GLOBAL EXPONENTIAL $\kappa$–DISSIPATIVE SEMIGROUPS AND EXPONENTIAL ATTRACTION

JIN ZHANG *

Department of Mathematics, College of Science, Hohai University
Nanjing 210098, China

PETER E. KLOEDEN AND MEIHUA YANG

School of Mathematics and Statistics
Huazhong University of Science & Technology
Wuhan 430074, China

CHENGKUI ZHONG

Department of Mathematics, Nanjing University
Nanjing 210093, China

(Communicated by Eduard Feireisl)

Abstract. Globally exponential $\kappa$–dissipativity, a new concept of dissipativity for semigroups, is introduced. It provides a more general criterion for the exponential attraction of some evolutionary systems. Assuming that a semigroup $\{S(t)\}_{t \geq 0}$ has a bounded absorbing set, then $\{S(t)\}_{t \geq 0}$ is globally exponentially $\kappa$–dissipative if and only if there exists a compact set $A^*$ that is positive invariant and attracts any bounded subset exponentially. The set $A^*$ need not be finite dimensional. This result is illustrated with an application to a damped semilinear wave equation on a bounded domain.

1. Introduction. It has long been considered that the global attractor is the appropriate concept to describe the long-time behavior of a dissipative infinite dimensional dynamical system. A set $A$ is called a global attractor of a semigroup $\{S(t)\}_{t \geq 0}$ on a complete metric space $(M,d)$ (the definition of the semigroup will be given later in Definition 2.6), if $A$ is compact, invariant, i.e., $S(t)A = A, \forall t \geq 0$, and attracts any bounded subset $B$ in $M$, i.e., $\forall B \subset M$ bounded,

$$\lim_{t \to \infty} \text{dist}_M(S(t)B,A) = 0.$$

By the definition, the global attractor contains all of the nontrivial dynamics of a dissipative system and is much smaller than the phase space. It was thought that the global attractor should have finite fractal dimension and that the dynamics restricted to the global attractor should also be finite fractal dimensional. The existence of the finite fractal dimensional global attractor, in fact, has been established for many classes of dissipative PDEs, see [2, 16, 24, 26, 29].

However, the global attractor may not be the ideal object to describe the long term behavior of some dissipative systems. Since the reduced dynamics given by

2010 Mathematics Subject Classification. Primary: 35B41; Secondary: 35K57, 37B55.

Key words and phrases. Exponential dissipativity, global attractor, measures of noncompactness.

* Corresponding author: Jin Zhang.
the Hölder–Mañé projection [13, 17, 20, 25] is only Hölder continuous, the projectors need to be Lipschitz to guarantee that the reduced dynamics on the attractor is finite dimensional. Moreover, the global attractor may attract the trajectories at a rather slow rate. In addition, it may be difficult to express the convergence rate in terms of the physical parameters of the system. This can be seen in the following example of a real Ginzburg–Landau equation in one dimension [2, 21]:

\[ u_t - \nu u_{xx} + u^3 - u = 0, \quad x \in [0, 1], \nu > 0, \quad u(0, t) = u(1, t) = -1, \quad t \geq 0. \]

Finally, the global attractor may be too sensitive to perturbations, which could prevent a clear understanding of the real system modelled by the given system, since a system is only an approximation of reality. Therefore, it is desirable to embed the global attractor into a proper smooth manifold containing a subset, which is robust under small perturbations, attracts trajectories at a fast rate, and moreover, has finite fractal dimension.

In [14], Foias, Sell and Temam proposed the notion of an inertial manifold, which is positive invariant, is represented by a finite dimensional Lipschitz graph, contains the global attractor and attracts the trajectories fast, i.e., at an exponential rate. The existence of inertial manifolds has since then been verified for many systems, see [4, 24, 26, 29]. However, for many systems, the existence of an inertial manifold still remains open. Moreover, the construction of an inertial manifold is based on the spectral gap condition, which is very restrictive and often difficult to verify or does not even hold for some systems, see [13, 27]. In addition, the inertial manifold has a smooth structure, but it is not always possible to embed the global attractor into a proper smooth finite dimensional manifold. All of these facts led Eden, Foias, Nicolaenko and Temam to propose the concept of exponential attractor (also called an inertial set) in [7].

**Definition 1.1.** Let \( X \) be a Banach space. A compact set \( \mathcal{M} \subset X \) is an exponential attractor for \( S(t) \) if

(i) it has finite fractal dimension, \( \dim_F \mathcal{M} < +\infty \),
(ii) it is positively invariant, i.e. for arbitrary \( t \geq 0 \), \( S(t)\mathcal{M} \subset \mathcal{M} \),
(iii) it attracts exponentially the bounded subsets of \( X \) in the following sense:

\[
\text{dist}(S(t)B, \mathcal{M}) \leq C(\|B\|_X)e^{-\alpha t}, \quad t \geq 0, \quad \forall B \subset E \text{ bounded}.
\]

Several methods have been proposed to construct the exponential attractor for a semigroup, see [1, 8, 9, 10, 12, 15, 22]. Generally speaking, exponential attractors can be constructed for dissipative systems which possess a certain kind of smoothing property. Actually, not only does the smoothing property provide us with an exponential attractive compact set \( \mathcal{M} \) (i.e. the exponential attractivity of the semigroup), but it also ensures the finite dimensionality of the set. However, for those systems that do not possess smoothing properties, the situation could be much more complicated, on the other hand, many semigroups generated by evolution equations have infinite dimensional global attractors, see, e.g., [31, 30].

In the present paper, we discuss the exponential attractivity of the semigroups which do not possess smoothing properties. We propose a new notion of global exponentially \( \kappa \)-dissipativity, which mainly focuses on the behavior of semigroups and pays no attention to the dimension of the attracting set. This concept involves on the Kuratowski measure of noncompactness (the definition of which will be given later in Definition 2.1).
Definition 1.2. Let \((M,d)\) be a complete metric space. A continuous semigroup \(\{S(t)\}_{t \geq 0}\) on \(M\) is called global exponentially \(\kappa\)-dissipative if for each bounded subset \(B \subset M\), there exist constants \(C\) and \(\alpha\), such that
\[
\kappa\left(\bigcup_{t \geq s} S(t)B\right) \leq Ce^{-\alpha s}, \quad \forall s \geq 0,
\]
where \(\kappa\) is the Kuratowski measure of noncompactness.

