Mutually unbiased bases and complementary spin 1 observables

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Abstract

The two observables are incompatible if they cannot be measured simultaneously; however, they become maximally incompatible (complementary) if their eigenstates are mutually unbiased. Only then does the measurement of one observable give no information about the other observable. The spin projection operators onto three mutually orthogonal directions are complementary only for spin $1/2$. For higher spin numbers the corresponding eigenstates are no longer unbiased. In this work we examine the properties of spin 1 mutually unbiased bases (MUB) and look for the physical meaning of the corresponding operators. We show that if the computational basis is chosen to be the eigenbasis of the spin projection operator along some direction $z$, then all the states, which are unbiased to this basis, have to be squeezed. Next, we study the generation and the measurement of MUB states by introducing the Fourier-like transform through spin squeezing. Finally, we try to ascribe some classical interpretation to the operators corresponding to MUB and study what information the observer gains while measuring them. Higher spin numbers are also considered.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Two $d$-dimensional orthonormal bases \{\ket{a_j}\} and \{\ket{b_k}\} are called unbiased if

$$\forall j,k |\langle a_j | b_k \rangle|^2 = \frac{1}{d},$$

and the set of more than two bases of this kind is called mutually unbiased if the above holds for every pair of bases from this set. In quantum mechanics this means that if the two observables $A$ and $B$ have unbiased eigenbases, then the measurement of the observable $A$
reveals no information about the possible outcomes of the measurement of the observable \( B \) and vice versa.

The study on mutually unbiased bases (MUB) was started by Schwinger [1] 50 years ago. The properties of MUB have been successfully applied in many areas of quantum physics: they provide the security of quantum key distribution protocols [2, 3] and the solution to the Mean King problem [4–6], they also minimize the number of von Neumann measurements required to determine a quantum state [7, 8] and are related to the discrete Wigner functions [9]. Despite their wide use, there is still an intriguing open question about the maximal number of such bases in non-prime power-dimensional complex spaces [10]. For continuous variables, such as position and momentum or electric and magnetic field, eigenbases of the corresponding operators are widely known to be MUB (for a study of MUB in infinite-dimensional Hilbert spaces see [11, 12]). In the case of discrete variables, such as spin, the physical meaning of MUB is clear only for spin 1/2. The three MUB of spin 1/2 have a good geometrical representation because they are simply the eigenbases of spin projection operators along any three mutually orthogonal directions. Moreover, the Hilbert space of spin 1/2 is isomorphic to the three-dimensional real space, which is the essence of the Bloch sphere picture. On the other hand, the Hilbert space of spin \( S \rangle > 1/2 \) is much richer than the three-dimensional real space that is why it would be naive to expect that MUB of higher spins have simple geometrical interpretations.

Most of the MUB research is devoted to the mathematical properties of the underlying Hilbert space; therefore, the intuitive physical picture behind the complex state vectors is somehow lost. We are trying to bring back this picture by studying the properties of MUB for spin 1 states. What motivates our work is the fact that knowledge of the complementary observables and the ability to generate and measure unbiased states for a certain system is necessary in order for that system to be fully exploited in quantum information processing tasks. In the following we find that if the computational basis is the eigenbasis of the spin projection operator onto any direction in space, then the remaining MUB states are squeezed. The eigenstates of the spin projection operator and their Fourier transforms, which can be interpreted as eigenstates of some kind of angle operator, are also briefly examined for higher spin numbers. Then, we show how to transform between different MUB by introducing the Fourier-like transform through spin squeezing. Next, we give the physical interpretation of the two bases which are complementary, one of them being the eigenbasis of the spin projection operator. In the end, we discuss our results in the context of recent studies on the biphoton implementation of a qutrit [15–18].

2. Spin 1 states unbiased to the \( S_z \) basis

The most common choice of the computational basis for spin 1 is the eigenbasis of the spin projection operator along some direction \( z \). Any state which is unbiased to all the states from the computational basis is of the form

\[
\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & e^{i\alpha} & e^{i\beta} \end{pmatrix},
\]

where \( \alpha \) and \( \beta \) are the arbitrary phases. The immediate property of the above state is the zero mean value of the \( z \) component of the mean spin vector \( \langle \vec{S} \rangle = (\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle) \), where \( S_j \)'s are the spin projection operators along the direction \( j \), which obey the cyclic commutation
relation \([S_j, S_k] = i\epsilon_{jkl} S_l\). This is because the operator
\(S_z\) is traceless and diagonal, and for

\[
\langle S_z \rangle = \frac{1}{3} \text{Tr}[S_z] = 0.
\]  

