Nonassociative Algebras and Nonperturbative Field Theory for Hierarchical Models

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Abstract

Hierarchical renormalization group (RG) transformations are related to nonassociative algebras. These algebras serve as a new basic tool for a rigorous treatment of global RG flows and the search of nontrivial infrared fixed points. Convergent expansion methods are presented and analyzed by the introduction of algebra-norms. It is shown that the infrared fixed points can be investigated by solving a quadratic equation with a finite number of unknowns. A continuous manifold of two-dimensional periodic nontrivial fixed points is given in terms of Theta-functions. The local Borel-summability of the $\epsilon$-expansion for $l_*$-well fixed points, $l_* \in \{2, 3, \ldots, \}$ is shown by using algebraic methods.

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In the renormalization group $[W71, WK74]$ approach one thinks of a Euclidean quantum field theory in terms of its renormalization group flow. Technically the idea is to split
an infinite dimensional path integral into finite dimensional portions corresponding to fluctuations on increasing length scales which can be performed stepwise one at a time. A single step then defines a transformation of a bare measure of function space into a renormalized one. The most important problem in this approach is to find the fixed points of this renormalization group transformation and to study the flow in their vicinity.

In constructive quantum field theory [GJS7], [R91] a rigorous version of the renormalization group plays the role of an organizing principle of convergent expansions. In particular polymer expansions [B84], [GK71] can be generated from a renormalization group equation [BY90]. The existing technology is however limited to models where one has a small expansion parameter at hand. Examples are asymptotically free theories where the expansion parameter is the running coupling constant [GK80]. A number of important problems remain out of reach of presently available methods. Among these are the infrared behavior of the two dimensional $O(N)$-invariant nonlinear $\sigma$-model [GK86], [PR91], [I91] and nonabelian gauge theories in four dimensions [B88a], [B88b].

The contribution of this report to the subject is a formulation of the renormalization group in terms of certain nonassociative algebras which will be called renormalization group algebras. The idea is to define a bilinear composition of measures on function space which equips it with the structure of a nonassociative algebra such that a renormalization group transformation becomes an algebraic operation. In particular the fixed point equation becomes algebraic in terms of this composition. We believe that a solution to the above problems can be given in terms of a structural theory of these algebras.

Our general goal is to device algebraic methods for nonperturbative studies of quantum fields. Nonassociative products are also underlying the tree formulas in the renormalization group approach of [G85]. In a nonassociative algebra there are different ways of multiplying a given number of elements. The different products are labelled by trees. For our renormalization group algebras these trees correspond to the ones of [G85].

In the case of hierarchical models this product is a convolution composed with a dilatation. The advantage of the algebraic point of view is that it disentangles algebraic and combinatoric aspects of a renormalization group analysis from analytic ones. In this report we present a general theory of hierarchical models based on nonassociative algebras. The formulation in terms of polymer algebras for the full models is under investigation.

### 0.1 Hierarchical Models

In this report we will restrict our attention to scalar hierarchical models and discrete block spin transformations. A generalization to $N$-component models is directly possible. It will be omitted for the sake of brevity. We expect also hierarchical gauge theories to be suited for a similar treatment.

Hierarchical models have been investigated by many authors [B72], [BS73], [CE77], [CE78], [D69a], [D69b], [P84], [CZ90], [P93], [PW91] in various setups. Here we will consider discrete hierarchical renormalization group transformations for Euclidean scalar lattice fields using
We start from a Euclidean scalar lattice field as a perturbation of a massless Gaussian measure. Let $\Lambda = \mathbb{Z}^d / L^N \mathbb{Z}^d$ be a unit lattice torus with side length $L$. Let $\mathcal{H}$ be the space of real valued functions $\Phi : \Lambda \to \mathbb{R}$ with mean zero and scalar product $(\Phi, \Psi) = \sum_{x \in \Lambda} \Phi(x)\Psi(x)$. Elements of $\mathcal{H}$ are called lattice fields. The space $\mathcal{H}'$ is given by $\sum_{x \in \Lambda} \Phi(x)$, and its elements are called lattice fields. A Euclidean scalar lattice field theory on $\Lambda$ is then defined by a massless Gaussian measure $d\mu(-\triangle)^{-1}(\Phi)$ on $\mathcal{H}'$ together with a local interaction $V(\Phi) = \sum_{x \in \Lambda} V(\Phi(x))$ such that $Z(\Phi) = e^{-V(\Phi)}$ is integrable with respect to $d\mu(-\triangle)^{-1}(\Phi)$. It is called polynomial if $V(\Phi)$ is a polynomial. The standard example is

$$V(\Phi) = \kappa + \mu^2 \Phi^2 + \lambda \Phi^4.$$  \hfill (1)

The parameter $\kappa$ is called bare vacuum energy, the parameters $\mu^2$ and $\lambda$ are called bare coupling constants. The massless covariance is given by its kernel

$$(-\triangle)^{-1}(x, y) = \frac{1}{|\Lambda|} \sum_{p \in \Lambda \setminus \{0\}} e^{ip(x-y)} \left( 2 \sum_{\mu=1}^d (1 - \cos p_{\mu}) \right)^{-1}. \hfill (2)$$

The problem is to construct and analyze the thermodynamic limit $N \to \infty$ of correlation functions in such a theory. The block spin renormalization group is a tool to perform such an analysis. A technical complication is the treatment of nonlocal terms in the effective actions. Hierarchical models are designed such that the effective actions remain local. They are obtained by replacing the massless covariance by a hierarchical version

$$(-\triangle)_{\text{hier}}^{-1}(x, y) = \sum_{M=0}^{N-1} L^{(2-d)M} \gamma \delta_{[L^{-M}x],[L^{-M}y]}.$$  \hfill (3)

Here $[L^{-M}x]$ denotes the integer part of $L^{-M}x$. $\gamma$ is a positive parameter. The full and the hierarchical model share a similar critical behavior. However in the hierarchical model translation invariance is manifestly broken and Osterwalder-Schrader positivity does not hold.

Hierarchical models are ideally suited for the renormalization group approach. Let us recall the definition of a hierarchical block spin transformation. Consider a second lattice $\tilde{\Lambda} = \mathbb{Z}^d / L^{N-1} \mathbb{Z}^d$ reduced in size by a factor of $L$. This second lattice is a rescaled block lattice. Let $\tilde{\mathcal{H}}$ be the corresponding space of functions $\tilde{\Phi} : \tilde{\Lambda} \to \mathbb{R}$. Let $C : \mathcal{H} \to \tilde{\mathcal{H}}$ be the linear operator given by $C\Phi(\tilde{x}) = \sum_{x \in \Lambda} C(\tilde{x}, x)\Phi(x)$ with

$$C(\tilde{x}, x) = \begin{cases} 
L^{-d} & \text{if } \tilde{x}^\mu \leq [L^{-1}x^\mu] < \tilde{x}^\mu + 1, \\
0 & \text{else.}
\end{cases} \hfill (4)$$

$C$ is the block average operator. Let $C^T : \tilde{\mathcal{H}} \to \mathcal{H}$ be the transpose of $C$. 

the terminology of [KW86, KW91]. Let us recall the construction of hierarchical models.
Let us rename the hierarchical massless covariance \(-\triangle \)\textsuperscript{\textit{hier}} on \(\Lambda\) into \(v\) and denote by \(\bar{v}\) the corresponding one on \(\bar{\Lambda}\). Then we have a splitting
\[
v(x, y) = L^{2+d} (C^T \bar{v} C)(x, y) + \Gamma(x, y),
\]
where \(\Gamma\) is given by
\[
\Gamma(x, y) = \gamma \delta_{x,y}.
\]
We are led to consider the renormalization group transformation given by
\[
\bar{Z}(\Phi) = \int d\mu(\zeta) Z(L^{1+\frac{d}{2}} C^T \phi + \zeta).
\]
The operator \(\Gamma\) is called the fluctuation covariance. Its most important property is that it is ultra local. Therefore the renormalization group transformation preserves the locality of an interaction. Suppose that \(Z(\Phi)\) factorizes into a product \(\prod_{x \in \Lambda} Z(\Phi(x))\). Then so does the effective interaction \(\bar{Z}(\Phi)\) into \(\prod_{\bar{x} \in \bar{\Lambda}} \bar{Z}(\Phi(\bar{x}))\). Note however that the effective interaction of a polynomial theory is not anymore polynomial. For a local theory the hierarchical renormalization group transformation is equivalent to the nonlinear transformation
\[
\bar{Z}(\Phi) = \left( \int d\mu_\gamma(\zeta) Z(L^{1-\frac{d}{2}} \phi + \zeta) \right)^{L^d}.
\]
of the local interaction Boltzmann factors. This transformation is a transformation on functions of a single variable. Remarkably it depends on the lattice geometry only through the dimension parameter \(d\) and the block length parameter \(L\). Both can be taken real valued extending hierarchical models to intermediate dimensions. In the following we will let \(L\) depend on \(d\) through \(L^d = 2\). This corresponds to the case when every block contains two lattice points. In the algebraic formulation the fixed point equation then becomes quadratic. The case when \(L^d\) is an integer larger than two also fits nicely into our scheme. In this case the fixed point equation becomes an algebraic equation involving higher powers. The critical properties are expected to be independent of \(L\) but do depend sensitively on the dimension \(d\). Let us further trade \(d\) for \(\beta = L^{1-\frac{d}{2}}\). Substituting \(\bar{Z}\) by \(\bar{Z}^{L^d}\) we then obtain
\[
\bar{Z}(\Phi) = \int d\mu_\gamma(\zeta) Z(\beta \phi + \zeta)^2.
\]
Let us then define a renormalization group operator \(\mathcal{R}\) depending on two parameters \(\beta\) and \(\gamma\) by
\[
\mathcal{R}_{\beta,\gamma} Z(\Phi) = \int d\mu_\gamma(\zeta) Z(\beta \phi + \zeta)^2.
\]
This will be the final form of the hierarchical renormalization group transformation here. Let us also introduce the notation
\[
L_X \mathcal{R}(Y)(\Phi) = 2 \int d\mu_\gamma(\zeta) X(\beta \phi + \zeta) Y(\beta \phi + \zeta)
\]
for the linearization of $\mathcal{R}$ at $X$.

We will consider two different choices for $\beta$ and $\gamma$. The first case is the one explained above with $\beta = 2^{2-d}$ and $\gamma = \frac{2}{2-d}$. The second case is related to the first by a transformation of the space of functions factorizing out a quadratic fixed point. This transformation leaves the form of the renormalization group transformation invariant and only changes the values of the parameters. In the second case $\beta$ is replaced by $L^{-2} \beta$ and $\gamma$ by $L^{-2} \gamma$. We will always assume that $d > 2$ unless it is explicitly stated differently.

### 0.2 Ultraviolet Fixed Point

One fixed point of eq. (10) is immediately found. It is simply

$$Z^{(0)}(\Phi) = 1.$$  \hspace{1cm} (12)

It corresponds to the pure massless field. This fixed point will be called ultraviolet fixed point. The linearization of $\mathcal{R}$ at this trivial fixed point is given by

$$L_{Z(0)} \mathcal{R}(Z)(\Phi) = 2 \int d\mu(\zeta) Z(\beta \Phi + \zeta).$$  \hspace{1cm} (13)

It can be diagonalized exactly. The eigenfunctions are normal ordered products

$$: \Phi^n : = \left(\frac{\gamma'}{2}\right)^n H_n \left(\frac{\Phi}{\sqrt{2\gamma'}}\right)$$  \hspace{1cm} (14)

given by rescaled Hermite polynomials. The normal ordering covariance is given by $\gamma' = (1 - \beta^2)^{-1} \gamma$. The corresponding eigenvalues are

$$2\lambda^{(0)}_n = 2^{1+n2^{-d}}.$$  \hspace{1cm} (15)

In the renormalization group language eigenfunctions with $\lambda^{(0)}_n > 0$ are called relevant, those with $\lambda^{(0)}_n = 0$ marginal, and the others irrelevant.

The number of relevant eigenfunctions depends on $d$. Let us restrict our attention to even eigenfunctions. One observes a sequence of critical dimensions $d_2 > d_3 > d_4 > \ldots > 2$ given by

$$d_n = \frac{2n}{n-1}.$$  \hspace{1cm} (16)

Below the critical dimension $d_n$ the eigenfunction $: \Phi^{2n} : \gamma'$ becomes relevant whereas it is irrelevant above $d_n$. In particular above $d = 4$ only 1 ($n = 0$) and $: \Phi^2 : \gamma'$ ($n = 1$) are relevant.

We think of the subspace spanned by the relevant eigenfunctions as a tangent space to an unstable manifold at the fixed point. Imagining normal coordinates points in a neighbourhood of the fixed point, points of the tangent space can be thought of as points on the manifold itself. The dimension of the unstable manifold of the ultraviolet fixed point changes as the dimension is lowered from the region above four.
### 0.3 High Temperature Fixed Point

A second fixed point is not hard to find either. It is given by

\[ Z^{(1)}(\Phi) = e^{-V^{(1)}(\Phi)} \]  

with  

\[ V^{(1)}(\Phi) = \frac{a}{2} \Phi^2 + b, \]  

where \( a = (L^2 - 1)/\gamma L^d \). It is called high temperature fixed point. The linearized transformation at the high temperature fixed point can also be diagonalized exactly. The eigenfunctions are given by

\[ :\Phi^n : \gamma'' Z^{(1)}(\Phi) \]  

with \( \gamma'' = (1 - L^{-2} \beta^2)^{-1} L^{-2} \gamma \). The corresponding eigenvalues are  

\[ L^{\lambda^{(1)}} = 2^{1-n+\frac{2d}{2d}}. \]  

Let us again consider even eigenfunctions only. We see that all eigenfunctions with \( n > 1 \) are irrelevant for the whole range \( d > 2 \) we are considering here. In this sense the high temperature fixed point is stable.

Let us remark that at any fixed point the fixed point itself is a relevant eigenfunction with eigenvalue \( \lambda = 1 \). This eigenfunction is called the volume eigenfunction. A perturbation by it does not change physical properties since it corresponds to a trivial rescaling.

For \( d \leq 4 \) there are no further fixed point given by functions \( Z(\Phi) \) which are integrable with respect to the Gaussian measure \( d\mu_\gamma(\Phi) \).

Not all integrable measures are however driven into the mentioned two fixed points upon iteration of renormalization group transformations. Noncritical theories in the unbroken phase are attracted by the high temperature fixed point. Those in the broken phase approach a singular fixed point in form of an infinitely deep double well. This observation suggests to widen the idea of fixed points to a broader class in order to incorporate fixed points which only exist asymptotically or do not correspond to integrable functions. We will not pursue this line of thoughts here but would like to mention that the notion of nonexistence of other fixed points refers to a certain class of admissible functions.

### 0.4 Nontrivial Fixed Points

For \( d \geq 4 \) no further fixed point exist besides the ultraviolet and the high temperature fixed point. As the dimension is lowered new fixed points appear sequentially at the critical dimensions \( d_n \).

The first threshold is \( d_2 = 4 \). Below four dimensions a nontrivial fixed point \( Z^{(2)}(\Phi) \) with double well shaped potential \( V^{(2)}(\Phi) \) exists. This fixed point is called infrared fixed point. At \( d = 3 \) it has been rigorously constructed in [KW86, KW91].
The next threshold is \( d_3 = 3 \) at which \( Z^{(3)}(\Phi) \) appears and so on. Below \( d_n \) a fixed point \( Z^{(n)}(\Phi) \) exists. The potential \( V^{(n)}(\Phi) \) has the shape of an \( n \)-well. The linearized transformation at \( Z^{(n)}(\Phi) \) has \( n \) relevant eigenfunctions including a trivial volume eigenfunction. In particular the infrared fixed point has two relevant eigenfunctions. They have not been calculated exactly. Numerical results can be found in [PPW94]. The corresponding eigenvalues are related to critical exponents. The nontrivial fixed point \( Z^{(n)}(\Phi) \) approaches \( Z^{(0)}(\Phi) \) as \( d \) goes to \( d_n \) from below. In this sense all nontrivial fixed points bifurcate from the ultraviolet fixed point as the dimension parameter is varied.

An interesting question is that of critical lines inbetween fixed points. The correlation length at a fixed point is either zero or infinite. A fixed point is called critical if the correlation length is infinite. The ultraviolet fixed point and all nontrivial fixed points are critical whereas the high temperature fixed point is noncritical. The critical theories form a submanifold within the set of all theories which contains the critical fixed points. On this critical manifolds the renormalization group flow always ends in a critical fixed point. Consider for instance the situation inbetween three and four dimensions where there are only two critical fixed points. The two fixed points are connected by a critical line of critical theories. Tangent to this line at the ultraviolet fixed point is the relevant eigenfunction \( \Phi^4 : \gamma \). The flow on this critical line goes from the ultraviolet to the infrared fixed point. In general for \( d_{n+1} \leq d \leq d_n \) there originate critical lines from \( Z^{(0)}(\Phi) \) to all fixed points \( Z^{(n)}(\Phi), \ldots, Z^{(2)}(\Phi) \) corresponding to the different relevant eigenfunctions of the linearized transformation at the ultraviolet fixed point. Note that there is also a line from \( Z^{(0)}(\Phi) \) to \( Z^{(1)}(\Phi) \) but the theories on this line are noncritical. There are further critical lines from \( Z^{(n)}(\Phi) \) to \( Z^{(n-1)}(\Phi), \ldots, Z^{(2)}(\Phi) \) corresponding to the unstable directions at the nontrivial fixed point \( Z^{(n)}(\Phi) \). Recall that the dimension of the unstable manifold of \( Z^{(n)}(\Phi) \) is \( n \)-dimensional. Excluding one direction corresponding to the volume eigenfunction and one yielding a line of noncritical theories leading to the high temperature fixed point we obtain a consistent picture.

A rigorous construction of all nontrivial fixed points has only been given within the infinitesimal version of the hierarchical renormalization group in the article of [F87]. A construction for the discrete case in the sense of [KW86, KW91] is still missing. The question of critical lines has so far only been investigated using heuristic methods. Most prominent among these is the \( \epsilon \)-expansion. The nontrivial fixed point \( Z^{(n)}(\Phi) \) and the linearization around it can be studied by \( \epsilon \)-expansion at \( d = d_n - \epsilon \). We will show in detail how this \( \epsilon \)-expansion can be done in the discrete hierarchical case.

### 0.5 Renormalization Group Algebra

For two functions \( X(\Phi) \) and \( Y(\Phi) \) we define a bilinear composition depending on two parameters \( \beta \) and \( \gamma \) by

\[
: X \times_{\beta, \gamma} Y : \gamma (\Phi) = \int d\mu_{\gamma(1-\beta^2)}(\zeta) : X :\gamma (\beta \Phi + \zeta) : Y :\gamma (\beta \Phi + \zeta) . \tag{21}
\]
This composition is commutative but not associative if $\beta \neq 1$. It becomes associative when $\beta = 1$. We call this operation the renormalization group product. In terms of it the fixed point equation takes the form

$$Z \times_{\beta, \gamma} Z(\Phi) = Z(\Phi).$$

(22)

Let us expand $X$ in terms of monomials

$$X(\Phi) = \sum_{n=0}^{\infty} X_n \Phi^{2n}.$$  

(23)

Again we consider only even functions. In terms of the expansion coefficients we find

$$(X \times_{\beta, \gamma} Y)_l = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \beta^{2l} \gamma^{n+m-l} c_{lm}^{mn} X_m Y_n.$$  

(24)

The $c_{lm}^{mn}$ are integer valued coefficients given by a decomposition formula for normal ordered products. This equation expresses the renormalization group product in the basis given by normal ordered products. We see that the dependence on $\gamma$ is trivial. It can be absorbed introducing

$$\tilde{X}_n = \gamma^n X_n.$$  

(25)

In terms of the rescaled coefficients we find

$$(\tilde{X} \times_{\beta} \tilde{Y})_l = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \beta^{2l} c_{lm}^{mn} \tilde{X}_m \tilde{Y}_n.$$  

(26)

The renormalization group product will be analyzed in this form below. In particular the fixed point equation becomes a quadratic equation involving infinitely many variables given by

$$(\tilde{Z} \times_{\beta} \tilde{Z})_l = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \beta^{2l} c_{lm}^{mn} \tilde{Z}_m \tilde{Z}_n.$$  

(27)

It will turn out that this representation is well adapted to an algebraic study of fixed points and linearizations around them. Let us remark that the product eq. (21) becomes associative when $\beta = 1$. But the covariance on the rhs of eq. (21) $\gamma(1 - \beta^2)$ becomes zero in this case.

0.6 Norms and Renormalization Group Algebra

We will consider a number of different norms in our analysis. Norms serve several purposes. Consider for instance an approximate study of the renormalization group flow of a hierarchical model. At the beginning we have to truncate the system to a finite number of degrees of freedom. The result will only be a good approximation if we can show that the
neglected remainders are small. This is most efficiently done in terms of norm estimates. In some sense the idea is to reduce the treatment of large fields to norm estimates.

Norm estimates also give information about invariant subspaces under renormalization group transformations. For instance with a suitable norm it can be shown that the flow converges to zero if the initial interaction is sufficiently small.

A norm $\| \cdot \|$ is called an algebra norm if

$$\| X \times Y \| \leq \| X \| \| Y \|. \quad (28)$$

We immediately find that a fixed point has to satisfy $\| Z \| \geq 1$.

Let us remark that a linear function $\omega$ from the space of effective interactions to $\mathbb{C}$ is called a weight if it has the property that $\omega(X \times Y) = \omega(X) \omega(Y)$. In this case fixed points would obey $\omega(Z) = 1$ and the subspace with unit weight would be invariant. Unfortunately, weights only exist for $\beta = 1$. For certain classes of nonassociative algebras in connection with genetic bases weights play an important rôle [B66].

Norms replace weights in the nonassociative case. Weights are nevertheless useful since they serve to construct certain prenorms in our case.

An algebra together with an algebra norm defines upon completion a Banach algebra. Banach algebras are very well studied objects in mathematics in the associative case.

A nice application of norm estimates is the determination of the large field behavior of the potential at the infrared fixed point [KW91]. A new application that will be given below is a proof of local Borel summability of the $\epsilon$-expansions for the nontrivial fixed points.

### 0.7 $\epsilon$-Expansion

The $\epsilon$-expansion is as old as the renormalization group [WK74]. We will present a version for the hierarchical case which does not rely on perturbation theory, see also [CE77]. The idea is to expand

$$Z^{(n)}(\Phi) = \sum_{k=0}^{\infty} Z^{(n)}_k(\Phi) \epsilon^k \quad (29)$$

at $d = d_n - \epsilon$. Numerical studies show that low order approximations already give the fixed points to a reasonable accuracy [ZJS93, PPW94]. In terms of the representation by normal ordered products the $\epsilon$-expansion will be written into recursive form which can be evaluated using computer algebra. The $\epsilon$-expansion for $Z^{(n)}(\Phi)$ does not converge. It will however be shown to be locally Borel summable.

