ONE-LOOP EFFECTIVE MULTI-GLUON LAGRANGIAN IN ARBITRARY DIMENSIONS

E. RODULFO* and R. DELBOURGO**

Department of Physics, University of Tasmania
G P O Box 252-21, Hobart 7001, Australia

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We exhibit the one-loop multi-gluon effective Lagrangian in any dimension for a field theory with a quasilocal background, using the background-field formalism. Specific results, including counter terms (up to 12 spacetime dimensions), have been derived, applied to the Yang-Mills theory and found to be in agreement with other string-inspired approaches.

1. Introduction

The work presented in this paper has been motivated by a reconsideration of photon-photon scattering, generalized to arbitrary dimensional spacetime and to any number of scattering photons. Though an extremely feebler process and therefore very difficult to observe, this nonlinear process of scattering photons by photons has attracted considerable attention since Euler first considered its low-energy limit in 1935. In this paper, we generalize to a non-Abelian analogue of this phenomenon: multi-gluon scattering in arbitrary dimensional spacetime which is not as weak a process as its Abelian counterpart, but is significantly richer in that gluons have classical self-interaction; also the gluon number need no longer be even, unlike photons. Multi-gluon scattering is mediated not only by meson and fermion virtual particles, but by vector and ghost loops too, quite aside from the tree-level contributions.

A process involving an arbitrary number of gluons may be represented by a sum over permutations of the Feynman diagram with those many legs (including pinched diagrams). Clearly, direct evaluation of the sum over Feynman amplitudes is very involved, so instead we shall attack this problem in the background-field formalism where, if appropriate quasilocal conditions are assumed, one can find exact expressions for the effective action without the tedious computation of Feynman diagrams. We shall revisit this method in section 2 where we shall be led to an expression for the one-loop effective Lagrangian for any gauge field theory with a quasilocal background in arbitrary dimensions. We shall then extract from this Lagrangian specific results including counterterms up to 12 spacetime dimensions and arrive at some conclusions. (We have limited ourselves to $D \leq 12$ since higher
numbers of dimensions have not gained wide currency.) In section 3, we apply the results of section 2 to the Yang-Mills theory with scalar bosons and Dirac fermions in the hope that this will provide greater insight into the structure of the effective Lagrangian associated with multi-gluon scattering in arbitrary dimensions, including string models and M-theory.

The requirement that the background be quasilocal (i.e. possesses a covariantly constant field strength tensor) does not mean that the results will be strictly valid only under that condition. In fact, as will be demonstrated, general results not constrained by the quasilocal assumption can be obtained for dimensions of practical importance, like \( D \leq 4 \). This is due to the fact that the covariant derivative of the field strength tensor of the background (which is set to zero in the quasilocal case) plays a significant role only in six and higher dimensions \( [12] \).

As a manifestation of the practical usefulness of working with a quasilocal background, we have successfully recovered the pure Yang-Mills Lagrangian (see eqn. 50) that portrays the curious disappearance of \( F_{\mu \nu}^2 \) one-loop divergences in 26 dimensions, as previously noted in \( [18] \) in respect of dimensionally reduced supersymmetric theories (and reproduced in reference \( [20] \) using open Bose string theory). It is interesting to note how precisely the same results follow from seemingly unrelated approaches. They tell us that new renormalization constants, involving higher powers of the curvature, must be introduced ab initio into quantum actions describing higher-dimensional gauge theories. What we and others have succeeded in doing is to calculate the (low-energy) one-loop contributions to such renormalization constants. In fewer space-time dimensions these terms of course represent finite calculable quantum corrections to the classical action.

Although our original motivation focuses on a particular process (multi-gluon scattering), the method and general results exhibited in this paper may be applied to any ordinary renormalizable theory with an appropriately chosen quasilocal background and seem readily extensible to gravity \( [22, 13] \).

2. The one-loop effective Lagrangian

The background-field procedure \( [1, 10, 11, 12, 13, 14] \) begins by replacing the field \( A \) in the original classical Lagrangian \( \mathcal{L}(A) \) by the sum \( A + h \), where \( A \) is now referred to as the background (or external) field and \( h \) the quantum (or internal) field. Working in a Euclidean formulation, so there is no distinction between upper and lower indices, the resulting Lagrangian is then expanded in \( h \) as follows.

\[
\mathcal{L}(A+h) = \mathcal{L}(A) + \frac{\partial \mathcal{L}(A)}{\partial A} h + \frac{1}{2} \partial_\mu h W_{\mu \nu}(A) \partial_\nu h + h N_\mu(A) \partial_\mu h + \frac{1}{2} h M(A) h + \mathcal{O}(h^3),
\]

\( (1) \)

\( ^1 \)The formal framework which relates string amplitudes to the background-field method has been explored fairly recently. See for example \( [21, 25] \).
where $W$, $N$ and $M$ are external spacetime-dependent functions which may be arranged to have the (anti)symmetry properties:

\[
W_{ij}^{\mu \nu} = W_{ji}^{\nu \mu} = W_{ij}^{\nu \mu},
\]

\[
N_{ij}^{\mu} = -N_{ji}^{\mu} \quad \text{and} \quad M_{ij} = M_{ji},
\]

by adding total spacetime derivatives to $L$. The relevant Lagrangian for loop diagrams is therefore given by \[10, 11\]

\[
L(A + h) - L(A) - \frac{\partial L(A)}{\partial A} = \frac{1}{2} \partial_{\mu} h W_{\mu \nu}(A) \partial_{\nu} h + h N_{\mu}(A) \partial_{\mu} h + \frac{1}{2} h M(A) h + O(h^3).
\]

One-particle-irreducible (1PI) loop diagrams are calculated by using the quantum fields $h$ as internal lines, while the background fields $A$ appear at external vertices.

