Einsteinian Strengths and Dynamical Degrees of Freedom for Alternative Gravity Theories

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Abstract

In the paper we present the results of the our calculations of the Einsteinian strengths $S_E(d)$ and numbers dynamical degrees of freedom $N_{DF}(d)$ for alternative gravity theories in $d \geq 4$ dimensions. At the first part we consider the numbers $S_E(d)$ and $N_{DF}(d)$ for metric-compatible and quadratic in curvature (or quadratic in curvature and in torsion) gravity. We show that in the all set of the metric-compatible quadratic gravity in $d \geq 4$ dimensions the 2-nd order Einstein-Gauss-Bonnet theory has the smallest numbers $S_E(d)$ and $N_{DF}(d)$, i.e., this quadratic theory of gravity has the strongest field equations. From the physical point of view this theory is the best one quadratic and metric-compatible theory of gravity in $d \geq 4$ dimensions.

At the second part of the paper we study the numbers $S_E(d)$ and $N_{DF}(d)$ in $d \geq 4$ dimensions for some other alternative gravity theories which are popular recently.

We finish our paper with some conclusions.
I. INTRODUCTION

The notion of “strength of the field equations” is a concept which enables us to compare different systems of the field equations. It was introduced by Einstein [1] in order to analyze systems of partial differential equations for physical fields. Later this notion was examined and effectively used in field theory by several authors [2–7]. In particular B.F. Schutz [3] pointed out that the Einsteinian strength of the field equations is closely connected with the number of the dynamical degrees of freedom which these equations admit in Cauchy problem.

The idea of “strength” is the following (see e.g. [2]). Suppose we have an analytic field function \( \Phi \) of \( d \) variables. We can expand it in Taylor series and the totality of its coefficients describe the field completely. Let us consider the terms of the \( n \)th-order of differentiation in the Taylor expansion; these are of the form

\[
\partial_{i_1} \partial_{i_2} \ldots \partial_{i_n} \Phi, \quad i_j = 1, 2, \ldots, d,
\]

and the total number of such coefficients is

\[
N_n(d) = \binom{d}{n} := \frac{(d + n - 1)!}{(d-1)!n!}.
\]

(2)

If the function \( \Phi \) satisfies some field equations or constraints, these give several relations \( M_n(d) \) between the \( n \)th-order coefficients, thereby reducing the number of coefficients left free to be assigned arbitrary values. Let us denote the number of coefficients left free by \( Z_n(d) \). By definition we have

\[
Z_n(d) = N_n(d) - M_n(d).
\]

(3)

One can prove (see e.g. [2]) that

\[
Z_n(d) = N_n(d) - M_n(d) = \binom{d}{n} (z_0 + \frac{z_1}{n} + \frac{z_2}{n^2} + \ldots).
\]

(4)
z_0$ gives the number of functions of $d$ variables ($x^1, x^2, ..., x^n$) left free. For the absolutely compatible systems $z_0 = 0$. $Z_n(d)$ is always $\geq 0$ for all $n$.

The coefficient of $\frac{1}{n}$, $z_1 =: S_E(d)$ give measure of the Einsteinian strength of the system of field equations under consideration. As one can easily see the Einsteinian strength $S_E(d)$ can also be defined as the coefficient of $1/n$ in the ratio $\frac{Z_0(d)}{n}$. From practical reasons we will use in the following this last definition of $S_E(d)$.

Larger the value of $S_E(d)$, weaker is the system. Of course, in the field theory we need the strongest systems.

The limit for large $n$ of

$$\frac{Z_n(d)}{\binom{d-1}{n}} \asymp \frac{S_E(d)}{(d-1)} =: N_{DF}(d)$$

is the number of free functions of $(d-1)$ variables in the theory [3–7] necessary to determine a solution locally. For hyperbolic systems this is the amount of Cauchy data, i.e., the number $N_{DF}(d)$ determines the amount of dynamical freedom in the system.

The symbol $\asymp$ means equality in the highest powers of $n$. $n \to \infty$.

