I. INTRODUCTION

Quantum entanglement has been found to be very useful in characterizing topological states of matter, which is not possible with conventional local order parameters. In particular, the entanglement spectrum (ES) is one such tool, and has been applied to many systems, such as quantum Hall fluids, topological order, topological insulators, fractional Chern insulators, symmetry-protected topological phases, quantum spin chains, and ladders, and other spin and fermionic systems.

A. Entanglement spectrum and edge-ES correspondence

The ES is defined as follows. Given a system’s ground state $|\Psi\rangle$, together with a bipartition of the full Hilbert space $\mathcal{H}$ into parts $L$ and $R$, so that $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$, one forms the reduced density matrix $\rho_L := \text{Tr}_R(|\Psi\rangle\langle\Psi|)$ on $L$. Because of hermiticity and positivity, the reduced density matrix can be written in the thermal form $\rho_L = \frac{1}{Z} e^{-H_{\text{ent.}}}$, where $H_{\text{ent.}}$ is the so-called entanglement Hamiltonian and $Z = \text{Tr}(e^{-H_{\text{ent.}}})$. The ES, is then simply the eigenenergies of the entanglement Hamiltonian.

This definition of the ES is an operational one. However, there exists a remarkable observation made by Li & Haldane for quantum Hall systems and by others in subsequent works for $(2+1)$-d topological phases: in the cases where the system possesses low energy states living near an open boundary of the manifold the system is placed on (i.e. edge states), it was found that the low-lying edge spectrum of the physical boundary Hamiltonian $H_{\text{ent.}}$ has a so-called edge-ES correspondence. (This correspondence should not be confused with the more established bulk-edge correspondence also used in the context of topological phases).

Analytic proofs of the edge-ES correspondence have been proposed, for example in Ref. [9] for $(2+1)$-d topological states whose edge states are described by a $(1+1)$-d CFT. In that work, a ‘cut and glue’ approach and methods of boundary CFT were used, and it was claimed that the edge and entanglement spectra should be equal up to rescaling and shifting in the low energy limit. However, this method is only applicable to chiral topological phases, where there are protected, physical chiral edge states appearing at an actual spatial boundary of a system. For the case of a non-chiral topological phase, it is unclear as to what information the ES will yield, or even if there is any form of the edge-ES correspondence that exists.
B. Edge-ES correspondence in non-chiral topological order

It is thus the purpose of this paper to explore the edge-ES correspondence in non-chiral topological phases. Specifically we consider the \(\mathbb{Z}_2\) Wen-plaquette model (unitarily equivalent to Kitaev’s Toric code model in the bulk), and ask if some form of correspondence exists. We choose to work on an infinite cylinder with a bipartition into two semi-infinite cylinders terminated with smooth edges. The model on this geometry has four topological sectors \(a\) \((a = 1, \ldots, 4)\) with four locally indistinguishable ground states (these are states with well-defined anyonic flux), and therefore the edge theory on the semi-infinite cylinder and ES of the full cylinder can be unambiguously defined within each topological sector. For the unperturbed Wen-plaquette model, there is in fact an exact edge-ES correspondence because the edge and entanglement spectra in each \(a\)-sector are flat and equally degenerate; thus, the two spectra agree perfectly up to rescaling and shifting. However, this correspondence is potentially lost in the presence of perturbations. Here, we present a detailed microscopic derivation of the edge and entanglement Hamiltonians of the Wen-plaquette model deformed by generic local perturbations, which allows us to compare the two spectra and hence explore the edge-ES correspondence. This calculation constitutes the main result of our paper.

From our calculation, we find the following.

(i) Our calculation shows that the edge states (belonging to the lowest energy eigenspace) of the unperturbed Wen-plaquette model on the semi-infinite cylinder are generated by so-called boundary operators, which can be mapped to \(\mathbb{Z}_2\) symmetric operators acting on a finite length (spin-1/2) spin chain, the effective low-energy degrees of freedom. The effects of generic local perturbations to the Wen-plaquette model are to lift this degeneracy - we find that the effective Hamiltonian in each topological sector \(a\) acting on these edge states, \(H^a_{\text{edge}}\), is a \(\mathbb{Z}_2\) symmetric Hamiltonian acting on the spin chain. The \(\mathbb{Z}_2\) symmetry can be understood as arising from the bulk topological order: it is generated by a Wilson loop operator wrapping around the cylinder.

(ii) We also find that the entanglement Hamiltonian \(H^a_{\text{ent}}\) in each \(a\)-sector acts on a (spin-1/2) spin chain of equal length, and is generated in part by the edge Hamiltonians \((H^a_{\text{edge}, L} + H^a_{\text{edge}, R})\) of the two halves of the bipartition \((L\) and \(R)\) and in part by \(V_{L,R}\), a perturbation spanning the cut. It is also \(\mathbb{Z}_2\) symmetric. However, \(H^a_{\text{ent}}\) is in general not equal to the edge Hamiltonians, being different in some arbitrary way. Thus, there is in general no edge-ES correspondence for generic perturbations, even in the low energy limit, i.e. the low lying values of the edge and entanglement spectra do not match.

(iii) We do find a mechanism in which an edge-ES correspondence is established, though. If we consider the Wen-plaquette model as a symmetry enriched topological phase (SET), by supplementing the \(\mathbb{Z}_2\) topological order with a global translational symmetry along the edge/entanglement cut, achieved by restricting perturbations to those that respect the symmetry, then there is a finite domain in Hamiltonian space in which both \(H^a_{\text{edge}}\) and \(H^a_{\text{ent}}\) realize the critical \((1+1)\)-d Ising model, which has the \(c = 1/2\) Ising CFT as its low energy effective theory. It is in this context that we have observed, in concrete examples, the edge-ES correspondence being realized. This happens because the global translational symmetry implies that the effective degrees of freedom of both the edge and entanglement cut are governed by Kramers-Wannier self-dual Hamiltonians, in addition to them being \(\mathbb{Z}_2\) symmetric, which is imposed by the topological order. The fact that the Hamiltonians have \(\mathbb{Z}_2\) symmetry and Kramers-Wannier self-duality then further guarantees that all perturbations about the \(c = 1/2\) Ising CFT must be irrelevant, giving us the result that the low lying values of the edge and entanglement spectra match upon shifting and rescaling. We therefore see that by considering the Wen-plaquette model as a SET, the topological order in the bulk together with the translation invariance of the perturbations along the edge/cut guarantee an edge-ES correspondence at least in some finite domain in Hamiltonian space.

C. Structure of paper

The rest of the paper is organized as follows. In section II we introduce the Wen-plaquette model and solve for its edge theory on the semi-infinite cylinder by identifying boundary operators and mapping them to \(\mathbb{Z}_2\) symmetric operators acting on a finite length (spin-1/2) spin chain. We also calculate the entanglement spectra on the infinite cylinder by deriving an effective spin ladder Hamiltonian whose ground states equal the ground states on the infinite cylinder. Next, in section III we consider the effects of perturbations to the Wen plaquette model. We present a quick summary of the Schrieffer-Wolff (SW) transformation, central to the derivation of our results. Then, we derive the edge theory and solve for the entanglement spectrum. This allows us to compare the edge-ES correspondence for the perturbed Wen-plaquette model. Then, in section IV we identify the mechanism to establish an edge-ES correspondence: we consider the Wen-plaquette model as a symmetry enriched topological phase (SET) with a global translational symmetry along the edge/entanglement cut, which forces the edge and entanglement Hamiltonians to be additionally Kramers-Wannier self-dual, resulting in the edge-ES correspondence. We also provide a numerical example of the correspondence where the perturbations are uniform magnetic fields acting on single spins. Lastly, in section V we discuss the implications of our findings and conclude.

Appendix A presents the Schrieffer-Wolff transformation and necessary formulae, appendix B presents the perturbation theory calculations for the entanglement Hamil-
tonian, specifically, $L'$, defined in section IV while appendix C presents the derivation of the edge and entanglement Hamiltonians of the Wen-plaquette model on an infinite cylinder perturbed by uniform single-site magnetic fields, as considered in section IV.

II. EXACT WEN-PLAQUETTE MODEL

A. Edge theory on semi-infinite cylinder

We first consider the unperturbed $\mathbb{Z}_2$ Wen-plaquette model on a semi-infinite cylinder, $L$ (left), terminated with a smooth boundary, with the periodic direction along the $y$-axis. The Hamiltonian is the sum of commuting plaquette terms,

$$H_L = -g \left( \sum_{\text{plaq.}} O_{\text{plaq.}} - c \right) = - \sum_{\text{pl.}} \left( \sum_{\text{pl.}} - c \right), \quad (1)$$

where $O_{\text{plaq.}} = \left\{ \sum_{\text{pl.}} - c \right\} = Z_1 X_2 Z_3 X_4$ is the four-spin plaquette operator, and $\{X_i, Y_i, Z_i\}$ the set of Pauli matrices acting on site $i$. The energy scale $g$ has been set to 1 and $c$ is a shift in energy such that the ground state energy of $H_L = 0$. The number of sites $L_y$ along $y$ is taken to be even, in order to avoid having a twist defect line (i.e. a consistent checkerboard coloring of the plaquettes to be even, in order to avoid having a twist defect line). The BOs individually commute with $O_{\text{plaq.}}$, square to 1, and thus have eigenvalues $Q_L$ and $Q_L$ respectively taking values $\pm 1$ each, giving the four topological sectors $a \simeq (Q_L, Q_L)$. States within each topological sector $a$ are said to have well-defined anyonic flux with respect to $\Gamma_L$ and $\tilde{\Gamma}_L$.

Ground state degeneracy. The ground state subspace $\mathcal{V}_{0,L}$ is however not four-dimensional. To find the ground state degeneracy, we need to find a maximal set of commuting and independent operators in addition to the plaquette operators. Besides $\Gamma_L$ and $\tilde{\Gamma}_L$, we have bound-

ary operators (BO) $S_{p,L} = \quad ; S_{\tilde{p},L} = \quad$, where $S_{p,L} = Z_{2p-1} X_{2p}$ and $S_{\tilde{p},L} = Z_{2\tilde{p}} X_{2\tilde{p}+1}$ are half-

plaquette operators or string operators acting on the spins on the boundary. The strings thus start and terminate outside the boundary of the semi-infinite cylinder (see fig. 1). There are $L_y/2$ of each kind of operators. The BOs individually commute with $O_{\text{plaq.}}$, $\Gamma$ and $\tilde{\Gamma}$. However, $[S_{p,L}, S_{\tilde{p},L}] \neq 0$ if $p$ neighbors $\tilde{p}$, so a choice of a maximal set of commuting and independent operators on the $L$ semi-infinite cylinder is

$$\{O_p, O_{\tilde{p}}, S_p, S_{\tilde{p}}, \Gamma\}_L = \left\{ \quad ; \quad ; \quad ; \quad \right\}_L. \quad (2)$$

Note that $\Gamma_L$ is not included in the set because it can be formed by the boundary operators: $\Gamma_L = \prod_{\tilde{p}} S_{\tilde{p},L}$. (Color online.) Semi-infinite cylinder that terminates in the $x$-direction, with the periodic direction along $y$. The grey plaquette and white plaquette operators are shown as blue and red circles respectively. The boundary operators (BO) can be thought of as half of the plaquette operators in the bulk (acting on the white ($p$) or grey ($\tilde{p}$) plaquettes). The green ellipses on the edge correspond to the virtual, boundary spin degree of freedom. The blue BO acts on a single virtual spin, while the red BO acts on a pair of nearest-neighbor virtual spins.
Thus $\mathcal{V}_{0,L}$ has four topological sectors $a$, such that $\mathcal{V}_{0,L} = \bigoplus_{a=1}^{4} \mathcal{V}^a_{0,L}$, each with $L_y/2 - 1$ states labeled by the eigenvalues of $S_{p,L}$, for a total dimensionality $\dim(\mathcal{V}_{0,L}) = 2^{L_y/2 + 1}$. Physically, these states can be thought of as having pairs of anyons ($e$ or $m$) that condense on the boundary - this process does not cost energy and thus the ground state degeneracy is given by the number of ways we can condense the anyons on the boundary. For this reason, these ground states can also be understood as edge states, and from now on, the terms 'lowest-energy states' and 'edge states' will be used interchangeably with 'ground states'. We will also define the projector $P^a_{0,L}$ onto the eigenspace $\mathcal{V}^a_{0,L}$ for future use.

At this point, we introduce a mapping of the boundary operators $S_{p,L}$ and $S_{\bar{p},L}$ to local operators acting on a finite length (spin-1/2) spin chain. This mapping will be important as it elucidates the tensor product structure of the edge theory on the semi-infinite cylinder. Consider the Wilson loop operator $\mathcal{L}_{L}$ partitioning $\mathcal{V}_{0,L}$ into two: $\mathcal{V}_{0,L} = \bigoplus_{Q=\pm 1} \mathcal{V}^{Q,L}_{0,L}$ labeled by $\hat{Q}_{L}$. Each $\hat{Q}_{L}$ operator has dimension $2^{L_y/2}$, which is isomorphic to a spin-1/2 chain of length $L_y/2$. Let us therefore associate a virtual spin-1/2 degree of freedom for each $\hat{p}$ plaquette (see fig. [1]). Here, we see that a (not unique) representation of the operators $S_{p,L}$ and $S_{\bar{p},L}$ acting on the $L$ spin-1/2 chain for each $\hat{Q}_{L}$ can be found:

\[
\begin{align*}
\hat{p} & \simeq \tau^z_{\hat{p},L} \text{ for } 1 \leq \hat{p} \leq \hat{L}_y/2, \\
\hat{\bar{p}} & \simeq \tau^z_{\hat{\bar{p}},L} \tau^z_{\hat{p}-1,L} \text{ for } 2 \leq \hat{p} \leq \hat{L}_y/2, \\
\end{align*}
\]

and $\hat{1}_{L} \simeq \hat{Q}_{L} \times \tau^z_{1,L} \tau^z_{L_y/2,L'}$ i.e. toroidal boundary conditions. One can check that the Pauli spin operators reproduce the canonical anticommutation algebra of $S_{p,L}$ and $S_{\bar{p},L}$ and that $\mathcal{L}_{L} = \hat{Q}_{L}$ is satisfied. Note that the Wilson loop $\mathcal{L}_{L}$ is mapped to the global spin-flip operator $\hat{Q}_{L} := \prod_{\hat{p}} \tau^z_{\hat{p},L}$, with eigenvalues $Q_{L} = \pm 1$. A similar representation can be found for the operators $S_{p,R}$ and $S_{\bar{p},R}$ acting on the spin-1/2 chain of $R$.

