A CONVERGENT APPROXIMATION OF THE PARETO OPTIMAL SET FOR FINITE HORIZON MULTIOBJECTIVE OPTIMAL CONTROL PROBLEMS (MOC) USING VIABILITY THEORY

A. GUIGUE

Abstract. The objective of this paper is to provide a convergent numerical approximation of the Pareto optimal set for finite-horizon multiobjective optimal control problems for which the objective space is not necessarily convex. Our approach is based on Viability Theory. We first introduce the set-valued return function $V$ and show that the epigraph of $V$ is equal to the viability kernel of a properly chosen closed set for a properly chosen dynamics. We then introduce an approximate set-valued return function with finite set-values as the solution of a multiobjective dynamic programming equation. The epigraph of this approximate set-valued return function is shown to be equal to the finite discrete viability kernel resulting from the convergent numerical approximation of the viability kernel proposed in [4, 5]. As a result, the epigraph of the approximate set-valued return function converges towards the epigraph of $V$. The approximate set-valued return function finally provides the proposed numerical approximation of the Pareto optimal set for every initial time and state. Several numerical examples are provided.

Key words. Multiobjective optimal control, Pareto optimality, Viability Theory, convergent numerical approximation, dynamic programming

AMS subject classifications. 49M2, 49L20, 54C60, 90C29

1. Introduction. Many engineering applications, such as trajectory planning for spacecraft [7] and robotic manipulators [13], continuous casting of steel [14], etc., can lead to an optimal control formulation where $p$ objective functions ($p > 1$) need to be optimized simultaneously. For a general optimization problem (GOP) with a vector-valued objective function, the definition of an optimal solution requires the comparison between elements in the objective space, which is the set of all possible values that can be taken by the vector-valued objective function. This comparison is generally provided by a binary relation, expressing the preferences of the decision maker. In applications, it is common to consider the binary relation defined in terms of a pointed convex cone $P \subset \mathbb{R}^p$ containing the origin [20]. However, in this paper, for simplicity, we will only consider the case $P = \mathbb{R}^p_+$, which yields the well-known Pareto optimality. The resolution of (GOP) therefore consists of finding the set of Pareto optimal elements in the objective space, or Pareto optimal set. In general, this set cannot be obtained analytically and we have to resort to numerical approximations. The main objective of this paper is therefore to propose a convergent numerical approximation of the Pareto optimal set for a general finite-horizon multiobjective optimal control problem (MOC). By general, we mean that we do not make any convexity assumption on the objective space $Y$ (or more generally, on the set $Y + \mathbb{R}^p_+$). Indeed, in the case where the objective space (or $Y + \mathbb{R}^p_+$) is convex, simple methods such the weighting method can be used to generate the entire Pareto optimal set (Theorem 3.4.4, [17, p. 72]).

When the objective space is not convex, very few approaches to find the Pareto optimal set have been proposed. An important line of research is to use evolutionary algorithms [8, 9], such as genetic algorithms. Also, very recently, an approach [15] inspired from the $\epsilon$-constraint method in nonlinear multiobjective optimization [10]
pp. 85–95] has been developed for multiobjective exit-time optimal control problems where \( P = \mathbb{R}^p \). In this approach, the \( n \)-dimensional state is augmented by \( p - 1 \) dimensions, which yields a new single objective optimal control problem. The Pareto optimal set of the original problem can be retrieved by inspecting the values of the return function of this new problem. The return function of the new problem is finally approximated by solving numerically the corresponding \((n + p - 1)\)-dimensional Hamilton-Jacobi-Bellman equation using a semi-Lagrangian "marching" method.

In this paper, instead of an exit-time optimal control problem, we consider an optimal control problem over a finite horizon \([0, T]\). The proposed approach fundamentally differs from [15]. Instead of augmenting the state space and solving the resulting augmented Hamilton-Jacobi-Bellman equation, we define the set-valued return function \( V(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \to 2^{\mathbb{R}^p} \) as the set-valued map associating with each time \( t \in [0, T] \) and state \( x \in \mathbb{R}^n \) the set of Pareto optimal elements in the objective space \( Y(t, x) \), where \( Y(t, x) \) is the set of all possible values that can be taken by the vector-valued objective function for trajectories starting at \( x \) at time \( t \). Hence, the Pareto optimal set for any time \( t \) and state \( x \) can be obtained just by evaluating \( V \) at \((t, x)\). We then derive a convergent approximation of \( V \) using Viability Theory [1, 6]. This approximation is a set-valued map with finite set-values, called the approximate set-valued return function. Hence, an approximation of the Pareto optimal set \( V(t, x) \) can be obtained just by evaluating the approximate set-valued return function at \((t, x)\). The advantage of using Viability Theory is that it provides a framework that allows to deal with problems with minimal regularity and convexity assumptions. Hence, it is expected that the proposed approach could be easily extended to more general classes of problems than the one considered in this paper, e.g., problems with state constraints, etc..

More precisely, the first step in the proposed approach is to show that the epigraph of \( V \), i.e., the graph of the set-valued map \( V + \mathbb{R}^p \), is equal to the viability kernel of a properly chosen closed set for some properly chosen dynamics. The next step is to introduce an approximate set-valued return function as the solution of a multiobjective dynamic programming equation. The epigraph of this approximate set-valued return function is shown to be equal to the finite discrete viability kernel resulting from the convergent numerical approximation of the viability kernel proposed in [4, 5]. From there, we easily obtain that the epigraph of the approximate set-valued return function converges in the sense of Painlevé-Kuratowski towards the epigraph of \( V \). The multiobjective dynamic programming equation obtained is very similar to the one obtained in [10], where no proof of convergence was provided.

This paper is organized as follows. In §2 we detail the class of multiobjective optimal control problems considered and define \( V \). In §3 we briefly discuss the concept of optimality in multiobjective optimization and present several useful properties related to Pareto optimal sets. In §4 we show that the epigraph of \( V \) is equal to the viability kernel of a properly chosen closed set for a properly chosen dynamics. Following §4 §5, we then propose in §6 a finite discrete approximation of this viability kernel. In §7 we show that the finite discrete viability kernel resulting from this approximation is equal to the epigraph of an approximate set-valued return function, defined as the solution of a multiobjective dynamic programming equation. From this multiobjective dynamic programming equation, we derive in §7 a numerical algorithm
to compute the approximate set-valued return function and therefore the approximate Pareto optimal set at \((t, x)\). Some numerical examples \[10\], for which the Pareto optimal set is analytically known, are provided in \[8\] and some conclusions are finally drawn in \[9\].

2. A multiobjective finite-horizon optimal control problem \([3]\) In this paper, we will take for \(\| \cdot \|\) in \(\mathbb{R}^p\) and \(\mathbb{R}^n\) the supremum norm. Let \(B\) be the closed unit ball.

Consider the evolution over a fixed finite time interval \(I = [0, T]\) \((0 < T < \infty)\) of an autonomous dynamical system whose \(n\)-dimensional state dynamics are given by a continuous function \(f(\cdot, \cdot) : \mathbb{R}^n \times U \to \mathbb{R}^n\), where the control space \(U\) is a nonempty compact subset of \(\mathbb{R}^m\). The function \(f(\cdot, u)\) is assumed to be Lipschitz, i.e., some \(K_f > 0\) obeys

$$\forall u \in U, \forall x_1, x_2 \in \mathbb{R}^n, \|f(x_1, u) - f(x_2, u)\| \leq K_f\|x_1 - x_2\|. \quad (2.1)$$

We also assume that the function \(f\) is uniformly bounded, i.e., some \(M_f > 0\) obeys

$$\forall x \in \mathbb{R}^n, \forall u \in U, \|f(x, u)\| \leq M_f. \quad (2.2)$$

A control \(u(\cdot) : I \to U\) is a bounded, Lebesgue measurable function. The set of controls is denoted by \(U\). The continuity of \(f\) and the Lipschitz condition \((2.1)\) guarantee that, given any \(t \in I\), initial state \(x \in \mathbb{R}^n\), and control \(u(\cdot) \in U\), the system of differential equations governing the dynamical system,

$$\begin{cases}
    \dot{x}(s) = f(x(s), u(s)), & t \leq s \leq T, \\
    x(t) = x,
\end{cases} \quad (2.3)$$

has a unique solution, called a trajectory and denoted \(s \to x(s; t, x, u(\cdot))\). Let \(F\) be the set-valued map defined from \(\mathbb{R}^n\) to \(\mathbb{R}^n\) by

$$F(x) = \bigcup_{u \in U} f(x, u).$$

The cost of a trajectory over \([t, T]\), \(t \in I\), is given by a \(p\)-dimensional vector function \(J(\cdot, \cdot, \cdot) : I \times \mathbb{R}^n \times U \to \mathbb{R}^p\),

$$J(t, x, u(\cdot)) = \int_t^T L(x(s; x, u(\cdot)), u(s)) \, ds, \quad (2.4)$$

where the \(p\)-dimensional vector function \(L(\cdot, \cdot) : \mathbb{R}^n \times U \to \mathbb{R}^p\), called the running cost, is assumed to be continuous. For simplicity, no terminal cost is included in \((2.4)\).

We assume that the function \(L\) is uniformly bounded, i.e., some \(M_L \geq 0\) obeys

$$\forall x \in \mathbb{R}^n, \forall u \in U, \|L(x, u)\| \leq M_L, \quad (2.5)$$

and that the function \(L(\cdot, u)\) satisfies a Lipschitz condition, i.e., some \(K_L \geq 0\) obeys

$$\forall u \in U, \forall x_1, x_2 \in \mathbb{R}^n, \|L(x_1, u) - L(x_2, u)\| \leq K_L\|x_1 - x_2\|. \quad (2.6)$$

Let \(L\) be the set-valued map defined from \(\mathbb{R}^n\) to \(\mathbb{R}^p\) by

$$L(x) = \bigcup_{u \in U} L(x, u).$$
The objective space \( Y(t, x) \) for (MOC) is defined as the set of all possible costs (2.4):

\[
Y(t, x) = \left\{ J(t, x, u(\cdot)), u(\cdot) \in \mathcal{U} \right\}.
\]

From (2.5), it follows that the set \( Y(t, x) \) is bounded (by \( M_L T \)), and also that

\[
Y(t, x) \subseteq \{ -(T - t) M_L 1 \} + \mathbb{R}_P^p.
\]

However, the set \( Y(t, x) \) is not necessarily closed.

