A large distance propagation in turbulent atmosphere results in disintegration of laser beam into speckles. We find that the most intense speckle approximately preserves both the Gaussian shape and the diameter of the initial collimated beam while loosing energy during propagation. One per 1000 of atmospheric realizations produces at 7km distance an intense speckle above 20% of the initial power. Such optimal realizations create effective extended lenses focusing the intense speckle beyond the diffraction limit of vacuum propagation. Atmospheric realizations change every several milliseconds. We propose to use intense speckles to greatly increase the time-averaged power delivery to the target plane by triggering the pulsed laser operations only at times of optimal realizations. Resulting power delivery and laser irradiance at the intense speckles well exceeds both intensity of diffraction-limited beam and intensity averaged over typical realizations.

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**Fig. 1.** (Color online) Distribution of laser irradiance $I$ in transverse plane (only 69x69 cm central part of target screen is shown) after $L = 7$ km propagation of the collimated Gaussian laser beam with the waist $w_0 = 1.5$ cm and the maximal intensity $I_{\text{max}} = 1$ through the turbulent atmosphere in the strong scintillation regime with $\sigma_I = 3.3$. Left panel: a typical atmospheric turbulence realization with $I_{\text{max}} = 0.04$ (61% of atmospheric realizations produce higher $I_{\text{max}}$). Right panel: a rare realization with $I_{\text{max}} = 0.19$ (0.16% realizations produce higher $I_{\text{max}}$). Dashed circles show $w_0$ and the waist of diffraction limited beam propagated in vacuum. The initial Gaussian beam disintegrates into several speckles with the width of the most intense speckle being about $w_0$. The intense speckles on left and right panels carry 4% and 19% of the total laser power, respectively.

Laser beam propagation though turbulent atmosphere results in disintegration of laser beam into speckles at the distances exceeding several kilometers (strong irradiance fluctuation regime) see Fig. 1 with examples of such propagation. At smaller distances (weak irradiance fluctuation regime) classic perturbative approaches well describe modification of laser beam propagation due to turbulence [2, 3], while statistically averaged beam propagation in strong scintillation regimes is addressed through semi-heuristic theory [4]. The strength of the fluctuations of the irradiance $I$ (laser beam intensity) at the target plane is characterized by the scintillation index $\sigma_I \equiv \langle I^2 \rangle / \langle I \rangle^2 - 1$. Here and below $\langle \ldots \rangle$ we denote an the average over the ensemble of atmospheric turbulence realizations. It was shown in Ref. [5] that a significant fraction of deviation between theoretical value of $\sigma_I$ [4] and simulations is due to rare large fluctuations of laser beam intensity. Here we study the structure of large fluctuations and propose to use them for the efficient delivery of laser energy over long distances by triggering the pulse laser operations only during the times of such rare fluctuations. Rear fluctuations which carry $\gtrsim 19\%$ of initial power, as in Fig. 1, occur in 0.16% realizations, and 0.1% realizations carry $\gtrsim 21\%$ of initial power. A temporal rate of change in atmospheric realizations is affected by atmosphere conditions. In typical conditions new atmospheric realization could occur each $\sim 10$ms [6]. Thus waiting for the optimal realization might take several seconds.

A propagation of a monochromatic beam with a single polarization through the turbulent media is described by the linear Schrödinger equation (LSE) (see e.g. [2, 3]):

$$i \frac{\partial}{\partial z} \psi + \frac{1}{2k} \nabla_2^2 \psi + k n_1(r, z) \psi = 0.$$  \hfill (1)
tric field, $\nabla_\perp \equiv \left( \frac{\partial }{\partial x}, \frac{\partial }{\partial y} \right)$, $k = 2\pi n_0 / \lambda_0$ is the wavenumber in medium, $\lambda_0$ is the wavelength in the vacuum, $n = n_0 + n_1$ is the linear index of refraction with the average value $n_0 = \langle n \rangle$ and the fluctuation $n_1(r, z, t)$. Time $t$ does not explicitly enters Eq. (1), thus serving as parameter distinguishing different atmospheric realizations so below we omit $t$ in arguments of all functions.

