A CRITICAL APPROACH TO TOTAL AND PARTIAL DERIVATIVES

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Abstract

In this brief note we critically examine the process of partial and of total differentiation, showing some of the problems that arise when we relate both concepts. A way to solve all the problems is proposed.

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1 Introduction

Our article is devoted to the discussion of the total derivative concept, a general and frequently applied concept of mathematical analysis. Indeed, the derivatives play a significant role in modern physical theories and are present in many basic physical laws.

Considering some of the basic statements of classical electrodynamics, one of the authors (A.E.Ch) paid attention to the fact of some inaccuracy of applying in physics the concept of the partial derivative of the many variable function (see [1,2]). L. Schwartz [3] warned prudently against this inaccuracy (the text after Eq.(I,2;5) in [3]): “In a short narrative one identifies sometimes \( f_1 \) and \( f \), saying, that is the same function (sic !), represented with the help of the variable \( x_1 \) instead of \( x \). Such a simplification is very dangerous and may result in very serious contradictions.”

Such an inaccuracy and its consequences in some problems of physics inspired our interest to the total derivative concept in the classical analysis under the condition of double dependence from a time variable \( t \): implicit and explicit ones. Such a situation is characteristic of many physical problems, first of all of classical mechanics (see for example, [4], where the author even introduces a new special term as “whole-partial derivative”) and classical electrodynamics (see, for instance, Section. 4 in [2] where it was considered in detail).

As a matter of fact, a concept of the partial derivative is habitually associated with a concept of the function of many variables, but a concept of the total derivative must be associated with another function, which is some restriction of the function of many variables. L. Schwartz did not introduce an additional co-term though he denoted repeatedly [3] the corresponding moment connected

\[ f = f(x) \text{ is the function determined in a set } \mathcal{E} \text{ with the values in a set } \mathcal{F}, \quad f_1 = f \circ u, \]

where \( u \) is the mapping of some set \( \mathcal{E}_1 \) into the set \( \mathcal{E} : x = u(x_1) \). L. Schwartz calls the function \( f_1 \) a foretype of the function \( f \) under a variable substitution.

\[ ^1 \text{Here } f = f(x) \text{ is the function determined in a set } \mathcal{E} \text{ with the values in a set } \mathcal{F}, \quad f_1 = f \circ u, \]

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with applying of the total derivative concept.

We are interested in the most important point, namely, in an originatio

Thus, let us consider functions $E$ which are determined as

$$E[x_1(t), \ldots, x_{n-1}(t), t] = \text{def} E[x(t), t], \quad (1)$$

$$E(x_1, \ldots, x_{n-1}, t) = \text{def} E(x, t), \quad (2)$$

thereby emphasizing the need to distinguish the different functions: $E[\ ]$ of one variable, $E( )$ is the function of $n$ variables.

These functions evaluated at different points of the globally defined manifold $O = R^{n-1} \times R$ are a source of confusion when we try to calculate total or partial derivatives, and can lead us to write down something meaningless. We shall explain this in detail in the next sections, using, to save writing, the notation introduced in (1) and (2) and a theoretical framework that will show that the problems and distinctions treated in this note have been not treated before.

Usually the functions $E[\ ]$ and $E( )$ represent the same physical value, being different functions in their mathematical origination.

Note that the authors neither in [2] nor in [4] do not distinguish this kind of function. Ambiguities in the “notation” for partial differentiation also have been remarked by Arnold [6] p. 226 (p. 258 in English translation) without further development.

Therefore, an unallowable identification of the functions $E[\ ]$ and $E( )$ happens quite often.

For instance, in the well-known physical formulae

$$\frac{d}{dt} E = (\mathbf{V} \cdot \nabla) E + \frac{\partial}{\partial t} E$$

$^2$as well as G.M.Fichtengoltz, who considers the case of double (explicit and implicit) dependence of functions on two variables, see [5], p. 388.
\[ \frac{df}{dt} = (H, f) + \frac{\partial f}{\partial t}, \]

where \( f \) is some dynamical value, \((H, f)\) is the Poisson bracket, \( H \) is the Hamilton function, the full derivatives in the \( lhs \) and the partial derivative in the \( rhs \) are applied to the different functions: there are the functions \( E[\ ] \) and \( f[\ ] \) in the \( lhs \), and there are the functions \( E(\ ) \) and \( f(\ ) \) in the \( rhs \) of the equation.

