A Family of non-Gaussian Martingales with Gaussian Marginals

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Abstract: We construct a family of non-Gaussian martingales the marginals of which are all Gaussian. We give the predictable quadratic variation of these processes and show they do not have continuous paths. These processes are Markovian and inhomogeneous in time, and we give their infinitesimal generators. Within this family we find a class of piecewise deterministic pure jump processes and describe the laws of jumps and times between the jumps.

AMS 2000 subject classifications: Primary 60G44, 60J25, 60J75; secondary 91B70.

Keywords and phrases: Martingales with Gaussian marginals, Time-inhomogeneous Markov processes.

1. Introduction

This paper is concerned with constructing an alternative process to the Brownian motion; a martingale with Gaussian marginals, yet not a Gaussian process. The construction of martingales with given marginals has significance in financial modelling (see for example [1], [2], [3] and [6]). In [4], the authors investigate the existence of an alternative model for which the Black-Scholes formula holds true. The case of the Bachelier formula raises the question of the existence of a non-Gaussian martingale with Gaussian marginals.

Our solution to this problem is based on the elementary observation that for $Y$ and $\xi$ independent standard Gaussian random variables, the distribution of $Z = \sqrt{r}Y + \sqrt{1-r}\xi$ is a standard Gaussian random variable for any value of $r \in [0, 1]$. This allows us to randomize $r$ and construct a family of Markovian martingales with Gaussian marginals.

The question of constructing (Markovian) martingales with given marginals has seen considerable interest in recent years, mostly initiated by the paper of Madan and Yor ([3]).

In [4], the authors give three different approaches: a continuous martingale, a time-changed Brownian motion, and a construction that uses Azéma-Yor’s solution to the Skorokhod embedding problem. These constructions are applied to a number of special cases including the case with Gaussian marginals. But out of the three constructions, only the Skorokhod embedding approach yields a
non-Gaussian martingale. The other two reduce to a construction of a Brownian motion. The continuous martingale approach looks for a process of the type

\[ X_t = \int_0^t \sigma(X_s, s) dW_s, \]

for which the marginal densities \( g(x, t) \) are \( N(0, t) \). Writing the forward equation for these densities, it follows that \( \sigma^2 \equiv 1 \) (see [6]), and \( X_t \) is a Brownian motion. In the time change approach

\[ X_t = B_{L_t} \]

where \( L \) is an increasing process — in fact \( L \) is assumed to be an increasing Markov process with inhomogeneous independent increments, independent of the Brownian motion \( B \). The assumption of Gaussian marginals implies

\[ \mathbb{E}[e^{i\lambda X_t}] = \mathbb{E}[e^{-(\lambda^2/2)L_t}] = e^{-(\lambda^2/2)t}, \]

and it follows that \( L_t = t \). But the Skorokhod embedding approach of Madan and Yor yields a discontinuous and time-inhomogeneous Markov process.

Our approach is different to the above. It uses basic principles, and has the advantage of producing an entire family of processes indexed by an (infinite) family of subordinators. The construction produces a family discontinuous and time-inhomogeneous Markov processes. We obtain the quadratic variation of these processes and infinitesimal generators in some cases. The richness of the family has the potential to allow for the imposition of specifications other than the marginal distributions.

Note that our method can be extended to include other types of marginal distributions, but for clarity of presentation we choose to focus solely on the Gaussian case.

Finally, all existing approaches yield discontinuous processes (barring the Brownian motion itself), and the question of the existence of a non-Gaussian continuous martingale with Gaussian marginals remains open.

2. A Family of non-Gaussian Martingales with Gaussian Marginals

In this section we construct a (non-Gaussian) Markov martingale \( X_t \) the marginals of which are Gaussian with mean zero and variance \( t \). The existence of such process is guaranteed by a Theorem of Kellerer (see [5] and [6]) that only requires the targeted marginal densities, \( g(x, t) \), be increasing in the convex order \( (\mathbb{E}[f(X_t)] \geq \mathbb{E}[f(X_s)] \) for \( s < t \) and \( f \) convex), and have means that do not depend on \( t \).

As eluded to in the introduction, the main idea of the proposed construction is the fact that, for any triple \( (R, Y, \xi) \) of independent random variables such that \( R \) takes values in \([0,1] \), \( \xi \) is standard Gaussian and \( Y \) is Gaussian with mean zero and variance \( \alpha^2 \), the random variable \( Z = \sigma(\sqrt{RY} + \alpha\sqrt{1-R}\xi) \) is Gaussian with mean zero and variance \( \sigma^2\alpha^2 \). However, the unconditional joint distribution of \((Y,Z)\) is not bivariate Gaussian, as can be verified by calculating
the fourth conditional moment of $Z$ given $Y = 0$. In fact, $(Y, Z)$ is a bivariate Gaussian pair if and only if $R$ is non-random. The martingale property of the two-step process $(Y, Z)$ holds if and only if

$$Y = \mathbb{E}[Z|Y] = \mathbb{E}[\sigma(\sqrt{RY} + \alpha\sqrt{1-R}\xi)|Y] = \sigma\mathbb{E}[\sqrt{R}|Y],$$

in other words,

$$\mathbb{E}[\sqrt{R}] = \frac{1}{\sigma}. \quad (1)$$

Furthermore, the conditional distribution of $Z$ given $Y$ is

$$F_{Z|Y=y}(dz) = \mathbb{P}[R = 1]\delta_y(dz) + \mathbb{E} \left[ \phi(\sigma\sqrt{Ry}, \alpha^2\sigma^2(1-R), z) 1_{R<1} \right] dz,$$

where $\delta_x$ is the Dirac measure at $x$ and $\phi(\mu, \sigma^2, \cdot)$ denotes the density of the Gaussian distribution with mean $\mu$ and variance $\sigma^2$.

