SIMPLICITY OF HIGHER RANK TRIPLET $W$-ALGEBRAS

SHOMA SUGIMOTO

Abstract. We show that the higher rank triplet $W$-algebra $W_{\sqrt{\pi}Q}$ is simple for $p \geq h - 1$. Furthermore, we show that the $W_{\sqrt{\pi}Q}$-module $W_{\sqrt{\pi(Q-\lambda_0)+\lambda_p}}$ introduced in [FT] is simple if $\sqrt{\pi}\lambda_p$ is in the closure of the fundamental alcove, and give the decomposition as a direct sum of simple $W_{p-\lambda}(g)$-modules.

1. Introduction

The triplet $W$-algebra (see, e.g., [AM1]-[AM3], [FGST1]-[FGST3], [NT], [TW]) is one of the most well-known examples of $C_2$-cofinite and irrational vertex operator algebra [FB, FHL, Kac], and related to many interesting topics. It has been widely believed that the higher rank triplet $W$-algebra $W_{\sqrt{\pi}Q}$ inherits properties of the triplet $W$-algebra such as simplicity, $C_2$-cofiniteness, irrationality, Kazhdan-Lusztig correspondence, etc. However, apart from the triplet $W$-algebra, very little is known about $W_{\sqrt{\pi}Q}$. The main purpose of this paper is to show the simplicity of $W_{\sqrt{\pi}Q}$.

Let $g$ be a finite-dimensional simply-laced simple Lie algebra and $h$ be the Coxeter number. For a fixed integer $p \geq h - 1$, let $\sqrt{\pi}Q$ be the rescaled root lattice of $g$. We consider the simple lattice vertex operator algebra $V_{\sqrt{\pi}Q}$ associated to $\sqrt{\pi}Q$, and the irreducible $V_{\sqrt{\pi}Q}$-module $V_{\sqrt{\pi}Q+\lambda}$ corresponding to $\lambda \in \Lambda = \frac{1}{\sqrt{\pi}}P/\sqrt{\pi}Q$ (see [D]). Each $V_{\sqrt{\pi}Q+\lambda}$ has a natural $B$-action, and thus, we obtain the homogeneous vector bundle $V_{\sqrt{\pi}Q+\lambda} = G \times_B V_{\sqrt{\pi}Q+\lambda}$ over the flag variety $G/B$. We define the higher rank triplet $W$-algebra as $W_{\sqrt{\pi}Q} = H^0(V_{\sqrt{\pi}Q})$ and $W_{\sqrt{\pi}Q}$-modules $W_{\sqrt{\pi}Q+\lambda} = H^0(V_{\sqrt{\pi}Q+\lambda})$ (see [FT]). In the previous article [S] we have shown that $W_{\sqrt{\pi}Q}$ is isomorphic to the intersection of the kernels of narrow screening operators on $V_{\sqrt{\pi}Q}$. Denote by $p$ and $\theta$ the Weyl vector and the highest root of $g$, respectively.

Theorem 1.1. The vertex operator algebra $W_{\sqrt{\pi}Q}$ is simple for any finite dimensional simply-laced simple Lie algebra $g$ and any integer $p \geq h - 1$. Moreover, $W_{\sqrt{\pi}Q+\lambda}$ is a simple $W_{\sqrt{\pi}Q}$-module for $\lambda \in \Lambda$ such that $(\sqrt{\pi}\lambda_p + \rho, \theta) \leq p$ (where $\lambda_p$ is given by (18)).

Theorem 1.1 has the following application to the representation theory of affine $W$-algebras. Denote by $W_{p-h}(g)$ the unique simple quotient of the affine $W$-algebra $W^{p-h}(g)$ at level $p-h$ ([FF]). In [S], for each $\lambda \in \Lambda$, we gave a natural $G \times W_{p-h}(g)$-module structure on $W_{\sqrt{\pi}Q+\lambda}$ and a $G \times W_{p-h}(g)$-module isomorphism

$$W_{\sqrt{\pi}Q+\lambda} \simeq \bigoplus_{\alpha \in P_+ \cap Q} L(\alpha + \lambda_0) \otimes W(-\sqrt{\pi}\alpha + \lambda) \subseteq V_{\sqrt{\pi}Q+\lambda}. \quad (1)$$

Here $P_+$ is the set of dominant integral weights, $\lambda_0 \in P_+$ satisfies $\lambda = -\sqrt{\pi}\lambda_0 + \lambda_p$ and $(\lambda_0, \rho) = 1$, $L(\beta)$ is the simple $g$-module with highest weight $\beta$, and each.
Theorem 1.2. Let \( h^{-1}H \) be a finite dimensional simply-laced simple Lie algebra, \( p \geq h - 1 \) and \( \lambda \in \Lambda \) such that \((\sqrt{p}\lambda_p + \rho, \theta) \leq p\). Then for any \( \alpha \in P_+ \cap Q \), we have

\[
W(\sqrt{p}\alpha + \lambda) \simeq L(\gamma - \sqrt{p}\alpha + \lambda - p\rho) \simeq T^{\rho}_{\sqrt{p}\lambda_p}\alpha + \lambda_0 \simeq T^{\rho}_{\alpha + \lambda_0, \sqrt{p}\lambda_p}
\]

as \( W_{p-h}(g) \)-modules (we used the Feigin-Frenkel duality \( W_{p-h}(g) \simeq W_{p-h}(g) \)) in the last isomorphism).

In particular, Theorem 1.2 gives an extension of [ArF, Theorem 2.2, 2.3].

Let us explain the outline of the proof of Theorem 1.1 and Theorem 1.2 briefly.

First, we prove the isomorphism

\[
W_{\sqrt{p}Q + \lambda} \simeq H^{l(w_0)}(\sqrt{p}Q, \gamma - \sqrt{p}\alpha + \lambda - p\rho) \simeq W_{\sqrt{p}Q - w_0(\lambda)}^\ast
\]

where \( w_0 \) is the longest element in the Weyl group \( W \) of \( g \) and \( M^\ast \) denotes the restricted dual of a \( W_{\sqrt{p}Q} \)-module \( M \). The first isomorphism in (2) is the Serre duality, which holds for any \( \lambda \in \Lambda \). The second isomorphism is derived using [S, Theorem 4.8] that holds under the condition \((\sqrt{p}\lambda_p + \rho, \theta) \leq p\).

The isomorphism (2) for \( \lambda_p = 0 \) implies that \( W_{\sqrt{p}Q} \) admits a non-degenerate \( W_{\sqrt{p}Q} \)-invariant bilinear form in the sense of [FHL]. Therefore, \( W_{\sqrt{p}Q} \) is simple. In turn, Theorem 1.2 for \( \lambda_p = 0 \) follows from the quantum Galois theory ([DLM, McR]). In particular, for \( \lambda_p = 0 \), we find that the character of \( W(\sqrt{p}\alpha + \lambda) \) coincides with that of the corresponding simple \( W_{p-h}(g) \)-module \( L(\gamma - \sqrt{p}\alpha + \lambda - p\rho) \) given in [Ar]. We can see that this coincidence of characters is also valid for \( \lambda_p \) satisfying the condition \((\sqrt{p}\lambda_p + \rho, \theta) \leq p\), which proves Theorem 1.2. Theorem 1.1 follows from (2) and Theorem 1.2.

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2. Preliminaries

2.1. \( O_X \)-vertex operator algebra and the Serre duality. In this paper, for a ringed space, the structure sheaf is of \( C \)-algebras.

Definition 2.1. Let \( R \) be a \( C \)-algebra and \( V \) be an \( R \)-module. We call \( V \) a vertex operator algebra over \( R \) if \( V \) satisfies the following conditions:

1. there exists a set \( \{V_n \mid n \in \mathbb{Z}\} \) of free \( R \)-modules of finite rank such that \( V = \bigoplus_{n \in \mathbb{Z}} V_n \) and \( V_n = 0 \) for \( n \ll 0 \),
2. there exist a vacuum vector \( |0\rangle \in V_0 \) and a conformal vector \( \omega \in V_2 \), and an \( R \)-module homomorphism \( Y_\omega : V \to \text{End}_R(V) \), \( a \mapsto a(\omega) \) for each \( n \in \mathbb{Z} \) such that
   a. for any \( a \in V_i \), \( n \in \mathbb{Z}, j \in \mathbb{Z} \), we have \( a(\omega)|V_j \rangle \in \text{Hom}_R(V_j, V_{j+i-n-1}) \),
Remark 2.2. In this paper, we will mainly deal with the case where $R = O_{G/B}(U)$ for some open subset $U$ of a flag variety $G/B$ over $\mathbb{C}$.

Remark 2.3. In this paper, the case where $L_0$ acts semisimply only appears.

For $\mathbb{C}$-algebras $R, R'$, a $\mathbb{C}$-algebra homomorphism $f : R \to R'$, and a vertex operator algebra $V$ over $R$, the $R'$-module $R' \otimes_R V$ has a structure of a vertex operator algebra over $R'$ in an obvious way. When $R$ is obvious, we write $\otimes$ instead of $\otimes_R$ for the tensor product over $R$. For a vertex operator algebra $V$ over $R$ and a $V$-module $M$, denote $V = \bigoplus_{n \in \mathbb{Z}} V_n$ and $M = \bigoplus_{\Delta \in \mathbb{C}} M_\Delta$ by the conformal grading of $V$ and $M$, respectively.

Let $(X, \mathcal{O}_X)$ be a ringed space. For open subsets $U_1, U_2$ of $X$ that $U_1 \subseteq U_2$, an $\mathcal{O}_X$-module $\mathcal{M}$ and $s \in \mathcal{M}(U_2)$, denote by $r_{U_1, U_2}^\mathcal{M}$ (or simply $r_{U_1, U_2}$) the restriction map from $\mathcal{M}(U_2)$ to $\mathcal{M}(U_1)$, and $s|_{U_1}$ the image $r_{U_1, U_2}(s)$ of $s$. Let us recall that for $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{N}$, a morphism $\phi : \mathcal{M} \to \mathcal{N}$ consists of $\mathcal{O}_X(U)$-module homomorphisms $\phi(U) : \mathcal{M}(U) \to \mathcal{N}(U)$ for each open subset $U$ of $X$, such that for any open subset $U_1$ of $U_2$, we have $r_{U_1, U_2}^\mathcal{M} \circ \phi(U_2) = \phi(U_1) \circ r_{U_1, U_2}^\mathcal{M}$. For an $\mathcal{O}_X$-module $\mathcal{F}$ and an open subset $U$ of $X$, we write $\mathcal{F}|_U$ for the restriction of $\mathcal{F}$ to $U$. Also, denote by $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ the $\mathbb{C}$-module consisting of all morphisms from $\mathcal{M}$ to $\mathcal{N}$. Then the $\mathcal{O}_X$-module $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$, so-called sheaf Hom, is defined by $U \mapsto \text{Hom}_{\mathcal{O}_X|_U}(\mathcal{M}|_U, \mathcal{N}|_U)$.

For a $\mathbb{C}$-algebra $R$ and a ringed space $(X, \mathcal{O}_X)$, denote $\text{Mod}_R$ and $\text{Mod}_{\mathcal{O}_X}$ by the categories of $R$-modules and $\mathcal{O}_X$-modules, respectively. We regard $\text{Mod}_{\mathcal{O}_X(X)}$ as a subcategory of $\text{Mod}_{\mathcal{O}_X}$ by the embedding $M \mapsto \mathcal{O}_X \otimes_{\mathcal{O}_X(X)} M$. Note that these categories are (co)complete abelian.

