Finite size emptiness formation probability of the XXZ spin chain at $\Delta = -\frac{1}{2}$

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Abstract

In this paper, we compute the emptiness formation probability of a (twisted-) periodic XXZ spin chain of finite length at $\Delta = -\frac{1}{2}$, thus proving the formulas conjectured by Razumov and Stroganov (2001 J. Phys. A: Math. Gen. 34 3185 and 5335–40). The result is obtained by exploiting the fact that the ground state of the inhomogeneous XXZ spin chain at $\Delta = -\frac{1}{2}$ satisfies a set of qKZ equations associated with $U_q(\hat{sl}_2)$.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The computation of correlation functions in quantum integrable systems is in general a quite difficult task. One paradigmatic example is the spin-$\frac{1}{2}$ XXZ spin chain [1]. Even in the study of the free fermion case, quite interesting mathematical structures have appeared [2]. Starting from the seminal papers of Razumov and Stroganov [3, 4], we know that the antiferromagnetic ground state of the XXZ spin chain at the value $\Delta = -\frac{1}{2}$ of the anisotropy parameter (or equivalently $q = e^{2\pi i/3}$) presents a remarkable combinatorial structure. The spin chain Hamiltonians with an odd number of spins $N = 2n + 1$ and periodic boundary conditions or even number of spins $N = 2n$ and twisted-periodic boundary conditions are related through a change of basis to the Markov matrix of a stochastic loop model [5]. In the loop basis, the ‘ground state’ is actually the steady state probability of the stochastic loop model. The most astonishing discovery was made by Razumov and Stroganov [6], who observed that once properly normalized, the components of the steady state are integer numbers enumerating fully packed loop configurations on a square grid. This conjecture has been eventually proven in [7].

Despite the XXZ spin chain at $\Delta = -\frac{1}{2}$ being a fully interacting system, several of its correlation functions have simple exact formulae even at finite size. This is the case of the emptiness formation probability (in short EFP), which is the probability that $k$ consecutive
spins are in the up direction in a chain of length \( N \). In their original papers \([3, 4]\), Razumov and Stroganov have conjectured exact factorized formulas for the EFP in terms of products of factorials. The aim of this paper is to prove these conjectures.

The XXZ spin chain with odd size has two ground states \( \Psi_{2n+1}^+ \) and \( \Psi_{2n+1}^- \), related by a spin flip on each site. Razumov and Stroganov have conjectured \([3]\) that \( E_{2n+1}^-(k) \), the EFP of a \( k \)-string of spins up in the state \( \Psi_{2n+1}^- \), satisfies

\[
\frac{E_{2n+1}^-(k-1)}{E_{2n+1}^-(k)} = \frac{(2k-2)!(2k-1)!(2n+k)!(n-k)!}{(k-1)!(3k-2)!(2n-k+1)!(n+k-1)!}.
\]

(1)

Strangely enough, Razumov and Stroganov did not provide the analogous formula for the state \( \Psi_{2n+1}^+ \), which reads

\[
\frac{E_{2n+1}^+(k-1)}{E_{2n+1}^+(k)} = \frac{(2k-2)!(2k-1)!(2n+k)!(n-k+1)!}{(k-1)!(3k-2)!(2n-k+1)!(n+k)!}.
\]

(2)

In particular, the probability of having a string of spins-up of length \( n \) (or \( n+1 \)) in a chain of length \( 2n+1 \) is equal to the inverse of \( A_{HT}(2n+1) \), the number of of half turn symmetric alternating sign matrices of size \( 2n+1 \) \([8]\):

\[
E_{2n+1}^-(n) = E_{2n+1}^+(n+1) = A_{HT}(2n+1)^{-1} = \prod_{j=1}^{n} \frac{(2j-1)!^2(2j)!^2}{(j-1)!^2(3j-1)!^2(3j)!^2}.
\]

(3)

In the case of a spin chain with even length and twisted boundary conditions, the ground state is unique. Razumov and Stroganov have conjectured \([4]\) that \( E_{2n}^+(k) \), the EFP of a \( k \)-string of spins up satisfies

\[
\frac{E_{2n}^+(k-1)}{E_{2n}^+(k)} = \frac{(2k-2)!(2k-1)!(2n+k-1)!(n-k)!}{(k-1)!(3k-2)!(2n-k)!(n+k-1)!}.
\]

(4)

The previous equation implies that in the case \( k = n \),

\[
E_{2n}^+(k) = A_{HT}(2n)^{-1}.
\]

(5)

Unlike the ground states of the odd size chains, whose components can be chosen to be real, the even size ground state has complex-valued components; therefore, we can also consider \( E_{2n}^-(k) \), a sort of ‘pseudo’ EFP obtained by sandwiching the ground state with itself (and not with its complex conjugate). The ratio of ‘pseudo’ EFPs corresponding to the same size of the spin chain has a factorized form given by

\[
\frac{E_{2n}^-(k-1)}{E_{2n}^-(k)} = -q \frac{(2k-2)!(2k-1)!(2n+k-1)!(n-k)!}{(k-1)!(3k-3)!(3k-1)(2n-k)!(n+k-1)!}.
\]

(6)

It turns out that apart from the factor \(-q\), the ratio in equation (6) can be written as a ratio of enumerations of \( k \)-punctured cyclically symmetric self-complementary plane partitions (PCSSCPP) of size \( 2n \), i.e. rhombus tilings of a regular hexagon of side length \( 2n \), which are symmetric under a \( \pi/3 \) rotation around the center of the hexagon and with a star-shaped frozen region of size \( k \), as exemplified in figure 1. Calling these enumerations CSSCPP\((2n, k)\), we have

\[
E_{2n}^-(k-1) = -q^{\text{CSSCPP}(2n, k-1)} E_{2n}^-(k).
\]

(7)

For \( k = n \) one obtains

\[
E_{2n}^-(n) = (-q)^n A_n^{-2} = (-q)^n \text{CSSCPP}(2n)\]

(8)
where \( A_n \) is the number of alternating sign matrices of size \( n \)
\[
A_n = \prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n + j)!},
\]
and \( \text{CSSCPP}(2n) \) is the number of cyclically symmetric self-complementary plane partitions
in a hexagon of size \( 2n \). The enumerations \( \text{CSSCPP}(2n, k) \) are easily computed by applying a
result of Ciucu [9] concerning enumerations of dimer coverings of planar graphs with reflection
symmetry. This will be explained briefly in appendix B.

Some partial results concerning the (pseudo)-norm and the EFP have been obtained in
the literature. In [10], by cleverly exploiting the relation between a natural degenerate scalar
product in the loop basis and the usual scalar product in the spin basis, Di Francesco and
collaborators have proven equations (3) and (8).

In the large \( N \) limit \( N \to \infty \), the EFP have been studied by Maillet and collaborators [11]. Of course the conjectures (1, 2, 4) must coincide and must give for the thermodynamic
limit of the EFP
\[
\lim_{N \to \infty} E_N(k) = \left( \frac{\sqrt{3}}{2} \right)^{2k} \prod_{j=1}^{k} \frac{\Gamma(j - 1/3) \Gamma(j + 1/3)}{\Gamma(j - 1/2) \Gamma(j + 1/2)}.
\]
This formula has been proven in [11] by specializing to \( \Delta = -\frac{1}{2} \), a multiple integral
representation for the correlation functions, which is valid for generic values of the anisotropy
parameter \( \Delta \).

The most effective technique which has allowed us to compute the (partial) sum of
components in the loop or in the spin basis has been pushed forward by Di Francesco and
Zinn-Justin [12] in the context of the periodic loop model with an even number of sites. They
introduced spectral parameters in the model in such a way as to preserve its integrability,
the original model being recovered once the spectral parameters are set to 1. In this way, the
components of the ground state become homogeneous polynomials in the spectral parameters,
which satisfy a certain relation under exchange or specialization of the spectral parameters.
As noticed first by Pasquier [13] and largely developed by Di Francesco and Zinn-Justin [14],
the exchange relations satisfied by the ground state are a special case \( q = e^{2\pi i/3} \) of the very
much studied quantum Knizhnik–Zamolodchikov equations (qKZ) [15]. In [16], the authors

![Figure 1](image-url) Domain tiled by \( k \)-punctured cyclically symmetric self-complementary plane partitions,
with an example of tiling for \( n = 3 \) and \( k = 2 \). The shadowed region indicates a fundamental
domain.
have applied this idea to the XXZ spin chain with spectral parameters and have shown that the properly normalized ground state of the spin chain with periodic or twisted periodic boundary conditions satisfies a special case of the \( U_q(\hat{sl}_2) \) qKZ equations at level 1. Here we employ this property in order to compute the emptiness formation probability. Our main idea is to consider a generalization of the EFP with spectral parameters EFP, which is constructed from the solution of the \( U_q(\hat{sl}_2) \) qKZ equations for generic \( q \) (see equations (48) and (49)). This ‘inhomogeneous’ EFP has certain symmetry and recursion properties that completely fix it in the same spirit as the recursion relations of the 6-vertex model with domain wall boundary conditions which completely determine its partition function. This will allow us to present an explicit determinantal formula for the inhomogeneous EFP valid at \( q = e^{2\pi i/3} \), and upon specialization of the spectral parameters will allow us to obtain the formulas (1), (2), (4) and (6).

The idea to use the solution of the qKZ equation to compute the inhomogeneous version of a correlation function can be in principle adapted to several other models like the XXZ spin chain with different boundary conditions, fused XXZ spin chain, \( U_q(\hat{sl}_n) \) spin chain or even XYZ spin chain, etc [18–21]. Indeed, in all these cases, by properly tuning the parameters (generalizing the relation \( q = e^{2\pi i/3} \), one has a so called combinatorial point at which the ground state energy per site does not get finite size corrections. By reasonings similar to the one in [16], one can argue that the ground state with spectral parameters satisfies a qKZ equation (or, in the case of the XYZ spin chain, an elliptic version of it).

Whether this idea could lead to other exact finite size formulae for some correlation function is an open question that in our opinion deserves further investigation.

The paper is organized as follows. In section 2, after having recalled some basic facts about the XXZ spin chain, following [16] we present the exchange equations satisfied by the ground state at \( \Delta = -\frac{1}{2} \), and then in section 2.2 we derive the recursion relations satisfied by the solutions of the \( U_q(\hat{sl}_2) \) qKZ equations at level 1. In section 3, we define the inhomogeneous version of the (pseudo) EFP, constructed using the solutions of the qKZ equations. We derive first its symmetries and then in section 3.1 we derive the recursion relations which completely determine it. In section 4, we will restrict to \( q = e^{2\pi i/3} \) and by showing that certain determinantal expressions satisfy the same recursion relations as the inhomogeneous EFP, we produce a representation of this inhomogeneous EFP whose homogeneous specialization is considered in section 4.1, where we prove the main conjectures. In appendix A, we compute the determinants of a family of matrices which are relevant for the computation of the homogeneous specialization considered in section 4.1. In appendix B, we compute the lozenge tilings enumerations CSSCPP(2n, k).

