TANGENTIAL WEAK DEFECTIVENESS AND GENERIC IDENTIFIABILITY

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Abstract. We investigate the uniqueness of decomposition of general tensors \( T \in \mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_r+1} \) as a sum of tensors of rank 1. This is done extending the theory developed in [Me06] to the framework of non twd varieties. In this way we are able to prove the non generic identifiability of infinitely many partially symmetric tensors.

1. Introduction

The decomposition of tensors \( T \in \mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_r+1} \) as a sum of simple tensors (i.e. tensors of rank 1) is a central problem for many applications from Multilinear Algebra to Algebraic Statistics, coding theory, blind signal separation and others, [DDL1], [DDL2], [DDL3], [KADL], [Si].

For statistical inference, it is meaningful to know if a probability distribution, arising from a model, uniquely determines the parameters that produced it. When this happens, the parameters are called identifiable. There are no useful models where all distributions are identifiable. Then the notion of generic identifiability for parametric models has been considered for instance in [AMR09] and in [SR12]. Conditions which guarantee the uniqueness of decomposition, for generic tensors in the model, are quite important in the applications. When generic identifiability holds, the set of non-identifiable parameters has measure zero, thus parameter inference is still meaningful. Notice that many decomposition algorithms converge to one decomposition, hence a uniqueness result guarantees that the decomposition found is the chased one. We refer to [KB09] and its huge reference list, for more details.

From a purely theoretical point of view, the study of unique decompositions, or canonical forms in the early XXth century dictionary, has connection with both invariant theory, [Hi], and projective geometry, [Pa] [Ri]. It is already over a decade, [Me06], that generic identifiability of symmetric tensors has shown its close connection to modern birational projective geometry and especially to the maximal singularities methods. In a series of papers, [Me06] [Me09] [GM], the generic identifiability problem for symmetric tensors has been completely solved.

The present paper is devoted to extend this theory to arbitrary tensors and can be considered as a first step, similar to [Me06], in this direction. As for the symmetric case it is expected that identifiability is very rare and our result support this convincement.

The main tool in [Me06] was the use, after [CC02], of non weakly defective varieties to study identifiability, see Section 2 for all the relevant definitions. Unfortunately it is very hard to determine the weak defectiveness of general tensors.

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This difficulty prevented, for many years, a straightforward application of the same techniques to them, see [Fo] for a similar approach in special cases.

In recent years the notion of tangential weakly defectiveness, introduced in [CO], has gradually substituted the weak defectivity and proved valid to study generic identifiability of subgeneric tensors, [CO] [BD+G] [BC] [BCO] [Kr] [CM]. In particular thanks to the main result in [CM] for the generic identifiability we may assume without loss of generality the non tangential weakly defectiveness under mild numerical assumptions.

Tangential weakly defectiveness does not behave as weakly defectiveness with respect to the maximal singularities method. Therefore in this paper we develop tools to plug in maximal singularities methods for non tangentially weakly defective varieties. In this way we are able to prove the non identifiability of many partially identifiable when \( d > n_i + 1 \) for any \( 1 \leq i \leq r \) and \( \frac{1}{2} \left( \sum n_i + 1 \right) > 2(\sum n_i) \).

The paper is organized as follows. After recalling notation and definitions we study in detail the singular loci of tangential linear systems for non tangentially weakly defective varieties. The main technical result is Theorem 23 where we prove that, under suitable hypothesis, these linear systems have not nested singularities. This result allow us to apply the standard Noether–Fano inequalities to show that some tangential projections are not birational, see Theorem 26. With this the non identifiability result is at hand following [Me06].

\section{Notation}

We work over the complex field. Let \( X \subset \mathbb{P}^N \) be an irreducible and reduced non-degenerate variety and \( X^{(h)} \) be the \( h \)-th symmetric product of \( X \). That is the variety parametrizing unordered sets of \( h \) points of \( X \). Let \( U^X_h \subset X^{(h)} \) be the smooth locus, given by sets of \( h \) distinct smooth points.

