Thermometric machine for ultraprecise thermometry of low temperatures

Ivan Henao, Karen V. Hovhannisyan, and Raam Uzdin

1 Fritz Haber Research Center for Molecular Dynamics, Institute of Chemistry, The Hebrew University of Jerusalem, Jerusalem 9190401, Israel
2 The Abdus Salam International Centre for Theoretical Physics (ICTP), Strada Costiera 11, 34151 Trieste, Italy
3 Institute of Physics and Astronomy, University of Potsdam, 14476 Potsdam, Germany

Estimating the temperature of a quantum system is a task of great importance in fundamental and applied research. In the study of fundamental quantum phenomena and the development of related applications, it is crucial to precisely prepare and characterize systems at temperatures near the absolute zero. For example, quantum computing based on superconducting qubits operates with quantum states close to the ground state present a significant challenge for quantum simulators near the absolute zero. For example, quantum computing based on superconducting qubits operates with quantum states close to the ground state present a significant challenge for quantum simulators near the absolute zero. For example, quantum computing based on superconducting qubits operates with quantum states close to the ground state present a significant challenge for quantum simulators near the absolute zero.

Thermal equilibrium states are exponentially hard to distinguish at very low temperatures, making equilibrium quantum thermometry in this regime a formidable task. We present a thermometric scheme that circumvents this limitation, by using a two-level probe that does not thermalize with the sample whose temperature is measured. This is made possible thanks to a suitable interaction that couples the probe to the sample and to an auxiliary thermal bath known to be at a higher temperature. Provided a reasonable upper bound on the temperature of the sample, the resulting “thermometric machine” drives the probe towards a steady state whose signal-to-noise ratio can achieve values as high as $O(1/T)$. We also characterize the transient state of the probe and numerically illustrate an extreme reduction in the number of measurements to attain a given precision, as compared to optimal measurements on a thermalized probe.

Quantum state estimation involves finding the operator $\rho(T)$, the most general strategy to estimate $T$ consists of performing a POVM (positive operator valued measurement) on $\rho(T)$, followed by a classical postprocessing that assigns temperature values to the different POVM outcomes. If the $j$th outcome occurs with probability $p_j(T)$, the Fisher information reads [26, 27]

$$F(T) = \sum_j p_j(T) \left( \frac{\partial \ln(p_j)}{\partial T} \right)^2,$$

where the sum runs over all the POVM outcomes. In this way, Eq. (1) limits the SNR for a temperature estimate drawn from $M$ independent realizations of the POVM. For a thermal state $\rho(T) = \rho_0 e^{\beta H}$, where $\beta = 1/T$ is the inverse temperature and we set the Boltzmann constant equal to one, the POVM that maximizes $F(T)$ corresponds to projective measurements of the Hamiltonian $H$. In such a case the bound (1) is also tight, for any value of $M$ [16].

In many-body systems, temperature measurements via small probes [5, 10, 11, 18, 28–34] constitute a more practical or convenient approach for thermometry, due to the destructive character or practical difficulty of direct measurements [7, 14, 18, 28, 35]. However, probes that thermalize with the system of interest (the ‘sample’) are rather limited in their probing capabilities [5, 6, 11, 13], in line with our initial observations. To understand this better consider the Fisher information $F^{th}(T)$, for energy measurements on a thermalized (hence the superscript ‘th’) probe. In what follows, the indices $P$ and $S$ will refer to the probe and the sample, respectively. Given an infinitesimal temperature variation $dT$, $F^{th}(T)$ quantifies the degree of distinguishability between two states $\rho_p^S(T)$ and $\rho_p^S(T + dT)$ [5, 27]. In the limit $T \to 0$, this distinguishability is suppressed by an exponential scaling $F^{th}(T) \sim O(\exp(-\beta(T - T_0)^2))$ [6, 8, 10, 11, 13, 36] that also affects the SNR (cf. Eq. (1)). Therefore, for $T \ll T_0^P - \epsilon_0^P$
thermometry using a (finite) thermal probe is exponentially inefficient.

If the sample itself is gapped (i.e., $\varepsilon_1^S - \varepsilon_0^S$ is finite), the exponential decay occurs even for probes that do not couple weakly to the sample and thus attain a non-Gibbsian state when equilibrated with the sample [11]. A gapless sample is characterized by a continuous spectrum above the ground state, which implies particularly that $\varepsilon_1^S \rightarrow \varepsilon_0^S$. The precision of a probe that strongly couples to such a sample is not exponentially suppressed anymore; at equilibrium, a generic probe will deliver only a polynomially decaying (with respect to $\beta$) SNR [11, 13].

In gapless harmonic systems, a free Brownian particle, when used as a probe, can yield an SNR that is as high as $O(1)$ [11]. The latter is the highest temperature scaling reported for equilibrium probe-based thermometry [37].

Motivated by the observation that non-equilibrium probes can provide better temperature sensing [15, 28, 38, 39], in this letter we design a non-equilibrium thermometric scheme that allows us to achieve a `divergent' SNR in the limit $T \rightarrow 0$. The key element behind this improvement is the use of an ancillary system [40] that interacts with the probe and the sample, and builds upon the recent finding that probe-based thermometry can be analytically enhanced [38]. These interactions are sequentially repeated with different sample constituents (see Fig. 1(a)), in the spirit of a collisional model [39, 41–43]. Crucially, the resulting evolution drives the probe to a non-thermal steady state, with a dependence on ancillary parameters that can be tuned to suppress the exponential scaling characteristic of low temperatures. Moreover, this mitigation can be enforced without any restriction on the energy gaps of the sample.

**Thermometric machine.** For the sake of simplicity, we will consider a sample composed of $N$ qubits. However, the results presented later for the steady state of the probe are generalizable to samples formed by $N$ identical and non-interacting systems, of arbitrary dimension and energy spectrum (see Sec. I of Supplemental Material (SM) [44]). The probe and the ancilla are two-level systems with Hamiltonians $H_P = \varepsilon_P |1_P\rangle\langle 1_P|$ and $H_v = \varepsilon_v |1_v\rangle\langle 1_v|$, respectively. Moreover, each sample qubit has Hamiltonian $H_S = \varepsilon_s |1_s\rangle\langle 1_s|$, and $H_S = \sum_{i=1}^N H_s^{(i)}$ is the total sample Hamiltonian. For a sample prepared in the thermal state $\rho_S = \rho_0^N$, where $\rho_S = \sum_{i=0}^1 \rho_i^S |i_S\rangle\langle i_S| = e^{-\beta S_i^+ S_i} / \text{tr}(e^{-\beta S_i^+ S_i})$, temperature information is transferred to the probe in the following way:

