Giant Shapiro resonances in a flux driven
Josephson junction necklace

G. Chu and Jorge V. José

Department of Physics, Northeastern University, Boston, MA 02115

We present a detailed study of the dynamic response of a ring of $N$ equally spaced
Josephson junctions to a time-periodic external flux, including screening current
effects. The dynamics are described by the resistively shunted Josephson junction
model, appropriate for proximity effect junctions, and we include Faraday’s law for
the flux. We find that the time-averaged $I–V$ characteristics show novel subharmonic
giant Shapiro voltage resonances, which strongly depend on having phase slips or not,
on $N$, on the inductance and on the external drive frequency. We include an estimate
of the possible experimental parameters needed to observe these quantized voltage
spikes.

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Motivated by significant advances in microfabrication techniques of Josephson junction (JJ) arrays, many contemporary studies have focused on understanding the dynamic response of two-dimensional JJ arrays [1-7]. One type of array with well-defined uniform properties is made of superconducting-normal-superconducting (SNS) proximity effect junctions. Several novel collective locked-in states have been discovered which are manifested in the current versus voltage ($I$ vs $V$) characteristics as plateaus. These giant Shapiro steps are a macroscopic manifestation of diverse underlying collective vortex oscillating states [1-5].

Most recent studies, experimental and theoretical [1-5,7], have considered current driven two-dimensional JJ arrays. In this paper we study the dynamic response of a ring of $N$ equally spaced JJ, referred to as a Josephson Junction Necklace (JJN), driven by a perpendicular external time-dependent flux $\Phi_e(t)$, including self-induced magnetic field effects. The motivation of this study is two-fold; first, to understand the dynamic response of a JJ array to a $\Phi_e(t)$, and second, to assess the importance of geometry in this response. As we show in this paper, the physics of the JJN is, in many respects qualitatively and quantitatively different from that of a current driven planar array.

The JJN is shown schematically in the inset of Fig. 1. This model is a natural extension of the standard rf SQUID to $N$ junctions instead of one. The external time-dependent flux considered here produces a Faraday voltage $V_e(t)$ of the form, $-\frac{d}{dt}\Phi_e(t) = V_e(t) = V_{dc} + V_{ac}\cos 2\pi\tilde{f}t$. We also include the self-induced magnetic flux $\Phi_i$ due to the screening currents, of importance in a proximity effect JJ. We model the dynamics of the order-parameter phase of the JJN in terms of the resistively shunted Josephson junction model (RSJ) defined by the current

$$I(t) = I_c \sin[\phi_i - \phi_{i+1} - \psi_{i,i+1}] + \frac{\hbar}{2eR} \frac{d}{dt}[\phi_i - \phi_{i+1} - \psi_{i,i+1}].$$

(1)

In this equation $\Delta_\mu \phi = \phi_i - \phi_{i+1}$ is the phase difference for the $\mu$-th junction denoted as $\times$ in the inset of Fig. 1. $I_c$ is its critical current and $R$ its shunt resistance, both assumed to be the same for all junctions in the necklace. Here $\psi_{i,i+1}(t) = \frac{2\pi}{\Phi_0} \int_j^{i+1} \vec{A} \cdot d\vec{\ell} = \frac{2\pi}{N} \frac{\Phi}{\Phi_0} \equiv \frac{2\pi}{N} f(t)$, where $\Phi(t)$ is the total flux through the ring of radius $r$, $f(t) = (\Phi_e(t) + \Phi_i(t))/\Phi_0 = f_e + f_i$. 

\[2\]
and $\Phi_0 = \frac{h}{2e}$ is the quantum of flux. The induced flux is given by $\Phi_i = LI(t)$, where $L$ is the geometrical inductance of the ring. It is convenient to write Eq(1) in dimensionless units with $i(\tau) = I(t)/I_c$, $\tau = t(2eRI_c/\hbar)$, $v = V/(RI_c)$, $\ell = L\frac{2\pi I_c}{\Phi_0}$, and $\nu_\phi = \frac{\hbar \nu}{2eRI_c}$. In these units Eq.(1) becomes

$$i(\tau) = \sin\left[\frac{2\pi}{N}\left([f(\tau)]_k - f(\tau)\right)\right] + \frac{2\pi}{N} \frac{d}{d\tau}\left([f(\tau)]_k - f(\tau)\right).$$

