LARGE DEVIATIONS FOR A CLASS SEMILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN ARBITRARY SPACE DIMENSION

LEILA SETAYESGHAR

Abstract. We prove the large deviation principle (LDP) for the law of the solutions to a class of parabolic semilinear stochastic partial differential equations (SPDEs) driven by multiplicative noise using the weak convergence method. The space dimension is arbitrary and the equation has nonlinearities of polynomial growth of any order. The class of SPDEs under study, contains as special cases, the stochastic Burgers and the reaction diffusion equations.

1. Introduction

Let $D \subset \mathbb{R}^d$, with $d \geq 1$ be a bounded convex domain with smooth boundary $\partial D$. We consider a family of nonlinear parabolic semilinear stochastic partial differential equations indexed by $\varepsilon > 0$

$$
\frac{\partial}{\partial t} u^\varepsilon(t, x) = \left[ \frac{\partial}{\partial x_i} \left( b_{ij}(x) \frac{\partial}{\partial x_j} u^\varepsilon(t, x) + g_i(t, x, u^\varepsilon(t, x)) \right) + f(t, x, u^\varepsilon(t, x)) \right] \\
+ \sqrt{\varepsilon} \sigma_j(t, x, u^\varepsilon(t, x)) \frac{d}{dt} B^j, \ t \geq 0, \ x \in D
$$

(1.1)

with Dirichlet boundary conditions $u^\varepsilon(t, x) = 0$, $t \geq 0, \ x \in \partial D$ and the initial condition $u^\varepsilon(0, x) = \xi(x)$, $x \in D$. Here $B = \{B^j(t), t \geq 0, j = 1, 2, \cdots, k\}$ is a $k$-dimensional Wiener process. The initial condition $\xi$, has a continuous stochastic modification and belongs to $L^p(D)$ (for the definition of $p$, see Theorem 2.3). The functions $f = f(t, x, r)$, $\sigma_i = \sigma_i(t, x, r)$, $i = 1, 2, \cdots, k$ are locally Lipschitz and have linear growth in $r \in \mathbb{R}$. The function $g_i = g_i(t, x, r)$, $i = 1, 2, \cdots, d$ is locally Lipschitz, and has polynomial growth of any order $\nu \geq 1$ in $r$. Therefore, our family of semilinear equations contains, as special cases, both the stochastic Burgers’ equation, and the stochastic reaction diffusion equation. The existence and uniqueness to equation (1.1) has been proven by Gyöngy and Rovira (2000) [18] via an approximation procedure. Our aim is to prove the large deviation principle for the law of the solutions to equation (1.1) by employing the weak convergence approach. In the context of stochastic differential equations (SDEs), the Freidlin-Wentzell theory [16], describes the asymptotic behavior of probabilities of the large deviations of the law of the solutions to a family of small noise finite dimensional SDEs, away from its law of large number limit. In this work we deal with the case where the driving noise is infinite dimensional. In [4], Budhiraja et al. (2008) use certain variational

2000 Mathematics Subject Classification. Primary 60H15, 60H10; Secondary 37L55.
Key words and phrases. large deviations, stochastic partial differential equations, infinite dimensional Brownian motion.
representations for infinite dimensional Brownian motions and show that, these representations provide a framework for proving large deviations for a variety of infinite dimensional systems, such as SPDEs. One of the advantages of their method is that the technical exponential probability estimates needed to justify certain approximations are bypassed. Instead, one is required to prove certain qualitative properties of the SPDE under study. The following is the main contribution of this paper and establishes the large deviation principle for the law of the solutions to equation (1.1).

**Theorem 1.1 (Main Theorem).** The processes \( \{u^\varepsilon(t) : t \in [0, T]\} \) satisfy the large deviation principle on \( C([0, T]; L^p(D)) \) with rate function \( I_\varepsilon \) given by (4.3).

The definition of the rate function is deferred to Section 3. In this work, we prove the Laplace principle [10] which is equivalent to the large deviation principle for Polish space random elements. To the best of our knowledge a large deviations principle for the law of the solutions to equation (1.1) has not been studied before.

**1.1. Outline of the paper.** In Section 2 we state some assumptions and preliminaries. The existence and uniqueness results for the family of semilinear SPDEs is also stated in this section. In Section 3 we state the large deviations theorem due to Budhiraja et al. ([4, Theorem 7]) which we exploit. In Section 4 we introduce the controlled and the skeleton equations and establish their existence and uniqueness. Section 5 is devoted to the proof of the main theorem. Establishing the large deviations principle for equation (1.1) hinges on proving the tightness and convergence properties of the controlled process. This is carried out in Theorem 5.1.

**1.2. Notations.** Unless otherwise noted, we adopt the following notations throughout the paper. The notation “\( \equiv \)” means by definition. \( C \) denotes a free constant which may take on different values, and depend upon other parameters. We use the notation \( |h(t, \cdot)|_p = |h(t)|_p \) to denote the \( L^p(\mathbb{R}^d) \)-norm of a function \( h = h(t, x) \) with respect to the variable \( x \in \mathbb{R}^d \). If \( h(t, x) \) is only defined for \( x \in D \), then \( |h(t)|_p \) denotes the \( L^p(D) \) norm. If \( h = h(t, x) \) is a random field and \( X \) assumes a value in a functional space, then saying that almost surely \( h \) is in \( X \) means that \( h \) has a stochastic modification which is in \( X \), almost surely.

### 2. Preliminaries

In this section we introduce a set of assumptions and preliminaries that are necessary for the formulation of the problem. Let \((\Omega, \mathcal{F}, \mathbb{P}, P)\) be a filtered probability space or stochastic basis carrying a \( k \)-dimensional Brownian motion \( \{B^j(t), t \geq 0, 1 \leq j \leq k\} \), with the filtration \( \mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]} \). The following are some main assumptions that are in effect throughout the paper:

(A1) The domain \( D \subset \mathbb{R}^d, d \geq 1 \) is a bounded convex set with smooth boundary.

