THE INTEGRAL CHOW RINGS OF THE STACKS OF HYPERELLIPTIC WEIERSTRASS POINTS

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ABSTRACT. We compute the integral Chow rings of the stacks $H^w_{g,n}$ parametrizing hyperelliptic curves with $n$ marked hyperelliptic Weierstrass points. We prove that the integral Chow ring of each of these stacks is generated as an algebra by any of the $\psi$-classes and that all relations live in degree one.

1. Introduction

The purpose of this paper is to compute the integral Chow rings of the stacks $H^w_{g,n}$ parametrizing smooth hyperelliptic curves with $n$ marked Weierstrass points. Aside from their intrinsic interest, products of these stacks naturally arise as boundary strata of proper moduli stacks of curves such as $H_g$, parametrizing stable hyperelliptic curves. Thus, any program to compute the integral Chow ring of $H_g$ via the stratification method of [8] will require our results as input.

We now describe the moduli stacks considered in this paper. Let $H_g$ denote the moduli stack of smooth hyperelliptic curves, and let $C_g \to H_g$ be the universal hyperelliptic curve. A result of Kleiman and Lønsted [7] implies that there is a smooth Cartier divisor $H_{g,w} \subset C_g$ parametrizing the hyperelliptic Weierstrass points in the fibers of $C_g \to H_g$. Moreover, if $\text{char } k \neq 2$, then $H_{g,w}$ is finite and étale of degree $2g + 2$ over $H_g$. The stack $H_{g,w}$ which we call the stack of hyperelliptic Weierstrass points parametrizes families of smooth hyperelliptic curves with a Weierstrass section.

Let $C_{g,w} \to H_{g,w}$ be the family of hyperelliptic curves obtained by base change along the finite étale morphism $H_{g,w} \to H_g$. The pullback of the Weierstrass divisor $H_{g,w}$ decomposes as the disjoint union of the universal Weierstrass section and a divisor $H^{w}_{g,2}$ which is finite and étale over $H^{w}_{g,1} := H_{g,w}$ of degree $2g + 1$. Iterating this construction, we obtain a tower of finite étale covers

$$H^{w}_{g,2g+2} \to H^{w}_{g,2g+1} \to \ldots \to H_{g,w} \to H_g$$

where $H^{w}_{g,n}$ parametrizes hyperelliptic curves with $n$-Weierstrass sections and the morphism $H^{w}_{g,n+1} \to H^{w}_{g,n}$ is finite and étale of degree $2g + 2 - n$. Each of the stacks $H^{w}_{g,n}$ is a $\mu_2$-gerbe over the stack $[\mathcal{M}_{0,2g+2}/S_{2g+2-n}]$ which parametrizes families of rational curves with $n$-marked points and a disjoint divisor of degree $2g + 2 - n$ which is étale over the base.

Putting this together we obtain the following cartesian diagram (1.1) where the horizontal arrows are $\mu_2$-gerbes and the vertical arrows are finite, étale and representable.
The integral Chow ring of $H_g$ was computed in the papers [3] (g-even), [2,5] (g-odd). In this paper we compute the integral Chow rings of the other stacks in diagram. Note that the rational Chow rings of the stacks $H_{g,n}^w$ are trivial because diagram (1.1) implies that they are gerbes over quotients of $M_{0,2g+2}$ which has trivial Chow groups. Thus, the problem of computing the integral Chow rings is all the more interesting.

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1.1. Statement of results.

Conventions and notation. Throughout this paper, we fix a natural number $g \geq 2$ and work over an algebraically closed field $k$ whose characteristic does not divide $2g + 2$. We will use the notation $A(N)$ to denote the affine space of binary forms of degree $N$. As such $A(N) \simeq \mathbb{A}^{N+1}$. There is a natural action of $\text{GL}_2$ on $A(N)$ given by $A \cdot f(x) = f(A^{-1}x)$. The kernel of this action is the diagonal subgroup $\mu_N$.

As noted above, the integral Chow ring of $H_g$ was computed by Edidin–Fulghesu, Fulghesu–Viviani and Di Lorenzo. In order to contrast it with our work we restate their results.

**Theorem 1.1.** The integral Chow ring of the stack of hyperelliptic curves is as follows.

- (Edidin–Fulghesu [3]) If $g$ is even, then
  $$A^*(H_g) = \mathbb{Z}[c_1, c_2]/(2(2g + 1)c_1, g(g - 1)c_1^2 - 4g(g + 1)c_2).$$

- (Fulghesu–Viviani [5], Di Lorenzo [2]) If $g$ is odd, then
  $$A^*(H_g) = \mathbb{Z}[\tau, c_2, c_3]/(4(2g + 1)\tau, 8\tau^2 - 2g(g + 1)c_2, 2c_3).$$

**Theorem 1.2.** The integral Chow ring of $[M_{0,2g+2}/S_{2g+2}]$ is

$$A^*([M_{0,2g+2}/S_{2g+2}]) = \frac{\mathbb{Z}[\tau, c_2, c_3]}{(2\tau^2 - 2g(g + 1)c_2, 2(2g + 1)\tau, 2c_3, p(\tau, c_2, c_3))}$$

where
• if $g$ is odd, then
  \[ p(\tau, c_2, c_3) = (g + 1)^2 \tau^g (\tau^2 + c_2) \frac{\tau + 1}{\tau} c_2 + \tau \frac{\tau + 1}{\tau} (\tau^3 + \tau c_2 - c_3) \frac{\tau + 1}{\tau}; \]

• if $g$ is even, then
  \[ p(\tau, c_2, c_3) = g(g + 2) \tau^{g+1} (\tau^2 + c_2) \frac{\tau}{\tau} c_2 + \tau^2 (\tau^3 + \tau c_2 - c_3) \frac{\tau + 2}{\tau}. \]

Once we mark at least one Weierstrass point, the presentation of the Chow rings no longer depends on the parity of the genus $g$.

**Theorem 1.3.** The integral Chow rings of $\mathcal{H}_{g,w}$ and $[\mathcal{M}_{0,2g+2}/S_{2g+1}]$ are

\[ A^*(\mathcal{H}_{g,w}) = \frac{\mathbb{Z}[\psi]}{(4g+1)(\psi)}, \quad A^*((\mathcal{M}_{0,2g+2}/S_{2g+1})) = \frac{\mathbb{Z}[l]}{(2g(2g+1)l)}, \]

respectively. Here $\psi$ is the restriction of the $\psi$-class on $\mathcal{H}_{g,1}$ to $\mathcal{H}_{g,w}$.

**Theorem 1.4.** The integral Chow rings of $\mathcal{H}^w_{g,2}$ and $[\mathcal{M}_{0,2g+2}/S_{2g}]$ are

\[ A^*(\mathcal{H}^w_{g,2}) = \frac{\mathbb{Z}[\psi]}{(4g\psi)}, \quad A^*((\mathcal{M}_{0,2g+2}/S_{2g})) = \frac{\mathbb{Z}[l]}{(2gl)}, \]

respectively. Here $\psi$ is the restriction of either of the two $\psi$-classes, $\psi_1, \psi_2$ on $\mathcal{H}_{g,2}$ to $\mathcal{H}^w_{g,2}$.

**Theorem 1.5.** For $3 \leq n \leq 2g + 2$, the integral Chow rings of $\mathcal{H}^w_{g,n}$ and $[\mathcal{M}_{0,2g+2}/S_{2g+2-n}]$ are

\[ A^*(\mathcal{H}^w_{g,n}) = \mathbb{Z}[\psi]/(2\psi), \quad A^*((\mathcal{M}_{0,2g+2}/S_{2g+2-n})) = \mathbb{Z}. \]

Here $\psi$ is the restriction of any of the $\psi$-classes on $\mathcal{H}_{g,n}$ to $\mathcal{H}^w_{g,n}$.

## 2. Proof of Theorem 1.2

2.1. **Description of $[\mathcal{M}_{0,2g+2}/S_{2g+2}]$ as a quotient by $\mathbb{G}_m \times \text{PGL}_2$.** The quotient stack $[\mathcal{M}_{0,2g+2}/S_{2g+2}]$ can be identified with the stack $\mathcal{D}_{2g+2}$ parametrizing pairs $(P \to S, D_{2g+2} \subset P)$ where $P \to S$ is a twisted $\mathbb{P}^1$ over $S$ and $D_{2g+2}$ is an effective Cartier divisor which is finite and étale over $S$ of degree $2g + 2$. By [6, Proposition 3.4], the algebraic stack $\mathcal{D}_{2g+2}$ can be identified with the quotient stack $[\mathbb{A}_{\text{sm}}(2g + 2)/(\text{GL}_2/\mu_{2g+2})]$, where $\mathbb{A}_{\text{sm}}(2g + 2)$ is the affine space of binary forms of degree $2g + 2$ with distinct roots, and the action on $\mathbb{A}_{\text{sm}}(2g + 2)$ is given by $A \cdot f(x) = f(A^{-1}x)$.

**Remark 2.1.** By [1, Theorem 4.1], the stack $\mathcal{H}_g$ of smooth hyperelliptic curves of genus $g$ is isomorphic to the quotient stack $[\mathbb{A}_{\text{sm}}(2g + 2)/(\text{GL}_2/\mu_{g+1})]$ with action defined by $A \cdot f(x) = f(A^{-1}x)$. Consider the short exact sequence

\[ 1 \to \mu_2 \to \mu_{2g+2} \to \mu_{g+1} \to 1 \]

where the second arrow is given by $\alpha \mapsto \alpha^2$. There is an induced map of quotient groups $\text{GL}_2/\mu_{g+1} \to \text{GL}_2/\mu_{2g+2}$ which is a $\mu_2$-torsor. This in turn induces a map of quotient stacks

\[ \mathcal{H}_g = [\mathbb{A}_{\text{sm}}(2g + 2)/(\text{GL}_2/\mu_{g+1})] \to \mathcal{D}_{2g+2} = [\mathbb{A}_{\text{sm}}(2g + 2)/(\text{GL}_2/\mu_{2g+2})] \]

which is a $\mu_2$-gerbe.

Since $2g + 2$ is always even, [1, Proposition 4.4] implies that the homomorphism $\text{GL}_2/\mu_{2g+2} \to \mathbb{G}_m \times \text{PGL}_2$, given by $[A] \mapsto (\det(A)^{g+1}, [A])$ is an isomorphism. Under this identification of groups, the action of $\mathbb{G}_m \times \text{PGL}_2$ on the affine space of binary forms $\mathbb{A}(2g + 2)$ is given by $(\lambda, [A]) \cdot f(x) = \lambda^{-1} \det(A)^{g+1} f(A^{-1}x)$. 
2.2. Computing the \( \mathbb{G}_m \times \text{PGL}_3 \)-equivariant Chow ring of \( A_{\text{sm}}(2g+2) \) using \( \mathbb{G}_m \times \text{GL}_3 \)-counterparts. In order to compute the Chow ring of the quotient stack \([A_{\text{sm}}(2g+2)/(\mathbb{G}_m \times \text{PGL}_2)]\), we follow the method used by Di Lorenzo in [2, Section 2.3] by finding a \( \mathbb{G}_m \times \text{GL}_3 \)-counterpart of the \( \mathbb{G}_m \times \text{PGL}_2 \)-scheme \( A_{\text{sm}}(2g+2) \) to resolve the issue that the group \( \text{PGL}_2 \) is non-special. Let us briefly recall the concepts from [2] here.