2. XXZ spin chain at \( \Delta = -\frac{1}{2} \) and qKZ equations

The Hamiltonian of the XXZ spin chain acts on a vector space \( \mathcal{H}_N = (\mathbb{C}^2)^{\otimes N} \) that consists of \( N \) copies of \( \mathbb{C}^2 \) each one labeled by an index \( i \). The Hamiltonian is written in terms of operators \( \sigma^\alpha_i \) which are Pauli matrices acting locally on the \( i \)th component \( C_2^i \):

\[
H_N(\Delta) = -\frac{1}{2} \sum_{i=1}^{N} \sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} + \Delta \sigma^z_i \sigma^z_{i+1}.
\] (11)

It is convenient to parametrize the anisotropy parameter as \( \Delta = \frac{q + q^{-1}}{2} \). The model is completely specified once the boundary conditions are provided. Here we will consider:

- periodic boundary conditions for odd values of the length of the spin chain, \( N = 2n + 1 \), i.e. \( \sigma^x_{N+1} = \sigma^x_{1} \),
\textbullet{} twisted periodic boundary conditions for even values of the length of the spin chain, \( N = 2n \), i.e. \( \sigma_N^z = \sigma_1^z \), while \( \sigma_{N+1}^z = e^{i\frac{2\pi}{N}} \sigma_N^z \), where \( \sigma^z = \sigma^+ \sigma^- + \sigma^- \sigma^+ \).

It is well known [1] that the Hamiltonian (11), for generic values of the parameter \( \Delta \) and of the twisting, is the logarithmic derivative of an integrable transfer matrix. In order to define the transfer matrix, we need the \( R \)-matrix and the twist matrix. In the present context, the \( R \)-matrix is an operator depending on a spectral parameter \( z \), which acts on a tensor product \( \mathbb{C}^2_i \otimes \mathbb{C}^2_j \). Introducing the basis of \( \mathbb{C}^2_i \)

\[
e^{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e^{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

we write \( R_{i,j}(z) \) in the basis \( \{ e^{\uparrow} \otimes e^{\downarrow}, e^{\downarrow} \otimes e^{\uparrow}, e^{\uparrow} \otimes e^{\downarrow}, e^{\downarrow} \otimes e^{\uparrow} \} \) of \( \mathbb{C}^2_i \otimes \mathbb{C}^2_j \) as

\[
R_{i,j}(z) = \begin{pmatrix}
a(z) & 0 & b(z) & c_1(z) & 0 \\
0 & b(z) & c_1(z) & 0 & 0 \\
0 & c_2(z) & b(z) & 0 & 0 \\
0 & 0 & 0 & a(z) & 0
\end{pmatrix}
\] (12)

with

\[
a(z) = \frac{qz - q^{-1}}{q - q^{-1}z}, \quad b(x) = \frac{z - 1}{q - q^{-1}z}, \quad c_1(z) = \frac{(q - q^{-1})z}{q - q^{-1}z}, \quad c_2(z) = \frac{(q - q^{-1})}{q - q^{-1}z}.
\] (13)

The twist matrix \( \Omega(\phi) \) acts on a single \( \mathbb{C}^2_i \) and in the basis \( \{ e^{\uparrow}, e^{\downarrow} \} \) it reads

\[
\Omega(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.
\] (14)

Using both the twist and the \( R \)-matrix, we construct the family of transfer matrices

\[
T_N(y|z_{1,1},...,z_{1,N},\phi) = \text{tr}_0[R_{0,1}(y/z_1)R_{0,2}(y/z_2)\ldots R_{0,N}(y/z_N)\Omega_0(\phi)]
\] (15)

depending on \( N \) ‘vertical’ spectral parameters \( z_{1,1},...z_{1,N} \)\(^1 \) and a single ‘horizontal’ spectral parameter \( y \). Thanks to the commutation relation

\[
[R_{i,j}(x), \Omega_0(\phi) \otimes \Omega_j(\phi)] = 0
\] (16)

and the Yang–Baxter equation

\[
R_{i,j}(z_i/z_j)R_{j,k}(z_j/z_k)R_{i,k}(z_i/z_k) = R_{j,k}(z_j/z_k)R_{i,k}(z_i/z_k)R_{i,j}(z_i/z_j)
\] (17)

the transfer matrices for different values of \( y \) commute:

\[
[T_N(y|z_{1,1},...,z_{1,N},\phi), T_N(y'|z_{1,1},...,z_{1,N},\phi)] = 0.
\] (18)

The Hamiltonian of the XXZ spin chain is given by

\[
\frac{1}{T_N(1|1,\phi)} \frac{dT_N(y|1,\phi)}{dy} \bigg|_{y=1} = -\frac{1}{q-q^{-1}} \left( H_N(\Delta) - \frac{3N}{2} \Delta \right).
\] (19)

At \( \Delta = -\frac{1}{2} \) and for generic values of the vertical spectral parameters, both in the odd size case with periodic boundary conditions and in the even size case with twisted boundary conditions, the transfer matrix has an eigenvalue equal to \( \lambda(y|z) = \prod_{i=1}^N (a(y/z) + b(y/z)) \).

\(^1 \) Our convention for an ordered string of variables labeled by an index is to use a bold character and a label for the ordered set of indices of the variables: \( x_{a_1,\ldots,a_N} = \{x_{a_1},\ldots,x_{a_N}\} \). Often, when clear from the context, we will omit the label \( \{a_1,\ldots,a_N\} \) and write \( x \) for \( x_{a_1,\ldots,a_N} \).
i + is equivalent to a set of exchange relations. Define the exchange operator 

\[ A_i \Psi_{2n+1}(z) = \pm \Psi_{2n+1}(z). \]  

(20)

The two eigenstates are related by a flipping of all the spins 

\[ \Psi_{2n+1}(z) = \prod_{i=1}^{N} \sigma^z_i \cdot \Psi_{2n+1}(z). \]  

(21)

- When \( N = 2n + 1 \), the eigenspace with eigenvalue 1 is twofold degenerate, \( \Psi_{2n+1}^+(z) \), corresponding to the two values \( \pm 1 \) of the total spin \( S^z = \frac{1}{2} \sum_{k=1}^{N} \sigma^z_k \).

\[ S^z \Psi_{2n+1}^+(z) = \pm \Psi_{2n+1}^+(z). \]  

(20)

These eigenstates reduce to the anti-ferromagnetic ground state(s) of the XXZ spin chain when the spectral parameters are specialized at \( z_i = 1 \).

2.1. Exchange relations at \( \Delta = -\frac{1}{2} \)

A crucial observation made in [16] was that for an appropriate choice of the normalization of \( \Psi_{2n}^+(z) \), the eigenvector equation 

\[ T_N(y(z, \phi)) \Psi_{2n}^+(z) = \lambda(y(z)) \Psi_{2n}^+(z) \]  

(23)

is equivalent to a set of exchange relations. Define the exchange operator \( P_{ij}(e^j_i \otimes e^j_i) = e^j_i \otimes e^j_i \), the left rotation operator \( \sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_{N-1} \otimes v_N) = v_2 \otimes \cdots \otimes v_{N-1} \otimes v_N \otimes v_1 \) and let \( \tilde{R}_{i,i+1}(z) = P_{i,i+1} R_{i,i+1}(z) \), then, \( \Psi_{2n}^+(z) \), as a polynomial of minimal degree in the spectral parameters \( z_i \), is determined up to a constant factor, by the following set of equations [16]: 

\[ \tilde{R}_{i,i+1}(z_i,z_{i+1}) \Psi_{2n}^+(z_1, \ldots, z_i, z_{i+1}, \ldots, z_N) = \Psi_{2n}^+(z_1, \ldots, z_{i+1}, z_{i}, \ldots, z_N) \]  

(24)

\[ \sigma_i \Psi_{2n}^+(z_1, z_2, \ldots, z_{N-1}, z_N) = D \Psi_{2n}^+(z_2, \ldots, z_{N-1}, z_N, z_{i+1}), \]  

(25)

with \( D = s = 1 \). These equations can be seen as the special case \( q = e^{2\pi i/3} \) of the level 1 qKZ equations [15], which corresponds to generic \( q, s = q^0 \) and \( D = q^{(N-1)/2} \). The solution of the level 1 qKZ equations can be normalized in such a way that they become polynomials in the variables \( z_i \) [16] of degree \( n - 1 \) in the case of even size \( N = 2n \), and degree \( n \) in the odd case \( N = 2n + 1 \).

Using the projectors \( P^\pm_i = \frac{1 \pm \sigma_i^z}{2} \) of the \( i \)th spin in the up/down direction, let us write the exchange equations (24) in components. If we have a spin up at site \( i \) and a spin down at site \( i + 1 \) or vice versa, we have 

\[ p_i^+ P_{i,i+1}^+ \Psi^+(z_i, z_{i+1}) = \sigma_i^+ \sigma_{i+1}^- (qz_i - q^{-1}z_{i+1}) \Psi^+(z_i, z_{i+1}, z_i) - (q - q^{-1})z_i \Psi^+(z_i, z_{i+1}) \]  

(26)

\[ p_i^- P_{i,i+1}^- \Psi^+(z_i, z_{i+1}) = \sigma_i^- \sigma_{i+1}^+ (qz_i - q^{-1}z_{i+1}) \Psi^+(z_i, z_{i+1}, z_i) - (q - q^{-1})z_i \Psi^+(z_i, z_{i+1}) \]

These equations form a triangular system. Starting from a given component, we can reconstruct all the others repeatedly using equations (26). Therefore, if we want to show that two \( a \ priori \) distinct solutions of the qKZ equations actually coincide, it is enough to check the equality of one of their components.

When there are two consecutive spins pointing in the same direction at positions \( i \) and \( i + 1 \), then the first of the qKZ equations reads 

\[ p_i^+ P_{i,i+1}^+ \Psi^+(\ldots, z_{i+1}, z_i, \ldots) = \frac{qz_{i+1} - q^{-1}z_i}{qz_i - q^{-1}z_{i+1}} p_i^+ \Psi^+(\ldots, z_i, z_{i+1}, \ldots) \]  

(27)
which means that the components having two consecutive spins up or down at positions \( i \) and
\( i + 1 \), \( p_i^\pm \bar{p}_i^\pm \Psi^\mu (\ldots, \, z_{i+1}, \, z_i, \ldots) \) have a factor \( q_{zi} = q^{-1}z_{i+1} \)
\[
p_i^\pm \bar{p}_i^\pm \Psi^\mu (\ldots, \, z_i, \, z_{i+1}, \ldots) = \left( q_{zi} - q^{-1}z_{i+1} \right) \bar{\Psi}_{i,i+1}^\mu (\ldots, \, z_i, \, z_{i+1}, \ldots)
\]
and the vectors \( \bar{\Psi}_{i,i+1}^\mu (\ldots, \, z_i, \, z_{i+1}, \ldots) \) are symmetric under exchange \( z_i \leftrightarrow z_{i+1} \). Another useful relation is obtained by considering the matrix \( e_i \), which is proportional to a projector and is a generator of the Temperley–Lieb algebra
\[
e_i = \tau e_i, \quad \tau = -q - q\bar{q}^{-1}
\]
The matrix \( e_i \) is preserved under multiplication by an \( \bar{R} \)-matrix for any value of the spectral parameter
\[
e_i \bar{R}_{i,i+1}(z) = \bar{R}_{i,i+1}(z) e_i = e_i.
\]
By applying \( e_i \) to the left of the first of the qKZ equations (24), we find
\[
e_i \Psi^\mu (\ldots, \, z_i, \, z_{i+1}, \ldots) = e_i \Psi^\mu (\ldots, \, z_i, \, z_{i+1}, \ldots).
\]
The components with most consecutive eqnarrayed spins have a completely factorized form
\[
\Psi^e (\uparrow, \, \ldots, \, \uparrow, \, \downarrow, \, \ldots, \, \downarrow) = \prod_{1 \leq i<j \leq n} \frac{q_{zi} - q^{-1}z_j}{q - q^{-1}} \prod_{n+1 \leq i<j \leq 2n} \frac{q_{zi} - q^{-1}z_j}{q - q^{-1}}
\]
\[
\Psi^+ (\uparrow, \, \ldots, \, \uparrow, \, \downarrow, \, \ldots, \, \downarrow) = \prod_{1 \leq i<j \leq n+1} \frac{q_{zi} - q^{-1}z_j}{q - q^{-1}} \prod_{n+2 \leq i<j \leq 2n+1} \frac{q_{zi} - q^{-1}z_j}{q - q^{-1}}
\]
\[
\Psi^- (\uparrow, \, \ldots, \, \uparrow, \, \downarrow, \, \ldots, \, \downarrow) = \prod_{1 \leq i<j \leq n+1} \frac{q_{zi} - q^{-1}z_j}{q - q^{-1}} \prod_{n+2 \leq i<j \leq 2n+1} \frac{q_{zi} - q^{-1}z_j}{q - q^{-1}}
\]
where the residual normalization ambiguity has been fixed by requiring these components to be equal to 1 for \( z_i = 1 \). From equations (32), we see that the maximally factorized components satisfy (among others) the following relations:
\[
\Psi^+ (\uparrow, \, \ldots, \, \uparrow, \, \downarrow, \, \ldots, \, \downarrow) = (1 - q^{-2})^n \Psi^e (\uparrow, \, \ldots, \, \uparrow, \, \downarrow, \, \ldots, \, \downarrow) \big|_{z_{2n+1}=0}
\]
\[
\Psi^e (\uparrow, \, \ldots, \, \uparrow, \, \downarrow, \, \ldots, \, \downarrow) = (1 - q^{-2})^n \lim_{z_{2n+1} \to -\infty} \Psi^+ (\uparrow, \, \ldots, \, \uparrow, \, \downarrow, \, \ldots, \, \downarrow) \big|_{z_{2n+1}=0}
\]

