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A delayed vaccinated epidemic model with nonlinear incidence rate and Lévy jumps

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A B S T R A C T
A stochastic susceptible–infectious–recovered epidemic model with nonlinear incidence rate is formulated to discuss the effects of temporary immunity, vaccination, and Lévy jumps on the transmission of diseases. We first determine the existence of a unique global positive solution and a positively invariant set for the stochastic system. Sufficient conditions for extinction and persistence in the mean of the disease are then achieved by constructing suitable Lyapunov functions. Based on the analysis, we conclude that noise intensity and the validity period of vaccination greatly influence the transmission dynamics of the system.

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1. Introduction

Epidemics exert considerable influence on human life. Controlling and eradicating infectious diseases are ongoing problems that have received increasing attention from numerous authors [1–6]. Various factors, such as vaccination, time delay, impulse and so on, are used to construct mathematical models and seek effective ways to eliminate infectious diseases. Time delay, which has great biologic meaning in epidemic systems, is often used [7–10]. Many scholars have also paid close attention to the effects of the temporary disease immunity of epidemic models, i.e., a fleeting immunity to a disease after recovery before becoming susceptible again. The phenomenon is common during the transmission of many epidemic diseases, such as influenza, Chlamydia trachomatis, Salmonella and so on. Thus, in current paper, we introduce temporal delays to make epidemic models more realistic and interesting.

Epidemic models are inevitably subject to environmental noise. Most epidemic models are driven by white noise, and many results have been achieved in this area [11–19]. However, under severe environmental perturbations, such as avian influenza, severe acute respiratory syndrome, volcanic eruptions, earthquakes, and hurricanes, the continuity of solutions may be broken; accordingly, a jump process should be introduced to prevent and control diseases [20–23].

In the present work, we consider the effects of vaccination and temporary immunity on a stochastic susceptible–infectious–recovered (SIR) epidemic model driven by Lévy noise. A generalized nonlinear incidence rate $f(S(t))g(I(t))$ is
also introduced. Based on the above factors, we formulate the delayed vaccinated SIR epidemic model as follows:

$$
\begin{align*}
\text{d}S(t) &= \left( \Lambda - \mu S(t) - pS(t) - \beta f(S(t))g(l(t)) + pS(t - \tau_1)e^{-\mu t_1} + \gamma I(t - \tau_2)e^{-\mu t_2} \right) \text{d}t \\
&\quad - f(S(t^-))g(l(t^-))\sigma_1 \text{d}B_1(t) + \int_Y \gamma(u)N(\text{d}t, \text{d}u), \\
\text{d}I(t) &= \left( \beta f(S(t))g(l(t)) - (\mu + \gamma)I(t) \right) \text{d}t + f(S(t^-))g(l(t^-))\sigma_1 \text{d}B_1(t) + \int_Y \gamma(u)N(\text{d}t, \text{d}u), \\
\text{d}R(t) &= (\gamma I(t) + pS(t) - \mu R(t) - pS(t - \tau_1)e^{-\mu t_1} - \gamma I(t - \tau_2)e^{-\mu t_2}) \text{d}t + \sigma_2 R(t) \text{d}B_2(t)
\end{align*}
$$

(1.1)

where $S(t), I(t),$ and $R(t)$ are the numbers of susceptible, infectious, and recovered populations, respectively. $\Lambda$ represents the constant recruitment of susceptible individuals, $\mu$ is the natural death rate of populations, $\beta$ is the transmission rate, $\gamma$ represents the recovery rate, $p$ denotes the proportional coefficient of the vaccinated for the susceptible, $\tau_1$ represents the validity period of the vaccination and $\tau_2$ is the length of the immunity period. All parameters are assumed to be positive constants.

Herein, $\sigma_i^2(t) (i = 1, 2)$ is the intensity of white noise and $\sigma_i > 0 \ (i = 1, 2); \text{and } B_i(t) (i = 1, 2)$ is the standard Brownian motions, which are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ satisfying the usual conditions [21]. $N$ is a Poisson counting measure with compensator $\hat{N}$ and characteristic measure $\lambda$ on a measurable subset $Y$ of $[0, \infty)$ which satisfies $\lambda(Y) < \infty$; $\lambda$ is assumed to be a lévy measure, such that $N(\text{d}t, \text{d}u) = N(\text{d}t, \text{d}u) - \lambda(\text{d}u)\text{d}t$; $\gamma : Y \times \Omega \rightarrow \mathbb{R}$ is bounded and continuous with respect to $\lambda$ and is $\mathcal{F}_Y \times \mathcal{F}_t$-measurable, where $\mathcal{B}(Y)$ is a $\sigma$-algebra with respect to the set $Y$, $\mathcal{F}_t$ is a sub $\sigma$-algebra of $\mathcal{F}_t$, and $\mathcal{F}$ is a $\sigma$-algebra of subsets of a given set $\Omega$ [22]. In this paper, $B_i$ and $N$ are assumed to be independent of each other.

As the first two equations do not depend on the last equation in system (1.1), therefore, we only consider the equations as follows:

$$
\begin{align*}
\text{d}S(t) &= \left( \Lambda - \mu S(t) - pS(t) - \beta f(S(t))g(l(t)) + pS(t - \tau_1)e^{-\mu t_1} + \gamma I(t - \tau_2)e^{-\mu t_2} \right) \text{d}t \\
&\quad - f(S(t^-))g(l(t^-))\sigma_1 \text{d}B_1(t) + \int_Y \gamma(u)N(\text{d}t, \text{d}u), \\
\text{d}I(t) &= \left( \beta f(S(t))g(l(t)) - (\mu + \gamma)I(t) \right) \text{d}t + f(S(t^-))g(l(t^-))\sigma_1 \text{d}B_1(t) + \int_Y \gamma(u)N(\text{d}t, \text{d}u).
\end{align*}
$$

(1.2)

The initial conditions of model (1.2) are

$$
S(\xi) = \phi_1(\xi) \geq 0, \quad I(\xi) = \phi_2(\xi) \geq 0, \quad \xi \in [-\tau, 0], \quad \phi_i(0) > 0, \quad i = 1, 2,
$$

(1.3)

where $(\phi_1(\xi), \phi_2(\xi)) \in L^1(\mathbb{R}^2_+; \mathbb{R}^2_+)$ is the space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^2_+$, herein, $\mathbb{R}^2_+ = \{(x_1, x_2) : x_i > 0, i = 1, 2, \tau = \max(\tau_1, \tau_2)\}$. According to the phenomena observed in nature, such as bee colonies, we assume that the self-regulating populations within the same species are strictly positive, that is,

**Assumption (H1)** $\gamma(u)$ is a bounded function and $|\frac{\partial}{\partial u} \gamma(u)| \leq \theta < 1, u \in Y$.

We also establish the following assumptions on functions $f(S)$ and $g(l)$:

**Assumption (H2)** $f(S)$ is a continuously differentiable function and monotonically increasing on $\mathbb{R}_+, f(0) = 0$. A constant $l > 0$ exists, such that $m_l \triangleq \inf_{0 < S < \xi} \frac{f(S)}{S} < \infty$, and $M_l \triangleq \sup_{0 < S < \xi} \frac{f(S)}{S} < \infty$.

**Assumption (H3)** $g(l)$ is twice continuously differentiable, and $\frac{g(l)}{l^2}$ is monotone decreasing on $\mathbb{R}_+, g(0) = 0$, and $g'(0) > 0$.

The aim of this paper is to prove the existence and uniqueness of a global positive solution. The extinction and persistence in the mean of the system are also discussed.

