Group theory for embedded random matrix ensembles

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Abstract. Embedded random matrix ensembles are generic models for describing statistical properties of finite isolated quantum many-particle systems. For the simplest spinless fermion (or boson) systems with say \( m \) fermions (or bosons) in \( N \) single particle states and interacting with say \( k \)-body interactions, we have EGUE(\( k \)) (embedded GUE of \( k \)-body interactions) with GUE embedding and the embedding algebra is \( U(N) \). In this paper, using EGUE(\( k \)) representation for a Hamiltonian that is \( k \)-body and an independent EGUE(\( t \)) representation for a transition operator that is \( t \)-body and employing the embedding \( U(N) \) algebra, finite-\( N \) formulas for moments up to order four are derived, for the first time, for the transition strength densities (transition strengths multiplied by the density of states at the initial and final energies). In the asymptotic limit, these formulas reduce to those derived for the EGOE version and establish that in general bivariate transition strength densities take bivariate Gaussian form for isolated finite quantum systems. Extension of these results for other types of transition operators and EGUE ensembles with further symmetries are discussed.

1. Introduction
Wigner introduced random matrix theory (RMT) in physics in 1955 primarily to understand statistical properties of neutron resonances in heavy nuclei [1, 2]. Depending on the global symmetry properties of the Hamiltonian of a quantum system, namely rotational symmetry and time-reversal symmetry, we have Dyson’s tripartite classification of random matrices giving the classical random matrix ensembles, the Gaussian orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensembles. In the last three decades, RMT has found applications not only in all branches of quantum physics but also in many other disciplines such as Econophysics, Wireless communication, information theory, multivariate statistics, number theory, neural and biological networks and so on [3, 4, 5, 6]. However, in the context of isolated finite many-particle quantum systems, classical random matrix ensembles are too unspecific to account for important features of the physical system at hand. One refinement which retains the basic stochastic approach but allows for such features consists in the use of embedded random matrix ensembles.

Finite quantum systems such as nuclei, atoms, quantum dots, small metallic grains, interacting spin systems modeling quantum computing core and ultra-cold atoms, share one common property - their constituents predominantly interact via two-particle interactions. Therefore, it is more appropriate to represent an isolated finite interacting quantum system, say with \( m \) particles (fermions or bosons) in \( N \) single particle (sp) states by random matrix models generated by random \( k \)-body (note that \( k < m \) and most often we have \( k = 2 \)) interactions and...
propagate the information in the interaction to many particle spaces. Then we have random matrix ensembles in \emph{m}-particle spaces - these ensembles are defined by representing the \emph{k}-particle Hamiltonian (\(H\)) by GOE/GUE/GSE and then the \emph{m} particle \(H\) matrix is generated by the \emph{m}-particle Hilbert space geometry. The key element here is the recognition that there is a Lie algebra that transports the information in the two-particle spaces to many-particle spaces. As a GOE/GUE/GSE random matrix ensemble in two-particle spaces is embedded in the \(m\)-particle \(H\) matrix, these ensembles are more generically called embedded ensembles (EE).

Embedded ensembles are proved to be rich in their content and wide in their scope. A book giving detailed discussion of the various properties and applications of a wide variety of embedded matrix ensembles is now available [7]. Significantly, the study of embedded random matrix ensembles is still developing. Partly this is due to the fact that mathematical tractability of these ensembles is still a problem and this is the topic of the present paper. A general formulation for deriving analytical results is to use the Wigner-Racah algebra of the embedding Lie algebra [7]. The focus in the present paper is on transition strengths. Note that transition strengths probe the structure of the eigenfunctions of a quantum many-body system.

Given a transition operator \(O\) acting on the \emph{m} particle eigenstates \(|\mathcal{E}\rangle\) of \(H\) will give transition matrix elements \(|\langle \mathcal{E}_f | O | \mathcal{E}_i \rangle|^2\) and the corresponding transition strength density (this will take into account degeneracies in the eigenvalues) is,

\[
I_O(E_i, E_f) = I(E_f) \left| \langle \mathcal{E}_f | O | \mathcal{E}_i \rangle \right|^2 I(E_i). \tag{1}
\]

In Eq. (1), \(I(E)\) are state densities normalized to the dimension of the \emph{m} particle spaces. Note that \(E_i\) and \(E_f\) belong to the same \emph{m} particle system or different systems depending on the nature of the transition operator \(O\). In the discussion ahead, we will consider both situations. Random matrix theory has been used in the past to derive the form of \(I(E)\), the state densities. In particular, exact (finite \(N\)) formulas for lower order moments \(\langle HP \rangle\), \(p = 2\) and \(4\) of \(I(E)\) are derived both for EGOE and EGUE ensembles using group theoretical methods directly [8] or indirectly [9]. Going beyond the eigenvalue densities, here we will apply group theoretical methods and derive for the first time some exact formulas for the lower order moments of the transition strength densities \(I_O(E_i, E_f)\). Using these, the general form of transition strength densities is deduced. We will restrict to fermion systems and at the end discuss the extension to boson systems and to other situations. Let us add that some results valid in the asymptotic \((N \to \infty)\) limit for both the state densities and transition strength densities are available in literature [10, 11, 12, 13].

In general, the Hamiltonian may have many symmetries with the fermions (or bosons) carrying other degrees of freedom such as spin, orbital angular momentum, isospin and so on. Also we may have in the system different types of fermions (or bosons) and for example in atomic nuclei we have protons and neutrons. In addition, a transition operator may preserve particle number and other quantum numbers or it may change them. Off all these various situations, here we have considered three different systems: (i) a system of \(m\) spinless fermions and a transition operator that preserves the particle number; (ii) a transition operator that removes say \(k_0\) number of particles from the \(m\) fermion system; (iii) a system with two types of spinless fermions with the transition operator changing \(k_0\) number of particles of one type to \(k_0\) number of other particles as in nuclear beta and double beta decay. In all these we will restrict to EGUE. Now we will give a preview.

