YORP torque as the function of shape harmonics

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ABSTRACT
The second-order analytical approximation of the mean Yarkovsky–O’Keefe–Radzievskii–Paddack (YORP) torque components is given as an explicit function of the shape spherical harmonics coefficients for a sufficiently regular minor body. The results are based upon a new expression for the insolation function, significantly simpler than in previous works. Linearized plane-parallel model of the temperature distribution derived from the insolation function allows us to take into account a non-zero conductivity. Final expressions for the three average components of the YORP torque related with rotation period, obliquity and precession are given in a form of the Legendre series of the cosine of obliquity. The series have good numerical properties and can be easily truncated according to the degree of the Legendre polynomials or associated functions, with first two terms playing the principal role.

Key words: radiation mechanisms: thermal – methods: analytical – celestial mechanics – minor planets, asteroids.

1 INTRODUCTION
The solar radiation affects rotation of minor bodies in the Solar system by two principal mechanisms: direct pressure and thermal re-radiation. The former has been shown to be of secondary importance (Nesvorny & Vokrouhlický 2008a) because its average effect is negligible. But the significance of the thermal re-radiation, known since the paper of Rubincam (2000) under the name of the YORP (Yarkovsky–O’Keefe–Radzievskii–Paddack) effect, is generally acclaimed and empirically demonstrated (Kaasalainen et al. 2007; Lowry et al. 2007; Taylor et al. 2007). A good account of the related problems can be found in Bottke et al. (2006).

The YORP torques acting on an irregular object are rather difficult to model. Until the last year, only the results of numerical simulations were available, with the torque values generated by summing the contributions from thousands of facets of a triangulated surface (Vokrouhlický & Čapek 2002; Čapek & Vokrouhlický 2004; Scheeres 2007). Although Scheeres (2007) provided a formula for the partially averaged YORP torque due to a single small triangular facet in terms of the elliptic integrals, his formulation remains of a semi-analytical type. The only known general relations between the body shape and the YORP torque consisted in the statement about the absence of YORP for spherical objects, and the ‘windmill asymmetry’ as the necessary condition for its appearance (Rubincam 2000). As it was shown by Breiter et al. (2007), if the thermal conductivity cannot be neglected, the notion of the ‘windmill asymmetry’ is wide enough to cover even the case of ellipsoids of revolution.

The first step towards the explanation of the YORP torque values in terms of an arbitrary body shape function was taken in the recent paper by Nesvorny & Vokrouhlický (2007). The body shape was expressed in terms of spherical harmonics and the general expressions for the mean component of the YORP torque responsible for the rotation period variations were derived within the second-order approximation, i.e. involving the products of shape harmonics coefficients. The resulting expressions are cumbersome, involving the products of the Wigner functions for the dependence on obliquity, and tabulated coefficients of power series in cosine of obliquity do not offer an explanation for the notion of the YORP order introduced in the paper. Recently, this line of research has been extended to the YORP torque component responsible for variations in obliquity, with a new element – the non-zero conductivity influence (Nesvorny & Vokrouhlický 2008b), but at the moment of submission we know only the preliminary version of this article, so our comments about its content have to be rather vague.

Our present contribution, developed in parallel to the work of Nesvorny and Vokrouhlický, is based upon similar patterns: shape description in terms of spherical harmonics, a linear, ‘plane-parallel’ temperature distribution model and the second-order approximation. Yet, we believe that a less complicated formulation of the insolation function, that we succeeded to obtain, leads to more convenient final expressions. The Wigner functions play the role of a ladder that becomes unnecessary after one climbs up to the final form of the mean YORP torque components, expressible in terms of the Legendre polynomials and associated functions of the order of 1. Formulating the mean torque

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as the Legendre series, we introduce the notion of the YORP degree, complementary to the YORP order of Nesvorný & Vokrouhlický (2007), directly related to the degree of the Legendre functions used, and helpful when it comes to truncating the final expressions.

2 GENERAL FORMULA FOR THE YORP TORQUE
Consider an infinitesimal, outwards oriented surface element $dS$ of a small body having temperature $T$. If the Lambert emission model is assumed, the thermal radiation of this element gives rise to the force (Bottke et al. 2006)

$$df = -\frac{2 \varepsilon_i \sigma}{3 v_c} T^4 dS,$$

(1)

where $\varepsilon_i$ is the surface element emissivity, $\sigma$ is the Stefan–Boltzmann constant and $v_c$ stands for the velocity of light. Acting on a surface element with the radius vector from the centre of mass $r$, the force produces an infinitesimal torque $dM = r \times df$. Assuming the common emissivity for all surface elements and integrating over the entire surface of the body, we obtain the net YORP torque

$$M = \frac{2 \varepsilon_i \sigma}{3 v_c} \int_S T^4 (r \times dS).$$

(2)

The temperature $T$ will be considered a continuous (but not necessarily smooth) function of the longitude and latitude of a surface element, and of the Sun position.

Obviously, if the body is spherical, the vectors $r$ and $dS$ are parallel, so the torque $M$ vanishes identically regardless of the temperature distribution function. On the other hand, using elementary identities of the vector calculus, we can rewrite equation (2), replacing the surface integral that involves $dS$ by an integral over the volume of the considered body

$$M = \frac{2 \varepsilon_i \sigma}{3 v_c} \int_V \left[ \nabla \times (T^4 r) \right] dV = \frac{2 \varepsilon_i \sigma}{3 v_c} \int_V \left( \nabla T^4 \right) \times r \, dV.$$

(3)

This form is not more useful for the purpose of the present paper than the usual equation (2), but it leads to a second basic fact about the YORP torque: $M$ vanishes if the temperature distribution is isotropic, regardless of the body shape. In other words, the mean value of the temperature over the surface is irrelevant for the computation of YORP.

3 BODY SHAPE MODEL AND INSOLATION FUNCTION

3.1 Surface equation in the body frame
Let us introduce the reference frame $Oxyz$ with the origin at the centre of mass and the axes aligned with the principal axes of inertia. In this ‘body frame’, we can express the object shape in the form of a parametric equation

$$r = a + a \sum_{l \geq 1} \sum_{m=0}^{l} \Theta_l^m (\cos \theta) \left[ C_{l,m} \cos m \lambda + S_{l,m} \sin m \lambda \right],$$

(4)

being the function of the colatitude $\theta$, longitude $\lambda$ and reference radius $a$. In further discussion, we will always use

$$u = \cos \theta, \quad w = \sin \theta = \sqrt{1 - u^2}.$$

(5)

The normalized Legendre functions $\Theta_l^m(u)$ are defined in Appendix A1. Normalized, dimensionless shape coefficients $C_{l,m}$ and $S_{l,m}$ will be assumed small quantities of the first order. Their small values should allow the treatment by means of the perturbation approach, and they should guarantee the convexity of figure, because no shadowing is allowed in our model, as long as the Sun remains above the local tangent plane.

The real form of equation (4) is instructive, but less advantageous in further transformations. For this reason, we pass to the complex form of the real-valued equation for $r$

$$r = a + a \sum_{l \geq 1} \sum_{m=-l}^{l} f_{l,m} Y_{l,m}(u, \lambda),$$

(6)

expressed in terms of spherical functions $Y_{l,m}$ (see Appendix A1), and of complex shape coefficients

$$f_{l,0} = C_{l,0}, \quad f_{l,m} = (C_{l,m} - i S_{l,m})/2, \quad \text{for } m > 0, \quad f_{l,-m} = (-1)^m (C_{l,m} + i S_{l,m})/2 = (-1)^m f_{l,m}^*. $$

(7)

Inverting equations (7), we find

$$C_{l,0} = f_{l,0}, \quad C_{l,m} = (f_{l,m} + (-1)^m f_{l,-m}) = f_{l,m} + f_{l,m}^*, \quad S_{l,m} = i \left( f_{l,-m} - (-1)^m f_{l,m} \right) = i \left( f_{l,m} - f_{l,m}^* \right),$$

(8)

in agreement with the properties (A11) of harmonic series.

For the sake of brevity, we introduce a symbol $\Psi$, so that

$$r = a (1 + \Psi),$$

(9)
\[ \Psi = \sum_{l \geq 1} \sum_{m=-l}^{l} f_{l,m} Y_{l,m}(u, \lambda). \] (10)

\( \Psi \) and its derivatives will be considered small quantities of the first order.

### 3.2 Normal vector approximation

In this paper, we will frequently use a modified set of spherical coordinates with associated unit vectors

\[ \hat{e}_r = \begin{pmatrix} w \cos \lambda \\ w \sin \lambda \\ u \end{pmatrix}, \quad \hat{e}_\lambda = \frac{1}{w} \frac{\partial \hat{e}_r}{\partial \lambda} = \begin{pmatrix} -\sin \lambda \\ \cos \lambda \\ 0 \end{pmatrix}, \quad \hat{e}_u = \frac{\partial \hat{e}_r}{\partial u} = \begin{pmatrix} -u \cos \lambda \\ -u \sin \lambda \\ w \end{pmatrix}, \] (11)

forming a right-handed orthonormal basis. Restricting the differentiation to the surface of the unit sphere \( S \), we define a spherical surface gradient \( \nabla_s \) as

\[ \nabla_s f \equiv \frac{1}{w} \frac{\partial f}{\partial \lambda} \hat{e}_\lambda + w \frac{\partial f}{\partial u} \hat{e}_u. \] (12)

Given the shape function (9), and setting \( r = \hat{e}_r \), we can compute the outward normal vector on the body surface

\[ N = \frac{\partial r}{\partial \lambda} \times \frac{\partial r}{\partial u} = a^2(1 + \Psi)(1 + \Psi) \hat{e}_r - w \frac{\partial \Psi}{\partial \lambda} \hat{e}_\lambda - w \frac{\partial \Psi}{\partial u} \hat{e}_u. \] (13)

The normal vector \( N \) given above is relatively easy to handle, but the unit normal vector \( \hat{n} \) derived from (13) has so complicated form that we will use its linear approximation with respect to the shape coefficients

\[ \hat{n} = \frac{N}{N} \approx \hat{e}_r - \frac{1}{w} \frac{\partial \Psi}{\partial \lambda} \hat{e}_\lambda - w \frac{\partial \Psi}{\partial u} \hat{e}_u. \] (14)

The first-order approximation is sufficient for our purpose because it will be multiplied by a first-order quantity inside the integral (2).

