Program schemes with binary write-once arrays and the complexity classes they capture

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Abstract
We study a class of program schemes, NPSB, in which, aside from basic assignments, non-deterministic guessing and while loops, we have access to arrays; but where these arrays are binary write-once in that they are initialized to ‘zero’ and can only ever be set to ‘one’. We show, amongst other results, that: NPSB can be realized as a vectorized Lindstr"om logic; there are problems accepted by program schemes of NPSB that are not definable in the bounded-variable infinitary logic \( L_{\omega_1^\omega} \); all problems accepted by the program schemes of NPSB have a zero-one law; and on ordered structures, NPSB captures the complexity class \( L^{NP} \). The class of program schemes NPSB is actually the union of an infinite hierarchy of classes of program schemes. When we amend the semantics of our program schemes slightly, we find that the classes of the resulting hierarchy capture the complexity classes \( \Sigma^p_i \) (where \( i \geq 1 \)) of the Polynomial Hierarchy \( PH \). Finally, we give logical equivalences of the complexity-theoretic question ‘Does \( NP \) equal \( PSPACE \)?’ where the logics (and classes of program schemes) involved define only problems with zero-one laws (and so do not define some computationally trivial problems).

1 Introduction

Finite model theory is essentially the study of logical definability over finite structures. An important sub-area of finite model theory is the relationship between the logical definability of classes of finite structures and computational complexity theory; that is, descriptive complexity theory. This relationship is best exemplified by Fagin’s seminal result that a problem, i.e., a class of finite structures over the same signature, can be defined by a sentence of existential second-order logic if, and only if, the problem (or, to be more precise, an encoding of it) can be accepted by a polynomial-time non-deterministic Turing machine.

In two recent papers [4, 30], we have studied logical definability in a slightly different context in that we have worked with classes of program schemes as opposed to more traditional logics. Program schemes are more computational in flavour than are formulae of traditional logics yet they remain amenable to logical manipulation.

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The concept of a program scheme originates from the 1970’s with work of, for example, Constable, Friedman, Gries, Hewitt and Paterson [5, 10, 21]; and complexity-theoretic considerations of such program schemes were subsequently studied by, for example, Harel, Jones, Muchnik, Peleg, Tiuryn and Urzyczyn [14, 17, 31]. Our analysis of program schemes differs from what has gone before in that we are always concerned with finite structures (and not infinite ones as was often the case previously) and we do not assume that the elements of our finite structures are necessarily linearly ordered. Our studies have exhibited an extremely close relationship between certain classes of program schemes and the more traditional logics from descriptive complexity theory, and our methodology has enabled us to prove new results extending known results from descriptive complexity theory. For example, in [4] we defined a hierarchy of classes of program schemes, NPSS, where these program schemes involve assignments, while loops and non-deterministic guessing and have access to a stack, before showing that this hierarchy is proper and (the union of it) has exactly the same expressive power as path system logic (a logic previously considered on ordered finite structures [13, 27] and which has been shown to be none other than stratified fixed point logic and stratified Datalog on the class of all finite structures [13, 18]). Furthermore, our hierarchy result translates to a strong hierarchy result about path system logic, a result which was previously unknown and not immediately derivable using known results and techniques from descriptive complexity theory (see [4] for more details).

As well as equating classes of program schemes with more mainstream logics from descriptive complexity theory, we have also shown how naturally defined classes of program schemes result in logics which previously have not arisen in descriptive complexity theory yet which have interesting properties. For example, in [20] we considered a hierarchy of classes of program schemes, NPSA, similar to the program schemes of NPSS but with arrays replacing the stack (with the levels of the hierarchy denoted by NPSA(i), for \(i \geq 1\)). We showed that (the union of) this hierarchy NPSA can be realized as a vectorized Lindström logic (many of the mainstream logics of descriptive complexity theory are explicitly defined as vectorized Lindström logics: see [7]), and that there are program schemes in this hierarchy (even in the first level) which accept problems not definable in the bounded-variable infinitary logic \(L^\omega_\infty\) (a fundamental and encompassing logic in finite model theory: see [7]). However, we also show that, like any problem definable in \(L^\omega_\infty\), every problem accepted by a program scheme of NPSA has a zero-one law. We remark that on ordered finite structures the class of program schemes NPSS captures the complexity class \(P\) and the class of program schemes NPSA captures the complexity class \(PSPACE\) (and in both cases the underlying hierarchy collapses to the first level).

In this paper, in an attempt to ‘bridge the gap’ between the polynomial-time world of NPSS and the polynomial-space world of NPSA, we modify the program schemes of NPSA so that all arrays are ‘binary write-once’ in the sense that all array elements are initially set at ‘zero’ and the only modification to any array element allowed is to set it to ‘one’. The resulting class of program schemes is denoted NPSB, with the levels of the underlying hierarchy being NPSB(i), for \(i \geq 1\). We show that NPSB retains some of the properties of NPSA: like NPSA, NPSB can be realized as a vectorized Lindström logic, and NPSB(1) contains problems not definable in \(L^\omega_\infty\) (every problem accepted by a program scheme of NPSB trivially has a zero-one law as NPSB is a sub-class of NPSA); but whereas both NPSA and NPSA(1) capture
**PSPACE** on ordered structures, NPSB(1) captures **NP** and NPSB captures \( L^{NP} \), with the NPSB hierarchy collapsing to NPSB(3).

We then amend the semantics of the class of program schemes NPSB in that we allow the current values of arrays to be ‘passed across’ to other program schemes (appearing as, what amounts to, subroutines in the main program scheme) in a computation (hitherto, the semantics has only allowed the current values of variables to be passed across). We denote the class of program schemes with this amended semantics as NPSB\(^p\) (the subscript reflects the polynomially many values passed across to the component program schemes). We show that on the class of all finite structures, NPSB\(^p\)(2\(i - 1\)) = NPSB\(^p\)(2\(i\)) and captures \( \Sigma^p_i \), i.e., the \( i \)th level of the Polynomial Hierarchy \( \text{PH} \), for \( i \geq 2 \); and so NPSB\(^p\) captures PH itself.

Finally, we compare the relative expressibilities of the classes of program schemes NPSA(1) and NPSB(1). Recall that any problem accepted by any program scheme of these classes has a zero-one law; and so there are complexity-theoretic trivial problems (like the problem consisting of all those structures of even size over some fixed signature) not accepted by such program schemes. We show that NPSA(1) = **PSPACE** \( \cap \text{EXT} \) and NPSB(1) = **NP** \( \cap \text{EXT} \), where \( \text{EXT} \) is the class of programs closed under extensions (and thus every non-trivial problem in \( \text{EXT} \) has a ‘one-law’); and thus that the question of whether **NP** is equal to **PSPACE** is equivalent to the question of whether the two classes of program schemes NPSB(1) and NPSA(1) accept the same class of problems (and also equivalent to whether two particular fragments of two vectorized Lindström logics are equally expressive) on the class of all finite structures.

This paper is structured as follows. In the next section, we outline the definitions relating to this paper before we define our classes of program schemes in Section 3. In Section 4, we identify the class of program schemes NPSB with a vectorized Lindström logic, before looking at our program schemes and logics on ordered structures in Section 5, where we also amend the original semantics of our program schemes. In Section 6, we compare the relative computational power of the classes of program schemes NPSB(1) and NPSA(1) (on the class of all finite structures) before presenting our conclusions and directions for further research in Section 7.

## 2 Preliminaries

The main reference texts for the basic concepts, notions and results of finite model theory are [5, 16] and it is to these books that we refer the reader for additional information. However, we give the definitions relating to this paper in some detail below as it is often the case that more generality is required than in [5, 16] and we also need some notions not defined in those texts.

Ordinarily, a signature \( \sigma \) is a tuple \( (R_1, \ldots, R_r, C_1, \ldots, C_c) \), where each \( R_i \) is a relation symbol, of arity \( a_i \), and each \( C_j \) is a constant symbol. First-order logic over the signature \( \sigma \), \( \text{FO}(\sigma) \), consists of those formulae built from atomic formulae over \( \sigma \) using \( \land, \lor, \neg, \forall \) and \( \exists \); and \( \text{FO} = \{\text{FO}(\sigma) : \sigma \text{ is some signature}\} \).

A finite structure \( \mathcal{A} \) over the signature \( \sigma \), or \( \sigma \)-structure, consists of a finite universe or domain \( |\mathcal{A}| \) together with a relation \( R_i \) of arity \( a_i \), for every relation symbol \( R_i \) of \( \sigma \), and a constant \( C_j \in |\mathcal{A}| \), for every constant symbol \( C_j \) (by an abuse of notation, we do not usually distinguish between constants and relations, \( C_j^A \) and \( R_i^A \)).
and constant and relation symbols, $C_j$ and $R_i$). A finite structure $A$ whose domain consists of $n$ distinct elements has size $n$, and we denote the size of $A$ by $|A|$ also (this does not cause confusion). We only ever consider finite structures of size at least 2, and the set of all finite structures of size at least 2 over the signature $\sigma$ is denoted $\text{STRUCT}(\sigma)$. A problem over some signature $\sigma$ consists of a subset of $\text{STRUCT}(\sigma)$ that is closed under isomorphism; that is, if $A$ is in the problem then so is every isomorphic copy of $A$. Throughout, all our structures are finite.

The class of problems defined by the sentences of FO is denoted by FO also, and we do likewise for other logics. It is widely acknowledged that, as a means for defining problems, first-order logic leaves a lot to be desired; especially when we have in mind developing a relationship between computational complexity and logical definability. For example, every first-order definable problem can be accepted by a log-space deterministic Turing machine (where structures are encoded as strings) yet there are problems in the complexity class $L$ (log-space) which cannot be defined in first-order logic (one such being the problem consisting of all those structures, over any signature, that have even size). Consequently, a number of methods have been developed so as to increase definability.

