Research Article

On Ordinary, Linear $q$-Difference Equations, with Applications to $q$-Sato Theory

Thomas Ernst

Department of Mathematics, Uppsala University, P.O. Box 480, 751 06 Uppsala, Sweden

Correspondence should be addressed to Thomas Ernst; thomas@math.uu.se

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The purpose of this paper is to develop the theory of ordinary, linear $q$-difference equations, in particular the homogeneous case; we show that there are many similarities to differential equations. In the second part we study the applications to a $q$-analogue of Sato theory. The $q$-Schur polynomials act as basis function, similar to $q$-Appell polynomials. The Ward $q$-addition plays a crucial role as operation for the function argument in the matrix $q$-exponential and for the $q$-Schur polynomials.

1. Introduction

We begin this paper with an introduction to $q$-difference equations. Since there is a well-known parallel approach to this theme, we quote some of the historical facts about this. Then we show an example of solutions to a $q$-difference equation with constant coefficients; the multiple root case can be solved in a similar way. When we know a solution to a homogeneous equation of order $n$, the equation can be transformed into another equation of order $n−1$; this is called reduction of order. The $q$-analogue of Euler’s differential equation is of particular importance in $q$-calculus because of its operational form. In a previous article [1] we introduced the concept $q$-analogues of matrix formulas. In this paper we continue on this theme; the main content of this paper is $q$-Sato theory, which is only one way to treat the theory of $q$-deformed solitons. Previously, articles on the $q$-KdV equation and $q$-Schur polynomials, for example, [2], were published; in this paper we define a quite different $q$-Schur polynomial, which is connected to the Ward $q$-addition.

We now start with the definitions; many of these can be found in the book [3].

Definition 1. Assume that $0 < q < 1$, $a \in \mathbb{R}$. The power function is defined by $q^a = e^{a \log(q)}$. Let $\delta > 0$ be an arbitrary small number. We will use the following branch of the logarithm: $−\pi + \delta < \text{Im} (\log(q)) \leq \pi + \delta$. This defines a simply connected space in the complex plane.

The variables $a, b, c, \ldots \in \mathbb{R}$ denote certain parameters. The variables $i, j, k, l, m, n, p$, and $r$ will denote natural numbers except for certain cases where it will be clear from the context that $i$ will denote the imaginary unit.

The $q$-analogue of a real number $a$ and the factorial function are defined by

$$\{a\}_q \equiv \frac{1−q^a}{1−q}, \quad \{n\}_q! \equiv \prod_{k=1}^{n} [k]_q, \quad \{0\}_q! \equiv 1. \quad (1)$$

The $q$-analogue of the derivate and the integral are given by

$$(D_q \varphi)(x) \equiv \frac{\varphi(x)−\varphi(qx)}{(1−q)x}, \quad (2)$$

$$\int_0^a f(t, q) d_q(t) \equiv a (1−q) \sum_{n=0}^{\infty} f(a q^n, q) q^n, \quad (3)$$

$$0 < |q| < 1, \quad a \in \mathbb{R}.$$
The inverse \( q \)-derivative is accordingly defined by
\[
D_q^{-1} : \varphi (x, q) \mapsto \int_0^x \varphi (t, q) \, d_q (t).
\] (4)

Let the Gauss \( q \)-binomial coefficient be defined by
\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!}, \quad k = 0, 1, \ldots, n.
\] (5)

If \( |q| > 1 \), or \( 0 < |q| < 1 \) and \( |z| < |1 - q|^{-1} \), the \( q \)-exponential function \( E_q (z) \) is defined by
\[
E_q (z) \equiv \sum_{k=0}^{\infty} \frac{1}{[k]_q!} z^k.
\] (6)

**Definition 2.** Let \( e \) denote the invertible operator \( \mathbb{R}[x] \mapsto \mathbb{R}[x] \) defined by
\[
e f (x) \equiv f (qx).
\] (7)

**Definition 3.** Let \( a \) and \( b \) be any elements with commutative multiplication. Then the NWA \( q \)-addition is given by
\[
(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \quad n = 0, 1, 2, \ldots.
\] (8)

There is a Ward number \( \overline{n}_q \)
\[
\overline{n}_q \equiv \overline{1}_q \overline{a}_q \overline{1}_q \overline{a}_q \overline{1}_q, \quad \text{where the number of } 1 \text{ on the RHS is } n.
\] (9)

The following theorem reminding of [4, page 258] shows how Ward numbers usually appear in applications.