It then follows that one-loop quantum effects will be governed only by terms bilinear in the quantum field $h$ and these are precisely the terms explicitly written on the right-hand-side of (3). Except for the theory of gravity which is not considered in this paper, we may safely assume a flat metric,

\[
W_{ij}^{\mu \nu} = -\delta_{\mu \nu} \delta^{ij}.
\]

If one further defines the tensor quantities

\[
X \equiv -m^2 + M - N_{\mu} N_{\mu},
\]

\[
Y_{\mu \nu} \equiv \partial_{\mu} N_{\nu} - \partial_{\nu} N_{\mu} + [N_{\mu}, N_{\nu}]
\]

which together with $h$ transform according to

\[
X \rightarrow e^{\Lambda(x)} X e^{-\Lambda(x)}
\]

\[
Y_{\mu \nu} \rightarrow e^{\Lambda(x)} Y_{\mu \nu} e^{-\Lambda(x)}
\]

\[
h \rightarrow e^{\Lambda(x)} h
\]

for some arbitrary antisymmetric matrix $\Lambda^{ij}(x)$, then the relevant bilinear Lagrangian may be cast in the manifestly gauge invariant form

\[
L = h[(\partial_{\mu} + N_{\mu})^2 + m^2 + X] h/2.
\]

The generating function for connected Green functions associated with this bilinear Lagrangian is given by

\[
\exp \frac{i}{\hbar} \int d^Dx L^{(1)} = \eta \int (dh) \exp \frac{i}{\hbar} \int d^Dx \frac{1}{2} h[(\partial_{\mu} + N_{\mu})^2 + m^2 + X] h
\]
where $\mathcal{L}^{(1)}$ is the one-loop effective Lagrangian and the constant $\eta$ is chosen so that

$$\mathcal{L}^{(1)} \xrightarrow{A=0} 0.$$  \hfill (12)

Differentiating (11) with respect to $X$ one finds that $\mathcal{L}^{(1)}$ is determined by the 2-point Green function evaluated at the same point.

$$\frac{\partial \mathcal{L}^{(1)}}{\partial X} = \frac{1}{2} \text{Tr}(h(x)h(x)).$$  \hfill (13)

Hence, one needs to solve the Green function equation associated with $\mathcal{L}$ and this is given by

$$[\partial^2 + m^2 + X(x) + \partial_\mu N_\mu(x) + 2N_\mu(x)\partial_\mu + N_\mu(x)N_\mu(x)] \frac{i}{\hbar} \langle h(x)h(x') \rangle = \delta^{D}(x,x')$$

(14)

But for an arbitrary background, this is clearly a nonlocal problem and one is obliged to consider perturbative methods. Brown and Duff \[15\] however, showed that by imposing appropriate restrictions on the background, one may obtain $\mathcal{L}^{(1)}$ in (13) exactly. We shall closely follow their method as we now assume a version of their quasilocal conditions on the background field:

$$\partial_\mu Y_{\nu\rho} = [Y_{\nu\rho}, N_\mu]$$ \hfill (15)

$$\partial_\mu X = [X, N_\mu]$$ \hfill (16)

It will be recognized that (15) is a non-Abelian analogue of the condition imposed by Schwinger \[4\] on the Maxwell field strength tensor in calculating one-loop effective Lagrangians for constant external electromagnetic fields. This restriction accommodates a non-Abelian background field with a covariantly constant field strength tensor \[26\]. The complementary condition (16) is the simplest restriction that still allows for non-Abelian gauge theories satisfying (15). It may be shown \[15\] that a tensor $Y_{\mu\nu}$ that satisfies (15) possesses commuting Lorentz components, or $[Y_{\mu\nu}, Y_{\rho\sigma}] = 0$. The field $N_\mu$ that satisfies (15) however, does not in general commute with itself (i.e., $[N_\mu, N_\nu] \neq 0$). It is in this sense that a quasilocal background specified by (15) and (16) may be taken non-Abelian.

The most general form of $N_\mu$ that satisfies (15) is given by \[15\]

$$N_\mu(x) = -Y_{\mu\nu}(x) x_\nu/2 + e^{\Lambda(x)} \partial_\mu e^{-\Lambda(x)}.$$ \hfill (17)

Exploiting the gauge invariance of $L$, one can work in a gauge described by the transformation \[15\]

$$N_\mu \rightarrow e^{-\Lambda(x)}(N_\mu + \partial_\mu)e^{\Lambda(x)}.$$ \hfill (18)

This brings us to a gauge where $Y_{\mu\nu}$ is constant: $Y_{\mu\nu}(x) = Y_{\mu\nu}(x')$, whereupon

$$N_\mu = -Y_{\mu\nu}(x')(x - x')_\nu/2.$$ \hfill (19)
It may be shown using (16) that in any gauge, $X$ commutes with $Y_{\mu\nu}$ and so in the gauge (18) $X$ is constant as well, i.e. $X(x) = X(x')$. With these simplifications, the Green function equation (14) reduces to

$$\left[\partial^2 + m^2 + X(x')\right] + (x - x')_\mu Y_{\mu\nu}(x')\partial_\nu - \frac{1}{4} (x - x')_\mu Y_{\mu\nu}(x')(x - x')_\nu$$

$$\frac{i}{\hbar} \langle h(x)h(x') \rangle = \delta^D(x,x').$$

(20)

