Recognition of finite exceptional groups of Lie type

Martin W. Liebeck
Department of Mathematics
Imperial College
London SW7 2BZ
UK

E.A. O’Brien
University of Auckland
Auckland
New Zealand

Abstract

Let $q$ be a prime power and let $G$ be an absolutely irreducible subgroup of $GL_d(F)$, where $F$ is a finite field of the same characteristic as $F_q$, the field of $q$ elements. Assume that $G \cong G(q)$, a quasisimple group of exceptional Lie type over $F_q$ which is neither a Suzuki nor a Ree group. We present a Las Vegas algorithm that constructs an isomorphism from $G$ to the standard copy of $G(q)$. If $G \not\cong 3D_4(q)$ with $q$ even, then the algorithm runs in polynomial time, subject to the existence of a discrete log oracle.

1 Introduction

Informally, a constructive recognition algorithm constructs an explicit isomorphism between a quasisimple group $G$ and a ‘standard’ copy of $G$, and exploits this isomorphism to write an arbitrary element of $G$ as a word in its defining generators. For a more formal definition, see [53, p. 192]. Such algorithms play a critical role in the ‘matrix group recognition project’ which aims to develop efficient algorithms for the investigation of subgroups of $GL_d(F)$ where $F$ is a finite field. We refer to the recent survey [50] for background related to this work. Such algorithms are available for classical groups; see, for example, [29, 30, 42]. Here we present constructive recognition algorithms for the finite exceptional groups of Lie type.

Let $G(q)$ denote a quasisimple exceptional group of Lie type over $F_q$, a finite field of size $q$. Howlett, Rylands & Taylor [35] provide defining matrices for a specific faithful irreducible representation of minimal dimension of the simply connected group of type $G(q)$: we call this representation the standard copy of type $G(q)$.

Our principal result is the following. In the statement, $V_d(F)$ denotes the underlying vector space of dimension $d$ over the field $F$.

Theorem 1 Let $q$ be a prime power and let $G$ be an absolutely irreducible subgroup of $GL_d(F)$, where $F$ is a finite field of the same characteristic as $F_q$. Assume that $G \cong G(q)$, a quasisimple group of exceptional Lie type over $F_q$ for $q > 2$, excluding Suzuki and Ree groups, and also $3D_4(q)$ with $q$ even. There is a Las Vegas algorithm that constructs an isomorphism from $G$ to the standard copy of type $G(q)$ modulo

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a central subgroup, and also constructs the inverse isomorphism; it also computes the high weight of \( V_d(F) \) as a \( G \)-module, up to a possible twist by a field or graph automorphism. The algorithm runs in polynomial time, subject to the existence of a discrete log oracle for extensions of \( \mathbb{F}_q \) of degree at most 3.

The possible central subgroups of the standard copy of type \( G(q) \) are trivial except when \( G(q) \) is of type \( E_6(q) \) (\( \epsilon = \pm 1 \)) or \( E_7(q) \), in which case they have order dividing \((3, q - \epsilon)\) and \((2, q - 1)\) respectively.

We now discuss how the isomorphisms in the statement of the theorem are realised. Let \( \hat{G}(q) \) denote the standard copy of type \( G(q) \). It has a Curtis-Steinberg-Tits presentation, which involves only those relations which arise from certain rank 2 subgroups of \( G(q) \): namely, the commutator relations among root elements corresponding to pairs of fundamental roots in the corresponding Dynkin diagram. Babai et al. [7, §4.2 and 6.1] reduce this presentation by running over root elements parametrised by an \( \mathbb{F}_p \)-basis for \( \mathbb{F}_q \) (where \( p \) is the characteristic of \( \mathbb{F}_q \)). Those root elements of \( \hat{G}(q) \) which satisfy this reduced Curtis-Steinberg-Tits presentation are the standard generators \( \hat{S} \) for \( \hat{G}(q) \).

Given a group \( G \) as in the statement of the theorem, described by a generating set \( X \), our algorithm produces a collection \( \hat{S} \) of generators of \( G \) (as words in \( X \)) which satisfy the reduced Curtis-Steinberg-Tits presentation. These are then used to construct the required isomorphisms \( \phi : G \to \hat{G}(q)/Z \) and \( \psi : \hat{G}(q)/Z \to G \) (where \( Z \) is a central subgroup), as follows. Cohen, Murray & Taylor [22] developed the generalised row and column reduction algorithm: in polynomial time, for a given high weight representation of \( G \cong G(q) \) with \( G(q) \) of untwisted type, this algorithm writes an arbitrary \( g \in G \) as a word \( w(S) \) in the standard generators; this has now been extended to twisted types in [23]. Now \( \phi(g) = w(S)Z \), the corresponding word in the standard generators of \( \hat{G}(q) \), defines the isomorphism \( \phi \). The inverse isomorphism \( \psi \) is defined similarly: for \( \hat{g} \in \hat{G}(q) \), the algorithms of [22, 23] express \( \hat{g} \) as a word \( w(\hat{S}) \), and we set \( \psi(\hat{g}Z) = w(S) \).

Together, the algorithms of Theorem 1 and of [22, 23] provide a solution to the constructive membership problem for \( G = \langle X \rangle \): namely, express an arbitrary \( g \in G \) as a word in \( X \).

Our algorithms to find standard generators in \( G \) begin by constructing \( SL_2 \) subgroups of \( G \) which can be placed as nodes in the Dynkin diagram so that they pairwise generate the appropriate group of rank 2, and these are then used to label root elements and toral elements of \( G \) relative to a fixed root system. We use the root elements to compute the high weight of the given representation of \( G \) on \( V_d(F) \), and then exploit the algorithms of [22, 23] to set up the isomorphisms explicitly.

To construct the \( SL_2 \) subgroups and label root elements, we use involution centralizers in \( G \). That such centralizers can be constructed in Monte Carlo polynomial time follows in odd characteristic from [52], and in even characteristic from [43]; see Section 2.3 for further discussion.

A distinguishing feature of our work is that the resulting algorithms are practical; this desire significantly influenced our design. Our algorithms are implemented and will be publicly available in MAGMA [12].

The excluded Suzuki and Ree groups of types \( ^2B_2(q), ^2G_2(q), \) and \( ^2F_4(q) \) were studied by Bäärnhielm [2–4]. His constructive recognition algorithms apply to conjugates of the standard copy of \( ^2B_2(q) \) and \( ^2G_2(q) \), and run in polynomial time subject
to the availability of a discrete log oracle. For the groups $^3D_4(q)$ ($q$ even), also excluded in the theorem, we provide a practical algorithm with running time $O(q)$. We also present practical algorithms for groups defined over $\mathbb{F}_2$, the only field not covered by Theorem 1. Where feasible, our theoretical results also include $\mathbb{F}_2$.

As stated, Theorem 1 applies to absolutely irreducible representations of quasisimple groups of exceptional Lie type. The principal motivation for stating it under this assumption is our application of the algorithms of [22, 23] to realise the isomorphisms between $G$ and $G(q)/Z$. Using the Meataxe and associated machinery [34, Chapter 7], the result can easily be reformulated to apply, with unchanged complexity, to all matrix representations (not necessarily irreducible) in defining characteristic. For all but $E_8(q)$ in even characteristic, our algorithms to construct the $SL_2$ subgroups and to label the root and toral elements are black-box provided that the algorithms employed in Theorem 2.2 for constructive recognition of small rank classical groups are black-box. Since algorithms are available for these tasks (see, for example, [30] and its references), a version of Theorem 1 could be formulated for black-box groups. We refrain from doing so.

Kantor & Magaard [39] presented black-box Las Vegas algorithms to recognise constructively the exceptional simple groups of Lie type and rank at least 2, other than $^2F_4(q)$, defined over a field of known size. These have complexity depending linearly on the size of the field. Dick [28] developed a polynomial-time algorithm, a modification of that proposed by [39], for $F_4(q)$ in odd characteristic. Relying as it does on centralizers of involutions, our work differs substantially from [39].

The structure of the paper is as follows. Section 2 records a number of results which underpin our algorithm. In Section 3 we prove results on probabilistic generation for certain groups of Lie type. In Sections 4-10 we present algorithms to construct $SL_2$ subgroups of a group $G$ as in Theorem 1 which correspond to the nodes in the associated Dynkin diagram. Sections 11-13 contain algorithms to label root elements and toral elements of $G$ relative to a fixed root system; to determine the high weight of the given representation of $G$ on $V_d(F)$; and to construct the standard generators for $G$. In Section 15 we present algorithms for the special case of groups defined over $\mathbb{F}_2$. In Section 16 we report on our implementation in MAGMA. Finally, for each group of exceptional Lie type and Lie rank at least 2, its reduced Curtis-Steinberg-Tits presentation on standard generators is listed explicitly in Appendix A.

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2 Background and preliminaries

A Monte Carlo algorithm is a randomised algorithm which always terminates but may return a wrong answer with probability less than any specified value. A Las Vegas algorithm is a randomised algorithm which never returns an incorrect answer,
but may report failure with probability less than any specified value.

Our algorithms usually search for elements of $G$ having a specified property. If $1/k$ is a lower bound for the proportion of such elements in $G$, then we can readily prescribe the probability of failure of the corresponding algorithm. Namely, to find such an element by random search with a probability of failure less than a given $\epsilon \in (0, 1)$ it suffices to choose (with replacement) a sample of uniformly distributed random elements in $G$ of size at least $\lceil -\log_e(\epsilon)k \rceil$.

Babai & Szemerédi [5] introduced the black-box group model, where group elements are represented by bit-strings of uniform length. The only group operations permissible are multiplication, inversion, and checking for equality with the identity element. Seress [53, p. 17] defined a black-box algorithm as one which does not use specific features of the group representation, nor particulars of how group operations are performed; it can only use the operations listed above.

Babai [6] present a Monte Carlo algorithm to construct in polynomial time independent nearly uniformly distributed random elements of a finite group. An alternative is the product replacement algorithm of Celler et al. [21], which runs in polynomial time by a result of [51]. For a discussion of both algorithms we refer to [53, pp. 26–30].

Often it is necessary to investigate the order of $g \in \text{GL}_d(F_q)$, which, due to integer factorisation, cannot be determined in polynomial time. We can, however, determine its pseudo-order, a good multiplicative upper bound for $|g|$, and the exact power of any specified prime that divides $|g|$, using a Las Vegas algorithm with complexity $O(d^3 \log d + d^2 \log d \log q)$. Our results sometimes assume the existence of an order oracle but, in our applications, it always suffices to use pseudo-order. A Las Vegas algorithm with the same complexity allows us to compute large powers $g^n$ where $0 \leq n < q^d$. We refer to [42, §2 and 10] for more details and references.

Leedham-Green & O’Brien [41] present Monte Carlo algorithms to generate random elements of the normal closure of a subgroup, and to determine membership in a normal subgroup of a black-box group having an order oracle. Babai & Shalev [9] prove that if the normal subgroup is simple and non-abelian, then the membership algorithm runs in Monte Carlo polynomial time. A consequence is a Monte Carlo black-box algorithm to prove that a group is perfect. This algorithm is used together with the black-box polynomial-time algorithm described in [53, pp. 38–40] to construct the derived series of a group.

To construct a direct factor of a semisimple group, we use the black-box algorithm, KILLFACTOR, of [8, Claim 5.3]; that it runs in polynomial time is a consequence of [11, Corollary 4.2].

If a matrix group acts absolutely irreducibly on its underlying vector space of dimension $d$, then we can determine the classical forms it preserves in $O(d^5)$ field operations (see [34, §7.5.4]). A hyperbolic basis for a vector space of dimension $d$ with a given non-degenerate bilinear form can be constructed in $O(d^3)$ field operations (see [16] for an algorithm to perform this task).

2.1 Recognition for classical groups

Babai et al. [10] proved the following.

**Theorem 2.1** Given a black-box group $G$ isomorphic to a simple group of Lie type
of known characteristic, the standard name of $G$ can be computed using a Monte Carlo polynomial-time algorithm.

Liebeck & O’Brien [43] present a Monte Carlo black-box polynomial-time algorithm to identify the defining characteristic. Kantor & Seress [38] give an alternative algorithm for absolutely irreducible matrix groups.

Kantor & Seress [36] developed the first black-box Las Vegas algorithms to recognise constructively classical groups; these have complexity depending linearly on the size of the field. More recently, Leedham-Green & O’Brien [42] developed algorithms for classical groups in natural representation and odd characteristic; those of Dietrich et al. [29] apply to even characteristic. Black-box equivalents appear in [30]. All run in time polynomial in the size of the input subject to the availability of a discrete log oracle.

Our algorithms for the labelling of root and toral elements rely on the availability of constructive recognition algorithms for the classical groups listed in the following theorem. As defined in [42], the standard copy of a classical group is its natural matrix representation, preserving a specified form.

**Theorem 2.2** Let $q$ be a prime power, and let $G$ be a subgroup of $GL_d(F)$, where $F$ is a field of the same characteristic as $\mathbb{F}_q$. Assume that $G$ is isomorphic to one of the following classical groups: $SL_2(q)$, $SL_3(q)$, $Sp_4(q)$, or $SU_4(q)$ in all characteristics; $SL_6(q)$, $Sp_6(q)$, $SU_6(q)$, $\Omega^+_8(q)$ or $\Omega^-_8(q)$ for even $q$. There is a Las Vegas algorithm that constructs an isomorphism from $G$ to its standard copy. Subject to the existence of a discrete log oracle, the algorithm runs in polynomial time.

This follows from [17], [18], [19], [26], [30], and [49].

### 2.2 Groups of Lie type

We use $SL^n_\epsilon(q)$ to denote $SL_n(q)$ for $\epsilon = 1$ and $SU_n(q)$ for $\epsilon = -1$; we adopt similar conventions for $D_4(q)$ and $2D_4(q)$; and for $E_6(q)$ and $2E_6(q)$. Dynkin diagrams for exceptional root systems are labelled as follows:

$$
\begin{align*}
E_l & \quad 1 & - & 3 & - & 4 & - & 5 & - & \cdots & - & l \\
& & & & & & & & & & & 2 \\
F_4 & \quad 1 & - & 2 & = & > & = & 3 & - & 4 \\
G_2 & \quad 1 & \equiv & > & = & 3 & - & 4 & \equiv & 2
\end{align*}
$$

where each node $i$ represents a simple root $\alpha_i$. This is the labelling of Bourbaki [13, p. 250], except for $G_2$, where $\alpha_1$ and $\alpha_2$ are interchanged.

Let $G = G(q)$ be an exceptional group of Lie type over $\mathbb{F}_q$; we exclude Suzuki and Ree groups. The root system of $G(q)$ is described in [20, Chapter 3]; if $G(q)$ is of twisted type, then we use the twisted root system of [20, Chapter 13]. For a long root $\alpha$ in the root system, we denote by $U_{\pm \alpha}$ the corresponding long root group, and a conjugate of $\langle U_{\pm \alpha} \rangle \cong SL_2(q)$ is a long $SL_2$ subgroup of $G(q)$. For a fixed
isomorphism between $\langle U_{\pm \alpha} \rangle$ and $SL_2(q)$, we denote by $h_\alpha(c)$ the element of $\langle U_{\pm \alpha} \rangle$ corresponding to the matrix $\text{diag}(c^{-1}, c)$; if $\alpha$ is a fundamental root $\alpha_i$, then we may write $h_i(c)$. If $\alpha_1, \ldots, \alpha_l$ are fundamental roots and $c_1, \ldots, c_l$ are integers, then $h_{c_1, \ldots, c_l}(\lambda) := h_1(\lambda^{c_1}) \cdots h_l(\lambda^{c_l})$.

An involution in a long $SL_2$ subgroup of $G(q)$ is a root involution. These involutions and their centralizers play a major role in our work. Proposition 2.3 lists the root involution centralizers; it appears in [33, 4.5] (for $q$ odd) and [1] (for $q$ even).

A subsystem subgroup is one generated by root groups corresponding to roots in a closed subsystem of the root system of $G(q)$.

**Proposition 2.3** Let $G = G(q)$ be an exceptional group of Lie type over $\mathbb{F}_q$, and let $t$ be a root involution. Let $D$ be a subsystem subgroup of $G$ as in the following table:

| $G$ | $G_2(q)$ | $^3D_4(q)$ | $F_4(q)$ | $E_6(q)$ | $E_7(q)$ | $E_8(q)$ |
|-----|----------|------------|----------|----------|----------|----------|
| $D$ | $A_1(q)$ | $A_1(q^q)$ | $C_3(q)$ | $A_5(q)$ | $D_6(q)$ | $E_7(q)$ |

(i) If $q$ is odd, then $C_G(t)$ has a subgroup $SL_2(q)D$ of index at most 2; the factors $SL_2(q)$ and $D$ commute (elementwise).

(ii) If $q$ is even, then $C_G(t) = QD$, where $Q$ is a normal 2-subgroup of $C_G(t)$.

If the root system of $G(q)$ has roots of different lengths, then a short $SL_2$ subgroup is one generated by a pair of opposite short root subgroups of $G(q)$; if $G(q)$ is of untwisted type, then these are isomorphic to $SL_2(q)$, otherwise they are isomorphic to $SL_2(q^2)$, or $SL_2(q^3)$ for $^3D_4(q)$.

Let $l$ be the rank of the root system of $G(q)$ and let $1, \ldots, l$ be the nodes of the Dynkin diagram. Let $K_1, \ldots, K_l$ be long (short) $SL_2$ subgroups of $G(q)$ which satisfy the following:

1. $K_i$ is long (short) if and only if node $i$ is a long (short) root;
2. if nodes $i, j$ are not joined then $K_i$ and $K_j$ commute;
3. if nodes $i, j$ are joined then $\langle K_i, K_j \rangle$ is the appropriate rank 2 group of Lie type: $A_2(q)$ or $A_2(q^2)$ if $i, j$ are joined by a single bond; $B_2(q)$ or $^2A_3(q)$ if joined by a double bond; $G_2(q)$ or $^3D_4(q)$ if joined by a triple bond.

We call such $K_1, \ldots, K_l$ basic $SL_2$ subgroups of $G(q)$.

### 2.3 Centralizers of involutions

The centralizer of an involution in a black-box group having an order oracle can be constructed using an algorithm of Bray [14]; he proved the following.

**Theorem 2.4** If $x$ is an involution in a group $H$, and $w$ is an arbitrary element of $H$, then $[x, w]$ either has odd order $2k + 1$, in which case $w[x, w]^k$ commutes with $x$, or has even order $2k$, in which case both $[x, w]^k$ and $[x, w^{-1}]^k$ commute with $x$. If $w$ is uniformly distributed among the elements of the group for which $[x, w]$ has odd order, then $w[x, w]^k$ is uniformly distributed among the elements of the centralizer of $x$. 

