Statistical mechanical assessment of a reconstruction limit of compressed sensing: Toward theoretical analysis of correlated signals

K. Takeda and Y. Kabashima

Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology
Yokohama 226-8502, Japan

received 31 August 2010; accepted in final form 23 May 2011
published online 21 June 2011

PACS 89.70.-a – Information and communication theory
PACS 75.10.Hr – Spin-glass and other random models
PACS 05.70.Fh – Phase transitions: general studies

Abstract – We provide a scheme for exploring the reconstruction limits of compressed sensing by minimizing the general cost function under the random measurement constraints for generic correlated signal sources. Our scheme is based on the statistical mechanical replica method for dealing with random systems. As a simple but non-trivial example, we apply the scheme to a sparse autoregressive model, where the first differences in the input signals of the correlated time series are sparse, and evaluate the critical compression rate for a perfect reconstruction. The results are in good agreement with a numerical experiment for a signal reconstruction.

Copyright © EPLA, 2011

Introduction. – Compressed sensing (CS) is a novel technique for data compression and has been drawing a lot of attention recently from the viewpoints of both theory and application. The key idea behind CS is to utilize the sparsity of the original input signals as the prior knowledge during the signal reconstruction stage, which can significantly reduce the number of signal measurements required for a perfect reconstruction. This setup is realistic because we often have to face situations where we have to handle sparse signals in the real world. A lot of effort has been paid and significant progress has been made in investigating the properties of CS [1–3]. After the pioneering works, contribution to CS problem from statistical mechanics analysis is now growing rapidly [4–10].

The measurement process of CS is summarized in the following linear equation:

$$\mathbf{y} = \mathbf{F}\mathbf{x}^0.$$ (1)

The vectors and matrices are denoted in bold in this article. The input signal vector $\mathbf{x}^0$ is $N$-dimensional and the compressed signal vector $\mathbf{y} \in \mathbb{R}^P$ is $P$-dimensional. $\mathbf{F}$ is a $P$-by-$N$ compression matrix. In this article, we particularly focus on random measurements, in which each $F_{\mu i}$ entry is independently and identically distributed (i.i.d.) from a Gaussian distribution of the zero mean and variance $N^{-1}$. The compression rate is defined by $\alpha \equiv P/N < 1$.

In earlier theoretical studies, the critical compression rate $\alpha_c$ for perfectly reconstructing $\mathbf{x}^0$ from $\mathbf{y}$ has been actively assessed for various reconstruction schemes under the assumption that the input signal vector $\mathbf{x}^0$ is sparsely modeled by the distribution,

$$P(x^0_i) = (1 - \rho)\delta(x^0_i) + \rho \tilde{P}(x^0_i),$$ (2)

within the large system limit of $N, P \to \infty$ keeping $\alpha = P/N$ constant [1–4]. Here, $\tilde{P}(x^0_i)$ is a given probabilistic distribution and $\rho$ denotes the density of the non-zero elements. In particular, the assessment for the reconstruction scheme for minimizing the so-called $\ell_1$-norm

$$\text{minimize } \sum_i |x_i| \text{ subject to } \mathbf{y}(= \mathbf{F}\mathbf{x}^0) = \mathbf{F}\mathbf{x},$$ (3)

which is termed the $\ell_1$-norm reconstruction hereafter, has drawn a lot of attention because of its computational feasibility and robustness to measurement noise. In this regard, it may be surprising that a mathematically rigorous method of combinatorial geometry [2] and the replica method for statistical mechanics [4] provide an identical $\alpha_c$ value although the methodological equivalence between the two schemes has not really been clarified yet. In addition, the value of $\alpha_c$ seems rather universal [4,11,12]; $\alpha_c$ is unchanged as long as $\mathbf{x}^0$ follows (2) and $\mathbf{F}^\top \mathbf{F}$, where $\mathbf{T}$ denotes the matrix transpose, asymptotically obeys a
rotational invariant ensemble. However, the necessary and sufficient condition for the universality is still also open.