The set-valued return function \( V(\cdot, \cdot) : I \times \mathbb{R}^n \rightarrow 2\mathbb{R}_P^p \) for (MOC) is defined as the set-valued map which associates with each time \( t \in I \) and initial state \( x \in \mathbb{R}^n \) the set of Pareto optimal elements in the objective space \( Y(t, x) \), where the definition of a Pareto optimal element is postponed to §3:

\[
V(t, x) = \mathcal{E}(\text{cl}(Y(t, x))).
\]  

(2.7)

The closure in (2.7) is used to guarantee the existence of Pareto optimal elements (Proposition 3.5, [12]). Hence,

\[
\forall t \in I, \forall x \in \mathbb{R}^n, V(t, x) \neq \emptyset.
\]

Remark 2.1. When \( p = 1 \), (2.7) takes the form

\[
V(t, x) = \left\{ \inf_{u(\cdot) \in \mathcal{U}} \int_t^T L(x(s; t, x, u(\cdot)), u(s)) \, ds \right\}.
\]

Hence, \( V(t, x) = \{ v(t, x) \} \), where \( v(\cdot, \cdot) \) is the value function for single objective optimal control problems [3, 19].

Finally, as \( V(t, x) \subseteq \text{cl}(Y(t, x)) \), we have

\[
V(t, x) \subseteq \{ -(T - t) M_L 1 \} + \mathbb{R}_P^p.
\]  

(2.8)

The objective of this paper is to find a convergent approximation to the Pareto optimal set \( V(0, x_0) \) where \( x_0 \in \mathbb{R}^n \) is some given initial state.

3. Multiobjective Optimization. For an optimization problem with a \( p \)-dimensional vector-valued objective function, the definition of an optimal solution requires the comparison of any two elements \( y_1, y_2 \) in the objective space, which is the set of all possible values that can be taken by the vector-valued objective function. This comparison is generally provided by a binary relation, expressing the preferences of the decision maker. In applications, it is common to consider the binary relation defined in terms of a pointed convex cone \( P \subset \mathbb{R}_P^p \) containing the origin [20].

Definition 3.1. Let \( y_1, y_2 \in \mathbb{R}_P^p \). Then, \( y_1 \preceq y_2 \) if and only if \( y_2 \in y_1 + P \).

The binary relation in Definition 3.1 yields the definition of generalized Pareto optimality.

Definition 3.2. Let \( S \) be a nonempty subset of \( \mathbb{R}_P^p \). An element \( y_1 \in S \) is said to be a generalized Pareto optimal element of \( S \) if and only if there is no \( y_2 \in S \) \( (y_2 \neq y_1) \) such that \( y_1 \in y_2 + P \), or equivalently, if and only if there is no \( y_2 \) such that \( y_1 \in y_2 + P \setminus \{0\} \). The set of generalized Pareto optimal elements of \( S \) is called the generalized Pareto optimal set and is denoted by \( \mathcal{E}(S, P) \). When \( P = \mathbb{R}_P^p \),
the generalized Pareto optimal elements are only referred to as Pareto optimal elements, and the set $E(S, P)$ is simply denoted $E(S)$.

An important role in this paper is played by the external stability or domination property [18, pp. 59-66].

**Definition 3.3 (External stability).** A nonempty subset $S$ of $\mathbb{R}^p$ is said to be externally stable if and only if

$$S \subset E(S, P) + P.$$ 

An immediate consequence of the external stability property is that $S + P = E(S, P) + P$. When $P$ is closed, a sufficient condition for a nonempty closed set $S$ to be externally stable is given in Proposition 3.4. Note that this condition is also sufficient to guarantee the existence of generalized Pareto optimal elements.

**Proposition 3.4 (Theorem 3.2.10, [18, p. 62]).** Let $S$ be a nonempty closed subset of $\mathbb{R}^p$. If $P$ is closed and $S$ is $P$-bounded [18, p. 52], i.e., $S^+ \cap -P = \{0\}$, then $S$ is externally stable.

**Corollary 3.5.** Let $K$ be a nonempty compact subset of $\mathbb{R}^p$. If $P$ is closed, then $K$ is externally stable.

In this paper, for simplicity, we will only consider the case $P = \mathbb{R}^p_n$.

4. **Characterization of the set-valued return function.** In this section, we show that the epigraph of the set-valued return function $V$, i.e., the graph of the set-valued map $V + R^p$, is equal to the viability kernel of a properly chosen closed set for some properly chosen dynamics.

Define the set-valued maps $FL^\sigma$ from $\mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}^p$ by

$$FL^\sigma(x) = \overline{co}\left( \bigcup_{u \in U} \{(f(x, u), \sigma L(x, u))\} \right),$$ 

and where $\overline{co}(S)$ denotes the closure of the convex hull of the set $S$. Observe that the set-valued map $FL^\sigma$ takes convex compact nonempty values. Moreover, $FL^\sigma$ is bounded by $M_{FL} = \max\{M_f, M_L\}$ and Lipschitz with Lipschitz constant $K_{FL} = \max\{K_f, K_L\}$. Denote $FL^+(x) = FL^{+1}(x)$ and $FL^-(x) = FL^{-1}(x)$. Define finally the expanded set-valued map $\phi$ from $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$ by

$$\phi(t, x, z) = \left\{ \begin{array}{ll}
\{1\} \times FL^-(x) & \text{if } t < T, \\
[0, 1] \times \overline{co}(FL^-(x) \cup \{0, 0\}) & \text{if } t \geq T.
\end{array} \right. \quad (4.1)$$

It is easy to see that $\phi$ is a Marchaud map (Definition 2.2, p. 184, [4]) bounded by $\max\{1, M_{FL}\}$.

Consider now the differential inclusion

$$(\dot{t}(s), \dot{x}(s), \dot{z}(s)) \in \phi(t(s), x(s), z(s)) \text{ a.e. } s \geq 0, \quad (4.2)$$
and the closed set $\mathcal{H} = \{(t, x, z), \ t \in [0, T], \ x \in \mathbb{R}^n, z \in \{-(T - t)M_L1\} + \mathbb{R}_+^p$. 

**Proposition 4.1.** The epigraph of the set-valued map $V$ is equal to the viability kernel of $\mathcal{H}$ for $\phi$, i.e.,

$$\text{Epi}(V) = \text{Viab}_\phi(\mathcal{H}),$$

where $\text{Epi}(V)$ is defined as $\text{Epi}(V) = \text{Graph}(V + \mathbb{R}_+^p)$.

**Proof.** First, we prove the inclusion

$$\text{Viab}_\phi(\mathcal{H}) \subset \text{Graph}(V + \mathbb{R}_+^p).$$

Take $(t, x, z) \in \text{Viab}_\phi(\mathcal{H})$. Hence, $(t, x, z) \in \mathcal{H}$, or $t \in I$ and $z \in \{-(T - t)M_L1\} + \mathbb{R}_+^p$.

- Assume that $t = T$. Then, as $V(T, x) = \{0\}$, it follows that $z \in \mathbb{R}_+^p = V(T, x) + \mathbb{R}_+^p$.
- Assume that $t \in [0, T)$. Let $(t(\cdot), x(\cdot), z(\cdot))$ be a solution to (4.2) with initial condition $(t, x, z)$ which remains in $\mathcal{H}$. By definition of $\phi$, $(x(\cdot), z(\cdot))$ is a solution to the differential inclusion

$$\begin{align*}
\begin{cases}
\dot{x}(s) + z(\cdot)
\end{cases} & \in FL^-(x(s)) \ a.e. \ s \in [0, T - t], \\
x(0) &= x, \\
z(0) &= z,
\end{align*}$$

and $t(s) = s + t$. Let $s' = s + t$. For $s' \in [t, T]$, define $x'(s') = x(s' - t)$ and $z'(s') = z(s' - t)$. Then, $(x'(\cdot), z'(\cdot))$ is a solution to the differential inclusion

$$\begin{align*}
\begin{cases}
\dot{x}'(s') + z'(\cdot)
\end{cases} & \in FL^-(x'(s')) \ a.e. \ s' \in [t, T], \\
x'(t) &= x, \\
z'(t) &= z,
\end{align*}$$

By the Relaxation Theorem (Theorem 2.7.2, [19, p. 96]), for $\epsilon > 0$, there exists $u(\cdot) \in \mathcal{U}$ such that

$$||x'(\cdot) - x(\cdot; t, x, u(\cdot))|| \leq \epsilon \text{ and } ||z'(\cdot) - z(\cdot; t, (x, z), u(\cdot))|| \leq \epsilon.$$

In particular, we get

$$z'(T) \leq z(T; t, (x, z), u(\cdot)) + \epsilon 1.$$

Moreover, as $(s', x'(s'), z'(s')) \in \mathcal{H}$ for all $s' \in [t, T]$, we have

$$z'(s') \in \{-(T - s')M_L1\} + \mathbb{R}_+^p,$$

or

$$z'(T) \in \mathbb{R}_+^p.$$

Hence,

$$z(T; t, (x, z), u(\cdot)) = z - \int_t^T L(x(s'; t, x, u(\cdot)), u(s')) \ ds' \geq z'(T) - \epsilon 1 \geq -\epsilon 1,$$

or

$$z + \epsilon 1 \geq \int_t^T L(x(s'; t, x, u(\cdot)), u(s')) \ ds',$$

which implies that $z \in \text{cl}(Y(t, x)) + \mathbb{R}_+^p = V(t, x) + \mathbb{R}_+^p$ by external stability.
Second, we prove the inclusion

\[ \text{Graph}(V + R^p_+) \subset \text{Viab}_x(H). \]

Take \((t, x, z) \in \text{Graph}(V + R^p_+).\) Hence, \(t \in [0, T],\) from \([2.8], z \in \{- (T - t)M_1 \} + R^p_+\).

- Assume that \(t = T.\) Then, \(z \in R^p_+\). Therefore, \((t(\cdot), x(\cdot), z(\cdot)) = (T, x, z)\) is a solution to \((4.2)\) viable in \(H.\)
- Assume that \(t \in [0, T).\) We have \(z = z' + d,\) where \(z' \in V(t, x)\) and \(d \in R^p_+\).

