JUMPS IN COHOMOLOGY AND FREE GROUP ACTIONS

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Abstract. A discrete group $G$ has periodic cohomology over $R$ if there is an
element in a cohomology group, cup product with which induces an isomorphism
in cohomology after a certain dimension. Adem and Smith showed if $R = \mathbb{Z}$, then
this condition is equivalent to the existence of a finite dimensional free-$G$-CW-
complex homotopy equivalent to a sphere. It has been conjectured by Olympia
Talelli, that if $G$ is also torsion-free then it must have finite cohomological di-
mension. In this paper we use the implied condition of jump cohomology over $R$
to prove the conjecture for $H\mathcal{F}$-groups and solvable groups. We also find neces-
sary conditions for free and proper group actions on finite dimensional complexes
homotopy equivalent to closed, orientable manifolds.

1. Introduction

Definition 1.1. A discrete group $G$ has periodic cohomology over $R$ if there is a
cohomology class $\alpha \in \text{Ext}_{RG}^*(R, R)$ with $|\alpha| > 0$ and an integer $n \geq 0$
such that the cup product map

$$\alpha \cup - : \text{Ext}_{RG}^i(R, M) \to \text{Ext}_{RG}^{i+|\alpha|}(R, M)$$

is an isomorphism for every $RG$-module $M$ and every integer $i \geq n$.

If $R = \mathbb{Z}$, then we simply say that $G$ has periodic cohomology. In this case Adem
and Smith [1] proved that this condition is equivalent to the existence of a finite
dimensional free-$G$-CW-complex homotopy equivalent to a sphere. If a group $G$
has periodic cohomology then it has periodic cohomology over any commutative ring
$R$ with a unit. This definition of periodicity using the cup product differs from
the general definition of periodic cohomology of a group. Namely, a group $G$ has
periodic cohomology after $k$-steps if there exists an integer $q \geq 0$ such that the
functors $H^i(G, -)$ and $H^{i+q}(G, -)$ are naturally equivalent for all $i > k$. We will
call this classical periodic cohomology. It is immediate that the cup product notion

Date: March 29, 2022.

Key words and phrases. spectral sequence, group cohomology.
of periodic cohomology implies the classical periodic cohomology. It is a conjecture by O. Talelli that the two notions are the same. She has proved that the conjecture holds in the case of the $H\mathcal{F}$-groups introduced by P. Kropholler in [7]. This is the smallest class of groups which contains all finite groups and which contains all groups $G$, whenever $G$ acts cellularly on a finite dimensional contractible CW-complex with all isotropy subgroups already in $H\mathcal{F}$. This is a large class of groups. Among others, it contains all countable linear groups, all countable solvable groups and all groups with finite virtual cohomological dimension.

The following was conjectured by Talelli in [11], Conjecture III:

**Conjecture (O. Talelli).** Every torsion-free discrete group $G$ with classical periodic cohomology after some steps has finite cohomological dimension.

The Conjecture was shown to be true for the class of $H\mathcal{F}$-groups by Mislin and Talelli in [8]. In our paper we investigate a weaker condition, implied by cohomological periodicity, for a discrete group $G$.

**Definition 1.2.** A discrete group $G$ has jump cohomology over $R$ if there exists an integer $k \geq 0$, such that if $H$ is any subgroup of $G$ with $\text{cd}_R(H) < \infty$, then $\text{cd}_R(H) \leq k$. The bound $k$ will be called the jump height over $R$. If $R = \mathbb{Z}$ then we say $G$ has jump cohomology and a jump height $k$.

By naturality of the cup product it follows that if a group $G$ has periodic cohomology over $R$, then $G$ has jump cohomology over $R$. In section 2 we prove the following proposition which relates the geometric aspects of the notion to the jump cohomology.

**Proposition 1.3.** Let $G$ be a discrete group that acts freely and properly on an $e$-dimensional CW-complex $E$. Suppose there is an integer $n \leq e$ such that $H_i(E, \mathbb{Z}) = 0$ for all $i > n$, and $H_n(E, \mathbb{Z}) \cong \mathbb{Z}$. Set $k = e - n$. If $H$ is any subgroup of $G$ with $\text{cd}(H) < \infty$ ($\text{hd}(H) < \infty$), then $\text{cd}(H) \leq k$ (respectively $\text{hd}(H) \leq k$).

In section 3 we investigate the classes of $H\mathcal{F}$-groups and solvable groups without $R$-torsion. We prove the following two theorems.