The main purpose of this article is to offer a methodological basis for exploring this universality using the replica method. For this objective, we evaluate $\alpha_c$ for general correlated distributions of $x^0$, where $P(x^0)$ is a joint distribution with sparsity and not necessarily factorizable to each $P(x^i)$, and the reconstruction schemes provided as

\[ \minimize E(x) \text{ subject to } y = Fx^0 = Fx, \quad (4) \]

where $E(x)$ is a generic cost function. For simplicity, we assume that each entry of $F$, $F_{\mu i}$, is an i.i.d. Gaussian random number of the zero mean and variance $N^{-1}$. However, as shown later, situations in which $x^0$ is expanded by the i.i.d. coefficients sampled from (2) using a certain basis $S$ can be cast to those of the correlated $F_{\mu i}$ for i.i.d. signals sampled from (2). Namely, our analysis practically covers correlated compression matrices as well [9].

In addition to the theoretical interest, exploring the above setting is also significant for practical relevance. In most real-world problems, the signals may be redundant in an information theoretic sense, but are not necessarily expressed as sparse upon first sight. In addition, in order to appropriately deal with such real-world signals, certain cost functions other than the naïve $\ell_1$-norm of (3), such as the total variation (TV) [13], are widely used in practice for reconstructing signals. Our generic assumptions concerning the correlated distributions of the signal sources and cost functions for the signal reconstruction are intended to extend the analysis of the performance measure of compressed sensing, $\alpha_c$, for more practically plausible scenarios beyond the simple cases of i.i.d. sparse sources and component-wise cost functions.

**Replica analysis: a general guideline.** – Here we sketch an outline of our analysis. This analysis is similar to that of the recent study regarding CS for correlated compression matrices [9] and that of the correlated channel in wireless telecommunication systems [14,15]. The technical details can be found in these references.

Following the basic scenario in [4], let us define the key quantity for our analysis, which plays the role of a correlation matrix in statistical mechanics and represents the typical value (per element) of the minimized cost (4) in the current context,

\[ C \equiv -\lim_{\beta \to \infty} \frac{1}{\beta N} \ln \langle Z(\beta, y) \rangle_{F,x^0} = -\lim_{\beta \to \infty} \lim_{n \to 0} \frac{1}{\beta N} \ln \langle Z^n(\beta, y) \rangle_{F,x^0}, \quad (5) \]

where $Z(\beta, y) \equiv \int dx \exp(-\beta E(x))\delta(F(x - x^0))$ is the partition function and $\ldots_{\beta}$ generally denotes the average with respect to random variable $X$. Taking the limit $\beta \to \infty$ works for singling out the solution of (4) in the partition function. Unfortunately, assessing $\langle Z^n(\beta, y) \rangle_{F,x^0}$ for $\forall n \in \mathbb{R}$ in (5) is technically difficult. For resolving this difficulty, we evaluate analytical expressions of $\langle Z^n(\beta, y) \rangle_{F,x^0}$ with respect to $\forall n \in \mathbb{N}$ using the identity

\[ Z^n(\beta, y) = \int \prod_{a=1}^n dx^a \exp(-\beta E(x^a))\delta(F(x^a - x^0)), \quad (6) \]

which is valid only for $n \in \mathbb{N}$, and employ the obtained expressions for assessment of (5) assuming that they hold for $\forall n \in \mathbb{R}$ as well. This is often termed the replica method as integration variables $x^a$ ($a = 1, 2, \ldots, n$) in (6) are regarded as $n$ “replicas” of the original state variable $x$.

For this, we analytically calculate the average of the right-hand side of (6) employing the saddle-point method with respect to macroscopic variables $q_{ab} = N^{-1}(x^a)^T x^b$ and $m_a = N^{-1}(x^0)^T x^a$, which is justified as $P, N \gg 1$. The intrinsic invariance of (6) under any permutations of replica indices $a = 1, 2, \ldots, n$ leads to the replica symmetric (RS) ansatz, which means that the dominant saddle point also possesses this property as $q_{ab} = Q$, $q_{ab} = q$ ($a \neq b$) and $m_a = m$. This reproduces the mathematically rigorous results for the basic model [4]. Therefore, we here also adopt this ansatz, validity of which will be checked later. The saddle point solution obtained under the RS ansatz seems to hold for $n \in \mathbb{R}$ as well. Employing this in the right-hand side of (5) yields an expression

\[ C = \text{Extr}_{q,m,\chi,Q,\tilde{m}} \left( \frac{\alpha (q - 2m + u)}{2\chi} + \left( \frac{\chi}{2} - \frac{Q}{2} + m \tilde{m} \right) \right) + \int dx^0 P(x^0) \int D\tilde{z} \phi(\omega, \tilde{Q}) \right). \quad (7) \]