The paper is organized as follows. In Chapter II we discuss the number $S_E(d)$ and $N_{DF}(d)$ for metric-compatible quadratic theories of gravity. We will divide these theories onto the following two classes:

1. The purely metric quadratic in curvature theories of gravity (PMQG) [8–15], and

2. The metric-compatible quadratic in curvature (or quadratic in curvature and torsion) gravity theories with torsion (PGT) [16–22]).

The PMQG theories give us a geometrization of the improved, symmetric energy-momentum tensor $T^{ik} = T^{ki}$ for matter in the framework of the Riemann geometry. The gravitational equations are here obtained by use Hilbert variational principle and have the following general form

---

$^1$ $z_0 = 0$ for any physically reasonable system (see e.g. [1,2,7])
\[
\frac{\delta(\sqrt{|g|L_g})}{\delta g^{ab}} = \frac{\delta(\sqrt{|g|L_{mat}})}{\delta g^{ab}} = 1/2\sqrt{|g|T_{ab}).
\]

(6)

Here we have 10 field equations of the 4th-order in general. In consequence we have here problems with ghosts and tachyons in weak field approximation, and with Newtonian limit.

The metric-compatible quadratic theories of gravity with torsion called “Poincaré gauge quadratic field theories of gravity” (PGT) give us a geometrization of the canonical pair \( ( T^{ik} \neq T^{ki}, cS^{ikl} = (-) cS^{kl}) \) of matter tensors in the framework of the more general Riemann-Cartan geometry. The gravitational field equations are here obtained by use the Palatini variational principle.

In the Palatini variational principle we take \( (g_{ik}; \Gamma^i_{kl}) \) as independent geometrical variables or, equivalently, an orthonormal tetrads field \( h^{(a)}(x) \) and Lorentz connection \( (\equiv \text{spin connection}) \Gamma_{(a)(b)}^i \), where the indices inside round brackets mean tetrad indices. This variational principle leads us to Einstein-Cartan-Sciama-Kibble (ECSK) theory of gravity for linear gravitational Lagrangian \( L_g = \chi R =: L_E \) and to its generalization – PGT for a quadratic in curvature (and in torsion) gravitational Lagrangian. Here we have 40 field equations of 2nd-order w.r.t. \( h^{(a)}_k \) and \( \Gamma_{(i)(k)}^i \) or, equivalently, 40 equations of 3rd-order w.r.t. metric and contortion \( K^i_{kl} \) where contortion is defined by the decomposition

\[
\Gamma^i_{kl} = \left\{ \begin{array}{c} i \\ kl \end{array} \right\} + K^i_{kl}. \quad (7)
\]

In this case we also have problems with ghosts and tachyons in weak field approximation, and with Newtonian limit.

The gravitational field equations have the form (if we take an orthonormal tetrads field \( h^{(a)}_k(x) \) and Lorentz connection \( \Gamma_{(i)(k)}^i \) as independent geometrical variables)

\[
\frac{\delta(\sqrt{|g|L_g})}{\delta h^{(a)}_k} = \frac{\delta(\sqrt{|g|L_{mat}})}{\delta h^{(a)}_k} = \sqrt{|g|cT_{(a)}^k})
\]

\[
\frac{\delta(\sqrt{|g|L_g})}{\delta \Gamma_{(i)(k)}^i} = \frac{\delta(\sqrt{|g|L_{mat}})}{\delta \Gamma_{(i)(k)}^i} = \sqrt{|g|cS_{(i)(k)}^l}, \quad (8)
\]

plus metricity constraints
\[ Dg_{kl} = 0, \quad (9) \]

where \(D\) means the exterior covariant derivative.

