Harnack Inequalities, Ergodicity, and Contractivity of Stochastic Reaction-Diffusion Equation in $L^p$

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Abstract. We derive Harnack inequalities for a stochastic reaction-diffusion equation with dissipative drift driven by additive rough noise in the $L^p(\Omega)$-space, for any $p \geq 2$, where $\Omega$ is a bounded, open subset of $\mathbb{R}^d$. These inequalities are used to study the ergodicity and contractivity of the corresponding Markov semigroup $(P_t)_{t \geq 0}$. The main ingredients of our method are a coupling by the change of measure and a uniform exponential moments' estimation $\sup_{t \geq 0} \mathbb{E}\exp(\epsilon \|\cdot\|_p)$ with some positive constant $\epsilon$ for the solution. Applying our results to the stochastic reaction-diffusion equation with a super-linear growth drift having negative leading coefficient, perturbed by a Lipschitz term, indicates that $(P_t)_{t \geq 0}$ possesses a unique and thus ergodic invariant measure and is supercontractive in $L^p$, which is independent of the Lipschitz term.

1. Introduction

We consider the reaction-diffusion equation

$$\frac{\partial X_t(\xi)}{\partial t} = \Delta X_t(\xi) + f(X_t(\xi)) + G \frac{\partial W_t(\xi)}{\partial t}, \quad (t, \xi) \in \mathbb{R}_+ \times \Omega,$$

(1.1)

driven by Gaussian noise. Here the homogeneous Dirichlet boundary condition on the bounded, open subset $\Omega$ of $\mathbb{R}^d$ is considered, the initial value $X_0 = x$ vanishes on the boundary $\partial \Omega$ of $\Omega$, the nonlinear drift function $f$ has polynomial growth and satisfies certain dissipative

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condition, and the random term $G \frac{\partial W_t(\xi)}{\partial t}$ denotes a Gaussian noise on a complete filtered probability space (see Section 2 for more details).

Let $x \in L^p = L^p(O), p \geq 2$, and denote by $(X^x_t)_{t \geq 0}$ a mild solution of the stochastic reaction-diffusion equation (1.1) with initial datum $X_0 = x$. Then $(X^x_t)_{t \geq 0}$ is a Markov process and it generates a Markov semigroup $(P_t)_{t \geq 0}$ defined as

$$(1.2) \quad P_t \phi(x) := \mathbb{E} \phi(X^x_t), \quad t \geq 0, \ x \in L^p, \ \phi \in B_b(L^p).$$

Our main concern in this paper is to investigate Harnack inequalities, ergodicity, and contractivity of the Markov semigroup $(P_t)_{t \geq 0}$ defined in (1.2) in the Banach space, $L^p$, for all $p \geq 2$.

The stochastic reaction-diffusion equation (1.1) has numerous applications in material sciences and chemical kinetics [ET89]. When $f(\xi) = \xi - \xi^3, \ \xi \in \mathbb{R}$, Eq. (1.1) is also called the stochastic Allen–Cahn equation or the stochastic Ginzburg–Landau equation. It is widely used in many fields, for example, the random interface models and stochastic mean curvature flows [Fun16]. There are many interesting and important properties for the solution of Eq. (1.1), which have been investigated in Hilbert setting. For example, the existence of invariant measures and ergodicity are studied in [BS, Hai02, Kaw05, WXX17], the large deviation principles are investigated in [BBP17, WRD12, XZ18], and sharp interface limits are derived in [KORVE07, Web10, Yip98]. See also [Bar15, BGJK, CH19, LQ20, LQ] and references therein for analysis in the numerical aspect.

In contrast to SPDEs in Hilbert spaces, only a few papers are treating the regularity, such as the ergodicity and contractivity, of the Markov semigroup $(P_t)_{t \geq 0}$ for SPDEs even with Lipschitz coefficients in Banach spaces. The authors in [BLSa10] studied invariant measures for SPDEs in M-type 2 Banach spaces, under Lipschitz and dissipativity conditions, driven by regular noise. Recently, their method was extended in [BK18] to an SPDE, arisen in stochastic finance, in a weighted $L^p$-space. We note that the authors in [BR16] showed the strong Feller property and irreducibility of $(P_t)_{t \geq 0}$, and thus the uniqueness of the invariant measure, if it exists, for the stochastic heat equation with white noise on $L^p(0, 1)$ with $p > 4$. See also [vN01] for the uniqueness of the invariant measure, if it exists, of the Ornstein–Uhlenbeck process (of the form (2.12)) using a pure analytical method.

For SPDEs with non-Lipschitz coefficients, the authors in [Cer03, Cer05] used the regularizing effect of $(P_t)_{t \geq 0}$ to derive the uniqueness of the invariant measure for Eq. (1.1) in the space $\mathcal{C} = \mathcal{C}(\bar{\mathcal{O}})$ of continuous functions on the closure $\bar{\mathcal{O}}$ of $\mathcal{O}$, taking advantage of the fact that a polynomial is uniformly continuous on bounded subsets of
Recently, [KN13] used the method developed in [KS02] to show the existence of a unique invariant measure on the space of continuous complex functions for the stochastic complex Ginzburg–Landau (Eq. (1.1) with \( f(u) = -i|u|^2u \), where \( i = \sqrt{-1} \)) driven by additive noise, relying on some good estimates of the solution in the Hilbert–Sobolev spaces \( \dot{H}^\beta \) with \( \beta > d/2 \), so that the noise is spatially regular enough.

To show the uniqueness of the invariant measure, these authors mainly formulated a Bismut–Elworthy–Li formula for the derivative \( DP_t \) to get a gradient estimate, which shows the strong Feller property of \( P_t \). Then the uniqueness of the invariant measure follows immediately by Khas’minskii and Doob theorems, provided the irreducibility holds. The difficulties for the study of the uniqueness of an invariant measure for SPDEs in Banach setting arise, mainly because the tools frequently used in the Hilbert space framework cannot be extended in a straightforward way to the Banach space setting [BR16].

In the past decade, Wang-type dimension-free inequalities have been a new and efficient tool to study diffusion semigroups. They were first introduced in [Wan97] for elliptic diffusion semigroups on non-compact Riemannian manifolds and in [Wan10] for heat semigroups on manifolds with boundary. Roughly speaking, such inequality for the Markov semigroup \( (P_t)_{t \geq 0} \) in a Banach space \( E \) is formulated as

\[
\Phi(P_t\phi(x)) \leq P_t(\Phi(\phi)(y)) \exp \Psi(t, x, y), \quad t > 0, \ x, y \in E, \ \phi \in B^+_b(E),
\]

where \( \Phi : [0, \infty) \to [0, \infty) \) is convex, \( \Psi \) is non-negative on \( [0, \infty) \times E \times E \) with \( \Psi(t, x, x) = 0 \) for all \( t > 0 \) and \( x \in E \), and \( B^+_b(E) \) denotes the family of all measurable and bounded, non-negative functions on \( E \).

There are two frequently used choices of \( \Phi \). One is given by a power function \( \Phi(\xi) = \xi^s, \ \xi \geq 0, \) for some \( s > 1 \), then (1.3) is reduces to

\[
(P_t\phi(x))^s \leq P_t\phi^s(y) \exp \Psi(t, x, y), \quad t > 0, \ x, y \in E, \ \phi \in B^+_b(E).
\]

Another is given by \( \Phi(\xi) = e^\xi, \ \xi \in \mathbb{R}, \) in which one may use \( \log \phi \) to replace \( \phi \), so that (1.3) becomes

\[
P_t\log \phi(x) \leq \log P_t\phi(y) + \Psi(t, x, y), \quad t > 0, \ x, y \in E, \ \phi \in B^+_b(E).
\]

The above inequalities (1.4) and (1.5) are called power-Harnack inequality and log-Harnack inequality, respectively. Both inequalities have been investigated extensively and applied to SODEs and SPDEs via coupling by the change of measure, see, e.g., [Kaw05, Wan07, WZ13, WZ14, Zha10], the monograph [Wan13], and references.
therein. Besides the gradient estimate which yields strong Feller property, these Harnack inequalities also have a lot of other applications. For example, they are used to study the contractivity of the Markov semigroup \((P_t)_{t \geq 0}\) in [DPRW09, Wan11, Wan17] and to derive almost surely strictly positivity of the solution for an SPDE in [Wan18].

For SPDEs with polynomial growth drift driven by rough noise in Hilbert setting, we are only aware [Kaw05, Xie19] investigating Harnack inequalities in a weighted \(L^2\)-space and a subspace of \(L^2\) consisting of all non-negative functions, respectively. When the noise is of trace-class, i.e., \(G\) appearing in Eq. (1.1) is a Hilbert–Schmidt operator, then the variational solution theory can be used and multiplicative noise can also be considered, as the solution is a semi-martingale so that Itô formula can be applied; see, e.g., [HLL20, Liu]. In the rougher white noise case, i.e., \(G\) coincides with the identity operator in \(L^2\), the variational solution would not exist; one needs to adopt the mild solution theory instead. Generally, the mild solution is not a semi-martingale, so that Itô formula is not available. We also note that, to derive Harnack inequalities for an SPDE with white noise, [WZ14] used finite-dimensional approximations to get a sequence of SODEs such that the arguments developed in [Wan11] for SODEs can be applied.

Our main idea to derive the Harnack inequalities (2.19) and (2.20) in the first main result, Theorem 2.1, for the Markov semigroup \((P_t)_{t \geq 0}\) defined by (1.2) of Eq. (1.1) on \((L^p)_{p \geq 2}\)-spaces, under the dissipativity condition (2.8) and the polynomial growth condition (2.9), are the construction of a coupling (see (3.10)) of the change of measure and a uniform pathwise estimate (see (3.18)) for this coupling. As a by-product, a gradient estimate and the uniqueness of the invariant measure for \((P_t)_{t \geq 0}\), if it exists, follows immediately. To the best of our knowledge, the Harnack inequalities (2.19) and (2.20) are the first two Harnack inequalities for SPDEs in Banach setting.

To show the existence of an invariant measure for Eq. (1.1) with super-linear growth and without strong dissipativity, e.g., \(q > 2\) and \(\lambda\) defined in (2.22) is non-positive, we derive a uniform estimate of \(\mu_n(\|\cdot\|_{q+p-2}^q)^2\) (see (4.6)) for a sequence of probability measures \((\mu_n)_{n \in \mathbb{N}_+}\) (defined in (4.4)). However, this is not strong enough to conclude the tightness of \((\mu_n)_{n \in \mathbb{N}_+}\), since the embedding \(L^{q+p-2} \subset L^p\) is not compact. To overcome this difficulty, we utilize a compact embedding (see (2.7)) by a Sobolev–Slobodeckii space. This forces us to derive a uniform estimate of \(\mu_n(\|\cdot\|_{\beta,p}^q)^2\) (see (4.7), with the \(\|\cdot\|_{\beta,p}\)-norm defined in (2.6)), where the aforementioned uniform estimate of \(\mu_n(\|\cdot\|_{q+p-2}^q)^2\) plays a key role. Then the tightness of \((\mu_n)_{n \in \mathbb{N}_+}\) follows and we get the
existence of an invariant measure for \((P_t)_{t \geq 0}\). In combination with the uniqueness result, we obtain the existence of a unique and thus ergodic invariant measure for \((P_t)_{t \geq 0}\) in the second main result, Theorem 2.2.

