A Divergence Median-based Geometric Detector with A Weighted Averaging Filter

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Abstract. To overcome the performance degradation of the classical fast Fourier transform (FFT)-based constant false alarm rate detector with the limited sample data, a divergence median-based geometric detector on the Riemannian manifold of Hermitian positive definite matrices is proposed in this paper. In particular, an autocorrelation matrix is used to model the correlation of sample data. This method of the modeling can avoid the poor Doppler resolution as well as the energy spread of the Doppler filter banks result from the FFT. Moreover, a weighted averaging filter, conceived from the philosophy of the bilateral filtering in image denoising, is proposed and combined within the geometric detection framework. As the weighted averaging filter acts as the clutter suppression, the performance of the geometric detector is improved. Numerical experiments are given to validate the effectiveness of our proposed method.

1. Introduction
Improving the performance of a detector in interference environment is an important subject in the field of signal processing. The classical fast Fourier transform (FFT) based constant false alarm rate (CFAR) detector [1] uses the FFT to obtain the correlation of sample data. Usually, the performance suffers from severe degradation with the limited sample data as the Doppler resolution is poor and the energy of Doppler filter banks spread. To overcome these drawbacks, F. Barbaresco has studied the structure of space of Hermitian positive-definite (HPD) matrix, and has proposed a generalized CFAR technique on a Riemannian manifold of Hermitian positive-definite (HPD) matrices, which was referred to as the Riemannian median-based geometric detector [2]. In the Riemannian median-based geometric detector, the radar received clutter data in each range cell in one CPI is modeled as an HPD matrix. In addition, the Riemannian median detector [3] are derived. This geometric detector has been used to monitor the turbulence of a plane [4], and target detection in coastal X-band and HF surface wave radars [5].

Many metrics can be used to measure the closeness between any two points on the Riemannian manifold of HPD matrices. Different measurements can reflect different structures of this space. Many divergences can be used as measurements. Mentioned a few, the square loss is used to measure the distance between the two states in the regression; the Bhattacharyya divergence has employed to medical image segmentation [6]; and the Kullback-Leibler (KL) divergence has been widely used to measure the information difference between two probability distributions [7]. These metrics have achieved good results in many applications. In our previous work [8], we have studied a geometric
detection method based on KL divergence. Experiments have shown that its performance outperforms the traditional FFT-CFAR detector.

In this paper, we study the geometric detector based on different metrics. In particular, the Log-Euclidean distance [9], the Bhattacharyya divergence [7], and the Hellinger distance [10] are used as replacements of the Riemannian distance in the geometric detector. Based on the three metrics, the Log-Euclidean median [9], the Bhattacharyya median [7], and the Hellinger median of a finite set of HPD matrices are derived. As a result, a divergence median-based geometric detector is developed. Moreover, we propose a weighted average filter which is combined within the geometric detector. This filter acts as a clutter suppression procedure, the detection performance can be improved.

2. Signal model and signal manifold

The radar usually sends several pulses to a moving target, and receives the return data which contains the phase information of this target. A certain model is used to capture the Doppler of target. In this paper, the Doppler is represented as the correlation of data $z = \{z_1, z_2, \ldots, z_n\}$, and model as a multivariate Gaussian process with zero mean, $z \sim \mathcal{CN}(\mathbf{0}, \mathbf{R})$ [2],

$$p(z|\mathbf{R}) = \frac{1}{\pi^* |\det(\mathbf{R})|} \exp\left\{-\frac{1}{2} z^\top \mathbf{R}^{-1} z\right\}$$

here, the matrix $\mathbf{R}$ is an HPD matrix, and it can be computed as [2],

$$\mathbf{R} = E \left[\mathbf{z} \mathbf{z}^\top\right] = \begin{bmatrix} r_0 & \overline{r}_1 & \cdots & \overline{r}_{n-1} \\ r_1 & r_0 & \cdots & \overline{r}_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n-1} & \cdots & \cdots & r_0 \end{bmatrix}, \quad r_k = E \left[z_k \overline{z}_{i+k}\right], \quad 0 \leq k \leq n-1, \quad 0 \leq i \leq n-1$$

where $r_k$ denotes the correlation coefficient of pulse data, and $\overline{r}_i$ is the complex conjugate of $r_i$. As there is not enough sample data to compute the statistical expectation $E \left[z_k \overline{z}_{i+k}\right]$, according to the ergodicity, it can be calculated by a finite time serial,

$$\hat{r}_k = \frac{1}{n-k} \sum_{j=0}^{n-k} z_j \overline{z}_{j+k}, \quad 0 \leq k \leq n-1$$

The pulse data $z = \{z_1, z_2, \ldots, z_n\}$ in each cell in one CPI is modeled by Equation (1) and (2), and represented by an HPD matrix $\mathbf{R}$. The pulse data $z$ is staying in the Euclidean space, and the HPD matrix $\mathbf{R}$ is viewed as a point in the manifold. Through this parameterization, the data $z$ is transformed into an $n$ dimensional non-liner manifold space,

$$\psi : \mathbb{C}^n \rightarrow P(n), \quad z \rightarrow \mathbf{R} \in P(n)$$

Here $P(n)$ forms a differentiable Riemannian manifold with non-positive curvature. Through this modelling for the radar echo, the target detection should be performed in the manifold. In particular, the structure of matrix space can be considered.