As a result, either the mean spin vector is \(\vec{0}\) or it lies in the \(XY\) plane. Indeed, the coordinates
of the mean spin vector are

\[
\langle S_x \rangle = \frac{\sqrt{2}}{3} \left( \cos \alpha + \cos(\beta - \alpha) \right);
\]

\[
\langle S_y \rangle = \frac{\sqrt{2}}{3} \left( \sin \alpha + \sin(\beta - \alpha) \right);
\]

\[
\langle S_z \rangle = 0.
\]  

Note that due to the rotational symmetry for the study of the physical properties of states (2) we
can consider the subclass of states for which \(\langle S_y \rangle = 0\). The rest of the states can be generated
by the rotation in the \(XY\) plane. It is easy to see that the subclass \(\langle S_y \rangle = 0\) corresponds to
\(\beta = 0\), or to \(\beta = 2\alpha - \pi\). In this case equation (2) becomes

\[
\frac{1}{\sqrt{3}} \begin{pmatrix}
1 \\
e^{i\alpha} \\
1
\end{pmatrix},
\]

or

\[
\frac{1}{\sqrt{3}} \begin{pmatrix}
1 \\
e^{i\alpha} \\
e^{-i2\alpha}
\end{pmatrix},
\]  

respectively.

We obtained a class of states unbiased to the computational basis, which are parameterized
only by \(\alpha\). The length of the corresponding mean spin vector is \(|(2\sqrt{2}/3) \cos \alpha|\) for states (5),
or zero for states (6). Since the maximum length of the mean spin vector is \(2\sqrt{2}/3 < 1\), any
state of the form (2), including (5) and (6), cannot be a coherent spin state. The coherent spin
state is the eigenstate of the spin projection operator along any direction with the corresponding
eigenvalue \(\pm S\) (see [19]). Observables whose eigenstates are of the form (2) have to be much
more sophisticated than simply spin projection operators. On the other hand, since the states
(6) are completely unpolarized, it may happen that at least some of them are the null projection
states—the eigenstates of the spin projection operators corresponding to eigenvalue 0.

Let us consider the uncertainties \(\Delta S^2_j, \Delta S^2_k, \Delta S^2_l\) of the above states. The uncertainty
relation for the spin projection operators onto any three mutually orthogonal directions yields

\[
\Delta S^2_j \Delta S^2_k \geq \frac{1}{4} |\langle S_l \rangle|^2.
\]  

Kitagawa and Ueda presented [19] that the above inequality is very sensitive to the choice
of the directions \(j, k\) and \(l\). In order to properly define the spin squeezing one should apply
(7) for the direction \(l\) lying along the mean spin vector. The spin squeezed state is defined
as the one for which the uncertainty \(\Delta S^2_j\) of the spin projection operator onto the direction
orthogonal to the mean spin vector is smaller than \(S/2\). In our case the state is squeezed if
there exists a direction orthogonal to the mean spin vector for which \(\Delta S^2_j < 1/2\). The mean
spin vector of the states (5) lies along the \(x\) direction; therefore, \(\Delta S^2_x = \langle S_x^2 \rangle\) and \(\Delta S^2_y = \langle S_y^2 \rangle\).

What is interesting is that both uncertainties do not depend on \(\alpha\):

\[
\Delta S^2_x = 1/3,
\]

\[
\Delta S^2_y = 2/3,
\]

thus according to the definition states (5) are squeezed in the \(y\) direction.
States (6) are unpolarized; however, in this case the variances are
\[
\Delta S_z^2 = \frac{2}{3} - \frac{1}{3} \cos 2\alpha, \\
\Delta S_x^2 = \frac{2}{3} + \frac{1}{3} \cos 2\alpha, \\
\Delta S_y^2 = \frac{2}{3},
\]
and one can conclude that these states are the null projection states along the direction
\[
\vec{n} = 1/\sqrt{3}(\sqrt{2} \cos \alpha, \sqrt{2} \sin \alpha, 1).
\]
The direction \(\vec{n}\) forms the tetrahedral angle \(\frac{1}{2} \arctan \sqrt{2} \approx 54.7^\circ\) with the \(z\) axis. Throughout the paper we will call two directions to be tetrahedral if they form the tetrahedral angle. If one wants to visualize spin 1 as a classical vector, the coherent states seem to be the most classical ones, since due to minimal uncertainty they are the closest approximation of a classical vector. On the other hand, the null projection states are considered to be the most non-classical ones. This due to two reasons. First of all, for the null projection state the joint state of the two spin 1/2 particles forming up spin 1 is maximally entangled. Secondly, the null projection state provides the maximal violation of the recent Bell-like inequality for spin 1 [20]. Moreover, these states can be considered as maximally squeezed states in the direction for which \(\Delta S_j^2 = 0\).

