Quantum Sturm-Liouville Equation, Quantum Resolvent, Quantum Integrals, and Quantum KdV: the Fast Decrease Case.

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Abstract

We construct quantum operators solving the quantum versions of the Sturm-Liouville equation and the resolvent equation, and show the existence of conserved currents. The construction depends on the following input data: the basic quantum field $O(k)$ and the regularization.

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Introduction.

To quantize nonlinear integrable systems, it was suggested by Faddeev et al. to quantize first the scattering data [1,2,6]. The scattering data are action-angle type fields, and their time dependence (in the Heinseberg formalism) for the case of fast decrease is trivial. At fixed time, scattering data are operators with some commutation relations. We will show, how to construct from scattering operators the observable fields, namely, the densities of the conserved currents. For simplicity, we consider the fast decrease case and the solitonless sector only. In the solitonless sector, the scattering data for the classical Sturm-Liouville equation are the transition coefficient $a(k)$ and the reflection coefficient $b(k)$, or their ratio $O(k) = b(k)/a(k)$. We will assume, that in the quantum theory we are given the operator $O(k)$, and we will proceed to define the observable fields in the theory. We will see, that we do not need specially to have a quadratic algebra for $a$ and $b$ in order to have the conserved currents in the theory.

In order to build the conserved currents, we use the Nonlinear Fourier Transform developed for the classical integrable equations in the paper [8]. In that paper, the authors investigated the nonlinear nonlocal change of variables, given by certain nonlinear functionals, with kernels given by homogeneous generalized functions. This nonlinear change of variables reduces the issue of integrability of the nonlinear equation to certain identities for rational symmetric functions. (Such property is similar to the usual Fourier Transform, which reduces the linear partial differential equation with constant coefficients to the polynomial algebraic equation). In the quantum theory, we use the functionals of the same type as we had in the classical case to express the conserved currents through ‘scattering data’ fields. Now the ‘scattering data’ are quantum fields, so one have to smear them to get rid of the singularities in the product, and give the prescription for the ordering. The best thing to do is to use the symmetric (Weyl) order. With the Weyl ordering, it is not necessary to know the commutation relations for the quantum field $O(k)$ in order to construct the conserved currents; in fact, we can start from any $O(k)$, and write the conserved currents as functionals of $O(k)$. (Of course, the correlation functions of such currents do depend on what $O(k)$ we have chosen, but the functionals, expressing the currents through $O(k)$, are universal.)
The construction of the conserved currents.

Let $O(k)$, “the basic field”, be some reasonable quantum field, defined as operator by matrix elements in a Hilbert space, or through bosonization in the style of [6]. Here $k \in \mathbb{R}$ (we consider the solitonless case, for the solitons one has to add finite number of points $\{i\kappa_m\}_{m=1}^M, \kappa_m \in \mathbb{R}^+\$).

Let $\{\Delta_n(k)\}_{n=1}^{\infty}$ be a sequence of $C^\infty$ test functions of fast decrease which converge to the $\delta$-function, say $\Delta_n(k) = \frac{1}{2\sqrt{\pi n^2}} \exp\left(-\frac{|k|^2}{4n^2}\right)$, $n = 1, 2, \ldots$ and $e_m(k)$ be a $C^\infty$ function which is equal to 1 for $|k| < m$ and of fast decrease for $|k| > 2m$, $m = 1, 2, \ldots$.

For $S = (s_1, s_2), s_1, s_2 = 1, 2, \ldots$ define the smeared field

$$O_S(k) = e_{s_1}(k) \int O(k_1) \Delta_{s_2}(k - k_1) dk_1.$$  \hspace{1cm} (1)

From the general axioms for quantum fields, and from the experience, we expect the products of smeared fields to be smooth in $\{k\}$ (including the diagonals). Let

$$(O_S(k_1)O_S(k_2)\ldots O_S(k_n))_W = \sum_{\{\sigma\}} \frac{1}{n!} O_S(k_{\sigma(1)})O_S(k_{\sigma(2)})\ldots O_S(k_{\sigma(n)})$$

be the Weyl ordered product of smeared fields.

Remarks.

1. Actually, we need only smooth behavior at points $k_i + k_j = 0$. If in the products $O(k_1)O(k_2)\ldots O(k_n)$ singularities are not at $k_i + k_j = 0$ we can take $\Delta(k) = \delta(k)$, and leave only the regularization by $e_S(k)$.

