GLOBAL EXISTENCE OF WEAK SOLUTIONS TO THE
COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH
TEMPERATURE-DEPENDING VISCOSITY COEFFICIENTS

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Abstract. This paper is devoted to the global existence of weak solutions to the three-dimensional compressible Navier-Stokes equations with heat-conducting effects in a bounded domain. The viscosity and the heat conductivity coefficients are assumed to be functions of the temperature, and the shear viscosity coefficient may vanish as the temperature goes to zero. The proof is to apply Galerkin method to a suitable approximate system with several parameters and obtain uniform estimates for the approximate solutions. The key ingredient in obtaining the required estimates is to apply De Giorgi’s iteration to the modified temperature equation, from which we can get a lower bound for the temperature not depending on the artificial viscosity coefficient introduced in the modified momentum equation, which makes the compactness argument available as the artificial viscous term vanishes.

1. Introduction and Main Result

In this paper, we study the global existence of weak solutions to the following three-dimensional compressible Navier-Stokes equations with temperature effects:

$$\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p &= \text{div} S, \\
\partial_t (\rho \vartheta) + \text{div}(\rho u \vartheta) + \text{div} q &= S : \nabla u - \vartheta p_\theta(\vartheta) \text{div} u.
\end{aligned}$$

(1.1)

Here \( \rho = \rho(t, x) \) is the density of the fluid, \( u = u(t, x) \) is the velocity field, \( \vartheta = \vartheta(t, x) \) is the temperature, \( p = p(t, x) \) is the pressure determined by the constitutive equation

$$p = p(\rho, \vartheta) = p_e(\rho) + \vartheta p_\theta(\vartheta),$$

(1.2)

with \( p_e \) being the elastic pressure and \( \vartheta p_\theta \) being the thermal pressure, \( q \) denotes the heat flux of the fluid satisfying the Fourier’s law

$$q = -\kappa(\vartheta) \nabla \vartheta,$$

with \( \kappa = \kappa(\vartheta) > 0 \) being the heat conductivity coefficient depending on the fluid temperature, and \( S \) denotes the viscous stress tensor

$$S = \mu(\nabla u + \nabla^T u) + \lambda \text{div} u I,$$
where \( \mu = \mu(\vartheta) \) and \( \lambda = \lambda(\vartheta) \) are the shear and bulk viscosity coefficients respectively depending on the temperature. We assume that

\[
\mu(\vartheta) \geq 0, \quad \lambda(\vartheta) + \frac{2}{3}\mu(\vartheta) \geq 0. 
\](1.3)

Note that \( \mu(\vartheta) \) is allowed to degenerate in the region of absolutely zero temperature.

We will consider system (1.1) in a smooth bounded domain \( \Omega \subset \mathbb{R}^3 \), and impose the following initial and boundary conditions

\[
(\rho, \rho u, \vartheta)(0, x) = (\rho_0, m_0, \vartheta_0)(x) \quad \text{in} \quad \Omega, 
\]

(1.4)

\[
u(t, x) = 0, \quad \nabla \vartheta(t, x) \cdot n(x) = 0 \quad \text{on} \quad [0, T] \times \partial \Omega, 
\]

(1.5)

where \( n(x) \) is the unit outward normal vector to the boundary at \( x \in \partial \Omega \).

There have been a huge number of works in the literature concerning the global existence of solutions to the compressible Navier-Stokes equations. In particular, results about the one-dimensional case are rather satisfactory, see \( 14, 18, 26, 27, 31 \) and the references therein. For the multi-dimensional case, Matsumura-Nishida \( 33 \) first showed the global existence of classical solutions with small initial data, and then Hoff \( 16 \) extended the result \( 33 \) to the discontinuous initial data case. See also \( 6, 7, 15 \) for the spherically symmetric case. For the large initial data which may contain vacuum, the global existence of weak solutions was first proved by Lions \( 30 \) for the isentropic case, i.e., \( p = A\rho^\gamma \) with \( \gamma \geq 3/2 \) in two dimensions and \( \gamma \geq 9/5 \) in three dimensions. This result was extended to \( \gamma > 1 \) for the spherically symmetric case by Jiang-Zhang \( 21 \) and \( \gamma > 3/2 \) for the general three-dimensional case by Feireisl-Novotný-Petzeltová \( 8 \).

In all the works mentioned above, the viscosity coefficients are assumed to be positive constants, which plays an essential role in obtaining fine estimates for the gradient of the velocity field. The global existence becomes more challenging if the viscosity coefficients are effective functions of the density due to the possible occurrence of vacuum. The one-dimensional case and the multi-dimensional case with spherically symmetric data were studied in \( 11, 13, 20, 22, 32, 34, 41 \) and the references therein. For the multi-dimensional case with general data, Bresch-Desjardins \( 3 \) established the global weak solutions, where a new entropy inequality (BD entropy) was obtained to yield more regularity for the density. Then, based on the compactness arguments in Mellet-Vasseur \( 35 \), Li-Xin \( 28 \) and Vasseur-Yu \( 40 \) proved the global existence of weak solutions to the compressible Navier-Stokes equations with the viscous Saint-Venant system for the shallow water contained. We also mention a very remarkable result by Vaigant-Kazhikhov \( 39 \) where the global well-posedness of classical solutions to a potential barotropic compressible model was obtained if the initial density is uniformly away from vacuum and the viscosity coefficients satisfy

\[
\mu = \text{constant} > 0, \quad \lambda(\vartheta) = \vartheta^\beta, \quad \beta > 3.
\]

This result was further developed by Jiu-Wang-Xin \( 23-25 \).

When the heat-conducting effects are considered and the viscosity coefficients are functions of the temperature, Feireisl \( 10 \) proved the global existence of “variational” solutions.
to the compressible Navier-Stokes equations in a bounded domain \( \Omega \subset \mathbb{R}^N, N = 2, 3 \), where the viscosity coefficients satisfy

\[
\mu(\vartheta) \geq \mu > 0, \quad \lambda(\vartheta) + \frac{2}{N} \mu(\vartheta) \geq 0,
\]

for some positive constant \( \mu > 0 \). The concept of “variational” solutions was first proposed by Feireisl [9], where the global existence of “variational” solutions was proved for the constant viscosity coefficients.

In the present paper, we consider the global existence of weak solutions to the initial-boundary problem (1.1), (1.4) and (1.5), with the viscosity coefficients \( \mu(\vartheta) \) and \( \lambda(\vartheta) \) satisfying (1.3). Note that the shear viscosity coefficient \( \mu(\vartheta) \) is degenerate and may vanish in the region of absolutely zero temperature. This assumption is based on the fact that zero viscosity may occur when the temperature is very low, as for some superfluids mentioned in [2].

As the shear viscosity coefficient vanishes, the parabolicity of the momentum equation (1.1) will degenerate. To overcome this difficulty, we add an artificial viscosity term \( \eta \Delta u \) \( (\eta > 0) \) in the momentum equation, which makes it possible to apply the Galerkin method to prove the global solvability of the approximate system. Besides, such an artificial viscosity term plays an essential role in obtaining the \( L^2(0,T; H^1_0(\Omega)) \) estimate for \( u \) in Sections 2 and 3. However, the artificial term also brings two key problems: first, suitable estimates on \( \nabla u \) independent of the parameter \( \eta > 0 \) are needed; second, possible density oscillation as the artificial viscosity term vanishes requires extra attention.

To obtain suitable estimates on \( \nabla u \) independent of \( \eta > 0 \), it suffices to get a uniform lower bound for the viscosity coefficient \( \mu(\vartheta) \) with respect to \( \eta > 0 \), which can be achieved by proving that the temperature is bounded away from zero by a positive constant. The required positive lower bound for the temperature is obtained in Section 4 by De Giorgi’s iteration, a useful method first established by De Giorgi [4] in obtaining Hölder regularity of solutions to elliptic equation with discontinuous coefficients, and then applied to the study of the compressible Navier-Stokes equations by Mellet and Vasseur [36]. It is worthy mentioning that the proof of [36] relies heavily on the following thermal energy inequality

\[
\int_{\Omega} \phi(\vartheta)(t,x)dx - \int_{s}^{t} \int_{\Omega} 2\mu\phi'(|D(u)|^2 dxdt - \int_{s}^{t} \int_{\Omega} \lambda\phi'(|\text{div}u|^2 dxdt
\]

\[
+ \int_{s}^{t} \int_{\Omega} \kappa\phi''(|\nabla\vartheta|^2 dxdt \leq -R \int_{s}^{t} \int_{\Omega} \rho\phi(\vartheta)|\text{div}u|^2 dxdt + \int_{\Omega} \rho\phi(\vartheta)(s,x)dx,
\]

for \( \phi(\vartheta) = \ln\left(\frac{C\vartheta}{\vartheta + \varepsilon}\right) \), which will be replaced by the modified temperature inequality (3.4) in our proof.

