Albanese Varieties of Singular Varieties over a Perfect Field

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Abstract

Let $X$ be a projective variety, possibly singular. A generalized Albanese variety of $X$ was constructed by Esnault, Srinivas and Viehweg for algebraically closed base field as a universal regular quotient of a relative Chow group of 0-cycles of degree 0 modulo rational equivalence. In this paper, we obtain a functorial description of the Albanese of Esnault-Srinivas-Viehweg over a perfect base field, using duality theory of 1-motives with unipotent part.

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0 Introduction

With a smooth projective variety $X$ over a field $k$ there is an associated Albanese variety $\text{Alb}(X)$, defined by a universal mapping property for rational maps from $X$ to abelian varieties. Equivalently, $\text{Alb}(X)$ can be considered as a universal quotient of the Chow group $\text{CH}_0(X)^0$ of 0-cycles of degree 0 on $X$ modulo rational equivalence. This second characterization was used by Esnault, Srinivas and Viehweg to define an Albanese variety $\text{Alb}(X, X_{\text{sing}})$ for possibly singular $X$ over an algebraically closed field $k$. Here $\text{CH}_0(X)$ was replaced by the relative Chow group of 0-cycles $\text{CH}_0(X, X_{\text{sing}})$ in the sense of Levine-Weibel [LW], which coincides with $\text{CH}_0(X)$ if $X_{\text{sing}} = \emptyset$. Then $\text{Alb}(X, X_{\text{sing}})$ is characterized by a universal mapping property for rational maps from $X$ to smooth connected commutative algebraic groups that factor through $\text{CH}_0(X, X_{\text{sing}})^0$.

The aim of this paper is to give a functorial description of the Albanese $\text{Alb}(X, X_{\text{sing}})$ of Esnault, Srinivas and Viehweg for possibly singular $X$ over a perfect field of arbitrary characteristic, see Theorems 0.1, 0.2 and 0.3. For this purpose we use the concept of generalized Albanese varieties over a perfect field and duality of 1-motives with unipotent part from [Ru2]. In this way we obtain a functor dual to $\text{Alb}(X, X_{\text{sing}})$ as well. The construction is explicit.

The case that the base field $k$ is of characteristic 0 and algebraically closed was done in [Ru1]. While the way of procedure appears analogous in parts, it was necessary to involve different methods, e.g. formal groups in positive characteristic behave distinctly from the characteristic 0 case and we cannot assume in general a resolution of singularities.

0.1 Leitfaden

We make use of the notions described in [Ru2], in particular categories of rational maps from a variety $X$ to algebraic groups [Ru2, Definition 2.7] and universal objects for such categories of rational maps [Ru2, Definition 2.11].

Let $X$ be a (singular) projective variety over an algebraically closed field $k$. The Albanese variety of Esnault-Srinivas-Viehweg $\text{Alb}(X, X_{\text{sing}})$ is by definition the universal object for the category $\text{Mr}^{\text{CH}}$ of rational maps from $X$ to smooth connected commutative algebraic groups factoring through $\text{CH}_0(X, X_{\text{sing}})^0$ (see Definition 3.1), therefore it is also called the universal regular quotient of $\text{CH}_0(X, X_{\text{sing}})^0$. Suppose $\pi: Y \to X$ is a projective resolution of singularities (we will later replace this by a morphism that is always available in any characteristic). We give a characterization of rational
maps factoring through $\chi^0_0(X, X_{\text{sing}})$ in terms of a formal subgroup of the functor $\text{Div}_Y$ of relative Cartier divisors on $Y$:

A rational map $\varphi : X \dashrightarrow G$ to a smooth connected commutative algebraic group induces a transformation $\tau_\varphi : L^\vee \to \text{Div}_Y$ (see [Ru2, Definition 2.4]), where $L$ is the affine part of $G$ and $L^\vee$ the Cartier dual. Let $\text{Mr}_{\text{Div}^0_{Y/X}}$ be the category of all those rational maps $\varphi$ from $Y$ to smooth connected commutative algebraic groups such that the image of the induced transformation $\tau_\varphi$ is contained in $\text{Div}^0_{Y/X}$, “the kernel of $\text{Div}_Y$ under push-forward from $Y$ to $X$”. The precise definition of $\text{Div}^0_{Y/X}$ (see Definitions 2.7 and 2.8) requires some technical issues: so-called formal divisors (see Definition 2.1) and restriction to Cartier curves. Then the category $\text{Mr}^\text{CH}$ is equal to $\text{Mr}_{\text{Div}^0_{Y/X}}$.

Using the result [Ru2, Theorem 2.12] about the existence of universal objects for categories of rational maps, we can re-prove the existence of the universal regular quotient $\text{Alb}(X, X_{\text{sing}})$ of $\chi^0_0(X, X_{\text{sing}})$ and obtain an explicit and functorial construction (cf. Section 5.1):

**Theorem 0.1.** Let $X$ be a projective variety over an algebraically closed field $k$, and let $\tilde{X} \to X$ be a projective resolution of singularities. Then the universal regular quotient $\text{Alb}(X, X_{\text{sing}})$ of $\chi^0_0(X, X_{\text{sing}})$ exists (as an algebraic group) and its dual (in the sense of 1-motives) represents the functor

$$\text{Div}^0_{X/X} \to \text{Pic}^0_{X}$$

that is the natural transformation of functors which assigns to a relative divisor the class of its associated line bundle. $\text{Pic}^0_{X}$ is represented by an abelian variety and $\text{Div}^0_{X/X}$ by a dual-algebraic formal group (i.e. a formal group whose Cartier dual is an algebraic affine group).

In the general case, i.e. if we do not assume a resolution of singularities, we can perform the same procedure, replacing resolution of singularities by a suitable blowing up of $X$. The construction is elementary, in particular it does not use alterations. We obtain (cf. Section 5.2):

**Theorem 0.2.** Let $X$ be a projective variety over an algebraically closed field $k$. Suppose $\pi : Y \to X$ is a birational projective morphism with the property that every rational map from $X$ to an abelian variety $A$ extends to a morphism from $Y$ to $A$, and $\text{Pic}^0_Y$ is an abelian variety. Then the functor $[\text{Div}^0_{Y/X} \to \text{Pic}^0_Y]$ is dual (in the sense of 1-motives) to the universal
regular quotient $\text{Alb}(X, X_{\text{sing}})$ of $\text{CH}_0(X, X_{\text{sing}})^0$. A morphism $\pi : Y \to X$ with the required properties does always exist and can be constructed as a blowing up of an ideal sheaf on $X$. In particular, $\text{Alb}(X, X_{\text{sing}})$ exists (as an algebraic group).

A descent of the base field yields (cf. Theorem 5.6):

**Theorem 0.3.** Let $X$ be a projective irreducible variety defined over a perfect field $k$. Let $\overline{k}$ be an algebraic closure of $k$. Then $\text{Alb} (X \otimes_k \overline{k}, (X \otimes_k \overline{k})_{\text{sing}})$ is defined over $k$.

Some hint for the reader: as the notations from [Ru2] are used throughout this paper, it might be helpful to know that there is a Glossary of Notations on page 24.

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1 Prerequisites

Throughout this note the following assumptions hold (unless stated otherwise). $k$ is an algebraically closed field (of arbitrary characteristic), except in No. 5.3 where $k$ is only assumed to be perfect. $X$ is a projective variety over $k$ and $\pi : Y \to X$ a resolution of singularities, i.e. $Y$ is regular and $\pi$ is a projective birational morphism which is an isomorphism over the regular points of $X$. In Nos. 5.2 and 5.3 we do not assume the existence of a resolution of singularities. $U \subset X_{\text{reg}}$ is an open dense subset of $X$, contained in the regular locus of $X$. We consider the category $\text{Mr}_{\text{CH}}$ of morphisms $\varphi : U \to G$ from $U$ to algebraic groups $G$ factoring through $\text{CH}_0(X, X_{\text{sing}})^0$ (see Definition 3.1). Here we assume algebraic groups $G$ always to be smooth, connected and commutative, unless stated otherwise.

1.1 Chow Group of Points on Singular Varieties

We recall the relative Chow group $\text{CH}_0(X, X_{\text{sing}})^0$ of 0-cycles of degree 0 modulo rational equivalence from [LW].

**Definition 1.1.** Let $S$ be a closed proper subset of $X$.

A Cartier curve in $X$ relative to $S$ is a curve $C \subset X$ satisfying

(a) $C$ is pure of dimension 1.

(b) No component of $C$ is contained in $S$.

(c) If $p \in S$ is a point of $C \cap S$, then the ideal of $C$ in $O_{X,p}$ is generated by a regular sequence (consisting of $\dim X - 1$ elements).

**Definition 1.2.** Let $C$ be a Cartier curve in $X$ relative to $X \setminus U$. Let $\gamma_1, \ldots, \gamma_r$ be the generic points of $C$. Let $O_{C,\Theta}$ be the semilocal ring on $C$ at the points in $\Theta = (C \setminus U) \cup (\gamma_1, \ldots, \gamma_r)$. Then define

$$K(C, U)^* = O^*_{C,\Theta}.$$

**Definition 1.3.** Let $C$ be a Cartier curve in $X$ relative to $X \setminus U$ and let $\nu : \tilde{C} \to C$ be its normalization. For $f \in K(C, U)^*$ and $p \in C(k)$ let

$$\text{ord}_p(f) = \sum_{\tilde{p} \to p} v_{\tilde{p}}(\tilde{f})$$

where $\tilde{f} = \nu^* f$ and $v_{\tilde{p}}$ is the discrete valuation attached to the point $\tilde{p} \in \tilde{C}(k)$ above $p \in C(k)$ (cf. [Fu, Exm. A.3.1]). Define the divisor of $f$ to be

$$\text{div}(f)_C = \sum_{p \in C} \text{ord}_p(f) [p].$$
Immediately from Definition 1.3 we get

Lemma 1.4. Let \( C \) be a Cartier curve in \( X \) relative to \( X \setminus U \) with normalization \( \nu : \tilde{C} \rightarrow C \). If \( \varphi : C \cap U \rightarrow G \) is a morphism from \( C \cap U \) to an algebraic group \( G \), then for each \( f \in K(C,U)^* \) it holds (with \( \varphi(\sum n_i p_i) := \sum n_i \varphi(p_i) \))

\[
\varphi(\text{div}(f)|_C) = (\varphi \circ \nu)(\text{div}(f \circ \nu)|_{\tilde{C}})
\]

Definition 1.5. Let \( Z_0(U) \) be the group of 0-cycles on \( U \), set

\[
\mathcal{R}_0(X,U) = \left\{(C,f) \mid C \text{ is a Cartier curve in } X \text{ relative to } X \setminus U \text{ and } f \in K(C,U)^* \right\}
\]

and let \( R_0(X,U) \) be the subgroup of \( Z_0(U) \) generated by elements \( \text{div}(f)|_C \) with \( (C,f) \in \mathcal{R}_0(X,U) \). Then define

\[
\text{CH}_0(X,X_{\text{sing}}) = Z_0(U)/R_0(X,U).
\]

Let \( \text{CH}_0(X,X_{\text{sing}})^0 \) be the subgroup of \( \text{CH}_0(X,X_{\text{sing}}) \) of cycles \( \zeta \) with \( \deg \zeta|_W = 0 \) for all irreducible components \( W \) of \( U \).

