On the solution of the inverse problem for a class of canonical systems corresponding to matrix string equations

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Abstract
We consider canonical systems (with $2p \times 2p$ Hamiltonians $H(x) \geq 0$), which correspond to matrix string equations. Direct and inverse problems are solved in terms of Titchmarsh–Weyl and spectral matrix functions and related $S$-nodes. Procedures for solving inverse problems are given.

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1 Introduction
Canonical systems have the form
\[ w'(x, \lambda) = i\lambda JH(x)w(x, \lambda), \quad J := \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}, \quad H(x) \geq 0, \quad (1.1) \]
where $w' := \frac{d}{dx}w$, $i$ is the imaginary unit ($i^2 = -1$), $\lambda$ is the so-called spectral parameter, $I_p$ is the $p \times p$ ($p \in \mathbb{N}$) identity matrix, $\mathbb{N}$ denotes the set of positive integer numbers, $H(x)$ is a $2p \times 2p$ matrix function (matrix valued function), and $H(x) \geq 0$ means that the matrices $H(x)$ are self-adjoint and the eigenvalues of $H(x)$ are nonnegative. Canonical systems are
important objects of analysis, being perhaps the most important class of the one-dimensional Hamiltonian systems and including (as subclasses) several classical equations. They have been actively studied in many already classical as well as in various recent works (see, e.g., [1, 4, 7, 8, 10–12, 15, 17, 19, 21, 22, 24–28, 30, 32, 35, 39–41] and numerous references therein).

In most works on canonical systems, a somewhat simpler case of $2 \times 2$ Hamiltonians $H(x)$ (i.e., the case $p = 1$) is dealt with. In particular, the trace normalisation $\text{tr} H(x) \equiv 1$ may be successfully used in the case $p = 1$. The cases with other values of $p$ ($p > 1$) are equally important but more complicated and less studied.

In this paper, we consider canonical systems $(1.1)$ with Hamiltonians $H(x)$ of the form

$$H(x) = \beta(x)^* \beta(x),$$

where $\beta(x)$ are $p \times 2p$ matrix functions. Systems $(1.1)$, $(1.2)$ are considered either on $[0, r]$ or on $[0, \infty)$. We assume that

$$\beta(x) \in U^{p \times 2p}[0, r], \quad U^{p \times q}[0, r] = \{ \mathcal{G} : \mathcal{G}'(x) \equiv \mathcal{G}'(0) + \int_0^x \mathcal{G}''(t) dt \},$$

where $\mathcal{G}' \in L_2^{p \times q}(0, r), \quad L_2^{p \times q}(0, r)$ stands for the class of $p \times q$ matrix functions with square integrable entries (i.e. the entries from $L_2(0, r)$) and $\mathcal{G}'$ is the standard derivative of $\mathcal{G}$. We say that $\mathcal{G}$ in $(1.3)$ is two times differentiable and that $\mathcal{G}''$ satisfying $\mathcal{G}'(x) \equiv \mathcal{G}'(0) + \int_0^x \mathcal{G}''(t) dt$ is the second derivative of $\mathcal{G}$. We also assume that

$$\beta(x) J \beta(x)^* \equiv 0, \quad \beta'(x) J \beta(x)^* \equiv i I_p.$$

We note that system $(1.1)$–$(1.4)$ (under some minor additional conditions) may be transformed into the matrix string equation [33, Appendix B]. Spectral theory of string equations is of great theoretical and applied interest (see [6, 13–20, 23, 39, 42] and various references therein). In particular, very interesting string equations appear in the study of nonlinear Camassa-Holm equation (see, e.g., [2, 3, 5]).
This paper is the third in a series of papers. A somewhat more general case of systems
\[ w'(x, \lambda) = i\lambda j H(x)w(x, \lambda), \quad j := \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad m_1, m_2 \in \mathbb{N} \]  
was studied in the previous two papers [33, 34].

Here, for the case of the canonical system (1.1)–(1.4) we present a solution of the inverse problem of recovery of the system (or, equivalently, of \( H(x) \)) from the Titchmarsh–Weyl (Weyl) matrix function. The uniqueness Theorem 2.2 from [39, p. 116] and the scheme of its proof are essential for our considerations, and we use several assertions from [33, 34] as well. Our result is new even for the case \( p = 1 \) since we do not require that the entries of \( H \) are real-valued.

The solution of the inverse problem for canonical systems corresponding to matrix string equations complements in an important way our solutions [10, 29, 31, 35] of the inverse problems for canonical systems corresponding to Dirac (Zakharov–Shabat) systems. Note that another interesting approach to Dirac systems is related to the Riemann-Hilbert problem formulation of the Zakharov–Shabat spectral problem in [43, 44]. For multicomponent spectral problems (such as problems with potentials taking values in matrix Lie algebras), further developments are summarised, for instance, in the review paper [9].

In the preliminary Section 2, we consider some linear similarity problems and operator identities which are necessary for our procedure. In section 3, we study Weyl functions of the system (1.1)–(1.4). The procedure to recover system from its Weyl functions is given in Section 4. Direct and inverse problems in terms of spectral functions are studied in Section 5.

