Fragile Complexity of Adaptive Algorithms

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Abstract. The fragile complexity of a comparison-based algorithm is $f(n)$ if each input element participates in $O(f(n))$ comparisons. In this paper, we explore the fragile complexity of algorithms adaptive to various restrictions on the input, i.e., algorithms with a fragile complexity parameterized by a quantity other than the input size $n$. We show that searching for the predecessor in a sorted array has fragile complexity $\Theta(\log k)$, where $k$ is the rank of the query element, both in a randomized and a deterministic setting. For predecessor searches, we also show how to optimally reduce the amortized fragile complexity of the elements in the array. We also prove the following results: Selecting the $k$th smallest element has expected fragile complexity $O(\log \log k)$ for the element selected. Deterministically finding the minimum element has fragile complexity $\Theta(\log(\text{Inv}))$ and $\Theta(\log(\text{Runs}))$, where Inv is the number of inversions in a sequence and Runs is the number of increasing runs in a sequence. Deterministically finding the median has fragile complexity $O(\log(\text{Runs}) + \log \log n)$ and $\Theta(\log(\text{Inv}))$. Deterministic sorting has fragile complexity $\Theta(\log(\text{Inv}))$ but it has fragile complexity $\Theta(\log n)$ regardless of the number of runs.

Keywords: Algorithms · Comparison based algorithms · Fragile complexity.

1 Introduction

Comparison-based algorithms have been thoroughly studied in computer science. This includes algorithms for problems such as Minimum, Median, Sorting, Searching, Dictionaries, Priority Queues, and many others. The cost measure analyzed is almost always the total number of comparisons performed by the algorithm, either in the worst case or the expected case. Recently, another type of cost measure has been introduced \cite{1} which instead considers how many comparisons each individual element is subjected during the course of the algorithm. In \cite{1}, a comparison-based algorithm is defined to have fragile complexity
If each individual input element participates in at most \( f(n) \) comparisons. The fragile complexity of a computational problem is the best possible fragile complexity of any comparison-based algorithm solving the problem.

This cost measure has both theoretical and practical motivations. On the theoretical side, it raises the question of to what extent the comparisons necessary to solve a given problem can be spread evenly across the input elements. On the practical side, this question is relevant in any real world situation where comparisons involve some amount of destructive impact on the elements being compared (hence the name of the cost measure). As argued in [1], one example of such a situation is ranking of any type of consumable objects (wine, beer, food, produce), where each comparison reduces the available amount of the objects compared. Here, an algorithm like QuickSort, which takes a single object and partitions the whole set with it, may use up this pivot element long before the algorithm completes. Another example is sports, where each comparison constitutes a match and takes a physical toll on the athletes involved. If a comparison scheme subjects one contestant to many more matches than others, both fairness to contestants and quality of result are impacted—finding a winner may not be very useful if this winner has a high risk of being injured in the process. The negative impact of comparisons may also be of non-physical nature, for instance when there is a privacy risk for the elements compared, or when bias grows if few elements are used extensively in comparisons.

\section{Previous work}

In [1], the study of algorithms’ fragile complexity was initiated and a number of upper and lower bounds on the fragile complexity for fundamental problems was given. The problems studied included MINIMUM, the SELECTION, SORTING, and HEAP CONSTRUCTION, and both deterministic and randomized settings were considered. In the deterministic setting, MINIMUM was shown to have fragile complexity \( \Omega(\log n) \) and SORTING to have fragile complexity \( O(\log n) \). Since SORTING can solve SELECTION, which can solve MINIMUM, the fragile complexity of all three problems is \( \Theta(\log n) \). The authors then consider randomized algorithms, as well as a more fine-grained notion of fragile complexity, where the objective is to protect selected elements such as the minimum or median (i.e., the element to be returned by the algorithm), possibly at the expense of the remaining elements. Among other results, it is shown in [1] that MINIMUM can be solved incurring expected \( O(1) \) comparisons on the minimum element itself, at a price of incurring expected \( O(n^\varepsilon) \) on each of the rest. Also a more general trade-off between the two costs is shown, as well as a close to matching lower bound. For SELECTION, similar results are given, including an algorithm incurring expected \( O(\log \log n) \) comparisons on the returned element itself, at a price of incurring expected \( O(\sqrt{n}) \) on each of the rest.

An earlier body of work relevant for the concept of fragile complexity is the study of sorting networks, started in 1968 by Batcher [5]. In sorting networks, and more generally comparator networks, the notion of depth (the number of layers, where each layer consists of non-overlapping comparators) and size (the
total number of comparators) correspond to fragile complexity and standard worst case complexity, respectively, in the sense that a network with depth \( f(n) \) and size \( s(n) \) can be converted into a comparison-based algorithm with fragile complexity \( f(n) \) and standard complexity \( s(n) \) by simply simulating the network.

