A procedure to predict the occurrence of periodic clusters in a system of globally coupled maps displaying a constant mean field is presented. The method employs the analogy between a system of globally coupled maps and a single map driven by a constant force. By obtaining the asymptotic orbits of the driven map, an associated coupling function can be constructed. This function allows to establish a direct connection between both systems. Some applications are shown.

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The phenomenon of emergence of nontrivial collective behaviors in dynamical systems of interacting chaotic elements has been a focus of recent attention [Kaneko, 1990; Chaté & Manneville, 1992(a), 1992(b); Chaté et al., 1996; Chawanya & Morita, 1998]. An important class of systems which can exhibit ordered collective effects are globally coupled nonlinear oscillators. Such systems arise naturally in the description of Josephson junctions arrays, charge density waves, multimode lasers, neural behaviors in dynamical systems of interacting chaotic elements has been a focus of recent attention [Kaneko, 1995; Perez et al., 1993]; and c) global quasiperiodic motion [Kaneko, 1991; Pikovsky & Kurths, 1994].

Usually, numerical simulations have been performed on the entire globally coupled system to search for the occurrence of various types of ordered collective behaviors. However, this direct procedure gives little information about the mechanism for the emergence of collective behaviors. In a previous work [Parravano & Cosenza, 1998] we have proposed an alternative method that allows to predict the emergence of given types of periodic collective behaviors in GCM systems exhibiting periodic clusters. In this article, we investigate the conditions for the emergence of a class of ordered collective behavior in a GCM characterized by the occurrence of \( K \) identical clusters, all of them running periodically among \( K \) states but out of phase with respect to each other in such a way that the mean field of the system remains constant during its evolution. This kind of collective behavior has been observed for globally coupled logistic maps as three identical out of phase clusters by Shinbrot [1994], who noticed that the addition of a constant term to a single local map has a similar dynamics as the GCM system. Here we use the similarity between a steadily driven map and a GCM to predict the conditions for the occurrence of general clustered collective behavior in GCM’s displaying a constant mean field.

We consider the globally coupled map system

\[
x_{t+1}^i = (1 - \epsilon)f(x_t^i) + \epsilon \langle x_t \rangle,
\]

with mean field coupling

\[
\langle x_t \rangle = \frac{1}{N} \sum_{i=1}^{N} x_t^i,
\]

where \( x_t^i (i = 1, \ldots, N) \) gives the state of the \( i \)th element at discrete time \( t \); \( N \) is the size of the system; \( \epsilon \) measures the strength of the coupling; and \( f(x) \) prescribes the local dynamics. Note that the mean field coupling affects all the elements in (Eq. 1) in exactly the same way at all times. This property of the coupling term allows to establish an analogy between the GCM system [Eq. 1] and an associated driven map

\[
s_{t+1} = (1 - \epsilon)f(s_t) + \epsilon L_t,
\]

where \( s_t \) is the state of the driven map at discrete time \( t \), \( f(s_t) \) is the same local dynamics as in Eq. 1, and \( L_t \) is the driving term. In general, \( L_t \) can be any function of the discrete time \( t \). Then, any element \( x_t^i \) in the GCM system behaves equivalently to a single driven map if \( L_t = \langle x_t \rangle, \forall t \), and their initial conditions are identical, i.e., \( x_o^i = s_o \). As it will be shown below, the analogy can also be drawn if \( L_t = \langle x_t \rangle \) only for asymptotic times. The basic idea is that, by studying the asymptotic dynamics of the associated driven map, one may infer if specific types of ordered collective behaviors can emerge in the GCM.

The simplest situation for which the associated driven map [Eq. 3] can be used to predict the emergence of ordered collective behavior in the GCM system [Eq. 1] occurs when the mean field remains constant, i.e., \( \langle x_t \rangle = C \). This behavior takes place in the GCM when \( K \) identical clusters, each having \( N/K \) elements and period \( K \), are
evolving out of phase in order to yield a constant \( \langle x_i \rangle \).

In this case, the behavior of any of such clusters in the GCM, can be emulated by an associated driven map subjected to a constant force \( L_1 = C \).

The steadily driven map will have a unique asymptotic orbit whenever the local map \( f \) satisfies Singer’s theorem. This is the case for the logistic map \( f(x) = 1 - ax^2 \), \( a \in (0, 2] \), which will be used as an example to explain our procedure. The unique asymptotic orbit of the driven map can be periodic on some regions of its space of parameters (i.e., in \((a, \epsilon, C)\) for the chosen \( f \)). Let us consider an orbit of period \( K \), and let the sequence of the points on this periodic orbit be \( \{\sigma_1, \sigma_2, \ldots, \sigma_K\} \). By using these \( K \) points on the periodic orbit of the driven map, the following associated coupling function can be defined

\[
\Theta(a, \epsilon, C) = \frac{1}{K} \sum_{j=1}^{K} \sigma_j. \tag{4}
\]

The associated function \( \Theta \) depends on the values \( \{\sigma_1, \sigma_2, \ldots, \sigma_K\} \), which themselves depend on the value of the constant drive \( C \), on the coupling parameter \( \epsilon \), and on the local map \( f(x) \). The function \( \Theta \) gives the value that the mean field in the GCM system \([\text{Eq. (4)}]\) would have at a given time if at this instant its elements are segregated in \( K \) identical clusters distributed in the states \( \{\sigma_1, \ldots, \sigma_K\} \). Note that the relation \( \Theta = \langle x_i \rangle \) will persist for subsequent times if the clusters in the GCM sequentially follow the values \( \{\sigma_1, \ldots, \sigma_K\} \) while remaining shifted one time step with respect to each other. The relation \( \langle x_i \rangle = \Theta \) is guaranteed for all times when the driven map satisfies the condition \( \Theta = C \). If \( C_* \) is the solution of \( \Theta(a, \epsilon, C) = C \), then the evolution of a set of \( K \) independent maps driven by the same constant force \( C_* \) describes a collective behavior in the corresponding GCM consisting of \( K \) identical, period \( K \) clusters which maintains \( \langle x_i \rangle = C_* \). The driven maps, as well as the clusters in the GCM, will cyclically commute over the states \( \{\sigma_1(C_*), \ldots, \sigma_K(C_*)\} \).