\textbf{Definition 2.} A point \( z \in U^X_h \) represents a set of \( h \) distinct points, say \( \{ z_1, \ldots, z_h \} \). We say that a point \( p \in \mathbb{P}^N \) is in the span of \( z, p \in \langle z \rangle \), if it is a linear combination of the \( z_i \).

\textbf{Definition 3.} The \emph{abstract \( h \)-Secant variety} is the irreducible and reduced variety of \( (z, p) \in U^X_h \times \mathbb{P}^N \) \( p \subset \langle z \rangle \), \( \subset X^{(h)} \times \mathbb{P}^N \).

Let \( \pi : X^{(h)} \times \mathbb{P}^N \to \mathbb{P}^N \) be the projection onto the second factor. The \emph{\( h \)-Secant variety} is

\[ \text{Sec}_h(X) := \pi(\text{Sec}_h(X)) \subset \mathbb{P}^N, \]

and \( \pi^X_h := \pi_{|\text{Sec}_h(X)} : \text{Sec}_h(X) \to \mathbb{P}^N \) is the \( h \)-secant map of \( X \).

The irreducible variety \( \text{Sec}_h(X) \) has dimension \( (hn + h - 1) \). One says that \( X \) is \( h \)-defective if

\[ \dim \text{Sec}_h(X) < \min\{\dim \text{Sec}_h(X), N\}. \]

For simplicity we will say that \( X \) is not defective if it is not \( h \)-defective for any \( h \).

\textbf{Definition 4.} Let \( X \subset \mathbb{P}^N \) be a non-degenerate subvariety. We say that a point \( p \in \mathbb{P}^N \) has rank \( h \) with respect to \( X \) if \( p \in \langle z \rangle \), for some \( z \in U^X_h \) and \( p \not\in \langle z' \rangle \) for any \( z' \in U^X_{h'} \), with \( h' < h \).
We call \( g := g(X) \) the rank of a general point and we say that \( X \subset \mathbb{P}^N \) is perfect if \( \frac{N + 1}{\dim X + 1} \in \mathbb{N} \).

**Definition 5.** A point \( p \in \mathbb{P}^N \) is \( h \)-identifiable with respect to \( X \subset \mathbb{P}^N \) if \( p \) is of rank \( h \) and \( (\pi_h^X)^{-1}(p) \) is a single point. The variety \( X \) is said to be \( h \)-identifiable if \( \pi_h^X \) is a birational map, that is the general point of Sec\(_h(X)\) is \( h \)-identifiable. For simplicity we will say that \( X \subset \mathbb{P}^N \) is generically identifiable if the generic point of \( \mathbb{P}^N \) is \( g \)-identifiable.

**Remark 6.** Note that \( \pi_h^X \) is generically finite if and only if \( X \) is perfect and not defective. These are therefore necessary condition for identifiability.

**Definition 7.** Let \( X \subset \mathbb{P}^N \) be a non-degenerate variety and \( \{x_1, \ldots, x_h\} \subset X \) general points. The variety \( X \) is \( h \)-weakly defective if the general hyperplane singular along \( h \) general points is singular along a positive dimensional subvariety passing through the points. Let \( H \in \mathcal{H}(h) := |I_{x_1^2 \ldots x_h^2}(1)| \) be a general section, we call \( \Gamma_h(H) \) its locus of tangency passing through \( x_1, \ldots, x_h \).

**Definition 8.** For a linear system \( \mathcal{H} \) we set \( \Gamma(\mathcal{H}) := \bigcap_{H \in \mathcal{H}} \text{Sing}(H) \) the common singular locus.

**Remark 9.** We want to stress that, by [CC02], if \( \Gamma_h(H) \) is zero dimensional in a neighborhood of \( \{x_1, \ldots, x_h\} \) then \( \Gamma_h(H) = \{x_1, \ldots, x_h\} \).

