GLOBAL INSTABILITY OF MULTI-DIMENSIONAL PLANE SHOCKS FOR ISOTHERMAL FLOW*

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Abstract In this paper, we are concerned with the long time behavior of the piecewise smooth solutions to the generalized Riemann problem governed by the compressible isothermal Euler equations in two and three dimensions. A non-existence result is established for the fan-shaped wave structure solution, including two shocks and one contact discontinuity which is a perturbation of plane waves. Therefore, unlike in the one-dimensional case, the multi-dimensional plane shocks are not stable globally. Moreover, a sharp lifespan estimate is established which is the same as the lifespan estimate for the nonlinear wave equations in both two and three space dimensions.

Key words Blow-up; global solution; instability; shock; contact discontinuity; Euler equations; isothermal; generalized Riemann problem; nonlinear wave equations

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1 Introduction

We are concerned with the non-existence of global solutions of the generalised Riemann problem governed by the compressible isothermal Euler equations. More precisely, we prove that the multi-dimensional (N = 2, 3) plane shocks are not stable in the global sense with respect to a smooth perturbation.

It is well-known that smooth solutions of the compressible Euler equations with some compression assumption will generate singularity in finite time no matter how small the initial data

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Therefore, it is natural and important to study the Cauchy problem with discontinuous initial data. For the one-dimensional case, a satisfactory theory on the global existence and stability of the Cauchy problem has been established by many mathematicians ([4–6]). However, the $B.V.$ space is not a well-posed space for the Cauchy problem in multidimensions. As a result, almost all efforts have been focused on the multidimensional Riemann problem ([7–16]) or the structural stability of important physical problems introduced in Courant-Friedrichs’ classic book [17], for example, the supersonic flow over a wedge or cone ([18–32]).

The multidimensional Riemann problem of the compressible Euler equations, which plays a prominent role in the theory of conservation laws, is one of the core and most challenging problems in the mathematical theory of conservation laws. One important problem is the generalized Riemann problem, which studies the Cauchy problem with discontinuous initial data along a smooth curve. If the data is assumed to be smooth up to the curve, then we expect the solution to be of a fan-shape structure. The generalized Riemann problem can also be regarded as the stability of the Riemann solutions of the Cauchy problem with two constant states separated by a hyperplane. There is a lot of literature on the local existence of the generalized Riemann problem, for example, Blokhin [33, 34] and Majda [35, 36] for the strong shock, Metivier [37] for the weak shock, Alinhac [38] for the rarefaction wave, Coulombel-Secchi [39] for the two-dimensional vortex sheet, and [40] for the two dimensional composited waves which can be shocks, rarefaction waves or vortex sheets.

A natural question arises is: what about the global existence of solutions of the generalized Riemann problem? As far as we know, there are few results on the global existence of those waves except for the ones on the unsteady potential flow equation in $n$-dimensional spaces ($n \geq 5$, see [41]) or in special space-time domains for the potential flow. It is therefore of great significance to study the global behaviour of the solutions of the generalized Riemann problem from both the mathematical and physical point of view. In this paper, we will show that the solutions of the generalized Riemann problem (if they exist locally and are a perturbation of plane shocks) cannot exist globally for the two and three dimensional cases, if the flow is isothermal. This means that the plane Riemann solutions are not stable globally with respect to a smooth perturbation. Based on this, in order to obtain the global stability, we should think about the generalized Riemann problem in a weak sense. Moreover, the lifespan estimate, which is consistent with the lifespan estimate for the nonlinear wave equations, is also obtained; the result is different from the one dimensional case, in which the global existence is established ([42, 43]).

2 Generalized Riemann Problem and Main Result

The multidimensional inviscid compressible flow is governed by the following Euler equation:

$$\begin{cases}
\rho_t + \text{div}(\rho \mathbf{u}) = 0, \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0.
\end{cases}$$

(2.1)

Here $\rho$, $p$ and $\mathbf{u}$ are density, pressure and velocity, respectively. For the isothermal flow, the pressure and density satisfy the thermodynamic relation that $p = \rho$. In this paper, we are concerned with the global stability/instability of solutions of the generalized Riemann problem governed by equations (2.1) for the isothermal flow. Since until now the local existence result...
for the vortex sheet has only been available for the two-dimensional case, we will consider the two-dimensional case first. The global non-stability for the three-dimensional case to the isothermal flow will be proved at the end of the paper, even though we do not know whether or not the local nonlinear existence can be obtained.

For the two dimensional case, equations (2.1) become

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\
(\rho u)_t + (\rho u^2)_x + (\rho uv)_y + \rho_x &= 0, \\
(\rho v)_t + (\rho uv)_x + (\rho v^2)_y + \rho_y &= 0,
\end{align*}
\]

where \((u, v)\) is the velocity with the initial data

\[
t = 0 : \quad \begin{cases} \\
\rho &= \begin{cases} 
\rho_r + \varepsilon \rho_0(x, y), & x > \varepsilon \Pi(y), \\
\rho_l + \varepsilon \rho_0(x, y), & x < \varepsilon \Pi(y),
\end{cases} \\
u &= \begin{cases} 
u_r + \varepsilon u_0(x, y), & x > \varepsilon \Pi(y), \\
u_l + \varepsilon u_0(x, y), & x < \varepsilon \Pi(y),
\end{cases} \\
v &= \varepsilon v_0(x, y),
\end{cases}
\]

where \(\rho_r, \rho_l, u_r, u_l\) are constants, and functions \(\Pi(y) \in C^\infty_0(\mathbb{R})\), \(\rho_0, u_0, v_0 \in C^\infty_0(\mathbb{R}^2)\) satisfy

\[
supp \Pi(y) \subset \{y : |y| \leq 1\}. \tag{2.4}
\]

and

\[
supp \rho_0, u_0, v_0 \subset B_1 = \{(x, y) : x^2 + y^2 \leq 1\}. \tag{2.5}
\]

Moreover, \(\varepsilon > 0\) is a small parameter.

