Boson star and dark matter

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Abstract Bound states of complex scalar fields (boson stars) have long been proposed as possible candidates for the dark matter in the universe. Considerable work has already been done to study various aspects of boson stars. In the present work, assuming a particular anisotropic matter distribution, we solve the Einstein-Klein-Gordon equations with a cosmological constant to obtain bosonic configurations by treating the problem geometrically. The results are then applied to problems covering a wide range of masses and radii of the boson stars and the relevant self interaction parameters are calculated. We compare our results with earlier treatments to show the applicability of the geometrical approach.

Keywords Exact solution · Einstein-Klein-Gordon equation · Boson star · Dark matter.

1 Introduction

According to recent cosmological observations, approximately 96% of the total material content of the universe is exotic in nature out of which 73% is gravitationally repulsive (dark energy) and the remaining 23% is attractive in nature but exists in the form of dark matter[123]. The quest for such exotic matter in the universe remains an outstanding problem in cosmology and astrophysics. In some models, massive axion of mass $m \sim 10^{-5}$ eV is considered as a standard dark matter particle (with no self-interaction). These massive axions may collapse to form compact objects known as
axion stars of masses comparable to the mass of a planet. The survival of such axion stars has been questioned by Seidel and Suen [4]. Moreover, these small objects cannot explain the observed abundance of dwarf galaxies and the dark matter density profiles in the core of galaxies. The view that dark matter in the galactic halos could be due to the presence of scalar field configurations has been considered by many [5, 6]. (see also Ref. [7] for a recent review). Hu et al [8] showed that the large scale structure of the galactic halo could be explained by dark matter composed of ultralight scalar particles of mass $m \sim 10^{-22}$ eV. Lee and Koh [9] have investigated boson stars with a self-interacting scalar field as the model for galactic halos. These results are consistent with the observation that the mass of a scalar particle should be greater than $10^{-28}$ eV not to disturb radial stability [10].

The large abundance of dark energy in the universe is also a field of active research. It appears that the dark energy could be well described by a cosmological constant [11]. In this paper we will accept this interpretation. The effect of the cosmological constant on the mass or radius of a compact boson star will be negligible. However, we will work in the general framework where the cosmological constant will be retained [12].

Historically, the concept of a gravitationally bound state of scalar particles, was proposed by Kaup [13] and Ruffini and Bonazzola [14]. Since a boson star is prevented from gravitational collapse by Heisenburg uncertainty principle, the maximum mass of a boson star with no self-interaction, is very small. However, Colpi et al [15] showed that a self-interaction of the scalar particles could yield compact objects of masses comparable to neutron stars. In the absence of direct observational evidence of fundamental scalar particles constituting a boson star, various models have been proposed to study the gross features of such hypothetical objects [16, 17, 18, 19, 20, 21, 22]. One of the main motivations to study these objects is to understand its possible role in explaining the dark matter in the universe, although boson stars made up of Higgs’s particles, dilatons and other scalar particles may also exist. The choice of the potential corresponding to the self-interaction is very important in the calculations of such models. A quartic type potential ($\Lambda \Phi^4$) is, in general, used for the self-interaction [15, 21], though some works on higher order self-interactions [22] and different forms such as cosh-type potential [1, 23] have also been reported in the literature. The motivation for choosing a certain potential form is to obtain configurations which are consistent with astrophysical and cosmological observations. For example, assuming a cosh-type potential of the form $V(\Phi) = V_0[\cosh(\lambda \sqrt{8\pi G\Phi}) - 1]$, Alcubierre et al [23] showed that one can construct a self-gravitating scalar object or oscillation of critical mass $M_{\text{crit}} \sim 10^{12} M_\odot$ which is comparable with the dark matter content of a galactic halo where, the parameters have values $\lambda \sim 20.28, V_0 \sim (3 \times 10^{-27} m_{\text{pl}})^4, m_\Phi \sim 1.1 \times 10^{-23}$ eV. Henriques and Mendes [18] considered a combined boson-fermion star and showed that it is possible to generate a wide variety of configurations ranging from objects of atomic sizes and masses of the order of $10^{-15} M_\odot$ to objects having galactic masses of the order of $10^{13} M_\odot$ and radii extending up to a few light years. Of particular interest is a configuration where the fermion core and the bosonic halo could form a supermassive compact object of mass $M \sim 10^{13} M_\odot$ for the bosonic particle mass $m_b \sim 10^{-32}$ GeV and fermionic particle mass $m_f \sim 10^{-5}$ GeV. The method that one usually follows in such calculations is that one makes a choice for the potential and then solves the Einstein-Klein-Gordon system numerically by fixing the values of the coupling constant and the value of the scalar field at the origin. It will be interesting if we assume the mass and radius of a boson star first and then look for the potential corresponding to the assumed configuration. In the present work, we explore this alternative approach,
where the mass of the scalar particle and the coupling constant get determined by the field equations. Essentially we will solve the Einstein-Klein-Gordon system with an energy momentum tensor corresponding to a self-interacting complex scalar field in the presence of a cosmological constant. Current observational studies done by High-Z Supernova Team\cite{2} and Supernova Cosmological Project group\cite{3} clearly indicate that, though small, we nevertheless need a positive cosmological constant to explain the current acceleration of the universe expansion. The current value of the cosmological constant is estimated to be $\Lambda_c \sim 3 \times 10^{-56}$ cm$^{-2}$\cite{24}. In our model, it helps us to construct a star with a proper boundary. In general, in the standard calculation of a boson star, one imposes the condition that the scalar field vanishes asymptotically which makes it difficult to define the radius properly (usually defined as where 90% of the mass is contained). However, a cosmological constant term makes it possible to define the radius in a conventional way (as in a fermion star), i.e., where the radial pressure vanishes. Similar observations may also be found in some other papers\cite{3,25}. In a recent paper\cite{26} the radius of the dark halo is described as the distance where pressure is equal to the cosmological constant $\Lambda_c$ of the order of magnitude $10^{-29}$ gm/cm$^3$ or alternatively, the cosmic microwave background radiation (CMBR) which is about $10^{-34}$ gm/cm$^3$ and the corresponding radii are 1 Mpc for $\Lambda_c$ and 100 Mpc for the CMBR. The exterior of the halo is described by the Schwarzschild de-Sitter spacetime. Note that boson stars have already been studied in de-Sitter as well as anti-de-Sitter universe\cite{27}.

