Turán-Type Results for Complete $h$-Partite Graphs in Comparability and Incomparability Graphs

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Abstract We consider an $h$-partite version of Dilworth’s theorem with multiple partial orders. Let $P$ be a finite set, and let $<_1, \ldots, <_r$ be partial orders on $P$. Let $G(P, <_1, \ldots, <_r)$ be the graph whose vertices are the elements of $P$, and $x, y \in P$ are joined by an edge if $x <_i y$ or $y <_i x$ holds for some $1 \leq i \leq r$. We show that if the edge density of $G(P, <_1, \ldots, <_r)$ is strictly larger than $1 - 1/(2h-2)^r$, then $P$ contains $h$ disjoint sets $A_1, \ldots, A_h$ such that $A_1 <_j \ldots <_j A_h$ holds for some $1 \leq j \leq r$, and $|A_1| = \ldots = |A_h| = \Omega(|P|)$. Also, we show that if the complement of $G(P, <)$ has edge density strictly larger than $1 - 1/(3h-3)$, then $P$ contains $h$ disjoint sets $A_1, \ldots, A_h$ such that the elements of $A_i$ are incomparable with the elements of $A_j$ for $1 \leq i < j \leq h$, and $|A_1| = \ldots = |A_h| = |P|^{1-o(1)}$. Finally, we prove that if the edge density of the complement of $G(P, <_1, <_2)$ is $\alpha$, then there are disjoint sets $A, B \subset P$ such that any element of $A$ is incomparable with any element of $B$ in both $<_1$ and $<_2$, and $|A| = |B| > n^{1-\gamma(\alpha)}$, where $\gamma(\alpha) \to 0$ as $\alpha \to 1$. We provide a few applications of these results in combinatorial geometry, as well.

Keywords Poset · Dilworth · Bipartite graph · Turan problem

1 Introduction

Let $k$ and $n$ be positive integers. A weak version of the widely used Dilworth’s theorem [2] states that every partially ordered set with $n$ elements either contains a chain of size $k$ or an antichain of size $\lceil n/k \rceil$. Applying Dilworth’s theorem multiple times, one can easily deduce the following result. Let $P$ be an $n$ element set, and let $<_1, \ldots, <_r$ be partial orders on $P$. There exists $H \subset P$ such that $|H| \geq \sqrt[3]{n}$, and $H$ is either a $<_i$-chain for some $1 \leq i \leq r$ or any two elements of $H$ are incomparable in any of the partial orders $<_1, \ldots, <_r$. 
Bipartite versions of Dilworth’s theorem have been considered in a series of papers by Fox, Pach and Tóth. Before we state their results, we introduce some notation.

Let \(<_1, \ldots, <_r\) be partial orders on a set \(P\). If \(a, b \in P\), write \(a \perp_i b\) if \(a\) and \(b\) are incomparable in \(<_i\). Also, write \(a \perp b\) if \(a \perp_i b\) holds for \(i = 1, \ldots, r\). If \(A, B \subseteq P\) and \(1 \leq i \leq r\), let \(A <_i B\) if for every \(a \in A\) and \(b \in B\) we have \(a <_i b\). Define \(A \perp_i B\) and \(A \perp B\) analogously.

In [4], Fox proved the following theorem for a single partial order.

**Theorem 1** ([4]) There exists \(n_0\) such that for all \(n > n_0\) and for all partially ordered sets \((P, <)\) on \(n\) elements, there exist \(A, B \subseteq P\) such that \(A\) and \(B\) are disjoint, 
\[|A| = |B| > \frac{n}{4 \log_2 n},\]
and either \(A < B\) or \(A \perp B\).

In [5], Fox and Pach generalized this result for multiple partial orders.

**Theorem 2** ([5]) Let \(r\) be a fixed positive integer and let \(<_1, \ldots, <_r\) be partial orders on the \(n\) element set \(P\). There exist \(A, B \subseteq P\) such that \(A\) and \(B\) are disjoint, 
\[|A| = |B| > \frac{n}{2(1+o(1)) (\log_2 \log_2 n)^r},\]
and either \(A <_i B\) holds for some \(1 \leq i \leq r\) or \(A \perp B\).

In [6], Fox, Pach and Tóth proved a Turán-type version of these results. Before we state it we introduce some further notation. If \(<_1, \ldots, <_r\) are partial orders on the set \(P\), let \(G(P, <_1, \ldots, <_r)\) be the graph whose vertex set is \(P\) and in which two elements \(a, b \in P\) are joined by an edge if \(a <_i b\) or \(b <_i a\) holds for some \(1 \leq i \leq r\). Call this graph the \(r\)-comparability graph of \((P, <_1, \ldots, <_r)\), and call the complement of \(G(P, <_1, \ldots, <_r)\) the \(r\)-incomparability graph of \((P, <_1, \ldots, <_r)\). Similarly, the directed comparability graph of \((P, <_1, \ldots, <_r)\) is \(\overrightarrow{G}(P, <_1, \ldots, <_r)\), in which \(\overrightarrow{xy}\) is an edge if \(x <_i y\) for some \(1 \leq i \leq r\). We note that it is allowed to have both \(\overrightarrow{xy}\) and \(\overrightarrow{yx}\) in the directed edge set.

For positive integers \(h, r, n, m\), define \(f^{C}_{r,h}(n, m)\) and \(f^{I}_{r,h}(n, m)\) as follows. Let \(f^{C}_{r,h}(n, m)\) be the maximal \(s\) such that if \(P\) is an \(n\) element set with partial orders \(<_1, \ldots, <_r\), and \(G(P, <_1, \ldots, <_r)\) has exactly \(m\) edges, then there exist \(1 \leq i \leq r\) and \(A_1, \ldots, A_h \subseteq P\) pairwise disjoint subsets such that \(|A_1| = \ldots = |A_h| = s\), and \(A_1 <_i \ldots <_i A_h\).

Similarly, let \(f^{I}_{r,h}(n, m)\) be the maximal \(s\) such that if \(P\) is an \(n\) element set with partial orders \(<_1, \ldots, <_r\), and the incomparability graph of \((P, <_1, \ldots, <_r)\) has exactly \(m\) edges, then there exist \(A_1, \ldots, A_h \subseteq P\) pairwise disjoint subsets such that \(|A_1| = \ldots = |A_h| = s\), and \(A_j \perp A_l\) for all \(1 \leq j < l \leq h\).

Here is the promised theorem by Fox, Pach and Tóth [6].

**Theorem 3** ([6])

(i) For every \(\epsilon > 0\), there exists \(c(\epsilon) > 0\) such that 
\[f^{C}_{1,2}(n, \left(\frac{1}{4} - \epsilon\right) n^2) < c(\epsilon) \log n.\]
For every $\epsilon > 0$,

$$f_{1,2}^C \left( n, \left( \frac{1}{4} + \epsilon \right) n^2 \right) > \frac{\epsilon n}{2}.$$ 

There is a constant $c_2 > 0$ such that for every $0 < \lambda < 1/2$,

$$f_{1,2}^I(n, \lambda n^2) > \frac{c_2 \lambda n}{\log n \log 1/\lambda}.$$ 

The aim of this paper is to generalize the previous theorem and to understand the behavior of the functions $f_{r,h}^C$ and $f_{r,h}^I$. Let us note a few things about Theorem 3. The functions $f_{1,h}^I$ and $f_{1,2}^C$ behave quite differently. As we can see, $f_{1,2}^C(n, m)$ has a large jump at $m/n^2 = 1/4$, and for $m/n^2 > 1/4$ the function $f_{1,2}^C(n, m)$ is linear in $n$. We show that $f_{r,h}^C$ has a similar behavior.

However, as we shall see, $f_{1,h}^I$ also jumps at some value of $m/n^2$ for $h > 2$.

Our paper is organized as follows. In the next section, we prove bounds on $f_{r,h}^C$ for arbitrary $r, h$ positive integers. We show that if $\alpha = 1/2 - 1/2(2h - 2)^r$, the function $f_{r,h}^C(n, m)$ jumps at the point $m/n^2 = \alpha$. If $m/n^2$ is strictly below the threshold $\alpha$, then $f_{r,h}^C(n, m)$ is $O(\log n)$, while above this point $f_{r,h}^C(n, m)$ becomes linear in $n$.

An $h$-partite graph is balanced if its classes have the same size. In Section 3, we investigate the largest balanced $h$-partite graph of the $1$-incomparability graph. We show that $f_{1,h}^I$ also jumps. If $m/n^2 < 1/2 - 1/2(h - 1)$, then $f_{1,h}^I(n, m) = 0$. However, for

$$\frac{m}{n^2} > \frac{1}{2} - \frac{1}{18(h - 1)} + \epsilon,$$

we have $f_{1,h}^I(n, m) = n^{1-o(1)}$.

In Section 4, we investigate the largest balanced bipartite graph of the $2$-incomparability graph. As we shall see, $f_{2,2}^I$ behaves quite differently as $f_{1,2}^I$. We show that $f_{2,2}^I(n, m)$ is approximately $n^\alpha$ for some $\alpha$ satisfying $\alpha \to 1$ as $m/n^2 \to 1/2$.

In the last section, we provide applications of these results for two problems in combinatorial geometry.

