Abstract. — We characterise the class of exponentiable \( \infty \)-toposes: \( X \) is exponentiable if and only if \( \mathcal{S}h(X) \) is a continuous \( \infty \)-category. The heart of the proof is the description of the \( \infty \)-category of \( \mathcal{C} \)-valued sheaves on \( X \) as an \( \infty \)-category of functors that satisfy finite limits conditions as well as filtered colimits conditions (instead of limits conditions purely); we call such functors \( \omega \)-continuous sheaves.

As an application, we show that when \( X \) is exponentiable, its \( \infty \)-category of stable sheaves \( \mathcal{S}h(X, \text{Sp}) \) is a dualisable object in the \( \infty \)-category of presentable stable \( \infty \)-categories.

CONTENTS

1 Introduction 2  
1.1 Exponentiability of \( \infty \)-toposes 2  
1.2 Conventions on sizes 3  
1.3 Acknowledgements 3  

2 \( \infty \)-toposes 3  
2.1 Definitions 3  
2.2 Affine \( \infty \)-toposes 4  
2.3 Tensor product of \( \infty \)-categories 5  

3 Coends for \( \infty \)-categories 7  
3.1 Definition and first properties 8  
3.2 Yoneda lemma 9  
3.3 Left Kan extensions as coends 9  

4 Exponentiable \( \infty \)-toposes 10  
4.1 Injective \( \infty \)-toposes and their points 10  
4.2 Continuous \( \infty \)-categories 12  
4.3 \( \omega \)-continuous sheaves 13  
4.3.1 \( \omega \)-continuous sheaves of spaces 14  
4.3.2 \( \mathcal{C} \)-valued sheaves 15  
4.3.3 \( \mathcal{C} \)-valued \( \omega \)-continuous sheaves 17  
4.4 Exponentiability theorem 19  
4.5 Glossary of maps between \( \infty \)-toposes 21  
4.6 Examples of exponentiable \( \infty \)-toposes 21  

5 Dualisability of the \( \infty \)-category of stable sheaves 23  
5.1 Stabilisation for presentable \( \infty \)-categories 23  
5.2 Dualisability in \( \mathfrak{Pres} \) 24  
5.3 Dualisability of stable sheaves 25  

References 25
§ 1. — Introduction

1.1 Exponentiability of ∞-toposes

An ∞-topos $\mathcal{X}$ is said to be exponentiable if the functor $\mathcal{Y} \mapsto \mathcal{X} \times \mathcal{Y}$ has a right adjoint: $\mathcal{Z} \mapsto \mathcal{Z}^{\mathcal{X}}$. The idea that exponentiability can be seen as a form of dualisability is made concrete by the following theorem.

Theorem 5.9. — Let $\mathcal{X}$ be an exponentiable ∞-topos, then the ∞-category $\text{Sh}(\mathcal{X}, \text{Sp})$ of stable sheaves on $\mathcal{X}$ is a dualisable object of the ∞-category of presentable stable ∞-categories.

In order to prove this dualisability result, we characterise the class of exponentiable ∞-toposes. The exponentiability of toposes was studied and understood in the 1981 article Continuous categories and exponentiable toposes by Johnstone and Joyal [1]. We obtain a characterisation of exponentiable ∞-toposes by following a similar proof. Another independent proof of the characterisation of exponentiable ∞-toposes has been written by Lurie in his book Spectral Algebraic Geometry [2, Theorem 21.1.6.12]. Our main addition here is the use of the tensor product of ∞-categories. In particular in theorem 2.15 we relate the tensor product of ∞-categories with the product in the ∞-category of ∞-toposes. The characterisation of exponentiable ∞-toposes ends up being the same as the one for toposes:

Theorem 4.33. — An ∞-topos $\mathcal{X}$ is exponentiable if and only if the ∞-category $\text{Sh}(\mathcal{X})$ is continuous i.e when the colimit functor $\text{Ind} (\text{Sh}(\mathcal{X})) \to \text{Sh}(\mathcal{X})$ has a left adjoint.

The pivot of the exponentiability proof is a rewriting of the ∞-category $\text{Sh}(\mathcal{X})$ in terms of finite limits and arbitrary colimits. When a presentable ∞-category is continuous it can be obtained as a coreflective localisation of an ∞-category of ind-objects:

$$\text{Ind}(\mathcal{D}) \xrightarrow{\beta} \text{Sh}(\mathcal{X}),$$

where $\epsilon$ is cocontinuous and $\beta$ is a fully faithful left adjoint to $\epsilon$. This allows us to describe the ∞-category of $\mathcal{C}$-valued sheaves on $\mathcal{X}$ as follows.

Theorem 4.32. — Let $\mathcal{X}$ be an exponentiable ∞-topos, and let $\mathcal{C}$ be an ∞-logos. Then there exists a finitely cocomplete subcategory $\mathcal{D} \subset \text{Sh}(\mathcal{X})$ and a bimodule $w : \mathcal{D}^{\text{op}} \times \mathcal{D} \to \mathcal{S}$ such that the ∞-category of $\mathcal{C}$-valued sheaves is equivalent to the ∞-category of left exact functors $F : \mathcal{D}^{\text{op}} \to \mathcal{C}$ satisfying the coend condition:

$$F(a) \simeq \int_{b \in \mathcal{D}} w(a, b) \otimes F(b), \text{ for all } a \in \mathcal{D}.$$ 

Such a description is what we call $\omega$-continuous sheaves. In fact, one of the first definitions of sheaves on a topological space $X$ involved Abelian groups associated to compact subsets of $X$. A sheaf was then a functor $\mathcal{F} : K \mapsto \mathcal{F}(K)$ commuting to finite limits and specific filtered colimits [3, ‘faisceaux continus’ in the chapter by Houzel]. Namely, a sheaf had to satisfy the additional condition:

$$\mathcal{F}(K) \simeq \lim_{K \to K'} \mathcal{F}(K'),$$

for all $K \in \mathcal{C}$. 


where $K \ll K'$ means that there exists an open subset $U$ such that $K \subset U \subset K'$.
A proof of the equivalence between sheaves on $X$ and $\omega$-continuous sheaves on $X$ is in HTT [4, Theorem 7.3.4.9], where they are called $\mathcal{K}$-sheaves by Lurie.

1.2 Conventions on sizes

Let $\omega \in U \in V \in W$ be three Grothendieck universes. To avoid heavy notations, we establish a dictionary: small means $U$-small, large means $V$-small and very large means $W$-small. By a limit or a colimit, we mean a small one.
By a category or an $\infty$-category, we mean a large one.

The large $\infty$-category of small spaces will be denoted $S$; its homotopy category is $H$. The very large $\infty$-category of large spaces is $bS$, with homotopy category $bH$. The large $\infty$-category of small $\infty$-categories is $\mathcal{C}at$; the very large one of large $\infty$-categories will be denoted $\mathcal{C}at$.

1.3 Acknowledgements

This work was supported by IBS-R008-D1.

§ 2. — $\infty$-toposes

A standard reference on $\infty$-toposes is HTT [4]. The reader may also have a look at Toposes and homotopy toposes [5] and Homotopical algebraic geometry I: topos theory [6].

2.1 Definitions

In this paragraph we recall the definition of an $\infty$-topos and introduce the terminology of $\infty$-logoses.

Definition 2.1. — We shall say that an $\infty$-category $\mathcal{L}$ is an $\infty$-logos if there exists a small $\infty$-category $D$ and an accessible left exact and reflective localization $P(D) \rightarrow L$.

The very large $\infty$-category $Log$ of $\infty$-logoses is the non-full subcategory of $\mathcal{C}at$ whose objects are the $\infty$-logoses and the morphisms are the left exact and cocontinuous functors. For $\mathcal{C}$ and $\mathcal{D}$ two $\infty$-categories, the $\infty$-category of left exact and cocontinuous functors shall be denoted $[\mathcal{C}, \mathcal{D}]_{lex}^{\text{op}}$.

Definition 2.2. — The very large $\infty$-category of $\infty$-toposes is defined by:

$$\text{Top} = \text{Log}^{\text{op}}.$$  

The isomorphism sends an $\infty$-topos $\mathcal{X}$ to its $\infty$-logos $Sh(\mathcal{X})$; a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is sent to $f^* : Sh(\mathcal{Y}) \rightarrow Sh(\mathcal{X})$.

Remark 2.3. — Manipulating $\infty$-toposes usually requires many back and forth between the $\infty$-category of $\infty$-toposes and its opposite. Distinquishing names and notations the two $\infty$-categories helps avoiding confusion, especially between the various types of morphisms.

Distinquishing between a category and its opposite is not new: the category of affine schemes is the opposite category of the category of rings; there the equivalence is denoted $A \leftrightarrow \text{Spec}(A)$. In the same way the category of
locales is the opposite category of the category of frames and the equivalence
is denoted \( X \mapsto \mathcal{O}(X) \).

Furthermore, there is a useful analogy between \( \infty \)-logoses and commutative
rings extending the analogy between colimits and sums; limits and produc-
ts; cocomplete categories and Abelian groups.

**Definition 2.4.** — Let \( D \) be a small category. Let \( \overline{D} \) be the free
category generated by \( D \) by finite limits i.e. \( (\overline{D})^{\text{op}} \) is the smallest
subcategory in \( \mathcal{P}(D^{\text{op}}) \) containing \( D^{\text{op}} \) and closed under finite
colimits. We shall call \( S[D] = \mathcal{P}(\overline{D}) \) the free \( \infty \)-logos
generated by \( D \).

**Proposition 2.5** (Universal property of free \( \infty \)-logoses). — Let \( D \) be a small
\( \infty \)-category and \( \mathcal{E} \) be an \( \infty \)-logos. Let \( i : D \to S[D] \) be the inclusion
functor. Then the restriction functor \( i^* \) induces an equivalence between the
\( \infty \)-category of cocontinuous left exact functors \( S[D] \to \mathcal{E} \) and the \( \infty \)-category of functors \( D \to \mathcal{E} \).

**Proof.** — Since \( \overline{D} \) is the free \( \infty \)-category with finite limits generated by \( D \)
and left Kan extensions of left exact \( \mathcal{E} \)-valued functors are still left exact, we
have the natural equivalences of \( \infty \)-categories:

\[
[D, \mathcal{E}] \cong [\overline{D}, \mathcal{E}]^{\text{lex}} \cong [\mathcal{P}(\overline{D}), \mathcal{E}]^{\text{lex}}
\]

induced by the inclusions \( D \subseteq \overline{D} \subseteq \mathcal{P}(\overline{D}) \).

**Proposition 2.6.** — An \( \infty \)-category \( \mathcal{E} \) is an \( \infty \)-logos if and only if it is a left exact
and accessible reflective localisation of a free \( \infty \)-logos:

\[
S[D] \xrightarrow{L} \mathcal{E}
\]

that is \( L \) is a left exact left adjoint and its right adjoint is fully faithful and accessible.

**Proof.** — By definition an \( \infty \)-logos \( \mathcal{E} \) is a left exact and accessible reflective
localisation of a presheaf \( \infty \)-category \( L : \mathcal{P}(D) \to \mathcal{E} \) with \( D \) a small \( \infty \)-cate-
gory. The proposition we want to prove is just a slight variation. Indeed for any small \( \infty \)-category \( D \), the Yoneda embedding \( D \hookrightarrow \mathcal{P}(D) \) extends
to a left exact and cocontinuous functor \( T : S[D] \to \mathcal{P}(D) \). Its right adjoint is the
left extension of the inclusion \( D \hookrightarrow \mathcal{P}(\overline{D}) = S[D] \), it is accessible
and fully faithful and \( LT : S[D] \to \mathcal{E} \) is the desired reflective localisation.

### 2.2 Affine \( \infty \)-toposes

The category of commutative rings is generated under colimits by free rings
\( \mathbb{Z}[x_1, \ldots, x_n] \), hence the category affine schemes is generated under limits by
the affine spaces \( \mathbb{A}^n \). We wish to prove the analogue statement for \( \infty \)-toposes.

**Definition 2.7.** — An affine \( \infty \)-topos is an \( \infty \)-topos \( \mathcal{X} \) such that \( \mathsf{Sh}(\mathcal{X}) \) is a
free \( \infty \)-logos. Let \( \mathsf{Aff} \) be the full subcategory of \( \mathcal{Top} \) whose objects are the affine
\( \infty \)-toposes.

We let \( \mathbb{A}^D \) be the affine \( \infty \)-topos with \( \infty \)-logos \( S[D] \), for a small \( \infty \)-category
\( D \). For convenience, we also let \( \mathbb{A} \) be the affine \( \infty \)-topos \( \mathbb{A}^\infty \); its \( \infty \)-logos will be
denoted \( S[X] \).

**Proposition 2.8.** — The \( \infty \)-category \( \mathcal{Top} \) is generated under pullbacks by affine
\( \infty \)-toposes.
Proof. — We are going to prove the dual statement that the \( \infty \)-category \( \mathcal{L}og \) is generated under pushouts by the free \( \infty \)-logoses. For any \( \infty \)-logos \( E \), there exists a free \( \infty \)-logos \( \mathcal{S}[D] \) and a left exact and accessible reflective localisation functor \( L : \mathcal{S}[D] \to \mathcal{S}h(X) \).