**Notations.** Some notations were already introduced in the introduction above. As usual, \( \mathbb{R} \) stands for the real axis, \( \mathbb{C} \) stands for the complex plane, the open upper half-plane is denoted by \( \mathbb{C}_+ \), and \( \overline{a} \) means the complex conjugate of \( a \). We set \( L_2^{p \times 1} = L_2^p \), \( L_2^1 = L_2 \) and \( \mathcal{U}^{p \times 1} = \mathcal{U}^p \). (\( L_2^p(0,r) \) and \( L_2(0,r) \) stand also for the corresponding Hilbert spaces of square summable functions.) The notation \( I \) stands for the identity operator. The norm \( \|A\| \) of the \( n \times n \) matrix \( A \) means the norm of \( A \) acting in the space \( \ell_n^2 \) of the sequences
of length \( n \). The class of bounded operators acting from the Hilbert space \( \mathcal{H}_1 \) into Hilbert space \( \mathcal{H}_2 \) is denoted by \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \), and we set \( \mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H}) \). The range of the operator \( A \) is denoted by \( \text{Im} (A) \).

2 Preliminaries

1. The matrices \( J \) introduced in (1.1) and \( j \) introduced in (1.5) are unitarily equivalent (in the case \( m_1 = m_2 = p \)). This equivalence is given by the relations

\[
J = \Theta j \Theta^*, \quad \Theta := \frac{1}{\sqrt{2}} \begin{bmatrix} I_p & -I_p \\ I_p & I_p \end{bmatrix}.
\] (2.1)

Therefore, we easily reformulate the statements from [33,34] (with \( j \)) into the corresponding statements here. Our next proposition is immediate from [33, Theorem C.1].

**Proposition 2.1** Let the \( p \times 2p \) matrix function \( \beta(x) \) satisfy (1.3) and (1.4). Introduce operators \( A \) and \( K \) acting in \( L^p_2(0, r) \) by the equalities

\[
A = \int_0^x (t - x) \cdot dt, \quad K = i \beta(x)J \int_0^x \beta(t)^* \cdot dt.
\] (2.2)

Then, \( K \) is linear similar to \( A \):

\[
K = VAV^{-1}, \quad V = u(x)(I + \int_0^x \mathcal{V}(x, t) \cdot dt),
\] (2.3)

where

\[
\begin{align*}
&u \in \mathcal{U}^{p \times p}[0, r], \quad u^* = u^{-1}, \quad u(0) = I_p, \\
&\sup \|\mathcal{V}(x, t)\| < \infty \quad (0 \leq t \leq x \leq r).
\end{align*}
\] (2.4)

and

\[
\begin{align*}
\sup \|\mathcal{V}(x, t)\| < \infty \quad (0 \leq t \leq x \leq r).
\end{align*}
\] (2.5)

We note that the operator \( A \) is equal to the minus squared integration:

\[
A = A^2, \quad A := i \int_0^x \cdot dt.
\] (2.6)
According to [34, Proposition A.1], the operator $V$, which was constructed in the proof of [33, Theorem C.1] and was mentioned in Proposition 2.1, has the following properties.

**Proposition 2.2** Let the conditions of Proposition 2.1 hold. Then, one may assume (without loss of generality) that the similarity transformation operators $V$ and $V^{-1}$ in (2.3) map vector functions $f \in U^p[0, r]$ into $U^p[0, r]$ (where $U^p[0, r] = U^{p\times 1}[0, r]$).

Partition $\beta$ into two $p \times p$ blocks $\beta_k$ ($k = 1, 2$). Further in the text, we suppose that

$$\det (\beta_2(0)) \neq 0 \quad (\beta =: \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}). \quad (2.7)$$

Introduce the operator $V_0 \in B(L^2_2(0, r))$ by the equalities

$$V_0 f = \beta_2(0) f + \int_0^x V_0(x-t)f(t)dt, \quad V_0 := (V^{-1}\beta_2)', \quad (2.8)$$

where $V^{-1}$ is applied to $\beta_2$ columnwise. In view of the additional condition (2.7), the proof of [34, Lemma A.2] works also in our case (of $J$ instead of $j$ in the corresponding relations) and we have the next proposition.

**Proposition 2.3** The operator $V_0$ is invertible and commutes with $A$:

$$V_0 A = AV_0. \quad (2.9)$$

Moreover, the operators $V_0$ and $V_0^{-1}$ map $U^p[0, r]$ into $U^p[0, r]$.

Propositions 2.1–2.3 yield the following theorem.

**Theorem 2.4** Let the $p \times 2p$ matrix function $\beta(x)$ satisfy (1.3), (1.4), and (2.7). Then, the operator $E$ given by the formula

$$E := VV_0, \quad Ef = u(x)\beta_2(0)f + \int_0^x \mathcal{E}(x,t)f(t)dt, \quad (2.10)$$

satisfies the equalities

$$K = EAE^{-1}, \quad E^{-1}\beta_2 \equiv I_p, \quad (2.11)$$

where $E^{-1}$ is applied to $\beta_2$ columnwise. Moreover, the operators $E$ and $E^{-1}$ map $U^p[0, r]$ into $U^p[0, r]$.
Proof. Relations (2.3), (2.9), and (2.10) yield the first equality in (2.11). The last statement in the theorem is immediate from the last statements in Propositions 2.2 and 2.3.

Finally, it follows from the last equalities in (2.3) and (2.4) and from (2.5) that

\[(V^{-1}\beta_2)(0) = \beta_2(0).\]  \hfill (2.12)

Taking into account (2.8) and (2.12), we derive

\[V_0I_p = \beta_2(0) + \int_0^x V(t)dt = (V^{-1}\beta_2)(x),\]  \hfill (2.13)

which implies the second equality in (2.11) ■

The following analog of [34, Remark A.5] is valid.

Remark 2.5 It follows from [34, Remark 2.5] and formulas (2.8) and (2.10) that the integral kernel \(E(x,t)\) (of \(E\)) in the domain \(0 \leq t \leq x \leq \ell < r\) is uniquely determined by \(\beta(x)\) on \([0, \ell]\) (and does not depend on the choice of \(\beta(x)\) for \(\ell < x < r\) and the choice of \(r \geq \ell\)).