One method is to extend first-order logic using a vectorized sequence of Lindström quantifiers corresponding to some problem $\Omega$; or, as we prefer, an operator $\Omega$ for short. Suppose that $\Omega$ is over the signature $\sigma$, where $\sigma = \langle R_1, \ldots, R_r, C_1, \ldots, C_c \rangle$, as above. The logic $(\pm \Omega)^*[\text{FO}]$ consists of those formulae built using the usual constructs of first-order logic and also the operator $\Omega$, where the operator $\Omega$ is applied as follows.

- Suppose that $\psi_1(x_1, y), \ldots, \psi_r(x_r, y)$ are formulae of $(\pm \Omega)^*[\text{FO}]$ such that:
  - each $x_i$ is a $k_{a_i}$-tuple of distinct variables, for some fixed $k \geq 1$;
  - $y$ is an $m$-tuple of distinct variables, for some $m \geq 0$, each of which is different from any variable of $x_1, \ldots, x_r$; and
  - all free variables of any $\psi_i$ are contained in either $x_i$ or $y$.
- Suppose that $d_1, \ldots, d_c$ are $k$-tuples of variables and constants (which need not be distinct).
- Then:
  $$\Omega[\lambda x_1 \psi(x_1, y), \ldots, x_r \psi_r(x_r, y)][d_1, \ldots, d_c]$$
  is a formula of $(\pm \Omega)^*[\text{FO}]$ whose free variables are the variables of $y$ together with any other variables appearing in $d_1, \ldots, d_c$.

If $\Phi$ is a sentence of the form $\Omega[\lambda x_1 \psi_1(x_1), \ldots, x_r \psi_r(x_r)](d_1, \ldots, d_c)$, as above, over some signature $\sigma'$ then we interpret $\Phi$ in a $\sigma'$-structure $A$ as follows (note that as $\Phi$ is a sentence, the variables of $y$ are absent and the tuples $d_1, \ldots, d_c$, which are only there if there are constant symbols in $\sigma$, consist entirely of constant symbols of $\sigma'$).

- The domain of the $\sigma$-structure $\Phi(A)$ is $|A|^k$.
- The relation $R_i$ of $\Phi(A)$ is defined via:
  - for any $u \in |\Phi(A)|^{a_i} \times |A|^{k_{a_i}}$, $R_i(u)$ holds in $\Phi(A)$ if, and only if, $\psi_i(u)$ holds in $A$. 

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• The constant $C_j$ of $\Phi(\mathcal{A})$ is defined via:
  
  - $C_j$ is the interpretation of the tuple of constants $d_j$ in $\mathcal{A}$.

We define that $\mathcal{A} \models \Phi$ if, and only if, $\Phi(\mathcal{A}) \in \Omega$ (the situation where $\Phi$ has free variables is similar except that $\Phi$ is interpreted in expansions of $\sigma'$-structures by an appropriate number of constants). We call logics such as $(\pm \Omega)^+[\text{FO}]$ vectorized Lindström logics. We shall also be interested in fragments of vectorized Lindström logics where: the formulae are such that the operator $\Omega$ does not appear within the scope of a negation sign, namely $\Omega^+[\text{FO}]$ (the positive fragment of $(\pm \Omega)^+[\text{FO}]$); and further the formulae are such that there are no nestings of the operator $\Omega$, namely $\Omega^+[\text{FO}]$ (the positive unnested fragment of $(\pm \Omega)^+[\text{FO}]$).

It can be the case that (a fragment of) a vectorized Lindström logic $(\pm \Omega)^+[\text{FO}]$ has a very straightforward normal form; a normal form which obviates the need to nest applications of the operator $\Omega$ and which tells us something about the ‘degree of difficulty’ of the particular problem $\Omega$ with respect to the class of problems defined by the sentences of the logic. For example, suppose that every problem in (a fragment of) $(\pm \Omega)^+[\text{FO}]$ can be defined by a sentence of the form $\Omega[\lambda x_1 \psi_1(x_1), \ldots, x_r, \psi_r(x_r)](d_1, \ldots, d_l)$, as above, except where each $\psi_i$ is quantifier-free first-order. Then we say that the problem $\Omega$ is complete for (the fragment of) $(\pm \Omega)^+[\text{FO}]$ via quantifier-free first-order translations. This is directly analogous to completeness for some complexity class via some resource-bounded reduction: in fact, as we shall see, such normal forms can often yield very strong complexity-theoretic completeness results.

Vectorized Lindström logics have been studied quite extensively in finite model theory and a whole range of complexity classes have been captured, i.e., characterized, by vectorized Lindström logics (see, for example, [2, 5, 22, 23] and the references therein). However, some (though not all) of these characterizations only hold in the presence of a built-in successor relation. Consider some vectorized Lindström logic $(\pm \Omega)^+[\text{FO}]$. To say that this logic has a built-in successor relation, which we denote by $(\pm \Omega)^+[\text{FO}_s]$, means that no matter which signature $\sigma'$ we are working over, there is always a binary relation symbol $\text{succ}$ and two constant symbols 0 and $\text{max}$ available (none of which is in $\sigma'$) such that $\text{succ}$ is always interpreted as a successor relation with least element 0 and greatest element $\text{max}$ in any $\sigma'$-structure. That is, for any $\sigma'$-structure of size $n$, $\text{succ}$ is always of the form $\{(0, u_1), (u_1, u_2), \ldots, (u_{n-2}, \text{max})\}$, where the elements of $\{0, u_1, u_2, \ldots, u_{n-2}, \text{max}\}$ are distinct. However, there is a further semantic stipulation on the sentences of $(\pm \Omega)^+[\text{FO}_s]$; we only consider as well-formed those sentences for which the interpretation in any structure is independent of the particular successor relation chosen. For example, define the problem TC over the signature $\sigma_{2++} = (E, C, D)$, where $E$ is a binary relation symbol and $C$ and $D$ are constant symbols, as consisting of all those $\sigma_{2++}$-structures for which, when considered as digraphs in the natural way, there is a directed path from the vertex $C$ to the vertex $D$. Then the following sentence is a well-formed sentence of $(\pm \text{TC})^+[\text{FO}_s]$ (as satisfiability in a given structure is invariant with respect to $\text{succ}$):

$$\text{TC}[\lambda(x_1, x_2), (y_1, y_2)](x_1 = 0 \land y_1 = \text{max} \land \text{succ}(x_2, y_2))$$
$$\lor (x_1 = \text{max} \land y_1 = 0 \land \text{succ}(x_2, y_2))(0, 0, \text{max}, \text{max}),$$

and it defines the problem over the empty signature consisting of those structures of even size. (Note that in [2], the mechanism by which a successor relation is introduced
into a logic is slightly different from how we have described here in that only problems on ordered structures are ever considered: see [1]. Nevertheless, the two approaches essentially amount to the same thing and we shall refer to the two mechanisms interchangeably). Note also that other relations can be built into logics in the same way as is a successor relation; or even just two distinct constants can be built in.)

From a logical perspective, there is a problem with our built-in successor relation in the following sense. Given a sentence of first-order logic in which the relation symbol \( \text{succ} \) appears (and in which other constant and relation symbols might appear), it is actually undecidable as to whether the sentence is invariant with respect to \( \text{succ} \). That is, there does not exist an effective enumeration of the well-formed sentences of \( \text{FO}_s \). Given this fact, it is highly debatable as to whether any ‘logic’ \( (\pm \Omega)^* \text{[FO}_s \text{]} \) should really be called a logic; and it is an open question currently occupying much research activity as to whether there actually exists a logic capturing the complexity class \( \text{P} \) (polynomial-time), or indeed any complexity class contained in \( \text{NP} \) (non-deterministic polynomial-time), where by ‘contained in’ we really mean ‘contained in but expected to be different from’ (as \( \text{NP} \) itself can be captured by a logic, one such being existential second-order logic). The reader is referred to [20] for more details on this and related points. (Notwithstanding the above discussion, we still refer to \( (\pm \Omega)^* \text{[FO}_s \text{]} \) as a logic on the grounds of convenience.)

Theorem 1, below, is an example of a normal form result. Define the problem CUB, over the signature \( \sigma_2 = \langle E \rangle \), where \( E \) is a binary relation symbol, as follows.

\[
\text{CUB} = \{ G \in \text{STRUCT}(\sigma_2) : \text{the graph } G \text{ has a subset of edges inducing a regular subgraph of degree 3} \}
\]

(think of a \( \sigma_2 \)-structure \( G \) as encoding an undirected graph via: ‘there is an edge \((u, v)\) if, and only if, \( u \neq v \) and \( E(u, v) \lor E(v, u) \) holds in \( G \)’). We shall need the following result later on.

**Theorem 1** [28] The complexity class \( \text{NP} \) is identical to the class of problems defined by the sentences of \( \text{CUB}^*[\text{FO}_s] \); and any problem in \( \text{NP} \) can be defined by a sentence of \( \text{CUB}^1[\text{FO}_s] \) of the form:

\[
\text{CUB}^1[\lambda x, y \psi(x, y)],
\]

where \( |x| = |y| = k, \) for some \( k \geq 1, \) and \( \psi \) is a quantifier-free formula of \( \text{FO}_s \). Hence, \( \text{CUB}^*[\text{FO}_s] = \text{CUB}^1[\text{FO}_s] = \text{NP} \) and \( \text{CUB} \) is complete for \( \text{NP} \) via quantifier-free first-order translations with successor. \( \square \)

Note that Theorem 1 subsumes the ‘traditional’ known complexity-theoretic result that \( \text{CUB} \) is complete for \( \text{NP} \) via log-space reductions (a result attributed to Chvátal in [3]).