Let \( S \) be the set of morphisms \( f \) in \( \mathcal{S}[D] \) such that \( L(f) \) is an equivalence in \( E \), then \( S \) is strongly saturated. Because both \( \mathcal{S}[D] \) and \( E \) are accessible \( \infty \)-categories, by proposition 5.5.4.2 in HTT \([4]\) there exists a small subset \( S_0 \subseteq S \) such that \( S_0 \) generates \( S \) as a strongly saturated class.

We can now identify \( E \) as \( S_0^{-1}\mathcal{S}[D] \). Let \( J \) be the \( \infty \)-category generated by two objects and one invertible arrow. We then obtain the following pushout in the \( \infty \)-category \( \mathcal{L}og \):

\[
\begin{array}{c}
\mathcal{S} \left[ \coprod_{S_0} \Delta^1 \right] \longrightarrow \mathcal{S}[D] \\
\downarrow \quad \downarrow r \\
\mathcal{S} \left[ \coprod J \right] \longrightarrow E.
\end{array}
\]

This ends the proof that any \( \infty \)-logos is a pushout of free \( \infty \)-categories of sheaves: morphisms \( f^* : \mathcal{S}h(X) \to \mathcal{S}h(Y) \) are canonically equivalent to morphisms \( g^* : \mathcal{S}[D] \to \mathcal{S}h(Y) \) such that \( g^*(s) \) is invertible for any \( s \in S_0 \). \( \square \)

2.3 Tensor product of \( \infty \)-categories

We gather useful facts from chapter 5.5 of HTT \([4]\), 1.4 and 4.8 of Higher Algebra \([7]\) on tensor products of \( \infty \)-categories and show that the coproduct of \( \infty \)-logoses is given by the tensor product of the underlying cocomplete \( \infty \)-categories.

Theorem 2.9 \([7, \text{Corollary 4.8.1.4}]\). — The very large \( \infty \)-category \( \widetilde{\mathcal{C}at}_{cc} \) of cocomplete \( \infty \)-categories and cocontinuous functors has a closed symmetric monoidal structure \( \otimes \) such that cocontinuous functors \( C \otimes D \to E \) canonically correspond to functors \( C \times D \to E \) cocomplete in each variable.

The unit object of \( \otimes \) is the cocomplete \( \infty \)-category \( \mathcal{S} \).

Given \( A, B, C \) three \( \infty \)-categories, we shall denote by \( [A, B]_{cc} \) the \( \infty \)-category of cocontinuous functors from \( A \) to \( B \). Like wise \( [A, B]^c \) is the \( \infty \)-category of continuous functors and \( [A \times B, C]^c_{cc} \) is the \( \infty \)-category of functors that are continuous in each variable.

It will be important to understand how \( C \otimes D \) is build as we need these technical details for future proofs. The basic idea is to force the commutation \( c \otimes \lim d_i \simeq \lim (c \otimes d_i) \) and then to add all the colimits of ‘pure tensor’ \( c \otimes d \). Because \( C \) and \( D \) are large, they are \( V \)-small so we get a reflective localisation functor:

\[
\left[ (C \times D)^{op}, \mathcal{S} \right] \longrightarrow \left[ (C \times D)^{op}, \mathcal{S} \right]^{cc}.
\]

By composition with the Yoneda embedding, we get a functor:

\[
C \times D \longrightarrow \left[ (C \times D)^{op}, \mathcal{S} \right]^{cc},
\]

which is cocontinuous in each variable. The tensor product \( C \otimes D \) is then the smallest cocomplete subcategory of \( \left[ (C \times D)^{op}, \mathcal{S} \right]^{cc} \) that contains the image of the Yoneda embedding.
Theorem 2.10 [7, Remark 4.8.1.18]. — Let $\mathcal{C}$ and $\mathcal{D}$ be two presentable $\infty$-categories, then $\mathcal{C} \otimes \mathcal{D}$ is presentable. Moreover $[\mathcal{C}, \mathcal{D}]_{cc}$ is also presentable, so that $\mathcal{Pres}$, the large $\infty$-category of presentable $\infty$-categories, inherits a closed symmetric monoidal structure from $\mathcal{Cat}_{cc}$.

Proposition 2.11 [7, Proposition 4.8.1.17]. — Let $\mathcal{C}$ and $\mathcal{D}$ be two presentable $\infty$-categories, then $\mathcal{C} \otimes \mathcal{D} \simeq [\mathsf{C}^{\mathsf{op}}, \mathcal{D}]$.

Proposition 2.12 [7, Proof of prop. 4.8.1.15]. — Let $\mathcal{A}$ and $\mathcal{B}$ be presentable $\infty$-categories. Let $\mathcal{A} \to S^{-1}\mathcal{A}$ and $\mathcal{B} \to T^{-1}\mathcal{B}$ be accessible and reflective localisations. Let $f : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ be the canonical map and denote by $S \boxtimes T$ the set of arrows in $\mathcal{A} \otimes \mathcal{B}$ of the form $f(s \times b)$ with $(s, b) \in S \times \mathcal{B}$ or $f(a \times t)$ with $(a, t) \in \mathcal{A} \times T$. Then the localisation of $\mathcal{A} \otimes \mathcal{B}$ along $S \boxtimes T$ exists, is reflective and accessible. In addition:

$$(S \boxtimes T)^{-1}\mathcal{A} \otimes \mathcal{B} \simeq (S^{-1}\mathcal{A}) \otimes (T^{-1}\mathcal{B}).$$

As a direct consequence of the universal property of the tensor product, we obtain the following corollary:

Corollary 2.13. — The following square is a pushout in $\mathcal{Cat}_{cc}$,

$$
\begin{array}{ccc}
\mathcal{A} \otimes \mathcal{B} & \longrightarrow & S^{-1}\mathcal{A} \otimes \mathcal{B} \\
\downarrow & & \downarrow r \\
\mathcal{A} \otimes T^{-1}\mathcal{B} & \longrightarrow & (S^{-1}\mathcal{A}) \otimes (T^{-1}\mathcal{B}).
\end{array}
$$

For the next theorem, we shall need a lemma that can be found in the online corrected version of HTT.

Lemma 2.14 [8, Lemma 6.3.3.4]. — Let $\mathcal{L}$ be an $\infty$-logos and let $F, G : \mathcal{L} \to \mathcal{L}$ be two accessible and left exact localisation functors. Then the intersection $F\mathcal{L} \cap G\mathcal{L}$ is a left exact and accessible localisation of $\mathcal{L}$.

We now describe the coproducts inside $\mathcal{Log}$. Notice that the following theorem is stated in HA [7, Example 4.8.1.19] but the proof is left to the reader as it has already been proved in HTT [4, Theorem 7.3.3.9] for the case where one of the two $\infty$-toposes is localic.

Theorem 2.15. — If $\mathcal{L}$ and $\mathcal{M}$ are two $\infty$-logoses, then $\mathcal{L} \otimes \mathcal{M}$ is a coproduct of $\mathcal{L}$ and $\mathcal{M}$ in $\mathcal{Log}$.

Proof. — Let $\mathcal{C}$ and $\mathcal{D}$ be two small $\infty$-categories, we first remark that

$$(S\mathcal{C}) \otimes (S\mathcal{D}) \simeq S(\mathcal{C} \sqcup \mathcal{D}).$$

To see this, we remark that

$$(S\mathcal{C}) \otimes (S\mathcal{D}) = \mathcal{P}(\mathcal{C}) \otimes \mathcal{P}(\mathcal{D}) \simeq \mathcal{P}(\mathcal{C} \times \mathcal{D}).$$

Now look at the finite completion functor $\mathcal{C} \mapsto \overline{\mathcal{C}}$. It starts from $\mathcal{Cat}$ and goes to $\mathcal{Cat}^{\text{lex}}$, the large $\infty$-category of finitely complete small $\infty$-categories with left exact functors. This functor is left adjoint to the forgetful functor. Hence it sends coproducts to coproducts. But in $\mathcal{Cat}^{\text{lex}}$ products and coproducts coincide, and because the forgetful functor preserves limits, we have:

$$\overline{\mathcal{C} \sqcup \mathcal{D}} \simeq \overline{\mathcal{C}} \times \overline{\mathcal{D}} \Rightarrow (S\mathcal{C}) \otimes (S\mathcal{D}) \simeq \mathcal{P}(\overline{\mathcal{C}} \times \overline{\mathcal{D}}) \simeq \mathcal{P}(\overline{\mathcal{C} \sqcup \mathcal{D}}) \simeq S(\mathcal{C} \sqcup \mathcal{D}).$$
By the universal property of free $\infty$-logoses, we deduce that $S[C \uplus D]$ a coproduct of $S[C]$ and $S[D]$.

Let $\mathcal{L}$ and $\mathcal{M}$ be two $\infty$-logoses, we will now show that $\mathcal{L} \otimes \mathcal{M}$ is an $\infty$-logos. There exist two small $\infty$-categories $C$ and $D$ together with two accessible left exact reflective localisation functors $G : S[C] \to \mathcal{L}$ and $H : S[D] \to \mathcal{M}$. Then both

$$G^{\mathcal{L}^p} : S[C]^{\mathcal{L}^p} \to \mathcal{L}^{\mathcal{L}^p} \quad \text{and} \quad H^{\mathcal{M}^p} : S[D]^{\mathcal{M}^p} \to \mathcal{M}^{\mathcal{M}^p}$$

are left exact and accessible reflective localisation functors. By corollary 2.13, we deduce that $\mathcal{L} \otimes \mathcal{M}$ is equivalent to the intersection $(\mathcal{L} \otimes S[D]) \cap (S[C] \otimes \mathcal{M})$ and is thus, by lemma 2.14, an accessible and left exact localisation of $S[C] \otimes S[D]$. As we have just shown above, $S[C] \otimes S[D]$ is equivalent to a free $\infty$-logos, so that $\mathcal{L} \otimes \mathcal{M}$ is indeed an $\infty$-logos.

Let $p^* : \mathcal{L} \to \mathcal{L}$ be a morphism of $\infty$-logoses (unique up to contrafectible choice) and let $q^* : \mathcal{S} \to \mathcal{M}$ be another. We claim that the maps $p^* \otimes Id_\mathcal{M} : \mathcal{M} \to \mathcal{L} \otimes \mathcal{M}$ and $Id_\mathcal{L} \otimes q^* : \mathcal{L} \otimes \mathcal{M}$ exhibit $\mathcal{L} \otimes \mathcal{M}$ as a pushout of $\mathcal{L}$ and $\mathcal{M}$ in $\mathcal{L}og$. Notice that both maps are left exact and cocontinuous: the first is the localisation along left exact functors of the left exact cocontinuous map $S[D] \to S[C] \otimes S[D] \simeq S[C \uplus D]$ induced by the canonical map $D \to C \uplus D$. For a symmetric reason, the second map is also a morphism of $\infty$-logoses.

For any $\infty$-logos $\mathcal{E}$, those two maps induce a commutative square

$$\begin{array}{ccc}
[L \otimes \mathcal{M}, \mathcal{E}]_{\mathcal{L}cc} & \longrightarrow & [\mathcal{L}, \mathcal{E}]_{\mathcal{L}cc} \times [\mathcal{M}, \mathcal{E}]_{\mathcal{L}cc} \\
\downarrow & & \downarrow \\
[S[C \uplus D], \mathcal{E}]_{\mathcal{L}cc} & \longrightarrow & [S[C], \mathcal{E}]_{\mathcal{L}cc} \times [S[D], \mathcal{E}]_{\mathcal{L}cc}.
\end{array}$$

In the above diagram, the vertical arrows are inclusions and the bottom one is an equivalence as $S[C \uplus D]$ is the coproduct $S[C] \uplus S[D]$.