For a piecewise \(C^1\) weak solution of (2.2) with \(C^1\)-discontinuities, by integration by parts, it is easy to know that the solution is a solution of (2.2) in the classic sense in each smooth subregion, and that across the discontinuities (\(\Pi\) for example) satisfies the Rankine-Hugoniot conditions

\[
\partial_t \Pi[\rho]^{\Pi^+}_{\Pi^-} - [\rho u]^\Pi^+_{\Pi^-} + \partial_y \Pi[\rho v]^{\Pi^+}_{\Pi^-} = 0 \tag{2.6}
\]
\[
\partial_t \Pi[\rho u]^{\Pi^+}_{\Pi^-} - [\rho u^2 + \rho]^\Pi^+_{\Pi^-} + \partial_y \Pi[\rho uv]^{\Pi^+}_{\Pi^-} = 0 \tag{2.7}
\]
\[
\partial_t \Pi[\rho v]^{\Pi^+}_{\Pi^-} - [\rho uv]^\Pi^+_{\Pi^-} + \partial_y \Pi[\rho v^2 + \rho]^\Pi^+_{\Pi^-} = 0, \tag{2.8}
\]

where \([\cdot]^\Pi^+_{\Pi^-}\) denotes the difference of the left hand side limit and the right hand side limit of the quantity concerned on the discontinuity \(x = \Pi(t, y)\).

For the corresponding one-dimensional Riemann problem, which is governed by the equation

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2)_x + \rho_x &= 0,
\end{align*}
\]

with the initial data

\[
t = 0 : \quad \begin{cases} \\
\rho &= \begin{cases} 
\rho_r, x > 0, \\
\rho_l, x < 0,
\end{cases} \\
u &= \begin{cases} 
\varepsilon u_0, x > 0, \\
u_l, x < 0,
\end{cases}
\end{cases}
\]

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it is well-known that if the constant vector \((\rho, u)\) lies in a cornered domain with boundaries being the wave curves starting from \((\rho_l, u_l)\), the Riemann problem of (2.9) and (2.10) admits a Riemann solution that consists of three constant states, \((\rho_l, u_l), (\rho_m, u_m), (\rho_r, u_r)\), separated by two shocks with shock speeds \(\sigma_+\) and \(\sigma_-\), respectively. Without loss of the generality, we assume that \(u_m = 0\), otherwise we can introduce the coordinate transformation that \(x \to x - u_m t\). In this case, across the shock, the following Rankine-Hugoniot conditions hold:

\[
\begin{align*}
\sigma_+(\rho_r - \rho_m) - \rho_r u_r &= 0, \\
\sigma_+ \rho_r u_r - \rho_r u_r^2 - \rho_r + \rho_m &= 0 \\
\sigma_-(\rho_l - \rho_m) - \rho_l u_l &= 0, \\
\sigma_- \rho_l u_l - \rho_l u_l^2 - \rho_l + \rho_m &= 0.
\end{align*}
\]

Moreover, the Riemann solution satisfies the following entropy condition:

\[
\rho_m > \rho_l, \quad \rho_r > \rho_l, \quad u_l > 0, \quad u_r < 0.
\]

In summary, the Riemann solution of equation (2.9) with initial data (2.10) is

\[
\rho = \begin{cases} 
\rho_r, & \text{if } x > \sigma_+ t \\
\rho_m, & \text{if } \sigma_- t < x < \sigma_+ t, \\
\rho_l, & \text{if } x < \sigma_- t
\end{cases}, \\
u = \begin{cases} 
0, & \text{if } \sigma_- t < x < \sigma_+ t, \\
u_r, & \text{if } x < \sigma_- t
\end{cases},
\]

The generalized Riemann problem (2.2) with initial data (2.3) can be regarded as a small perturbation of the Riemann solution (2.14) when \(\varepsilon\) is sufficiently small.

Under a condition for the speed of the initial data (a condition similar to the one in [39] for the vortex sheet), Chen and Li [40] established the local existence of a piecewise smooth solution to equations (2.2) with initial data (2.3) containing all three waves (i.e., shock wave, rarefaction wave and contact discontinuity). Their existence result also includes the case in which the generalized Riemann solution consists of the 1-shock wave \(x = \Pi_-(t, y)\) corresponding to the first eigenvalue, the 3-shock wave \(x = \Pi_+(t, y)\) corresponding to the third eigenvalue, and the contact discontinuity \(x = \Pi_0(t, y)\), with the condition that

\[
\varepsilon \Pi(y) = \Pi_-(0, y) = \Pi_0(0, y) = \Pi_+(0, y),
\]

and that

\[
0 < \rho_* < \rho < \rho^* < \infty,
\]

where \(\rho_*\) and \(\rho^*\) are two constants.

The aim of this paper is to prove that, for the isothermal case, such a piecewise smooth solution of equation (2.2) with initial data (2.3) obtained in [40] cannot be global in time in general. This result is different from the one for the one dimensional case in which the global existence is established ([42, 43]). We also obtain the lifespan estimate.

**Theorem 2.1** For the given initial data (2.3), assume that

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (e^y + e^{-y}) \rho_0 dx dy + (\rho_l - \rho_r) \int_{-\infty}^{+\infty} (e^y + e^{-y}) \Pi_0 dy \geq 0,
\]

and that there exists a constant \(C > 0\) such that

\[
\int_{-\infty}^{+\infty} \int_{\varepsilon \Pi(y)}^{+\infty} (e^y - e^{-y}) (\rho_r + \varepsilon \rho_0) v_0 dx dy + \int_{-\infty}^{+\infty} \int_{-\infty}^{-\varepsilon \Pi(y)} (e^y - e^{-y}) (\rho_l + \varepsilon \rho_0) v_0 dx dy \geq C.
\]
If there exist positive constants $R$ and $t_0$ such that solution $(\rho, u, v)$ of equation (2.2) satisfies
\[ |(\rho, u, v) - (\rho, u, v)_{t,m,r}|_{L^\infty} \leq C\varepsilon, \quad \text{when } x^2 + y^2 \geq R \text{ and } t \geq t_0, \]
where the constant $C$ in the above two inequalities is positive and does not depend on $\varepsilon$, then the piecewise smooth solution (whose discontinuities consist of two shock waves $x = \Pi_\pm(t, y)$ and a contact discontinuity $x = \Pi_0(t, y)$) for the generalized Riemann problem (2.2) of isothermal compressible Euler equations with initial data (2.3) will blow up in a finite time. Moreover, there exists a positive constant $C$ independent of $\varepsilon$ such that the upper bound of the lifespan satisfies the estimate
\[ T(\varepsilon) \leq C\varepsilon^{-2}. \quad (2.19) \]

**Remark 2.1** The lifespan estimate (2.19) is consistent with the lifespan estimate of smooth solutions of nonlinear wave equations in two space dimensions.