Finally, we establish a uniform exponential moments’ estimation of 
\[ \sup_{t \geq 0} \mathbb{E} \exp(\epsilon \| \cdot \|_p^p) \] with some positive constant \(\epsilon\) for the solution (see (3.6) in Lemma 3.2). This shows the Gaussian concentration property for the distribution of the solution, which will be used to derive the contractivity property of \((P_t)_{t \geq 0}\). It is worthy pointed out that such exponential moments’ estimation in existing literature has the form 
\[ \sup_{t \geq 0} \mathbb{E} \exp(\epsilon \| \cdot \|_2^2) \] on the underlying Hilbert space \((H, \| \cdot \|_H)\), while our estimate includes a higher index \(p\) when \(p > 2\). This enables us to obtain the supercontractivity of \((P_t)_{t \geq 0}\) in the last main result, Theorem 2.3.

The rest of the paper is organized as follows. Some preliminaries, assumptions, and main results are given in the next section. We derive a uniform pathwise estimate to get the existence of a unique global solution to Eq. (1.1), and establish an exponential moments’ estimation, in Section 3. In another part of Section 3, we construct the coupling and derive a uniform pathwise estimation for this coupling. These estimations ensure the well-posedness of the coupling and will be used in Section 4 to derive Harnack inequalities. The proofs of the main results, Theorems 2.1, 2.2, and 2.3, are given in the last section.

2. Preliminaries and Main Results

Let \(O \subset \mathbb{R}^d, d \geq 1\), be an open, bounded Lipschitz domain. Throughout \(p \geq 2\) is a fixed constant. Denote by \(L^p = L^p(O)\) the usual Lebesgue space on \(O\) with norm \(\| \cdot \|_p\), \(B_b(L^p)\) the class of bounded measurable functions on \(L^p\), and \(B^+_b(L^p)\) the set of positive functions in \(B_b(L^p)\). For a function \(\phi \in B_b(L^p)\), define
\[
\|\phi\|_\infty = \sup_{x \in \Omega} |\phi(x)|, \quad \|\nabla \phi\|_\infty = \sup_{x \in \Omega} |\nabla \phi|(x),
\] where \(|\nabla \phi|(x) = \limsup_{y \to x} \frac{|\phi(y) - \phi(x)|}{\|y - x\|}, x \in L^p\).

Let \((U, \| \cdot \|_U, \langle \cdot, \cdot \rangle_U)\) be a separable Hilbert space and \((W_t)_{t \geq 0}\) be a \(U\)-valued cylindrical Wiener process with respect to a complete filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) satisfying the usual condition, i.e., there exists an orthonormal basis \((e_k)_{k=1}^\infty\) of \(U\) and a family of independent standard real-valued Brownian motions \((\beta_k)_{k=1}^\infty\) such that
\[
W_t = \sum_{k=1}^\infty e_k \beta_k(t), \quad t \geq 0.
\]
Define by $F$ the Nemytskii operator associated with $f$, i.e.,
\begin{equation}
F(x)(\xi) = f(x(\xi)), \quad x \in L^p, \; \xi \in \mathcal{O}.
\end{equation}

Then Eq. (1.1) can be rewritten as the stochastic evolution equation
\begin{equation}
dX_t = (AX_t + F(X_t))dt + GdW_t,
\end{equation}
with the initial datum $X_0 = x \in L^p$, where $A$ is the Dirichlet Laplacian operator on $L^p$, $F$ is the Nemytskii operator defined in (2.2) associated with $f$, and $W$ is a $U$-valued cylindrical Wiener process given in (2.1).

It is well-known that the Dirichlet Laplacian operator $A$ in Eq. (2.3) generates an analytic $C_0$-semigroup in $L^p$, denoted by $(S_t^p)_{t \geq 0}$, for all $p \geq 2$. These semigroups are consistent, in the sense that $S_t^{p_1}.x = S_t^{p_2}x$, for all $t \geq 0$, $x \in L^{p_1} \cap L^{p_2}$, and $p_1, p_2 \geq 2$. Then we shall denote all $(S_t^p)_{t \geq 0}, p \geq 2$, by $(S_t)_{t \geq 0}$, if there is no confusion. It is known that the following ultracontractivity of $(S_t)_{t \geq 0}$ holds:
\begin{equation}
\|S_t u\|_{\beta,r} \leq C e^{-\lambda_1 t} t^{-|\beta|\frac{d}{2} + \frac{dr-q}{2r}} \|u\|_s, \quad t > 0, \; u \in L^s,
\end{equation}
for all $\beta \geq 0$ and $1 \leq s \leq r \leq \infty$, where $\lambda_1 > 0$ is the first eigenvalue of $A$. For convenience, here and what follows, we frequently use the generic constant $C$, which may be different in each appearance. When $p = 2$, the following Poincaré inequality holds:
\begin{equation}
\|\nabla u\|_2 \geq \lambda_1 \|u\|_2, \quad u \in H^1_0,
\end{equation}
where $H^1_0 = H^1_0(\mathcal{O})$ denotes the space of weakly differentiable functions, vanishing on the boundary $\partial \mathcal{O}$, whose derivatives belong to $L^2$. In Section 4, we also need the Sobolev–Slobodeckij space $W^{\beta,p}$ with $\beta \in (0,1)$, and $W^{\beta,p}_0 := \{ \phi \in W^{\beta,p} : \phi|_{\partial \mathcal{O}} = 0 \}$, whose norm is defined by
\begin{equation}
\|\phi\|_{\beta,p} := \left( \|\phi\|_{\beta}^p + \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|\phi(\xi) - \phi(\eta)|^p}{|\xi - \eta|^{d+\beta p}} \mathrm{d}\xi \mathrm{d}\eta \right)^{\frac{1}{p}}.
\end{equation}

It is known that the following compact embedding holds true (see, e.g., [Agr15, Theorem 2.3.4]):
\begin{equation}
W^{\beta,p}_0 \subset L^p, \quad \beta \in (0, d/p).
\end{equation}

### 2.1. Main Assumptions and Results.
Let us give the following assumptions on the data of Eq. (1.1). We begin with the conditions on the drift function $f$.

**Assumption 2.1.** There exist constants $L_f \in \mathbb{R}$, $\theta, L'_f > 0$, and $q \geq 2$ such that for all $\xi, \eta \in \mathbb{R}$,
\begin{align}
(f(\xi) - f(\eta))(\xi - \eta) &\leq L_f |\xi - \eta|^2 - \theta |\xi - \eta|^q, \quad (2.8) \\
|f(\xi) - f(\eta)| &\leq L'_f (1 + |\xi|^{q-2} + |\eta|^{q-2}) |\xi - \eta|. \quad (2.9)
\end{align}
Remark 2.1. A motivating example of $f$ such that Assumption 2.1 holds true is a polynomial of odd order $q - 1$ with a negative leading coefficient (for the stochastic Allen–Cahn equation, $q = 4$), perturbed with a Lipschitz continuous function; see, e.g., [DPZ14, Example 7.8].

Remark 2.2. It follows from (2.8) that the Nemytskii operator $F$, defined in (2.2), associated with $f$ is a well-defined, continuous operator from $L^p$ to $L^{p'}$, with $p' = p/(p - 1)$, such that

$$
\langle F(u) - F(v), u - v \rangle \leq L_f \|u - v\|^2 - \theta \|u - v\|^{p'}_p, \quad u, v \in L^p,
$$

where $\langle \cdot, \cdot \rangle$ is the dualization between $L^{p'}$ and $L^p$ with respect to $L^2$. Then it is clear that $\langle u, v \rangle = (u, v)$ for any $u \in L^2$ and $v \in L^p$.

To perform the assumption of the noise part, as we consider the Banach spaces $(L^p)_{p \geq 2}$, let us first recall the required materials of stochastic calculus in Banach spaces, especially the martingale-type (M-type) $2$ spaces and the $\gamma$-radonifying operators. It is known that the stochastic calculations in Banach space depend heavily on the geometric structure of the underlying spaces.

We first recall the definitions of the M-type for a Banach space. Let $E$ be a Banach space and $(\epsilon_n)_{n \in \mathbb{N}}$ be a Rademacher sequence in a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, i.e., a sequence of independent random variables taking the values $\pm 1$ with the equal probability $1/2$. Then $E$ is called of M-type 2 if there exists a constant $\tau_M \geq 1$ such that

$$
\|f_N\|_{L^2(\Omega'; E)} \leq \tau_M \left( \|f_0\|_{L^2(\Omega'; E)}^2 + \sum_{n=1}^{N} \|f_n - f_{n-1}\|_{L^2(\Omega'; E)}^2 \right)^{1/2}
$$

for all $E$-valued square integrable martingales $(f_n)_{n=0}^N$.

For stochastic calculus in Banach spaces, the so-called $\gamma$-radonifying operators play the important roles instead of Hilbert–Schmidt operators in Hilbert setting. Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence of independent $\mathcal{N}(0, 1)$-random variables in a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. Denote by $\mathcal{L}(U, E)$ the space of linear operators from $U$ to $E$. An operator $R \in \mathcal{L}(U, E)$ is called $\gamma$-radonifying if there exists an orthonormal basis $(h_n)_{n \in \mathbb{N}}$ of $U$ such that the Gaussian series $\sum_{n \in \mathbb{N}} \gamma_n Rh_n$ converges in $L^2(\Omega'; E)$. In this situation, it is known that the number

$$
\|R\|_{\gamma(U, E)} := \left\| \sum_{n \in \mathbb{N}} \gamma_n Rh_n \right\|_{L^2(\Omega'; E)}
$$

does not depend on the sequence $(\gamma_n)_{n \in \mathbb{N}}$ and the basis $(h_n)_{n \in \mathbb{N}}$, and it defines a norm on the space $\gamma(U, E)$ of all $\gamma$-radonifying operators from $U$ to $E$. In particular, if $E$ reduces to a Hilbert space, then
\( \gamma(U, E) \) coincides with \( \mathcal{L}_2(U, E) \) isometrically, where \( \mathcal{L}_2(U, E) \) denotes the space of all Hilbert–Schmidt operators from \( U \) to \( E \).