In order to solve (20), we substitute the trial solution

$$\langle h(x)h(x') \rangle = \bar{h} \int_0^\infty ds \int \frac{d^Dp}{(2\pi i)^D} e^{-(m^2 + X)s - P(s) - [i(x-x') + Q(s)]p - pR(s)p/2},$$

(21)

where $P(s)$, $Q_\mu(s)$ and the symmetric nonsingular matrix $R_{\mu\nu}(s)$ are to be determined in terms of $X$ and $Y$ subject to the conditions ensuring consistency as the background field vanishes,

$$\lim_{A \to 0} \begin{pmatrix} P(s) \\ Q(s) \\ R(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2s \end{pmatrix}.$$  

(22)

$P$, $Q$ and $R$ are found to satisfy first order differential equations [16, 17] which may readily be solved (see Appendix). Then (13) can be integrated with respect to $X$ and the integration constant determined by (12). The resulting one-loop effective Lagrangian may then be expressed as

$$\mathcal{L}^{(1)} = \pm \frac{\hbar}{2(4\pi)^{D/2}} \int_0^\infty ds \, s^{1-D/2} e^{-m^2 s} \text{Tr} \sum_{p=0}^\infty \frac{(-1)^p}{p!} \left\{ X s + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{4n(2n)!} \text{tr}(Y s)^{2n} \right\}^p - [X(0)s]^p \right\}$$

(23)

where the overall sign is (+) for bosons and (−) for fermions. Tr denotes a trace over internal indices (including possibly spinor indices) while tr stands for a purely Lorentz trace. $X(0)$ is $X$ evaluated at zero background and $B_n$ are Bernoulli numbers. The result (23) which may be continued to arbitrary $D$, summarizes the contributions from all one-loop Feynman diagrams possessing arbitrarily many legs. It should also be noted that (23) remain valid in strong (but quasilocal) fields [26] as it contains all orders of the coupling constant which enter the expression through $X$ and $Y$.

Let us now extract some results from the one-loop effective Lagrangian (23). First, let us denote by $\mathcal{L}^{(1)}(D,p)$ the part of $\mathcal{L}^{(1)}$ which is of order $q$ in $X$ and order $r$ in $Y$ such that $q + r = p$, i.e.

$$\mathcal{L}^{(1)}(D,p) \sim \sum_{q+r=p} X^q Y^r.$$  

(24)
The first six non-vanishing $L^{(1)}(D,p)$’s are

\begin{align}
L^{(1)}(D,1) &= \frac{\hbar}{2(4\pi)^{D/2}} \frac{\Gamma(1-D/2)}{m^{2-D}} \text{Tr}\{-[X - X(0)]\} \\
L^{(1)}(D,2) &= \frac{\hbar}{2(4\pi)^{D/2}} \frac{\Gamma(2-D/2)}{m^{4-D}} \text{Tr}\left\{\frac{1}{2}[X^2 - X(0)^2] - \frac{1}{12}\text{tr}Y^2\right\} \\
L^{(1)}(D,3) &= \frac{\hbar}{2(4\pi)^{D/2}} \frac{\Gamma(3-D/2)}{m^{6-D}} \text{Tr}\left\{-\frac{1}{6}[X^3 - X(0)^3] + \frac{1}{12}X^2\text{tr}Y^2\right\} \\
L^{(1)}(D,4) &= \frac{\hbar}{2(4\pi)^{D/2}} \frac{\Gamma(4-D/2)}{m^{8-D}} \text{Tr}\left\{\frac{1}{24}[X^4 - X(0)^4] - \frac{1}{24}X^2\text{tr}Y^2 \\
&\quad + \frac{1}{288}(\text{tr}Y^2)^2 + \frac{1}{360}\text{tr}Y^4\right\} \\
L^{(1)}(D,5) &= \frac{\hbar}{2(4\pi)^{D/2}} \frac{\Gamma(5-D/2)}{m^{10-D}} \text{Tr}\left\{-\frac{1}{120}[X^5 - X(0)^5] + \frac{1}{72}X^3\text{tr}Y^2 \\
&\quad - \frac{1}{288}X(\text{tr}Y^2)^2 - \frac{1}{360}X\text{tr}Y^4\right\} \\
L^{(1)}(D,6) &= \frac{\hbar}{2(4\pi)^{D/2}} \frac{\Gamma(6-D/2)}{m^{12-D}} \text{Tr}\left\{\frac{1}{120}[X^6 - X(0)^6] - \frac{1}{288}X^4\text{tr}Y^2 + \\
&\quad \frac{1}{576}X^2(\text{tr}Y^2)^2 + \frac{1}{720}X^2\text{tr}Y^4 - \frac{1}{10368}(\text{tr}Y^2)^3 - \frac{1}{4320}\text{tr}Y^2\text{tr}Y^4 - \frac{1}{5670}\text{tr}Y^6\right\}. \\
\end{align}

The divergent part of $L^{(1)}(D,p)$ is given by $L^{(1)}(D \to 2p, p)$. For the results (25) to (30), the corresponding divergent parts (as $D$ approaches an integer value $2p$) are picked out as