The way to deal with possible density oscillation is mostly based on the weak continuity property of the effective viscous pressure

\[ P_{eff} = p - (\lambda + 2\mu)\text{div}u, \]

which was first introduced by Lions [30] for the barotropic case with \( \mu \) and \( \lambda \) being positive constants, and then generalized by Feireisl [10] to the case that \( \mu \) and \( \lambda \) depend on the
temperature with \( \mu(\vartheta) \geq \mu > 0 \) for some positive constant \( \mu \). The assumption in [10] that \( \mu(\vartheta) \) has a positive lower bound is essential to ensure the weak continuity property of \( P_{\text{eff}} \), and this may fail for our degenerate case (1.3). However, if we add an artificial viscosity term \( \eta \Delta u \) in the momentum equation, then the term \( \mu(\vartheta) + \eta \) can be viewed as a new shear viscosity coefficient with a positive lower bound \( \eta \), and therefore the weak continuity property of \( P_{\text{eff}} \) still holds. In Section 4 (\( \eta \to 0 \)), we will prove that the temperature is bounded away from zero by De Giorgi’s method, which combined with the strictly increasing assumption of the shear viscosity coefficient with respect to the temperature in Theorem 1.1 implies that there exists a positive constant \( \mu > 0 \) independent of \( \eta \) such that

\[
\mu(\vartheta) \geq \mu > 0.
\]

(1.6)

Based on [10], the bound (1.6) implies the strong convergence of the density as the artificial viscosity term \( \eta \Delta u \) vanishes.

Another difficulty in establishing the global existence of weak solutions lies in the temperature concentration, which as in [9] can be tackled by the renormalization of the temperature equation (1.1) 3. Concretely, multiplying (1.1) 3 by \( h(\vartheta) \) for some suitable function \( h \), we have

\[
\partial_t(\varrho H(\vartheta)) + \text{div}(\varrho uH(\vartheta)) - \Delta \mathcal{K}_h(\vartheta) = h(\vartheta)\mathcal{S} : \nabla u - h'(\vartheta)\kappa(\vartheta)|\nabla \vartheta|^2 - h(\vartheta)\vartheta p\vartheta\text{div}u,
\]

where

\[
H(\vartheta) = \int_0^\vartheta h(z)dz, \quad \mathcal{K}_h(\vartheta) = \int_0^\vartheta \kappa(z)h(z)dz.
\]

The idea of renormalization was first proposed by DiPerna and Lions in [5], where they replaced the continuity equation (1.1) 1 by a renormalized version

\[
\partial_t b(\varrho) + \text{div}(b(\varrho)u) + (b'(\varrho)\varrho - b(\varrho))\text{div}u = 0,
\]

for suitable functions \( b = b(\varrho) \), and then such a method was applied by Lions [29] and Feireisl [9] to overcome the temperature concentration.

The weak solutions to the initial-boundary value problem (1.1), (1.4) and (1.5) are defined as follows:

**Definition 1.1.** We call \((\varrho, u, \vartheta)\) a weak solution to the initial-boundary value problem (1.1), (1.4) and (1.5) if

(i) the density \( \varrho \geq 0 \) satisfies

\[
\varrho \in L^\infty(0, T; L^\gamma(\Omega)) \cap C([0, T]; L^1(\Omega)),
\]

the velocity \( u \) belongs to \( L^2(0, T; H^1_0(\Omega)) \), and \((\varrho, u)\) solves the continuity equation (1.1) 1 in the sense of distributions, that is, for any \( \Phi \in C_c^\infty((0, T) \times \Omega) \)

\[
\int_0^T \int_\Omega \varrho \partial_t \Phi dxdt + \int_0^T \int_\Omega \varrho u \cdot \nabla \Phi dxdt = 0;
\]
(ii) the momentum equation \([1.1]\) holds in \(D'((0,T) \times \Omega)\), that means,
\[
\int_0^T \int_{\Omega} \varrho u \cdot \partial_t \Phi + \varrho u \otimes u : \nabla \Phi + p \text{div}\Phi dx dt = \int_0^T \int_{\Omega} \mathbb{S} : \nabla \Phi dx dt,
\]
for any \(\Phi \in C^\infty_c((0,T) \times \Omega)\). Moreover, \(\varrho u \in C([0,T]; L^{2\gamma}_{weak}(\Omega))\) satisfies the initial condition \([1.4]\);

(iii) the temperature \(\vartheta \geq 0\) satisfies
\[
\vartheta \in L^2(0,T; H^1(\Omega)), \quad \varrho \vartheta \in L^\infty(0,T; L^1(\Omega)),
\]
and \(\vartheta(t,\cdot) \to \vartheta_0\) in \(D'(\Omega)\), as \(t \to 0^+\), that is, for any \(\chi \in C^\infty_c(\Omega)\), it holds
\[
\lim_{t \to 0^+} \int_{\Omega} \vartheta(t,x)\chi(x) dx = \int_{\Omega} \vartheta_0(x)\chi(x) dx.
\]
Furthermore, the following temperature inequality holds
\[
\int_0^T \int_{\Omega} \varrho \vartheta \partial_t \varphi dx dt + \int_0^T \int_{\Omega} (\varrho u \vartheta \cdot \nabla \varphi + \mathcal{K}(\vartheta) \Delta \varphi) dx dt 
\leq \int_0^T \int_{\Omega} (\varrho \varrho_0 \text{div} u - \mathbb{S} : \nabla u) \varphi dx dt - \int_0^T \int_{\Omega} \varrho_0 \vartheta_0 \varphi(0) dx,
\]
for any \(\varphi \in C^\infty_c([0,T] \times \Omega)\) satisfying
\[
\varphi \geq 0, \quad \varphi(T,\cdot) = 0, \quad \nabla \varphi \cdot n|_{\partial \Omega} = 0,
\]
where
\[
\varrho \mathbb{S} = \varrho \left[ \mu(\vartheta) (\nabla u + \nabla^T u) + \lambda(\vartheta) \text{div} u \right]
\]
and
\[
\mathcal{K}(\vartheta) = \int_0^\vartheta \kappa(z) dz;
\]
(iv) the energy inequality holds, that is, for a.e. \(t \in (0,T)\)
\[
E[\varrho, u, \vartheta](t) \leq E[\varrho, u, \vartheta](0),
\]
where
\[
E[\varrho, u, \vartheta](t) = \int_{\Omega} \varrho \left( \frac{1}{2} |u|^2 + P_e(\varrho) + \vartheta \right)(t) dx,
\]
and
\[
E[\varrho, u, \vartheta](0) = \int_{\Omega} \left( \frac{1}{2} \frac{|m_0|^2}{\varrho_0} + \varrho_0 P_e(\varrho_0) + \varrho_0 \vartheta_0 \right) dx,
\]
with
\[
P_e(\varrho) = \int_1^\varrho \frac{p_e(z)}{z^2} dz.
\]

Remark 1.1. The weak solutions in Definition \([1.1]\) are similar to the “variational” solutions in \([9]\).
Remark 1.2. As pointed out in [12], the reason for introducing the function $K(\vartheta) = \int_0^\vartheta \kappa(z)dz$ with $\nabla K(\vartheta) = \kappa(\vartheta)\nabla \vartheta = -q$ is that we are unable to deduce $\kappa(\vartheta)\nabla \vartheta$ is locally integrable by a priori estimates. However, we can deduce $K(\vartheta) \in L^1((0,T) \times \Omega)$ by constructing proper approximate equations and requiring suitable growth restrictions on $\kappa(\vartheta)$.

Remark 1.3. As will be shown later, the bounds on the velocity that we can obtain from a priori estimates fail to ensure the convergence of the term $S: \nabla u$ in the sense of distributions. Therefore, as in [9, 10], we replaced the temperature equation (1.1d) by the following two inequalities in Definition 1.1:

\begin{equation}
\partial_t (\varrho \vartheta) + \text{div}(\varrho u \vartheta) - \Delta K(\vartheta) \geq S: \nabla u - \vartheta p_0(\varrho) \text{div} u, \tag{1.7}
\end{equation}

and

\begin{equation}
E[\varrho, u, \vartheta](t) \leq E[\varrho, u, \vartheta](0). \tag{1.8}
\end{equation}

Before giving our main result, we need to state our assumptions on the pressure. Throughout this paper, we assume that the pressure $p$ satisfies one of the following two conditions:

1. $p$ has the form

\begin{equation}
p(\varrho, \vartheta) = p_e(\varrho) + \vartheta p_0(\varrho), \tag{1.9}
\end{equation}

where $p_e$ and $p_0$ satisfy

\begin{equation}
\begin{cases}
p_e \in C[0, \infty) \cap C^1(0, \infty), \\
p_e(0) = 0, p_e'(\varrho) \geq a_1 \varrho^{\gamma-1} - b & \text{for all } \varrho > 0, \\
p_e(\varrho) \leq a_2 \varrho^{\gamma} + b & \text{for all } \varrho \geq 0,
\end{cases} \tag{1.10}
\end{equation}

\begin{equation}
\begin{cases}
p_0 \in C[0, \infty) \cap C^1(0, \infty), \\
p_0(0) = 0, p_0'(\varrho) \geq 0 & \text{for all } \varrho > 0, \\
p_0(\varrho) \leq c(1 + \varrho^{\gamma/3}) & \text{for all } \varrho \geq 0,
\end{cases} \tag{1.11}
\end{equation}

where $\gamma > \frac{3}{2}$ and $a_1, a_2, b, c$ are positive constants;

2. $p$ has the form

\begin{equation}
p(\varrho, \vartheta) = p_e(\varrho) + R \vartheta \varrho, \tag{1.12}
\end{equation}

where $p_e(\varrho)$ satisfies (1.10) with $a_1, a_2, b$ being positive constants, $\gamma > 3$ and $R$ is a positive constant.

It can be easily checked that the above two conditions are independent.