Linear absorption (results in exponential decay of laser intensity with propagation distance) is straightforward to include into Eq. (1). Kerr nonlinearity can be also added to Eq. (1) resulting in nonlinear Schrödinger Eq. which describes the catastrophic self-focusing (collapse) for laser powers $P$ above critical power $P_c$ ($P_c \sim 3$ GW for $\lambda_0 = 1064$nm) [8] and multiple filamentation for $P \gg P_c$ [9]. At distances well below the nonlinear length, one can consider Kerr nonlinearity as perturbation (see e.g. Ref. [10]) combining it with the effect of atmospheric turbulence. Nonlinear beam combining in atmosphere can be also considered to fight with turbulence [11][12]. Such nonlinear analysis is however beyond the scope of this Letter.

We solve Eq. (1) by the standard method of random phase screens [3] which is based on the approximation of statistically independent optical pulse phase fluctuations at each screen [3]. This method is a version of split-step numerical method [13, 14] which separates Eq. (1) into the exactly solvable refraction, $\partial_\psi \psi'' = i(k_1)(r, z)\psi^R$, and diffraction, $\partial_\psi \psi'' = i(k_2)(r, z)\psi^D$, parts. The exact solutions at the distance $\Delta z$ are given by $\psi^R(r, z + \Delta z) = \psi^R(r, z)exp(i\Delta z)$ and $\psi^D_{k_1}(z + \Delta z) = \psi^D_{k_1}(z)(-i\frac{\Delta z}{k_2}\Delta z)$, respectively. Here $S \equiv \int k^2 + \Delta z n_1(r, z')d^2z'$ is the phase shift and $\psi^D_{k_1} = (2\pi)^{-2} \int \psi(r, z)e^{-i(r\cdot k_1 + \chi z)}d^2r$ is the Fourier transform (FT) for the transverse wavevector $k_\perp = (k_x, k_y)$. Sequential combining both solutions at each step $\Delta z$ (requires performing both FT and inverse FT), while decreasing $\Delta z$ ensures convergence to the solution of Eq. (1).

The method of random phase screen approximates FT of the phase shift at the refraction step as $\hat{S}_{k_1} = \hat{\xi}_{k_1}k\sqrt{2\pi \hat{\Phi}_{k_1, \kappa=0} \Delta z}$, where $\hat{\Phi}_{k_1, \kappa=0} = \frac{2\pi}{\alpha} \int D(r, z)e^{-i(r\cdot k_\perp + \chi z)}d^2r$ is the FT over $\rho \equiv (r, z)$ of the structure function, $D(\rho) \equiv \langle n_1(r, z) - n_1(0, 0) \rangle^2$, evaluated at the zero component $\kappa = 0$ of the wavevector in $z$ direction [15]. The Kolmogorov-Obukhov law $D(\rho) \approx C_n^2 \rho^{5/3}$ is valid for the atmospheric turbulence (at $l_0 \ll \rho \ll L_0$) which implies $\hat{\Phi}_{k_1, \kappa=0} = 0.033 C_n^2 k_1^{-11/3}, \left| k_\perp \right| = k_1$. Here $l_0$ is the inner scale of turbulence, typically a few mm, and $L_0$ is the outer scale typically ranging from hundreds meters to kilometers. The modification of $\hat{\Phi}_{k_1, \kappa=0}$ for both $k_1 \gtrsim 2\pi / L_0$ and $k_1 \lesssim 2\pi / L_0$ is straightforward to implement [15][15]. We found in agreement with Ref. [5], that the simplest numerical cutoff described below does not affect the results of simulation for our range of parameters. These parameters include the size of the square computational domain (the transverse screen size) $L = L_x = L_y = 276.5$ cm with the uniformly distributed $N \times N$ points in that domain and $N = 1024$. It implies that $-\pi N / L \leq k_x \left( k_y \right) \leq \pi N / L$ which defines the upper cut-off in $k_x$ variable, while the elementary step $\Delta k = 2\pi / L$ of the numerical grid $\mathbf{k}_\perp \equiv \Delta k(j_x, j_y)$, $-N/2 \leq j_x(j_y) \leq N/2$ in $k_\perp$ determines the lower cut-off. Also $\mathbf{k}_\perp$ are the uncorrelated complex Gaussian random variables on the grid $\mathbf{k}_\perp$, such that $\langle \hat{\xi}_{k_1} \rangle = \langle \hat{\xi}_{k_1}^\ast \hat{\xi}_{k_1} \rangle = 0$ for $j_1 \neq j_2$ and $\langle \hat{\xi}_{k_1} \hat{\xi}_{j_1} \rangle = (\Delta k)^2$. Here $\hat{\xi}$ means complex conjugation and the real values of $S$ are ensured by the condition $\xi_{k_1} = \xi_{k_1}^\ast$. This numerical method is similar to Ref. [3], except that Ref. [3] used top-hat probability density function (PDF) for $\hat{\xi}_{k_1}$ instead of Gaussian PDF. We also verified that top-hat PDF produces essentially the same results (nearly visually indistinguishable on the plots below) in comparison with Gaussian PDF which is expected from the central limit theorem [16] for $N \gg 1$.

