Factorization scheme analysis of $F_2^\gamma(x, Q^2)$ and parton distributions functions of the photon

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Factorization scheme analysis of $F_2^\gamma(x, Q^2)$ in the next-to-leading order QCD is revisited. It is emphasized that the presence of the inhomogeneous term in the evolution equations for quark distribution functions of the photon implies subtle but important difference in the way factorization mechanism works in photon–hadron and photon–photon collisions as compared to the hadronic ones. It is argued that none of the existing NLO analyses of $F_2^\gamma(x, Q^2)$ takes this difference properly into account. The source of the ensuing incompleteness is traced back to the misinterpretation of the behaviour of $q^\gamma(x, M)$ as a function of $\alpha_s(M)$. Parton model interpretation of the so-called “constant terms” in the LO photonic coefficient function $C_\gamma^{(0)}(x)$ is given and smooth transition between the properties of virtual and real photon analyzed. Finally phenomenological consequences of this analysis are discussed.

1 Introduction

Observed from a large distance the photon behaves as a neutral structureless object governed by the laws of Quantum Electrodynamics. However, when probed at short distances it exhibits also some properties characteristic of hadrons. This “photon structure” is quantified, similarly as in the case of hadrons, in terms of parton distribution functions (PDF), satisfying certain evolution equations. Because of the direct coupling photons to quark–antiquark pairs these evolution equations are, contrary the case of hadrons, inhomogeneous. This inhomogeneity has important implications for the way factorization of mass singularities operates in collisions involving photons, implications that have not been properly taken into account in existing NLO analyses of $F_2^\gamma(x, Q^2)$. The primary aim of this paper is to remove this shortcoming.

Secondly, we shall address several issues concerning the structure of the virtual photon: transition between the properties of real and virtual photon, properties and role of the longitudinal virtual photon, and parton model interpretation of the so-called “constant terms” in LO photonic coefficient function $C_\gamma^{(0)}(x)$.

The paper is organized as follows. In the next Section basic facts and notation concerning PDF of the photon are reviewed and the properties of the pointlike part of quark distribution function of the photon critically reanalyzed. In Section 3, which contains the main result of this paper, we discuss in detail the factorization scale and scheme dependence of $F_2^\gamma(x, Q^2)$ at the NLO and point out the ingredients that must be included to make this analysis complete. The properties of the virtual photon and the transition of PDF of the virtual photon to those of the real one are analyzed in Section 4. Phenomenological implications of the present analysis are discussed in Section 5.

1Supported by the Grant Agency of ASCR under grant No. A1010602
2 Structure of the real photon

Despite the recent progress in investigation of the structure of the photon \(^2\) our knowledge of the properties of the photon still lags behind that of the nucleon. We shall be primarily interested in strong interaction effects, but as the basic ideas and formalism of the partonic structure of the photon have a close analogy in QED, the latter will serve as a guide in some of the following considerations.

2.1 Notation and basic facts

In QCD the coupling of quarks and gluons is characterized by the renormalized colour coupling (“couplant” for short) \(\alpha_s(\mu)\), depending on the renormalization scale \(\mu\) and satisfying the equation

\[
\frac{d\alpha_s(\mu)}{d \ln \mu^2} = \beta(\alpha_s(\mu)) = -\frac{\beta_0}{4\pi} \alpha_s^2(\mu) - \frac{\beta_1}{16\pi^2} \alpha_s^3(\mu) + \cdots,
\]

where, in QCD with \(n_f\) massless quark flavours, the first two coefficients, \(\beta_0 = 11 - 2n_f/3\) and \(\beta_1 = 102 - 38n_f/3\), are unique, while all the higher order ones are ambiguous. As we shall stay in this paper within the NLO, only the first two, unique, terms in (1) will be taken into account in the following. Nevertheless, even for a given r.h.s. of (1) its solution \(\alpha_s(\mu)\) is not a unique function of \(\mu\), because there is an infinite number of solutions of (1), differing by the initial condition. This so-called renormalization scheme (RS) ambiguity \(^3\) can be parameterized in a number of ways. One of them makes use of the fact that in the process of renormalization another dimensional parameter, denoted usually \(\Lambda\), inevitably appears in the theory. This parameter depends on the RS and at the NLO actually fully specifies it: RS\(=\{\Lambda_{\text{RS}}\}\). For instance, \(\alpha_s(\mu)\) in the familiar MS and \(\overline{\text{MS}}\) RS are solutions of the same equation (1), but are associated with different \(\Lambda_{\text{RS}}\)\(^4\). In this paper we shall work in the standard MS RS of the couplant.

In QCD “dressed” PDF \(^5\) result from the resummation of multiple parton emissions off the corresponding “bare” parton distributions. As a result of this resummation PDF acquire dependence on the factorization scale \(M\). In parton model this scale defines the upper limit on some measure \(t\) of the off-shellness of partons included in the definition of \(D(x,M)\)

\[
D_i(x,M) \equiv \int_{t_{\text{min}}}^{M^2} dt_i(x,t), \quad i = q, \overline{q}, G,
\]

where the unintegrated PDF \(d_i(x,t)\) describe distribution functions of partons with the momentum fraction \(x\) and fixed off-shellness \(t\). Parton virtuality \(\tau \equiv |p^2 - m^2|\) or transverse mass \(m_T^2 \equiv p_T^2 + m^2\), are two standard choices of such a measure. Because at small \(t\), \(d_i(x,t) = \mathcal{O}(1/t^k), k = 1,2\), the dominant part of the integral (2) comes from the region of small off-shellness \(t\). Varying the upper bound \(M^2\) in (2) has therefore only a small effect on the integral (3), leading to weak (at most logarithmic) scaling violations.

The factorization scale dependence of PDF of the photon \(^6\) is determined by a system of coupled inhomogeneous evolution equations

\[
\frac{d\Sigma(x,M)}{d \ln M^2} = k_q + P_{qq} \otimes \Sigma + P_{qG} \otimes G,
\]

\(^2\)For recent theoretical and experimental reviews see \([1]\) and \([2]\), respectively.
\(^3\)In higher orders this ambiguity includes also the arbitrariness of the coefficients \(\beta_i, i \geq 2\) in \([1]\).
\(^4\)The variation of both the renormalization scale \(\mu\) and the renormalization scheme \(\text{RS} = \{\Lambda_{\text{RS}}\}\) is actually redundant. It suffices to fix one of them and vary the other, but I will stick to the common habit of considering both of them as free parameters.
\(^5\)In the following the adjective “dressed” will be dropped.
\(^6\)If not stated otherwise all distribution functions in the following concern the photon.
\[
\begin{align*}
\frac{dG(x, M)}{d\ln M^2} &= k_G + P_{Gq} \otimes \Sigma + P_{GG} \otimes G, \\
\frac{dq_{NS}^i(x, M)}{d\ln M^2} &= \sigma_{NS}^i + P_{NS} \otimes q_{NS}^i,
\end{align*}
\]

where \(\sigma_{NS}^i \equiv (\epsilon_i^2/\epsilon^2 - 1)/n_f\) and the convolution \(\otimes\) is defined in a standard way as

\[
P \otimes q(x) \equiv \int_x^1 \frac{dy}{y} P(x/y)q(y).
\]

The singlet and nonsinglet quark distribution functions \(\Sigma\) and \(q_{NS}^i\) are given as

\[
\Sigma(x, M) \equiv \sum_{i=1}^{n_f} \left[ q_i(x, M^2) + \overline{q}_i(x, M^2) \right] \equiv \sum_{i=1}^{n_f} q_i^+(x, M^2), \tag{7}
\]

\[
q_{NS}^i(x, M) \equiv q_i^+(x, M^2) - \Sigma(x, M)/n_f. \tag{8}
\]