Before we start, we introduce some of the standard notation we use. As usual, $[n]$ denotes the set $\{1, \ldots, n\}$. If $G$ is a graph, $V(G)$ is the vertex set of $G$, $E(G)$ is the edge, $e(G) = |E(G)|$ is the number of edges, and $d(G) = e(G)/(\binom{|V(G)|}{2})$ is the edge density of $G$. If $X, Y \subseteq V(G)$, $G[X]$ is the subgraph of $G$ induced on $X$, and $G[X, Y]$ is the induced bipartite subgraph of $G$ with vertex classes $X$ and $Y$. Also, $K_s$ is the complete graph on $s$ vertices and $K_{s,t}$ is complete bipartite graph with vertex classes having sizes $s$ and $t$.

A linear extension of a partial order $<$ is a total order $<^*$ such that $x < y$ implies $x <^* y$. Also, the dual of $<$ is $<^d$, where $<^d$ is defined such that $x <^d y$ if $y < x$.

To avoid clutters, we omit floors and ceilings whenever they are not crucial.

## 2 The $r$-Comparability Graph

In this section, generalizing part (i) and (ii) of Theorem 3, we prove the following result about the behaviour of $f_{r,h}^C$. 

[2] Springer
Let $h, r, n$ be positive integers and $0 < \epsilon < 1/(2h - 2)^r$.

(i) We have

$$f_{r,h}^C\left(n, \left(\frac{1}{2} - \frac{1}{2(2h - 2)^r} - \epsilon\right)n^2\right) < 2\epsilon^{-1}(2h - 2)^r \log n.$$

(ii) There exists a constant $c(r, h, \epsilon) > 0$ such that

$$f_{r,h}^C\left(n, \left(\frac{1}{2} - \frac{1}{2(2h - 2)^r} + \epsilon\right)n^2\right) > c(r, h, \epsilon)n.$$  \hspace{1cm} (*)

Also, for $h = 2$, we have

$$f_{r,2}^C\left(n, \left(\frac{1}{2} - \frac{1}{2^{r+1}} + \epsilon\right)n^2\right) > \frac{\epsilon n}{r2^{r+1}}.$$  \hspace{1cm} (**)

**Proof of (i).** Let $G = (A, B, E)$ be a bipartite graph with

$$|A| = |B| = \frac{n}{(2h - 2)^r},$$

and $|E| > |A||B|(1 - \epsilon)$ such that $G$ does not contain $K_{r,t}$ with $t > 2\epsilon^{-1} \log n$. A random bipartite graph, where the edges are chosen with probability $(1 - \epsilon/2)$, has this property with high probability, see [1].

Define $(P, <_1, \ldots, <_r)$ as follows. Let $\{P_t\}_{t \in [2h - 2]^r}$ be a partition of the $n$-element set $P$ into $(2h - 2)^r$ equal sized parts, and let $f^r_t : P_t \rightarrow A$ and $g^r_t : P_t \rightarrow B$ be arbitrary bijections. Let $\overline{t} = (t_1, \ldots, t_r)$ and $\overline{u} = (u_1, \ldots, u_r)$ be two different elements of $[2h - 2]^r$ and suppose that the first coordinate they differ in is the $q$-th coordinate. Without loss of generality, $t_q < u_q$. If $t_q + 1 < u_q$, let $x <_q y$ for all $x \in P_t$ and $y \in P_u$. If $t_q + 1 = u_q$, let $x <_q y$ if $f^r_t(x)g^r_t(y) \in E$.

One can easily check that the relations $<_1, \ldots, <_r$ we have defined are partial orders. Also, $G(P, <_1, \ldots, <_r)$ contains at least

$$\left(\frac{(2h - 2)^r}{2}\right) \frac{(1 - \epsilon)n^2}{(2h - 2)^{2r}} > \left(\frac{1}{2} - \frac{1}{2(2h - 2)^r} - \epsilon\right)n^2$$

edges.

Suppose that $A_1, A_2, \ldots, A_h$ are disjoint subsets of $P$ such that $|A_1| = \ldots = |A_h| = t$ and $A_1 <_q \ldots <_q A_h$ with some $q \in [r]$. Then there exist $\overline{t}_1, \ldots, \overline{t}_h$ such that for $i = 1, \ldots, h$, we have

$$|P_{t_i} \cap A_i| > \frac{t}{(2h - 2)^r}.$$  

Also, the $q$-th coordinates of $\overline{t}_1, \ldots, \overline{t}_h$ are strictly monotone increasing. Hence, there exists $1 \leq j < h$ such that the difference between the $q$-th coordinate of $\overline{t}_j$ and $\overline{t}_{j+1}$ is 1. But then $f^r_{t_j}(A_j \cap P_{t_j})$ and $g^r_{t_{j+1}}(A_{j+1} \cap P_{t_{j+1}})$ span a complete bipartite graph in $G$, so $t/(2h - 2)^r < 2\epsilon^{-1} \log n$. Hence,

$$f_{r,h}^C\left(n, \left(\frac{1}{2} - \frac{1}{2(2h - 2)^r} - \epsilon\right)n^2\right) < 2(2h - 2)^r \epsilon^{-1} \log n.$$

In the rest of this section, we shall prove part (ii) of the theorem. We are going to deduce part (ii) from a Turán-type result for multicolored directed graphs. But first, we need some definitions.
A directed graph $D = (V, E)$ is a $k$-diamond, if $V = \{a, a', b_1, ..., b_k\}$ and
\[ E = \{ \overrightarrow{abi} : i = 1, ..., k \} \cup \{ \overrightarrow{b_ia'} : i = 1, ..., k \}. \]
Call the vertex $a$ the bottom of $D$ and $a'$ the top of $D$.

The directed graph $S = (V', E')$ is an $h$-part spiral, if its vertex set can be partitioned as $V' = \{a_1, ..., a_{h-1}\} \cup B_1 \cup ... \cup B_h$ such that $|B_1| = ... = |B_h|$ and
\[ E' = \{ \overrightarrow{bd_i} : i = 1, ..., h-1; b \in B_i \} \cup \{ \overrightarrow{a_ib'} : i = 1, ..., h-1; b' \in B_{i+1} \}. \]
Call $|B_1|$ the width of the spiral and $B_1, ..., B_h$ the classes of the spiral.

Also, a directed graph $R = (V'', E'')$ is an $h$-part rooted spiral, if its vertex set can be partitioned as $V'' = \{a_1, ..., a_h\} \cup B_2 \cup ... \cup B_{h+1}$ such that $|B_1| = ... = |B_{h+1}|$ and
\[ E'' = \{ \overrightarrow{ai} : i = 1, ..., h; a \in B_{i+1} \} \cup \{ \overrightarrow{ba_j} : j = 2, ..., h; b \in B_j \}. \]
Call $a_1$ the root and $|B_1|$ the width of the rooted spiral (Fig. 1).

It is clear that if the directed comparability graph of a partially ordered set $(P, <)$ contains an $h$-part rooted spiral with classes $B_1, ..., B_h$, then $B_1 < ... < B_h$. Hence, it is enough to find an $h$-part spiral with large width in the directed comparability graph. To prove such a result, we need the following lemma first.

**Lemma 5** Let $\epsilon > 0$ and $q, n$ be positive integers. Let $G = (V, E)$ be a directed graph with $|V| = n$, $|E| > (1/2 - 1/2^{q+1} + \epsilon)n^2$. Let $\chi : E \to [q]$ be a $q$ coloring of the edges. Then $G$ contains a monochromatic $k$-diamond with
\[ k > \frac{\epsilon^2 n}{q^2 2^{2q+2}}. \]

**Proof** Let $\lambda = \epsilon/q^{2q+1}$. For $W \subset V$, $x \in V$ and $i = 1, ..., q$, let
\[ U^W_i(x) = \{ y \in W : \overrightarrow{xy} \in E, \chi(\overrightarrow{xy}) = i \}, \]
and let
\[ D^W_i(x) = \{ z \in W : \overrightarrow{zx} \in E, \chi(\overrightarrow{zx}) = i \}. \]
For simplicity, write $U^V_i(x) = U^V_i(x)$ and $D^V_i(x) = D^V_i(x)$. Also, for all $H \subset [q]$, let
\[ V_H = \{ x \in V(G) : |U_i(x)| > \lambda n \iff i \in H \}. \]
The sets \( \{V_H\}_{H \subset \{q\}} \) partition \( V \) into \( 2^q \) parts. The number of edges connecting two different parts in this partition is at most

\[
\sum_{H_1, H_2 \subset \{q\} : H_1 \neq H_2} |V_{H_1}||V_{H_2}| \leq \left( \frac{2^q}{2} \right) \left( \frac{n}{2^q} \right)^2 = \left( \frac{1}{2} - \frac{1}{2^{q+1}} \right) n^2.
\]

Hence, there exists \( F \subset \{q\} \) such that \( G[V_F] \) contains at least \( \epsilon n^2/2^q \) edges. Let \( E' \) be the set of edges in \( G[V_F] \) whose color is in \( F \). Note that for every \( x \in V_F \) there are at most \( q \lambda n \) edges \( e \) containing \( x \) such that \( \chi(e) \notin F \). Thus,

\[
|E'| > \left( \frac{\epsilon}{2^q} - q \lambda \right) n^2 = \frac{\epsilon n^2}{2^{q+1}}.
\]

But then there exists \( p \in F \) such that \( G[V_F] \) contains at least \( \epsilon n^2/q2^{q+1} \) edges of color \( p \). So there exists \( a \in V_F \) with

\[
|U^V_F(a)| > \frac{\epsilon n}{q2^{q+1}}.
\]

Let \( A = U^V_F(a) \). There are at least

\[
\lambda n |A| > \frac{\epsilon^2 n^2}{q^{2^22(q+1)}}
\]

good edges of color \( p \) connecting an element of \( A \) with an element of \( V \), as every element of \( A \) has at least \( \lambda n \) edges of color \( p \) containing it. Hence, there exists \( a' \in V \) with

\[
|D^A_p(a')| > \frac{\epsilon^2 n}{q^{2^22(q+1)}}.
\]

Then the vertex set \( \{a, a'\} \cup D^A_p(a') \) spans a \( p \)-colored \( k \)-diamond with

\[
k > \frac{\epsilon^2 n}{q^{2^22(q+1)}}.
\]

Now we are ready to prove our key result about spirals.