Given a \( \text{PGL}_2 \)-scheme \( X \) of finite type over \( \text{Spec} \, k \), a \( \text{GL}_3 \)-counterpart of \( X \) is another scheme \( Y \) equipped with a \( \text{GL}_3 \)-action such that \([Y/\text{GL}_3] \cong [X/\text{PGL}_2]\). In [2, Theorem 1.4], Di Lorenzo proves that a \( \text{GL}_3 \)-counterpart always exists for a given \( \text{PGL}_2 \)-scheme. Indeed, with the adjoint representation of \( \text{PGL}_2 \), we have a morphism of algebraic groups \( \text{PGL}_2 \to \text{GL}_3 \). It follows that the quotient \( \text{GL}_3 / \text{PGL}_2 \) is a \( \text{GL}_3 \)-counterpart of \( \text{Spec} \, k \). Pulling back \([A_{\text{sm}}(2g+2)/\text{PGL}_2]\) along the \( \text{GL}_3 \)-torsor \( \text{GL}_3 / \text{PGL}_2 \to [\text{Spec} \, k / \text{PGL}_2] \cong \mathcal{M}_0 \) will in turn give a \( \text{GL}_3 \)-counterpart of the \( \text{PGL}_2 \) action on \( A_{\text{sm}}(2g+2) \). The \( \mathbb{G}_m \times \text{GL}_3 \)-counterpart can be defined similarly.

Let \( \Delta' \) denote the complement of \( A_{\text{sm}}(2g+2) \) in \( A(2g+2) \), i.e. the discriminant locus of \( A(2g+2) \). Because the weight of the action of \( \mathbb{G}_m \) in the quotient stack \([A_{\text{sm}}(2g+2)/(\mathbb{G}_m \times \text{PGL}_2)]\) presenting \([\mathcal{M}_{0,2g+2}/S_{2g+2}]\) is different from the weight of the \( \mathbb{G}_m \) action that Di Lorenzo used in his computation of \( A^*(\mathcal{H}_g) \) for \( g \) odd [2], our next proposition describes the \( \mathbb{G}_m \times \text{GL}_3 \)-counterpart for a \( \mathbb{G}_m \) action with an arbitrary weight. Precisely, if \( \lambda \cdot f = \chi(\lambda)f \) for some character \( \chi \) of \( \mathbb{G}_m \), we consider the \( \mathbb{G}_m \times \text{GL}_3 \)-counterparts for \( A(2g+2) \) and \( \Delta' \), where \( \mathbb{G}_m \times \text{PGL}_2 \) acts on \( A(2g+2) \) by

\[
(\lambda, A) \cdot f = \chi(\lambda)(\det A)^{g+1}f(A^{-1}x).
\]

**Proposition 2.2.** [2, Proposition 2.3 & 2.6] Let \( V_{g+1} \) be the scheme parametrizing pairs \((q, f)\) where \( q \) is a global section of \( \mathcal{O}_{\mathbb{P}^2}(2) \) with zero locus \( Q \), and \( f \) is a global section of \( \mathcal{O}_Q(g+1) \). Let \( D' \subset V_{g+1} \) be the singular locus inside \( V_{g+1} \). Then

- \( V_{g+1} \) is a \( \mathbb{G}_m \times \text{GL}_3 \)-counterpart of \( A(2g+2) \) with the action of \( \mathbb{G}_m \times \text{GL}_3 \) given by
  
  \[
  (\lambda, A) \cdot (q, f) := (\det(A)q(A^{-1}x), \chi(\lambda)f(A^{-1}x)).
  \]

- \( D' \) is a \( \mathbb{G}_m \times \text{GL}_3 \)-counterpart of \( \Delta' \).

Applying Proposition 2.2, we see that to compute \( A^*([\mathcal{M}_{0,2g+2}/S_{2g+2}]) \) we must compute \( A_{g_m \times \text{GL}_3}^*(V_{g+1} \setminus D') \) where \( \mathbb{G}_m \times \text{GL}_3 \) acts by

\[
(\lambda, A) \cdot (q, f) = (\det(A)q(A^{-1}x), \lambda^{-1}f(A^{-1}x)).
\]

This computation is almost identical to the computation made by Di Lorenzo in [2, Section 4]. The only difference is when passing to the projectivization of \( V_{g+1} \) in order to trivialize the global \( \mathbb{G}_m \)-action. Let \( \sigma_0 \) denote the \( \text{GL}_3 \)-counterpart of \( 0 \in A(2g+2) \), which turns out to be the zero section. The map \( V_{g+1} \setminus \sigma_0 \to \mathbb{P}(V_{g+1}) \) is a \( \mathbb{G}_m \)-torsor over \( \mathbb{P}(V_{g+1}) \) with the associated line bundle \( V^{-1} \otimes \mathcal{O}(-1) \). Here \( V \) is the standard character of \( \mathbb{G}_m \). By contrast in [2] the associated line bundle is \( V^{-2} \otimes \mathcal{O}(-1) \).

Following [2], we denote by \( D \) the projectivization of \( D' \). The \( \mathbb{G}_m \)-torsor \( V_{g+1} \setminus D' \to \mathbb{P}(V_{g+1}) \setminus D \) induces a surjective pullback on the equivariant Chow groups \( A^*_{\mathbb{G}_m \times \text{GL}_3}(\mathbb{P}(V_{g+1}) \setminus D) \to A^*_{\mathbb{G}_m \times \text{GL}_3}(V_{g+1} \setminus D') \) with kernel generated by \( c_1(V^{-1} \otimes \mathcal{O}(-1)) = -\tau - h_{g+1} \), where \( \tau \) is the first Chern class of the standard representation of \( \mathbb{G}_m \), and \( h_{g+1} \) is the first Chern class of \( \mathcal{O}(1) \).

The same method used in the calculation of \( A^*(\mathcal{H}_g) \) for odd genus \( g \) in [2, Section 4] yields the following result.
Proposition 2.3. The integral Chow ring of \([(V_{g+1} \setminus \mathcal{D}^t)/\left(\mathbb{G}_m \times \text{GL}_2\right)]\) is a quotient ring
\[ \mathbb{Z}[\tau, c_2, c_3]/(2\tau^2 - 2g(g + 1)c_2, 2(2g + 1)\tau, 2c_3, p_{g+1}(-\tau)) \]
where \(p_{g+1}(-\tau) = p_{g+1}(h_{g+1})\) is the monic polynomial of degree \(2g + 3\), which vanishes in \(A^*_{GL_2}(\mathbb{P}(V_{g+1}))\), obtained by applying the GL_3-equivariant projective bundle theorem to \(\mathbb{P}(V_{g+1})\).

Finally, checking that the monic polynomial \(p_{g+1}(-\tau)\) is not in the ideal \((2\tau^2 - 2g(g + 1)c_2, 2(2g + 1)\tau, 2c_3)\) by using [5, Proposition 6.5] completes the proof of Theorem 1.2.

3. Description of \(\mathcal{H}_{g,w}\) and \([\mathcal{M}_{0,2g+2}/\mathcal{S}_{2g+1}]\) as quotients by Borel subgroups of rank 2

In this section we give a description of the stacks \(\mathcal{H}_{g,w}\) and \([\mathcal{M}_{0,2g+2}/\mathcal{S}_{2g+1}]\) as quotients by Borel subgroups of GL_2/\(\mu_{g+1}\) and GL_2/\(\mu_{2g+2}\) respectively.

Recall that Pernice [9, Proposition 1.3] proved that the stack \(\mathcal{H}_{g,1}\) is equivalent to the stack \(\mathcal{H}'_{g,1}\) parametrizing the following data
\[(P \to S, \mathcal{L}, i: \mathcal{L}^2 \hookrightarrow \mathcal{O}_S, \sigma_P, j)\]
where \(P \to S\) is a twisted \(\mathbb{P}^1\), \(\mathcal{L}\) is a line bundle on \(P\) which restricts to a line bundle of degree \(-(g + 1)\) on the fibers of \(P \to S\), \(\sigma_P: S \to P\) is a section and \(j: \sigma_P^* \mathcal{L} \to \mathcal{O}_S\) satisfies \(j \circ \sigma = \sigma_P^* (i)\).

Given this data Pernice follows [1] to obtain a family of pointed hyperelliptic curves by taking the double cover \(C = \text{Spec}_{\mathcal{O}_P}(\mathcal{O}_P \oplus \mathcal{L}) \to P\) where the \(\mathcal{O}_P\)-algebra structure on \(\mathcal{O}_P \oplus \mathcal{L}\) is induced by \(i\). If \(f: C \to S\) is the corresponding family of hyperelliptic curves, then Pernice shows that the pair \((\sigma_P, j)\) determines a section \(\sigma: S \to C\) such that \(\sigma_P = f \circ \sigma\).

Now the section \(\sigma\) is a Weierstrass section if and only if \(\sigma\) is in the ramification divisor of the double cover \(C \to P\). This is equivalent to the condition that \(j = 0\) as a map \(\sigma_P^* \mathcal{L} \to \mathcal{O}_S\).

Putting this together we obtain the following description of \(\mathcal{H}_{g,w}\).

Proposition 3.1. The stack \(\mathcal{H}_{g,w}\) is equivalent to the stack \(\mathcal{H}'_{g,w}\) parametrizing the data
\[(P \to S, \mathcal{L}, i: \mathcal{L}^2 \hookrightarrow \mathcal{O}_S, \sigma_P, 0)\]

Let \(\mathbb{H}(2g + 2, 1) = \{(f, s) \mid f(0, 1) = s^2\} \subset \mathbb{A}(2g + 2) \times \mathbb{A}^1\) and let \(\mathbb{H}^{sm}(2g + 2, 1)\) be the open set where \(f\) has distinct roots. Likewise let \(\mathbb{H}^w(2g + 2, 1)\) be the divisor defined by the equation \(s = 0\), and let \(\mathbb{H}^{smw}(2g + 2, 1)\) be its intersection with \(\mathbb{H}^{sm}(2g + 2, 1)\). By [5, Proposition 1.5], the stack \(\mathcal{H}'_{g,1}\) is equivalent to the quotient stack \([\mathbb{H}^{smw}(2g + 2, 1)/(B_2/\mu_{g+1})]\) where \(B_2\) is the Borel subgroup of lower triangular matrices in GL_2, and the action is given as follows. If \(A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in B_2/\mu_{g+1}\) then
\[ A \cdot (f(x), s) = (f(A^{-1}x), c^{-g+1}s) \]
Note that our notation \(\mathbb{H}^{smw}(2g + 2, 1)\) differs from that of [9].