It is conceivable that at least in the hierarchical case an $\epsilon$-expansion with remainder can be turned into a rigorous construction of fixed points. Again remainder terms will have to be estimated using norms.
0.8 Critical Dimensions and Renormalization Group Algebra

The critical dimensions have been introduced as thresholds where eigenfunctions of the linearized renormalization group transformation at the ultraviolet fixed point become relevant. They coincide with the critical dimensions at which new fixed points bifurcate from the ultraviolet fixed point. This can be seen by the following reasoning. Consider the fixed point equation $Z \times_\beta Z = Z$ and put $Z = Z_0 + H$ with $Z_0$ the high temperature fixed point. For $Z$ to be fixed point the remainder has to satisfy $(I - 2Z_0) \times_\beta H = H \times_\beta H$. Here $I$ is a unit element adjoined to the algebra. Treating $H$ as a perturbation of $Z_0$ we obtain $H \in \ker((I - 2Z_0) \times_\beta)$ as first order condition. But in the basis given by normal ordered products we have $((I - 2Z_0) \times_\beta) = \text{diag}(1 - 2\beta^{2l})_{l=0,1,...}$. Thus the kernel is nontrivial iff there is an $l$ such that $1 - \beta^{2l} = 0$. This is equivalent to $d = \frac{2l}{l+1}$. 
Chapter 1

Introduction to Nonassociative Algebras

1.1 Definition of the Nonassociative Algebra

We recall in this section some basic definitions for nonassociative algebras [B66, S66].

Let $\mathcal{B}$ be an infinite-dimensional vector space over $\mathbb{C}$ with basis $e_0, e_1, e_2, \ldots$. Let a multiplication $\times$ on $\mathcal{B}$ be defined by constants of multiplication $B_{lmn} \in \mathbb{C}$ for $l, m, n \in \mathbb{N}$ and the table of multiplication

$$e_m \times e_n = \sum_l B_{lmn}^l e_l.$$ (1.1)

We suppose that the $\times$-multiplication obeys the distributive laws

$$a \times (b + c) = a \times b + a \times c$$

$$(a + b) \times c = a \times c + b \times c$$ (1.2)

and

$$(\lambda a) \times b = a \times (\lambda b) = \lambda a \times b$$ (1.3)

for all $a, b, c \in \mathcal{B}$ and $\lambda \in \mathbb{C}$.

We call an element $a \in \mathcal{B}$ finite of degree $n$ if $a_l = 0$ for all $l \neq n$. Let $\mathcal{B}_n$ be the subspace of $\mathcal{B}$ consisting of elements of degree $n$. The algebra $\mathcal{B}$ is called graded if

$$\mathcal{B}_m \times \mathcal{B}_n \subseteq \mathcal{B}_{m+n}.$$ (1.4)

The direct sum $\bigoplus_{n=0}^{\infty} \mathcal{B}_n$ is the subalgebra of $\mathcal{B}$ consisting of sums of elements of finite degree. In this chapter we will only consider $\bigoplus_{n=0}^{\infty} \mathcal{B}_n$. Later we will also consider closures of $\bigoplus_{n=0}^{\infty} \mathcal{B}_n$ with respect to certain norms.
Consider two elements \( a, b \in \mathcal{B} \) defined by

\[
a := \sum_{m : m \geq 0} a_m e_m, \quad b := \sum_{n : n \geq 0} b_n e_n.
\]  

(1.5)

Then, we have

\[
a \times b = \sum_{l : l \geq 0} c_l e_l,
\]

where

\[
c_l = \sum_{m,n : m,n \geq 0} B_{l}^{mn} a_m b_n.
\]

(1.7)

The vector space \( \mathcal{B} \) together with the multiplication \( \times \) is called an algebra \( (\mathcal{B}, \times) \) (with basis \( e_0, e_1, e_2, \ldots \)). We call the algebra \( (\mathcal{B}, \times) \) commutative iff

\[
a \times b = b \times a
\]

(1.8)

for all \( a, b \in \mathcal{B} \). This is equivalent to the following condition for the constants of multiplication

\[
B_{l}^{mn} = B_{l}^{nm}
\]

(1.9)

for all \( l, m, n \in \mathbb{N} \). We call the algebra \( (\mathcal{B}, \times) \) associative iff

\[
a \times (b \times c) = (a \times b) \times c,
\]

(1.10)

for all \( a, b, c \in \mathcal{B} \). This is equivalent to the following condition for the constants of multiplication

\[
\sum_{l : l \geq 0} B_{l}^{mn} B_{k}^{ql} = \sum_{l : l \geq 0} B_{l}^{ml} B_{k}^{nq}.
\]

(1.11)

We consider now a change of the basis of the algebra \( \mathcal{B} \). Let \( f_0, f_1, f_2, \ldots \) be another basis of the vector space \( \mathcal{B} \). Suppose that

\[
e_n = \sum_{l = 0}^{\infty} U_{nl} f_l
\]

(1.12)

and that the matrix \( U \) is invertible, i. e. , \( U^{-1} \) exists, such that

\[
\sum_{l = 0}^{\infty} U_{nl} (U^{-1})_{lk} = \delta_{nk} = \sum_{l = 0}^{\infty} (U^{-1})_{nl} U_{lk}.
\]

(1.13)

Then the table of multiplication reads

\[
f_m \times f_n = \sum_{l} C_{l}^{mn} f_l,
\]

(1.14)
where the constants of multiplication are

\[ C_{lm}^{mn} = \sum_{m',n',l'} \left( (U^{-1})_{m'm} (U^{-1})_{nn'} U_{n'l'} B_{l'm}^{mn} \right). \]  

(1.15)

Below we will consider examples where the transformation matrix \( U \) is diagonal, i.e.,

\[ U_{nl} = \lambda_n \delta_{nl}. \]  

(1.16)

Then the constants of multiplication are related by

\[ C_{lm}^{mn} = \frac{\lambda_l}{\lambda_m \lambda_n} B_{l'm}^{mn}. \]  

(1.17)

Define a new multiplication \( \tilde{x} \) for the basis elements \( e_n \) by

\[ e_m \tilde{x} e_n = \sum_l C_{lm}^{mn} e_l. \]  

(1.18)

Then eq. (1.15) implies

\[ U(a \times b) = U(a) \tilde{x} U(b), \]  

(1.19)

for all \( a, b \in \mathcal{B} \), i.e., \( U : (\mathcal{B}, \times) \rightarrow (\mathcal{B}, \tilde{x}) \) is an algebra-isomorphism.

For \( a = \sum_{n=0}^{\infty} a_n e_n \), \( b = \sum_{n=0}^{\infty} b_n e_n \in \mathcal{B} \) define the canonical scalar product

\[ \langle a, b \rangle = \sum_{n=0}^{\infty} a_n b_n. \]  

(1.20)

Consider the function \( F : \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \rightarrow \mathbb{R} \) defined by

\[ F(a \otimes b \otimes c) = \langle a, b \times c \rangle. \]  

(1.21)

We call the algebra symmetric iff \( F \) is a symmetric function. The algebra is symmetric if the constant of multiplication \( B_{lm}^{mn} \) is symmetric under permutations of \( l, m \) and \( n \).

In the following we will introduce the notion of weight and genetic basis for the algebra \( \mathcal{B} \) (cf. [B66]). We call a linear mapping \( \omega : \mathcal{B} \rightarrow \mathbb{C} \) a weight iff \( \omega \neq 0 \) and \( \omega(u \times v) = \omega(u)\omega(v) \) for all \( u, v \in \mathcal{B} \). If there exists a weight \( \omega \) for the algebra \( \mathcal{B} \) we call \( \mathcal{B} \) a weighted algebra. We can easily see that \( \omega(\mathcal{B}) = \mathbb{C} \), i.e., \( \omega \) is surjective. Therefore

\[ \dim(\mathcal{B} - \ker \omega) = 1. \]  

(1.22)

\( \omega \) is a one-dimensional representation of the algebra \( \mathcal{B} \). Let \( \{ e_m, m \in \mathbb{N} \} \) be a basis of \( \mathcal{B} \). We call this basis genetic iff

\[ \sum_l B_{lm}^{mn} = 1, \]  

(1.23)

for all \( m, n \in \mathbb{N} \). \( B_{lm}^{mn} \) are the constants of multiplication with respect to the basis \( \{ e_m, m \in \mathbb{N} \} \).
Lemma 1.1.1 Let $\mathcal{B}$ be an algebra. Then there exists a genetic basis $\{e_m, m \in \mathbb{N}\}$ iff there exists a weight $\omega$ of $\mathcal{B}$.

Proof: “$\Rightarrow$” Suppose there exists a genetic basis $\{e_m, m \in \mathbb{N}\}$ of $\mathcal{B}$. Consider $a = \sum_{m=1}^{N} a_m e_m$, $u_m \in \mathbb{C}$. Define

$$\omega(a) := \sum_{m=1}^{N} a_m. \quad (1.24)$$

Obviously $\omega: \mathcal{B} \to \mathbb{C}$ is linear and non-zero. We have, using eq. (1.23),

$$\omega(u \times v) = \sum_{l,m,n} B_{lmn}^m u_m v_n = \sum_{m,n} u_m v_m = \omega(u) \omega(v). \quad (1.25)$$

Thus, $\omega$ is a weight.

“$\Leftarrow$” Suppose that there exists a weight $\omega: \mathcal{B} \to \mathbb{C}$. Since $\mathcal{B} = (\mathcal{B} - \ker \omega) \oplus \ker \omega$ and $\dim(\mathcal{B} - \ker \omega) = 1$ there exists a basis $\{u_m, m \in \mathbb{N}\}$ such that $u_0 \in \mathcal{B} - \ker \omega$, $\omega(u_0) = 1$, and $u_1, u_2, \ldots \in \ker \omega$. Define a new basis $\{e_m, m \in \mathbb{N}\}$ by

$$e_0 := u_0, \quad e_n := u_0 + u_n, \quad (1.26)$$

for all $n \geq 1$. Then, we have

$$\omega(e_m \times e_n) = \omega(e_m) \omega(e_n) = 1. \quad (1.27)$$

Since

$$\omega(e_m \times e_n) = \sum_{l} B_{lmn}^m, \quad (1.28)$$

we see that eq. (1.23) holds. Therefore, the basis $\{e_m, m \in \mathbb{N}\}$ is genetic. \hfill \Box

For two subsets $U, V \subseteq \mathcal{B}$ define a subset $U \times V$ of $\mathcal{B}$ by

$$U \times V := \{u \times v | u \in U, \ v \in V\}. \quad (1.29)$$

We call a subspace $U$ of the vector space $\mathcal{B}$ a subalgebra of $\mathcal{B}$ iff $U \times U \subseteq U$. Obviously, $(U, \times)$ is itself an algebra, where the $\times$-multiplication is restricted to $U$. We call a subspace $I$ of the vector space $\mathcal{B}$ a left (right) ideal of $\mathcal{B}$ iff $\mathcal{B} \times I \subseteq I$ ($I \times \mathcal{B} \subseteq I$). Let us remark that in the RG context the subalgebras (ideals) play the rôle of invariant subspaces with respect to the (linearized) RG transformation.

If $\mathcal{B}$ does not contain a unit element $\mathbf{1}$ we may adjunct an unit element $\mathbf{1}$ by

$$(\alpha \mathbf{1} + a) \times (\beta \mathbf{1} + b) = (\alpha \beta) \mathbf{1} + \alpha b + \beta a + a \times b, \quad (1.30)$$

where $\alpha, \beta \in \mathbb{C}$ and $a, b \in \mathcal{B}$. 

14
Let \( \mathcal{B} \otimes \mathcal{B} \) be the tensor product. A multiplication on \( \mathcal{B} \otimes \mathcal{B} \) can be defined by
\[
(a \otimes b) \times (c \otimes d) = (a \times c) \otimes (b \times d).
\] (1.31)

Let us denote the basis elements of the tensor product \( e_m \otimes e_{m'} \) by \( e_{mm'} \). Then the constants of multiplication \( B_{lm',nn'}^{mm'} \) of the tensor-algebra \( \mathcal{B} \otimes \mathcal{B} \) are given by the following table of multiplication
\[
e_{mm'} \times e_{nn'} = \sum_{l,l'} B_{ll',nn'}^{mm'} e_{ll'}
\] (1.32)
or explicitly
\[
B_{ll',nn'}^{mm'} = B_{lm}^{mm'} B_{n'n}^{l'm'}.
\] (1.33)

Let \( \| \cdot \| \) be a norm on \( \mathcal{B} \). We call \( \| \cdot \| \) an algebra-norm on \( (\mathcal{B}, \times) \) iff
\[
\|a \times b\| \leq \|a\| \cdot \|b\|.
\] (1.34)

Let us remark that if \( U : (\mathcal{B}, \times) \rightarrow (\mathcal{B}, \tilde{\times}) \) is an algebra-isomorphism defined by \( U(a) = \lambda a \), \( \lambda \in \mathbb{C} \) and \( \| \cdot \| \) is a norm on \( \mathcal{B} \) obeying
\[
\|a \times b\| \leq \lambda \|a\| \cdot \|b\|.
\] (1.35)
we have
\[
\|a \tilde{\times} b\| \leq \|a\| \cdot \|b\|,
\] (1.36)
i. e. \( \| \cdot \| \) is an algebra-norm on \( (\mathcal{B}, \tilde{\times}) \).

A **Banach-algebra** is an algebra \( (\mathcal{B}, \times) \) with algebra-norm \( \| \cdot \| \) such that the normed space \( (\mathcal{B}, \| \cdot \|) \) is complete, i. e. , every Cauchy sequence converges.

We will now introduce the notion of regular quasi-representation for nonassociative algebras. The regular quasi-representation for nonassociative algebras correspond to the linearized RG transformations. For an algebra \( (\mathcal{B}, \times) \) and \( a \in \mathcal{B} \) define the linear operator \( L_{\times}(a) : \mathcal{B} \rightarrow \mathcal{B} \) by
\[
L_{\times}(a)(b) = a \times b.
\] (1.37)
The matrix elements are given by
\[
L_{\times}(a)_{ln} := \sum_m B_{lm}^{nd} a_m.
\] (1.38)

We omit the index \( \times \) of \( L_{\times} \) if there arises no ambiguities. For nonassociative algebras we call \( L_{\times} \) a **regular quasi-representation**. Let us remark that \( L_{\times} \) is not a representation in the nonassociative case. Supposing that \( (\mathcal{B}, \times) \) is an associative algebra we may conclude
\[
L(a \times b) = L(a) \circ L(b).
\] (1.39)
Thus $L : (\mathcal{B}, \times) \to (\text{Lin}(\mathcal{B}, \mathcal{B}), \circ)$ is a representation of the algebra $(\mathcal{B}, \times)$ which is called regular representation of $(\mathcal{B}, \times)$. We will use the notation $L(a) = (a \times)$. If the algebra is associative and commutative, we have

$$[L(a), L(b)] = 0,$$

(1.40)

for all $a, b \in \mathcal{B}$. If $L(a)$ and $L(b)$ commute they have the common eigenvectors. We call $c \neq 0$ an eigenvector of $L(a)$ if there exists $\lambda(a, c) \in \mathbb{C}$ such that

$$L(a) c = \lambda(a, c) c.$$

(1.41)

Using relation (1.39) we obtain

$$\lambda(a \times b, c) = \lambda(a, c) \lambda(b, c).$$

(1.42)

Thus $\lambda(\cdot, c) : (\mathcal{B}, \times) \to (\mathbb{C}, \cdot)$ is a 1-dimensional representation of $(\mathcal{B}, \times)$ for any eigenvector $c$. Let us remark that for an idempotent element $z \in \mathcal{B}$, i.e., obeying $z \times z = z$, we have $\lambda(z, z) = 1$. This equation holds also for the nonassociative case, i.e., $z$ is an eigenvector for $L(z)$ with eigenvalue 1.

In this paper we are concerned with algebras where the commutative and associative multiplication $\times_1$ is deformed by a linear operator. Define a $\times_h$-multiplication by

$$a \times_h b = S_h(a \times_1 b),$$

(1.43)

for all $a, b \in \mathcal{B}$. $S_h : \mathcal{B} \to \mathcal{B}$ is a linear operator defined by

$$S_h(e_m) := (m) e_m,$$

(1.44)

where $h : \mathbb{N} \to \mathbb{C}$. The cases we are mostly interested in are $h(m) = \lambda^m$ and $h(m) = \lambda^{m^2}$. This can be viewed as a 1-parameter deformation of the algebra $\mathcal{B}$ giving up the associativity axiom.

Furthermore, suppose that $(\mathcal{B}, \times)$ is isomorphic to a symmetric algebra. We call such algebras renormalization group algebras. For $h \neq 1$ the algebra is commutative and nonassociative. If the constants of multiplication of $(\mathcal{B}, \times_1)$ are given by $B_{i}^{mn}$ the constants of multiplication of $(\mathcal{B}, \times)$ are given by $h(l) B_{i}^{mn}$. Supposing that $e_0$ is the unit element of $(\mathcal{B}, \times_1)$ we see that $B_{i}^{0n} = \delta_{ln}$. This is equivalent to $L_{\times_1}(e_0) = 1$. Thus the eigenvalues of $L_{\times_1}$ are equal to 1 for all $a \in \mathcal{B}$. Obviously,

$$L_{\times}(e_0) = \text{diag} (h(0), h(1), h(2), \ldots).$$

(1.45)

The vector space $\mathcal{B}$ can be identified by its dual space $\mathcal{B}'$ if we define a dual basis $e'_m$ of $\mathcal{B}'$ by

$$e'_m(e_n) := \delta_{mn}.$$  

(1.46)
Then we see that $L(B) \subseteq B \otimes B$ and

$$L(a) = \sum_{l,m,n} B_{lmn}^m a_l \otimes e_n.$$  \hfill (1.47)

In the following we are interested in commutative but nonassociative algebras. For $x \in B$ the subalgebra $[x]$ generated by $x$ consists of all products of $x$. If this subalgebra $[x]$ is associative for all $x \in B$ then $B$ is called a power algebra. Unfortunately, nonassociative algebras defined by RG are not power algebras. Therefore we have to introduce the notion of powers and show how to calculate with these powers. This point will be studied in the next sections.

### 1.2 Powers and Tree Graphs, Forms and Power Series

If $B$ is not a power algebra the power $x^n$ is not uniquely defined for $x \in B$ and $n \in \mathbb{N}$. In this section we will generalize the notion of powers for general nonassociative algebras (cf. [B66].)

We will see that the different ways of multiplying $n$ copies of an element $x$ correspond to binary trees. These trees do also occur in renormalization theory of field theoretic models. These trees are related to a non-associative algebra structure.

Let $\mathcal{F}$ be a set provided with an addition $+$ and containing a zero element such that $\mathcal{F}$ is generated by a unit element $1 \neq 0$. The addition is neither assumed to be commutative nor associative. $\mathcal{F}$ is called the set of all non-associative integers. The elements of $\mathcal{F}$ consist of sums of the unit element $1$, e. g.,

$$(1 + 1) + 1, \ (1 + 1) + (1 + 1), \ (1 + (1 + 1)) + 1.$$  \hfill (1.48)

Let $\mathcal{P}(B)$ be the set of all products of elements of $B$. Define a homomorphism

$$h : \mathcal{P}(B) \rightarrow \mathcal{F},$$  \hfill (1.49)

where in $h(P)$ is defined by replacing in the product $P$ each factor by $1$ and each multiplication $\times$ by addition $+$. We define $h(1) := 0$. $h(P)$ is called the form of the product $P$ (cf. [B66]). We may identify each element $S \in \mathcal{F}$ containing $n$ unit elements $1$ by a rooted tree with $n$ final vertices such that all vertices which are not final have 2 successors. Denote the set of all such defined (binary) trees by $T_n^{(2)}$.

**Lemma 1.2.1** For $n \geq 1$, the number of all binary trees with $n$ final vertices is

$$|T_n^{(2)}| = \frac{(2n - 2)!}{(n - 1)!n!}.$$  \hfill (1.50)
Proof: We have the following recursive relations

\[ |T^{(2)}_1| = 1, \quad |T^{(2)}_n| = \sum_{k=1}^{n-1} |T^{(2)}_{n-k}||T^{(2)}_k|, \quad n \geq 2. \quad (1.51) \]

The generating function \( F \) of the number of binary trees \( |T^{(2)}_n| \)

\[ F(x) := \sum_{n: n \geq 1} |T^{(2)}_n| x^n \quad (1.52) \]

obeys

\[ F^2(x) = F(x) - x. \quad (1.53) \]

This implies, supposing \(|4x| < 1\),

\[ F(x) = \frac{1}{2}(1 - \sqrt{1 - 4x}) = -\frac{1}{2} \sum_{n: n \geq 1} \left(\frac{1}{n}\right)(-4)^n x^n. \quad (1.54) \]

Since

\[ \left(\frac{1}{n}\right) = \frac{\frac{1}{2}(-\frac{1}{2}) \cdots (\frac{1}{2} - n + 1)}{n!} = (-1)^{n-1} \frac{2^{n-1}}{2^n n!} \cdot 1 \cdot 3 \cdot 5 \cdots (2n - 3), \quad (1.55) \]

eq. (1.54) implies

\[ |T^{(2)}_n| = \frac{2^{n-1}}{n!} \cdot 1 \cdot 3 \cdot 5 \cdots (2n - 3) = \frac{(2n - 2)!}{n!(n-1)!}. \quad \square \quad (1.56) \]

For a form \( S \in \mathcal{F} \) the degree \( \delta(S) \) of \( S \) is the number of unit elements in the sum \( S \). Denote

\[ \mathcal{F}_n := \{ S \in \mathcal{F} | \delta(S) = n \} \quad (1.57) \]

By Lemma 1.2.1 we have \(|\mathcal{F}_n| = \frac{(2n-2)!}{n!(n-1)!}\).

Let us define an equivalence class in \( \mathcal{B} \) by

\[ a \sim b \iff h(a) = h(b), \quad (1.58) \]

for all \( a, b \in \mathcal{B} \). Consider \( x \in \mathcal{B} \). Each equivalence class can be represented by a form \( S \in \mathcal{F} \) where each factor in the product is equal to \( x \). Denote this product by \( x^S \). Define for two forms \( S_1, S_2 \in \mathcal{F} \) the power \( x^{S_1S_2} = (x^{S_1})^{S_2} \) by replacing \( x \) by \( x^{S_1} \) in the product \( x^{S_2} \). This product is associative but not commutative

\[ (S_1S_2)S_3 = S_1(S_2S_3), \quad S_1S_2 = S_2S_1. \quad (1.59) \]

For a symbol \( X \) consider formal power series

\[ F(X) := \sum_{S: S \in \mathcal{F}} f(S) X^S, \quad f(S) \in \mathbb{C}. \quad (1.60) \]
Denote the set of all such formal power series by $\mathcal{F}_F[[X]]$. For $F, G \in \mathcal{F}_F[[X]]$ define addition and multiplication by

$$(F + G)(X) := \sum_{S : S \in F} (f(S) + g(S)) X^S$$

$$(FG)(X) := \sum_{T : T \in F} \left( \sum_{S_1, S_2 \in F, T = S_1 + S_2} f(S_1) g(S_2) \right) X^T.$$  \hfill (1.61)

Replacing $X$ in $F(X)$ by an element $x \in B$ we get a formal power series denoted by $F(x)$. We have

$$(FG)(x) = F(x) \times G(x).$$  \hfill (1.62)

In the next section we will see how to express quasi-inverses and quasi-roots in terms of power series.