To order \(\alpha\) the splitting functions \(P_{ij}(x, M)\) and \(k_i(x, M)\) are given as power expansions in \(\alpha_s(M)\):

\[
k_q(x, M) = \frac{\alpha}{2\pi} \left[ k_q^{(0)}(x) + \frac{\alpha_s(M)}{2\pi} k_q^{(1)}(x) + \left( \frac{\alpha_s(M)}{\pi} \right)^2 k_q^{(2)}(x) + \cdots \right], \tag{9}
\]

\[
k_G(x, M) = \frac{\alpha}{2\pi} \left[ \frac{\alpha_s(M)}{2\pi} k_G^{(1)}(x) + \left( \frac{\alpha_s(M)}{\pi} \right)^2 k_G^{(2)}(x) + \cdots \right], \tag{10}
\]

\[
P_{ij}(x, M) = \frac{\alpha_s(M)}{2\pi} P_{ij}^{(0)}(x) + \left( \frac{\alpha_s(M)}{\pi} \right)^2 P_{ij}^{(1)}(x) + \cdots, \tag{11}
\]

where the leading order splitting functions \(k_q^{(0)}(x) = 3e_q^2(x^2 + (1-x)^2)\) and \(P_{ij}^{(0)}(x)\) are unique, while all higher order ones \(k_q^{(j)}(x), k_G^{(j)}, P_{ij}^{(j)}, j \geq 1\) depend on the choice of the factorization scheme (FS)\(^7\). The photon structure function \(F_2^\gamma(x, Q^2)\), measured in deep inelastic scattering experiments on the (slightly off–shell) photons \(\gamma\), is given as the convolution \(\otimes\)

\[
F_2^\gamma(x, Q^2) = \sum_q 2xe_q^2 \left[ q(M) \otimes C_q(Q/M) + G(M) \otimes C_G(Q/M) + C_\gamma^q(Q/M) \right] \tag{12}
\]

of photonic PDF and coefficient functions \(C_q(x), C_G(x), C_\gamma^q(x)\) admitting perturbative expansions

\[
C_q(x, Q/M) = \delta(1-x) + \frac{\alpha_s(\mu)}{2\pi} C_q^{(1)}(x, Q/M) + \cdots, \tag{13}
\]

\[
C_G(x, Q/M) = \frac{\alpha_s(\mu)}{2\pi} C_G^{(1)}(x, Q/M) + \cdots, \tag{14}
\]

\[
C_\gamma^q(x, Q/M) = \frac{e_q^2 \alpha}{2\pi} C_\gamma^{(0)}(x, Q/M) + \frac{e_q^2 \alpha}{2\pi} C_\gamma^{(1)}(x, Q/M) + \cdots. \tag{15}
\]

The renormalization scale \(\mu\), used as argument of the expansion parameter \(\alpha_s(\mu)\), is in principle independent of the factorization scale \(M\). Note that despite the presence of \(\mu\) as argument of \(\alpha_s(\mu)\) in \((13)\) \((15)\), the coefficient functions \(C_q, C_G\) and \(C_\gamma^q\) are actually independent of \(\mu\) because the \(\mu\)–dependence of \(\alpha_s(\mu)\) is cancelled by explicit dependence of \(C_q^{(i)}, C_G^{(i)}, C_\gamma^{(i)}, i \geq 2\) on \(\mu\)\(^8\). On the

\(^7\)We can turn this statement around and consider any factorization scheme FS to be specified by the corresponding set of functions \(k_q^{(j)}, k_G^{(j)}, P_{ij}^{(j)}, j \geq 1\).

\(^8\)The factor 2 accounts for the inclusion of antiquarks in the sum.
other hand, PDF and the coefficient functions $C_q, C_G$ and $C_\gamma$ do depend on both the factorization scale $M$ and factorization scheme $\text{FS} = \{k_i^{(1)}, k_i^{(2)}, P_{ij}^{(1)}\}, i \geq 1$, but in such a correlated manner that physical quantities, like $F_2^q$, are independent of both $M$ and the FS, provided expansions (9–11) and (13–15) are taken to all orders in $\alpha_s(M)$ and $\alpha_s(\mu)$. In practical calculations based on truncated forms of (9–11) and (13–15) this invariance is, however, lost and the choice of both $M$ and FS makes numerical difference even for physical quantities. At the NLO $\text{RS} = \{\Lambda_{\text{RS}}\}$ and $\text{FS} = \{k_i^{(1)}, k_i^{(2)}, P_{ij}^{(1)}\}$.

The expressions for $C_q^{(1)}, C_G^{(1)}$ given in (8) are usually claimed to correspond to “$\overline{\text{MS}}$ factorization scheme”

As argued in [9], this denomination is, however, incomplete. The adjective “$\overline{\text{MS}}$” concerns exclusively the choice of the RS of the couplant $\alpha_s$ and has nothing to do with the choice of the splitting functions $P_{ij}^{(1)}$. The choices of the renormalization scheme of the couplant $\alpha_s(M)$ and of the factorization scheme of PDF are two completely independent decisions, concerning two different and in general unrelated redefinition procedures. Both are necessary in order to specify uniquely the results of fixed order perturbative calculations, but we may combine any choice of the RS of the couplant with any choice of the FS of PDF. Note that the coefficient functions $C_q^{(1)}, C_G^{(1)}, C_\gamma^{(1)}$ depend on both of them, while the splitting functions depend only on the FS of PDF. The results given in (8) correspond to $\overline{\text{MS}}$ RS of the couplant but to the “minimal subtraction” FS of PDF

It is therefore more appropriate to call this full specification of the renormalization and factorization schemes as “$\overline{\text{MS}} + \text{MS}$ scheme”. Although the phenomenological relevance of treating $\mu$ and $M$ as independent parameters has been demonstrated [11], we shall follow the usual practice and set $\mu = M$.

### 2.2 Properties of the pointlike solutions

The general solution of the evolution equations (8, 9) for the quark distribution functions $q(x, M)$ can be written as the sum of a particular solution of the full inhomogeneous equation, called pointlike part, and the general solution of the corresponding homogeneous equation, called hadronic part:

$$q(x, M) = q^{\text{PL}}(x, M) + q^{\text{HAD}}(x, M).$$  \hspace{1cm} (16)

Having separated $q(x, M)$ into its hadronic and pointlike parts we can insert it into the r.h.s. of the evolution equation (4) for the gluon distribution function and write its solutions as a sum of the hadronic and pointlike parts. At the LO, where $k_G = 0$, both of these components satisfy standard homogeneous evolution equation

$$\frac{dG^k(x, M)}{d \ln M^2} = P_{Gq} \otimes \Sigma^k + P_{GG} \otimes G^k, \quad k = \text{HAD, PL}.$$  \hspace{1cm} (17)

However, as there is an infinite number of pointlike solutions $q^{\text{PL}}(x, M)$, which differ by terms satisfying the homogeneous evolution equation, the above separation of the quark and gluons distribution functions into their pointlike and hadronic parts is not unique and consequently these concepts have separately no physical meaning. To see the most important feature of the pointlike part of quark distribution functions that will be crucial for the following analysis, we now consider in detail the case of nonsinglet quark distribution function $q_{\text{NS}}^i(x, M)$ at the LO (which is sufficient for our purposes) and, moreover, drop the superscript $i$. A subset of all pointlike solutions of (8) may be characterized by the value of the initial scale $M_0$ at which they vanish. At the LO and in terms of moments such solutions are given explicitly as

$$q_{\text{NS}}^{\text{PL}}(n, M_0, M) = \frac{4\pi}{\alpha_s(M)} \left[ 1 - \left( \frac{\alpha_s(M)}{\alpha_s(M_0)} \right)^{-2P_{qq}^{(0)}(n)/\beta_0} \right] a_{\text{NS}}(n),$$  \hspace{1cm} (18)

\[9\text{See Section 2.6 of [11], in particular eq. (2.31), for discussion of this point.} \]
where 
\[ a_{NS}(n) \equiv \frac{\alpha}{2\pi\beta_0} \frac{k^{(0)}_{NS}(n)}{1 - 2P^{(0)}_{qq}(n)/\beta_0}, \]  
(19)

and result from resummation of infinite series of diagrams in the upper part of Fig. 1:

\[ q^{PL}_{NS}(x, M_0, M) = \frac{\alpha}{2\pi} k^{(0)}_{NS}(x) \int_{M_0^2}^{M^2} \frac{d\tau}{\tau} + \int_x^1 \frac{dy}{y} P^{(0)}_{qq} \left( \frac{x}{y} \right) \int_{M_0^2}^{M^2} \frac{d\tau_1}{\tau_1} \alpha_s(\tau_1) \frac{\alpha}{2\pi} k^{(0)}_{NS}(y) \int_{M_0^2}^{\tau_1} \frac{d\tau_2}{\tau_2} + \int_x^1 \frac{dy}{y} P^{(0)}_{qq} \left( \frac{x}{y} \right) \int_y^1 \frac{dw}{w} P^{(0)}_{qq} \left( \frac{y}{w} \right) \int_{M_0^2}^{M^2} \frac{d\tau_1}{\tau_1} \alpha_s(\tau_1) \int_{M_0^2}^{\tau_1} \frac{d\tau_2}{\tau_2} \alpha_s(\tau_2) \frac{\alpha}{2\pi} k^{(0)}_{NS}(w) \int_{M_0^2}^{\tau_2} \frac{d\tau_3}{\tau_3} + \cdots \]  
(20)

which, as illustrated in Fig. 2, softens substantially the \( x \)-dependence of \( a_{NS}(x) \) with respect to the first term in (20), proportional to \( k_{NS}(x) \). This construction is similar to that of hadrons with the first term in (20), \( (\alpha/2\pi) k^{(0)}_{NS}(x) \ln(M^2/M_0^2) \), playing the role of “bare” quark distribution function. In the case of the pointlike part (20) this bare distribution does, however, depend on the scale \( M \), and its derivative with respect to \( \ln M^2 \) generates the inhomogeneous term in (5).

Formally \( q^{PL}_{NS}(x, M_0, M) \) as given in (18,20) can be considered also for \( M < M_0 \), but is negative there. For \( M/M_0 \to \infty \) the second term in brackets of (18) vanishes and therefore all the pointlike solutions share the same large \( M \) behaviour

\[ q^{PL}_{NS}(x, M_0, M) \to \frac{4\pi}{\alpha_s(M)} a_{NS}(x) \equiv q^{AP}_{NS}(x, M) \propto \ln \frac{M^2}{\Lambda^2}, \]  
(21)

defining the so called asymptotic pointlike solution \( q^{AP}_{NS}(x, M) \) \([12, 13]\). Note that this asymptotic pointlike solution is a very special case of the general pointlike one (18), because for this solution the lower integration limit \( M_0 \) in (20) has been identified with \( \Lambda \), i.e. \( M_0 = \Lambda! \) The fact that for the asymptotic pointlike solution (21) \( \alpha_s(M) \) appears in the denominator of (21) has been the source of claims (see, for instance, [1]) that \( q(x, M) = O(1/\alpha_s) \). This claim is wrong for two reasons. First, it is obviously invalid for those of the currently used parameterizations that distinguish the pointlike and hadronic components of photonic PDF. For instance, the widely used Schuler–Sjöstrand sets SaS1 and SaS2, which take \( M_0 = 0.6 \) GeV and \( M_0 = 2 \) GeV, respectively are not asymptotic pointlike solutions and therefore manifestly do not behave as \( \alpha/\alpha_s \). However, as argued in [14] this
claim is misleading even for the asymptotic pointlike solution. To see the point consider the limit \( \Lambda \to 0 \) for fixed \( M_0 \). We easily see that

\[
q_{\text{PL}}(n, M, M_0) \to \frac{\alpha}{2\pi} k_{\text{NS}}(n, M) \ln \frac{M^2}{M_0^2},
\]

i.e. (18) reduces to moments of the first term in the expansion (20), corresponding to pure QED splitting \( \gamma \to q\bar{q} \). The fact that the asymptotic pointlike solution (21), for which \( M_0 = \Lambda \), diverges when \( \Lambda \to 0 \) is then a direct consequence of the fact that for this (and only this) pointlike solution the decrease of the coupling \( \alpha_s(M/\Lambda) \) as \( \Lambda \to 0 \) is overridden by the simultaneous extension of the integration region as \( M_0 \to 0 \)! In other words, if QCD is switched off by sending \( \Lambda \to 0 \) without simultaneously extending the integration region, i.e. for fixed \( M_0 \), there is no trace of QCD left and we get back the simple QED formula.

In summary, the pointlike part \( q_{\text{PL}}^{\text{NS}}(x, M) \) of the quark distribution function of the photon is of the order \( \alpha \) and not of the order \( \alpha/\alpha_s \), as often claimed. One can use the latter only as a shorthand for the specification of large \( M \) behaviour as expressed in (21). In fact, this is what one finds in the original papers [12, 13] which do not contain any explicit claim that \( q(x, M) = \mathcal{O}(1/\alpha_s) \). To take this behaviour literally leads, as shown in the next Section, to incorrect conclusions.

3 Factorization scheme analysis of \( F_2^\gamma(x, Q^2) \)

First a general comment. All the splitting and coefficient functions discussed below can be calculated using one of two different approaches

- Ultraviolet renormalization of composite operators within OPE.

\[\text{\footnotesize See, for instance, eq. (3.20) of [13].}\]
Infrared renormalization technique anchored in the framework of parton model and based on analysis of Feynman diagrams \[10\].

For the discussion of the transition between the properties of real and virtual photon, discussed in the next Section, the second approach has clear advantage in its simplicity and physical transparency and we shall therefore adopt it also throughout the paper. Moreover we shall work with nonzero quark masses \( m_q \), which provide natural regulators of parallel singularity associated with the primary \( \gamma \rightarrow q \bar{q} \) splitting. A simple and straightforward analysis of this splitting (see next Section) yields

\[
C_{\gamma}^{(0)}(x, Q/M) = 3 \left[ (x^2 + (1 - x)^2) \left( \ln \frac{M^2}{Q^2} + \ln \frac{1 - x}{x} \right) + 8x(1 - x) - 1 \right] \tag{23}
\]

Note that when evaluated via ultraviolet renormalization of composite operators using dimensional regularization \( C_{\gamma}^{(0)} \) contains also a term proportional to \( \ln \frac{4\pi}{\gamma_E} \). However, this term has nothing to do with QCD and is related exclusively to the renormalization of electromagnetic couplant \( \alpha_1 \).

The reason why the existing NLO analyses of \( F_{\gamma}^2(x, Q^2) \), like \[1, 15, 16, 17, 18, 19\], are incomplete is that they all take literally the relation \( q \propto \frac{1}{\alpha_s} \). This, in turn, leads to several incorrect conclusions:

- The term \( C_q^{\gamma} \) in \[12\], which starts at the order \( O(\alpha) \), is claimed to be of the NLO in \( \alpha_s \) with respect to \( q(x, M) \), while in fact it is of the same order as \( q(x, M) \).
- The term in \[15\] proportional to \( \alpha C_{\gamma}^{(0)} \) is retained (though misleadingly assigned to NLO), while the term proportional to \( \alpha \alpha_s C_{\gamma}^{(1)} \) is discarded, while in complete NLO analysis it must be retained as well.
- Similarly in the case of splitting functions \[9–11\]: while in the homogeneous splitting functions terms proportional to \( \alpha_2 \) \( P^{(1)}_{ij} \) are retained, the inhomogeneous splitting functions of the same order, \( \alpha_2 k^{(2)}_q, \alpha_2 G^{(2)}_q \) are discarded, while they should be kept as well.