By definition of \(V(t, x),\) there exists a sequence \(u_n(\cdot) \in U\) such that

\[
\lim_{n \to +\infty} \int_t^T L(x_n(s; t, x, u_n(\cdot)), u_n(s)) \, ds = z'. \tag{4.3}
\]

Using the Compactness of Trajectories theorem (Theorem 2.5.3, [19, p. 89]) and by passing to a subsequence if necessary, we can therefore assume that there exists \((x(\cdot), z(\cdot))\) solution on \([t, T]\) to the differential inclusion

\((\dot{x}(s'), \dot{z}(s')) \in FL^+(x(s')) \text{ a.e. } s' \in [t, T]\)

with initial condition \((x, 0)\) such that

\[
\lim_{n \to +\infty} \|x_n(\cdot; t, x, u_n(\cdot)) - x(\cdot)\| = 0,
\]

and

\[
\lim_{n \to +\infty} \|z_n(\cdot; t, (x, 0), u_n(\cdot)) - z(\cdot)\| = 0. \tag{4.4}
\]

From \((4.3)\) and \((4.4),\) we deduce that \(z(T) = z'.\)

Define now \((x'(\cdot), z'(\cdot))\) as follows:

\[
(t(s), x'(s), z'(s)) = \begin{cases} 
(s + t, x(s + t), z - z(s + t)) & s \in [0, T - t], \\
(T, x(T), d) & s > T - t.
\end{cases}
\]

As \(z'(T - t) = z - z(T) = z - z' = d,\) it follows that \((t(\cdot), x'(\cdot), z'(\cdot))\) is a solution of \((4.2).\) It remains to check that \((t(s), x'(s), z'(s)) \in H\) for all \(s \geq 0.\) For \(s > T - t,\) as \(d \in R^p_+\), this is obvious. For \(s \in [0, T - t],\) using the definition of \(z,\) \((4.3),\) and \((4.4),\) we have

\[
z'(s) = \lim_{n \to +\infty} \int_t^T L(x_n(s; t, x, u_n(\cdot)), u_n(s)) \, ds + d - z_n(s + t; t, (x, 0), u_n(\cdot)).
\]

As

\[
z_n(s + t; t, (x, 0), u_n(\cdot)) = \int_t^{s+t} L(x_n(s; t, x, u_n(\cdot)), u_n(s)) \, ds,
\]

we get

\[
z'(s) = d + \lim_{n \to +\infty} \int_{s+t}^T L(x_n(s; t, x, u_n(\cdot)), u_n(s)) \, ds.
\]
As
\[
\int_{s+t}^{T} L(x_n(s; t, x, u_n(\cdot)), u_n(s)) \, ds \geq -(T - (s + t))M_L 1
\]
and \(d \in \mathbb{R}^n_+\), we finally obtain \(z'(s) \geq -(T - t(s))M_L 1\). Hence, \((t(s), x'(s), z'(s)) \in H\) and \((t(\cdot), x'(\cdot), z'(\cdot))\) is viable in \(H\).

\[\square\]

Remark 4.1. Proposition 4.1 remains valid if we take for \(H\) the closed set \(\{(t, x, z), t \in [0, T], x \in \mathbb{R}^n, z \in \{-(T - t)M_L 1\} + \mathbb{R}^p_+\}\), where \(M_L > M_L\). This remark will be used in \(\square\).

5. Approximation of \(\text{Viab}_\phi(H)\). In this section, we approximate \(\text{Viab}_\phi(H)\) by finite discrete viability kernels. A preliminary step is to approximate \(\text{Viab}_\phi(H)\) by discrete viability kernels. Our developments closely follow \(\square\).

5.1. Approximation of \(\text{Viab}_\phi(H)\) by discrete viability kernels. In \(\square\) (Theorem 2.14, p. 190), it is shown that \(\text{Viab}_\phi(H)\) can be approximated by discrete viability kernels in the sense of Painlevé-Kuratowski by considering an approximation \(\phi_\epsilon\) of \(\phi\) satisfying the following three properties:

\((\text{H}_0)\) \(\phi_\epsilon\) is an upper semicontinuous set-valued map from \(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p\) to \(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p\) which takes convex compact nonempty values.

\((\text{H}_1)\)
\[
\text{Graph}(\phi_\epsilon) \subset \text{Graph}(\phi) + g(\epsilon) B \text{ where } \lim_{\epsilon \to 0^+} g(\epsilon) = 0^+.
\]

\((\text{H}_2)\)
\[
\forall(t_\epsilon, x_\epsilon, z_\epsilon) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \quad \bigcup_{\|((t, x, z) - (t_\epsilon, x_\epsilon, z_\epsilon))\| \leq M \epsilon} \phi(t, x, z) \subset \phi_\epsilon(t_\epsilon, x_\epsilon, z_\epsilon),
\]

where \(M = \max\{1, M_{FL}\}\) denotes a bound for \(\phi\) and \(\epsilon > 0\) is the time step discretization.

Define the set-valued map \(\phi_\epsilon\), \(\epsilon > 0\), from \(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p\) to \(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p\) by

\[
\phi_\epsilon(t, x, z) = \begin{cases} 
\{1\} \times (FL^{-}(x) + \epsilon K MB) & \text{if } t < T - M \epsilon, \\
[0, 1] \times \overline{\text{co}}((FL^{-}(x) + \epsilon K MB) \cup \{(0, 0)\}) & \text{if } t \geq T - M \epsilon.
\end{cases}
\]

Theorem 5.1. The set-valued map \(\phi_\epsilon\) satisfies \((\text{H}_0)\), \((\text{H}_1)\), and \((\text{H}_2)\).

Proof.
\((\text{H}_0)\) This follows from the properties of the set-valued map \(FL^{-}\) and the fact that \(FL^{-}(x) + \epsilon K MB \subset \overline{\text{co}}((FL^{-}(x) + \epsilon K MB) \cup \{(0, 0)\})\).

\((\text{H}_1)\) This relation holds with \(g(\epsilon) = \epsilon \max\{1, K\} M\).

Let \((t_\epsilon, x_\epsilon, z_\epsilon) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p\) and \((s_\epsilon, f_\epsilon, l_\epsilon) \in \phi_\epsilon(t_\epsilon, x_\epsilon, z_\epsilon)\). Consider the following two cases:
• $t_s < T - M\epsilon$. Then, $s_e = 1$ and $(f_s, l_s) \in FL^-(x_e) + \epsilon KMB$. Hence, $g(\epsilon) = \epsilon K M$.

• $t_s \geq T - M\epsilon$. We have $s_e \in [0, 1]$ and $(f_s, l_s) \in \overline{\varphi}((FL^-(x_e) + \epsilon KMB) \cup \{(0, 0)\})$. There exists $t \geq T$ such that $|t - t_s| \leq M\epsilon$. We have

$$
(f_s, l_s) \in \overline{\varphi}((FL^-(x_e) + \epsilon KMB) \cup \{(0, 0)\}),
\subset \overline{\varphi}(FL^-(x_e) \cup \{(0, 0)\}) + \epsilon KMB.
$$

Hence, $g(\epsilon) = \epsilon \max\{1, K\} M$.

(H2) Let $(t_s, x_s, z_s) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$. Take $(t, x, z) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$ such that $\|(t, x, z) - (t_s, x_s, z_s)\| \leq M\epsilon$. In particular, $\|x - x_*\| \leq M\epsilon$. Consider the two following cases:

• $t_s < T - M\epsilon$. This implies $t < T$. The Lipschitz property of $FL^-$ yields:

$$
\phi(t, x, z) = \{1\} \times FL^-(x) \subset \{1\} \times (FL^-(x_e) + \epsilon KMB) = \phi_e(t_s, x_s, z_s).
$$

• $t_s \geq T - M\epsilon$. Then, either $t < T$. In such a case, as above,

$$
\phi(t, x, z) = \{1\} \times FL^-(x) \subset \{1\} \times (FL^-(x_e) + \epsilon KMB) \subset \phi_e(t_s, x_s, z_s).
$$

Either $t \geq T$. In such a case, using the Lipschitz property of $FL^-$, we have

$$
FL^-(x) \cup \{(0, 0)\} \subset (FL^-(x_e) + \epsilon KMB) \cup \{(0, 0)\}.
$$

Hence,

$$
\phi(t, x, z) = [0, 1] \times \overline{\varphi}(FL^-(x) \cup \{(0, 0)\}) \subset \phi_e(t_s, x_s, z_s).
$$

5.2. Approximation of $\text{Viab}_\phi(\mathcal{H})$ by discrete finite viability kernels. In [4] (Theorem 2.19, p. 195), it is shown that $\text{Viab}_\phi(\mathcal{H})$ can be approximated by finite discrete viability kernels in the sense of Painlevé-Kuratowski by considering an approximation $\Gamma_{e,h}$ of $G_e(\cdot)$ satisfying the following two properties:

(H3) \quad \text{Graph}(\Gamma_{e,h}) \subset \text{Graph}(G_e) + \psi(\epsilon, h)B$ where $\lim_{\epsilon \to 0^+, \frac{\psi(\epsilon, h)}{\epsilon} \to 0^+}.$

(H4) $\forall (t_h, x_h, z_h) \in R_h \times R^n_h \times R^p_h$, 

$$
\bigcup_{\|(t_s, x_s, z_s) - (t_h, x_h, z_h)\| \leq h} (G_e(t_e, x_e, z_e) + hB) \cap R_h \times R^n_h \times R^p_h \subset \Gamma_{e,h}(t_h, x_h, z_h),
$$

where $h > 0$ is the state step discretization, $R_h$ is an integer lattice of $R$ generated by segments of length $h$, $G_e$ is the set-valued map from $R \times R^n \times R^p$ to $R \times R^n \times R^p$ defined by

$$
G_e(t_e, x_e, z_e) = \{(t_e, x_e, z_e)\} + \epsilon \phi_e(t_e, x_e, z_e),
$$

and $\Gamma_{e,h}$ is the set-valued map from $R_h \times R^n_h \times R^p_h$ to $R_h \times R^n_h \times R^p_h$ defined as follows.
We assume that $\epsilon > 0$.