1. After preparing the ancilla in a thermal state $\rho_v = \sum_{i=0}^1 \rho_i^v |i_v\rangle\langle i_v| = e^{-\beta \varepsilon_v} / \text{tr}(e^{-\beta \varepsilon_v})$, where $\beta_v < \beta$, it is coupled to the probe and to a single sample qubit via the three-body interaction $H_I = \varepsilon_I (|0_P0_1_v\rangle\langle 1_P1_0_v| + h.c.)$ (where h.c. stands for Hermitian conjugate and $|i_P j_v k_v\rangle = |i_P\rangle \otimes |j_v\rangle \otimes |k_v\rangle$). In addition, we assume that $\varepsilon_v = \varepsilon_P + \varepsilon_s$, which implies

$$[H_I, H_P + H_S + H_v] = 0. \quad (3)$$

2. The interaction $H_I$ is switched off and the ancilla is rethermalized.

Each repetition of Step (1) gives rise to a “collision” between the probe, the ancilla, and the sample [41]. If $H_I$ is switched on during a time $\pi / 2 \varepsilon_I$, the total Hamiltonian $H_{Psv} = H_P + H_S + H_v + H_I$ yields the unitary evolution (apart from unimportant global phases, see Sec. II of SM [44] for further details)

$$U_{Psv} |0_P0_1_v\rangle = |1_P1_0_v\rangle, \quad (4)$$

$$U_{Psv} |1_P1_0_v\rangle = |0_P0_1_v\rangle. \quad (5)$$

Furthermore, $U_{Psv}$ acts as the identity on any other eigenstate of $H_P + H_S + H_v$.

**Tuning of $\varepsilon_v$.** Since $\varepsilon_s$ is the energy gap above the ground state of the system, the corresponding low-temperature regime can be characterized by temperatures $T$ in an interval $(0, T_{\max})$ such that $T_{\max} \ll \varepsilon_s$. Given a fixed temperature $T_v$, this information can be harnessed to properly choose the energy gap $\varepsilon_v$. To that end, we consider the prior temperature

$$\tilde{T} \equiv T_{\max} / 2, \quad (6)$$

which encapsulates the knowledge that $T \in (0, T_{\max})$.

In this way, the goal of the thermometric machine is to estimate with as much precision as possible the specific value of $T$. 

![Image](image-url)
Let $\rho^P_k(T)$ denote the state of the probe after it has interacted with $k$ qubits, and let $T/\Delta_k T$ be the SNR that results from $M$ (independent) energy measurements performed on $\rho^P_k(T)$. In Sec. II of SM [44] we show that this state is diagonal in the eigenbasis of the probe Hamiltonian, i.e. $\rho^P_k(T) = \sum^\infty_{i=0} p^P_{i,k}(T)(i|p)(i|p)$. Accordingly, the Fisher information and the corresponding Cramer-Rao bound for the SNR are maximized by performing energy measurements.

The statistics obtained from a two-level system belong to the so-called exponential family [45] and therefore the associated Cramer-Rao bound is tight. For the sake of completeness, we also present a detailed proof based on the error-propagation formula in Sec. III of SM [44]. Hence, the exact steady-state SNR of the probe, defined by the limits $N \to \infty$ and $k \to \infty$, is given by

$$T \frac{\Delta \xi}{\Delta T} = \sqrt{M (p^P_{0,\infty}(T)p^P_{1,\infty}(T))} \frac{\varepsilon_s}{T} = \sqrt{M e^{-\frac{1}{2} \beta \varepsilon_s (1 - \frac{\varepsilon_s}{\max})}} \frac{T}{1 + e^{-\beta \varepsilon_s (1 - \frac{\varepsilon_s}{\max})}},$$

where the prior temperature $\tilde{T}$ is used to tune the ancillary energy gap as $\varepsilon_v = T \frac{\varepsilon_s}{\tilde{T}}$.

Assuming that $T_v \geq T_{\max}$, Eq. (8) implies that $\varepsilon_v \geq 2\varepsilon_s$, and from the condition $\varepsilon_P = \varepsilon_v - \varepsilon_s$ it also follows that $\varepsilon_P \geq \varepsilon_s$. Since this implies that $\varepsilon_s = \min\{\varepsilon_s, \varepsilon_v, \varepsilon_P\}$, for the tuning $\varepsilon_v = \varepsilon_s$ and $T_v \geq T_{\max}$ precise estimation of temperatures in the ultracold regime $T \ll \varepsilon_s$ is possible even if the energy gaps of the ancilla and the probe are arbitrarily larger than $T$. Hence, the advantage provided by the machine does not vanish for any gapless system.

**Steady-state regime.** The red curves in Fig. 1(b) depict steady-state SNRs for $M = 1$ and energy gaps $\varepsilon_v$ that satisfy Eq. (8). The prior temperatures $T$ labeling each curve are given by $T_1 = \varepsilon_s/4$, $T_2 = \varepsilon_s/8$, $T_3 = \varepsilon_s/12$, and $T_4 = \varepsilon_s/16$. Hence, the respective tasks consist of estimating temperatures that belong to the intervals $(0, \varepsilon_s/2)$, $(0, \varepsilon_s/4)$, $(0, \varepsilon_s/6)$, and $(0, \varepsilon_s/8)$. The red curves intersect the black dashed line at $T = T_v$, where $T = T_{\max}/2$. According to Eqs. (6) and (7), in this case

$$T \frac{\Delta \xi}{\Delta T} = \frac{1}{2} \frac{\varepsilon_s}{T} = \frac{1}{2} \frac{\varepsilon_s}{\tilde{T}}.$$  

Therefore, for $T = \tilde{T}$ the SNR diverges as $T$ tends to zero.

As we show in Sec. VI of SM [44], the scaling $O(\varepsilon_s/\tilde{T})$ in Eq. (9) is robust under the practical constraint of a not perfectly known temperature $T_v$. Specifically, in the interval $T \in (0, T_{\max})$ the steady-state SNR can attain a value $\frac{1}{2} \left(1 + \frac{\Delta T}{\varepsilon_s} T \right) \frac{\varepsilon_s}{\tilde{T}}$ if the relative error in the estimation of $T_v$ satisfies $\Delta T_v / T_v \leq 1/2$. Since $T_v$ can be much larger than $T$ the temperature $T_v$ can be efficiently estimated, e.g., by performing energy measurements on a thermalized two-level system [46].

The blue dotted curve in Fig. 1(b) shows the thermal SNR, obtained for $T_v = T$. In this case, the probe equilibrates to the thermal state associated with $T$. This is not surprising, since $H_1$ commutes with the total free Hamiltonian $H_P + H_s + H_v$ (cf. Eq. (3)) and therefore the thermal state $\rho_{p,sv}(T) = \frac{e^{-\beta(H_s + H_v)}}{\text{Tr}[e^{-\beta(H_s + H_v)}]}$ is invariant under $U_{p,sv}$. By applying Eq. (1) to $\rho_P(T) = \text{Tr}_{sv}[\rho_{p,sv}(T)]$ and assuming $\varepsilon_P = \varepsilon_s$, it follows that

$$T \frac{\Delta \xi}{\Delta T} = \sqrt{M} \frac{1}{1 + e^{-\beta \varepsilon_s}} \frac{\varepsilon_s}{T},$$

which is plotted in Fig. 1(b) for $M = 1$. Note that for temperatures $T \ll \varepsilon_s$ this SNR follows an exponential scaling $O\left(e^{-\frac{1}{2} \beta \varepsilon_s}\right)$.