We recall that the Kuratowski measure of noncompactness has been used to describe the asymptotic compactness of a semigroup (such as the \(\omega\)-limit compactness property \([18]\)), and then used to prove the existence of the global attractor in \([28, 32]\). However, the notion was only used to obtain an attracting compact set without consideration about the rate of the attraction. Here, we shall prove that the dissipative rate of the measure of non-compactness does provide information about this rate of attraction. We shall provide several sufficient conditions that can be used to determine if a semigroup is globally exponentially \(\kappa\)-dissipative. In addition, we shall prove that if a globally exponentially \(\kappa\)-dissipative semigroup admits a bounded absorbing set, then there exists a positively invariant compact subset which attracts any bounded subset exponentially.

The paper is organized as follows. In Section 2, we recall some definitions and useful results that will be used later. In Section 3 and Section 4, we provide several sufficient conditions and also a property of global exponential \(\kappa\)-dissipative semigroup. In Section 5, we give a simple application of our abstract result. In particular, we show that the solution semigroup \(\{S(t)\}_{t \geq 0}\) of a damped semilinear wave equation is globally exponentially \(\kappa\)-dissipative.

2. The measure of noncompactness and some preliminary results. We recall the concept of measure of noncompactness and its basic properties and refer the reader to \([5, 18]\) for more details.

Definition 2.1. Let \((M,d)\) be a metric space and let \(A\) be a bounded subset of \(M\). The measure of noncompactness \(\kappa(A)\) is defined by
\[
\kappa(A) = \inf\{\delta > 0 \mid A \text{ admits a finite cover by sets of diameter } \leq \delta\}.
\]

Lemma 2.2. Let \((M,d)\) be a complete metric space and \(\kappa\) be the measure of noncompactness. Then
\begin{enumerate}
  \item \(\kappa(B) = 0\), if and only if \(B\) is compact;
  \item if \(M\) is a Banach space, then \(\kappa(B_1 + B_2) \leq \kappa(B_1) + \kappa(B_2)\);
  \item \(\kappa(B_1) \leq \kappa(B_2)\) whenever \(B_1 \subset B_2\);
  \item \(\kappa(B_1 \cup B_2) \leq \max\{\kappa(B_1), \kappa(B_2)\}\);
  \item \(\kappa(B) = \kappa(B)\).
\end{enumerate}

Lemma 2.3. Let \(M\) be an infinite dimensional Banach space and let \(B(\varepsilon)\) be a ball of radius \(\varepsilon\), then \(\kappa(B(\varepsilon)) = 2\varepsilon\).

Lemma 2.4. Let \(X\) be an infinite dimensional Banach space with the following decomposition:
\[
X = X_1 \oplus X_2, \quad \dim X_1 < \infty.
\]
Let \(P : X \to X_1\), \(Q : X \to X_2\) be the canonical projectors and let \(A\) be a bounded subset of \(X\). If the diameter of \(QA\) is less than \(\varepsilon\), then \(\kappa(A) < \varepsilon\).
The next lemma will be used later in the proof of Theorem 3.2.

**Lemma 2.5.** Let \((M, d)\) be a complete metric space, let \(A\) and \(B\) be two subsets of \(M\), where \(A\) is compact. If \(\text{dist}(B, A) \leq \gamma\), then \(\kappa(B) \leq 2\gamma\).

**Proof.** It is sufficient to prove that \(\kappa(B) \leq 2\gamma + \varepsilon\) for any \(\varepsilon > 0\). In fact, for each \(x \in B\), there exists \(y_x \in A\), such that
\[
d(x, y_x) < \gamma + \varepsilon/4,
\]
and thus
\[
B \subset \bigcup_{x \in B} N^{\gamma + \varepsilon/4}(y_x),
\]
where
\[
N^{\gamma + \varepsilon/4}(y_x) := \{ z \in M \mid d(z, y_x) < \gamma + \varepsilon/4 \}.\]

Due to the compactness of \(A\), there exist \(y_1, y_2, \ldots, y_m \in A\) such that
\[
\{y_x \mid x \in B\} \subset \bigcup_{i=1}^m N^{\varepsilon/4}(y_i),
\]
which implies that
\[
\bigcup_{x \in B} N^{\gamma + \varepsilon/4}(y_x) \subset \bigcup_{i=1}^m N^{\gamma + \varepsilon/4}(y_i).
\]
Combining this with (2.1), we have
\[
B \subset \bigcup_{i=1}^m N^{\gamma + \varepsilon/4}(y_i).
\]
Bearing in mind that \(\text{diam } N^{\gamma + \varepsilon/4}(y_i) = 2\gamma + \varepsilon\), the proof is completed. \(\Box\)

The concept of \(\omega\)-limit compactness of a semigroup, which is an important necessary and sufficient condition for the existence of the global attractor (see [18]), is now recalled along with the definition of a continuous semigroup.

**Definition 2.6.** Let \((M, d)\) be a complete metric space, a family of maps \(\{S(t)\}_{t \geq 0}\) from \(M\) to \(M\) is called a semigroup if it satisfies:
\[
S(0) = I, \quad S(t)S(s) = S(s)S(t) = S(t + s).
\]
Furthermore, we say a semigroup \(\{S(t)\}_{t \geq 0}\) is a \(C^0\) semigroup or continuous semigroup if
\[
S(t)x_0 \text{ is continuous in } x_0 \in M \text{ and } t \in R.
\]

**Definition 2.7.** Let \(\{S(t)\}_{t \geq 0}\) be a \(C^0\) semigroup in a complete metric space \((M, d)\). A subset \(B_0 \subset M\) is called an absorbing set in \(M\), if for any bounded subset \(B\), there exists some \(t_0 \geq 0\) such that \(S(t)B \subset B_0\) for all \(t \geq t_0\).

