MULTIPLE ARRANGEMENTS

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Abstract. This paper surveys the theory of multiple packings and coverings. The study of multiple arrangements started in the 60s of the last century, and it was restricted mostly to lattice arrangements on the plane or of general arrangements of balls. We emphasize two new topics which were intensively investigated recently: decomposition of multiple coverings into simple coverings and characterization of multiple tilings.

We say that a family of sets is a \( k \)-fold packing if every point of the space belongs to the interior of at most \( k \) sets. Quite analogously, we say that a family of sets forms a \( k \)-fold covering if every point of the space belongs to the closure of at least \( k \) sets. Let \( \delta_k(K) \) and \( \vartheta_k(K) \), respectively, denote the densities of the densest \( k \)-fold packing and the thinnest \( k \)-fold covering of the space with congruent copies of the convex body \( K \). Similarly we use the notation \( \delta^T_k(K) \), \( \vartheta^T_k(K) \), \( \delta^L_k(K) \) and \( \vartheta^L_k(K) \) for the optimum densities of the corresponding \( k \)-fold translative and lattice arrangements (compare the corresponding definitions of \( \delta \), \( \delta_T \), \( \delta_L \), \( \vartheta \), \( \vartheta_T \) and \( \vartheta_L \) above).

1. Multiple arrangements on the plane

The literature on multiple packing and covering is relatively extensive, and it mostly deals with arrangements of congruent copies of the circular disk \( B^2 \). The values of \( \delta_k(B^2) \) and \( \vartheta_k(B^2) \) are not known for any \( k > 1 \). We know that \( \delta^2(B^2) > 2\delta^1(B^2) \) and \( \vartheta^2(B^2) < 2\vartheta^1(B^2) \) as was shown by Heppes [1957] and Danzer [1960], respectively. G. Fejes Tóth [1976a] established the bounds

\[
\delta^k(B^2) \leq k \frac{\pi}{6} \cot \frac{\pi}{6k} \quad \text{and} \quad \vartheta^k(B^2) \geq k \frac{\pi}{3} \csc \frac{\pi}{3k}.
\]

Observe that \( \frac{\pi}{6} \cot \frac{\pi}{6k} \) and \( \frac{\pi}{3} \csc \frac{\pi}{3k} \) are equal to the density of a disk with respect to the circumscribed and inscribed regular \( 6k \)-gon. For \( k = 1 \) these inequalities are sharp as they coincide with Thue’s and Kershner’s theorems.

Bolle [1976] proved that there are positive constants \( c_i \) such that

\[
k - c_1 k^{\frac{\pi}{6}} \leq \delta^k_L(B^2) \leq k - c_2 k^{\frac{\pi}{6}}
\]

and

\[
k + c_3 k^{\frac{\pi}{6}} \leq \vartheta^k_L(B^2) \leq k + c_4 k^{\frac{\pi}{6}}
\]
and showed in [1984] that the exponent $\frac{1}{4}$ is best possible in these inequalities. In [1989] he proved that for convex disks $K$ with piecewise twice differentiable boundary there are positive constants $c(K)$ and $C(K)$ such that

$$\delta_k^L(K) \geq k - c(K)k^{\frac{1}{2}}$$

and

$$\vartheta_k^L(K) \leq k + C(K)k^{\frac{1}{2}}.$$  

Moreover, for a polygon $P$ the stronger inequalities

$$\delta_k^L(P) \geq k - c(P)k^{\frac{1}{3}}$$

and

$$\vartheta_k^L(P) \leq k + C(P)k^{\frac{1}{3}}$$

hold.

The exact values of $\delta_k^L(B^2)$ have been found for $k \leq 10$ (see Heppes [1959], Blundon [1963], Bolle [1970], Yakovlev [1983], Temesvári [1991] and Temesvári and Végh [1998]). The values of $\delta_k^L(B^2)$ are known for $k \leq 8$ (see Blundon [1957], Haas [1975], Subak [1960] and Temesvári [1984a, 1992a, 1992b, 1992c]. Linhart [1983] described an algorithmic approach for approximating the values of $\delta_k^L(B^2)$ with arbitrarily high accuracy. Elaborating on results by Yakovlev [1984] Temesvári, Hörváth and Yakovlev [1987] described a method for finding the densest $k$-fold lattice packing with circles. They reduced this task to a finite number of optimization problems, each over an explicitly given compact domain. A similar method for the thinnest $k$-fold lattice covering with circles was given by Temesvári [1988].

For a triangle $T$, Sriamorn [2014] determined $\delta_k^L(T)$ and $\vartheta_k^L(T)$ for all $k$. Moreover, Sriamorn [2016] showed that

$$\delta_k^L(T) = \delta_1^L(T) = \frac{2k^2}{2k + 1}$$

and Sriamorn and Wetayawanich [2015] showed that

$$\vartheta_k^L(T) = \vartheta_1^L(T) = \frac{2k + 1}{2}$$

for all $k$.

