DIRECTIONAL DYNAMICS OF $\mathbb{Z}_+ \times \mathbb{Z}$-ACTIONS GENERATED BY 1D-CA AND THE SHIFT MAP

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ABSTRACT. In this short paper, we compute the directional sequence entropy for of $\mathbb{Z}_+ \times \mathbb{Z}$-actions generated by cellular automata and the shift map. Meanwhile, we study the directional dynamics of this system. As a corollary, we prove that there exists a sequence such that for any direction, some of the systems above have positive directional sequence entropy. Moreover, with help of mean ergodic theory for directional weak mixing systems, we obtain a result of number theory about combinatorial numbers.

1. Introduction

In the late 1940s, cellular automata (CA) were created by John von Neumann [4], who was inspired by biological applications. A cellular automaton is a sort of dynamical system invented by Ulam [3] and von Neumann [4] as a model for self-production. 1-dimensional CA (1D-CA) is made up of an infinite lattice with finite states and a mapping called the local rule. Hedlund [5] takes a methodical approach to CA from a mathematical standpoint. Since 1940, CA have continued to be the focus of extreme attention by many researchers [9].

The entropy of a system has been studied extensively for various purposes in fields such as computer science, mathematics, physics, chemistry, information theory. It is well known that the entropy measures the chaoticity or unpredictability of a system. In ergodic theory, there exist numerous notions of entropy of the measure-preserving transformation on the probability space (e.g., measure entropy, topological entropy, directional entropy, rotational entropy, etc.) [6, 7, 8]. Ward [6] investigated the topological entropy of 1-dimensional linear CA (1D-LCA). If the local rule $f$ is defined by $f(x_{-k}, \cdots, x_k) = \sum_{i=-k}^{k} x_i \pmod{m}$ for $k \in \mathbb{N}$, the uniform Bernoulli measure, as proven by Akın [7], is a measure with maximum entropy. In Ref. [8], the author calculated the measure-theoretic entropy and directional entropy of $\mathbb{Z} \times \mathbb{N}$-actions generated by some linear 1D-LCA and the shift map. In Ref. [10], the author derived a formula to compute the topological directional entropy of $\mathbb{Z}^2$-action using the coefficients of the local rule. Akın et al. [11] studied the quantitative behavior of 1-D linear cellular automata (LCAs) and proved that the Hausdorff of the limit set of a LCA is the unique root of Bowen’s equation. Chang, and Akın [15] proved that every invertible 1D-LCA is a Bernoulli automorphism without making use of the natural extension.

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Since Milnor [12] introduced a notion of directional dynamics, the study of directional dynamics has led to other productive lines of research, the most notable among these being Boyle and Lind’s work on expansive sub-dynamics [16]. In [1] Johnson and Şahin studied directional recurrence properties for \( \mathbb{Z}^d \)-actions. Using classical sequence entropy [17, 18, 19] and directional entropy [8, 12], Liu and Xu [21] introduced directional sequence entropy for \( \mathbb{Z}^d \)-actions. Moreover, they [21, 22] have defined directional properties intrinsically and studied their relationship to spectrum. Dennunzio et al. [14] provided a circumstantial classification for a class of the LCA by taking into account non-trivial examples to study some of Sablik’s categorization classes and extended studies of directional dynamics to include factor languages and attractors.

In the present paper, we consider \( \mathbb{Z}_+ \times \mathbb{Z} \)-actions generated by 1D-CA and the shift map on space \( \mathbb{Z}_a \), consisting two-sided infinite sequences, where \( \mathbb{Z}_a \) is a finite ring. In Section 2, we recall some notions and results that we will use. In Section 3, we compute the directional sequence entropy for \( \mathbb{Z}_+ \times \mathbb{Z} \)-actions. Meanwhile, we study the directional dynamics behavior of this system. We prove that \( \mathbb{Z}_+ \times \mathbb{Z} \)-systems are directional weak mixing along any direction \( \vec{v} \in S^1 \), under given conditions in Section 4. Moreover, we take advantage of directional version of mean ergodic theory introduced by the second author [20] to obtain a number theory result about combinatorial numbers. That is, for \( m \)-a.e. \( x \in (0, 1) \), the probability that the number 0 or 1 appears in the sequence \( \left\{ \sum_{n=0}^{\infty} \binom{n}{r} x_{n+1} \mod 2 \right\}_{n=1}^{\infty} \) is 1/2, where \( \binom{n}{r} \) is the combinatorial number, \( 0.x_1x_2 \ldots \) is the 2-adic development of \( x \) and \( m \) is the Lebesgue measure on \( [0, 1] \). In fact, we prove a stronger result in Section 5.