It should be noticed that for a given GCM the associated coupling function \( \Theta \) can be constructed only when the associated driven map displays periodic asymptotic responses for some values of \( L_1 = C \). As an example, Fig. 1 shows the bifurcation diagram of \( s_t \) as function of \( \epsilon \) for the local map \( f(x) = 1 - ax^2 \) with \( a = 2 \), driven by a constant term \( C = 0.07 \). Figure 1 also shows the time average \( \bar{s} \) of the iterates \( s_t \) as a function of \( \epsilon \). The statistical quantity \( \bar{s} \) gives the value of \( \Theta \) on the periodic regions of the bifurcation diagram. The intersection of the curve \( \bar{s} \) with the line \( C = 0.07 \) in the period three window gives the value of \( \epsilon \) that satisfies the condition \( \Theta(a = 2, \epsilon, C = 0.07) = 0.07 \). Consequently, for this value of \( \epsilon \) \((\approx 0.065)\) and for \( a = 2 \), a GCM system coupled through its mean field can exhibit a collective behavior consisting of three identical out of phase clusters that yield a constant \( \langle x_i \rangle = 0.07 \).

Note that this prediction is made without direct numerical simulation on the GCM system. However, the solution of the equation \( \Theta = C \) does not tell which initial conditions on the GCM will conduct towards this particular collective behavior. We have verified that the GCM system coupled through the mean field reaches the predicted behavior starting from initial conditions \( \{x_{i0}\} \) consisting of three groups of \( N/3 \) elements, each group distributed around one of the states \( \{\sigma_1, \sigma_2, \sigma_3\} \) with some dispersion. For \( N = 1200 \), the system reaches the expected behavior when the dispersion is less than 0.007. For larger dispersions or for initially uniform distributed elements on the interval \([-1, 1]\), the GCM system tends to split into three similar but not identical clusters resulting in a period three mean field that oscillates around the value 0.07 with a small amplitude. The later case was observed by Shinbrot [1994] in a GCM system with a quadratic local map and mean field coupling. In general, oscillations of the mean field arise in the clustered collective behavior of a GCM when \( N \) is not a multiple of \( K \).

Figure 2 shows \( \Theta(a = 2, \epsilon, C) \) as a function of \( C \) for several values of the coupling strength \( \epsilon \). The continuous curves correspond to values of \( \Theta \) in periodic regions of the indicated values of \( \epsilon \). In the chaotic regions \( \Theta \) cannot be calculated; however we show \( \bar{s} \) for the three smaller values of \( \epsilon \) for which chaos can occur. The intersections of the curves with the diagonal give the solutions to the equations \( \Theta(a = 2, \epsilon, C) = C \) for each \( \epsilon \) shown.

Figure 3 shows the solution of the equation \( \Theta(a = 2, \epsilon, C) = C \) on the plane of parameters \((\epsilon, C)\) for various regions where the associated driven map has periodic orbits. The labels on the different portions of the solution curve indicate the period \( K \) of the orbits. For each value of \( \epsilon \) and \( C \) on this curve a collective behavior comprising \( K \) periodic clusters that result in a constant mean field \( \langle x_i \rangle = C \) can take place in the corresponding GCM for appropriate initial conditions. The inserts in Fig. 3 are magnifications of the solution \( \Theta = C \) corresponding to two different periodic windows of the driven map.

In summary, based on the analogy between a GCM system and a single driven map, we have presented a method to predict the parameter values for the existence of periodic clustered collective behavior in GCM’s displaying a constant mean field. By obtaining the asymptotic orbits of the driven map, the associated coupling function \( \Theta \) can be constructed. This function allows to establish a direct connection between a GCM and its associated driven map. The solutions of equation \( \Theta(a, \epsilon, C) = C \) represent surfaces in the space of parameters \((a, \epsilon, C)\) of the driven map. For a set of parameter values \((a, \epsilon, C)\) on these surfaces, the corresponding GCM can have a mean field \( \langle x_i \rangle = C(a, \epsilon) \) resulting from a periodic clustered behavior. The method also gives the states \( \{\sigma_1(C), \ldots, \sigma_K(C)\} \) sequentially visited by the clusters.
Results were shown for the logistic map with \( a = 2 \), however, the procedure can be applied for other local maps and for any global coupling function invariant to permutations of its \( N \) arguments.

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FIGURE CAPTIONS

Fig. 1: Bifurcation diagram for Eq. (3) as a function of \( \epsilon \), with fixed \( a = 2 \) and \( L_t = C = 0.07 \). The horizontal line corresponds to the constant value of \( C \). The continuous curve is the time average \( \bar{s} \), which gives the function \( \Theta \) on the periodic regions of the diagram.

Fig. 2: Associated coupling function \( \Theta \) (continuous curves) on periodic regions of Eq. (3), as a function of \( C \), for several values of \( \epsilon \) and fixed \( a = 2 \). The numbers in parenthesis indicate the periodicity \( K \). The dots correspond to values of the time average \( \bar{s} \) on chaotic regions.

Fig. 3: The solution of \( \Theta(a = 2, \epsilon, C) = C \) for various regions where Eq. (3) has periodic orbits. Labels indicate the period \( K \) of the orbits. The inserts are magnifications of the indicated portions of the curve corresponding to two different periodic windows of the driven map.