This construction of a two-step process can be extended to that of a continuous time Markov process. Indeed, let $R_{s,t}$ be a family of random variables indexed by $0 < s \leq t$. We assume that $R_{s,t}$ takes values in $[0, 1]$, has distribution that depends on $(s, t)$ only through $\sqrt{t/s}$, and has moment of order $1/2$ equal to $\sqrt{s/t}$. We denote the distribution of $R_{s,t}$ by $G_{\sqrt{t/s}}(dr)$. We shall also need a family $\xi_{s,t}$ of standard Gaussian random variables.

The process $X_t$ is constructed as a Markov process with the following almost sure representation of $X_t$ in terms of $X_s$, $s < t$,

$$X_t = \sqrt{\frac{t}{s}} \left( \sqrt{R_{s,t}X_s} + \sqrt{s}\sqrt{1-R_{s,t}}\xi_{s,t} \right). \quad (2)$$

We assume given the usual set-up of a probability space endowed with a filtration $\mathcal{F}_t$ to which $X_t$ is adapted. In the representation (2), $R_{s,t}$ and $\xi_{s,t}$ are assumed to be independent of each other, $\mathcal{F}_t$-measurable and independent of $\mathcal{F}_s$.

In the sequel, we shall often write $\alpha$ for $\sqrt{s}$, $\sigma$ for $\sqrt{t/s}$, $\tau$ for $\sqrt{u/t}$ and whenever independence between the variables involved need not be emphasized, $R$ for $R_{s,t}$ and $R_\tau$ for $R_{t,u}$.

**Definition 1** The family $(G_{\sigma})_{\sigma \geq 1}$ is a log-convolution semi-group if the the distribution of the product of any two independent random variables with distributions $G_\sigma$ and $G_\tau$, is $G_{\sigma\tau}$.

Define, for $\sigma \geq 1$ and $R_\sigma$ distributed as $G_\sigma$, $U_\sigma = -\ln R_\sigma$, and, for $p \geq 0$, $V_p = U_{\sigma p}$. If $K_p$ denotes the distribution of $V_p$, then $(G_{\sigma})_{\sigma \geq 1}$ is a log-convolution semi-group if and only if $(K_p)_{p \geq 0}$ is a convolution semi-group:

$$K_0 = \delta_0, \quad K_p * K_q = K_{p+q}.$$

**Proposition 1** Define, $P_{s,t}(x, dy)$ as,

$$P_{0,t}(x, dy) = \frac{1}{\sqrt{2\pi\sqrt{t}}} \exp \left( -\frac{(y-x)^2}{2t} \right) dy. \quad (3)$$
and for $s > 0$,
\[ P_{s,t}(x,dy) = \gamma(\sigma)x + \int \frac{1}{\sqrt{2\pi t(1-r)}} \exp \left( -\frac{(y - \sigma \sqrt{t}x)^2}{2t(1-r)} \right) G_\sigma(dr)dy \]
\[ = \gamma(\sigma)x + \mathbb{E} \left[ \varphi(\sigma \sqrt{t}x, \sigma^2(1-R_\sigma), y) 1_{R_\sigma < 1} \right] dy, \]
where $R_\sigma$ is distributed as $G_\sigma$ and $\gamma(\sigma) = G_\sigma(\varphi)$. If $(G_\sigma)_{\sigma \geq 1}$ is a log-convolution semi-group then for any $u > t > s > 0$ and any $x$,
\[ \int P_{s,t}(x,dy)P_{t,u}(y,dz) = P_{s,u}(x,dz) \] (5)
and, for any $u > t > 0$,
\[ \int P_{0,t}(0,dy)P_{t,u}(y,dz) = P_{0,u}(0,dz). \] (6)

**Proof:** We prove (5) and (6) by showing that the almost sure formulation (2) is consistent.

\[ X_u = \tau \left( \sqrt{R_{t,u}}X_t + \sigma \sqrt{1 - R_{t,u}} \xi_{t,u} \right) \]
\[ = \tau \left( \sqrt{R_{t,u}} \sigma \left( \sqrt{R_{s,t}}X_s + \alpha \sqrt{1 - R_{s,t}} \xi_{s,t} \right) + \sigma \alpha \sqrt{1 - R_{t,u}} \xi_{t,u} \right) \]
\[ = \sigma \tau \left( \sqrt{R_{s,t}R_{t,u}}X_s + \alpha \left( \sqrt{(1 - R_{s,t})R_{t,u}} \xi_{s,t} + \sqrt{1 - R_{t,u}} \xi_{t,u} \right) \right) \]