Now we are ready to state our main result.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. Assume that

(i) the pressure $p$ satisfies (1.9) or (1.12);

(ii) the heat conductivity coefficient $\kappa(\vartheta) \in C^1[0, \infty)$ and satisfies

\begin{equation}
\kappa(1 + \vartheta^2) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^2), \tag{1.13}
\end{equation}

for positive constants $\kappa$ and $\overline{\kappa}$;
(iii) the viscosity coefficients $\mu$ and $\lambda$ are globally Lipschitz continuous functions on $[0, \infty)$ and $\mu$ is strictly increasing, moreover, $\mu$ and $\lambda$ satisfy

$$2\mu(\vartheta) + 3\lambda(\vartheta) \geq \nu(\vartheta) > 0 \quad \text{for all } \vartheta,$$

$$\nu(\vartheta) \geq C\vartheta \quad \text{for small } \vartheta,$$

for some function $\nu : [0, +\infty) \to \mathbb{R}$ and positive constant $C$;

(iv) the initial data satisfy

$$\begin{cases} 
\rho_0 \in L^\gamma(\Omega), & \rho_0 \geq \underline{\rho} > 0 \quad \text{on } \Omega, \\
\varrho_0 \in L^1(\Omega), & \varrho_0 \geq \underline{\varrho} > 0 \quad \text{on } \Omega, \\
\frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega)
\end{cases}$$

for positive constants $\rho$ and $\varrho$. Then for any given $T > 0$, the initial-boundary value problem (1.1), (1.4) and (1.5) admits a global weak solution $(\rho, \mathbf{u}, \vartheta)$ in the sense of Definition 1.1.

**Remark 1.4.** Zero viscosity only occurs when the temperature is very low in superfluids as in [2]; Otherwise, the viscosity of all fluids is positive by the second law of thermodynamics. Thus our restrictions (1.14) and (1.15) in Theorem 1.1 are physical and reasonable.

**Remark 1.5.** The strictly increasing assumption of the shear viscosity coefficient $\mu$ in Theorem 1.1 is reasonable. In fact, gases are generally considered as compressible fluid, and in most cases the shear viscosity of gases increases as the temperature increases; See [38] for example.

We give some comments on Theorem 1.1 by comparing it with two closely related results of the full compressible Navier-Stokes equations in [10] and [36]. In [10], Feireisl proved the global existence of the so-called “variational” solutions where the viscosity coefficients are functions of the temperature satisfying

$$\mu(\vartheta) \geq \underline{\mu} > 0, \quad \kappa(\vartheta) + \frac{2}{3}\mu(\vartheta) \geq 0,$$

and the pressure is given by (1.9) with $p_e$ and $p_\vartheta$ satisfying (1.10) and (1.11). By contrast, Theorem 1.1 does not require that the shear viscosity has a positive lower bound and allows another choice for the pressure (i.e., (1.9) and (1.12)), and thus can be regarded as an improvement of Feireisl’s result. The other related result is obtained by Mellet and Vasseur [36], where they used De Giorgi’s method to show that the temperature is uniformly positive in any finite time interval provided that the initial temperature has a lower bound away from zero. The assumptions on $\rho, \mathbf{u}$ and $p$ in [36] are

$$\rho \in L^\infty(0, T; L^p(\Omega)), \quad \text{for some } p > 3,$$

$$\mathbf{u} \in L^2(0, T; H^1_0(\Omega)),$$

$$p = p(\rho, \vartheta) = \tilde{p}_e(\rho) + R\rho\vartheta,$$

where $\tilde{p}_e$ can be any function of $\rho$. These assumptions on solutions are independent of ours when we impose the first kind of conditions on $p$, that is, (1.9), (1.10) and (1.11).
When $p$ satisfies the second kind of conditions, that is, (1.10) and (1.12), the assumptions (1.17)-(1.19) are weaker than ours, but it seems that they are not enough to get the global existence.

Our paper is organized as follows. In Section 2, we construct a suitable approximate system (2.1)-(2.9) with three parameters $\varepsilon, \eta$ and $\delta$ and obtain its global solvability by means of a modified Galerkin method. In Section 3, we let $\varepsilon \to 0$ for the approximate solutions constructed in Section 2. In Section 4, we apply De Giorgi’s iteration to obtain a positive lower bound for the temperature not depending on the parameter $\eta$ and then pass to the limit $\eta \to 0$. In Section 5, we let $\delta \to 0$ to finish the proof of Theorem 1.1.

2. CONSTRUCTION OF APPROXIMATE SOLUTIONS

First, we construct the following approximate system:

(i) continuity equation with vanishing viscosity:

$$\partial_t \varrho + \text{div}(\varrho u) = \varepsilon \Delta \varrho, \quad \varepsilon > 0,$$

$$\nabla \varrho \cdot n = 0 \quad \text{on } \partial \Omega,$$

$$\varrho(0, \cdot) = \varrho_{0,\delta} \quad \text{in } \Omega; \quad (2.1)$$

(ii) momentum equation with artificial pressure and artificial viscosity:

$$\partial_t (\varrho u) + \text{div}(\varrho u \otimes u) + \nabla (p(\varrho, \vartheta)) + \delta \varrho^3 = \eta \Delta u + \text{div}S, \quad \eta, \delta > 0,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

$$(\varrho u)(0, \cdot) = m_{0,\delta} \quad \text{in } \Omega; \quad (2.4)$$

(iii) regularized temperature equation:

$$\partial_t ((\delta + \varrho) \vartheta) + \text{div}(\varrho u \vartheta) - \Delta K(\vartheta) + \delta \vartheta^3 = (1 - \delta) S : \nabla u - \varrho p_\delta(\varrho) \text{div}u,$$

$$\nabla \vartheta \cdot n = 0 \quad \text{on } \partial \Omega,$$

$$\vartheta(0, \cdot) = \vartheta_{0,\delta} \quad \text{in } \Omega. \quad (2.7)$$

Note that the construction of the above approximate system is motivated by but different from [8]. Moreover, the modified initial data are required to satisfy the following conditions:

$$\begin{cases}
\varrho_{0,\delta} \in C^{2+\nu}(\bar{\Omega}), \nu > 0, & \nabla \varrho_{0,\delta} \cdot n|_{\partial \Omega} = 0, \quad 0 < \delta \leq \varrho_{0,\delta} \leq \delta^{-1/2\beta}; \\
\varrho_{0,\delta} \to \varrho_0 \text{ in } L^1(\Omega), & |\{x \in \Omega \mid \varrho_{0,\delta} < \varrho_0\}| \to 0, \quad \text{as } \delta \to 0; \\
\vartheta_{0,\delta} \in C^{2+\nu}(\bar{\Omega}), & \nabla \vartheta_{0,\delta} \cdot n|_{\partial \Omega} = 0, \quad 0 < \vartheta_{0,\delta} \leq \vartheta_0 \leq \vartheta; \\
\vartheta_{0,\delta} \to \vartheta_0 \text{ in } L^1(\Omega), & \text{as } \delta \to 0; \\
m_{0,\delta} = \begin{cases} m_0, & \text{if } \varrho_{0,\delta} \geq \varrho_0, \\
0, & \text{if } \varrho_{0,\delta} < \varrho_0, \end{cases} \quad (2.10)
\end{cases}$$

where the positive constant $\vartheta_0$ is independent of $\delta > 0$. In particular, the regularized initial value of the total energy

$$E(0) = E_\delta(0) = \int_\Omega \left( \frac{1}{2} \frac{|m_{0,\delta}|^2}{\varrho_{0,\delta}} + \varrho_{0,\delta} P_\varepsilon(\varrho_{0,\delta}) + \frac{\delta}{\beta - 1} \varrho_{0,\delta}^\beta + \varrho_{0,\delta} \vartheta_{0,\delta} \right) \, dx \quad (2.11)$$
is bounded by a constant independent of $\delta > 0$.

**Remark 2.1.** Roughly speaking, the extra term $\varepsilon \Delta \varrho$ is introduced in (2.1) to convert the hyperbolic equation (1.1) into a parabolic one from which one can obtain better regularity property of the density $\varrho$. The quantity $\varepsilon \nabla u \nabla \varrho$ is added to (2.4) to eliminate the term related to $\varepsilon \Delta \varrho$ in the energy inequality. The new quantity $\eta \Delta u$ represents the artificial viscosity which ensures the parabolic property of the momentum equation (2.4) and the term $\delta \nabla \varrho^3$ represents an artificial pressure which will play an essential role in obtaining estimates for the density $\varrho$. The term $\delta \vartheta^3$ is introduced to improve the integrability of the temperature. The other terms related to the parameter $\delta > 0$ are introduced to avoid technicalities in the temperature estimates.

Similarly to Chapter 7 in [9], for fixed positive parameters $\varepsilon$, $\eta$ and $\delta$, the global solvability of the approximate system (2.1)-(2.9) can be obtained by Galerkin method and the result is as follows.