Our paper is organized as follows. The boson star model is developed in Section 2 where, by making use of an ansatz for one of the metric potentials for an anisotropic matter distribution, we generate a solution for the Einstein-Klein-Gordon system. In Section 3, we utilize this solution to obtain different boson star configurations and determine the nature of the self interaction. We discuss our results in Section 4 and outline the physical implications of our studies along with some comments on the earlier work on the possibility of the formation of a boson star\cite{28} and on its stability\cite{29} under radial perturbations.

2 Boson star configuration

We write the energy-momentum tensor for a scalar field in the form

$$T_{ij} = \frac{1}{2} (\Phi^*_i \Phi_j + \Phi_i \Phi^*_j) - \frac{1}{2} g_{ij} \left[ g^{mn} \Phi^*_m \Phi_n + V(|\Phi|^2) \right],$$

where $\Phi(r, t)$ is a complex scalar field and $V(|\Phi|^2)$ is the potential of self-interaction of the scalar field. The scalar field is assumed to be of the form

$$\Phi(r, t) = \phi(r)e^{-i\omega t},$$

which guarantees a spherically symmetric static matter distribution. We write the metric for the spherically symmetric space-time in the standard coordinates

$$ds^2 = -e^{2\gamma(r)}dt^2 + e^{2\nu(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$
where \(\gamma(r)\) and \(\mu(r)\) are to be determined. The Einstein’s field equations (with \(c = 1\)) are then obtained as

\[
8\pi G \rho \equiv \frac{1 - e^{-2\mu}}{r^2} + \frac{2\mu'e^{-2\mu}}{r} + A_c = \frac{1}{2} \left[ \omega^2 e^{-2\gamma} \phi^2 + \phi'^2 e^{-2\mu} + V(\phi^2) \right],
\]

(4)

\[
8\pi G p_r \equiv \frac{2\gamma'e^{-2\mu}}{r} - \frac{(1 - e^{-2\mu})}{r^2} + A_c = \frac{1}{2} \left[ \omega^2 e^{-2\gamma} \phi^2 + \phi'^2 e^{-2\mu} - V(\phi^2) \right],
\]

(5)

\[
8\pi G p_\perp \equiv \left[ e^{-2\mu} \left( \gamma'' + \gamma' \mu' - \frac{\gamma'}{r} - \frac{\mu'}{r} - \frac{(1 - e^{-2\mu})}{r^2} + A_e \right) \right] = \frac{1}{2} \left[ \omega^2 e^{-2\gamma} \phi^2 - \phi'^2 e^{-2\mu} - V(\phi^2) \right],
\]