6
Thus if the odd order case occurs sufficiently often, then we can construct random elements of the involution centralizer in Monte Carlo polynomial time.

Parker & Wilson [52, Theorems 1-4] proved the following two results.

**Theorem 2.5** There is an absolute constant $c > 0$ such that if $H$ is a finite simple group of Lie type of Lie rank $r$ defined over a field of odd characteristic, and $x$ is an involution in $H$, then the proportion of $h \in H$ such that $[x, h]$ has odd order is at least $c/r$.

**Theorem 2.6** There is an absolute constant $c > 0$ such that if $H$ is a finite simple group of Lie type of Lie rank $r$ defined over a field of odd characteristic, and $C$ is a conjugacy class of involutions in $H$, then the proportion of elements of $H$ which power up to an element of $C$ is at least $c/r^3$.

By Theorem 2.6, an involution in a specified class of $H$ can be constructed in polynomial time by searching for an element of even order and computing a suitable power. By Theorem 2.5, random elements of the centralizer of this involution can be constructed, and a bounded number of these generate the centralizer (see [46]).

We shall make frequent use of the following lemma, also proved by Parker & Wilson [52, Lemma 26].

**Lemma 2.7** Let $H$ be a finite group of Lie type, $T$ a maximal torus in $H$ and let $C$ be a conjugacy class of $H$. Assume that at least a proportion $k$ of the regular semisimple elements of $T$ power to a member of $C$. Then at least a proportion $k/|N_H(T)/T|$ of the elements of $H$ power to an element of $C$.

Liebeck & O’Brien [43] proved the following.

**Theorem 2.8** If $H$ is a finite group of Lie type over a field of characteristic 2, and $t$ is a root involution of $H$ such that $C_H(t)$ is not soluble, then the proportion of $h \in H$ such that $[t, h]$ has odd order is at least $1/4$.

Hence random elements of the centralizer of a root involution can be constructed in polynomial time, and a bounded number of these will generate the centralizer (see [43, Lemma 3.10]).

### 2.4 The Formula

Variations of the following lemma, sometimes known as the “Formula”, have been used in algorithms for some years – see, for example, [48, Section 4.10]. A proof can be found in [29, 7.1].

**Lemma 2.9** Let $K = H \ltimes M$ where $M$ has exponent 2. Suppose $h \in H$ has odd order and acts fixed point freely on $M$. If $k = am \in K$ where $a \in C_H(h)$ and $m \in M$, then $a = hk(hh^k)^{(h|-1)/2}$.

The lemma sometimes allows us to construct a complement to a normal 2-subgroup in a semidirect product. We apply it when $H = \langle h \rangle \times D$ for quasisimple $D$. Now the lemma enables us to construct random elements of $H$ (namely, $hk(hh^k)^{(h|-1)/2}$ for random $k \in K$). Since, by [46], $D$ may be generated by two random elements, we can thus construct a generating set for $D$. 


3 Probabilistic generation of certain groups

Our algorithms rely on various results on probabilistic generation; these we now present.

Proposition 3.1 Let \( G = D_4^+(q) \) with \( \epsilon = \pm \) and \( q > 2 \) even, and let \( x \) be an element of order \( q + 1 \) in a long \( SL_2 \) subgroup of \( G \). For random \( g \in G \), the probability that \( \langle x, x^g \rangle = G \) is positive, and is at least \( 1 - c/q \), where \( c \) is an absolute constant.

Proof. Let \( P \) be the probability that \( \langle x, x^g \rangle = G \) for random \( g \in G \). If \( \langle x, x^g \rangle \neq G \), then \( x, x^g \in M \) for some maximal subgroup \( M \) of \( G \). Given a maximal subgroup \( M \) containing \( x \), the probability that \( x^g \) lies in \( M \) is \( |x^G \cap M|/|x^G| \). It is well-known (see [44, 2.5]) that

\[
\frac{|x^G \cap M|}{|x^G|} = \frac{\text{fix}_{G/M}(x)}{|G : M|},
\]

where \( \text{fix}_{G/M}(x) \) denotes the number of fixed points of \( x \) in the action of \( G \) on the cosets of \( M \). Also, the number of conjugates of \( M \) containing \( x \) is \( \text{fix}_{G/M}(x) \). Hence, if \( M \) is a set of representatives of the conjugacy classes of maximal subgroups of \( G \), then

\[
1 - P \leq \sum_{M \in \mathcal{M}} \frac{\text{fix}_{G/M}(x)^2}{|G : M|}.
\]

The maximal subgroups of \( G \) are determined in [40] (for \( \epsilon = + \)) and are listed in [15, Tables 8.52–8.53] (for \( \epsilon = - \)). In Tables 1 and 2, we list those maximal subgroups \( M \) which contain a conjugate of \( x \), together with the values of \( \text{fix}_{G/M}(x) \) and \( |G : M| \).

The notation is standard: \( P_i \) denotes a parabolic subgroup, the stabilizer of a totally singular \( i \)-space; and \( N_i^\pm (\delta = \pm) \) is the stabilizer of a nonsingular subspace of dimension \( i \) and type \( \delta \). Note that we omit \( N_2^+ \) from the tables: if \( x, x^g \) are contained in \( N_2^+ \) then they lie in a subgroup of \( N_2^+ \) which is contained in \( P_1 \). In Table 1 we comment if a row covers 3 classes of maximal subgroups; in each case these are permuted by a triality automorphism of \( G \).

| \( M \) | \( \text{fix}_{G/M}(x) \) | \( |G : M| \) | Comment |
|----|----------------|-------------|--------|
| \( P_1 \) | \((q+1)^2\) | \(\frac{q^3(q-1)(q+1)}{q-1}\) | 3 classes of subgroups \( M \) |
| \( P_2 \) | \(3(q+1)\) | \(\frac{q^3(q-1)(q^2+1)^2}{q^2}\) | |
| \( N_1 \) | \(q(q^2-1)\) | \(q^3(q^2-1)\) | 3 classes of subgroups \( M \) |
| \( N_2^- \) | \(\frac{1}{2}q(q-1)(q^2-2q+2)\) | \(\frac{q^3(q-1)(q^2-1)}{2(q+1)}\) | 3 classes of subgroups \( M \) |
| \( N_4^+ \) | \(\frac{1}{2}q^3(q-1)^3\) | \(\frac{q^3(q^2-1)(q^2+1)^2}{4q^2} \) | |
| \( N_4^- \) | \(\frac{1}{2}q^3(q+1)(q^2-1)\) | \(\frac{1}{3}q^8(q^6-1)(q^2-1)\) | 3 classes of subgroups \( M \) |

Table 1: Fixed points of \( x \) for \( G = \Omega_8^+(q) \)

The values of \( \text{fix}_{G/M}(x) \) given in the tables are calculated reasonably routinely; we give a sketch. Let \( G \) have natural module \( V_8 \). Regard \( x \) as acting on an orthogonal decomposition \( V_8 = V_4 + V_4' \), where \( V_4, V_4' \) are non-degenerate subspaces of type \( O_4^+ \) and \( O_4^- \) respectively; \( x \) acts trivially on \( V_4 \), and as an element in one of the \( SL_2(q) \) factors of \( \Omega_4^+(q) = SL_2(q) \otimes SL_2(q) \) on \( V_4' \). For \( M = P_1 \) (or \( N_1 \)), \( \text{fix}_{G/M}(x) \) is the
Let Proposition 3.2 but is zero for \( \epsilon \in V \) number of singular (or nonsingular) 1-spaces in \( V \); for \( M = P_2 \), \( \text{fix}_{G/M}(x) \) is the sum of the numbers of singular 2-spaces fixed by \( x \) in \( V_4 \) and \( V'_4 \); and for \( M = P_3 \), the singular 3-spaces in \( V_8 \) fixed by \( x \) are spanned by one of the \( q + 1 \) fixed 2-spaces in \( V'_4 \) together with a fixed 1-space in \( V_4 \). For \( M = N_2^- \), the 2-spaces of type \( O^-_3 \) fixed by \( x \) either lie in \( V_4 \) or in \( V'_4 \), and there are \( q(q - 1) \) of the latter, as can be seen using (1). Likewise, \( N_4^\pm \)-spaces in \( V_8 \) fixed by \( x \) are sums of fixed nonsingular 2-spaces in \( V_4 \) and \( V'_4 \), and it is straightforward to count these. Finally, the cases where \( \epsilon = - \) and \( M = \Omega^-_4(q^2) \) or \( U_3(q) \) are handled using (1). In the first case, \( x^G \cap M \) is a class of elements of order \( q + 1 \) in \( \Omega^-_4(q^2) \cong L_2(q^4) \), so has size \( q^4(q^4 + 1) \). The second case arises from the adjoint representation of \( U_3(q) \), in which \( x \) acts as \( \text{diag}(\alpha, \alpha, \alpha^{-2}) \) for some scalar \( \alpha \) of order \( q + 1 \) or its inverse. Hence \( |x^G \cap M| = 2|SU_3(q) : GU_2(q)| = 2q^2(q^2 - q + 1) \), from which \( \text{fix}_{G/M}(x) \) follows using (1).

The lower bound \( 1 - c/q \) in the statement of the proposition follows from the information in the tables together with (2). These also imply that \( 1 - P \) is less than 1 for \( q \geq 8 \), giving the positivity statement for these values of \( q \). For \( q = 4 \) we can verify computationally that \( G \) is generated by two conjugates of \( x \).

For \( q = 2 \) the probability in the previous proposition remains positive for \( \epsilon = - \), but is zero for \( \epsilon = + \).

**Proposition 3.2** Let \( G \) be one of \( F_4(q), E_6(q), E_7(q) \) or \( E_8(q) \) with \( q \) even, and let \( x \) be an element of order \( q + 1 \) in a long \( SL_2 \) subgroup of \( G \). For random \( g \in G \), the probability that \( \langle x, x^g \rangle \) is a subsystem subgroup \( D_4^\epsilon(q) \) (for some \( \epsilon \in \{+, -\} \)) is positive, and is at least \( 1/6 - c/q \), where \( c \) is an absolute constant.

**Proof.** That the probability is positive follows immediately from Proposition 3.1 (and the ensuing remark for the case \( q = 2 \)). Let \( D \cong D_4^\epsilon(q) \) be a fixed subsystem subgroup of \( G \) which contains a long \( SL_2 \) subgroup containing \( x \), and define

\[
\Delta = \{ D^g : g \in C_G(x) \}.
\]

Observe that \( |\Delta| = |C_G(x) : C_G(x) \cap N_G(D)| \) since \( C_G(x) \) acts transitively on \( \Delta \).
We consider first the case where $\epsilon = +$. Here $N_G(D) = D.S_3, DT_2.S_3, DA_1(q^3).S_3$ or $DD_4(q).S_3$ according as $G$ is of type $F_4, E_6^2, E_7$ or $E_8$, where $T_2$ denotes a rank 2 torus (see [45, Table 5.1]). The table below gives $C_G(x)$, $C_G(x) \cap N_G(D)$ and $|\Delta|$.

| $G$        | $C_G(x)$                  | $C_G(x) \cap N_G(D)$ | $|\Delta| \sim$ |
|------------|---------------------------|----------------------|-----------------|
| $F_4(q)$   | $\langle x \rangle$ $C_3(q)$ | $\langle \langle x \rangle A_1(q)^3 \rangle.S_3$ | $q^{12}/6$      |
| $E_6^\pm(q)$ | $\langle x \rangle A_5^\pm(q)$ | $\langle \langle x \rangle A_1(q)^3 T_2 \rangle.S_3$ | $q^{24}/6$      |
| $E_7(q)$   | $\langle x \rangle D_6(q)$ | $\langle \langle x \rangle A_1(q)^6 \rangle.S_3$ | $q^{48}/6$      |
| $E_8(q)$   | $\langle x \rangle E_7(q)$ | $\langle \langle x \rangle D_4(q)A_1(q)^3 \rangle.S_3$ | $q^{96}/6$      |

The number of pairs $(x^g, E)$ with $x^g \in E \in \Delta$ is of the order of $|\Delta| \cdot |D : (q + 1)A_1(q^3)| \sim |\Delta|q^{18}$. Given $E$, the proportion of conjugates $x^g \in E$ such that $\langle x, x^g \rangle = E$ is at least $1 - c/q$ by Proposition 3.1. Clearly $E$ is the unique member of $\Delta$ containing such $x^g$. Hence the number of conjugates $x^g$ such that $\langle x, x^g \rangle$ is a member of $\Delta$ is at least $(1 - c/q)|\Delta|q^{18}$. The number of conjugates of $x$ in $G$ is $|G : C_G(x)|$, where $C_G(x)$ is as in the above table. Hence the probability that $\langle x, x^g \rangle$ is a member of $\Delta$ is at least

$$\frac{(1 - \frac{c}{q})|\Delta|q^{18}}{|G : C_G(x)|} \geq \frac{1}{6} - \frac{c}{q}.$$ 

This completes the proof for $\epsilon = +$. The proof for $\epsilon = -$ is similar. \hfill \Box

**Proposition 3.3** Let $G = G_2(q)$ with $q > 2$ even; let $A_1$ and $\tilde{A}_1$ denote commuting $SL_2(q)$ subgroups of $G$ generated by long and short root groups respectively; let $x, y$ be elements of order 3 in $\tilde{A}_1, A_1$ respectively; and let $t$ be an involution in $A_1$.

(i) $C_G(x) \cong SL_3(q)$ and $C_G(y) = (q - \epsilon) \times \tilde{A}_1$, where $q \equiv \epsilon \mod 3$.

(ii) For random $g \in G$, the probability that $\langle x, x^g \rangle$ is a conjugate of $\tilde{A}_1$ is positive, and is at least $1 - c_1/q$ where $c_1$ is an absolute constant.

(iii) For random $g \in G$, the probability that $\langle y, y^g \rangle$ is a conjugate of $A_1$ is $\frac{1}{q^3(q^3 + q^{2+1})}$.

(iv) For random $g \in G$, the probability that $\langle A_1, t^g \rangle \cong SL_3(q)$ is positive, and is at least $1/2 - c_2/q$ where $c_2$ is an absolute constant.

**Proof.** (i) If $L$ denotes the Lie algebra of type $G_2$, then

$$L \downarrow A_1 \tilde{A}_1 = L(A_1 \tilde{A}_1) \oplus (V(1) \otimes V(3)),$$

where $V(i)$ denotes the irreducible module of high weight $i$ (see, for example, [47, 11.12(ii)]). It follows that $C_L(x), C_L(y)$ have dimensions 8 and 6 respectively, so the centralizers of $x$ and $y$ in the algebraic group $G_2$ are connected reductive subgroups of types $A_2$ and $T_1A_1$.

(ii) This is similar to the proof of Proposition 3.2. Define $\Delta = \{\tilde{A}_1^g : g \in C_G(x)\}$. Then

$$|\Delta| = |C_G(x) : C_G(x) \cap N_G(\tilde{A}_1)| = |SL_3(q) : A_1 \cdot (q - \epsilon)| \sim q^4.$$ 

The number of pairs $(x^g, E)$ with $x^g \in E \in \Delta$ is of the order of $|\Delta| \cdot |\tilde{A}_1 : (q - \epsilon)| \sim q^6$. Arguing as in Proposition 3.1, we see that, given $E$, the proportion of conjugates $x^g \in E$ such that $\langle x, x^g \rangle = E$ is at least $1 - c/q$. Clearly $E$ is the unique member
of $\Delta$ containing such $x^g$. Hence the number of conjugates $x^g$ such that $\langle x, x^g \rangle$ is a member of $\Delta$ is at least $1 - c/q)q^6$, so the probability that $\langle x, x^g \rangle$ is a member of $\Delta$ is at least

$$\frac{(1 - \frac{c}{q})q^6}{|G : C_G(x)|} \geq 1 - \frac{c}{q}.$$  

The positivity statement in (ii) follows from the fact that $\tilde{A}_1 \cong SL_2(q)$ can be generated by two conjugates of $x$, which can be proved using (2). Indeed, let $P$ be the probability that $(x, x^g) = \tilde{A}_1$ for random $g \in \tilde{A}_1$. The maximal subgroups of $\tilde{A}_1$ containing $x$ are $N := N_{\tilde{A}_1}((x)) \cong D_{2(q-\epsilon)} (\epsilon = \pm 1)$ and conjugates of $SL_2(q_0)$ for maximal subfields $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$. Observe that $\text{fix}_{\tilde{A}_1/N}(x) = 1$. If $M = SL_2(q_0)$ then $x\tilde{A}_1 \cap M = x^M$, so

$$\text{fix}_{\tilde{A}_1/M}(x) = \frac{|\tilde{A}_1 : M|}{|x\tilde{A}_1|} = \frac{|x^M|}{|x\tilde{A}_1|} = \frac{q - \epsilon}{q_0 - \epsilon_0},$$

where $\epsilon_0 = \pm 1$ is such that $q_0 \equiv \epsilon_0$ mod 3. Hence by (2)

$$1 - P \leq \frac{2}{q(q+\epsilon)} + \sum_{q_0} \left( \frac{q - \epsilon}{q_0 - \epsilon_0} \right)^2 \frac{q_0(q_0^2 - 1)}{q(q^2 - 1)},$$

which is less than 1.

(iii) If $A^h_1$ is a conjugate of $A_1$ containing $y$, then $y, y^{h^{-1}} \in A_1$, so there exists $a \in A_1$ such that $y^{h^{-1}} = y^a$. Then $ah \in C_G(y)$ which is contained in $A_1A_1$ by (i), so $A^h_1 = A_1$. In other words, the only conjugate of $A_1$ containing $y$ is $A_1$. Thus the probability in (ii) is $|y^G \cap A_1|/|y^G|$. The conclusion follows.

(iv) This is similar to the proof of (ii). Let $S$ be a subsystem subgroup $SL_3(q)$ containing $A_1$, and let $\Delta = \{S^g : g \in \tilde{A}_1\}$. Then $|\Delta| = |\tilde{A}_1 : \tilde{A}_1 \cap N_G(S)|$ which is at most $|A_1 : (q + 1)2| \sim q^2/2$. The number of pairs $(t^g, E)$ with $t^g \in E \in \Delta$ is of the order of $|\Delta|q^2 \sim q^6/2$, and a proportion of at least $1 - c/q$ of these satisfy $\langle A_1, t^g \rangle = E$. Since $|t^G| \sim q^6$, the lower bound in the conclusion follows. The positivity statement follows from the next proposition.

Proposition 3.4 Let $G = ^3D_4(q)$ with $q$ even, and let $A$ be a long $SL_2(q)$ subgroup of $G$. Let $x$ and $t$ be elements of order $q + 1$ and 2 in $A$, respectively. For random $g \in G$, the probability that $\langle x, t^g \rangle$ is a subsystem subgroup $SL_3(q)$ is positive, and is at least $1 - c/q$, where $c$ is an absolute constant.

