THE POLYNOMIAL SIEVE AND EQUAL SUMS OF LIKE POLYNOMIALS

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Dedicated to Étienne Fouvry on his sixtieth birthday

Abstract. A new “polynomial sieve” is presented and used to show that almost all integers have at most one representation as a sum of two values of a given polynomial of degree at least 3.

1. Introduction

Suppose that we are given a set \( \mathcal{A} \subset \mathbb{Z}^m \). A primary goal in sieve theory is to estimate how many elements of \( \mathcal{A} \) have components belonging to a particular sequence of integers, such as squares, for example. Let \( w : \mathbb{Z}^m \to \mathbb{R}_{\geq 0} \) be a non-negative weight function such that
\[
\sum_{n \in \mathbb{Z}^m} w(n) < \infty.
\]
Let \( f(x; y) \in \mathbb{Z}[x, y] \) be a polynomial, with \( y = (y_1, \ldots, y_m) \), which we suppose takes the shape
\[
f(x; y) = c_0(y)x^d + \cdots + c_d(y),
\]
for polynomials \( c_0, \ldots, c_d \in \mathbb{Z}[y] \) such that \( c_0 \) does not vanish identically. In particular \( f(x; y) \) has degree \( d \) with respect to \( x \).

We seek an upper bound for the sum
\[
S(\mathcal{A}) = \sum_{\substack{n \in \mathcal{A} \\ f(x; n) \text{ soluble}}} w(n),
\]
where for \( n \in \mathcal{A} \) solubility of \( f(x; n) \) means that there exists \( x \in \mathbb{Z} \) such that \( f(x; n) = 0 \). In order to prevent this condition being vacuous, it is natural to restrict attention to \( n \in \mathcal{A} \) for which \( f(x; n) \) does not vanish identically, Moreover, we will introduce extra flexibility into our bound for \( S(\mathcal{A}) \) by allowing \( w \) to be supported away from the zeros of a given auxiliary polynomial. Our work is inspired by Heath-Brown’s

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square sieve \[6\], which corresponds to the special case \(m = 1\) and \(f(x; y) = x^2 - y\).

**Theorem 1.1.** Let \(\mathcal{P}\) be a set of primes, with \(P = \#\mathcal{P}\). Let \(\alpha \in \mathbb{Z}_{>0}\) and let \(g \in \mathbb{Z}[y]\) be a non-zero polynomial. For each \(p \in \mathcal{P}\) and \(n \in \mathbb{Z}^m\), let

\[ h(n) = \gcd(c_0(n), \ldots, c_d(n)) \]

and

\[ \nu_p(n) = \#\{ x \pmod{p} : f(x; n) \equiv 0 \pmod{p} \}. \]

Suppose that \(w(n) = 0\) if \(g(n)h(n) = 0\) or if \(|n| \geq \exp(P)\). Then we have

\[ S(\mathcal{A}) \ll \frac{1}{P^2} \sum_{p, q \in \mathcal{P}} \left| \sum_{i,j \in \{0,1,2\}} c_{i,j}(\alpha) S_{i,j}(p, q) \right|, \]

with

\[ S_{i,j}(p, q) = \sum_{\substack{n \in \mathcal{A} \\gcd(pq, g(n)h(n)) = 1}} w(n)\nu_p(n)^i\nu_q(n)^j \]

and

\[ c_{i,j}(\alpha) = \begin{cases} 
(\alpha - d)^2, & \text{if } (i, j) = (0, 0), \\
\alpha + (\alpha - 1)d - d^2, & \text{if } (i, j) = (1, 0) \text{ or } (0, 1), \\
(1 + d)^2, & \text{if } (i, j) = (1, 1), \\
-\alpha + d, & \text{if } (i, j) = (2, 0) \text{ or } (0, 2), \\
-1 - d, & \text{if } (i, j) = (2, 1) \text{ or } (1, 2), \\
1, & \text{if } (i, j) = (2, 2). 
\end{cases} \]

This result will be established in [2]. The implied constant is allowed to depend on the polynomials \(f \in \mathbb{Z}[x, y]\) and \(g \in \mathbb{Z}[y]\).

The parameter \(\alpha \geq 1\) in Theorem 1.1 should be thought of as bounded absolutely in terms of \(d\) and \(m\). Our upper bound for \(S(\mathcal{A})\) leads us to study the sums \(S_{i,j}(p, q)\) for suitable primes \(p\) and \(q\). In favourable circumstances it will be possible to get an asymptotic formula for each of these sums, with appropriate main terms \(M_{i,j}(p, q)\). The idea would then be to choose \(\alpha \geq 1\) in such a way that the sum \(\sum_{i,j} c_{i,j}(\alpha)M_{i,j}(p, q)\) vanishes.

Theorem 1.1 is a generalisation of the square sieve of Heath-Brown [6]. To see this we take \(m = 1\), \(f(x; y) = x^2 - y\) and \(g(y) = 2y\) in our result. Then \(d = 2\), \(h(n) = 1\) and \(\nu_p(n) = 1 + \left(\frac{n}{p}\right)\) if \(p > 2\). A direct
calculation shows that
\[ \sum_{i,j \in \{0,1,2\}} c_{i,j} (\alpha) \nu_p(n)^i \nu_q(n)^j \]
\[ = (\alpha - 1)^2 + (\alpha - 1) \left\{ \left( \frac{n}{p} \right) + \left( \frac{n}{q} \right) \right\} + \left( \frac{n}{pq} \right), \]
if \( \gcd(pq, 2n) = 1 \). We are led to take \( \alpha = 1 \) in Theorem 1.1. Then, if \( p = q \) is an odd prime in \( \mathcal{P} \), we deduce that
\[ \sum_{i,j} c_{i,j}(1) S_{i,j}(p, q) \leq \sum_{n \in \mathcal{A}} w(n). \]
It therefore follows that
\[ S(\mathcal{A}) \ll \frac{1}{P} \sum_{n \in \mathcal{A}} w(n) + \frac{1}{P^2} \sum_{p \neq q \in \mathcal{P}} \left| \sum_{n \in \mathcal{A}} w(n) \left( \frac{n}{pq} \right) \right|, \]
which recovers [6, Thm. 1] exactly. In a similar fashion, by taking \( m = 1, f(x; y) = x^d - y \) and \( g(y) = dy \), it is possible to deduce the power sieve of Munshi [12, Lemma 2.1] from Theorem 1.1.

We will illustrate Theorem 1.1 by investigating the numbers that can be represented as the sum of two values of a given polynomial. Let \( f \in \mathbb{Z}[x] \) be a polynomial of degree \( d \geq 3 \) with positive leading coefficient. Consider the arithmetic function
\[ r_f(n) = \#\{(y, z) \in \mathbb{Z}^2_{\geq 0} : n = f(y) + f(z)\}. \]
The average behaviour of \( r_f(n) \) is easily understood with recourse to the geometry of numbers, with the outcome that there is a constant \( c_f > 0 \) such that
\[ \sum_{n \leq N} r_f(n) \sim c_f N^{2/d}, \quad (N \to \infty). \] (1.1)
The following result provides an estimate for its second moment.

**Theorem 1.2.** We have
\[ \sum_{n \leq N} r_f(n)^2 \sim 2c_f N^{2/d}, \quad (N \to \infty). \]
There are asymptotically \( \frac{1}{2}c_f N^{2/d} \) integers \( n \leq N \) for which \( r_f(n) \neq 0 \), and almost all of these have essentially just one representation.

In fact this result may be further quantified in the following manner. For \( B \geq 1 \), let \( E_f(B) \) denote the number of positive integers \( y_1, y_2, y_3, y_4 \leq B \) such that
\[ f(y_1) + f(y_2) = f(y_3) + f(y_4), \] (1.2)
with \( \{y_1, y_2\} \neq \{y_3, y_4\} \). The sum in Theorem 1.2 counts solutions of (1.2) in positive integers \( y_1, \ldots, y_4 \) with \( f(y_1) + f(y_2) \leq N \). Any solution in which \( y_3, y_4 \) are not a permutation of \( y_1, y_2 \) will be counted by \( E_f(B) \), for \( B \) of order \( N^{1/d} \). Amongst the trivial solutions, there will be \( \Theta(N^{1/d}/d) \) in which \( y_1 = y_2 \), whence

\[
\sum_{n\leq N} r_f(n)^2 = 2 \sum_{n\leq N} r_f(n) + O(N^{1/d} + E_f(cN^{1/d}))
\]

for an appropriate constant \( c > 0 \). The first part of Theorem 1.2 will therefore follow from (1.1), if we are able to show that \( E_f(B) = o(B^2) \). The second part is standard (see the deduction of Theorem 2 from Theorem 1 in [7], for example, which deals with the case \( f(x) = x^3 \).

Assuming that \( d \geq 3 \), we would like to show that there exists \( \delta > 0 \) such that

\[
E_f(B) = O_f(B^{2-\delta}),
\]

which clearly suffices for the first part of Theorem 1.2. It is in the special case \( f(x) = x^d \) that this quantity has received the most attention. Although there have been subsequent refinements by many authors, it follows from work of Hooley [8, 10] that one can take any \( \delta < 1/3 \) in (1.3) when \( f(x) = x^d \). For general polynomials \( f \in \mathbb{Z}[x] \) of degree \( d \geq 3 \), progress has not been so fluid. For \( d = 3 \), Wooley [14] has shown that any \( \delta < 1/3 \) is admissible in (1.3). For \( d \geq 7 \), previous work of the author [2, Thm. 1] shows that any \( \delta < 5/6 - 2/\sqrt{7} = 0.077\ldots \) is admissible. This was extended in joint work of the author with Heath-Brown [3, Cor. 3], where for \( d \geq 5 \) any \( \delta < 3/4 - \sqrt{5}/3 = 0.044\ldots \) is shown to be admissible in (1.3). It therefore remains to deal with the case \( d = 4 \).

**Theorem 1.3.** Let \( \varepsilon > 0 \) and let \( f \in \mathbb{Z}[x] \) be a non-zero quartic polynomial. Then we have \( E_f(B) \ll_{\varepsilon, f} B^{2-1/6+\varepsilon} \).