**Theorem 5.2.** The set-valued map $\Gamma_{\epsilon,h}$ satisfies $(H_3)$ and $(H_4)$.

**Proof.**

$(H_3)$ This relation holds with $\psi(\epsilon,h) = 4h + \epsilon h K$, which verifies

$$\lim_{\epsilon \to 0^+, \frac{h}{\epsilon} \to 0^+} \frac{(4 + \epsilon K)h}{\epsilon} = 0^+.$$ 

Let $(t_h, x_h, z_h) \in \mathbb{R}_h \times \mathbb{R}_h^p \times \mathbb{R}_h^p$ and $(s_h, f_h, l_h) \in \Gamma_{\epsilon,h}(t_h, x_h, z_h)$. Consider the following two cases:

- $t_h < T - M \epsilon - h$, $\Gamma_{\epsilon,h}(t_h, x_h, z_h) =$
  $$[t_h + \epsilon - 2h, t_h + \epsilon + 2h] \cap \mathbb{R}_h \times \{(x_h, z_h)\} + \epsilon \mathbb{F}^{-}(x_h) + \alpha_{\epsilon,h} B \cap \mathbb{R}_h^p \times \mathbb{R}_h^p.$$  

- $t_h \geq T - M \epsilon - h$, $\Gamma_{\epsilon,h}(t_h, x_h, z_h) =$
  $$[t_h, t_h + \epsilon + 2h] \cap \mathbb{R}_h \times \{(x_h, z_h)\} + \epsilon \mathbb{F}^{-}(x_h) + \alpha_{\epsilon,h} B \cup \{(x_h, z_h)\} + 2h B \cap \mathbb{R}_h^p \times \mathbb{R}_h^p,$$

where $\alpha_{\epsilon,h} = 2h + \epsilon h K + \epsilon^2 K M$.

Hence, $(s_h, f_h, l_h) \in (t_h, x_h, z_h) + \epsilon \phi_e(t_h, x_h, z_h) + \max\{2h, 2h + \epsilon h K\} B = G_{\epsilon}(t_h, x_h, z_h) + (2h + \epsilon h K) B$.

- $t_h \geq T - M \epsilon - h$. We have $s_h \in [t_h, t_h + \epsilon + 2h] \cap \mathbb{R}_h$ and $(f_h, l_h) \in \mathbb{F}^{-}(x_h, z_h) + \epsilon \phi_e(t_h, x_h, z_h) + \max\{2h, 2h + \epsilon h K\} B$. There exists $t_e \geq T - M \epsilon$ such that $0 \leq t_e - t_h \leq h$. We have
  $$s_h \in [t_h, t_h + \epsilon + 2h] \cap \mathbb{R}_h,$$
  $$\subset [t_h, t_h + \epsilon + 2h],$$
  $$\subset t_e + \epsilon [0, 1] + [-2h, 2h].$$

Moreover,

$$(f_h, l_h) \in \mathbb{F}^{-}(x_h, z_h) + \epsilon \phi_e(t_h, x_h, z_h) + \max\{2h, 2h + \epsilon h K\} B = G_{\epsilon}(t_e, x_h, z_h) + (4h + \epsilon h K) B.$$
Consider the two following cases:

- $t_h < T - M\varepsilon - h$. This implies $t_\varepsilon < T - M\varepsilon$. The Lipschitz property of $\text{FL}^-$ yields:
  \[
  G_\varepsilon(t_\varepsilon, x_\varepsilon, z_\varepsilon) + h\mathbf{B} = (t_\varepsilon, x_\varepsilon, z_\varepsilon) + \varepsilon\phi_\varepsilon(t_\varepsilon, x_\varepsilon, z_\varepsilon) + h\mathbf{B},
  \]
  with
  \[
  \begin{align*}
  & (t_\varepsilon + \varepsilon + [-h, h]) \times (\{(x_\varepsilon, z_\varepsilon)\} + \varepsilon\text{FL}^-(x_\varepsilon) + \varepsilon^2 KMB + h\mathbf{B}), \\
  & \subset (t_h + \varepsilon + [-2h, 2h]) \times (\{(x_h, z_h)\} + h\mathbf{B} + \varepsilon\text{FL}^-(x_h) + \varepsilon hK\mathbf{B} + \varepsilon^2 KM + h\mathbf{B}), \\
  & = (t_h + \varepsilon + [-2h, 2h]) \times (\{(x_h, z_h)\} + \varepsilon\text{FL}^-(x_h) + \alpha_{\varepsilon, h}\mathbf{B}).
  \end{align*}
  \]
  Hence,
  \[
  (G_\varepsilon(t_\varepsilon, x_\varepsilon, z_\varepsilon) + h\mathbf{B}) \cap \mathbb{R}_h \times \mathbb{R}^n_h \times \mathbb{R}^p_h \subset \Gamma_{\varepsilon, h}(t_h, x_h, z_h).
  \]

- $t_h \geq T - M\varepsilon - h$. Then, either $t_\varepsilon < T - M\varepsilon$. In such a case, as above,
  \[
  G_\varepsilon(t_\varepsilon, x_\varepsilon, z_\varepsilon) + h\mathbf{B} \subset (t_h + \varepsilon + [-2h, 2h]) \times (\{(x_h, z_h)\} + \varepsilon\text{FL}^-(x_h) + \alpha_{\varepsilon, h}\mathbf{B}).
  \]
  As $\varepsilon > 2h$,
  \[
  t_h + \varepsilon + [-2h, 2h] \subset [t_h, t_h + \varepsilon + 2h].
  \]
  Moreover,
  \[
  \{(x_h, z_h)\} + \varepsilon\text{FL}^-(x_h) + \alpha_{\varepsilon, h}\mathbf{B} \subset \text{co}(\{(x_h, z_h)\} + \varepsilon\text{FL}^-(x_h) + \alpha_{\varepsilon, h}\mathbf{B}) \cup (\{(x_h, z_h)\} + 2h\mathbf{B}).
  \]
  Hence,
  \[
  (G_\varepsilon(t_\varepsilon, x_\varepsilon, z_\varepsilon) + h\mathbf{B}) \cap \mathbb{R}_h \times \mathbb{R}^n_h \times \mathbb{R}^p_h \subset \Gamma_{\varepsilon, h}(t_h, x_h, z_h).
  \]

Either, $t_\varepsilon \geq T - M\varepsilon$. In which case, using the Lipschitz property of $\text{FL}^-$, we have

- \[
  G_\varepsilon(t_\varepsilon, x_\varepsilon, z_\varepsilon) + h\mathbf{B} = (t_\varepsilon, x_\varepsilon, z_\varepsilon) + \varepsilon([0, 1] \times \text{co}(\{(x_\varepsilon, z_\varepsilon)\} + \varepsilon\text{FL}^-(x_\varepsilon) + \varepsilon^2 KMB) \cup \{(0, 0)\}) + h\mathbf{B},
  \]
  with
  \[
  \begin{align*}
  & (\{t_\varepsilon - h, t_\varepsilon + \varepsilon + h\}) \times (\text{co}(\{(x_\varepsilon, z_\varepsilon)\} + \varepsilon\text{FL}^-(x_\varepsilon) + \varepsilon^2 KMB) \cup \{(x_\varepsilon, z_\varepsilon)\}) + h\mathbf{B}, \\
  & \subset \text{co}(\{(x_h, z_h)\} + h\mathbf{B} + \text{FL}^-(x_h) + hK\mathbf{B} + \varepsilon^2 KMB) \cup (\{(x_h, z_h)\} + h\mathbf{B}) + h\mathbf{B}, \\
  & \subset \text{co}(\{(x_h, z_h)\} + \varepsilon\text{FL}^-(x_h) + \alpha_{\varepsilon, h}\mathbf{B}) \cup (\{(x_h, z_h)\} + 2h\mathbf{B}).
  \end{align*}
  \]
  Hence,
  \[
  (G_\varepsilon(t_\varepsilon, x_\varepsilon, z_\varepsilon) + h\mathbf{B}) \cap \mathbb{R}_h \times \mathbb{R}^n_h \times \mathbb{R}^p_h \subset \Gamma_{\varepsilon, h}(t_h, x_h, z_h).
  \]

\[\square\]

6. Convergent approximation of the set-valued return function $V$. In this section, we first introduce a sequence of approximate set-valued return functions with finite set-values recursively defined by a multiobjective dynamic programming equation \cite{11,12}. We then show that the epigraphs of these approximate set-valued return functions are equal to the sets involved in the calculation of the finite discrete
viability kernels of the discrete set $\mathcal{H}_h = (\mathcal{H} + h\mathbf{B}) \cap \mathbb{R}_h \times \mathbb{R}_h^n \times \mathbb{R}_h^p$ for the finite discrete dynamics $\Gamma_{\epsilon,h}$ (Proposition 2.18, p. 195, [4]). This allows us to conclude that the sequence of approximate set-valued return functions is finite and that the epigraph of the final approximate set-valued return function of this sequence converges in the sense of Painlevé-Kuratowski towards the epigraph of the set-valued return function $V$.

Recall the definition of $\mathcal{H} = \{(t, x, z), t \in [0, T], x \in \mathbb{R}^n, z \in \{- (T - t)M_L 1\} + \mathbb{R}^p_+\}$. Here, we take $\overline{M}_L > M_L$ (Remark 4.1) in the definition of $\mathcal{H}$. Hence, $\mathcal{H} = \{(t, x, z), t \in [0, T], x \in \mathbb{R}^n, z \in \{- (T - t)M_L 1\} + \mathbb{R}^p_+\}$. Let $I_h = (I + [-h, h]) \cap \mathbb{R}_h$. We define the finite-valued set-valued map $\mathcal{V}_h$ from $I_h \times \mathbb{R}_h^n$ to $\mathbb{R}_h^p$ such that

$$\text{Graph}(\mathcal{V}_h^0 + \mathbb{R}_h^p) = \mathcal{H}_h,$$

where $\mathcal{H}_h = (\mathcal{H} + h\mathbf{B}) \cap \mathbb{R}_h \times \mathbb{R}_h^n \times \mathbb{R}_h^p$. We recursively define now the finite set-valued maps $\mathcal{V}^k_{\epsilon,h}$, $k \geq 1$, from $I_h \times \mathbb{R}_h^n$ to $\mathbb{R}_h^p$ as follows:

- If $t_h < T - M\epsilon - h$, $\mathcal{V}^{k+1}_{\epsilon,h}(t_h, x_h) = \mathcal{V}^k_{\epsilon,h}(t_h, x_h) + \mathbb{R}^p_+$

where $\mathcal{V}^k_{\epsilon,h}$ is a multiple of $\mathcal{V}_h^k$ and $\mathcal{V}_h^k$ is not required anymore as the sets involved are finite.