Let \( T > 0 \) and \( (E, \| \cdot \|_E) \) be an M-type 2 space. For any adapted process \( \Phi \in L^s(\Omega; L^2(0, T; \gamma(U, E))) \) with \( s \geq 2 \), the following one-sided Burkholder inequality for the \( E \)-valued stochastic integral
\[
\mathbb{E} \sup_{t \in [0, T]} \left\| \int_0^t \Phi_r dW_r \right\|_E^s \leq C \mathbb{E} \left( \int_0^T \| \Phi_r \|_{\gamma(U, E)}^2 dt \right)^{\frac{s}{2}}
\] (2.11)

Coming back to our case, it is known that \( (L^p)_{p \geq 2} \) are M-type 2 spaces. For more details about definitions and properties of M-type 2 spaces and \( \gamma \)-radonifying operators, we refer to [Brz97].

With these preliminaries, now we can give our assumptions on \( G \) appearing in Eq. (2.3). Let \( \mathcal{L}_S(U, L^p) \) be the set of all densely defined closed linear operators \( (L, \text{Dom}(L)) \) from \( U \) to \( L^p \) such that for every \( s > 0 \), \( S_s L \) extends to a unique operator in \( \gamma(U, L^p) \), which is again denoted by \( S_s L \). Assume that \( G \in \mathcal{L}_S(U, L^p) \) and denote by \( W_A \) the stochastic convolution, also known as the Ornstein–Uhlenbeck process, i.e.,
\[
W_A(t) = \int_0^t S_{t-r} G dW_r, \quad t \geq 0.
\] (2.12)

It is clear that \( W_A \) is the mild solution of the linear equation
\[
dZ_t = AZ_t dt + GdW_t, \quad Z_0 = 0,
\] (2.13)
i.e., Eq. (2.3) with \( F = 0 \), with vanishing initial datum. We always assume that, for each fixed \( T > 0 \), \( \int_0^T \| S_t G \|_{\gamma(U, L^p)}^2 dt < \infty \), so it follows from the Burkholder inequality (2.11) that
\[
\mathbb{E} \| W_A(T) \|_p^2 \leq C \int_0^T \| S_t G \|_{\gamma(U, L^p)}^2 dt < \infty,
\]
which shows that \( W_A(T) \) possesses the bounded second moment in \( L^p \). Moreover, we perform the following stronger assumption, as we shall handle the polynomial drift function \( f \) which satisfies Assumption 2.1.

**Assumption 2.2.** \( W_A \) has a continuous version in \( L^{q+p-2} \) such that
\[
\sup_{t \geq 0} \mathbb{E} \| W_A(t) \|_{q+p-2}^2 < \infty,
\] (2.14)
and for sufficiently large \( \epsilon_0 > 0 \),
\[
\sup_{t \geq 0} \mathbb{E} \exp(\epsilon_0 \| W_A(t) \|_{q+p-2}^2) < \infty.
\] (2.15)
The above Assumption 2.2 on the Ornstein–Uhlenbeck process $W_A$ (2.12) is necessary for the study of the well-posedness, Harnack inequalities, ergodicity, and contractivity, in Sections 3 and 4. To indicate that Assumption 2.2 is natural, we remark that the condition (2.14) on the Ornstein–Uhlenbeck process can use the Burkholder inequality (2.16) which coincides with the identity operator on $L^2$. In various of applications, even when $G$ is an unbounded operator.

**Example 2.1.** Consider Eq. (2.3) driven by 1D rough noise, e.g., $\mathcal{O} = (0, 1), U = L^2(0, 1)$ (with a uniformly bounded orthonormal basis $(e_k = \sqrt{2}\sin(k\pi x))_{k \in \mathbb{N}_+}$), and $G : L^2 \to L^p$ is given by

$$Gx = \begin{cases} x, & x \in L^p; \\ 0, & x \in L^2 \setminus L^p, \end{cases}$$

which coincides with the identity operator on $L^2$ if $p = 2$. Then one can use the Burkholder inequality (2.11) to show that $W_A$ belongs to $C([0, \infty); L^s(\Omega; L^r))$ for any $s, r \geq 2$, following an argument used in [LQ20, (2.17) in Lemma 2.2]. Indeed, for any $t \geq 0$,

$$E\|W_A(t)\|^s_r = E\left\|\int_0^t S_{t-r}GdW_r\right\|^s_r \leq C\left(\int_0^t \|S_rG\|_{\gamma(L^2, L^r)}^2 dr\right)^{\frac{s}{2}}.$$ 

We note that $L^r, r \geq 2$, is a Banach function space with finite cotype, so $\Phi \in \gamma(L^2, L^r)$ if and only if $(\sum_{k=1}^{\infty} (\Phi e_k)^2)^{1/2}$ belongs to $L^r$ and there exist a constant $C > 0$ such that (see [vNVW08, Lemma 2.1])

$$\frac{1}{C}\left\|\sum_{k=1}^{\infty} (\Phi e_k)^2\right\|_{r/2} \leq \|\Phi\|^2_{\gamma(L^2, L^r)} \leq C\left\|\sum_{k=1}^{\infty} (\Phi e_k)^2\right\|_{r/2}, \quad \Phi \in \gamma(L^2, L^r).$$

It follows that

$$\sup_{t \geq 0} E\|W_A(t)\|^s_r \leq C\left(\int_0^\infty \left\|\sum_{k=1}^{\infty} e^{-2\lambda_k r} e_k^2\right\|_{r/2} dt\right)^{\frac{s}{2}} \leq C\left(\sum_{k=1}^{\infty} \|e_k\|^2_r \left(\int_0^\infty e^{-2\lambda_k r} dt\right)^{\frac{s}{2}} \leq C\left(\sum_{k=1}^{\infty} \frac{1}{2\lambda_k}\right)^{\frac{s}{2}} = \left(\frac{1}{12}\right)^{\frac{s}{2}} < \infty,$$

where $\lambda_k = \pi^2 k^2, k \in \mathbb{N}_+$. This shows (2.14) with $s = r = q + p - 2$. Moreover, the above estimate and the dominated convergence theorem yield (2.15) with $s = r = q + p - 2$ and any $\epsilon_0 > 0$:

$$\sup_{t \geq 0} E\exp(\epsilon_0\|W_A(t)\|^s_r) = \sup_{t \geq 0} \sum_{k=0}^{\infty} \frac{\epsilon_0 k^2}{k!} E\|W_A(t)\|^k_s = C \exp\left(\frac{\epsilon_0}{\sqrt{12}}\right).$$
Remark 2.3. It should be pointed out that the operator $G$ defined in (2.16), for $p > 2$, is unbounded. One can also give other examples, e.g., $G$ coincides with fractional Laplacian operators with certain indexes, such that the conditions (2.14) and (2.15) hold.

In the derivation of the existence of an invariant measure in Theorem 4.2 when $q > 2$, we need the following Sobolev regularity of the Ornstein–Uhlenbeck process $(W_A(t))_{t \geq 0}$.

Assumption 2.3. There exists a sufficiently small $\beta_0 > 0$ such that

$$\sup_{t \geq 0} \mathbb{E} \|W_A(t)\|_{\beta_0,p} < \infty.$$  

Example 2.2. As in Example 2.1,

$$\sup_{t \geq 0} \mathbb{E} \|W_A(t)\|_{\beta_0,r}^s \leq C \left( \int_0^\infty \|(-A)^{\frac{\beta_0}{2}} S_t G\|_{\gamma(L^2, L^r)}^2 dr \right)^{\frac{s}{2}} \leq C \left( \int_0^\infty \left| \sum_{k=1}^\infty \lambda_k^{\beta_0} \|e^{-2\lambda_k t} e_k^2\|_{r/2} \right|^2 dt \right)^{\frac{s}{2}} \leq C \left( \sum_{k=1}^\infty \frac{1}{\lambda_k^{1-\beta_0}} \right)^{\frac{s}{2}},$$

which is convergent if and only if $\beta_0 < 1/2$. This shows (2.17) with $r = p$ and $s = 1$ for any $\beta_0 \in (0, 1/2)$.

To derive Wang-type Harnack inequalities, ergodicity, and supercontractivity for $(P_t)_{t \geq 0}$ in Section 4, we need the following standard elliptic condition.

Assumption 2.4. $GG^*$ is invertible, with inverse $(GG^*)^{-1}$, such that $G^{-1} := G^*(GG^*)^{-1}$ is a bounded linear operator from $L^p$ to $U$:

$$\|G^{-1}\|_\infty := \|G^{-1}\|_{\mathcal{L}(L^p, U)} < \infty.$$  

Remark 2.4. The operator $G : L^2 \rightarrow L^p$ defined in (2.16) is invertible with inverse $G^{-1} : L^p \rightarrow L^2$ given by $G^{-1}x = x$, $x \in L^p$. It is clear that $G^{-1}$ is bounded through the continuous embedding $L^p \subset L^2$.

2.2. Main results. Now we are in the position to present our main results. Let us first recall some definitions. The Markov semigroup $(P_t)_{t \geq 0}$ defined in (1.2) is called strong Feller if $P_T \phi \in C_b(L^p)$ for any $T > 0$ and $\phi \in \mathcal{B}_b(L^p)$. A probability measure $\mu$ on $L^p$ is said to be an invariant measure of $(P_t)_{t \geq 0}$ or of Eq. (2.3), if

$$\int_{L^p} P_t \phi(x) \mu(dx) = \int_{L^p} \phi(x) \mu(dx), \quad \phi \in \mathcal{B}_b(L^p), \ t \geq 0.$$
If \((P_t)_{t \geq 0}\) has an invariant measure \(\mu\), then \((P_t)_{t \geq 0}\) is called hypercontractive if \(\|P_t\|_{2 \to 4} = 1\) for some \(t > 0\), where \(\cdot\|_{r \to s}\) denotes the operator norm from \(L^r(\mu)\) to \(L^s(\mu)\): \(\|P_t\|_{r \to s} = \sup\{\|P_t \phi\|_{L^s(\mu)} : \mu(\phi^r) \leq 1\}\), and \((P_t)_{t \geq 0}\) is called supercontractive if \(\|P_t\|_{2 \to 4} < \infty\) for all \(t > 0\).

Our first main result is the following log-Harnack inequality and power-Harnack inequality, from which the uniqueness of the invariant measure, if it exists, for \((P_t)_{t \geq 0}\) follows.

**Theorem 2.1.** Let Assumptions 2.1 and 2.4, and the condition (2.14) hold. For any \(T > 0, s > 1, x, y \in L^p\), and \(\phi \in B^+_b(L^p)\),

\[
\tag{2.19}
P_T \log \phi(y) \leq \log P_T \phi(x) + \frac{\lambda \|G^{-1}\|_\infty^2}{e^{2\lambda T} - 1} \|x - y\|_p^2,
\]

\[
\tag{2.20}
(P_T \phi(y))^s \leq P_T \phi^s(x) \exp \left(\frac{s \lambda \|G^{-1}\|_\infty^2 \|x - y\|_p^2}{(s - 1)(e^{2\lambda T} - 1)}\right).
\]

Consequently, \((P_t)_{t \geq 0}\) has at most one invariant measure.

The next main result is the existence of an invariant measure for \((P_t)_{t \geq 0}\). In combination with the uniqueness result in Theorem 2.1, \((P_t)_{t \geq 0}\) possesses exactly one ergodic invariant measure.