**Theorem 1.4.** Let $G$ be a discrete group without $R$-torsion with jump cohomology of height $k$ over $R$. If $G$ is an $H\mathcal{F}$-group, then $\text{cd}_R(G) \leq k$. In particular, any $H\mathcal{F}$-group $G$ has jump cohomology of height $k$ over $\mathbb{Q}$ if and only if $\text{cd}_\mathbb{Q}(G) \leq k$. 

Theorem 1.5. Let $G$ be a solvable group that acts freely and properly on an $e$-dimensional CW-complex $E$. Suppose there is an integer $n \leq e$ such that $H_i(E, \mathbb{Z}) = 0$ for all $i > n$, and $H_n(E, \mathbb{Z}) \cong \mathbb{Z}$, then $h(G) \leq e - n$. If $G$ is also $R$-torsion-free, $hd_R(G) \leq e - n$.

Our results give affirmative answers for the case of $HF$-groups, and when $R$ is a domain of characteristic zero for the case of solvable groups to the conjecture we believe is a natural generalization of the conjecture by Talelli.

Conjecture 1.6. For every discrete group $G$ without $R$-torsion the following are equivalent.

(1) $G$ has jump cohomology of height $k$ over $R$.

(2) $G$ has periodic cohomology over $R$ starting in dimension $k + 1$.

(3) $cd_R(G) \leq k$

This conjecture in particular would imply Talelli’s Conjecture. It also says that every group with periodic cohomology over $\mathbb{Q}$ must have $cd_\mathbb{Q}(G) < \infty$. Periodic or jump cohomology is a rather strong condition for the cohomology of $G$. We present the example of the Thompson group $F = \langle x_0, x_1, x_2, ... | x_i^{-1}x_n x_i = x_{n+1} \text{ for all } i < n \text{ and } n \in \mathbb{N} \rangle$, which is of type $FP_\infty$. It has periodic cohomology with integral coefficients and vanishing cohomology with $\mathbb{Z}F$-module coefficients, but it does not have jump cohomology over any ring $R$.

Next we apply Theorem 1.5 to groups that can act freely and properly on a finite dimension complex homotopy equivalent to a closed, orientable manifold.

Corollary 1.7. If $G$ is a solvable group, that acts freely and properly on an $(m+n)$-dimensional complex homotopy equivalent to closed, orientable manifold of dimension $n$, then the Hirsch rank $h(G) \leq m$.

Corollary 1.8. If $G$ has a non-countable, torsion-free, solvable subgroup, then $G$ cannot act freely and properly on a finite dimensional complex homotopy equivalent to a closed, orientable manifold.

Lastly, in section 4, we find an obstruction for a countable group acting on certain complexes to be free. We apply the next proposition to groups that admit free and proper actions on finite dimension complexes homotopy equivalent to closed, orientable manifolds.
Proposition 1.9. Let $G$ be a finitely generated torsion-free group that acts freely and properly on an $n$-dimensional CW-complex $X$ such that the $G$-invariant submodule $(H_{n-1}(X, R))^G \neq 0$. Then $G$ is a free group if and only if $\text{Ext}^2_{RG}(R, H_n(X, R))$ is trivial.

Theorem 1.10. If $G$ is a torsion-free countable group that acts freely and properly on an $n$-dimensional complex homotopy equivalent to a closed, orientable manifold of dimension $n - 1$, then $G$ must be free.

I would like to thank Alejandro Adem, Donald Passman and Olympia Talelli for their conversations and advice. Alejandro Adem and Olympia Talelli read an earlier version of the paper and made useful suggestions.

2. Periodicity and Boundedness

Unless otherwise specified, $R$ will denote any commutative ring with a unit.

A group $G$ is said to be $R$-torsion-free if for every finite subgroup $H$ of $G$, $n \cdot 1_R$ is an invertible element in $R$, where $n$ is the order of $H$. One should observe that this is equivalent to the following definition, which will be used in Lemma 3.4.

$G$ is $R$-torsion-free if and only if for each prime $p$, such that there exists a non-identity element $g \in G$ with $g^p = 1$, $p \cdot 1_R$ is an invertible element in $R$.

Definition 2.1. Cohomological dimension of a discrete group $G$ over $R$, denoted $\text{cd}_R(G)$, is defined as

$$\inf \{ n : \text{Ext}^i_{RG}(R, -) = 0 \text{ for } i > n \}$$

and the homological dimension, $\text{hd}_R(G)$, of $G$ over $R$ is

$$\inf \{ n : \text{Tor}^i_{RG}(R, -) = 0 \text{ for } i > n \}$$

An immediate consequence of $G$ having finite cohomological or homological dimension is that $G$ must be $R$-torsion-free.