**Definition 2.8.** A continuous semigroup \(\{S(t)\}_{t \geq 0}\) in a complete metric space \((M, d)\) is called \(\omega\)-limit compact, if for any bounded subset \(B\) and any \(\varepsilon > 0\), there exists a constant \(t_1 \geq 0\) such that
\[
\kappa\left(\bigcup_{t \geq t_1} S(t)B\right) \leq \varepsilon.
\]
Lemma 2.9. Let \( \{S(t)\}_{t \geq 0} \) be a continuous semigroup in a complete metric space \((M, d)\). Then \( S(t) \) has a global attractor \( A \) in \( M \) if and only if

1. there is a bounded absorbing set \( B \subset M \),
2. \( \{S(t)\}_{t \geq 0} \) is \( \omega \)-limit compact.

3. Sufficient conditions for global exponential \( \kappa \)-dissipative semigroups.

In this section, we consider several different sufficient conditions for a semigroup to be globally exponential \( \kappa \)-dissipative are considered here.

Theorem 3.1. Suppose that a continuous semigroup \( \{S(t)\}_{t \geq 0} \) has a bounded absorbing set \( B_0 \) in a complete metric space \((M, d)\) and is uniformly \( \kappa \)-contracting, i.e., there exist \( t_0 > 0 \) and \( \beta \in [0, 1) \) such that for each bounded subset \( B \subset M \),

\[
\kappa(S(t_0)B) \leq \beta \kappa(B) \tag{3.1}
\]

Then \( \{S(t)\}_{t \geq 0} \) is global exponentially \( \kappa \)-dissipative.

Proof. Since \( S(t) \) has a bounded absorbing set \( B_0 \), then for each bounded subset \( B \subset M \) there exists some \( s_0 \) such that \( \bigcup_{t \geq s_0} S(t)B \subset B_0 \) is bounded. More precisely, there exists a constant \( C > 0 \) such that

\[
\kappa \left( \bigcup_{t \geq s_0} S(t)B \right) \leq C.
\]

For any \( s \geq s_0 \), set \( s - s_0 = nt_0 + \tau \), where \( \tau \in [0, t_0) \). Then

\[
\bigcup_{t \geq s} S(t)B = S(s - s_0) \bigcup_{t \geq s_0} S(t)B = S(nt_0) \bigcup_{t \geq s_0+s} S(t)B \subset S(nt_0) \bigcup_{t \geq s_0} S(t)B,
\]

it follows from (3.1) that

\[
\kappa \left( \bigcup_{t \geq s} S(t)B \right) \leq \beta^n \kappa \left( \bigcup_{t \geq s_0} S(t)B \right) \leq C \beta^n,
\]

let \( \beta = e^{-\alpha} \), we have

\[
\kappa \left( \bigcup_{t \geq s} S(t)B \right) \leq Ce^{-\alpha n}.
\]

The proof is finished by noticing that \( nt_0 > s - s_0 - t_0 \), so

\[
\kappa \left( \bigcup_{t \geq s} S(t)B \right) \leq Ce^{-\alpha \frac{s - s_0 - t_0}{t_0}} \leq (Ce^\alpha)^{\frac{s - s_0}{t_0}} \leq (Ce^\alpha)^{\frac{s - s_0}{t_0}}
\]

is valid for every \( s \geq s_0 \).

Corollary 1. Any uniformly compact semigroup \( \{S(t)\}_{t \geq 0} \) on a complete metric space is a global exponential \( \kappa \)-dissipative semigroup.
Remark 1. This conclusion shows that \( \kappa \)-contracting and uniformly compact semigroups are global exponentially \( \kappa \)-dissipative. Intuitively, by the contraction property \( \kappa (S(t_0)B) \leq \beta \kappa (B) \), the measure of noncompactness \( \kappa (S(t_0)B) \) should decay to zero exponentially fast (related to \( \beta \)). The \( \kappa \)-contracting property (or uniformly compact) is actually a relatively strong condition and can be replaced by other more general conditions.

Now it will be shown that the existence of an exponentially attracting compact set is a sufficient condition for the global exponential \( \kappa \)-dissipativity. In Theorem 4.1 of the following section it will be proved that this condition is also a necessary condition for global exponential \( \kappa \)-dissipativity when the semigroup has a bounded absorbing set.

Theorem 3.2. Let \( \{S(t)\}_{t \geq 0} \) be a continuous semigroup on a complete metric space \((M,d)\). If there is a compact subset \( A \subset M \) such that for each bounded subset \( B \subset M \), there exist positive constants \( C \) and \( \alpha \) such that
\[
\text{dist} \left( \bigcup_{t \geq s} S(t)B, A \right) = \text{dist} \left( S(s) \left( \bigcup_{t \geq 0} S(t)B \right), A \right) \leq Ce^{-\alpha s}.
\]
then \( \{S(t)\}_{t \geq 0} \) is global exponentially \( \kappa \)-dissipative.

Proof. By the assumption, for each bounded subset \( B \), \( \bigcup_{t \geq 0} S(t)B \) is bounded in \( M \), so there exist positive constants \( C \) and \( \alpha \) such that
\[
\text{dist} \left( \bigcup_{t \geq s} S(t)B, A \right) = \text{dist} \left( S(s) \left( \bigcup_{t \geq 0} S(t)B \right), A \right) \leq Ce^{-\alpha s}.
\]
Recalling Lemma 2.5, it follows that
\[
\kappa \left( \bigcup_{t \geq s} S(t)B \right) \leq Ce^{-\alpha s}.
\]
Combining with Lemma 2.2(v), we complete the proof.

Applications to PDEs usually involve semigroups acting on Banach spaces. In order to prove that these semigroups are global exponential \( \kappa \)-dissipative semigroups, we present some useful methods.

In particular, let \( X \) be a Banach space with the decomposition
\[
X = X_1 \oplus X_2, \quad \text{dim } X_1 < \infty
\]
and denote projections by \( P : X \to X_1 \) and \( (I - P) : X \to X_2 \). In addition, let \( \{S(t)\}_{t \geq 0} \) be a continuous semigroup on \( X \).