(6)

where \(A_c\) is a cosmological constant term. A prime (’) here denotes differentiation with respect to the radial coordinate \(r\). The Klein-Gordon equation

\[
- \frac{dV}{d|\Phi|^2} \Phi = 0,
\]

(7)

for the line element takes the form

\[
\phi'' + \left( \frac{2}{r} + \gamma' - \mu' \right) \phi' + e^{2(\mu-\gamma)} \omega^2 \phi = e^{2\mu} \frac{dV}{d|\phi|^2}. \tag{8}
\]

Eq. (4)-(6) together with Eq. (8) comprise the system of equations describing the boson star.

Note that by analogy with an anisotropic fluid, we may identify Eq. (4)-(6) as density \(\rho\), radial pressure \(p_r\) and tangential pressure \(p_\perp\) equations, respectively. It then follows that in the interior of a boson star, pressure is anisotropic due to the term \(\phi'^2 e^{-2\mu}\). Clearly \(p_r > p_\perp\) in this model. If \(\Delta = 8\pi G (p_r - p_\perp)\) is the measure of pressure anisotropy, i.e.,

\[
\Delta = \phi'^2 e^{-2\mu}, \tag{9}
\]

then Eq. (5) and Eq. (6) may be combined to yield

\[
\gamma'' + \gamma' \mu' - \frac{\gamma'}{r} - \frac{\mu'}{r} - \frac{(1 - e^{-2\mu})}{r^2} + A_e e^{2\mu} = 0, \tag{10}
\]

which is a non linear second order differential equation with two coupled functions \(\gamma(r)\) and \(\mu(r)\). To solve Eq. (10) we first make use of an ansatz for one of the metric potentials given by Vaidya and Tikekar in the form

\[
e^{2\mu} = \frac{1 + kr^2/R^2}{1 - r^2/R^2}, \tag{11}
\]

where \(R\) and \(k\) are dimensionless parameters. Eq. (10) then gets the form

\[
(1 + k - kx^2)\Psi_{xx} + kx\Psi_x + k(k + 1)\Psi + \frac{\Delta R^2(1 + k - kx^2)^2}{(1 - x^2)} \Psi = 0, \tag{12}
\]

where we introduced the following transformations

\[
\Psi = e^{\gamma}, \quad x^2 = 1 - \frac{r^2}{R^2}.
\]
To solve Eq. (12) the radial dependence of the anisotropic parameter $\Delta$ needs to be specified. We make a choice for the anisotropic parameter \[ \Delta = \frac{\alpha k^2 (1 - z^2)}{R^2 (1 + k - k x^2)^2}, \] (13)
where $\alpha > 0$ is a constant. Eq. (12) then gets the form \[(1 - z^2) \Psi_{zz} + z \Psi_z + [k (1 + \alpha) + 1] \Psi = 0, \] (14)
where $z = \sqrt{k/(k + 1)} x$. Using the properties of Gegenbauer function and Tschebyshev polynomial, a general solution of Eq. (14) may be obtained as \[ \Psi(z) = e^\gamma = A \left[ \cos[(\beta + 1) \zeta + \delta] - \cos[(\beta - 1) \zeta + \delta] \right], \] (15)
where, $\beta = \sqrt{k(1 + \alpha) + 2}$, $\zeta = \cos^{-1} z$, and $A$ and $\delta$ are constants. The energy density ($\rho$), radial pressure ($p_r$), tangential pressure ($p_\perp$) and the anisotropic parameter ($\Delta$) are then obtained as
\[
\rho = \frac{1}{8 \pi G R^2 (1 - z^2)} \left[ 1 + \frac{2}{(k + 1)(1 - z^2)} \right] - \frac{\Lambda_c}{8 \pi G}, \] (16)
\[
p_r = \frac{1}{8 \pi G R^2 (1 - z^2)} \left[ 1 + \frac{2 z \Psi_z}{(k + 1) \Psi} \right] + \frac{\Lambda_c}{8 \pi G}, \] (17)
\[
p_\perp = p_r - \Delta, \] (18)
\[
\Delta = \frac{\alpha k}{8 \pi G R^2} \left[ \frac{(k + 1)(1 - z^2) - 1}{(k + 1)^2 (1 - z^2)^2} \right]. \] (19)
Thus all the physical parameters can be obtained once the geometry of the star is specified.