### 1.3 Quasi-Roots and Quasi-Inverses

In algebras without unit elements the definition of an inverse element is not possible. In this case the notion of an inverse can be replaced by the notion of a quasi-inverse.

Let $B$ be an algebra without unit element. We call an element $x$ a quasi-inverse of $y$ iff

$$x \circ y := x + y + x \times y = 0.$$  \hfill (1.63)

The multiplication $\circ$ is called quasi-product. This definition can be motivated by the fact that if $x$ is a quasi-inverse of $y$ then $1 + x$ is the inverse of $1 + y$ in $B_1$, where $B_1$ is the algebra $B$ adjunced with a unit element $1$ defined by eq.$(1.30)$.

We call $y \in B$ a quasi-root of $x \in B$ iff

$$y \circ y = 2y + y \times y = x.$$  \hfill (1.64)

This definition can be motivated by $(1 + y) \times (1 + y) = 1 + x$.

**Lemma 1.3.1** Let $B$ be a commutative algebra and $y \in B$ be the quasi-inverse of $x \in B$. Then we have the following formal power series expansion

$$y = \sum_{n=1}^{\infty} (-1)^n x^n,$$  \hfill (1.65)

where the forms are recursively defined by

$$1_1 := 1, \quad 1_{n+1} := 1_n + 1,$$  \hfill (1.66)

for all $n \geq 1$. Furthermore, suppose that there exists a norm $\| \cdot \|$ in $B$ and $q \in \mathbb{R}_+$ such that

$$\|a \times b\| \leq q \|a\| \|b\|$$  \hfill (1.67)
for all \(a, b \in \mathcal{B}\). Suppose that \(q \parallel x \parallel < 1\). Then, we have
\[
\|y\| \leq \frac{\|x\|}{1 - q \|x\|}.
\] (1.68)

We get an analogous lemma for the quasi-root.

**Lemma 1.3.2** Let \(\mathcal{B}\) be an algebra and \(y \in \mathcal{B}\) be the quasi-root of \(x \in \mathcal{B}\). Then we have the following formal power series expansion
\[
y = \sum_{S : S \in \mathcal{F} - \{0\}} u(S) x^S,
\] (1.69)
where the coefficients \(u(S)\) are recursively determined by
\[
u(1) = \frac{1}{2}, \quad u(T) = -\frac{1}{2} \sum_{S_1, S_2 \in \mathcal{F} - \{0\} : T = S_1 + S_2} u(S_1) u(S_2)
\] (1.70)
for \(T \in \mathcal{F} - \{0, 1\}\). Furthermore,
\[
u_n := \sum_{T : T \in \mathcal{F}_n} |u(T)| \leq \frac{1}{2} \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 3) \frac{2^n n!}{n! (n - 1)!} = 2^{-2n+1} \frac{(2n - 2)!}{n! (n - 1)!}.
\] (1.71)
We have \(\nu_n \leq \frac{1}{2(n-1)}\) for \(n \geq 2\).

**Proof:** Let \(y\) be the quasi-root of \(x\). Then we have
\[(1 + y) \times (1 + y) = 1 + x.
\] (1.72)
For \(c \in \mathbb{R}\), we have
\[
(1 + \sum_{m: m \geq 1} q_m c^m)^2 = 1 + c, \quad q_m := \left(\frac{1}{2}\right)^m = \frac{(-1)^{n-1}}{2^n n!} \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 3).
\] (1.73)
Thus
\[
q_1 = \frac{1}{2}, \quad q_n = -\frac{1}{2} \sum_{m=1}^{n-1} q_{n-m} q_m.
\] (1.74)
We use induction in \(n\). Suppose that \(u_m \leq (-1)^{m-1} q_m\) for all \(m < n\). By eq. (1.70), we have
\[
u_n = \sum_{T : T \in \mathcal{F}_n} |u(T)| \leq \frac{1}{2} \sum_{T : T \in \mathcal{F}_n} |u(T)| \sum_{S_1, S_2 \in \mathcal{F} - \{0\} : T = S_1 + S_2} |u(S_1)| |u(S_2)|
\leq \frac{1}{2} \sum_{m=1}^{n-1} \left( \sum_{S_1 : S_1 \in \mathcal{F}_n} |u(S_1)| \right) \left( \sum_{S_2 : S_2 \in \mathcal{F}_n} |u(S_2)| \right) = \frac{1}{2} \sum_{m=1}^{n-1} \nu_{n-m} \nu_m.
\] (1.75)
By induction hypothesis we have

\[ u_n \leq \frac{(-1)^{n-1}}{2} \sum_{m=1}^{n-1} q_{n-m} q_m = (-1)^{n-1} q_n. \]  

(1.76)

This proves the assertion. \( \square \)

**Lemma 1.3.3** Consider \( F(X) = \sum_{S: S \in \mathcal{F}(\emptyset)} f(S) X^S \in \mathbb{C}[X] \) and suppose that there exists a norm \( \| \cdot \| \) in \( \mathcal{B} \) and \( q \in \mathbb{R}_+ \) such that

\[ \|a \times b\| \leq q \|a\| \|b\| \]  

(1.77)

for all \( a, b \in \mathcal{B} \). Then, we have for \( x \in \mathcal{B} \)

\[ \|F(x)\| \leq \frac{1}{q} \left[ \sum_{n: n \geq 1} f_n (q\|x\|)^n \right], \]  

(1.78)

where

\[ f_n := \sum_{S: S \in \mathcal{F}_n} |f(S)|. \]  

(1.79)

Let \( y \in \mathcal{B} \) be the quasi-inverse of \( x \in \mathcal{B} \). Suppose that \( q\|x\| < 1 \). Then, we have

\[ \|y\| \leq \frac{\|x\|}{2} (1 + \ln(1 - q\|x\|)). \]  

(1.80)

**Proof:** We have

\[ \|F(x)\| \leq \sum_{n: n \geq 1} \sum_{S \in \mathcal{F}_n} |f(S)| q^{n-1} \|x\|^n. \]  

(1.81)

This and definition (1.79) proves eq. (1.78). By Lemma 1.3.2 and the bound eq. (1.78) we see

\[ \|y\| \leq \frac{1}{q} \sum_{n: n \geq 1} u_n (q\|x\|)^n \leq \frac{1}{2q} \left( q\|x\| + \sum_{n: n \geq 2} \frac{(q\|x\|)^n}{n-1} \right). \]  

(1.82)

This proves eq. (1.80). \( \square \)
Chapter 2

Introduction to Renormalization Group Algebras

2.1 Examples of Hierarchical Renormalization Group Algebras

In this section we study examples of hierarchical renormalization group algebras. We will show how these examples can be related to hierarchical renormalization group transformations. Fixed points of the renormalization group transformation (RGT) are the idempotents of the renormalization group algebra.

In the following we consider the space $\mathcal{B} := L^2(e^{-\frac{\Phi^2}{2\gamma}} d\Phi)$. For $L > 1$, $\beta$, $\gamma > 0$ and the (not normalized) hierarchical renormalization group transformation $R_{\gamma,L}^\beta$ in $d$ dimensions is defined by

$$R_{\gamma,L}^\beta(F)(\Psi) = \int d\mu_{\gamma}(\Phi) F(\Phi + \beta \Psi)^L,$$

where $d\mu_{\gamma}(\Phi)$ is a Gaussian measure with mean zero defined by

$$d\mu_{\gamma}(\Phi) := (2\pi\gamma)^{-1/2} d\Phi \exp\left\{-\frac{1}{2\gamma}\Phi^2\right\}.$$

The RGT depends on the three parameters $\beta$, $L^d$ and $\gamma$. $\beta$ is called the scaling parameter, $L^d$ the volume factor, and $\gamma$ the (free) covariance of the RG transformation. The volume factor and the scaling parameter obey the relation

$$\beta = L^{1-\frac{d}{2}}.$$  \(2.3\)

Define a mapping $E_{\gamma} : \mathcal{B} \rightarrow \mathcal{B}$ by $E_{\gamma} := R_{\gamma,1}^1$, i.e.

$$E_{\gamma}(F)(\Phi) := \int d\mu_{\gamma}(\zeta) F(\zeta + \Phi).$$  \(2.4\)
and the Gaussian expectation value \(<\gamma>\) by

\[
< F >_{\gamma} := \mathbb{E}_{\gamma}(F)(0) = \int d\mu_{\gamma}(\zeta)F(\zeta).
\] (2.5)

We are interested in the existence and construction of fixed points \(F^*\) of the RG transformation (2.1)

\[\mathcal{R}_{\gamma,L}^\beta(F^*) = F^*.\] (2.6)

Three fixed points are immediately found. The zero fixed point \(F_0 := 0\), the ultraviolet fixed point \(F_{UV} := 1\) and the high temperature fixed point \(F_{HT}\) defined by

\[
F_{HT}(\Psi) := L^{\frac{1}{2\gamma L^d}} \exp\left\{-L^2 - \frac{1}{2\gamma L^d} \Psi^2\right\}.
\] (2.7)

The high temperature fixed point turns out to be stable. The most important problem is that of finding non-trivial unstable fixed points. These will be called infrared fixed points. They are related to phase transitions in our hierarchical model in the infrared limit.

Thus we are looking for some other nontrivial fixed points \(F_{IR}\). That \(F_{HT}\) is a fixed point can be shown as an application of the following Lemma

**Lemma 2.1.1** For all \(c \neq -\frac{1}{2\gamma}\) and \(F: \mathbb{R} \rightarrow \mathbb{R}\), such that the following Gaussian integrals exist, we have

\[
\mathbb{E}_{\gamma}(e^{-c(\cdot)^2}F(\cdot))(\Psi) = L^{\frac{1}{2\gamma c}} \exp\{-cL_c\Psi^2\} \mathbb{E}_{L_c\gamma}(F)(L\Psi),
\] (2.8)

where \(L_c := (1 + 2\gamma c)^{-1}\).

The foregoing Lemma 2.1.1 implies the following Corollary.

**Corollary 2.1.1** Define

\[
\mathcal{U}_c^N(F)(\Phi) := \mathcal{N} \exp\{c\Phi^2\} F(\Phi).
\] (2.9)

For all \(c \neq -\frac{1}{2\gamma L^d}\) we have

\[
\mathcal{U}_c^N \circ \mathcal{R}_{\gamma,L}^\beta \circ (\mathcal{U}_c^N)^{-1} = \mathcal{U}_c^{L^{\frac{d}{(1-L^d L^\beta^2)}} \circ \mathcal{R}_{L_c} L_c \circ \mathcal{R}_{\gamma,L}^\beta L^d}.
\] (2.10)

where

\[
L_c := \frac{1}{1 + 2\gamma L^d c}.
\] (2.11)
Proof: The assertion is proven by

\[
R_{\gamma,L}^{\beta} \circ (U_{c}^{N})^{-1}(F)(\Psi) = \int d\mu(\Phi)[N^{-1} \exp\{-c(\Phi + \beta\Psi)^2\}F(\Phi + \beta\Psi)]^{L^d}
\]

\[
= N^{-L^d} \int d\mu(\Phi) \exp\{-cL^d(\Phi + \beta\Psi)^2\}F^{L^d}(\Phi + \beta\Psi)
\]

\[
= L^{\frac{d}{2}} N^{-L^d} \exp\{-cL^d\beta^2\} \int d\mu(\Phi)F^{L^d}(\Phi + \beta\Psi)
\]

\[
= U L^{\frac{d}{2}} N^{-L^d} \circ R_{L^d}^{\beta L}(F)(\Psi). \quad \square
\]

We introduce a similarity transformation which preserves the form of the renormalization group transformation eq. (2.1) but changes the parameters in a favorable manner.

For a transformation \(S : \mathcal{B} \to \mathcal{B}\), define the equivalent renormalization group transformation

\[
T_S[R] := S \circ R \circ S^{-1}.
\]

The following special similarity transformation were used by Koch and Wittwer [KW86, KW91] for an analysis of the double-well fixed point in 3 dimensions by means of a rigorous beta-function technique.

\[
S_{HT} : \mathcal{B} \to \mathcal{B}, \quad S_{HT}(F)(\Phi) := \frac{F(\Phi)}{F_{HT}(\Phi)} = Z(\Phi).
\]

Then Corollary 2.1.1 implies

\[
T_{S_{HT}}[R_{\gamma,L}^\beta ] = R_{\gamma',L^d}^\beta,
\]

where

\[
\beta' := L^{-1-\frac{d}{2}}, \quad \gamma' := L^{-2}\gamma,
\]

if \(\beta\) is defined by eq. (2.3). The result of the transformation \(T_{S_{HT}}\) is that the scaling parameter \(\beta\) becomes smaller. Furthermore, the logarithm of fixed points becomes more convex which leads to a simpler treatment of the large field problem.

In the following, we will study the transformed RG transformation \(R_{\gamma',L^d}^\beta\) for the case \(L^d = 2\).

Let us remark that the restriction to \(L^d = 2\) is not necessary for the use of methods in this paper. We will use only the condition \(L^d = 2\) in order to keep the notations simpler. All results can be easily generalized to the case \(L^d \in \{2, 3, \ldots\}\).

We will write \(R_{\gamma'}^\beta = R_{\gamma',1}^\beta\). Obviously, \(R_{\gamma',2}^\beta(F) = R_{\gamma'}^\beta(F^2)\). The fixed points of the original RGT are in one to one correspondence with the fixed points of the new RGT. The infrared fixed points \(F_{IR}\) of \(R_{\gamma}^\beta\) and \(Z_{IR}\) are related by

\[
F_{IR}(\Phi) = F_{HT}(\Phi) Z_{IR}(\Phi).
\]
The RG transformation $R_{\gamma'}^{\beta'}$, $\beta' := L^{-1 - \frac{d}{2}}$, has the high temperature fixed point $Z_{HT} = 1$ and the UV fixed point $Z_{UV}$, where

$$Z_{UV}(\Psi) := L^{-\frac{1}{Ld - 1}} \exp\{\frac{1 - L^{-2}}{2\gamma' Ld} \Psi^2\} = 2^{-\frac{1}{d}} \exp\{\frac{1 - 2\beta'^2}{4\gamma'} \Psi^2\}. \quad (2.18)$$

The high temperature fixed point $F_{HT}$ of $R_{\gamma}$ reads

$$F_{HT}(\Phi) = 2^{\frac{1}{d}} \exp\{-\frac{2\beta^2 - 1}{4\gamma} \Phi^2\} \quad (2.19)$$

and we have

$$R_{\gamma}^{\beta'} ((F_{HT} \cdot A) \cdot (F_{HT} \cdot B)) = F_{HT} R_{\gamma'}^{\beta'} (A \cdot B). \quad (2.20)$$

In the following let us define some multiplication on $B$. For $A, B \in B$ define a $\cdot$-product by

$$A \cdot B(\Phi) := A(\Phi) B(\Phi). \quad (2.21)$$

($B, \cdot$) defines a commutative and associative algebra.

Let $S_\delta : B \to B$ be the scaling transformation

$$S_\delta(\Phi) = F(\delta\Phi). \quad (2.22)$$

For $A, B \in B$ define a $\ast$-product by

$$A \ast_{\beta, \gamma} B := S_\beta E_{\gamma}(A \cdot B) = R_{\gamma'}^{\beta'} (A \cdot B). \quad (2.23)$$

($B, \ast$) defines a commutative algebra. We will omit the second index $\gamma$ in $\ast_{\beta, \gamma}$ if no confusion can arise.

Let us introduce Wick-ordered functions by

$$: F(\Phi) :_{\gamma} := \exp\{-\frac{\gamma}{2} \frac{\partial^2}{\partial \Phi^2}\} F(\Phi). \quad (2.24)$$

Note that $E_{-\gamma}(F) = : F :_{\gamma}$ and

$$E_{\gamma_1 + \gamma_2} = E_{\gamma_1} \circ E_{\gamma_2}, \quad (2.25)$$

where $\circ$ denotes the composition of two functions.

The Wick-ordered monomials are expressed by a sum of monomials in the following Lemma.

**Lemma 2.1.2** For $n \in \mathbb{N}$ we have

$$: \Phi^n :_{\gamma} = \sum_{k : n - k \in 2\mathbb{N}} \frac{\gamma}{2}^{\frac{n-k}{2}} \binom{n}{k} \frac{(n - k)!}{(\frac{n}{2})!} \Phi^k. \quad (2.26)$$
Proof: We have

\[ :\Phi^n :_\gamma = \exp\left\{ \frac{\gamma}{2} \frac{\partial^2}{\partial \Phi^2} \right\} \Phi^n \]

\[ = \sum_{m=0}^n (-\frac{\gamma}{2})^m \frac{n(n-1) \cdots (n-2m+1)}{m!} \Phi^{n-2m} \]

\[ = \sum_{k: n-k \leq 2\mathbb{N}} \left( -\frac{\gamma}{2} \right)^{\frac{n-k}{2}} \binom{n}{k} \frac{(n-k)!}{(\frac{n-k}{2})!} \Phi^k. \]  

This implies the assertion. \(\square\)

Lemma 2.1.3 For \(c \neq -\frac{1}{2\gamma}\) we have

\[ : \exp\{c\Phi^2\} G(\Phi) :_\gamma = \mathcal{L}_c^\frac{1}{2} \exp\{c\mathcal{L}_c \Phi^2\} : \mathcal{L}_c :_\gamma (\mathcal{L}_c \Phi), \]

where

\[ \mathcal{L}_q := (1 + 2q)^{-1}. \]

Lemma 2.1.4 For all \(m, n \in \mathbb{N}\) we have the orthogonality relation

\[ : \Phi^m :_\gamma : \Phi^n :_\gamma = \delta_{m,n} \gamma^m m!. \]

Furthermore, for all \(a_i \in \mathbb{C}, i \in \{1, \ldots, n\}\), we have

\[ : \prod_{i=1}^n \exp\{a_i \Phi^2\} :_\gamma : = \prod_{i,j: i<j} \exp\{\gamma a_i a_j\} \]

and

\[ : \exp\{a\Phi^2\} :_\gamma : \exp\{b\Phi^2\} :_\gamma : = (1 - 4\gamma^2 ab)^{\frac{1}{2}}. \]

Proof: First of all, we will prove eq. (2.31). By the definition of Wick-ordered products, we obtain

\[ : \prod_{i=1}^n \exp\{a_i \Phi\} :_\gamma : = \left( \prod_{i=1}^n \exp\left\{ -\frac{\gamma}{2} a_i^2 \right\} \right) : \exp\left\{ \sum_{j=1}^n a_j \Phi \right\} :_\gamma . \]

Since, for all \(a \in \mathbb{C}\),

\[ : \exp\{a \Phi\} :_\gamma : = \exp\left\{ \frac{\gamma}{2} a^2 \right\}, \]

(2.34)
eq. (2.33) implies
\[
\langle \prod_{i=1}^{n} \exp\{a_i \Phi\} : \gamma \rangle = \exp\{-\frac{\gamma}{2} \left[ \sum_{i=1}^{n} a_i^2 - \left( \sum_{j=1}^{n} a_j \right)^2 \right] \}.
\] (2.35)

This proves eq. (2.31). We will prove eq. (2.30). Using eq. (2.31) we obtain
\[
\langle : \Phi^m : \Phi^n : \gamma \rangle = \exp\{\gamma ab\} \bigg|_{a=b=0} = \delta_{m,n} \gamma^m m!.
\] (2.36)

We will prove eq. (2.32). Using Lemma 2.1.3, we obtain
\[
\langle : \exp\{a \Phi\} : \gamma : \exp\{b \Phi\} : \gamma \rangle = \left( \mathcal{L}_a \mathcal{L}_b \right)^{\frac{1}{2}} \int d\mu(\Phi) \exp\{(a \mathcal{L}_a + b \mathcal{L}_b)\Phi^2\} = \left( \mathcal{L}_a \mathcal{L}_b \mathcal{L}_-(a \mathcal{L}_a + b \mathcal{L}_b) \right)^{\frac{1}{2}} = \left[ (1 + 2\gamma a)(1 + 2\gamma b) \left( 1 - \frac{2\gamma a}{1 + 2\gamma a} - \frac{2\gamma b}{1 + 2\gamma b} \right) \right]^{-\frac{1}{2}} = (1 - 4\gamma^2 ab)^{\frac{1}{2}}. \quad \square
\] (2.37)

We have the following scaling relation for Wick ordered functions
\[
: F : \beta \gamma (\delta \Phi) = : S_\beta(F) : \gamma (\Phi).
\] (2.38)

Eq. (2.25) implies
\[
: E_\gamma(F) : \gamma = E_\gamma( : F : \gamma) = F.
\] (2.39)

Thus \( E_\gamma(\cdot) \) is the inverse of Wick-ordering \( (\cdot) : \gamma \). For two functions \( A, B \in \mathcal{B} \) define a product
\[
A \times_\beta B := S_\beta E_\gamma( : A : \gamma \cdot : B : \gamma).
\] (2.40)

Let us remark that eq. (2.40) is equivalent to
\[
: S_\beta^{-1}(A \times_\beta B) : \gamma = : A : \gamma \cdot : B : \gamma.
\] (2.41)

**Lemma 2.1.5** For \( A, B \in \mathcal{B} \), we have
\[
: A \times_\beta \gamma B : \gamma = R_\gamma^{(1-\beta^2)}( : A : \gamma \cdot : B : \gamma).
\] (2.42)
Proof: We have,

\[ R_\gamma^{(1-\beta^2)}(A : \gamma \cdot : B : \gamma) = R_\gamma^{(1-\beta^2)}(A \times_1 B : \gamma). \] (2.43)

Therefore,

\[ R_\gamma^{(1-\beta^2)}(A : \gamma \cdot : B : \gamma)(\Phi) =: A \times_1 B : \beta^2(\beta \Phi). \] (2.44)

This implies, using relation (2.38), the assertion eq. (2.42).  

\( (\mathcal{B}, \times) \) defines a commutative algebra. This algebra is associative for the case \( \beta = 1 \) and nonassociative for the case \( \beta \neq 1 \).

The following Lemma shows a relation of the \( \times \)-and \( * \)-product.

**Lemma 2.1.6** For \( \beta \neq 1 \), we have

\[ : A \times_{\beta, \gamma} B : \gamma =: A : \gamma *_{(1-\beta^2)} : B : \gamma. \] (2.45)

**Proof:** We have, using Lemma 2.1.5

\[ : A \times_{\beta, \gamma} B : \gamma = R_\gamma^{(1-\beta^2)}(A : \gamma \cdot : B : \gamma). \] (2.46)

The definition eq. (2.23) implies the assertion.  

The UV fixed point for \( R_{\gamma(1-\beta^2)}^{-2+2d} \) reads

\[ Z_{UV}^{(\Phi)} = 2^{-\frac{1}{\beta}} \exp\{c_\Phi^2\}, \] (2.47)

where

\[ c_\Phi := \frac{1 - 2\beta^2}{4\gamma(1 - \beta^2)}, \quad \beta := 2^{-\frac{2+2d}{2d}}. \] (2.48)

Consider now the case \( \beta = 1 \). The mapping \( E_{-\gamma} : (\mathcal{B}, \times) \rightarrow (\mathcal{B}, \cdot) \) is an algebra-isomorphism, i. e.,

\[ E_{-\gamma}(A \times B) = E_{-\gamma}(A) \cdot E_{-\gamma}(B), \] (2.49)

for all \( A, B \in \mathcal{B} \). This algebra-isomorphism shows that \( (\mathcal{B}, \times) \) is associative for the case \( \beta = 1 \). The function \( 1 \in \mathcal{B} \) defined by

\[ 1(\Phi) := 1 \] (2.50)

is the unit element of \( (\mathcal{B}, \times) \) resp. \( (\mathcal{B}, \cdot) \). For \( \beta \neq 1 \) there is no unit element of \( (\mathcal{B}, \times) \).

