THE LERAY-HIRSCH THEOREM FOR EQUIVARIANT ORIENTED COHOMOLOGY OF FLAG VARIETIES

J. MATTHEW DOUGLASS AND CHANGLONG ZHONG

Abstract. We use the formal affine Demazure algebra to construct an explicit Leray-Hirsch Theorem for torus equivariant oriented cohomology of flag varieties. We then generalize the Borel model of such theory to partial flag varieties.

1. Introduction

The Leray Hirsch Theorem is a fundamental result from algebraic topology: Suppose $p: e \to B$ is fibre bundle with fibre $F$ and fibre embedding $i: F \to E$. Then one has $\pi^*: H^*(B) \to H^*(E)$ and $i^*: H^*(E) \to H^*(F)$. Under reasonable hypotheses, including that the map $i^*$ is surjective, the conclusion of the Leray-Hirsch Theorem is that if $j: H^*(F) \to H^*(E)$ is a right inverse of $i^*$, then the mapping $H^*(B) \otimes H^*(F) \to H^*(E)$ given by $b \otimes f \mapsto \pi^*(b) \smile j(f)$ is an isomorphism.

Examples that are relevant in Schubert calculus and the representation theory of groups of Lie type are the fibrations that arise as projections from a complete flag variety to a partial flag variety. Suppose $G$ is a reductive algebraic group and $B$ and $P$ are a Borel subgroup and a parabolic subgroup, respectively, of $G$. Then the natural projection $\pi: G/B \to G/P$ is a fibration with fibre $P/B$, and $P/B$ is the flag variety of a Levi factor $L$ of $P$. In this case, the Leray-Hirsch Theorem provides an explicit isomorphism

\[ H^*(G/P) \otimes H^*(P/B) \cong H^*(G/B), \]

once it has been shown that mapping $i^*: H^*(G/B) \to H^*(P/B)$ is surjective and a right inverse, $j: H^*(P/B) \to H^*(G/B)$, has been chosen.

One choice of a right inverse of $j$ and a direct proof of the isomorphism \(1\) is given in [D94]. More generally, one can replace cohomology by $T$-equivariant cohomology, where $T$ is a maximal torus in $G$. Drellich and Tymoczko [DT17] have described a choice of a right inverse of $j$ using Goresky-Kottwitz-MacPherson (GKM) theory and given a direct proof of the isomorphism \(1\) in $T$-equivariant homology. They also show that the isomorphism in equivariant cohomology induces an isomorphism in cohomology. For oriented cohomology, the Leray-Hirsch Theorem was proved in

---

2010 Mathematics Subject Classification. Primary 14M15, Secondary 14F43.

Key words and phrases. equivariant oriented cohomology, flag varieties, Leray-Hirsch Theorem.

J.M. Douglass would like to acknowledge that this material is based upon work supported by (while serving at) the National Science Foundation. However, any opinion, finding, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

C. Zhong would like to thank Gufang Zhao for helpful conversations.
In this paper, we extend these results to torus-equivariant, oriented cohomology theories.

Oriented cohomology theories, in the sense of Levine-Morel [LM07], are contravariant functors from the category of smooth varieties to the category of commutative rings that satisfy certain axioms. Examples include Chow groups and K-theory of varieties. One important property of such theories is that they are covariant for proper morphisms. Moreover, Chern classes can be defined, and the first Chern class of tensor products of line bundles are determined by a formal group law. As with generalized cohomology theories in algebraic topology, there is a universal oriented cohomology, called algebraic cobordism, whose associated formal group law is the universal formal group law over the Lazard ring.

More generally, torus-equivariant oriented cohomology theories have been constructed geometrically in [Des09, HM13, Kr12]. Suppose $h^T$ is an equivariant oriented cohomology theory. Based on seminal work of Demazure and Kostant-Kumar, with $G$ and $T$ as above, an algebraic construction of the $T$-equivariant oriented cohomology of partial flag varieties, $h^T(G/P)$, has been developed in [CZZ16, CZZ19, CZZ15, LZZ16]. By carefully studying this algebraic approach, we are able to construct a Leray-Hirsch isomorphism for the fibre bundle $\pi: G/B \to G/P$.

**Theorem 1.1** (Theorem 3.3). Let $S = h^T(pt)$. There is an isomorphism of $S$-modules

$$h^T(G/P) \otimes h^T(pt) \cong h^T(G/B).$$

We want to stress that our isomorphism is constructed explicitly on certain (algebraically defined) bases on both sides.

The proof of the theorem follows from the twisted Leibniz rule of the formal Demazure (divided difference) operators $\Delta_\alpha$, which is defined as follows: for any simple root $\alpha$, define

$$\Delta_\alpha(p) = \frac{p - s_\alpha(p)}{x_\alpha}, \quad p \in S.$$ 

The twisted Leibniz says that, for any $p_1, p_2 \in S$, one has

$$\Delta_\alpha(p_1 p_2) = \Delta_\alpha(p_1) p_2 + s_\alpha(p_1) \Delta_\alpha(p_2).$$

Following from the machinery of [CZZ16], this rule induces a coproduct structure on the algebra (the formal affine Demazure algebra) $D$ generated by these formal Demazure operators. The $S$-dual of this algebra $D$ is then a commutative algebra as well, which is isomorphic to the ring $h^T(G/B)$. Hence, the coefficients appearing in the twisted Leibniz rule will give structure constants of certain classes of $h^T(G/B)$ (see [GZ20] for more details). Moreover, by carefully studying these coefficients, one obtains the isomorphism of Theorem 3.3.

As an application, we generalize the Borel isomorphism [CZZ16] to the parabolic case. More precisely, if the torsion primes of $P$ are invertible in $h(pt)$, then we construct an explicit isomorphism of $S$-modules:

$$h^T(G/P) \otimes h^T(pt)^{W_P} \cong h^T(G/P)^{W_P \otimes}.$$ 

Here $W$ is the Weyl group of $G$, $W_P$ is the Weyl group of $P$, and $\otimes$ is certain (non-$S$-linear) action of $W_P$ on $h^T(P/B)$, defined in §4.5.
The paper is organized as follows: In Section 2 we recall basic facts of algebraic construction of equivariant oriented cohomology of flag varieties. We prove the main result in Section 3, and prove the generalization of Borel model in Section 4.

**Notation**

We follow the notations used in [CZZ16, CZZ19, CZZ15]. Let $\Sigma \hookrightarrow \Lambda^\vee$, $\alpha \mapsto \alpha^\vee$ be a root datum in the sense of [SGA3, Exp. XXI, Definition 1.1.1]. That is, $\Lambda$ is a lattice of rank $n$, $\Sigma$ is a non-empty finite subset of $\Lambda$, called the set of roots. The root lattice $\Lambda_r$ is the subgroup of $\Lambda$ generated by elements of $\Sigma$, and the weight lattice is defined as $\Lambda_w = \{ \lambda \in \mathbb{Q} \otimes \mathbb{Z} \mid \omega \in \mathbb{Z} \text{ for all } \alpha \in \Sigma \}$. Assume the root datum is semisimple. Let $\Pi = \{\alpha_1, ..., \alpha_n\}$ be a fixed set of simple roots, and $s_i = s_{\alpha_i}$ be the simple reflection determined by $\alpha_i$. The Weyl group $W$ is generated by $s_i, i = 1, ..., n$.

Let $G$ be a split semi-simple linear algebraic group with Borel subgroup $B$ and maximal torus $T$. Let the group of characters of $T$ be $\Lambda$, and the associated root datum coincide with $\Sigma \hookrightarrow \Lambda^\vee$. For each subset $J \subset [n]$, denote by $P_J$ the parabolic subgroup corresponding to $J$, $W_J < W$ the subgroup, and $W^J$ the set of minimal length representations of the left cosets $W/W_J$. Let $\Sigma_J = \{\alpha \in \Sigma \mid s_{\alpha} \in W_J\}$.