2. Preliminaries

In this section, we list some notations, definitions and lemmas. First, we denote

$$
\begin{align*}
f^{\mu} &= \sup_{t \geq 0} f(t), \quad f^l = \inf_{t \geq 0} f(t), \quad (f(t)) = \frac{1}{t} \int_0^t f(s) \text{d}s, \quad f^* = \lim_{t \to +\infty} \sup f(t), \quad f_* = \lim_{t \to +\infty} \inf f(t),
\end{align*}
$$

where $f(t)$ is a continuous and bounded function defined on $[0, +\infty)$.

Then, we give the Itô’s formula for general stochastic differential equations. Define the $n$-dimensional stochastic equation [24]:

$$
\text{d}X = f(t, X) \text{d}t + g(t, X) \text{d}B(t)
$$

(2.1)

with initial value $X(t_0) = x_0$. Here, $f(t, X) = (f_1(t, X), f_2(t, X), \ldots, f_n(t, X))$ is a $n$-dimensional vector function, $(g(t, X))_{n \times l}$ is a $n \times l$ matrix function and $B(t) = (B_1(t), B_2(t), \ldots, B_l(t))$ is a $l$-dimensional standard Brownian motion defined on
the probability space \((\Omega, F, P)\). Define the differential operator \(L\) associated with Eq. (2.1)

\[
L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(t, X) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=1}^{l} g_{ik}(t, X) g_{jk}(t, X) \frac{\partial^2}{\partial x_i \partial x_j}.
\]

If function \(V(t, X) \in C^2(\mathbb{R}^n \times \mathbb{R})\), then we have

\[
LV(t, X) = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} f_i(t, X) \frac{\partial V}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=1}^{l} g_{ik}(t, X) g_{jk}(t, X) \frac{\partial^2 V}{\partial x_i \partial x_j}.
\]

Thus, the Itô’s formula is listed as follows

**Lemma 2.1** ([24]). Let \(X(t)\) satisfy Eq. (2.1) and function \(V(t, X) \in C^2(\mathbb{R}^n \times \mathbb{R})\). Then

\[
dV(t, X) = LV(t, X) dt + V_t(t, X) g(t, X) dB(t),
\]

where, \(V_t(t, X) = \left( \frac{\partial V(t, X)}{\partial x_1}, \frac{\partial V(t, X)}{\partial x_2}, \ldots, \frac{\partial V(t, X)}{\partial x_n} \right)\).

**Lemma 2.2** ([25]). Suppose that \(x(t) \in C[\Omega \times [0, T], \mathbb{R}^n]\) here \(\mathbb{R}^n_+ := \{a | a > 0, a \in \mathbb{R}\}\). \(1 + \gamma(u) > 0, u \in Y\) and there exists a positive constant \(C\) such that \(\int_{Y} (1 + \gamma(u))^2 \lambda(du) < \infty\), then

(1) If positive constants \(\lambda_0, T\) and \(\lambda \geq 0\) exist such that

\[
\ln x(t) \leq \lambda t - \lambda_0 \int_0^t x(s) ds + \sum_{i=1}^{n} \beta_i B_i(t) + \sum_{i=1}^{n} \int_0^t \int_Y (1 + \gamma(u)) \tilde{N}(dt, du),
\]

for all \(t \geq T\), where \(\beta_i\) is a constant, \(1 \leq i \leq n\), then

\[
\begin{cases}
\langle x \rangle^* \leq \lambda/\lambda_0 \text{ a.s.,} & \text{if } \lambda_0 \geq 0; \\
\lim_{t \to \infty} x(t) = 0 \text{ a.s.,} & \text{if } \lambda_0 < 0.
\end{cases}
\]

(2) If positive constants \(\lambda_0, T\) and \(\lambda > 0\) exist such that

\[
\ln x(t) \geq \lambda t - \lambda_0 \int_0^t x(s) ds + \sum_{i=1}^{n} \beta_i B_i(t) + \sum_{i=1}^{n} \int_0^t \int_Y (1 + \gamma(u)) \tilde{N}(dt, du),
\]

for all \(t \geq T\), then, \(\langle x \rangle^* \geq \lambda/\lambda_0 \text{ a.s.}\)

3. Existence and uniqueness of the global solution

Denote

\[
I' = \{S, I) \in \mathbb{R}_+^2 : S + I \leq \frac{A}{\mu} \equiv N_0\}
\]

and then we discuss the existence and uniqueness of the global positive solution and the positive invariant of system (1.2) with positive initial value (1.3).

**Theorem 3.1.** If Assumption (H1) hold, then for any initial value \((S(0), I(0)) \in L^1([-\tau, 0]; \mathbb{R}_+^2),\) a unique solution \((S(t), I(t)) \in \mathbb{R}_+^2\) of system (1.2) exists on \(t \geq -\tau\) and the solution will remain in \(\mathbb{R}_+^2\) with probability one.

**Proof.** According to the local Lipschitz condition of system (1.2), we obtain that for any initial value \(X_0 = (S(0), I(0)) \in \mathbb{R}_+^2\), a unique local solution \((S(t), I(t))\) exists on \([-\tau, \tau_e]\), herein, \(\tau_e\) represents the explosion time. To prove that the solution is global, it is required to obtain that \(\tau_e = \infty\) a.s. Then, we suppose that \(k_0 \geq 1\) is sufficiently large such that \(S(0)\) and \(I(0)\) lie within the interval \([1/k_0, k_0]\). For each integer \(k > k_0\), we define the stopping time \(\tau_k = \inf\{t \in [-\tau, \tau_e] : S(t) \not\in (1/k, k)\}\) or \(I(t) \not\in (1/k, k)\). Then, \(\tau_k\) increases as \(k \to \infty\). Denote \(\tau_{\infty} = \lim_{k \to +\infty} \tau_k\), thus \(\tau_{\infty} \leq \tau_e\). In the following, we need to show that \(\tau_{\infty} = \infty\). If not, there are constants \(T > 0\) and \(\varepsilon \in (0, 1)\) satisfying \(P(\tau_{\infty} < \infty) > \varepsilon\). Thus, an integer \(k_1 \geq k_0\) exists such that \(P(\tau_k \leq T) \geq \varepsilon\), for all \(k > k_1\). Construct a \(C^2\)-function \(V : \mathbb{R}_+^2 \to \mathbb{R}_+\) by

\[
V(S, I) = (S - a - aln S/a) + (I - 1 - ln I) + e^{-\mu_1 t} \int_{t-\tau_1}^{t} S(s) ds + e^{-\mu_2 t} \int_{t-\tau_2}^{t} I(s) ds,
\]
where \( a \) is a constant that will be given later. By virtue of Itô’s formula, we derive