**Theorem 6** Let \( r, h \) be positive integers and \( \epsilon > 0 \). There exists \( c(r, h, \epsilon) > 0 \) with the following property. Let \( G = (V, E) \) be a directed graph with \( |V| = n \) and

\[
|E| > \left( \frac{1}{2} - \frac{1}{2(2h - 2)r} + \epsilon \right) n^2,
\]

and let \( \chi : E \to [r] \) be an \( r \)-coloring of the edges of \( G \). Then \( G \) contains a monochromatic \( h \)-part spiral of width at least \( c(r, h, \epsilon)n \).

**Proof** Let \( \lambda \) be the unique solution of the quadratic equation

\[
\sqrt{\epsilon/h^r(\epsilon/h^r - r \lambda)^2} = \lambda
\]

satisfying \( \lambda < \epsilon/h^r \). We shall prove that \( G \) contains an \( h \)-part spiral of width at least \( \lambda n \).

Suppose to the contrary that \( G \) does not contain an \( h \)-part spiral of width at least \( \lambda n \). For \( W \subset V, x \in V \) and \( i \in [r] \), define \( U^W_i(x) \) and \( D^W_i(x) \) as in the previous proof. For \( x \in V \) and \( i \in [r] \), let \( l_i(x) \) be the largest \( l \) such that \( G \) contains an \( l \)-part rooted spiral with root \( x \) and width \( \lambda n \) in color \( i \). Note that if there exists \( x \in V \) and \( i \in [r] \) with \( l_i(x) \geq h \), we are
done as an $h$-part rooted spiral of width $\lambda n$ trivially contains an $h$-part spiral of width $\lambda n$. Hence, we can suppose that $0 \leq l_i(x) < h$.

For $\overline{t} = (t_1, \ldots, t_r) \in \{0, \ldots, h-1\}^r$, define

$$V_{\overline{t}} = \{x \in V : l_i(x) = t_i, i \in [r]\}.$$ 

The sets $\{V_{\overline{t}}\}_{\overline{t} \in \{0, \ldots, h-1\}^r}$ partition $V$ into $h^r$ parts. Let $n_{\overline{t}} = |V_{\overline{t}}|$. Also, let

$$I(\overline{t}) = \{i \in [r] : t_i \not\in \{0, h-1\}\},$$

and $\epsilon' = \epsilon / h^r$. We show that $G[V_{\overline{t}}]$ contains at most

$$\left(\frac{1}{2} - \frac{1}{2|I(\overline{t})|+1}\right)n_{\overline{t}}^2 + \epsilon' n^2$$

edges.

Suppose that $G[V_{\overline{t}}]$ has more than

$$\left(\frac{1}{2} - \frac{1}{2|I(\overline{t})|+1}\right)n_{\overline{t}}^2 + \epsilon' n^2$$

edges. First of all, this forces $n_{t_i}$ to be at least $\sqrt{\epsilon' n}$, as $G[V_{\overline{t}}]$ has more than $\epsilon' n^2$ edges. If $t_i = 0$ for some $i$, then the number of edges of color $i$ in $G[V_{\overline{t}}]$ is at most $\lambda n^2$. Otherwise, there exists $x \in G[V_{\overline{t}}]$ with $|U_i(x)| > \lambda n$, and $x \cup U_i(x)$ spans a 1-part rooted spiral of width $\lambda n$, contradicting $t_i = 0$.

Similarly, if $t_i = h-1$ for some $i$, then the number of edges of color $i$ in $G[V_{\overline{t}}]$ is also at most $\lambda n^2$, otherwise there exist $x \in G[V_{\overline{t}}]$ with $|D_i(x)| > \lambda n$. Taking the union of $D_i(x)$ and an $(h-1)$-part rooted spiral with root $x$ and width $\lambda n$, we get an $h$-part spiral of width $\lambda n$.

Hence, the number of edges in $G[V_{\overline{t}}]$ with color in $I(\overline{t})$ is at least

$$\left(\frac{1}{2} - \frac{1}{2|I(\overline{t})|+1}\right)n_{\overline{t}}^2 + (\epsilon' - r\lambda)n^2 > \left(\frac{1}{2} - \frac{1}{2|I(\overline{t})|+1} + \epsilon' - r\lambda\right)n_{\overline{t}}^2.$$

Applying Lemma 5 with $q = |I(\overline{t})|$, we get that there exists a monochromatic $k$-diamond in $G[V_{\overline{t}}]$ with color in $p \in I(\overline{t})$, where

$$k > \frac{(\epsilon' - r\lambda)^2n_{t_i}}{q^2r^{2q+2}} > \frac{(\epsilon' - r\lambda)^2\sqrt{\epsilon' n}}{r^22^{2q+2}} = \lambda n.$$

Let $a, a', b_1, \ldots, b_k \in V_{\overline{t}}$ be the vertices of this $k$-diamond, where the vertex $a$ is the bottom and $a'$ is the top of the diamond. Let $S$ be a $t_p$-part rooted spiral with root $a'$ and width $\lambda n$, then taking the union of this $k$-diamond and $S$, we get a $p$ colored $t_p + 1$-part rooted spiral with root $a$ and width $\lambda n$, contradicting $t_p(a) = t_p$.

So far, we showed that the graph induced on $V_{\overline{t}}$ can contain at most

$$\left(\frac{1}{2} - \frac{1}{2|I(\overline{t})|+1}\right)n_{\overline{t}}^2 + \epsilon' n^2$$

edges. Hence, the complement of $G$ contains at least

$$-\epsilon n^2 + \sum_{\overline{t}\in\{0,\ldots, h-1\}^r} \frac{n_{\overline{t}}^2}{2|I(\overline{t})|+1}$$
edges. Using the Cauchy-Schwarz inequality, we have
\[
\sum_{t \in \{0, \ldots, h-1\}} r_n \frac{n_t}{2^{t(h)+1}} \geq \left( \sum_{t \in \{0, \ldots, h-1\}} n_t \right)^2 \left( \sum_{t \in \{0, \ldots, h-1\}} 2^{t(h)+1} \right)^{-1} = \frac{n^2}{2(2h-2)^r}.
\]
Hence, \( G \) contains less than
\[
\left( \frac{1}{2} - \frac{1}{2(2h-2)^r} + \epsilon \right) n^2
\]
edges, which is a contradiction.

Solving the quadratic equation in the beginning of the proof yields
\[
c(r, h, \epsilon) = \Omega \left( \frac{\epsilon^{5/2}}{r^{2}2^{2r}h^{5r/2}} \right).
\]
However, in the case \( h = 2 \), we can get a better bound. In this special case, while we repeat the previous proof, we do not need to use Lemma 5 at any point. We can deduce the following result.

**Proposition 7** Let \( r \) be a positive integer and \( \epsilon > 0 \). Let \( G = (V, E) \) be a directed graph with \( |V| = n \) and \( |E| > (1/2 - 1/2^{r+1} + \epsilon)n^2 \). Any \( r \) coloring of the edges of \( G \) contains a monochromatic 2-part spiral of width at least \( \epsilon n / r^{2^{r+1}} \).

**Proof** We shall proceed similarly as in the previous proof and in the proof of Lemma 5. Let \( \lambda = \epsilon / r^{2^{r+1}}. \) For any \( H \subset [r] \) let
\[
V_H = \{ x \in V : |U_i(x)| \geq \lambda n \iff i \in H \}.
\]
The set system \( \{ V_H \}_{H \subset [r]} \) partitions \( V \) into \( 2^r \) parts. Thus, the number of edges connecting two different parts is at most \( (1/2 - 1/2^{r+1})n^2 \). Hence, there exists \( H_0 \subset [r] \) such that \( e(G[V_{H_0}]) > \epsilon n^2 / 2^r \). Let \( f \) be the number of edges of \( G[V_{H_0}] \) whose color is not in \( H_0 \). Then
\[
f < (r - |H_0|)|V_{H_0}| \lambda n < r \lambda n^2.
\]
Hence, the number of edges of \( G[V_{H_0}] \) whose color is in \( H_0 \) is at least
\[
\left( \frac{\epsilon}{2^r} - r \lambda \right) n^2 = r \lambda n^2.
\]
But then, there exists \( i \in H_0 \) and \( v \in V_{H_0} \) such that
\[
|D_i(v)| \geq |D^V_{H_0}(v)| > \lambda n.
\]
Setting \( B_1 = D_i(v), a_1 = v \) and \( B_2 = U_i(x) \), the set \( \{a_1\} \cup B_1 \cup B_2 \) spans a 2-spiral of width \( \lambda n \) of color \( i \) in \( G \). \( \square \)

After these preparations, the proof of Theorem 4 is immediate.