2. Taking into account the definition of \(K\) in (2.2), it is easy to see that

\[K - K^* = i\beta(x)J \int_0^\ell \beta(t)^* \cdot dt.\]  \hfill (2.14)

Hence, the first equality in (2.11) yields the operator identity

\[AS - SA^* = i\Pi J\Pi^*,\]  \hfill (2.15)

where

\[S = E^{-1}(E^*)^{-1} > 0, \quad \Pi h = \Pi(x)h, \quad \Pi(x) := (E^{-1}\beta)(x),\]  \hfill (2.16)

\[\Pi \in \mathcal{B}(\mathbb{C}^2, L_p^p(0, r)), \quad \Pi(x) \in L^{p \times 2p}(0, r), \quad h \in \mathbb{C}^2.\]  \hfill (2.17)

The triple of bounded operators \(\{A, S = S^*, \Pi\}\), such that the operator identity (2.15) holds and \(J = J^* = J^{-1}\), is called a symmetric \(S\)-node. (In our case, \(J\) is given in (1.1).) We partition \(\Pi\) into the the blocks \(\Phi_1, \Phi_2 \in \mathbb{C}^2\).
\[ \mathcal{B}(\mathbb{C}^p, L^p_2(0, r)) \text{ and the matrix function } \Pi(x) \text{ into the corresponding } p \times p \text{ blocks } \Phi_1(x) \text{ and } \Phi_2(x): \]

\[ \Pi = [\Phi_1 \Phi_2], \quad \Pi(x) = [\Phi_1(x) \Phi_2(x)]. \quad (2.18) \]

Note that the last relations in (2.11) and (2.16) imply that

\[ \Phi_2(x) \equiv I_p. \quad (2.19) \]

Next, we introduce the projectors \( P_\ell \in \mathcal{B}(L^p_2(0, r), L^p_2(0, \ell)) \):

\[ (P_\ell f)(x) = f(x) \quad (0 < x < \ell, \quad \ell \leq r), \quad (2.20) \]

and set

\[ S_\ell := P_\ell S \Pi_\ell^*, \quad E_\ell := P_\ell E \Pi_\ell^*, \quad A_\ell := P_\ell A \Pi_\ell^*, \quad \Pi_\ell(x) = \Pi_\ell(x)g. \quad (2.21) \]

A more detailed version of the following considerations is contained in [33, Chapter 2] and [34, Chapter 3]. Since \( E \) is a triangular operator, \( E^{-1} \) is triangular as well and we have \( P_\ell E^{-1} = P_\ell E^{-1} \Pi_\ell^* P_\ell \). It follows that

\[ E_\ell^{-1} = P_\ell E^{-1} \Pi_\ell^*, \quad S_\ell = E_\ell^{-1}(E_\ell^*)^{-1}, \quad (2.22) \]

We also have \( P_\ell A = P_\ell A \Pi_\ell^* P_\ell \). Thus, the operator identity (2.15) and relations (2.16), (2.21), and (2.22) yield

\[ A_\ell S_\ell - S_\ell A_\ell^* = i \Pi_\ell J \Pi_\ell^*, \quad \Pi_\ell(x) = (E_\ell^{-1} \beta)(x) \quad (0 < x < \ell). \quad (2.23) \]

The transfer matrix function (in Lev Sakhnovich form [36, 37, 39]) is given by the formula

\[ w_A(\ell, \lambda) = I_{2p} - i \Pi_\ell \Pi_\ell^* S_\ell^{-1}(A_\ell - \lambda I)^{-1} \Pi_\ell. \quad (2.24) \]

**Remark 2.6** According to Remark 2.5, \( E_\ell \) may be constructed in the same way as \( E \), and so \( E_\ell, S_\ell, \Pi_\ell, \text{ and } w_A(\ell, \lambda) \) do not depend on the choice of \( \beta(x) \) for \( \ell < x < r \) and the choice of \( r \geq \ell \). In particular, \( w_A(\ell, \lambda) \) is uniquely defined on the semi-axis \( 0 < \ell < \infty \) for \( \beta(x) \) considered on the semi-axis \( 0 \leq x < \infty \).
The fundamental solution of the canonical system (1.1)–(1.4) may be expressed via the transfer functions \( w_A(\ell, \lambda) \) using continuous factorisation theorem [39, p. 40] (see also [35, Theorem 1.20] as a more convenient for our purposes presentation). Recall also our assumption that that (2.7) holds for system (1.1)–(1.4).

**Theorem 2.7** Let the Hamiltonian of the canonical system (1.1) have the form (1.2). Assume that \( \beta(x) \) in (1.2) belongs \( U^{p \times 2p}[0, r] \), satisfies (1.4), and that \( \det(\beta_2(0)) \neq 0 \). Then, the fundamental solution \( W(x, \lambda) \) of the canonical system, normalised by

\[
W(0, \lambda) = I_{2p},
\]

admits representation

\[
W(\ell, \lambda) = w_A(\ell, \frac{1}{\lambda}).
\]

If theorem’s conditions hold for each \( 0 < r < \infty \), then (2.26) is valid for each \( \ell \) on the semi-axis \((0, \infty)\).

The proof of Theorem 2.7 coincides with the proof of Theorem 2.2 in [33].