The difference between the functions:

\[ E[x_1(t), \ldots, x_{n-1}(t), t] = \text{def} E[x(t), t], \quad E(x_1, \ldots, x_{n-1}, t) = \text{def} E(x, t) \]

is usually not remarked in the literature, and for this reason we can often write down meaningless symbols like:

\[ \frac{\partial}{\partial t} E[x(t), t], \quad (3) \]

and

\[ \frac{d}{dt} E(x, t). \quad (4) \]

The symbols (3) and (4) are meaningless, because the process denoted by the operator of partial differentiation can be applied only to functions of several independent variables and \( E[x(t), t] \) is not such a function. Meanwhile, the operator of total differentiation with respect to a given variable can be formally applied to functions of one variable only. However, we have a well-known formula to relate both concepts:

\[ \frac{d}{dt} E = (\nabla \cdot \nabla)E + \frac{\partial}{\partial t} E \quad (5) \]

(here \( \nabla = \frac{dx}{dt} \)).

Let us show that, in this form, Eq. (5) cannot be correct. What is the correct argument for the symbol \( E \) on both sides? If we say that the correct argument for both sides is \( [x(t), t] \) we get the chain of symbols (3), but in this case, the
operator of a partial differentiation would indicate that we must construct a new function in the form \( \partial E/\partial t \), hence we use the following procedure:

\[
\lim_{\Delta t \to 0} \left\{ \frac{E \left[ x(t) + \Delta t \frac{dx(t)}{dt}, t + \Delta t \right] - E[x(t), t]}{\Delta t} \right\}.
\]

But this is the definition of total differentiation! Thus, the symbols of total and of partial differentiation denote the same process, therefore, because \( E \) is the same function on both sides of the equation, we get:

\[
(\mathbf{V} \cdot \nabla)E[x(t), t] = 0
\]

always. But even if the procedure which we followed were correct (which it is not, of course!), this equation is not correct for \( E \) as a function of the functions \( x(t) \), because the partial differentiation would involve increments of the functions \( x(t) \) in the form \( x(t) + \Delta x(t) \) and we do not know how we must interpret this increment because we have two options: either \( \Delta x(t) = x(t) - x^*(t) \), or \( \Delta x(t) = x(t) - x(t^*) \). Both are different processes because the first one involves changes in the functional form of the functions \( x(t) \), while the second involves changes in the position along the path defined by \( x = x(t) \) but preserving the same functional form. Hence, it is clear that we have here different concepts.

If we remember the definition of partial differentiation, we can see where the mistake is: “the symbol: \( \frac{\partial}{\partial t} E(x, t) \) means that we take the variations of \( t \) when the values of \( x \) are constant”. It means that we make the only change \( t + \Delta t \) in the function. But this is only possible if the coordinates \( x \) are independent from \( t \). Hence, we can see that the correct argument cannot be \( [x(t), t] \), because, as we have shown, this supposition leads to the incorrect result (5). If we make the other supposition, that the correct argument is \( (x, t) \) we can get the same conclusion, i.e., equation (5). Hence, none of these suppositions is correct.

What is the solution, then? Actually, in the equation (5) we have two different functions: on the left hand side we have the function \( E[x(t), t] \) defined on a
curve in a $n$-surface and on the right hand side we have the function $E(x, t)$ defined on the all $n$-surface, which obviously are quite different functions, while we have a limiting procedure to get a unification of concepts in the realm of functions of one variable.

2 Theoretical framework

We shall begin describing the elements that we shall use in the development of the problem’s explanation. The globally defined coordinates of our manifold $O$ are given by $(x, t)$, we define a function $E : O \rightarrow \mathbb{R}$ where $\mathbb{R}$ is the real line. Hence the values of the function $E$ on any point $O$ are given as $E(x, t)$. However, we shall be interested in the 1-dimensional subsets of $O$, hence we denote any of them by $T$. To describe this set (a path) we must introduce a function of the form:

$$p(t) = \langle x(t), t \rangle$$

Otherwise we can introduce this function in the form:

$$p(s) = \langle x(s), t(s) \rangle$$

Parametrization (6) for the path is a special case of (7) when we choose the function: $t(s) = s = t$. We suppose that $T \subset O$ is 1-dimensional, hence a path in $O$. This path can be the integral curve of a set of ordinary differential equations (ODE’s), we mean, it defines the orbit of a 1-parametric group action over $O$. If this action is a free action we get that for any pair $p, q \in O$ there exists an integral curve joining them when we define one of them as an initial value. In this way the paths cover the manifold $O$ defining a foliation by 1-dimensional sheets. The whole previous construction is better understood if we introduce the tangent vector space at each point of $O$. If the tangent vector field is defined
at all the points of $O$ by the equation:

$$
\sum_{i=1}^{n-1} f_i(x, t) \frac{\partial}{\partial x_i} + \frac{\partial}{\partial t},
$$

we can define the tangent vectors at each point of the path with the help of the set of ODE’s:

$$
\frac{dx_i(t)}{dt} = f_i[x(t), t]; \quad i = 1, \ldots, n - 1,
$$

whose integral curves allow us to construct the 1-dimensional foliation sheets.