Section 2 gives some basic results for EGUE(\(k\)) for spinless fermion systems as derived in [8] and some of their extensions. Using these results, formulas for the lower order bivariate moments of the transition strength densities for the situation (i) above are derived and they are presented in detail in Section 3. Using these, results in the asymptotic limit are derived and they are presented in Section 4. Some basic results for the situations (ii) and (iii) above.
are given in Section 5. Finally, in Section 6 briefly discussed are some open group theoretical problems in the embedded ensembles theory for transition strength densities.

2. Basic EGUE(k) results for a spinless fermion system

Let us consider $m$ spinless fermions in $N$ degenerate $sp$ states with the Hamiltonian $\hat{H}$ a $k$-body operator,

$$\hat{H} = \sum_{i,j} V_{ij}(k) A_i^\dagger(k) A_j(k), \quad V_{ij}(k) = \langle k, i \mid \hat{H} \mid k, j \rangle .$$

(2)

Here $A_i^\dagger(k)$ is a $k$ particle (normalized) creation operator and $A_i(k)$ is the corresponding annihilation operator (a hermitian conjugate). Also, $i$ and $j$ are $k$-particle indices. Note that the $k$ and $m$ particle space dimensions are $\binom{N}{k}$ and $\binom{N}{m}$ respectively. We will consider $\hat{H}$ to be EGUE($k$) in $m$-particle spaces. Then $V_{ij}$ form a GUE with $V$ matrix being Hermitian. The real and imaginary parts of $V_{ij}$ are independent zero centered Gaussian random variables with variance satisfying,

$$V_{ab}(k) V_{cd}(k) = V_H^2 \delta_{ad} \delta_{bc} .$$

(3)

Here the ‘over-line’ indicates ensemble average. From now on we will drop the hat over $H$ and denote when needed $H$ by $H(k)$. Let us add that in physical systems, $k = 2$ is of great interest and in some systems such as atomic nuclei it is possible to have $k = 3$ and even $k = 4$ [14, 15].

The $U(N)$ algebra that generates the embedding, as shown in [8], gives formulas for the lower order moments of the one-point function, the eigenvalue density $I(E) = \langle \langle \delta(\hat{H} - E) \rangle \rangle$ and also for the two-point function in the eigenvalues. In particular, explicit formulas are given in [8, 7] for $\langle H^P \rangle^m$, $P = 2, 4$ and $\langle H^P \rangle^m \langle H^Q \rangle^m$, $P + Q = 2, 4$. Used here is the $U(N)$ tensorial decomposition of the $H(k)$ operator giving $\nu = 0, 1, \ldots, k$ irreducible parts $B^{\nu, \omega_\nu}(k)$ and then,

$$H(k) = \sum_{\nu = 0}^{k} W_{\nu, \omega_\nu}(k) B^{\nu, \omega_\nu}(k) .$$

(4)

With the GUE($k$) representation for the $H(k)$ operator, the expansion coefficients $W$’s will be independent zero centered Gaussian random variables with, by an extension of Eq. (3),

$$W_{\nu_1, \omega_{\nu_1}}(k) W_{\nu_2, \omega_{\nu_2}}(k) = V_H^2 \delta_{\nu_1, \nu_2} \delta_{\omega_{\nu_1} \omega_{\nu_2}} .$$

(5)

For deriving formulas for the various moments, the first step is to apply the Wigner-Eckart theorem for the matrix elements of $B^{\nu, \omega_\nu}(k)$. Given the $m$-fermion states $\{ f_m v_i \}$, we have with respect to the $U(N)$ algebra, $f_m = \{1^m\}$, the antisymmetric irreducible representation in Young tableau notation and $v_i$ are additional labels. Note that $\nu$ introduced above corresponds to the Young tableaux $\{2^{\nu} 1^{N-2\nu}\}$ and $\omega_\nu$ are additional labels. Now, Wigner-Eckart theorem gives

$$\langle f_m v_f \mid B^{\nu, \omega_\nu}(k) \mid f_m v_i \rangle = \langle f_m \mid B^{\nu}(k) \mid f_m \rangle C^{\nu, \omega_\nu}_{f_m v_f , f_m v_i} .$$

(6)

Here, $\langle - - \mid - - \mid - - \rangle$ is the reduced matrix element and $C^{\nu, \omega_\nu}_{- - , - - }$ is a $U(N)$ Clebsch-Gordan (C-G) coefficient [note that we are not making a distinction between $U(N)$ and $SU(N)$]. Also, $\{ f_m v_i \}$ represent a $m$ hole state (see [8] for details). In Young tableau notation $f_m = \{1^{N-m}\}$. Definition of $B^{\nu, \omega_\nu}(k)$ and the $U(N)$ Wigner-Racah algebra will give,

$$|\langle f_m \mid B^{\nu}(k) \mid f_m \rangle|^2 = \Lambda^\nu(N, m, m - k) ,$$

$$\Lambda^\nu(N', m', r) = \left( \begin{array}{c} m' - \mu \\ r \end{array} \right) \left( \begin{array}{c} N' - m' + r - \mu \\ r \end{array} \right) .$$