**Proof of Theorem 4, part (ii).** Let \( <_1, \ldots, <_r \) be partial orders on the \( n \) element set \( P \). Define the directed graph \( G = (P, E) \) and the coloring \( \chi : E \rightarrow [r] \) as follows: if \( x, y \in P \) are comparable in at least one of the partial orders \( <_1, \ldots, <_r \), then choose one of them, say \( <_i. \) Without loss of generality, \( x <_i y \). Let \( xy \in E \) and \( \chi(xy) = i \). By Theorem 6, there
exists a color \( p \) such that the directed graph \( G \) contains a \( p \)-colored \( h \)-part spiral of width \( c(r, h, \epsilon)n \), let its vertex set be \( \{a_1, ..., a_{h-1}\} \cup B_1 \cup ... \cup B_h \). But then \( B_1 <_p ... <_p B_h \) and \( |B_1| = ... = |B_h| > c(r, h, \epsilon)n \). Hence, (*) is proved.

In case \( h = 2 \), we repeat the proof of (*), but we use Proposition 7 instead of Theorem 6. This yields

\[
\begin{align*}
\frac{f_{r,2}(n, \left(\frac{1}{2} - \frac{1}{2^{r+1}} + \epsilon\right)n^2)}{\frac{r}{2^{r+1}}} > \frac{\epsilon n}{r^{2^r+1}}.
\end{align*}
\]

\[ \blacksquare \]

3 Balanced Complete \( h \)-Partite Subgraph in the Incomparability Graph

In this section, we prove a result about large balanced complete \( h \)-partite subgraphs in the incomparability graph of \( (P, <) \). Note that if \( P \) is the disjoint union of \( h - 1 \) chains, each of size \( n/(h - 1) \), then there is no \( K_h \) in the incomparability graph of \( (P, <) \). Hence, the incomparability graph of \( (P, <) \) needs to have density at least \( 1 - 1/(h - 1) \) if we hope to find a large balanced complete \( h \)-partite graph in it. Our next result shows that if we are slightly above this density, we do find a large balanced complete \( h \)-partite graph in the incomparability graph.

**Theorem 8** Let \( h \geq 2 \) be a positive integer and let \( s = \lceil\log_2 h\rceil \).

(i) For \( m < (1/2 - 1/2(h - 1))n^2 \), we have \( f_{1,h}^I(n, m) = 0 \).

(ii) For every \( \epsilon > 0 \), there exists \( c(h, \epsilon) > 0 \) such that

\[
\begin{align*}
f_{1,h}^I \left(n, \left(\frac{1}{2} - \frac{1}{18(h - 1)} + \epsilon\right)n^2\right) > \frac{c(h, \epsilon)n}{(\log n)^s}.
\end{align*}
\]

In the proof, we shall use the following easy corollary of Theorem 3 and Theorem 4.

**Proposition 9** Let \( h, n \) be positive integers. Let \( s \) be the smallest integer such that \( h \leq 2^s \). There exists \( c(h) > 0 \) with the following property. Let \( < \) be a partial order on the \( n \) element set \( P \). If \( n \) is sufficiently large, then either

(i) there exist \( A_1, ..., A_h \subset P \) disjoint sets such that

\[
|A_1| = ... = |A_s| > \frac{c(h)n}{(\log n)^s},
\]

and \( A_i \perp A_j \) for \( 1 \leq i < j \leq h \);

(ii) or there exist \( B_1, B_2, B_3 \subset P \) disjoint sets such that

\[
|B_1| = |B_2| = |B_3| > \frac{c(h)n}{(\log n)^s},
\]

and \( B_1 < B_2 < B_3 \).

**Proof** Let \( c = c(1, 3, 1/16) \), where \( c(r, h, \epsilon) \) is the constant defined in Theorem 4. If the comparability graph of a poset \( (Q, <) \), with \( |Q| = m \) has more than \( 7m^2/16 \) edges, then by Theorem 4 there exists \( B_1, B_2, B_3 \subset Q \) satisfying \( |B_1| = |B_2| = |B_3| > cm \) and \( B_1 < B_2 < B_3 \). Hence, we can suppose that the comparability graph of \( P \) does not contain a subgraph of size at least \( n/(\log n)^s \) with edge density larger than \( 7/8 \), otherwise (ii) holds.
if $c(h) < c$. But then, applying Theorem 3, every subgraph of size $n' > n/\log n$ contains two sets, $A$ and $A'$ such that $|A| = |A'| > c_0 n'/\log n'$ with a suitable constant $c_0 > 0$, and $A \perp A'$.

For $k = 0, \ldots, s$ and $i = 1, \ldots, 2^k$, we shall define the sets $X_{k,1}, \ldots, X_{k, 2^k} \subset P$ with the following properties: $X_{0,1} = P$; $|X_{k,1}| = \ldots = |X_{k, 2^k}| > \varepsilon_0 n/\log n^k$, and $X_{i,k} \perp X_{j,k}$ for $1 \leq i < j \leq 2^k$. Suppose that $X_{k,1}, \ldots, X_{k, 2^k}$ are already defined satisfying those properties. We define $X_{k+1,1}, \ldots, X_{k+1, 2^k+1}$ as follows. As $|X_{i,k}| > \varepsilon_0 n/\log n^k > n/\log n^k$ if $n$ is sufficiently large, there exist $X_{k+1, 2i-1}, X_{k+1, 2i} \subset X_{i,k}$ such that

$$|X_{k+1, 2i-1}| = |X_{k+1, 2i}| > \frac{c_0 |X_{i,k}|}{\log |X_{i,k}|} > \frac{\varepsilon_0^{k+1} n}{\log n^{k+1}},$$

and $X_{k+1, 2i-1} \perp X_{k+1, 2i}$. Then $X_{k+1, 1}, \ldots, X_{k+1, 2^k+1}$ also satisfy the properties. Set $A_i = X_{x,i}$ for $i = 1, \ldots, h$. Then (i) holds.

**Proof of Theorem 8.** We shall prove part (ii) of the theorem. Let $(P, \prec)$ be a partially ordered set $n$ elements such that

$$e(G(P, \prec)) < \left(\frac{1}{18(h-1)} - \varepsilon\right) n^2.$$

Let $k = \lceil 2\varepsilon^{-1}\rceil$. Let $\prec'$ be any linear extension of $\prec$, and let $x_1 \prec' \ldots \prec' x_n$ be the enumeration of the elements of $P$ by $\prec'$. Partition $P$ into $k$ equal $\prec'$ intervals $P_1, \ldots, P_k$. Namely, for $i = 1, \ldots, k$, let $P_i = \{x_{(i-1)n/k+1}, \ldots, x_{in/k}\}$.

Let $c_0 = c(h)$ be the constant defined in Proposition 9, and set $c(h, \varepsilon) = c_0 \varepsilon/k$. Also, let $z = c(h, \varepsilon)n/(\log n)^k$. Suppose that $P$ does not contain $A_1, \ldots, A_h$ disjoint sets such that

$$|A_1| = \ldots = |A_h| > z,$$

and $A_i \perp A_j$ for $1 \leq i < j \leq h$. By Proposition 9, every subset of $P$ of size at least $\varepsilon n/k$ contains three sets $B_1, B_2, B_3$ of size $z$ such that $B_1 < B_2 < B_3$. Let

$$m = \frac{(1 - \varepsilon)n}{3kz}. \quad (1)$$

Picking greedily, for $i = 1, \ldots, k$, we can find $3m$ disjoint sets

$$\{B_{i,j,t}\}_{j=1,\ldots,m; t=1,2,3}$$

in $P_i$, such that $|B_{i,j,t}| = z$ and $B_{i,j,1} < B_{i,j,2} < B_{i,j,3}$.

Define a new graph $H = ([k] \times [m], E)$ as follows: join $(i, j)$ and $(i', j')$ by an edge if $i = i'$ or there is an edge in $G(P, \prec)$ between $B_{i,j,2}$ and $B_{i', j', 2}$.

Suppose $H$ has $d$ edges. If $(i, j)$ and $(i', j')$ are joined by an edge, where $i < i'$, then $G(P, \prec)$ contains every edge between $B_{i,j,1}$ and $B_{i', j', 3}$. This is true as there exists $x \in B_{i,j,2}$ and $y \in B_{i', j', 2}$ with $x < y$, so for any $x' \in B_{i,j,1}$ and $y' \in B_{i', j', 3}$, we have $x' < x < y < y'$. The number of edges of $H$ of the form $\{(i, j), (i', j')\}$ is $k(\binom{m}{2})$. Hence, the number of edges $\{(i, j), (i', j')\}$ of $H$ with $i \neq i'$ correspond to at least

$$\binom{d - k \binom{m}{2}}{2} z^2$$

edges in $G(P, \prec)$. But $G(P, \prec)$ has at most $1/18(h-1) - \varepsilon)n^2$ edges, so

$$dz^2 - k z^2 \binom{m}{2} < \left(\frac{1}{18(h-1)} - \varepsilon\right) n^2.$$
Here, \( kz^2/m^2 < n^2/18k < \epsilon n^2/2 \). Hence, we have

\[
dz^2 < \left( \frac{1}{18(h - 1)} - \frac{\epsilon}{2} \right) n^2.
\]

Thus, using Eq. 1, we get

\[
d < 9k^2m^2 \left( \frac{1}{18h} - \frac{\epsilon}{2} \right) n^2(1 - \epsilon)^{-2} < \left( \frac{1}{2(h - 1)} - \epsilon \right) (km)^2.
\]

Applying Turán’s theorem [13] to \( H \) there is a complete graph on \( h \) vertices in the complement of \( H \). Let the vertices of this \( Kh \) be \((i_1, j_1), \ldots, (ih, jh)\). For \( l = 1, \ldots, h \), let \( A_l = B_{il, jl} \). Then \(|A_1| = \ldots = |Ah| = c(h, \epsilon) n/(\log n)^s\), and \( A_l \perp A_{l'} \) for \( 1 \leq l < l' \leq h \), which is a contradiction.