Now, letting $R_{s,u} = R_{s,t}R_{t,u}$ and
\[ \xi_{s,u} = \left( \frac{\sqrt{(1 - R_{s,t})R_{t,u}}}{\sqrt{1 - R_{s,u}}} 1_{R_{s,u} < 1} + \frac{\sqrt{1 - R_{t,u}}}{\sqrt{1 - R_{s,u}}} 1_{R_{s,u} = 1} \right) \xi_{s,t} + \frac{\sqrt{1 - R_{t,u}}}{\sqrt{1 - R_{s,u}}} 1_{R_{s,u} < 1} \xi_{t,u}, \]
we see that
\[ X_u = \tau \sigma \left( \sqrt{R_{s,u}}X_s + \alpha \sqrt{1 - R_{s,u}} \xi_{s,u} \right) \]
with $R_{s,u}$ distributed as $G_\tau(dr)$. Also, the unconditional distribution of $\xi_{s,u}$ as well as its conditional distribution given $R_{s,t}$ and $R_{t,u}$ are standard Gaussian. This in turn implies that $\xi_{s,u}$ is independent of $R_{s,u}$. \( \square \)

**Proposition 2 (Lévy-Khinchin Theorem)** Assume that the family $(G_\sigma)_{\sigma \geq 1}$ is a log-convolution semi-group and let $(R_\sigma)_{\sigma \geq 1}$ be independent random variables with laws $(G_\sigma)_{\sigma \geq 1}$. Let $L_\sigma(\lambda) = \mathbb{E} \left[ e^{\lambda \ln R_\sigma} \right] = \mathbb{E} \left[ (R_\sigma)\lambda \right]$ be the moment generating function of the (positive) random variable $U_\sigma = -\ln R_\sigma$.

For any $\sigma \geq 1$, $U_\sigma$ is infinitely divisible, and
\[ \ln L_\sigma(\lambda) = -\beta \lambda + \int_0^{\infty} \left( 1 - e^{-\lambda x} \right) \nu(dx) \ln \sigma \] (7)
where the Lévy measure $\nu(dx)$ satisfies $\nu(\{0\}) = 0$ and $\int_0^\infty (1 \wedge x) \nu(dx) < \infty$.

In what follows, we denote by $\psi$ the Laplace exponent of the log-convolution semi-group $(G_\sigma)_{\sigma \geq 1}$:

$$\psi(\lambda) = \beta \lambda + \int_0^\infty (1 - e^{-\lambda x}) \nu(dx).$$

The following theorem follows from the above and the Chapman-Kolmogorov existence result.

**Theorem 3** Assume that the family $(G_\sigma)_{\sigma \geq 1}$ is a log-convolution semi-group with Laplace exponent

$$\psi(\lambda) = \beta \lambda + \int_0^\infty (1 - e^{-\lambda x}) \nu(dx).$$

If $\psi(1/2) = 1$, then there exists a Markov martingale $X_t$ starting at zero with transition probabilities $P_{s,t}(x,dy)$ given by (5) and (6) the marginal distributions of which are Gaussian with mean zero and variance $t$.

### 3. Path properties

**Theorem 4** The process $X_t$ is continuous in probability:

$$\forall c > 0, \lim_{s \to t} \mathbb{P}[|X_t - X_s| > c] = 0.$$

**Proof:** Using Lemma 5 below, we write,

$$\mathbb{P}[|X_t - X_s| > c] \leq \frac{1}{c^2} \mathbb{E}[(X_t - X_s)^2] = \frac{1}{c^2} [t - t^{1-\delta} s^{\delta} + t^{1-\delta} s^{\delta} - s] = \frac{t - s}{c^2}.$$

**Lemma 5** Let $\delta = \psi(1)/2$ so that $L_\sigma(1) = \sigma^{-2\delta}$. Then

$$\mathbb{E}[(X_t - X_s)^2|X_s] = t - t^{1-\delta} s^{\delta} + t^{1-\delta} s^{1+\delta} X_s^2.$$

**Proof:** Using representation (5), we see that

$$\mathbb{E}[(X_t - X_s)^2|X_s] = \mathbb{E} \mathbb{E}[\mathbb{E}((X_t - X_s)^2|X_s, R_\sigma)|X_s]$$

$$= \alpha^2 \sigma^2 \mathbb{E}[1 - R_\sigma] + \mathbb{E} \left[ \sigma \sqrt{R_\sigma} - 1 \right] X_s^2$$

$$= \alpha^2 \sigma^2 (1 - L_\sigma(1)) + \left( \sigma \mathbb{E}[R_\sigma] - 1 \right) X_s^2$$

$$= \alpha^2 \sigma^2 (1 - L_\sigma(1)) + \left( \sigma^2 L_\sigma(1) - 1 \right) X_s^2$$

$$= \alpha^2 \sigma^2 - \alpha^2 \sigma^{2-2\delta} + \sigma^{2-2\delta} X_s^2 - X_s^2.$$
Theorem 6 The \( (\text{predictable}) \) quadratic variation of \( X_t \) is
\[
\langle X, X \rangle_t = \delta t + (1 - \delta) \int_0^t \frac{X_s^2}{s} \, ds,
\]
where \( \delta = \psi(1)/2 \). Furthermore, it can be obtained as a limit in probability,
\[
\langle X, X \rangle_t = \mathbb{P} \lim_{n \to \infty} \sum_{k=0}^{n-1} \mathbb{E} \left[ (X_{t_{k+1}} - X_{t_k})^2 \big| X_{t_k} \right]
\]
where \( t_0 < t_1 < \ldots < t_n \) is a subdivision of \([0, t]\).