The stack \(\mathcal{H}_{g,1}\) is the quotient by \(B_2/\mu_{g+1}\) of the divisor \(\mathbb{H}^{smw}(2g + 2, 1)\) defined by the equation \(s = 0\) in \(\mathbb{H}^{sm}(2g + 2, 1)\).

By [1, Proposition 4.4], the group \(G = B_2/\mu_{g+1}\) is isomorphic to \(B_2\) if \(g\) is even, and if \(g\) is odd to \(\mathbb{G}_m \times (B_2/\mathbb{G}_m)\) where \(B_2/\mathbb{G}_m\) is the group of lower triangular matrices in PGL_2. Under these identifications, the action of \(B_2/\mu_{g+1}\) on \(\mathbb{H}^{sm}(2g + 2, 1)\) is given by
If \( g \) is even, \( G = B_2 \) acts by

\[
A \cdot (f(x), s) := \left( (\det A)^g f(A^{-1}x), a^2 c e^{-\frac{2g+2}{2}} s \right), \quad A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.
\]

If \( g \) is odd, \( G = G_m \times (B_2/G_m) \) acts by

\[
(\alpha, A) \cdot (f(x), s) := \left( \alpha^{-2} \det(A)^{g+1} f(A^{-1}x), \alpha^{-1} a^{\frac{g+1}{2}} c^{-\frac{g+1}{2}} s \right)
\]

where \( A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \) is a lower triangular matrix in \( \text{PGL}_2 \). Note that the expression \( \det(A)^{g+1} f(A^{-1}x) \) is invariant under homotheties so this gives a well-defined action of \( \text{PGL}_2 \).

Now consider the quotient stack \( [\mathcal{M}_{0,2g+2}/S_{2g+2-n}] \).

For any \( n \geq 1 \), the quotient stack \( [\mathcal{M}_{0,2g+2}/S_{2g+2-n}] \) parametrizes the data \( (P \to S, D_{2g+2}, \sigma_1, \ldots, \sigma_n) \) where \( P \to S \) is a twisted \( \text{P}^1 \), \( D_{2g+2} \subset P \) is an effective Cartier divisor which is finite and étale over \( S \) of degree \( 2g+2 \), and \( \sigma_1, \ldots, \sigma_n \) are distinct sections of \( D_{2g+2} \to S \).

The methods used above also realize the stack \( [\mathcal{M}_{0,2g+2}/S_{2g+1}] \) as a quotient of \( \mathbb{H}^w_{s_m}(2g+1) \) by \( B_2/\mu_{2g+2} \). Recall that we identify \( [\mathcal{M}_{0,2g+2}/S_{2g+1}] \) as the stack parametrizing triples \( (P \to S, D_{2g+2}, \sigma) \) where \( P \to S \) is a twisted \( \text{P}^1 \), \( D_{2g+2} \subset P \) is finite and étale of degree \( 2g+2 \) over \( S \), and \( \sigma : S \to D_{2g+2} \) is a section. Taking the marked point to be \((0 : 1) \in \text{P}^1 \) when rigidifying, we have an equivalence \( [\mathcal{M}_{0,2g+2}/S_{2g+1}] = [\mathbb{H}^w_{s_m}(2g+1)/(B_2/\mu_{2g+2})] \). Since \( 2g+2 \) is always even, the quotient \( B_2/\mu_{2g+2} \) is always isomorphic to \( G_m \times (B_2/G_m) \). In terms of this isomorphism, the action on \( \mathbb{H}^w_{s_m}(2g+1) \) is given by

\[
(\alpha, A) \cdot f(x) = \alpha^{-1} (\det A)^{g+1} f(A^{-1}x).
\]

3.1. Description of the Chow rings of \( \mathcal{H}_{g,w} \), \( [\mathcal{M}_{0,2g+2}/S_{2g+1}] \), in terms of torus equivariant Chow groups. Since \( B_2 \) is an unipotent extension of the group of diagonal matrices \( D_2 \), \( B_2/\mu_{g+1} \) and \( B_2/\mu_{2g+2} \) are unipotent extensions of the two-dimensional quotient tori \( D_2/\mu_{g+1} \) and \( D_2/\mu_{2g+2} \) respectively. This implies that for any algebraic space \( X \), the map of stacks \( [X/(D_2/\mu_{g+1})] \to [X/(B_2/\mu_{g+1})] \) (resp. \( [X/(D_2/\mu_{2g+2})] \to [X/(B_2/\mu_{2g+2})] \)) is an isomorphism. It follows that the flat pullback on equivariant Chow groups induces an isomorphism \( A^*_{B_2/\mu_{g+1}}(X) \simeq A^*_{D_2/\mu_{g+1}}(X) \) (resp. \( A^*_{B_2/\mu_{2g+2}}(X) \simeq A^*_{D_2/\mu_{2g+2}}(X) \)). Hence \( A^*(\mathcal{H}_{g,1}) = A^*_T(\mathbb{H}_{s_m}(2g+2,1)) \), \( A^*(\mathcal{H}_{g,w}) = A^*_T(\mathbb{H}^w_{s_m}(2g+2,1)) \) and \( A^*([\mathcal{M}_{0,2g+2}/S_{2g+1}]) = A^*_T(\mathbb{H}^w_{s_m}(2g+2,1)) \) for the actions of the two-dimensional tori \( T = D_2/\mu_{g+1} \) and \( T = D_2/\mu_{2g+2} \) respectively. We now describe these actions.

- If \( g \) is even, then using coordinates \((t_0, t_1)\) on \( T \), the action on \( \mathbb{H}(2g+2,1) \) is given by

\[
(t_0, t_1) \cdot (f(x_0, x_1), s) = \left( (t_0 t_1)^g f(x_0/t_0, x_1/t_1), t_0^{\frac{g+2}{2}} t_1^{\frac{g+2}{2}} s \right).
\]

- If \( g \) is odd, then using coordinates \((\alpha, \rho)\) on \( T \), the action on \( \mathbb{H}(2g+2,1) \) is given by

\[
(\alpha, \rho) \cdot (f(x_0, x_1), s) = \left( \alpha^{-2} \rho^{g+1} f(x_0/\rho, x_1), \alpha^{-1} \rho^{\frac{g+4}{2}} s \right).
\]

- For the stack \([\mathcal{M}_{0,2g+2}/S_{2g+1}]\), using coordinates \((\alpha, \rho)\) on \( T \), the action on \( \mathbb{H}^w(2g+2,1) \) is given by

\[
(\alpha, \rho) \cdot f(x_0, x_1) = \alpha^{-1} \rho^{g+1} f(x_0/\rho, x_1).
\]
Consider the inclusion \( \mathbb{A}(2g + 1) \to \mathbb{A}(2g + 2), h \mapsto x_0 h \). The variety \( \mathbb{H}_{sm}^{w}(2g + 2, 1) \) can be identified with the open set \( U_w := (\mathbb{A}(2g + 1) \setminus \Delta') \setminus L \) where \( \Delta' \) is the discriminant locus of the space \( \mathbb{A}(2g + 1) \) as defined in the previous section, and \( L \) is the linear subspace of forms satisfying \( h(0, 1) = 0 \). By choosing a suitable action of the rank-two torus \( T \) on \( \mathbb{A}(2g + 1) \), we can ensure that the map \( \mathbb{A}(2g + 1) \to \mathbb{A}(2g + 2) \) is \( T \)-equivariant for the various actions of \( T \) on \( \mathbb{A}(2g + 2) \). In particular, we may calculate the Chow rings of the stacks \( \mathcal{H}_{g,w} \) and \([\mathcal{M}_{0,2g+2}/S_{2g+1}]\) as the \( T \)-equivariant Chow rings of \( U_w \) for different actions of the torus \( T \). This is summarized in the following proposition.

**Proposition 3.2.**

- If \( g \) is even, then \( A^{*}(\mathcal{H}_{g,w}) = A_{T}^{*}(U_{w}) \) where the action of \( T \) is given by
  \[
  (t_0, t_1) \cdot h(x_0, x_1) = t_0^{g-1} t_1^{q} h(x_0/t_0, x_1/t_1).
  \]
- If \( g \) is odd, then \( A^{*}(\mathcal{H}_{g,w}) = A_{T}^{*}(U_{w}) \) where the action of \( T \) is given by
  \[
  (\alpha, \rho) \cdot h(x_0, x_1) = \alpha^{-2} \rho^{g} h(x_0/\rho, x_1).
  \]
- For any genus \( g \), \( A^{*}([\mathcal{M}_{0,2g+2}/S_{2g+1}]) = A_{T}^{*}(U_{w}) \) where \( T \) acts by
  \[
  (\alpha, \rho) \cdot h(x_0, x_1) = \alpha^{-1} \rho^{g} h(x_0/\rho, x_1).
  \]

4. Description of \( \mathcal{H}^{w}_{g,2}, \mathcal{H}^{w}_{g,3}, [\mathcal{M}_{0,2g+2}/S_{2g}], [\mathcal{M}_{0,2g+2}/S_{2g-1}] \) as quotients by tori

4.1. The 2-Weierstrass pointed case. Let \( \mathcal{H}^{o}_{g,2} \) be the open substack of \( \mathcal{H}_{g,2} \) parametrizing two-pointed hyperelliptic curves \( (C \to S, \sigma_1, \sigma_2) \) such that for every \( s \in S \), \( \sigma_1(s) + \sigma_2(s) \) does not equal to the \( g_2 \). The construction of \( \mathcal{H}^{o}_{g,2} \) can be extended to give a presentation for \( \mathcal{H}^{o}_{g,2} \) as a quotient stack. The stack \( \mathcal{H}^{o}_{g,2} \) is equivalent to the stack \( (\mathcal{H}^{o}_{g,2})' \) parametrizing the following data

\[
(P \to S, \mathcal{L}, i: \mathcal{L}^2 \subseteq \mathcal{O}_{P}, \sigma_{P,1}, \sigma_{P,2}, j_1, j_2)
\]

where \( P \to S \) is a twisted \( \mathbb{P}^{1} \), \( \mathcal{L} \) is a line bundle on \( P \) which restricts to a line bundle of degree \(-(g + 1)\) on the fibers of \( P \to S \), \( \sigma_{P,1}, \sigma_{P,2}: S \to P \) are distinct sections and \( j_{i}: \sigma_{P,i}^{*}\mathcal{L} \to \mathcal{O}_{S} \) satisfies \( j_{i}^{*}\mathcal{L}^2 = \sigma_{P,i}^{*}(i) \) for \( \ell = 1, 2 \).