Definition 2.2. A discrete group $G$ has periodic cohomology over $R$ if there is a cohomology class $\alpha \in \text{Ext}^*_R(G, R)$ with $|\alpha| > 0$ and an integer $n \geq 0$ such that the cup product map

$$\alpha \cup - : \text{Ext}^i_{RG}(R, M) \to \text{Ext}^{i+|\alpha|}_{RG}(R, M)$$
is an isomorphism for every $RG$-module $M$ and every integer $i \geq n$. The smallest degree of such $\alpha$ will be the period of the periodic cohomology of $G$. If $R = \mathbb{Z}$, we simply say that $G$ has periodic cohomology.

The above notion of periodicity was used by Adem and Smith in [1]. They prove a generalization of Wall’s conjecture [8] assuming that the periodicity is induced by the cup product with a cohomology class in $H^*(G, \mathbb{Z})$.

**Theorem 2.3.** (Adem-Smith). A discrete group $G$ has periodic cohomology if and only if $G$ acts freely and properly on a finite dimensional complex homotopy equivalent to a sphere.

Let us investigate a weaker cohomological condition, which arises naturally, when a group $G$ has a periodic cohomology over a ring $R$. It will be our main interest in section 3, where we will investigate specific types of groups satisfying the condition.

**Definition 2.4.** A discrete group $G$ has jump cohomology (homology) over $R$ if there exists an integer $k \geq 0$, such that if $H$ is any subgroup of $G$ with $\text{cd}_R(H) < \infty$ ($\text{hd}_R(H) < \infty$), then $\text{cd}_R(H) \leq k$ ($\text{hd}_R(H) \leq k$). The bound $k$ will be called the jump height over $R$. If $R = \mathbb{Z}$ then we simply say $G$ has jump cohomology (homology) and a jump height $k$.

This next proposition gives a more geometric meaning to the jump height.

**Proposition 2.5.** Let $G$ be a discrete group acting freely and properly on an $e$-dimensional CW-complex $E$. Suppose there is an integer $n \leq e$ such that $H_i(E, \mathbb{Z}) = 0$ for all $i > n$, and $H_n(E, \mathbb{Z}) \cong \mathbb{Z}$. Set $k = e - n$. Then $G$ has jump cohomology and homology of height $k$.

**Proof.** We will show the proof of the proposition concerning cohomology. The proof of the homological version is similar.

Suppose $H$ is a subgroup of $G$ with finite cohomological dimension $m$. We can assume that the induced action of $H$ on $H_n(E, \mathbb{Z})$ is trivial, otherwise we can pass to an index two subgroup of $H$ and use Serre’s Theorem, [9].

We get the following fibration

$$E \rightarrow E \times_H EH \rightarrow BH$$
where $E \times_H EH$ is the Borel Construction and it is homotopy equivalent to $E/H$. Therefore, by the Leray-Serre spectral sequence we have

$$E_2^{p,q} = H^p(BH, H^q(E, M)) \implies H^{p+q}(E/H, M)$$

There exists a $\mathbb{Z}G$-module $F$ such that $H^m(H, F) \neq 0$. By the usual corner argument of the sequence it follows

$$H^{m+n}(E/H, F) \cong H^m(BH, H^n(E, F))$$

By the Universal Coefficient Theorem we have the following isomorphism of $\mathbb{Z}H$-modules

$$H^n(E, F) \cong \text{Hom}(H_n(E, \mathbb{Z}), F) \oplus N \cong F \oplus N$$

where $N = \text{Ext}(H_{n-1}(E, \mathbb{Z}), F)$. This shows that

$$H^{m+n}(E/H, F) \cong H^m(BH, F) \oplus H^m(BH, N)$$

Therefore, $H^{m+n}(E/H, F) \neq 0$ and $m + n \leq e$. □

**Remark 2.6.** If there exist a finite dimensional free-$G$-CW-complex $E$ homotopy equivalent to a sphere $S^n$, then $G$ has a periodic cohomology over any ring $R$. For we can always construct the join $E*E$ with the diagonal $G$-action. This complex is homotopy equivalent to $S^{2n+1}$ and $H_*(E*E, \mathbb{Z})$ has the trivial $G$-action. Therefore the induced $G$-action on $H^*(E*E, R)$ is also trivial. Then the Gysin exact sequence of cohomology over $R$ shows that the cup product with the Euler class of the spherical fibration induces an isomorphism in cohomology of $G$ after dimension $2k$, where $2k$ is the jump height over $R$. This, together with Theorem 2.3, yields that, if a group $G$ has periodic cohomology, then it has periodic cohomology over any ring $R$.