**Theorem 2.2.** Let Assumption 2.1 and the condition (2.14) hold. Assume that \(q > 2\) such that

\[
\tag{2.21}
d < \frac{2p(q + p - 2)}{(p - 1)(q - 2)}
\]

and Assumption 2.3 holds, or \(\lambda\) defined by

\[
\tag{2.22}
\lambda := -L_f + \theta \chi_{q=2,p\neq2} + \lambda_1 \chi_{q\neq2,p=2} + (\lambda_1 + \theta) \chi_{q=p=2},
\]

is positive. Then \((P_t)_{t \geq 0}\) has an invariant measure \(\mu\) with full support on \(L^p\) such that \(\mu(\|\cdot\|_{q+p-2}) < \infty\). Assume furthermore that Assumption 2.4 holds, then \(\mu\) is the unique invariant measure of \((P_t)_{t \geq 0}\).

The last main result is the contractivity of \((P_t)_{t \geq 0}\).

**Theorem 2.3.** Let Assumptions 2.1 and 2.4, and the condition (2.15) hold for a sufficiently large \(\epsilon_0 > 0\).

(1) Assume that \(\lambda\) defined in (2.22) is positive, then \((P_t)_{t \geq 0}\) is hypercontractive. Consequently, \((P_t)_{t \geq 0}\) is compact on \(L^2(\mu)\) for large \(t > 0\), and there exist constants \(C, \lambda_0 > 0\) such that for all \(t \geq 0\),

\[
\tag{2.23}
\mu(P_t \phi \log P_t \phi) \leq Ce^{-\lambda_0 t} \mu(\phi \log \phi), \quad f \in B^+_b(L^p), \quad \mu(\phi) = 1,
\]

\[
\tag{2.24}
\|P_t \phi - \mu(\phi)\|_{L^2(\mu)} \leq Ce^{-\lambda_0 t} \|\phi - \mu(\phi)\|_{L^2(\mu)}, \quad \phi \in L^2(\mu).
\]
(2) Let $q > 2$ and Assumptions 2.1 and 2.4 hold. Assume that (2.15) holds for all $\epsilon_0 > 0$ if $p = 2$ or holds for a sufficiently large $\epsilon_0 > 0$ if $p > 2$, then $(P_t)_{t \geq 0}$ is supercontractive.

3. Coupling and Moments’ Estimations

The main aims of this section are to show the existence of a unique global solution to Eq. (1.1) and to construct a well-defined coupling process for this solution process. We also derive several uniform a priori estimates on algebraic and exponential moments of these two processes, which will be used in Section 4 to derive Wang-type Harnack inequalities, ergodicity, and contractivity of $(P_t)_{t \geq 0}$.

3.1. Well-posedness and moments’ estimations. To show the well-posedness of Eq. (2.3), let us recall that an $L^p$-valued process $(X_t)_{t \in [0,T]}$ is called a mild solution of Eq. (2.3) with initial datum $X_0 = x$ if $\mathbb{P}$-a.s.

\begin{equation}
\text{d}X_t = S_t x + \int_0^t S_{t-r} F(X_r) \text{d}r + W_A(t), \quad t \in [0,T].
\end{equation}

From Remark 2.2, the deterministic convolution in Eq. (3.1) makes sense. Define $Z = X - W_A$. It is clear that $X$ is a mild solution if and only if $Z$ is a mild solution of the random PDE

\begin{equation}
\partial_t Z_t = \Delta Z_t + F(Z_t + W_A(t)), \quad Z_0 = x.
\end{equation}

The following results show the existence of a unique mild solution of Eq. (2.3) which is a Markov process and depends on the initial data continuously in a pathwise sense.

**Lemma 3.1.** Let $T > 0$, $x \in L^p$, Assumption 2.1, and the condition 2.14 hold. Eq. (1.1) with initial datum $X_0 = x$ possesses a unique mild solution, in $C([0,T]; L^p) \cap L^{q+p-2}(0,T; L^{q+p-2}) \mathbb{P}$-a.s., which is a Markov process. Moreover, there exists a constant $\lambda$ defined in (2.22) such that

\begin{equation}
\|X_t^x - X_t^y\|_p \leq e^{-\lambda t} \|x - y\|_p, \quad t \geq 0, \quad x, y \in L^p.
\end{equation}

**Proof.** From (2.9), $f$ is locally Lipschitz continuous, so it is clear that both Eq. (3.2) with $Z = X - W_A$ and Eq. (2.3) exist local solutions on $[0, T_0]$ for some $T_0 \in (0,T]$. To extend this local solution to the whole time interval $[0,T]$, it suffices to give a priori uniform estimations for $Z$ and $X$. 
Testing $p|Z_t|^{p-2}Z_t$ on Eq. (3.2) with $t \in [0, T_0)$, and using the conditions (2.8)-(2.9) and Young inequality, we obtain

$$
\partial_t \|Z_t\|_p^p + p(p-1) \int_\Omega |Z_t|^{p-2} |\nabla Z_t|^2 d\xi
= p\langle |Z_t|^{p-2}Z_t, F(Z_t + W_A(t)) - F(W_A(t)) \rangle + p\langle |Z_t|^{p-2}Z_t, F(W_A(t)) \rangle
\leq pL_f \|Z_t\|_p^p - p\theta \|Z_t\|_{q+p-2}^{q+p-2} + p\langle |Z_t|^{p-2}Z_t, F(W_A(t)) \rangle
\leq pL_f \|Z_t\|_p^p - p\theta \|Z_t\|_{q+p-2}^{q+p-2} + C \|F(W_A(t))\|_{\frac{q}{q+p-2}}^{\frac{q}{q+p-2}}
\leq C(1 + \|W_A(t)\|_{q+p-2}^{q+p-2}) + pL_f \|Z_t\|_p^p - p\theta_1 \|Z_t\|_{q+p-2}^{q+p-2},
$$

where $\theta_1$ could be chosen as any positive number which is smaller than $\theta$. This yields

$$
(3.4)
$$

$$
\|Z_t\|_p^p + p\theta_1 \int_0^t \|Z_r\|_{q+p-2}^{q+p-2}dr + p(p-1) \int_0^t \|Z_t|^{p-2} |\nabla Z_t|^2 d\xi dr
\leq \|x\|_p^p + C \int_0^t (1 + \|W_A(r)\|_{q+p-2}^{q+p-2}) dr + pL_f \int_0^t \|Z_t\|_p^p dr.
$$

Using Grönwall inequality, we obtain

$$
\|Z_t\|_p^p + p\theta_1 \int_0^t \|Z_r\|_{q+p-2}^{q+p-2}dr + p(p-1) \int_0^t \|Z_t|^{p-2} |\nabla Z_t|^2 d\xi dr
\leq e^{pL_f} \left( \|x\|_p^p + C \int_0^t (1 + \|W_A(r)\|_{q+p-2}^{q+p-2}) dr \right).
$$

The above uniform estimate, in combination with the regularity (2.14) of $W_A$ in Assumption 2.2, implies the global existence of a mild solution $Z$ to Eq. (3.2) on $[0, T]$ in $C([0, T]; L^p) \cap L^{q+p-2}(0, T; L^{q+p-2})$ $\mathbb{P}$-a.s. Taking into account the relation $X = Z + W_A$ and the condition (2.14), we obtain a global mild solution $X$ to Eq. (2.3).

To show the continuous dependence (3.3), let us note that

$$
\partial_t (X_t^x - X_t^y) = A(X_t^x - X_t^y) + F(X_t) - F(Y_t).
$$

Testing $p|X_t^x - X_t^y|^{p-2}(X_t^x - X_t^y)$ on the above equation, using integration by parts formula, and applying the condition (2.8), we obtain

$$
\|X_t^x - X_t^y\|_p^p + p\theta \int_0^t \|X_r^x - X_r^y\|_{q+p-2}^{q+p-2} dr
+ p(p-1) \int_0^t \int_\Omega |X_r^x - X_r^y|^{p-2} |\nabla (X_r^x - X_r^y)|^2 d\xi dr
$$
We conclude (3.3) with \( \lambda = -L_f \) by Grönwall inequality. When \( q = 2 \), then (3.3) holds with \( \lambda = \theta - L_f \), as one can subtract the first integral on the left-hand side of the above inequality; while \( p = 2 \), then using the Poincaré inequality (2.5) yields (3.3) with \( \lambda = \lambda_1 - L_f \). Similarly, (3.3) holds with \( \lambda = \lambda_1 + \theta - L_f \) when \( q = p = 2 \). These statements show (3.3) with \( \lambda \) given by (2.22).

The pathwise continuous dependence clearly implies the uniqueness of the solution to Eq. (2.3). One can also show the Markov property for this solution using a standard method, see, e.g., [DPZ14, Theorem 9.21]. This completes the proof.

**Remark 3.1.** The pathwise estimate (3.3) immediately yields the following contraction-type estimate between any two solutions in \( r \)-Wasserstein distance for any \( r \geq 1 \):

\[
W_r(\mu^1_t, \mu^2_t) := \inf(E\|X^1_t - X^2_t\|_r^r)^{\frac{1}{r}} \leq e^{-\lambda t}W_r(\mu^1_0, \mu^2_0), \quad t \geq 0,
\]

where \( (\mu^1_0)_{i=1}^2 \) are two measures on \( L^p \), \( (X^i_t)_{i=1}^2 \) are the solutions to Eq. (2.3) starting from \( (x^i_0)_{i=1}^2 \) of laws \( (\mu^i_0)_{i=1}^2 \), and the infimum runs over all random variables \( (X^i_t)_{i=1}^2 \) with laws \( (\mu^i_t)_{i=1}^2, \ t \geq 0 \), respectively. Similar contraction-type estimate in \( 2 \)-Wasserstein distance on \( \mathbb{R}^d \) had been investigated in [BGG12].

Under the stronger condition (2.15) instead of (2.14), we have the following exponential moments’ estimation, also known as exponential integrability, which is related to Fernique theorem and logarithmic Sobolev inequality [AMS94, BG99]. It will be used to show the Gaussian concentration property for the invariant measure of \((P_t)_{t \geq 0}\), if it exists, in Section 4.

**Lemma 3.2.** Let \( x \in L^p \) and Assumptions 2.1 and (2.2) hold for sufficiently large constant \( \epsilon_0 > 0 \). Assume that \( q > 2 \) or \( \lambda \) defined in (2.22) is positive, then there exists a constant \( \epsilon_1 > 0 \) such that

\[
\sup_{t \geq 0} E \exp(\epsilon_1 \|X^x_t\|_p^p) < \infty.
\]

If (2.15) is valid for all \( \epsilon_0 > 0 \), then (3.6) holds for all \( \epsilon_1 > 0 \).

