Orthogonal and symplectic Yangians - representations of the quadratic evaluation

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Orthogonal and symplectic Yangians - representations of the quadratic evaluation

D Karakhanyan\textsuperscript{1} and R Kirschner\textsuperscript{2}

\textsuperscript{1} Yerevan Physics Institute, 2 Alikhanyan br., 0036 Yerevan, Armenia
\textsuperscript{2} Institut für Theoretische Physik, Universität Leipzig, 04009 Leipzig, Germany

E-mail: roland.kirschner@itp.uni-leipzig.de

Abstract. Orthogonal or symplectic Yangians are defined by the Yang-Baxter $RLL$ relation involving the fundamental $R$ matrix with $so(n)$ or $sp(2m)$ symmetry. The conditions on the evaluation of the second order are investigated with respect to the restrictions implied on the representation weights.

1. Orthogonal and symplectic Yangians

Yang-Baxter relations are known as the basis of the treatment of integrable quantum systems\cite{1, 2, 3, 4, 10, 11, 5}. The formulation of the Yangian algebra of type $G$ can be based on the fundamental Yang-Baxter $R$ matrix. In the case of symmetry with respect to the Lie algebra $G = so(n)$ or $G = sp(n)$, $n$ even, it has the form \cite{5, 6, 8, 9}

$$R^{a_1a_2}_{b_1b_2}(u) = u(u + \beta)R^{a_1a_2}_{b_1b_2} + (u + \beta)P^{a_1a_2}_{b_1b_2} - \epsilon u K^{a_1a_2}_{b_1b_2},$$

(1)

where

$$I^{a_1a_2}_{b_1b_2} = \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}, \quad P^{a_1a_2}_{b_1b_2} = \delta^{a_1}_{b_2}\delta^{a_2}_{b_1}, \quad K^{a_1a_2}_{b_1b_2} = \epsilon^{a_1a_2}\epsilon^{b_1b_2}, \quad \beta = \frac{n}{2} - \epsilon.$$

(2)

Here $\epsilon_{ab}$ is a non-degenerate invariant metric in the space $V$ of the fundamental representation. The sign factor $\epsilon$ distinguishes the orthogonal ($\epsilon = +1$) and the symplectic ($\epsilon = -1$) cases.

The generators $(L^{(k)})^{a}_{b}$ of the extended Yangian algebra $\mathcal{Y}(G)$ appear in the expansion of the $L$ operator

$$L^{a}_{b}(u) = \sum_{k=0}^{\infty} \frac{(L^{(k)})^{a}_{b}}{u^k}, \quad L^{(0)} = I,$$

(3)

which satisfies the Yang-Baxter $RLL$-relations

$$P^{a_1a_2}_{b_1b_2}(u-v)L^{a_1}_{c_1}(u)L^{a_2}_{c_2}(v) = L^{a_2}_{c_2}(v)L^{a_1}_{c_1}(u)P^{a_1a_2}_{b_1b_2}(u-v).$$

(4)

$L(u)$ is an algebra valued matrix depending on the spectral parameter $u$ and plays the role of the generating function of the Yangian algebra generators, $L \in EndV \otimes \mathcal{Y}$. Factorizing central elements leads to the Yangian algebra definition equivalent to the original definitions by Drinfeld \cite{10, 11}. The Yangians of orthogonal and symplectic types have been considered in \cite{16}.
and their algebraic structure and representation theory have been considered in [17]. Proofs of the equivalence are given in [18, 19].

The truncation of the expansion (3) results in constraints, restricting the Lie algebra representations. In the case of quadratic evaluation $L$ can be written as

$$L(u) = u^2 I + uG + H.$$ 

The matrix elements of $G$ are the generators obeying the Lie algebra commutation relations. In general, the third term $H$ contains further generators. The RLL relations imply a number of constraints relating $G$ and $H$ [14].