Now, suppose $G$ has periodic cohomology over $R$ starting in dimension $k+1$. If $H$ is any subgroup of $G$ with finite cohomological dimension $h$ over $R$, using Shapiro’s Lemma, we have

$$\text{Ext}_{RH}^{k+1}(R, M) \cong \text{Ext}_{RG}^{k+1}(R, \text{Coind}_H^GM) \overset{\alpha}{\longrightarrow} \text{Ext}_{RG}^{k+h|\alpha|+1}(R, \text{Coind}_H^GM) \cong \text{Ext}_{RH}^{k+h|\alpha|+1}(R, M) = 0$$

for any $RH$-module $M$. This establishes the following lemma.

**Lemma 2.7.** If a group $G$ has periodic cohomology over $R$ starting in dimension $k+1$, then $G$ has a jump cohomology of height $k$ over $R$. 
Therefore, among the three cohomological conditions stated in the conjecture, jump cohomology is the weakest.

3. Some classes of groups

The following class of groups was introduced by P. Kropholler in [7]. Our next theorem shows that they satisfy the conjecture stated in section 1.

Definition 3.1. Let $\mathcal{X}$ denote a class of groups. Define $H\mathcal{X}$ to be the smallest class of groups containing $\mathcal{X}$ with the property: if a group $G$ acts cellularly on a finite dimensional contractible CW-complex with all isotropy subgroups in $H\mathcal{X}$, then $G$ is in $H\mathcal{X}$.

In [7] many properties of these classes of groups, such as subgroup and extension closure, closure under countable direct unions and free product, were shown. The main interest to us is the hierarchical description of $H\mathcal{X}$-groups defined by operations $H_\alpha$ for each ordinal $\alpha$ inductively:

$H_0\mathcal{X} = \mathcal{X}$

For each $\beta > 0$,

$H_\beta\mathcal{X}$ is the class of groups $G$ which act cellularly on a finite dimensional contractible CW-complex $X$ such that for each cell $\sigma$ of $X$ the isotropy group $G_\sigma$ is in $H_\alpha\mathcal{X}$ for some $\alpha < \beta$.

It is immediate that a group $G$ is a $H\mathcal{X}$-group if and only if there is an $\alpha$ such that $G$ is in $H_\alpha\mathcal{X}$. Let $\mathcal{F}$ denote the class of all finite groups. We obtain the following result pertaining to $H\mathcal{F}$-groups.

Theorem 3.2. Let $G$ be a discrete group without $R$-torsion with jump cohomology of height $k$ over $R$. If $G$ is an $H\mathcal{F}$-group, then $cd_R(G) \leq k$. In particular, any $H\mathcal{F}$-group $G$ has jump cohomology of height $k$ over $\mathbb{Q}$ if and only if $cd_\mathbb{Q}(G) \leq k$.

First we need a lemma.

Lemma 3.3. If a discrete group $G$ acts cellularly on an $n$-dimensional contractible complex $X$, such that for any cell stabilizers of $X$, $cd_R(G_\sigma) \leq k$, then $cd_R(G) \leq k+n$. 
Proof. By considering the double complex $\text{Hom}_{RG}(P_*, C_R^\alpha(X, M))$, where $P_*$ is a projective resolution of $R$ over $RG$ and $C_R^\alpha(X, M)$ is the cellular cochain complex of $X$ with $RG$-module coefficients $M$ (see for example [3]) we can derive the first-quadrant spectral sequence

$$E_1^{p,q} = \text{Ext}^q_{RG}(R, C_R^p(X, M)) \implies \text{Ext}^{p+q}_{RG}(R, C_R^\alpha(X, M))$$

where $\text{Ext}^\alpha_{RG}(R, C_R^\alpha(X, M))$ is the cohomology of the total complex associated to the double complex.

For each $p$-cell $\sigma$ of $X$ there exists a $G_\sigma$-module $R_\sigma$. This module is isomorphic to $R$ additively, and $G_\sigma$ acts on it through the orientation character. Let $M_\sigma = \text{Hom}_R(R_\sigma, M)$. Let $X_p$ denote the collection of all the $p$-cells and let $\Sigma_p$ be a set of representatives of all the $G$-orbits in $X_p$. We have the following decomposition

$$C_R^p(X, M) = \text{Hom}_R(C_R^p(X), M) = \bigoplus_{\sigma \in X_p} \text{Hom}_R(R_\sigma, M) = \bigoplus_{\sigma \in \Sigma_p} \text{Hom}_R(\text{Ind}_{G_\sigma}^GR_\sigma, M)$$

Now, by Shapiro’s lemma,

$$\text{Ext}^q_{RG}(R, C_R^p(X, M)) \cong \bigoplus_{\sigma \in \Sigma_p} \text{Ext}^q_{R[G_\sigma]}(R, M_\sigma).$$