Proof. Let $S$ be a subsystem subgroup $SL_3(q)$ containing $A$, and let $\Delta = \{S^g : g \in C_G(x)\}$. Note that $C_G(x) = \langle x \rangle SL_2(q^3)$ and $|\Delta| = |SL_2(q^3) : (q^3 + 1)| \sim q^6$. The number of pairs $(t^g, E)$ with $t^g \in E \in \Delta$ is of the order of $q^{10}$, and also $|t^G| \sim q^{10}$. The lower bound follows as in the previous propositions.

For the positivity statement, let $S$ be a subsystem subgroup $SL_3(q)$ containing $x$. The maximal subgroups of $S$ appear in [15, Tables 8.3–8.4], from which we deduce that $x$ lies in just two maximal subgroups $P_1$, $P_2$, stabilizers of 1- and 2-spaces, respectively. These have structure $(\mathbb{F}_q^2).(SL_2(q) \times (q - 1))$, and each contains $q^3 - 1$ involutions. Since the total number of involutions in $S$ is $(q^3 - 1)(q + 1)$, there is an involution $t$ such that $S = \langle x, t \rangle$, as required.
4 Basic $SL_2$ subgroups in $SL_3(q)$ and $SL_6(q)$, $q$ odd

Recall the definition of basic $SL_2$ subgroups in Section 2.2. As components for our subsequent work, we require algorithms to construct two basic $SL_2$ subgroups in a given $SL_3(q)/Z$, and five basic $SL_2$ subgroups in a given $SL_6(q)/Z$; here $q$ is odd and $Z$ is a central subgroup.

In these and subsequent algorithms, we assume that our input group $G$ is described by a collection of generators in $GL_d(F)$ for some field $F$ of the same characteristic as $F_q$.  

4.1 Algorithm for $SL_3(q)$

Let $G$ be isomorphic to $SL_3(q)/Z$ with $q$ odd and $Z$ a central subgroup. The algorithm to construct two basic $SL_2$ subgroups in $G$ is the following.

1. Find an involution $t_1 \in G$ by random search.
2. Construct $C_G(t_1)$ and $K_1 = C_G(t_1)' \cong SL_2(q)$.
3. Find an involution $t_2 \in C_G(t_1)$ which does not commute with $K_1$, and compute $K_2 = C_G(t_2)' \cong SL_2(q)$.

Now $K_1$ and $K_2$ are the required basic $SL_2$ subgroups of $G$.

Lemma 4.1 The algorithm is Monte Carlo, has probability greater than a positive absolute constant (independent of $q$) of finding the required involutions, and runs in polynomial time.

Proof. That we can both construct the involution $t_1$ and its centralizer with positive probability independent of $q$ follows from Section 2.3. Now consider the second involution $t_2$. There is a maximal torus $T$ of order $(q - 1)^2/|Z|$ in $C := C_G(t_1)$, and at least $1/4$ of its regular elements power into the conjugacy class in $C$ of a suitable involution $t_2$. The number of non-regular elements in $T$ is at most $3(q - 1)$, and $|N_C(T) : T| = 2$. By Lemma 2.7, the proportion of elements of $C$ which power to a suitable involution $t_2$ is at least $\frac{1}{2}(\frac{1}{4} - \frac{3}{q - 1})$. Therefore this is a lower bound for the probability of finding $t_2$ and is positive for every $q$ since $t_2$ exists. In Section 2 we cite polynomial-time algorithms to perform the other tasks.

4.2 Algorithm for $SL_6(q)$

Let $G$ be isomorphic to $SL_6(q)/Z$, where $q$ is odd and $Z$ is a central subgroup. Involutions in $G$ have centralizers with derived groups $SL_4(q) \circ SL_2(q)$, $SL_5(q)$, $SL_3(q) \circ SL_3(q)$ or $SL_3(q^2)/Z$. In our algorithm we consider only involutions having centralizers of the first type, and we call such a centralizer “good”; we can inspect orders of random elements in the involution centralizer to determine whether the centralizer is good.

The algorithm to construct five basic $SL_2$ subgroups in $G$ is the following.
1. Find an involution $t_1 \in G$ with good centralizer, so $C_G(t_1)' = SL_4(q) \circ SL_2(q)$. Use \textsc{KillFactor} (see Section 2) to construct the factor $K_1 \cong SL_2(q)$ of this centralizer.

2. Find an involution $t_2 \in C_G(t_1)$ with good centralizer such that $[t_2, K_1] \neq 1$. Construct $K_2$, the $SL_2(q)$ factor of $C_G(t_2)'$.

3. Find an involution $t_3 \in C_G(t_1, t_2)$ with good centralizer such that $t_3 t \notin Z(G)$ for all $t \in \langle t_1, t_2 \rangle$, and $[t_3, K_1] = 1$, $[t_3, K_2] \neq 1$. Construct $K_3 = C_G(t_1, t_2, t_3)' \cong SL_2(q)$.

4. Find an involution $t_4 \in C_G(t_1, t_2, t_3)$ with good centralizer such that $t_4 t \notin Z(G)$ for all $t \in \langle t_1, t_2, t_3 \rangle$, and $[t_4, K_1] = [t_4, K_2] = 1$, $[t_4, K_3] \neq 1$. Construct $K_4$, the $SL_2(q)$ factor in $C_G(t_4)'$.

5. Let $t_5 = t_1 t_3$ and construct $K_5$, the $SL_2(q)$ factor in $C_G(t_5)'$.

With respect to a suitable basis of $V_6(q)$, $\pm t_1 = (-1, -1, 1, 1, 1, 1)$, $\pm t_2 = (1, -1, -1, 1, 1, 1)$, $\pm t_3 = (1, 1, -1, -1, 1, 1)$, and $\pm t_4 = (1, 1, 1, -1, 1, 1)$. Hence $K_1, \ldots, K_5$ are the required basic $SL_2$ subgroups.

**Lemma 4.2** The algorithm is Monte Carlo, has probability greater than a positive absolute constant of finding the required involutions, and runs in polynomial time.

**Proof.** In Steps 2, 3 and 4 we use a maximal torus of order $(q - 1)^5$ and Lemma 2.7 to estimate the probabilities of finding suitable involutions $t_2, t_3, t_4$. We illustrate the calculation for $t_2$. Write $t_1 = (-1, -1, 1, 1, 1, 1)$ as above, and let $T$ consist of the diagonal matrices $(\alpha_1, \ldots, \alpha_6)$ where $\alpha_i \in \mathbb{F}_q^*$ and $\prod \alpha_i = 1$. Let $Q$ be the subgroup of index 2 in $\mathbb{F}_q^*$. If we take $\alpha_2 \in \mathbb{F}_q^* \setminus Q$, $\alpha_1 \in Q$ and the other $\alpha_i$ arbitrary, then this element of $T$ powers to a suitable involution $t_2$, and the number of such elements in $T$ is $|T|/4$, of which at most $f(q)$ are non-regular, for some polynomial $f(q)$ of degree at most 4. Also $|N_G(T) : T| = 48$, where $C = C_G(t_1)$. Hence Lemma 2.7 shows that the proportion of elements of $C$ powering to a suitable involution $t_2$ is at least $1/192 - c/q$ for some absolute constant $c$. \hfill \Box

**5 Basic $SL_2$ subgroups in $E_6(q)$, $E_7(q)$ and $E_8(q)$, $q$ odd**

Let $G$ be isomorphic to one of the quasisimple groups of type $G(q) = E_6(q)$, $E_7(q)$ or $E_8(q)$ with $q$ odd. We first present algorithms to construct basic $SL_2$ subgroups of $G$ and later justify them. Each algorithm starts with the construction of an involution centralizer; these are described in Proposition 2.3.

As usual, we assume that our input group $G$ is described by a collection of generators in $GL_d(F)$ for some field $F$ of the same characteristic as $\mathbb{F}_q$.

**5.1 $E_6(q), q$ odd**

Let $G(q) = E_6(q)$, $q$ odd.

1. Find an involution $t_0 \in G$ with $C_G(t_0)$ of type $A_1 A_5$. Construct the factors $K_0 \cong SL_2(q)$ and $D \cong A_5(q)$ of $C_G(t_0)'$.  

2. Find an involution \( t_2 \in C_G(t_0) \) such that \( C_D(t_2)' = E \cong SL_3(q) \circ SL_3(q) \). Construct the two \( SL_3(q) \) factors.

3. Construct basic \( SL_2 \) subgroups \( K_1, K_3 \) in the first factor \( SL_3(q) \) of \( E \), and \( K_5, K_6 \) in the second factor. Let \( Z(K_i) = \langle t_i \rangle \).

4. Construct \( K_4 \), the \( SL_2(q) \) factor of \( C_D(t_1, t_6)' \cong SL_2(q)^3 \) which centralizes \( K_1 K_6 \). Now \( K_1, K_3, K_4, K_5, K_6 \) are basic \( SL_2 \) subgroups in \( D \cong SL_6(q) \).

5. The centralizer \( C_G(t_2) \) is of type \( A_1 A_5 \); construct \( K_2 \), the \( SL_2(q) \) factor of \( C_G(t_2) \).

We now have the six basic \( SL_2 \) subgroups \( K_1, \ldots, K_6 \) in the \( E_6 \) Dynkin diagram:

\[
\begin{array}{cccccc}
K_1 & - & K_3 & - & K_4 & - \quad K_5 & - \quad K_6 \\
 & | & & & & \\
 & K_2 & & & & \\
\end{array}
\]

### 5.2 \( E_7(q), q \) odd

This algorithm is similar to that for \( E_6 \).

1. Find an involution \( t_0 \in G \) such that \( C_G(t_0) \) is of type \( A_1 D_6 \), and construct the factors \( K_0 \cong SL_2(q) \) and \( D \cong D_6(q) \) of \( C_G(t_0)' \).

2. Find an involution \( t_1 \in C_G(t_0) \) such that \( C_D(t_1)' = E \cong A_5(q) \) and \( C_G(t_1) \) is of type \( A_1 D_6 \).

3. Construct basic \( SL_2 \) subgroups \( K_2, K_4, K_5, K_6, K_7 \) in \( E \). Let \( t_i \) be the central involution in \( K_i \).

4. The element \( t_1 \) is a root involution; construct the factor \( K_1 \cong SL_2(q) \) of \( C_G(t_1) \).

5. The element \( t_3 = t_0 t_5 t_7 \) is a root involution; construct the factor \( K_3 \cong SL_2(q) \) of \( C_G(t_3) \).

We now have the seven basic \( SL_2 \) subgroups \( K_1, \ldots, K_7 \) in the \( E_7 \) Dynkin diagram:

\[
\begin{array}{cccccc}
K_1 & - & K_3 & - & K_4 & - \quad K_5 & - \quad K_6 & - \quad K_7 \\
 & | & & & & \\
 & K_2 & & & \quad K_7 & & \quad K_6 & & \quad K_5 & & \quad K_4 & & \quad K_3 & & \quad K_1 & & \\
\end{array}
\]

### 5.3 \( E_8(q), q \) odd

This algorithm is similar to that for \( E_6 \) and \( E_7 \).

1. Find an involution \( t_0 \in G \) with \( C_G(t_0) \) of type \( A_1 E_7 \), and construct the factors \( K_0 \cong SL_2(q), D \cong E_7(q) \) of \( C_G(t_0)' \).

2. Find an involution \( t_8 \in C_G(t_0) \) such that \( C_D(t_8)' = E \cong E_6(q) \).

3. Construct basic \( SL_2 \) subgroups \( K_1, \ldots, K_6 \) in \( E \).
4. The element $t_8$ is a root involution; construct the factor $K_8 \cong SL_2(q)$ of $C_G(t_8)$.

5. The element $t_7 = t_0t_2t_5$ is a root involution; construct the factor $K_7 \cong SL_2(q)$ of $C_G(t_7)$.

We now have the eight basic $SL_2$ subgroups $K_1,\ldots,K_8$ in the $E_8$ Dynkin diagram:

$$
K_1 - K_3 - K_4 - K_5 - K_6 - K_7 - K_8
\uparrow
K_2
$$

### 5.4 Justification

**Proposition 5.1** The algorithms for $E_6(q)$, $E_7(q)$ and $E_8(q)$ for odd $q$ described above are Monte Carlo and run in polynomial time.

**Proof.** We first prove the correctness of the algorithm for $E_6(q)$. In Step 1, finding the involution $t_0$ and constructing its centralizer is justified by the results of Section 2.3. The factors $K_0$ and $D$ of $C_G(t_0)'$ are constructed using the algorithm KillFactor, referred to in Section 2.

Now consider Step 2 of the algorithm: find an involution $t_2 \in C_G(t_0)$ with centralizer containing $SL_3(q) \circ SL_3(q)$. We show that there is a positive lower bound (independent of $q$) for the probability of finding such an involution. This does not follow directly from Theorem 2.6, but follows from the method of its proof in [52]. Namely, there is a maximal torus of $C_G(t_0) \cong (SL_2(q) \circ A_5(q)) \circ Z(2)$ of order $(q^3 - 1)^2/2$, and at least $1/8$ of the elements of this torus power to involutions which have the desired centralizer structure; thus Lemma 2.7 gives the required conclusion. The centralizer can be computed by Section 2.3, and the $SL_3(q)$ factors extracted using KillFactor.

Step 3 of the algorithm is justified in Section 4. In Step 4, the construction of $C_D(t_1)$, and of the centralizer of $t_6$ within this group, is justified using Section 2.3. Observe that if $t_4 = Z(K_4)$, then $t_4 = t_1t_6t_0$.

Step 5 requires a little more argument. Recall that $E \cong SL_3(q) \circ SL_3(q)$, a central product of two $SL_3(q)$ subsystem subgroups of $G$. From the subsystem $A_2^3$ of the $E_6$ root system, we see that $C_G(E)$ is isomorphic to $Z(E)SL_3(q)$, so $t_2$ and $K_0$ are contained in $C := C_G(E)' \cong SL_3(q)$. Hence $t_2$ is a root involution and we let $K_2$ be the $SL_2(q)$ factor of $C_G(t_2)$.

Finally, we show that $K_1,\ldots,K_6$ pairwise generate either their direct product or $SL_3(q)$ according to their positions in the Dynkin diagram. Observe that $C_G(t_2)' = K_2S$ with $S \cong SL_6(q)$, and clearly $E < S$. Hence $K_2 \leq C_G(E)' = C$. Therefore $K_2$ centralizes each of $K_1,K_3,K_5,K_6$ and $\langle K_2,K_0 \rangle = C$. Hence $K_0,K_2$ and $K_4$ are contained in $C_G(K_1K_6)' \cong SL_4(q)$. The central involutions $t_0,t_2,t_4$ commute: $t_2$ commutes with $t_4$ since it commutes with $t_0,t_1,t_6$ and $t_4 = t_1t_6t_0$. Working in $SL_4(q)$ relative to a basis diagonalizing these three involutions, we see that $\langle K_2,K_4 \rangle \cong SL_3(q)$. This justifies the algorithm for $E_6(q)$.

For $E_7(q)$ the proof is similar, with the following additional observations. In Step 2, such an involution $t_1$ can be found with positive probability by the usual
argument using Lemma 2.7 and a maximal torus of order \((q - 1)(q^6 - 1)\) in \(C_G(t_0)\). In Step 4, observe that \(K_1\) commutes with \(E\) and \(\langle K_0, K_1 \rangle = C_G(E) \cong SL_3(q)\). For Step 5 and the Dynkin diagram generation of the \(K_1\), observe the equation between toral elements \(h_0(-1) = h_{2234321}(-1) = h_3(-1)h_5(-1)h_7(-1)\) (using the notation of Section 2.2). Since \(t_i = h_i(-1)\), the involution in the centre of the final \(SL_2(q)\) to complete the Dynkin diagram must be \(t_3 = t_0t_5t_7\).

Finally consider \(E_8(q)\). To justify Step 2 we take a maximal torus of order \((q + 1)^2(q^6 + q^3 + 1)\) in \(C_G(t_0)\) and use Lemma 2.7 as usual. In Step 5 the equation \(h_0(-1) = h_{2346532}(-1) = h_2(-1)h_5(-1)h_7(-1)\) justifies our definition \(t_7 = t_0t_5t_5\), and implies that the factor \(K_7 \cong SL_2(q)\) of its centralizer completes the Dynkin diagram as claimed.

### 6 Basic \(SL_2\) subgroups in \(E_6(q)\), \(E_7(q)\) and \(E_8(q)\), \(q\) even

In this section we assume that \(G\) is described by a collection of generators in \(GL_d(F)\), and \(G\) is quasisimple and isomorphic to one of \(G(q) = E_6(q), E_7(q)\) or \(E_8(q)\), where \(F\) and \(\mathbb{F}_q\) are both finite fields of characteristic 2, and \(q > 2\). We first present algorithms to construct basic \(SL_2\) subgroups of \(G\), and later justify them.

Throughout, \(\omega\) denotes a generator for the multiplicative group of \(\mathbb{F}_q\).

#### 6.1 \(E_6(q)\), \(q\) even

Assume \(G\) is isomorphic to \(E_6(q)\) with \(q\) even and \(q > 2\).

1. Find \(y \in G\) of order \((q + 1)(q^5 - 1)/d\) where \(d = (3, q - 1)\), and define \(x = y^{(q^5 - 1)/d}\).
2. Find \(g \in G\) such that \(X := \langle x, x^g \rangle\) is isomorphic to \(D_4(q)\) (where \(\epsilon = \pm\)).
3. Construct an isomorphism \(\phi\) from \(X\) to the standard copy of \(D_4(q) = \Omega(V)\), where \(V = V_5(q)\).
4. Find a standard basis \(e_1, e_2, e_3, f_3, f_2, f_1\) for a non-degenerate subspace of \(V\) of type \(O^+_7\). In the \(SL_3(q)\) subgroup of \(X\phi\) fixing \(\langle e_1, e_2, e_3 \rangle\) and \(\langle f_1, f_2, f_3 \rangle\), write down six elements acting on \(\langle e_1, e_2, e_3 \rangle\) as \(v_1, u_1^+, u_1^- \ (i = 1, 2)\), where

\[
\begin{align*}
v_1 &= \begin{pmatrix} \omega^{-1} & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, & u_1^+ &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & u_1^- &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^{-1} & 0 \\ 0 & 0 & \omega \end{pmatrix}, & u_2^+ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, & u_2^- &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

Abusing notation, write also \(v_i, u_i^+, u_i^-\) for the inverse images of these elements under \(\phi\). Define \(K_0 = \langle v_1, u_1^+, u_1^- \rangle\) and \(K_2 = \langle v_2, u_2^+, u_2^- \rangle\), basic \(SL_2\) subgroups in \(X\).
5. Construct the involution centralizer \(C_G(u_1^+)\).
6. Apply Lemma 2.9 and the ensuing remark to \(N := \langle C_G(u_1^+), v_1 \rangle = Q(D \times \langle v_1 \rangle)\), where \(D = C_G(K_0) \cong A_5(q)\) and \(Q = O_2(N)\). This constructs \(C_N(v_1) = (Z(Q) \times D) \langle v_1 \rangle\). Construct its second derived group, \(D\).
7. Construct an isomorphism $\psi$ from $D$ to the standard copy of $SL_6(q) = SL(V)$ modulo a central subgroup $Z$ of order either 1 or $(3, q - 1)$. (Here $|Z| = 3$ if and only if $3|q - 1$ and $Z(G) = 1$.)