We have

\[
\epsilon T_i^k = h^{(a)}_{i\epsilon} T_{(a)}^k \\
\epsilon S_{ik}^l = h^{(a)}_i h^{(b)}_k S_{(a)(b)}^l. \quad (10)
\]

In Chapter III we analyze \(S_E(d)\) and \(N_{DF}(d)\) for other alternative gravity theories, which are popular recently. Namely, we discuss these numbers for:

1. Teleparallel equivalent of general relativity (TEGR) [23–25],

2. Teleparallel new general relativity (TNGR) of Hayashi and Shirafuji [26–27],

3. General relativity + scalar field (equiv to scalar-tensor theories of gravity), (see e.g. [28–29]),

and

4. The most general metric-affine gauge theory of gravity (MAG) developed by F.W. Hehl et al., (see e.g. [30–33]).

We would like to remark that the gravitational Lagrangians for TEGR and TNGR are quadratic in metric-compatible torsion, and the gravitational Lagrangian \(L_g\) for MAG is quadratic in irreducible parts of curvature, nonmetricity and torsion.

We finish our paper with some Conclusions. Our main conclusions are the following:

1. The 2nd-order Einstein-Gauss-Bonnet theory of gravity is the best one quadratic and metric-compatible gravity theory in \(d \geq 4\),

2. MAG has to week field equations (and too much free parameters – 28) in order to be a reasonable alternative theory of gravity.
II. STRENGTHS AND DYNAMICAL DEGREES OF FREEDOM FOR QUADRATIC GRAVITY THEORIES IN $D \geq 4$ DIMENSIONS

Here we will present the results of calculations of the Einsteinian strengths $S_E(d)$, $d \geq 4$, and numbers of dynamical degrees of freedom $N_{DF}(d)$, $d \geq 4$ for a typical PMQG (4th-order in general) which follows from the Lagrangian

$$L_g = \chi R + c_0 R^2 + c_1 |Ric|^2 + c_2 |Riem|^2,$$

where $\chi$, $c_0$, $c_1$, $c_2$ are some dimensional constants, and for a typical PGT with torsion.

The gravitational Lagrangian $L_g$ for PGT can only be quadratic in curvature like (11), but admitting torsion, or can contain terms quadratic in curvature like (11) plus terms quadratic in irreducible components of torsion. The most general Lagrangian $L_g$ for PGT was given, e.g., by Hayashi and Shirafuji [17].

We have the following

**Proposition 1**

The number $Z_n(d)$ of the free coefficients of order $n$ in Taylor expansion of an analytic solution to the vacuum field equations which follow from (11) is equal

$$Z_n(d) \simeq \left[ \frac{d}{n} \right] \frac{12}{\left( \frac{d}{3} \right)} \simeq 2d(d-2)\left[ \frac{d-1}{n} \right],$$

where

$$\left[ \frac{d}{n} \right] := \left( \frac{n+d-1}{n} \right) = \frac{(n+d-1)!}{n!(d-1)!},$$

and the symbol $\simeq$ means equality in the highest powers of $n$.

**Proof**

Since we are working with purely metric theory of gravity then we have $\frac{d(d+1)}{2} = \left( \frac{d+1}{2} \right)$ unknown metric functions which determine an analytic solution. $\left( \frac{d+1}{2} \right)$ functions of $d$ variables ($d$ coordinates $x^0, x^1, ..., x^{d-1}$) give the total number $\left( \frac{d+1}{2} \right)\left[ \frac{d}{n} \right]$ of the nth-order coefficients. But our theory is generally covariant. This means that the action integral
\[ S = \int_{\Omega} \sqrt{|g|} L_g d\Omega \quad (14) \]

and the vacuum field equations
\[ L_{ab} := \frac{\delta (\sqrt{|g|} L_g)}{\delta g^{ab}} = 0 \quad (15) \]

are invariant under the groupoid \( Diff M_d \) of coordinate transformations.