Proof: First note that \( X_t \) is a square integrable martingale on any finite interval \([0, T]\). In fact \( \sup_{t \leq T} \mathbb{E}[X_t^2] = T \). Also,
\[
\mathbb{E}[X_t^2 | \mathcal{F}_s] = \mathbb{E} \left[ (t - R_s) + \sigma^2 R_s X_s^2 | X_s \right] = t(1 - L_s(1)) + \sigma^2 L_s(1) X_s^2.
\]
Since \( L_s(1) = \sigma^{-2\delta} = \sigma^{-2\delta} \), we find
\[
\mathbb{E}[X_t^2 | \mathcal{F}_s] = t - t^{1-\delta}s^{\delta} + t^{1-\delta}s^{-1+\delta}X_s^2.
\]
It follows that
\[
\mathbb{E} \left[ (1 - \delta) \int_0^t \frac{X_u^2}{u} \, du \big| \mathcal{F}_s \right] = (1 - \delta) \int_0^s \frac{X_u^2}{u} \, du + (1 - \delta) \int_s^t (1 - u^{-\delta}s^{\delta} + u^{-\delta}s^{-1+\delta}X_s^2) \, du
\]
\[
= (1 - \delta) \int_0^s \frac{X_u^2}{u} \, du + (1 - \delta)(t - s) - s^{\delta}(1 - s^{-1}X_s^2)(t^{1-\delta} - s^{1-\delta})
\]
\[
= (1 - \delta) \int_0^s \frac{X_u^2}{u} \, du + (1 - \delta)(t - s) - s^{\delta}t^{1-\delta} + s + t^{1-\delta}s^{-1+\delta}X_s^2 - X_s^2
\]
and
\[
\mathbb{E} \left[ X_t^2 - \delta t - (1 - \delta) \int_0^t \frac{X_u^2}{u} \, du \big| \mathcal{F}_s \right] = t - t^{1-\delta}s^{\delta} + t^{1-\delta}s^{-1+\delta}X_s^2 - \delta t - (1 - \delta) \int_0^s \frac{X_u^2}{u} \, du - (1 - \delta)(t - s) + s^{\delta}t^{1-\delta} - s - t^{1-\delta}s^{-1+\delta}X_s^2 + X_s^2
\]
\[
= t - \delta t - (1 - \delta) \int_0^s \frac{X_u^2}{u} \, du - (1 - \delta)(t - s) - s + X_s^2
\]
\[
= X_s^2 - \delta s - (1 - \delta) \int_0^s \frac{X_u^2}{u} \, du.
\]
\[\square\]

The next result states that the only continuous process that can be constructed in the the way described in Section 2 is the Brownian motion.
Theorem 7 If \( R_\sigma \) is not degenerate \( (R_\sigma \neq \sigma^{-2}) \), \( X_t \) is not continuous.

Proof: We proceed by contradiction and assume that \( X_t \) is continuous. Itô’s formula for \( e^{i\lambda X_t} \) gives

\[
e^{i\lambda X_t} = 1 + M_t - \frac{\lambda^2}{2} \int_0^t e^{i\lambda X_s} d\langle X, X \rangle_s,
\]

where \( M_t = \int_0^t i\lambda e^{i\lambda X_s} dX_s \) is a true martingale. In fact

\[
E [\langle M, M \rangle_t] = E \left[ -\int_0^t \lambda^2 e^{i2\lambda X_s} d\langle X, X \rangle_s \right]
\]

\[
= E \left[ -\int_0^t \lambda^2 e^{i2\lambda X_s} (\delta ds + (1 - \delta) \frac{X^2_s ds}{s}) \right]
\]

\[
\leq \delta \lambda^2 t + (1 - \delta) \lambda^2 \int_0^t E[X^2_s] \frac{ds}{s}
\]

\[
= \delta \lambda^2 t + (1 - \delta) \lambda^2 t = \lambda^2 t,
\]

since \( X_s \) is \( N(0, s) \) and \( E[X^2_s] = s \).

Taking expectations in 8, we obtain that \( \theta(\lambda, t) = E[e^{i\lambda X_t}] = e^{-\lambda^2 t/2} \) must satisfy

\[
\theta(\lambda, t) = 1 - \frac{\lambda^2}{2} \left[ \delta \int_0^t \theta(\lambda, s) ds + (1 - \delta) \int_0^t E[X^2_s e^{i\lambda X_s}] ds \right]
\]

\[
= 1 - \frac{\lambda^2}{2} \left[ \delta \int_0^t \theta(\lambda, s) ds - (1 - \delta) \int_0^t \frac{\partial^2 \theta}{\partial \lambda^2}(\lambda, s) ds \right].
\]

Differentiating in \( t \), we get that \( \theta(\lambda, t) \) must satisfy

\[
-\frac{\lambda^2}{2} \theta(\lambda, t) = -\frac{\lambda^2}{2} \left[ \delta \theta(\lambda, t) - (1 - \delta) \frac{\partial^2 \theta}{\partial \lambda^2}(\lambda, t) \right],
\]

that is,

\[
-\frac{\lambda^2}{2} = -\frac{\lambda^2}{2} \left[ \delta - (1 - \delta)t(\lambda^2 t - 1) \right].
\]

This, of course, can only occur if \( \delta = 1 \), which corresponds to \( L_\sigma(1) = \sigma^{-2} \) and \( R_\sigma \) being non-random equal to \( \sigma^{-2} \).