Next, let us consider the three dimensional case. We denote the coordinates as $(x, y_1, y_2)$, so the three dimensional compressible isothermal Euler system is
\[
\begin{aligned}
(r)_t + (ru)_x + (rv_1)_y + (rv_2)_y &= 0, \\
(ru)_t + (ru^2)_x + (rpu_1)_y + (rpuv_2)_y + \rho &= 0, \\
(rv_1)_t + (rvu_1)_x + (rv_2)_y + (rv_1v_2)_y + \rho v_1 &= 0, \\
(rv_2)_t + (rvu_2)_x + (rv_1v_2)_y + (rv_2^2)_y + \rho v_2 &= 0,
\end{aligned}
\quad (2.20)
\]
where $(u, v_1, v_2)$ are the velocity. In order to make the notations consistent with the ones for the two dimensional case, let $v := (v_1, v_2)$.

**Theorem 2.2** Assume that the initial datum satisfies that
\[
\int_{|y| = 1} \int_{\mathbb{R}^2} e^{y_1^2 + y_2^2} \rho_0(x, y) dx dy d\sigma + (\rho_t - \rho_r) \int_{|y| = 1} \int_{\mathbb{R}^2} e^{y_1^2 + y_2^2} \Pi(y) dy d\sigma \geq 0
\quad (2.21)
\]
and that there exists a constant $C > 0$ such that
\[
\begin{aligned}
\int_{|y| = 1} \int_{\mathbb{R}^2} e^{y_1^2 + y_2^2} (\rho_t + \varepsilon \rho_0) v_0(y) \cdot \omega dx dy \\
+ \int_{|y| = 1} \int_{\mathbb{R}^2} e^{y_1^2 + y_2^2} (\rho_r + \varepsilon \rho_0) v_0(y) \cdot \omega dx dy \geq C.
\end{aligned}
\quad (2.22)
\]
Then it is impossible that there exists a global piecewise smooth solution of the generalized Riemann problem for the compressible isothermal Euler system (2.20) with initial data (2.3) and consisting of two shocks and one contact discontinuity such that
\[ |(\rho, u, v) - (\rho, u, v)_{t,m,r}|_{L^\infty} \leq C\varepsilon, \]
where the constant $C$ in the above two inequalities is positive and does not depend on $\varepsilon$. Furthermore, we have the following lifespan estimate:
\[ T(\varepsilon) \leq \exp \left( C\varepsilon^{-1} \right). \quad (2.23) \]

**Remark 2.2** Although there is no result on the local existence of solutions of the generalized Riemann problem due to the nonlinear vortex sheet in three dimensions, we can show that the three dimensional solutions of the generalized Riemann problem of isothermal compressible Euler equations cannot exist globally even if one could show the local existence.
Remark 2.3 The lifespan estimate (2.23) is consistent with the lifespan estimate of smooth solutions of nonlinear wave equations in three space dimensions.

We will show Theorem 2.1 in Section 3, and Theorem 2.2 in Section 4.

3 Proof of Theorem 2.1: Two Dimensional Case

To show Theorem 2.1, we will rewrite the first and third equations in (2.2) in four subdomains separated by the shocks and contact discontinuity, by subtracting the background solution. Next, we introduce the multiplier $e^y + e^{-y}$ for the first equation and $e^y - e^{-y}$ for the third equation. Then we can derive an ordinary differential system for two quantities, which are integrals of the solutions with respect to the space variables, by using the Rankine-Hugoniot conditions (2.6)–(2.8). By a delicate analysis of the obtained ordinary differential system, we obtain a blow-up result for the new quantity. Finally, the desired lifespan estimate will be established too.

First, let us introduce a technical lemma. Letting
\[ X(t) = \int_{-\infty}^{+\infty} (e^y + e^{-y}) \left[ \int_{\Pi_+(t,y)}^{+\infty} (\rho - \rho_r)dx + \int_{\Pi_0(t,y)}^{\Pi_+(t,y)} (\rho - \rho_m)dx \right. \]
\[ + \int_{\Pi_+(t,y)}^{\Pi_0(t,y)} (\rho - \rho_m)dx + \int_{-\infty}^{\Pi_-(t,y)} (\rho - \rho_l)dx \right] dy \]
\[ + (\rho_m - \rho_r) \int_{-\infty}^{+\infty} (e^y + e^{-y}) [\Pi_+(t,y) - \sigma_+ t] dy \]
\[ + (\rho_m - \rho_l) \int_{-\infty}^{+\infty} (e^y + e^{-y}) [\sigma_- t - \Pi_-(t,y)] dy \]
and letting
\[ Y(t) = \int_{\mathbb{R}^2} (e^y - e^{-y}) \rho v dx dy, \]
we have

Lemma 3.1 For the solutions of equations (2.2), we have the following identities:

\[ X'(t) = Y(t) \]

(3.3)

and

\[ Y'(t) = X(t) + \int_{\mathbb{R}^2} (e^y + e^{-y}) \rho v^2 dx dy. \]

(3.4)