The invariance of the action integral leads us to \( d \) generalized Bianchi identities of the form
\[ \nabla^a L_{ab} \equiv 0. \quad (16) \]

The invariance of the field equations means that if \( g_{ij}(x) \) are representatives of the components of the metric tensor then so are the other representatives related to \( g_{ij}(x) \) by a coordinate transformation
\[ g_{i'j'}(x') = \frac{\partial x^i}{\partial x'^i} \frac{\partial x^j}{\partial x'^j} g_{ij}(x). \quad (17) \]

The last equation tells us that from the number of the nth-order arbitrary coefficients of \( g_{ij} \) one should subtract \( d \left[ \begin{array}{c} d \\ n+1 \end{array} \right] \), i.e., gauge freedom establishes \( d \left[ \begin{array}{c} d \\ n+1 \end{array} \right] \) coefficients.

Let us now consider the number of relations between the nth-order coefficients due to the vacuum field equations (15). These field equations form the system \( \frac{d(d+1)}{2} = \left( \begin{array}{c} d+1 \\ 2 \end{array} \right) \) equations of the 4th-order (in general). These therefore give \( \left( \begin{array}{c} d+1 \\ 2 \end{array} \right) \left[ \begin{array}{c} d \\ n-4 \end{array} \right] \) relations between the nth-order coefficients of \( g_{ij} \). But not all these relations are independent because the field equations (15) satisfy \( d \) differential identities (16) which are of the 5th-order. So, only
\[ \left( \begin{array}{c} d+1 \\ 2 \end{array} \right) \left[ \begin{array}{c} d \\ n-4 \end{array} \right] - d \left[ \begin{array}{c} d \\ n-5 \end{array} \right] \quad (18) \]

relations between nth-order coefficients of \( g_{ij} \) are independent.

Summing up, we have the following number \( Z_n(d) \) of the free coefficients of order \( n \) in Taylor’s expansion of an analytic solution \( g_{ij}(x) \) of the vacuum equations (15)
\[ Z_n(d) = \left( \begin{array}{c} d+1 \\ 2 \end{array} \right) \left[ \begin{array}{c} d \\ n \end{array} \right] - d \left[ \begin{array}{c} d \\ n+1 \end{array} \right] \]
\[ - \left\{ \left( \begin{array}{c} d+1 \\ 2 \end{array} \right) \left[ \begin{array}{c} d \\ n-4 \end{array} \right] - d \left[ \begin{array}{c} d \\ n-5 \end{array} \right] \right\}. \quad (19) \]
By use the asymptotic formulas

\[
\begin{bmatrix}
  n \to \infty \\
  n-k
\end{bmatrix} = \begin{bmatrix}
  d \\
  n
\end{bmatrix} \left( 1 - \frac{k(d-1)}{n} + O\left(\frac{1}{n^2}\right) \right),
\]

(20)

\[
\begin{bmatrix}
  d \\
  n
\end{bmatrix} = \frac{n}{(d-1)} \begin{bmatrix}
  d-1 \\
  n
\end{bmatrix} \{1 + O\left(\frac{1}{n}\right)\},
\]

(21)

one can easily obtain from (19)

\[
Z_n(d) \asymp \begin{bmatrix}
  d \\
  n
\end{bmatrix} \frac{12d}{n}
\]

\[
\asymp 2d(d-2) \begin{bmatrix}
  d-1 \\
  n
\end{bmatrix}.
\]

(22)

QED

One can read from (22) that the Einsteinian strength \( S_E(d) \) for the quadratic gravity with gravitational Lagrangian (11) reads

\[
S_E(d) = 12 \binom{d}{3},
\]

(23)

while the number dynamical degrees of freedom

\[
N_{DF}(d) = \frac{S_E(d)}{(d-1)} = 2d(d-2).
\]

(24)

If \( d = 4 \) then we have from the two last formulas

\[
S_E(4) = 48, \quad N_{DF}(4) = 16.
\]

(25)

In the special cases the numbers \( S_E(d) \) and \( N_{DF}(d) \) can be smaller. For example, in the case \( 3c_0 + c_1 = 0 \) (≡ Bach-Weyl-Einstein theory) we have

\[
Z_n(d) \asymp \begin{bmatrix}
  d \\
  n
\end{bmatrix} \frac{2(d-1)[(d-1)^2 - 2]}{n}
\]

\[
\asymp \begin{bmatrix}
  d-1 \\
  n
\end{bmatrix} 2[(d-1)^2 - 2],
\]