Remark 1.6. The definition of \( \text{CH}_0(X,X_{\text{sing}}) \) and \( \text{CH}_0(X,X_{\text{sing}})^0 \) is independent of the choice of the dense open subscheme \( U \subset X_{\text{reg}} \) (see [ESV, Corollary 1.4]).

Remark 1.7. By our terminology a curve is always reduced, in particular a Cartier curve. Allowing non-reduced Cartier curves (which is the common definition) does not change the groups \( \text{CH}_0(X,X_{\text{sing}}) \) and \( \text{CH}_0(X,X_{\text{sing}})^0 \), see [ESV, Lemma 1.3] for more explanation.

Remark 1.8. The definition of \( \text{CH}_0(X,X_{\text{sing}}) \) is made in such a way that for any (possibly singular) curve \( C \) the Chow group of points coincides with the Picard group: \( \text{CH}_0(C,C_{\text{sing}}) = \text{Pic}(C) \) and \( \text{CH}_0(C,C_{\text{sing}})^0 = \text{Pic}^0(C) \), see [LW, Proposition 1.4].

1.2 Functors of Relative Cartier Divisors

Relative Cartier divisors will be used for the description of (the duals of) generalized Albanese varieties. For the sake of functoriality we need not only to consider Cartier divisors on \( X \), but also their infinitesimal deformations. Of great relevance will be the completion \( \widehat{\text{Div}}_X \) of the functor \( \text{Div}_X \) of relative Cartier divisors on \( X \) (see [Ru1, No. 2.1]), which assigns to a finite (i.e. finite dimensional) \( k \)-algebra \( R \) the abelian group

\[
\widehat{\text{Div}}_X(R) = \Gamma(X \otimes_k R, (\mathcal{K}_X \otimes_k R)^*/(\mathcal{O}_X \otimes_k R)^*)
\]
where the star * denotes the unit groups. The functor $\widehat{\text{Div}}_X : \text{Art}/k \to \text{Ab}$ from the category of finite $k$-algebras to the category of abelian groups is a formal group (see [Ru2 Proposition 2.1]).

Let $\psi : V \to X$ be a morphism of varieties.

**Proposition 1.9.** Let $\widehat{\text{Dec}}_{X,V}$ (cf. [Ru1, Definition 2.11]) be the subfunctor of $\widehat{\text{Div}}_X$ consisting of those relative Cartier divisors on $X$ whose support (see [Ru2 Definition 2.2]) intersects $\psi(V)$ properly. Then $\widehat{\text{Dec}}_{X,V} : \text{Art}/k \to \text{Ab}$ is a formal $k$-group.

**Proof.** Analogous to [Ru2 Proposition 2.1]. ■

**Proposition 1.10.** The pull-back of Cartier divisors from $X$ to $V$ induces a homomorphism of formal groups $\cdot V : \widehat{\text{Dec}}_{X,V} \to \widehat{\text{Div}}_V$.

**Proof.** Straightforward. ■

2 Formal Divisors

For Cartier divisors on a $k$-scheme $X$ there is a notion of pull-back, but not a push-forward. If $X$ is a normal scheme, the group of Cartier divisors on $X$ can be identified with the subgroup of locally principal Weil divisors, and there is a push-forward of Weil divisors. But for Weil divisors it does not make sense to speak about infinitesimal deformations, because Weil divisors are formal sums of prime divisors and prime divisors are always reduced.

We introduce a formal group functor $F\text{Div}_X$ of formal divisors on $X$ that admits a push-forward and a natural transformation $F \text{Div}_X$ from $\widehat{\text{Div}}_X$. The group of formal infinitesimal divisors $\text{LDiv}(X)$ from [Ru1, No. 3.3] is the Lie algebra of $F\text{Div}_X$.

In this section we consider the case that $X$ is a curve $Z$.

**Functor of Formal Divisors**

Let $Z$ be a curve over $k$.

**Definition 2.1.** Define the formal $k$-group functor of formal divisors on $Z$

$$F\text{Div}_Z : \text{Art}/k \to \text{Ab}$$

from the category of finite $k$-algebras to the category of abelian groups by

$$F\text{Div}_Z = \bigoplus_{q \in Z(k)} \text{Hom}_{\text{Ab}/k} \left( \hat{O}_{Z,q}^*, k^* \right)$$
where $\text{Hom}_{Ab/k}$ denotes the $k$-group functor of homomorphisms of $k$-group functors, and $\mathcal{A}^*$ stands for the $k$-group functor $\mathbb{G}_m (?) \hat{\otimes}_k \mathcal{A}$ for any profinite $k$-algebra $\mathcal{A}$.

As Weil restriction is right-adjoint to base extension, cf. [Ru2, 1.1.2], the $R$-valued points of $F\text{Div}_Z$ are given by

$$F\text{Div}_Z(R) = \bigoplus_{q \in Z(k)} \text{Hom}_{Ab/k} \left( \mathbb{G}_m (?) \hat{\otimes}_k \widehat{O}_{Z,q}, \mathbb{G}_m (?) \otimes_k R \right).$$

**Proposition 2.2.** The functor $F\text{Div}_Z$ of formal divisors on $Z$ is a formal $k$-group.

**Proof.** The formal $k$-group functor

$$\text{Hom}_{Ab/k} \left( \widehat{O}_{Z,q}^*, k^* \right) = \text{Hom}_{Ab/k} \left( \widehat{O}_{Z,q}, ? \right) \circ \mathbb{G}_m \circ (\_ \otimes_k ?)$$

as a composition of left-exact functors (the tensor product is over a field), is left-exact, hence a formal group (see [Ru2, Prop. 1.7]). Finite direct sums of formal groups are formal groups. The finite subsets $E \subset Z(k)$ form a directed inductive system, ordered by inclusion. Thus

$$F\text{Div}_Z = \varinjlim_{E \subset Z(k)} \bigoplus_{q \in E} \text{Hom}_{Ab/k} \left( \widehat{O}_{Z,q}^*, k^* \right)$$

is the limit of a directed inductive system of formal groups, hence a formal group. ■

**Remark 2.3.** The $k$-group functor $\widehat{O}_{Z,q}^* = \mathbb{G}_m (?) \hat{\otimes}_k \widehat{O}_{Z,q}$ is an affine group by Lemma 2.4 below, since the completion $\widehat{O}_{Z,q}$ of the local ring at $q \in Z$ is a profinite $k$-algebra.

**Lemma 2.4.** Let $k$ be a ring and $\mathcal{A}$ a profinite $k$-algebra which is a projective $k$-module. Let $G$ be an $\mathcal{A}$-functor which is represented by an affine $\mathcal{A}$-scheme. Then the induced $k$-functor $G (?) \hat{\otimes}_k \mathcal{A}$ is represented by an affine $k$-scheme.

**Proof.** We adapt the proof of [DG, I, § 1, Proposition 6.6] to the situation where $\mathcal{A}$ is, instead of a finite $k$-algebra, a profinite $k$-algebra which is a projective $k$-module.

First assume that $G = \text{Spec}_\mathcal{A} S$, where $S = \text{Sym}_\mathcal{A} (V \otimes_k \mathcal{A})$ is the symmetric algebra of a $k$-module of the form $V \otimes_k \mathcal{A}$. Let $\{A_i\}$ be a projective system of discrete finite quotients of $\mathcal{A}$ such that $\mathcal{A} = \varprojlim A_i$, and let $\mathcal{H} = \text{Hom}^\text{cont}_k(\mathcal{A}, k)$. 

8
For $R \in \text{Alg}/k$

\[
G \left( R \hat{\otimes}_k A \right) = \text{Hom}_{\text{Alg-}A} \left( S, R \hat{\otimes}_k A \right) = \text{Hom}_A \left( V \otimes_k A, R \hat{\otimes}_k A \right) = \text{Hom}_k \left( V, \lim_\leftarrow \left( R \otimes_k A_i \right) \right) = \lim_\leftarrow \text{Hom}_k \left( V, R \otimes_k A_i \right) = \lim_\leftarrow \text{Hom}_k \left( V \otimes_k \text{Hom}_k \left( A_i, k \right), R \right) = \text{Hom}_k \left( V \otimes_k \text{Hom}_k \left( A_i, k \right), R \right) = \text{Hom}_k \left( V \otimes_k \text{Hom}_k \left( A_i, k \right), R \right) = \text{Hom}_{k-\text{alg}} \left( \text{Sym}_k \left( V \otimes_k \mathcal{H} \right), R \right)
\]

where we used in the fourth equation

\[
R \hat{\otimes}_k \lim_\leftarrow A_i = \lim_\leftarrow \left( R \otimes_k A_i \right) = \left\{ \sum_i r_i \otimes a_i \ \middle| \ r_i \in R, \ a_i \in A \text{ s.t. } \overline{a_i} \in \ker \left( A_i \to A_j \right) \text{ for } i > j \right\}
\]

and in the eighth equation, according to [A] Chapter 2, Exercise 20, p. 33]

\[
V \otimes_k \lim_\rightarrow H_i = \lim_\rightarrow \left( V \otimes_k H_i \right).
\]

Thus one has

\[
\hat{\Pi}_{A/k} \text{Spec}_A \left( \text{Sym}_A \left( V \otimes_k A \right) \right) = \text{Spec}_k \left( \text{Sym}_k \left( V \otimes_k \mathcal{H} \right) \right)
\]

where $\hat{\Pi}_{A/k} : \text{Fctr}(\text{Alg}/A, \text{Set}) \to \text{Fctr}(\text{Alg}/k, \text{Set})$ is the functor given by $G \mapsto G \left( ? \hat{\otimes}_k A \right)$.