(7)
Note that $\Lambda^\nu(N, m, k)$ is nothing but, apart from a $N$ and $m$ dependent factor, a $U(N)$ Racah coefficient [8]. This and the various properties of the $U(N)$ Wigner and Racah coefficients give two formulas for the ensemble average of a product any two $m$ particle matrix elements of $H$,

$$
\langle f_{m,v_1} \mid H(k) \mid f_{m,v_2} \rangle \langle f_{m,v_3} \mid H(k) \mid f_{m,v_4} \rangle = V_H^2 \sum_{\nu_\nu \nu \nu} \Lambda^\nu(N, m, m - k) C^{\nu,\omega_v}_{f_{m,v_1}, f_{m,v_2}} C^{\nu,\omega_v}_{f_{m,v_3}, f_{m,v_4}},
$$

and also

$$
\langle f_{m,v_1} \mid H(k) \mid f_{m,v_2} \rangle \langle f_{m,v_3} \mid H(k) \mid f_{m,v_4} \rangle = V_H^2 \sum_{\nu_\nu \nu \nu} \Lambda^\nu(N, m, k) C^{\nu,\omega_v}_{f_{m,v_1}, f_{m,v_2}} C^{\nu,\omega_v}_{f_{m,v_3}, f_{m,v_4}}.
$$

Eq. (9) follows by applying a Racah transform to the product of the two C-G coefficients appearing in Eq. (8). Let us mention two important properties of the $U(N)$ C-G coefficients that are quite useful,

$$
\sum_{\nu_\nu \nu \nu} C^{\nu,\omega_v}_{f_{m,v_1}, f_{m,v_3}} = \sqrt{\binom{N}{m}} \delta_{\nu,0}, \quad C^{0,0}_{f_{m,v_1}, f_{m,v_2}} = \binom{N}{m}^{-1/2} \delta_{v_1,v_2}.
$$

From now on we will use the symbol $f_m$ only in the C-G coefficients, Racah coefficients and the reduced matrix elements. However, for the matrix elements of an operator we will use $m$ implying totally antisymmetric state for fermions (symmetric state for bosons).

Starting with Eq. (4) and using Eqs. (5), (9) and (10) will immediately give the formula,

$$
\langle \langle |H(k)|^2 \rangle \rangle^m = \binom{N}{m}^{-1} \sum_{\nu_\nu} \langle mv_i \mid |H(k)|^2 \mid mv_i \rangle = V_H^2 \Lambda^0(N, m, k).
$$

Similarly, for $\langle \langle |H^4(k)| \rangle \rangle^m$ first the ensemble average is decomposed into 3 terms as,

$$
\langle \langle |H^4(k)| \rangle \rangle^m = \sum_{\nu_\nu} \langle mv_i \mid |H^4(k)\rangle \mid mv_i \rangle
$$

$$
= \sum_{\nu_\nu \nu \nu} \langle mv_i \mid H(k) \mid mv_j \rangle \langle mv_j \mid H(k) \mid mv_i \rangle \langle mv_i \mid H(k) \mid mv_j \rangle \langle mv_j \mid H(k) \mid mv_i \rangle
$$

$$
+ \langle mv_i \mid H(k) \mid mv_j \rangle \langle mv_j \mid H(k) \mid mv_i \rangle \langle mv_i \mid H(k) \mid mv_j \rangle \langle mv_j \mid H(k) \mid mv_i \rangle
$$

$$
+ \langle mv_i \mid H(k) \mid mv_j \rangle \langle mv_j \mid H(k) \mid mv_i \rangle \langle mv_i \mid H(k) \mid mv_j \rangle \langle mv_j \mid H(k) \mid mv_i \rangle.
$$

Note that the trace $\langle \langle H^4(k) \rangle \rangle^m = \binom{N}{m} \langle H^4 \rangle^m$. It is easy to see that the first two terms simplify to give $2\langle H^2 \rangle^m \langle H^2 \rangle^m$ and the third term is simplified by applying Eq. (8) to the first ensemble average and Eq. (9) to the second ensemble average. Then, the final result is

$$
\langle \langle |H^2| \rangle \rangle^m = 2 \langle \langle H^2 \rangle \rangle^m + V_H^4 \binom{N}{m}^{-1} \sum_{\nu_\nu \nu \nu} \Lambda^\nu(N, m, k) \Lambda^\nu(N, m, m - k) d(N : \nu); \quad (13)
$$

$$
\quad d(N : \nu) = \binom{N}{\nu}^2 - \binom{N}{\nu - 1}^2.
$$

(14)
Also, a by-product of Eqs. (9) and (10) is
\[
\sum_{v_i} (mv_i \mid H(k) \mid mv_j) (mv_j \mid H(k) \mid mv_k) = \langle [H(k)]^2 \rangle^m \delta_{v_i v_k}
\] (15)
and we will use this in Section 3. Now we will discuss the results for the moments of the transition strength densities generated by a transition operator \( \mathcal{O} \).

3. Lower-order moments of transition strength densities: \( H \) EGUE\((k)\) and \( \mathcal{O} \) an independent EGUE\((t)\)

For a spinless fermion system, similar to the \( H \) operator, we will consider the transition operator \( \mathcal{O} \) to be a \( t \)-body and represented by EGUE\((t)\) in \( m \)-particle spaces. Then, the matrix of \( \mathcal{O} \) in \( t \)-particle spaces will be a GUE with the matrix elements \( \mathcal{O}_{ab}(t) \) being zero centered independent Gaussian variables with the variance satisfying,
\[
\mathcal{O}_{ab}(t) \mathcal{O}_{cd}(t) = V_0^2 \delta_{ad} \delta_{bc}.
\] (16)

Further, we will assume that the GUE representing \( H \) in \( k \) particle space and the GUE\((t)\) representing \( \mathcal{O} \) in \( t \) particle space are independent (this is equivalent to the statement that \( \mathcal{O} \) do not generate diagonal elements \( \langle E_i \mid \mathcal{O} \mid E_j \rangle \); see [11]). It is important to mention that in the past there are attempts to study numerically transition strengths using the eigenvectors generated by a EGOE (with symmetries) for \( H \) for nuclear systems and a realistic transition operator with results in variance with those obtained from realistic interactions [16]. As described in the present paper, a proper random matrix theory for transition strengths has to employ ensemble representation for both the Hamiltonian and the transition operator.