**Theorem 4.** Assume that \( n, k \in \mathbb{N} \). Then
\[
(\overline{n}_q)^k = \sum_{m_1 + \cdots + m_n = k} \binom{k}{m_1, \ldots, m_n} q,
\] (11)

where each partition of \( k \) is multiplied with its number of permutations.

A table of some \( \overline{n}_q^k \) is given in [3, page 109].

**Definition 5.** The notation \( \sum_m \) denotes a multiple summation with the indices \( m_1, \ldots, m_n \) running over all nonnegative integer values.

Given an integer \( k \), the formula
\[
m_0 + m_1 + \cdots + m_j = k
\] (12)
determines a set \( J_{m_0, \ldots, m_j} \in \mathbb{N}^{j+1} \).

Then if \( f (x) \) is the formal power series \( \sum_{a} a_j x^j \), its \( k \)th NWA-power is given by
\[
\left( \Theta_{q=0} a_j x^j \right)^k \equiv \left( a_0 \Theta_{q=1} a_1 x \Theta_{q=2} x^2 \cdots \right)^k
\equiv \sum_{\left| k \right| = k} \prod_{j=0}^{m_i} \left( a_j x^j \right)^{m_j} \binom{k}{m_j}.
\] (13)

Difference equations are mathematical models describing real life situations in many applied sciences. For an excellent introduction to this subject see Nørlund 1924 [5].

**Theorem 6.** Let \( y \) be a function of the continuous variable \( x \). The homogeneous linear \( q \)-difference equation of order \( n \) is of the form
\[
\sum_{k=0}^{n} p_{n+k} (x) D^k_q y = 0,
\] (14)
or even
\[
\sum_{k=0}^{n} p_{n-k} (x) x^k D^k_q y = 0.
\] (15)

Instead of studying (15), we can study an equation
\[
\sum_{j=0}^{n} a_j (x) y^{(q^{-j} x)} = 0,
\] (16)

where the \( a_j (x) \) are known functions of \( x, j \).

**Proof.** Begin with formula (16), and use the formula [3, page 211 (6.101)]
\[
f (q^n x) = \sum_{k=0}^{n} (q-1)^k x^k q^{\lfloor \frac{k}{2} \rfloor} \binom{n}{k}_q D^k_q (f) (x).
\] (17)

This gives
\[
\sum_{j=0}^{n} a_j (x) \sum_{k=0}^{n-j} (q-1)^k x^k q^{\lfloor \frac{k}{2} \rfloor} \binom{n-j}{k}_q D^k_q y
= \sum_{k=0}^{n} \sum_{j=0}^{n-k} a_j (x) (q-1)^k x^k q^{\lfloor \frac{k}{2} \rfloor} \binom{n-j}{k}_q D^k_q y.
\] (18)

The last expression is equivalent to the LHS of (15). \( \Box \)

The first steps from (16) to an investigation of the linear \( q \)-difference equation (15) were taken in two dissertations by Smith 1911 [6] and Nørlund’s student Ryde 1921 [7], who generalized the method of Frobenius for solving linear differential equations.

Equation (16) was first studied by Carmichael 1912 [8]. He distinguished between the two cases \( |q| \neq 1 \) and \( |q| = 1 \) and, according to Trjitzinsky [9], treated the case \( |q| = 1 \) satisfactorily.

In 1915 Mason [10] proved two theorems about \( q \)-difference equations with entire function coefficients. He also introduced the notion of characteristic equation for a \( q \)-difference equation.