The stack \( \mathcal{H}^{o}_{g,2} \) is equivalent to the closed substack \( (\mathcal{H}^{o}_{g,2})' \) in \( (\mathcal{H}^{w}_{g,2})' \) parametrizing tuples where both \( j_{1}, j_{2} \) are the zero map. The same argument of \( \mathcal{H}^{o}_{g,2} \) can be realized as the quotient stack \([\mathbb{H}_{sm}(2g + 2, 2)/D_{2}/\mu_{g+1}]\). Here

\[
\mathbb{H}_{sm}(2g + 2, 2) = \{(f, s_{0}, s_{\infty})| f(0, 1) = s_{0}^{2}, f(1, 0) = s_{\infty}^{2}\} \subset \mathbb{A}_{sm}(2g + 2) \times \mathbb{A}^{2}
\]

and \( D_{2}/\mu_{g+1} \) is the torus in \( GL_{2}/\mu_{g+1} \) that fixes \((1 : 0)\) and \((0 : 1)\) in \( \mathbb{P}^{1} \). The action of \( D_{2}/\mu_{g+1} \) is given by

\[
A \cdot (f(x), s_{0}, s_{\infty}) = (f(A^{-1}x), c^{-(g+1)} s_{0}, a^{-(g+1)} s_{\infty}), \quad A = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}.
\]

With this identification \( \mathcal{H}^{w}_{g,2} \) is the quotient by \( D_{2}/\mu_{g+1} \) of the codimension-two subvariety \( \mathbb{H}_{sm}^{w}(2g + 2, 2) = \{ f \in \mathbb{A}_{sm}(2g + 2) | f(0, 1) = f(1, 0) = 0 \} \subset \mathbb{A}_{sm}(2g + 2) \).

A similar analysis shows that \([\mathcal{M}_{0,2g+2}/S_{2g}]\) is equivalent to the quotient stack \([\mathbb{H}_{sm}^{w}(2g + 2, 2)/D_{2}/\mu_{g+2}]\). Since the quotient \( D_{2}/\mu_{g+1} \) (resp. \( D_{2}/\mu_{2g+2} \)) is a rank-two torus \( T \), we may compute \( A^{*}(\mathcal{H}^{o}_{g,2}), A^{*}(\mathcal{H}^{w}_{g,2}) \) (resp. \( A^{*}([\mathcal{M}_{0,2g+2}/S_{2g}]) \)), by computing the \( T \)-equivariant Chow groups. The \( T \)-actions are described as follows.
• If \( g \) is even, then using coordinates \((t_0, t_1)\) on \( T \), the action on \( \mathbb{H}(2g + 2, 2) \) is given by
  \[
  (t_0, t_1) \cdot (f(x_0, x_1), s_0, s_\infty) = \left( (t_0 t_1)^g f(x_0/t_0, x_1/t_1), t_0^{-g} t_1^{g+2} s_0, t_0^{-g} t_1^{g+2} s_\infty \right).
  \]

• If \( g \) is odd, then using coordinates \((\alpha, \rho)\) on \( T \), the action on \( \mathbb{H}(2g + 2, 2) \) is given by
  \[
  (\alpha, \rho) \cdot (f(x_0, x_1), s_0, s_\infty) = \left( \alpha^{-2} \rho^{g+1} f(x_0/\rho, x_1), \alpha^{-1} \rho^{g+1} s_0, \alpha^{-1} \rho^{g+1} s_\infty \right).
  \]

• For the stack \([\mathcal{M}_{0, 2g+2}/S_{2g}]\), using coordinates \((\alpha, \rho)\) on \( T \) the action on \( \mathbb{H}^w(2g + 2, 2) \) is given by
  \[
  (\alpha, \rho) \cdot f(x_0, x_1) = \alpha^{-1} \rho^{g+1} f(x_0/\rho, x_1).
  \]

As in the one-pointed case we can consider the map \( A(2g) \to \mathbb{H}(2g + 2, 2) \), \( h \mapsto x_0 x_1 h \), and we see that \( \mathbb{H}^w_{sm}(2g + 2, 2) \) is the image of the open set \( U_{w,2} \subset A(2g) \) where \( U_{w,2} := (\mathbb{A}(2g) \setminus \Delta^g) \setminus (L_0 \cup L_\infty) \) is the open set parametrizing forms \( h(x_0, x_1) \) of degree 2g with distinct roots such that both \( h(1,0) \) and \( h(0,1) \) are non-zero.

For suitable choices of \( T \)-action on \( A(2g) \), we can make the map \( U_{w,2} \to \mathbb{H}^w_{sm}(2g + 2, 2) \) \( T \)-equivariant with respect to the various actions on \( \mathbb{H}^w_{sm}(2g + 2, 2) \). This can be summarized as follows.

**Proposition 4.1.**

- If \( g \) is even, then \( A^*(\mathcal{H}_{g,2}^w) = A^*_T(U_{w,2}) \) where the action of \( T \) is given by
  \[
  (t_0, t_1) \cdot h(x_0, x_1) = (t_0 t_1)^g h(x_0/t_0, x_1/t_1).
  \]

- If \( g \) is odd, then \( A^*(\mathcal{H}_{g,2}^w) = A^*_T(U_{w,2}) \) where the action of \( T \) is given by
  \[
  (\alpha, \rho) \cdot h(x_0, x_1) = \alpha^{-2} \rho^g h(x_0/\rho, x_1).
  \]

- For any genus \( g \), \( A^*([\mathcal{M}_{0, 2g+2}/S_{2g}]) = A^*_T(U_{w,2}) \) where \( T \) acts by
  \[
  (\alpha, \rho) \cdot h(x_0, x_1) = \alpha^{-1} \rho^g h(x_0/\rho, x_1).
  \]

### 4.2. The 3-Weierstrass-pointed case.

A similar construction can be used to present \( \mathcal{H}_{g,3}^w \) as closed quotient substack of the stack \( \mathcal{H}_{g,3} \) parametrizing 3-pointed hyperelliptic curves where no two of the points sum to a \( g \). In this case \( \mathcal{H}_{g,3}^w = \mathbb{H}_{sm}(2g + 2, 3)/((Z(\text{GL}_2/\mu_{g+1}))); \)

where \( \mathbb{H}_{sm}(2g + 2, 3) = \{(f, s_0, s_1, s_\infty)|f(0,1) = s_0^2, f(1,1) = s_1^2, f(1,0) = s_\infty^2\} \subset \mathbb{A}_{sm}(2g + 2) \times \mathbb{A}^3 \) and \( Z(\text{GL}_2/\mu_{g+1}) = \mathbb{G}_m \) is the center of \( \text{GL}_2/\mu_{g+1} \). Similarly, we also have that \( [\mathcal{M}_{0, 2g+2}/S_{2g-1}] \) is equivalent to the quotient \( \mathbb{H}_{sm}^w(2g + 2, 3)/\mathbb{G}_m \) where \( \mathbb{H}_{sm}^w(2g + 2, 3) = \{f \in \mathbb{A}_{sm}(2g + 2) \mid f(0,1) = f(1,1) = f(1,0) = 0\} \) is a codimension-three subvariety in \( \mathbb{A}_{sm}(2g + 2) \). The \( \mathbb{G}_m \)-action is described as follows.

- If \( g \) is even, then the action on \( \mathbb{H}(2g + 2, 3) \) is given by
  \[
  t \cdot (f(x_0, x_1), s_0, s_1, s_\infty) = \left( t^{2g} f(x_0/t, x_1/t), t^{-1} s_0, t^{-1} s_1, t^{-1} s_\infty \right).
  \]

- If \( g \) is odd, then the action on \( \mathbb{H}(2g + 2, 3) \) is given by
  \[
  t \cdot (f(x_0, x_1), s_0, s_1, s_\infty) = \left( t^{-2} f(x_0, x_1), t^{-1} s_0, t^{-1} s_1, t^{-1} s_\infty \right).
  \]

- For the stack \([\mathcal{M}_{0, 2g+2}/S_{2g-1}]\), the action on \( \mathbb{H}^w(2g + 2, 3) \) is given by
  \[
  t \cdot f(x_0, x_1) = t^{-1} f(x_0, x_1).
  \]
The stack $\mathcal{H}_{g,3}^w$ is the quotient of the closed codimension-three subset where all of the three $s_i$’s are 0. This in turn can be realized as $[U_{w,3}/\mathbb{G}_m]$ where $U_{w,3} := (\mathbb{A}(2g-1) \setminus \Delta') \setminus (L_0 \cup L_1 \cup L_\infty)$ is the parameter space of binary forms of degree $2g-1$ with distinct roots which do not vanish at $(0 : 1), (1 : 1), (1 : 0)$ in $\mathbb{P}^1$, with the induced action of $\mathbb{G}_m$ described in the following proposition.

**Proposition 4.2.**
- For any genus $g$, $A^*(\mathcal{H}_{g,3}^w) = A^*_{\mathbb{G}_m}(U_{w,3})$ where the action of $\mathbb{G}_m$ is given by
  $$t \cdot h(x_0, x_1) = t^{-2}h(x_0, x_1).$$
- For any genus $g$, $A^*([ \mathcal{M}_{0,2g+2}/S_{2g-1}]) = A^*_{\mathbb{G}_m}(U_{w,3})$ where $\mathbb{G}_m$ acts by
  $$t \cdot h(x_0, x_1) = t^{-1}h(x_0, x_1).$$

**Remark 4.3.** The integral Chow ring of $\mathcal{H}_{g,3}^w$ can also be computed using Proposition 6.2. However, the advantage of working with the presentation $[U_{w,3}/\mathbb{G}_m]$ is that it will allow us to identify a natural generator for $\text{Pic}(\mathcal{H}_{g,3}^w)$.

## 5. Computing the Chow rings

In this section we prove Theorem 1.3 and 1.4 by computing the appropriate $T$-equivariant Chow rings as indicated in Proposition 3.2 and 4.1. We begin with a general result on the $T$-equivariant Chow ring of the set of binary forms of degree $N$ with distinct roots.

### 5.1. The equivariant Chow ring of $\mathbb{A}(N) \setminus \Delta'$.

Suppose that we are given an arbitrary action of $T = (\mathbb{G}_m)^2$ on $\mathbb{A}(N)$, the affine space of binary forms of degree $N$ in $x_0, x_1$, by

$$
(t_0, t_1) \cdot f(x_0, x_1) = t_0^a t_1^b f(t_0^{-a_0} t_1^{-a_1} x_0, t_0^{-b_0} t_1^{-b_1} x_1).
$$

The goal of this section is to give a presentation for the equivariant Chow ring $\mathbb{A}^*_T(\mathbb{A}(N) \setminus \Delta')$ where $\Delta'$ is the discriminant locus of $\mathbb{A}(N)$ as defined before, in terms of the weights of the action.