With EGUE representation, \( \mathcal{O} \) is Hermitian and hence \( \mathcal{O}^\dagger = \mathcal{O} \). Moments of the transition strength densities \( I_0(E_i, E_f) \) are defined by
\[
M_{PQ}(m) = \langle \mathcal{O}^\dagger H Q \mathcal{O} H^P \rangle^m = \langle \mathcal{O} H Q \mathcal{O} H^P \rangle^m.
\] (17)
Here the ensemble average is w.r.t. both EGUE\((k)\) and EGUE\((t)\). Now we will derive formulas for \( M_{PQ} \) with \( P + Q = 2 \) and 4; the moments with \( P + Q \) odd will vanish by definition.

Firstly, the unitary decomposition of \( \mathcal{O}(t) \) gives,
\[
\mathcal{O}(t) = \sum_{\nu=0}^t U_{\nu,\omega,\nu}(t) B^{\nu,\omega,\nu}(t).
\] (18)
The \( U \)’s satisfy a relation similar to Eq. (5). Now, for \( P = Q = 0 \) we have using Eq. (11),
\[
\langle \mathcal{O} \mathcal{O} \rangle^m = V_0^2 \Lambda^0(N,m,t).
\] (19)
Also, we have the relations
\[
\langle \mathcal{O} \mathcal{O} H^P \rangle^m = \langle \mathcal{O} \mathcal{O} \rangle^m \langle H^P \rangle^m = \langle \mathcal{O} H^P \mathcal{O} \rangle^m.
\] (20)
For the proof consider \( \langle \mathcal{O} \mathcal{O} H^P \rangle^m = {N \choose m}^{-1} \sum_{v_i,v_j} (mv_i \mid \mathcal{O} \mathcal{O} \mid mv_j) \langle mv_j \mid H^P \mid mv_i \rangle \). Now, applying Eq. (15) gives \( [mv_i \mid \mathcal{O}(t) \mathcal{O}(t) \mid mv_j] = \langle \mathcal{O} \mathcal{O} \rangle^m \delta_{v_i v_j} \) and substituting this will give the first equality in Eq. (20). Finally, \( \langle \mathcal{O} \mathcal{O} H^P \rangle^m = \langle \mathcal{O} H^P \mathcal{O} \rangle^m \) follows from the cyclic invariance of the \( m \)-particle average. Eq. (20) gives,
\[
M_{20}(m) = M_{02}(m) = \langle \mathcal{O} \mathcal{O} \rangle^m \langle H^2 \rangle^m, \quad M_{40}(m) = M_{04}(m) = \langle \mathcal{O} \mathcal{O} \rangle^m \langle H^4 \rangle^m.
\] (21)
Formulas for $\langle \mathcal{O}\mathcal{O} \rangle^m$, $\langle H^2 \rangle^m$ and $\langle H^4 \rangle^m$ follow from Eqs. (19), (11) and (13). Thus, the non-trivial moments $M_{PQ}$ for $P + Q \leq 4$ are $M_{11}$, $M_{13} = M_{31}$ and $M_{22}$.

It is easy to recognize that the bivariate moment $M_{11}$ has same structure as the third term in Eq. (12) for $\langle H^4 \rangle^m$ as the $\mathcal{O}$ and $H$ ensembles are independent. Then, the formula for $M_{11}$ follows directly from the second term in Eq. (13). This gives,

$$
M_{11}(m) = \frac{\langle \mathcal{O}(t) H(k) \mathcal{O}(t) H(k) \mathcal{O}(t) \mathcal{O}(t) \rangle^m}{\langle \mathcal{O}(t) \mathcal{O}(t) \mathcal{O}(t) \mathcal{O}(t) \rangle^m} = 
V_{\mathcal{O}}^2 V_{H}^2 \left( \frac{N}{m} \right)^{-1} \sum_{\nu=0}^{\min(m,m-k)} \Lambda^{\nu}(N,m,\nu) \Lambda^{\nu}(N,m-m-\nu) d(N:\nu). 
$$

(22)

This equation has the correct $t \leftrightarrow \nu$ symmetry. Using Eqs. (22) and (11), we have for the bivariate correlation coefficient $\xi$, the formula

$$
\xi(m) = \frac{\sqrt{M_{0}(m) M_{02}(m)}}{\sqrt{M_{0}(m) M_{02}(m)}} = \frac{\sum_{\nu=0}^{\min(m,m-k)} \Lambda^{\nu}(N,m,\nu) \Lambda^{\nu}(N,m-m-\nu) d(N:\nu)}{\sum_{\nu=0}^{\min(m,m-k)} \Lambda^{\nu}(N,m,\nu) \Lambda^{\nu}(N,m-m-\nu) d(N:\nu)}. 
$$

(23)

Turning to $M_{PQ}$ with $P + Q = 4$, the first trivial moment is $M_{13} = M_{31}$. For $M_{31}$ we have,

$$
\langle \mathcal{O}(t) H(k) \mathcal{O}(t) \mathcal{O}(t) \mathcal{O}(t) \rangle^m = \sum_{m_{1},m_{2},k_{1},v_{1},v_{2}} \langle m_{1} | \mathcal{O}(t) | m_{2} \rangle \langle m_{2} | \mathcal{O}(t) | m_{1} \rangle \langle m_{1} | H(k) | m_{2} \rangle \langle m_{2} | H(k) | m_{1} \rangle \langle m_{1} | [H(k)]^{3} | m_{2} \rangle \langle m_{2} | [H(k)]^{3} | m_{1} \rangle.
$$