**Higher energy subspaces.** Higher energy subspaces $\mathcal{V}_{a>0,L}$ are spanned by states for which the plaquette condition $\hat{q}_{L} = +1$ is violated. As such, there is a spectral gap of at least +1 separating the ground state subspace from the higher energy subspaces. These violations are generated by string operators that have at least one end-point in the bulk.

For each higher subspace $\mathcal{V}_{a>0,L}$, we can further define a notion of topological sectors $a$, where $a = 1, \cdots, 4$, in the following way: The subspace $\mathcal{V}_{a>0,L}^a$, is the space spanned by all states in $\mathcal{V}_{a,L}^a$ which are acted upon by all possible products of finite-length (i.e. local) string operators such that the number of end-points of these string operators in the bulk of $L$ is $a$. These sectors are called topological because the matrix element of generic local operators using states belonging to different topological sectors vanishes.

**Edge theory.** The edge theory in each topological sector $a$ is defined to be the Hamiltonian $H_{\text{edge},L}^{a}$ acting on the $2^{L_y/2-1}$ edge states of the subspace $\mathcal{V}^a_{0,L}$ that give rise to the different states’ energy levels. However, all states in $\mathcal{V}_{0,L}$ have the exact same energy, and hence the edge Hamiltonian for the exact Wen-plaquette model in each topological sector is identically 0.

Let us summarize what we have learned:

The effective low energy degrees of freedom $(\mathcal{V}^a_{0,L})$ at the boundary of the $L$ semi-infinite cylinder of length $L_y$ is a spin chain made of $L_y/2$ virtual spin-1/2 degrees of freedom, for each topological $\hat{Q}_{L}$ sector. The spin chain is generated by the boundary operators $S_{p,L}$ and $S_{\bar{p},L}$ which are half-plaquette operators in the bulk. In the effective spin chain language, these boundary operators are mapped to $Z_2$ symmetric spin-operators $\tau^z_{\hat{p},L} \tau^z_{\hat{p}-1,L}$ and $\tau^z_{\bar{p},L}$. Without perturbations, $H_{\text{edge},L}^{a}$ is identically 0. With perturbations, there will be dynamics on this effective spin chain, generated by these boundary operators. A similar situation arises for the $R$ semi-infinite cylinder.

**B. Entanglement spectrum**

Let us now solve for the four ground states of the exact Wen-plaquette model on the infinite cylinder, and compute their entanglement spectrum for a bipartition of the infinite cylinder into two semi-infinite cylinders. We can do this by putting two semi-infinite cylinders $L$ and $R$ together, and gluing them with the plaquette terms that act on the strip of plaquettes spanning the two cylinders. That is, we solve:

\[
H = H_L + H_R + H_{LR},
\]

where $H_L$ is given by eqn. [1] (and correspondingly for $H_R$), acting on the $L$ and $R$ semi-infinite cylinders respectively, while $H_{LR} = -\sum_{\text{plaquet}, \text{strip}} O_{\text{plaq}}$. The entanglement cut is naturally taken to be through the strip of plaquettes that divides the system into the $L$ and $R$ subsystems, such that the full Hilbert space $\mathcal{H}$ is the tensor product of the two semi-infinite cylinders: $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$. Figure 2 shows the gluing process.

Now, $\mathcal{H}$ can also be written as $\mathcal{H} = \bigoplus_{\alpha, \beta} \mathcal{V}_{\alpha,L} \otimes \mathcal{V}_{\beta,R}$. Note that a plaquette operator in $H_{LR}$ is comprised of two matching boundary operators acting on the $L$ and $R$ cylinders, for example $O_{\text{plaq}, \text{strip}, (p)} = S_{p,L} \otimes S_{p,R}$ (note: $p$ is the same for $L$ and $R$) and similar for $\bar{p}$, which implies that its action must be such that the tensor product structure is preserved, $H_{LR} : \mathcal{V}_{\alpha,L} \otimes \mathcal{V}_{\beta,R} \rightarrow \mathcal{V}_{\alpha,L} \otimes \mathcal{V}_{\beta,R}$. Since $H_{LR}$ is the sum of mutually commuting terms, we can simply focus on its action on the
The effective Hamiltonian, i.e. the projection of $H_{LR}$ into $\mathcal{V}^{Q}_{0,L} \otimes \mathcal{V}^{Q}_{0,R}$, is therefore

$$H_{\text{eff}} = - \sum_{p=1}^{L_y/2} (\tau_{p,L}^x \tau_{p,R}^x + \tau_{p,L}^y \tau_{p-1,R}^x \tau_{p,R}^x \tau_{p-1,L}^x).$$

(6)

This is a spin-ladder system (see figure 2), of length $L_y/2$, with $O_a$ and $O_{\tilde{p}}$ coupling the two chains. It also has a $\mathbb{Z}_2$ symmetry generated by the global spin flip operator $Q_L := \prod_{\tilde{p}} \tau_{p,L}^{z}$ (whose eigenvalue is $Q_L$). Solving eqn. (4) for both $Q$ will give us all four ground states of the system.

Let us fix a $\tilde{Q}$. Define the (unique) state $|\uparrow\rangle_L$, as the state that is a ground state of $H_L$ with $\tilde{Q}_L = \tilde{Q}$ and furthermore satisfies $\tau_{p,L}^z = +1$. Then we see that the other $2^{L_y/2}$ states spanning $\mathcal{V}^{Q}_{0,L}$ comprise of all other possible spin configurations $|\uparrow\rangle_L$, generated by $\tau_{p,L}^z$ acting on $|\uparrow\rangle_L$. The same is also true for $R$. The ground state of eqn. (3), or equivalently of the full Hamiltonian, in this $\tilde{Q}$ sector with $Q = Q_L = Q_R$ is therefore given by

$$|\tilde{Q},Q\rangle = \mathcal{N} \prod_{\tilde{p}} \left( \frac{1 + \tau_{\tilde{p},L}^x \tau_{\tilde{p},R}^x}{2} \right) (|\uparrow\rangle_L |\uparrow\rangle_R + Q |\bar{\uparrow}\rangle_L |\bar{\uparrow}\rangle_R)$$

$$= \frac{1}{\sqrt{2^{L_y/2}}} \frac{1}{\sqrt{2}} \sum_{\tau} (|\tau\rangle_L |\tau\rangle_R + Q |\bar{\tau}\rangle_L |\bar{\tau}\rangle_R)$$

$$= \frac{1}{\sqrt{2^{L_y/2}-1}} \sum_{\tau} \mathbb{P}_{Q,L} |\tau\rangle_L |\tau\rangle_R, \quad (7)$$

where $\mathbb{P}_{Q,L} := (1 + Q_L \tilde{Q}_L)/2$ is the $\mathbb{Z}_2$ projector onto the $Q_L = q_L$ sector in the $L$ chain. Thus, in total, there are four ground states on the infinite cylinder as claimed, each with well-defined anyonic flux through the cylinder.

Lastly, from eqn. (7), it can be readily seen that the entanglement spectrum of $|\tilde{Q},Q\rangle$ is flat: the reduced density matrix on $L$ is

$$\rho^{\infty}_{L}(\tilde{Q},Q) := \text{Tr}_R |\tilde{Q},Q\rangle \langle\tilde{Q},Q| = \frac{1}{2^{L_y/2}-1} \mathbb{P}_{Q,L}, \quad (8)$$

with $2^{L_y/2}-1$ non-zero eigenvalues all of value $1/2^{L_y/2}-1$. The entanglement Hamiltonian $H_{\text{ent.}}^a := -\log(\rho^{\infty}_{L})$ is therefore also flat and acts on a virtual spin chain of length $L_y/2$.

In summary:

The entanglement Hamiltonian of the exact Wen-plaquette model in topological sector $a$ is flat and acts on a virtual spin-1/2 spin chain of length $L_y/2$, similar to the case of the edge Hamiltonian. Each entanglement Hamiltonian gives an ES with $2^{L_y/2}-1$ finite values. The entanglement Hamiltonians can be derived by considering an effective Hamiltonian acting on a spin-ladder system with length $L_y/2$, for each $\tilde{Q} = \tilde{Q}_L = \tilde{Q}_R$ sector. Solving the effective Hamiltonian in each $\tilde{Q}$ sector gives two ground states distinguished by the eigenvalue $Q = Q_L = Q_R$, for a total of four ground states overall.
C. Edge-ES correspondence

Since the edge Hamiltonian is identically 0, and the entanglement Hamiltonian flat, the edge-ES correspondence is exact in this case: the two spectra are equal up to rescaling and shifting. However, that is not to say that the flat entanglement spectrum is uninteresting: for example, the entanglement entropy in each topological sector can be readily calculated, yielding

\[ S_a = \text{Tr}(\rho_L^a \log \rho_L^a) = \left( \hat{I}_y/2 - 1 \right) \log 2. \]  

(9)

From there the topological entanglement entropy, defined to be the universal sub-leading piece of the entanglement entropy, \( S_a = \alpha_a L - \gamma_a + \cdots \), can be extracted:

\[ \gamma_a = \log 2, \]

(10)
in agreement with the fact that \( \{\hat{Q},\hat{Q}\} \) are the so-called minimum entangled states on the infinite cylinder.

We conclude:

The edge-ES correspondence for the exact Wen-plaquette model is exact: all 2\( L^2/2-1 \) levels of the edge Hamiltonian flat, the edge-ES correspondence for the exact Wen-plaquette model. We begin this section by introducing the Schrieffer-Wolff (SW) transformation

\[ H_0 = P_0 H_0 P_0 + P_1 H_0 P_1, \]

(12)

where \( P_\alpha \) are projectors to \( \mathcal{V}_\alpha, \alpha = 0, 1 \). Let us now add a small perturbation to the system \( \epsilon V \) that does not commute with \( H_0 \). We assume that the perturbations are weak enough such that the new Hamiltonian can be written as

\[ H = H_0 + \epsilon V = \tilde{P}_0 H_0 \tilde{P}_0 + \tilde{P}_1 H_0 \tilde{P}_1, \]

(13)

where there are still low-energy \( \tilde{\mathcal{V}}_0 \) and high-energy \( \tilde{\mathcal{V}}_1 \) subspaces separated by the spectral gap. \( \mathcal{V}_\alpha \) and \( \tilde{\mathcal{V}}_\alpha \) are assumed to have the same dimensionality, and have significant overlap. Then, we can find a unique direct rotation (i.e. unitary) \( U \) between the old and new subspaces (\( UP_\alpha U^\dagger = P_\alpha \)) such that we can rotate \( H \) to a new Hamiltonian \( H' \) with eigenspaces \( \mathcal{V}_\alpha \):

\[ H' := U H U^\dagger = P_0 U H U^\dagger P_0 + P_1 U H U^\dagger P_1. \]

(14)

This is the so-called Schrieffer-Wolff transformation. It will be useful to define \( U \) can be written uniquely as \( U = e^S \), where \( S \) is an antihermitian and block off-diagonal (in both \( \mathcal{V}_\alpha \) and \( \tilde{\mathcal{V}}_\alpha \) ) operator, and can be constructed perturbatively in \( \epsilon : S = \sum_{n \geq 1} e^n S_n, S_n = -S_n \). The exact formulas for \( S_n \) can be found in Ref. 33 and we reproduce them in appendix A.

Though our discussion above has been limited to Hamiltonians with only two invariant subspaces (low and high), the SW transformation can be readily generalized to Hamiltonians which have many invariant subspaces each separated by a spectral gap, such that the Hilbert space \( \mathcal{H} = \bigoplus_{\alpha \geq 0} \mathcal{V}_\alpha \), see Ref. 36. \( S \) will still be block off-diagonal.

The generalized Schrieffer-Wolff transformation is also referred to as the effective Hamiltonian method. To second order in perturbation theory, \( H' \), which by con-
struction is block-diagonal in $V_{s}$, is given explicitly by

$$
\langle i, \alpha | H' | j, \alpha \rangle = E_{i}^{\alpha} \delta_{ij} + \epsilon \langle i, \alpha | V | j, \alpha \rangle + \frac{\epsilon^{2}}{2} \sum_{k \neq \alpha} \left( \frac{1}{E_{i}^{\alpha} - E_{k}^{\alpha}} + \frac{1}{E_{j}^{\alpha} - E_{k}^{\alpha}} \right),
$$
(15)

where $\langle i, \alpha \rangle \in V_{s}$, $| j, \beta \rangle \in V_{s}$ and $E_{i}^{\alpha}$ is the energy of $| i, \alpha \rangle$. See appendix A for the full perturbative series of the effective Hamiltonian.

### B. Edge theory on semi-infinite cylinder

As mentioned before, the SW transformation is suitable for use on the perturbed Wen-plaquette model on the semi-infinite cylinder, $H_{L} + \epsilon V_{L}$, because the unperturbed Hamiltonian $H_{L}$ is block-diagonal with spectral gaps separating the different energy eigenspaces. To find the edge theory, $H_{\text{edge}, L}$, we want to evaluate eqn. (15) (with $V \rightarrow V_{L}$) for the $L$ semi-infinite cylinder with states belonging to $V_{0,L}$, the lowest energy subspace. Furthermore, we can make use of the fact that states with different $Q_{L}$ do not mix at any order in perturbation theory, as the perturbations are local and cannot generate a global term that mixes $Q_{L}$ sectors, so we can consider eqn. (15) restricted to states belonging to $V_{0,L}^{Q_{L}}$.

However, since by construction $H_{\text{edge}, L}^{Q_{L}}$ acts only on $V_{0,L}^{Q_{L}}$, $H_{\text{edge}, L}^{Q_{L}}$ is generated solely from virtual processes in the perturbative series (eqn. (15)) that map states in the ground state subspace back to itself. These virtual processes are simply products of boundary operators and plaquette operators on $L$.