Our proof of Theorem 1.3 will follow the strategy of Hooley [8, 10] for the case \( f(x) = x^d \), except that we invoke Theorem 1.1 rather than the generalised Selberg sieve adopted by Hooley. While this doesn’t afford stronger results it does result in a more straightforward exposition. The lack of homogeneity that comes from treating general polynomials \( f(x) \) leads to several additional complications when estimating the emergent exponential sums. This ultimately leads to a weaker exponent in Theorem 1.3 compared with Hooley’s exponent \( 5/3 + \varepsilon \) when \( f(x) = x^4 \). However, in this special case, our argument can easily be modified to recover this exponent.
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2. Proof of Theorem 1.1

Our argument is a generalisation of the proof of [6, Thm. 1]. It will be convenient to write \( \nu_p = \nu_p(n) \) in what follows, for each \( p \in \mathcal{P} \). We begin by considering the expression

\[
\Sigma = \sum_{n \in \mathcal{A}} w(n) \left( \sum_{\substack{p \in \mathcal{P} \\ p \nmid g(n)h(n)}} \{ \alpha + (\nu_p - 1)(d - \nu_p) \} \right)^2.
\]

Each \( n \) is clearly counted with non-negative weight. Suppose now that \( n \in \mathcal{A} \) is such that \( f(x; n) \) is soluble and \( g(n)h(n) \neq 0 \). Then \( 1 \leq \nu_p \leq d \) for every \( p \in \mathcal{P} \) such that \( p \mid h(n) \). Hence it follows that

\[
\alpha + (\nu_p - 1)(d - \nu_p) \geq \alpha \geq 1
\]

in the summand, whence

\[
\sum_{\substack{p \in \mathcal{P} \\ p \nmid g(n)h(n)}} \{ \alpha + (\nu_p - 1)(d - \nu_p) \} \geq \sum_{\substack{p \in \mathcal{P} \\ p \nmid g(n)h(n)}} 1 \geq P - \sum_{p \mid g(n)h(n)} 1,
\]

if \( f(x; n) \) is soluble. But

\[
\sum_{p \mid N} 1 \ll \frac{\log N}{\log \log 3N},
\]

for any \( N \in \mathbb{Z}_{>0} \). It follows from our assumptions on the support of \( w \) that \( \Sigma \gg P^2 S(\mathcal{A}) \), with an implied constant that depends on the polynomials \( f \in \mathbb{Z}[x, y] \) and \( g \in \mathbb{Z}[y] \).

A companion estimate for \( \Sigma \) is achieved by expanding the square, giving the upper bound

\[
\sum_{p, q \in \mathcal{P}} \sum_{\substack{n \in \mathcal{A} \\ \gcd(pq, h(n)g(n)) = 1}} w(n) \{ \alpha + (\nu_p - 1)(d - \nu_p) \} \{ \alpha + (\nu_q - 1)(d - \nu_q) \}.
\]

Multiplying out the summand and then comparing this with our lower bound for \( \Sigma \), we easily arrive at the statement of Theorem 1.1.
3. Proof of Theorem 1.3 — preliminaries

Throughout the proof of Theorem 1.3 we will allow all implied constants to depend in any way upon $f$. Any further dependencies will be indicated explicitly by appropriate subscripts. Suppose that $f(x) = a_0 x^4 + \cdots + a_4$ for $a_0, \ldots, a_4 \in \mathbb{Z}$ and $a_0 > 0$. Note that

$$4^4 a_0^3 f(x) = (4a_0 x + a_1)^4 + b_2 (4a_0 x + a_1)^2 + b_3 (4a_0 x + a_1) + b_4,$$

for $b_2, b_3, b_4 \in \mathbb{Q}$ depending on $a_0, \ldots, a_4$. After a possible change of variables it therefore suffices to establish Theorem 1.3 for the monic polynomial

$$f(x) = x^4 + ax^2 + bx,$$

for given $a, b \in \mathbb{Z}$. Furthermore, we may henceforth assume that $(a, b) \neq (0, 0)$, since otherwise Theorem 1.3 is a consequence of work of Greaves [5], which shows that Theorem 1.3 holds with exponent $2 - \frac{1}{4} + \varepsilon$.

In any given point $y = (y_1, \ldots, y_4)$ counted by $E_f(B)$ we may assume without loss of generality that $\max_i y_i = y_1$ and $y_3 \geq y_4$. It follows that $y_1 > y_3 \geq y_4 > y_2 \geq 0$. Our starting point will be the factorisation properties of the equivalent equation

$$f(y_1) - f(y_3) = f(y_4) - f(y_2).$$

Through the substitutions

$$u_1 = y_1 - y_3, \quad v_1 = y_4 - y_2,$$
$$u_2 = y_1 + y_3, \quad v_2 = y_4 + y_2,$$

this equation transforms into

$$u_1 (u_2^3 + u_1^2 u_2 + 2au_2 + 2b) = v_1 (v_2^3 + v_1^2 v_2 + 2av_2 + 2b).$$

(3.1)

We observe that $u_1, u_2, v_1, v_2$ are positive integers of size at most $2B$. Moreover, $u_2 \neq v_2$ since otherwise we would have $y_4 + y_2 - y_3 = y_1 > y_3$, from which it would follow that $2y_3 > y_4 + y_2 > 2y_3$, which is impossible. We may further assume that $u_1 \neq v_1$, since the remaining contribution is $O(1)$. Indeed, if $u_1 = v_1$ then our equation becomes

$$u_2^2 + u_2 v_2 + v_2^2 + u_1^2 = -2a,$$

since $u_2 \neq v_2$. This has $O(1)$ solutions in positive integers $u_1, u_2, v_2$.

We will analyse the Diophantine equation (3.1) by drawing out common factors between $u_1$ and $v_1$. Given the extra symmetry inherent when $b = 0$, we will also need to draw out common factors between $u_2$ and $v_1$. Let us write

$$h_1 = \gcd(u_1, v_1), \quad h_2 = \gcd(u_2, v_1/h_1).$$
We then make the change of variables 

\[(r, s) = (u_1/h_1, u_2/h_2), \quad (\varrho, \sigma) = (v_1/(h_1h_2), v_2),\]

with \(\gcd(r, h_2\varrho) = \gcd(s, \varrho) = 1.\) Moreover, since \(u_i \neq v_i\) for \(i = 1, 2\) we may assume that \(r \neq h_2\varrho\) and \(h_2s \neq \sigma\) in any solution. These variables satisfy the new equation

\[r(h_2^3s^3 + h_1^2h_2r^2s + 2ah_2s + 2b) = h_2\varrho(\sigma^3 + h_1^2h_2^2\varrho^2\sigma + 2a\sigma + 2b).\]

In particular, \(h_2 | 2b\) since \(\gcd(r, h_2) = 1.\) Let us write \(2b = h_2c,\) for \(c \in \mathbb{Z}.\) Then we have

\[r(h_2^2s^3 + h_1^2r^2s + 2as + c) = \varrho(\sigma^3 + h_1^2h_2^2\varrho^2\sigma + 2a\sigma + h_2c). \quad (3.2)\]

Here \(h_1h_2 \leq 2B\) and \(r, s, \varrho, \sigma\) are positive integers satisfying

\[\gcd(r, h_2\varrho) = \gcd(s, \varrho) = 1 \quad \text{and} \quad (r - h_2\varrho)(h_2s - \sigma) \neq 0,\]

together with the inequalities

\[r \leq \frac{2B}{h_1}, \quad s \leq \frac{2B}{h_2}, \quad \varrho \leq \frac{2B}{h_1h_2}, \quad \sigma \leq 2B.\]

Define the number

\[A = h_1^2h_2^2\varrho^2 + 2a. \quad (3.3)\]

When \(|A|\) is small or \(\max\{h_1, h_2\}\) is large, we will use work of Bombieri and Pila \(1\) to estimate the corresponding contribution. In the alternative case, we will ultimately apply Theorem \(\text{[13]}\) Let \(C \geq 1\) and let \(1 \leq H \leq 2B.\) Let \(N_1(B, C; H)\) (resp. \(N_2(B, C; H)\)) denote the total contribution to \(E_f(B)\) from solutions with \(|A| > C\) and \(\max\{h_1, h_2\} \leq H\) (resp. \(|A| \leq C\) or \(\max\{h_1, h_2\} > H\)). Then our work so far implies that

\[E_f(B) \leq N_1(B, C; H) + N_2(B, C; H) + O(1).\]

The treatment of the second term is relatively straightforward.

**Lemma 3.1.** Let \(\varepsilon > 0.\) Then

\[N_2(B, C; H) \ll \varepsilon C^{1+\varepsilon}B^{4/3+\varepsilon} + H^{-1}B^{7/3+\varepsilon}.\]

**Proof.** One way to estimate the number of solutions to \((3.2)\) is to first fix some of the variables, viewing the resulting equation as something of smaller dimension. Let \(C_{h, \varrho, r} \subset A_Q^2\) denote the affine cubic curve which arises when \(h = (h_1, h_2)\) and \(\varrho, r\) are fixed. Let us put \(A' = h_1^2r^2 + 2a,\) for ease of notation. Then we claim that \(C_{h, \varrho, r}\) is absolutely irreducible unless

\[c = 0 \quad \text{and} \quad h_2\varrho^2A^3 = r^2A'^3, \quad (3.4)\]

with \(A \neq 0.\) To prove this we suppose that \(C_{h, \varrho, r}\) is not absolutely irreducible. Then it must contain a line defined over \(\mathbb{Q}.\) We may assume
that this line is given parametrically by \((s, \sigma) = (t, \alpha t + \beta)\) for \(\alpha, \beta \in \mathbb{Q}\). Making this substitution into (3.2) and equating coefficients of \(t\), we deduce that

\[
\beta = c = 0 \quad \text{and} \quad rh_2^2 = q\alpha^3 \quad \text{and} \quad rA' = qA\alpha,
\]

with \(A \neq 0\), since \(h_1h_2\rho(r \pm h_2\rho) \neq 0\). Eliminating \(\alpha\) easily leads to the claim.