### 3 Program schemes

Program schemes are more ‘computational’ means for defining classes of problems than are logical formulae. A program scheme \( \rho \in \text{NPSA}(1) \) involves a finite set \( \{ x_1, x_2, \ldots, x_k \} \) of variables, for some \( k \geq 1, \) and is over a signature \( \sigma. \) It consists of a finite sequence of instructions where each instruction, apart from the first and the last, is one of the following:
• an assignment instruction of the form ‘\(x_i := y\)’, where \(i \in \{1, 2, \ldots, k\}\) and where \(y\) is a variable from \(\{x_1, x_2, \ldots, x_k\}\), a constant symbol of \(\sigma\) or one of the special constant symbols 0 and \(\text{max}\) which do not appear in any signature;

• an assignment instruction of the form ‘\(x_i := A[y_1, y_2, \ldots, y_d]\)’ or ‘\(A[y_1, y_2, \ldots, y_d] := y_0\)’, for some \(i \in \{1, 2, \ldots, k\}\), where each \(y_j\) is a variable from \(\{x_1, x_2, \ldots, x_k\}\), a constant symbol of \(\sigma\) or one of the special constant symbols 0 and \(\text{max}\) which do not appear in any signature, and where \(A\) is an array symbol of dimension \(d\);

• a guess instruction of the form ‘guess \(x_i\)’, where \(i \in \{1, 2, \ldots, k\}\); or

• a while instruction of the form ‘while \(\varphi\) do \(\alpha_1; \ldots; \alpha_q\) od’, where \(\varphi\) is a quantifier-free formula of \(\text{FO}(\sigma \cup \{0, \text{max}\})\) whose free variables are from \(\{x_1, x_2, \ldots, x_k\}\), and where each of \(\alpha_1, \alpha_2, \ldots, \alpha_q\) is another instruction of one of the forms given here (note that there may be nested while instructions).

The first instruction of \(\rho\) is ‘input\((x_1, x_2, \ldots, x_l)\)’ and the last instruction is ‘output\((x_1, x_2, \ldots, x_l)\)’, for some \(l\) where \(1 \leq l \leq k\). The variables \(x_1, x_2, \ldots, x_l\) are the input-output variables of \(\rho\), the variables \(x_{l+1}, x_{l+2}, \ldots, x_k\) are the free variables of \(\rho\) and, further, any free variable of \(\rho\) never appears on the left-hand side of an assignment instruction nor in a guess instruction. Essentially, free variables appear in \(\rho\) as if they were constant symbols.

A program scheme \(\rho \in \text{NPSA}(1)\) over \(\sigma\) with \(t\) free variables, say, takes a \(\sigma\)-structure \(A\) and \(t\) additional values from \(|A|\), one for each free variable of \(\rho\), as input; that is, an expansion \(A'\) of \(A\) by adjoining \(t\) additional constants. The program scheme \(\rho\) computes on \(A'\) in the obvious way except that:

• execution of the instruction ‘guess \(x_i\)’ non-deterministically assigns an element of \(|A|\) to the variable \(x_i\);

• the constants 0 and \(\text{max}\) are interpreted as two arbitrary but distinct elements of \(|A|\); and

• initially, every input-output variable and every array element is assumed to have the value 0.

Note that throughout a computation of \(\rho\), the value of any free variable does not change. The expansion \(A'\) of the structure \(A\) is accepted by \(\rho\), and we write \(A' \models \rho\), if, and only if, there exists a computation of \(\rho\) on this expansion such that the output-instruction is reached with all input-output variables having the value \(\text{max}\) (in particular, some computations might not be terminating). We can easily build the usual ‘if’ and ‘if-then-else’ instructions using while instructions (see, for example, [24]). Henceforth, we shall assume that these instructions are at our disposal.

We want the sets of structures accepted by our program schemes to be problems, i.e., closed under isomorphism, and so we only ever consider program schemes \(\rho\) where a structure is accepted by \(\rho\) when 0 and \(\text{max}\) are given two distinct values from the universe of the structure if, and only if, it is accepted no matter which pair of distinct values is chosen for 0 and \(\text{max}\). This is analogous to how we build two constant symbols into a logic. Furthermore, we can build a successor relation into the program.
schemes of NPSA(1) so as to obtain the class of program schemes NPSA_s(1). As with our logics, we write NPSA(1) and NPSA_s(1) to also denote the class of problems accepted by the program schemes of NPSA(1) and NPSA_s(1), respectively (and do likewise with other classes of program schemes).

We have two remarks. First, our notation NPSA(1) reflects the fact that NPSA(1) is the first level of an infinite hierarchy of classes of program schemes, as we shall see presently. Second, as the definition of our class of program schemes NPSA(1) stands, we do not know whether the program schemes in this class can be recursively enumerated. However, we are prepared to live with this (possible) inconvenience as we could have defined the program schemes of NPSA(1) to be devoid of the constant symbols 0 and max and be such that initially every variable and array element is non-deterministically assigned the same element of the input structure (this would result in the same class of problems). Such a definition would mean that the class of structures accepted by such a program scheme is always closed under isomorphism (hence, recursive enumerability would not be an issue). However, there are three real reasons for having the constant symbols 0 and max in our program schemes. First, we can use 0 to initialize all variables and arrays, with the result that we never have to worry about whether an assignment involves an uninitialized variable or array element. Second, having two distinct constants around is useful when it comes to programming. Third, we shall soon use the constant symbol max to enable us to study ‘binary write-once’ arrays (that is, arrays where the elements can only be set to max and thereafter remain unchanged).

Henceforth, we think of our program schemes as being written in the style of a computer program. That is, each instruction is written on one line and while instructions (and, similarly, if and if-then-else instructions) are split so that ‘while \( \varphi \) do’ appears on one line, ‘\( \alpha_1 \)’ appears on the next, ‘\( \alpha_2 \)’ on the next, and so on (of course, if any \( \alpha_i \) is a while, if or if-then-else instruction then it is split over a number of lines in the same way). The instructions are labelled 1, 2, and so on, according to the line they appear on. In particular, every instruction is considered to be an assignment, a guess or a test. An instantaneous description (ID) of a program scheme on some input consists of a value for each variable, the number of the instruction about to be executed and values for all array elements. A partial ID consists of just a value for each variable and the number of the instruction about to be executed. One step in a program scheme computation is the execution of one instruction, which takes one ID to another, and we say that a program scheme can move from one ID to another if there exists a sequence of steps taking the former ID to the latter.

As we hinted at above, the class of program schemes NPSA(1) is but the first level of an infinite hierarchy of program schemes. Suppose that we have defined a class of program schemes NPSA(2m−1), for some \( m \geq 1 \), and that any program scheme has associated with it: a set of input-output variables; a set of free variables; and a set of bound variables (this is certainly the case when \( m = 1 \), where the associated set of bound variables is empty).

**Definition 2** Let the program scheme \( \rho \in \text{NPSA}(2m-1) \) be over the signature \( \sigma \). Suppose that \( \rho \) has: input-output variables \( x_1, x_2, \ldots, x_k \); free variables \( x_{k+1}, x_{k+2}, \ldots, x_{k+s} \); and bound variables \( x_{k+s+1}, x_{k+s+2}, \ldots, x_{k+s+t} \). Let \( x_{i_1}, x_{i_2}, \ldots, x_{i_p} \) be
Obviously, NPSB(1) ⊆ NPSB(2) ⊆ ... ⊆ ∪{NPSB(i) : i = 1, 2, ...} = NPSA

(we use the inclusion relation between consecutive classes because this is how they are related as classes of problems). It is easy to see that, for one thing, FO ⊆ NPSA.

In this paper, we are primarily interested in some sub-classes of program schemes of NPSA, namely the sub-classes NPSB(i), for i = 1, 2, ..., and the union of these classes NPSB, where the only allowed assignment instructions with an array element on the left-hand side are of the form A[x_1, x_2, ..., x_k] := max; that is, the only values array elements can have are 0 and max, and once an array element is set to max then it remains at max thereafter (the notation reflects the binary nature of these arrays). Obviously, NPSB(i) ⊆ NPSA(i), for all i = 1, 2, ...; and NPSB ⊆ NPSA.

Results concerning the program schemes of NPSA have already been obtained, and some of these results relevant to this paper are stated below. A problem Ω, over
some signature σ and where the domain of any σ-structure of size n is taken to be \{1, 2, \ldots, n\}, for which the function f(n), defined as the number of structures in Ω of size n divided by the number of σ-structures of size n, is such that the limit as n tends to infinity exists and is 0 or 1 is said to have a zero-one law.

**Theorem 4**

(i) There exists a problem Ωa, involving reachability in Petri nets, for which

\[ NPSA = (\pm \Omega_a)^*[FO], \]

and the class of problems NPSA has a zero-one law.

(ii) There is a quantifier-free first-order translation with 2 constants from any problem in NPSA(1) to the problem Ωa; and so Ωa is complete for NPSA(1) via quantifier-free first-order translations with 2 constants.

(iii) The problem CUB is in NPSA(1) but not definable in the logic \(\mathcal{L}_{\omega}^{\omega}\).

(iv) In the presence of a built-in successor relation, the hierarchy NPSA_s collapses to the first level, NPSA_s(1), and captures the complexity class \(\text{PSPACE}\).

It is worth mentioning the role of the logic \(\mathcal{L}_{\omega}^{\omega}\) in finite model theory. This logic is an important logic for a number of reasons, one of which is that it subsumes many of the logics from finite model theory (including transitive-closure logic, least fixed point logic and partial fixed point logic) in that these logics can be realized as fragments of \(\mathcal{L}_{\omega}^{\omega}\). Furthermore, \(\mathcal{L}_{\omega}^{\omega}\) has a zero-one law and so any logic subsumed by \(\mathcal{L}_{\omega}^{\omega}\) has a zero-one law. It is particularly interesting that NPSA(1) (and so also NPSA) can not be realized as a fragment of \(\mathcal{L}_{\omega}^{\omega}\) (as CUB is a problem in NPSA(1) that is not in \(\mathcal{L}_{\omega}^{\omega}\); a result proven in \([29]\)).