We may adjoin to \( (\mathcal{B}, \times) \) a unit element \( 1 \) by defining the multiplication of two elements \( a1 + A, b1 + B \), for \( a, b \in \mathcal{C} \) and \( A, B \in \mathcal{B} \) by

\[ (a1 + A) \times (b1 + B) := (ab)1 + (aB + bA + A \times B). \] (2.51)

This defines the algebra \( (\mathcal{B} \oplus \{1\}, \times) \) with unit element \( 1 \).
2.2 Constants of Multiplication

In this section we present the constants of multiplication for the algebras defined in section 3.1. Useful estimations and relations are derived for the constants of multiplication. The form of the constants of multiplication depends on the basis chosen. We will consider different bases. The first basis is given by simple powers of \( \Phi \). Later we will also consider normal ordered products.

Consider for the algebra \( B \) a basis \( \{ f_m | m \in \mathbb{N} \} \) defined by

\[
f_m := \Phi^m \gamma^m_2.
\] (2.52)

The constants of multiplication \( M_{mn}^l \) for the \( \cdot \)-multiplication are given by

\[
f_m \cdot f_n = \sum_{l \in \mathbb{N}} M_{mn}^l f_l
\] (2.53)

where

\[
M_{mn}^l = \delta_{m+n,l}.
\] (2.54)

The constants of multiplication \( S_{lmn}^l \) for the \( * \)-multiplication are defined by

\[
f_m * f_n = \beta^l \sum_{l \in \mathbb{N}} S_{lmn}^l f_l.
\] (2.55)

**Lemma 2.2.1** For a triple \((l,m,n) \in \mathbb{N}^3\) such that \( m + n - l \in 2\mathbb{N} \), we have

\[
S_{lmn}^l = 2 \frac{m + n}{2} \frac{(m + n)!}{[\frac{1}{2}(m + n - l)]!}.
\] (2.56)

and \( S_{lmn}^l = 0 \) if \( m + n - l \in 2\mathbb{N} + 1 \) or \( l > m + n \).

**Proof:** We have,

\[
< (\Phi + \beta \Psi)^{m+n} >_\gamma = \sum_l \beta^l \binom{m+n}{l} < \Phi^{m+n-l} \Psi^l >_\gamma.
\] (2.57)

Since

\[
< \Phi^{m+n-l} >_\gamma = \frac{1}{2} \frac{m + n - l}{2} \frac{(m + n - l)!}{[\frac{1}{2}(m + n - l)]!},
\] (2.58)

for \( m + n - l \in 2\mathbb{N} \), and = 0 otherwise, the assertion follows. \( \square \)

We consider the subalgebra \( B^{(e)} \subset B \) of even functions. A basis \( \{ v_m | m \in \mathbb{N} \} \) is defined by \( v_m := f_{2m} \). For this subalgebra of \( B \) the constants of multiplication are

\[
S_{lmn}^l := S_{2l}^{2m}.
\] (2.59)
For the following Lemma define

\[ A_{lm}^{mn} := \sqrt{\frac{(2l)!}{(2m)!(2n)!}} S_{lm}^{mn} \]  

(2.60)

**Lemma 2.2.2** \( A_{lm}^{mn} \) obeys the bound

\[ A_{lm}^{mn} \leq \sqrt{\left(\frac{2(m+n)}{2m}\right)\binom{m+n}{l}}. \]  

(2.61)

**Proof:** We obtain

\[ A_{lm}^{mn} = 2^{-2(m+n-l)} \left(\frac{m+n}{l}\right)^2 \left(\frac{2(m+n)}{2m}\right) [2(m+n)]! \frac{l!}{(m+n)!} \frac{l!}{(2l)!} \]  

(2.62)

For \( k \in \mathbb{N} \) we have

\[ \frac{k!^2}{4(2k)!} \leq \frac{(k+1)^2}{(2k+1)(2k+2)(2k)!} = \frac{(k+1)!^2}{[2(k+1)]!}. \]  

(2.63)

Therefore, since \( l \leq m+n \),

\[ \frac{l!^2}{(2l)!} \leq \frac{4^{m+n-l} (m+n)!^2}{[2(m+n)]!}. \]  

(2.64)

Insertion of the estimation eq. (2.64) on the rhs of eq. (2.62) proves the assertion. □

**Lemma 2.2.3** For all \( a, b \in \mathbb{C} \), we have

\[ \exp\{a\Phi\} \ast_{\beta} \exp\{b\Phi\} = \exp\left\{\frac{\beta}{2} (a+b)^2\right\} \exp\{\beta (a+b)\Phi\} \]  

(2.65)

and

\[ \sum_{m,n:l \leq m+n} S_{lm}^{mn} \frac{a^m b^n}{m! n!} = \frac{(a+b)^l}{l!} \exp\left\{\frac{(a+b)^2}{2}\right\}. \]  

(2.66)

**Suppose that** \( a, b \in \mathbb{R} \), \( 2(a+b) < 1 \). **Then we have**

\[ \exp\{a\Phi^2\} \times_{\beta} \exp\{b\Phi^2\} = \frac{1}{\sqrt{1-2(a+b)}} \exp\left\{\frac{a+b}{1-2(a+b)}\beta^2 \Phi^2\right\} \]  

(2.67)

and

\[ \sum_{m,n:l \leq m+n} S_{lm}^{mn} \frac{a^m b^n}{m! n!} = \frac{1}{\sqrt{1-2(a+b)}} \frac{[\frac{a+b}{1-2(a+b)}]^l}{l!}. \]  

(2.68)
Consider for the algebra \((\mathcal{B}, \times)\) a basis \(\{e_m | m \in \mathbb{N}\}\) defined by
\[
e_m := \Phi^m \gamma^{\frac{m^2}{2}}.
\]
(2.69)

Using Lemma 2.1.2 we obtain

**Lemma 2.2.4** For \(n, k \in \mathbb{N}\) and \(\alpha \in \mathbb{R}\) define
\[
E_{nk}(\alpha) := \begin{cases}
\left(\frac{\alpha}{2}\right)^{\frac{n-k}{2}} \left(\frac{n-k}{k}\right)! : n - k \in 2\mathbb{N} \\
0 : n - k \notin 2\mathbb{N}
\end{cases}
\]
(2.70)

Then we have
\[
e_n = \sum_{k : n - k \in 2\mathbb{N}} E_{nk}(-1) f_k,
\]
\[
f_n = \sum_{k : n - k \in 2\mathbb{N}} E_{nk}(1) e_k.
\]
(2.71)

The constants of multiplication \(C^{mn}_l\) are defined by
\[
e_m \times \beta e_n = \beta^l \sum_{l \in \mathbb{N}} C^{mn}_l e_l.
\]
(2.72)

For a triple \((l, m, n) \in \mathbb{N}^3\) define the “dual” triple \((l', m', n')\) by
\[
l' := m + n - l, \quad m' := l + n - m, \quad n' := m + l - n.
\]
(2.73)

**Lemma 2.2.5** For a triple \((l, m, n) \in \mathbb{N}^3\) such that the “dual” triple \((l', m', n') \in 2\mathbb{N}^3\) has even entries, we have
\[
C^{mn}_l = \frac{m! n!}{(\frac{1}{2} l')! (\frac{1}{2} m')! (\frac{1}{2} n')!}
\]
(2.74)

and \(C^{mn}_l = 0\) if \((l', m', n') \notin 2\mathbb{N}^3\) or \(l < |m - n|\) or \(l > m + n\). Furthermore, we have
\[
\sum_{l \in \mathbb{N}} C^{mn}_l C^{lq}_k = \sum_{l \in \mathbb{N}} C^{mq}_l C^{ml}_k,
\]
(2.75)
i. e., the algebra \((\mathcal{B}, \times)\) is associative for \(\beta = 1\).

**Proof:** We have, using eq. (2.72),
\[
\langle : \Phi^l : \cdots : \Phi^m : \cdots : \Phi^n : \rangle = \sum_{k \geq 0} C^{mn}_k \langle : \Phi^k : \cdots : \Phi^l : \rangle = \sum_{k \geq 0} C^{mn}_k \langle : \Phi^k : \cdots : \Phi^l : \rangle.
\]
(2.76)
The orthogonality relation of Lemma 2.1.4, eq. (2.30), yields
\[ \langle \Phi^i : \gamma \rangle : \Phi^m : \gamma \cdot : \Phi^n : \gamma \rangle = C_{il}^{mn} \gamma^l! \] (2.77)

Furthermore, Lemma 2.1.4, eq. (2.31), yields
\[ \langle \Phi^i : \gamma \rangle : \Phi^m : \gamma \cdot : \Phi^n : \gamma \rangle = \frac{\partial^{l+m+n} e^{\gamma(ab+bc+ac)}|_{a=b=c=0}}{\partial a^l \partial b^m \partial c^n} \gamma^{m+n+l} \frac{1}{m!n!l!} (ab)^n (bc)^l (ac)^m |_{a=b=c=0} \] (2.78)

where the sum over \( m', n', l' \), is restricted by
\[ l = m' + n', \quad m = l' + n', \quad n = l' + m'. \] (2.79)

Eq. (2.77) and eq. (2.78) imply the assertion. \( \square \)

We consider the subalgebra \( B(e) \subset B \) of even functions. \( \{ e_{2m} | m \in \mathbb{N} \} \) is a basis of \( B(e) \). For this subalgebra of \( B \) the constants of multiplication are
\[ C_{il}^{mn} := C_{2l}^{2m, 2n}. \] (2.80)

We will show that there exists a genetic basis for the algebra \( (B(e), \times) \). By Lemma 1.1.1 this implies that there exists a weight \( \omega \). Obviously, \( \{ : \Phi^{2m} :_1, m \in \mathbb{N} \} \) is a basis. We have
\[ : \Phi^{2m} :_1 = 2^{-m} H_{2m}(\frac{\Phi}{\sqrt{2}}), \] (2.81)
where \( H_n \) is the \( n \)th Hermite-polynomial defined by
\[ \sum_{n=0}^{\infty} \frac{H_n(\Phi)}{n!} z^n = \exp\{-z^2 + 2z\Phi\}. \] (2.82)

Thus,
\[ H_{2m}(0) = (-1)^m \frac{(2m)!}{m!}. \] (2.83)

Therefore,
\[ : \Phi^{2m} :_1 : \Phi^{2n} :_1 = \sum_l C_l^{mn} : \Phi^{2l} :_1 \] (2.84)
implies, for \( \Phi = 0 \),
\[ \frac{1}{(-2)^m} \frac{(2m)!}{m!} \frac{1}{(-2)^n} \frac{(2n)!}{n!} = \sum_l C_l^{mn} \frac{1}{(-2)^l} \frac{(2l)!}{l!}. \] (2.85)
Therefore, we see

$$\sum_l \frac{\gamma(m) \gamma(n)}{\gamma(l)} C_{lmn} = 1,$$  \hspace{1cm} (2.86)

for all $m, n \in \mathbb{N}$, where

$$\gamma(m) := \left(\frac{-2}{m!}\right)^m (2m)!.$$  \hspace{1cm} (2.87)

This implies that \{\gamma(m) e_{2m}, \ m \in \mathbb{N}\} is a genetic basis with constants of multiplication $\frac{\gamma(m) \gamma(n)}{\gamma(l)} C_{lmn}$.

A weight $\omega : \mathcal{B} \rightarrow \mathbb{C}$ can be defined by

$$\omega(x) = \langle \omega, x \rangle = \sum_m \omega m x m,$$  \hspace{1cm} (2.88)

for all $x = \sum_m x_m e_{2m}$, where

$$\omega m := \gamma(m)^{-1}.$$  \hspace{1cm} (2.89)

Suppose that the series on the rhs of eq. (2.88) converges. $\omega$ obeys the “dual” fixed point equation

$$\sum_l \omega l C_{lmn} = \omega m \omega n.$$  \hspace{1cm} (2.90)

Let us remark that the algebra $(\mathcal{B}, \times \beta)$ is not weighted if $\beta \neq 1$. This can be shown as follows. Suppose that there exists a weight $\omega$. Consider

$$\omega(e_0 \times \beta e_{2m}) = \beta^{2m} \omega(e_{2m}) = \omega(e_0) \cdot \omega(e_{2m})$$  \hspace{1cm} (2.91)

for all $m \in \mathbb{N}$. This implies $\omega(e_{2m}) = 0$ for all $m \geq 1$. Using

$$\omega(e_{2m} \times \beta e_{2n}) = \sum_l C_{lmn}^m \omega(e_{2l}) = C_{0mn}^m = 0$$  \hspace{1cm} (2.92)

we get a contradiction to $C_{0mn}^m \neq 0$. Thus, there exists no weight $\omega$ for $(\mathcal{B}, \times \beta)$, $\beta \neq 1$.

For $z \in \mathcal{B}$ and the weight $\omega$ defined by eq. (2.88) above we get the formal series

$$\omega(z) = \sum_m \frac{(2m)!}{m!} \left(-\frac{1}{2}\right)^m z_m, \quad z = \sum_m z_m e_{2m}.$$  \hspace{1cm} (2.93)

If $z$ is a fixed point of $(\mathcal{B}, \times \beta)$, i. e., $z \times \beta z = z$, the following equation is valid

$$\omega(S_{\beta^{-1}}(z)) = \omega^2(z).$$  \hspace{1cm} (2.94)

For the following Lemma define

$$B(l, m, n) := \sqrt{\frac{(2l)!}{(2m)! (2n)!} C_{lmn}^{mn}}.$$  \hspace{1cm} (2.95)

We see that $B(l, m, n)$ is a symmetric function.

The following estimate is due to [KW91].
Lemma 2.2.6 For all \( l, m, n \in \mathbb{N} \) such that \(|m - n| \leq l \leq m + n\) we have

\[
B(l, m, n) \leq \min_{(l', m', n') \in \{(l, m, n), (m, n, l), (n, l, m)\}} \left( \frac{2l'}{l' + m' - n'} \right) \left( \frac{m' + n'}{l'} \right).
\] (2.96)

**Proof:** We obtain

\[
B^2(l, m, n) = \frac{(2l)!}{(2m)!(2n)!} (C_l^{mn})^2 = \frac{(2l)!}{(2m)!(2n)!} \frac{(2m)!^2(2n)!^2}{(l + m - n)!^2(l + n - m)!^2(l + n - m)!^2(m + n - l)!^2}.
\] (2.97)

This implies

\[
B^2(l, m, n) = \left( \frac{2l}{l + m - n} \right) \frac{(2m)!^2(2n)!}{(l + m - n)!^2(l + n - m)!^2(m + n - l)!^2}.
\] (2.98)

Thus

\[
B^2(l, m, n) = \left( \frac{2l}{l + m - n} \right) \left( \frac{m + n}{l} \right)^2 \frac{l!^2}{(m + n)!^2(l + m - n)!(l + n - m)!}.
\] (2.99)

We have to prove that

\[
I_{m,n,l} := \frac{(2m)!^2(2n)!}{(m + n)!^2(l + m - n)!(l + n - m)!} \leq 1
\] (2.100)

For \( l \geq k \), define the function

\[
f_k(l) := \frac{l!^2}{(l - k)!(l + k)!}.
\] (2.101)

Since

\[
f_k(l) = \frac{(l + 1 - k)(l + 1 + k)}{(l + 1)^2} f_k(l + 1) = \frac{(l + 1)^2 - k^2}{(l + 1)^2} f_k(l + 1) \leq f_k(l + 1),
\] (2.102)

and

\[
I_{m,n,l} = \frac{f_{m-n}(l)}{f_{m-n}(m + n)}, \quad l \leq m + n
\] (2.103)

we see that eq. (2.100) holds. This proves the assertion. □
Lemma 2.2.7 For all $a, b \in \mathbb{C}$, we have
\[
\exp(a\Phi) : \gamma \cdot : \exp(b\Phi) : \gamma = \exp(\gamma ab) : \exp((a + b)\Phi) : \gamma
\] (2.104)
and, for $4\gamma^2 ab < 1$, $\mathcal{L}_q := (1 + 2\gamma q)^{-1}$,
\[
\exp(a\Phi^2) : \gamma \cdot : \exp(b\Phi^2) : \gamma = (\mathcal{L}_a\mathcal{L}_b\mathcal{L}_c^{-1}(a\mathcal{L}_a + b\mathcal{L}_b))^{\frac{1}{2}} : \exp((a\mathcal{L}_a + b\mathcal{L}_b)\mathcal{L}_c^{-1}(a\mathcal{L}_a + b\mathcal{L}_b)\Phi^2) : \gamma
\] (2.105)

Lemma 2.2.8 For $a, b \in \mathbb{C}$ define
\[
f_m(a) := \exp\left(\frac{a^2}{2}\right) \frac{a^m}{m!}, \quad g_n(a) := \exp\left(-\frac{a^2}{2}\right) \frac{(-a^2)^m}{(2m)!}, \quad h_m(a) := \exp\left(\frac{a^2}{2}\right) \frac{a^{2m}}{(2m)!}.
\]
Then we have
\[
\sum_{|m-n| \leq l \leq m+n} f_m(a) f_n(b) C^{mn}_l = f_l(a + b),
\] (2.106)
for all $a, b \in \mathbb{C}$;
\[
\sum_{|m-n| \leq l \leq m+n} g_m(a) g_n(b) C^{mn}_l = g_l(a + b),
\] (2.107)
for all $a, b \in \mathbb{R}$; and
\[
\sum_{|m-n| \leq l \leq m+n} h_m(a) h_n(b) C^{mn}_l = \frac{1}{2} [h_l(a + b) + h_l(a - b)],
\] (2.108)
for all $a, b \in \mathbb{C}$. Furthermore, we have for all $a, b \in \mathbb{R}$,
\[
\sum_{|m-n| \leq l \leq m+n} \frac{a^m b^n}{m! n!} C^{mn}_l = \mathcal{N}(a, b) \frac{c(a, b)^l}{l!},
\] (2.109)
where
\[
\mathcal{N}(a, b) := \frac{1}{\sqrt{1 - 4ab}}, \quad c(a, b) := \frac{a + b - 4ab}{1 - 4ab}.
\] (2.110)
Generally, we have for all $N \geq 2$, $a_1, \ldots, a_N \in \mathbb{R}$,
\[
\sum_{l_1, l_2, \ldots, l_{N-2}, m_1, m_2, \ldots, m_N} \mathcal{C}^{m_1 m_2} C_{l_1 l_2}^{m_3} \ldots C_{l_N}^{m_{N-2} m_N} \frac{a_1^{m_1} \cdots a_N^{m_N}}{m_1! \cdots m_N!} \mathcal{N}(a_1, \ldots, a_N) c(a_1, \ldots, a_N)^l,
\] (2.111)
where
\[
\mathcal{N}(a_1, \ldots, a_N) := (\mathcal{L}_{a_1} \cdots \mathcal{L}_{a_N} \mathcal{L}_{-(a_1 c_{a_1} + \cdots + a_N c_{a_N})})^{\frac{1}{2}},
\]
\[
c(a_1, \ldots, a_N) := (a_1 \mathcal{L}_{a_1} + \cdots + a_N \mathcal{L}_{a_N} \mathcal{L}_{-(a_1 c_{a_1} + \cdots + a_N c_{a_N})})
\] (2.112)
and $L_q = (1 + 2q)^{-1}$. Moreover,
\begin{equation}
\sum_{m,n} \frac{a^{m+n}}{m!n!} C_l^{mn} = \frac{1}{l!} \frac{1}{\sqrt{1 - 4a^2}} \left( \frac{2a}{1 + 2a} \right)^l. \tag{2.114}
\end{equation}

Proof: Eq. (2.104) of Lemma 2.2.7 implies
\begin{equation}
\sum_{m,n} \frac{a^m b^n}{m!n!} C_l^{mn} = \frac{(a + b)^l}{l!} \exp\{ab\}. \tag{2.115}
\end{equation}

From this equation follows eq. (2.107). Eq. (2.108) can be implied by replacing $a, b \in \mathbb{R}$ by $ia, ib$ in eq. (2.115) and taking the real part. Using eq. (2.115) and change the signs of $a$ and $b$ and add the four resulting equalities we obtain
\begin{equation}
\sum_{m,n} \frac{a^{2m} b^{2n}}{(2m)! (2n)!} C_l^{mn} = \frac{1}{2(2l)!} \left[ \exp\{ab\}(a + b)^{2l} + \exp\{-ab\}(a - b)^{2l} \right]. \tag{2.116}
\end{equation}

This implies Eq. (2.103). Eq. (2.110) follows from eq. (2.105) of Lemma 2.2.7. The generalization eq. (2.112) follows also immediately from eq. (2.105) of Lemma 2.2.7. Furthermore, eq. (2.116) follows from eq. (2.110) if we change the signs of $a$ and $b$ and add the four resulting equalities. Eq. (2.114) follows eq. (2.110) if we set $a = b$. □

For $C_l^{mn}$ there exists recursive equations.

Lemma 2.2.9 We have for all $l, m, n \geq 0$
\begin{equation}
l C_l^{mn} = m C_{l-1}^{m-1,n} + n C_{l-1}^{m,n-1}. \tag{2.117}
\end{equation}

Proof: Set $\gamma = 1$. Partial integration gives
\begin{equation}
\langle F(\Phi) : \Phi^l \rangle = \langle \frac{\partial}{\partial \Phi} F(\Phi) : \Phi^{l-1} \rangle. \tag{2.118}
\end{equation}

Then, we obtain,
\begin{equation}
\langle \Phi^m : \Phi^n : \Phi^l \rangle = m \langle \Phi^{m-1} : \Phi^n : \Phi^{l-1} \rangle + n \langle \Phi^m : \Phi^{n-1} : \Phi^{l-1} \rangle. \tag{2.119}
\end{equation}

Using
\begin{equation}
\langle \Phi^m : \Phi^n : \Phi^l \rangle = C_l^{mn} \langle \Phi^l : \Phi^l \rangle = C_l^{mn} l!, \tag{2.120}
\end{equation}

we obtain
\begin{equation}
C_l^{mn} l! = C_{l-1}^{m-1,n} m (l - 1)! + C_{l-1}^{m,n-1} n (l - 1)!. \tag{2.121}
\end{equation}

This implies the assertion. □

36
2.3 Norms and Scalar Products

When we follow a renormalization group flow, we only want to keep track explicitly of relevant and marginal degrees of freedom. We want to control remainders using norm estimates. It turns out that the norms have to be well adjusted to the class of functions obtained by RGTs. Roughly speaking we treat the large field problem with the help of norms. From the Banach algebra point of view we obtain a number of interesting estimates useful in a structural analysis.