The system (5) is the answer to a very important question that we must keep in mind all along the work. The question is:

(AA) How can we construct sheets $T$ such that its tangent vectors are given by the vector field with components $f_i$?

We cannot overestimate the question. Hence our starting point is a situation of total lack of knowledge regarding the form of the 1-dimensional sheet $T$. We just have the form of the vector field $f_i$, that is, we know the distribution of vector fields in the space. In [7] we have called this situation the $\{f\}$-case. In any usual treatment of the subject of differential geometry any distribution of vector spaces if the system (9) is solvable is called “integrable”. Substantially the same is done to construct foliation in more dimensions, the obvious change is that we don’t want 1-dimensional sheets, instead, we want to construct $m < n - 1$ dimensional submanifolds at each point. The basic question is, again (AA). We must remark something very important, when we pass from (8) to (9) we have changed the functions $f_i(x, t)$ evaluated at any point on the manifold $O$, by the functions $f_i(x(t), t)$ which are evaluated on the sheets of the foliation. We have supposed, to do this operation, that the sheets exist. To prove this supposition, it is usual to impose a Holder condition on the vector fields, this is sometimes enough to prove the existence of solutions by fixed point arguments. However, when we can construct solutions for the system, a formal proof may be avoided.
This change from the globally defined manifold to the local integral sheets of the foliation is done noncritically. This criticism is what we shall carry out in the next sections inside the framework described in this section.

3 The problem

We give the curve $T$ using the parametrization (6) but we write down:

$$g(t) = E \circ p(t)$$

where $g(t)$ is 1-variable function, $E$ and $p$ denote an $n$-variable function defined on $O$ and a curve on $O$, respectively. This way of writing down the functions involved is more precise than the usual notations (1) and (2). For this reason only the use of notations like (1) and (2) should be suppressed. The really important task becomes apparent when we try to differentiate totally or partially the functions (1) and (2). If we want to partially differentiate (2) we have no problem, because the usual definition of a partial derivative requires that we must change one of the variables while keeping the rest constant. In the same way, if we want to derive with respect to $t$ the expression (1) we should not have any problem, because it is a differentiation of a one variable function when we know the forms of $E$ and $p$. Let’s show now the problems.

(A) If we want to differentiate totally (2) in any variable without using, for the moment, any path, a moment of reflection shows us that we really employ the definition of a partial derivative. Hence our use of the symbol $dE(x, t)/dt$ is wrong. It is so because the function is an $n$-variable function and our conventions for the use of symbols (the syntactical rules) tell us that for $n$-variable functions the correct notation is $\partial$. Hence, the symbol $dE(x, t)/dt$ is not correct, but it is very easy to write down the right symbols.
Now, if we want to partially differentiate (1) with respect to \( t \), a moment of reflection shows us that when we try to give an increment to \( t \), while keeping constant the other set of variables, this last condition cannot be fulfilled because if \( t \) is incremented by an amount \( \Delta t \) the other variables are incremented by an amount \( (dx(t)/dt)\Delta t \). Hence we cannot keep them constant and we have a problem here, because we cannot apply the usual definition of partial differentiation to expression (1).

To our knowledge, this situation has not been pointed out before in the usual treatises. However, a brief search in the corresponding literature gives us the paper by K. Brownstein [4] where the concept of so called “whole partial derivative” is introduced. Let’s see if it gives us some light. We shall use the framework outlined in section 2. Brownstein starts with a function of the form:

\[
G[\Phi(q_1, q_2, q_3, q_4), q_1, q_2, q_3, q_4],
\]

which falls within the scope of our general framework. To see this we write down the following coordinate cover:

\[
\langle \Phi_1(q_1, q_2, q_3, q_4), \Phi_2(q_1, q_2, q_3, q_4), \Phi_3(q_1, q_2, q_3, q_4), q_1, q_2, q_3, q_4 \rangle,
\]