4. Explicit Constructions

Before we engage in the explicit construction of the processes outlined in the previous sections, let us observe that these fall into one of two subclasses according to whether or not \( G_\sigma(\{1\}) \) is nil, uniformly in \( \sigma \).
Indeed,
\[ \gamma(\sigma) = G_\sigma(\{1\}) = \lim_{\lambda \uparrow \infty} L_\sigma(\lambda) = \lim_{\lambda \uparrow \infty} \exp(-\psi(\lambda) / \ln \sigma) \]
and
\[ \gamma(\sigma) = 0 \Leftrightarrow \lim_{\lambda \uparrow \infty} \psi(\lambda) = +\infty. \]

4.1. The Case \( \gamma(\sigma) > 0 \)

In this section we apply our construction to the case where \( \gamma(\sigma) = G_\sigma(\{1\}) > 0 \). The processes thus obtained are piecewise deterministic pure jump process in the sense that between any two consecutive jumps, the process behaves according to a deterministic function. Examples of such processes include the case where \( G_\sigma \) is an inverse log-Poisson distribution.

The interpretation of these processes as piecewise deterministic pure jump processes requires the computation of the infinitesimal generator.

**Proposition 8** Let \( G_\sigma \) be a log-convolution semi-group for which \( \gamma(\sigma) = G_\sigma(\{1\}) > 0 \), \( \gamma \) is differentiable at 1 and \( \lim_{\lambda \downarrow 0} \psi(\lambda) = 0 \). Then the infinitesimal generator of \( X_t \) on the set of \( C_0^2 \)-functions is given by

\[
A_0 f(x) = \frac{1}{2} f''(x) \text{ and for } s > 0,
\]

\[
A_s f(x) = \frac{x}{2s} f'(x)
\]

\[
-\gamma'(1) \frac{1}{2s} \int [f(x + z) - f(x)] \int_{[0,1]} \phi((\sqrt{r} - 1)x, s(1 - r), z) G(dr)dz,
\]

where

\[
\bar{G}(dr) = \lim_{\sigma \downarrow 1} \frac{G_\sigma(dr \cap [0,1])}{G_\sigma([0,1])}
\]

is a probability measure on \([0,1]\), and the limit is understood in the weak sense.

Thus the process \( X \) starts off as a Brownian motion and, when in \( x \) at time \( s \), drifts at the rate of \( x/(2s) \), and jumps at the rate of \( -\gamma'(1)/(2s) \). The size of the jump from \( x \) has density \( \int_{[0,1]} \phi((\sqrt{r} - 1)x, s(1 - r), z) G(dr)dz \), the mean of which is \( \int_{[0,1]} (\sqrt{r} - 1)G(dr)x \). In other words, while in positive territory, \( X_t \) continuously drifts upwards and has jumps that tend to be negative. In negative region, the reverse occurs; \( X_t \) drifts downwards and has (on average) positive jumps.

**Proof:** First note that the conditional moment generating function of \( U_\sigma \) given \( U_\sigma > 0 \) is

\[
L_\sigma^*(\lambda) = \frac{L_\sigma(\lambda) - \gamma(\sigma)}{1 - \gamma(\sigma)}
\]
and converges to
\[ \lim_{\sigma \downarrow 1} L_s^\sigma(\lambda) = 1 + \frac{\psi(\lambda)}{\gamma'(1)}. \]

By the (Laplace) continuity theorem, if \( \lim_{\lambda \downarrow 0} \psi(\lambda) = 0 \) then there exists a probability measure on \([0, 1), \bar{G}(dr)\), such that
\[ \bar{G}(dr) = \lim_{\sigma \downarrow 1} \frac{G_\sigma(dr \cap [0, 1))}{G_\sigma([0, 1))}. \]

Next,
\[
\begin{align*}
\frac{1}{t-s} \left( \mathbb{E}[f(X_t)|X_s = x] - f(x) \right) &= \frac{1}{s} \left[ \frac{f(\sigma x)\gamma(\sigma) - f(x)}{\sigma^2 - 1} \right. \\
&\quad + \frac{1}{\sigma^2 - 1} \int f(y) \int_{[0,1)} \phi(\sigma \sqrt{r}x, t(1-r), y) G_\sigma(dr)dy \\
&= \frac{1}{s} \left[ \frac{f(\sigma x)\gamma(\sigma) - f(x)}{\sigma^2 - 1} \right. \\
&\quad + \left. \frac{1}{\sigma^2 - 1} \int f(y) \int_{[0,1)} \phi(\sigma \sqrt{r}x, t(1-r), y) \frac{G_\sigma(dr)}{1 - \gamma(\sigma)} dy \right].
\end{align*}
\]