Slightly modifying the proof above, one can show that we can replace \( 1/2 - 1/18(h - 1) \) in (ii) with \( 1/2 - 1/8(h - 1) \). However, we conjecture that \( 1/2 - 1/2(h - 1) \) is the sharp threshold.

**Conjecture 10** Let \( h \) be a positive integer, \( \epsilon > 0 \). There exists \( c(h, \epsilon) > 0 \) such that

\[
f\left( n, \left( \frac{1}{2} - \frac{1}{2(h - 1)} + \epsilon \right) n^2 \right) > \frac{c(h, \epsilon)n}{(\log n)^s}
\]

holds.

## 4 Balanced Complete Bipartite Graph in the 2-Incomparability Graph

In this section, we investigate the size of the largest balanced complete bipartite graph in the 2-incomparability graph of \((P, <_1, <_2)\).

Fix a positive integer \( h \). By our previous results, if the edge density of the incomparability graph of \((P, <)\) exceeds some threshold strictly less than 1, we have a complete balanced \( h \)-partite graph of size \( n^{1-o(1)} \) in the incomparability graph. However, as we shall see, this is no longer true for the 2-incomparability graph, or in general, for the \( r \)-incomparability graph, where \( r \geq 2 \).

However, we show that if the incomparability graph of \((P, <_1, <_2)\) has edge density \((1 - \epsilon + o(1))\), there is a complete balanced bipartite graph of size \( n^{\beta(\epsilon)} \), where \( \beta(\epsilon) \rightarrow 1 \) as \( \epsilon \rightarrow 0 \). This is still much larger than the size of the largest balanced complete bipartite graph of a random graph, whose edges are chosen with probability \( 1 - \epsilon \). With high probability, such a graph has edge density \((1 - \epsilon + o(1))\), and its largest balanced bipartite graph has size \( O(\epsilon^{-1} \log n) \).

We prove the following result.

**Theorem 11** (i) For every \( 0 < \epsilon < 1 \) and positive integer \( k \geq 2 \), we have

\[
f^{I}_{2,2} \left( n, \left( \frac{1}{2} - \frac{1}{2k} - \epsilon \right) n^2 \right) < 2\epsilon^{-1}kn^{1-1/(k-1)} \log n.
\]

(ii) For every \( \delta > 0 \), if \( n \) is a sufficiently large positive integer, there exists \( \gamma(\delta) > 0 \) such that

\[
f^{I}_{2,2} \left( n, \left( \frac{1}{2} - \gamma(\delta) \right) n^2 \right) > n^{1-\delta}.
\]
The proof of part (i) is a probabilistic construction. We shall only briefly sketch the idea, the reader can find more about random graphs in [1].

Proof of (i). Our task is to construct partial orders \( \prec_1, \prec_2 \) on an \( n \) element set \( P \), such that the complement of \( G(P, \prec_1, \prec_2) \) does not contain a large complete bipartite graph.

For any positive integer \( N \), let \( G_N = (X_N, Y_N, E_N) \) be a bipartite graph with the following properties:

1. \( |X_N| = |Y_N| = N; \)
2. for every \( x \in X_N \cup Y_N \) we have \( \deg(x) < \epsilon N^{1/(k-1)} \);
3. the complement of \( G \) does not contain a \( K_{t, t} \) with \( t > 2e^{-1}N^{1/(k-1)} \log n \);
4. \( G_N \) has a complete matching \( M_N \).

If the edges of \( G \) are chosen independently with probability \( \epsilon N^{1/(k-1)} - 1/2 \), then with positive probability \( G \) satisfies conditions (2), (3) and (4).

Let \( A_1, \ldots, A_k \) be disjoint sets of size \( n/k \), and let \( P = A_1 \cup \ldots \cup A_k \). Let \( \prec_1 \) be any partial order such that \( A_1, \ldots, A_k \) are \( \prec_1 \)-chains, and \( A_i \perp_1 A_j \) for \( 1 \leq i < j \leq k \).

Now define \( \prec_2 \) as follows: for \( i = 1, \ldots, k \), let \( f_i : A_i \to X_{n/k} \) and \( g_i : A_i \to Y_{n/k} \) be arbitrary bijections. Define the relation \( <_{2}^{*} \) such that for any \( a \in A_i \) and \( b \in A_{i+1} \), where \( 1 \leq i \leq k - 1 \), we have \( a <_{2}^{*} b \) if \( f_i(a)g_{i+1}(b) \in E_{n/k} \). Let \( \prec_2 \) be the partial order induced by the relation \( <_{2}^{*} \).

First of all, we shall bound the number of edges of \( G(P, \prec_1, \prec_2) \) from above. Note that

\[
e(G(P, \prec_1)) = k \binom{n/k}{2} < n^2/2k.
\]

Also, \( e(G(P, \prec_2)) < \epsilon n \). This is true as for every \( 1 \leq i < j \leq k \) and \( x \in A_i, y \in A_j \), we have \( x \prec y \) iff there exists a sequence \( x_0, \ldots, x_{j-i} \) such that \( x_0 = a, x_{j-i} = y, x_l \in X_{i+l} \) for \( l = 1, \ldots, j-i-1 \), and \( (x_{i+l-1})g_{i+l-1}(x_{i+l}) \in E(G_{n/k}) \) for \( l' = 0, \ldots, j-i-1 \).

As every vertex in \( G_{n/k} \) has degree less than \( \epsilon N^{1/(k-1)} \), the number of such sequences with \( x_0 = a \) is at most

\[
e^{i-j/2} \binom{n/k}{l-j/(k-1)} < \epsilon n.
\]

Hence, for every \( x \in P \) there are at most \( \epsilon n \) elements \( y \in P \) such that \( x \prec y \). Thus,

\[
e(G(P, \prec_2)) < \epsilon n^2.
\]

We deduce that \( e(G(P, \prec_1, \prec_2)) < (1/2k + \epsilon)n^2 \).

Also, let \( X, Y \subseteq P \) be disjoint sets such that \( X \perp Y \) and \( |X| = |Y| \). Then, there exist positive integers \( t \) and \( u \) such that \( 1 \leq t, u \leq k \), \( |X \cap A_t| \geq |X|/k \) and \( |Y \cap A_u| \geq |Y|/k \).

We cannot have \( t = u \), otherwise, there exist \( x \in X \cap A_t \) and \( y \in Y \cap A_u \) with \( x \prec y \) or \( y \prec x \), contradicting \( X \perp Y \). Hence, \( t \neq u \). Without loss of generality, suppose that \( t < u \).

Let \( H \) be the bipartite subgraph of \( G(P, \prec_2) \) induced on \( A_t \cup A_u \). We show that \( H \) contains a subgraph isomorphic to \( G_{n/k} \). Let \( x \in A_t \), arbitrary, and let \( a_0(x), \ldots, a_{u-t-1}(x) \) be the unique sequence such that \( a_0(x) = x, a_l(x) \in A_{t+l} \) for \( l = 1, \ldots, u-t-1 \), and \( f_{t+l'}(a_l'(x))g_{t+l'+1}(a_{l+1}(x)) \in M_{n/k} \). As \( M_{n/k} \) is a complete matching, every \( a_l : A_t \to A_{t+l} \) is a bijection. Also, the subgraph of \( G(P, \prec_2) \) induced on \( A_{u-1} \cup A_u \) is isomorphic to \( G_{n/k} \). If \( x' \in A_{u-1} \) and \( x'' \in A_u \) with \( x' \prec x'' \), then \( a_{u-1}^{-1}(x') \prec x'' \). Hence, the subgraph of \( G(P, \prec_2) \) induced on \( A_t \cup A_u \) contains a subgraph isomorphic to \( G_{n/k} \).
Thus, the complement of $H$ does not contain $K_{t,t}$ with
\[ t > 2\varepsilon^{-1}N^{1-1/(k-1)}\log n, \]
so
\[ |X| = |Y| < 2kN^{1-1/(k-1)}\log n < 2\varepsilon^{-1}kn^{1-1/(k-1)}\log n. \]

Our next aim is to prepare the proof of part (ii) of Theorem 11. It turns out, our proof would be simpler if $<_1$ and $<_2$ had a common linear extension, which is not the case in general. However, the next lemma shows that we can find a constant number of large subsets in our poset such that between these subsets $<_1$ and $<_2$ behave as if they had a common linear extension.