Batcher, as well as a number of later authors \([9,16,17,20]\), gave sorting networks with \( O(\log^2 n) \) depth and \( O(n \log^2 n) \) size. For a long time it was an open question whether better results were possible. In 1983, Ajtai, Komlós, and Szemerédi \([2,3]\) answered this in the affirmative by constructing a sorting network of \( O(\log n) \) depth and \( O(n \log n) \) size. This construction is quite complex and involves expander graphs \([22,23]\). It was later modified by others \([8,12,18,21]\), but finding a simple, optimal sorting network, in particular one not based on expander graphs, remains an open problem. Comparator networks for other problems, such as selection and heap construction have also been studied \([4,6,15,19,26]\).

While comparator networks are related to fragile complexity in the sense that results for comparator networks can be transferred to the fragile complexity setting by simple simulation, it is demonstrated in \([1]\) that the two models are not equivalent: there are problems where one can construct fragile algorithms with the same fragile complexity, but with strictly lower standard complexity (i.e., total number of comparisons) than what is possible by simulation of comparison networks. These problems include Selection and Heap Construction.

### 1.2 Our Contribution

In many settings, the classical worst case complexity of comparison-based algorithms can be lowered if additional information on the input is known. For instance, sorting becomes easier than \( \Theta(n \log n) \) if the input is known to be close to sorted. Another example is searching in a sorted set of elements, which becomes easier than \( O(\log n) \) if we know an element of rank close to the element searched for. Such algorithms may be described as adaptive to input restrictions (using the terminology from the sorting setting \([10]\)). Given that the total number of comparisons can be lowered in such situations, the question arises whether also reductions in the fragile complexity are possible under these types of input restrictions.

In this paper, we expand the study of the fragile complexity of comparison-based algorithms to consider the impact of a number of classic input restrictions. We show that searching for the predecessor in a sorted array has fragile complexity \( \Theta(\log k) \), where \( k \) is the rank of the query element, both in a randomized and a deterministic setting. For predecessor searches, we also show how to optimally reduce the amortized fragile complexity of the elements in the array. We also prove the following results: Selecting the \( k \)th smallest element has expected fragile complexity \( O(\log \log k) \) for the element selected. Deterministically finding the minimum element has fragile complexity \( \Theta(\log \text{Inv}) \) and \( \Theta(\log \text{Runs}) \), where Inv is the number of inversions in a sequence and Runs is the number of increasing runs in a sequence. Deterministically finding the median has fragile complexity \( O(\log \text{Runs} + \log \log n) \) and \( \Theta(\log \text{Inv}) \). Deterministic sorting has
fragile complexity $\Theta(\log(\text{Inv}))$ but it has fragile complexity $\Theta(\log n)$ regardless of the number of runs.

2 Searching

The problem of predecessor searching is, given a sorted array $A$ with $n$ elements, $A[0]..A[n-1]$, answer queries of the form “What is the index of the largest element in $A$ smaller than $x$?” Binary search is the classic solution to the predecessor search problem. It achieves $\log n$ fragile complexity for $x$, and fragile complexity at most one for each element of $A$. We can improve on this in two ways. The first is where we try to keep the fragile complexity of $x$ small, which is possible if we know something about the rank of $x$. We show that the optimal dependency on the rank of $x$ is $\Theta(\log k)$ where $k$ is its rank, both for deterministic and randomized algorithms. The second setting is where we are concerned with the fragile complexity of the other elements. While there is no way to improve a single search, classical deterministic binary search will always do the first comparison with the same element (typically the median). Hence we consider deterministic algorithms that improve the amortized fragile complexity of any element of the array $A$ over a sequence of searches.

2.1 Single search

**Theorem 1.** Let $A$ be a sorted array. Determining the predecessor of an element $x$ within $A$ has fragile complexity $\Theta(\log k)$ for deterministic and randomized algorithms, where $k$ is the rank of $x$ in $A$.

**Proof.** The upper bound follows from standard exponential search [11]: We compare $x$ to $A[2], A[4], A[8], \ldots$ until we find the smallest $i$ such that $x < A[2^i]$. We perform a binary search with the initial interval $[2^{i-1}, 2^i]$. If $x$ has the predecessor $A[k]$, this requires $O(\log k)$ comparisons.

For the lower bound assume we have a deterministic algorithm to determine the rank of an element $x$. If the answer of the algorithm is $k$, let $B_k$ be the bit-string resulting from concatenating the sequence of the outcomes of the comparisons performed by the algorithm, the $i$-th bit $B_k[i] = 0$ for $x < A[k]$, otherwise it is 1. Because the algorithm is deterministic and correct, all these bit-strings are different and they are a code for the numbers 1, $\ldots$, $n$. Now, for any $k$, consider the uniform distribution on the numbers 0, $\ldots$, $k-1$, a distribution with entropy $\log k$. By Shannon’s source coding theorem, the average code length must be at least $\log k$, i.e., $\sum_{i=0}^{k-1} |B_i| \geq k \log k$.

For a contradiction, assume there would be an algorithm with $|B_i| \leq \log i$ (the binary logarithm itself). Then for $k > 1$, $\sum_{i=0}^{k-1} |B_i| < k \log k$, in contrast to Shannon’s theorem.