Define the semigroup $\hat{W} = (\hat{s}_i | i = 1, ..., n)$ subject to the braid relations and $\hat{s}_i^2 = \hat{s}_i$. We extend the Bruhat order of $W$ to $\hat{W}$. We can identify the two sets $\hat{W}$ and $W$ with $\hat{w} \mapsto w$.

For an integer $k$, Denote $[k] = \{1, ..., k\}$. For each sequence $I = (i_1, ..., i_k)$, denote $\ell(I) = k$, and $w(I) = s_{i_1} \cdots s_{i_k} \in W$. Note that $\ell(w(I)) \leq \ell(I)$, and the equality holds if and only if $I$ is a reduced sequence. For a subset $E \subset [k]$, denote $I|_E$ be the subsequence of $I$ consisting $i_j$ with $j \in E$. Sometimes we maybe write $s_i \in I$ for $i \in I$. Denote $\hat{w}(I) = \hat{s}_{i_1} \cdots \hat{s}_{i_k}$. If $I$ is reduced, then $\hat{w}(I) = w(I)$. We denote by $I^{-1}$ the sequence obtained from $I$ by reversing the order.

We always fix a set of reduced sequences $\{I_w\}_{w \in W}$. For any two sequences $I_1, I_2$, we denote by $I_1 \sqcup I_2$ the sequence obtained by adjoining $I_2$ to the end of $I_1$. When the context is clear, we may even write $I_1 I_2$ for $I_1 \sqcup I_2$. We say a set of reduced sequences $\{I_w\}_{w \in W}$ is $J$-compatible if for any $w = uv, u \in W^J, v \in W_J$, we have $I_w = I_u \sqcup I_v$. Note that this always possible. Indeed, for any $w \in W$, one has the decomposition $w = uv$ with $u \in W^J, v \in W_J$. So by fixing a set of reduced decompositions for elements in $W^J$ and $W_J$, one then define $I_w = I_u \sqcup I_v$. This will give a set of $J$-compatible reduces sequences.

Let $h$ be an oriented cohomology of Levine-Morel, whose associated formal group law is denoted by $F$ over the coefficient ring $R = h(pt)$. It is classical from Levine-Morel that each oriented cohomology theory uniquely determines a formal group law $F$. For example, if $h$ is the Chow group/singular cohomology (resp. K-theory), then $F$ is the additive formal group law $F_a = x + y$ (resp. the multiplicative formal group law $F_m = x + y - xy$). Throughout this paper, we assume that the root system and the coefficient ring $R$ satisfy the regularity condition of [CZZ19, Lemma 2.7]. This is satisfied, for instance, when $\mathbb{Z} \subset R$ and the root datum does not contain a component of simply connected $C_n$ (otherwise one needs to assume 2 is invertible in $R$).
2. EQUIVARIANT ORIENTED COHOMOLOGY OF FLAG VARIETIES

In this section, we recall the algebraic construction of the formal affine Demazure algebra, and its relation with equivariant oriented cohomology of flag varieties. Such construction is done in a series of papers \cite{CZZ16, CZZ19, CZZ15, LZ16}.

2.1. Let $S$ be the formal group algebra defined in \cite{CPZ13}. It is defined as

$$R[[x_\lambda | \lambda \in \Lambda]]/\mathcal{J}_F,$$

where $\mathcal{J}_F$ is the closure of the ideal generated by

$$x_0, x_{\lambda + \mu} - F(x_\lambda, x_\mu), \forall \lambda, \mu \in \Lambda.$$

For example, if $\{t_1, \ldots, t_n\}$ is a basis of $\Lambda$, then $S \sim = R[[x_{t_1}, \ldots, x_{t_n}]]$. It is proved in \cite[Theorem 3.3]{CZZ15} that $S \sim = h_T(pt), x_\lambda \mapsto c_1(L_\lambda)$, where $L_\lambda$ is the associated line bundle. The $W$-action on $\Lambda$ induces a $W$-action on $S$.

Denote $x_{\pm i} = x_{\pm \alpha_i}$. For any root $\alpha$, let $\kappa_\alpha = \frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}}$, which belongs to $S$.

**Example 2.1.** If $F = F_a$, then $h$ is singular cohomology, and $S$ is the completion of the symmetric algebra of $\Lambda$ at the augmentation ideal. We have $\kappa_i = 0$ for all $i$, and the coefficient $-\frac{x_1}{x_i}$ in Example 3.8 equals 1. If $F = F_m$, then $h$ is the K-theory, and $S \equiv Z[\Lambda], x_\lambda \mapsto 1 - e^{-\lambda}$. In this case, $\kappa_i = 1$ for all $i$, and the coefficient $-\frac{x_1}{x_i}$ in Example 3.8 equals $e^{-\alpha_i}$.

Let $Q = S[\frac{1}{x_\alpha} | \alpha \in \Sigma^+]$. Define $Q_W = Q \otimes_R R[W]$, which is a $Q$-module with basis $\delta_w, w \in W$. Define the product as

$$q \delta_{w} q' \delta_{w'} = qw(q') \delta_{ww'}, \quad q, q' \in Q, w, w' \in W.$$

There is an action of $Q_W$ on $Q$, given by

$$q \delta_w \cdot q' = qw(q'), \quad q, q' \in Q, w \in W.$$

Define the Demazure operator

$$\Delta_i(u) = \frac{u - s_i(u)}{x_i}, \quad u \in Q.$$

Then if $u \in S$, we have $\Delta_i(u) \in S$ by \cite[Corollary 3.3]{CPZ13}.

**Example 2.1.** If $F = F_a$, then $h$ is singular cohomology, and $S$ is the completion of the symmetric algebra of $\Lambda$ at the augmentation ideal. We have $\kappa_i = 0$ for all $i$, and the coefficient $-\frac{x_1}{x_i}$ in Example 3.8 equals 1. If $F = F_m$, then $h$ is the K-theory, and $S \equiv Z[\Lambda], x_\lambda \mapsto 1 - e^{-\lambda}$. In this case, $\kappa_i = 1$ for all $i$, and the coefficient $-\frac{x_1}{x_i}$ in Example 3.8 equals $e^{-\alpha_i}$.

Let $Q = S[\frac{1}{x_\alpha} | \alpha \in \Sigma^+]$. Define $Q_W = Q \otimes_R R[W]$, which is a $Q$-module with basis $\delta_w, w \in W$. Define the product as

$$q \delta_{w} q' \delta_{w'} = qw(q') \delta_{ww'}, \quad q, q' \in Q, w, w' \in W.$$

There is an action of $Q_W$ on $Q$, given by

$$q \delta_w \cdot q' = qw(q'), \quad q, q' \in Q, w \in W.$$

Define the Demazure operator

$$\Delta_i(u) = \frac{u - s_i(u)}{x_i}, \quad u \in Q.$$

Then if $u \in S$, we have $\Delta_i(u) \in S$ by \cite[Corollary 3.3]{CPZ13}.

2.2. For any $\alpha \in \Sigma$, define the formal Demazure element $X_\alpha$ and the formal push-pull element $Y_\alpha$ to be

$$X_\alpha = \frac{1}{x_\alpha} - \frac{1}{x_\alpha} \delta_s, \quad Y_\alpha = \kappa_\alpha - X_\alpha = \frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha} \delta_s.$$

For simplicity, denote $X_i = X_{\alpha_i}, Y_i = Y_{\alpha_i}$. The following identities can be found in \cite[Proposition 3.2]{Z15}:

1. $X_i q = \Delta_i(q) + s_i(q) X_i, \quad q \in Q$ (Bernstein relation).
2. $X_i^2 = \kappa_i X_i, \quad Y_i^2 = \kappa_i X_i$. 