It is worth mentioning that $\delta_k^L(B^2) = k\delta_1^L(B^2)$ for $k = 2, 3$ and $4$, and also $\vartheta_k^L(B^2) = 2\vartheta_1^L(B^2)$. These equalities for the very same multiplicities have been extended to an arbitrary centrally symmetric convex disk in place of the circle by Dumir and Hans-Gill [1972a, 1972b] and G. Fejes Tóth [1984a]. The equality $\vartheta_k^L(B^2) = 2\vartheta_1^L(B^2)$ was further generalized by Temesvári, who proved in [1984a] that the density of a 2-periodic double covering by circles is at most $2\vartheta_1^L(B^2)$, and in [1994d] proved the analogous result for centrally symmetric convex disks. Recall from XX that an $m$-periodic arrangement is the union of $m$ translates of a lattice arrangement.

2. Decomposition of multiple arrangements

The equalities $\delta_k^L(K) = 3\delta_1^L(K)$ and $\delta_k^L(K) = 4\delta_1^L(K)$ for centrally symmetric disks $K$ were derived by noticing that every 3-fold lattice packing by such a disk is the union of three simple (1-fold) packings, and every such 4-fold packing is the union of two 2-fold packings. This observation belongs to the topic concerning
decompositions of multiple arrangements into simple ones, problems and results that focus on the combinatorial structure of such arrangements. Research in this direction was initiated by Pach [1985]. He proved, among other things, that every 2-fold packing with positively homothetic copies of a convex disk can be decomposed into four (simple) packings. For coverings, he made the conjecture that for every convex disk $K$ there exists a minimal natural number $m(K)$ such that every $m(K)$-fold covering of the plane by translates of $K$ can be decomposed into two coverings. In [1986] he proved this conjecture for centrally symmetric polygons. New interest arose in the topic after Tardos and Tóth [2007] proved the conjecture for triangles. Soon after, Pálvölgyi and Tóth [2010] proved the conjecture for every convex polygon $P$. Unfortunately, the number $m(K)$ increases with the number of sides of $P$, thus the attempt to extend the result to all convex disks through polygonal approximation fails. Still, it came as a surprise when Pálvölgyi [2013] (see also Pach and Pálvölgyi [2016]) disproved Pach’s conjecture by showing that it does not hold for the circle. For subsequent developments on decomposition of multiple arrangements we refer the reader to the survey article of Pach, Pálvölgyi and Tóth [2013].

3. Multiple arrangements in space

The densest 2-fold lattice packing and the thinnest 2-fold lattice covering of balls in three dimensions were determined by Few and Kanagasabapathy [1969] and Few [1967], respectively. Purdy [1973] constructed a threefold lattice packing of balls which he conjectured to be of maximum density. He supported the conjecture by proving that it provides a local maximum of the density among threefold lattice packings of balls.

Adapting Blichfeldt’s idea, Few [1964] gave the following upper bound for the $k$-fold packing density of the $n$-dimensional ball:

$$\delta_k(B^n) \leq (1 + n^{-1})(n + 1)^n k^{(k-1)n+1}.\]

This is better than the trivial bound $k$ only for large values of $n$ compared to $k$. By a further elaboration on the same idea for $k = 2$, Few [1968] obtained the stronger inequality

$$\delta_2(B^n) \leq \frac{4}{3} (n + 2) \left(\frac{2}{3}\right)^{n/2}.$$

G. Fejes Tóth [1979] gave a non-trivial upper bound for $\delta_k(B^n)$, as well as a non-trivial lower bound for $\vartheta_k(B^n)$ for every $n$ and $k$.

For multiple lattice arrangements of balls, Bolle [1979, 1982] established sharper estimates. He proved that there are positive constants $c_n$ and $C_n$ such that

$$\frac{\delta_k(B^n)}{k} \leq 1 - c_n k^{\frac{n+1}{2}} \quad \text{and} \quad \frac{\vartheta_k(B^n)}{k} \geq 1 + c_n k^{\frac{n+3}{2n}}$$

when $n \equiv 1 \pmod{4}$ and

$$\frac{\delta_k(B^n)}{k} \leq 1 - c_n k^{\frac{n+3}{2n}} \quad \text{and} \quad \frac{\vartheta_k(B^n)}{k} \geq 1 + c_n k^{\frac{n+3}{2n}}$$

when $n \equiv 1 \pmod{4}$.