So we only need to show that if $(\varphi, \psi) \in [S[C], \mathcal{E}]_{\mathcal{L}cc} \times [S[D], \mathcal{E}]_{\mathcal{L}cc}$ factorises through $\mathcal{L}$ and $\mathcal{M}$ then the associated map $\varphi \uplus \psi$ factorises through $\mathcal{L} \otimes \mathcal{M}$. Let $S$ be a set of arrows of $S[C]$ such that $\mathcal{L} \simeq S^{-1}S[C]$ and let be $T$ such that $\mathcal{M} \simeq T^{-1}S[D]$. If $\varphi$ and $\psi$ factorise, it means that $\varphi$ sends arrows in $S$ to equivalences and $\psi$ sends arrows in $T$ to equivalences. Let $S \otimes T$ be the set of arrows of the form $s \otimes x$ for $s \in S, x \in S[D]$ or $y \otimes t$ with $t \in T, y \in S[C]$, in $S[C] \otimes S[D]$. By the proof that $S[C] \otimes S[D] \simeq S[C \uplus D]$ above, we have that the map from $S[C \uplus D]$ to $\mathcal{E}$ associated to $(\varphi, \psi)$ is equivalent to the map $\varphi \otimes \psi : S[C] \otimes S[D] \to \mathcal{E}$. But if $\varphi \otimes \psi$ sends arrows in $S \otimes T$ to equivalences so it factorises through $\mathcal{L} \otimes \mathcal{M} \simeq (S \otimes T)^{-1}S[C] \otimes S[D]$. \hfill $\square$

§ 3. — COENDS FOR $\infty$-CATEGORIES

As a prerequisite for the study of dualisable objects in $\mathcal{C}at_{\mathcal{L}cc}$ and the $\infty$-category of $\omega$-continuous sheaves, we must develop the theory of coends in the $\infty$-setting. Traditional references on ends and coends for categories include MacLane [9] and Kelly [10]. An introduction is given by Uemier [11]. The beginning of the theory of coends for quasi-categories has been developed by Cranch [12] and Glasman [13]. In his thesis Cranch develops the definition of dinatural transformations between bifunctors and proves it extends the usual definition for categories, while Glasman proves that the space of natural transformations can be written as an end.
The tensor product of ∞-category of presheaves allow us to develop the theory of coends in a straightforward way. It also does not depend on a particular model for ∞-categories.

In this section we let $D$ be a small ∞-category and $\mathcal{C}$ be a cocomplete ∞-category.

### 3.1 Definition and first properties

**Definition 3.1.** — The left Kan extension of the map functor $\text{Map}_D : D^{\text{op}} \times D \to \mathcal{S}$ to $\mathcal{P}(D^{\text{op}} \times D)$ is called the coend functor and is denoted:

$$\int_D : \mathcal{P}(D^{\text{op}} \times D) \to \mathcal{S}.$$  

**Remark 3.2.** — By definition $\int_D$ is cocontinuous.

**Proposition 3.3.** — The functor $\mathcal{P}(D^{\text{op}}) \times \mathcal{P}(D) \to \mathcal{S}$ defined by:

$$(F, G) \mapsto \int_{c \in \mathcal{C}} F(c) \times G(c),$$

is cocontinuous in each variable.

**Proof.** — This functor is the composition of the cocontinuous functor $\int_D$ with the canonical map $\mathcal{P}(D^{\text{op}}) \times \mathcal{P}(D) \to \mathcal{P}(D^{\text{op}}) \otimes \mathcal{P}(D) \cong \mathcal{P}(D^{\text{op}} \times D)$ which is cocontinuous in each variable. □

Thanks to the tensor product it is possible to extend the definition of the coend to bimodules with values in any cocomplete ∞-category $\mathcal{E}$.

**Proposition 3.4.** — The coend functor induces a cocontinuous functor $\int_D : [D \times D^{\text{op}}, \mathcal{E}] \to \mathcal{E}$ still called the coend functor.

**Proof.** — The map is obtained by tensoring with $\text{Id}_\mathcal{E}$. We then have a cocontinuous functor $\int_D \otimes \text{Id}_\mathcal{E} : \mathcal{P}(D^{\text{op}} \times D) \otimes \mathcal{E} \to \mathcal{E}$. But the tensor product $\mathcal{P}(D^{\text{op}} \times D) \otimes \mathcal{E}$ is canonically equivalent to the ∞-category $[D \times D^{\text{op}}, \mathcal{E}]$. □

**Proposition 3.5 (Fubini).** — Let $\mathcal{C}$ and $\mathcal{D}$ be two small ∞-categories and $\mathcal{E}$ be a cocomplete ∞-category. For any functor $F : C^{\text{op}} \times C \times D^{\text{op}} \times D \to \mathcal{E}$ we have:

$$\int_{c \in \mathcal{C}} \int_{d \in \mathcal{D}} \left( \int_{c \in \mathcal{C}} \int_{d \in \mathcal{D}} F(-, -, d, d)(c, c) \right)(d, d).$$

**Proof.** — Thanks to the equivalences

$$[C^{\text{op}} \times C \times D^{\text{op}} \times D, \mathcal{E}] \cong \mathcal{P}(C^{\text{op}}) \otimes [D^{\text{op}} \times D, \mathcal{E}] \cong \mathcal{P}(D \times D^{\text{op}}) \otimes [C^{\text{op}} \times C, \mathcal{E}],$$

the following coend square commutes:

$$\begin{array}{ccc}
\mathcal{P}(D \times D^{\text{op}}) \otimes \mathcal{E} & \xrightarrow{\int_D} & \mathcal{P}(C \times C^{\text{op}}) \otimes \mathcal{E} \\
\int_{\mathcal{C}} & & \int_{\mathcal{C}} \\
\mathcal{P}(D \times D^{\text{op}}) \otimes \mathcal{E} & \xrightarrow{\int_D} & \mathcal{E}.
\end{array}$$

□
3.2 Yoneda lemma

Let us recall the definition of tensoring of a cocomplete $\infty$-category over $S$. Let $(K, c) \in S \times \mathcal{C}$. Then the cotensor $c^K$ defined by:

$$K \otimes c = \lim_{\rightarrow} c.$$

It is a cocontinuous functor in each variable.

**Proposition 3.6 (Yoneda).** — *Let $F : D \to \mathcal{C}$ be a functor. Then for any $c \in D$,*

$$F(c) \simeq \int_{d \in \mathcal{C}} [d, c] \otimes F(d),$$

*where $[d, c]$ is a shorthand notation for the $\text{Map}(d, c)$.*

**Proof.** — Let’s prove the case where $\mathcal{C} = S$. Let $F : D \to S$ be any functor and let $y : D^{op} \to \mathcal{P}(D^{op})$ be the Yoneda embedding.

By cocontinuity of the coend functor, we have for $c \in D$,

$$\int_{d \in D} [d, c] \times F(d) \simeq \lim_{x \in \text{el}(F)} \int_{d \in D} [d, c] \times [x, d].$$

But by definition of the coend functor $\int_{d \in D} [d, c] \times [x, d] \simeq x(c)$ And the formula is proved:

$$\int_{d \in D} [d, c] \times F(d) \simeq \lim_{x \in \text{el}(F)} [x, c] \simeq F(c).$$

Hence the functor $F \mapsto \int_{d \in D} [d, -] \times F(d)$ is homotopic to the identity. Tensoring it with the identity of $\mathcal{C}$, we obtain an endofunctor of $[D, \mathcal{C}]$ $F \mapsto \int_{d \in D} [d, -] \otimes F(d)$ homotopic to the identity, which proves the formula. □

3.3 Left Kan extensions as coends

When a bimodule $D^{op} \times D \to \mathcal{C}$ is given by the tensor product of two functors, the coend is easily expressible in terms of colimits. In return, we are able to calculate left Kan extension along the Yoneda embedding with the coend functor.

**Proposition 3.7.** — *Let $G$ be an object of $\mathcal{P}(D)$ and let $F : D \to \mathcal{C}$ be a functor. Then:*

$$\int_{d \in D} G(d) \otimes F(d) \simeq \lim_{d \in \text{el}(G)} F(d).$$
Proof. — The functor $\mathcal{P}(D) \times [D, \mathcal{C}] \to [\mathcal{D}^{\mathcal{P}} \times D, \mathcal{C}]$ sending $(G, F)$ to $G \otimes F$ is cocontinuous in the first variable and the coend functor is cocontinuous. We then have that $G \simeq \lim_{c \in \text{el}(G)} [\cdot, c]$ implies:

$$
\int_{d \in D} G(d) \otimes F(d) \simeq \int_{d \in D} \left( \lim_{c \in \text{el}(G)} [c, d] \otimes F(c) \right)
$$

$$
\simeq \int_{d \in D} \lim_{c \in \text{el}(G)} [d, c] \otimes F(c) \simeq \lim_{c \in \text{el}(G)} \int_{d \in D} [d, c] \otimes F(c)
$$

$$
\simeq \lim_{d \in \text{el}(G)} F(d). \quad \text{(Yoneda)}
$$

\[\square\]

Corollary 3.8. — Let $F : D \to \mathcal{C}$ be a functor. Then the left Kan extension of $F$ along the inclusion $i : D \to \mathcal{P}(D)$ is given by:

$$
\text{Lan}_i F : G \mapsto \int_{d \in D} G(d) \otimes F(d).
$$

Proof. — For any functor $G \in \mathcal{P}(D)$, write $\text{el}(G)$ for its set of elements. Then $G \simeq \lim_{c \in \text{el}(G)} d$ in $\mathcal{P}(D)$, so the left Kan extension is given by:

$$
(\text{Lan}_i F)(G) \simeq \lim_{d \in \text{el}(G)} F(d).
$$

Then apply proposition 3.7. \[\square\]

Remark 3.9. — Dualising the proofs, one can obtain the results of the theory of ends.

§ 4. — Exponentiable $\infty$-Toposes

In this section we prove that exponentiable $\infty$-toposes $\mathcal{X}$ are those whose $\infty$-logos $\text{Sh}(\mathcal{X})$ is continuous. This result is an $\infty$-version of the theorem of Johnstone and Joyal [1, Theorem 4.10].

Definition 4.1. — Let $\mathcal{X}$ be an $\infty$-topos, we will say that $\mathcal{X}$ is exponentiable if the functor $\mathcal{Y} \mapsto \mathcal{Y} \times \mathcal{X}$ has a right adjoint.

For an $\infty$-topos $\mathcal{Y}$ we will say that the particular exponential $\mathcal{Y}^{\mathcal{X}}$ exists if there is an $\infty$-topos $\mathcal{Y}^{\mathcal{X}}$ and a map $\mathcal{X} \times \mathcal{Y}^{\mathcal{X}} \to \mathcal{Y}$ such that the induced map $\text{Map}(\mathcal{Z}, \mathcal{Y}^{\mathcal{X}}) \to \text{Map}(\mathcal{Z} \times \mathcal{X}, \mathcal{Y})$ is an isomorphism in $\mathcal{K}$ for every $\mathcal{Z} \in \text{Top}$.

Remark 4.2. — By proposition 5.2.2.12 in HTT [4], an $\infty$-topos $\mathcal{X}$ is exponentiable if and only if for any $\mathcal{Y} \in \text{Top}$, the particular exponential $\mathcal{Y}^{\mathcal{X}}$ exists.

4.1 Injective $\infty$-Toposes and their Points

Definition 4.3. — We shall say that $f : \mathcal{X} \to \mathcal{Y}$ is an inclusion or that $\mathcal{X}$ is a subtopos of $\mathcal{Y}$ if $f^*$ has a fully faithful right adjoint.

Definition 4.4. — An $\infty$-topos $\mathcal{X}$ is injective if for every subtopos $m : \mathcal{Y} \to \mathcal{Z}$, the composition morphism $\text{Map}(\mathcal{Z}, \mathcal{X}) \to \text{Map}(\mathcal{Y}, \mathcal{X})$ has a section.
Remark 4.5. — This notion of injective ∞-topos corresponds to the notion of weakly injective topos defined in *Sketches of an Elephant* [14]. We do not investigate the notions of complete injective and strongly injective ∞-toposes.

Proposition 4.6. — All affine ∞-toposes are injective. Furthermore an ∞-topos is injective if and only if it is a retraction in Top of an affine ∞-toposes.

Proof. — Let \( \mathcal{X} \) be an injective ∞-topos, then by definition, there exists an inclusion \( \mathcal{X} \to \mathcal{A}^D \) with \( D \) a small ∞-category. Because \( \mathcal{X} \) is injective, this morphism must split.

On the contrary we will prove that any affine ∞-topos is injective: let \( \mathcal{F} = \text{Sh}(\mathcal{Y}) \) and \( \mathcal{G} = \text{Sh}(\mathcal{Z}) \) be two ∞-toposes and \( f : \mathcal{Y} \to \mathcal{Z} \) be an inclusion of ∞-toposes. Thanks to the universal property of affine ∞-toposes, we have the following equivalences \( \text{Map}(\mathcal{Y}, \mathcal{A}^D) \simeq \text{core}([D, \mathcal{F}]) \) and \( \text{Map}(\mathcal{Z}, \mathcal{A}^D) \simeq \text{core}([D, \mathcal{G}]) \), where core designates the maximal subgroupoid of an ∞-category. Then the reflective localisation \( f^* \) gives the desired reflective localisation \( (f^*)^D \).

Finally, let’s prove that a retraction of an injective ∞-topos is still injective: let \( r : \mathcal{X} \to \mathcal{X}' \) be a retraction in Top with \( \mathcal{X} \) injective and \( s : \mathcal{X}' \to \mathcal{X} \) a section. Let \( i : \mathcal{Y} \to \mathcal{Z} \) be an inclusion and \( f : \mathcal{Y} \to \mathcal{X}' \) be any map. Then \( sf : \mathcal{Y} \to \mathcal{X} \) can be extended in \( g : \mathcal{Z} \to \mathcal{X} \) because \( \mathcal{X} \) is injective. Then \( rg : \mathcal{Z} \to \mathcal{X} \) extends \( f \).