Here $\omega = (\omega_i) \equiv \tilde{m} x^0 + \sqrt{\chi} z$, $\text{Extr}_{\ldots}$ denotes the extremization of $\ldots$ with respect to $\Theta$, $P(x^0)$ is the generic $N$-dimensional distribution of the original signal $x^0$, and $u = N^{-1} \int dx^0 P(x^0)|x^0|^2$ denotes the second moment (per element) of the original signal. $D\tilde{z}$ stands for the $N$-dimensional Gaussian measure $(2\pi)^{-N/2} \prod_{i=1}^N dz_i \exp(-z_i^2/2)$. The function $\phi(h, \tilde{Q})$ is defined by the minimization including the $N$ variables as

\[ \phi(h, \tilde{Q}) \equiv \frac{1}{N} \text{min}_x \left\{ \frac{Q}{2} x^T x - h^T x + E(x) \right\}. \quad (8) \]

With regard to the final expressions (7) and (8), three points are worthwhile to note. First, the right-hand side of (8), in conjunction with substitution of $h = \omega$ and $\tilde{Q}$ as provided by (7), stands for the problem statistically equivalent to the original one (4). This means that random constraints $y = Fx$ of (4), in which multiple variables are coupled with one another, can be handled as a bunch of decoupled extra random costs ($\tilde{Q}/2)x_i^2 - \omega_i x_i$ ($i = 1, 2, \ldots, N$) in the performance assessment of large systems. Such correspondence is sometimes termed “decoupling principle” in information theory literature [16].

Second, the values of $q$ and $m$ determined by the extremization condition of the right-hand side of (7) represent the typical values of the averages of $N^{-1} x^T x$ and $N^{-1}(x^0)^T x$ with respect to the uniform distribution of the solutions
of (4), respectively. If and only if the solutions typically accorded to $x^0$ allowing negligible errors per component in $N \to \infty$, the solution for $q=m=u$ is thermodynamically dominant, implying that the reconstruction is typically successful. Therefore, one can characterize $\alpha_\epsilon$ as a transition condition at which the successful solution $q=m=u$ loses its thermodynamic dominance. When $E(x)$ is convex downward, which is often the case in practice, this can be examined by assessing the local stability of $q=m=u$ since (4) is guaranteed to possess a unique solution. It might also be noteworthy that our criterion for a successful reconstruction is different from that of earlier mathematical studies [1–3] in which no errors were permitted. However, we expect that such differences are irrelevant in the $\alpha_\epsilon$ assessment as was the case for the basic problems of (2) and (3) [4].

The final point is the computational cost for carrying out the above assessment. Although the average with respect to $F$ has already been analytically taken into account, those with respect to $x^0$ and auxiliary random numbers $\tilde{z}$ still remain in the expression (7). In practice, this should be assessed using a Monte Carlo sampling method for sufficiently large $N$ and $P$, which in principle can offer arbitrarily accurate estimates of the averages in the large system limit $N,P \to \infty$ (under the assumption that a certain thermodynamic limit exists). Therefore, the computational cost for performing the Monte Carlo sampling practically determines the feasibility. There are two possible sources for the computational difficulty. The first one is the computational cost for generating $x^0$ following $N$-dimensional distribution $P(x^0)$, which generally grows exponentially with respect to $N$. However, when $x^0$ can be expressed as $x^0 = Sx'$, where $S$ and $x'$ are a fixed matrix and a vector sampled from a computationally feasible distribution, respectively, generating $x^0$ is not a crucial problem for standard computational resources to date. This is also the case for $\tilde{z}$. The other difficulty could come out in numerically performing a minimization with respect to $x$ in (8). However, when $E(x)$ is convex, which we are assuming, the cost function on the right-hand side of (8) is guaranteed to be convex as well. This indicates that one can also avoid a computational explosion using various schemes known for convex optimization [17,18] in assessing (8). Furthermore, when the variable dependence of $E(x)$ is pictorially expressed as a graph free from cycle, one may be able to use more efficient algorithms for the minimization [19]. These imply that although performing the developed method is generally computationally difficult, it is still practically useful in the performance analysis for certain non-trivial classes of CS problems. In the next part, this is illustrated through application to time series data signals that are characterized by the sparsity concerning the difference between signals of successive times.