**Proposition 2.1.** For fixed positive parameters $\varepsilon$, $\eta$ and $\delta$, under the hypotheses of Theorem 1.1 and the assumptions imposed on the initial data (2.10), if the exponent

$$\beta > \max\{4, \gamma\},$$

then the approximate system (2.1)-(2.9) admits a global weak solution $(\varrho, u, \vartheta)$ satisfying the following properties:

(i) the density $\varrho \geq 0$ satisfies

$$\partial_t \varrho, \Delta \varrho \in L^p((0,T) \times \Omega) \quad \text{for a certain } p > 1,$$

the velocity $u$ belongs to the space $L^2((0,T); H^1_0(\Omega))$, $(\varrho, u)$ solves the modified continuity equation (2.1) a.e. on $(0,T) \times \Omega$, and the boundary condition (2.2) together with the initial condition (2.3) are satisfied in the sense of traces. Moreover,

$$\delta \int_0^T \int_{\Omega} \varrho^{\beta+1} dx dt \leq C(\varepsilon, \delta),$$

$$\varepsilon \int_0^T \int_{\Omega} |\nabla \varrho|^2 dx dt \leq C,$$

with $C$ independent of $\varepsilon > 0$;

(ii) $(\varrho, u, \vartheta)$ solves the modified momentum equation (2.4) in $\mathcal{D}'((0,T) \times \Omega)$. Moreover,

$$\varrho u \in C([0,T]; L_{weak}^1(\Omega))$$

satisfies the initial condition (2.6);

(iii) the temperature $\vartheta \geq 0$ satisfies

$$\vartheta \in L^3((0,T) \times \Omega), \quad \vartheta \in L^2(0,T; H^1(\Omega)).$$
and the renormalized temperature inequality holds in $\mathcal{D}'((0, T) \times \Omega)$, that is,
\[
\int_0^T \int_\Omega (\delta + \varrho) H(\vartheta) \partial_t \varphi \, dx \, dt \\
+ \int_0^T \int_\Omega (\varrho H(\vartheta) \mathbf{u} \cdot \nabla \varphi + K_h(\vartheta) \Delta \varphi - \delta \vartheta^3 h(\vartheta) \varphi) \, dx \, dt \\
\leq \int_0^T \int_\Omega ((\delta - 1) \mathbf{S} : \nabla \mathbf{u} h(\vartheta) + h'(\vartheta) \kappa(\vartheta) |\nabla \vartheta|^2) \varphi \, dx \, dt \\
+ \int_0^T \int_\Omega h(\vartheta) \vartheta \partial \varrho \mathbf{d} \, \mathbf{u} \varphi \, dx \, dt + \varepsilon \int_0^T \int_\Omega \nabla \vartheta \cdot \nabla ((H(\vartheta) - \vartheta h(\vartheta)) \varphi) \, dx \, dt \\
- \int_\Omega (\delta + \varrho_0) H(\vartheta) \varphi(0) \, dx
\] (2.12)
for any $\varphi \in C_c^\infty([0, T] \times \Omega)$ satisfying
\[
\varphi \geq 0, \quad \varphi(T, \cdot) = 0, \quad \nabla \varphi \cdot \mathbf{n}|_{\partial \Omega} = 0,
\] (2.13)
where $H(\vartheta) = \int_\vartheta^0 h(z) \, dz$ and $K_h(\vartheta) = \int_0^\vartheta \kappa(z) h(z) \, dz$, with the non-increasing $h \in C^2([0, \infty))$ satisfying
\[
0 < h(0) < \infty, \quad \lim_{z \to \infty} h(z) = 0,
\] (2.14)
and
\[
h''(z) h(z) \geq 2(h'(z))^2 \text{ for all } z \geq 0;
\] (2.15)
(iv) the energy inequality
\[
\int_0^T \int_\Omega (-\partial_t \psi) \left( \frac{1}{2} |\mathbf{u}|^2 + \varrho P_m(\varrho) + \frac{\delta}{\beta - 1} \varrho^\beta + (\delta + \varrho) \vartheta \right) \, dx \, dt \\
+ \int_0^T \int_\Omega \psi \left( \delta \mathbf{S} : \nabla \mathbf{u} + \eta |\nabla \mathbf{u}|^2 + \delta \vartheta^3 \right) \, dx \, dt \\
\leq \int_\Omega \left( \frac{1}{2} \frac{|\mathbf{m}_{0, \delta}|^2}{\varrho_{0, \delta}} + \varrho_{0, \delta} P_m(\varrho_{0, \delta}) + \frac{\delta}{\beta - 1} \varrho_{0, \delta}^\beta + (\delta + \varrho_{0, \delta}) \vartheta_{0, \delta} \right) \, dx \\
- \int_0^T \int_\Omega p_b(\varrho) \text{div} \mathbf{u} \psi \, dx \, dt
\] (2.16)
hold for any $\psi \in C^\infty([0, T])$ satisfying
\[
\psi(0) = 1, \quad \psi(T) = 0, \quad \partial_t \psi \leq 0,
\] (2.17)
where the elastic pressure component $p_e$ has been written in the form
\[
p_e(\varrho) = p_m(\varrho) - p_b(\varrho),
\]
with $p_m$ non-decreasing, $P_m(\varrho) = \int_1^\varrho \frac{p_m(z)}{z^2} \, dz$, and
\[
p_b \in C^2([0, \infty)), \quad p_b \geq 0,
\]
compactly supported in $[0, \infty)$.
Remark 2.2. As proved in [9] for the constant viscosity coefficients case, the hypothesis (2.15) is imposed to ensure the convex and weakly lower semi-continuous property of the function

$$(\vartheta, \nabla u) \mapsto h(\vartheta) \nabla u,$$  \hspace{1cm} (2.18)

which is still valid for the temperature-depending viscosity coefficients case (cf. [19]).

3. Passing to the limit for $\varepsilon \rightarrow 0$

In this section, our main task is to pass the limit $\varepsilon \rightarrow 0$ in the approximate system (2.1)-(2.9). For clarity, we denote by $(\rho_\varepsilon, u_\varepsilon, \vartheta_\varepsilon)$ the weak solutions constructed in Proposition 2.1. Following Section 4 in [10], the main difficulty lies in obtaining the strong convergence of the density

$$\rho_\varepsilon \rightarrow \rho \quad \text{in} \quad L^1((0, T) \times \Omega),$$  \hspace{1cm} (3.1)

called usually the effective viscous pressure. This method was first proposed and proved by Lions [30] for the barotropic case with constant viscous coefficients. More specifically, if

$$\begin{cases} 
\rho_\varepsilon \rightharpoonup \rho & \text{weakly in} \ L^1((0, T) \times \Omega), \\
p(\rho_\varepsilon, \vartheta_\varepsilon) \rightharpoonup \overline{p} & \text{weakly in} \ L^1((0, T) \times \Omega), \\
u_\varepsilon \rightharpoonup u & \text{weakly in} \ L^2(0, T; H^1_0(\Omega)),
\end{cases}$$

then under certain hypotheses it holds

$$(p(\rho_\varepsilon, \vartheta_\varepsilon) - (\lambda + 2\mu)\text{div}u_\varepsilon) \rho_\varepsilon \rightharpoonup (\overline{p} - (\lambda + 2\mu)\text{div}u) \rho$$

weakly in $L^1((0, T) \times \Omega)$. Then, Feireisl [10] extended the above result to the case where $\mu$ and $\lambda$ are functions of the temperature and satisfy

$$\mu(\vartheta) \geq \underline{\mu} > 0, \quad \lambda(\vartheta) + \frac{2}{3}\mu(\vartheta) \geq 0,$$

for some positive constant $\underline{\mu} > 0$.

For fixed $\eta, \delta > 0$, the effective viscous pressure corresponding to the approximate system (2.1)-(2.9) are as follows

$$p + \delta \vartheta^\beta - (\lambda(\vartheta) + 2\mu(\vartheta) + \eta)\text{div}u.$$ 

Based on Section 4 in [10] and under assumptions of Theorem 1.1, we can show that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \varphi \left( p(\rho_\varepsilon, \vartheta_\varepsilon) + \delta \vartheta_\varepsilon^\beta - (\lambda(\vartheta_\varepsilon) + 2\mu(\vartheta_\varepsilon) + \eta)\text{div}u_\varepsilon \right) \rho_\varepsilon \, dx \, dt$$

$$\quad = \int_0^T \int_\Omega \varphi \left( \overline{p} - (\lambda(\vartheta) + 2\mu(\vartheta) + \eta)\text{div}u \right) \rho \, dx \, dt, \hspace{1cm} (3.2)$$
for any $\varphi \in C_c^\infty((0,T) \times \Omega)$ provided that

\begin{align*}
  &\begin{cases}
    p(\varrho_\varepsilon, \vartheta_\varepsilon) + \delta \varrho_\varepsilon^\beta \to \varphi \quad \text{weakly in } L^{(\beta+1)/\beta}((0,T) \times \Omega), \\
    \varrho_\varepsilon \to \varrho \quad \text{in } C([0,T]; L^\beta(\Omega)), \\
    u_\varepsilon \to u \quad \text{weakly in } L^2(0,T; H^1_0(\Omega)), \\
    \vartheta_\varepsilon \to \vartheta \quad \text{in } L^2((0,T) \times \Omega).
  \end{cases}
\end{align*}

With the strong convergence of the density $\varrho_\varepsilon$ (3.1), we conclude our result as follows.