Injective ∞-toposes have the particular property of being characterised by their ∞-categories of points. That is, knowing \( \text{pt}(\mathcal{X}) \), we can recover \( \mathcal{X} \) in the case where \( \mathcal{X} \) is injective.

Remark 4.7. — Generally there are several ways to build the opposite category of an \((\omega,2)\)-category depending on whether one would like to ‘op’ 1-arrows and/or 2-arrows. The definition of the ∞-category of points of an ∞-topos here reflects the choice of a definition \( \text{Top} = \text{Log}^{\omega\text{p}} \) where we choose to ‘op’ only 1-arrows. Having said that, the definition of the ∞-category of points of an ∞-topos \( \mathcal{X} \) is just \( \text{pt}(\mathcal{X}) = [\ast, \mathcal{X}]_{\text{Top}} \) in this \((\omega,2)\)-categorical framework.

In the ∞-category of points, morphisms correspond to ‘generalisation’ of points. Morphisms in the opposite ∞-category would correspond to ‘specialisation’ of points.

Definition 4.8. — Let \( \mathcal{Inj} \) be the full subcategory of \( \text{Top} \) made of injective ∞-toposes.

Let us also define \( \text{pt}(\mathcal{Inj}) \) as a non-full subcategory of \( \widehat{\text{Cat}}_{\infty} \). Its objects are presheaves ∞-categories \( \mathcal{P}(D) \) with \( D \) a small ∞-category and their retractions by \( \omega \)-continuous functors; its morphisms are the \( \omega \)-continuous functors.

Proposition 4.9. — The functor of points \( \text{pt} : \mathcal{Inj} \to \text{pt}(\mathcal{Inj}) \) is an equivalence of ∞-categories.

Proof. — Let \( D \) be a small ∞-category, then \( \text{pt}(\mathcal{A}^D) \simeq \mathcal{P}(\mathcal{D}^{\ast/}) \) so, by Proposition 4.6, \( \text{pt} \) is a well defined functor from \( \mathcal{Inj} \) to \( \text{pt}(\mathcal{Inj}) \).

We build a new functor \( \psi : \mathcal{A} \mapsto [\mathcal{A}, \mathcal{S}]_{\omega/} \), where \( [\mathcal{A}, \mathcal{S}]_{\omega/} \) is the ∞-category of \( \omega \)-continuous functors between \( \mathcal{A} \) and \( \mathcal{S} \).

We claim that \( \psi \) is a functor from \( \text{pt}(\mathcal{Inj}) \) to \( \mathcal{Inj}^{op} \). For this, let \( m : \mathcal{A} \to \mathcal{B} \) be an \( \omega \)-continuous functor, then it induces a functor \( m^* : [\mathcal{B}, \mathcal{S}]_{\omega/} \to [\mathcal{A}, \mathcal{S}]_{\omega/} \). Because filtered colimits are left exact in \( \mathcal{S} \), we see that finite limits and all colimits in \( \psi(\mathcal{A}) \) and \( \psi(\mathcal{B}) \) are computed pointwise, so \( m^* \) is cocontinuous and...
left exact. Finally, \( \psi(\mathcal{P}(D^{op})) \cong S[D] = Sh(A^D) \), so that by proposition 4.6 \( \psi \) is well defined.

By the above computation, the functor of points \( pt \) induces an equivalence on the subcategory of affine \( \infty \)-toposes to the subcategory of \( pt(J_nj) \) made of presheaves \( \infty \)-categories. This equivalence extends to their Cauchy-completion.

\[ \square \]

### 4.2 Continuous \( \infty \)-categories

The definition of a continuous category was first given in Continuous categories and exponentiable toposes [1]. We shall prove here the same propositions in the \( \infty \)-setting.

**Definition 4.10.** — Let \( \mathcal{C} \) be an \( \infty \)-category with filtered colimits. We will say that \( \mathcal{C} \) is continuous if the colimit functor \( \epsilon : \text{Ind}(\mathcal{C}) \to \mathcal{C} \) has a left adjoint \( \beta : \mathcal{C} \to \text{Ind}(\mathcal{C}) \).

The \( \infty \)-category \( \text{Ind}(\mathcal{C}) \) is not presentable even when \( \mathcal{C} \) is. We will now focus on continuous categories with smallness properties. Namely, we wish to replace \( \text{Ind}(\mathcal{C}) \) by a presentable \( \infty \)-category \( \text{Ind}(\mathcal{D}) \).

**Definition 4.11.** — Let \( \mathcal{C} \) be a continuous \( \infty \)-category. If there exists a small full subcategory \( \mathcal{D} \subset \mathcal{C} \), such that \( \mathcal{D} \) is stable in \( \mathcal{C} \) under finite limits and colimits and such that the evaluation functor \( \text{Ind}(\mathcal{D}) \to \mathcal{C} \) has a fully faithful left adjoint, then we call the triple adjunction:

\[
\text{Ind}(\mathcal{D}) \xrightarrow{\beta} \mathcal{C} \xleftarrow{\alpha} \text{Ind}(\mathcal{C})
\]
a standard presentation.

**Proposition 4.12.** — Let \( \mathcal{C} \) be a presentable and continuous \( \infty \)-category. Then \( \mathcal{C} \) has a standard presentation.

**Proof.** — Because \( \mathcal{C} \) is presentable, there exists a small and dense full subcategory \( \mathcal{D} \subset \mathcal{C} \). We can then take \( \mathcal{D}' \) the smallest full subcategory of \( \mathcal{C} \) containing \( \mathcal{D} \) and closed in \( \mathcal{C} \) under finite limits and colimits. As such, \( \mathcal{D}' \) is dense in \( \mathcal{C} \) so that the evaluation functor \( \epsilon : \text{Ind}(\mathcal{D}') \to \mathcal{C} \) has a fully faithful right adjoint \( \alpha : \mathcal{C} \to \text{Ind}(\mathcal{D}') \).

As \( \epsilon : \text{Ind}(\mathcal{C}) \to \mathcal{C} \) is continuous and \( \text{Ind}(\mathcal{D}') \subset \text{Ind}(\mathcal{C}) \) commutes with limits, we deduce that \( \epsilon : \text{Ind}(\mathcal{D}') \to \mathcal{C} \) is continuous and then has a left adjoint \( \beta \) because \( \text{Ind}(\mathcal{D}') \) and \( \mathcal{C} \) are presentable.

\[ \square \]

**Proposition 4.13.** — Let \( \mathcal{D} \) be an \( \infty \)-category. Then \( \text{Ind}(\mathcal{D}) \) is continuous.

**Proof.** — We denote a generic object of \( \text{Ind}(\text{Ind}(\mathcal{D})) \) as \( \left( \lim_{\to i} \lim_{\to j} \right) d_{ij} \).

Then, the functor \( \alpha : \text{Ind}(\mathcal{D}) \to \text{Ind}(\text{Ind}(\mathcal{D})) \) right adjoint to \( \epsilon \) is given by \( \left( \lim_{\to i} \right) d_i \mapsto \left( \lim_{\to j} \right) d_{ij} \).

We claim that the left adjoint \( \beta \) is given by sending \( \left( \lim_{\to i} \right) d_i \) in \( \text{Ind}(\mathcal{D}) \) on \( \left( \lim_{\to i} \right) d_i \) in \( \text{Ind}(\text{Ind}(\mathcal{D})) \), that is \( \beta = \text{Ind}(\alpha) \). We have a unit transformation of \( \langle \beta, \epsilon \rangle : 1 \cong \varphi \). So we can check the adjunction on mapping spaces. Any \( d \in \mathcal{D} \) is an \( \omega \)-compa\( \tilde{\beta}t \) object of \( \text{Ind}(\mathcal{D}) \), so that for any \( \left( \lim_{\to i} \right) d_{ij} \), \( \beta(d) \) is a formal colimit of \( \omega \)-compa\( \tilde{\beta}t \) objects of \( \text{Ind}(\mathcal{D}) \). Let \( a = \left( \lim_{\to i} \right) \left( \lim_{\to j} \right) d_{ijk} \).

Then we have:

\[
\text{Map}(\beta(d), a) \cong \lim_{\to j} \lim_{\to k} \text{Map}(d_{ij}, d_{ijk}) \cong \text{Map}(d, \varphi(a)).
\]
Proposition 4.14. — Any retraction by $\omega$-continuous functors of a continuous
$\infty$-category is continuous.

Proof. — Let $r : C \to D$ be a retraction by $\omega$-continuous functors and sup-
pose $C$ is continuous. Let $s$ be an $\omega$-continuous section of $r$. Because both
commute with filtered colimits, we have $\varepsilon_D \circ \text{Ind}(r) \simeq r \circ \varepsilon_C$ and
$s \circ \varepsilon_D \simeq \varepsilon_C \circ \text{Ind}(s)$. This means we get the following retraction
diagram:

$$
\begin{array}{c}
\text{Ind}(D) \xrightarrow{\varepsilon_D} \text{Ind}(C) \xrightarrow{\varepsilon_C} \text{Ind}(D) \\
\downarrow \quad \downarrow \quad \downarrow \\
D \quad \varepsilon_D \quad r \quad D.
\end{array}
$$

Let $\theta = \text{Ind}(r) \circ \beta_C \circ s$. The functor $\theta$ is a good candidate to be the left
adjoint to $\varepsilon_D$. Indeed, from the unit transformation $\text{Id} \simeq \varepsilon_C \circ \beta_C$ we get
$\theta : \text{Id} \simeq \varepsilon_D \circ \varepsilon_D$. From the counit transformation $\beta_D \circ \varepsilon_C \to \text{Id}$ we also get a
unit transformation $k : \theta \circ \varepsilon_D \to \text{Id}$. Finally $k \theta \circ \theta u : \theta \to \text{Id}$ is homotopic
to the identity transformation. Unfortunately $\varepsilon_D k \circ u \varepsilon_D : \varepsilon_D \to \varepsilon_D$ is not
homotopic to the identity transformation (in this case, one would call $\theta$ a
weak adjoint). Instead $\varepsilon_D k$ is idempotent.

Fortunately, the category $[D, \text{Ind}(D)]$ has all filtered colimits; thus idem-
potents split [4, corollary 4.4.5.16]. Let $\theta \xrightarrow{s} \beta \xrightarrow{\circ} \theta$ be such a splitting.
We get a new counit map $k' = k \circ (\sigma_{\varepsilon_D}) : \beta \varepsilon_C \to \text{Id}$ and a new unit map
$u' = (\varepsilon_D t) \circ u : \text{Id} \simeq \varepsilon_D \beta$. This time $\varepsilon_D k' \circ u' \varepsilon_D$ is homotopic to the
unit transformation, as well as $k' \beta \circ \beta u'$.

So $\beta$ is a left adjoint to $\varepsilon_D$, hence $D$ is a continuous $\infty$-category. \qed

Proposition 4.15. — A presentable $\infty$-category $C$ is the $\infty$-category of points of
an injective $\infty$-topos $X$ if and only if $C$ is continuous.

Proof. — Suppose $C \in \text{pt}(\text{Inj})$, then by proposition 4.9, we know that $C$ is a
retraction by $\omega$-continuous functors of an $\infty$-category of presheaves $\mathcal{P}(D)$. But
$\mathcal{P}(D)$ is finitely presentable, so it is continuous by proposition 4.13, and $C$ is continuous by
proposition 4.14.

Conversely, assume that $C$ is continuous. Because it is smally presentable, by
proposition 4.12, we get a standard presentation $\text{Ind}(D) \to C$. In particular, $C$ is a retraction by
$\omega$-continuous functors of $\text{Ind}(D)$, and $\text{Ind}(D)$ is itself such a retraction of $\mathcal{P}(D)$, so that $C \in \text{pt}(\text{Inj})$. \qed

Corollary 4.16. — If $X$ is an exponentiable $\infty$-topos, then the $\infty$-category
$\text{Sh}(X)$ is continuous.

Proof. — By proposition 4.6 the $\infty$-topos $\mathcal{A}$ is injective and the fun-
cctor $(-) \times X$ preserves inclusions, so $\mathcal{A}^X$ is also injective. Now, by definition of $\mathcal{A}$, we have the equivalence of $\infty$-categories $\text{pt}(\mathcal{A}^X) \simeq \text{Sh}(X)$ which implies the result. \qed

4.3 $\omega$-continuous sheaves

Let $X$ be a locally quasi-compa$\epsilon$ and Hausdorff topological space and let $C$ be an
$\infty$-category where filtered colimits are left exact. Then the $\infty$-category of
$C$-valued sheaves on $X$ has an alternative description [4, Corollary 7.3.4.10];
it is the $\infty$-category of functors $F : X^{\text{op}} \to C$, where $X$ is the poset of compa$\epsilon$
subsets of $X$, that preserve finite limits and some filtered colimits. We call such sheaves $\omega$-continuous sheaves.

We wish to prove that, more generally the category of $C$-valued sheaves on an exponentiable $\infty$-topos can be described with small colimits and finite limits condition instead of small limits conditions.

Let us recall the following definition.

