THE SEMICLASSICAL EXPANSION OF THE T-J MODEL

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1. INTRODUCTION

The discovery of high temperature superconductivity has spurred intense investigations of the two dimensional doped antiferromagnet. In the strong coupling limit, the t-J Hamiltonian, derived from the large-U Hubbard model, is often used to describe the low lying excitations. At zero doping, it directly reduces to the quantum antiferromagnetic Heisenberg model (QHM). Substantial progress has been recently achieved in understanding the Heisenberg limit, both theoretically and experimentally [C1]. The effects of doping, however, are still highly controversial. As demonstrated in the other lectures of this Winter School, the problem of interacting spins and charges requires novel theoretical approaches.

Let us start from the simplest one-band form of the Hubbard Model:

\[ H = - \sum_{\langle i,j \rangle,s} t c_{is}^\dagger c_{js} + U \sum_i n_{i\uparrow} n_{i\downarrow} \tag{1.1} \]

where \( c_{is}^\dagger \) creates an electron at site \( i \) with spin index \( s \), and \( \langle i; j \rangle \) denotes a summation over all sites and their nearest neighbors on the square lattice. The large-U limit of Eq. (1.1) can be derived using second order perturbation theory in \( t/U \). In the restricted Hilbert space of no double occupancies, the lowest order effective Hamiltonian is the t-J model:

\[ H^{tJ} = P_s \left[ - \sum_{\langle i,j \rangle,s} t c_{is}^\dagger c_{js} \right. \]

\[ \left. - \frac{J}{4} \sum_{\langle i,j,k \rangle} (c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger - c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger)(c_{j\downarrow} c_{k\uparrow} - c_{j\uparrow} c_{k\downarrow}) \right] P_s + O(t^3/U^2) \tag{1.2} \]

where \( J = 4t^2/U \) is the “superexchange” constant, and \( P_s \) projects onto the states with no double occupancies. \( \langle i; j k \rangle \) are triads of nearest neighbors. At half filling, (one electron per site), Eq. (2) reduces to the antiferromagnetic Quantum Heisenberg Model (QHM) on the square lattice:

\[ H^{tJ} \bigg|_{n_i=1} \rightarrow H^{QHM} = \frac{J}{2} \sum_{\langle i,j \rangle} (S_i \cdot S_j - \frac{1}{4}) \tag{1.3} \]
where
\[ S_i^\alpha \equiv \frac{1}{2} \sum_{ss'} c_{is}^\dagger \sigma_{ss'}^\alpha c_{is'}^\dagger, \quad \alpha = 1, 2, 3 \] (1.4)

are spin 1/2 operators in the subspace of one electron per site, and \( \sigma^\alpha \) are the three Pauli matrices. Charge fluctuations are completely suppressed at half filling, and we are left with the spin excitations of the Heisenberg model. The two dimensional quantum Heisenberg model has been successfully addressed by two complementary methods: the semiclassical approximation (large-S expansion) [C1], and the Schwinger boson mean field theory (large-N expansion) [A1]. There are also Fermion large-N expansions which do not recover the known properties of the QHM on the square lattice, and we shall not review them here. In order to address the doped system in a formulation than readily recovers the Heisenberg limit, we introduce two commuting Schwinger bosons \( a_i, b_i \), and a slave fermion \( f_i \), to represent the allowed states of the projected Hilbert Space. The operators obey
\[
[a_i, a_j^\dagger] = \delta_{ij}, \quad [f_i, f_j^\dagger] = \delta_{ij}, \quad [a_i, b_j^\dagger] = 0, \quad \text{etc.}
\] (1.5)

and satisfy the local constraints
\[
a_i^\dagger a_i + b_i^\dagger b_i + f_i^\dagger f_i = 1 \quad \forall \ i.
\] (1.6)

The t-J model is faithfully represented by
\[
\mathcal{H}^{tJ} = t \sum_{\langle i; j \rangle} f_i^\dagger f_j F_{ij} - \frac{J}{4} \sum_{\langle i; j k \rangle} \left( \delta_{ik} - f_k^\dagger f_i \right) A_{ij}^\dagger A_{kj} (1 - f_j^\dagger f_j),
\] (1.7)

where
\[
A_{ij}^\dagger = a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger, \quad F_{ij}^\dagger = a_j^\dagger a_i + b_j^\dagger b_i.
\] (1.8)