In the absence of arrays, when the resulting class of program schemes is denoted NPS, and additionally in the presence of a stack, when the resulting class of program schemes is denoted NPSS, there are results analogous to parts (i), (ii) and (iv) of Theorem 4 (see \([4]\)) in that: both NPS and NPSS can be realized as vectorized Lindström logics so that the problems corresponding to the operators involved in these logics are complete for NPS(1) and NPSS(1) via quantifier-free first-order translations with 2 constants; and on ordered structures, the complexity classes captured are \(\text{NL}\) (non-deterministic log-space) and \(\text{P}\), respectively. However, unlike NPSA, both NPS and NPSS can be realized as fragments of \(\mathcal{L}_{\omega}^{\omega}\). Furthermore, the underlying hierarchies of NPS and NPSS are proper at every level (even if we restrict to problems only involving trees) whereas, as we shall affirm later, all that is known as regards NPSA is that NPSA(1) ⊂ NPSA(2) ⊂ NPSA(3).

### 4 Partitioned Petri nets

We begin by describing a generalization of the digraph reachability problem to a scenario where the moves between nodes depend upon the availability and utilization of external resources. We first describe the basic decision problem in an everyday fashion before we consider a manifestation of it as a class of structures over a given
signature and see how this problem is related to computation in the program schemes of NPSB(1).

Consider the following scenario. We are given a direct graph \( G = (V, E) \), where \(|V| = n\), with a source vertex \( \text{source} \) and a sink vertex \( \text{sink} \), but where each edge is labelled with a (possibly empty) set of labels with each label being of one of the following forms:

- ‘user resource \( r_i \) is unused’;
- ‘system resource \( s_j \) is available’;
- ‘user resource \( r_i \) is unused and this move uses this resource but makes the system resource \( s_j \) available (if it wasn’t available previously)’.

There is a polynomial number of different user resources \( \{r_i : i = 1, 2, \ldots, p(n)\} \), which are either in the state ‘used’ or the state ‘unused’; and a polynomial number of system resources \( \{s_i : i = 1, 2, \ldots, q(n)\} \), which are either in the state ‘available’ or the state ‘unavailable’ (for some polynomials \( p \) and \( q \)). A move in the digraph from vertex \( u \) to vertex \( v \) via the edge \((u, v)\) can only be made if either no labels label the edge \((u, v)\) or at least one of the labels labelling the edge \((u, v)\) is satisfied (with a resulting change in the state of a user resource, and possibly a system resource, if the label is of the third type). The question we ask is, given the initial state where all user resources are unused and no system resources are available, is it possible to move from \( \text{source} \) to \( \text{sink} \) in our given environment? That is, can the user use his or her resources wisely so as to enable a traversal in the digraph from the source to the sink?

Note that whether a move can be made depends only on certain predicates involving the states of the resources: for example, there are no moves dependent upon the state of a user resource being ‘used’ or of a system resource being ‘unavailable’. The situation is as it is as this decision problem arises naturally from our consideration of our program schemes; but we comment further on this problem and related problems in the Conclusion.

We encode the above decision problem as a problem, i.e., class of finite structures, involving Petri nets. Our encoding is natural and has certain properties which we shall utilize later. The reader is referred to [8] for the basic notions and concepts relating to Petri nets (this reference also gives details of numerous complexity-theoretic results concerning fundamental problems in Petri nets).

**Definition 5** Define \( \sigma_b = \langle P, Q, T_1, T_2, T_3, C, D \rangle \) where \( P, Q, T_1, T_2 \) and \( T_3 \) are relation symbols of arities 1, 1, 2, 3 and 4, respectively, and \( C \) and \( D \) are constant symbols. Let \( \mathcal{P} \) be a \( \sigma_b \)-structure. We can think of the elements of \(|\mathcal{P}|\) as being the places of a Petri net and the relations \( P \) and \( Q \) as describing two partitions of these places. We can think of:

- the relation \( T_1 \) as describing the set of transitions
  \[ \{\{u\}, \{v\} : u, v \in P \text{ and } T_1(u, v) \text{ holds}\}; \]
- the relation \( T_2 \) as describing the set of transitions
  \[ \{\{u, i\}, \{v, i\} : u, v \in P, i \notin P, i \in Q \text{ and } T_2(u, v, i) \text{ holds}\} \]
  \[\cup\{\{u, j\}, \{v, j\} : u, v \in P, j \notin P, Q \text{ and } T_2(u, v, j) \text{ holds}\}; \]
and

- the relation $T_3$ as describing the set of transitions

$$\{(\{u, i\}, \{v, j\}) : u, v \in P, i, j \notin P, i \in Q, j \notin Q \text{ and } T_3(u, v, i, j) \text{ holds}\}.$$  

Furthermore, the initial marking of our Petri consists of the place $C$ and the places not in $P$ but in $Q$. We define the problem $\Omega_b$ as

$$\{P \in \text{STRUCT}(\sigma_b) : \text{there is a marking reachable from the initial marking in which there is at least one token on the place } D\}.$$  

Note that the transitions encoded within a $\sigma_b$-structure $P$ are of one of four types, as depicted in Fig. 1, and that the relations $T_1$, $T_2$ and $T_3$ of $P$ might have additional tuples in them that do not affect how we think of $P$ as a Petri net.

![Figure 1](image)

Figure 1: The different types of transitions.

With reference to our decision problem presented earlier, it should be clear that: the places in $P$ correspond to the vertices $V$ of our digraph $G = (V, E)$, with $C$ corresponding to the source vertex and $D$ the sink vertex; the places not in $P$ but in $Q$ correspond to the user resources; and the places not in $P$ and not in $Q$ correspond to the system resources (henceforth, we shall use this terminology to describe the places of our Petri net). Additionally, the transitions described by $T_1$ correspond to edges of $E$ with no labels; the transitions described by $T_2$ yield edges labelled with labels of the form ‘user resource $r_i$ is unused’ and ‘system resource $s_j$ is available’; and the transitions described by $T_3$ yield edges labelled with labels of the form ‘user resource $r_i$ is unused and this move uses this resource but makes the system resource $s_j$ available’. We interpret a user resource as being in the state ‘unused’, if there is a token on it, and as being in the state ‘used’ otherwise (such places only ever have at most one token on them). It may be the case, in a reachable marking, that a system resource has more than one token on it. However, tokens can not be removed from
such places. Thus, it is only ever important as to whether a system resource has no tokens on it, when we think of it being in the state ‘unavailable’, or at least one token on it when, we think of it being in the state ‘available’.

The proof of the following theorem is similar to those in [30] although there are additional complications caused by only having assignments which set array values to max.

**Theorem 6** There is a quantifier-free first-order translation with 2 constants from any problem in NPSB(1) to the problem $\Omega_b$. Hence, $\Omega_b$ is complete for NPSB(1) via quantifier-free first-order translations with 2 constants.

**Proof** Let $\rho$ be a program scheme of NPSB(1) over some signature $\sigma$ in which if and if-then-else instructions might occur. W.l.o.g., we may assume that array symbols only appear in assignment instructions, that there is only one array symbol, $B$, and that this array symbol has dimension $d \geq 1$. We assume that the variables involved in $\rho$ are $x_1, x_2, \ldots, x_k$.

Let $\mathcal{A}$ be a $\sigma$-structure of size $n \geq 2$. An element $u = (u_0, u_1, \ldots, u_k)$ of $\{1, 2, \ldots, l\} \times |\mathcal{A}|^k$ encodes a partial ID of $\rho$ on input $\mathcal{A}$ via: a computation of $\rho$ on $\mathcal{A}$ is about to execute instruction $u_0$ and the variables $x_1, x_2, \ldots, x_k$ currently have the values $u_1, u_2, \ldots, u_k$, respectively. Henceforth, we identify partial IDs of $\rho$ and the elements of $\{1, 2, \ldots, l\} \times |\mathcal{A}|^k$.

We now build a Petri net $\mathcal{P}$, as in Definition 5, using $\rho$ and $\mathcal{A}$. Our Petri net $\mathcal{P}$ has a set of places consisting of the set $\{1, 2, \ldots, l\} \times |\mathcal{A}|^k$ in union with the set $\{w_0, w_m : w \in |\mathcal{A}|^d\}$. The sets of places $P$ and $Q$ are

$$P = \{1, 2, \ldots, l\} \times |\mathcal{A}|^k \text{ and } Q = \{w_0 : w \in |\mathcal{A}|^d\},$$

respectively. Hence, the user resources are $\{w_0 : w \in |\mathcal{A}|^d\}$ and the system resources $\{w_m : w \in |\mathcal{A}|^d\}$. We shall use a token on the user resource $(w_1, w_2, \ldots, w_d)_0$ to signify that the current value of $B[w_1, w_2, \ldots, w_d]$ is 0; and a token on the system resource $(w_1, w_2, \ldots, w_d)_m$ to signify that the current value of $B[w_1, w_2, \ldots, w_d]$ is max. Obviously, we have to take care to ensure that a marking does not yield contradictory interpretations.

Let $u \in \{1, 2, \ldots, l\} \times |\mathcal{A}|^k$.

Suppose that the instruction $u_0$ does not involve the array symbol $B$ and it is possible for $\rho$ on input $\mathcal{A}$ to move from any ID whose partial ID is $u$ to an ID whose partial ID is $v$ in one step. Then the transition $\{(\{u\}, \{v\})\}$ is in $T_1$ (more precisely, the pair $(u, v)$ is in $T_1$).