2. The only property that we need from the product of operators $(O_S(k_1)O_S(k_2)\ldots O_S(k_n))_W$ is that it should be symmetric in $\{k\}$. Therefore, if $O(k)$ would be a free field, we could use the Wick order, rather than the Weyl order.
Consider the following operator-valued series

$$
\Psi_{S,\lambda}(k, x) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int \left( O_S(k_1)O_S(k_2) \ldots O_S(k_n) \right)_W \frac{1}{(k + k_1 + i0)(k_1 + k_2 + i0) \ldots (k_{n-1} + k_n + i0) \cdot \exp(2i(k_1 + \ldots + k_n)x) \frac{dk_1 \ldots dk_n}{(2\pi i)^n}}
$$

$$
i_{S,\lambda}(0, x) \equiv -\frac{1}{2}
$$

$$
i_{S,\lambda}(p, x) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int \left( O_S(k_1)O_S(k_2) \ldots O_S(k_n) \right)_W \frac{k_1^{2p-1} + k_2^{2p-1} + \ldots + k_n^{2p-1}}{(k_1 + k_2 + i0)(k_2 + k_3 + i0) \ldots (k_{n-1} + k_n + i0)} \frac{\exp(2i(k_1 + \ldots + k_n)x) \frac{dk_1 \ldots dk_n}{(2\pi i)^n}}{p = 1, 2, \ldots}
$$

$$
u_{S,\lambda}(x) = 4i_{S,\lambda}(1, x)
$$

Each term in the series is well-defined, because $\left( O_S(k_1)O_S(k_2) \ldots O_S(k_n) \right)_W$ is smooth in $k_1, k_2, \ldots, k_n$ and of fast decrease, and $\frac{1}{\prod_{j=1}^{n-1}(k_j + k_{j+1} + i0)}$ is a distribution in $n$ variables.

In the classical limit, $O(k)$ is some function (scattering data), and

$$
\Psi(k, x) = \int_0^{\infty} e^{-\lambda} \Psi_\lambda(k, x) d\lambda
$$

$$
i(p, x) = \int_0^{\infty} e^{-\lambda} i_\lambda(p, x) d\lambda, \quad p = 1, 2, \ldots
$$

are, respectively, the Jost function and the densities of integrals of motion, expressed through scattering data, which were considered in [8].

In the quantum case, the integral over $\lambda$ should be regularized to get the convergent answer. For our purposes, the precise prescription to take the final integral over $\lambda$ is not important.
Remark. We assumed that the matrix elements of integrals in (2), (3) grow with \( n \) not faster than \( (n-1)! \) for large \( n \), and therefore, the series (2), (3) are convergent; the operators (4) are related to operators (2), (3) via the (regularized) Borel transform. In fact, if we take some particular \( O(k) \), we may need the generalized Borel transform to ensure convergence; namely, we have to evaluate the large \( n \) dependence of the integrals and to change the coefficient in front of the \( n \)-th term integral from \( \lambda^n n! \) to some \( c(n, \lambda) \) in all the formulas in such a way that the series, with the coefficients \( c(n, \lambda) \) instead of \( \lambda^n n! \), are convergent. Afterwards, we introduce the measure \( d\mu(\lambda) \); the regularized operators \( \Psi_S(k, x) \) and \( i_S(p, x) \) are defined as

\[
\Psi_S(k, x) = \int d\mu(\lambda)\Psi_{S,\lambda}(k, x)
\]

\[
i_S(p, x) = \int d\mu(\lambda)i_{S,\lambda}(p, x), \quad p = 1, 2, \ldots
\]

Definition.
Let us have 2 quantum fields,

\[
A_{S,\lambda}(x) := \sum_n \frac{\lambda^n}{n!} \int (O_S(k_1)O_S(k_2) \ldots O_S(k_n))W f_n(k_1, k_2, \ldots, k_n, x) \frac{dk_1 \ldots dk_n}{(2\pi i)^n}
\]

\[
B_{S,\lambda}(x) := \sum_m \frac{\lambda^m}{m!} \int (O_S(k_1)O_S(k_2) \ldots O_S(k_m))W g_n(k_1, k_2, \ldots, k_m, x) \frac{dk_1 \ldots dk_m}{(2\pi i)^n}
\]

where \( \{f_n\}, \{g_m\} \) are some distributions.