Definition 2.4. For a ringed space $(X, \mathcal{O}_X)$, an $\mathcal{O}_X$-module $\mathcal{V}$ is called an $\mathcal{O}_X$-vertex operator algebra if

1. there exists a set $\{ \mathcal{V}_n \mid n \in \mathbb{Z} \}$ of locally free of finite rank $\mathcal{O}_X$-modules such that $\mathcal{V} = \bigoplus_{n \in \mathbb{Z}} \mathcal{V}_n$ as $\mathcal{O}_X$-modules and $\mathcal{V}_n = 0$ for $n \ll 0$,
2. for any open subset $U \subseteq X$, $\mathcal{V}(U)$ is a vertex operator algebra over $\mathcal{O}_X(U)$ and $\mathcal{V}_n(U) = \mathcal{V}(U)_n$ for any $n \in \mathbb{Z}$,
(3) for any open subsets $U_1 \subseteq U_2 \subseteq X$, the restriction map $r_{U_1,U_2}: \mathcal{V}(U_2) \to \mathcal{V}(U_1)$ defines a vertex operator algebra homomorphism.

**Definition 2.5.** For a ringed space $(X, \mathcal{O}_X)$ and an $\mathcal{O}_X$-vertex operator algebra $\mathcal{V}$, an $\mathcal{O}_X$-module $\mathcal{M}$ is called a $\mathcal{V}$-module if

1. there exists a set $\{\mathcal{M}_\Delta | \Delta \in \mathbb{C}\}$ of locally free of finite rank $\mathcal{O}_X$-modules such that $\mathcal{M} = \bigoplus_{\Delta \in \mathbb{C}} \mathcal{M}_\Delta$ as $\mathcal{O}_X$-modules and $\mathcal{M}_{\Delta-n} = 0$ for any $\Delta \in \mathbb{C}$ and $n \gg 0$,
2. for any open subset $U \subset X$, $\mathcal{M}(U)$ is a $\mathcal{V}(U)$-module,
3. for any open subsets $U_1 \subseteq U_2 \subseteq X$, the restriction map $r_{U_1,U_2}^{\mathcal{M}}: \mathcal{M}(U_2) \to \mathcal{M}(U_1)$ defines the linear map such that for any $a \in \mathcal{V}(U_2)$ and $n \in \mathbb{Z}$,

$$r_{U_1,U_2}^{\mathcal{M}} \circ a(n) = r_{U_1,U_2}(a(n)) \circ r_{U_1,U_2}^{\mathcal{M}}.$$

By abuse of notation, when $\mathcal{M}$ is an $\mathcal{O}_X \otimes \mathcal{O}_X$-module for some vertex operator algebra $V$ over $\mathcal{O}_X(X)$, we simply call $\mathcal{M}$ a $V$-module.

**Lemma 2.6.** Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{V}$ be an $\mathcal{O}_X$-vertex operator algebra, and $\mathcal{M}$ be a $\mathcal{V}$-module. Then $\mathcal{M}$ has a structure of $H^0(\mathcal{V})$-module.

**Proof.** For any open subset $U \subset X$, $\mathcal{M}(U)$ is a $\mathcal{V}(U)$-module, and $\mathcal{M} \in \mathcal{O}_X(U)$, and $n \in \mathbb{Z}$, set $(f \otimes s)^{\mathcal{V}(U)} = f(s|_{U(n)})$. Then the action satisfies the axioms in Definition 2.5.

**Definition 2.7.** Let $V$ be a vertex operator algebra over a $\mathbb{C}$-algebra $R$. Then for $a = \sum_{\Delta \in \mathbb{C}} a_{\Delta} \in V$, $a_{\Delta} \in V_{\Delta}$ and $n \in \mathbb{Z}$, the operator $(a^\dagger(n))^M \in \text{End}_R(M)$ is defined by

$$(a^\dagger(n))^M = \sum_{\Delta \in \mathbb{C}} (-1)^{\Delta} \sum_{m \geq 0} (\frac{m!}{m^2})^M a_{\Delta}^{(-n-m+2\Delta-2)}.$$

**Lemma 2.8.** Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{V}$ be an $\mathcal{O}_X$-vertex operator algebra, $\mathcal{M}$ be a $\mathcal{V}$-module, and $F$ be a $\mathbb{C}$-additive functor from a (co)complete abelian subcategory $\mathcal{C}$ of $\text{Mod}_{\mathcal{O}_X} \to \text{Mod}_{\mathcal{O}_X}$ such that if $\mathcal{N}$ is locally free of finite-rank, then so is $F(\mathcal{N})$.

1. If $F$ is covariant, then $F(\mathcal{M})$ has the $H^0(\mathcal{V})$-module structure as follows: for an open subset $U \subseteq X$, $a \in H^0(\mathcal{V})$, $n \in \mathbb{Z}$, and $\Delta \in \mathbb{C}$, we have $F(\mathcal{M})^U = F(a)(\mathcal{M})^U$, $a^F(n)^M = F(a^\dagger(n))^M$.
2. If $F$ is contravariant, then $F(\mathcal{M})$ has the $H^0(\mathcal{V})$-module structure as follows: for an open subset $U \subseteq X$, $a \in H^0(\mathcal{V})$, $n \in \mathbb{Z}$, and $\Delta \in \mathbb{C}$, we have $F(\mathcal{M})^U = F(a)$, $a^F(n)^M := F((a^\dagger(n))^M)$.

**Proof.** It is easily checked that the axioms in Definition 2.5 are satisfied.

**Corollary 2.9.** Let $V$ be a vertex operator algebra over a $\mathbb{C}$-algebra $R$, and let $M$ be a $V$-module. Then

$$M^* = \bigoplus_{\Delta \in \mathbb{C}} M^*_\Delta = \bigoplus_{\Delta \in \mathbb{C}} \text{Hom}_R(M, R)$$

has the $V$-module structure defined by

$$(a^\dagger(n))\phi(v) = \phi(a^\dagger(n)v),$$

where $a \in V$, $\phi \in M^*$, $v \in M$, and $a^\dagger(n)$ is given in Definition 2.7. We call $M^*$ the restricted dual $V$-module of $M$ (see [FHL]).
Proof. Let us consider the case where $X = \text{Spec } R$, $\mathcal{V} = \mathcal{O}_X \otimes V$, $\mathcal{M} = \mathcal{O}_X \otimes M$, $\mathcal{C} = \text{Mod}_R$, and the contravariant functor $F: \mathcal{C} \to \text{Mod}_{\mathcal{O}_X}$ is defined by
\begin{equation}
F(M) = M^*, \quad F(f) = ? \circ f
\end{equation}
for any $M \in \text{Mod}_R$ and $R$-module homomorphism $f$. Then by Lemma 2.8, the assertion is proved. \[\square\]

Corollary 2.10. Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{V}$ be an $\mathcal{O}_X$-vertex operator algebra, $\mathcal{M}$ be a $\mathcal{V}$-module. Then
\begin{equation}
\mathcal{M}^* := \bigoplus_{\Delta \in \mathcal{C}} \mathcal{M}_\Delta := \bigoplus_{\Delta \in \mathcal{C}} \text{Hom}_{\mathcal{O}_X}(\mathcal{M}_\Delta, \mathcal{O}_X)
\end{equation}
has the $\mathcal{V}$-module structure defined by
\begin{equation}
s(n)\phi(U) = \{ \phi(U_0) \circ \tau_{U_0, U}(s)^{\uparrow} \mid U_0 \text{ is an open subset of } U \},
\end{equation}
where $U$ is an open subset of $X$, $s \in \mathcal{V}(U)$, $n \in \mathbb{Z}$ and $\phi \in \mathcal{M}^*(U)$. We call $\mathcal{M}^*$ the restricted dual of $\mathcal{M}$.

Proof. It is easy to check the axioms in Definition 2.5. \[\square\]

Theorem 2.11. Let $X$ be a projective Cohen-Macaulay scheme of pure dimension $N$ over $\mathbb{C}$, $\mathcal{V}$ be an $\mathcal{O}_X$-vertex operator algebra, and $\mathcal{M}$ be a $\mathcal{V}$-module. Then the Serre duality (see, e.g. [H, III, (7.7)])
\begin{equation}
H^n(\mathcal{M}) = \bigoplus_{\Delta \in \mathcal{C}} H^n(\mathcal{M}_\Delta) \simeq \bigoplus_{\Delta \in \mathcal{C}} H^{N-n}(\mathcal{M}_\Delta^* \otimes \omega_X)^* = H^{N-n}(\mathcal{M}^* \otimes \omega_X)^*
\end{equation}
provides an $H^0(\mathcal{V})$-module isomorphism, where $\omega_X$ is the dualizing sheaf of $X$.

Proof. By Lemma 2.8 and Corollary 2.10, $\mathcal{M}^*$ and $\mathcal{M}^* \otimes \omega_X$ are $H^0(\mathcal{V})$-modules. Moreover, because right derived functors are $\mathbb{C}$-additive (see [H, III, (1.1 A)]), by Lemma 2.8, $H^n(\mathcal{M})$, $H^{N-n}(\mathcal{M}^* \otimes \omega_X)^*$, $\text{Ext}^n(\mathcal{M}^* \otimes \omega_X, \omega_X)$ and $\text{Ext}^n(\mathcal{O}_X, \mathcal{M})$ have the $H^0(\mathcal{V})$-module structures. Because (6) is defined by the isomorphisms [H, III, (6.3), (6.7), (7.6)], it is enough to show that they are $H^0(\mathcal{V})$-module isomorphisms. Since the isomorphisms [H, III, (6.3), (7.6)] are natural, namely, defined by the natural isomorphisms between the functors, they are $H^0(\mathcal{V})$-module isomorphisms.

Let us show that [H, III, (6.7)] is the $H^0(\mathcal{V})$-module isomorphism. For $n \geq 0$, $\phi_n$ denotes the natural isomorphism between the functors $\text{Ext}^n(\mathcal{M}^* \otimes \omega_X, ?)$ and $\text{Ext}^n(\mathcal{O}_X, \mathcal{M} \otimes \omega_X \otimes ?)$ in [H, III, (6.7)]. For any $s \in H^0(\mathcal{V})$ and $m \in \mathbb{Z}$, the $\mathcal{O}_X$-module endomorphism $s(m)$ on $\mathcal{M}$ induces the natural transformations from $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}^* \otimes \omega_X, ?)$ and $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M} \otimes \omega_X \otimes ?)$ to themselves, respectively. We use the same letter $s(m)$ for these natural transformations. Then, by [H, II, Ex.5.1], we have $s(m) \circ \phi_0 = \phi_0 \circ s(m)$. By the universality of $\text{Ext}^n(\mathcal{M}^* \otimes \omega_X, ?)$ and $\text{Ext}^n(\mathcal{O}_X, \mathcal{M} \otimes \omega_X \otimes ?)$, we have $s(m) \circ \phi_n = \phi_n \circ s(m)$, where by abuse of notations, these $s(m)$ mean the natural transformations from $\text{Ext}^n(\mathcal{M}^* \otimes \omega_X, ?)$ and $\text{Ext}^n(\mathcal{O}_X, \mathcal{M} \otimes \omega_X \otimes ?)$ to themselves, respectively, that are uniquely determined by the universality. Thus, [H, III, (6.7)] gives the $H^0(\mathcal{V})$-module isomorphism
\begin{equation}
\text{Ext}^n(\mathcal{M}^* \otimes \omega_X, \omega_X) \simeq \text{Ext}^n(\mathcal{O}_X, \mathcal{M} \otimes \omega_X^* \otimes \omega_X) = \text{Ext}^n(\mathcal{O}_X, \mathcal{M})
\end{equation}
for any $n \geq 0$. \[\square\]
Remark 2.12. Let us give (6) more explicitly in the case of $n = 0$. We have the natural perfect pairing (see [H, III, (7.6)])

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{M}^* \otimes \omega_X, \omega_X) \times H^N(\mathcal{M}^* \otimes \omega_X) \rightarrow H^N(\omega_X) \simeq \mathbb{C}$$

and the linear isomorphism

$$\phi : H^0(\mathcal{M}) \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{M}^* \otimes \omega_X, \omega_X), \ s \mapsto \phi(s),$$

where for any open subset $U \subseteq X$, $f \in \mathcal{M}^*(U)$ and $x \in \omega_X(U)$, $\phi(s)$ is defined by

$$\phi(s)(U)(f \otimes x) := f(U)(s|_U)x.$$

By Lemma 2.8, we give the $H^0(\mathcal{Y})$-module structure on $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}^* \otimes \omega_X, \omega_X)$ as follows: for any open subset $U \subseteq X$, $a \in H^0(\mathcal{Y})$ and $n \in \mathbb{Z}$,

$$(a(n), \phi(s))(U) := (a(n))(U) \circ a_{(n)}|_U.$$ 

Then (9) is the $H^0(\mathcal{Y})$-module isomorphism because

$$\phi(a(n), s)(U)(f \otimes x) = f(U)(a(n)s|_U)x = (a_{(n)}^\dagger)(U)(s|_U)x = (a_{(n)}(s))(U)(f \otimes x).$$

Thus, we have

$$R^N\Gamma(\phi(a(n), s)) = R^N\Gamma(\phi(s)) \circ a_{(n)}^\dagger = a_{(n)}R^N\Gamma(\phi(s)),$$

where the $a_{(n)}^\dagger$ in the second term is the $H^0(\mathcal{Y})$-action on $H^N(\mathcal{M})$, and the $a_{(n)}$ in the third term is that on $H^N(\mathcal{M})^*$. Thus, by (8) and (12), the linear isomorphism

$$H^0(\mathcal{M}) \simeq H^N(\mathcal{M}^* \otimes \omega_X)^*, \ s \mapsto R^N\Gamma(\phi(s))$$

is also the $H^0(\mathcal{Y})$-module isomorphism.