It is now possible to generalize the t-J model to \( S > 1/2 \) by replacing the constraint (1.6) by
\[
a_i^\dagger a_i + b_i^\dagger b_i + f_i^\dagger f_i = 2S \quad \forall \ i.
\] (1.9)

In the half-filled (undoped) case, \( f_i^\dagger f_i = 0 \), the Schwinger bosons describe states of spin \( S \), and their bilinear forms yield the matrix elements of the standard spin operators
\[
\frac{1}{2} (a_i^\dagger, b_i^\dagger) \sigma^\alpha \left( \begin{array}{c} a_i \\ b_i \end{array} \right) = S_i^\alpha, \quad \alpha = 1, 2, 3.
\] (1.10)

In the presence of a hole at site \( i \) the spin size at that site gets reduced by 1/2. We shall see that the large-S limit yields a non trivial classical \( (S \to \infty) \) theory provided we hold the scaled coupling constants \( J, t \) fixed, where
\[
\bar{J} = 4JS^2, \quad \bar{t} = 2tS.
\] (1.11)
It is also possible to generalize the t-J model to large-N by introducing \( N \) flavors of bosons per site [A2]. The large-N mean field theory predicts spiral magnetic phases at finite doping concentrations [J1]. Recently however, the fluctuation determinant has been found to be unstable (negative) in a range of momenta. The offending fluctuations were identified as local enhancements of the spiral distortion. Clearly, the holes drive strong perturbations of the spins on the lattice constant scale. These are difficult to treat by direct application of continuum and mean field approximations on the Hubbard and t-J models. This is why we shall concentrate on the large-S expansion.

2. SPIN-HOLE COHERENT STATES

Spin coherent states have been fruitfully used by Haldane to map the QHM onto the nonlinear sigma model [H1]. The construction of the spin coherent states path integral allowed a simple derivation of the topological Berry phases. The large-S expansion of this path integral provides a unified semiclassical treatment of the ordered and disordered phases of the quantum antiferromagnet. In the ordered phase, one obtains the usual spin wave expansion about the Néel state, which yields a very good approximation for the ground state staggered magnetization and uniform susceptibility even for spin \( 1/2 \) [C1].

In this lecture we generalize this approach to represent the t-J model by defining “spin-hole coherent states”. This will allow us to treat the short range interactions carefully, while observing the local constraints. We derive a semiclassical expansion of the ground state and low excitations in the presence of holes. Although the expansion is formally controlled by the large spin size \( S \), at low doping we may rely on the success of this approximation for the \( S=1/2 \) QHM.

Spin coherent states of spin \( S \) are defined as

\[
|\hat{\Omega}_S(\theta, \phi)\rangle = (2S)!^{-1/2} \left( ua^\dagger + vb^\dagger\right)^{2S}|0,0\rangle ,
\]

(2.1)

where \( \hat{\Omega} \) is a unit vector, and

\[
u = \cos(\theta/2)e^{i\phi/2} , \quad v = \sin(\theta/2)e^{-i\phi/2} .
\]

(2.2)

\( \theta, \phi \) are the latitude and longitude angles on the unit sphere.

The spin-hole coherent states are defined as follows:

\[
|\hat{\Omega}, \xi\rangle_S = |\hat{\Omega}_S\rangle |0\rangle + |\hat{\Omega}_{S-\frac{1}{2}}\rangle \xi f^\dagger |0\rangle .
\]

(2.3)

where \( \xi \) is an anticommuting Grassman variable. The states (2.3) allow a resolution of the identity, while conserving the constraint (1.9) for any given \( S \)-sector:

\[
\frac{2S}{4\pi} \int d\phi \, d\cos \theta \, d\xi^* d\xi \exp \left[ -\alpha_S \, \xi^* \xi \right] |\hat{\Omega}, \xi\rangle \langle \hat{\Omega}, \xi| = I .
\]

