Random Points, Convex Bodies, Lattices

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Abstract

Assume \( K \) is a convex body in \( \mathbb{R}^d \), and \( X \) is a (large) finite subset of \( K \). How many convex polytopes are there whose vertices come from \( X \)? What is the typical shape of such a polytope? How well the largest such polytope (which is actually \( \text{conv} X \)) approximates \( K \)? We are interested in these questions mainly in two cases. The first is when \( X \) is a random sample of \( n \) uniform, independent points from \( K \) and is motivated by Sylvester’s four-point problem, and by the theory of random polytopes. The second case is when \( X = K \cap \mathbb{Z}^d \) where \( \mathbb{Z}^d \) is the lattice of integer points in \( \mathbb{R}^d \). Motivation comes from integer programming and geometry of numbers. The two cases behave quite similarly.

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1. Sylvester’s question

In the 1864 April issue of the Educational Times J. J. Sylvester [26] posed the innocent looking question that read: “Show that the chance of four points forming the apices of a reentrant quadrilateral is 1/4 if they be taken at random in an indefinite plane.” It was understood within a year that the question is ill-posed. (The culprit is, as we all know by now, the “indefinite plane” without a properly defined probability measure on it.) So Sylvester modified the question: let \( K \subset \mathbb{R}^2 \) be a convex body (that is, a compact, convex set with nonempty interior) and choose four random, independent points uniformly from \( K \), and write \( P(K) \) for the probability that the four points form the apices of a reentrant quadrilateral, or, in more modern terminology, that their convex hull is a triangle. How large is \( P(K) \), and for what \( K \) is \( P(K) \) the largest and the smallest? This question became known

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as Sylvester’s four-point problem. It took fifty years to find the answer: Blaschke [16] showed that for all convex bodies $K \subset \mathbb{R}^2$

$$P(\text{disk}) \leq P(K) \leq P(\text{triangle}).$$

Assume now, more generally, that $X_n = \{x_1, \ldots, x_n\}$ is a random sample of $n$ uniform, independent points from the convex body $K$ and write $p(n, K)$ for the probability that $X_n$ is in convex position, that is, no $x_i$ is in the convex hull of the others. Sylvester’s question is just the complementary problem for $n = 4$: $P(K) = 1 - p(4, K)$. The probability $p(n, K)$ has been determined in various special cases (see [22, 17, 9, 27]). The following result from [7] describes the asymptotic behaviour of $p(n, K)$.

**Theorem 1.1.** For every convex body $K \subset \mathbb{R}^2$ of unit area

$$\lim_{n \to \infty} n^2 \sqrt{p(n, K)} = \frac{e^2}{4} A^3(K)$$

where $A(K)$ is the supremum of the affine perimeter of all convex sets $S \subset K$.

The affine perimeter, $AP(K)$ can be defined in many ways (see [23]), for instance $AP(K) = \int_{\partial K} \kappa^{1/3} ds$ where $\kappa$ is the curvature and integration goes by arc-length. (This definition works for smooth convex bodies, the extension for all convex bodies can be found in [23].) Theorem 1 of [6] says that there is a unique convex compact set $K_0 \subset K$ with $AP(K_0) = A(K)$. The proof of Theorem 1.1 gives more than just the asymptotic behaviour of $p(n, K)$, namely, if the random points $x_1, \ldots, x_n$ are in convex position, then their convex hull is, with high probability, very close to $K_0$. For the precise formulation see [7].

Define $Q(X_n)$ as the collection of all convex polygons spanned by the points of $X_n$, that is, $P \in Q(X_n)$ iff $P = \text{conv}\{x_{i_1}, \ldots, x_{i_k}\}$ for some $k$-tuple of points from $X_n$ that is in convex position ($k \geq 3$). Clearly, $Q(X_n)$ is a random collection as it depends on the random sample $X_n$. How many polygons are there in $Q(X_n)$? The answer is given in [7]. Write $E|Q(X_n)|$ for the expectation of the size of $Q(X_n)$.

