Automodel solutions for superdiffusive transport by Lévy walks

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Abstract
The method of approximate automodel solution for the Green’s function of time-dependent superdiffusive (nonlocal) transport equations (Kukushkin A B and Sdvizhenskii P A 2016 J. Phys. A: Math. Theor. 49 255002) is extended to the case of a finite velocity of carriers. This corresponds to an extension from Lévy flight based transport to transport of a type which belongs to the class of ‘Lévy walks + rests’, to allow for retardation effects in Lévy flights. This problem covers the cases of transport by resonant photons in astrophysical gases and plasmas, heat transport by electromagnetic waves in plasmas, migration of predators, and other applications. We treat a model case of 1D transport on a uniform background with a simple power-law step-length probability distribution function (PDF).
A solution for the arbitrary superdiffusive PDF is suggested, and verification of the solution for a particular power-law PDF, which corresponds, e.g., to Lorentzian wings of atomic spectral line shapes for the emission of photons, is carried out using the computation of the exact solution.

Keywords: Lévy walks, Lévy flights, superdiffusive transport, automodel (self-similar) solutions

(Some figures may appear in colour only in the online journal)

1. Introduction

A wide range of problems require the description of transport in the medium for a finite velocity of carriers. The processes of nonlocal transport, which significantly differ from conventional diffusion, are of special interest (see, e.g., the surveys in [1, 2]). Energy transfer by photons in spectral lines of atoms and ions in plasmas and gases in astrophysical objects, nonlocal heat transport by electromagnetic waves in plasmas, and migration of predators belong to such processes. These phenomena have a superdiffusive character and have to be described by an integral equation in spatial coordinates, irreducible to a diffusion differential equation. The latter makes the numerical simulation of superdiffusive transport a formidable task.

The phenomenon of superdiffusion is closely related to the concept of Lévy flights [3–7]. A known example of such a phenomenon is radiative transfer in plasmas and gases in the Biberman–Holstein model [8–11]. This model considers resonance photon scattering by an atom or ion with complete redistribution over the frequency in absorption and re-emission. Here, rare distant flights of photons (‘jumps’), which correspond to emission/absorption in the ‘wings’ of the spectral line, dominate over the contribution of frequent close displacements, which produce diffusive (Brownian) motion and correspond to emission/absorption in the core of the spectral line. Distant flights caused by long-tailed (e.g. power-law) wings of the integral operator (i.e. of the step-length probability distribution function (PDF)) in the transport equation are shown [12] to be Lévy flights. The dominant contribution of long-free-path photons to radiative transfer in spectral lines has been recognized in [13, 14]. Simple models based on this dominance have been developed for quasi-steady-state transport, now known as escape probability methods [15–17].

For time-dependent superdiffusive transport by Lévy flights, recently a large class of transport on a uniform background was shown [18–23] to possess an approximate automodel solution. Solutions for the Green’s function were constructed using scaling laws for the propagation front (i.e. the time dependence of the relevant-to-superdiffusion average displacement of the carrier) and asymptotic solutions far beyond
and far ahead of the propagation front. The validity of the suggested automodel solutions was proved by their comparison with exact numerical solutions, in the 1D case of the transport equation with a simple long-tailed PDF with various power-law exponents, and in the case of the Biberman–Holstein equation of 3D resonance radiative transfer for various (Doppler, Lorentz, Voigt and Holtsmark) spectral line shapes.

The present work extends the method [18] to the case of a finite velocity of carriers. This corresponds to an extension of Lévy flight based transport to transport of a type which belongs to the class of ‘Lévy walks + rests’ (see figure 1 in [11]), to allow for retardation effects in the Lévy flights. As in [18], we treat a model case of 1D transport on a uniform background with a simple power-law step-length PDF (section 2). A solution for an arbitrary superdiffusive PDF is suggested (section 3), which uses asymptotic solutions far beyond and far ahead of the propagation front. The solution for this particular power-law PDF, which corresponds, e.g., to Lorentzian wings of atomic spectral line shapes for the emission of photons and uses asymptotic solutions far beyond and far ahead of the propagation front [24, 25], is presented in section 4. Verification of the solution is performed in section 5, using a numerical simulation of the exact solution [24] of the transport equation. A modification of the solution and its verification are presented in section 6.