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6 For simplicity of exposition, we assume the rank is close to one, but the result clearly holds for rank distance to other positions in $A$. 
The bound $\sum_{i=0}^{k-1} |B_i| \geq k \log k$ also holds for randomized algorithms if the queries are drawn uniformly from $[1, \ldots, k]$, following Yao’s principle: Any randomized algorithm can be understood as a collection of deterministic algorithms from which the ‘real’ algorithm is drawn according to some distribution. Now each deterministic algorithm has the lower bound, and the average number of comparisons of the randomized algorithm is a weighted average of these. Hence the lower bound also holds for randomized algorithms. \hfill \Box

2.2 Sequence of searches
As mentioned, in binary search, the median element of the array will be compared with every query element. Our goal here is to develop a search strategy so as to ensure that data far away from the query will only infrequently be involved in a comparison. Data close to the query must be queried more frequently. While we prove this formally in Theorem 3, it is easy to see that predecessor and successor of a query must be involved in comparisons with the query in order to answer the query correctly.

**Theorem 2.** There is a search algorithm that for any sequence of predecessor searches $x_1, x_2, \ldots, x_m$ in a sorted array $A$ of size $n$ the number of comparisons with any $y \in A$ is $O\left(\log n + \sum_{i=1}^{m} \frac{1}{d(x_i, y)}\right)$ where $d(x, y)$ is the number of elements between $x$ and $y$ in $A$, inclusive. The runtime is $O(\log n)$ per search and the structure uses $O(n)$ bits of additional space.

**Proof.** We use the word interval to refer to a contiguous range of $A$; when we index an interval, we are indexing $A$ relative to the start of the interval. Call an aligned interval $I$ of $A$ of rank $i$ to be $(A[k \cdot 2^i] \ldots A[(k+1) \cdot 2^i])$ for some integer $k$, i.e., the aligned intervals of $A$ are the dyadic intervals of $A$. There are $O(n)$ aligned intervals of $A$, and for each aligned interval $I$ of rank $i$ we store an offset $I$.offset which is in the range $[0, 2^i)$, and it is initialized to 0.

The predecessor search algorithm with query $x$ is a variant of recursive binary search, where at each step an interval $I_q$ of $A$ is under consideration, and the initial recursive call considers the whole array $A$. Each recursive call proceeds as follows: Find the largest $i$ such that there are at least three rank-$i$ aligned intervals in $I_q$, use $I_m$ to denote the middle such interval (or an arbitrary non-extreme one if there are more than three), and we henceforth refer to this recursive call as a rank-$i$ recursion. Compare $I_m[I_m$.offset$]$ with $x$, and then increment $I_m$.offset modulo $2^i$. Based on the result of the comparison, proceed recursively as in binary search. The intuition is by moving the offset with every comparison, this prevents a single element far from the search from being accessed too frequently. We note that the total space used by the offsets is $O(n)$ words, which can be reduced to $O(n)$ bits if the offsets are stored in a compact representation.

First, several observations:
1. In a rank-$i$ recursion, $I_q$ has size at least $3 \cdot 2^i$ (since there must be at least three rank-$i$, size $2^i$ aligned intervals in $I_q$) and at most $8 \cdot 2^i$, the latter being true as if it was this size there would be three rank-$i + 1$ intervals in $I_q$, which would contradict $I_m$ having rank $i$. 


2. If \( I_q \) has size \( k \) then if there is a recursive call, it is called with an interval of size at most \( \frac{2}{7} k \). This is true by virtue of \( I_m \) being rank-\( i \) aligned with at least one rank-\( i \) aligned interval on either side of \( I_m \) in \( i \). Since \( I_q \) has size at most \( 8 \cdot 2^i \), this guarantees an eighth of the elements of \( I_q \) will be removed from consideration as a result of the comparison in any recursive call.

3. From the previous two points, one can conclude that for a given rank \( i \), during any search there are at most 7 recursions with of rank \( i \). This is because after eight recursions any rank-\( i \) search will be reduced below the minimum for rank \( i \): \( 8 \cdot 2^i \cdot \left( \frac{2}{7} \right) ^8 < 3 \cdot 2^i \).

For the analysis, we fix an arbitrary element \( y \) in \( A \) and use the potential method to analyse the comparisons involving \( y \). Let \( \mathcal{I}_y = \{ I_1^y, I_2^y, \ldots \} \) be the \( O(\log n) \) aligned intervals that contain \( y \), numbered such that \( I^i \) has rank \( i \). Element \( y \) will be assigned a potential relative to each aligned interval \( I^i \) in \( \mathcal{I}_y \) which we will denote as \( \varphi_y(I^i) \). Let \( t_y(I^i) \) be number of times \( I^i \) offset needs to be incremented before \( I^i[I^i, \text{offset}] = y \), which is in the range \([0, 2^i]\). The potential relative to \( I^i \) is then defined as \( \varphi_y(I^i) := \frac{2^i - t_y(I^i)}{2^i} \), and the potential relative to \( y \) is defined to be the sum of the potentials relative to the intervals in \( \mathcal{I}_y \): \( \varphi_y := \sum_{I^i \in \mathcal{I}_y} \varphi_y(I^i) \).