### 3.3 Solar position

Although the shape of an object has been described in the body frame, we will temporarily use another reference frame, where the description of the solar position and the terminator equation is more convenient. Thus, we introduce the \( Ox'y'z' \) orbital frame, with the \( Ox' \) axis pointing towards the Sun, \( Oy' \) axis directed along the orbital momentum vector of the minor body and the \( Oz' \) axis completing the orthogonal, right-handed triad.

In terms of the usual 3-1-3 Euler angles and rotation matrices \( R_i \), the transformation from the body frame \( Oxyz \) to the orbital frame \( Ox'y'z' \) requires (Fig. 1)

\[ r' = R_3(\psi)R_2(\epsilon)R_3(-\Omega) r. \] (15)

However, a 3-2-3 rotation sequence is preferred in physics and we will use it in the present discussion. As a matter of fact, the two sets of Euler angles are very closely related and

\[ r' = R_3(\psi')R_2(\epsilon)R_3(\phi) r \] (16)

![Figure 1. Reference frames and orientation angles used in the paper.](https://academic.oup.com/mnras/article-abstract/388/2/927/978971)
is fully equivalent to equation (15) if we set
\[ \phi = -\Omega + \frac{\pi}{2}, \quad \psi = \vartheta - \frac{\pi}{2}. \]  
(17)

Assuming the circular, Keplerian motion approximation, we use a constant obliquity angle \( \varepsilon \), whereas \( \Omega \) reflects the 'diurnal' rotation, and the argument of latitude \( \vartheta \) is an angle that varies according to the 'yearly' orbital motion. The unit vector towards the Sun \( \hat{n}_0 \) and the vector normal to orbital plane \( \hat{h}_0 \) are given by trivial expression in the orbital frame
\[ \hat{n}_0 = (1, 0, 0)^T = \hat{e}_z, \quad \hat{h}_0 = (0, 0, 1)^T = \hat{e}_x. \]  
(18)

Applying the inverse (transposed) rotation matrix (16), we transform \( \hat{n}_0 = \hat{e}_x \) to the body frame, obtaining
\[ \hat{n}_b = (c \cos \varphi \cos \psi - \sin \varphi \sin \psi) \hat{e}_x + (c \sin \varphi \cos \psi + \cos \varphi \sin \psi) \hat{e}_y + s \cos \psi \hat{e}_z. \]  
(19)

Throughout the text, we use the symbols
\[ c = \cos \varepsilon, \quad s = \sin \varepsilon, \]  
(20)
related with the obliquity angle \( \varepsilon \). Both \( c \) and \( s \) are considered constant and the motion of the Sun in the body frame consists of the yearly motion described by \( \psi \) and the apparent daily motion reflected in \( \phi \).

### 3.4 Insolation function

The so-called insolation function is primarily defined in terms of the unit normal vector of the body surface \( \hat{n} \) and the unit vector directed to the Sun \( \hat{n}_0 \) as
\[ \mathcal{E} = (1 - A) \Phi \max (0, \hat{n} \cdot \hat{n}_0), \]  
(21)
where \( A \) designates the Bond albedo and \( \Phi = \Phi_0 a_0^{-2} \) is the ratio of the solar constant \( \Phi_0 \approx 1366 \, \text{W m}^{-2} \) to the square of the orbital semimajor axis \( a_0 \) expressed in astronomical units. \( \mathcal{E} \) determines the energy flux received by a given surface element of an illuminated body.

According to equation (14), we study an approximate insolation function involving
\[ \hat{n} \cdot \hat{n}_0 \approx \hat{e}_z \cdot \hat{n}_0 - \frac{1}{w^2} \frac{\partial \psi}{\partial \lambda} \frac{\partial (\hat{e}_z \cdot \hat{n}_0)}{\partial \lambda} = -w^2 \frac{\partial \psi}{\partial u} \frac{\partial (\hat{e}_z \cdot \hat{n}_0)}{\partial u}, \]  
(22)
because \( \hat{n}_0 \) is independent of \( u \) and \( \lambda \).

The function \( \max (0, \hat{n} \cdot \hat{n}_0) \) is actually equivalent to a product \( H(\hat{n} \cdot \hat{n}_0) \hat{n} \cdot \hat{n}_0 \), where \( H(x) \) is the Heaviside theta function. The values of \( H(\hat{n} \cdot \hat{n}_0) \) are as follows: 1 on the sunny side of the body, 0 on the dark side and 1/2 on the terminator. As it was demonstrated by Nesvorný & Vokrouhlický (2007), replacing the actual terminator by the terminator of a spherical body leads to an error of the second order. Thus, within the framework of the first-order approximation, we can replace \( H(\hat{n} \cdot \hat{n}_0) \) by \( H(\hat{e}_z \cdot \hat{n}_0) \). Moreover, thanks to the properties of the Heaviside function, we can replace \( H(y)(dy/dx) \) by \( (H(y) y)/dx \), so
\[ \max (0, \hat{n} \cdot \hat{n}_0) \approx \max (0, \hat{e}_z \cdot \hat{n}_0) \hat{e}_z \cdot \hat{n}_0 \approx H(\hat{e}_z \cdot \hat{n}_0) \hat{e}_z \cdot \hat{n}_0 - \frac{H(\hat{e}_z \cdot \hat{n}_0)}{w^2} \frac{\partial \psi}{\partial \lambda} \frac{\partial (\hat{e}_z \cdot \hat{n}_0)}{\partial \lambda} = -w^2 \frac{\partial \psi}{\partial u} \frac{\partial (\hat{e}_z \cdot \hat{n}_0)}{\partial u}, \]  
(23)
where
\[ \Xi = H(\hat{e}_z \cdot \hat{n}_0) \hat{e}_z \cdot \hat{n}_0. \]  
(24)

In order to expand \( \Xi \) in spherical harmonic series, we benefit from the rotational invariance of the scalar product and consider \( \Xi' \) in the orbital frame, where
\[ \Xi' = H(\hat{e}_{\varphi} \cdot \hat{e}_{\varphi}') \hat{e}_{\varphi} \cdot \hat{e}_{\varphi}' = H(w' \cos \lambda') w' \cos \lambda'. \]  
(25)

Using the rule (A10), we obtain \( \Xi' = \sum_{l,k} \xi_{l,k} Y_{l,k}(\varphi, \lambda') \) with the coefficients
\[ \xi_{l,k} = \int_{-1}^{1} du' \int_{-\pi}^{\pi} H(w' \cos \lambda') w' \cos \lambda' Y_{l,k}(u', \lambda') \, d\lambda' = \int_{-1}^{1} du' \int_{-\pi}^{\pi} w' \cos \lambda' Y_{l,k}^*(u', \lambda') \, d\lambda' = \frac{H_k}{\sigma_{l,k}} w_{l,k}^{1,1}, \]  
(26)
expressed in terms of the overlap integrals \( w_{l,k}^{1,1} \) described in Appendix A3, and coefficients \( H_k \) defined as
\[ H_k = \int_{-\pi}^{\pi} \cos L \, e^{iL} \, dL, \]  
(27)
or, explicitly
\[ H_k = H_{k\pm} = \begin{cases} (1 + (-1)^k) \frac{(-1)^{k+1}}{k^2 - 1} & \text{for } k^2 \neq 1, \\ \pi k^2 & \text{for } k^2 = 1, \end{cases} \]  
(28)
with the particular case \( H_0 = 2 \) and \( H_{3k+1} = 0 \) for all \( |k| > 0 \).
Note that, apart from \( l = 1 \), only even degree \( l \) and order \( k \) terms will have non-zero coefficients \( \xi'_{l,k} \), but we will not use this information explicitly for the time being.

The transformation of \( \mathbf{Z} \) back to the body frame is performed by means of the usual Wigner function apparatus (see Appendix A2). So, we obtain

\[
\mathbf{Z} = \sum_{l \geq 0} \sum_{k = l} \sum_{m = -l} \xi'_{l,k} D_{m,k}^{l} (\phi, \varepsilon, \psi) Y_{l,m}(\alpha, \lambda) = \sum_{l \geq 0} \sum_{k = l} \sum_{m = -l} \left[ -\sum_{k = l} H_k w_{l,k}^{1,1} D_{m,k}^{l} \right] Y_{l,m} = \sum_{l \geq 0} \sum_{m = -l} \xi_{l,m} Y_{l,m}.
\]

(29)
Throughout the main body of the article, we will use the convention that \( D_{m,k}^{l} \) without an explicit argument always designates \( D_{m,k}^{l}(\phi, \varepsilon, \psi) \), and \( Y_{l,m} \) without an argument is a function of the body frame coordinates \( Y_{l,m}(\alpha, \lambda) \).

If we substitute \( \mathbf{Z} \) from equations (29) into (23), we can write the insolation function as a sum

\[
\mathcal{E} \approx \mathcal{E}^{(0)} + \mathcal{E}^{(1)},
\]

(30)

\[
\mathcal{E}^{(0)} = -(1 - A) \Phi \sum_{l \geq 0} \sum_{k = l} \sum_{m = -l} \frac{H_k w_{l,k}^{1,1}}{\sigma_{l,1}} D_{m,k}^{l} Y_{l,m},
\]

(31)

\[
\mathcal{E}^{(1)} = -(1 - A) \Phi \sum_{l \geq 0} \sum_{m = -l} \sum_{j} \sum_{k = j} f_{j,k} \xi_{l,m} \left( \nabla_{i} Y_{j,k} \right) \cdot \left( \nabla_{i} Y_{l,m} \right),
\]

(32)
where we introduced the explicit form (10) of the non-spherical shape function \( \Psi \).

Using a vector calculus identity for the scalar product of two gradients on the unit sphere

\[
\Delta_{s}(F_1 F_2) = \nabla_{i} \cdot \nabla_{i}(F_1 F_2) = F_1 \Delta_{s} F_2 + 2(\nabla_{i} F_1) \cdot (\nabla_{i} F_2) + F_2 \Delta_{s} F_1,
\]

(33)
we achieve the transformation

\[
\left( \nabla_{i} Y_{j,k} \right) \cdot \left( \nabla_{i} Y_{l,m} \right) = -\frac{1}{2} \left[ Y_{j,k} \Delta_{s} Y_{l,m} + Y_{l,m} \Delta_{s} Y_{j,k} - \Delta_{s}(Y_{j,k} Y_{l,m}) \right] = \frac{1}{2} \left[ (l (l + 1) + j (j + 1)) Y_{j,k} Y_{l,m} + \Delta_{s}(Y_{j,k} Y_{l,m}) \right],
\]

(34)
where we used the fact that a spherical harmonic \( Y_{l,m} \) is the eigenfunction of the Laplacian operator on the unit sphere \( \Delta_{s} \), with the eigenvalue \(-l (l + 1)\). Now, if we convert the products \( Y_{j,k} Y_{l,m} \) into Clebsch–Gordan series (A33), we obtain

\[
\left( \nabla_{i} Y_{j,k} \right) \cdot \left( \nabla_{i} Y_{l,m} \right) = \frac{1}{2} \sum_{p,q} (-1)^{p} \left[ (l (l + 1) + j (j + 1) - p (p + 1)) G_{p,j}^{m,k,-q} \right] Y_{p,q},
\]

(35)
where the Gaunt coefficients \( G_{p,j}^{m,k,-q} \) are non-zero only if \( q = m + k \), according to their properties (A30).