2. Basic equation and general solution

The nonstationary equation for the Green’s function \( f(x, t) \) of the 1D superdiffusive (nonlocal) transport of excitation in a homogeneous medium, with allowance for a finite velocity of the motion of carriers, has the form (the derivation of this equation may be found in section 2 of [24]):

\[
\frac{\partial f(x, t)}{\partial t} = -\left( \frac{1}{\tau} + \sigma \right) f(x, t) + \frac{1}{\tau} \int_{-\infty}^{\infty} dx' W(|x - x'|) \times f(x', t - \frac{|x - x'|}{c}) \delta \left( t - \frac{|x - x'|}{c} \right) + \delta(x) \delta(t),
\]

where \( W(\rho) \) is the step-length PDF, which describes the probability density for the process of carrier starts (‘emission’...
of the carrier by the medium) and subsequent stops ('absorption' of the carrier by the medium) after passing the distance $\rho$:

$$\int_{-\infty}^{+\infty} W(|x - x'|)dx' = 1;$$

(2)

\(\tau\) is the average 'waiting time', i.e. the time between the moments of stopping and starting the carrier (average lifetime of the medium's excitation); \(c\) is the (constant) velocity of carriers; \(\sigma\) is the average inverse lifetime of the de-excitation of the medium by all mechanisms other than the emission of the carrier; \(\theta(x)\) is the Heaviside function; and \(\delta(x)\) is the Dirac delta-function.

This problem covers the cases of energy transport by resonant photons in astrophysical gases and plasmas and heat transport by electromagnetic waves in plasmas.

In the case of resonant photon transport in astrophysical gases and plasmas, the function \(f\) is the density of excited atoms (equation (1) corresponds to a two-level model of an atom or ion), \(\tau\) is the lifetime of excited atoms with respect to the spontaneous radiative decay of the excited atomic state, \(W(\rho)\) is the probability of a process where a photon emitted by an excited atom at some point in the homogeneous medium is absorbed by a non-excited atom (i.e. one in the ground atomic state) residing at a distance \(\rho\) from the emitting atom, and \(\sigma\) is the average inverse lifetime of the radiationless de-excitation of an atom or ion (e.g., the de-excitation of an atom or ion in plasmas by the electron impact). The effect of the finite velocity of photons is important to note when the lifetime of a photon between its emission and absorption is comparable with (or exceeds) the lifetime of an excited atom with respect to spontaneous radiative decay. The latter may be of importance for astrophysical gases and plasmas (see, e.g., [26, 27]) where one has to allow for the retardation of the light, in contrast to the case of radiative transfer in laboratory plasmas and gases. Resonance radiative transfer in spectral lines with a long-tailed spectral line shape (including Doppler, Lorentz, and Voigt spectral line shapes) was shown in [12] to be described in terms of the Lévy flight type trajectories of photons. Therefore, the extension of Biberman–Holstein-type transport [8–23] to the case of a finite velocity of photons corresponds to a transition to photon trajectories which belong, according to [1, 2], to the class of 'Lévy walks + rests'.

In the case of heat transport by electromagnetic waves in plasmas, the nonlocal transport model based on the equation, integral in space coordinates, with a long-tailed kernel and, respectively, dominance of carriers with a long free path, was
shown to describe the steady-state transport of heat in magnetized plasmas by electron/ion Bernstein waves [28] and electron cyclotron waves [29], and the transport of temperature perturbation by electron Bernstein waves [30] using the Biberman–Holstein equation for electron temperature perturbation. An extension to account for the finite group velocity of waves for these and other waves is needed for the time-dependent superdiffusive transport of the heat.

Note that even for zero annihilation the volume-integrated excitation density is not conserved in time because the conservation law holds true only for the sum of volume-integrated values of the medium’s excitation density and the carriers’ density (if the transport problem is applied to the dynamics of objects of the same type, for example, the search for food by animals, the total number of animals in motion and at rest will be a constant value).

We consider a model step-length PDF of the power-law type. The dimensionless PDF has the form

\[ W(\rho) = 0.5 \gamma / (1 + \rho)^{\gamma + 1}, \quad 0 < \gamma < 2, \]

where \( \rho \) is the space variable normalized by a characteristic length of the free path (e.g., for radiative transfer in resonance spectral lines, this is the inverse value of the absorption coefficient in the center of the spectral line, \( 1/\kappa_0 \)). The power-law PDF is known to describe the contribution of distant flights (i.e. Lévy flights) in the case of resonant photon transport for various spectral line shapes, including Doppler, Lorentz, Voigt and Holtsmark spectral line shapes (see [11, 12, 18–23]).

