UPPER BOUNDS FOR INVERSE DOMINATION IN GRAPHS

ELLIOT KROP, JESSICA MCDONALD, AND GREGORY J. PULEO

Abstract. In any graph \( G \), the domination number \( \gamma(G) \) is at most the independence number \( \alpha(G) \). The Inverse Domination Conjecture says that, in any isolate-free \( G \), there exists pair of vertex-disjoint dominating sets \( D, D' \) with \( |D| = \gamma(G) \) and \( |D'| \leq \alpha(G) \). Here we prove that this statement is true if the upper bound \( \alpha(G) \) is replaced by \( \frac{3}{2} \alpha(G) - 1 \) (and \( G \) is not a clique). We also prove that the conjecture holds whenever \( \gamma(G) \leq 5 \) or \( |V(G)| \leq 16 \).

1. Introduction

In this paper all graphs are simple. A dominating set for a graph \( G \) is a set of vertices \( D \) such that every vertex of \( G \) either lies in \( D \) or has a neighbor in \( D \). The domination number of \( G \), written \( \gamma(G) \), is the size of a smallest dominating set in \( G \). Note that a maximum independent set is a dominating set, so \( \gamma(G) \leq \alpha(G) \), where \( \alpha(G) \) is the independence number of \( G \).

If a graph \( G \) has no isolates and \( D \) is a minimum dominating set in \( G \), then \( V(G) - D \) is also a dominating set in \( G \) (owing to the minimality of \( D \)); this was first observed by Ore [10]. In general we say that a dominating set \( D' \) is an inverse dominating set for a graph \( G \) if there is some minimum dominating set \( D \) such that \( D \cap D' = \emptyset \). A graph with isolates cannot have an inverse dominating set, but otherwise, given Ore’s observation, we can define the inverse domination number of a graph \( G \), written \( \gamma^{-1}(G) \), as the smallest size of an inverse dominating set in \( G \). The Inverse Domination Conjecture asserts that \( \gamma^{-1}(G) \leq \alpha(G) \) for every isolate-free \( G \).

The Inverse Domination Conjecture originated with Kulli and Sigarkanti [9], who in fact provided an erroneous proof. Discussion of this error and further consideration of the conjecture first appeared in a paper of Domke, Dunbar, and Markus [3]. It has since been shown by Driscoll and Krop [4] that the weaker bound of \( \gamma^{-1}(G) \leq 2 \alpha(G) \) holds in general, and Johnson, Prier and Walsh [7] showed that the conjecture itself holds whenever \( \gamma(G) \leq 4 \). Johnson and Walsh [8] have also proved two fractional analogs of the conjecture, and Frendrup, Henning, Randerath and Vestergaard [5] have shown that the conjecture holds for a number of special families, including bipartite graphs and claw-free graphs.

In this paper we prove two main results in support of the Inverse Domination Conjecture. The first is an improvement on the \( 2 \alpha(G) \) approximation to the conjecture.

Theorem 1.1. If \( G \) is a graph with no isolated vertices and \( G \) is not a clique, then \( \gamma^{-1}(G) \leq \frac{3}{2} \alpha(G) - 1 \).

The second author is supported in part by NSF grant DMS-1600551.
Note that if \( G \) is a clique and \( G \neq K_1 \), then trivially \( \gamma^{-1}(G) = \alpha(G) = 1 \), which is why we must exclude cliques in Theorem 1.1.

Our second main result improves the range of \( \gamma(G) \) for which the conjecture is known.

**Theorem 1.2.** If \( G \) is a graph with no isolated vertices and \( \gamma(G) \leq 5 \), then \( \gamma^{-1}(G) \leq \alpha(G) \).

As a corollary of Theorem 1.2 we are also able to obtain the following.

**Corollary 1.3.** If \( G \) is a graph with no isolated vertices and \( |V(G)| \leq 16 \), then \( \gamma^{-1}(G) \leq \alpha(G) \).

It is worth noting that Asplund, Chaffee, and Hammer [2] have formulated a stronger form of the Inverse Domination Conjecture. In the strengthened version one requires, for every minimum dominating set \( D \), the existence of a dominating set \( D' \) with \( D \cap D' = \emptyset \) and \( |D'| \leq \alpha(G) \). It is not hard to see that our proof for Theorem 1.1 also works for this stronger conjecture. However, the same is not true for Theorem 1.2 where we pick our minimum dominating set \( D \) very carefully.