(2.4)

The factor \( \alpha_S = (2S + 1)/(2S) \) is required for normalizing the matrix elements to unity. We shall be able to replace \( \alpha \) by unity, by renormalizing the chemical potential \( \mu \) in the grand canonical ensemble.
By repeatedly inserting the resolution of the identity (2.4) in the Trotter product expansion of the density matrix, we following the standard procedure \[H1\] and construct the path integral for the partition function:

\[
Z = \int \mathcal{D} \hat{\Omega} \mathcal{D} \xi^* \mathcal{D} \xi \exp \left[ \int_0^\beta d\tau \sum_i \left( i(2S - \xi_i^* \xi_i)A(\hat{\Omega}_i) \cdot \dot{\hat{\Omega}}_{i} + \xi_i^* \dot{\xi}_i \right) - H^{tJ}[\hat{\Omega}, \xi^*, \xi] \right],
\]

where

\[
H^{tJ} = \frac{\langle \hat{\Omega}, \xi | H^{tJ} | \hat{\Omega}, \xi \rangle}{\langle \hat{\Omega}, \xi | \hat{\Omega}, \xi \rangle}
\]

(2.5)
is a function of the variables \(\theta_i, \phi_i, \xi_i\), and

\[
A \cdot \dot{\hat{\Omega}}_i = (1 - \cos \theta_i(\tau)) \dot{\phi}_i(\tau)
\]

(2.7)
is the spin-kinetic term. The coordinate invariant notation uses the gauge potential \(A(\Omega)\) which describes the vector potential of a unit magnetic monopole at the center of the sphere. The integral of the (2.7) over a closed loop on the sphere yields the solid angle enclosed by that loop.

The Grassman “time derivative” is defined in its discrete form:

\[
\dot{\xi} = (\xi(\tau) - \xi(\tau - \epsilon))/\epsilon,
\]

(2.8)
where \(\epsilon\) is the infinitesimal timestep.

\(H^{tJ}\) has quadratic and quartic terms in the Grassmann variables. It is possible to integrate out the Grassmans exactly only for quadratic hamiltonians. Therefore, we decouple the four-fermion terms by a Hartree-Fock approximation. For our purposes this approximation will be justified by the following arguments: (i) hole-correlation corrections to this Hartree-Fock decoupling are higher order in hole density and (ii) the quartic terms vanish in the ferromagnetically correlated regions, where the hole density will turn out to be significant.

Thus we arrive at a model which is quadratic in the Grassmann variables:

\[
H^{tJ} \approx \tilde{H}^J + \sum_{ij} \left( \tilde{H}_{ij}^f - \delta_{ij} \mu \right) \xi_i^* \xi_j
\]

\[
\tilde{H}^J = - \frac{J}{8} \sum_{\langle i,j,k \rangle} (\delta_{ik} - e^{i\psi_{ik}} \rho_{ik} \rho_{jj}) \sqrt{(1 - \hat{\Omega}_j \cdot \hat{\Omega}_k)(1 - \hat{\Omega}_i \cdot \hat{\Omega}_j)}
\]

\[
\tilde{H}_{ij}^f = \frac{\bar{t}}{\sqrt{2}} \sqrt{1 + \hat{\Omega}_i \cdot \hat{\Omega}_j} \, e^{i\gamma_{ij}}
\]

(2.9)

\[
\gamma_{ij} \text{ and } \psi_{ij} \text{ are the phases of } u_i^+ u_j + v_i^* v_j \text{ and } (u_i^+ v_k^* - v_i^* u_k^*)(u_j v_k - v_j u_k) \text{ respectively.}
\]

The hole density matrix \(\rho_{ij}[\hat{\Omega}] = \langle f_i^+ f_j \rangle\) is to be determined self-consistently, by solving for the Fermion ground state in the presence of a spin configuration \(\{\hat{\Omega}\}\).
The hole hamiltonian $\bar{H}^f$ describes two distinct hopping processes: inter-sublattice hopping ("t-terms") and intra-sublattice hopping ("J-terms"). When the spins $\hat{\Omega}_i$ have short range antiferromagnetic order the $\gamma$ phases in the t-terms fluctuate wildly, while $\psi_{ij} = \eta_i A^N \cdot (x_i - x_j)$ represents a slowly varying Néel gauge field $A^N(x)$ which obeys

$$\nabla \times A_i^N = \frac{1}{2} n_x \times n_y \cdot n .$$

(2.10)