**Theorem 1.2.** For every convex body $K \subset \mathbb{R}^2$ of unit area

$$\lim_{n \to \infty} n^{-1/3} \log E|Q(X_n)| = 3 \cdot 2^{-2/3} A(K).$$

Further, there is a limit shape to the polygons in $Q(X_n)$, meaning that all but a small fraction of the polygons in $Q(X_n)$ are very close to $K_0$. We use $\delta(S, T)$ to denote the Haussdorf distance of $S, T \subset \mathbb{R}^2$.

**Theorem 1.3.** For every convex body $K \subset \mathbb{R}^2$ and for every $\varepsilon > 0$

$$\lim_{n \to \infty} \frac{E\{|P \in Q(X_n): \delta(P, K_0) > \varepsilon\}}{E|Q(X_n)|} = 0.$$

We will see in Section 3 that similar phenomena hold for the lattice case. In general, lattice points and random points, in relation to convex bodies, behave very much alike. Quite often one understands in the random case what to expect for lattice points, or the other way around. The proofs are quite different and are omitted in this survey.
2. Higher dimensions

Much less is known in higher dimensions. One reason is that the unicity of the convex subset of $K$ with maximal affine surface area is not known. It is a mystery, for instance, which convex subset of the unit cube in $R^3$ has maximal affine surface area. But there are other reasons as well, connected to the lack of the multiplicative rule (5.3) from [7]. Yet one can prove the following asymptotic formula [8]. Here $C^d$ denotes the set of all convex bodies in $R^d$, and $p(n, K)$ denotes, as before, the probability that the random sample $X_n = \{x_1, \ldots, x_n\}$ from $K$ is in convex position, that is, no $x_i$ is in the convex hull of the others.

Theorem 2.1. For every $K \in C^d$, and for all $n \geq n_0$

$$c_1 < n^{2/(d-1)} \sqrt[p(n, K)] < c_2$$

where $n_0, c_1, c_2$ are positive constants that depend only on $d$.

With Vinogradov’s convenient $\ll$ or $\ll_d$ notation this says that

$$1 \ll_d n^{2/(d-1)} \sqrt[p(n, K)] \ll_d 1.$$ 

From this one can estimate the size of $E|Q(X_n)|$ when $K \in C^d$:

$$n^{(d-1)/(d+1)} \ll_d \log E|Q(X_n)| \ll_d n^{(d-1)/(d+1)}.$$ 

For comparison let us have a look at lattice polytopes contained in some fixed $K \subset C^d$. So let $Z^d$ be the lattice of the integers in $R^d$ and consider, for a large integer $m$, the lattice $\frac{1}{m}Z^d$. Assume $K$ contains $n$ points from this lattice. As $m$ is large, $n = (1 + o(1))m^d \text{Vol} K$. Write $P_m(K)$ for the collection of all $\frac{1}{m}Z^d$-lattice polytopes contained in $K$. The next theorem, which follows easily from the results of [12], shows a very strong analogy between $P_m(K)$ and $Q(X_n)$.

Theorem 2.2. For every $K \in C^d$

$$n^{(d-1)/(d+1)} \ll_d \log |P_m(K)| \ll_d n^{(d-1)/(d+1)}.$$ 

The result shows that when $K \in C^d$ contains $n$ lattice points, these lattice points span (essentially) $\exp\{cn^{(d-1)/(d+1)}\}$ convex polytopes, the same number as in the random case. Lattice points and random points in convex bodies behave similarly: this is the moral.

3. Lattice polygons and limit shape

In the plane Theorem 2.2 can be proved in stronger form (see [5], [6], and [28]):

Theorem 3.1. For every $K \in C^2$

$$\lim n^{-2/3} \log |P_n(K)| = 3 \frac{\sqrt[3]{\zeta(3)}}{4\zeta(2)} A(K).$$
Here $\zeta(.)$ stands for Riemann $\zeta$ function. Note that this result is in complete analogy with Theorem 1.2: just the constant is different. (Also, $n$ is in power $-2/3$ instead of $-1/3$ as $K$ contains $(1 + o(1))n^2$ Area $K$ lattice points.) The analogy carries over to Theorem 1.3 as well:

**Theorem 3.2.** For every convex body $K \in C^d$ and for every $\varepsilon > 0$

$$\lim \frac{|\{ P \in P_n(K) : \delta(P, K_0) > \varepsilon \}|}{|P_n(K)|} = 0.$$  

This shows again that all but a tiny fraction of the polygons in $P_n(K)$ are very close to $K_0$. In other words, these polygons have a limit shape. Theorems of this type were first proved by Bárány [5], Vershik [28] (for the case when $K$ is the unit square). Sinai [25] found a different proof which uses probability theory and gives a central limit theorems about how small that tiny fraction of polygons is. This has been generalized by Vershik and Zeitouni [29] to all convex bodies in $R^2$. A central limit theorem of this type holds for the random sample case as well, see [14] for the precise statement.

4. The integer convex hull

The integer convex hull, $I(K)$, of a convex body $K \in C^d$ is, by definition, the convex hull of the lattice points contained in $K$:

$$I(K) = \text{conv}(Z^d \cap K).$$

$I(K)$ is clearly a convex polytope. How many vertices does it have? Motivation for the question comes from integer programming, classical enumeration questions (like the circle problem), and from the theory of random polytopes. In integer programming one wants to know that $I(K)$ does not have too many vertices, assuming, say, that $K$ is a nice rational polytope. The latter means that $K$ can be given by $m$ inequalities with integral coefficients; the size of such an inequality is the number of bits necessary to encode it as a binary string. Then the size of the rational polytope is the sum of the sizes of the defining inequalities. Strengthening earlier results by Shevchenko [24], and Hayes and Larman [21], Cook, Hartman, Kannan, and McDiarmid [18] showed that for a rational polytope $K$ of size $\phi$

$$f_0(I(K)) \leq 2m^d(12d^2\phi)^{d-1}.$$  

Here, as usual, $f_i(P)$ stands for the number of $i$-dimensional faces of the polytope $P$. Thus $f_0(P)$ is the number of vertices of $P$. Most likely, the same inequality holds for all $i = 0, 1, \ldots, d - 1$:

$$f_i(I(K)) \ll \phi^{d-1}$$

where the implied constant depends on $d$ and $m$ as well.
The above inequality for $f_0(I(K))$ is best possible, as is shown Bárány, Howe, and Lovász in [11]:

**Theorem 4.1.** For fixed $d \geq 2$ and for every $\phi > 0$ there exists a rational simplex $P \subset \mathbb{R}^d$ of size at most $\phi$ such that $I(P)$ has $\gg_d \phi^{d-1}$ vertices.

The construction uses algebraic number theory. It shows further that the estimate $f_i(I(K)) \ll \phi^{d-1}$ for all $i$ is best possible, if true.

What about the integer convex hull of other convex bodies? Balog and Bárány [3] considered case $K = rB^2$ where $B^2$ is the Euclidean unit ball centered at the origin and $r$ is large and showed that

$$0.3 r^{2/3} < f_0(I(rB^2)) < 5.5 r^{2/3}.$$

Later Balog and Deshouilliers [4] determined the average of $f_0(I(rB^2))$ on an interval $[R, R + H]$ which turned out to be very close to $3.453 R^{2/3}$ as $R$ goes to infinity ($H$ has to be large). Bárány and Larman [13] determined the order of magnitude of $f_i(I(rB^2))$. (The method, and the result, apply not only to the unit ball but to smooth enough convex bodies as well.)

**Theorem 4.2.** For every $d \geq 2$ and every $i = 0, 1, \ldots, d - 1$

$$r^{d(d-1)/(d+1)} \ll_d f_i(I(rB^2)) \ll_d r^{d(d-1)/(d+1)}.$$

This result is related to a beautiful theorem of G. E. Andrews [1] stating that a lattice polytope $P$ in $\mathbb{R}^d$ with volume $V > 0$ has $\ll_d V^{(d-1)/(d+1)}$ vertices. The above theorem shows that Andrews’ estimate is best possible (apart from the constant implied by $\ll_d$). A similar (perhaps less compact) example was given earlier by V. I. Arnol’d [2].