2.2. Our setting. Unless otherwise noted, the ground field is the complex number field $\mathbb{C}$ below. Let $\mathfrak{g}$ be a simply-laced simple Lie algebra of rank $l$, and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ the triangular decomposition, $\mathfrak{h}$ the Cartan subalgebra, $\mathfrak{b} = \mathfrak{n}_- \oplus \mathfrak{h}$ the Borel subalgebra, $G$ and $B$ the semisimple, simply-connected, complex algebraic groups corresponding to $\mathfrak{g}$ and $\mathfrak{b}$, respectively. We adopt the standard numbering for the simple roots $\{\alpha_1, \ldots, \alpha_l\}$ of $\mathfrak{g}$ as in [B] and $\{\omega_1, \ldots, \omega_l\}$ denotes the corresponding fundamental weights. Denote by $\Delta^\pm$ the sets of positive roots and negative roots of $\mathfrak{g}$, respectively. Let $Q$ be the root lattice of $\mathfrak{g}$, $P$ the weight lattice of $\mathfrak{g}$, $P_+$ the set of dominant integral weights of $\mathfrak{g}$. Denote by $\langle \cdot, \cdot \rangle$ the invariant form of $\mathfrak{g}$ normalized as $\langle \alpha_i, \alpha_i \rangle = 2$ for any $1 \leq i \leq l$, $W$ the Weyl group of $\mathfrak{g}$ generated by the simple reflections $\{\sigma_i\}_{i=1}^l$, $(c^{ij})$ the inverse matrix to the Cartan matrix of $\mathfrak{g}$, $\rho$ the half sum of positive roots, $\theta$ the highest root of $\mathfrak{g}$, $h$ the Coxeter number of $\mathfrak{g}$. We choose the set $\Lambda_0$ of representatives of generators of the abelian group $P/Q$ in $P_+$ as [Kac2], that is, $\lambda_0 \in \Lambda_0 \subseteq P_+$ satisfies $\langle \lambda_0, \theta \rangle = 1$. For $\mu \in \mathfrak{h}^*$ and the longest element $w_0$ in $W$, we use the notation $\mu'$ for $-w_0(\mu)$. For $\sigma \in W$, $l(\sigma)$ denotes the length of $\sigma$. Let $h_1, \ldots, h_l$ be the basis of $\mathfrak{h}$ corresponding to the simple roots by $\langle \cdot, \cdot \rangle$. Denote by $L(\beta)$ the simple $\mathfrak{g}$-module with highest weight $\beta$. For $\mu \in P, \mathbb{C}(\mu)$ denotes the one-dimensional $\mathfrak{b}$-module such that the action of $\mathfrak{n}_-$ is trivial and $h_i$ acts as $\langle \alpha_i, \mu \rangle id$ for $1 \leq i \leq l$. For a $B$-module $V$ and $\mu \in P$, we use the letter $V(\mu)$ for the $B$-module $V \otimes \mathbb{C}(\mu)$. For a vertex operator algebra $V$, $\mathcal{U}(V)$ denotes the universal enveloping algebra of $V$. For a $V$-module $M$ and $m \in M$, $\mathcal{U}(V)m$ means the $V$-submodule of $M$ generated by $m$. 
We fix an integer $p \in \mathbb{Z}_{\geq 2}$. Set

$$\Lambda_p = \{ \sum_{i=1}^{l} \frac{s_i}{\sqrt{p}} \omega_i \mid 0 \leq s_i \leq p - 1 \} \subseteq \frac{1}{\sqrt{p}} P_+.$$  

For $\mu \in \frac{1}{\sqrt{p}} P$, denote by $\mu_0$ and $\mu_p$ the elements of $P$ and $\Lambda_p$, respectively, such that $\mu = -\sqrt{p}\mu_0 + \mu_p$. Let

$$V_{\sqrt{p}Q} = \bigoplus_{\alpha \in Q} F(-\sqrt{p}\alpha)$$

be the simple lattice vertex operator algebra associated to the $\sqrt{p}Q$. Here, for $\mu \in \mathfrak{h}^*$, $F(\mu)$ denotes the Fock space over the rank $l$ Heisenberg vertex operator algebra $F(0)$ with highest weight $\mu$.

Remark 2.13. Our notation $F(\mu)$ corresponds to $\pi_{\kappa + h, \sqrt{\kappa + \mu}}$ in [ACL, Section 3], where $\kappa$ is a complex number such that $\kappa \neq -h$. Namely, we have $F(\mu) \simeq \pi_{\kappa + h, \sqrt{\kappa + \mu}}$ as $F(0)$-modules, where the $F(0)$-module structure on $\pi_{\kappa + h, \sqrt{\kappa + \mu}}$ is given by $(at^{-1}(0))_{(\kappa)} = \frac{1}{\sqrt{\kappa + \mu}} at^n$ for $a \in \mathfrak{h}$, $n \in \mathbb{Z}$, and the vacuum vector $|0\rangle$ of $F(0)$. In this paper, we mainly consider the cases where $\kappa + h = p$ or $\frac{1}{p}$.

We choose the conformal vector $\omega$ of $V_{\sqrt{p}Q}$ as

$$\omega = \frac{1}{2} \sum_{1 \leq i, j \leq l} c^{ij}(\alpha_i)(-1)\alpha_j + Q_0(\rho)(-2)|0\rangle \in F(0) \subseteq V_{\sqrt{p}Q},$$

where $Q_0 = \sqrt{p} - \frac{1}{\sqrt{p}}$. The central charge $c$ of $\omega$ is given by

$$c = l - 12|Q_0\rho|^2 = l - Q_0^2 h \dim \mathfrak{g}.$$  

The parameter set $\Lambda$ of simple $V_{\sqrt{p}Q}$-modules is given by

$$\Lambda = \{ \lambda = -\sqrt{p}\lambda_0 + \lambda_p \mid \lambda_0 \in \Lambda_0, \lambda_p \in \Lambda_p \}. $$

By [D], the set of $V_{\sqrt{p}Q}$-modules

$$\{ V_{\sqrt{p}Q + \lambda} = \bigoplus_{\alpha \in Q} F(-\sqrt{p}\alpha + \lambda) = \bigoplus_{\alpha \in Q} F(-\sqrt{p}(\alpha + \lambda_0) + \lambda_p) \}_{\lambda \in \Lambda}$$

provides a complete set of isomorphism classes of simple $V_{\sqrt{p}Q}$-modules.

We also define the generalized lattice vertex operator algebra and their modules

$$V_{\sqrt{p}P} = \bigoplus_{\lambda_0 \in \Lambda_0} V_{\sqrt{p}(\lambda_0)}, \quad V_{\sqrt{p}P + \lambda_p} = \bigoplus_{\lambda_0 \in \Lambda_0} V_{\sqrt{p}(\lambda_0) + \lambda_p}$$

in the same manner as $V_{\sqrt{p}Q}$ and $V_{\sqrt{p}Q + \lambda}$ (e.g., see [BK, DL, McR]).

For $\alpha \in Q$ and $\lambda \in \Lambda$, $| - \sqrt{p}\alpha + \lambda \rangle$ denotes the lattice point vector in $V_{\sqrt{p}Q + \lambda}$, so that $F(-\sqrt{p}\alpha + \lambda) = U(\mathcal{F}(0))| - \sqrt{p}\alpha + \lambda \rangle$. For $\mu \in \frac{1}{\sqrt{p}} P$, the conformal weight $\Delta_\mu$ of $|\mu\rangle$ is given by

$$\Delta_\mu = \frac{1}{2} |\mu - Q_0\rho|^2 + \frac{c - l}{24} = \frac{1}{2} |\mu|^2 - Q_0(\mu, \rho) \in \mathbb{Q}.$$  

In particular, for any $\lambda \in \Lambda$, we have $V_{\sqrt{p}Q + \lambda} = \bigoplus_{\Delta \in \mathbb{Q}} \Delta \bigoplus_{\lambda \in \Lambda} (V_{\sqrt{p}Q + \lambda})_{\Delta}$. 

Simplicity of higher rank triplet $W$-algebras
Lemma 2.15 ([S, Corollary A.4]). For \( \lambda_p \in \Lambda_p \) such that \((\sqrt{p}\lambda_p + \rho, \theta) \leq p\), we have \(\epsilon_{\lambda_p} (w_0) = -\rho\).

Lemma 2.16 ([S, Lemma 3.7]). Let \( w_0 = \sigma_{i(\omega_0)} \ldots \sigma_{i_1} \) be a minimal length expression of the longest element \( w_0 \) of \( W \). Then for \( \lambda_p \in \Lambda_p \), the following conditions are equivalent:

1. For any \( 1 \leq n \leq l(w_0) - 1 \), we have \((\epsilon_{\lambda_p} (\sigma_{i_n} \ldots \sigma_{i_1}), \alpha_{i_{n+1}}) = 0\),
2. We have \((\sqrt{p}\lambda_p + \rho, \theta) \leq p\).

For \( 1 \leq i \leq l, \alpha \in P, \lambda \in \Lambda \) such that \( 0 \leq s_i := (\sqrt{p}\lambda_p, \alpha_i) \leq p - 2 \), the narrow screening operator \( S_{i,\lambda_p} \in \text{Hom}_C(F(-\sqrt{p}\alpha + \lambda), F(-\sqrt{p}\alpha + \sigma_i \lambda )) \) is defined by

\[
S_{i,\lambda_p} = \int_{[\Gamma_{i+1}]} S_i(z_1) \ldots S_i(z_{s_i+1}) dz_1 \ldots dz_{s_i+1},
\]

where \( S_i(z) = | - \frac{1}{\sqrt{p} \alpha_i} (z) \) (see [CRW, NT, S] for the precise definition of \( S_i(z) \)), and the cycle \([\Gamma_{s_i+1}]\) is given in [NT, Proposition 2.1]. By [NT, Theorem 2.7, 2.8], we have \( S_{i,\lambda_p} \neq 0 \). For convenience, we set \( S_{i,\lambda_p} = 0 \) for \( \lambda \in \Lambda \) such that \( s_i = p - 1 \).