**Application: a sparse autoregressive model.**

**Model definition.** For illustrating the utility of the developed scheme, we focus on the time series data signals generated from the use of the autoregression process of the first order with sparsity (sparse AR(1) model, denoted by SAR(1) in the following). A SAR(1) process is defined by the stochastic recurrence equation

$$x^0_{i+1} = \begin{cases} \rho x^0_i + \sqrt{1-\rho^2} \eta_i & \text{with prob. } \rho, \\ x^0_i & \text{with prob. } 1-\rho, \end{cases}$$

where $0 \leq \rho \leq 1$. We assume that random variable $\eta_i$ at each time $i$, including the first signal $x^0_1$, is independently drawn from the normal Gaussian distribution $\mathcal{N}(0,1)$. Equivalently, this process is represented by the conditional probability of the signal at time $i$ given a state at time $i-1$, $x^0_{i-1}$, as

$$P(x^0_i|x^0_{i-1}) = (1-\rho)\delta(x^0_i-x^0_{i-1}) + \frac{\rho}{\sqrt{2\pi(1-\rho^2)}} \exp \left( -\frac{(x^0_i-x^0_{i-1})^2}{2(1-\rho^2)} \right).$$

The CS of this process has already been investigated from algorithmic point of view [20]. Here we address the critical compression rate $\alpha_\epsilon$ of the signals from this process by the replica analysis. Although for simplicity reasons we focus on SAR(1) in the current article, extending the following argument to that of the $k$-th order, SAR($k$), is straightforward.

This model is considered as a special Gaussian mixture transition distribution model proposed by Le et al. for handling the non-Gaussian and nonlinear features of a time series in a unified framework [21,22]. In (9) and (10), $r$ represents a parameter of the autoregression satisfying $0 \leq r \leq 1$, while $\rho$ (0 $\leq \rho \leq 1$) stands for a density parameter with respect to the difference in signals between successive times. An example of the signals from SAR(1) is depicted in fig. 1. For $\rho = 1$ this process is reduced to a normal autoregressive model of the first order, and for $\rho < 1$ the signal at time $i$ pauses for the same state as the one in the previous time step $i-1$ with a finite probability $1-\rho$. Therefore, SAR(1) of $\rho < 1$ typically generates a time series that has a lot more pausing states than usual autoregressive models. This property may be suitable for modeling various kinds of time series data such as acoustic
is also a certain random matrix, the ensemble of \( \mathbf{F} \) is sparse but the signals themselves are dense. This indicates that using the naïve \( \ell_1 \)-norm as a cost function for the signal reconstruction is not promising for improving the reconstruction performance. Instead, it may be reasonable to choose the cost function \( E(\mathbf{x}) \) as \( \ell_1 \)-norm for the signal differences, namely

\[
\min \sum_{i} |x_{i+1} - x_i| \quad \text{subject to} \quad y = \mathbf{Fx},
\]

in terms of striking a balance between the statistical accordance to the original signals and computational feasibility.

Defining a vector of the signal differences as \( x'_i = x_i - x_{i-1} \) and \( x'_0 = x_1 \) formally converts (11) into an expression of the naïve \( \ell_1 \)-norm reconstruction for \( x' = (x'_i) \) subject to the constraint \( y = \mathbf{F}'x' \) offered by a modified compression matrix \( \mathbf{F}' = \mathbf{FS} \), where \( S = (S_{ij}) \) is provided as \( S_{ij} = 1 \) for \( i > j \) and vanishes, otherwise. Although \( \mathbf{F}' \) is also a certain random matrix, the ensemble of \( \mathbf{F}' \) is no more rotationally invariant as \( \{(\mathbf{F}'^T \mathbf{F}')_p = S^T S \} \) holds true. Therefore, one cannot apply the results of earlier studies for the basic settings \([1–4]\) to the analysis of \( \text{SAR}(1) \), as was pointed out in [9].