The rest of the paper is organized as follows. In Section 2 we introduce the notion of an independent set of representatives, or ISR, and explore the connections between ISRs and inverse domination. (In this section, we also obtain, as a corollary, the inequality \( \gamma^{-1}(G) \leq b(G) \) for graphs without isolated vertices, where \( b(G) \) is the largest number of vertices in an induced bipartite subgraph of \( G \).) In Section 3 we prove Theorem 1.4. In Section 4 we leverage the machinery of Section 2 to prove Theorem 1.2 and Corollary 1.3.

## 2. ISRs and Inverse Domination

If \((X_1, \ldots, X_k)\) is a collection of sets, a set of representatives for \((X_1, \ldots, X_k)\) is a set \(\{x_1, \ldots, x_k\}\) such that \(x_i \in X_i\) for each \(i\). If \(G\) is a graph and \(V_1, \ldots, V_k\) are subsets of \(V(G)\), an independent set of representatives, or ISR, for \((V_1, \ldots, V_k)\) is a set of representatives for the sets \(V_1, \ldots, V_k\) that is also an independent set in \(G\). A partial ISR for \(V_1, \ldots, V_k\) is an ISR for any subfamily of \(V_1, \ldots, V_k\).

Several authors have proved various sufficient conditions guaranteeing the existence of ISRs; many of the proofs are topological in nature. See [11] for a collection of such results. A fundamental result on ISRs is the following sufficient condition due to Haxell [6]. In what follows, given a graph \(G\) and a set \(A \subseteq V(G)\), \(G[A]\) denotes the subgraph of \(G\) induced by \(A\). Given a collection of sets \((V_1, \ldots, V_k)\) and \(J \subseteq [k]\), we write \(V_J\) for the union \(\bigcup_{j \in J} V_j\).

**Theorem 2.1** (Haxell [6]). Let \(G\) be a graph and let \(V_1, \ldots, V_n\) be a partition of \(V(G)\). If, for all \(S \subseteq [n]\),

\[
\gamma(G[V_S]) \geq 2|S| - 1,
\]

then \(G\) has an independent set \(v_1, \ldots, v_n\) such that \(v_i \in V_i\) for each \(i\) (that is, \((V_1, \ldots, V_n)\) has an ISR).

Our basic idea for using Theorem 2.1 to obtain results on inverse domination is to apply it to a specific partition of vertices outside \(D\) (where \(D\) is a minimum dominating set), namely to what we’ll call a standard partition.

Let \(G\) be a graph and suppose that \(X, Y\) are disjoint sets of vertices where \(X\) dominates \(Y\). The standard partition of \(Y\), subject to a given ordering \((v_1, \ldots, v_n)\)
of $X$, is the partition $(V_1, \ldots, V_n)$ with
\[ V_i = N_Y(v_i) \setminus \bigcup_{j<i} V_j, \]
where $N_Y(v_i)$ indicates those neighbors of $v_i$ that are in $Y$. Consider a minimum dominating set $D$, and the standard partition of $V(G) - D$ with respect to any ordering of $D$. If this partition has an ISR, then the ISR is an independent set disjoint from $D$ that dominates $D$. Expanding this independent set to a maximal independent set in $G - D$ would give an independent dominating set disjoint from $D$, implying that $\gamma^{-1}(G) \leq \alpha(G)$. However, we cannot always find an ISR for a standard partition of $G - D$. Instead, we obtain more technical results.

In the following, given disjoint sets $X_1, \ldots, X_k$ and $S \subseteq X_1 \cup \cdots \cup X_k$, we write $i(S)$ for the set $\{j : S \cap X_j \neq \emptyset\}$. When $S = \{v\}$, we'll denote the unique element of $i(S)$ by $i(v)$.

**Theorem 2.2.** Let $G$ be a graph, let $D$ be a minimum dominating set in $G$, and let $F$ be a maximal independent set in $D$. Let $(d_1, \ldots, d_n)$ be any ordering of $D - F$, and let $(V_1, \ldots, V_n)$ be the standard partition of $G - D - N(F)$ subject to this ordering. Then there exist two partial ISRs $R_1, R_2$ of $(V_1, \ldots, V_n)$ such that $i(R_1) \cap i(R_2) = \emptyset$ and $i(R_1) \cup i(R_2) = [n]$.