3. Spin 1 MUB

The four MUB in the three-dimensional complex space are most commonly known to be the eigenvectors of four unitary operators from the Weyl–Heisenberg group: \(U, V, UV\) and \(UV^2\). All four operators have eigenvalues 1, \(\omega = e^{i\pi/3}\) and \(\omega^2\), and obey the relation \(AB = q BA\), where \(q\) is either \(\omega\) or \(\omega^2\). The bases, up to normalization, are given by

\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}; \\
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}, \begin{pmatrix}
\omega \\
\omega \\
\omega^*
\end{pmatrix}, \begin{pmatrix}
1 \\
\omega^* \\
\omega
\end{pmatrix}; \\
\begin{pmatrix}
\omega \\
1 \\
1
\end{pmatrix}, \begin{pmatrix}
\omega^* \\
\omega^* \\
\omega
\end{pmatrix}, \begin{pmatrix}
1 \\
\omega \\
\omega
\end{pmatrix}; \\
\begin{pmatrix}
\omega^* \\
\omega^* \\
\omega
\end{pmatrix}, \begin{pmatrix}
1 \\
\omega \\
\omega
\end{pmatrix}, \begin{pmatrix}
1 \\
\omega^* \\
\omega
\end{pmatrix}.
\]

Note that the second basis (12) is the Fourier transform of the computational basis. If basis (11) is the eigenbasis of \(S_z\), then the operator \(\exp(-i S_z t) = \text{diag}\{1, e^{-i\omega}, e^{-i\omega^2}\}\) generates a permutation inside bases (12)–(14) for times being the multiple of \(2\pi/3\). This is an analogy to the \(\pi\) rotation about the \(z\) axis for spin 1/2, which generates the swap operation in the \(S_x\) and \(S_y\) bases. The transition between different bases, other than the computational one, can be obtained by an application of the operator \(\text{diag}\{1, \omega, 1\}\) generated by the Hamiltonian proportional to \(S_z^2\), which is a one-axis twisting squeezing generator (see [19] and below).

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The states in the first column of bases (12)–(14) are of the form (5); therefore, the corresponding mean spin vectors point in the $\vec{x}$ direction and due to the rotational symmetry the other states from these bases have the same physical properties. The lengths of the mean spin vectors of MUB (12)–(14) are $\frac{2\sqrt{2}}{3}$, $\frac{2\sqrt{2}}{3}$, and $\frac{2\sqrt{2}}{3}$, respectively. The vectors lie in the $XY$ plane and point in the directions $0$, $\frac{2\pi}{3}$, and $-\frac{2\pi}{3}$ for (12), or $\pi$, $\frac{\pi}{3}$, and $-\frac{\pi}{3}$ for (13) and (14) (see figure 1 left). The degree of squeezing is related to the length of the mean spin vector, yet equation (8) states that for all states of the form (5) the uncertainties $\Delta S_z^2$ and $\Delta S_y^2$ do not depend on the state, suggesting that all these states are equally squeezed, which seems in contrary to the fact that the mean spin vectors of (13) and (14) are shorter than the corresponding vectors of (12). However, it is possible that states with a shorter mean spin vector are squeezed in the direction other than $y$. This idea is depicted in figure 1 on the right. Indeed, it is easy to check that the first states in (12)–(14) are squeezed along $y$, the direction tilted by an angle $\approx -\frac{\pi}{6}$ and $\approx \frac{\pi}{6}$, respectively. For the last two bases uncertainties drop slightly below 0.06. Note that complex conjugation changes between bases (13) and (14). The physical consequence is that complex conjugation changes the sign of $S_y$, and when applied to the spin vector $\vec{S} = \{S_x, S_y, S_z\}$, it generates mirror reflection with respect to the $XZ$ plane. This is visible in uncertainties and in the distribution of mean spin vectors (see figure 1).