Using the triangularity of the equations (26) we can conclude that the equations (33) induce equalities between components of \( \Psi^e_{2n+1}(z) \) or \( \psi^e_{2n}(z) \) and components of \( \psi^e_{2n+1}(z) \) or \( \Psi^e_{2n+1}(z) \) with the last spin down, i.e.
\[
\Psi^+_{2n+1}(z) \otimes e^e_{2n+2} = (1 - q^{-2})^n p^-_{2n+2} \psi^e_{2n+2}(z) \big|_{z_{2n+1}=0}
\]
\[
\psi^e_{2n}(z) \otimes e^e_{2n+1} = (1 - q^{-2})^n \lim_{z_{2n+1} \to -\infty} e^e_{2n+1} p^-_{2n+1} \psi^e_{2n+1}(z) \big|_{z_{2n+1}=0}.
\]
2.2. Recursion relation

We claim that, upon specialization $z_{i+1} = q^2 z_i$, the solution of the qKZ equation for $N$ spins reduces to the solution of the same system of equations for $N - 2$ spins. In order to make the previous statement more precise, we need to introduce some notation. Let $v_i$ be the vectors which are in the image of the projectors proportional to the generator of the Temperley–Lieb algebra $e_i$:

$$v_i = e_i^\dagger \otimes e_{i+1}^\dagger - q^{-1} e_i^\dagger \otimes e_{i+1}^\dagger, \quad e_i v_i = -(q + q^{-1}) v_i. \quad (35)$$

Introduce the injective map

$$\Phi_N^{(i)} : (\mathbb{C}^2)^{\otimes N} \to (\mathbb{C}^2)^{\otimes N+2}, \quad (36)$$

which inserts the vector $v_i$ at position $(i, i + 1)$ and shift by two steps the indices of the sites with $j \geq i$, i.e. on a basis

$$\Phi_N^{(i)} (e_1^\dagger \otimes \cdots \otimes e_{i-1}^\dagger \otimes e_i^\dagger \otimes \cdots \otimes e_j^\dagger) = e_1^\dagger \otimes \cdots \otimes e_{i-2}^\dagger \otimes v_i \otimes e_j^\dagger \otimes \cdots \otimes e_N^\dagger. \quad (37)$$

Then, we claim that

**Proposition 1.** The solutions of the exchange equations (24), with the 'boundary conditions' given by equations (32), satisfy the following recursion relations:

$$\Psi_N^\mu (\mathbf{z}) |_{z_{i+1}=q^2 z_i} = (-q)^{f_N(i)} (q^2 z_i)^{\delta(\mu)} \prod_{j=1}^{i-1} \frac{q z_j - q^{-1} z_{j+1}}{q - q^{-1}} \prod_{j=i+1}^{N} \frac{q^2 z_i - q^{-1} z_j}{q - q^{-1}} \Phi_N^{(i)} (\Psi_N^{\mu - 2}(\mathbf{z}_{[i,i+1]})) \quad (38)$$

with $\delta(-) = 0$, $\delta(+) = 1$, $f_N(i) = i - \lfloor \frac{i}{2} \rfloor$ and $\mathbf{z}_{[i,i+1]} = \{z_1, \ldots, z_i, z_{i+1}, \ldots, z_N\}$, i.e. the ordered set $\mathbf{z}$ from which the variables $z_i$ and $z_{i+1}$ are removed.

**Proof.** If $z_{i+1} = q^2 z_i$, then $\tilde{R}_{i,i+1}(z_i/z_{i+1})$ becomes a projector proportional to a generator of the Temperley–Lieb algebra

$$\tilde{R}_{i,i+1}(q^{-2}) = \tau^{-1} e_i. \quad (39)$$

Therefore, by specializing the qKZ equations to $z_{i+1} = q^2 z_i$, we deduce

$$\Psi_N^\mu (z_i, z_{i+1} = q^2 z_i) = \tilde{R}_{i,i+1}(q^{-2}) \Psi_N^\mu (q^2 z_i, z_i) = \tau^{-1} e_i \Psi_N^\mu (q^2 z_i, z_i). \quad (40)$$

In particular, $\Psi_N^\mu (z_i, z_{i+1} = q^2 z_i)$ lies in the image of $\Phi_N^{(i)}$ and, by the injectivity of this map, there is a unique $\tilde{\Psi}_N^\mu (z_i, z_{[i,i+1]})$ such that

$$\Psi_N^\mu (z_i, z_{i+1} = q^2 z_i) = \Phi_N^{(i)} (\tilde{\Psi}_N^\mu (z_i, z_{[i,i+1]})). \quad (40)$$

In order to determine the equations satisfied by $\tilde{\Psi}_N^\mu (z_i, z_{[i,i+1]})$, we make use of the following relations among $R$-matrices:

$$e_i \tilde{R}_{i-1}(z_{i+2}/z_i) \tilde{R}_{i+1}(z_{i+2}/z_{i+1}) \tilde{R}_{i-1}(z_{i+2}/z_i) = \frac{(q z_{i+2} - q^{-1} z_i)(q z_i - q^{-1} z_{i+1})}{(q z_{i-1} - q^{-1} z_i)(q z_i - q^{-1} z_{i+2})} e_i \tilde{R}_{i-1,i+2}(z_{i+2}, z_i). \quad (41)$$

Applying both sides to $\Psi_N^\mu (z_i, z_{i+1} = q^2 z_i)$ and using that

$$\tilde{R}_{i-1,i+2}(z_{i+2}, z_i) \Phi_i = \Phi_i \tilde{R}_{i-1,i}(z_{i+2}, z_i), \quad (42)$$
we get that \( \tilde{\Psi}_N^\mu(z_i, z_{[i,i+1]}^+) \) satisfies
\[
\tilde{\Psi}_N^\mu(z_i, z_{[i,i+1]}^+) = \frac{(qz_{i+2}^N - q^{-1}z_i)(qz_i - q^{-1}z_{i-1})}{(qz_{i-1}^N - q^{-1}z_i)(qz_i - q^{-1}z_{i+2})} \tilde{\Phi}_{i-1, i}(z_{[i+1]}, z_i) \tilde{\Psi}_N^\mu(z_i, z_{[i,i+1]}^+) \tag{43}
\]
and the vector
\[
(q)^{-\delta \mu(i)} (q^2 z_i)^{\mu(i)} \prod_{j=1}^{i-1} \frac{q - q^{-1}}{q - q^{-1}z_i} \prod_{j=i+1}^{N} \frac{q - q^{-1}}{q^{-1}z_i - q^{-1}z_j} \tilde{\Psi}_N^\mu(z_i, z_{[i,i+1]}^+) \tag{44}
\]
satisfies all the qKZ equations at size \( N - 2 \). In order to check that it coincides with \( \tilde{\Psi}_N^\mu(z_{[i,i+1]}^+) \), it is enough to check that the components with most eqnarrayed consecutive spins starting from position \( i + 1 \) coincide, which is indeed the case. \( \Box \)

3. (Pseudo)-EFP with spectral parameters

The formal definition of the emptiness formation probability makes use of the natural scalar product \( \langle , \rangle_N \) on \( \mathcal{H}_N \), induced by the scalar product on \( \mathbb{C}^N \) where \( \{ e_1^+, e_1^- \} \) form an orthonormal basis\(^2\). The (pseudo)-EFPs read
\[
E_{2n+1}^\pm(k) = \frac{\langle \Psi_1^\pm(k), \prod_{i=1}^{n+1} p_i^+ \cdot \Psi_{2n+1}^\pm(1) \rangle}{\langle \Psi_{2n+1}^\pm(1), \Psi_{2n+1}^\pm(1) \rangle}
\]
\[
E_{2n}^c(k) = \frac{\langle (\Psi_{2n}^c(1))^*, \prod_{i=1}^{n} p_i^+ \cdot \Psi_{2n}^c(1) \rangle}{\langle \Psi_{2n}^c(1), \Psi_{2n}^c(1) \rangle}
\]
\[
E_{2n}^\varepsilon(k) = \frac{\langle \Psi_{2n}^\varepsilon(1), \prod_{i=1}^{n} p_i^+ \cdot \Psi_{2n}^\varepsilon(1) \rangle}{\langle \Psi_{2n}^\varepsilon(1), \Psi_{2n}^\varepsilon(1) \rangle}.
\]

Our strategy to compute the EFPs is to consider an inhomogeneous version of these quantities which is obtained, roughly speaking, by substituting in equations (45) \( \Psi_{2n}^c(1) \) with \( \Psi_{2n}^\varepsilon(k) \) solution of the qKZ equations. We will see that if the substitution is done in the proper way, then the inhomogeneous EFPs turn out to be symmetric polynomials in the spectral parameters and satisfy certain recursion relations which completely characterize these functions among the polynomials of the same degree in the variable \( z_i \). When defining the inhomogeneous EFP for \( k \) eqnarrayed spins up, it is convenient to extract the factor \( \prod_{1 \leq i < j \leq k} (qy_i - q^{-1}y_j) \) from \( \prod_{i=1}^{k} p_i^+ \Psi_N(y_{[1...k]}; z_{[1...N-k]}) \) and to introduce the vectors \( \Psi_N(k; y_{[1...k]}; z_{[1...N-k]}) \in \mathcal{H}_{N-k} \):
\[
\left( \bigotimes_{i=1}^{k} e_i^+ \right) \otimes \Psi_N(k; y_{[1...k]}; z_{[1...N-k]}) = \frac{\prod_{i=1}^{k} p_i^+ \Psi_N(y_1, \ldots, y_k, z_1, \ldots, z_{N-k})}{\prod_{1 \leq i < j \leq k} (qy_i - q^{-1}y_j)}. \tag{46}
\]