**Remark 2.8** The second equality in (2.23) implies that \( E_\ell \Pi_\ell g = \beta(x)g \) \((g \in \mathbb{C}^{2p})\) which, in view of the second equality in (2.22), yields

\[
H(\ell) = \frac{d}{d\ell}(\Pi_\ell S_\ell^{-1} \Pi_\ell).
\]

**Remark 2.9** It is easy to see that \( S_r = S, \Pi_r = \Pi \) and

\[
w_A(\lambda) := w_A(r, \lambda) = I_{2p} - i\Pi^* S^{-1}(A - \lambda I)^{-1}\Pi.
\]

**3 Weyl functions**

Recall that \( W(x, \lambda) \) is the normalised fundamental solution of the canonical system (1.1)–(1.4) and set:

\[
\mathcal{W}(r, \lambda) = \{\mathcal{W}_{ik}(r, \lambda)\}_{i,k=1}^2 = W(r, \bar{\lambda})^* = w_A(1/\bar{\lambda})^*,
\]

(3.1)
where $\mathcal{W}_{ik}$ are the $p \times p$ blocks of $\mathcal{W}$.

Pairs of meromorphic in $\mathbb{C}_+$, $p \times p$ matrix functions $\mathcal{P}_k(\lambda)$ ($k = 1, 2$) such that

$$\mathcal{P}_1(\lambda)^* \mathcal{P}_1(\lambda) + \mathcal{P}_2(\lambda)^* \mathcal{P}_2(\lambda) > 0, \quad \begin{bmatrix} \mathcal{P}_1(\lambda)^* & \mathcal{P}_2(\lambda)^* \end{bmatrix} \begin{bmatrix} \mathcal{P}_1(\lambda) \\ \mathcal{P}_2(\lambda) \end{bmatrix} \geq 0$$  \hspace{1cm} (3.2)

(where the first inequality holds in one point (at least) of $\mathbb{C}_+$ and the second inequality holds in all the points of analyticity of $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathbb{C}_+$), are called nonsingular, with property-J.

**Notation 3.1** The notation $\mathcal{N}(r)$ stands for the set of matrix functions of the form

$$\phi(r, \lambda) = i(\mathcal{W}_{11}(r, \lambda)\mathcal{P}_1(\lambda) + \mathcal{W}_{12}(r, \lambda)\mathcal{P}_2(\lambda))$$

$$\times (\mathcal{W}_{21}(r, \lambda)\mathcal{P}_1(\lambda) + \mathcal{W}_{22}(r, \lambda)\mathcal{P}_2(\lambda))^{-1},$$  \hspace{1cm} (3.3)

where the pairs $\{\mathcal{P}_1, \mathcal{P}_2\}$ are nonsingular, with property-J.

Functions $\phi \in \mathcal{N}(r)$ for general type canonical systems have been studied in [35, Appendix A] (see also references therein). They are called Titchmarsh–Weyl (Weyl) functions of the canonical system (1.1) on $[0, r]$.

**Notation 3.2** The class of $p \times p$ analytic matrix functions $\phi(\lambda)$ ($\lambda \in \mathbb{C}_+$), such that

$$i(\phi(\lambda)^* - \phi(\lambda)) \geq 0,$$  \hspace{1cm} (3.4)

is denoted by $\mathcal{H}$ (Herglotz class).

The matrix functions of Herglotz class admit well-known Herglotz representation

$$\phi(\lambda) = \mu \lambda + \nu + \int_{-\infty}^{\infty} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\tau(t), \quad \mu \geq 0, \quad \nu = \nu^*,$$  \hspace{1cm} (3.5)

where $\tau(t)$ is a $p \times p$ matrix function such that $\tau(t_1) \geq \tau(t_2)$ for $t_1 > t_2$ (i.e., $\tau$ is monotonically increasing) and

$$\frac{d\tau(t)}{1 + t^2} < \infty.$$  \hspace{1cm} (3.6)

The following proposition shows that $\mathcal{N}(r)$ is well defined.
Proposition 3.3 Let $W(x, \lambda)$ be the fundamental solution of the canonical system (1.1) such that the relations (1.2)–(1.4) and (2.7) are valid, let $W$ be expressed via $W$ using formula (3.1), and let the pair $\{P_1, P_2\}$ be nonsingular, with property-J. Then, we have

$$\det (W_{21}(r, \lambda)P_1(\lambda) + W_{22}(r, \lambda)P_2(\lambda)) \neq 0$$

(3.7)
in the domain of nonsingularity of $\{P_1, P_2\}$. Moreover, the class $\mathcal{N}(r)$ belongs to Herglotz class

$$\mathcal{N}(r) \subset \mathbb{H}.$$  

(3.8)

Proof. Inequality (3.7) is proved (for the $S$-node case) in [38, p. 11]. In order to make the paper self-contained we prove it here (in a slightly more direct way). Namely, we suppose that the determinant in (3.7) equals zero, and so there is some $g$ such that

$$\begin{bmatrix} 0 & I_p \end{bmatrix} W(x, \lambda) \begin{bmatrix} P_1(\lambda) \\ P_2(\lambda) \end{bmatrix} g = 0, \quad g \in \mathbb{C}^p, \quad g \neq 0.$$  

(3.9)

It follows from the expression for $W$ via $w_A$ in (3.1) and from the equality [37, (2.8), p.24] that

$$W(r, \lambda)^* JW(r, \lambda) = J - i(\lambda - \overline{\lambda})J \Pi^* S^{-1}(I - \overline{\lambda} A)^{-1} S(I - \lambda A^*)^{-1} S^{-1} \Pi J.$$  

(3.10)