For the case $G = \text{Spec}_A S$, where $S$ is an arbitrary $A$-algebra, let $\mathcal{I}$ be the kernel of the canonical homomorphism $\text{Sym}_A \left( S \otimes_k A \right) \to S$. Then $S$ is identified in $\text{Alg}/A$ with the amalgamated sum due to the diagram

\[
\text{Sym}_A \left( S \otimes_k A \right) \xleftarrow{\iota} \text{Sym}_A \left( \mathcal{I} \otimes_k A \right) \xrightarrow{\varepsilon} A
\]

where $\iota$ is the canonical homomorphism and $\varepsilon(\mathcal{I} \otimes_k A) = 0$. The functor $\hat{\Pi}_{A/k}$ is left-exact, i.e. it commutes with finite projective limits. Then $\hat{\Pi}_{A/k} \text{Spec}_A(S)$ is the fibre product obtained from the transformation of the amalgamated sum by $\hat{\Pi}_{A/k} \circ \text{Spec}_A$. Thus $\hat{\Pi}_{A/k} G$ is represented by an affine $k$-scheme, since it is given by the fibre product of affine $k$-schemes. ■
Definition 2.5. Let $\pi : Z \rightarrow C$ be a finite morphism of curves over $k$. The push-forward of formal divisors

$$\pi_* : \mathcal{FDiv}_Z \rightarrow \mathcal{FDiv}_C$$

is the homomorphism of formal $k$-group functors induced by the homomorphisms

$$\text{Hom}_{\text{Ab}/k}(\hat{\mathcal{O}}^*_{Z,q}, \mathcal{O}^*_{C,\pi(q)}) \rightarrow \text{Hom}_{\text{Ab}/k}(\hat{\mathcal{O}}^*_{Z,q}, \mathcal{O}^*_{C,\pi(q)})$$

where $q \in Z(k)$ and $\hat{\pi}^#_q : \hat{\mathcal{O}}_{C,\pi(q)} \rightarrow \hat{\mathcal{O}}_{Z,q}$ is the local homomorphism of complete local rings associated with $\pi$, giving $(\hat{\pi}^#_q)^* = \mathbb{G}_m(\hat{\mathcal{O}}_{Z,q} \otimes_k \hat{\mathcal{O}}_{C,\pi(q)})$.

Proposition 2.6. Let $Z$ be a normal curve over $k$. There is a homomorphism of formal $k$-groups $\text{fml} : \hat{\mathcal{Div}}_Z \rightarrow \mathcal{FDiv}_Z$ for each finite $k$-algebra $R$ given by

$$\text{fml}(R) : \Gamma\left(Z \otimes R, \mathcal{K}^*_Z \otimes_{R/k} \mathcal{O}_{Z,R}^* \otimes_{R/k} \mathcal{O}_{Z,R} \right) \rightarrow \bigoplus_{q \in Z(k)} \text{Hom}_{\text{Ab}/k}(\hat{\mathcal{O}}^*_{Z,q}, \mathcal{O}^*_R)$$

$$\mathcal{D} \mapsto \sum_{q \in Z(k)} (\mathcal{D}, ?)_q$$

where $(\varphi, ?)_q : \mathcal{K}^*_Z \times Z \rightarrow G$ is the local symbol associated with a rational map $\varphi$ from $Z$ to an algebraic group $G$.

Proof. The first task is to show that for $\mathcal{D} \in \hat{\mathcal{Div}}_Z(R)$ and $q \in Z(k)$ the expression $(\mathcal{D}, ?)_q : \hat{\mathcal{O}}^*_{Z,q} \rightarrow \mathcal{O}^*_R$ yields a well defined homomorphism of abelian groups. After that the naturality will be shown.

Let $\delta \in \Gamma\left(U_q \otimes R, \mathcal{K}^*_Z \otimes_{R/k} \mathcal{O}_{Z,R}^* \otimes_{R/k} \mathcal{O}_{Z,R} \right)$ be a local section representing $\mathcal{D}$ in a neighbourhood $U_q \subset Z$ of $q$. The restriction of $\delta$ to a suitable open dense subset $U \subset U_q$ gives an element of $\Gamma\left(U \otimes R, \mathcal{O}_{Z,R}^* \otimes_{R/k} \mathcal{O}_{Z,R} \right) = \mathbb{G}_m(\mathcal{O}_Z(U) \otimes_{k} \mathcal{O}_Z(U))$. Thus $\delta$ can be seen as a rational map $\delta : Z \rightarrow \mathbb{L}_R$.

Let $\delta' \in \Gamma\left(U'_q \otimes R, \mathcal{K}^*_Z \otimes_{R/k} \mathcal{O}_{Z,R}^* \otimes_{R/k} \mathcal{O}_{Z,R} \right)$ be another local section representing $\mathcal{D}$ in a neighbourhood $U'_q \subset Z$ of $q$. Then $\delta$ and $\delta'$ differ at $q$ by a unit $\gamma \in \Gamma\left(U'_q \otimes R, \mathcal{O}_{Z,R}^* \otimes_{R/k} \mathcal{O}_{Z,R} \right)$, i.e. $\delta' = \delta \cdot \gamma$. The corresponding rational map of
this unit $\gamma : Z \twoheadrightarrow \mathbb{L}_R$ is regular at $q$. Therefore for each $f \in \hat{O}_{Z,q}^*$ we compute $(\gamma, f)_q = \gamma(q)^{\nu_q(f)} = 1$ since $\nu_q(f) = 0$. By [Ru1, Proposition 3.15] one obtains $(\delta', f)_q = (\delta \gamma, f)_q = (\delta, f)_q : (\gamma, f)_q = (\delta, f)_q$.

This shows that the expression $(\mathcal{D}, ?)_q := (\delta, ?)_q : \hat{O}_{Z,q}^* \to R^*$ is independent of the choice of local representative $\delta$ at $q$.

Let $h : R \to S$ be a homomorphism of $k$-algebras. Then

$$
(\hat{\text{Div}}_Z(h)(\mathcal{D}), ?)_q = (\mathbb{L}_h \circ ?)_q = \hat{\text{Div}}_Z(h) \circ \text{fml}.
$$

which implies $\text{fml} \circ \hat{\text{Div}}_Z(h) = \hat{\text{FDiv}}_Z(h) \circ \text{fml}$. ■

**Definition 2.7.** Let $C$ be a projective curve over $k$, and let $\pi : Z \to C$ be its normalization. Then define the formal subgroup $\hat{\text{Div}}_{Z/C}$ of $\hat{\text{Div}}_Z$ to be the kernel of the composition $\pi_* \circ \text{fml}$

$$
\text{Div}_{Z/C} = \ker \left( \hat{\text{Div}}_Z \xrightarrow{\text{fml}} \hat{\text{FDiv}}_Z \xrightarrow{\pi_*} \hat{\text{FDiv}}_C \right).
$$

**Definition 2.8.** Let $X$ be a projective variety over $k$, and $\pi : Y \to X$ a birational morphism. Then define the formal subgroup $\hat{\text{Div}}_{Y/X}$ of $\hat{\text{Div}}_Y$ as follows:

$$
\text{Div}_{Y/X} = \bigcap_C \left( ? \cdot \tilde{C} \right)^{-1} \hat{\text{Div}}_{\tilde{C}/C}
$$

where $C$ ranges over all Cartier curves in $X$ relative to $X \setminus U$ (see Def. [11]), where $\nu_C : \tilde{C} \to C$ is the normalization of $C$ and $? \cdot \tilde{C} : \text{Dec}_Y \tilde{C} \to \hat{\text{Div}}_{\tilde{C}}$ is the pull-back of relative Cartier divisors from $Y$ to $\tilde{C}$ (see Propositions [1.9] and [1.10]). Note that $\tilde{C} \to X$ factors through $Y$: let $C^Y$ denote the proper transform of $C$, i.e. the closure in $Y$ of $\pi^{-1}(C \cap U)$. As $\pi|_{C^Y} : C^Y \to C$ is a birational morphism, the normalization $\tilde{C}^Y$ of $C^Y$ is isomorphic to $\tilde{C}$ and hence $\nu : \tilde{C} \to C$ factors through a morphism $\mu : \tilde{C} \to C^Y$.

**Remark 2.9.** For $\text{char}(k) = 0$, Definition 2.7 of $\text{Div}_{Z/C}$ coincides with the definition of $\hat{\text{Div}}_{Z/C}$ given in [Ru1, Proposition 3.23]. Likewise, Definition 2.8 of $\text{Div}_{Y/X}$ coincides with the definition of $\hat{\text{Div}}_{Y/X}$ given in [Ru1, Proposition 3.24].

**Proof.** The second statement follows from the first by definition. For the first statement, observe that we used in Definition 2.7 the fact that a relative Cartier divisor $\mathcal{D} \in \hat{\text{Div}}_Z(R)$ induces a rational map $\delta : Z \twoheadrightarrow \mathbb{L}_R$.
locally around each \( q \in \mathbb{Z}(k) \) and we associated with \( D \) the local symbols \((\delta, ?)_q\). For \( \text{char}(k) = 0 \), we are reduced to consider the cases \( L_R = \mathbb{G}_m \) and \( L_R = \mathbb{G}_a \). Then the explicit descriptions of the local symbol for \( \mathbb{G}_m \) resp. \( \mathbb{G}_a \) (see [Ru1, Example 3.13 resp. 3.14]) yield the coincidence of the two definitions. □

3 Category of Rtl. Maps Factoring through Rtl. Equivalence

We keep the notation fixed at the beginning of this note on page 5: \( X \) is a projective variety over \( k \) which admits a resolution of singularities \( \pi : Y \rightarrow X \), and \( U \subset X_{\text{reg}} \) a dense open subset, which we identify with its inverse image in \( Y \). Algebraic groups are assumed to be smooth, connected and commutative, unless stated otherwise.

**Definition 3.1.** Let \( \text{Mr}^{\text{CH}} \) be the category of rational maps from \( X \) to algebraic groups defined as follows: The objects of \( \text{Mr}^{\text{CH}} \) are morphisms \( \varphi : U \rightarrow G \) whose associated map on 0-cycles of degree 0

\[
\sum l_i p_i \mapsto \sum l_i \varphi(p_i)
\]

factors through a homomorphism of groups \( \text{CH}_0(X, X_{\text{sing}})^0 \rightarrow G(k) \).