\begin{align}
L^{(1)}(D \to 2,1) &= \frac{\hbar}{4\pi(2-D)} \text{Tr}\{-[X - X(0)]\} \\
L^{(1)}(D \to 4,2) &= \frac{\hbar}{16\pi^2(4-D)} \text{Tr}\left\{\frac{1}{2}[X^2 - X(0)^2] - \frac{1}{12}\text{tr}Y^2\right\} \\
L^{(1)}(D \to 6,3) &= \frac{\hbar}{64\pi^3(6-D)} \text{Tr}\left\{-\frac{1}{6}[X^3 - X(0)^3] + \frac{1}{12}X\text{tr}Y^2\right\} \\
L^{(1)}(D \to 8,4) &= \frac{\hbar}{256\pi^4(8-D)} \text{Tr}\left\{\frac{1}{24}[X^4 - X(0)^4] - \frac{1}{24}X^2\text{tr}Y^2 \\
&\quad + \frac{1}{288}(\text{tr}Y^2)^2 + \frac{1}{360}\text{tr}Y^4\right\}.
\end{align}
\[ \mathcal{L}^{(1)}(D \to 10, 5) = \frac{\hbar}{1024\pi^5(10 - D)} \text{Tr} \left\{ -\frac{1}{120} [X^5 - X(0)^5] + \frac{1}{72} X^3 \text{tr} Y^2 - \frac{1}{288} X(\text{tr} Y^2)^2 - \frac{1}{360} X\text{tr} Y^4 \right\} \]

\[ \mathcal{L}^{(1)}(D \to 12, 6) = \frac{\hbar}{4096\pi^6(12 - D)} \text{Tr} \left\{ \frac{1}{720} [X^6 - X(0)^6] - \frac{1}{288} X^4 \text{tr} Y^2 + \frac{1}{576} X^2(\text{tr} Y^2)^2 + \frac{1}{720} X^2 \text{tr} Y^4 - \frac{1}{10368}(\text{tr} Y^2)^3 - \frac{1}{4320} \text{tr} Y^2 \text{tr} Y^4 - \frac{1}{5670} \text{tr} Y^6 \right\} . \]

Any higher dimensional Lagrangians are probably irrelevant to the most popular physical models.

One-loop counterterms are usually defined as the negative of our \( \mathcal{L}^{(1)}(D \to 2p, p) \). The results (31) and (32) supply the divergent Lagrangians in \( D = 2 \) and \( D = 4 \), respectively. These results are valid in general (even in the nonquasilocal case) as may be checked with references [18, 19, 10, 13] because the covariant derivatives \( D Y \) (where \( D_\mu \equiv \partial_\mu + [N_\mu, \cdot] \)), only begin to appear in the invariants for \( D \geq 6 \) [19]. However, the results (33) to (36) for \( D = 6, 8, 10, 12 \) are valid only in the quasilocal case. The results for \( D = 6, 8, 10 \) may still be compared only with references [18, 19] provided the quasilocal conditions (15) and (16) are imposed on their results. The presence of covariant derivatives however, provides some degree of arbitrariness in the choice of invariants due to the Bianchi identities and possible partial integrations on the covariant derivatives. For instance, the invariant \( Y_{\mu\nu}Y_{\nu\rho}Y_{\rho\mu} \) appears in the results of [18, 19] but does not appear in the quasilocal case (33) because the Bianchi identities allow us to write (up to a total divergence)

\[ Y_{\mu\nu}Y_{\nu\rho}Y_{\rho\mu} = \frac{1}{2}(D_\mu Y_{\mu\nu})^2 + \frac{1}{2}(D_\mu Y_{\nu\rho})(D_\nu Y_{\rho\mu}), \]

which clearly vanishes in the quasilocal case. Our results for \( D \geq 6 \) may be viewed as unique quasilocal limiting expressions of more general frameworks since all arbitrariness in the choice of invariants disappears in this limit.

One may also indirectly compare the results of this section with those of Avramidi [26], who calculated the asymptotic coefficients of the heat kernel up to \( D = 16 \) essentially, although some work is needed before the results can be immediately compared.

### 3. Yang-Mills Theory

In order to apply the results of section 2 to a specific field theory, one needs to determine for the theory in question the relevant second order operator

\[ \Delta \equiv (\partial_\mu + N_\mu)^2 + m^2 + X \]
which appears in the bilinear Lagrangian (10) and from this identify \( N_\mu, m^2 \) and \( X \). \( Y_{\mu \nu} \) follows immediately from \( N_\mu \) through (6). Once these expressions are correctly identified, the results of section 2 can be easily applied. As will be demonstrated in this section in the case of Yang-Mills theory, the term \( m^2 + X \) in (38) plays the role of a generic ‘source’ which determines the type of virtual particle loop that mediates the interaction. We begin by considering the pure Yang-Mills theory in section 3.1. Scalar bosons and Dirac fermions in a Yang-Mills background are discussed in sections 3.2 and 3.3. Results for the Yang-Mills theory incorporating the mesons and fermions close this section.

3.1 Pure Yang-Mills theory

Performing the background-field replacement \( A_\mu \rightarrow A_\mu + a_\mu \) in the bare Lagrangian for the pure Yang-Mills theory \cite{10, 16}

\[
\mathcal{L} = -\frac{1}{4} F^a_{\mu \nu} F^{a}_{\mu \nu} \tag{39}
\]

where

\[
F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{40}
\]

and using the Feynman-background gauge \cite{10, 16, 12}

\[
\mathcal{L}_{fix} = -\frac{1}{2} \left[ (\partial_\mu \delta^{ac} + g f^{abc} A^b_\mu) a^c_\mu \right]^2 \tag{41}
\]

one finds that the relevant bilinear Lagrangian is

\[
2L_{YM} = a^2 \left[ (\partial_\mu \delta^{ab} \delta_{\alpha \beta} - g f^{abc} A^c_\mu \delta_{\alpha \beta})^2 - 2g f^{abc} A^c_\mu F^a_{\alpha \beta} \right] a^b_\mu + \eta^{\alpha}_{\mu} (\partial_\mu \delta^{ab} - g f^{abc} A^c_\mu) \eta^b_{\mu} \tag{42}
\]