Define the \( \circ \circ \circ \) product of \( A_{S,\lambda}(x) \) and \( B_{S,\lambda}(x) \) as follows:

\[
\circ \circ \circ A_{S,\lambda}(x)B_{S,\lambda}(x) = \sum_n \sum_{m_1, m_2} \frac{\lambda^n}{n!} \int (O_S(k_1)O_S(k_2) \ldots O_S(k_n))W f_{m_1}(k_1, k_2, \ldots, k_{m_1}, x) g_{m_2}(k_{m_1+1}, k_{m_1+2}, \ldots, k_n, x) \frac{dk_1 \ldots dk_n}{(2\pi i)^n}
\]

Remark.
Suppose that the integrals

\[
A_S(x) := \int_0^\infty e^{-\lambda}A_{S,\lambda}(x) d\lambda =
\]

\[
\sum_n \int (O_S(k_1)O_S(k_2) \ldots O_S(k_n))W f_n(k_1, k_2, \ldots, k_n, x) \frac{dk_1 \ldots dk_n}{(2\pi i)^n}
\]
Lemma 1 ("Quantum Sturm-Liouville equation").

Operator $\Psi_{S,\lambda}(k, x)$ satisfies the following equation:

$$
\left( \frac{d^2}{dx^2} + 2ik \frac{d}{dx} \right) \Psi_{S,\lambda}(k, x) = -\circ \circ \circ u_{S,\lambda}(x) \Psi_{S,\lambda}(k, x),
$$

where

$$
\circ \circ \circ u_{S,\lambda}(x) \Psi_{S,\lambda}(k, x) = 4 \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int (O_S(k_1)O_S(k_2)\ldots O_S(k_n))_W
$$

$$
\frac{k_1 + k_2 + \ldots + k_n}{(k_1 + k_2 + i0)(k_2 + k_3 + i0)\ldots(k_{n-1} + k_n + i0)}
$$

$$
+ \sum_{m=1}^{n-1} \frac{1}{(k + k_1 + i0)(k_1 + k_2 + i0)\ldots(k_{m-1} + k_m + i0)}
$$

$$
\frac{(k_{m+1} + k_{m+2} + \ldots + k_n)}{(k_{m+1} + k_{m+2} + i0)(k_{m+2} + k_{m+3} + i0)\ldots(k_{n-1} + k_n + i0)}
$$

$$
\exp(2i(k_1 + \ldots + k_n)x) \frac{dk_1\ldots dk_n}{(2\pi i)^n}
$$
Lemma 2 ("Quantum Resolvent Equation").

Operators $i_{S,\lambda}(p, x)$ satisfy the following equation:

$$-4 \frac{d}{dx} i_{S,\lambda}(p + 1, x) = \frac{d^3}{dx^3} i_{S,\lambda}(p, x) + 4^\circ u_{S,\lambda}(x) \left( \frac{d}{dx} i_{S,\lambda}(p, x) \right)^\circ,$$

where for $p = 1, 2, \ldots$

$$^\circ u_{S,\lambda}(x) \left( \frac{d}{dx} i_{S,\lambda}(p, x) \right)^\circ = 4 \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \int (O_S(k_1)O_S(k_2)\ldots O_S(k_n)) W$$

$$\sum_{m=1}^{n-1} \frac{(k_1 + k_2 + \ldots + k_m)}{(k_{m+1} + k_{m+2} + \ldots + k_n)} \cdot \frac{(k_{m+1} + k_{m+2} + i0)(k_{m+2} + k_{m+3} + i0)\ldots(k_{m-1} + k_n + i0)}{(k_{m+1} + k_{m+2} + i0)(k_{m+2} + k_{m+3} + i0)\ldots(k_{m-1} + k_n + i0)} \exp(2i(k_1 + \ldots + k_n)x) \frac{dk_1 \ldots dk_n}{(2\pi i)^n};$$

and

$$\left( \frac{d}{dx} u_{S,\lambda}(x) \right)^\circ i_{S,\lambda}(p, x)^\circ = 4 \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \int (O_S(k_1)O_S(k_2)\ldots O_S(k_n)) W$$

$$\sum_{m=1}^{n-1} \frac{(k_1 + k_2 + \ldots + k_m)^2}{(k_{m+1} + k_{m+2} + \ldots + k_n)^2} \cdot \frac{(k_{m+1} + k_{m+2} + i0)(k_{m+2} + k_{m+3} + i0)\ldots(k_{m-1} + k_n + i0)}{(k_{m+1} + k_{m+2} + i0)(k_{m+2} + k_{m+3} + i0)\ldots(k_{m-1} + k_n + i0)} \exp(2i(k_1 + \ldots + k_n)x) \frac{dk_1 \ldots dk_n}{(2\pi i)^n}.$$  

Lemma 3 ("Quantum Resolvent").