**Transient regime.** In some situations the transient state of a system that probes a thermal bath may be more informative than its equilibrium state [5, 29, 47, 48]. In our case, this occurs if the probe starts in the ground state, see Fig. 2. The transient state $\rho^P_k(T)$ is completely characterized by the ground population $p^P_0$, derived analytically in Sec. III of SM [44]. Defining $r \equiv p^P_1/k + p^P_0/k$, we have that

$$p^P_{0,k} = \left[1 - (1 - r)^k\right] p^P_{0,\infty} + (1 - r)^k p^P_{0,0},$$

where $p^P_{0,0}$ is the initial ground population and $p^P_{0,\infty} = \frac{1}{1 + (\xi_s/\varepsilon_s + v/T)}$ is the steady-state value. Keeping in mind Eq. (11), the transient SNR is given by

$$T \frac{\Delta \xi}{\Delta T} = \sqrt{M} \frac{T \lambda^P_{0,k}(T)}{\sqrt{p^P_{0,k}(T)p^P_{1,k}(T))}},$$

where

$$\lambda^P_{0,k} = \frac{\partial p^P_{0,k}}{\partial T} = \left[1 - (1 - r)^k\right] \lambda^P_{0,\infty} + (k + 1) \left(p^P_{0,\infty} - p^P_{0,0}\right) \frac{\partial r}{\partial T} (1 - r)^k.$$
purple curves stand for temperatures \( T \) such that \( T < T_0 \), and \( T > T_0 \), respectively. These plots show that a high thermometric precision can also be attained in the transient regime. For \( p_{0,0}^P = 1/2 \), maximum precision is achieved for \( T = \varepsilon_s/4.5 < T_0 \) (Fig. 2(a)) and \( T = \varepsilon_s/10.5 < T_0 \) (Fig. 2(b)). If \( p_{0,0}^P = 1 \) we also see that the maximum SNR takes place before reaching the steady state, as mentioned before.

**Measurement cost and relation with the number of interactions.** We define the “measurement cost” as the number of measurements \( M \) required to achieve a given SNR. Figure 3 shows the thermal SNR (cyan dashed curve) and the transient SNR obtained with the machine (solid curves), for fixed parameters specified in the caption. In order to examine the corresponding measurement costs, the variable \( k \) has two possible meanings. For the cyan dashed curve, \( k = M \) is the number of measurements that must be performed on a thermalized probe to achieve the corresponding SNR. In the case of the solid curves, \( k \) is the number of interactions undergone by the probe before it is measured. The purple curve represents the SNR obtained from a single measurement (\( M = 1 \)) of the probe after \( k \) interactions. The red curve is the SNR obtained from \( M = 2 \) measurements. Since this requires to measure the same state \( \rho_{0}^{(k)}(T) \) two times, the whole process must be repeated. In other words, after the first measurement the probe should be reset to its initial state and then undergo \( k \) new interactions to be measured again. Remarkably, 20000 measurements performed on the thermal probe are not enough to surpass the SNR obtained with only 2 measurements on a probe that is prepared in the non-equilibrium state \( \rho_{0}^{(k)}(T) \).

This analysis provides a more operational characterization of the thermometric enhancement associated with the machine, in terms of an extreme reduction of the measurement cost. Now, we discuss the “interaction cost” (i.e. the number of interactions) to achieve a given precision. After \( k \) interactions, a measurement of the probe state \( \rho_{0}^{(k)}(T) \) represents a suboptimal POVM performed on \( k \) sample qubits. Hence, the associated SNR satisfies

\[
\frac{T}{\Delta_{k}T} \leq \frac{\sqrt{ke^{-\beta \varepsilon_s} \varepsilon_s}}{1 + e^{-\beta \varepsilon_s} T},
\]  

where the r.h.s. is the optimal SNR, obtained from an energy measurement of the \( k \) qubits.

Since this bound provides the ultimate precision, we also stress that it must be obeyed by any probing scheme. In the low-temperature regime a similar bound holds if the qubits are replaced by systems of arbitrary Hamiltonian \( H_s = \sum_i \varepsilon_i^{*} |i_s\rangle \langle i_s| \), as long as the energy gap above the ground state satisfies \( \varepsilon_i^+ - \varepsilon_i^0 \gg T \). This is a consequence of the universal scaling \( \mathcal{F}_s^{th}(T) \sim \mathcal{O}(e^{-\beta (\varepsilon_i^+ - \varepsilon_i^0)}) \) that characterizes gapped systems. In particular, it implies that any sequential probing of \( k \) non-interacting systems with energy gap \( \varepsilon_i^+ - \varepsilon_i^0 \gg T \) requires at least \( k \sim e^{\beta (\varepsilon_i^+ - \varepsilon_i^0)} \) interactions to attain a SNR \( \sim \mathcal{O}(1/T) \).

**Sample perturbation.** Reducing the back action of temperature measurements in the sample is a very de-
Consider a probe that is initialized in the fully mixed state. After $k$ interactions, the maximum heating of the sample is less than a single quantum of energy $\varepsilon_s$. Namely, $\lim_{k \to \infty} Q_S^k = \varepsilon_s (p_{0,0}^p - p_{0,\infty}^p) [1 - (1 - r)^k]$. Remarkably, we see that for $p_{0,0}^p > p_{0,\infty}^p$, the maximum heating of the sample is less than a single quantum of energy $\varepsilon_s$. Which are more experimentally friendly. In Sec. IX of [44], we also show that the transient SNR (12) can be obtained using machines based on two-body interactions, which is arguably a less harmful disturbance. We numerically illustrate this situation in Sec. VIII of [44], by considering a probe that is initialized in the fully mixed state. In contrast, for $p_{0,0}^p = 1$ (initial ground state) the sample is inevitably heated up.

Conclusions. Heat engines harness a temperature difference for work extraction, a thermodynamic task that would otherwise be impossible. In this letter, we propose a thermometric machine that employs a high-temperature thermal bath to efficiently encode the ultracold temperature of a thermal sample in a probe. This is manifested both in the transient state of the probe and its steady state, whose maximum SNR grows as $O(1/T)$ [49] and beats the exponential scaling that hinders low-temperature equilibrium thermometry in gapped systems.

The machine operates through an energy-conserving three-body interaction that characterizes absorption refrigerators [50–52] and other autonomous thermodynamic devices [53]. In a future, it would be interesting to study if similar thermometric enhancements can be obtained using machines based on two-body interactions, which are more experimentally friendly. In Sec. IX of SM [44], we also show that the transient SNR (12) can remain above $\sim 38\%$ of the SNR corresponding to an optimal coupling between the probe and the sample [28]. This result is noteworthy, keeping in mind the simplicity of the three-body interaction that drives our machine. Finally, we remark that the use of thermometric machines to probe samples comprising interacting particles is also an open subject of investigation.