Since $X$ is contractible the two cochain complexes $C_R^\alpha(X, M)$ and $C_R^\alpha(pt., M)$ are homotopy equivalent. Hence,

$$\text{Ext}^\alpha_{RG}(R, C_R^\alpha(X, M)) \cong \text{Ext}^\alpha_{RG}(R, C_R^\alpha(pt., M)) = \text{Ext}^\alpha_{RG}(R, M).$$

The spectral sequence then becomes

$$E_1^{p,q} = \bigoplus_{\sigma \in \Sigma_p} \text{Ext}^q_{R[G_\sigma]}(R, M_\sigma) \implies \text{Ext}^{p+q}_{RG}(R, M)$$

It follows that $E_1^{p,q} = 0$ if $p > n$ or $q > k$. We can infer inductively for $i = 0, 1, 2, \ldots$ that the differentials $d_i^{p,q} = 0$ and the terms of the spectral sequence $E_i^{p,q} = 0$ if $p > n$ or $q > k$. Therefore, $E_\infty^{p,q} \cong E_1^{p,q} = 0$ if $p > n$ or $q > k$ and $\text{Ext}^{p+q}_{RG}(R, M) = 0$ if $p > n$ and $q > k$. Thus $\text{cd}_R(G) \leq n + k$.

□

Proof of Theorem. We will show that $\text{cd}_R(G) \leq k$ by transfinite induction, using the hierarchical description of $H\mathcal{F}$-groups.
If $G$ is in $H_0\mathcal{F} = \mathcal{F}$, then $G$ is a finite group. So, $cd_R(G) = 0$. Assume now that for a fixed $\beta$, and all $\alpha < \beta$, $cd_R(H) \leq k$ for any subgroup $H$ of $G$ in $H_\alpha\mathcal{F}$. Suppose $G$ is an $H_\beta\mathcal{F}$-group. By assumption there is a finite dimensional contractible $G$-CW-complex $X$ such that for any cell $\sigma$ of $X$ the isotropy group $G_\sigma$ is an $H_\alpha\mathcal{F}$-group for some $\alpha < \beta$. So, by induction, $cd(G_\sigma) \leq k$. Let $n$ be the dimension of $X$. Now, by the lemma, $cd_R(G) \leq k + n$. Hence, by hypothesis, $cd_R(G) \leq k$.

□

Our next goal is to study solvable groups with jump cohomology. For this we need to assume $R$ is an integral domain. Note that by the previous theorem the conjecture stated in section 1 is true for countable $R$-torsion-free solvable groups, for they are $HF$-groups, for an arbitrary commutative ring $R$. What proceeds extends this to uncountable solvable groups when $R$ is a domain. In fact, our arguments show that if a torsion-free solvable group has jump cohomology, then it is countable, since it must have finite Hirsch rank. We denote by $\mathbb{F}$ the fraction field of $R$, so $\mathbb{F}$ is a field of an arbitrary characteristic.

Lemma 3.4. Let $1 = G_0 \subset G_1 \subset \ldots \subset G_n = G$ be an upper central series of $G$. If $G$ is an $R$-torsion-free group, then so are the factor groups $G_i/G_{i-1}$, $i = 1, \ldots, n$.

Proof. Recall that an upper central series for $G$ can be defined inductively by letting $G_i$ to be the normal subgroup of $G$ such that $G_i/G_{i-1} = Z(G/G_{i-1})$, where $Z(G/G_{i-1})$ is the center of the quotient group $G/G_{i-1}$.

Let us proceed by induction. Suppose, for all $j \leq i$, $G_j/G_{j-1}$ are $R$-torsion-free. We need to show that the same is true for the quotient group $G_{i+1}/G_i$. Suppose $\bar{x} \in G_{i+1}/G_i$ such that $\bar{x}^p = 1$ for a prime number $p$, and $p \cdot 1_R$ is not invertible in the ring $R$. This shows that $x^p \in G_i$. Now for any element $g$ of $G$, $[g, x]^p \equiv [g, x^p] \equiv 1$ mod $G_{i-1}$. Since $[g, x] \in G_i$, the induction shows $[g, x] \in G_{i-1}$. So, in the quotient group $G/G_{i-1}$, $[\bar{g}, \bar{x}] = 1$ for all $\bar{g} \in G/G_{i-1}$. Therefore, by definition, $\bar{x} \in G_i/G_{i-1}$ and $x \in G_i$. It follows that $\bar{x}$ is the identity element in $G_{i+1}/G_i$.