We call the formal series

$$R_{S,\lambda}(k, x) := 1 - 2 \sum_{n=1}^{\infty} \frac{1}{k^{2n}} i_{S,\lambda}(n, x)$$

the Quantum Resolvent of the Sturm-Liouville Equation. Here $i_{S,\lambda}(n, x)$ are defined in (3); by Lemma 2, these operators $i_{S,\lambda}(n, x)$ satisfy (6).
There is the following relation between $R$ and $\Psi$:

$$R_{S,\lambda}(k, x) = \circ \Psi_{S,\lambda}(k, x) \Psi_{S,\lambda}(-k, x) \circ,$$

(7)

where, from the definition of the $\circ \circ$ product (5)

$$\circ \Psi_{S,\lambda}(k, x) \Psi_{S,\lambda}(-k, x) \circ = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int (O(k_1)O(k_2)\ldots O(k_n))_W$$

$$\frac{1}{(k + k_1 + i0)(k_1 + k_2 + i0)\ldots(k_{n-1} + k_n + i0)}$$

$$+ \frac{1}{(-k + k_1 + i0)(k_1 + k_2 + i0)\ldots(k_{n-1} + k_n + i0)}$$

$$+ \sum_{m=1}^{n-1} \frac{1}{(k_1 + k_2 + i0)(k_2 + k_3 + i0)\ldots(k_m + k + i0)}$$

$$\frac{1}{(-k + k_{m+1} + i0)(k_{m+1} + k_{m+2} + i0)\ldots(k_{n-1} + k_n + i0)}$$

$$\exp(2i(k_1 + \ldots + k_n)x) \frac{dk_1 \ldots dk_n}{(2\pi i)^n}$$

The equation (7) should be understood in the following way: the right-hand side of (7) depends from $k$ only through the denominator. We have to expand it formally in the inverse powers of $k$. The series we obtain in this way is $R_{S,\lambda}(k, x)$.

**Lemma 4**

There is the following relation for the operators $i_{S,\lambda}(n, x)$:
\[ 2 \circ i_{S,\lambda}(p_1, x) \frac{d}{dx} i_{S,\lambda}(p_2, x)^\circ - \frac{d}{dx} \left( (\circ i_{S,\lambda}(p_1, x) i_{S,\lambda}(p_2, x)^\circ) \right) \]

\[ := 4i \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \int (O_S(k_1) O_S(k_2) \ldots O_S(k_n))_W \]
\[ \cdot \sum_{m=1}^{n-1} \frac{k_1^{2p_1-1} + k_2^{2p_1-1} + \ldots + k_m^{2p_1-1}}{k_{m+1}^{2p_2-1} + k_{m+2}^{2p_2-1} + \ldots + k_n^{2p_2-1}} \]
\[ \frac{(k_{m+1} + k_{m+2} + i0)(k_{m+2} + k_{m+3} + i0) \ldots (k_{n-1} + k_n + i0)}{(k_1 + k_2 - k_3 - \ldots - k_m + k_{m+1} + \ldots + k_n)} \]
\[ \exp(2i(k_1 + k_2 + \ldots + k_n)x) \frac{dk_1 \ldots dk_n}{(2\pi i)^n} \]

\[ = -2i \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int (O_S(k_1) O_S(k_2) \ldots O_S(k_n))_W \]
\[ \cdot \left( (k_1^{2p_1+1} + k_2^{2p_1+1} + \ldots + k_n^{2p_1+1}) \cdot (k_1^{2p_2-1} + k_2^{2p_2-1} + \ldots + k_n^{2p_2-1}) \right) - \]
\[ \left( (k_1^{2p_1-1} + k_2^{2p_1-1} + \ldots + k_n^{2p_1-1}) \cdot (k_1^{2p_2+1} + k_2^{2p_2+1} + \ldots + k_n^{2p_2+1}) \right) \]
\[ \frac{\exp(2i(k_1 + \ldots + k_n)x)}{(k_1 + k_2 + i0)(k_2 + k_3 + i0) \ldots (k_{n-1} + k_n + i0)} \frac{dk_1 \ldots dk_n}{(2\pi i)^n}. \]
The field \( O_S(k) \) is of action-angle type, and its time evolution is trivial

\[
-i \frac{\partial}{\partial t(l)} O_S^{(l)}(k, t(l)) = 2k^{2l-1} O_S^{(l)}(k, t(l)), \quad l = 2, 3, \ldots
\]