To clarify this statement, let us show this for a particular second order term in a fixed $Q_{L}$ sector. Let $v_{1,L}$ and $v_{2,L}$ be two local perturbations, each of which are not sums of perturbations, coming from $V_{L}$ (which is generally a sum of local perturbations) that give rise to a non-zero contribution in the matrix element, i.e. $\langle i, 0 | v_{1,L} | k, \beta \rangle \langle k, \beta | v_{2,L} | j, 0 \rangle \neq 0$ for some $i, j, k, \beta$. $v_{2,L}$ is comprised of products of string operators with end points in the bulk of the semi-infinite cylinder. The number of end points in $L$ determines $\beta$, i.e. $v_{2,L}$ can link a ground state to a state with $\beta$ excitations in $L$. This is a state which lives in $V_{s,L}$. Furthermore, this state is unique. Indeed, $| k, \beta \rangle = v_{2,L} | j, 0 \rangle$, and so $\langle l, \gamma | v_{2,L} | j, 0 \rangle = 0$ for any other state such that $| l, \gamma \rangle \neq | k, \beta \rangle$. Now, since $\langle i, 0 | v_{1,L} | k, \beta \rangle \neq 0$, $v_{1,L}$ must be such that it creates the same bulk excitations as $v_{2,L}$, in order to cancel out the excitations of $| k, \beta \rangle$, modulo products of plaquette and boundary operators. The presence of the boundary operators makes $| i, 0 \rangle$ potentially not equal to $| j, 0 \rangle$. Putting all these facts together, we see that replacing $| k, \beta \rangle | k, \beta \rangle$ with the sum over all possible states in the Hilbert space, i.e. $\sum_{l, \gamma} | l, \gamma \rangle \langle l, \gamma | = 1$ for this $v_{1,L}, v_{2,L}$ does not change the value of the matrix element, yielding the desired assertion:

$$
\langle i, 0 | v_{1,L} | k, \beta \rangle \langle k, \beta | v_{2,L} | j, 0 \rangle = \langle i, 0 | v_{1,L} | v_{2,L} | j, 0 \rangle = \langle i, 0 | \prod_{\{p, \bar{p}\}} \left( \sum_{p, \bar{p}} \right) \prod_{\{p, \bar{p}\}} | j, 0 \rangle.
$$
(16)

A similar argument can be made for terms of other orders in perturbation theory.

Since $\prod_{p, \bar{p}} = 1$ for the ground states, we therefore see that $H_{\text{edge}, L}^{Q_{L}}$ must be a function of products of $S_{p, L}$ and $S_{\bar{p}, L}$ projected down into $V_{0,L}^{Q_{L}}$:

$$
H_{\text{edge}, L}^{Q_{L}} = f_{Q_{L}, L} \left( \sum_{p, \bar{p}} \right)
$$
(17)

However, recall the mapping given by eqn. (3), which allows us to interpret the edge Hamiltonian in terms of a more physical picture: a Hamiltonian acting on a spin chain.

The edge Hamiltonian in a $Q_{L}$-sector at finite order in perturbation theory is therefore a local Hamiltonian acting on a spin-$1/2$ chain of length $L_{y}/2$:

$$
H_{\text{edge}, L}^{Q_{L}} = f_{Q_{L}, L} \left( \sum_{p, \bar{p}} \right)
$$
(18)

with toroidal boundary conditions given by $Q_{L}$ (i.e. $\tau_{0,L}^{z} = Q_{L}^{z} \tau_{L_{y}/2,L}^{z}$). Since it is constructed perturbatively, it generically appears at order $\epsilon$. Importantly, we also see that the edge Hamiltonian is always $\mathbb{Z}_{2}$ symmetric regardless of the type of perturbation. This is not a surprising result because the two sectors of $\mathbb{Z}_{2}$ in the spin language, given by the generator $Q_{L} := \prod_{\bar{p}} \tau_{\bar{p}}^{z}$, correspond to the two topological sectors $Q_{L} = \pm 1$ of the Wilson loop $\langle \bar{p} \rangle$, which are preserved under any local perturbations. The edge Hamiltonian in each topological sector $a$ then arises from projecting $H_{\text{edge}, L}^{Q_{L}}$ into the relevant sector $H_{\text{edge}, L}^{Q_{L}}$.

In summary:

The edge Hamiltonian of the perturbed Wen-plaquette model for the $L$ semi-infinite cylinder is given by $H_{\text{edge}, L}^{Q_{L}}$, eqn. (18), in each $Q_{L}$ sector. At any finite order in perturbation theory, it is a $\mathbb{Z}_{2}$ symmetric local Hamiltonian on the virtual spin chain. This is guaranteed if the perturbations of the Wen-plaquette model are themselves local on the cylinder. To further find the edge Hamiltonians in each topological sector $a \simeq (Q_{L}, Q_{L})$, we project into the $Q_{L}$ sector:

$$
H_{\text{edge}, L}^{Q_{L}} = \mathbb{P}_{Q_{L}} H_{\text{edge}, L}^{Q_{L}} \mathbb{P}_{Q_{L}}
$$
(19)

where $\mathbb{P}_{Q_{L}} = (1 + Q_{L} \hat{Q}_{L})/2$. A similar result holds for the $R$ semi-infinite cylinder.
C. Entanglement spectrum

We now find the ground states in each topological sector $a$ of the perturbed Wen-plaquette model on the infinite cylinder,

$$H = H_L + H_R + H_{LR} + \epsilon(V_L + V_R + V_{LR}).$$  \hfill (20)

The precise definition of each term can be found in eqn. \textbf{(11)}.

Let us fix a sector $a \simeq (\tilde{Q},Q)$. There will now be corrections to the ground state leading to an ES with dispersion. However, there now exists a difficulty in comparing the edge and entanglement spectra. From eqn. \hfill (19), we see that $H_{\text{edge}}$ still has $2^{L_y/2-1}$ eigenvalues; but, there will be many more entanglement energies than the $2^{L_y/2-1}$ finite ones as found in eqn. \hfill (7). What does it mean to compare the two spectra then? The resolution can be found by looking at the general form of the perturbed ground state. Dropping the normalization constant for now, for a fixed $\tilde{Q}$, the perturbed ground state can be written as

$$|\tilde{Q},Q\rangle = \sum_{\tau_1,\tau_2} (P_Q - \Lambda)_{\tau_1,\tau_2} |\tau_1\rangle_L |\tau_2\rangle_R +$$

$$\sum_{\tau_1,\tau_2,\alpha,\beta} (\Theta_{\tau_1,\alpha,\beta} |\tau_2\rangle_L |i,\alpha\rangle_R + \Xi_{\tau_1,\alpha,\beta} |i,\alpha\rangle_L |\tau_2\rangle_R)$$

$$+ \sum_{\alpha,\beta} \Omega_{\alpha,\beta} |i,\alpha\rangle_L |j,\beta\rangle_R,$$ \hfill (21)

where $(P_Q)_{\tau_1,\tau_2}$ are the matrix elements of the matrix $P_Q = (1 + Q[I,\ldots,I])^2$, written in the $\tau^z$ basis. $\Lambda, \Theta, \Xi, \Omega$ are coefficient matrices linking a state in $L$ with another state in $R$ and are small in magnitude ($\epsilon$ or higher). Here $|\tau_1\rangle_L \in \mathcal{V}_{0,L}^Q$, and $|i,\alpha\rangle \in \mathcal{V}_{\alpha,L}$ for $\alpha \geq 1$. The same holds true for states in $R$.

We have written the ground state in a way to emphasize the size in the entanglement structure. Our claim is that for generic local perturbations, $\Lambda$ has the form $\Lambda = P_{Q} \Lambda' P_{Q}$ and is an order $\epsilon$ correction which is related to $H_{\text{ent}}$. Furthermore, it is generated by the terms $H'_{\text{edge},L}, H'_{\text{edge},R}$ and $(P_{0,L} \otimes P_{0,R}) V L R (P_{0,L} \otimes P_{0,R})$. We see that $P_{Q} \Lambda' P_{Q}$ is a $2^{L_y/2} \times 2^{L_y/2}$ matrix with $2^{L_y/2-1}$ eigenvalues that have eigenvectors with $P_{Q} = 1$, linking states in $\mathcal{V}_{0,L}^Q$ with $\mathcal{V}_{0,R}^Q$, which is spanned by $|\tau_1\rangle_L \otimes |\tau_2\rangle_R$. $\Theta$ and $\Xi$ are generically order $\epsilon$ corrections and links states in $\mathcal{V}_{0,L}^Q$ with $\mathcal{V}_{\alpha,R}$ and $\mathcal{V}_{\alpha,L}^Q$ with $\mathcal{V}_{0,R}^Q$, respectively. $\Omega$ represents order $\epsilon$ corrections to the ground state in the space $\mathcal{V}_{\alpha \geq 0,L} \otimes \mathcal{V}_{\beta \geq 0,R}$.

We sketch here why $\Lambda = P_{Q} \Lambda' P_{Q}$ is related to the entanglement Hamiltonian. If we form the un-normalized reduced density matrix on $L$, $\rho_L^Q = \text{Tr}_R[Q,Q]/\langle Q,Q \rangle$, it will have a dominant piece that looks like

$$\langle \rho_L^Q \rangle_{\text{dom}} = \sum_{\tau_1,\tau_2} (P_Q - P_Q(\Lambda' + \Lambda') P_Q + \cdots)_{\tau_1,\tau_2} |\tau_1\rangle_L \langle \tau_2|,$$ \hfill (22)

where $\cdots \cdots$ refers to higher order terms. Since in a suitable basis $P_Q = I_a$ (for a fixed $Q$), the expression above can be approximated with an exponential $(1 - x \approx \exp(-x)$ for small $x$), and so the entanglement Hamiltonian is unitarily equivalent to $P_Q(\Lambda' + \Lambda') P_Q$.

The original, flat, non-zero $2^{L_y/2-1}$ eigenvalues of $P_Q$ therefore become the non-zero $2^{L_y/2-1}$ eigenvalues of $\exp[-P_Q(\Lambda' + \Lambda') P_Q]$, i.e.

$$1 \rightarrow \text{eig}(\exp[-P_Q(\Lambda' + \Lambda') P_Q]),$$ \hfill (23)

where these are near $1$ in magnitude. The entanglement spectrum $\xi_{\text{ent}}$, associated with these eigenvalues is then calculated by taking $-1$ times the logarithm of the eigenvalues of the reduced density matrix: $\xi_{\text{ent}} \approx \text{eig}(P_Q(\Lambda' + \Lambda') P_Q)$. Of course, there will be other levels in the entanglement spectrum coming from the sub-dominant part of $\rho_L^Q$, but they ‘flow down from infinity’, as those $\xi_{\text{ent}} \sim -\log(\epsilon) \sim \infty$. These are not the levels of interest and will therefore be ignored. Thus, we aim to derive only the new eigenvalues of the reduced density matrix that are perturbed from the original non-zero values, up to leading order corrections. These values are defined to be the ones that give rise to the relevant part of the entanglement spectrum, and are the values of interest when comparing the edge to the ES in the edge-ES correspondence.

Of course, the discussion above was to sketch the flow of the logic of our argument, and we have to be rigorous in our derivation. They key step to solve for the ground states of the perturbed system is to rewrite eqn. \textbf{(11)} in a way which makes obtaining the form of eqn. \textbf{(21)} manifest - in other words, we want to reorder the perturbative
series which gives the tensor product structure automatically. The key is the Schrieffer-Wolff transformation. We can rewrite $H_L + \epsilon V_L$ within each topological sector (letting $a \simeq (\tilde{Q}_L, Q_L)$ also refer to the topological sector on the $L$ semi-infinite cylinder) as:

$$e^{-S_L^0}e^{S_L^0}P_L^a(H_L + \epsilon V_L)P_L^a e^{-S_L^0}e^{S_L^0},$$  \hspace{1cm} (24)

where $P_L^a = \sum_{\alpha \geq 0} P_{\alpha, L}$, a projector onto the topological sector $a$. $P_{\alpha, L}$ are the projectors onto $V_{\alpha, L}$ (see section \textbf{II}A). This can be done since local perturbations do not connect topological sectors $a$ at any order in perturbation theory.

The BCH expansion then allows us to expand it as

$$e^{-S_L^0}(H_{\text{edge}, L} + P_{\alpha, L}^0 + H_{\alpha, L}^0 + 2P_{\alpha, L}^0 + H_{\alpha, L}^0 + \cdots) e^{S_L^0}$$

$$= \sum_{\alpha \geq 1} \sum_{\text{large}} \langle -S_L^0, H_{\text{edge}, L}^0 + \sum_{\alpha \geq 1} H_{\alpha, L}^0 + \sum_{\alpha \geq 1} [-S_L^0, \alpha P_{\alpha, L}] + \sum_{\text{small}} [-S_L^0, H_{\text{edge}, L}^0] + \sum_{\alpha \geq 1} [-S_L^0, H_{\alpha, L}^0] + \sum_{\text{small}} [-S_L^0, \alpha P_{\alpha, L}] + \cdots, \hspace{1cm} (25)$$

and similarly for the theory on the $R$ semi-infinite cylinder, $H_R + \epsilon V_R$, using $a' \simeq (\tilde{Q}_R, Q_R)$ as the topological sector label. $H_{\alpha, L}^0$ are (at least) order $\epsilon$ effective Hamiltonians acting in $V_{\alpha, L}$ that generate the dynamics over $\alpha P_{\alpha, L}$ upon the addition of perturbations $\epsilon V_L$ to the system. All terms labelled ‘large’ are of order 1, while all terms labelled ‘small’ are at least of order $\epsilon$. To help illustrate the meaning of the terms that appear in the first line of the equation above between the exponentials, we have schematically plotted the energy spectrum of the Wen plaquette model under perturbations in figure 3. The Schrieffer-Wolff transformation simply rotates the perturbed Hamiltonian into a Hamiltonian that is block diagonal in the old eigenspaces (so that $P_\alpha = \Pi_\alpha$), and adds small dynamics due to the perturbation on top of each space ($H_{\alpha}^0 \simeq O(\epsilon)$).

Now, we see that $H_0 = H_{LR} + \sum_{\alpha \geq 1} \alpha(\sum_{\alpha} P_{\alpha, L}^0 + \sum_{\alpha'} P_{\alpha', R}^0)$ is nothing but the original, unperturbed Wen plaquette Hamiltonian on the full cylinder (c.f. eqn. (28)) whose ground states are given by eqn. (1), with $a = a' \equiv (\tilde{Q}, Q)$ (hence justifying the use of a single label $a$). The small terms in eqn. (25) (and similarly for $R$) can therefore be thought of as perturbations to the large Hamiltonian, $H_0$, and their corrections to the non-degenerate ground state (within each a sector) can be calculated in normal non-degenerate wavefunction perturbation theory.

Similarly, we can decompose $\epsilon V_{LR}$ as

$$\epsilon V_{LR} = \epsilon \sum_{\alpha \geq 0} (P_{\alpha, L}^0 \otimes P_{\alpha, R}^0) V_{LR}(P_{\alpha, L}^0 \otimes P_{\alpha, R}^0) \hspace{1cm} (26)$$

because $V_{LR}$ is local and cannot mix different $\tilde{Q}_L$ sectors (and similarly for $\tilde{Q}_R$).