**Proposition 3.1.** For fixed positive parameters $\eta$ and $\delta$, under the hypotheses of Theorem 1.2, the initial-boundary value problem (1.1), (1.4) and (1.5) with parameters $\eta$ and $\delta$ admits an approximate solution $(\varrho, u, \vartheta)$, which is also the limit of the weak solution constructed in Proposition 2.1 when $\varepsilon \to 0$, satisfying

(i) the density $\varrho \geq 0$ satisfies

$$\varrho \in C([0,T]; L^\beta(\Omega)) \cap L^{\beta+1}((0,T) \times \Omega)$$

and the initial condition (2.3). The velocity $u$ belongs to the space $L^2(0,T; H^1_0(\Omega))$, and $(\varrho, u)$ solves the continuity equation (1.1) in the sense of distributions;

(ii) $(\varrho, u, \vartheta)$ solves a modified momentum equation

$$\partial_t(\varrho u) + \text{div}(\varrho u \otimes u) + \nabla(p(\varrho, \vartheta) + \delta \varrho^\beta) = \text{div}S + \eta \Delta u$$

in $\mathcal{D}'((0,T) \times \Omega)$, where the viscous stress tensor $S$ is given by

$$S = \mu(\vartheta)(\nabla u + \nabla^T u) + \lambda(\vartheta)\text{div}uI.$$

Moreover, $\varrho u \in C([0,T]; L^{2\gamma}_{\text{weak}}(\Omega))$ satisfies the initial condition (2.6);

(iii) the temperature $\vartheta \geq 0$ satisfies

$$\vartheta \in L^3((0,T) \times \Omega), \quad \vartheta^{\frac{3-\omega}{2}} \in L^2(0,T; H^1(\Omega)), \quad \omega \in (0,1),$$

and the renormalized temperature inequality holds in $\mathcal{D}'((0,T) \times \Omega)$, that is,

\begin{align*}
  &\int_0^T \int_\Omega (\delta + \varrho)H(\vartheta)\partial_t \varphi \, dx \, dt \\
  &+ \int_0^T \int_\Omega (\delta H(\vartheta)u \cdot \nabla \varphi + \kappa_h(\vartheta)\Delta \varphi - \delta \vartheta^3 h(\vartheta)\varphi) \, dx \, dt \\
  &\leq \int_0^T \int_\Omega (\delta - 1)S : \nabla \varphi \to \nabla u(\vartheta) + h'(\vartheta)\kappa(\vartheta) |\nabla \vartheta|^2 \varphi \, dx \, dt \\
  &\quad + \int_0^T \int_\Omega h(\vartheta) \varrho \partial \varphi \, dx \, dt - \int_\Omega (\delta + \varrho_0)H(\vartheta_0)\varphi(0) \, dx
\end{align*}

for any $\varphi \in C_c^\infty([0,T] \times \Omega)$ satisfying

$$\varphi \geq 0, \varphi(T, \cdot) = 0, \nabla \varphi \cdot n|_{\partial \Omega} = 0.$$
where \( H(\vartheta) = \int_0^\vartheta h(z)dz \) and \( K_h(\vartheta) = \int_0^\vartheta \kappa(z)h(z)dz \), with the non-increasing \( h \in C^2([0, \infty)) \) satisfying
\[
0 < h(0) < \infty, \quad \lim_{z \to \infty} h(z) = 0,
\]
and
\[
h''(z)h(z) \geq 2(h'(z))^2 \text{ for all } z \geq 0;
\]
(iv) the energy inequality
\[
\int_0^T \int_\Omega (-\partial_t \psi) \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho P_e(\varrho) + \frac{\delta}{\beta - 1} \varrho^\beta + (\delta + \varrho)\vartheta \right) \, dxdt
+ \int_0^T \int_\Omega \psi \left( \delta \mathbf{S} : \nabla \mathbf{u} + \eta |\nabla \mathbf{u}|^2 + \delta \vartheta^2 \right) \, dxdt \leq \int_\Omega \left( \frac{1}{2} \frac{|\mathbf{m}_{0,\delta}|^2}{\varrho_{0,\delta}} + \varrho_{0,\delta} P_e(\varrho_{0,\delta}) + \frac{\delta}{\beta - 1} \varrho_{0,\delta}^\beta + (\delta + \varrho_{0,\delta})\vartheta_{0,\delta} \right) \, dx
\]
holds for any \( \psi \in C^\infty([0, T]) \) satisfying
\[
\psi(0) = 1, \quad \psi(T) = 0, \quad \partial_t \psi \leq 0.
\]

4. Passing to the limit for \( \eta \to 0 \)

In this section, our goal is to eliminate the artificial viscosity \( \eta \Delta \mathbf{u} \) in the modified momentum equation (3.3). To this end, we denote by \( (\varrho_\eta, \mathbf{u}_\eta, \vartheta_\eta) \) the weak solutions constructed in Proposition 3.1. Note that, for any fixed \( \eta > 0 \), \( \sqrt{\eta} \nabla \mathbf{u} \) is bounded in \( L^2((0, T) \times \Omega) \), which plays an essential role in obtaining the compactness of weak solutions as \( n \to \infty \) and \( \varepsilon \to 0 \). However, this estimate is not uniform on \( \eta > 0 \). Therefore, to obtain the uniform bound \( \nabla \mathbf{u}_\eta \in L^2((0, T) \times \Omega) \) with respect to \( \eta \), we need to show that the temperature-depending viscosity coefficient \( \mu(\vartheta_\eta) \) is bounded below from zero, which can be achieved by obtaining a uniform positive lower bound for the temperature \( \vartheta_\eta \).

4.1. A positive bound from below for the temperature.

We first give our result about the positive bound for the temperature \( \vartheta_\eta \).

**Proposition 4.1.** Let \( (\varrho_\eta, \mathbf{u}_\eta, \vartheta_\eta) \) be a weak solution constructed in Proposition 3.1. Assume that the viscosity coefficients satisfy the assumptions in Theorem 1.1, and the initial temperature satisfies the assumptions in (2.10), that is,
\[
\vartheta_\eta(0) = \vartheta_{0,\delta} \geq \underline{\vartheta} > 0.
\]
Then there exists a constant \( \tilde{\vartheta} > 0 \) such that
\[
\vartheta_\eta(t, x) \geq \tilde{\vartheta} > 0
\]
for all \( t \in [0, T] \) and almost all \( x \in \Omega \).

**Remark 4.1.** It should be pointed out that the constant \( \tilde{\vartheta} \) does not depend on the positive parameters \( \eta \) and \( \delta \), which is essential to take the limit \( \eta \to 0 \) and \( \delta \to 0 \).
Before proving Proposition 4.1, we introduce two important lemmas.

**Lemma 4.1.** (37) Let $U_k$ be a sequence satisfying

(i) $0 \leq U_0 \leq C$;
(ii) for some constants $A \geq 1, 1 < \beta_1 < \beta_2$ and $C > 0$,

\[ 0 \leq U_k \leq C \frac{A^k}{K} (U_{k-1}^{\beta_1} + U_{k-1}^{\beta_2}). \]  

(4.3)

Then there exists some $K_0$ such that for every $K > K_0$, the sequence $U_k$ converges to 0 when $k$ goes to infinity.

**Lemma 4.2.** (9) Let $\gamma$ be a non-negative function such that $0 < M_1 \leq \int_\Omega \gamma \, dx, \int_\Omega \gamma^2 \, dx \leq M_2$, with $M_2 > 6/5$.

Then there exists a positive constant $C$ depending only on $M_1, M_2$ such that

\[ \|v\|_{H^1(\Omega)} \leq C \left( \|\nabla v\|_{L^2(\Omega)} + \int_\Omega |\gamma| v \, dx \right). \]  

(4.5)

Our proof is in the spirit of the work of Mellet and Vasseur (36), where they obtained a bound from below for the temperature in the compressible Navier-Stokes equations by De Giorgi’s method. Since the case that the pressure $p$ satisfies the second kind of conditions, that is, (1.10) and (1.12) has been included in (36), we only give the proof of the case that $p$ satisfies (1.9), (1.10) and (1.11).

**Proof of Proposition 4.1.** For clarity, we divide the proof into four steps.

**Step 1.** According to the renormalized temperature inequality (3.4), $(\vartheta_\eta, u_\eta, \vartheta_\eta)$ satisfies

\[ \partial_t ((\delta + \vartheta_\eta) H(\vartheta_\eta)) + \text{div}(\vartheta_\eta u_\eta H(\vartheta_\eta)) - \Delta K_h(\vartheta_\eta) - h'(\vartheta_\eta) \kappa(\vartheta_\eta) |\nabla \vartheta_\eta|^2 \]
\[ \leq \delta \partial_\eta^3 h(\vartheta_\eta) - (1 - \delta) \mathcal{S}_h : \nabla u_\eta h(\vartheta_\eta) + \partial_\eta p_\vartheta(\vartheta_\eta) \text{div} u_\eta h(\vartheta_\eta), \]  

in $\mathcal{D}'((0, T) \times \Omega)$, where $H(\vartheta) = -\int_0^\vartheta h(z) \, dz$ and $\mathcal{K}_h(\vartheta) = -\int_0^\vartheta \kappa(z) h(z) \, dz$, with the non-increasing $h \in C^2([0, \infty))$ satisfying

\[ 0 < h(0) < \infty, \lim_{z \to \infty} h(z) = 0, \]

and

\[ h''(z) h(z) \geq 2(h'(z))^2 \text{ for all } z \geq 0. \]

In particular, the function $h(z) = \frac{1}{z+\omega} 1_{[z+\omega \leq C]}$ with $\omega > 0$ satisfies all the conditions, thus the inequality (4.1) holds with

\[ H(\vartheta_\eta) = \begin{cases} -\ln(\vartheta_\eta + \omega) + \ln \omega, & \text{if } \vartheta_\eta + \omega \leq C, \\ -\ln C + \ln \omega, & \text{if } \vartheta_\eta + \omega > C. \end{cases} \]  

(4.7)

Taking

\[ \phi(\vartheta_\eta) = H(\vartheta_\eta) + \ln C - \ln \omega = \left[ \ln \left( \frac{C}{\vartheta_\eta + \omega} \right) \right]_+, \]  

(4.8)
we deduce for any $0 \leq s \leq t \leq T$

$$
\int_{\Omega} \left( (\delta + \varrho_\eta) \phi(\vartheta_\eta) \right)(t) dx - 2(1 - \delta) \int_s^t \int_{\Omega} \mu(\vartheta_\eta)|D(u_\eta)|^2 \phi'(\vartheta_\eta) dx d\tau
$$

$$
- (1 - \delta) \int_s^t \int_{\Omega} \lambda(\vartheta_\eta)|\nabla u_\eta|^2 \phi'(\vartheta_\eta) dx d\tau + \int_s^t \int_{\Omega} \phi''(\vartheta_\eta) \kappa(\vartheta_\eta)|\nabla \vartheta_\eta|^2 dx d\tau
$$

$$
\leq \int_{\Omega} \left( (\delta + \varrho_\eta) \phi(\vartheta_\eta) \right)(s) dx - \delta \int_s^t \int_{\Omega} \varrho_\eta^3 \phi'(\vartheta_\eta) dx d\tau
$$

$$
- \int_s^t \int_{\Omega} \varrho_\eta \varrho \phi(\vartheta_\eta) dx d\tau,
$$

where $D(u) = \frac{1}{2} (\nabla u + \nabla^T u)$.