2.3. \( \mathcal{V}_{\sqrt{pq}} \) and \( \mathcal{V}_{\sqrt{pq}+\lambda}(\mu) \). For \( 1 \leq i \leq l, \lambda \in \Lambda, \mu \in P \), we consider the following operators

\[
f_i = |\sqrt{p}\alpha_i \rangle (0),
\]

\[
h_{i,\lambda_p} (\mu) = -\frac{1}{\sqrt{p}} (\alpha_i (0) + \sqrt{p} \mu) \text{id}
\]

acting on \( V_{\sqrt{pq}+\lambda} \otimes \mathbb{C}(\mu) \) (\( f_i \) is called the screening operator). When \( \mu = 0 \), we use the notation \( h_{i,\lambda_p} \) instead of \( h_{i,\lambda_p} (0) \). For any \( \lambda \in \Lambda, \mu \in P \) and \( 1 \leq i \leq l \), the operators \( f_i \) and \( h_{i,\lambda_p} (\mu) \) act on \( V_{\sqrt{pq}+\lambda} \otimes \mathbb{C}(\mu) \) as differentials: namely, we have

\[
f_i (a_{(n)} v) = (f_i a)_{(n)} v + a_{(n)} f_i v,
\]

\[
h_{i,\lambda_p} (\mu) (a_{(n)} v) = (h_{i,\mu} a)_{(n)} v + a_{(n)} h_{i,\lambda_p} (\mu) v
\]

for \( a \in V_{\sqrt{pq}}, \ v \in V_{\sqrt{pq}+\lambda} \otimes \mathbb{C}(\mu) \) and \( n \in \mathbb{Z} \). It is straightforward to show that

\[
[f_i, L_n] = [h_{i,\lambda_p} (\mu), L_n] = [S_{i,\lambda_p}, L_n] = 0, \ [f_i, S_{j,\lambda_p}] = 0
\]

for \( 1 \leq i, j \leq l \) and \( n \in \mathbb{Z}, \lambda \in \Lambda \) (see [S]). In particular, all \( f_i, h_{i,\lambda_p} (\mu) \) and \( S_{i,\lambda_p} \) preserve the conformal grading.

Lemma 2.17 ([FT, S]). The operators \( \{ f_i, h_{i,\lambda_p} \}_{i=1}^l \) give rise to an integrable action of \( b \) on \( V_{\sqrt{pq}+\lambda} \). More generally, the operators \( \{ f_i, h_{i,\lambda_p} (\mu) \}_{i=1}^l \) give rise to an integrable action of \( b \) on \( V_{\sqrt{pq}+\lambda} \otimes \mathbb{C}(\mu) \) that preserves the conformal grading.
Lemma 2.18. For any \( \lambda \in \Lambda \), we have \( V_{pQ+\lambda}^{\ast} \simeq V_{pQ+\mu} \ast V_{\sqrt{pQ}} \) as \( V_{pQ} \)-modules.

Proof. For any \( \lambda \in \Lambda \), since \( V_{pQ+\lambda} \) is a simple \( V_{pQ} \)-module, by [FHL, Proposition 5.3.2], \( V_{pQ+\lambda}^{\ast} \) is isomorphic to a simple \( V_{\sqrt{pQ}} \)-module \( V_{\sqrt{pQ}+\lambda} \) for some \( \lambda^\ast \in \Lambda \). We have

\[
\rho^{V_{\sqrt{pQ}+\lambda} \ast}(b)(a(\alpha)v) = (\rho^{V_{\sqrt{pQ}+\lambda}}(b)a(\alpha)\rho^{V_{\sqrt{pQ}+\lambda} \ast}(b)v).
\]

Moreover, since the sheaf of sections of \( G/B \) is \( \mathcal{O}_{G/B} \) and \( \mathcal{O}_{G/B} \) is in the center of \( V_{pQ} \), for any \( s \in H^0(\mathcal{V}_{pQ}) \) and \( n \in \mathbb{Z} \), \( s(n) \) defines an \( \mathcal{O}_{G/B} \)-module homomorphism. Denote by \( \pi' \) for the projection from \( G \) to \( G/B \). For any open subset \( U \subseteq G/B \), we give the vertex operator algebra \( \mathcal{O}_G(\pi'^{-1}(U)) \otimes V_{\sqrt{pQ}} \) over

By Lemma 2.17, we obtain the \( B \)-module \( V_{\sqrt{pQ}+\lambda}^\ast(\mu) \). For \( \lambda \in \Lambda \) and \( \mu \in P \), let

\[
\rho^{V_{\sqrt{pQ}+\lambda} \ast}(\mu) : B \to GL_C(V_{\sqrt{pQ}+\lambda}^\ast(\mu))
\]

denote the action of \( B \) on \( V_{\sqrt{pQ}+\lambda}^\ast(\mu) \) given by Lemma 2.17. By (27) and (28), the \( B \)-action \( \rho^{V_{\sqrt{pQ}+\lambda} \ast}(\mu) \) defines the automorphisms of \( V_{\sqrt{pQ}+\lambda}(\mu) \) as \( V_{\sqrt{pQ}} \)-module. Namely, for \( b \in B \), we have

\[
\rho^{V_{\sqrt{pQ}+\lambda} \ast}(\mu)(b)(a(\alpha)v) = (\rho^{V_{\sqrt{pQ}+\lambda}}(b)a(\alpha)\rho^{V_{\sqrt{pQ}+\lambda} \ast}(\mu)(b)v).
\]
$O_G(\pi^{t-1}(U))$ defined by $(f_1 \otimes v_1)(f_2 \otimes v_2) = f_1 f_2 \otimes v_1 v_2$. Then by (31), the orbifold $(O_G(\pi^{t-1}(U)) \otimes V_{\sqrt{pQ}})^B$ of the $B$-action defined by

$$b(f(g) \otimes v) = f(gb^{-1}) \otimes \rho^V_{\sqrt{pQ}}(b)v$$

on $O_G(\pi^{t-1}(U)) \otimes V_{\sqrt{pQ}}$ is the vertex operator algebra over $(O_G(\pi^{t-1}(U)))^B = O_G/B(U)$. In the same manner, $(O_G(\pi^{t-1}(U)) \otimes V_{\sqrt{pQ} + \lambda}(\mu))^B$ is the $(O_G(\pi^{t-1}(U)) \otimes V_{\sqrt{pQ}})^B$-module for any $\lambda \in \Lambda$ and $\mu \in P$. Since

$$\mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu)(U) = (O_G(\pi^{t-1}(U)) \otimes V_{\sqrt{pQ} + \lambda}(\mu))^B$$

and the compatibility between the action of $\mathcal{Y}_{\sqrt{pQ}}$ and the restriction maps of $\mathcal{Y}_{\sqrt{pQ}}$ and $\mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu)$ is clear, we obtain the following (here, let us recall Definition 2.4).

**Lemma 2.19.** The sheaf of sections $\mathcal{Y}_{\sqrt{pQ}}$ is an $O_G/B$-vertex operator algebra. Moreover, for $\lambda \in \Lambda$ and $\mu \in P$, $\mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu)$ is a $\mathcal{Y}_{\sqrt{pQ}}$-module.

Let us define the main object in the present paper.

**Definition 2.20.** By Lemma 2.19, the sheaf cohomology $H^0(\mathcal{Y}_{\sqrt{pQ}})$ is a vertex operator algebra, and we call $H^0(\mathcal{Y}_{\sqrt{pQ}})$ the higher rank triplet $W$-algebra associated to $\sqrt{pQ}$.

**Corollary 2.21.** For $0 \leq i \leq l(u_0)$, $H^i(\mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu))$ has a natural $H^0(\mathcal{Y}_{\sqrt{pQ}})$-module and $G$-module structure. Moreover, the $G$-action on $H^i(\mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu))$ gives the automorphisms of $H^0(\mathcal{Y}_{\sqrt{pQ}})$-module.

**Proof.** For each $0 \leq i \leq l(u_0)$, we consider the right derived functor $R^i \Gamma$ of the global section functor $\Gamma$ from $\text{Mod}_{O_G/B}$ to $\text{Mod}_C = \text{Mod}_{\text{O}_G/B(G/B)} \subseteq \text{Mod}_{\text{O}_G/B}$. By Lemma 2.6 and Lemma 2.19, $\mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu)$ is an $H^0(\mathcal{Y}_{\sqrt{pQ}})$-module. Hence by Lemma 2.8 and (38), $H^i(\mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu))$ is the $H^0(\mathcal{Y}_{\sqrt{pQ}})$-module and the set of morphisms

$$\{R^i \Gamma(s_{(n)})\}_{s \in H^0(\mathcal{Y}_{\sqrt{pQ}}); n \in \mathbb{Z}}$$

defines the $H^0(\mathcal{Y}_{\sqrt{pQ}})$-module structure on $H^i(\mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu))$. For $g \in G$, we have the isomorphism of sheaves $g_* \mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu) \cong \mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu)$ by

$$g : g_* \mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu)(U) \to \mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu)(U), \ t \mapsto gt$$

for any open subset $U \subseteq G/B$, where $g_* \mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu)(U) := \mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu)(g^{-1}U)$ and for $x \in U$, $(gt)(x) := t(g^{-1}x)$. Then we obtain the $G$-action on $H^i(\mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu))$ by $R^i \Gamma(g)$. On the other hands, by definition, we have

$$(gs)_{(n)}g = gs_{(n)} : g_* \mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu) \to \mathcal{Y}_{\sqrt{pQ} + \lambda}(\mu).$$

By sending these morphisms by $R^i \Gamma$, the remaining claim is proved. \qed

By (29), for any subset $I' \subseteq \{1, \ldots, l\}$, $\bigcap_{i \in I'} \ker f_i |_{F(0)}$ and $\bigcap_{i \in I'} \ker S_{i, 0} |_{V_{\sqrt{pQ}}}$ are vertex operator full subalgebras of $F(0)$ and $V_{\sqrt{pQ}}$, respectively (namely, they have the conformal vector $\omega$ in (16)).

**Definition 2.22.** For $p \in \mathbb{Z}_{\geq 2}$, $\lambda \in \Lambda$ and $\alpha \in P_+ \cap Q$, set

$$W(-\sqrt{p} \alpha + \lambda) := \bigcap_{i=1}^l (U(\ker f_i |_{F(0)}) - \sqrt{p} \alpha + \lambda) \subseteq F(-\sqrt{p} \alpha + \lambda).$$
Then

\[ W(0) = \bigcap_{i=1}^{l} \ker f_i|_{F(0)} \]

is a vertex operator full subalgebra of \( F(0) \), and \( W(-\sqrt{p}\alpha + \lambda) \) are \( W(0) \)-modules.

**Lemma 2.23** ([S, Main Theorem (2), Theorem 4.14, Lemma 4.19]). For any \( p \in \mathbb{Z}_{\geq 2} \) and \( \lambda \in \Lambda \), we have an embedding

\[ H^0(\mathcal{V}_{\rho_{\lambda}+\lambda}) \hookrightarrow \bigcap_{i=1}^{l} \ker S_{i,\lambda}|_{V_{\rho_{\lambda}+\lambda}}, \quad s \mapsto s(id B). \]

When \( \lambda = 0 \), (46) gives the vertex operator algebra isomorphism. Moreover, for \( \lambda \in \Lambda \) such that \( (\sqrt{p}\lambda + \rho, \theta) \leq p \), (46) gives the \( H^0(\mathcal{V}_{\rho_{\lambda}}) \)-module isomorphism.

**Definition 2.24.** For \( \lambda \in \Lambda \), \( W_{\sqrt{p}\rho_{\lambda}+\lambda} \) denotes the image of \( H^0(\mathcal{V}_{\rho_{\lambda}+\lambda}) \) by (46).

