CORRELATION OF ARITHMETIC FUNCTIONS OVER $\mathbb{F}_q[T]$

OFIR GORODETSKY AND WILL SAWIN

Abstract. For a fixed polynomial $\Delta$, we study the number of polynomials $f$ of degree $n$ over $\mathbb{F}_q$ such that $f$ and $f+\Delta$ are both irreducible, an $\mathbb{F}_q[T]$-analogue of the twin primes problem. In the large-$q$ limit, we obtain a lower-order term for this count if we consider non-monic polynomials, which depends on $\Delta$ in a manner which is consistent with the Hardy-Littlewood Conjecture. We obtain a saving of $q$ if we consider monic polynomials only and $\Delta$ is a scalar. To do this, we use symmetries of the problem to get for free a small amount of averaging in $\Delta$. This allows us to obtain additional saving from equidistribution results for $L$-functions. We do all this in a combinatorial framework that applies to more general arithmetic functions than the indicator function of irreducibles, including the Möbius function and divisor functions.

1. Introduction

Let $q$ be a prime power, $\mathbb{F}_q$ be the finite field with $q$ elements and $\mathbb{F}_q[T]$ be the polynomial ring over $\mathbb{F}_q$ in indeterminate $T$.

- We denote by $A_{n,q}$ the set of polynomials of degree $n$ in $\mathbb{F}_q[T]$ and by $A_{q} = \cup_{n \geq 0} A_{n,q}$ the set of non-zero polynomials in $\mathbb{F}_q[T]$.
- Similarly, we denote by $M_{n,q} \subseteq A_{n,q}$ the set of monic polynomials of degree $n$ in $\mathbb{F}_q[T]$ and by $M_{q} = \cup_{n \geq 0} M_{n,q}$ the set of all monic polynomials in $\mathbb{F}_q[T]$.
- We denote by $P_{n,q} \subseteq M_{n,q}$ the set of monic irreducible polynomials of degree $n$ in $\mathbb{F}_q[T]$ and by $P_{q} = \cup_{n \geq 1} P_{n,q}$ the set of all monic irreducible polynomials in $\mathbb{F}_q[T]$.
- Given $\Delta \in A_{q}$, we denote by $a_{\Delta,q}$ the number of distinct roots of $\Delta$ in $\mathbb{F}_q$:
  \[ a_{\Delta,q} = \# \{ a \in \mathbb{F}_q : \Delta(a) = 0 \}. \]

There are several important arithmetic functions $\mathbb{F}_q[T] \setminus \{0\} \to \mathbb{C}$ which are used to study polynomials. Two well-known examples include the von Mangoldt function $\Lambda_q$, defined on monics by

\[ \Lambda_q(f) = \begin{cases} \deg P & \text{if } f = P^k, \text{ where } P \in P_{q}, k > 0, \\ 0 & \text{otherwise}, \end{cases} \]

and the Möbius function $\mu_q$, defined on monics by

\[ \mu_q(f) = \begin{cases} (-1)^k & \text{if } f = \prod_{i=1}^{k} P_i, \text{ where } P_i \in P_{q} \text{ are distinct}, \\ 0 & \text{otherwise}. \end{cases} \]

We extend these functions to non-monic by $\alpha(c \cdot f) = \alpha(f)$ for all $c \in \mathbb{F}_q^\times, f \in M_{q}$. The mean value of $\Lambda_q$ has a well-known closed form [28 Prop. 2.1]:

\[ \frac{\sum_{f \in A_{n,q}} \Lambda_q(f)}{q^n(q-1)} = \frac{\sum_{f \in M_{n,q}} \Lambda_q(f)}{q^n} = 1, \]

and this is an analogue of the Prime Number Theorem. For $\Delta \in \mathbb{F}_q[T] \setminus \{0\}$, the asymptotics of the mean values of $\Lambda_q(f)\Lambda_q(f+\Delta)$ over either $f \in A_{n,q}$ or $f \in M_{n,q}$, that is,

\[ \frac{\sum_{f \in A_{n,q}} \Lambda_q(f)\Lambda_q(f+\Delta)}{q^n(q-1)} \quad \text{or} \quad \frac{\sum_{f \in M_{n,q}} \Lambda_q(f)\Lambda_q(f+\Delta)}{q^n}, \]

are much less understood. Such asymptotic gives us quantitative information about pairs of primes in $\mathbb{F}_q[T]$ whose difference is $\Delta$. The Hardy-Littlewood Conjecture predicts precise asymptotics for these mean values when $q^n \to \infty$, see [28, 13]. In this paper, we are interested in improving asymptotic results for the mean value of $\Lambda_q(f)\Lambda_q(f+\Delta)$, and other ‘shifted correlation’ sums, in the limit $q \to \infty$. This limit is also known in the literature as the ‘large-$q$ limit’ or ‘large finite field limit’. In particular, we think of $n$ as fixed, and...
Δ ∈ F_q[T] \ {0} is always of degree less than n. For results on twin primes in F_q[T] where q is fixed and n → ∞, see the work of Castillo, Hall, Lenke Oliver, Pollack and Thompson [8] and the results of the second author and Shusterman [32] obtained while this article was in press.

Pollack [25, Thm. 2], Bender and Pollack [4, Thm. 1.3], Bary-Soroker [2, Thm. 1.1] and Carmon [6, §6] have shown that

\[ \sum_{f \in M_{n,q}} \Lambda_q(f) \Lambda_q(f + \Delta) q^n - 1 = O_n \left( \frac{1}{\sqrt{q}} \right), \]

from which it follows by the first part of Lemma 1 below that

\[ \sum_{f \in A_{n,q}} \Lambda_q(f) \Lambda_q(f + \Delta) q^n (q - 1) - 1 = O_n \left( \frac{1}{\sqrt{q}} \right) \]

as well. The proofs of (1.1) involve the calculation of the Galois groups of certain generic polynomials, and an application of a function field analogue of Chebotarev Density Theorem. These methods give an implied constant of order n!^2 (see the statement of [4, Thm. 1.3]).

Using methods from L-functions, Pollack has shown that [24, Thm. 1]

\[ \sum_{f \in A_{n,q}} \Lambda_q(f) \Lambda_q(f + \Delta + \Delta) q^n (q - 1) - 1 = O \left( \frac{|\Delta| \phi(\Delta)}{\phi(\Delta) \sqrt{q}} \right), \]

where φ is Euler’s totient function. As \[ \frac{|\Delta|}{\phi(\Delta)} \leq 2^{\deg \Delta} \leq 2^{n-1}, \] the dependence of the error term in (1.3) on n is much better than in (1.2) and (1.1). More importantly for us, the dependence on q is better in (1.3) compared to (1.2) and (1.1), and in fact the Hardy-Littlewood Conjecture predicts that the left hand side of (1.3) is, in general, \( \Theta_n \left(\frac{1}{q}\right) \), see [5.4]. Thus, the power of q appearing in Pollack’s result is the best possible.

When \( \Delta = 1 \), Keating and Roditty-Gershon [21, 22] have improved (1.3) in the large-q limit, namely they have shown that

\[ \sum_{f \in A_{n,q}} \Lambda_q(f) \Lambda_q(f + 1) q^n (q - 1) - 1 = -\frac{1}{q} + O_n \left( \frac{1}{q^{3/2}} \right), \]

see the case \( k = 0 \) of [21, Thm. 1.3]. Finally, we mention the work of Bary-Soroker and Stix, giving precise answers for \( n = 3 \) and \( \Delta = 1 \) [3]. In [5, 1] we prove the following theorem, which is a corollary of the general results presented in [1.2].

**Theorem 1.** Let n be a positive integer. Let \( \Delta \) be a squarefree polynomial in \( A_q \) which is either of degree \( \leq n - 5 \) or of degree \( n - 1 \). We have, for \( n \geq 5 \),

\[ \sum_{f \in A_{n,q}} \Lambda_q(f) \Lambda_q(f + \Delta) q^n (q - 1) - 1 = -\frac{1 + a_{\Delta,q}}{q} + O_n \left( \frac{1}{q^{3/2}} \right). \]

If \( \Delta \in F_q^* \) and \( n \geq 4 \) then

\[ \sum_{f \in M_{n,q}} \Lambda_q(f) \Lambda_q(f + \Delta) q^n - 1 = O_n \left( \frac{1}{q} \right). \]

The first part of Theorem 1 is the first result showing a dependence of the mean value of \( \Lambda_q(f) \Lambda_q(f + \Delta) \) on \( \Delta \) (namely, on the linear factors of \( \Delta \)). In particular, it is the first result that gives us a glimpse into lower-order terms of the Hardy-Littlewood constant, see [5.4] for more details, where we show that \( 1 + (-1 + a_{\Delta,q})/q \) is a first-order approximation for the Hardy-Littlewood constant.

The second part of Theorem 1 is the first result that gives the correct error term for the left hand side of (1.4), that is, for the shifted correlation of \( \Lambda_q \) over monic polynomials.

We now discuss the Möbius function. The mean value of \( \mu_q \) is also well known [5, Eq. (5)]:

\[ \sum_{f \in A_{n,q}} \mu_q(f) \frac{1}{q^n (q - 1)} = \sum_{f \in M_{n,q}} \mu_q(f) q^n = 0. \]

Their result estimates \( \sum_{f \in M_{n,q}} \sum_{c \in F_q^*} \Lambda_q(f) \Lambda_q(f + c) \), which is the same quantity as \( \sum_{f \in A_{n,q}} \Lambda_q(f) \Lambda_q(f + 1) \) by [2, Thm. 10].
for all $n \geq 2$. This again may be considered as an analogue of the Prime Number Theorem. A conjecture of Chowla [9], for which Sarnak has found deep interpretations [29], asserts in particular that

$$\lim_{x \to \infty} \frac{\sum_{n \leq x} \mu(n)\mu(n + h)}{x} = 0$$

for all $h \geq 1$. Here $\mu$ is the usual Möbius function, defined on the positive integers. In the function field setting, Rudnick and Carmon [7] and Carmon [6] used an algebro-geometric argument to show that for any $\Delta \in \mathbb{F}_q[T] \setminus \{0\}$ of degree $< n$, we have

$$\sum_{f \in \mathcal{A}_{n,q}} \mu_q(f)\mu_q(f + \Delta) = O\left(\frac{n^2}{\sqrt{q}}\right),$$

from which it follows by the first part of Lemma [1] that

$$\sum_{f \in \mathcal{A}_{n,q}} \mu_q(f)\mu_q(f + \Delta) = O\left(\frac{n^2}{\sqrt{q}}\right)$$

as well. This result is a large-$q$ analogue of Chowla Conjecture. When $\Delta = 1$, Keating and Roditty-Gershon have improved (1.5) in the large-$q$ limit, namely they have shown that [21, Thm. 4.4]

$$\sum_{f \in \mathcal{A}_{n,q}} \mu_q(f)\mu_q(f + 1) = O\left(\frac{1}{q^{3/2}}\right).$$

For a recent breakthrough on the function field Chowla Conjecture in the large-$n$ limit with fixed $q$, see the results of the second author and Shusterman [32], obtained while this article was in press.

In §5.2 we prove the following theorem, which is again a corollary of the general results presented in §1.2.

**Theorem 2.** Let $n$ be a positive integer. Let $\Delta$ be a squarefree polynomial in $\mathcal{A}_q$ which is either of degree $\leq n - 5$ or of degree $n - 1$. We have, for $n \geq 5$,

$$\sum_{f \in \mathcal{A}_{n,q}} \mu_q(f)\mu_q(f + \Delta) = O\left(\frac{1}{q^{3/2}}\right).$$

If $\Delta \in \mathbb{F}_q^\times$ and $n \geq 4$ then

$$\sum_{f \in \mathcal{A}_{n,q}} \mu_q(f)\mu_q(f + \Delta) = O\left(\frac{1}{q}\right).$$

For $\Delta \neq 1$, (1.6) gives an additional saving of $q$ compared to previous results. The estimate (1.7) is the first estimate which give a saving of $q$ when the average is over monic polynomials.

In Theorems 1 and 2 the condition that $\Delta$ is squarefree comes only in one part of the proof, where we use Theorem 8 an equidistribution result which currently requires squarefree-ness. This should not be a fundamental condition, and the general case is expected to be true, although challenging. The same goes also to the range of the degree of $\Delta$, for which the current results do not allow the values $n - 4$, $n - 3$ and $n - 2$.

### 1.1. Arithmetic functions on $\mathbb{F}_q[T]$ and previous work.

An arithmetic function on $\mathbb{F}_q[T]$ is any function $\alpha : \mathbb{F}_q[T] \setminus \{0\} \to \mathbb{C}$. If $f \in \mathbb{F}_q[T] \setminus \{0\}$ has prime factorization $c \cdot \prod_{i=1}^k P_i^{e_i}$ where $c \in \mathbb{F}_q^\times$ and $P_i$ distinct primes in $\mathcal{P}_q$, then its extended factorization type is the multiset

$$\lambda_f = \{(\deg P_i, e_i) : 1 \leq i \leq k\}.$$

Let EFT be the set of all extended factorization types. Following Rodgers [27, §2B], we say that an arithmetic function $\beta : \mathbb{F}_q[T] \setminus \{0\}$ is a factorization function if the value $\beta(f)$ is determined by $\lambda_f$, i.e. if there is a function $b$ : EFT $\to \mathbb{C}$ such that $\beta(f) = b(\lambda_f)$ for all $f \in \mathbb{F}_q[T] \setminus \{0\}$. The function $b$ is not unique, since for instance the value of $b$ on the multiset of $q + 1$ (1,1)-s can be chosen arbitrarily (as there are only $q$ distinct linear polynomials).

With any function $\alpha : \mathbb{F}_q[T] \to \mathbb{C}$ and any prime power $q$, we may associate a factorization function $\alpha_q : \mathbb{F}_q[T] \setminus \{0\} \to \mathbb{C}$ by letting

$$\alpha_q(f) = \alpha(\lambda_f).$$

\[\text{Sometimes we use the domain } \mathbb{F}_q[T] \text{ instead of } \mathbb{F}_q[T] \setminus \{0\}, \text{ but we shall never use the value of } \alpha \text{ at 0.}\]
An arithmetic function $\beta: \mathbb{F}_q[T] \to \mathbb{C} \setminus \{0\}$ is said to be even if $\beta(c \cdot f) = \beta(f)$ for any $c \in \mathbb{F}_q^*$ and any $f \in \mathbb{F}_q[T] \setminus \{0\}$. Any factorization function is even.

From now on we reserve the notation $\alpha_q$ for a factorization function on $\mathbb{F}_q[T]$ which comes from $\alpha: \text{EFT} \to \mathbb{C}$. Although for any specific $q$, $\alpha_q$ does not determine a unique function $\alpha$ such that (1.8) holds, we do have a unique $\alpha$ once we look at an infinite number of $q$-s, that is: if $\alpha(\lambda_f) = \beta(\lambda_f)$ for all $f \in \mathbb{F}_q[T] \setminus \{0\}$ for infinitely many $q$-s, we must have $\alpha = \beta$. In particular, a family of functions $\{\alpha_q: q \text{ a prime power}\}$ which come from $\alpha$ determines $\alpha$ uniquely.

For functions $\alpha, \beta: X \to \mathbb{C}$ and a non-empty finite subset $S \subseteq X$, we denote the mean value of $\alpha$ over $S$ by

$$E_S \alpha = \frac{\sum_{f \in S} \alpha(f)}{|S|}$$

and the covariance of $\alpha$ and $\beta$ over $S$ by

$$\text{Cov}_S (\alpha, \beta) = E_S (\alpha \overline{\beta}) - E_S \alpha \cdot E_S \overline{\beta} = E_S ((\alpha - E_S \alpha)(\overline{\beta} - E_S \overline{\beta})).$$

Let $\alpha, \beta$ be arithmetic functions on $\mathbb{F}_q[T]$. Many important questions of number theory are encoded in the following covariances:

$$\text{Cov}_{\mathcal{M}_q}(\alpha, \beta; n, \Delta) = \text{Cov}_{f \in \mathcal{M}_{n,q}} (\alpha(f), \beta(f + \Delta))$$

and

$$\text{Cov}_{\mathcal{A}_q}(\alpha, \beta; n, \Delta) = \text{Cov}_{f \in \mathcal{A}_{n,q}} (\alpha(f), \beta(f + \Delta)),$$

where $n$ is a positive integer and $\Delta$ is a non-zero polynomial of degree $< n$. Let

$$\max(\alpha; n) = \max_{f \in \mathcal{M}_{n,q}} |\alpha(f)|.$$

Andrade, Bary-Soroker and Rudnick \cite{1} Thm. 1.4 have shown that

$$\text{Cov}_{\mathcal{M}_q}(\alpha, \beta; n, \Delta) = O_{n, \max(\alpha; n), \max(\beta; n)} \left( \frac{1}{\sqrt{q}} \right)$$

for any pair of factorization functions $\alpha, \beta$. From (1.9) and the first part of Lemma \cite{1} we obtain that

$$\text{Cov}_{\mathcal{A}_q}(\alpha, \beta; n, \Delta) = O_{n, \max(\alpha; n), \max(\beta; n)} \left( \frac{1}{\sqrt{q}} \right)$$

holds as well. Estimate (1.9) extends the results of Pollack, Bender and Pollack, Bary-Soroker, Carmon and Rudnick, and Carmon concerning the shifted correlation of $\Lambda_q$ and $\mu_q$. We remark that by applying the methods of Pollack \cite{27} carefully for general factorization functions (by borrowing the combinatorial ideas in Rodgers \cite{27}), one can in fact obtain

$$\text{Cov}_{\mathcal{A}_q}(\alpha, \beta; n, \Delta) = O_{n, \max(\alpha; n), \max(\beta; n)} \left( \frac{1}{q} \right).$$

1.2. Main results. Our first theorem is a determination of the main term of $\text{Cov}_{\mathcal{A}_q}(\alpha_q, \beta_q; n, \Delta)$ in the limit $q \to \infty$ for most choices of $\Delta$. To state the theorem, we need the notion of Fourier expansion of factorization functions \cite{27} \S 2B, which we now explain.