In any case the involved algebra-valued matrices can be decomposed into a trace contribution (proportional to $I$), a $\varepsilon$-antisymmetric (marked by overbar) and a $\varepsilon$-symmetric traceless parts.

In this respect we have

$$G = gI + \bar{G}, \quad H = kI + \bar{H} + \frac{1}{2}(\bar{G}^2 + \beta\bar{G}).$$

The vanishing of the $\varepsilon$-symmetric traceless part in $G$ and the particular form of the $\varepsilon$-symmetric traceless part in $H$ follows from the RLL relation (4). Further (4) implies that $g, k$ are central, the matrix $\bar{G}$ involves the generators obeying the Lie algebra relations,

$$[\bar{G}_1 + P_{12} - \varepsilon K_{12}, \bar{G}_2] = 0$$

$$[\bar{G}_{ab}, \bar{G}_{cd}] = -\varepsilon_{cb}\bar{G}_{ad} + \varepsilon_{ad}\bar{G}_{cb} + \varepsilon_{ac}\bar{G}_{bd} - \varepsilon_{db}\bar{G}_{ca}. \quad (5)$$

The latter extends to the $m$-th power of $\bar{G}$, $\bar{G}^m_{ab} = (\bar{G}^m)_{ab}$, as

$$[\bar{G}_{ab}, \bar{G}^m_{cd}] = -\varepsilon_{cb}\bar{G}^m_{ad} + \varepsilon_{ad}\bar{G}^m_{cb} + \varepsilon_{ac}\bar{G}^m_{bd} - \varepsilon_{db}\bar{G}^m_{ca}. \quad (6)$$

The commutation relations of the adjoint action hold also for the further generators involved in the matrix $\bar{H}$

$$[\bar{G}_1 + P_{12} - \varepsilon K_{12}, \bar{H}_2] = 0$$

$$[\bar{G}_{ab}, \bar{H}_{cd}] = -\varepsilon_{cb}\bar{H}_{ad} + \varepsilon_{ad}\bar{H}_{cb} + \varepsilon_{ac}\bar{H}_{bd} - \varepsilon_{db}\bar{H}_{ca}. \quad (7)$$

In [14] the relations of the quadratic evaluation of the Yangian of orthogonal or symplectic type have been derived from the RLL relation (4) in terms of 8 constraints. That work was a continuation of the studies in [12, 13].

We shall refer to the constraint labels and use the notations of [14]. The algebra relations are completed by the commutation relation of the generators in $\bar{H}$ (related to the 7th constraint),

$$[\bar{H}_1, \bar{H}_2] + \frac{1}{16}\{\bar{G}^2_1 + \bar{G}^2_2, [P_{12} - \varepsilon K_{12}, \bar{G}_1 - \bar{G}_2]\} +$$

$$+ \frac{1}{4}[P_{12} - \varepsilon K_{12}, (2h + \frac{\beta^2}{2})(\bar{G}_1 - \bar{G}_2) - 2g(\bar{H}_1 - \bar{H}_2) + \beta(\bar{G}^2_1 - \bar{G}^2_2)] = 0 \quad (8)$$

and the constraint on the products of $\bar{G}$ and $\bar{H}$ (related to the 6th constraint)

$$[\bar{H}, \bar{G}] + 2\beta\bar{H} - g(\bar{G}^2 + \beta\bar{G}) = c^{(2,6)}I, \quad (9)$$

and the constraint on the square of $\bar{H}$,

$$\bar{H}^2 = c^{(2,8)}I + \frac{1}{4}\bar{G}^4 - g\beta\bar{H} + \beta\bar{G}^3 + (\frac{5}{4}\beta^2 + \beta g + h)\bar{G}^2 + (\frac{\beta^3}{2} + \frac{3\beta^2 g}{2} + 2h\beta)\bar{G}. \quad (10)$$
The eigenvalues $c^{(2,6)}$ and $c^{(2,8)}$ are central. The last relation is related to the 8th constraint. We notice that actually its $\varepsilon$-symmetric part follows from that constraint, but its $\varepsilon$-antisymmetric part has been derived from the 7th constraint. The relation of the 6th constraint is $\varepsilon$-symmetric.