**Proof.** Let $H$ be a graph consisting of two disjoint copies of $G - D - N(F)$, and let $W_1, \ldots, W_n$ be a partition of $V(H)$ obtained by letting each $W_i$ consist of both copies of each vertex in $V_i$.

We will use Theorem 2.1 to obtain an ISR of $(W_1, \ldots, W_n)$. Let $S$ be any subset of $[n]$, and let $H' = H[W_S]$. We will show that $\gamma(H') \geq 2|S|$.

Observe that $H'$ consists of two disjoint copies of the subgraph $G' := G[V_S]$, so that any dominating set in $H'$ must dominate each of those copies. If $\gamma(H') < 2|S|$, then let $C$ be a minimum dominating set of $H'$. We can partition $C$ into $C = C_1 \cup C_2$, where $C_1$ dominates one copy of $G'$ and $C_2$ dominates the other copy. Without loss of generality $|C_1| \leq |C_2|$, and since $|C| < 2|S|$, this implies $|C_1| < |S|$. Let $C' = C_1 \cup C_2$, and let $D' = \{d : i \in S\} \cup C'$. We know that $D'$ dominates $V(G) - D$, and moreover since $F \subseteq D'$ and $F$ dominates $D - F$, we see that $D'$ is a dominating set of $G$. Since $|D'| < |D|$, this contradicts the minimality of $D$.

Thus $(W_1, \ldots, W_n)$ has some ISR $R$. We can partition $R = R_1 \cup R_2$ where $R_1$ consists of the $R$-vertices in one copy of $G'$ and $R_2$ consists of the $R$-vertices in the other copy of $G'$. Now $R_1$ and $R_2$ are each independent subsets of $G'$, and since $R$ is an ISR we see that $i(R_1) \cap i(R_2) = \emptyset$ and $i(R_1) \cup i(R_2) = [n]$. \hfill \Box

As an immediate and useful corollary to Theorem 2.2 we get the following.

**Corollary 2.3.** Let $G$ be a graph, let $D$ be a minimum dominating set in $G$, and let $F$ be a maximal independent set in $D$. Let $(d_1, \ldots, d_n)$ be any ordering of $D - F$, and let $(V_1, \ldots, V_n)$ be the standard partition of $G - D - N(F)$ subject to this ordering. Then $(V_1, \ldots, V_n)$ has a partial ISR of size at least $n/2$.

Observe that if $D$ is a minimum dominating set in a graph $G$ without isolates, then each vertex in $D$ has a neighbor in $G - D$. These neighbors can be used to help build inverse dominating sets, and our first use of this will be in the following corollary.
Corollary 2.4. Let $G$ be a graph without isolated vertices and let $D$ be a minimum dominating set in $G$. If $b(G)$ is the largest number of vertices in an induced bipartite subgraph of $G$, then $\gamma^{-1}(D) \leq b(G)$.

Proof. Let $F$ be a maximal independent set in $D$, and let $R_1, R_2$ be partial ISRs as in Theorem 2.2. As $R_1$ and $R_2$ are each independent and $R_1 \cap R_2 = \emptyset$, $R_1 \cup R_2$ induces a bipartite subgraph of $G$. Since $i(R_1) \cup i(R_2) = [n]$, the set $R_1 \cup R_2$ dominates $D - F$. Expand $R_1 \cup R_2$ to a maximal set $B \subseteq G - D$ inducing a bipartite subgraph.

The maximality of $B$ implies that $B$ dominates $G - F$. Let $F_0 = F - N(B)$, so that $B$ dominates $G - F_0$. Observe that $B \cup F_0$ still induces a bipartite graph, so that $b(G) \geq |B| + |F_0|$. On the other hand, each vertex $v \in F_0$ has some neighbor $v' \in V(G) - D$. Augmenting $B$ by adding in such a vertex $v'$ for each $v \in F_0$ yields a inverse dominating set of size at most $|B| + |F_0|$, which is at most $b(G)$.