Thus, the first-order approximation of the insolation function is given by

\[
\mathcal{E} \approx \sum_{l \geq 0} \sum_{m = -l} \left( \mathcal{E}_{l,m}^{(0)} + \mathcal{E}_{l,m}^{(1)} \right) Y_{l,m},
\]

(36)
where

\[
\mathcal{E}_{l,m}^{(0)} = -(1 - A) \Phi \sum_{k = l} H_k w_{l,k}^{1,1} D_{m,k}^{l} = -(1 - A) \Phi \sum_{k = l} \frac{H_k w_{l,k}^{1,1}}{\sigma_{l,1}} d_{m,k}^{l} e^{-i(m \phi + k \psi)},
\]

(37)

\[
\mathcal{E}_{l,m}^{(1)} = -(1)^{m} \left( 1 - A \right) \Phi \sum_{j \geq 0} \sum_{k = j} \sum_{l = j} \sum_{p \geq 0} \gamma_{p,j}^{l} G_{p,j}^{m-k,-m} \sum_{q = -p} H_{q} w_{l,q}^{1,1} d_{m-k,q}^{l} e^{-i(m-k \phi + q \psi)},
\]

(38)
\[
\gamma_{p,j}^{l} = \frac{1}{2} \left[ p (p + 1) + j (j + 1) - l (l + 1) \right].
\]

(39)
Unless otherwise stated, a Wigner d-function \( d_{m,k}^{l} \) without an explicit argument is \( d_{m,k}^{l}(\varepsilon) \) defined in equation (A18). Both equations (37) and (38) contain many terms that actually vanish in virtue of the properties possessed by functions \( H_k, w_{l,k}^{1,1} \) and \( G_{p,j}^{m-k,-m} \), but we delay the task of purging the series until the last stage of the YORP torque derivation, preferring rather a compact notation in intermediate transformations.

### 4 Temperature Model

If the body is treated as a perfectly non-conducting matter, with the thermal conductivity \( K = 0 \), the temperature of a flat surface element is described by

\[
T^i = \frac{\mathcal{E}}{\xi_{i} \sigma},
\]

(40)
where \( \mathcal{E} \) is the insolation function. This idealized temperature function leads to the models of YORP called Rubincam’s approximation (Rubincam 2000; Vokrouhlický & Čapek 2002).
A common practice in modelling the YORP torques for bodies with non-zero conductivity is to assume a so-called ‘plane-parallel’ temperature model (Čapek & Vokrouhlický 2004; Scheeres & Mirrahimi 2008; Nesvorný & Vokrouhlický 2008b; Mysen 2008). It is based on a one-dimensional (radial) heat diffusion equation

$$\frac{\partial \bar{T}}{\partial \tau} = \chi \frac{\partial^2 \bar{T}}{\partial \zeta^2},$$

(41)

where $\zeta$ measures the depth ($\zeta = 0$ on the surface and $\zeta > 0$ inside the body). The coefficient $\chi$ is the ratio of the thermal conductivity $K$ to the product of the density $\rho$ and the specific heat capacity $c_p$.

$$\chi = \frac{K}{\rho c_p}. \tag{42}$$

The boundary conditions on the body surface are related with the conservation of energy, which implies

$$\varepsilon \sigma T^4 - K \left[ \frac{\partial \bar{T}}{\partial \zeta} \right]_{\zeta=0} - E = 0. \tag{43}$$

The first term is the re-radiated energy, proportional to the fourth power of $T$, and the product of the Boltzmann constant $\sigma$ and the surface emissivity $\varepsilon$. The second term is responsible for the heat transport towards the body's centre, and the last term is the insolation function $E$, providing the incoming radiation energy. Moreover, we search only the steady-state solution that is a quasi-periodic function of time, and such that $\lim_{\tau \to \infty} \frac{\partial \bar{T}}{\partial \tau} = 0$.

Non-linear boundary conditions are linearized around the constant value $T_0$, given by

$$\varepsilon \sigma T^4_0 = \varepsilon^{(0)} \frac{(1-A) \Phi}{4} \tag{44}$$

leading to the linear model

$$T \approx T_0 - \bar{T}, \tag{45}$$

where $\bar{T}$ is given by

$$4 \varepsilon \sigma T^4_0 \bar{T} - K \left[ \frac{\partial \bar{T}}{\partial \zeta} \right]_{\zeta=0} = E - \varepsilon^{(0)}. \tag{46}$$

A more detailed description of this model can be found in Bertotti, Farinella & Vokrouhlický (2003) or Vokrouhlický (1998). For our purpose, we simply state an operational rule that each term of the insolation function $E$ having a form

$$B_{pq}(u, \lambda) e^{i(p\phi+q\psi)}$$

creates a term $\bar{T}_{pq}$ defined by

$$4 \varepsilon \sigma T^4_0 \bar{T}_{pq} = B_{pq}(u, \lambda) K_{pq} e^{i(p\phi+q\psi+q\psi_0)} \tag{47}$$

where, assuming a constant rotation rate $\omega = -\dot{\phi}$, and a circular orbit with mean motion $n_s = \dot{\psi}$, the thermal conductivity effects are described in terms of

$$K_{pq} = \frac{1}{\sqrt{1 + 2 F_{pq} + F_{pq}^2}}, \tag{48}$$

$$\sin \varphi_{pq} = sgn(p \omega - q n_s) F_{pq} K_{pq}, \tag{49}$$

$$\cos \varphi_{pq} = (1 + F_{pq}) K_{pq}, \tag{50}$$

$$F_{p,q} = \frac{\rho c_p}{4 \varepsilon \sigma T^4_0} \sqrt{\frac{|p \omega - q n_s|}{\bar{T}}}. \tag{51}$$

Note that $K_{-p,-q} = K_{pq}$, but $\varphi_{-p,-q} = -\varphi_{pq}$. If the second subscript is 0, we will use an abbreviated form

$$K_{p0} = K_p, \quad \varphi_{p0} = \varphi_p, \tag{52}$$

with important special cases $K_0 = 1$ and $\varphi_0 = 0$.

If we apply the rule (47) to the insolation function series (36–38) and split $\bar{T}$ into the spherical part $T^{(0)}$ and non-spherical correction $T^{(1)}$, we find

$$\varepsilon \sigma T^4 \approx \varepsilon \sigma T^4_0 + 4 \varepsilon \sigma T^4_0 \left( T^{(0)} + T^{(1)} \right) = \frac{(1-A) \Phi}{4} + \sum_{l \geq 1} \sum_{m=-l}^l t^{(0)}_{l,m} Y_{l,m}(u, \lambda) + \sum_{l \geq 0} \sum_{m=-l}^l t^{(1)}_{l,m} Y_{l,m}(u, \lambda), \tag{53}$$

with

$$t^{(0)}_{l,m} = -(1-A) \Phi \sum_{k=0}^{l} \frac{w^{1,k}_{l} H_{k}}{\sigma_{1,1}} K_{m,k} d_{m,k} e^{-i(m \phi + k \psi + \varphi_{m,k})}, \tag{54}$$

$$t^{(1)}_{l,m} = (1-A) \Phi \sum_{j=1}^{l} \sum_{p=0}^{l} \sum_{k=-p}^{p} \sum_{m=-l}^{l} f_{j,k} (-1)^m g_{j,m} \tilde{G}_{p,j,m} \frac{w^{1,q}_{j} H_{q}}{\sigma_{1,1}} d_{m-k,q} K_{m-k,q} e^{-i(m \phi + q \psi + \varphi_{m-k,q})}. \tag{55}$$
This temperature model is accurate up to the first power of the shape coefficients \( f_{l,k} \) not only because of the approximate insolation function used to derive it, but also because of \( T_0 \) defined according to the spherical model.

## 5 TORQUE INTEGRATION OVER THE SURFACE

### 5.1 Approximate integrand

The integral (2) involves the factor \((r \times dS)\). In terms of the modified spherical coordinates in the body frame,
\[
dS = N \, d\lambda,
\]
where the normal vector \( N \) is given by equation (13). Substituting equation (9), we find
\[
r \times N = a^3 (1 + \Psi^2) \hat{e}_z \times [(1 + \Psi) \hat{e}_r - \nabla_{\Psi} \Psi] = -a^3 (1 + \Psi^2) \hat{e}_r \times \nabla_{\Psi} \Psi = -a^3 (1 + \Psi^2) L^z \Psi.
\]
where \( L^z = -i \hat{e}_r \times \nabla_{\Psi} \) is the angular momentum operator (Jackson 1967; Biedenharn & Louck 1981) reduced to the unit sphere. Up to the second order, we can approximate \( r \times N \) as
\[
r \times N \approx -i a^3 (1 + 2 \Psi) L^z \Psi = a^3 (1 + 2 \Psi) F.
\]

According to the action of \( L^z \) on spherical harmonics in the complex, body frame-fixed basis
\[
\hat{e}_- = \frac{1}{\sqrt{2}} (\hat{e}_q - i \hat{e}_s), \quad \hat{e}_+ = \frac{1}{\sqrt{2}} (\hat{e}_q + i \hat{e}_s), \quad \hat{e}_0 = \hat{e}_t,
\]
the vector \( F = -i L^z \Psi \) is a sum
\[
F = \sum_{l \geq 1} \sum_{m=-l}^l f_{l,m} F_{l,m},
\]
where the vector coefficients
\[
F_{l,m} = -i \left( \sigma_{l,m}^{-} Y_{l,m+1} \hat{e}_+ + \sigma_{l,m}^{+} Y_{l,m-1} \hat{e}_+ + m Y_{l,m} \hat{e}_0 \right)
\]
involve auxiliary symbols
\[
\sigma_{l,m}^{-} = \sqrt{\frac{(l-m+1)}{2}} = \sigma_{l,m}^{+}, \quad \sigma_{l,m}^{-} = \sqrt{\frac{(l+m+1)}{2}} = \sigma_{l,m}^{+}.
\]
Note that from the point of view of a usual scalar product \( \hat{e}_+ \cdot \hat{e}_0 = 0 \), and \( \hat{e}_0 \cdot \hat{e}_0 = 1 \), but \( \hat{e}_+ \cdot \hat{e}_+ = \hat{e}_- \cdot \hat{e}_- = 0 \) and \( \hat{e}_- \cdot \hat{e}_+ = 1 \).