We aim in the following two propositions to prove that

$$\text{Graph}(\mathcal{V}^{k+1}_{\epsilon,h} + \mathbb{R}_h^p) \subset \text{Graph}(\mathcal{V}^k_{\epsilon,h} + \mathbb{R}_h^p),$$

and

$$\text{Graph}(\mathcal{V}^k_{\epsilon,h} + \mathbb{R}_h^p) = A^k,$$

where the sets $A^k$ are recursively defined (Proposition 2.18, p. 195, [4]) from $A^0 = \mathcal{H}_h$ and the relation

$$A^{k+1} = \{(t_h, x_h, z_h) \in A^k \text{ s.t. } \Gamma_{\epsilon,h}(t_h, x_h, z_h) \cap A^k \neq \emptyset\}.$$

To simplify the proof of Proposition 6.1, we will assume that $T$ is a multiple of $h$ and that $\epsilon - 2h > 2h$, which guarantees that for all $t_h \in I_h$, $t_h + \epsilon - 2h > h$. We will also assume that for all $(t_h, x_h) \in I_h \times \mathbb{R}_h^n$ such that $t_h \geq h$,

$$\mathcal{V}^0_{\epsilon,h}(t_h, x_h) = \{(-(T + h - t_h)\overline{M}_L - h)1\}. $$

This guarantees that, for all $(t_h, x_h) \in I_h \times \mathbb{R}_h^n$ for all $(f, l) \in \mathcal{F}_h$, for all $\bar{t}_h \in (t_h + \epsilon + [-2h, 2h]) \cap \mathbb{R}_h$ for all $\bar{x}_h \in (x_h + \epsilon f + \alpha_{\epsilon,h}) \cap \mathbb{R}_h^n$

$$\mathcal{V}^0_{\epsilon,h}(\bar{t}_h, \bar{x}_h) = \{(-(T + h - \bar{t}_h)\overline{M}_L - h)1\}. $$

...
We will finally assume that \( \epsilon, h \), and \( \overline{M}_L \) have been chosen such that
\[
\epsilon M_L + \alpha \epsilon, h \leq (\epsilon - 2h) \overline{M}_L.
\] (6.3)

**Proposition 6.1.**
\[
\forall k, \quad \text{Graph}(V_{e,h}^{k+1} + R_{h,+}^p) \subset \text{Graph}(V_{e,h}^k + R_{h,+}^p).
\]

**Proof.** We start by proving that this relation holds for \( k = 0 \). Take \((t_h, x_h, z_h)\) \(\in\) \(\text{Graph}(V_{e,h}^1 + R_{h,+}^p)\). Hence, \(z_h \in V_{e,h}^1(t_h, x_h) + R_{h,+}^p\). We have two cases to consider:

1. If \( t_h \geq T - M \epsilon - h \). From (6.2), we directly get \( z_h \in V_{e,h}^0(t_h, x_h) + R_{h,+}^p \).
2. Otherwise, \( t_h < T - M \epsilon - h \). By (6.1),
\[
V_{e,h}^1(t_h, x_h) = \mathcal{E} \left( \left\{ (\epsilon + \alpha \epsilon, h) \cap R_{h}^p + V_{e,h}^0((t_h + \epsilon + [-2h,2h]) \cap R_{h}, (x_h + \epsilon f + \alpha \epsilon, h) \cap R_{h}^p), (f, 1) \in \text{FL}^+(x_h) \right\} \right).
\]

Let \( \tilde{t}_h \in (t_h + \epsilon + [-2h,2h]) \cap R_{h}, \tilde{x}_h \in (x_h + \epsilon f + \alpha \epsilon, h) \cap R_{h}^p, \tilde{z}_h \in V_{e,h}^0(\tilde{t}_h, \tilde{x}_h) \), and \( \tilde{z}_h \in (\epsilon + \alpha \epsilon, h) \cap R_{h}^p \). Hence, \( \tilde{z}_h \geq -(\epsilon M_L + \alpha \epsilon, h)1 \). Moreover, from above, we have
\[
z_h = (-(T + h - \tilde{t}_h)M_L - h)1 \geq (-(T + h - t_h - (\epsilon - 2h)M_L - h))1 = (-(T + h - t_h)M_L - h)1 + (\epsilon - 2h)M_L 1.
\]
Hence, by (6.3),
\[
z_h \geq (-(T + h - t_h)M_L - h)1 + (\epsilon - 2h)M_L 1 - (\epsilon M_L + \alpha \epsilon, h)1 + (\epsilon M_L + \alpha \epsilon, h)1 \geq (-(T + h - t_h)M_L - h)1 + (\epsilon M_L + \alpha \epsilon, h)1.
\]
Therefore,
\[
\tilde{z}_h + z_h \in V_{e,h}^0(t_h, x_h) + R_{h,+}^p.
\]

Hence,
\[
V_{e,h}^1(t_h, x_h) \subset V_{e,h}^0(t_h, x_h) + R_{h,+}^p
\]
from which we get that
\[
z_h \in V_{e,h}^1(t_h, x_h) + R_{h,+}^p \subset V_{e,h}^0(t_h, x_h) + R_{h,+}^p + R_{h,+}^p = V_{e,h}^0(t_h, x_h) + R_{h,+}^p.
\]

Assume now that the relation holds up to \( k \). We aim to prove that it holds for \( k + 1 \), i.e.,
\[
\text{Graph}(V_{e,h}^{k+2} + R_{h,+}^p) \subset \text{Graph}(V_{e,h}^{k+1} + R_{h,+}^p).
\]

Take \((t_h, x_h, z_h)\) \(\in\) \(\text{Graph}(V_{e,h}^{k+2} + R_{h,+}^p)\). Hence, \(z_h \in V_{e,h}^{k+2}(t_h, x_h) + R_{h,+}^p\). We have two cases to consider:

1. If \( t_h \geq T - M \epsilon - h \). From (6.2), we directly get \( z_h \in V_{e,h}^{k+1}(t_h, x_h) + R_{h,+}^p \).
2. Otherwise, \( t_h < T - M \epsilon - h \). By (6.1), for some \((f, 1) \in \text{FL}^+(x_h)\), there exist \( \tilde{t}_h \in (t_h + \epsilon + [-2h,2h]) \cap R_{h}, \tilde{x}_h \in (x_h + \epsilon f + \alpha \epsilon, h) \cap R_{h}^p \), and \( \tilde{z}_h \in (\epsilon + \alpha \epsilon, h) \cap R_{h}^p \) such that
\[
z_h \in \tilde{z}_h + V_{e,h}^{k+1}(\tilde{t}_h, \tilde{x}_h) + R_{h,+}^p.
\]
From the induction assumption, we get:
\[ z_h \in \tilde{z}_h + V_{e,h}^k (\tilde{t}_h, \tilde{x}_h) + R_{h,+}^p, \]
or,
\[ z_h \in \left\{ (1+\alpha_{e,h} B)\cap R_h^p + V_{e,h}^k ((t_h+\epsilon+[-2h,2h])\cap R_h, (x_h+\epsilon f+\alpha_{e,h} B)\cap R_h^n), (f, l) \in FL^+(x_h) \right\} + R_{h,+}^p. \]
Applying external stability to
\[ S = \left\{ (1+\alpha_{e,h} B)\cap R_h^p + V_{e,h}^k (t_h+\epsilon+[-2h,2h])\cap R_h, (x_h+\epsilon f+\alpha_{e,h} B)\cap R_h^n), (f, l) \in FL^+(x_h) \right\}, \]
and using (6.1) yields \[ z_h \in V_{e,h}^{k+1}(t_h, x_h) + R_{h,+}^p. \]

\[ \Box \]

**Proposition 6.2.**
\[ \forall k, \text{ Graph}(V_{e,h}^k + R_{h,+}^p) = A^k. \]

**Proof.** This relation holds for \( k = 0 \) by definition. Assume now that the relation holds up to \( k \). We aim to prove that it holds for \( k + 1 \), i.e.,
\[ \text{ Graph}(V_{e,h}^{k+1} + R_{h,+}^p) = A^{k+1}. \]
First, we prove the inclusion
\[ \text{ Graph}(V_{e,h}^{k+1} + R_{h,+}^p) \subset A^{k+1}. \] (6.4)
Take \( (t_h, x_h) \in I_h \times R_h^p \) and \( z_h \in V_{e,h}^{k+1}(t_h, x_h) + R_{h,+}^p \). We have two cases to consider:

1. If \( t_h \geq T - M\epsilon - h \), then by (6.2), we have \( (t_h, x_h, z_h) \in \text{ Graph}(V_{e,h}^k + R_{h,+}^p) \).
   Therefore, from the induction assumption, \( (t_h, x_h, z_h) \in A^k \). Moreover, by (6.3), we also have \( (t_h, x_h, z_h) \in \Gamma_{e,h}(t_h, x_h, z_h) \). Hence, \( (t_h, x_h, z_h) \in \Gamma_{e,h}(t_h, x_h, z_h) \cap A^k \), or \( \Gamma_{e,h}(t_h, x_h, z_h) \cap A^k \neq \emptyset \), which shows that \( (t_h, x_h, z_h) \in A^{k+1} \).