8. Consider $v_2 \in K_2$. This element acts on $D$. Compute $T \in GL_6(q)$ such that $(g^{v_2})^T = (gv)^T$ for all $g \in D$.

9. Diagonalise $T$ to find a basis of $V$ with respect to which $T = (\omega I_3, \omega^{-1}I_3)$. Let this basis be $x_1, \ldots, x_6$.

10. For $1 \leq i \leq 5$, let $a_i, b_i, c_i$ be the inverse images under $\psi$ of matrices fixing $x_j$ for $j \neq i, i + 1$ and such that

\[
\begin{align*}
a_i \psi &: x_i \rightarrow \omega^{-1}x_i, x_{i+1} \rightarrow \omega x_{i+1}, \\
b_i \psi &: x_i \rightarrow x_i + x_{i+1}, x_{i+1} \rightarrow x_{i+1}, \\
c_i \psi &: x_i \rightarrow x_i, x_{i+1} \rightarrow x_i + x_{i+1}.
\end{align*}
\]

Define $K_1 = \langle a_1, b_1, c_1 \rangle$, and for $i = 3, \ldots, 6$, define $K_i = \langle a_{i-1}, b_{i-1}, c_{i-1} \rangle$. Then $K_1, K_3, K_4, K_5, K_6$ are basic $SL_2$ subgroups in $D \cong SL_6(q)$.

We now have the six basic $SL_2$ subgroups $K_1, \ldots, K_6$ in the $E_6$ Dynkin diagram.

### 6.2 $E_7(q)$, $q$ even

Assume $G$ is isomorphic to $E_7(q)$ with $q$ even and $q > 2$.

1-6. These steps are as for the $E_6$ algorithm, with the following modifications. In Step 1, we find an element $y$ of order $(q + 1)(q^5 - 1)$ and define $x = y^{q^5 - 1}$; the basic $SL_2$ subgroups constructed in Step 4 are $K_0$ and $K_1$; in Step 6, we construct $D = C_G(K_6)$ which is isomorphic to $D_6(q)$.

7. Construct an isomorphism $\psi$ from $D$ to the standard copy of $D_6(q) = \Omega_{12}^+(q) = \Omega(V)$, where $V$ has associated bilinear form $(, )$.

8. Consider $v_2 \in K_1$. This element acts on $D$. Compute $T \in GL_{12}(q)$ such that $(g^{v_2})^T = (gv)^T$ for all $g \in D$.

9. Diagonalise $T$ to find a basis of $V$ with respect to which $T = (\omega I_6, \omega^{-1}I_6)$. Choose a basis $e_1, \ldots, e_6$ for the $\omega$-eigenspace, and a basis $f_1, \ldots, f_6$ for the $\omega^{-1}$-eigenspace such that $(e_i, f_j) = \delta_{ij}$.

10. For $1 \leq i \leq 5$, let $a_i, b_i, c_i \in D$ be the inverse images under $\psi$ of the matrices in $\Omega(V)$ fixing $e_j, f_j$ for $j \neq i, i + 1$ and such that

\[
\begin{align*}
a_i \psi &: e_i \rightarrow \omega^{-1}e_i, e_{i+1} \rightarrow \omega e_{i+1}, f_i \rightarrow \omega f_i, f_{i+1} \rightarrow \omega^{-1}f_{i+1}, \\
b_i \psi &: e_i \rightarrow e_i + e_{i+1}, e_{i+1} \rightarrow e_i + e_{i+1}, f_i \rightarrow f_i, f_{i+1} \rightarrow f_{i+1} + f_i, \\
c_i \psi &: e_i \rightarrow e_i, e_{i+1} \rightarrow e_i + e_{i+1}, f_i \rightarrow f_i + f_{i+1}, f_{i+1} \rightarrow f_{i+1}.
\end{align*}
\]

Define $K_2 = \langle a_1, b_1, c_1 \rangle$, and for $i = 4, \ldots, 7$, define $K_i = \langle a_{i-2}, b_{i-2}, c_{i-2} \rangle$. Finally, define $K_3 = \langle a_6, b_6, c_6 \rangle$, where these elements are the inverse images of the matrices fixing $e_j, f_j$ for $j \geq 3$ and such that

\[
\begin{align*}
a_6 \psi &: e_1 \rightarrow \omega e_1, e_2 \rightarrow \omega e_2, f_1 \rightarrow \omega^{-1}f_1, f_2 \rightarrow \omega^{-1}f_2, \\
b_6 \psi &: e_1 \rightarrow e_1 + f_2, e_2 \rightarrow e_2 + f_1, f_1 \rightarrow f_1, f_2 \rightarrow f_2, \\
c_6 \psi &: e_1 \rightarrow e_1, e_2 \rightarrow e_2, f_1 \rightarrow f_1 + e_2, f_2 \rightarrow f_2 + e_1.
\end{align*}
\]
Then $K_2, K_3, K_4, K_5, K_6, K_7$ are basic $SL_2$ subgroups in $D \cong D_6(q)$.

We now have the seven basic $SL_2$ subgroups $K_1, \ldots, K_7$ in the $E_7$ Dynkin diagram.

### 6.3 $E_8(q), q$ even

Assume $G$ is isomorphic to $E_8(q)$ with $q$ even and $q > 2$.

1-6. These steps are as for the $E_6$ algorithm, with the following modifications. In Step 1, we find an element $y$ of order $(q + 1)(q^2 - 1)$ and define $x = y^{q^2 - 1}$; the basic $SL_2$ subgroups constructed in Step 4 are $K_0$ and $K_8$; in Step 6, we construct $D = C_G(K_0)$ which is isomorphic to $E_7(q)$.

7. Using the algorithm of Section 6.2, construct basic $SL_2$ subgroups $K_1, \ldots, K_7$ of $D$; label root elements of $D$ as in Section 11.1. Construct an isomorphism $\psi$ from $D$ to the standard copy of $E_7(q)$ (a group of $56 \times 56$ matrices).

8. Consider $v_2 \in K_8$. This element acts on $D$. Compute $T \in D\psi$ such that $(g^{v_2})\psi = (g\psi)^T$ for all $g \in D$.

9. Compute $g \in D\psi$ such that $T^g = h_{2346543}(\lambda)$ for some $\lambda \in \mathbb{F}_q$. Replace $K_8$ by $K_8^{g\psi^{-1}}$.

We now have the eight basic $SL_2$ subgroups $K_1, \ldots, K_8$ in the $E_8$ Dynkin diagram.

### 6.4 Justification

#### Proposition 6.1

The algorithms for $E_6(q)$, $E_7(q)$ and $E_8(q)$ for even $q$ described above are Monte Carlo and run in polynomial time.

**Proof.** In Step 1 of each algorithm, the justification for being able to find an element $y$ of the specified order is standard. Consider, for example, the $E_6$ case: in the simple group $G := E_6(q)$ there is a cyclic maximal torus $T$ of order $t := (q + 1)(q^2 - 1)/d$ (see [37, §2]); the number of generators of $T$ is $\phi(t) > t/c \log \log q$ where $c$ is an absolute constant; hence the proportion of elements of order $t$ in $G$ is at least $1/(c \log \log q \cdot |N_G(T) : T|)$.

Observe that $y$ lies in a maximal torus of $G$ contained in a subsystem subgroup of type $A_1A_5$, $A_1D_6$ or $A_1E_7$ (see [37, §2]). Hence the power $x$ of $y$ must lie in a long $SL_2(q)$ subgroup of $G$. By Proposition 3.2, there is a positive lower bound independent of $q$ for the probability that, for random $g \in G$, $\langle x, x^g \rangle$ is $D_4^*(q)$, a subsystem subgroup, as required for Step 2.

In Step 3, the construction of an isomorphism $X \rightarrow D_4^*(q)$ is justified by Theorem 2.2.

In Step 5, the element $u_1^+$ is a root involution so the construction of the involution centralizer $C_G(u_1^+)$ is justified by Theorem 2.8 and the ensuing remark. The structure of $C_G(u_1^+)$ is given in [1]: $C_G(u_1^+) = QD$, where $D = C_G(K_0) \cong A_5(q)$, $D_6(q)$ or $E_7(q)$, and $Q \cong q^{1+20}, q^{1+32}$ or $q^{1+56}$, when $G$ is of type $E_6, E_7, E_8$ respectively. The element $v_1$ normalizes $C_G(u_1^+)$ (which is $C_G(U)$ where $U$ is a root group
of $K_0$ containing $u_1^+$), centralizes $D$, and acts fixed point freely on $Q/Z(Q)$. Lemma 2.9 and the ensuing remark gives a construction of $(Z(Q) \times D)\langle v_1 \rangle$, as claimed in Step 6.

For $E_6(q)$ (or $E_7(q)$), the isomorphism in Step 7 from $D$ to $A_5(q)$ (or $D_6(q)$) is justified by Theorem 2.2. The remaining steps ensure that $v_2$ acts as $h_2(\omega)$ (or $h_1(\omega)$) on $D$. Hence we choose the remaining $SL_2(q)$ subgroups in $D$ to fit in with the subgroups $K_0, K_2$ (or $K_0, K_1$) already defined. That they pairwise generate the correct groups is established as in the proof of Proposition 5.1. In Step 8, the computation of the matrix $T$ involves solving linear equations in the entries of $T$ of the form $TA = BT$, where $A = (g^{\psi_2})$, $B = g\psi$ for generators $g$ of $D$; such systems of equations over $\mathbb{F}_q$ can be solved in polynomial time.

For the $E_6$ case, in $SL_3(q) \cong \langle K_2, K_4 \rangle$, $K_2$ and $K_4$ satisfy the correct picture of being the subgroups $(X, 1)$ and $(1, X)$, for $X \in SL_2(q)$, relative to some basis of the natural module $V = V_3(q)$: indeed, we constructed $K_4$ so that it is stabilized by $v_2 \in K_2$, which implies that $v_2$ stabilizes $C_V(K_4)$; thus $C_V(K_4) \subseteq [V, K_2]$ as required. Similar remarks apply in the $E_7$ case to $SL_3(q) \cong \langle K_1, K_3 \rangle$.

For $E_8(q)$, Steps 7–9 are more complex. In Step 7, we construct an isomorphism $\psi$ from $D$ to the standard copy of $E_7(q)$, a specific group of $56 \times 56$ matrices with standard generators $\hat{S}$. Specifically, we find, as in Sections 13.1 a set $S$ of standard generators of $D$ which satisfies the reduced Curtis-Steinberg-Tits presentation of $E_7(q)$. For $x \in D$, we use the algorithm of [22], applied to the action of $D$ on an absolutely irreducible composition factor of $V_d(F) | D$, to express $x$ as a word $w(S)$; now $\psi$ is defined to send $x$ to $w(\hat{S})$. That this step can be performed in polynomial time follows from this proposition (already proved for $E_7(q)$), together with the algorithms of Section 13.1 and [22].

Since $D = C_G(v_1)'$, the element $v_2$ acts on $D$, and induces an inner automorphism. In Step 8, we use linear algebra to find $T' \in GL_{56}(q)$ such that $(g^{\psi_2})T = (gv)^T$ for all generators $g$ of $D$. Some scalar multiple, $T$, of $T'$ of determinant 1 must lie in $D\psi$; we use [22] to determine $T$.

The centralizer of $T$ in $D\psi$ contains the image under $\psi$ of $C_D(K_8) = C_G(K_0, K_8) \cong E_6(q)$. It follows that $T$ is $D\psi$-conjugate to the toral element $h := h_{2346543}(\lambda)$ for some eigenvalue $\lambda$ of $T$ on $V_56(q)$. We compute $g \in D\psi$ conjugating $T$ to $h$ as follows.

1. Map $D\psi$ to its action on the Lie algebra $L$ of type $E_7$ over $\mathbb{F}_q$. Call this map $\phi$.
2. In each of $C_L(T\phi)$ and $C_L(h\phi)$, compute a split Cartan subalgebra by taking the centralizer of a random semisimple element. We claim that this is a split Cartan subalgebra with probability at least $c(1 - |\Phi|/q)$, where $\Phi := \Phi(E_6)$, the $E_6$ root system, and $c$ is a positive absolute constant. To prove the claim, observe that
$$C_L(h\phi) = \langle z \rangle \oplus L(E_6) = H \oplus \sum_{\alpha \in \Phi(E_6)} L_\alpha$$
where $z$ is semisimple, $H$ is a split Cartan subalgebra of $L$, and the $L_\alpha$ are 1-dimensional root spaces for $\alpha \in \Phi$. If $v \in H$ satisfies $\alpha(v) \neq 0$ for all $\alpha$, then $C_{C_L(h\phi)}(v) = H$—that is, $v$ is regular semisimple in $C_L(h\phi)$. The number of such $v \in H$ is at least $|H|/(1 - |\Phi|/q)$, so the total number of
regular semisimple elements in \( C_L(h\phi) \) is at least this number multiplied by the number of conjugates of \( H \) under the group \( C_{D\psi\phi}(h\phi) \cong E_6(q) \circ (q - 1) \).

For large \( q \), the stabilizer of \( H \) in the latter group is the normalizer of a Cartan torus, of order \( (q - 1)^7 |W(E_6)| \). It follows that the number of regular semisimple elements in \( C_L(h\phi) \) is at least

\[
(|E_6(q) : (q - 1)^6 W(E_6)|) \cdot |H|(1 - |\Phi|/q),
\]

which is at least \( c(1 - |\Phi|/q) \cdot |C_L(h\phi)| \). This proves the claim. That we have found a split Cartan subalgebra can be verified in polynomial time by the argument of [24, 5.2].

3. Use the polynomial-time algorithm of [25, Theorem 1] to compute Chevalley bases \( B_T, B_h \) of \( L \) with respect to the Cartan subalgebras constructed in Step 2.

4. The element \( g' \) of \( GL(L) \) conjugating \( B_T \) to \( B_h \) lies in \( D\psi\phi \), and conjugates \( T\phi \) to an element of a Cartan torus of \( D\psi\phi \) containing \( h\phi \).

5. Adjust \( g' \) by a computation in the Weyl group of \( E_7(q) \) to an element \( g'' \) of \( D\psi\phi \) conjugating \( T\phi \) to \( h\phi \). Take \( g = g''^{-1} \in D\psi \), as required.

For convenience, we now abuse notation and write \( g \) instead of \( g\psi^{-1} \). To complete the proof, we argue that replacing \( K_8 \) by \( K_8'' \) provides a set \( K_1, \ldots, K_8 \) of basic \( SL_2 \) subgroups. For this, we need only to check that \( K_8'' \) centralizes \( K_1, \ldots, K_6 \) and \( \langle K_7, K_8'' \rangle \cong SL_3(q) \). First observe that \( v_2^g \in N_G(D) = DK_0 \), so \( v_2^g = hk_0 \) with \( k_0 \in K_0 \). Also \( C_G(K_8^g) = C_G(v_2^g) \geq C_D(h) \), and this contains \( K_1, \ldots, K_6 \). Finally \( C_G(K_7) = C_G(h_7) \) where \( h_7 = h_{a_7}(\omega) \), so \( C_G(K_7, K_8^g) = C_G(h_7, hk_0) \). We claim that this centralizer is of type \( E_6(q) \). Indeed, we can label the \( E_8 \) root system so that \( k_0 = h_{a_0}(\mu) = h_{23465432}(\mu) \) for some \( \mu \in F_q^* \); the fact that \( hk_0 \) is conjugate to \( v_2 \) forces \( \mu = \lambda^{-1} \) or \( \lambda^{-3} \) (recall that \( h = h_{23465432}(\lambda) \)). Now considering \( h_7 \) and \( hk_0 \) as elements of the subsystem subgroup \( A_3 \) corresponding to the roots \( a_7, a_8, a_0 \), we see that they lie in an \( A_2 \) subgroup, and hence centralize an \( E_6 \) subsystem in \( E_8 \). This proves the claim. Hence \( \langle K_7, K_8^g \rangle \leq C_G(E_6(q)) \cong SL_3(q) \). Since \( \langle K_7, K_8^g \rangle \) contains \( \langle h_7, hk_0 \rangle \), a toral subgroup of rank 2, it follows that \( \langle K_7, K_8^g \rangle \cong SL_3(q) \) as required.

\[ \square \]

7 Basic \( SL_2 \) subgroups in \( F_4(q) \)

7.1 \( F_4(q), q \text{ odd} \)

Let \( G \) be isomorphic to \( F_4(q) \) with \( q \) odd. We present an algorithm to construct basic \( SL_2 \) subgroups in \( G \).

1. Find an involution \( t_0 \in G \) such that \( C_G(t_0) \cong (SL_2(q) \circ Sp_6(q)) \).
2. Construct the factors \( K_0 \cong SL_2(q) \) and \( D \cong Sp_6(q) \) of the centralizer.
3. Find an involution \( t_1 \in C_G(t_0) \) such that \( C := C_D(t_1) \cong SL_3(q) \).
4. Construct basic \( SL_2 \) subgroups \( K_3, K_4 \) in \( C \). Let \( t_i \) be the involution in \( K_i \).
4. Let \( t_2 = t_0 t_4 \), a root involution; construct \( K_2 \), the \( SL_2(q) \) factor in \( C_G(t_2) \).

5. Also \( t_1 \) is a root involution; construct \( K_1 \), the \( SL_2(q) \) factor of its centralizer.

We now have the four basic \( SL_2 \) subgroups \( K_1, K_2, K_3, K_4 \) in the Dynkin diagram:

\[
K_1 - K_2 = \Rightarrow K_3 - K_4
\]

**Proposition 7.1** The above algorithm for \( F_4(q) \) for odd \( q \) is Monte Carlo and runs in polynomial time.

**Proof.** Finding the involutions and centralizers in Steps 1 and 2 is justified in the usual way using Lemma 2.7 and Section 2.3. For Step 4, with respect to a suitable basis for the natural 6-dimensional module for \( D \cong Sp_6(q) \), \( t_3 = (-1, -1, -1, -1, 1, 1) \), \( t_4 = (-1, -1, 1, 1, -1, -1) \) and \( t_0 = -I \); hence \( t_2 = t_0 t_4 \) is a root involution.