3. Proof of Theorem 1.1

Theorem 3.1. Let $G$ be a graph, and let $D$ be a minimum dominating set in $G$. There is a set $T \subset V(G) - D$ such that $T$ is a dominating set in $G$ and $|T| \leq \alpha(G) + \left\lfloor \frac{\gamma(G) - 1}{2} \right\rfloor$.

Proof. Let $F$ be a maximal independent set in $D$, and write $D - F$ as $\{d_1, \ldots, d_n\}$. Let $(V_1, \ldots, V_n)$ be the standard partition of $N(D - F)$.

Let $R$ be a largest possible partial ISR for $(V_1, \ldots, V_n)$. By Corollary 2.3, we have $|R| \geq n/2$. Expand $R$ to a maximal independent set $S$ in $G - D$. The set $S$ dominates every vertex of $V(G) - D$ and at least $n/2$ vertices of $D - F$. We now expand $S$ to dominate the rest of $D$.

Let $F' = F - N(S)$. Observe that $S \cup F'$ is an independent set, so $|S| + |F'| \leq \alpha(G)$. Expand $S$ to a set $S_1$ by adding an arbitrary $(G - D)$-neighbor of $v'$ for each $v' \in F'$; we have $|S_1| \leq |\alpha(G)|$. Next, expand $S_1$ to a set $T$ by adding an arbitrary $(G - D)$-neighbor of $w$ for each $w \in D - F - N(S_1)$; note that $|D - F - N(S_1)| \leq n/2$, so $|T| \leq \alpha(G) + n/2$. As $n \leq \gamma(G) - 1$ and $|T|$ is an integer, this implies that

$$|T| \leq \alpha(G) + \left\lfloor \frac{\gamma(G) - 1}{2} \right\rfloor.$$ 

Since $T$ is a dominating set in $G$, the theorem is proved.

The following lemma is more general than is necessary for proving Theorem 1.1, but stating it in this generality will be useful for later results.

Lemma 3.2. If a graph $G$ has a minimum dominating set $D$ and an independent set $S$ such that $S - D$ dominates $D - S$, then $\gamma^{-1}(G) \leq \alpha(G)$.

Proof. Let $S_1 = S - D$ and let $S_2 = S \cap D$. Expand $S_1$ to a maximal independent set $S_1'$ of $G - D$. Now $S_1'$ dominates $G - D$. Let $S_2'$ be the set of vertices in $D$ not dominated by $S_1'$. Observe that $S_2' \subseteq S_2$, since by hypothesis $S_1$ dominates $D - S_2$. Hence $S_1' \cup S_2'$ is an independent set, so that $\alpha(G) \geq |S_1'| + |S_2'|$.

Since $D$ is a minimum dominating set of $G$ and $G$ has no isolated vertices, each vertex of $D$ has a neighbor outside of $D$. Let $T$ be the vertex set obtained from $S_1'$ by adding in, for each $v \in S_2'$, a neighbor of $v$ outside of $D$. Now $T$ is a dominating set in $G$ and $|T| \leq |S_1'| + |S_2'| \leq \alpha(G)$. Hence $\gamma^{-1}(G) \leq \alpha(G)$.
The proof of Theorem 1.1 now follows easily. If \( G \) has a minimum dominating set \( D \) that is independent, then we can choose \( S = D \) to vacuously meet the hypothesis of Lemma 3.2 and hence \( \gamma^{-1}(G) \leq \alpha(G) \leq (3/2)\alpha(G) - 1 \). Otherwise, \( \gamma(G) \leq \alpha(G) - 1 \), so by Theorem 3.1 we have

\[
\gamma^{-1}(G) \leq \alpha(G) + \left\lfloor \frac{\alpha(G) - 1}{2} \right\rfloor \leq \alpha(G) + \left\lfloor \frac{\alpha(G) - 2}{2} \right\rfloor \leq \frac{3}{2} \alpha(G) - 1.
\]