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* ivan.henao@mail.huji.ac.il
† karen.hovhannisyan@uni-potsdam.de
‡ raam@mail.huji.ac.il

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Consider a sample formed by \( N \) systems of dimension \( d_s \geq 2 \), each of which is characterized by the Hamiltonian \( H_s = \sum_{i=0}^{d_s-1} \varepsilon_{s}^{(i)} |i_s\rangle \langle i_s| \). Our goal is to show that the thermometric machine presented in the main text can be straightforwardly adapted to this kind of sample. More specifically, we will demonstrate that: highly precise (probe) thermometry for \( d_s = 2 \) implies highly precise thermometry for arbitrary dimension \( d_s \) and energy spectrum \( \{\varepsilon_{s}^{(i)}\} \), in the steady-state regime of the probe.

The idea is to substitute the interaction Hamiltonian \( H_I \) in the main text by another (three-body) Hamiltonian that couples the probe and the ancilla to a single pair of energy levels in the system. Associating these energy levels with eigenstates \( |j_s\rangle \) and \( |j'_s\rangle \), and eigenenergies \( \varepsilon_{s}^{(j')} > \varepsilon_{s}^{(j)} \), we consider the interaction Hamiltonian

\[
H_I = |j_{pjs}1_{v_s}\rangle \langle j_{pjs}0_{v_s}| + \text{h.c.} \tag{S1}
\]

In this way, we will show that the repeated switching of \( H_I \) drives the probe through “probabilistic collisions” with the ancilla and an effective qubit of energy gap \( \varepsilon_{s}^{(j')} - \varepsilon_{s}^{(j)} \), equilibrated at the temperature of the sample. This dynamics leads the probe to a steady state that allows us to translate the results obtained for dimension \( d_s = 2 \) to arbitrary dimension.

For \( d_s \) arbitrary, the total free Hamiltonian \( H_{\text{free}} = H_P + H_s + H_v \) has eigenstates \( \{|0_P\}, \{|1_P\}\} \otimes \{|i_s\} \otimes |0_v, 1_v\rangle \). Since \( H_I \) transforms eigenstates outside the subspace span\( \{|0_P\}, \{|1_P\}\} \otimes \{|j_s\}, |j'_s\rangle \otimes |0_v, 1_v\rangle \) to the zero vector, these eigenstates are also eigenstates of the total Hamiltonian \( H = H_{\text{free}} + H_I \). Based on this property, we can conveniently express the evolution generated by \( H \) on an initial state of the form \( \rho = \rho_P \otimes \rho_s \otimes \rho_v \), where \( \rho_s \) is a thermal state and \( \rho_P \) and \( \rho_v \) are both diagonal in the corresponding energy bases.

To this end, we write \( \rho_s \) as \( \rho_s = \sum_{i=0}^{d_s-1} p_i^s(T)|i_s\rangle \langle i_s| \), where \( p_i^s(T) = \frac{\sigma}{(1 - \sigma)} \) are thermal populations at temperature \( T \), and the total state as

\[
\rho = (1 - p_j^s(T) - p_{j'}^s(T))\sigma + (p_j^s(T) + p_{j'}^s(T))\rho_{j,j'}, \tag{S2}
\]

where

\[
\sigma = \frac{\rho_P \otimes \rho_v \otimes \sum_{i \neq j,j'} p_i^s(T)|i_s\rangle \langle i_s|}{(1 - p_j^s(T) - p_{j'}^s(T))},
\]

\[
\rho_{j,j'} = \frac{p_j^s(T)|j_s\rangle \langle j_s| + p_{j'}^s(T)|j'_s\rangle \langle j'_s|}{p_j^s(T) + p_{j'}^s(T)}. \tag{S3}
\]

As anticipated at the beginning of this appendix, we see that \( \rho_{j,j'} \) is the thermal state (at temperature \( T \)) of an effective two-level level system characterized by the Hamiltonian \( H^\text{eff}_s = \varepsilon_{s}^{(j)} |j_s\rangle \langle j_s| + \varepsilon_{s}^{(j')} |j'_s\rangle \langle j'_s| \). Defining the map

\[
\mathcal{E}_{P,j,j'}(\rho_P) \equiv \text{Tr}_{sv}\left[ e^{-iH_t} \rho_P \otimes \rho_v \otimes \rho_{s,j,j'} e^{iH_t} \right], \tag{S4}
\]

it follows that a collision with the probe is described by the map

\[
\mathcal{E}_P(\rho_P) \equiv \text{Tr}_{sv}\left[ e^{-iH_t} \rho_P \otimes \rho_v \otimes \rho_{s} e^{iH_t} \right] = (1 - p_j^s(T) - p_{j'}^s(T)) e^{-iHp_t} \rho_P e^{iHp_t} + (p_j^s(T) + p_{j'}^s(T)) \mathcal{E}_{P,j,j'}(\rho_P), \tag{S5}
\]

which characterizes a transformation where a collision with the aforementioned two-level system occurs with probability \( p_j^s(T) + p_{j'}^s(T) \).

In contrast, the maps \( \mathcal{E}_P \) and \( \mathcal{E}_{P,j,j'} \) possess a common (and unique) fixed point \( \rho_{P,j,j'}^{(\infty)} \), which implies that collisions with the effective two-level system fully determine the steady state of the probe. This follows from the fact that \( \rho_{P,j,j'}^{(\infty)} \) is diagonal in the energy basis of the probe, as shown in the next appendix. In such a case, a direct substitution of \( \rho_P \) by \( \rho_{P,j,j'}^{(\infty)} \) in Eq. (S5) shows that

\[
\mathcal{E}_P(\rho_{P,j,j'}^{(\infty)}) = \mathcal{E}_{P,j,j'}(\rho_{P,j,j'}^{(\infty)}). \tag{S6}
\]

Note also that this is possible because \( e^{-iHp_t} \rho_P e^{iHp_t} = \rho_P \) for any \( \rho_P \) diagonal.

In the next appendix we derive the probe evolution corresponding to sample subsystems of dimension \( d_s = 2 \), for which we find that the steady state is diagonal in the eigenbasis of \( H_P \). Since this evolution amounts to repeated applications of the map \( \mathcal{E}_{P,j,j'} \), the diagonal character of \( \rho_{P,j,j'}^{(\infty)} \) follows. In this way we prove the main claim of this appendix. Namely, that for the interaction (S1) the corresponding steady state can be described in terms of collisions with the ancilla and thermal quibts.

\section{S-II. Derivation of the Global (Probe-Sample-Ancilla) Evolution}

In this appendix we solve the Schrödinger equation for the total Hamiltonian \( H = H_P + H_s + H_v + H_I \).