Definition 4.17. — By an idempotent comonad on an $\infty$-category $C$, we mean the following data: an endofunctor $W : C \to C$ together with a natural transformation $\varepsilon : W \Rightarrow \text{Id}_C$ such that both $iW : W^2 \Rightarrow W$ and $W\varepsilon : W^2 \Rightarrow W$ are point-wise equivalences of endofunctors. This data is equivalent to the data of the coreflective subcategory of fixed points of $W$ inside $C$ [4, Proposition 5.2.7.4].

4.3.1 $\omega$-continuous sheaves of spaces

Given an exponentiable $\infty$-topos $X$, as $\mathcal{Sh}(X)$ is a continuous $\infty$-category we have a standard presentation:

$$\text{Ind}(D) \xrightarrow{\beta} \mathcal{Sh}(X).$$

We then obtain an idempotent cocomplete comonad $W = \beta\varepsilon$ on $\text{Ind}(D)$ and an identification between $\mathcal{Sh}(X)$ and the $\infty$-category $\text{Fix}(W)$ of fixed points of $W$ in $\text{Ind}(D)$.

Definition 4.18. — The idempotent comonad $W : \text{Ind}(D) \to \text{Ind}(D)$ is continuous, we write $w : D \Rightarrow D$ for the corresponding bimodule. That is for $(a, b) \in D^{\text{op}} \times D$, we set $w(a, b) = \text{Map}_D(a, \beta\varepsilon b)$.

Remark 4.19. — An object of $w(a, b)$ is what is called a wavy arrow and denoted $a \rightsquigarrow b$ in Continuous categories and exponentiable toposes [1].

Proposition 4.20. — Let $X$ be an exponentiable $\infty$-topos together with a standard presentation of its associated $\infty$-logos. Then $\mathcal{Sh}(X)$ is equivalent to the $\infty$-category of left exact functors $\mathcal{F} : D^{\text{op}} \to \infty$ satisfying the condition:

$$\mathcal{F}(a) \cong \int_{b \in D} w(a, b) \times \mathcal{F}(b),$$

for all $a \in D$.

Proof. — Let $i : \text{Ind}(D) \to \mathcal{P}(D)$ be the canonical embedding and write $(w \otimes -)$ for the left Kan extension of $w : D \Rightarrow \mathcal{P}(D)$ along $D \to \mathcal{P}(D)$. That is for $\mathcal{F} : D^{\text{op}} \to \infty$ we have:

$$w \otimes \mathcal{F} = \int_{b \in D} w(-, b) \times \mathcal{F}(b).$$

Now suppose $\mathcal{F}$ is a left exact functor, then we claim that $w \otimes \mathcal{F} \simeq iW\mathcal{F}$. Indeed, the comonad $W$ is cocomplete, hence it coincides with the left Kan extension of its own restriction to $D$. Furthermore, the embedding $i$ commutes with filtered colimits and $D$ generates Ind($D$) under filtered colimits, hence $iW$ is also a left Kan extension of its restriction to $D$.

The next step is to show that the two functors $(w \otimes -)$ and $iW$ coincide on $D$. This is true by definition as $w(-, b) = iWb$. The conclusion is that $W$ is a restriction to Ind($D$) of the functor $(w \otimes -)$. Because of this, we can deduce that $i(\text{Fix}(W)) = \text{Fix}(w \otimes -) \cap i(\text{Ind}(D))$ which proves the theorem: the functor $\beta : \mathcal{Sh}(X) \to \text{Fix}(w \otimes -) \cap i(\text{Ind}(D))$ is an equivalence of $\infty$-categories. 

\qed
Definition 4.21. — Let \((w \otimes -)\) be the left Kan extension of \(w : D \to \mathcal{P}(D)\) along the Yoneda embedding \(y : D \to \mathcal{P}(D)\). We call an object of \(\text{Fix}(w_i) \cap i(\text{Ind}(D))\) an \(\omega\)-continuous sheaf of (of spaces). In other words, an \(\omega\)-continuous sheaf of spaces is a left exact functor \(D^{op} \to \mathcal{S}\) such that:

\[
\mathcal{F}(a) \simeq \int_{b \in D} w(a, b) \times \mathcal{F}(b),
\]

for all \(a \in D\).

4.3.2 \(\mathcal{C}\)-valued sheaves

Let \(\mathcal{X}\) be an \(\omega\)-topos and \(\mathcal{C}\) be any \(\omega\)-category. The usual definition of \(\mathcal{C}\)-valued sheaves on \(\mathcal{X}\) is the following: \(\text{Sh}(\mathcal{X}, \mathcal{C}) = [\text{Sh}(\mathcal{X})^{op}, \mathcal{C}]\). However, in the case where \(\mathcal{C}\) is a bicomplete \(\omega\)-category, we wish to show there is another useful expression to work with: \(\text{Sh}(\mathcal{X}, \mathcal{C}) \simeq \text{Sh}(\mathcal{X}) \otimes \mathcal{C}\).

This result is a slightly different version of proposition 2.11 where the assumptions on the two \(\omega\)-categories are weakened; essentially by replacing the presentability condition by a small generation one. We begin with the simplest case.

Lemma 4.22. — Let \(D\) be a small \(\omega\)-category and \(\mathcal{C}\) be a bicomplete \(\omega\)-category, then \([\mathcal{P}(D)^{op}, \mathcal{C}]^\omega \simeq \mathcal{P}(D) \otimes \mathcal{C}\).

Proof. — By theorem 5.6, \(\mathcal{P}(D)\) is a dualisable object of \(\mathcal{C}\text{al}_{cc}\); its dual is \(\mathcal{P}(D^{op})\). Because \(\mathcal{C}\) is supposed to be bicomplete, we now have the equivalences:

\[
\mathcal{P}(D) \otimes \mathcal{C} \simeq [\mathcal{P}(D^{op}), \mathcal{C}]_{cc} \simeq [D^{op}, \mathcal{C}] \simeq \mathcal{P}(D)^{op}, \mathcal{C}]^\omega.
\]

Definition 4.23. — A cocomplete \(\omega\)-category \(\mathcal{C}\) shall be called smally generated if it admits a small and dense subcategory \(D \subset \mathcal{C}\). Equivalently, \(\mathcal{C}\) is smally generated when it is a reflective localisation of an \(\omega\)-category of presheaves on a small \(\omega\)-category.

Proposition 4.24. — Let \(\mathcal{A}\) and \(\mathcal{B}\) be two \(\omega\)-categories. Suppose that \(\mathcal{A}\) is cocomplete and smally generated and \(\mathcal{B}\) is bicomplete, then:

\[
\mathcal{A} \otimes \mathcal{B} \simeq [\mathcal{A}^{op}, \mathcal{B}]^\omega.
\]

Proof. — Let \(D\) be a small \(\omega\)-category and let \(S\) be a large set or arrows of \(\mathcal{P}(D)\) such that \(\mathcal{A}\) is the subcategory of \(\omega\)-local objects of \(\mathcal{P}(D)\). Let \(f : \mathcal{P}(D) \times \mathcal{B} \to \mathcal{P}(D) \otimes \mathcal{B}\) be the canonical map and let \(T\) be the large set of all morphisms in \(\mathcal{P}(D) \otimes \mathcal{B}\) having the form \(f(s \times \text{Id}_b)\) for every \(s \in S\) and \(b \in \mathcal{B}\). Then by proposition 2.12, we have:

\[
T^{-1}(\mathcal{P}(D) \otimes \mathcal{B}) \simeq \mathcal{A} \otimes \mathcal{B}.
\]

By the previous lemma we have \(\mathcal{P}(D) \otimes \mathcal{B} \simeq [\mathcal{P}(D)^{op}, \mathcal{B}]^\omega\) so that \(\mathcal{A} \otimes \mathcal{B}\) correspond to the \(\omega\)-category of \(\omega\)-local objects of \([\mathcal{P}(D)^{op}, \mathcal{B}]^\omega\), where \(T'\) is the large set of all morphisms of the form \(f'(s \times \text{Id}_b)\) with \(f' : \mathcal{P}(D) \times \mathcal{B} \to [\mathcal{P}(D)^{op}, \mathcal{B}]^\omega\) the corresponding canonical map.
We only need to check that the ∞-category \([A^{op}, B]^c\) is the subcategory of \([P(D)^{op}, B]^c\) made of \(T\)'-local objects. For this, we draw the following commutative diagram:

\[
\begin{array}{ccc}
{P(D)^{op}, B}^c & \xrightarrow{\varphi} & {P(D)^{op} \times B^{op}, S}^c \\
\uparrow & & \uparrow \\
{A^{op}, B}^c & \xrightarrow{\psi} & {A^{op} \times B^{op}, S}^c,
\end{array}
\]

where all arrows are fully faithful. Let \(T''\) be the large set of objects of \(\varphi(T')\) and let \(F\) be an object of \([P(D)^{op}, B]^c\). The morphism \(F\) is \(T''\)-local if and only if \(\varphi(F)\) is \(T''\)-local and \(T''\)-local objects of \([P(D)^{op}, B]^c\) are precisely the objects of \([A^{op} \times B^{op}, S]^c\) by direct computation (use Yoneda lemma and the proof of proposition 5.5.4.20 of HTT [4]). Hence, \(\varphi(F)\) is \(T''\)-local if and only if it lies in the image of \(\psi\). We have proved the desired equivalence. 

**Corollary 4.25.** — Let \(X\) be an ∞-topos and \(\mathcal{C}\) be a bicomplete ∞-category, then:

\[
\operatorname{Sh}(X, \mathcal{C}) \simeq \operatorname{Sh}(X) \otimes \mathcal{C}.
\]

**Corollary 4.26.** — Let \(A\) and \(B\) be two cocomplete and smallly generated ∞-categories and let \(\mathcal{C}\) be a bicomplete ∞-category. Then for every cocontinuous functor \(f : A \rightarrow B\), the cocontinuous functor \(f' = f \otimes \text{Id}_\mathcal{C}\) has a right adjoint \(f^* : [A^{op}, \mathcal{C}]^c \rightarrow [B^{op}, \mathcal{C}]^c\) given by precomposition by \(f^{op}\).

**Proof.** — For this proof, we need to understand concretely how \(f'\) is built. We draw the diagram:

\[
\begin{array}{ccc}
A \otimes \mathcal{C} & \xrightarrow{f^{op}} & [A^{op} \times \mathcal{C}^{op}, \mathcal{S}]^c \\
\downarrow Lf & & \downarrow f^* \\
B \otimes \mathcal{C} & \xrightarrow{f'} & [B^{op} \times \mathcal{C}^{op}, \mathcal{S}]^c.
\end{array}
\]

With the horizontal arrows going to the right being fully faithful.

By left Kan extension, we get the functor \(f'\); localising it we have \(Lf\). Then by construction of the tensor product, \(Lf\) sends the subcategory \(A \otimes \mathcal{C}\) to \(B \otimes \mathcal{C}\); the restriction of \(Lf\) to \(A \otimes \mathcal{B}\) is the desired \(f'\).

Meanwhile, \(f^*\) is well defined on the right and restricts to the central column. The key point is that it can also be restricted to the first column thanks to **proposition 4.24**.

By proposition 4.3.3.7 in HTT [4], \(f_i\) is left adjoint to \(f^*\). This implies that \(Lf_i\) is left adjoint to \(f^*\) and because the restriction of an adjunction is still an adjunction, we deduce that \(f'\) is left adjoint to \(f^*\).

**Remark 4.27.** — Corollaries 4.25 and 4.26 imply in particular that for every topological space \(X\), there always exists a sheafification functor adjoint to the natural inclusion:

\[
\operatorname{PSh}(X) \otimes \mathcal{C} \xleftarrow{\sim} \operatorname{Sh}(X) \otimes \mathcal{C}.
\]

where \(\operatorname{PSh}(X)\) denotes the ∞-category of presheaves in \(\mathcal{S}\) on \(X\), as long as \(\mathcal{C}\) is bicomplete. However this sheafification functor is usually not left exact.
Corollary 4.28. — Let $D$ be a small $\infty$-category which has all finite colimits and let $\varphi : \mathcal{C} \to \mathcal{E}$ be a left exact and cocontinuous functor between bicomplete $\infty$-categories. Then the functors $\text{Id}_{\text{Ind}(D)} \otimes \varphi : \text{Ind}(D) \otimes \mathcal{C} \to \text{Ind}(D) \otimes \mathcal{E}$ and $(\varphi \circ -) : \mathcal{D}^{\text{op}}_{\text{lex}} \to \mathcal{C}^{\text{lex}}$ are canonically equivalent. In particular both are left exact and cocontinuous.

Proof. — We know from Proposition 4.24 that the functors $(\text{Ind}(D) \otimes -)$ and $[\mathcal{D}^{\text{op}}, -]_{\text{lex}}$ are equivalent when applied to bicomplete $\infty$-categories.