Let $\alpha_q: \mathbb{F}_q[T] \to \mathbb{C}$ be a family of factorization functions which come from $\alpha: \text{EFT} \to \mathbb{C}$ and let $n$ be a positive integer. Let $S_n^\#$ be the set of conjugacy classes of $S_n$, identified as usual with partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n$ ($\lambda_i$ is always non-increasing). We may embed $S_n^\#$ in EFT by identifying $\lambda$ with the multiset

$$\{(\lambda_i, 1) : 1 \leq i \leq k\}.$$  

Under this identification, $\alpha|_{S_n^\#}$ is a class function on $S_n$. We may expand this function in the basis of irreducible characters of $S_n$, which are also indexed by partitions and we denote them as usual by $\chi_\lambda$:

$$\alpha|_{S_n^\#}(\pi) = \sum_{\lambda \vdash n} \hat{\alpha}_\lambda \chi_\lambda(\pi).$$

The coefficients $\hat{\alpha}_\lambda$ are called the Fourier coefficients of $\alpha$. 

Theorem 3. Let $\alpha_q, \beta_q$ be factorization functions which come from $\alpha, \beta$. Let $n \geq 5$ be an integer and let $\Delta$ be a squarefree polynomial in $A_q$ which is either of degree $\leq n - 5$ or of degree $n - 1$. Then
\[
\text{Cov}_{A_q}(\alpha_q, \beta_q; n, \Delta) = \frac{(a_{\Delta, q} - 1)\hat{\alpha}_{(n-1,1)}\beta_{(n-1,1)}}{q} + O_{n, \max(\alpha;n), \max(\beta;n)} \left( \frac{1}{q^{3/2}} \right).
\]

For an arithmetic interpretation of $\hat{\alpha}_{(n-1,1)}$, see Lemma 9. From Theorem 3 and Lemma 1 we immediately have the following corollary.

Corollary 1. Under the assumptions of Theorem 3 and the following additional assumptions:

1. $\Delta$ is of the form $c(T + a)^k$ $(c \in \mathbb{F}_q^*, a \in \mathbb{F}_q, k \geq 0)$, and
2. $\gcd(q - 1, n - \deg \Delta) = 1$, or $\text{char}(\mathbb{F}_q) > 2$, $\alpha = \bar{\beta}$ and $\gcd(\frac{q-1}{2}, n - \deg \Delta) = 1$,
we have
\[
\text{Cov}_{M_q}(\alpha_q, \beta_q; n, \Delta) = \frac{(a_{\Delta, q} - 1)\hat{\alpha}_{(n-1,1)}\beta_{(n-1,1)}}{q} + O_{n, \max(\alpha;n), \max(\beta;n)} \left( \frac{1}{q^{3/2}} \right).
\]

In particular, Corollary 1 shows that under additional assumptions, the results of Theorems 3 and 2 on $\text{Cov}_{M_q}(\alpha_q, \Lambda_q; n, \Delta)$, $\text{Cov}_{M_q}(\mu_q, \nu_q; n, \Delta)$ may be extended and improved.

Remark 1. In view of the condition in Theorem 3 that $\Delta$ should be squarefree, $k$ in Corollary 1 is forced to be 0 or 1. As mentioned, we expect that the squarefree-ness will be removed eventually.

There are three main parts in the proof of Theorem 3.

First, we consider the case when $f$ is coprime to $\Delta$. In this case, we take advantage of the averaging over the leading coefficient of $f$ to write (in Proposition 2) the sum as a sum over monic polynomials involving $\mathbb{F}_q$-multiples of the shift $\Delta$, and hence as a covariance of the sums of $\alpha_q$ and $\beta_q$ in short arithmetic progressions of length $q$ with step size $\Delta$. We can detect membership in these progressions by combining Dirichlet characters modulo $\Delta$ with characters ramified at $\infty$, and this allows us to replace this sum with a sum over Dirichlet character in Proposition 2. (We must have $f$ coprime to $\Delta$ to take advantage of Dirichlet characters mod $\Delta$.)

Second we attack this sum over Dirichlet characters. We relate the term corresponding to a Dirichlet character to the $L$-function of that Dirichlet character. When the factorization function is any of the divisor function, the Möbius function, or the von Mangoldt function, this is a standard manipulation in analytic number theory, but in the general case it requires combinatorial ideas of Rodgers 27. To estimate these sums, expressed in terms of $L$-functions, we need a new equidistribution result (Theorem 8) as the existing ones do not quite cover the types of Dirichlet characters we need.

Third, we consider the case when $f$ is not coprime to $\Delta$. The largest contribution comes when $\gcd(f, \Delta)$ is a linear polynomial $L$. We can relate this case to a correlation sum of $f/L$ with shift $\Delta/L$. Because this sum is shorter, we can get by with a weaker estimate, which is provided already by the result 15,11 of Andrade, Bary-Soroker and Rudnick, which we apply in Lemma 4. Because their result involves Galois-theoretic methods, our final result involves a combination of these ideas with the $L$-function methods of Pollack, Keating and Roditty-Gershon used in the first two parts of our argument. Both Galois groups and $L$-functions are by this point common approaches to function field analytic number theory problems, but are usually used separately.

We conjecture that $\text{Cov}_{M_q}(\alpha_q, \beta_q; n, \Delta)$ and $\text{Cov}_{A_q}(\alpha_q, \beta_q; n, \Delta)$ share the same asymptotics:

Conjecture 1. Let $\alpha_q, \beta_q$ be factorization functions which come from $\alpha, \beta$. Let $n \geq 5$ be an integer. Let $\Delta$ be a non-zero polynomial of degree < $n$. Then
\[
\text{Cov}_{M_q}(\alpha_q, \beta_q; n, \Delta) = \frac{(a_{\Delta, q} - 1)\hat{\alpha}_{(n-1,1)}\beta_{(n-1,1)}}{q} + O_{n, \max(\alpha;n), \max(\beta;n)} \left( \frac{1}{q^{3/2}} \right).
\]

Conjecture 1 differs from Corollary 1 only in that it removes the additional assumptions made in the corollary.

Our second result is an improvement of 15,11 in the case that $\Delta$ is a scalar.

Theorem 4. Let $\alpha_q, \beta_q$ be factorization functions which come from $\alpha, \beta$. Let $n \geq 4$. Let $\Delta \in \mathbb{F}_q^\times$. Then
\[
\text{Cov}_{M_q}(\alpha_q, \beta_q; n, \Delta) = O_{n, \max(\alpha;n), \max(\beta;n)} \left( \frac{1}{q} \right).
\]
The main ideas of the proof of Theorem \[ \ref{Thm:Main} \] are completely new. We introduce an $L$-function formula for the correlation of general arithmetic functions, which relates an average over polynomials to an average over certain Dirichlet characters (Proposition \[ \ref{Prop:Main} \]). This falls into the general framework in analytic number theory where we replace an identity, in this case $f_2 = f_1 + \Delta$, with an average over characters. This case may be surprising because we are detecting an additive identity using multiplicative Dirichlet characters. However, using Dirichlet characters ramified at primes dividing $\Delta$ and at $\infty$, it is possible to do this. The contribution of a given character is closely related to the Dirichlet $L$-function of that character.

When $\Delta$ is a scalar, we can compose a character with a ring automorphism of $\mathbb{F}_q[T]$ to get a new character ramified at the same points, which will have the same Dirichlet $L$-function. This gives an additional symmetry of the average over characters (Proposition \[ \ref{Prop:Symmetry} \]) which we are able to use in order to get improved asymptotics by first summing over compositions of a given character using elementary Gauss sum estimates, getting some cancellation, and then getting additional cancellation by summing over all characters using $L$-function equidistribution results.

Both Theorem \[ \ref{Thm:Main} \] and Theorem \[ \ref{Thm:Shifted} \] rely on new equidistribution results, discussed in Appendix \[ \ref{App:Equidistribution} \].

1.3. Further results. For an integer $k \geq 2$, the $k$-th divisor function $d_{k,q} : \mathbb{F}_q[T] \to \mathbb{C}$ is defined on monics by

$$d_{k,q}(f) = \#\{(f_1, f_2, \ldots, f_k) : f_1 f_2 \cdots f_k = f, f_i \in \mathcal{M}_q\}.$$  

We extend $d_{k,q}$ to non-monic by $d_{k,q}(c \cdot f) = d_{k,q}(f)$ for all $c \in \mathbb{F}_q^\times$, $f \in \mathcal{M}_q$. The mean value of $d_{k,q}$ is given by \[ \ref{Lem:Mean} \] Lem. 2.1]

$$E_{\mathcal{M}_{n,q}} d_{k,q} = E_{\mathcal{A}_{n,q}} d_{k,q} = \left(\frac{n + k - 1}{n}\right).$$

The estimate \[ \ref{Estimate:Shifted} \], which was in fact motivated by the ‘shifted divisor problem’, implies that

$$\text{Cov}_{\mathcal{M}_q}(d_{k,q}, d_{l,q}; n, \Delta) = O_n\left(\frac{1}{\sqrt[4]{q}}\right),$$

which, by Lemma \[ \ref{Lem:Mean} \] implies that

$$\text{Cov}_{\mathcal{A}_q}(d_{k,q}, d_{l,q}; n, \Delta) = O_n\left(\frac{1}{\sqrt[4]{q}}\right)$$

as well. For $k = l = 2$, Keating and Roditty-Gershon have improved \[ \ref{Estimate:Shifted} \] for $\Delta = 1$ \[ \ref{Thms:Keating} \] Thms. 4.2:

$$\text{Cov}_{\mathcal{A}_q}(d_{2,q}, d_{2,q}; n, 1) = -\left(\frac{n - 1}{q}\right) + O_n\left(\frac{1}{q^{3/2}}\right).$$

In \[ \ref{Thms:Keating} \] we prove the following theorem, which is again a corollary of the general results presented in \[ \ref{Thms:Main} \].

In particular, we obtain the main term of $\text{Cov}_{\mathcal{A}_q}(d_{k,q}, d_{l,q}; n, \Delta)$ for most $\Delta$-s, which turns out to be an interesting combinatorial expression.

**Theorem 5.** Let $n$ be a positive integer. Let $\Delta$ be a squarefree polynomial in $\mathcal{A}_q$ which is either of degree $\leq n - 5$ or of degree $n - 1$. For any $k, l \geq 2$ and $n \geq 5$ we have

$$\text{Cov}_{\mathcal{A}_q}(d_{k,q}, d_{l,q}; n, \Delta) = \frac{(a_{\Delta,q} - 1)(n - 1)^2(n + k - 2)(n + l - 2)}{q} \cdot O_{n,k,l}\left(\frac{1}{q^{3/2}}\right).$$

If $\Delta \in \mathbb{F}_q^\times$ and $n \geq 4$ then

$$\text{Cov}_{\mathcal{M}_q}(d_{k,q}, d_{l,q}; n, \Delta) = O_{n,k,l}\left(\frac{1}{q}\right).$$

In the setting of integers, the asymptotics of $(\sum_{n \leq x} d_k(n)d_l(n + h))/x$ as $x \to \infty$ are known only in the case $k = l = 2$, which is due to Ingham \[ \ref{Ingham} \], and $k = 2$, $l > 2$, which is due to Linnik \[ \ref{Linnik} \] Ch. 3. If $k$ and $l$ are both greater than 2, there are complicated conjectures for the asymptotics, which are due to Ivić \[ \ref{Ivic} \] and Conrey and Gonek \[ \ref{ConreyGonek} \]. **Theorem 5** can be interpreted as recovering first-order approximation for the arithmetic constants in these conjectures.

In \[ \ref{Thms:Ingham} \] we prove the following.
Theorem 6. Let $\alpha_q, \beta_q$ be factorization functions which come from $\alpha, \beta$. Let $n \geq 5$ be an integer. Let $h$ be an integer such that $0 \leq h \leq n - 5$. Then

$$\sum_{\Delta \in A_q} \text{Cov}_{E_q}(\alpha_q, \beta_q; n, \Delta) = -\sum_{\Delta^{h-1} = n-h-1} \hat{\alpha}_q \hat{\beta}_q + O_n(1 + \frac{1}{\sqrt{q}}).$$

Several special cases of Theorem 6 were proved in [21], namely $\alpha_q = \beta_q = \Lambda_q$, $\alpha_q = \beta_q = d_{2q}$ and $\alpha_q = \beta_q = \mu_q$:

$$\sum_{\Delta \in A_q} \text{Cov}_{E_q}(\Lambda_q, \Lambda_q; n, \Delta) = -1 + O_n\left(\frac{1}{\sqrt{q}}\right),$$

$$\sum_{\Delta \in A_q} \text{Cov}_{E_q}(d_{2,q}, d_{2,q}; n, \Delta) = -(n - 2h - 1)^2 \cdot \mathbf{1}_{h \leq \frac{n}{2}} - 1 + O_n\left(\frac{1}{\sqrt{q}}\right),$$

$$\sum_{\Delta \in A_q} \text{Cov}_{E_q}(\mu_q, \mu_q; n, \Delta) = O_n\left(\frac{1}{\sqrt{q}}\right).$$

1.4. Discussion. It is natural to ask what kind of technical improvements are needed in order to obtain additional lower order terms, or better error terms, in our main results. We can obtain, without much effort, additional lower order terms in most of the lemmas and propositions appearing in the proofs of Theorems 3 and 4. The two exceptions are the following. Firstly, we do not know how to obtain lower order terms in (3.22), which is proved using a Chebotarev Density Theorem for function fields. Secondly, we do not know how to obtain lower order terms in the equidistribution results of Appendix A which are proved using Deligne’s equidistribution theorem. Once one is able to obtain lower order terms in these estimates, one additional change is needed. Instead of working in the general setting of factorization functions, we should restrict to the class of arithmetic functions of von Mangoldt type, introduced by Hast and Matei [14, Def. 4.3]. This class is still quite general – any factorization function can be written as a sum of a function of von Mangoldt type and a function supported on non-squarefree polynomials [14, Prop. 4.5]. In addition, the von Mangoldt function, the M"obius function and the divisor functions are all of von Mangoldt type. These functions have the advantage that there is no error term in (3.32) for most characters $\chi$, which allows us to make use of lower order terms in the equidistribution results.

2. Preliminaries

2.1. Hayes characters. Here we review the function field analogue of Dirichlet characters, first introduced by Hayes in the paper [15] which is based on his thesis. We call these characters “Hayes characters”, or sometimes “generalized arithmetic progression characters”. Unless otherwise stated, the proofs of the statements in this section appear in Hayes’ original paper. The main difference between Hayes characters and Dirichlet characters is that in the function field setting we can also consider characters modulo the prime at infinity.

2.1.1. Equivalence relation. Let $\ell$ be a non-negative integer and $M \in A_q$. We define an equivalence relation $R_{\ell,M}$ on $\mathcal{M}_q$ by saying that $A \equiv B \mod R_{\ell,M}$ if and only if $A$ and $B$ have the same first $\ell$ next-to-leading coefficients and $A \equiv B \mod M$. We adopt throughout the following convention: The $j$-th next-to-leading coefficient of a polynomial $f(T) \in \mathcal{M}_q$ with $j > \deg f$ is considered to be 0. It may be shown that there is a well-defined quotient monoid $\mathcal{M}_q/R_{\ell,M}$, where multiplication is the usual polynomial multiplication. An element of $\mathcal{M}_q$ is invertible modulo $R_{\ell,M}$ if and only if it is coprime to $M$. The units of $\mathcal{M}_q/R_{\ell,M}$ form an abelian group, having as identity element the equivalence class of the polynomial 1. We denote this unit group by $(\mathcal{M}_q/R_{\ell,M})^\times$. We note that $(\mathcal{M}_q/R_{\ell,M})^\times$ is isomorphic to

$$\left(1 + T \mathbb{F}_q[T]\right)/\left(1 + T^{\ell+1} \mathbb{F}_q[T]\right) \oplus \left(\mathbb{F}_q[T]/M \mathbb{F}_q[T]\right)^\times,$$

and its size is given by

$$\left| (\mathcal{M}_q/R_{\ell,M})^\times \right| = q^{\phi(M)}.$$

We note that the relation $R_{\ell,c,M}$ depends only on $\ell$ and the ideal generated by $M$, that is, $R_{\ell,c,M}$ yields the same relation for any $c \in \mathbb{F}_q^\times$. 
2.1.2. Representative sets. A set of polynomials in \(\mathcal{M}_q\) is called a representative set modulo \(R_{\ell,M}\) if the set contains one and only one polynomial from each equivalence class of \(R_{\ell,M}\). The set \(\{f \in \mathcal{M}_{\ell+\deg M,q} : \gcd(f,M) = 1\}\) is a representative set modulo \(R_{\ell,M}\). More generally, if \(n \geq \ell + \deg M\), then \(\{f \in \mathcal{M}_{n,q} : \gcd(f,M) = 1\}\) is a disjoint union of \(q^{n-\ell-\deg M}\) representative sets modulo \(R_{\ell,M}\).

2.1.3. Characters. For every character \(\chi\) of the finite abelian group \((\mathcal{M}_q/R_{\ell,M})^\times\), we define \(\chi^\dagger\) with domain \(\mathcal{M}_q\) as follows: If \(A\) is invertible modulo \(R_{\ell,M}\) and if \(c\) is the equivalence class of \(A\), then \(\chi^\dagger(c) = \chi(c)\); If \(A\) is not invertible, then \(\chi^\dagger(A) = 0\).