4. Proof of Theorem 1.2

Our proof of Theorem 1.2 relies on a careful choice of minimum dominating set. For shorthand, it will be convenient to speak of the independence number of a dominating set \( D \) to refer to the independence number of the induced subgraph \( G[D] \), and likewise to write \( \alpha(D) \) for \( \alpha(G[D]) \). We will consider a dominating set \( D \) in a graph \( G \) to be optimal if it is of minimum size and, among minimum-size dominating sets, has greatest independence number and, subject to that, has the fewest edges in the induced subgraph \( G[D] \). In order to build inverse dominating sets in a graph \( G \), we previously used the fact that any vertex \( v \) in a minimum dominating set \( D \) has a neighbor in \( G - D \) (provided \( G \) is isolate-free). In some arguments, it is helpful if such a neighbor is private with respect to \( D \); that is, if we are able to choose \( w \in V(G) - D \) with \( N(w) \cap D = \{v\} \). In fact, the choice of a private neighbor for \( v \) is always possible when \( D \) is a minimum dominating set, unless \( v \) is isolated in \( G[D] \). The following lemma tells us that if \( D \) is optimal, we can improve on this.

**Lemma 4.1.** Let \( G \) be an isolate-free graph and let \( D \) be an optimal dominating set in \( G \). If \( v \in D \) is not an isolated vertex in \( G[D] \), then \( v \) has at least 2 private neighbors with respect to \( D \).

**Proof.** Let \( G_v \) be the subgraph of \( G \) induced by the private neighbors of \( v \). We in fact show \( \gamma(G_v) > 1 \). Suppose to the contrary that \( G_v \) has a dominating vertex \( w \). Let \( D' = (D - v) \cup \{w\} \). Every vertex of \( G - D' \) is either \( v \) itself, hence dominated by \( w \), or a private neighbor of \( v \), hence dominated by \( w \), or a vertex of \( G - D \) that is not a private neighbor of \( D \), hence dominated by \( D - v \). Thus, \( D' \) is a dominating set. Furthermore, as \( w \) was a private neighbor of \( v \), the vertex \( w \) is an isolated vertex in \( D' \). In particular, for any maximum independent set \( S \) in \( D \), we see that \( (S - v) \cup \{w\} \) is also a maximum independent set in \( D' \), so \( D' \) has at least as large an independence number as \( D \) did. As \( w \) is isolated in \( D' \) but \( v \) was not isolated in \( D \), we see that \( D' \) has fewer edges than \( D \), contradicting the optimality of \( D \). \( \square \)

**Lemma 4.2.** Let \( G \) be an isolate-free graph and let \( D \) be an optimal dominating set in \( G \). Suppose that the number of isolates in \( G[D] \) is \( a \). Then either \( G \) has an independent set \( S \) such that \( S - D \) dominates \( D - S \), or all of the following are true:

\[
\begin{align*}
(1) \quad a + 1 & \leq \alpha(D) \leq |D| - 3, \\
(2) \quad |V(G)| + a & \geq 3|D|, \text{ and} \\
(3) \quad |D| & \geq a + 5.
\end{align*}
\]

**Proof.** Assuming that \( G \) has no such independent set \( S \), we prove each part of the conclusion separately.

1. If \( D \) is an independent set, then taking \( S = D \) gives the desired independent set. Hence \( a + 1 \leq \alpha(D) \leq |D| - 1 \), and we may choose a vertex \( d^* \in D \) that is not
graph \( \gamma \) is a dominating set of \( G \) for graphs \( \gamma(G) \leq 5 \). In light of the following lemma, it will suffice to prove the conjecture for graphs with domination number exactly 5.

**Lemma 4.3.** Let \( k \) be a positive integer. If \( \gamma^{-1}(G) \leq \alpha(G) \) for every isolate-free graph \( G \) with \( \gamma(G) = k \), then \( \gamma^{-1}(G) \leq \alpha(G) \) for every isolate-free graph \( G \) with \( \gamma(G) \leq k \).
Proof. Let $G$ be an isolate-free graph with $\gamma(G) \leq k$, and let $t = k - \gamma(G)$. Let $G'$ be the disjoint union of $G$ and $t$ copies of $K_2$. Now $\gamma(G') = \gamma(G) + t = k$, so by hypothesis, $\gamma^{-1}(G') \leq \alpha(G') = \alpha(G) + t$. In particular, in $G'$ we can choose a minimum dominating set $D'$ and a second disjoint dominating set $T'$ with $|T'| \leq \alpha(G')$. Observe that $D'$ and $T'$ must each contain one vertex from every added copy of $K_2$. Hence, letting $D = D' \cap V(G)$ and $T = T' \cap V(G)$, we see that $|D| = |D'| - t = \gamma(G)$ and $|T| \leq \alpha(G') - t = \alpha(G)$. Furthermore, $D$ and $T$ are dominating sets in $G$. Hence, $\gamma^{-1}(G) \leq \alpha(G)$. \hfill $\Box$

We wish to strengthen the conclusion of Theorem 2.2 by eliminating the maximal independent set $F$ inside $D$, and instead finding a pair of ISRs that jointly dominate the entire minimum dominating set $D$. When $\gamma(G) = 5$ and $\alpha(D) \leq 2$, we are able to do this.