This decomposition allows us to identify the origins of the corrections of $\Lambda, \Theta, \Xi$ and $\Omega$, at least to the first order process to the corrections to the ground states, eqn. (7). For a fixed topological sector $a$, standard non-degenerate wavefunction perturbation theory tells us to correct the ground states in the first order process as

$$[\tilde{Q}, Q] = [\tilde{Q}, Q]_0 + \sum_{\text{exc.}} \frac{\langle \text{exc.} | O_{\text{small}} | \tilde{Q}, Q \rangle_0}{\Delta E} [\text{exc.}], \hspace{1cm} (27)$$

where $O_{\text{small}}$ are small corrections to the unperturbed Hamiltonian, $|\text{exc.}\rangle$ excited eigenstates of the exact Wen plaquette model, and $\Delta E$ the energy difference of the excited state and ground state which is always negative.

Thus we see that since $H_{\text{edge}, L}^0 + H_{\text{edge}, R}^0 : V_{\alpha, L}^0 \otimes V_{\alpha, R}^0 \rightarrow V_{\alpha, L}^0 \otimes V_{\alpha, R}^0$, it gives rise to part of the correction $\Lambda$. The other correction in $\Lambda$ comes from $(P_{\alpha, L}^0 \otimes P_{\alpha, R}^0) V_{LR}(P_{\alpha, L}^0 \otimes P_{\alpha, R}^0)$. Furthermore, both terms which contribute are symmetric under the $Z_2$ generators $\tilde{Q}_L$ and $\tilde{Q}_R$ and so do not mix the $Q_L$ and $Q_R$ topological sectors. Together, $\Lambda$ must be therefore $Q$ symmetric, i.e. $\Lambda = \mathbb{P}_Q \Lambda^T \mathbb{P}_Q$, for some $\Lambda'$, as claimed. The exact derivation of $\Lambda'$ is left to appendix \textbf{B}. It turns out that $\Lambda'$ is a Hermitian matrix, so that $\Lambda = \Lambda^T$.

Next, since $S_L^0$ is block diagonal, $[-S_L^0, n P_{n, L}^0] : V_{\alpha, L}^0 \otimes V_{\alpha, R}^0 \rightarrow V_{\alpha, L}^0 \otimes V_{\alpha, R}^0$, and generates $\Xi$ (the $R$ equivalent generators $\Theta$). Similarly, other terms in the decomposition of $V_{LR}$ also contribute to $\Theta, \Xi$ and $\Omega$.

Obviously, it is impossible to compute corrections to the ground state exactly for arbitrary local perturbations. However, it is still possible to say something concrete for generic arbitrary perturbations. In general, $\Lambda$ is of order $\epsilon$. We can then ignore $\Theta$ and $\Xi$ as they give rise to $\epsilon^2$ corrections in the eigenvalues of the reduced density matrix. Since we are only concerned to corrections to the ground state which give the leading order corrections to the entanglement energies (which are order $\epsilon$ from $\Lambda$), we only need to keep corrections in the dominant part of the reduced density matrix. The perturbed wavefunction, considered to leading order, is then

$$[\tilde{Q}, Q] = \frac{1}{\sqrt{Z}} \sum_{\tau, \tau'} (\mathbb{P}_Q - \mathbb{P}_Q \Lambda^T \mathbb{P}_Q)_{\tau, \tau'} |\tau \rangle_L |\tau' \rangle_R, \hspace{1cm} (28)$$

$Z$ being a normalization factor. The reduced density
matrix to leading order is then given by eqn. \(22\),
\[
\rho^a_L = \frac{1}{2} \exp[-H^a_{\text{ent.}}] = \text{Tr}_R(\hat{Q}, \hat{Q}) / \langle \hat{Q}, \hat{Q} \rangle \\
\approx \frac{1}{Z} (\mathbb{P}_Q - 2\mathbb{P}_Q \Lambda' \mathbb{P}_Q) \\
\approx \frac{1}{Z} \exp[-2\mathbb{P}_Q \Lambda' \mathbb{P}_Q],
\]
leading to the identification
\[
H^a_{\text{ent.}} \equiv 2\mathbb{P}_Q \Lambda' \mathbb{P}_Q.
\]
The entanglement spectrum, \(\xi^a_{\text{ent.}}\), is given by
\[
(\xi^a_{\text{ent.}})_n = 2 \text{eig}(\mathbb{P}_Q \Lambda' \mathbb{P}_Q)_n
\]
where \(n = 1, 2, \cdots, 2L_{y/2} - 1\) label the eigenvalues of eigenvectors that have \(\mathbb{P}_Q = 1\).

To conclude:

\[
\text{The entanglement Hamiltonian, } H^a_{\text{ent.}}, \text{ generically appearing at order } \epsilon^2, \text{ is given by eqn. } (30), \text{ where } \Lambda' \text{ is generated at lowest order by } H^a_{\text{edge}, L} + H^a_{\text{edge}, R} \text{ and } V_{LR} (\text{see appendix B}). \text{ It is } \mathbb{Z}_2 \text{ symmetric and acts on a virtual spin chain of } L_y/2 \text{ sites, similar to the edge Hamiltonians.}
\]

\subsection*{D. Edge-ES correspondence}

From the calculation of \(\Lambda'\) in appendix B and eqn. (31), we see that within each topological sector \(a\), the edge Hamiltonian \(H^a_{\text{edge}, L} + H^a_{\text{edge}, R}\) and the entanglement Hamiltonian \(H^a_{\text{ent.}}\) differ from each other in two ways: (i) terms in the edge Hamiltonian are reproduced in the entanglement Hamiltonian but with term-dependent rescaling factors, and (ii) additional terms arising from \(V_{LR}\) appear in the entanglement Hamiltonian. Since this means that \(H^a_{\text{edge}, L} + H^a_{\text{edge}, R}\) and \(H^a_{\text{ent.}}\) can differ in an arbitrary fashion, there is no reason to expect that the edge spectrum will match the entanglement spectrum, even for the low energy values. We therefore conclude that there is no edge-ES correspondence in general.

However, we note that both Hamiltonians have remarkably similar structure: they both act on a spin-1/2 chain of length \(L_y/2\), and are \(\mathbb{Z}_2\) invariant, i.e. they commute with the global spin flip operator \(\prod_i \tau_i^z\), which is the Wilson loop operator in the bulk. The \(\mathbb{Z}_2\) symmetry can therefore be understood as being enforced by the bulk topological order.

To summarize:

\[
\text{There is no edge-ES correspondence, even in the low energy limit, for generic local perturbations. However, both the edge and entanglement Hamiltonians are } \mathbb{Z}_2 \text{ symmetric and act on a virtual spin chain of } L_y/2 \text{ sites.}
\]

\section*{IV. MECHANISM FOR CORRESPONDENCE: TRANSLATIONAL SYMMETRY AND KRAMERS-WANNIER DUALITY}

In this section, we present a mechanism which ensures an edge-ES correspondence, at least in a finite domain in Hamiltonian space. We consider the Wen-plaquette model as a symmetry enriched topological phase (SET) with the global symmetry being translational invariance along the edge/entanglement cut. That is, we restrict ourselves to perturbations that are translationally invariant along the width of the cylinder. In that case, both the edge and entanglement Hamiltonians will be Kramers-Wannier (KW) self-dual.

It is not difficult to understand why this is so for the edge Hamiltonian. For the Wen-plaquette model and for an even \(L_y\) cylinder, we have assigned a consistent checkerboard coloring of the plaquettes (see fig. 1), in which \(e\) quasiparticles live on one color and \(m\) quasiparticles live on the other color. However, this coloring is not unique, and we could have swapped the two colors, effectively exchanging \(e \leftrightarrow m\) quasiparticles, a so-called electromagnetic duality. One can check that the fusion rules obeyed by the anyons are invariant under this swap. This swap can also be thought of being effected by simply translating the Wen-plaquette model by one site around the circumference of the cylinder, while keeping the underlying checkerboard coloring.

\section*{E. Remarks}

One might ask: what happens if \(\Lambda\) appears at order \(\epsilon^n\), \(n \geq 2\) instead of at order \(\epsilon^2\)? Of course, this situation is a result of fine-tuning the perturbations to the system. For example, perturbing the Wen-plaquette model with only single-site magnetic fields (so that \(V_{LR} = 0\)) will yield an edge Hamiltonian at order \(\epsilon^2\), and therefore \(\Lambda\) at order \(\epsilon^2\) as well. One has to now account for the additional contributions from \(\Theta, \Xi\) in eqn. (21) as they will lead to \(\epsilon^2\) corrections in the dominant part of the reduced density matrix, thereby potentially modifying the entanglement spectrum.

However, the procedure to account for these additional contributions is clear: we simply perform non-degenerate wavefunction perturbation theory on \(\hat{Q}, \hat{Q}\) to a desired order consistently in the reduced density matrix, using the decomposition eqn. (25) and eqn. (26).

An explicit example showing how this is done is given in the next section (also refer to appendix C), where we consider a mechanism to achieve an edge-ES correspondence in the low energy limit. We look at the case of uniform single-site magnetic fields as perturbations and present the perturbative calculations to order \(\epsilon^2\) explicitly. In that case, we will see that the terms \(\Theta, \Xi\) simply lead to a constant shift in the entanglement energies of the entanglement spectrum, and so the relation that \(H^a_{\text{ent.}} \equiv 2\mathbb{P}_Q \Lambda' \mathbb{P}_Q\) still holds up to a constant shift.
In terms of boundary operators, one sees that this swaps \( S_{p,L} \) with \( S_{\tilde{p},L} \), which necessarily leaves the physics of the edge or entanglement Hamiltonian invariant. However, recall that in the spin chain language in the edge Hamiltonian with the same coefficient, the mass gap. On the other hand, for a \( \mathbb{Z}_2 \) preserving theory on \( \mathcal{T} = \pm 1 \), then there will only be a single ground state with \( \mathcal{S} = +1 \). The difference in energies of these ground states with \( \mathcal{T} = \pm 1 \) will be exponentially small.

Now, let us take the limit as the length of the chain becomes infinite. In this limit, the boundary conditions do not matter, and we should only consider sectors with different \( \mathcal{S} \) of the original Hilbert space and different \( \mathcal{S} \) of the dual Hilbert space. From our exposition above, if the original Hamiltonian breaks the \( \mathbb{Z}_2 \) symmetry, then it will have two degenerate ground states labeled by \( \mathcal{S} = \pm 1 \) with some energy \( E \). However, after the KW transformation, the dual Hamiltonian will now have only one state near \( E \) (it has \( \mathcal{S} = +1 \)) - all other states have higher energies, with an energy difference of at least the mass gap. Thus we see that the dual Hamiltonian does not break the dual \( \mathbb{Z}_2 \) symmetry. Similarly, if the original Hamiltonian does not break the \( \mathbb{Z}_2 \) symmetry, it has only one state with \( \mathcal{S} = +1 \) that has lowest energy \( E' \); all other states are separated in energy by the mass gap. However, its dual Hamiltonian will have two states near energy \( E' \), with \( \mathcal{S} = \pm 1 \). Thus, we see that the dual Hamiltonian spontaneously breaks the dual \( \mathbb{Z}_2 \) symmetry. This thereby concludes the proof of our claim.

Next, let us be given a Kramers-Wannier \( \mathbb{Z}_2 \) symmetric self-dual Hamiltonian \( H_0 \), and deform it by a small \( \mathbb{Z}_2 \) symmetric perturbation \( H_1 \) which is odd under KW duality:

\[
H = H_0 + hH_1,
\]

where a KW transformation maps \( h \to -h \). If the deformed Hamiltonian \( H \) is gapped and breaks the \( \mathbb{Z}_2 \) symmetry spontaneously for some sign of \( h \), it will not break it for the opposite sign, and vice versa. Thus the theory must have a phase transition at \( h = 0 \). This therefore concludes the proof that \( H_{\text{edge}}^\mathcal{T} \) and \( H_{\text{ent}}^\mathcal{T} \) must be sitting at a phase transition.

Furthermore, it is natural to expect the phase transition to be generically of second order. If that was the case, the KW self-dual Hamiltonians would be gapless (i.e. critical). Now, a stronger statement can be made: if the dominant part of \( H_{\text{edge}}^\mathcal{T} \) and \( H_{\text{ent}}^\mathcal{T} \) are the transverse field Ising model ((1 + 1)-d Ising model), then the low energy spectra of both Hamiltonians must be that of the \( c = 1/2 \) Ising CFT. This is guaranteed because there
are no relevant deformations to the critical (1 + 1)-d Ising model that are both $Z_2$ invariant and KW-even. To see this is true, list all relevant operators in the theory (i.e. with scaling dimension $\Delta < 2$): the spin primary field $\sigma$ and the energy density primary field $\epsilon$. Now $\sigma$ is $Z_2$ even, so this deformation does not appear in the edge and entanglement Hamiltonians. On the other hand, $\epsilon$ is $Z_2$ even but is KW-odd, so it cannot appear either. Any deformations in the edge and entanglement Hamiltonians must therefore be irrelevant - all renormalization flows are towards the $c = 1/2$ Ising CFT.

This therefore shows that that we there is a finite domain in Hamiltonian space such that the edge and entanglement Hamiltonians both realize the $c = 1/2$ Ising CFT as their low energy effective theory. It is therefore seen that in this case, the Wen-plaquette model, considered as an SET with the global symmetry being translational invariance along the edge/entanglement cut, realizes an edge-ES correspondence: the low lying values of both the edge and entanglement spectra will match.

However, a note of caution should be pointed out here. There is no guarantee that the edge and entanglement Hamiltonians will be both near the critical (1 + 1)-d Ising models, even though it is natural to assume they should be. There are other models which are also $Z_2$ symmetric and Kramers-Wannier self-dual, such as the tricritical Ising model or even $Z_2$ symmetric Hamiltonians which break the KW-duality spontaneously (i.e. the model realizes a first order transition, which the critical Ising model and tricritical Ising model do not).

Thus, one cannot conclude that the edge and entanglement Hamiltonians must both be the $c = 1/2$ Ising CFT. For example, it could be that the edge Hamiltonian is a critical Ising model while the entanglement Hamiltonian is instead a tricritical Ising model. In that case, there is no edge-ES correspondence in the way that we have defined, as the low energy spectra clearly do not match. However, what is still guaranteed with the global translational symmetry is that both the edge and entanglement Hamiltonians will be $Z_2$ symmetric and Kramers-Wannier dual; thus, a weaker form of edge-ES correspondence holds in which both Hamiltonians belong to the same class of Hamiltonians, whether or not the specific form of the Hamiltonian is the critical Ising, tricritical Ising, or a first order phase transition Hamiltonian.