Let $\lambda_1 : T \to \mathbb{G}_m, (t_0, t_1) \mapsto t_0$ and $\lambda_2 : T \to \mathbb{G}_m, (t_0, t_1) \mapsto t_1$ be the standard generators for the character group of the torus $T$. Set $c_1(\lambda_1) = l_1$ and $c_1(\lambda_2) = l_2$ in $A^*(BT)$. Define the characters $\chi_1, \chi_2$ by $\chi_1(t) = t_0^{\alpha_0} t_1^{\alpha_1}, \chi_2(t) = t_0^{\beta_0} t_1^{\beta_1}$, and set

$$
T_1 := c_1(\chi_1) = c_1(\lambda_1^{a_0} \lambda_2^{a_1}) = \alpha_0 l_1 + \alpha_1 l_2,
$$

$$
T_2 := c_1(\chi_2) = c_1(\lambda_1^{\beta_0} \lambda_2^{\beta_1}) = \beta_0 l_1 + \beta_1 l_2.
$$

Let $\mathbb{B}(N) = \mathbb{P}(\mathbb{A}(N))$ be the projectivization of the affine space $\mathbb{A}(N)$ of binary forms of degree $N$. We denote by $\Delta$ the image of the discriminant locus $\Delta'$ under the projectivization map.

**Proposition 5.1.** The equivariant Chow ring $\mathbb{A}^*_T(\mathbb{A}(N) \setminus \Delta')$ is a quotient ring

$$
\mathbb{Z}[l_1, l_2, \xi]/(\alpha_{1,0}(\xi), \alpha_{1,1}(\xi), a_l l_1 + b_l - \xi, p(\xi))
$$
where $\xi = c_1(\mathcal{O}_{\mathbb{P}(N)}(1))$ is the hyperplane class and

$$
\alpha_{1,0} = 2(N - 1)\xi - N(N - 1)(T_1 + T_2)
= 2(N - 1)\xi - N(N - 1)[(\alpha_0 + \beta_0)l_1 + (\alpha_1 + \beta_1)l_2],
$$

$$
\alpha_{1,1} = \xi^2 - (T_1 + T_2)\xi - N(N - 2)T_1T_2
= \xi^2 - [(\alpha_0 + \beta_0)l_1 + (\alpha_1 + \beta_1)l_2]\xi - N(N - 2)(\alpha_0l_1 + \alpha_1l_2)(\beta_0l_1 + \beta_1l_2),
$$

$$
p(\xi) = \prod_{i=0}^{N}[\xi - (N - i)T_1 - iT_2]
= \prod_{i=0}^{N}[\xi - (N - i)(\alpha_0l_1 + \alpha_1l_2) - i(\beta_0l_1 + \beta_1l_2)].
$$

**Proof.** Choose coordinates $(X_0 : X_1 : \ldots : X_N)$ on $\mathbb{P}(N)$ so that the coordinate function $X_i$ is the coefficient of $x_0^{N-i}x_1$ in a binary form of degree $N$. Then there is an induced action of $T$ on $\mathbb{P}(N)$ given by

$$
t \cdot (X_0 : \ldots : X_N) = (\chi_i^{-N}(t)X_0 : \ldots : \chi_i^{-(N-i)}(t)X_i : \ldots : \chi_N^{-N}(t)X_N).
$$

Following [13] Lemma 2.3 which is proved for $G_m$-actions in [13] Section 3.3, we obtain that $A^*_T(\mathbb{P}(N)) = \mathbb{Z}[l_1, l_2, \xi]/p(\xi)$ where $\xi = c_1(\mathcal{O}_{\mathbb{P}(N)}(1))$ is the hyperplane class, and $p(\xi) = \prod_{i=0}^{N}(\xi - (N - i)T_1 - iT_2)$ is a degree $N+1$ monic polynomial since $\{-(N - i)T_1 - iT_2\}_{i=0}^{N}$ is the set of equivariant Chern roots of the representation $\mathbb{A}(N)$.

By [13] Proposition 4.2 the ideal generated by the image of $A^*_T(\Delta) \to A^*_T(\mathbb{P}(N))$ is computed, and we get

$$
A^*_T(\mathbb{P}(N) \setminus \Delta) = A^*_T(\mathbb{P}(N))/((\alpha_{1,0}(\xi), \alpha_{1,1}(\xi)).
$$

Now we have a $G_m$-torsor $\mathbb{A}(N) \setminus \Delta' \to \mathbb{P}(N) \setminus \Delta$ associated to the line bundle $\lambda^a_1 \otimes \lambda^b_2 \otimes \mathcal{O}(-1)$. An argument similar to the one used in the proof of [13] Lemma 3.2] shows that the pullback $A^*_T(\mathbb{P}(N) \setminus \Delta) \to A^*_T(\mathbb{A}(N) \setminus \Delta')$ is surjective with kernel generated by $al_1 + bl_2 - \xi$. Substituting $\xi = al_1 + bl_2$ yields the presentation of the proposition. \qed

5.2. Integral Chow ring of $H^w_{g,n}$ for even genus $g$ and $n = 1, 2$.

In this section, we will apply the results of the previous section to compute the integral Chow rings of $H^w_{g,n}$ for even genus $g$ and $n = 1, 2$.

5.2.1. 1-Weierstrass-pointed case. Recall that by Proposition 5.2 if the genus $g$ is even, then $A^*(H_{g,w}) = A^*_T(U_w)$ where $U_w = (\mathbb{A}(2g+1) \setminus \Delta') \setminus L$ with the action of $T = G^2_m$ on $\mathbb{A}(2g+1)$ defined by

$$(t_0, t_1) \cdot f(x_0, x_1) = t_0^{a-1}t_1^bf(x_0/t_0, x_1/t_1).$$

Then by Proposition 5.1 with $N = 2g + 1$, $a = g - 1$, $b = g$ and $(\alpha_0, \alpha_1) = (1, 0)$, $(\beta_0, \beta_1) = (0, 1)$ we have

$$
A^*_T(\mathbb{A}(2g+1) \setminus \Delta') = \mathbb{Z}[l_1, l_2][\xi]/(\alpha_{1,0}(\xi), \alpha_{1,1}(\xi), (g - 1)l_1 + gl_2 - \xi, p(\xi))
$$

where

$$
p(\xi) = \prod_{i=0}^{2g+1}[(\xi - (2g - i + 1)l_1 - il_2)].
$$
Let \( i : L \setminus (\Delta' \cap L) \hookrightarrow \mathbb{A}(2g + 1) \setminus \Delta' \) be the \( T \)-equivariant closed embedding. By the localization sequence for equivariant Chow groups, \( A^*_T((\mathbb{A}(2g + 1) \setminus \Delta') \setminus L) \) is the quotient of \( A^*_T(\mathbb{A}(2g + 1) \setminus \Delta') \) by

\[
\text{Im}(i_* : A^*_T(L \setminus (\Delta' \cap L)) \to A^*_T(\mathbb{A}(2g + 1) \setminus \Delta')).
\]

Since \( L \) and \( \mathbb{A}(2g + 1) \) are linear subspaces, the pullback \( i^* : A^*_T(\mathbb{A}(2g + 1) \setminus \Delta') \to A^*_T(L \setminus (\Delta' \cap L)) \) is necessarily surjective because it factors the surjective map \( A^*_T(\text{Spec } k) \to A^*_T(L \setminus (\Delta' \cap L)) \) through the surjection \( A^*_T(\text{Spec } k) \to A^*_T(\mathbb{A}(2g + 1) \setminus \Delta') \). By the projection formula \( i_* i^*[\alpha] = [\alpha] \cdot [L \setminus (\Delta' \cap L)] \) for any class \( [\alpha] \in A^*(\mathbb{A}(2g + 1) \setminus \Delta') \). Thus the image of \( i_* \) is generated by the image of the fundamental class of the hyperplane \( [L]_T \) under the composite \( i_* j^* \) where \( j : L \setminus (\Delta' \cap L) \hookrightarrow L \) is the open immersion.

Thus to finish obtaining a presentation for \( A^*_T(U_w) \) we need only compute the \( T \)-equivariant fundamental class of the hyperplane \( L \). This hyperplane is defined by the vanishing of the coefficient of \( x_1^{2g+1} \) in a binary form and its \( T \)-equivariant fundamental class is

\[
[L]_T = c_1 \left( \lambda_1^{-1} \lambda_2^{g-(2g+1)} \right) = (g-1)l_1 - (g+1)l_2.
\]

Thus we get

\[
A^*_T(\mathbb{A}(2g + 1) \setminus \Delta' \setminus L) = \frac{\mathbb{Z}[l_1, l_2]}{\langle \alpha_{1,0}(\xi), \alpha_{1,1}(\xi), p(\xi), (g-1)l_1 - (g+1)l_2 \rangle}
\]

where

\[
\begin{align*}
\xi &= (g-1)l_1 + gl_2, \\
\alpha_{1,0}(\xi) &= -[2g(l_1 + l_2) + 4gl_1], \\
\alpha_{1,1}(\xi) &= (g-1)(g-2)l_1^2 + g(g-1)l_2^2 - 2(g^2 + 2g - 1)l_1l_2.
\end{align*}
\]

**Proposition 5.2.** The polynomial \( p((g-1)l_1 + gl_2) \) is in the ideal generated by \( \alpha_{1,0}((g-1)l_1 + gl_2), \alpha_{1,1}((g-1)l_1 + gl_2) \) and \( (g-1)l_1 - (g+1)l_2 \).

**Proof.** We can write

\[
p((g-1)l_1 + gl_2) = \prod_{i=0}^{2g+1} [(-g + i - 2)l_1 + (g - i)l_2].
\]

Notice that the \((i = 2g + 1)\)-th term on the right hand side of the above equality is

\[
(g-1)l_1 + (-g - 1)l_2,
\]

thus it is contained in the ideal generated by \((g-1)l_1 - (g+1)l_2\).

**Proposition 5.3.** \( \alpha_{1,1}((g-1)l_1 + gl_2) \) is in the ideal generated by \( \alpha_{1,0}((g-1)l_1 + gl_2) \) and \((g-1)l_1 - (g+1)l_2\).

**Proof.** For simplicity, let us denote by \( g_1, g_2 \) the two degree one generators \( \alpha_{1,0}((g-1)l_1 + gl_2) \) and \((g-1)l_1 - (g+1)l_2\) respectively. It is easy to see that

\[
\alpha_{1,1}((g-1)l_1 + gl_2) = (l_2) \cdot g_1 + [(g-2)l_1 - gl_2] \cdot g_2.
\]

Therefore for even genus \( g \),

\[
A^*(\mathcal{H}_{g,w}) = \frac{\mathbb{Z}[l_1, l_2]}{(6gl_1 + 2gl_2, (g-1)l_1 - (g+1)l_2)}.
\]
Note that there is a natural ring homomorphism $A^*(\mathcal{H}_g) \to A^*(\mathcal{H}_{g,w})$, and it can be given explicitly by sending $c_1 \mapsto l_1 + l_2$ and $c_2 \mapsto l_1 l_2$. More precisely,

$$A^*(\mathcal{H}_g) = \frac{\mathbb{Z}[c_1, c_2]}{(2(2g+1)c_1, g(g-1)c_1^2 - 4g(g+1)c_2)}$$

$$\downarrow$$

$$A^*(\mathcal{H}_{g,w}) = \frac{\mathbb{Z}[l_1, l_2]}{(6gl_1 + 2gl_2, (g-1)l_1 - (g+1)l_2)}.$$

We can check that the images of the generators of ideal for $A^*(\mathcal{H}_g)$ are indeed vanishing in the Chow ring $A^*(\mathcal{H}_{g,w})$. Namely, let $f_1(c_1) = 2(2g+1)c_1$ and $f_2(c_1, c_2) = g(g-1)c_1^2 - 4g(g+1)c_2$, then

$$f_1(l_1 + l_2) = 2(2g+1)(l_1 + l_2)$$

$$= 1 \cdot g_1 + (-2) \cdot g_2,$$

and

$$f_2(l_1 + l_2, l_1 l_2) = g(g-1)(l_1 + l_2)^2 - 4g(g+1)l_1 l_2$$

$$= (-l_2) \cdot g_1 + [g(l_1 - l_2)] \cdot g_2,$$

where $g_1, g_2$ denote two degree one generators of ideal of $A^*(\mathcal{H}_{g,w})$ as above.