□

Proposition 3.5. Let $G$ be a nilpotent group without $R$-torsion with jump homology of height $k$ over $R$. Then $hd_R(G) = h(G) \leq k$ and $\text{Tor}_{h(G)}^R(R, \mathbb{F}) \cong \mathbb{F}$, where $h(G)$ is the Hirsch rank of $G$. 

Proof. Let \( 1 = G_0 \subset G_1 \subset \ldots \subset G_n = G \) be a central series of \( G \) with \( R \)-torsion-free factor groups. We use induction on \( n \).

Let us assume \( h(G_i) = hd_R(G_i) = h_i \), and \( \text{Tor}^R_{h_i}(G_i, F) \cong F^{\omega_i} \), where \( F^{\omega_i} \) is additively isomorphic to \( F \) with a trivial \( G/G_i \)-action. Let \( A \subset G_{i+1}/G_i \) and \( A \cong \mathbb{Z}^m \). Let \( G_{i+1}' = \varphi^{-1}(A) \), where \( \varphi : G_{i+1} \to G_{i+1}/G_i \) is the canonical quotient map. So, there exists an extension, \( 1 \to G_i \to G_{i+1}' \to A \to 1 \).

By the corner argument of the Lyndon-Hochschild-Serre spectral sequence of the extension above it follows,

\[
\text{Tor}^R_{m+h_i}(G_i, F) \cong \text{Tor}^R_{m}(A, \text{Tor}^R_{h_i}(G_i, F)) \cong \text{Tor}^R_{m}(A, F^{\omega_i})
\]

Since \( A \cong \mathbb{Z}^m \),

\[
\text{Tor}^R_{m}(A, F^{\omega_i}) \cong \bigwedge^m(A \otimes F^{\omega_i}) \cong \bigwedge^m(\mathbb{Z}^m \otimes F^{\omega_i}) \cong F
\]

where \( \bigwedge^m(A \otimes F^{\omega_i}) \) is the exterior algebra of the \( F \)-vector space \( A \otimes F^{\omega_i} \). This shows \( \text{Tor}^R_{m+h_i}(G_i, F) \neq 0 \). On the other hand, since \( hd_R(G_i) = h_i \) and \( hd_R(A) = m \), the corner argument of the spectral sequence also shows, \( hd_R(G_{i+1}') \leq m + h_i \). Thus \( hd_R(G_{i+1}') = m + h_i \). So, by the hypothesis \( m + h_i \leq k \), showing \( m \leq k \). This proves that \( G_{i+1}/G_i \) is a finite-rank abelian group. Thus, \( G_{i+1} \) has finite Hirsch rank, and since \( G_{i+1}/G_i \) is also \( R \)-torsion-free, \( h(G_{i+1}/G_i) = hd_R(G_{i+1}/G_i) \). So, again by the corner argument of the spectral sequence associated to the extension, \( 1 \to G_i \to G_{i+1} \to G_{i+1}/G_i \to 1 \), we have \( hd_R(G_{i+1}) \leq h(G_{i+1}) \).

Let \( m = hd_R(G_{i+1}/G_i) \). The spectral sequence gives

\[
\text{Tor}^R_{m+h_i}(G_i, F) \cong \text{Tor}^R_{m}(G_{i+1}/G_i, \text{Tor}^R_{h_i}(G_i, F)) \cong \text{Tor}^R_{m}(G_{i+1}/G_i, F^{\omega_i}) \cong \bigwedge^m(G_{i+1}/G_i \otimes F^{\omega_i})
\]

The last isomorphism follows from the fact that \( G_{i+1}/G_i \) is an \( R \)-torsion-free abelian group and \( F^{\omega_i} \) has trivial \( G_{i+1}/G_i \)-action. Also, since \( G_{i+1}/G_i \) is central in \( G/G_i \), the induced action of \( G/G_{i+1} \) on \( \bigwedge^m(G_{i+1}/G_i \otimes F^{\omega_i}) \) is trivial. So the module \( \bigwedge^m(G_{i+1}/G_i \otimes F^{\omega_i}) \) is \( F^{\omega_i+1} \), and \( hd_R(G_{i+1}) = h(G_{i+1}) \). This completes the induction.

\[ \square \]

Our next result requires a version of a theorem of Stammbach \[10\]. For the reader’s convenience we outline its proof which can be also found in \[2\].

**Proposition 3.6.** (Stammbach) Let \( G \) be an \( R \)-torsion-free solvable group with the Hirsch rank \( h \), where \( R \) is an integral domain of characteristic zero. Then
hd_R(G) = h and Tor_h^G(F, A) \cong F, for some FG-module A isomorphic to F as an F-module.