How does \( \varphi_y(I^i) \) change during a search? First, if there is no rank-\( i \) recursive call during the search to an interval containing \( y \), it does not change as \( I^i \cdot \text{offset} \) is unchanged. Second, observe from point 3 that a search can increase \( \varphi_y(I^i) \) by only \( \frac{2}{7} \). Furthermore if \( y \) was involved in a comparison during a rank-\( i \) recursion, there will be a loss of \( 1 - \frac{2}{7} \) units of potential in \( \varphi_y(I^i) \) as the offset of \( I^i \) changes from 0 to \( 2^i - 1 \).

Following standard potential-based amortized analysis, the amortized number of comparisons involving \( y \) during a search is the actual number of comparisons (zero or one) plus the change in the potential \( \varphi_y \). Let \( i_{\min} \) be the smallest value of \( i \) for which there was a rank-\( i \) recursion that included \( y \). As the maximum gain telescopes, the potential gain is at most \( \frac{14}{2^{i_{\min}}} \), minus 1 if \( y \) was involved in a comparison. Thus the amortized number of comparisons with \( y \) in the search is at most \( \frac{14}{2^{i_{\min}}} \).

Observe that if there was a rank-\( i \) recursion that included \( y \), that \( d(x, y) \) is at most \( 8 \cdot 2^i \) by point 1. This gives \( d(x, y) \leq 8 \cdot 2^i \leq 8 \cdot 2^{m_{\min}} \). Thus the amortized cost can be restated as being at most \( \frac{14}{2^{i_{\min}}} \leq \frac{14}{112} d(x, y) \).

To complete the proof, the total number of comparisons involving \( y \) over a sequence of searches is the sum of the amortized costs plus any potential loss. As the potential \( \varphi_y \) is always nonnegative and at most \( [\log n] \) (1 for each \( \varphi_y(I^i) \)), this gives the total cost as \( O \left( \log n + \sum_{i=1}^{m} \frac{1}{d(x, y)} \right) \).

Note that the above proof was designed for easy presentation and not an optimal constant. Also note that this theorem implies that if the sequence of searches is uniformly random, the expected fragility of all elements is \( O(\log n / n) \), which is asymptotically the best possible since random searches require \( \Omega(\log n) \) comparisons in expectation.
2.3 Lower Bounds.

It is well-known that comparison-based searching requires $\Omega(\log n)$ comparisons per search. In our method, taking a single search $x_i$ summing over the upper bound on amortized cost of the number of comparisons with $y_i, \frac{d}{d_{(x_i,y)}}$, for all $y$ yields a harmonic series which sums to $O(\log n)$. But we can prove something stronger:

**Theorem 3.** There is a constant $c$ such that if a predecessor search algorithm has an amortized number of comparisons of $f(d(x_i,y))$ for an arbitrary $y$ for every sequence of predecessor searches $x_1, x_2, \ldots x_m$, then $\sum_{k=1}^{p} f(k) \geq c \log p$ for all $p \leq n$.

**Proof.** This can be seen by looking at a random sequence of predecessor searches for which the answers are uniform among $A[0] \ldots A[p-1]$, if the theorem was false, similarly to the proof of Theorem 1, this would imply the ability to execute such a sequence in $o(\log p)$ amortized time per operation. \hfill $\square$

This shows that a flatter asymptotic tradeoff between $d(x_i, y)$ and the amortized comparison cost is impossible; more comparisons are needed in the vicinity of the search than farther away. For example, a flat amortized number of comparisons of $\frac{\log n}{n}$ for all elements would sum up to $O(\log n)$ amortized comparisons over all elements, but yet would violate this theorem.

2.4 Extensions.

Here we discuss extensions to the search method above. We omit the proofs as they are simply more tedious variants of the above.

One can save the additional space used by the offsets of the intervals through the use of randomization. The offsets force each item in the interval to take its turn as the one to be compared with, instead one can pick an item at random from the interval. This can be further simplified into a binary search where at each step one simply picks a random element for the comparison amongst those (in the middle half) of the part of the array under consideration.

To allow for insertions and deletions, two approaches are possible. The first is to keep the same array-centric view and simply use the packed-memory array [14, 24, 25] to maintain the items in sorted order in the array. This will give rise to a cost of $O(\log^2 n)$ time which is inherent in maintaining a dynamic collection of items ordered in an array [7] (but no additional fragility beyond searching for the item to insert or delete as these are structural changes). The second approach would be to use a balanced search tree such as a red-black tree [13]. This will reduce the insertion/deletion cost to $O(\log n)$ but will cause the search cost to increase to $O(\log^2 n)$ as it will take $O(\log n)$ time to move to the item in each interval indicated by the offset, or to randomly choose an item in an interval. The intervals themselves would need to allow insertions and deletions, and would, in effect be defined by the subtrees of the red-black tree. It remains open whether there is a dynamic structure with the fragility results of Theorem 2 where insertions and deletions can be done in $O(\log n)$ time.
3 Selection

In this section we consider the problem of finding the \( k \)-th smallest element of an unsorted array. There is a randomized algorithm that selects the \( k \)-th smallest element with expected fragile complexity of \( O(\log \log n) \) for the selected element \([1]\). We consider the question if this complexity can be improved for small \( k \). In this section we define a sampling method that, combined with the algorithm given in \([1]\), selects the \( k \)-th smallest element with expected \( O(\log \log k) \) comparisons.