Condition (C*): For any bounded set \( B \subset X \) there exist positive constants \( t_0, C \) and \( \alpha \) such that for any \( \varepsilon > 0 \) there exists a finite dimensional subspace \( X_1 \subset X \) satisfying
\[
\{\|PS(t)B\|\}_{t \geq 0} \text{ is bounded, and} \quad \|(I - P)S(t)B\| < Ce^{-\alpha t} + \varepsilon \quad \text{for } t \geq t_0,
\]
where \( P : X \to X_1 \) is a bounded projection.

Theorem 3.3. Let \( X \) be a Banach space and let \( \{S(t)\}_{t \geq 0} \) be a continuous semigroup on \( X \) for which Condition (C*) holds. Then \( \{S(t)\}_{t \geq 0} \) is a global exponentially \( \kappa \)-dissipative semigroup.
Proof. By Condition \((C^*)\), for each bounded subset \(B \subset X\) there exists \(t_0 > 0\) such that \(\bigcup_{t \geq t_0} S(t)B\) is also bounded. This combines with the conclusions in Lemmas \(2.2, 2.3\) gives
\[
\kappa \left( \bigcup_{t \geq t_0} S(t)B \right) \leq \kappa \left( P \left( \bigcup_{t \geq t_0} S(t)B \right) \right) + \kappa \left( (I - P) \left( \bigcup_{t \geq t_0} S(t)B \right) \right)
\leq \kappa \left( (I - P) \left( \bigcup_{t \geq t_0} S(t)B \right) \right)
< \kappa \left( Ne^{-\alpha t} + \varepsilon(0) \right)
= 2Ce^{-\alpha t} + 2\varepsilon
\]
for all \(t > t_1 (\cup_{t \geq t_0} S(t)B)\). Taking \(\varepsilon \to 0\) gives the assertion.

In \([29]\) the semigroup decomposition method was used to prove the asymptotic compactness, this motivates the following decomposition condition for a semigroup to be global exponentially \(\kappa\)-dissipative.

**Theorem 3.4.** Let \(\{S(t)\}_{t \geq 0}\) be a continuous semigroup on a Banach space \(X\). Suppose, for each \(t > 0\), the semigroup \(S(t)\) admits a decomposition \(S(t) = S_1(t) + S_2(t)\), where the operators \(S_1(t)\) are uniformly compact for \(t\) large and \(S_2(t)\) is a continuous mapping which satisfies:

for every bounded set \(B \subset X\) there exist constants \(C, \alpha\) such that
\[
\|S_2(t)B\| \leq Ce^{-\alpha t} \quad \text{for every } t > 0.
\]

In addition, suppose that \(\{S(t)\}_{t \geq 0}\) has a bounded absorbing set \(B_0 \subset X\), then \(\{S(t)\}_{t \geq 0}\) is a global exponential \(\kappa\)-dissipative semigroup.

Proof. We omit the proof which similar to Theorem 3.3.

4. Existence of exponentially attracting set for global exponential \(\kappa\)-dissipative semigroups. An important property for the long-time behavior of global exponential \(\kappa\)-dissipative semigroups will now be established.

**Theorem 4.1.** Let \(\{S(t)\}_{t \geq 0}\) be a continuous semigroup on a Banach space \(X\). Suppose, for each \(t > 0\), the semigroup \(S(t)\) admits a decomposition \(S(t) = S_1(t) + S_2(t)\), where the operators \(S_1(t)\) are uniformly compact for \(t\) large and \(S_2(t)\) is a continuous mapping which satisfies:

for every bounded set \(B \subset X\) there exist constants \(C, \alpha\) such that
\[
\|S_2(t)B\| \leq Ce^{-\alpha t} \quad \text{for every } t > 0.
\]

In addition, suppose that \(\{S(t)\}_{t \geq 0}\) has a bounded absorbing set \(B_0 \subset X\), then \(\{S(t)\}_{t \geq 0}\) is a global exponential \(\kappa\)-dissipative semigroup.

Proof. Since \(\{S(t)\}_{t \geq 0}\) is \(\omega\)-limit compact, the \(\omega\)-limit set of its bounded absorbing set \(B_0\),
\[
\mathcal{A} := \bigcap_{s \geq 0} \bigcup_{t \geq s} S(t)B_0
\]
is the global attractor of \(\{S(t)\}_{t \geq 0}\). In addition, since \(\{S(t)\}_{t \geq 0}\) is a global exponential \(\kappa\)-dissipative semigroup, there exist positive constants \(C_0\) and \(\alpha\) such
that
\[ \kappa \left( \bigcup_{t \geq s} S(t)B_0 \right) \leq C_0 e^{-\alpha s}. \]

Set \( s_1 = 1 \), it follows from Definition 2.1 that there exist \( a_1^{(1)}, a_2^{(1)}, \ldots, a_{k_1}^{(1)} \in B_0 \) and \( t_1^{(1)}, t_2^{(1)}, \ldots, t_{k_1}^{(1)} \geq s_1 \) such that
\[ \bigcup_{t \geq s_1} S(t)B_0 \subset \bigcup_{i=1}^{k_1} \mathcal{N}^{\varepsilon_1} \left( S(t_1^{(1)})a_1^{(1)} \right) \subset \bigcup_{i=1}^{k_1} \bigcup_{t \geq 0} \mathcal{N}^{\varepsilon_1} \left( S(t)S(s_1)a_i^{(1)} \right), \]
where \( \varepsilon_1 := C_0 e^{-\alpha s_1} \). Thus
\[ \text{dist} \left( \bigcup_{t \geq s_1} S(t)B_0, \bigcup_{i=1}^{k_1} \bigcup_{t \geq 0} S(t)S(s_1)a_i^{(1)} \right) \leq \varepsilon_1. \]

Proceeding inductively, set \( s_m = m \). Then there exist \( a_1^{(m)}, a_2^{(m)}, \ldots, a_{k_m}^{(m)} \in B_0 \) and \( t_1^{(m)}, t_2^{(m)}, \ldots, t_{k_m}^{(m)} \geq s_m \) such that
\[ \text{dist} \left( \bigcup_{t \geq s_m} S(t)B_0, \bigcup_{i=1}^{k_m} \bigcup_{t \geq 0} S(t)S(s_m)a_i^{(m)} \right) \leq \varepsilon_m := C_0 e^{-\alpha s_m}. \quad (4.1) \]

Next, we will show that the set defined
\[ \mathcal{A}^* := \mathcal{A} \bigcup \left( \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{k_m} \bigcup_{t \geq 0} S(t)S(s_m)a_i^{(m)} \right) \]
is the compact subset we desire.