A general solution of equation (1), written in dimensionless form, for the step-length PDF (3) was obtained in [24]:

\[
 f(x, t, R_c) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} d\rho \int_{-\infty}^{+\infty} e^{-s} ds \cdot \sigma \int_{0}^{+\infty} \frac{e^{-\alpha R_c cos(\rho u)}}{(1 + u)^{\gamma + 1}} du,
\]
where time $t$ is in the units of $\tau$, the space coordinate $x$ is in the units of the characteristic free path length $1/\kappa_0$ ($\kappa_0$ is the characteristic value of the absorption coefficient), and the retardation parameter $R_c = c\tau\kappa_0$ is the ratio of the average waiting time to the average time of flight for the characteristic value of the free path, namely, for an absorption length equal to $1/\kappa_0$. In what follows we consider the case $\sigma = 0$.

The asymptotics of the dimensionless Green’s function far ahead of the propagation front was derived in [24]:

$$f(\rho \to \ell R_c = 0, t, R_c) = \left(1 - \frac{\rho}{R_c}\right)W(\rho)\theta \left(1 - \frac{\rho}{R_c}\right),$$

$$\rho = |x|.$$  \tag{5}

In the case of an infinite velocity of carriers ($R_c \to \infty$), it coincides with the respective asymptotics in [18] (see equation (6) therein).

3. Approximate automodel solution for arbitrary superdiffusive PDF

Following the principles of the method in [18], we construct the following approximate automodel solution (for details see the appendix):

$$f_{\text{auto}}(x, t, R_c) = \left(1 - \frac{\rho}{R_c}\right)W(\rho)\theta \left(1 - \frac{\rho}{R_c}\right),$$

where the automodel function $g$ has a known asymptotic behavior,

$$g(s) = \begin{cases} 1, & s = s_{\text{min}} = \rho_{\text{fr}}(t, R_c)/(R_c t), \\ s, & s \gg s_{\text{min}}, \end{cases}$$  \tag{6}

where $\rho_{\text{fr}}(t, R_c)$ is the propagation front defined by a relation which equates the exact solutions in the alternative limits (cf equation (25) in [23], instead of equation (5) in [18]), namely the asymptotics far ahead, equation (5), and the asymptotics far behind the propagation front, which is a plateau-like function, dependent on the time variable only:

$$\left(1 - \frac{\rho_{\text{fr}}}{R_c}\right)W(\rho_{\text{fr}})\theta \left(1 - \frac{\rho_{\text{fr}}}{R_c}\right) = f(0, t, R_c).$$  \tag{7}

It is easy to prove that the solution of (6)–(8) tends to the exact solution in both alternative limits.

As in [18], we introduce the function $Q$ needed for determination of the automodel function $g$ in the intermediate range of values of the automodel variable $s$:

$$\left(1 - \frac{\rho_{\text{fr}}}{R_c}\right)Q(\rho, t, R_c)W(\rho)\theta \left(1 - \frac{\rho_{\text{fr}}}{R_c}\right) = f(0, t, R_c).$$  \tag{8}

To analyze the accuracy of the approximate automodel solution one has to show a weak dependence of the $Q_1$ and $Q_2$.
functions on, respectively, the space coordinate and time:

\[ Q(\rho, t(\rho, s, R_c), R_c) \equiv Q_1(s, \rho, R_c) \approx g(s, R_c), \]  
\[ Q(\rho(t, s, R_c), t, R_c) \equiv Q_2(s, t, R_c) \approx g(s, R_c), \]  

where the functions \( t(\rho, s) \) and \( \rho(t, s) \) are determined by the relation

\[ s = \rho(t, R_c), \]

(12)

Note that the definition (8) of the propagation front is a partial case of the definition of \( \rho(t, s) \) with (8) and (12): \( \rho^a(t) = \rho(t, s = 1) \).