2. Preliminaries

2.1. Cellular automaton. Let \( \mathbb{Z}_a = \{0, 1, \ldots, a-1\} \) be a finite ring. The compact topological space \( \mathbb{Z}_a^{\mathbb{Z}} \) consists of two-sided infinite sequences denoted as \( x = (x_n)_{n=\infty}^{\infty} \), where \( x_n \in \mathbb{Z}_a \). Let \( T_{f[-r,r]} : \mathbb{Z}_a^{\mathbb{Z}} \to \mathbb{Z}_a^{\mathbb{Z}} \) be a map that acts locally on the set of two-sided infinite sequences, where \( f : \mathbb{Z}_a^{2r+1} \to \mathbb{Z}_a \) is called local rule.

**Definition 2.1.** Let us define the local rule \( f \) by

\[
(2.1) \quad f(x_{-r}, \ldots, x_r) = \sum_{i=-r}^{r} \lambda_i x_i \mod a,
\]

where at least one of \( \lambda_{-r}, \ldots, \lambda_r \) is nonzero. The cellular automaton \( T_{f[-r,r]} \) generated by \( f \) given in (2.1) is defined as;

\[
(2.2) \quad (T_{f[-r,r]}(x)) := (y_n)_{n=\infty}^{\infty}, y_n = f(x_{n-r}, \ldots, x_{n+r}) = \sum_{i=-r}^{r} \lambda_i x_{n+i} \mod a,
\]

where the positive integer \( r \) is called the radius of the local rule \( f \).

We consider the semigroup \( \mathbb{Z}_+ \times \mathbb{Z} \)-action \( \Phi \) defined by

\[
(2.3) \quad \Phi^{(m,n)} := T_{f[-r,r]}^m \circ \sigma^n, \quad m \in \mathbb{Z}_+ \text{ and } n \in \mathbb{Z},
\]
where $\sigma$ is the shift map from $\mathbb{Z}_a$ to $\mathbb{Z}_a$ given by $(\sigma(x))_i = x_{i+1}$ for $i \in \mathbb{Z}$, $x \in \mathbb{Z}_a$. Note that the shift map $\sigma$ is one of the simplest examples of CA.

2.2. Directional sequence entropy and directional weak mixing. To study directional system, Liu and Xu [21] introduced a new invariant directional sequence entropy. We recall it as follows.

Let $(\mathbb{Z}_a, \mathcal{X}, \mu, \Phi)$ be a $\mathbb{Z}^2$-measure preserving system ($\mathbb{Z}^2$-m.p.s. for short) and $\vec{v} = (1, \beta) \in \mathbb{R}^2$ be a direction vector, where $\Phi$ is a $\mathbb{Z}_+ \times \mathbb{Z}$-action given in (2.3), $\mathcal{X}$ is a $\sigma$-algebra generated by the cylinder sets and $\mu$ is a product measure on $\mathbb{Z}_a$ (see [13] for details). For the sake of simplicity, we write the vector $\vec{v}$ by $(1, \beta)$. Someone can show that all results in this paper are true for $\vec{v} = (0, 1)$ since this is reduced to the special case of $\mathbb{Z}$-actions. We put

$$\Lambda^{\vec{v}}(b) = \{(m, n) \in \mathbb{Z}^2 : \beta m - b/2 \leq n \leq \beta m + b/2\}$$

and write $\Lambda^{\vec{v}}(b) = \Lambda^{\vec{v}}(b) \cap ([0, k - 1] \times \mathbb{Z})$.