2. Otherwise, \( t_h < T - M\epsilon - h \). By (6.1), for some \( (f, l) \in FL^+(x_h) \), there exist \( \tilde{t}_h \in (t_h + \epsilon + [-2h, 2h]) \cap R_h \), \( \tilde{x}_h \in (x_h + \epsilon f + \alpha_{e,h} B) \cap R_h^n \), and \( \tilde{z}_h \in (1+\alpha_{e,h} B) \cap R_h^p \) such that
\[ z_h \in \tilde{z}_h + V_{e,h}^k (\tilde{t}_h, \tilde{x}_h) + R_{h,+}^p. \]
Hence, from the induction assumption, \( (\tilde{t}_h, \tilde{x}_h, z_h - \tilde{z}_h) \in A^k \). To get that \( (t_h, x_h, z_h) \in A^{k+1} \), it remains to prove that \( (t_h, x_h, z_h) \in A^k \) and \( (t_h, x_h, z_h - \tilde{z}_h) \in \Gamma_{e,h}(t_h, x_h, z_h) \) where \( \Gamma_{e,h}(t_h, x_h, z_h) \) is given by (6.2). \( (t_h, x_h, z_h) \in A^k \) comes from Proposition 6.1 and the induction assumption, i.e.,
\[ (t_h, x_h, z_h) \in \text{ Graph}(V_{e,h}^{k+1} + R_{h,+}^p) \subset \text{ Graph}(V_{e,h}^k + R_{h,+}^p) = A_k. \]
Moreover, we have:
(a) \( \bar{t}_h \in (t_h + \epsilon + [-2h, 2h]) \cap R_h \),
(b) \( \bar{x}_h \in (x_h + \epsilon f + \alpha_{\epsilon,h} B) \cap R^n_h \),
(c) and
\[
\bar{z}_h \in (\epsilon I + \alpha_{\epsilon,h} B) \cap R^p_h \Rightarrow -\bar{z}_h \in (-\epsilon I + \alpha_{\epsilon,h} B) \cap R^p_h \\
\Rightarrow z_h - \bar{z}_h \in (z_h - \epsilon I + \alpha_{\epsilon,h} B) \cap R^p_h .
\]

Hence, \( \Gamma_{\epsilon,h}(t_h, x_h, z_h) \cap A_k \neq \emptyset \) and (6.4) is proved.

Conversely, we prove the inclusion
\[
A^{k+1} \subset \text{Graph}(V_{\epsilon,h}^{k+1} + R^p_{h,+}). \tag{6.5}
\]

Take \((t_h, x_h, z_h) \in A^{k+1} \). We have two cases to consider:

1. If \( t_h \geq T - M \epsilon - h \). By definition of \( A^{k+1} \), \((t_h, x_h, z_h) \in A^k \). Hence, from the induction assumption, we get \( z_h \in V_{\epsilon,h}^{k}(t_h, x_h) + R^p_{h,+} \), and from (6.2), \( z_h \in V_{\epsilon,h}^{k+1}(t_h, x_h) + R^p_{h,+} \).

2. Otherwise, \( t_h < T - M \epsilon - h \). Then, there exist \((\bar{t}_h, \bar{x}_h, \bar{z}_h) \in A^k \) such that \((\bar{t}_h, \bar{x}_h, \bar{z}_h) \in \Gamma_{\epsilon,h}(t_h, x_h, z_h)\), or:
\[
\begin{align*}
(a) & \quad \bar{t}_h \in (t_h + \epsilon + [-2h, 2h]) \cap R_h , \\
(b) & \quad \bar{x}_h \in (x_h + \epsilon f + \alpha_{\epsilon,h} B) \cap R^n_h , \\
(c) & \quad \bar{z}_h \in (z_h - \epsilon I + \alpha_{\epsilon,h} B) \cap R^p_h \Rightarrow z_h \in \bar{z}_h + (\epsilon I + \alpha_{\epsilon,h} B) \cap R^p_h ,
\end{align*}
\]

for some \((f, l) \in FL^-(x_h)\). From the induction assumption, we have \( \bar{z}_h \in V_{\epsilon,h}^{k}(\bar{t}_h, \bar{x}_h, \bar{z}_h) + R^p_{h,+} \). Hence,
\[
\begin{align*}
z_h & \in (\epsilon I + \alpha_{\epsilon,h} B) \cap R^p_h + V_{\epsilon,h}^{k}(\bar{t}_h, \bar{x}_h) + R^p_{h,+} .
\end{align*}
\]

Applying external stability to
\[
S = \left\{ (\epsilon I + \alpha_{\epsilon,h} B) \cap R^p_h + V_{\epsilon,h}^{k}(t_h + \epsilon + [-2h, 2h]) \cap R_h , (x_h + \epsilon f + \alpha_{\epsilon,h} B) \cap R^n_h \right\} , (f, l) \in FL^+(x_h),
\]

and using (6.1) yields \( z_h \in V_{\epsilon,h}^{k+1}(t_h, x_h) + R^p_{h,+} \).

\( \square \)

**Corollary 6.3.** The sequence of approximate set-valued return functions \( V_{\epsilon,h}^{k} \) is finite.

**Proof.** This follows from Proposition 6.2 and the fact that the sequence \( A_k \) is finite (Proposition 2.18, p. 195, [4]).

We denote \( k(\epsilon, h) \) the last element of this sequence.

**Corollary 6.4.** The epigraph of the approximate set-valued return function \( V_{\epsilon,h}^{k(\epsilon, h)} \) converges in the sense of Painlevé-Kuratowski towards the epigraph of the set-valued return function \( V \), i.e.,
\[
\text{Graph}(V + R^p_h) = \lim_{\epsilon \to 0+, \frac{h}{2} \to 0^+} \text{Graph}(V_{\epsilon,h}^{k(\epsilon, h)} + R^p_{h,+}).
\]
Proposition 7.1, to find $I$ over their entire domain. Moreover, from Proposition 4.1, we have

$$\text{Graph}(V + R^p) = \text{Viab}_p(H).$$

Finally, from Proposition 6.2 and [4] (Proposition 2.18, p. 195), we have

$$\text{Graph}(V^{k(e,h)} + R^p_{h,+}) = A^{k(e,h)} = \text{Viab}_p(H).$$

The desired result follows.

7. A General Numerical Algorithm. In this section, we present a general algorithm to approximate the set-valued return function $V^{k(e,h)}$. As shown in Proposition 7.1, it is not needed to compute $V^{k}_{e,h}$, $k = 0, \ldots, k(e, h)$ over their entire domain $H = \mathbb{R}^n_h$.

**Proposition 7.1.** \(\forall k \geq 0, \forall t_h \in I_h, t_h \geq T - M \epsilon - h - k(\epsilon - 2h), \forall x_h \in \mathbb{R}^n_h,

$$V^{k+1}_{e,h}(t_h, x_h) = V^k_{e,h}(t_h, x_h).$$

**Proof.** For $k = 0$, (7.1) directly follows from (6.2). Assume now that (7.1) holds for $k > 0$. We prove that (7.1) also holds for $k + 1$. Let $t_h \in I_h$, $t_h \geq T - M \epsilon - h - (k + 1)(\epsilon - 2h)$ and $\bar{t}_h \in (t_h + \epsilon + [-2h, 2h]) \cap \mathbb{R}^n_h$. Then,

$$\bar{t}_h \geq t_h + (\epsilon - 2h) \geq T - M \epsilon - h - (k + 1)(\epsilon - 2h) + (\epsilon - 2h) = T - M \epsilon - h - k(\epsilon - 2h).$$

From the induction assumption, we get $\forall x_h \in \mathbb{R}^n_h, \forall (f, l) \in FL^+(x_h), \forall x_h \in (x_h + \epsilon f + \alpha_{e,h}B) \cap \mathbb{R}^n_h$,

$$V^{k+1}_{e,h}((\bar{t}_h, x_h)) = V^k_{e,h}((\bar{t}_h, x_h)).$$

Hence, using (6.1), we obtain $V^{k+2}_{e,h}(t_h, x_h) = V^{k+1}_{e,h}(t_h, x_h)$, which completes the proof.

We now present a very general numerical algorithm to approximate the set $V(0, x_0)$, where for simplicity, we take the initial state $x_0$ in $\mathbb{R}^n_h$. From Corollary 6.4, the suggested approximation to $V(0, x_0)$ is given by the finite set $V^{k(e,h)}_{e,h}(-h, x_0)$. The proposed numerical algorithm is composed of two stages. In the first stage (Algorithm 1), the computational domain is determined using the bound on the dynamics. Once the computational domain has been determined, in the second stage (Algorithm 2), $V^{k(e,h)}_{e,h}(-h, x_0)$ is calculated using the multiple dynamic programming equation (6.1)-(6.2) together with Proposition 7.1.

Choose for example $\epsilon_i = 1/2^i$ and $h_i = 1/2^{2i}$. Let $J$ be the number of discretization steps in $h_i$ for the interval $[-h_i, T + h_i]$, i.e., $J = (T + 2h_i)/h_i + 1$ (for simplicity, we assume that $T$ is a multiple of $h_i$) and let $t_j = -h_i + j \ast h_i$. First, we need to determine the computational domains $\Omega_j, j = 0, \ldots, J - 1$.\}
Algorithm 1

Initialization $\forall t_j, -h_i \leq t_j < \epsilon_i - 3h_i$, $\Omega_j = \{x_0\}$. Otherwise, $\Omega_j = \emptyset$.

Main loop

1 Set $j = 0$.
2 Repeat
   2.1 Set $x_h$ to the first grid point in $\Omega_j$.
   2.2 Repeat
      2.3 For all $t_j$, $t_j + \epsilon_i - 2h_i \leq t_j \leq t_j + \epsilon_i + 2h_i$,
      \[\Omega_j' = \Omega_j' \cup \{(x_h + \epsilon_i f + \alpha, B) \cap R^2_h, (f, l) \in FL^+(x_h)\}\] (2.3)
   2.4 Until all the grid points in $\Omega_j$ have been visited.
3 Until $j = J - 1$.

Algorithm 2

Initialization $\forall t_j, T - M \epsilon_i - h_i \leq t_j \leq T + h_i$, $\forall x_h \in \Omega_j$, $V^{k(\epsilon_i, h_i)}(t, x_h) = \emptyset$. Otherwise, $V^{k(\epsilon_i, h_i)}(t, x_h) = \emptyset$.

Let $j^*$ be the largest index such that $t_j < T - M \epsilon_i - h_i$.