Let us denote by $L : \mathcal{P}(D) \to \text{Ind}(D)$ the left adjoint to the natural inclusion of ind-objects inside presheaves. Then by the previous corollary, the functor $L \otimes \mathcal{C} : \mathcal{P}(D) \otimes \mathcal{C} \simeq [\mathcal{D}^{\text{op}}, \mathcal{C}] \to \text{Ind}(D) \otimes \mathcal{C} \simeq [\mathcal{D}^{\text{op}}, \mathcal{C}]_{\text{lex}}$ has a right adjoint given by the inclusion $[\mathcal{D}^{\text{op}}, \mathcal{C}]_{\text{lex}} \subset [\mathcal{D}^{\text{op}}, \mathcal{C}]$. The same is true for $\mathcal{E}$. By functoriality of the tensor product, we know that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{P}(D) \otimes \mathcal{C} & \xrightarrow{\text{Id}_{\mathcal{P}(D)} \otimes \varphi} & \mathcal{P}(D) \otimes \mathcal{E} \\
L \otimes \text{Id}_{\mathcal{C}} & \downarrow & L \otimes \text{Id}_{\mathcal{E}} \\
\text{Ind}(D) \otimes \mathcal{C} & \xrightarrow{\text{Id}_{\text{Ind}(D)} \otimes \varphi} & \text{Ind}(D) \otimes \mathcal{E}.
\end{array}
$$

Under the usual identification, the top map is equivalent to the composition functor $(\varphi \circ -) : [\mathcal{D}^{\text{op}}, \mathcal{C}] \to [\mathcal{D}^{\text{op}}, \mathcal{E}]$. Since $\varphi$ is assumed to be left exact, we know it sends left exact functors to left exact functors. From this we get that the restriction of $(\varphi \circ -)$ to the subcategories of left exact functors coincide with the tensor product map $\text{Id}_{\text{Ind}(D)} \otimes \varphi$. □

4.3.3 $\mathcal{C}$-valued $\omega$-continuous sheaves

Going back to an exponentiable $\infty$-topos $\mathcal{X}$ and a standard presentation:

$$
\text{Ind}(D) \xrightarrow{\beta} \text{Sh}(\mathcal{X}),
$$

we let $\mathcal{C}$ be a bicomplete $\infty$-category. Tensoring the standard presentation with $\mathcal{C}$ we get another triple adjunction:

$$
\text{Ind}(D) \otimes \mathcal{C} \xrightarrow{\beta'} \text{Sh}(\mathcal{X}) \otimes \mathcal{C}.
$$

The $\infty$-category $\text{Ind}(D) \otimes \mathcal{C}$ is canonically equivalent to $[\mathcal{D}^{\text{op}}, \mathcal{C}]_{\text{lex}}$. In the same way $\text{Sh}(\mathcal{X}) \otimes \mathcal{C}$ can be identified with $\text{Sh}(\mathcal{X}, \mathcal{C})$. The functors $\beta'$ and $\epsilon'$ are given by $\beta' = \beta \otimes \text{Id}_{\mathcal{C}}$, $\epsilon' = \epsilon \otimes \text{Id}_{\mathcal{C}}$. And we also identify $\epsilon'$ with $\alpha'$ with $\epsilon'$.

Exactly as in the case of $\omega$-continuous sheaves of spaces we obtain an idempotent cocontinuous comonad $W'$ on $\text{Ind}(D) \otimes \mathcal{C}$ by letting $W = \beta' \epsilon'$, as well as an identification between the $\infty$-category of $\mathcal{C}$-valued sheaves on $\mathcal{X}$ and the $\infty$-category of fixed points of $W'$.

Note $L : \mathcal{P}(D) \to \text{Ind}(D)$ the localisation functor adjoint to $i : \text{Ind}(D) \to \mathcal{P}(D)$, and $L' = L \otimes \text{Id}_{\mathcal{C}} : \mathcal{P}(D) \otimes \mathcal{C} \to \text{Ind}(D) \otimes \mathcal{C}$. Note $i'$ the right adjoint to $L'$. By construction, in the proof of Proposition 4.20, the left Kan extension $w_i : \mathcal{P}(D) \to \mathcal{P}(D)$ satisfies $Lw_i \simeq WL$. And as $w_i$ is cocontinuous, also by construction, if $w_i'$ is defined as $w_i \otimes \text{Id}_{\mathcal{C}}$, we get $L'w_i' \simeq W'L'$. This guarantees the following inclusion: $\text{Fix}(w_i') \cap i'([\mathcal{D}^{\text{op}}, \mathcal{C}]_{\text{lex}}) \subset i'(\text{Fix}(W'))$.

We define $\mathcal{C}$-valued $\omega$-continuous sheaves as:
**Definition 4.29.** — Let $\mathcal{X}$ be an exponentiable $\infty$-topos together with a standard presentation with generators $\mathcal{D}$ and let $\mathcal{C}$ be a cocomplete and finitely complete $\infty$-category. We define the $\infty$-category $\mathcal{S}h_{\omega}(\mathcal{D}, \mathcal{C})$ of $\mathcal{C}$-valued $\omega$-continuous sheaves on $\mathcal{X}$ as the $\infty$-category of left exact functors $\mathcal{F}: \mathcal{D}^{\mathcal{C}} \to \mathcal{C}$ such that:

$$\mathcal{F}(a) = \int_{b} w(a, b) \otimes \mathcal{F}(b),$$

for all $a \in \mathcal{D}$. Where $\otimes$ denotes the canonical tensoring of the cocomplete $\infty$-category $\mathcal{C}$ over $\mathcal{S}$.

We deduce the following proposition.

**Proposition 4.30.** — Let $\mathcal{X}$ be an exponentiable $\infty$-topos, $\mathcal{D}$ an $\infty$-category of generators of a standard presentation and $\mathcal{C}$ a bicomplete $\infty$-category, then $\mathcal{C}' : \mathcal{S}h_{\omega}(\mathcal{D}, \mathcal{C}) \to \mathcal{S}h(\mathcal{X}, \mathcal{C})$ is fully faithful.

The remaining key proposition is to show that $(w \otimes -)$ sends left exact functors to left exact functors. For this, we need to make the assumption that $\mathcal{C}$ is an $\infty$-logos.

**Definition 4.31.** — Let $\mathcal{C}$ be an $\infty$-logos. Let us denote by $\mid - \mid : \mathcal{S} \to \mathcal{C}$ the (essentially unique) morphism of $\infty$-logoses between $\mathcal{S}$ and $\mathcal{C}$.

**Theorem 4.32.** — Let $\mathcal{X}$ be an exponentiable $\infty$-topos together with $\mathcal{D}$ an $\infty$-category of generators of a standard presentation for $\mathcal{S}h(\mathcal{X})$. Let $\mathcal{C}$ be an $\infty$-logos, then the embedding $\mathcal{S}h_{\omega}(\mathcal{D}, \mathcal{C}) \to \mathcal{S}h(\mathcal{X}, \mathcal{C})$ is an equivalence of $\infty$-categories. Besides, this equivalence is functorial along morphisms of $\infty$-logoses.

**Proof.** — From the discussion above, we only need to prove that the coend:

$$(w \otimes -)_{\mathcal{D}} : \mathcal{F} \mapsto w \otimes \mathcal{F} = \int_{b \in \mathcal{D}} \mid w(-, b) \mid \times \mathcal{F}(b),$$

sends left exact functors $\mathcal{F}: \mathcal{D}^{\mathcal{C}} \to \mathcal{C}$ to left exact functors. The idea of the proof is exactly the same as the one of proposition 4.20, modulo a change of base $\infty$-logos to $\mathcal{C}$. Using the cocontinuous and left exact map $\mid - \mid : \mathcal{S} \to \mathcal{C}$, the $\infty$-category $\mathcal{D}$ becomes enriched over $\mathcal{C}$. Given an object $d \in \mathcal{D}$, the functor $\mid \text{Map}_{\mathcal{D}}(-, d) \mid : \mathcal{D}^{\mathcal{C}} \to \mathcal{C}$ defines an object of $\mid \mathcal{D}^{\mathcal{C}} \mid_{\text{lex}}$ and using the $\mathcal{C}$-enriched version of the Yoneda embedding, we get a fully faithful functor:

$$\mathcal{D} \leftrightarrow [\mathcal{D}^{\mathcal{C}}]_{\text{lex}} \cong \text{Ind}(\mathcal{D}) \otimes \mathcal{C},$$

that describes $\text{Ind}(\mathcal{D}) \otimes \mathcal{C}$ as the $\mathcal{C}$-enriched $\infty$-category freely generated by $\mathcal{D}$ under filtered $\mathcal{C}$-colimits: $\text{Ind}(\mathcal{D}) \otimes \mathcal{C} \cong \text{Ind}(\mathcal{D})$. Moreover the embedding:

$$\text{Ind}(\mathcal{D}) \cong [\mathcal{D}^{\mathcal{C}}]_{\text{lex}} \leftrightarrow [\mathcal{D}^{\mathcal{C}}] \cong \mathcal{S}(\mathcal{D}) \otimes \mathcal{C},$$

commutes with filtered $\mathcal{C}$-colimits as they are computed pointwise.

As a consequence, we need only to check that $(w \otimes -)$ sends representable functors to left exact ones. Let $d$ be an object of $\mathcal{D}$, when:

$$\int_{b \in \mathcal{D}} \mid w(-, b) \mid \times \mid \text{Map}_{\mathcal{D}}(b, d) \mid \cong \int_{b \in \mathcal{D}} w(-, b) \times \text{Map}_{\mathcal{D}}(b, d) \cong \mid w(-, d) \mid,$$

which is left exact as a composite of left exact functors.

The functoriality of the equivalence is a consequence of the functoriality of the tensor product coupled with corollary 4.28.
4.4 Exponentiability theorem

In corollary 4.16, we have seen that the \( \infty \text{-category} \mathcal{S}h(X) \) is continuous for an exponentiable \( \infty \text{-topos} X \). In the next theorem we wish to show the reciprocal statement.

**Theorem 4.33.** — An \( \infty \text{-topos} X \) is exponentiable if and only if the \( \infty \text{-category} \mathcal{S}h(X) \) is continuous.

The proof of this theorem will follow naturally from the lemmas below.

**Lemma 4.34.** — An \( \infty \text{-topos} X \) is exponentiable if and only if the particular exponentials \( (A^D)^X \) exists for every \( A^D \in \text{Aff} \).

**Proof.** — By remark 4.2, we only need to show that the particular exponentials \( y^X \) exist for every \( y \in \text{Top} \). But by proposition 2.8, any \( y \in \text{Top} \) is a limit of affine \( \infty \text{-toposes} \) i.e \( y \simeq \lim_{i \in I} A^D_i \). As every exponential \( (A^D_i)^X \) exists, we get a map:

\[
\mathcal{X} \times \lim_{i \in I} (A^D_i)^X \to \lim_{i \in I} A^D_i,
\]

that exhibits \( \lim_{i \in I} (A^D_i)^X \) as the exponential \( y^X \).

**Lemma 4.35.** — Let \( X \) be an \( \infty \text{-topos} \) for which the exponential \( A^X \) exists, then all exponentials \( (A^D)^X \) exist for every affine \( \infty \text{-topos} \) \( A^D \).

**Proof.** — The first part of the proof consists in showing that the \( \infty \text{-topos} \) \( \mathcal{D} \) defined by \( \mathcal{P}(D) = \mathcal{S}h([D]) \) is exponentiable.

For this, we will show that \( \mathcal{P}(D) \) is coexponentiable in \( \text{Log} \). The map \( \mathcal{S}[C] \to \mathcal{S}[C \times D^{op}] \) gives the unit map \( \mathcal{S}[C] \to \mathcal{S}[C \times D^{op}] \otimes \mathcal{P}(D) \). For every \( \mathcal{L} \in \text{Log} \), we then have a map:

\[
\text{Map}_{\text{Log}}(\mathcal{S}[C \times D^{op}], \mathcal{L}) \to \text{Map}_{\text{Log}}(\mathcal{S}[C], \mathcal{L} \otimes \mathcal{P}(D)),
\]

which is an isomorphism in \( \mathcal{H} \). Hence by lemma 4.34, \( \mathcal{D} \) is exponentiable and by the calculation we have just done \( (A^C)^{D^{op}} \simeq A^{C \times D^{op}} \).

We shall end the proof by noticing that the particular exponential \( (A^D)^X \) can be defined as \( (A^X)^{D^{op}} \) for any small \( \infty \text{-category} D \). The evaluation map \( \mathcal{X} \times A^X \to A \) gives:

\[
\mathcal{X}^{D^{op}} \times (A^X)^{D^{op}} \to A^D.
\]

Using the map \( \mathcal{X} \to \mathcal{X}^{D^{op}} \) (from exponential of the first projection \( \mathcal{X} \times \mathcal{D}^{op} \to \mathcal{X} \)), we end up with the evaluation map:

\[
\mathcal{X} \times (A^X)^{D^{op}} \to A^D.
\]

Finally for every \( \infty \text{-topos} \) \( y \), we get the following equivalences of mapping spaces in \( \mathcal{T} \):

\[
[y, (A^X)^{D^{op}}] \simeq [y \times D^{op} \times \mathcal{X}, A] \simeq [y \times \mathcal{X}, A^D].
\]

Using lemma 4.34 again, \( X \) is exponentiable. \( \square \)
Lemma 4.36. — Let $\mathcal{X}$ be an $\infty$-topos such that $\text{Sh}(\mathcal{X})$ is a continuous $\infty$-category, then the exponential $\mathbb{A}^{\mathcal{X}}$ exists in $\text{Top}$. 