Proof We rewrite the first equation in (2.2) as

\[
\begin{cases}
  (\rho - \rho_r)_t + (\rho u - \rho_r u_r)_x + (\rho v)_y = 0, & \text{when } \Pi_+(t,y) < x < \infty, \\
  (\rho - \rho_m)_t + (\rho u)_x + (\rho v)_y = 0, & \text{when } \Pi_-(t,y) < x < \Pi_0(t,y), \\
  (\rho - \rho_l)_t + (\rho u - \rho_l u_l)_x + (\rho v)_y = 0, & \text{when } -\infty < x < \Pi_-(t,y),
\end{cases}
\]

(3.5)

where $x = \Pi_+(t,y)$ and $x = \Pi_-(t,y)$ are the right and left shocks, respectively, and $x = \Pi_0(t,y)$ is the contact discontinuity.
By the first equation in (3.5) and by integration by parts, we have
\[
\frac{d}{dt} \int_{-\infty}^{+\infty} \int_{\Pi_+(t,y)} (\rho - \rho_r)(e^y + e^{-y}) \, dx \, dy \\
= - \int_{-\infty}^{+\infty} \int_{\Pi_+(t,y)} \partial_t \Pi_+(t,y)(\rho - \rho_r)(e^y + e^{-y}) \, dx \, dy \\
+ \int_{-\infty}^{+\infty} \int_{\Pi_+(t,y)} (\rho - \rho_r)x(e^y + e^{-y}) \, dx \, dy \\
- \int_{-\infty}^{+\infty} \int_{\Pi_+(t,y)} (\rho u - \rho_r u_r) e^y \, dx \, dy \\
= \int_{-\infty}^{+\infty} \int_{\Pi_+(t,y)} (\rho v)(e^y - e^{-y}) \, dx \, dy, \tag{3.6}
\]
where \(\Pi_+\) denotes that the value taken at \(x = \Pi_+(t,y)\) is the limit from the right hand. Next, in the region \(\Pi_0(t,y) < x < \Pi_+(t,y)\), by the second equation in (3.5), we have
\[
\frac{d}{dt} \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)} (\rho - \rho_m)(e^y + e^{-y}) \, dx \, dy \\
= \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)} \partial_t \Pi_0(t,y)(\rho - \rho_m)(e^y + e^{-y}) \, dx \, dy \\
- \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)} \partial_t \Pi_0(t,y)(\rho - \rho_m)(e^y + e^{-y}) \, dx \, dy \\
+ \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)} [(\rho u) e^y + (\rho v) e^y] \, dx \, dy \\
= \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)} (\rho v)(e^y - e^{-y}) \, dx \, dy, \tag{3.7}
\]
where \(\Pi_+\) and \(\Pi_0^+\), in a fashion similar to as above, denote that the values taken are the limit from the left hand side of \(\Pi_+\) and from the right hand side of \(\Pi_0\), respectively.

For the integration in the regions \(\Pi_-(t,y) < x < \Pi_0(t,y)\) and \(-\infty < x < \Pi_-(t,y)\), similarly, we have the following results:
\[
\frac{d}{dt} \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)} (\rho - \rho_m)(e^y + e^{-y}) \, dx \, dy \\
= \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)} \partial_t \Pi_0(t,y)(\rho - \rho_m)(e^y + e^{-y}) \, dx \, dy \\
- \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)} \partial_t \Pi_0(t,y)(\rho - \rho_m)(e^y + e^{-y}) \, dx \, dy \\
= \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)} (\rho v)(e^y - e^{-y}) \, dx \, dy.
\]
and

\[\frac{d}{dt} \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)}^{\Pi_0(t,y)} (\rho v)(e^y - e^{-y})dxdy = \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)}^{\Pi_0(t,y)} (\rho v)(e^y + e^{-y})dxdy\]

We omit the details for brevity, since the arguments for the two results above are similar to the ones for (3.6) and (3.7). It follows, by adding (3.6)–(3.9) together, that

\[
\frac{d}{dt} \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)}^{\Pi_0(t,y)} (\rho - \rho_r)dx + \int_{\Pi_0(t,y)}^{\Pi_+(t,y)} (\rho - \rho_m)dx \\
+ \int_{\Pi_0(t,y)}^{\Pi_-(t,y)} (\rho - \rho_m)dx + \int_{\Pi_0(t,y)}^{\Pi_-(t,y)} (\rho - \rho_l)dx \right\} (e^y + e^{-y})dy = \int_{\mathbb{R}^2} \int_{-\infty}^{+\infty} (e^y - e^{-y})dy \left\{ \vartheta \Pi_+ [\rho] \big|_{\Pi^+_e} - [\rho u] \big|_{\Pi^+_e} + \vartheta \Pi_+ [\rho v] \big|_{\Pi^+_e} \right\} dy

where \([f]_{\Pi^+_e} \big|_{\Pi^-_e} \] denotes the jump difference between the left hand side limit and the right hand side limit of the function \( f \) on the curve \( x = \Pi_A(t,y) \) for \( A \in \{+,0,-\} \). By Rankine-Hugoniot conditions (2.6) and (2.11)–(2.12), we obtain from (3.10) that

\[
\frac{d}{dt} \int_{-\infty}^{+\infty} \int_{\Pi_0(t,y)}^{\Pi_0(t,y)} (\rho - \rho_r)dx + \int_{\Pi_0(t,y)}^{\Pi_+(t,y)} (\rho - \rho_m)dx \\
+ \int_{\Pi_0(t,y)}^{\Pi_+(t,y)} (\rho - \rho_m)dx + \int_{\Pi_0(t,y)}^{\Pi_-(t,y)} (\rho - \rho_l)dx \right\} (e^y + e^{-y})dy = \int_{\mathbb{R}^2} \int_{-\infty}^{+\infty} (e^y + e^{-y})dy \left\{ \vartheta \Pi_+ (\rho_m - \rho_r) - \rho_r u_r \right\} dy