Suppose that \(h, \rho, r\) do not satisfy (3.4). It then follows from a result of Bombieri and Pila \([1]\) that

\[
\#\{s, \sigma \leq 2B : (s, \sigma) \in C_{h, \rho, r}(\mathbb{Z})\} = O(\varepsilon(B^{1/3+\varepsilon}), \quad (3.5)
\]

for any \(\varepsilon > 0\). The implied constant is independent of \(h, \rho, r\) and depends only on \(\varepsilon\). Alternatively, if \(h, \rho, r\) do satisfy (3.4) then we have the trivial bound \(O(B/h_2)\) for the number of points in \(C_{h, \rho, r}(\mathbb{Z})\), which arises from noting that there are at most 3 choices of \(\sigma\) associated to a given choice of \(s\).

We may now handle the contribution from \(|A| \leq C\), in which case \(h_1h_2\rho \ll C\). There are \(O(\varepsilon(C^{1+\varepsilon})\) choices for \(h_1, h_2, \rho\) satisfying this bound, by the trivial estimate for the divisor function. When \(h, \rho, r\) do not satisfy (3.4) we will apply (3.5). This case therefore gives an overall contribution \(O(\varepsilon(C^{1+\varepsilon}B^{4/3+\varepsilon}))\). Alternatively, when \(h, \rho, r\) do satisfy (3.4) there are at most 8 choices for \(r\) when \(h_1, h_2, \rho\) are fixed. This case therefore makes the smaller overall contribution \(O(\varepsilon(C^{1+\varepsilon}B))\).

Next, let us consider the contribution from \(h_1 > H\). We fix a choice of \(h, r\) and \(\rho\) in (3.2). When (3.4) fails we may apply (3.5). This leads to the contribution

\[
\ll \varepsilon B^{1/3+\varepsilon} \sum_{h_1 > H} \sum_{h_2 \leq 2B} \frac{B^2}{h_1^2h_2} \ll \varepsilon H^{-1}B^{7/3+\varepsilon} \log B.
\]

Taking \(\log B = O(B^{\varepsilon})\) and redefining the choice of \(\varepsilon > 0\), this is satisfactory for the lemma. Alternatively, when (3.4) is satisfied we apply the bound \(O(B/h_2)\) for the number of \(s, \sigma\). But then \(\rho, r\) are restricted by the equation \(h_2^2 \rho^2 A^3 = r^2 A^3\), which once reduced modulo \(h_1^2\) implies that

\[
8a^3(h_2^2 \rho^2 - r^2) \equiv 0 \pmod{h_1^2}.
\]

We must have \(a \neq 0\) since \(c = 0\) and we are assuming that \((a,b) \neq (0,0)\). Let \(q = h_1^2 / \gcd(h_1^2, 8a^3)\). Then this congruence becomes \(h_2^2 \rho^2 \equiv r^2 \pmod{q}\). Write \(q' = q / \gcd(q, 2)\). Since \(\gcd(r, h_2\rho) = 1\) we deduce that \(r \equiv h_2\rho \pmod{q'}\) or \(r \equiv -h_2\rho \pmod{q'}\). In particular we must have \(q' \ll B/h_1\), since \(0 \neq r \pm h_2\rho \ll B/h_1\). In either case, given \(h, \rho\)
we see that the number of $r$ that can possibly contribute is
\[ \ll \frac{B}{h_1 q'} \ll \frac{B}{h_1^3}, \]
and to each of these is associated at most 8 choices for $q$. This case therefore leads to the overall contribution
\[ \ll B \sum_{h_1 > H} \sum_{h_2 \leq 2B} \frac{B^2}{h_1^2 h_2} \ll H^{-2} B^2 \log B, \]
which is satisfactory.

It remains to consider the contribution from $h_2 > H$. This is handled in a completely analogous fashion, by first fixing a choice of $h, s$ and $\varrho$ and considering the affine cubic curve $D_{h, \varrho, s} \subset \mathbb{A}_3^2$. In this case, on writing $A'' = h_2^2 s^2 + 2a$, one finds that $D_{h, \varrho, s}$ is absolutely irreducible unless
\[ c = 0 \quad \text{and} \quad h_1^2 \varrho^2 A^3 = s^2 A'^3, \]
with $A \neq 0$. When $D_{h, \varrho, s}$ is absolutely irreducible one applies the analogue of (3.5). When it fails to be absolutely irreducible one applies the trivial bound $O(B/h_1)$ for the number of points in $D_{h, \varrho, s}(\mathbb{Z})$ The remainder of the argument runs just as before. This concludes the proof of the lemma.

The estimation of $N_1(B, C; H)$ is much more awkward. The remainder of this paper is dedicated to proving the following result.

**Lemma 3.2.** Let $\varepsilon > 0$ and assume that $C \gg 1$. Then we have
\[ N_1(B, C; H) \ll \varepsilon H^{1/2} B^{3/2+\varepsilon} + B^{2-1/6+\varepsilon}. \]

Once this result is combined with Lemma 3.1 we see that the choices $C \ll 1$ and $H = B^{1/2}$ are sufficient to establish Theorem 1.3.

We now begin the proof of Lemma 3.2. Our plan will be to fix choices of $h_1, h_2$ and $\varrho$, and then to count the number of $r, s, \sigma$ that contribute to $N_1(B, C; H)$. Define the cubic polynomials
\begin{align*}
F(u, v) &= h_2^2 v^3 + (h_1^2 u^2 + 2a)v + c, \\
G(u, v) &= v^3 + (h_1^2 h_2^2 u^2 + 2a)v + h_2 c,
\end{align*}
where we recall that $2b = h_2 c$. Then (3.2) can be written
\[ rF(r, s) = \varrho G(\varrho, \sigma). \]

Recall the definition (3.3) of $A$. We are proceeding under the assumption that $|A| > C \geq 1$. In particular $A \neq 0$. Part of our work will
lead us to consider the homogeneous quartic polynomial
\[
K(Z, X, Y, W) = W^4 h_1 h_2 g G(\rho, Z/W) - 2\{W^4 f(X/W) - W^4 f(Y/W)\},
\]
where \(G\) is given by (3.7). The condition on \(C\) in Lemma 3.2 comes from the following result.

**Lemma 3.3.** Assume that \(C \gg 1\). Then \(K\) is non-singular.

**Proof.** We recall that \(h_1 h_2 \rho \neq 0\) and \(|A| > C\). Returning to (3.9) we see, by taking partial derivatives, that any singular point on the projective surface \(K = 0\) must satisfy
\[
W(3Z^2 + AW^2) = 0
\]
and
\[
4X^3 + aXW^2 + bW^3 = 0, \quad 4Y^3 + 2aYW^2 + bW^3 = 0,
\]
in addition to \(\partial K/\partial W = 0\). A short calculation shows that the latter constraint is equivalent to the equation
\[
h_1 h_2 \rho F(Z, W) = 2\{2a(X^2 - Y^2)W + 3b(X - Y)W^2\},
\]
where \(F(Z, W) = Z^3 + 3AZW^2 + 4h_2 cW^3\). There can be no singular points with \(W = 0\). Hence it follows that there are at most 18 singular points on \(K = 0\), and these all take the shape \([\xi, \eta, \eta', 1]\), where
\[
\xi = \pm \sqrt{-A/3}
\]
and \(\eta, \eta'\) are roots of the cubic equation \(4t^3 + 2at + b = 0\). In particular, it follows that \(h_1 h_2 \rho F(\xi, 1) \ll 1\), which is impossible provided that \(C\) is taken to be sufficiently large in our lower bound \(|A| > C\). Hence there are no singular points, which thereby establishes the lemma. \(\Box\)

We proceed to indicate how the polynomial sieve will be brought to bear on the proof of Lemma 3.2. The structure of our argument is modelled on that of Hooley [10], corresponding to the special case \(f(x) = x^4\). We shall assume that \(C \gg 1\) for the remainder of the proof, so that Lemma 3.3 applies and \(K\) is non-singular. Since \(\gcd(r, \rho) = 1\), it follows from (3.8) that \(\rho \mid F(r, s)\) in any solution to be counted. We therefore have
\[
N_1(B, C; H) \leq \sum_{h_1, h_2 \leq H} \sum_{\rho \leq 2B/(h_1 h_2)} N_1(B; H; \rho, \rho),
\]
(3.10)
where $A$ is given by (3.3) and $N_1(B; H; h, \varrho)$ is equal to
\[
\sum_{\substack{r \leq 2B/h_1, \ s \leq 2B/h_2 \\ \gcd(rs, \varrho) = 1 \\ F(r, s) \equiv 0 \pmod{\varrho}}} 1, \quad \text{if } \exists \sigma \in \mathbb{Z} \text{ s.t. } \varrho G(\varrho, \sigma) = rF(r, s), \\
0, \quad \text{otherwise}.
\]

This is now in a form suitable for an application of Theorem 1.1.

To be precise, we take $A = \{(r, s) \in \mathbb{Z}^2 : r, s \leq 2B/h_1 \times 2B/h_2, \gcd(rs, \varrho) = 1, F(r, s) \equiv 0 \pmod{\varrho}\}$ and $w$ to be the indicator function for this set. We take $f(x; r, s) = \varrho G(\varrho, x) - rF(r, s)$ and $g(r, s) = 1$. Recalling (3.7) we have $d = 3$ and $h(r, s) \mid \varrho$ in Theorem 1.1. In particular $f(x; r, s)$ never vanishes identically, for any $(r, s) \in \mathcal{A}$. Let $\alpha \geq 1$ and $Q \geq 1$ be parameters. Let $D_K$ be the discriminant of the quartic form $K$ in (3.9). Then $D_K$ is a non-zero integer since $K$ is non-singular. We let
\[
\mathcal{P} = \{\text{primes } p \leq Q : \ p \nmid 6h_1h_2\varrho AD_K\}.
\]

In particular $K$ remains non-singular modulo any prime $p \in \mathcal{P}$. For any $p \in \mathcal{P}$ and $(r, s) \in \mathcal{A}$, we put
\[
\nu_p(r, s) = \#\{x \pmod{p} : \varrho G(\varrho, x) \equiv rF(r, s) \pmod{p}\}.
\]

We will always assume that $Q$ satisfies $B^{1/100} \leq Q \leq B$. In particular
\[
\#\mathcal{P} = \pi(Q) - \#\{p \leq Q : p \mid 6h_1h_2\varrho AD_K\} \sim \frac{Q}{\log Q},
\]

by the prime number theorem. It now follows from Theorem 1.1 that
\[
N_1(B; H; h, \varrho) \ll \frac{\log^2 Q}{Q^2} \sum_{p, q \in \mathcal{P}} \left| \sum_{i, j \in \{0, 1, 2\}} c_{i, j}(\alpha) S_{i, j} \right|,
\]

with
\[
S_{i, j} = \sum_{(r, s) \in \mathcal{A}} \nu_p(r, s)^i \nu_q(r, s)^j.
\]
and
\begin{equation}
    c_{i,j}(\alpha) = \begin{cases} 
    (\alpha - 3)^2, & \text{if } (i, j) = (0, 0), \\
    4(\alpha - 3), & \text{if } (i, j) = (1, 0) \text{ or } (0, 1), \\
    16, & \text{if } (i, j) = (1, 1), \\
    3 - \alpha, & \text{if } (i, j) = (2, 0) \text{ or } (0, 2), \\
    -4, & \text{if } (i, j) = (2, 1) \text{ or } (1, 2), \\
    1, & \text{if } (i, j) = (2, 2).
    \end{cases}
\end{equation}
(3.14)

We will ultimately be led to take \( \alpha = 1 \) in §6.