Equation (16) has also been studied by Adams [11], who generalized the results of Carmichael and Mason. He assumed the coefficient functions \( a_j (x) \) to be analytic or to have poles of finite order at the origin. Adams also studied partial \( q \)-difference equations.
In 1933 Trjitzinsky [9] solved an inhomogeneous first order linear \( q \)-difference equation and studied the solutions of linear \( q \)-difference equations.

There is an alternative approach, called timescales; this is just another dialect of \( q \)-calculus, with completely different and more general definitions. This generality leads to many general theorems, but the \( q \)-analogues are far from easy to find. For instance, timescales have another \( q \)-Laplace transform than the one the author is going to use later.

2. The Ordinary, Linear Case

Some of the results in this section have previously occurred in a paper on internet by Bangerezako [12]. We refer to him in each case and to the page number.

A \( q \)-difference equation of order \( n \), containing powers of operator (2), is said to be linear if it is linear in the dependent variable \( y \) and the \( q \)-difference \( D_q y, D_q^2 y, \ldots, D_q^n y \). The most general linear nonhomogeneous \( q \)-difference equation of order \( n \) is of the form

\[
\sum_{k=0}^n p_{n-k} (x) D_q^k y = Ly = f(x),
\]

where \( L \) is a linear sum of \( q \)-differential operators.

We assume that since the equation is of order \( n \), its general solution will depend on \( n \) distinct arbitrary constants and proceed to consider the mode of this dependence.

Suppose that two distinct particular solutions of (19) are known; say \( y = y_1 \) and \( y = y_2 \). Then

\[
Ly_1 = f(x), \quad Ly_2 = f(x);
\]

that is,

\[
Ly_1 - Ly_2 = 0.
\]

Thus if \( u \) represents the difference between any two solutions of (19), \( u \) will satisfy the homogeneous equation

\[
Lu = 0,
\]

which contains no term free from \( u \) or a \( q \)-difference operator of \( u \). The general solution of (19) will be the sum of two components:

1. the general solution of the homogeneous equation involving \( n \) arbitrary constants and known as the complementary function,
2. a particular solution involving no arbitrary constants.

Theorem 7 (see [12, page 38]). The homogeneous linear \( n \)-order \( q \)-difference equation

\[
\sum_{k=0}^n a_k D_q^k y(x) = 0, \quad a_k \in \mathbb{R}, \quad 0 \leq k \leq n,
\]

has the general solution

\[
y(x) = \sum_{k=1}^n C_x E_q (r_k x),
\]

where \( \{ r_k \}_{k=1}^n \) are solutions of the characteristic equation

\[
\sum_{k=0}^n a_k x^k = 0.
\]

We have assumed that (25) has no multiple roots.

Proof. Similar to the ordinary case, use the chain rule for \( D_q \).

Example 8. Compare with [12, page 17]. Consider the equation

\[
D_q^2 y + 2D_q y + 5y = 0.
\]

The corresponding differential equation has solutions \( y = Ae^{-x} \sin 2x \) and \( y = Be^{-x} \cos 2x \), and we find that (26) has the solutions

\[
y = aE_q ((-1+2i)x) + bE_q ((-1-2i)x).
\]

These solutions can be rephrased in the form

\[
y = A \sum_{k=0}^{\infty} x^k \left[ (k-1)/2 \right] \sum_{m=0}^{\infty} \frac{k}{2(m+1)} (-1)^{k-m} 2^{2m+1} \]

\[
+ B \sum_{k=0}^{\infty} x^k \left[ k/2 \right] \sum_{m=0}^{\infty} \frac{k}{2m} (-1)^{k-m} 2^{2m}.
\]

It is obvious that we can continue this process to find \( q \)-analogues of any homogeneous, linear differential equation with constant coefficients, which has an exact solution in terms of sums of exponential functions.