We refer to objects of \( \text{Mr}^{\text{CH}} \) as rational maps from \( X \) to algebraic groups *factoring through rational equivalence* or *factoring through \( \text{CH}_0(X, X_{\text{sing}})^0 \).*

**Theorem 3.2.** The category \( \text{Mr}^{\text{CH}} \) is equal to the category \( \text{Mr}_{\text{Div}^{0,\text{red}}_{Y/X}} \), i.e. to the category of those rational maps from \( Y \) to algebraic groups which induce a transformation to \( \text{Div}^{0,\text{red}}_{Y/X} \) (see Definitions 2.7, 2.8).

**Proof.** A rational map from \( Y \) to an algebraic group induces a transformation to \( \text{Div}^{0,\text{red}}_{Y/X} \) if and only if it induces a transformation to \( \text{Div}_{Y/X} \) by [Ru2, Lemma 2.6]. Such a rational map is necessarily regular on \( U \), since all \( D \in \text{Div}_{Y/X} \) have support only on \( Y \setminus U \). This follows from the fact that \( \pi : Y \rightarrow X \) is an isomorphism on \( U \), and \( \text{Div}_{Y/X} \) is the kernel of the push-forward \( \pi_* \). Then according to Definitions 1.5 and [Ru2, Definition 2.10] the task is to show that for a morphism \( \varphi : U \rightarrow G \) from \( U \) to an algebraic

---

1 A category of rational maps to algebraic groups is defined already by its objects, according to [Ru2, Remark 2.8].
group $G$ with canonical decomposition $0 \to L \to G \to A \to 0$ the following conditions are equivalent:

(i) $\varphi \left( \text{div}(f)_C \right) = 0 \quad \forall (C, f) \in \mathcal{R}_0(X, U),$

(ii) $\tau_\varphi(\lambda) \in \text{Div}_{Y/X}(R) \quad \forall R \in \text{Art}/k, \lambda \in L^\vee(R),$

where $\tau_\varphi : L^\vee \to \hat{\text{Div}}_{Y^\vee}$ is the induced transformation from [Ru2, Definition 2.4]. Now if $h_\lambda : G \to \lambda_*G$ denotes the push-out via $\lambda \in L^\vee(R) = \text{Hom}_{\text{Ab}/k}(L, \mathbb{L}_R)$ and $\iota := \text{id}_{\mathbb{L}_R} \in \text{Hom}_{\text{Ab}/k}(\mathbb{L}_R, \mathbb{L}_R) = \mathbb{L}_R^\vee(R)$, we obtain $\tau_\varphi(\lambda) = \tau_{h_\lambda \circ \varphi}(\iota) =: \text{div}_R(h_\lambda \circ \varphi)$. This was explained in the proof of [Ru2, Theorem 2.12]. By [Ru2, Lemma 1.12] obviously (i) is equivalent to

(i') $(h_\lambda \circ \varphi) \left( \text{div}(f)_C \right) = 0 \quad \forall (C, f) \in \mathcal{R}_0(X, U), \forall R \in \text{Art}/k, \lambda \in L^\vee(R).$

Then the equivalence of (i) and (ii) follows from Lemma 3.3 below, if we replace $\varphi$ in Lemma 3.3 by $h_\lambda \circ \varphi$.

**Lemma 3.3.** Let $\varphi : U \to G$ be a morphism from $U$ to a smooth connected algebraic group $G \in \text{Ext}(A, \mathbb{L}_R)$, where $A$ is an abelian variety and $\mathbb{L}_R$ the linear group associated with $R \in \text{Art}/k$ (see [Ru2, Definition 1.11]). Then the following conditions are equivalent:

(i) $\varphi \left( \text{div}(f)_C \right) = 0 \quad \forall (C, f) \in \mathcal{R}_0(X, U),$

(ii) $\text{div}_R(\varphi) \in \hat{\text{Div}}_{Y/X}(R)$

where $\text{div}_R(\varphi) := \tau_\varphi(\iota)$ for $\tau_\varphi : \mathbb{L}_R^\vee \to \hat{\text{Div}}_{Y^\vee}$ the induced transformation from [Ru2, Definition 2.4] and $\iota := \text{id}_{\mathbb{L}_R} \in \text{Hom}_{\text{Ab}/k}(\mathbb{L}_R, \mathbb{L}_R) = \mathbb{L}_R^\vee(R)$.

**Proof.** (i)$\iff$(ii) Let $C$ be a Cartier curve in $X$ relative to $X \setminus U$, and let $\nu : \bar{C} \to C$ be its normalization. Then Lemma 3.4 below asserts that the following conditions are equivalent:

(i) $\varphi_C \left( \text{div}(f) \right) = 0 \quad \forall f \in \mathcal{K}(C, U)^*,$

(ii) $\nu_\ast \left( \text{div}_R(\varphi)_C \right) = 0.$

We have $\text{div}_R(\varphi)_C = \text{div}_R(\varphi) \cdot \bar{C}$, where $? \cdot \bar{C} : \text{Dec}_{Y_{/X}} \bar{C} \to \hat{\text{Div}}_{Y_{/X}}$ is the pull-back of Cartier divisors from $Y$ to $\bar{C}$ (see Propositions 1.9 and 1.10). Then condition (i) is equivalent to

(iii) $\left( \nu_\ast(\text{div}_R(\varphi)) \right)_C = 0 \quad \forall \text{Cartier curves } C \text{ relative to } X \setminus U.$

Conditions (iii) and (ii) are equivalent by definition of $\hat{\text{Div}}_{Y/X}$ (see Defs. 2.7 and 2.8).

**Lemma 3.4.** Let $C$ be a projective curve and $\nu : \bar{C} \to C$ its normalization.

Let $\varphi : C \to G$ be a rational map from $C$ to a smooth connected algebraic group $G \in \text{Ext}(A, \mathbb{L}_R)$, i.e. $G$ is an $\mathbb{L}_R$-bundle over an abelian variety $A$, where $R$ is a finite $k$-algebra. Suppose that $\varphi$ is regular on a dense open subset $U_C \subset C_{\text{reg}}$, which we identify with its preimage in $\bar{C}$. Then the following conditions are equivalent:
U be a local trivialization of the induced \( L \) since \( \phi \) all

It remains to remark that for some \( h \), if and only if \( \sum_{q \to p} (\phi, g) = 0 \) for all \( g \in \hat{O}_{C,p}^* \). Indeed, for \( f \) and given \( g \in \hat{O}_{C,p}^* \), the approximation theorem yields an element \( b_p \in K_C \) such that \( b_p \equiv g \mod D \) at \( \nu^{-1}(p) \) and \( b_p \equiv 1 \mod D \) at \( s \) for all \( s \in S \setminus \nu^{-1}(p) \). Then \( b_p \) can be chosen such that \( b_p = \tilde{f}_p \) for some \( f_p \in K(C, U_C)^* \). This means that \( g = f_p + h = f_p (1 + f_p^{-1} h) \) for some \( h \in \hat{m}_{C,p} \) with \( \tilde{h} \in \hat{m}_{C,q}^n \) for each \( q \to p \). As \( f_p \in O_{C,p}^* \), we have

Let \( D \) be a modulus for \( \phi \). For each \( p \in \nu(S) \), each \( g \in \hat{O}_{C,p}^* \), there is a rational function \( f_p \in K(C, U_C)^* \) such that \( \tilde{g} / \tilde{f}_p \equiv 1 \mod D \) at \( \nu^{-1}(p) \) and \( \tilde{f}_p \equiv 1 \mod D \) at \( s \) for all \( s \in S \setminus \nu^{-1}(p) \), by the approximation theorem. Indeed, for \( D = \sum_{q \in S} n_q s \) and given \( g \in \hat{O}_{C,p}^* \), the approximation theorem yields an element \( b_p \in K_C \) such that \( b_p \equiv g \mod D \) at \( \nu^{-1}(p) \) and \( b_p \equiv 1 \mod D \) at \( s \) for all \( s \in S \setminus \nu^{-1}(p) \). Then \( b_p \) can be chosen such that \( b_p = \tilde{f}_p \) for some \( f_p \in K(C, U_C)^* \). This means that \( g = f_p + h = f_p (1 + f_p^{-1} h) \) for some \( h \in \hat{m}_{C,p} \) with \( \tilde{h} \in \hat{m}_{C,q}^n \) for each \( q \to p \). As \( f_p \in O_{C,p}^* \), we have

Then \( (\phi, \tilde{f}_p)_q = (\phi, \tilde{g})_q \) for all \( q \in \nu^{-1}(p) \) and \( (\phi, \tilde{f}_p) = 0 \) for all \( s \in S \setminus \nu^{-1}(p) \), by \( \text{Se}^3 \) III, No. 1, Definition 2 (i), (ii). Hence \( \phi(\text{div}(f_p)) = 0 \) if and only if \( \sum_{q \to p} (\phi, \tilde{f}_p)_q = \sum_{q \to p} (\phi, \tilde{g})_q = 0 \). Thus \( \phi(\text{div}(f_c)) = 0 \) for all \( f \in K(C, U_C)^* \) if and only if \( \sum_{q \to p} (\phi, \tilde{g})_q = 0 \) for all \( g \in \hat{O}_{C,p}^* \), \( p \in \nu(S) \). It remains to remark that \( (\phi, h)_c = 0 \) for all \( h \in \hat{O}_{C,c}^* \). Since \( \phi \) and \( h \) are both regular at \( c \).

(ii) \( \iff \) (iii) Let \( p \in C \). For each \( q \in \nu^{-1}(p) \) let \( \Phi_q : U_q \times L \to \text{G}_Y \) be a local trivialization of the induced \( L \)-bundle over \( C \) in a neighbourhood \( U_q \ni q \). For all \( f \in O_{C,q}^* \), we have

\[
(\phi, f)_q = (|\phi|_{\Phi_q}, f)_q = (\text{div}_R(\phi), f)_q
\]
Then \( \sum_{q \rightarrow p} (\text{div}_R(\varphi), \mathbf{?)})_q = 0 \) for all \( g \in \hat{\mathcal{O}}_{C,p} \) is equivalent to the condition that the image \( \sum_{q \rightarrow p} (\text{div}_R(\varphi), \mathbf{?)})_q \) of \( \text{div}_R(\varphi) \) in \( \bigoplus_{q \rightarrow p} \text{Hom}_{Ab/k}(\hat{\mathcal{O}}_{C,q}^*, R^*) \) vanishes on \( \hat{\mathcal{O}}_{C,p}^* \), which says

\[
0 = \sum_{q \rightarrow p} (\text{div}_R(\varphi), \mathbf{?)})_q \circ (\nu_q^#)^* \in \bigoplus_{q \rightarrow p} \text{Hom}_{Ab/k}(\hat{\mathcal{O}}_{C,q(\nu)}, R^*).
\]

This is true for all \( p \in C \) if and only if \( (\nu_\ast \circ \text{fm} (\text{div}_R(\varphi))) = 0 \) by definition of the push-forward for formal divisors (see Definition 2.5).