The first term in (42) gives the vector part while the second gives the contribution of the two fictitious fields \( \eta_i, i = 1, 2 \). Hence, the relevant second order vector and ghost operators are, respectively

\[
(\Delta_{\text{vector}})^{ab}_{\alpha \beta} = (\partial_\mu \delta^{ab} \delta_{\alpha \beta} - g f^{abc} A^c_\mu \delta_{\alpha \beta})^2 - 2g f^{abc} F^c_{\alpha \beta} \tag{43}
\]

\[
(\Delta_{\text{ghost}})^{ab}_{\mu} = (\partial_\mu \delta^{ab} - g f^{abc} A^c_\mu)^2 \tag{44}
\]

Comparison with (38) immediately yields:

\[
(N_{\text{vector}})^{ab}_{\mu \alpha \beta} = -g f^{abc} A^c_\mu \delta_{\alpha \beta} \tag{45}
\]

\[
(Y_{\text{vector}})^{ab}_{\mu \alpha \beta} = -g f^{abc} F^c_{\mu \nu} \delta_{\alpha \beta} \tag{46}
\]

\[
(X_{\text{vector}})^{ab}_{\alpha \beta} = -2g f^{abc} F^c_{\alpha \beta} \tag{47}
\]

\[
(N_{\text{ghost}})^{ab}_{\mu} = -g f^{abc} A^c_\mu \tag{48}
\]

\[
(Y_{\text{ghost}})^{ab}_{\mu \nu} = -g f^{abc} F^c_{\mu \nu} \tag{49}
\]
together with the vanishing expressions $m_{\text{vector}} = m_{\text{ghost}} = X_{\text{ghost}} = 0$. Note that the ghost effective Lagrangian acquires an overall factor of $-2$ resulting from the two "fermionic" fields $\eta_1$ and $\eta_2$.

Using the results of section 2 one finds that the first few nonvanishing $\mathcal{L}_{pYM}^{(1)}(D, p)$'s are

\[
\mathcal{L}_{pYM}^{(1)}(D, 2) = \frac{\hbar g^2}{2(4\pi)^{D/2}} \left( \frac{26 - D}{12} \right) \int_0^\infty ds \, s^{1-D/2} \, CF_{\mu\nu}^a F_{\mu\nu}^a
\]

\[
\mathcal{L}_{pYM}^{(1)}(D, 4) = \frac{\hbar g^4}{2(4\pi)^{D/2}} \int_0^\infty ds \, s^{3-D/2} \, f^{abc} f^{bcd} f^{daeh} \left\{ \frac{238 + D}{360} F_{\mu\nu}^a F_{\rho\sigma}^a F_{\tau\lambda}^b + \frac{50 + D}{288} F_{\mu\nu}^a F_{\rho\sigma}^a F_{\tau\lambda}^b + \frac{4}{15} F_{\mu\nu}^a F_{\rho\sigma}^b F_{\tau\lambda}^c \right\}
\]

\[
\mathcal{L}_{pYM}^{(1)}(D, 5) = -\frac{\hbar g^5}{2(4\pi)^{D/2}} \int_0^\infty ds \, s^{4-D/2} \, f^{abf} f^{bch} f^{cdi} f^{efj},
\]

\[
\mathcal{L}_{pYM}^{(1)}(D, 6) = \frac{\hbar g^6}{2(4\pi)^{D/2}} \int_0^\infty ds \, s^{5-D/2} \, f^{abc} f^{bcd} f^{def} f^{fgi} f^{fjl} \left\{ \frac{506-D}{5670} F_{\mu\nu}^a F_{\rho\sigma}^a F_{\tau\lambda}^b F_{\gamma\delta}^c F_{\epsilon\zeta}^d - \frac{214+D}{4320} F_{\mu\nu}^a F_{\rho\sigma}^a F_{\tau\lambda}^b F_{\gamma\delta}^c F_{\epsilon\zeta}^d \right\}
\]

where $C_{\delta\cd} = f^{abc} f^{abd}$ is the Casimir of the adjoint representation. The result (50) clearly exhibits the curious absence of $F_{\mu\nu}^2$ one-loop divergences in the pure Yang-Mills theory in 26 spacetime dimensions, as noted in references [18, 20].

The divergent part of (50) is the well known result [10, 23, 24]

\[
\mathcal{L}_{pYM}^{(1)}(D \to 4, 2) = \frac{\hbar g^2}{32\pi^2 (4-D)} \frac{11}{3} CF_{\mu\nu}^a F_{\mu\nu}^a
\]

which is of course valid even in the nonquasilocal case. The divergent part of the Lagrangian in 8 dimensions follows immediately from (51):

\[
\mathcal{L}_{pYM}^{(1)}(D \to 8, 4) = \frac{\hbar g^4}{256\pi^4 (8-D)} f^{abc} f^{bcd} f^{daeh} \left( \frac{41}{60} F_{\mu\nu}^a F_{\rho\sigma}^a F_{\tau\lambda}^b F_{\gamma\delta}^c - \frac{7}{48} F_{\mu\nu}^a F_{\rho\sigma}^a F_{\tau\lambda}^b F_{\gamma\delta}^c \right)
\]