As a result, all existing analyses write \( F_{\gamma}^2 \) at the LO in the form

\[
F_{\gamma}^{\gamma,\text{LO}}(x, Q^2) = \sum_q 2x e_q^2 q(x, M), \tag{24}
\]

where \( q(x, M) \) satisfy LO evolution equations, which take into account only \( k_q^{(0)} \) and \( P_{ij}^{(0)} \) splitting functions and at the NLO use the expression

\[
F_{\gamma}^{\gamma,\text{NLO}}(x, Q^2) = \sum_q 2x e_q^2 q(x, M) + \sum_q 2x e_q^2 \left( \frac{\alpha_s(M)}{2\pi} \left[ q(M) \otimes C_{\gamma}^{(1)}(Q/M) + G(M) \otimes C_{\gamma}^{(1)}(Q/M) \right] + \frac{\alpha}{2\pi} e_q^2 C_{\gamma}^{(0)}(x, Q/M) \right), \tag{25}
\]

where the evolution equations for quark and gluon distribution functions include terms up to \( k_q^{(1)}, k_G^{(1)} \) and \( P_{ij}^{(1)} \). However, taking into account that \( q(x, M) = O(\alpha) \), correct LO and NLO expressions for \( F_{\gamma} \) read instead as follows

\[
F_{\gamma}^{\gamma,\text{LO}}(x, Q^2) = \sum_q 2x e_q^2 \left( q(x, M) + \frac{\alpha}{2\pi} e_q^2 C_{\gamma}^{(0)}(x, Q/M) \right), \tag{26}
\]

\[11\] See also discussion following eq. \[28\] below.
\[ F_2^{\gamma,{\text{NLO}}} (x, Q^2) = \sum_q 2xe_q^2 \left( q(x, M) + \frac{\alpha_s}{2\pi} e_q^2 C^{(0)}_s (x, Q/M) \right) + \]
\[ \sum_q 2xe_q^2 \left( C^{(1)}_s (Q/M) + G(M) \right) \]  
\[ + \frac{\alpha_s}{2\pi} \frac{\alpha_s(M)}{2\pi} e_q^2 C^{(1)}_G (Q/M) \],

where PDF in (27) satisfy evolution equations with splitting functions (9–11) up to the order \( \alpha_s^2 (M) \), i.e. including terms proportional to \( k_q^{(2)} \), \( k_G^{(2)} \). The difference of (24–25) with respect to (24 25) is threefold:

- The term proportional to \( C^{(0)}_s \) is part of the LO expression.

- The NLO expression in (27) contains, beside the usual partonic convolutions \( q \otimes C^{(1)}_q \) and \( G \otimes C^{(1)}_G \) also the NLO direct term proportional to \( C^{(1)}_s \). This additional term is crucial for the consistency of the NLO approximation. As shown below, it also cancels part of the factorization scale and scheme dependence of the NLO hadronic terms and thus contributes to theoretical stability of NLO calculations.

- In the NLO evolution equations inhomogeneous splitting functions \( k_i^{(2)} \), \( i = q, G \) of the order \( \alpha_s^2 \) are retained.

We note that in the case of the proton the expressions for \( F_2^p (x, Q^2) \) at LO and NLO differ from those in (26–27) merely by the absence of the terms proportional to \( C^{(0)}_s \) and \( C^{(1)}_s \). For \( F_2^p (x, Q^2) \) factorization scale and scheme dependence of parton distribution functions is cancelled by explicit factorization scale and scheme dependence of the coefficient functions \( C^{(i)}_q, C^{(i)}_G, i \geq 1 \). Concretely, at the NLO the scale and scheme dependence of quark distribution functions in the first (i.e. LO) term on the r.h.s. of (27) is cancelled to order \( \alpha_s \) by the corresponding dependence of NLO coefficient coefficients \( C^{(1)}_q, C^{(1)}_G \). This reflects the fact that logarithmic derivatives of PDF of the proton with respect to \( \ln M \) and FS start at the order \( \alpha_s \) and also implies that at the LO the relation between \( F_2^p (x, Q^2) \) and quark distribution functions is unique. Moreover, a precise physical interpretation of the factorization scale \( M \) is of little relevance because any redefinition of \( M \) can be compensated by appropriate change of the NLO splitting functions \( F_{kl}^{(1)} \).

For photon the inhomogeneous term in evolution equations for quark distribution functions modifies this cancellation mechanism because part of the factorization scale dependence driven by this inhomogeneous term is compensated already at the LO by the term in \( C^{(0)}_s \) proportional to \( k_q^{(0)} \ln(Q^2/M^2)! \) The presence of such a term in \( C^{(0)}_s \), as well as the fact that \( C^{(1)}_s \) contains the term proportional to \( k_q^{(1)} \), are general consequences of the RG invariance of \( F_2^\gamma (x, Q^2) \). For photon the interpretation of factorization scale \( M \) therefore does matter, as it determines LO coefficient \( C^{(0)}_s \). Note that \( M^2 \) in the (28) must be interpreted as maximal transverse momentum squared of the virtual quark included in the resummation (23).

To see explicitly how the cancelation of factorization scheme dependence works at the NLO and why it requires the inclusion of the term \( C^{(1)}_s \), let us follow the standard procedure (1) of redefining quark distribution functions of the photon, but include terms up to the order \( \alpha \alpha_s \):
\[ \tilde{q}(x, M) \equiv q(x, M) + \frac{\alpha}{2\pi} e_q^2 \gamma (x) + \frac{\alpha_s(M)}{2\pi} e_q^2 E_\gamma (x). \]  

It is straightforward to show that provided \( q(x, M) \) satisfies the evolution equation with inhomogeneous splitting functions \( k_q^{(0)}, k_q^{(1)} \) and \( k_q^{(2)} \), the tilded distribution function \( \tilde{q}(x, M) \) satisfies the
same evolution equation, but with tilded inhomogeneous splitting and coefficient functions given as:

\[
\begin{align*}
\tilde{k}_q^{(0)} &= k_q^{(0)}, \\
\tilde{k}_q^{(1)} &= k_q^{(1)} - e_q^2 P^{(0)}_{qq} \otimes D_\gamma, \\
\tilde{k}_q^{(2)} &= k_q^{(2)} - e_q^2 \left( P^{(1)}_{qq} \otimes D_\gamma + P^{(1)}_{q\gamma} \otimes E_\gamma + \beta_0 E_\gamma/2 \right), \\
\tilde{C}_q^{(0)} &= C_q^{(0)} - D_\gamma, \\
\tilde{C}_q^{(1)} &= C_q^{(1)} - C_q^{(1)} \otimes D_\gamma - E_\gamma.
\end{align*}
\]

On the other hand, NLO homogeneous splitting functions \(P_{ij}^{(1)}\) are unchanged by the redefinition (28). The relations (30–31) imply that instead of the functions \(D_\gamma(x)\) and \(E_\gamma(x)\) this redefinition can equally well be parameterized by the inhomogeneous splitting functions \(k_q^{(1)}, k_q^{(2)}\).

Despite the fact that \(D_\gamma\) is related by (33) to the splitting function \(k_q^{(1)}\) standing in \(\Gamma\) by \(\alpha_s\), only the term proportional to \(\alpha_s E_\gamma\) is related to genuine QCD effects. This is indicated already by the observation that solutions of the equation

\[
\frac{dq(x, M)}{d\ln M^2} = \frac{\alpha}{2\pi} k_q^{(0)}(x)
\]

to which the evolution equation (6) reduces in the limit \(\alpha_s \to 0\) are determined up to an arbitrary function of \(x\), i.e. just like the term in (28) proportional to \(D_\gamma(x)\). The fact that this term in (28) has nothing to do with QCD follows also from the fact that it can be fully included in the redefinition of the lower integration bound \(M_0^2\) in (28) by setting \(M_0^2(x) = M_0^2 \exp(-D_\gamma(x)/(x^2 + (1 - x)^2))\). With such a choice of the lower integration bound the result of the resummation in (28) does not vanish at any fixed initial scale, but still solves the same inhomogeneous evolution equation and therefore represents legitimate pointlike solution.

It is also illustrative to see how the redefinition (28) affects separately the pointlike and hadronic parts of quark distribution functions. For the term containing \(D_\gamma\) the answer can be taken from the analysis in \([13]\), where explicit expressions for moments \(F_{\gamma}(n, M^2)\) as functions of \(k_q^{(1)}(n)\) are given. One finds that the variation of \(k_q^{(1)}\) by \(\delta k_q^{(1)} \equiv k_q^{(1)} - k_q^{(1)} = -e_q^2 P^{(0)}_{qq} \otimes D_\gamma\) modifies the pointlike part \(q^{PL}(n, M)\) at all scales \(M\) by the term (31)

\[
\delta q^{PL}(n, M) = -\frac{\alpha}{2\pi} \left[ 1 - \left( \frac{\alpha_s(M)}{\alpha_s(M_0)} \right)^{-d(n)} \right] \frac{\delta k_q^{(1)}}{P^{(0)}_{qq}(n)} = \frac{\alpha}{2\pi} \left[ 1 - \left( \frac{\alpha_s(M)}{\alpha_s(M_0)} \right)^{-d(n)} \right] e_q^2 D_\gamma(n),
\]

while in the hadronic part only the initial condition at \(M_0\) is changed

\[
\delta q^{HAD}(n, M_0) = -\frac{\alpha}{2\pi} \frac{\delta k_q^{(1)}}{P^{(0)}_{qq}(n)} = \frac{\alpha}{2\pi} e_q^2 D_\gamma(n).
\]

The NLO splitting functions remain, however, unchanged. Summing (35) with (36) multiplied by the evolution factor \((\alpha_s(M)/\alpha_s(M_0))^{-d(n)}\) yields the second term on the r.h.s. of (28).