Next, we define the filtering method \( \text{ReSet} \) in a tail-recursive fashion. The idea of this procedure is the following: First, we build a random half size sample \( A_1 \) from the input set \( X \). Later, we continue recursively constructing a random half sample \( A_i \) from the previous sample \( A_{i-1} \) until we get a random sample \( A_\ell \) of size \( O(k+1) \). Once \( A_\ell \) is given, then a set \( A'_\ell \) of size \( O(k) \) is given for the previous recursive call. Using such set, a new subset \( A'_{\ell-1} \) is given from the previous sample \( A_{\ell-1} \) where its expected size is \( O(k) \). This process continuous until a final subset \( C \) is given from the input set \( X \) such that its expected size \( O(k) \) and it contains the \( k \)-th smallest element of \( X \).

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1: procedure \( \text{ReSet}(X,k) \)  \( \triangleright \) Returns a small subset \( C \) of \( X \) that contains the \( k \)-th element
2:     Let \( n = |X| \) and \( C = \emptyset \)
3:     if \( k \geq n^\frac{1}{2} - 1 \)  \( \triangleright \) The set has size \( O(k+1) \)
4:         Let \( A' = X \)
5:     else  \( \triangleright \) Recursively construct a sample of expected size \( O(k+1) \)
6:         Sample \( A \) uniformly at random from \( X \), \( |A| = \frac{n}{2} \)
7:         Let \( A' = \text{ReSet}(A,k) \)
8:         Choose the \( (k+1) \)-th smallest element \( z \) from \( A' \) (by standard linear time selection)
9:     Let \( C = \{ x \in X : x \leq z \} \)
10:    return \( C \)
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In the following theorem we show that the combination of the \( \text{ReSet} \) procedure and the randomized selection algorithm in \([1]\), results in expected \( O(\log \log k) \) comparisons for the selected element.

**Theorem 4.** Randomized selection is possible in expected fragile complexity \( O(\log \log k) \) in the selected element.

**Proof.** Let us show that the following procedure for selecting the \( k \)-th element in a set \( X \) with \( |X| = n \), gives an expected fragile complexity \( O(\log \log k) \) in the \( k \)-th element:

If \( k > n^{\frac{1}{2\log n}} \), then let \( S' = X \). If \( k \leq n^{\frac{1}{2\log n}} \), then sample uniformly at random \( S \) from \( X \), where \( |S| = \frac{n}{2} \). Let \( C = \text{ReSet}(S,k) \) and select the \( k+1 \)-th smallest element \( z \) from \( C \) by standard linear time selection. Let \( S' = \{ x \in X : x \leq z \} \). Finally, apply to \( S' \) the randomized selection algorithm of \([1]\).

Let \( x_k \) denote the \( k \)-th smallest element in \( X \) and let \( f_k \) denote the fragile complexity of \( x_k \). Note that if \( x_k \in S \), then, before constructing \( S' \), \( f_k \) is given by the fragile complexity of \( x_k \) in \( \text{ReSet}(S,k) \) plus \( O(|C|) \) when finding the
$(k + 1)$-th smallest element in $C$. Otherwise, $x_k$ is not compared until $S'$ is constructed. On the other hand, recall that the expected $f_k$ in the algorithm in [1] is $O(\log \log m)$ where $m$ is the size of the input set. Hence, the expected $f_k$ after selecting the $k + 1$-th element in $C$ is 1 when creating $S'$ plus the expected $f_k$ in the randomized selection algorithm in [1] that is $\sum_{\ell=1}^{k} O(\log \log |S'|) = E[O(\log \log |S'|)]$. Thus, $E[f_k] = (E[f_k] \in \text{ReSet}|x_k \in S \in \bigcup |C|) + \sum_{\ell=1}^{k} E[O(\log \log |S'|)]$. Since the logarithm is a concave function, $E[O(\log \log |S'|)] \leq O(\log \log (E[|S'|]))$. Therefore, if we prove that: (i) the expected fragile complexity of $x_k$ before creating $S'$ is $O(1)$ and (ii) $E[|S'|] = c'k^c$ for some constants $c$ and $c'$. Then, we obtain that $E[f_k] \leq O(1) + 1 + O(c \log \log k + \log k) = O(\log \log k)$, as desired. In order to prove (i) and (ii) we consider 2 cases: 

(1) $k > n^{\frac{1}{100}}$, (2) $k \leq n^{\frac{1}{100}}$.

Case 1) $S'$ is $X$ and it makes no previous comparisons in any element, proving (i). In addition, $S'$ has size less than $k^{100}$. Thus, (ii) holds.

Case 2) $S$ is a sample of $X$ with size $\frac{n}{k}$ and $S' = \text{ReSet}(S, k)$.