(The quasilocal results (51) and (55) may be compared with equation (2.9) of reference [20].) For completeness, let us also write down the quasilocal divergent
Lagrangians in ten and twelve dimensions:

\[
\mathcal{L}^{(1)}_{pYM}(D \to 10, 5) = -\frac{\hbar g^5 f_{abf} f_{bcg} f_{cdh} f_{dei} f_{eaj}}{1024\pi^5(10 - D)} F^f_{\mu\nu} F^g_{\rho\sigma} F^i_{\lambda\sigma} F^j_{\lambda\mu} \tag{56}
\]

\[
\mathcal{L}^{(1)}_{pYM}(D \to 12, 6) = \frac{\hbar g^6 f_{abg} f_{bch} f_{cdi} f_{efk} f_{fal}}{4096\pi^6(12 - D)} \left( \frac{247}{2835} F^g_{\mu\nu} F^h_{\rho\sigma} F^i_{\lambda\sigma} F^k_{\lambda\tau} F^l_{\tau\mu} \right) + \frac{113}{2160} F^g_{\mu\nu} F^h_{\mu\nu} F^i_{\lambda\sigma} F^k_{\lambda\tau} F^l_{\tau\lambda} \tag{57}
\]

### 3.2 Scalar bosons in a Yang-Mills background

The Lagrangian for scalar bosons in a classical background gauge field \( A_\mu \) takes the bilinear form \[25\]

\[
L_{\text{scalar}} = \phi^\dagger \left( [\partial_\mu - ig T^a s A^a_\mu]^2 + m^2 \right) \phi \tag{58}
\]

where we normalize the generators according to

\[
\text{Tr} \left( T^a s T^b s \right) = T_s \delta^{ab}. \tag{59}
\]

The relevant second order operator (38) for this theory is therefore

\[
\Delta_{\text{scalar}} = (\partial_\mu - ig T^a s A^a_\mu)^2 + m^2 \tag{60}
\]

and one immediately identifies:

\[
(N_{\text{scalar}})_\mu = -ig T^a s A^a_\mu \tag{61}
\]

\[
(Y_{\text{scalar}})_{\mu\nu} = -ig T^a s F^a_{\mu\nu} \tag{62}
\]

\[
X_{\text{scalar}} = 0. \tag{63}
\]

The results of the previous section now allow us to list down the first few non-vanishing \( \mathcal{L}^{(1)}_{\text{scalar}}(D, p) \)'s:

\[
\mathcal{L}^{(1)}_{\text{scalar}}(D, 2) = \frac{\hbar g^2}{2(4\pi)^D/2} \frac{\Gamma(2 - D/2)}{m^{4-D}} \frac{1}{2} T_s F^a_{\mu\nu} F^a_{\mu\nu} \tag{64}
\]

\[
\mathcal{L}^{(1)}_{\text{scalar}}(D, 4) = \frac{\hbar g^4}{2(4\pi)^D/2} \frac{\Gamma(4 - D/2)}{m^{8-D}} \text{Tr} \left( T^a s T^b s T^c s T^d \right) \left\{ \frac{1}{360} F^a_{\mu\nu} F^b_{\mu\nu} F^c_{\rho\sigma} F^d_{\sigma\mu} + \frac{1}{288} F^a_{\mu\nu} F^b_{\mu\nu} F^c_{\rho\sigma} F^d_{\sigma\mu} \right\}. \tag{65}
\]
The corresponding divergent Lagrangians are

\[
L_{\text{scalar}}^{(1)}(D \to 4, 2) = \frac{\hbar g^2}{32\pi^2(4-D)} \frac{1}{6} T_s F_{\mu\nu}^a F_{\mu\nu}^a
\]

\[
L_{\text{scalar}}^{(1)}(D \to 8, 4) = \frac{\hbar g^4}{256\pi^4(8-D)} \frac{1}{6} T_s T_a T_b T_c T_d \left\{ \frac{1}{360} F_{\mu\nu}^{a} F_{\rho\sigma}^{b} F_{\tau\lambda}^{c} F_{\tau\lambda}^{d} + \frac{1}{288} F_{\mu\nu}^{a} F_{\rho\sigma}^{b} F_{\sigma\rho}^{c} F_{\tau\lambda}^{d} \right\}
\]

\[
L_{\text{scalar}}^{(1)}(D \to 12, 6) = \frac{\hbar g^6}{4096\pi^6(12-D)} \frac{1}{6} T_s T_a T_b T_c T_d \left\{ \frac{1}{5670} F_{\mu\nu}^{a} F_{\rho\sigma}^{b} F_{\tau\lambda}^{c} F_{\tau\lambda}^{d} F_{\sigma\rho}^{e} F_{\tau\lambda}^{f} + \frac{1}{4320} F_{\mu\nu}^{a} F_{\rho\sigma}^{b} F_{\tau\lambda}^{c} F_{\tau\lambda}^{d} F_{\sigma\rho}^{e} F_{\tau\lambda}^{f} + \frac{1}{10368} F_{\mu\nu}^{a} F_{\rho\sigma}^{b} F_{\tau\lambda}^{c} F_{\tau\lambda}^{d} F_{\sigma\rho}^{e} F_{\tau\lambda}^{f} \right\}
\]

\[\] (66)

3.3 Dirac fermions in a Yang-Mills background

The Lagrangian for Dirac fermions in a classical background gauge field \(A_\mu\) is usually written in a form that involves a first order operator

\[
L_{\text{fermion}} = \psi^\dagger \left( i \gamma_\mu \left( \partial_\mu - ig T^a_\mu A^a_\mu \right) - m \right) \psi
\]

where \(\gamma_\mu\) represents a \(2^{[D/2]} \times 2^{[D/2]}\) Dirac matrix normalized as usual by

\[
\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} 1
\]

\[
\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]
\]

\[
\text{Tr} 1 = 2^{[D/2]}
\]

A valid second order operator \([21, 25]\) is obtained through squaring:

\[
\Delta_{\text{fermion}} = \left[ -i \gamma_\mu \partial_\mu - ig T^a_\mu A^a_\mu \right] - m \right] \psi
\]