The set of functions \(\chi^\dagger\) defined in this way are called the characters modulo \(R_{\ell,M}\), or sometimes “characters modulo \(R_{\ell,M}\)”. We shall for notational reasons abuse language somewhat and write \(\chi\) instead of \(\chi^\dagger\) to indicate a character of the relation \(R_{\ell,M}\) derived from the character \(\chi\) of the group \((\mathcal{M}_q/R_{\ell,M})^\times\). Thus we write \(\chi_0\) for the character of \(R_{\ell,M}\) which has the value 1 when \(A\) is invertible and the value 0 otherwise. We denote by \(G(R_{\ell,M})\) the set \(\{\chi^\dagger : \chi \in (\mathcal{M}_q/R_{\ell,M})^\times\}\). If \(\chi_1, \chi_2 \in G(R_{\ell,M})\), then

\[
\frac{1}{q^{\ell+\deg M}} \sum_{F \in \mathcal{M}_{\ell,M}} \chi_1(F)\chi_2(F) = \begin{cases} 
0 & \text{if } \chi_1 \neq \chi_2, \\
1 & \text{if } \chi_1 = \chi_2,
\end{cases}
\]

\(F\) running through a representative set modulo \(R_{\ell,M}\). In particular, if \(n \geq \ell + \deg M\) and \(\chi_2 = \chi_0\), we have

\[
\frac{1}{q^{n-\deg M}} \phi(M) \sum_{F \in \mathcal{M}_{n,q}} \chi(F) = \begin{cases} 
0 & \text{if } \chi \neq \chi_0, \\
1 & \text{if } \chi = \chi_0.
\end{cases}
\]

We have also

\[
\frac{1}{q^{\ell+\deg M}} \sum_{\chi \in G(R_{\ell,M})} \chi(A)\chi(B) = \begin{cases} 
1 & \text{if } A \equiv B \mod R_{\ell,M}, \\
0 & \text{otherwise}.
\end{cases}
\]

We also call the elements of \(G(R_{\ell,M})\) “generalized arithmetic progression characters”, because for any \(n \geq \ell\) and \(A \in \mathcal{M}_{n,q}\), \(\chi \in G(R_{\ell,M})\) is constant on the set

\[
\{f \in \mathcal{M}_{n,q} : f \equiv A \mod R_{\ell,M}\} = \{f \in \mathbb{F}_q[T] : f \equiv A \mod M\} \cap \{f \in \mathcal{M}_{n,q} : \deg(f-A) < n - \ell\}
\]

which is an intersection of an arithmetic progression and a short interval. We set, for future use,

\[
\text{GAP}(n,A;R_{\ell,M}) = \{f \in \mathcal{M}_{n,q} : f \equiv A \mod R_{\ell,M}\}.
\]

A character \(\chi\) modulo \(R_{\ell,M}\) is said to be “primitive modulo \(R_{\ell,M}\)”, or just “primitive”, if \(\chi \notin G(R_{\ell-1,M})\) and if for any proper divisor \(Q \mid M\), \(\chi\) is not of the form \(\chi_0^Q\) times a character from \(G(R_{\ell,Q})\). The number of non-primitive characters in \(G(R_{\ell,M})\) is bounded from above by

\[
\sum_{P | M, \ell | P} |G(R_{\ell,M}/P)| + 1_{\ell > 0} |G(R_{\ell-1,M})| \leq |G(R_{\ell,M})| \cdot (\sum_{P | M} 1_{P | P}) = O_{\deg M}(\frac{|G(R_{\ell,M})|}{q}).
\]

If \(\ell = 0\) and \(\deg M > 0\), we can identify \(\mathbb{F}_q^\times\) naturally with a subgroup of \((\mathcal{M}_q/R_{\ell,M})^\times \cong (\mathbb{F}_q[T]/M)^\times\) and \(\chi\) modulo \(R_{\ell,M}\) is said to be “even” if \(\chi\) is trivial on \(\mathbb{F}_q^\times\) and “odd” otherwise. When either \(\ell > 0\) or \(\deg M = 0\), we consider all characters modulo \(R_{\ell,M}\) to be odd. Thus, the number of even characters in \(G(R_{\ell,M})\) is 0 if \(\deg M = 0\) or \(\ell > 0\), and is \(\frac{|G(R_{\ell,M})|}{q-1}\) otherwise. In particular,

\[
\#\{\chi \in G(R_{\ell,M}) : \chi\text{ primitive and odd}\} = |G(R_{\ell,M})|\left(1 + O_{\ell,\deg M}(\frac{1}{q})\right).
\]

2.1.4. Structure of \(G(R_{\ell,M})\). Any character \(\chi \in G(R_{\ell,M})\) is of the form \(\chi_1 \cdot \chi_2\) where \(\chi_1 \in G(R_{\ell,1})\) and \(\chi_2 \in G(R_{0,M})\). This follows, for instance, from counting considerations. Characters modulo \(R_{\ell,1}\) are called “short interval characters” and characters modulo \(R_{0,M}\) are called “(usual) Dirichlet characters”.

2.1.5. $L$-functions. Let $\chi \in G(R_{\ell,M})$. The $L$-function of $\chi$ is the following series in $u$:

$$L(u, \chi) = \sum_{f \in \mathcal{M}_q} \chi(f) u^{\deg f},$$

which also admits the Euler product

$$(2.6)\quad L(u, \chi) = \prod_{P \in \mathcal{P}_q} (1 - \chi(P) u^{\deg P})^{-1}.$$ 

If $\chi$ is the trivial character $\chi_0$ of $G(R_{\ell,M})$, then

$$L(u, \chi_0) = \frac{\prod_{P|\mathcal{M}} (1 - u^{\deg P})}{1 - qu}.$$ 

Otherwise, the orthogonality relation $(2.3)$ implies that $L(u, \chi)$ is a polynomial in $u$ of degree at most $\ell + \deg M - 1$.

The first one to realize that Weil’s proof of the Riemann Hypothesis for Function Fields [34 Thm. 6, p. 134] implies the Riemann Hypothesis for the $L$-functions of $\chi \in G(R_{\ell,M})$ was Rhin [26 Thm. 3] in his thesis (cf. [11 Thm. 5.6] and the discussion following it). Hence we know that if we let $a(\chi)$ count the multiplicity of the root $u = 1$ in $L(u, \chi)$, and factor $L(u, \chi)$ as

$$(2.7)\quad L(u, \chi) = (1 - u)^{a(\chi)} \prod_{i=1}^{\deg L(u, \chi) - a(\chi)} (1 - \gamma_i(\chi) u),$$

then the $\gamma_i(\chi)$’s are $q$-Weil numbers of weight 1, i.e. they are algebraic numbers such that

$$(2.8)\quad |\gamma_i(\chi)| = \sqrt{q},$$

for all $i$, and $(2.8)$ is true for the conjugates of $\gamma_i(\chi)$ as well. It is known (for instance, by the functional equation) that if $\chi$ is primitive modulo $R_{\ell,M}$ then

$$\deg L(u, \chi) = \ell + \deg M - 1.$$ 

If $\chi$ is not trivial, we denote by $\Theta_\chi$ the conjugacy class of the matrix $\text{diag}(\sqrt[q]{\gamma_1(\chi)}, \ldots, \sqrt[q]{\gamma_{\deg L(u, \chi) - a(\chi)}(\chi)})$ in the unitary group $U(\deg L(u, \chi) - a(\chi))$. We sometimes abuse notation and think of $\Theta_\chi$ as a specific matrix. Thus,

$$L(u, \chi) = (1 - u)^{a(\chi)} \det(I - u\sqrt[q]{\Theta_\chi}).$$

Taking the logarithmic derivatives of $(2.6)$ and $(2.7)$ and comparing coefficients, we obtain

$$\sum_{f \in \mathcal{M}_{n,q}} \Lambda_q(f) \chi(f) = -\text{Tr}(\Theta_\chi^*) q^{\frac{3}{2}} - a(\chi),$$

for all $\chi_0 \neq \chi \in G(R_{\ell,M})$, from which the bound

$$(2.9)\quad \left| \sum_{f \in \mathcal{M}_{n,q}} \Lambda_q(f) \chi(f) \right| \leq (\ell + \deg M - 1) q^{\frac{3}{2}}$$

for all $\chi_0 \neq \chi \in G(R_{\ell,M})$ follows. If $\chi$ is odd and primitive then $a(\chi) = 0$ and $\Theta_\chi \in U(\ell + \deg M - 1)$.

2.2. Relations between $\text{Cov}_{A_\chi}(\alpha, \beta; n, \Delta)$ and $\text{Cov}_{M_q}(\alpha, \beta; n, \Delta)$. 

Lemma 1. Let $\alpha, \beta : \mathbb{F}_q[T] \to \mathbb{C}$ be two even arithmetic functions. Let $n$ be a positive integer. Let $\Delta \in \mathbb{F}_q[T]$ be a non-zero polynomial of degree $< n$.

1. We have

$$(2.10)\quad \text{Cov}_{A_\chi}(\alpha, \beta; n, \Delta) = \frac{\sum_{c \in \mathbb{F}_q} \text{Cov}_{M_q}(\alpha, \beta; n, c : \Delta)}{q - 1}.$$
(2) Let $c_1 \in \mathbb{F}_q^\times$, $c_2 \in \mathbb{F}_q$. Then

\begin{equation}
(2.11) \quad \text{Cov}_{M_q}(\alpha, \beta; n, \Delta(T)) = \text{Cov}_{M_q}(\alpha, \beta; n, \frac{\Delta(c_1 T + c_2)}{c_1^n}).
\end{equation}

If $\alpha = \beta$, we also have

\begin{equation}
(2.12) \quad \text{Cov}_{M_q}(\alpha, \beta; n, \Delta(T)) = \text{Cov}_{M_q}(\alpha, \beta; n, \frac{\Delta(c_1 T + c_2)}{c_1^n}).
\end{equation}

(3) Let $\Delta = a(T + b)^k$ ($\alpha \in \mathbb{F}_q^\times$, $b \in \mathbb{F}_q, 0 \leq k < n$). If $\gcd(n - k, q - 1) = 1$ then

\begin{equation}
\text{Cov}_{A_q}(\alpha, \beta; n, \Delta) = \text{Cov}_{M_q}(\alpha, \beta; n, \Delta).
\end{equation}

The same conclusion holds if the following conditions hold simultaneously: $\text{char}(\mathbb{F}_q) > 2, \alpha = \beta, \gcd(n - k, \frac{q - 1}{2}) = 1$.

Proof. (1) We rewrite the right hand side of (2.10) as follows:

\begin{equation}
\sum_{c \in \mathbb{F}_q^\times} \text{Cov}_{M_q}(\alpha, \beta; n, c \cdot \Delta) = \sum_{c \in \mathbb{F}_q^\times} \sum_{f \in M_{n,q}} \alpha(f) \overline{\beta}(f + c \cdot \Delta) = \sum_{c \in \mathbb{F}_q^\times} \frac{E_{M_{n,q}}(f \cdot \overline{\beta})(f + \Delta)}{(q - 1)q^n} = E_{M_{n,q}}(f \cdot \overline{\beta})(f + \Delta).
\end{equation}

(2) To show that (2.11) holds it suffices to show that

\begin{equation}
\sum_{f \in M_{n,q}} \alpha(f) \overline{\beta}(f + \Delta) = \sum_{f \in M_{n,q}} \alpha(f) \overline{\beta}(f + \frac{\Delta(c_1 T + c_2)}{c_1^n}).
\end{equation}

Note that $g(T) \mapsto g(c_1 T + c_2)/(c_1^n)$ is a permutation of $M_{n,q}$ (indeed, its inverse is given by $g(T) \mapsto c_1^n g(T - c_1^{-1})$). Applying this permutation to the left hand side of (2.13), we get the right hand side of (2.13). If $\alpha = \beta$, we also have

\begin{equation}
\sum_{f \in M_{n,q}} \alpha(f) \overline{\beta}(f + \frac{\Delta(c_1 T + c_2)}{c_1^n}) = \sum_{f \in M_{n,q}} \alpha(f + \frac{\Delta(c_1 T + c_2)}{c_1^n}) \overline{\beta}(f) = \sum_{f \in M_{n,q}} \alpha(f) \overline{\beta}(f - \frac{\Delta(c_1 T + c_2)}{c_1^n}),
\end{equation}

which establishes (2.12).

(3) First assume that $\gcd(n - k, q - 1) = 1$, a condition which ensures that $c \mapsto c^{n-k}$ is a permutation on $\mathbb{F}_q^\times$. From the two previous parts of the lemma, it suffices to show that for any $c \in \mathbb{F}_q^\times$ we have $c_1 \in \mathbb{F}_q^\times$ and $c_2 \in \mathbb{F}_q$ such that

\begin{equation}
a(T + b)^k = \frac{c \cdot (a(c_1 T + c_2 + b)^k)}{c_1^{n-k}}.
\end{equation}

We may take $c_1$ such that

\begin{equation}
c_1^{n-k} = c
\end{equation}

and

\begin{equation}
c_2 = b(c_1 - 1).
\end{equation}

We now assume instead that $\text{char}(\mathbb{F}_q) > 2, \alpha = \beta$ and $\gcd(n - k, \frac{q - 1}{2}) = 1$. The condition $\gcd(n - k, \frac{q - 1}{2}) = 1$ ensures that the subgroup of $\mathbb{F}_q^\times$ generated by $\{-1, g^{n-k}\}$ (where $g$ is a generator of $\mathbb{F}_q^\times$) is all of $\mathbb{F}_q^\times$. Indeed, $-1 = g^{q-1}$ and so the subgroup is in fact generated by $g^{\gcd(n-k, \frac{q-1}{2})}$, which is itself a generator if and only if $\gcd(n - k, \frac{q-1}{2}) = 1$. 
From the two previous parts of the lemma, it suffices to show that for any \( c \in \mathbb{F}_q^\times \) we have \( c_1 \in \mathbb{F}_q^\times \), \( c_2 \in \mathbb{F}_q \) and \( \varepsilon \in \{ \pm 1 \} \) such that
\[
a(T + b)^k = \varepsilon \frac{c_1 (a(T + c_2 + b)^k)}{c_1^n}.
\]
We may take \( c_1 \) and \( \varepsilon \) such that
\[
c_1^n - k = \varepsilon c
\]
and
\[
c_2 = b(c_1 - 1).
\]
□

2.3. Some Fourier expansions. The Fourier expansions of various arithmetic functions were calculated by Rodgers [27, §9].

**Proposition 1** (Rodgers). Let \( \mu, \Lambda, d_k : \text{EFT} \to \mathbb{C} \) be the functions with which \( \mu_q, \Lambda_q, d_{k,q} \) are associated, respectively. Let \( n \geq 2 \) be an integer and assume that \( k \geq 2 \).

1. The Fourier coefficients of \( \mu|_{S^\# n} \), defined in (1.11), are given by
\[
\hat{\mu}_\lambda = \begin{cases} 
(-1)^n & \lambda = (1^n), \\
0 & \text{otherwise}.
\end{cases}
\]

2. The Fourier coefficients of \( \Lambda|_{S^\# n} \), defined in (1.11), are given by
\[
\hat{\Lambda}_\lambda = \begin{cases} 
(-1)^{n-r} & \lambda = (r, 1^{n-r}) \text{ for some } 1 \leq r \leq n, \\
0 & \text{otherwise}.
\end{cases}
\]

3. The Fourier coefficients of \( d_k|_{S^\# n} \), defined in (1.11), are given by
\[
(\hat{d}_k)_\lambda = \begin{cases} 
s_\lambda \left( \underbrace{1, \ldots, 1}_k \right) & \ell(\lambda) \leq k, \\
0 & \text{otherwise},
\end{cases}
\]
where \( s_\lambda \) is the usual Schur function and \( \ell(\lambda) \) is the number of parts in \( \lambda \). Moreover, if \( \lambda = (\lambda_1, \ldots, \lambda_r) \) with \( r \leq k \), then
\[
s_\lambda \left( \underbrace{1, \ldots, 1}_k \right) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i},
\]
with the convention that \( \lambda_{r+1} = \lambda_{r+2} = \ldots = \lambda_k = 0 \).

**Corollary 2.** Let \( n \geq 3 \) and let \( \lambda = (n-1, 1) \) be a partition of \( n \). In the notation of Proposition 4 we have
\[
\hat{\mu}_\lambda = 0, \hat{\Lambda}_\lambda = -1, (\hat{d}_k)_\lambda = \binom{n+k-2}{k-2}(n-1).
\]

3. Proof of Theorem 3

3.1. Identities. For any arithmetic function \( \alpha \) on \( \mathbb{F}_q[T] \), any Hayes character \( \chi \) and any positive integer \( n \), we set
\[
(3.1) \quad S(n, \alpha, \chi) = \sum_{f \in \mathcal{M}_{n,q}} \alpha(f) \chi(f).
\]
For any positive integer \( n \) and any non-zero polynomial \( \Delta \) of degree \( < n \), we set
\[
(3.2) \quad \mathcal{M}_{n,q,\Delta} = \{ f \in \mathcal{M}_{n,q} : \gcd(f, \Delta) = 1 \}.
\]
Proposition 2. Let $n$ be a positive integer. Let $\Delta \in \mathcal{A}_q$ be a polynomial of degree $< n$. Let $h$ be an integer such that $n \geq h \geq \deg \Delta$. Let $\alpha, \beta : \mathbb{F}_q[T] \to \mathbb{C}$ be two arithmetic functions. Define

$$\text{Cov}_{M_q}(\alpha, \beta; n, \Delta, n-h) = \text{Cov}_{A \in (M_q/R_{n-h, \Delta})} \left( \sum_{f \equiv A \mod R_{n-h, \Delta}, f \in M_{n,q}} \alpha(f), \sum_{f \equiv A \mod R_{n-h, \Delta}, f \in M_{n,q}} \beta(f) \right).$$

Then the following identities hold.