This amounts to (3.3), based on the observation that

\[
\int_{-\infty}^{+\infty} (e^y + e^{-y}) \left( \vartheta \Pi_+ (\sigma_+ - \vartheta) \right) dy = \frac{d}{dt} \left( \int_{-\infty}^{+\infty} (e^y + e^{-y}) (\Pi_+ - \sigma_+) dy \right)
\]
As above, we divide the ensuing quantity into four parts such that
\[
\int_{-\infty}^{+\infty} (e^y + e^{-y}) (\sigma_\gamma - \partial_t \Pi_-) \, dy = \frac{d}{dt} \left( \int_{-\infty}^{+\infty} (e^y + e^{-y}) (\sigma_\gamma - \Pi_-) \, dy \right).
\]

Now we are going to show (3.4). As in (3.5), we rewrite the third equation in (2.2) as follows:
\[
\begin{align*}
&(\rho v)_t + (\rho u v)_x + (\rho v^2)_y + (\rho - \rho_r)_y = 0, \quad \text{when } \Pi_+(t, y) < x < \infty, \\
&(\rho v)_t + (\rho u v)_x + (\rho v^2)_y + (\rho - \rho_m)_y = 0, \quad \text{when } \Pi_-(t, y) < x < \Pi_0(t, y), \\
&(\rho v)_t + (\rho u v)_x + (\rho v^2)_y + (\rho - \rho_l)_y = 0, \quad \text{when } -\infty < x < \Pi_-(t, y).
\end{align*}
\]
(3.12)

As above, we divide the ensuing quantity into four parts such that
\[
\frac{d}{dt} \int_{-\infty}^{+\infty} \int_{\Pi_+(t, y)}^{+\infty} (e^y - e^{-y}) \rho v \, dx \, dy
\]
\[
= \frac{d}{dt} \int_{-\infty}^{+\infty} \left( \int_{\Pi_+}^{\Pi_0} + \int_{\Pi_0}^{\Pi_+} + \int_{\Pi_-}^{\Pi_0} + \int_{-\infty}^{\Pi_-} \right) (e^y - e^{-y}) \rho v \, dx \, dy.
\]
The first integration is on the region \( \Pi_+(t, y) < x < \infty \). By integration by parts, we have
\[
\frac{d}{dt} \int_{-\infty}^{+\infty} \int_{\Pi_+(t, y)}^{+\infty} (\rho v)_t (e^y - e^{-y}) \, dx \, dy
\]
\[
= - \int_{-\infty}^{+\infty} \partial_t \Pi_+(t, y) (\rho v)(e^y - e^{-y}) \bigg|_{x=\Pi_+}^{+\infty} \, dy + \int_{-\infty}^{+\infty} (\rho v)_t (e^y - e^{-y}) \, dx \, dy
\]
\[
= - \int_{-\infty}^{+\infty} \partial_t \Pi_+(t, y) (\rho v)(e^y - e^{-y}) \bigg|_{x=\Pi_+}^{+\infty} \, dy
\]
\[
- \int_{-\infty}^{+\infty} \int_{\Pi_+(t, y)}^{+\infty} (\rho u v (e^y - e^{-y}))_x + (\rho v^2 (e^y - e^{-y}))_y + ((\rho - \rho_r)(e^y - e^{-y}))_y \, dx \, dy
\]
\[
+ \int_{-\infty}^{+\infty} \int_{\Pi_+(t, y)}^{+\infty} (\rho v^2 + \rho - \rho_r)(e^y + e^{-y}) \, dx \, dy,
\]
\[
= \int_{-\infty}^{+\infty} (e^y - e^{-y}) \left[ - \partial_t \Pi_+(\rho v) + (\rho u v - \partial_y \Pi_+(\rho v^2 + \rho - \rho_r)) \right] \bigg|_{x=\Pi_+}^{+\infty} \, dy
\]
\[
+ \int_{-\infty}^{+\infty} \int_{\Pi_+(t, y)}^{+\infty} (\rho v^2 + \rho - \rho_r)(e^y + e^{-y}) \, dx \, dy.
\]
(3.13)

Similarly, for the other three integrations, after straightforward computations, we have that
\[
\frac{d}{dt} \int_{-\infty}^{+\infty} \int_{\Pi_0(t, y)}^{\Pi_+} (\rho v)_t (e^y - e^{-y}) \, dx \, dy
\]
\[
= \int_{-\infty}^{+\infty} (e^y - e^{-y}) \left[ \partial_t \Pi_0(\rho v) - (\rho u v) + \partial_y \Pi_0(\rho v^2 + \rho - \rho_m) \right] \bigg|_{x=\Pi_+}^{+\infty} \, dy
\]
\[
- \int_{-\infty}^{+\infty} (e^y - e^{-y}) \left[ \partial_t \Pi_0(\rho v) - (\rho u v) + \partial_y \Pi_0(\rho v^2 + \rho - \rho_m) \right] \bigg|_{x=\Pi_0}^{+\infty} \, dy
\]
\[
+ \int_{-\infty}^{+\infty} \int_{\Pi_0(t, y)}^{\Pi_+(t, y)} (\rho v^2 + \rho - \rho_m)(e^y + e^{-y}) \, dx \, dy,
\]
(3.14)
\[
\frac{d}{dt} \int_{-\infty}^{+\infty} \int_{\Pi_0(t, y)}^{\Pi_-} (\rho v)(e^y - e^{-y}) \, dx \, dy
\]
Therefore, it follows, by adding (3.13)-(3.16) together, that

\[
\int_{-\infty}^{+\infty} (e^y - e^{-y}) \left[ \partial_t \Pi_0 (\rho v) - (\rho uv) + \partial_y \Pi_0 (\rho v^2 + \rho - \rho_m) \right] \bigg|_{x=\Pi_0^+} dy
\]

\[- \int_{-\infty}^{+\infty} (e^y - e^{-y}) \left[ \partial_t \Pi_- (\rho v) - (\rho uv) + \partial_y \Pi_- (\rho v^2 + \rho - \rho_m) \right] \bigg|_{x=\Pi_-^+} dy
\]

\[+ \int_{-\infty}^{+\infty} \int_{\Pi_-^+ (t,y)} (\rho v^2 + \rho - \rho_m)(e^y + e^{-y}) dx dy,
\]