First, let us show (i). If $x_k \notin S$, then there are no previous comparisons. Hence, the expected fragile complexity of $x_k$ before constructing $S'$ is given by $E[f_k] = \sum_{\ell=1}^{k} \frac{n}{k} E[|S'|] = E[|S'|] = O(k)$, which gives an expectation of $O(k)\frac{n}{k} = O(1)$, proving (i). So, let us show that $\sum_{\ell=1}^{k} \frac{n}{k} E[|S'|] = O(k)$. Let $A_0 = S$ and let $A_1$ be the sample of $A_0$ when passing through line 6 in \text{ReSet}. Similarly, denote by $A_i$ to the sample of $A_{i-1}$ in the $i$-th recursive call of \text{ReSet} and let $A_k = \text{ReSet}(A_i, k)$. Note that by definition $A_k = C$. Let $\ell+1$ be the number of recursive calls in \text{ReSet}(S, k).

Since $A_i$ is a uniform random sample of size $\frac{A_{i-1}}{2^i}$ for all $i \geq 1$, $E|x \in A_i|x \in A_{i-1}| = 2^{-i}$ and $P|x \in A_i|x \notin A_{i-1}| = 0$. Hence, $P|x_k \in A_i| = P|x_k \in \bigcap_{i=0}^{\ell-1} A_i| = 2^{-i}$. Note that the number of comparisons of $x_k$ in \text{ReSet} is given by the number of times $x_k$ is compared in lines 8 and 9. Thus, for each $i$-th recursive call: if $x_k \in A_i$, then $x_k$ is compared once in line 9; and if $x_k \in A_i \cup A_i'$, then $x_k$ is compared at most $|A_i'|$ times in line 8. Otherwise, $x_k$ is not compared in that and the next iterations. Thus, $E[f_k] = \sum_{\ell=1}^{k} \frac{n}{k} E[|S'|] = \sum_{i=0}^{\ell} \frac{n}{k} 2^{-i}(1 + E[|A_i'|]) = 2(1 + E[|A_i'|])$. Let us compute $E[|A_i'|]$. Since the $(\ell+1)$-th iteration \text{ReSet}(A_i, k) passes through the if in line 3, there is no new sample from $A_i$. Thus, $A_i'$ is given by the $k+1$ smallest elements of $A_i$. Therefore, $E[|A_i'|] = k + 1$ Denote by $a_j^i$ to the $j$-th smallest element of $A_i$.

For the case of $0 \leq i < \ell$, we have $A_i' = \{ x \in A_{i+1} : x \leq a_{i+1}^i \}$. Hence, $E[|A_i'|] = E[\{ x \in A_{i+1} : x \leq a_{i+1}^i \}] = E[\bigcup_{j=1}^{k} \{ x \in A_{i+1} : a_{j+1}^i < x \leq a_{j+1}^i \}] \leq \sum_{j=1}^{k} \sum_{i=1}^{\infty} t^{2^{-i}(2^{-(i-1)})} = 2(k + 1)$. Therefore, $E[f_k] = \sum_{\ell=1}^{k} \frac{n}{k} E[|S'|] = \sum_{i=0}^{\ell} \frac{n}{k} 2^{-i}(1 + E[|A_i'|]) = 2 + 2E[|A_i'|] = O(k)$ proving (i). Finally, let us show (ii): For simplicity, let $c_j$ denote the $j$-th smallest element of $C$. Then, $E[|S'|] = E[\{ x \in X : x \leq c_1 \}] + \sum_{j=1}^{k} E[\{ x \in X : c_j \leq x \leq c_{j+1} \}] \leq \sum_{j=0}^{k} \sum_{i=0}^{\infty} jk^{-1}(1 - k^{-2})^{j-1} = k(k + 1) = O(k^2)$, proving (ii).
4 Sorting

When the input is known to have some amount of existing order, sorting can be done faster than $\Theta(n \log n)$. Quantifying the amount of existing order is traditionally done using measures of disorder [10], of which Inv and Runs are two classic examples. A sorting algorithm is adaptive to a measure of disorder if it is faster for inputs with a smaller value of the measure. For the above measures, run times of $O(n \log(\text{Inv}/n))$ and $O(n \log(\text{Runs}))$ can be achieved. These results are best possible for comparison-based sorting, by standard information-theoretic arguments based on the number of different inputs having a given maximal value of the measure.

The fact [3, 1] that we can sort all inputs in $\Theta(n \log n)$ time and $\Theta(\log n)$ fragile complexity can be viewed as being able to distribute the necessary comparisons evenly among the elements such that each element takes part in at most $\Theta(\log n)$ comparisons. Given the running times for adaptive sorting stated above, it is natural to ask if for an input with a given value of Inv or Runs we are able to sort in a way that distributes the necessary comparisons evenly among the elements, i.e., in a way such that each element takes part in at most $O(\log(\text{Inv}))$ or $O(\log(\text{Runs}))$ comparisons, respectively. In short, can we sort in fragile complexity $O(\log(\text{Inv}))$ and $O(\log(\text{Runs}))$? Or more generally, what problems can we solve with fragile complexity adaptive to Inv and Runs? In this section, we study the fragile complexity of deterministic algorithms for MINIMUM, MEDIAN, and SORTING and essentially resolve their adaptivity to Inv and Runs.