Now, introducing a sequence of real numbers

$$
C_k = e^{-M[1-2^{-k}]} \quad \text{for all positive integers } k,
$$

where $M$ is a positive number to be chosen later. Define $\phi_{k,\omega}$ by

$$
\phi_{k,\omega}(\vartheta_\eta) = \ln \left( \frac{C_k}{\vartheta_\eta + \omega} \right),
$$

it is easy to check that

$$
\phi'_{k,\omega}(\vartheta_\eta) = -\frac{1}{\vartheta_\eta + \omega} 1_{\{\vartheta_\eta + \omega \leq C_k\}},
$$

$$
\phi''_{k,\omega}(\vartheta_\eta) \geq \frac{1}{(\vartheta_\eta + \omega)^2} 1_{\{\vartheta_\eta + \omega \leq C_k\}}.
$$

Next define $U_{k,\omega}$ by

$$
U_{k,\omega} := \sup_{T_k \leq t \leq T} \left( \int_{\Omega} (\delta + \varrho_\eta) \phi_{k,\omega}(\vartheta_\eta) dx \right)
$$

$$
+ (1 - \delta) \int_{T_k}^T \int_{\Omega} \frac{\nu(\vartheta_\eta)}{\vartheta_\eta + \omega} 1_{\{\vartheta_\eta + \omega \leq C_k\}} |D(u_\eta)|^2 dx dt
$$

$$
+ \int_{T_k}^T \int_{\Omega} \frac{\kappa(\vartheta_\eta)}{(\vartheta_\eta + \omega)^2} 1_{\{\vartheta_\eta + \omega \leq C_k\}} |\nabla \vartheta_\eta|^2 dx dt,
$$

where $\{T_k\}$ is a sequence of non-negative numbers. Note that $U_{k,\omega}$ depends on $\eta$, $\delta$ and $\omega$, that is, $U_{k,\omega} = U_{k,\eta,\delta,\omega}$, and for convenience, we still write it as $U_{k,\omega}$.

If $T_k = 0$ for all $k \in \mathbb{N}$, by (4.9) and (4.14), then we claim that

$$
U_{k,\omega} \leq \int_{\Omega} (\delta + \varrho_0,\delta) \phi_{0,\omega}(\vartheta_0,\delta) dx + \delta \int_{\Omega} \int_0^T \frac{\varrho_\eta^3}{\vartheta_\eta + \omega} 1_{\{\vartheta_\eta + \omega \leq C_k\}} dx dt
$$

$$
+ \int_0^T \int_{\Omega} \frac{\varrho_\eta}{\vartheta_\eta + \omega} 1_{\{\vartheta_\eta + \omega \leq C_k\}} \varrho \phi(\vartheta_\eta) |\nabla u_\eta| dx dt.
$$

(4.15)
In fact, taking $0 \leq T_{k-1} \leq s \leq T_k \leq t \leq T$ in (4.9) and combining (4.12), (4.13) with (1.14), we have

$$
\int_{\Omega} \left((\delta + \varrho \eta) \phi_{k,\omega}(\vartheta \eta)\right)(s)dx + (1 - \delta) \int_{T_k} \int_{\Omega} \frac{\nu(\vartheta \eta)}{\vartheta \eta + \omega} 1_{\{\vartheta \eta + \omega \leq C_k\}} |D(u)\eta|^2 dx d\tau \\
+ \int_{T_k} \int_{\Omega} \frac{\kappa(\vartheta \eta)}{(\vartheta \eta + \omega)^2} 1_{\{\vartheta \eta + \omega \leq C_k\}} |\nabla \vartheta \eta|^2 dx d\tau \\
\leq \int_{\Omega} \left((\delta + \varrho \eta) \phi_{k,\omega}(\vartheta \eta)\right)(s)dx + \delta \int_{T_{k-1}} \int_{\Omega} \frac{\partial^3}{\partial \vartheta \eta + \omega} 1_{\{\vartheta \eta + \omega \leq C_k\}} d\tau dx \\
+ \int_{T_{k-1}} \int_{\Omega} \frac{p_\vartheta(\vartheta \eta)}{\vartheta \eta + \omega} 1_{\{\vartheta \eta + \omega \leq C_k\}} |\text{div} u\eta| d\tau dx dt.
$$

Taking the supremum over $t \in [T_k, T]$ on both sides of (4.16), one deduces that

$$
U_{k,\omega} \leq \int_{\Omega} \left((\delta + \varrho \eta) \phi_{k,\omega}(\vartheta \eta)\right)(s)dx + \delta \int_{T_{k-1}} \int_{\Omega} \frac{\partial^3}{\partial \vartheta \eta + \omega} 1_{\{\vartheta \eta + \omega \leq C_k\}} d\tau dx \\
+ \int_{T_{k-1}} \int_{\Omega} \frac{p_\vartheta(\vartheta \eta)}{\vartheta \eta + \omega} 1_{\{\vartheta \eta + \omega \leq C_k\}} |\text{div} u\eta| d\tau dx dt,
$$

which implies (4.15) provided $T_k = 0$ for all $k \in N$.

**Step 2.** In this step, we prove that the second term on the right-hand side of (4.15) can be controlled by $U_{k-1,\omega}^\sigma$ for some $\sigma > 1$. More precisely, we have

$$
\delta \int_0^T \int_{\Omega} \frac{\partial^3}{\partial \vartheta \eta + \omega} 1_{\{\vartheta \eta + \omega \leq C_k\}} d\tau dx dt \leq C \frac{2^{k\alpha}}{M^\alpha} U_{k-1,\omega}^\sigma,
$$

for some $\sigma > 1$ and $\alpha > 1$, where the constant $C$ is independent of $\eta, \delta > 0$.

Indeed, if $\vartheta \eta + \omega \leq C_k$, then for any $\omega > 0$

$$
\frac{\partial^3}{\vartheta \eta + \omega} \leq 1
$$

by taking $M$ large enough such that $C_k$ is small enough, and

$$
\phi_{k-1,\omega}(\vartheta \eta) = \ln \left(\frac{C_{k-1}}{\vartheta \eta + \omega}\right) \geq \ln \frac{C_{k-1}}{C_k},
$$