Physical parameters for our simulations are $\lambda = 1.064$um, the propagation distance $z_{\text{final}} = 7$ km with $\Delta z = 500$m, $C_n^2 = 10^{-14} \text{m}^{-2/3} = 4.64 \times 10^{-16} \text{cm}^{-2/3}$ and a collimated input Gaussian laser beam $\psi(r, 0) \equiv \exp(-r^2 / w_0^2)$ with the waist $w_0 = 1.5$ cm of unit intensity. Examples of simulations are shown in Fig. [4]. The size of ensemble is typically $4 \cdot 10^4$ atmospheric realizations. The averaged maximum of irradiance $I_{\text{max}} = 5.10433 \cdot 10^{-2}$ (here $l_{\text{max}}$ is the maximum intensity in the target plane) and the averaged irradiance $I_{\text{center}} = 2.86788 \cdot 10^{-3}$ at the center of the target plane with $z_{\text{final}} = 7$ km. Increase of either $C_n^2$ or $w_0$ requires decrease of $\Delta z$ to keep high numerical precision.

It was shown in Ref. [5] that the accurate calculation (or measurement from experiment) of $\sigma_r$ requires the ensemble of $\gtrsim 10^5$ realizations (because of giant fluctuations of laser intensity) which is unpractical because atmospheric conditions are usually not stationary at the timescale required for measurements of such large ensembles (hours), i.e. the time dependence of $C_n$ becomes essential. We argue that in this case $\sigma_r$ turns to be of limited usefulness because it assumes the approximation of stationary stochastic process which is not valid due to the time dependence of $C_n$. Instead, we focus on the study of individual large fluctuations of laser intensity which qualitatively could be interpreted as looking into optimal realizations of atmospheric turbulence through the optimal fluctuation theory. That idea was pioneered in Ref. [17] for condensed matter, reinvented in field theory in Ref. [18] and found in many applications ranging from fluid turbulence to [19][21] to nonlinear optics [9][22][23].