This section introduces some norms which will be useful in hierarchical renormalization group studies. We define a norm \( \| \cdot \|_{\rho} \) for the representations of the algebra by functions. Then we introduce supremums norm \( \| \cdot \|_{\rho,\infty} \) on \( \mathbb{R}^\infty \) which will be useful for \( \epsilon \)-expansions and other fixed point methods. Furthermore, we define a norm \( \| \cdot \|_{\rho,0,0} \). We show that the norm \( \| \cdot \|_{\rho,0,0} \) of the UV fixed point \( Z_{UV} \) is infinite. It was shown by Koch and Wittwer [KW91] that the norm \( \| \cdot \|_{\rho,0,0} \) of the infrared fixed point \( Z_{IR} \) is finite for \( \beta = L^{-1-\frac{2}{d}} \), in \( d = 3 \) dimensions. Furthermore, we define a norm \( \| \cdot \|_{\langle \gamma \rangle} \) from a scalar product \( \langle \cdot , \cdot \rangle \).

This norm is useful for deriving an upper bound for the (double-well) infrared fixed points.

2.3.1 The norm \( \| \cdot \|_{\rho} \)

Our first norm is a supremum norm with a large field regulator. Let \( F \) be a real analytic function and \( \rho \in \mathbb{R} \). Define a norm by

\[
\| F \|_{\rho} := \sup_{\Phi \in \mathbb{R}} \exp\{-\rho \Phi^2\} |F(\Phi)|. \tag{2.122}
\]

Denote the vector space of all real analytic functions \( Z \) with finite norm \( \| Z \|_{\rho} < \infty \) by \( \mathcal{B}_\rho \). Obviously,

\[
\mathcal{B}_{\rho_1} \subseteq \mathcal{B}_{\rho_2}, \quad \text{if } \rho_1 \leq \rho_2 \tag{2.123}
\]

and

\[
\| A \cdot B \|_{\rho_1+\rho_2} \leq \| A \|_{\rho_1} \cdot \| B \|_{\rho_2}. \tag{2.124}
\]

Lemma 2.1.1 implies

\[
\| \mathcal{R}_{\gamma,L^d}^\beta (F) \|_{\frac{1}{1-2L^d \gamma \rho}} \leq (1 - 2\gamma \rho)^{-\frac{L^d}{2}} \| F \|_{\rho} \tag{2.125}
\]

for \( \rho < \frac{1}{2\gamma} \). Therefore, we see that

\[
\mathcal{R}_{\gamma,L^d}^\beta (\mathcal{B}_\rho) \subseteq \mathcal{B}_{F(\rho)}, \tag{2.126}
\]

where

\[
F(\rho) := \frac{L^d \beta^2 \rho}{1 - 2L^d \gamma \rho}. \tag{2.127}
\]
For
\[ \rho^* := \frac{1 - L^d \beta^2}{2L^d \gamma}, \]
we have \( F(\rho^*) = \rho^* \). Since \( \mathcal{B}_{\rho_1} \subseteq \mathcal{B}_{\rho_2} \) if \( \rho_1 \leq \rho_2 \) and \( F(\rho) \leq \rho \) if \( \rho \leq \rho^* \), we obtain
\[ \mathcal{R}_\gamma L^d (\mathcal{B}_{\rho}) \subseteq \mathcal{B}_{\rho}, \tag{2.129} \]
for \( \rho \leq \rho^* \). Thus we see that the normed space \( \mathcal{B}_{\rho^*} \) is invariant under RGTs. For the special case \( \beta = L^{1 - \frac{d}{2}} \), we have \( \rho^* = \rho^*_{HT} \), where
\[ \rho^*_{HT} := -\frac{L^2 - 1}{2\gamma L^d} = \frac{1 - L^d \beta^2}{2\gamma L^d}. \tag{2.130} \]
For the special case \( \beta' = L^{-1 - \frac{d}{2}} \), we have \( \rho^* = \rho^*_{IR} \),
\[ \rho^*_{IR} := \frac{1 - L^{-2}}{2\gamma L^d} = \frac{1 - L^d \beta'^2}{2\gamma L^d}. \tag{2.131} \]

**Lemma 2.3.1** Suppose that \( \rho \in \mathbb{R}^+, \rho < \frac{1}{4\gamma}, 2\beta^2 < 1 \). The product \( * \) defines a mapping \( \mathcal{B}^2_{\rho} \to \mathcal{B}^\prime_{\rho} \), where
\[ \rho' := \frac{2\rho \beta^2}{1 - 4\gamma \rho}, \tag{2.132} \]
and \( \mathcal{B}^\prime_{\rho} \subseteq \mathcal{B}_{\rho} \) if \( \rho \leq \rho^* := \frac{1 - 2\beta^2}{4\gamma} \).

**Proof:** Suppose that \( A, B \in \mathcal{B}_{\rho} \). Then
\[ : A : \gamma : B : \gamma \in \mathcal{B}_{2\rho}. \tag{2.133} \]
This implies
\[ E_\gamma (A : B) \in \mathcal{B}_{\frac{1 - 2\rho \beta^2}{4\gamma \rho}}. \tag{2.134} \]
Furthermore,
\[ S_\beta E_\gamma (A : B) \in \mathcal{B}_{\frac{1 - 2\rho \beta' \rho^2}{4\gamma \rho}}. \tag{2.135} \]
\( \rho \leq \rho^* \) implies \( \rho' \leq \rho \). Thus by eq. (2.133) \( A * B \in \mathcal{B}_{\rho'} \). \( \Box \)

In the Banach algebra picture the following norm takes best care of the particular form of the multiplication table. Define a norm \( \| \cdot \|_\rho^{(\infty)} \) on \( \mathbb{R}^\infty \) by
\[ \| a \|_\rho^{(\infty)} := \sup_{t \in \mathbb{N}} (|a_t| \rho^t!), \tag{2.136} \]
Define a function $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$G(\rho) := 2\beta^2 \rho - 2.$$  \hfill (2.137)

For $a, b \in \mathbb{R}^\infty$ define

$$(a \times_\beta b)_l := \beta^{2l} \sum_{|m-n| \leq l \leq m+n} \mathcal{C}^mn_l m \ a_n \ b_m.$$ \hfill (2.138)

**Lemma 2.3.2** For $\rho \in \mathbb{R}_+$, $a, b \in \mathbb{R}^\infty$ we have

$$\|a \times_\beta b\|_{(\infty)} \leq \frac{1}{\sqrt{1 - (\frac{2}{\rho})^2}} \|a\|_{(\infty)} \|b\|_{(\infty)}.$$ \hfill (2.139)

**Proof:** We have

$$|(a \times_\beta b)_l| \rho^l \overset{!}{\leq} (\beta^2 \rho)^l \sum_{|m-n| \leq l \leq m+n} \left(\frac{1}{\rho}\right)^m \left(\frac{1}{\rho}\right)^n l! \mathcal{C}^mn_l ||a||_{(\rho)} ||b||_{(\rho)}.$$ \hfill (2.140)

Using Lemma 2.2.8, eq. (2.114) we obtain

$$|(a \times_\beta b)_l| \rho^l \overset{!}{\leq} (\beta^2 \rho)^l \frac{1}{\sqrt{1 - (\frac{2}{\rho})^2}} \left(\frac{\rho}{1 + \rho}\right)^l ||a||_{(\rho)} ||b||_{(\rho)}.$$ \hfill (2.141)

This implies

$$|(a \times_\beta b)_l| (G^{-1}(\rho))^l \rho^l \overset{!}{\leq} \frac{1}{\sqrt{1 - (\frac{2}{\rho})^2}} ||a||_{(\rho)} ||b||_{(\rho)}.$$ \hfill (2.142)

Replacing $\rho$ by $G(\rho)$ we obtain the assertion. \hfill \Box

The function $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ has a fixed point

$$\rho^*_\infty := \frac{2}{2\beta^2 - 1}.$$ \hfill (2.143)

We have $G(\rho) \leq \rho$ for $\rho \leq \rho^*_\infty$ and $G(\rho) \geq \rho$ for $\rho \geq \rho^*_\infty$. This and Lemma 2.3.2 proves the following lemma.

**Lemma 2.3.3** For $\rho \leq \rho^*_\infty$, $(\mathbb{R}^\infty_\rho, \times_\beta)$ is a Banach algebra and for $\beta \in (\frac{1}{\sqrt{2}}, 1)$ we have

$$\|a \times_\beta b\|_{(\rho^*_\infty)} \leq \frac{1}{2\beta \sqrt{1 - \beta^2}} \|a\|_{(\rho^*_\infty)} \|b\|_{(\rho^*_\infty)}.$$ \hfill (2.144)
Suppose that for $a = \sum_m a_m e_{2m}$, $e_{2m} =: \Phi^{2m} _1$, the weight $\omega(a)$ of $a$ is defined by eq. (2.88).

**Lemma 2.3.4** If $\|a\|_{\rho_{\infty}}^{(\infty)} < \infty$ then the series expansion of $\omega(a)$ is convergent for all $\beta = 2^d n \frac{d}{n}$, $d > 2$, and

$$|\omega(a)| \leq \frac{\|a\|_{\rho_{\infty}}^{(\infty)}}{4(1 - \beta^2)}. \quad (2.145)$$

**Proof:** $\|a\|_{\rho_{\infty}}^{(\infty)} < \infty$ implies

$$|a_l| < \frac{C}{l!} (\rho_{\infty}^*)^{-l}, \quad C := \|a\|_{\rho_{\infty}}^{(\infty)}. \quad (2.146)$$

Thus, using $\frac{(2m)!}{m!^2} \leq \frac{1}{2} 4^m$,

$$|\omega(a)| \leq \sum_m \frac{(2m)!}{m!} 2^{-m} |a_m| \leq \sum_m \frac{(2m)!}{m!} \frac{C}{(2\rho_{\infty}^*)^m} \leq C \frac{2}{2} \sum_m (2\beta^2 - 1) m = \frac{C}{4(1 - \beta^2)}, \quad \square \quad (2.147)$$

**2.3.2 The norm $\| \cdot \|_{\rho,\alpha,\delta}$**

Define a norm $\| \cdot \|_{\rho,\alpha,\delta}$ on $\mathbb{R}^\infty$ by

$$\|(a_0, a_1, a_2, \ldots)\|_{\rho,\alpha,\delta} := \sum_{n=0}^{\infty} |a_n| n!^\alpha (n + 1)^\delta \rho^n, \quad (2.148)$$

for all $\alpha \in \mathbb{R}$, $\rho \in \mathbb{R}_+$, $a := (a_0, a_1, a_2, \ldots) \in \mathbb{R}^\infty$. Then

$$\mathbb{R}^{\infty}_{\rho,\alpha,\delta} := \{a \in \mathbb{R}^\infty| \|a\|_{\rho,\alpha,\delta} < \infty\}, \| \cdot \|_{\rho,\alpha,\delta} \quad (2.149)$$

is a Banach space. We will use the notation $\| \cdot \|_{\rho,\alpha,\delta} = \| \cdot \|_{\rho}$ for $\alpha = \delta = 0$. For $a, b \in \mathbb{R}^\infty$ define a product

$$(a * b)_l := \sum_{\substack{m, n \in \mathbb{N} \colon m - n \leq l \leq m + n}} \beta^{l+m+n} A_{l}^{mn} a_m b_n, \quad (2.150)$$

where

$$A_{l}^{mn} := \sqrt{\frac{(2l)!}{(2m)! (2n)!}} \Omega_{l}^{mn}. \quad (2.151)$$

Consider the function $g : \mathbb{R}^\infty \to \mathcal{B}$ defined by

$$g(a)(\Phi) := a(\Phi) := \sum_{n, n \in \mathbb{N}} \frac{\beta^n}{\gamma^n} \frac{a_n}{\sqrt{(2n)!}} \Phi^{2n}. \quad (2.152)$$
$g$ is a $\ast_{\beta}$-homomorphism of $(\mathbb{R}^\infty, \ast_{\beta})$ into $(\mathcal{B}, \ast_{\beta})$, i.e.

$$g(a \ast_{\beta} b)(\Phi) = a(\Phi) \ast_{\beta} b(\Phi).$$

(2.153)

Define a function $G_\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$G_\alpha(\rho) := 2^{1+\alpha}(\beta + \beta^2 \rho).$$

(2.154)

**Lemma 2.3.5** For $\alpha, \rho \in \mathbb{R}_+, \delta \geq 0$ and $a, b \in \mathbb{R}^\infty$, we have

$$\|a \ast_{\beta} b\|_{\rho, \alpha, \delta} \leq \|a\|_{G_\alpha(\rho), \alpha, \delta} \|b\|_{G_\alpha(\rho), \alpha, \delta}.$$  (2.155)

**Proof:** Using Lemma 2.2.2 and $(m + n)! \leq 2^{m+n} m! n!$ we obtain

$$\|a \ast_{\beta} b\|_{\rho, \alpha, \delta} = \sum_l \left| \sum_{m,n \geq l \leq m+n} \beta^{l+m+n} a_m b_n A_l^{mn} |l|^{\alpha} (l+1)^{\delta} \rho^l \right|$$

$$\leq \sum_l \left| \sum_{m,n \geq l \leq m+n} |a_m||b_n| \beta^{l+m+n} \left( \frac{2(m+n)}{2m} \right) \left( \frac{m+n}{l} \right) \right|$$

$$\leq \sum_l \left| \sum_{m,n \geq l \leq m+n} |a_m||b_n| (m+1)^{\delta} (n+1)^{\delta} \rho^l \right|$$

$$\leq \sum_l \left| \sum_{m,n \geq l \leq m+n} |a_m||b_n| (2\beta)^{m+n-l} \right|$$

$$\leq \sum_{m,n} |a_m||b_n| (m+1)^{\delta} (2\beta)^{m+n-l} (2\beta^2 \rho)^l$$

$$\leq \|a\|_{G_\alpha(\rho), \alpha, \delta} \|b\|_{G_\alpha(\rho), \alpha, \delta}.$$

(2.156)

This proves the assertion. □

The function $G_\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ has a fixed point

$$\rho_\ast^\alpha := \frac{2^{1+\alpha}\beta}{1-2^{1+\alpha}\beta^2}.$$  (2.157)

We have $G_\alpha(\rho) \leq \rho$ for $\rho \geq \rho_\ast^\alpha$ and $G_\alpha(\rho) \geq \rho$ for $\rho \leq \rho_\ast^\alpha$. This and Lemma 2.3.5 proves the following Lemma.

**Lemma 2.3.6** For $\rho \geq \rho_\ast^\alpha$ and $\beta < 2^{-\frac{1+\alpha}{2}}, (\mathbb{R}^\infty_{\rho, \alpha, \delta}, \ast)$ is a Banach algebra.
Lemma 2.3.7 For all $n \geq 1$, we have
\[
\frac{(2n)!}{n!^2} = q_n 2^{2n},
\] (2.158)
where
\[
q_n := \prod_{k=1}^{n} \frac{2k - 1}{2k}, \quad \frac{1}{\sqrt{n}} \leq q_n \leq \frac{1}{2}. \tag{2.159}
\]
Furthermore,
\[
\lim_{n \to \infty} q_n \sqrt{n} = \frac{1}{\sqrt{\pi}} \tag{2.160}
\]
and
\[
\sqrt{2n - 1} q_n \leq \sqrt{\frac{2}{\pi}} \exp\{-\frac{1}{4n}\}. \tag{2.161}
\]

**Proof:** Wallis’ formula gives
\[
\sqrt{\frac{\pi}{2}} = \lim_{n \to \infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n - 2)}{3 \cdot 5 \cdot 7 \cdots (2n - 1)} \sqrt{2n}. \tag{2.162}
\]
Thus
\[
\sqrt{\frac{\pi}{2}} = \lim_{n \to \infty} \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n - 2)^2}{(2n - 1)!} \sqrt{2n} = \lim_{n \to \infty} \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots 2n^2}{(2n)!2n} \sqrt{2n}. \tag{2.163}
\]
This implies
\[
\sqrt{\pi} = \lim_{n \to \infty} \frac{n! 2n}{(2n)! \sqrt{n}} = \lim_{n \to \infty} \frac{1}{q_n \sqrt{n}}. \tag{2.164}
\]
We have
\[
((2n - 1)q_n^2)^{-1} = \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots \frac{(2n - 2)^2}{(2n - 3) \cdot (2n - 1)} = \frac{2^2}{2^2 - 1} \cdot \frac{4^2}{4^2 - 1} \cdots \frac{(2n - 2)^2}{(2n - 2)^2 - 1} = \frac{1}{1 - \frac{1}{2^2}} \cdot \frac{1}{1 - \frac{1}{4^2}} \cdots \frac{1}{1 - \frac{1}{(2n - 2)^2}}. \tag{2.165}
\]
This implies
\[
\lim_{n \to \infty} \frac{(2n - 1)q_n^2}{(2n - 1)q_n^2} = \frac{1}{1 - \frac{1}{(2n)^2}} \cdot \frac{1}{1 - \frac{1}{(2n+2)^2}} \cdots. \tag{2.166}
\]
Therefore
\[
\frac{2}{\pi} \cdot \frac{1}{(2n - 1)q_n^2} = \frac{1}{1 - \frac{1}{(2n)^2}} \cdot \frac{1}{1 - \frac{1}{(2n+2)^2}} \cdots \tag{2.167}
\]
Since
\[
\frac{1}{(2n)^2} + \frac{1}{(2n+2)^2} + \ldots \geq \frac{1}{2n}
\] (2.168)
we obtain
\[
(2n - 1)q_n^2 = \frac{2}{\pi} \cdot (1 - \frac{1}{(2n)^2}) \cdot (1 - \frac{1}{(2n+2)^2}) \cdot \ldots
\]
\[
= \frac{2}{\pi} \exp\left\{-\frac{1}{(2n)^2} - \frac{1}{(2n+2)^2} - \ldots \right\} \leq \frac{2}{\pi} \exp\left\{-\frac{1}{2n}\right\}.
\] (2.169)
This proves the assertion. \(\square\)

Recall that the constants of multiplication \(S_{lm}^{mn}\) arise in the basis given by simple powers of fields. In terms of normal ordered products we found the constant of multiplication \(C_{lm}^{mn}\). Similar norm estimates to those above can also be proved in the normal ordered case.

For \(a = (a_0, a_1, a_2, \ldots), b = (b_0, b_1, b_2, \ldots) \in \mathbb{R}^\infty\) define a product
\[
(a \times b)_l := \sum_{m,n \in \mathbb{N} : |m-n| \leq l \leq m+n} \beta^{l+m+n} B(l, m, n) a_m b_n,
\] (2.170)
where
\[
B(l, m, n) := \sqrt{\frac{(2l)!}{(2m)! (2n)!}} C_{lm}^{mn}.
\] (2.171)
Consider the function \(f : \mathbb{R}^\infty \to B\) defined by
\[
f(a) := a(\Phi) := \sum_{n \in \mathbb{N}} \left(\frac{\beta}{\gamma}\right)^n \frac{a_n}{\sqrt{(2n)!}} : \Phi^{2n} : \gamma.
\] (2.172)
f obeys
\[
f(a \times b) = a(\Phi) \times \beta b(\Phi).
\] (2.173)
Define a function \(F_\alpha : \mathbb{R}_+ \to \mathbb{R}_+\) by
\[
F_\alpha(\rho) := 2^\alpha (\beta + 2\beta^2 \rho).
\] (2.174)

**Lemma 2.3.8** For \(\alpha \in \mathbb{R}, \rho \in \mathbb{R}_+, \delta \geq 0\) and \(a, b \in \mathbb{R}^\infty\), we have
\[
\|a \times b\|_{\rho, \alpha, \delta} \leq \|a\|_{F_\alpha(\rho), \alpha, \delta} \|b\|_{F_\alpha(\rho), \alpha, \delta}.
\] (2.175)

43
Proof: Using Lemma 2.2.6 and \((m + n)! \leq 2^{m+n+m!} \) we obtain

\[
\|a \tilde{\times} b\|_{\rho, \alpha, \delta} = \sum_{l} \left| \sum_{m, n: |m-n| \leq l \leq m+n} a_m b_n B(l, m, n) \right| \rho l \alpha (l + 1) \delta \rho^l
\]

\[
\leq \sum_{l} \sum_{m, n: |m-n| \leq l \leq m+n} |a_m| |b_n| \beta^{l+m+n} \sqrt{\frac{(2l)!}{l!}} (m+n)^{l+n} \beta^{m+n-l} \]

\[
2^{\alpha(m+n)} m! n! (m+1) \gamma (n+1) \delta \rho^l
\]

\[
\leq \sum_{l} \sum_{m, n: |m-n| \leq l \leq m+n} |a_m| |m!|^\alpha (m+1) \delta |b_n| |n!|^\alpha (n+1) \delta \beta^{m+n-l} \]

\[
2^{\alpha(m+n)} \left( \frac{m+n}{l} \right)^{(2\beta^2)^l} \rho^l
\]

\[
\leq \sum_{m, n} |a_m| |m!|^\alpha (m+1) \delta |b_n| |n!|^\alpha (n+1) \delta [2^{\alpha(m+n)} (\beta + 2\beta^2 \rho)]^{m+n}
\]

\[
= \|a\|_{F_{\alpha}(\rho), \alpha, \delta} \|b\|_{F_{\alpha}(\rho), \alpha, \delta}. \quad (2.176)
\]

This proves the assertion. \( \square \)

The function \( F_{\alpha} : \mathbb{R}_+ \to \mathbb{R}_+ \) has a fixed point

\[
\rho^* := \frac{2^\alpha \beta}{1 - 2^{1+\alpha} \beta^2}. \quad (2.177)
\]

We have \( F_{\alpha}(\rho) \leq \rho \) for \( \rho \geq \rho^* \) and \( F_{\alpha}(\rho) \geq \rho \) for \( \rho \leq \rho^* \). This and Lemma 2.3.8 proves the following Lemma.

**Lemma 2.3.9** For \( \rho \geq \rho^* \), and \( \beta < 2^{-\frac{1+\alpha}{2}} \) \((\mathbb{R}^{\infty}_{\rho, \alpha, \delta}; \tilde{\times}) \) is a Banach algebra.

In the following we will only consider the case \( \alpha = \delta = 0 \) and write \( \mathbb{R}^{\infty}_{\rho} = \mathbb{R}^{\infty}_{\rho, 0, 0} \). Suppose that for \( a = \sum_m a_m e_{2m}, \) \( e_{2m} := \Phi^{2m} : 1, \) the weight \( \omega(a) \) of \( a \) is defined by eq. (2.88).