**Lemma 5** ("Quantum Integrals").

The quantities

\[
i_{S,\lambda}(p_1, l, x, t(l)) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int (O_S(k_1)O_S(k_2)\ldots O_S(k_n))_W \\
k_1^{2p_1-1} + k_2^{2p_1-1} + \ldots + k_n^{2p_1-1} \\
(k_1 + k_2 + i0)(k_2 + k_3 + i0)\ldots(k_{n-1} + k_n + i0) \\
\exp \left( 2i \left( (k_1 + \ldots + k_n)x + (k_1^{2l-1} + k_2^{2l-1} + \ldots + k_n^{2l-1})t(l) \right) \right) \frac{dk_1 dk_2 \ldots dk_n}{(2\pi i)^n}
\]

are conserved currents, namely, there is an operator \( j_{S,\lambda}(p_1, l, x, t(l)) \) which is a \( \circ \circ \)-ordered product of \( i_{S,\lambda}(p, l, x, t(l)) \) and its derivatives, with \( p = 1, 2, \ldots \), such that

\[
\frac{d}{dt(l)} i_{S,\lambda}(p_1, l, x, t(l)) = \frac{d}{dx} j_{S,\lambda}(p_1, l, x, t(l))
\]

The proof is essentially the same as in [7]:

For a fixed \( l \) let us introduce the following notations:

\[
i(p) := i_{S,\lambda}(p, l, x, t(l));
\]

\[
A(x) \sim B(x), \quad \text{if} \quad A(x) - B(x) = \frac{d}{dx} C(x), \quad \text{where} \quad C(x) \text{ is a } \circ \circ \circ \text{ polynomial}
\]

in \( i_{S,\lambda}(p, l, x, t(l)) \), \( p = 1, 2, \ldots \), and their derivatives;

The \( \circ \circ \circ \) product, defined in (5), has the properties

\[
\circ \circ \circ AB = \circ \circ \circ BA,
\]

\[
\circ \circ \circ (\circ \circ \circ AB)C = \circ \circ \circ A \circ \circ \circ BC
\]

\[
\frac{d}{dx} \circ \circ \circ AB = \circ \circ \circ \left( \frac{d}{dx} A \right) B + \circ \circ \circ A \frac{d}{dx} B
\]

10
therefore, from Lemma 2,
\[ \circ i(p_1) \frac{d}{dx} i(p_2 + 1) \circ + \circ \left( \frac{d}{dx} \circ i(p_1 + 1) i(p_2) \circ \right) \]
\[ = - \frac{1}{4} \frac{d}{dx} \left( \circ i(p_1) \frac{d^2}{dx^2} i(p_2) + i(p_2) \frac{d^2}{dx^2} \circ i(p_1) - \frac{d}{dx} i(p_1) \frac{d}{dx} i(p_2) + 4u \cdot i(p_1) i(p_2) \circ \right) \sim 0 \]

From this relation, \( \circ i(p_1) \frac{d}{dx} i(p_2 + 1) \circ \sim - \circ \left( i(p_1 + 1) \frac{d}{dx} i(p_2) \circ \right) \); we also have \( \frac{d}{dx} i(0) = 0 \), and therefore,
\[ \circ i(p_1) \frac{d}{dx} i(p_2) \circ \sim 0 \]

Taking the derivative of \( i(p) \) with respect to \( t_l \) and comparing with Lemma 4, we get
\[ \frac{d}{dt_l} i(p) \sim \circ i(p) \frac{d}{dx} i(l - 1) + i(p + 1) \frac{d}{dx} i(l - 2) + \ldots + i(p + l - 2) \frac{d}{dx} i(1) \circ \sim 0 \]

**Fermionic case** Let \( \eta(k) \) be a grassman quantum field, and \( \eta_S(k) \) the smeared grassman field. For grassman fields, we cannot just symmetrize the product because the result would be zero. But we can multiply polynomial in \( \eta \) and a function of \( \{k\} \) in such a way that the product is symmetric in \( \{k\} \). For example, one can substitute
\[ (O_S(k_1) O_S(k_2) \ldots O_S(k_n))_W := \eta_S(k_1) \eta_S(k_2) \ldots \eta_S(k_n) \]
\[ \cdot \det \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ k_1 & k_2 & k_3 & \ldots & k_n \\ k_1^2 & k_2^2 & k_3^2 & \ldots & k_n^2 \\ k_1^3 & k_2^3 & k_3^3 & \ldots & k_n^3 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ k_1^{n-1} & k_2^{n-1} & k_3^{n-1} & \ldots & k_n^{n-1} \end{vmatrix} \]

or, alternatively, \( (O_S(k_1) \ldots O_S(k_n))_W = \epsilon_{i_1 \ldots i_n} \eta_S(k_{i_1}) \ldots \eta_S(k_{i_n}) \) in all the formulas, \( n = 2, 3, \ldots \).