According to the conclusions drawn from equation (3), we can drop \( l = m = 0 \) terms from our temperature model (53) because they have no effect on the YORP torque. Hence, we will use
\[
\varepsilon_\star T^4 = \sum_{l \geq 1} \sum_{m=-l}^l \left( t_{l,m}^{(0)} + t_{l,m}^{(1)} \right) Y_{l,m}
\]
in further considerations. Thus, we have reached the stage where the problem of the YORP torque is reduced to an integral over the surface of the unit sphere
\[
M \approx -\frac{2 a^3}{3 v_c} \int_0^{2\pi} d\lambda \sum_{l \geq 1} \sum_{m=-l}^l \int_0^1 \, du \int_0^{2\pi} Y_{l,k} F_{l,m} \, d\lambda,
\]
where the first term in the square bracket is a first-order YORP torque and remaining two define the second-order torque. Any second-order term \( t_{l,m}^{(2)} \) in temperature function, depending on the products of shape coefficients, would result in a third-order contribution after being multiplied by the first-order vector \( F \). This justifies the use of the first-order approximation of \( T^4 \) in the second-order YORP torque solution.

### 5.2 Integrated torque in the body frame

The first-order part of equation (64) is
\[
M^{(1)} = -\frac{2 a^3}{3 v_c} \sum_{l \geq 1} \sum_{m=-l}^l \sum_{j \geq 1} \sum_{k=-j}^j f_{l,m} \int_0^1 \, du \int_0^{2\pi} Y_{l,j} F_{l,m} \, d\lambda.
\]
If we substitute \( Y_{l,k} = (-1)^k Y_{l,-k} \) and use the orthogonality of spherical functions (A8), we easily find
\[
M^{(1)} = -\frac{2 a^3}{3 v_c} \sum_{l \geq 1} \sum_{m=-l}^l f_{l,m} \int_0^1 \, du \int_0^{2\pi} \left[ \sigma_{l,m}^{-} t_{l,m+1}^{(0)} \hat{e}_+ + \sigma_{l,m}^{+} t_{l,m-1}^{(0)} \hat{e}_+ + m t_{l,m}^{(0)} \hat{e}_0 \right] Y_{l,m} \, d\lambda.
\]

In the second order, we have \( M^{(2)} = M^{(02)} + M^{(11)} \), with
\[
M^{(11)} = -\frac{2 a^3}{3 v_c} \sum_{l \geq 1} \sum_{m=-l}^l \sum_{j \geq 1} \sum_{k=-j}^j f_{l,m} \int_0^1 \, du \int_0^{2\pi} Y_{l,j} F_{l,m} \, d\lambda.
\]
resembling $M^{(1)}$, and

$$M^{(02)} = \frac{4 a^3}{3 \nu c} \sum_{l,j,p} \sum_{m=-l}^{l} \sum_{q=-p}^{p} \sum_{r=-1}^{1} \sum_{m=-j}^{j} \sum_{q=-p}^{p} f_{l,m} f_{l,p,q} t_j^{(0)} Y_{l,j} Y_{l,p,q} d\lambda.$$

(68)

Of course, the final form of $M^{(11)}$ is similar to equation (66), with all $t_j^{(0)}$ replaced by $t_j^{(1)}$.

$$M^{(11)} = -\frac{2 i a^3}{3 \nu c} \sum_{l,j,p} \sum_{m=-l}^{l} \sum_{q=-p}^{p} f_{l,m} f_{l,p,q} \left[ \sigma_{l,m} t_j^{(1)} e_\phi + \sigma_{l,m} t_j^{(1)} e_\psi - m t_j^{(1)} e_\theta \right],$$

(69)

whereas $M^{(02)}$ looks different, but still the integral of three spherical functions is directly given in terms of the Gaunt integrals (A28). In our case, we obtain $G_{l,m,q,k}^{m,q,k+1}$ and $G_{l,m,q,k}^{m,q,k-1}$, that are non-zero only for $m + q + k$ equal to 0, −1 and 1, respectively (see Appendix A3). This property helps us to remove one sum, and

$$M^{(02)} = \frac{4 i a^3}{3 \nu c} \sum_{l,j,p} \sum_{m=-l}^{l} \sum_{q=-p}^{p} f_{l,m} f_{l,p,q} \left( \sigma_{l,m} G_{l,m,q,k}^{m+1,q,-m-q-1} t_j^{(0)} e_\phi + \sigma_{l,m} G_{l,m,q,k}^{m-1,q,-m-q+1} t_j^{(0)} e_\psi + m G_{l,m,q,k}^{m,q,-m-q} t_j^{(0)} e_\theta \right).$$

(70)

Further restrictions of summation range, implied by properties (A30), will be introduced later.

The equations derived in this section provide a tool to compute the complete second-order approximation of the YORP torque

$$M = M^{(1)} + \mu M^{(02)} + M^{(11)}$$

(71)

if only the spherical harmonics expansion of the temperature distribution function is known up to the first order in shape coefficients. We introduce a symbol $\mu = 1$ as a marker that will help us to locate the contribution of the $M^{(02)}$, neglected by Nesvorný & Vokrouhlický (2008b), in the final expressions.

6 ELEMENTS OF DYNAMICS

Given an object on a circular orbit, rotating around its principal axis of the maximum inertia, we can describe its dynamics under the action of a torque $M$ in terms of four differential equations related to the 3-1-3 Euler angles $\Omega = -\phi + \pi/2, \vartheta = \psi + \pi/2$, and to the angular rotation rate $\omega$:

$$\dot{\omega} = \frac{M \cdot \hat{e}_z}{J_3},$$

(72)

$$\dot{\epsilon} = \frac{M \cdot \hat{e}_y}{\omega J_3},$$

(73)

$$\dot{\psi} = n_s - \frac{M \cdot \hat{e}_y}{\omega J_3 s},$$

(74)

$$\dot{\phi} = \omega - \frac{c M \cdot \hat{e}_z}{\omega J_3 s} = \omega + c (\dot{\theta} - n_s),$$

(75)

where $J_3$ is the maximum moment of inertia, $n_s$ is the orbital mean motion and, as usual, $c = \cos \epsilon, s = \sin \epsilon$. The three unit vectors in the above equations define an inertial reference frame attached to the body’s equator–equinox system: $\hat{e}_1$ – directed along the projection of $\hat{e}_z$ (normal to the orbital plane) on the equatorial plane $Oxy$, $\hat{e}_2$ – pointing to the equinox (directed along $\hat{e}_z \times \hat{e}_x$), and $\hat{e}_3$ – directed to the north pole. The components of this right-handed, orthonormal basis in the body frame are given by

$$\hat{e}_1 = -\cos \phi \hat{e}_x - \sin \phi \hat{e}_z = -\frac{1}{\sqrt{2}} \left[ e^{-i \phi} \hat{e}_+ + e^{i \phi} \hat{e}_- \right],$$

$$\hat{e}_2 = \sin \phi \hat{e}_x - \cos \phi \hat{e}_z = \frac{i}{\sqrt{2}} \left[ e^{-i \phi} \hat{e}_+ - e^{i \phi} \hat{e}_- \right],$$

$$\hat{e}_3 = \hat{e}_z = \hat{e}_0.$$

(76)

7 MEAN VALUES OF THE YORP TORQUE PROJECTIONS

The YORP torques belong to the class of very weak, dissipative factors that may produce significant effects only through their mean values that act systematically. If we exclude any commensurability between $n_s$ and $\omega$, we can identify the averaging with respect to time with

$$\langle M_j \rangle = \frac{1}{4 \pi^2} \int_0^{2\pi} \int_0^{2\pi} M_j \, d\phi \, d\psi,$$

(77)

rejecting all purely periodic terms of $\phi$ and $\psi$ in

$$M_j = M \cdot \hat{e}_j, \quad j = 1, 2, 3.$$

(78)
7.1 Spin component $M_3$

According to equation (72), the mean component $\langle M_3 \rangle = M \cdot \mathbf{\hat{e}}_3$ plays the principal role in the YORP effect, being responsible for the systematic change in rotation period. Besides, it can also be described with a simpler set of formulae than $M_1$ or $M_2$. For these reasons, we consider $M_3$ prior to the remaining components.

7.1.1 First order

The first-order terms in $M_3$, to be labelled as $M_3^{(1)}$, result from $M^{(1)}$ defined in equation (66). But

$$\langle M_3^{(1)} \rangle = \langle M^{(1)} \cdot \mathbf{\hat{e}}_3 \rangle = -\frac{2i a^3}{3 v_c} \sum_{l \geq 1} \sum_{m=-l}^l f_{l,m} m \langle \tilde{t}_{l,m}^{(0)} \rangle = 0$$  \hspace{1cm} (79)

because, according to equation (54), $\langle \tilde{t}_{l,m}^{(0)} \rangle = 0$ for all $m \neq 0$, whereas for $m = 0$, we have $m \langle \tilde{t}_{l,m}^{(0)} \rangle = 0$. Thus, as demonstrated by Nesvorný & Vokrouhlický (2007, 2008b), the spin-up or spin-down of an asteroid is a second-order effect, involving the products of the shape coefficients.