4 Structure of the virtual photon

4.1 Equivalent photon approximation

All present knowledge of the structure of the photon comes from experiments at the ep and e\(^+\)e\(^-\) colliders, where incoming leptons act as sources of transverse and longitudinal virtual photons.\(^{12}\)

\(^{12}\)In a typical “photoproduction” experiment at HERA the average photon virtuality \(\langle P^2 \rangle \approx 10^{-2} - 10^{-3}\) GeV\(^2\).
To order $\alpha$ their respective unintegrated fluxes are given as

$$f_{T}^{\gamma}(y, P^2) = \frac{\alpha^2}{2\pi} \left( \frac{1 + (1 - y)^2}{y} \frac{1}{P^2} - \frac{2m^2e^2y}{P^4} \right),$$  \hspace{1cm} (37)$$

$$f_{L}^{\gamma}(y, P^2) = \frac{\alpha^2}{2\pi} \frac{2(1 - y)}{y} \frac{1}{P^2}. \hspace{1cm} (38)$$

The transverse and longitudinal fluxes thus coincide for $y = 0$, while at $y = 1$, $f_{L}^{\gamma}$ vanishes. Note that while for $P^2 \gg m^2$ the second term in (37) is negligible with respect to $1/P^2$, for $P^2$ close to $P^2_{\text{min}} = m^2y^2/(1 - y)$ their ratio is finite and approaches $2(1 - y)/(1 + (1 - y)^2)$. 

4.2 From virtual to real photons

The choice of a way mass singularities resulting from the primary $\gamma^{*} \rightarrow q\bar{q}$ splitting are regularized is crucial for smooth and physically transparent transition between the properties of virtual and real photons. As far as perturbative calculations of coefficient functions (hard scattering cross-sections in general) are concerned, there is no principal difference in this respect between QED and QCD. For PDF of the photon the situation is, however, more complicated. The point is that in QCD the transition from virtual to real photon is expected to be determined by nonperturbative parameters related to colour confinement, rather than the quark masses, which govern such transition in pure QED. Nevertheless, it is illustrative to see how this transition is realized in QED, where the “lepton distribution functions” of the photon, are explicitly calculable.

In QED masses of charged fermions, in particular electron, play a fundamental role. For real photon parallel singularity associated with the purely QED splitting $\gamma \rightarrow q\bar{q}$ is naturally regulated by quark and lepton masses. For virtual photon also its virtuality $P^2$ shields off this singularity but quark and lepton masses are indispensable for proper limiting behaviour of lepton (quark) distribution functions of the virtual photon as $P^2 \rightarrow 0$. The effects of nonzero photon virtuality $P^2$ are threefold:

- longitudinally polarized photons must be taken into account,
- unintegrated PDF of transversally polarized photons contain terms proportional to $P^2$,
- parton level cross-sections obtain contributions from terms proportional to $P^2$. 

Figure 3: Kinematics of the basic $\gamma^{*}(P^2) \rightarrow q\bar{q}$ splitting in QED.
4.3 Parton model interpretation of constant terms in $C_\gamma^{(0)}$

The evaluation of quark distribution functions of the virtual photon in pure QED serves as a useful guide to parton model interpretation of the so called “constant terms” in LO coefficient function $C_\gamma^{(0)}$ and leads to explicit formulae for the virtuality dependence of quark distribution functions of the photon.

Consider, for definiteness, the production of a heavy scalar particle $H$ with mass $M_H$ in electron–proton collisions. The relevant parton level hard scattering subprocess

$$\gamma(P^2, k_a) + q(p) \rightarrow e(k_c) + H(p_H) \quad (39)$$

is described by the sum of amplitudes corresponding to diagrams in Fig. 3a,b. In pure QED and to order $\alpha$ the probability of finding inside the photon of virtuality $P^2$ a quark with mass $m_q$, electric charge $e_q$, momentum fraction $x$ and virtuality $\tau = m_q^2 - k_b^2$ up to $M^2$ is given by the t–channel diagram in Fig. 3a.\(^3\)

$$q_{\text{QED}}(x, m_q^2, P^2, M^2) = \left( \frac{\alpha}{2\pi} \frac{3e_q^2}{4} \int_{\tau_{\text{min}}}^{M^2} \frac{W_i(x, m_q^2, P^2)}{\tau^2} \, d\tau \right) \quad (40)$$

In the collinear kinematics, which is relevant for finding the lower limit on $\tau$, the values of $m_q$, $x$, $\tau$ and $p_T$ are related to initial photon virtuality $P^2$ as follows

$$\tau = xP^2 + \frac{m_q^2}{1-x} + p_T^2, \quad \Rightarrow \quad \tau_{\text{min}} = xP^2 + \frac{m_q^2}{1-x}. \quad (41)$$

In terms of $M_H$ and $s \equiv (k_a + p)^2 = M_H^2/s$. $W(x, m_q^2, P^2, M^2)$ can in general be written as

$$W(x, m_q^2, P^2, M^2) = f(x) \tau + \frac{\tau^2}{s} + \cdots$$

where the dots indicate term of the type $\tau^{k+1}/s^k, k \geq 1$. There is no term proportional to $s$ as it would violate unitarity. The functions $f(x), g(x)$ and $h(x)$ are unique functions that can be determined from the analyses of the vertex $\gamma \rightarrow q\overline{q}$ in collinear kinematics. On the other hand, $c(x)$ is a process dependent. The terms in $\text{(41)}$ proportional to $f(x), g(x)$ and $h(x)$ are dominated by small quark virtualities and have therefore clear parton model interpretation: so long as $\tau \ll M_H^2$, eq. (41) describes the flux of quarks that are almost collinear with the incoming photon and “live” long with respect to the production time of the heavy particle $H$ in the lower vertex in Fig. 3a.

Substituting (41) into (40) and performing the integration gives, in units of $3e_q^2\alpha/2\pi$,

$$q_{\text{QED}}(x, m_q^2, P^2, M^2) = f(x) \ln \left( \frac{M^2}{\tau_{\text{min}}} \right) + f(x) + \frac{g(x)m_q^2 + h(x)P^2}{\tau_{\text{min}}} \left( 1 - \frac{\tau_{\text{min}}}{M^2} \right) + c(x) \frac{M^2}{s}. \quad (43)$$

In practical applications the factorization scale $M$ is identified with some kinematical variable characterising hardness of the interaction, like $Q^2$ in DIS or $M_H$ in our process $\text{(39)}$. As a result $M^2/s$ becomes a function of $x$ and we can write $c(x)M^2/s = \kappa(x)$. For $\tau_{\text{min}} = xP^2 + m_q^2/(1-x) \ll M^2$ (43) simplifies to

$$q_{\text{QED}}(x, m_q^2, P^2, M^2) = f(x) \ln \left( \frac{M^2}{xP^2 + m_q^2/(1-x)} \right) - f(x) + \frac{g(x)m_q^2 + h(x)P^2}{xP^2 + m_q^2/(1-x)} + \kappa(x). \quad (44)$$

The s–channel diagram in Fig. 3b is, however, crucial for preserving gauge invariance in the case of the longitudinal photon where it cancels one of the terms originating from Fig. 3a that does not vanish for $P^2 \rightarrow 0$.\(^3\)
This basic result, which takes into account nonzero quark mass \( m_q \) as well as initial photon virtuality \( P^2 \), reduces for \( x(1-x)P^2 \gg m^2 \) to

\[
q_{\text{QED}}(x,0,P^2,M^2) = f(x) \ln \left( \frac{M^2}{xP^2} \right) - f(x) + \frac{h(x)}{x} + \kappa(x),
\]

(45)

whereas for \( P^2/m_q^2 \rightarrow 0 \) it approaches

\[
q_{\text{QED}}(x,m_q^2,0,M^2) = f(x) \ln \left( \frac{M^2(1-x)}{m_q^2} \right) - f(x) + g(x)(1-x) + \kappa(x),
\]

(46)

describing distribution of quarks inside the real photon. The fact that (45) requires nonzero quark mass reflects the fact that in QED masses of charged fermions play vital role and cannot be zero.