**Lemma 2.3.10** If \( \|a\|_{\rho^*} < \infty \) then the series expansion of \( \omega(a) \) is convergent for all \( \beta = 2^{-\frac{2+\ell}{d}}, d > 2 \), and

\[
|\omega(a)| \leq \frac{\|a\|_{\rho^*}}{4(1 - \beta^2)}. \quad (2.178)
\]

**Proof:** \( \|a\|_{\rho^*} < \infty \) implies

\[
|a_l| < \frac{C}{\sqrt{(2l)!}} (1 - 2\beta^2)^l, \quad C := \|a\|_{\rho^*}. \quad (2.179)
\]
Thus, using \( \frac{(2m)!}{m!^2} \leq \frac{1}{2} 4^m \),

\[
|\omega(a)| \leq C \sum_m \frac{\sqrt{(2m)!}}{2^m m!^2} (1 - 2\beta^2)^m \leq \frac{C}{2} \sum_m (1 - 2\beta^2)^m
\]

\[
\leq \frac{C}{4(1 - \beta^2)}.
\]

(2.180)

### 2.3.3 The norm \( \| \cdot \|_{(\gamma)} \)

In the following we will write \( \| \cdot \|_\rho := \| \cdot \|_{\rho,0,0} \). For two functions \( F, G : \mathbb{R} \to \mathbb{R} \) define a scalar product by

\[
\langle F, G \rangle_{(\gamma)} := \int d\mu_\gamma(\Phi) : F(\Phi) : \gamma : G(\Phi) : \gamma
\]

and a norm

\[
\|F\|_{(\gamma)} := \sqrt{\langle F, F \rangle_{(\gamma)}}.
\]

(2.181)

Define a function \( e_b : \mathbb{R} \to \mathbb{R} \) by

\[
e_b(\Phi) := \exp\{b\Phi\}.
\]

(2.182)

Lemma 2.3.11 For all \( \Psi \in \mathbb{R} \), we have

\[
\langle F, e_{\Psi} \rangle_{(\gamma)} = F(\gamma \Psi)
\]

(2.183)

and

\[
\|e_{\Psi}\|_{(\gamma)}^2 = \exp\{\gamma \Psi^2\}.
\]

(2.184)

**Proof:** Using integration by parts we obtain

\[
\langle (\cdot)^n, e_{\Psi} \rangle_{(\gamma)} = \int d\mu_\gamma(\Phi) : (\Phi^n) : \gamma : \exp\{\Phi \Psi\} : \gamma
\]

\[
= \gamma^n \int d\mu_\gamma(\Phi) : \frac{\partial^n}{\partial \Phi^n} \exp\{\Phi \Psi\} : \gamma
\]

\[
= \gamma^n (\gamma \Psi)^n.
\]

(2.185)

This implies eq. (2.184). Eq. (2.185) follows since

\[
\|e_{\Psi}\|_{(\gamma)}^2 = \langle e_{\Psi}, e_{\Psi} \rangle_{(\gamma)} = e_{\Psi}(\gamma \Psi) = \exp\{\gamma \Psi^2\}.
\]

(2.186)

This proves the assertion. \( \square \)
Lemma 2.3.12 For $f := (f_0, f_1, \ldots) \in \mathbb{R}_c^\infty$, $c, \beta, \gamma \in \mathbb{R}$ define

$$F(\Phi) := \sum_{n=0}^{\infty} \frac{\beta^n}{\gamma^n} \frac{f_n}{\sqrt{(2n)!}} \Phi^{2n}.$$  \hfill (2.188)

Then we have

$$\|F\|_{(c)} \leq \|f\|_{c\beta\gamma}.$$  \hfill (2.189)

Proof: Consider the orthonormal basis $c_n \Phi^{2n} \sqrt{(2n)!}$. Then we have

$$\|F\|_{(c)}^2 = \sum_{n=0}^{\infty} \langle F, \Phi^{2n} \rangle_c < \langle \Phi^{2n}, F \rangle_c \frac{c^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} f_n^2 \frac{c^2}{\gamma^n} \leq \|f\|_{2\beta\gamma}^2. \quad \square$$  \hfill (2.190)

Lemma 2.3.13 For $F : \mathbb{R} \to \mathbb{R}$, $c, \Psi \in \mathbb{R}$, we have

$$|F(\Psi)| \leq \|F\|_{(c)} \exp\{\frac{\Psi^2}{c}\}.$$  \hfill (2.191)

Proof: Lemma 2.3.11 yields

$$F(\Psi) = \langle F, e_{\Psi} \rangle_c > c.$$  \hfill (2.192)

Thus

$$|F(\Psi)| \leq \|F\|_{(c)} \cdot \|e_{\Psi}\|_{(c)}.$$  \hfill (2.193)

Lemma 2.3.11, eq. 2.185, yields the assertion.  \square

The following Lemma presents an upper bound for the infrared fixed points. This bound shows the large field behaviour of the fixed points with finite norm. The infrared fixed point in three dimensions belongs to the class of functions with finite norm (cf. [KW91]).

Lemma 2.3.14 Suppose that $F(\Phi) = \sum_{n=0}^{\infty} f_n \Phi^{2n}$ and $\|f\|_{\rho} < \infty$ for some $\rho > \rho^*$, $f \ast_{\beta} f = f$ then there exists a positive constant $K$ such that

$$|F(\Psi)| \leq \exp\{K \Psi^2 \rho^* \}$$  \hfill (2.194)

for all $\Psi \in \mathbb{R}$.  

46
Proof: We have, using Lemma 2.3.3, \( \rho^*_+ = \frac{2^3}{1 + 2\beta^2} \)

\[
\|f\|_{\rho^*_+}^\alpha \leq \|f\|_{\rho^*_+ + 2\beta^2}^\alpha. 
\] (2.195)

Recall that we have set \( \alpha = \delta = 0 \) and \( \rho^*_+ \) is the fixed point of eq. (2.154) of the function \( G_\alpha \). Define a function \( H_f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) by

\[
H_f(\rho) = \|f\|_{\rho^*_+ + \rho}. 
\] (2.196)

By eq. (2.195) follows

\[
H_f(\rho) \leq H_f((2\beta^2)^n\rho) = 2^n 
\] (2.197)

for all \( n \in \mathbb{N} \). Define for \( q \in \mathbb{R}_+ \)

\[
c_q := \sup_{\rho \in [1, \frac{1}{2\beta^2}]} \rho^{-q} \ln H_f(\rho). 
\] (2.198)

Then, we have

\[
H_f(\rho) \leq \exp\{c_q \rho^q\} 
\] (2.199)

for all \( \rho \in [1, \frac{1}{2\beta^2}] \). Consider \( \rho \in [1, \infty) \). There exists \( n \in \mathbb{N} \) such that

\[
(2\beta^2)^n \rho \in [1, \frac{1}{2\beta^2}]. 
\] (2.200)

By eq. (2.197) and eq. (2.199) follows

\[
H_f(\rho) \leq \exp\{c_q 2^n(2\beta^2)^n q^q\}. 
\] (2.201)

For \( q = \frac{d}{2} \) we have

\[
2(2\beta^2)^q = 2 \cdot 2^{-\frac{2d}{2}} = 1.
\] (2.202)

Thus eq. (2.201) implies

\[
\|f\|_{\rho^*_+ + \rho} \leq \exp\{c_q \rho^q\} 
\] (2.203)

for all \( \rho \in [1, \infty) \). Using Lemma 2.3.13, we obtain

\[
|F(\Psi)| \leq \exp\{K\Psi^{\frac{2d}{d+2}}\} 
\] (2.204)

and Lemma 2.3.12 yields

\[
|F(\Psi)| \leq \|f\|_{\gamma} \exp\left\{ \frac{\Psi^2}{c} \right\}. 
\] (2.205)

By eq. (2.203) follows

\[
|F(\Psi)| \leq \exp\{c_q (\frac{\beta}{\gamma} - \rho^*_+)^q + \frac{\Psi^2}{c} \} 
\] (2.206)
for $\frac{c \beta}{\gamma} \geq 1$. Define
\[ g(c) := c q \left( \frac{c \beta}{\gamma} - \rho^*_\gamma \right)^q + \frac{\Psi^2}{c}. \] (2.207)

Let $x$ be defined such that
\[ c = \frac{\gamma}{\beta} (x + \rho^*_\gamma). \] (2.208)

Since $\rho^*_\gamma \geq 1$, we have $\frac{c \beta}{\gamma} \geq 1$. Thus eq. (2.207) implies
\[ g(c) \leq c q x^q + \beta \Psi^2 \frac{\gamma x}{\beta}. \] (2.209)

The minimal value of the rhs of inequality (2.209) reads
\[ x = \left( \frac{\beta \Psi^2}{\gamma q c_q} \right)^{\frac{1}{q+1}}. \] (2.210)

For this value of $x$ we get
\[ g(c) \leq K(\beta, \gamma, d) \Psi^{2d} \frac{\gamma q}{\beta}, \] (2.211)

where
\[ K(\beta, \gamma, d) := c q \left( \frac{\beta}{\gamma q c_q} \right)^{\frac{4d}{q+2}} + \frac{\beta}{\gamma} \left( \frac{\beta}{\gamma q c_q} \right)^{\frac{2d}{q+2}}. \] (2.212)

This proves the assertion. □

The following Lemma gives a bound on functions with a quadratic potential. The statement is that the norm is finite provided that the modulus of the (negative) mass squared is sufficiently small. The ultraviolet fixed point turns out to be on the border of the convergent region.

**Lemma 2.3.15** Consider $Q_c(\Phi) := \exp\{c \Phi^2\}$ for $c \in \mathbb{R}$. Then $Q_c(\Phi)$ is contained in the Banach algebra $(\mathbb{R}^\infty_{\rho^*_c}, \ast)$, iff $c < c_* := \frac{1-2\beta^2}{4\gamma}$. Then we have, for $\beta = 2^{-\frac{2+4}{d}}$ and $\rho^*_c = \frac{2\beta}{1-2\beta^2}$,
\[ \|Q_c\|_{\rho^*_c} \leq 1 + \left( \frac{2}{\pi} \right)^{\frac{1}{4}} \frac{2c \rho^*_c}{\beta} \frac{2c \rho^*_c}{\beta} \frac{-2c \rho^*_c}{\beta}. \] (2.213)

Let $Z_{UV}$ be the UV fixed point of the RG transformation $\mathcal{R}^\beta_{\gamma,2}$. Then $Z_{UV}$ is not contained in the Banach algebra $(\mathbb{R}^\infty_{\rho^*_c}, \ast)$.

**Proof:** Using the definition of the UV fixed point $Z_{UV}$ we obtain
\[ Z_{UV}(\Phi) = 2^{-\frac{1}{4}} \exp\{c_* \Phi^2\}. \] (2.214)
Consider
\[ Q_c = (a_0, a_1, a_2, \ldots) \in \mathbb{R}_\rho^\infty, \]
where
\[ a_n := \left(\frac{\gamma}{\beta}\right)^n \sqrt{\frac{(2n)!}{n!^2}} c^n = \left(\frac{2\gamma c}{\beta}\right)^n \sqrt{q_n}. \]
(2.215)

Then, using Lemma 2.3.7,
\[ a_n = \left(\frac{2\gamma c}{\beta}\right)^n \left(\pi n\right)^{-\frac{1}{4}} + r_n, \]
(2.216)

where \( r_n \geq 0 \) and \( \lim_{n \to \infty} r_n \sqrt{n} = 0 \). Thus
\[ \|a\|_{\rho^*} = \sum_{n : n \in \mathbb{N}} \left(\frac{2\gamma \rho^*}{\beta}\right)^n \left(\pi n\right)^{-\frac{1}{4}} + r_n. \]
(2.217)

The series is convergent if
\[ c < \frac{\beta}{2\gamma \rho^*} = \frac{1 - 2\beta^2}{4\gamma} = c_*, \]
and divergent if
\[ c \geq \frac{\beta}{2\gamma \rho^*} = \frac{1 - 2\beta^2}{4\gamma} = c_. \]
(2.218)

For the convergent case we have, using 2.3.7
\[ \|a\|_{\rho^*} \leq 1 + \sum_{n : n \in \mathbb{N}} \left(\frac{2c\rho^*}{\beta}\right)^n \left(\pi n\right)^{-\frac{1}{4}} \exp\left\{-\frac{1}{8n}\right\} (2n - 1)^{-\frac{1}{4}}. \]
(2.219)

This implies
\[ \|a\|_{\rho^*} \leq 1 + \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \frac{2c\rho^*}{\beta} \frac{\beta}{1 - 2c\rho^*}. \]
(2.220)

This proves the assertion. \( \Box \)

**Lemma 2.3.16** Consider \( Q_c(\Phi) := \exp\{c\Phi^2\} \) for \( c \in \mathbb{R} \). Then \( Q_c(\Phi) \) is contained in the Banach algebra \( (\mathbb{R}_\rho^\infty, \tilde{\times}) \), if \( c < \frac{\beta}{2\gamma \rho^*} \). Then we have, for \( \beta = 2^{-\frac{244}{4\gamma}} \) and \( \rho^* = \frac{\beta}{1 - 2\beta^2} \),
\[ \|Q_c\|_{\rho^*} \leq 1 + \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \frac{2c\rho^*}{\beta} \frac{\beta}{1 - 2c\rho^*}. \]
(2.221)

Let \( Z_{UV} \) be the UV fixed point of the RG transformation \( R_{\gamma(1-\beta^2)} \). Then \( Z_{UV} \) is not contained in the Banach algebra \( (\mathbb{R}_\rho^\infty, \tilde{\times}) \).

49
Proof: The UV fixed point $Z_{UV}$ of the RG transformation $\mathcal{R}_\gamma^{\beta}$ reads

$$Z_{UV}(\Phi) = 2^{-\frac{1}{2}} \exp\{c_s \Phi^2\}, \quad (2.224)$$

where

$$c_s := \frac{1 - 2\beta^2}{4\gamma(1 - \beta^2)}. \quad (2.225)$$

Consider

$$Q_c = (a_0, a_1, a_2, \ldots) \in \mathbb{R}_\rho^\infty, \quad (2.226)$$

where

$$a_n = \left(\frac{\gamma}{\beta}\right)^n \frac{\sqrt{(2n)!}}{n!} c^n. \quad (2.227)$$

Lemma 2.3.7 shows that

$$a_n = \left(\frac{2\gamma c}{\beta}\right)^n \left((\pi n)^{-\frac{1}{4}} + r_n\right), \quad (2.228)$$

where $r_n \geq 0$ and $\lim_{n \to \infty} r_n \sqrt{n} = 0$. Thus

$$\|a\|_{\rho} = \sum_{n: n \in \mathbb{N}} \left(\frac{2\gamma c}{\beta}\right)^n \left((\pi n)^{-\frac{1}{4}} + r_n\right). \quad (2.229)$$

The series is convergent, for $\rho = \rho^*$, if

$$c < \frac{1 - 2\beta^2}{2\gamma} \quad (2.230)$$

and divergent if

$$c \geq \frac{1 - 2\beta^2}{2\gamma}. \quad (2.231)$$

For the convergent case we have, using Lemma 2.3.7

$$\|a\|_{\rho^*} = 1 + \sum_{n: n \in \mathbb{N}} \left(\frac{2\gamma c}{\beta}\right)^n \left(\frac{2}{\pi}\right)^\frac{1}{4} \exp\left(\frac{1}{8n}\right) (2n - 1)^{-\frac{1}{4}}. \quad (2.232)$$

This implies

$$\|a\|_{\rho^*} \leq 1 + \left(\frac{2}{\pi}\right)^\frac{1}{4} \frac{2\gamma c}{\beta} \left(\frac{2\gamma c}{\beta}\right)^n \quad (2.233)$$

Since

$$\exp\{c_s \Phi^2\} : \gamma = L_{\gamma}^{\frac{1}{2}} \exp\{\frac{c_s}{1 + 2\gamma c_s} \Phi^2\}, \quad (2.234)$$

we see that $f^{-1}(Z_{UV}) \in \mathbb{R}_\rho^\infty$ iff

$$\frac{c_s}{1 - 2\gamma c_s} < \frac{1 - 2\beta^2}{2\gamma}. \quad (2.235)$$
This is equivalent to
\[
c_* < \frac{1 - 2\beta^2}{4\gamma(1 - \beta^2)}.
\] (2.236)
By eq. (2.225) this is not valid. This proves the assertion. \(\Box\)

Let us consider an approximate RGT where we only keep track of the first \(N\) terms in the expansion
\[
A(\Phi) = \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{(2n)!}} \Phi^{2n} \gamma^n.
\] (2.237)
This leads us to define the following truncated product depending on the order \(N\) of truncation.

For \(a, b \in \mathbb{R}\) and \(N \in \mathbb{N}\) define a product
\[
(a \bullet_{\beta} b)_l := \sum_{m,n \in \mathbb{N}, l \leq m+n} a_m b_n A_N(l,m,n),
\] (2.238)
where \(A_N(l,m,n) := A(l,m,n)\) if \(l \geq N\) and \(= 0\) if \(l < N\).

**Lemma 2.3.17** For \(\alpha, \rho \in \mathbb{R}_+, \delta \geq 0, N \in \mathbb{N}, q_N \) defined by eq. (2.159) and \(a, b \in \mathbb{R}\), we have
\[
\|a \bullet_{\beta} b\|_{\rho,\alpha,\delta} \leq \sqrt{q_N} \|a\|_{G_{\alpha}(\rho),\alpha,\delta} \|b\|_{G_{\alpha}(\rho),\alpha,\delta}.
\] (2.239)

Suppose that \(\omega : \mathcal{B} \to \mathbb{C}\) is a weight of the algebra \((\mathcal{B}, \times_1)\). For \(z \in \mathcal{B}\) define \(p(z) \in \mathbb{R}_+\) by
\[
p(z) := |\omega(z)|.
\] (2.240)
\(p\) is a pre-norm of \(\mathcal{B}\) which satisfies
\[
p(x \times_1 y) = p(x) p(y).
\] (2.241)
From a pre-norm we obtain a norm by the following canonical construction. Define an equivalence relation on \(\mathcal{B}\) by \(x \sim y\) iff \(x - y \in \ker \omega\). Then we conclude that \(p\) is a norm on \(\mathcal{B}/\sim\).

Define
\[
u_m := \frac{e_{2m}}{\sqrt{(2m)!}} = \frac{\Phi_{2m}}{\gamma^m \sqrt{(2m)!}},
\]
\[
u_m := \frac{f_{2m}}{\sqrt{(2m)!}} = \frac{\Phi_{2m}}{\gamma^m \sqrt{(2m)!}}.
\] (2.242)
We have
\[
u_m = \sum_{n=0}^{m} U_{mn} u_n,
\] (2.243)
where

\[ U_{mn} = U_{mn}(1), \quad U_{mn}(\alpha) := \sqrt{\frac{(2n)!}{(2m)!}} E_{2m,2n}(\alpha) = \sqrt{\frac{(2m)!}{(2n)!}} \frac{\alpha^{m-n}}{(m-n)!}. \]  

(2.244)

**Lemma 2.3.18** For all \( m, n \in \mathbb{N}, m \geq n \) we have

\[ |U_{mn}(\alpha)| \leq \binom{m}{n} (2\alpha)^{m-n}. \]  

(2.245)

**Proof:** We have, for \( m \geq n \),

\[ |U_{mn}(\alpha)| \leq \binom{m}{n} \sqrt{\frac{(2m)!}{m!^2}} \left( \sqrt{\frac{(2n)!}{n!^2}} \right)^{-1} \alpha^{m-n} = \binom{m}{n} \sqrt{\frac{\alpha_m}{\alpha_n}} (2\alpha)^{m-n} \leq \binom{m}{n} (2\alpha)^{m-n}. \]  

(2.246)

This proves the assertion. \( \square \)

**Lemma 2.3.19** Consider for \( \delta \in \mathbb{R}_+ \)

\[ G(\Phi) = \sum_m a_m u_m, \quad : G_\delta(\Phi) = \sum_n b_n u_n. \]  

(2.247)

We have

\[ b_n = \sum_{m: m \geq n} a_m U_{mn}(\frac{-\delta}{\gamma}) \]  

(2.248)

and

\[ \|b\|_\rho \leq \|a\|_{\rho + 2\frac{\delta}{\gamma}}. \]  

(2.249)

**Proof:** We have

\[ \|b\|_\rho = \sum_n |b_n|\rho^n = \sum_n \sum_{m: m \geq n} |a_m U_{mn}(\frac{-\delta}{\gamma})|\rho^n \leq \sum_m \sum_{n=0}^m |U_{mn}(\frac{-\delta}{\gamma})| |a_m|\rho^n \leq \sum_m \sum_{n=0}^m \binom{m}{n} (2\frac{\delta}{\gamma})^{m-n} \rho^n |a_m| \leq \sum_m |a_m| (\rho + 2\frac{\delta}{\gamma})^m = \|a\|_{\rho + 2\frac{\delta}{\gamma}}. \]  

(2.250)

This proves the assertion. \( \square \)
2.4 Twodimensional Fixed Points

In this section the case $\beta = 1$ will be studied. Because of the singularity of the RGT $R^\beta_{-\beta,2}$ at $\beta = 1$ the $\times_1$-multiplication cannot be used for the study of RG fixed points. Therefore we will use the multiplication $\ast$ instead. We will show that there exists a continuum of periodic fixed points in this case.

The $\ast$-multiplication of two exponential functions is

$$\exp\{a\Phi\} \ast \exp\{b\Phi\} = \exp\{\frac{\gamma}{2}(a + b)^2\} \exp\{(a + b)\Phi\}. \quad (2.251)$$

Define, for $m \in \mathbb{Z}, q \in \mathbb{R},$

$$u_m := \exp\{2\pi im \frac{q}{\sqrt{\gamma}}\Phi\}. \quad (2.252)$$

Then

$$u_m \ast u_n = \exp\{-2\pi^2 q^2(m + n)^2\} u_{m+n}. \quad (2.253)$$

We see that the multiplication rule eq. (2.253) defines a nonassociative algebra, since

$$(u_m \ast u_n) \ast u_l = \exp\{-2\pi^2 q^2[(m + n)^2 + (m + n + l)^2]\} u_{l+m+n} \neq u_m \ast (u_n \ast u_l). \quad (2.254)$$

But the algebra defined by eq. (2.253) does define a power algebra. For

$$A := \sum_{m \in \mathbb{Z}} a_m u_m, \quad B := \sum_{n \in \mathbb{Z}} b_n u_n, \quad (2.255)$$

the $\ast$-multiplication of $A$ and $B$ yields

$$A \ast B = \sum_{l \in \mathbb{Z}} c_l u_l, \quad (2.256)$$

where

$$c_l := \exp\{-2\pi^2 q^2 l^2\} \sum_{m, n \in \mathbb{Z}: m+n=l} a_m b_n. \quad (2.257)$$

Thus, the fixed point equation

$$A \ast A = A \quad (2.258)$$

is equivalent to

$$a_l = \exp\{-2\pi^2 q^2 l^2\} \sum_{m, n \in \mathbb{Z}: m+n=l} a_m a_n. \quad (2.259)$$

The solution of the fixed point eq. (2.258) can be expressed in terms of the Theta-functions.