The notion of tangentially weakly defective varieties has been introduced in [CO]. Here we follow the notations of [BBC].

For a subset \( A = \{x_1, \ldots, x_h\} \subset X \) of general points we set \( \mathcal{M}_A := \bigcup_i \mathcal{T}_{x_i}X \).

By Terracini Lemma the space \( \mathcal{M}_A \) is the tangent space to Sec\(_h(X)\) at a general point in \( \langle A \rangle \).

**Definition 10.** The tangential \( h \)-contact locus \( \Gamma_h(A) \) is the closure in \( X \) of the union of all the irreducible components which contain at least one point of \( A \), of the locus of points of \( X \) where \( \mathcal{M}_A \) is tangent to \( X \). We will write \( \gamma_h := \dim \Gamma_h(A) \). We say that \( X \) is \( h \)-twd (\( h \)-tangentially weakly defective) if \( \gamma_h > 0 \).

**Remark 11.** Note that in general it is difficult to predict the behavior of \( \Gamma(\mathcal{H}(h)) \) for non \( h \)-twd varieties. By definition \( \Gamma(\mathcal{H}(h)) \) is zero dimensional in a neighborhood of the assigned singular points but not much is known about singular components away from these. Our Proposition 21 is a first attempt to study this problem, under strong hypothesis.

For what follows it is useful to introduce also the notion of tangential projection.

**Definition 12.** Let \( X \subset \mathbb{P}^N \) be a variety and \( A = \{x_1, \ldots, x_h\} \subset X \) a set of general points. The \( h \)-tangential projection (from \( A \)) of \( X \) is \( \tau_h : X \dashrightarrow \mathbb{P}^M \) the linear projection from \( \mathcal{M}_A \). That is, by Terracini Lemma, the projection from the tangent space of a general point \( z \in \langle A \rangle \) of Sec\(_h(X)\) restricted to \( X \).

**Remark 13.** By Terracini’s Lemma \( \tau_h \) is the rational map associated to the linear system \( \mathcal{H}(h) = |I_{x_1^2 \ldots x_h^2}(1)| \).
3. Properties of contact locus for non twd varieties

In this section we study properties of the contact loci $\Gamma_{g-1}(H)$ (for a general $H \in \mathcal{H}(g-1)$) of projective varieties that are non defective and not $(g-1)$-twd. In particular in view of applications to Noether–Fano inequalities we are interested in studying the infinitesimally near singularities of $\mathcal{H}$.

We start recalling [CC02] Proposition 3.6] and its generalization to twd. This Proposition will be useful to reduce the study of $\Gamma_{g-1}(H)$ to the special case of $g = 2$.

**Proposition 14.** Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non degenerate variety. Assume that $X$ is not $h$-defective and $h(\dim X + 1) - 1 < N$. Let $X_s = \tau_s(X)$ be a general tangential projection.

i) $X$ is $h$-weakly defective if and only if $X_s$ is $(h-s)$-weakly defective.

ii) $X$ is $h$-twd if and only if $X_s$ is $(h-s)$-twd.

**Proof.** Point i) is [CC02 Proposition 3.6]. Point ii) is a simple adaptation of point i) substituting weakly defectivity with twd. □

For future reference we observe the following fact.

**Lemma 15.** Let $Z \subset \mathbb{P}^a$ be a reduced projective variety of dimension $\dim(Z) = a$. Then $\text{codim}(\mathcal{I}_Z(1)) \geq a + 1$ and equality is fulfilled only by linear spaces.