As in the case of the photon fluxes (28,29) the leading logarithmic term, dominant for large \( M^2 \), as well as the “constant” terms proportional to \( f(x), g(x) \) and \( h(x) \) come entirely from the integration region close to \( \tau_{\text{min}} \) and are therefore unique. At \( \tau = \tau_{\text{min}} \) both types of the singular terms in (11), i.e. \( 1/\tau \) and \( m_q^2/\tau^2 \) or \( P^2/\tau^2 \), are of the same order but the faster fall-off of the \( 1/\tau^2 \) terms implies that for large \( M^2 \) the integral in (11) is dominated by the weaker singularity \( 1/\tau \). In other words, while the logarithmic term is dominant at large \( M^2 \), the constant terms resulting from nonzero \( m^2 \) and/or \( P^2 \) come from the kinematical configurations which are even more collinear than those giving the logarithmic term. On the other hand, not all constant terms are of this origin, as exemplified by the term proportional to \( \kappa(x) \) which comes from the integration over the whole phase space and is therefore process dependent. These terms have no parton model interpretation and belong therefore naturally to the coefficient function \( C^{(0)}_\gamma \).

The analysis of the vertex \( \gamma(P^2) \rightarrow q\bar{q} \) in collinear kinematics or the explicit evaluation of the diagrams in Fig. 3 yields the following results

\[
\begin{align*}
f_T(x) &= x^2 + (1-x)^2, & g_T(x) &= \frac{1}{1-x}, & h_T(x) &= 0, \\
f_L(x) &= 0, & g_L(x) &= 0, & h_L(x) &= 4x^2(1-x).
\end{align*}
\]

(47)

The vanishing of \( f_L(x) \) and \( g_L(x) \) is a consequence of gauge invariance, which guarantees that \( \gamma_L \) decouples in the limit \( P^2 \rightarrow 0 \). The fact that \( h_T(x) = 0 \) is due to helicity conservation, which permits violation by quark mass terms only. The expressions (43-44) exhibit explicitly the smooth transition between the quark distribution functions of the virtual and real photon in pure QED, governed by the ratio \( P^2/m_q^2 \).

For virtual photon eq. (12) holds separately for both transverse and longitudinal polarizations. For \( x(1-x)P^2 \gg m^2 \) the finite terms \( C^{(0)}_{\gamma,T},C^{(0)}_{\gamma,L} \) are given as (20)

\[
\begin{align*}
C^{(0)}_{\gamma,T}(x,P^2,Q/M) &= 3 \left( x^2 + (1-x)^2 \right) \left( \ln \frac{M^2}{Q^2} + \ln \frac{1}{x^2} \right) + 8x(1-x) - 2, \\
C^{(0)}_{\gamma,L}(x,P^2,Q/M) &= 4x(1-x),
\end{align*}
\]

(48)

(49)

while for \( P^2 = 0 \)

\[
\begin{align*}
C^{(0)}_{\gamma,T}(x,0,Q/M) &= 3 \left( x^2 + (1-x)^2 \right) \left( \ln \frac{M^2}{Q^2} + \ln \frac{1}{x} \right) + 8x(1-x) - 1, \\
C^{(0)}_{\gamma,L}(x,0,Q/M) &= 0
\end{align*}
\]

(50)

(51)

In combination with (43) and (47) these results imply \( \kappa(x) = -1 + 6x(1-x) \). The origins of the nonlogarithmic parts of \( C^{(0)}_\gamma \) in (48,50) can then be identified as follows:
Real photon:
\[
-1 + 8x(1 - x) = \underbrace{-2 + 8x(1 - x)}_{\text{for massless quark}} + \underbrace{\frac{1}{g^T(x)(1-x)}}_{\text{nonuniversal part}}
\]
\[
= \underbrace{-1 + 6x(1 - x)}_{\text{nonuniversal part}} - \underbrace{\left(x^2 + (1 - x)^2\right)}_{f^T(x)} + \underbrace{1}_{g^T(x)(1-x)}
\]
(52)

Virtual photon:
\[
-2 + 8x(1 - x) = -1 + 6x(1 - x) - \left(x^2 + (1 - x)^2\right)
\]
(53)

As the virtuality \(P^2\) (or more precisely \(\tau_{\text{min}}\)) increases toward the factorization scale \(M^2\), the expression (13) for the quark distribution functions of the virtual photon approaches zero. We emphasize that this holds not only for the logarithmic term but for all terms that have parton model interpretation.

4.4 The real world

In realistic QCD nonperturbative effects, in particular those connected with the confinement, are expected to determine the structure of the real photon as well as its virtuality dependence. For instance, within the Schuler–Sjöstrand set of parameterizations [21] the role of such parameter is played by vector meson masses for the hadronic component and by the initial \(M_0\) for the pointlike one. Nevertheless, the analysis of the previous subsections is still relevant for the coefficient function \(C^{(0)}_\gamma\) as well as the discussion of virtuality dependence of the pointlike part of quark distribution functions.

5 Phenomenological implications

The results of previous Sections have several implications for phenomenological analyses of hard scattering processes with photons in the initial state. Some of them are discussed below.

5.1 LO and NLO analysis of \(F^\gamma_2(x, Q^2)\) for the real photon

As argued in detail in Section 3, the existing analyses of \(F^\gamma_2(x, Q^2)\) are incomplete due to the fact that they do not take into account

- the NLO photonic coefficient function \(C^{(1)}_\gamma(x)\) in expression (27) for \(F^\gamma_2\) an
- the NLO inhomogeneous splitting functions \(k^{(2)}_q\) and \(k^{(2)}_{G}\) in evolution equations (3-5).

There is no problem to remedy the first shortcoming, because the coefficient function \(C^{(1)}_\gamma(x)\) is actually known. As argued in [22] it can be deduced from the \(O(\alpha_s^2)\) gluonic coefficient function calculated in [23]. For illustration of its numerical importance, \((\alpha_s(M)/\pi)C^{(1)}_\gamma(x)\) in \(\overline{\text{MS}} + \text{MS}\) scheme is compared in Fig. 4a to \(2C^{(0)}_\gamma(x)\) as well as to \((4\pi/\alpha_s^2)u(x)\) and \((4\pi/\alpha_s^2)u(x)\) corresponding to SaS1M parameterization. In this comparison we took \(\alpha_s(M)/2\pi = 0.03\), which is approximately the value of \(\alpha_s\) at \(M = 10\) GeV. Note that the contribution of \(C^{(1)}_\gamma\) to \(F^\gamma_2(x, Q^2)\) is comparable to that of \(C^{(0)}_\gamma\). As in most experiments \(Q^2\) is well below 100 GeV^2, the relative importance of \(C^{(1)}_\gamma\) is
even bigger. We also see that for $x \sim 0.4$ sum of the contributions of coefficient functions $C^{(0)}$ and $C^{(1)}$ to $F^2_{\gamma}$ is almost the same as the contributions of quark distribution functions themselves.

As far as the NLO inhomogeneous splitting functions $k^{(2)}_q(x)$ and $k^{(2)}_G(x)$ are concerned, the situation is different. These functions are unknown as they cannot be derived from the existing NLO calculations. To obtain them requires complete NNLO calculation, similar to that in [24] for moments of nucleon structure functions. No such calculation is, however, in sight. In the absence of $k^{(2)}_q(x)$ and $k^{(2)}_G(x)$ a complete NLO analysis of $F^2_{\gamma}$ is impossible, but we may investigate what happens if at least the effects of $C^{(1)}$ are taken into account. The simplest way to estimate these effects is to evaluate the difference

$$q'(x, M) \equiv q(x, M) - \frac{\alpha_s(M)}{2\pi} \frac{\alpha_s(M)}{2\pi} e_q^2 C^{(1)}(x),$$

(54)

defining quark distribution functions which when inserted into (27) including the NLO coefficient function $C^{(1)}$ return the same $F^2_{\gamma}$ as $q(x, M)$ inserted into (27) without the $C^{(1)}$ term. For $u$-quark the effect of the redefinition (54) is shown in Fig. 4b, using SaS1M parameterization of $u(x, M)$ at $M = 10$ GeV. The reduction of $u(x, M)$ due to the inclusion of $C^{(1)}$ is substantial for all $0.05 \leq x \leq 0.9$.