which implies

$$
1_{\{\vartheta \eta + \omega \leq C_k\}} \leq \left[\ln \frac{C_{k-1}}{C_k}\right]^{-\alpha} \phi_{k-1,\omega}(\vartheta \eta)^\alpha, \text{ for any } \alpha > 0.
$$
Combining (4.18) with (4.19), we have
\[
\delta \int_0^T \int_\Omega \frac{\partial^3 \eta}{\theta + \omega} 1_{\{\theta + \omega \leq C_k\}} \, dx dt \\
\leq \delta^{1-\beta} \left[ \ln \frac{C_{k-1}}{C_k} \right]^{-\alpha} \int_0^T \int_\Omega (\delta + \varrho_\eta)^\beta \phi_{k-1,\omega}(\theta_\eta)^\alpha \, dx dt \\
\leq C \delta^{1-\beta} \left[ \ln \frac{C_{k-1}}{C_k} \right]^{-\alpha} T^{1/p'} |\Omega|^{1/q'} \left\| (\delta + \varrho_\eta)^\beta \phi_{k-1,\omega}(\theta_\eta)^\alpha \right\|_{L^{p}(0,T;L^{q}(\Omega))},
\]
with \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \frac{1}{q} + \frac{1}{q'} = 1 \). By Lemma 4.2 and the growth restriction imposed on \( \kappa(\theta) \) (1.13), the last term on the right-hand side of (4.20) can be controlled by
\[
\left\| (\delta + \varrho_\eta)^\beta \phi_{k-1,\omega}(\theta_\eta)^\alpha \right\|_{L^p(0,T;L^q(\Omega))} \\
= \left\| ((\delta + \varrho_\eta)\phi_{k-1,\omega}(\theta_\eta))^{\beta/\alpha} \phi_{k-1,\omega}(\theta_\eta)^{1-\beta/\alpha} \right\|_{L^{p\alpha}(0,T;L^{q\alpha}(\Omega))}^\alpha \\
\leq \left\| ((\delta + \varrho_\eta)\phi_{k-1,\omega}(\theta_\eta))^{\beta/\alpha} \right\|_{L^{\infty}(0,T;L^{\alpha/\beta}(\Omega))} \left\| \phi_{k-1,\omega}(\theta_\eta)^{1-\beta/\alpha} \right\|_{L^{1-\beta/\alpha}(0,T;L^{1-\beta/\alpha}(\Omega))}^\alpha \\
= \left\| (\delta + \varrho_\eta)\phi_{k-1,\omega}(\theta_\eta) \right\|_{L^{\infty}(0,T;L^1(\Omega))}^{\beta} \left\| \phi_{k-1,\omega}(\theta_\eta) \right\|_{L^{2}(0,T;L^6(\Omega))}^{\alpha-\beta} \\
\leq \left\| (\delta + \varrho_\eta)\phi_{k-1,\omega}(\theta_\eta) \right\|_{L^{\infty}(0,T;L^1(\Omega))}^{\beta} \left( \left\| \varrho_\eta \phi_{k-1,\omega}(\theta_\eta) \right\|_{L^{\infty}(0,T;L^1(\Omega))} + \left\| \nabla \phi_{k-1,\omega}(\theta_\eta) \right\|_{L^{2}(0,T;\Omega)} \right)^{\alpha-\beta} \\
\leq CU_{k-1,\omega}^{\beta} \left( U_{k-1,\omega} + U_{k-1,\omega}^{1/2} \right)^{\alpha-\beta} \\
\leq C \left( U_{k-1,\omega}^{\alpha} + U_{k-1,\omega}^{\frac{\alpha+\beta}{2}} \right)^{\alpha-\beta},
\]
with the coefficients \( p, q, \alpha \) and \( \beta \) satisfying
\[
\frac{1}{p\alpha} = \frac{1 - \beta/\alpha}{2}, \quad \frac{1}{q\alpha} = \frac{\beta}{\alpha} + \frac{1 - \beta/\alpha}{6},
\]
or equivalently,
\[
p = \frac{2}{\alpha - \beta}, \quad q = \frac{6}{\alpha + 5\beta}.
\]
Substituting (4.21) into (4.20), we deduce
\[
\delta \int_0^T \int_\Omega \frac{\partial^3 \eta}{\theta + \omega} 1_{\{\theta + \omega \leq C_k\}} \, dx dt \leq C \delta^{1-\beta} \left[ \ln \frac{C_{k-1}}{C_k} \right]^{-\alpha} \left( U_{k-1,\omega}^{\alpha} + U_{k-1,\omega}^{\frac{\alpha+\beta}{2}} \right). \tag{4.22}
\]
By (4.10), we have
\[
\left[ \ln \frac{C_{k-1}}{C_k} \right]^{-\alpha} = \frac{2^{k\alpha}}{M^\alpha}. \tag{4.23}
\]
Moreover, we can choose \( \alpha > 1 \) and \( 0 < \beta < 1 \) such that
\[
\sigma := \min \left( \frac{\alpha + \beta}{2}, \alpha \right) > 1,
\] (4.24)
and
\[
\delta^{1-\beta} \leq 1.
\] (4.25)
Combining (4.22)-(4.25) together, we obtain (4.17).

**Step 3.** In this step, we prove that the last term on the right-hand side of (4.15) can be controlled by \( U_{k-1, \omega}^{\sigma} \) for the same \( \sigma > 1 \) as Step 2. To be precise, we have
\[
\int_0^T \int_\Omega \frac{\partial \eta}{\partial t} \chi_{\{\phi_{\eta} + \omega \leq C_k\}} p_\theta(\eta) |\text{div} u_\eta| dx dt \leq \frac{1}{2} U_{k, \omega} + C \frac{\sigma^2}{M_\alpha} U_{k-1, \omega}^{\sigma},
\] (4.26)
for the same \( \sigma > 1 \) and \( \alpha > 1 \) as Step 2, where the constant \( C \) is independent of \( \eta, \delta > 0 \).

In fact, by Cauchy-Schwarz and Young’s inequality, we have
\[
\int_0^T \int_\Omega \frac{\partial \eta}{\partial t} \chi_{\{\phi_{\eta} + \omega \leq C_k\}} p_\theta(\eta) |\text{div} u_\eta| dx dt \leq \frac{1}{8} \int_0^T \int_\Omega \nu(\eta) \chi_{\{\phi_{\eta} + \omega \leq C_k\}} |D(u_\eta)|^2 dx dt
+ C \int_0^T \int_\Omega \frac{\partial \eta}{\partial t} \chi_{\{\phi_{\eta} + \omega \leq C_k\}} |p_\theta(\eta)|^2 dx dt
\]
(4.27)
\[
\leq \frac{1}{2} U_{k, \omega} + C \int_0^T \int_\Omega \chi_{\{\phi_{\eta} + \omega \leq C_k\}} |p_\theta(\eta)|^2 dx dt,
\]
where we have used (1.15) in the last inequality. Then, by (4.19), the last term on the right-hand side of (4.27) can be controlled by
\[
\int_0^T \int_\Omega \chi_{\{\phi_{\eta} + \omega \leq C_k\}} |p_\theta(\eta)|^2 dx dt
\]
\[
\leq \left[ \ln \frac{C_{k-1}}{C_k} \right]^{-\alpha} \int_0^T \int_\Omega \phi_{k-1, \omega}(\eta) \alpha (\delta + \eta)^\beta (\delta + \eta)^{-\beta} |p_\theta(\eta)|^2 dx dt
\]
(4.28)
\[
\leq \left[ \ln \frac{C_{k-1}}{C_k} \right]^{-\alpha} \| (\delta + \eta)^\beta \phi_{k-1, \omega}(\eta) \|_{L^p(0,T; L^q(\Omega))} \| (\delta + \eta)^{-\beta} (p_\theta(\eta))^2 \|_{L^{p'}(0,T; L^{q'}(\Omega))}
\]
with \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \frac{1}{q} + \frac{1}{q'} = 1 \), where in accordance with (4.21), the second term on the right-hand side can be controlled by
\[
\| (\delta + \eta)^\beta \phi_{k-1, \omega}(\eta) \|_{L^p(0,T; L^q(\Omega))} \leq C U_{k-1, \omega}^{\sigma}, \text{ with } \sigma > 1,
\] (4.29)
provided \( \sigma = \min \{ \frac{\alpha + \beta}{2}, \alpha \} > 1 \) and the coefficients \( p, q, \alpha \) and \( \beta \) satisfy
\[
p = \frac{2}{\alpha - \beta}, \quad q = \frac{6}{\alpha + 5\beta}.
\]
Thus, to complete the proof of (4.26), it suffices to prove
\[ \| (\delta + \varrho_{\eta})^{-\beta} (p\varrho(\varrho_{\eta}))^2 \|_{L^{p'}(0,T;L^{q'}(\Omega))} \leq C. \] (4.30)
Note that \( \frac{\alpha + \beta}{2} > 1 \), \( \alpha > 1 \) and \( 0 < \beta < 1 \) implies
\[ p < \frac{1}{1 - \beta}, \quad q < \frac{3}{1 + 2\beta}, \]
which means
\[ p' > \frac{1}{\beta}, \quad q' > \frac{3}{2(1 - \beta)}. \]
To prove (4.30), it is enough to prove
\[ \| (\delta + \varrho_{\eta})^{-\beta} (p\varrho(\varrho_{\eta}))^2 \|_{L^{p'}(0,T;L^{q'}(\Omega))} \leq C \]
using (4.32). Substituting (4.17), (4.26) and (4.32) into (4.15), we obtain
\[ U_{k,\omega} \leq C \frac{2^{\alpha\omega}}{M^\alpha} U_{k-1,\omega} \] (4.33)
which in accordance with \( \varrho_{\eta} \in L^{\infty}(0,T;L^{\gamma}(\Omega)), \gamma > 3/2 \) can be achieved by choosing \( \beta \) close enough to 0.

**Step 4.** In this step, we are ready to complete the proof of Proposition 4.1. By virtue of the assumption \( \varrho_{0,\delta} \geq \varrho > 0 \), we can choose \( M \) large enough such that \( e^{-M^2/2} < \varrho \), which implies for any \( \omega > 0 \)
\[ \phi_{k,\omega}(\varrho_{0,\delta}) = \ln \left( \frac{e^{-M[1-2^{-k}]}}{\varrho_{0,\delta} + \omega} \right) = 0. \] (4.32)

Substituting (4.17), (4.26) and (4.32) into (4.15), we obtain
\[ U_{k,\omega} \leq C \frac{2^{\alpha\omega}}{M^\alpha} U_{k-1,\omega} \] (4.33)
Therefore, by Lemma 4.1 for \( M \) large enough (independently on \( \eta, \delta \) and \( \omega \)), we deduce
\[ \lim_{k \to \infty} U_{k,\omega} = 0, \] (4.34)
which with help of (4.12) and the definition of \( U_{k,\omega} \) (4.14) yields
\[ \int_0^T \int_{\Omega} \kappa(\varrho_{\eta}) \left\| \nabla \ln \left( \frac{e^{-M}}{\varrho_{\eta} + \omega} \right) \right\|^2 \, dx \, dt = 0, \] (4.35)
and
\[ \int_{\Omega} (\delta + \varrho_{\eta}) \left[ \ln \left( \frac{e^{-M}}{\varrho_{\eta} + \omega} \right) \right]_+ \, dx = 0. \] (4.36)
By the growth restriction imposed on $\kappa(\vartheta)$ (1.13), (4.35) implies
\[
\ln\left(\frac{e^{-M}}{\vartheta + \omega}\right)_+ \text{ is constant in } \Omega \text{ for all } t \in [0, T]. 
\] (4.37)
Furthermore, by (4.36), we have
\[
\ln\left(\frac{e^{-M}}{\vartheta + \omega}\right)_+ = 0,
\]
This yields
\[
\vartheta + \omega \geq e^{-M}
\]
for any $\omega > 0$, which completes our proof. \(\square\)

In accordance with Proposition 4.1 and the strictly increasing property of $\mu(\vartheta)$, there exists a constant $\mu > 0$ independent of the positive parameters $\eta$ and $\delta$ such that
\[
\mu(\vartheta \eta + \omega) \geq \mu > 0.
\]
To conclude, we have the following proposition.