**Proof.** If $Z$ is a linear space there is nothing to prove. Assume that $Z$ is not a linear space, then $\dim(Z) > \dim Z$. We have

$$\text{codim}(\mathcal{I}_Z(1)) = \text{codim}(\mathcal{I}_Z(1)) = \dim(Z) + 1 > \dim Z + 1.$$ □

**Definition 16.** Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non-degenerate and non $h$-defective variety. Let $\{x_1, \ldots, x_h\} \subset X$ be a set of general point and $\mathcal{H}(h) = |x_1, \ldots, x_h(1)|$ the linear system of hyperplane sections singular in $\{x_1, \ldots, x_h\}$. Set

$\mathcal{W}_h := \{(H,x)|H \in \mathcal{H}(h), x \in \Gamma_h(H)\} \subset \mathcal{H} \times X$

and $\pi^h_1 : \mathcal{W}_h \rightarrow \mathcal{H}(h), \pi^h_2 : \mathcal{W}_h \rightarrow X$ the two canonical projections. We denote with $W_h := \pi^h_1(\mathcal{W}_h) \subset \mathcal{H}(h)$.

It is clear that $W_s \subset |x_1^{g-1}, \ldots, x_s^{g-1}(1)|$ for any $h < s$. Then we may identify $W_s$ as a subvariety of $W_h$ for any $h \leq s$. Our next aim is to prove, in some cases, a more precise result.

**Proposition 17.** Assume that $X$ is perfect and not defective with general rank $g$. Set $\mathcal{H} := \mathcal{H}(g-2) = |x_1^{g-1}, \ldots, x_{g-2}^{g-2}(1)|$ and assume

$$\dim(\Gamma_{g-1}(H)) = a,$$

for $H \in \mathcal{H}(g-1)$. Then we have $\text{codim}_{\mathcal{H}(g-2)}(W_{g-1}) = a + 1$.

**Proof.** The variety $X$ is not defective, then $\dim(\mathcal{H}(g-2)) = 2n + 1$. By a parameter count we have $\dim W_{g-1} = 2n$.

By definition for a general $[H] \in W_{g-1}$ we have

$$\dim(\pi^{-1}_1(H)) = \dim\{x \in X | x \in \Gamma_{g-1}(H)\} = \dim \Gamma_{g-1}(H)$$

therefore we conclude that

$$\dim(W_{g-1}) = \dim(W_{g-1}) - \dim(\pi^{-1}_1(H)) = 2n - a$$

yielding $\text{codim}_{\mathcal{H}(g-2)}(W_{g-1}) = a + 1$. □
Lemma 19. Let $X \subset \mathbb{P}^{2 \dim X + 1}$ be an irreducible, reduced non-degenerate variety. Assume that $X$ is not defective and not $1$-twd. Then for a general tangent hyperplane $H \in \mathcal{H}(1)$, the tangential locus $\Gamma_1(H)$ is a linear space. In particular, under the hypothesis, also $\Gamma(\mathcal{H}(1))$ is a linear space.

Proof. If $X$ is not $1$-weakly defective, by Remark 9, $\Gamma_1(H)$ is a point. Assume that $X$ is $1$-weakly defective and $\dim \Gamma_1(H) = a$. Let $x \in X$ be a general point and $H \in \mathcal{H}(1)$ a general tangent section in $x$. Let us consider the variety

$$W_1 \subset \mathcal{O}(1) =: \mathcal{H}$$

parametrizing singular hyperplane sections. Proposition 17 yields $\operatorname{codim}_\mathcal{H}(W_1) = a + 1$ and so $\operatorname{codim}(T_H|W_1) = a + 1$. On the other hand, by the infinitesimal Bertini’s theorem [CC02, Thm 2.2], we have

$$T_H|W_1 \subset \mathcal{H}(-\operatorname{Sing}(H))$$

and so $\operatorname{codim}_\mathcal{H}(\mathcal{H}(-\Gamma_1(H))) \leq a + 1$.

Hence we conclude by Proposition 15 that $\Gamma_1(H)$ is a linear space. \qed

Lemma 19. Let $X \subset \mathbb{P}^N$ be an irreducible, reduced non-degenerate projective variety. Assume that $X$ is $1$-weakly defective with $\dim(\Gamma_1(H)) = a$, for $H \in \mathcal{H}(1)$ a general tangent hyperplane. Then a general hyperplane section $X'$ of $X$ satisfies $\dim(\Gamma_1(H')) = a - 1$, for $H'$ a general tangent hyperplane to $X'$.