Hence, the assumption (3.9) together with the second inequality in (3.2) yield

$$g^* \begin{bmatrix} P_1(\lambda)^* & P_2(\lambda)^* \end{bmatrix} W(r, \lambda)^* JW(r, \lambda) \begin{bmatrix} P_1(\lambda) \\ P_2(\lambda) \end{bmatrix} g$$

$$= 0 \geq i(\lambda - \overline{\lambda})g^* \begin{bmatrix} P_1(\lambda)^* & P_2(\lambda)^* \end{bmatrix} J \Pi^* S^{-1}(I - \overline{\lambda} A)^{-1} S(I - \lambda A^*)^{-1} S^{-1} \Pi J$$

$$\times \begin{bmatrix} P_1(\lambda) \\ P_2(\lambda) \end{bmatrix} g.$$  

(3.11)

Since $S > 0$, the right-hand side of (3.11) equals zero, which implies

$$\Pi J \begin{bmatrix} P_1(\lambda) \\ P_2(\lambda) \end{bmatrix} g = 0.$$  

(3.12)
From the equality $W(r, \lambda) = w_A(1/\lambda)^*$, where $w_A$ is given by (2.28), and from formula (3.12), we derive

$$W(r, \lambda) \begin{bmatrix} P_1(\lambda) \\ P_2(\lambda) \end{bmatrix} g = \begin{bmatrix} P_1(\lambda) \\ P_2(\lambda) \end{bmatrix} g.$$  \hspace{1cm} (3.13)

Using (3.9) and (3.13), we obtain

$$P_2(\lambda)g = 0.$$  \hspace{1cm} (3.14)

It follows from (2.19), (3.12) and (3.14) that

$$\Pi J \begin{bmatrix} P_1(\lambda) \\ P_2(\lambda) \end{bmatrix} g = \Phi_2(x) P_1(\lambda)g = P_1(\lambda)g = 0.$$  \hspace{1cm} (3.15)

However, $P_1(\lambda)g \neq 0$ in the domain of nonsingularity of $\{P_1, P_2\}$ and we arrive at a contradiction. Thus, (3.7) is proved.

Note that according to (3.10) (and the second inequality in (3.2)) we have

$$\begin{bmatrix} P_1(\lambda)^* & P_2(\lambda)^* \end{bmatrix} W(r, \lambda)^* J W(r, \lambda) \begin{bmatrix} P_1(\lambda) \\ P_2(\lambda) \end{bmatrix} \geq 0.$$  \hspace{1cm} (3.16)

Taking into account relations (3.7) and (3.16) as well as the definition (3.3) of $\phi \in \mathcal{N}(r)$, we obtain

$$\begin{bmatrix} i\phi(\lambda)^* & I_p \end{bmatrix} J \begin{bmatrix} -i\phi(\lambda) \\ I_p \end{bmatrix} \geq 0,$$  \hspace{1cm} (3.17)

and (3.4) follows. Thus, $\mathcal{N}(r) \subseteq \mathbf{H}$. In the next section, we will show that $\mu = 0$ in Herglotz representation of $\phi \in \mathcal{N}(r)$ (see (4.1)), and so $\mathcal{N}(r) \subseteq \mathbf{H}$ (i.e., (3.8) is valid).

**Proposition 3.4** Assume that $r_2 > r_1 > 0$. Let canonical system (1.1), such that relations (1.2)–(1.4) are valid on $[0, r_2]$ and $\det(\beta_2(0)) \neq 0$, be given. Then, we have

$$\mathcal{N}(r_2) \subseteq \mathcal{N}(r_1),$$  \hspace{1cm} (3.18)
that is, the families $\mathcal{N}(\ell)$ are embedded.

If the canonical system is given on $[0, \infty)$ (and the relations (1.2)–(1.4) hold for each $r > 0$), there is a matrix function $\varphi(\lambda)$ such that

$$\varphi \in \bigcap_{r>0} \mathcal{N}(r),$$

(3.19)

and the intersection of the families $\mathcal{N}(r)$ is nonempty. Moreover, the following inequality is valid for $\varphi$ satisfying (3.19):

$$\int_0^\infty \left[ I_p \ i\varphi(\lambda)^* \right] W(x, \lambda)^* H(x) W(x, \lambda) \left[ I_p \ -i\varphi(\lambda) \right] dx < \infty \quad (\lambda \in \mathbb{C}_+).$$

(3.20)

**Definition 3.5** Matrix functions $\varphi(\lambda)$ satisfying (3.19) are called Weyl functions of the canonical system on $[0, \infty)$.

**Remark 3.6** The inequality (3.20) is often used as another definition of the Weyl function.

**Proof of Proposition 3.4.** Let $\phi \in \mathcal{N}(r_2)$, that is, let $\phi$ admit representation (3.3) where $r = r_2$. In view of (1.1) and (3.1), we can factorise $W(r_2, \lambda)$:

$$W(r_2, \lambda) = W(r_1, \lambda) \tilde{W}(r_2, \lambda),$$

(3.21)

where

$$\frac{d}{dx} \tilde{W}(x, \lambda)^* = i\lambda JH(x) \tilde{W}(x, \lambda)^* \quad (x \geq r_1), \quad \tilde{W}(r_1, \lambda) = I_{2p}. \quad (3.22)$$

Hence, we have

$$\frac{d}{dx} (\tilde{W}(x, \lambda)J\tilde{W}(x, \lambda)^*) = i(\overline{\lambda} - \lambda)\tilde{W}(x, \lambda)H(x)\tilde{W}(x, \lambda)^* \geq 0 \quad (\lambda \in \mathbb{C}_+).$$

(3.23)