## 4 Kernel of the Push-forward of Divisors

The functor \( \text{Div}_{Y/X} \) was introduced in Section 2 as “the kernel of the push-forward of relative divisors” (see Definitions 2.7, 2.8). The goal of this section is to show that \( \text{Div}^{0,\text{red}}_{Y/X} := \text{Div}_{Y/X} \times_{\text{Pic}^0_{Y}} \text{Pic}^0_{Y} \) is represented by a dual-algebraic formal group.

### Notation 4.1.

For any subfunctor \( F \) of \( \text{Div}_{Y} \) we denote by \( F^{0,\text{red}} \) the induced subfunctor of \( \text{cl}^{-1} \text{Pic}^0_{Y} \):

\[
F^{0,\text{red}} = F \times_{\text{Pic}^0_{Y}} \text{Pic}^0_{Y}.
\]

### Remark 4.2.

If \( F \) is a formal group, then \( F^{0,\text{red}} \) is again a formal group, since \( F^{0,\text{red}} = F \times_{\text{Pic}^0_{Y}} \text{Pic}^0_{Y} \) and the category \( \mathcal{G}f/k \) of formal groups admits fibre-products (see [Fol 1, 4.9, p. 35]).

### Lemma 4.3.

Let \( Z \) be a normal curve over \( k \). Then the natural transformation \( \text{fm} : \text{Div}_Z \rightarrow \text{FDiv}_Z \) induces a monomorphism of formal \( k \)-groups \( \text{Div}_Z^{0,\text{red}} \rightarrow \text{FDiv}_Z \).

#### Proof.

Let \( R \) be a finite \( k \)-algebra and \( D \in \text{Div}_Z^{0,\text{red}}(R) \) with \( \text{fm} (D) = 0 \). Let \( G_{\mathbb{L}_R}(D) \in \text{Ext}_{Ab/k}(\text{Alb}(Z), \mathbb{L}_R) \cong \text{Pic}^0_{\text{Alb}(Z)}(R) \) (see [Ru2, Proposition 1.19]) be the image of \( \mathcal{O}_{Z \otimes R}(D) \) under the homomorphism

\[
alb : \text{Pic}^0_Z \rightarrow \text{Alb} (\text{Pic}^0_Z) = \text{Pic}^0 (\text{Pic}^0_Z) = \text{Pic}^0 (\text{Alb}(Z)).
\]

Then \( P_{\mathbb{L}_R}(D) := G_{\mathbb{L}_R}(D) \times_{\text{Alb}(Z)} Z \) is the \( \mathbb{L}_R \)-bundle on \( Z \) associated with \( \mathcal{O}_{Z \otimes R}(D) \). The canonical 1-section of \( \mathcal{O}_{Z \otimes R}(D) \) induces a rational map

\[
\varphi^D : Z \rightarrow P_{\mathbb{L}_R}(D) = G_{\mathbb{L}_R}(D) \times_{\text{Alb}(Z)} Z \rightarrow G_{\mathbb{L}_R}(D)
\]
s.t. \( D = \tau_{\varphi^D}(i) \), where \( \tau_{\varphi^D} : \mathbb{L}_R \rightarrow \widehat{\text{Div}}_Z \) is the induced transformation from [Ru2] Definition 2.4 and \( i := \text{id}_{\mathbb{L}_R} \in \text{Hom}_{\text{Ab}/k}(\mathbb{L}_R, \mathbb{L}_R) = \mathbb{L}_R(R) \). If \( g \in \widehat{O}^*_Z \), for some \( q \in Z(k) \), then by [Ru1] Lemma 3.16 the local symbol \((\varphi^D, g)_q\) lies in the fibre of \( G_{\mathbb{L}_R} \) over \( 0 \in \text{Alb}(Z) \), which is \( \mathbb{L}_R \), and \((\varphi^D, g)_q = (D, g)_q\). As \( \text{fnl}(D) = 0 \), it holds \( 0 = (D, g)_q = (\varphi^D, g)_q \) for all \( q \in Z(k) \) and all \( g \in \widehat{O}^*_Z \). Lemma 3.4 implies that \( 0 = \varphi^D(\text{div}(f)) \) for all \( f \in K(Z, Z_{\text{reg}})^* \), i.e. \( \varphi^D \) factors through \( \text{CH}_0(Z, Z_{\text{sing}})^0 \). But \( \text{CH}_0(Z, Z_{\text{sing}})^0 = \text{Alb}(Z) \) is an abelian variety, since \( Z_{\text{sing}} = \emptyset \), thus \( \varphi^D \) extends to a morphism defined on all of \( Z \). The transformation \( \tau_{\varphi^D} \) has the property that \( \text{Supp}(\text{im} \tau_{\varphi^D}) \subset Z \) is the locus where \( \varphi^D \) is not defined. Then \( \text{im}(\tau_{\varphi^D}) = 0 \), hence \( D = \tau_{\varphi^D}(i) = 0 \).

**Theorem 4.4.** Let \( C \) be a projective curve over \( k \), and let \( \pi : Z \rightarrow C \) be its normalization. Then the functor \( \text{Div}^{0,\text{red}}_{Z/C} \) is represented by a dual-algebraic formal group.

**Proof.** First note that the normalization \( \pi : Z \rightarrow C \) is an isomorphism on the preimage \( \pi^{-1} C_{\text{reg}} \) of the regular locus of \( C \). Hence the push forward \( \pi_* \) is an isomorphism on formal divisors supported on \( \pi^{-1} C_{\text{reg}} \). Therefore the formal divisors in \( \ker (\text{FDiv}_Z \rightarrow \text{FDiv}_C) \) have support only on the inverse image \( S_Z \) of the singular locus \( S \) of \( C \), which is finite.

\[
\ker \left( \text{FDiv}_Z \rightarrow \text{FDiv}_C \right)
\]

\[
= \ker \left( \bigoplus_{q \in Z(k)} \text{Hom}_{\text{Ab}/k} \left( \widehat{O}^*_Z, k^* \right) \rightarrow \bigoplus_{p \in C(k)} \text{Hom}_{\text{Ab}/k} \left( \widehat{O}^*_C, k^* \right) \right)
\]

\[
= \bigoplus_{p \in C(k)} \ker \left( \bigoplus_{q \rightarrow p} \text{Hom}_{\text{Ab}/k} \left( \widehat{O}^*_Z, k^* \right) \rightarrow \text{Hom}_{\text{Ab}/k} \left( \widehat{O}^*_C, k^* \right) \right)
\]

\[
= \bigoplus_{p \in C(k)} \text{Hom}_{\text{Ab}/k} \left( \widehat{O}^*_C, \widehat{O}^*_C, k^* \right)
\]

\[
\bigoplus_{p \in S(k)} \text{Hom}_{\text{Ab}/k} \left( \widehat{O}^*_Z, \widehat{O}^*_C, k^* \right)
\]

\[
\bigoplus_{p \in S(k)} \text{Hom}_{\text{Ab}/k} \left( O^*_Z / O^*_C, k^* \right)
\]

Now \( O^*_Z / O^*_C = \prod_{p \in S} O^*_Z / O^*_C \) is a sheaf of abelian groups over \( \text{Alg}/k \).
with Lie-algebra $\text{Lie}(\mathcal{O}_Z^*/\mathcal{O}_C^*) = \mathcal{O}_Z/\mathcal{O}_C = \prod_{p \in S} \mathcal{O}_{Z,p}/\mathcal{O}_{C,p}$. This is a coherent sheaf concentrated on finitely many points $p \in S$, which implies that $\text{Lie}(\mathcal{O}_Z^*/\mathcal{O}_C^*)$ is finite dimensional. Then the group sheaf $\mathcal{O}_Z^*/\mathcal{O}_C^*$ is represented by an affine algebraic group $L \cong \prod_{p \in S} \mathbb{T}_p \times \mathbb{U}_R$ for $p \in S$, where $\mathbb{T}_p$ denotes the torus $(\mathbb{G}_m)^p/\mathbb{G}_m$ and $\mathbb{U}_R$ is the unipotent group from [Ru2, 1.1.6] associated with the finite $k$-algebra $R_p = k+(\bigoplus_{q \to p} m_{Z,q})/m_{C,p}$.

Then $\ker(\text{FDiv}_Z \to \text{FDiv}_C) = \text{Hom}_{\mathbb{A}b/k}((\mathcal{O}_Z^*/\mathcal{O}_C^*), k^*) = L^\vee$ is the Cartier-dual of an affine algebraic group, hence a dual-algebraic form al group.

$\hat{\text{Div}}_Z$ is a formal group by [Ru2, Proposition 2.1], hence $\hat{\text{Div}}_Z$ is again a formal group, according to Remark 4.2. Due to Lemma 4.3 the transformation $\text{fm}\_{0,\text{red}}: \text{Div}_{Z/C}^{0,\text{red}} \to \text{FDiv}_Z$ is a monomorphism of formal groups.

Then the fibre-product of formal groups $\text{Div}_{Z/C}^{0,\text{red}} = \text{Div}_{Z}^{0,\text{red}} \times_{\text{FDiv}_Z} L^\vee$ is a subgroup of the dual-algebraic formal group $L^\vee$, hence dual-algebraic by [Ru2, Lemma 3.15]. ■

**Theorem 4.5.** Let $X$ be a projective variety over $k$ that admits a resolution of singularities $\pi: Y \to X$. Then the functor $\text{Div}_{Y/X}^{0,\text{red}}$ is represented by a dual-algebraic formal group.

The proof of Theorem 4.5 is similar to the proof of [Ru1, Proposition 3.24]. But instead of considering $k$-valued points and Lie-algebra of the functor $\text{Div}_{Y/X}^{0,\text{red}}$ separately (which makes sense only for $\text{char}(k) = 0$), we can control the values of this functor for all finite rings $R$ by a formal group $\mathcal{F}_{Y,D}$ (see [Ru2, Definition 3.13]) associated with a sufficiently large effective divisor $D$ on $Y$ with support on the inverse image $S_Y$ of the singular locus $S = X_{\text{sing}}$ of $X$. This theorem will be fundamental for an independent proof of the existence of $\text{Alb}(X, X_{\text{sing}})$. Conversely, Theorem 4.5 will follow from the existence of $\text{Alb}(X, X_{\text{sing}})$ (proven in [ESV, Theorem 1]) and Proposition 5.1.