Suppose that the instruction $u_0$ is of the form $x_j := B[x_{i_1}, x_{i_2}, \ldots, x_{i_d}]$ and it is possible for $\rho$ on input $A$ to move from any ID whose partial ID is $u$ to an ID whose partial ID is $v$ in one step (because the value of $B[x_{i_1}, x_{i_2}, \ldots, x_{i_d}]$ is such that $\rho$ on input $A$ can move from an ID whose partial ID is $u$ to an ID whose partial ID is $v$ in one step). Then both of the transitions $\{(u, (u_{i_1}, u_{i_2}, \ldots, u_{i_d}))_0, (v, (u_{i_1}, u_{i_2}, \ldots, u_{i_d}))_0\}$ and $\{(u, (u_{i_1}, u_{i_2}, \ldots, u_{i_d}))_m, (v, (u_{i_1}, u_{i_2}, \ldots, u_{i_d}))_m\}$ are in $T_2$ (of course, in the former transition, $v_j$ is 0, and in the latter $v_j$ is max, with $u_i = v_i$, for all $i = 1, 2, \ldots, k$ different from $j$).
Suppose that the instruction \( u_0 \) is of the form \( B[x_{i_1}, x_{i_2}, \ldots, x_{i_d}] := \text{max} \) and it is possible for \( \rho \) on input \( A \) to move from an ID whose partial ID is \( u \) to an ID whose partial ID is \( v \) in one step. Then the transition \( \{ \{ u, (u_{i_1}, u_{i_2}, \ldots, u_{i_d})_m \}; \{ v, (u_{i_1}, u_{i_2}, \ldots, u_{i_d})_m \} \} \) is in \( T_2 \) and the transition \( \{ \{ u, (u_{i_1}, u_{i_2}, \ldots, u_{i_d})_0 \}; \{ v, (u_{i_1}, u_{i_2}, \ldots, u_{i_d})_m \} \} \) is in \( T_3 \) (of course, in both transitions \( u_i = v_i \), for \( i = 1, 2, \ldots, k \).

Our initial marking of \( P \) is such that there is one token on each place of \( \{ w_0 : w \in |A|^d \} \) and one token on the place \( (1, 0) \in \{ 1, 2, \ldots, l \} \times |A|^k \), which we define to be \( C \); and we define \( D \) as the place \( (l, \text{max}) \in \{ 1, 2, \ldots, l \} \times |A|^k \).

It is not difficult to see that our Petri net \( P \) (that is, our \( \sigma_0 \)-structure \( P \)) can be described in terms of the \( \sigma \)-structure \( A \) using quantifier-free first-order formulae (in which 0 and \( \text{max} \) appear: explicit descriptions of structures by quantifier-free first-order formulae are given in, for example, \cite{25}). Consequently, in order for the result to follow we need to show that: \( A \models \rho \) if, and only if, \( P \in \Omega_0 \) and that \( \Omega_0 \in \text{NPSB}(1) \).

Suppose that \( A \models \rho \). Then there is a sequence \( \pi \) of (full, not partial) IDs starting at the initial ID (where all variables have the value 0, where the instruction to be executed is instruction 1 and where the array \( B \) has the value \( 0 \) throughout) and ending in a final ID (where all variables have the value \( \text{max} \) and where the instruction to be executed is instruction \( l \)) such that \( \rho \) moves from one ID in \( \pi \) to the next in one step. As hinted earlier, we can mirror any ID with a set of markings of our Petri net \( P \) as follows. If the ID consists of the partial ID \( u \in \{ 1, 2, \ldots, l \} \times |A|^k \) together with some valuation on the array \( B \) then the place \( u \) is marked with one token as are the places of \( \{ w_0 : w \in |A|^d, B[w] = 0 \} \), and the places of \( \{ w_m : w \in |A|^d, B[w] = \text{max} \} \) are marked with at least one token. This accounts for all tokens. Note that the initial ID of \( \rho \) corresponds to the initial marking of \( P \). A simple analysis yields that if \( \rho \) on input \( A \) moves from one ID to another in one step then the Petri net can fire a transition to move from the marking corresponding to the first ID to a marking corresponding to the subsequent ID; and conversely (as remarked earlier, as regards the system resources, it does not matter how many tokens reside on them but only whether or not at least one token resides). Hence, \( A \models \rho \) if, and only if, \( P \in \Omega_0 \).

All that remains is to show that \( \Omega_0 \in \text{NPSB}(1) \). There are two essential difficulties in deriving a program scheme to accept \( \Omega_0 \). First, a \( \sigma_0 \)-structure \( P \) might be such that a reachable marking involves more than one token on some system resource; and we need to cater for this event when we simulate a sequence of transitions in \( P \) by an execution of a program scheme on input \( P \). Second, we need to keep track of where tokens are in a way which avoids us modelling the fact that a token is on a place simply by using an array indexed by the place names; for we are not allowed to register that a token has moved from a place by assigning some array element the value \( 0 \) (recall, the only assignment instruction allowed on an array element is to set that element to \( \text{max} \)).

Our Petri net \( P \) is such that initially there is one token, call it \( t \), on the place \( C \) of \( P \) and there is one token on every user resource (we assume that the place \( C \) is indeed in \( P \): otherwise, our program scheme simply rejects the input \( P \)). No other tokens are involved in the initial marking. Also, transitions are such that we can imagine the token \( t \) as being moved from place to place amongst the places of \( P \), and we can imagine every other token either staying where it is, after some transition, or being moved from user resource to a system resource, and then staying where it is thereafter.
As regards our first difficulty, we do not need to actually monitor how many tokens lie on any system resource but only whether there is at least one token such a place. This obviates the need to count tokens. As regards our second difficulty, in order to decide whether (at least) one token lies on some system resource $s$, we use a dedicated array $B_1$, of dimension 1, so that whenever a token is placed on such a $s$ then $B_1[s]$ is set at $\max$: once $B_1[s]$ has been set to $\max$ we know that there will be a token on $s$ thereafter. In order to decide whether a token lies on some user resource $r$, we use an array $B_2$, of dimension 1, to register when the token originally on the place $r$ is first moved from $r$ by setting $B_2[r]$ equal to $\max$ at this point. Consequently, if we wish to know whether there is a token on such a place $r$, we test to see whether $B_2[r] = 0$ holds. Finally, we model the movement of the solitary token $t$ by using a dedicated variable, $x$ say: that is, the token $t$ is on place $p$ if, and only if, $x$ has the value $p$. Given the above discussion, it is straightforward to see that the problem $\Omega_b$ can be accepted by a program scheme of NPSB(1), and so the result follows. 

In essence, Theorem 3 tells us that any problem accepted by a program scheme of NPSB(1) can be described by a sentence of the form

$$\Omega_b[\lambda x \psi_P(x), x \psi_Q(x), x, y \psi_1(x, y), x, y, z \psi_3(x, y, z), x, y, z, w \psi_3(x, y, z, w)](u, v),$$

where: $|x| = |y| = |z| = |w| = k$, for some $k \geq 1$, and all variables are distinct; $\psi_P$, $\psi_Q$, $\psi_1$, $\psi_2$ and $\psi_3$ are quantifier-free first-order formulae over $\sigma_b \cup \{0, \max\}$; and $u$ and $v$ are $k$-tuples of constant symbols (in fact, we can actually take $u$ to be 0 repeated $k$ times and $v$ to be $\max$ repeated $k$ times: moreover, the sentence is such that whether it is true in some given structure is independent of the distinct values chosen for 0 and $\max$).

Similarly to as in [30], Theorem 3 allows us to relate the class of problems accepted by the program schemes of NPSB with the class of problems defined by the sentences of the logic $(\Omega_b)^*[\FO]$. For each $m \geq 1$, we define the fragment $\Omega_b(m)$ of $(\Omega_b)^*[\FO]$ as follows.

- $\Omega_b(1)$ consists of all formulae of the form

$$\Omega_b[\lambda x \psi_P, x \psi_Q, x, y \psi_1, x, y, z \psi_2, x, y, z, w \psi_3](u, v),$$

where: $\psi_P$, $\psi_Q$, $\psi_1$, $\psi_2$ and $\psi_3$ are quantifier-free first-order formulae over $\sigma_b \cup \{0, \max\}$; $u$ and $v$ are $k$-tuples of constant symbols or variables; there may be other free variables; and the truth of any interpretation of the formula (over a relevant structure and with values given for any free variables) is independent of the pair of distinct values chosen for 0 and $\max$.

- $\Omega_b(m + 1)$, for odd $m \geq 1$, consists of the universal closure of $\Omega_b(m)$; that is, the set of formulae of the form $\forall z_1 \forall z_2 \ldots \forall z_k \psi$, where $\psi$ is a formula of $\Omega_b(m)$.

- $\Omega_b(m + 1)$, for even $m \geq 2$, consists of the set of formulae of the form

$$\Omega_b[\lambda x (\psi_1 \lor \neg \psi_2), x(\psi_1 \lor \neg \psi_2), x, y(\psi_1 \lor \neg \psi_2), x, y, z(\psi_1 \lor \neg \psi_2), x, y, z, w(\psi_1 \lor \neg \psi_2)](u, v),$$

where: $\psi_1$, $\psi_2$, $\psi_3$ are quantifier-free first-order formulae over $\sigma_b \cup \{0, \max\}$; $u$ and $v$ are $k$-tuples of constant symbols or variables; there may be other free variables; and the truth of any interpretation of the formula (over a relevant structure and with values given for any free variables) is independent of the pair of distinct values chosen for 0 and $\max$. 

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where: $\psi_1^P, \psi_2^P, \psi_1^Q, \psi_2^Q, \psi_1^\ast, \psi_2^\ast, \psi_1^3, \psi_2^3$ and $\psi_3^3$ are formulae of $\pm \Omega_b(m)$; $u$ and $v$ are tuples of constant symbols or variables; there may be other free variables; and the truth of any interpretation of the formula (over a relevant structure and with values given for any free variables) is independent of the pair of distinct values chosen for 0 and $max$.