**Lemma 12** Let $r, h \geq 2$ be positive integers. There exists $c(r, h) > 0$ with the following property. Let $<_1^0, \ldots,<_r^0$ be partial orders on the $n$ element set $P$, and for $s = 1, \ldots, r$, let $<_s^0$ be the dual of $<_s^0$. There exist $A_1, \ldots, A_h \subset P$ pairwise disjoint sets and $\alpha_1, \ldots, \alpha_r \in \{0, 1\}$ such that
\begin{enumerate}[(i)]
  \item $|A_1| = \ldots = |A_h| > c(r, h)n$;
  \item if $x \in A_i$ and $y \in A_j$ with $1 \leq i < j \leq h$, and $x$ and $y$ are comparable in $<_s$, then $x < _s^{\alpha_s} y$.
\end{enumerate}

**Proof** For $s = 1, \ldots, r$, let $<_s^r$ be a linear extension of $<_s^0$. It is enough to prove our lemma for $<_1^r, \ldots,<_r^r$ instead of $<_1^0, \ldots,<_r^0$. We shall deduce Lemma 12 from the following claim. \hfill \Box

**Claim 13** Let $p$ and $r$ be positive integers. There exists $c'(p, r) > 0$ with the following property. Let $<_1, \ldots,<_r$ be total orders on the $n$ element set $P$. There exist $B_1, \ldots, B_p \subset P$ pairwise disjoint subsets such that
\begin{enumerate}[(i)]
  \item $|B_1| = \ldots = |B_p| > c'(p, r)n$;
  \item for $s = 1, \ldots, r$ and $1 \leq i < j \leq r$, we have either $B_i <_s B_j$ or $B_j <_s B_i$.
\end{enumerate}

**Proof** We shall proceed by induction on $r$. In case $r = 1$, the statement is trivial with $c'(p, 1) = 1/p$. Let $r \geq 2$ and suppose the statement holds for $r - 1$ instead of $r$. Let $C_1, \ldots, C_p \subset P$ be disjoint sets such that
\[ |C_1| = \ldots = |C_p| > c'(p, r - 1)n, \]
and for every $1 \leq i < j \leq p$ and $s = 1, \ldots, r - 1$, we have $C_i <_s C_j$ or $C_j <_s C_i$. Let $P' = \bigcup_{i=1}^p C_i$, and for $x \in P'$, let $\tau(x)$ be the position of $x$ in the order $<_r$ in $P'$. For $i = 1, \ldots, p$, let
\[ D_j = \left\{ x \in P : \frac{(j-1)|P'|}{p} < \tau(x) \leq \frac{j|P'|}{p} \right\}. \]
We have $D_i <_r D_j$ for any $1 \leq i < j \leq p$. Our $B_1, \ldots, B_p$ are going to be suitable subsets of $C_1, \ldots, C_p$ and $D_1, \ldots, D_p$. 

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Let $S, T$ be two disjoint copies of $[p]$, and define the bipartite graph $G = (S, T, E)$ as follows: for $i \in S$ and $j \in T$, let $ij \in E$ if

$$|C_i \cap D_j| > \frac{|P'|}{p^2(p + 1)}.$$ 

We show that $G$ has a complete matching. By Hall’s theorem [7], we only need to check if Hall’s condition holds. Let $X \subset [p]$ be arbitrary and let $\Gamma(X)$ denote the set of neighbours of $X$ in $G$. Let $U = \bigcup_{i \in X} D_i$, then

$$|U| = \frac{|X||P'|}{p}.$$ 

Also, the elements of $\Gamma(X)$ cover at most $|\Gamma(X)||P'|/p$ elements in $U$, while the elements not in $\Gamma(X)$ cover at most $p|X|(|P'|/p^2(p + 1)) = |P'||X|/p(p + 1)$ elements in $U$. Hence, we obtain

$$\frac{|X||P'|}{p} \leq \frac{|\Gamma(X)||P'|}{p} + \frac{|P'||X|}{p(p + 1)}.$$ 

Thus, we have

$$|X|\left(1 - \frac{1}{(p + 1)}\right) \leq |\Gamma(X)|.$$ 

But $|X|$ and $|\Gamma(X)|$ are integers not larger than $p$. Hence, $|X| \leq |\Gamma(X)|$ also holds. So, Hall’s condition is satisfied and there exists a complete matching in $G$. Let the edge set of such a matching be $\{ix_i : i \in S\}$. Setting $B_i = C_i \cap D_{x_i}$ and $c'(p, r) = c'(p, r - 1)/p^2(p + 1)$, we have both (i) and (ii) satisfied. \hfill \Box

Let $p = (h - 1)^{2r-1} + 1$ and let $B_1, ..., B_p \subset P$ be disjoint sets such that $|B_1| = \cdots = |B_p| > c'(p, r)n$, and for $1 \leq i < j \leq p$ and $1 \leq s \leq r$, we have either $B_i <_s B_j$ or $B_j <_s B_i$. Define the partial orders $\{<_\pi\}_{\pi \in [0, 1]^{r-1}}$ on $[p]$ as follows: for $i, j \in [p]$ and $\pi \in [0, 1]^{r-1}$, let $i <_\pi j$ if $B_i <_s B_j$, and for $s = 1, ..., r - 1$, we have $B_i <_s B_j$ in case $v_s = 0$, and $B_j <_s B_i$ in case $v_s = 1$. Then any two different elements of $[p]$ are comparable in at least one of the partial orders $\{<_\pi\}_{\pi \in [0, 1]^{r-1}}$. Hence, by repeated applications of Dilworth’s theorem, there exist $\pi \in \{0, 1\}^{r-1}$ and $C \subset [p]$ such that

$$|C| \geq [p^{1/2r-1}] = h,$$

and $C$ is a $<_\pi$ chain. Let $i_1 <_\pi \cdots <_\pi i_h$ be $h$ elements of this chain, and for $j = 1, ..., h$, let $A_j = B_{i_j}$. Also, for $s = 1, ..., r$, let $a_s = w_j$. Finally, let $c(r, h) = c'(h, p)$. Then the conditions of the theorem are satisfied. \hfill \Box

Before we start the proof of part (ii) of Theorem 11, we still need the following two lemmas.

**Lemma 14** Let $A_0, ..., A_k$ be pairwise disjoint sets of size $m$, and let

$$P = \bigcup_{i=1}^k A_i.$$ 

Let $<$ be a partial order on $P$ such that whenever $x < y$ for some $x \in A_i$ and $y \in A_j$, then $i < j$. Suppose that $G(P, <)\{A_0, A_k\}$ has less than $m^2/4$ edges. There exist $0 \leq l \leq k - 1$ and $X \subseteq A_l, Y \subseteq A_{l+1}$ such that $|X|, |Y| > m^{1-1/k}$, and $X \perp Y$. 

\hfill Springer
Proof For any $X \subset P$ and $i = 1, ..., k$, let

$$U_i(X) = \{ y \in A_i : \exists x \in X, x < y \}. $$

Let $B = \{ x \in A_0 : |U_k((x))| < m/2 \}$. Then $|B| > m/2$, otherwise $G(P, <)[A_0, A_k]$ has more than $m^2/4$ edges. Suppose that there is no $l \in \{0, ..., k - 1\}$ and subsets $X \subset A_l, Y \subset A_{l+1}$ such that $|X| = |Y| > m^{1-1/k}$, and $X \perp Y$.

We show that we can find a decreasing sequence of sets $B \supseteq B_1 \supseteq \ldots \supseteq B_k$ with the following properties: $|B_i| = 2^{k-i}m^{1-1/k}$, and $|U_i(B_i)| > m/2$. Note that $B_k$ is a one element set. Hence, writing $x$ for that one element, we have

$$|U_k((x))| > \frac{m}{2},$$

contradicting $x \in B$, finishing our proof.

We shall define our sets $B_1, ..., B_k$ recursively. Let $B_1$ be any subset of $B$ of size $2^{k-1}m^{1-1/k}$. If $|U_1(B_1)| \leq m/2$, then choosing $X = B_1$ and $Y = A_1 \setminus U_1(B_1)$, we have $X \perp Y$ and $|X|, |Y| > m^{1-1/k}$. Hence, we have $|U_1(B_1)| \leq m/2$. \qed

Suppose that $B_i$ is already defined satisfying $|B_i| = 2^{k-i}m^{1-1/k}$ and $|U_i(B_i)| > m/2$.

Claim 15 For any positive integer $t \leq |B_i|$, we can choose a set $C \subset B_i$ such that $|C| = t$ and $|U_i(C)| \geq |U_i(B)|t/|B_i|.$

Proof Let $x_1, ..., x_p$ be the elements of $B_i$. Let $S_1, ..., S_p$ be a partition of $U(B_i)$ such that $S_j \subset U_i((x_j))$ for $j = 1, ..., p$. Without the loss of generality, $|S_1| \geq \ldots \geq |S_p|$. Set $C = \{x_1, ..., x_t\}$, then

$$|U_i(C)| \geq |S_1| + \ldots + |S_t| \geq \frac{|U_i(C)|t}{|B_i|}. $$

\qed

Setting $t = 2^{k-i}m^{1-(i+1)/k}$, we get a set $C$ such that

$$|C| = 2^{k-i}m^{1-(i+1)/k},$$

and $|U_i(C)| \geq m^{1-1/k}$. If $|U_{i+1}(C)| \leq m/2$, then set $X = C$ and $Y = A_{i+1} \setminus U_{i+1}(C)$. Then, we have $X \perp Y$ and $|X|, |Y| > m^{1-1/k}$, which is a contradiction. Hence, $|U_{i+1}(C)| > m/2$, and $B_{i+1} = C$ satisfies our conditions. \qed

We also need the following easy corollary of Theorem 2, which we shall state without proof.