In summary:

Considered as an SET, the symmetry protection (translational invariant perturbations) and the bulk topological order ensure that the edge and entanglement Hamiltonians of the Wen-plaquette model are Kramers-Wannier self-dual and are $Z_2$ symmetric. If both theories are close to the critical (1 + 1)-d Ising model, then there are no relevant KW-even and $Z_2$ symmetric perturbations to the model, this implies that there is a finite domain in Hamiltonian space in which their low energy physics is a $c = 1/2$ Ising CFT. There is therefore an edge-ES correspondence in such a scenario (the low energy spectra match). However, there is no guarantee that both Hamiltonians will always be critical Ising models, as there are other $Z_2$ symmetric models which realize the Kramers-Wannier self-duality, such as the tricritical Ising model or a Hamiltonian sitting at a first order phase transition. Thus, a weaker form of the edge-ES correspondence instead holds, in which both the edge and entanglement Hamiltonians belong to the same class of Kramers-Wannier even and $Z_2$ symmetric Hamiltonians.

A. Analytical example: uniform single-site magnetic fields

We present an analytic and numerical illustration of our claim of the edge-ES correspondence, in which both the edge and entanglement Hamiltonians realize the critical (1 + 1)-d Ising models and thus have the $c = 1/2$ Ising CFT as their low energy effective theory. Let us consider the case of perturbations being uniform single-site magnetic fields:

$$\epsilon V = \epsilon \sum_i h_X X_i + h_Y Y_i + h_Z Z_i,$$

(33)

where $(h_X, h_Y, h_Z)$ is a vector with entries of order 1. This perturbation is translationally invariant along the edge/cut of the cylinder. On physical grounds it is the simplest possible local perturbation of the Toric code, and on theoretical grounds, it is the deformation of the Toric code that is the most well studied, see existing literature on the subject, e.g. Ref. [38–40].

Such a uniform single-site magnetic field generates an edge and entanglement Hamiltonian with the critical (1 + 1)-d Ising model as their dominant term at order $\epsilon^2$: eqn. (19) (appendix C) gives the detailed calculations. This tells us that the edge Hamiltonians of each topological sector are the critical periodic ($Q = +1$) /antiperiodic ($Q = -1$) transverse field Ising Hamiltonians projected into the $Z_2$ sectors ($Q = \pm 1$). Explicitly, the critical (1 + 1)-d Ising model on a spin chain of length $L_y/2$, also called the transverse field Ising model (TFIM), is given
by
\[ H_{\text{TFIM}}^{Q} = -\sum_{p=1}^{L_y/2} (\tau_p^x \tau_{p-1}^x + \tau_p^z \tau_{p-1}^z), \]

with toroidal boundary conditions \( \tau_0 = \bar{Q} \tau_{L_y/2} \), and its decomposition into its \( \mathbb{Z}_2 \) charge sectors as follows:
\[ H_{\text{TFIM},Q} = \mathbb{P}_Q H_{\text{TFIM}}^{Q} \mathbb{P}_Q, \]

where \( \mathbb{P}_Q = \prod_{p=1}^{L_y/2} \tau_p^z \). This model is the lattice realization of the \( c = 1/2 \) Ising CFT.

On the other hand, for the entanglement Hamiltonians (refer to appendix C2 for the calculation of the entanglement Hamiltonians), up to rescaling and shifting, we have the following identification between states in the topological phase (left) and their entanglement Hamiltonians to leading order (\( c^2 \)) (right):
\[
|\text{1}\rangle \leftrightarrow H_{\text{TFIM},+1}^{+1} \\
|\text{c}\rangle \leftrightarrow H_{\text{TFIM},-1}^{+1} \\
|m\rangle \leftrightarrow H_{\text{TFIM},-1}^{-1} \\
|\varepsilon\rangle \leftrightarrow H_{\text{TFIM},-1}^{-1},
\]

where the label \( \{I,c,m,\varepsilon = c \times m\} \) indicates that the states carry the corresponding anyonic flux.

Thus, we see that there is an edge-ES correspondence in this case: both the edge and entanglement Hamiltonians have the \( c = 1/2 \) Ising CFT as their low energy effective theory.

### B. Numerical example: uniform single-site magnetic fields

To check our predictions, we numerically solve for the ground states of the Wen-plaquette model on an infinite cylinder with even \( L_y \) circumference, \( L_y = 20 \), with perturbations of the form eqn. (33), using DMRG for infinite cylinder (13). Here, \( e_{hX} = -0.01 \), \( e_{hY} = -0.015 \), and \( e_{hZ} = -0.02 \). Starting from a random initialization of the MPS, it was found that the DMRG algorithm converged to four orthonormal states, which are the four ground states with well-defined anyonic flux through the cylinder. We then find the entanglement spectrum associated with each ground state, and plot it against momenta around the cylinder. Figure 4 shows the entanglement spectra in each topological sector \( a \) of the Wen-plaquette model with the parameters described above.

Each spectrum has been (i) shifted and then (ii) rescaled by the same value across all topological sectors \( a \). The shift has been chosen so that the lowest entanglement energy across all entanglement spectra is at entanglement energy 0, and the rescaling chosen so that in the topological sector which contains the lowest overall entanglement energy, the lowest entanglement energy at momentum \( K = \pm 2 \) is at entanglement energy 2. In doing so, we fix the identity primary (scaling dimension \( \Delta = 0 \), momentum \( K = 0 \)) and the holomorphic and antiholomorphic stress-energy tensors (scaling dimension \( \Delta = 2 \), momentum \( K = \pm 2 \)), which are always present in a CFT without defects. Remarkably, this common shift and rescaling of the entanglement spectra across topological sectors \( a \) reproduces the \( c = 1/2 \) Ising CFT spectra in the different charge sectors accurately: from figure 4(a), we can identify the identity primary \( 1 \) (conformal weights \( (h_1, \bar{h}_1) = (0, 0) \)) and the energy-density primary \( \epsilon \sim (1/2, 1/2) \) with its descendants, which belong to the \( (Q, Q) = (+1, +1) \) charge sector of the TFIM. In the CFT language this is the \( \mathbb{Z}_2 = +1 \) sector of the usual diagonal \( c = 1/2 \) Ising CFT. From 4(b), we identify the spin primary \( \sigma \sim (1/16, 1/16) \) with its descendants, which belong to the \( (Q, \bar{Q}) = (+1, -1) \) charge sector of the TFIM. This corresponds to the \( \mathbb{Z}_2 = -1 \) sector of the Ising CFT. From 4(c), we identify the disorder primary \( \mu \sim (1/16, 1/16) \) with its descendants, which is identical to the spectrum of \( \sigma \), belonging to the \( (Q, Q) = (-1, +1) \) sector of the TFIM. In this case, this corresponds to the \( \mathbb{Z}_2 = +1 \) sector of \( c = 1/2 \) Ising CFT with anti-periodic boundary conditions. Lastly, we identify the Majorana fermions \( \psi \sim (1/2, 0) \) and \( \bar{\psi} \sim (0, 1/2) \) and its descendants which belong to the \( (Q, \bar{Q}) = (-1, -1) \) sector of the TFIM. This corresponds to the \( \mathbb{Z} = -1 \) sector of the Ising CFT with anti-periodic boundary conditions. This is in agreement with the theoretical prediction from our calculation, eqn. (36).

Lastly, even though our analysis in this paper was restricted to the Wen-plaquette model on an infinite cylinder with even \( L_y \) circumference, we numerically solve for the ground states of the perturbed Wen-plaquette model on an infinite cylinder with odd \( L_y \) circumference. The perturbation is as before, eqn. (33), with values as in the even cylinder case. We take \( L_y = 19 \). In this case, a consistent checkerboard coloring of the plaquettes cannot be done. If one insists on placing a checkerboard coloring on the cylinder, there is necessarily a line of topological defects where an \( e \) quasiparticle (as measured locally) transmutes into an \( m \) quasiparticle upon crossing the line defect (14).

In this case, we obtain two ground states. This makes sense as the only Wilson loop operator that exists is a string that wraps around the cylinder twice. The two ground states can be taken to be eigenvectors of this Wilson loop operator. Figure 5 shows the plots of the entanglement energies against momenta along a \( L_y = 19 \) cylinder for both ground states, with a common shifting and rescaling as follows. We shift the spectra of the two ground states so that the lowest entanglement energy of one sector is at 1/16, and we rescale both ES so that the next lowest entanglement energy in that sector is at 9/16. The two ES exactly match the spectrum of the boundary.
FIG. 4. (Color online). Rescaled and shifted entanglement spectra in each topological sector \(a\) for Wen-plaquette model on an \(L_y = 20\) (even), infinite cylinder, for \(\epsilon_h X = -0.01, \epsilon_h Y = -0.015\) and \(\epsilon_h Z = -0.02\). The data points are marked by blue crosses. We have also plotted the \(c = 1/2\) Ising CFT spectra in each sector for comparison (scaling dim. versus momenta), with the identification that entanglement energy equals scaling dimension. They are given by the green squares and circles. Sectors and circles represent conformal towers labeled by different primaries at the bottom of the towers, and double circles denote two-fold degenerate states. (a) The ES of the ground state \(|\cdot\rangle\) with the identity anyonic flux through the cylinder. This corresponds to the \((\tilde{Q}, Q) = (+1, +1)\) sector of the TFIM, whose low energy effective theory is the \(c = 1/2\) Ising CFT in the sector that the identity primary & its descendants (green circles), and energy density primary \(\epsilon\) & its descendants belong to. (b) ES of \(|e\rangle\), the state with the electric anyonic flux. This gives the \((\tilde{Q}, Q) = (+1, -1)\) sector of the TFIM. This corresponds to the sector of the Ising CFT which the spin primary \(\sigma\) and its descendants belong to (green circles). (c) ES of \(|m\rangle\), the state with the magnetic anyonic flux. This gives the \((\tilde{Q}, Q) = (-1, +1)\) sector of the TFIM, which in turn corresponds to the sector of the Ising CFT (with a \(D_1\) defect insertion) which the disorder primary \(\mu\) and its descendants belong to (green circles). The spectra of the conformal families from \(\sigma\) and \(\mu\) coincide, and so do the entanglement spectra. (d) ES of \(|\epsilon\rangle\), the state with the fermionic anyonic flux. This gives the \((\tilde{Q}, Q) = (-1, -1)\) sector of the TFIM. This corresponds to the sector of the Ising CFT (with a \(D_1\) defect insertion) which the two Majorana fermion primaries \(\psi, \bar{\psi}\) and their descendants belong to (green squares and circles).

All these results can be understood from a CFT point-of-view, using the language of conformal defects. The \(c = 1/2\) Ising CFT can be given different twisted boundary conditions by insertions of conformal defect lines \(X\), where the spectrum is then given by the generalized twisted partition function

\[
Z_X = \text{tr} \left( X q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \right).
\]

In order to be able to move the defects around without energy cost, we need \(X\) to commute with the energy-momentum tensor, or equivalently \([L_n, X] = [\bar{L}_n, X] = 0\) where \(L_n\) and \(\bar{L}_n\) are the Virasoro algebra generators. The conformal defects are then classified by representations of the Virasoro algebra, for the \(c = 1/2\) CFT we have three defects: \(D_1, D_\epsilon\) and \(D_\mu\). Crossing the \(D_\epsilon\) defect \(\sigma\) goes to \(-\sigma\) and it thus implement anti-periodic boundary conditions, while for
$D_\sigma$ we have $\sigma \rightarrow \mu$ which is nothing but the Kramers-Wannier duality. The spectra with these defect insertions are given by \cite{41,42}

$$
\begin{align*}
Z_1 &= |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2 + |\chi_{\frac{1}{2}}|^2, \\
Z_\sigma &= |\chi_{\frac{1}{2}}|^2 + \chi_0 \bar{x}_{\frac{1}{2}} + \chi_{\frac{1}{2}} \bar{x}_0, \\
Z_\sigma &= (\chi_0 + \chi_{\frac{1}{2}}) \bar{x}_{\frac{1}{2}} + \chi_{\frac{1}{2}}(\bar{x}_0 + \bar{x}_{\frac{1}{2}}).
\end{align*}
$$

The spectra in figure 4(a) and 4(b) correspond to the insertion of the trivial (identity) conformal defect $D_1$ in the partition function, while the spectra figure 4(c) and 4(d) correspond to the insertion of the conformal epsilon defect $D_\sigma$. These two cases are distinguished by the absence and presence of a Wilson flux line in the ground states of the bulk respectively, as measured by \cite{41}. This Wilson flux line terminates at the boundary and is responsible for implementing anti-periodic BC, and thus can be thought of as ‘inserting’ the conformal defect $D_\sigma$ which has the same function. Next, the spectra in figure 4 correspond to the insertion of the conformal duality defect $D_\sigma$ in the partition function. This duality defect is nothing but the endpoint of the topological line defect in the bulk which comes from electro-magnetic duality. Thus this is completely consistent with the following correspondence between bulk and boundary: \textit{electro-magnetic duality} $\leftrightarrow$ \textit{Kramers-Wannier duality}, as seen several times in this paper. Perhaps this hints that one can understand defects in a CFT as arising from defects in a topological phase of one higher dimension, which has the CFT as its boundary theory.\cite{41}

Next, if one can find a mechanism to protect the edge-ES correspondence in such cases, then one can construct lattice realizations of CFTs with defects, by constructing the low-energy effective edge Hamiltonian using the Schrieffer-Wolff transformation and identifying a mapping from boundary operators acting in the bulk to local operators acting on some effective low-energy degrees of freedom. In particular, for the case of the Wen-plaquette model on the odd cylinder, we find by direct perturbative calculations the lattice realization of a duality-twisted Ising model, which was only written down very recently by Ref. \cite{47} eqn. (37).

V. CONCLUSION AND DISCUSSION

In this paper, we have studied the edge-ES correspondence in a non-chiral topological phase, specifically concentrating on a phase with $\mathbb{Z}_2$ topological order, the Wen-plaquette model on an infinite cylinder. Our main result of the paper is a detailed, microscopic calculation of both the edge and entanglement Hamiltonians, which exhibits the mechanism for the absence or presence of the correspondence. We find through our calculation that the correspondence, i.e., that the edge and entanglement spectra agree in the low energy limit up to rescaling and shifting, is exact for the unperturbed Wen-plaquette model. However, for generic local perturbations, there is no edge-ES correspondence.