Verification of the automodel solution (6)–(8) should be performed in the following way:

\[ Q(s, t, R_c) = Q(\rho^a(t, R_c) / s, t, R_c) = \frac{1}{36 \rho^a(t, R_c) R_c^2 f_{exact}^2(\rho^a(t, R_c) / s, t, R_c)} \left[ \cos \left( \frac{\pi}{6} + \frac{1}{3} \arctg \left[ \frac{1}{216 R_c^2 f_{exact}^2(\rho^a(t, R_c) / s, t, R_c) - 1 - i R_c} \left( 1 + \frac{1}{108 R_c^2 f_{exact}^2(\rho^a(t, R_c) / s, t, R_c) - 1 - i R_c} \right) \right] \right] \right]^{2} \]

(15)

The automodel function (9) may be expressed explicitly in terms of the automodel variable \( s = \rho^a(t, R_c) / \rho \):

\[ f_{auto}(x, t, R_c) = \left( t - \frac{\rho}{R_c} Q(\rho, t, R_c) \right) \frac{1}{4[1 + \rho Q(x, t, R_c)]^{1/2}}, t > \frac{\rho}{R_c} Q(\rho, t, R_c). \]

(16)

4. Automodel solution for \( \gamma = 0.5 \) power-law PDF

In what follows we consider the partial case \( \gamma = 0.5 \) which corresponds, e.g., to Lorentzian wings of atomic spectral line shapes for the emission of photons (see, e.g., the asymptotics of the Holstein function, equation (38) in [11]). The PDF (3) takes the form:

\[ W(\rho) = \frac{1}{4(1 + \rho)^{3/2}}. \]

(13)

Solving equation (8), we obtain the explicit expression for the propagation front \( \rho^a(t, R_c) \):

\[ \rho^a(t, R_c) = \frac{1}{36 R_c^2 f^2(0, t, R_c)} \left\{ \cos \left( \frac{\pi}{6} + \frac{1}{3} \arctg \left[ \frac{1}{216 R_c^2 f_{exact}^2(0, t, R_c) - 1 - i R_c} \left( 1 + \frac{1}{108 R_c^2 f_{exact}^2(0, t, R_c) - 1 - i R_c} \right) \right] \right] \right\}^{2} \]

(14)

The asymptotics far behind the propagation front for large retardation parameters and even larger values of time \( t \gg R_c \rightarrow \infty \) was calculated in [25]:

\[ f(x \rightarrow 0, t \gg R_c, R_c \rightarrow \infty) = \frac{1}{R_c^{1/2} \pi^{1/2}} \left[ \frac{2}{\pi^2} \arctg \left( 2\sqrt{2 + \sqrt{3}} \right) \right]^{1/2} = 0.0930 \]

Equation (17) gives the scaling, which coincides with that of equation (19) in [31], and specifies the numerical coefficient.

5. Verification of automodel solution for \( \gamma = 0.5 \) power-law PDF

We start the verification of the automodel solution (6)–(8) with the calculation of the exact solution (4) to determine the automodel function in the range of values of \( s \sim 1 \), where the automodel solution is expected to be most sensitive to the interpolation between the limits (7) of high and low values of \( s \);

- analysis of the accuracy of the self-similarity of the function \( Q \), defined by equation (9), in view of equation (10) and/or (11); and
- analysis of the accuracy of the automodel solution (6)–(8) with respect to the exact solution (4).
The results of the numerical calculation of the exact solution (4) for time moments $t = 100, 300, 1000$ and retardation parameters $R_c = 1, 10$ are given in figures 1–6.

Using the results of calculating the exact solution, we can determine the function $Q$ (15) and analyze the accuracy of its self-similarity in view of equation (11). The respective results are presented for $t = 100, 300, 1000$, and $R_c = 1$ (figures 7, 8) and $R_c = 10$ (figures 9, 10).

It is seen from figures 7–10 that the accuracy of the self-similarity of function (15), as a function of the automodel variable $s$ only (i.e. the accuracy of the applicability of relation (11) which assumes the independence of (15) from the time variable $t$), is high: the relative deviation of (15) for $t = 100$ and 300 from its value for $t = 1000$ does not exceed $2 \times 10^{-3}$ for $R_c = 1$ and $5 \times 10^{-2}$ for $R_c = 10$.

Now we are ready to take the next step: knowing the automodel function (15), we can construct the approximate automodel solution (6) and compare it with the calculated exact solution (4). The automodel solution for the Green’s function is not applicable for simultaneously very small values of the time and space coordinates: indeed, the automodel solution does not assume a description of the evolution of the system immediately after the action of an instant point source. Therefore, for self-similarity analysis we will take the exact solution for $t = 1000$ as a reference solution. This means that the automodel function (15), determined from comparison of the exact and automodel solutions for $t = 1000$, is used in the automodel solutions for other values of time. The respective comparison of the automodel and exact solutions for $t = 100$ and 300 is presented in figures 11(a) and (b) for $R_c = 1$ and in figures 12(a) and (b) for $R_c = 10$.