(24)

and the last equality follows from the result that the EGU’s representing $H$ and $\mathcal{O}$ are independent. The ensemble average of the product of two $\mathcal{O}$ matrix elements follows easily from Eq. (9) giving

$$
\langle m_{1} | \mathcal{O}(t) | m_{2} \rangle \langle m_{2} | \mathcal{O}(t) | m_{1} \rangle = V_{\mathcal{O}} \sum_{\nu,\omega} \alpha^{\nu}(N,m,\nu) \alpha^{\nu}(N,m,m-\nu) d(N:\nu).
$$

(25)

The ensemble average of the product of a $H$ matrix element and $H^{3}$ matrix element appearing in Eq. (24) is,

$$
\langle m_{1} | H(k) | m_{2} \rangle \langle m_{2} | [H(k)]^{3} | m_{1} \rangle = 
\sum_{m_{1},m_{2},k_{1},v_{1},v_{2}} \langle m_{1} | H(k) | m_{2} \rangle \langle m_{2} | H(k) | m_{1} \rangle \langle m_{1} | [H(k)]^{3} | m_{2} \rangle \langle m_{2} | [H(k)]^{3} | m_{1} \rangle.
$$

(26)

The first two terms in Eq. (26) combined with Eq. (24) will give $2\langle [H(k)]^{3} \rangle^{m} M_{11}(m)$. For the third term we can use Eqs. (8) and (9) giving

$$
\langle m_{1} | H(k) | m_{2} \rangle \langle m_{2} | H(k) | m_{1} \rangle \langle m_{1} | [H(k)]^{3} | m_{2} \rangle \langle m_{2} | [H(k)]^{3} | m_{1} \rangle = 
V_{H}^{4} \sum_{\nu_{1},\nu_{2}}^{k} \sum_{\nu=0}^{\min(m,m-k)} \Lambda^{\nu_{1}}(N,m,\nu_{1}) \Lambda^{\nu_{2}}(N,m_{2} \nu_{2})
$$

(27)

$$
\times C_{\nu_{1}}^{\nu_{1}}(\nu_{1}) C_{\nu_{2}}^{\nu_{2}}(\nu_{2}) C_{\nu_{1}}^{\nu_{1}}(\nu_{1}) C_{\nu_{2}}^{\nu_{2}}(\nu_{2}) C_{\nu_{1}}^{\nu_{1}}(\nu_{1}) C_{\nu_{2}}^{\nu_{2}}(\nu_{2}) C_{\nu_{1}}^{\nu_{1}}(\nu_{1}) C_{\nu_{2}}^{\nu_{2}}(\nu_{2}).
$$
Combining this with Eq. (25) and applying the orthonormal properties of the C-G coefficients will give the final formula for $M_{31}$,

$$M_{31}(m) = \langle \mathcal{O}(t) H(k) \mathcal{O}(t) | H(k) \rangle^3 |^m =$$

$$V_3^2 V_H^4 \left( \begin{array}{c} N \cr m \end{array} \right)^{-1} \sum_{\nu=0}^{\min(k,m-t)} 2\Lambda^0(N,m,k) \Lambda^\nu(N,m,m-k) d(N : \nu)$$

$$+ \sum_{\nu=0}^{\min(k,m-k,m-t)} \Lambda^\nu(N,m,t) \Lambda^\nu(N,m,k) \Lambda^\nu(N,m,m-k) d(N : \nu) \right\} . \tag{28}$$

For deriving the formula for $M_{22}$, we will make use of the decompositions similar to those in Eqs. (24) and (26). Then it is easy to see that $M_{22}$ will have three terms and simplifying these will give (details are not given here due to lack of space),

$$M_{22}(m) = \langle \mathcal{O}(t) \mathcal{O}(t) \rangle^m \left\{ \langle H(k) H(k) \rangle^m \right\}^2$$

$$+ V_2^2 \left( V_H^4 \right) \left( \begin{array}{c} N \\
^m \end{array} \right)^{-1} \sum_{\nu=0}^{\min(t,m-k)} \Lambda^\nu(N,m,m-t) [\Lambda^\nu(N,m,k)]^2 d(N : \nu)$$

$$+ \left( \begin{array}{c} N \\
^m \end{array} \right)^{-1} \sum_{\nu=0}^{\min(k+t,m-k)} \Lambda^\nu(N,m,k) d(N : \nu) \sum_{\nu_1=0}^t \sum_{\nu_2=0}^k \sum_{\rho} |\langle f_m || [B^{\nu_1}(t) B^{\nu_2}(k)]^\nu. \rho || f_m \rangle|^2 \right\} . \tag{29}$$

Here $\rho$ labels multiplicity of the irrep $\nu$ in the Kronecker product $\nu_1 \times \nu_2 \rightarrow \nu$. The third term above simplifies to,

$$\left( \begin{array}{c} N \\
^m \end{array} \right)^{-2} \sum_{\nu=0}^{\min(k+t,m-k)} \sum_{\nu_1=0}^t \sum_{\nu_2=0}^k \Lambda^\nu(N,m,k) \Lambda^{\nu_1}(N,m,m-t) \Lambda^{\nu_2}(N,m,m-k)$$

$$\times d(N : \nu_1) d(N : \nu_2) \sum_{\rho} \left[ U(f_m \nu_1 f_m \nu_2 ; f_m \nu) \right]^2 .$$

The $U$ or Racah coefficient here is with respect to $U(N)$. Deriving formulas for this $U$-coefficient is an important open problem.