$n \approx \eta_i \hat{\Omega}_i$ is the staggered magnetization (Néel) field, where $\eta_i = +1, (-1)$ on sublattice A (B) is the “sublattice charge”. $\nabla \times A_i^N$ is the “topological charge density” whose integral for continuos fields on a compact surface, is invariant [P1].

Weigmann, Wen, Shankar and Lee have studied Lagrangians which contain similar intra-sublattice $A^N$-coupled hopping terms in the context of high $T_c$ superconductivity [W1]. Their starting model differed however from the t-J model in that it did not contain the inter-sublattice hopping terms (t-terms). These terms are not $A^N$-gauge invariant, and do not conserve the sublattice charges. The t-terms cannot be neglected, especially in the $\bar{t}/\bar{J} > 1$ regime. We shall see, however, that in the large-$S$ limit they are dynamically eliminated from the low energy and long wavelength Lagrangian.

We begin by integrating out the fermions to obtain a purely spin partition function

$$Z^s = \int D\hat{\Omega} \exp \left[ \int_0^\beta d\tau \left( i \sum_i (2S - \rho_i) A(\hat{\Omega}_i) \cdot \dot{\hat{\Omega}}_i - \bar{H}^J[\hat{\Omega}] - E^f[\hat{\Omega}] \right) \right] ,$$

(2.11)

where

$$E^f[\hat{\Omega}] = -\beta^{-1} \text{Tr} \left[ 1 + T_\tau \exp \left[ -\int_0^\beta d\tau (H^J[\hat{\Omega}] - \mu) \right] \right]$$

(2.12)

is the fermion free energy in the presence of a general spin history $\hat{\Omega}(\tau)$. $T_\tau$ is the time ordering operator. Here we concentrate on the zero temperature case $\beta = \infty$. Eq. (2.11) is a useful starting point for the semiclassical approximation. In the classical limit, $S \rightarrow \infty$, the kinetic term is so large, that the only important configurations in the path integral are classical, i.e. the spins are frozen and $\langle \hat{\Omega} \rangle = 0$. One has therefore to minimize $\bar{H}^J + E^f$ with respect to $\hat{\Omega}$ for a given chemical potential or a specified number of holes. The second step is include the semiclassical fluctuations whose dynamics are controlled by the kinetic term. We discuss the single hole and the many hole cases seperately.
3. RESULTS

In Ref. [A3], a Lanczos algorithm on the Connection Machine was used to minimize the energy for $128 \times 128$ spins. The following results were found:

**The single hole:**

In the regime $\bar{t}/\bar{J} > 0.87$, we have found that the “polaron” which is a *local* alignment of spins, yields a lower energy than any of the possible uniform states, including: the Néel state, spiral states and canted states. This result helps to explain why the uniform spiral states were found to be unstable against short wavelength distortions in the mean field theory [A2]. The polaron variational parameters were chosen to describe a ferromagnetic core, a transition region, and a far field antiferromagnetic tail. The latter is completely determined by the boundary condition $\delta \theta$ and the pure Heisenberg model (i.e. the Laplace equation).

Our results are quite simple. For $1 < \bar{t}/\bar{J} < 4.1$ the single hole energy is minimized by the five-site polaron (one flipped spin), depicted in Fig. 1.

**Fig. 1:** The five site polaron. The hole density $\rho_{ii}$ is primarily concentrated on the sites of the unfilled arrows. The circular arrows represent an allowed tunneling path, where the polaron hops two lattice constants to the left.