This kind of question can be considered in a more general setting. Let $G$ be the group of all isometries of $\mathbb{R}^d$ with translations by elements of $\mathbb{Z}^d$ factored out. $G$ is a compact topological group with a Haar measure which is a unique invariant probability measure when normalized properly. Assume $g \in G$ is chosen according to this probability measure. Then $gK$ is a random copy of $K$ and we can talk about the expectation of the random variable $f_0(I(gK))$.

For the next result we assume $K \in \mathcal{C}^d$ and define the function $u: K \to R$ by

$$u(x) = \text{Vol}(K \cap (2x - K)),$$

that is, $u(x)$ is the volume of the intersection of $K$ with $K$ reflected about $x$. Set, finally, $K(u < t) = \{ x \in K : u(x) < t \}$. The following is an unpublished result of Bárány and Matoušek:

**Theorem 4.3.** Consider all $K \in \mathcal{C}^d$ with the ratio of the radii of the smallest circumscribed and the largest inscribed balls to $K$ bounded by $D$. Then, as $\text{Vol} K$ goes to infinity,

$$\text{Vol} K(u < 1) \ll E f_0(I(gK)) \ll \text{Vol} K(u < 1)$$
where the constants implied by \( \ll \) depend only on \( d \) and \( D \).

It follows easily from Minkowski’s classical theorem that all vertices of \( I(K) \) belong to \( K(\mu < 2^d) \). (This is the first step in proving the upper bound.) It is not hard to see that \( \text{Vol} K(\mu < 2^d) \ll \text{Vol} K(\mu < 1) \). So the meaning of the theorem is that the average number of vertices of \( I(gK) \) is essentially the volume of \( K(\mu < 2^d) \). Probably the same is true for the expected number of \( i \)-dimensional faces of \( I(gK) \) but there is no proof in sight.

The behaviour of \( \text{Vol} K(\mu < 1) \) is more or less known (from [10], say, but more precise results are known as well): it is of order \( (\text{Vol} K)^{(d-1)/(d+1)} \) for smooth enough convex bodies and of order \( (\log \text{Vol} K)^{d-1} \) for polytopes, and it is between these bounds for all convex bodies.

We mention further that Theorem 4.3 is quite analogous to a result in the theory of random polytopes. Given \( K \in \mathcal{C}^d \), and a random sample of \( n \) points, \( X_n \), from \( K \), \( K_n = \text{conv} X_n \) is called a random polytope on \( n \) points. It is shown in [10] that, assuming \( \text{Vol} K = n \) (which is the proper scaling for comparison with Theorem 4.3), for all \( i = 0, 1, \ldots, d-1 \)

\[
\text{Vol} K(\mu < 1) \ll \mathbb{E} f_i(K_n) \ll \text{Vol} K(\mu < 1)
\]

where the implied constants depend only on dimension.

Note that, unlike Theorem 4.3, this result works for all \( i = 0, \ldots, d-1 \) (without any condition on the ratio of radii of the circumscribed and inscribed balls). Most likely, Theorem 4.3 also holds for all \( i \), which would make the analogy even more complete.

There is, however, a point here where the analogy breaks down. Let \( K \subset \mathbb{R}^2 \) be the square of area \( n \), so \( K_n \) is a random polytope, and \( I(gK) \) is the integer hull of a random copy of \( K \). The expectation of \( \text{Area}(K \setminus K_n) \) is of order \( \log n \) (see [10], say), while the expectation of \( \text{Area}(K \setminus I(gK)) \) is of order \( (\log n)^2 \). (The latter result comes again from the unpublished work of Bárány and Matoušek.) The reason is that the boundary of \( K_n \) contains no points from \( X_n \) apart from its vertices, while the boundary of \( I(gK) \) does. A further reason is that what we are measuring here is a metric property, and not a combinatorial one. We think that the same phenomena is bound to happen in higher dimension.