We have managed to identify a mechanism to establish the edge-ES correspondence though, by considering the Wen-plaquette model as a SET with the global symmetry being translational invariance along the edge/entanglement cut. That is, if we deform the Wen-plaquette model by perturbations that are restricted to be invariant under translation by one site along the edge/cut, then both the edge and entanglement Hamiltonians are Kramers-Wannier self-dual. There is then a finite domain in Hamiltonian space such that both the edge and entanglement Hamiltonians have the $c = 1/2$ Ising CFT as their low energy effective theory, as there are no KW-even and $\mathbb{Z}_2$ symmetric perturbations to the critical $(1+1)$-d Ising model that are relevant. Thus, the edge-ES correspondence is achieved in such a scenario. However, there is no guarantee that both the edge and

![FIG. 5. (Color online). Rescaled and shifted entanglement spectra in each topological sector for the perturbed Wen-plaquette model on an $L_y = 19$ (odd), infinite cylinder. Shown also are the $c = 1/2$ CFT spectra with a conformal duality defect $D_\sigma$ (see figure 4 for details on labels). (a) The ES corresponds to the spectrum of the duality-twisted Ising model with primary fields $(0, 1/16), (1/2, 1/16)$. (b) The ES corresponds to the spectrum of the duality-twisted Ising model with primary fields $(1/6, 0), (1/6, 1/2)$.](image_url)
entanglement Hamiltonians will be near the critical Ising models, as there are other models which are Kramers-Wannier even and $\mathbb{Z}_2$ symmetric as well. Thus, a weaker form of the edge-ES correspondence holds, in which both Hamiltonians belong to the same class of Hamiltonians, even though the low energy spectra do not match.

Our approach of deriving an effective spin ladder Hamiltonian spanning the entanglement cut can potentially be extended to prove some form of the edge-ES correspondence in other non-chiral topological phases that are defined by a fixed point Hamiltonian consisting of a sum of mutually commuting local terms. One would have to find a set of maximally commuting operators on one half of the system, and also find a representation of the algebra of the operators acting on the edge to derive an analogous spin ladder system. In particular, we note that the extension of our calculations to the direct generalization of $\mathbb{Z}_2$ topological phases, $\mathbb{Z}_N$ topological phases, is straightforward, and we leave it to future work to explore the edge-ES correspondence in those cases.

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Appendix A: Schrieffer-Wolff transformation

In this appendix we introduce the Schrieffer-Wolff transformation which reproduces the work of Ref. [35].

Let $H_0$ be a Hamiltonian that has a low-energy eigenspace $\mathcal{V}_0$ and a high-energy eigenspace $\mathcal{V}_1$ separated by a spectral gap. $H_0$ can be written as

$$H_0 = P_0 H_0 P_0 + P_1 H_0 P_1,$$

where $P_\alpha$ are projectors onto $\mathcal{V}_\alpha$, where $\alpha = 0, 1$. Let us add a small perturbation $\epsilon V$ which does not commute with $H_0$. Assuming the perturbation is weak enough, the new Hamiltonian will still have low and high-energy eigenspaces, $\mathcal{V}_0$ and $\mathcal{V}_1$, which have the same dimension as $\mathcal{V}_0$ and $\mathcal{V}_1$ respectively. That is,

$$H = H_0 + \epsilon V = \tilde{P}_0 H \tilde{P}_0 + \tilde{P}_1 H \tilde{P}_1.$$  

Then there exists a unique direct rotation (i.e. unitary) $U$ that rotates the old and new subspaces

$$U \tilde{P}_\alpha U^\dagger = P_\alpha.$$  

By direct rotation, we mean the ‘minimal’ rotation that maps $\tilde{P}_\alpha$ to $P_\alpha$: among all unitary operators $U$ satisfying $U \tilde{P}_\alpha U^\dagger = P_\alpha$, the direction rotation $U$ differs least from the identity in the Frobenius norm. See Ref. [35] for the construction of a direct rotation. In that case, we can rotate $H$ to a new Hamiltonian $H'$ with eigenspaces $\mathcal{V}_\alpha$:

$$H' := U H U^\dagger = P_0 U H U^\dagger P_0 + P_1 U H U^\dagger P_1.$$  

This is the so-called Schrieffer-Wolff transformation.

There exists a unique antihermitian and block-off diagonal (in both $\mathcal{V}_\alpha$ and $\hat{V}_\alpha$) operator $S$ such that $U = e^S$ and $||S|| < \pi/2$. It is constructed perturbatively as follows.

First we introduce some notation. Decompose an operator $X$ on the Hilbert space in its block-diagonal $X_d$ and block-off-diagonal $X_{od}$ parts:

$$X_d = P_0 X P_0 + (1 - P_0) X (1 - P_0),$$

$$X_{od} = P_0 X (1 - P_0) + (1 - P_0) X P_0.$$  

Also given $Y$, an operator acting on the Hilbert space, define the linear map which is the adjoint action of $Y$ on other operators $X$ that act on the Hilbert space:

$$\hat{Y}(X) = [Y, X].$$  

Lastly, define

$$\mathcal{L}(X) = \sum_{i,j} \langle i | X_{od} | j \rangle \frac{E_i - E_j}{E_i} | i \rangle \langle j |,$$

where $| i \rangle$ is an eigenvector of $H_0$ with eigenvalue $E_i$ and similarly for $j$. Note that $\mathcal{L}(X)$ is by construction block off-diagonal.

Now, $S$ can be written perturbatively as

$$S = \sum_{n=1}^\infty \epsilon^n S_n, \quad S_n^\dagger = -S_n,$$

where

$$S_1 = \mathcal{L}(X_{od}),$$

$$S_2 = -\hat{\mathcal{L}}(S_1),$$

$$S_n = -\hat{\mathcal{L}}(S_{n-1}) + \sum_{j \geq 1} a_{2j} \mathcal{L}(S_{od})^{2j}(V_{od})_{n-1},$$

for $n \geq 3$.  

(A9)

Here the coefficients $a_m$ come from the Taylor series

$$x \coth(x) = \sum_{n=0}^\infty a_{2n} x^{2n}, \quad a_m = \frac{2^m B_m}{m!},$$

(A10)
where $B_m$ are the Bernoulli numbers. We have also used the shorthand
\begin{equation}
\hat{S}^k_{(\text{od})m} = \sum_{n_1, \ldots, n_k \geq 1 \atop n_1 + \cdots + n_k = m} \hat{S}_{n_1} \cdots \hat{S}_{n_k}, \tag{A11}
\end{equation}
From here, the low-energy effective Hamiltonian of $H'$ is given by
\begin{equation}
H_{\text{eff}} := P_0 H' P_0 = P_0 H_0 P_0 + \epsilon P_0 V P_0 + \sum_{n=2}^\infty \alpha^n H_{\text{eff},n}, \tag{A12}
\end{equation}
where
\begin{equation}
H_{\text{eff},n} = \sum_{j \geq 1} b_{2j-1} P_0 \hat{S}^{2j-1}_{(\text{od})n-1} P_0. \tag{A13}
\end{equation}
$b_{2j-1}$ are the Taylor coefficients of $\tanh(x/2)$, i.e.
\begin{equation}
\tanh(x/2) = \sum_{n=1}^{\infty} b_{2n-1} x^{2n-1}, \quad b_{2n-1} = \frac{2(2^{2n} - 1) B_{2n}}{(2n)!}. \tag{A14}
\end{equation}
To second order in perturbation theory, and with a basis of states $|i, \alpha\rangle \in V_\alpha$, the matrix elements of $H'$ projected into $V_\alpha$ is given explicitly
\begin{equation}
\langle i, \alpha \mid H' \mid j, \alpha \rangle = E_i^\alpha \delta_{ij} + \epsilon \langle i, \alpha \mid V \mid j, \alpha \rangle + \frac{\epsilon^2}{2} \sum_{\beta \neq \alpha} \langle i, \alpha \mid V \mid k, \beta \rangle \left( \frac{1}{E_i^\alpha - E_k^\beta} + \frac{1}{E_j^\alpha - E_k^\beta} \right), \tag{A15}
\end{equation}
where $E_i^\alpha$ is the energy of $|i, \alpha\rangle$.

The generalized Schrieffer-Wolff transformation to Hamiltonians that have many invariant subspaces, each separated by a spectral gap to the next subspace, such that the Hilbert space $\mathcal{H} = \bigoplus_{\alpha \geq 0} V_\alpha$, is given in Ref. [36]. The generator of rotation $\hat{S}$ is still a block off-diagonal hermitian operator, with its first term in its expansion given by
\begin{equation}
S_1 = \sum_{i,j,\alpha,\beta \atop \alpha \neq \beta} \langle i, \alpha \mid V_{\text{od}} \mid j, \beta \rangle \frac{E_i^\alpha}{E_i^\alpha - E_j^\beta} |i, \alpha\rangle \langle j, \beta|, \tag{A16}
\end{equation}
where $|i, \alpha\rangle \in V_\alpha$, $|j, \beta\rangle \in V_\beta$ and $E_i^\alpha$ is the energy of $|i, \alpha\rangle$. We will use this term in the derivation of the entanglement Hamiltonian.

Appendix B: Calculation of $\Lambda'$ of the entanglement spectrum

In this appendix, we calculate $\Lambda'$, (see section III C eqn. [21]), obtained from standard non-degenerate wavefunction perturbation theory. Our starting point is the perturbed Wen-plaquette Hamiltonian on the infinite cylinder, $H = H_L + H_R + H_{LR} + \epsilon(V_L + V_R + V_{LR})$, written according to the decompositions given by eqn. (25) and eqn. (26) using the Schrieffer-Wolff transformation. This reorganizes the perturbative series to make the tensor product structure of the new ground state manifest. We have:
\begin{equation}
H = H_{LR} + \sum_{\alpha \geq 1} \alpha \left( \sum_a P_{\alpha,L}^a + \sum_{a'} P_{\alpha,R}^{a'} \right) \tag{B1}
\end{equation}
wherethe large part is simply the unperturbed Wen-plaquette model $H_L + H_R + H_{LR}$, and the other part is small (order $\epsilon$ or higher) which comes from the perturbations $\epsilon(V_L + V_R + V_{LR})$.

From section IIIB, it suffices to solve $H_{LR}$ for the four ground states of the Wen-plaquette model. The projection of $H_{LR}$ onto a $\tilde{Q}$ sector, under the mapping given by eqn. (11), becomes a spin-ladder Hamiltonian:
\begin{equation}
H_{\text{eff}} = - \sum_{p-1} \left( \tau_{p,L}^x \tau_{p,R}^x + \tau_{p,L}^z \tau_{p-1,L}^z \tau_{p,R}^z \tau_{p-1,R}^z \right), \tag{B2}
\end{equation}
whose ground states (in each topological sector $a \simeq (\tilde{Q}, Q)$) are given by eqn. (7):
\begin{equation}
|\tilde{Q}, Q\rangle = \frac{1}{\sqrt{2^{L_z}/2-1}} \sum_{\tau} \Psi_Q(\tau) \tau_{L} \tau_{R}, \tag{B3}
\end{equation}
where $|\tau\rangle_L \in V_{0,L}$ is a spin configuration on the $L$ spin chain. A similar statement holds for $|\tau\rangle_R$ on the $R$ spin chain. Let us call the operator $\tau_{p,L}^x \tau_{p,R}^x$ the ‘$x$-run’ operator and $\tau_{p,L}^z \tau_{p-1,L}^z \tau_{p,R}^z \tau_{p-1,R}^z$ the ‘$z$-run’ operator.

For each topological sector $a$, we wish to calculate the corrections to the ground state of eqn. (B2) in its $V_{0,L} \otimes V_{0,R}^Q$ tensor product structure, generated by the small part of eqn. (B1). To leading order in $\epsilon$, the relevant terms in $H$ for each $\tilde{Q}$ sector that generate the changes in the two ground states of $\tilde{Q}$ are
\begin{equation}
H_{\text{edge},L}^\tilde{Q} + H_{\text{edge},R}^\tilde{Q} + \epsilon (P_{0,L}^\tilde{Q} \otimes P_{0,L}^\tilde{Q}) V_{LR}(P_{0,L}^\tilde{Q} \otimes P_{0,L}^\tilde{Q}). \tag{B4}
\end{equation}
From the mapping to virtual spin operators acting on the two spin chains, we can rewrite the above as

\[
   f_{Q,L}(\tau^x_{p,L}, \tau^z_{p,L} \tau^x_{p-1,L}) + f_{Q,R}(\tau^x_{p,R}, \tau^z_{p,R} \tau^x_{p-1,R}) + g_Q(\tau^x_{p,L}, \tau^z_{p,L} \tau^x_{p-1,L} + \tau^z_{p,R} \tau^x_{p-1,R}),
\]

where \(f_{Q,L}\) and \(f_{Q,R}\) are as in eqn. (18) and \(g_Q\) the function associated with \((P^Q_{0, L} \otimes P^Q_{0, R})V_{LR}(P^Q_{L, L} \otimes P^Q_{L, R})\). There are also toroidal boundary conditions for both chains given by \(Q_L\) (i.e. \(\tau^z_{0,L} = -\tau^z_{L/2,L}\)) on the \(L\) chain and similarly for the \(R\) chain.

Since both the original, unperturbed Hamiltonian eqn. (B2) and the perturbation eqn. (B5) are symmetric under \(\mathbb{Z}_2 \times \mathbb{Z}_2\), generated by \(Q_L = \prod_p \tau^z_{p,L} \) and \(Q_R = \prod_p \tau^x_{p,R}\), it suffices to perform standard, non-degenerate wavefunction perturbation theory to the ground states eqn. (B3), which have \(Q_L = Q_R = Q\). We therefore need to solve for all the eigenstates of \(H_{\text{eff}}\).

**Eigenstates of \(H_{\text{eff}}\).** \(H_{\text{eff}}\) is exactly solvable because all terms in it commute. However there is one constraint: \(\prod_p \tau^z_{p,L} \tau^x_{p-1,L} \tau^z_{p,R} \tau^x_{p-1,R} = 1\), and one term, \(Q_L\), not present in \(H_{\text{eff}}\) that commutes with it. Thus, all eigenstates of \(H_{\text{eff}}\) can be uniquely labeled by the eigenvalues of the set of commuting operators

\[
   \left\{ \{\tau^x_{p,L} \tau^z_{p,R}\}_{p=1}^{L_y/2}, \{\tau^x_{p,L} \tau^z_{p-1,L} \tau^x_{p,R} \tau^z_{p-1,R}\}_{p=2}^{L_y/2}, Q_L \right\}.
\]

(B6)

Note the choice of the \(z\)-rung operators from \(p = 2\) to \(L_y/2\) only. The ground states of \(H_{\text{eff}}\) satisfy \(\tau^z_{p,L} \tau^z_{p,R} = +1\) and \(\tau^x_{p,L} \tau^x_{p,R} \tau^z_{p-1,L} \tau^z_{p-1,R} = +1\). There are two ground states given by eqn. (B3) with \(Q = Q_L\).