which is more general than the one used by Brownstein in function (11). In this case we are over a 4-dimensional manifold represented on a 7-dimensional manifold. We can generate 4-dimensional sheets such that the manifold be covered by them if we can integrate the following system of first order coupled partial differential equations:

\[
\frac{\partial \Phi(q)}{\partial q_j} = F_i(\Phi_1(q), \Phi_2(q), \Phi_3(q), q_1, q_2, q_3, q_4); \quad i = 1, \ldots, 3, j = 1, \ldots, 4.
\]
The generated group of transformations is a 4-parametric group. Just like before, the functions $F_i$ are the components of a known vector field over the manifold. And of course, here the same noncritical change has been realized, because we have started at the manifold, and we have finalized at its sub-manifold. A very important feature, which we shall express later in the 1-dimensional case, appears here. The partial differential equations (13) must be compatible differential equations, that is: their cross derivatives must coincide. However, to establish the conditions we must partially differentiate the functions at the right of (13), but this is not possible because of the argument already used (B). The cross differentiation condition leads us to the usual condition of commutativity of vector fields, or in general grounds to the generators of a Lie algebra, basic to the Frobenius’ theory of integrability. However, we see that we don’t know how to calculate this condition because of the argument (B).

Coming back to Brownstein’s case let’s discover again the same difficulties as in the 1-dimensional. If we suppose, as Brownstein does, that we can partially differentiate the function $G$ in (11) with respect to any $q$ variable, we fall again within the argument (B): the definition of the partial derivative requires that we must change one of the $q$s only. But if we change any $q$ by an amount $\Delta q$ we change the variable $\Phi$ by an amount $(\partial \Phi / \partial q) \Delta q$. Hence, Brownstein’s concept is arguable because we believe he makes the same mistakes that we are pointing out here. Brownstein’s mistake is as follows: he must first define the partial derivatives which appear in his formula (10) [4] derivatives which are in doubt because of the argument (B). So we can conclude that he achieves his goal: to introduce a new symbol and a new name, but based on noncritical concepts.

3 Eq. (10) in [4]:

$$\frac{\partial G}{\partial q^3} = \frac{\partial G}{\partial \Phi} \frac{\partial \Phi}{\partial q^3} + \frac{\partial G}{\partial q^3}$$
4 A solution

Let’s continue with our critical analysis. For this we shall write down a highly incorrect (because of the argument (B)), but nonetheless, very popular, expression:

\[
\frac{dE[x(t), t]}{dt} = \sum_{i=1}^{n-1} \frac{dx_i}{dt} \frac{\partial E[x(t), t]}{\partial x_i} + \frac{\partial E[x(t), t]}{\partial t}.
\] (14)

One of the first mistakes is the following: it is supposed that the function \( E[x(t), t] \) is the same on both sides of the equation. Let’s apply the (B) argument to (14):

1. At the right of the equation we see formations like \( \frac{\partial E[x(t), t]}{\partial t} \), which by the use of the argument (B), have been shown to be wrongly defined.

2. At the left we see the symbol \( \frac{dE[x(t), t]}{dt} \) which is not defined because its “definition”, the right side, is wrongly defined, and we have no other definition for \( \frac{dE[x(t), t]}{dt} \). Hence, we don’t know how to calculate it.

Thus, a solution is required. This can be obtained with the help of equation (10) and some distinctions based upon it. The first and most important thing is to suppose that \( g(t) = E \circ p(t) \) is a 1-variable function only and that it is known. Hence, the usual definition of 1-variable derivation is available. This supposition implies that we must know the path \( p \) and the functional form of \( E \). We have analyzed this supposition in detail in another paper [7], and so we shall not repeat it. Hence, it is the case that on the left side of (14) \( \frac{dg(t)}{dt} \) must appear and not the function \( E \) defined along the path. On the right side the function \( E[x, t] \) must appear to get a partial derivative using the usual definition. Finally, as it is the case that the function \( \frac{dX(t)}{dt} \) is defined on one common point of a class of paths and not all over the space \( O \), as is the case for
\( E(x, t) \) we shall write instead of \( dx(t)/dt \) the functions \( f_i(x, t) \) defined all over \( O \) to get on the right hand side the expression:

\[
\sum_{i=1}^{n-1} f_i(x, t) \frac{\partial E(x, t)}{\partial x_i} + \frac{\partial E(x, t)}{\partial t}.
\] (15)

But what is the relation between \( dg(t)/dt \) with the expression (15)? We cannot make them equal all over the space \( O \), because this is not correct, we shall fall in previous mistakes again. However both expressions must be the same over a path, hence we approximate the expression (15) to the points of one path with the help of a limiting procedure:

\[
\frac{dg(t)}{dt} = \lim_{x \to p} \left( \sum_{i=1}^{n-1} f_i(x, t) \frac{\partial E(x, t)}{\partial x_i} + \frac{\partial E(x, t)}{\partial t} \right)
\] (16)

We have discussed in detail several ways to use this expression in [7] using the supposition that, in fact, we have integrability. Here we shall just discuss the uses in the non-integrable case. But first let’s remark the advantages of (16):

1.- On the left hand side we have a function of just one variable, hence the definition of derivative is clear.