We have considered in [15] the conditions on the representation weights in the more restricted cases of the linear evaluation and the Lie algebra resolution with the particular examples of spinorial and Jordan-Schwinger type representations.

In this contribution we study the implication of the quadratic evaluation Yangian algebra relations of the representation weights in general.

2. Highest weight representations

We separate in the extended Yangian algebra the Cartan and Borel subalgebras. For this aim it is convenient to change the index range and to specify the metric

$$\varepsilon_{ab} = \varepsilon_a \delta_{a,-b}, \quad a, b = -\frac{n}{2}, \ldots, -1, +1, \ldots \frac{n}{2},$$

$$\varepsilon_a = 1$$ in the case $so(n)$ and $\varepsilon_a = sgn(a)$ in the case $sp(n)$. We restrict ourselves to even $n$. $\frac{n}{2}$ is the rank of the Lie algebra. We use the index notations $i, j, k = 1, \ldots, \frac{n}{2}$ for the positive index range.

The highest weight vector of a Yangian algebra representation obeys

$$L(u)_{-ij}|0\rangle = 0, \quad (i < j), \quad L(u)_{-i,-j}|0\rangle = 0, \quad L(u)_{-i,i}|0\rangle = h_i(u)|0\rangle.$$ 

In the quadratic evaluation case this means in particular

$$\tilde{G}_{-ij}|0\rangle = 0, \quad (i < j), \quad \tilde{G}_{-i,-j}|0\rangle = 0, \quad \tilde{G}_{-i,i}|0\rangle = h_i|0\rangle,$$

$$\tilde{H}_{-ij}|0\rangle = 0, \quad (i < j), \quad \tilde{H}_{-i,-j}|0\rangle = 0, \quad \tilde{H}_{-i,i}|0\rangle = \tilde{h}_i|0\rangle,$$

and the action of the central elements $g, k$ on $|0\rangle$ results into numbers which we denote by the same letters for simplicity.

It turns out that any power of the matrix $\tilde{G}$ obeys analogous relations,

$$G^m_{-ij}|0\rangle = 0, \quad (i < j), \quad G^m_{-i,-j}|0\rangle = 0, \quad (i < j), \quad G^m_{-i,i}|0\rangle = 0,$$

$$G^m_{i,-i}|0\rangle = h^{(m)}_i|0\rangle, \quad G^m_{i,-i}|0\rangle = h^{(m)}_{-i}|0\rangle.$$ 

The eigenvalues $h^{(m)}_i$ are function of the weights $h_i$ and can be calculated iteratively using the relations

$$h^{(m+1)}_i = h^{(m)}_i (eh_i - 2\beta + i - \epsilon) - (1 - \epsilon)h^{(m)}_{-i} + \sum_{k<i} eh^{(m)}_{-k} + \sum_{k>i} eh^{(m)}_{-k} + h^{(m)}_{+k}.$$

$$h^{(m+1)}_{-i} = -eh^{(m)}_{-i} h_i - (i - 1)h^{(m)}_{-i} + \sum_{k<i} h^{(m)}_{-k}.$$

We have the relations for the Yangian weight functions

$$h_i(u) = u^2 \left( 1 + \frac{h_i + g}{u} + \frac{1}{2u^2} (2k + 2\tilde{h}_i + h^{(2)}_{+i} + \beta h_i) \right).$$
3. Conditions from the 6th constraint

We derive the consequences of the 6th constraint (9) on the weights. To this end we consider the action of the matrix elements of \( \{ \bar{H}, \bar{G} \}_{-ij} \).