(3) $X_i$ satisfy the twisted braid relations. More precisely, if $(s_is_j)^{m_{ij}} = 1$, then

$$
X_i X_j X_i \cdots X_j X_i - \sum_{v \leq s_is_j} \cdots c_v X_i X_j = \sum_{v \leq s_is_j} \cdots c_v X_i X_j X_i \cdots
$$

Moreover, $c_v = 0$ and $c_v = 0$ if $\ell(v) = m_{ij} - 1$. Indeed, $c_v = 0$ for all $v \leq s_is_j \cdots$ if $F = x + y + dxy, d \in R$. In other words, in this case, the braid relations are satisfied.

For any sequence $I = (i_1, \ldots, i_k)$, we define

$$
X_I = X_{i_1} \cdots X_{i_k}, \quad Y_I = Y_{i_1} \cdots Y_{i_k}.
$$

Since we have fixed a set of reduced sequences $\{I_w, w \in W\}$, so $X_{I_w}, Y_{I_w}$ make sense. Moreover, by the twisted braid relations, they depend on the choice of $I_w$, unless $F = x + y + dxy, d \in R$.

2.3. Let $D := D_F$ be the $R$-subalgebra of $Q_W$ generated by elements of $S$ and the elements $X_\alpha, \alpha \in \Sigma$. This is called the formal affine Demazure algebra. Similarly, for each $J \subset \Pi$, let $D_J$ be the $R$-subalgebra of $Q_W$ generated by elements of $S$ and $X_\alpha, \alpha \in \Sigma_J$. To distinguish it from $X_\alpha$ in $D$, we sometimes denote $X_\alpha \in D_J$ by $X^J_\alpha$. For each $w \in W$, we fix a reduced sequence $I_w$. It is proved in [CZZ16] that $D$ (resp. $D_J$) is a free $S$-module with basis $\{X_{I_w} | w \in W\}$ (resp. $\{X_{I_w} | w \in W_J\}$).

Denote

$$
\delta_w = \sum_{v \leq w} b_{v,I_w} X_{I_w}, \quad \delta_u = \sum_{v \leq w \leq u} b^J_{u,v} X^J_{I_w}, \quad v \in W, \quad u \in W_J,
$$

then we have $b_{v,I_w}, b^J_{u,v} \in S$. Moreover, if $v = u \in W_J$, then $b_{v,I_w} = b^J_{v,I_w}$.

2.4. There is a coproduct structure on the (left)-$Q$-module $Q_W$ defined by

$$
\Delta : Q_W \rightarrow Q_W \otimes_Q Q_W, \quad q\delta_w \mapsto q\delta_w \otimes \delta_w,
$$

with counit

$$
Q_W \rightarrow Q, \quad q\delta_w \mapsto q.
$$

It induces a coproduct structure on the left $S$-module $D$, and on $D_J$. Indeed, it comes from the Leibniz rule

$$
X_i \cdot (pq) = \Delta_i(pq) = \Delta_i(p)q + s_i(p)\Delta_i(q), \quad p, q \in Q.
$$

See [CZZ16, Proposition 9.5] and [GZ20] for more details.

Replacing $Q$ by $S$ above, we obtain the commutative co-algebra $S_W$ with $S$-basis $\delta_w, w \in W$. Moreover, we have embeddings:

$$
S_W \subset D \subset Q_W.
$$
2.5. Let $Q_W' = \text{Hom}_Q(Q_W, Q)$, $D^* = \text{Hom}_S(D, S)$, $D_J^* = \text{Hom}_S(D_J, S)$ be the duals. We have inclusions

$$D^* \subset S_W^* \subset Q_W'^*,$$

Let $f_\nu \in Q_W'^*$ be the basis dual to $\delta_\nu$. Then $Q_W'^*$ becomes a ring with product

$$f_\nu \cdot f_\nu = \delta_{\nu, \nu} f_{\nu}, \quad \nu, \nu' \in W.$$

The identity is

$$1 := \sum_{w \in W} f_w.$$

Let $X_{f_\nu}^* \in D^*$ be the $S$-basis of $D^*$ dual to $X_{f_\nu} \in D$, which is also a $Q$-basis of $Q_W'$ dual to $X_{f_\nu}$. Let $Y_{f,w}^* \in D^*$ be the basis dual to $Y_{f,w} \in D$. Similarly, denote $\{X_{f_\nu}^*, w \in W\}$ be the basis of $D_J^*$ dual to $D_J$. It follows from \cite{CZZ19} Lemma 7.4] that $1 = X_{f}^* \in D^*$.

There is an action of $Q_W$ on $Q_W'^*$ defined by

$$(z \bullet f)(z') = f(z'z), \quad z, z' \in Q_W, f \in Q_W'^*.$$ 

It is easy to obtain

$$(p\delta_\nu) \bullet (qf_\nu) = qv_\nu^{-1}(p)f_{v_\nu^{-1}}, \quad p, q \in Q.$$ 

For each $J \subset \Pi$, define

$$x_J = \prod_{\alpha \in \Sigma_J, \alpha < 0} x_\alpha \in S.$$ 

For example, $x_{\Pi} = \prod_{\alpha < 0} x_\alpha$. Define

$$\text{pt}_w = x_{\Pi} \bullet f_w = w(x_{\Pi})f_w \in Q_W'^*, \quad Y_J = \sum_{w \in W_J} \delta_w \frac{1}{x_J} \in Q_W'^*.$$ 

It is proved in \cite{CZZ19} Lemma 10.3] that $\text{pt}_w \in D^*$ and in \cite{CZZ19} Lemma 10.12] that $Y_J \in D$.

2.6. Denote the map $\pi_J : G/B \to G/P_J$, $i_w : w \mapsto G/B$. Then we have the following conclusion relating geometry:

**Theorem 2.2.**

1. \cite{CZZ15} Theorem 8.2] There is a commutative diagram

$$
\begin{array}{ccc}
D^* & \xrightarrow{Y_J} & (D_J)^*_{W_J} \\
\Theta & \sim & \Theta \\
\downarrow & & \downarrow \\
\wp_T(G/B) & \xrightarrow{\pi_J} & \wp_T(G/P_J) \\
\end{array}
$$

Moreover, the left $\Theta$ maps $Y_{f_\nu^{-1}} \bullet \text{pt}_\nu$ into the Bott-Samelson classes $\zeta_{f_\nu}$, and the right $\Theta$ maps $i_w(1)$ into $\text{pt}_w$. One also identifies $D_J^*$ with $\wp_T(P_J/B)$ and $S_{W_J}^*$ with $\wp_T(W_J)$, respectively. Denoted the isomorphism $D_J^* \to \wp_T(P_J/B)$ by $\Theta_J$.

2. \cite{CZZ15} Corollary 8.7] There is a commutative diagram

$$
\begin{array}{ccc}
D^* & \xrightarrow{Y_J^*} & (D_J^*)^*_{W_J} \\
\Theta & \sim & \Theta \\
\downarrow & & \downarrow \\
\wp_T(G/B) & \xrightarrow{\pi_J^*} & \wp_T(G/P_J)^* \\
\end{array}
$$

Moreover, the left $\Theta$ maps $Y_{f_\nu^{-1}} \bullet \text{pt}_\nu$ into the Bott-Samelson classes $\zeta_{f_\nu}$, and the right $\Theta$ maps $i_w(1)$ into $\text{pt}_w$. One also identifies $D_J^*$ with $\wp_T(P_J/B)$ and $S_{W_J}^*$ with $\wp_T(W_J)$, respectively. Denoted the isomorphism $D_J^* \to \wp_T(P_J/B)$ by $\Theta_J$.
Here \((D^*)^{W_J}\) is the \(W_J\)-invariant subring via the \(\cdot\)-action of \(W_J \subset W\).