If \( \varepsilon_v = \varepsilon_s + \varepsilon_P \), the interaction Hamiltonian \( H_I = |0_P 0_s 1_v\rangle \langle 1_P 1_s 0_v| + \text{h.c.} \) commutes with the free Hamiltonian \( H_{\text{free}} = H_P + H_s + H_v \). This follows from the

\[\text{PROBE-SAMPLE-ANCILLA EVOLUTION}\]
commutation relations

\[ [H_s, H_I] = \varepsilon_s \varepsilon_I (|0_p 0_s 1_v⟩⟨1_p 1_s 0_v| + \text{h.c.}), \]  
\[ [H_P, H_s] = \varepsilon_p \varepsilon_s (|0_p 0_s 1_v⟩⟨1_p 1_s 0_v| + \text{h.c.}), \]  
\[ [H_v, H_s] = \varepsilon_v \varepsilon_I (|0_p 0_s 1_v⟩⟨1_p 1_s 0_v| - \text{h.c.}), \]  

which, when added together, yield a commutator \([H_{\text{free}}, H_I]\) proportional to \(\varepsilon_v - \varepsilon_s - \varepsilon_P\). Therefore, the evolution during a time \(t\) is given by the global unitary

\[ U_{Psv}(t) = e^{-iHt} = e^{-iH_{\text{free}}t}e^{-iH_{\text{int}}t}. \]  

The eigenstates of \(U_{Psv}(t)\) can be obtained by diagonalizing \(H_I\). In the eigenbasis of \(H_{\text{free}}\) is a matrix with all the entries equal to zero except for \(|0_p 0_s 1_v⟩⟨1_p 1_s 0_v| = (1_p 1_s 0_v| H_I |0_p 0_s 1_v⟩ = \varepsilon_I\). The straightforward diagonalization leads to

\[ H_I |+⟩ = \varepsilon_I |+⟩, \]  
\[ H_I |−⟩ = -\varepsilon_I |−⟩, \]  

where \(|±⟩ = \frac{1}{\sqrt{2}} (|0_p 0_s 1_v⟩ ± |1_p 1_s 0_v⟩)\).

Expressing \(|0_p 0_s 1_v⟩\) as \(|0_p 0_s 1_v⟩ = \frac{1}{\sqrt{2}} (|+⟩ + |−⟩)\) and \(|1_p 1_s 0_v⟩\) as \(|1_p 1_s 0_v⟩ = \frac{1}{\sqrt{2}} (|+⟩ − |−⟩)\), we have that

\[ e^{-iH_{\text{int}}t} |0_p 0_s 1_v⟩ = \frac{1}{\sqrt{2}} \left(e^{-i\varepsilon_I t} |+⟩ + e^{i\varepsilon_I t} |−⟩\right), \]  
\[ e^{-iH_{\text{int}}t} |1_p 1_s 0_v⟩ = \frac{1}{\sqrt{2}} \left(e^{-i\varepsilon_I t} |+⟩ − e^{i\varepsilon_I t} |−⟩\right). \]  

Accordingly, for \(t = \frac{\pi}{2\varepsilon_I}\), \(e^{-iH_{\text{int}}t} |0_p 0_s 1_v⟩ = -i |1_p 1_s 0_v⟩\) and \(e^{-iH_{\text{int}}t} |1_p 1_s 0_v⟩ = i |0_p 0_s 1_v⟩.\) Since \(|0_p 0_s 1_v⟩\) and \(|1_p 1_s 0_v⟩\) are both eigenstates of \(H_{\text{free}}\), Eq. (S9) yields

\[ U_{Psv}(\pi/2\varepsilon_I) |0_p 0_s 1_v⟩ = |1_p 1_s 0_v⟩, \]  
\[ U_{Psv}(\pi/2\varepsilon_I) |1_p 1_s 0_v⟩ = |0_p 0_s 1_v⟩. \]  

On the other hand, \(H_I |ip_j k_v⟩ = 0\) for any eigenstate of \(H_{\text{free}}\) different from \(|0_p 0_s 1_v⟩\) and \(|1_p 1_s 0_v⟩\), and thus

\[ U_{Psv}(\pi/2\varepsilon_I) |ip_j k_v⟩ \propto |ip_j k_v⟩. \]  

If \(U_{Psv}(\pi/2\varepsilon_I)\) is applied on any state diagonal in the eigenbasis of \(H_{\text{free}}\), the phase factors implicit in Eqs. (S14)-(S16) cancel out. This is the case in particular for the initial state \(\rho_P^{(0)} \otimes \rho_s(T) \otimes \rho_v\), where \(\rho_P^{(0)}, \rho_s(T)\) and \(\rho_v\) are diagonal in the corresponding energy bases. Moreover, any application of \(U_{Psv}(\pi/2\varepsilon_I)\) yields another diagonal state. This implies that for any state \(\rho_P^{(k)} \otimes \rho_s(T) \otimes \rho_v\) (where \(\rho_P^{(k)}\) is the probe state after \(k\) cycles, see main text), the aforementioned phases can be ignored in the subsequent evolution generated by \(U_{Psv}(\pi/2\varepsilon_I)\).

S-III. PROBE STATE AFTER \(k\) INTERACTIONS

As mentioned above, the state \(\rho_P^{(k)}\) is diagonal in the eigenbasis of \(H_P\). Hence, it suffices to compute the population \(p_{0,k}^{P}\) to fully characterize this state. For the sake of brevity, in the following we will write \(U_{Psv}(\pi/2\varepsilon_I)\) as \(U_{\text{psv}}\). The state \(\rho_P^{(k+1)}\) is related to the previous state \(\rho_P^{(k)}\) through the expression:

\[ \rho_P^{(k+1)} = \text{Tr}_{sv} \left[ U_{Psv} \left( \rho_P^{(k)} \otimes \rho_s \otimes \rho_v \right) U_{Psv}^\dagger \right]. \]  

From Eqs. (S14)-(S16), \(U_{Psv}\) exchanges the populations of the eigenstates \(|0_p 0_s 1_v⟩\) and \(|1_p 1_s 0_v⟩\), and keeps any other population unmodified. Therefore,

\[ p_{0,k+1}^P = p_{0,k}^P + \Delta_{k+1} p, \]  

where \(\Delta_{k+1} p = p_{0,k+1}^P - p_{0,k}^P\) is the population variation of the eigenstate \(|0_p 0_s 1_v⟩\), resulting from the \((k + 1)\)th cycle. Explicitly,

\[ \Delta_{k+1} p = p_{1,k}^P (p_{0,k}^P - p_{0,k}^P (p_{0,k}^P - p_{0,k}^P) = p_{1,k}^P - r p_{0,k}^P, \]  

where

\[ r = p_{1,k}^P + p_{0,k}^P \]  

and the second line follows by writing \(p_{1,k}^P = 1 - p_{0,k}^P\).