Let us moreover introduce the operator
\[
\mathcal{P}_N(z) = \prod_{i=1}^{N} (z_i p_i^+ + p_i^-) \tag{47}
\]
\(^2\) In the following, most of the time it will be clear from the context which Hilbert space we are considering and therefore we will omit the label \( N \) in the scalar product \( \langle , \rangle_N \).
properties, but for the moment let us just point out that, for the moment, it is evident that for $\mu = +$ as will be shown below. We also define the inhomogeneous version of the pseudo-EFP
\[ E^\mu_N (k; y; z) = \prod_{i=1}^{N-k} z_i^{\delta(\mu)} (P_{N-k} (z) (\Psi_N^\mu (k; q^{-1} y_{k+1,...,2k}; z))^*) \cdot \Psi_N^\mu (k; y_{1,...,k}; z), \] (48)
where $\delta(e) = \delta(-) = 0$, while $\delta(+) = 1$ and $N$ has the parity corresponding to $\mu$. Actually the polynomiality is true also for the case $\mu = +$ as will be shown below. We also define the inhomogeneous version of the pseudo-EFP
\[ E^\mu_N (k; y; z) = \prod_{i=1}^{N-k} z_i^{\delta(\mu)} (P_{N-k} (z) (\Psi_N^\mu (k; q^{-1} y_{k+1,...,2k}; z))^*) \cdot \Psi_N^\mu (k; y_{1,...,k}; z), \] (49)

The choice to multiply the variables $y_{k+1}, \ldots, y_{2k}$ by $q^{-6}$ is motivated by the fact that in this way $E^\mu_N (k; y; z)$ turns out to be symmetric under exchange $y_i \leftrightarrow y_j$ for all $1 \leq i, j \leq 2k$, as will be shown at the end of section 3.1. The polynomials (48), (49) have other remarkable properties, but for the moment let us just point out that, for $q = e^{2\pi i/3}$ and $z_i = y_i = 1$, these functions provide the homogeneous (pseudo)-EFPs as defined in equations (45):
\[ E^\mu_N (k; 1; 1) = E^\mu_{N+2k} (k; 1; 1). \] (50)

This follows from the fact that
\[ \prod_{i=1}^{2n+1} z_i^{\delta(\mu)} (P_{2n+1} (z) (\Psi_{2n+1}^\mu (z)))^* = \prod_{i=1}^{2n+1} z_i^{\delta(\mu)} (P_{2n+1} (z) (\Psi_{2n+1}^\mu (z)))^*. \] (52)

To prove equations (52), we observe that the vector $\Psi_N^\mu (z^{-1})$ satisfies the exchange equation
\[ \Psi_N^\mu (z_i^{-1}, z_i^{-1}) = \tilde{R}_{i+1} (z_i/z_i+1) \Psi_N^\mu (z_i^{-1}, z_i^{-1}). \] (53)

The same equation holds also for the vector $P_N (z) (\Psi_N^\mu (z))^*$, i.e.
\[ P_N (z_{i+1}, z_i) (\Psi_N^\mu (z_{i+1}, z_i))^* = \tilde{R}_{i+1} (z_i/z_i+1) P_N (z_i, z_{i+1}) (\Psi_N^\mu (z_i, z_{i+1}))^*. \] (54)

This is a consequence of the following commutation relation among the $\tilde{R}$-matrix and the operator $(p_i^+ + z_i p_i^-) (p_{i+1}^+ + z_{i+1} p_{i+1}^-)$:
\[ \tilde{R}_{i+1} (z_{i+1}/z_i) (z_i p_i^- + p_i^+) (z_{i+1} p_{i+1}^- + p_{i+1}^+) = (z_{i+1} p_i^+ + p_i^-) (z_{i+1} p_{i+1}^+ + p_{i+1}^-) \tilde{R}_{i+1} (z_{i+1}/z_i), \] which implies
\[ \tilde{R}_{i+1} (z_{i+1}/z_i) P (z_i, z_{i+1}) = P (z_{i+1}, z_i) \tilde{R}_{i+1} (z_{i+1}/z_i) \] (55)
and equation (54). Therefore, to conclude equations (52), it is sufficient to check that they hold for the components with most eqarrayed spins.

We are left with only four different inhomogeneous EFP and as a bonus we have also shown that $E^{+}_{2n+1}(k; y; z)$ is a polynomial of its variables.

**Symmetry** under $z_i \leftrightarrow z_j$

The inhomogeneous (pseudo)-EFP $E^{\mu}_{N}(k; y; z)$ is obviously symmetric under exchange $y_i \leftrightarrow y_j$ for $1 \leq i, j \leq k$ and $k + 1 \leq i, j \leq 2k$. Using equations (53) and (54), it is easy to show that it is symmetric also under exchange $z_i \leftrightarrow z_j$. Indeed

$$E^{\mu}_{N}(k; y; z) = \left( (\mathcal{P}(z_i, z_{i+1}) \Psi^{\mu}_{N}(k; z_i, z_{i+1})) \right)^{\star},$$

where in the third equality we have used the fact that the $\tilde{R}$-matrix is symmetric while the fourth equality follows from equation (54). The proof of the symmetry of the pseudo-EFP under $z_i \leftrightarrow z_j$ is completely analogous.

**Factorized cases**

Using equations (32) we can provide the value of $E^{\mu/\mu}_{N}(k; y; z)$ corresponding to the maximal number of consecutive eqarrayed spins. They coincide for the true and for the pseudo EFP and read

$$E^{\pm/\pm}_{2k}(k; y; z) = \prod_{1 \leq i < j \leq k} \frac{(qz_i - q^{-1}z_j)(qz_j - q^{-1}z_i)}{(q - q^{-1})^2}$$

$$E^{+/-}_{2k+1}(k + 1; y; z) = \prod_{1 \leq i < j \leq k} \frac{(qz_i - q^{-1}z_j)(qz_j - q^{-1}z_i)}{(q - q^{-1})^2} \prod_{i=1}^{k} z_i^2$$

$$E^{-/^+}_{2k+1}(k; y; z) = \prod_{1 \leq i < j \leq k+1} \frac{(qz_i - q^{-1}z_j)(qz_j - q^{-1}z_i)}{(q - q^{-1})^2}.$$  \hspace{1cm} (57)

We will see in the following that the first and the third of these equations will provide the starting point of a recursion which will be worked out in the next section and which completely characterize the inhomogeneous EFP.

### 3.1. Recursion relation for the inhomogeneous EFP

We begin this section by presenting some relations among the EFP at different parities which are obtained by setting one of the spectral parameters to zero or sending it to infinity

$$E^{+}_{2n+1}(k; y; z) = (-1)^n(q - q^{-1})^{-2n} \left( \prod_{i=1}^{2n+1-k} z_i \right) E^{+}_{2n+1}(k; y; z) \big|_{z_{2n+2} = 0}$$ \hspace{1cm} (58)

$$E^{-}_{2n}(k; y; z) = (-1)^n(q - q^{-1})^{2n} \lim_{z_{2n+1} \to \infty} z_{2n+1}^{-1} E^{-}_{2n+1}(k; y; z).$$ \hspace{1cm} (59)

The first of these equations follows from equations (34) and by noting that

$$\mathcal{P}(z) \big|_{z_{2n+2} = 0} = \mathcal{P}(z \setminus z_{2n+2}) p_{2n+2}^{-}.$$ \hspace{1cm} (60)

For the second one we note that, writing

$$E^{-}_{2n+1}(k; y; z) = \left( (\Psi^{\mu+1}_{2n+1}(k; z) \big|_{z_{2n+2} = 0})^\star, \mathcal{P}_{2n}(z \setminus z_{2n+1}) p_{2n+1}^+ \Psi^{\mu}_{2n+1}(k; z) \big|_{z_{2n+2} = 0} \right)$$

$$+ \left( (\Psi^{\mu+1}_{2n+1}(k; z) \big|_{z_{2n+2} = 0})^\star, \mathcal{P}_{2n}(z \setminus z_{2n+1}) p_{2n+1}^+ \Psi^{\mu}_{2n+1}(k; z) \big|_{z_{2n+2} = 0} \right)$$ \hspace{1cm} (61)
the first term on the rhs is a polynomial in $z_{2n+1}$ of degree $2n-1$ while the second is of degree $2n$; therefore, in the limit only the second one survives and we can use again equations (34).

**Specialization**

$z_i = q^{k_i} z_i$

The inhomogeneous EFP satisfies a recursion relation inherited from the recursion relation among solutions of the qKZ equations, equation (38):

$$
\mathcal{E}^\mu_N(k; y; z)|_{z_{i+1} = q^2 z_i} = \tau^{-1}\left(\left(\Psi^\mu_N(k; z_i, z_{i+1} = q^2 z_i)\right)^*\right),
\mathcal{P}(z_{i+1} = q^2 z_i) e_i \Psi^\mu_N(k; z_i, z_{i+1} = q^2 z_i)
\tau^{-1}(e_i \mathcal{P}(z_{i+1} = q^2 z_i) (\Psi^\mu_N(k; z_i, z_{i+1} = q^2 z_i))^*)_i \cdot \Psi^\mu_N(k; z_i, z_{i+1} = q^2 z_i).
$$

(62)

A simple computation shows that $e_i(p_i^+ + z_i p_i^-)(p_i^+ + q^2 z_i p_i^-) = \frac{1}{r} e_i(p_i^+ + z_i p_i^-)(p_i^+ + q^2 z_i p_i^-) e^*_i$ which means

$$
e_i \mathcal{P}(z_{i+1} = q^2 z_i) = \frac{1}{r} e_i \mathcal{P}(z_{i+1} = q^2 z_i) e^*_i
$$

(63)

and we can substitute it into the last line of equation (62) obtaining

$$
\tau^{-2}(e_i \mathcal{P}(z_{i+1} = q^2 z_i) (e_i \Psi^\mu_N(k; z_i, z_{i+1} = q^2 z_i))^*)_i \cdot \Psi^\mu_N(k; z_i, z_{i+1} = q^2 z_i).N.
$$

(64)

Now use equation (31) in order to exchange the variables $z_i$ and $z_{i+1}$ on the lhs of the scalar product

$$
\tau^{-2}(e_i \mathcal{P}(z_{i+1} = q^2 z_i) (e_i \Psi^\mu_N(k; z_i, z_{i+1} = q^2 z_i))^*)_i \cdot \Psi^\mu_N(k; z_i, z_{i+1} = q^2 z_i).N.
$$

(65)

Therefore, we can apply to both sides of the scalar product the recursion relation (38) and find

$$
\frac{\mathcal{E}^\mu_N(k; y; z)|_{z_{i+1} = q^2 z_i}}{\mathcal{E}^\mu_{N-2}(k; y; z \setminus \{z_i, z_{i+1}\})} = (-1)^k(1 + q^2)z_i \prod_{j=1}^{2k} \frac{q y_j - q^{-1} z_i}{q - q^{-1}} \prod_{l \in \{1, \ldots, N\} \setminus \{i, i+1\}} \frac{q z_j - q^{-1} z_i}{q - q^{-1}}.
$$

(66)

The case of the pseudo EFP at even size is analogous

$$
\mathcal{E}^\mu_{2n}(k; y; z)|_{z_{i+1} = q^2 z_i} = (\Psi^\mu_{2n}(k; z_{i-1}, z_{i+1} = q^2 z_i))
\mathcal{E}^\mu_{2n}(k; z_i, z_{i+1} = q^2 z_i)_{2n} = (e_i \Psi^\mu_{2n}(k; z_{i-1}, z_{i+1} = q^2 z_i))_{2n}
\Psi^\mu_{2n}(k; z_i, z_{i+1} = q^2 z_i)_{2n}
$$

(67)

and again we can apply the recursion at the level of vectors to both sides of the scalar product finding

$$
\frac{\mathcal{E}^\mu_{2n}(k; y; z)|_{z_{i+1} = q^2 z_i}}{\mathcal{E}^\mu_{2n-2}(k; y; z \setminus \{z_i, z_{i+1}\})} = (-1)^k(1 + q^2) \prod_{j=1}^{2n} \frac{q y_j - q^{-1} z_i}{q - q^{-1}} \prod_{l \in \{1, \ldots, N\} \setminus \{i, i+1\}} \frac{q z_j - q^{-1} z_i}{q - q^{-1}}.
$$

(68)

Let us look at $\mathcal{E}^\mu(k; y; z)$ as polynomials in $z_1$. Their degrees are in both cases less than $2n - 1$. The recursion relations (equations (66) and (68)) provide the value of $\mathcal{E}^\mu_{2n}(k; y; z)$ for $2(2n - k - 1)$ distinct values of $z_1$ (i.e. for $z_1 = q^{2k} z_i$). Therefore, for $n > k$, by Lagrange interpolation these specializations determine uniquely $\mathcal{E}^\mu_{2n}(k; y; z)$ once $\mathcal{E}^\mu_{2n-2}(k; y; z)$ is known.
As a first consequence, we can argue that \( \mathcal{E}_{2n}^{\ell/2}(k; y; z) \) is symmetric under exchange \( y_i \leftrightarrow y_j \) for all \( 1 \leq i, j \leq 2k \). Indeed for the case when \( n = k \), we have explicit expressions for \( \mathcal{E}_{2n=2k}^{\ell/2}(k; y; z) \), given by equations (57), from which we can read that they are even independent from \( y \). The recursion relations (66) and (68) are symmetric under exchange \( y_i \leftrightarrow y_j \) and therefore by induction if \( \mathcal{E}_{2n-2}^{\ell/2}(k; y; z) \) is symmetric, then also \( \mathcal{E}_{2n}^{\ell/2}(k; y; z) \) is symmetric.