In the situation when full NLO calculation of $F^2_{\gamma}$ is impossible due to the lack of knowledge of $k^{(2)}_q$ and $k^{(2)}_G$ and Fig. 4 indicates the importance of $C^{(1)}$ term, LO analyses of $F^2_{\gamma}$ and other quantities involving initial photons is all we can consistently do. In such analyses we can exploit the freedom in the choice of the factorization scale and scheme. Contrary to the case of hadrons, these choices influence the relation between PDF and $F^2_{\gamma}$ (and other physical quantities) already at the LO through factorization scale and scheme dependence of $C^{(0)}$. It is therefore not true, as claimed for instance in [25], that the SaS1M and SaS2M sets of parameterizations are “theoretically inconsistent” because they combine in LO expression for $F^2_{\gamma}$ the “NLO” quantity $C^{(0)}$ with the LO quark distribution functions. The MS sets of SaS parameterizations are as legitimate definitions of the LO quark distribution functions as their DIS companions and there is no theoretical reason why, as suggested in [25], they should not be used in phenomenological analyses.
5.2 What is measured in DIS on virtual photons?

In experiments [3, 4, 5, 6] at e+e− colliders structure of the photon was investigated via standard DIS on the photon with small but nonzero virtuality \( P^2 \). The resulting data were used in [18, 19] to determine PDF of the virtual photon in LO and NLO approximations. In these analyses \( C_{\gamma}^{(0)} \) was taken in the form

\[
C_{\gamma}^{(0)}(x, P^2, Q/M) = 3 \left[ (x^2 + (1-x)^2) \left( \ln \frac{M^2}{Q^2} + \ln \frac{1}{x^2} \right) + 6x(1-x) - 2 \right],
\]

(55)

which, however, does not correspond to the structure function that is really measured in e+e− collisions, but to the following combination

\[
F_{\gamma,2}^\gamma(x, P^2, Q^2) \equiv F_{\gamma,2,T}^\gamma(x, P^2, Q^2) - \frac{1}{2} F_{\gamma,2,L}^\gamma(x, P^2, Q^2)
\]

(56)

of structure functions corresponding to transverse and longitudinal polarizations of the target photon. This combination results after averaging over the target photon polarizations by means of contraction with the tensor \(-g_{\mu\nu}/2\). The term \(-2 + 6x(1-x)\) follows also directly from considerations of the previous Section:

\[
-2 + 6x(1-x) = -2 + 8x(1-x) - 2x(1-x)
\]

(57)

Because the fluxes \( f_{\gamma,T}^\gamma(y, P^2) \) and \( f_{\gamma,L}^\gamma(y, P^2) \) of transverse and longitudinal photons are different functions of \( y \), any complete analysis of experimental data in terms of the structure functions \( F_{\gamma,2,T}^\gamma(x, P^2, Q^2) \) and \( F_{\gamma,2,L}^\gamma(x, P^2, Q^2) \) at fixed \( x, P^2, Q^2 \) requires combining data for different \( y \). This is in principle possible, but experimentally difficult to accomplish. The situation is simpler at small \( y \), where \( f_{\gamma,T}^\gamma(y, P^2) = f_{\gamma,L}^\gamma(y, P^2) = f_{\gamma}^\gamma(y, P^2) \), and the data therefore correspond to the convolution of \( f_{\gamma}^\gamma(y, P^2) \) with the sum \( F_{\gamma,2,T}^\gamma + F_{\gamma,2,L}^\gamma \). The nonlogarithmic term in \( C_{\gamma}^{(0)} \) appropriate to this combination is, however, not \(-2 + 6x(1-x)\), as in [35] and [18, 19] but \(-2 + 12x(1-x)\), the sum of nonlogarithmic terms corresponding to transverse and longitudinal photons. Numerically the difference between these two expressions is quite important.

5.3 Longitudinal gluons inside hadrons?

If there are longitudinal photons inside leptons, what about longitudinal gluons inside hadrons? Although real longitudinal gluons decouple, as do real longitudinal photons, gluons as well as quarks inside hadrons are off–shell and therefore there in no reason not to introduce distribution functions of transverse and longitudinal gluons as well. The problem is, however, that while in QED the fluxes \( f_{\gamma,T}^\gamma(x, P^2) \) and \( f_{\gamma,L}^\gamma(x, P^2) \) are known, we do not know how to calculate analogous fluxes of transverse and longitudinal gluons inside hadrons. In fact it would be sufficient to know the relative size of these fluxes, but even this is not calculable. Nevertheless, guided by the situation for photon fluxes, we can expect them to be similar, in particular at low \( x \). If this is assumed, the considerations of the preceding subsection applied to gluons imply that the NLO gluonic contribution to \( F_{\gamma}^{\gamma}(x, Q^2) \) comes from the sum of contributions of transverse and longitudinal gluons. This in turn means that the finite nonlogarithmic term in the gluonic coefficient function \( C_{G}^{(1)}(x) \) should be taken as \(-2 + 12x(1-x)\) rather than the usual \(-2 + 6x(1-x)\).
5.4 Virtuality dependence of initial conditions

There are currently two approaches to introducing virtuality dependence of PDF of the photon. Schuler–Sjöstrand base their parameterizations on the idea, suggested in [26], to use dispersion relations in $P^2$ written for moments of $F^\gamma_2(x, P^2, Q^2)$. Their parameterizations do not satisfy the same evolution equations as those of the real photon, but as the difference is formally of power correction type this is no principal drawback.

In a different approach pursued in [13, 14], PDF of the virtual photon are assumed to satisfy the same evolution equations as those of the real photon, and their virtuality dependence is introduced via the virtuality dependence of the initial conditions at $Q^2 = P^2$. This dependence is assumed to interpolate between the standard boundary conditions for the real photon, which in the GRV approach are defined at a very low scale $Q^2 = \mu^2 < 1$ GeV$^2$, and the predictions of perturbation theory

$$D(x, P^2, \bar{P}^2) = \eta(P^2)D_{NP}(x, \bar{P}^2) + [1 - \eta(P^2)]D_{PT}(x, \bar{P}^2), \quad D = q, \bar{q}, G,$$

where $\bar{P}^2 = \max(P^2, \mu^2)$, $\eta(P^2) = (1 + P^2/m_\rho^2)^{-2}$ and

$$D_{NP}(x, \bar{P}^2) = \kappa(2\pi\alpha/f_\rho) \begin{cases} f^\pi(x, P^2), & P^2 > \mu^2 \\ f^\pi(x, \mu^2), & 0 \leq P^2 \leq \mu^2 \end{cases},$$

$$q_{PT}(x, \bar{P}^2) = \kappa(2\pi\alpha/f_\rho) \begin{cases} 0, & \text{LO} \\ 2e^2 \frac{\pi}{2f_\rho^2} \left[ (x^2 + (1 - x)^2) \ln \frac{1}{x} - 2 + 6x(1 - x) \right], & \text{NLO} \end{cases},$$

$$G_{PT}(x, \bar{P}^2) = 0,$$

where $\kappa, f_\rho, \mu^2, f^\pi(x, \mu^2)$ are given in [27, 28]. The above boundary conditions are assumed to be valid in the DIS$_\gamma$ factorization scheme (see next subsection). The difference in (60) between the form of boundary conditions on quark distribution functions at LO and NLO follows again from misassignment of the coefficient function $C_q^{(0)}(x)$ to the NLO. As argued in previous sections this term is actually of the LO and therefore if taken into account in boundary conditions, it must be present in both LO and NLO analyses.