\[
dV(S, t) = \left(1 - \frac{a}{S}\right) \left[(\lambda - \mu S - pS - \beta f(S)g(I) + pS(t - \tau_1)e^{-\mu t_1} + \gamma I(t - \tau_2)e^{-\mu t_2})dt + \sigma f(S)g(I)dB_1(t)\right]
- \frac{a^2 f^2(S)g^2(I)}{2S^2}dt
- a \int_y \left[\ln(1 - \frac{f(S)}{S}g(I)\gamma(u)) + \frac{f(S)}{S}g(I)\gamma(u)\right] \lambda(du)dt
- \int_y \left[a \ln(1 - \frac{f(S)}{S}g(I)\gamma(u)) + f(S)g(I)\gamma(u)\right] \tilde{N}(dt, du)
+ (1 - \frac{1}{I}) \left[(\beta f(S)g(I) - (\mu + \gamma)I)dt + \sigma f(S)g(I)dB_1(t)\right] + \frac{\sigma^2 f^2(S)g^2(I)}{2I^2}dt
- \int_y \left[\ln(1 + f(S)\frac{g(I)}{I}\gamma(u)) - f(S)\frac{g(I)}{I}\gamma(u)\right] \lambda(du)dt
+ \int_y \left[f(S)g(I)\gamma(u) - \ln(1 + f(S)\frac{g(I)}{I}\gamma(u))\right] \tilde{N}(dt, du)
+ ps\gamma e^{-\mu t_1}dt - pS(t - \tau_1)e^{-\mu t_1}dt + \gamma le^{-\mu t_2}dt - \gamma l(t - \tau_2)e^{-\mu t_2}dt
= \left(1 - \frac{a}{S}\right) \left[(\Lambda - \mu S - pS - \beta f(S)g(I) + pS(t - \tau_1)e^{-\mu t_1} + \gamma I(t - \tau_2)e^{-\mu t_2})\right]dt
+ \left[\frac{a^2 f^2(S)g^2(I)}{2S^2} - a \int_y \left[\ln(1 - \frac{f(S)}{S}g(I)\gamma(u)) + \frac{f(S)}{S}g(I)\gamma(u)\right] \lambda(du)\right]dt
+ (1 - \frac{1}{I}) \left[(\beta f(S)g(I) - (\mu + \gamma)I) + \frac{\sigma^2 f^2(S)g^2(I)}{2I^2}\right]dt
- \int_y \left[\ln(1 + f(S)\frac{g(I)}{I}\gamma(u)) - f(S)\frac{g(I)}{I}\gamma(u)\right] \lambda(du)dt
+ (ps\gamma e^{-\mu t_1} - pS(t - \tau_1)e^{-\mu t_1} + \gamma le^{-\mu t_2} - \gamma l(t - \tau_2)e^{-\mu t_2})dt
- \sigma_1(1 - \frac{a}{S})f(S)g(I)dB_1(t) + \sigma_1(1 - \frac{1}{I})f(S)g(I)dB_1(t)
- \int_y \left[a \ln(1 - \frac{f(S)}{S}g(I)\gamma(u)) + f(S)g(I)\gamma(u)\right] \tilde{N}(dt, du)
+ \int_y \left[-\ln(1 + f(S)\frac{g(I)}{I}\gamma(u)) + f(S)g(I)\gamma(u)\right] \tilde{N}(dt, du)
= LV(S, t)dt - \sigma_1(S - a)\frac{f(S)}{S}g(I)dB_1(t) + \sigma_1(I - 1)\frac{f(S)}{I}g(I)dB_1(t)
- \int_y \left[a \ln(1 - \frac{f(S)}{S}g(I)\gamma(u)) + f(S)g(I)\gamma(u)\right] \tilde{N}(dt, du)
+ \int_y \left[-\ln(1 + f(S)\frac{g(I)}{I}\gamma(u)) + f(S)g(I)\gamma(u)\right] \tilde{N}(dt, du).

Here, \( LV : \mathbb{R}^+_0 \to \mathbb{R}_+ \) is defined as follows

\[
LV(S, t) = \left(1 - \frac{a}{S}\right) \left[(\Lambda - \mu S - pS - \beta f(S)g(I) + pS(t - \tau_1)e^{-\mu t_1} + \gamma I(t - \tau_2)e^{-\mu t_2})\right]
- a \int_y \left[\ln(1 - \frac{f(S)}{S}g(I)\gamma(u)) + \frac{f(S)}{S}g(I)\gamma(u)\right] \lambda(du)\right]dt
+ \left[\frac{1}{I} \left(\beta f(S)g(I) - (\mu + \gamma)I\right) + \frac{\sigma^2 f^2(S)g^2(I)}{2I^2}\right]dt
- \int_y \left[\ln(1 + f(S)\frac{g(I)}{I}\gamma(u)) - f(S)\frac{g(I)}{I}\gamma(u)\right] \lambda(du)dt
+ (ps\gamma e^{-\mu t_1} - pS(t - \tau_1)e^{-\mu t_1} + \gamma le^{-\mu t_2} - \gamma l(t - \tau_2)e^{-\mu t_2})dt
\leq (\Lambda + a\Lambda + pa + \mu + \gamma) - (\mu + p(1 - e^{-\mu t_1}))S - \frac{a\Lambda}{S} + [a\beta f(S)g(I) - \mu - \gamma(1 - e^{-\mu t_2})]t
+ \frac{a^2 f^2(S)g^2(I)}{2S^2} + \frac{\sigma^2 f^2(S)g^2(I)}{2I^2}.
Then applying Taylor formula to function $\ln(1 - t)$ here $t = \frac{(S)}{S} g(I)\gamma(u)$ and Assumption (H1) to $\varphi_1$, we have that

$$\varphi_1 = -\ln\left(1 - \frac{(S)}{S} g(I)\gamma(u)\right) - \frac{(S)}{S} g(I)\gamma(u)$$

$$= \frac{(S)}{S} g(I)\gamma(u) + \frac{(\frac{(S)}{S} g(I)\gamma(u))^2}{2(1 - \delta \frac{(S)}{S} g(I)\gamma(u)^2)} - \frac{(S)}{S} g(I)\gamma(u)$$

$$\leq \frac{(M_{g0} g'(0))\gamma(u)^2}{2(1 - \delta M_{g0} g'(0)\gamma(u)^2)} \leq \frac{(M_{g0} g'(0))^2 \theta^2}{2(1 - M_{g0} g'(0)\theta)^2}$$

where $\delta \in (0, 1)$ is an arbitrary number. Similarly, we can obtain that

$$\varphi_2 = -\ln\left(1 + f(S)g(I)\gamma(u)\right) + f(S)\frac{g(I)}{I} \gamma(u) \leq \frac{(M_{g0} g'(0))^2 \theta^2}{2(1 - M_{g0} g'(0)\theta)^2}$$

Then

$$LV(S, I) \leq (A + \mu a + pa + \mu + \gamma) + \frac{a n^2 K^2}{2} M_{g0}^2 (g'(0))^2 + \sigma^2 M_{g0}^2 (g'(0))^2 K^2 + \frac{(a + 1)M_{g0}^2 (g'(0))^2 \theta^2}{2(1 - M_{g0} g'(0)\theta)^2} \triangleq \tilde{K}$$
Therefore, we achieve that
\[
\begin{align*}
dV(S, l) & \leq \tilde{K}dt - \sigma_1(S - a)\frac{f(S)}{S}g(I)dB_1(t) + \sigma_1(l - 1)f(S)\frac{g(l)}{l}dB_1(t) \\
& - \int_0^l [a\ln(1 - \frac{f(S)}{S}g(I))\gamma(u) + f(S)g(l)\gamma(u)\tilde{N}(dt, du)] \\
& + \int_0^l [-ln(1 + f(S)\frac{g(l)}{l}\gamma(u) + f(S)g(l)\gamma(u))\tilde{N}(dt, du)].
\end{align*}
\]
(3.4)