Combining equations (9), (11) and (19) and integrating we also obtain a solution for the scalar wave function in the form
\[ \sigma = \sigma_0 + \frac{\sqrt{\alpha k}}{2} \sin^{-1} z, \] (20)
where $\sigma_0$ is an integration constant.

2.1 Determination of the Potential

To analyze the form of the potential, we now express Eq. (1) - (6) and Eq. (5) in dimensionless forms as
\[
\bar{\rho} \equiv \frac{1 - e^{-2 \mu}}{r^2_s} + \frac{2 \mu e^{-2 \mu}}{r_s} - \Lambda_s = \Omega^2 e^{-2 \gamma} \sigma^2 + \sigma'^2 e^{-2 \mu} + \frac{4 \pi G}{m^2} V(\sigma), \] (21)
\[
\bar{p_r} \equiv \frac{2 \gamma e^{-2 \mu}}{r_s} - \frac{(1 - e^{-2 \mu})}{r^2_s} + \Lambda_s = \Omega^2 e^{-2 \gamma} \sigma^2 + \sigma'^2 e^{-2 \mu} - \frac{4 \pi G}{m^2} V(\sigma), \] (22)
\[
\bar{p_\perp} \equiv e^{-2 \mu} \left( \gamma'' + \gamma'^2 - \gamma' \mu' + \gamma' \frac{\mu'}{r_s} \right) + \Lambda_s = \Omega^2 e^{-2 \gamma} \sigma^2 + \sigma'^2 e^{-2 \mu} - \frac{4 \pi G}{m^2} V(\sigma), \] (23)
\[
\sigma'' + (\gamma' - \mu' + \frac{2}{r_*})\sigma' + \Omega^2 e^{2(\mu-\gamma)}\sigma = \frac{1}{2} e^{-2\mu} \left( \frac{4\pi G}{m^2} \right) \frac{dV}{ds},
\]

where we have made the following rescaling: \(r_* = rm, \; R_* = Rm, \; \sigma = \sqrt{4\pi G\Phi}, \; \Omega = \omega/m, \; \Lambda_* = \Lambda m^2, \; \bar{\rho} = 8\pi G\bar{\rho}, \; \bar{p}_r = 8\pi G\bar{p}_r, \) and \(\bar{p}_{\perp} = 8\pi G\bar{p}_{\perp}.\) Here \(m\) is the mass of the scalar particle and a prime now denotes differentiation with respect to \(r_*\).

We rewrite the Eq. (24) in the form

\[
\frac{1-nz^2}{nz^2R_*^2} \sigma'' + \left[ -\frac{1}{nz^3} - \frac{2}{nzR_*^2} + \frac{(1-nz^2)}{n^2z^2R_*^2} \gamma_z - \frac{(1-nz^2)}{n^2z^2} \mu_z \right] \sigma + \Omega^2 e^{2(\mu-\gamma)} \sigma = \frac{1}{2} e^{-2\mu} \left( \frac{4\pi G}{m^2} \right) \frac{dV}{ds},
\]

where \(n = (k+1)/k.\) Defining the potential in dimensionless form as

\[
\bar{V} = 4\pi GR^2 V = \frac{4\pi R_*^2}{m^2 m_{\text{pl}}} V,
\]

we write Eq. (25) in the form

\[
\frac{1-nz^2}{nz^2} \sigma'' + \left[ -\frac{1}{nz^3} - \frac{2}{nz} + \frac{(1-nz^2)}{n^2z^2} \gamma_z - \frac{(1-nz^2)}{n^2z^2} \mu_z \right] \sigma + \hat{\Omega}^2 e^{2(\mu-\gamma)} \sigma = \frac{1}{2} e^{-2\mu} \frac{d\bar{V}}{d\sigma},
\]

where, \(\hat{\Omega} = \Omega R_*\). The physical quantities are then obtained as

\[
\bar{\rho} = \hat{\Omega}^2 e^{-2\gamma} \sigma^2 + \frac{(1-nz^2)}{n^2z^2} e^{-2\mu} \sigma_z^2 + \bar{V}, \tag{28}
\]

\[
\bar{p}_r = \hat{\Omega}^2 e^{-2\gamma} \sigma^2 + \frac{(1-nz^2)}{n^2z^2} e^{-2\mu} \sigma_z^2 - \bar{V}, \tag{29}
\]

\[
\bar{p}_{\perp} = \hat{\Omega}^2 e^{-2\gamma} \sigma^2 - \frac{(1-nz^2)}{n^2z^2} e^{-2\mu} \sigma_z^2 - \bar{V}, \tag{30}
\]

\[
\hat{\Delta} = 2 \frac{(1-nz^2)}{n^2z^2} e^{-2\mu} \sigma_z^2. \tag{31}
\]