\[
\{ \bar{H}, \bar{G} \}_{-ij} = \\
\sum_{k<i} \left( \bar{H}_{-i-k} \bar{G}_{k-j} + \epsilon \bar{H}_{-i-k} \bar{G}_{-k-j} + \bar{G}_{-i-k} \bar{H}_{k-j} + \epsilon \bar{G}_{-i-k} \bar{H}_{-k-j} \right) + \\
\bar{H}_{-i-j} \bar{G}_{ij} + \epsilon \bar{H}_{-i-j} \bar{G}_{-ij} + \bar{G}_{-i-j} \bar{H}_{ij} + \epsilon \bar{G}_{-i-j} \bar{H}_{-ij} + \\
\sum_{k>i} \left( \bar{H}_{-i-k} \bar{G}_{k-j} + \epsilon \bar{H}_{-i-k} \bar{G}_{-k-j} + \bar{G}_{-i-k} \bar{H}_{k-j} + \epsilon \bar{G}_{-i-k} \bar{H}_{-k-j} \right).
\]

The action on the highest weight vector for \( i < j \) is

\[
\{ \bar{H}, \bar{G} \}_{-ij} |0 \rangle = \sum_{k<i} \left( [ \bar{H}_{-i-k}, \bar{G}_{k-j} ] + [ \bar{G}_{-i-k}, \bar{H}_{k-j} ] \right) |0 \rangle + \\
\left( [ \bar{H}_{-i-j}, \bar{G}_{ij} ] + [ \bar{G}_{-i-j}, \bar{H}_{ij} ] \right) |0 \rangle + \\
\sum_{k<i} \left( [ \bar{H}_{-i-k}, \bar{G}_{k-j} ] + [ \bar{G}_{-i-k}, \bar{H}_{k-j} ] + \epsilon [ \bar{H}_{-i-k}, \bar{G}_{-k-j} ] + \epsilon [ \bar{G}_{-i-k}, \bar{H}_{-k-j} ] \right) |0 \rangle
\]

We have to calculate the commutators by using (7) All of them turn out to be proportional to \( \bar{H}_{-ij} \). This implies that the matrix element \( (-i, j) \), \( i < j \) of the 6th constraint annihilates \( |0 \rangle \). The same holds for the matrix elements \( (-i, -j) \). In general, the matrix elements of the constraints corresponding to the Borel subalgebra of the lowering generators annihilate the highest weight vector and do not imply conditions on the representation weights. The conditions on the weights follow from the matrix elements corresponding to the Cartan subalgebra of generators, \( (-i, i) \) and \( (i, -i) \).

The considered constraint (9) is \( \epsilon \)-symmetric. Therefore it is sufficient to consider the matrix elements \( (i, -i) \).

\[
\{ \bar{H}, \bar{G} \}_{i-i} = \\
\sum_{k<i} \left( \bar{H}_{i-k} \bar{G}_{k-i} + \epsilon \bar{H}_{i-k} \bar{G}_{-k-i} + \bar{G}_{i-k} \bar{H}_{k-i} + \epsilon \bar{G}_{i-k} \bar{H}_{-k-i} \right) + \\
\left( \bar{H}_{i-i} \bar{G}_{i-i} + \epsilon \bar{H}_{i-i} \bar{G}_{-i-i} + \bar{G}_{i-i} \bar{H}_{i-i} + \epsilon \bar{G}_{i-i} \bar{H}_{-i-i} \right) + \\
\sum_{k>i} \left( \bar{H}_{i-k} \bar{G}_{k-i} + \epsilon \bar{H}_{i-k} \bar{G}_{-k-i} + \bar{G}_{i-k} \bar{H}_{k-i} + \epsilon \bar{G}_{i-k} \bar{H}_{-k-i} \right).
\]

The action on the highest weight vector results in a few non-vanishing terms only,

\[
2\bar{h}_i h_i + \sum_{k<i} \left( [ \bar{H}_{i-k}, \bar{G}_{k-i} ] + [ \bar{G}_{i-k}, \bar{H}_{k-i} ] \right) |0 \rangle.
\]