(A2) The matrix \( b_{ij}(x) \in C^2(\bar{D}) \) is symmetric for every \( x \in D \), and satisfies the uniform ellipticity condition, i.e.,

\[
\frac{1}{\kappa} |\gamma|^2 \geq b_{ij}(x)\gamma_i \gamma_j \geq \kappa |\gamma|^2, \ \forall \ \gamma \in \mathbb{R}^d, \ x \in D
\]
(A3) The functions $g_i$ are of the form $g_i(t, x, r) = g_{i1}(x, t, r) + g_{i2}(t, r)$, where $g_{i1}$ and $g_{i2}$ are Borel functions of $(t, x, r) \in \mathbb{R}_+ \times D \times \mathbb{R}$ and of $(t, r) \in \mathbb{R}_+ \times \mathbb{R}$, respectively. Moreover, for every $T \geq 0$ there is a constant $K$ such that

$$|g_{i1}(t, x, r)| \leq K(1 + |r|), \quad |g_{i2}(t, r)| \leq K(1 + |r|^\nu),$$

for all $t \in [0, T]$, $x \in D$, $r \in \mathbb{R}$, with some $\nu \geq 1$, and for all $i = 1, \ldots, d$.

(A4) The functions $f = f(t, x, r)$, $\sigma_j = \sigma_j(t, x, r)$, $j = 1, \ldots, k$ are Borel functions and have linear growth in $r$, i.e.,

For every $T \geq 0$ there exists a constant $L$ such that

$$\sum_j |\sigma_j(t, x, r)|^2 \leq L(|r|^2 + 1), \quad j = 1, \ldots, k,$$

$|f(t, x, r)| \leq L(|r| + 1),$

for all $t \in [0, T]$, $x \in D$, and $r, s \in \mathbb{R}$.

(A5) For every $T \geq 0$, there exists a constant $L$ such that

$$\sum_j |\sigma_j(t, x, r) - \sigma_j(t, x, s)|^2 \leq L(|r - s|^2), \quad j = 1, \ldots, k,$$

$$|f(t, x, r) - f(t, x, s)| \leq L|r - s|,$$

$$|g_i(t, x, r) - g_i(t, x, s)| \leq L(1 + |r|^\nu - 1 + |s|^\nu - 1)|r - s|, \quad i = 1, \ldots, d,$$

for all $t \in [0, T]$, $x \in D$, $r, s \in \mathbb{R}$.

**Definition 2.1 (Mild Solution).** A random field $u^\varepsilon \doteq \{u^\varepsilon(t, x) : t \in [0, T], x \in D\}$ is called a mild solution of equation (1.1) with initial condition $\xi$ if $(t, x) \rightarrow u^\varepsilon(t, x)$ is continuous a.s., and $u^\varepsilon(t, x)$ is $\{\mathcal{F}_t\}$-measurable for any $t \in [0, T]$, and $x \in D$, and if

$$u^\varepsilon(t, x) = \int_D G_t(x, y)\xi(y)dy + \varepsilon \int_0^t \int_D G_{t-s}(x, y)\sigma_j(s, u^\varepsilon(s))(y)dydB^j(s)$$

$$- \int_0^t \int_0^1 \partial_y G_{t-s}(x, y)g_i(s, u^\varepsilon(s))(y)dyds$$

$$+ \int_0^t \int_D G_{t-s}(x, y)f(s, u^\varepsilon(s))(y)dyds.$$

The function $G_t(x, y)$, $t \geq 0$, $x, y \in D$ is the Green’s function or the heat kernel associated with the following linear equation

$$\frac{\partial}{\partial t} u^\varepsilon(t, x) = \frac{\partial}{\partial x_i} \left( b_{ij}(x) \frac{\partial}{\partial x_j} u^\varepsilon(t, x) \right),$$

with Dirichlet’s boundary condition.
where \( b_{ij} \in C^2(\bar{D})\), and \( \partial D \) is Lipschitz. We now state some estimates on the Dirichlet heat kernel ([18, Proposition 3.5], [22]).

### 2.1. Estimates on the Dirichlet Heat Kernel

There exist Borel functions \( a, b, d \) and some constants \( K, C > 0 \) such that for some \( p \geq 1 \) and for all \( 0 \leq s < t \leq T, x, y \in D \)

\[
|\partial_s G_{t-s}(x,y)| \leq C(t-s, x-y), \quad |a(t, \cdot)|_{p} \leq K_p t^{-1+\lambda_p},
\]

\(
(E1)\)

\[
\frac{\partial}{\partial x_i} G_{t-s}(x,y) \leq b(t-s, x-y), \quad |b(t, \cdot)|_{p} \leq K_p t^{-1-v_p+\lambda_p},
\]

\(
(E2)\)

\[
\frac{\partial}{\partial s} G_{t-s}(x,y) \leq c(t-s, x-y), \quad |c(t, \cdot)|_{p} \leq K_p t^{-1-\varepsilon_p+\lambda_p},
\]

\(
(E3)\)

where \( \lambda_p \leq 1 \) and \( v_p, \varepsilon_p \) and \( K_p \) are nonnegative constants.

\[
|D^i_i D^j_j G_{t-s}(x,y)| \leq K t^{-(d+2n+|\gamma|)/2} \exp \left(-C \frac{|x-y|^2}{t-s} \right)
\]

\(\text{for } 2n + |\gamma| \leq 3, \text{ where } D^i_i = \partial_i / \partial t^n, \quad D^j_j = \partial_j / \partial x^n_1, \cdots \partial_j / \partial x^n_d, \quad \gamma = (\gamma_1, \cdots \gamma_d) \text{ is a multi-index, } |\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_d.\)

\(\text{(E4)}\)

**Remark 2.2.** Due to \( A = (\partial / \partial x_i)(b_{ij}(x) \partial / \partial x_j) \approx A^*, \quad G_{t-s}(x,y) \) is symmetric in \( x, y \). Therefore \( (E4) \) also holds with \( D_y \) in place of \( D_x \).