**Proof of Thm 4.5.** The functor $\text{Div}_{Y/X}^{0,\text{red}} = \bigcap_{\mathcal{D} \in \text{Div}_Y(R)} \text{Dec}_{Y,C} \times \text{Div}_{C/C}^\mathcal{D}$ as a projective limit of formal groups, is again a formal group. (A formal group functor is a formal group if and only if it commutes with finite projective limits, and projective limits commute.) It remains to show that $\text{Div}_{Y/X}^{0,\text{red}}$ is dual-algebraic. According to [Ru2, Proposition 3.21] it is sufficient to show that there exists an effective divisor $D$ on $Y$ such that $\text{Div}_{Y/X}^{0,\text{red}} \subset \mathcal{F}_{Y,D}$.

Let $\mathcal{D} \in \text{Div}_Y(R)$ be a non-trivial relative Cartier divisor on $Y$ for some $R \in \text{Art}/k$. Assume $\text{Supp}(\mathcal{D})$ is not contained in the inverse image $S_Y = S \times_X Y$ of the locus $S = X \setminus U$. Then $\pi(\text{Supp}(\mathcal{D}))$ on $X$ is not contained
in $S$. Let $\mathcal{L}$ be a very ample line bundle on $X$, consider the space $|\mathcal{L}|^{d-1}$, where $d = \dim X$, of complete intersection curves $C = H_1 \cap \ldots \cap H_{d-1}$ with $H_i \in |\mathcal{L}| = \mathbb{P}(H^0(X, \mathcal{L}))$ for $i = 1, \ldots, d - 1$. For Cartier curves $C$ in $|\mathcal{L}|^{d-1}$ (with $C = C \times_X Y$) the following properties are open and dense:

(a) $C$ intersects $\pi(\text{Supp}(\mathcal{D})) \cap U$ properly.

(b) $\mathcal{D} \cdot C_Y$ is a non-trivial divisor on $C \cap \pi^{-1}U$.

(c) $C_Y$ is regular in a neighbourhood of $\text{Supp}(\mathcal{D} \cdot C_Y) \cap \pi^{-1}U$.

(a) is due to the fact that $\mathcal{L}$ is very ample, (b) follows from (a) and the fact that $\text{Supp}(\mathcal{D})$ is locally a prime divisor at almost every $q \in Y$ and (c) is a consequence of the Bertini theorems. Therefore there exists a Cartier curve $C$ in $X$ satisfying the conditions (a)–(c). As the normalization $\nu : \tilde{C} \to C$ coincides with $\pi|_{C_Y}$, it is an isomorphism on $C_Y \cap U$. Thus $\nu_*(\mathcal{D} \cdot \tilde{C}) \neq 0$.

This implies $\mathcal{D} \notin \text{Div}_{Y/X}^0(R)$. Hence $\text{Supp}(\text{Div}_{Y/X}^0)$ is contained in $S_Y$.

Let $E$ be the reduced effective divisor associated with the sum of the components of $S_Y$ of codimension 1 in $Y$. If $C$ is a Cartier curve in $X$ relative to $S$, then the normalization $\nu : \tilde{C} \to C$ factors through the proper transform $C^\nu$. By Lemma 4.6 one finds a family $T \subset |\mathcal{L}|^{d-1}$ of Cartier curves $C$ such that the set $\bigcup_{C \in T} C^\nu \cap E$ contains an open dense subset $W$ of $E$. Since $\text{Div}_{\tilde{C}/C}^0$ is dual-algebraic, by [Ru2, Proposition 3.21] there exists a natural number $n_C \in \mathbb{N}$ such that $\text{Div}_{\tilde{C}/C}^0 \subset \mathcal{F}_{\tilde{C}, n_C E, \tilde{C}}$. As $\text{Div}_{\tilde{C}/C}^0$ is Cartier dual to the affine part of $\text{Pic}_C^0$, the bound $n_C$ is controlled by the dimension of the unipotent part of $\text{Pic}_C^0$. (It is sufficient that $n_C$ satisfies $m_{C,q}^{n_C} \subset m_{C,p}$ for all $p \in C$, $q \in \nu^{-1}(p)$, see BLR Section 9.2, proof of Proposition 9). By upper semi-continuity of $\dim \text{Pic}_C^0$ for the curves $C$ in $|\mathcal{L}|^{d-1}$, we may assume that there exists a common bound $n \geq n_C$ for all $C \in T$. If $\text{char}(k) = p > 0$ and $p$ divides $n$, replace $n$ by $n+1$. Now $\mathcal{D} \in \text{Div}_{Y/X}^0(R)$ implies by definition $\mathcal{D} \cdot \tilde{C} \in \text{Div}_{\tilde{C}/C}^0(R)$ for all Cartier curves $C$. Then $\mathcal{D} \cdot \tilde{C} \in \mathcal{F}_{\tilde{C}, n C, E}$ for all $C \in T$. By construction of $T \subset |\mathcal{L}|^{d-1}$ and $n$, this yields $\mathcal{D} \in \mathcal{F}_{Y,n E}$. ■

**Lemma 4.6.** Let $X$ be a projective variety over $k$ which admits a projective resolution of singularities $\pi : Y \to X$. Then $\bigcup_C C^\nu = Y$, i.e. the union of the proper transforms of the curves $C$ in $X$ covers $Y$, where $C$ ranges over all Cartier curves in $X$ relative to $X \setminus U$.

**Proof.** According to [Ha, II, Theorem 7.17] we may assume that $\pi : Y \to X$ is given as the blowing up of an ideal sheaf on $X$, in particular $\pi_*\mathcal{O}_Y(1) = \mathcal{J}$ is an ideal sheaf on $X$. Let $\mathcal{L}$ be an ample line bundle on $X$. Then $\mathcal{A} := \mathcal{O}_Y(1) \otimes \pi^*\mathcal{L}^N$ for sufficiently large integer $N$ is a very ample
line bundle on $Y$ over $X$ (see [Ha II, Proposition 7.10 (b)]) and hence over $k$, since $X$ is projective.

For a given closed point $q \in Y$ we are now going to construct a Cartier curve $C$ on $X$ such that $q \in C^Y$. Choose a tangent vector $t \in T_q Y \setminus T_q S_Y$, where $T_q Y$ resp. $T_q S_Y$ denotes the tangent space of $Y$ resp. $S_Y$ at $q$, and $S_Y$ is the inverse image in $Y$ of the locus of $S = X \setminus U$. Since $A$ is very ample, one finds a complete intersection curve $C_{q,t} \in |A|^{d-1}$, where $d = \dim X$, through $q$ such that $t \in T_q C_{q,t}$. The curve $C_{q,t} = \bigcap_{i=1}^{d-1} Z(s_i)$ is given as intersection of the divisors of zeroes $Z(s_i)$ for

$$s_i \in \Gamma(Y, A) = \Gamma(X, \pi_\ast A) = \Gamma\left(X, \pi_\ast (\pi^\ast L^N \otimes_{O_Y} O_Y(1))\right) = \Gamma\left(X, L^N \otimes_{O_X} \pi_\ast O_Y(1)\right) = \Gamma\left(X, L^N \otimes_{O_X} J\right) \subset \Gamma\left(X, L^N\right).$$

Thus $C_{q,t}$ is the proper transform of the curve $C = \bigcap_{i=1}^{d-1} Z(s_i) \in |L^N|^{d-1}$ on $X$, which is a Cartier curve by construction.

5 Universal Reg. Quotient of the Chow Group

The results obtained up to now provide the necessary foundations for a functorial description of the generalized Albanese variety $\text{Alb}(X, X_{\text{sing}})$ of Esnault-Srinivas-Viehweg and its dual, which was the initial intention of this work.

5.1 Existence and Construction

The Albanese $\text{Alb}(X, X_{\text{sing}})$ of Esnault-Srinivas-Viehweg and the rational map $\text{alb}_{X, X_{\text{sing}}}: X \to \text{Alb}(X, X_{\text{sing}})$ are by definition the universal object for the category $\text{Mr}_{\text{CH}}^{\text{reg}}$ of morphisms from $U \subset X_{\text{reg}}$ factoring through $\text{CH}_0(X, X_{\text{sing}})^0$ (see Definition 3.1). Therefore $\text{Alb}(X, X_{\text{sing}})$ is often called the universal regular quotient of $\text{CH}_0(X, X_{\text{sing}})^0$.

**Proposition 5.1.** The following conditions are equivalent:

(i) $\text{Alb}(X, X_{\text{sing}})$ exists as an algebraic group.

(ii) $\text{Di}_{Y/X}^{0, \text{red}}$ is represented by a dual-algebraic formal group.
Proof. The proof of [Ru2, Theorem 2.12] shows that a category $\mathcal{M}r$ of rational maps from $Y$ to algebraic groups satisfying $[\mathcal{R} u_2, 2.3.1, (\Diamond 1, 2)]$ admits a universal object if and only if the formal group $\mathcal{F} \subset \hat{\text{Div}}_Y$ generated by $\bigcup_{\varphi \in \mathcal{M}r} \text{im}(\tau_\varphi)$ is dual-algebraic and $\mathcal{M}r = \mathcal{M}r_\mathcal{F}$. By Theorem 3.2 the category $\mathcal{M}r^{CH}$ is equal to the category $\mathcal{M}r_{\hat{\text{Div}}^{0,\text{red}}_Y}$ of rational maps from a desingularization $Y$ of $X$ to algebraic groups which induce a transformation to the formal group $\hat{\text{Div}}^{0,\text{red}}_{Y/X}$. Thus $\mathcal{M}r^{CH}$ admits a universal object if and only if $\hat{\text{Div}}^{0,\text{red}}_{Y/X}$ is dual-algebraic. ■

Proof of Theorem 0.1. The existence of $\text{Alb}(X, X_{\text{sing}})$ was proven in [ESV, Theorem 1]. Alternatively we showed the existence from a direct proof of Theorem 4.5 in combination with Proposition 5.1.