A second important consequence is that any family of polynomials labeled by \( n \) and \( k \), which satisfy the following conditions:

- they are symmetric in the spectral parameters,
- the degree in each spectral parameter is less than \( 2n - 1 \),
- they coincide with \( \mathcal{E}_{2k}^{\ell/2}(k; y; z) \) for \( n = k \),
- they satisfy the recursion relations equations (66) and (68),

must coincide with \( \mathcal{E}_{2n}^{\ell/2}(k; y; z) \). This line of reasoning will be adopted in section 4 where we will provide a determinantal representation of \( \mathcal{E}_{2n}^{\ell/2}(k; y; z) \) at \( q = e^{2\pi i/3} \).

The same arguments hold also for \( \mathcal{E}_{2n+1}^{\ell/2}(k; y; z) \), because the degree is less than \( 2n + 1 \) and we have always enough specialization in order to apply the Lagrange interpolation and reconstruct all the \( \mathcal{E}_{2n+1}^{\ell/2}(k; y; z) \) starting from the initial conditions \( \mathcal{E}_{2k+1}^{\ell/2}(k; y; z) \). The case of \( \mathcal{E}_{2n+1}^{+}(k; y; z) \) is slightly different. Again the degree is bounded by \( 2n + 1 \) and this allows us to fix \( \mathcal{E}_{2n+1}^{+}(k; y; z) \) starting from \( \mathcal{E}_{2k+1}^{+}(k; y; z) \), but the problem is that we do not have an explicit formula \( \mathcal{E}_{2k+1}^{+}(k; y; z) \), being available only for \( n = k - 1 \). This apparent problem is bypassed using relations (58).

4. Inhomogeneous EFP at \( q = e^{2\pi i/3} \)

In order to introduce the expression of \( \mathcal{E}^{\mu}_{2n}(k; y; z) \) and of \( \mathcal{E}^{\mu}_{2n}^{+}(k; y; z) \) which is best suited for taking the specialization \( z_i = y_n = 1 \), we analyze first the case \( k = 0 \), in which there are no variables \( y \). Let us introduce the Young diagrams

\[
\lambda(m, r) = \left\{ \frac{r}{2}, \frac{r+1}{2}, \ldots, \frac{r+m-1}{2}, \frac{r+m}{2} \right\}.
\]

Then we find that the inhomogeneous version of the squared norm or of the sum of the square of the components is given in terms of the product of two Schur polynomials

\[
\begin{align*}
\mathcal{E}^{\mu}_{2n}(0; z) & = 3^{-\frac{1}{2}}(\frac{1}{2}!)^{1-1}S_{\lambda(N,0)}(z_1, \ldots, z_N)S_{\lambda(N,1)}(z_1, \ldots, z_N), \\
\mathcal{E}^{\mu}_{2n}^{+}(0; z) & = 3^{-m(m-1)}S_{\lambda(2m,0)}(z_1, \ldots, z_{2n})^2.
\end{align*}
\]

The proof of equations (70) is quite simple and follows the pattern discussed at the end of the previous section. Equations (70) are trivially true for \( N = 1, 2 \) (or \( n = 1 \)); moreover their rhs are polynomials in \( z_i \) of degree at most \( 2\left\lfloor \frac{N}{2} \right\rfloor - 1 \). The Schur polynomials \( S_{\lambda(m,r)}(z_1, \ldots, z_m) \) satisfy a recursion relation when one specializes \( z_i = q^\pm z_j \) (see for example appendix B of [17])

\[
S_{\lambda(m,r)}(z)|_{z_i = q^\pm z_j} = (-q^\mp z_j)^m \prod_{\ell=1}^{m} (z_\ell - q^\mp z_j) S_{\lambda(m-2, r)}(z \setminus [z_i, z_j]).
\]

This implies that the rhs of equations (70) satisfy the recursion relations (66) and (68) and therefore equations (70) hold.
Generic value of $k$

The recursion relation (71) for the Schur functions $S_{\lambda(m,r)}$ suggests a possible representation also in the case $k \neq 0$. For the sake of clarity, let us focus for a moment on $E_{2n}(k; y; z)$. It is easy to see that any product of the kind $S_{(m,0)}(y, z)S_{(n,1)}(y, z)$, with $I \subset \{1, \ldots, 2k\}$ and $F = \{1, \ldots, 2k\} \setminus I$, satisfies the recursion relations (66), but with a ‘wrong’ initial condition. It is reasonable to hope that an appropriate linear combination of terms with different choices of $I$ could provide the right initial condition and hence $E_{2n}(k; y; z)$.

In order to present how this idea actually works, it is convenient to introduce a bit of notation. Let $\tilde{\rho}, \tilde{\sigma}$ be strictly increasing infinite sequences of nonnegative integers; then, consider the following family of matrices:

$$
\mathcal{M}^{(\tilde{\rho}, \tilde{\sigma})}(r, s; y; z) = \begin{pmatrix}
\tilde{\rho}_1 y_1^2 & \tilde{\rho}_2 y_1^2 & \cdots & \tilde{\rho}_r y_1^2 & 0 & 0 & \cdots & 0 \\
\tilde{\rho}_1 y_2^2 & \tilde{\rho}_2 y_2^2 & \cdots & \tilde{\rho}_r y_2^2 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \tilde{\rho}_1 y_s^2 & \tilde{\rho}_2 y_s^2 & \cdots & \tilde{\rho}_r y_s^2 \\
0 & 0 & \cdots & 0 & \tilde{\rho}_1 y_{s+1}^2 & \tilde{\rho}_2 y_{s+1}^2 & \cdots & \tilde{\rho}_r y_{s+1}^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \tilde{\rho}_1 y_n^2 & \tilde{\rho}_2 y_n^2 & \cdots & \tilde{\rho}_r y_n^2 \\
\end{pmatrix}
$$

(72)

and let us define the following polynomials:

$$
S^{(\tilde{\rho}, \tilde{\sigma})}(r, s; y; z) = \frac{\det \mathcal{M}^{(\tilde{\rho}, \tilde{\sigma})}(r, s; y; z)}{\prod \limits_{1 \leq i < j \leq r} (z_i - z_j)^2 \prod \limits_{1 \leq i < j \leq 2s} (y_i - y_j) \prod \limits_{1 \leq i \leq r} (z_i - y_j)}.
$$

(73)

The divisibility of $\det \mathcal{M}^{(\tilde{\rho}, \tilde{\sigma})}(r, s; y; z)$ by $\prod \limits_{1 \leq i < j \leq r} (z_i - z_j)^2$ and by $\prod \limits_{1 \leq i < j \leq 2s} (y_i - y_j)$ is immediate. If we set $z_i = y_j$ for some $i, j$, then we subtract from the row corresponding to $y_j$ the two rows corresponding to $z_i$ getting a null row. This means that $\det \mathcal{M}^{(\tilde{\rho}, \tilde{\sigma})}(r, s; y; z)$ is also divisible by $(z_i - y_j)$.

Using the Laplace expansion along the first $r + s$ columns, we can write $S^{(\tilde{\rho}, \tilde{\sigma})}(r, s; y; z)$ as a bilinear in Schur polynomials

$$
S^{(\tilde{\rho}, \tilde{\sigma})}(r, s; y; z) = \sum \limits_{I \subset \{1, \ldots, 2s\}} (-1)^{|I|} \prod \limits_{i \in I, j \in \{r+1, \ldots, 2s\}} (y_i - y_j) \prod \limits_{i \in I, j \in \{1, \ldots, r\}} (y_i - y_j) S_{\rho(r+s)}(z, y; z) S_{\sigma(r+s)}(z, y; z),
$$

(74)

where $\rho(m)$ and $\sigma(m)$ are Young diagrams of length $m$, whose entries are $\rho(m)_i = \tilde{\rho}_i - i + 1$, $\sigma(m)_i = \tilde{\sigma}_i - i + 1$. In particular, note that when $s = 0$, then $S^{(\tilde{\rho}, \tilde{\sigma})}(r, 0; z)$ factorizes as a product of two Schur polynomials.

Now let us introduce the following family of integer sequences:

$$
\tilde{\lambda}_i(r) = \left\lceil \frac{3i - 3 + r}{2} \right\rceil, \quad \tilde{\lambda}_i(0) = \{0, 1, 3, 4, 6, 7, \ldots\} \quad \tilde{\lambda}_i(1) = \{0, 2, 3, 5, 6, 8, \ldots\} \quad \tilde{\lambda}_i(2) = \{1, 2, 4, 5, 7, 8, \ldots\} ... \quad (75)
$$
Then, we claim that
\[ E_{2n+1}^-(k; y; z) = 3^{-n^2+k(k-1)/2} S^{(0),(1)}(2n+1-k, k; y; z) \] (76)
\[ E_{2n+1}^+(k; y; z) = 3^{-n^2+k(k-1)/2} \left( \prod_{j=1}^{2n-k+1} z_j^{-1} \right) S^{(1),(2)}(2n+1-k, k; y; z) \] (77)
\[ E_2^n(k; y; z) = 3^{-n(n-1)+k(k-1)/2} S^{\tilde{(0),(0)}}(2n-k, k; y; z) \] (78)
\[ E_2^n(k; y; z) = 3^{-n(n-1)+k(k-1)/2} \left( \prod_{j=1}^{2n-k+1} z_j^{-1} \right) S^{(0),(2)}(2n-k, k; y; z). \] (79)

These formulas reduce to equations (70) for \( k = 0 \). Using the relations among inhomogeneous EFP with different parities (equations (58) and (59)) and the explicit form of the matrices \( M^{(1),(1)}(1) \), we see easily that equation (77) follows from equation (78), which in turn follows from equation (76). Therefore, it remains to prove only equations (76) and (79).