Combining the two sets of equations, i.e., Eq. (119) - (110) and Eq. (28) - (31), we get

\[
\bar{\rho} \equiv \frac{1}{(1-z^2)} \left[ 1 + \frac{2}{(1+k)(1-z^2)} \right] - \hat{\Lambda}_c = \hat{\Omega}^2 e^{-2\gamma} \sigma^2 + \frac{(1-nz^2)}{n^2z^2} e^{-2\mu} \sigma_z^2 + \bar{V}, \tag{32}
\]

\[
\bar{p}_r \equiv -\frac{1}{(1-z^2)} \left[ 1 + \frac{2\gamma z}{(1+k)} \right] + \hat{\Lambda}_c = \hat{\Omega}^2 e^{-2\gamma} \sigma^2 + \frac{(1-nz^2)}{n^2z^2} e^{-2\mu} \sigma_z^2 - \bar{V}, \tag{33}
\]

\[
\bar{p}_{\perp} \equiv \bar{p}_r - \hat{\Delta} = \hat{\Omega}^2 e^{-2\gamma} \sigma^2 - \frac{(1-nz^2)}{n^2z^2} e^{-2\mu} \sigma_z^2 - \bar{V}, \tag{34}
\]

\[
\hat{\Delta} \equiv ak \left[ \frac{(k+1)(1-z^2) - 1}{(k+1)^2(1-z^2)} \right] = 2 \frac{(1-nz^2)}{n^2z^2} e^{-2\mu} \sigma_z^2 \tag{35}
\]

where \(\hat{\Lambda}_c = 8\pi GR^2 \Lambda_c.\) Eq. (32) - (35) thus help us to obtain our physical quantities from two different perspectives - one from the geometry and the other from the energy-momentum part of the field equations.
2.2 Boundary conditions

To calculate the physical quantities of the geometrical part of the field equations, we impose the following boundary conditions.

1. At the boundary of the star \((r = b)\) the interior solution is matched to the external Schwarzschild-de Sitter metric which gives

\[
e^{2\gamma(r=b)} = \left(1 - \frac{2M}{b} - \frac{\Lambda c b^2}{3}\right),
\]

\[
e^{2\mu(r=b)} = \left(1 - \frac{2M}{b} - \frac{\Lambda c b^2}{3}\right)^{-1}.
\]

We note that the effect of \(\Lambda c\) for highly compact objects like neutron stars is negligible in this model. However, for large objects whose compactness is very low, the contribution due to \(\Lambda c\) cannot be ignored.

2. The radial pressure \(\tilde{p}_r\) should vanish at the boundary which gives

\[
\frac{\Psi_z(b)}{\Psi(b)} = -\frac{1 + k}{2z_b} \left[1 - \tilde{\Lambda}_c \left(1 - z_b^2\right)^{1/2}\right],
\]

where \(z_b^2 = \frac{k}{k + 1} \left(1 - \frac{b^2}{R^2}\right)\).

For given values of mass and radius, Eq. (36) - (38) may be used to calculate the values of the constants \(\tilde{\Lambda}, R\) and \(\delta\) for assumed values of the curvature parameter \(k\) and the anisotropic parameter \(\alpha\). Thus all the physical quantities can be obtained once the geometry is specified.

3 Numerical results

Using the results of Section 2, we now choose different bosonic configurations and calculate the interaction parameters. The method is the following.

For a given mass \(M\) and radius \(b\), we calculate the constants from the boundary conditions, viz., \(R\) (from Eq. (37)), \(\delta\) (from Eq. (38)) and \(A\) (from Eq. (36)). We combine Eq. (32) and (33) to obtain

\[
\tilde{\rho} + \tilde{p}_r = \frac{1}{1 - z^2} \left[1 + \frac{2}{(1 + k)(1 - z^2)}\right] - \frac{1}{1 - z^2} \left[1 + \frac{2\gamma z}{(1 + k)}\right] = 2\tilde{\Omega}^2 e^{-2\gamma} \sigma^2 + \frac{2(1 - nz^2)}{n^2 z^2} e^{-2\mu} \sigma_z^2,
\]

where there are two unknown parameters, \(\tilde{\Omega}\) and \(\sigma_0\). We plot \((\tilde{\rho} + \tilde{p}_r)\) against the radial coordinate \(r\) from the centre to the boundary of the star using the geometry part (left hand side) of Eq. (39). By suitably choosing the values of \(\tilde{\Omega}\) and \(\sigma_0\), we fit this curve with the one obtained from the left hand side of Eq. (39). This has been shown in Figure 1.