(3.15)

and

\[d \int_{-\infty}^{+\infty} (e^y - e^{-y}) (\partial_y \Pi_0 (\rho v) - (\rho uv) + \partial_y \Pi_0 (\rho v^2 + \rho - \rho_1)) \bigg|_{x=\Pi_-^+} dy
\]

\[+ \int_{-\infty}^{+\infty} \int_{\Pi_-^+ (t,y)} (\rho v^2 + \rho - \rho_1)(e^y + e^{-y}) dx dy.
\]

(3.16)

Therefore, it follows, by adding (3.13)-(3.16) together, that

\[
d \int_{\mathbb{R}^2} (e^y - e^{-y}) \rho v dx dy
\]

\[= \int_{-\infty}^{+\infty} \left[ \int_{\Pi_+^+ (t,y)} (\rho - \rho_1) dx + \int_{\Pi_0^+ (t,y)} (\rho - \rho_m) dx \right.
\]

\[+ \int_{\Pi_-^+ (t,y)} (\rho - \rho_m) dx + \int_{\Pi_-^+ (t,y)} (\rho - \rho_1) dx \bigg] (e^y + e^{-y}) dy + \int_{\mathbb{R}^2} (e^y + e^{-y}) \rho v^2 dx dy
\]

\[- \int_{-\infty}^{+\infty} (e^y - e^{-y}) \left\{ \partial_t \Pi_0 [\rho v]_{\Pi_+^+} - [\rho uv]_{\Pi_+^+} + \partial_y \Pi_0 [\rho v^2 + \rho]_{\Pi_+^+} \right\} dy
\]

\[- \int_{-\infty}^{+\infty} (e^y - e^{-y}) \left\{ \partial_t \Pi_0 [\rho v]_{\Pi_-^+} - [\rho uv]_{\Pi_-^+} + \partial_y \Pi_0 [\rho v^2 + \rho]_{\Pi_-^+} \right\} dy
\]

\[- \int_{-\infty}^{+\infty} (e^y - e^{-y}) \left\{ \partial_t \Pi_- [\rho v]_{\Pi_-^+} - [\rho uv]_{\Pi_-^+} + \partial_y \Pi_- [\rho v^2 + \rho]_{\Pi_-^+} \right\} dy
\]

\[- (\rho_m - \rho_r) \int_{-\infty}^{+\infty} \partial_y \Pi_+ (t,y)(e^y - e^{-y}) dy
\]

\[+ (\rho_m - \rho_l) \int_{-\infty}^{+\infty} \partial_y \Pi_- (t,y)(e^y - e^{-y}) dy.
\]

(3.17)

For the last two integrals in (3.17), by integration by parts, we have that

\[ (\rho_m - \rho_r) \int_{-\infty}^{+\infty} \partial_y \Pi_+ (t,y)(e^y - e^{-y}) dy
\]

\[= (\rho_m - \rho_r) \int_{-\infty}^{+\infty} \partial_y [\Pi_+ (t,y) - \sigma_+ t] (e^y - e^{-y}) dy
\]

\[= - (\rho_m - \rho_r) \int_{-\infty}^{+\infty} [\Pi_+ (t,y) - \sigma_+ t] (e^y + e^{-y}) dy,
\]

\[(\rho_m - \rho_l) \int_{-\infty}^{+\infty} \partial_y \Pi_- (t,y)(e^y - e^{-y}) dy
\]

\[= (\rho_m - \rho_l) \int_{-\infty}^{+\infty} \partial_y [\Pi_- (t,y) - \sigma_- t] (e^y - e^{-y}) dy.
\]
Thus, by the Rankine-Hugoniot conditions (2.8) on $\Pi_+$, $\Pi_0$, and $\Pi_-$, if follows from (3.17)–(3.18) that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} (e^y - e^{-y}) \rho v dx dy
\]
\[
= \int_{\mathbb{R}^2} (e^y + e^{-y}) \rho v^2 dx dy + \int_{-\infty}^{+\infty} (e^y + e^{-y}) \left[ \int_{\Pi_+(t,y)}^{+\infty} (\rho - \rho_r) dx + \int_{\Pi_0(t,y)}^{\Pi_+(t,y)} (\rho - \rho_m) dx \right] dy
\]
\[
+ \int_{\Pi_0(t,y)}^{\Pi_-(t,y)} (\rho - \rho_m) dx + \int_{-\infty}^{\Pi_-(t,y)} (\rho - \rho_l) dx \right] dy
\]
\[
+ (\rho_m - \rho_r) \int_{-\infty}^{+\infty} (e^y + e^{-y}) [\Pi_+(t,y) - \sigma_+ t] dy
\]
\[
+ (\rho_m - \rho_l) \int_{-\infty}^{+\infty} (e^y + e^{-y}) [\sigma_- t - \Pi_-(t,y)] dy.
\]
Therefore, from (3.11) and (3.19) we obtain (3.3) and (3.4).

Based on Lemma 3.1, we can now show the proof of Theorem 2.1.

**Proof of Theorem 2.1** Based on the entropy condition (2.13) and the Rankine-Hugoniot conditions (2.12), we know that
\[
1 + u_r < \sigma_+ < 1 + u_m = 1 \quad \text{and} \quad -1 = -1 + u_m < \sigma_- < -1 + u_l,
\]
where we have used the assumption that $u_m = 0$. Therefore if $\varepsilon$ is small, the propagation speed of waves is smaller than 1. Due to the assumption of the support of the initial data in (2.4) and (2.5), there exist constants $t_0$ and $C_0$ such that the support of the solution $v(t, x, y)$ satisfies
\[
\text{supp } v(t, x, y) \subseteq \{(x, y)|x^2 + y^2 \leq (t - t_0 + C_0 t_0 + 1)^2\}
\]
\[
\triangleq \{(x, y)|x^2 + y^2 \leq (t + C_1)^2\} \quad t \geq t_0,
\]
where the constant $C_1 = (C_0 - 1)t_0 + 1$ is independent of $\varepsilon$.