**Proof.** By (3.4) and the chain rule, we have

\[
\partial_t \exp(\epsilon \|Z_t\|_p^p) = \epsilon \exp(\epsilon \|Z_t\|_p^p) \partial_t \|Z_t\|_p^p
\]

\[
\leq \epsilon \exp(\epsilon \|Z_t\|_p^p) \left( -p(p-1) \int_\partial \|Z_t\|^{p-2} \|
\nabla Z_t\|^2 d\xi
\right)
\]

\[
\|Z_t\|_p^p + pL_f \int_0^t \|X^x_r - X^y_r\|_p^p dr.
\]

(3.5)
+ C(1 + \|W_A(t)\|_{q+p-2}^{q+p-2}) + pL_f \|Z_t\|_p^p - p\theta_1 \|Z_t\|_{q+p-2}^{q+p-2},

where \(\theta_1\) could be chosen as any positive number smaller than \(\theta\). When \(q > 2\), by the continuous embedding \(L^{q+p-2} \subset L^p\) and Young inequality, we have

\[-p(p-1) \int_{\mathcal{O}} |Z_t|^{p-2} |\nabla Z_t|^2 d\xi + pL_f \|Z_t\|_p^p - p\theta_1 \|Z_t\|_{q+p-2}^{q+p-2} \leq C - \theta_2 \|Z_t\|_p^p,

for some positive constant \(\theta_2\). This is also true when \(q = 2\) and \(L_f < \theta\), with \(\theta_2 = \theta_1 - L_f > 0\), as one can choose \(\theta_1\) as close as possible to \(\theta\).

When \(q = p = 2\), the Poincaré inequality yields that

\[-p(p-1) \int_{\mathcal{O}} |Z_t|^{p-2} |\nabla Z_t|^2 d\xi + pL_f \|Z_t\|_p^p - p\theta_1 \|Z_t\|_{q+p-2}^{q+p-2} \leq -2(\lambda_1 + \theta_1 - L_f) \|Z_t\|_p^p.

Therefore, when \(q > 2\) or \(\lambda > 0\), there exists \(\theta_3 > 0\) such that

\[\partial_t \exp(\epsilon \|Z_t\|_p^p) \leq \epsilon \exp(\epsilon \|Z_t\|_p^p) [C(1 + \|W_A(t)\|_{q+p-2}^{q+p-2}) - \theta_3 \|Z_t\|_p^p] + C_3\]

Using the elementary inequality: for any constants \(\epsilon, C_1, C_2 > 0\), there exists a constant \(C_3\) such that

\[\epsilon e^{\epsilon x} (C_1 - C_2 x) \leq C_3 - e^{\epsilon x}, \quad x \geq 0,

we obtain

\[\partial_t \exp(\epsilon \|Z_t\|_p^p) \leq C_{\epsilon,\theta} \exp(C_{\epsilon,\theta} \|W_A(t)\|_{q+p-2}^{q+p-2}) - \exp(\epsilon \|Z_t\|_p^p),

and thus

\[\exp(\epsilon \|Z_t\|_p^p) \leq C_{\epsilon,\theta} \exp(C_{\epsilon,\theta} \|W_A(t)\|_{q+p-2}^{q+p-2}).\]

It follows from the relation \(Z = W_A\) and the above inequality that

\[\mathbb{E} \exp(\epsilon \|X_t\|_p^p) \leq \sqrt{\mathbb{E} \exp(\epsilon 2p \|Z_t\|_p^p)} \cdot \sqrt{\mathbb{E} \exp(\epsilon 2p \|W_A(t)\|_p^p)} \leq C_{\epsilon,\beta,p} \mathbb{E} \exp(C_{\epsilon,\beta,p} \|W_A(t)\|_{q+p-2}^{q+p-2}).\]

Using the condition (2.15), we get (3.6). If (2.15) is valid for all \(\epsilon_0 > 0\), then the above estimate yields (3.6) for any \(\epsilon_1 > 0\).

We note that the restrictions on the numbers \(q\) and \(p\), the dissipative constants \(L_f\) and \(\theta\) in (2.8)-(2.9), and the Poincaré constant \(\lambda_1\) in (2.5), appearing in Lemma 3.2, will be frequently used in the rest of the present paper.

Remark 3.2. When \(q > 2\), i.e., the dissipative drift function \(f\) in (2.8) and (2.9) has super-linear growth, the Gaussian concentration property (3.6) for Eq. (2.3) is always valid. Otherwise, one needs
to ensure that the constant $\lambda$, given in (2.22), is positive. This corresponds to the cases $q = 2$, $p > 2$, and $\theta > L_f$, or $q = 2$, $p = 2$, and $\lambda_1 + \theta > L_f$. In particular, in the latter case with $p = 2$, one can utilize the dissipativity of the Dirichlet Laplacian operator in the Hilbert space $L^2$ to reduce the requirement of the dissipative constants in (2.8): $L_f < \lambda_1 + \theta$. If $p \neq 2$, the term $\int_0^t \int_0^r \left| Z_t \right|^{p-2} \left| \nabla Z_t \right|^2 d\xi dr$, involving the Dirichlet Laplacian operator, has no dissipativity, so that the stronger restriction $L_f < \theta$ has to be imposed.

3.2. Construction of coupling and moments’ estimations.

Our main idea is the construction of a coupling by the change of measure. Let $T > 0$ be fixed throughout the rest of Section 3 and set

\begin{equation}
\gamma_t := \int_0^{T-t} e^{2\lambda r} dr = \frac{e^{2\lambda(T-t)} - 1}{2\lambda}, \quad t \in [0, T],
\end{equation}

where $\lambda$ is given in (2.22). For convention, if $\lambda = 0$, we set $\frac{e^{2\lambda t} - 1}{2\lambda} = t$ for $t \in [0, T]$. Then $\gamma$ is smooth, strictly positive, and strictly decreasing on $[0, T]$ (with $\gamma_T = 0$) such that

\begin{equation}
\gamma_t' + 2\lambda \gamma_t + 1 = 0.
\end{equation}

Moreover, the integral of $(\gamma_t^{-1})_{t \in [0, T]}$ on $[0, T]$ diverges:

\begin{equation}
\int_0^T \frac{1}{\gamma_t} dt = \infty.
\end{equation}

Now we can define the coupling $Y$ of $X$ as the mild solution of the coupling equation

\begin{equation}
\text{d}Y_t = ((AX_t + F(Y_t) + \gamma_t^{-1}(X_t - Y_t)) dt + GdW_t,
\end{equation}

with an initial datum $Y_0 = y \in L^p$. Since the additional drift term $\gamma_t^{-1}(X_t - Y_t)$ is Lipschitz continuous for each fixed $t \in [0, T)$ and $\omega \in \Omega$, one can use similar arguments in Lemma 3.1 to show that the coupling $(X_t, Y_t)$ is a well-defined continuous process on $[0, T)$.

**Remark 3.3.** For $t \in [0, T)$, $\gamma_t^{-1}$ is continuous and thus integrable. So one can use the arguments in Lemma 3.1 to extend the local solution to $[0, T)$. However, in the right end-point case, $\gamma_T^{-1}$ is singular and, due to (3.9), $\gamma_T^{-1}$ is not integrable on $[0, T]$. Therefore, it is difficult to get a uniform a priori estimation, following the idea in Lemma 3.1, to conclude the well-posedness of $Y$ at the terminal time $T$.

For each $s \in [0, T)$, we set

\begin{equation}
v_s := \frac{G_s^{-1}(X_s - Y_s)}{\gamma_s}, \quad \tilde{W}_s := W_s + \int_0^s v_r dr,
\end{equation}

and define
\begin{equation}
M_s := \exp \left( - \int_0^s \langle v_r, dW_r \rangle_U - \frac{1}{2} \int_0^s \|v_r\|_2^2 dr \right).
\end{equation}

From (3.11) and (3.12), $R$ can also be rewritten as
\begin{equation}
M_s := \exp \left( - \int_0^s \langle v_r, d\tilde{W}_r \rangle_U + \frac{1}{2} \int_0^s \|v_r\|_2^2 dr \right), \quad s \in [0, T).
\end{equation}

It is clear that $Q_s := M_s P$ is a probability measure. By the representation (3.11) and the non-degenerate condition (2.18) in Assumption 2.4, we have
\begin{equation}
\frac{1}{2} \int_0^s \|v_r\|^2_2 dr \leq \frac{\|G^{-1}\|_{\infty}^2}{2} \int_0^s \|X_r - Y_r\|^2_\gamma dr.
\end{equation}

Due to the well-posedness of $X$ and $Y$ in the space $C([0, T]; L^p) \cap L^{q+p-2}(0, T; L^{q+p-2})$ P-a.s., we have $\frac{1}{2} \int_0^s \|v_r\|^2_2 dr < \infty$ P-a.s. for any $s \in [0, T)$. Therefore, the Girsanov theorem yields that $(\tilde{W}_t)_{t \in [0,s]}$ is a $U$-valued cylindrical Wiener process under the probability measure $Q_s$.

Next, we will give a uniform pathwise estimate to $\int_0^s \gamma_r^{-2} \|X_r - Y_r\|^2_\gamma dr$ and two uniform moments’ estimations for certain functionals of $(M_s)$ for all $s \in [0, T)$, so that a Novikov condition, see (3.19), holds true. This fact will indicate that $(M_s)$ defined in (3.13) is indeed a uniformly integrable martingale and $(\tilde{W}_s)$ defined in (3.11) is a cylindrical Wiener process, on $[0, T]$, under certain probability measure.

**Lemma 3.3.** Let Assumptions 2.1 and 2.4, and the condition (2.14) hold. For any $T > 0$ and $x, y \in L^p$,
\begin{equation}
\sup_{s \in [0,T]} \mathbb{E}[M_s \log M_s] \leq \frac{\lambda \|G^{-1}\|_{\infty}^2}{e^{2\lambda T} - 1} \|x - y\|_p^2.
\end{equation}

Consequently, $M_T := \lim_{s \uparrow T} M_s$ exists such that $(M_s)_{s \in [0,T]}$ is a uniformly integrable martingale and $(\tilde{W}_t)_{t \in [0,T]}$ is a $U$-valued cylindrical Wiener process under the probability measure $Q = M_T P$.