The rhs of equations (76) and (79) are polynomials in \( z_j \) respectively of degree \( 2n+1 \) and \( 2n-2 \). Moreover, using equation (71) and the form of \( S^{(r_1),(r_2)}(m, k; y; z) \) expressed by equation (74), we can easily obtain the recursion relation
\[ S^{(r_1),(r_2)}(m, k; y; z) |_{z_j=q^z z_j} = (-q^{\mp} z_j)^{r_1+r_2} \prod_{\ell=1}^{m} (z_\ell - q^\mp z_j)^2 \times \prod_{a=1}^{2k} (y_a - q^\mp z_j) S^{(r_1),(r_2)}(m-2, k; y; z \setminus [z_i, z_j]). \] (80)

In order to conclude, as explained at the end of section 3.1, it remains to show that equations (76) and (79) hold for \( n = k \), i.e. we need to prove that
\[ S^{(1),(2)}(k+1, k; y; z) = \prod_{1 \leq i < j \leq k+1} (z_i^2 + z_i z_j + z_j^2) \]
\[ S^{(0),(2)}(k, k; y; z) = \prod_{1 \leq i < j \leq k} (z_i^2 + z_i z_j + z_j^2). \] (81)

We proceed by factor exhaustion. A preliminary remark is that both \( S^{(1),(2)}(2n-k+1, k; y; z) \) and \( S^{(0),(2)}(2n-k, k; y; z) \), as polynomials in \( y_i \) of degree \( n-k \) and in particular they vanish as soon as \( k > n \). Therefore, using the recursion relation (80) we conclude that
\[ S^{(1),(2)}(k+1, k; y; z) |_{z_j=q^z z_j} = S^{(0),(2)}(2n-k, k; y; z) |_{z_j=q^z z_j} = 0. \] (82)

Since their degree as polynomials in \( z_j \) is respectively \( k \) and \( k-1 \), this means that we have proven equations (82) up to a numerical constant. Such a constant will be fixed to be equal to \( 1 \) in the following section, where we will compute explicitly the specialization of the inhomogeneous EFP for \( z_i = t^{l_i-1} \) and \( y_j = t^{n-k+j-i-1} \).

### 4.1. Homogeneous limit

In this section, we arrive at last to the computation of the homogeneous (pseudo) EFP using equations (76)–(79). We only need to consider a last intermediate step by setting \( z = z(t) \) and \( y = t^{n-k} y(t) \) with
\[ z(t)_i = t^{l_i-1} \quad \text{and} \quad y(t)_j = t^{l_j-1}. \]
The matrices $\mathcal{M}_{s}^{(r),\lambda(t)}(m, k; z(t); t^n y(t))$ with $r, s$ and $m$ as in equations (76)–(79) have noticeable structure as columns matrices. Let us look at a concrete example

\[
\mathcal{M}_{s}^{(r),\lambda(t)}(3, 2; z(t); t^3 y(t)) = \begin{pmatrix}
t^0 & t^1 & t^2 & t^3 & 0 & 0 & 0 & 0 \\
t^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t^0 & t^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
t^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & t_0^0 & t_2^0 & t_3^0 & t_4^0 \\
0 & 0 & 0 & 0 & t_0^1 & t_1^1 & t_2^1 & t_3^1 \\
0 & 0 & 0 & 0 & t_0^2 & t_1^2 & t_2^2 & t_3^2 \\
0 & 0 & 0 & 0 & t_0^3 & t_1^3 & t_2^3 & t_3^3 \\
0 & 0 & 0 & 0 & t_0^4 & t_1^4 & t_2^4 & t_3^4 \\
0 & 0 & 0 & 0 & t_0^5 & t_1^5 & t_2^5 & t_3^5 \\
0 & 0 & 0 & 0 & t_0^6 & t_1^6 & t_2^6 & t_3^6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\] (83)

The entries of the $j$th column (apart for the zeros) are consecutive powers of some $v_j$, where $v_j$ is itself a power of $t$ which depends on the column index $j$. In the precedent example, $v_j = \{1, t, t^3, t^4, t^6, 1, t^2, t^3, t^6\}$. Moreover, some $v_j$ appear twice, once in the first half of the columns and once in the second half (in the example $1, t^3, t^6$), while the remaining $v_j$ are of the form $a_1\lambda^i$ in the first half of the columns and $a_2\lambda^i$ in the second half (in the example $\lambda = t^3, a_1 = t, a_2 = t^3$).

By these considerations, we are led to introduce the following families of $2(\ell + r + s) \times 2(\ell + r + s)$ matrices $\mathcal{G}^{(r,s)}(v; \lambda, a_1, a_2)$ that are made of six blocks of rectangular matrices as follows:

\[
\mathcal{G}^{(r,s)}(v; \lambda, a_1, a_2) = \begin{pmatrix}
D_{r+r+\ell}^{(0)}(v) & D_{r+r+\ell}^{(0)}(a_1\lambda) & 0 & 0 \\
0 & 0 & D_{r+r+\ell}^{(0)}(v) & D_{r+r+\ell}^{(0)}(a_2\lambda)
\end{pmatrix},
\] (84)

where $v = \{v_1, \ldots, v_\ell\}$, $a_1\lambda = \{a_i\lambda^0, \ldots, a_\ell\lambda^{r+s-1}\}$ and the blocks consist of the following rectangular matrices:

\[
D_{m,l}^{(r)}(v) = \begin{pmatrix}
v_{1}^{l} & v_{2}^{l} & \cdots & v_{\ell}^{l} \\
v_{1}^{l+1} & v_{2}^{l+1} & \cdots & v_{\ell}^{l+1} \\
\vdots & \vdots & \ddots & \vdots \\
v_{1}^{l+m-1} & v_{2}^{l+m-1} & \cdots & v_{\ell}^{l+m-1}
\end{pmatrix}.
\] (85)

Apart from a trivial reordering of the columns we have

\[
\mathcal{M}_{s}^{(r),\lambda(t)}(2n - k, k; z(t); y(t)) = \mathcal{G}^{(n-k,k)}((1^{2n-k}; t^3, t^2))
\]

\[
\mathcal{M}_{s}^{(r),\lambda(t)}(2n - k, k; z(t); y(t)) = \mathcal{G}^{(n-k,k)}((1^{2n-k}; t^3, t^2))
\]

\[
\mathcal{M}_{s}^{(r),\lambda(t)}(2n + 1 - k, k; z(t); y(t)) = \mathcal{G}^{(n, n-k, k)}((1^{2n-k}; t^3, t^2))
\]

\[
\mathcal{M}_{s}^{(r),\lambda(t)}(2n + 1 - k, k; z(t); y(t)) = \mathcal{G}^{(n, n-k, k)}((1^{2n-k}; t^3, t^2))
\] (86)

Therefore, by calling

\[
\mathcal{E}^{s}_{2n}(k; t) := \mathcal{E}^{s}_{2n}(k; y(t); z(t))
\] (87)

we have

\[
\mathcal{E}^{s}_{2n}(k; t) = 3^{n(n-1)+k(k-1)/2} \frac{\det(\mathcal{G}^{(n, n-k, k)}((1^{2n-k}; t^3, t^2)))}{\prod_{1 \leq i < j \leq 2n-k} (t^{i-1} - t^{j-1}) \prod_{1 \leq i < j \leq 2n+k} (t^{i-1} - t^{j-1})}
\]
Then, from equations (89)–(91) we find that for generic values of

\[ r_n = r_{n+1} = 1 \]

These facts are proved in detail in appendix A.

Proposition A.9 shows that for both sides we have

\[ \det \mathcal{G}^{(r, r)}(\lambda; a_1, a_2) = \prod_{1 \leq i, j \leq r} (\lambda^{j-1}a_1 - \lambda^{i-1}a_2) \mathcal{D}^{(r, r)}(\lambda) \]

and

\[ \mathcal{D}^{(r, r)}(\lambda) = (-1)^{r(r+1)}\lambda^{(r-\ell)}(\ell^2 \prod_{1 \leq i, j \leq r} (\lambda^{j-1} - \lambda^{i-1}) \prod_{1 \leq i, j \leq r+2}(\lambda^{j-1} - \lambda^{i-1}) \]

These facts are proved in detail in appendix A.

Before proceeding to the computation of the rhs of equations (88), we come back for a moment to the argument we interrupted at the end of section 4. Equations (88) have been proven up to a constant independent of the difference \( n - k \). To show that the constant is 1, it is enough to check that the equations for \( \mathcal{E}_{2n}^-(k; t) \) and for \( \mathcal{E}_{2n+1}^-(k; t) \) hold true in the case \( n = k \). For the lhs we use equations (57) with \( q = e^{2\pi i/3} \) and \( z_t = t^{-1} \), while for the rhs we use equations (89)–(91). Rather than directly comparing the two sides of the equations, it is more convenient to compute the double ratio. A tedious but straightforward computation using proposition A.9 shows that for both sides we have

\[ \begin{align*}
\left( \frac{\mathcal{E}_{2k+3}^-(k+1; t)}{\mathcal{E}_{2k+1}^-(k; t)} \right) / & \left( \frac{\mathcal{E}_{2k-1}^-(k-1; t)}{\mathcal{E}_{2k-3}^-(k-1; t)} \right) = 3^{-1}4^{-k}3^{k+1} - 1 \\
\left( \frac{\mathcal{E}_{2k+3}^+(k+1; t)}{\mathcal{E}_{2k+1}^+(k; t)} \right) / & \left( \frac{\mathcal{E}_{2k-1}^+(k-1; t)}{\mathcal{E}_{2k-3}^+(k-1; t)} \right) = 3^{-1}t^{2(k-1)}t^3 - 1
\end{align*} \]

which, combined with the direct verification for \( n = k = 1, 2 \), gives the desired result.

Proofs of the conjectures

Taking the limit \( t \to 1 \) directly in equations (88) is not easy. Instead we consider the ratios \( \mathcal{E}_{2n}^-(k; t) / \mathcal{E}_{2n}^+(k; t) \) which are easier to compute, and from them recover equations (1), (2), (4), (6). Let us explain the computation for the case \( \mathcal{E}_{2n}^-(k; t) \), the other case being dealt with in the same manner. Using the first of equations (88), we get

\[ \mathcal{E}_{2k}^+(k-1; t) / \mathcal{E}_{2k}^+(k; t) = 3^{-1}4^{-k}\prod_{i=1}^{k-1}4^{2n-k} - t^{i-1} \times \frac{\det \mathcal{G}^{(n, n-k, k-1)}(k; t, t^2)}{\det \mathcal{G}^{(n, n-k, k)}(k; t, t^2)} \]

Then, from equations (89)–(91) we find that for generic values of \( \nu, \lambda \) and \( a_i \) the ratio \( \det \mathcal{G}^{(r, r)}(\nu; a_1, a_2) / \det \mathcal{G}^{(r+1, r+2)}(\nu; a_1, a_2) \) does not depend on \( \nu \) and is given by a very simple formula

\[ \frac{\det \mathcal{G}^{(r, r)}(\nu; a_1, a_2)}{\det \mathcal{G}^{(r+1, r+2)}(\nu; a_1, a_2)} = \lambda^{(r-1)(3r-2)/2} \prod_{j=(r-1)}^{r-1} (\lambda^{j-1}a_1 - \lambda^{j-1}a_2) \mathcal{D}^{(r, r)}(\lambda) / \mathcal{D}^{(r+1, r+2)}(\lambda) \]
\[
\frac{D^{(r,s)}(t)}{D^{(r+1,s+1)}(t)}(\lambda) = (-1)^{r+s} \frac{\prod_{i=1}^{r+2s} (\lambda^i - 1) \prod_{i=1}^{r+s} (\lambda^i - 1)^2}{\prod_{i=1}^{r+1} (\lambda^i - 1) \prod_{i=1}^{r+2s} (\lambda^i - 1) \prod_{i=1}^{2r-2} (\lambda^i - 1)}. \tag{95}
\]

At this point we make use of these equations in equation (93) and substitute $\lambda = r^3$, $a_1 = t$ and $a_2 = r^2$. Repeating the same steps with the proper modifications for the other EFPs, we finally obtain

\[
\begin{align*}
\mathcal{E}_{2n}^\pm(k-1; t) & = \kappa^\pm_{\nu, n,k}(\frac{[3]\,t}{3})^{k-1} \left[ 2n + k - 1, n - k, \{2k - 1\}, \{2k - 2\} \right]_t! \\
\mathcal{E}_{2n+1}^\pm(k-1; t) & = \kappa^\pm_{\nu, n,k}(-q)(\frac{[3]\,t}{3})^{k-1} \left[ 2n + k + 1, n - k, \{2k - 1\}, \{2k - 2\} \right]_t! \\
\end{align*}
\tag{96}
\]

where we have introduced the usual $t$-numbers and $t$-factorials

\[
\begin{align*}
[n]_t! & = \prod_{i=1}^n [i]_t \quad \text{and} \quad [i]_t! = t^{i-1}. \\
\end{align*}
\]

The powers of $t$ on the rhs of equations (96) do not concern us because we are actually interested in the specialization $t = 1$, which at this point is immediate and reproduces the conjectured formulas (1), (2), (4) and (6).