By Hölder’s inequality and (3.21), for $t \geq t_0$, we have that
\[
Y^2(t) \leq \int_{x^2 + y^2 \leq (t + C_1)^2} (e^y + e^{-y}) \rho v dx dy \int_{\mathbb{R}^2} (e^y + e^{-y}) \rho v^2 dx dy.
\]
Note that
\[
\int_{x^2 + y^2 \leq (t + C_1)^2} (e^y + e^{-y}) \rho v dx dy
\]
\[
\leq \rho^* \int_{|y| \leq t + C_1} (e^y + e^{-y}) \int_{|x| \leq \sqrt{(t + C_1)^2 - |y|^2}} dx dy
\]
\[
\leq C(t + C_1)^{2\frac{1}{p}} \int_{|y| \leq t + C_1} e^{-t + C_1} (e^y + e^{-y}) \sqrt{t + C_1 - |y|} dy
\]
\[
\leq C(t + 1)^{2\frac{1}{p}} e^t,
\]
where the last inequality can be obtained by using the variable transformation $\tau = t - |y|$. Here and afterwards, $C$ denotes a generic positive constant which is independent of $\varepsilon$. Therefore, it follows from (3.3), (3.4), (3.22) and (3.23) that
\[
Y'(t) \geq X(t) + CY^2(t) e^{-t} (t + 1)^{-\frac{1}{2}}
\]

\[ \geq X(0) + \int_{0}^{t} Y(\tau)d\tau + CY^2(t)e^{-t}(t + 1)^{-\frac{1}{2}}, \quad t \geq t_0. \] (3.24)

Let
\[ e^t Z = \int_{0}^{t} Y(\tau)d\tau. \]

As was done for (3.24), it is easy to get from (3.3), (3.4) and (3.27) below that
\[ Z'' + 2Z' \geq e^{-t} X(0) + Ce^{-t}Y^2(t) \left( \int_{x^2 + y^2 \leq (C_0 t + 1)^2} (e^y + e^{-y})\rho dx dy \right)^{-1}. \] (3.25)

In fact, by the finite propagation speed of waves, we also know that the support of solution \( v(t, x, y) \) satisfies that, for all \( t \geq 0 \),
\[ \text{supp } v(t, x, y) \subset \{ (x, y) \mid x^2 + y^2 \leq (C_0 t + 1)^2 \}. \] (3.26)

Thus, by Hölder’s inequality, we also have, for all \( t \geq 0 \), that
\[ Y^2(t) \leq \int_{x^2 + y^2 \leq (C_0 t + 1)^2} (e^y + e^{-y})\rho dx dy \int_{\mathbb{R}^2} (e^y + e^{-y})\rho v^2 dx dy. \] (3.27)

Then (3.27) follows by exactly the same argument as the one for (3.24).

By assumption (2.17), we know that \( X(0) \geq 0 \). Then it follows from (3.25) that
\[ (e^{2t}Z')' \geq 0, \quad t \geq 0. \] (3.28)

By assumption (2.18), we know that \( Z'(0) = Y(0) \geq C\varepsilon \). This means that \( Z'(0) = Y(0) \geq 0 \), so (3.28) implies, for \( t \geq 0 \), that
\[ e^{2t}Z' \geq 0. \]

Therefore for \( t \geq 0 \), we have that
\[ Z' \geq 0, \quad \text{and} \quad Z + Z' \geq Z + \frac{Z'}{2}. \] (3.29)

Also, (3.24) yields that
\[ Z'' + 2Z' \geq e^{-t} X(0) + C(Z + Z')^2(t + 1)^{-\frac{1}{2}}, \quad t \geq t_0. \] (3.30)

Letting \( W = Z' + 2Z \), we finally get from (3.30) and (3.29) that, for \( t \geq t_0 \),
\[ W' \geq CW^2(1 + t)^{-\frac{1}{2}} + e^{-t} X(0) \geq CW^2(1 + t)^{-\frac{1}{2}}. \] (3.31)

We note that for \( t_0 \geq 1 \), we have
\[ W(t_0) = Z'(t_0) + 2Z(t_0) \geq Z(1) = e^{-1} \int_{0}^{1} Y(t)dt \geq CY(0), \]
because from (2.18), we know that \( Y(0) \geq C\varepsilon \). Therefore by (3.31), we know that
\[ \frac{W'}{W^2} \geq \frac{C}{(1 + t)^{\frac{3}{2}}}. \]

Then
\[ W(t) \geq \frac{1}{W(t_0)} - 2C(1 + t)^{\frac{3}{2}}. \]

Therefore, \( W(t) \) will blow up before a time \( C\varepsilon^{-2} \). This is the lifespan estimate (2.19). \( \square \)
4 Proof of Theorem 2.2: Three Dimensional Case

In this section, we will prove Theorem 2.2. In order to do this, instead of the test function $e^y \pm e^{-y}$ used in two dimensions, we introduce the following test function:

$$F(y) = \int_{\omega^1_1 + \omega^2_2} e^{y_1 + y_2 \omega_2} d\sigma.$$  

Test function $F(y)$ is radially symmetric and satisfies the following properties:

$$\begin{align*}
\Delta_y F(y) &= F(y), \\
0 &\leq F(y) \leq C r^{-\frac{4}{3}} e^r, \quad \text{where } r = |y| = \sqrt{y_1^2 + y_2^2}.
\end{align*}$$  

(4.1)

One can refer to [44] for more details regarding the properties of the test function $F(y)$. Based on the test function $F(y)$, we can now prove Theorem 2.2.