Lemma 4.4. Let $D$ be an optimal dominating set in an isolate-free graph $G$. Suppose that $|D| = 5$, that $\alpha(D) \leq 2$, and that $G[D]$ has no isolated vertices. Then there is an ordering $(d_1, \ldots, d_5)$ of $D$ and a pair of independent sets $R_1$ and $R_2$ such that $R_1$ is an ISR for $(V_1, V_2, V_3)$ and $R_2$ is an ISR for $(V_4, V_5)$, where $(V_1, \ldots, V_5)$ is the standard partition of $G - D$ with respect to this ordering.

Proof. Choose $d_1, d_2 \in D$ so that $\{d_1, d_2\}$ is an independent set, if possible. (Thus, $d_1, d_2 \in E(G)$ only if $D$ is a clique.) Note that since $\alpha(D) \leq 2$, the set $\{d_1, d_2\}$ contains a maximal independent set in $D$, hence dominates $D - \{d_1, d_2\}$. This implies that there are at least 3 edges from $\{d_1, d_2\}$ to the rest of $D$.

First we argue that there is an independent set $\{r_1, r_2\}$ with $r_i \in V_i$. If not, then $V_1$ and $V_2$ are joined by a complete bipartite graph. Let $v_1^* \in V_1$ and $v_2^* \in V_2$ be private neighbors of $d_1$ and $d_2$ respectively. Observe that $\{v_1^*, v_2^*\} \cup (D \setminus \{d_1, d_2\})$ is a dominating set of $D$. Furthermore, there are no edges between $\{v_1^*, v_2^*\}$ and $D \setminus \{d_1, d_2\}$. This implies that $|E(D')| \leq |E(D)| - 2$, contradicting the optimality of $D$. (Note that $\alpha(D') \geq \alpha(D)$ since $\alpha(D) \leq 2$.)

Now, since $D$ is a minimal dominating set of $G$, there is some vertex $r_3 \in V(G)$ not dominated by $\{d_1, d_2, r_1, r_2\}$. As $\{d_1, d_2\}$ dominates $D$, we have $r_3 \in V(G) - D$. Choose $d_3$ to be a neighbor of $r_3$ in $D$. Let $R_1 = \{r_1, r_2, r_3\}$, and let $d_4$ and $d_5$ be the remaining vertices of $D$, ordered arbitrarily. Observe that $R_1$ is an ISR for $(V_1, V_2, V_3)$ in the standard partition of $V(G) - D$ with respect to this ordering. It remains to find the desired $R_2$.

We claim that there are nonadjacent vertices $r_4, r_5$ each with $r_i \in V_i$. If not, then $V_4$ and $V_5$ are joined by a complete bipartite graph. Let $v_4^* \in V_4$ and $v_5^* \in V_5$ be private neighbors of $d_4$ and $d_5$ respectively. Now $D' = \{d_1, d_2, d_3, v_4^*, v_5^*\}$ is a dominating set in $D$. Furthermore, since $\{d_1, d_2\}$ is a dominating set in $D$, there are at least two edges in the cut $\{(d_1, d_2, d_3), \{d_4, d_5\}\}$, while by contrast there are no edges joining $\{v_4^*, v_5^*\}$ with $\{d_1, d_2, d_3\}$. Hence $|E(D')| \leq |E(D)| - 2$, contradicting the optimality of $D$. (Again $\alpha(D') \geq \alpha(D)$ since $\alpha(D) \leq 2$.) \hfill $\Box$

Theorem 4.5. If $G$ is an isolate-free graph with $\gamma(G) = 5$, then $G$ has a minimum dominating set $D$ such that $\gamma^{-1}(D) \leq \alpha(G)$.