Proof. Let $x \in X$ be a general point, $H \in |\mathcal{I}_x(1)|$ a general hyperplane section singular at $x$ and $L \in |\mathcal{I}_x(1)|$ a general hyperplane section passing through $x$. The divisor $L$ is smooth in a neighborhood of $x$ and $\mathcal{B}_S|\mathcal{I}_x(1)| = \{x\}$. Hence, by Bertini’s theorem,

$$\dim(\operatorname{Sing}(H) \cap L)) = \dim \Gamma_1(H) - 1 = a - 1$$

To conclude observe that $H\vert_x$ is a general tangent section of $L$ at $x$. \qed

Let $(z_1, \ldots, z_n)$ be a system of local coordinates at the point $(x \in X) \cong ((0, \ldots, 0) \in \mathbb{C}^n)$. Every divisor $H \in |\mathcal{I}_x(1)|$ can be expressed locally as

$$H = (Q_H(z_1, \ldots, z_n) + \sum_{d \geq 3} F_d(z_1, \ldots, z_n) = 0)$$

where $Q_H(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]_2$ is a Quadric and $F_d$ are homogeneous polynomials of degree at least 3. The rank of the double point $x \in H$ is by definition the rank of the Quadric $Q_H$. The singular locus $A = \operatorname{Sing}(Q_H)$ is a linear space $A \subset \mathbb{C}^n$ of dimension $\dim(A) = \dim(X) - \operatorname{rank}(Q_H)$. It is called the asymptotic space of $H$ at the point $x$. Let $\nu : X' \to X$ be the blow up of $X$ at $x$ with exceptional divisor $E$. Under the identification $E = \mathbb{P}((T_x X)^*) = \mathbb{P}^{n-1}$ we have that $\nu_*^{-1}(H) \cap E = \mathbb{P}(Q_H)$ and $\operatorname{Sing}(\nu_*^{-1}(H)) \cap E \subset \mathcal{O}(A)$. Note further that to every point $y \in E$ we can associate uniquely a line $l_y \subset T_x X$ corresponding to the tangent direction represented by $y$.

With this notation in mind we are going to improve Proposition 18.

Proposition 20. Let $X \subset \mathbb{P}^{2 \dim X + 1}$ be an irreducible, reduced non-degenerate projective variety. If $X$ is not defective $\mathbb{P}(\operatorname{Sing}(Q_H)) = \nu_*^{-1}(\Gamma_1(H)) \cap E$.

Proof. Let $H \in \mathcal{H}(1)$ be a generic hyperplane section singular at $x$. If $\dim(\Gamma_1(H)) = 0$, by [CC02, Theorem 1.4], $x$ is an ordinary double point of $H$. Thus $Q_H$ is a Quadric of maximal rank.
Assume \( \dim(\Gamma_1(H)) = a > 0 \). By Proposition 18 it is enough to prove that \( \text{rank}(Q_H) = \dim X - a + 1 \). Let \( \nu : X' \to X \) be the blow up of \( X \) at the general point \( x \in X \), with exceptional divisor \( E \), and \( H' = \nu^{-1}_x(H) \) the strict transform of \( H \). We have

\[
\nu^{-1}_x(\Gamma_1(H)) \cap E \subseteq \text{Sing}(H')
\]

We already observed that \( \text{Sing}(H') \cap E \subseteq \mathbb{P}(\text{Sing}(Q_H)) \) hence

\[
\nu^{-1}_x(\Gamma_1(H)) \cap E \subseteq \mathbb{P}(\text{Sing}(Q_H)).
\]

This leads to

\[
\text{rank}(Q_H) \leq \dim(X) - a + 1.
\]

Let \( H_1, \ldots, H_a \in \mathcal{H}(1) \) be general sections. Then Lemma 19 yields that \( X^a := H_1 \cap \ldots \cap H_a \) is not \( 1 \)-weakly defective. Hence, by the first part of the proof, we conclude

\[
\text{rank}(Q_H) \geq \dim(X) - a + 1
\]

and finish the proof. \( \square \)

We take the opportunity to stress a property of \( \Gamma(\mathcal{H}(g-1)) \) for non twd varieties, recall Remark 11.

