particles can “unwind”) and second, there is no uniform in space bound for the ratio \( u(t, x)/|x| \). Rather, taking \( x' = 0 \) in (1.3) shows that the quantity may diverge like \( C\log(1/|x|) \) as \( |x| \to 0 \), which is indeed sharp (e.g.
1.1 Main result

We consider the initial vorticity given by the patch on some bounded measurable set \( \Omega_0 \) of unit strength, whose solution is identified with moving domains \( \Omega_t \).

We assume that the corresponding velocity satisfies

\[
\|u(t,0)\| = 0, \quad t \geq 0. 
\]

(1.5)

Remark 1.1. The above condition (1.5) is guaranteed, for instance, when \( \Omega_0 \) satisfies the following symmetry assumption for some integer \( m \geq 2 \):

\[
\Omega_0 = R_j^m(\Omega_0), \quad 1 \leq j \leq m - 1,
\]

where \( R_j^m \) is the counter-clockwise rotation in \( \mathbb{R}^2 \) with angle \( \frac{2\pi j}{m} \). In particular, our result applies to nearly circular Kirchhoff ellipses. In this case, a detailed information about the particle trajectories is available in [13].

Together with (1.2) and (1.3), we have that

\[
\|u(t)\|_{L^\infty} \leq C(|\Omega_0| + 1), \quad t \geq 0 
\]

(1.6)

and

\[
|u(t,x)| \leq C(|\Omega_0| + 1)|x| \ln \frac{1}{|x|}, \quad |x| \leq \frac{1}{2}, \quad t \geq 0.
\]

(1.7)

Moreover, we shall assume that \( \Omega_0 \) is a small perturbation of the unit disc \( D \) (this is without loss of generality thanks to the scaling of the Euler equations). We are now ready to state our main result on winding number for disk-like patches.

**Theorem 1.2.** For any \( R > 0 \), there exist constants \( \delta_0 \in (0,1) \) and \( C > 0 \) such that if

\[
\Omega_0 \subset B_R \quad \text{and} \quad |D\Delta \Omega_0| < \delta_0,
\]

and if the solution \( \omega(t) = \mathbf{1}_{\Omega_t} \) of (1.1) for the initial data \( \omega_0 = \mathbf{1}_{\Omega_0} \) satisfies the assumption (1.5), then, by denoting \( \delta := |D\Delta \Omega_0| \), for any \( T > 0 \), there exists a set \( H_T \subset \Omega_0 \) such that

\[
|H_T| \geq |\Omega_0| - C\delta^{1/8}
\]

and

\[
\left| \frac{N_x(T)}{T} - \frac{1}{4\pi} \right| \leq C\delta^{1/8} \log \frac{1}{\delta}, \quad x \in H_T.
\]

In addition, there exists a set \( H \subset \Omega_0 \) such that

\[
|H| \geq |\Omega_0| - C\delta^{1/8}
\]

and

\[
\liminf_{t \to \infty} \left| \frac{N_x(t)}{t} - \frac{1}{4\pi} \right| \leq C\delta^{1/8} \log \frac{1}{\delta}, \quad x \in H.
\]
Recalling (1.4), the above result states that most particles wind around the origin roughly the same number of times as particles in the case of the disc. The proof is based on the stability result of Sideris and Vega [15] (see also [12, 17]) which shows that for disc-like patches, most particles which start from the initial patch stays in the patch most of the time.

Notations

We collect the notations here that are used throughout the paper.

- Given $r > 0$, $B_r = \{ x \in \mathbb{R}^2 \mid |x| < r \}$. We denote the unit disc by $D := B_1$. The complement is denoted by $D^c$.
- Given $\epsilon > 0$ and $i \in \mathbb{N} \cup \{0\}$, we set $B_\epsilon^i = B_{2-\epsilon^i}$ and $A_\epsilon^i = B^i \setminus B^{i+1}$.
- The radial and tangential part of the velocity around the origin are defined by
  \[ u = u_{\text{rad}} \frac{x}{|x|} + u_{\text{tan}} \frac{x^\perp}{|x|}. \] (1.8)

Here we denote $x^\perp = (-x_2, x_1)$ for $x = (x_1, x_2)$.

2 Proof

Proof of Theorem 1.2. Fix $R > 0$ and let $\Omega_0 \subset B_R$ such that the solution $\omega(t) = 1_{\Omega_t}$ of (1.1) for the initial data $\omega_0 = 1_{\Omega_0}$ satisfies the assumption (1.5). Take any $\delta_0 > 0$ satisfying $\delta_0^{1/8} \leq 1/2$. Assume
  \[ \delta := |\Omega_0 \triangle D| < \delta_0. \]

If $\delta = 0$, then we can take $H_T, H = D \setminus \{0\}$ by (1.4). From now on, we assume $\delta > 0$.