Define Theta-functions $\Theta_2(z, q)$ and $\Theta_3(z, q)$, for $|q| < 1, z \in \mathbb{C}$, by

$$\Theta_2(z, q) := \sum_{n \in \mathbb{Z}} q^n \exp\{2niz\} = 1 + 2 \sum_{n \in \mathbb{N}} q^n \cos\{2nz\},$$

$$\Theta_3(z, q) := - \sum_{n \in \mathbb{Z} + \frac{1}{2}} q^n \exp\{2niz\} = -2 \sum_{n \in \mathbb{N}} q^{(n+\frac{1}{2})^2} \cos\{(2n + 1)z\}. \quad (2.260)$$
Let us remark that \( \Theta_2 \) and \( \Theta_3 \) are even functions in \( z \). Furthermore, define

\[
\Theta_+(q) := \Theta_3(0, q^4) = \sum_{n \in 2\mathbb{Z}} q^{n^2},
\]

\[
\Theta_-(q) := -\Theta_2(0, q^4) = -\sum_{n \in 2\mathbb{Z}+1} q^{n^2},
\]

\[
\Theta_{1/2}(q) := \sum_{n : n \in 2\mathbb{Z}+\frac{1}{2}} q^{n^2}.
\] (2.261)

**Lemma 2.4.1** A solution of eq. (2.259) is given by

\[
a_l := \begin{cases} 
N_1 p^l & : l \text{ even} \\
N_2 p^l & : l \text{ odd}, 
\end{cases}
\] (2.262)

where

\[
p := \exp\{-4\pi^2 q^2\}, \quad N_1 := \frac{1}{\Theta_{1/2}(p^2)}, \quad N_2 := \frac{\Theta_{1/2}(p^2) - \Theta_+(p^2)}{\Theta_-(p^2) \Theta_{1/2}(p^2)}.\] (2.263)

**Proof:** For \( l \) even we have

\[
\sum_{m \in 2\mathbb{Z}} a_{l-m} a_m = N_1^2 \sum_{m \in 2\mathbb{Z}} p^{(l-m)^2+m^2} + N_2^2 \sum_{m \in 2\mathbb{Z}+1} p^{(l-m)^2+m^2}
\] (2.264)

and for \( l \) odd

\[
\sum_{m \in 2\mathbb{Z}} a_{l-m} a_m = N_1 N_2 \left[ \sum_{m \in 2\mathbb{Z}} p^{(l-m)^2+m^2} + \sum_{m \in 2\mathbb{Z}+1} p^{(l-m)^2+m^2} \right].
\] (2.265)

Shift \( m \to m + \frac{l}{2} \) we obtain, for \( l \) even

\[
\sum_{m \in 2\mathbb{Z}} p^{(l-m)^2+m^2} = \begin{cases} 
\Theta_+(p^2) p^l & : \frac{l}{2} \text{ even} \\
\Theta_-(p^2) p^l & : \frac{l}{2} \text{ odd}, 
\end{cases}
\] (2.266)

and

\[
\sum_{m \in 2\mathbb{Z}+1} p^{(l-m)^2+m^2} = \begin{cases} 
\Theta_-(p^2) p^l & : \frac{l}{2} \text{ even} \\
\Theta_+(p^2) p^l & : \frac{l}{2} \text{ odd}. 
\end{cases}
\] (2.267)

For \( l \) odd we obtain

\[
\sum_{m \in 2\mathbb{Z}} p^{(l-m)^2+m^2} = p^l \sum_{m \in 2\mathbb{Z}} p^{2m^2} = \Theta_{1/2}(p^2) p^l
\] (2.268)

and

\[
\sum_{m \in 2\mathbb{Z}+1} p^{(l-m)^2+m^2} = p^l \sum_{m \in 2\mathbb{Z}} p^{2m^2} = \Theta_{1/2}(p^2) p^l.
\] (2.269)
Therefore, we get
\[
\sum_{m \in \mathbb{Z}} a_{l-m}a_m = \begin{cases} 
 p^l [N_1^2 \Theta_+(p^2) + N_2^2 \Theta_-(p^2)] & : l \text{ even} \\
 p^l N_1 N_2 \Theta_{1/2}(p^2) & : l \text{ odd},
\end{cases}
\] (2.270)

Insertion of eq. (2.270) into eq. (2.259) yields
\[
N_1 = N_1^2 \Theta_+ + N_2^2 \Theta_- \\
N_2 = N_1 N_2 \Theta_{1/2}.
\] (2.271)

Eq. (2.271) is fulfilled by definition of $N_1$ and $N_2$. This proves the assertion. \qed

Define
\[
Z_*(\Phi) = \sum_{l : l \in \mathbb{Z}} a_l \exp\{2\pi il q \sqrt{\gamma} \Phi\},
\] (2.272)

where $a_l$ is defined in Lemma 2.4.1. Then we have
\[
Z_*(\Phi) = N_1 \sum_{l : l \in 2\mathbb{Z}} p^l \exp\{2\pi il q \sqrt{\gamma} \Phi\} + N_2 \sum_{l : l \in 2\mathbb{Z}+1} p^l \exp\{2\pi il q \sqrt{\gamma} \Phi\}.
\] (2.273)

Using the definitions of the Theta-functions we get, for $p := \exp\{-4\pi^2 q^2\}$, the following solutions of the fixed point equation
\[
Z_*(\Phi) = \frac{\Theta_3(4\pi q \sqrt{\gamma} \Phi, p)}{\Theta_1(\Phi)} - \frac{\Theta_2(4\pi q \sqrt{\gamma} \Phi, p)(\Theta_{1/2}(p^2) - \Theta_+(p^2))}{\Theta_-(p^2) \Theta_{1/2}^2(p^2)}.
\] (2.274)

**Lemma 2.4.2** $Z_*(q)(\Phi)$ defined by eq. (2.274) is an even periodic solution of $\mathcal{R}_{\gamma,2}^\beta(Z) = Z$, for all $q \in \mathbb{R}$, $|q| < 1$.

In the following we will consider the algebra of even periodic functions. Suppose that
\[
u_{m} = \nu_{m}.
\] (2.275)

Then
\[
\nu_{m} * \nu_{n} = g(m + n)\nu_{m+n},
\] (2.276)

where
\[
g(k) := \exp\{-2\pi^2 q^2 k^2\}.
\] (2.277)

Define
\[
o_{m} := \cosh\{2\pi q \sqrt{\gamma} m \Phi\}.
\] (2.278)

Then, we have
\[
o_{m} * o_{n} = \frac{1}{2} (g(m + n)o_{m+n} + g(m - n)o_{m-n}).
\] (2.279)
For
\[ A = \sum_{m \in \mathbb{N}} a_m o_m \]  
the fixed point equation \( A \ast A = A \) is equivalent to
\[ a_l = \frac{1}{2} \left( \sum_{m,n \in \mathbb{N}, \ m+n=l} g(m+n)a_m a_n + \sum_{m,n \in \mathbb{N}, \ m-n=l} g(m-n)a_m a_n \right). \]

### 2.5 Gauss-Hermite-Formula for Constants of Multiplication

In this section we want to find a coordinate transformation such that the constants of multiplication \( B_{mn}^l \) become diagonal. Diagonality means here that \( B_{mn}^l = 0 \) if \( m \neq n \).

In the following we will explain the Gaussian method for interpolating integrals. For two continuous functions \( f, g \) define the scalar product
\[ (f, g) := \int_{-\infty}^{\infty} dx \exp\{-x^2\} f(x) g(x). \]

Hermite polynomials \( H_n(x) \) are orthogonal with respect to this scalar product
\[ (H_m, H_n) = (H_m, H_m) \delta_{m,n}. \]

The generating function \( S(\Phi, z) \) for Hermite polynomials reads
\[ S(\Phi, z) = \sum_{n=0}^{\infty} \frac{H_n(\Phi)}{n!} z^n = \exp\{-z^2 + 2z\Phi\}. \]

Therefore
\[ : \exp\{z\Phi\} :_\gamma = \exp\{-\frac{\gamma}{2} z^2 + z\Phi\} = S\left(\frac{\Phi}{\sqrt{2\gamma}}, \sqrt{\frac{\gamma}{2}} z\right). \]

Thus
\[ : \Phi^n :_\gamma = \left(\frac{\gamma}{2}\right)^{\frac{n}{2}} H_n\left(\frac{\Phi}{\sqrt{2\gamma}}\right). \]

**Theorem 2.5.1** Let \( x_1, x_2, \ldots, x_n \) be the zeroes of \( H_n(x) \) and \( w_1, \ldots, w_n \) be the solutions of
\[ \sum_{i=1}^{n} H_k(x_i) w_i = \begin{cases} (H_0, H_0), & k = 0, \\ 0, & k \in \{1, \ldots, n-1\}. \end{cases} \]

If \( P \) is a polynomial with grade \( \leq 2n - 1 \), we have
\[ \int_{-\infty}^{\infty} dx e^{-x^2} P(x) = \sum_{i=1}^{n} w_i P(x) \]
and for $f \in C^{2n}$ there exists $\zeta \in \mathbb{R}$ such that
\[
\int_{-\infty}^{\infty} dx \, e^{-x^2} f(x) = \sum_{i=1}^{n} w_i f(x) + \frac{f^{(2n)}(\zeta)}{(2n)!} (H_n, H_n).
\] (2.289)

The coefficients $w_i$ are given by
\[
w_i = \frac{2^{n+1} n! \sqrt{\pi}}{|H_{n+1}(x_i)|^2}.
\] (2.290)

Lemma 2.1.4 eq. (2.30) yields
\[
<: \Phi^l : \gamma \cdots \Phi^m : \gamma \cdots \Phi^m : \gamma > = C_i^{mn} \, \gamma^{l+m+n} \, l!.
\] (2.291)

Thus, using eq. (2.286), $\gamma = \frac{1}{2}$,
\[
C_i^{mn} = 2^{l+m+n} \, l!^{-1} < H_l \cdot H_m \cdot H_n > \gamma.
\] (2.292)

Theorem 2.5.1 implies
\[
C_i^{mn} = \frac{1}{2^{l+m+n} \sqrt{\pi} l!} \int_{-\infty}^{\infty} d\Phi \, \exp\{-\Phi^2\} \, H_l(\Phi)H_m(\Phi)H_n(\Phi).
\] (2.293)

**Corollary 2.5.1** Let $x_1, x_2, \ldots, x_N$ be the zeroes of $H_n(\Phi)$ and $w_i := \frac{2^{N+1} N! \sqrt{\pi}}{|H_{N+1}(x_i)|^2}$. Then, for $m + n + l \leq 2N - 1$, we have
\[
C_i^{mn} = \frac{1}{2^{l+m+n} \sqrt{\pi} l!} \sum_{i=1}^{N} w_i H_l(x_i)H_m(x_i)H_n(x_i).
\] (2.294)

For $l_{\text{max}} \in \mathbb{N}$ and $N \in \mathbb{N}$, $N \geq \frac{1}{2}(3l_{\text{max}} + 1)$ the equations
\[
a_l = \sum_{m,n=0}^{l_{\text{max}}} \beta^l C_i^{mn} a_m a_n, \quad l \in \{0, \ldots, l_{\text{max}}\}
\] (2.295)

imply
\[
b_j = \sum_{i=1}^{N} Q_{ji} \, b_i^2, \quad j \in \{1, \ldots, N\},
\] (2.296)

where
\[
Q_{ji} := \sum_{i=0}^{l_{\text{max}}} \frac{2^{-l} \beta^l}{l! \sqrt{\pi}} \, w_i \, H_l(x_i)H_l(x_j)
\] (2.297)

and
\[
b_j := \sum_{l=0}^{l_{\text{max}}} 2^{-\frac{1}{2}} H_l(x_j) \, a_l.
\] (2.298)
Proof: Eq. (2.294) follows immediately from eq. (2.293) and Theorem 2.5.1. Eq. (2.298) and eq. (2.295) yield
\[ b_j = \sum_{l=0}^{l_{max}} 2^{-\frac{l}{2}} H_l(x_j) \sum_{m,n=0}^{l_{max}} \beta^l C_{lmn}^l a_m a_n. \] (2.299)
Insertion of eq. (2.294) into the rhs of eq. (2.299) gives
\[ b_j := \sum_{l=0}^{l_{max}} 2^{-\frac{l}{2}} H_l(x_j) \sum_{m,n=0}^{l_{max}} \beta^l \sum_{i=1}^{N} w_i H_l(x_i) H_m(x_i) H_n(x_i) a_m a_n \]
\[ = \sum_{l=0}^{l_{max}} \frac{2^{-l} \beta^l}{l! \sqrt{\pi}} \sum_{i=1}^{N} b_i^2 w_i H_l(x_i) H_l(x_j) \]
\[ = \sum_{i=1}^{N} Q_{ji} b_i^2. \] (2.300)

Lemma 2.5.1 For \( i, j \in \{1, \ldots, N\} \) let \( Q_{ij} \) be defined by eq. (2.297). Then, we have
\[ \sum_{i=1}^{N} Q_{ji} = 1, \] (2.301)
for all \( j \in \{1, \ldots, N\} \).

Proof: Use Corollary 2.5.1 and the solution \( a_l = \delta_{l,0} \) of eq. (2.293). Insertion of this solution into eq. (2.298) and the use of \( H_0(x) = 1 \) yields \( b_j = 1 \), for all \( j \in \{1, \ldots, N\} \). Thus eq. (2.296) implies the assertion. \( \Box \)

The matrix for the linearized RG equation reads
\[ U_{nl} := 2\beta^l \sum_m C_{lmn}^l a_m. \] (2.302)

Lemma 2.5.2 The eigenvalue problem
\[ U(a)u = \lambda u \] (2.303)
is equivalent to
\[ V(b)v = \lambda v, \] (2.304)
where
\[ V_{ji}(b) := 2Q_{ji}b_i, \quad i, j \in \{1, \ldots, N\} \] (2.305)
and \( b \) is related to \( a \) by eq. (2.298) and correspondingly \( v \) is related to \( u \).
Lemma 2.5.3 For \( j \in \{1, \ldots, N\} \), \( m \in \{1, \ldots, l_{max}\} \), we have
\[
\sum_{i=1}^{N} Q_{ji} H_m(x_i) = \beta^m H_m(x_j).
\] (2.306)

2.6 \( \epsilon \)-Expansion

In this section we study the \( \epsilon \)-expansion for the infrared fixed points. For dimensions \( d_* = \frac{2l_*}{l_* - 1} \), \( l_* \in \{2,3,\ldots\} \) the ultraviolet fixed point \( Z_{UV} \) is a bifurcation point. In \( d_* \) dimensions the UV fixed point bifurcates into the UV fixed point and the \( l_* \)-well fixed point if the dimension is lowered. We will use the algebra with base \( \{u_m, m = 0, 1, 2, \ldots\} \) and multiplication
\[
u_m \times \beta \nu_n = \sum_{|m-n| \leq l \leq m+n} \beta^{2l} C_{l,m} \nu_l,
\] (2.307)
where \( \beta := 2^{\frac{2-d}{2d}} \).

Lemma 2.6.1 Suppose that \( l_* \in \{2,3,\ldots\} \) and \( d_* := \frac{2l_*}{l_* - 1} \), \( \beta_* := 2^{\frac{2-d_*}{2d_*}} \). For \( d := d_* - \epsilon \) and \( \beta := 2^{\frac{2-d}{2d}} \) consider the \( \epsilon \)-expansion
\[
\beta^{-2l} = \beta_*^{-2l} \sum_{m=0}^{\infty} (\beta^{-2l})^m \epsilon^m.
\] (2.308)

Furthermore, suppose that \( z = \sum_{m=0}^{\infty} z_m u_m \) is a fixed point, i.e.
\[
z = z \times \beta z.
\] (2.309)

This is equivalent to
\[
S_{\beta^{-1}}(z) = z \times_1 z,
\] (2.310)
where \( S_{\beta^{-1}} \) is a linear operator defined by
\[
S_{\beta^{-1}}(u_m) = \beta^{-2m} u_m,
\] (2.311)
for all \( m \in \mathbb{N} \). Consider the \( \epsilon \)-expansion
\[
z = u_0 + \sum_{k=1}^{\infty} \epsilon^k z^{(k)}.
\] (2.312)

Then we have
\[
z^{(1)} \in \ker[(1 - 2u_0)\times_{\beta_*}] = \{\lambda u_m \mid \lambda \in \mathbb{C}\} =: \mathcal{B}_{l_*}.
\] (2.313)

Let \( P_{l_*} : \mathcal{B} \to \mathcal{B} \) be the projection operator defined by \( P_{l_*}(\mathcal{B}) = \mathcal{B}_{l_*} \). Consider the \( \epsilon \)-expansion of \( S_{\beta^{-1}} \)
\[
S_{\beta^{-1}} = \sum_{m=0}^{\infty} \epsilon^m S_{\beta^{-1}} \circ S_{\beta^{-1}}^m,
\] (2.314)
where the linear operator \( S_{\beta}^{(m)} : \mathcal{B} \to \mathcal{B} \) is defined by
\[
S_{\beta}^{(m)}(u_l) = (\beta^{-2l})^m u_l.
\] (2.315)

Then we have
\[
z^{(1)} = \alpha u_l, \quad \alpha := \frac{2(\beta^{-2l})^{(1)}}{c_{l+1}^{(1)}}
\] (2.316)

and
\[
(1 - P_l) z^{(k)} = \left[ (1 - 2u_0) \times m \right]^{-1} \sum_{m=1}^{k-1} [-S_{\beta}^{(k-m)}((1 - P_l) z^{(m)})] + (1 - P_l) (z^{(k-m)} \times m z^{(m)}),
\] (2.317)

and for \( k \geq 2 \)
\[
P_l z^{(k)} = \frac{1}{(\beta^{-2l})^{(1)}} \left[ \sum_{m=1}^{k-1} (\beta^{-2l})(k+1-m) P_l z^{(m)} - \sum_{m=2}^{k-1} P_l (z^{(k+1-m)} \times m z^{(m)}) + 2P_l (z^{(1)} \times m (1 - P_l) z^{(k)}) \right].
\] (2.318)

**Proof:** The fixed point equation gives
\[
\sum_{m=0}^{\infty} e^m S_{\beta}^{(m)}(u_0 + \sum_{k=1}^{\infty} e^k z^{(k)}) = u_0 + 2u_0 \times \sum_{k=1}^{\infty} e^k z^{(k)} + \sum_{k_1, k_2=1}^{\infty} e^{k_1+k_2} z^{(k_1)} \times m z^{(k_2)}.
\] (2.319)

Since \( S_{\beta}^{(0)} = 1 \) and
\[
S_{\beta}^{(m)}(u_0) = \delta_{m,0} u_0,
\] (2.320)
we obtain
\[
(1 - 2u_0) \times m z^{(k)} = \sum_{m=1}^{k-1} (-S_{\beta}^{(k-m)}(z^{(m)}) + z^{(k-m)} \times m z^{(m)}).
\] (2.321)

For \( k = 1 \), we obtain
\[
(1 - 2u_0) \times m z^{(1)} = 0.
\] (2.322)

This shows eq. (2.313). For \( k = 2 \) we get
\[
(1 - 2u_0) \times m z^{(2)} = -S_{\beta}^{(1)}(z^{(1)}) + z^{(1)} \times m z^{(1)}.
\] (2.323)

Thus
\[
S_{\beta}^{(1)}(P_l(z^{(1)})) = P_l(z^{(1)} \times m z^{(1)}).
\] (2.324)
Therefore, $z^{(1)} = \alpha u_*$,

$$\alpha(\beta^{-2l_\ast})^{(1)} = \beta^{2l_\ast}C^l_{l_\ast} \alpha^2. \quad (2.325)$$

Since $2\beta^{2\ast} = 1$ we see eq. (2.316). Applying the projection operator $P_\ast$ to eq. (2.321) we obtain

$$0 = \sum_{m=1}^{k-1} [-S^{(k-m)}(P_\ast(z^{(m)})) + P_\ast(z^{(k-m)} \times \beta_* z^{(m)})]. \quad (2.326)$$

For $k \geq 3$, eq. (2.326) implies

$$-2P_\ast(z^{(k-1)} \times \beta_* z^{(1)}) + S^{(1)}(P_\ast(z^{(k-1)})) = - \sum_{m=1}^{k-2} S^{(k-m)}(P_\ast(z^{(m)})) + \sum_{m=2}^{k-2} P_\ast(z^{(k-m)} \times \beta_* z^{(m)}). \quad (2.327)$$

Since

$$S^{(1)}(u_*) = (\beta^{-2l_\ast})^{(1)}$$

we obtain

$$S^{(1)}(P_\ast(z^{(k-1)})) = \sum_{m=1}^{k-2} S^{(k-m)}(P_\ast(z^{(m)})) - \sum_{m=2}^{k-2} P_\ast(z^{(k-m)} \times \beta_* z^{(m)}) + 2P_\ast(z^{(1)} \times \beta_* (1 - P_\ast)(z^{(k-1)})) \quad (2.329)$$

Using $S^{(1)}(u_*) = (\beta^{-2l_\ast})^{(1)}$ and replacing $k$ by $k+1$ in eq. (2.329) we obtain eq. (2.318). Applying projection operator $1 - P_\ast$ to eq. (2.321) we obtain eq. (2.317).

**Lemma 2.6.2** Let $z^{(k)}$ be the coefficients of $\epsilon^k$ in the $\epsilon$-expansion of a fixed point $z$ in $d_* - \epsilon$ dimensions, $d_* = \frac{2l_\ast}{1 - \beta^{-2l_\ast}}$. Then, we have

$$z^{(k)}_l = 0, \quad \text{if } l > l_* k. \quad (2.330)$$

In the following we will present a bound on the coefficients $z^{(k)}$ of the $\epsilon$-expansion. For the proof of this bound we will need the following Lemma.