Taking into account (3.22) and (3.23), we derive $\tilde{W}(x, \lambda)J\tilde{W}(x, \lambda)^* \geq J$, which yields (see, e.g., [35, Corollary E.3])

$$\tilde{W}(x, \lambda)^*J\tilde{W}(x, \lambda) \geq J.$$ 

(3.24)
Recall that the pair \( \{P_1, P_2\} \) generates \( \phi(\lambda) \in \mathcal{N}(r_2) \) via (3.3) where \( r = r_2 \). According to (3.2) and (3.24), the pair \( \{\tilde{P}_1, \tilde{P}_2\} \) given by

\[
\begin{bmatrix}
\tilde{P}_1(\lambda) \\
\tilde{P}_2(\lambda)
\end{bmatrix} = \tilde{\mathcal{W}}(r_2, \lambda)
\begin{bmatrix}
P_1(\lambda) \\
P_2(\lambda)
\end{bmatrix}
\]  

(3.25)

is nonsingular, with property-J. It follows from (3.21) and (3.25), that the same \( \phi(\lambda) \) is generated via (3.3) (where \( r = r_1 \)) by the pair \( \{\tilde{P}_1, \tilde{P}_2\} \). Thus, \( \phi(\lambda) \in \mathcal{N}(r_1) \) and (3.18) is proved.

The existence of holomorphic \( \varphi \) satisfying (3.19) is proved similar to the analogous fact in [33, Appendix A]. However, here we use Fundamental normality test (stronger Montel’s theorem) instead of Montel’s theorem. Since \( \mathcal{N}(r) \) belongs to Herglotz class, according to stronger Montel’s theorem there is a sequence \( \{\phi_k(\lambda)\} \), where

\[
\phi_k(\lambda) \in \mathcal{N}(r_k), \quad r_k \to \infty \quad \text{for} \quad k \to \infty,
\]

which converges uniformly on all compact subsets of \( \mathbb{C}_+ \) to a holomorphic function \( \varphi(\lambda) \).

On the other hand, taking into account (3.3), we see that

\[
\mathcal{W}(r, \lambda)^{-1} \begin{bmatrix}
-i\phi(r, \lambda) \\
I_p
\end{bmatrix} = \begin{bmatrix}
P_1(\lambda) \\
P_2(\lambda)
\end{bmatrix} (\mathcal{W}_{21}(r, \lambda)P_1(\lambda) + \mathcal{W}_{22}(r, \lambda)P_2(\lambda))^{-1}.
\]

(3.26)

Therefore, \( \phi(\lambda) \in \mathcal{N}(r) \) is equivalent to the inequality

\[
[i\phi(\lambda)^* I_p] \mathfrak{A}(r, \lambda) \begin{bmatrix}
-i\phi(\lambda) \\
I_p
\end{bmatrix} \geq 0, \quad \mathfrak{A}(r, \lambda) := (\mathcal{W}(r, \lambda)^{-1})^* J \mathcal{W}(r, \lambda)^{-1}
\]

(3.27)

for all \( \lambda \in \mathbb{C}_+ \). For any \( r > 0 \), there is some \( k_r \) such that the relation \( \phi_k(\lambda) \in \mathcal{N}(r) \) \( (k \geq k_r) \) holds. Therefore, (3.27) holds for these \( \phi_k(\lambda) \), and so it holds also for the limit function \( \varphi(\lambda) \). Hence, (3.19) is valid.

Finally, it is easy to see that

\[
\frac{d}{dx} (W(x, \overline{x})^* J W(x, \lambda)) = i(\lambda - \mu)W(x, \overline{x})^* H(x)W(x, \lambda),
\]

(3.28)
which implies, in particular, that
\[ W(r, \lambda)JW(r, \lambda) \equiv W(r, \lambda)\ast JW(r, \lambda) \equiv J, \quad W(r, \lambda) = JW(r, \lambda)^{-1}J. \]  
(3.29)

Using (3.28), (3.29), and the definition of \( \mathfrak{A} \) in (3.27), we obtain
\[
\int_0^r W(x, \lambda)^* H(x) W(x, \lambda) \, dx = \frac{i}{\lambda - \bar{\lambda}} (J - W(r, \lambda)^* JW(r, \lambda)) = \frac{i}{\lambda - \bar{\lambda}} (J - J\mathfrak{A}(r, \lambda)J). \]  
(3.30)

Hence, inequality (3.27) yields
\[
[I_p \quad i\phi(\lambda)^*] \int_0^r W(x, \lambda)^* H(x) W(x, \lambda) \, dx \left[ I_p \quad -i\phi(\lambda) \right] \leq \frac{\phi(\lambda) - \phi(\lambda)^*}{\lambda - \bar{\lambda}} \]  
(3.31)
for all \( \phi(\lambda) \in \mathcal{N}(r) \). It follows from (3.31) that the inequality (3.20) is valid for \( \varphi(\lambda) \) satisfying (3.19).

4 Inverse problem

We will consider canonical system (1.1)–(1.4) on \([0, r]\) and corresponding symmetric \( S \)-node \( \{ A, S, \Pi \} \) introduced in Section 2. For this purpose, some properties of the \( S \)-nodes [37–39] will be recalled in the first paragraph.

1. Let a symmetric \( S \)-node be given, where \( A, S \in \mathcal{B}(\mathcal{H}), \, \Pi = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}, \, \Phi_1, \Phi_2 \in \mathcal{B}(\mathcal{C}^p, \mathcal{H}) \) and \( \mathcal{H} \) is a Hilbert space. By definition of the symmetric \( S \)-node, we have (2.15) (and we assume that \( J \) for (2.15) is given in (1.1)).