We calculate the commutators using (7).

\[
[ \bar{G}_{k-i}, \bar{H}_{i-k} ] = -\epsilon \bar{H}_{-i-k} + \epsilon \bar{H}_{-k} \bar{H}_{i-k}, \quad [ \bar{G}_{i-k}, \bar{H}_{k-i} ] = \epsilon \bar{H}_{-i-i} - \epsilon \bar{H}_{-k-k}
\]
and obtain

\[ 2\bar{h}_i h_i + 2\epsilon (i - 1)\bar{h}_i - 2\epsilon \sum_{k=1}^{i-1} \bar{h}_k. \]

Adding the remaining contribution to the 6th constraint we obtain the conditions on the weights

\[ 2\bar{h}_i h_i + 2\epsilon (i - 1)\bar{h}_i - 2\epsilon \sum_{k=1}^{i-1} \bar{h}_k - 2\beta \epsilon \bar{h}_i - g(h^{(2)}_{\ell i} - \beta \epsilon h_i) - \epsilon^{(2,6)} = 0. \]

We calculate \( h^{(2)}_{\ell i} \) by (14) and obtain the final form of the conditions on the weights implied by the 6th constraint

\[ (2\bar{h}_i - gh_i)(i - 1 + \epsilon h_i) - 2\beta \bar{h}_i - \sum_{\ell < i} (2\bar{h}_\ell - gh_\ell) - \epsilon^{(2,6)} = 0 \quad (15) \]

4. The \( \bar{H} \) commutators

The commutators of the generators are important for deriving the conditions on the weights. In the above case of the 6th constraint the commutators \([\bar{G}_1, \bar{G}_2]\) and \([\bar{G}_1, \bar{H}_2]\) were sufficient. In the case of the 8th constraint the \([\bar{H}_1, \bar{H}_2]\), the 7th constraint, is needed. As a preparation we rewrite (8) in index form analogous to (5, 7).

Indeed,

\[ [G_1, J_2] = -[P - \epsilon K, J_2] \]

has the index form

\[ [G_{ab}, J_{cd}] = -J_{ad}\epsilon_{cb} + J_{cb}\epsilon_{ad} + \epsilon_{ac}J_{bd} - J_{ca}\epsilon_{db} \]

Therefore

\[ [P - \epsilon K, J_1 - J_2]_{abcd} = 2\epsilon_{ad}J_{cb} - 2\epsilon_{cb}J_{ad} + \epsilon_{ac}(J_{bd} - \epsilon_{db}) - \epsilon_{db}(J_{ca} - \epsilon_{ac}) \quad (16) \]

We consider the second term in (8)

\[ \frac{1}{2}(\bar{G}_1^2 + \bar{G}_2^2, [P_1 - \epsilon K_1, \bar{G}_1 - \bar{G}_2]) = [P - \epsilon K, (G_1^3 - G_2^3)] + [G_1, G_1(P - \epsilon K)G_1] - [G_2, G_2(P - \epsilon K)G_2] \]

For the second term on r.h.s we obtain

\[ [\bar{G}_1, G_1(P - \epsilon K)\bar{G}_1] = \bar{G}_1^2 \bar{G}_2 P - \bar{G}_1 \bar{G}_2^2 P - \epsilon \bar{G}_1^2 K \bar{G}_1 + \epsilon \bar{G}_1 K \bar{G}_1^2 \]

\[ [\bar{G}_1, G_1(P - \epsilon K)\bar{G}_1]_{abcd} = \bar{G}_1^2 \bar{G}_2 G_{cb} - \bar{G}_1 \bar{G}_2^2 G_{cb} + \epsilon \bar{G}_1 K \bar{G}_1^2 G_{cb} \]

The third term is obtained by the 1 ↔ 2 parity, or its index form by interchanging \( a \leftrightarrow c, b \leftrightarrow d \).