Proof. Let $D$ be an optimal dominating set in $G$. By Lemma 2.2 and by parts (1) and (3) of Lemma 4.2, we may assume that $\alpha(D) \leq 2$ and that $D$ has no isolated vertices. In particular, since $D$ is not an independent set, we have $\alpha(G) \geq 6$, a fact we will use later.
By Lemma 4.2, we see that there is an ordering \((d_1, \ldots, d_5)\) of \(D\) and a pair of independent sets \(R_1, R_2\) such that \(R_1\) is an ISR for \((V_1, V_2, V_3)\) and \(R_2\) is an ISR for \((V_4, V_5)\). \(R_1, R_2\) is the standard partition of \(G - D\) for the given ordering. Among all such pairs \((R_1, R_2)\), choose \(R_1\) and \(R_2\) to minimize the number of edges from \(R_1\) to \(R_2\).

If \((V_1, \ldots, V_5)\) has a partial ISR of size 4, then we immediately get the desired conclusion: taking \(R\) to be such an ISR, we see that \(R\) dominates all of \(D\) except possibly for a single vertex \(w \in D\), so we win by letting \(S = R \cup \{w\}\) (or \(S = R\)) and applying Lemma 3.2.

Thus, \((V_1, \ldots, V_5)\) has no partial ISR of size 4, which implies that \(R_1\) is a maximal partial ISR of this family, and so \(R_1\) dominates \(V_4 \cup V_5\).

Let \(T\) be the set of vertices in \(G\) that are not dominated by \(R_1 \cup R_2\). If \(T = \emptyset\) then we immediately have the desired conclusion, as \(R_1 \cup R_2\) is an inverse dominating set of size \(\gamma\). Thus we may assume that \(T\) is a nonempty subset of \(V(G) - D\), and in particular, \(T \subseteq V_1 \cup V_2 \cup V_3\).

Write \(R_1 = \{r_1, r_2, r_3\}\) with \(r_i \in V_i\). We claim that if \(T\) intersects \(V_j\) for some \(j \in \{1, 2, 3\}\), then the corresponding vertex \(r_j\) is not adjacent to any vertex of \(R_2\). Otherwise, let \(r'_j \in T \cap V_j\), and let \(R'_1 = (R_1 \setminus \{r_j\}) \cup \{r'_j\}\). Now \(R'_1\) is an ISR of \((V_1, V_2, V_3)\), and since \(r'_j\) is not dominated by \(R_1 \cup R_2\), there are fewer edges between \(R'_1\) and \(R_2\) than there were between \(R_1\) and \(R_2\). This contradicts the choice of \(R_1 \cup R_2\), establishing the claim.

In particular, the above claim implies that \(|i(T)| = 1\), since if \(|i(T)| \geq 2\), then taking distinct \(j, k \in i(T)\), we see that \(R_2 \cup \{r_j, r_k\}\) is a partial ISR of \((V_1, \ldots, V_5)\) having size 4, contradicting our earlier claim that the largest such partial ISR has size 3.

Let \(k\) be the unique index in \(i(T)\). Let \(R^* = (R_1 \cup R_2) \setminus \{r_k\}\). We next claim that any vertex of \(\bigcup_{j \neq k} V_j\) not dominated by \(R^*\) is adjacent to all of \(T\). Otherwise, let \(v_j\) be such a vertex that is not adjacent to all of \(T\), with \(v_j \in V_j\).

Let \(v_k\) be a vertex of \(T\) not adjacent to \(v_j\). If \(j \in \{1, 2, 3\}\), then let \(R'_2 = R_2 \cup \{v_j, v_k\}\). Now \(R'_2\) is an independent set, since \(R_2 \subset R^*\) and neither \(v_j\) nor \(v_k\) is dominated by \(R^*\) (by choice of \(v_j\) and because \(v_k \in T\)). As \(i(R_2) = \{4, 5\}\) this implies that \(R'_2\) is a partial ISR of \((V_1, \ldots, V_5)\) having size 4, contradicting the earlier claim that the largest such ISR has size 3. If instead \(j \in \{4, 5\}\), then taking \(R'_1 = (R_1 \setminus \{r_k\}) \cup \{v_j, v_k\}\) gives the same contradiction.