4. Asymptotic results for the bivariate moments for $H$ EGUE($k$) and $O$ an independent EGUE($t$)

Lowest order (sufficient for most purposes) shape parameters of the bivariate strength density are the bivariate reduced cumulants of order four, i.e. $k_{rs}$, $r + s = 4$. The $k_{rs}$ can be written in terms of the normalized central moments $\tilde{M}_{PQ}$ where $\tilde{M}_{PQ} = M_{PQ}/M_{00}$. Then, the scaled moments $\mu_{PQ}$ are

$$\mu_{PQ} = \left\{ \left[ \tilde{M}_{20} \right]^{P/2} \left[ \tilde{M}_{02} \right]^{Q/2} \right\}^{-1} \tilde{M}_{PQ} , \ P + Q \geq 2 . \tag{30}$$

Note that $\mu_{20} = \mu_{02} = 1$ and $\mu_{11} = \xi$. Now the fourth order cumulants are,

$$k_{40} = \mu_{40} - 3, \ k_{04} = \mu_{04} - 3, \ k_{31} = \mu_{31} - 3 \xi, \ k_{13} = \mu_{13} - 3 \xi, \ k_{22} = \mu_{22} - 2 \xi^2 - 1 . \tag{31}$$

The $|k_{rs}| \sim 0$ for $r + s \geq 3$ implies that the transition strength density is close to a bivariate Gaussian (note that in our EGUE applications, $k_{rs} = 0$ for $r + s$ odd by definition). We have
verified in large number of numerical examples, obtained using the formulas in Section 3 for some typical values of $N$, $m$, $k$ and $t$, that the cumulants $|k_{rs}|$ with $r+s=4$ are in general very small implying that for EUGE, transition strength densities approach bivariate Gaussian form.

For a better understanding of this result, it is useful to derive expressions for $\mu_{PQ}$ and thereby for $k_{PQ}$, using Eq. (31), in the asymptotic limit defined by $N \to \infty$, $m \to \infty$, $m/N \to 0$ and $k$ and $t$ fixed. First we will consider $N \to \infty$ and $m$ fixed with $k, t < m$. We will use,

$$\left(\frac{N - p}{r}\right)^{\rho/|N \to 0|} \text{ and } d(N : \nu) \mathcal{N}^{2\nu} \left|_{\nu \to 0} \right. \frac{N^{2\nu}}{|\nu|}.$$  

Note that $d(N : \nu)$ was defined by Eq. (14). Let us start with the formula for $\xi$ given by Eq. (23). Applying Eq. (32) it is easily seen that only the term with $\nu = t$ in the sum in Eq. (23) will contribute in the asymptotic limit. Using this, applying Eqs. (32) and the formula for $\Lambda^\nu$ given by Eq. (7) will lead to,

$$\xi(m) \to \frac{m!N^{2t}}{N^{m}(t!)} \frac{(m-k)}{(m-t)} \frac{(N-m-t)}{(N-m+k)} \frac{(N-2t)}{(N-k)} \to \frac{(m-t)}{(m-k)}.$$  

Similarly, $\mu_{40}$ will be, using Eqs. (21), (13) and (11),

$$\mu_{40}(m) \to 2 + \frac{m!N^{2k}}{N^{m}(k!)^2} \frac{(m-k)}{(m-t)} \frac{(N-m)}{(m-t)} \frac{(N-m+k)}{(N-k)} \to 2 + \frac{(m-t)}{(m-k)}.$$

Turning to $\mu_{31}$, it is easy to see from Eq. (28) that $M_{31}$ has two terms. The first term is $(2\xi O(t)O(t))^{m} [(H(k)H(k))]^{m}$ and in the sum in the second term only $\nu = k$ will survive in the asymptotic limit. These then will give,

$$\mu_{31}(m) = (2\xi + \frac{N}{m})^{-1} \frac{(N-m)}{(m-t)} \frac{(N-m+k)}{(N-m+k)} \frac{(N-m-k)}{(N-k)} \to \xi \left[2 + \frac{(m-t)}{(m-k)} \right] = \xi \mu_{40}.$$  

Note that in the final simplifications we have used Eq. (32). Finally, let us consider $\mu_{22}$. Firstly, $M_{22}$ has three terms as seen from Eq. (29) and then, there will be corresponding three terms in $\mu_{22}$. Let us call them $T1, T2$ and $T3$. It is seen that $T1 = 1$ and $T2$ is (in the corresponding sum in $M_{22}$, only $\nu = t$ term will contribute in the asymptotic limit),

$$T2 \to \left(\frac{N}{m}\right)^{-1} \frac{(N-m)}{(m-t)} \frac{(N-m+k)}{(N-m+k)} \frac{(N-m-k)}{(N-k)} \to \left(\frac{m}{k}\right)^{-2} \left(\frac{m-t}{k}\right)^{2}.$$  

Similarly, $T3$ in the asymptotic limit will be

$$T3 \to \frac{\Lambda^{t+k}(N,m,k)}{(N,m,t)} \to \left(\frac{m}{k}\right)^{-2} \left(\frac{m-t}{k}\right)^{2}.$$  

Here we have used Eq. (33) for $\xi$ in the $N \to \infty$ limit. Now combining $T1, T2$ and $T3$ we have

$$\mu_{22}(m) \to 1 + \left(\frac{m}{k}\right)^{-2} \left(\frac{m-t}{k}\right)^{2} + \left(\frac{m-t}{k}\right)^{-2} \left(\frac{m-t}{k}\right)^{2}.$$  

Comparing Eq. (37) with the formula for $T3$, it is easy to see that $[U(f_{m \, t} f_{m \, k} : f_{m \, t+k}]^{2} \frac{\text{asympt}}{(m-k)^{-2}}$. The asymptotic formulas given by Eqs. (33), (34), (35) and (38) show that the finite $N$ results derived in Section 3 reduce in the asymptotic limit exactly to those derived before using binary correlation approximation [11, 7]. More importantly, they show that the cumulants tend to zero in the dilute (or asymptotic) limit.
5. Lower-order moments of transition strength densities: results for particle transfer operator and beta decay type operator

In this Section we will consider transition operators that are non-hermitian and that are of great interest for example in nuclear physics. These are particle removal (or addition) operators and beta decay type operators. Let us begin with particle removal operator $O$ and say it removes $k_0$ number of particles when acting on a $m$ fermion state. Then the general form of $O$ is,

$$O = \sum_{\alpha_0} V_{\alpha_0} \ A_{\alpha_0}(k_0) \ .$$  \hspace{1cm} (39)