**Theorem 2.25** ([S]). For \( \lambda \in \Lambda \), we have a \( G \times W(0) \)-module isomorphism

\[ H^0(\mathcal{V}_{\rho_{\lambda}+\lambda}) \simeq W_{\sqrt{p}\rho_{\lambda}+\lambda} \simeq \bigoplus_{\alpha \in \mathbb{P}_+ \cap Q} L(\alpha + \lambda_0) \otimes W(-\sqrt{p}\alpha + \lambda). \]

**Remark 2.26.** The lowest conformal weight of \( W_{\sqrt{p}\rho_{\lambda}+\lambda} \) is \( \Delta_{\lambda} \) in (21), and we have \((W_{\sqrt{p}\rho_{\lambda}+\lambda})_{\Delta_{\lambda}} = L(\lambda_0) \otimes \mathbb{C}|\lambda\rangle\).

For \( \lambda \in \Lambda \), denote by \( \mathcal{V}_{\rho_{\lambda}+\lambda} \) the sheaf of sections of the dual bundle \( G \times_B V^\ast_{\sqrt{p}\rho_{\lambda}+\lambda} \) over \( G/B \), where the \( B \)-action on \( V^\ast_{\sqrt{p}\rho_{\lambda}+\lambda} \) is given by the contragredient representation \((\rho^\ast_{\sqrt{p}\rho_{\lambda}+\lambda})^\ast \) of \( \rho^\ast_{\sqrt{p}\rho_{\lambda}+\lambda} \). Then for \( 1 \leq i \leq l \), we have

\[ (\rho^\ast_{\sqrt{p}\rho_{\lambda}+\lambda})^\ast(h_i) = \rho_{\sqrt{p}w_{i0}+\lambda_i'}(h_i) + \text{id}, \]

\[ (\rho^\ast_{\sqrt{p}\rho_{\lambda}+\lambda})^\ast(f_i) = \rho_{\sqrt{p}w_{i0}+\lambda_i'}(f_i). \]

Here we identify \( V^\ast_{\sqrt{p}\rho_{\lambda}+\lambda} \) with \( V_{\sqrt{p}w_{i0}+\lambda_i'} \) by Lemma 2.18, and (48) and (49) follow from (34) and \(|\sqrt{p}\alpha_i\rangle_{(0)} = -|\sqrt{p}\alpha_i\rangle_{(0)}\), respectively. Thus, we obtain the following.

**Lemma 2.27.** For any \( \lambda \in \Lambda \), we have \( \mathcal{V}_{\rho_{\lambda}+\lambda}(-2\rho) \simeq \mathcal{V}_{\sqrt{p}w_{i0}+\lambda_i'}(-\rho) \) as \( \mathcal{V}_{\sqrt{p}\rho_{\lambda}} \)-modules. Moreover, for \( 0 \leq i \leq l(w_0) \), we have

\[ H^i(\mathcal{V}_{\rho_{\lambda}+\lambda}(-2\rho)) \simeq H^i(\mathcal{V}_{\sqrt{p}w_{i0}+\lambda_i'}(-\rho)) \]

as \( H^0(\mathcal{V}_{\sqrt{p}\rho_{\lambda}}) \)-modules.

3. PROOFS OF THEOREM 1.1 AND THEOREM 1.2

3.1. **Proof of** (2). By Lemma 2.11 and Lemma 2.19, we obtain the following.

**Proposition 3.1.** For any \( \lambda \in \Lambda \), we have \( H^0(\mathcal{V}_{\rho_{\lambda}}) \simeq H^0(\mathcal{V}_{\sqrt{p}\rho_{\lambda}+\lambda}(-2\rho))^\ast \)

as \( W_{\sqrt{p}\rho_{\lambda}} \)-modules.

For \( 1 \leq i \leq l \), let \( P_i \) be the minimal parabolic subgroup of \( G \) corresponding to \( \alpha_i \). For \( \lambda \in \Lambda \) and \( \mu \in \mathbb{P} \), we consider the homogeneous vector bundle \( P_i \times_B V_{\sqrt{p}\rho_{\lambda}+\lambda}(\mu) \) over \( P_i/B \simeq \mathbb{C}P^1 \), and the sheaf of sections (by abuse of notations, we use the same letter). In the same manner above, \( P_i \times_B V_{\sqrt{p}\rho_{\lambda}} \) is an \( \mathcal{O}_{P_i/B} \)-vertex operator algebra, and \( P_i \times_B V_{\sqrt{p}\rho_{\lambda}+\lambda}(\mu) \) is a \( P_i \times_B V_{\sqrt{p}\rho_{\lambda}} \)-module. In [S, Lemma 4.10], we proved that \( \ker S_{0,0}|_{V_{\sqrt{p}\rho_{\lambda}}} \simeq H^0(P_i \times_B V_{\sqrt{p}\rho_{\lambda}}) \) as vertex operator algebras. We induce the
ker $S_{i,n} |_{V/\mathfrak{g}}$-module and $P_i$-module structure (and thus, $W_{\sqrt{\mathfrak{g}}} \mathfrak{q}$-module structure) on $H^n(P_i \times_B V_{\sqrt{\mathfrak{g}}\mathfrak{q} + \lambda}(\mu))$ in the same manner as Corollary 2.21.

By Lemma 2.8, for $\lambda \in \Lambda$, $\mu \in P$, $p_1, p_2 \in \mathbb{Z}$ and the projection $\pi'_i$ from $G/B$ to $G/P_i$, we induce the $W_{\sqrt{\mathfrak{g}}} \mathfrak{q}$-module structure on

$$(R^{p_1} \Gamma \circ R^{p_2} \pi'_i)(\mathfrak{g}^{\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda}(\mu)) = H^{p_1}(G \times p_1, H^{p_2}(P_i \times B V_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \lambda)(\mu)).$$

**Proposition 3.2.** For $\lambda \in \Lambda$ such that $(\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda, \rho, \theta) \leq p$, we have $H^0(\mathfrak{g}^{\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda}) \simeq H^{(i\sigma)}(\mathfrak{g}^{\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda})(-2\rho))$ as $W_{\sqrt{\mathfrak{g}}} \mathfrak{q}$-modules.

**Proof.** Let us take a minimal expression $w_0 = \sigma_i(w_0) \cdots \sigma_1$ of $w_0$. In [S, Theorem 4.8], when $\lambda \in \Lambda$ satisfies the condition Lemma 2.16 (1), we proved that for any $\sigma = \sigma_{i_{n-1}} \cdots \sigma_{i_1}$, $2 \leq n \leq l(w_0)$, we have the $W_{\sqrt{\mathfrak{g}}} \mathfrak{q}$-module and $P_{i_n}$-module isomorphism

$$H^1(P_{i_n} \times_B V_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \sigma_{i_n}(\epsilon_{\lambda_p}(\sigma))) \simeq H^0(P_{i_n} \times_B V_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \sigma_{i_n}(\epsilon_{\lambda_p}(\sigma))) \simeq 0,$$

$$H^1(P_{i_n} \times_B V_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \sigma_{i_n}(\epsilon_{\lambda_p}(\sigma))) \simeq H^1(P_{i_n} \times_B V_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \sigma_{i_n}(\epsilon_{\lambda_p}(\sigma))) \not\simeq 0.$$

Then we obtain the $W_{\sqrt{\mathfrak{g}}} \mathfrak{q}$-module and $G$-module isomorphism

$$H^{(i)}(\mathfrak{g}^{\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda}(\epsilon_{\lambda_p}(\sigma)))$$

$$\simeq H^0(G \times p_{i_n} H^0(P_{i_n} \times_B V_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \sigma_{i_n}(\epsilon_{\lambda_p}(\sigma))))$$

$$\simeq H^1(G \times p_{i_n} H^1(P_{i_n} \times_B V_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \sigma_{i_n}(\epsilon_{\lambda_p}(\sigma))))$$

$$\simeq H^{(i+1)}(\mathfrak{g}^{\sqrt{\mathfrak{g}} \mathfrak{q} + \sigma_{i_n}(\epsilon_{\lambda_p}(\sigma))),$$

where the first and third isomorphisms in (53) also follow from (51) and (52), namely, the fact that the Leray spectral sequences have the trivial filtrations (see [S, Theorem 4.8]). By continuing (53) for $1 \leq n \leq l(w_0)$ and using Lemma 2.15 and Lemma 2.27 lastly, we obtain that

$$H^0(\mathfrak{g}^{\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda}) \simeq \cdots \simeq H^{(i\sigma)}(\mathfrak{g}^{\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda + \rho}(-\rho)) \simeq H^{(i\sigma)}(\mathfrak{g}^{\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda}(-2\rho))$$

as $W_{\sqrt{\mathfrak{g}}} \mathfrak{q}$-modules and $G$-modules. □

**Corollary 3.3.** For $\lambda \in \Lambda$ such that $(\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda, \rho, \theta) \leq p$, we have a $W_{\sqrt{\mathfrak{g}}} \mathfrak{q}$-module isomorphism

$$W_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \lambda \simeq W^*_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \lambda'.$$

**Proof.** By [FHL, Proposition 5.3.1], we have

$$H^{(i\sigma)}(\mathfrak{g}^{\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda}(-2\rho)) \simeq (H^{(i\sigma)}(\mathfrak{g}^{\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda}(-2\rho))^*)$$

as $W_{\sqrt{\mathfrak{g}}} \mathfrak{q}$-modules. Thus, by Proposition 3.1 and Proposition 3.2, we have

$$W_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \lambda \simeq H^{(i\sigma)}(\mathfrak{g}^{\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda}(-2\rho)) \simeq (H^{(i\sigma)}(\mathfrak{g}^{\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda}(-2\rho))^*) \simeq W^*_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \lambda'.$$

□

**Remark 3.4.** When $g = \mathfrak{sl}_2$, $W_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \lambda$ is self-dual for any $\lambda \in \Lambda$ (see [NT, Proposition 3.12]). In fact, all $\lambda \in \Lambda$ satisfy $(\sqrt{\mathfrak{g}} \mathfrak{q} + \lambda, \rho, \theta) \leq p$ and $\lambda = \lambda'$.

If $W_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \lambda' \simeq W_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \lambda$ as $W_{\sqrt{\mathfrak{g}}} \mathfrak{q}$-modules, then by [FHL, Section 5.3], there exists a non-degenerate bilinear form

$$\langle , \rangle : W_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \lambda' \times W_{\sqrt{\mathfrak{g}}} \mathfrak{q} + \lambda \to \mathbb{C}.$$
such that for any \( a \in W_{\mathcal{P}}Q, u \in W_{\mathcal{P}}Q + \lambda, v \in W_{\mathcal{P}}Q + \lambda \) and \( n \in \mathbb{Z} \), we have
\[
\langle a^{(n)}u, v \rangle = \langle u, a^{(n)}v \rangle. \tag{58}
\]
Namely, \( \langle \cdot, \cdot \rangle \) is a non-degenerate \( W_{\mathcal{P}} \)-invariant bilinear map. In particular, \( \langle \cdot, \cdot \rangle \) is non-degenerate on \((W_{\mathcal{P}}Q + \lambda)_{\Delta} \times (W_{\mathcal{P}}Q + \lambda)_{\Delta} \) for each \( \Delta \in \mathbb{Q} \), and we have \( \langle (W_{\mathcal{P}}Q + \lambda)_{\Delta}, (W_{\mathcal{P}}Q + \lambda)_{\Delta'} \rangle = 0 \) if \( \Delta \neq \Delta' \).

**Lemma 3.5.** Let us fix an integer \( p \geq 2 \) and \( \lambda \in \Lambda \). If \( W_{\mathcal{P}}Q + \lambda \simeq W_{\mathcal{P}}^*Q + \lambda \) as \( W_{\mathcal{P}} \)-modules and \( W_{\mathcal{P}}Q + \lambda \) is generated by its lowest conformal weight vectors (see Remark 2.26) as \( W_{\mathcal{P}}Q \)-module, then for any nonzero \( W_{\mathcal{P}}Q \)-submodule \( S \subseteq W_{\mathcal{P}}Q + \lambda \), we have \( S \cap (W_{\mathcal{P}}Q + \lambda)_{\Delta} \neq \{0\} \).