(2)

In writing this equation we have used the fact that the current is conserved in the ring which, together with the periodic boundary conditions (p.b.c.) $\phi_{N+1} = \phi_1$, implies that the phase difference $\Delta\phi = \frac{2\pi}{N}[f(\tau)]_k$, where $[f]_k$ stands for the nearest integer function defined as $[f(\tau)]_k = \text{int}(f + \frac{1}{2})$. The specific form of the applied external flux considered in this paper is $f_e = \Phi_e/\Phi_0 = -(f^0_e + v_{dc}/2\pi\tau + \frac{v_{ac}}{2\pi\nu_\phi}\sin 2\pi
\nu_\phi \tau)$, with $f^0_e \equiv f_e(\tau = 0)$. Substituting this expression for $f_e$ in Eq(2) we get

$$\frac{\ell}{N} \frac{d}{d\tau}i(\tau) + \left\{i(\tau) - \sin\left[\frac{2\pi}{N}\left([f(\tau)]_k + 1\right)\right] - 2\pi f^0_e + v_{dc}\tau + \frac{v_{ac}}{2\pi\nu_\phi}\sin 2\pi
\nu_\phi \tau - \ell i(\tau)\right\}

= \frac{1}{N}(v_{dc} + v_{ac} \cos 2\pi\nu_\phi \tau) + \frac{2\pi}{N} \frac{d}{d\tau}\left([f(\tau)]_k\right).$$

(3)

Eq(3) is the general equation describing the response of the JJN to a time-dependent external flux, and understanding the structure of its time-dependent solutions is the central goal of this paper. To find $i(\tau)$ from Eq(3) we need to know $[f(\tau)]_k$. As shown below, we can use the ground state properties ($f_e = f^0_e$) of the JJN to obtain solutions for $[f(\tau)]_k$ in the relevant physical regimes. We start then by considering the ground state properties of the JJN including inductive effects, which appear not to have been discussed before for $N \geq 2$. From the p.b.c. and current conservation it follows that the phase difference for each and all junctions is $\Delta\phi = (2\pi k)/N$, with $k = 0, 1, 2, 3, ..., N - 1$. When the self-inductance is neglected, the ground state energy per junction $E_g/N$, in normalized units, is $\epsilon_N = \frac{2\pi E_e}{\Phi_0 I_c N} = -\cos[\frac{2\pi}{N}([f^0_e]_k - f^0_e)]$, with the corresponding normalized ground state current $i_N$ obtained from $i_N = \frac{-\partial \epsilon_N}{\partial f^0_e} = \sin\left[\frac{2\pi}{N}([f^0_e]_k - f^0_e)\right]$. For a periodic JJN we note the important symmetry of the ground state energy $f^0_e \rightarrow f^0_e + 1$ and $f^0_e \rightarrow f^0_e - 1$, which allows the analysis to be restricted to $f^0_e \epsilon[0, 1]$. In Fig. 1 we show $\epsilon_N(\ell = 0)$ (solid line).
for \( N = 10 \). The difference between curves I and II is that in II there is a discontinuity in \( \epsilon_N \) at \( f_e^0 = 1/2 \), whereas curve I is continuous. Physically, as \( f_e^0 \) increases from zero to \( f_e^0 = (1/2)^- \) the JJN can either follow curve I or curve II. Following II entails absorbing a quantum of flux (\( k = 1 \)) while following I implies no change in the flux trapped by the JJN. Of course, in equilibrium the system will prefer to follow II rather than I, which is a higher energy state. However, if there is an external \( f_e(\tau) \) present the JJN will be allowed to follow curve I as well. We shall call the process following curve II a phase slip (PS) case while the one that follows I the no phase slip (NPS) process. For the \( \ell \neq 0 \) case, the ground state energy becomes \( \epsilon_N = -\cos[(\pi/2)(f_k - f)] + \frac{2\pi^2 \ell^2}{\epsilon N} \) and the corresponding normalized current is \( i_N = \sin[(\pi/2)(f_k - f)] \). From these equations one finds that the total flux in the ring is given by \( f = f_e^0 + \frac{\ell}{2\pi} \sin[(\pi/2)(f_k - f)] \), which is a self-consistent transcendental equation that, for a given set of values for \( f_e^0 \) and \( \ell \), can be solved numerically. We note that in contrast to the \( N = 1 \) case, where there is a critical value \( \ell_c = 1 \) that separates non-hysteretic (\( \ell > 1 \)) from hysteretic (\( \ell < 1 \)) behavior, the JJN with \( N \geq 2 \) is always hysteretic. The size of the hysteresis loop grows as \( \ell \) increases from zero until \( \ell \geq N \), when the hysteresis loop covers the whole \( f_e^0 \epsilon[0, 1] \) range. The metastability of the JJN is evident from the history-dependent ground state energy shown as a dashed line in Fig. 1. In the PS case we see that there are two values of \( f_e^0 \) for which \( \epsilon_N \) has a discontinuity, one while ramping up \( f_e^0 \) the other when decreasing it. The specific \( f_e^0 \) values at which the discontinuities take place depend on \( \ell \) and \( N \). The NPS case is, again, represented by continuous processes.