Proof. — Let $\mathcal{X}$ be an $\infty$-topos such that $\text{Sh}(\mathcal{X})$ is continuous. To show that $\mathbb{A}^{\mathcal{X}}$ exists, we have to find an injective $\infty$-topos $\mathcal{J}$ and functorial isomorphisms $\text{Map}_{\mathcal{Log}}(\text{Sh}(\mathcal{J}), \mathcal{L}) \to \text{Map}_{\mathcal{Log}}(\mathcal{F}[\mathcal{X}], \mathcal{L} \otimes \text{Sh}(\mathcal{X}))$ in $\mathcal{K}$.

First we build $\mathcal{J}$. For this take a standard presentation of $\text{Sh}(\mathcal{X})$:

$$\text{Ind}(D) \xrightarrow{\beta} \text{Sh}(\mathcal{X}).$$

Let $W = \beta \iota$. Now because $\beta$ and $\iota$ are adjoint and that $\beta$ is fully faithful, we have that $W$ is an idempotent cocontinuous comonad on $\text{Ind}(D)$ and $\beta$ induces an equivalence between $\text{Sh}(\mathcal{X})$ and the fixed points of $W$.

Let $w : D^{\text{op}} \times D \to \mathcal{S}$ be the corresponding bimodule. Notice that because $W$ has its values in ind-objects, the bimodule $w$ is left exact in the first variable. Moreover the idempotent comonad structure of $W$ can be rewritten in the following way: the bimodule $w$ bears a bimodule map $w \Rightarrow \text{Map}_D$ inducing the following formula:

$$\int_c w(a, c) \times w(c, b) \simeq w(a, b).$$

Let us denote by $(- \otimes w) : \mathcal{P}(D^{\text{op}}) \to \mathcal{P}(D^{\text{op}})$ the functor defined by:

$$G \otimes w = \int_c G(c) \times w(c, -).$$

Since $w$ is left exact in the first variable, the endofunctor $(- \otimes w)$ (obtained by left extension from $w$) is cocontinuous and left exact; it also bears the structure of an idempotent comonad. We shall call $\mathcal{P}$ its $\infty$-category of fixed points. We end up with the following presentation:

$$\mathcal{P}(D^{\text{op}}) \xrightarrow{\gamma} \mathcal{P},$$

where $\gamma \simeq (- \otimes w)$, both $\gamma$ and $\iota$ are fully faithful and $\iota$ is left exact. From this presentation we deduce immediately that $\mathcal{P}$ is an $\infty$-logos and as it is a retrait in the $\infty$-category of $\infty$-logoses of $\mathcal{P}(D^{\text{op}})$, its associated $\infty$-topos is injective. This is our $\mathcal{J}$.

Let $\mathcal{L}$ be any $\infty$-logos. We will show that $[\mathcal{P}, \mathcal{L}]^{\text{lex}}_{\text{cc}}$ and $\text{Sh}(\mathcal{X}) \otimes \mathcal{L}$ are equivalent by contemplating their respective descriptions.

The $\infty$-category $[\mathcal{P}, \mathcal{L}]^{\text{lex}}_{\text{cc}}$ is equivalent, by definition of $\mathcal{P}$, to the $\infty$-category of cocontinuous and left exact functors $\mathcal{F} : \mathcal{P}(D^{\text{op}}) \to \mathcal{L}$ such that:

$$\mathcal{F} \circ (- \otimes w) \simeq \mathcal{F}.$$ 

But since $\mathcal{F}$ is cocontinuous and left exact, this $\infty$-category is also equivalent to the $\infty$-category of left exact functors $\mathcal{F} : D^{\text{op}} \to \mathcal{L}$ such that:

$$w \otimes \mathcal{F} \simeq \mathcal{F}.$$ 

In other words, $[\mathcal{P}, \mathcal{L}]^{\text{lex}}_{\text{cc}}$ is equivalent to $\text{Sh}_w(D, \mathcal{L})$. Moreover this equivalence is functorial in $\mathcal{L}$. 


Using theorem 4.32 (one only needs a continuous ∞-category to use the conclusions of the theorem), we are also given functorial equivalences of ∞-categories between $\text{Sh}_\omega(D, L)$ and $\text{Sh}(X) \otimes L$, so that we obtain equivalences in $\mathcal{T}$: $\text{Map}_{\mathcal{T}}(\mathcal{Y}, \mathcal{J}) \cong \text{Map}_{\mathcal{T}}(\mathcal{Y} \times X, \mathcal{A})$ which are functorial in $\mathcal{Y}$. This proves the existence of $\mathbb{A}^X$.

4.5 Glossary of maps between ∞-toposes

Given a morphism of ∞-toposes $f : X \to Y$, we shall say that:

- the ∞-topos $X$ has trivial $Y$-shape if $f^*$ is fully faithful;
- the morphism $f$ is essential if $f^*$ has a left adjoint;
- the morphism $f$ is proper if it satisfies the stable Beck-Chevalley condition [4, Definition 7.3.1.4];
- the morphism $f$ is cell-like if $f$ is proper and $X$ has trivial $Y$-shape;
- the morphism $f$ is étale if there exists $U \in \text{Sh}(Y)$ such that $f^* : \text{Sh}(Y) \to \text{Sh}(Y)/U$ is the product by $U$;
- the ∞-topos $X$ is an open (resp. closed, resp. locally closed) subtopos of $Y$ if $f$ is an étale inclusion (resp. proper inclusion, resp. the intersection of an étale inclusion and a proper inclusion).

4.6 Examples of exponentiable ∞-toposes

**Proposition 4.37.** Let $X$ be an ∞-topos and suppose that the ∞-category $\text{Sh}(X)$ is $\omega$-presentable. Then $X$ is exponentiable.

**Proof.** If $\text{Sh}(X)$ is $\omega$-presentable, then there exists a small ∞-category $D$ such that $\text{Sh}(X) \cong \text{Ind}(D)$ and by proposition 4.13, the ∞-category $\text{Sh}(X)$ is continuous.

In particular all the affine ∞-toposes $\mathbb{A}^D$ are exponentiable. Also all ∞-toposes $X$ such that $\text{Sh}(X)$ is a presheaf ∞-category. In particular if $G$ is a discrete group then $\text{Sh}G$ is an exponentiable ∞-topos. Another class of examples is given by the locally coherent $n$-toposes.

**Definition 4.38.** Let $C$ be an $n$-category which admits finite limits. We will say that a Grothendieck topology on $C$ is finitary if for every object $c \in C$ and every covering sieve $C^{(0)}_{/c} \subset C_{/c}$ there exists a finite collection of morphisms $\{e_i \to c\}_{i \in I}$ in $C^{(0)}_{/c}$ which generates the sieve $C^{(0)}_{/c}$.

**Definition 4.39.** Let $n < \infty$, an $n$-topos is locally coherent if it is an $n$-topos associated to a finitary $n$-site.

**Proposition 4.40.** Let $n < \infty$ and $X$ be a locally coherent $n$-topos, then $X$ is exponentiable.

**Proof.** If $C$ is a finitary $n$-site, then $\text{Sh}(C)$ is $\omega$-presentable. Indeed the sheaf condition for $F \in \text{Sh}(C)$ boils down to finite limit conditions. All sieves are generated by finite collections $\{e_i \to c\}_{i \in I}$ so the sheaf condition:

$$
\prod_i F(e_i) \iff \prod_{i \to j} F(e_j)
$$
involves only finite products at each level and there is only a finite number of levels because in $S_{\leq n-1}$ limits of cosimplicial objects can be computed after being truncated at level $n$.

The consequence is that the inclusion $\mathcal{S}h(C) \hookrightarrow \mathcal{P}(C)$ commutes with filtered colimits, which means that the reflective localisation $\mathcal{P}(C) \to \mathcal{S}h(C)$ is $\omega$-accessible, so $\mathcal{S}h(C)$ is $\omega$-presentable and $X$ is exponentiable by proposition 4.37.

The following two propositions are trivial properties of exponentiable objects in an $\infty$-category with finite limits.

**Proposition 4.41.** — Let $X$ and $Y$ be two exponentiable $\infty$-toposes, then $X \times Y$ is exponentiable.

**Proposition 4.42.** — Let $X$ be an exponentiable $\infty$-topos and $r : X \to Y$ a retraction. Then $Y$ is also exponentiable.

**Proposition 4.43.** — Let $X \to Y$ be an étale morphism. If $Y$ is exponentiable, so is $X$. In particular open subtoposes of $Y$ are exponentiable.

*Proof.* — The $\infty$-category $\mathcal{S}h(Y)(U)$ is continuous because colimits in the slice $\infty$-topos can be computed using the projection $\pi_U : Y(U) \to Y$.

**Corollary 4.44.** — Let $X$ be a locally quasi-compact and quasi-separated topological space, then its associated $\infty$-topos is an exponentiable $\infty$-topos.

*Proof.* — If $X$ is locally quasi-compact and quasi-separated, then the frame $O(X)$ is a retract of $\text{Ind}(O(X))_X$. Passing to the associated $\infty$-toposes and using proposition 4.37 proves the corollary.

**Remark 4.45.** — This corollary implies in particular that $\infty$-toposes associated to locally quasi-compact and Hausdorff topological spaces are exponentiable. An independent proof of that statement is given in HTT [4, Theorem 7.3.4.9].

The following proposition encompasses some of the previous ones.

**Proposition 4.46.** — Let $f : I \to \mathcal{Top}$ be a small diagram of exponentiable $\infty$-toposes. Suppose also that for any arrow $i \to j$ in $I$, the following square commutes:

\[
\begin{array}{ccc}
\text{Ind}(\mathcal{S}h(X_j)) & \xrightarrow{\text{Ind}(f_{ij})} & \text{Ind}(\mathcal{S}h(X_i)) \\
\beta_i & & \beta_j \\
\mathcal{S}h(X_j) & \xrightarrow{f_{ij}^*} & \mathcal{S}h(X_i).
\end{array}
\]

Then,

- the colimit of $f$ is exponentiable;
- if $I$ is cofiltered, the limit of $f$ is exponentiable.

*Proof.* — By sections 6.3.2 and 6.3.3 in HTT [4], limits and filtered colimits of $\infty$-categories of sheaves can be computed in $\mathcal{C}at$. By direct computation, $\varprojlim \text{Ind}(\mathcal{S}h(X_i)) \cong \text{Ind}(\varprojlim \mathcal{S}h(X_i))$ and thanks to the commuting squares we requested, we get a functor $\beta : \varprojlim \mathcal{S}h(X_i) \to \text{Ind}(\mathcal{S}h(X_i))$ left adjoint to the evaluation functor, so that $\varprojlim \mathcal{S}h(X_i)$ is continuous.

In the same way, if $I$ is cofiltered, then $\varprojlim \text{Ind}(\mathcal{S}h(X_i)) \cong \text{Ind}(\varprojlim \mathcal{S}h(X_i))$ and we end up with the same conclusion.
Remark 4.47. — As a consequence, a colimit of a diagram of exponentiable ∞-toposes with étale maps is exponentiable.

Corollary 4.48. — Let \( f : I \to \text{Top} \) be a small cofiltered diagram of exponentiable ∞-toposes. Assume that for every arrow \( i \to j \), the corresponding morphism \( f(i) \to f(j) \) is proper and that the ∞-logos associated to each \( f(i) \) is \( \omega \)-accessible. Then the limit of \( f \) is also an exponentiable ∞-topos.

Proof. — Let \( f : X \to Y \) be an arrow in such a diagram. Then by assumption both \( \text{Sh}(X) \) and \( \text{Sh}(Y) \) are ∞-categories of ind-objects. In addition since \( f \) is proper, by remark 7.3.1.5 in HTT [4], \( f^* \) is \( \omega \)-continuous so that \( f^* \) preserves \( \omega \)-compact objects. We can then apply proposition 4.46.

We now describe subtoposes of an exponentiable ∞-topos.

Proposition 4.49. — Let \( X \) be an exponentiable ∞-topos and \( i : Y \subset X \) be a subtopos. If the reflective localisation \( i^* : \text{Sh}(X) \to \text{Sh}(Y) \) is \( \omega \)-accessible, then \( Y \) is exponentiable.

Proof. — If the right adjoint to \( i^* \) is \( \omega \)-continuous, then \( \text{Sh}(Y) \) becomes a retract by \( \omega \)-continuous functors and we conclude with proposition 4.14.

Corollary 4.50. — Let \( X \hookrightarrow Y \) be a closed subtopos of \( Y \). Suppose \( Y \) is exponentiable, then \( X \) is also exponentiable.

Proof. — By remark 7.3.1.5 in HTT [4], if \( f \) is proper, the functor \( f^* \) is \( \omega \)-continuous.