Here, $A_{\alpha_0}(k_0)$ is a $k_0$ particle annihilation operator and $\alpha_0$ are indices for a $k_0$ particle state. We will choose $V_{\alpha_0}$ to be zero centered independent Gaussian random variables with variance satisfying $V_{\alpha}V_{\beta} = V^2_{\alpha} \delta_{\alpha\beta}$. Then we have,

$$\langle O^\dagger O \rangle^m = V^2_{\alpha} \binom{m}{k_0}, \quad \langle O O^\dagger \rangle^m = V^2_{\alpha} \binom{N-m}{k_0} \ .$$  \hspace{1cm} (40)

Eq. (15) also gives the important relations,

$$\langle O^\dagger O H \nu \rangle^m = \langle O^\dagger O \rangle^m \langle H \nu \rangle^m, \quad \langle O^\dagger H \nu O \rangle^m = \langle O^\dagger O \rangle^m \langle H \nu \rangle^{m-k_0} \ .$$  \hspace{1cm} (41)

Another useful result that follows from Eq. (40) is,

$$\left\langle m \mid A^\dagger(k_0) \mid m-k_0 \right\rangle \left\langle m-k_0 \mid A(k_0) \mid m \right\rangle = \binom{N-k_0}{m-k_0} \ .$$  \hspace{1cm} (42)

Following the procedure used in Section 3 it is possible to derive formulas for the lower order moments of the transition strength density generated by $O$ given by Eq. (39). Eqs. (41) and (40) along with Eqs. (11) and (13) will give $M_{20}$, $M_{02}$, $M_{10}$ and $M_{04}$. Then, the first non-trivial moment is $M_{11}$. Formula for $M_{11} = \langle O^\dagger H O H \rangle$ is derived by starting with the definition of $\langle O^\dagger H O H \rangle^m$,

$$\binom{N}{m} \langle O^\dagger H O H \rangle^m = \sum_{v_1,v_2,v_3,v_4} \left\langle m,v_1 \mid O^\dagger \mid m-k_0,v_2 \right\rangle \left\langle m-k_0,v_3 \mid O \mid m,v_4 \right\rangle \langle m-k_0,v_3 \mid H \mid m-k_0,v_4 \rangle \left\langle m,v_4 \mid H \mid m,v_1 \right\rangle \ .$$  \hspace{1cm} (43)

Applying Eqs. (4) - (7) and the Wigner-Eckart theorem along with Eq. (42) will give,

$$M_{11}(m) = \langle O^\dagger H(k)O H(k) \rangle = V^2_O V^2_H \binom{N}{m}^{-1} \binom{N-k_0}{k_0}^{1/2} \sum_{\nu=0}^{k} \left[ d(N : \nu) \Lambda^\nu(N,m-k_0,m-k_0-k) \Lambda^\nu(N,m,m-k) \right]^{1/2} (-1)^{\phi} U(f_m f_{m-k_0} f_m f_{m-k_0} ; f_{k_0} \nu) \ .$$  \hspace{1cm} (44)

The phase factor $\phi$ depends on $\nu$. Formula for the $U$-coefficient appearing in Eq. (44) was available in [17]. It is possible to proceed forward and derive formulas for $M_{31}$, $M_{13}$ and $M_{22}$ (these will be reported elsewhere). Now we will turn to beta decay type operators.

Let us consider a system with $m_1$ fermions in $N_1$ sp states and $m_2$ fermions in $N_2$ sp states with $H$ preserving $(m_1,m_2)$. Then, the $H$ operator, assumed to be k-body, is given by,

$$H(k) = \sum_{i+j=k} \sum_{\alpha \beta} V_{\alpha \beta}(i,j) A^\dagger_\alpha(i) A_\beta(i) A^\dagger_\beta(j) A_\alpha(j) \ ,$$

$$V_{\alpha \beta}(i,j) = \langle i, \alpha : j, a \mid H \mid i, \beta : j, b \rangle \ .$$  \hspace{1cm} (45)
Here we are using Greek labels $\alpha, \beta, \ldots$ to denote the many particle states generated by fermions occupying the orbit with $N_1$ sp states and the Roman labels $a, b, \ldots$ for the many particle states generated by the fermions occupying the orbit with $N_2$ sp states. For each $(i, j)$ pair with $i + j = k$, we have a matrix $V(i, j)$ in the $k$-particle space and we assume that the $V(i, j)$ matrices are represented by independent GUE’s with their matrix elements being zero centered with variance,

$$V_{aa; \beta}(i, j) V_{a' a''; \beta'}(i', j') = V^2_H(i, j) \delta_{i i'} \delta_{j j'} \delta_{\alpha \alpha'} \delta_{\beta \beta'} \delta_{a a'}. \quad (46)$$

It is important to note that the embedding algebra for the EGUE generated by the action of the $H(k)$ operator on $|m_1, v_\alpha : m_2, v_\alpha \rangle$ states is the direct sum algebra $U(N_1) \oplus U(N_2)$. Thus we have $\text{EGUE}(k) = [U(N_1) \oplus U(N_2)]$ ensemble. For this system, we will consider a beta decay $(k_0 = 1)$ type transition operator,

$$O = \sum_{\alpha, \beta} O_{\alpha \alpha} A^\dagger_{\alpha}(k_0) A_{\alpha}(k_0) ; \quad O_{\alpha \alpha} = \langle k_0, \alpha | O | k_0, \alpha \rangle \quad (47)$$

and assume a GUE representation for the $O$ matrix in the defining space giving $O_{\alpha \beta} O_{\beta \alpha} = V^2_O \delta_{\alpha \beta} \delta_{\alpha \beta}$. Note that in general the $O$ matrix is a rectangular matrix. Now, the ensemble averaged bivariate moments of the transition strength density are $M_{PQ}(m_1, m_2) = \langle O^\dagger H O H P \rangle^{(m_1, m_2)}$. Note that $O$ takes $(m_1, m_2)$ to $(m_1 + k_0, m_2 - k_0)$.