With the above information we are now ready to discuss the dynamic properties of the JJN. We start by introducing the important parameter \( \kappa^2 \equiv \nu_\Phi/\nu_\phi \), with \( \nu_\Phi \) the characteristic frequency for the relaxation of magnetic fields. In the linearized regime of the Josephson term in Eq(3), \( \nu_\Phi = R/L \), and \( \kappa = \sqrt{\frac{\Phi_0}{2\pi i \epsilon L}} \equiv \sqrt{1/\ell} \). This parameter measures the importance of screening effects: when \( \kappa = \infty \) they are negligible while for \( \kappa \leq 1 \) they are all important. Thus we want to study the dynamics of the JJN as a function of \( \kappa \) in the two extreme regimes of PS and NPS. For simplicity we shall write \( \ell \) instead of \( \kappa \) in our
results, although the connection between \( \kappa = \sqrt{1/\ell} \) is only valid in the linearized regime. Most of the time the value of \([f(\tau)]_k\) is constant except when \(f(\tau)\) passes through the phase slip points of Fig. 1. In this case \([f(\tau)]_k\) is discontinuous and its derivative \(\frac{d}{d\tau}[f(\tau)]_k\) has a \(\delta\)-function character. We use this information in evaluating the time-averaged current given in Eq(3). We start by considering the simplest case when \(\ell = 0\), i.e. when there is no screening. In the NPS case \(<i_N>\) can be calculated analytically since \(\frac{d}{d\tau}[f(\tau)]_k = 0\), and the integral \(<i_N> = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{\tau/2}^{\tau} d\tau \sin[(\frac{2\pi}{N})(2\pi k - 2\pi f_0 e + \nu_{dc} s + \frac{2\pi}{2\pi \nu_{\phi}} \sin 2\pi \nu_{\phi} s)] + v_{dc}/N\), gives \(<i_N> = v_{dc}/N\), when \(v_{dc} \neq m\nu_{\phi}\), and \(<i_N> = \sin[\frac{2\pi}{N}(k - f_0)] J_{-m}(\frac{\nu_{dc}}{2\pi \nu_{\phi}}) + v_{dc}/N\), whenever \(v_{dc} = m\nu_{\phi}\), with \(m = 0, 1, 2, \ldots\). In deriving this result we have used the identity \(e^{iz\sin\phi} = \sum_{-\infty}^{\infty} J_m(z) e^{-im\phi}\), with \(J_m(z)\) the Bessel function of integer order. Note that when \(v_{dc} = m\nu_{\phi}\) the initial condition \(f_0\) plays a crucial role since \(<i_N> = <i_N> (f_0)\), and thus when \(v_{dc} = m\nu_{\phi}\) there are finite intervals of \(<i_N>\) for each value of \(v_{dc}\). These giant Shapiro resonances (GSR) are shown in Fig. 2(a) for \(N = 10, \nu_{\phi} = 0.01\) and \(v_{ac} = 1\). We note that the size of the current intervals for which there are resonances is comparable to one. The averaged current in the PS can be evaluated analytically following a similar logic as in ref. [6]. The qualitative result is that there are subharmonic resonances for \(v_{dc} = \frac{m}{n}\nu_{\phi}\), with \(n = 1, 2, \ldots\). The analytic analysis does not give, however, the order of magnitude of these subharmonic resonances. To resolve this question we evaluated \(<i_N>\) in the PS case numerically. The results for \(N = 10, \nu_{\phi} = .01, v_{ac} = 1\) are shown in Fig. 2(c). These results were obtained as follows: first the time interval was divided in \(\Delta\tau\) segments, with \(\nu_{\phi}\Delta\tau = 10^{-4}\), and the \(v_{dc}\) interval into 24 pieces. Next, for each value of \(v_{dc}\) we took 100 initial conditions for \(f_0\). Whenever \(f_{e}(\tau)\) reaches a discontinuity point, which is determined with a precision of \(10^{-6}\), we changed the value of \([f(\tau)]_k\). Note that the scale in Fig. 2(c) is about an order of magnitude smaller than in Fig. 2(a).