(3) The push-forward to base field map \(\mathbb{h}_T(G/B) \to \mathbb{h}_T(pt)\) is given by \(Y_{1w} \cdot \cdot\cdot\):
\[
D^* \mapsto (D^*)^{W} \cong S.
\]
This defines the Poincaré pairing, with \(Y_{1w}\) (resp. \(X_{1w}^*)\) dual to \(Y_{1w}^*\) (resp. \(X_{1w}^{\ast}\)).

(4) \([\text{CZZ19}, \text{Corollary 8.4}]\) If \(\{I_{w}, w \in W\}\) is \(J\)-compatible, then \(\{X_{1w}^*|u \in W^J\}\) is a basis of \((D^*)^{W_J}\).

Because of Theorem \(2.2\) (4), we will focus on \(X_{1w}^*, w \in W\) in this paper.

**Remark 2.3.** Let \(X(w) = BwB/B\) be the Schubert variety and \(Y(w) = B^{-1}wB/B\) be the opposite Schubert variety. As in Theorem \(2.2\) we know \(Y_{1w} \cdot \cdot\ \) coincides with the Bott-Samelson class, whose Poincaré dual is \(Y_{1w}^* \in D^*\). We do not have a geometrical explanation of \(X_{1w}^*\) or \(X_{1w}^{\ast}\).

On the other hand, if \(F = F_a\) (so \(\mathbb{h}\) is singular cohomology or Chow group), then \(Y_{1w} \cdot \cdot\ \) is the Schubert class \([X(w)]\) with dual class \(Y_{1w}^* = [Y(w)]\). Also \(X_{1w} = (-1)^{\ell(w)}Y_{1w}^*\).

Now let \(F = F_m\) (so \(\mathbb{h}\) is the K-theory), then \(X = 1 - Y_i\). Then
\[
Y_{1w} \cdot \cdot\ \cdot pt_e = [O_X(w)], \quad Y_{1w}^* = X_{1w}^{-1, 0} \cdot pt_e = [O_Y(w)],
\]
\[
X_{1w} \cdot \cdot\ \cdot pt_e = [O_X(w)\langle 0, X(w)\rangle], \quad X_{1w}^* = Y_{1w}^{-1, 0} \cdot pt_e = [O_Y(w)].
\]

These identifications can be proved by using the argument of \([\text{SZZ17}, \text{Remark 5.8}]\), and comparing with the usual definition of various Schubert classes, for examples, in \([\text{AMSS19}, \text{§3.1}]\).

**2.7.** We now try to relate \(\mathbb{h}_T(G/B)\) with \(\mathbb{h}_T(P_J/B)\). The closed embedding \(j : P_J/B \to G/B\) induces two maps, \(j^* : \mathbb{h}_T(G/B) \to \mathbb{h}_T(P_J/B)\) and \(j_* : \mathbb{h}_T(P_J/B) \to \mathbb{h}_T(G/B)\). We want to consider their algebraic replacements.

Algebraically, the embeddings \(i : Q_{W_J} \to Q_W\) and \(i : D_J \to D\) satisfy
\[
i(\delta_v) = \delta_v, \quad i(X_{1v}^J) = X_{1v}, \quad v \in W_J,
\]
so they are maps of co-algebras. Therefore, they induce surjections of rings on the duals \(i^* : Q_{W_J}^* \to Q_W^*\), \(i^* : D^* \to D_J^*\), satisfying
\[
i^*(f_w) = \delta_{w,v}f_v, \quad i^*(X_{1w}^{J}) = \delta_{w,v}X_{1v}^{J}, \quad w \in W, v \in W_J.
\]

There is a commutative diagram, where the horizontal maps are surjective:
\[
\begin{array}{ccc}
D^* & \overset{i^*}{\longrightarrow} & D_J^*\\
\downarrow & & \downarrow \\
Q_{W_J} & \overset{i}{\longrightarrow} & Q_W.
\end{array}
\]

It is not difficult to see that \(j^*\) is compatible with the map \(i^*\) above, i.e., there is a commutative diagram:
\[
\begin{array}{ccc}
D^* & \overset{i^*}{\longrightarrow} & D_J^*\\
\overset{\sim}{\downarrow} & & \overset{\sim}{\downarrow} \\
\mathbb{h}_T(G/B) & \overset{j^*}{\longrightarrow} & \mathbb{h}_T(P_J/B).
\end{array}
\]

In particular, the map \(j^*\) is surjective.
On the other hand, we define an embedding of $S$-modules

$$i_* : D_j^* \to D_j^*, \quad X_{i_w}^* \mapsto X_{i_w}^*, \forall w \in W_J.$$  

It is clear that $i_*$ is a section of $i^*$. The map $i_*$ is the algebraic replacement of the push-forward $j_* : \mathbb{H}_T(P_J/B) \to \mathbb{H}_T(G/B)$. It will be interesting to identify the two. Since the geometric meaning of the basis $X_{i_w}^*$ is not clear, we do not have an answer to it. Instead, we define

$$j'_* := \Theta \circ i_* \circ \Theta_J^{-1} : \mathbb{H}_T(P_J/B) \to \mathbb{H}_T(G/B).$$  

Then $j'_*$ is a section of the surjection $j^*$.

### 3. The Leray-Hirsch Theorem

In this section, we prove the main result, that is, we give an explicit construction of the Leray-Hirsch Theorem by using the basis $\{X_{i_w}^*, w \in W\}$, assuming that $\{I_w, w \in W\}$ is $J$-compatible.

#### 3.1. From \textit{[CZZ16] Theorem 9.2}, we know that $D$ is a cocommutative coalgebra. Let $P_{E_1, E_2}^I$ be the coefficients in \textit{[CZZ16] Lemmas 4.8 and 9.5}. That is, if $I = (i_1, \ldots, i_k)$, then

$$\Delta(X_I) = \sum_{E_1, E_2 \subseteq [k]} P_{E_1, E_2}^I X_{I|E_1} \otimes X_{I|E_2},$$

where $P_{E_1, E_2}^I = B_1 \circ B_2 \circ \cdots \circ B_k(1)$ with $B_j : S \to S$ defined by

$$B_j = \left\{ \begin{array}{ll}
-x_{ij} s_{ij}, & \text{if } j \in E_1 \cap E_2, \\
\Delta_{ij}, & \text{if } j \in (E_1 \cup E_2)^c, \\
s_{ij}, & \text{if } j \in (E_1 \setminus E_2) \cup (E_2 \setminus E_1).
\end{array} \right.$$  

Note that

$$s_i(r) = r, \quad \Delta_i(r) = 0, \quad \forall r \in R.$$  

Denote the $B$-operators by $B^I, B^{II}, B^{III}$. From the definition of $P_{E_1, E_2}^I$ and (2), we know that $P_{E_1, E_2}^I$ is equal to 0 if there is a $B_j^I$ with $j$ greater than any $j'$ of $B_j^I$; is constant-free (or 0) if there is a $B_j^I$ with $j$ less than any $j'$ of $B_j^I$; and is equal to 1 if all operators are of the form $B_j^I$. As a summary, we have

**Lemma 3.1.** For any $E_1, E_2 \subseteq [\ell(I)]$, we have

1. $P_{E_1, E_2}^I = 0$ if $\max(E_1 \cup E_2)^c > \max(E_1 \cap E_2)$.
2. In particular, $P_{E_1, E_2}^I = 0$ if $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 \neq [\ell(I)]$.
3. $P_{E_1, E_2}^I$ is a constant-free power series or 0 if $\min(E_1 \cap E_2) < \min(E_1 \cup E_2)^c$.
4. In particular, $P_{E_1, E_2}^I$ is a constant-free power series or 0 if $E_1 \cup E_2 = [\ell(I)]$ and $E_1 \cap E_2 \neq \emptyset$.
5. $P_{E_1, E_2}^I = 1$ if $E_1 \cup E_2 = [\ell(I)]$ and $E_1 \cap E_2 = \emptyset$.