According to Theorem 3.2 it holds $\text{Alb}(X, X_{\text{sing}}) = \text{Alb}_{\hat{\text{Div}}^{0,\text{red}}_{Y/X}}(Y)$, and the latter was constructed as the dual 1-motive of $[\hat{\text{Div}}^{0,\text{red}}_{Y/X} \rightarrow \text{Pic}^{0,\text{red}}_Y]$ (see [Ru2, Remark 2.14]). This gives the existence and an explicit construction of the universal regular quotient of $\text{CH}_0(X, X_{\text{sing}})^0$, as well as a description of its dual. The proof of Theorem 0.1 is thus complete. ■

5.2 Case without Desingularization

If the characteristic of the base field $k$ is positive, it is not known whether there exists always a desingularization of the given projective variety $X$. Therefore we want to get rid of this assumption, which was used in the functorial description of the universal regular quotient of $\text{CH}_0(X, X_{\text{sing}})^0$. In this subsection we show that we can replace the desingularization of $X$ by any birational projective morphism $Y \rightarrow X$ where the variety $Y$ is normal and has the property that every rational map from $Y$ to an abelian variety is regular on all of $Y$. Then we construct a variety $Y$ with the required properties by blowing up an ideal sheaf on $X$.

Theorem 5.2. Let $\pi : Y \rightarrow X$ be a projective morphism with the properties

(Y 1) $\pi : Y \rightarrow X$ is birational.

(Y 2) Every rational map from $X$ to an abelian variety $A$ extends to a morphism from $Y$ to $A$.

(Y 3) $\text{Pic}^{0,\text{red}}_Y$ is an abelian variety.

Then the functor $[\hat{\text{Div}}^{0,\text{red}}_{Y/X} \rightarrow \text{Pic}^{0,\text{red}}_Y]$ is dual (in the sense of 1-motives) to the universal regular quotient $\text{Alb}(X, X_{\text{sing}})$ of $\text{CH}_0(X, X_{\text{sing}})^0$.

Proof. Let $\text{Fct}(X, \mathcal{A}V)$ denote the category of rational maps from $X$ to
abelian varieties, $\text{Mor}(X, AV)$ the category of morphisms from $X$ to abelian varieties. Let $\text{Alb}_{\text{fct}}(X)$ denote the classical Albanese variety of $X$ in the sense of functions, i.e. the universal object for $\text{Fct}(X, AV)$ (see [Lg]), and $\text{Alb}_{\text{mor}}(X)$ the Albanese variety of $X$ in the sense of morphisms, i.e. the universal object for $\text{Mor}(X, AV)$ (see [Se1]). Conditions (Υ 1) and (Υ 2) imply that $\text{Fct}(X, AV)$ is equivalent to $\text{Mor}(Y, AV)$, hence $\text{Alb}_{\text{fct}}(X) = \text{Alb}_{\text{mor}}(Y)$. Since $\text{Pic}^{0,\text{red}}_Y$ is an abelian variety by condition (Υ 3), the functoriality of $\text{Pic}^{0,\text{red}}_Y$ yields a universal mapping property of $\text{Pic}^{0,\text{red}}_Y$ for $\text{Mor}(Y, AV)$, hence $\text{Alb}_{\text{mor}}(Y)$ is the dual of $\text{Pic}^{0,\text{red}}_Y$.

Now the concept of categories of rational maps from $Y$ to algebraic groups can be set up in the same way as it was done for smooth proper varieties in [Ru2 Subsection 2.2]: A rational map $\varphi : Y \dashrightarrow G$ from $Y$ to an algebraic group $G$ with affine part $L$ and abelian quotient $A$ induces a transformation $\tau_{\varphi} : L^\vee \to \hat{\text{Div}}_Y$ as in [Ru2 Subsection 2.2.1], here we only needed $A(K_Y) = A(\mathcal{O}_{Y,q})$ for every point $q \in Y$ (which is nothing else than to say that every rational map from $Y$ to $A$ is regular at every $q \in Y$, and this is satisfied by conditions (Υ 1) and (Υ 2)). Then the existence and construction of universal objects for categories of rational maps from $Y$ to algebraic groups as in [Ru2 Subsection 2.2.2], the equality of the categories $\text{Mr}^{\text{CH}}$ and $\text{Mr}^{\text{Div}^{0,\text{red}}_{Y/X}}$ as in Theorem 3.2 and the representability of $\text{Div}^{0,\text{red}}_{Y/X}$ as in Theorem 1.5 carry over literally. Hence the proof of Theorem 0.1 from above is valid in this situation as well.

**Theorem 5.3.** To a given projective variety $X$ over $k$ there exists always a projective morphism $\pi : Y \to X$ satisfying the properties (Υ 1 – 3) from Theorem 5.2.

**Proof.** Consider the universal rational map $\alpha : X \dashrightarrow \text{Alb}_{\text{fct}}(X)$ from $X$ to the Albanese of $X$ in the sense of functions. As $\text{Alb}_{\text{fct}}(X)$ is an abelian variety, it is projective (see [Po] II, Theorem 8.12]). Let $\iota : \text{Alb}_{\text{fct}}(X) \to \mathbb{P}^r$ be an embedding of $\text{Alb}_{\text{fct}}(X)$ into projective space. Then the composition $\iota \circ \alpha : X \dashrightarrow \mathbb{P}^r$ lifts to a morphism $\beta : V \to \mathbb{P}^r$, where $\sigma : V \to X$ is the blowing up of a suitable ideal sheaf $\mathcal{J}$ on $X$ (see [Ha] II, Example 7.17.3)). The closed subscheme corresponding to $\mathcal{J}$ has support equal to $X \setminus U$, where $U$ is the maximal open set on which $\alpha$ is regular.

$$
\begin{array}{ccc}
V & \xrightarrow{\beta} & \mathbb{P}^r \\
\sigma \downarrow & & \downarrow \iota \\
X \xrightarrow{\alpha} & \text{Alb}_{\text{fct}}(X) & \\
\end{array}
$$
The image of $\beta$ is the closure of the image of $\imath \circ \alpha$ in $\mathbb{P}^r$, which is contained in the image of $\text{Alb}_{\text{ft}}(X)$ in $\mathbb{P}^r$:

$$\beta(V) \subset \overline{\beta(\sigma^{-1} U)} = \overline{\imath \alpha(U)} \subset \imath(\text{Alb}_{\text{ft}}(X)).$$

Thus $\beta$ induces a morphism $\alpha_V : V \to \text{Alb}_{\text{ft}}(X)$ extending $\alpha \circ \sigma|_{\sigma^{-1} U}$. Since $\alpha : X \to \text{Alb}_{\text{ft}}(X)$ is defined by the universal mapping property for rational maps from $X$ to abelian varieties, every rational map $\varphi : X \to A$ factors as $\varphi = h \circ \alpha : X \to \text{Alb}_{\text{ft}}(X) \to A$ and hence extends to a morphism $\varphi_V = h \circ \alpha_V : V \to \text{Alb}_{\text{ft}}(X) \to A$.

Proof of Theorem 0.2. Follows directly from Theorem 5.2 and Theorem 5.3.

5.3 Descent of the Base Field

Let $k$ be a perfect field. Let $\overline{k}$ be an algebraic closure of $k$. Let $X$ be a projective variety defined over $k$. Write $\overline{X} = X \otimes_k \overline{k}$.

We are going to show that the field of definition of the universal regular quotient $\text{Alb}(\overline{X}, \overline{X}_{\text{sing}})$ of $\text{CH}_0(\overline{X}, \overline{X}_{\text{sing}})^0$ descends to $k$. When we do not assume that $X$ is endowed with a $k$-rational point, the Albanese of Esnault-Srinivas-Viehweg for $X$ exists only as a $k$-torsor $\text{Alb}^{(1)}(X, X_{\text{sing}})$ for an algebraic $k$-group $\text{Alb}^{(0)}(X, X_{\text{sing}})$.

Definition 5.4. If $\varphi : X \to P$ is a rational map to a torsor $P$ for an algebraic group $G$ we define $\varphi^{(-)} : X \times X \to G$ to be the rational map which assigns to $(p, q) \in X \times X$ the unique $g \in G$ such that $g \cdot \varphi(p) = \varphi(q)$. We set $\varphi^{(1)} := \varphi$ and $\varphi^{(0)} := \varphi^{(-)}$. 22
**Definition 5.5.** Let \( \varphi : X \rightarrow P \) be a rational map defined over \( k \) to a \( k \)-torsor \( P \) for a \( k \)-group \( G \). Then \( \varphi \otimes_k k : X \otimes_k k \rightarrow P \otimes_k k \) over \( k \) maps 0-cycles of degree 0 to \( G(k) \): a cycle \( z \in Z_0(X \otimes_k k)^0 \) decomposes into a sum of 0-cycles of the form \( q - p \). By the property of \( P \otimes_k k \) of being a torsor for \( G \otimes_k k \), one may consider \( (\varphi \otimes_k k)(z) \) as an element of \( G(k) \).

We say that \( \varphi \) factors geometrically through rational equivalence, if the base-changed map \( \varphi \otimes_k k \) over \( k \) factors through rational equivalence in the sense of Definition 3.1.

**Theorem 5.6.** There exists a \( k \)-torsor \( \text{Alb}^{(1)}(X, X_{\text{sing}}) \) for an algebraic \( k \)-group \( \text{Alb}^{(0)}(X, X_{\text{sing}}) \) and rational maps defined over \( k \)

\[
\text{alb}_{X, X_{\text{sing}}}^{(i)} : X^{2-i} \rightarrow \text{Alb}^{(i)}(X, X_{\text{sing}})
\]

for \( i = 1, 0 \), satisfying the following universal property:

If \( \varphi : X \rightarrow G^{(1)} \) is a rational map defined over \( k \) to a \( k \)-torsor \( G^{(1)} \) for an algebraic \( k \)-group \( G^{(0)} \), factoring geometrically through rational equivalence (see Definition 5.5), there exist a unique affine homomorphism of \( k \)-torsors \( h^{(1)} \) and a unique homomorphism of algebraic \( k \)-groups \( h^{(0)} \) defined over \( k \), such that

\[
\varphi^{(i)} = h^{(i)} \circ \text{alb}_{X, X_{\text{sing}}}^{(i)}
\]

for \( i = 1, 0 \). Here \( \varphi^{(i)} \) are the rational maps from Definition 5.4.

**Proof.** Since \( X \) and the singular locus \( X_{\text{sing}} \) are defined over \( k \), the absolute Galois group \( \text{Gal}(\overline{k}/k) \) acts on \( \overline{X} \) and on the group \( \text{CH}_0(\overline{X}, \overline{X}_{\text{sing}})^0 \).