We now consider the general case with \(\ell \neq 0\). To evaluate the time-averaged current in this case we need to solve the differential equation given in Eq(3). This is not a simple ordinary differential equation since there is an implicit transcendental dependence of \(i_N(\tau)\)
through the sine function. The equation is solved by using a fourth order Runge-Kutta method following a similar procedure as in the $\ell = 0$ case, for parameter values similar to those of Fig. 2(c) but with $\ell = 50.0$, or $\kappa = 0.1414$. For the NPS case the results are shown in Fig. 2(b). There we see subharmonic resonances but of magnitude larger than in Fig. 2(c). We call these resonances then giant subharmonic Shapiro resonances (GSSR). In the PS case and for the same parameter values as in Fig. 2(b), Fig. 2(d) shows, in contrast, that the subharmonic resonances remain of the same order of magnitude as in the $\ell = 0$ limit. We also calculated the spectral function, not shown here, defined as $S(2\pi\nu_\phi) = \lim_{\tau \to \infty} \left| \frac{1}{T} \int_0^T i_N(s)e^{i2\pi\nu_\phi s}ds \right|^2$. It is found that at a resonance the current as a function of time is a periodic function but with a complicated subharmonic structure.

The results presented in Fig. 2 show that the JJN is capable of exhibiting truly GSSR in the NPS case although they are also present, but of a smaller magnitude, in the PS case. To further understand the properties of these resonances we calculated the magnitude of the resonance spikes widths $\Delta < i_N >$ as a function of $\nu_\phi, \sqrt{\ell} = 1/\kappa, v_{ac}$ and $N$. The corresponding results are shown in Fig. 3(a-d) for the 1/2 resonance, for both the NPS case (○) and the PS case (×). The significant characteristic of Figs. 3(a-c) is that for the NPS case there are maximum or optimal $\Delta < i_N >$ values as a function $\nu_\phi, \sqrt{\ell}$ and $N$. The situation is less clear for the PS case although small maxima can be seen in Figs. 3(a-c). These specific maximum values are listed in the figure caption. Note that as a function of $\ell$, and as seen in Figs. 2, $\Delta < i_N >$ is larger for the NPS case in the large $\ell$ limit but it is slightly smaller in the PS case in the small $\ell$ regime. A similar suppression of the 1/2 step width for large $\nu_\phi$ was found in [3,7] for a current driven square array. The suppression was not found, however, for integer steps. In contrast, we found that the $n = 1$ step width as a function of $\nu_\phi$ behaves in a qualitatively similar way as in the 1/2 case. In Fig. 3(d) we show the behavior of $\Delta < i_N >$ as a function of $v_{ac}$. Contrary to what is seen in one JJ or in the 2D-array calculations, $\Delta < i_N >$ grows monotonically up to about $v_{ac} = 3$ and then it appears to become periodic for the NPS case but is still aperiodic for the PS limit. We checked that these results are stable against an increase in the number of periods used.