2.- On the right hand side we have only usual partial derivatives and n-variable functions, hence the usual definition of partial derivative is clear.

The meaning of the limiting procedure is very simple: on the globally defined manifold \( O \) we shall make that the variables that describe it tend to the point of the path in some specified way. Of course this can be done in many ways and depends on the topological properties of the manifold. In any simple connected manifold the way in which we get the points of the path should be not important. The most common case of this approximation procedure is the one which answer the question (AA), that is, we approximate the tangent vectors of the path to the vectors given by the vector field of components \( f_i \).
5 Some uses of the formula (16)

Let’s show how the formula (16) can be used in differential geometry. For the sake of completeness we shall expose what is commonly considered as the right procedure, and then we shall show that it can be done with our methods, too. Take an abstract manifold $M$ of dimension $N$ and define over it a path $p$. Hence its coordinate representatives are given by:

$$(x_1[p(t)], \ldots, x_N[p(t)]).$$  \hfill (17)

The usual goal is to define in an intrinsic manner the tangent vector $s$, that is, in such a way that they depend on the points of $M$ only and not in the space in which $M$ is contained. This can be done by defining the tangent vectors in terms of the path $p$ in the following way: we define the equivalence class of $p$ as

$$(p) = \text{def} \left\{ p^* \left| \frac{dx_i[p(t)]}{dt} \right|_{t=0} = \left| \frac{dx_i[p^*(t)]}{dt} \right|_{t=0} \right\},$$ \hfill (18)

in words: a path $p$ is equivalent to a path $p^*$ if and only if at the common point $p(0) = p^*(0)$ they have the same tangent vector. With this definition it is a usual matter to prove that the directional derivative of the path, directed along the tangent vectors is independent of the selected path. To do this it is necessary to use the chain rule to write down the following two expressions:

$$\frac{df[p(t)]}{dt} = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} \frac{dx_i[p(t)]}{dt}$$ \hfill and \hfill $$\frac{df[p^*(t)]}{dt} = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} \frac{dx_i[p^*(t)]}{dt}.$$ \hfill (19)

We can see that in the limit $t \to 0$ both expressions are the same, hence the derivative is independent of the underlying path. If we remember the argument (B) we cannot write equations (19), hence, we must use the correct expressions. For that, we take two paths which have only one common point, hence we have two functions $g$ and $g^*$ equal in at least one point. The expression at the right
of (16) under the sign of limit does not depend on the path, and thus remains the same, however the limit change because the process of approximation must be done considering two different paths, hence we write:

\[
\frac{dg(t)}{dt} = \lim_{x \to p} \sum_{i=1}^{N} V_i \frac{\partial f}{\partial x_i} \quad \text{and} \quad \frac{dg^*(t)}{dt} = \lim_{x \to p^*} \sum_{i=1}^{N} V_i \frac{\partial f}{\partial x_i}.
\] (20)

If we suppose that the tangents to the paths tend to the vector field \( \mathbf{V} \) we can write down the right hand sides of (20) as the right hand sides of (19). But this is equivalent to writing down:

\[
\frac{dx_i[p(t)]}{dt} = V_i\{x[p(t)]\} \quad \text{and} \quad \frac{dx_i[p^*(t)]}{dt} = V_i\{x[p^*(t)]\}.
\] (21)

Now, if we suppose that as our initial value we have \( p(0) = p^*(0) = p_0 \) we shall get the same path by invoking the usual theorems. Hence the underlying paths are not important and our process is well defined over the equivalence classes. But we have a bonus, when we cannot use the equations (21), which is the case when our limiting procedure does not converge to the tangent vectors, that is, when we cannot find a 1-dimensional foliation of the manifold by 1-dimensional integral paths of the vector field, we can express this condition in a quite simple manner

\[
\lim_{x \to p} V_i(x_1, \ldots, x_N) \neq \frac{dx_i}{dt}
\] (22)

Which, of course, makes the integrability a part of the topological properties of the manifold.