Throughout the paper, $C$ denotes a positive constant which may depend on $R > 0$, may change from line to line, but it is independent of $\delta_0, \delta > 0$.

2.1 $L^1$-stability of a disk patch

We recall the $L^1$-stability result for the circular vortex patch from [15]:

**Lemma 2.1** ([15, Theorem 3]). For any bounded open set $\Omega_0 \subset \mathbb{R}^2$ and for any $r > 0$, we have
  \[ \|1_{\Omega_t} - 1_{B_r}\|_{L^1}^2 \leq 4\pi \sup_{x \in \Omega_0 \triangle B_r} ||x|^2 - r^2| \cdot \|1_{\Omega_0} - 1_{B_r}\|_{L^1} \quad \text{for any } t > 0. \]

Note that $\|1_A - 1_B\|_{L^1} = \int_{A \triangle B} 1 \, dx = |A \triangle B|$. Applying the above lemma with our $\Omega_0 \subset B_R$ and $1_D$, we obtain that
  \[ |\Omega_t \triangle D| \leq C \sqrt{\delta}, \quad t > 0. \] (2.1)

From this $L^1$ bound, we obtain that (e.g. see [5, 7]).
Lemma 2.2. Under the above assumptions on $\Omega_0$, we have with $u(t) = K \ast 1_{\Omega(t)}$ and $u_D = K \ast 1_{D}$ that

$$\|u(t) - u_D\|_{L^\infty} \leq C\delta^{1/4}$$

for all $t \geq 0$. In particular, we have

$$\left| \frac{u_{\tan}(t, x)}{|x|} - \frac{1}{2} \right| \leq C\delta^{1/4}/|x|, \quad x \in D \setminus \{0\}. \quad (2.2)$$

Proof. From the estimate $|K(x)| \leq C/|x|$, we have

$$\|K \ast f\|_{L^\infty} \leq C\|f\|_{L^1}^{1/2} \|f\|_{L^\infty}^{1/2}$$

for any $f \in (L^1 \cap L^\infty)(\mathbb{R}^2)$ (e.g. see Lemma 2.1. in [9]). Thus, we can estimate, for any $x \in \mathbb{R}^2$,

$$|u(t, x) - u_D(x)| \leq \|K \ast (1_{\Omega(t)} - 1_{D})\|_{L^\infty} \leq C\|1_{\Omega(t)} - 1_{D}\|_{L^2}^{1/2} \|1_{\Omega(t)} - 1_{D}\|_{L^\infty}^{1/2} \leq C|\Omega(t)\triangle D|^{1/2}.$$

Thus, by (2.1), we get the first estimate. Taking the tangential components and using that $u_{D,\tan} = |x|/2$ for $x \in D \setminus \{0\}$, we obtain (2.2). \qed

Next, we recall a simple lemma from [5, Lemma 2.2]:

Lemma 2.3. Under the above assumptions on $\Omega_0$, for any $T > 0$, we have

$$\int_{\Omega_0} \left( \frac{1}{T} \int_0^T 1_{B_r}(\phi_x(t))dt \right) dx \leq Cr^2, \quad r > 0$$

and

$$\int_{\Omega_0} \left( \frac{1}{T} \int_0^T 1_{D^c}(\phi_x(t))dt \right) dx \leq C\sqrt{\delta}.$$

Proof. We compute, using that the flow map $x \mapsto \phi_x(t)$ is area-preserving, for $r > 0$ and $t > 0$,

$$\int_{\Omega_0} 1_{B_r}(\phi_x(t))dx = \int_{\Omega_0 \cap B_r} 1dx \leq |B_r| = \pi r^2,$$

and, by (2.1),

$$\int_{\Omega_0} 1_{D^c}(\phi_x(t))dx = \int_{\Omega_0 \cap D^c} 1dx \leq \int_{\Omega_0 \triangle D} 1dx = |\Omega_0 \triangle D| \leq C\sqrt{\delta}.$$

Now the desired bounds follow from integrating in time and applying Fubini’s theorem. \qed
2.2 Estimate of winding number

Denote \( \epsilon := \delta^{1/8} > 0 \).