As in [28], a straightforward induction yields that:

- for every odd $m \geq 1$, every formula in the closure of $\pm \Omega_b(m)$ under $\land, \lor$ and $\exists$ is logically equivalent to a formula of $\pm \Omega_b(m)$; and

- for every even $m \geq 1$, every formula in the closure of $\pm \Omega_b(m)$ under $\land, \lor$ and $\forall$ is logically equivalent to a formula of $\pm \Omega_b(m)$.

Consequently, $(\pm \Omega_b)^[FO] = \cup \{\Omega_b(m) : m \geq 1\}$.

**Corollary 7** In the presence of 2 built-in constant symbols, $\pm \Omega_b(m) = NPSB(m)$, for each $m \geq 1$; and so $(\pm \Omega_b)^[FO] = NPSB$.

**Proof** We proceed by induction on $m$ almost identically to the proof of Corollary 10 from [30]. The base case, when $m = 1$, follows by Theorem 3.

Note that $(\pm \Omega_b)^[FO] = NPSB$ even in the absence of our 2 built-in constant symbols as we can ‘build them ourselves’ using existential quantification.

We end this section by showing that NPSB can not be realized as a fragment of $L_{\omega}^\omega$ (unlike NPS and NPSS).

**Lemma 8** The problem CUB can be accepted by a program scheme of NPSB(1).

**Proof** It was shown in [30] that CUB is in NPSA(1): however, the program scheme used there to accept CUB is not in NPSB(1). Nevertheless, the basic approach can be amended to yield a program scheme of NPSB(1).

Let $G$ be a $\sigma_2$-structure. We begin by ‘guessing’ a set of distinct edges in the graph $G$. We use two 3-dimensional array symbols, $B_1$ and $B_2$, to store these guessed edges. In particular, if our first guessed edge is $(u_1, v_1)$, having checked that $(u_1, v_1)$ is indeed an edge of $G$, we set $B_1[0, 0, u_1] = max$ and $B_2[0, 0, v_1] = max$. Next, we guess an edge $(u_2, v_2)$, check to see whether this edge is indeed an edge of $G$ and then check to see whether this edge is different from $(u_1, v_1)$. If so then we set $B_1[u_1, v_1, u_2] = max$ and $B_2[u_1, v_1, v_2] = max$: otherwise, we set $B_1[u_1, v_1, max] = max$ and $B_2[u_1, v_1, max] = max$ and stop guessing. We continue in this fashion until the guessing stage stops whence we have a list of distinct edges of $G$.

Finally, we check to see whether the guessed set of edges induces a regular subgraph of $G$ of degree 3. It is clear that this whole process can be implemented by a program scheme of NPSB(1); hence, the result follows.

The facts that the problem CUB can not be defined in $L_{\omega}^\omega$ (see [29]) and that non-recursive problems can be defined in $L_{\omega}^\omega$ (see [5]) immediately yield the following result.

**Corollary 9** There are problems definable in NPSB(1) (and so NPSB) which are not definable in $L_{\omega}^\omega$; and there are problems definable in $L_{\omega}^\omega$ which are not definable in NPSB.
5 Ordered structures and amended semantics

Given our characterization, in the preceding section, of the class of problems accepted by the program schemes of NPSB, we now consider the class of problems accepted by these program schemes when we restrict ourselves to ordered structures.

Using Theorem 1, we can easily modify the program scheme implicit in the proof of Lemma 8 so that, in the presence of a built-in successor relation, it accepts any given problem in \( \text{NP} \). Conversely, any problem in \( \text{NPSB}_s(1) \) is in \( \text{NP} \). Theorem 6 then yields the following result.

**Corollary 10** As classes of problems, \( \text{NP} = \text{NPSB}_s(1) \); and \( \Omega_b \) is complete for \( \text{NP} \) via quantifier-free first-order translations with successor. □

By Corollary 9, \( \text{NPSB}_s = (\pm \Omega_b)^{\ast}[\text{FO}_s] \); and by Corollary 5.5 of [23] and Corollary 11, \( (\pm \Omega_b)^{\ast}[\text{FO}_s] = (\pm \text{HP})^{\ast}[\text{FO}_s] \), where \( \text{HP} \) is the problem over the signature \( \sigma_{2^{++}} \) consisting of all those \( \sigma_{2^{++}} \)-structures \( A \) which, when considered as digraphs with edge relation \( E^A \) and two given vertices \( C^A \) and \( D^A \), are such that there is a Hamiltonian path from \( C^A \) to \( D^A \). Furthermore, by Corollary 3.2.2 of [26], \( (\pm \text{HP})^{\ast}[\text{FO}_s] = \text{L}_{\text{NP}} \) (the class of problems accepted by a log-space deterministic oracle Turing machine with access to an \( \text{NP} \) oracle), and every problem in \( (\pm \text{HP})^{\ast}[\text{FO}_s] \) can be defined by a sentence of the form:

\[
\exists z_1 \exists z_2 \ldots \exists z_m (\text{HP}[\lambda x, y\psi(x, y, z)](0, \text{max}) \land \neg\text{HP}[\lambda x, y\varphi(x, y, z)](0, \text{max})),
\]

where: \( x \) and \( y \) are \( k \)-tuples of variables, for some \( k \); \( \psi \) and \( \varphi \) are quantifier-free first-order formulae (with successor); and \( 0 \) (resp. \( \text{max} \)) is the constant symbol \( 0 \) (resp. \( \text{max} \)) repeated \( k \) times. Hence, translating this normal form into a program scheme yields that any problem in \( \text{NPSB}_s \) can actually be accepted by a program scheme of \( \text{NPSB}_s(3) \). Furthermore, any problem accepted by a program scheme \( \forall z_1 \forall z_2 \ldots \forall z_m \rho \) of \( \text{NPSB}_s(2) \) can be accepted by a program scheme of \( \text{NPSB}_s(1) \): we simply replace the universal quantification by code within a program scheme of \( \text{NPSB}_s(1) \) which uses a while instruction and the successor relation to check whether a structure is accepted by \( \rho \) for every valuation of the free variables \( z_1, z_2, \ldots, z_m \). Hence, we have the following result.

**Theorem 11** \( \text{NPSB}_s(1) = \text{NPSB}_s(2) = \text{NP} \) and \( \text{NPSB}_s(3) = \text{NPSB}_s = \text{L}_{\text{NP}} \). □
fact that a polynomial number of values is passed across in an evaluation (rather than just a constant number in the standard semantics). Allowing a polynomial number of values to be passed across to an evaluation drastically changes the expressibility of the resulting class of program schemes (modulo the usual complexity-theoretic qualifications). The complexity class $\text{PH}$ is the Polynomial Hierarchy; that is, $\text{PH} = \bigcup_{i=1}^{\infty} \Sigma^p_i$, where $\Sigma^p_1 = \text{NP}$ and where, for each $i \geq 2$, $\Sigma^p_i = \text{NP} \Sigma^p_{i-1}$ (the class of problems accepted by a polynomial-time non-deterministic oracle Turing machine with access to a $\Sigma^p_{i-1}$ oracle).

**Theorem 12** $\text{NPSB}^p(1) = \text{NPSB}(1)$, $\text{NPSB}^p(2) = \text{NPSB}(2)$ and for every $i \geq 2$, $\text{NPSB}^p(2i-1) = \text{NPSB}^p(2i) = \Sigma^p_i$. Consequently, $\text{NPSB}^p = \text{PH}$.

**Proof** Similarly to the proof (elucidated immediately prior to Theorem 11) that $\text{NPSB}_s(1) = \text{NPSB}_s(2)$, so we can show that $\text{NPSB}^p(2i-1) = \text{NPSB}^p(2i)$, for all $i \geq 2$. Obviously, $\text{NPSB}^p(1) = \text{NPSB}(1)$ and $\text{NPSB}^p(2) = \text{NPSB}(2)$ (as our original semantics and our amended semantics do not differ in these cases).

We now show how to build our own successor relation using a program scheme of $\text{NPSB}^p(3)$. Essentially, we guess a successor relation and store it in the array $S$, of dimension 2, via the following code:

```
x := 0
while x $\neq$ max do
  guess y
  if x $\neq$ y then
    S[x, y] := max
    x := y
  fi
od
```

Then we check, using an if-then-else instruction with the test a program scheme of $\text{NPSB}^p(2)$, that every value appears in the guessed relation $S$ and that no value appears more than once. Consequently, by Corollary 10, any problem in $\text{NP}$ can be accepted by some program scheme of $\text{NPSB}^p(3)$.

Notwithstanding the above remark, we would like to explicitly simulate a non-deterministic polynomial-time Turing machine computation using a program scheme of $\text{NPSB}^p(3)$. We can use arrays to store the work-tape of any such Turing machine and our successor relation, held in $S$, to mirror the movement of the tape heads. Our only restriction to this simulation is that we can only set array values at `max`: we cannot reset them to 0. Hence, the obvious means of simulation is doomed to failure given that, in general, the contents of a cell of a Turing machine work-tape fluctuate and that if we simulate a cell of the work tape using a fixed number of array elements then we can only register a constant number of changes to the cell contents. However, we can get round this difficulty by using the fact that any (accepting) computation of our Turing machine has length polynomial in the size of the input structure: hence, we can use an array to store the complete history of changes to the contents of a Turing machine work-tape cell as follows.