2.2.1. Directional sequence entropy. For a finite measurable partition $\alpha$ of $\mathbb{Z}_a$, let

$$H_\mu(\alpha) = -\sum_{A \in \alpha} \mu(A) \log \mu(A).$$

Let us consider any infinite subset $S = \{(m_i, n_i)\}_{i=1}^\infty$ of $\Lambda^{\vec{v}}(b)$ such that $\{m_i\}_{i=1}^\infty$ is strictly monotone, from the definition of sequence entropy [17, 18, 19] we obtain

$$h_\mu^S(\Phi, \alpha) = \lim_{k \to \infty} \frac{1}{k} H_\mu \left( \bigvee_{i=1}^k \Phi^{-(m_i, n_i)} \alpha \right).$$

Then one can define the directional sequence entropy of $\Phi$ for the subset $S$ by

$$h_\mu^S(\Phi) = \sup_{\alpha} h_\mu^S(\Phi, \alpha),$$

where the supremum is taken over all finite measurable partitions of $\mathbb{Z}_a$.

2.2.2. Directional weak mixing. Let $\mathcal{A}_\vec{v}^\Phi(b)$ be the collection of $f \in \mathcal{H} := L^2(\mathbb{Z}_a, \mathcal{X}, \mu)$ such that

$$\left\{ U_\Phi^{(m,n)} f : (m, n) \in \Lambda^{\vec{v}}(b) \right\}$$

is compact in $L^2(\mathbb{Z}_a, \mathcal{X}, \mu)$, where $U_\Phi^{(m,n)} : L^2(\mathbb{Z}_a, \mathcal{X}, \mu) \to L^2(\mathbb{Z}_a, \mathcal{X}, \mu)$ is the unitary operator such that

$$U_\Phi^{(m,n)} f = f \circ \Phi^{(m,n)}$$

for all $f \in L^2(\mathbb{Z}_a, \mathcal{X}, \mu)$.

One can easily show that $\mathcal{A}_\vec{v}^\Phi(b)$ is a $U_{\Phi^{\vec{v}}} \in \text{inv}$ for any vector $\vec{w} \in \mathbb{Z}_a$ and conjugation-invariant subalgebra of $\mathcal{H}$. Then there exists a $\Phi$-invariant sub-$\sigma$-algebra $\mathcal{K}_\vec{v}(b)$ of $\mathcal{X}$ such that

(2.4) $\mathcal{A}_\vec{v}^\Phi(b) = L^2(\mathbb{Z}_a, \mathcal{K}_\vec{v}(b), \mu)$.

Directly from (2.4), we define the $\vec{v}$-directional Kronecker algebra of $(\mathbb{Z}_a, \mathcal{X}, \mu, \Phi)$ by

$$\mathcal{K}_\vec{v}(b) = \left\{ B \in \mathcal{X} : \left\{ U_\Phi^{(m,n)} 1_B : (m, n) \in \Lambda^{\vec{v}}(b) \right\} \text{ is compact in } L^2(\mathbb{Z}_a, \mathcal{X}, \mu) \right\}.$$
Note that the definition of $K_{\mu}^\vec{v}(b)$ is independent of the selection of $b \in (0, \infty)$ (see [21, Proposition 3.1]). So we omit $b$ in $K_{\mu}^\vec{v}(b)$ and write it as $K_{\mu}^\vec{v}$. We say $\mu$ is $\vec{v}$-weak mixing if $K_{\mu}^\vec{v} = \{X, \emptyset\}$.

3. Computation of directional sequence entropy

In this section, we will compute the sequence entropy and directional sequence entropy of $\mathbb{Z}_+ \times \mathbb{Z}$-actions obtained by 1D-CA and the shift map. We need the following lemma (see [17] for $\mathbb{Z}$-actions).

**Lemma 3.1.** Let $(\mathbb{Z}_+^Z, X, \mu, \Phi)$ be a $\mathbb{Z}_+^2$-m.p.s., where $\Phi$ is a function given in (2.3), $\{\xi_k\}$ be a sequence of measurable partitions such that $\xi_1 \leq \xi_2 \leq \cdots \leq \xi_n \leq \cdots$, and $\bigvee_{i=1}^\infty \xi_i = \epsilon$ (where $\epsilon$ denotes the partition into points of $(\mathbb{Z}_+^Z, \mu)$). Then for any $S$ and $\Phi$, $h_S(\Phi) = \lim_{k \to \infty} h_S(\Phi, \xi_k)$.