Now consider excited states of \(H_{\text{eff}}\) which we will need in perturbation theory. Such states can be built up from the ground states by acting products of the following mutually commuting operators on them (they also commute with \(Q_L\)):

\[
   s^z_p := \tau^z_{p,R}, \text{ for } p = 1, \ldots, L_y/2 \text{ and } s^z_p := \prod_{2 \leq q \leq p} \tau^z_{q,L}, \text{ for } p = 2, \ldots, L_y/2,
\]

(B7)

to give \(\prod s^{z/x}_p |\tilde{Q}, Q\rangle\). These \(s\)-operators violate the \(x\)-rung or \(z\)-rung = +1 conditions with an energy cost of \(2q + 4g\) above the ground states respectively.

**Ground state perturbation; calculation of \(\Lambda^\prime\).** We are now ready to find the four perturbed ground states of the Wen-plaquette model on the infinite cylinder. Let us find the correction to the ground state \(|Q, Q\rangle\) for fixed \(|\tilde{Q}, Q\rangle\).

Consider first the contribution from \(H^a_{\text{edge,L}} = \mathbb{P}_{Q,L} Q_L (\tau^x_{p,L} \tau^z_{p,L} \tau^x_{p-1,L}) \mathbb{P}_{Q,L}\) first, \(f_{Q,L}\) contains products of \(\tau^x_{L,p}\) and \(\tau^z_{L,p} \tau^x_{p-1,L}\). Let us consider a generic first order term in \(f_{Q,L}\) given by \(\epsilon(c \prod \tau^x_{p,L} \prod \tau^z_{p,L} \tau^x_{p-1,L})\), where \(c\) is the coefficient of the term (necessarily making it hermitian). From the first order process in perturbation theory, the correction to the ground state from this term is an order \(\epsilon\) correction given by

\[
   \epsilon \sum_{\text{prod}} \frac{\langle \tilde{Q}, Q \rangle \prod s^{z/x}(\prod \tau^x_{p,L} \prod \tau^z_{p,L} \tau^x_{p-1,L}) |\tilde{Q}, Q\rangle}{\Delta(\prod s^{z/x})} \times \prod s^{z/x} |\tilde{Q}, Q\rangle.
\]

(B8)

An explanation is in order. The sum is over all possible products \(\prod s^{z/x}\) which generate all possible excited states \(\prod s^{z/x} |\tilde{Q}, Q\rangle\) (in the same topological sector), with the prime denoting the exclusion of the trivial product. \(\Delta(\prod s^{z/x})\) is the energy gap between the ground state \(|Q\rangle\) and the excited state defined by the product, which is always negative: \(\Delta < 0\). We have also made use of the fact that \(\mathbb{P}_{Q,L} = 1\) on the particular ground state we are working with.

It is not hard to see that the only contributions arise if \(\prod s^{z/x}(\prod \tau^x_{p,L} \prod \tau^z_{p,L} \tau^x_{p-1,L}) = 1\) or \(Q_L\). That is, the excited states cancel the excitations over the ground states from \(H^a_{\text{edge,L}}\). In other words, we have

\[
   \prod s^{z/x} = \prod \tau^x_{p,L} \prod \tau^z_{p,L} \tau^x_{p-1,L} \text{ or } \prod s^{z/x} = \tilde{Q}_L \left( \prod \tau^x_{p,L} \prod \tau^z_{p,L} \tau^x_{p-1,L} \right).
\]

(B9)

For the first case, the matrix element is 1, and so the excitation induced from \(H^a_{\text{edge,L}}\) on the ground state is reproduced in the correction to the ground state, up to a negative rescaling that depends on the energy difference of this excited state with the ground state. For the second case, the matrix element is \(Q\) (since \(Q_L = Q\)), and the correction to the ground state is

\[
   \tilde{Q}_L \left( \prod \tau^x_{p,L} \prod \tau^z_{p,L} \tau^x_{p-1,L} \right) |\tilde{Q}, Q\rangle \\
   = \tilde{Q}_L \prod \tau^x_{p,L} \prod \tau^z_{p,L} \tau^x_{p-1,L} |Q\tilde{Q}_L |\tilde{Q}, Q\rangle \\
   = Q \left( \prod \tau^x_{p,L} \prod \tau^z_{p,L} \tau^x_{p-1,L} \right) |\tilde{Q}, Q\rangle,
\]

(B10)

where in the second equality we have commuted \(\tilde{Q}_L\) past the excitations. The \(Q\) from the matrix element cancels the \(Q\) from the correction to the ground statement, and so this correction once again reproduces the excitation induced from \(H^a_{\text{edge,L}}\) on the ground state, up to a negative rescaling \(1/\Delta(\prod s^{z/x})\).

Interestingly, we therefore see that each term in the edge Hamiltonian \(H^a_{\text{edge,L}}\) is reproduced as excitations to the ground state acting on the \(L\) spin chain, albeit with a negative term-dependent rescaling \(1/\Delta(\prod s^{z/x}) < 0\). That is,

\[
   H^a_{\text{edge,L}} \text{ induces corr. } (H^a_{\text{edge,L}})_L |\tilde{Q}, Q\rangle \\   = - \sum_{\tau, \tau'} \left( \mathbb{P}_Q H^a_{\text{edge,L}} \mathbb{P}_Q \right)_{\tau, \tau'} |\tau\rangle_L |\tau\rangle_R.
\]

(B11)
where the bar on $\hat{H}_{\text{edge}}^Q$ signifies that we reproduce each term in $H_{\text{edge}}^Q$ but with each term scaled by a positive rescaling: $1/|\Delta(\prod s^{z/x})|$. In the above, we have introduced notation using double subscripts (two $L$s). The $L$ in $\hat{H}_{\text{edge}}^Q$ corresponds to the form of the Hamiltonian acting on the $L$ semi-infinite cylinder, while the $L$ of the parenthesis around it corresponds to operators acting on the $L$ spin-chain. Explicitly this means

$$\left(\hat{H}_{\text{edge},L}^Q\right)_{\xi} := \hat{H}_{\text{edge},L}(\tau_+^{x}_{\tilde{p},\xi}, \tau_+^{z}_{\tilde{p}-1,\xi}, \tau_+^{x}_{\tilde{p},\xi})$$

(B12)

where $\xi, \xi \in \{L, R\}$, and in the above case we have $\xi = \xi = L$. We will use this notation below.

It will be helpful to provide examples of both cases to make our discussion concrete. Consider an example of the first case: a term $\hat{\tau}_{\tilde{p},L}$ of $H_{\text{edge}}^Q$ such that $\tilde{p} \neq 1$. Then, there exists either a single operator ($s^z_2$) or a product of two adjacent operators ($s^z_2 s^x_1$) which is the inverse of $\hat{\tau}_{\tilde{p},L}$, i.e. itself. The energy difference $\Delta = -4$ in this case. Consider next an example of the second case: $\hat{\tau}_{\tilde{p},L}$ in $H_{\text{edge}}^Q$. One needs to multiply by $s^x_{L/2}$ to get $\hat{Q}_L$. Now, $\langle \hat{Q}_L, \hat{Q}_L \rangle_{\tilde{s}^x_{\tilde{p},L}} = |\tilde{\tau}_{\tilde{p},L}|^2 \langle \hat{Q}_L, \hat{Q}_L \rangle = \tilde{\tau}_{\tilde{p},L}^2 \langle \hat{Q}_L, \hat{Q}_L \rangle$. The excited state involved in the process also has an energy difference of $\Delta = -4$ from the ground state. Concluding, we have the result that any $\hat{\tau}_{\tilde{p},L}$ term that appears in $H_{\text{edge}}^Q$ shows up also in the correction to the ground state:

$$\tau_{\tilde{p},L}^{x} \overset{\text{induces corr.}}{\longrightarrow} -\frac{1}{4} \left(\tau_{\tilde{p},L}^{x}\right)_L |\tilde{Q}, Q\rangle$$

$$= -\frac{1}{4} \sum_{\tau', \tau} \left(P_Q \tau_{\tilde{p},L}^{x} \tau_{\tilde{p},L}^{x} P_Q\right)_{\tau', \tau} |\tau', \tau\rangle_L |\tau\rangle_R$$

(B13)

Next let us consider the corrections from $H_{\text{edge},R}^Q$. By making use of the fact the ground states satisfy the x-rung and z-rung operators $= +1$, we can convert $H_{\text{edge},R}^Q$ acting on the $R$ spin chain to it acting on the $L$ spin chain: $\prod \tau_{\tilde{p},R}^{x} \prod \tau_{\tilde{p},R}^{z} \langle \hat{Q}, Q\rangle = \prod \tau_{\tilde{p},L}^{x} \prod \tau_{\tilde{p},L}^{z} \langle \hat{Q}, Q\rangle$, and our above analysis holds. We therefore have

$$H_{\text{edge},R}^a \overset{\text{induces corr.}}{\longrightarrow} -\left(\hat{H}_{\text{edge},R}^Q\right)_L |\tilde{Q}, Q\rangle$$

$$= -\sum_{\tau', \tau} \left(P_Q \hat{H}_{\text{edge},R}^Q P_Q\right)_{\tau', \tau} |\tau', \tau\rangle_L |\tau\rangle_R$$

(B14)

where we remind the reader once again that it is the modified (term-dependent rescaled) $R$ edge Hamiltonian acting on the $L$ spin degrees of freedom of the ground state.

Lastly, consider the contribution from $\epsilon(P_{0,L} \otimes P_{0,L}^Q) V_{LR} (P_{0,L} \otimes P_{0,L})$, which can be written as $g_{\tilde{Q}}(\tau^{x}_{\tilde{p},L}, \tau^{x}_{\tilde{p}-1,L}, \tau^{z}_{\tilde{p},R}, \tau^{z}_{\tilde{p}-1,L})$ in the spin chain language. Like above, if $g_{\tilde{Q}}$ acts on the ground state $|\tilde{Q}, Q\rangle$, then we can convert terms that act on the $R$ spin chain to act on the $L$ spin chain, so that the overall contributions from $V_{LR}$ act only on the $L$ spin chain. For example, $\tau^{x}_{\tilde{p},L} \tau^{x}_{\tilde{p}-1,L} \tau^{z}_{\tilde{p},R} \tau^{z}_{\tilde{p}-1,L} |\tilde{Q}, Q\rangle = |\tilde{Q}, Q\rangle$. Thus,

$$V_{LR} \overset{\text{induces corr.}}{\longrightarrow} -(\hat{V}_{LR})_{L} |\tilde{Q}, Q\rangle$$

$$= -\sum_{\tau', \tau} \left(P_Q V_{LR} P_Q\right)_{\tau', \tau} |\tau', \tau\rangle_L |\tau\rangle_R$$

(B15)

where $(\hat{V}_{LR})_{L}$ is a term-dependent rescaled, $R \rightarrow L$ operators swapped version of $V_{LR}$.

Therefore, we have calculated $\Lambda$ and hence $\Lambda'$:

$$\Lambda = P_Q (\hat{H}_{\text{edge},L}^Q + \hat{H}_{\text{edge},R}^Q + \hat{V}_{LR}) \mathbb{P}_Q \equiv \mathbb{P}_Q \Lambda' \mathbb{P}_Q.$$  

(B16)

**Appendix C: Derivation of $H^a_{\text{edge},L}$ and $H^a_{\text{ent.}}$ for uniform magnetic fields as perturbations**

In this appendix we calculate $H^a_{\text{edge},L}$ and $H^a_{\text{ent.}}$ for the case of perturbations being uniform single-site magnetic fields:

$$\epsilon V = \epsilon \sum_i h_X X_i + h_Y Y_i + h_Z Z_i.$$  

(C1)

This can be written as $V = V_L + V_R$ where $V_L$ are the perturbations acting on the $L$ semi-infinite cylinder and $V_R$ are the perturbations acting on the $R$ semi-infinite cylinder.

1. **Calculation of $H^a_{\text{edge},L}$**

We calculate $H^a_{\text{edge},L}$ of the $L$ semi-infinite cylinder according to the Schrieffer-Wolff transformation, eqn. (A12) and eqn. (A15), for the perturbation $V_L$, to lowest non-trivial order.

The zeroth order term is identically 0, since the unperturbed Wen-plaquette model on the semi-infinite cylinder has a flat edge theory. Next, the first order ($\epsilon$) term of the edge Hamiltonian in each topological sector, $\epsilon P_{0,L} V_L P_{0,L}^*$, is also identically 0 because single site magnetic fields cannot connect states in $H^a_{0,L}$ to $H^a_{0,L}$. Thus, we have to go to second order in $\epsilon$. 

At this order, there can now be virtual processes that connect $V_{0,L}^a$ to $V_{0,L}^b$. They are mediated by excited states that are one unit of energy above the ground states. We can therefore simplify the the notation in eqn. (A15) for the case of the Wen-plaquette model, in each $Q$ sector:

$$\langle i, 0, \tilde{Q} | H' | j, 0, \tilde{Q} \rangle = -e^2 \sum_k \langle i, 0, \tilde{Q} | V_L | k, 1 \rangle \langle k, 1 | V_L | j, 0, \tilde{Q} \rangle - e^2 \langle i, 0, \tilde{Q} | V_L V_L | j, 0, \tilde{Q} \rangle,$$

for $|i, 0, \tilde{Q} \rangle \in V_{0,L}^Q$ and $|k, 1 \rangle \in V_{1,L}$. Now, there are only two ways in which the above matrix element is non-zero. The first case is when a term in the first $V_L$ annihilates a term in the second $V_L$. However, this simply contributes to a diagonal matrix element whose value is the same for all states. In other words, such a process just renormalizes the energy of the system. We will hence disregard the contributions of this matrix element. The second case is when a term in the first $V_L$ combines with a term in the second $V_L$ to form a boundary operator $S_{p,L}$ or $S_{p,R}$. This is now not a diagonal matrix element, and gives the lowest order non-trivial dynamics to the edge Hamiltonian. Dropping the label $\{0\}$ for notational convenience, the edge Hamiltonian to second order in $e^2$ simplifies to

$$\langle i, \tilde{Q} | H' | j, \tilde{Q} \rangle = -e^2 h_Z h_X \sum_{l=1}^{L_y} \left( \langle i, \tilde{Q} | Z_l X_{l+1} | j, \tilde{Q} \rangle + \langle i, \tilde{Q} | X_l Z_l | j, \tilde{Q} \rangle \right)$$

$$= -2 e^2 h_Z h_X \sum_{l=1}^{L_y} \langle i, \tilde{Q} | Z_l X_{l+1} | j, \tilde{Q} \rangle$$

$$= -2 e^2 h_Z h_X \langle i, \tilde{Q} | \left( \sum_{p=1}^{L_y/2} \sqrt{L_p} - \sum_{p=1}^{L_y/2} \sqrt{L_p} \right) | j, \tilde{Q} \rangle. \quad \text{(C3)}$$

From the mapping, eqn. (3), we can read off the edge Hamiltonian in each $Q$ sector at order $e^2$:

$$H_{\text{edge},L}^{\tilde{Q}} = -2 e^2 h_Z h_X \sum_{p=1}^{L_y/2} \left( \sqrt{L_p} \tau_{p,L}^z + \sqrt{L_p} \tau_{p,L}^z \right), \quad \text{(C4)}$$

with toroidal boundary conditions $\tau_{0,L} = \tilde{Q} \tau_{L_y/2,L}$. This is nothing but the critical periodic ($\tilde{Q} = +1$) or antiperiodic ($\tilde{Q} = -1$) $(1+1)$-d Ising model, which is $Z_2$ symmetric and is clearly at the Kramers-Wannier self-dual point.