**Example 3.2.** We consider the $A_2$ case. If $I = (1, 2, 1)$, then

$$P_{(1,2,3), (1)}^{1} = (x_1 s_1) \circ s_2 \circ s_1(1) = x_1, \quad P_{(1,2), (1)}^{1} = (x_1 s_1) \circ s_2 \circ \Delta_1(1) = 0, \quad P_{(1,2), (3)}^{1} = s_1 \circ s_2 \circ s_1(1) = 1.$$
Lemma 3.4. Let \( u : X_{I_u} \to X_{I_w} \) be an isomorphism. The map\( u \circ \Phi \) is an isomorphism. Consequently, the map \( \pi^*_j \circ j^*_w \) is an isomorphism.

Theorem 3.3. The map \( \Phi \) is an isomorphism. Consequently, the map \( \pi^*_j \circ j^*_w \) is an isomorphism.

We prove some technical lemmas first. The first one is well-known.

Lemma 3.4. Let \( u = u_1 u_2 \) and \( w = w_1 w_2 \) with \( u_1, w_1 \in W^J, u_2, w_2 \in W_J \), then \( u \geq w \) implies that \( u_1 \geq w_1 \).

Lemma 3.5. (1) If \( I \) is reduced, then \( c_{1,I_u} = 1 \) for \( u = w(I) = \tilde{w}(I) \).

(2) If \( I \neq \emptyset \), then \( c_{1,e} = 0 \).
(3) Let \( w \in W^J \), and \( I \) be a non-empty sequence consisting of \( s_i \in W_J \). Then in the expression

\[
X_{I_0} = \sum_{u_2 \in W_J} c_{I_0, I_0, I_2} X_{I_0 I_2},
\]

we have \( c_{I_0, I_0, I_2} = 0 \).

Proof. (1) and (2): See \cite[Lemma 7.6]{CZZ16} and \cite[Lemma 7.4]{CZZ19}, respectively.

(3) By using the Bernstein relation in \cite{CZZ19} recursively, for any \( p \in S \), we have

\[
X_{I_0} p = \sum_{u_1 \in W^J, u_2 \in W_J, u_1 u_2 \leq w} \phi_{I_0, I_0, I_2}(p) X_{I_0 I_2}, \quad \text{for some } \phi_{I_0, I_0, I_2}(p) \in S.
\]

From Part (1) we know

\[
X_{I_0} = \sum_{e \neq v \in W_J} c_{I, I, I_0} X_{I_0},
\]

so

\[
X_{I_0} X_I = \sum_{e \neq v \in W_J} X_{I_0} c_{I, I, I_0} X_{I_0} = \sum_{e \neq v \in W_J} \sum_{u_2 \in W_J, u_2 \in W^J, u_1 u_2 \leq w} \phi_{I_0, I_0, I_2}(c_{I, I, I_0}) X_{I_0 I_2} X_{I_0}.
\]

We still need to express each \( X_{I_0 I_2} X_{I_0} \) as a linear combination of \( X_I, t \in W \), but the only possible case for \( t = w \) to appear is when \( u_1 = w \), in which case \( u_2 = e \) (since \( u_1 u_2 \leq w \)). In other words, we only need to consider the linear combination of \( X_{I_0} X_I \). But \( v \neq e \), so \( X_{I_0} \) does not appear, which proves the conclusion. \( \Box \)

**Lemma 3.6.** Let \( w \in W^J, u, v \in W_J \). Then

1. \( p_{I_0, I_0} \) is invertible, and
2. \( p_{I_0, I_0} = 0 \) unless \( u \leq v \).

Proof. (1). Denote \( I = I_{w, w} \). We have

\[
p_{I_0, I_0} = \sum_{E_1, E_2 \subseteq [k]} P_{E_1, E_2} \mathbb{1}_{E_1} I_{E_1, I_0} \mathbb{1}_{E_2} I_{I_0, E_2}.
\]

Note that the \( c \)-coefficients will vanish unless \( \bar{w}(I_{E_1}) \geq w \) and \( \bar{w}(I_{E_2}) \geq v \). Moreover, \( \bar{w}(I_{E_1}) \geq w \) implies that \( I_{E_1} \supset I_w \). In other words, \( I_{E_1} \) contains the first part of \( I = I_w \cup I_w \) (that is, \( I_{E_1} \supset I_w \)). Then Lemma 3.5(3) implies that \( c_{I_{E_1}, I_0} = 0 \) unless \( I_{E_1} = I_w \). So assume \( I_{E_1} = I_w \) in the remaining part of the proof, in which case \( c_{I_{E_1}, I_0} = 1 \).

We then study \( I_{E_2} \), which consists of two cases:

i) If in addition \( I_{E_2} = I_w \), then by Lemma 3.1(5), we have

\[
P_{E_1, E_2} = 1 = c_{I_{E_1}, I_0} = c_{I_{E_2}, I_0}.
\]

ii) If \( I_{E_2} \neq I_w \) (in other words, \( I_{E_2} \) does not coincide with the second part of \( I = I_w \cup I_w \)), then \( I_{E_1} \cap E_2 \) is contained in \( I_w \) (i.e., the first part of \( I \)), so \( \min(E_1 \cap E_2) \leq \min(E_1 \cup E_2) \), which, by Lemma 3.5(3), implies that \( P_{E_1, E_2} \) is constant-free, which implies that \( P_{E_1, E_2} \mathbb{1}_{E_1} I_{E_1, E_2} \mathbb{1}_{I_0, E_2} \) is constant-free (or zero).

Therefore, \( p_{I_0, I_0} \) is invertible.

(2). Denote \( I' = I_{wu} \). Similarly as above, we just need to consider \( I_{E_1} = I_w \) and \( \bar{w}(I_{E_2}) > v \).
If \( u \not\leq v \), then for any \( E_1, E_2 \subset [\ell(I')] \), all elements of \( E_1 \cap E_2 \) (which is contained in \( I_{|E_1} = I_w \)) is smaller than elements of \((E_1 \cup E_2)^c\), so by Lemma 3.6(1), \( P_{E_1, E_2}^{I_w} = 0 \). Therefore, \( P_{I_{|E_1}I_w}^{I_w} = 0 \). □

**Theorem 3.7.** For any \( w \in W^J, v \in W_J \), we have

\[
X^*_I_{I_w} \cdot X^*_I_{I_v} = \sum_{w \leq u \in W^J, u \geq u_2 \in W_J} p_{I_{I_w}, I_{I_v}}^{I_uI_{I_u}I_{I_v}I_{I_v}}.
\]

Moreover, the coefficient \( p_{I_{I_w}, I_{I_v}}^{I_uI_{I_u}I_{I_v}I_{I_v}} \) is invertible.