The following Theorem ([18, Theorem 2.1]) asserts the existence of a unique solution to equation (1.1).

**Theorem 2.3** (Existence and Uniqueness of Solution Mapping). Assume the set of Hypotheses \( (H) \). Then there exists \( \rho_0 = \rho_0(\nu, d) \), such that for every \( \rho > \rho_0 \) equation (1.1) has a unique \( L^p(D) \)-valued solution, provided \( \xi \) is an \( \mathcal{F}_0 \)-measurable, \( L^p(D) \)-valued random element. Moreover, if \( \xi \) has a continuous stochastic modification, then \( u(t, x) \) has a stochastic modification which is continuous in \( (t, x) \in [0, \infty) \times D \).

**Remark 2.4.** The proof of the above theorem is based on an approximation procedure where the nonlinear functions \( f, g_i \) in equation (1.1) are approximated by bounded Lipschitz functions. It is proven via an energy inequality [18, Lemma 4.3] that the solutions to the approximations of equation (1.1) are bounded in probability [18, Proposition 4.4]. This yields the tightness of the approximated solutions in \( C([0,T]; L^p(D)) \). Using the Skorokhod’s representation theorem, it is then shown that the sequence of approximated solutions converges in probability to the unique solution of equation (1.1).

The following two Lemmas are used in proving the main theorem.

**Lemma 2.5** ([18, Corollary 3.6]). Set

\[
I(\phi^\varepsilon)(t, x) \doteq \int_0^t \int_D G_{t-s}(x,y)\phi^\varepsilon(t,y)dydB(s), \quad t \in [0, T], \quad x \in D
\]

For a sequence of random fields \( \phi^\varepsilon = \{\phi^\varepsilon(t, x) : t \in [0, T], x \in D\} \). If for \( \rho > d \) we have a constant \( C \) such that \( |\phi^\varepsilon|_{\rho} \leq C \) for all \( t \in [0, T] \) and for all \( \varepsilon \), then \( I(\phi^\varepsilon) \) is tight in \( C([0, T] \times D) \), uniformly in \( \varepsilon \).
Let $q \geq 1$ and $R(r, t; x, y) = \partial_y G_{r-t}(x, y)$ or $G(r, t; x, y)$ for $t \in [0, T]$ and $x \in D$. For $v \in L^\infty([0, T]; L^q(D))$, define the linear operator $J$ by

$$J(v)(t, x) = \int_0^t \int_D R(r, t; x, y)v(r, y)dydr, \ t \in [0, T], \ x \in D$$

$$J(v)(t, x) = 0 \text{ if } x \notin D$$

provided the integral exists.

**Lemma 2.6** ([18, Corollary 3.2]). Assume (E1)-(E3) with $\lambda_p > 0$. Let $\zeta_n(t, x)$ be a sequence of random fields on $[0, T] \times D$ such that almost surely

$$|\zeta_n(t, \cdot)|_p \leq \theta_n, \text{ for all } t \in [0, T],$$

where $\theta_n$ is a finite random variable for every $n$. Assume that the sequence $\theta_n$ is bounded in probability, i.e.

$$\lim_{C \to \infty} \sup_{n} P(\theta_n \geq C) = 0$$

Then for $0 \leq \alpha < \min(\lambda_p/\varepsilon_p, \lambda_p)$, $J(\zeta_n)$ is uniformly tight in $C^\alpha([0, T]; L^p(D))$. In the case $p = \infty$ the sequence $J(\zeta_n)$ is tight in the $C^{\alpha,\beta}([0, T]; L^p(D))$ for $0 \leq \alpha < \min(\lambda_p/\varepsilon_p, \lambda_p)$, $0 \leq \beta < \min(\lambda_p/v_p, 1)$.

### 3. The Large Deviation Principle

In this section, we state Theorem 3.1 [4, Theorem 6], which asserts the uniform Laplace principle for a family of functionals of a cylindrical Wiener process under two main assumptions. Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be the filtered probability space introduced as before. Let $\vartheta: \Omega \times [0, T] \to L^2(\mathbb{R}^k)$, be an $L^2(\mathbb{R}^k)$-valued predictable process. Define

$$A_2 = \left\{ \vartheta : \int_0^T |\vartheta|^2 ds < \infty, \text{a.s.} \right\}$$

For any $N > 0, N \in \mathbb{N}$, define

$$\Lambda^N = \left\{ \tau \in L^2([0, T]; \mathbb{R}^k) : \int_0^T |\tau|^2 ds \leq N \right\},$$

$$A_2^N = \left\{ \eta(\omega) \in \Lambda^N : \eta \in A_2 : P - \text{a.s.} \right\},$$

where $| \cdot |$ is the norm in $\mathbb{R}^k$. Note that $\Lambda^N$ is a compact metric space equipped with the weak topology from $L^2([0, T]; \mathbb{R}^k)$. The space $A_2$ is the space of controls, and plays an important role in the weak convergence method to the theory of large deviations. Let $\mathcal{E}_0$ and $\mathcal{E}$ be Polish spaces, and let the initial condition $\xi$ take values in a compact subspace of $\mathcal{E}_0$. For every $\varepsilon \in (0, 1)$, let $\mathcal{H}^\varepsilon: \mathcal{E}_0 \times C[0, T]; \mathbb{R}^\infty, \to \mathcal{E}$ be a family of measurable maps, and define $Y^\varepsilon = \mathcal{H}^\varepsilon(\xi, \sqrt{\varepsilon}B)$. For a control $\varphi \in A_2$, and under the measure $Q$ defined by

$$\frac{dQ}{dP} = \exp \left\{ -\frac{1}{\sqrt{\varepsilon}} \int_0^T \langle \varphi(s), dB(s) \rangle - \frac{1}{2\varepsilon} \int_0^T |\varphi(s)|^2 ds \right\}.$$
Girsanov’s theorem implies that the process
\[ \tilde{B}(t) = B(t) + \varepsilon^{-1/2} \int_0^t \varphi(s) ds \]
is a \( k \)-dimensional Wiener process. The following is the standing assumption of Theorem 3.1, the large deviation principle of [4].