**Theorem 3.2.** Let $\mu$ be the uniform Bernoulli measure on $\mathbb{Z}_+^Z$,

$$f(x_{n-r}, \cdots, x_{n+r}) = \sum_{i=-r}^r \lambda_i x_{n+i} \pmod{a}$$

with $\gcd(\lambda_-a) = 1$ and $\gcd(\lambda_+, a) = 1$, and $\Phi$ be a $\mathbb{Z}_+ \times \mathbb{Z}$-action defined in (2.3). Then for the $\mathbb{Z}_+ \times \mathbb{Z}$-m.p.s. $(\mathbb{Z}_+^Z, X, \mu, \Phi)$ and $S = \{(m_i, n_i)\}_{i=1}^\infty \subset \mathbb{Z}_+ \times \mathbb{Z}$ such that $\{m_i\}_{i=1}^\infty$ is a strictly monotone increasing syndetic set with gap $N \in \mathbb{N}$ and $m_i > n_i$, one has

$$h^S(\Phi) = 2r \log a \cdot \limsup_{l \to \infty} \frac{m_l}{l}.$$ 

**Proof.** If we choose $M \in \mathbb{N}$ large enough then $\xi(-M, M) \vee \Phi^{-\{m_1, n_1\}} \xi(-M, M)$ consists of all cylinder sets in the form

$$-(rm_1 + M) - n_1 \left[j_{-rm_1 - M}, \cdots, jrm_1 + M\right][(rm_1 + M) - n_1].$$

Since $2(rm_1 + M) + 1 > 2rm_2 + 1$ it follows that

$$\xi(-M, M) \vee \Phi^{-\{m_1, n_1\}} \xi(-M, M) \vee \Phi^{-\{m_2, n_2\}} \xi(-M, M)$$

consists of all cylinder sets in the form

$$-(rm_2 + M) - n_2 \left[j_{-rm_2 - M}, \cdots, jrm_2 + M\right][(rm_2 + M) - n_2].$$

By the same way we can prove that $\bigvee_{i=0}^l \Phi^{-\{m_i, n_i\}} \xi(-M, M)$ consists of all cylinder sets in the form

$$-(rm_1 + M) - n_l \left[j_{-rm_1 - M}, \cdots, jrm_l + M\right][(rm_l + M) - n_l].$$
Thus, we get
\[ h^S(\Phi, \xi(-M, M)) = \limsup_{l \to \infty} \frac{1}{l+1} H_\mu(\sum_{i=0}^{l} \Phi^{-m_i,n_i} \xi(-M, M)) \]
\[ = \limsup_{l \to \infty} \frac{1}{l+1} a^{2(rm_i+M)-1} \mu\left(-\begin{array}{c} rm_i+M \\ \vdots \\ \hat{m_i}+M \end{array}, M \right) \cdot \log \mu\left(-\begin{array}{c} rm_i+M \\ \vdots \\ \hat{m_i}+M \end{array}, M \right) \]
\[ = \limsup_{l \to \infty} \frac{1}{l+1} \log a^{-2(rm_i+M)-1} = \left(\limsup_{l \to \infty} \frac{2(rm_i+M)+1}{l+1}\right) \cdot \log a \]
\[ = 2r \log a \cdot \limsup_{l \to \infty} \frac{m_l}{l}. \]

By Lemma 3.1 and the fact that \( V_{i=M}^\infty \xi(-i,i) = \varepsilon \), one has
\[ h^S(\Phi) = 2r \log a \cdot \limsup_{l \to \infty} \frac{m_l}{l}. \]

This finishes the proof.

**Corollary 3.3.** For the \( \mathbb{Z}_+ \times \mathbb{Z} \)-m.p.s. \( (\mathbb{Z}_+^Z, \mathcal{X}, \mu, \Phi) \) defined in Theorem 3.2, if \( \bar{v} = (x,y) \in S^1 \) with \( x > y \), then the directional sequence entropy is only depends on the gap of the first coordinate, not direction.