For simplicity, assume that we wish to hold the contents of $n$ Turing machine work-tape cells (where the input structure has size $n$) using some arrays and that
these contents are only ever 0 or 1. Furthermore, assume that the time taken by our Turing machine to accept (if it does) is \(n\). The general case where a cell can contain more symbols, where there is a polynomial number of work-tape cells to deal with and where the Turing machine accepts in a polynomial number of steps can be handled similarly by increasing the dimensions of our arrays. Let \(A\) and \(B\) be array symbols of dimension 2. Using our successor relation (constructed earlier), we use the array cells \(A[u, 1], A[u, 2], \ldots, A[u, n]\) (we think of the elements of our input structure as being named \(\{1, 2, \ldots, n\}\) with the names reflecting our successor relation) to register the first change of the contents of the work-tape cell \(u\), the second change of the work-tape cell \(u\), the third change of the work-tape cell \(u\), and so on; and the array cells \(B[u, 1], B[u, 2], \ldots, B[u, n]\) to register the value of work-tape cell \(u\) after the first change, the value of work-tape cell \(u\) after the second change, the value of work-tape cell \(u\) after the third change, and so on.

If \(A[u, i] = \text{max}\) then this is interpreted as meaning that there have been at least \(i\) changes of contents; and if \(B[u, i] = 0\) (resp. \(B[u, i] = \text{max}\)) then this is interpreted as meaning that after the \(i\)th change, the contents of work-tape cell \(u\) is 0 (resp. 1). Note that when the work-tape cell \(u\) changes from 1 to 0, on the \(i\)th change, say, in order to register this change we need only set \(A[u, i] = \text{max}\) and leave \(B[u, i]\) alone (as it has been initialized to 0). Furthermore, with this representation, and using our successor relation, we can easily determine the current contents of any work-tape cell: we simply cycle down the array \(A\) to find the last change of contents and then ascertain the current contents using \(B\). Thus, it should be clear how we can explicitly simulate our Turing machine computation using a program scheme of NPSB\(^p\)(3).

Now, consider a polynomial-time non-deterministic oracle Turing machine \(M\) consulting an \(\text{NP}\) oracle. By Corollary 10 and using an array to hold the contents of the oracle tape, we can simulate an oracle call of \(M\) by an if-then-else instruction where the test is a program scheme of NPSB\(^p\)(2) (exactly because we are allowed, in our modified semantics, to pass the values of arrays over to the evaluation of a test). Hence, we have essentially proven that any problem in \(\text{NP}^{\text{NP}}\) can be accepted by a program scheme of NPSB\(^p\)(3). Conversely, it is straightforward to see that any problem accepted by a program scheme of NPSB\(^p\)(3) can be accepted by a polynomial-time non-deterministic oracle Turing machine with an oracle in \(\text{NP}\) (the only point worthy of note in this regard is that we must ensure that the contents of all arrays in the program scheme are written on the simulating Turing machine’s oracle tape). Hence, NPSB\(^p\)(3) = \(\text{NP}^{\text{NP}}\).

The general result now follows by a simple induction: for example, any polynomial-time non-deterministic oracle Turing machine consulting an oracle in \(\text{NP}^{\text{NP}}\) can be explicitly simulated; and by above the oracle calls can be simulated by if-then-else instructions where the tests are program schemes from NPSB\(^p\)(4).

### 6 Some relative computational capabilities

We now turn to the relative computational capabilities of the classes of program schemes NPSB(1) and NPSA(1) on the class of all finite structures (we have more to say about comparing the classes NPSB and NPSA in the Conclusion).

The following definitions are essential to what follows. Let \(\sigma\) be some signature and let \(\mathcal{A}\) and \(\mathcal{B}\) be \(\sigma\)-structures. If \(|\mathcal{A}| \subseteq |\mathcal{B}|\) and:
for every relation symbol $R$ of $\sigma$, $R^A$ is $R^B$ restricted to $|A|$; and

for every constant symbol $C$ of $\sigma$, $C^A = C^B$,

then we say that $A$ is a sub-structure of $B$ and write $A \subseteq B$. If the problem $\Omega$ over $\sigma$ is such that for all $\sigma$-structures $A$ and $B$ for which $A \subseteq B$, it is necessarily the case that $A \in \Omega$ implies $B \in \Omega$, then we say that $\Omega$ is closed under extensions. Let $\mathcal{EXT}$ be the class of all problems that are closed under extensions.

**Lemma 13** Every problem in NPSA(1) is closed under extensions.

**Proof** Let $\Omega$ be a problem over the signature $\sigma$ accepted by the program scheme $\rho$ of NPSA(1). Let $A$ and $B$ be $\sigma$-structures such that $A \subseteq B$, and suppose that $A \models \rho$. Consider the program scheme $\rho$ on input $B$ where 0 and $max$ are chosen to be distinct elements of $|A|$. By ‘mirroring’ an accepting computation of $\rho$ on input $A$, with the chosen 0 and $max$, we obtain an accepting computation of $\rho$ on input $B$ (the fact that all tests in while, if and if-then-else instructions are quantifier-free first-order enables us to do this). Hence, $B \in \Omega$.

**Theorem 14** $\text{PSPACE} \cap \mathcal{EXT} = \text{NPSA}(1)$ and $\text{NP} \cap \mathcal{EXT} = \text{NPSB}(1)$.

**Proof** Let $\Omega$ be some problem in $\text{PSPACE} \cap \mathcal{EXT}$. By [24], there exists a program scheme $\rho \in \text{NPSA}_s(1)$ accepting $\Omega$. Modify $\rho$ to obtain the program scheme $\rho' \in \text{NPSA}(1)$ as follows. In $\rho'$, begin by guessing a successor relation; that is, when $A$ is some input structure, guess elements $u_1, u_2, \ldots, u_m \in |A|$ so that

$M[0] = u_1, M[u_1] = u_2, \ldots, M[u_m] = max$,

where $M$ is a new one-dimensional array symbol and where the elements of $\{0, u_1, u_2, \ldots, u_m, max\}$ are distinct (this latter condition can be checked as we guess). Replace any atomic relation of the form $\text{succ}(x, y)$ in $\rho$ with the formula $y = M[x]$, and replace any instruction of the form $\text{guess } x$ with the following fragment of code:

```plaintext
guess x
  goodx := 0
  ok := 0
  while ok = 0 do
    if ($x = \text{goodx}$ $\lor$ $\text{goodx} = \text{max}$) then
      ok := max
    else
      goodx := $M[\text{goodx}]$
    fi
  od
  if $x \neq \text{goodx}$ then 'loop forever' fi
```

(where $\text{goodx}$ and $\text{ok}$ are new variables). Note that this fragment of code essentially limits our guesses to elements appearing in the domain of our guessed successor relation. We need to show that acceptance by the program scheme $\rho'$ is invariant with respect to 0 and max and that it accepts the problem $\Omega$. 

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Suppose that $A \in \Omega$. Then $A$ is accepted by $\rho$ no matter which successor relation is chosen for $\text{succ}$ in $\rho$. Choose distinct $0'$ and $\text{max}'$ in $|A|$ and a successor relation $\text{succ}'$ on $|A|$ (with minimal and maximal elements the chosen elements $0'$ and $\text{max}'$).

In particular, $\rho$ accepts $A$ with these constants and this successor relation. Consider a computation of $\rho'$ on input $A$ where the guessed successor relation is $\text{succ}'$. Then there exists a computation of $\rho'$ mirroring any accepting computation of $\rho$ on input $A$ with this particular successor relation. That is, $A$ is accepted by $\rho'$ and acceptance does not depend upon the chosen constants $0$ and $\text{max}$.

Conversely, suppose that there is a guessed successor relation, call it $\text{succ}'$ (whose domain need not be all of $|A|$), with minimal and maximal elements $0'$ and $\text{max}'$, yielding an accepting computation of $\rho'$ on input $A$. Let $B \subseteq |A|$ be the domain of this successor relation and let $B$ be the restriction of $A$ to $B$. Then $B$ is accepted by $\rho$ when the successor relation is taken as $\text{succ}'$ (note that the domain of $\text{succ}'$ is the whole of $|B|$). Hence, $B \in \Omega$. However, $\Omega$ is closed under extensions and so $A \in \Omega$. But we have seen from above that if $A \in \Omega$ then $A$ is accepted by $\rho'$ and acceptance does not depend upon the chosen constants $0$ and $\text{max}$. Thus, acceptance by $\rho'$ is invariant with respect to $0$ and $\text{max}$; and $\text{PSPACE} \cap \text{EXPTIME} \subseteq \text{NPSA}(1)$. The fact that every problem in $\text{NPSA}(1)$ can be solved by a polynomial-space algorithm is straightforward; and every problem in $\text{NPSA}(1)$ is closed under extensions by Lemma 13.

Now consider a problem $\Omega \in \text{NP} \cap \text{EXPTIME}$ accepted by the program scheme $\rho \in \text{NPSB}(1)$. We proceed as above, and define a program scheme $\rho' \in \text{NPSB}(1)$, except with the following amendment. In NPSB(1), we are only allowed assignments to array elements of the form $M[x_1, x_2, \ldots, x_k] := \text{max}$ and so we need some way of encoding our guessed successor relation. We encode our relation as:

$$M[0, u_1] = \text{max}, M[u_1, u_2] = \text{max}, \ldots, M[u_m, \text{max}] = \text{max},$$

where $M$ is a new array symbol of dimension 2. Of course, we ensure that the elements of $\{0, u_1, u_2, \ldots, u_m, \text{max}\}$ are distinct as we guess. Note that we need to remember the previously guessed element, $u_i$, so that we know to set $M[u_i, u_{i+1}]$ equal to $\text{max}$. We also need to modify our code so that an atomic relation of the form $\text{succ}(x, y)$ is replaced by the formula $M[x, y] = \text{max}$. Arguing as above yields the result. 

One view of Theorem 14 is that it provides syntactic characterizations (via the classes of program schemes $\text{NPSA}(1)$ and $\text{NPSB}(1)$) of semantically defined complexity classes (namely, $\text{PSPACE} \cap \text{EXPTIME}$ and $\text{NP} \cap \text{EXPTIME}$).