**ASSUMPTION:** There exists a measurable map \( \mathcal{H}^0 : \mathcal{E}_0 \times L^2([0,T];\mathbb{R}^k) \to \mathcal{E} \), such that:

1. **(S1)** For every \( M < \infty \) and compact set \( K \subset \mathcal{E}_0 \), the set
   \[ \Gamma_{M,K} \doteq \left\{ \mathcal{H}^0\left( \xi, \int_0^t \varphi(s) ds \right) : \varphi \in S^M, \xi \in K \right\}, \]
is a compact subset of \( \mathcal{E} \).
2. **(S2)** Consider \( M < \infty \) and the families \( \{ \varphi^\varepsilon \} \subset A^M_2 \) and \( \{ \xi^\varepsilon \} \subset \mathcal{E}_0 \), such that \( \varphi^\varepsilon \to \varphi \), and \( \xi^\varepsilon \to \xi \) in distribution, as \( \varepsilon \to 0 \). Then
   \[ \mathcal{H}^\varepsilon\left( \xi^\varepsilon, \sqrt{\varepsilon} B + \int_0^t \varphi^\varepsilon(s) ds \right) \to \mathcal{H}^0\left( \xi, \int_0^t \varphi(s) ds \right), \]
in distribution as \( \varepsilon \to 0 \).

For \( \psi \in \mathcal{E} \), and \( \xi \in \mathcal{E}_0 \), define the rate function
\[ I_\xi(\psi) \doteq \inf_{\{ \beta \in L^2([0,T];\mathbb{R}^k) : \psi \in \mathcal{H}^0(\xi,\int_0^T \beta(s) ds) \}} \left\{ \frac{1}{2} \int_0^T |\beta(s)|^2 ds \right\}. \tag{3.1} \]
The following theorem states the uniform Laplace principle for the family \( \{ Y^\varepsilon_\xi \} \).

**Theorem 3.1** ([4, Theorem 6]). Let \( \mathcal{H}^0 : \mathcal{E}_0 \times C([0,T];\mathbb{R}^k) \to \mathcal{E} \) be a measurable map satisfying assumptions (S1) and (S2). Suppose that for all \( f \in \mathcal{E} \), \( \xi \mapsto I_\xi(f) \) is a lower semi-continuous map from \( \mathcal{E}_0 \) to \( [0,\infty] \). Then for every \( \xi \in \mathcal{E}_0 \), \( I_\xi(\psi) : \mathcal{E} \to [0,\infty] \), is a rate function on \( \mathcal{E} \) and the family \( \{ I_\xi, \xi \in \mathcal{E}_0 \} \) of rate functions has compact level sets on compacts. Furthermore, the family \( \{ Y^\varepsilon_\xi \} \) satisfies the uniform Laplace principle on \( \mathcal{E} \) with rate function \( I_\xi \), uniformly in \( \xi \) on compact subsets of \( \mathcal{E}_0 \).

**Remark 3.2.** In order to prove Theorem 1.1, it suffices to verify assumptions (S1) and (S2) with \( \mathcal{E}_0 = L^\rho(D) \) and \( \mathcal{E} = C([0,T];L^\rho(D)) \).

### 4. The Controlled and the Skeleton Equations

Recall that the solution map of equation (1.1) is \( u^\varepsilon = \mathcal{H}^\varepsilon(\xi, \sqrt{\varepsilon} B) \). The map \( v^\varepsilon_{\xi,\varphi} = \mathcal{H}^\varepsilon(\xi, \sqrt{\varepsilon} B + \int_0^t \varphi(s) ds) \) is the solution map of the stochastic controlled equation, with mild solution
where the infimum is taken over all \( \beta \), then define the solution mapping, and let \( \psi \).

Theorem 4.1 (Existence and Uniqueness of the Controlled Process).

The following Theorem ([4, Theorem 10]), asserts the existence and uniqueness of the controlled process. The map \( v^{0,\varphi} = H^0(\xi, f_0^t \varphi(s) ds) \) is the solution map of the skeleton equation whose mild solution is

\[
v^{0,\varphi}(t, x) = \int_D G_t(x, y) \xi(y) dy - \int_0^t \int_D \partial_y G_{t-s}(x, y) g_i(s, v^{0,\varphi}(s))(y) dy ds \\
+ \int_0^t \int_D G_{t-s}(x, y) f(s, v^{0,\varphi}(s))(y) dy ds \\
+ \int_0^t \int_D G_{t-s}(x, y) \sigma_j(s, v^{0,\varphi}(s))(y) \varphi_j(s) dy ds
\]  

(4.2)

4.1. The Rate Function. Let \( \psi \in C([0, T]; L^\beta(D)) \), for every \( t \in [0, T] \), and \( x \in D \). Define the following rate function (or action functional)

\[
I_\xi(\psi) = \frac{1}{2} \inf_{\beta} \int_0^T |\beta(s)|^2 ds,
\]  

(4.3)

where the infimum is taken over all \( \beta \in L^2([0, T]; \mathbb{R}^K) \) such that

\[
\psi(t, x) = \int_D G_t(x, y) \xi(y) dy - \int_0^t \int_D \partial_y G_{t-s}(x, y) g_i(s, \psi(s))(y) dy ds \\
+ \int_0^t \int_D G_{t-s}(x, y) f(s, \psi(s))(y) dy ds \\
+ \int_0^t \int_D G_{t-s}(x, y) \sigma_j(s, \psi(s))(y) \varphi_j(s) dy ds
\]  

(4.4)

4.2. Existence and Uniqueness of the Controlled Process. The following Theorem ([4, Theorem 10]), asserts the existence and uniqueness of the controlled process, with the main ingredient of proof being the Girsonov’s theorem.