### 4. Directional weak mixing

In this section, we are going to prove several and important new results. To study the complexity of directional systems, Liu [20] introduced directional weak mixing. Since this paper considers \( \mathbb{Z}_+ \times \mathbb{Z} \)-actions obtained by cellular automata and the shift map, we only recall results in [20] for it. In fact, the following results holds for all \( \mathbb{Z}^2 \)-actions.

Now we will prove a class of systems with \( \mathbb{Z}_+ \times \mathbb{Z} \)-actions generated by 1D-CA and the shift map is directional weak mixing along any direction \( \bar{v} \in S^1 \). To prove this result let us begin the following lemma [20].

**Lemma 4.1.** Let \( (\mathbb{Z}_+^Z, \mathcal{X}, \mu, \Phi) \) be a \( \mathbb{Z}^2 \)-m.p.s. and \( \bar{v} = (1, \beta) \in \mathbb{R}^2 \) be a direction vector. Then the following statements are equivalent.

(a) \( (\mathbb{Z}_+^Z, \mathcal{X}, \mu, \Phi) \) is \( \bar{v} \)-weak mixing.

(b) There exists \( b > 0 \) such that
\[ \lim_{k \to \infty} \frac{1}{\#(A_k^{\bar{v}}(b))} \sum_{(m,n) \in A_k^{\bar{v}}(b)} |\langle U_{\Phi}^{(m,n)} 1_B, 1_C \rangle - \mu(B)\mu(C)| = 0, \]
for any \( B, C \in \mathcal{X} \).

(c) For any \( b > 0 \),
\[ \lim_{k \to \infty} \frac{1}{\#(A_k^{\bar{v}}(b))} \sum_{(m,n) \in A_k^{\bar{v}}(b)} |\langle U_{\Phi}^{(m,n)} 1_B, 1_C \rangle - \mu(B)\mu(C)| = 0, \]
for any \( B, C \in \mathcal{X} \).
Considering lemma 4.1, we have the following result.

**Theorem 4.2.** Let $\mu$ be the uniform Bernoulli measure on $\mathbb{Z}^a_+$, 
\[ f(x_0, \cdots, x_{n+r}) = \sum_{i=0}^{r} \lambda_i x_{n+i} (\text{mod } a) \]
and $\Phi$ be a $\mathbb{Z}_+ \times \mathbb{Z}$-action given in (2.3). Then the system $(\mathbb{Z}^a_+, X, \mu, \Phi)$ is directional weak mixing along any direction $\vec{v} \in S^1$.

**Proof.** Note that for any $M, N \in \mathbb{N}$, there exists $n > N + M$ such that for any $m \in \mathbb{Z}_+$, one has 
\[ \mu(\Phi^{-\langle m, n \rangle} \chi(-M, M) \cap \chi(-N, N)) = \mu(\Phi^{-\langle m, n \rangle} \chi(-M, M)) \mu(\chi(-N, N)). \]
We can easily show that the system associated with (2.3) satisfies (b) of Lemma 4.1 along any $\vec{v} \in S^1$. Hence we immediately obtain that the above system is directional weak mixing along any direction $\vec{v} \in S^1$. □

With the help of [20, Theorem 1.3], we immediately obtain that another result about directional sequence entropy for the system defined in Theorem 4.2.

**Corollary 4.3.** Let $(\mathbb{Z}^a_+, X, \mu, \Phi)$ be the $\mathbb{Z}^2$-m.p.s which is defined in Theorem 4.2. For any direction vector $\vec{v} \in S^1$, there exists a sequence $S = \{(m_i, n_i)\}_{i=1}^{\infty}$ of $\Lambda^\vec{v}(b)$ satisfying equality 
\[ h^\vec{v}_\mu(\Phi, \alpha) = H_\mu(\alpha) \]
for any finite measurable partition $\alpha$ of $\mathbb{Z}^a_+$.