Proof. Let 1 = G_0 \subset G_1 \subset \ldots \subset G_n = G be a derived series of G. Let S_i = G_i/G_{i-1} and rk(S_i) = h_i. Set F_i = Tor_{h_i}^G(F, F). From Proposition 3.5 it follows that F_i is additively isomorphic to F.

Denote by F_i^{op} the additive group of F_i with the inverse G-action. Let A = \bigotimes_{i=1}^n F_i^{op}. We use induction on n to prove Tor_h^G(F, A) \cong F. For n = 1 the result is clear. Suppose n \geq 2, then G/G_1 has derived length n - 1. Thus, by the Lyndon-Hochschild-Serre spectral sequence of the extension 1 \to G_1 \to G \to G/G_1 \to 1, we have

\[
\text{Tor}_h^G(F, A) \cong \text{Tor}_h^{G/G_1}(F, \text{Tor}_{h_1}^{G/G_1}(F, A)) \cong \\
\text{Tor}_{h_1}^{G/G_1}(F, \text{Tor}_{h_1}^{G/G_1}(F, F) \otimes A) \cong \text{Tor}_{h_1}^{G/G_1}(F, \otimes_{i=2}^n F_i^{op}) \cong F.
\]

Now the assertion that hd_R(G) = h follows from the fact that h \leq hd_F(G) \leq hd_R(G) \leq h, when G is R-torsion-free. \qed

**Theorem 3.7.** Let G be a solvable group which acts freely and properly on an e-dimensional CW-complex E. Suppose there is an integer n \leq e such that H_i(E, \mathbb{Z}) = 0 for all i > n, and H_n(E, \mathbb{Z}) \cong \mathbb{Z}, then h(G) \leq e - n. If G is also without R-torsion, hd_R(G) \leq e - n.

Proof. Suppose h(G) > e - n. Then, there exists a subgroup H of G which has finite Hirsch rank larger than e - n. Now by 3.6 it follows that hd_Q(H) = h(H).

Therefore the homological dimension of H over Q is finite and larger than e - n, which contradicts Proposition 2.5. \qed

**Example 3.8.** Let F denote the Thompson group defined by the presentation

\[\langle x_0, x_1, x_2, \ldots | x_i^{-1} x_i x_{i+1} = x_{i+2} \text{ for all } i < n \text{ and } n \in \mathbb{N} \rangle\]

This group is finitely presented for it also has the presentation

\[\langle x_0, x_1 | x_2^{x_0} = x_3, x_3^{x_1} = x_4 \rangle\]

Brown and Goeghegan \[4\] showed that it is also of type FP_\infty, i.e. it has a projective \(ZG\)-resolution of \(Z\), where each module is finitely generated. They also proved \(H_n(F, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}\) for \(n \geq 1\) and \(H^n(F, \mathbb{Z}F) = 0\) for all \(n\). This may suggest that this group has periodic cohomology. But in fact, \(F\) cannot have periodic or jump cohomology over any ring, since it has an infinite rank abelian subgroup \(\mathbb{Z}^\infty\). Namely, \(\langle x_0 x_1^{-1}, x_2 x_3^{-1}, x_4 x_5^{-1}, \ldots \rangle \subseteq F\).
This reinstates our belief that periodicity and jumps are rather special conditions on the cohomology of a discrete group, since countable groups containing the Thompson group are the only class of groups, known to us, not to be $H.F$.

**Corollary 3.9.** If $G$ is a solvable group that acts freely and properly on an $(m+n)$-dimensional complex homotopy equivalent to a closed, orientable manifold of dimension $n$, then the Hirsch rank $h(G) \leq m$.

□

**Corollary 3.10.** If $G$ has a non-countable, torsion-free, solvable subgroup, then $G$ cannot act freely and properly on a finite dimensional complex homotopy equivalent to a closed, orientable manifold.

*Proof.* Let $H$ be a non-countable, torsion-free, solvable subgroup of $G$. By [2], Lemma 7.9, p. 100, $H$ has infinite Hirsch rank. Therefore it cannot act freely and properly on a finite dimensional complex homotopy equivalent to a closed, orientable manifold. □

4. **Actions of Height One**

These next results apply only to countable torsion-free groups. We find an obstruction for a group to be free in a specific setting. It would be interesting to know whether the following proposition holds for not necessarily countable groups and what implications there are when the obstruction does not vanish.

**Proposition 4.1.** Let $G$ be a finitely generated torsion-free group that acts freely and properly on an $n$-dimensional CW-complex $X$ such that the $G$-invariant submodule $\left(H_{n-1}(X, R)\right)^G \neq 0$. Then $G$ is a free group if and only if $\text{Ext}_R^2(R, H_n(X, R)) = 0$.