Taking integral on the above inequality from 0 to \( \tau \), then
\[
\begin{align*}
\int_0^{\tau} dV(S, l) & \leq \int_0^{\tau} \tilde{K}dt - \int_0^{\tau} \sigma_1(S - a)\frac{f(S)}{S}g(I)dB_1(t) + \sigma_1(l - 1)f(S)\frac{g(l)}{l}dB_1(t) \\
& - \int_0^{\tau} \int_0^l [a\ln(1 - \frac{f(S)}{S}g(I))\gamma(u) + f(S)g(l)\gamma(u)]\tilde{N}(ds, du) \\
& + \int_0^{\tau} \int_0^l [-ln(1 + f(S)\frac{g(l)}{l}\gamma(u) + f(S)g(l)\gamma(u))\tilde{N}(ds, du),
\end{align*}
\]
where \( \tau = \min\{\tau_k, T\} \). Consequently,
\[
EV(S(\tau_k \wedge T), I(\tau_k \wedge T)) \leq V(S(0), I(0)) + \bar{K}E(\tau_k \wedge T) \leq V(S(0), I(0)) + \bar{K}T.
\]
Let \( \Omega_k = \{\tau_k \leq T\} \), then \( P(\Omega_k) \geq \varepsilon \). For each \( \omega \in \Omega_k \), \( S(\tau_k, \omega) \), or \( I(\tau_k, \omega) \) equals either \( k \) or \( 1/k \), and
\[
V(S(\tau_k, \omega), I(\tau_k, \omega)) \geq \min\{k - 1 - \ln k, 1/k - 1 + \ln k\}.
\]
Thus,
\[
\begin{align*}
V(S(0), I(0)) + KT & \geq E[1_{\Omega_k}(\omega)V(S(\omega), I(\omega))] \\
& \geq \varepsilon \min\{k - 1 - \ln k, 1/k - 1 + \ln k\},
\end{align*}
\]
(3.5)

where \( 1_{\Omega_k} \) is the indicator function of \( \Omega_k \). Letting \( k \to \infty \), we obtain the contradiction.

The proof is completed.

Next, we prove that \( \Gamma \) is a positively invariant set of system (1.2).

**Theorem 3.2.** The region \( \Gamma \) is almost surely positive invariant of system (1.2).

**Proof.** Suppose \((S(\theta), I(\theta)) \in \Gamma, \theta \in [-\tau, 0]\) and \( n_0 \geq 0 \) be sufficiently large such that \( S(\theta) \in (\frac{1}{n_0}, \frac{A}{n}) \) and \( I(\theta) \in (\frac{1}{n_0}, \frac{A}{n}) \). For each integer \( n \geq n_0 \), the stopping times are defined as follows
\[
\tau_n = \inf\{t > 0 | (S(t), I(t)) \in X(t) \in \Gamma, (S(t), I(t)) \notin \left[\frac{1}{n}, \frac{A}{n}\right] \},
\]
\[
\tau = \inf\{t > 0 | (S(t), I(t)) \notin \Gamma\}.
\]
We need to show that \( P(\tau < t) = 0 \) for all \( t > 0 \).

Notice that \( P(\tau < t) \leq P(\tau_n < t) \), then we have to prove lim sup\( n \to +\infty \) \( P(\tau_n < t) = 0 \). Define the function
\[
W(S, l) = \frac{1}{S} + \frac{1}{l},
\]
then
\[
\begin{align*}
dW(S, l) = LW(S, l)dt + \frac{\sigma_1(S)g(S)}{S^2}dB_1(t) - \frac{\sigma_1(S)g(S)}{S^2}dB_1(t) \\
+ \int_0^l \left[\frac{f(S)g(l)\gamma(u)}{S(l + f(S)g(l)\gamma(u))} - \frac{f(S)g(l)\gamma(u)}{l(l + f(S)g(l)\gamma(u))}\right]d\tilde{N}(dt, du),
\end{align*}
\]
here,
\[
LW(S, l) = -\frac{A}{S^2} + \mu + \frac{p}{S} + \frac{f(S)g(S)}{S^2} - \frac{p\gamma(t - \tau_1)e^{-\tau_2/\gamma}}{S^2} - \gamma I(t - \tau_2)e^{-\tau_2/\gamma} \\
+ \frac{\sigma_1^2 f^2(S)g^2(l)}{S^3} + \int_0^l \frac{f^2(S)g^2(l)\gamma(u)}{S(1 - \frac{f(S)}{S}g(l)\gamma(u))}du \\
- \frac{\beta f(S)g(l)}{l^2} + \mu + \gamma + \frac{\sigma_1^2 f^2(S)g^2(l)}{l^3} + \int_0^l \frac{f^2(S)g^2(l)\gamma(u)}{l(1 + f(S)g(l)\gamma(u))}du.
\]
Taking integral and expectation on both sides of (3.7) and by virtue of Fubini's Theorem, then we derive

\[
\begin{align*}
\text{Suppose } &
\text{Theorem 4.1.}
\end{align*}
\]

Thus, (4.1) the extinction of diseases

\[
\text{Proof is completed.}
\]

By (3.8) and (3.9), we obtain that

\[
\begin{align*}
\frac{\sigma_f(S)g(I)}{S^2} + \frac{f^2(S)g^2(I)}{I^2} + \frac{\int f^2(S)g^2(I)\gamma^2(u)}{\int I(1 + f(S)g(I)\gamma(u))} &
\end{align*}
\]

Thus, (3.7)

\[
\begin{align*}
dW & \leq \eta W(X)dt + \frac{\sigma_f(S)g(I)}{S^2}dB_1(t) - \frac{\sigma_f(S)g(I)}{I^2}dB_1(t) \\
& + \int \left[ \frac{f(S)g(I)\gamma(u)}{S(S - f(S)g(I)\gamma(u))} - \frac{f(S)g(I)\gamma(u)}{I(1 + f(S)g(I)\gamma(u))} \right] \hat{N}(dt, du)
\end{align*}
\]

where

\[
\begin{align*}
\eta &= \max \left\{ u + p + \frac{\beta f(S)g(I)}{S} + \frac{\sigma_f^2(S)g^2(I)}{S^2} + \int \frac{f^2(S)g^2(I)\gamma^2(u)}{I(1 + f(S)g(I)\gamma(u))} \lambda(du); \right. \\
& \left. \mu + \gamma + \frac{\sigma_f^2(S)g^2(I)}{I^2} + \int \frac{f^2(S)g^2(I)\gamma^2(u)}{I(1 + f(S)g(I)\gamma(u))} \lambda(du) \right\}
\end{align*}
\]

Taking integral and expectation on both sides of (3.7) and by virtue of Fubini's Theorem, then we derive

\[
\int_0^t \int 0 E(W(X(x)))d\xi.
\]

Applying Gronwall's Lemma, we obtain that

\[
\begin{align*}
E(W(X(s))) & \leq W(x_0)e^{\eta s}.
\end{align*}
\]

for all \( s \geq 0, t \geq \tau_n \). Thus, (3.8)

\[
\begin{align*}
E(W(X(t \wedge \tau_n))) & \leq W(x_0)e^{\eta(t \wedge \tau_n)} \leq W(x_0)e^{\eta t}, \quad t \geq 0.
\end{align*}
\]

In consideration of \( W(X(t \wedge \tau_n)) > 0 \) and some component of \( X(t \wedge \tau_n) \) being less than or equal to \( \frac{1}{n} \), we achieve that (3.9)

\[
\begin{align*}
E(W(X(t \wedge \tau_n))) & \geq E(W(X(t \wedge \tau_n))))1_{\{\tau_n < t\}} \\
& \geq nP(\tau_n < t).
\end{align*}
\]

By (3.8) and (3.9), we obtain that

\[
\begin{align*}
P(\tau_n < t) & \leq \frac{W(x_0)e^{\eta t}}{n},
\end{align*}
\]

for all \( t \geq 0 \). Therefore,

\[
\limsup_{n \to +\infty} P(\tau_n < t) = 0.
\]

The proof is completed.