(26)

i.e., here we have

\[
S_E(d) = 2(d-1)[(d-1)^2 - 2], \quad N_{DF}(d) = 2[(d-1)^2 - 2].
\]

(27)
Iff $d = 4$, then we have for the Bach-Weyl-Einstein theory

$$S_E(4) = 42, \quad N_{DF}(4) = 14. \tag{28}$$

The other special case is given by $c_1 = c_2 = 0$, i.e., by $L_g = \chi R + c_0 R^2$. In this case we have

$$S_E(d) = (d - 1)^2(d - 2), \quad N_{DF}(d) = (d - 1)(d - 2), \tag{29}$$
i.e., we have in this case the same values of the numbers $S_E(d)$ and $N_{DF}(d)$ as in the case of the so-called scalar-tensor theories of gravity with linear gravitational Lagrangian $L_g$ (See Section III.D).

Iff $d = 4$, then we have in the last case

$$S_E(4) = 18, \quad N_{DF}(4) = 6. \tag{30}$$

These results are not surprising because the higher-order gravity theories with $L_g = f(R), \ f'(R) \neq 0$ are dynamically equivalent general relativity (GR) plus a new scalar field $\Psi$ (see e.g [15]).

We would like to emphasize that there exist an interesting example of the PMQG called Einstein–Gauss–Bonnet theory (EGBT) [34–41] which has gravitational Lagrangian $L_g$ of the form

$$L_g = L_E + L_{GB} = L_E$$
$$+ \alpha (R^{iklm}R_{iklm} - 4R^{ik}R_{ik} + R^2), \tag{31}$$
where $\alpha$ is a new coupling constant.

The Lagrangian $L_{GB}$ is called Gauss–Bonnet or Lovelock Lagrangian.

The field equations of the EGBT are of the 2nd-order for $d \geq 4$ (iff $d = 4$, then these field equations are simply Einstein equations) and this quadratic theory of gravity admits no ghosts or tachyons in its linear approximation.

For the EGBT we have
1. \(\binom{d+1}{2}\) unknown metric functions,

2. \(\binom{d+1}{2}\) field equations 2nd-order (but not quasilinear)

3. \(d\) differential identities of the 3rd-order,

4. gauge freedom 
\[g_{ij}'(x') = \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} g_{kl}(x).\]

Repeating reasoning given in the proof of the Proposition 1 we easily get in the case

\[Z_n(d) = \left(\frac{d+1}{2}\right) \left[\frac{d}{n}\right] - d \left[\frac{d}{d+1}\right]
- \left\{ \left(\frac{d+1}{2}\right) \left[\frac{d}{n-2}\right] - d \left[\frac{d}{n-3}\right] \right\}.\]  

(32)

By use the asymptotic formulas (20)-(21) one can easily obtain from the last formula

\[Z_n(d) \simeq \left[\frac{d}{n}\right] \frac{d(d-1)d-3}{n}
\simeq d(d-3) \left[\frac{d-1}{n}\right].\]  

(33)

We see that in this case

\[S_E(d) = d(d-1)(d-3), \quad N_{DF}(d) = d(d-3),\]  

(34)

i.e., we have in the case the same numbers \(S_E(d)\) and \(N_{DF}(d)\) as in GR.

Thus the EGBT has the *smallest* numbers \(S_E(d)\) and \(N_{DF}(d)\) among the PMQG, i.e., this quadratic theory of gravity has the *strongest* field equations among PMQG.

Following Einstein [1] the EGBT is the *best one theory* from the all set of the PMQG because it has the strongest field equations.