Working in \( D \), we see that \([K_2, K_4] = 1\) and \( \langle K_2, K_3 \rangle \cong Sp_4(q) \). Now \( K_0 \) and \( t_1 \) lie in \( C_G(C) \), which is an \( SL_3(q) \) generated by long root groups in \( G \). Arguing as in Proposition 5.1 for \( E_6(q) \), we deduce that \( t_1 \) is a root involution. If \( K_1 \) is the \( SL_2(q) \) factor of its centralizer, then \( K_1 \) centralizes \( K_3 \) and \( K_4 \); also \( \langle K_1, K_2 \rangle = C_G(C) \cong SL_3(q) \).

\[ \blacksquare \]

### 7.2 \( F_4(q) \), \( q \) even

Assume \( G \) is isomorphic to \( F_4(q) \), where \( q \) is even and \( q > 2 \). We present an algorithm to construct basic \( SL_2 \) subgroups in \( G \). Recall that \( \omega \) denotes a generator of the multiplicative group of \( \mathbb{F}_q \).

1-6. These steps are as for the \( E_6 \) algorithm in Section 6.1 with the following modifications. In Step 1, we find an element \( y \) of order \((q + 1)(q^2 - 1)\) and define \( x = y^{q^2 - 1} \); the basic \( SL_2 \) subgroups constructed in Step 4 are \( K_0 \) and \( K_1 \); in Step 6, we construct \( D = C_G(K_0) \) which is isomorphic to \( Sp_6(q) \).

7. Construct an isomorphism \( \psi \) from \( D \) to the standard copy of \( Sp_6(q) = Sp(V) \).

8. Consider \( v_2 \in K_1 \). This element acts on \( D \). Compute \( T \in GL_6(q) \) such that \((g^{v_2}) \psi = (g \psi)^T \) for all \( g \in D \).

9. Diagonalise \( T \) to find a basis of \( V \) with respect to which \( T = (\omega I_3, \omega^{-1} I_3) \).

Choose a basis \( e_1, e_2, e_3 \) for the \( \omega \)-eigenspace, and a basis \( f_1, f_2, f_3 \) for the \( \omega^{-1} \)-eigenspace such that \((e_i, f_j) = \delta_{ij} \).

11. Define \( a_0, b_0, c_0 \in D \) to be the inverse images under \( \psi \) of the elements in \( Sp(V) \) fixing \( e_2, e_3, f_2, f_3 \) and acting on \( e_1, f_1 \) as follows:

\[
\begin{align*}
a_0 \psi &: e_1 \mapsto \omega^{-1} e_1, f_1 \mapsto \omega f_1, \\
b_0 \psi &: e_1 \mapsto e_1 + f_1, f_1 \mapsto f_1, \\
c_0 \psi &: e_1 \mapsto e_1, f_1 \mapsto e_1 + f_1.
\end{align*}
\]

For \( i = 1, 2 \) let \( a_i, b_i, c_i \in D \) be the inverse images under \( \psi \) of the matrices in \( Sp(V) \) fixing \( e_j, f_j \) for \( j \neq i, i + 1 \) and such that

\[
\begin{align*}
a_i \psi &: e_i \mapsto \omega^{-1} e_i, e_{i + 1} \mapsto \omega e_{i + 1}, f_i \mapsto \omega f_i, f_{i + 1} \mapsto \omega^{-1} f_{i + 1}, \\
b_i \psi &: e_i \mapsto e_i + e_{i + 1}, e_{i + 1} \mapsto e_{i + 1}, f_i \mapsto f_i, f_{i + 1} \mapsto f_{i + 1} + f_i, \\
c_i \psi &: e_i \mapsto e_i, e_{i + 1} \mapsto e_i + e_{i + 1}, f_i \mapsto f_i + f_{i + 1}, f_{i + 1} \mapsto f_{i + 1}.
\end{align*}
\]
Proposition 7.2 The above algorithm for $SL_8$ Basic is Monte Carlo and runs in polynomial time.

The proof is similar to that of Proposition 6.1.

8 Basic $SL_2$ subgroups in $G_2(q)$

8.1 $G_2(q)$, $q$ odd

Let $G$ be isomorphic to $G_2(q)$ with $q$ odd. We present an algorithm to construct basic $SL_2$ subgroups in $G$.

1. Find an involution $t_0 \in G$ and compute its centralizer $C_G(t_0) \cong (SL_2(q) \circ \langle t \rangle).2$. Construct $S_1$ and $S_2$, the two $SL_2(q)$ factors.

2. If $q \equiv 1 \mod 4$, then find an involution $t_1 \neq t_0$ with $t_1 \in C_G(t_0)' = S_1S_2$; if $q \equiv 3 \mod 4$, then find an involution $t_1 \in C_G(t_0)\setminus S_1S_2$.

3. Construct the two $SL_2(q)$ factors of $C_G(t_1)$. For one of them – call it $S$ – either
   
   (a) $\langle S, S \rangle \cong SL_3(q)$, $\langle S, t \rangle = G$, or
   
   (b) $\langle S, S \rangle \cong SL_3(q)$, $\langle S, t \rangle = G$.

Assume (a) holds; relabel as $K_0 = S_1$, $K_1 = S$, $K_2 = S_2$. Now $K_0$ and $K_2$ are basic $SL_2$ subgroups, and we can place $K_0$, $K_1$, and $K_2$ in the extended $G_2$ Dynkin diagram as follows:

\[ K_0 \rightarrow K_1 \equiv \equiv K_2 \]

Proposition 8.1 The above algorithm for $G_2(q)$ for odd $q$ is Monte Carlo and runs in polynomial time.

Proof. Finding the involutions and centralizers in Steps 1 and 2 is justified as usual using Lemma 2.7 and Section 2.3. We next prove the claim in Step 3. First we show that conclusion (a) or (b) in that step holds for at least one involution in $C_G(t_0)$ satisfying the condition in Step 2 on being inside or outside the derived group. Let $\alpha_1, \alpha_2$ be fundamental roots with $\alpha_1$ long, and let $\alpha_0 = 2\alpha_1 + 3\alpha_2$ be the highest root. We choose notation so that, interchanging $S_1$ and $S_2$ if necessary, $S_1 = \langle U_{\pm\alpha_1} \rangle$, $S_2 = \langle U_{\pm\alpha_2} \rangle$. Let $t_1$ be the involution in the centre of $\langle U_{\pm\alpha_1} \rangle$. If $q \equiv 1 \mod 4$ then $t_1 \in C_G(t_0)'$, since it equals $h_0(i)h_2(-i) \in S_1S_2$. If $q \equiv 3 \mod 4$ then $t_1 \not\in C_G(t_0)'$; indeed, Bruhat decomposition implies that if $t_1 = s_1s_2$ with $s_i \in S_i$, then $s_1$ is in $B$ or $Bn_{\alpha_2}B$, and $s_2$ is in $B$ or $Bn_{\alpha_2}B$. Since $t_1 = h_1(-1) \in B$, the only possibility is that $s_1, s_2 \in B$, which leads to $h_1(-1) = u_{\alpha_0}(a)u_{\alpha_2}(a')h_0(b)h_2(b')$ for some $a, a', b, b' \in F_q$. This is impossible since the only involution of the form $h_0(b)h_2(b')$ is $t_0 = h_2(-1)$. When $q \equiv 1 \mod 4$, all non-central involutions in $C_G(t_0)' = S_1S_2$ are conjugate; when $q \equiv 3 \mod 4$, all outer involutions in $C_G(t_0) \setminus C_G(t_0)'$ are conjugate. The claim in Step 3 follows.
8.2 \( G_2(q), q \text{ even} \)

Assume \( G \) is isomorphic to \( G_2(q) \), where \( q \) is even and \( q > 2 \). We present an algorithm to construct basic \( SL_2 \) subgroups in \( G \). Recall that \( \omega \) denotes a generator of the multiplicative group of \( \mathbb{F}_q \).

1. Find \( y \in G \) of order \( 3(q-\epsilon) \), where \( \epsilon = \pm 1 \) and \( q \not\equiv \epsilon \mod 3 \). Define \( x = y^{q-\epsilon} \), an element of order 3.
2. If, after \( O(1) \) random selections, we fail to find \( g \in G \) with the property that \( \langle x, x^q \rangle = K_2 \cong SL_2(q) \) then go to Step 1.
3. Construct an isomorphism \( \phi \) from \( K_2 \) to the standard copy of \( SL_2(q) \). In \( K_2 \), write down
   \[
   u = \phi^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad v = \phi^{-1} \begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{pmatrix}.
   \]
4. If \( q = 4 \) then compute \( K_0 := C_G(K_2) \). Otherwise, construct the involution centralizer \( C_G(u) \), and \( N := \langle C_G(u), v \rangle \); apply Lemma 2.9 to \( N \) to construct \( K_0 = C_N(v) \), a long \( SL_2 \) subgroup centralizing \( K_2 \); now \( K_0K_2 = A_1(q)\tilde{A}_1(q) \) in \( G \).
5. Construct an isomorphism from \( K_0 \) to the standard copy of \( SL_2(q) \), and hence write down an involution \( t \in K_0 \).
6. Find \( g \in G \) such that \( \langle K_0, t^q \rangle = Y \cong SL_3(q) \).
7. Construct an isomorphism \( \phi \) from \( Y \) to \( SL_3(q) = SL(V) \). Compute \( \langle v_1, v_2 \rangle = [V, K_0 \phi] \) and \( \langle v_3 \rangle = C_V(K_0 \phi) \). Construct \( K_1 \cong SL_2(q) \) in \( Y \) generated by the preimages under \( \phi \) of generators for \( K_1 \): with respect to the basis \( v_1, v_2, v_3 \), these are
   \[
   \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^{-1} & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.
   \]

We now have the three \( SL_2(q) \) subgroups \( K_0, K_1, K_2 \) in the extended \( G_2 \) Dynkin diagram.

Proposition 8.2 The above algorithm for \( G_2(q) \) for even \( q \) is Monte Carlo and runs in polynomial time.

Proof. In Step 1, \( y \) lies in a maximal torus of \( G \) contained in a subsystem subgroup \( A_1(q)A_1(q) \), where the first factor is generated by long root subgroups of \( G \) and the second by short root subgroups. Hence the element \( x = y^{q-\epsilon} \) of order 3 lies in \( A_1(q) \) or \( \tilde{A}_1(q) \). If \( x \in \tilde{A}_1(q) \) then, by Proposition 3.3(ii), there is a positive lower bound independent of \( q \) for the probability that \( \langle x, x^q \rangle \) is a conjugate of \( \tilde{A}_1(q) \). If \( x \in A_1(q) \) then Proposition 3.3(iii) shows that the probability that \( \langle x, x^q \rangle \) is a conjugate of \( A_1(q) \) is very small, justifying Step 2.

Consider Step 4. The construction of \( C_G(u) \) is justified as in [43, Theorem 3.9], and the structure of this involution centralizer is given by [1]: \( C_G(u) = QK_0 \), where \( Q \) is abelian of order \( q^3 \). If \( U = C_{K_2}(u) \), then \( C_G(u) = C_G(U) \) and \( N_G(U) = \)
\((C_G(U), v) \cong Q(K_0 \times \langle v \rangle)\). The element \(v\) acts fixed point freely on \(Q\) for \(q > 4\), so we can apply Lemma 2.9.

Finally, Proposition 3.3 justifies Step 6: for random \(g \in G\), there is a positive lower bound independent of \(q\) for the probability that \(\langle K_0, t^g \rangle = Y \cong SL_3(q)\).

\section{Basic \(SL_2\) subgroups in \(2E_6(q)\)}

\subsection{\(2E_6(q)\), \(q\) odd}

Let \(G\) be isomorphic to the quasisimple group \(G(q) = 2E_6(q)\) with \(q\) odd.

Since \(G(q)\) is a twisted group, we construct basic \(SL_2\) subgroups and root elements relative to the twisted root system, which is of type \(F_4\) (see \([33, 2.4]\)). Thus we aim to find \(SL_2\) subgroups \(K_1, \ldots, K_4\) forming the diagram

\[
K_1 \rightarrow K_2 \rightarrow K_3 \rightarrow K_4
\]

where \(K_i \cong SL_2(q)\) for \(i = 1, 2; K_i \cong SL_2(q^2)\) for \(i = 3, 4\); \(\langle K_1, K_2 \rangle \cong SL_3(q)\); \(\langle K_2, K_3 \rangle \cong SU_4(q)\); and \(\langle K_3, K_4 \rangle \cong SL_3(q^2)\) or \(PSL_3(q^2)\).

We present an algorithm to construct basic \(SL_2\) subgroups in \(G\).

1. Find an involution \(t_0 \in G\) such that \(C_G(t_0)\) is of type \(A_1 \in 2A_5\). Construct the factors \(K_0 \cong SL_2(q)\) and \(D \cong 2A_5(q)\) of the centralizer.

2. Find an involution \(t_1 \in C_G(t_0)\) such that \(C := C_D(t_1)' \cong SL_3(q^2)\) or \(PSL_3(q^2)\).

3. Using Section 4.1, construct basic \(SL_2\) subgroups \(K_3, K_4\) in \(C\). Let \(t_3, t_4\) be the involutions in \(K_3, K_4\).

4. Let \(t_2 = t_0t_4\), a root involution; construct \(K_2\), the \(SL_2(q)\) factor in \(C_G(t_2)\).

5. Also \(t_1\) is a root involution; construct \(K_1\), the \(SL_2(q)\) factor in \(C_G(t_1)\).

We now have the four basic \(SL_2\) subgroups \(K_1, \ldots, K_4\) in the Dynkin diagram.

\textbf{Proposition 9.1} The above algorithm for \(2E_6(q)\) for odd \(q\) is Monte Carlo and runs in polynomial time.

The proof is similar to that of Proposition 7.1.

\subsection{\(2E_6(q)\), \(q\) even}

Assume \(G\) is isomorphic to \(2E_6(q)\), where \(q\) is even and \(q > 2\). We present an algorithm to construct basic \(SL_2\) subgroups in \(G\). Recall that \(\omega\) denotes a generator of the multiplicative group of \(\mathbb{F}_q\).

1-6. These steps are as for the \(E_6\) algorithm in Section 6.1, with the following modifications. In Step 1, we find an element \(y\) of order \((q^6 - 1)/(3, q + 1)\) and define \(x = y^{(q^6-1)/(q+1)}\); the long \(SL_2\) subgroups constructed in Step 4 are \(K_0\) and \(K_1\); in Step 6, we construct \(D = C_G(K_0)\) which is isomorphic to \(SU_6(q)/Z\), where \(Z\) is a central subgroup of order 1 or \((3, q + 1)\).
Construct an isomorphism $\psi$ from $D$ to the standard copy of $SU_6(q)$ modulo a central subgroup.

Consider $v_2 \in K_1$. This element acts on $D$. Compute $T \in GL_6(q^2)$ such that $(g^{v_2})\psi = (g\psi)^T$ for all $g \in D$.

Diagonalise $T$ to find a basis of $V$ with respect to which $T = (\omega I_3, \omega^{-1} I_3)$. Choose a basis $e_1, e_2, e_3$ for the $\omega$-eigenspace, and a basis $f_1, f_2, f_3$ for the $\omega^{-1}$-eigenspace such that $(e_i, f_j) = \delta_{ij}$.

Now define three basic $SL_2$ subgroups in $D$ as follows. Define $a_0, b_0, c_0 \in D$ to be the inverse images under $\psi$ of the elements in $SU(V)$ fixing $e_2, e_3, f_2, f_3$ and acting on $e_1, f_1$ as follows:

For $i = 1, 2$ let $a_i, b_i, c_i \in D$ be the inverse images under $\psi$ of the matrices in $SU(V)$ fixing $e_j, f_j$ for $j \neq i, i + 1$ and such that

- $a_i \psi : e_i \mapsto \omega^{-1} e_i, f_i \mapsto \omega f_i$,
- $b_i \psi : e_i \mapsto e_i + e_{i+1}, f_i \mapsto f_i + f_{i+1}$,
- $c_i \psi : e_i \mapsto e_i, f_i \mapsto f_i + f_{i+1}$.

where $\nu$ is a primitive element of $\mathbb{F}_{q^2}$ and $\bar{\nu} = \nu^q$. Define $K_2 = (a_0, b_0, c_0) \cong SL_2(q)$, and for $i = 3, 4$, define $K_i = (a_{i-2}, b_{i-2}, c_{i-2}) \cong SL_2(q^2)$. Then $K_2, K_3$ and $K_4$ are basic $SL_2$ subgroups in $D$.

We now have the four basic $SL_2$ subgroups $K_1, \ldots, K_4$ in the Dynkin diagram.

**Proposition 9.2** The above algorithm for $2E_6(q)$ for even $q$ is Monte Carlo and runs in polynomial time.

The proof is similar to that of Proposition 6.1.

## 10 Basic $SL_2$ subgroups in $3D_4(q)$

### 10.1 $3D_4(q), q$ odd

Let $G$ be isomorphic to $3D_4(q)$ with $q$ odd. The twisted root system is of type $G_2$, and we must construct basic subgroups $SL_2(q)\cap SL_2(q^3)$.

1. Find an involution $t_0 \in G$ and compute its centralizer $C_G(t_0) \cong (SL_2(q) \cap SL_2(q^3)).2$. Construct the two $SL_2$ factors $K_0 \cong SL_2(q)$ and $K_2 \cong SL_2(q^3)$.

2. If $q \equiv 1 \mod 4$, then find an involution $t_1 \neq t_0$ with $t_1 \in C_G(t_0)' = K_0K_2$; if $q \equiv 3 \mod 4$, then find an involution $t_1 \in C_G(t_0)\setminus K_0K_2$.

3. Construct the factor $K_1 \cong SL_2(q)$ of $C_G(t_1)$.

We now have the three basic $SL_2$ subgroups $K_0, K_1, K_2$ in the extended $G_2$ Dynkin diagram.
Proposition 10.1 The above algorithm for $3D_4(q)$ for odd $q$ is Monte Carlo and runs in polynomial time.

The proof is similar to that of Proposition 8.1.

10.2 $3D_4(q)$, $q$ even

Let $G$ be isomorphic to $3D_4(q)$ with $q$ even. This case differs from the others: our algorithm to construct basic $SL_2$ subgroups employs an $O(q)$ search for an involution.