**Proof.** Let \( S \subseteq W_{\mathcal{P}}Q + \lambda \) be a nonzero \( W_{\mathcal{P}} \)-submodule in \( W_{\mathcal{P}}Q + \lambda \). We take a nonzero vector \( v = \sum v_{\Delta} \in S \), where \( v_{\Delta} \in (W_{\mathcal{P}}Q + \lambda)_{\Delta} \). Let \( \Delta' \) be the conformal weight such that \( v_{\Delta'} \neq 0 \) and \( \Delta' \geq \Delta \) for all \( \Delta \) such that \( v_{\Delta} \neq 0 \). Then by the non-degeneracy of \( \langle \cdot, \cdot \rangle \) on \((W_{\mathcal{P}}Q + \lambda)_{\Delta} \times (W_{\mathcal{P}}Q + \lambda)_{\Delta} \), there exists an element
\[
a = \sum (a_{i_{1}})_{(-n_{1})} \cdots (a_{i_{m}})_{(-n_{m})}(x \otimes |\lambda') \in (W_{\mathcal{P}}Q + \lambda_{\Delta})_{\Delta},
\]
where \( x \in L(\lambda_{0}) \) and \( \sum (a_{i_{1}})_{(-n_{1})} \cdots (a_{i_{m}})_{(-n_{m})} \in \mathcal{U}(W_{\mathcal{P}}Q) \), such that
\[
0 \neq \langle a, v_{\Delta'} \rangle = \langle x \otimes |\lambda', \sum (a_{i_{1}})_{(-n_{1})} \cdots (a_{i_{m}})_{(-n_{m})}v_{\Delta'} \rangle. \tag{59}
\]
Since \( W_{\mathcal{P}}Q + \lambda = \bigoplus_{\Delta \geq \Delta_{\lambda}} (W_{\mathcal{P}}Q + \lambda)_{\Delta} \) and \( \Delta' = \Delta_{\lambda} \), we obtain that
\[
\sum (a_{i_{m}})_{(-n_{m})} \cdots (a_{i_{1}})_{(-n_{1})}v = \sum (a_{i_{m}})_{(-n_{m})} \cdots (a_{i_{1}})_{(-n_{1})}v_{\Delta'} \in L(\lambda_{0}) \otimes |\lambda|
\]
is nonzero. Thus, \( S \) contains a nonzero element of \( L(\lambda_{0}) \otimes |\lambda| = (W_{\mathcal{P}}Q + \lambda)_{\Delta_{\lambda}} \). \( \square \)

**3.2. Proof of Theorem 1.1 for \( \lambda_{p} = 0 \).** We introduce a generalized vertex operator algebra
\[
W_{\mathcal{P}} := \bigcap_{i=1}^{l} \ker S_{i,0}|_{V_{\mathcal{P}}} = \bigoplus_{\lambda_{0} \in \Lambda_{0}} W_{\mathcal{P}}Q - \lambda_{0} \simeq H^{0}(G \times B V_{\mathcal{P}}),
\]
and for \( \lambda_{p} \in \Lambda_{p} \), the \( W_{\mathcal{P}}Q \)-modules
\[
W_{\mathcal{P}}Q + \lambda_{p} := \bigoplus_{\lambda_{0} \in \Lambda_{0}} W_{\mathcal{P}}Q - \lambda_{0} + \lambda_{p} \simeq H^{0}(G \times B V_{\mathcal{P}} + \lambda). \tag{62}
\]
In the same manner as Corollary 3.3, we obtain the following:

**Theorem 3.6.** We have the \( W_{\mathcal{P}}Q \)-module isomorphism
\[
W_{\mathcal{P}}Q + \lambda_{p} \simeq W_{\mathcal{P}}^*Q + \lambda_{p}. \tag{63}
\]

**Corollary 3.7.** For \( p \geq h - 1 \), \( W_{\mathcal{P}} \) is simple.

**Proof.** The condition \( p \geq h - 1 \) is equivalent to \( (\rho, \theta) \leq p \), namely, \( (\sqrt{p}\lambda_{p} + \rho, \theta) \leq p \) for the case \( \lambda_{p} = 0 \). By Theorem 3.6, \( W_{\mathcal{P}}Q \) is self-dual, and thus, has a non-degenerate \( W_{\mathcal{P}}Q \)-invariant bilinear form \( W_{\mathcal{P}}Q \times W_{\mathcal{P}}Q \to \mathbb{C} \). Since \( W_{\mathcal{P}}Q \) is generated by the vacuum vector \( |0\rangle \), which is the unique eigenvector of the (lowest) conformal weight 0 in \( W_{\mathcal{P}}Q \), in the same manner as Lemma 3.5, \( W_{\mathcal{P}} \) is simple. \( \square \)
From now on, $\kappa$ and $\kappa'$ mean complex numbers such that $\kappa \neq -h$, $\kappa' \neq -h$, and $\kappa' = \frac{1}{\kappa + h} - h$. Denote by $W^\kappa(g)$ and $W_{\kappa}(g)$ the affine $W$-algebra [FF] of level $\kappa$ and the unique simple quotient, respectively. Then we have the Feigin-Frenkel duality [FF1, FF2, ACL]

\[
W^\kappa(g) \simeq W^{\kappa'}(g).
\]
From now on, we use the letters $k = p - h$ and $k' = \frac{1}{p} - h$.

**Theorem 3.8.** For $p \geq h - 1$ and $\lambda_p = 0$, Theorem 1.1 is true and $W(-\sqrt{p}\alpha + \lambda)$ is a simple $W_k(g)$-module for any $\alpha \in P_+ \cap Q$.

**Proof.** By Corollary 3.7, $W_{\sqrt{p}P}$ is simple. By applying [McR, Theorem 3.2] to $W_{\sqrt{p}P}$, for any $\alpha \in P_+ \cap Q$ and $\lambda_0 \in \Lambda_0$, we obtain that $W(-\sqrt{p}(\alpha + \lambda_0))$ is simple as $W(0)$-module. In particular, $W(0)$ is simple. On the other hand, by [S, Theorem 1.2], we have $W(0) \simeq W^k(g)$ for $p \geq h - 1$. Thus, for the case where $\lambda_p = 0$, $W(-\sqrt{p}\alpha + \lambda)$ is simple as $W_k(g)$-module. Finally, by applying [McR, Proposition 2.26] to $W_{\sqrt{p}P}$, we obtain that $W_{\sqrt{p}(Q - \lambda_0)}$ is simple as $W_{\sqrt{p}Q}$-module for each $\lambda_0 \in \Lambda_0$. □

By combining (64) with the simplicity of $W^k(g) \simeq W(0)$ in Theorem 3.8, for $p \geq h - 1$, we obtain that

\[
W_k(g) \simeq W^k(g) \simeq W^{k'}(g) \simeq W_{k'}(g).
\]

### 3.3. Proof of Theorem 1.2

Let us recall the notations in [Kac2, Chapter 6]. In this paper, we consider the affine Lie algebras of type $A^{(1)}_l$, $D^{(1)}_l$, $E^{(1)}_6$, $E^{(1)}_7$, $E^{(1)}_8$. For an affine Lie algebra $\hat{g}$ corresponding to the finite-dimensional simply-laced simple Lie algebra $g$, the Cartan subalgebra $h$ and its dual $h^*$ are decomposed as $h = h^0 \oplus (\mathbb{C}K + \mathbb{C}d)$ and $h^* = h^0 \oplus (\mathbb{C}d + \mathbb{C}\Lambda_0)$, respectively (there would be no risk of confusion with $\Lambda_0 \subseteq P_+$ in Section 2.2). For $\mu \in h^*$, we use the letter $\bar{\mu} \in h^*$ for the classical part of $\mu$. By this notation, we have the decomposition

\[
\mu = \bar{\mu} + \langle \mu, K \rangle \Lambda_0 + (\mu, \Lambda_0) d.
\]

The affine Weyl group $\hat{W}$ of $\hat{g}$ is given by $\hat{W} = W \ltimes Q$ (see [Kac2, Proposition 6.5]). Here, the action

\[
\hat{W} = W \ltimes Q \rightarrow \text{GL}(h^*), \quad (\sigma, \beta) \mapsto \sigma t_\beta
\]
is defined by the following: for $\mu \in h^*$, we have

\[
\sigma t_\beta(\mu) = \sigma(\bar{\mu} + (\mu, K)\beta) + (\mu, K)\Lambda_0 + ((\mu, \Lambda_0 - \beta) - \frac{1}{2}(|\beta|^2(\mu, K))d).
\]

We also define the $\circ$ action of $\hat{W}$ on $h^*$ by

\[
\sigma t_\beta \circ \mu = \sigma t_\beta(\mu + (\rho + h\Lambda_0) - (\rho + h\Lambda_0).
\]

By [Kac2, Proposition 6.3], the set of positive real roots of $\hat{g}$ is given by

\[
\hat{\Delta}^+_re = \{\bar{\gamma} + n\delta \mid \bar{\gamma} \in \Delta^\pm, n > 0\} \cup \Delta^+,
\]
where $\Delta^+$ and $\Delta^-$ are sets of positive and negative roots of $g$, respectively. Set

\[
\check{\Delta}^+ = \{\mu \in h^* \mid (\gamma, \mu + \rho + h\Lambda_0) \geq 0\text{ for any }\gamma \in \hat{\Delta}^+_re\}.
\]

By calculation, we obtain the following lemma.
Lemma 3.9. For \( \lambda \in \Lambda, \sigma \in W \) and \( \beta \in Q \), we have
\[
\sigma t_{\beta} \circ (-p(\alpha + \lambda_0 + \rho) + \sqrt{p} \lambda_p + k \Lambda_0) \in \mathcal{C}^+
\]
if and only if
\[
0 \leq (\sigma^{-1}(\gamma), p(\beta - (\alpha + \lambda_0 + \rho)) + \sqrt{p} \lambda_p + \rho) \leq p \text{ for any } \gamma \in \Delta^+.
\]

When \( \lambda \in \Lambda \) satisfies \( (\sqrt{p} \lambda_p + \rho, \theta) \leq p \), a pair \((\sigma, \beta)\) satisfying (73) is given by the following.

Lemma 3.10. Let \( \alpha \in P_+ \cap Q \) and \( \lambda \in \Lambda \) such that \( (\sqrt{p} \lambda_p + \rho, \theta) \leq p \). Then there exist \( \omega_{\lambda_0} \in \Lambda_0 \) and \( \sigma_{\lambda_0} \in W \) such that \( \beta_{\lambda_0} = \alpha + \lambda_0 + \rho - \omega_{\lambda_0} \in Q \) and
\[
\sigma_{\lambda_0}^{-1}(\Delta^+) \cap \Delta^\pm = \{ \pm \gamma \in \Delta^\pm | (\gamma, \omega_{\lambda_0}) = \frac{\pm 1}{2} \}.
\]
Moreover, the pair \((\sigma_{\lambda_0}, \beta_{\lambda_0})\) satisfies (73) for \( \lambda \).