5. Random 0-1 polytopes

Finally we mention a recent development, prompted by a question of K. Fukuda and G. M. Ziegler [30]. They asked how many facets a 0-1 polytope in \( \mathbb{R}^d \) can have; a 0-1 polytope is a polytope whose vertices only have 0 or 1 coordinates. So such a polytope is the convex hull of a subset of the vertices of the unit cube, \( Q^d \), in \( \mathbb{R}^d \). 0-1 polytopes play an important role in combinatorial optimization where the target is, very often, a concise description of the facets of the polytope. This task has turned out to be difficult for several classes of 0-1 polytopes.
Write $G(d)$ for the maximal number of facets a 0-1 polytope can have. It is not hard to see that $2^d \leq G(d) \leq 2d!$. The upper and lower bounds have been improved slightly: the lower bound by a construction of Christoff (see [30]), and the upper bound by Fleiner, Kaibel, and Rote [20].

The vertices of every 0-1 polytope are on a sphere (centered at $(1/2, \ldots, 1/2)$). There is a formula (see for instance [9]) for the expected number of facets of a random polytope with $n$ uniform independent points from the (unit) sphere in $R^d$. It says that, in the range when $2d < n < 2^d$, the expected number of facets is of order $(\log n/d)^{d/2}$.

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The proof of Theorem 5.1 is based on the fact that for all small $\varepsilon > 0$ and large enough $d$ $P(1 - \varepsilon) \subset P(\varepsilon)$, of course, but the drop from $1 - \varepsilon$ to $\varepsilon$ is very abrupt: $P(\varepsilon)$ is in a small neighbourhood of $P(1 - \varepsilon)$. This shows that $P(1 - \varepsilon) \subset K_n$ with high probability. But only a tiny fraction of $K_n$ lies outside $P(\varepsilon)$: most of the boundary of $P(\varepsilon)$ is outside $K_n$. Thus most of the boundary of $P(\varepsilon)$ is cut off by facets of $K_n$. These facets lie outside $P(1 - \varepsilon)$. Comparing the surface area of $P(\varepsilon)$ with the amount a facet can cut off from it gives the lower bound.

The actual proof is technical, difficult, and makes extensive use a beautiful result of Dyer, Füredi, and McDiarmid [19]. Their target was to determine the threshold $n = n(d)$ such that $K_n$ contains most of the volume of $Q^d$. As they prove, this happens at $n = (2/\sqrt{\pi})^d$. Their method describes where $p(x, n)$ drops from one to zero as $d \to \infty$. The analysis carries over for other values of $n$. In our case higher precision is required as we need a good estimate on how fast $p(x, n)$ drops from one to zero. We were able to control this only where the curvature of the boundary of $P(\varepsilon)$ behaves nicely. This is perhaps the spot where the exponent $d/2$ (for the random spherical polytope) is lost and we only get $d/4$ for $K_n$. 

**Theorem 5.1.** There is a constant $c > 0$ such that for all $d \geq 2$

$$G(d) \gg \left(\frac{\varepsilon d}{\log d}\right)^{d/4}.$$ 

The construction giving this estimate is random. Write $K_n$ for the convex hull of $n$ random, uniform, and independent 0-1 vectors. Assume $x$ is a point from $Q^d$, and define

$$p(x, n) = \text{Prob}[x \in K_n].$$

General principles would tell that, for most $x \in Q^d$, $p(x, n)$ is either close to one or close to zero. To be more specific, set

$$P(t) = \{x \in Q^d : p(x, n) \geq t\}.$$ 

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The vertices of every 0-1 polytope are on a sphere (centered at $(1/2, \ldots, 1/2)$). There is a formula (see for instance [9]) for the expected number of facets of a random polytope with $n$ uniform independent points from the (unit) sphere in $R^d$. It says that, in the range when $2d < n < 2^d$, the expected number of facets is of order $(\log n/d)^{d/2}$. So if the analogy between random points and lattice points carries over the 0-1 case one should expect $G(d)$ to be of order $d^{d/2}$. This is too much to ask for at the moment, yet the following is true (see [15]).
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