It can be seen that, despite the small difference between the automodel functions (15) at various time moments, the deviation of the proposed approximate automodel solution (6) from the exact one is as large as a factor of a few units in the region of the propagation front. Therefore, further modification of the approximate automodel solution (6) is necessary to improve the interpolation between the known asymptotics of the exact solution.

Figure 8. Characterization of self-similarity of function (15) as a function of automodel variable $s$ only, for $R_c = 1$: relative deviation of (15) for $t = 100$ and 300 from its value for $t = 1000$.

Figure 9. The same as figure 7 but for $R_c = 10$. 
6. Modification of automodel solution for $\gamma = 0.5$

power-law PDF

To improve the interpolation between the known asymptotics of the exact solution, we introduce a free parameter $\alpha$ into the definition of the propagation front (8):

$$\left( t - \frac{\rho_0}{R_c} \right) W(\rho_0) = a f_{\text{exact}}(0, t, R_c).$$  \hspace{1cm} (18)

For $W(\rho)$ (13), equation (18) gives the modification of the explicit expression (14) for the dependence of the propagation front on the time variable, which, in turn, determines the maximum allowable value of the parameter $\alpha$,

$$a_{\text{max}}(t, R_c) = \frac{1}{R_c f_{\text{exact}}(0, t, R_c) \sqrt{108 (1 + tR_c)}}$$  \hspace{1cm} (19)

and the respective value of the propagation front,

$$\lim_{\alpha \to a_{\text{max}}} \rho_0(t, R_c, \alpha) = \rho_0(t, R_c, a_{\text{max}}) = \frac{3}{4} tR_c - \frac{1}{4}. $$  \hspace{1cm} (20)
It appears that the minimum of the logarithmic derivative
\[
\frac{d \ln f_{\text{auto}}(s, t, R_c, \alpha)}{d \ln Q_1} = \left[ \frac{t R_c s}{\rho(t, R_c, \alpha) Q_1(s, t, R_c, \alpha)} - 1 \right]^{-1} - \frac{3}{2} \left[ \frac{s}{\rho(t, R_c, \alpha) Q_1(s, t, R_c, \alpha)} + 1 \right]^{-1}
\]
(21)
in the range \(s \sim 1\) is reached at \(\alpha = \alpha_{\text{max}}(t, R_c)\).

The modification of the propagation front definition leads to a modification of the approximate automodel solution:
\[
f_{\text{auto}}(s, t, R_c, \alpha_{\text{max}}(t, R_c)) = \left( t - \frac{\rho(t, R_c, \alpha_{\text{max}}(t, R_c))}{\rho} \right)
\times W(pg(\rho(t, R_c, \alpha_{\text{max}}(t, R_c)) / \rho))
\times \frac{1}{\alpha_{\text{max}}(t, R_c) + (1 - \alpha_{\text{max}}(t, R_c))} \frac{R_c}{t R_c}
\times \theta(t - \frac{\rho(t, R_c, \alpha_{\text{max}}(t, R_c))}{\rho}), \tag{22}
\]
where the asymptotics of \(g(s)\) is not changed and is defined by equation (7). The automodel function (15) is also modified to take the form
\[
Q_1(s, t, R_c, \alpha) = Q(\rho(t, R_c) / s, t, R_c, \alpha_{\text{max}}(t, R_c))
= \frac{1}{36 \rho(t, R_c, \alpha_{\text{max}}(t, R_c)) R_c^2} \frac{1}{s} \left( \frac{1}{1 + t R_c} \left( \frac{216 R_c^2}{108 R_c^2} \frac{1}{\rho(t, R_c, \alpha_{\text{max}}(t, R_c)) / s, t, R_c, \alpha_{\text{max}}(t, R_c)) + (1 - \alpha_{\text{max}}(t, R_c))} \right)^{2} - \frac{1}{2} \right)^{2}
\times \cos \left( \frac{\pi}{6} + \frac{1}{3} \arctg \left( \frac{1}{(1 + t R_c) \left( \frac{216 R_c^2}{108 R_c^2} \frac{1}{\rho(t, R_c, \alpha_{\text{max}}(t, R_c)) / s, t, R_c, \alpha_{\text{max}}(t, R_c)) + (1 - \alpha_{\text{max}}(t, R_c))} \right)^{2} - \frac{1}{2} \right) \right) - \frac{1}{s} \rho(t, R_c, \alpha_{\text{max}}(t, R_c)). \tag{23}
\]

The dependence of the automodel function (23) on \(s\) for \(t = 100, 300, 1000\) and the characterization of the self-similarity of the function (23), as a function of the automodel variable \(s\) only, are shown for \(R_c = 10\) in figures 13 and 14, respectively, and for \(R_c = 10\) in figures 15 and 16.