In the limit \(k \to \infty\), the population \(p_{0,k}^P\) reaches the steady-state value and thus \(\Delta_{k+1} p = 0\). Hence, from Eq. (S19) we obtain:

\[ p_{0,\infty}^P = \frac{p_{1,0}^P}{r} = \frac{1}{1 + e^{\beta\varepsilon_v - \beta\varepsilon_s}}. \]  

Using this expression we can also rewrite Eq. (S19) as

\[ \Delta_{k+1} p = r \left( p_{0,\infty}^P - p_{0,k}^P \right), \]  
\[ p_{0,k+1}^P = (1 - r) p_{0,k}^P + rp_{0,\infty}^P. \]  

Let us see now that (for any \(k \geq 0\)) the recurrence relation (S22) is satisfied by the expression

\[ p_{0,k+1}^P = \sum_{j=0}^{k} (1 - r)^j r p_{0,\infty}^P + (1 - r)^{k+1} p_{0,0}^P \]

\[ = \left[ 1 - (1 - r)^{k+1} \right] p_{0,\infty}^P \]  
\[ + (1 - r)^{k+1} p_{0,0}^P. \]  

(S23)

The second line is obtained by computing the geometric series \(\sum_{j=0}^{k}(1 - r)^j\). This can be straightforwardly proved by induction. For \(k = 0\), Eq. (S23) yields \(p_{0,1}^P = r p_{0,\infty}^P + (1 - r) p_{0,0}^P\), in agreement with Eq. (S22). Moreover, the substitution of \(p_{0,k+1}^P\) (as given in Eq. (S23)) into \(p_{0,k+2}^P = (1 - r) p_{0,k+1}^P + r p_{0,\infty}^P\) (cf. (S22)) yields

\[ p_{0,k+2}^P = (1 - r) \left[ 1 - (1 - r)^{k+1} \right] p_{0,\infty}^P \]  
\[ + (1 - r)^{k+2} p_{0,0}^P \]

\[ = \left[ 1 - (1 - r)^{k+2} \right] p_{0,\infty}^P \]  
\[ + (1 - r)^{k+2} p_{0,0}^P. \]  

(S24)

This means that if Eq. (S23) holds for \(k\), then it also holds for \(k + 1\). Since it is valid for \(k = 0\), it also follows that it is valid for any \(k \geq 0\).
S-IV. SIGNAL-TO-NOISE RATIO (SNR) FOR ENERGY MEASUREMENTS ON $\rho_{p}^{(k)}$

Here we derive the thermometric precision corresponding to energy measurements on the probe state $\rho_{p}^{(k)}(T)$. As a byproduct, we show also that these measurements saturate the associated Cramér-Rao bound. If $\Delta_{k}T$ denotes the (absolute) error for the estimation of $T$ on $\rho_{p}^{(k)}(T)$, the aforementioned saturation means that

$$\Delta_{k}T = \frac{1}{\sqrt{MF_{p}^{(k)}(T)}}$$  \hspace{1cm} (S25)

where

$$F_{p}^{(k)}(T) = \sum_{i=0}^{\infty} p_{i,k}(T) \left( \frac{\partial \ln(p_{i,k}(T))}{\partial T} \right)^{2}$$

$$= \frac{1}{p_{0,k}(T)p_{1,k}(T)} \left( \frac{\partial p_{1,k}(T)}{\partial T} \right)^{2}$$ \hspace{1cm} (S26)

is the Fisher information that results from energy measurements on $\rho_{p}^{(k)}(T)$.

Consider the error propagation formula [16]

$$\Delta T = \sqrt{\frac{\text{Var}(O)}{M \left| \frac{\partial \langle O \rangle}{\partial T} \right|}},$$ \hspace{1cm} (S27)

which provides an expression for $\Delta T$, given that $T$ is estimated from $M$ (independent) measurements of an observable $O$. In this formula, $\text{Var}(O) = \text{Tr}[\langle O^2 \rho(T) \rangle - \langle O \rangle^2]$ and $\langle O \rangle = \text{Tr}[\langle O \rho(T) \rangle]$ are respectively the variance and mean value of $O$, with respect to a state $\rho(T)$ where the temperature $T$ has been encoded. If $O = H_{p} = \varepsilon_{p}|1_{p}\rangle\langle 1_{p}|$ and $\rho(T) = \rho_{p}^{(k)}(T)$, we have that

$$\text{Var}(H_{p}) = p_{0,k}(T)p_{1,k}(T)\varepsilon_{p}^2,$$ \hspace{1cm} (S28)

$$\frac{\partial}{\partial T}(H_{p}) = \frac{\partial p_{1,k}(T)}{\partial T} \varepsilon_{p}$$

$$= -\frac{\partial p_{0,k}(T)}{\partial T} \varepsilon_{p},$$ \hspace{1cm} (S29)

where the third line follows from probability conservation: $\frac{\partial}{\partial T}(p_{0,k}(T)+p_{1,k}(T)) = 0$.

Let us now define the "sensitivity"

$$\lambda_{0,k}^{p}(T) = \frac{\partial p_{0,k}(T)}{\partial T}.$$ \hspace{1cm} (S30)

In this way, the substitution of Eqs. (S28) and (S29) into Eq. (S27) lead to

$$\Delta T = \frac{\sqrt{p_{0,k}(T)p_{1,k}(T)}}{\sqrt{M |\lambda_{0,k}^{p}(T)|}}.$$ \hspace{1cm} (S31)

By comparing Eqs. (S26) and (S31), we see that the saturation of the Cramér-Rao bound (cf. Eq. (S25)) follows.

S-V. STEADY-STATE SNR (LIMITS $N \rightarrow \infty$ AND $k \rightarrow \infty$)

The steady state SNR is obtained by taking the limit $\lim_{k \rightarrow \infty} \Delta_{k}T$ (cf. Eq. (S31)), in the expression $T/\Delta_{k}T$. From Eqs. (S30) and (S21), it follows that

$$\lambda_{0,\infty}^{p}(T) = \frac{\partial p_{0,\infty}}{\partial T}$$

$$= \frac{e^{\beta \varepsilon_{s} - \beta_{u} \varepsilon_{u}}}{(1 + e^{\beta \varepsilon_{s} - \beta_{u} \varepsilon_{u}})^{2} T^{2}}$$

$$= p_{0,\infty}(T)p_{1,\infty}(T) \frac{\varepsilon_{s}}{T^{2}}.$$ \hspace{1cm} (S32)

Therefore,

$$\frac{T}{\Delta_{\infty}T} = \sqrt{\frac{M \left(p_{0,\infty}(T)p_{1,\infty}(T)\right) \varepsilon_{s}}{T}}$$

$$= \sqrt{Me^{\beta \varepsilon_{s} - \beta_{u} \varepsilon_{u}} \frac{\varepsilon_{s}}{T}}$$

$$= \sqrt{M e^{-\frac{1}{(\beta_{s} - \beta_{u} \varepsilon_{u})}} \frac{\varepsilon_{s}}{T}}.$$ \hspace{1cm} (S33)