**Proposition 21.** Let \( X \subset \mathbb{P}^N \) be a non defective, perfect, irreducible, reduced and non-degenerate variety with general rank \( g \). Assume that \( X \) is not \((g-1)\)-twd. Then \( \langle T_{x_1}X, \ldots, T_{x_{g-1}}X \rangle \) is tangent only along a zero dimensional scheme.

**Proof.** Let \( W \subset \langle T_{x_1}X, \ldots, T_{x_{g-1}}X \rangle \cap X \) be an irreducible component where \( \langle T_{x_1}X, \ldots, T_{x_{g-1}}X \rangle \) is tangent to \( X \). By Proposition 14 we have that

\[
X_{g-2} := \tau_{g-2}(X)
\]

is not 1-twd and not defective, where \( \tau_{g-2} \) is the linear projection from \( \langle T_{x_2}X, \ldots, T_{x_{g-1}}X \rangle \).

**Claim 1.** \( \tau_{g-2}(W) = \tau_{g-2}(x_1) \)

**Proof.** Let \( y = \tau_{g-2}(x_1) \) and \( H \in \mathcal{I}_g^r(1) \) a general tangent hyperplane section. By Proposition 14 \( X_{g-2} \) is not 1-twd and by Proposition 15 \( \Gamma_1(H) \) is a linear space, therefore

\[
\Gamma_1(H) \cap T_y \times X_{g-2} = y.
\]

On the other hand, by construction, we have

\[
\tau_{g-2}(W) \subset \Gamma_1(H),
\]

and this proves the claim. \( \square \)

The variety \( X \) is not defective and \( y = \tau_{g-2}(x_1) \) is a general point of \( X_{g-2} \). Therefore \( \tau_{g-2}^{-1}(y) \) is a finite scheme and we conclude by the Claim that \( W \) is 0-dimensional. \( \square \)

**Remark 22.** It would be very interesting to understand if the result in Proposition 24 is true for smaller values of the rank. Unfortunately our proof is based on Proposition 15 and cannot be extended in this direction.

The following is the main result of this section.

**Theorem 23.** Let \( X \subset \mathbb{P}^N \) be a projective irreducible, reduced and non-degenerate variety of general rank \( g \). Let \( \{x_1, \ldots, x_{g-1}\} \) be general points on \( X \) and \( \mathcal{H} = \mathcal{H}(g-1) \). Assume that:

- \( X \) is perfect and non defective
- \( X \) is not \((g-1)\)-twd
Claim 2. \( \Gamma(\mathcal{H}) \) is empty.

Proof of the Claim. By Proposition 18 the tangential locus \( \Gamma_1(H) \) is a linear space. The variety \( X_{g-2} \) is not 1-twd therefore \( \Gamma(H') = \Gamma(\mathcal{H}) \). This is enough to show that \( \Gamma(\mathcal{H}) \subset E \).

Assume that there is a point \( z \in \Gamma(\mathcal{H}) \cap E \) and denote by \( l_z \subset \mathbb{P}^{\dim X+1} \) the corresponding line in the projective space. By Proposition 20 this forces \( l_z \subset \Gamma_1(H) \) and the contradiction follows.