The edge Hamiltonian in each topological sector $\alpha$ is then found by projecting the Ising model down into the relevant $Z_2 = Q_L$ sector:

$$H_{\text{edge},L}^{\tilde{Q},Q} = \mathbb{P}_Q L H_{\text{edge},L}^{\tilde{Q}} \mathbb{P}_Q L, \quad \text{(C5)}$$

where $\mathbb{P}_Q = (1 + Q_L \hat{Q}_L)/2$, $\hat{Q}_L := \prod_y \tau_{\hat{p},L}$.

2. Calculation of $H_{\text{ent}}^Q$. We calculate $H_{\text{ent}}^Q$ of the ground states of the Wen-plaquette model perturbed by uniform single-site magnetic fields to lowest order in $\epsilon$. We start with the calculation of $\Lambda'$ in eqn. (B16) of Appendix B. Firstly note that $V_{LR} = 0$, so $\hat{V}_{LR} = 0$ as well. Secondly note that as discussed in Appendix B, each term in the both edge Hamiltonians is reproduced in $\Lambda'$, acting on the $L$ spin degrees of freedom, with a term-dependent rescaling that depends on the energy difference between the ground state of $H_{\text{eff}}$ in eqn. (B2) and the excited states of $H_{\text{eff}}$ generated by the edge Hamiltonians. However, both edge Hamiltonians only contain the terms $\tau_{p,L/2}^z$ and $\tau_{p,L/2}^z \tau_{p-1,L/2}^z$, and the excited states generated by these terms acting on the ground states $|\tilde{Q}, Q \rangle$ give an energy difference of $-4$ (they violate 2 rung operators or 2 2 rung operators respectively). Thus, in this case, the term-dependent rescaling becomes a single, overall rescaling, and we have the result

$$\Lambda' = \frac{1}{4} \left( H_{\text{edge},L}^{Q} + H_{\text{edge},R}^{Q} \right)_L = \frac{1}{2} \left( H_{\text{edge},L}^{\tilde{Q}} \right)_L, \quad \text{(C6)}$$

where in the second equality we made use of the fact that $(H_{\text{edge},R})_L = (H_{\text{edge},L})_L$.

At this point, it would be tempting to conclude from eqn. (30) and eqn. (31) that the entanglement Hamiltonian is then precisely proportional to the edge Hamiltonian in the lowest order in $\epsilon$. However, the identification $H_{\text{ent}}^Q = 2 \mathbb{P} Q \Lambda^Q \mathbb{P} Q$ is only valid if $\Lambda'$ appears at order $\epsilon$. In the case we are considering here, it appears at order $\epsilon^2$, and so we cannot directly identify the entanglement Hamiltonian.

To be consistent, we want to calculate all $\epsilon^2$ corrections to the reduced density matrix $\rho_L^Q$. This implies that in eqn. (21), we have to calculate $\Lambda$ to second order in $\epsilon$, and $\Theta$ and $\Xi$ to first order in $\epsilon$. Thus our calculation of $\Lambda'$ in eqn. (C6) is not entirely correct as it is only a part of the $\epsilon^2$ correction. Also, it might be the case that $\Theta$ and $\Xi$ give rise to undesirable $\epsilon^2$ contributions in the dominant part of the reduced density matrix that modifies the entanglement spectrum from that of the spectrum of the edge Hamiltonians. However, we will show that in this case, (i) the other contribution to $\Lambda'$ is simply a constant, (ii) the contributions from $\Theta$ and $\Xi$ simply contribute shift the entanglement Hamiltonian. This calculation also explicitly shows how to perform calculations in our perturbative framework to order $\epsilon^2$, and by extension, to arbitrary order in $\epsilon$. 


Result. We first state the result. To order $\epsilon^2$,

$$\Lambda = \mathbb{P}_{Q,L} \Lambda^\dagger \mathbb{P}_{Q,L} = \mathbb{P}_{Q,L} \left( \frac{1}{2} \left( H^Q_{\text{edge},L} \right)_L + \mathcal{O}(\epsilon^2) \right) \mathbb{P}_{Q,L}$$

$$\Theta = \epsilon \mathbb{P}_{Q,L} \otimes \tilde{w}_1$$

$$\Xi = \epsilon \mathbb{P}_{Q,L} \otimes \tilde{w}_2$$

$$\Omega \sim \mathcal{O}(\epsilon^2) \Omega'$$  \hspace{1cm} (C7)

where $\tilde{w}_1$ and $\tilde{w}_2$ are (long) column vectors of unit strength denoting coefficients in front of a state $|\tau\rangle_L|\alpha\rangle_R$ ($\alpha \geq 1$), and $|i,\alpha\rangle_L|\tau\rangle_R$ respectively. The exact expression or length of $\tilde{w}_1$ and $\tilde{w}_2$ are unimportant here. $\Omega$ is a matrix that appears at $\epsilon^2$ as it requires at least a second order virtual process to create an excitation to the ground state outside of the space of two spin chains $\nu^Q_{0,L} \otimes \nu^Q_{0,R}$.

With this result, we form the reduced density matrix $\rho^a_{\tau,\tau} = \text{Tr}_R(\tilde{Q}, Q)|\langle Q, Q|$. To order $\epsilon^2$, we have the unnormalized reduced density matrix

$$\rho^a_{\tau,\tau} = \sum_{\tau,\alpha} \left( 1 + \mathcal{O}(\epsilon^2) + \tilde{w}_1^\dagger \tilde{w}_1 + \tilde{w}_2^\dagger \tilde{w}_2 \right) \rho^a_{\tau,\tau}$$

$$- \mathbb{P}_{Q,L} \left( H^Q_{\text{edge},L} \right)_L \mathbb{P}_{Q,L}$$

$$+ \sum_{\tau,\alpha} \left( \mathbb{P}_{Q,L} \otimes \tilde{w}_2 |\tau\rangle_L (i,\alpha|L + \text{h.c.}) \right)$$

$$+ \sum_{\tau,\alpha} \cdots \mathcal{O}(\epsilon^2) \cdots |i,\alpha\rangle_L |j,\beta\rangle_L.$$  \hspace{1cm} (C8)

To extract the entanglement spectrum, we find the eigenvalues of $\rho^a_{\tau,\tau}$. This can be calculated in standard matrix perturbation theory, yielding the unnormalized eigenvalues

$$\text{eig.}(\rho^a_{\tau,\tau}) =$$

$$\left( 1 + \mathcal{O}(\epsilon^2) + \tilde{w}_1^\dagger \tilde{w}_1 + \tilde{w}_2^\dagger \tilde{w}_2 \right) - \text{eig.} \left( \mathbb{P}_{Q,L} \left( H^Q_{\text{edge},L} \right)_L \mathbb{P}_{Q,L} \right)$$  \hspace{1cm} (C9)

Since $\tilde{w}_1^\dagger \tilde{w}_1$ and $\tilde{w}_2^\dagger \tilde{w}_2$ are just numbers of order $\epsilon^2$, we see that we can absorb the order $\epsilon^2$ constants in the first term of the above expression into a constant shift of the second term, which is nothing but the entanglement Hamiltonian. Thus, we have

$$H^a_{\text{ent.}} = \mathbb{P}_{Q,L} \left( H^Q_{\text{edge},L} \right)_L \mathbb{P}_{Q,L} + \text{const.},$$  \hspace{1cm} (C10)

at order $\epsilon^2$. Therefore, it is clear that in this case

$$H^a_{\text{ent.}} = H^a_{\text{edge},L} \text{ up to shifting and rescaling}.$$  \hspace{1cm} (C11)

at order $\epsilon^2$, an edge-ES correspondence. The edge/ES Hamiltonians calculated in this case are the critical $(1+1)$-d Ising models, or the transverse field Ising model, eqn. (34), projected into the different charge sectors ($\mathbb{Z}_2$ labeled by $Q$ and toroidal boundary conditions labeled by $\tilde{Q}$). We have the following identification between states in the topological phase (left) and their edge/ES Hamiltonians (right):

$$|1\rangle \leftrightarrow H^1_{\text{TFIM},+1}$$

$$|\epsilon\rangle \leftrightarrow H^1_{\text{TFIM},-1}$$

$$|m\rangle \leftrightarrow H^1_{\text{TFIM},+1}$$

$$|\epsilon\rangle \leftrightarrow H^1_{\text{TFIM},-1}. \hspace{1cm} (C12)$$

where the label $\{1, m, \epsilon = \epsilon \times m\}$ indicates that the states carry the corresponding anyonic flux.

Proof. We present the proof of our assertion, eqn. (C7). First let us find the order $\epsilon$ corrections in $\Theta$ and $\Xi$. We identify the relevant terms in eqn. (B1) that contribute. Let us concentrate on the contribution from perturbations on the right semi-infinite cylinder, $V_R$. The term that contributes is $[-S^a_{\tau,\sigma}, \alpha P^a_{\alpha, \tau}]$, specifically, the order-$\epsilon$ term of $S^a_{\tau,\sigma}$, which is given by eqn. (A16). By virtue of the fact that $V_R$ is a sum of single-site magnetic fields which can only connect the subspace $\nu^{V^a_{0,R}}$ to $\nu^{V^a_{1,R}}$ through a single virtual process, we can further distill the relevant term:

$$-\epsilon (P^a_{0,R} V_R P^a_{1,R} + \text{h. c.}) \hspace{1cm} (C13)$$

is the term that contributes to $\Theta$ to order $\epsilon$. The correction induced is

$$\epsilon \sum_{\tau,\sigma} \langle \tau |_{L} (i, \alpha | R P_{1,R} V_R P_{0,R} | Q, Q) \times | \tau |_{L} | i, \alpha \rangle_R$$

$$= \epsilon \sum_{\tau,\sigma} \langle \tau |_{L} (i, \alpha | R V_R (P^Q_{L} | \sigma \rangle_{L} | \sigma \rangle_{R}) \times | \tau |_{L} | i, \alpha \rangle_R$$

$$= \epsilon \sum_{\tau,\sigma} (P^Q_{L})_{\tau,\sigma} (i, \alpha | R V_R | \sigma \rangle \times | \tau |_{L} | i, \alpha \rangle_R. \hspace{1cm} (C14)$$

where $\alpha = 1$. Now, $V_R = \sum_s (w^s_1), v^s_R$, where $v^s_R$ are single-site magnetic fields. For each $v^s_R$, $\epsilon v^s_R$ acting on $| \sigma \rangle_R$ creates a unique excited state $v^s_R | \sigma \rangle_R \in \nu^1 |$ which is unique - there is no other $v^s_R$ such that $v^s_R | \sigma \rangle_R = v^s_R | \sigma \rangle_R$. Thus, the label $s$ identifies a unique excited state $| \sigma^s \rangle_R \equiv v^s_R | \sigma \rangle_R$. Using this result, we can write the correction as

$$\epsilon \sum_{\tau,\sigma} (P^Q_{L})_{\tau,\sigma} (w^s_1) | \tau | \times | \sigma^s \rangle_R,$$  \hspace{1cm} (C15)

from which we read off

$$\Theta = \epsilon \mathbb{P}_{Q,L} \otimes \tilde{w}_1.$$  \hspace{1cm} (C16)

as claimed. A similar analysis for the contributions from the perturbation $V_L$ on the $L$ semi-infinite cylinder will yield

$$\Xi = \epsilon \mathbb{P}_{Q,L} \otimes \tilde{w}_2.$$  \hspace{1cm} (C17)
Next, we show that $\Lambda$ has the asserted form. We have already accounted for the $H_{\text{edge}}^Q$ term, as it appears from the edge Hamiltonians, and so we only have to account for the $O(\epsilon^2)$ shift in $\Lambda'$. This second order process corrects for the $V$ term, which involves two $V$s. As it is a process which involves two $V$s, it gives rise to contributions of at least order $\epsilon^2$. In our case, the unperturbed Hamiltonian is the exact Wen-plaquette model $H_L + H_R + H_{LR}$, and $|n^{(0)}\rangle$ is the ground state of the exact Wen-plaquette model on the infinite cylinder in each topological sector, eqn. \((C18)\). Note that the second term evaluates to 0 because $\langle n^{(0)}|V|n^{(0)}\rangle = 0$.

Now, there are two contributions to $O(\epsilon^2)$. The third term in eqn. \((C18)\) simply rescales $|n^{(0)}\rangle = |Q, Q\rangle$ - this gives one source of the shift $O(\epsilon^2)$ in $\Lambda'$. The other source comes from the $[-S^\alpha_R, [-S^\alpha_R, \alpha P^\alpha_{\alpha, R}]]$ term in eqn. \((C19)\) (and also the R term), specifically with the first order term $S^\alpha_R$, and $S^\alpha_R$. Expanding the two commutators and focusing on the relevant term on the $R$ cylinder gives us

$$[-S^\alpha_R, [-S^\alpha_R, \alpha P^\alpha_{\alpha, R}]] \sim \epsilon^2 P^\alpha_{\alpha, R} V_{LR} P^\alpha_{\alpha, R} V_{LR} P^\alpha_{\alpha, R}.$$  \((C19)\)

However making use of the fact that the each term in $V_{LR} = \sum_i (w_i)^2 r_i^2$ creates a unique excited state above any given state in $\mathcal{V}^\alpha_{0, R}$, it must be that the above term is simply proportional (at order $\epsilon^2$) to $P^\alpha_{\alpha, R}$, i.e.

$$[-S^\alpha_R, [-S^\alpha_R, \alpha P^\alpha_{\alpha, R}]] \sim \epsilon^2 P^\alpha_{\alpha, R}.$$ \((C20)\)

This then contributes $\sim \epsilon^2|\tilde{Q}, Q\rangle$ in perturbation theory as well. A similar story holds for the $L$ term. Thus, we have identified the sources of the $O(\epsilon^2)$ shift in $\Lambda'$ of eqn. \((C7)\). This concludes the proof of our claim.

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