Theorem 4.1 (Existence and Uniqueness of the Controlled Process). Let \( H^\varepsilon \) denote the solution mapping, and let \( \varphi \in A_2^N \) for some \( N \in \mathbb{N} \). For \( \varepsilon > 0 \) and \( \xi \in L^\beta(D) \) define

\[
v^{\varepsilon,\varphi}_\xi \equiv H^\varepsilon \left( \xi, \sqrt{\varepsilon} B + \int_0^t \varphi(s) ds \right),
\]  

then \( v^{\varepsilon,\varphi}_\xi \) is the unique solution of equation (4.1).
4.3. Existence and Uniqueness of the Skeleton. The next theorem shows the existence and uniqueness of the skeleton equation whose proof is almost verbatim to that of Theorem 2.3, and thus omitted.

**Theorem 4.2 (Existence and Uniqueness of the Skeleton).** Fix $\xi \in L^\rho(D)$ and $\varphi \in L^2([0,T]; \mathbb{R}^d)$. Then there exists a unique function $\psi \in C([0,T]; L^\rho(D))$ which satisfies equation (4.4).

5. Proof of Theorem 1.1

The following theorem plays a key role in proving Theorem 1.1. It leads to the verification of assumptions (S1) and (S2). To this purpose, let

**Theorem 5.1 (Convergence of the Controlled Process).** Let $M < \infty$, and suppose that $\xi^\varepsilon \to \xi$ and $\varphi^\varepsilon \to \varphi$ in distribution as $\varepsilon \to 0$ with $\{\varphi^\varepsilon\} \subset A^M_2$. Then $v_{\xi^\varepsilon}^{(\cdot),\varphi^\varepsilon} \to v_{\xi}^{0,\varphi}$ in distribution.

**Proof.** We carry out the proof in two steps.

**Step 1: Tightness**

Note that

$$v_{\xi^\varepsilon}^{(\cdot),\varphi^\varepsilon}(t, x) = \int_D G_t(x, y)\xi^\varepsilon(y)dy + \sqrt{\varepsilon} \int_0^t \int_D G_{t-s}(x, y)\sigma_j(s, v_{\xi^\varepsilon}^{(\cdot),\varphi^\varepsilon}(s))(y)dB^j(s)$$

$$- \int_0^t \int_D \partial_y G_{t-s}(x, y)g_i(s, v_{\xi^\varepsilon}^{(\cdot),\varphi^\varepsilon}(s))(y)dyds$$

$$+ \int_0^t \int_D G_{t-s}(x, y)f(s, v_{\xi^\varepsilon}^{(\cdot),\varphi^\varepsilon}(s))(y)dyds$$

$$+ \int_0^t \int_D G_{t-s}(x, y)\sigma_j(s, v_{\xi^\varepsilon}^{(\cdot),\varphi^\varepsilon}(s))(y)\varphi^\varepsilon_j(s)dyds$$

$$= Z_1^\varepsilon + Z_2^\varepsilon + Z_3^\varepsilon + Z_4^\varepsilon + Z_5^\varepsilon \quad (5.1)$$

We show tightness of $Z_\ell^\varepsilon$ for $\ell = 1, 2, 3, 4, 5$ in $C([0,T]; L^\rho(D))$, and therefore assert the claim. Since $\xi^\varepsilon \in L^\rho(D)$, the tightness of $Z_1^\varepsilon$ follows by the following lemma

**Lemma 5.2 ([5, Lemma A.2]).** Let $\xi \in L^\rho(D)$. Then $(t \to G_t \xi)$ belongs to $C([0,T]; L^\rho(D))$, and

$$\xi \to \{t \to G_t \xi\},$$

is a continuous map in $\xi$.

As for the tightness of $Z_2^\varepsilon$, we employ Lemma 2.1. Note that since

$$|\sigma(s, \cdot)|_\rho \leq |v_{\xi^\varepsilon}^{(\cdot),\varphi^\varepsilon}(s, \cdot)|_\rho \leq C, \quad \text{for all } s \in [0,T], \text{ and for all } \varepsilon \in (0,1),$$
the assumption of Lemma 2.1 is satisfied, and so $Z_2^\varepsilon$ is tight in $C([0,T]; L^\rho(D))$.

As for the tightness of $Z_3^\varepsilon$, we mainly use Lemma 2.6. Note that $g_i(t, x, r) = g_{i1}(t, x, r) + g_{i2}(t, r)$. Therefore

$$Z_3^\varepsilon = \int_0^t \int_D \partial_y G_{t-s}(x, y) g_i(s, \varphi^\varepsilon(s))(y) dy ds$$

$$= \int_0^t \int_D \partial_y G_{t-s}(x, y) g_{i1}(s, \varphi^\varepsilon(s))(y) dy ds$$

$$+ \int_0^t \int_D \partial_y G_{t-s}(x, y) g_{i2}(\varphi^\varepsilon(s))(y) dy ds = Z_{3,1}^\varepsilon + Z_{3,2}^\varepsilon \quad (5.2)$$

$g_{i1}$ satisfies the linear growth condition:

$$\sup_{t \in [0, T]} \sup_{x \in [0, 1]} |g_{i1}(t, x, r)| \leq K(1 + |r|).$$

In Lemma 2.6, let $\zeta^\varepsilon(t, x) \equiv g_{i1}(t, \varphi^\varepsilon(t))(x)$. We have

$$\sup_{t \in [0, T]} |g_{i1}(s, \varphi^\varepsilon(t))|_1 \leq K + \sup_{t \in [0, T]} |\varphi^\varepsilon(t)|_\rho$$

Let $\theta^\varepsilon = K + \sup_{t \in [0, T]} |\varphi^\varepsilon(t)|_\rho$. We have

$$\lim_{C \to \infty} \sup_{\varepsilon} P(K + K \sup_{t \in [0, T]} |\varphi^\varepsilon(t)|_\rho \geq C) \leq \lim_{C \to \infty} \sup_{\varepsilon} P(\sup_{t \in [0, T]} |\varphi^\varepsilon(t)|_\rho \geq \frac{C}{2})$$

Clearly the first term on the RHS of the immediate above display is equal to zero. As for the second term, it suffices to show that

$$\sup_{t \in [0, T]} |\varphi^\varepsilon(t)|_\rho$$

is bounded in probability, i.e.