S-VI. STEADY-STATE SNR FOR IMPRECISE ANCILLARY TEMPERATURE

Previously we have computed the SNR assuming that the temperature $T_{v}$ is perfectly known. However, in practice this temperature must also be estimated in order to evaluate the ancillary energy gap $\varepsilon_{v}$ (cf. (8) in the main text), and such an estimation carries a finite error $\Delta T_{v}$. Denoting the estimated ancillary temperature as $T_{v}^{\text{est}}$, and the actual value as $T_{v}$, we thus have that $T_{v}^{\text{est}} = T_{v} \pm \Delta T_{v}$. The purpose of this appendix is to show that the steady-state SNR for $T$ is quite robust to the error $\Delta T_{v}$, if the corresponding SNR satisfies $T_{v}/\Delta T_{v} \geq 2$. Since $T_{v}$ can be larger than $T$, the exponential inefficiency in the estimation of $T$ does not carry over the estimation of $T_{v}$. For example, one could estimate $T_{v}$ via measurements of the Hamiltonian $H_{u} = \varepsilon_{u}|1_{u}\rangle\langle 1_{u}|$ of a two-level system that has been thermalized at temperature $T_{v}$ and whose energy gap is such that $T_{v}/\Delta T_{v}$ is maximized. Specifically, the thermal SNR corresponding to $M$ energy measurements on $\rho_{u} = \frac{e^{-\lambda_{u} H_{u}}}{\text{Tr}[e^{-\lambda_{u} H_{u}}]}$ reads

$$\frac{T_{v}}{\Delta T_{v}} = \sqrt{M \frac{e^{-\frac{1}{2} \beta_{s} \varepsilon_{u}} \varepsilon_{u}}{1 + e^{-\beta_{s} \varepsilon_{u}} T_{v}}}.$$ \hspace{1cm} (S34)

This quantity attains its maximum $\max_{\varepsilon_{u}} T_{v}/\Delta T_{v} \sim \sqrt{M}$ for $\beta_{s} \varepsilon_{u} = 2.5$. Hence, $M = 16$ measurements suffice to yield an SNR $T_{v}/\Delta T_{v} \sim 2.6 > 2$. 


To analyze the impact of the error $\Delta T_v$ on the SNR that characterizes the estimation of $T$ it is convenient to recall the error-free expression (Eq. (7) in the main text)

$$\frac{T}{\Delta \infty T} = \frac{\sqrt{M}e^{-\frac{1}{2} \beta \varepsilon_s (1 - \frac{1}{\tilde{T}})}}{1 + e^{-\beta \varepsilon_s (1 - \frac{1}{\tilde{T}})}} \frac{\varepsilon_s}{T}, \quad (S35)$$

where we have written $T_{\text{max}}$ as $2\tilde{T}$ ((6) in the main text). We note that for $0 < T < 2\tilde{T}$, which is the interval where the temperature $T$ is known to be, the factor $1 - \frac{T}{2\tilde{T}}$ varies in the interval $(-1, 1)$. Hence, for the extremal points -1 and 1 of this interval we have that $e^{-\frac{1}{2} \beta \varepsilon_s (1 - \frac{1}{\tilde{T}})} = e^{-\frac{1}{1 + e^{-\beta \varepsilon_s (1 - \frac{1}{\tilde{T}})}}}$ and consequently Eq. (S35) yields the thermal SNR corresponding to temperatures $T \to 0$ and $T = 2\tilde{T}$. On the other hand, for all $T \in (0, 2\tilde{T})$ the ratio $e^{-\frac{1}{2} \beta \varepsilon_s (1 - \frac{1}{\tilde{T}})}$ is strictly larger than $e^{-\frac{1}{1 + e^{-\beta \varepsilon_s (1 - \frac{1}{\tilde{T}})}}}$, attaining a maximum of $1/2$ if $T = \tilde{T}$. This occurs because the factor $1 - \frac{T}{2\tilde{T}}$ “damps” the exponent $\beta \varepsilon_s$ and prevents that the exponential $e^{-\frac{1}{2} \beta \varepsilon_s (1 - \frac{1}{\tilde{T}})}$ vanishes even if $\beta \varepsilon_s \gg 1$. A similar situation occurs if $\Delta T_v > 0$ and $T_v/\Delta T_v \geq 2$, as we discuss below.

If $\Delta T_v > 0$ the SNR is modified via the error in the tuning of the energy gap $\varepsilon_v$, which is chosen according to $T_v^\text{est}$ instead of $T_v$:

$$\varepsilon_v = \frac{\varepsilon_v}{T} T_v = \frac{\varepsilon_v}{T} (T_v \pm \Delta T_v). \quad (S36)$$

In this way, instead of the ideal expression (S35) the steady-state SNR $\frac{\sqrt{M}e^{-\frac{1}{2} \beta \varepsilon_s [1 - \frac{T}{\tilde{T}}]} \varepsilon_s}{1 + e^{-\beta \varepsilon_s [1 - \frac{T}{\tilde{T}}]}} \frac{\varepsilon_s}{T}$ (cf. Eq. (S33)) takes the form

$$\frac{T}{\Delta \infty T} = \frac{\sqrt{M}e^{-\frac{1}{2} \beta \varepsilon_s [1 - \frac{T}{\tilde{T}}] \varepsilon_s}}{1 + e^{-\beta \varepsilon_s [1 - \frac{T}{\tilde{T}}] \varepsilon_s}} \frac{\varepsilon_s}{T} \quad (S37)$$

Here, the factor $x_T \equiv 1 - \frac{T}{\tilde{T}} \left(1 \pm \frac{\Delta T_v}{\tilde{T}} \right)$ is what determines the damping of the exponent $\beta \varepsilon_s$, and as long as $e^{-\frac{1}{2} \beta \varepsilon_s x_T} \varepsilon_s$ remains finite in the limit $\beta \varepsilon_s \gg 1$ we can guarantee that the non-ideal SNR (S37) is not exponentially suppressed. In particular, for $x_T = 0$ we have that

$$T = \frac{\tilde{T}}{1 \pm \frac{\Delta T_v}{\tilde{T}}} \quad (S38)$$

and

$$\frac{T}{\Delta \infty T} = \frac{\sqrt{M}}{2} \left(1 \pm \frac{\Delta T_v}{T_v} \right) \frac{\varepsilon_s}{T}. \quad (S39)$$

This expression indicates that the error $\Delta T_v$ does not alter the scaling $\mathcal{O}(\varepsilon_s/\tilde{T})$ previously found for the ideal SNR (as the only difference is the constant $1 \pm \frac{\Delta T_v}{\tilde{T}}$). However, it is worth noting that if $T = \frac{T_v}{1 \pm \frac{\Delta T_v}{T_v}}$ then $T \in (0, 2\tilde{T})$ if and only if $T_v/\Delta T_v \geq 2$. Keeping in mind that the plus and minus signs in Eqs. (S38) and (S39) correspond respectively to estimated temperatures $T_v + \Delta T_v$ and $T_v - \Delta T_v$, the condition $T_v/\Delta T_v \geq 2$ ensures that if $T_v^\text{est} = T_v - \Delta T_v$, the SNR $\frac{\sqrt{M}}{2} \left(1 - \frac{\Delta T_v}{T_v} \right) \frac{\varepsilon_s}{T}$ can be obtained for some temperature in the prior interval $(0, 2\tilde{T})$. Finally, we note that the SNR (S37) also decays to exponentially small values as $T$ approaches the extrema 0 and $2\tilde{T}$. However, this is not different from the ideal SNR (corresponding to $\Delta T_v = 0$), as discussed before.