**Saddle point equation and critical condition.** Let us evaluate the critical reconstruction limit of \( \text{SAR}(1) \) by using the scheme developed in the preceding part for (10) and (11). Note that (11) is described by a spin chain with random fields. Similar problems have been analyzed in \([26,27]\). Extremization of (7), in conjunction with the substitution of \( P(\mathbf{x}) = \prod_{i=1}^{N} P(x'_i | x'_{i-1}) \), where \( P(x'_i | x'_{i-1}) = \exp(-x'_i^2/2)/\sqrt{2\pi} \), yields a set of saddle point equations, as

\[
\hat{\mathbf{F}} = \hat{m} = \alpha \chi, \hat{\mathbf{Q}} = \alpha(\mathbf{q} - 2m + u)/\chi^2,
\]

\[
q = \int \prod_{i=1}^{N} \mathcal{D} \mathbf{z} \mathcal{D} x'_0 P(x'_0 | x'_{0-1}) \left( \frac{1}{N} \sum_j \left( z_j(\mathbf{Q}, \hat{\mathbf{Q}}) \right)^2 \right),
\]

\[
m = \int \prod_{i=1}^{N} \mathcal{D} \mathbf{z} \mathcal{D} x'_0 P(x'_0 | x'_{0-1}) \left( \frac{1}{N} \sum_j x'_j(\mathbf{Q}, \hat{\mathbf{Q}}) \right),
\]

\[
\chi = \frac{1}{\sqrt{X}} \int \prod_{i=1}^{N} \mathcal{D} \mathbf{z} \mathcal{D} x'_0 P(x'_0 | x'_{0-1}) \left( \frac{1}{N} \sum_j \hat{z}_j x'_j(\mathbf{Q}, \hat{\mathbf{Q}}) \right),
\]

(12)

where \( \mathcal{D}z \equiv \exp(-z^2/2)/\sqrt{2\pi} \) and the \( N \)-dimensional vector \( x' = (x'_j(\mathbf{Q}, \hat{\mathbf{Q}})) \) is determined by

\[
\frac{\partial}{\partial x'_i} \phi(\mathbf{Q}, \hat{\mathbf{Q}}) = (\hat{Q} x'_i - \hat{m} x'_0) - \sqrt{X} \hat{z}_i
\]

\[
+ \text{sgn}(x'_i - x'_{i+1}) + \text{sgn}(x'_i - x'_{i-1}) = 0 \quad (i = 1, 2, \ldots, N),
\]

which corresponds to the minimization condition of \( \phi(\mathbf{Q}, \hat{\mathbf{Q}}) \),\( \text{sgn}(x) = x/|x| \) for \( x \neq 0 \).

For a sufficiently large \( \alpha \) given \( r \) and \( \rho \), the set of equations (12) allows for the following solution: \( \chi \to +0, \hat{Q} = \hat{m} \to +\infty, Q = m \to u \) and \( \hat{\chi} \sim O(1) \). This is because the third to fifth terms in (13) are negligible compared to the first and second ones if \( |x'_i - x'_0| \sim O(1) \) as \( Q = \hat{m} \to +\infty \) while \( \hat{\chi} \) is kept at \( O(1) \), and therefore, \( x'_j(\mathbf{Q}, \hat{\mathbf{Q}}) \to x'_0 \) \( (j = 1, 2, \ldots, N) \) holds in (13). This solution represents nothing but a successful reconstruction.

The critical reconstruction rate \( \alpha_c \) is determined by the local instability condition of this solution, which is summarized as the condition for preventing the behavior of \( \chi \to +0 \). In order to accurately evaluate this, we pay attention to the infinitesimal differences between \( x'_i \) and \( x'_0 \) by introducing the novel variables \( \hat{x}_i(\sqrt{\chi}) \equiv \lim_{\chi \to +0}(\alpha/\chi)(x'_i - x'_0) \) \( (i = 1, 2, \ldots, N) \). Rewriting (12) using these variables within the limit of \( \chi \to +0 \) and exploring the local stability condition of \( \chi \to +0 \) yield a set of equations for determining the reconstruction limit \( \alpha_c \),

\[
\hat{\chi} = \frac{1}{\alpha} \int \prod_{i=1}^{N} D\hat{z}_i Dx'_0 P(x'_0 \mid x'_{0-1}) \left( \frac{1}{N} \sum_j (\hat{z}_j(\sqrt{\chi}) \hat{z}_j) \right)^2,
\]

\[
\alpha_c = \frac{1}{\sqrt{\chi}} \int \prod_{i=1}^{N} D\hat{z}_i Dx'_0 P(x'_0 \mid x'_{0-1}) \left( \frac{1}{N} \sum_j \hat{z}_j x'_j(\sqrt{\chi}) \right),
\]