Finally combining the results we have on open and closed subtoposes, we get the following proposition:

Proposition 4.51. — Every locally closed subtopos of an exponentiable ∞-topos is exponentiable.

Proposition 4.52. — Let \( f : X \to Y \) be a map between two ∞-toposes. Suppose moreover that \( f \) is cell-like or that \( f \) is essential with \( X \) having trivial \( Y \)-shape. In such circumstances, if \( X \) is exponentiable, then \( Y \) is also exponentiable.

Proof. — In both cases, \( f^* \) is fully faithful with a (left or right) adjoint that commutes with filtered colimits. Then apply proposition 4.14.

Remark 4.53. — By a result of Scott [15], every exponentiable locale has enough points. This is no longer the case for ∞-toposes [4, Example 6.5.4.5].

§ 5. — Dualisability of the ∞-Category of Stable Sheaves

In this section we prove that when an ∞-topos is exponentiable, its ∞-category of stable sheaves is dualisable.

5.1 Stabilisation for Presentable ∞-Categories

We shall recall the definition of the stabilisation functor and its properties. Our reference for this topic is Higher Algebra [7, Ch. 1 & Sec. 4.8]. We shall denote by \( \text{Sp} \) the ∞-category of spectra.

Definition 5.1. — Let \( \text{Pres}_\text{sp} \) denote the full subcategory of \( \widehat{\text{Cat}_{\infty}} \) whose objects are the presentable and stable ∞-categories.
Theorem 5.2 [7, 4.8.1.23 & 4.8.2.18]. — The ∞-category \( \mathcal{P}_{\text{res}} \) inherits a closed symmetric monoidal structure from \( \mathcal{P} \). Furthermore, the inclusion functor \( \mathcal{P}_{\text{res}} \hookrightarrow \mathcal{P} \) has a left adjoint, the stabilisation functor: \( \mathcal{P} \to \text{Sp}(\mathcal{C}) \simeq \mathcal{C} \otimes \text{Sp} \) making \( \mathcal{P}_{\text{res}} \) a symmetric monoidal reflective localisation of \( \mathcal{P} \).

5.2 Dualisability in \( \mathcal{P} \)

We start by recalling the notion of dualisable objects in a symmetric monoidal \( \infty \)-category \( (\mathcal{C}, \otimes) \) [7, Ch. 4.6.1].

Definition 5.3. — An object \( X \) of \( \mathcal{C} \) is dualisable if there exists another object \( X^\vee \in \mathcal{C} \) with two maps \( \eta: 1_{\mathcal{C}} \to X \otimes X^\vee \) and \( \varepsilon: X^\vee \otimes X \to 1_{\mathcal{C}} \). where \( 1_{\mathcal{C}} \) is the unit of \( \mathcal{C} \), such that the composite maps:

\[
\begin{align*}
X & \xrightarrow{\eta \otimes \text{Id}_{X}} X \otimes X^\vee \otimes X \xrightarrow{\text{Id}_{X} \otimes \varepsilon} X; \\
X^\vee & \xrightarrow{\text{Id}_{X^\vee} \otimes \eta} X^\vee \otimes X \otimes X^\vee \xrightarrow{\varepsilon \otimes \text{Id}_{X^\vee}} X^\vee,
\end{align*}
\]

are homotopic to the identities on \( X \) and \( X^\vee \) respectively.

Remark 5.4. — In the case where \( \mathcal{C} \) is a closed symmetric monoidal \( \infty \)-category, a dualisable object \( X \) has its dual given by \( X^\vee = [X, 1] \) where \([-, -]\) is the internal hom associated to the monoidal structure and \( 1 \) is the monoidal unit.

Lemma 5.5. — In a closed symmetric monoidal \( \infty \)-category, any retraction of a dualisable object is dualisable.

Proof. — Let \( r: X \to Y \) be a retraction with \( X \) a dualisable object and let \( s: Y \to X \) be a section. Set \( Y^\vee = [Y, 1_{\mathcal{C}}] \) an let's show that \( Y^\vee \) has the right property. Because \( r: X \to Y \) is a retraction, the same is true for \( s^\vee : Y^\vee \to Y \). We are then supplied with maps \( \eta_Y = (r \otimes s^\vee) \eta_X \) and \( \varepsilon_Y = \varepsilon_X (r^\vee \otimes s) \). The composition \( (\text{Id}_Y \otimes \varepsilon_Y) \circ (r^\vee \otimes \text{Id}_X) : Y \to Y \) is then a retraction of \( \text{Id}_X \), hence homotopic to the identity itself. The same is true for the other composition.

Theorem 5.6. — The \( \infty \)-categories of the form \( \mathcal{P}(D) \) with \( D \) a small \( \infty \)-category and their retracts are dualisable objects of \( \mathcal{C}(\text{res}) \). Moreover, they are exactly the dualisable objects of \( \mathcal{P} \).

Proof. — Let \( D \) be a small \( \infty \)-category, then if \( \mathcal{P}(D) \) has a dual, it has to be \( \mathcal{P}(D^{\text{op}}) \), so let’s introduce \( \mathcal{P}(D^{\text{op}} \times D) \) the \( \infty \)-category of bimodules on \( D \); we have \( \mathcal{P}(D^{\text{op}}) \otimes \mathcal{P}(D) \simeq \mathcal{P}(D^{\text{op}} \times D) \).

Then let \( \eta: S \to \mathcal{P}(D^{\text{op}} \times D) \) be the cocontinuous functor sending the point \( * \in S \) to the map-bimodule \([-, -]_D \). And finally, let \( \varepsilon: \mathcal{P}(D^{\text{op}} \times D) \to S \) be the coend functor.

The composition \( (\text{Id} \otimes \varepsilon)(\eta \otimes \text{Id}) \) is given by the formula for presheaves:

\[
\left[ F \in D^{\text{op}} \right] F(b) \times [a, b] = F(a) \text{ for a functor } F \in \mathcal{P}(D).
\]

The composition formula comes from \( \left[ F \in D^{\text{op}} \right] F(b) \times F(a) = F(b) \) for a functor \( F \in \mathcal{P}(D^{\text{op}}) \). Because a retraction of a dualisable object is dualisable by lemma 5.5, we are done for the first half.

Let \( \mathcal{C} \) be a dualisable presentable \( \infty \)-category, \( D \subset \mathcal{C} \) be a small and dense subcategory and let \( L: \mathcal{P}(D) \to \mathcal{C} \) be the associated reflective localisation functor. The dual map \( L^\vee: \mathcal{C}^\vee \simeq [\mathcal{C}, S] \to \mathcal{P}(D^{\text{op}}) \) is fully faithful because \( L \) is a reflective localisation functor, it is also cocontinuous. It has a left adjoint.
which is the left Kan extension along \( L \). As a consequence \( \mathcal{C}' \) is a retractive by cocontinuous functors of \( \mathcal{P}(D^{op}) \). Finally because \( \mathcal{C} \simeq (\mathcal{C}')' \) we deduce that \( \mathcal{C} \) is a retractive of \( \mathcal{P}(D) \).

\[ \]

5.3 Dualisability of \( \infty \)-stable sheaves

The \( \infty \)-logos of an exponentiable \( \infty \)-topos is not dualisable in general in \( \mathcal{P}_{\text{res}} \), as in general an \( \infty \)-category of ind-objects is not dualisable. This is no longer the case in \( \mathcal{P}_{\text{res}_{\text{st}}} \).

**Theorem 5.7.** — The dualisable objects of \( \mathcal{P}_{\text{res}_{\text{st}}} \) are the \( \infty \)-categories of the form \( \mathcal{P}(D) \otimes \text{Sp} \) and their retractiles.

**Proof.** — Since the stabilisation functor \( (\text{Sp} \otimes -) : \mathcal{P}_{\text{res}} \to \mathcal{P}_{\text{res}_{\text{st}}} \) is symmetric monoidal, it sends dualisable objects to dualisable objects. Hence we know that the \( \infty \)-categories of the form \( \mathcal{P}(D) \otimes \text{Sp} \) and their retractiles are dualisable by **Theorem 5.6**.

Let \( \mathcal{C} \) be a dualisable presentable \( \infty \)-category, then there exists a small \( \infty \)-category \( D \) and a reflective localisation \( \mathcal{P}(D) \to \mathcal{C} \) which induces a reflective localisation \( \mathcal{P}(D) \otimes \text{Sp} \to \mathcal{C} \otimes \text{Sp} \simeq \mathcal{C} \). We end the proof with the same arguments as in **Theorem 5.6**.

**Lemma 5.8.** — Let \( D \) be a small \( \infty \)-category with finite colimits. Then \( \text{Ind}(D) \otimes \text{Sp} \) is a retractive of \( \mathcal{P}(D) \otimes \text{Sp} \) in \( \mathcal{P}_{\text{res}_{\text{st}}} \).

**Proof.** — Since \( D \) has small colimits, then by **proposition 2.11** the \( \infty \)-category \( \text{Ind}(D) \otimes \text{Sp} \) is equivalent to the \( \infty \)-category of left exact functors \( [D^{op}, \text{Sp}]^{\text{lex}} \) and \( \mathcal{P}(D) \otimes \text{Sp} \) is equivalent to [\( D^{op}, \text{Sp} \)].

Because \( \text{Sp} \) is \( \infty \)-stable and colimits in functor \( \infty \)-categories are computed pointwise, the embedding \( [D^{op}, \text{Sp}]^{\text{lex}} \hookrightarrow [D^{op}, \text{Sp}] \) commutes with all limits and colimits. It then has a left adjoint such that \( \text{Ind}(D) \otimes \text{Sp} \) is a retractive of \( \mathcal{P}(D) \otimes \text{Sp} \) by cocontinuous functors.

**Theorem 5.9.** — Let \( X \) be an exponentiable \( \infty \)-topos, then \( \text{Sh}(X) \otimes \text{Sp} \), the \( \infty \)-category of \( \infty \)-stable sheaves on \( X \), is a dualisable object of \( \mathcal{P}_{\text{res}_{\text{st}}} \).

**Proof.** — After tensoring by \( \text{Sp} \) a standard presentation of \( \text{Sh}(X) \), we get a retractive in \( \mathcal{P}_{\text{res}_{\text{st}}} \):

\[
\text{Ind}(D) \otimes \text{Sp} \xleftarrow{\mathcal{C}'} \text{Sh}(X) \otimes \text{Sp}.
\]

We conclude using **Lemma 5.8** and **Theorem 5.7**.

This theorem can be compared to a result of Niefield and Wood [16]:

**Theorem.** — An \( R \)-ring \( A \) is coexponentiable if and only if \( A \) is projective and finitely generated as an \( R \)-module.

**References**

[1] Johnstone P. and Joyal A., ‘Continuous categories and exponentiable toposes’, *Journal of Pure and Applied Algebra* 25 no. 3, (1982) 255–296.

[2] Lurie J., ‘Spectral Algebraic Geometry’. Online book ‘under construction’, Feb., 2018.
REFERENCES

[3] Kashiwara M. and Schapira P., *Sheaves on Manifolds*, vol. 292 of *Grundlehren der mathematischen Wissenschaften*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1990. With a chapter in French by Christian Houzel.

[4] Lurie J., *Higher Topos Theory*, vol. 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, July, 2009.

[5] Rezk C., ‘Toposes and Homotopy Toposes’. Lecture notes, 2005.

[6] Toën B. and Vezzosi G., ‘Homotopical algebraic geometry I: topos theory’, *Advances in Mathematics* 193 no. 2, (June, 2005) 257–372.

[7] Lurie J., ‘Higher Algebra’. Online book, Sept., 2017.

[8] Lurie J., ‘Higher Topos Theory’. Corrected online version, Apr., 2017.

[9] Mac Lane S., *Categories for the Working Mathematician*, vol. 5 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 1978.

[10] Kelly G. M., *Basic Concepts of Enriched Category Theory*, vol. 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, New York, 1982.

[11] Upmeier H., ‘Ends and Coends’. Lecture notes.

[12] Cranch J., *Algebraic Theories and (Infinity, 1)-Categories*. PhD thesis, University of Sheffield, Nov., 2010. *arXiv:1011.3243 [math.AT]*.

[13] Glasman S., ‘A spectrum-level Hodge filtration on topological Hochschild homology’, *Selecta Mathematica* 22 no. 3, (2016) 1583–1612.

[14] Johnstone P. T., *Sketches of an Elephant: a Topos Theory Compendium*. Vol. 2, vol. 44 of *Oxford Logic Guides*. Oxford University Press, Oxford, 2002.

[15] Scott D., ‘Continuous lattices’, in *Toposes, Algebraic Geometry and Logic: Dalhousie University, Halifax, January 16–19, 1971*, Lawvere F. W., ed., pp. 97–136. Springer Berlin Heidelberg, Berlin, Heidelberg, 1972.

[16] Niefield S. and Wood R., ‘Coexponentiability and Projectivity: Rigs, Rings, and Quantales’, *Theory and Applications of Categories* 32 no. 36, (2017) 1222–1228.