To analyse \( S_{i,j} \) we will break the sum into congruence classes modulo \( pq \). Let \( Y \geq 1 \) and let \( N \in \mathbb{Z} \) with \( |N| \leq pq/2 \). Then we have
\begin{equation}
    \Gamma(Y, N) = \sum_{y \leq Y} e_{pq}(Ny) \ll \min\left\{ Y, \frac{pq}{|N|} \right\}.
\end{equation}
(3.15)

Given \( r \in \mathbb{Z} \) the orthogonality of characters yields
\begin{align*}
    \#\{ x \leq 2B/h_1 : x \equiv r \pmod{pq} \} &= \frac{1}{pq} \sum_{m (\text{mod } pq)} e_{pq}(mr) \sum_{x \leq 2B/h_1} e_{pq}(-mx) \\
    &= \frac{1}{pq} \sum_{-pq/2 < m \leq pq/2} e_{pq}(mr) \Gamma \left( \frac{2B}{h_1}, m \right),
\end{align*}
and similarly for \( \#\{ y \leq 2B/h_2 : y \equiv s \pmod{pq} \} \). Hence
\begin{equation}
    S_{i,j} = \frac{1}{(pq)^2} \sum_{-pq/2 < m, n \leq pq/2} \Gamma \left( \frac{2B}{h_1}, m \right) \Gamma \left( \frac{2B}{h_2}, n \right) \Psi_{i,j}(m, n), \tag{3.16}
\end{equation}
with
\begin{equation*}
    \Psi_{i,j}(m, n) = \sum_{(r,s) (\text{mod } pq) \gcd(q,rs)=1} \nu_p(r, s)^i \nu_q(r, s)^j e_{pq}(mr + ns).
\end{equation*}

It therefore remains to understand the exponential sums \( \Psi_{i,j}(m, n) \). For typical values of \( m, n \) we want to show that there is enough cancellation in the sum to make its modulus rather small. Recall from the definition (3.11) of \( \mathcal{P} \) that \( \gcd(pq, \varrho) = 1 \). Using this, we are able to establish the following factorisation property.

**Lemma 3.4.** Suppose that \( p \neq q \) and choose \( p', q', \varrho, \overline{\varrho} \in \mathbb{Z} \) such that \( p\varrho + \overline{\varrho} = 1 \) and \( pp' + qq' = 1 \). Then we have
\begin{equation*}
    \Psi_{i,j}(m, n) = \sum_i(p; \overline{\varrho} m, \overline{\varrho} q' n) \sum_j(q; \overline{\varrho} p' m, \overline{\varrho} p n) \Phi(\varrho; \overline{\varrho} m, \overline{\varrho} q n),
\end{equation*}
where
\[
\begin{align*}
\Sigma_t(p; M, N) &= \sum_{(r,s) \pmod{p}} \nu_p(r, s)^t e_p(Mr + Ns), \\
\Phi(g; M, N) &= \sum_{(r,s) \pmod{g}} e_g(Mr + Ns).
\end{align*}
\]

Suppose that \( p = q \) and choose \( \overline{p}, \overline{q} \in \mathbb{Z} \) such that \( \overline{p}q + \overline{q}p = 1 \). Then we have
\[
\Psi_{i,j}(m, n) = \begin{cases} 
p^2 \Sigma_{i+j}(p; \overline{m'}, \overline{n'}) \Phi(g; \overline{m'}, \overline{n'}) & \text{if } (m, n) = (p(m', n'), \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. The proof of this result is standard. The first part is obtained by making the substitution
\[
r = (r_0qq' + r_1pp')\overline{p}q + r_2pqq', \quad s = (s_0qq' + s_1pp')\overline{p}q + s_2pqq,
\]
for \( r_0, s_0 \pmod{p}, r_1, s_1 \pmod{q} \) and \( r_2, s_2 \pmod{q} \), with \( r_2s_2 \) coprime to \( q \). For the second part we make the substitution
\[
r = r_1\overline{p}q + r_2(p\overline{p})^2, \quad s = s_1\overline{p}q + s_2(p\overline{p})^2,
\]
for \( r_1, s_1 \pmod{p^2} \) and \( r_2, s_2 \pmod{q} \), with \( r_2s_2 \) coprime to \( q \). This leads to the expression
\[
\Psi_{i,j}(m, n) = \Phi(g; \overline{m}^2, \overline{n}^2) \sum_{(r_1, s_1) \pmod{p^2}} \nu_p(r_1, s_1)^{i+j} e_p(\{mr_1 + ns_1\}),
\]
in the notation of the lemma. Writing \( r_1 = r_1' + pr_1'' \) for \( r_1', r_1'' \pmod{p} \), and similarly for \( s_1, \) the second factor becomes
\[
\sum_{(r_1', s_1') \pmod{p}} \nu_p(r_1', s_1')^{i+j} e_p(\{mr_1' + ns_1'\}) \sum_{(r_1'', s_1'') \pmod{p}} e_p(\{mr_1'' + ns_1''\}).
\]
But the inner sum is zero unless \( p \mid \gcd(m, n) \), in which case it is \( p^2 \). This completes the proof of the lemma. \( \square \)

We have reduced our task to a detailed analysis of the exponential sums \( \Sigma_t(p; M, N) \) and \( \Phi(g; M, N) \), for \( 0 \leq t \leq 4 \) and given \( M, N \in \mathbb{Z} \). This will be the object of the following two sections. The trivial bound for \( \Sigma_t(p; M, N) \) is \( O(p^2) \). Likewise, in the special case that \( g \) is a prime, the trivial bound for \( \Phi(g; M, N) \) is \( O(\sqrt{q}) \). In our work we will show that for generic choices of \( M, N \) these sums actually satisfy square-root cancellation. We will do so using the Weil bound for curves and the Deligne bound for surfaces, combined with an elementary treatment of \( \Phi(g; M, N) \) when \( q \) is a higher prime power. We prepare the ground
by framing some basic tools that will be common to both. Given any sum over residue classes, we will use $\sum^*$ to mean a sum in which all the variables of summation are coprime to the modulus.

Our primary means of estimating the exponential sums for prime modulus will be the “method of moments” developed by Hooley [9], as summarised in the following result.

**Lemma 3.5.** Let $F$ and $G_1, \ldots, G_k$ be polynomials over $\mathbb{Z}$, of degree at most $d$, and let

$$S = \sum_{x \in \mathbb{F}_p^n} e_p(F(x)),$$

for any prime $p$. For each $j \geq 1$ and $\tau \in \mathbb{F}_p^n$, write

$$N_j(\tau) = \# \{ x \in \mathbb{F}_p^n : G_1(x) = \cdots = G_k(x) = 0, F(x) = \tau \},$$

and suppose that there exists $N_j \in \mathbb{R}$ such that

$$\sum_{\tau \in \mathbb{F}_p^n} |N_j(\tau) - N_j|^2 \ll_{d,k,n} p^{\kappa j},$$

(3.19)

where $\kappa \in \mathbb{Z}$ is independent of $j$. Then $S \ll_{d,k,n} p^{\kappa/2}$.

Let $q$ be a prime power. We shall need to be able to count $\mathbb{F}_q$-points on certain varieties. In dimension 2 we will call upon the work of Deligne [4] and in dimension 1 we will use work of Weil [13]. The facts that we need are summarised in the following two results.

**Lemma 3.6.** Let $W \subset \mathbb{P}^n_{\mathbb{F}_q}$ be a non-singular complete intersection of dimension 2 and degree $d$. Then

$$\# \{ x \in \mathbb{F}_q^n : [x] \in W \} = q^3 + O_d(n(q^2)).$$

**Lemma 3.7.** Let $V \subset \mathbb{A}^n_{\mathbb{F}_q}$ be an absolutely irreducible curve of degree $d$. Then

$$\# V(\mathbb{F}_q) = q + O_d(n(q^{1/2})).$$

4. The exponential sums $\Sigma_t(\scriptstyle{p; M, N})$

In this section we examine the exponential sum $\Sigma_t = \Sigma_t(\scriptstyle{p; M, N})$ in (3.17) for a prime $p$ that $6h_1h_2gA\Delta K$, where $A$ is given by (3.3) and $D_K$ is the non-zero discriminant of the quartic form (3.9). For $i = 1, 2$, we let $\overline{h}_i \in \mathbb{Z}$ be such that $h_i\overline{h}_i \equiv 1 \pmod{p}$. Recall the definitions (3.6), (3.7) of the cubic polynomials $F$ and $G$. Reversing the changes of variables leading to these, one easily checks that

$$h_1h_2uF(u,v) = 2 \left\{ f \left( \frac{h_1u + h_2v}{2} \right) - f \left( \frac{-h_1u + h_2v}{2} \right) \right\},$$

(4.1)
where \( f(x) = x^4 + ax^2 + bx \). We will argue according to the value of \( t \).