2.1. The Multiple Root Case. For [12, page 38] we illustrate the general technique with an example.

Example 9. Try to find a \( q \)-difference equation satisfied by the homogeneous solution

\[
y_h = (C_1 x + C_2) E_q (-x).
\]

The differential equation (with multiple root characteristic equation)

\[
y'' + 2y' + y = 0
\]

has solution \( y_h = (C_1 x + C_2) e^{-x} \), and (29) is a \( q \)-analogue of this. Let \( r \in \mathbb{R} \), and let \( P(x) \in \mathbb{R}[x] \). Consider the space of functions

\[
\Psi_{q,x} (r) \equiv E_q (rx) P(x),
\]

and let \( e_p \) denotes the invertible operator \( \mathbb{R}[x] \mapsto \mathbb{R}[x] \) defined by

\[
e_p P(x) = P(qx).
\]

We find that \( D_q f(x) = (-C_1 xq - C_2 + C_1) E_q (-x) \) and \( D_q f(x) = (C_1 xq^2 + C_2 - C_1 - C_1 q) E_q (-x) \).

We try with the equation

\[
e_p^{-1} D_q^2 f + [2]_q D_q f + e_p q f = 0,
\]

which indeed solves the problem.
In [12, page 31] for nonhomogeneous $q$-difference equations, the solution will be $y = y_h + y_p$, where $y_p$ denotes a particular solution.

**Example 10.** Find a particular solution to

$$D_q^2 y + 3D_q y + 2y = 3x.$$  \quad (34)

We try with $y_p(x) = Ax + B$. This gives $A = 3/2, B = -9/4$. The particular solution is the same as that in the ordinary case.

In general, we can find particular solutions very similar to the ordinary case $q = 1$ by replacing integers by $q$-integers and solving the resulting system of equations.

### 2.2. Reduction of Order

When any solution of a homogeneous equation of order $n$ is known, the equation can be transformed into another (also linear and reduced) of order $n - 1$. If the known solution is $u_1$, the transformation is

$$u = u_1 \int^x v(t) \, d_q(t),$$  \quad (35)

where $v$ is a new dependent variable. For simplicity consider an equation of the third order (the general proof is similar)

$$\sum_{k=0}^{3} p_{3-k}(x) D_q^k y = 0,$$  \quad (36)

where the $p_k$ are functions of $x$ or constants. By substituting

$$u(x) = u_1(x) \int^x v(t) \, d_q(t),$$

$$D_q(u(x)) = D_q(u_1(x)) \int^x v(t) \, d_q(t) + u_1(qx) \, v(x);$$

$$D_q^2 u(x) = D_q^2(u_1(x)) \int^x v(t) \, d_q(t) + (1 + q) D_q(u_1(qx)) \, v(x) + u_1(q^2x) D_q(v(x)),$$

$$D_q^3 u(x) = D_q^3(u_1(x)) \int^x v(t) \, d_q(t) + (1 + q + q^2) D_q^2(u_1(qx)) \, v(x) + (1 + q + q^2) D_q(u_1(q^2x)) D_q(v(x)) + u_1(q^3x) D_q^3 v(x),$$  \quad (37)

and rearranging we have

$$\left( p_0 D_q^3(u_1(x)) + p_1 D_q^2(u_1(x)) + p_2 D_q(u_1(x)) \right) \int^x v(t) \, d_q(t) + p_3 u_1(x) \int^x v(t) \, d_q(t) + \left( (1 + q + q^2) p_0 D_q^2(u_1(qx)) + (1 + q) p_1 D_q(u_1(qx)) + p_2 u_1(qx) \right) v(x)$$

$$\times \left( (1 + q + q^2) p_0 D_q^2(u_1(q^2x)) + (1 + q) p_1 u_1(q^2x) \right) D_q v(x) + p_2 u_1(q^2x) D_q^2 v(x) = 0.$$  \quad (38)

Since $u_1$ is a solution of (36) the first term disappears, leaving a homogeneous linear equation of the second order in $v$.