**Proof.** Let $s \in [0, T)$ be fixed. By the construction (3.11), we can rewrite Eq. (2.3) and Eq. (3.10) on $[0, s]$ as
\begin{align}
\text{(3.16) } & \quad dX_t = (AX_t + F(X_t) - \gamma^{-1}_t (X_t - Y_t)) dt + Gd\tilde{W}_t, \\
\text{(3.17) } & \quad dY_t = (AX_t + F(Y_t)) dt + Gd\tilde{W}_t,
\end{align}

with initial values $X_0 = x$ and $Y_0 = y$, respectively. Under $Q_s$, we have
\begin{equation}
\partial_t (X_t - Y_t) = A(X_t - Y_t) + F(X_t) - F(Y_t) - \gamma^{-1}_t (X_t - Y_t).
\end{equation}
As in the proof of the inequality (3.5), we test \( p|X_t - Y_t|^{p-2}(X_t - Y_t) \) on the above equation, use integration by parts formula, and apply the condition (2.8) to obtain
\[
\partial_t \|X_t - Y_t\|_p^p + p(p-1) \int |X_t - Y_t|^{p-2} |\nabla(X_t - Y_t)|^2 d\xi \\
\leq pL_f \|X_t - Y_t\|_p^p - p\theta \|X_t - Y_t\|_{q+p-2}^{q+p-2} - p\gamma_t^{-1} \|X_t - Y_t\|_p^p.
\]
It follows from the chain rule that
\[
\partial_t \|X_t - Y_t\|_p^2 = \frac{2}{p} \|X_t - Y_t\|_p^{2-p} \partial_t \|X_t - Y_t\|_p^{p}
\]
\[
\leq -2(p-1) \|X_t - Y_t\|_p^{2-p} \int |X_t - Y_t|^{p-2} |\nabla(X_t - Y_t)|^2 d\xi \\
+ 2L_f \|X_t - Y_t\|_p^2 - 2\theta \|X_t - Y_t\|_p^{2-p} \|X_t - Y_t\|_{q+p-2}^{q+p-2} - 2\gamma_t^{-1} \|X_t - Y_t\|_p^2,
\]
and thus
\[
\partial_t \|X_t - Y_t\|_p^2 \leq -2\lambda \|X_t - Y_t\|_p^2 - 2\gamma_t^{-1} \|X_t - Y_t\|_p^2.
\]
The product rule of differentiation and the equality (3.8) yield that
\[
\partial_t (\gamma_t^{-1} \|X_t - Y_t\|_p^2) = \gamma_t^{-1} \partial_t \|X_t - Y_t\|_p^2 - \gamma_t^{-2} \gamma_t' \|X_t - Y_t\|_p^2 \\
\leq -\gamma_t^{-2} (\gamma_t' + 2\lambda \gamma_t + 2) \|X_t - Y_t\|_p^2 \\
= -\gamma_t^{-2} \|X_t - Y_t\|_p^2.
\]
Integrating on both sides from 0 to \( s \), we obtain
\[
(3.18) \quad \frac{\|X_s - Y_s\|_p^2}{\gamma_s} + \int_0^s \frac{\|X_t - Y_t\|_p^2}{\gamma_t^2} dt \leq \frac{\|x - y\|_p^2}{\gamma_0}, \quad Q_s\text{-a.s.}
\]
This pathwise estimate particularly implies the Novikov condition
\[
(3.19) \quad \sup_{s \in [0,T]} \mathbb{E}_s \exp \left( \frac{1}{2} \int_0^s \|v_t\|_2^2 dt \right) \leq \exp \left( \frac{\|G^{-1}\|_2^2 \|x - y\|_p^2}{2\gamma_0} \right) < \infty,
\]
so that, by the martingale convergence theorem, \( M_T := \lim_{t \uparrow T} M_t \) exists such that \((M_t)_{t \in [0,T]} \) is a martingale, and by Girsanov theorem, \((\tilde{W}_t)_{t \in [0,T]} \) is a cylindrical Wiener process under the probability measure \( Q \).

Finally, it follows from the equality (3.13), the estimate (3.14), and the above pathwise inequality (3.18) that
\[
\log M_s = -\int_0^s \langle v_r, d\tilde{W}_r \rangle_U + \frac{1}{2} \int_0^s \|v_r\|_U^2 dr \\
\leq -\int_0^s \langle v_r, d\tilde{W}_r \rangle_U + \frac{\|G^{-1}\|_2^2 \|x - y\|_p^2}{2\gamma_0}.
\]
Then we arrive at
\[
\mathbb{E}[M_s \log M_s] = \mathbb{E}_s \log M_s \leq \frac{\|G^{-1}\|_\infty^2 \|x - y\|_p^2}{2\gamma_0}, \quad s \in [0, T),
\]
and thus obtain (3.15), noting that \(\gamma_0\) is given in (3.7).

**Lemma 3.4.** Let Assumptions 2.1 and 2.4, and the condition (2.14) hold. For any \(T > 0, x, y \in L^p\), and \(s > 1\),
\[
\sup_{s \in [0, T]} \mathbb{E} M_s^{\frac{s-1}{s}} \leq \exp \left( \frac{s\lambda\|G^{-1}\|_\infty^2 \|x - y\|_p^2}{(s - 1)^2(e^{2\lambda T} - 1)} \right).
\]

**Proof.** Denote by \(v_r^s := -\frac{1}{s-1}v_r\) for \(r \in [0, s] \subset [0, T]\). The representation (3.13) and the pathwise estimate (3.18) with \(\gamma_0\) given in (3.7) yield that \(\mathbb{Q}_s\)-a.s.
\[
M_s^{\frac{1}{s-1}} = \exp \left( -\frac{1}{s-1} \int_0^s \langle v_r, d\tilde{W}_r \rangle_U + \frac{1}{2(s - 1)} \int_0^s \|v_r\|_U^2 \, dr \right)
\]
\[
= \exp \left( \int_0^s \langle v_r^s, d\tilde{W}_r \rangle_U - \frac{1}{2} \int_0^s \|v_r^s\|_U^2 \, dr \right)
\]
\[
\times \exp \left( \frac{s}{2(s - 1)^2} \int_0^s \|v_r\|_U^2 \, dr \right)
\]
\[
\leq \exp \left( \int_0^s \langle v_r^s, d\tilde{W}_r \rangle_U - \frac{1}{2} \int_0^s \|v_r^s\|_U^2 \, dr \right)
\]
\[
\times \exp \left( \frac{s\lambda\|G^{-1}\|_\infty^2 \|x - y\|_p^2}{(s - 1)^2(e^{2\lambda T} - 1)} \right).
\]

It follows that
\[
\mathbb{E} M_s^{\frac{s-1}{s}} = \mathbb{E}_s M_s^{\frac{1}{s-1}}
\]
\[
\leq \exp \left( \frac{s\lambda\|G^{-1}\|_\infty^2 \|x - y\|_p^2}{(s - 1)^2(e^{2\lambda T} - 1)} \right)
\]
\[
\times \mathbb{E}_s \exp \left( \int_0^s \langle v_r^s, d\tilde{W}_r \rangle_U - \frac{1}{2} \int_0^s \|v_r^s\|_U^2 \, dr \right).
\]

Taking into account the fact that
\[
\mathbb{E}_s \exp \left( \int_0^s \langle v_r^s, d\tilde{W}_r \rangle_U - \frac{1}{2} \int_0^s \|v_r^s\|_U^2 \, dr \right) = 1, \quad s \in [0, T],
\]
we obtain (3.21). \(\square\)
4. Harnack Inequalities, Ergodicity, and Contractivity

In the last section, we derive Harnack inequalities, ergodicity, and contractivity for the Markov semigroup \((P_t)_{t \geq 0}\), in the following three parts. At each of these three parts, we give the proof of our main results, Theorems 2.1, 2.2, and 2.3, respectively. Other applications, including several estimates for the density of \((P_t)_{t \geq 0}\), are also derived.

4.1. Harnack inequalities. We begin with the following Harnack inequalities.

**Theorem 4.1.** Let Assumptions 2.1 and 2.4, and the condition (2.14) hold. Then (2.19) and (2.20) hold for any \(T > 0, s > 1, x, y \in L^p\), and \(\phi \in B^+_b(L^p)\).

**Proof.** We first show that \(X_T = Y_T\) Q-a.s. From Lemma 3.3, \((M_t)_{t \in [0,T]}\) is a uniformly integrable martingale and \((\tilde{W}_t)_{[0,T]}\) is a cylindrical Wiener process under the probability measure Q. So \(Y_t\) can be solved up to time \(T\). Let

\[
\tau := \inf\{t \in [0, T] : X_t = Y_t\} \quad \text{with} \quad \inf \emptyset = \infty.
\]

Suppose that for some \(\omega \in \Omega\) such that \(\tau(\omega) > T\), then the continuity of the process \(X - Y\), in Lemma 3.1 and Remark 3.3, yields

\[
\inf_{t \in [0, T]} \|X_t - Y_t\|^2_\rho(\omega) > 0 \quad \text{so that} \quad \int_0^T \frac{\|X_t - Y_t\|^2_\rho}{\gamma^2} \, dt = \infty
\]

holds on the set \((\tau > T) := \{\omega : \tau(\omega) > T\}\), by the divergence relation (3.9). So we can conclude \(Q(\tau > T) = 0\), from the pathwise estimate (3.18), and thus \(X_T = Y_T\) Q-a.s. by the definition of \(\tau\).

Therefore, we get a coupling \((X, Y)\) by the change of measure, with changed probability \(Q = M_T \mathbb{P}\), such that \(X_T = Y_T\) Q-a.s. Consequently, the inequalities (2.19) and (2.20) follow from the following known inequalities (see, e.g., [Wan13, Theorem 1.1.1]):

\[
P_T \log \phi(y) \leq \log P_T \phi(x) + \mathbb{E}[M_t \log M_t],
\]

\[
(P_T \phi(y))^s \leq (P_T \phi^s(x))(\mathbb{E}M_t^{\frac{s}{s-1}})^{s-1},
\]

and the estimations (3.15) and (3.21), in Lemmas 3.3 and 3.4, respectively. □

The log-Harnack inequality (2.19) imply the following gradient estimate and regularity properties for the Markov semigroup \((P_t)_{t \geq 0}\).
Corollary 4.1. Let Assumptions 2.1 and 2.4, and the condition (2.14) hold. For any $T > 0$ and $\phi \in B_0(L^p)$,

\begin{equation}
\|DP_T \phi\| \leq \sqrt{\frac{2\lambda\|G^{-1}\|^2_\infty}{e^{2\lambda T} - 1}} \sqrt{PT \phi^2 - (P_T \phi)^2}.
\end{equation}

Consequently, $(P_t)_{t \geq 0}$ is strong Feller and has at most one invariant measure, and if it has one, the density of $(P_t)_{t \geq 0}$ with respect to the invariant measure is strictly positive.

**Proof.** The gradient estimate (4.1) and the uniqueness of the invariant measure for $(P_t)_{t \geq 0}$ with a strictly positive density, if it exists, are direct consequence of the log-Harnack inequality (2.19), see Proposition 1.3.8 and Theorem 1.4.1 in [Wan13], respectively. Finally, the strong Feller property of $(P_t)_{t \geq 0}$ follows easily from the gradient estimate (4.1). \hfill $\square$

Remark 4.1. Under the conditions in Corollary 4.1, there exist constants $C, T_0 > 0$ such that

\begin{equation}
\|\mathcal{L}(X^x_t) - \mathcal{L}(X^y_t)\|_{TV} \leq Ce^{-\lambda t}\|x - y\|_p, \quad t \geq T_0, \quad x, y \in L^p,
\end{equation}

where $\mathcal{L}(X)$ denotes the distribution of $X$ on $L^p$, $\lambda$ is given in (2.22), and $\|\cdot\|_{TV}$ denotes the total variation norm of a signed measure, i.e.,

$\|\mu - \nu\|_{TV} := \sup_{\|\phi\|_\infty \leq 1} |\int_{L^p} \phi d\mu - \int_{L^p} \phi d\nu|$ for two measures $\mu$ and $\nu$.