Firstly, conclusion (i) is valid since \( \mathcal{A} \) and \( \bigcup_{t \geq 0} S(t)S(s_m)a_i^{(m)} \) are positively invariant for each fixed \( m \) and \( i \).

To prove the compactness of set \( \mathcal{A}^* \), consider an arbitrary sequence \( \{x_n\} \) in \( \mathcal{A}^* \), it will be shown that \( \{x_n\} \) has a convergent subsequence in \( \mathcal{A}^* \). In fact, if \( \{x_n\} \) has a subsequence in \( \mathcal{A} \), which is a compact subset, then \( \{x_n\} \) has a convergent subsequence in \( \mathcal{A} \). Hence, without loss of generality, the proof reduces to the case that
\[ \{x_n\} \subset \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{k_m} \bigcup_{t \geq 0} S(t)S(s_m)a_i^{(m)}. \]

Then there exist \( m_n, t_n \) and \( 1 \leq i_n \leq k_{m_n} \) for \( n = 1, 2, \ldots \) such that
\[ x_n = S(t_n)S(s_{m_n})a_{i_n}^{(m_n)}, \quad n = 1, 2, \ldots. \]

On one hand, if \( t_n \to +\infty \) or \( s_{m_n} \to +\infty \), the existence of the convergent subsequence \( \{x_n\} \) in \( \mathcal{A} \) can be guaranteed by the \( \omega \)-limit set property. On the other hand, when \( \{t_n\} \) and \( \{s_{m_n}\} \) are both bounded, there exists a constant \( N_0 \) such that
\[ \{x_n\} \subset \bigcup_{m=1}^{N_0} \bigcup_{i=1}^{k_{m_n}} \bigcup_{t=0}^{N_0} S(t)S(s_{m_n})a_{i_n}^{(m_n)}, \]
for all \( n = 1, 2, \ldots, t_n \leq N_0 \) and \( s_{m_n} \leq N_0 \), i.e., \( \{x_n\} \) is contained in a compact set and thus has a convergent subsequence.
It remains to prove conclusion (ii). For any bounded subset $B \subset M$, since \( \{S(t)\}_{t \geq 0} \) has a bounded absorbing set $B_0$, then there exists a time $t_0 > 0$ depending on $B$ such that

\[ S(t)B \subset B_0 \quad \text{for all } t \geq t_0, \]

this implies that

\[
\bigcup_{t \geq t_0 + s} S(t)B = \bigcup_{t \geq s} S(t)S(t_0)B \subset \bigcup_{t \geq s} S(t)B_0.
\]

Set $s = m + \tau$, where $\tau \in [0, 1)$, then

\[
\bigcup_{t \geq s} S(t)B_0 = \bigcup_{t \geq m + \tau} S(t)B_0 \subset \bigcup_{t \geq m} S(t)B_0
\]

and hence

\[
\text{dist} \left( \bigcup_{t \geq t_0 + s} S(t)B, A^* \right) \leq \text{dist} \left( \bigcup_{t \geq m} S(t)B_0, A^* \right) \leq \text{dist} \left( \bigcup_{t \geq m} S(t)B_0, \bigcup_{i=1}^{k_m} \bigcup_{t \geq 0} S(t)S(m)a_i^{(m)} \right). \tag{4.2}
\]

Finally, the combination of estimates (4.1) and (4.2) gives

\[
\text{dist} \left( \bigcup_{t \geq t_0 + s} S(t)B, A^* \right) \leq C_0e^{-\alpha m} \leq (C_0e^\alpha)e^{-\alpha s}.
\]

The proof is completed.

**Remark 2.** It follows from the proof of Theorem 4.1 that, the sequence \( \{a_i^{(m)}\} \), $i = 1, 2, \cdots, k_m$, $m = 1, 2, \cdots$ can be chosen in any subset $B_1$ which is dense in $B_0$. In particular, we can choose regular points \( \{a_i^{(m)}\} \) such that the orbits $S(t)a_i^{(m)}$ are smooth. In fact, for any dense set $B_1$ of the absorbing set $B_0$, we know that there exist points \( \{b_i^{(m)}\} \in B_1 \), such that

\[
\bigcup_{t \geq s_m} S(t)B_0 \subset \bigcup_{i=1}^{k_m} \mathcal{N}^{s_m} \left( S(t_i^{(m)})a_i^{(m)} \right) 
\subset \bigcup_{i=1}^{k_m} \mathcal{N}^{2s_m} \left( S(t_i^{(m)})b_i^{(m)} \right).
\]

Similarly to what we did in the proof of Theorem 4.1, the conclusion of Theorem 4.1 is also valid for the set

\[
A^* = A \cup A_1 = A \cup \left( \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{k_m} \bigcup_{t \geq 0} S(t)S(s_m)b_i^{(m)} \right),
\]

where $A_1$ is the set of positive trajectories of some points in the dense set $B_1$. Therefore, the attracting set $A^*$ can be constructed as the union of global attractor and some smooth forward trajectories.