As a result we obtain the index form of the 7th constraint (8) as

\[ [\bar{H}_{ab}, \bar{H}_{cd}] + \frac{1}{8}(\bar{G}_{ad}^2 \bar{G}_{cb} - \bar{G}_{ad} \bar{G}_{cb}^2 + \bar{G}_{ac}^2 \bar{G}_{bd} - \bar{G}_{ca} \bar{G}_{db}^2) \quad (17) \]

\[ -\bar{G}_{cb} \bar{G}_{ad} + \bar{G}_{db} \bar{G}_{ad}^2 - \bar{G}_{ca} \bar{G}_{db} \bar{G}_{ac} + \bar{G}_{ca} \bar{G}_{db} \bar{G}_{ac} + [P - \epsilon K, J_1 - J_2]_{abcd} = 0 \]

where in the last term on l.h.s \( J \) is to be substituted by

\[ 4J = \frac{1}{2} G^3 + (2h + \frac{1}{2}\beta^2)\bar{G} + \beta\bar{G}^2 - 2g\bar{H}, \quad (18) \]

and (16) is to be applied.
5. Conditions from the 8th constraint
We write the constraint (10) in the form

\[ H^2 + \beta g H = Q, \quad Q = c^{(2,8)} I + \frac{1}{4} G^2 + \beta \bar{G}^2 + \left( \frac{5}{4} \beta^2 + \frac{\beta g}{2} + k \right) G^2 + \left( \frac{\beta^3}{2} + \frac{\beta^2 g}{2} + 2 k \beta \right) G \]

The matrix elements \((-i, j), i < j\), and \((-i, -j)\) of the 8th constraint expression annihilate the highest weight vector \(|0\rangle\) in analogy to the previous case. Consider the matrix element \((i, -i)\). For the r.h.s. we have

\[ Q_{i-i} = c^{(2,8)} + \frac{1}{4} h^{(4)}_{-i} + \beta h^{(3)}_{-i} + \left( \frac{5}{4} \beta^2 + \frac{\beta g}{2} + k \right) \beta h^{(2)}_{-i} - \epsilon \left( \frac{\beta^3}{2} + \frac{\beta^2 g}{2} + 2 k \beta \right) h_i \]

\(h^{(m)}_{-i}\) are to be calculated by (14) in terms of the weights \(h_k, k = 1, \ldots, \frac{n}{2}\).

We write \(\bar{H}^2\) in terms of the matrix elements of \(\bar{H}\),

\[ \bar{H}^2_{i-i} = \sum_{k<i} \bar{H}_{i-k} \bar{H}_{k-i} + \epsilon \bar{H}_{i-k} \bar{H}_{-k-i} \]

\[ \bar{H}_{i-i} \bar{H}_{i-i} + \epsilon \bar{H}_{i-k} \bar{H}_{-k-i} + \sum_{k>i} \bar{H}_{i-k} \bar{H}_{k-i} + \epsilon \bar{H}_{i-k} \bar{H}_{-k-i} \]

and act on the highest weight vector

\[ \bar{H}^2_{i-i} |0\rangle = \bar{h}_i^2 + \sum_{k<i} |\bar{H}_{i-k} \bar{H}_{k-i}| |0\rangle \]

We have to consider the corresponding matrix element of the 7th constraint.

\[ [\bar{H}_{i-k}, \bar{H}_{k-i}] + \frac{1}{8} (G_{i-k}^2 G_{k-i} - G_{k-k}^2 G_{i-i} + G_{k-k} G_{i-i} - G_{i-i} G_{k-k}) + 2 (J_{k-i} - J_{i-k}) = 0, \]

\[ [\bar{H}_{i-k}, \bar{H}_{k-i}] |0\rangle \left|_{i<k} \right. = \frac{\epsilon}{4} (h^{(2)}_{-k} h_k - h^{(2)}_{k-i} h_i) - \frac{1}{4} (h^{(3)}_{-k} - h^{(3)}_{k-i}) - \frac{\beta}{2} (h^{(2)}_{-k} - h^{(2)}_{k-i}) + \frac{\epsilon}{2} (2 h + 1) (h_k - h_i) + \epsilon g (h_k - h_i) \]