The final step of verification of the modified approximate automodel solution (22) is presented in figures 17 and 18, quite similar to figures 11 and 12 for the verification of the automodel solution (6). It appears that for \(R_c = 10\) we have substantial improvement of the accuracy of the automodel solution, while for \(R_c = 1\) the accuracy is slightly worse. In general, the approximate automodel solution (22) seems to be better than that from equation (6).

7. Conclusions

It is shown that the method [18] of an approximate automodel solution for the Green’s function of time-dependent superdiffusive (nonlocal) transport on a uniform background may be extended to the case of a finite velocity of carriers. This corresponds to an extension from Lévy flight based transport to transport of a type which belongs to the class of ‘Lévy walks + rests’ [1] and is characterized by trajectories with a finite average waiting time and a finite velocity of carriers. A solution for the arbitrary superdiffusive step-length PDF is suggested, and verification of the solution for a particular power-law PDF, which corresponds, e.g., to Lorentzian wings of atomic spectral line shapes for the emission of photons, is carried out using the computation of the exact solution. Success in identifying such solutions is based on the identification of the dominant role of long-free-path carriers in all three scaling laws used to construct the automodel solution, namely, the scaling laws for the propagation front (i.e. relevant-to-superdiffusion average displacement) and asymptotic solutions far beyond and far ahead of the propagation front. Detailed analysis of the automodel solutions for two values of the characteristic retardation parameter \(R_c\) (the ratio of the average waiting time to the average time of flight) enables us to evaluate the accuracy of defining the automodel function \(g\) (i.e. the accuracy of its self-similarity), which is no worse than a few percent in a wide space-time region, and the resulting accuracy of the improved automodel solution, which is no worse than several tens of percent.

The results of the modification of the simplest automodel solution via modification of the propagation front in the framework of optimizing the parameters of interpolation between the known asymptotics of the exact solution show that there is much freedom for such modifications to achieve the main goal of approximate automodel solutions for Lévy flight based transport and of various extensions of the principles [18], namely the construction of approximate solutions
of superdiffusive transport problems with high enough accuracy with essential savings of computation time. Indeed, as shown in [22, 23] for the case of Lévy flight transport, obtaining automodel (self-similar) solutions in the entire space of independent variables requires mass numerical simulations (distributed computing); nevertheless, their total volume is significantly reduced due to the self-similarity of the solutions.

**Figure 13.** Function (23) of automodel variable \( s \) for \( R_c = 1 \) and various time moments, \( t = 100, 300, 1000 \), in various ranges of \( s \).

**Figure 14.** Characterization of self-similarity of function (23) as a function of automodel variable \( s \) only, for \( R_c = 1 \): relative deviation of (23) for \( t = 100 \) and \( 300 \) from its value for \( t = 1000 \).

**Figure 15.** The same as figure 13 but for \( R_c = 10 \).
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Appendix

The construction of an approximate automodel solution in the case of a finite velocity of carriers should generalize the respective original procedure in the case of infinite velocity, as suggested in [18] and verified in [18, 21–23]. Such a solution should be based on the interpolation between two limiting cases, namely far ahead of and far behind the propagation front of the medium’s excitation, with a proper
choice of the propagation front which defines the region where the above asymptotics should be stitched.

1. At large distances from the source and just after the arrival of the first signal, namely at \( \rho \sim R_t t - 0, \rho \gg 1, \rho = |x| \), the suggested automodel solution, equation (6), coincides with the asymptotics far ahead of the propagation front, equation (5). This asymptotics is derived from the exact solution [24].

   Note that in the case of radiative transfer in spectral lines this condition corresponds to very high values of the optical length of the distance from the instant point source of photons and to the closeness of the observed point to the event horizon determined by the light cone. It is easy to understand this asymptotics in the case of an infinite velocity of carriers. Indeed, at distant points from the source at small values of dimensionless time an observer sees the source as an almost steady-state point source. Therefore, the observer merely accumulates absorbed photons linearly with time. This is because after a distant flight (Lévy flight) of a photon the excitation of the medium is trapped in Brownian motion around the absorption point. Here, the frequent exchange of excitation between close neighboring points of the medium has a negligible effect as compared with the absorption of photons from the original source of photons. In the case of a finite velocity, the result for this asymptotics appears to differ only by the allowance for the retardation (the first factor in (5)) and by the explicit formal effect of the light cone condition (the third factor in (5)).