1. Find an element of even order in $G$ that powers to a root involution $t \in G$.
2. Find $y \in G$ of order $(q + 1)(q^3 - 1)$, and let $x = y^{q^3 - 1}$.
3. Find $g \in G$ such that $Y := \langle x, t^g \rangle \cong SL_3(q)$.
4. Construct an isomorphism from $Y$ to the standard copy of $SL_3(q)$, and hence write down $K_0$ and $K_1$, basic $SL_2$ subgroups in $Y$. In $K_0$ write down the preimages of
   $$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{pmatrix}$$
   where $\omega$ denotes a generator of the multiplicative group of $\mathbb{F}_q$.
5. Construct $N := \langle C_G(u), v \rangle$.
6. For $q > 2$, use Lemma 2.9 to construct $K_2 := C_N(v)^\sigma \cong SL_2(q^3)$. For $q = 2$ construct $K_2 := C_G(K_0)$.

We now have the three basic $SL_2$ subgroups $K_0, K_1, K_2$ in the extended $G_2$ Dynkin diagram.

Proposition 10.2 The above algorithm for $3D_4(q)$ for even $q$ is Monte Carlo and has complexity $O(q)$.

Proof. By [43, Theorem 3.8], the proportion of elements of even order in $G$ that power to a root involution is at least $1/8q$. In Step 2, $y$ lies in a subgroup $SL_2(q) \times SL_2(q^3)$, so $x$ lies in the $SL_2(q)$ factor. In Step 3, by Proposition 3.4, there is a positive lower bound independent of $q$ for the probability that $Y := \langle x, t^g \rangle \cong SL_3(q)$, a subsystem group. Step 5 yields $N = \langle C_G(u), v \rangle \cong Q.(SL_2(q^3) \times \langle v \rangle)$, where $Q \cong q^{1+8}$ and $v$ acts fixed point freely on $Q/Z(Q)$. Hence Lemma 2.9 can be applied in Step 6. Since $K_2$ centralizes $K_0$, this completes the extended Dynkin diagram. ■

11 Labelling root and toral elements

Assume that $G$ is described by a collection of generators in $GL_d(F)$, and $G$ is isomorphic to a quasisimple exceptional group of type $G(q)$, where $F$ and $\mathbb{F}_q$ are finite fields of the same characteristic, and $q > 2$. Assume also that $G(q)$ is neither a Suzuki nor a Ree group. In previous sections we have shown how to construct a
family of basic $SL_2$ subgroups $K_r$ of $G$ as in the Dynkin diagram. We now show how to label root and toral elements in these subgroups consistently: we define root elements $x_{\pm r}(c_i)$ and toral elements $h_r(\omega)$ in each $K_r$, where $c_i$ runs over an $\mathbb{F}_q$-basis of $\mathbb{F}_q$ or an extension field, and $\omega$ is a primitive element of the field. Our labelling algorithms are largely independent of the characteristic $p$.

We use these root and toral elements in Section 12 to compute the high weight of the representation of $G$ on $V = V_d(F)$ when $V$ is irreducible, and in Section 13 to construct standard generators of $G$.

We summarise the result of this section.

**Proposition 11.1** Let $G$ be a subgroup of $GL_d(F)$, where $F$ is a finite field of the same characteristic as $\mathbb{F}_q$, and assume that $G \cong G(q)$, a quasisimple group of exceptional Lie type over $\mathbb{F}_q$ for $q > 2$, excluding Suzuki and Ree groups. Assume also that generators are given for a family of basic $SL_2$ subgroups of $G$ as in the Dynkin diagram. Subject to the availability of a discrete log oracle, there is a Las Vegas polynomial-time algorithm to label root and toral elements in each of the basic $SL_2$ subgroups.

The algorithm is described and justified in the remainder of this section. We make frequent use of the algorithms to construct isomorphisms to various low-dimensional classical groups given by Theorem 2.2.

**11.1 Labelling $E_6(q)$, $E_7(q)$ and $E_8(q)$**

Here we assume that $G \cong G(q) = E_l(q)$, $l = 6, 7$ or 8. In Sections 5 and 6, we constructed basic $SL_2$ subgroups $K_1, \ldots, K_l$ of $G$.

1. Construct an isomorphism $\phi$ from $\langle K_1, K_3 \rangle$ to $SL_3(q) = SL(V)$. Choose a basis $v_1, v_2, v_3$ of $V$ such that $v_1 \in C_V(K_3)$, $v_2 \in [V, K_1] \cap [V, K_3]$ and $v_3 \in C_V(K_1)$. Write all matrices with respect to this basis. Define

$$x_1(c_i) = \phi^{-1}\begin{pmatrix} 1 & c_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_3(c_i) = \phi^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_i \\ 0 & 0 & 1 \end{pmatrix},$$

$$x_{-1}(c_i) = \phi^{-1}\begin{pmatrix} 1 & 0 & 0 \\ c_i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{-3}(c_i) = \phi^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_i & 1 \end{pmatrix},$$

where $c_i$ runs over an $\mathbb{F}_p$-basis of $\mathbb{F}_q$, and let

$$h_1(\omega) = \phi^{-1}(\omega^{-1}, \omega, 1), \quad h_3(\omega) = \phi^{-1}(1, \omega^{-1}, \omega).$$

2. Construct an isomorphism $\psi$ from $\langle K_3, K_4 \rangle$ to $SL_3(q) = SL(V)$. Choose a basis $v_1, v_2, v_3$ as in Step 1. Compute $\lambda$ such that $h_3(\omega) = (\lambda^{-1}, \lambda, 1)$ and define $h_4(\omega) = (1, \lambda^{-1}, \lambda)$. Compute $\mu_i$ such that

$$x_{\pm 3}(c_i)\psi = \begin{pmatrix} 1 & \mu_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ \mu_i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

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and define

\[ x_{\pm 4}(c_i) = \psi^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu_i \\ 0 & 0 & 1 \end{pmatrix}, \psi^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mu_i & 1 \end{pmatrix}, \]

taking the plus terms to be both upper or both lower triangular, and similarly for the minus terms.

3. Repeat Step 2 in turn for \( \langle K_i, K_j \rangle \) with \((i, j) = (2, 4), (4, 5), (5, 6), \ldots, (l-1, l)\).

The justification for the above labelling is largely self-evident. In Step 2, observe that the root elements \( x_{\pm 3}(c_i) \psi \) are as claimed, since the root groups generated by these elements are normalized by \( h_3(\omega) \).

**11.2 Labelling \( F_4(q) \)**

Here we assume that \( G \cong G(q) = F_4(q) \). In Section 7 we constructed basic \( SL_2 \) subgroups \( K_1, \ldots, K_4 \) of \( G \).

1. Working in \( \langle K_1, K_2 \rangle \cong SL_3(q) \), label \( x_{\pm 1}(c_i), x_{\pm 2}(c_i) \) and \( h_1(\omega), h_2(\omega) \) as in Step 1 of Section 11.1.

2. Construct an isomorphism \( \psi \) from \( \langle K_2, K_3 \rangle \) to \( Sp_4(q) = Sp(V) \), and let \((, )\) be the associated symplectic form on \( V \). Let \( U = C_V(K_2 \psi) \), \( W = [V, K_2 \psi] \) and choose \( e_1, f_1 \in V \) such that \( e_1 \in C_W(x_{-2}(1) \psi), f_1 \in C_W(x_2(1) \psi) \) and \((e_1, f_1) = 1\). Let \( X = \langle e_1^{K_3} \psi \rangle, Y = \langle f_1^{K_3} \psi \rangle \), so that \( X \) and \( Y \) are 2-spaces with \( V = X \oplus Y \). Choose \( e_2 \in U \cap X, f_2 \in U \cap Y \) such that \((e_2, f_2) = 1\). Write all matrices relative to the basis \( e_1, e_2, f_2, f_1 \) of \( V \).

3. Compute \( \lambda_i, \mu_i \) such that

\[ x_{2}(c_i) \psi = \begin{pmatrix} 1 & 0 & 0 & \lambda_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, x_{-2}(c_i) \psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mu_i & 0 & 0 & 1 \end{pmatrix}, \]

and define

\[ x_{3}(c_i) = \psi^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda_i & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda_i & 1 \end{pmatrix}, x_{-3}(c_i) = \psi^{-1} \begin{pmatrix} 1 & \mu_i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\mu_i \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Compute \( \lambda \) such that \( h_2(\omega) \psi = (\lambda^{-1}, 1, 1, \lambda) \), and define \( h_3(\omega) = \psi^{-1}(\lambda, \lambda^{-1}, \lambda, \lambda^{-1}) \).

4. Working in \( \langle K_3, K_4 \rangle \cong SL_3(q) \), label \( x_{\pm 4}(c_i) \) and \( h_4(\omega) \) as in Step 2 of Section 11.1.

Observe that the choice of basis in Step 2 ensures that the root elements \( x_{\pm 2}(c_i) \psi, x_{\pm 3}(c_i) \psi \) are as claimed in Step 3.
11.3 Labelling \( G_2(q) \) and \( ^3D_4(q) \)

Here we assume that \( G \cong G(q) \) is isomorphic to either \( G_2(q) \) or \( ^3D_4(q) \). In Sections 8 and 10 we constructed basic \( SL_2 \) subgroups \( K_0, K_1, K_2 \) of \( G \), with \( K_0, K_1 \cong SL_2(q) \) and \( K_2 \cong SL_2(q^3) \).

1. Working in \( \langle K_0, K_1 \rangle \cong SL_3(q) \), label \( x_{\pm 0}(c_i), x_{\pm 1}(c_i) \) and \( h_0(\omega), h_1(\omega) \) as in Step 1 of Section 11.1.

2. Construct an isomorphism \( \psi \) from \( K_2 \) to \( SL(V) = SL_2(q) \) or \( SL_2(q^3) \). Compute \( T \in SL(V) \) such that \( (x^{h_1(\omega)})\psi = (x\psi)^T \) for all \( x \in K_2 \). Choose a basis of \( V \) consisting of eigenvectors of \( T \), and write all matrices relative to this basis.

3. Let \( q = p^a \). For \( q > 3 \) let \( \Lambda = \{ \omega^{\pm p^j} : 0 \leq j \leq a - 1 \} \), a subset of \( \mathbb{F}_q \) of size \( 2\log_q p \). For \( \lambda \in \Lambda \) define \( d_2(\lambda) = \psi^{-1}(\lambda^{-1}, \lambda) \). Find \( \lambda \) such that \( h_1(\omega)^2d_2(\lambda) \in K_0 \) and define
\[
h_2(\omega) = d_2(\lambda).
\]

4. If \( q > 4 \), then \( \lambda = \omega^{\epsilon p^j} \) where \( \epsilon = \pm 1 \). Define
\[
y_+ = \psi^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y_- = \psi^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]
Find \( \delta = \pm \) such that \( \langle x_1(1)^{K_2}, y_0^{K_0}, K_1 \rangle \subset G \). (For the opposite \( \delta \) this subgroup is \( G \).) Now \( \delta = \epsilon \), except possibly for \( q = 5 \) or 9. For \( q = 5 \) or 9, if \( \delta \neq \epsilon \), then replace \( h_2(\omega) \) by \( h_2(-\omega) \) or \( h_2(\omega) \) (respectively), \( \epsilon \) by \( -\epsilon \), and \( j \) by 0 or 1 (respectively).

If \( q = 4 \), find \( \delta \) as for \( q > 4 \); compute \( i \in \{ 1, 2 \} \) such that \( \lambda = \omega^i \), and choose \( j \in \{ 0, 1 \} \) such that \( \delta = (-1)^{i+j+1} \). If \( q = 3 \), set \( \delta = \epsilon = + \) and \( j = 0 \).

5. Let \( c_i \) \((1 \leq i \leq a)\) be an \( \mathbb{F}_q \)-basis of \( E \) where \( E = \mathbb{F}_q \) (or \( \mathbb{F}_q^3 \) for \( ^3D_4(q) \)), and let \( \nu \) be a primitive element of \( E \) (so \( \nu = \omega \) if \( E = \mathbb{F}_q \)). If \( \delta = + \), then define \( h_2(\nu) = \psi^{-1}(\nu^{-p^j}, \nu^p) \) and
\[
x_2(c_i) = \psi^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_{-2}(c_i) = \psi^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]
If \( \delta = - \), then define \( h_2(\nu) = \psi^{-1}(\nu^p, \nu^{-p}) \) and
\[
x_2(c_i) = \psi^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad x_{-2}(c_i) = \psi^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

We now justify the above labelling. In Step 2, the computation of the matrix \( T \) involves solving linear equations of the form \( TA = BT \), where \( A = (x^{h_1(\omega)})\psi, B = x\psi \), for generators \( x \) of \( K_2 \); this system can be solved in polynomial time. In Step 3, \( \Lambda \) is introduced because the isomorphism \( \psi \) could change eigenvalues of elements of \( K_2 \) by a field automorphism or inversion. We define \( h_2(\omega) \) as in Step 3 to ensure that \( h_1(\omega)^2h_2(\omega) = h_{2a+3\beta}(\omega) \in K_0 = \langle x_{\pm 0}(c) : c \in \mathbb{F}_q \rangle \). Membership of \( h_1(\omega)^2d_2(\lambda) \) in \( K_0 \) can be decided readily: \( K_0 = C_G(K_2) \), so we simply check whether the given element commutes with the generators of \( K_2 \). For \( q \) odd, \( h_1(-\omega)^2h_2(-\omega) \in K_0 \);
this causes a complication only for \( q = 5 \) or \( 9 \): if \( q \) is not one of these values, then \(-\omega \not\in \Lambda\), so Step 3 defines \( h_2(\omega) \) uniquely. However, if \( q = 5 \) or \( 9 \) then \(-\omega = \omega^{-1} \) or \( \omega^{-3} \) respectively, so we may need to replace \( h_2(\omega) \) by \( h_2(-\omega) \), \( \epsilon \) by \(-\epsilon\), and \( j \) by \( 0 \) or \( 1 \), as indicated in Step 4. Similar observations apply for \( q \) even: non-uniqueness occurs only for \( q = 4 \) when \( \omega^2 = \omega^{-1} \). In Step 4, the \( y_\pm \) are elements of a root group normalized by \( h_1(\omega) \) (this is the reason for Step 2), hence can be taken as root elements \( x_{\pm 2}(1) \). The choice of \( \delta \) ensures that \( y_3 \) is \( x_{+2}(1) \) rather than the negative, hence justifying the parametrization of root elements in Step 5.

11.4 Labelling \( ^2E_6(q) \)

Here we assume that \( G \cong G(q) = ^2E_6(q) \). In Section 9 we constructed basic \( SL_2 \) subgroups \( K_1, \ldots, K_4 \) of \( G \), with \( K_1, K_2 \cong SL_2(q) \) and \( K_3, K_4 \cong SL_2(q^2) \).

1. Construct an isomorphism \( \phi \) from \( \langle K_2, K_3 \rangle \) to \( SU_4(q) = SU(V) \), let \( (, ) \) be the associated hermitian form on \( V \), and write \( \alpha = \alpha^q \) for \( \alpha \in \mathbb{F}_q \). Let \( U = C_V(K_2\phi), W = [V, K_2\phi] \). Write \( V \downarrow K_3\phi = X \oplus Y \) with \( X, Y \) 2-spaces. Choose \( e_1 \in X \cap W, e_2 \in X \cap U, f_1 \in Y \cap W, f_2 \in Y \cap U \) such that \( (e_1, f_1) = (e_2, f_2) = \lambda \), where \( \lambda \) is a fixed element of \( \mathbb{F}_q^2 \) such that \( \lambda + \bar{\lambda} = 0 \).

Write all matrices with respect to the basis \( e_1, e_2, f_2, f_1 \) of \( V \).

2. Let \( c_i \) \( (1 \leq i \leq a) \) be a \( \mathbb{F}_p \)-basis of \( \mathbb{F}_q \), and extend it to a basis \( d_i \) \( (1 \leq i \leq 2a) \) of \( \mathbb{F}_q^2 \) over \( \mathbb{F}_p \). Let \( \omega, \nu \) be primitive elements of \( \mathbb{F}_q, \mathbb{F}_q^2 \), respectively. For each \( i \), define

\[
\begin{align*}
x_2(c_i) &= \phi^{-1} \begin{pmatrix} 1 & 0 & 0 & c_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & x_{-2}(c_i) &= \phi^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c_i & 0 & 0 & 1 \end{pmatrix}, \\
x_3(d_i) &= \phi^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ d_i & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -d_i & 1 \end{pmatrix}, & x_{-3}(d_i) &= \phi^{-1} \begin{pmatrix} 1 & d_i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_i \\ 0 & 0 & 0 & 1 \end{pmatrix},
\end{align*}
\]

and set \( h_2(\omega) = \phi^{-1}(\omega^{-1}, 1, 1, \omega), h_3(\nu) = \phi^{-1}(\nu, \nu^{-1}, \nu, -\bar{\nu}^{-1}) \).

3. Working in \( \langle K_1, K_2 \rangle \cong SL_3(q) \), label \( x_{\pm 1}(c_i) \) and \( h_1(\omega) \) as in Step 2 of Section 11.1.

4. Working in \( \langle K_3, K_4 \rangle \cong SL_3(q^2) \) or \( PSL_3(q^2) \), label \( x_{\pm 4}(d_i) \) and \( h_4(\nu) \) as in Step 2 of Section 11.1.

12 Determining the high weight of a representation

Let \( G \) be an absolutely irreducible subgroup of \( GL_d(F) \) that is isomorphic to a quasisimple exceptional group \( G(q) \) of Lie type over \( \mathbb{F}_q \), where \( F \) and \( \mathbb{F}_q \) have the same characteristic. Assume also that \( G(q) \) is neither a Suzuki nor a Ree group.

Write \( V = V_d(F) \). In this section we describe a simple algorithm to compute the high weight of the absolutely irreducible \( FG \)-module \( V \). That is, we compute the non-negative integers \( n_r (1 \leq r \leq l) \) such that \( V = V(\lambda) \), the irreducible module of
high weight \( \lambda = \sum_{r=1}^{l} n_r \lambda_r \), where \( l \) is the rank of the corresponding simple algebraic group and \( \lambda_r \) are the fundamental dominant weights. Unlike previous sections, the algorithm applies for all values of \( q \) including 2.

First consider the case where \( G(q) \) is of untwisted type. The algorithm is the following. Using the work of previous sections, construct the root and toral elements \( x_{\pm r}(c_i), h_r(\omega) \) of \( G \). Construct the maximal unipotent subgroup \( U \) generated by all the positive root elements \( x_r(c_i) \). (For \( F_4(2) \) additional generators are required – see Section 15.1.) A consequence of [27, Theorem 4.3(c)] is that \( C_V(U) \) is a 1-dimensional space, spanned by a maximal vector \( v \). Since the \( h_r(\omega) \) normalize \( U \), they fix \( C_G(U) \). Thus, for each \( r \in \{1, \ldots, l\} \), there exists \( n_r \in \{0, \ldots, q-1\} \) such that \( \omega^{n_r} \in F \) and

\[
v^{h_r(\omega)} = \omega^{n_r} v.
\]

These are the required integers \( n_r \); to compute them, use a discrete log oracle in \( F_q \). The only ambiguity occurs when \( v^{h_r(\omega)} = v \), in which case \( n_r \) can be 0 or \( q-1 \). To distinguish between them, compute the spin \( \langle v^{K_r} \rangle \) of \( v \) under \( K_r \): if this is a 1-dimensional (trivial) module for \( K_r \), then \( n_r = 0 \); otherwise \( n_r = q-1 \).