1. We have

$$\text{(3.3) Cov}_{M_q}(\alpha, \beta; n, \Delta, n-h) = \left( \frac{1}{q^{n-h} \phi(\Delta)} \right)^2 \sum_{\chi_0 \neq \chi \mod R_{n-h, \Delta}} S(n, \alpha, \chi) S(n, \beta, \chi).$$

2. Assume further that $h \geq \deg \Delta + 1$. Then

$$\frac{\text{Cov}_{M_q}(\alpha, \beta; n, \Delta, n-h)}{q^{h-\deg \Delta}} - \frac{\text{Cov}_{M_q}(\alpha, \beta; n, \Delta, n-h+1)}{q^{h-\deg \Delta-1}} = \sum_{\delta \in \mathcal{A}_{n-\deg \Delta-1, \mathcal{A}_q}} \text{Cov}_{g \in M_{n,q, \Delta}} (\alpha(f), \beta(f + \delta \Delta)).$$

Proof. (1) For any $A \in (M_q/R_{n-h, \Delta})^\times$, (3.4) implies that

$$\text{(3.5) } \mathbb{I}_{g \in \text{GAP}(n,A;R_{n-h, \Delta})} = \frac{\sum_{\chi \in \mathcal{G}(R_{n-h, \Delta})} \chi(A) \chi(g)}{q^{n-h} \phi(\Delta)}$$

for all $g \in M_{n,q}$. From (3.5) we obtain

$$\text{(3.6) } \sum_{g \in \text{GAP}(n,A;R_{n-h, \Delta})} \alpha(g) = \sum_{g \in M_{n,q}} \alpha(g) \cdot \mathbb{I}_{g \in \text{GAP}(n,A;R_{n-h, \Delta})} = \frac{1}{q^{n-h} \phi(\Delta)} \sum_{g \in M_{n,q}} \alpha(g) \left( \sum_{\chi \in \mathcal{G}(R_{n-h, \Delta})} \chi(A) \chi(g) \right) = \frac{1}{q^{n-h} \phi(\Delta)} \sum_{\chi \in \mathcal{G}(R_{n-h, \Delta})} \chi(A) \left( \sum_{g \in M_{n,q}} \alpha(g) \chi(g) \right) = \frac{\sum_{\chi \in \mathcal{G}(R_{n-h, \Delta})} \chi(A) S(n, \alpha, \chi)}{q^{n-h} \phi(\Delta)}.$$

The term corresponding to $\chi = \chi_0$ in (3.6) is

$$\text{(3.7) } \frac{S(n, \alpha, \chi_0)}{q^{n-h} \phi(\Delta)} = q^{h-\deg \Delta} E_{M_{n,q, \Delta}} \alpha.$$  

From (3.6) and (3.7), we have

$$\text{(3.8) } \text{Cov}_{M_q}(\alpha, \beta; n, \Delta, n-h) = \frac{1}{q^{n-h} \phi(\Delta)} \sum_{A \in (M_q/R_{n-h, \Delta})^\times} \left( \sum_{f \in \text{GAP}(n,A;R_{n-h, \Delta})} \alpha(f) - q^{h-\deg \Delta} E_{M_{n,q, \Delta}} \alpha \right) \cdot \left( \sum_{f \in \text{GAP}(n,A;R_{n-h, \Delta})} \beta(f) - q^{h-\deg \Delta} E_{M_{n,q, \Delta}} \beta \right) = \frac{1}{q^{n-h} \phi(\Delta)} \sum_{\chi_0 \neq \chi_1 \in \mathcal{G}(R_{n-h, \Delta})} \chi_1(A) S(n, \alpha, \chi_1) \cdot \frac{\sum_{\chi_0 \neq \chi_2 \in \mathcal{G}(R_{n-h, \Delta})} \chi_2(A) S(n, \beta, \chi_2)}{q^{n-h} \phi(\Delta)} = \left( \frac{1}{q^{n-h} \phi(\Delta)} \right)^2 \sum_{\chi_1, \chi_2 \in \mathcal{G}(R_{n-h, \Delta}) \setminus \{\chi_0\}} S(n, \alpha, \chi_1) S(n, \beta, \chi_2) \sum_{A \in (M_q/R_{n-h, \Delta})^\times} \chi_1(A) \chi_2(A) \frac{\sum_{A \in (M_q/R_{n-h, \Delta})^\times} \chi_1(A) \chi_2(A)}{q^{n-h} \phi(\Delta)}.$$  

We conclude the proof of (3.8) by applying the orthogonality relation (2.2) to the right hand side of (3.5).
Since Euler’s totient function is multiplicative, we have
\begin{equation}
\frac{\text{Cov}_{\mathcal{M}_q}(\alpha, \beta; n, \Delta, n - h)}{q^{h - \deg \Delta}} = \frac{1}{q^{n - \deg \Delta} \phi(\Delta)} \sum_{\alpha \in (\mathcal{M}_q/R_{n-h, \Delta})^\times} \sum_{f, g \in \text{GAP}(n, A; R_{n-h, \Delta})} \alpha(f) \overline{\beta(g)} - q^{h - \deg \Delta} E_{\mathcal{M}_q, \alpha, \beta}. \tag{3.10}
\end{equation}

Instead of summing over pairs of polynomials \(f, g \in \text{GAP}(n, A; R_{n-h, \Delta})\) in (3.10), we sum over \(f \in \text{GAP}(n, A; R_{n-h, \Delta})\) and \(\delta := \frac{q - f}{h}\), an arbitrary polynomial of degree \(\leq h - \deg \Delta - 1\):
\begin{equation}
\frac{\text{Cov}_{\mathcal{M}_q}(\alpha, \beta; n, \Delta, n - h)}{q^{h - \deg \Delta}} = \frac{1}{q^{n - \deg \Delta} \phi(\Delta)} \sum_{f \in \mathcal{M}_q, \delta: \deg \delta \leq h - \deg \Delta - 1} \sum_{\alpha \in (\mathcal{M}_q/R_{n-h, \Delta})^\times} \alpha(f) \overline{\beta(f + \delta)} - q^{h - \deg \Delta} E_{\mathcal{M}_q, \alpha, \beta}. \tag{3.11}
\end{equation}

Similarly, putting \(h - 1\) for \(h\) in (3.11), we get
\begin{equation}
\frac{\text{Cov}_{\mathcal{M}_q}(\alpha, \beta; n, \Delta, n - h + 1)}{q^{h - 1 - \deg \Delta}} = \frac{1}{q^{n - \deg \Delta} \phi(\Delta)} \sum_{f \in \mathcal{M}_q, \delta: \deg \delta \leq h - \deg \Delta - 2} \sum_{\alpha \in (\mathcal{M}_q/R_{n-h, \Delta})^\times} \alpha(f) \overline{\beta(f + \delta)} - q^{h - \deg \Delta - 1} E_{\mathcal{M}_q, \alpha, \beta}. \tag{3.12}
\end{equation}
(Note: If \(h - \deg \Delta - 1 = 0\), then the only polynomial \(\delta\) of degree \(\leq h - \deg \Delta - 2\) is the zero polynomial, whose degree is defined to be \(-\infty\).) Subtracting (3.11) from (3.10), we get
\begin{equation}
\frac{\text{Cov}_{\mathcal{M}_q}(\alpha, \beta; n, \Delta, n - h)}{q^{h - \deg \Delta}} - \frac{\text{Cov}_{\mathcal{M}_q}(\alpha, \beta; n, \Delta, n - h + 1)}{q^{h - 1 - \deg \Delta}} = \sum_{\delta: \deg \delta = h - \deg \Delta - 1} \left( E_{f \in \mathcal{M}_q, \alpha} \overline{\beta(f + \delta)} - E_{\mathcal{M}_q, \alpha} \overline{\beta(f + \delta)} \right),
\end{equation}
which establishes (3.4). \(\Box\)

3.2. Estimates.

Lemma 2. Let \(\Delta \in \mathcal{A}_q\). We have
\[
\frac{|\Delta|}{\phi(\Delta)} = 1 + \frac{a \Delta q}{q} + O_{\deg \Delta} \left( \frac{1}{q^2} \right).
\]

Proof. Since Euler’s totient function is multiplicative, we have
\[
\frac{|\Delta|}{\phi(\Delta)} = \prod_{P | |\Delta|} \left( 1 + \frac{1}{|P| - 1} \right) = \sum_{D \mid \Delta, D \in \mathcal{M}_q} \prod_{P \mid D} \left( \frac{\mu_q^2(D)}{|P| - 1} \right) = 1 + \frac{a \Delta q}{q - 1} + \sum_{D \mid \Delta, D \in \mathcal{M}_q} \frac{\mu_q^2(D)}{|P| - 1}.
\]

In particular, since \(|P| - 1 \geq |P|/2\) for \(P \in \mathcal{P}_q\),
\begin{equation}
\left| \frac{|\Delta|}{\phi(\Delta)} \right| - \left( 1 + \frac{a \Delta q}{q - 1} \right) \leq \sum_{D \mid \Delta, D \in \mathcal{M}_q} \frac{\mu_q^2(D)}{|D|} \leq \frac{d^2_q(\Delta)}{|q|}, \tag{3.12}
\end{equation}
where in the last inequality we have used $|D| \geq q^2$ and the fact that the number of summands is, by definition, bounded by $d_{2,q}(\Delta)$. If $\Delta = e \cdot \prod_i P_i^{e_i}$ is the prime factorization of $\Delta$, then

\begin{equation}
    d_{2,q}(\Delta) = \prod_i (e_i + 1) \leq 2\sum_i e_i \leq 2^{\deg \Delta}.
\end{equation}

The lemma follows from (3.12) and (3.13). □

Later we sometimes use

\begin{equation}
    \frac{|\Delta|}{\phi(\Delta)} = 1 + O_{\deg \Delta} \left( \frac{1}{q} \right),
\end{equation}

a weaker version of Lemma 2.

**Lemma 3.** Let $\alpha : \mathbb{F}_q[T] \to \mathbb{C}$ be a factorization function. Let $n$ be a positive integer. Let $\Delta$ be a non-zero polynomial of degree $< n$. Then

\begin{equation}
    \sum_{f \in M_{n,q} : \gcd(f, \Delta) \neq 1} \alpha(f) = a_{\Delta, q} \sum_{f \in M_{n-1,q}} \alpha(f) + O_{\max(\alpha; n), \deg \Delta} (q^{n-2}).
\end{equation}

**Proof.** We write

\[
\sum_{f \in M_{n,q} : \gcd(f, \Delta) \neq 1} \alpha(f) = S_1 + S_2,
\]

where

\[
S_1 = \sum_{f \in M_{n,q} : \deg \gcd(f, \Delta) = 1} \alpha(f),
\]

\[
S_2 = \sum_{f \in M_{n,q} : \deg \gcd(f, \Delta) \geq 2} \alpha(f).
\]

We have

\[
|S_2| \leq \max(\alpha; n) \cdot \sum_{f \in M_{n,q} : \deg \gcd(f, \Delta) \geq 2} 1 = \max(\alpha; n) \cdot \sum_{D \mid \Delta, D \in M_q} \sum_{\deg D \geq 2} \frac{1}{D \mid D, D \in M_q, \deg D \geq 2}
\]

\[
\leq \max(\alpha; n) \cdot q^{n - \deg D} \leq q^{n-2} \cdot \max(\alpha; n) \cdot d_{2,q}(\Delta).
\]

As in the proof of Lemma 2 we have $d_{2,q}(\Delta) \leq 2^{\deg \Delta}$ and so

\begin{equation}
    |S_2| = O_{\max(\alpha; n), \deg \Delta} (q^{n-2}).
\end{equation}

Let $X = \{T - a : a \in \mathbb{F}_q, \Delta(a) = 0\}$. By inclusion-exclusion, we have

\[
S_1 = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\{L_1, \ldots, L_i\} \subseteq X} \sum_{f \in M_{n,q}} \alpha(f)\left| L_i \right| f
\]

\[
= S_3 + S_4
\]

where

\[
S_3 = \sum_{T - a \in X} \sum_{g \in M_{n-1,q}} \alpha(g(T) \cdot (T - a)),
\]

\[
S_4 = \sum_{i=2}^{n} (-1)^{i-1} \sum_{\{L_1, \ldots, L_i\} \subseteq X} \sum_{f \in M_{n,q}} \alpha(f)\left| L_i \right| f
\]

We estimate $S_4$ as follows:

\begin{equation}
    |S_4| \leq \max(\alpha; n) \cdot \sum_{i=2}^{n} \binom{|X|}{i} q^{n-i} \leq \max(\alpha; n) \cdot q^{n-2} \cdot 2^{|X|}
\]

\[
\leq \max(\alpha; n) \cdot q^{n-2} \cdot 2^{\deg \Delta} = O_{\max(\alpha; n), \deg \Delta} (q^{n-2}).
\]
Since \( \alpha \) is a factorization function, and \( g(T) \cdot (T-a), g(T+a) \cdot T \) have the same extended factorization type, we have

\[
S_3 = a_{\Delta, q} \sum_{f \in M_{n-1,q}} \alpha(f \cdot T).
\]

From (3.16), (3.17) and (3.18) we obtain (3.15) as needed. \( \square \)

The previous two lemmas were elementary. The following lemma requires a deeper result on the cycle structure of polynomials over finite fields.

**Lemma 4.** Let \( \alpha, \beta : \mathbb{F}_q[T] \to \mathbb{C} \) be factorization functions. Let \( n \) be a positive integer. Let \( \Delta \) be a non-zero polynomial of degree < \( n \). Then

\[
\sum_{f \in M_{n,q} : \gcd(f, \Delta) \neq 1} \alpha(f) \beta(f + \Delta) = a_{\Delta, q} \sum_{\gamma \in \mathbb{F}_q : \gamma \neq 0} \alpha(f) \beta(g \cdot T) q^{n-1} + O_{\max(\beta, n), \max(\alpha, n), n}(q^{n-2}).
\]

**Proof.** The same reasoning as in the proof of Lemma 3 shows that

\[
\sum_{f \in M_{n,q} : \gcd(f, \Delta) \neq 1} \alpha(f) \beta(f + \Delta) = \sum_{a \in \mathbb{F}_q : \gamma \neq 0} \sum_{g \in M_{n-1,q}} \alpha((T-a) \cdot g) \beta((T-a) \cdot (g + \frac{\Delta}{T-a})) q^{n-2}.
\]

We fix \( a \in \mathbb{F}_q \) such that \( \Delta(a) = 0 \). Let \( \lambda \) be a partition of \( n-1 \). We say that a polynomial \( g \) of degree \( n-1 \) is of type \( \lambda \) if it is squarefree and the degrees of its factors coincide with the parts of \( \lambda \). For any factorization function \( \gamma \), we let \( \gamma(\lambda) \) be the common value of \( \gamma \) on polynomials of type \( \lambda \) (if there is no polynomial of type \( \lambda \) in \( M_{n-1,q} \), we set \( \gamma(\lambda) = 0 \)).

We introduce two probabilities: \( P_{S_{n-1}}(\lambda) \) is the probability that a uniformly chosen element in \( S_{n-1} \) has cycle structure given by \( \lambda \), while \( P_{M_{n-1,q}}(\lambda) \) is the probability that a uniformly chosen element in \( M_{n-1,q} \) has type \( \lambda \). It is well known that [1, Lem. 2.1]

\[
P_{M_{n-1,q}}(\lambda) = P_{S_{n-1}}(\lambda) + O_n \left( \frac{1}{q} \right).
\]

Andrade, Bary-Soroker, and Rudnick [1, Thm. 1.4] proved that for any partitions \( \lambda_1, \lambda_2 \) of \( n-1 \), we have

\[
P_{g \in M_{n-1,q}}(g, g + \frac{\Delta}{T-a}) \text{ are of type } (\lambda_1, \lambda_2) = P_{S_{n-1}}(\lambda_1)P_{S_{n-1}}(\lambda_2) + O_n \left( \frac{1}{\sqrt{q}} \right).
\]

Denote by \( (\lambda_i, 1) \) the partition of \( n \) obtained by adjoining to the partition \( \lambda_i \) a part of size 1. Let

\[
p_{q, \lambda_1, \lambda_2} := P_{g \in M_{n-1,q}}((T-a)g, (T-a)(g + \frac{\Delta}{T-a})) \text{ are of type } (\lambda_1, 1), (\lambda_2, 1).
\]

We have

\[
p_{q, \lambda_1, \lambda_2} = p_{q, \lambda_1, \lambda_2, 1} + p_{q, \lambda_1, \lambda_2, 2},
\]

where

\[
p_{q, \lambda_1, \lambda_2, 1} = P_{g \in M_{n-1,q}}((T-a)g, (T-a)(g + \frac{\Delta}{T-a}) \text{ are of type } (\lambda_1, 1), (\lambda_2, 1),
\]

and \( g \) and \( g + \frac{\Delta}{T-a} \), are coprime with \( T-a \),

\[
p_{q, \lambda_1, \lambda_2, 2} = P_{g \in M_{n-1,q}}((T-a)g, (T-a)(g + \frac{\Delta}{T-a}) \text{ are of type } (\lambda_1, 1), (\lambda_2, 1),
\]

\[\gcd(T-a, g(g + \frac{\Delta}{T-a})) \neq 1).\]

Since the probability that both \( g, g + \frac{\Delta}{T-a} \) are coprime to \( T-a \) is \( 1 + O_n \left( \frac{1}{\sqrt{q}} \right) \), we have from (3.22)

\[
p_{q, \lambda_1, \lambda_2, 2} = O_n \left( \frac{1}{q} \right), \quad p_{q, \lambda_1, \lambda_2, 1} = P_{S_{n-1}}(\lambda_1)P_{S_{n-1}}(\lambda_2) + O_n \left( \frac{1}{\sqrt{q}} \right),
\]
and so
\[ p_{\alpha,\beta} = P_{S_{n-1}}(\lambda_1)P_{S_{n-1}}(\lambda_2) + O_n\left(\frac{1}{\sqrt{q}}\right). \]

Since \( \{P_{S_{n-1}}(\lambda_1)P_{S_{n-1}}(\lambda_2)\}_{\lambda_1,\lambda_2 \nmid n-1} \) sum to 1, the probabilities \( p_{\alpha,\beta} \) sum to \( 1 + O_n\left(\frac{1}{\sqrt{q}}\right) \), and we obtain
\[
\frac{1}{q^{n-1}} \sum_{g \in M_{n-1,q}} \alpha((T-a) \cdot g) \overline{\beta}(T-a) \cdot (g + \frac{\Delta}{T-a}) = \sum_{\lambda_1,\lambda_2 \nmid n-1} \left(P_{S_{n-1}}(\lambda_1)P_{S_{n-1}}(\lambda_2) + O_n\left(\frac{1}{\sqrt{q}}\right)\right) \\
\cdot \alpha((\lambda_1,1)) \overline{\beta}((\lambda_2,1)) + O_{n,\max(\alpha;n),\max(\beta;n)}\left(\frac{1}{\sqrt{q}}\right) \\
= \left(\sum_{\lambda_1 \nmid n-1} P_{S_{n-1}}(\lambda_1)\alpha((\lambda_1,1))\right) \left(\sum_{\lambda_2 \nmid n-1} P_{S_{n-1}}(\lambda_2)\overline{\beta}((\lambda_2,1))\right) \\
+ O_{n,\max(\alpha;n),\max(\beta;n)}\left(\frac{1}{\sqrt{q}}\right).
\]

By (3.21) and an argument similar to the above we have
\[
\frac{1}{q^{n-1}} \sum_{g \in M_{n-1,q}} \alpha((T-a) \cdot g) = \sum_{\lambda_1 \nmid n-1} P_{S_{n-1}}(\lambda_1)\alpha((\lambda_1,1)) + O_{n,\max(\alpha;n)}\left(\frac{1}{q}\right),
\]
and the same holds with \( \beta \) in place of \( \alpha \). From (3.20), (3.23) and (3.24), we obtain (3.19) as needed.