We now choose the scalar wave potential in the form

\[
V = V_0 + m^2 |\Phi|^2 + \lambda |\Phi|^4,
\]
where the constant potential term $V_0$, the coupling constant $\lambda$ and the mass of the scalar particle $m$ are yet to be fixed. To this end, we rewrite the potential $V$ in dimensionless form as

$$\tilde{V} = 4\pi G R^2 V = \tilde{V}_0 + R^2 \sigma^2 + A R^2 \sigma^4,$$  \hspace{1cm} (40)

where $A = \lambda/4\pi G m^2$ and $\tilde{V}_0 = 4\pi G R^2 V_0$. Differentiating Eq. (40) we get

$$\frac{d\tilde{V}}{d\sigma} = 2 R^2 \sigma + 4 A R^2 \sigma^3.$$  \hspace{1cm} (41)

We now plot $d\tilde{V}/d\sigma$ against $\sigma$ using Eq. (27) and Eq. (41) as shown in Figure 2. This fixes the values of $R_+$ and $A$. The mass of the scalar particle is then obtained from the relation $m = R_+/R$.

Finally making a suitable choice for the constant $\tilde{V}_0$, the energy density $\tilde{\rho}$ and the radial pressure $\tilde{p}_r$ are plotted against the radius $r$ as shown in Figure 3 and Figure 4 respectively. The variation of radial pressure with density (EOS) both from the geometric part and the matter part are shown in Figure 5.

Thus we get a complete description of the boson star. Following the technique, we have considered a wide variety of bosonic configurations and the results have been compiled in Table 1.

4 Discussion

The geometric approach developed in this paper helps us to study the form of the potential of a wide variety of bosonic configurations. The key features of our studies are given below:

| $\alpha$ | $\delta$ | $A$ | $\tilde{\sigma}_{0}$ | $\Omega$ | $R_+$ | $\lambda$ | $V_0$ | $V_0$(MeV/fm$^3$) | $m$(eV) |
|---------|---------|-----|----------------------|--------|-------|---------|-------|-----------------|--------|
| 0.5     | 2.738   | 40.094 | 6.993 | 24.710 | 39.5 | 42.2 | 27.4 | 1.6 x 10$^{-14}$ |
| 1.0     | 2.644   | 48.748 | 9.965 | 24.948 | 18.168 | 6.648 | 60.8 | 39.5 | 1.2 x 10$^{-14}$ |
| 2.0     | 2.490   | 62.550 | 14.229 | 23.179 | 9.970 | 1.572 | 90.7 | 58.9 | 6.3 x 10$^{-14}$ |

Table 1 Results of different bosonic configurations.

Case II: $k = 100, M = 1M_\odot, b = 10 km$ and $R = 155.68 km$

| 0.2 | 2.647 | 31.133 | 4.462 | 33.320 | 19.194 | 14.67 | 27.8 | 69.3 | 3.6 x 10$^{-14}$ |
| 0.5 | 2.615 | 35.962 | 6.943 | 24.651 | 19.670 | 3.883 | 38.1 | 94.8 | 2.5 x 10$^{-14}$ |
| 1.0 | 2.511 | 43.231 | 9.899 | 19.733 | 14.090 | 1.055 | 54.8 | 136.5 | 1.8 x 10$^{-14}$ |
| 1.5 | 2.386 | 49.661 | 12.196 | 18.536 | 9.786 | 1.392 | 69.6 | 173.5 | 1.2 x 10$^{-14}$ |
| 2.0 | 2.257 | 55.446 | 14.149 | 18.234 | 3.638 | 11.305 | 81.6 | 203.4 | 4.6 x 10$^{-14}$ |

Case III: $k = 100, M = 1.5M_\odot, b = 10 km$ and $R = 113.25 km$

| 0.5 | 2.378 | 29.789 | 6.822 | 14.787 | 10.679 | 4.348 | 34.6 | 162.9 | 1.8 x 10$^{-14}$ |
| 1.0 | 2.252 | 35.337 | 9.738 | 12.079 | 5.874 | 4.083 | 49.1 | 231.7 | 1.0 x 10$^{-14}$ |
| 1.2 | 2.195 | 37.384 | 10.700 | 11.592 | 3.185 | 10.789 | 54.8 | 258.0 | 5.6 x 10$^{-14}$ |