Main loop

1 Set $j = j^*$.
2 Repeat
   2.1 Set $x_h$ to the first grid point in $\Omega_j$ and $A = \emptyset$.
   2.2 Repeat
      2.3 For all $t_j$, $t_j + \epsilon_i - 2h_i \leq t_j \leq t_j + \epsilon_i + 2h_i$,
      \[A = A \cup \{(\epsilon_i f + \alpha, B) \cap R^2_h + V^{k(\epsilon_i, h_i)}(t_j, (x_h + \epsilon_i f + \alpha, B) \cap R^2_h), (f, l) \in FL^+(x_h)\}\]
   2.4 Set $V^{k(\epsilon_i, h_i)}(t, x_h) = \mathcal{E}(A)$.
   2.5 Until all the grid points in $\Omega_j$ have been visited.
3 Until $j = 0$.

To reduce the size of the set $A$ in Algorithm 2, it is possible to only keep the Pareto optimal elements at each iteration in Step 2.3. This procedure is justified by the two following lemmas.

Lemma 7.2. Let $S_1$ and $S_2$ be two finite subsets of $\mathbb{R}^p$. Then, $\mathcal{E}(S_1 \cup S_2, P) = \mathcal{E}(S_1 \cup \mathcal{E}(S_2), P)$.

Proof. Take $z_1 \in \mathcal{E}(S_1 \cup S_2, P)$. Assume for contradiction that $z_1 \notin \mathcal{E}(S_1 \cup \mathcal{E}(S_2, P), P)$. Then, by external stability, there exists $z_2 \in \mathcal{E}(S_1 \cup \mathcal{E}(S_2, P), P) \subset S_1 \cup S_2$ such that $z_1 \in z_2 + P \setminus \{0\}$. But, this contradicts $z_1 \in \mathcal{E}(S_1 \cup S_2, P)$.
Conversely, take $z_1 \in \mathcal{E}(S_1 \cup \mathcal{E}(S_2), P)$. Assume for contradiction that $z_1 \notin \mathcal{E}(S_1 \cup S_2, P)$. Then, by external stability, there exists $z_2 \in \mathcal{E}(S_1 \cup S_2, P) \subset S_1 \cup S_2$ such that $z_1 \in z_2 + P \setminus \{0\}$. Assume that $z_2 \notin S_2$. Then, necessarily $z_2 \in S_1 \cup \mathcal{E}(S_2, P)$. But, this contradicts $z_1 \in \mathcal{E}(S_1 \cup \mathcal{E}(S_2, P), P)$. Assume now that $z_2 \in S_2$. Then, by external stability, there exists $z_3 \in \mathcal{E}(S_2, P)$ such that $z_2 \in z_3 + P \setminus \{0\}$. Hence, $z_1 \in z_3 + P \setminus \{0\}$ with $z_3 \in S_1 \cup \mathcal{E}(S_2, P)$. But, this again contradicts $z_1 \in \mathcal{E}(S_1 \cup \mathcal{E}(S_2, P), P)$. \[ \square \]

**Proposition 7.3.** Let $S_1, \ldots, S_l$ be finite subsets of $\mathbb{R}^p$. Then,

$$\mathcal{E}\left( \bigcup_{i=1}^l S_i, P \right) = E_I,$$

where $E_I$ is recursively defined by $E_1 = \mathcal{E}(S_1, P)$ and the relation

$$E_{i+1} = \mathcal{E}(S_{i+1} \cup E_i, P).$$

**Proof.** We proceed by induction. For $I = 1$, this is by definition. Assume now that the relation holds up to $I$. We aim to prove that it holds for $I+1$, i.e.,

$$\mathcal{E}\left( \bigcup_{i=1}^{I+1} S_i \right) = E_{I+1}.$$ 

Apply Lemma 7.2 to $\bigcup_{i=1}^l S_i$ and $S_{I+1}$. Then, we get

$$\mathcal{E}\left( \bigcup_{i=1}^{I+1} S_i, P \right) = \mathcal{E}\left( S_{I+1} \cup \mathcal{E}\left( \bigcup_{i=1}^l S_i, P \right), P \right).$$

Using the induction assumption, this yields

$$\mathcal{E}\left( \bigcup_{i=1}^{I+1} S_i, P \right) = \mathcal{E}(S_{I+1} \cup E_I, P) = E_{I+1}. \quad \square$$

Using Proposition 7.3 we can change Step 2.3 in Algorithm 2 to Step 2.3’ as follows:

**2.1** Set $x_h$ to the first point in $\Omega_j$ and $A = \emptyset$.

**2.2**

**2.3’** For all $t_j$, $t_j + \epsilon_i - 2h_i \leq t_j' \leq t_j + \epsilon_i + 2h_i$,

$$A = \mathcal{E}(A \cup \{(\epsilon_i h_i + B) \cap R^n_{h_i} + V^{k(\epsilon_i, h_i)}_{\epsilon_i, h_i}(t_j', (x_h + \epsilon_i f + \alpha \epsilon_i, h_i) B) \cap R^n_{\epsilon_i, h_i}), (f, l) \in FL^+(x_h)\}).$$

**2.4** Set $V^{k(\epsilon_i, h_i)}_{\epsilon_i, h_i}(t_j, x_h) = \mathcal{E}(A)$.

**2.5** Until all the points in $\Omega_j$ have been visited.

**8. Numerical Examples.** In this section, the algorithms from [7] are applied to a simple class of optimal control problems for which the set-valued return function $V$ can be obtained analytically. The convergence of $V^{k(\epsilon_i, h_i)}_{\epsilon_i, h_i}(-h_i, x_0)$ towards $V(0, x_0)$ is investigated. Recall that incrementing $i$ by 1 means dividing the time step discretization by 2 and the state step discretization by 4.
8.1. Description. Consider the following simple autonomous biobjective \((p = 2)\) optimal control problem. The one-dimensional \((n = 1)\) dynamics is simply
\[
\dot{x}(s) = u(s), \quad s \in [0, T],
\]
with \(U = \{-1, 1\}\) and initial condition \(x_0\). The cost of a trajectory \(x(\cdot)\) over \(I\) is given by
\[
J_1(0, x_0, u(\cdot)) = \int_0^T P(x(s))u(s) \, ds \quad \text{and} \quad J_2(0, x_0, u(\cdot)) = \int_0^T u(s) \, ds,
\]
where \(P(\cdot)\) is a given polynomial.

The objective space \(Y(0, x_0)\) for the problem above can be easily determined. Let \(\alpha = \mu\{t \in I, \, u(t) = 1\}/T\) where \(\mu(\cdot)\) denotes the Lebesgue measure, then
\[
J_2(0, x_0, u(\cdot)) = \int_0^T u(s) \, ds = \alpha T - (1 - \alpha)T = (2\alpha - 1)T,
\]
and
\[
J_1(0, x_0, u(\cdot)) = \int_0^T P(x(s))u(s) \, ds = \int_0^T P(x(s))\dot{x}(s) \, ds = [Q(x(t))]_0^T,
\]
where \(Q(\cdot)\) is an antiderivative of \(P(\cdot)\). With
\[
x(T) = x(0) + \int_0^T u(s) \, ds = x_0 + (2\alpha - 1)T,
\]
and defining \(\delta = (2\alpha - 1)T\), we get
\[
J_1(0, x_0, u(\cdot)) = Q(x_0 + \delta) - Q(x_0) \quad \text{and} \quad J_2(0, x_0, u(\cdot)) = \delta.
\]
The set \(Y(0, x_0)\) can finally be obtained by varying \(\delta\) between \(-T\) and \(T\).

It is possible to derive a systematic procedure that gives an interval \((\subset [-T, T])\) such that for all \(\delta\) in this interval, the corresponding element \((J_1(0, x_0, u(\cdot)), J_2(0, x_0, u(\cdot))) = (Q(x_0 + \delta) - Q(x_0), \delta)\) in the objective space is a Pareto optimal element. We will not present this procedure here, and therefore assume that the Pareto optimal set \(V(0, x_0)\) is known.

8.2. Results. We consider four different polynomials \(P(\cdot)\) and initial conditions \(x_0\) with \(T = 0.5\), which yield four problems \((\text{MOC}1), (\text{MOC}2), (\text{MOC}3), \text{and} (\text{MOC}4)\). We have chosen these polynomials such that the Pareto optimal sets \(V(0, x_0)\) present different characteristics. For each problem and for \(i = 3, i = 4, \text{and} i = 5\), we compute \(V_{i, h_i}^{k(e_i, h_i)}(-h_i, x_0)\) using Algorithm 1 and Algorithm 2 from \[7\] provide the cardinality of \(V_{i, h_i}^{k(e_i, h_i)}(-h_i, x_0)\), i.e., \(|V_{i, h_i}^{k(e_i, h_i)}(-h_i, x_0)|\), calculate the Hausdorff distance \[2\] p. 365 \(H(V_{i, h_i}^{k(e_i, h_i)}(-h_i, x_0), V(0, x_0))\) between \(V_{i, h_i}^{k(e_i, h_i)}(-h_i, x_0)\) and \(V(0, x_0)\), and finally generate a "normalized" Hausdorff distance
\[
\overline{H}(V_{i, h_i}^{k(e_i, h_i)}(-h_i, x_0), V(0, x_0)) = \frac{H(V_{i, h_i}^{k(e_i, h_i)}(-h_i, x_0), V(0, x_0))}{H(V_{i, h_i}^{k(e_i, h_i)}(-h_i, x_0), V(0, x_0))}.~
\]
Our results are summarized in Tables 8.1, 8.2, 8.3, and 8.4 and also in Figures 8.1, 8.2, 8.3, and 8.4. The first number corresponds to the total number of grid points, i.e., the cardinality of all the sets $\Omega_i$. The second number corresponds to the total number of successors for all the grid points, where a successor to a grid point $x_h$ is defined as a grid point that can be reached from $x_h$. Using Graph Theory terminology, the size of a problem would correspond to the number of nodes and vertices respectively. Figures 8.1, 8.2, 8.3, and 8.4 display the objective space, the Pareto optimal set, and the approximate Pareto optimal sets for $i = 3$, $i = 4$, and $i = 5$ or each problem.

(MOC1) $P(x) = x - 1$, $x_0 = 1$. The set $Y(0, x_0) + R^2_+$ is convex. Hence, it is possible to obtain every element of $V(0, x_0)$ using the weighting method.

(MOC2) $P(x) = -x + 1$, $x_0 = 1.5$. The set $Y(0, x_0) - R^2_+$ is convex. Only the two Pareto optimal elements for $\delta = -T$ and $\delta = T$ can be obtained using the weighting method.