**Proposition 3.** Let \( \alpha, \beta : \mathbb{F}_q[T] \to \mathbb{C} \) be factorization functions. Let \( n \) be a positive integer. Let \( \Delta \) be a non-zero polynomial of degree \( < n \) and \( c \in \mathbb{F}_q^\times \). Then
\[
\sum_{f \in M_{n,q}, \Delta} \text{Cov}_{f \in M_{n,q}}(\alpha(f), \beta(f+c\Delta)) = q \cdot \text{Cov}_{A_q}(\alpha, \beta; n, \Delta) \\
- a_{\Delta,q} \left( E_{A_{n,q}} \alpha - E_{f \in A_{n-1,q}} \alpha(f \cdot T) \right) \\
\cdot \left( E_{A_{n,q}} \beta - E_{f \in A_{n-1,q}} \overline{\beta}(f \cdot T) \right) \\
+ O_{n,\max(\alpha;n),\max(\beta;n)}\left(\frac{1}{\sqrt{q}}\right).
\]

**Proof.** Fix \( c \in \mathbb{F}_q^\times \). We have, from Lemmas 2 and 3
\[
E_{f \in M_{n,q},\Delta} \alpha(f) \overline{\beta}(f+c\Delta) = \frac{\left|\Delta\right|}{\phi(\Delta)} \cdot \frac{\sum_{f \in M_{n,q}} \alpha(f) \overline{\beta}(f+c\Delta) - \sum_{f \in M_{n,q}, \gcd(f,\Delta) \neq 1} \alpha(f) \overline{\beta}(f+c\Delta)}{q^n} \\
= (1 + \frac{a_{\Delta,q}}{q}) \left( E_{f \in M_{n,q}} \alpha(f) \overline{\beta}(f+c\Delta) \\
- \frac{a_{\Delta,q}}{q} E_{f \in M_{n-1,q}} \alpha(f \cdot T) \cdot E_{f \in M_{n-1,q}} \overline{\beta}(f \cdot T) \right) \\
+ O_{n,\max(\alpha;n),\max(\beta;n),n} \left(\frac{1}{q^{\frac{3}{2}}}\right) \\
= (1 + \frac{a_{\Delta,q}}{q} E_{f \in M_{n,q}} \alpha(f) \overline{\beta}(f+c\Delta) - \frac{a_{\Delta,q}}{q} E_{f \in M_{n-1,q}} \alpha(f \cdot T) \cdot E_{f \in M_{n-1,q}} \overline{\beta}(f \cdot T) \\
+ O_{n,\max(\alpha;n),\max(\beta;n),n} \left(\frac{1}{q^{\frac{3}{2}}}\right).}
\]
Similarly, from Lemmas 2 and 3 we have
\[
E_{M_{n,q},\Delta} \alpha = (1 + \frac{a_{\Delta,q}}{q}) E_{M_{n,q}} \alpha - \frac{a_{\Delta,q}}{q} E_{f \in M_{n-1,q}} \alpha(f \cdot T) + O_{n,\max(\alpha;n),n} \left(\frac{1}{q^{2}}\right).
\]
From (3.25) and (3.26) we obtain
\[ (3.27) \]
\[
E_{f \in M_{n,q}}(f + c\Delta) - E_{M_{n,q}}\alpha \cdot E_{M_{n,q}}\beta = (1 + \frac{a\Delta_q}{q})(E_{f \in M_{n,q}}(f + c\Delta) - E_{M_{n,q}}\alpha \cdot E_{M_{n,q}}\beta)
\]
\[
- \frac{a\Delta_q}{q}(E_{M_{n,q}}\alpha - E_{f \in M_{n-1,q}}\alpha(f \cdot T))
\]
\[
\cdot (E_{M_{n,q}}\beta - E_{f \in M_{n-1,q}}\beta(f \cdot T))
\]
\[
+ O_{\max(\alpha,n),\max(\beta,n)}(q^{-\frac{1}{2}}).
\]
Summing (3.27) over \( c \in \mathbb{F}_q^\times \) and applying (2.10), we have
\[
\sum_{c \in \mathbb{F}_q^\times} \text{Cov}_{f \in M_{n,q}}(\alpha(f), \beta(f + c\Delta)) = (1 + \frac{a\Delta_q}{q})(q - 1)\text{Cov}_{A_q}(\alpha, \beta; n, \Delta)
\]
\[
- \frac{a\Delta_q}{q}(E_{A_{n,q}}\alpha - E_{f \in A_{n-1,q}}\alpha(f \cdot T))
\]
\[
\cdot (E_{A_{n,q}}\beta - E_{f \in A_{n-1,q}}\beta(f \cdot T))
\]
\[
+ O_{\max(\alpha,n),\max(\beta,n)}(\frac{1}{\sqrt{q}}).
\]

From (3.28) and (1.10) we conclude the proof of the proposition. \( \square \)

3.3. Additive decomposition of character sums, after Rodgers. The following results are generalizations of results of Rodgers [27]. Since the proofs are very similar, we refer to Rodgers’ work when appropriate.

**Lemma 5.** Let \( \alpha : \mathbb{F}_q[T] \rightarrow \mathbb{C} \) be a factorization function. Let \( \chi \) be a Hayes character in \( G(R_{\ell,M}) \setminus \{\chi_0\} \).

1. We have
\[
(3.29) \quad |S(n, \alpha, \chi)| = O_{\max(\alpha,n),n,\ell+\deg M}(q^{\frac{n-1}{2}}).
\]
2. If \( \alpha \) is supported on non-squarefree polynomials and \( \chi^2 \neq \chi_0 \), then
\[
(3.30) \quad S(n, \alpha, \chi) = O_{\max(\alpha,n),n,\ell+\deg M}(q^{\frac{n-1}{2}}).
\]

**Proof.** In the special case \( \ell = 0, M = T^k \), the bounds (3.29) and (3.30) were proved in [27 Lem. 6.3], [27 Lem. 6.2] respectively. The proofs work for any \( \ell \) and \( M \). The dependence on \( \ell + \deg M \) comes from (2.9), which generalizes [27] Eq. (26)]. \( \square \)

The next lemma requires some notation and definitions. For any partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of \( n \), let \( \chi_\lambda \) be the irreducible character of \( S_n \) associated with \( \lambda \). Let \( \chi_{\lambda,q} : \mathbb{F}_q[T] \rightarrow \mathbb{C} \) be the factorization function defined as follows (cf. [27 §2B]):
\[
\chi_{\lambda,q}(f) = \begin{cases} 
\chi_{\lambda}(\pi_f) & \text{if } f \text{ is squarefree of degree } n, \\
0 & \text{otherwise},
\end{cases}
\]
where \( \pi_f \) is any element of \( S_n \) in the conjugacy class associated with \( f \). We denote by \( s_{\lambda} \) the usual Schur function, and recall that its definition may be extended to unitary matrices, by evaluating \( s_{\lambda} \) at the corresponding eigenvalues (cf. [27 §5B, §5D]). We let \( \lambda' \) denote the partition of \( n \) conjugate to \( \lambda \).

**Lemma 6.** Let \( \chi \in G(R_{\ell,M}) \) such that \( \chi^2 \neq \chi_0 \). Let \( \lambda \vdash n \). Then
\[
(3.31) \quad S(n, \chi_{\lambda,q}, \chi) = q^{\frac{n}{2}}(-1)^n s_{\lambda'}(\Theta_{\chi}) + O_{n,\ell+\deg M}(q^{\frac{n-1}{2}}).
\]
Moreover, if \( \alpha_q \) is a factorization function which comes from \( \alpha \), then
\[
(3.32) \quad S(n, \alpha_q, \chi) = q^{\frac{n}{2}} \sum_{\lambda \vdash n} \hat{\alpha}_{\lambda} s_{\lambda'}(\Theta_{\chi}) + O_{\max(\alpha_q,n),n,\ell+\deg M}(q^{\frac{n-1}{2}}).
\]
Proof. In the special case \( \ell = 0, M = T^k \), the estimate (3.31) was proven in [27] Thm. 7.1. The proof works as is for any \( \ell \) and \( M \).

Let \( \text{EFT}_n \subseteq \text{EFT} \) consist of those elements of \( \text{EFT} \) that correspond to polynomials of degree \( n \). To prove (3.32), note that both \( S(n, \alpha_q, \chi) \) and \( \sum_{\lambda \vdash n} \hat{\alpha}_\lambda \sum_{\nu \vdash n} \Theta_{\nu} \) depend linearly on \( \alpha_{\text{EFT}_n} \), and so it suffices to prove (3.32) for a set of \( \alpha \)-s whose restriction to \( \text{EFT}_n \) spans the vector space of functions \( \text{EFT}_n \rightarrow \mathbb{C} \). Such a set is given by

\[
\{ \chi_\lambda \}_{\lambda \vdash n} \cup \{ \mathbb{I}_\nu : \nu \in \text{EFT}_n, \nu \text{ corresponds to non-squarefree polynomials} \}.
\]

We immediately verify (3.32) for these functions using (3.31) and (3.30).

Lemma 7. The number of character \( \chi \in \text{G}(R_{\ell, M}) \) such that \( \chi^2 = \chi_0 \) is \( O_{\deg M}(1) \) if \( q \) is odd, and \( O(q^{\deg M} + \deg M) \) if \( q \) is even.

Proof. We start with the case \( \ell = 0 \). From (2.1), the 2-torsion in \( \text{G}(R_{0, M}) \cong \text{G}(R_{0,1}) \) corresponds to solutions of

\[
x^2 \equiv 1 \mod M, \quad \deg x < \deg M.
\]

If \( q \) is odd, each such \( x \) defines a unique monic divisor of \( M \) via \( \gcd(x-1, M) \). By (3.13), the number of monic divisors of \( M = \prod_{i=1}^{m} P_i^{e_i} \) is bounded by \( O_{\deg M}(1) \), as needed. If \( q \) is even, (3.34) becomes \( M \mid (x+1)^2 \). We may write \( M = M_0 \cdot M_1^2 \) where \( M_0, M_1 \) are coprime and \( M_0 \) squarefree. The condition \( M \mid (x+1)^2 \) becomes \( M_0M_1 \mid x+1 \), which bounds the size of the 2-torsion by \( O(q^{\deg M} + \deg M) \).

We now proceed to prove the lemma in the case \( M = 1 \). By (2.1), \( \text{G}(R_{\ell,1}) \cong \text{G}(R_{1,1}) \cong (1 + T \mathbb{F}_q[T])/(1 + T^{\ell+1} \mathbb{F}_q[T]) \). In particular, \( R_{1,1} \) is of order \( q^\ell \). If \( q \) is odd, the 2-torsion is trivial. If \( q \) is even, the 2-torsion corresponds to polynomials of the form

\[
1 + T \cdot f, \quad \deg f < \ell,
\]

such that

\[
T^{\ell+1} \mid (1 + T \cdot f)^2 - 1.
\]

Since

\[
(1 + T \cdot f)^2 - 1 = T^2 f^2,
\]

we actually count polynomials \( f \) of degree \( < \ell \) such that \( T^{\ell+1} \mid f \), and their number is \( q^{\frac{\ell+1}{2}} \).

The isomorphism

\[
\text{G}(R_{\ell, M}) \cong \text{G}(R_{0, M}) \oplus \text{G}(R_{\ell,1}),
\]

proves the case of general \( \ell \) and \( M \) from the last two cases. \( \square \)

We have the following proposition, which relies crucially on the equidistribution result Theorem 8.

Proposition 4. Let \( \alpha_q, \beta_q \) be factorization functions which come from \( \alpha, \beta \). Let \( (\ell, M) \in \mathbb{N} \times \mathcal{A}_q \) where \( M \) is squarefree and either \( \ell \geq 4 \) or \( \ell = 0 \) and \( \deg M \geq 2 \). Then

\[
\sum_{\chi \in \text{G}(R_{\ell, M}) \backslash \{ \chi_0 \}} q^{-n} S(n, \alpha_q, \chi) S(n, \beta_q, \chi) = \sum_{\lambda \vdash n} \hat{\alpha}_\lambda \hat{\beta}_\lambda
\]

\[
\frac{1}{|G(R_{\ell, M})|} = \sum_{\lambda : \ell \mid \deg M - 1} \hat{\alpha}_\lambda \hat{\beta}_\lambda
\]

\[+ O_{\max(\alpha_q; n), \max(\beta_q; n), \ell, \deg M} \left( \frac{1}{\sqrt{q}} \right). \tag{3.35}\]

Proof. As in the proof of Lemma 6, both \( \sum_{\chi \in \text{G}(R_{\ell, M}) \backslash \{ \chi_0 \}} S(n, \alpha_q, \chi) S(n, \beta_q, \chi) \) and \( \sum_{\lambda \vdash n} \hat{\alpha}_\lambda \hat{\beta}_\lambda \) depend linearly on \( \alpha_{\text{EFT}_n} \) and conjugate-linearly on \( \beta_{\text{EFT}_n} \). Hence it suffices to prove (3.35) for \( \alpha, \beta \) in the set \( \{ \chi_\lambda \} \).

First assume that \( \alpha = \mathbb{I}_\nu \) for some \( \nu \in \text{EFT}_n \) which corresponds to non-squarefree polynomials. Then \( \hat{\alpha}_\lambda = 0 \) for all \( \lambda \vdash n \). Thus we need to prove that

\[
\sum_{\chi \in \text{G}(R_{\ell, M}) \backslash \{ \chi_0 \}} q^{-n} S(n, \mathbb{I}_\nu, \chi) S(n, \beta_q, \chi) = O_{\max(\beta_q; n), \ell, \deg M} \left( \frac{1}{\sqrt{q}} \right). \tag{3.36}\]


By breaking the sum over $G(R_{t,M}) \setminus \{\chi_0\}$ in the left hand side of (3.36) into two sums, one over $\chi \neq \chi_0$ and another over the rest of the $\chi$-s, and then applying Lemma 5 and Lemma 7, we obtain the right hand side of (3.36). The argument also works if $\beta = 1$.

We now assume that $\alpha = \chi_{\lambda(1)}, \beta = \chi_{\lambda(2)}$ for two partitions $\lambda^{(i)} \vdash n$. In this case we need to prove that

$$\sum_{\chi \in G(R_{t,M}) \setminus \{\chi_0\}} q^{-n} S(n, \chi_{\lambda(1)}, \chi) S(n, \chi_{\lambda(2)}, \chi) \left| G(R_{t,M}) \right| = 1 \chi_{\lambda(1)} = \chi_{\lambda(2)} + O_{n,\ell,\deg M \leq 1} \left( \frac{1}{\sqrt{q}} \right).$$

We write

$$\sum_{\chi \in G(R_{t,M}) \setminus \{\chi_0\}} q^{-n} S(n, \chi_{\lambda(1)}, \chi) S(n, \chi_{\lambda(2)}, \chi) \left| G(R_{t,M}) \right| = S_1 + S_2,$$

where

$$S_1 = \sum_{\chi \in G(R_{t,M}), \text{ primitive, odd and } \chi \neq \chi_0} q^{-n} S(n, \chi_{\lambda(1)}, \chi) S(n, \chi_{\lambda(2)}, \chi) \left| G(R_{t,M}) \right|,$$

$$S_2 = \sum_{\chi \in G(R_{t,M}) \setminus \{\chi_0\}, \text{ even or non-primitive or } \chi \neq \chi_0 \neq \chi_0} q^{-n} S(n, \chi_{\lambda(1)}, \chi) S(n, \chi_{\lambda(2)}, \chi) \left| G(R_{t,M}) \right|.$$

From Lemma 2 and Lemma 7 we see that the number of characters appearing in the sum (3.39) is $O_{\deg M \left( \frac{1}{\sqrt{q}} \right)^2}$. Thus, we may use the first part of Lemma 5 to bound $S_2$ by

$$S_2 = O_{n,\ell,\deg M \left( \frac{1}{q} \right)}.$$

We estimate $S_1$ as follows. From Lemma 3 we may rewrite $S_1$ as

$$S_1 = \sum_{\chi \in G(R_{t,M}), \text{ primitive, odd and } \chi \neq \chi_0} q^{-n} s_{\lambda^{(1)}}(\chi) s_{\lambda^{(2)}}(\chi) \left| G(R_{t,M}) \right| + O_{n,\ell,\deg M \left( \frac{1}{\sqrt{q}} \right)}.$$

Defining

$$S_3 = \sum_{\chi \in G(R_{t,M}), \text{ primitive, odd and } \chi \neq \chi_0} s_{\lambda^{(1)}}(\chi) s_{\lambda^{(2)}}(\chi) \left| G(R_{t,M}) \right|,$$

$$S_4 = \sum_{\chi \in G(R_{t,M}), \text{ primitive, odd and } \chi \neq \chi_0} s_{\lambda^{(1)}}(\chi) s_{\lambda^{(2)}}(\chi) \left| G(R_{t,M}) \right|,$$

we have

$$S_1 = S_3 - S_4 + O_{n,\ell,\deg M \left( \frac{1}{\sqrt{q}} \right)}.$$

The sum $S_4$ is easily bounded by Lemma 2 and Lemma 7 as

$$S_4 = O_{n,\ell,\deg M \left( \frac{1}{q} \right)}.$$

We now turn to $S_3$. The function $\Theta \mapsto s_{\lambda^{(1)}}(\Theta) s_{\lambda^{(2)}}(\Theta)$ is a class function on $PU(\ell + \deg M - 1)$, which is also a Laurent polynomial in the eigenvalues of $\Theta$, and so it can be written as a linear combination of irreducible characters of $PU(\ell + \deg M - 1)$:

$$s_{\lambda^{(1)}}(\Theta) s_{\lambda^{(2)}}(\Theta) = \sum_{\rho \text{ irreducible rep. of } PU(\ell + \deg M - 1)} a_{\rho,\lambda^{(1)},\lambda^{(2)},\ell,\deg M - 1} \cdot \text{tr}(\rho(\Theta)),$$

where $a_{\rho,\lambda^{(1)},\lambda^{(2)},\ell,\deg M - 1} = 0$ for all but finitely many $\rho$-s. We denote the trivial representation of $PU(\ell + \deg M - 1)$ by $1$. Since $\int_{PU(\ell + \deg M - 1)} \chi_1(\Theta) d\Theta = 1_{\rho = 1}$, we can calculate the coefficient of $1(\Theta)$ in the right hand side of (3.36) by integrating over $PU(\ell + \deg M - 1)$:

$$a_{1,\lambda^{(1)},\lambda^{(2)},\ell,\deg M - 1} = \int_{PU(\ell + \deg M - 1)} s_{\lambda^{(1)}}(\Theta) s_{\lambda^{(2)}}(\Theta) d\Theta = 1_{\lambda^{(1)} = \lambda^{(2)}},$$

where $\lambda^{(1)} = \lambda^{(2)}$.
where the last passage is a special case of orthogonality of irreducible characters \[12\text{ Eq. (3.8)}\]. Now, from (3.33), (3.34) and Theorem 8 we obtain
\[ S_3 - 1 \leq \sum_{1 \neq \rho \text{ irred. rep. of } PU(\ell + \text{deg } M - 1)} |a_{\rho, \lambda^{(1)}, \lambda^{(2)}, \ell + \text{deg } M - 1}| \]
\[ (3.45) \]

\[ \cdot \left| \sum_{\chi \in G(R_{\ell, M}) \text{ primitive and odd}} \text{tr}(\rho(\Theta_1)) \right| \cdot (1 + O_{\ell, \text{deg } M}(1/q)) \]
\[ \leq (1 + O_{\ell, \text{deg } M}(1/q)) \sum_{1 \neq \rho \text{ irred. rep. of } PU(\ell + \text{deg } M - 1)} |a_{\rho, \lambda^{(1)}, \lambda^{(2)}, \ell + \text{deg } M - 1}| C(\rho) \]
for a constant \(C(\rho)\) depending only on \(\rho\), where in the first inequality we used (2.5). Combining (3.38), (3.40), (3.41), (3.42) and (3.45), we obtain (3.37), as needed. \(\square\)

The following theorem is a generalization of [27] Thm. 10.1, which corresponds to the special case \(\Delta = 1\).