We require also that \( S \) is strictly positive, that its inverse is bounded, that \( \Phi_2 g = 0 \) yields \( g = 0 \), and that \( A \) has only one point of spectrum, namely, zero. (In fact, it would suffice that the spectrum of \( A \) consists of no more than a countable set of points as supposed in [38].)

By \( \mathcal{N}(w_A) \) we denote the class of matrix functions (linear fractional transformations) (3.3) where \( \mathcal{W}(\lambda) = w_A(1/\lambda)^* \) and the pairs \( \{ P_1, P_2 \} \) are again nonsingular, with property-\( J \). Thus, \( \mathcal{N}(r) \) is a particular case of \( \mathcal{N}(w_A) \), which corresponds to the \( S \)-node introduced in Section 2.
Under our conditions on the $S$-node (mentioned at the beginning of the paragraph), the proof of Proposition 3.3 works for the case of $\mathcal{N}(w_A)$. Therefore, we have $\mathcal{N}(w_A) \in H$ and (3.7) holds. It follows that matrix functions $\phi$ given by (3.3) are well defined and admit Herglotz representation (3.5). Moreover, the $S$-node and any $\phi \in \mathcal{N}(w_A)$ satisfy also the conditions of Theorem 1.4.2 and Proposition 1.3.1 from [38]. According to [38, Proposition 1.3.1], the range of $\Phi_2 \mu$ (for $\mu$ from (3.5)) belongs to the range of $A$. Assuming additionally $\operatorname{Im} (A) \cap \operatorname{Im} (\Phi_2) = 0$, we derive
\[ \mu = 0 \quad \text{for} \quad \phi \in \mathcal{N}(w_A). \quad (4.1) \]

For each $\phi \in \mathcal{N}(w_A)$ or, equivalently, for its Herglotz representation, we construct operators
\[ \mathcal{S} = \int_{-\infty}^{\infty} (I - t A)^{-1} \Phi_2 (d \tau (t)) \Phi_2^* (I - t A^*)^{-1}, \quad (4.2) \]
\[ \mathcal{\Phi}_1 = -i \int_{-\infty}^{\infty} \left( A (I - t A)^{-1} + \frac{t}{1 + t^2} I \right) \Phi_2 d \tau (t) + i \Phi_2 \nu. \quad (4.3) \]

Theorem 1.3.1 and Theorem 1.4.2 from [38] show that the integrals in (4.2) and (4.3) weakly converge (for $\phi \in \mathcal{N}(w_A)$).

**Notation 4.1** The set $\mathcal{N}(S, \Phi_1)$ is the set of functions $\phi \in H$ such that $\mu = 0$ in Herglotz representation (3.5), that the integrals in (4.2) and (4.3) weakly converge for $\tau$ from the Herglotz representation of $\phi$, and that the following equalities hold:
\[ S = \mathcal{S}, \quad \Phi_1 = \mathcal{\Phi}_1. \quad (4.4) \]

We will need Theorem 2.4 from [39, p. 57] (see below), which is an important corollary of Proposition 1.3.2 and Theorem 1.4.2 from [38].

**Theorem 4.2** Let a symmetric $S$-node satisfy five conditions:
\begin{enumerate}
\item[a)] the operator $S$ is positive and bounded together with its inverse;
\item[b)] the spectrum of $A$ is concentrated at zero;
\item[c)] zero is not an eigenvalue of $A$;
\item[d)] $\operatorname{Im} (A) \cap \operatorname{Im} (\Phi_2) = 0$;
\item[e)] $\Phi_2 g = 0$ yields $g = 0$.
\end{enumerate}

Then, we have
\[ \mathcal{N}(S, \Phi_1) = \mathcal{N}(w_A). \quad (4.5) \]
2. The operators $A$, $S$ and $\Phi_2$ corresponding to system (1.1)–(1.4) on $[0, r]$ are given by the equalities (2.6), (2.16), and (2.19), respectively. Hence, it is immediate that the conditions of Theorem 4.2 are satisfied. Recall that (in view of (3.1)) in the particular case of the $S$-node corresponding to system (1.1)–(1.4) on $[0, r]$ we have

$$\mathcal{N}(w_A) = \mathcal{N}(r).$$  \hfill (4.6)

Equalities (4.5) and (4.6) provide a procedure to solve inverse problem.

**Theorem 4.3** Assume that $\phi(\lambda)$ is a Weyl function of the canonical system (1.1)–(1.4) on $[0, r]$ (where $\det (\beta_2(0)) \neq 0$), that is, $\phi(\lambda) \in \mathcal{N}(r)$.

Then, $\phi(\lambda) \in \mathcal{H}$ and admits Herglotz representation (3.5) (with $\mu = 0$). Using $\tau$ and $\nu$ from this Herglotz representation and relations (4.2) and (4.3), we recover $S = \tilde{S}$, $\Phi_1 = \tilde{\Phi}_1$, and $\Pi = [\Phi_1 \ \Phi_2]$, where $\Phi_2 g \equiv g$ ($g \in \mathbb{C}^p$). Finally, the Hamiltonian $H$ is recovered by the formula

$$H(\ell) = \frac{d}{d\ell} (\Pi_\ell^* S_\ell^{-1} \Pi_\ell), \quad 0 < \ell \leq r; \quad S_\ell := P_\ell S P_\ell^*, \quad \Pi_\ell := P_\ell \Pi.$$ \hfill (4.7)

**Proof.** Equalities (4.5) and (4.6) show that (4.4) holds, that is, $S = \tilde{S}$ and $\Phi_1 = \tilde{\Phi}_1$. The expression for $\Phi_2$ in the theorem is immediate from (2.19). Formula (4.7) follows from Remark 2.8. \hfill $\blacksquare$

In view of Remark 2.6, $H(\ell)$ recovered in (4.7) does not depend on the choice of $r > \ell$, which yields the next corollary.