Next, we consider the following question: if the global attractor $A$ is finite dimensional, whether the dimension of set $A^*$ is also finite?
Lemma 4.2. Let $(M,d)$ be a complete metric space, and let $\{S(t)\}_{t \geq 0}$ be a continuous semigroup on $M$. If the semigroup $\{S(t)\}_{t \geq 0}$ is Lipschitz continuous (or Hölder continuous) respect to $t$ in a set $B \subset M$, and the global attractor of $\{S(t)\}_{t \geq 0}$ satisfies $d_H(A) = d_0 < \infty$, where $d_H(A)$ denotes the Hausdorff dimension of the set $A$. Then, for any point $u_0 \in B$,

$$d_H \left( \bigcup_{t \geq 0} S(t)u_0 \right) \leq d_0 + 1.$$ 

Proof. It follows from the definition of Hausdorff dimension, for any $\varepsilon > 0$, $\delta \in (0,1)$ and $d > 0$, there exist balls $\{B(x_i,r_i)\}_{i=1}^{\infty} \subset A$ cover the global attractor $A$, where $r_i \leq \delta < 1$, $i = 1, 2, \ldots$, and

$$\sum_{i=1}^{\infty} r_i^{d_0+d} < \varepsilon.$$ 

Due to the compactness of $A$, the number of the covering balls is finite, we will denote them by $\{B(x_i,r_i)\}_{i=1}^{n_0} \subset A$. In addition, it follows from the definition of a global attractor that, for any $u_0 \in B$, there exists a time $t_0$ such that

$$\text{dist} \left( S(t)u_0, A \right) < \text{dist} \left( \bigcup_{i=1}^{n_0} B(x_i,r_i), A \right) \quad \forall t > t_0.$$ 

By Lipschitz (or Hölder) continuity, the length of the trajectory $S(t)u_0$ is finite in time $t \in [0,t_0]$, so it can be covered by finite balls $\{B(y_i,\tilde{r}_i)\}_{i=1}^{\tilde{n}_0}$, where $\tilde{r}_i \leq \delta$, $i = 1, 2, \ldots, \tilde{n}_0$ and

$$\bigcup_{i=1}^{\tilde{n}_0} B(y_i,\tilde{r}_i) \supset \bigcup_{t=0}^{t_0} S(t)u_0, \quad \sum_{i=1}^{\tilde{n}_0} r_i^{d_0+1+d} < \varepsilon.$$ 

Thus,

$$\sum_{i=1}^{\tilde{n}_0} r_i^{d_0+1+d} + \sum_{i=1}^{\tilde{r}_i} r_i^{d_0+1+d} < 2\varepsilon.$$ 

Finally, the property of Hausdorff dimension gives

$$d_H \left( \bigcup_{t \geq 0} S(t)u_0 \right) \leq d_0 + 1,$$

which completes the proof. \qed

Combining the above Lemma with the countable stable property of Hausdorff dimension, we can obtain the following theorem.

Theorem 4.3. Let $(M,d)$ be a complete metric space, and let $\{S(t)\}_{t \geq 0}$ be a continuous semigroup on $M$. If $\{S(t)\}_{t \geq 0}$ is a global exponential $\kappa-$dissipative semigroup and Lipschitz continuous (or Hölder continuous) respect to $t$ in some dense subset $B_1$ of the absorbing set, and the global attractor $A$ of $\{S(t)\}_{t \geq 0}$ satisfies $d_H(A) = d_0 < \infty$. Then, the compact set $A^*$ is finite Hausdorff dimensional.

Since the fractal dimension does not have countable stability, so the above theorem does not hold for the fractal dimension. Indeed, let $S(t) : \ell^2 \to \ell^2$ is the semigroup associated with the system
\[
\frac{dx_n^j}{dt} = -a(x_n^j)x_n^j, \quad x(0) = \bar{x} \in \ell^2,
\]
in the Banach space
\[
\ell^2 = \left\{ x = (x_1^1, x_2^1, x_3^1, \ldots, x_1^2, x_2^2, \ldots, x_n^1, \ldots, x_2^n, \ldots) \mid \sum_{n=1}^{\infty} \sum_{j=1}^{2n} |x_n^j|^2 < \infty \right\},
\]
where
\[
a(x_n^j) = \begin{cases} 
1 & \text{if } |x_n^j| \geq \frac{1}{2^n m}, \\
2^n \cdot n \cdot |x_n^j| & \text{if } |x_n^j| < \frac{1}{2^n m}.
\end{cases}
\]

Obviously, the global attractor \( A = \{0\} \) is a single point set. On the other hand, the set \( \{x \in \ell^2 \mid |x_n^j| \leq \frac{1}{2^n m}\} \) is compact and exponentially attracts any bounded subset of \( \ell^2 \), so the semigroup \( S(t) \) is global exponentially \( \kappa \)-dissipative. However, this can be seen that the fractal dimension of any exponentially attracting set \( A^\ast \) constructed as in Theorem 4.1 is infinite. Therefore, the finite fractal dimension of global attractor and \( \kappa \)-exponential dissipative are not insufficient to give the finite fractal dimensionality of exponential attracting set. More conditions are needed.

Note that the above semigroup \( S(t) \) is not Fréchet differentiable at the point set 0. Motivated by the method in \([34]\), we assume that a semigroup is Fréchet differentiable in a neighborhood of global attractor \( A \).

Following the process in \([34]\), we provide the following theorem.

**Theorem 4.4.** Let \( \{S(t)\}_{t \geq 0} \) be a continuous semigroup on a Banach space \( X \), if \( \{S(t)\}_{t \geq 0} \) is a global exponential \( \kappa \)-dissipative semigroup and satisfies

(i) the fractal dimension of global attractor is finite, i.e., \( d_F(A) < \infty \),

(ii) there exists a constant \( \epsilon > 0 \), such that for any \( T^* > 0 \),

\[
S = S(T^*) : N^\epsilon(A) \to N^\epsilon(A)
\]

is a \( C^1 \) map.

Then the fractal dimension of set \( A^\ast \) is finite, and \( A^\ast \) is the exponential attractor of \( \{S(t)\}_{t \geq 0} \).

**Sketch of proof.** Since \( \{S(t)\}_{t \geq 0} \) is a global exponentially \( \kappa \)-dissipative semigroup, thus by Theorem 4.1 the compact set \( A^\ast \) exponentially attracts the set \( N^\epsilon(A) \). Therefore, for any \( \lambda > 0 \), there exists a time \( T^* > 0 \), such that for all \( x \in N^\epsilon(A) \), the linear map \( D_x S(x) = D_x S(T^*)(x) \) admits a decomposition \( D_x S(x) = K + C \), where \( K \) is compact and \( \|C\| < \lambda \).