Spin states are sometimes presented as cones in the three-dimensional space. The surface of the cone represents an area covered by the spin vector due to uncertainty principle, and its dilation angle is related to uncertainties. In figure 2 we suggest how one may visualize spin states corresponding to bases (11) and (12).

3.1. Higher spin numbers

Let us also consider two MUB for higher spin numbers, namely the eigenbasis of $S_z$ and its Fourier transform. The number of MUB for non-prime power dimensions is still unknown [10]; therefore, we study only these two cases. The Hilbert space is $d = 2s + 1$ dimensional. Mean spin vectors of all states unbiased to the $S_z$ basis are lying in the $XY$ plane due to similar reason as before (recall equation (3)). Moreover, a $2\pi/d$ rotation about $z$ causes a cyclic permutation of the second basis. Once again, the rotational symmetry allows us to examine only the state
The mean spin vector corresponding to the above state points in the $x$ direction. This is because $S_x$ is an antisymmetric matrix and an expectation value of any matrix with respect to the state (15) equals the sum of all matrix elements divided by $d$; thus $\langle S_x \rangle = 0$. The length of the mean spin vector is given by

$$\langle S_x \rangle = \frac{1}{d} \sum_{j=1}^{d-1} \sqrt{j(d-j)} \leq \frac{d-1}{2} = s.$$  

It is shorter than the length of the mean spin vector of the coherent state, which indicates that the state (15) is squeezed. Indeed,

$$\Delta S_x^2 = \frac{d^2 - 1}{12} = \frac{s(s+1)}{3} > \frac{s}{2},$$

but

$$\Delta S_z^2 = \frac{d^2 - 1}{12} - \frac{d^2}{2} \sum_{j=1}^{d-1} \frac{j(j-1)}{d} \leq \frac{d^2}{2} \sum_{j=1}^{d-1} \frac{j(j-1)}{d} \leq \frac{d-1}{2} = s.$$  

4. Generation and measurement of spin 1 MUB

In order to prepare unbiased states and to perform suitable measurements one has to know how to transform between different MUB. In the case of spin 1 some states are more accessible.
as well as some measurements are easier to perform; however, in order to implement certain information processing tasks, the ability of preparing and measuring all desired spin states is required (see [22]). The simplest measurements of spin are the one of the Stern–Gerlach type, although generalized Stern–Gerlach measurements have also been proposed [23]. Since the measurement can also be considered as a preparation procedure, the coherent states and the null projection states seem to be the most accessible ones. It is also obvious that the most natural choice of the computational basis should be the eigenbasis of $S_z$ for some reference axis $z$. Transformations between different coherent states are relatively easy to perform, since they require only a spin rotation via an application of a linear magnetic field. On the other hand, a transformation between coherent and null projection states is more complicated, since it requires nonlinear effects.

There are three distinct kinds of operations that can be done on the spin 1 system: rotation, one-axis twisting and two-axis countertwisting. The second and the third one generate spin squeezing. It is well known that the generators of rotations are given by spin matrices. In particular, the rotation about the direction $\vec{n}$ is generated by $S_\vec{n} = \vec{n} \cdot \vec{S}$, where $\vec{S} = (S_x, S_y, S_z)$. The generator of one-axis twisting is the square of the rotation generator $S_\vec{n}^2$. Finally, the generator of two-axis countertwisting is given by $S_\vec{n}^2 - S_\vec{m}^2$, where $\vec{n}$ and $\vec{m}$ are orthogonal. These two types of spin squeezing were first proposed by Kitagawa and Ueda [19].

We already mentioned that if the computational basis corresponds to $S_z$, then transformations within the three remaining MUB can be obtained by a rotation about $z$. These transformations are simply analogs of translation or boost. Transformations between MUB (12)–(14) are done via one-axis twisting generated by $\exp(-i S^2_{\vec{n}} t)$, for which the change between the bases occurs for times being a multiple of $2\pi/3$. The above operator is of course nonlinear and requires quadrupole effects. Still, the most important question is how to obtain the three bases from the $S_z$ basis. Mathematically, MUB states can be generated via the Fourier transform of the computational basis. Let us look for a more general transformation, a Fourier-like transform, which takes computational basis states to states (2). This transformation is represented by a unitary matrix whose entries are equal up to modulus, i.e. a complex Hadamard matrix (Hadamard for short). It is important to mention that the complex Hadamards play a crucial role in MUB studies (for a review see [14]). For spin 1 such a transformation can be obtained for one-axis twisting about an axis tetrahedral to $z$, whose direction is given by equation (10). The corresponding operator $\exp(-i S^2_{\vec{n}_\phi} t)$ becomes a Hadamard for $t = 2\pi/3$ (and $4\pi/3$), which up to a global phase is given by

$$\frac{1}{\sqrt{3}} \begin{pmatrix} e^{i\phi} & e^{-ip} & e^{-12p} \\ e^{ip} & e^{i\phi} & e^{-12p} \\ e^{12p} & e^{-ip} & e^{i\phi} \end{pmatrix}.$$  