**Proof.** We have

\[
X^*_I_{I_w} \cdot X^*_I_{I_v} = \sum_{u_1 \in W^J, u_2 \in W_J} p_{I_{I_w}, I_{I_v}}^{I_uI_{I_u}I_{I_v}I_{I_v}}
\]

with \( p_{I_{I_w}, I_{I_v}}^{I_uI_{I_u}I_{I_v}I_{I_v}} \) determined in (3).

Firstly, we know that \( c_{I_{I_w}, I_{I_v}} = 0 \) unless \( w = \tilde{w}(I_w) \leq \tilde{w}(I_{I_v}) \leq \tilde{w}(I) = u_1u_2 \). Then Lemma 3.6 implies that \( w \leq u_1 \). So (4) is proved.

The remaining part is proved in Lemma 3.6. □

**Example 3.8.** We consider the case of \( A_2 \), with two simple roots \( \alpha_1, \alpha_2 \). Denote \( \delta_{ij} = \delta_i \delta_j \), and \( x_{i+j} = x_{\alpha_i + \alpha_j} = F(x_i, x_j) \). Similarly, \( X_{ij} = X_iX_j \). For simplicity, we will use \( X_w \) to denote \( X_{I_w} \) in this section. Let \( W_J = \{1, s_1\} \), then \( W^J = \{e, s_2, s_1s_2, s_2s_1, s_1s_2s_1\} \).

Then Theorem 3.7 gives the following identities:

\[
\begin{align*}
X^*_1X^*_2 &= X^*_1X^*_2 = x_1X^*_1 - \frac{x_1}{x_2}X^*_2, \\
X^*_2X^*_2 &= X^*_2X^*_1 = X^*_2 + X^*_1, \\
X^*_eX^*_e &= X^*_eX^*_1 = X^*_e.
\end{align*}
\]

Note that \( \frac{x_1}{x_2} \in S \) is invertible. We give the lexicographic order to the set \( W = W^J \cdot W_J = \{wv | w \in W^J, v \in W_J\} \), where \( W_J \) has the Bruhat order and \( W^J \) has the opposite Bruhat order. The transition matrix from the basis

\[
\{X^*_1X^*_2X^*_eX^*_1X^*_2X^*_e\}
\]

to the basis

\[
\{X^*_1X^*_eX^*_2X^*_eX^*_1X^*_2X^*_eX^*_eX^*_1X^*_2X^*_eX^*_1\}
\]

is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{-x_1}{x_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & -x_1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
Proof of Theorem 3.3. We know \((D^*)^{W_J}\) is a free \(S\)-module of rank \(|W_J|\), \(D^*_J\) is a free \(S\)-module of rank \(W_J\). So \((D^*)^{W_J} \otimes_S D^*_J\) is a free \(S\)-module of rank \(|W| = |W_J| \cdot |W_J|\). Therefore, it suffices to show that \(\Phi\) is surjective, in other words, to show that \(X^*_w \in \text{Im } \Phi\) for any \(w \in W_J\), \(v \in W_J\). First, if \(w \in W_J\) is maximal and \(v = e\), from \(X^*_e = 1\), we have
\[
X^*_w = X^*_w \cdot X^*_e = X^*_w \cdot i_*(X^*_e) \in \text{Im } \Phi.
\]

If \(w \in W_J\) is maximal and \(\ell(v) \geq 1\), from Theorem 3.7 we know that
\[
X^*_w \cdot i_*(X^*_e) = X^*_w \cdot X^*_v = \sum_{u \leq v} W^I_{w,u,v} X^*_w.
\]

Since \(W^I_{w,u,v}\) is invertible, by induction on \(\ell(v)\), we see that \(X^*_w \in \text{Im } \Phi\).

Now assume the conclusion holds for any \(w \in W_J\) with \(\ell(w) = l + 1\) and any \(v \in W_J\). We prove it for \(w \in W_J\) with \(\ell(w) = l\) and \(v \in W_J\). If \(\ell(v) = 0\), it again follows from the fact \(X^*_e = 1\). Now assume \(X^*_w \in \text{Im } \Phi\) for \(\ell(v) \leq k\), and consider \(v \in W_J\) of length \(k + 1\). We have
\[
X^*_w \cdot i_*(X^*_w) = X^*_w \cdot X^*_v = \sum_{u \leq v} \sum_{w < u_1 \in W_J, u_2 \in W_J} W^I_{w,u_1,u_2,v} X^*_w + \sum_{u_3 < v} \sum_{w < u_1 \in W_J, u_2 \in W_J} W^I_{w,u_1,u_2,v} X^*_w + \sum_{w < u_1 \in W_J, u_2 \in W_J} W^I_{w,u_1,u_2,v} X^*_w.
\]

By induction, \(X^*_w \cdot i_*(X^*_w) \in \text{Im } \Phi\) for any \(w < u_1 \in W_J, u_2 \in W_J\), and \(X^*_w \in \text{Im } \Phi\) for any \(u_3 < v\). Moreover, \(W^I_{w,u,v}\) is invertible, so \(X^*_w \in \text{Im } \Phi\). The proof is finished. \(\Box\)

3.3. We state some consequences of the main theorem.

Corollary 3.9. The ring \(D^*\) is a free module over \((D^*)^{W_J}\). If \(\{I_w, w \in W\}\) is \(J\)-compatible, then the set \(\{X^*_w | v \in W_J\}\) is a basis. Geometrically, via the embedding \(\pi^*_J : \mathbb{V}(G/P_J) \to \mathbb{V}(G/B)\), we can view \(\mathbb{V}(G/B)\) as a free \(\mathbb{V}(G/P_J)\)-module with basis \(\{\Theta(X^*_w) | v \in W_J\}\).

The following corollary generalizes [D],[T],[T],[T].

Corollary 3.10. Let \(\chi_J\) be the character of the \(\bullet\)-action of \(W_J\) on \(\mathbb{V}(G/B)\), and \(\chi\) be the character of the \(\bullet\)-action of \(W_J\) on \(\mathbb{V}(P_J/B)\), then
\[
\chi_J = |W_J| \cdot \chi.
\]

Proof. If follows from [CZZ1], Lemma 4.3] that \(\delta_w \bullet (f \cdot f') = (\delta_w \bullet f) \cdot (\delta_w \bullet f')\) for any \(f, f' \in Q^I_W\). Since \(X^*_w \in (D^*)^{W_J}\), we have \(X^*_w \in (D^*)^{W_J}\) for \(w \in W_J\), so we have \(\delta_u \bullet (X^*_w \cdot X^*_u) = X^*_w \cdot (\delta_u \bullet X^*_u)\), \(v, u \in W_J\).

The conclusion then follows similarly as that in [C],[T],[T], Theorem 4.7]. \(\Box\)

Recall from [CZZ1], §11 that we have
\[
\mathbb{B}(G/B) \cong \mathbb{B}(pt) \otimes_{\mathbb{V}(pt)} \mathbb{V}(G/B) \cong R \otimes_S D^*.
\]

Here we view \(R\) as a \(S\)-algebra via the augmentation map \(\epsilon : S \to R, x \mapsto 0\). Similar isomorphisms hold for \(G/P_J\) and \(P_J/B\) as well. Denote the basis \(1 \otimes X^*_w \in R \otimes_S D^*\) by \(e X^*_w\), and similarly denote \(e X^*_v \in R \otimes_S D^*_J\). Denote \(i_1 : R \otimes_S D^*_J \to R \otimes_S D^*\). We then have the following ordinary version of Leray-Hirsch Theorem:
Corollary 3.11. We have an isomorphism of $R$-modules

$$R \otimes_S (D^*)^{W_J} \otimes_S D^*_J \to R \otimes_S D^*,$$

which induces an isomorphism of $R$-modules

$$h(G/P_J) \otimes h(p) \to h(G/P_J) \otimes h(p).$$

4. Borel model

In this section, we apply Theorem 3.3 to obtain a Leray-Hirsch Theorem in terms of the Borel model.