\[
\hat{\chi} - \sqrt{\chi} \hat{z}_i + \text{sgn}(\hat{\chi} - \hat{z}_{i+1} + x'_0 - x'_i) + \text{sgn}(\hat{\chi} - \hat{z}_{i-1} + x'_0 - x'_i) = 0,
\]

where \( \epsilon > 0 \) is a sufficiently small positive constant. We also checked the local stability against the disturbance that breaks the replica symmetry [28], which gives the stability condition as

\[
\frac{\alpha}{\chi^2} \int \prod_{i=1}^{N} D\hat{z}_i Dx'_0 P(x'_0 \mid x'_{0-1}) \left( \frac{1}{N} \sum_{j,k} \left( \frac{\partial x'_j(\mathbf{Q}, \hat{\mathbf{Q}})}{\partial \omega_k} \right) \right)^2 < 1.
\]

(15)

It may be noteworthy that this accords to that for the dynamical stability of the successful solution \( x_* = x'_0 \) concerning a belief propagation based algorithm for solving (11). A similar accordance has been observed in another system before [29].

Unfortunately, the off-diagonal contributions of \( (\partial x'_j(\mathbf{Q}, \hat{\mathbf{Q}})/\partial \omega_k)^2 \) \( (j \neq k) \) always prevent the solution of (14) from satisfying (15). This implies the necessity of exploring the replica symmetry breaking (RSB) solutions for accurately assessing \( \alpha_c \). However, we still speculate that the RS estimate at least offers a fairly good approximation since the deviation from the results of numerical experiments shown later is considerably small. This speculation is also supported by the fact that the RS assessment provides the correct estimate of \( \alpha_c \) of the \( l_0 \) recovery scheme for the basic model in spite that the RS solution is locally unstable for the RSB disturbance [4].
Statistical mechanical assessment of a reconstruction limit of compressed sensing

The reconstruction limit as a function of $\rho$ for a larger $N$ is observed as expected from the replica analysis for $\rho = 0.5$, the value of the reconstruction limit for $r = 0$, which corresponds to cases where there were no time correlations except for the pausing, is larger (bottom panel). This implies that the reconstruction limit does depend on the types of sparsity and that the sparsity of the signal differences is not as useful as that for the signals themselves in reducing the data size. With regard to the autoregression parameter $r$, a decrease in $\alpha_c$ or the equivalent improvement of the reconstruction performance is observed as $r$ is increased (bottom panel). This is plausible because the correlations generally decrease the information quantity of the signals, which in principle makes it possible to reduce the data size.

To verify the obtained results, we also conducted numerical experiments. In the experiments, $\alpha_c$ was numerically assessed as follows: In a trial, we first prepared an $N \times N$ random compression matrix $F$, and deleted the rows of the matrix one-by-one until the signal reconstruction failed. A failure was judged when $|x_s - x^0| > 10^{-4}$, where $x_s$ is the reconstructed vector, was first satisfied, and the value $P_c = P + 1$, where $P$ is the number of rows when the reconstruction failure, was recorded. We used the convex optimization package for MATLAB developed in [17,18] to search for $x_s$. For each $N$, this trial was repeated $10^5$ times, and the typical reconstruction limit for a finite $N$, $\alpha_c(N)$, was assessed as $\alpha_c(N) = P_c/N$, where $\overline{P}$ denotes the arithmetic average over the $10^5$ trials. Finally, the critical value of $N \rightarrow \infty$ was evaluated by using the quadratic fitting with respect to $N^{-1}$ to $\alpha_c(N)$.

The results are summarized in fig. 3, where the dependence of $\alpha_c(N)$ on the signal length $N$ is depicted for $r = 0$ and $r = 0.5$ with $\rho = 0.5$. A decrease of the reconstruction limit $\alpha_c$ (or improvement of reconstruction performance) for a larger $r$ is observed as expected from the replica analysis. In order to compare this with the reconstruction limit from the replica analysis, we also performed

Fig. 2: Reconstruction limit $\alpha_c$ for signal from SAR(1) as a function of $\rho$ with $r = 0.5$ (top) or $r$ with $\rho = 0.5$ (bottom). In the top figure, the dependence on $\rho$ (cross) is almost the same as the basic setting examined in [4] (dotted curve).