Applying a process similar to that in \([34]\), we have

\[
d_F(A^\ast \cap N^\epsilon(A)) < \infty.
\]

On the other hand, it is obvious that \( d_F(A^\ast \setminus N^\epsilon(A)) < \infty \), since the set is the union of finite trajectories with finite length. Thus, we have

\[
d_F(A^\ast) < \infty.
\]

**Remark 3.** When the fractal dimension of attractor is finite, the exponential attractor is powerful concept for studying the rate of attracting (see \([21]\) page 133). Exponential attractors are as general as global attractors. To the best of our knowledge, exponential attractors exist for all equations of mathematical physics for which we can prove the existence of the finite fractal dimensional global attractors. When
the fractal dimension of the attractor is infinite, our concept of κ—exponential dissipativity allows us to study the existence of a compact positive invariant exponential attracting set.

5. Application to the damped semilinear wave equation. In this section, we will show that the damped semilinear wave equations generate global exponentially κ—dissipative semigroups.

Consider the equation

$$\begin{cases}
u_{tt} + u_t - \Delta u + f(u) = g, & x \in \Omega, \ t \geq 0, \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), & x \in \Omega, \\ u(x, t) = 0, & x \in \partial \Omega, \ t \geq 0, 
\end{cases}$$

where \( \Omega \) is a bounded subset of \( \mathbb{R}^3 \) with a smooth boundary \( \partial \Omega \). Let \( V = H_0^1(\Omega) \) and \( H = L^2(\Omega) \), and denote their scalar products and the norms by \((\cdot, \cdot), \| \cdot \|\) and \((\cdot, \cdot), | \cdot |\), respectively. In addition, the nonhomogeneous term \( g \in L^2(\Omega) \) is given and the nonlinear operator \( f : V \to H \) is assumed to be continuous and compact (e.g., \( f(u) = u^r \) with \( 1 < r < 3 \)).

The following lemmas from [3], [28] or [29] are valid for above initial boundary value problem.

**Lemma 5.1.** Under the above assumptions, if \( u_0 \in V \) and \( u_1 \in H \), then the initial boundary value problem \([5.1]\) has a unique weak solution

\[ u \in C([0, T]; V) \text{ and } u_t \in C([0, T]; H), \]

which generates the semigroup

\[ S(t)[u_0, u_1] = [u(t), u_t(t)] \]

on the Banach space \( E_0 = V \times H \).

Furthermore, if \( u_0 \in D(A) \) and \( u_1 \in V \), then the weak solution \( u \) satisfies

\[ u \in C([0, T]; D(A)) \text{ and } u_t \in C([0, T]; V), \]

**Lemma 5.2.** Under the above assumptions, if \( u_0 \in V \) and \( u_1 \in H \), then the semigroup \( \{S(t)\}_{t \geq 0} \) has a bounded absorbing set \( B_0 \) in \( E_0 \). Furthermore, if \( u_0 \in D(A) \) and \( u_1 \in V \), then \( \{S(t)\}_{t \geq 0} \) has a bounded absorbing set \( B_1 \) in \( E_1 = D(A) \times V \).

**Lemma 5.3.** Under the above assumptions, the semigroup \( \{S(t)\}_{t \geq 0} \) possesses a global attractor \( A \) with finite Hausdorff dimension and finite fractal dimension.

The main goal of this section is to apply Theorem [3.3] to show that the semigroup \( \{S(t)\}_{t \geq 0} \) is global exponentially κ—dissipative.

**Theorem 5.4.** The semigroup \( S(t) : E_0 \to E_0 \) associated with the equation \([5.1]\) is global exponentially κ—dissipative.

**Proof.** It is sufficient to verify the Condition \((C^*)\). For this purpose, consider the complete orthonormal basis \( \{\omega_j\}_{j \in \mathbb{N}} \) of \( H \) such that

\[-\Delta \omega_j = \lambda_j \omega_j, \quad (\omega_j, \omega_k) = \delta_{jk}, \text{ for } j, k = 1, 2, \ldots, \]

\[0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to \infty \text{ as } j \to \infty.\]

In addition, let

\[ H_1^m = \text{span}\{\omega_1, \omega_2, \ldots, \omega_m\} \quad \text{and} \quad H_2^m = (H_1^m)^\perp. \]
Then, each $u \in H$ can be decomposed as

$$u = P_m u + (I - P_m) u := u^{(1)} + u^{(2)}, \quad u^{(1)} \in H^m_1, \quad u^{(2)} \in H^m_2,$$

where $P_m : H \to H^m_1$ is the orthogonal projector.

In view of Lemma 5.2, $\{S(t)\}_{t \geq 0}$ has a bounded absorbing set $B_0$ of $E_0$, i.e., for any bounded set $B \subset E_0$, there exists $t_0 > 0$ such that

$$[u(t), u_t(t)] = S(t)[u_0, u_1] \in B_0$$

for any $t \geq t_0$ and $[u_0, u_1] \in B$. Thus, there exists a positive constant $\rho > 0$ such that,

$$\|u(t)\|^2 + |u_t(t)|^2 \leq \rho^2, \quad \text{for any } t \geq t_0.$$

The inner product of equation (5.1) in $H$ with $u_t^{(2)} + \frac{1}{2} u^{(2)}$ gives

$$\frac{1}{2} \frac{d}{dt} \left( |u_t^{(2)}|^2 + \|u^{(2)}\|^2 + (u^{(2)}, u_t^{(2)}) \right) + \frac{1}{2} \left( |u_t^{(2)}|^2 + \|u^{(2)}\|^2 + (u_t^{(2)}, u^{(2)}) \right)$$

$$= \left( g, u_t^{(2)} + \frac{1}{2} u^{(2)} \right) - \left( f(u), u_t^{(2)} + \frac{1}{2} u^{(2)} \right).$$