Using (71) and (72), this may be written as

\[
\Delta_{\text{fermion}} = \left( \partial_\mu - ig T^a_\mu A^a_\mu \right)^2 1 + m^2 1 - g \sigma_{\mu\nu} T^a_\mu F_{\mu\nu}^a / 2.
\]

\[\] (75)
This may now be compared with (38) and one finds

\[ (N_{\text{fermion}})_{\mu} = -igT^a_{\mu}A^a_{\mu}, \]  

(76)

\[ (Y_{\text{fermion}})_{\mu\nu} = -igT^a_{\mu}F^a_{\mu\nu}, \]  

(77)

\[ (X_{\text{fermion}}) = -g\sigma_{\mu\nu}T^a_{\mu}F^a_{\mu\nu}/2. \]  

(78)

The first nonvanishing \( L_{\text{fermion}}^{(1)}(D, p) \) follows from (26) and is

\[ L_{\text{fermion}}^{(1)}(D, 2) = -\frac{\bar{h}g^2}{2(4\pi)^{D/2}} \frac{\Gamma(2 - D/2)}{m^{4-D}} \frac{2^{D/2}}{6} T_f F^a_{\mu\nu} F^a_{\mu\nu} \]  

(79)

where \( \text{Tr}(T^a_{\mu}T^b_{\nu}) = T_f \delta^{ab} \) has been used. The corresponding divergent Lagrangian is given by

\[ L_{\text{fermion}}^{(1)}(D \to 4, 2) = \frac{\bar{h}g^2}{32\pi^2(4 - D)} \frac{4}{3} T_f F^a_{\mu\nu} F^a_{\mu\nu} \]  

(80)

Up to this order and as a check on our work, let us combine the results for the divergent Lagrangians of the pure Yang-Mills (54), the scalar boson (67) and the Dirac fermion (80).

\[ L_{YM}^{(1)}(D \to 4, 2) = \frac{\bar{h}g^2}{256\pi^4(8 - D)} \left( \frac{11}{3} C + \frac{1}{6} T_s - \frac{4}{3} T_f \right) F^a_{\mu\nu} F^a_{\mu\nu}. \]  

(81)

The result (81) is the one-loop divergent Lagrangian for the Yang-Mills theory with scalar bosons and Dirac fermions and is not restricted to the quasilocal cases (15) and (16). Except for the scalar sector, one may compare (81) with reference [14] which gives the \( \beta \) function for Yang-Mills with Dirac fermions.

The quasilocal assumption causes \( L_{\text{fermion}}^{(1)}(D, 3) \) to vanish. The next nonvanishing Lagrangian is of order \( F^4 \),

\[ L_{\text{fermion}}^{(1)}(D \to 4, 4) = -\frac{\bar{h}g^4}{2(4\pi)^{D/2}} \frac{\Gamma(4 - D/2)}{m^{8-D}} 2^{D/2} \text{Tr} \left( T^a_{\mu}T^b_{\nu}T^c_{\rho}T^d_{\sigma} \right) \]  

\[ \left( \frac{8}{18} F^a_{\mu\nu}F^b_{\rho\sigma}F^c_{\tau\sigma}F^d_{\tau\rho} - \frac{7}{180} F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\sigma}F^d_{\nu\sigma} \right), \]  

(82)

whose divergent part in 8 dimensions is

\[ L_{\text{fermion}}^{(1)}(D \to 8, 4) = \frac{\bar{h}g^4}{256\pi^4(8 - D)} \text{Tr} \left( T^a_{\mu}T^b_{\nu}T^c_{\rho}T^d_{\sigma} \right) \]  

\[ \left( \frac{8}{3} F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\sigma}F^d_{\sigma\mu} - \frac{28}{45} F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\sigma}F^d_{\mu\nu} \right), \]  

(83)

Finally, let us collect the results for the pure Yang-Mills (55), scalar (68) and fermion (83) sectors and exhibit the quasilocal divergent Lagrangian in 8 dimensions.

\[ L_{YM}^{(1)} = \frac{\bar{h}g^4}{256\pi^4(8 - D)} \left( \left\{ \frac{7}{48} f^{ef}_{\mu\nu} f^{gb}_{\rho\sigma} f^{hc}_{\sigma\mu} + \frac{1}{288} \text{Tr} (T^a_{\mu}T^b_{\nu}T^c_{\sigma}T^d_{\rho} \right) \right) \]
\[
\frac{8}{9} \text{Tr}(T^a T^b T^c T^d) F^a_{\mu \nu} F^b_{\nu \rho} F^c_{\rho \sigma} F^d_{\sigma \mu} + \frac{41}{60} \epsilon^{efg} \epsilon^{fgh} \epsilon^{jkl} \] 
\[
\frac{1}{360} \text{Tr}(T^a T^b T^c T^d) + \frac{28}{45} \text{Tr}(T^a T^b T^c T^d) \right] F^a_{\mu \nu} F^b_{\nu \rho} F^c_{\rho \sigma} F^d_{\sigma \mu} \right) \tag{84}
\]

The ten and twelve dimensional results are too complicated to write down and are probably not worth exposing. Nevertheless, the existence of higher dimensional divergent Lagrangians suggests that any theoretical model done in \(D\)-spacetime dimensions must incorporate the \(F^{D/2}\)-invariant (together with properly chosen \(DF\)-invariants of the same order in the non-quasilocal case) into the bare Lagrangian. This is entirely in keeping with the dimensional nature of the gauge field \(A\) and coupling \(g\), namely

\[
[A] = M^{D/2-1}, \quad [g] = M^{2-D/2},
\]

and accords perfectly with results (50) - (57). To reiterate, it is not enough to assume that the field theory starts off with a bare Lagrangian such as (39). In quantum field theory one should at the very least include extra powers of \(F\), as encapsulated in the divergent parts (82) - (84), etc.