**Lemma 2.6.3** Let $(\beta^{-2l})^{(m)}$ be defined by eq. (2.308). Then we have

$$|(\beta^{-2l})^{(m)}| \leq \frac{1}{2} \left(\frac{2}{d_*}\right)^m (2\frac{d_*}{2} - 1). \quad (2.331)$$

Furthermore,

$$\sup_{l,l \neq l_*} \frac{1}{1 - 2\beta^{2l}} = \frac{1}{1 - 2\beta^{2l}}. \quad (2.332)$$
Proof: An explicit calculation yields

\[(\beta - 2l)^{(m)} = \sum_{k=0}^{m-1} \frac{1}{(m-k)!} \left( -\frac{2l \ln 2 \ d^*}{d^*} \right)^{m-k} \binom{m-1}{k} d_{-}^{m-k}. \tag{2.333} \]

Using \((m-1) \leq \frac{1}{2} 2^m\), we obtain eq. (2.331) By definition of \(\beta^*\) we have \(2\beta^*_* = 2 \frac{\lambda}{l^*}\). Therefore, \(1 - 2\beta^*_*\) is minimal for \(l = l^*_* \pm 1\). Since

\[2 \frac{\lambda}{l^*}_- - 1 > 1 - 2^{-\frac{\lambda}{l^*}}, \tag{2.334} \]

the assertion eq. (2.332) follows. \(\square\)

**Lemma 2.6.4** Let \(z^{(k)}\), for all \(k \in \mathbb{N}\), be the coefficients of the \(\epsilon\)-expansion. Then there exists constants \(K, C\) such that

\[|z_i^{(k)}| \leq K C^k (\rho^*_\infty)^{-l} \frac{(k-1)!}{l!}, \tag{2.335} \]

for all \(l, k \in \mathbb{N}\), where

\[\rho^*_\infty := 2 \frac{\lambda}{2\beta^2 - 1} > 1. \tag{2.336} \]

Proof: The recursive equations (2.318) and (2.317) for the coefficients of the \(\epsilon\)-expansion are written in components, for all \(l \neq l^*_*\),

\[z_i^{(k)} = [(1 - 2u_0) \times \beta^*_*]^{-1} \sum_{m=1}^{k-1} (-1)(\beta - 2l)^{(k-m)} z_i^{(m)} + (z^{(k-m)} \times \beta^*_* z^{(m)})_!], \tag{2.337} \]

and

\[z_{l^*_*}^{(k)} = \frac{1}{(\beta - 2l^*_*)!} \sum_{m=1}^{k-1} (\beta - 2l^*_*)^{(k+1-m)} z_{l^*_*}^{(m)} - \sum_{m=2}^{k-1} (z^{(k+1-m)} \times \beta^*_* z^{(m)})_! + 2P_{l^*_*} (z^{(1)} \times \beta^*_* (1 - P_{l^*_*})(z^{(k)}))]. \tag{2.338} \]

We proceed by induction. Suppose that eq. (2.333) holds for all \(l' < l\). Consider the case \(l \neq l^*_*\). Since \(z_i^{(k)} = 0\) if \(l > l^*_* k\) (see Lemma 2.6.2), we may suppose \(l \leq l^*_* k\). We want to find a better estimate of \(|(\beta - 2l)^{(m)}|\). For that, we distinguish two cases.

a) \(l \leq \frac{md^*}{2ln 2}\): By eq. (2.333) follows

\[| (\beta - 2l)^{(m)} | \leq \frac{1}{2} \left( \frac{2}{d^*} \right)^m \sum_{k=0}^{m-1} \frac{1}{k!} \left( \frac{2l \ln 2 d^*}{d^*} \right)^k. \tag{2.339} \]

62
Thus
\[ |(\beta^{-2l})^{(m)}| \leq \frac{1}{2} \left( \frac{2}{d_*} \right)^m 2 \frac{m}{m^2} = \frac{1}{2} \left( \frac{2e}{d_*} \right)^m. \] (2.340)

b) \( l > \frac{md}{2 \ln 2} \): By eq. (2.333) follows
\[ |(\beta^{-2l})^{(m)}| \leq \frac{1}{2} \left( \frac{2}{d_*} \right)^m \frac{m}{(m-1)!} \left( \frac{2l \ln 2}{d_*} \right)^{m-1} \] (2.341)
for all \( m \geq 2 \). Define
\[ I := \sum_{m=1}^{k-1} (\beta^{-2l})^{(k-m)} K C^m (m-1)!. \] (2.342)
Then we obtain, using eq. (2.341),
\[ I \leq K \sum_{m=1}^{k-1} \left( \frac{2}{d_*} \right)^m \frac{1}{(k-m-2)!} \left( \frac{2l \ln 2}{d_*} \right)^{m-1} C^{k-m} (m-1)!. \] (2.343)
Replacing \( m \) by \( k - m \) we obtain
\[ I \leq K \sum_{m=1}^{k-1} \left( \frac{2}{d_*} \right)^m \frac{1}{(m-2)!} \left( \frac{2l \ln 2}{d_*} \right)^{m-1} C^{k-m} (k-m-1)!. \] (2.344)
Using \( l \leq l_*k \) we get
\[ I \leq K C^k (k-1)! \frac{d_*^2}{4e \ln 2} \sum_{m=1}^{k-1} \frac{1}{(m-2)!} \left( \frac{4l_* \ln 2}{C d_*^2} \right)^m \frac{k^{m-1}}{(k-1) \cdots (k-m)}. \] (2.345)
Therefore
\[ \sum_{m=1}^{k-1} (\beta^{-2l})^{(k-m)} K C^m (m-1)! \leq K C^k (k-1)! F(C), \] (2.346)
where \( F \) is a function obeying
\[ \lim_{C \to \infty} F(C) = 0. \] (2.347)
Thus, eqs. (2.337), (2.340) and (2.346) imply
\[ |z_l^{(k)}| \rho_\infty l! \leq \frac{1}{1 - 2 \frac{1}{l_*}} \left[ \sum_{m=1}^{k-1} \left( \frac{1}{2} \left( \frac{2e}{d_*} \right)^m K C^m (m-1)! \right) + C^k (k-1)! F(C) + \frac{K^2}{2 \beta \sqrt{1 - \beta^2}} C^k (k-m-1)! (m-1)! \right]. \] (2.348)
By eqs. (2.348) follows eq. (2.335) for the case \( l \neq l_* \).
Consider now the case $l = l_*$: Using Lemma 2.6.3 and Lemma 2.3.3 of section 2.3 we obtain

$$
|z^{(k)}_{l_*}| \rho^{d_*}_{\infty} l_*! \leq \frac{1}{\sqrt{|(\beta - 2\beta_*)^{(1)}}|} \left[ \sum_{m=1}^{k-1} \frac{1}{4} \left( \frac{2}{d_*} \right)^m (2l_* - 1) \left( \frac{K}{C} \right)^m (m - 1)! \right] + \frac{K^2}{2\beta \sqrt{1 - \beta^2}} C^{k+1} (k - m)! (m - 1)!
$$

(2.349)

By eqs. (2.349) and (2.348) follows the assertion. \(\square\)

### 2.7 Convergent Approximation Methods

Infrared fixed points are solutions of quadratic equations with an infinite number of unknowns. For practical calculations a truncation of these equations to equation with a finite number of unknowns is necessary. For a RG analysis it is sufficient to study the truncated finite-dimensional case. The corresponding error made by truncation can be estimated. In this section such methods will be studied.

Firstly, we will explain the beta-function method (cf. [KW91], [P93]). By beta-function methods the RG flow of an infinite number of partition functions is reduced to a flow of a finite number of coupling constants. Consider the RG transformation in algebraic representation

$$
z' = z \times z.
$$

(2.350)

Consider a projection operator $P : \mathcal{B} \rightarrow \mathcal{B}$ such that the space $P(\mathcal{B})$ is finite dimensional. Suppose that an element $z_0 \in P(\mathcal{B})$ can be parametrized by $N$ variables $\gamma_0, \ldots, \gamma_{N-1}$, i.e., $z_0 = z_0(\gamma_0, \ldots, \gamma_{N-1})$. Define two multiplications $\circ$ and $\bullet$ by

$$
a \circ b := P(a \times b),
a \bullet b := (1 - P)(a \times b).
$$

(2.351)

Obviously,

$$
a \times b = a \circ b + a \bullet b.
$$

(2.352)

We split a partition function $z$ into a relevant part $z_0 := P(z) \in P(\mathcal{B})$ and an irrelevant part $r := (1 - P)(z) \in P^c(\mathcal{B}) := \mathcal{B} - P(\mathcal{B})$

$$
z = z_0 + r.
$$

(2.353)

Consider the RG flow of the irrelevant part $r \mapsto r'$

$$
r' = (z_0 + r) \bullet (z_0 + r).
$$

(2.354)
For \( \hat{\gamma} := (\gamma_0, \ldots, \gamma_{N-1}) \), \( z_0 = z_0(\hat{\gamma}) \) we want to define the fixed point \( r_* = r_*(\hat{\gamma}) \) of eq. (2.354), i.e.,
\[
r_* = z_0 \bullet z_0 + 2z_0 \bullet r_* + r_* \bullet r_*.
\]
(2.355)

Then, the beta-function \( b : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is defined by
\[
z_0(b(\hat{\gamma})) = z_0(\hat{\gamma}) \circ z_0(\hat{\gamma}) + 2z_0(\hat{\gamma}) \circ r_*(\hat{\gamma}) + r_*(\hat{\gamma}) \circ r_*(\hat{\gamma}).
\]
(2.356)

The fixed points of the RG eq. (2.350) are determined by the fixed points \( \hat{\gamma} \) of the beta-
function \( b \), i.e.,
\[
b(\hat{\gamma}^*) = \hat{\gamma}^*.
\]
(2.357)

For this fixed point \( \hat{\gamma}^* \), we see that
\[
z^* := z_0(\hat{\gamma}^*) + r_*(\hat{\gamma}^*)
\]
(2.358)
is a fixed point of eq. (2.350).

**Lemma 2.7.1** For \( N \in \mathbb{N} \) define a projection operator \( P_N : \mathcal{B} \rightarrow \mathcal{B} \) by
\[
P_N((z_0, z_1, z_2, \ldots)) := (z_0, z_1, z_2, \ldots, z_N, 0, 0, \ldots).
\]
(2.359)

Define a function \( F_{z_0} : \mathcal{B} \rightarrow \mathcal{B} \) by
\[
F_{z_0}(r) := z_0 \bullet z_0 + 2z_0 \bullet r + r \bullet r,
\]
(2.360)
for all \( z_0 \in P(\mathcal{B}) \). Define, for all \( x \in \mathcal{B} \), a norm by
\[
\|x\| := \sum_{n=0}^{\infty} \|x_n\| \rho^n, \quad \rho := \frac{\beta}{1 - 2\beta^2}.
\]
(2.361)

Let \( \kappa > 0 \) be a positive constant such that
\[
4\kappa \sqrt{q_N} < 1, \quad q_N := \prod_{k=1}^{N} \frac{2k - 1}{2k}\n\]
(2.362)
and define
\[
\epsilon(N, \kappa) = \epsilon := \frac{1}{2\sqrt{q_N}} \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdots (2n - 3)}{2^n n!} (4\kappa \sqrt{q_N})^n.
\]
(2.363)

Then, we have
\[
F_{z_0}(U_{\epsilon}) \subseteq U_{\epsilon}, \quad U_{\epsilon} := \{x \in \mathcal{B} \|x\| \leq \epsilon\}
\]
(2.364)
and \( F_{z_0} \) satisfies in \( U_{\epsilon} := \{x \in \mathcal{B} \|x\| < \epsilon\} \) the Lipschitz condition with \( K := 2\sqrt{q_N}(\kappa + \epsilon)\), i.e.,
\[
\|F_{z_0}(r_1) - F_{z_0}(r_2)\| < K \|r_1 - r_2\|.
\]
(2.365)

65
Suppose that $K < 1$. Define recursively
\[
    r^{(0)} := 0,
    r^{(n+1)} := F_{z_0}(r^{(n)}).
\] (2.366)

Then
\[
    \lim_{n \to \infty} r^{(n)} = \sum_{n=1}^{\infty} (r^{(n)} - r^{(n-1)}) = r_\ast
\] (2.367)
exists and is a fixed point of $F_{z_0}$, i.e., $F_{z_0}(r_\ast) = r_\ast$. Furthermore,
\[
    \|r_\ast\| \leq \frac{\sqrt{q_N} \kappa^2}{1 - K}.
\] (2.368)

Proof: By Lemma 2.3.17 we have for all $r \in U_\epsilon$
\[
    \|F_{z_0}(r)\| \leq 2 \sqrt{q_N} (\kappa + \epsilon)^2 = \epsilon.
\] (2.369)
This proves eq. (2.364). Moreover,
\[
    \|F_{z_0}(r_1) - F_{z_0}(r_2)\| = \|2_0 \bullet (r_1 - r_2) + r_1 \bullet (r_1 - r_2) + r_2 \bullet (r_1 - r_2)\|
\leq 2 \sqrt{q_N} (\kappa + \epsilon) \|r_1 - r_2\|.
\] (2.370)
This proves the Lipschitz condition eq. (2.365). By induction we can prove
\[
    \|r^{(n)} - r^{(n-1)}\| < K^{n-1} \sqrt{q_N} \kappa^2,
\] (2.371)
for all $n \in \mathbb{N}$, $n \geq 1$. Thus the series $\sum_{n=1}^{\infty} (r^{(n)} - r^{(n-1)})$ is convergent and
\[
    \|r_\ast\| \leq \sum_{n=1}^{\infty} \|r^{(n)} - r^{(n-1)}\| < \sum_{n=1}^{\infty} K^{n-1} \sqrt{q_N} \kappa^2 = \frac{\sqrt{q_N} \kappa^2}{1 - K}.
\] (2.372)
This proves the assertion. □

Since $\lim_{N \to \infty} q_N = 0$, the suppositions of Lemma 2.7.1 can be fulfilled for $N$ large enough.

**Corollary 2.7.1** For $\kappa \in \mathbb{R}_+$ let $N$ be large enough such that the suppositions of Lemma 2.7.1 are fulfilled. For $z_0 \in P_N(U_\kappa)$ let $r_\ast \in (1 - P_N)(B)$ be the solution of $F_{z_0}(r) = r$. Let $z_0$ be the solution of
\[
    z_0 = z_0 \circ z_0 + 2z_0 \circ r_\ast + r_\ast \circ r_\ast.
\] (2.373)
Then $z_\ast = z_0 + r_\ast$ is the solution of $z = z \times z$ and
\[
    \|z_\ast - z_0\| < \frac{\sqrt{q_N} \kappa^2}{1 - 2 \sqrt{q_N} (\kappa + \epsilon)},
\] (2.374)
where $\epsilon$ is defined by eq. (2.363).
Lemma 2.7.2 Suppose that for $z_0 \in P_N(B)$ such that
\[ z_0 = z_0 \circ z_0 \] (2.375)
the linear operator $(1 - 2z_0)\times$ is invertible. Define
\[ \kappa := \|z_0\|, \quad M := \|O_{z_0}\|, \] (2.376)
where $O_{z_0} := [(1 - 2z_0)\times]^{-1}$. Suppose that
\[ 4M^2\kappa^2 \sqrt{q_N} < 1 \] (2.377)
and define
\[ \epsilon := \frac{1}{2M} \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdots (2n - 3)}{2^n n!} \left(4\sqrt{q_N}M^2\kappa^2\right)^n. \] (2.378)
Define $G_{z_0} : B \to B$ by
\[ G_{z_0}(r) := O_{z_0}(z_0 \bullet z_0 + r \times r). \] (2.379)
Then we have
\[ G_{z_0}(\overline{U}_\epsilon) \subseteq \overline{U}_\epsilon \] (2.380)
and $G_{z_0}$ satisfies on $U_\epsilon$ the Lipschitz condition with $K := 2M\epsilon$. Suppose that $K < 1$. Define recursively
\[ r^{(0)} := 0 \]
\[ r^{(n+1)} := G_{z_0}(r^{(n)}). \] (2.381)
Then
\[ \lim_{n \to \infty} r^{(n)} = \sum_{n=1}^{\infty} (r^{(n)} - r^{(n-1)}) = r_* \] (2.382)
extists and is a fixed point of $G_{z_0}$, i.e. $G_{z_0}(r_*) = r_*$. Furthermore,
\[ \|r_*\| \leq \frac{M\sqrt{q_N}\kappa^2}{1 - K} \] (2.383)
and
\[ z = z_0 + r_* \] (2.384)
fulfills
\[ z = z \times z. \] (2.385)
Proof: For $r \in U_\epsilon$ we have, using Lemma 2.3.17,
\[ \|G_{z_0}(r)\| \leq M(\sqrt{q_N}\kappa^2 + \epsilon^2) = \epsilon. \] (2.386)
This proves eq. (2.380). Since
\[
\|G_{z_0}(r_1) - G_{z_0}(r_2)\| = \|U_{z_0}(r_\ast \times (r_1 - r_2) + r_2 \times (r_1 - r_2))\| \\
\leq 2M\epsilon \|r_1 - r_2\| \tag{2.387}
\]
we see that $G_{z_0}$ satisfies on $U_\epsilon$ the Lipschitz condition with $K = 2M\epsilon$. By induction we can prove
\[
\|r^{(n)} - r^{(n-1)}\| \leq K^{n-1} \sqrt{qN} M\kappa^2, \tag{2.388}
\]
for all $n \in \mathbb{N}$, $n \geq 1$. Thus the series $\sum_{n=1}^{\infty} (r^{(n)} - r^{(n-1)})$ is convergent and the bound (2.383) follows. Furthermore, $G_{z_0}(r_\ast) = r_\ast$ implies
\[
r_\ast = 2z_0 \times r_\ast + z_0 \cdot z_0 + r_\ast \times r_\ast. \tag{2.389}
\]
Using
\[
z_0 \times z_0 = z_0 \circ z_0 + z_0 \cdot z_0 = z_0 + z_0 \cdot z_0 \tag{2.390}
\]
we obtain
\[
z_0 + r_\ast = (z_0 + r_\ast) \times (z_0 + r_\ast). \tag{2.391}
\]
This proves the assertion. $\square$
Chapter 3

Conclusion

The main conclusion is that the underlying structure of the renormalization group is that of a nonassociative algebra. Different presentations of renormalization group transformations correspond to different choices of bases in this algebra. For instance the usual form given by Gauss integrable functions is a special choice. For theories which are not asymptotically free other choices of bases are better. A good choice is distinguished by the property that few parameters suffice to approximate well the renormalization group flow in a vicinity of fixed points.

The algebra is infinite dimensional. Nontrivial fixed points turned out to be in the closure of the algebra consisting of elements of finite degree. The closure is defined as completion with respect to a certain norm.

In this report we have been concerned with scalar hierarchical models. Our methods can also be applied to more general classes of models such as scalar models with many components, $O(N)$ invariant models, matrix models, gauge theories, and, we believe, to any quantum field theory. The first step in the analysis of a general model is the determination of the constants of multiplication encoding the algebra structure.

The algebra approach suggests an immediate generalization of quantum fields which are not perturbations of Gaussian measures. For instance we could start from any system of orthogonal polynomials with respect to any given non-Gaussian measure.

Certain physical aspects have direct counterparts in the language of the renormalization group algebra. Subalgebras correspond to universality classes. Idempotents are fixed points. Critical indices are given by eigenvalues of the linearized transformation at a fixed point. Invariant subspaces under the linearized renormalization again group correspond to algebra ideals.

In this report we have considered the case $L^d = 2$. More general we could treat the situation when $L^d$ is a positive integer larger than two. But physical properties and the general picture of fixed points and critical behavior are expected to be independent of the value of $L^d$.

The algebra approach is also useful for computational purposes. For example we have
presented a version of the $\epsilon$-expansion which can be computed iteratively, each step consisting of simple algebraic calculations. Inspecting the Banach algebra we have seen that the construction of fixed points can be reduced to a finite dimensional problem. These considerations lead to convergent expansions where error terms are controlled using norm estimates. An example of such a construction is the $\beta$-function technique. In terms of special basis we found exact solutions of the fixed point equation in two dimensions given by $\Theta$-functions. A possibility which we have touched upon in the second chapter is given by calculations with trees. Tree formulas are useful in various aspects. They can be used to derive improved convergent expansions. They can also be used to actually compute for example fixed points. Tree formulas are well known in the theory of renormalization. It seems not to have been realized before that the reason for their appearance is nonassociativity of renormalization group algebras.

An important aspect which we have left out is the construction of Greens functions. An amusing fact is that they can be computed exactly at the fixed points. At the fixed point also operator product expansions can be calculated exactly and be expressed in terms of the algebra. Outside the fixed points Greens functions also serve to define the critical manifold. An interesting open problem is the determination of the critical manifold in terms of the algebra.

We expect our methods to be generalizable to the full (non-hierarchical) model. There the main technical problem is to combine the renormalization group algebra with polymer algebras and thereby treat the appearance of nonlocal terms. An important difference to the hierarchical model is the wave function term. Work along these lines is in progress.

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We thank Klaus Pinn and Peter Wittwer for sharing their insights with us. Support by the Deutsche Forschungsgemeinschaft is gratefully acknowledged.
Chapter 4

Appendices

Appendix 1: Fixed Point Equation

We present the fixed point equation

\[ a_l = \beta^{2l} \sum_{m,n} C_{lm}^m a_m a_n, \quad (4.1) \]

for the special case \( d = 3 \), \( \beta = 2^{-\frac{d+2}{2d}} \) and \( l_{max} = 4 \):

\( a_0 = a_0^2 + 2a_1^2 + 24a_2^2 + 720a_3^2 + 40320a_4^2 \)
\[ a_1 = \frac{\sqrt{2}a_0 a_1}{2} + \sqrt{2}a_1^2 + 6\sqrt{2}a_1 a_2 + 24\sqrt{2}a_2^2 + 180\sqrt{2}a_2 a_3 + 1080\sqrt{2}a_3^2 \]
\[ + 10080\sqrt{2}a_3 a_4 + 80640\sqrt{2}a_4^2 \]
\( a_2 = \frac{2^{2/3}a_0 a_2}{8} + \frac{2^{2/3}a_1 a_2^2}{16} + 2^{2/3}a_1 a_2 + \frac{152^{2/3}a_1 a_3}{4} + \frac{92^{2/3}a_2 a_3^2}{2} + 60 2^{2/3}a_2 a_3 \]
\[ + 2102^{2/3}a_2 a_4 + 6752^{2/3}a_3^2 + 50402^{2/3}a_3 a_4 + 352802^{2/3}a_4^2 \]
\( a_3 = \frac{a_0 a_3}{16} + \frac{a_1 a_2}{16} + \frac{3}{4} a_1 a_3 + \frac{7}{2} a_1 a_4 + \frac{a_2^2}{2} + \frac{45 a_2 a_3}{4} + 84 a_2 a_4 + 75 a_3^2 \]
\[ + 1575 a_3 a_4 + 11760 a_4^2 \]
\( a_4 = \frac{\sqrt{2}a_0 a_4}{64} + \frac{\sqrt{2}a_1 a_3}{64} + \frac{\sqrt{2}a_1 a_4}{4} + \frac{\sqrt{2}a_2 a_3}{128} + \frac{3}{8} \sqrt{2}a_2 a_3 \]
\[ + \frac{21\sqrt{2}a_2 a_4}{4} + \frac{225 a_3 a_4}{64} + 105 \sqrt{2}a_3 a_4 + \frac{3675 \sqrt{2}a_4^2}{4} \]
Appendix 2: Constants of Multiplication

We present a table for all constants of multiplication $B(l, m, n)$ for $l, m, n \leq 4$. Coefficients which do not occur in the table are either permutations of the arguments of $B$ or are equal to zero.

| $l$ | $m$ | $n$ | $B(l, m, n)$ |
|-----|-----|-----|-------------|
| 0   | 0   | 0   | $=1.00000000$ |
| 1   | 1   | 0   | $=1.00000000$ |
| 1   | 1   | 1   | $2 \sqrt{2} = 2.82842712$ |
| 2   | 1   | 1   | $\sqrt{6} = 2.44948974$ |
| 2   | 2   | 0   | $=1.00000000$ |
| 2   | 2   | 1   | $4 \sqrt{2} = 5.65685425$ |
| 2   | 2   | 2   | $6 \sqrt{6} = 14.69693846$ |
| 3   | 2   | 1   | $\sqrt{15} = 3.87298335$ |
| 3   | 2   | 2   | $8 \sqrt{5} = 17.88854382$ |
| 3   | 3   | 0   | $=1.00000000$ |
| 3   | 3   | 1   | $6 \sqrt{2} = 8.48528137$ |
| 3   | 3   | 2   | $15 \sqrt{6} = 36.74234615$ |
| 3   | 3   | 3   | $40 \sqrt{5} = 89.44271912$ |
| 4   | 2   | 2   | $70 = 8.36660027$ |
| 4   | 3   | 1   | $2 \sqrt{7} = 5.29150262$ |
| 4   | 3   | 2   | $8 \sqrt{7} = 36.66060557$ |
| 4   | 3   | 3   | $15 \sqrt{70} = 125.49900400$ |
| 4   | 4   | 0   | $=1.00000000$ |
| 4   | 4   | 1   | $8 \sqrt{7} = 11.31370850$ |
| 4   | 4   | 2   | $28 \sqrt{6} = 68.58571280$ |
| 4   | 4   | 3   | $112 \sqrt{5} = 250.43961350$ |
| 4   | 4   | 4   | $70 \sqrt{70} = 585.66201860$ |

Table 4.1: The coefficients $B(l, m, n)$ for $l, m, n \leq 4$.  

72
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