**Theorem 7.** Let \(\alpha_q, \beta_q\) be factorization functions which come from \(\alpha, \beta\). Let \(n\) be a positive integer. Let \((\Delta, h) \in A_q \times \mathbb{N}\) such that \(\Delta\) is squarefree and either \(n - 4 \geq h \geq \text{deg } \Delta\), or \(\text{deg } \Delta \geq 2\) and \(h = n\). Then
\[ \text{Cov}_{\Delta_q}(\alpha_q, \beta_q; n, \Delta, n - h) = q^{h - \text{deg } \Delta} \sum_{\lambda_1 \leq n - h + \text{deg } \Delta - 1} \hat{\alpha}_\lambda \overline{\beta}_\lambda + O_{\text{max}(\alpha_q; n), \text{max}(\beta_q; n), n}(q^{h - \deg \Delta - 1}). \]

**Proof.** From Proposition 4 with \(\ell = n - h\), \(M = \Delta\) and (3.38), we have
\[ \text{Cov}_{\Delta_q}(\alpha_q, \beta_q; n, \Delta, n - h) = \left(\frac{q^{n - h} \phi(\Delta)}{\phi(\Delta)}\right)^2 \sum_{\chi_0 \neq \chi \mod R_{n - h, \Delta}} S(n, \alpha_q, \chi) \overline{S(n, \beta_q, \chi)} \]
\[ = q^h \frac{\sum_{\chi \in G(R_{n - h, \Delta})} \chi_0^{-n}}{|G(R_{n - h, \Delta})|} q^{-n} S(n, \alpha_q, \chi) \overline{S(n, \beta_q, \chi)} \]
\[ = q^h \left(\sum_{\lambda_1 \leq n - h + \text{deg } \Delta - 1} \hat{\alpha}_\lambda \overline{\beta}_\lambda + O_{\text{max}(\alpha_q; n), \text{max}(\beta_q; n), n - h, \text{deg } \Delta}(1/q)\right), \]
from which (3.46) follows by invoking (3.14). \(\square\)

Although we do not use it, the next proposition shows that in the case \(h = \text{deg } \Delta\), the error term in Theorem 7 can be improved, and the proof is elementary.

**Proposition 5.** Let \(\alpha_q, \beta_q\) be factorization functions which come from \(\alpha, \beta\). Let \(n\) be a positive integer and let \(\Delta\) be a polynomial in \(A_q\) of degree \(\leq n\). Then
\[ \text{Cov}_{\Delta_q}(\alpha_q, \beta_q; n, \Delta, n - \text{deg } \Delta) = \sum_{\lambda_1 \leq n - 1} \hat{\alpha}_\lambda \overline{\beta}_\lambda + O_{\text{max}(\alpha_q; n), \text{max}(\beta_q; n), n}(1/q). \]

**Proof.** From (3.14) we have
\[ \text{Cov}_{\Delta_q}(\alpha_q, \beta_q; n, \Delta, n - \text{deg } \Delta) = \sum_{f \in \mathcal{M}_{n, q}} \alpha_q(f) \beta_q(\bar{f}) \frac{q^n}{q^n} - \sum_{f \in \mathcal{M}_{n, q}} \alpha_q(f) \sum_{f \in \mathcal{M}_{n, q}} \beta_q(\bar{f}) \frac{q^n}{q^n} + O_{\text{max}(\alpha_q; n), \text{max}(\beta_q; n), n}(1/q). \]
(3.47)

We use the notation of the proof of Lemma 4 with \(n\) in place of \(n - 1\). In particular, for a partition \(\lambda\) of \(n\), let \(p_\lambda\) the probability that a uniformly chosen element in \(S_n\) has cycle structure given by \(\lambda\). Let \(\gamma_q\) be a factorization function which comes from \(\gamma\). By (3.21) we have
\[ \sum_{f \in \mathcal{M}_{n, q}} \gamma_q(f) \frac{q^n}{q^n} = \sum_{\lambda \vdash n} p_\lambda \cdot \gamma_q(\lambda) + O_{n, \text{max}(\gamma_q; n)}(1/q) = \frac{1}{n!} \sum_{\pi \in S_n} \gamma(\pi) + O_{n, \text{max}(\gamma_q; n)}(1/q). \]
(3.48)
If $\gamma|_{S_n^{\mathbb{Z}}}(\pi) = \sum_{\lambda | n} \hat{\gamma}(\lambda) \chi_{S_n}(\pi)$ is the Fourier expansion of $\gamma$, then \textbf{[3.48]} may be expressed as
\begin{equation}
(3.49) \quad \sum_{f \in \mathcal{M}_{n,q}} \gamma(f) \frac{1}{q^n} = \hat{\gamma}(n) + O_{n,\max(\gamma,n)}\left(\frac{1}{q}\right).
\end{equation}
From \textbf{[3.47]} and \textbf{[3.49]} with $\gamma = \alpha, \beta$, we have
\begin{equation}
\text{Cov}_{\mathcal{A}_q}(\alpha, \beta; n, \Delta, n - \deg \Delta) = E_{\mathbb{Z}_q} \alpha \beta - \hat{\alpha}(n) \cdot \hat{\beta}(n) + O_{\max(\alpha,n),\max(\beta,n)}\left(\frac{1}{q}\right).
\end{equation}
The proof is concluded by noting that
\begin{equation}
E_{\mathbb{Z}_q} \alpha \beta = \sum_{\lambda | n} \hat{\alpha}(\lambda) \hat{\beta}(\lambda),
\end{equation}
a direct consequence of Plancherel theorem for the group $S_n$. \hfill \Box

We also need the following identity.

\textbf{Lemma 8.} Let $\alpha_q$ be a factorization function which comes from $\alpha$. Let $n$ be a positive integer. Then
\begin{equation}
\hat{\alpha}(n-1,1) = E_{f \in \mathcal{A}_{n-1,q}} \alpha_q(f \cdot T) - E_{A_{n,q}} \alpha_q + O_{n,\max(\alpha_q,n)}\left(\frac{1}{q}\right).
\end{equation}

\textbf{Proof.} Let $x \in \mathbb{F}_q$. Observe that, whenever $f \in \mathcal{A}_{n-1,q}$ is squarefree and coprime to $T$ and $T-x$, $\alpha_q(f \cdot T) = \alpha_q(f \cdot (T-x))$. Hence
\begin{equation}
E_{f \in \mathcal{A}_{n-1,q}} \alpha_q(f \cdot T) = E_{f \in \mathcal{A}_{n-1,q}, x \in \mathbb{F}_q} \alpha_q(f \cdot (T-x)) + O_{n,\max(\alpha_q,n)}\left(\frac{1}{q}\right).
\end{equation}
Now
\begin{equation}
E_{f \in \mathcal{A}_{n-1,q}, x \in \mathbb{F}_q} \alpha_q(f \cdot (T-x)) = \frac{1}{q^n(q-1)} \sum_{f \in \mathcal{A}_{n-1,q}, x \in \mathbb{F}_q} \alpha_q(f \cdot (T-x)) = \frac{1}{q^n(q-1)} \sum_{g \in \mathcal{A}_{n,q}} \alpha_q(g) \sum_{x \in \mathbb{F}_q : T-x \mid g} \alpha_q(g)\# \{x \in \mathbb{F}_q : g(x) = 0\}.
\end{equation}
Hence
\begin{equation}
E_{f \in \mathcal{A}_{n-1,q}} \alpha_q(f \cdot T) - E_{A_{n,q}} \alpha_q = \frac{1}{q^n(q-1)} \sum_{g \in \mathcal{A}_{n,q}} \alpha_q(g) (\# \{x \in \mathbb{F}_q : g(x) = 0\} - 1) + O_{n,\max(\alpha_q,n)}\left(\frac{1}{q}\right).
\end{equation}

Now for $g$ a squarefree polynomial, the function $(\# \{x \in \mathbb{F}_q : g(x) = 0\} - 1)$ simply counts the fixed points of the corresponding permutation and subtracts one. This is also the trace of the standard $(n-1)$-dimensional representation of $S_n$. By \textbf{[1]} Lemma 2.1, the average value of $\alpha_q$ times the trace of the standard representation over squarefree polynomials is equal, to within $O_{n,\max(\alpha_q,n)}\left(1/q\right)$, of its average over permutations, which is equal, by character theory, to the multiplicity $\hat{\alpha}(n-1,1)$ of the standard representation within $\alpha$, as needed. \hfill \Box

\textbf{3.4. Conclusion of the proof of Theorem 3} Recall that $n \geq 5$ is an integer and that $\Delta$ is a squarefree polynomial in $\mathcal{A}_q$ which is either of degree $\leq n - 5$ or of degree $n - 1$. From the second part of Proposition \textbf{[2]} with $h = \deg \Delta + 1$, we obtain
\begin{equation}
(3.50) \quad \frac{\text{Cov}_{\mathcal{A}_q}(\alpha, \beta; n, \Delta, n - \deg \Delta - 1)}{q} = \sum_{c \in \mathbb{F}_q} \text{Cov}_{f \in \mathcal{M}_{n,q,\Delta}}(\alpha(f), \beta(f + c\Delta)).
\end{equation}
From \([3.50]\) and Proposition 3 we obtain
\[
\frac{\text{Cov}_{M_q}(\alpha, \beta; n, \Delta, n - \deg \Delta - 1)}{q} - \text{Cov}_{M_q}(\alpha, \beta; n, \Delta, n - \deg \Delta)
\]  
(3.51)
\[
= q \cdot \text{Cov}_{M_q}(\alpha, \beta; n, \Delta) - a_{\Delta, q} (E_{A_n, q} \alpha - E_{f \in A_n, q} \alpha(f \cdot T)) \cdot (E_{A_n, q} \overline{\beta} - E_{f \in A_n, q} \overline{\beta}(f \cdot T))
\]
\[
+ O_{\max(\alpha; n), \max(\beta; n), n}(\frac{1}{\sqrt{q}}).
\]
From (3.51), Lemma 8 and Theorem 7 with \(h = 1 + \deg \Delta\) and \(h = \deg \Delta\), we obtain
\[
- \delta_{(n,1)} \beta_{(n,1)} + O_{\max(\alpha; n), \max(\beta; n), n}(\frac{1}{\sqrt{q}})
\]
(3.52)
\[
= q \cdot \text{Cov}_{A_q}(\alpha, \beta; n, \Delta) - a_{\Delta, q} \delta_{(n,1)} \beta_{(n,1)} + O_{\max(\alpha; n), \max(\beta; n), n}(\frac{1}{\sqrt{q}}).
\]
By isolating the term \(\text{Cov}_{A_q}(\alpha, \beta; n, \Delta)\) in (3.52), we conclude the proof of the theorem. \(\square\)

4. PROOF OF THEOREM 11

4.1. Fundamental identity. The following proposition uses the definitions of \(S(n, \alpha, \chi)\) and \(M_{n,q,\Delta}\), recall (3.1) and (3.2).

**Proposition 6.** Let \(\alpha, \beta : M_q \to \mathbb{C}\) be two arithmetic functions. Let \(n\) be a positive integer and let \(\Delta\) be a polynomial in \(A_q\) of degree \(< n\). Let \(c \in \mathbb{F}_q^*\) be the leading coefficient of \(\Delta\), and \(g\) be the unique element of \((M_q/R_{n-\deg \Delta, 1})^*\) such that
\[
g \equiv 1 \mod \Delta, \quad g \equiv T^{n-\deg \Delta} + c \mod R_{n-\deg \Delta, 1}.
\]
We have
\[
E_{f \in M_{n,q,\Delta}} \alpha(f) \beta(f + \Delta) = \frac{1}{q^{n-\deg \Delta} \phi^2(\Delta)} \sum_{\chi \in G(R_{n-\deg \Delta, \Delta})} \chi(g) S(n, \alpha, \chi) \overline{S(n, \beta, \chi)}.
\]

**Proof.** The orthogonality relation (2.4) implies that for any \(f \in M_{n,q,\Delta}\) we have
\[
(4.1) \quad \alpha(f) = \frac{1}{q^{n-\deg \Delta} \phi(\Delta)} \sum_{g_1 \in M_{n,q,\Delta}} \sum_{\chi \in G(R_{n-\deg \Delta, \Delta})} \alpha(g_1) \chi(g_1) \overline{\chi(f)},
\]
and similarly we have
\[
(4.2) \quad \beta(f + \Delta) = \frac{1}{q^{n-\deg \Delta} \phi(\Delta)} \sum_{g_2 \in M_{n,q,\Delta}} \sum_{\chi \in G(R_{n-\deg \Delta, \Delta})} \beta(g_2) \chi(g_2) \overline{\chi(f + \Delta)}.
\]
From (4.1) and (4.2) we obtain
\[
E_{f \in M_{n,q,\Delta}} \alpha(f) \beta(f + \Delta)
\]
(4.3)
\[
= \frac{1}{q^{n-\deg \Delta} \phi^2(\Delta)} \sum_{g_1, g_2 \in M_{n,q,\Delta}} \sum_{\chi_1, \chi_2 \in G(R_{n-\deg \Delta, \Delta})} \alpha(g_1) \chi_1(g_1) \beta(g_2) \chi_2(g_2) \sum_{f \in M_{n,q,\Delta}} \chi_1(f) \chi_2(f + \Delta).
\]
By the definition of \(g\), we have
\[
f + \Delta \equiv g \cdot f \mod R_{n-\deg \Delta, \Delta}.
\]
for all \(f \in M_{n,q,\Delta}\). For any pair of characters \(\chi_1, \chi_2 \in G(R_{n-\deg \Delta, \Delta})\) we have, by (4.3) and the orthogonality relation (2.2) with \(F = M_{n,q,\Delta}\),
\[
(4.4) \quad \sum_{f \in M_{n,q,\Delta}} \chi_1(f) \chi_2(f + \Delta) = \chi_2(g) \sum_{f \in M_{n,q,\Delta}} \chi_1(f) \chi_2(f)
\]
\[
= \chi_2(g) \cdot q^{n-\deg \Delta} \phi(\Delta) \cdot 1_{\chi_1 = \chi_2}.
\]
Plugging (4.5) in (4.3), we conclude the proof. \(\square\)
4.2. Estimates.