Furthermore, the above ring map is injective in codimension one. In order to see this, for integers $A, B, a \in \mathbb{Z}$ such that

$$A(6gl_1 + 2gl_2) + B[(g-1)l_1 - (g+1)l_2] = a(l_1 + l_2),$$

we need to prove that the minimum integer $a$ with the above equality holds has to be $2(2g+1)$. It can be seen by comparing the coefficients of $l_1, l_2$ on both sides, so we have

$$A = \frac{a}{2(2g+1)}, \quad B = \frac{-a}{2g+1},$$

and thus it implies the ring homomorphism is injective on Picard groups.

In particular, by computing the Smith normal form of the two degree one relations, we can write

$$A^*(\mathcal{H}_{g,w}) = \frac{\mathbb{Z}[l_1, l_2]}{(l_2 + (4g^2 - 2g + 1)l_1, 4g(2g+1)l_1)}$$

$$\cong \frac{\mathbb{Z}[l_1]}{(4g(2g+1)l_1)}$$

which means the integral Picard group $\text{Pic}(\mathcal{H}_{g,w})$ for even genus $g$ is cyclic of order $4g(2g+1)$.

**Natural generator for $A^*(\mathcal{H}_{g,w})$.** There are multiple integral linear combinations of $l_1, l_2$ which generate the integral Picard group of $\mathcal{H}_{g,w}$. We conclude our proof by showing that the restriction to $\mathcal{H}_{g,w}$ of $\psi$-class on $\mathcal{H}_{g,1}$ generates $A^*(\mathcal{H}_{g,w})$.

**Corollary 5.4.** $A^*(\mathcal{H}_{g,w})$ is generated by $\psi$, and consequently

$$A^*(\mathcal{H}_{g,w}) = \mathbb{Z}[\psi]/(4g(2g+1)\psi).$$
Proof. According to the degree one relations, we can rewrite these two relations as follows
\[
\begin{align*}
(\ell_1 - \ell_2) + (-2) \left( \left( 1 - \frac{g}{2} \right) \ell_1 + \frac{g}{2} \ell_2 \right) &= 0, \\
4g(2g + 1) \left( \left( 1 - \frac{g}{2} \right) \ell_1 + \frac{g}{2} \ell_2 \right) &= 0.
\end{align*}
\]
By setting
\[
l_3 := \left( 1 - \frac{g}{2} \right) \ell_1 + \frac{g}{2} \ell_2 + \frac{(g - 1)\ell_1 - (g + 1)\ell_2}{2},
\]
we get a new generator of \( \text{Pic}(H_{g,w}) \) with order \( 4g(2g + 1) \).

Claim. The pullback of \( \psi \) to \( H_{g,w} \) has Chow class \(-\frac{g}{2}\ell_1 + \frac{q+2}{2}\ell_2\).

Proof of Claim. The restriction of \( \psi \) to \( H_{g,w} \) is the Chow class which associates to the Weierstrass-pointed hyperelliptic curves \( C \to S, \sigma_w \) the class \( \sigma_w^*(c_1(\mathcal{O}(-\sigma_w))) \). On the other hand, we may also consider the class \( c_1(\mathcal{O}_{H_{g,1}}(H_{g,w})) \). Its restriction to a family \( (C \to S, \sigma_w) \) is the class \( \sigma_w^*(c_1(\mathcal{O}(\sigma_w))) = -\psi \). On the other hand, we know that \( H_{g,w} \) is defined by the equation \( s = 0 \) in the quotient stack \( \mathbb{H}_{sm}(2g + 2, 1)/B_2 \). This equation has \( T \)-weight \( t_0^2 t_1^{-\frac{2g+2}{2}} \). Thus, \( c_1(\mathcal{O}_{H_{g,1}}(H_{g,w})) = \frac{g}{2}\ell_1 - \frac{g+2}{2}\ell_2 \in A_T^*(\mathbb{H}_{sm}^w(2g + 2, 1)) = A^*(H_{g,w}) \). The claim now follows. \( \Box \)

5.2.2. 2-Weierstrass-pointed case. By Proposition \ref{prop:2-weierstrass-pointed}, \( A^*(H_{g,2}^w) = A_T^*(U_{w,2}) \) where \( U_{w,2} = (\mathbb{A}(2g) \setminus \Delta') \setminus (L_0 \cup L_\infty) \) and \( T \) acts on \( \mathbb{A}(2g) \) by
\[
(t_0, t_1) \cdot h(x_0, x_1) = (t_0 t_1)^{g-1} h(x_0/t_0, x_1/t_1).
\]
By Proposition \ref{prop:H2w} with \( N = 2g \), \( a = b = g - 1 \) and \( (\alpha_0, \alpha_1) = (1, 0) \), \( (\beta_0, \beta_1) = (0, 1) \) we have that
\[
A_T^*(\mathbb{A}(2g) \setminus \Delta') = \frac{\mathbb{Z}[l_1, l_2][\xi]}{(\alpha_{1,0}(\xi), \alpha_{1,1}(\xi), p(\xi), (g - 1)(l_1 + l_2) - \xi)}
\]
where
\[
p((g - 1)(l_1 + l_2)) = \prod_{i=0}^{2g}((-g + i - 1)l_1 + (g - i - 1)l_2),
\]
\[
\alpha_{1,0}((g - 1)(l_1 + l_2)) = -2(2g - 1)(l_1 + l_2),
\]
\[
\alpha_{1,1}((g - 1)(l_1 + l_2)) = (g - 1)(g - 2)(l_1 + l_2)^2 - 4g(g - 1)l_1 l_2.
\]
The hyperplanes \( L_0, L_\infty \) are defined by the vanishing of the coefficients of \( x_1^{2g}, x_0^{2g} \) respectively, and their equivariant fundamental classes are
\[
[L_0]_T = (g - 1)l_1 - (g + 1)l_2,
\]
and
\[
[L_\infty]_T = -(g + 1)l_1 + (g - 1)l_2.
\]
By a similar argument used in the one-pointed case, we conclude that
\[
A_T^*(U_{w,2}) = \frac{\mathbb{Z}[l_1, l_2]}{(\alpha_{1,0}(\xi), \alpha_{1,1}(\xi), p(\xi), (g - 1)(l_1 + l_2) - (g + 1)l_1 + (g - 1)l_2)}
\]
where \( \xi = (g - 1)(l_1 + l_2) \).
Proposition 5.5. The polynomials $p((g-1)(l_1+l_2))$, $\alpha_{1,0}((g-1)(l_1+l_2))$ and the degree two relation $\alpha_{1,1}((g-1)(l_1+l_2))$ are all in the ideal generated by the classes of two hyperplanes $[L_0]_T, [L_\infty]_T$.

Proof. Notice that $[L_0]_T + [L_\infty]_T = -2(l_1+l_2)$ so $\alpha_{1,0}((g-1)(l_1+l_2)) = (2g-1)([L_0]_T + [L_\infty]_T)$. The first term in the product of $p((g-1)(l_1+l_2))$ associated to $i = 0$ is $-(g+1)l_1 + (g-1)l_2 = [L_0]_T$. To see that $\alpha_{1,1}$ is in the ideal generated by $[L_0]_T, [L_\infty]_T$ note that

$$\alpha_{1,1}((g-1)(l_1+l_2)) = (g-1)(g-2)(l_1+l_2)^2 - 4g(g-1)l_1l_2$$

$$= \left[\frac{(g-1)(g-2)}{2}\right] 2(l_1 + l_2)^2 + (g-1)^2l_2[L_\infty]_T + (g^2 - 1)l_2[L_0]_T. \qed$$

The integral Chow ring of $H_{g,2}^w$ can be expressed as

$$A^*(H_{g,2}^w) = \frac{\mathbb{Z}[l_1, l_2]}{((g-1)l_1 - (g+1)l_2, 2(l_1 + l_2))}.$$ 

In particular, the Picard group of $H_{g,2}^w$ can be computed as

$$\text{Pic}(H_{g,2}^w) = \mathbb{Z}l_1 \oplus \mathbb{Z}l_2 \cong \frac{\mathbb{Z}l_2}{(4gl_2)} \cong \frac{\mathbb{Z}l_1}{(4gl_1)},$$

meaning that the integral Picard group is cyclic of order $4g$. To obtain a natural generator of the Picard group we can rewrite the degree one relations as

$$\left\{ -2 \left( l_1 + \frac{g}{2}(l_2 - l_1) \right) + (l_1 - l_2) = 0, \right.$$ 

$$\left. 4 \left( l_1 + \frac{g}{2}(l_2 - l_1) \right) + 2(g-1)(l_1 - l_2) = 0. \right.$$ 

Thus

$$l_3 := \left( l_1 + \frac{g}{2}(l_2 - l_1) \right) + (g-1)l_1 - (g+1)l_2$$

$$= \frac{g}{2}l_1 - \frac{g + 2}{2}l_2$$

can also be realized as a generator of the Picard group. Once again we can check that $-l_3 = \psi_\infty$ where $\psi_\infty$ is the $\psi$-class of the section which corresponds to the section $s_\infty$ in the presentation of $H_{g,2}^w$ as a quotient of the closed subvariety

$$H_{2,\text{sm}}^w(2g+2, 2) \subset H_{\text{sm}}(2g+2, 2) = \{(f, s_0, s_\infty)| f(0, 1) = s_0^2, f(1, 0) = s_\infty^2 \}$$

described in Section 4.11. Moreover, by symmetry, it is easy to check that $l_3' = -\frac{g+2}{2}l_1 + \frac{g}{2}l_2$ also generates the Picard group. Now we have $-l_3' = \psi_0$ where $\psi_0$ corresponds to the section $s_0$.

5.3. Integral Chow ring of $H_{g,n}^w$ for odd genus $g$ and $n = 1, 2$.

We will then finish the computation of the integral Chow ring of $H_{g,n}^w$ for odd genus $g$ and $n = 1, 2$. 