Proof. Let us fix \( \lambda \in \Lambda \) such that \( (\sqrt{p} \lambda_p + \rho, \theta) \leq p \). First we prove the assertion in the case of \( (\sqrt{p} \lambda_p + \rho, \theta) < p \). By [KT, Lemma 2.10], there exist \( \sigma \in W \) and \( \beta = \alpha + \lambda_0 + \rho - \sum_{i=1}^{l} n_i \omega_i \in Q \) satisfying (73) for \( \lambda \) (where \( n_i \in \mathbb{Z} \)). Set
\[
I_+ = \{ i \mid i \in \Lambda^+ \}
\]
Clearly, we have \( I := \{ 1, \ldots, l \} = I_+ \cup I_- \). For \( x \in \Delta_+ \), if there exists, we write \( \bar{\gamma}_x \) for the positive root such that \( \sigma^{-1}(\bar{\gamma}_x) = x \). For each \( i \in I_+ \), by considering the case \( \bar{\gamma} = \bar{\gamma}_{\alpha_i} \) in (73), we have
\[
0 \leq (\alpha_i, -p \sum_{j \in I} n_j \omega_j + \sqrt{p} \lambda_p + \rho) = -pm_i + (\sqrt{p} \lambda_p + \rho, \alpha_i) \leq p.
\]
By (76) and the assumption \( (\sqrt{p} \lambda_p + \rho, \theta) < p \), we have \( n_i = 0 \) for any \( i \in I_+ \). Similarly, for each \( i \in I_- \), by considering the case \( \bar{\gamma} = \bar{\gamma}_{-\alpha_i} \) in (73), we have \( n_i = 1 \) for any \( i \in I_- \). Thus, we obtain that
\[
\beta = \alpha + \lambda_0 + \rho - \sum_{i \in I_-} \omega_i.
\]

Let us note that one of \( \bar{\gamma}_{\theta} \) or \( \bar{\gamma}_{-\theta} \) always exists. If there exists \( \bar{\gamma}_{\theta} \), then by considering the case \( \bar{\gamma} = \bar{\gamma}_{\theta} \) in (73) with the assumption \( (\sqrt{p} \lambda_p + \rho, \theta) < p \), we have \( \theta, \sum_{i \in I_-} \omega_i = 0 \). Since \( \theta \geq \alpha_i \) for any \( i \in I \), we obtain that \( I_- = \emptyset \). Thus, we have \( \beta = \alpha + \lambda_0 + \rho \) (namely, \( \omega_{\lambda_0} = 0 \)) and \( \sigma_{\lambda_0} := \sigma = \text{id} \), and they satisfy (74).

On the other hand, if there exists \( \bar{\gamma}_{-\theta} \), similarly we obtain that \( I_- = \{ \omega_i \} \) for some \( i \in I \) such that \( (\omega_i, \theta) = 1 \). In particular, \( \omega_i \in \Lambda_0 \). Thus, for \( \omega_{\lambda_0} := \omega_i \) and \( \sigma_{\lambda_0} := \sigma \), it is easy to check that they satisfy (74) by (73).

Finally, let us extend the results above to the case of \( (\sqrt{p} \lambda_p + \rho, \theta) \leq p \). Since \( \omega_{\lambda_0} \) and \( \sigma_{\lambda_0} \) do not depend on the choice of \( \lambda_p \) and they satisfy (74), by applying (74) to (73), we obtain the assertion. \( \square \)

Remark 3.11. It is easy to check that \( \omega_{\lambda_0} \in \Lambda_0 \) and \( \sigma_{\lambda_0} \in W \) in Lemma 3.10 satisfy (73) if and only if \( (\sqrt{p} \lambda_p + \rho, \theta) \leq p \).

Let us recall notations in [Ar]. For an affine Lie algebra \( \hat{g} \) and \( \mu \in \hat{h}^* \), denote by \( \hat{L}(\mu) \) and \( \hat{M}(\mu) \) the corresponding irreducible highest weight module and Verma module of \( \hat{g} \) with highest weight \( \mu \in \hat{h}^* \). On the other hand, for \( \bar{\mu} \in \hat{h}^* \), let \( \gamma_{\bar{\mu}} : Z(\hat{g}) \rightarrow \mathbb{C} \) be the evaluation at the Verma module \( \hat{M}(\bar{\mu}) \) of \( \hat{g} \) with highest weight \( \bar{\mu} \), where \( Z(\hat{g}) \) is the center of \( \mathcal{U}(\hat{g}) \). We write \( L_n(\gamma_{\bar{\mu}}) \) and \( M_n(\gamma_{\bar{\mu}}) \) for
the corresponding irreducible highest weight module and Verma module of $W^\kappa(\mathfrak{g})$, respectively. Then $L_\kappa(\gamma_\bar{\mu})$ is the simple quotient of $M_\kappa(\gamma_\bar{\mu})$ (see [Ar, Section 5.3]). When $\kappa = k$ or $k'$, we will denote $L_\kappa(\gamma_\bar{\mu})$ and $M_\kappa(\gamma_\bar{\mu})$ simply by $L(\gamma_\bar{\mu})$ and $M(\gamma_\bar{\mu})$, respectively. By [ArF, Lemma 4.2], we obtain the following lemma.

**Lemma 3.12.** Under the identification (64), for $\nu, \mu \in P$, we have

$$L_\kappa(\gamma_\nu - (\kappa + h)(\mu + \rho)) \simeq L_{\kappa'}(\gamma_\mu - (\kappa' + h)(\nu + \rho)).$$

For a $W^\kappa(\mathfrak{g})$-module $M$, we write $[M]$ for the element corresponding to $M$ in the Grothendieck group of the category of $W^\kappa(\mathfrak{g})$-modules. By combining Lemma 3.10 with the main results of [Ar] and [KT], we obtain the following lemma.

**Lemma 3.13.** For $\alpha \in P_+ \cap Q$ and $\lambda \in \Lambda$ such that $(\sqrt{p}\lambda_\alpha + \rho, \theta) \leq p$, we have

$$[L(\gamma_{-p(\alpha + \lambda_0) + \rho} + \sqrt{p}\lambda_\alpha)] = \sum_{y \in y_\alpha, \lambda_0} a_{y, y_\alpha, \lambda_0} [M(\gamma_{y(y_\alpha)})],$$

where $\gamma$ is the Bruhat order, $\sigma_{\lambda_0}$ and $\omega_{\lambda_0}$ are defined in Lemma 3.10,

$$y_\alpha, \lambda_0 = t_{\omega_{\lambda_0} - (\alpha + \lambda_0 + \rho)}^{-1} \sigma_{\lambda_0}^{-1} \in \hat{W},$$

$$a_{y, y_\alpha, \lambda_0} = (-1)^{t(y)}(y_\alpha, \lambda_0)Q_{y, y_\alpha, \lambda_0}(1),$$

$$\mu_\lambda = \sigma_{\lambda_0}(-p\omega_{\lambda_0} + \sqrt{p}\lambda_\rho + \rho) - \rho + k\lambda_0 \in \hat{C}^+,$$

and $Q_{y', \rho}(z)$ is the inverse Kazhdan-Lusztig polynomial. In particular, we have

$$\text{tr}L(\gamma_{-p(\alpha + \lambda_0) + \rho} + \sqrt{p}\lambda_\alpha) q^{L_0 - \frac{p}{2}} = \sum_{y \in y_\alpha, \lambda_0} a_{y, y_\alpha, \lambda_0} \frac{q^{\frac{1}{2}(y(y_\alpha))} - q^{\frac{1}{2}((y(y_\alpha) + \rho))}}{y(q)^{\frac{p}{2}}}.$$
and $F(-\sqrt{p}\alpha + \lambda)\Delta_{\lambda} + \lambda = \mathbb{C} - \sqrt{p}\alpha + \lambda$, we have
\[
W(-\sqrt{p}\alpha + \lambda)_{\text{top}} = \mathbb{C} - \sqrt{p}\alpha + \lambda = F(-\sqrt{p}\alpha + \lambda)_{\text{top}}.
\]
Thus, by (87) and (89), we obtain (84). The last assertion follows from (84), [Ar, Theorem 5.3.1], and Theorem 3.8.

Let us recall the following result in [S].

**Theorem 3.15** ([ArF, S]). For any $\alpha \in P_+ \cap Q$ and $\lambda \in \Lambda$ such that $(\sqrt{p}\lambda + \rho, \theta) \leq p$, we have the character formula
\[
\text{tr}_{W(-\sqrt{p}\lambda + \lambda)} q^{L_0 - \frac{c}{24}} = \sum_{\sigma \in W} (-1)^{(\sigma)} \frac{q^{\frac{1}{2} \sqrt{p}\sigma(\alpha + \lambda_0 + \rho) - \lambda_p - \frac{1}{2} p \rho^2}}{\eta(q)^{\frac{1}{2}}}.
\]
where $\eta(q)$ is the Dedekind eta function. Furthermore, (90) coincides with the characters of the $W_k(g)$-modules $T_{\sqrt{p}\lambda + \lambda}$ and $\hat{T}_{\lambda + \lambda_0, \sqrt{p}\lambda}$ defined in [ArF], where $T_{\lambda + \lambda_0, \sqrt{p}\lambda} = H_+^0(\hat{V}(\nu + \lambda\lambda_0))$, $\hat{V}(\nu + \lambda\lambda_0)$ is the Weyl module over $\hat{g}$ of level $\kappa$ induced from $L(\nu)$, and $H_+^0(\nu)$ is the functor in [ArF, Section 2.1].

When $\mu = 0$, the functor $H_+^0(\nu)$ is the “+” reduction $H_+^0(\nu)$ in [Ar, Section 6.4].

**Remark 3.16.** By [ArF, Section 4.3], for $\nu, \mu \in P_+$ and $\kappa \in \mathbb{C}$ such that $\kappa \neq -h$, we have $\text{tr}_{\hat{T}_{\lambda + \lambda_0, \sqrt{p}\lambda}} q^{L_0 - \frac{c}{24}} = \text{tr}_{\hat{T}_{\lambda + \lambda_0, \sqrt{p}\lambda}} q^{L_0 - \frac{c}{24}}$.

**Lemma 3.17.** For any $\alpha \in P_+ \cap Q$ and $p \geq h$, we have
\[
[T_{\alpha + \lambda_0, 0}^\perp] = \sum_{\sigma \in W} (-1)^{(\sigma)} [M(\gamma y_{\sigma} (\alpha + \lambda_0) - \lambda_0)],
\]
where $y_{\sigma} = t_{\sigma(\omega_{\lambda_0})} - (\alpha + \lambda_0 + \rho)\sigma^{-1}$ in $\hat{W}$.

**Proof.** As in [ArF], we have the contragredient BGG resolution $C^\perp(\mu)$ of $L(\mu)$ and the induced reduction $\hat{C}^\perp(\mu)$ of $\hat{g}$-module of level $\kappa'$ such that
\[
\hat{C}^\perp(\mu) = \bigoplus_{\sigma \in W} \hat{M}(\sigma \circ \mu)^*,
\]
where $M(\xi)^*$ is the contragredient Verma module over $\hat{g}$ with highest weight $\xi \in \mathfrak{h}^*$, and $\hat{M}(\xi + \kappa\lambda_0)^*$ is the corresponding induced $\hat{g}$-module of level $\kappa'$. Let us consider the case where $\kappa = k$ and $\mu = \alpha + \lambda_0$. Because $p \geq h$, $\alpha + \lambda_0 + k'\lambda_0$ satisfies the condition [Ar, (385)]. By [Ar, Theorem 9.1.3], the “+” reduction $H_+^0(\nu)$ is the exact functor from $O_{k'}$ (see [Ar, Section 6.1]) to the full subcategory of the category of graded $W_k(g)$-modules consisting of admissible $W_k(g)$-modules. Thus, by sending the induced BGG resolution $\hat{C}^\perp(\alpha + \lambda_0)$ by $H_+^0(\nu)$, we obtain the resolution $H_+^0(\hat{C}^\perp(\alpha + \lambda_0))$ of $T_{\alpha + \lambda_0, 0}^\perp$. Since $[H_+^0(\hat{M}(\sigma \circ (\alpha + \lambda_0 + k'\lambda_0)^*))] = [M(\gamma y_{\sigma} (\alpha + \lambda_0) - \lambda_0)]$, we obtain the assertion.

From now on, for $q$-series $A(q), B(q) \in q^\mathbb{Z}[[q]], x \in \mathbb{Q}$, the notion $A(q) \geq B(q)$ means that $A(q) - B(q) \in q^\mathbb{Z}_{\geq 0}[[q]]$. 

\[\text{Simplicity of higher rank triplet } W\text{-algebras}\]
Remark 3.18. For a vertex operator algebra $V$, a $V$-module $M$ and a $V$-submodule $N$ of $M$, we have $\text{tr}_M q^{L_0 - \frac{c}{2}} \geq \text{tr}_N q^{L_0 - \frac{c}{2}}$ and $\text{tr}_M q^{L_0 - \frac{c}{2}} \geq \text{tr}_{M/N} q^{L_0 - \frac{c}{2}}$, and the equal signs hold if and only if $M = N$ and $N = 0$, respectively.