7.1.2 Second order

The second-order part of $M_3$ will consist of terms derived from $M^{(11)}$ and $M^{(02)}$ defined in equations (69) and (70), where we substitute $\tilde{t}_{l,m}^{(0)}$ from equation (54) and $t_{l,m}^{(0)}$ from equation (55). We begin with $M_3^{(11)}$ and after the substitution of temperature coefficients (55) with $k = m$ and $q = 0$, we readily obtain the mean value

$$\langle M_3^{(11)} \rangle = \langle M^{(11)} \cdot \mathbf{\hat{e}}_3 \rangle = -\frac{2i a^3}{3 v_c} \sum_{l \geq 1} \sum_{m=-l}^l f_{l,m} m \langle \tilde{t}_{l,m}^{(0)} \rangle = -i \alpha \sum_{l,j,p,m} (-1)^m m f_{l,m} f_{j,m} \tilde{w}_{l,j,m}^{(1)} H_0 \sigma_{1,l} G_{l,j,m}^{-m,m,0} W_p d_{0,p}^2 \Phi_{l,j,m}^{(0)} \epsilon_{l,j,m}^{(0)},$$  \hspace{1cm} (80)

where

$$\alpha = \frac{2a^3(1-A)\Phi}{3v_c}.$$  \hspace{1cm} (81)

Recalling that $\varphi_{0,0} = 0, K_{0,0} = 1, H_0 = 2$ and $w_{l,j,m}^{(0)} = 0$ when $p$ is odd, we reduce $[M_3^{(11)}]$ to

$$\langle M_3^{(11)} \rangle = -i 2 \alpha \sum_{l,j,p \geq 1} \sum_{m=-l}^l (-1)^m m f_{l,m} f_{j,m} \tilde{w}_{l,j,m}^{(1)} G_{l,j,m}^{-m,m,0} W_p d_{0,p}^2,$$  \hspace{1cm} (82)

using the special case of the overlap integral $W_p$ defined in equation (A27).

In agreement with the results of Nesvorný & Vokrouhlický (2007), we find that the remaining term $M_3^{(02)}$ has no systematic effect on the spin rate. Indeed,

$$\langle M_3^{(02)} \rangle = i \frac{4a^3}{3v_c} \sum_{l,j,p,m,q} f_{l,m} f_{p,q} G_{l,p,j,m,q}^{-m,m,q} \langle \tilde{t}_{l,m,q}^{(0)} \rangle = i 4 \alpha \sum_{l,j,p \geq 1} \sum_{m=-l}^l m (-1)^m f_{l,m} f_{j,m} \tilde{w}_{l,j,m}^{(1)} G_{l,j,m}^{-m,m,0} W_p d_{0,p}^2 = 0,$$  \hspace{1cm} (83)

thanks to the symmetries between the terms with the indices $(l,j,m) = (q_1, q_2, \pm q_3)$ and $(q_2, q_1, \pm q_3)$. For similar reasons, we can actually replace $\tilde{g}_{2,p}^{(l)}$ in equation (82) by $\tilde{g}_{0,l}^{(l)}$.

Thus, the complete mean YORP torque $\langle M_3 \rangle$ is given by

$$\langle M_3 \rangle = \langle M_3^{(11)} \rangle = -i 2 \alpha \sum_{l,j,p \geq 1} \sum_{m=-l}^l m (-1)^m f_{l,m} f_{j,m} \tilde{w}_{l,j,m}^{(1)} G_{l,j,m}^{-m,m,0} W_p d_{0,p}^2.$$  \hspace{1cm} (84)

Equation (82) can be converted into an explicitly real form if we combine the positive and negative values of the summation index $m$ and use the parity properties of shape coefficients, Wigner d-functions and Gaunt coefficients. The result, with vanishing terms dropped and with optimised summation with respect to $l$ and $j$, becomes

$$\langle M_3 \rangle = -2 \alpha \sum_{l \geq 1} \sum_{j \geq 1} \sum_{m=-l}^l (-1)^m m j (2l + 2j + 1) \left(C_{l,m} S_{l,j,m} - S_{l,m} C_{l,j,m} \right) \sum_{p=j}^{l+j} W_p G_{l,j+p,2p}^{-m,m,0} d_{0,p}^2.$$  \hspace{1cm} (85)

Two remarkable properties known from previous works (Čapek & Vokrouhlický 2004; Nesvorný & Vokrouhlický 2007; Scheeres 2007; Scheeres & Mirrahimi 2008) are worth noting: $\langle M_3 \rangle$ is an even function of $(\pi/2 - \epsilon)$, and it is independent of thermal conductivity. The former property is a direct consequence of the fact that $d_{0,p}^2$ is actually the Legendre polynomial $P_2(\cos \epsilon)$. A good argument for the independence of $\langle M_3 \rangle$ on conductivity can be found in Nesvorný & Vokrouhlický (2008b).
7.2 Obliquity component $M_1$

7.2.1 First order

Using equation (66), we project $M^{(1)}$ on $\hat{e}_i$ defined by equation (76) in terms of $\hat{e}_\pm$. Then, recalling that $\hat{e}_\pm \cdot \hat{e}_0 = \hat{e}_+ \cdot \hat{e}_- = 0$, and $\hat{e}_\pm \cdot \hat{e}_\pm = 1$, we find that the average value of the first-order terms is given by

$$
\begin{align*}
\langle M_1^{(1)} \rangle &= i \sqrt{\frac{2}{3}} \int \frac{d^3 \mathbf{v}_c}{2\pi^2} \sum_{l \geq 1} \sum_{m=1}^l f_{1m}^* \left( \sigma_{1,m}^+ \langle \phi_{l,m-1} e^{-i\phi} \rangle + \sigma_{1,m}^- \langle \phi_{l,m+1} e^{i\phi} \rangle \right). 
\end{align*}
$$

(86)

Substitution of the spherical temperature function from equation (54), followed by the rejection of purely periodic terms (in both components it amounts to setting $m = k = 0$), leads to

$$
\begin{align*}
\langle M_1^{(1)} \rangle &= -i \frac{\alpha}{\sqrt{2 \sigma_{1,1}^j}} \int \frac{d^3 \mathbf{v}_c}{2\pi^2} \sum_{l \geq 1} f_{1,0}^* \left( K_{1,0} \sigma_{1,0}^+ \phi_{l,0} e^{-i\phi} + K_{-1,0} \sigma_{1,0}^- \phi_{l,0} e^{i\phi} \right).
\end{align*}
$$

(87)

Using the particular values of the subscripted symbols, namely $\varphi_{-1,0} = -\varphi_1$, $K_{-1,0} = K_1$, $H_0 = 2$, $d_{l,0}^0 = -d_{l,0}^0$, $\sigma_{l,0}^+ = \sqrt{l(l+1)/2}$, $f_{l,0}^* = C_{l,0}$, we find the first-order contribution in an explicitly real form

$$
\begin{align*}
\langle M_1^{(1)} \rangle &= -2 \alpha K_1 \sin \varphi_1 \sum_{l \geq 1} C_{l,0} \sqrt{l(l+1)} d_{l,0}^0.
\end{align*}
$$

(88)

depending only on zonal harmonics coefficients $C_{l,0}$. Moreover, the properties of overlap integrals (A24) imply that only the even degree zonals remain in the final formula, so — introducing the special function $W_l$ from equation (A27) — we finally obtain

$$
\begin{align*}
\langle M_1^{(1)} \rangle &= -2 \alpha K_1 \sin \varphi_1 \sum_{l \geq 1} C_{l,0} \sqrt{2l(2l+1)} W_l d_{l,0}^0.
\end{align*}
$$

(89)

This expression confirms the presence of the YORP effect in obliquity even for such regular objects as spheroids, provided they have non-zero conductivity (Breiter et al. 2007).

7.2.2 Second order

In the second order, we first consider $M_1^{(1)}$. According to equation (69), the departure point is similar to $M_1^{(1)}$, i.e.

$$
\begin{align*}
\langle M_1^{(1)} \rangle &= i \sqrt{\frac{2}{3}} \int \frac{d^3 \mathbf{v}_c}{2\pi^2} \sum_{l \geq 1} \sum_{m=1}^l f_{1m} \left( \sigma_{1,m}^+ \langle \phi_{l,m-1} e^{-i\phi} \rangle + \sigma_{1,m}^- \langle \phi_{l,m+1} e^{i\phi} \rangle \right), 
\end{align*}
$$

(90)

although the temperature coefficient $\phi_{l,m-1}$ from equation (55) is more complicated. The $\phi$ and $\psi$ independent part, resulting from $k = m$ and $q = 0$, is

$$
\begin{align*}
\langle M_1^{(1)} \rangle &= i \sqrt{2} \alpha K_1 \int \frac{d^3 \mathbf{v}_c}{2\pi^2} \sum_{l \geq 1} \sum_{p,m,q} f_{1m,p} \sigma_{1,m}^+ \langle \phi_{l,m-1} e^{-i\phi} \rangle W_p d_{l,0}^p. 
\end{align*}
$$

(91)

The treatment of $M_1^{(2)}$ is comparable to $M_1^{(1)}$. The result becomes quite similar to $M_1^{(1)}$ because in this case equations (70) and (54) lead to

$$
\begin{align*}
\langle M_1^{(2)} \rangle &= i \sqrt{2} \alpha K_1 \int \frac{d^3 \mathbf{v}_c}{2\pi^2} \sum_{l \geq 1} \sum_{p,m,q} f_{1m,p} \sigma_{1,m}^+ \langle \phi_{l,m-1} e^{-i\phi} \rangle W_p d_{l,0}^p.
\end{align*}
$$

(92)

In order to add equations (92) to (91), we manipulate the former, interchanging the indices $(l, p, j)$, permuting the indices in the Gaunt coefficients, and making use of $f_{1,m} = (-1)^m f_{1,m,0}$, and $\sigma_{1,m} = \sigma_{1,m,0}$. The resulting total second-order torque is

$$
\begin{align*}
\langle M_1^{(2)} \rangle = \langle M_1^{(1)} + \mu M_1^{(2)} \rangle = i \sqrt{2} \alpha K_1 \int \frac{d^3 \mathbf{v}_c}{2\pi^2} \sum_{l \geq 1} \sum_{p,m,q} f_{1m,p} \sigma_{1,m}^+ \langle \phi_{l,m-1} e^{-i\phi} \rangle W_p d_{l,0}^p.
\end{align*}
$$

(93)

The marker $\mu = 1$ introduced in equation (71) helps us to trace the influence of the $M_1^{(2)}$.