Hence, any vertex of \(\bigcup_{j \neq k} V_j\) not dominated by \(R^*\) is adjacent to all of \(T\). If there is any vertex of \(\bigcup_{j \neq k} V_j\) not dominated by \(R^*\), then let \(w\) be such a vertex; now \(R_1 \cup R_2 \cup \{w\}\) is an inverse dominating set of size 6, where \(\alpha(G) \geq 6\), and we are done. Hence, we may assume that \(R^*\) dominates \(\bigcup_{j \neq k} V_j\).

In this case, let \(D' = R^* \cup \{d_k\}\). Since \(R^*\) dominates \(\bigcup_{j \neq k} V_j\), we see that \(D'\) is a dominating set of \(G\). Since \(k \leq 3\), the set \(\{d_k, r_4, r_5\}\) is an independent set: if \(d_k\) were adjacent to \(r_4\), this would imply \(r_4 \in V_k\), contradicting \(r_4 \in V_4\), and likewise for \(r_5\). This contradicts the optimality of \(D\).

**Corollary 4.6.** If \(|V(G)| \leq 16\) then \(\gamma^{-1}(G) \leq \alpha(G)\).

**Proof.** Let \(G\) be some graph with \(\gamma^{-1}(G) > \alpha(G)\), and let \(D\) be an optimal dominating set in \(G\). Let \(\alpha\) be the number of isolated vertices in \(G[D]\). By Lemma 3.2 there cannot be any independent set \(S\) such that \(S - D\) dominates \(D - S\), so by Lemma 4.2 we have:
By Theorem 4.5 and Lemma 4.3 we have $\gamma(G) \geq 6$, so that $|D| \geq 6$. If $a = 0$ then (2) yields $|V(G)| \geq 18$. Otherwise, $a \geq 1$, and then (2) combined with (3) yields $|V(G)| \geq 2a + 15 \geq 17$. \qed

References

1. Ron Aharoni, Eli Berger, and Ran Ziv, Independent systems of representatives in weighted graphs, Combinatorica 27 (2007), no. 3, 253–267. MR 2345810
2. J. Asplund, J. Chaffee, and J.M. Hammer, Some inverse domination results and extensions, Manuscript.
3. Gayla S. Domke, Jean E. Dunbar, and Lisa R. Markus, The inverse domination number of a graph, Ars Combin. 72 (2004), 149–160. MR 2069054
4. K. Driscoll and E. Krop, Some results related to the Kulli-Sigarkanti conjecture, Congressus Numerantium (accepted).
5. Allan Frendrup, Michael A. Henning, Bert Randerath, and Preben Dahl Vestergaard, On a conjecture about inverse domination in graphs, Ars Combin. 97A (2010), 129–143. MR 2683740
6. PE Haxell, A note on vertex list colouring, Combinatorics, Probability & Computing 10 (2001), no. 4, 345.
7. P. D. Johnson, Jr., D. R. Prier, and M. Walsh, On a problem of Domke, Dunbar, Haynes, Hedetniemi, and Markus concerning the inverse domination number, AKCE Int. J. Graphs Comb. 7 (2010), no. 2, 217–222. MR 2768345
8. P. D. Johnson, Jr. and M. Walsh, Fractional inverse and inverse fractional domination, Ars Combin. 87 (2008), 13–21. MR 2414002
9. V. R. Kulli and S. C. Sigarkanti, Inverse domination in graphs, Nat. Acad. Sci. Lett. 14 (1991), no. 12, 473–475. MR 1181626
10. Oystein Ore, Theory of graphs, American Mathematical Society Colloquium Publications, Vol. XXXVIII, American Mathematical Society, Providence, R.I., 1962. MR 0150753

(Jessica McDonald, Gregory J. Puleo) DEPARTMENT OF MATHEMATICS AND STATISTICS, AUBURN UNIVERSITY, AUBURN, ALABAMA, USA 36849
E-mail address, Jessica McDonald: mcdonald@auburn.edu
E-mail address, Gregory J. Puleo: gjp0007@auburn.edu

(Elliot Krop) DEPARTMENT OF MATHEMATICS, CLAYTON STATE UNIVERSITY, MORROW, GEORGIA, USA 30260
E-mail address, Elliot Krop: elliotkrop@clayton.edu