Case IV: $k = 2, M = 1M_\odot, b = 10 km$ and $R = 28.58 km$

| 0.5 | 2.301 | 1.099 | 0.207 | 3.590 | 2.000 | 1.72 | 1.66 | 122.73 | 1.4 x 10$^{-14}$ |
| 1.0 | 2.274 | 1.234 | 0.316 | 2.928 | 1.740 | 2.49 | 1.94 | 143.43 | 1.2 x 10$^{-14}$ |
| 2.0 | 2.157 | 1.551 | 0.693 | 2.621 | 1.089 | 2.17 | 2.58 | 190.75 | 7.5 x 10$^{-14}$ |
| 2.2 | 2.128 | 1.561 | 0.753 | 2.664 | 0.907 | 2.90 | 2.70 | 199.62 | 6.3 x 10$^{-14}$ |
| 2.5 | 2.098 | 1.638 | 0.840 | 2.590 | 0.529 | 8.11 | 2.87 | 212.19 | 3.6 x 10$^{-14}$ |
Fig. 1 $(\tilde{\rho}+\tilde{p}_r)$ plotted against radius $r$ (km). The solid curve originates from the geometry and the dotted curve is derived from the matter part of the field equations. We took, $M = 1\ M_\odot$, $b = 10\ km$ and $\alpha = 1$.

Fig. 2 $\frac{d\tilde{V}}{d\sigma}$ plotted against $\sigma$. The solid curve originates from the geometry and the dotted curve is derived from the matter part of the field equations. We took, $M = 1\ M_\odot$, $b = 10\ km$ and $\alpha = 1$.

- Considering a boson star of mass $M = 1\ M_\odot$ and radius $b = 10\ km$, we have shown the variation of scalar field $\sigma$ against radius $r$ for three different values of the anisotropic parameter $\alpha$ as shown in Figure 6. The scalar field decreases smoothly towards the boundary from the centre and is nodeless, as desired. Possible radial excitations of the scalar field configurations have not been considered here.

- For given values of $M$, $b$ and $k$, the mass of the scalar particle $m$ decreases and the constant potential term $V_0$ increases with the increase of the anisotropic factor $\alpha$. The coupling constant $\Lambda$ first decreases, remains almost constant for a specific range of $\alpha$, and then increases. These variations are shown in Figure 7.

- Note that $\alpha$ cannot be made arbitrarily large. In our model, the maximum value of $\alpha$ decreases with increasing compactness for a given $k$. However, for the same compactness it increases if $k$ is made small.
In this model we have considered a quartic type potential only, however, higher order terms in the potential may also be considered. Variations of the potential $\tilde{V}$ against the scalar field $\sigma$ and with respect to $r$ inside the stellar interior have been shown in Figure 8 and Figure 9 respectively.

Our method can be used to study a wide variety of bosonic configurations. In Table 2 we have compiled some of our results which are comparable with the results obtained by earlier workers.

**Boson star of mass of the order of a planet:** The model can be used to describe a boson star of mass of the order of a planet. For example, if we consider a star of mass $M = 10^{-6} \, M_{\odot}$ and radius $b = 10^{-5} \, \text{km}$, the mass of the scalar particle turns out to be $m \sim 10^{-5} \, \text{eV}$. In CDM model, it is observed that massive...
axions of similar order of masses can collapse to form compact object of mass $M \sim 0.6 m_P^2 / m \sim 10^{-6} M_\odot$.[1]

- **Axion star**: Stable axion stars have masses less than 0.846 $M_\odot$ and radii greater than 20.5 km with mass of the scalar field $m \sim 10^{-10}$ eV, close to the lower bound of axions.[33]. For a star of mass 0.8 $M_\odot$ and radius 20.5 km, we get the mass of the scalar particle in the same order.

- **Boson star in galactic centre**: Some galactic centres are supposed to contain super-massive compact dark objects. It is believed that these ultra-compact objects are either supermassive black holes or very compact clusters of stellar size black holes.[34]. Observational data seem to allow the possibility that these could well be bosonic stars. In a boson star, the radius $b \geq M (m^{-2})$.[35]. Data collected for the central object of a galaxy by Genzel et al.[34] show that for a boson star model as in Ref.[35], the mass and radius of the central object should be $M \sim 10^6 M_\odot$ and $b \sim 10^7$ km, respectively. The mass of the corresponding scalar particle turns out to be $m \sim 10^{-17}$ eV. These results are consistent with our calculations.