3. Divergence measures and divergence medians

In this Section, a brief description about the divergence measure and the divergence median are given in the following.

3.1. Divergence measures on the Riemannian manifold of HPD matrices

For any two points $\mathbf{R}_1, \mathbf{R}_2$ on the HPD matrix space $P(n)$, a natural distance which is the Riemannian distance can be given by,

$$d^\mathcal{R}_n(\mathbf{R}_1, \mathbf{R}_2) = \left\| \log_{\mathcal{R}} \left(\mathbf{R}_1^{-\mathcal{R}} \mathbf{R}_2 \mathbf{R}_1^{-1}\mathcal{R}\right) \right\|_F = \sum_{k=1}^{n} \log^2 (\lambda_k)$$
where \( \lambda_k \) is the \( k \)th eigenvalue of \( R_1^{1/2} R_2 R_1^{1/2} \), and \( \logm(\cdot) \) is the logarithmic map on the Riemann manifold.

In addition to the Riemannian distance, many divergences can be used to measure the dissimilarity between two HPD matrices. In the following, three geometric measures are presented in Table 1.

| Divergence measures | Formulation |
|---------------------|-------------|
| Log-Euclidean [9]   | \( d_{LE}(R_1, R_2) = \left\| \logm(R_1) - \logm(R_2) \right\|_F \) |
| Bhattacharyya [7]   | \( d_{bh}(R_1, R_2) = 2 \log \left( \frac{R_1 + R_2}{\sqrt{|R_1| |R_2|}} \right) \), \( R_1, R_2 \in \mathcal{P}(n) \) |
| Hellinger [10]      | \( d_H(R_1, R_2) = \sqrt{2 - 2 \left( |R_1|^{1/4} |R_2|^{1/4} \left( \frac{1}{2} (R_1 + R_2) \right)^{1/4} \right) \)}, \( R_1, R_2 \in \mathcal{P}(n) \) |

### 3.2. Divergence medians for a finite set of HPD matrices

The divergence median has not received much attention in the research field. Based on the divergence metric, many medians have been derived. Here, four divergence-based medians are listed in Table 2.

| Divergence median | Iteration formulation |
|-------------------|-----------------------|
| Riemannian [3]    | \( \hat{R}_{m+1} = \hat{R}_m^{\varphi} \exp \left( \sum_{k=1}^{m} \frac{C_k}{G_k} \hat{R}_k^{\varphi} \right) \) |
|                   | \( C_k = \logm \left( \hat{R}_k^{\varphi} R_k \hat{R}_k^{\varphi} \right), \ G_k = \{k / \hat{R}_k \neq \hat{R}_i \} \) |
| Log-Euclidean [9] | \( \hat{R}_{m+1} = \expm \left( \sum_{i=1}^{m} \frac{1}{d_{LE}(\hat{R}, R)} \left( \sum_{i=1}^{m} d_{LE}(\hat{R}, R) \logm(R) \right) \right) \) |
| Bhattacharyya [7] | \( \hat{R}_{m+1} = \left( \sum_{i=1}^{m} \frac{R_i^{-1} (\hat{R}, R)}{d_{bh}(\hat{R}, R)} \right)^{1/1} \left( \sum_{i=1}^{m} \frac{1}{d_{bh}(\hat{R}, R)} \right)^{-1} \) |
| Hellinger         | \( \hat{R}_{m+1} = \left( \sum_{i=1}^{m} \frac{|R_i|^{1/4} |R|^{1/4} \left( \frac{1}{2} (R_i + R) \right)^{1/4} \left( \frac{1}{2} (R_i + R) \right)^{1/4} \right)^{1/2} \right)^{1/2} \left( \sum_{i=1}^{m} \frac{|R_i|^{1/4} |R|^{1/4} \left( \frac{1}{2} (R_i + R) \right)^{1/4} \left( \frac{1}{2} (R_i + R) \right)^{1/4} \right)^{1/2} \right)^{1/2} \) |

### 4. Divergence median-based geometric detector

As mentioned above, the divergence measures and their corresponding medians are presented. The pulse data reflected from a moving target is modeled as an HPD matrix. The element of this matrix
denotes the correlation of data. Then, a target detection algorithm is designed similar to the framework of Barbaresco’s work [2] according to these HPD matrices in this paper. As the detection algorithm is performed on the space of HPD matrix, and considers the structure of this space which is essentially a manifold. In this sense, our designed detection algorithm can be viewed as a geometric detector. The main differences between our designed detection algorithm and the Riemannian median-based geometric detector are twofold: 1) the different structures are considered owing to different metrics utilized; and 2) a weighted averaging filter which is carried out prior to the target detection is combined within our detection framework. This filter is borrowed from the bilateral filtering [11] in image denoising, and acts as a clutter suppressed procedure.