On the other hand for a standard PGT\(^2\) we have the following Proposition 2

\(^2\)With or without terms quadratic in torsion in its Lagrangian \(L_g\).
\[ Z_n(d) = \frac{d(d+1)}{2} \left[ \frac{d}{n} \right] + \frac{d(d-1)}{2} d \left[ \frac{d}{n-1} \right] - d \left[ \frac{d}{n+1} \right] - \left\{ d^2 \left[ \frac{d}{n-2} \right] + \frac{d(d-1)}{2} d \left[ \frac{d}{n-3} \right] - \frac{d(d-1)}{2} \left[ \frac{d}{n-4} \right] - d \left[ \frac{d}{n-3} \right] \right\} \times \left[ \frac{d}{n} \right] \frac{(d+1)(d-1)(d-2)}{n} \lesssim d(d+1)(d-2) \left[ \frac{d-1}{n} \right], \]  

(35)

i.e., here we have

\[ S_E(d) = d(d+1)(d-1)(d-2), \quad N_{DF}(d) = d(d+1)(d-2). \]  

(36)

If \( d = 4 \) then we get from the last equation

\[ S_E(4) = 120, \quad N_{DF}(4) = 40. \]  

(37)

The proof of the Proposition 2 is very like to the proof of the Proposition 1. Namely, in the formula (35), likely as it was in the formula (19), the first two terms on the right give the total number of the nth-order coefficients and the other terms before the sign \( \lesssim \) give numbers of independent conditions imposed on these nth-order coefficients: \( d \left[ \frac{d}{n+1} \right] \) follow from gauge freedom and \( \left\{ d^2 \left[ \frac{d}{n-2} \right] + d \left[ \frac{d}{n-3} \right] - \frac{d(d-1)}{2} \left[ \frac{d}{n-4} \right] - d \left[ \frac{d}{n-3} \right] \right\} \) conditions follow from the vacuum field equations and from differential identities which are satisfied by them (see e.g. [6]). By use of the asymptotic formulas (20)-(21) one can easily obtain the above given expressions on \( S_E(d) \) and \( N_{DF}(d) \) for a typical PGT.

Comparing (23), (24) with (36) we see that a typical 3rd-order PGT has much more greater strength and number dynamical degrees of freedom than a typical 4th-order PMQG, i.e., a typical PMQG has much more stronger field equations than a typical PGT.

Note also that the formal limes

\[ \lim_{d \to \infty} \frac{S_E^{PGT}(d)}{S_E^{PMQG}(d)} = \infty, \]  

(38)

i.e., it is infinite. This means that if \( d \) is growing then the vacuum field equations of a PMQG become more and more stronger in comparison with the vacuum field equations of a typical PGT.

We will finish this Section with the following conclusions:
1. A typical PMQG obtained by use Hilbert variational principle has much more stronger field equations than a competitive PGT with torsion obtained by use Palatini variational principle.

2. In $d \geq 4$ dimensions the 2nd-order EGBT is physically distinguished among the all set of the metric-compatible and quadratic in curvature gravity theories (no tachyons and ghosts, the strongest field equations).

### III. STRENGTHS $S_E(D)$ AND NUMBERS DYNAMICAL DEGREES OF FREEDOM $N_{DF}(D)$ FOR OTHER ALTERNATIVE GRAVITY THEORIES IN $D \geq 4$ DIMENSIONS

In order to our paper was not too longer, we give in this Section only the results of (rather simple) calculations without details. We will begin with Einstein-Cartan-Sciama-Kibble (ECSK) theory of gravity.

#### A. ECSK theory of gravity

As it is commonly known that this theory of gravity has the same linear gravitational Lagrangian $L_g$ as GR has, i.e, in this theory $L_g = L_E = \chi R$ [42,43,44]; but we admit in ECSK theory the metric-compatible connection with nonzero torsion.