**Theorem 5.** MINIMUM has fragile complexity $\Theta(\log(\text{Runs}))$.

**Proof.** For the upper bound: identify the runs in $O(1)$ fragile complexity by a scan of the input. Then, use a tournament on the heads of the runs since the minimum is the minimum of the heads of the runs. For the lower bound: apply the logarithmic lower bound for MINIMUM [1] on the heads of the runs. \( \square \)

**Theorem 6.** SORTING has fragile complexity $\Theta(\log n)$, no matter what value of Runs is assumed for the input.

**Proof.** The upper bound follows from general sorting. For the lower bound: the input consisting of a run $R$ of length $n-1$ and one more element $x$ has Runs = 2, but $\log n$ comparisons on $x$ can be forced by an adversary before the position of $x$ in $R$ is determined. \( \square \)

**Theorem 7.** MEDIAN has fragile complexity $O(\log(\text{Runs}) + \log \log n)$.

**Proof.** Assume that $4 \cdot \text{Runs} \cdot \log n < n/2$, since otherwise the claimed fragile complexity is $O(\log n)$ for which we already have a median algorithm [1]. Consider the rank space $[1, n]$ (i.e., the indices of the input elements in the total
fragile complexity by a scan. A run will be considered short if the run consists of fewer than \(7 \cdot \log n\) elements and it will be considered long otherwise. A step of the algorithm proceeds by first performing a type B removal followed by a type A removal. A type B removal consists of removing all short runs that are present. The short runs that are removed will be reconsidered again at the end once the number of elements under consideration by the algorithm is less than \(64 \cdot \text{Runs} \cdot \log n\).

Once a type B removal step is completed, only long runs remain under consideration. We now describe a type A removal step. Note that a long run may become short after a type A removal step, in which case it will be removed as part of the next type B removal step. Each run can become short (and be removed by a type B removal) only once, hence the total number of elements removed by type B removals will be at most \(7 \cdot \text{Runs} \cdot \log n\), as claimed.

In the following, let \(N\) denote the elements under consideration just before a type A removal (i.e., the elements of the remaining long runs), and let \(N = |N|\). The algorithm stops when \(N \leq 64 \cdot \text{Runs} \cdot \log n\).

To execute the type A removal step, the algorithm divides each long run \(R\) into blocks of length \(\log n\). The blocks of a run are partitioned by a partitioning block. The partitioning block has the property that there are at least \(|R|/7\) elements of \(R\) whose values are less than the values in the partitioning block and at least \(5|R|/7\) elements of \(R\) whose value are greater than the elements in the partitioning block. One element \(x_R\) is selected from the partitioning block. We will refer to this element as a partitioning element. These partitioning elements are then sorted into increasing order, which incurs a cost of \(O(\log(\text{Runs}))\) fragile complexity on each of the partitioning elements. The runs are then arranged in the same order as their partitioning elements. Label this sequence of runs as \(R_1, R_2, \ldots, R_k\), and let \(t\) be the largest index such that \(\sum_{i=1}^{t-1} |R_i| < N/8\).

Since the partitioning element \(x_{R_t}\) is smaller than all the elements in the blocks with values greater than their respective partitioning blocks in \(R_1, R_{t+1}, \ldots, R_k\), we have that \(x_{R_t}\) is smaller than \((7/8)(5N/7) = 5N/8\) of the remaining elements. Hence in rank it is at least \(N/8\) below the median of the remaining
elements. By the invariant on the position in rank space of this median and the fact that \( N > 64 \cdot \text{Runs} \cdot \log n \), we note that \( x_{R_t} \) has a rank below \( a \). We also note that all the elements below the partitioning blocks in \( R_1, R_2, \ldots, R_t \) have value less than \( x_{R_t} \). This constitutes at least \((1/8)(N/7) = N/56\) elements in \( \mathcal{N} \) with rank below \( a \). Therefore, we can remove \( N/56 \) elements with rank below \( a \).

In a similar manner, we can find at least \( N/56 \) elements in \( \mathcal{N} \) with rank above \( b \). Removal of these \( 2N/56 = N/28 \) elements in \( \mathcal{N} \) constitutes a type A removal step.

Since the number of elements under consideration, i.e. \( N \), decreases by a constant factor at each step, the algorithm performs \( O(\log n) \) type A and type B removal steps before we have \( N \leq 64 \cdot \text{Runs} \cdot \log n \). Since each block under consideration in a type A removal step has size \( \log n \), we can guarantee that each element in a partitioning block only needs to be selected as a partitioning element \( O(1) \) times. This implies that a total cost of \( O(\log(\text{Runs})) \) fragile complexity is incurred on each element once we have that \( N \leq 64 \cdot \text{Runs} \cdot \log n \).

We now describe the final step of the algorithm. At this point, the algorithm combines the last \( N \) elements with all the short runs removed during its execution up to this point, forming the set \( \mathcal{S} \). This set is the original elements subjected to a series of type A removals, each of which are balanced and outside the rank interval \([a, b]\) Hence, the median of \( \mathcal{S} \) is the global median. As \(|\mathcal{S}| = O(\text{Runs} \cdot \log n)\), we can find this median in \( O(\log(\text{Runs} \cdot \log n)) = O(\log(\text{Runs}) + \log \log n) \) fragile complexity \([1]\), which dominates the total fragile complexity of the algorithm.