Now consider the twisted groups. For \( ^2E_6(q) \), as in Section 11.4, compute the root elements \( x_{\pm r}(c_i) \), and also the toral elements \( h_1(\omega), h_2(\omega), h_3(\nu), h_4(\nu) \), where \( \omega \) and \( \nu \) are primitive elements for \( F_q \) and \( F_q^2 \) respectively. Construct the maximal unipotent group \( U \) generated by the positive root elements \( x_r(c_i) \), and compute \( C_V(U) = \langle v \rangle \). Using a discrete log oracle, find \( 0 \leq a, b, c, d, e, f \leq q-1 \) such that

\[
v^{h_1(\omega)} = \omega^a v, \ v^{h_2(\omega)} = \omega^b v, \ v^{h_3(\nu)} = \nu^c + dq v, \ v^{h_4(\nu)} = \nu^e + fq v.
\]

The high weight of \( V \) relative to the \( E_6 \) Dynkin diagram is

\[
e c b d f
\]

\[
a
\]

Similarly, for \( ^3D_4(q) \), compute \( 0 \leq a, b, c, d \leq q-1 \) such that

\[
v^{h_1(\omega)} = \omega^a v, \ v^{h_2(\nu)} = \nu^{b+cq+4dq^2} v
\]

where \( \nu \) is now a primitive element for \( F_q^3 \). The high weight of \( V \) relative to the \( D_4 \) Dynkin diagram is \( bacd \). In both twisted cases, we distinguish between the possibilities 0 and \( q-1 \) as in the untwisted case.

Since the labelling of the root and toral elements is only determined up to an automorphism of \( G \), the same is true of the high weight.

We have now justified the following result.

**Proposition 12.1** Subject to the availability of a discrete log oracle, the above algorithm determines in polynomial time the high weight of the absolutely irreducible \( FG \)-module \( V \), up to a twist by a field or graph automorphism of \( G \).

## 13 Constructing the standard generators

Assume \( G \) is described by a collection of generators in \( GL_d(F) \), where \( F \) is a finite field of the same characteristic as \( F_q \), and \( G \) is isomorphic to an exceptional group \( G(q) \) of Lie type over \( F_q \). Assume also that \( G(q) \) is neither a Suzuki nor a Ree group.
In previous sections we showed how to construct a family of basic $SL_2$ subgroups $K_r$ of $G$ as in the Dynkin diagram, and how to label root elements $x_{\pm r}(c_i)$ and toral elements $h_r(\omega)$ in each $K_r$.

In this section, we use commutators among these root elements to construct additional root elements in rank 2 subsystems, guided by the Chevalley commutator relations [20, 5.2.2]. The root elements constructed in $G$ correspond to the generators of the reduced Curtis-Steinberg-Tits presentation for $G(q)$ as in [7, §4.2 and 6.1]: namely, the standard generators of $G$.

We list these presentations on standard generators explicitly in Appendix A. They are used to verify the correctness of the output of the algorithms: namely, the elements $x_{\pm r}(c_i)$ and $h_r(\omega)$.

We summarise the result of this section.

**Proposition 13.1** Let $G$ be a subgroup of $GL_d(F)$, where $F$ is a finite field of the same characteristic as $\mathbb{F}_q$ and $q > 2$, and assume that $G \cong G(q)$, a quasisimple group of exceptional Lie type over $\mathbb{F}_q$ which is neither a Suzuki nor a Ree group. Assume also that generators are given for a family of basic $SL_2$ subgroups of $G$ as in the Dynkin diagram. Subject to the availability of a discrete log oracle, there is a Las Vegas polynomial-time algorithm to construct the standard generators of $G$.

The proposition is justified in the following sections.

### 13.1 Standard generators of $E_6(q)$, $E_7(q)$ and $E_8(q)$

These are the most straight-forward cases. Let $l$ be the rank of $G(q)$ (so $l = 6, 7$ or 8). From Proposition 11.1 we know fundamental root elements $x_{\pm r}(c_i) \in G$ for $1 \leq r \leq l$ and $c_i$ in an $\mathbb{F}_p$-basis of $\mathbb{F}_q$. For each edge $r, s$ in the Dynkin diagram with $r < s$, define additional root elements $x_{\pm rs}(c_i)$ by

$$x_{rs}(c_i) = [x_r(c_i), x_s(1)], \quad x_{-rs}(c_i) = [x_{-r}(c_i), x_{-s}(-1)].$$

The reduced Curtis-Steinberg-Tits presentation has generators $x_{\pm r}(c_i)$, $x_{\pm rs}(c_i)$ for all relevant $r, s, i$, the relations being the Chevalley commutator relations among these elements, together with the relations expressing that all generators have order $p$. This presentation defines the simply connected group $E_l(q)$. We give an explicit version in Appendix A.1.

If we require a presentation for the simple group $G/Z(G)$, then an additional relation may be needed to kill the centre. This only applies for $l = 6$ or 7, as the simply connected group $E_8(q)$ is simple. We know the toral elements $h_r(\omega) \in K_r$. If $Z(G) \neq 1$, then $q - 1$ is divisible by 3 if $G(q) = E_6(q)$; or by 2 if $G(q) = E_7(q)$; and $Z(G) = \langle z \rangle$ where

$$z = \begin{cases} h_1(\lambda^2)h_3(\lambda)h_5(\lambda^2)h_6(\lambda), & \text{if } G(q) = E_6(q) \\ h_2(-1)h_5(-1)h_7(-1), & \text{if } G(q) = E_7(q) \end{cases}$$

and $\lambda$ is a cube root of unity. Each $h_r(\omega)$ can be expressed in terms of the generators $x_{\pm r}(c_i)$ using the expression

$$h_r(\omega) = n_r(\omega^{-1})n_r(1)^{-1},$$

where $n_r(c) := x_r(c)x_{-r}(-c^{-1})x_r(c)$. Hence the relation $z = 1$, where $z$ is as above, completes a presentation of the simple group $G/Z(G)$.
13.2 Standard generators of $F_4(q)$

Suppose $G(q) = F_4(q)$. From Proposition 11.1 we know fundamental root elements $x_{\pm r}(c_i) \in G$ for $1 \leq r \leq 4$ and $c_i$ in an $F_p$-basis of $F_q$. We define additional root elements as follows. For $1 \leq r \leq 4$, let

$$n_r = x_r(1)x_{-r}(-1)x_r(1).$$

Now define

$$x_{12}(c_i) = x_1(c_i)^{n_2}, \quad x_{-12}(c_i) = x_{-1}(c_i)^{n_2},$$
$$x_{23}(c_i) = x_3(-c_i)^{n_2}, \quad x_{-23}(c_i) = x_{-3}(c_i)^{n_2},$$
$$x_{34}(c_i) = x_3(c_i)^{n_4}, \quad x_{-34}(c_i) = x_{-3}(c_i)^{n_4},$$
$$x_{23^2}(c_i) = x_2(c_i)^{n_3}, \quad x_{-23^2}(c_i) = x_{-2}(c_i)^{n_3}$$

(where $23^2$ denotes the root $\alpha_2 + 2\alpha_3$ and so on). For the definition of $x_{\pm 23^2}(c_i)$ we have used the $F_4$ structure constants in [31].

This defines all the root elements in rank 2 subsystems. The reduced Curtis-Steinberg-Tits presentation of $G$ defines the simple group $G \cong F_4(q)$, since this group is simply connected. We give an explicit version in Appendix A.2.

13.3 Standard generators of $^2E_6(q)$

Suppose $G(q) = ^2E_6(q)$. This is very similar to the $F_4(q)$ case, except that for short roots we define root elements over $F_{q^2}$ rather than $F_q$. From Proposition 11.1 we know fundamental root elements $x_{\pm r}(c_i)$ for $r = 1, 2$ and $x_{\pm s}(d_i)$ for $s = 3, 4$, where $c_i$ and $d_i$ run over bases for $F_q$ and $F_{q^2}$ over $F_p$, respectively. We define additional root elements $x_{\pm 12}(c_i), x_{\pm 23}(d_i), x_{\pm 34}(d_i), x_{\pm 23^2}(c_i)$ using exactly the same equations as in Section 13.2 for $F_4(q)$.

This defines all the root elements in rank 2 subsystems. We give an explicit version of the reduced Curtis-Steinberg-Tits presentation of the simply connected version of $G$ in Appendix A.3. This is a variant of the presentation given in [7, §6.1]. To get a presentation for the simple group $G/Z(G)$, we add the relation $z = 1$, where $z = h_3(\lambda)h_5(\lambda^2)$; the $h_r$ are expressed in terms of $x_{\pm r}(c)$ as in Section 13.1, and $\lambda$ is a cube root of unity.

13.4 Standard generators of $G_2(q)$

Suppose $G(q) = G_2(q)$. From Proposition 11.1 we know fundamental root elements $x_{\pm r}(c_i)$ in $G$ for $r = 1, 2$. It is convenient to change notation at this point. Let $\alpha, \beta$ be fundamental roots in the $G_2$ root system with $\alpha$ long, $\beta$ short, and write $x_{\pm 1}(c_i) = x_{\pm \alpha}(c_i), x_{\pm 2}(c_i) = x_{\pm \beta}(c_i)$. We define additional root elements as follows. Let

$$n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1), \quad n_\beta = x_\beta(1)x_{-\beta}(-1)x_\beta(1).$$

Now define

$$x_{\alpha+\beta}(c_i) = x_\beta(-c_i)^{n_\alpha}, \quad x_{-\alpha-\beta}(c_i) = x_\beta(-c_i)^{n_\alpha},$$
$$x_{\alpha+2\beta}(c_i) = x_{\alpha+\beta}(c_i)^{n_\beta}, \quad x_{-\alpha-2\beta}(c_i) = x_{-\alpha-\beta}(c_i)^{n_\beta},$$
$$x_{\alpha+3\beta}(c_i) = x_\alpha(c_i)^{n_\beta}, \quad x_{-\alpha-3\beta}(c_i) = x_{-\alpha}(c_i)^{n_\beta},$$
$$x_{2\alpha+3\beta}(c_i) = x_{\alpha+3\beta}(-c_i)^{n_\alpha}, \quad x_{-2\alpha-3\beta}(c_i) = x_{-\alpha-3\beta}(-c_i)^{n_\alpha}$$
This defines all the root elements. The reduced Curtis-Steinberg-Tits presentation of $G$ defines the simple group $G \cong G_2(q)$, since this group is simply connected. We give an explicit version in Appendix A.4.

### 13.5 Standard generators of $^3D_4(q)$

Suppose $G(q) = ^3D_4(q)$. This is similar to $G_2(q)$, except that for short roots we define root elements over $\mathbb{F}_{q^3}$ rather than $\mathbb{F}_q$. From Proposition 11.1 we know root elements $x_{\pm 1}(c_i)$ for $c_i$ in an $\mathbb{F}_q$-basis of $\mathbb{F}_q$, and $x_{\pm 2}(d_i)$ for $d_i$ in an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^3}$. As for $G_2(q)$ in Section 13.4, relabel these as $x_{\pm \alpha}(c_i)$, $x_{\pm \beta}(d_i)$ respectively. We define additional root elements $x_{\pm (\alpha + \beta)}(d_i)$, $x_{\pm (\alpha + 2\beta)}(d_i)$, $x_{\pm (\alpha + 3\beta)}(c_i)$, $x_{\pm (2\alpha + 3\beta)}(c_i)$ using exactly the same equations as in Section 13.4 for $G_2(q)$.

This defines all the root elements. The reduced Curtis-Steinberg-Tits presentation of $G$ defines the simple group $G \cong ^3D_4(q)$, since this group is simply connected. We give an explicit version in Appendix A.5.

### 14 Completion of proof of Theorem 1

Theorem 1 is an immediate consequence of the results summarised in Section 2, and of the algorithms presented and justified in Sections 5–13. Babai et al. [7, Corollary 4.4] prove that the reduced Curtis-Steinberg-Tits presentation for a universal Chevalley group $G$ of rank at least 2 has length $O(\log^3 |G|)$, so evaluation of the relations takes polynomial time. That the resulting constructive recognition algorithm is Las Vegas is established by verifying that the standard generators satisfy these presentations, which are given explicitly in Appendix A.

### 15 Algorithms for $q = 2$

Our algorithms to construct basic $SL_2$ subgroups fail when $q = 2$: the critical elements $v_1$ and $v_2$ constructed in Step 4 of Section 6.1 are now both the identity; the algorithms to construct standard generators in Section 13 also fail in some cases.

Since it is desirable to have practical recognition algorithms for exceptional groups over $\mathbb{F}_2$, we now provide such. We often exploit the fact that explicit computations can be performed readily in some of their subgroups using standard machinery; for details of such, see, for example, [34, Chapter 4]. We omit $G_2(2)$ since it is isomorphic to the almost simple classical group $U_3(3).2$.

#### 15.1 $E_6(2), F_4(2)$ and $^2E_6(2)$

Assume $G$ is isomorphic to one of $E_6(2), F_4(2)$ or $^2E_6(2)$.

1. Apply Steps 1-4 of Sections 6.1, 7.2 and 9.2. These construct basic $SL_2$ subgroups $K_0, K_i$ where $i = 2$ for $E_6(2)$ and $i = 1$ for $F_4(2)$ and $^2E_6(2)$. They also find a root involution $u := u_1^+ \in K_0$.

2. Construct $C_G(u) = QD$, where $D = C_G(K_0)$ and $Q$ is a normal 2-subgroup.
3. Construct $Q$, the soluble radical of $C_G(u)$. Now construct $D$ as follows. Find involutions $s \in D \setminus Q$ such that $|Q : C_Q(s)| \leq 2^{12}$. Note that $C_Q(s)$ can be computed from the action of $s$ on the vector space $Q/Z(Q)$ over $\mathbb{F}_2$. Search for sufficient $Q$-conjugates of $s$ lying in $C_G(K_0)$ to generate $D$.

4. Find an involution $t \in D$ such that $\langle K_i, t \rangle \cong SL_3(2)$. Compute $D \cap \langle K_i, t \rangle \cong SL_2(2)$, and call it $K_j$, where $j = 4$ for $G \cong E_6(2)$ and $j = 2$ otherwise.

5. Construct $T := C_D(K_i)$ which is isomorphic to $SL_3(2)^2$, $SL_3(2)$ or $SL_3(4)$ for $G \cong E_6(2)$, $F_4(2)$ or $2^2E_6(2)$ respectively.

6. Compute $C_T(K_j)$ which is isomorphic to $SL_2(2)^2$, $SL_2(2)$ or $SL_2(4)$ respectively. Define its $SL_2$ factors to be $K_1K_6$, $K_4$ or $K_4$, respectively.

7. Search in $T$ for the remaining basic $SL_2$ subgroups $K_3K_5 \cong SL_2(2)^2$, $K_3 \cong SL_2(2)$ or $K_3 \cong SL_2(4)$. We now know all the basic $SL_2$ subgroups in $G$.

8. The labelling of fundamental root elements is carried out as in Section 11.

9. The construction of the standard generators is as in Sections 13.1, 13.2 and 13.3.

### 15.2 $E_7(2)$

Assume $G \cong E_7(2)$.

1. Construct $K_0$, $K_1$ and $D = C_G(K_0)$ as in the previous section. Find an involution $t \in D$ such that $\langle K_1, t \rangle \cong SL_3(2)$, and compute $K_3 = D \cap \langle K_1, t \rangle$.

2. Step 5 of the previous section is too expensive to apply in $D \cong \Omega^{+}_{12}(2)$. Instead, we first compute $C_D(K_3) \cong SL_2(2) \times \Omega^+_8(2)$. Name the direct factors as $K_2$ and $E$ respectively.

3. Compute $C_E(K_1) \cong SL_4(2)$, and construct $K_5, K_6, K_7$, basic $SL_2$ subgroups in $C_E(K_1)$.

4. In $C_D(K_6, K_7) \cong SL_4(2)$, search for the remaining basic $SL_2$ subgroup $K_4 \cong SL_2(2)$, satisfying $[K_1, K_4] = 1$ and $\langle K_4, K_i \rangle \cong SL_3(2)$ for $i = 2, 3, 5$.

We now know all of the basic $SL_2$ subgroups in $G$. The labelling of fundamental root elements and the construction of standard generators is unchanged from Sections 11.1 and 13.1.

### 15.3 $E_8(2)$

An approach modelled on the previous algorithms is too expensive to apply to $E_8(2)$. Instead, we present a different algorithm which recognises the group only in its 248-dimensional adjoint representation.

Assume that $G \leq GL_{248}(F)$, where $F$ is a finite field of characteristic 2, and $G \cong E_8(2)$.

1. Using [32], find a basis of $V = V_{248}(F)$ with respect to which the action of $G$ is realised over $\mathbb{F}_2$. Replace $V$ by the $\mathbb{F}_2$-span of this basis, and $F$ by $\mathbb{F}_2$.  

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2. Construct $K_0$ and $D = C_G(K_0) \cong E_7(2)$ as in Steps 1-3 of Section 15.1.

3. Apply the algorithm of Section 15.2 to construct basic $SL_2$ subgroups and root elements $x_{\pm r}(1) \ (1 \leq r \leq 7)$ in $D$. Let $x_{\pm 0}(1)$ be two involutions generating $K_0$.

4. Let $\hat{G}$ be the standard copy of $E_8(2)$ in $GL_{248}(2)$, with fundamental root element generators $\hat{x}_{\pm r}(1) \ (1 \leq r \leq 8)$. Let $\hat{x}_{\pm 0}(1)$ be root elements in $\hat{G}$ corresponding to the longest root $\alpha_0$ in the root system.