4. The extinction of diseases

In this section, to discuss the extinction of the disease, we define

\[
R_0 = \frac{\beta M_{n_0}g(0)\Lambda}{(\mu + \gamma)(\mu + p(1 - e^{-\mu \tau_n}))},
\]

and for the sake of simplicity, we denote \( (\bar{s}(t)) = \frac{1}{t} \int_0^t \bar{s}(s)ds, \) then the following theorem is obtained.

**Theorem 4.1.** Suppose \((S(t), I(t))\) be any solution of system (1.2) with an initial value (1.3). Thus (1) If \( \sigma^2_1 > \frac{\beta^2 M_{n_0}^2}{2m_{n_0}(\mu + \gamma)} \), then

\[
\limsup_{t \to \infty} \frac{\ln(I(t))}{t} = \frac{\beta^2 M_{n_0}^2}{2\sigma^2_1 m_{n_0}} - (\mu + \gamma) < 0 \text{ a.s.;}
\]
(2) If $R_0 - 1 < \frac{\Lambda^2 \sigma^2 g^2(0) m_0}{2(\mu + \gamma)(\mu + p(1 - e^{-\mu t}))^2}$ and $\hat{\sigma}_1^2 \leq \frac{\beta(\mu + p(1 - e^{-\mu t})) m_0}{2(\mu + \gamma)(\mu + p(1 - e^{-\mu t}))^2}$, then
\[
\limsup_{t \to \infty} \frac{\ln l(t)}{t} \leq (\mu + \gamma)(R_0 - 1 - \frac{\Lambda^2 g^2(0) \hat{\sigma}_1^2 m_0}{2(\mu + \gamma)(\mu + p(1 - e^{-\mu t}))^2}) < 0 \quad \text{a.s.}
\]
where $\hat{\sigma}_1^2 = \sigma_1^2 + \int_Y \frac{\gamma^2(u)}{1 + M_0 e^S (0)^2} \lambda(du)$.

**Proof.** Applying Itô's formula, we derive that
\[
dln(t) = (\beta f(S(t)) \frac{g(I(t))}{I(t)} - (\mu + \gamma) - \frac{\sigma_1^2}{2} \frac{g^2(I(t))}{I(t)^2} f^2(S(t))) dt + \sigma f(S(t)) \frac{g(I(t))}{I(t)} dB(t)
\]
\[
+ \int_Y \ln[1 + f(S(s^-)) \frac{g(I(s^-))}{I(s^-)} \gamma(u)] - f(S(s^-)) \frac{g(I(s^-))}{I(s^-)} \gamma(u)] \lambda(du) dt
\]
\[
+ \int_Y \ln[1 + f(S(s^-)) \frac{g(I(s^-))}{I(s^-)} \gamma(u)] N(du, dt).
\]
Then
\[
\frac{\ln l(t)}{t} = \ln \frac{l(0)}{t} + \beta f(S(t)) \frac{g(I(t))}{I(t)} - (\mu + \gamma) - \frac{\sigma_1^2}{2} \frac{g^2(I(t))}{I(t)^2} f^2(S(t))) dt + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln l(0)}{t}
\]
(4.1)

Here, $M_1(t) = \int_0^t \sigma f(S(s^-)) \frac{g(I(s^-))}{I(s^-)} dB(s)$ and $M_2(t) = \int_0^t \int_Y \ln[1 + f(S(s^-)) \frac{g(I(s^-))}{I(s^-)} \gamma(u)] N(du, ds)$.

On the other hand, we have
\[
d(S + l + pe^{-\mu t}) \int_{t_1}^t S(s) ds + \gamma e^{-\mu t} \int_{t_1}^t l(s) ds
\]
\[
= [\Lambda - (\mu + p(1 - e^{-\mu t})) S - (\mu + \gamma(1 - e^{-\mu t}) l)] dr.
\]
(4.3)

then,
\[
S + l + pe^{-\mu t} \int_{t_1}^t S(s) ds + \gamma e^{-\mu t} \int_{t_1}^t l(s) ds
\]
\[
= \frac{S(0) + l(0) + pe^{-\mu t} \int_{t_1}^t S(s) ds + \gamma e^{-\mu t} \int_{t_1}^t l(s) ds}{t}
\]
\[
= \Lambda - (\mu + p(1 - e^{-\mu t})) S(t) - (\mu + \gamma(1 - e^{-\mu t})) l(t)
\]
(4.4)

Therefore,
\[
\langle S(t) \rangle = \frac{\Lambda}{\mu + p(1 - e^{-\mu t})} - \frac{\mu + \gamma(1 - e^{-\mu t})}{\mu + p(1 - e^{-\mu t})} l(t) - \langle \phi(t) \rangle
\]
(4.5)

and here, $\phi(t) = \frac{S + l + pe^{-\mu t} \int_{t_1}^t S(s) ds + \gamma e^{-\mu t} \int_{t_1}^t l(s) ds}{t} - \frac{S(0) + l(0) + pe^{-\mu t} \int_{t_1}^t S(s) ds + \gamma e^{-\mu t} \int_{t_1}^t l(s) ds}{t}$, thus $\lim_{t \to \infty} \phi(t) = 0$.

According to (4.5), we obtain that
\[
\frac{\ln l(t)}{t} \leq \beta M_0 g'(0) (S(t^-)) - (\mu + \gamma) - \frac{g^2(0) \hat{\sigma}_1^2 m_0}{2} (S^2(t^-)) + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln l(0)}{t}
\]
\[
= \beta M_0 g'(0) \frac{\Lambda}{\mu + p(1 - e^{-\mu t})} - (\mu + \gamma) - \frac{g^2(0) \hat{\sigma}_1^2 m_0}{2} (\mu + p(1 - e^{-\mu t}))^2
\]
\[
- \beta M_0 g'(0) \frac{\mu + \gamma(1 - e^{-\mu t})}{\mu + p(1 - e^{-\mu t})} l(t) + \frac{S(0) + l(0)}{t} + pe^{-\mu t} \int_{t_1}^t S(s) ds + \gamma e^{-\mu t} \int_{t_1}^t l(s) ds
\]
\[
+ \frac{\Lambda}{\mu + p(1 - e^{-\mu t})} \langle \phi(t) \rangle - \frac{g^2(0) \hat{\sigma}_1^2 m_0}{2} \frac{(\mu + \gamma(1 - e^{-\mu t}))^2}{(\mu + p(1 - e^{-\mu t}))^2} l(t)
\]
\[
- \frac{g^2(0) \hat{\sigma}_1^2 m_0}{2} \frac{1}{(\mu + p(1 - e^{-\mu t}))^2} \langle l(t) \rangle \langle \phi(t) \rangle + \frac{\ln l(0)}{t}
\]

According to (4.5), we obtain that
\[
\frac{\ln l(t)}{t} \leq \beta M_0 g'(0) \frac{\Lambda}{\mu + p(1 - e^{-\mu t})} - (\mu + \gamma) - \frac{g^2(0) \hat{\sigma}_1^2 m_0}{2} (\mu + p(1 - e^{-\mu t}))^2
\]