The matrix element \(i - i\) of the 8th constraint implies the weight relations

\( \bar{h}_i^2 - \epsilon g \beta \bar{h}_i + \epsilon g \sum_{\ell<i} (h_{\ell} - \bar{h}_i) = -\frac{\epsilon}{4} \sum_{\ell<i} (h^{(2)}_{\ell} h_k - h^{(2)}_{i-k} h_{\ell}) + c^{(8,2)} + \frac{1}{4} h^{(4)}_{i-k} + \beta h^{(3)}_{i-k} \)

\[ \left( \frac{5}{4} \beta^2 + \frac{\beta g}{2} + k \right) h^{(2)}_{i-k} - \epsilon \left( \frac{\beta^3}{2} + \frac{\beta^2 g}{2} + 2 k \beta \right) h_i + \sum_{\ell<i} \left( \frac{1}{4} h^{(3)}_{i-k} + \frac{\beta}{2} h^{(2)}_{i-k} - \frac{1}{2} \epsilon (2 k + 1) \right) h_{\ell} \]

\[ - (i - 1) \left( \frac{1}{4} h^{(3)}_{i-k} + \frac{\beta}{2} h^{(2)}_{i-k} - \frac{1}{2} \epsilon (2 k + 1) \right) h_i \].

The iteration of (14) results in the explicit forms of \(h^{(r)}_{-i}, r = 2, 3, 4\) to be substituted above.
6. Distinguished Lie algebra representations

The subset of representation weights $h_i$ characterises a representation of the underlying Lie algebra. As in the restricted cases of the linear evaluation and the Lie algebra resolution of the quadratic evaluation studied earlier [15] only distinguished Lie algebra representations can be embedded in the a representation of the quadratic evaluation. Indeed the obtained conditions of the weights $h_i, \bar{h}_i$ imply that the values of $h_i$ are restricted.

We write the obtained weight conditions for $i = 1$. From the 6th constraint (15)

$$(2\bar{h}_1 - gh_1)(ch_1 - \beta) - c^{(2,6)} = 0$$

and from the 8th constraint (19)

$$\bar{h}_1^2 - \epsilon\beta\bar{h}_1 = c^{(2,8)} + \frac{1}{4}h_1^2 - \epsilon\beta h_1^4 + (\frac{5}{4}\beta^2 + \frac{\beta g}{2} + h)h_1^2 - \epsilon(\frac{\beta^3}{2} + \beta^2 g + 2h\beta)h_1 = 0$$

We can simplify the relations by imposing $g = 0$ without restriction of generality. Indeed to the equivalence transformation of the Yangian $L(u) \rightarrow \frac{u^2}{u}\bar{L}(u)$ allows to remove $g$. The last relation fixes $\bar{h}_1$ in terms of $h_1, c^{(2,8)}, k$ up to a sign. The first relation fixes $\bar{h}_1$ in terms of $h_1, c^{(2,6)}$ Fixing that sign relates $c^{(2,6)}, c^{(2,8)}, k$.

After $\bar{h}_1$ and these central element values are fixed the other $\bar{h}_i, i > 1$ can be calculated iteratively from the consequence of the 6th constraint (15),

$$\bar{h}_i(i - 1 + ch_i - \beta) = \bar{h}_{i-1}(i - 1 + ch_{i-1} - \beta)$$

Thus the sequence of second order weights $\bar{h}_i$ turns into functions of the first order weights $h_i$ and the central element values. Substituting these functions into the consequence of the 8th constraint (19) we obtain the restrictions on the first order weights and in this way on the distinguished Lie algebra representation which allow the embedding into the second order truncated Yangian representation.

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