## 5. A Result in Number Theory

In this section, we will obtain a result about combinatorial mathematics. To complete the proof, we need to recall some results. In [20], the second author obtained a mean ergodic theory for directional weak mixing, which is the key to prove our result, and we restated it as follows. We remark that the following results, that is, Theorem 5.1 and Theorem 5.2 hold for any $\mathbb{Z}^d$-m.p.s.

**Theorem 5.1.** Let $(\mathbb{Z}^a_+, X, \mu, \Phi)$ be a $\mathbb{Z}^2$-m.p.s., $\vec{v} = (1, \beta) \in \mathbb{R}^2$ be a directional vector and $b \in (0, \infty)$. Then the following statements are equivalent.

(a) $(\mathbb{Z}^a_+, X, \mu, \Phi)$ is $\vec{v}$-weak mixing.

(b) For any infinite subset $Q = \{(m_i, n_i)\}_{i=1}^{\infty}$ of $\Lambda^\vec{v}(b)$ with $\liminf_{n \to \infty} \frac{\#(Q \cap \Lambda^\vec{v}(b))}{\#(\Lambda^\vec{v}(0))} > 0$, one has 
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} U_{\Phi}^{(m_i, n_i)} g - \int_{\mathbb{Z}^a_+} g \, d\mu \|_2 = 0 \]
for all $g \in L^2(\mathbb{Z}^a_+, \mathbb{X}, \mu)$, where $\#(A)$ is the number of elements in finite subset $A$.

Moreover, for direction $\vec{v} = (m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, one has the following version Birkhoff ergodic theory. We remark that the proof follows methods in [2].
Theorem 5.2. Let \((\mathbb{Z}_a^2, \mathcal{X}, \mu, \Phi)\) be a \(\mathbb{Z}^2\)-m.p.s. and \(\vec{v} = (m, n) \in \mathbb{Z}^2\) be a direction vector. If \((\mathbb{Z}_a^2, \mathcal{X}, \mu, \Phi)\) is \(\vec{v}\)-weak mixing then for \(\mu\)-a.e. \(x \in \mathbb{Z}_a^2\)

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} U_{\Phi}^{(km, kn)}(x) = \int_{\mathbb{Z}_a^2} g \, d\mu
\]

for all \(g \in L^1(\mathbb{Z}_a^2, \mathcal{X}, \mu)\).

Proof. Let \(A_N g(x) = \frac{1}{N} \sum_{k=0}^{N-1} g(\Phi^{(km, kn)} x)\). Given \(\phi \in L^1(\mathbb{Z}_a^2, \mathcal{X}, \mu)\), let

\[
M_N \phi = \max \left\{ \sum_{j=0}^{k-1} \phi \circ \Phi^{(jm, jn)} : 1 \leq k \leq N \right\}
\]

It is easy to see that \(A_N \phi \leq \frac{1}{N} M_N \phi\). Let

\[
A(\phi) = \left\{ x \in \mathbb{Z}_a^2 : \sup_N M_N \phi(x) = \infty \right\}.
\]

Then is \(A(\phi)\) is a \(\Phi^{(m, n)}\)-invariant set. By Theorem 5.1, one has \(\mu(A(\phi)) = 0\) or 1. Given \(g \in L^1(\mathbb{Z}_a^2, \mathcal{X}, \mu)\) and \(\epsilon > 0\), let \(\phi = g - \int gd\mu - \epsilon\). Suppose to the contrary that \(\mu(A(\phi)) = 1\). Then

\[
0 < \int_{A(\phi)} \phi \, d\mu = \int_{A(\phi)} \left( g - \int gd\mu - \epsilon \right) \, d\mu = -\epsilon \mu(A) \leq 0
\]

a contradiction. Hence \(\mu(A(\phi)) = 0\). Note for any \(x \in A(\phi)^c\), one has \(\limsup_{n \to \infty} A_n \phi(x) \leq 0\). So, one gets

\[
\limsup_{n \to \infty} A_n \phi(x) \leq 0
\]

for \(\mu\)-a.e. \(x \in \mathbb{Z}_a^2\). Since \(A_n \phi = A_n (g - \int gd\mu - \epsilon)\), one has

\[
\limsup_{n \to \infty} A_n g(x) \leq \int gd\mu + \epsilon.
\]

Finally, do same thing for \(-g\), one has

\[
\liminf_{n \to \infty} A_n g(x) \geq \int gd\mu - \epsilon.
\]

As \(\epsilon > 0\) is arbitrary, we obtain that for \(\mu\)-a.e. \(x \in \mathbb{Z}_a^2\)

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} U_{\Phi}^{(km, kn)}(x) = \int_{\mathbb{Z}_a^2} g \, d\mu
\]

for all \(g \in L^1(\mathbb{Z}_a^2, \mathcal{X}, \mu)\). \(\square\)

With help of Theorem 5.2, we obtain an interesting result about number theory.