According to (4.5), we obtain that
\[
\frac{\ln l(t)}{t} \leq \beta M_0 g'(0) \frac{\Lambda}{\mu + p(1 - e^{-\mu t})} - (\mu + \gamma) - \frac{g^2(0) \hat{\sigma}_1^2 m_0}{2} (\mu + p(1 - e^{-\mu t}))^2
\]
\[
- \beta M_0 g'(0) \frac{\mu + \gamma(1 - e^{-\mu t})}{\mu + p(1 - e^{-\mu t})} l(t) + \frac{S(0) + l(0)}{t} + pe^{-\mu t} \int_{t_1}^t S(s) ds + \gamma e^{-\mu t} \int_{t_1}^t l(s) ds
\]
\[
+ \frac{\Lambda}{\mu + p(1 - e^{-\mu t})} \langle \phi(t) \rangle - \frac{g^2(0) \hat{\sigma}_1^2 m_0}{2} \frac{(\mu + \gamma(1 - e^{-\mu t}))^2}{(\mu + p(1 - e^{-\mu t}))^2} l(t)
\]
\[
- \frac{g^2(0) \hat{\sigma}_1^2 m_0}{2} \frac{1}{(\mu + p(1 - e^{-\mu t}))^2} \langle l(t) \rangle \langle \phi(t) \rangle + \frac{\ln l(0)}{t}
\]
\[
\begin{align*}
\ln(1 + \gamma(M_0g(0)) - M_0g'(0) + \Lambda - g^2(0)\sigma_1^2m_{N_0}\mu
\leq (\mu + \gamma)\left(\beta M_{N_0}g'(0) + \frac{\Lambda}{\mu + \gamma}\right) - M_0g'(0) + \Lambda - g^2(0)\sigma_1^2m_{N_0}\mu
\end{align*}
\]

Thus, By virtue of the condition (2) and (4.6), we achieve that

\[
\begin{align*}
\ln(1 + \gamma(M_0g(0)) - M_0g'(0) + \Lambda - g^2(0)\sigma_1^2m_{N_0}\mu
\leq (\mu + \gamma)\left(\beta M_{N_0}g'(0) + \frac{\Lambda}{\mu + \gamma}\right) - M_0g'(0) + \Lambda - g^2(0)\sigma_1^2m_{N_0}\mu
\end{align*}
\]

where

\[
\psi(t) = g^2(0)\sigma_1^2m_{N_0}\frac{\Lambda}{\mu + \gamma} - M_0g'(0) + \Lambda - g^2(0)\sigma_1^2m_{N_0}\mu
\]

In addition,

\[
\begin{align*}
\langle M_1, M_1 \rangle_t &= \sigma_1^2 \int_0^t f(\bar{S}(s^-)) \frac{g^2(I(s^-))}{I(s^-)} ds,
\langle M_2, M_2 \rangle_t &= \int_0^t \int_Y \ln(1 + f(\bar{S}(s^-))) \frac{g(I(s^-))}{I(s^-)} \varphi(s^-) \lambda(du) ds,
\end{align*}
\]

and

\[
\ln(1 - g'(0)m_{N_0}\theta) \leq \ln(1 + g'(0)m_{N_0}\theta) \leq \ln(1 + g'(0)m_{N_0}\theta)
\]

Then, we have that

\[
\langle M_2, M_2 \rangle_t \leq \max((\ln(1 + g'(0)m_{N_0}\theta))^2, (\ln(1 - g'(0)m_{N_0}\theta))^2)\lambda(Y)t,
\]

and

\[
\lim_{t \to \infty} \frac{\langle M_1, M_1 \rangle_t}{t} = \sigma_1^2 \lim_{t \to \infty} \frac{1}{t} \int_0^t f(\bar{S}(s^-)) \frac{g^2(I(s^-))}{I(s^-)} ds \leq \sigma_1^2 M_0^2 g^2(0) \frac{\Lambda}{\mu} < \infty \quad \text{a.s.}
\]

\[
\lim_{t \to \infty} \frac{\langle M_2, M_2 \rangle_t}{t} \leq \max((\ln(1 + g'(0)m_{N_0}\theta))^2, (\ln(1 - g'(0)m_{N_0}\theta))^2)\lambda(Y) < \infty, \quad \text{a.s.}
\]

Thus,

\[
\lim_{t \to \infty} \frac{M_i(t)}{t} = 0 \quad (i = 1, 2) \quad \text{and} \quad \lim_{t \to \infty} \psi(t) = 0.
\]

By virtue of the condition (2) and (4.6), we achieve that

\[
\lim_{t \to \infty} \frac{\ln(I(t))}{t} \leq (\mu + \gamma)(R_0 - 1 - \frac{\Lambda}{2(\mu + \gamma)(\mu + p(1 - e^{-\mu_2}))^2}) < 0 \quad \text{a.s.}
\]

Moreover, by (4.6), we have that

\[
\ln(I(t))/t \leq \beta M_{N_0}g(0)\bar{S}(t^-) - (\mu + \gamma) - \frac{g^2(0)\sigma_1^2m_{N_0}}{2} \bar{S}(t^-)^2 + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln(I(t))}{t}
\]

\[
= - \frac{g^2(0)\sigma_1^2m_{N_0}}{2} \left[ \left( \frac{\beta M_{N_0}}{\sigma_1^2g'(0)m_{N_0}} \right)^2 - \left( \frac{\beta M_{N_0}}{\sigma_1^2g'(0)m_{N_0}} \right)^2 \right] - (\mu + \gamma)
\]

\[
\leq - \frac{g^2(0)\sigma_1^2m_{N_0}}{2} \left[ \left( \frac{\beta M_{N_0}}{\sigma_1^2g'(0)m_{N_0}} \right)^2 - \left( \frac{\beta M_{N_0}}{\sigma_1^2g'(0)m_{N_0}} \right)^2 \right] - (\mu + \gamma)
\]

\[
+ \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln(I(t))}{t}
\]
According to the condition (1) and (4.8), we obtain that
\[
\frac{\ln I(t)}{t} \leq - (\mu + \gamma) + \frac{\beta^2 M_0^2}{2 \sigma_1^2 m^0} + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln I(0)}{t}
\]  
(4.8)

That is \(\lim_{t \to \infty} I(t) = 0\). Moreover, we have that
\[
\lim_{t \to \infty} \left( S(t) \right) = \frac{\lambda}{\mu + p(1 - e^{-\mu t})} - \frac{\mu + \gamma(1 - e^{-\mu t})}{\mu + p(1 - e^{-\mu t})} \lim_{t \to \infty} I(t) = \lim_{t \to \infty} \phi(t) = \frac{\lambda}{\mu + p(1 - e^{-\mu t})}.
\]

The conclusion is proven.

**Remark 1.** From Theorem 4.1, we show that if condition (i) \(\tilde{\sigma}_1^2 > \frac{\beta^2 M_0^2}{2 \mu p(1 - e^{-\mu t})}\), or (ii) \(\sigma_1^2 \leq \frac{g(0)}{2 \mu p(1 - e^{-\mu t})}\) hold, then for an arbitrary solution \((S(t), I(t))\) of system (1.2), we have \(\lim_{t \to \infty} I(t) = 0\), which implies that the disease is extinct.

**5. Persistence in the mean of system**

Now we are in the position to discuss the persistence in the mean of the disease and before that some notations are presented in the following.