As illustrated in Figure 1, the correlation of pulse data $z$ returned from a moving target in each cell is modelled as an HPD matrix $R$. All these HPD matrices constitute a Riemannian manifold. A weighted averaging filter is carried on according to the similarity of these matrices on manifold. Specially, each estimated matrix is replaced by a weighted average of its surrounding matrices. Based on these estimated matrices, the detection statistic is defined as the distance between the HPD matrix $D_R$ of cell under test and the mean matrix $\bar{R}$ calculated by the matrices of reference cells. The mean matrix denotes the clutter power level. Finally, the decision is made by comparing the statistic in each cell with an adaptive threshold $\gamma$.

For an HPD matrix $R_i$ in the $i^{th}$ cell in one CPI, and its surrounding $m$ matrices are given as \{ $R_1, R_2, ..., R_m$ \}. A weighted strategy is employed, and the weighted matrix $R_{wi}$ can be estimated as follow,

$$R_{wi} = \sum_{i=1}^{m} w_i R_i, \quad 0 \leq w_i \leq 1, \quad \sum_{i=1}^{m} w_i = 1$$

where $w_i$ is the weight of $i^{th}$ HPD matrix. Similar to the bilateral filtering in image processing, the weight here is defined according the similarity between the matrix $R_i$ and a surrounding matrix $R_j$. In general, there are many ways to formulate the similarity between two elements. For instance, the reciprocal, the exponent, and the logarithm can be utilized into the definition. Here, we use the exponent of distance between $R_i$ and $R_j$, as,

$$w_i = \frac{1}{W} \exp \left\{ -d^2(R_i, R_j)/h^2 \right\}$$

where $d(R_i, R_j)$ is the Riemannian distance. $W = \sum_{i=1}^{m} \exp \left\{ -d^2(R_i, R_j)/h^2 \right\}$ is a normalizing factor, and $h$ is a filtering parameter, which controls the exponential decay. $m$ and $h$ are two free parameters, and they are chosen according to the context.

5. Application of the target detection method

![Figure 1. Divergence median-based geometric detector](image-url)
In this section, numerical experiments and real clutter data are performed to evaluate the detection performances of our proposed three divergence medians-based geometric detectors with the weighted averaging filter, namely the Log-Euclidean median-based geometric detector (LogEMGD), the Bhattacharyya median-based geometric detector (BhatMGD), and the Hellinger median-based geometric detector (HelMGD). The performances of these detectors are compared with the Riemannian median-based geometric detector (RiemMGD) [3] and the classical FFT-CFAR detector [1].

The performances of: 1) the LogEMGD detector, the BhatMGD detector, and the HelMGD detector, 2) the RiemMGD detector, (3) the FFT-CFAR detector are compared via Monte Carlo simulations. The data samples are from N = 7 received pulses. In our simulations, the radar received echo contains 7 pulses. The radar central frequency fc is 9 GHz, and the pulse repetition frequency (PRF) is 1000 Hz. The target signal model is

\[ a_p \]

where \( r_T \) is the pulse repetition interval, and \( d_f \) is the Doppler frequency. A target has an approximate constant velocity \( v = 5 \) m/s. As few pulses are considered, \( \alpha \) is assumed to be constant. \( M = 16 \) range cells are considered as reference cells around cell under test, and used for averaging. In the simulation, the clutter is assumed as the K distribution.

Figure 2. \( P_d \) versus SCR in K distribution with different parameter values
A series of Monte Carlo simulation experiments are used to compare the performances between our proposed detector, the Riem MG D detector and the FFT-CFAR detector. Since there is not enough prior information, an analytical expression for the detection threshold cannot be obtained. In the experiment, the detection threshold is obtained by Monte Carlo experiment. Particularly, the detection statistic in the absence of the target is calculated by the $10^6$ experiment, and the threshold is determined according to a given false alarm probability $P_{fa}$. To ensure the detection probability is accurately estimated, we repeat 200 times simulations to estimate the detection probability under different SCRs. The detection probability is estimated by the relative frequencies.

The $P_d$ vs SCRs under different $P_{fa}$ are given in Figure 2. The SCR varies from $-10$ to $10$ dB, and the $P_{fa}$ is set to $10^{-5}$. The parameter values $m$ and $h$ are chosen as $m=11$, $h=1$; $m=11$, $h=1.5$; $m=13$, $h=1$. From Figure 2 we can know that the proposed three detectors have better detection performance than the Riem MG D detector. Moreover, the detection performances of these geometric detectors are better than that of the FFT-CFAR detector. These results prove the advantage of our proposed geometric detection method sufficiently.

6. Conclusion
In this paper, we have proposed a divergence median-based geometric detector. In particular, we have derived the Hellinger median for a set of HPD matrices. The three metrics, namely the log-Euclidean distance, the Bhattacharyya divergence, and the Hellinger distance, are utilized in our detector. These metrics can achieve different structures of Riemannian manifold. In addition, a pre-processing procedure which is borrowed from the bilateral filtering in image processing is carried out before the target detection. As it acts as a clutter suppression function, the performance can be improved. At the analysis stage, we have assessed the detection performance by giving plots of $P_d$ vs SCR in simulated experiments. These results have shown that our proposed geometric detectors have better performance than the Riemannian mean-based geometric detector and the FFT-CFAR detector.

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