After a tedious procedure of combining the terms with $(i, j, m) = (q_1, q_2, \pm q_3) \text{ and } (i, j, m) = (q_2, q_3, \pm q_4)$, we obtain the explicitly real form

$$
\begin{align*}
\langle M_1^{(2)} \rangle &= \alpha K_1 \cos \varphi_1 \int \frac{d^3 \mathbf{v}_c}{2\pi^2} \sum_{l \geq 1} \sum_{j \geq 0} \sum_{m=0}^l (-1)^m \left( C_{l,m} S_{l+j,m} - C_{l+j,m} S_{l,m} \right) \frac{2}{\sqrt{2}} \left( \sigma_{1,m}^+ \langle \phi_{l,j} e^{-i\phi} \rangle - \sigma_{1,m}^- \langle \phi_{l,j} e^{i\phi} \rangle \right) W_p d_{l,0}^p \\
&+ \alpha K_1 \sin \varphi_1 \int \frac{d^3 \mathbf{v}_c}{2\pi^2} \sum_{l \geq 1} \sum_{j \geq 0} \sum_{m=0}^l (-1)^m \left( C_{l,m} C_{l+j,m} + S_{l,m} S_{l+j,m} \right) \frac{2}{\sqrt{2} \sqrt{2}} \left( \sigma_{1,m}^+ \langle \phi_{l,j} e^{-i\phi} \rangle - \sigma_{1,m}^- \langle \phi_{l,j} e^{i\phi} \rangle \right) W_p d_{l,0}^p.
\end{align*}
$$

(94)
Using the recurrence relation (A32), we were able to simplify the sum of two Gaunt integrals, but it did not help for their difference, which explains asymmetry of the two parts of equation (94). Note that $\mu$ is present only in the terms factored by $\sin \varphi_1$, i.e. in the part that does vanish in Rubincam’s approximation.

### 7.3 Precession component $M_2$

Probably, the YORP-induced precession will meet rather limited interest, being negligible when compared to the precession resulting from much stronger gravitational torques. Fortunately, the similarity between $\hat{e}_1$ and $\hat{e}_2$, resulted in a very simple rule: it suffices to apply a formal substitution $(\sin \varphi_1, \cos \varphi_1) \rightarrow (-\cos \varphi_1, \sin \varphi_1)$ to turn $M_1^{(1)}$ or $M_1^{(2)}$ into $M_2^{(1)}$ or $M_2^{(2)}$, respectively. Of course, this observation is based on the complete derivation that we omit for the sake of brevity, but it can also be deduced from the equations of Scheeres (2007) or Scheeres & Mirrahimi (2008).

### 7.4 Summary

In order to collect the final equations for the average YORP torque in a possibly compact form, revealing their actual structure, we first introduce a few auxiliary symbols

$$S_{l,m} = C_{l,m} S_{l+2,m} - S_{l,m} C_{l+2,m},$$

and

$$C_{l,m} = C_{l,m} S_{l+2,m} + S_{l,m} S_{l+2,m}$$

and

$$s_1 = K_1 \sin \varphi_1, \quad c_1 = K_1 \cos \varphi_1$$

that should not be confused with $c = \cos \varepsilon$ and $s = \sin \varepsilon$.

Further, we note that only the special cases of Wigner d-functions are required: $d^{2p}_{0,0}$ and $d^{2p}_{2,0}$, which are simply related with the Legendre polynomials $P_{2p}(c)$ and the Legendre functions $P_{2p}(c)$, respectively. We will also limit the summation according to a practical assumption that shape harmonics are known only up to some finite degree and order $N$. Thus, using equation (A22), we conclude this section with the following set of formulae:

$$\langle M_1 \rangle = -2\alpha \left( \frac{s_1}{c_1} \right) \sum_{l=1}^{E[N/2]} C_{2l,0} W_l \| P_{2l}^{(2)}(c) \|$$

and

$$\langle M_2 \rangle = -2\alpha \left( \frac{s_1}{c_1} \right) \sum_{l=1}^{E[N/2]} C_{2l,0} W_l \langle p \rangle U_{l,m,j,p}^1 \| P_{2p}^{(2)}(c) \|.$$

$$\langle M_3 \rangle = \alpha \sum_{l=1}^{N-2} \sum_{j=1}^{E[N/2]} \sum_{m=1}^{E[N/2]} \sum_{p=1}^{E[N/2]} S_{l,m,j,p}^1 V_{l,m,j,p}^1 \| P_{2p}^{(2)}(c) \|.$$

where

$$V_{l,m,j,p} = -2(-1)^m j m (2l + 2j + 1) \| G_{l,j+2,j}^{m,m,0} \| W_p,$$

$$U_{l,m,j,p}^{(c)} = (-1)^m \frac{2 - \delta_{ij}}{2 - \delta_{0m}} \| G_{l,j+2,j}^{l+2,0} \| W_{p,(2p+1)}$$

and

$$\frac{2}{(2p+1)} \| H_{l,m,j,p} \| W_p.$$

$$H_{l,m,j,p} = \sqrt{(l + m)(l - m + 1)} \| G_{l,j+2,j,2p}^{m+1,-m,1} - \sqrt{(l - m)(l + m + 1)} \| G_{l,j+2,j,2p}^{m-1,-m,-1}.$$
and
\[ T_{l,m,j,p} = \sqrt{\frac{(2l+1)(2l+2j+1)(4p+1)}{2\pi}} \left[ \begin{array}{cccc} l & l+2j & 2p & 0 \\ 0 & 0 & 0 & -1 \end{array} \right] \left( \begin{array}{c} \sigma^i_{jm} \\ m-1 & -m & 1 \\ m+1 & -m & -1 \end{array} \right) \]  \tag{106}

The 3-j symbols and Gaunt coefficients admit a number of recurrence relation and their computation is a routine task in physics (e.g. Luscombe & Luban 1998; Xu 1996; Sébilleau 1998). A reliable software for the 3-j coefficients can be found in the open source srtoolks\(^1\) package, although major computer algebra systems implement the 3-j symbols as one of standard functions. The values of \(s_{i,j}^{m,m,0} \) and \( T_{l,m,j,p} \) decrease with the growing \( p \), but they do not reveal any monotonicity with respect to \( l, n \) or \( j \).

The presence of even degree Legendre polynomials \( c \) in \( M_1 \) and of the Legendre functions \( P_{2p} \) \( (c) \) in \( M_2 \) \( (\varepsilon) \) explains the symmetries well known from Rubincam (2000) or Čapek & Vokrouhlický (2004): \( M_1 \) \( (\varepsilon) \) \( = M_3 \) \( (\pi - \varepsilon) \) and \( M_2 \) \( (\varepsilon) \) \( = - M_2 \) \( (\pi - \varepsilon) \). Similarly, \( M_3 \) \( (\varepsilon) \) \( = - M_1 \) \( (\pi - \varepsilon) \). The situation is much better in the present formulation because the monotonicity properties of \( W_p \) alone (even fortified in the products \( q_{i,j}^{m,m,0} W_p \) or \( T_{l,m,j,p} \) \( W_p \)) suggests to attach the major importance to the terms with the lowest \( p \) values. In this context, we can justify the partition of the YORP torque terms into subsequent ‘orders’ of \( j \), proposed by Nesvorný & Vokrouhlický (2007), because the index \( j \) establishes the lower bound on the range of \( p \). Truncating with respect to \( j \) is also consistent with the approach of Mysen (2008), who imposed it at the beginning of his derivation, using an insolation function that actually is restricted to the first two Legendre polynomials \( P_1 \) and \( P_2 \), with the effect of \( P_1 \) disappearing after the averaging.

The sums presented in the previous section can be easily rearranged from the shape coefficient combinations a sum of the Legendre functions, to an alternative form of the Legendre series with a sum of coefficients attached to each Legendre function. Simple manipulations lead to
\[ M_1 = \alpha \sum_{q=1}^{N} (s_1 X_q^1 + s_2 X_q^2 + c_1 Z_q) \]  \tag{107}
\[ M_2 = \sum_{q=1}^{N} (-c_1 X_q^1 - c_1 X_q^2 + s_1 Z_q) \]  \tag{108}
\[ M_3 = \sum_{q=1}^{N-1} A_q P_{2q}(c) \]  \tag{109}
with the coefficients given as the sums
\[ X_q^1 = -2 C_q W_q, \quad X_q^2 = \sum_{j=0}^{q-N-2j} \sum_{l=1}^{j} C_{j,m}^{l} U_{l,m,j,q}^c, \quad Z_q = \sum_{j=1}^{q-N-2j} \sum_{l=1}^{j} \sum_{m=1}^{j} S_{j,m}^{l} U_{l,m,j,q}, \quad A_q = \sum_{j=1}^{q-N-2j} \sum_{l=1}^{j} \sum_{m=1}^{j} S_{j,m}^{l} V_{l,m,j,q} \]  \tag{110}
where the lower summation bound is \( l = \max(1, q-j) \).

These Legendre series suggest to introduce a notion of the ‘YORP degree’, complementary to the ‘YORP order’ proposed by Nesvorný & Vokrouhlický (2007). A term of the YORP degree \( q \) is a sum of terms with YORP orders \( j \leq q \). Truncating the YORP series, one should first decide on the maximum degree \( q_{\text{max}} \), and then either maintain the maximum order \( j \) naturally bounded by the current value of \( q \) or additionally impose some smaller value of \( j_{\text{max}} \). According to our experience, a decent approximation level is reached already at \( q_{\text{max}} = 2 \), so a qualitative pattern of the \( \varepsilon \) dependence is established in the second-degree truncated series.

9 TEST APPLICATIONS

9.1 General remarks

The basic assumptions of our solution are similar to the ones of Nesvorný & Vokrouhlický (2007, 2008b): almost spherical homogeneous bodies with constant albedo and small thermal penetration depth combined with low conductivity. The shape restriction is quite severe, ruling

\(^1\) Available at http://www.ipgp.jussieu.fr/~wieczor/SHTOOLS/SHTOOLS.html
out many irregularly shaped objects like (433) Eros. A good account of this difficulty was given in section 8 of Nesvorný & Vokrouhlický (2007). Our experience confirms the statement of Nesvorný & Vokrouhlický (2007) that qualitatively correct results can be obtained even for such irregular bodies, although the main cause of discrepancies is not the magnitude of the shape coefficients as such but rather the occurrence of major concavities in the figure of a considered body that generates large shadow zones treated as illuminated in the analytical insolation function models. So, we test our solution on a spheroid, that is obviously convex, and on the asteroid 1998 KY26 – a reasonably regular object.