Let \( Y \) be the completion of the Cartesian square

\[
\begin{array}{ccc}
  Y & \xrightarrow{\nu} & Z \\
  \downarrow & & \downarrow \\
  X & \xrightarrow{\tau_{g-2}} & X_{g-2}
\end{array}
\]

and \( \mathcal{H}_Y = \nu_*^{-1}(\mathcal{H}) = \eta_*^{-1}(\mathcal{H}) \) the strict transform linear system. By construction and Claim 2 the set \( \Gamma(\mathcal{H}_Y) \) is contained in the locus where \( \eta \) is not an isomorphism. On the other hand, by monodromy, the same should be true for a different choice of \( (g-2) \) points in \( \{x_1, \ldots, x_{g-1}\} \). Thanks to the general choice of the points \( \{x_1, \ldots, x_{g-1}\} \) this is enough to show that \( \Gamma(\mathcal{H}_Y) \) is empty. In particular for any \( y \in Y \) there are divisors \( H \in \mathcal{H}_Y \) with \( \text{mult}_y H \leq 1 \). To conclude we construct the divisor

\[
D = \frac{1}{M} \sum_{i=1}^{M} H_i,
\]

for \( H_i \in \mathcal{H}_Y \) general. The locus \( \Gamma(\mathcal{H}_Y) \) is empty therefore we may assume that for any \( y \in Y \) there are at most dim \( \mathcal{H}_Y \) divisors in \( \{H_i\}_{i=1, \ldots, M} \) singular in \( y \). This is enough to conclude. \( \square \)

4. Noether–Fano inequalities and generic identifiability

In this section we apply the previous results on the singular locus of linear system \( \mathcal{H}(g-1) \) to produce non generic identifiability statements. We start recalling two results in this area.

Theorem 24 (M06). Let \( X \subseteq \mathbb{P}^N \) be a projective, irreducible non-degenerate variety. Suppose that \( X \) is generically identifiable. Then the \((g(X) - 1)\)-tangential projection \( \tau_{g(X)-1} : X \dashrightarrow \mathbb{P}^{\dim(X)} \) is birational.

Theorem 25 (Noether–Fano Inequalities [Co]). Let \( \pi : X \to X' \) and \( \rho : Y \to Y' \) be two Mori fiber spaces and \( \varphi : X \dashrightarrow Y \) a birational, not biregular, map
Choose a very ample linear system $H_Y$ in $Y$ and let $H_X = \varphi^{-1}_s (H_Y)$. Let $a \in \mathbb{Q}$ such that $H_X \equiv -aK_X + \pi^*(A)$ for some divisor $A \in \text{Pic}(S)$.

Then either $(X, \frac{1}{a}H_X)$ has not canonical singularities or $K_X + \frac{1}{a}H_X$ is not NEF.

We are ready to connect the contact loci properties and the Noether–Fano inequalities to produce a tool for non identifiability statements.

**Theorem 26.** Let $X^n \subset \mathbb{P}^N$ be a projective smooth non-degenerate variety and $\tau_{g-1}: X \dashrightarrow \mathbb{P}^{\dim X}$ be a general tangential projection, associated to the linear system $H := H(g-1)$. Assume that

- $\pi: X \to S$ is a structure of a Mori fiber space such that $H \equiv -aK_X + \pi^*(A)$ with $a > 1$ a rational number and $A \in \text{Pic}(S)$
- The $\mathbb{Q}$-divisor $K_X + \frac{1}{a}H$ is NEF
- $X$ is not $(g-1)-twd$

Then $\tau_{g-1}$ is not birational, in particular $X$ is not generically identifiable.

**Proof.** If $\pi^N_g : sec_g(X) \to \mathbb{P}^N$ is of fiber type then $\tau_{g-1}$ is of fiber type, see for instance [CM, Lemma 16 (i)], and we conclude, by Theorem 24, that $X$ is not identifiable.

Then we may assume that $X$ is perfect and not defective. In particular $\tau_{g-1}$ is a not biregular map onto $\mathbb{P}^{\dim X}$.