Lemma 3.19. For $\nu, \mu \in P_+$ and $\kappa \in \mathbb{C}$ such that $\kappa \neq -h$, if

$$\text{tr}_{T^\kappa_{\nu, \mu}} q^{L_0 - \frac{c}{2}} = \text{tr}_{L(\gamma_{\nu-(\kappa+h)(\mu+\rho)})} q^{L_0 - \frac{c}{2}},$$

then $T^\kappa_{\nu, \mu} \simeq L(\gamma_{\nu-(\kappa+h)(\mu+\rho)}) \simeq L(\gamma_{\mu-(\kappa'+h)(\nu+\rho)}) \simeq T^\kappa_{\mu, \nu}$ as $W^\kappa(g)$-modules.

Proof. By Lemma 3.12 and Remark 3.16, it is enough to show that $T^\kappa_{\nu, \mu} \simeq L(\gamma_{\nu-(\kappa+h)(\mu+\rho)})$ by combining the injective homomorphism $\tilde{V}(\nu + \kappa \Lambda_0) \hookrightarrow \tilde{M}(\nu + \kappa \Lambda_0)$ in the BGG resolution (93) with the canonical homomorphism $\tilde{M}(\nu + \kappa \Lambda_0)^* \rightarrow \tilde{W}(\nu + \kappa \Lambda_0)$, where $\tilde{W}(\nu + \kappa \Lambda_0)$ is the Wakimoto module of highest weight $\nu$ and level $\kappa$ (see [ArF, Section 3]), we obtain the $g$-module homomorphism

$$\phi : \tilde{V}(\nu + \kappa \Lambda_0) \rightarrow \tilde{W}(\nu + \kappa \Lambda_0),$$

which is isomorphic on the eigenspaces with lowest conformal weights (see [ArF, Proposition 3.4]). In particular, $\phi$ sends the highest weight vector $v_\nu$ of $\tilde{V}(\nu + \kappa \Lambda_0)$ to the highest weight vector $v'_\nu$ of $\tilde{W}(\nu + \kappa \Lambda_0)$. Denote by $[v_\nu] \in H^0_{\nu}(\tilde{V}(\nu + \kappa \Lambda_0)) = T^\kappa_{\nu, \mu}$ and $[v'_\nu] \in H^0_{\nu}(\tilde{W}(\nu + \kappa \Lambda_0))$ the equivalence classes of $v_\nu \otimes 1_\mu$ and $v'_\nu \otimes 1_\mu$, respectively (see the definition of $H^0_{\nu}(\cdot)$ in [ArF, Section 2.1]). Then, clearly we have $H^0_{\nu}(\phi)([v_\nu]) = [v'_\nu]$. On the other hand, by [ArF, Lemma 3.2] and Remark 2.13, we obtain the $W^\kappa(g)$-module isomorphism

$$\phi' : H^0_{\mu}(\tilde{W}(\nu + \kappa \Lambda_0)) \simeq \pi_{\kappa, \nu-(\kappa+h)\mu} \simeq F\left(\frac{\nu}{\sqrt{K + h}} - \sqrt{K + h}\mu\right),$$

which sends $[v'_\nu]$ to the highest weight vector $[\frac{\nu}{\sqrt{K + h}} - \sqrt{K + h}\mu]$ of $F\left(\frac{\nu}{\sqrt{K + h}} - \sqrt{K + h}\mu\right)$. Thus, the $W^\kappa(g)$-module homomorphism $\phi'' = \phi' \circ H^0_{\mu}(\phi)$ sends $[v_\nu]$ to $[\frac{\nu}{\sqrt{K + h}} - \sqrt{K + h}\mu]$, and we have

$$T^\kappa_{\nu, \mu} / \ker \phi'' \simeq \text{Im} \phi'' \simeq \mathcal{U}(W^\kappa(g))[\frac{\nu}{\sqrt{K + h}} - \sqrt{K + h}\mu].$$

By combining (86) and (97) with Remark 3.18, we have

$$\text{tr}_{T^\kappa_{\nu, \mu}} q^{L_0 - \frac{c}{2}} \geq \text{tr}_{T^\kappa_{\nu, \mu} / \ker \phi''} q^{L_0 - \frac{c}{2}} \geq \text{tr}_{L(\gamma_{\nu-(\kappa+h)(\mu+\rho)})} q^{L_0 - \frac{c}{2}}.$$

The assertion follows from the assumption (94), (98), and Remark 3.18.

Remark 3.20. When $\kappa = k'$, $p \geq h$, $\nu = \alpha + \lambda_0$, and $\mu = 0$, Lemma 3.19 follows from [Ar, Theorem 9.1.4] and the exactness of $H^0_{\mu}(\cdot)$ in the proof of Lemma 3.17.

Theorem 3.21. Theorem 1.2 is true.

Proof. If $\lambda_0 = 0$, then the assertion follows from Lemma 3.14, Theorem 3.15, and Lemma 3.19. In particular, when $p = h - 1$, we have $(\sqrt{p} \lambda_0 + \rho, \theta) \leq p$ if and only if $\lambda_0 = 0$, and thus the claim is proved.

Let us consider the case $p \geq h$. By combining Lemma 3.13 and Lemma 3.17 with Theorem 1.2 for the case of $\lambda_0 = 0$, we have

$$\sum_{\sigma \in \mathbb{W}} (-1)^{\ell(\sigma)} [M(\gamma_{y_0, \lambda_0 - \kappa + \rho})] = \sum_{y \geq y_0, \lambda_0} a_{y, y_0, \lambda_0} [M(\gamma_{y, y_0 - \kappa + \rho})].$$
Let us recall the fact that
\[ [M(y_{\mu_{\alpha}})] = [M(y_{\mu_{\beta}})] \iff y' \in W y \]
for \( y, y' \in \hat{W} \). Set the equivalence relation on \( \hat{W} \) by \( y \sim y' \) when \( y' \in W y \). Then by (99), for any \( y \in \hat{W}/\sim \), we have
\[
\sum_{y' \geq y_{\alpha, \lambda_0}, y' \sim y} a_{y', y_{\alpha, \lambda_0}} = \begin{cases} (-1)^{(\sigma)} \ y = y_\sigma \text{ for some } \sigma \in W, \\ 0 \quad \text{otherwise.} \end{cases}
\]

By applying (101) to (83), we have
\[
\text{Proof of Theorem 1.1.}
\]

3.4. For \( \beta \in P_+ \), denote by \( v_\beta \) the highest weight vector of \( L(\beta) \). For \( \alpha \in P_+ \cap Q \), \( \lambda_0 \in \Lambda_0 \) and \( A_{\alpha+\lambda_0} \in \mathbb{C} v_{\alpha+\lambda_0} \otimes W(\sqrt{\mu_{\alpha}} + \lambda) \), we use the notation
\[
x \otimes A_{\alpha+\lambda_0} = \sum f_{i_1} \cdots f_{i_n} A_{\alpha+\lambda_0},
\]
where \( x = \sum f_{i_1} \cdots f_{i_n} v_{\alpha+\lambda_0} \in L(\alpha + \lambda_0) \) and \( f_{i_j} \) in the right hand side of (107) are the screening operators (25).
Lemma 3.22. Let us take $\nu_p \in \Lambda_p$ such that $W_{\sqrt{p}\beta+\nu_p} \simeq W_{\sqrt{p}\beta+\nu_p}$ as $W_{\sqrt{p}\beta}$-modules. Let us also assume that for any $\beta \in P_+$, $W(-\sqrt{p}\beta+\nu_p)$ and $W(-\sqrt{p}\beta+\nu'_p)$ are generated by $|\sqrt{p}\beta+\nu_p)$ and $|\sqrt{p}\beta+\nu'_p)$ as $W(0)$-modules, respectively. Then $W_{\sqrt{p}\beta+\nu_p}$ is simple as $W_{\sqrt{p}\beta}$-module. Moreover, for any $\lambda \in \Lambda$ such that $\lambda_p = \nu_p$ and $W_{\sqrt{p}\lambda+\lambda'} \simeq W_{\sqrt{p}\lambda+\lambda'}$ as $W_{\sqrt{p}\lambda}$-modules, $W_{\sqrt{p}\lambda+\lambda}$ is the simple $W_{\sqrt{p}\lambda}$-module.

Proof. Because $W(-\sqrt{p}\beta+\nu_p) = U(W(0)) - \sqrt{p}\beta + \nu_p$ and

$$f_1 \cdots f_n |\sqrt{p}\beta+\nu_p) = (f_1 \cdots f_n |\sqrt{p}\beta) (\Delta_{\sqrt{p}\beta+\nu_p} - \Delta_{\sqrt{p}\beta} - 1)|\nu_p)$$

(see (27)), $W_{\sqrt{p}\beta+\nu_p}$ is generated by $|\nu_p)$ as $W_{\sqrt{p}\beta}$-module. Similarly, $W_{\sqrt{p}\beta+\nu_p}$ is generated by $|\nu'_p)$ as $W_{\sqrt{p}\beta}$-module. By $(W_{\sqrt{p}\beta}$-version of) Lemma 3.5, $W_{\sqrt{p}\beta+\nu_p}$ and $W_{\sqrt{p}\beta+\nu'_p}$ are simple $W_{\sqrt{p}\beta}$-modules.

Let us take $\lambda \in \Lambda$ such that $\lambda_p = \nu_p$, and let $S$ be a nonzero $W_{\sqrt{p}\lambda}$-submodule in $W_{\sqrt{p}\lambda+\lambda}$. In the same manner as above, $W_{\sqrt{p}\lambda+\lambda'}$ is generated by $L(\lambda_0 \otimes |\lambda)$ as $W_{\sqrt{p}\lambda}$-module. Then by Lemma 3.5, $S$ contains a nonzero element $y \otimes |\lambda$ in $L(\lambda_0 \otimes |\lambda)$. Since $W_{\sqrt{p}\beta+\nu_p}$ is simple as $W_{\sqrt{p}\beta}$-module, by [DM, Corollary 4.2], for any $x \in L(\lambda_0)$, there exist elements $\{a_{n,x} \in W_{\sqrt{p}\beta}\}_{n \in \mathbb{Z}}$ such that

$$x \otimes |\lambda) = \sum_{n \in \mathbb{Z}} (a_{n,x})_{(n)} y \otimes |\lambda)$$

(109)

However, since $x, y \in L(\lambda_0)$, we can assume that $a_{n,x} \in V_{\sqrt{p}\lambda} \cap W_{\sqrt{p}\beta} = W_{\sqrt{p}\lambda}$ for any $x \in L(\lambda_0)$ and $n \in \mathbb{Z}$. Thus, we have

$$L(\lambda_0 \otimes |\lambda) \subseteq U(W_{\sqrt{p}\lambda}) (y \otimes |\lambda) \subseteq S$$

and the claim is proved.

Remark 3.23. [DM, Proposition 4.1, Corollary 4.2] are claims about not generalized vertex operator algebras, but vertex operator algebras. However, it is easily checked that their proofs hold for generalized vertex operator algebras as well.

Theorem 3.24. Theorem 1.1 is true.

Proof. By Theorem 1.2, Corollary 3.3 and Theorem 3.6, all $\lambda \in \Lambda$ such that $(\sqrt{p}\beta + \rho, \theta) \leq p$ satisfy the assumptions in Lemma 3.22. Thus, the claim is proved.

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KYUSHU UNIVERSITY, FACULTY OF MATHEMATICS, FUKUOKA 819-0386 JAPAN
Email address: sugimoto.shoma.657@m.kyushu-u.ac.jp