The hole density is approximately $1/2$ and $1/8$ on the central and neighboring sites respectively, with a small amount of leakage (due the $J$-terms in (2.9)) to sites further away. For $4.1 < \bar{t}/\bar{J} < 6.6$, the polaron has two flipped spins (diagonally across a plaquette), and at larger values the core radius increases slowly as $R_c \sim (\bar{t}/\bar{J})^{1/4}$, and the energy goes asymptotically as $\epsilon_h + 4\bar{t} \sim (\bar{J}\bar{t})^{1/2}$. The most important fact is that the small polarons *do not distort the Néel background*. In particular, the configurations centered on a bond [A4] are considerably higher in energy. We also find that the polarons have no tails [S1], i.e. $\delta \theta = 0$, throughout
the regime discussed above. This follows from competing contributions of order \( \pm J(\delta \theta)^2 \) of \( \tilde{H}^J \) and \( E^f \). Since in addition, the density \( \rho \) is exponentially localized near the polaron sites, we conclude that the classical interactions between polarons are short ranged.

The polaron breaks the lattice translational symmetry. This symmetry is restored by tunneling events, where two spins \( i \) and \( k \) simultaneously flip their orientation (see Fig. 1). The tunneling matrix element \( \Gamma_{ik} \) (the polaron’s hopping rate), is non-perturbative in \( S^{-1} \):

\[
\Gamma_{ik} = \Gamma_0 \exp \left( -\sum_{i'} \int d\phi_i \bar{S}^{z}_{i'} \right) \approx S^{1/2} \beta_{ik} \bar{t} \exp(-S \alpha_{ik}), \tag{3.1}
\]

Eq. (3.1) can be computed as follows: The azimuthal coordinates are analytically continued \( \phi \to \tilde{\phi} = \phi' + i\phi'' \), while their canonical momenta \( S^z_i = (2S - \rho_i) \cos \theta_i \) are kept real. It can be readily verified that \( \sum_i S^z_i \), and \( \tilde{H}^J + E^f \) are conserved along the tunneling path \( \bar{S}^z_i(\tilde{\phi}) \) which minimizes the action. As a result of these conservation laws, we obtain a selection rule: tunneling can only take place between sites on the same sublattice! This, in effect, amounts to the elimination of the inter-sublattice “t-terms”.

\( \alpha_{ik}, \beta_{ik} \) are slowly varying dimensionless functions of \( \bar{t} \) and \( \bar{J} \). For five site polarons, the number of spins involved in the tunneling path is at least three. For \( S = \frac{1}{2} \) we estimate the exponent to be roughly unity, but a fuller treatment of the multidimensional tunneling problem is required for a quantitative determination of \( \Gamma \) and the polaron’s effective mass.

The single polaron in a perfect Néel background occupies a Bloch wave of dispersion \( \epsilon_k = 2\Gamma_c(\cos(2k_x) + \cos(2k_y)) + 2\Gamma_b(\cos(k_x + k_y) + \cos(k_x - k_y)) \), where \( c, b \) denote the site of the other flipped spin as labelled in Fig. 1. By energetic arguments, \( \Gamma_b \leq \Gamma_c \). A Berry’s phase calculation of the tunneling matrix elements for \( S=1/2 \), yields an overall positive sign for \( \Gamma_c, \Gamma_b > 0 \). Thus the single polaron energy is minimized at \( k = (\pi/2, \pi/2) \). This result agrees with other studies of the single hole spectral function in the t-J model [S2]. For small deviations of the background spins from antiferromagnetic order the tunneling rate \( \Gamma_{ik} \) is modulated by the overlap of the background and the perfectly antiferromagnetic configurations. This overlap is just \( \exp[i\eta_i A^N(x_i - x_k)] \). \( A^N \) and \( \eta_i \) are the aforementioned Néel gauge field and sublattice charge respectively. We notice that \( A^N \) couples in a gauge invariant way to the polarons, and that the sublattice charges are conserved in the hopping.

**Interactions:**

The interactions between two polarons were computed in the regime \( \bar{t}/\bar{J} = 1 - 4 \). We define \( U^p_{ij} = \frac{1}{2} e_{ij} - 2e_h \), where \( e_{ij} \) is the relaxed energy of a two-hole polaron with flipped spins at sites \( i \) and \( j \). \( U^p_{ij} \) is repulsive, and of order \( 0.6\bar{J} - 2.6\bar{J} \). The intersite interactions, for neighboring polarons at sites (a)-(d) (see Fig. 1.) are plotted in Fig. 2.