To include it would, however, go against the very essence of the parton model interpretation of virtuality dependent PDF. To retain clear physical interpretation of the factorization scale $Q$, $q(x, P^2, Q^2)$ must vanish as the lower bound on the quark virtuality $\tau^{\min}(x) \propto P^2$ approaches $Q^2$ from below. This occurs explicitly in the expression for all terms, whether logarithmic or not, that have parton model interpretation. The term $\tau^{\min}/Q^2$, which guarantees the vanishing of the nonlogarithmic part of $f^\pi(x, \mu^2)$ as $\tau^{\min} \to Q^2$, is formally of the higher twist and therefore invisible in any other considerations within the leading twist approximation.

5.5 Why DIS$_\gamma$ factorization scheme?

For hadrons the LO relation between $F^\gamma_2$ quark distribution functions is the same in all FS and is also identical to that of the parton model. DIS FS was introduced to retain this simple relation also at the NLO, where different FS correspond to different NLO splitting functions $P_{kl}^{(1)}(x)$ and hard scattering cross–sections (coefficient functions for $F^\gamma_2$). However, once the latter is chosen, the former is uniquely determined (or vice versa). For proton the DIS FS amounts to setting $C_q^{(1)} = 0$.

For photon the situation is different. The DIS$_\gamma$ FS was introduced for real photon to get rid of the troubling $\ln(1-x)$ term in the LO expression for $C_{\gamma}^{(0)}(x, 0, 1)$. It implies setting $D_{\gamma} = C_{\gamma}^{(0)}$.

---

\[14\]To retain the notation of [18, 19] I use in this subsection $Q^2$ instead of $M^2$ to denote the factorization scale.
Figure 5: Exact as well as approximate bounds on the ratio $\tau/m_q^2$ for real photons and massive quarks (a) and on the ratio $\tau/P^2$ for virtual photons coupled to massless quarks (b). In both cases solid and dashed curves correspond to exact and approximate bounds respectively, plotted for four values of the ratio $Q^2/m^2$ in a) and $Q^2/P^2$ in b).

and $E_\gamma = 0$ in (28) and thus modifies the inhomogeneous splitting functions $k^{(1)}$ and $k^{(2)}$, but does not change the NLO homogeneous splitting functions $F^{(1)}_{kl}$. As emphasized in Section 3 this redefinition of quark distribution functions has in fact little to do with QCD. In parton model derivation of Section 4, the troubling term $\ln(1-x)$ is a direct consequence of the fact that for real photon parallel singularity associated with the splitting $\gamma \to q\bar{q}$ is regulated by means of quark masses. This is natural in QED, but as emphasized in subsection 4.2 not in QCD, where nonperturbative properties of hadrons (with the exception of pseudoscalar mesons) and the photon are expected to result from the dynamics of confinement and we can therefore start from massless quarks.

Note that the term $\ln(1-x)$ in $C^{(0)}(x,0,1)$ causes problems primarily because it appears there decoupled from the value of the quark mass $m_q$ with which it originally entered the considerations in the expression (11) for $\tau_{\text{min}}$ and thus persists there even in the limit $m_q \to 0$. It comes from the lower bound on the quark virtuality $\tau$ in collinear kinematics

$$\tau_{\text{coll min}} \equiv \frac{m_q^2}{1-x} \leq \tau \leq \tau_{\text{coll max}} \equiv \frac{Q^2}{x}.$$  

The ratio $Q^2(1-x)/xm_q^2$ of upper an lower integration limits in (22) equals unity (and thus the integral vanishes) at $x_m \equiv 1/(1 + m_q^2/Q^2)$, which for light quarks is very close to 1. For $x \to 1$ collinear kinematics is no longer appropriate and should be replaced with the exact one

$$\frac{y 1 - \sqrt{1-z}}{2x} \leq \frac{\tau}{m_q^2} \leq \frac{y 1 + \sqrt{1+z}}{2x},$$

where $y \equiv Q^2/m_q^2$ and $z \equiv 4yx/(1-x)$. These exact bounds are shown, for several values of $y$, in Fig. 3, by the solid curves, together with the dashed ones, corresponding to the approximate bounds as given in (22). Also shown by the dotted curves are the functions $1/x$ and $1/(1-x)$.

On the other hand when the term $\ln((1-x)/x)$ alone is added (as part of the expression for $C^{(0)}_\gamma$) to the pointlike solution of the inhomogeneous evolution equation with initial conditions at
\( M_0 \gg m_q^2 \), the sum vanishes already at \( x_M \equiv 1/(1 + M_0^2/Q^2) \ll x_m \) and becomes independent of \( m_q \) down to \( m_q = 0 \)!

However, it would be strange to retain in \( C_{\gamma}^{(0)} \) the term that follows directly from assuming nonzero quark masses, but which when combined with the PDF is then unrelated to their values. In the presence of confinement it seems more appropriate to start from the very beginning with massless quarks coupled to off-shell photons and then construct the limit \( P^2 \to 0 \).

This approach is even more natural taking into account that all current data on the structure of the “real” photon actually come from interactions of virtual photons, albeit with small virtuality.

For them the bounds on quark virtuality in collinear kinematics

\[
xP^2 \leq \tau \leq \frac{Q^2}{x}
\]

are only slightly changed using exact kinematics

\[
1 + \frac{(1 - x)y (1 - \sqrt{1 + z})}{x} \leq \frac{\tau}{P^2} \leq 1 + \frac{(1 - x)y (1 + \sqrt{1 + z})}{x}
\]

where now \( y \equiv Q^2/P^2 \) and \( z \equiv 1/(1 - x) - xy/(1 - x) \). The modification of (64) is so small that some of the solid curves in Fig. 3b are indistinguishable from the corresponding dashed one. Our suggestion is therefore to use also for the real photon the expression

\[
C_{\gamma}^{(0)}(x, 0, 1) = \left[ x^2 + (1 - x)^2 \right] \ln \frac{1}{x^2} - 2 + 8x(1 - x),
\]

derived for the transverse photon with virtuality \( P^2 \gg m_q^2 \). This term is different from (55) normally used for the virtual photon, as the latter corresponds to \( F_{2,\gamma}^{\perp} - F_{2,\perp}^{\perp}/2 \).

6 Conclusions

We have discussed the factorization mechanism for \( F_2^\gamma(x, Q^2) \) and pointed out the differences with respect to the case of proton structure function \( F_p^2(x, Q^2) \) which are due to the presence of the inhomogeneous term in evolution equations for \( q^\gamma \). On the one hand, this term allows us to calculate the asymptotic behaviour of \( F_2^\gamma(x, Q^2) \) as \( Q^2 \to \infty \) but on the other hand the same term also implies that a complete NLO analysis of \( F_2^\gamma(x, Q^2) \) requires the inclusion of the \( \mathcal{O}(\alpha_s^2) \) terms \( k_q^{(2)} \), \( k_G^{(2)} \) in the inhomogeneous splitting functions \( k_q \), \( k_G \), as well as of the \( \mathcal{O}(\alpha_s) \) term \( C_{\gamma}^{(1)} \) term in photonic coefficient function \( C_{\gamma}^q \). Unfortunately, none of them has been included in the existing phenomenological analyses. There is no problem to include the latter as the necessary calculations are available, and we have shown that its numerical importance may be quite large. On the other hand, \( k_q^{(2)} \) and \( k_G^{(2)} \) are not known as their evaluation requires three loop QCD calculation. Consequently, at the present time a complete NLO analysis of \( F_2^\gamma(x, Q^2) \) is impossible to perform.

As far as the structure of the virtual photon is concerned, we have discussed the question of what is actually measured in DIS on the virtual photon and emphasized the role of the longitudinal photon in these considerations.

We have also analyzed parton model interpretation of the constant terms in \( C_{\gamma}^{(0)} \) and discussed their implications for the specification of initial conditions imposed on the PDF of the virtual photon. We have argued that the presence of colour confinement washes out the difference between the regularization of mass singularities in the case of real and virtual photons. This offers us a
simple way of avoiding the problems with the term $\ln(1 - x)$ appearing in the standard expression for $C_\gamma^{(0)}$ in the case of the real photon.

Acknowledgments

I am grateful to W. van Neerven, E. Laenen and S. Larin for correspondence concerning higher order QCD calculations of photonic coefficient and splitting functions.

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