Lemma 9. Let $\alpha \colon \mathcal{M}_q \to \mathbb{C}$. Let $n$ be a positive integer and let $\Delta$ be a polynomial in $\mathcal{A}_q$ of degree $< n$. We have

$$E_{\mathcal{M}_{n,\Delta}} \alpha = E_{\mathcal{M}_{n,\Delta}} \alpha + O_{n,\max(\alpha; n)} \left( \frac{1}{q} \right).$$

Proof. We have

(4.6) $$E_{\mathcal{M}_{n,\Delta}} \alpha - E_{\mathcal{M}_{n,\Delta}} \alpha = S_1 + S_2,$$

where

$$S_1 = \left( \frac{1}{q^{n-\deg \phi(\Delta)}} - \frac{1}{q^n} \right) \sum_{f \in \mathcal{M}_{n,\Delta}} \alpha(f), \quad S_2 = -\frac{1}{q^n} \sum_{f \in \mathcal{M}_{n,\Delta} \setminus \mathcal{M}_{n,\Delta}} \alpha(f).$$

We have, by (3.4),

(4.7) $$|S_1|, |S_2| \leq \left( 1 - \frac{\phi(\Delta)}{|\Delta|} \right) \max(\alpha; n) = O_{n,\max(\alpha; n)} \left( \frac{1}{q} \right).$$

We conclude the proof from (4.6) and (4.7). \qed

Lemma 10. Let $\alpha, \beta \colon \mathcal{M}_q \to \mathbb{C}$. Let $n$ be a positive integer and let $\Delta$ be a polynomial in $\mathcal{A}_q$ of degree $< n$. Let $g \in (\mathcal{M}_q/R_{n-\deg \Delta, \Delta})^\times$. We have

$$\frac{1}{q^{2(n-\deg \Delta)\phi(\Delta)}} \sum_{\chi \in G(R_{n-\deg \Delta, \Delta})} \chi(g)S(n, \alpha, \chi)\overline{S(n, \beta, \chi)}$$

(4.8) $$= E_{\mathcal{M}_{n,\Delta}} \alpha \cdot E_{\mathcal{M}_{n,\Delta}} \beta$$

$$+ \frac{1}{q^{2(n-\deg \Delta)\phi(\Delta)}} \sum_{\chi \in G(R_{n-\deg \Delta, \Delta}) \text{ odd and primitive}} \chi(g)S(n, \alpha, \chi)\overline{S(n, \beta, \chi)}$$

$$+ O_{n,\max(\alpha; n),\max(\beta; n)} \left( \frac{1}{q} \right).$$

Proof. The term corresponding to $\chi = \chi_0$ in the left hand side of (4.8) contributes, by Lemma 9

$$E_{\mathcal{M}_{n,\Delta}} \alpha \cdot E_{\mathcal{M}_{n,\Delta}} \beta = E_{\mathcal{M}_{n,\Delta}} \alpha \cdot E_{\mathcal{M}_{n,\Delta}} \beta + O_{n,\max(\alpha; n),\max(\beta; n)} \left( \frac{1}{q} \right).$$

The terms corresponding to $\chi \in G(R_{n-\deg \Delta, \Delta}) \setminus \{ \chi_0 \}$ in the left hand side of (4.8), which are even or not primitive, contribute, by (2.1.3) and the first part of Lemma 5

$$O_{n,\max(\alpha; n),\max(\beta; n)} \left( \frac{1}{q} \right).$$

These two estimates conclude the proof of the lemma. \qed

4.3. Hidden symmetry. The following key proposition introduces an action of $\mathbb{F}_q^\times$ on $G(R_{\ell,1})$, which preserves primitivity and $L$-functions.

Proposition 7. Let $\ell$ be a positive integer. Let $\chi \in G(R_{\ell,1})$ be a primitive character. For any $c \in \mathbb{F}_q^\times$, define a function $\chi_c \colon \mathcal{M}_q \to \mathbb{C}$ by

$$\chi_c(f) = \chi(f(cT)/c^{\deg f}).$$

Then $\chi_c$ is well-defined on $\mathcal{M}_q/R_{\ell,1}$ and in fact is a primitive character in $G(R_{\ell,1})$. Moreover,

$$\Theta \chi = \Theta \chi_c.$$

Proof. Fix $c \in \mathbb{F}_q^\times$. Let $f_1, f_2 \in \mathcal{M}_q$ be polynomials such that $f_1 \equiv f_2 \mod R_{\ell,1}$. Then $f_1, f_2$ have the same first $\ell$ next-to-leading coefficients. The $i$-th next to leading coefficient of $f_j(cT)/c^{\deg f_j}$ ($j \in \{1, 2\}$) is the $i$-th next-to-leading coefficient of $f_j(T)$, divided by $c^{i}$. Thus, $f_1(cT)/c^{\deg f_1} \equiv f_2(cT)/c^{\deg f_2} \mod R_{\ell,1}$. This shows that $\chi_c$ can be regarded as a function of $\mathcal{M}_q/R_{\ell,1}$. By definition, $\chi_c$ is multiplicative, and it takes 1 to 1, so $\chi_c \in G(R_{\ell,1})$.

We now establish $\Theta \chi = \Theta \chi_c$. The coefficients of $u^i$ in $L(u, \chi)$ and $L(u, \chi_c)$ are given by $\sum_{f \in \mathcal{M}_{i,\chi}} \chi(f(T))$ and $\sum_{f \in \mathcal{M}_{i,\chi_c}} \chi(f(cT)/c^i)$, respectively. The map $f \mapsto f(cT)/c^i$ is a permutation of $\mathcal{M}_{i,\chi}$, whose inverse
Lemma 11. Let \( \ell \) be a positive integer. Let \( \chi \in G(R_{\ell,1}) \). Let \( c \in \mathbb{F}_q^\times \). For any factorization function \( \alpha \), we have

\[
S(n, \alpha, \chi) = S(n, \alpha, \chi_c).
\]

Proof. Since \( f(T), f(cT)/c^{\deg f} \) have the same extended factorization type for any \( c \in \mathbb{F}_q^\times \), and the inverse of \( f \mapsto f(cT)/c^{\deg f} \) is \( f \mapsto f(T)/c^{\deg f} \), we have

\[
S(n, \alpha, \chi_c) = \sum_{f \in \mathcal{M}_{n,q}} \alpha(f)\chi(f(cT)/c^n) = \sum_{f \in \mathcal{M}_{n,q}} \alpha(f(T)/c^n)\chi(f) = \sum_{f \in \mathcal{M}_{n,q}} \alpha(f)\chi(f) = S(n, \alpha, \chi),
\]

as needed. \( \square \)

4.4. Conclusion of proof. Applying Proposition \([B] \) with \( \Delta \in \mathbb{F}_q^\times \), we find that

\[
E_{f \in \mathcal{M}_{n,q}} \alpha_q(f)\beta_q(f + \Delta) = \frac{1}{q^{2n}} \sum_{\chi \in G(R_{n,1})} \chi(g)S(n, \alpha_q, \chi)S(n, \beta_q, \chi),
\]

where \( g \) may be taken to be \( g = T^n + \Delta \). Applying Lemma \([10] \) to the right hand side of \((4.9) \), we find that

\[
\text{Cov}_{\mathcal{M}_{n}}(\alpha_q, \beta_q; n, \Delta) = \frac{1}{q^{2n}} \sum_{\chi \in G(R_{n,1})} \text{primitive} \chi(T^n + \Delta)S(n, \alpha_q, \chi)S(n, \beta_q, \chi) + O_{n, \max(\alpha_q; n), \max(\beta_q; n)}\left(\frac{1}{q}\right).
\]

We claim that the multiset \( A = \{\chi_c : c \in \mathbb{F}_q^\times, \chi \in G(R_{n,1})\} \) consists of \( q - 1 \) copies of \( G(R_{n,1}) \). Indeed, the map \( \chi \mapsto \chi_c \) is a bijection for any \( c \in \mathbb{F}_q^\times \). Thus, in \((4.10) \) we may sum over primitive characters in \( A \) and divide by \( q - 1 \), instead of summing over primitive characters in \( G(R_{n,1}) \), and obtain from Lemma \([11] \)

\[
\text{Cov}_{\mathcal{M}_{n}}(\alpha_q, \beta_q; n, \Delta) = \frac{\sum_{\chi \in G(R_{n,1})} \text{primitive} \chi(T^n + \Delta)S(n, \alpha_q, \chi)S(n, \beta_q, \chi)}{q^{2n}} + O_{n, \max(\alpha_q; n), \max(\beta_q; n)}\left(\frac{1}{q}\right).
\]

When \( \chi \in G(R_{n,1}) \), \( \psi_\chi(x) := \chi(T^n + x) \) is an additive character of \( \mathbb{F}_q \), since \((T^n + x_1)(T^n + x_2) \equiv T^n + x_1 + x_2 \mod R_{n,1} \) and \((T^n + x)^p \equiv 1 \mod R_{n,1} \). Moreover, we claim that if \( \chi \) is primitive then \( \psi_\chi \) is non-trivial. Otherwise, whenever \( f \equiv g \mod R_{n,1} \), we may write \( f = g \cdot (T^n + x) \mod R_{n,1} \) for some \( x \in \mathbb{F}_q \), and then \( \chi(f) = \chi(g) \chi(T^n + x) = \chi(g) \), implying \( \chi \) is not primitive, a contradiction. Thus, if we set

\[
A(\chi, \Delta) = \frac{\sum_{c \in \mathbb{F}_q^\times} \chi_c(T^n + \Delta)}{\sqrt{q}} = \frac{\sum_{c \in \mathbb{F}_q^\times} \chi(T^n + \frac{\Delta}{c^n})}{\sqrt{q}} = \frac{\sum_{c \in \mathbb{F}_q^\times} \psi_\chi(\frac{\Delta}{c^n})}{\sqrt{q}} = \frac{\sum_{c \in \mathbb{F}_q^\times} \psi_\chi(\Delta_{c^n})}{\sqrt{q}},
\]

then \( |A(\chi, \Delta)| \leq n \) by Weil’s bound on additive character sums \([32] \) Thm. 2E]. We express \((4.11) \) as

\[
\text{Cov}_{\mathcal{M}_{n}}(\alpha_q, \beta_q; n, \Delta) = \frac{\sum_{\chi \in G(R_{n,1})} \text{primitive} A(\chi, \Delta)S(n, \alpha_q, \chi)S(n, \beta_q, \chi)}{q^{2n}} + O_{n, \max(\alpha_q; n), \max(\beta_q; n)}\left(\frac{1}{q}\right).
\]

The number of characters over which we sum and satisfy \( \chi^2 = \chi_0 \) is \( O(q^{-\frac{1}{2}}) \) by Lemma \([4] \) and so by Lemma \([5] \) their total contribution to the right hand side of \((4.12) \) is \( O_{\max(\alpha_q; n), \max(\beta_q; n)}(q^{-\frac{1}{2}}) \), which can be absorbed in the error term. From now on we ignore these characters when it will be convenient for us.

If either \( \alpha_q \) or \( \beta_q \) is supported on non-squarefrees, then \( S(n, \alpha_q, \chi)S(n, \beta_q, \chi) = O_{\max(\alpha_q; n), \max(\beta_q; n)}(q^{-\frac{1}{2}}) \) by Lemma \([5] \) and the right hand side of \((4.12) \) is \( O_{\max(\alpha_q; n), \max(\beta_q; n), n}(\frac{1}{q}) \), as needed. Thus, since both \( E_{f \in \mathcal{M}_{n,q}} \alpha_q(f)\beta_q(f + \Delta) \) and \( S(n, \alpha_q, \chi)S(n, \beta_q, \chi) \) are linear in \( \alpha_q \) and conjugate-linear in \( \beta_q \), it suffices to consider the case that \( \alpha_q = \chi_{\lambda_1, q} \) and \( \beta_q = \chi_{\lambda_2, q} \), where \( \lambda_1, \lambda_2 \vdash n \). We then have, by Lemma \([9] \)
\[ S(n, \alpha_q, \chi) = q^{n/2}S_{\chi'}(\Theta_\chi) + O_q(n^{n-1}) \quad \text{and} \quad S(n, \beta_q, \chi) = q^{n/2}S_{\chi'}(\Theta_\chi) + O_q(n^{n-1}), \]
and it remains to show that
\[
q^{-n} \sum_{\chi \in \mathcal{G}(\mathbb{R}_{+}, 1)} A(\chi, \Delta)(\Theta_\chi) S_{\chi'}(\Theta_\chi) = O_{n, \max(n, \alpha), \max(\beta; n)} \left( \frac{1}{q} \right).
\]

By decomposing \( s_{\chi'} s_{\chi''} \) as a linear combination of irreducible characters of \( \text{PU}(n - 1) \), we may apply Theorem 5 with \( \ell = n \) and conclude that \((5.13)\) holds, which concludes the proof of the theorem. 

5. Applications

5.1. Proof of Theorem 1 From Theorems 3 and 4 applied to the functions \( \alpha_q = \beta_q = \Lambda_q \), together with the calculation of the constants in Corollary 2 we obtain the results of Theorem 1. 

5.2. Proof of Theorem 2 From Theorems 3 and 4 applied to the functions \( \alpha_q = \beta_q = \mu_q \), together with the calculation of the constants in Corollary 2 we obtain the results of Theorem 2. 

5.3. Proof of Theorem 5 From Theorems 3 and 4 applied to the functions \( \alpha_q = d_{k, q} \) and \( \beta_q = d_{\ell, q} \), together with the calculation of the constants in Corollary 2 we obtain the results of Theorem 5. 

5.4. Consistency of Theorem 1 with the Hardy-Littlewood Conjecture. Let \( \Lambda \) be the usual von Mangoldt function, defined on the positive integers. The Hardy-Littlewood Conjecture 13 states that for any even, non-zero integer \( \Delta \),
\[
\sum_{n \leq x} \Lambda(n)\Lambda(n + \Delta) \sim \mathfrak{S}_\Delta, \quad (x \to \infty),
\]
where the constant \( \mathfrak{S}_\Delta \) is defined as the following product over primes, which converges to a positive number:
\[
\mathfrak{S}_\Delta = \prod_{p \mid \Delta} \prod_{\Delta} \frac{1 - \frac{1}{p}}{(1 - \frac{1}{p})^2} \prod_{\Delta} \frac{1 - \frac{2}{p}}{(1 - \frac{2}{p})^2}.
\]
In the function field setting, for a prime power \( q > 2 \), the same heuristics that suggest \((5.1)\) also suggest that
\[
\frac{\sum_{f \in \mathcal{M}_{n, q}} \Lambda_q(f)\Lambda_q(f + \Delta)}{q^n} \sim \mathfrak{S}_{\Delta, q}, \quad (q^n \to \infty),
\]
for all \( \Delta \in \mathcal{A}_q \), where \( \mathfrak{S}_{\Delta, q} \) is a product over prime polynomials:
\[
\mathfrak{S}_{\Delta, q} = \prod_{\Delta} \prod_{\Delta} \frac{1 - \frac{1}{|p|}}{(1 - \frac{1}{|p|})^2} \prod_{\Delta} \frac{1 - \frac{2}{|p|}}{(1 - \frac{2}{|p|})^2}.
\]
Since \( \mathfrak{S}_{\Delta, q} \) does not change if we multiply \( \Delta \) by a non-zero scalar, \((5.2)\) implies that
\[
\frac{\sum_{f \in \mathcal{M}_{n, q}} \Lambda_q(f)\Lambda_q(f + \Delta)}{q^n(q - 1)} \sim \mathfrak{S}_{\Delta, q}, \quad (q^n \to \infty)
\]
for all \( \Delta \in \mathcal{A}_q \). We have the following estimate for \( \mathfrak{S}_{1, q} \):
\[
\mathfrak{S}_{1, q} = \prod_{p \in \mathcal{P}_q} \frac{1 - \frac{2}{|p|}}{(1 - \frac{1}{|p|})^2} \prod_{p} \left( 1 - \frac{1}{|p|^2} + O \left( \frac{1}{|p|^3} \right) \right)
\]
\[
= \prod_{i \geq 1} \prod_{p \mid \deg p = i} \left( 1 - \frac{1}{q^{2i}} + O \left( \frac{1}{q^{3i}} \right) \right) = (1 - \frac{1}{q^2} + O \left( \frac{1}{q^3} \right))^{\prod_{i \geq 2} (1 + O(q^{-2i}))}^{O(q)}
\]
\[
= (1 - \frac{1}{q} + O \left( \frac{1}{q^2} \right))(1 + O \left( \frac{1}{q^2} \right)) = 1 - \frac{1}{q} + O \left( \frac{1}{q^2} \right),
\]
from which we deduce that

\[
\mathcal{G}_{\Delta,q} = \prod_{p|\Delta} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^2}} \cdot \mathcal{G}_{1,q} = \prod_{p|\Delta, p > 1} \left(1 + O\left(\frac{1}{q^2}\right)\right) \quad \prod_{p|\Delta, \deg p = 1} \frac{1 - \frac{1}{q}}{1 - \frac{1}{q}} \cdot \left(1 - \frac{1}{q} + O\left(\frac{1}{q^2}\right)\right)
\]

(5.3)

\[= (1 + O_{\deg} \Delta(q^{-2}))(1 + \frac{a_{\Delta,q}}{q} + O_{\deg} \Delta(q^{-2}))(1 - \frac{1}{q} + O\left(\frac{1}{q^2}\right))\]

\[= 1 + \frac{-1 + a_{\Delta,q}}{q} + O_{\deg} \Delta(q^{-2}).\]

The first two terms of the Taylor series of $\mathcal{G}_{\Delta,q}$, given in (5.3), agree with the main term given in Theorem 1 for $\Lambda_q$.

5.5. Proof of Theorem B. Plugging $\Delta = 1$ and $h$ in place of $h + 1$ in the second part of Proposition 2, we obtain

(5.4)

\[\sum_{\delta \in \mathcal{A}_{\delta}} \text{Cov}_{\mathcal{M}_n}(\alpha; \beta; n, \delta) = \frac{\text{Cov}_{\mathcal{M}_n}(\alpha, \beta; n, 1, n - h - 1)}{q^{h+1}} - \frac{\text{Cov}_{\mathcal{M}_n}(\alpha, \beta; n, 1, n - h)}{q^h}\]

for any $0 \leq h \leq n - 1$. The proof is concluded by using Theorem 7 to simplify the right hand side of (5.4). □

Appendix A. Equidistribution results

**Theorem 8.** Let $(\ell, M) \in \mathbb{N} \times \mathcal{A}_q$ such that $M$ is squarefree and either $\ell \geq 4$ or $\ell = 0$ and $\deg M \geq 2$. Let $\rho$ be an irreducible non-trivial representation of $\text{PU}(\ell + \deg M - 1)$. Then there exists a positive constant $C(\rho)$, depending only on $\rho$, such that

\[\left| \frac{\sum_{\chi \in G(R_{\ell,M}) \text{ primitive and odd}}}{\# \{\chi \in G(R_{\ell,M}) : \chi \text{ primitive and odd} \} \text{tr}(\rho(\Theta))} \right| \leq \frac{C(\rho)}{\sqrt{q}}.\]

The cases $\ell = 0$ and $M = 1$ are due to Katz [19, Cor. 6.9], [20, §8].