Observe \( \epsilon = \delta^{1/8} < \delta_0^{1/8} \leq 1/2 \). Fix \( T > 0 \) and define

\[
G_i^T(x) := \frac{1}{T} \int_0^T 1_{B_i^T}(\phi_x(t))dt, \quad i \geq 0
\]

(recall the notation \( B_i^T := B_{2^{-i} \epsilon} \)) and

\[
G_{-1}^T(x) := \frac{1}{T} \int_0^T 1_{D}(\phi_x(t))dt.
\]

Lemma 2.3 states that

\[
\int_{\Omega_0} G_i^T(x)dx \leq C\left(2^{-i} \epsilon\right)^2, \quad i \geq 0,
\]

and, from Chebyshev’s inequality, we obtain

\[
|\{ x \in \Omega_0 \mid G_i^T(x) \geq 2^{-i} \epsilon \}| \leq C2^{-i} \epsilon = C2^{-i} \delta^{1/8}, \quad i \geq 0.
\]

Similarly, we have

\[
|\{ x \in \Omega_0 \mid G_{-1}^T(x) \geq 2 \epsilon \}| \leq C\frac{\sqrt{\delta}}{\epsilon} = C\delta^{3/8} \leq C\delta^{1/8}.
\]

Now we define

\[
H_T := \{ x \in \Omega_0 \mid G_i^T(x) < 2^{-i} \epsilon, \, \forall i \geq -1 \}.
\]

Then we have

\[
|H_T| \geq |\Omega_0| - C\delta^{1/8}.
\]

For any fixed \( x \in H_T \), we split the time integral as follows:

\[
\int_0^T \frac{u_{\tan}(t, \phi_x(t))}{|\phi_x(t)|}dt = \int_{[0,T] \setminus (\cup_{i \geq -1} I_i)} \frac{u_{\tan}(t, \phi_x(t))}{|\phi_x(t)|}dt + \sum_{i \geq -1} \int_{I_i} \frac{u_{\tan}(t, \phi_x(t))}{|\phi_x(t)|}dt,
\]

where

\[
I_i := \{ t \in [0, T] \mid \phi_x(t) \in A_i^i \}, \quad i \geq 0
\]

(recall the notation \( A_i^i := B_i^i \setminus B_{i+1}^i \)) and

\[
I_{-1} := \{ t \in [0, T] \mid \phi_x(t) \in D^c \}.
\]
For each $i \geq -1$, we have that $|I_i| \leq T G^T_i(x) \leq T 2^{-i}\epsilon$
so that we obtain $\sum_{i\geq-1} |I_i| \leq CT\epsilon$.

For $i \geq 0$, we estimate, using the log-Lipschitz bound (1.7),
\[
\left| \int_{I_i} \frac{u_{tan}(t, \phi_x(t))}{|\phi_x(t)|} dt \right| \leq \int_{I_i} \frac{|u(t, \phi_x(t))|}{|\phi_x(t)|} dt \leq C(|\Omega_0| + 1)|I_i| \ln(2^{(i+1)}/\epsilon) \leq CT2^{-i}\epsilon \ln(2^{(i+1)}/\epsilon),
\]
where the last inequality follows from $|\Omega_0| \leq |D| + |\Omega_0 \Delta D| \leq C + \delta \leq C$.

In the case $i = -1$, we instead use (1.6) to get, for $|\phi_x(t)| \geq 1$,
\[
\left| \frac{|u_{tan}(t, \phi_x(t))|}{|\phi_x(t)|} \right| \leq |u(t, \phi_x(t))| \leq C(1 + |\Omega_0|) \leq C.
\]
It gives
\[
\left| \int_{I_{-1}} \frac{u_{tan}(t, \phi_x(t))}{|\phi_x(t)|} dt \right| \leq CT\epsilon.
\]
Hence we obtain
\[
\left| \sum_{i\geq-1} \int_{I_i} \frac{u_{tan}(t, \phi_x(t))}{|\phi_x(t)|} dt \right| \leq CT\epsilon \left( 1 + \sum_{i\geq0} 2^{-i} \ln(2^{(i+1)}/\epsilon) \right) \leq CT\epsilon(1 + \ln \frac{1}{\epsilon}) \leq CT\epsilon \ln \frac{1}{\epsilon}.
\]
Note that the above constant $C > 0$ is independent of $x \in H_T$.