Proposition 16 Let $<_1, <_2$ be partial orders on the $n$ element set $P$. At least one of the following holds:

(i) there exist $A_1, A_2 \subset P$ such that $|A_1| = |A_2| > n^{1-o(1)}$, and $A_1 \perp A_2$;  
(ii) there exist $B_1, B_2, B_3 \subset P$ such that $|B_1| = |B_2| = |B_3| > n^{1-o(1)}$, and $B_1 <_1 B_2 <_2 B_3$ or $B_1 <_1 B_2 <_1 B_3$.

Proof of Theorem 11, (ii). We have to prove that there exists a constant $\gamma(\delta)$ such that if $P$ is a set with $n$ elements, and $<_1, <_2$ are partial orders on $P$ satisfying $e(G(P, <_1, <_2)) < \gamma(\delta)n^2$, then $P$ contains two disjoint subsets $A, B$ of size at least $n^{1-\gamma}$ such that $A \perp B$. For simplicity, let $G_1 = G(P, <_1)$ and $G_2 = G(P, <_2)$.
Suppose that \( P \) does not contain two disjoint subsets \( A, B \) of size at least \( n^{1-\delta} \) such that \( A \perp B \). Let \( k = \lfloor 2/\delta \rfloor \) and \( h = 128k \), and let \( c_1 = c(2, h) \), where \( c(r, h) \) is the constant defined in Lemma 12. Then there exist \( L_1, \ldots, L_h \subset P \) pairwise disjoint sets with the following properties: \( |L_1| = \ldots = |L_h| = c_1n \); replacing \( <2 \) with its dual if necessary, if \( x \in L_i \) and \( y \in L_j \) for some \( 1 \leq i < j \leq h \), and \( x, y \) are comparable in \( <1 \) or \( <2 \), then \( x < y \) or \( y < x \), respectively.

Let \( m = n^{1-\delta/2} \). By Proposition 16, if \( n \) is sufficiently large, every subset of \( P \) of size at least \( c_1n/2 \) contains three disjoint subsets \( B_1, B_2, B_3 \) of size \( m \) such that \( B_1 < B_2 < B_3 \) or \( B_1 < B_2 < B_3 \). Hence, we can cover at least half of \( L_i \) with disjoint triples of subsets such that each set has size \( m \) and each triple spans a balanced complete 3-partite graph in \( G_1 \) or in \( G_2 \).

More precisely, let \( s = c_1n/2m \). Then, for \( i = 1, \ldots, h \), there is a system of disjoint sets \( \{B_{i,j,l}\}_{j=1, \ldots, s_i=1, 2, 3} \) such that \( B_{i,j,l} \subset L_i \), \( |B_{i,j,l}| = m \), and \( B_{i,j,1} < B_{i,j,2} < B_{i,j,3} \) or \( B_{i,j,1} < B_{i,j,2} < B_{i,j,3} \). Call the pair \((i, j)\) in \([h] \times [s] \) type 1, if \( B_{i,j,1} < B_{i,j,2} < B_{i,j,3} \), and call it type 2 otherwise. Without the loss of generality, we can suppose that there are at least \( sh/2 \) type 1 pairs in \([h] \times [s] \), and let \( S \) be the set of such pairs.

Let \( H = (S, E) \) be the complete graph on \( S \), and let \( w \) be a weight function defined on \( E \) as follows. Let \((i, j), (i', j') \in S\), and let \( f \) be the edge joining \((i, j)\) and \((i', j')\). If \( i = i' \), or there exist \( x \in B_{i,j,2} \) and \( y \in B_{i',j',2} \) such that \( x < y \), then let \( w(f) = 1 \); otherwise, let

\[
w(f) = \frac{e(G_2[B_{i,j,2}, B_{i',j',2}])}{m^2}.
\]

Note that if there exist \( x \in B_{i,j,2} \) and \( y \in B_{i',j',2} \) such that \( x < y \), then \( B_{i,j,1} < B_{i',j',1} \). Hence, there are at least \( m^2 \) edges between \( B_{i,j,1} \cup B_{i,j,2} \cup B_{i,j,3} \) and \( B_{i',j',1} \cup B_{i',j',2} \cup B_{i',j',3} \) in \( G_1 \). Thus, if \( i \neq i' \), there are at least \( w(f)m^2 \) edges between \( B_{i,j,1} \cup B_{i,j,2} \cup B_{i,j,3} \) and \( B_{i',j',1} \cup B_{i',j',2} \cup B_{i',j',3} \). Also, the number of edges \( \{(i, j), (i', j')\} \) in \( H \), where \( i = i' \), is at most

\[
h\left(\frac{s}{2}\right) < hs^2.
\]

Let \( w(E) = \sum_{f \in E} w(f) \). Then the number of edges of \( G(P, <1, <2) \) is at least

\[
(w(E) - hs^2)m^2.
\]

Let \( t \) be the number of edges \( f \in E \) such that \( w(f) \leq 1/4 \). We show that

\[
t \leq |S|^2 \left(\frac{1}{2} - \frac{1}{2k}\right).
\]

Suppose that \( t > |S|^2(1/2 - 1/2k) \). Consider the graph \( H' \) with vertex set \( S \), and edge set \( E' = \{f \in E : w(f) \leq 1/4\} \). By Turán’s theorem [13], there exists \( T \subset S \) of size \( k + 1 \) such that \( H'[T] \) is a complete graph. Let \((i_0, j_0), \ldots, (i_k, j_k)\) be the elements of \( T \) and suppose that \( i_0 < \ldots < i_k \). First, note that \( A_{i_l,j_l,1} \perp A_{i_l',j_l',2} \) for all \( 0 \leq l < l' \leq k \), as the weight of the edge \( \{(i_l, j_l), (i_l', j_l')\} \) is less than 1.

Set \( A_l = B_{i_l,j_l,2} \) for \( l = 0, \ldots, k \). Then \( e(G_2[A_0, A_k]) \leq m^2/4 \). Hence, by Lemma 14, there exist \( 0 \leq l \leq k - 1 \) and \( X \in A_l, Y \in A_{l+1} \) such that \( |X| = |Y| = m^{1-1/k} \), and \( X \perp Y \). But then \( X \perp Y \), and

\[
m^{1-1/k} > n(1-\delta/2)^2 > n^{1-\delta},
\]
contradiction. Thus, we must have

\[ t \leq |S|^2 \left( \frac{1}{2} - \frac{1}{2k} \right). \]

Then

\[ w(E) = \sum_{f \in E} w(f) > \frac{|E| - t}{4} > \]

\[ > \frac{1}{4} \left( \left( \frac{|S|}{2} \right) - |S|^2 \left( \frac{1}{2} - \frac{1}{2k} \right) \right) = \frac{|S|^2}{8k} - \frac{|S|}{8} > \frac{|S|^2}{16k}, \]

where the last inequality holds if \( n \) is sufficiently large. Plugging this result in Eq. 2, we get the following lower bound on the number of edges of \( G(P, <_1, <_2) \):

\[ e(G(P, <_1, <_2)) > (w(E) - hs^2)m^2 > \left( \frac{|S|^2}{16k} - hs^2 \right) m^2 > \]

\[ \left( \frac{h^2s^2}{64k} - hs^2 \right) m^2 = \frac{h^2s^2m^2}{128k} > 256c^2_1n^2\delta^{-1}. \]

Thus, setting \( \gamma(\delta) = 256c^2_1\delta^{-1} \) finishes the proof of the theorem.

We remark that if \( <_1 \) and \( <_2 \) have a common linear extension, which is often the case in applications, then we do not need to use Lemma 12 in the previous proof and we can simply write \( 1/h \) instead of \( c_1 \). Then we get the bound \( \gamma(\delta) = \delta/256 \), which almost matches the constant of part (i) in Theorem 11. However, we conjecture that an even stronger bound holds in general.

**Conjecture 17** Let \( k \) be a positive integer. If \( 1 - 1/k \leq \alpha < 1 - 1/(k + 1) \), we have

\[ f^{l_1}_{2,2}(n, \alpha n^2/2) = n^{1-1/k+o_\alpha(1)}, \]

where \( o_\alpha(1) \) is some function of \( n \) satisfying \( o_\alpha(1) \to 0 \) as \( n \to \infty \), with \( \alpha \) fixed.

We also conjecture that \( f^{l_1}_{r,h}(n, m) \) has a similar growth as \( f_{2,2}(n, m) \) for \( r \geq 3 \) or \( r = 2 \) and \( h \geq 3 \), but we cannot even quantify a precise conjecture for these cases.