4.1. Recall that the torsion primes of the root datum are prime factors of the torsion index, defined in [Dem73, §5]. For example, we have the following table

| Type | $A_n$ | $B_n$ | $C_n$ | $D_n, n \geq 4$ | $E_6$ | $E_7$ | $E_8$ |
|------|-------|-------|-------|----------------|-------|-------|-------|
| Torsion primes | $\emptyset$ | $2$ | $\emptyset$ | $2$ | $2,3$ | $2,3$ | $2,3,5$ |

4.2. Recall from [CZZ16, §11] that there is the so-called characteristic map, which is a ring homomorphism

$$c : S \to D^*, \quad p \mapsto p \cdot 1.$$ 

Another way to define the map $c$ is as follows: define

$$ev_p : D \to S, \quad ev_p(z) = z \cdot p \in S,$

then it is proved in [CZZ16, §11] that $ev_p = c(p) \in D^*$. Consequently, we have

$$c(p) = \sum_{w \in W} w(p)f_w = \sum_{w \in W} \Delta_I(w(p))X_{I_w}^*.$$

Indeed, this generalizes the formula of [GSZ12, Proposition 6.1] (in the case of $F_a$, the $\epsilon(u)\tau_u$ is our $X_u^*$, and $\partial_w$ is our $\Delta_w$). Geometrically, this is the map

$$S \mapsto h_T(G/B), \quad x_\lambda \mapsto c_1(L_\lambda),$$

where $c_1(L_\lambda)$ is the first Chern class of the line bundle $L_\lambda$ on $G/B$ associated to the character $\lambda$.

This map commutes with the Weyl group action, i.e., we have commutative diagram

$$
\begin{array}{c}
S \xrightarrow{\delta_w} S \\
\downarrow c \hspace{2cm} \downarrow c \\
D^* \xrightarrow{\delta_w} D^*
\end{array}
$$

Therefore, it restricts onto $(D^*)^{W_J}$, that is, we have a commutative diagram

$$
\begin{array}{c}
S^{W_J} \xrightarrow{c} (D^*)^{W_J} \\
\downarrow c \hspace{2cm} \downarrow c \\
S \xrightarrow{c} D^*
\end{array}
$$

(5)

We can similarly define the characteristic map for $P_J/B$ (with $1_J := \sum_{v \in W_J} f_v = X_{\ell_J}^* \in D_J^*$)

$$c^J : S \to D_J^* \cong h_T(P_J/B), \quad p \mapsto p \cdot 1_J = \sum_{v \in W_J} \Delta_{I_v}(p)X_{I_v}^*.$$
4.3. The following theorem is the so-called Borel model. It generalizes \([\text{KK86}, \text{Theorem 4.4}], \text{[KK90, Theorem 11.5.15]}, \text{and [GHZ06, Theorem 2.6]}\).

**Theorem 4.1.** [\text{CZZ16, Theorem 11.4}]  
If the torsion primes of \(G\) are invertible in \(R\), or if \(F = F_m\) is the multiplicative formal group law (equivalently, if \(\mathfrak{h}\) is the \(K\)-theory), then we have an isomorphism

\[
\rho : S \otimes_{SW} S \rightarrow \mathcal{D}^*, \quad p \otimes q \mapsto pc(q) = \sum_{w \in W} p\Delta_{I_w}(q)X_{I_w}^*.
\]

**Corollary 4.2.** Assume the torsion primes of \(G\) are invertible in \(R\), or \(F = F_m\). Let \(\{p_w | w \in W\} \subset S\). Then \(\{p_w | w \in W\}\) is a basis of \(S\) over \(S^W\) if and only if \(\{c(p_w) | w \in W\}\) is a basis of \(\mathcal{D}^*\) over \(S\).

**Proof.** This follows from the definition of \(\rho\). \(\square\)

Recall from [\text{CPZ13, Theorem 6.7}] that there exists \(u_0 \in S\) such that \(\{\Delta_{I_w}(u_0) | w \in W\}\) is a basis of \(S\) over \(S^W\). Therefore, we have

**Corollary 4.3.** If the torsion primes of \(G\) are invertible in \(R\), or if \(F = F_m\), then

\[
\{\sum_{v \in W} \Delta_{I_v}\Delta_{I_w}(u_0)X_{I_w}^* | w \in W\}
\]

is a basis of \(\mathcal{D}^*\) over \(S\).

4.4. Theorem 4.1 applies to \(P_J/B\) as well, that is, we have the ring homomorphism

\[
\rho_J : S \otimes_{SW_J} S \rightarrow \mathcal{D}^*_J, \quad p \otimes q \mapsto pc(q) = \sum_{v \in W_J} pv(q)f_v = \sum_{w \in W_J} p\Delta_{I_w}(q)X_{I_w}^{J*}.
\]

It becomes an isomorphism if the torsion primes of \(P_J\) is invertible in \(R\), or if \(F = F_m\). As an application of Theorem 3.3, we prove the following theorem, which generalizes [\text{GSZ13, Theorem 4.2}] from equivariant cohomology to equivariant oriented cohomology.

**Theorem 4.4.** Assume the torsion primes of \(P_J\) are invertible in \(R\), or \(F = F_m\). Then we have an isomorphism \(\psi\) of \(S\)-modules, determined by the following triangle:

\[
\begin{array}{ccc}
(D^*)^{W_J} \otimes_{S^{W_J}} S & \xrightarrow{\sim} & (D^*)^{W_J} \otimes_{S} (S \otimes_{S^{W_J}} S) \\
\psi & \sim & \phi (1 \otimes p_J) \\
\otimes & \sim & \rightarrow \\
\mathcal{D}^* & \rightarrow & \mathcal{D}^* \\
\end{array}
\]

Here the top map is the canonical isomorphism between tensors, the map \(j\) is the canonical embedding of \((D^*)^{W_J}\) into \(D^*\), and \(\psi\) is the composite of the horizontal and vertical maps.

**Proof.** The horizontal maps is an isomorphism by standard property of tensor product, and the vertical map is the isomorphism in Theorem 3.3 together with the Borel model \((6)\). The conclusion then follows. \(\square\)

**Remark 4.5.** There is another map, which is a ring homomorphism

\[
(D^*)^{W_J} \otimes_{S^{W_J}} S \rightarrow \mathcal{D}^*, \quad f \otimes p \mapsto f \cdot c(p).
\]

Here we view \((D^*)^{W_J}\) as a \(S^{W_J}\)-module via the map \(c\) in \((5)\). We do not know whether it is an isomorphism, even when the torsion primes of \(G\) or \(P_J\) are invertible in \(R\).
4.5. Recall from [LZZ16 §3] that there are two actions of $Q_W$ on $Q'_W$. The first one being the $\bullet$-action. The other one can be denoted by $\ast$, defined as follows. From [CZZ19 Theorem 10.13] we know that via the $\bullet$-action, $Q'_W$ is a free $Q_W$-module of rank one with basis $\text{pt}_e$ (which corresponds to the class of the identity point), and $D^*$ is a free $D$-module of rank one with basis $\text{pt}_e$. That is,

$$Q'_W = Q_W \bullet \text{pt}_e, \quad D^* = D \bullet \text{pt}_e.$$  

We then use this to define the $\ast$-action. For each $f \in D^*$, write it as $z \bullet \text{pt}_e \in D \bullet \text{pt}_e = D^*$, then define

$$f \ast z' = (z \bullet \text{pt}_e) \ast z' = (zz') \bullet \text{pt}_e, \quad z' \in D.$$  