#### 2.3. A $q$-Analogue of the Euler Equation

The $q$-difference operator

$$\theta_q \equiv \frac{1 - \epsilon}{1 - q}$$  \quad (39)

is a $q$-analogue of $x(d/dx)$. The operator $\theta_q$ maps the polynomial $x^n$ to $[n]_q x^n; \theta_q$ keeps the degree of a polynomial and is very important in $q$-calculus.

This implies that the equation

$$\sum_{k=0}^n \theta_q^k y(x) = b(x)$$  \quad (40)

is a $q$-analogue of the Euler equation. Our investigations show that the regularity theorems of Adams [11], Carmichael [8], and Mason [10] are also valid for the regularity of solutions to the generalized Euler equation (15).

The following two formulas from [3, page 179] are of particular interest in this context:

$$\theta_q^n = \sum_{k=0}^n S(n, k) \theta_q^{(k)} D_q^k,$$

$$q^{(\frac{k}{2})} x^n D_q = \sum_{k=1}^n s(n, k) \theta_q^{(k)},$$  \quad (41)

where $S(n, k)$ and $s(n, k)$ are $q$-Stirling numbers, inverse to each other.

### 3. First Matrix Calculations

We now come to the main content of this paper, which is a continuation of [1]. We start with a short repetition. The definition of letters in an alphabet and the corresponding linear functional is found in [1]. In our case, the alphabet is the reals.

**Definition 11.** Matrix elements will always be denoted $(i, j)$. Here $i$ denotes the row and $j$ denotes the column. The matrix elements range from 0 to $n - 1$. This holds both for real numbers (linear functional) and for the letters in the matrix. Juxtaposition of matrices (like in (53)) will always be interpreted as matrix multiplication. If $A$ and $B$ are commuting matrices of the same dimension (belonging to the alphabet), one defines $A \Theta_q B$ as a matrix with matrix elements...
(i.e., letters) $A(i, j)\sigma_q B(i, j)$. If $A$ and $B$ are commuting matrices of the same dimension, one defines $A\sigma_q B$ as a matrix with matrix elements $A(i, j)\sigma_q B(i, j)$.

**Definition 12.** Let $A$ be an $n \times n$ matrix, $0 < |q| < 1$, and $\|A\| < |1 - q|^{-1}$. Then
\[
E_q(A) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} A^k,
\]
\[
E_{1/q}(A) = \sum_{k=0}^{\infty} \frac{q^k}{[k]_{1/q}!} A^k.
\]

3.1. $q$-Sato Theory: In Sato theory, infinite-dimensional matrices and pseudodifferential operators are used to solve differential equations, with applications to soliton theory and the KdV equation. The following polynomial is used in the computations.

**Definition 13.** Given an integer $n$, the formula
\[
k_1 + 2k_2 + 3k_3 + \cdots + mk_m = n
\]
determines a set $\mathbb{N}^n$.

Then the elementary Schur polynomial $p_n$ is defined by the following equation:
\[
p_n(x_1, x_2, \ldots) = \prod_{k_i \geq 0} \frac{x_1^{k_i}}{k_i!}.
\]

These polynomials satisfy the equation
\[
\frac{\partial p_n}{\partial x_m} = p_{n-m}, \quad (p_n = 0 \text{ for } n < 0).
\]

We now begin with the $q$-deformations. The following definition is slightly different from [13, page 213], where it was assumed that $w_k \in \mathbb{R}[[x]]$ (formal power series).

**Definition 14** (see [14, page 60]). Define the following pseudo-$q$-differential operator
\[
W_{m,q} \equiv 1 + \sum_{k=1}^{m} w_k D_q^{-k}, \quad w_k \in \mathbb{R},
\]
where $D_q^{-k}$ is defined by iterating (4).