Lastly, for $t \in [0, T] \setminus (\cup_{i\geq-1} I_i)$ we have that
\[
\epsilon \leq |\phi_x(t)| < 1,
\]
from which it follows by (2.2) that
\[
\left| \frac{u_{tan}(t, \phi_x(t))}{|\phi_x(t)|} \right| - \frac{1}{2} \leq C \frac{\delta^{1/4}}{|\phi_x(t)|} \leq C \frac{\delta^{1/4}}{\epsilon} \leq C \delta^{1/8}.
\]
Therefore, we obtain the lower bound
\[
\int_{[0, T] \setminus (\cup_{i\geq-1} I_i)} \frac{u_{tan}(t, \phi_x(t))}{|\phi_x(t)|} dt \geq T(1 - C\epsilon) \left( \frac{1}{2} - C \delta^{1/8} \right).
\]
Collecting all the estimates, we obtain
\[
\frac{1}{T} \int_0^T \frac{u_{tan}(t, \phi_x(t))}{|\phi_x(t)|} dt \geq (1 - C\epsilon) \left( \frac{1}{2} - C \delta^{1/8} \right) - C\epsilon \log \frac{1}{\epsilon}
\geq \frac{1}{2} - C \delta^{1/8} \left( 1 + \delta^{1/8} + \log \frac{1}{\delta} \right) \geq \frac{1}{2} - C \delta^{1/8} \log \frac{1}{\delta}.
\]
This gives the lower bound
\[ \frac{N_r(T)}{T} \geq \frac{1}{4\pi} - C\delta^{1/8} \log \frac{1}{\delta}, \]
while the upper bound can be obtained similarly. The proof for the first statement is now complete. To see the second statement, we just observe that with \( T_n := n \)
\[ H := \limsup_{n} H_{T_n} = \bigcap_{m \geq 1} \bigcup_{n \geq m} H_{T_n} \]
has measure greater than \(|\Omega_0| - C\delta^{1/8}\). \(\square\)

**Remark 2.4.** Since Yudovich’s log-Lipschitz estimate in principle allows for arbitrarily fast rotation, it seems like an interesting problem to ask whether there is an initial data \( \omega_0 \in L^1 \cap L^\infty \) which has a trajectory winding around the origin arbitrarily many times in a fixed time interval. This is related to the (difficult) question of whether instantaneous spiraling is possible for vortex patches. We refer the interested readers to discussions in [6, Sections 5–6], [8, 14].

**Remark 2.5.** It is also interesting to ask a similar question about winding number (or travel distance) on well-known solutions other than a disk, whose certain stability has been known. It might include Kirchhoff’s ellipse [16], Lamb dipoles [1], Shear flows [3] and rectangles in a strip [4], and etc.

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**References**

[1] Ken Abe and Kyudong Choi. Stability of Lamb dipoles. preprint, arXiv:1911.01795.

[2] H. Bahouri and J.-Y. Chemin. Équations de transport relatives à des champs de vecteurs non-lipschitziens et mécanique des fluides. Arch. Rational Mech. Anal., 127(2):159–181, 1994.

[3] Jacob Bedrossian and Nader Masmoudi. Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. Publ. Math. Inst. Hautes Études Sci., 122:195–300, 2015.

[4] Jennifer Beichman and Sergey Denisov. 2D Euler equation on the strip: stability of a rectangular patch. Comm. Partial Differential Equations, 42(1):100–120, 2017.
[5] Kyudong Choi. On the estimate of distance traveled by a particle in a disk-like vortex patch. Appl. Math. Lett., 97:67–72, 2019.

[6] Tarek M. Elgindi and In-Jee Jeong. On singular vortex patches, I: Well-posedness issues. Memoirs of the AMS, to appear, arXiv:1903.00833.

[7] Tarek M. Elgindi and In-Jee Jeong. On singular vortex patches, II: Long-time dynamics. Transactions of the AMS, to appear, arXiv:1909.13555.

[8] V. Elling and M. V. Gnann. Variety of unsymmetric multibranched logarithmic vortex spirals. European J. Appl. Math., 30(1):23–38, 2019.

[9] Dragoș Iftimie, Thomas C. Sideris, and Pascal Gamblin. On the evolution of compactly supported planar vorticity. Comm. Partial Differential Equations, 24(9-10):1709–1730, 1999.

[10] V. I. Judovič. Non-stationary flows of an ideal incompressible fluid. Ž. Vyčisl. Mat. i Mat. Fiz., 3:1032–1066, 1963.

[11] Andrew J. Majda and Andrea L. Bertozzi. Vorticity and incompressible flow, volume 27. Cambridge University Press, Cambridge, 2002.

[12] Carlo Marchioro and Mario Pulvirenti. Mathematical theory of incompressible nonviscous fluids, volume 96 of Applied Mathematical Sciences. Springer-Verlag, New York, 1994.

[13] T. B. Mitchell and L. F. Rossi. The evolution of Kirchhoff elliptic vortices. Physics of Fluids, 20(5):054103, 2008.

[14] D. I. Pullin. On similarity flows containing two-branched vortex sheets. In Mathematical aspects of vortex dynamics (Leesburg, VA, 1988), pages 97–106. SIAM, Philadelphia, PA, 1989.

[15] Thomas C. Sideris and Luis Vega. Stability in $L^1$ of circular vortex patches. Proc. Amer. Math. Soc., 137(12):4199–4202, 2009.

[16] Yun Tang. Nonlinear stability of vortex patches. Trans. Amer. Math. Soc., 304(2):617–638, 1987.

[17] Y. H. Wan and M. Pulvirenti. Nonlinear stability of circular vortex patches. Comm. Math. Phys., 99(3):435–450, 1985.