9.2 Spheroids

As a first test, we confronted the present results with YORP torque formulae for spheroids provided by Breiter et al. (2007). Although the results for spheroids used a simplistic thermal lag model of ‘delayed Sun’, the final formulae from section 3.5 of Breiter et al. (2007) can be easily generalized to a more realistic thermal model by replacing \( \sin \delta \) with \( s_1 \) of the present paper. Then, with the shape model

\[
C_{2,0} \approx - \frac{2}{3} \sqrt{\frac{\pi}{5}} \left( e^2 + \frac{11}{21} e^4 \right), \quad C_{4,0} \approx \frac{2 \sqrt{\pi}}{35} e^4, \tag{111}
\]

derived for an oblate spheroid

\[
r = a_c \sqrt{1 - e^2} \frac{\sqrt{1 - e^2}}{e^2 w^2}, \tag{112}
\]

with the semi-axis \( a_c \) related to the mean radius \( a \) by the series in eccentricity \( e = \sqrt{1 - c^2 a_c^2} \)

\[
a \approx a_c \left( 1 - \frac{e^2}{6} - \frac{11 e^4}{120} \right), \tag{113}
\]

we find from equation (107)

\[
\langle M_1 \rangle = \alpha s_1 \left( X_1^1 P_1^1(c) + X_1^1 P_4^1(c) + X_2^1 P_1^1(c) \right) \approx - \frac{2(1 - A) \Phi a_3^3}{3 v_c} \pi s c (16 e^2 + (7 + 5 c^2) e^4). \tag{114}
\]

On the other hand, according to Breiter et al. (2007),

\[
\langle M_1 \rangle \approx - \frac{\pi}{12} \left( 1 - A \right) \Phi a_3^3 \frac{s c}{v_c} \sin \delta \left[ e^2 + \frac{1}{4} - \frac{5}{16} s^2 \right] e^4. \tag{115}
\]

Replacing \( \sin \delta \) by \( s_1 \) and substituting equation (113), we obtain a perfect agreement between equations (114) and (115).

The comparison is meaningful only up to \( e^4 \), because \( C_{2,0} = O(e^4) \) marks the limit of our present second-order approximation. For the same reasons, we dropped \( C_{6,0} = O(e^6) \) and higher degree zonal coefficients.

9.3 Asteroid 1998 KY26

From the collection of numerically simulated mean YORP torques presented by Čapek & Vokrouhlický (2004), we selected the asteroid 1998 KY26, considered also by Scheeres (2007) and Scheeres & Mirrahimi (2008). The same object was used by Nesvorný & Vokrouhlický (2007, 2008b) as an example for their second-order model. The authors were quite optimistic about this body, and, indeed, a side view presented in their paper looks like well fitting the assumption of the theory. However, the polar view (Fig. 2) reveals a source of possible problems when the obliquity \( \varepsilon \) is close to \( 0 \) or \( \pi \). Some regions, qualified as sunlit by the spherical terminator criterion, are actually in shadow, whereas

![Figure 2. Asteroid 1998 KY26 seen from the north pole with a simulated lighting from the equator. The regions where our insolation function model fails are labelled A (self-shadowing) and B (terminator cuts off a sunlit region).](https://academic.oup.com/mnras/article-abstract/388/2/927/978971)
some sunlit regions are neglected and considered dark. Apparently, these two effects cancel out in the numerical simulation, but the analytical model tends to sharpen their influence.

Nevertheless, we constructed the spherical harmonics model up to degree and order \( N = 26 \) using the least-squares adjustment on the grid of 2048 surface points determined by Ostro et al. (1999), reduced to the reference frame of the centre of mass and principal axes of inertia. The standard deviation of our harmonics set on the grid points was 0.2 mm, quite satisfactory for an object with the mean radius \( a = 13.1913 \) m. Interested rather in comparison than in the dynamics of the asteroid itself, we used the same, partially fictitious values of physical parameters that were applied by Čapek & Vokrouhlický (2004): rotation period 6 h, density \( \rho = 2.5 \) g cm\(^{-3}\), orbital semi-axis \( a_o = 2.5 \) au, heat capacity \( c_p = 680 \) J (kg K\(^{-1}\)) and albedo \( A = 0 \). The maximum moment of inertia, that we derived during the reduction to the principal axes, was \( J_3 = 1.7273 \times 10^9 \) kg m\(^2\).

The agreement of \( \dot{\omega} \) computed from equation (109) shown in Fig. 3 with the results of Čapek & Vokrouhlický (2004) is as good as the one obtained by Nesvorný & Vokrouhlický (2007). It is not surprising because we expanded our expressions in power series of \( c \) and compared them with table I of Nesvorný & Vokrouhlický (2007), concluding a perfect agreement of the numerical coefficients. Yet, when we pass to the obliquity variations \( \dot{\delta} \), we find much more distortion with respect to smooth curves taken from Čapek & Vokrouhlický (2004), although the orders of magnitude and qualitative dependence on obliquity and thermal conductivity may be qualified as correct. In our opinion, the degraded accuracy at small values of \( \varepsilon \) or \( (\pi - \varepsilon) \) should be attributed to the north–south directed elevated patterns in the topography of 1998 KY\(_{26}\). We have observed that the departure from the sine-like curve of Čapek & Vokrouhlický (2004) strongly depends on the maximum degree and order \( N \) of the accounted shape harmonics. As a matter fact, a sufficiently inaccurate shape model, with the elevated regions smoothed out, gives better shapes of the \( \dot{\delta} \) curve, albeit producing a smaller amplitude.

## 10 CONCLUSIONS

Although we used the same assumptions and general strategy as Nesvorný and Vokrouhlický (2007, 2008b), we have obtained a significantly simpler form of the final expressions of the mean YORP torque. In particular, the final formulae are expressed in terms of the Legendre polynomials and associated functions, as compared to the products of Wigner functions that can be found in Nesvorný & Vokrouhlický (2007). The advantage comes from the way we expand the insolation function – more straightforward than the one exposed in appendix B of Nesvorný & Vokrouhlický (2007). Our formulation is well suited for the partition into the terms of primary and secondary influence according to the YORP degree, directly related with the degree of the Legendre functions present in the final expressions. Moreover, it has significantly better numerical properties than the power series of \( c \). We confirm the basic facts established in recent years, like the independence of the spin-related torque component on conductivity (Čapek & Vokrouhlický 2004; Nesvorný & Vokrouhlický 2008b) within the plane-parallel temperature model, or the presence of YORP effect in obliquity for the bodies with rotational symmetry (Breiter et al. 2007). The explicit appearance of \( P_2(c) \) in equation (109) and the remarks about the properties of \( W_p \) neatly explain the fact that the spin related YORP torques tend to vanish close to the zeros of \( P_2(c) \) (Nesvorný & Vokrouhlický 2007). Similarly, we can predict that if the obliquity component \( \langle M_1 \rangle \) has zeros different than \( c = 0 \) or \( s = 0 \) [implied by each Legendre function \( P_2^0(c) \)], the new zeros should be located in the domain \( 40^\circ < \varepsilon < 60^\circ \) (or \( 120^\circ < \varepsilon < 140^\circ \)) implied by \( P_2^1(c) \pm P_2^3(c) = 0 \). This conjecture is confirmed by a large sample of Gaussian random spheres simulated in Čapek & Vokrouhlický (2004), as well as by the observed ‘Slivan states’ in the Koronis family (Slivan 2002; Vokrouhlický, Nesvorný & Bottke 2003).

The assumptions about the insolation model seem to be the main source of divergence between numerical models and a completely analytical formulation. In this context, the approach taken by Mysen (2008) seems to be a good compromise: in his paper, the insolation function is derived from numerical simulation and then incorporated into an analytical treatment. As usual, however, such semi-analytical approach breaks the direct link between the shape coefficients and the YORP torque magnitude. It is a hard duty of an analytical solution to show such link, even if the final relation becomes as cumbersome as the one derived here.
Notably, the YORP effect in Rubincam’s approximation is a second-order effect in terms of the shape coefficients. Which of the harmonics products will play the leading role depends on the particular body shape because there is no monotonicity of the numerical coefficients accompanying a particular harmonics product. It is easier to point out non-significant harmonics: for example, the spin torque \( \langle M_3 \rangle \) is independent of the zonal harmonics coefficients \( C_{l,0} \). Although the presence of the non-zero conductivity brings in the first-order terms to the \( \langle M_1 \rangle \) and \( \langle M_2 \rangle \), in most cases they will be less significant than the second-order contribution, because \( s_1 \) – factoring the zonal coefficients – is typically much smaller than \( c_1 \). Thus, the zonal harmonics coefficients are generally of minor importance, unless the body has a highly regular shape, close to a rotationally invariant solid.

The modular structure of the present work allows a modification of particular elements of the solution without restarting the entire work from the scratch. Our next goal is the application of a more advanced temperature solution. The results are promising and will soon be announced in a separate paper.

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APPENDIX A: SPECIAL FUNCTIONS

Generally speaking, we use the definitions of special functions according to Biedenharn & Louck (1981). The only exception from this rule is the inclusion of the Condon–Shortley phase \((-1)^m\) in the definition of associated Legendre functions (ALFs) \( P^m_l \). All formulae from other sources have been adjusted to fit this convention.

A1 Spherical harmonics

If \( P_l(u) \) stands for the usual Legendre polynomial of degree \( l \), we define the ALFs as

\[
P^m_l(u) = \frac{(-1)^m}{2^l l!} \left( 1 - u^2 \right)^{l/2} \frac{d^{l+m}u(2u^2 - 1)^{l/2}}{du^{l+m}} = (-1)^m (1 - u^2)^{l/2} \frac{d^m P_l(u)}{du^m}.
\] (A1)

Introducing the normalization factor \( \sigma_{l,m} \)

\[
\sigma_{l,m} = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}},
\] (A2)

we define the normalized ALF

\[
\Theta^m_l(u) = \sigma_{l,m} P^m_l(u).
\] (A3)
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Spherical harmonics $Y_{lm}$ used in this paper are

$$Y_{lm}(u, \lambda) = \Theta_l^m(u) e^{im\lambda}, \quad (A4)$$

where $u$ is the sine of latitude (i.e. cosine of colatitude) and $\lambda$ is the longitude of the point on a unit sphere.

The symmetries of the Legendre functions

$$P_{l-m}^m(u) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(u), \quad (A5)$$

$$\Theta_{l-m}^m(u) = (-1)^m \Theta_l^m(u) \quad (A6)$$

lead to

$$Y_{l-m}^m(u, \lambda) = (-1)^m Y_{l-m}(u, \lambda), \quad (A7)$$

where an asterisk indicates a complex conjugate.

The following orthogonality relations hold

$$\int_0^{2\pi} d\lambda \int_{-1}^1 Y_{l,j}(u, \lambda) Y_{l,m}^*(u, \lambda) du = \delta_{j,l} \delta_{k,m} \quad (A8)$$

with $\delta_{pq}$ designating the Kronecker delta function equal to 1 if $p = q$ and zero otherwise.