We identified that the optimal laser fluctuation at large propagation distance $z \gtrsim 3$ km is reasonably well approximated by the Gaussian beam (we called it optimal beam (OB) below) in the general approximate form $\psi_{\text{optimal}}(r, z) = I_{\text{max}}(z)^{1/2} \exp(-|r - r_0(z)|^2 / w(z)^2 + i|z - r_0(z)|^2 \alpha(z) + i k_0(z) \cdot r + i \phi(z))$. Here a maxi-
Fig. 2. (Color online) Averaged intensity profiles $I_{\text{avg}}(r)$ of OB centered at the location $r = r_0$ of intensity maximum, $I(r = r_0) = I_{\text{max}}$, for propagation to $z = z_{\text{final}} = 3.5\,\text{km}$ and $7\,\text{km}$. Each solid line is normalized to 1 at maximum and represents averaging over angles and over 50 OB realizations with the same final value of $I_{\text{max}}(z_{\text{final}})$. The final value is chosen within the corresponding bins (the bin width is 0.1 of $I_{\text{max}}$) of the histograms for $I_{\text{max}}(z_{\text{final}})$ realizations in the insets. The bins are selected near the maximum of PDF of $I_{\text{max}}(z_{\text{final}})$, in the tail and the far tail. Total number of realizations in each histogram is $4 \times 10^5$ (the total number of realizations in each beam is listed in the inset). The standard deviation of OB profiles are shown by corresponding short-dashed lines. Dotted line represents the initial Gaussian beam and thick dashed line shows $I = 1/\cosh(2|r - r_{\text{max}}|/w_0)$ for comparison. Typically, the waist of OB is $\sim 20\%$ narrower than $w_0$. Similar results were also obtained in simulations with $w_0 = 3\,\text{cm}$.

At larger distances, $z \gtrsim 3\,\text{km}$, the beam disintegrates into speckles with the most intense speckle forming OB. At larger distances, $z \gtrsim 3\,\text{km}$, OB amplitude fluctuates about approximately $z$-independent value $I_{\text{max}}(z)$, as seen in Fig. 3. According to Fig. 2 OB waist fluctuates about $w_0$, so the optical power in OB is also approximately constant at $z \gtrsim 3\,\text{km}$. Qualitatively we interpret this behavior as random multiple focusing-defocusing events of OB at random screens which compensate the diffraction in average.

To explain why the intensity profile of OB is close to Gaussian we recall that each random phase screen modifies $\psi$ into $\psi e^{i S(r)}$. Neglecting the effect of small scale fluctuations of $S(r)$ on OB dynamics, we expand $S(r)$ near the center of OB into Taylor series as $S(r) = S(r_0) + (r - r_0) \cdot \nabla S(r_0) + \sum_{l,m=1}^2 (1/2)(x_l - x_{l,0})(x_m - x_{m,0})\nabla_l \nabla_m S(r_0) + O(|r - r_0|^3)$, where $(x_1, x_2) \equiv (x, y)$ and $\nabla_l \equiv \partial/\partial_{x_l}$. Each derivative of $S$ is the Gaussian random variable. Then the linear term $(r - r_0) \cdot \nabla S(r_0)$ ensures a small random reorientation of OB about $z$-direction at each phase screen. The quadratic form $\sum_{l,m=1}^2 (1/2)(x_l - x_{l,0})(x_m - x_{m,0})\nabla_l \nabla_m S(r_0)$ can be diagonalized by the linear transform of $r - r_0$ and is responsible for the change of curvature of OB front. Both linear and quadratic terms can be qualitatively interpreted as multiple thin lenses located in the plane of each phase screen as schematically shown in Fig. 4. Linear terms are responsible for the random shift of the center of lenses in transverse plane or small tilt with respect.
to the transverse plane. Either mechanism results in the random change of slopes and wander of OB in transverse direction as shown in Fig. 3a and Fig. 3. The quadratic terms can be interpreted as the action of multiple small random change of slopes and wander of OB in transverse to the transverse plane. Either mechanism results in the approximate statistical steady state for $z \gtrsim 3$ km. On these propagation distances, the small fraction of power lost from OB to higher Hermite-Gaussian modes at each random screen is approximately compensated by the power returned to OB from surrounding non-small ripples. This effect is however small between neighboring screens, which explains why OB needs to be close to Gaussian form with $w \approx w_0$.

In conclusion, we found that OB carry $\gtrsim 21\%$ of initial power in 1 per 1000 realizations. One can identify optimal atmospheric realizations by a lower power laser beam which continuously illuminates target plane. When target camera/telescope detects an optimal realization on the target, it triggers the pulse operation of the high-power laser. The typical transverse displacement of OB is $\sim 10$ cm as seen in Fig. 3. If higher precision of OB location is required, for instance for space-debris cleaning [11], then one, in addition, can continuously scan a lower power laser beam over angles to find optimal realization for transverse OB location.

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