When \( t = 0 \) we trivially have

\[
\Sigma_0 = \begin{cases} 
p^2, & \text{if } p \mid \gcd(M, N), \
0, & \text{otherwise.} 
\end{cases}
\]

(4.2)

Next, when \( t = 1 \), we open up the function \( \nu(p, r, s) \) given by (3.12) to conclude that

\[
\Sigma_1 = \sum_{(r, s, x) \in \mathbb{F}_p^3} e(p(M + Ns)).
\]

We will show that

\[
\Sigma_1 = O(p \gcd(p, M, N)).
\]

(4.3)

On carrying out the non-singular change of variables implicit in (4.1), we obtain

\[
\Sigma_1 = S_p(Mh_1 + Nh_2, -Mh_1 + Nh_2),
\]

(4.4)

where for \( c = (c_1, c_2) \in \mathbb{Z}^2 \) we set

\[
S_p(c) = \sum_{(x, y) \in \mathbb{F}_p^3} e_p(c_1 x + c_2 y).
\]

Inserting the second part of the following result into (4.4) establishes (4.3).

**Lemma 4.1.** We have

\[
S_p(0, 0) = p^2 + O(p) \quad \text{and} \quad S_p(c) = O(p \gcd(p, c_1, c_2)).
\]

**Proof.** We begin by establishing the first part of the lemma. We convert the problem into one involving projective varieties via the identity

\[
S_p(0, 0) = \frac{1}{p - 1} \# \{(z, x, y, w) \in \mathbb{F}_p^4 : K(z, x, y, w) = 0, \ w \neq 0\},
\]

where \( K \) is given by (3.9). Combining Lemmas 3.6 and 3.3 we see that

\[
\# \{(z, x, y, w) \in \mathbb{F}_p^4 : K(z, x, y, w) = 0\} = p^3 + O(p^2),
\]

since \( K \) is non-singular over \( \mathbb{F}_p \). But \( K(Z, X, Y, 0) = Y^4 - X^4 \). Hence

\[
\# \{(z, x, y) \in \mathbb{F}_p^3 : K(z, x, y, 0) = 0\} = O(p^2).
\]

Putting these two estimates together gives \( S_p(0, 0) = p^2 + O(p) \).

Turning to \( S_p(c) \) for general \( c \in \mathbb{Z}^2 \), we may assume that \( p \gg 1 \) and \( p \nmid (c_1, c_2) \), else the bound follows from the first part of the lemma. Let \( j \gg 1 \) and put \( q = p^j \). Since \( K \) is non-singular over \( \mathbb{F}_q \), it follows
that $K(z, x, y, 1)$ must be absolutely irreducible over $\mathbb{F}_q$. We apply Lemma 3.5 with $k = 1$ and $n = 3$. Then, for $\tau \in \mathbb{F}_q$, we must consider
\[
N_j(\tau) = \# \{(x, y, z) \in \mathbb{F}_q^3 : K(z, x, y, 1) = 0, \ c_1 x + c_2 y = \tau\}.
\]
We reduce $c_1, c_2$ modulo $p$ and view them as elements of $\mathbb{F}_q$. Without loss of generality we assume that $c_1 \neq 0$, eliminating $x$ to get
\[
N_j(\tau) = \# \{(y, z) \in \mathbb{F}_q^2 : K(z, -c_1^{-1} c_2 y + c_1^{-1} \tau, y, 1) = 0\}.
\]
It follows from Hilbert’s irreducibility theorem that there are $O(1)$ values of $\tau \in \mathbb{F}_q$ for which $K(z, -c_1^{-1} c_2 y + c_1^{-1} \tau, y, 1)$ fails to be absolutely irreducible. For these we employ the trivial bound $N_j(\tau) = O(q)$. For the remaining values of $\tau$, Lemma 3.7 yields $N_j(\tau) = q + O(q^{1/2})$, uniformly in $\tau$. Taking $N_j = q$ in Lemma 3.5 therefore permits the choice $\kappa = 2$ in (3.19), which completes the proof of the lemma.

We now turn to the case $t = 2$. Define the quadratic polynomial
\[
H(Z_1, Z_2, W) = Z_1^2 + Z_1 Z_2 + Z_2^2 + AW^2,
\]
where $A$ is given by (3.3). This quadratic form is non-singular modulo $p$, since $p \nmid A$ for any $p \in \mathcal{P}$. Next, observe that
\[
\rho G(\rho, x_1) - \rho G(\rho, x_2) = \rho(x_1 - x_2) H(x_1, x_2, 1).
\]
Opening up $\nu_p(r, s)^2$ in $\Sigma_2$ gives
\[
\Sigma_2 = \Sigma_1 + \sum_{(r, s, x_1, x_2) \in \mathbb{F}_p^4} e_p(Mr + Ns),
\]
where $\Sigma_1$ is the contribution from $x_1 - x_2 = 0$. We will show that
\[
\Sigma_2 = O\left( p \gcd(p, M, N) \right).
\]
We may remove the condition $x_1 - x_2 \neq 0$ in the second sum of (4.6) at the expense of an additional error term $O(p)$. Hence, on making the change of variables implicit in (4.1), we obtain
\[
\Sigma_2 = \Sigma_1 + T_p(M\overline{h}_1 + N\overline{h}_2, -M\overline{h}_1 + N\overline{h}_2) + O(p),
\]
where for $c = (c_1, c_2) \in \mathbb{Z}^2$ we set
\[
T_p(c) = \sum_{(x, y, z_1, z_2) \in \mathbb{F}_p^4} e_p(c_1 x + c_2 y),
\]
where $K$ is given by (3.9). The estimate (4.7) will follow on combining Lemma 4.1 with the second part of the following result in (4.8).
Lemma 4.2. We have

\[ T_p(0, 0) = p^2 + O(p) \quad \text{and} \quad T_p(c) = O(p \gcd(p, c_1, c_2)). \]

Proof. The proof of this result is similar to our argument in Lemma 4.1.

Let \( p = p^j \) for \( j \geq 1 \). We may clearly assume that \( p \gg 1 \), since the estimates are trivial otherwise. We begin with the first estimate, converting the problem into one involving projective varieties by noting that \( T_p(0, 0) \) is equal to

\[ \frac{1}{p-1} \# \left\{ (x, y, z_1, z_2, w) \in \mathbb{F}_p^5 : \begin{array}{l}
H(z_1, z_2, w) = K(z_1, x, y, w) = 0 \\
w \neq 0
\end{array} \right\}, \]

The desired conclusion will follow from Lemma 3.6 provided we can show that \( H = K = 0 \) defines a non-singular surface in \( \mathbb{P}^4_{\mathbb{F}_q} \).

Delaying this for the moment, we move to an analysis of \( T_p(c) \) for general \( c \in \mathbb{Z}^2 \), with \( p \nmid (c_1, c_2) \). Still under the assumption that the projective variety \( H = K = 0 \) is non-singular, it follows that \( H(z_1, z_2, 1) = K(z_1, x, y, 1) = 0 \) defines an absolutely irreducible affine variety over \( \mathbb{F}_q \). We apply Lemma 3.5 with \( k = 2 \) and \( n = 4 \). On assuming without loss of generality that \( c_1 \neq 0 \), we must consider

\[ N_j(\tau) = \# \left\{ (y, z_1, z_2) \in \mathbb{F}_q^3 : \begin{array}{l}
H(z_1, z_2, 1) = 0 \\
K(z_1, -c_1^{-1} c_2 y + c_1^{-1} \tau, y, 1) = 0
\end{array} \right\}, \]

for \( \tau \in \mathbb{F}_q \). By Hilbert’s irreducibility theorem there are \( O(1) \) values of \( \tau \) for which the equations in \( N_j(\tau) \) fail to define an absolutely irreducible curve in \( \mathbb{A}^3_{\mathbb{F}_q} \). For these we take \( N_j(\tau) = O(q) \). For the remaining \( \tau \), Lemma 3.7 yields \( N_j(\tau) = q + O(q^{1/2}) \), uniformly in \( \tau \). Taking \( N_j = q \) in Lemma 3.5 therefore permits the choice \( \kappa = 2 \) in (3.19), which leads to the claimed bound for \( T_p(c) \).

It remains to show that the equations \( H = K = 0 \) produce a non-singular variety in \( \mathbb{P}^4_{\mathbb{F}_q} \). For this we consider the existence of a non-zero point \((Z_1, Z_2, X, Y, W)\) such that

\[ H = K = 0, \quad \lambda \nabla H = \mu \nabla K, \]

with \( (\lambda, \nu) \neq (0, 0) \). We have already remarked that \( H \) and \( K \) are non-singular over \( \mathbb{F}_q \). Hence we must have \( \lambda \mu \neq 0 \) in any such solution. Moreover, \( W \neq 0 \) in any such solution, since for \( W = 0 \) the equation for \( K \) implies that \( X = Y = 0 \) and the remaining constraints force \( Z_1 = Z_2 = 0 \). Next we observe that \( \partial H / \partial Z_2 = 0 \), in any solution. Likewise, on replacing \( K(Z_1, X, Y, W) \) by \( K(Z_2, X, Y, W) \), we may adjoin to this the equation \( \partial H / \partial W = 0 \). Finally, since \( H = 0 \) and \( W \neq 0 \), an application of Euler’s identity implies that \( \partial H / \partial W = 0 \), which is...
impossible since $H$ is non-singular. Hence there are no singular points, as claimed.

Suppose now that $t \geq 3$ and write $\mathbf{x} = (x_1, \ldots, x_t)$. Opening up $\nu_p(r, s)^t$ in $\Sigma_t$ gives

$$\Sigma_t = \sum_{(r, s, x) \in F_p^{t+2}} e_p(Mr + Ns),$$

via (4.5). Let $\sigma(\mathbf{x})$ denote the number of distinct elements in the set $\{x_1, \ldots, x_t\}$. Clearly $1 \leq \sigma(\mathbf{x}) \leq t$. The contribution from those $(r, s, \mathbf{x})$ for which $\sigma(\mathbf{x}) = 1$ is $\Sigma_1$. This event can only arise in one way. The contribution from those $(r, s, \mathbf{x})$ for which $\sigma(\mathbf{x}) = 2$ is $\Sigma_2 - \Sigma_1$, by (4.6). There are $c_t$ ways in which this can arise, for an appropriate constant $c_t$ depending on $t$. Next, consider the contribution from $(r, s, \mathbf{x})$ for which $\sigma(\mathbf{x}) = 3$. Suppose, for example, that $\mathbf{x} = (x, y, z, x, \ldots, x)$ with $(x - y)(x - z)(y - z) \neq 0$. In this case $x, y, z$ will satisfy

$$0 = H(x, y, 1) = H(x, z, 1),$$

whence in fact $x + y + z = 0$. In view of (4.6), the contribution from this case is therefore found to be

$$\sum_{(r, s, x, y) \in F_p^4} e_p(Mr + Ns) = \Sigma_2 - \Sigma_1 + O(p).$$

This situation arises in $d_t$ ways, say. Finally, our argument shows that there can be no contribution from $(r, s, \mathbf{x})$ for which $\sigma(\mathbf{x}) \geq 4$. It follows that

$$\Sigma_t = \{1 - c_t - d_t\} \Sigma_1 + \{c_t + d_t\} \Sigma_2 + O_t(p), \quad (4.9)$$

for $t \geq 3$. We are now ready to record the following result, which summarises our investigation in this section.