**S-VII. TRANSIENT SNR**

The transient SNR is obtained by substituting Eq. (S23) into (S31) and evaluating explicitly $\lambda^P_{0,k}(T)$. To compute $\lambda^P_{0,k}(T)$ we only have to take the derivative of (S31) with respect to $T$:

$$\begin{align*}
\lambda^P_{0,k+1} &= \frac{\partial P_{0,k+1}}{\partial T} \\
&= \left[1 - (1 - r)^{k+1}\right] \lambda^P_{0,\infty} \\
&+ (k + 1) \left(p^0_{0,\infty} - p^0_{0,0}\right) \frac{\partial r}{\partial T} (1 - r)^k. \quad (S40)
\end{align*}$$

Here, $\lambda^P_{0,\infty}$ is obtained from Eq. (S21) and $\frac{\partial r}{\partial T}$ is obtained from Eq. (S20).

**S-VIII. PERTURBATION OF THE SAMPLE**

In this appendix we present analytical and numerical results concerning the perturbation of the sample and ancillary states during the transient dynamics. These results indicate that preparing the probe in a completely mixed state generates less perturbation, as compared to preparing it in the ground state. Moreover, the sample is always heated up in the second case, while it can some times be cooled down if the probe starts in the fully mixed state.

We first apply Eq. (S23) to derive the total heat transferred to the sample and to the ancillary bath after $k$ interactions. Due to the form of $H_I$, at the $(k + 1)$th cycle two situations can take place. If the ground state of the probe gains population $\Delta k_{+1} p = p^0_{0,k+1} - p^0_{0,k}$, the corresponding sample qubit releases heat $\Delta k_{+1} p_{\varepsilon_s}$ and the ancilla absorbs heat $\Delta k_{+1} p_{\varepsilon_a}$, which equals the heat absorbed by the ancillary bath in the subsequent thermalization. The opposite situation occurs if $\Delta k_{+1} p < 0$.

After $k$ cycles, the sample and the ancilla have absorbed heat $Q^k_s$ and $Q^k_a$, respectively. The heat intake to the sample corresponding to the $jth$ cycle reads
FIG. S1. Perturbation of the sample and the ancilla, generated by the joint interaction with the probe (color coding in the bottom charts). The two upper plots show the ground population of the \((k+1)\)th sample qubit and the two lower plots show the ground population of the ancilla, after the \((k+1)\)th interaction. The black dashed lines correspond to the steady-state values. Fixed parameters in (a) and (b) are respectively the same in Figs. 2(a) and 2(b) of the main text. Thus, the prior temperatures are \(T = \varepsilon_s/4\) for (a) and \(T = \varepsilon_s/10\) for (b).

\[-\Delta_{j+1} p \varepsilon_s.\] Moreover, the ancilla absorbs heat \(\Delta_{j+1} p \varepsilon_v.\) Using Eq. \eqref{eq:Q_s}, we obtain the total heats

\[
Q_s^k = -\sum_{j=0}^{k-1} \Delta_{j+1} p \varepsilon_s = \varepsilon_s (p^p_{0,0} - p^p_{0,\infty}) [1 - (1 - r)^k],
\]

\[
Q_v^k = \sum_{j=0}^{k-1} \Delta_{j+1} p \varepsilon_v = \varepsilon_v (p^p_{0,\infty} - p^p_{0,0}) [1 - (1 - r)^k].
\]

These equations show consistently that whenever the sample is heated up the ancilla is cooled down and vice versa. The plots in Fig. S2 also corroborate this behavior. To obtain these graphs, we consider fixed physical parameters identical to those of Fig. 2 in the main text. Specifically, for Fig. 4(a) the prior temperature is \(T = \varepsilon_s/4\) and we consider two sample temperatures \(T_s = \varepsilon_s/4.5\) and \(T_s = \varepsilon_s/3.5.\) Figure 4(b) shows plots for the prior temperature is \(T = \varepsilon_s/10\) and two sample temperatures \(T_s = \varepsilon_s/10.5\) and \(T_s = \varepsilon_s/9.5.\)

The plots in Fig. S1 depict the ground population of the sample qubits and the ancilla as a function of the number of interactions with the probe. For \(k\) sufficiently large, the probe approaches its steady state and hence the perturbations that it causes become negligible. As a consequence, the populations approach their unperturbed values given by the black dashed lines. From these plots we can draw the following insights:

1. Preparing the probe in the fully mixed state generally causes less perturbation, as indicated by the red and orange curves.

2. If the probe starts in its ground state (blue and purple curves), the ancilla is always cooled down and the sample is always heated up. On the other hand, the red curves show that the opposite effect can occur if the probe starts in the fully mixed state. Since cooling back the sample is naturally more difficult than reheating it, this shows that to minimize the disturbance of the sample it is preferable to initialize the probe in the fully mixed state.

3. For larger temperatures (Fig. S1(a)) the perturbation of both the sample and the ancilla is larger. While the corresponding population variations are between orders of \(10^{-3}\) and \(10^{-2}\), the order of the variations in Fig. S1(b) is \(10^{-5}\). This is noteworthy because it suggests that colder samples are subject to less (absolute) perturbation.

S-IX. COMPARISON BETWEEN THE TRANSIENT SNR (EQ. \(12\) IN THE MAIN TEXT) AND THE SNR FOR AN OPTIMAL PROBE-SAMPLE INTERACTION

Reference [28] studies optimal POVMs for coarse-grained thermometry, where the number of outcomes of the POVM is smaller than the dimension of the measured sample. In particular, a generic two-outcome POVM can be implemented by letting the sample to undergo an arbitrary interaction (possibly involving ancillas) with a two-level probe. Given this equivalence between coarse-grained and probe-based thermometry, we can apply the results of [28] to compare the performance of the thermometric machine with the maximum SNR that results from such an optimal interaction. This comparison is performed in Fig. S2, for fixed parameters \(\varepsilon_s/T = 8\) and \(\varepsilon_s/T = 7.\)

The red solid curve in Fig. S2 shows the ratio between the optimal probe SNR and the sample SNR (obtained via an energy measurement of \(k\) qubits). This ratio tends asymptotically towards the value \(\sqrt{2}/\pi\), as indicated by the black dashed line. On the other hand, the blue solid curve depicts the ratio between the transient SNR of our machine’s probe and the sample SNR. For the considered parameters \(k \approx 6000\) interactions allow to approximate very well the steady-state SNR \(\sim 3.54\). Clearly, more interactions cannot increase this value and therefore the blue curve is monotonically decreasing. However, it is remarkable that in the interval \(k \in [1, 6000]\) the machine produces an SNR that is at least \(\sim 38\%\) of the optimal
FIG. S2. Comparison between the transient SNR achieved with the machine and the maximum probe SNR corresponding to an optimal probe-sample interaction. Fixed parameters are \( \varepsilon_s/T = 8 \) and \( \varepsilon_s/\tilde{T} = 7 \). In all the cases the SNRs are “normalized” with respect to the sample SNR and computed for a single measurement \( M = 1 \).

one, even though in the worst case scenario this would require a \((k + 1)\)-body interaction.