(19)

Straightforward calculations show that the above operation generates three squeezed states which are symmetrically distributed in the $XY$ plane and whose lengths are equal $\sqrt{2/3}$ regardless of $\phi$. The angle $\phi$ is only affecting the orientation of mean spin vectors in the plane. In order to obtain the complete Fourier transform one has to follow the above operation by one-axis twisting and a rotation, both about $z$. However, it is enough to apply only (19) to generate a basis which is unbiased to $S_z$. Such a basis is suitable to perform certain tasks such as quantum cryptography on a three level system. The other two bases can be obtained by one-axis twisting $\pm 2\pi/3$ $z$-pulses.

If one knows how to reverse the above operations, one is capable of measuring in all four bases using only typical Stern–Gerlach magnets. Therefore, a $\pm 2\pi/3$ rotations about $z$ and a $\pm 2\pi/3$ one-axis twisting $z$-pulses, the Fourier-like transform (19), its inverse and a typical
Stern–Gerlach measurement constitute the full set of operations needed to perform the three level quantum cryptography or a tomography of spin 1.

5. Complementary observables

Let us examine the physical meaning of four unitary operators from the Weyl–Heisenberg group: $U, V, UV$ and $UV^2$. $U$ is a diagonal phase shift operator and $V$ is a permutation of the computational basis

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. $$

The eigenbasis of $U$ was taken to be the eigenbasis of $S_z$. It corresponds to states invariant under rotations about the $z$ axis. The states of the second basis, namely the Fourier transform of the $S_z$ basis, are dial states and it is natural to interpret them as eigenstates of the angle operator, which does not exist in quantum mechanics [21]. Indeed, we already noted that their uncertainties are minimal for three directions in the $XY$ plane corresponding to the angles $0$, $2\pi/3$ and $-2\pi/3$ (see figure 2). However, it is difficult to interpret the physical meaning of $V$ in terms of valid spin observables and natural operations, such as rotation, since it causes a cyclic permutation of $S_z$ eigenstates

$$|S_z = -1 \rangle \rightarrow |S_z = 0 \rangle \rightarrow |S_z = 1 \rangle \rightarrow |S_z = -1 \rangle.$$ 

This permutation cannot be a simple spin rotation, since it takes one coherent state to the null projection state, then to the other coherent state and then back to the initial coherent state—it has to be a nontrivial combination of squeezing and rotation. It is even harder to interpret $UV$ and $UV^2$.

To understand what is the relation between the last three Weyl–Heisenberg operators and the physical properties of spin, one can expand $V, UV$ and $UV^2$ in the spin operator basis. In fact, one can do this for any operator whose eigenbasis is one of (12)–(14) bases. However, since any operator can be written as a linear combination of projectors, it is sufficient to expand only respective projectors. Moreover, it is enough to do this only for the first states in (12)–(14), since the remaining projectors can be obtained by a simple rotation about $z$, due to the rotational symmetry.

The projector onto the first dial state (12) is given by

$$\Pi_V = \frac{1}{4} I + \frac{\sqrt{7}}{\sqrt{3}} S_z + \frac{1}{8}(S_x^2 - S_y^2). \quad (20)$$

where $I$ is the identity. It means that (20) corresponds to a rotation and two-axis countertwisting squeezing, as already predicted. The projector onto the first state of the basis (13) yields

$$\Pi_{UV} = \frac{1}{4} I - \frac{1}{8\sqrt{2}} S_x + \frac{1}{8} S_z^2 - \frac{1}{4} S_y^2 + \frac{1}{\sqrt{6}}(S_z S_x + S_z S_y); \quad (21)$$

however, it can be expressed in terms of distinct rotation and squeezing generators as

$$\Pi_{UV} = \alpha I - \beta S_x - \gamma S_y^2 + \delta (S_z^2 - S_y^2), \quad (22)$$