Theorem 5.3. Given \(N \in \mathbb{Z}\), \(k \in \mathbb{N}\) and \(j \in \{0, 1, \ldots, k-1\}\), for \(m\)-a.e. \(x \in (0, 1)\), the probability that the number \(j\) appears in the sequence \(\{\sum_{l=0}^{N} \binom{nN}{l} x_{l+1} (\text{mod } k)\}_{n=1}^\infty\) is \(1/k\), where \(\binom{nN}{l}\) is the combinatorial number, \(0.x_1x_2\ldots\) is the \(k\)-adic development of \(x\) and \(\nu\) is the Lebesgue measure on \([0, 1]\).
Proof. For simplicity, we only prove the case of $k = 2$, because other cases are similar to prove.

Let the local rule $f$ be given as $f(x_{-1}, x_0, x_1) = x_0 + x_1 \pmod{2}$. We consider the semigroup $\mathbb{Z}_+ \times \mathbb{Z}_+$-action $\Phi$ defined by $\sigma \circ T_{[1,1]}^s = \Phi(s,t)$, $s, t \in \mathbb{Z}_+$, where $\sigma$ is the shift map from $\{0,1\}^{\mathbb{Z}_+}$ to $\{0,1\}^{\mathbb{Z}_+}$. Suppose $\mu$ is the uniform Bernoulli measure on the product space $\{0,1\}^{\mathbb{Z}_+}$ and let $\mathcal{X}$ be the $\sigma$-algebra generated by the cylinder sets. Then we obtain a $\mathbb{Z}_{2\mathbb{Z}}$-m.p.s. $(\{0,1\}^{\mathbb{Z}_+}, \mathcal{X}, \mu, \Phi)$.

Let $Y$ be all real numbers with unique 2-adic development. Then $\nu(Y) = 1$. For any $x \in Y$ with 2-adic development $x = 0.x_0x_1 \ldots$, we identify it as $(x_i)_{i=0}^{\infty} \in \{0,1\}^{\mathbb{Z}_+}$. Let us consider cylinder set $A = [0,1]$ and $h(x) = 1_A(x)$. Then, we have

$$h(\Phi^{(n,n)}x) = \begin{cases} 1, & \sum_{l=0}^{nN} (nN) x_{l+1} = 1 \pmod{2}, \\ 0, & \sum_{l=0}^{nN} (nN) x_{l+1} = 0 \pmod{2}. \end{cases}$$

By Theorem 4.2, we know that for $\nu = (1,N)$, $(\{0,1\}^{\mathbb{Z}_+}, \mathcal{X}, \mu, \Phi)$ is $\nu$-weak mixing. Therefore one has, by Theorem 5.2,

$$\lim_{p \to \infty} \frac{1}{p} \# \{ n \in [0,p-1] : \sum_{l=0}^{nN} (nN) x_{l+1} = 0 \pmod{2} \} = \lim_{p \to \infty} \frac{1}{p} \sum_{n=0}^{p-1} h(\Phi^{(n,n)}x)$$

$$= \lim_{p \to \infty} \frac{1}{p} \sum_{n=0}^{p-1} U_{\Phi}^{(n,n)} h(x) = \int_{[0,1]} h(x) d\nu(x) = 1/2.$$

This finishes the proof of Theorem 5.3.

Remark 5.4. In particular, if $N = 1$ and $k = 2$, then $\sum_{l=0}^{n} \binom{n}{l} x_{l+1}$ is the sum of the combinatorial number, which may be applied in combinatorial mathematics.

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