**Lemma 4.3.** We have $\Sigma_t(p; M, N) = O_t(p \gcd(p, M, N))$ for $t \geq 0$, and

$$\Sigma_t(p; 0, 0) = \max\{1, t\} p^2 + O(p)$$

for $0 \leq t \leq 2$.

**Proof.** The first part follows from (4.2), (4.3), (4.7) and (4.9). Turning to the second part, with $M = N = 0$, the case $t = 0$ follows directly from (4.2) and the case $t = 1$ follows from (4.4) and Lemma 4.1. Finally, the case $t = 2$ follows from (4.8) and Lemmas 4.1 and 4.2. \qed
5. The exponential sum \( \Phi(\rho; M, N) \)

Recall the definition (3.18) of the exponential sum \( \Phi(\rho; M, N) \) for \( \rho \in \mathbb{Z}_{>0} \) and \( M, N \in \mathbb{Z} \). It will be convenient to define

\[
\Delta(M, N) = h_2^2 M^2 + h_1^2 N^2.
\] (5.1)

Suppose that \( \rho_1, \rho_2 \) are coprime integers and let \( \overline{\rho}_1, \overline{\rho}_2 \in \mathbb{Z} \) be such that \( \rho_1 \overline{\rho}_1 + \rho_2 \overline{\rho}_2 = 1 \). Then arguing as in the proof of Lemma 3.4 it is easy to see that

\[
\Phi(\rho_1 \rho_2; M, N) = \Phi(\rho_1; \overline{\rho}_2 M, \overline{\rho}_2 N) \Phi(\rho_2; \overline{\rho}_1 M, \overline{\rho}_1 N). \] (5.2)

This renders it sufficient to study

\[
\Phi(p^k) = \Phi(p^k; M, N) = \sum_{(r,s) \text{ (mod } p^k)} \star e_{p^k}(Mr + Ns)
\]

for a given prime power \( p^k \), where \( F \) is given by (3.6).

We begin by examining the case \( k = 1 \). Suppose that \( p > 2 \) and that \( \ell, m \in \mathbb{Z} \), with \( p \nmid \ell \). We will use the familiar formula

\[
\sum_{x \text{ (mod } p)} e_p(\ell x^2 + mx) = \varepsilon_p \sqrt{p} \left( \frac{\ell}{p} \right) e_p(-4\ell m^2),
\] (5.3)

for the Gauss sum, where \( \varepsilon_p = 1 \) (resp. \( \varepsilon_p = i \)) if \( p \equiv 1 \pmod{4} \) (resp. if \( p \equiv 3 \pmod{4} \)). We may now establish the following result.

**Lemma 5.1.** We have \( \Phi(p) \ll p^{1/2} \gcd(p, \Delta(M, N))^{1/2} \).

**Proof.** Recall that \((a, b) \neq (0, 0)\). In view of the bound \( |\Phi(p)| \leq p^2 \), we may assume that \( p \nmid 2 \gcd(a, c) \). Next we observe that

\[
|\Phi(p)| \leq \sum_{r \text{ (mod } p)} \#\{s \text{ (mod } p) : h_2^2 s^3 + (h_1^2 r^2 + 2a)s + c \equiv 0 \pmod{p} \} \leq 3p,
\]

for \( p \nmid 2 \gcd(a, c) \). When \( p \nmid h_2 \) this follows since there are at most 3 solutions of the congruence \( s^3 + As + B \equiv 0 \pmod{p} \), for given \( A, B \in \mathbb{Z} \). When \( p \mid h_2 \), but \( p \nmid (h_1^2 r^2 + 2a) \), there is a unique choice of \( s \) for given \( r \), which is satisfactory. Finally, when \( p \mid h_2 \) and \( p \mid h_1^2 r^2 + 2a \) we must have \( p \nmid h_1 \), since \( p \nmid 2 \gcd(a, c) \). Then there at most \( p \) choices for \( s \) but only at most 2 for \( r \), which is also satisfactory.

We may assume that \( p \nmid 2 \Delta(M, N) \gcd(a, c) \) for the remainder of the proof. Suppose that \( p \mid h_1 \). In particular \( \Delta(M, N) \equiv h_2^2 M^2 \pmod{p} \)
and so \( p \nmid h_2M \). We have

\[
\Phi(p) = \sum_{\ell \equiv \frac{h_2}{2} \pmod{p}}^* e_p(Mr + Ns)
\]

\[
= c_p(M) \sum_{s \pmod{p}}^* e_p(Ns),
\]

where \( c_p(M) \) is the Ramanujan sum. Our argument in the preceding paragraph shows that \( |\Phi(p)| \leq 3 \gcd(p, M) = 3 \) when \( p \nmid h_1 \), which is satisfactory for the lemma.

Suppose next that \( p \nmid \gcd(c, M) \). Then it follows that \( p \nmid 2ah_1N \), since \( p \nmid 2\Delta(M, N) \gcd(a, c) \). Replacing \( h_1 \) by \( r \) and using additive characters to detect the congruence, we have

\[
\Phi(p) = \sum_{r^2 \equiv -h_2^2 s^2 - 2a \pmod{p}}^* e_p(Ns) \sum_{(r, s) \pmod{p}}^* e_p(\ell(r^2 + h_2^2 s^2 + 2a + Ns)).
\]

The contribution to the sum from \( \ell \equiv 0 \pmod{p} \) is easily seen to be \( O(1) \). Moreover, we may assume that \( p \nmid h_2 \) in the remaining sum, else we get \( \Phi(p) = O(1) \) overall. Replacing \( h_2 s \) by \( s \), we get

\[
\Phi(p) = \frac{1}{p} \sum_{\ell \pmod{p}} e_p(2a\ell) \sum_{r \pmod{p}}^* e_p(\ell r^2) \sum_{s \pmod{p}}^* e_p(\ell s^2 + h_2^2 Ns) + O(1).
\]

Applying (5.3), we see that

\[
\sum_{\ell \pmod{p}}^* e_p(\ell r^2) = \varepsilon_p \sqrt{p} \left( \frac{\ell}{p} \right) - 1
\]

and

\[
\sum_{s \pmod{p}}^* e_p(\ell s^2 + h_2^2 Ns) = \varepsilon_p \sqrt{p} \left( \frac{\ell}{p} \right) e_p(-4lh_2^2 N^2) - 1.
\]

Hence

\[
\Phi(p) = \varepsilon_p^2 \sum_{\ell \pmod{p}} e_p(2a\ell) e_p(-4lh_2^2 N^2) + O(p^{1/2}).
\]
But this is $O(p^{1/2})$ by the Weil bound for the Kloosterman sum. This is satisfactory and so we can henceforth assume that $p \not| \gcd(c, M)$ and $p \not| 2h_1 \Delta(M, N) \gcd(a, c)$.

We have

$$\Phi(p) = \sum_{(r, s) \in \mathbb{F}_p^2 \atop F(r, s) = 0} e_p(Mr +Ns) + O(1),$$

with $F$ given by (3.6). We will use Lemma 3.5 to estimate the sum, with $k = 1$ and $n = 2$. Let $j \geq 1$ and put $q = p^j$. Then for $\tau \in \mathbb{F}_q$ we must consider

$$N_j(\tau) = \# \{(r, s) \in \mathbb{F}_q^2 : \tau = Mr +Ns\},$$

where we reduce $M$ and $N$ modulo $p$ and view them as elements of $\mathbb{F}_q$. If $M \neq 0$ then, on recalling our expression (3.6) for $F$, we eliminate $r$ to get

$$N_j(\tau) = \# \{s \in \mathbb{F}_q : g(s) = 0\},$$

where $g$ is a polynomial of degree 3 with non-zero leading coefficient $h_2^2 + M^2 - 2h_2 N^2$. Hence $N_j(\tau) \leq 3$ in this case. Suppose next that $M = 0$. In particular $c \neq 0$ and $N \neq 0$. We may eliminate $s$ to get

$$N_j(\tau) = \# \{r \in \mathbb{F}_q : h_2^2 N^2 r^2 + h_2^2 N^2 - 3r^3 + 2aN^{-1}r + c = 0\},$$

Hence $N_j(\tau) \leq 2$ if $\tau \neq 0$ and $N_j(0) = 0$. Taking $N_j = 0$ in Lemma 3.5 therefore allows us to take $\kappa = 1$ in (3.19), whence $\Phi(p) \ll p^{1/2}$, as required to conclude the proof of the lemma.

It remains to consider the general case $k \geq 2$. It will be useful to collect together some basic estimates for the number of solutions to various polynomial congruences. Let $\nu \geq 1$ and let $A, B, C, D \in \mathbb{Z}$. Beginning with quadratic congruences, we observe that

$$\# \{x \pmod{p^{\nu}} : x^2 + D \equiv 0 \pmod{p^{\nu}}\} \leq 2p^{\min\{\nu p(D), \nu\}/2}.$$  

Let $\xi \geq 0$ and assume that $2\xi \leq \nu$. It follows from this that

$$\# \{x \pmod{p^{\nu}} : p^{2\xi} x^2 + D \equiv 0 \pmod{p^{\nu}}\}$$

$$\begin{cases} \leq 2p^{\xi + \min\{\nu p(D), \nu\}/2}, & \text{if } 2\xi \leq \nu p(D), \\ = 0, & \text{otherwise.} \end{cases} \quad (5.4)$$

Next, if $p \not| \gcd(A, B, C)$, we have

$$\# \{x \pmod{p^{\nu}} : Ax^3 + Bx + C \equiv 0 \pmod{p^{\nu}}\} \leq 3p^{\nu p(\delta(A, B, C))/2}, \quad (5.5)$$

where $\delta(A, B, C) = -(4AB^3 + 27A^2C^2)$ is the underlying discriminant. This is established by Huxley [11], for example. We are now ready to establish the following result.
Lemma 5.2. Suppose that \( k \geq 2 \). Then \( \Phi(p^k) \ll p^k \gcd(p^{[k/2]}, h_1) \).