Firstly, recognize that $A^\dagger_{\alpha}(k_0)$ and $A_{\beta}(k_0)$ transform as the $U(N_1)$ tensors $f_{k_0} = \{1_{k_0}\}$ and $\{ f_{k_0} \} = \{1^{N_1 - k_0}\}$ and similarly $A^\dagger_{\alpha}(k_0)$ and $A_{\beta}(k_0)$ with respect to $U(N_2)$. These will give easily the results,

$$\langle O^\dagger O \rangle^{(m_1, m_2)} = V^2_O \left( N_1 - m_1 \right) \left( m_2 \right) \left( k_0 \right) , \quad \langle O O^\dagger \rangle^{(m_1, m_2)} = V^2_O \left( N_2 - m_2 \right) \left( m_1 \right) \left( k_0 \right) . \quad (48)$$

For deriving formulas for the bivariate moments with $P$ and/or $Q \neq 0$, unitary decomposition of $H$ is carried out with respect to the $U(N_1) \oplus U(N_2)$ algebra. This gives,

$$\langle H^2 \rangle^{(m_1, m_2)} = \sum_{i + j = k} V^2_H(i, j) \Lambda^0(N_1, m_1, i) \Lambda^0(N_2, m_2, j) . \quad (49)$$

It is also easy to show that $\langle O^\dagger O H P \rangle^{(m_1, m_2)} = \langle O^\dagger O \rangle^{(m_1, m_2)} \langle H P \rangle^{(m_1 + k_0, m_2 - k_0)}$ and $\langle O^\dagger H P O \rangle^{(m_1, m_2)} = \langle O^\dagger O \rangle^{(m_1, m_2)} \langle H P \rangle^{(m_1 + k_0, m_2 - k_0)}$. Also, the first nontrivial bivariate moment $M_{11}(m_1, m_2) = \langle O^\dagger H O H \rangle^{(m_1, m_2)}$ is given by,

$$M_{11}(m_1, m_2) = V^2_O \left( N_1 - k_0 \right) \left( m_2 - k_0 \right) \left( k_0 \right) \left[ \left( N_1 \right) \left( m_2 \right) \right]^{-1} \sqrt{\left( N_1 \right) \left( k_0 \right) \left( N_2 \right) \left( k_0 \right) \sum_{i + j = k} V^2_H(i, j) \times \prod_{\nu = 0}^j \left[ \prod_{\nu' = 0}^j \left[ d(N_1 : \nu) d(N_2 : \nu') \Lambda^\nu(N_1, m_1, m_1 - i) \Lambda^\nu(N_1, m_1 + k_0, m_1 + k_0 - i) \right]^{1/2} \right]^{1/2} \left[ \Lambda^\nu(N_2, m_2, m_2 - j) \Lambda^\nu(N_2, m_2 - k_0, m_2 - k_0 - j) \right]^{1/2} \left( -1 \right)^{\phi_1(\nu) + \phi_2(\nu')} \times U(f_{m_1 + k_0} f_{m_1}, f_{m_1 + k_0} f_{m_1} f_{k_0} \nu) U(f_{m_2} f_{m_2 - k_0} f_{m_2} f_{m_2 - k_0} : f_{k_0} \nu). \quad (50)$$

Here, $\phi_1$ and $\phi_2$ are phase factors. Also, $\mathcal{F}_r = \left\{ 1^{N-r} \right\}$ and $\nu = \{ 2^\nu, 1^{N-2\nu} \}$ with $N = N_1$ or $N_2$ as appropriate. Formula for the $U$-coefficients in Eq. (50) is available in [17].
6. Conclusions and future outlook
In this paper we have presented for the first time exact (finite $N$) results for the moments of the transition strength densities using $U(N)$ Wigner-Racah algebra for EGUE random matrix ensembles. In particular, formulas for the moments up to fourth order are derived in detail for the Hamiltonian $a_{\text{EGUE}(k)}$ and the transition operator $a_{\text{EGUE}(t)}$ for spinless fermion systems. Numerical results on one hand and the asymptotic results derived from the exact results on the other, showed that the fourth order cumulants approach zero in the dilute limit implying that the strength densities approach bivariate Gaussian form. As discussed in Section 5, the formulation given in Sections 2 and 3 extends to transition operators that are particle removal (or particle addition) operators and also to beta decay (also neutrinoless double beta decay) type operators. Complete results for these (i.e. for $MPQ$ with $P + Q = 4$) will be presented elsewhere. All these results are useful in nuclear spectroscopy [11, 7]. Another important extension is to EGUE with $U(\Omega) \times SU(r)$ embedding discussed in [18]. With applications in mesoscopic systems, it is important to derive formulas for the bivariate moments of transition strength densities for systems with $r = 2$. For fermions, this corresponds to spin degree of freedom and then we have spin scalar and spin vector transition operators. For this system, formulas for the Racah coefficients that appear, even for the fourth moment of the state densities, are not yet available [7] and an exploration of the asymptotic methods suggested in [19] could prove to be fruitful. Let us mention that the results presented in Sections 3 and 5 extend to boson systems by using the $N \to -N$ and $N \to m$ symmetries discussed in [8, 7]. In future, it is also important and useful to extend the present work to EGOE and EGSE ensembles. Finally, future in the subject of embedded random matrix ensembles in quantum physics is exciting with enormous scope for developing new group theoretical methods for their analysis and with the possibility of their applications in a variety of isolated finite quantum many-particle systems.

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