(5.2)

Since $g \in H$ and $f : V \to H$ is compact, for any $\varepsilon > 0$, there exists an $m \in \mathbb{N}$ such that, for any $u \in B_V(0, \rho)$,

$$|(I - P_m)g|^2 < \varepsilon, \quad |(I - P_m)f(u)|^2 < \varepsilon \quad \text{and} \quad \lambda_{m+1} > 1.$$

Then, the first term in the right hand side of (5.2) can be estimated as

$$\left( g, u_t^{(2)} + \frac{1}{2} u^{(2)} \right) = \left( (I - P_m)g, u_t^{(2)} + \frac{1}{2} u^{(2)} \right)$$

$$\leq \| (I - P_m)g \| \cdot \left| u_t^{(2)} + \frac{1}{2} u^{(2)} \right|$$

$$\leq 2 \| (I - P_m)g \|^2 + \frac{1}{8} \left| u_t^{(2)} + \frac{1}{2} u^{(2)} \right|^2$$

$$\leq 2 \varepsilon + \frac{1}{8} \left( |u_t^{(2)}|^2 + \frac{1}{4} |u^{(2)}|^2 + (u^{(2)}, u^{(2)}) \right)$$

$$\leq 2 \varepsilon + \frac{1}{8} \left( |u_t^{(2)}|^2 + \frac{1}{4 \lambda_{m+1}} \| u^{(2)} \|^2 + (u^{(2)}, u^{(2)}) \right)$$

$$\leq 2 \varepsilon + \frac{1}{8} \left( |u_t^{(2)}|^2 + \| u^{(2)} \|^2 + (u^{(2)}, u^{(2)}) \right),$$

(5.3)

and by the similar techniques, the second term can be estimated as

$$\left| \left( f(u), u_t^{(2)} + \frac{1}{2} u^{(2)} \right) \right| \leq 2 \varepsilon + \frac{1}{8} \left( |u_t^{(2)}|^2 + \| u^{(2)} \|^2 + (u^{(2)}, u^{(2)}) \right).$$

(5.4)

Combining equation (5.2) and two estimates (5.3), (5.4) then yields

$$\frac{1}{2} \frac{d}{dt} \left( |u_t^{(2)}|^2 + \| u^{(2)} \|^2 + (u^{(2)}, u_t^{(2)}) \right) + \frac{1}{4} \left( |u_t^{(2)}|^2 + \| u^{(2)} \|^2 + (u_t^{(2)}, u^{(2)}) \right) \leq 4 \varepsilon.$$

Finally, the application of the Gronwall’s inequality gives

$$|u_t^{(2)}(t)|^2 + \| u^{(2)}(t) \|^2 + (u^{(2)}(t), u_t^{(2)}(t)) \leq 2 \rho^2 e^{-\frac{8}{5} t} + 8 \varepsilon \quad \text{for } t \geq t_0.$$
The proof is completed by taking
\[
\left| \left( u(t)^2, u(t)^2 \right) \right| \leq \frac{1}{2} |u(t)^2|^2 + \frac{1}{2} |u(t)^2|^2 \leq \frac{1}{2} |u(t)^2|^2 + \frac{1}{2\lambda_{m+1}} \|u(t)^2\|^2
\]
\[
\leq \frac{1}{2} |u(t)^2|^2 + \frac{1}{2} \|u(t)^2\|^2.
\]
\[\square\]

It follows from the conclusions in Theorem 4.1 and Theorem 5.4 that, the equation (5.1) has a compact and positive invariant set \(\mathcal{A}^*\), which attracts any bounded subset \(\mathcal{B}\) of \(E_0\) exponentially. Next, we are aiming to prove that the dimension of the set \(\mathcal{A}^*\) is finite, to this end, the following results is necessary.

**Lemma 5.5.** There exists a constant \(L\), such that for any \(\{u_0, u_1\} \in B_1\), the solution \(u(t)\) of the equation (5.1) satisfies
\[
|u(t_1) - u(t_2)|^2 + \|u(t_1) - u(t_2)\|^2 \leq L|t_1 - t_2|^2, \quad \forall t_1, t_2 \geq 0.
\]

**Proof.** Recalling Lemma 5.2, there exists a constant \(C\) such that
\[
\|u(t)\|^2 \leq C, \quad |u(t)|^2 \leq C,
\]
\[
|\Delta u(t)|^2 \leq C, \quad \|u(t)\| \leq C, \quad \forall t \geq 0.
\]

It follows from \(\|u(t)\| \leq C\) that
\[
\|u(t_1) - u(t_2)\|^2 \leq C|t_1 - t_2|^2.
\]

Moreover, since \(u_{tt} = -u_t + \Delta u - f(u) + g\), we have
\[
|u_{tt}|^2 \leq |u_t|^2 + |\Delta u|^2 + |f(u)|^2 + |g|^2
\]
\[
\leq 2C + M,
\]
thus
\[
|u(t_1) - u(t_2)|^2 \leq (2C + M)|t_1 - t_2|^2, \quad \forall t_1, t_2 \geq 0.
\]

Setting \(L = 3C + M\), the proof is completed. \[\square\]

By the conclusions of Lemma 5.3 and Theorem 4.3, the attracting set \(\mathcal{A}^*\) is finite Hausdorff dimensional. The fractal dimension of \(\mathcal{A}^*\) is also finite can be obtained by Theorem 4.4 and the following lemma in [29].

**Lemma 5.6.** For any \(t > 0\), the mapping \(S(t)\) is Fréchet differentiable on \(E_0\).

**Acknowledgments.** We would like to express our sincere thanks to the anonymous referee for his/her valuable comments and suggestions which led to an important improvement of our original manuscript. This work was supported by the National Science Foundation of China Grant (11031003, 11571125 and 11601117).

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Received July 2015; revised January 2017.

E-mail address: zhangjin86@hhu.edu.cn
E-mail address: kloeden@mathematik.uni-kl.de
E-mail address: yangmeih@hust.edu.cn
E-mail address: ckzhong@nju.edu.cn