2. In the alternative limiting case of small distances and long time, namely \( \rho \ll R_t t, t \gg 1 \), the automodel solution appears to be dependent only on the time variable, \( f(0, t, R_t) \).

   The asymptotics far behind the propagation front for an infinite velocity of carriers is found to be a plateau-like function, dependent on the time variable only [18]:

\[
  f(x, t) \sim \frac{1}{2 \rho_t} \theta(\rho_t(t) - \rho), \quad 0 \leq \rho \ll \rho_t(t), \tag{A1}
\]

where \( \rho_t(t) \) naturally defines the dimensionless coordinate of the propagation front, and \( \theta(y) \) is the Heaviside function. In the case of a finite velocity of carriers, weak dependence on the coordinate at small distances from the original source appears to be present as well; however the relation to the propagation front is more complicated, being less explicit as compared with equation (A1).

3. The definition of the propagation front in the case of superdiffusive transport is known to be very sensitive to the tail of the step-length PDF. The variance (i.e. the mean value of the squared distance from an instant point source) for the PDF (3) diverges. It is this property that characterizes the large class of PDFs called Lévy stable distributions (see [1–7]). In the theory of radiative transfer in resonance spectral lines for an infinite velocity of photons, an alternative definition has been proposed in [32]:

\[
  \tau = \int_0^\infty \int_0^\infty 4\pi \rho_t^2 f(\rho_t, t) d\rho_t dt. \tag{A2}
\]

For most interesting spectral line shapes (Doppler and Lorentz cases) this definition is approximately reduced to a unified simple law (see [11, 14]):

\[
  (t + 1) \cdot T(\rho_t(t)) = 1, \tag{A3}
\]

where \( T(\rho) \) is the probability of the carrier passing, without any absorption, a distance not exceeding a certain value, \( \rho \). This function may be expressed in terms of the PDF (3). In the 1D case one has:

\[
  W(\rho) = -\frac{dT(\rho)}{d\rho}, \quad T(0) = 1. \tag{A4}
\]

Note that equation (A3) is valid for all dimensions (for the 3D case see equation (75) in [11]); its applicability for the 3D case has been shown, e.g., via a massive numerical analysis of exact solutions, carried out in [22] for the PDF (3) in the 1D case and for Doppler and Lorentz line shapes in the 3D case. However, the definition (A3) works worse for the PDF in the case of Voigt line shape which combines the line shapes with substantially different levels of superdiffusivity (namely, a stronger, power-law tail for Lorentz line shape versus an exponential tail for Doppler line shape, and different values of the exponent in the power-law tail of the \( T(\rho) \) function: \( \gamma = 1/2 \) for Lorentz and approximately \( \gamma = 1 \) for the Doppler case; cf equation (75) in [11]). It appears that the definition of the propagation front may be improved if it is represented by a relation which equates the exact solutions in the alternative limits (cf equation (25) in [23]):

\[
  f_{\text{exact}}(0, t) = t W(\rho_t(t)). \tag{A5}
\]

For a finite velocity of carriers, the definition (A5) is extended and takes the form of equation (8).

4. Items 1–3 describe the three main elements of the approximate automodel solution. Generalization of the automodel solution [18] to the case of a finite velocity of carriers may be made by using again the asymptotics far ahead of the propagation front, where the propagation front is generalized in the form of equation (8). Thus, the solution (6)–(8) tends to the exact solution of equation (1) in both alternative limits (namely, far ahead of the propagation front in the form of equation (5) and the exact solution far behind the propagation front) and stitches these asymptotics with the help of the equation for the propagation front which defines the location and the manner of stitching.
Note that the constructed approximate automodel solution requires numerical solution of the transport equation, equation (1), only in the limited part of the entire space \([\rho, t]\) of the problem, namely far behind the propagation front to find the function \(f(0, t, R)\) and in the region of stitching, i.e. close to the propagation front. This gives substantial savings of computational time needed to obtain the solution in a much larger volume of the space with good accuracy (cf the massive numerical analysis in [22, 23] in the case of an infinite velocity of carriers).

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