We find both attractive and repulsive interactions, and it is interesting to note that for \( \bar{t}/\bar{J} < 1.8 \) there is a near neighbour attraction of antiferromagnetically
correlated spins. We also consider the possibility of polaron condensation into hole-rich domains [I1]. The condensation energy per hole is determined by minimizing it with respect to the spin configuration, and the density. The spins in the hole-rich domains align ferromagnetically, and the energy per hole is given by 
\[ e_{fm} = -4\tilde{t} + 4\sqrt{B\tilde{J}\tilde{t}}\pi. \]
This results coincides with that of Emery, Kivelson and Lin [I1], except that their quantum correction factor \( B=0.584 \) is here set to 1/2. In Fig. 2, the condensation energy \( \Delta e_c = e_{fm} - e_h \) is plotted. We find that it becomes negative at \( \tilde{t}/\tilde{J} = 2.7 \), above which phase-separation will occur for large \( S \).

Attractive interactions and negative classical condensation energies may yield charge density waves or superconductivity in the ground state of the quantum model. However, if realistic intersite Coulomb repulsions are added to the t-J model, the short range interactions may change sign. In particular, phase separation may be suppressed, or pushed to higher values of \( \tilde{t}/\tilde{J} \).

The information given above allows us to write the effective Lagrangian for a dilute system of small polarons:

\[
\mathcal{L}^{s-p} = \sum_i \left[ i(2S - p_i^* p_i) A(\hat{\Omega}_i) \cdot \hat{\Omega}_i + p_i^* p_i \right] + \frac{\tilde{J}}{8} \sum_{\langle ij \rangle} \hat{\Omega}_i \cdot \hat{\Omega}_j + \sum_i (e_h - \mu) p_i^* p_i + \sum_{\langle ij \rangle} \Gamma_{ik} e^{\eta A^N \cdot x_{ik}} p_i^+ p_k + \sum_{ij} U_{ij}^p p^*_i p^*_j p_j p_j
\]

Eq. (3.2) is the most important result. \( \mathcal{L}^{s-p} \) describes a two charge system of spinless Fermions \( p_i \) with short range interactions \( U_{ij}^p \) coupled to Heisenberg spins. The formation of polarons can be viewed as a strong short-wavelength dressing of the original f-holes by the spins. As a consequence, the uncomfortable \( t \)-terms have been conveniently eliminated, and the effects of holes on the spin background is short ranged. This Lagrangian describes the low lying excitations of the so-called t'-J Hamiltonian

\[
\mathcal{H}^{t'-J} = \frac{\tilde{J}}{2} \sum_{\langle ij \rangle} S_i \cdot S_j + 4 \sum_{\langle i:j:k \rangle} \Gamma_{ik} p_i^+ p_k A_{ij}^\dagger A_{jk}
\]

Fig. 2: Classical interactions between polarons, in units of \( J \).
Lines (a)-(d) represent the second polaron positions as labelled in Fig. 1. The solid line represents the relative condensation energy per hole of the hole-rich phase (see text).
A major advantage of the $t'$-J model over the $t$-J model is that in the small concentration limit $\delta << 1$, (3.2) is amenable to the continuum approximation. Following Haldane [H1] the spin interactions can be relaxed by the $(2+1)$ dimensional non linear sigma model, with $\delta$ dependent renormalized stiffness constant and spin wave velocity. The precise evaluation of the sigma model parameters for finite $\delta$ is beyond the scope of this paper, but we expect that above some critical density $\delta > \delta_c$ the ground state is disordered [C2], i.e. a “spin liquid”. In the massive “spin liquid” phase, Eq. (3.2) reduces to Wiegmann, Wen, Shankar and Lee’s model [W1]:

\[
L^{WWSL} = \sum_{\eta=\pm} \left[ p_\eta^\dagger (\partial_\tau + i\eta A^{N}_0) p_\eta + \frac{1}{2m} p_\eta^\dagger |\nabla + i\eta A^{N}_0|^2 p_\eta + \frac{1}{4\kappa} (F_{\mu\nu})^2 \right] + O(p^\dagger p p^\dagger p) \ldots
\]  