One has to do this change both in the definition of \( \Psi_{S,\lambda} \), \( i_{S,\lambda} \) and in the definition of the \( \circ \circ \) product. After such substitution, the conserved currents
could be obtained by the same construction as before. In fact, in this construction $i_{S,\lambda}$ and $\Psi_{S,\lambda}$ is a sum of even and odd in $\eta$ terms. Therefore, for the conserved quantities we will have $\frac{d}{dt}i_{\text{odd}} = \frac{d}{dx}j_{\text{odd}}$ and $\frac{d}{dt}i_{\text{even}} = \frac{d}{dx}j_{\text{even}}$.

**Conclusion.**

Starting from the basic field $O_S(k)$, we defined the observable fields $u_{S,\lambda}(x, l, t(l))$, $i_{S,\lambda}(p, l, x, t(l))$, and the regularized product $\circ$. It is rather interesting to notice, that, say, for $l = 2$, we have the following identity:

$$-4 \cdot \frac{\partial}{\partial t} u - \frac{\partial^3}{\partial x^3} u = 3 \frac{\partial}{\partial x} \circ u (x, t),$$

where $\circ u (x, t)$ is a new observable field, obtained as a $\circ$ product $\circ uu$. If $u(x, t)$ were a function, and with standard multiplication of functions, rather than the $\circ$ one, this would be the KdV equation. In the quantum theory, we need some regularization to multiply operators at the same point. With our regularization, we have the "Quantum KdV" relation.

We also have, with our definitions of $u$, $i$, and $\circ$, that

$$i_{S,\lambda}(2, x) = -\frac{1}{16} \circ (3u^2 + u'') (x)$$

$$i_{S,\lambda}(3, x) = \frac{1}{64} \circ (10u^3 + 10uu'' + 5(u')^2 + u''') (x)$$

\[\ldots\]
Remark. Notice, that in the noncommutative case, there is an alternative way to get the KdV equation. Let \( O(k) \) be an element of a free algebra (which means, that we didn’t impose any commutation relations). Consider the formal series
\[
\hat{u}(x, t) = 4 \sum_{n=1}^{\infty} \int O(k_1)O(k_2) \ldots O(k_n) \frac{k_1 + k_2 + \ldots + k_n}{(k_1 + k_2 + i0)(k_2 + k_3 + i0) \ldots (k_{n-1} + k_n + i0)} \\
\exp \left( 2i \left( (k_1 + \ldots + k_n)x + (k_1^3 + k_2^3 + \ldots + k_n^3)t \right) \right) \frac{dk_1dk_2 \ldots dk_n}{(2\pi i)^n}
\]
(no symmetrization in the product)

Then, we have the following relation for the formal series:
\[
-4 \cdot \frac{\partial}{\partial t} \hat{u} - \frac{\partial^3}{\partial x^3} \hat{u} = 3 \frac{\partial}{\partial x}(\hat{u} \hat{u}),
\]
(here the product is in the algebra)

Discussion

To prove lemmas 1–5 we need only the identities for symmetric functions in \( n \) variables \( k_1, \ldots, k_n, \ n = 1, 2, \ldots \). The properties of the basic quantum field \( O(k) \) are, essentially, not important (except for the convergence of the series). We need only that the products of smeared (regularized) basic field make sense and not singular on the diagonals.

We postponed taking the Borel transform (4), and therefore our construction depend on parameters, \( \lambda \), which is, in fact the coupling constant, and \( S = (s_1, s_2) \) (“the cutoffs”). Therefore, if the operator \( O(k) \) is known, one might be able to study a “renormalization flow” \( \lambda(S) \). At the “fixed points of the flow” \( \lambda(S)O_S(k) \) is dimensionless, and the operators \( i(p, x), p = 1, 2, \ldots \) are of (formal) dimension \( 2p \). However, we do not know the exact relationship between the fields \( i(p, x) \) and operators in conformal field theory “at the fixed point.”
We feel that our construction is quite general, and similar approach will work for other nonlinear models. However, we, in fact, didn’t finish the quantization. For the quantization to be complete, one have to tell, what is the basic field $O$ as operator, and what regularization to choose.

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