**Proof of Theorem 2.1.** Theorem 2.1 follows from Theorem 4.1 and Corollary 4.1. \hfill $\square$

4.2. Ergodicity. In this part, we show the existence of an invariant measure for the Markov semigroup $(P_t)_{t \geq 0}$. In combination with the uniqueness of the invariant measure, as shown in Corollary 4.1, we derive the existence of a unique and thus ergodic invariant measure.

**Theorem 4.2.** Let Assumptions 2.1 and 2.3, and the condition (2.14) hold. Assume that $q > 2$ and

\begin{equation}
d < \frac{2p(q + p - 2)}{(p - 1)(q - 2)}.
\end{equation}

Then $(P_t)_{t \geq 0}$ has an invariant measure $\mu$ with full support on $L^p$ such that $\mu(\|\cdot\|_{q+p-2}^q < \infty$. Assume furthermore that Assumption 2.4 holds, then $\mu$ is the unique and thus ergodic invariant measure of $(P_t)_{t \geq 0}$.

**Proof.** The uniqueness of the invariant measure and the strong Feller Markov property for $(P_t)_{t \geq 0}$ have been shown in Corollary 4.1. Thus, to show the existence of an invariant measure, by Krylov–Bogoliubov
that there exists a constant $C$ measures $(\mu_n)_{n \in \mathbb{N}_+}$ defined by

\begin{equation}
\mu_n := \frac{1}{n} \int_0^n \delta_0 P_t dt, \quad n \in \mathbb{N}_+,
\end{equation}

where $\delta_0 P_t$ is the distribution of $X_t^0$, the solution of Eq. (2.3) with initial datum $X_0 = 0$.

It follows from the relation $X = Z + W_A$, the estimate (3.4) with $x = 0$, and Young inequality that

\begin{equation}
\|X_t^0\|_p^p \leq 2^{p-1} \|Z_t^0\|_p^p + 2^{p-1} \|W_A(t)\|_p^p
\end{equation}

\begin{align*}
&\leq C \int_0^t (1 + \|W_A(r)\|_{q+p-2}^{q+p-2}) dr + 2^{p-1} p L \int_0^t \|Z_r^0\|_p^p dr \\
&\quad - 2^{p-1} p \theta_1 \int_0^t \|Z_r^0\|_{q+p-2}^{q+p-2} dr + 2^{p-1} \|W_A(t)\|_p^p
\end{align*}

\begin{align*}
&\leq C \int_0^t (1 + \|W_A(r)\|_{q+p-2}^{q+p-2}) dr - \theta_4 \int_0^t \|Z_r^0\|_{q+p-2}^{q+p-2} dr + 2^{p-1} \|W_A(t)\|_p^p
\end{align*}

\begin{align*}
&\leq C \int_0^t \|X_r^0\|_{q+p-2}^{q+p-2} dr - \theta_5 \int_0^t \|X_r^0\|_{q+p-2}^{q+p-2} dr + 2^{p-1} \|W_A(t)\|_p^p;
\end{align*}

for some constants $\theta_4, \theta_5 > 0$, where we have used the elementary inequality $|\xi - \eta|^r \geq 2^{1-r} \xi^r - \eta^r$ for $\xi, \eta \geq 0$ and $r \geq 1$, in the last inequality. Then we have

$$
\theta_5 \int_0^t \|X_r^0\|_{q+p-2}^{q+p-2} dr \leq C \int_0^t (1 + \|W_A(r)\|_{q+p-2}^{q+p-2}) dr + 2^{p-1} \|W_A(t)\|_p^p.
$$

The above estimate, in combination with the condition (2.14), yields that there exists a constant $C$ such that for all $n \geq 1$,

$$
\mu_n(\| \cdot \|_{q+p-2}) = \frac{1}{n} \int_0^n \mathbb{E} \|X_r^0\|_{q+p-2}^{q+p-2} dr
\begin{equation}
\leq \frac{C}{\theta_5} \left(1 + \frac{\mathbb{E}\|W_A(n)\|_p^p}{n} + \frac{1}{n} \int_0^n \mathbb{E} \|W_A(r)\|_{q+p-2}^{q+p-2} dr \right) \leq C.
\end{equation}

It follows from the ultracontractivity (2.4), with $r = p$ and $s = \frac{q+p-2}{q-1}$, and Young convolution inequality that

\begin{align*}
&\int_0^n \left\| \int_0^t S_{t-r} F(X_r^0) dr \right\|_{\beta,p} dt \\
&\leq C \int_0^n \int_0^t e^{-\lambda_1(t-r)(t-r) - \alpha} (1 + \|X_r^0\|_{q+p-2}^{q+p-2}) dr dt
\end{align*}
This shows that the transition kernel of \((\mu_\alpha)\) with respect to \(\mu\) is continuous with respect to \(\mu\) provided that \(\beta > 0\) is sufficiently small, since \(d < \frac{d(p-1)(q-2)}{2p(q+p-2)}\). The fact that 
\[
\sup_{n \geq 1} \left( \int_0^n e^{-\lambda_1 t} t^{-\alpha} dt \right) \leq \int_0^\infty e^{-\lambda_1 t} t^{-\alpha} dt < \infty,
\]
for all \(\lambda_1 > 0\) and \(\alpha \in (0,1)\), and Young inequality imply that
\[
\int_0^n \left\| \int_0^t S_{t-r} F(X_r^0) dr \right\|_{\beta,p} dt \leq C \int_0^n (1 + \|X_t^0\|_0^{q+p-2}) dt.
\]
By Fubini theorem, the estimate (4.6), and the condition (2.17), we arrive at

(4.7) 
\[
\mu_n(\| \cdot \|_{\beta,p}) = \frac{1}{n} \int_0^n \mathbb{E}\|X^0_r\|_{\beta,p} dr 
\leq \frac{1}{n} \int_0^n \left\| \int_0^t S_{t-r} F(X_r^0) dr \right\|_{\beta,p} dr + \frac{1}{n} \int_0^n \mathbb{E}\|W_A(t)\|_{\beta,p} dr 
\leq \frac{C}{n} \int_0^n (1 + \|X_t^0\|_0^{q+p-2}) dt + \frac{1}{n} \int_0^n \mathbb{E}\|W_A(t)\|_{\beta,p} dr \leq C < \infty,
\]
for all \(n \geq 1\) and \(\beta < (1 - \frac{d(p-1)(q-2)}{2p(q+p-2)})\). For any fixed \(p \geq 2\), we take \(\beta < (1 - \frac{d(p-1)(q-2)}{2p(q+p-2)}) \land \beta_0 \land \frac{d}{p^*}\), so that the embedding \(W_0^{\beta,p} \subset L^p\) in (2.7) is compact. Consequently, the above estimate (4.7) shows that \(\{u \in L^p : \|u\|_{\beta,p} \leq N\}\) is relatively compact in \(L^p\) for any \(N > 0\), and thus \((\mu_n)_{n \in \mathbb{N}}\) is tight. This shows the existence of an invariant measure, denoted by \(\mu\), of \((P_t)_{t \geq 0}\).

To show that the invariant measure \(\mu\) has full support on \(L^p\), let us choose \(s = 2\), \(\phi = \chi_\Gamma\), in (2.20), with \(\Gamma\) being a Borel set in \(L^p\), and get

\[
(P_T \chi_\Gamma(x))^2 \int_{L^p} \exp \left( - \frac{2\lambda\|G^{-1}\|_\infty^2}{\epsilon^2 - 1} \|x - y\|_p^2 \right) \mu(dy) 
\leq \int_{H} P_T \chi_\Gamma(y) \mu(dy) = \int_{H} \chi_\Gamma(y) \mu(dy) = \mu(\Gamma), \quad T > 0, \ x \in L^p.
\]
This shows that the transition kernel of \((P_t)_{t \geq 0}\) is absolutely continuous with respect to \(\mu\) so that it has a density \(p_T(x,y)\). Suppose that \(\text{supp } \mu \neq L^p\), then there exist \(x_0 \in L^p\) and \(r > 0\) such that \(\mu(B(x_0,r)) = 0\), where \(B(x_0,r)\) is a ball in \(L^p\) with radius \(r\) and center \(x_0\). Then \(p_T(x_0,B(x_0,r)) = 0\) and \(\mathbb{P}(\|X_T - x_0\|_p \leq r) = 0\) for all
$T > 0$. This contracts with the fact that $X^x_T$ is a continuous process on $L^p$ as shown in Lemma 3.1.

Similarly to (4.6), we have (with $n = 1$ and $X_0 = x$)

$$
\int_0^1 P_t \cdot \|x\|^{q+p-2}_{q+p-2}(x)dt = \int_0^1 E\|X^x_t\|^{q+p-2}_{q+p-2}dt \leq C(1 + \|x\|^p_p).
$$

Integrating on $L^p$ with respect to the invariant measure $\mu$ and using Fubini theorem, we obtain

$$
\mu(\| \cdot \|^q_{q+p-2}) = \int_0^1 \int_H P_t \cdot \|x\|^{q+p-2}_{q+p-2}(dx)dt \leq C(1 + \mu(\| \cdot \|^p_p)) < \infty.
$$

This shows that $\mu(\| \cdot \|^q_{q+p-2}) < \infty$ and completes the proof. \hfill \Box

Remark 4.2. In the case $q > 2 = p$, the condition (4.3) is equivalent to $d < 4 + 8/(q - 2)$, which will be always valid in $d = 1, 2, 3$-dimensional cases.

Remark 4.3. Under Assumption 2.1 and the condition (2.14), if $\lambda$ defined in (2.22) is positive, one can use the standard remote control method to show that $(P_t)_{t \geq 0}$ has an invariant measure $\mu$, once we derive similar estimates as (3.4) and (3.3), without the restriction (4.3); see, e.g., [DPZ96, Theorem 6.3.2]. In this case, $\mu$ also has full support on $L^p$ such that $\mu(\| \cdot \|^p_{p+q-2}) < \infty$ holds.

Remark 4.4. Under the conditions of Theorem 4.2 or Remark 4.3, $(P_t)_{t \geq 0}$ has a unique invariant measure $\mu$ with full support on $L^p$, which shows that $(P_t)_{t \geq 0}$ is irreducible, i.e., $P_T \chi_\Gamma(x) > 0$ for arbitrary non-empty open set $\Gamma \subset L^p$, $x \in L^p$, and $T > 0$. Indeed, the power-Harnack inequality (2.20) with $f = \chi_\Gamma$ yields that

$$
(P_T \chi_\Gamma(y))^s \leq P_T \chi_\Gamma(x) \exp\left(\frac{s\lambda\|G^{-1}\|_\infty^2\|x - y\|^2_p}{(s - 1)(e^{2sT} - 1)}\right), \quad y \in L^p.
$$

The facts that $\mu$ is $P_T$-invariant and has full support on $L^p$ imply

$$
\int_{L^p} P_T \chi_\Gamma(y)\mu(dy) \leq \int_{L^p} \chi_\Gamma(y)\mu(dy) = \mu(\Gamma) > 0,
$$

which shows that there is a $y \in L^p$ such that $P_T \chi_\Gamma(y) > 0$. Then (4.8) yields that $P_T \chi_\Gamma(x) > 0$ for all $x \in L^p$, so that the irreducibility holds.