5. Compute all matrices $g \in GL_{248}(2)$ such that $x_{\pm r}(1)^g = \hat{x}_{\pm r}(1)$ for $0 \leq r \leq 7$. There are precisely two such matrices. To see this, observe that $\langle x_{\pm r}(1) : 0 \leq r \leq 7 \rangle = DK_0$, so any two such matrices $g$ differ by an element of $C_{GL(V)}(DK_0)$. Now $V \downarrow DK_0 = W_1 \oplus W_2 \oplus W_3$, a sum of indecomposables of dimensions 2, 112 and 134. Here $W_1$ and $W_2$ are irreducible and $W_3$ is uniserial with socle series having irreducible factors of dimensions 1, 132, 1. Let $M$ be the maximal submodule of $W_3$, and let $\langle s \rangle$ be the socle of $M$. Then $\text{Hom}_{DK_0}(V,V)$ has dimension 4, with basis $1, \pi_1, \pi_2, \phi$ where $\pi_i$ is the projection $V \to W_i$ and $\phi$ sends $w_1 + w_2 + w_3$ to 0 if $w_3 \in M$ and to $s$ if $w_3 \notin M$. The only invertible elements of this space are 1 and $1 + \phi$. Hence, as asserted, there are exactly two such matrices $g$. Call them $g_1, g_2$.

6. For $i = 1, 2$ define $x_{\pm 8}^{(i)} = \hat{x}_{\pm 8}(1)^{g_i^{-1}}$. Decide for which value of $i$ the group $\langle D, x_{\pm 8}^{(i)} \rangle$ is isomorphic to $E_8(2)$; the other is a “large” subgroup of $GL_{248}(2)$ containing elements of order much larger than those of $E_8(2)$. For this value of $i$, define $x_{\pm 8}(1) = x_{\pm 8}^{(i)}$.

We have now labelled all fundamental root elements $x_{\pm r}(1) \ (1 \leq r \leq 8)$ in $G$. The construction of the standard generators is unchanged from Section 13.

15.4 $3D_4(2)$

Assume $G \cong 3D_4(2)$. Let $\alpha, \beta$ be fundamental long and short roots in the root system, as in Section 13.4.

1. Construct subgroups $K_0, K_1 \cong SL_2(2)$ and $K_2 \cong SL_2(8)$ as in Section 10.2.

2. Construct an isomorphism from $K_2$ to $SL_2(8)$. In $K_2$, define $x_\beta(c), x_{-\beta}(c)$ (for $c \in \mathbb{F}_8$) to be the preimages of \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ respectively.

3. Let $X_{\pm \beta} = \{ x_{\pm \beta}(c) : c \in \mathbb{F}_8 \}$. Find $g \in K_2$ and involutions $x_\epsilon \in K_1 \ (\epsilon = \pm)$ such that each of $\langle x_{\epsilon K_2}, X_{\beta g}^g, K_1 \rangle$ is a proper subgroup of $G$. Define $x_{\pm \alpha}(1) = x_{\pm}$ and replace $x_{\pm \beta}(c)$ by $x_{\pm \beta}(c)^g$. Now we have labelled the fundamental root elements of $G$.

4. Construct the remaining standard generators of $G$ as in Section 13.5.
16 Implementation and performance

We have implemented these algorithms in MAGMA. We use the product replacement algorithm [21] to generate random elements; our implementations of [14], [17], [26], [30], and [49]; and Brooksbank’s implementations of his algorithm [19] for constructive recognition of $Sp_4(q)$.

The computations reported in Table 3 were carried out using MAGMA V2.19 on a 2.8 GHz processor. We list the CPU time $t_1$ in seconds taken to construct standard generators in a random conjugate of the standard copy of dimension $d_1$ of an exceptional group of type $G(q)$; sometimes, we list $t_2$, the time taken to perform the same task in an irreducible representation of dimension $d_2$. The time is averaged over three runs.

We use Taylor’s implementation of [22,23] to write an element of $G(q)$ as a word in the standard generators. As one illustration, it takes 17 seconds to write an element of $E_8(5^2)$ as a word in its standard generators.

| Group      | $d_1$ | $t_1$ | $d_2$ | $t_2$ |
|------------|-------|-------|-------|-------|
| $E_6(2^3)$ | 27    | 8     | 78    | 51    |
| $E_6(5^2)$ | 27    | 15    | 78    | 119   |
| $E_7(2^4)$ | 56    | 35    | 133   | 158   |
| $E_7(5^2)$ | 56    | 53    | 133   | 301   |
| $E_8(2^3)$ | 248   | 978   | –     | –     |
| $E_8(5^2)$ | 248   | 520   | –     | –     |
| $F_4(2)$   | 26    | 11    | 246   | 235   |
| $F_4(2^3)$ | 26    | 14    | 246   | 607   |
| $F_4(5^3)$ | 26    | 30    | 52    | 248   |
| $G_2(2^3)$ | 6     | 1     | 14    | 2     |
| $G_2(5^3)$ | 7     | 2     | 14    | 3     |
| $^2E_6(2^3)$ | 27 | 99 | 78 | 790 |
| $^2E_6(5^2)$ | 27 | 102 | 78 | 865 |
| $^3D_4(2^6)$ | 8 | 15 | 26 | 122 |
| $^3D_4(5^5)$ | 8 | 6 | 28 | 90 |

Table 3: Time to construct standard generators

A Reduced Curtis-Steinberg-Tits presentations

The Curtis-Steinberg-Tits presentations are well known; the reduced versions using only an $\mathbb{F}_p$-basis of the field $\mathbb{F}_q$ (and extensions) are described in [7]. Since we know of no explicit versions listing the constants in the Chevalley relations, which we need for our work, we include such here. The constants are calculated using [20, 5.2.2] together with the $N_{\alpha\beta}$ structure constants for the $G_2$ and $F_4$ Lie algebras from [31].

In all cases, the generators are the root elements we have constructed in Section 13, namely the elements $x_r(c_i)$ for roots $r$ in subsystems spanned by two non-orthogonal fundamental roots, and elements $c_i$ in an $\mathbb{F}_p$-basis of $\mathbb{F}_q$ (or an extension
field). In every case, the presentation contains the following relations:

\[ x_r(c_i)^p = 1, \]
\[ [x_r(c_i), x_s(d_i)] = 1 \text{ if } r + s \text{ is not a root}. \]

For \( c = \sum k_i c_i \in \mathbb{F}_q \) (or an extension field) with \( k_i \in \mathbb{F}_p \), we set \( x_r(c) = \prod x_r(c_i)^{k_i} \).

We present the remaining relations for each type below.

### A.1 Relations for \( E_6(q), E_7(q), E_8(q) \)

The relations for these types are simple: for each edge \( rs \) in the Dynkin diagram with \( r < s \), and for \( c_i, d_i \) in the \( \mathbb{F}_p \)-basis of \( \mathbb{F}_q \),

1. \( [x_r(c_i), x_s(d_i)] = x_{rs}(c_id_i) \)
2. \( [x_{r}(c_i), x_{s}(d_i)] = x_{rs}(-c_id_i) \)
3. \( [x_{r}(c_i), x_{rs}(d_i)] = x_{s}(-c_id_i) \)
4. \( [x_{s}(c_i), x_{rs}(d_i)] = x_{r}(c_id_i) \)
5. \( [x_{r}(c_i), x_{rs}(d_i)] = x_{s}(c_id_i) \)
6. \( [x_{s}(c_i), x_{rs}(d_i)] = x_{r}(-c_id_i) \)

### A.2 Relations for \( F_4(q) \)

For the edges 12 and 34 in the Dynkin diagram of \( F_4 \) we have the relations (1)-(4) of the previous section. The remaining relations are the following:

1. \( [x_2(c_i), x_3(d_i)] = x_{23}(c_id_i)x_{232}(c_i d_i^2) \)
2. \( [x_2(c_i), x_{23}(d_i)] = x_{3}(-c_id_i)x_{-232}(-c_i d_i^2) \)
3. \( [x_{-2}(c_i), x_{23}(d_i)] = x_{3}(c_id_i)x_{232}(-c_i d_i^2) \)
4. \( [x_{-2}(c_i), x_{-3}(d_i)] = x_{23}(-c_id_i)x_{-232}(c_id_i^2) \)
5. \( [x_{232}(c_i), x_{-3}(d_i)] = x_{23}(c_id_i)x_{2}(c_i d_i^2) \)
6. \( [x_{232}(c_i), x_{-23}(d_i)] = x_{3}(-c_id_i)x_{-2}(-c_i d_i^2) \)
7. \( [x_{-232}(c_i), x_{23}(d_i)] = x_{3}(c_id_i)x_{2}(-c_i d_i^2) \)
8. \( [x_{-232}(c_i), x_{3}(d_i)] = x_{23}(-c_id_i)x_{2}(-c_i d_i^2) \)
9. \( [x_{23}(c_i), x_{3}(d_i)] = x_{232}(2c_id_i) \)
10. \( [x_{-23}(c_i), x_{-3}(d_i)] = x_{-232}(-2c_id_i) \)
11. \( [x_{23}(c_i), x_{-3}(d_i)] = x_{2}(2c_id_i) \)
12. \( [x_{3}(c_i), x_{-23}(d_i)] = x_{-2}(2c_id_i) \)

### A.3 Relations for \( ^2E_6(q) \)

Here the Dynkin diagram is \( F_4 \). For the edge 12 we have relations (1)-(4) of Section A.1 with \( c_i, d_i \) in an \( \mathbb{F}_p \)-basis of \( \mathbb{F}_q \), and for edge 34 we have these relations for \( c_i, d_i \) in an \( \mathbb{F}_p \)-basis of \( \mathbb{F}_q^2 \). The remaining relations are (1)-(12) in Appendix A.2 with the following adjustments:

(a) in relations (1)-(8), \( c_i \in \mathbb{F}_q, d_i \in \mathbb{F}_q^2 \), and in (9)-(12) \( c_i, d_i \in \mathbb{F}_q^2 \);

(b) in relations (1)-(4), \( c_i d_i^2 \) is replaced by \( c_i d_i \overline{d}_i \) (where \( \overline{d}_i = d_i^q \));

(c) in relations (5)-(8), \( c_id_i \) is replaced by \( c_i \overline{d}_i \), and \( c_i d_i^2 \) is replaced by \( c_i d_i \overline{d}_i \);
(d) in relations (9)-(10), $2c_id_i$ is replaced by $c_i\bar{d}_i + \bar{c}_id_i$;

(e) in relations (11)-(12), $2c_id_i$ is replaced by $c_id_i + \bar{c}_id_i$.

Note that this is a variant of the presentation described in [7, 6.1].

**A.4 Relations for $G_2(q)$**

As in Section 13.4, we let $\alpha, \beta$ be fundamental roots in the $G_2$ root system, and define root elements $x_\alpha(c_i)$ for $r$ one of the long roots $\pm \alpha, \pm(\alpha + 3\beta), \pm(2\alpha + 3\beta)$, or one of the short roots $\pm \beta, \pm(\alpha + \beta), \pm(\alpha + 2\beta)$ and $c_i$ in a $\mathbb{F}_p$-basis of $\mathbb{F}_q$. The relations are the following:

1. $[x_\alpha(c_i), x_\beta(d_i)] = x_{\alpha + \beta}(c_id_i)x_{\alpha + 2\beta}(-c_id_i^2)x_{\alpha + 3\beta}(c_id_i^3)x_{2\alpha + 3\beta}(c_id_i^2d_i^2)$
2. $[x_\alpha(c_i), x_{-\alpha - \beta}(d_i)] = x_{-\beta}(-c_id_i)x_{-\alpha - 2\beta}(c_id_i^2)x_{-2\alpha - 3\beta}(c_id_i^4)x_{-\alpha - 3\beta}(-c_id_i^2d_i^2)$
3. $[x_{\alpha + 3\beta}(c_i), x_{\beta}(d_i)] = x_{\beta}(-c_id_i)x_{\alpha + \beta}(-c_id_i^2)x_{\alpha + 2\beta}(c_id_i^3)x_{2\alpha + 3\beta}(-c_id_i^4d_i^2)$
4. $[x_{\alpha + 3\beta}(c_i), x_{-\alpha - 2\beta}(d_i)] = x_{-\beta}(-c_id_i)x_{-\alpha - \beta}(-c_id_i^2d_i)x_{-2\alpha - 3\beta}(c_id_i^4)d_i^2$)
5. $[x_{\alpha + 2\beta}(c_i), x_{-\alpha - \beta}(d_i)] = x_{-\beta}(-c_id_i)x_{-\alpha - \beta}(c_id_i^2)x_{\alpha + 3\beta}(c_id_i^3)d_i^2$)
6. $[x_{\alpha + 3\beta}(c_i), x_{-\alpha - \beta}(d_i)] = x_{\alpha}(c_id_i)x_{\alpha + \beta}(c_id_i^3)x_{\alpha + 2\beta}(c_id_i^2d_i^2)x_{\alpha}(-c_id_i^2d_i^2)$)
7. $[x_{-\alpha}(c_i), x_{-\beta}(d_i)] = x_{-\alpha}(c_id_i)x_{-\alpha - 2\beta}(-c_id_i^2)x_{-\alpha - 3\beta}(-c_id_i^3)x_{2\alpha - 3\beta}(c_id_i^2d_i^2)$
8. $[x_{-\alpha}(c_i), x_{\alpha + \beta}(d_i)] = x_{\beta}(c_id_i)x_{\alpha + 2\beta}(c_id_i^3)x_{\alpha + 3\beta}(c_id_i^2d_i^2)$
9. $[x_{-\alpha - \beta}(c_i), x_{\beta}(d_i)] = x_{-\alpha}(c_id_i)x_{\alpha + \beta}(-c_id_i^2d_i)x_{\alpha}(-c_id_i^2d_i^2)$
10. $[x_{-\alpha - 3\beta}(c_i), x_{\alpha + 2\beta}(d_i)] = x_{-\beta}(-c_id_i)x_{\alpha + \beta}(-c_id_i^2d_i)x_{\alpha}(-c_id_i^2d_i^2)$
11. $[x_{-\alpha - 3\beta}(c_i), x_{\alpha + \beta}(d_i)] = x_{-\alpha}(c_id_i)x_{\alpha + 2\beta}(c_id_i^2)x_{\alpha}(-c_id_i^2d_i^2)$
12. $[x_{-\alpha - 3\beta}(c_i), x_{\alpha + 2\beta}(d_i)] = x_{-\alpha}(c_id_i)x_{\alpha + \beta}(-c_id_i^2d_i)x_{\alpha}(-c_id_i^2d_i^2)$
13. $[x_{\beta}(c_i), x_{\alpha + \beta}(d_i)] = x_{\alpha + 2\beta}(2c_id_i)x_{\alpha + 3\beta}(-3c_id_i^2)x_{2\alpha + 3\beta}(-3c_id_i^3)$
14. $[x_{\beta}(c_i), x_{-\alpha - 2\beta}(d_i)] = x_{-\alpha}(c_id_i)x_{\alpha}(-3c_id_i^2)x_{\alpha}(-3c_id_i^3)$
15. $[x_{\alpha + \beta}(c_i), x_{-\alpha - 3\beta}(d_i)] = x_{\alpha}(c_id_i)x_{\alpha}(-3c_id_i^2)x_{\alpha}(-3c_id_i^3)$
16. $[x_{\alpha + 2\beta}(c_i), x_{-\alpha - \beta}(d_i)] = x_{\alpha}(c_id_i)x_{\alpha}(-3c_id_i^2)x_{\alpha}(-3c_id_i^3)$
17. $[x_{\alpha + 2\beta}(c_i), x_{-\alpha - \beta}(d_i)] = x_{\alpha}(c_id_i)x_{\alpha}(-3c_id_i^2)x_{\alpha}(-3c_id_i^3)$
18. $[x_{\alpha}(c_i), x_{-\alpha - \beta}(d_i)] = x_{\alpha}(c_id_i)x_{\alpha}(-3c_id_i^2)x_{\alpha}(-3c_id_i^3)$
19. $[x_{\alpha}(c_i), x_{\alpha + 3\beta}(d_i)] = x_{\alpha}(c_id_i)$
20. $[x_{\alpha}(c_i), x_{-\alpha - 3\beta}(d_i)] = x_{\alpha}(c_id_i)$
21. $[x_{\alpha + 3\beta}(c_i), x_{\alpha - 2\beta}(d_i)] = x_{\alpha}(c_id_i)$
22. $[x_{\alpha + 3\beta}(c_i), x_{\alpha - \beta}(d_i)] = x_{\alpha}(c_id_i)$
23. $[x_{\alpha + 3\beta}(c_i), x_{\alpha - 3\beta}(d_i)] = x_{\alpha}(c_id_i)$
24. $[x_{\beta}(c_i), x_{-\alpha - \beta}(d_i)] = x_{-\alpha}(c_id_i)$
25. $[x_{\beta}(c_i), x_{\alpha + \beta}(d_i)] = x_{\alpha}(c_id_i)$
26. $[x_{\beta}(c_i), x_{\alpha - \beta}(d_i)] = x_{\alpha}(c_id_i)$
27. $[x_{\beta}(c_i), x_{\alpha + 2\beta}(d_i)] = x_{\alpha}(c_id_i)$
28. $[x_{\beta}(c_i), x_{-\alpha - \beta}(d_i)] = x_{\alpha}(c_id_i)$
29. $[x_{-\alpha}(c_i), x_{\alpha - 2\beta}(d_i)] = x_{-\alpha - 2\beta}(c_id_i)$
30. $[x_{-\alpha - \beta}(c_i), x_{\alpha - 2\beta}(d_i)] = x_{\alpha - 2\beta}(c_id_i)$

**A.5 Relations for $^3D_4(q)$**

Here the Dynkin diagram is $G_2$. The relations are (1)-(30) in Appendix A.4 with the following adjustments:
(a) in relations (1)-(12), $c_i \in \mathbb{F}_q$, $d_i \in \mathbb{F}_{q^3}$; in (13)-(18), $c_i, d_i \in \mathbb{F}_{q^3}$; in (19)-(24), $c_i, d_i \in \mathbb{F}_q$; and in (25)-(30), $c_i, d_i \in \mathbb{F}_{q^3}$;

(b) in relations (1)-(12), $c_i d_i^2$ is replaced by $c_i \bar{d}_i \bar{d}_i$ (where $\bar{d}_i = d_i^q$), $c_i d_i^3$ by $c_i d_i \bar{d}_i \bar{d}_i$, and $c_i^2 d_i^3$ by $c_i^2 d_i \bar{d}_i \bar{d}_i$;

(c) in relations (13)-(18), $2 c_i d_i$ is replaced by $\bar{c}_i \bar{d}_i + \bar{c}_i \bar{d}_i$, $3 c_i^2 d_i$ by $c_i \bar{c}_i \bar{d}_i + c_i \bar{c}_i \bar{d}_i + c_i \bar{c}_i \bar{d}_i$, and $3 c_i d_i^2$ by $c_i d_i \bar{d}_i + c_i d_i \bar{d}_i + \bar{c}_i d_i \bar{d}_i$;

(d) in relations (25)-(30), $3 c_i d_i$ is replaced by $c_i \bar{d}_i + \bar{c}_i \bar{d}_i$.

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