For the convenience, we denote
\[
\tilde{\rho}_0 = \frac{\beta m g(0) \Lambda}{(\mu + \gamma)(\mu + p(1 - e^{-\mu t}))},
\]
\[
\tilde{\sigma}_1 = \sigma_1^2 g^2(0) M_0 + \int \frac{g^2(0) M_0^2 \gamma^2(u)}{(1 - m_0 g'(0) \sigma)^2} \lambda(du),
\]
\[
\lambda^* = (\mu + \gamma) \left( R_0 - 1 - \frac{\sigma_1^2 g'(0) M_0 \Lambda^2}{2(\mu + \gamma)(\mu + p(1 - e^{-\mu t}))^2} \right),
\]
\[
\lambda_0 = g'(0) M_0 \frac{\mu + \gamma(1 - e^{-\mu t})}{\mu + p(1 - e^{-\mu t})} \left( \beta - \frac{\sigma_1^2 g'(0) M_0 \Lambda}{\mu_0 (\mu + p(1 - e^{-\mu t}))} \right),
\]
\[
\tilde{\lambda}_0 = \frac{\lambda^*}{\lambda_0}, \quad \tilde{\Gamma} = \frac{(\mu + p(1 - e^{-\mu t}))(\mu + \gamma)(R_0 - 1) - \frac{\tilde{\lambda}_0 A^2}{2 \mu^2}}{\beta g'(0) m_0 (\mu + \gamma(1 - e^{-\mu t}))},
\]
\[
I^* = \frac{\Lambda}{\mu + p + \beta M g(0) N_0}.
\]

**Theorem 5.1.** Suppose that Assumption (H1) hold and then for the solution \((S(t), I(t))\) of model (1.2),

(1) If \(\tilde{\rho}_0 - 1 > \frac{\tilde{\lambda}_0 A^2}{2 \mu p(1 - e^{-\mu t})}\), we have
\[
\liminf_{t \to \infty} I(t) \geq I^*, \quad \liminf_{t \to \infty} I(t) \geq \tilde{\Gamma};
\]

(2) If \(\tilde{\rho}_0 - 1 < \frac{\tilde{\lambda}_0 A^2}{2 \mu p(1 - e^{-\mu t})}\) and \(\tilde{\lambda}_0 \tilde{\sigma}_1^2 < \frac{\beta M_0 (\mu + p(1 - e^{-\mu t}))}{\Lambda g'(0) m_0}\), we have
\[
\limsup_{t \to \infty} S(t) \leq I_*, \quad \limsup_{t \to \infty} I(t) \leq \tilde{I}_*.
\]

**Proof.** Since \(\lim_{t \to 0} \frac{g(l)}{l} = g(0)\), then a constant \(\varepsilon > 0\) exists satisfying \(g(l) > (g'(0) - \varepsilon)l\) for all \(0 < l \leq \varepsilon\). By virtue of (4.6), we have that
\[
\frac{\ln I(t)}{t} \leq (\mu + \gamma) \left( R_0 - 1 - \frac{g^2(0) \tilde{\sigma}_1^2 m_0 \Lambda^2}{2(\mu + \gamma)(\mu + p(1 - e^{-\mu t}))^2} \right) + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} - \frac{M_0 g'(0) \mu + \gamma(1 - e^{-\mu t})}{\mu + p(1 - e^{-\mu t})} \left( \beta - \frac{g'(0) \tilde{\lambda}_0 \tilde{A} m_0}{\mu_0 (\mu + p(1 - e^{-\mu t}))} \right) I(t) + \psi(t)
\]
(5.1)
Then

\[
\ln I(t) \leq (\mu + \gamma) \left( R_0 - 1 - \frac{g'(0)\sigma_1^2 R_0 \lambda^2}{2(\mu + \gamma)\mu + p(1 - e^{-\mu t})} \right) + \frac{M_1(t) + M_2(t)}{t} \\
- M_{N_0} \frac{g'(0)\mu + \gamma(1 - e^{-\mu t})}{\mu + p(1 - e^{-\mu t})} \left( \beta - \frac{g'(0)\sigma_2^2 R_0 \lambda^2}{M_{N_0}(\mu + p(1 - e^{-\mu t}))} \right) \int_0^t I(s)ds + \psi(t) \\
= (\mu + \gamma) \left( R_0 - 1 - \frac{g'(0)\sigma_1^2 R_0 \lambda^2}{2(\mu + \gamma)\mu + p(1 - e^{-\mu t})} \right) + \frac{M_1(t) + M_2(t)}{t} \\
- M_{N_0} \frac{g'(0)\mu + \gamma(1 - e^{-\mu t})}{\mu + p(1 - e^{-\mu t})} \left( \beta - \frac{g'(0)\sigma_2^2 R_0 \lambda^2}{M_{N_0}(\mu + p(1 - e^{-\mu t}))} \right) \int_0^t I(s)ds + F(t) \\
= \lambda^* t - \lambda_0 \int_0^t I(s)ds + F(t),
\]

(5.2)

here, \( F(t) = M_1(t) + M_2(t) + \psi(t) \).

Considering \( \lim_{t \to \infty} \frac{F(t)}{t} = 0 \), then for an arbitrary \( \xi > 0 \), there exists a \( T_1 = T_1(\omega) > 0 \) and a set \( \Omega_k \) such that \( \frac{F(t)}{t} \leq \xi \) and \( P(\Omega_k) \geq 1 - \xi \) for all \( t \geq T_1, \omega \in \Omega_k \). Let \( \tilde{T} = \max(T, T_1) \), then according to Lemma 2.2 and Theorem 3 in Ref. [22], we achieve that

\[
\limsup_{t \to \infty} I(t) \leq \frac{\lambda^*}{\lambda_0} \tilde{T}.
\]

(5.3)

On the other hand, by (4.2) and (4.5), we obtain that

\[
\frac{\ln I(t)}{t} = \beta f(S) \frac{g'(l)}{l} - (\mu + \gamma) - \sigma_1^2 \frac{g'(l(S^{-}))}{f(l(S^{-}))} g'(l(S^{-})) + \frac{M_1(t) + M_2(t)}{t} + \frac{\ln I(0)}{t} \\
+ \frac{1}{t} \int_0^t \left( \ln(1 + f(S(s^{-}))) g'(l(S^{-})) \right) \left( u = f(S(s^{-})) \right) g'(l(S^{-})) \left( \lambda^* \right) ds \\
\geq \beta (g'(0) - \epsilon) m_{N_0} (S(t)) - (\mu + \gamma) - \frac{\sigma_1^2 \Lambda}{2 \mu} + \frac{M_1(t) + M_2(t)}{t} + \frac{\ln I(0)}{t} \\
= \beta (g'(0) - \epsilon) m_{N_0} \frac{\Lambda}{\mu + p(1 - e^{-\mu t})} - \beta (g'(0) - \epsilon) m_{N_0} \frac{(\mu + \gamma)(1 - e^{-\mu t})}{\mu + p(1 - e^{-\mu t})} \left( I(t) \right) \\
- (\mu + \gamma) - \frac{\sigma_1^2 \Lambda}{2 \mu} - \beta (g'(0) - \epsilon) m_{N_0} \phi(t) + \frac{M_1(t) + M_2(t) + \ln I(0)}{t} \\
= (\mu + \gamma) \left( R_0 - 1 \right) - \frac{\sigma_1^2 \Lambda}{2 \mu} - \frac{\beta (g'(0) - \epsilon) m_{N_0} (\mu + \gamma)(1 - e^{-\mu t})}{\mu + p(1 - e^{-\mu t})} \left( I(t) \right) \\
- \beta (g'(0) - \epsilon) m_{N_0} \phi(t) + \frac{M_1(t) + M_2(t) + \ln I(0)}{t} \\
= (\mu + \gamma) \left( \frac{R_0 - 1}{\mu + p(1 - e^{-\mu t})} - \frac{\beta (g'(0) - \epsilon) m_{N_0} (\mu + \gamma)(1 - e^{-\mu t})}{\mu + p(1 - e^{-\mu t})} \left( I(t) \right) \\
- \beta (g'(0) - \epsilon) m_{N_0} \phi(t) + \frac{M_1(t) + M_2(t) + \ln I(0)}{t} \right)
\]