Letting \( \sigma \) decrease to 1, we see that
\[ A_s f(x) = \lim_{t \downarrow s} \frac{1}{t-s} \left( \mathbb{E}[f(X_t)|X_s = x] - f(x) \right) \]
\[
\begin{align*}
&= \frac{1}{s} \left[ \frac{xf'(x) + \gamma'(1)f(x)}{2} - \frac{\gamma'(1)}{2} \int f(y) \int_{[0,1)} \phi(\sqrt{r}x, s(1-r), y) G(dr)dy \right] \\
&= \frac{x}{2s} f'(x) + \frac{\gamma'(1)}{2s} \int [f(y) - f(x)] \int_{[0,1)} \phi(\sqrt{r}x, s(1-r), y) G(dr)dy \\
&= \frac{x}{2s} f'(x) + \frac{\gamma'(1)}{2s} \int [f(x + z) - f(x)] \int_{[0,1)} \phi((\sqrt{r} - 1)x, s(1-r), z) G(dr)dz.
\end{align*}
\]

Note that the domain of \( A_s \) can be extended to include functions that do not vanish at infinity, such as \( f(x) = x^2 \). Indeed by Theorem 6, \( g_s(x) = \delta + (1 - \delta) \frac{x^2}{s} \) solves the martingale problem for \( f(x) = x^2 \).

The next proposition immediately follows from the observation that the process \( X \) does not jump between times \( s \) and \( t \) if and only if \( X_u = \sqrt{\frac{u}{s}} X_s \) for \( u \in (s, t) \).
Proposition 9 Let $T_s$ denote the first jump time after $s > 0$. Then, for any $t > s$,

$$\mathbb{P}[T_s > t] = \gamma(\sigma),$$

where as before, $\sigma = \sqrt{t/s}$.

4.2. The Poisson Case: $\gamma(\sigma) = \sigma^{-c}$

In this case $\beta = 0$, $\nu(dx) = c\delta_1(dx)$ with $c = \frac{1}{1 - e^{-1/2}}$, and $\psi(\lambda) = c(1 - e^{-\lambda})$. In other words $U_{\sigma} = -\ln R_{\sigma}$ has a Poisson distribution with mean $c \ln \sigma$.

The assumptions of Proposition 9 are clearly satisfied with $\gamma(\sigma) = \sigma^{-c}$, $\gamma'(1) = -c$, $\lim_{\sigma \downarrow 1} L_{\sigma}^c(\lambda) = e^{-\lambda}$ and $\bar{G}(dr) = \varepsilon_{e^{-1}}(dr)$, so that $X_t$ has infinitesimal generator

$$A_s f(x) = \frac{x}{2s} f'(x) + \frac{c}{2s} \int [f(x + z) - f(x)] \phi(-x/c, s(1 - e^{-1}), z) dz.$$ 

It jumps at the rate of $\frac{c}{2s}$ with a size distributed as a Gaussian random variable with mean $-\frac{x}{c}$ and variance $s(1 - e^{-1})$. The graph below shows a simulation of a path of such a process.

Furthermore, the law of the first jump time after $s$ is given by

$$\mathbb{P}[T_s > t] = \gamma(\sigma) = \frac{s^{c/2}}{t^{c/2}}.$$ 

In other words, $T_s$ is Pareto distributed (with location parameter $s$ and scale parameter $c/2 \sim 1.27$). In particular, 

$$\mathbb{E}[T_s] = \frac{cs}{c - 2} \text{ and } \mathbb{E}[T_s^2] = \infty.$$
4.3. The Case $\gamma(\sigma) = 0$

When $\gamma(\sigma) = 0$, we are only able to compute the infinitesimal generator for functions of a specific type. Examples of such functions include polynomials.

**Proposition 10** Assume that $\beta = 0$ so that

$$\psi(\lambda) = \int_0^\infty (1 - e^{-\lambda x}) \nu(dx).$$

Let $f$ be a $C^1$-function with the following property: there exist a function $N_f$ and a (signed) finite measure $M_f$ such that

$$f(\sigma e^{-u/2}x + \sqrt{t}1 - e^{-u}z) = N_f(\sigma) \int_0^\infty e^{-\lambda u}M_f(s, x, z, d\lambda), \quad u > 0,$$

and

$$\lim_{\sigma \downarrow 1} N_f(\sigma) = 1.$$

Then, for any $s > 0$,

$$A_s f(x) = \frac{x}{2s} f'(x) + \frac{1}{2s} \int [f(x + y) - f(x)] \int_0^{+\infty} \phi((e^{-\omega/2} - 1)x, s(1 - e^{-\omega}), y) \nu(d\omega) dy.$$

**Proof:** Let

$$C_\sigma f(u) = C_\sigma f(s, x, z, u) = f(\sigma e^{-u/2}x + \sqrt{t}1 - e^{-u}z).$$