to calculate the averages. All the results shown in Fig. 3 indicate that the qualitative and quantitative properties of the $JNJ$ are inherently different from those obtained previously for one JJ or for 2D-arrays.

These results allow us to make a rough estimate of the appropriate experimental parameters for which these resonances can be seen experimentally taking the typical values from experiments [12]. For a lattice constant $a \sim 10 \mu m$ and with separation between junctions of $b \sim 2 \mu m$, with the optimal value of $N = 16$, we get a diameter of $d \sim 61 \mu m$. The critical currents are between $1 \times 10^{-2}$ to $10 \ mA$, so that using the estimate for the inductance for a ring [12] of $L = 1.25 \mu_0d$, one gets the values of $\sqrt{\ell}$ between $1.73 \sim 54.84$ which are within the range of maximum values shown in Fig. 3(b).

In conclusion, we have presented a detailed analysis of the response of a Josephson junction necklace to an external time-dependent flux. The main result of our analysis is that there are giant subharmonic Shapiro resonances, mainly in the no phase slip or “fast” regime. Furthermore, we found that these resonances have a strong dependence on the number of junctions $N$, normalized frequency $\nu_\phi$ and effective inductance $\ell$. An estimate was also provided for the possible experimental conditions under which these resonances may be seen.

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FIGURES

FIG. 1. Normalized Ground state energy for \( N = 10 \) as a function of \( f_0^0 \). Solid line \( \ell = 0 \) and dashed line \( \ell = 5.0 \). The curves \( I \) and \( II \) represent the no-phase-slip (NPS) and phase-slip (PS) processes, respectively. The inset gives a schematic representation of the \( JJN \) model studied here.

FIG. 2. The time-averaged normalized current \( \langle i_N \rangle \) vs \( v_{dc} \) for \( N = 10, \nu_\phi = 0.01 \) and \( v_{ac} = 1.0 \). (a) the NPS and (c) PS cases, both with \( \ell = 0 \). (b) the NPS and (d) PS cases with \( \ell = 50.0 \). The results were obtained as described in the text. The \( \frac{1}{2} \) and \( \frac{1}{3} \) subharmonic resonances are indicated by arrows.

FIG. 3. Renormalized current width \( \Delta \langle i_N \rangle \) for the \( 1/2 \)-resonance as a function of (a) \( \nu_\phi \), (b) \( \sqrt{\ell} \), (c) \( N \) and (d) \( v_{ac} \) in the NPS (●) and PS (×) cases. In (a) \( v_{ac} = 1, N = 10 \) and \( \ell = 50.0 \). In (b) \( \nu_\phi = 0.01, N = 10 \) and \( v_{ac} = 1 \). In (c) \( v_{ac} = 1, \ell = 50.0 \) and \( \nu_\phi = .01 \). In (d) \( \nu_\phi = 0.01, \ell = 50.0 \) and \( N = 10 \). The approximate maxima in Figs. 3(a-d) are located at \( \nu_\phi^{max} \sim 0.0251, \sqrt{\ell^{max}} \sim 5.012 \) and \( N^{max} \sim 16 \), respectively.