**Theorem 15.** The homogeneous, linear $q$-difference equation
\[
W_{m,q} D_q^m f(x) = \left(D_q^m + \sum_{k=1}^{m} w_k D_q^{-k}\right) f(x) = 0
\]
has $m$ linearly independent solutions $\{f_q^{(k)}\}^{m}_{k=1}$, which are all analytic; that is,
\[
f_q^{(k)}(x) = \sum_{l=0}^{\infty} \frac{g_q^{(k)} x^l}{[l]_q!}, \quad k = 1, 2, \ldots, m.
\]
The constants $\xi^{(k)}$ are uniquely determined by the initial values of the function $f$. The solutions form an $m$-dimensional vector space.

**Proof.** According to the fundamental theorem of algebra, the corresponding characteristic equation has $m$ complex roots. This gives $m$ solutions like in (27). When there are multiple roots, we multiply by a suitable polynomial, like in formula (33).

The rank of the $\infty \times m$ Wronskian matrix
\[
\Xi = \begin{pmatrix}
\xi_{0,q}^{(1)} & \xi_{0,q}^{(2)} & \cdots & \xi_{0,q}^{(m)} \\
\xi_{1,q}^{(1)} & \xi_{1,q}^{(2)} & \cdots & \xi_{1,q}^{(m)} \\
\vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]
is $m$ and we have
\[
W_{m,q} D_q^m \left(1, \frac{x}{[1]_q}, \frac{x^2}{[2]_q}, \ldots \right) \Xi = 0.
\]
The shift operator $\Lambda$ (not to be confused with the Polya-Vein matrix from [1]) is defined by
\[
\Lambda \equiv \begin{pmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & \cdots & 1
\end{pmatrix}
\]
This implies
\[
E_q(x \Lambda) = \begin{pmatrix}
1 & x & \frac{x^2}{[2]_q} & \frac{x^3}{[3]_q} & \cdots \\
1 & x & \frac{x^2}{[2]_q} & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
Introduce the following notation $H_q(x)$:
\[
H_q(x) \equiv E_q(x \Lambda) \Xi = \begin{pmatrix}
f_q^{(1)} & f_q^{(2)} & \cdots & f_q^{(m)} \\
D_q f_q^{(1)} & D_q f_q^{(2)} & \cdots & D_q f_q^{(m)} \\
D_q^2 f_q^{(1)} & D_q^2 f_q^{(2)} & \cdots & D_q^2 f_q^{(m)} \\
\vdots & \vdots & \ddots & \vdots
\end{pmatrix}.
\]
We will now try to determine $W_{m,q}$ from the $m$ solutions $f_q^{(k)}$. By (47)
\[
(D_q^{m-1} f_q^{(1)}) w_1 + (D_q^{m-2} f_q^{(1)}) w_2 + \cdots + f_q^{(1)} w_m = -D_q^{m} f_q^{(1)}
\]
\[
\vdots
\]
\[
(D_q^{m-1} f_q^{(m)}) w_1 + (D_q^{m-2} f_q^{(m)}) w_2 + \cdots + f_q^{(m)} w_m = -D_q^{m} f_q^{(m)}.
\]

**Theorem 16.** A formula for the pseudo-$q$-differential operator $W_{m,q}$ as a quotient of determinants is
\[
W_{m,q} = \left| \begin{array}{cccc}
D_q^{m} & D_q^{m-1} & \cdots & D_q^{1} \\
D_q^{m} & D_q^{m-1} & \cdots & D_q^{1} \\
\vdots & \vdots & \ddots & \vdots \\
D_q^{m} & D_q^{m-1} & \cdots & D_q^{1} \\
D_q^{m} & D_q^{m-1} & \cdots & D_q^{1} \\
\end{array} \right|.
\]

The entries of the matrices are functions, except for the last column of the numerator, which consists of pseudo-$q$-differential operators.