- **Boson star of galactic scale**: In this model, we find that for a scalar particles of mass $m \sim 10^{-21}$ eV and constant potential $V_0 \sim 10^{-22}$ Mev/fm$^3$, a boson star of mass $M \sim 10^{12} M_\odot$ and radius $b \sim 10^{13}$ km may be obtained. These values are comparable to the scalar field dark matter model of Ref.[23], where $m_\phi \sim 1.1 \times 10^{-25}$ eV and $V_0 \sim 10^{-25}$ Mev/fm$^3$ for a potential of the form $V(\phi) = V_0 [\cosh(\lambda \sqrt{8 \pi G \phi}) - 1]$. The size of the galaxy of mass $\sim 10^{12} M_\odot$ was about 10$^{13}$ km. Thus the results are in good agreement with the present model.

We have shown that a wide variety of boson stars may be obtained in this model by either scaling or by considering different compactness. Obviously not all of these stars will be realistic. One also needs to consider the stability of such configurations under radial perturbations. This may provide constraints on the mass of the scalar field[9], and therefore, the mass of the boson star. For radial stability, the mass of boson should be $m \geq 10^{-28}$ eV[9][10]. In this model, with such scalar particle masses, one can describe a boson star of mass and radius as high as $M \sim 10^{18} M_\odot$ and $b \sim 10^{19}$ km, respectively.

![Fig. 5 Radial pressure $p_r$ plotted against density $\rho$. The solid curve originates from the geometry and the dotted curve is derived from the matter part of the field equations. We took, $M = 1 M_\odot$, $b = 10$ km and $\alpha = 1$.](image)
Table 2 Variations of scalar particle masses for different boson star configurations with $\alpha = 0.5$ and $k = 100$.

| $M$ ($M_\odot$) | $b$ (km) | $V_0$ (Mev/fm$^3$) | $\Lambda$ | $m$ (eV) |
|-----------------|---------|---------------------|---------|---------|
| $10^{-6}$       | $10^{-5}$ | $9.619 \times 10^{13}$ | 2.08    | $2.46 \times 10^{-5}$ |
| 0.8             | 20.5    | 8.4                 | 1.34    | $10^{-11}$ |
| $10^6$          | $10^7$  | $9.619 \times 10^{-11}$ | 2.08    | $2.46 \times 10^{-17}$ |
| $10^{12}$       | $10^{13}$ | $9.619 \times 10^{-23}$ | 2.08    | $2.46 \times 10^{-23}$ |
| $10^{18}$       | $10^{19}$ | $9.619 \times 10^{-35}$ | 2.08    | $2.46 \times 10^{-29}$ |

Fig. 6 Scalar field $\sigma$ plotted against radius $r$ (km). Dotted, solid and dashed curves are for $\alpha$ equal to 0.5, 1 and 1.5 respectively.

Fig. 7 Variations of $\Lambda$ (solid line), $V_0/10$ in Mev/fm$^3$ (dotted line) and $m/10^{-11}$ in eV (long dashed line) with $\alpha$. 
Fig. 8 Scalar field potential $\tilde{V}$ plotted against scalar field $\sigma$. Dotted, solid and dashed curves are for $\alpha$ equal to 0.5, 1 and 1.5 respectively.

Fig. 9 Scalar field potential $\tilde{V}$ plotted against radius $r$. Dotted, solid and dashed curves are for $\alpha$ equal to 0.5, 1 and 1.5 respectively.

To conclude, although the present model is based on a particular geometry, depending on two parameters (Vaidya-Tikekar model), and a simple anisotropic distribution, the resulting configurations, at least a class of them, may be physically relevant. This is confirmed by comparing the results of our geometrical approach with earlier results, obtained by direct numerical integration. Whether boson stars can actually account for the dark matter, or at least a part of it, will require extensive analysis of the observational results of the CMB radiation, the Supernova data and other relevant astronomical results. The stability of the boson star configurations under radial perturbations also remains an interesting issue. For scalar fields without self-interaction, extensive numerical calculations done by Seidel and Suen [28] have shown interesting stability properties of the configurations for finite perturbations. Whether boson stars with self-interacting fields have similar behaviour remains to be checked. It may be also be pointed out that Seidel and Suen [29] have also suggested a mechanism, the gravitational cooling, which permits the scalar field to get rid of the excess kinetic energy so
that a bound configuration can be formed (instead of a virialised cloud). These results make the possibility of the formation of boson stars look more likely. We hope to take up these issues elsewhere.

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