**Proof.** Because the case $\ell = 0$ is due to Katz, it suffices to handle the case $\ell \geq 4$.

We apply the isomorphism (2.1). We can write the average over $\chi$ as an iterated average over, first, characters $\chi_1$ of $(\mathbb{F}_q[T]/M\mathbb{F}_q[T])^\times$ of, second, an average over characters $\Lambda$ of $(1 + T\mathbb{F}_q[T])/(1 + T^{\ell+1}\mathbb{F}_q[T])$ and it suffices to prove the same bound for the average over $\Lambda$. We can view $\chi_1$ as a character of the idele class group of $\mathbb{F}_q(T)$ unramified away from $M\mathbb{F}_q^\times$ (with at most tame ramification at $\infty$), hence a character of the Galois group of $\mathbb{F}_q(T)$ unramified away from $M\mathbb{F}_q^\times$, which we view as a rank one Galois representation $V_{\chi_1}$. Let $\mathcal{F}_{\chi_1}$ be the rank one middle extension sheaf on $\mathcal{A}_q^1$ associated to $V_{\chi_1}$.

In [30, Theorem 1.2 and Theorem 1.3], families of conjugacy classes $\varphi_\Lambda$ in the unitary group $U(N) = U(\ell + \deg M - 1)$ associated, respectively, to $V_{\chi_1}$ and $\mathcal{F}_{\chi_1}$ are defined. It is proven in [30, §3, proof of Theorem 1.2] that these are equal, so we will use them interchangeably. Moreover, these conjugacy classes match the ones $\Theta_\chi$ we have defined, i.e.

$\varphi_\Lambda = \Theta_\chi$

up to conjugacy. This is because conjugacy classes in $U(\ell + \deg M - 1)$ are uniquely determined by their characteristic polynomial, both conjugacy classes are defined such that their characteristic polynomials match certain $L$-functions, and these $L$-functions agree because the $L$-function of a Galois representation associated to a character equals the $L$-function of the character. Hence we can apply the equidistribution result of [30, Theorem 1.3]. More precisely, we will apply its proof.

Because $\rho$ is a non-trivial representation of a projective unitary group, it is not one-dimensional. In [30, §6, proof of Theorem 1.3], it is shown that

\[\left| \frac{\sum_{\Lambda \in \mathcal{S}_q} \text{tr}(\rho(\varphi_\Lambda))}{\# \{\chi \in G(R_{\ell,M}) : \chi \text{ primitive and odd} \}} \right| = O\left(\frac{1}{\sqrt{q}}\right)\]

where $\mathcal{S}_q$ is the set of primitive characters of

$$(1 + T\mathbb{F}_q[T])/(1 + T^{\ell+1}\mathbb{F}_q[T]),$$

and, as mentioned earlier, $\varphi_\Lambda = \theta_\chi$. 
However, it is not proved in [30, §6] that the constant in the big $O$ is uniform in the characteristic or the choice of $\chi_1$. We do this now using an analogue of the argument in [31, Lemmas 2.7, 2.8, and 2.9].

The constant arises as a sum of Betti numbers of

$$H_c^i(U_{F_q}, V)$$

where $V$ is a sheaf constructed from the representation $\rho$ and $U_{F_q}$ is the open subset of $\mathbb{A}^\ell$ defined in [30, Definition 4.5]. By [30, Lemma 4.4] because $V_{\chi_1}$ is tamely ramified at infinity, and thus $F_{\chi_1}$ is as well, the subset $U$ consists of the primitive Dirichlet characters.

To check this Betti number boundedness we must dig into the weeds of étale cohomology. First note that the associated sheaf $G$ arises in [30, Definition 4.2] as $R^1pr_2(\rho^1 F \otimes L_{\text{univ}})$ for $L_{\text{univ}}$ a certain lisse sheaf of rank one. We can check that the dual sheaf is $R^1pr_2(\rho^1 F^\vee \otimes L_{\text{univ}}^*)$ because they are each lisse pure sheaves and have the same trace function. So it remains to bound the Betti numbers of

$$H_c^{i+2m} \left( U_{F_q}, (Rpr_2(\rho^1 F \otimes L_{\text{univ}}))^\otimes m \otimes (Rpr_2(\rho^1 F^\vee \otimes L_{\text{univ}}^*))^\otimes m \right).$$

Observe that $U$ is an open subset of $\mathbb{A}^\ell$, the parameter space of characters with Swan conductor $\leq \ell$, with complement $\mathbb{A}^{\ell-1}$, parameterizing characters with Swan conductor $\leq \ell - 1$. By excision, the compactly supported Betti numbers of $U$ with coefficients in the complex $(Rpr_2(\rho^1 F \otimes L_{\text{univ}}))^\otimes m \otimes (Rpr_2(\rho^1 F^\vee \otimes L_{\text{univ}}^*))^\otimes m$ are at most the sum of the Betti numbers of these two spaces with coefficients in the same complex. Because the two cases are equivalent with an index shifted by one, it suffices to bound the Betti numbers of $\mathbb{A}^\ell$.

To do this, we apply the Küneth formula and the projection formula, reducing us to

$$H_c^{i+2m} \left( (\mathbb{A}^1_{F_q})^\times \times \mathbb{A}^\ell_{F_q}, pr_1^*(F^\otimes m \boxtimes F^\vee \otimes m) \otimes L_{\text{univ}}^\otimes m \otimes L_{\text{univ}}^\vee \otimes m \right).$$

Applying the projection formula again, this is

$$H_c^{i+2m} \left( (\mathbb{A}^1_{F_q})^\times \times \mathbb{A}^\ell_{F_q}, F^\otimes m \boxtimes F^\vee \otimes m \otimes Rpr_1!(L_{\text{univ}}^\otimes m \otimes L_{\text{univ}}^\vee \otimes m) \right).$$

In [31, proof of Lemma 2.7, second equation]

$$Rpr_1!(L_{\text{univ}}^\otimes m \otimes L_{\text{univ}}^\vee \otimes m) = i_!Q_{ev}[-2\ell](-\ell)$$

where $i$ is the inclusion of the closed set $Z$ in $\left( \mathbb{A}^1_{F_q} \right)^\times$ where the first $\ell$ elementary symmetric polynomials in the first $m$ variables equal the first $m$ elementary symmetric polynomials in the last $m$ variables.

By a final application of the projection formula, we end up with

$$H_c^{i+2m-2\ell} \left( Z, i^*(F^\otimes m \boxtimes F^\vee \otimes m) \right).$$

Now $Z$ is a closed set in $\mathbb{A}^{2m}$ defined by $\ell$ equations of degree at most $\ell$. Because $M$ is squarefree, over a field extension in which it splits, $\chi_1$ is a product of at most $\deg M$ tame characters ramified at one of the roots of $M$ and $\infty$, so $F$ is a tensor product of at most degree $M$ tame character sheaves $L_{\rho_j}(x - a_j)$, for $a_j$ the roots of $M$.

The Betti numbers are now bounded by Theorem 12 of [18], with $N = 2m, r = \ell, s = 2m \deg m, \delta = 0, d_i = i$ for $i$ from 1 to $\ell, e_1, \ldots, e_s = 1, f = 0, F_1, \ldots, F_s$ the defining equations of $Z, G_1, \ldots, G_s$ the linear functions $x - a_j$ in the $2m$ variables with $a_j$ the roots of $M$. The Betti number bound is now given by Katz as

$$3(4m \deg M + \ell + 2)^{2m+\ell}$$

which has all the desired uniformity properties (noting that $m$ depends only on $\rho$).
Theorem 9. Let \( \ell \geq 3 \), and if \( \ell = 3 \) assume that the characteristic is not 2 or 5. For any \( \chi \in G(R_{\ell,1}) \) and \( \Delta \in \mathbb{F}_q^\times \), set
\[
\psi_{\Delta}(\chi) = \sum_{c \in \mathbb{F}_q^\times} \chi(T^\ell + \Delta c^\ell) / \sqrt{q}.
\]
Let \( \rho \) be an irreducible representation of \( PU(\ell - 1) \). Then there exists a positive constant \( D(\rho) \), depending only on \( \rho \), such that
\[
\left| \sum_{\chi \in G(R_{\ell,1}) \text{ primitive}} \frac{\text{tr}(\rho(\Theta_{\chi})))\psi_{\Delta}(\chi)}{\# \{ \chi \in G(R_{\ell,1}) : \chi \text{ primitive} \}} \right| \leq \frac{D(\rho)}{\sqrt{q}}.
\]

Proof. Let \( \exp \) be a fixed additive character of \( q \). Because \( x \mapsto \chi(T^\ell + x) \) is an additive character of \( \mathbb{F}_q \), it is \( \exp(a_{\chi}x) \) for some \( a_{\chi} \) in \( \mathbb{F}_q \). From basic properties of characters, each \( a_{\chi} \) occurs equally often, and \( \chi \) is primitive if and only if \( a_{\chi} \neq 0 \). We have the Gauss sum relation
\[
\psi_{\Delta}(\chi) = \sum_{c \in \mathbb{F}_q^\times} \frac{\exp(a_{\Delta}c^\ell)}{\sqrt{q}} = -\frac{1}{\sqrt{q}} + \sum_{\chi' : \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times \chi' = 1} \frac{G(\chi'^{-1},\psi)}{q} \chi'(a_{\chi} \Delta).
\]

It follows that
\[
\sum_{\chi \in G(R_{\ell,1}) \text{ primitive}} \frac{\text{tr}(\rho(\Theta_{\chi})))\psi_{\Delta}(\chi)}{\# \{ \chi \in G(R_{\ell,1}) : \chi \text{ primitive} \}} = \sum_{\chi' : \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times \chi' = 1} \frac{G(\chi'^{-1},\psi)}{q} \chi'(\Delta) \sum_{\chi \in G(R_{\ell,1}) \text{ primitive}} \frac{\text{tr}(\rho(\Theta_{\chi})))\chi'(a_{\chi})}{\# \{ \chi \in G(R_{\ell,1}) : \chi \text{ primitive} \}} - \sum_{\chi \in G(R_{\ell,1}) \text{ primitive}} \frac{\text{tr}(\rho(\Theta_{\chi})))\chi'(\Delta)}{\# \{ \chi \in G(R_{\ell,1}) : \chi \text{ primitive} \}}
\]

Because \( |\text{tr}(\rho(\Theta_{\chi})))| \leq \dim \rho \), we have
\[
\left| \sum_{\chi \in G(R_{\ell,1}) \text{ primitive}} \frac{\text{tr}(\rho(\Theta_{\chi})))\chi'(\Delta)}{\# \{ \chi \in G(R_{\ell,1}) \}} \right| \leq \frac{\dim \rho}{\sqrt{q}}.
\]

Because \( |G(\chi'^{-1},\psi)| = \sqrt{q} \) and \( |\chi'(\Delta)| = 1 \), it suffices to show that the averages
\[
\left| \sum_{\chi \in G(R_{\ell,1}) \text{ primitive}} \frac{\text{tr}(\rho(\Theta_{\chi})))a_{\chi}'}{\# \{ \chi \in G(R_{\ell,1}) : \chi \text{ primitive} \}} \right| \leq \frac{C(\rho)}{\sqrt{q}}
\]
for some constant \( C(\rho) \) (and then take \( D(\rho) = (\ell - 1)C(\rho) + \dim \rho \)).

First let us handle the case where \( \rho \) is the trivial representation. In this case, the bound follows immediately from the fact that each \( a_{\chi} \) occurs equally often, so all the non-trivial \( \chi' \) cancel completely.

Next let us handle the case where \( \rho \) is non-trivial and \( p > 2\ell + 1 \). In this case, as explained in [20, Remark 6.3] the characters \( \chi \) are associated to Artin-Schreier sheaves arising from polynomials of degree \( \ell \). From [20] the formula in the proof of Lemma 6.1] we can see that \( a_{\chi} \) is simply the top degree term of the polynomial times \( \ell \).

In the proof of [20, Theorem 8.2], Katz shows that the average of \( \text{tr}(\rho(\theta_{\chi})) \) over all polynomials of degree \( \ell \) with every term but the linear term fixed is \( O(1/\sqrt{q}) \) for all but a fraction of \( \leq 1/q \) of possible fixed choices for the high-degree terms (in fact, the problematic leading terms occur only for \( \ell \geq 5 \), and then they occupy a fraction at most \( q^{-(\ell-3)/2} \) of the possible leading terms). Since \( a_{\chi} \) is constant on these sets of polynomials, the same cancellation holds for \( \text{tr}(\rho(\theta_{\chi}))\chi'(a_{\chi}) \). Summing over all possible choices of leading terms, we get the desired bound.

Next let us handle the case where \( p \) is small. Here we cannot use the argument of Katz as a black box and must do some geometry. However, all the geometry is only a minor variant of the geometry done by Katz.

Let \( \text{Prim}_{\ell} \) be the space of primitive characters defined by Katz. He defined a sheaf \( L_{\text{univ}} \) on \( \text{Prim}_{\ell} \) [20, §4] whose Frobenius conjugacy class at a point corresponding to a character \( \chi \) is \( \theta_{\chi} \) [20, Lemma 4.1]. By
composing its monodromy representation with \( \rho \), we obtain a sheaf \( \rho(L_{\text{univ}}) \) whose Frobenius trace at a point is \( \text{tr}(\rho(\Theta_\chi)) \).

Let us in addition define a sheaf whose Frobenius trace at a point is \( \chi'(a_\chi) \). To do this, we check that \( a_\chi \) is a polynomial function on \( \text{Prim}_q \). Let \( \ell_0 \) be the largest prime-to-\( \ell \) divisor of \( \ell \).

To check this, observe that in the isomorphism defined in \([20, \S 2]\) between \((1 + TF_q[T])/(1 + T^{\ell+1}F_q[t])\) and \( \prod_{1 \leq m, m \text{ prime to } p, m \leq n} W_{l(m, \ell)}(\mathbb{F}_q) \), because \( 1 + xT^\ell \) is the Artin-Hasse exponential of \(-xT^\ell\), it is sent to a product which is 0 in every factor except \( m = \ell_0 \) and \( (0, \ldots, 0, \ell_0x) \) in the factor with \( m = \ell_0 \).

The character of this group associated to a tuple of Witt vectors is defined in \([20, \S 3]\) by elementwise multiplying Witt vectors, taking the trace to the Witt vectors of \( \mathbb{F}_p \), and applying a character of \( W_{l(m, \ell)}(\mathbb{F}_p) \). The product of the Witt vector \((0, \ldots, 0, \ell_0x)\) with another Witt vector depends only on the first coordinate \( a_0 \) of that other Witt vector, and taking the trace and applying a character of \( W_{v_p(\ell_0)+1}(\mathbb{F}_p) \) is the same as taking \( \exp(\ell_0xa_0) \). Hence we can take \( a_\chi = \ell_0a_0 \).

Then \( \chi'(a_\chi) \) is the trace function of the Artin-Schreier sheaf \( L_{\chi'}(a_\chi) \). Thus we must show cancellation in

\[
\sum_{x \in \text{Prim}_q} \text{tr}(\text{Frob}_q, \rho(L_{\text{univ}}), x) \text{tr}(\text{Frob}_q, L_{\chi'}(a_\chi), x).
\]

By the Lefschetz fixed point formula, this is

\[
2\ell \sum_{i=0}^{2\ell} (-1)^i \text{tr}(\text{Frob}_q, H^i_c(\text{Prim}_q, \mathbb{F}_q), \rho(L_{\text{univ}}) \otimes L_{\chi'}(a_\chi)).
\]

Because \( \rho(L_{\text{univ}}) \) and \( L_{\chi'}(a_\chi) \) are both pure of weight 0, eigenvalues of Frobenius acting on \( H^i \) have norm at most \( q^{i/2} \). We will show that \( H^{2\ell} \) vanishes, so each trace is at most \( q^{\ell - 1/2} \) times the dimension of \( H^i \).

Next we will show that the dimensions of the \( H^i \) are uniformly bounded. Thus the sum of traces will be \( O(q^{\ell - 1/2}) \) and dividing by the denominator will be \( O(q^{-1/2}) \), as desired.

We handle vanishing of the top cohomology first. By \([20, \text{Theorem 5.1}]\), under these assumptions, the monodromy of \( L_{\text{univ}} \) is a subgroup of \( GL_{2\ell-1} \) which contains \( SL_{2\ell-1} \) and thus maps surjectively onto \( PGL_{2\ell-1} \).

In particular, \( \rho(L_{\text{univ}}) \) is irreducible. Furthermore, \( L_{\chi'}(a_\chi) \) is lisse of rank one, so \( \rho(L_{\text{univ}}) \otimes L_{\chi'}(a_\chi) \) is irreducible. If \( \dim \rho \neq 1 \) then \( \rho(L_{\text{univ}}) \otimes L_{\chi'}(a_\chi) \) has an irreducible monodromy representation of dimension greater than one and so is non-trivial. If \( \dim \rho = 1 \) then \( \rho \) is the trivial representation and so these components are simply the monodromy representations of the Kummer sheaf, which are non-trivial because \( a_\chi \) is an affine coordinate of \( \text{Prim}_q \) under its isomorphism with an open subset of affine space. Hence in all cases the monodromy representation is irreducible and non-trivial, so it has no monodromy invariants, and thus the top cohomology vanishes.

To bound the Betti numbers, we can simply observe that in each characteristic, the sheaves in question can be defined only in terms of \( \ell, \rho, p \) and not the finite field \( \mathbb{F}_q \), so their Betti numbers are independent of \( \mathbb{F}_q \). Because there are only finitely many \( p \) left to consider, the Betti numbers are bounded in terms only of \( \ell, \rho \).

\[\square\]

**Acknowledgments**

We wish to thank Lior Bary-Soroker and Ze’ev Rudnick for comments on an earlier version of the manuscript.

The research of OG was supported by the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no 320755.

This research was partially conducted during the period WS served as a Clay Research Fellow, and partially conducted during the period he was supported by Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zurich Foundation.

**References**

[1] J. C. Andrade, L. Bary-Soroker, and Z. Rudnick. Shifted convolution and the Titchmarsh divisor problem over \( \mathbb{F}_q[t] \). *Philos. Trans. Roy. Soc. A*, 373(2040):20140308, 18, 2015.

[2] L. Bary-Soroker. Hardy-Littlewood tuple conjecture over large finite fields. *Int. Math. Res. Not. IMRN*, 2014(2):568–575, 2014.