**Proof.** By Cramer’s rule we have
\[
w_1 = \frac{-D_q^{m} f_q^{(1)} - D_q^{m-2} f_q^{(1)} - \cdots - f_q^{(1)}}{-D_q^{m} f_q^{(1)} - D_q^{m-2} f_q^{(1)} - \cdots - f_q^{(1)}}.
\]
\[
\vdots
\]
\[
w_j = \frac{-D_q^{m} f_q^{(1)} - D_q^{m-2} f_q^{(1)} - \cdots - f_q^{(1)}}{-D_q^{m} f_q^{(1)} - D_q^{m-2} f_q^{(1)} - \cdots - f_q^{(1)}}.
\]

By combining (46) and the above two equations we obtain a formula for $W_{m,q}$. An expansion of the numerator of (55) along the last column completes the proof.

4. **Time Evolution**

We now assume that $w_j$ also depend on an infinite number of time variables $t_i$. This implies that the solutions of (47), $f_q^{(k)}(x)$, also depend on $t_i$:
\[
f_q^{(k)}(x,t) = f_q^{(k)}(x; t_1, t_2, \ldots),
\]
and $H_q(x)$ given by (53) can be written as $H(x,t,q)$. We assume that $H(x,t,q)$ evolves in time as
\[
H(x,t,q) = QE(x)\LambdaQE(\eta(t,\Lambda))\Xi,
\]
where
\[
\eta(t,\Lambda) \equiv \left( \Phi_{q,n=1}^{x\Lambda} t_n \Lambda^n \right), \quad t_1 \equiv x\Phi_q f_1.
\]
We find that the $q$-Schur polynomial $P_{n,q}$ is defined by the following equation:
\[
P_{n,q} \left( x\Phi_q f_1, t_2, t_3, \ldots \right) \equiv \sum_{k_1+k_2+\cdots+k_m=0} \left( \Phi_q f_1 \right)^{k_1} \prod_{k_i \in M_{k_1,\ldots,k_m}} t_i^{k_i},
\]
where $M_{k_1,\ldots,k_m}$ is defined by (44). Or equivalently
\[
\sum_{n=0}^{\infty} P_{n,q} z^n = E_q \left( \eta(t,z) \right).
\]
The first $P_{n,q}$ are
\[
P_{0,q} = 1,
\]
\[
P_{1,q} = x + t_1,
\]
\[
P_{2,q} = \left( x\Phi_q f_1 \right)^2 \prod_{[2]} t_1 + t_2,
\]
\[
P_{3,q} = \left( x\Phi_q f_1 \right)^3 \prod_{[3]} t_1 + (x + t_1) t_2 + t_3.
\]

**Remark 17.** These $q$-Schur polynomials are completely different than those in [2, 15] and give richer $q$-differential properties, due to the NWA $q$-addition.

**Theorem 18.** These polynomials satisfy the equations
\[
D_{q,t_n} P_{n,q} = P_{n-m,q}, \quad P_{n,q} = 0 \text{ for } n < 0,
\]
\[
D_{q,x} P_{n,q} = P_{n-1,q}.
\]

**Proof.** Operate with $D_{q,t_n}$ on (61), and write the right hand side as a product of $q$-exponentials. After performing the $q$-differentiation to the right, multiply both sides by $z^{-m}$.
Consider Theorem 19.

We have the following theorem for the entries of $H(x, t, q)$.

**Theorem 19.** Consider

$$h^{(j)}_{0,q}(x; 0) = f^{(j)}_{q}(x).$$

(65)

$$h^{(j)}_{n,q}(x; t) = D_{q,t}h^{(j)}_{0,q}(x; t) = D_{q,t}^{n}h^{(j)}_{0,q}(x; t).$$

By formula (66) we find

$$w_{j}(q, x, t) = \frac{\left| \begin{array}{c} h(m-1,q) - h(m,q) - h(q) \\ \vdots \\ h(m-1)^{(m)} - h(m,q)^{(m)} - h(q)^{(m)} \\ \vdots \\ h(m-1)^{(m)} - h(m-j,q)^{(m)} - h(q)^{(m)} \\ \vdots \\ h(0)^{(m)} - h(0)^{(m)} \\ h(m)^{(m)} - h(m)^{(m)} \\ \vdots \\ h(m-1)^{(m)} - h(m-1)^{(m)} \end{array} \right|}{\left| \begin{array}{cc} D_{q} \vdots & D_{q}^{m} \\ \vdots & \vdots \\ D_{q}^{m} & D_{q}^{m} \end{array} \right|},$$