One can easily calculate that in this case (see e.g. [6])

$$Z_n(d) = \frac{d(d + 1)}{2} \left[ \frac{d}{n} \right] - d \left[ \frac{d}{n + 1} \right] + \frac{d^2(d - 1)}{2} \left[ \frac{d}{n - 1} \right]$$

$$- \left\{ d^2 \left[ \frac{d}{n - 2} \right] + \frac{d^2(d - 1)}{2} \left[ \frac{d}{n - 1} \right] - \frac{d(d - 1)}{2} \left[ \frac{d}{n - 2} \right] - d \left[ \frac{d}{n - 3} \right] \right\}$$

$$\approx d(d - 1)(d - 3) \left[ \frac{d}{n} \right] \approx d(d - 3) \left[ \frac{d - 1}{n} \right].$$

(39)

We see from the above formula that

$$S_E(d) = d(d - 1)(d - 3), \quad N_{DF}(d) = d(d - 3)$$

(40)
in the case.

If \( d = 4 \) then we have \( S_E(4) = 12 \), \( N_{DF}(4) = 4 \).

So, in the framework of the ECSK theory we have the same numbers \( S_E(d) \) and \( N_{DF}(d) \) as in GR. It is not surprising because we have here the same vacuum field equations as in GR.

**B. Teleparallel equivalent of GR (TEGR)**

TEGR is a formal rephrasing, step-by-step, the all formalism of GR in terms of the teleparallel Weitzenböck connection \( \Gamma^i_{kl} \) and its torsion \( T^i_{kl} := \Gamma^i_{lk} - \Gamma^i_{kl} \) (see e.g. [23-25]). The Weitzenböck teleparallel connection and torsion are determined by a distinguished tetrads (or other anholonomic frames) field. The gravitational Lagrangian \( L_g \) of the TEGR is the Einsteinian Lagrangian of GR \( L_g = \chi R \) formally rewritten in terms of Weitzenböck torsion and it is quadratic function of this torsion.

In this case we also have the same values of the numbers \( S_E(d) \) and \( N_{DF}(d) \) as in GR and in ECSK theory, i.e., we have

\[
S_E(d) = d(d-1)(d-3), \quad N_{DF}(d) = \frac{S_E(d)}{d-1} = d(d-3).
\] (41)

If \( d = 4 \) then we obtain from the above expressions \( S_E(4) = 12 \), \( N_{DF}(4) = 4 \).

**C. Teleparallel “new general relativity” (TNGR) of Hayashi and Shirafuji**

It is also the theory of gravitation in the Weitzenböck spacetime [26,27] which is determined by a quadruplet of linearly independent parallel vector fields \( h_{(a)}^i \). Gravitational Lagrangian \( L_g \) is quadratic in irreducible parts of teleparallel torsion (determined by \( h_{(a)}^i \)) with respect to the group of global Lorentz transformations and contains three free parameters. But this Lagrangian is constructed *independently* of the Einstein Lagrangian \( L_E = \chi R \), i.e., TNGR is *not a simple refrasing* of GR, like TEGR.

One can easily calculate that in this case we have
We see that in this case

\[ S_E(d) = 2d(d-1)(d-2), \quad N_{DF}(d) = 2d(d-2). \]  

(43)

For \( d = 4 \) this gives \( S_E(4) = 48, \quad N_{DF}(4) = 16. \)

One can easily see that in this case we have the same values of the numbers \( S_E(d) \) and \( N_{DF}(d) \) as for a typical PMQG in general case. We conclude from the above interesting fact that probably a typical PMQG can be refrased in terms of the Weitzenböck teleparallel connection with \( L_g \) quartic in teleparallel torsion (like reformulation of GR onto TNGR), i.e., we have some kind of the dynamical equivalence between PMQG and TNGR.

D. Scalar-tensor gravity theories \( \equiv \) GR + scalar field

The theories of gravity of such a kind follow e.g., from low energy limit of superstrings [28,29] and contain, as a special case, Jordan-Brans-Dicke theory.