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Appendix A. A determinant evaluation

In this appendix, we evaluate the determinants of a family of matrices that appeared in the final step of the computation of the EFP in section 4. The matrices we are interested in are labeled by three indices $\ell, r, s$ and are made of blocks of rectangular matrices

\[
G^{(\ell,r,s)}(\nu; \lambda, a_1, a_2) = \begin{pmatrix}
D^{(0)}_{\ell+r,t} (\nu) & D^{(0)}_{\ell+r,t+1} (a_1, \lambda) & 0 & 0 \\
0 & D^{(0)}_{\ell+r,t} (\nu) & D^{(0)}_{\ell+r,t+1} (a_2, \lambda) & 0 \\
D^{(e+r)}_{2s+\ell} (\nu) & D^{(e+r)}_{2s+\ell} (a_1, \lambda) & D^{(e+r)}_{2s+\ell} (\nu) & D^{(e+r)}_{2s+\ell} (a_2, \lambda) \\
\end{pmatrix}, \tag{A.1}
\]

where $\nu = \{v_1, \ldots, v_{\ell}\}, a_1 = \{a_1, a_2, \ldots, a_{\ell+1}\}$ and each block consists of the following rectangular matrices:

\[
D^{(0)}_{m,1} (\nu) = \begin{pmatrix}
v_1^j & v_1^{j+1} & \cdots & v_1^{j+m-1} \\
v_2^j & v_2^{j+1} & \cdots & v_2^{j+m-1} \\
\vdots & \vdots & \ddots & \vdots \\
v_{\ell}^j & v_{\ell}^{j+1} & \cdots & v_{\ell}^{j+m-1} \\
\end{pmatrix}. \tag{A.2}
\]
The matrix \( G^{(\ell,r,s)}(v; \lambda, a_1, a_2) \) has the total size \( 2(\ell + r + s) \times 2(\ell + r + s) \). Here is an example

\[
G^{(2,1,1)}(v; \lambda, a_1, a_2) = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & v_1 & v_2 & a_1 \lambda a_1 & 0 & 0 & 0 & 0 \\
v_1^2 & v_1^2 & a_1^2 \lambda (\lambda a_1)^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & v_1 & v_2 & a_2 \lambda a_2 \\
v_1^2 & v_1^2 & a_1^2 \lambda (\lambda a_1)^2 & 0 & 0 & 0 & 0 & 0 \\
v_1^3 & v_1^3 & a_1^3 \lambda (\lambda a_1)^3 & v_1^3 & a_2^3 \lambda (\lambda a_2)^2 & (\lambda a_2)^2 \\
v_1^3 & v_1^3 & a_1^3 \lambda (\lambda a_1)^3 & v_1^3 & a_2^3 \lambda (\lambda a_2)^2 & (\lambda a_2)^2 \\
\end{pmatrix}.
\] (A.3)

We are interested in the determinant of \( G^{(\ell,r,s)}(v; \lambda, a_1, a_2) \) or equivalently in the determinant of the matrix \( \tilde{G}^{(\ell,r,s)}(v; \lambda, a_1, a_2) \), obtained from \( G^{(\ell,r,s)}(v; \lambda, a_1, a_2) \) through some simple row and column manipulations

\[
\tilde{G}^{(\ell,r,s)}(v; \lambda, a_1, a_2) = \begin{pmatrix}
D^{(0)}_{\ell+r,\ell}(v) & D^{(0)}_{\ell+r+1,\ell+1}(\lambda a_1) & 0 & 0 \\
0 & D^{(0)}_{\ell+r+2,\ell+1}(\lambda a_1) & D^{(0)}_{\ell+r+2,\ell+1}(v) & D^{(0)}_{\ell+r+2,\ell+1}(a_2) \\
\end{pmatrix}.
\] (A.4)

From the defining equation (A.4), we see that the first \( \ell \) columns of \( \tilde{G}^{(\ell,r,s)}(v; \lambda, a_1, a_2) \) have rank \( \ell + r \) and therefore \( \det \tilde{G}^{(\ell,r,s)}(v; \lambda, a_1, a_2) = 0 \) for \( r < 0 \). For \( r \geq 0 \), \( \det \tilde{G}^{(\ell,r,s)}(v; \lambda, a_1, a_2) \) factorizes nicely. This property is the content of the following three propositions.

**Proposition 2.** For \( r \geq 0 \), we have

\[
\det G^{(\ell,r,s)}(v; \lambda, a_1, a_2) = \prod_{1 \leq i,j \leq \ell} (v_i - v_j)^2 \prod_{0 \leq a_1, a_2 \leq \ell} (v_i - \lambda^{j-1} a_a) \det G^{(0,r,s)}(\lambda, a_1, a_2).
\] (A.5)

**Proof.** The proof is made by induction on \( \ell \). Let us look at \( \det \tilde{G}^{(\ell,r,s)}(v; \lambda, a_1, a_2) \) as a polynomial in \( v_1 \). Using the matrix \( \tilde{G}^{(\ell,r,s)}(v; \lambda, a_1, a_2) \) to compute the determinant, we see that its degree is at most \( 2(\ell + r + s) - 1 \). The presence of all the factors \( (v_i - v_j)^2 \), \( (v_i - \lambda^{j-1} a_a) \) is obvious using the matrix \( \tilde{G}^{(\ell,r,s)}(v; \lambda, a_1, a_2) \). Since these factors exhaust the total degree, it only remain to determine the term \( D^{(\ell,r,s)}(v \setminus v_1; \lambda, a_1, a_2) \) constant in \( v_1 \):

\[
\det \tilde{G}^{(\ell,r,s)}(v; \lambda, a_1, a_2) = \prod_{j=0}^{r+s-1} (v_1 - \lambda^j a_a) \prod_{i=0}^{\ell} (v_i - v_1) \det \tilde{G}^{(\ell-1,r,s)}(v \setminus v_1; \lambda, a_1, a_2).
\] (A.6)

We can just set \( v_1 = 0 \) in \( \tilde{G}^{(\ell,r,s)}(v; \lambda, a_1, a_2) \) and compute its determinant. We get immediately

\[
\det \tilde{G}^{(\ell,r,s)}(v; \lambda, a_1, a_2)|_{v_1=0} = \lambda^2 \frac{(v_1)^{\ell+r+s}}{a_1^2 a_2^{\ell+s}} \prod_{i=0}^{\ell} v_i^2 \det \tilde{G}^{(\ell-1,r,s)}(v \setminus v_1; \lambda, a_1, a_2).
\] (A.7)

Comparing equation (A.6) for \( v_1 = 0 \) and (A.7), we get

\[
D^{(\ell,r,s)}(v \setminus v_1; \lambda, a_1, a_2) = \det \tilde{G}^{(\ell-1,r,s)}(v \setminus v_1; \lambda, a_1, a_2)
\] (A.8)

which provides the recursion we were looking for.

It only remains to establish the evaluation of \( \det G^{(0,r,s)}(\lambda, a_1, a_2) \), but let us first deal with the particular case \( r = 0 \), which is quite simple and was useful in section 4.1. When \( r = 0 \), the matrix \( G^{(0,0,s)}(\lambda, a_1, a_2) \) has the form of a Vandermonde matrix and we find easily the following.
Proposition 3.
\[ \det(\mathcal{G}^{(0,0,t)}(\lambda; a_1; a_2)) = (a_1 a_2)^{j t} \prod_{1 \leq i < j \leq s} (\lambda^{j-i} - \lambda^{i-j}) \prod_{1 \leq i \leq s} (\lambda^{j-i} a_1 - \lambda^{i-j} a_2). \] (A.9)

The case when \( r \) is generic is a bit more subtle and is dealt with in the following:

Proposition 4.
\[ \det(\mathcal{G}^{(0,r,s)}(\lambda; a_1; a_2)) = (a_1 a_2)^{(r+s) t} \prod_{1 \leq i < j \leq s} (\lambda^{j-i} a_1 - \lambda^{i-j} a_2) \mathcal{D}^{(0,r,s)}(\lambda) \] (A.10)

with
\[ \mathcal{D}^{(0,r,s)}(\lambda) = \frac{(-1)^{(r+s)} \lambda^s (\lambda - 1) \prod_{1 \leq i < j \leq r} (\lambda^{j-i} - \lambda^{i-j}) \prod_{1 \leq i \leq s} \left(\lambda^{i-r+1} - \lambda^{r-i}\right)}{\prod_{1 \leq i \leq s} \left(\lambda^{i+r-s} - \lambda^{r-i}\right)}. \] (A.11)

**Proof.** The proof is made by factor identification. For \( \ell = 0 \) the matrix \( \mathcal{G}^{(0,r,s)}(\lambda; a_1; a_2) \) reads as follows:
\[
\begin{pmatrix}
\mathcal{D}^{(0)}_{r+r+s}(a_1 \lambda) & 0 \\
\mathcal{D}^{(0)}_{r+2s,r+r+s}(a_1 \lambda) & \mathcal{D}^{(0)}_{r+2s,r+r+s}(a_2 \lambda)
\end{pmatrix}.
\] (A.12)

From the Laplace expansion of \( \det(\mathcal{G}^{(0,r,s)}(\lambda; a_1; a_2)) \) with respect to the first \( r + s \) columns, we can easily deduce that all the determinants of the minors coming from the first \( r + s \) columns are divided by \( \mathcal{D}^{(0)}_{r+r+s}(a_1 \lambda) \), while all the determinants of the minors coming from the last \( r + s \) columns are divided by \( \mathcal{D}^{(0)}_{r+2s,r+r+s}(a_2 \lambda) \). In this way, we have obtained the factor \( (a_1 a_2)^{(r+s) t} \) in equation (A.10). In order to determine the total degree of \( \det(\mathcal{G}^{(0,r,s)}(\lambda; a_1; a_2)) \) as a function of \( a_1 \) and \( a_2 \), we note that the only nonvanishing contributions in the Laplace expansion of \( \det(\mathcal{G}^{(0,r,s)}(\lambda; a_1; a_2)) \) with respect to the first \( r + s \) columns come from the minor in the first \( r + s \) columns containing the first \( r \) rows and \( s \) among the last \( 2s \) rows, while the minor in the last \( r + s \) columns containing the rows from \( r + 1 \) to \( 2r \) and \( s \) among the last \( 2s \) rows. Therefore, the term of highest degree in \( a_1 \) corresponds to the minor containing the last \( s r \) and its degree is \((r + s)^t + s^2\).