(70)

By applying the operator $D_{q,t}$ to (70) and employing (66), we obtain

$$D_{q,t}W_{n,m}D_{q}^{m} + \left( e_{n}W_{m,q}\right) D_{q}^{m} = B_{n,q}W_{m,q}D_{q}^{m},$$

(72)

where $B_{n,q}$ is a certain $q$-difference operator. After applying $D_{q,t}^{m}W_{n,m}$ from the right, we obtain

$$B_{n,q} = D_{q,t}W_{n,m}D_{q}^{m} + e_{n}W_{m,q}D_{q}^{m},$$

(74)

By a similar reasoning as in the case $q = 1$, we have

$$B_{n,q} = \left( W_{m,q}D_{q}^{m}W_{n,m}\right)^{+},$$

(75)

where $(\cdot)^{+}$ denotes the $q$-difference part of the operator. This implies that the time evolution of $W_{n,m}(x; t)$ is governed by

$$D_{q,t}W_{q} = B_{n,q}W_{q} - e_{n}W_{q}D_{q}^{m},$$

(76)

which we will call the $q$-Sato equation.

5. Conclusion

We have found a $q$-analogue of a simplified and more mathematical form of Sato theory. We hope that this paper will have many applications for $q$-difference equations and in soliton theory. A further paper on $q$-Laplace transformations is in preparation.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.
References

[1] T. Ernst, "An umbral approach to find $q$-analogues of matrix formulas," *Linear Algebra and its Applications*, vol. 439, no. 4, pp. 1167–1182, 2013.

[2] L. Haine and P. Iliev, "The bispectral property of a $q$-deformation of the Schur polynomials and the $q$-KdV hierarchy," *Journal of Physics, A: Mathematical and General*, vol. 30, no. 20, pp. 7217–7227, 1997.

[3] T. Ernst, *A Comprehensive Treatment of q-Calculus*, Birkhäuser, 2012.

[4] M. Ward, "A calculus of sequences," *American Journal of Mathematics*, vol. 58, no. 2, pp. 255–266, 1936.

[5] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer, Berlin, Germany, 1924.

[6] E. R. Smith, *Zur Theorie der Heineschen Reihe und ihrer Verallgemeinerung [Dissertationen]*, Universität München, 1911.

[7] F. Ryde, "A contribution to the theory of linear homogeneous geometric difference equations ($q$-difference equations)," Dissertation Lund, 1921.

[8] R. D. Carmichael, "The general theory of linear $q$-difference equations," *American Journal of Mathematics*, vol. 34, no. 2, pp. 147–168, 1912.

[9] W. J. Tritzinsky, "Analytic theory of linear $q$-difference equations," *Acta Mathematica*, vol. 61, no. 1, pp. 1–38, 1933.

[10] T. E. Mason, "On properties of the solutions of linear $q$-difference equations with ENTire function coefficients," *American Journal of Mathematics*, vol. 37, no. 4, pp. 439–444, 1915.

[11] C. R. Adams, "On the linear ordinary $q$-difference equation," *Annals of Mathematics: Second Series*, vol. 30, no. 1–4, pp. 195–205, 1928.

[12] G. Bangerezako, *q-Difference Equations*, Preprint.

[13] Y. Ohta, J. Satsuma, D. Takahashi, and T. Tokihiro, "An elementary introduction to Sato theory," *Progress of Theoretical Physics: Supplement*, no. 94, pp. 210–241, 1988.

[14] E. Druitt, *Hirota’s direct method and Sato’s formalism in soliton theory [Honour Thesis]*, The University of Melbourne, Melbourne, Australia, 2005.

[15] R. Carroll, "Hirota formulas and $q$-hierarchies," *Applied Analysis*, vol. 82, no. 8, pp. 759–786, 2003.
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