Then, since $\gamma(\sigma) = 0$, $U_\sigma$ is almost surely strictly positive and,

$$\frac{1}{t - s} (E[f(X_t)|X_s = x] - f(x))$$

$$= \frac{1}{s \sigma^2 - 1} \int \left( E \left[ f \left( \sigma e^{-U_\sigma/2}x + \sqrt{t}1 - e^{-U_\sigma}z \right) \right] - f(x) \right) \phi(z) dz.$$

$$= \frac{1}{s \sigma^2 - 1} \left\{ \int \left( E [C_\sigma f(U_\sigma)] - C_\sigma f(0) \right) \phi(z) dz + (f(\sigma x) - f(x)) \right\}$$

$$= \frac{1}{s \sigma^2 - 1} \left\{ N_f(\sigma) \int_0^\infty \left( e^{-\lambda u} - 1 \right) M_f(d\lambda) \phi(z) dz + (f(\sigma x) - f(x)) \right\}$$

$$= \frac{1}{s \sigma^2 - 1} \left\{ \frac{N_f(\sigma) \ln \sigma}{\sigma - 1} \int_0^\infty \frac{e^{-\psi(\lambda) \ln \sigma} - 1}{\ln \sigma} M_f(d\lambda) \phi(z) dz + \frac{f(\sigma x) - f(x)}{\sigma - 1} \right\}.$$

Taking the limit as $\sigma \downarrow 1$ (that is $t \downarrow s$), we get

$$A_s f(x) = \frac{x}{2s} f'(x) - \frac{1}{2s} \int_0^\infty \psi(\lambda) M_f(s, x, z, d\lambda) \phi(z) dz.$$
Since
\[ \psi(\lambda) = \int_0^\infty (1 - e^{-\lambda \omega}) \, \nu(d\omega), \]

\[ A_s f(x) = x^2 s f'(x) - \frac{1}{2} \int_0^\infty \int_0^\infty \left[ \int_0^\infty (1 - e^{-\lambda \omega}) \, \nu(d\omega) \right] M_f(s, x, z, d\lambda) \phi(z) \, dz \]

\[ = \frac{x}{2s} f'(x) - \frac{1}{2s} \int_0^\infty \int_0^\infty \left[ \int_0^\infty (1 - e^{-\lambda \omega}) M_f(s, x, z, d\lambda) \right] \nu(d\omega) \phi(z) \, dz \]

\[ = \frac{x}{2s} f'(x) + \frac{1}{2s} \int_0^\infty \int_0^\infty \left[ f \left( e^{-\omega/2} x + \sqrt{s} \sqrt{1 - e^{-\omega}} z \right) - f(x) \right] \nu(d\omega) \phi(z) \, dz \]

and the proof is completed by a change of variables in \( z \).

**Lemma 11** Let \( f(x) = x^n \), then
\[ f \left( \sigma e^{-u/2} x + \sqrt{t} \sqrt{1 - e^{-u}} z \right) = \sigma^n \int_0^\infty e^{-\lambda u} M_f(s, x, z, d\lambda) \]

where
\[ M_f(s, x, z, d\lambda) = \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{n!}{k! j! (n-k-j)!} (-1)^j s^{k} z^{n-k} \left( \epsilon_{k/2} * m_j \right)(d\lambda) \]

and \( m_j(d\lambda) \) is the \( j \)-order convolution of the probability measure
\[ m(d\lambda) = \frac{1}{2 \sqrt{\pi}} \sum_{n=1}^{+\infty} \frac{\Gamma(n-1/2)}{n!} \xi_n(d\lambda). \]

**Proof:** First, write the Taylor series of the (analytic on \((0, 1)\)) function \( 1 - \sqrt{1-x} \),
\[ 1 - \sqrt{1-x} = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{\Gamma(n-1/2)}{n!} \Gamma(1/2) x^n. \]

It immediately follows that,
\[ 1 - \sqrt{1-e^{-u}} = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{\Gamma(n-1/2)}{n!} \Gamma(1/2) e^{-nu} = \int_0^\infty e^{-\lambda u} m(d\lambda), \]

where \( m(d\lambda) = \frac{1}{2 \sqrt{\pi}} \sum_{n=1}^{+\infty} \frac{\Gamma(n-1/2)}{n!} \xi_n(d\lambda) \) is a probability measure. Now,
\[ f \left( \sigma e^{-u/2} x + \sqrt{t} \sqrt{1 - e^{-u}} z \right) \]
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\[ \sigma^n \sum_{k=0}^{n} \binom{n}{k} e^{-ku/2} x^k s^{(n-k)/2}(1 - e^{-u})(n-k)/2 z^{n-k} \]

\[ = \sigma^n \sum_{k=0}^{n} \binom{n}{k} e^{-ku/2} x^k s^{(n-k)/2} \left[ 1 - \left(1 - \sqrt{1 - e^{-u}}\right) \right]^{n-k} z^{n-k} \]

\[ = \sigma^n \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{n!}{k! j! (n-k-j)!} (-1)^j x^k s^{(n-k)/2} z^{n-k} e^{-ku/2} \left(1 - \sqrt{1 - e^{-u}}\right)^j \]

The proof is ended by observing that

\[ e^{-ku/2} \left(1 - \sqrt{1 - e^{-u}}\right)^j = \int_0^\infty e^{-\lambda u} (\varepsilon_{k/2} \ast m_j) (d\lambda). \]

\[ \square \]

The following theorem is now proven.