In the evaluation of the reconstruction limit $\alpha_c$, multiple integrals in the first and the second equations in (14) should be performed. This can be done in practice by using a Monte Carlo method. Particularly in the current case, this scheme works very efficiently because the subroutine for determining $\tilde{x}(\sqrt{\chi\tilde{z}})$, which is expressed as the third equation, can be carried out by using only the $O(N)$ computational cost for a given pair of $x^0$ and $\tilde{z}$ with making use of the belief propagation (equivalently, transfer matrix method or dynamic programming) [19].

Monte Carlo assessment of $\alpha_c$ and experimental validation. We evaluated the reconstruction limit $\alpha_c$ by iteratively solving (14). For numerical stability, we solved the equation by converting the coordinates of the variables as $x' = S^{-1}x$. We set the length of $x^0$ to $N = 2 \times 10^3$ and took $10^3$ (fig. 2) or $10^4$ (fig. 3) sample averages for the numerical evaluation of $\alpha_c$. Making $N$ much larger is practically difficult due to the slow convergence of the iteration under the sample fluctuations. However, we judged that the signal length of $N = 2 \times 10^3$ was large enough for the evaluation of $\alpha_c$ because the change in $\alpha_c$ evaluated for $N = 1 \times 10^3$ was smaller than the value of the typical sample fluctuations.

The reconstruction limit as a function of $\rho$ and $r$ is depicted in fig. 2. For a fixed $r$ (top panel), $\alpha_c$ behaves as a convex upward function of $\rho$ similar to that for the case of the basic setting (dotted curve) [4]. When comparing this with the results from the basic setting of the i.i.d. sparse signals in [4], where $\alpha_c = 0.8312 \ldots$ is evaluated for $\rho = 0.5$, the value of the reconstruction limit for $r = 0$, which corresponds to cases where there were no time correlations except for the pausing, is larger (bottom panel). This implies that the reconstruction limit does depend on the types of sparsity and that the sparsity of the signal differences is not as useful as that for the signals themselves in reducing the data size. With regard to the autoregression parameter $r$, a decrease in $\alpha_c$, or the equivalent improvement of the reconstruction performance is observed as $r$ is increased (bottom panel). This is plausible because the correlations generally decrease the information quantity of the signals, which in principle makes it possible to reduce the data size.

Fig. 3: Reconstruction limit $\alpha_c$ for SAR(1) of finite dimension $N$ from reconstruction experiment. The results for $r = 0$ (+) and $r = 0.5$ (×) when $\rho = 0.5$ are shown. The curves indicate the result of the quadratic function regression for the dependence of (inverse of) input signal dimension $N$ (solid for $r = 0$ and broken for $r = 0.5$). The two horizontal lines indicate the results of a replica analysis for $r = 0$ (solid) and $r = 0.5$ (broken), respectively.
a scaling analysis using a quadratic function regression and extrapolated the result to $N \to \infty$, which gives $\alpha_c = 0.8485(3)$ for $r = 0$ and $\alpha_c = 0.8406(3)$ for $r = 0.5$. The reconstruction limits for $N \to \infty$ from the extrapolation are reasonably close to the values from the replica analysis ($\alpha_c = 0.8491(2)$ for $r = 0$ and $\alpha_c = 0.8412(1)$ for $r = 0.5$ respectively), considering possible biases which come out due to influences of higher-order terms of $N^{-1}$ in the data fitting, which validates our analysis based on the statistical mechanical scheme.

Summary and discussion. – In summary, we have developed a scheme to assess the typical reconstruction limit of compressed sensing problems that are defined by the generic signal sources and cost functions under the assumption of random measurements. Although the scheme is computationally difficult in general, it is still of practical utility when the source distribution is computationally feasible and the cost function is convex downward. As an example for showing the utility, we have taken up the problem of sparse autoregression and have examined how $\alpha_c$ depends on two system parameters that specify the autoregression process. Our investigation has indicated that the sparsity of the signal differences between successive times is not as useful as that of the signals themselves for compressing the data size.

In earlier studies [4,11,12], the universality of $\alpha_c$ has been observed for i.i.d. sparse sources as long as the cross correlation matrix $(\mathbf{F})^T \mathbf{F}$ of the random compression matrix $\mathbf{F}$ asymptotically obeys a rotationally invariant ensemble. The problem of the sparse autoregression of vanishing correlation parameter ($r = 0$) can be cast to the cases of the i.i.d. sources in which the $(\mathbf{F})^T \mathbf{F}$ ensemble is not asymptotically rotationally invariant. Our result indicates that applying the theoretical results obtained for random compression matrices and i.i.d. sources to realistic problems requires a certain care because either/both $\mathbf{F}$ or/and the original signals can contain non-negligible correlations in most real-world problems.