5.3.1. 1-Weierstrass-pointed case. Recall that by Proposition 3.2 if the genus $g$ is odd, then $A^*(\mathcal{H}_{g,w}) = \mathcal{A}^*_T(U_w)$ where $U_w = (\mathbb{A}(2g+1 \setminus \Delta') \setminus L$ with the action of $T = \mathbb{G}_m^2$ on $\mathbb{A}(2g+1)$ defined by
\[
(\alpha, \rho) \cdot f(x_0, x_1) = \alpha^{-2} \rho^g f(x_0/\rho, x_1).
\]

Then by Proposition 5.1 with $N = 2g + 1$, $a = -2, b = g$ and $(\alpha_0, \alpha_1) = (0, 1), (\beta_0, \beta_1) = (0, 0)$ we have
\[
A^*_T(\mathbb{A}(2g+1 \setminus \Delta')) = \frac{\mathbb{Z}[l_1, l_2][\xi]}{(\alpha_{1,0}(\xi), \alpha_{1,1}(\xi), p(\xi), -2l_1 + gl_2 - \xi)}
\]
where
\[
p(\xi) = \prod_{i=0}^{2g+1} [\xi - (2g - i + 1)l_2].
\]

By substituting $\xi$ by $-2l_1 + gl_2$, we can write the generators of ideal as
\[
\alpha_{1,0}(-2l_1 + gl_2) = -[2g(l_1 + l_2) + 6gl_1],
\]
\[
\alpha_{1,1}(-2l_1 + gl_2) = (-2l_1 + gl_2)^2 - l_2(-2l_1 + gl_2).
\]
The hyperplane $L$ is defined by the vanishing of the coefficient of $x_1^{2g+1}$ and with the given weights its equivariant fundamental class is
\[
[L]_T = -2l_1 + gl_2.
\]
Therefore,
\[
A^*_T(U_w) = \frac{\mathbb{Z}[l_1, l_2]}{(\alpha_{1,0}(-2l_1 + gl_2), \alpha_{1,1}(-2l_1 + gl_2), -2l_1 + gl_2, p(-2l_1 + gl_2))}.
\]

But the following proposition can be easily checked.

**Proposition 5.6.** The polynomial $p(-2l_1 + gl_2)$ and $\alpha_{1,1}$ are both in the ideal generated by $\alpha_{1,0}(-2l_1 + gl_2)$ and $-2l_1 + gl_2$.

**Proof.** It is obvious that $\alpha_{1,1}$ is in the ideal generated by $-2l_1 + gl_2$ since $\alpha_{1,1} = \xi^2 - l_2\xi$. For $p(\xi)$, we can write
\[
p(-2l_1 + gl_2) = \prod_{i=0}^{2g+1} [-2l_1 + (-g + i - 1)l_2]
\]
\[
= (-2l_1 + gl_2) \prod_{i=0}^{2g} [-2l_1 + (-g + i - 1)l_2],
\]
and thus it is also in the ideal generated by $-2l_1 + gl_2$.

Therefore for odd genus $g$,
\[
A^*(\mathcal{H}_{g,w}) = \frac{\mathbb{Z}[l_1, l_2]}{(8gl_1 + 2gl_2, -2l_1 + gl_2)}.
\]
Likewise, we have the natural ring homomorphism

\[ A^*(\mathcal{H}_g) = \frac{\mathbb{Z}[\tau, c_2, c_3]}{(4(2g+1)\tau, 8\tau^2 - 2(g^2 - 1)c_2, 2c_3)} \]

\[ \xrightarrow{\tau \mapsto l_1, c_2 \mapsto -l_2^2, c_3 \mapsto 0} \]

\[ A^*(\mathcal{H}_{g,w}) = \frac{\mathbb{Z}[l_1, l_2]}{(8gl_1 + 2gl_2, -2l_1 + gl_2)}. \]

It can be checked that the image of generators \( f_1(\tau) = 4(2g+1)\tau, f_2(\tau, c_2) = 8\tau^2 - 2(g^2 - 1)c_2 \)
under the ring homomorphism are both vanishing in \( A^*(\mathcal{H}_{g,w}) \) since

\[ f_1(l_1) = 4(2g+1)l_1 = 1 \cdot g_1 + (-2) \cdot g_2 \]

and

\[ f_2(l_1, -l_2^2) = 8l_1^2 + 2(g^2 - 1)l_2^2 = l_2 \cdot g_1 + (-4l_1 + 2gl_2) \cdot g_2 \]

where \( g_1 := 8gl_1 + 2gl_2 \) and \( g_2 := -2l_1 + gl_2 \) are the generators of the ideal for \( A^*(\mathcal{H}_{g,w}) \).

Moreover it can be checked similarly that the ring map is injective on Picard groups.

In particular, by computing the Smith normal form of the degree one relations and changing one of the generators to

\[ l'_3 := l_1 - \left(\frac{g-1}{2}\right)l_2, \]

we can express the integral Picard group of \( \mathcal{H}_{g,w} \) as follows

\[ \text{Pic}(\mathcal{H}_{g,w}) = \frac{\mathbb{Z}l_2 \oplus \mathbb{Z}l'_3}{(l_2 - 2l'_3, 4g(2g+1)l'_3)} \cong \frac{\mathbb{Z}l'_3}{(4g(2g+1)l'_3)} \]

which means that the integral Picard group of \( \mathcal{H}_{g,w} \) for odd genus \( g \) is also cyclic of order \( 4g(2g+1) \).

**Natural generator for \( A^*(\mathcal{H}_{g,w}) \).** There are multiple integral linear combinations of \( l_1, l_2 \)
which generate the integral Picard group of \( \mathcal{H}_{g,w} \). We now show that the restriction to \( \mathcal{H}_{g,w} \)
of \( \psi \)-class on \( \mathcal{H}_{g,1} \) generates \( A^*(\mathcal{H}_{g,w}) \).

**Corollary 5.7.** \( A^*(\mathcal{H}_{g,w}) \) is generated by \( \psi \) and consequently

\[ A^*(\mathcal{H}_{g,w}) = \frac{\mathbb{Z}[\psi]}{4g(2g+1)\psi}. \]

**Proof.** Set

\[ l_3 := l'_3 + \frac{(-2)l_1 + gl_2}{\text{relation}} \]

\[ = (-1)l_1 + \left(\frac{g+1}{2}\right)l_2, \]

**Claim.** The pullback of \( \psi \) to \( \mathcal{H}_{g,w} \) has Chow class \( l_1 - \frac{g+1}{2}l_2 \).
Proof of Claim. As in the even genus case, the restriction of \( \psi \) to \( \mathcal{H}_{g,w} \) is the Chow class which associates to the Weierstrass-pointed hyperelliptic curves \( (C \to S, \sigma_w) \) the class \( \sigma_w^*(c_1(\mathcal{O}(\sigma_w))) \). On the one hand, we may also consider the class \( c_1(\mathcal{O}_{\mathcal{H}_{g,1}}(\mathcal{H}_{g,w})) \). Its restriction to a family \( (C \to S, \sigma_w) \) is the class \( \sigma_w^*(c_1(\mathcal{O}(\sigma_w))) = -\psi \). On the other hand we know that \( \mathcal{H}_{g,w} \) is defined by the equation \( s = 0 \) in the quotient stack \( \mathbb{H}_{g,-}(2g + 2, 1)/\mathbb{G}_m \). This equation has \( T \)-weight \( \alpha^{-1} \rho^{\frac{g+1}{2}} \). Thus, \( c_1(\mathcal{O}_{\mathcal{H}_{g,1}}(\mathcal{H}_{g,w})) = -l_1 + \frac{g+1}{2}l_2 \in A^*_T(\mathbb{H}_{g,1}) = A^*(\mathcal{H}_{g,w}) \). The claim now follows.

5.3.2. 2-Weierstrass-pointed case. By Proposition 4.1 when \( g \) is odd, we have \( A^*(\mathcal{H}^w_{g,2}) = A^*_T(U_{w,2}) \) where \( U_{w,2} = (\mathbb{A}(2g) \setminus \Delta') \setminus (L_0 \cup L_\infty) \) and \( T \) acts on \( \mathbb{A}(2g) \) by

\[
(\alpha, \rho) \cdot h(x_0, x_1) = \alpha^{-2} \rho^gh(x_0/\rho, x_1).
\]

By Proposition 5.1 with \( N = 2g, a = -2, b = g \) and \( (\alpha_0, \alpha_1) = (0, 1), (\beta_0, \beta_1) = (0, 0) \) we have that

\[
A^*_T(\mathbb{A}(2g) \setminus \Delta') = \frac{\mathbb{Z}[l_1, l_2][\xi]}{(\alpha_{1,0}(\xi), \alpha_{1,1}(\xi), \rho(\xi), -2l_1 + gl_2 - \xi)}
\]

where by substituting \( \xi = -2l_1 + gl_2 \), we have

\[
p(-2l_1 + gl_2) = \prod_{i=0}^{2g}[-2l_1 - (g - i)l_2],
\]

\[
\alpha_{1,0}(-2l_1 + gl_2) = -4(2g - 1)l_1,
\]

\[
\alpha_{1,1}(-2l_1 + gl_2) = 4l_1^2 + g(g - 1)l_2^2 - 2(2g - 1)l_1l_2.
\]

The hyperplanes \( L_0, L_\infty \) defined by setting the coefficients of \( x_1^{2g}, x_0^{2g} \) to zero respectively have the following equivariant fundamental classes.

\[
[L_0]_T = -2l_1 + gl_2,
\]

\[
[L_\infty]_T = -2l_1 - gl_2.
\]

It can be checked that \( p(-2l_1 + gl_2) \) and \( \alpha_{1,0}, \alpha_{1,1} \) are all in the ideal generated by the hyperplanes \( [L_\infty]_T, [L_0]_T \). Thus

\[
A^*(\mathcal{H}^w_{g,2}) = \frac{\mathbb{Z}[l_1, l_2]}{(-2l_1 + gl_2, -2l_1 - gl_2)}.
\]

In particular, the integral Picard group of \( \mathcal{H}^w_{g,2} \) can be computed as

\[
\text{Pic}(\mathcal{H}^w_{g,2}) = \frac{\mathbb{Z}l_3}{4g l_3},
\]

where \( l_3 \) can be taken either \( l_3 = -l_1 - \frac{g+1}{2}l_2 \) or \( l'_3 = -l_1 + \frac{g+1}{2}l_2 \), and both have order \( 4g \). Once again it is easy to check that \(-l_3\) and \(-l'_3\) are the pullbacks of the two \( \psi \)-classes from \( \mathcal{H}_{g,2} \).
5.4. Integral Chow ring of $[\mathcal{M}_{0,2g+2}/S_{2g+2-n}]$ for $n = 1, 2$.

We will now complete the proofs of Theorem 1.3 and 1.4.