(5.4)

As \( 0 < S + I \leq N_0 \), then we derive that \( -\infty < \ln I(t) < \ln(N_0) \). Thus,

\[
(I(t)) \geq \frac{\mu + p(1 - e^{-\mu t})}{\beta g'(0) - \epsilon) m_{N_0} (\mu + \gamma)(1 - e^{-\mu t})} \left( R_0 - 1 \right) - \frac{\sigma_1^2 \Lambda}{2 \mu} \left( I(t) \right) \\
- \beta (g'(0) - \epsilon) m_{N_0} \phi(t) + \frac{M_1(t) + M_2(t) + \ln I(0)}{t}
\]

(5.5)

By virtue of the conclusion \( \lim_{t \to \infty} \phi(t) = 0 \) and the arbitrariness of \( \epsilon \), we obtain that

\[
\liminf_{t \to \infty} I(t) \geq \frac{(\mu + \gamma)(1 - e^{-\mu t}) - \frac{\sigma_1^2 \Lambda}{2 \mu}}{\beta g'(0) - \epsilon) m_{N_0} (\mu + \gamma)(1 - e^{-\mu t})} \left( \tilde{T} \right) \leq I_0.
\]

(5.6)

In addition, by virtue of (4.5) and (5.6), we have that

\[
\limsup_{t \to \infty} S(t) \leq \frac{\Lambda}{\mu + p(1 - e^{-\mu t})} - \frac{\mu + \gamma(1 - e^{-\mu t})}{\mu + p(1 - e^{-\mu t})} \left( \tilde{T} \right) \leq I_0.
\]

(5.7)
In the following, we prove
\[
\liminf_{t \to \infty} \bar{S}(t) \geq \bar{l}^*.
\]
By assumptions (H2) and (H3), a constant \( T_2 \geq T_1 > 0 \) exists satisfying
\[
f(S(t))g(I(t)) \leq M_{0g}(0)N_0 S(t),
\]
for all \( t \geq T_2 \).

Using the first equation of model \( (1.2) \), we achieve that
\[
\frac{S(t) - S(0)}{t} = \Lambda - \frac{\beta}{t} \int_0^t f(S(s))g(I(s))ds - \frac{\mu + p}{t} \int_0^t S(s)ds + \frac{\gamma}{t} I(s - \tau_2)e^{-\mu t_2}ds + \frac{\sigma_1}{t} \int_0^t f(S(s))g(I(s))d\beta(s)
\]
\[
+ \frac{1}{t} \int_0^t f(S(s))g(I(s)) \int_y \gamma(u)\tilde{N}(ds, du)
\]
\[
\geq \Lambda - \frac{1}{t} \int_0^{T_2} [\beta f(S(s))g(I(s)) + (\mu + p)S(s)]ds - \frac{\sigma_1}{t} \int_0^t f(S(s))g(I(s))d\beta(s)
\]
\[
- \frac{1}{t} \int_0^t f(S(s))g(I(s)) \int_y \gamma(u)\tilde{N}(ds, du)
\]
\[
- \frac{1}{t} \int_{T_2}^t [\beta M_{0g}(0)N_0 + \mu + p]S(s)ds.
\]

Therefore, by Theorem 4.1 and applying the arbitrariness of \( \eta \), we have
\[
\liminf_{t \to \infty} \bar{S}(t) \geq \frac{\Lambda}{\mu + p + \beta M_{0g}(0)N_0} \triangleq \bar{l}^*.
\]
The desired result is obtained.

Remark 2. From Theorem 5.1, we obtain that if \( \hat{R}_0 > 1 + \frac{\sigma_1^2A^2}{2\mu^2(\mu + \gamma)} \), and choose \( A_1 = \min\{l_*, \hat{\lambda}_*\} \), then the solution \((S(t), I(t))\) of system \( (1.2) \) with an initial condition \( (1.3) \) is persistent in the mean. Moreover, denote \( A_2 = \max\{l_*, \bar{l}_*\} \), we can also obtain the condition for the permanence in the mean of the system, that is,
\[
A_1 \leq \liminf_{t \to \infty} S(t) \leq \limsup_{t \to \infty} S(t) \leq A_2,
\]
and
\[
A_1 \leq \liminf_{t \to \infty} I(t) \leq \limsup_{t \to \infty} I(t) \leq A_2.
\]

6. Numerical simulations

In this section, we will perform some numerical simulations to illustrate our theoretical results by Euler numerical approximation [26].

1. Choose the parameter values in model \( (1.2) \) as follows:
\[
\Lambda = 0.6, \quad \beta = 0.2, \quad \mu = 0.2, \quad \gamma = 0.2, \quad p = 0.05, \quad f(S) = S, \quad g(I) = \frac{1}{1 + I},
\]
\[
\tau_1 = 0.5, \quad \tau_2 = 0.5, \quad S(0) = 2, \quad I(0) = 0.35, \quad Y = (0, +\infty), \quad \lambda(Y) = 1,
\]
and the only difference between conditions of Figs. 1 and 2 is the different values \( \gamma(u) \).

In Fig. 1, the intensities of the noises are \( \sigma_1 = 0.08 \) and \( \gamma(u) = 0.7 \), then we have that
\[
\sigma_1^2 = 0.02 > \frac{\beta^2 M_{0g}^2}{2m_{0g}(\mu + \gamma)} = 0.008,
\]
and the condition (1) of Theorem 4.1 is satisfied. Thus, the disease \( I \) goes to extinction with probability one and Fig. 1 confirms it. Moreover, by Fig. 1, we achieve that the disease is extinct whereas the corresponding deterministic system is persistent because of the effect of the jump noise. In Fig. 2, the intensities of the noises are \( \sigma_1 = 0.08 \) and \( \gamma(u) = 0.2 \), then we obtain that
\[
\hat{R}_0 = 1.4651 > 1 + \frac{\sigma_1^2A^2}{2\mu^2(\mu + \gamma)} \triangleq 1,
\]
(2) Let $\mu = 0.26$ and the other parameters are the same as those in Fig. 1. Considering different values of $\tau_1$ and $\tau_2$, then we can observe the effects of the validity period of the vaccination $\tau_1$ to system (1.2) (See Fig. 3).

From Figs. 1–3, we achieve that the intensity of Lévy noise $\gamma(u)$ and the validity period of the vaccination $\tau_1$ can greatly influence the extinction and persistence of the disease $I$.

7. Conclusion

A stochastic delay epidemic model is proposed with vaccination and generalized nonlinear incidence rate. The effects of Lévy jumps are considered in this model. The existence of a unique global solution is proven. Sufficient conditions that guarantee diseases to be extinct and persistent in the mean are given. From the analysis and discussion, we derive that noise intensity and the validity period of vaccination greatly influence the extinction and persistence of diseases.
Finally, some interesting issues merit further investigations. In this paper, we obtain sufficient conditions for the persistence and extinction of the model. However, whether the threshold value could be derived is an interesting issue. In addition, if we also consider the effects of non-autonomous environment to the proposal of epidemic model, how will the properties change? We will investigate these questions in our future work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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