## 5 Applications

Partial orders naturally arise in some geometric problems. The intersection graph of a set system \( C \) is the graph \( G = G(C, E) \), where \( A, B \in C \) forms an edge if \( A \cap B \neq \emptyset \). The intersection graph of convex sets in the plane was investigated in a series of papers. Larman et. al. [8], and Pach and Törőcsik [10] showed that the intersection graph of convex sets is a 4-incomparability graph. Hence, by an immediate application of Dilworth’s theorem yields that amongst \( n \) convex sets there are always at least \( n^{1/5} \) such that they are pairwise disjoint, or any two of them intersects. Also, as it was noted in [5], Theorem 2 implies a bipartite version of this theorem, namely that if \( C \) is a family of \( n \) convex sets, then there are \( A, B \subseteq C \) of size \( n^{1-o(1)} \) such that for any \( A \in A \) and \( B \in B \), we have \( A \cap B = \emptyset \), or for any \( A \in A \) and \( B \in B \), we have \( A \cap B \neq \emptyset \). In [6], this result was improved, showing that we can find two linear sized families \( A, B \subseteq C \) with the same property.

Call a set in \( \mathbb{R}^2 \) vertically convex, if every vertical line intersects the set in an interval. The intersection graph of connected, vertically convex sets is also a 4-incomparability graph.
Hence, Theorem 2 also implies the existence of a complete bipartite graph of size \( n^{1-o(1)} \) either in the intersection graph or its complement. However, we can no longer guarantee a linear sized complete bipartite graph in the intersection graph or in its complement. In [9], it is shown that for any \( \epsilon > 0 \), there is a collection of \( n \) continuous functions on \([0, 1]\) such that the largest bipartite graph in the intersection graph has size \( O(n/ \log n) \), and the largest complete bipartite graph in its complement has size \( O(n^\epsilon) \).

Nevertheless, Theorem 4 immediately implies that if the intersection graph of vertically convex sets is sparse enough, then we have a linear sized complete bipartite graph in its complement.

**Theorem 18** Let \( \epsilon > 0 \) and let \( C \) be a collection of \( n \) connected, vertically convex sets in the plane. If the number of unordered pairs \( \{A, B\} \in C(2) \) with \( A \cap B \neq \emptyset \) is less than \( n^2(1/32 - \epsilon) \), then there are \( A, B \subset C \) such that

\[
|A| = |B| > \frac{\epsilon n}{128},
\]

and for every \( A \in \mathcal{A}, B \in \mathcal{B} \) we have \( A \cap B = \emptyset \).

**Proof** As we aim for the self-containment of the paper, we shall define the 4-partial orders on \( C \), whose incomparability graph is the intersection graph. For any \( C \in C \), let

\[
l(C) = \inf\{x \in \mathbb{R} : \exists y : (x, y) \in C\}
\]

and let

\[
r(C) = \sup\{x \in \mathbb{R} : \exists y : (x, y) \in C\}.
\]

Define the relations \( <_1, <_2, <_3 \) on \( C \) as follows:

- \( C <_1 D \), if \( l(C) \leq l(D) \) and \( r(C) \leq l(D) \);
- \( C <_2 D \), if \( l(C) \leq l(D) \) and \( r(D) \leq r(C) \);
- \( C <_3 D \), if for every vertical line \( l \) which intersects both \( C \) and \( D \), the interval \( l \cap C \) is below \( l \cap D \).

Note that \( <_1, <_2, <_3 \) are not partial orders, as it is possible that \( C <_i D \) and \( D <_i C \) both hold.

However, define the relations \( <_1, <_2, <_3, <_4 \) on \( C \) as follows:

- \( C <_1 D \), if \( C <_1 D \) and \( C <_3 D \);
- \( C <_2 D \), if \( C <_1 D \) and \( D <_3 C \);
- \( C <_3 D \), if \( C <_2 D \) and \( C <_3 D \);
- \( C <_4 D \), if \( C <_2 D \) and \( D <_3 C \).

One can easily check that \( <_1, <_2, <_3, <_4 \) are partial orders on \( C \), and \( C \) and \( D \) are comparable in some \( <_i \) if and only if \( C \) and \( D \) are disjoint.

Now, if there are less than \( (1/32 - \epsilon)n^2 \) unordered pairs \( \{A, B\} \in C(2) \) such that \( A \) and \( B \) intersect, then \( G(P, <_1, <_2, <_3, <_4) \) has more than

\[
\left( \frac{1}{2} - \frac{1}{32} + \epsilon \right)n^2
\]

edges. Hence, by Theorem 4, there exists \( i \in [4] \) and \( A, B \subset C \) such that \( |A| = |B| > \frac{\epsilon n}{128} \) and \( A <_i B \). But then for every \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), we have \( A \cap B = \emptyset \).

We note that these results have no analogue in higher dimensions: Tietze [11] proved that any graph can be realized as the intersection graph of 3-dimensional convex sets.
We also show an application of Theorem 4 for a variant of a classical problem of Erdős: let $\alpha < \pi$ be a positive real. Let $g_d(\alpha)$ be the smallest integer $m$ such that in any configuration of $m$ points in the $d$-dimensional space there is an angle larger than $\alpha$. Erdős and Szekeres [3] proved that

$$g_2\left(\pi - \frac{\pi}{r} + \epsilon\right) = 2^r + 1,$$

if $r \geq 2$ is an integer and $\epsilon > 0$ is sufficiently small. They also proved that

$$2^{(1/\beta)^{d-1}} < g_d(\pi - \beta) < 2^{(4/\beta)^{d-1}}$$

for any $0 < \beta < \pi$.

The author of this paper [12] considered the following generalization of this problem: given $0 < \alpha < \pi$ and positive integers $m$ and $d$, what is the maximal $s$ such that any configuration of $m$ points in the $d$-dimensional space contains $s$ points, where every triangle has an angle larger than $\alpha$. It was proved that any configuration of $t+1$ points in the plane contains $t+1$ points, where every triangle has an angle larger than $\pi - \pi/r$.

We prove a tripartite version of this result.

**Theorem 19** Let $0 < \alpha < \pi$ and let $d$ be a positive integer. There exists a constant $c(\alpha, d) > 0$ with the following property. Suppose that $n$ is a sufficiently large positive integer and $S$ is a configuration of $n$ points in the $d$-dimensional space. There exist three pairwise disjoint sets $A, B, C \subset S$ such that

$$|A| = |B| = |C| > c(\alpha, d)n,$$

and for every $X \in A$, $Y \in B$, $Z \in C$, the angle $XYZ \angle$ is larger than $\alpha$.

**Proof** Let $s = \lceil 1/(\pi - \alpha) \rceil$, and let $V$ be a finite set of unit vectors with the property that for any $w \in \mathbb{R}^d$, there exists $v \in V$ such that the angle of $v$ and $w$ is less than $(\pi - \alpha)/2$. Such $V$ trivially exists. For each $v \in V$, define the relation $<v$ on $\mathbb{R}^d$ as follows: if $X, Y \in \mathbb{R}^d$, then $X <v Y$ if the angle of $v$ and $\overrightarrow{XY}$ is less than $(\pi - \alpha)/2$. Then $<v$ is a partial order: $X <v Y$ is equivalent to the inequality

$$\langle v, \overrightarrow{XY} \rangle > |\overrightarrow{XY}| \sin(\alpha/2).$$

Hence, if $X <v Y$ and $Y <v Z$, then

$$\langle v, \overrightarrow{XZ} \rangle = \langle v, \overrightarrow{XY} \rangle + \langle v, \overrightarrow{YZ} \rangle > (|\overrightarrow{XY}| + |\overrightarrow{YZ}|) \sin(\alpha/2) \geq |\overrightarrow{XZ}| \sin(\alpha/2),$$

so $X <v Z$.

Also, if $X <v Y <v Z$ holds for some $X, Y, Z \in \mathbb{R}^d$, then by elementary geometry, the angle $XYZ \angle$ is larger than $\alpha$.

Let $S \subset \mathbb{R}^d$, $|S| = n$. Then $G(S, \{<v\}_{v \in V})$ is the complete graph on $n$ vertices, because we choose $V$ such that for any $X, Y \in \mathbb{R}^d$, there exists $v \in V$ with $X <v Y$. Let

$$c(\alpha, d) = c(|V|, 3, 2^{-2|V|-2}),$$

where $c(r, h, \epsilon)$ is the constant defined in Theorem 4. If $n$ is sufficiently large, then

$$\binom{n}{2} > \left(\frac{1}{2} - \frac{1}{22|V|+2}\right)n^2.$$
Hence, by Theorem 4, there exist \( v \in V \) and \( A, B, C \subset S \) such that \( A, B, C \) are pairwise disjoint, \( |A| = |B| = |C| > c(\alpha, d)n \), and \( A <_v B <_v C \). But then, for any \( X \in A, Y \in B \) and \( Z \in C \), we also have \( XYZ \angle > \alpha \). □

Consider the case \( d = 2 \) and \( \alpha < \pi - \pi/r \). In the proof above, we can choose \( V \) to be a \( 2r \) element set, so using the bound in the remark after Theorem 6, we can show that \( c(\alpha, 2) > e^{-cr} \) with some constant \( c > 0 \). However, we conjecture that an even stronger bound holds.

**Conjecture 20** Let \( 0 < \alpha < \pi - \pi/r \) and let \( n \) be a positive integer. Let \( S \) be a set of \( n \) points in the plane. There exist \( A, B, C \subset S \) disjoint subsets such that \( |A| = |B| = |C| = \Omega(\sqrt{n/r}) \), and for every \( X \in A, Y \in B, Z \in C \), we have \( XYZ \angle > \alpha \).

Taking \( S \) to be the \( \lfloor \sqrt{n} \rfloor \times \lfloor \sqrt{n} \rfloor \) square grid, one can easily show that the dependence on \( r \) in the conjecture cannot be improved.

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