This can be defined similarly as a right action of $Q_W$ on $Q'_W$. To make it into a left action, we introduce an anti-involution

$$\iota : Q_W \to Q_W, \quad \iota(q \delta_w) := \delta_{w^{-1}q} x_{11}.$$  

It also restricts to an anti-involution $\iota : D \to D$, since $\iota(X_i) = X_i$. We then define the left action

$$z \circ f := f \ast \iota(z).$$  

We have

$$z \circ (z' \bullet \text{pt}_e) = (z' \bullet \text{pt}_e) \circ \iota(z) = (z \iota(z')) \bullet \text{pt}_e, \quad z, z' \text{ belong to } D \text{ or } Q_W.$$  

It follows from the definition that the $\bullet$-action and the $\circ$-action commute. Moreover, viewing $D^*$ as a left $D$-module via the $\bullet$-action, there is an isomorphism of rings

$$\pi : D \mapsto \text{End}_D(D^*), \quad \pi(z) : f \mapsto z \circ f, \quad f \in D^*.$$  

Using the basis $f_w \in Q'_W$, we have the following formulas:

$$(p \delta_w) \bullet (q f_w) = q v w^{-1}(p) f_{w^{-1}}, \quad (p \delta_w) \circ (q f_w) = p w(q) f_{w}, \quad p, q \in Q, v, w \in W.$$  

So the $\circ$-action is not $S$-linear. We also have the following commutative diagrams:

$$
\begin{align*}
S \otimes_{SW} S & \xrightarrow{\rho} D^* \quad , \quad S \otimes_{SW} S \xrightarrow{\rho} D^* \\
(z \circ -) \otimes 1 & \xrightarrow{z \circ -} 1 \otimes (z \circ -) \quad , \\
S \otimes_{SW} S & \xrightarrow{\rho} D^* \quad , \quad S \otimes_{SW} S \xrightarrow{\rho} D^*.
\end{align*}
$$

In other words, via the Borel model, the $\bullet$-action comes from the Weyl group action on the second component, and the $\circ$-action comes from the Weyl group action on the first component. Indeed, for cohomology and $K$-theory, a detailed discussion of these actions can be found in [MNS20].

One has a similar picture on the ring $D_j^*$. We use $(D_j^*)^{(W_j, \circ)}$ (resp. $(S \otimes_{SW_j} S)^{(W_j, \circ)}$) to denote the $W_j$-invariant subring of $D_j^*$ (resp. $S \otimes_{SW_j} S$) via the $\circ$-action. We then have

$$S \cong S^{W_j} \otimes_{SW_j} S \cong (S \otimes_{SW_j} S)^{(W_j, \circ)} \xrightarrow{\rho_j} (D_j^*)^{(W_j, \circ)} \cong \mathbb{H}_T(P_j/B)^{(W_j, \circ)}.$$  

Moreover, this map can be written down explicitly as:

$$(7) \quad S \to (D_j^*)^{(W_j, \circ)}, \quad p \mapsto \rho_j(1 \otimes p) = c_j(p) \in D_j^*.$$
4.6. Together with Theorem 4.4 we get the following:

**Theorem 4.6.** Assume the torsion primes of $P_j$ are invertible in $R$, or $F = F_m$. We have an isomorphism of $\mathbb{H}_T(\text{pt})$-modules

$$\mathbb{H}_T(G/P_j) \otimes_{\mathbb{H}_T(\text{pt})} \mathbb{H}_T(P_j/B)^{(W_j, \circ)} \cong \mathbb{H}_T(G/B).$$

**Proof.** We have the following commutative diagram, where all maps are isomorphisms (of modules or rings):

$$
\begin{array}{ccc}
(D^*)^{W_j} \otimes_{S^{W_j}} S & \xrightarrow{\theta} & (D^*)^{W_j} \otimes_S (S \otimes_{S^{W_j}} S) \\
\downarrow \psi_4 & & \downarrow \psi_3 \\
(D^*)^{W_j} \otimes_{S^{W_j}} (D^*_j)^{(W_j, \circ)} & \xrightarrow{\theta \otimes \theta_j} & (D^*)^{W_j} \otimes_S D^*_j \\
\downarrow \psi_2 & & \downarrow \psi_1 \\
\mathbb{H}_T(G/P_j) \otimes_{\mathbb{H}_T(\text{pt})} \mathbb{H}_T(P_j/B)^{(W_j, \circ)} & \xrightarrow{\theta \otimes \theta_j} & \mathbb{H}_T(G/P_j) \otimes_{\mathbb{H}_T(\text{pt})} \mathbb{H}_T(P_j/B) \cong \mathbb{H}_T(G/B).
\end{array}
$$

Here $\psi_1$ is the canonical map between tensors, $\psi_2$ and $\psi_3$ are induced by the embeddings on the second component, and $\psi_4$ is induced by the map in (7). The conclusion then follows. \qed

**References**

[AMSS19] P. Aluffi, L. Mihalcea, J. Schurmann, C. Su, Motivic Chern classes of Schubert cells, Hecke algebras, and Applications to Casselman’s Problem, Preprint, arXiv:1902.10101v1.

[CPZ13] B. Calmès, V. Petrov, K. Zainoulline, Invariants, torsion indices and oriented cohomology of complete flags, *Invent. Math.*, 17: 3886-3910, 2013.

[DGZ20] R. Goldin, C. Zhong, Structure constants in equivariant oriented cohomology of flag varieties, *Ann. Sci. École Norm. Sup. (4)*, 53(1): 1728, 2017.

[HM13] J. Heller and J. Malagón-López, Equivariant algebraic cobordism, *J. Reine Angew. Math.*, 684: 87–112, 2013.
[KK86] B. Kostant, S. Kumar, The nil Hecke ring and cohomology of $G/P$ for a Kac-Moody group $G^*$, Adv. Math. 62: 187–237, 1986.

[KK90] B. Kostant, S. Kumar, $T$-equivariant $K$-theory of generalized flag varieties, J. Differential geometry 32: 549–603, 1990.

[Kr12] A. Krishna, Equivariant cobordism of schemes, Doc. Math., 17: 95–134, 2012.

[KiKr13] V. Kiritchenko and A. Krishna, Equivariant cobordism of flag varieties and of symmetric varieties, Transform. Groups, 18(2): 391–413, 2013.

[LZZ16] C. Lenart, K. Zainoulline, C. Zhong, Parabolic Kazhdan-Lusztig basis, Schubert classes and equivariant oriented cohomology, J. Inst. Math. Jussieu, to appear.

[LM07] M. Levine and F. Morel, Algebraic cobordism, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2007.

[MNS20] L. Mihalcea, H. Naruse, C. Su, Left Demazure-Lusztig operators on equivariant (quantum) cohomology and K-theory, arXiv:2008.12670 2020.

[SZZ17] C. Su, G. Zhao, C. Zhong, On K-theoretic stable bases of Springer resolutions, Ann. Sci. Éc. Norm. Supér., to appear.

[Z15] C. Zhong, On the formal affine Hecke algebra, J. Inst. Math. Jussieu, 14(4): 837-855, 2015.

DIVISION OF MATHEMATICAL SCIENCES, NATIONAL SCIENCE FOUNDATION, 2415 EISENHOWER AVE, ALEXANDRIA, VA 22314, USA

E-mail address: mdouglas@nsf.gov

STATE UNIVERSITY OF NEW YORK AT ALBANY, 1400 WASHINGTON AVE, ES 110, ALBANY, NY 12222

E-mail address: czhong@albany.edu