**Corollary 4.4** Let a canonical system (1.1) be given on $[0, \infty)$, let relations (1.2)–(1.4) hold for each $r > 0$, and let $\det (\beta_2(0)) \neq 0$.

Then, formula (4.7) provides a unique solution of the inverse problem to recover Hamiltonian $H(x)$ on $[0, \infty)$ from a Weyl function $\varphi(\lambda)$ (i.e., from $\varphi(\lambda)$ satisfying (3.19)).

5 Spectral matrix functions: direct and inverse problems

Consider monotonically increasing $p \times p$ matrix functions $\tau(t)$ (i.e., $\tau(t_1) \geq \tau(t_2)$ for $t_1 \geq t_2$), which are defined on $\mathbb{R}$. The space $L_2(d\tau)$ is the space of
vector functions (mapping $\mathbb{R}$ into $\mathbb{C}^p$) with the scalar product

$$(f_1, f_2)_{L_2(d\tau)} = \int_{-\infty}^{\infty} f_2(t)^* (d\tau(t)) f_1(t). \quad (5.1)$$

The space $L_2(r, H)$, where $H$ is the Hamiltonian of the canonical system (1.1) on $[0, r]$ is the space of vector functions mapping $[0, r]$ into $\mathbb{C}^p$ with the scalar product

$$(f_1, f_2)_{L_2(r, H)} = \int_0^r f_2(t)^* H(t) f_1(t) dt. \quad (5.2)$$

The definition of the spectral function below corresponds to a canonical system with the boundary condition $[I_p \ 0] \ w(0, \lambda) = 0$. A simple connection between this case and more general boundary conditions is given in [39, Ch. 4] and in [35, Appendix A].

**Definition 5.1** A monotonically increasing $p \times p$ matrix function $\tau(t) \ (t \in \mathbb{R})$ is called a spectral matrix function (spectral function) of the canonical system (1.1) on $[0, r]$ if the operator $U$:

$$U f = \int_0^r [0 \ I_p] \mathcal{W}(x, \lambda) H(x) f(x) dx \quad (5.3)$$

maps $L_2(r, H)$ isometrically into $L_2(d\tau)$.

**Theorem 5.2** The spectral functions of the canonical system (1.1)–(1.4) on $[0, r]$ (where $\det (\beta_2(0)) \neq 0$) coincide with the set of matrix functions $\tau$ in Herglotz representations of all $\phi \in \mathcal{N}(r)$.

**Proof.** According to theorems from [39, pp. 55 and 57] (that is, to [39, Theorem 2.2, p. 55] and Theorem 4.2 from our previous section), the matrix functions $\tau$ in Herglotz representations of all $\phi \in \mathcal{N}(r)$ are spectral functions of the corresponding canonical system.

Now, assume that $\tau$ is a spectral function of the given canonical system (1.1)–(1.4) on $[0, r]$. Then, in view of [39, Theorem 2.3, p. 56] and [39, Corollary 2.4, p. 59], we have the weak convergence of the integral in (4.2) (that is, $\tilde{S}$ is well defined) and the equality $S = \tilde{S}$. It follows from [35,
Theorem A.7] that a sufficient condition for the inequality (3.6) to hold for spectral functions is the condition that the identity
\[ \Pi^* S^{-1} (A - zI)^{-1} \Phi_2 g \equiv 0 \] (5.4)
yields \( g = 0 \). Taking into account the series expansion of (5.4) for \( z \to \infty \) and the inequality \( \Phi_2^* S^{-1} \Phi_2 > 0 \), one can see that (5.4) yields, indeed, \( g = 0 \). Thus, (3.6) is valid.

We proved above that \( \tilde{S} \) is well defined and (3.6) holds. Using these properties, it is derived on [38, p. 2] that the integral in (4.3) weakly converges, that is, \( \tilde{\Phi}_1 \) is well defined. (Moreover, formula [38, (1.1.10)] and the assumptions c)–e) in Theorem 4.2, which are fulfilled in our case, show that our \( \tilde{S} \) and \( \tilde{\Phi}_1 \) coincide with \( \tilde{S} \) and \( \tilde{\Phi}_1 \) in [38]). Next, by virtue of [38, Lemma 1.1.2] we see that \( S = \tilde{S} \) implies that \( \Phi_1 = \tilde{\Phi}_1 \) for some \( \nu = \nu^* \). Therefore, \( \phi \) of the form (3.5), where \( \mu = 0 \), \( \tau \) is a spectral function and \( \nu \) is determined by (4.3) and the equality \( \Phi_1 = \tilde{\Phi}_1 \), belongs to \( N(S, \Phi_1) \). Finally, equalities (4.5) and (4.6) show that this \( \phi \) belongs \( N(r) \). In other words, any spectral function may be obtained from the Herglotz representation of some \( \phi \in N(r) \). ■

**Remark 5.3** Since \( N(r) = N(S, \Phi_1) \), there is a unique \( \phi(\lambda) \in N(r) \) corresponding to each spectral function \( \tau \). Indeed, \( \nu \) in the Herglotz representation of \( \phi \) is uniquely determined by the equality \( \Phi_1 = \tilde{\Phi} \).

**Corollary 5.4** Relations (4.2)–(4.4) and (4.7) give a procedure to recover canonical system (1.1)–(1.4) on \([0, r]\) (where \( \det (\beta_2(0)) \neq 0 \)) from a spectral function \( \tau \) and corresponding matrix \( \nu = \nu^* \).

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