where the above coefficients are $\alpha = \frac{\sqrt{7}}{2} \approx 0.88$, $\beta = \frac{1}{3\sqrt{2}} \approx 0.23$, $\gamma = \frac{\sqrt{7} - \sqrt{3} - \sqrt{3}}{\sqrt{6}} \approx 0.7$, $\delta = \frac{\sqrt{3} - \sqrt{7}}{\sqrt{6}} \approx 0.24$ and

$$\vec{n} = \left(0, -\frac{\sqrt{7} + 1}{\sqrt{2}\sqrt{7}}, \frac{\sqrt{7} - 1}{\sqrt{2}\sqrt{7}}\right) \approx (0, -0.83, 0.56).$$

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The projector \( \Pi_{UV} \) corresponds to a rotation and both types of squeezing. Finally, the projector onto the first state in (14) is just a complex conjugation of the projector \( \Pi_{UV} \):

\[
\Pi_{UV}^* = \Pi_{UV}^T.
\]

(23)

Complex conjugation changes the sign of \( S_n \), which corresponds to a mirror reflection with respect to the \( XZ \) plane. Therefore, the direction of squeezing is now given by \( \vec{n} \approx (0, 0.83, 0.56) \).

Standard quantization procedures associate classical canonical variables, such as position and momentum, with complementary operators. In our case, it is the other way around. We are given a set of complementary observables and we try to ascribe some classical interpretation to its elements. We have already identified (11) and (12) with the spin projection along \( z \) and the angle in the \( XY \) plane; however, the interpretation of the remaining two bases is somehow problematic. States (12)–(14) are bizarre in the sense that they are neither coherent nor maximally squeezed null projection states and the physical meaning of the corresponding observables is not as simple as of the standard spin projection operator \( S_z \). Moreover, (13) and (14) cannot be considered as angle eigenstates, because, although the corresponding mean spin vectors are symmetrically distributed in the \( XY \) plane, the minimal uncertainty is not given for the direction confined in this plane (see figure 1).

At this point it is convenient to introduce an alternative set of spin matrices. Since the system that is under consideration poses rotational symmetry, the well-known set of spin matrices \( S_x, S_y \) and \( S_z \), which is natural for the Cartesian coordinate system, can be changed into one which is more suitable for spherical coordinates. Moreover, the interesting property of the above triple, namely its complementarity, occurs only for \( S = 1/2 \). The new triple can be \( S_r, S_\theta \) and \( S_\phi \), where \( S_r \) corresponds to the length of the spin vector, \( S_\theta \) to the angle it forms with the \( z \) axis and \( S_\phi \) to the angle its projection onto the \( XY \) plane forms with the \( x \) axis. Actually, the first two operators are nothing new, since \( S_r = \sqrt{S_x^2 + S_y^2 + S_z^2} \), which is the well-known spin operator with eigenvalues \( \sqrt{S(S+1)} \), which is proportional to identity on the spin \( 1 \) subspace, and \( S_\theta \) is related to \( S_r \) via the simple function \( S_\theta = \cos(S_\theta) \). On the other hand, \( S_\phi \) is new in this set and is given by the Fourier transform of \( S_r \). Moreover, as was already shown, \( S_\theta \) and \( S_\phi \) are complementary for all spin numbers.

Let us consider another pair of MUB. Among all the states unbiased to the \( S_z \) basis there is a class of maximally squeezed states given by equation (6) which corresponds to the null projection states along all directions tetrahedral to \( z \). In this class there are infinitely many triples of mutually orthogonal states forming a basis which is mutually unbiased to the \( S_z \) one. These states can be represented in the real space as three mutually orthogonal planes (see figure 3). In general, any three null projection states along three arbitrary mutually orthogonal directions form a spin 1 basis.