We note that for \( \text{Runs} = 2 \), we can improve the above result to \( O(1) \) fragile complexity as follows. Let the two runs be \( R_1 \) and \( R_2 \), with \(|R_1| \leq |R_2|\). Compare their middle elements \( x \) and \( y \) and assume \( x \leq y \). Then the elements in the first half of \( R_1 \) are below \( n/2 \) other elements, and hence are below the median. Similarly, the elements in the last half of \( R_2 \) are above the median. Hence, we can remove \(|R_1|/2\) elements on each side of the median by removing that many elements from one end of each run. The median of the remaining elements is equal to the global median. By recursion, we in \( \log |R_1| \) steps end up with \( R_1 \) reduced to constant length. Then \( O(1) \) comparisons with the center area of \( R_2 \) will find the median. Because both runs lose elements in each recursive step, both \( x \) and \( y \) will be new elements each time. The total fragile complexity of the algorithm is therefore \( O(1) \).

**Theorem 8.** Minimum has fragile complexity \( \Theta(\log(\text{Inv})) \).

**Proof.** Lower bound: For any \( k \), consider the instances composed of \( \sqrt{k} \) elements in random order followed by \( n - \sqrt{k} \) larger elements in sorted order. These instances have \( \text{Inv} \leq k \). Finding the minimum is equal to finding the minimum on the first \( \sqrt{k} \) elements, which has a lower bound \([1]\) of \( \log \sqrt{k} = \Omega(\log k) \) on its fragile complexity.

For the upper bound, we will remove a subset \( I \) of size \( O(\text{Inv}) \) which leaves a single sorted run \( R \). We can find the minimum in \( I \) in \( O(\log(\text{Inv})) \) fragile complexity by a tournament tree, which can then be compared to the head of \( R \) for the final answer. \( \square \)
We find \( I \) and \( R \) in \( O(1) \) fragile complexity during a scan of the input as follows, using \( R \) as a stack. For each new element \( e \) scanned, we compare it to the current top element \( f \) of \( R \). If \( e \) is larger, we push \( e \) to the top of \( R \). If \( e \) is smaller, it forms an inversion with \( f \), and we include \( e \) in \( I \). We also put a mark on \( f \). If an element on the stack gets two marks, we pop it, include it in \( I \), remove one of its marks (which will account for its inclusion in \( I \)) and move the second mark to the new top element \( f' \) of \( R \). If \( f' \) now has two marks, this process continues until an element with only a single mark is created (or the stack gets empty).

An element is compared with exactly one element residing earlier in the input (when the element is scanned). To count comparisons with elements residing later in the input, call such a comparison large or small, depending on whether the other element is larger or smaller. It is easy to see that elements on the stack always have between zero and one marks, that an element with zero marks has participated in one large comparison, and that an element with one mark has either participated in at most two larger comparisons or one smaller comparison. Hence, the fragile complexity of the process is \( O(1) \). By the accounting scheme, \( I \) is no larger than \( \text{Inv} \) plus the number of marks, which is also at most \( \text{Inv} \).

\[
\text{Theorem 9.} \quad \text{Median has fragile complexity } \Theta(\log(\text{Inv})).
\]

\textit{Proof.} As Median solves Minimum via padding with \( n \) elements of value \(-\infty\), the lower bound follows from the lower bound on Minimum. For the upper bound, find \( R \) and \( I \) as in the upper bound for Minimum, sort \( I \) in fragile complexity \( O(\log(\text{Inv})) \) and use the algorithm for Median for \( \text{Runs} = 2 \). \( \Box \)

\[
\text{Theorem 10.} \quad \text{Sorting has fragile complexity } \Theta(\log(\text{Inv})).
\]

\textit{Proof.} The lower bound follows from the lower bound on Minimum. For the upper bound, find \( R \) and \( I \) as in the upper bound for Minimum and let each element recall its position in the input. Divide the sorted sequence \( R \) into contiguous blocks of size \( |I| \) and let \( R_i \) be the set of \( i \)'th elements of all blocks. With the \( i \)'th element of \( I \) we perform an exponential search on \( R_i \), starting from the block where the element’s position in the input is. If the search moves a distance \( k \), the element from \( I \) participated in at least \( k \) inversions in the input, so \( k \leq \text{Inv} \) and hence the incurred fragile complexity for the element is \( O(\log k) = O(\log(\text{Inv})) \). A fragile complexity of \( O(1) \) is incurred on the elements of \( R \) as each \( R_i \) is used once. After this, each element from \( I \) knows its position in \( R \) within a window of size \( |I| \). If the window of an element and a block overlaps, we call the element and the block associated. Each block of \( R \) is associated with at most \( |I| = O(\text{Inv}) \) elements, and each element is associated with at most two blocks. For each block in turn, we now sort its associated elements and merge them into the block (except for tail elements overlapping the next block). The sorting incurs \( O(\log(\text{Inv})) \) fragile complexity, as does the merging if we use exponential merging \( \Box \). We remove all inserted elements from any association with the next block. Then we continue with the next block. \( \Box \)
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