Now we show that as a function of \( a_1 \), \( \det(\mathcal{G}^{(0,r,s)}(\lambda; a_1; a_2)) \) has zeros of order \( s - |j| \) at \( a_1 = \lambda^j a_2 \) for \( 0 \leq |j| \leq s - 1 \). Let for simplicity \( j \geq 0 \) (the case \( j < 0 \) being dealt with in the same manner). If we set \( a_1 = \lambda^j a_2 \) then the first \( r + s - j \) columns of \( \mathcal{D}^{(0)}_{r+2s,r+r+s}(\lambda^j a_2 \lambda) \) are equal to the last \( r + s - j \) columns of \( \mathcal{D}^{(0)}_{r+2s,r+r+s}(a_2 \lambda) \). Therefore, if in \( \mathcal{G}^{(0,r,s)}(\lambda; a_1 = \lambda^j a_2; a_2) \) we subtract the last \( r + s - j \) columns of the matrix \( \mathcal{G}^{(0,r,s)}(\lambda; a_1; a_2) \) from the \( j \)th column, for \( 1 \leq i \leq r - s - j \), we obtain a matrix whose first \( r + s - j \) columns have rank \( r \). This means that the determinant of the original matrix has a zero of order at least \( s - j \) at \( a_1 = \lambda^j a_2 \).

Since we have determined all the zeros of \( \det(\mathcal{G}^{(0,r,s)}(\lambda; a_1; a_2)) \) as a function of \( a_1 \) and \( a_2 \), we have established equation (A.10) up to the unknown factor \( \mathcal{D}^{(r,s)}(\lambda) \) which does not depend on \( a_1 \) and \( a_2 \). In order to determine such a factor, we specialize \( a_1 = \lambda^l a_2 \). We find that the first \( r \) columns of \( \mathcal{D}^{(0)}_{r+2s,r+r+s}(\lambda^l a_2 \lambda) \) are equal to the last \( r \) columns of \( \mathcal{D}^{(0)}_{r+2s,r+r+s}(a_2 \lambda) \). Hence, by subtracting the last \( r + 2s \) columns of the matrix \( \mathcal{G}^{(0,r,s)}(\lambda; a_1; a_2) \) from its \( i \)th column, for \( 1 \leq i \leq r \), we obtain the following matrix:
\[
\begin{pmatrix}
\mathcal{D}^{(0)}_{r,r}(\lambda^l a_2 \lambda) & * \\
0 & \mathcal{D}^{(0)}_{r+2s,r+r+s}(a_2 \lambda^r, \lambda^{r+s+1}, \ldots, \lambda^{r+2s-1}, 1, \lambda, \ldots, \lambda^{r+s-1})
\end{pmatrix}.
\] (A.13)
whose determinant is simply the product of the determinants of the two diagonal blocks
det \( D_r^{0} \) and det \( D_{r+2s+2}^{0} \):

\[
\det G(0, r; a_1 = \lambda t, a_2) = \det D_r^{0} \det D_{r+2s+2}^{0} = (-1)^{r(r+2)} \lambda^t \binom{t}{2} a_2^2 \prod_{1 \leq i < j \leq r} (\lambda^j - \lambda^i) \prod_{1 \leq i < j \leq r+2s} (\lambda^j - \lambda^i)
\]

By comparing equation (A.14) with equation (A.10) specialized at \( a_1 = \lambda^t a_2 \), we obtain equation (A.11).

**Corollary 1.**

\[
\frac{\det G(r, r; \lambda, a_1, a_2)}{\det G(r+1, r+1; \lambda, a_1, a_2)} = \lambda^{(t-1)(3t-2)/2} \prod_{j=0}^{r-1} (\lambda^j - a_1 - a_2) \frac{D(r, r; \lambda)}{D(r+1, r+1; \lambda)}
\]

\[
\frac{D(r, r; \lambda)}{D(r+1, r+1; \lambda)} = (-1)^{r+r} \prod_{i=1}^{r+2s-1} (\lambda^i - 1) \prod_{i=1}^{r-1} (\lambda^i - 1)^2
\]

**Appendix B. Plane partitions/dimer coverings**

A plane partition can be seen as a tiling of a regular hexagon of side length \( N \) with the following three types of rhombi of unit side length

A \( k \)-punctured cyclically symmetric self-complementary plane partition (PCSSCPP) of size \( 2n \) is a plane partition symmetric under a \( \pi/3 \) rotation around the center of the hexagon of side length \( 2n \) and with a star-shaped frozen region of size \( k \) (see figure 1).

**Figure B1.** Left: the hexagonal graph corresponding to a PCSSCPP. The number of edges is \( 2n-k \), while the number of boundary vertices on the bottom side is \( 2n \). The shaded region is a fundamental domain. Right: the graph \( G(n, k) \) obtained by restricting to the fundamental domain and making the identifications of the edges imposed by the cyclic symmetry.
Figure B2. (a) The domain $\tilde{G}(n, k)$, obtained from $\bar{G}(n, k)$ by removing the edges incident to the symmetry axis and lying on its right. The dimers on the edges which lie on the symmetry axis have weight $1/2$. (b) Another presentation of the graph $\bar{G}(n, k)$ and in dashed the corresponding domain $\bar{D}(n, k)$. (c) The domain $D(n, k)$. In dashed is the square lattice on which run the NILPs. (d) A redrawing of the square lattice of the NILPs. Each path that goes through a vertical edge incident to the bottom-right corners gets a weight $1/2$.

Following closely Ciucu [9] we compute the enumeration of $k$-PCSSCPP of size $2n$, that we call CSSCPP$(2n, k)$. It is well known that plane partitions can be seen also as dimer coverings of an hexagonal graph. In the case of the $k$-PCSSCPP, the graph is reported in figure B1. Thanks to the symmetry under a rotation of $\pi/3$, it is sufficient to consider dimer coverings of the fundamental domain with ‘periodic boundary conditions’ which are nothing else than dimer covering of a graph obtained by cutting the fundamental domain and joining the opposite cutted edges as on the right of figure B1. Let us call $G(n, k)$ this graph. Note that $G(n, k)$ is a planar graph symmetric under reflection along the vertical axis. In [9], Ciucu has proven that $M(G)$, the enumeration of dimer coverings of a planar graph $G$ with reflection symmetry, is related to the weighted enumerations of dimers covering of different graph $\tilde{G}$, obtained from $G$ by removing the edges incident to the symmetry axis and lying on its right (see figure B2). The relation reads

$$M(G) = 2^r M(\tilde{G}),$$

where $r$ is the number of edges lying on the symmetry axis, and the weighted enumeration $M(\tilde{G})$ is obtained by assigning a weight $1/2$ to each dimer lying on the symmetry axis.
In the case of PCSSCPP, we are led to consider weighted dimer covering of the graph $\tilde{G}(n, k)$ reported in figure B2, which can also be seen as rhombus tilings of the domain $D(n, k)$ or as non-intersecting lattice paths starting from the right boundary of $D(n, k)$ and ending on its northwest boundary. This last representation allows to use the Lindström–Gessel–Viennot theorem and find

$$CSSCPP(2n, k) = 2^{n-k} M(\tilde{G}(n, k)) = \det[Q_{i+k,j+k}]_{1 \leq i, j \leq n-k}$$  \hspace{1cm} (B.1)$$

with

$$Q_{i,j} = 2^i \binom{i+j-2}{j-1} + \binom{i+j-2}{2j-i-1}.$$  \hspace{1cm} (B.2)$$

The determinant of $Q_{i+k,j+k}$ can be evaluated using theorem 40 of [22] and we get

$$CSSCPP(2n, k) = \prod_{j=1}^{n-k} \frac{(j-1)!(j+2k-1)!(3j+3k-2)!}{(2j+k-1)!(2j+k-2)!(2j+3k-1)!(2j+3k-2)!}$$  \hspace{1cm} (B.3)$$

from which one easily finds

$$\frac{CSSCPP(2n, k-1)}{CSSCPP(2n, k)} = \frac{(2k-2)!(2k-1)!(2n+k-1)!(n-k)!}{(k-1)!(3k-3)!(3k-2)!(2n+k-1)!(n+k-1)!}.$$  \hspace{1cm} (B.4)$$

References

[1] Baxter R J Exactly Solved Models in Statistical Mechanics (London: Academic) (New York: Dover)
[2] Barouch E and McCoy B M 1971 Phys. Rev. A. 3 786
[3] Razumov A V and Stroganov Yu G 2001 J. Phys. A: Math. Gen. 34 3185 (arXiv:cond-mat/0012141)
[4] Razumov A V and Stroganov Yu G 2001 J. Phys. A: Math. Gen. 34 5335–40 (arXiv:cond-mat/0102247)
[5] Batchelor M T, de Gier J and Nienhuis B 2001 J. Phys. A: Math. Gen. 34 L265–70 (arXiv:cond-mat/0101385)
[6] Razumov A V and Stroganov Yu G 2004 Theor. Math. Phys. 138 333–7
Razumov A V and Stroganov Yu G 2004 Teor. Mat. Fiz. 138 395–400 (arXiv:math.CO/0104216)
Razumov A V and Stroganov Yu G 2005 Theor. Math. Phys. 142 237–43
Razumov A V and Stroganov Yu G 2005 Teor. Mat. Fiz. 142 284–92 (arXiv:math-ph/0501013)
[7] Cantini L and Sportiello A 2011 J. Comb. Theory A 118 123–46 (arXiv:1003.3376 [math])
[8] Kuperberg G 2002 Ann. of Math 156 835–66 (arXiv:math/0008184)
[9] Ciucu M 1997 J. Comb. Theory A 77 67–97
[10] Francesco P Di, Zinn-Justin P and Zuber J-B 2006 J. Stat. Mech. P08011 (arXiv:math-ph/0603009)
[11] Kitanine N, Maillet J M, Slavnov NA and Terras V 2002 J. Phys. A: Math. Gen. 35 L385–91
[12] Di Francesco P and Zinn-Justin P 2005 E. J. Comb. 12 R6 (arXiv:math-ph/0410061)
[13] Pasquier V 2006 Ann. Henri Poincare 7 397–421 (arXiv:cond-mat/0506075)
[14] Di Francesco P and Zinn-Justin P 2005 J. Phys. A: Math. Gen. 38 L815–22 (arXiv:math-ph/0508059)
[15] Frenkel I B and Reshetikhin N 1992 Commun. Math. Phys. 146 1–60
[16] Razumov A, Stroganov Yu and Zinn-Justin P 2007 J. Phys. A: Math. Theor. 40 11827–47 (arXiv:0704.3542)
[17] Biane P, Cantini L and Sportiello A 2011 arXiv:1101.3427
[18] Batchelor M T, de Gier J and Nienhuis B 2001 J. Phys. A: Math. Gen. 34 L265–70 (arXiv:cond-mat/0101385)
[19] Zinn-Justin P 2007 Commun. Math. Phys. 272 661 (arXiv:math-ph/0603018)
[20] Di Francesco P and Zinn-Justin P 2005 J. Phys. A: Math. Gen. 38 L815–22 (arXiv:math-ph/0508059)
[21] Bazhanov V V and Mangazeev V V 2005 J. Phys. A: Math. Gen. 38 L145–53
Bazhanov V and Mangazeev V V 2006 J. Phys. A: Math. Gen. 39 12235 (arXiv:hep-th/0602122)
Mangazeev V V and Bazhanov V V 2010 J. Phys. A: Math. Theor. 43 085206 (arXiv:0912.2163)
Razumov A V and Stroganov Yu G 2010 Theor. Math. Phys. 164 977–91 (arXiv:0911.5030)
Fendley P and Hagendorf C 2010 J. Phys. A: Math. Theor. 43 402004
Hagendorf C and Fendley P 2011 arXiv:1109.4090
[22] Krattenthaler C 1999 Séminaire Lotharingien Combin. 42 (‘The Andrews Festschrift’) B42q
(arXiv:math/9902004)