A function $F(u, \lambda)$ defined on a unit sphere can be expanded in spherical harmonic series

$$F(u, \lambda) = \sum_{l \geq 0} \sum_{m=-l}^l a_{l,m} Y_{l,m}(u, \lambda), \quad (A9)$$

$$a_{l,m} = \int_{-1}^1 du \int_0^{2\pi} d\lambda F(u, \lambda) Y_{l,m}^*(u, \lambda) \delta_\lambda. \quad (A10)$$

If $F$ is real-valued, then the coefficients $a_{l,m}$ admit the property similar to (A7)

$$a_{l,m}^* = (-1)^m a_{l,-m}. \quad (A11)$$

### A2 Wigner functions (generalized spherical harmonics)

Consider two Cartesian reference frames $Oxyz$ and $O\hat{x}'\hat{y}'\hat{z}'$ having the same origin $O$, and related by rotation defined in terms of the 3-2-3 Euler angles $\alpha, \beta, \gamma$

$$r' = R_3(\gamma) R_2(\beta) R_1(\alpha) r, \quad (A12)$$

with the standard rotation matrices

$$R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix}, \quad R_2(\phi) = \begin{bmatrix} \cos \phi & 0 & -\sin \phi \\ 0 & 1 & 0 \\ \sin \phi & 0 & \cos \phi \end{bmatrix}, \quad R_3(\phi) = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (A13)$$

The same transformation can also be specified in terms of more common 3-1-3 Euler angles as

$$r' = R_3(\gamma - \pi/2) R_1(\beta) R_1(\alpha + \pi/2) r = R_3(\gamma + \pi/2) R_1(\beta) R_1(\alpha - \pi/2) r. \quad (A14)$$

According to the transformation laws of spherical harmonics for the 3-2-3 Euler angles sequence, each spherical function of polar coordinates $(\theta, \lambda)$ in $Oxyz$ frame becomes a combination of the same degree spherical functions of polar coordinates $\theta'$ and $\lambda'$ in $O\hat{x}'\hat{y}'\hat{z}'$.

Thus, for each harmonic

$$Y_{l,m}(\cos \theta', \lambda') = \sum_{k=-l}^l D_{l,k}^m(\alpha, \beta, \gamma) Y_{l,k}(\cos \theta, \lambda), \quad (A15)$$

or, conversely,

$$Y_{l,m}(\cos \theta, \lambda) = \sum_{k=-l}^l [D_{l,k}^m(\alpha, \beta, \gamma)]^* Y_{l,k}(\cos \theta', \lambda'), \quad (A16)$$

where the general definition of Wigner D-matrix elements (or Wigner D-functions) is (Biedenharn & Louck 1981)

$$D_{l,k}^m(\alpha, \beta, \gamma) = \hat{d}_{l,k}^m(\beta) \ e^{-(im\phi + ik\gamma)}. \quad (A17)$$

The symbols $\hat{d}_{l,k}^m(\beta)$ designate so-called Wigner d-functions

$$\hat{d}_{l,k}^m(\beta) = \sqrt{\frac{(l+k)!(l-k)!}{(l+m)!(l-m)!}} \ {\sin(\beta/2)}^{l+k} \ {\cos(\beta/2)}^{l-k} \ P_{l-k}^{l+k,m}(\cos \beta). \quad (A18)$$
The definition of d-functions in equation (A18) involves the Jacobi polynomials

\[ p_{n}^{(a,b)}(x) = \sum_{m=0}^{n} \binom{n + a}{m} \binom{n + b}{n - m} \left( \frac{x - 1}{2} \right)^{n-m} \left( \frac{x + 1}{2} \right)^{m}. \]  

(A19)

Wigner d-functions \( d_{m,k} \) admit some parity properties like

\[ d_{m,k}^l = (-1)^{k+m} d_{m,k}. \quad d_{m,k}^l = (-1)^{k+m} d_{-m,-k}. \]  

(A20)

Using these two identities, one can always reduce the \( d_{m,k}^l \) function to the case when \( k + m \geq 0 \) and \( k - m \geq 0 \), avoiding the appearance of negative powers of sines and cosines in equation (A18).

As it follows from (A20), the conjugate of \( D_{m,k} \) is

\[ [D_{m,k}^{(\alpha, \beta, \gamma)}]^* = (-1)^{n-k} D_{-m,-k}^{(\alpha, \beta, \gamma)}. \]  

(A21)

In the special case \( d_{0,0}^l \) or \( d_{0,k}^l \), Wigner d-functions reduce to AFL

\[ d_{0,0}^l(\beta) = (-1)^{l} d_{0,0}^l(\beta) = \sqrt{\frac{(l-k)!}{(l+k)!}} \rho_l(\cos \beta) = \sqrt{\frac{4\pi}{2l+1}} \Theta_l(\cos \beta). \]  

(A22)

### A3 Overlap integrals and Gaunt coefficients

The Wigner 3-j symbols are numerical coefficients defined as a sum

\[ \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix} = \frac{\left[(j_1 - j_2 - j_3)! (j_1 - j_2 + j_3)! (j_1 + j_2 + j_3)!\right]^{1/2}}{(j_1 + m_1)! (j_1 - m_1)! (j_2 + m_2)! (j_2 - m_2)! (j_3 + m_3)! (j_3 - m_3)!} \times \sum_{k} \frac{(-1)^{k+j_1-j_2-m_3}}{k! (j_1 - j_2 - k)! (j_1 - m_1 - k)! (j_2 + m_2 - k)! (j_3 - j_2 + m_1 + k)! (j_3 - j_1 - m_2 + k)!} \]  

where the index \( k \) takes all integer values leading to non-negative factorial arguments.

The so-called overlap integral is an integral of the product of two ALFs with the integration limits \( \pm 1 \). We use a normalized version of this function

\[ u_{m_1,n_2}^{m_1,n_2} = \int_{-1}^{1} \Theta_{m_1}(u) \Theta_{n_2}(u) \, du. \]  

(A24)

Using a formula of Dong & Lemus (2002), converted to the normalized version, we can express the overlap integrals in terms of the Wigner 3-j symbols

\[ u_{m_1,n_2}^{m_1,n_2} = \frac{(-1)^{\min(m_1,m_2)}}{\pi} 2^{m_1-m_2-1} |m_1 - m_2| \sqrt{(2m_1 + 1)(2m_2 + 1)} \times \sum_{l = \max(m_1-m_2,m_1,m_2)}^{n_1+n_2} (2l+1) G([m_1-m_2], l) \left[ \begin{array}{ccc} n_1 & n_2 & l \\ 0 & 0 & 0 \\ -m_1 & m_2 & m_1-m_2 \end{array} \right] \]  

(A25)

\[ G(m, l) = (1 + (-1)^{m+l}) \sqrt{\frac{(l-m)!}{(l+m)!}} \Gamma((l+1)/2) \Gamma((l+m+1)/2) \Gamma((l+3)/2). \]  

(A26)

Obviously, the overlap integrals are symmetric with respect to the indices \( u_{m_1,n_2}^{m_1,n_2} = u_{m_2,n_1}^{m_2,n_1} \), and they vanish either if \( |m_1| > n_1 \) or if the integrand in equation (A24) is an odd function of \( u \), i.e., when \( (-1)^{m_1+n_1+m_2+n_2} = -1 \). Moreover, because of the orthogonality of the ALFs, we have \( u_{m_1,n_2}^{m_1,n_2} = 0 \), for \( n_1 \neq n_2 \).

Because of a frequent appearance of \( u_{1,2p}^{1,2p} \sigma_{1,1} \) in the final formulae, we introduce the special case of the overlap integral

\[ W_{p} = \frac{u_{1,2p}^{1,2p}}{\sigma_{1,1}} = \frac{\sqrt{\pi}}{4\pi} \left( 2(p-1)! \right)^{2} \frac{\sqrt{4p+1}}{(p+1)(2p-1)} = \frac{\sqrt{\pi}}{4\pi} \left( 2(p-1)! \right)^{2} \frac{\sqrt{4p+1}}{(p+1)(2p-1)} \left( P_{2p}(0) \right)^{2} \]  

(A27)

The Gaunt coefficients (or Gaunt integrals) are defined as an integral of three spherical functions over the unit sphere surface

\[ G_{l_1,l_2,l_3}^{m_1,m_2,m_3} = \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{\pi} Y_{l_1,m_1}(\alpha, \lambda) Y_{l_2,m_2}(\alpha, \lambda) Y_{l_3,m_3}(\alpha, \lambda) \, d\lambda. \]  

(A28)

Their expression in terms of the Wigner 3-j symbols is much simpler than for the overlap integrals

\[ G_{l_1,l_2,l_3}^{m_1,m_2,m_3} = \frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi} \left[ \begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \\ m_1 & m_2 & m_3 \end{array} \right]. \]  

(A29)

Gaunt coefficients are non-zero if the following three conditions are satisfied:

\[ (-1)^{l_1+l_2+l_3} = 1, \quad m_1 + m_2 + m_3 = 0, \quad |l_i - l_j| \leq l_i \leq l_i + l_j. \]  

(A30)
Note that the signs of all $m_i$ superscripts can be simultaneously inverted, and the pairs $l_i, m_i$ can be arbitrarily permuted
\begin{equation}
G_{l_1,l_2,l_3}^{m_1,m_2,m_3} = G_{l_3,l_2,l_1}^{-m_1,-m_2,-m_3} = G_{l_2,l_1,l_3}^{m_2,m_1,m_3} = G_{l_3,l_1,l_2}^{-m_2,-m_1,-m_3}, \text{ etc.} \tag{A31}
\end{equation}
From the three-term recurrence identities related with the Wigner 3-j symbols (Luscombe & Luban 1998), we can deduce useful relation
\begin{equation}
(l_3(l_3 + 1) - l_2(l_2 + 1) - l_1(l_1 + 1)) G_{l_1,l_2,l_3}^{m,0,-m} = \sqrt{l_2(l_2 + 1)} \left[ \sqrt{(l_1 - m)(l_1 + m + 1)} G_{l_1,l_2,l_3}^{m,-1,-m} + \sqrt{(l_1 + m)(l_1 - m + 1)} G_{l_1,l_2,l_3}^{m+1,-1,-m} \right]. \tag{A32}
\end{equation}
The Gaunt coefficients appear in the transformation rule for the product of spherical harmonics (Clebsch–Gordan series). Applying equations (A28) to (A10), we find
\begin{equation}
Y_{j_1,k_1} Y_{j_2,k_2} = \sum_{l = \max(0,j_1+j_2,|j_1-k_1|)}^{j_1+j_2} (-1)^{k_1+k_2} G_{j_1,j_2,l}^{k_1,k_2,-k_1-k_2} Y_{l,k_1+k_2}, \tag{A33}
\end{equation}
where the sum $j_1 + j_2 + l$ must be even.

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