Extending the result of Schmidt [1961] to multiple arrangements, Florian [1978a] proved that $\delta_k(K) < k$ and $\vartheta_k(K) > k$ for every smooth convex body $K$ without establishing a concrete bound.
By a Blichfeldt-type argument Few [1964] proved
\[ \delta^k(B^n) \geq \delta(B^n) \left( \frac{2k}{k + 1} \right)^{n/2}. \]
In [1971] Few studied the multiplicity of partial coverings of space, and, as an application of a general theorem, obtained a better lower bound for \( \delta^k(B^n) \) for large values of \( k \) and \( n \).

Groemer [1986a] proved lower bounds for the \( k \)-fold lattice packing density of a convex body \( K \) involving the intrinsic volumes of \( K \). From his results follows the existence of positive constants \( c_n \) such that
\[ \delta^k_L(K) \geq k - c_n k^{(n-1)/n} \]
for every convex body \( K \in \mathbb{E}^n \).

Cohn [1976] proved that
\[ \vartheta^k_L(K) < [(k + 1)^{1/n} + 8n]^n = k(1 + O(n^2 k^{-1/n})) \]
as \( k \to \infty \) for every \( n \)-dimensional convex body \( K \).

The best known upper bound for \( \vartheta^k(K) \) is due to Naszódi and Polyanskii [2018] who, improving slightly on an earlier result by Frankl, Nagy and Naszódi [2018], proved that
\[ \vartheta^k(K) \leq 3.153(1 + o(1)) \max\{n \ln n, k\} \]
for every convex body \( K \in \mathbb{E}^n \).

Blachman and Few [1963] gave bounds for the density of multiple packings of spherical caps. Also L. Fejes Tóth [1966a], Galiev [1996], Blinovsky [1999] and Blinovsky and Litsyn [2011] investigated multiple ball packings in spherical spaces.

4. Multiple tiling

A system of bodies forms a \( k \)-fold tiling if each point of the space is covered exactly \( k \) times, except perhaps the boundary points of the bodies. There are centrally symmetric polytopes that admit a translational \( k \)-fold tiling, but no simple tiling. The simplest example is perhaps the regular octagon of side-length 1, whose translates by the unit square lattice form a 7-fold tiling.

Bolle [1994] proved that a convex polygon that admits a \( k \)-fold tiling of the plane by translations is centrally symmetric. He also gave a characterization of those convex polygons that admit a \( k \)-fold lattice-tiling. Kolountzakis [2021] gave an algorithm which decides for a centrally symmetric convex polygon if it can tile the plane by translations at some level. His algorithm runs in polynomial time in the number of sides of the polygon. Yang and Zong [2019, 2021] characterized those convex polygons that admit two-, three-, four- or five-fold translational tiling. Only parallelograms and centrally symmetric hexagons admit a two-, three- or four-fold translational tiling. There are two more classes of polygons admitting five-fold tilings: the affine images of a special octagon and of a decagon.

Gravin, Robins and Shiryaev [2012] proved that if translates of a convex polytope form a \( k \)-fold tiling of \( \mathbb{E}^n \), then it is centrally symmetric and its facets are centrally symmetric as well. This generalizes a theorem of Minkowski [1897] concerning simple tilings. For the three-dimensional case this means that only zonotopes admit a translational \( k \)-fold tiling. For rational polytopes Gravin, Robins
and Shiryaev also proved the converse of their above mentioned theorem: Every rational polytope in \( E^n \) that is centrally symmetric and has centrally symmetric facets admits a \( k \)-fold lattice tiling for some positive integer \( k \). For zonotopes, this was proved earlier by Groemer [1978].

A quasi-periodic set is a finite union of translated lattices, not necessarily of the same lattice. Kolountzakis [2000] proved that a \( k \)-fold tiling by translates of a convex polygon other than a parallelogram is quasi-periodic. Gravin, Kolountzakis, Robins and Shiryaev [2013] proved an analogous theorem for the three-dimensional case: A \( k \)-fold tiling by translates of a polytope that is not a two-flat zonotope is quasi-periodic. A two-flat zonotope is the Minkowski sum of two 2-dimensional symmetric polygons one of which may degenerate into a single line segment.

Gravin, Robins and Shiryaev [2012] raised the problem whether the following generalization of the Venkov-McMullen theorem holds: If translates of a polytope \( P \) form a \( k \)-fold tiling of \( E^n \), then \( P \) also admits an \( m \)-fold lattice tiling for some, possibly different, multiplicity. The two-dimensional case of this conjecture was confirmed independently by Liu [2021] and Yang [2019]. A further step in the direction of proving the conjecture was made by Chan [2015], who proved it for certain quasi-periodic \( k \)-fold tilings. Lev and Liu [2000] gave a characterization of those polytopes in \( E^n \) that tile with some multiplicity \( k \) by translations along a given lattice.

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