Proof. Define the integer \( \Delta = 2^3a^3 + 3^3b^2 \). Our argument will differ according to whether or not \( \Delta \) vanishes. Throughout our argument we put \( \xi_i = v_p(h_i) \), for \( i = 1, 2 \), with \( h_i = p^{\xi_i}h_i' \).

Our starting point is an analysis of the quantity

\[
M(\nu) = \#\{s \pmod{p^\nu} : p \nmid s, \; g(s) \equiv 0 \pmod{p^\nu}\},
\]

for \( \nu \geq 1 \), where \( g(s) = p^{2\xi_2}h_2'^2s^3 + 2as + c \). We will show that

\[
M(\nu) \ll \begin{cases} 
1, & \text{if } \Delta \neq 0, \\
p^{\nu/2}, & \text{if } \Delta = 0.
\end{cases}
\]

Suppose first that \( \Delta = 0 \). Then we must have \((a, b) = (-6t^2, 8t^3)\), for some non-zero integer \( t \). In particular \( h_2 = O(1) \) and we observe that

\[
M(\nu) \leq \#\{s \pmod{p^\nu} : (h_2s + 4t)(h_2s - 2t)^2 \equiv 0 \pmod{p^\nu}\} \ll p^{\nu/2},
\]

as required.

Turning to the case \( \Delta \neq 0 \), we suppose that \( b = 0 \). Then \( a \neq 0 \) and we now have

\[
M(\nu) \leq \#\{s \pmod{p^\nu} : p^{2\xi_2}h_2'^2s^2 + 2a \equiv 0 \pmod{p^\nu}\}.
\]

If \( 2\xi_2 \leq \nu \), it now follows from (5.4) that \( M(\nu) \leq 2p^{v_p(2a)} \ll 1 \). If \( 2\xi_2 > \nu \), then we trivially have \( M(\nu) \ll 1 \) since then \( \nu \leq v_p(2a) \). We now suppose that \( b \neq 0 \). In particular \( h_2 = p^{\xi_2}h_2' = O(1) \). Write

\[
\gamma = \min\{2\xi_2, v_p(2a), v_p(c)\},
\]

so that also \( p^\gamma = O(1) \). We may assume that \( \gamma < \nu \), since otherwise \( M(\nu) \ll 1 \) is trivial. Let us write \( 2a = p^\gamma a' \) and \( c = p^\gamma c' \), so that

\[
M(\nu) \leq p^\gamma \#\{s \pmod{p^{\nu-\gamma}} : p^{2\xi_2-\gamma}h_2'^2s^3 + a's + c' \equiv 0 \pmod{p^{\nu-\gamma}}\}.
\]

Since the cubic polynomial now has content coprime to \( p \), we may apply (5.5) to deduce that \( M(\nu) \ll 3p^{\gamma/2+\xi_2+v_p(d)/2} \), where \( d \) is the integer

\[
d = 2^3a'^3 + 3^3p^{2\xi_2-\gamma}h_2'^2c'^2
= p^{-3\gamma}\{2^3a'^3 + 3^3h_2'^2c'^2\}
= 4p^{-3\gamma}\Delta.
\]

Since \( \Delta \neq 0 \) it follows that \( p^{v_p(d)} \ll 1 \) and so \( M(\nu) \ll 1 \), as required to complete the proof of (5.6).
We are now ready to establish the bound for \( \Phi(p^k) \) in the lemma, observing that \( |\Phi(p^k)| \) is at most
\[
\# \{ r, s \pmod{p^k} : p \nmid rs, h_2^2 s^3 + (h_1^2 r^2 + 2a) s + c \equiv 0 \pmod{p^k} \}.
\]
If \( 2\xi_1 \geq k \) then it follows from (5.6) that \( \Phi(p^k) \ll p^{k+[k/2]} \), which is satisfactory. Alternatively, if \( 2\xi_1 < k \), we deduce from (5.4) and (5.6) that
\[
|\Phi(p^k)| \leq 2 \sum_{\substack{r \pmod{p^k} \\ 2\xi_1 \leq \ell \leq \min \{ v_p(g(s)), k \} / 2}} p^{\ell_1 + \min \{ v_p(g(s)), k \} / 2}
\leq 2h \sum_{2\xi_1 \leq j < k} p^{k-j/2} M(j) + 2p^{\xi_1+k/2} M(k)
\ll p^{k+\xi_1},
\]
which is also satisfactory. This completes the proof of the lemma. \( \square \)

We now collect together our work so far to deduce a general bound for the exponential sum \( \Phi(\varrho; M, N) \) using the multiplicativity property (5.2). Given \( \varrho \in \mathbb{Z}_{>0} \), we will write \( \varrho = uvw^2 \), where
\[
u = \prod_{p \mid \varrho} p, \quad v = \prod_{\substack{p \mid \varrho \mid \varrho \\ell \\ j \geq 2, 2j}} p.
\]
Clearly \( v \) divides \( w \). Drawing together Lemmas 5.1 and 5.2 we easily arrive at the following result.

**Lemma 5.3.** There exists an absolute constant \( A > 0 \) such that
\[
\Phi(\varrho; M, N) \leq A^{\omega(\varrho)} u^{1/2} v w^2 \gcd(u, \Delta(M, N))^{1/2} \gcd(w, h_1),
\]
where \( \Delta \) is given by (5.1).

### 6. Conclusion

It is now time to bring everything together in (3.16). Using the basic estimate \( [\theta] = \theta + O(1) \), the contribution to \( S_{i,j} \) from the term \( m = n = 0 \) is seen to be
\[
\frac{\Psi_{i,j}(0,0)}{(pq\varrho)^2} \left[ \frac{2B}{h_1} \right] \left[ \frac{2B}{h_2} \right] = M_{i,j} + O \left( \frac{\Psi_{i,j}(0,0) B}{\min \{ h_1, h_2 \} (pq\varrho)^2} \right),
\]
with
\[
M_{i,j} = \frac{4\Psi_{i,j}(0,0) B^2}{h_1 h_2 (pq\varrho)^2}.
\]
Hence it follows from (3.15) that
\[ S_{i,j} = M_{i,j} + O \left( \frac{\Psi_{i,j}(0,0)B}{\min\{h_1, h_2\}(pq\varrho)^2} \right) + O(E_{i,j}), \] (6.2)
where
\[ E_{i,j} = \sum_{-pq\varrho/2 < m, n < pq\varrho/2 \atop (m,n) \neq (0,0)} \min \left\{ \frac{B}{h_1}, \frac{pq\varrho}{|m|} \right\} \min \left\{ \frac{B}{h_2}, \frac{pq\varrho}{|n|} \right\} \frac{|\Psi_{i,j}(m,n)|}{(pq\varrho)^2}. \]

We now come to the estimation of $\Psi_{i,j}(m, n)$, writing $\varrho = uvw^2$, with $u, v, w$ as in (5.7). In particular $\gcd(pq, \varrho) = 1$. It will be convenient to put
\[ \lambda(\varrho) = u^{1/2}vw^3 \gcd(w, h_1). \] (6.3)
Drawing together Lemmas 4.3 and 5.3 in Lemma 3.4, we deduce that $\Psi_{i,j}(m, n) \ll A^{\omega(\varrho)}pq\lambda(\varrho) \gcd(pq, m, n) \gcd(u, \Delta(m, n))^{1/2}$, if $p \neq q$ and
$\Psi_{i,j}(m, n) \ll A^{\omega(\varrho)}p_1(m,n)^3\lambda(\varrho) \gcd(p, m', n') \gcd(u, \Delta(m, n'))^{1/2}$, if $p = q$, where $(m', n') = (m, n)/p$ and
$1_{p/(m,n)} = \begin{cases} 1, & \text{if } p \mid (m, n), \\ 0, & \text{otherwise}. \end{cases}$

If $p = q$ then $1_{p/(m,n)^3}\gcd(p, m', n') \leq pq \gcd(pq, m, n)$. Recall that $A^{\omega(n)} = O_{A,\varepsilon}(n^\varepsilon)$ for any $\varepsilon > 0$. In particular we have $A^{\omega(\varrho)} = O_{\varepsilon}(B^\varepsilon)$ in these estimates, since $\varrho \leq 2B$. We therefore conclude that
$\Psi_{i,j}(m, n) \ll \varepsilon B^{3/2}pq\lambda(\varrho) \gcd(pq, m, n) \gcd(u, \Delta(m, n))^{1/2}$, (6.4)
for any $p, q \in \mathcal{P}$. In particular, taking $m = n = 0$, it follows that
$\Psi_{i,j}(0,0) \ll \varepsilon B^{3/2}(pq)^2 \varrho \gcd(w, h_1).$ (6.5)

The following result will be useful when it comes to summing (6.4) over the relevant $\varrho$.