(MOC3) $P(x) = -2x^3 - 15/4x^2 + 2/75x + 1/5$, $x_0 = 0$. The set $Y(0, x_0) + R^2_+$ is nonconvex and the set $V(0, x_0)$ is nonconnected. More precisely, $V(0, x_0)$ is the union of two sets.

(MOC4) $P(x) = -3/2x - 1/8$, $x_0 = 0$. This problem is similar to (MOC3).

**Remark 8.1.** To reduce the size of the problems, we have proceeded to some simplifications in Algorithm 1 and Algorithm 2. First, we have individually computed $\alpha_{\epsilon_i, h_i}$ for the dynamics $f(\cdot, \cdot)$ and each component of the running cost $L(\cdot, \cdot)$. Second, we have reduced the interval $[t_j + \epsilon_i - 2h_i, t_j + \epsilon_i + 2h_i]$ to the single time $t_j + \epsilon_i - 2h_i$. Finally, for any given $l$, we have reduced the set $(\epsilon_l 1 + \alpha_{\epsilon_i, h_i} B) \cap R^2_+$ to a single element, i.e., the closest lattice element to $\epsilon_l 1$, which somehow corresponds to setting $\alpha_{\epsilon_i, h_i} = 0$ for the running cost. Hence, we have set $M$ to $\max\{1, M_l\} = 1$.

**Table 8.1**

| Table 8.1 | Results for (MOC1). |
|-----------|---------------------|
| i=3       | (306,5897)          | (1630,65093)   |
| i=4       | (10422,856445)      |
| i=5       |                     |
| $|V_{\epsilon_i, h_i}|-h_i, x_0)\rangle$ | 10  | 33  | 130  |
| $H(V_{\epsilon_i, h_i}^-(-h_i, x_0), V(0, x_0))$ | 0.001227 | 0.046550 | 0.022605 |
| $\bar{H}(V_{\epsilon_i, h_i}^-(-h_i, x_0), V(0, x_0))$ | 1  | 0.5103 | 0.2478 |

**Table 8.2**

| Table 8.2 | Results for (MOC2). |
|-----------|---------------------|
| i=3       | (306,10961)         | (1630,132125) |
| i=4       | (10422,1826357)     |
| i=5       |                     |
| $|V_{\epsilon_i, h_i}|-h_i, x_0)\rangle$ | 34  | 130  | 514  |
| $H(V_{\epsilon_i, h_i}^-(-h_i, x_0), V(0, x_0))$ | 0.051067 | 0.033192 | 0.016627 |
| $\bar{H}(V_{\epsilon_i, h_i}^-(-h_i, x_0), V(0, x_0))$ | 1  | 0.65  | 0.3256 |
Table 8.3
Results for \((\text{MOC}3)\).

| i = 3 | i = 4 | i = 5 |
|-------|-------|-------|
| Size  | (306,6529) | (1630,66613) | (10422,834285) |
| \(|V_{\epsilon_i,h_i}^k(-h_i,x_0)|\) | 3 | 21 | 99 |
| \(\mathcal{H}(V_{\epsilon_i,h_i}^k(-h_i,x_0), V(0,x_0))\) | 0.765685 | 0.054420 | 0.035360 |
| \(\overline{\mathcal{H}}(V_{\epsilon_i,h_i}^k(-h_i,x_0), V(0,x_0))\) | 1 | 0.0711 | 0.0462 |

Table 8.4
Results for \((\text{MOC}4)\).

| i = 3 | i = 4 | i = 5 |
|-------|-------|-------|
| Size  | (306,7553) | (1630,85213) | (10422,1134221) |
| \(|V_{\epsilon_i,h_i}^k(-h_i,x_0)|\) | 9 | 33 | 129 |
| \(\mathcal{H}(V_{\epsilon_i,h_i}^k(-h_i,x_0), V(0,x_0))\) | 0.033857 | 0.028646 | 0.014031 |
| \(\overline{\mathcal{H}}(V_{\epsilon_i,h_i}^k(-h_i,x_0), V(0,x_0))\) | 1 | 0.8461 | 0.4144 |

Fig. 8.1. (\text{MOC1}): Objective space (plain line), Pareto optimal set \(V(0,x_0)\) (bold line) and approximate Pareto optimal set \(V_{\epsilon_i,h_i}^k(-h_i,x_0)\) for \(i = 3\) (o), \(i = 4\) (+), and \(i = 5\) (·).
Fig. 8.2. (MOC2): Objective space (plain line), Pareto optimal set $V(0, x_0)$ (bold line) and approximate Pareto optimal set $V_{k_i h_i}(-h_i, x_0)$ for $i = 3$ (o), $i = 4$ (+), and $i = 5$ (·).

Fig. 8.3. (MOC3): Objective space (plain line), Pareto optimal set $V(0, x_0)$ (bold lines) and approximate Pareto optimal set $V_{k_i h_i}(-h_i, x_0)$ for $i = 3$ (o), $i = 4$ (+), and $i = 5$ (·).
Fig. 8.4. (MOC4): Objective space (plain line), Pareto optimal set \( V(0, x_0) \) (bold lines) and approximate Pareto optimal set \( V_{\epsilon_i, h_i}^{k_i}(-h_i, x_0) \) for \( i = 3 \) (o), \( i = 4 \) (+), and \( i = 5 \) (·).
8.3. Discussion. Because the dynamics and the final time $T$ are the same for the four problems, it is normal that the total number of grid points remains the same. On the other hand, the running cost is different for the four problems, hence for a given grid point $x_h$, the sets $FL^+(x_h)$ differ, which explains why the number of successors varies between the four problems.

The large value, i.e., 0.765685, in Table 8.3 for (MOC3) comes from the fact that the approximate Pareto optimal set is not able for $i = 3$ to capture the upper part of the Pareto optimal set. However, as $i$ increases, the approximate Pareto optimal set now captures the upper part of the Pareto optimal set, and as result, the Hausdorff distance $H(V_{e_i}^{k(e_i,h_i)}(-h_i,x_0),V(0,x_0))$ decreases considerably.

As $i$ increases, as expected, for the four problems, a better approximation of the Pareto optimal set is obtained. The proposed approach also works well regardless whether the set $Y(0,x_0) + R^P_+$ is convex or not.

9. Conclusion. In this paper, we have derived a convergent approximation of the Pareto optimal set for finite-horizon multiobjective optimal control problems. Several techniques such as domain decomposition [3] or dynamic grid refinement [4] could be considered to improve the computational complexity of the proposed approach. Let us also mention the idea of clustering recently proposed in [10], which consists of keeping only a subset (with fixed cardinality) of the approximate Pareto optimal set at each grid point during the resolution of the multiobjective dynamic programming equation (6.1)-(6.2).

A direct extension to this work would be to consider the general case of a pointed closed convex cone $P$ instead of the nonnegative orthant $R^P_+$ and constraints on the state. Also, the proposed approach could be considered for other classes of optimal control problems, such as multiobjective exit-time optimal control problems [15].

REFERENCES

[1] J.-P. Aubin, Viability theory, Birkhauser, Boston, 1991.
[2] J.-P. Aubin and H. Frankowska, Set-Valued Analysis, Birkhauser, Boston, 1990.
[3] M. Bardi and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhauser, Boston, 1997.
[4] P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre, Set-valued numerical analysis for optimal control and differential games, Stochastic and differential games : Theory and numerical methods. Annals of the international Society of Dynamic Games, M. Bardi, T.E.S. Raghavan, T. Parthasarathy Eds., Birkhauser, Boston, (1999), pp. 177–247.
[5] ———, Numerical schemes for discontinuous value functions of optimal control, Set-Valued Analysis, 8 (2000), pp. 111–126.
[6] ———, Differential games through viability theory: Old and recent results, Advances in Dynamic Game Theory. Annals of the international Society of Dynamic Games, S. Jorgensen, M. Quincampoix, T. L. Vincent, T. Basar Eds., Birkhauser, Boston, (2007), pp. 3–35.
[7] V. Coverstone-Carroll, J. W. Hartmann, and W. J. Mason, Optimal multi-objective low-thrust spacecraft trajectories, Comput. Methods Appl. Mech. Engrg., 186 (2000), pp. 387–402.
[8] K. Deb, Multi-objective optimization using evolutionary algorithms, John Wiley & Sons, Chichister, 2001.
[9] P. J. Fleming and R. C. Purshouse, Evolutionary algorithms in control systems engineering: a survey, Control Engineering Practice, 10 (2002), pp. 1223–1241.
[10] A. Guigue, An approximation method for multiobjective optimal control problems application to a robotic trajectory planning problem., Submitted to Optim. Eng., (2010).
[11] ———, Set-valued return function and generalized solutions for multiobjective optimal control problems (moc), Submitted to SIAM J. Control Optim., (2011).
[12] A. Guigue, M. Ahmadi, M. J. D. Hayes, and R. G. Langlois, A discrete dynamic programming approximation to the multiobjective deterministic finite horizon optimal control problem, SIAM J. Control Optim., 48 (2009), pp. 2581–2599.
[13] A. Guigue, M. Ahmadi, R. G. Langlois, and M. J. D. Hayes, Pareto optimality and multiobjective trajectory planning for a 7-dof redundant manipulator, IEEE Transactions on Robotics, 26 (2010), pp. 1094–1099.
[14] B.-Z. Guo and B. Sun, Numerical solution to the optimal feedback control of continuous casting process, J. Glob. Optim., 39 (1998), pp. 171–195.
[15] A. Kumar and A. Vladimirsky, An efficient method for multiobjective optimal control and optimal control subject to integral constraints, J. Comp. Math., 28 (2010), pp. 517–551.
[16] K. M. Miettinen, Nonlinear Multiobjective Optimization, Kluwer Academic Publishers, Boston, 1999.
[17] Y. Sawaragi, H. Nakayama, and T. Tanino, Theory of Multiobjective Optimization, Academic Press, Inc., Orlando, 1985.
[18] T. Tanino, Sensitivity analysis in multiobjective optimization, J. Optim. Theory Appl., 56 (1988), pp. 479–499.
[19] R. Vinter, Optimal Control, Birkhauser, Boston, 2000.
[20] P. L. Yu, Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives, J. Optim. Theory Appl., 14 (1974), pp. 319–377.