$$\lim_{C \to \infty} \sup_{\varepsilon} P(\sup_{t \in [0, T]} |\varphi^\varepsilon(t)|_\rho \geq C) = 0. \quad (5.3)$$

The proof of (5.3) is similar to that in [23] but we include it here for the convenience of the reader. Recall the class of stochastic semilinear equations (1.1) which we rewrite here

$$\frac{\partial}{\partial t} u^\varepsilon(t, x) = \left[ \frac{\partial}{\partial x_i} \left( b_{ij}(x) \frac{\partial}{\partial x_j} u^\varepsilon(t, x) + g_i(t, x, u^\varepsilon(t, x)) \right) + f(t, x, u^\varepsilon(t, x)) \right]$$

$$+ \sqrt{\varepsilon} \sigma_j(t, x, u^\varepsilon(t, x)) \frac{d}{dt} B^j, \quad t > 0, \ x \in D \quad (5.4)$$
Note that the controlled equation (5.1) can be recovered from the above equation. In [17], Gyöngy and Rovira (2000) prove the existence and uniqueness of the solutions to the above class of stochastic semi-linear equations, by an approximation procedure. Let \( f_n(t, x, r) \) and \( g_n(t, x, r) \) be Borel functions for every integer \( n \), such that they are globally Lipschitz in \( r \in \mathbb{R} \), and \( f_n = f, g_n = g_i \) and \( \sigma_j \) for \( |r| \leq n, f_n = g_n = \sigma_{jn} = 0 \) for \( |r| \geq n + 1 \). Moreover, \( f_n, g_n \) and \( \sigma_{jn} \) satisfy the same growth conditions as \( f, g_i \) and \( \sigma_j \), respectively. We have, by ([17, Proposition 4.1]), that there exists a unique solution, say \( v^{\gamma(e), \varphi^e}_{\xi, n} \), to the semi-linear equation (1.1) with \( f, g_i \) and \( \sigma_j \) replaced by \( f_n, g_{in} \) and \( \sigma_{jn} \). That is, \( v^{\gamma(e), \varphi^e}_{\xi, n} \) is the unique solution to the truncated equation. Furthermore, \( v^{\gamma(e), \varphi^e}_{\xi, n} \) converges to \( v^{\gamma(e), \varphi^e}_{\xi} \) in \( C([0, T]; L^p(D)) \) in probability, uniformly in \( \varepsilon \), as \( n \) approaches infinity. It has been demonstrated in ([17, Proposition 4.4]) that, for every \( n \geq 1 \)

\[
\lim_{C \to \infty} \sup_{\varepsilon \in (0, 1)} P(\sup_{t \leq T}|v^{\gamma(e), \varphi^e}_{\xi, n}(t, \cdot)|_\rho \geq C) = 0. \tag{5.5}
\]

Observe that

\[
\sup_{\varepsilon \in (0, 1)} P(\sup_{t \leq T}|v^{\gamma(e), \varphi^e}_{\xi, n}|_\rho \geq C) \leq \sup_{\varepsilon \in (0, 1)} P(\sup_{t \leq T}|v^{\gamma(e), \varphi^e}_{\xi} - v^{\gamma(e), \varphi^e}_{\xi, n}|_\rho \\
+ \sup_{t \leq T}|v^{\gamma(e), \varphi^e}_{\xi, n}|_\rho \geq C) \\
\leq \sup_{\varepsilon \in (0, 1)} P(\sup_{t \leq T}|v^{\gamma(e), \varphi^e}_{\xi} - v^{\gamma(e), \varphi^e}_{\xi, n}|_\rho \geq \frac{C}{2}) \\
+ \sup_{\varepsilon \in (0, 1)} P(\sup_{t \leq T}|v^{\gamma(e), \varphi^e}_{\xi, n}|_\rho \geq \frac{C}{2}). \tag{5.6}
\]

By letting \( C \) approach infinity, and exploiting the boundedness in probability of \( |v^{\gamma(e), \varphi^e}_{\xi, n}|_\rho \), we get

\[
\lim_{C \to \infty} \sup_{\varepsilon \in (0, 1)} P(\sup_{t \leq T}|v^{\gamma(e), \varphi^e}_{\xi}|_\rho \geq C) \leq \lim_{C \to \infty} \sup_{\varepsilon \in (0, 1)} P(\sup_{t \leq T}|v^{\gamma(e), \varphi^e}_{\xi} - v^{\gamma(e), \varphi^e}_{\xi, n}|_\rho \geq \frac{C}{2}).
\]

Now by letting \( n \) tend to infinity, due the convergence in probability of \( v^{\gamma(e), \varphi^e}_{\xi, n} \) to \( v^{\gamma(e), \varphi^e}_{\xi} \), we conclude that

\[
\lim_{C \to \infty} \sup_{\varepsilon \in (0, 1)} P(\sup_{t \leq T}|v^{\gamma(e), \varphi^e}_{\xi, n}(t, \cdot)|_\rho \geq C) = 0. \tag{5.7}
\]