**Theorem 5.8.** The following results hold

- If $n = 1$, then
  $$A^*([\mathcal{M}_{0,2g+2}/S_{2g+1}]) = \frac{\mathbb{Z}[l]}{(2g(2g+1)l)}$$
  where we can take the generator $l$ to be, for example $l = -l_1 + (g+1)l_2$.
- If $n = 2$, then
  $$A^*([\mathcal{M}_{0,2g+2}/S_{2g}]) = \frac{\mathbb{Z}[l]}{(2gl)}$$
  where we can take the generator $l$ to be, for example either $l = -l_1 + (g+1)l_2$ or $l = -l_1 - (g+1)l_2$.

**Proof.** The proof is essentially the same as the computation of the integral Chow ring of $\mathcal{H}^w_{g,n}$ for odd genus $g$ when $n = 1, 2$ given the $T$-actions on $U_w$ and $U_{w,2}$. \hfill \Box

6. Proof of Theorem 1.5

Let’s now consider the quotient stack $[\mathcal{M}_{0,2g+2}/S_{2g+2-n}]$ for $n \geq 3$.

**Proposition 6.1.** If $n \geq 3$, the stack $[\mathcal{M}_{0,2g+2}/S_{2g+2-n}]$ is represented by an open subset of an affine space. In particular $A^*([\mathcal{M}_{0,2g+2}/S_{2g+2-n}]) = \mathbb{Z}$ if $n \geq 3$.

**Proof.** We recall that $\mathcal{M}_{0,2g+2}$ is represented by the scheme $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^{2g-1} \setminus \Delta$ where $\Delta$ is the big diagonal. Under this identification a point $(\mathbb{P}^1, s_1, \ldots, s_{2g+2}) \in \mathcal{M}_{0,2g+2}$ maps to $(p_4, \ldots, p_{2g+2})$ where $(0, 1, \infty, p_4, \ldots, p_{2g+2})$ is the image of the $(2g+2)$-tuple $(s_1, \ldots, s_{2g+2})$ under the unique automorphism of $\mathbb{P}^1$ that takes $(s_1, s_2, s_3)$ to $(0, 1, \infty)$. The action of $S_{2g+2}$ on $\mathcal{M}_{0,2g+2}$ which permutes the sections acts on $(2g-1)$-tuple $(p_4, \ldots, p_{2g+1})$ as follows. First let $\sigma \in S_{2g+2}$ act on $(0, 1, \infty, p_4, \ldots, p_{2g+2})$ by permutation and then by apply the unique element of $\text{PGL}_2$ which sends $(\sigma(0), \sigma(1), \sigma(\infty))$ to $(0, 1, \infty)$ to obtain a tuple $(0, 1, \infty, q_4, \ldots, q_4)$. Then $\sigma(p_4, \ldots, p_{2g+2}) = (q_4, \ldots, q_{2g+2})$.

For any $n$, we identify $S_{2g+2-n}$ to be subgroup of $S_{2g+2}$ which fixes the first $n$ points. It follows from our description of the action that if $n \geq 3$ then actions of $S_{2g+2-n}$ act by permuting the tuple $(p_{n+1}, \ldots, p_{2g+2}) \in (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{2g-2-n}$. Since the points are distinct, the action of $S_{2g+2-n}$ is free and the quotient is the variety $((\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \times \text{Sym}^{2g+2-n}(\mathbb{P}^1 \setminus \{0, 1, \infty\}))/\Delta$ which is an open subvariety of $\mathbb{A}^{2g-1}$. \hfill \Box

6.1. 3-Weierstrass-pointed case. For both $g$ even and odd, Proposition 4.2 states that $A^*(\mathcal{H}^w_{g,3}) = A^*_\mathbb{C}_m(U_{w,3})$ where $\mathbb{G}_m$ acts on $\mathbb{A}(2g-1)$ by

$$t \cdot h(x_0, x_1) = t^{-2}h(x_0, x_1).$$

With this action each of the hyperplanes $L_0, L_1, L_{\infty}$ has the same equivariant fundamental class which is $-2l$ where $l = c_1(\lambda)$. It follows easily that

$$A^*(\mathcal{H}^w_{g,3}) = \frac{\mathbb{Z}[l]}{(2l)}.$$

Moreover, the class $(-l)$ is the pullback any of the $\psi$-classes from $\mathcal{H}_{g,3}$ and so can be viewed as the generator of the integral Chow ring of $\mathcal{H}^w_{g,3}$.
6.2. The \( n \)-Weierstrass pointed case for \( n > 3 \). It remains to find a presentation for \( A^*(\mathcal{H}^w_{g,n}) \) for \( n > 3 \). In fact, we will compute \( A^*(\mathcal{H}^w_{g,n}) \) for \( n \geq 3 \), but we need our earlier presentation of \( A^*(\mathcal{H}^w_{g,3}) \) to obtain a natural generator when \( n > 3 \).

Recall that the map \([\mathcal{M}_{0,2g+2}/S_{2g+1}] \to \mathcal{D}_{2g+2}\) realizes the map \([\mathcal{M}_{0,2g+2}/S_{2g+2}] \to [\mathcal{M}_{0,2g+2}/S_{2g+2}]\) as the universal effective Cartier divisor \( B \to \mathcal{D}_{2g+2} \) which is finite and étale of degree \( 2g+2 \). Let \( \mathcal{M} \) be the pullback of the line bundle \( \mathcal{O}(-B) \) to \([\mathcal{M}_{0,2g+2}/S_{2g+1}]\).

**Proposition 6.2.** (Dan Petersen) The universal Weierstrass divisor \( \mathcal{H}^w_{g,w} \) is the \( \mu_2 \)-gerbe obtained by taking the square root of \( \mathcal{M} \). For \( n \geq 1 \) the gerbes \( \mathcal{H}^w_{g,n} \to [\mathcal{M}_{0,2g+2}/S_{2g+2-n}] \) are the gerbes associated to the square root of the pullback of the line bundle \( \mathcal{M} \) to \([\mathcal{M}_{0,2g+2}/S_{2g+2-n}]\).

**Proof.** The data \((P \to S, D_{2g+2})\) where \( P \to S \) is a twisted \( \mathbb{P}^1 \) and \( D_{2g+2} \) is a Cartier divisor which is finite and étale of degree \( 2g+2 \) over \( S \) corresponds to a morphism \( S \to [\mathcal{M}_{0,2g+2}/S_{2g+2}] \). Let \( \mathcal{P} \) be the \( \mu_2 \)-gerbe parametrizing the square root of \( \mathcal{O}(-D_{2g+2}) \). Precisely, the line bundle \( \mathcal{O}(-D_{2g+2}) \) corresponds to the morphism \( P \to \mathcal{B}G \) and \( \mathcal{P} \) is the gerbe obtained by base change along the square root gerbe \( \mathcal{B}G_m \to \mathcal{B}G_n \) induced by map \( \mathcal{G}_m \to \mathcal{G}_n, \lambda \mapsto \lambda^2 \). By \( [1] \) we know that to give a \( \mu_2 \)-cover of \( P \) branched along \( D_{2g+2} \) (i.e. a hyperelliptic curve) is equivalent to giving a section \( s: P \to \mathcal{P} \).

The map \( S \to [\mathcal{M}_{0,2g+2}/S_{2g+2}] \) lifts to a morphism \([\mathcal{M}_{0,2g+2}/S_{2g+1}]\) when the étale covering \( D_{2g+2} \to S \) admits a section \( \sigma: S \to D_{2g+2} \). Putting this together we see that the data of a section of \( D_{2g+2} \to S \) together with a section \( D_{2g+2} \to \mathcal{P} \) is equivalent to the data of double cover of \( C \to P \) branched along \( D_{2g+2} \) together with a Weierstrass section; i.e. a section of \( \mathcal{H}^w_{g,w}(S) \).

The second statement follows from base change. \( \square \)

**Proposition 6.3.** The \( \mu_2 \)-gerbe \( \mathcal{H}^w_{g,n} \to [\mathcal{M}_{0,2g+2}/S_{2g+2-n}] \) is trivial if \( n \geq 3 \).

**Proof.** The \( \mu_2 \)-gerbe \( \mathcal{H}^w_{g,n} \to [\mathcal{M}_{0,2g+2}/S_{2g+2-n}] \) is the gerbe associated to the square root of a line bundle on \([\mathcal{M}_{0,2g+2}/S_{2g+2-n}]\) by Proposition 6.2. Since \([\mathcal{M}_{0,2g+2}/S_{2g+2-n}]\) is represented by an open subvariety of an affine space, and line bundles over open sets of an affine space are trivial, it proves the result. \( \square \)

It follows that if \( n \geq 3 \), \( \mathcal{H}^w_{g,n} = [\mathcal{M}_{0,2g+2}/S_{2g+2-n}] \times B\mu_2 \) by Proposition 6.3. Hence \( A^*(\mathcal{H}^w_{g,n}) = A^*(B\mu_2) = \mathbb{Z}[l]/(2l) \) for \( n \geq 3 \).

**Natural generator for** \( A^*(\mathcal{H}^w_{g,n}) \) **when** \( n > 3 \). As in the case when \( n = 1, 2, 3 \), we will show that any restriction of \( \psi \)-class on \( \mathcal{H}^w_{g,n} \) to \( \mathcal{H}^w_{g,n} \) generates the integral Chow ring \( A^*(\mathcal{H}^w_{g,n}) \).

**Corollary 6.4.** For \( n \geq 3 \), \( A^*(\mathcal{H}^w_{g,n}) \) is generated by \( \psi \) where \( \psi \) is the pullback of any \( \psi \)-class from \( \mathcal{H}^w_{g,n} \). Consequently \( A^*(\mathcal{H}^w_{g,n}) = \mathbb{Z}[\psi]/(2\psi) \).

**Proof.** The proof proceeds by induction and it suffices to prove that \( \psi \neq 0 \) in \( \text{Pic} (\mathcal{H}^w_{g,n}) \). We have already proved the statement for \( n = 3 \). Assume that the statement has been proved for \( n = k + 3 \) with \( k > 0 \). By Proposition 6.3, we have the following cartesian diagram

\[
\begin{array}{ccc}
\mathcal{H}^w_{g,k+4} \xrightarrow{g_2} [(\mathcal{M}_{0,2g+2}/S_{2g+2-k})/\mu_2] \\
\downarrow f \\
\mathcal{H}^w_{g,k+3} \xrightarrow{g_1} [(\mathcal{M}_{0,2g+2}/S_{2g+1-k})/\mu_2]
\end{array}
\]
where the map $f$ corresponds to forgetting any of the marked Weierstrass points and the horizontal maps $g_1, g_2$ are both isomorphisms and $\mu_2$ acts trivially on both schemes $\mathcal{M}_{0,2g+2}/S_{2g-1-k}$ and $\mathcal{M}_{0,2g+2}/S_{2g-2-k}$. The pullback map on the Chow rings induced by the right vertical map is the identity map so $f^*$ is an isomorphism and in particular it is not the zero map. By induction we know that $\psi \neq 0$ in $A^*(Hw_{g,k+3})$. Hence $f^*\psi$ is non-zero in $A^*(Hw_{g,k+4})$. Since the pullback of a $\psi$-class under the map $f$ is a $\psi$-class the result follows. \qed

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