In this case we have (in Einstein frame or in Jordan frame)

\[
Z_n(d) = d^2 \left[ \frac{d}{n} - d \left\lfloor \frac{d}{n+1} \right\rfloor - \left( d^2 \left\lfloor \frac{d}{n-2} \right\rfloor - d \left\lfloor \frac{d}{n-3} \right\rfloor \right) \right] \\
\sim \left\lfloor \frac{2d(d-1)(d-2)}{n} \right\rfloor \sim 2d(d-2) \left[ \frac{d-1}{n} \right].
\]  

(44)

It follows from the last formula

\[ S_E(d) = (d-1)^2(d-2), \quad N_{DF}(d) = (d-1)(d-2). \]  

(45)

If \( d = 4 \) then we have \( S_E(4) = 18, \quad N_{DF}(4) = 6. \)

Note that in these theories of gravity exist two distinguished kind of frames: Jordan frame and Einstein frame which are not equivalent physically in general in presence of matter. [45].
These frames are connected by conformal rescalling of the metric. In our opinion the general physical non-equivalenece of these two distinguished frames is a defect of the scalar-tensor theories of gravity.

E. The most general metric-affine gauge thery of gravity (MAG)

In MAG the spacetime geometry is a metric-affine geometry with the gravitational field strengths nonmetricity $Dg_{ik}$, torsion $T^i$ and curvature $\Omega^i_k$ (see e.g. [30–33]). The independent geometrical variables are: metric $g_{ik}$, tetrad $h^{(a)}_i$ and Lorentz connection $\Gamma^{(a)}_{(b)i}$. The most general parity conserving quadratic Lagrangian $L_g$ is expressed in terms of the irreducible pieces of the nonmetricity $Dg_{ik}$, torsion $T^i$ and curvature $\Omega^i_k$ and contains 28 dimensionless free parameters (apart from $\chi$, cosmological constant $\Lambda$ and strong coupling $\rho$). The gravitational field equations are here of 2nd-order with respect to metric, tetrad and Lorentz connection. The total number of these equations is equal $\frac{d(d+1)}{2} + d^2 + d^3$.

One can obtain (by use Lagrange multipliers technics) the all considered in this paper theories of gravity (including GR and ECSK theory) as special cases of MAG.

It is easy to calculate that in the case of MAG we have

$$Z_n(d) = \binom{d}{2} \left[ \frac{\alpha}{n} + \frac{\beta}{n+1} \right] + \binom{d-2}{n-2} \left[ \frac{\alpha}{n-2} + \frac{\beta}{n-1} \right] - \binom{d-2}{n-3} \left[ \frac{\alpha}{n-3} + \frac{\beta}{n-2} \right] - \binom{d}{n-1} \frac{(2d^2 + 2d - 3)}{n} \sim \binom{d}{n} \frac{(2d^2 + 2d - 3)}{n}.$$  \hspace{1cm} (46)

It follows from the above expression that we have for MAG

$$S_E(d) = d(d - 1)(2d^2 + 2d - 3), \quad N_{DF}(d) = d(2d^2 + 2d - 3).$$  \hspace{1cm} (47)

Iff $d = 4$ then we get $S_E(4) = 444, \quad N_{DF}(4) = 148$.

One can see from these results that the vacuum field equations of MAG are very week.
IV. CONCLUSIONS

We would like to finish our paper with the following conclusions:

1. The PMQG theories obtained by use Hilbert variational principle have much more stronger field equations then the competitive PGT obtained by use Palatini variational principle.

Among the all metric-compatible quadratic gravity in $d \geq 4$ dimensions the 2nd-order EGBT has the strongest field equations, i.e., this theory is the best one quadratic and metric-compatible theory of gravity.

2. If we confine to the metric-compatible theories of gravity with linear (in curvature) gravitational Lagrangian $L_g$ then we will have the two strongest theories of gravity: GR and ECSK theory. It seems to us that the experimental evidence and Ockham razor favorize GR.

3. Probably any PMQG can be reformulated (like Einsteinian GR) into a suitable teleparallel theory of gravity with a gravitational Lagrangian quartic in torsion.

4. MAG has too week field equations and too much free parameters (28) in order to be a useful model of a quadratic theory of gravity.

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3Of course the same strength and number dynamical freedom have any gravity theory with $L_g = \chi R +$ term quadratic in torsion whose vacuum equations reduce to the vacuum Einstein equations.
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