**Corollary 15** $\text{NP} = \text{PSPACE}$ if, and only if, $\text{NPSA}(1) = \text{NPSB}(1)$.

**Proof** If $\text{NP} = \text{PSPACE}$ then $\text{NP} \cap \text{EXPTIME} = \text{PSPACE} \cap \text{EXPTIME}$; and so $\text{NPSA}(1) = \text{NPSB}(1)$ by Theorem 14. Conversely, if $\text{NPSA}(1) = \text{NPSB}(1)$ then $\text{NPSA}(1) = \text{NPSB}(1)$; and so $\text{NP} = \text{PSPACE}$ by [24] and Corollary 10. 

Corollary 15 is somewhat surprising given that every problem in $\text{NPSA}$ (and so $\text{NPSB}$) has a zero-one law. In fact, any problem in $\text{EXPTIME}$, apart from the empty problem (over some signature), has a ‘1-law’ (the meaning of ‘1-law’ should be obvious); and so any non-trivial problem in $\text{NPSA}(1)$ (and $\text{NPSB}(1)$) has a 1-law. Note that
any class of problems each of which has a zero-one law can not contain, for example, the computationally trivial problem consisting of all those structures of even size.

Corollary 15 can be extended slightly in that we can obtain some additional equivalences involving fragments of certain vectorized Lindström logics. Referring back to Theorem 4, the problem mentioned in that theorem is actually defined as follows.

**Definition 16** Let the signature $\sigma_a = \langle T_1, T_3, M, C \rangle$, where $M$ is a unary relation symbol, $T_1$ is a binary relation symbol, $T_3$ is a relation symbol of arity 4 and $C$ is a constant symbol. We can envisage a $\sigma_a$-structure $A$ as a Petri net whose places are given by $|A|$ and whose transitions are given by $T_1$ and $T_2$ via:

- there is a transition $(\{x\}, \{y\})$ whose input place is $\{x\}$ and whose output place is $\{y\}$ if, and only if, $T_1(x, y)$ holds; and
- there is a transition $(\{x_1, x_2\}, \{y_1, y_2\})$ whose input places are $\{x_1, x_2\}$ and whose output places are $\{y_1, y_2\}$ if, and only if, $T_3(x_1, x_2, y_1, y_2)$ holds, where $x_1 \neq x_2$ and $y_1 \neq y_2$.

The relation $M$ can be seen as providing an initial marking (with one token on place $p$ if, and only if, $M(p)$ holds) and the constant $C$ as providing a final marking (consisting of one token on the place $C$).

A $\sigma_a$-structure $A$, i.e., a Petri net, complete with initial and final markings, where every transition has either 2 input places and 2 output places or 1 input place and 1 output place, is in the problem $\Omega_a$ if, and only if, there is a marking covering the final marking that is reachable from the initial marking, i.e., there is a reachable marking in which there is at least one token on the place $C$.

**Corollary 17** The following are equivalent.

(a) $\text{NP} = \text{PSPACE}$.

(b) $\text{NPSB}(1) = \text{NPSA}(1)$.

(c) $\Omega^1_b[\text{FO}] = \Omega^1_a[\text{FO}]$.

(d) The problems $\Omega_b$ and $\Omega_a$ are equivalent via quantifier-free first-order translations with 2 constants.

**Proof** Corollary 15 implies (a) $\iff$ (b). Theorems 4 and 6 imply (b) $\iff$ (d). It is trivially the case that (c) $\implies$ (a) and that (d) $\implies$ (c).

We end by returning to an earlier remark concerning the NPSB hierarchy on the class of all finite structures. We include the following result here as we can utilize results of this section, and this result also applies to the NPSA hierarchy.

**Proposition 18** On the class of all finite structures,

$\text{NPSB}(1) \subset \text{NPSB}(2) \subset \text{NPSB}(3)$. 

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By Lemma 13, every problem in NPSB(1) is closed under extensions; and so, trivially, NPSB(1) ⊂ NPSB(2).

Consider the following first-order sentence over the signature σ₂ = ⟨E, C⟩, where E is a binary relation symbol and C is a constant symbol:

$$\exists x(E(C,x) \land \exists y(E(x,y) \land \forall z(E(x,z) \Rightarrow z = y)))$$.

There is clearly a program scheme of NPSB(3) accepting the problem Ω defined by this sentence. For any k ≥ 1, consider the digraphs, A_k and B_k, depicted in Fig. 2 (note that B_k only differs from A_k by having an extra vertex and edge). No matter what the value of k, A_k ∈ Ω but B_k ∉ Ω. We shall show that for any program scheme ρ of NPSB(2), there exists some k such that A_k |= ρ implies that B_k |= ρ. This will yield our result.

Let ρ be a program scheme of NPSB(2) of the form $\forall x_1 \forall x_2 \ldots \forall x_k \rho'$, for some program scheme $\rho'$ of NPSB(1), and let $\bar{B}_k$ be an extension of $B_k$ by k constants (one for each variable $x_i$). There is an extension of $A_k$, denote it $\bar{A}_k$, such that $\bar{A}_k$ is embeddable into $\bar{B}_k$ via a one-to-one mapping: call the mapping $\pi$. Suppose that there is an accepting computation of $\rho'$ on input $\bar{A}_k$. We can ‘mirror’ this computation by a computation of $\rho'$ on $\bar{B}_k$ by making guesses according to the mapping $\pi$ (after having chosen our constants 0 and max, again according to $\pi$). The two computations of $\rho'$, on $\bar{A}_k$ and $\bar{B}_k$, proceed in tandem (in that their flows of control are identical) and because the computation of $\rho'$ on $\bar{A}_k$ leads to acceptance, so must the computation of $\rho'$ on $\bar{B}_k$ (recall, any tests are quantifier-free first-order and so only ever refer to the current values of variables). Our result follows.

We add that the proof of Proposition 18 suffices to show that, on the class of all finite structures, NPSA(1) ⊂ NPSA(2) ⊂ NPSA(3).

7 Conclusions

In this paper, we have examined the computational capabilities of different classes of program schemes, based around ‘binary write-once arrays’, on the class of finite structures, the class of ordered finite structures and with respect to different semantics. We now discuss some potential directions for future research.

Perhaps the most obvious unanswered question is as regards the NPSB hierarchy: ‘Is it the case that, like the NPS and NPSS hierarchies, the NPSB hierarchy is proper
at every level?’ (the same question can be asked for the NPSA hierarchy). So far, we have not been able to answer this question (beyond Proposition 18). The main reason for the lack of progress is that whereas in [4] we were able to ‘re-use’ domain elements so as to ‘mirror’ computations of program scheme of NPS and NPSS (in the style of the proof of Proposition 18), the existence of arrays means that we can ‘remember the values already used’ in a computation and consequently it is not clear that domain elements can be re-used in a suitably anonymous fashion (the reader is referred to [4], and the proofs therein, in order to make more sense of this remark). The fact that working with program schemes of NPSB takes us outside the ‘bounded-variable world’ of the logic $L_{\omega}^{\omega}$ (see Corollary 9), whereas this is not the case with the program schemes of NPS and NPSS, is particularly intriguing in this respect.

The results in Section 6, relating the computational capabilities of the classes of program schemes NPSB(1) and NPSA(1), are in the style of Abiteboul and Vianu [1, 2], Abiteboul, Vianu and Vardi [3] and Dawar [6]. However, we would prefer to have determined similar results but regarding the classes NPSB and NPSA (or, equivalently, the logics $(\pm \Omega_b)^*[\text{FO}]$ and $(\pm \Omega_a)^*[\text{FO}]$). So far, we have been unable to extend the results of Section 6 to these classes of program schemes. There are some very straightforward implications to be made however. For instance (on the class of all finite structures):

- by Corollary 17 if $\text{NP} = \text{PSPACE}$ then NPSB = NPSA (and, equivalently, $(\pm \Omega_b)^*[\text{FO}] = (\pm \Omega_a)^*[\text{FO}]$);
- by Theorems 4 and 11, if NPSB = NPSA (or, equivalently, $(\pm \Omega_b)^*[\text{FO}] = (\pm \Omega_a)^*[\text{FO}]$) then $L^{\text{NP}} = \text{PSPACE}$; and
- by Theorems 4 and 11, if $\Omega_b^*[\text{FO}] = \Omega_a^*[\text{FO}]$ then $\text{NP} = \text{PSPACE}$ (as any problem in $\Omega_b^*[\text{FO}]$ can easily be seen to be in $\text{NP}$).

We would like to be able to equate the questions: ‘Is $L^{\text{NP}}$ equal to $\text{PSPACE}$?’ ‘Is NPSB equal to NPSA?’ and ‘Is $(\pm \Omega_b)^*[\text{FO}]$ equal to $(\pm \Omega_a)^*[\text{FO}]$?; as well as the questions: ‘Is $\text{NP}$ equal to $\text{PSPACE}$?’ and ‘Is $\Omega_b^*[\text{FO}]$ equal to $\Omega_a^*[\text{FO}]$?’ As yet, we have been unable to do so.

Finally, let us return to the decision problem described at the beginning of Section 4 involving the traversal of a digraph subject to the utilization of user and system resources. We feel that this problem, and its variations, are very relevant in the study of the complexity of agent-based systems. Essentially, an agent-based system is an environment within which an agent must successfully accomplish a task. Agents interact with the environment by performing actions and these actions can result in a change of state of the environment. The reader is referred to [32] for some basic definitions and complexity-theoretic results in agent-based systems. Our resource-dependent digraph traversal problem can easily be viewed as an agent-based system, and we intend to investigate exactly how the study of program schemes and logics can impact upon that of agent-based systems in a future paper.

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