Therefore

\[
\lim_{C \to \infty} \sup_{\varepsilon} P(\theta^\varepsilon \geq C) = 0,
\]

and the assumption of Lemma 2.6 is satisfied. This establishes the tightness of \( Z^\varepsilon_3 \). The proof of tightness for \( Z^\varepsilon_1 \) follows by the same analogy as \( Z^\varepsilon_3 \), and thus omitted.
As for the tightness of $Z^\varepsilon_5$, we have

$$
\sup_{\varepsilon \in (0,1)} |Z^\varepsilon_5| \leq \sup_{\varepsilon \in (0,1)} \left| \int_0^t \varphi^\varepsilon(s) \left[ \int_D G_{t-s}(x,y) \sigma_i(s, v^\gamma;\varphi^\varepsilon(s))(y) dy \right] ds \right|
$$

$$
\leq \sup_{\varepsilon \in (0,1)} \left( \int_0^t (\varphi^\varepsilon)^2 ds \right)^{1/2}
\times \sup_{\varepsilon \in (0,1)} \left( \int_0^t \left[ \int_D G_{t-s}(x,y) \sigma_i(s, v^\gamma;\varphi^\varepsilon(s))(y) dy \right]^2 ds \right)^{1/2},
$$

where Hölder’s inequality has been used. Note that the first term on the RHS of the immediate above display is bounded by the properties of controls. As for the second term, we have

$$
\int_D G_{t-s}(x,y) \sigma_j(s, v^\gamma;\varphi^\varepsilon(s))(y) dy \leq \left( \int_D (G_{t-s}(x,y))^{\rho_1} dy \right)^{1-1/\rho}
\times \left( \int_D (\sigma_j(s, v^\gamma;\varphi^\varepsilon(s))(y) dy \right)^{1/\rho} < C(T)
$$

where the linear growth condition of $\sigma_j$ (A4), and estimate (E4) have been used. This leads to

$$
\sup_{\varepsilon \in (0,1)} |Z^\varepsilon_5| \leq C(T),
$$

and hence establishes the tightness of $Z^\varepsilon_5$ in $C([0,T]; L^\rho(D))$. Therefore, the tightness of $v^\gamma;\varphi^\varepsilon$ in $C([0,T]; L^\rho(D))$ is concluded.

**Step 2: Convergence**

Having the tightness of $Z^\varepsilon$ for $\ell = 1, 2, 3, 4, 5$ at hand, by Prohorov’s theorem, we can extract a subsequence along which each of the aforementioned processes and $v^\gamma;\varphi^\varepsilon$ converge in distribution to $Z^0_\ell$ and $v^0;\varphi(t, x)$ in $C([0,T]; L^\rho(D))$. We aim to show that the respective limits are as follows:

$$Z^0_1 = \int_0^1 G_t(s,y)\xi(y)dy,$$

$$Z^0_2 = 0,$$

$$Z^0_3 = -\int_0^t \int_0^1 \partial_y G_{t-s}(x,y)g_i(s, v^0;\varphi^\varepsilon(s))(y) dy ds,$$

$$Z^0_4 = \int_0^t \int_0^1 G_{t-s}(x,y)f(s, v^0;\varphi^\varepsilon(s))(y) dy ds,$$

$$Z^0_5 = \int_0^t \int_0^1 G_{t-s}(x,y)\sigma_j(s, v^0;\varphi^\varepsilon(s))(y) \varphi_j(s) dy ds.$$

The case $\ell = 1$ follows from lemma (5.2). For $\ell = 2$, let
\[
\mathcal{J}^\varepsilon = \int_0^t \int_D G_{t-s}(x,y) \sigma_j(s, v_{\xi}^{(\varepsilon), \varphi^\varepsilon}(s))(y) dB^i(s).
\]

Note that \(\mathcal{J}^\varepsilon\) is tight by Lemma 2.1. As a result, convergence of \(Z_2^\varepsilon\) to zero (as \(\varepsilon \to 0\)) follows readily.

As for \(\ell = 3\), we invoke the Skorokhod Representation Theorem [11]. Denote the RHS of \(Z_3^0\) by \(\bar{Z}_3^0\). We have

\[
|Z_3^\varepsilon - \bar{Z}_3^0| \leq \int_0^t \int_D |\partial_y G_{t-s}| \left|\frac{|v_{\xi}^{(\varepsilon), \varphi^\varepsilon}|^{\nu-1} + |v_{\xi}^{0, \varphi}|^{\nu-1}}{\nu} v_{\xi}^{(\varepsilon), \varphi^\varepsilon} - v_{\xi}^{0, \varphi}\right| dy ds
\]

where the Lipschitz property of \(g_t\) with linearly growing constant (A5), and Hölder’s inequality have been used. Note that by estimate (E4) the integrals on the RHS of (5.9) are finite. Therefore, since \(v_{\xi}^{(\varepsilon), \varphi^\varepsilon}\) converges to \(v_{\xi}^{0, \varphi}\) as \(\varepsilon \to 0\), the RHS of (5.9) also converges to zero. By the fact that the limit is unique, and that \(\bar{Z}_3^0\) is a continuous random field (by Theorem 4.2) we conclude that \(Z_3^0 = \bar{Z}_3^0\).

For \(\ell = 4\), we invoke the Skorokhod Representation Theorem [11] again.

\[
|Z_4^\varepsilon - \bar{Z}_4^0| \leq \int_0^t \int_D |\partial_y G_{t-s}| |v_{\xi}^{(\varepsilon), \varphi^\varepsilon} - v_{\xi}^{0, \varphi}| dy ds
\]

Note that the RHS of (5.10) converges to zero as \(\varepsilon \to 0\) since \(v_{\xi}^{(\varepsilon), \varphi^\varepsilon} \to v_{\xi}^{0, \varphi}\), and

\[
\int_0^t \int_D |\partial_y G_{t-s}(x,y)| dy ds \leq C(T),
\]

by estimate (E4). For \(\ell = 5\), we invoke the Skorokhod Representation Theorem [11] again. Denote the RHS of \(Z_5^0\) by \(\bar{Z}_5^0\). We have

\[
|Z_5^\varepsilon - \bar{Z}_5^0| \leq \int_0^t \int_D |\partial_y G_{t-s}| |\sigma_j(s, v_{\xi}^{(\varepsilon), \varphi^\varepsilon}(s))(y)\sigma_j(s, v_{\xi}^{0, \varphi}(s))(y)| |\varphi_j^\varepsilon(s)||dy ds
\]

\[
+ \int_0^t \int_D |\partial_y G_{t-s}| |\sigma_j(s, v_{\xi}^{0, \varphi}(s))(y)\varphi_j^\varepsilon(s) - \varphi_j(s)dy ds dy ds
\]