Suppose that the computational basis is a basis made of three null projection states along the \( x, y \) and \( z \) directions, which in the \( S_z \) basis are given by

\[
|x\rangle = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \quad |y\rangle = \begin{pmatrix} i \frac{1}{2} \\ 0 \\ -i \frac{1}{2} \end{pmatrix}, \quad |z\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

Usually, the state \( |y\rangle \) is written as \( \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)^T \), but there is a reason why we multiplied it by \( i \). Interestingly, any state of the form

\[
|\theta, \phi\rangle = \sin \theta \cos \phi |x\rangle + \sin \theta \sin \phi |y\rangle + \cos \theta |z\rangle
\]

(24)

is also a null projection state along the \( \vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) direction; therefore, all real linear combinations of \( |x\rangle, |y\rangle \) and \( |z\rangle \) resemble the vectors in the Euclidean space \( \mathbb{R}^3 \).
Another interesting fact is that the Fourier transform of the above computational basis is the eigenbasis of the spin projection operator $S_m$ along the $\vec{m} = (1, -1, -1)$ direction, which is tetrahedral to $z$. $S_m$ generates a rotation about $\vec{m}$, which for an angle being a multiple of $2\pi/3$ causes a cyclic permutation of the basis states—the $2\pi/3$ rotation about $\vec{m}$ transforms the $XY$ plane into the $XZ$ plane, etc.

Let us find what information we gain while measuring spin in the new computational basis. The information related to measuring $|j\rangle$ is that the spin definitely does not lie along direction $j$; therefore, the measurement gives an answer to the question: *Along which one of the three mutually orthogonal axes does the spin not lie?* Note that the uncertainty principle forbids the spin to lie definitely along one axis that is why the question we can ask sounds a little bit odd. Eventually, one can ask two complementary questions:

- *Along which one of the three mutually orthogonal axes $j$, $k$ and $l$ does the spin not lie?*
- *What is the spin projection onto one of the four axes tetrahedral to $j$, $k$ and $l$?*

Despite the fact that we have chosen different computational basis, the remaining two MUB (13) and (14) are still squeezed, though not maximally squeezed, i.e. they are not the null projection states. The corresponding observables, which are linear combinations of projectors onto MUB states, are generators of transformations which are combinations of rotations and squeezing. Once more, it is hard to identify them as some simple physical quantities or to associate them with reasonable questions one can ask about spin. However, it would be very interesting to find the physical meaning of observables whose eigenstates are partially squeezed and we leave this as an open question.

6. Conclusions

Some physical aspects of spin 1 complementary observables were discussed. In particular, we presented that spin squeezed states can play an important role in MUB studies. Moreover, we proposed methods of generation and measurement of different spin 1 MUB, which is necessary for spin 1 quantum information processing. Since experimental quantum optics is rapidly developing, it is important to mention how our work is related to this field. One of the most promising optical implementations of qutrit is realized via polarization states of a
biphoton—joint polarization states of two indistinguishable photons. Still, even for a biphoton, a generation of all possible states requires nonlinearities from either nonlinear crystals [15, 16] or a measurement-induced state filtering [17]. These nonlinearities somehow resemble a quadrupole nonlinearity needed to obtain spin squeezed states. It seems that the nature truly reveals its quantum behavior through nonlinear effects.

State of spin 1 can be represented as a product state of two 1/2 spins. The Hilbert space of spin 1/2 is isomorphic to a photon polarization space; therefore, the product representation of spin 1 is isomorphic to the biphoton polarization representation. The computational basis of a biphoton is spanned by the following three Fock states: $|2, 0\rangle$, $|1, 1\rangle$ and $|0, 2\rangle$, where $|n_h, n_v\rangle$ denotes the number $n_h$ ($n_v$) of horizontally (vertically) polarized photons. In this case, the Weyl–Heisenberg operators $U$ and $V$ have the following interpretation. $U$ is a diagonal operator whose generator can be given by the number of horizontally (or vertically) polarized photons $n_h = a_h^\dagger a_h$. This operation can be performed within linear optics. On the other hand, $V$ generates permutation $|2, 0\rangle \rightarrow |1, 1\rangle \rightarrow |0, 2\rangle \rightarrow |2, 0\rangle$, which requires nonlinearities, exactly like in the case of spin 1. The same stands for $UV$ and $UV^2$.

Recently, two cryptographic protocols have been designed for biphotons [18]. One of them, the so-called umbrella protocol, uses only two bases, which resemble the bases discussed in this work. The first basis contains only maximally entangled photon states, which correspond to our basis made of null projection states, while the second one, tetrahedral to the first one, consists of two product states and one maximally entangled state, which in our case is spin projection basis made of two coherent states and one null projection state. The authors of the protocol proved that the two-MUB umbrella protocol is more efficient than a three-MUB generalization of BB84 for qubits. We have already presented that one can change between different unbiased states of spin 1 by applying linear and quadrupole magnetic fields; therefore, it is clear that the umbrella protocol can be implemented for spin 1.

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