(3.4)

where $m$ is the effective mass at $k = (\pi/2, \pi/2)$, and the “electromagnetic” Néel-fields are $F_{\nu\mu} = (\partial_\mu A^N_\nu - \partial_\nu A^N_\mu)$. $\kappa$ is the inverse spin correlation length, which is also the coupling constant of the gauge field. Previous analyses [W1] have concluded that the ground state of (3.4) is most likely an RVB-type superconductor. Lee argued that the pairing is caused by two effects: (i) attraction between the opposite charges induced by the Néel gauge field, and (ii) suppression of coherent single particle propagation due to fluctuating Bohm-Aharonov phases, while the pairs $\langle p^\dagger p p^\dagger p \rangle$ propagate as free bosons. Both (i) and (ii) are only valid in the magnetically disordered phase, a pleasing feature which agrees with the phase diagrams of the copper oxide superconductors.

Aside from the mechanism of superconductivity, the small polaron theory could be checked numerically on finite lattice Monte-Carlo simulations, and experimentally in the copper-oxides and other doped antiferromagnets. For example: the polaron size can be estimated by NMR techniques [M1], and its internal excitations could be probed by optical absorption. In the frozen moments regime, one expects the polarons to exhibit conductivity typical of weakly localized semiconductors [C3].

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REFERENCES

[A1] Auerbach, A. and Arovas, D.P., Phys. Rev. Lett. 61, 617 (1988); Jour. Appl. Phys. 67, 5734 (1990).
[A2] Auerbach, A. and Larson, B.E., Phys. Rev. B43, 7800 (1991).
[A3] Auerbach, A. and Larson, B.E., Phys. Rev. Lett. 66, 2262 (1991).
[A4] Aharony, A., Birgeneau, R.J., Coniglio, A., Kastner, M.A. and Stanley, H.E., Phys. Rev. Lett.60, 1330 (1988).
[C1] Chakravarty S., Proc. of 1989 Symposium on High Tc Superconductivity, Ed. Bedell, K.S. et. al., (Addison-Wesley, 1990) and references therein.
[C2] Spin wave theory about a single polaron yields a finite $O(\delta)$ correction to the ground state magnetization. This suggests that $\delta_c \neq 0$.
[C3] Chen C.Y. et. al, Phys. Rev. Lett.63, 2307 (1989).
[H1] Haldane, F.D.M., “Two Dimensional Strongly Correlated Electron Systems”, Eds. Gan, Z.Z. and Su, Z.B., (Gordon and Breach, 1988), pp. 249-261;Phys. Rev. Lett.61, 1029 (1988).
[I1] Ioffe, L.B. and Larkin, A.I., Phys. Rev. B37, 5730 (1988); Emery, V.J., Kivelson, S.A. and Lin, H.Q., Phys. Rev. Lett. 64, 475, (1990); Marder, M., Papanicolau, N. and Psaltakis, G.C., Phys. Rev. B41, 6920 (1990).
[J1] Jayaprakash, C., Krishnamurthy, H.R., and Sarker, S., Phys. Rev. B40, 2610, (1989); Kane, C.L., Lee, P.A., Ng, T.K., Chakraborty, B. and Read, N., Phys. Rev. B41, 2653 (1990).
[M1] Mendels, P., et. al. Physica C 171, 429 (1990).
[P1] Polyakov, A. M., “Gauge Field And Strings”, (Harwood, 1987), P. 140.
[S1] Schraiman, B.I. and Siggia, E.D., Phys. Rev. Lett.62, 156 (1989).
[W1] Wiegmann, P. B., Phys. Rev. Lett.60, 821 (1988); Wen, X-G., Phys. Rev. B39, 7223 (1989); Shankar, R., Phys. Rev. Lett.63, 203 (1989); Lee, P.A., Phys. Rev. Lett.63, 690, (1989).