**Lemma 6.1.** Let $\Delta \in \mathbb{Z}_{>0}$, $\varepsilon > 0$ and let $\delta \in \{0, 1\}$. We have
\[ \sum_{\varrho \leq 2B/(h_1h_2)} \frac{\lambda(\varrho) \gcd(u, \Delta)}{\varrho^\delta} \ll \varepsilon (\Delta B)^{\delta} \left( \frac{B}{h_1h_2} \right)^{3/2-\delta}, \]
where $\lambda(\varrho)$ is given by (6.3) and $\varrho = uvw^2$, with $u, v, w$ as in (5.7).
Proof. Let \( S_\delta \) denote the sum in the lemma, for \( \delta \in \{0, 1\} \). It suffices to handle the case \( \delta = 1 \), since \( S_0 \ll (h_1 h_2)^{-1} BS_1 \). We will make use of the fact that

\[
\sum_{n \leq N} \gcd(n, \Delta) = \sum_{d \mid \Delta} d \#\{n \leq N : d \mid n\} \ll \tau(\Delta) N,
\]

where \( \tau \) is the divisor function, together with results that follow from it using partial summation. Recalling (6.3), we note that

\[
\lambda(\g) \frac{\g}{\nu_1^{1/2}} = \gcd(w, h_1) \frac{\gcd(u, \Delta)^{1/2}}{u_1^{1/2}}.
\]

Hence we have

\[
S_1 \leq \sum_{w \leq 2B/(h_1h_2)} \gcd(w, h_1) \sum_{v \mid w} \sum_{u \leq 2B/(h_1h_2v^2)} \gcd(u, \Delta)^{1/2} \frac{1}{u_1^{1/2}} \ll \tau(\Delta) B^{1/2+\varepsilon} \frac{B^{1/2+\varepsilon}}{(h_1h_2)^{1/2}}.
\]

We complete the proof of the lemma by taking \( \tau(\Delta) = O_\varepsilon(\Delta^\varepsilon) \). \qed

We now turn to an upper bound for \( N_1(B, C; H) \), following (3.10) and (3.13). We start by analysing the main term \( M_{i,j} \) in (6.1). Suppose that \( p \neq q \). Then it follows from Lemmas 3.4 and 4.3 that

\[
M_{i,j} = 4\Sigma_i(p; 0, 0)\Sigma_j(q; 0, 0)\Phi(\g; 0, 0)B^2 \leq 4 \max\{1, i\} \max\{1, j\} \Phi(\g; 0, 0)B^2 + O\left( \frac{B^{1/2+\varepsilon}}{(h_1h_2)^{1/2}} \right).
\]

for \( i, j \in \{0, 1, 2\} \). Recalling (3.14), we deduce that

\[
\sum_{i,j \in \{0,1,2\}} c_{i,j}(\alpha) M_{i,j} = 4\Phi(\g; 0, 0)B^2 \frac{\Phi(\g; 0, 0)}{h_1h_2\min\{p, q\}g^2} \leq 4\Phi(\g; 0, 0)B^2 \frac{\Phi(\g; 0, 0)}{h_1h_2\min\{p, q\}g^2} \leq \Phi(\g; 0, 0)B^2 \frac{\Phi(\g; 0, 0)}{h_1h_2\min\{p, q\}g^2}.
\]

if \( p \neq q \). Taking \( \alpha = 1 \) therefore eliminates the main term in this expression. Suppose next that \( p = q \in \mathcal{P} \). Then, returning to (6.1), we deduce from Lemma 3.4 that

\[
M_{i,j} = 4\Sigma_{i+j}(p; 0, 0)\Phi(\g; 0, 0)B^2 \leq \Phi(\g; 0, 0)B^2 \frac{\Phi(\g; 0, 0)}{h_1h_2\min\{p, q\}g^2} \leq \Phi(\g; 0, 0)B^2 \frac{\Phi(\g; 0, 0)}{h_1h_2\min\{p, q\}g^2}.
\]
for \( i, j \in \{0, 1, 2\} \). It now follows that

\[
\sum_{h_1, h_2 \leq H} \frac{\log^2 Q}{Q^2} \sum_{\rho \leq 2B/(h_1 h_2)} \sum_{p, q \in \mathcal{P}} \sum_{i, j \in\{0, 1, 2\}} c_{i,j}(1) M_{i,j} \leq B^2 \log^2 H \log^2 Q \sum_{\rho \leq 2B} \Phi(\rho; 0, 0) \Psi(\rho; 0, 0),
\]

where \( \Psi = \sum_{p \neq q \in \mathcal{P}} \min\{p, q\}^{-1} + \sum_{p \in \mathcal{P}} 1 \leq Q \). Next, Lemma 5.3 implies that \( \Phi(\rho; 0, 0) \ll \epsilon \min\{p, q\} \), whence

\[
\sum_{\rho \leq 2B} \Phi(\rho; 0, 0) \ll_B \sum_{\rho \leq 2B} \gcd(w, h_1) \ll_B \gcd(w, h_2).
\]

But

\[
\sum_{\rho \leq 2B} \frac{\gcd(w, h_1)}{2^\rho} \ll_B \sum_{\rho \leq 2B} \frac{\gcd(w, h_1)}{2^\rho} \sum_{v \mid w} \frac{1}{v} \sum_{u \leq 2B/(vu^2)} \frac{1}{u} \ll_B \epsilon B^{2+\epsilon},
\]

on recalling the decomposition \( \psi = uvw^2 \) from (5.7) and employing the bound \( \sum_{1 \leq n \leq N} 1/n \ll \log N \ll N^\epsilon \). We conclude that

\[
\sum_{h_1, h_2 \leq H} \frac{\log^2 Q}{Q^2} \sum_{\rho \leq 2B/(h_1 h_2)} \sum_{i, j \in\{0, 1, 2\}} c_{i,j}(1) M_{i,j} \ll_{\epsilon} B^{2+\epsilon},
\]

on redefining the choice of \( \epsilon \).

We now turn to the error terms in (6.2). Firstly, it follows from (6.5) that

\[
\Psi_{i,j}(0, 0) B \min\{h_1, h_2\} (pq) \ll \epsilon \gcd(w, h_1) B^{1+\epsilon} \min\{h_1, h_2\} \psi.
\]

Next, we note from (5.1) that \( \Delta(m, n) = 0 \) if and only if \( m = n = 0 \). In view of (6.4) we see that the contribution to the sum \( E_{i,j} \) in (6.2) from \( m = 0 \), in which case \( \Delta(0, n) = h_1^2 n^2 \), is

\[
\ll \frac{B}{h_1} \sum_{n \neq 0} \frac{pq \Psi_{i,j}(0, n)}{|n| (pq)^2} \ll \epsilon \frac{B^{1+\epsilon}}{h_1} \sum_{n \neq 0} \frac{\gcd(pq, n) \gcd(u, h_1^2 n^2)^{1/2}}{|n|}.
\]
Similarly, the contribution from \( n = 0 \) can be bounded by the same quantity, in which \( h_1 \) is replaced by \( h_2 \). We therefore conclude that terms with \( mn = 0 \) and \((m, n) \neq (0, 0)\) give an overall contribution

\[
\ll \frac{\varrho^{-1} \lambda(\varrho) B^{1+\varepsilon}}{\min\{h_1, h_2\}} \sum_{-p\varrho/2 < k \leq p\varrho/2 \atop k \neq 0} \frac{\gcd(u, h_1h_2k)}{|k|}
\]

in (6.2).

Next we consider the contribution to \( E_{i,j} \) in (6.2) from \( mn \neq 0 \). Applying (6.4) we see that this is

\[
\ll \varepsilon B^2 pq \lambda(\varrho) \sum_{-p\varrho/2 < m, n \leq p\varrho/2 \atop mn \neq 0} \frac{\gcd(pq, m, n) \gcd(u, \Delta(m, n))^{1/2}}{|mn|}
\]

\[
\ll \varepsilon B^2 pq \lambda(\varrho) \sum_{-p\varrho/2 < m, n \leq p\varrho/2 \atop mn \neq 0} \frac{\gcd(u, \Delta(m, n))}{|mn|},
\]

since \( \gcd(pq, u) = 1 \).

Combining this with our estimates so far we conclude that

\[
|S_{i,j} - M_{i,j}| \ll \varepsilon \frac{\gcd(w, h_1) B^{1+\varepsilon}}{\min\{h_1, h_2\}} + \frac{\varrho^{-1} \lambda(\varrho) B^{1+\varepsilon}}{\min\{h_1, h_2\}} \sum_{-p\varrho/2 < k \leq p\varrho/2 \atop k \neq 0} \frac{\gcd(u, h_1h_2k)}{|k|}
\]

\[
+ \frac{pq \lambda(\varrho) B^2}{B^{3/2}} \sum_{-p\varrho/2 < m, n \leq p\varrho/2 \atop mn \neq 0} \frac{\gcd(u, \Delta(m, n))}{|mn|}.
\]

We would now like to introduce a summation over \( \varrho \leq 2B/(h_1h_2) \). For the first term we use (6.6). For the remaining two terms we apply Lemma 6.1. This leads to the conclusion that

\[
\sum_{\varrho \leq 2B/(h_1h_2)} |S_{i,j} - M_{i,j}| \ll \varepsilon B^{5\varepsilon} \frac{B}{\min\{h_1, h_2\}} + \frac{B^{3/2}}{\min\{h_1, h_2\}(h_1h_2)^{1/2}}
\]

\[
+ \frac{(pq)^{1+\varepsilon} B^{3/2}}{(h_1h_2)^{3/2}}.
\]
Now
\[
\sum_{h_1, h_2 \leq H} \frac{1}{\min\{h_1, h_2\}} \ll H \log H
\]
and
\[
\sum_{h_1, h_2 \leq H} \frac{1}{\min\{h_1, h_2\}(h_1h_2)^{1/2}} \ll H^{1/2}, \quad \sum_{h_1, h_2 \leq H} \frac{1}{(h_1h_2)^{3/2}} \ll 1.
\]

Using these estimates it follows that
\[
\sum_{h_1, h_2 \leq H} \frac{\log^2 Q}{Q^2} \sum_{p, q \in \mathcal{P}} \sum_{i, j \in \{0, 1, 2\}} \sum_{\varphi \leq 2B/(h_1h_2)} |S_{i,j} - M_{i,j}|
\ll \varepsilon B^{6\varepsilon} \left\{ HB + (H^{1/2} + Q^2)B^{3/2} \right\}.
\]

By assumption \( H \leq 2B \). Hence \( HB \ll H^{1/2}B^{3/2} \). Returning to (3.10) and (3.13), and recalling (6.7), we now conclude that
\[
N_1(B; H) \ll \varepsilon B^{2+\varepsilon} + (H^{1/2} + Q^2)B^{3/2+6\varepsilon}.
\]

Taking \( Q = B^{1/6} \), we conclude the proof of Lemma 3.2 on redefining the choice of \( \varepsilon \).

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