Proof of Theorem 2.2. Theorem 2.2 follows from Theorem 4.2 and Remark 4.3. \hfill \Box
4.3. Contractivity and estimates of density. Finally, we use the Harnack inequalities (2.19) and (2.20) to derive an estimate of the density, denoted by \( p_T(x, y) \), with respect to the invariant measure \( \mu \) of \( (P_t)_{t \geq 0} \), and establish contractive property for \( (P_t)_{t \geq 0} \).

**Corollary 4.2.** Let Assumption 2.1 and the condition (2.14) hold. Assume that \( q > 2 \) such that (4.3) hold and Assumption 2.3 hold, or \( \lambda \) defined in (2.22) is positive. Then for all \( T > 0 \), \( x \in L^p \), and \( s > 1 \),

\[
(4.9) \quad \|p_T(x, \cdot)\|_{L^s(\mu)} \leq \left( \int_H \exp \left( - \frac{s \lambda \|G^{-1}\|^2_\infty \|x - y\|^2_p}{e^{2\lambda T} - 1} \right) \mu(dy) \right)^{-\frac{s-1}{s}}.
\]

**Proof.** Using (2.20) with \( s \) replaced by \( \frac{s}{s-1} \), and noting that \( \mu \) is \( (P_t)_{t \geq 0} \)-invariant, we obtain

\[
\|p_T(x, \cdot)\|_{L^s(\mu)} = \sup \{ \langle p_T(x, \cdot), \phi \rangle_\mu : \phi \in B_0^+(L^p), \mu(\phi^\frac{s}{s-1}) \leq 1 \}
\]

\[
= \sup \{ P_T \phi(x) : \phi \in B_0^+(L^p), \mu(\phi^\frac{s}{s-1}) \leq 1 \}
\]

\[
\leq \left( \int_H \phi^\frac{s}{s-1}(y) \mu(dy) \right)^{\frac{s-1}{s}} \left( \int_H \exp \left( - \frac{s \lambda \|G^{-1}\|^2_\infty \|x - y\|^2_p}{e^{2\lambda T} - 1} \right) \mu(dy) \right)^{\frac{s-1}{s}}
\]

\[
= \mu(\phi^\frac{s}{s-1}) \left( \int_H \exp \left( - \frac{s \lambda \|G^{-1}\|^2_\infty \|x - y\|^2_p}{e^{2\lambda T} - 1} \right) \mu(dy) \right)^{\frac{s-1}{s}}.
\]

This shows the density estimate (4.9). \( \square \)

**Remark 4.5.** According to [Wan10, Proposition 2.4], the log-Harnack inequality (2.19) and the power-Harnack inequality (2.20) are equivalent to the following two heat kernel inequalities, respectively, provided \( P_T \) have a strictly positive density \( p_T(x, y) \) with respect to a Radon measure of \( P_T \):

\[
\int_{L^p} p_T(x, z) \log \frac{p_T(x, z)}{p_T(y, z)} \mu(dz) \leq \frac{\lambda \|G^{-1}\|^2_\infty}{e^{2\lambda T} - 1} \|x - y\|^2_p,
\]

\[
\int_{L^p} p_T(x, z) \left( \frac{p_T(x, z)}{p_T(y, z)} \right)^\frac{1}{s-1} \mu(dz) \leq \frac{s \lambda \|G^{-1}\|^2_\infty}{(s-1)^2(1 - e^{-2\lambda T})} \|x - y\|^2_p.
\]

Under the conditions in Corollary 4.2, \( (P_t)_{t \geq 0} \) has a unique invariant measure \( \mu \) such that \( p_T(x, y) \) is strictly positive. Then the above two heat kernel inequalities are direct consequence of Theorem 4.1, Theorem 4.2, and Remark 4.3.

Finally, we study the contractivity property of \( (P_t)_{t \geq 0} \).

**Corollary 4.3.** Let Assumptions 2.1 and 2.4, and the condition (2.15) hold for a sufficiently large \( \epsilon_0 > 0 \). Assume that \( \lambda \) defined
in (2.22) is positive, then \((P_t)_{t \geq 0}\) is hypercontractive. Consequently, \((P_t)_{t \geq 0}\) is compact in \(L^2(\mu)\) for large \(t > 0\), and there exist constants \(C, \lambda_0 > 0\) such that (2.23) and (2.24) hold for all \(t \geq 0\).

**Proof.** According to [Wan17, Theorem 2.1], to show the hypercontractivity for some \(t > 0\) and compactness in \(L^2(\mu)\) for large \(t > 0\) of \((P_t)_{t \geq 0}\) such that (2.23) and (2.24) hold, it suffices to show that there exists a measurable function \(\phi : [0, \infty) \rightarrow (0, \infty)\) with \(\lim_{t \uparrow \infty} \phi(t) = 0\) and constants \(T_0, C_0, \epsilon > 0\) such that the following conditions hold:
\[
\|X_t^x - X_t^y\|_p \leq \phi(t)\|x - y\|_p, \quad t \geq 0, \quad x, y \in L^p;
\]
\[
(P_{T_0} f(y))^2 \leq P_{T_0} f^2(x) \exp(C_0 \|x - y\|_p^2), \quad x, y \in L^p, \quad \phi \in \mathcal{B}_b(L^p);
\]
\[
\mu(\exp(\epsilon \|X_t^x\|_p^2)) < \infty.
\]

The first coupling property follows from (3.3) with \(\phi(t) = e^{-\lambda t}\). The second condition is ensured by the power-Harnack inequality (2.20) with \(s = 2\) and \(C_0 = 2\lambda \|G^{-1}\|_\infty^2 (e^{2\lambda T_0} - 1)^{-1}\) for any \(T_0 > 0\) and \(\phi \in \mathcal{B}_b^+(L^p)\) (and, clearly, for all \(\phi \in \mathcal{B}_b(L^p)\)). Finally, by the dominated convergence theorem and the fact \(\mu_t^0 \rightarrow \mu\) weakly as \(t \rightarrow \infty\), we have
\[
\mu(\exp(\epsilon \|X_t^x\|_p^2)) = \lim_{N \rightarrow \infty} \mu(N \wedge \exp(\epsilon \|X_t^x\|_p^2)) = \lim_{N \rightarrow \infty} \mu_t^0(N \wedge \exp(\epsilon \|X_t^x\|_p^2)) = \lim_{N \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{E}[N \wedge \exp(\epsilon \|X_t^0\|_p^2)] \leq \sup_{t \geq 0} \mathbb{E}[\exp(\epsilon \|X_t^0\|_p^2)].
\]

Then the last concentration property follows from (4.10) and (3.6). \(\square\)

In the case \(q > 2\), even \(\lambda\) defined in (2.22) is non-positive, the existence of an invariant measure for \((P_t)_{t \geq 0}\) has been shown in Theorem 4.2. In this case, we have the following supercontractivity of \((P_t)_{t \geq 0}\).

**Corollary 4.4.** Let \(q > 2\) and Assumptions 2.1 and 2.4 hold. Assume that (2.15) holds for all \(\epsilon_0 > 0\) if \(p = 2\) or holds for a sufficiently large \(\epsilon_0 > 0\) if \(p > 2\), then \((P_t)_{t \geq 0}\) is supercontractive.

**Proof.** The power-Harnack inequality (2.20) with \(s = 2\) yields that
\[
(P_T \phi(y))^2 \leq P_T \phi^2(x) \exp\left(\frac{2\lambda \|G^{-1}\|_\infty^2}{e^{2\lambda T} - 1}\|x - y\|_p^2\right), \quad T > 0, \quad \phi \in \mathcal{B}_b(L^p).
\]
Then for any \(\phi\) with \(\mu(\phi^2) = 1\), we have
\[
(P_T \phi(y))^2 \int_{L^p} \exp\left(-\frac{2\lambda \|G^{-1}\|_\infty^2}{e^{2\lambda T} - 1}\|x - y\|_p^2\right) \mu(dx)
\]
\[
\leq \int_{L^p} P_T \phi^2(x) \mu(dx) = \int_{L^p} f^2(x) \mu(dy) = \mu(\phi^2) = 1.
\]

It is clear that
\[
\int_{L^p} \exp \left( -\frac{2\lambda \|G^{-1}\|_\infty^2}{e^{2\lambda T} - 1} \|x - y\|_p^2 \right) \mu(dx)
\geq \int_{B(0, 1)} \exp \left( -\frac{2\lambda \|G^{-1}\|_\infty^2}{e^{2\lambda T} - 1} (1 + \|y\|_p^2) \right) \mu(dx)
\geq \mu(B(0, 1)) \exp \left( -\frac{2\lambda \|G^{-1}\|_\infty^2}{e^{2\lambda T} - 1} (1 + \|y\|_p^2) \right).
\]

Then we arrive at
\[
P_T \phi(y) \leq \frac{\exp(T^{-1}\|G^{-1}\|_\infty^2)}{\sqrt{\mu(B(0, 1))}} \exp(T^{-1}\|G^{-1}\|_\infty^2\|y\|_p^2).
\]

This estimate yields
\[
\|P_T\|_{2 \to 4} \leq \|P_T\|_4 \leq \frac{\exp(T^{-1}\|G^{-1}\|_\infty^2)}{\sqrt{\mu(B(0, 1))}} \sqrt{\mu(\exp(4T^{-1}\|G^{-1}\|_\infty^2\|y\|_p^2))}.
\]

When \( p = 2 \), similar arguments in (4.10) and the estimate (3.6) with any \( \epsilon_1 \), which is valid provided (2.15) holds for any \( \epsilon_0 \), imply \( \|P_T\|_{2 \to 4} < \infty \) for any \( T > 0 \). When \( p > 2 \), by Young inequality and (3.6) with some \( \epsilon_1 \), which is valid provided (2.15) holds with some \( \epsilon_0 \), we have
\[
\|P_T\|_{2 \to 4} \leq C_{\epsilon_0} \frac{\exp(T^{-1}\|G^{-1}\|_\infty^2)}{\sqrt{\mu(B(0, 1))}} \sqrt{\sup_{t \geq 0} \mathbb{E}\exp(\epsilon_1\|X_t\|_p^2)} < \infty.
\]

This shows the supercontractivity of \((P_t)_{t \geq 0}\). \( \square \)

**Proof of Theorem 2.3.** Theorem 2.3 follows from Corollary 4.3 and Corollary 4.4. \( \square \)

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