In particular, for the conjugates by means of \( \sigma \in \text{Gal}(\overline{k}/k) \) one has the equalities \( \overline{X}^\sigma = \overline{X} \) and \( (\text{CH}_0(\overline{X}, \overline{X}_{\text{sing}})^0)^\sigma = \text{CH}_0(\overline{X}, \overline{X}_{\text{sing}})^0 \).

Let \( (\text{alb}_{X, X_{\text{sing}}}^\sigma)^\sigma : \overline{X} \rightarrow \text{Alb}(\overline{X}, \overline{X}_{\text{sing}})^\sigma \) be the transform of \( \text{alb}_{X, X_{\text{sing}}} : \overline{X} \rightarrow \text{Alb}(\overline{X}, \overline{X}_{\text{sing}}) \) by means of \( \sigma \in \text{Gal}(\overline{k}/k) \). Since \( (\text{alb}_{X, X_{\text{sing}}}^\sigma)^\sigma \) factors through \( (\text{CH}_0(\overline{X}, \overline{X}_{\text{sing}})^0)^\sigma = \text{CH}_0(\overline{X}, \overline{X}_{\text{sing}})^0 \), it is an object of \( \text{Mr}^{\text{CH}}(\overline{X}) \). Then the map \( (\text{alb}_{X, X_{\text{sing}}}^\sigma)^\sigma \) factors as \( (\text{alb}_{X, X_{\text{sing}}}^\sigma)^\sigma = h_\sigma \circ \text{alb}_{X, X_{\text{sing}}}^\sigma \), where \( h_\sigma : \text{Alb}(\overline{X}, \overline{X}_{\text{sing}}) \rightarrow \text{Alb}(\overline{X}, \overline{X}_{\text{sing}})^\sigma \) is an affine homomorphism (i.e. a homomorphism of algebraic groups composed with a translation). Now we can apply the machinery of Galois descent, which is described e.g. in [Se3, V, § 4], cf. [Ru2, 2.3.3]. In particular, the proof of Theorem 5.6 is the same as in [Se3, V, No. 22].
Glossary

Categories

Set \( \text{sets} \) \([\text{Ru2} \ 1.1.1]\)
Ab \( \text{abelian groups} \) \([\text{Ru2} \ 1.1.1]\)
\( \text{Alg} / k \) \( k \)-algebras \([\text{Ru2} \ 1.1.1]\)
Art /\( k \) \( k \)-algebras of finite length \([\text{Ru2} \ 1.1.1]\)
Fctr(A, B) functors from \( A \) to \( B \) \([\text{Ru2} \ 1.1.1]\)
\( \text{Ab} / k \) \( k \)-group functors \( (= \text{Fctr(Alg} / k, \text{Ab})) \) \([\text{Ru2} \ 1.1.1]\)
\( \text{Ab} / k \) \( k \)-group sheaves (for fppf-topology) \([\text{Ru2} \ 1.1.5]\)
\( \text{G} / k \) \( k \)-groups \( (= \text{k-group schemes}) \) \([\text{Ru2} \ 1.1.3]\)
\( \text{Ga} / k \) affine \( k \)-groups \([\text{Ru2} \ 1.1.3]\)
\( \text{Ga} / k \) affine algebraic \( k \)-groups \([\text{Ru2} \ 1.1.3]\)
\( \text{Gf} / k \) formal \( k \)-groups \([\text{Ru2} \ 1.1.4]\)
d\( \text{Gf} / k \) dual-algebraic formal \( k \)-groups \([\text{Ru2} \ 1.2.1]\)

Functors

\( \hat{F} \) completion of \( F \in \text{Ab} / k \) \([\text{Ru2} \ 1.1.1]\)
\( \mathcal{F}_{\text{et}} \) \( \mathcal{F} \circ \text{red} \) étale part of \( \mathcal{F} \in \mathcal{G} / k \) \([\text{Ru2} \ 1.1.4]\)
\( \mathcal{F}_{\text{inf}} \) \( \ker (\mathcal{F} \to \mathcal{F}_{\text{et}}) \) infinitesimal part of \( \mathcal{F} \in \mathcal{G} / k \) \([\text{Ru2} \ 1.1.4]\)
\( L_R \) \( \mathcal{G}_{\text{m}} (?, \otimes R) \) linear group ass. to \( R \in \text{Alg} / k \) \([\text{Ru2} \ 1.1.6]\)
\( T_R \) \( L_{R_{\text{red}}} \) torus ass. to \( R \in \text{Alg} / k \) \([\text{Ru2} \ 1.1.6]\)
\( U_R \) \( \ker (L_R \to T_R) \) unipotent group ass. to \( R \in \text{Alg} / k \) \([\text{Ru2} \ 1.1.6]\)
\( \mathcal{F}_{X,D} \subset \text{Div}_X \) formal group of modulus \( D \) \([\text{Ru2} \ 3.2.1]\)
\( \text{Pic}_X \) Picard functor of \( X \) \([\text{Ru2} \ 2.1]\)
\( \text{Pic}_{X,0,\text{red}} \) reduced identity component of \( \text{Pic}_X \) \([\text{Ru2} \ 2.1]\)
\( \text{Div}_{X,0,\text{red}} \) \( \text{Div}_{X,0,\text{red}} = c_1^{-1} \text{Pic}_{X,0,\text{red}} \) \([\text{Ru2} \ 2.1]\)
\( \text{Div}_{X,0,\text{red}} \) flat Cartier divisors on \( X \) \([\text{Ru2} \ 2.1]\)
\( \text{Dec}_{X,V} \subset \text{Div}_X \) intersecting \( V \) properly \([\text{Ru2} \ 2.1]\)
\( \text{FDiv}_{X,C} \) formal divisors on a curve \( C \) \([\text{Ru2} \ 2.1]\)
\( \text{Div}_{Z/C} \) \( \ker (\tilde{\text{Div}}_Z \to \text{FDiv}_Z \to \text{FDiv}_{X,C}) \) \([\text{Ru2} \ 2.1]\)
\( \text{Div}_{Y/X} = \bigcap_C \left( ? \cdot \tilde{C} \right)^{-1} \tilde{\text{Div}}_{\tilde{C}/C} \) if \( Y \to X \) is birational \([\text{Ru2} \ 2.1]\)
\( \text{Div}_{Y/X} = \bigcap_C \left( ? \cdot \tilde{C}^Y \right)^{-1} \tilde{\text{Div}}_{\tilde{C}^Y/C} \cdot \tilde{C}^Y \)
**Algebraic groups**

| Symbol   | Description                                      | Reference |
|----------|--------------------------------------------------|-----------|
| $J(C)$   | Jacobian of a curve $C$                          | [Ru2, 3.3]|
| $J(C, D)$| Jacobian of $C$ of modulus $D$                   | [Ru2, 3.3]|
| $L(C, D)$| Affine part of $J(C, D)$                        | [Ru2, 3.3]|
| $T(C, D)$| Torus part of $J(C, D)$                         | [Ru2, 3.3]|
| $U(C, D)$| Unipotent part of $J(C, D)$                     | [Ru2, 3.3]|
| $\text{Pic}_X$ | Picard scheme of $X$     | [Ru2, 2.1.2]|
| $\text{Pic}_X^{\text{red}}$ | Picard variety of $X$             | [Ru2, 2.1.2]|
| $\text{Pic} (C, D)$ | Affine part of $J(C, D)$ | [Ru2, 3.3]|
| $\text{T}(C, D)$ | Torus part of $J(C, D)$ | [Ru2, 3.3]|
| $\text{U}(C, D)$ | Unipotent part of $J(C, D)$ | [Ru2, 3.3]|
| $\text{Alb}(X)$ | Albanese variety of $X$ | [Ru2, 2.3.1]|
| $\text{Alb}_F(X)$ | Universal object for $M_{rF}$ | [Ru2, 2.3.1]|
| $\text{Alb}^{(1)}(X)$ | Universal torsor for $M_{rF}$ | [Ru2, 2.3.3]|
| $\text{Alb}^{(0)}(X)$ | Universal group for $M_{rF}$ | [Ru2, 2.3.3]|
| $\text{Alb}(X, D)$ | Albanese variety of $X$ of modulus $D$ | [Ru2, 3.2.1]|
| $\text{Alb}(X, X_{\text{sing}})$ | Universal regular quotient of $\text{CH}_0(X, X_{\text{sing}})$ | 5.1|
| $\text{Alb}^{(1)}(X, X_{\text{sing}})$ | Torsor descended from $\text{Alb}(\overline{X}, \overline{X}_{\text{sing}})$ | 5.3|
| $\text{Alb}^{(0)}(X, X_{\text{sing}})$ | Group descended from $\text{Alb}(\overline{X}, \overline{X}_{\text{sing}})$ | 5.3|

**Chow Groups of 0-cycles**

| Symbol   | Description                                      | Reference |
|----------|--------------------------------------------------|-----------|
| $\text{CH}_0(X, D)$ | Relative Chow group of $X$ of modulus $D$ | [Ru2, 3.4]|
| $\text{CH}_0(X, D)^0$ | $= \ker (\deg : \text{CH}_0(X, D) \to \mathbb{Z})$ | [Ru2, 3.4]|
| $\text{CH}_0(X, X_{\text{sing}})$ | Relative Chow group by Levine-Weibel | 1.1|
| $\text{CH}_0(X, X_{\text{sing}})^0$ | $= \ker (\deg : \text{CH}_0(X, X_{\text{sing}}) \to \mathbb{Z})$ | 1.1|

**Rational Maps**

| Symbol   | Description                                      | Reference |
|----------|--------------------------------------------------|-----------|
| $M_{r}$  | A category of rational maps                      | [Ru2, 2.2]|
| $M_{r, \text{CH}}$ | Rational maps factoring through $\text{CH}_0(X, X_{\text{sing}})^0$ | 3|
| $M_{r, \text{CH}_0(X, D)^0}$ | Rational maps factoring through $\text{CH}_0(X, D)^0$ | [Ru2, 3.4]|
| $M_{r, X, D}$ | $= \{ \phi \mid \text{mod}(\phi) \leq D \}$ | [Ru2, 3.2.1]|
| $M_{r, F}$ | $= \{ \phi \mid \text{im}(\tau_{\phi}) \subset F \}$ | [Ru2, 2.2]|
| $\tau_{\phi}$ | $: L^r \to \text{Div}_X$ induced transformation of $\phi$ | [Ru2, 2.2]|
| $\text{mod}(\phi)$ | Modulus of rational map $\phi$ | [Ru2, 3.2]|

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