**Theorem 12** Assume that \( \beta = 0 \). For any polynomial \( f \) and any \( s > 0 \),

\[ A_s f(x) = \frac{x}{2s} f'(x) + \frac{1}{2s} \int [f(x+y) - f(x)] \int_0^{+\infty} \phi((e^{-\omega/2} - 1)x, s(1 - e^{-\omega}), y) \nu(d\omega) dy, (9) \]

### 4.4. The Gamma Case: \( \gamma(\sigma) = 0 \)

Here \( \beta = 0, \nu(dx) = ax^{-1} e^{-bx} dx \) with \( a = \frac{1}{\ln (1 + \frac{\lambda}{b})} \) and \( \psi(\lambda) = a \ln \left(1 + \frac{\lambda}{b}\right); \) that is \( U_\sigma \) has a gamma distribution with density

\[ h_\sigma(u) = \frac{b^{a \ln \sigma}}{1(a \ln \sigma)} u^{a \ln \sigma - 1} e^{-bu}, \quad u > 0, \]

and \( R_\sigma \) has an inverse log-gamma distribution with density

\[ g_\sigma(r) = \frac{b^{a \ln \sigma}}{1(a \ln \sigma)} (-\ln r)^{a \ln \sigma - 1} r^{-b}, \quad 0 < r < 1. \]
In this case it is possible to compute the generator for a much wider class of functions.

**Proposition 13** Let $G_\sigma$ be the log-convolution semi-group of the inverse log-gamma distributions. Then (9) holds for any bounded function with bounded first derivative.

**Proof:** In the proof of Proposition 10 we write that

$$\frac{1}{t-s} \left( E[f(X_t)|X_s=x] - f(x) \right) = \frac{1}{s} \frac{1}{\sigma^2 - 1} \int \left( E[C_\sigma f(U_\sigma)] - f(x) \right) \phi(z) dz.$$

Denote by $\theta(u)$ the quantity $e^{-u^2/2} \sqrt{s\sqrt{1-e^{-u^2}}}$. Then, inserting $E[C_1 f(U_\sigma)] = E[f(\theta(U_\sigma))]$ we get,

$$\frac{1}{t-s} \left( E[f(X_t)|X_s=x] - f(x) \right) = \frac{1}{s} \frac{1}{\sigma + 1} \int \left\{ \frac{E[C_\sigma f(U_\sigma)] - E[C_1 f(U_\sigma)]}{\sigma - 1} + \frac{E[C_1 f(U_\sigma)] - f(x)}{\sigma - 1} \right\} \phi(z) dz. \tag{10}$$

Since

$$\frac{C_\sigma f(U_\sigma) - C_1 f(U_\sigma)}{\sigma - 1} = \frac{f(\sigma \theta(U_\sigma)) - f(\theta(U_\sigma))}{\sigma - 1} = \frac{\theta(U_\sigma) f'(\eta_\sigma)}{\sigma - 1},$$

for some $\eta_\sigma$ between $\theta(U_\sigma)$ and $\sigma \theta(U_\sigma)$, $\theta$ and $f'$ being bounded, we obtain that

$$\lim_{\sigma \downarrow 1} \int \frac{E[C_\sigma f(U_\sigma)] - E[C_1 f(U_\sigma)]}{\sigma - 1} \phi(z) dz = xf'(x).$$

To compute the limit of the second term in (10), we use Lemma 14 below, which shows that

$$\lim_{\sigma \downarrow 1} \int \frac{E[C_1 f(U_\sigma)] - f(x)}{\sigma - 1} \phi(z) dz.$$
\[
\begin{align*}
&= a \int \int_0^\infty f \left( e^{-u/2}x + \sqrt{s\sqrt{1-e^{-u}}}z \right) - f(x)e^{-bu}du \phi(z)dz \\
&= \int [f(x + y) - f(x)] \int_0^\infty \phi(x(e^{-u/2} - 1), s(1 - e^{-u}), y)ae^{-bu}u dudy.
\end{align*}
\]

Note that since \( \nu((0, \infty)) = +\infty \), \( \int_0^\infty \phi(x(e^{-u/2} - 1), s(1 - e^{-u}), y)\nu(du)du \) cannot be re-scaled to produce a density for the jumps of the process.

**Lemma 14** Let \( V_p \) have a gamma distribution with density:

\[
h_p(v) = \frac{b^p}{\Gamma(p)} v^{p-1} e^{-bv}, \quad v > 0.
\]

Let \( g \) be such that \( g(0) = 0 \) and \( g(v)/v \) is bounded. Then

\[
\lim_{p \downarrow 0} \frac{1}{p} \mathbb{E}[g(V_p)] = \int_0^\infty \frac{g(v)}{v} e^{-bv} dv.
\]

**Proof:** First observe that

\[
\frac{1}{p} \mathbb{E}[g(V_p)] = \frac{1}{b} \mathbb{E} \left[ \frac{g(V_{p+1})}{V_{p+1}} \right].
\]

taking the limit as \( p \downarrow 0 \), we obtain by dominated convergence

\[
\lim_{p \downarrow 0} \frac{1}{p} \mathbb{E}[g(V_p)] = \frac{1}{b} \mathbb{E} \left[ \frac{g(V_1)}{V_1} \right] = \int_0^\infty \frac{g(v)}{v} e^{-bv} dv.
\]

**Acknowledgement:** This research was supported by the Australian Research Council. The authors would like to thank Boris Granovsky for very fruitful discussions during his visit to Monash University.

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