Exploring a more realistic time series modeled by SAR($k$) ($k \geq 2$), two-dimensional signals (images) is included in our future plan. Besides, compressed sensing with noise is also significant for application. Its performance can be analyzed by the generalization of our formalism, which is also a promising future work.

***

Support by KAKENHI Nos. 22300003, 22300098, The Mitsubishi Foundation and the JSPS GCOE “CompView” is acknowledged (YK).

REFERENCES

[1] Candès E. J., Romberg J. and Tao T., IEEE Trans. Inf. Theory, 52 (2006) 489.
[2] Donoho D. L., IEEE Trans. Inf. Theory, 52 (2006) 1289.
[3] Candès E. J. and Tao T., IEEE Trans. Inf. Theory, 52 (2006) 5406.
[4] Kabashima Y., Wadayama T. and Tanaka T., J. Stat. Mech. (2009) L09003.
[5] Donoho D. L., Maleki A. and Montanari A., Proc. Natl. Acad. Sci. U.S.A., 106 (2009) 18914.
[6] Rangan S., Fletcher A. K. and Goyal V. K., in Proceedings of Advances in Neural Information Systems (NIPS), edited by Bengio Y. et al., Vol. 22 (NIPS Foundation) 2009, p. 1545.
[7] Ganguli S. and Sompolinsky H., Phys. Rev. Lett., 104 (2010) 188701.
[8] Bayati M. and Montanari A., Proceedings of the International Symposium on Information Theory (ISIT) (IEEE) 2010, p. 1528.
[9] Takeda K. and Kabashima Y., Proceedings of the International Symposium on Information Theory (ISIT) (IEEE) 2010, p. 1538.
[10] Tanaka T. and Raymond J., Proceedings of the International Symposium on Information Theory (ISIT) (IEEE) 2010, p. 1598.
[11] Donoho D. L. and Tanner J., Philos. Trans. R. Soc. London, Ser. A, 367 (2009) 4273.
[12] Donoho D. L. and Tanner J., Discrete Comput. Geom., 43 (2010) 522.
[13] Rudin L. I., Osher S. and Fatemi E., Physica D, 60 (1992) 259.
[14] Takeda K., Uda S. and Kabashima Y., Europhys. Lett., 76 (2006) 1193.
[15] Hatibu H., Takeda K. and Kabashima Y., Phys. Rev. E, 80 (2009) 061124.
[16] Guo D. and Verdú S., IEEE Trans. Inf. Theory, 51 (2005) 1983.
[17] Grant M. C. and Boyd S. P., CVX: Matlab software for disciplined convex programming (web page and software), 2009, http://stanford.edu/~boyd/cvx.
[18] Grant M. C. and Boyd S. P., in Recent Advances in Learning and Control, edited by Blondel V. D., Boyd S. P. and Kimura H. (Springer-Verlag, London) 2008, p. 95.
[19] Pearl J., Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference (Morgan Kaufmann, San Francisco) 1988.
[20] Saligrama V. and Zhao M., Proc. SPIE, 7446 (2009) 744609.
[21] Le N. D., Martin R. D. and Raftery A. E., J. Am. Stat. Assoc., 91 (1996) 1504.
[22] Wong C. S. and Li W. K., J. R. Stat. Soc. B, 62 (2000) 95.
[23] Cemgil A. T., Kappen H. J. and Barber D., IEEE Trans. Audio Speech Lang. Process., 14 (2006) 679.
[24] Takahashi H., Horibe N., Shimada M. and Ikegami T., J. Phys. Soc. Jpn., 77 (2008) 084802.
[25] Sazuka N., Ohira T., Marumo K., Shimizu T., Takayasu M. and Takayasu H., Physica A, 324 (2003) 366.
[26] Derrida B. and Hilhorst H., J. Phys. A, 16 (1983) 2641.
[27] Weigt M. and Monasson R., Europhys. Lett., 36 (1996) 209.
[28] de Almeida J. R. L. and Thouless D. J., J. Phys. A, 11 (1978) 983.
[29] Kabashima Y., J. Phys. A, 36 (2003) 11111.