*Proof.* If $G$ is a free group then $cd_R(G) \leq 1$, so $\text{Ext}_R^2(R, H_n(X, R))$ must be trivial.

Conversely, suppose $\text{Ext}_R^2(R, H_n(X, R)) = 0$. Let $(C_*(X), d_*)$ be the chain complex of $X$, where $C_i(X)$ is the free $R$-module generated by the $i$-cells of $X$. We proceed by induction on the number of generators of $G$ with the case of $G$ having exactly one generator understood.

There exists an exact sequence of $ZG$-modules

$$0 \rightarrow d_n(C_n(X)) \rightarrow \ker(d_{n-1}) \rightarrow H_{n-1}(X, R) \rightarrow 0$$
and an associated long exact sequence

\[
\text{Ext}^0_{RG}(R, \ker(d_{n-1})) \to \text{Ext}^0_{RG}(R, H_{n-1}(X, R)) \to \text{Ext}^1_{RG}(R, d_n(C_n(X))) \to \text{Ext}^1_{RG}(R, \ker(d_{n-1}))
\]

The first term of the long exact sequence is the \(G\)-invariant submodule \((\ker(d_{n-1}))^G\). This module is trivial, for it is a submodule of the \(G\)-invariant submodule \((C_{n-1}(X))^G\). But \(G\) is infinite and \(C_{n-1}(X)\) is a free \(RG\)-module, hence \((C_{n-1}(X))^G = 0\). Thus the second term, the \(G\)-invariant submodule \((H_{n-1}(X, R))^G\), of the long exact sequence injects into \(\text{Ext}^1_{RG}(R, d_n(C_n(X)))\). This shows that \(\text{Ext}^1_{RG}(R, d_n(C_n(X))) \neq 0\).

On the other hand there exists an exact sequence of \(RG\)-modules

\[
0 \to H_n(X, R) \to C_n(X) \to d_n(C_n(X)) \to 0
\]

The long exact sequence associated to this gives:

\[
\text{Ext}^1_{RG}(R, H_n(X, R)) \to \text{Ext}^1_{RG}(R, C_n(X)) \to \text{Ext}^1_{RG}(R, d_n(C_n(X))) \to \text{Ext}^2_{RG}(R, H_n(X, R))
\]

According to our assumption \(\text{Ext}^2_{RG}(R, H_n(X, R)) = 0\), so the term \(\text{Ext}^1_{RG}(R, C_n(X))\) surjects onto the module \(\text{Ext}^1_{RG}(R, d_n(C_n(X)))\), which is nontrivial. Therefore \(\text{Ext}^1_{RG}(R, C_n(X)) \neq 0\). Now since \(G\) is finitely generated and \(C_n(X)\) is a free \(RG\)-module, we must have \(\text{Ext}^1_{RG}(R, RG) \neq 0\). So by [5] \(G\) must split as a nontrivial free product of groups.

Suppose then \(G = H \ast K\) where \(H, K \neq G\). For any \(RG\)-module \(M\),

\[
\text{Ext}^2_{RG}(R, M) = \text{Ext}^2_{RH}(R, M) \oplus \text{Ext}^2_{RK}(R, M)
\]

It then follows that,

\[
\text{Ext}^2_{RH}(R, H_n(X, R)) = \text{Ext}^2_{RK}(R, H_n(X, R)) = 0.
\]

Also by Grushko’s Theorem [5] there exist a fewer number of generators for each of the groups \(H\) and \(K\) than the number of generators of \(G\). So we can apply induction on these subgroups to show that they are free.

\[\square\]

**Theorem 4.2.** If \(G\) is a torsion-free countable group that acts freely and properly on an \(n\)-dimensional complex homotopy equivalent to a closed, orientable manifold of dimension \(n - 1\), then \(G\) must be free.
Proof. First, let us assume $G$ is finitely generated. If the induced action of $G$ on $H_{n-1}(X) = \mathbb{Z}$ is nontrivial then $G$ must contain an index two subgroup $H$ which acts trivially on $H_{n-1}(X)$. Thus, there always exists such $H \subseteq G$ acting trivially, with $[G : H] \leq 2$. Also, since $H_n(X) = 0$, $H^2(H, H_n(X)) = 0$. By Proposition 4.1 it follows that $H$ is free. Since $H$ is a finite index subgroup, $G$ must be free.

In general, since $G$ is a countable group, it is a countable union of an ascending chain of finitely generated subgroups. By the Theorem of Berstein, [6], we have $cd(G) \leq 2$. Using Proposition 2.5, it follows that $cd(G) \leq 1$. Therefore $G$ is free.

□

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