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Appendix

To find the functions \(P(s), Q(s)\) and \(R(s)\), let us preoperate the Green function equation (20) by \(\int d^Dx e^{ip \cdot (x-x')}\) and substitute the trial solution (21).

\[
\int d^Dx e^{ip \cdot (x-x')} \begin{bmatrix} \partial^2 + m^2 + X(x') + (x-x')_\mu Y_{\mu \nu}(x') \partial_\nu - Y^2_{\mu \nu}(x') (x-x')_\nu / 4 \end{bmatrix}
\]

\[
\times i \int_0^\infty ds \int \frac{d^Dp'}{(2\pi)^D} e^{-(m^2+X)s-P(s)-[i(x-x')+Q(s)]p'-p'R(s)p'/2}
\]

\[
= \int d^Dx e^{ip \cdot (x-x')} \delta^D(x, x').
\]

Using the delta function representation \(\int d^Dx e^{i(p-p') \cdot x} = (2\pi)^D \delta^D(p-p')\), this yields

\[
\int_0^\infty ds \begin{bmatrix} -p^2 + m^2 + X(x') + Y_{\mu \nu}(x') p_\nu \partial_{p_\mu} + \frac{1}{4} Y^2_{\mu \nu}(x') \partial_{p_\mu}^2 \end{bmatrix}
\]
\[ e^{-(m^2 + X)s - P(s) - Q(s)} p - \frac{1}{2} p R(s) - p = i^{D-1}. \]

From the antisymmetry of \( Y \) and the symmetry of \( R \) one can prove that
\[ p_\mu Y_{\mu\nu} R_{\nu\rho} p_\rho = \frac{1}{2} [Y, R]_{\mu\rho} p_\rho. \]

But the complementary quasilocal condition (16) implies that \( P, Q \) and \( R \) will depend only on \( Y \) and will therefore commute with \( Y \). Hence, the right-hand-side above vanishes and whenever \( Y_{\mu\nu} R_{\nu\rho} \) is encountered, we may set it to zero.

Performing the \( p \)-differentions leads to
\[
\int_0^\infty ds \left[ m^2 + X + \frac{1}{4} (Q \cdot Y^2 \cdot Q - \text{tr} Y^2 R) + Q \cdot (-Y + \frac{1}{2} Y^2 R) \cdot p + \frac{1}{2} (p \cdot (-2 + \frac{1}{2} R Y^2 R) \cdot p) \right] \times e^{(m^2 + X)s - P(s) - Q(s) \cdot p - \frac{1}{2} p R(s) - p} = i^{D-1},
\]
which is clearly integrable provided \( P, Q \) and \( R \) satisfy the first order differential equations,
\[
\frac{\partial}{\partial s} P(s) = \frac{1}{4} [Q(s) \cdot Y^2 \cdot Q(s) - \text{tr} Y^2 R(s)],
\]
\[
\frac{\partial}{\partial s} Q(s) = Q(s) \cdot \left[ -Y + \frac{1}{2} Y^2 R(s) \right],
\]
\[
\frac{\partial}{\partial s} R(s) = -2 + \frac{1}{2} R(s) Y^2 R(s).
\]

By this means,
\[
e^{-4(m^2 + X) s - P(s) - Q(s) \cdot p - p R s \cdot p/2} \bigg|_0^\infty = -i^{D-1}
\]

The differential equations for \( P, Q, R \) subject to the boundary condition (22) give us the following solutions:
\[
P(s) = -\text{tr}[\ln \sec(i Y s)]/2
\]
\[
Q(s) = 0
\]
\[
R(s) = 2 Y^{-1} \tan(i Y s)
\]

Performing the \( p \)-integration in the trial solution (21) using the formula
\[
\int d^D p e^{-P \cdot Q - \frac{1}{2} p R} = i^{D - D/2} s^{-D/2} e^{-P + \frac{1}{2} Q R - \frac{1}{2} \text{tr} \ln(-R/2s)}
\]
then setting \( x = x' \), one finds that the Green function evaluated at zero separation is
\[
\langle h(x) h(x) \rangle = \frac{\hbar}{(4\pi)^{D/2}} \int_0^\infty ds s^{-D/2} e^{-(m^2 + X) s - \text{tr} \ln[(i Y s)^{-1} \sin(i Y s)]/2}.
\]
Substituting this into (13) and integrating with respect to $X$ while remembering the limiting value (12), one finds \textsuperscript{15, 16}

$$\mathcal{L}^{(1)} = \frac{\hbar}{2(4\pi)^{D/2}} \int_{0}^{\infty} \frac{ds}{s^{1+D/2}} e^{-m^2 s} \text{Tr} \left\{ e^{-X s - \text{tr} \ln[(iY s)^{-1} \sin(iY s)]/2} - e^{-X(0)s} \right\}.$$  

Last but not least, the Taylor expansion

$$\ln \left[ \frac{\sin(z)}{z} \right] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^{2n}B_{2n}}{2n(2n)!} z^{2n},$$

where $B_n$ are the Bernoulli numbers, allows one to rewrite the last integral in the form (23) which is suitable for the purpose of obtaining $\mathcal{L}^{(1)}$ to any specific order in the fields $X$ and $Y$.

**References**

[*] Email: rodulfo@oberon.phys.utas.edu.au

[**] Email: delbourg@oberon.phys.utas.edu.au

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