Proof of Theorem 2.2 Let

$$X(t) = \int_{\mathbb{R}^2} F(y) \left[ \int_{\Pi_+ (t, y)}^{+\infty} (\rho - \rho_r) dx + \int_{\Pi_0 (t, y)}^{\Pi_+ (t, y)} (\rho - \rho_m) dx \right.$$

$$+ \int_{\Pi_0 (t, y)}^{\Pi_0 (t, y)} (\rho - \rho_m) dx + \int_{-\infty}^{\Pi_+ (t, y)} (\rho - \rho_l) dx \right] dy$$

$$+ (\rho_m - \rho_r) \int_{\mathbb{R}^2} F(y) [\Pi_+ (t, y) - \sigma, t] dy$$

$$+ (\rho_m - \rho_l) \int_{\mathbb{R}^2} F(y) [\sigma - t - \Pi_+ (t, y)] dy,$$

and

$$Y(t) = \int_{|\omega| = 1} \int_{\mathbb{R}^2} \int_{-\infty}^{+\infty} e^{y_1 + y_2 \omega_2} \rho v_\omega dxdyd\sigma,$$

where

$$v_\omega (t, x, y_1, y_2) = \omega_1 v_1 (t, x, y_1, y_2) + \omega_2 v_2 (t, x, y_1, y_2).$$

Here $(u, v_1, v_2)$ represent the velocity.

Multiplying the third and fourth equations in (2.20) by $\omega_1$ and $\omega_2$, respectively, and then adding them together, we come to a new system:

$$\begin{align*}
(\rho)_t + (\rho u)_x + (\rho v_1)_y + (\rho v_2)_y & = 0, \\
(\rho u)_t + (\rho u^2)_x + (\rho uv_1)_y + (\rho uv_2)_y + \rho_x & = 0, \\
(\rho v_\omega)_t + (\rho uv_\omega)_x + (\rho v_1 v_\omega)_y + (\rho v_2 v_\omega)_y + \omega_1 \rho y_1 + \omega_2 \rho y_2 & = 0.
\end{align*}$$  

(4.2)

In a fashion similar to the two dimensional case, by a straightforward computation the same as for the one used in the proof of Lemma 3.1 (we omit the long and tedious details for the sake of brevity), we can establish the following ordinary differential system:

$$X'(t) = Y(t),$$

$$Y'(t) = X(t) + \int_{|\omega| = 1} \int_{\mathbb{R}^2} \int_{-\infty}^{+\infty} e^{y_1 + y_2 \omega_2} \rho v_\omega^2 dxdyd\sigma.$$  

(4.3)

Let

$$e^t Z(t) = \int_0^t Y(\tau) d\tau.$$
Then, by Hölder’s inequality and (4.3), we get that
\[
Z'' + 2Z' = e^{-t}X(0) + e^{-t}&\int_{|\omega|=1} \int_{x^2 + r^2 \leq (t+R)^2} e^{y_1\omega_1 + y_2\omega_2} \rho v_\omega^2 dxdy\sigma \\
\geq e^{-t}X(0) + e^{-t} &Y^2(t) \left( \int_{|\omega|=1} \int_{x^2 + r^2 \leq (t+R)^2} e^{y_1\omega_1 + y_2\omega_2} \rho dxdy\sigma \right)^{-1}.
\]

(4.4)

For the last inequality above we use the properties that \(v_1\) and \(v_2\) are compactly supported and we employ the finite propagation speed. More precisely, by the entropy condition, the shock for the last inequality above we use the properties that \(v\) is sufficiently small, we know that there exists \(R > 0\) large enough, not depending on the data, such that the support of \(v_1\) and \(v_2\) satisfies that \(x^2 + r^2 \leq (t+R)^2\), where \(r^2 = y_1^2 + y_2^2\).

By (2.21), we know that \(X(0) \geq 0\), so (4.4) implies that
\[
(e^{2t}Z')' \geq 0, \quad t \geq 0.
\]

By (2.22), we know further that that \(Z'(0) = Y(0) \geq 0\). Thus it follows from the inequality above that
\[
e^{2t}Z' \geq 0, \quad t \geq 0.
\]

Therefore,
\[
Z' \geq 0, \quad t \geq 0
\]
\[
Z + Z' \geq Z + \frac{Z'}{2}, \quad t \geq 0. \quad (4.5)
\]

Next, we need to estimate the last term in (4.4) for when \(t \geq \tilde{t}_0\) for some fixed time \(\tilde{t}_0\) which is independent of \(\varepsilon\). However, unlike for the two dimensional case, things here have to be done in a different way. By (4.1), we have
\[
\int_{|\omega|=1} \int_{x^2 + r^2 \leq (t+R)^2} e^{y_1\omega_1 + y_2\omega_2} \rho dxdy\sigma \\
= \int_{x^2 + r^2 \leq (t+R)^2} F \rho dxdy \\
\leq C(t + 1)^\frac{1}{2} \int_{r \leq \sqrt{(t+R)^2 - |x|^2}} \rho Fr \sqrt{t + R - r} dr \\
\leq C(t + 1)^\frac{1}{2} e^t \int_{r \leq \sqrt{(t+R)^2 - |x|^2}} e^{-t-R+r} \sqrt{t + R - r} dr \\
\leq C(t + 1)^e, \quad t \geq \tilde{t}_0. \quad (4.6)
\]

Finally, letting \(W = Z' + 2Z\), from (2.21), (4.4) and (4.5), we have that
\[
W' \geq CW^2(1+t)^{-1} + e^{-t}X(0) \geq CW^2(1+t)^{-1}, \quad t \geq \tilde{t}_0. \quad (4.7)
\]

Without loss of the generality, we can assume that \(\tilde{t}_0 \geq 1\). Thus, by (4.5), we have that
\[
W(\tilde{t}_0) = Z'(\tilde{t}_0) + 2Z(\tilde{t}_0) \geq Z(1) = e^{-1} \int_0^1 Y(t) dt \geq CY(0).
\]
By (2.22), we know that there exists a constant $C$ which does not depend on the data such that $Y(0) \geq C\varepsilon$. Then it follows from (4.7) that

$$W(t) \geq \frac{1}{\frac{1}{W(t_0)} - \ln \frac{1+t}{1+t_0}} \geq \frac{1}{C\varepsilon - \ln \frac{1+t}{1+t_0}}.$$ 

Therefore, $W(t)$ becomes infinite before the time $\exp(C\varepsilon^{-1})$. □

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