The first term on the RHS of (5.11) can be bounded above by

\[
M \left( \int_0^t \int_0^1 |G_{t-s}|^2 |\sigma_j(s, v_{\xi}^{(\varepsilon), \varphi^\varepsilon}(s))(y) - \sigma_j(s, v_{\xi}^{0, \varphi}(s))(y)|^2 dy ds \right)^{1/2}
\]

\[
\leq C(T) \sup_{x, t} \left|v_{\xi}^{(\varepsilon), \varphi^\varepsilon} - v_{\xi}^{0, \varphi}\right|
\]

(5.12)
Large deviations for a class of semilinear SPDEs 13

where the Cauchy-Schwartz inequality, properties of the controls, estimate (E4), and (A5) have been used. The first term on the RHS of (5.11) thus converges to zero, since $v_{\xi}^{(r,\varphi)}(\varepsilon) \to v_{\xi}^{0,\varphi}$ as $\varepsilon \to 0$. The second term on the RHS of (5.11) also converges to zero as $\varepsilon \to 0$, since $\varphi_{\varepsilon} \to \varphi$, and

$$\int_0^t \int_D (G_{t-s}(x,y)\sigma_j(s,v_{\xi}^{0,\varphi}(s))(y))^2 dy ds \leq \left( \int_0^t \int_D (G_{t-s}(x,y))^{2\rho} dy ds \right)^{1-2/\rho} \times \left( \int_0^t \int_D (\sigma_j(s,v_{\xi}^{0,\varphi}(s))(y))^\rho dy ds \right)^{2/\rho} \leq C(T)$$

where (A4), and estimate (E4) have been used. Again, by the fact that the limit is unique, and that $Z_0^0$ is a continuous random field (by Theorem 4.2) we conclude that $Z_0^0 = \bar{Z}_0^0$. Thus, we have proven that along a subsequence, the controlled process converges to the skeleton equation.

5.1. Verification of Assumption (S1). Assumption (S1) follows by Theorem 4.2, and applying Theorem 5.1 with $\gamma = 0$.

5.2. Verification of Assumption (S2). Assumption (S2) follows by applying Theorem 5.1 with $\gamma(r) = r$, $r \in [0,1)$.

This concludes the proof of Theorem 1.1.

References

1. Bertini, L., Cancrini, N., and Jona-Lasinio, L., 1994 The Stochastic Burgers’ Equation. Commun. Math. Phys. 165, 211-232.
2. Boué, M., and Dupuis, P., 1998. A variational representation for positive functional of infinite dimensional Brownian motions. Ann. Probab. 26, 1641-1659.
3. Budhiraja, A., and Dupuis, P., 2000. A variational representation for positive functional of infinite dimensional Brownian motions. Probab., Math. Statist. 20, 39-61.
4. Budhiraja, A., Dupuis, P., and Maroulas, V., 2008. Large deviations for infinite dimensional stochastic dynamical systems. Ann. Probab. 36, 1390-1420.
5. Cardon-Weber, C., 1999. Large Deviations for a Burgers’-type SPDE. Stochastic Process. Appl. 84, 53-70.
6. Cerrai, S., and Debussche, A., Large deviations for two-dimensional stochastic Navier-Stokes equation with vanishing noise correlation. arXiv: 1603.02527.
7. Cerrai, S., and Debussche, A., Large deviations for the dynamic $\Phi^4_3$ model. arXiv: 1705.00541.
8. Da Prato, G., Debussche, A., and Temam, R., 1995. Stochastic Burgers’ Equation. Nonlinear Differential Equations and Applications. 1, 389-402.
9. Da Prato, G., and Gatarek, D., 1995. Stochastic Burgers’ equation with correlated noise, Stochastic Stochastics Rep. 52, 29-41.
10. Dupuis, P., and Ellis, R. S., 1997. A Weak Convergence Approach to the Theory of Large Deviations. John Wiley & Sons, New York.
11. Dupuis, P., and Ellis, R. S., 1997. A Weak Convergence Approach to the Theory of Large Deviations. John Wiley & Sons, New York.
12. Da Prato, G., and Zabczyk, J., 1992. Stochastic Equations in Infinite Dimensions. Cambridge Univ. Press, Cambridge, UK.
13. Eidelman, S.D., 1956. On the fundamental solutions of parabolic systems I. Mat. Sb. 38, 52-92 (in Russian)
14. Eidelman, S.D., 1961. On the fundamental solutions of parabolic systems II. Mat. Sb. 53, 73-136 (in Russian)
15. Friedman, A., 1964. Partial differential equations of parabolic type. Prentice-Hall, Englewood Cliffs, NJ.
16. Freidlin, M.I., Da Prato, G., and Wentzell, A.D., 1984. Random Perturbations of Dynamical Systems. Springer, New York.
17. Gyöngy, I., 1998. Existence and uniqueness results for semi-linear stochastic partial differential equations. Stochastic Process. Appl. 73, 271-299.
18. Gyöngy, I., and Rovira, C., 2000. On $L^p$-solutions of semilinear stochastic partial differential equations. Stochastic Process. Appl. 90, 83-108.
19. Gyöngy, I., and Nualart, D.: On the Stochastic Burgers’ Equation in the Real Line, Ann. Probab. 27 (1999) 782-802.
20. Hairer, M., A theory of regularity structures. Inventiones Mathematicae 198 (2014) 269-504.
21. Hairer, M., and Weber, H.: Large deviations for white-noise driven, nonlinear stochastic PDEs in two and three dimensions, Ann. Fac. Sci. Toulouse Math. 24 (2015) 55-92.
22. Ladyzhenskaya, O.A., Solonnikov, N.A., Ural’tseva, N.N., 1968. Linear and quasilinear equations of parabolic type. Transactions of Mathematical Monographs. Vol. 23. AMS, Providence, RI.
23. L. Setayeshgar., 2014. Large deviations for a stochastic Burgers’ equation. Communications on Stochastic Analysis (COSA). 8, 141-154.

Leila Setayeshgar: Department of Mathematics and Computer Science, Providence College, Providence, RI 02918 , USA
E-mail address: lsetayes@providence.edu