6 On the integrability concept

The notion of integrability which we have reached in the previous section must be compared with the most usual notion based on cross differentiation, that is, the generation of a Lie algebra structure for the generators of a distribution \( \mathbf{E} \) of vector spaces along any manifold. For the case of differential 1-forms the
conclusions about integrability can be obtained with the use of its associated complex, the De Rham complex, where the integrability condition of a differential 1-form $w$ can be expressed with the notation: $dw \neq 0$ because if it is zero, by the use of the usual local Poincaré’s lemma we get a local integral $f$ of the form $w = df$. Another way to express the same condition is with the use of the integral invariants of Cartan. For the case of differential 1-forms we have that, if $f$ is such a scalar invariant, hence the distribution of vector fields $E$ along the space is such that: $E \subseteq \ker(df)$ [8]. If we use $n$-forms, $\varphi$, the condition is $E \subseteq \ker(\varphi) \cap \ker(d\varphi)$ (Hence the $n$-form is an integral invariant of Cartan).

However this notion is based on one idea: the cross differentiation of expressions like the functions at the right hand side of (13) process which we have showed to be meaningless because we don’t have a way to compute it. But with the development of the free coordinate tensor formalism many things were left aside, and it was possible to express the conditions in free coordinate terms which avoid the explicit use of operations like the usual total differentiation. In this sense, we believe that in the free coordinate tensor formalism such problems like the one treated in this note cannot be found, except in the cases in which the total derivatives appear. Coming back to the notion of integrability, even in the tensor formalism it is based on the idea of Lie algebra, which is a formal reconstruction of the idea of cross differentiation, but the notion of integrability given by the equation (22) is not based on this notion as a primitive notion, instead it is based on the idea of appropriation to a given curve as a primitive concept.

7 Summary and discussion

The brief treatment given here suggests that a profound, case by case, investigation of the uses of the formalism introduced in differential geometry and topology
is necessary. However, probably the usual tools must be complemented with a
critical view of the subject involved, because our representations of the underly-
ing processes may not be the same and this is the origin of the ignorance of the
problem. Really, the problem arises in the language, not in the usual formalism,
because we take seriously the idea that a partial derivative can be defined only
when all the variables are constants except one. Trying to respect this definition
is the source of everything. This definition defines two syntactical rules of the
form:

\[
d/dt : C(\mathbb{R}, \mathbb{R}^m) \to C(\mathbb{R}, \mathbb{R}^m); \quad \partial/\partial q : C(\mathbb{R}^n, \mathbb{R}^m) \to C(\mathbb{R}^n, \mathbb{R}^m) \quad (23)
\]
one for each operator. The functor \(C\) should be taken as adequate for each
case. The syntactical rules are, of course, that the symbol \(d/dt\) can only operate
when the set represented by \(\mathbb{R}\) doesn’t appear as a cartesian product, that is,
its exponent can be only 1, which means 1-variable function. For the operator
of partial differentiation, the set must have a cartesian exponent different from
0 and 1, that is, we consider only \(n\)-variable functions \((n \neq 0, 1)\). Hence, if we
look more closely at the equation \(g = E \circ p\) we can write down the sequence:
\(\mathbb{R} \to \mathbb{R}^n \to \mathbb{R}^n\) which shows that any derivative of the function \(g\) must be a
d/dt. The composition operation is of the syntactical form:

\[
\circ : C(\mathbb{R}^m, \mathbb{R}^m) \times C(\mathbb{R}, \mathbb{R}^n) \to C(\mathbb{R}, \mathbb{R}^m), \quad (24)
\]
which shows that its action produces a 1-variable function. Usually this kind
of rule is not taken into account, and people proceed with heuristic arguments
based on one or other representation of the subject, of even without representa-
tion just by operating on the symbols. This is not really wrong, or at least
that is our opinion. However, if one tries to take the propositions seriously a
moment of reflection over our own concepts and the use which we made of them
is necessary. This is the heart of our attempt in this note.
The consistency of mathematical analysis it is the mayor problem which motivated the new approaches. See, for example, very interesting paper by K. Brownstein [4], although we criticize some aspects of this work. After our paper has been already submitted to this Journal we have discovered a brilliant work by R.M. Santilli “Nonlocal-integral isotopies of differential calculus, mechanics and geometries” [9] where the author settles a new approach to differential calculus (see, e.g., [9] p. 19, 1.5 “Isodifferential calculus”).

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