By Theorem 23 there is a variety $Y$ and a birational map $\nu: Y \to X$ with the following property: for any $\epsilon > 0$ there is a $\mathbb{Q}$-divisor $D$, with $D \equiv \nu^{-1}_s H(g-1)$ such that for any point $y \in Y$

\[ \text{mult}_y D < 1 + \epsilon. \]

In particular $(Y, \frac{1}{a}\nu^{-1}_s (H(g-1)))$ and henceforth $(X, \frac{1}{a}H_X)$ have canonical singularities. Then, by Theorem 25 applied to the diagram

\[ \xymatrix{ X \ar[r] \ar[d]_{\pi} & \mathbb{P}^{\dim X} \ar[d] \ar[r]_{\nu} & Y' \ar[d]_{\rho} \ar[r] & Y \ar[d]_{\pi'} \ar[r] & Y' \ar[d]_{\rho'} \ar[r] & Y'' \ar[d]_{\pi''} \ar[r] & \mathbb{P}^N } \]

the map $\tau_{g-1}$ cannot be birational and therefore $X$ is not generically identifiable by Theorem 24. \qed

We are ready to prove the non identifiability statement announced in the introduction.

**Definition 27.** Let $n = (n_1, \ldots, n_r)$ and $d = (d_1, \ldots, d_r)$ be two $r$–uples of positive integers. The Segre-Veronese variety $SV^n_d$ is the embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \subset \mathbb{P}^{\Pi^{(n_1+d_1)}}$ via the complete linear system $|\pi^*_1 \mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \otimes \cdots \otimes \pi^*_r \mathcal{O}_{\mathbb{P}^{n_r}}(d_r)|$ where $\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to \mathbb{P}^{n_i}$ are the canonical projections.
Theorem 28. Fix two multindexes \( n = (n_1, \ldots, n_r) \) and \( d = (d_1, \ldots, d_r) \). Let \( X = SV_d^n \) the corresponding Segre-Veronese variety. Assume that \( d_i > n_i + 1 \), for \( i = 1, \ldots, r \), and

\[
\left\lceil \frac{\prod (n_i + d_i)}{\sum n_i + 1} \right\rceil > 2\left( \sum n_i \right).
\]

Then \( X \) is not generically identifiable.

Proof. If \( X \) is defective or non perfect the statement is clear, recall Remark [5]. Assume that \( X \) is not defective and perfect. Then \( \tau_{g-1} : X \to \mathbb{P}^{\dim X} \) is generically finite. The numerical assumption reads

\[
g(X) = \left\lceil \frac{\prod (n_i + d_i)}{\sum n_i + 1} \right\rceil > 2\left( \sum n_i \right) = 2 \dim(X)
\]

and, by [CM Corollary 22], the variety \( X \) is not \((g-1)-\text{twd.}\).

After reordering the indexes we may assume that

\[
\frac{n_i + 1}{d_i} \geq \frac{n_i + 1}{d_i}, \quad \text{for any } i.
\]

(1) Let \( p : X \to Y \) be the canonical projection onto the Segre-Veronese \( Y = SV_{d_1, \ldots, d_r}^n \) and \( a = \frac{d_1}{n_1 + 1} > 1 \). Then \( p \) is a Mori fiber Space and

\[
K_X + \frac{1}{a} H(g-1) = 0.
\]

Further note that the cone of effective divisor of \( X \) is spanned by the lines in the factors \( \mathbb{P}^n \) and, by Equation (1), we have

\[
K_X + \frac{1}{a} H(g-1) \cdot l_i = -(n_i + 1) + \frac{n_i + 1}{d_i} d_i \geq 0,
\]

This shows that \( K_X + \frac{1}{a} H(g-1) \) is NEF and, by Theorem [26] we prove that \( X \) is not generically identifiable. \( \square \)

Remark 29. In recent years the Secant varieties of Segre-Veronese varieties have been studied intensively, see for instance [AB], [AMR], [BB1], [BCC]. However, to the best of our knowledge, this is the first result regarding non generic identifiability for infinite classes of Segre-Veronese varieties with \( r \geq 2 \).

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