**Proposition 4.2.** For fixed $\delta > 0$, under the hypotheses of Theorem 1.1, the initial-boundary value problem (1.1), (1.4) and (1.5) with the parameter $\delta > 0$ admits an approximate solution $(\varrho, \mathbf{u}, \vartheta)$, which is also the limit of the weak solution constructed in Proposition 3.1 when $\eta \to 0$, satisfying

(i) the density $\varrho \geq 0$ satisfies
\[
\varrho \in C([0, T]; L^\beta_{weak}(\Omega)) \cap L^{\beta+1}((0, T) \times \Omega)
\]
and the initial condition (2.3). The velocity $\mathbf{u}$ belongs to the space $L^2(0, T; H^1_0(\Omega))$, and $(\varrho, \mathbf{u})$ solves the continuity equation (1.1) in the sense of distributions;

(ii) $(\varrho, \mathbf{u}, \vartheta)$ solves a modified momentum equation
\[
\partial_t (\varrho \mathbf{u}) + \text{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla(p(\varrho, \vartheta) + \delta \vartheta^3) = \text{div}\mathbb{S} 
\] (4.38)
in $\mathcal{D}'((0, T) \times \Omega)$, where the viscous stress tensor $\mathbb{S}$ is given by
\[
\mathbb{S} = \mu(\vartheta)(\nabla \mathbf{u} + \nabla^T \mathbf{u}) + \lambda(\vartheta)\text{div}\mathbb{I}.
\]
Moreover, $\varrho \mathbf{u} \in C([0, T]; \frac{1}{2\omega} L^\infty_{weak}(\Omega))$ satisfies the initial condition (2.6);

(iii) the temperature $\vartheta \geq 0$ satisfies
\[
\vartheta \in L^3((0, T) \times \Omega), \quad \vartheta^{\frac{1-\omega}{2}} \in L^2(0, T; H^1(\Omega)), \quad \omega \in (0, 1),
\]
and the initial condition \(21\) is satisfied in the sense of distributions. Furthermore, the renormalized temperature inequality holds in \(D'(\(0, T\) \times \Omega)\), that is,

\[
\begin{align*}
\int_0^T \int_\Omega (\delta + \varrho) H(\vartheta) \partial_t \varphi \, dx \, dt &+ \int_0^T \int_\Omega (\varrho H(\vartheta) u \cdot \nabla \varphi + K_h(\vartheta) \Delta \varphi - \delta \varrho^3 h(\vartheta) \varphi) \, dx \, dt \\
&\leq \int_0^T \int_\Omega ((\delta - 1) S : \nabla u h(\vartheta) + h'(\vartheta) \kappa(\vartheta)|\nabla \vartheta|^2) \varphi \, dx \, dt \\
&+ \int_0^T \int_\Omega h(\vartheta) \partial \varrho(\vartheta) \text{div} u \varphi \, dx \, dt - \int_\Omega (\delta + \varrho_0, \delta) H(\vartheta_0, \delta) \varphi(0) \, dx
\end{align*}
\]  

\((4.39)\)

for any \(\varphi \in C^\infty([0, T] \times \Omega)\) satisfying

\[\varphi \geq 0, \ \varphi(T, \cdot) = 0, \ \nabla \varphi \cdot n|_{\partial \Omega} = 0,\]

where \(H(\vartheta) = \int_0^\vartheta h(z) \, dz\) and \(K_h(\vartheta) = \int_0^\vartheta \kappa(z) h(z) \, dz\), with the non-increasing \(h \in C^2([0, \infty))\) satisfying

\[0 < h(0) < \infty, \ \lim_{z \to \infty} h(z) = 0,\]

and

\[h''(z) h(z) \geq 2(h'(z))^2 \text{ for all } z \geq 0;\]

(iv) the energy inequality

\[
\begin{align*}
\int_0^T \int_\Omega \left( -\partial_t \psi \left( \frac{1}{2} \varrho |u|^2 + \varrho P_e(\varrho) \right) + \frac{\delta}{\beta - 1} \varrho^3 + (\delta + \varrho) \varrho \right) \, dx \, dt &+ \int_0^T \int_\Omega \psi \delta \left( S : \nabla u + \varrho^3 \right) \, dx \, dt \\
&\leq \int_\Omega \left( \frac{1}{2} |m_0|^2 + \frac{\delta}{\beta - 1} \varrho_{0, \delta}^\beta + \varrho_{0, \delta} P_e(\varrho_{0, \delta}) + (\delta + \varrho_{0, \delta}) \varrho_{0, \delta} \right) \, dx
\end{align*}
\]  

\((4.40)\)

holds for any \(\psi \in C^\infty([0, T])\) satisfying

\[\psi(0) = 1, \ \psi(T) = 0, \ \partial_t \psi \leq 0.\]

5. Passing to the limit for \(\delta \to 0\)

The final step is to deal with terms related to the parameter \(\delta > 0\) in \((4.38)-(4.40)\). To this end, we denote by \((\varrho_\delta, u_\delta, \vartheta_\delta)\) the weak solutions constructed in Proposition 4.2. Observe that estimates for \(\varrho_\delta\) are similar to the previous sections, and estimates for \(u_\delta\) can be deduced after some calculations, thus our main task in this section is to deal with terms related to \(\vartheta_\delta\).
5.1. Estimates independent of $\delta > 0$.

For convenience, in the rest of this section, we denote $C$ a generic positive constant independent of $\delta > 0$.

First, by (2.11) and the energy inequality (4.40), we have the following estimates
\[
\|\sqrt{\varrho_\delta} u_\delta\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{5.1}
\]
\[
\|\varrho_\delta\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq C, \tag{5.2}
\]
\[
\|\left(\delta + \varrho_\delta\right) \vartheta_\delta\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \tag{5.3}
\]
\[
\delta \int_0^T \int_\Omega \delta_\delta : \nabla u_\delta dx dt \leq C, \tag{5.4}
\]
\[
\delta \int_0^T \int_\Omega \vartheta_\delta^3 dx dt \leq C. \tag{5.5}
\]

Then, taking $\varphi(t,x) = \psi(t)$ satisfying $0 \leq \psi \leq 1$, $\psi \in C^\infty_c(0,T)$ and $h(\vartheta) = \frac{\xi}{\xi + \vartheta}$ with $0 < \xi < 1$ in (4.39), we have
\[
\int_0^T \int_\Omega \left( \frac{1 - \delta}{\xi + \vartheta_\delta} \delta_\delta : \nabla u_\delta + \frac{\kappa(\vartheta_\delta)}{(\xi + \vartheta_\delta)^2} |\nabla \vartheta_\delta|^2 \right) \psi + \varrho_\delta \ln(\xi + \vartheta_\delta) \partial_t \psi dx dt
\]
\[
\leq \delta \int_0^T \int_\Omega \frac{\vartheta_\delta^3}{\xi + \vartheta_\delta} \psi dx dt + \int_0^T \int_\Omega \frac{\varrho_\delta(\varrho_\delta)}{\xi + \vartheta_\delta} \text{div} u_\delta \psi dx dt. \tag{5.6}
\]

By virtue of (5.5), we take the limit for $\xi \to 0$ in (5.6) to deduce
\[
\int_0^T \int_\Omega \left( \frac{1 - \delta}{\vartheta_\delta} \delta_\delta : \nabla u_\delta + \frac{\kappa(\vartheta_\delta)}{\vartheta_\delta^2} |\nabla \vartheta_\delta|^2 \right) \psi + \varrho_\delta \ln \vartheta_\delta \partial_t \psi dx dt
\]
\[
\leq C \left( 1 + \int_0^T \int_\Omega p_\phi(\varrho_\delta) \text{div} u_\delta \psi dx dt \right), \tag{5.7}
\]
where by the continuity equation (1.11), the last term on the right-hand side can be rewritten as
\[
\int_0^T \int_\Omega p_\phi(\varrho_\delta) \text{div} u_\delta \psi dx dt = \int_0^T \int_\Omega \varrho_\delta P_\phi(\varrho_\delta) \partial_t \psi dx dt,
\]
with
\[
P_\phi(\varrho) = \int_1^\varrho \frac{p_\phi(z)}{z^2} dz.
\]

Thus, by assumption (1.11), the growth restriction imposed on $\kappa(\vartheta)$ (1.13) and estimate (5.2), we have
\[
\|\varrho_\delta \ln \vartheta_\delta\|_{L^\infty(0,T;L^1(\Omega))} \leq C,
\]
\[
\|\nabla \ln \vartheta_\delta\|_{L^2((0,T) \times \Omega)} \leq C,
\]
\[
\|\nabla \vartheta_\delta\|_{L^2((0,T) \times \Omega)} \leq C,
\]
which combined with Lemma 4.2 implies
\[
\|\ln \vartheta_\delta\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{5.8}
\]
\[
\|\vartheta_\delta\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{5.9}
\]
Next, taking \( h(\vartheta) = \frac{1}{(1+\vartheta)^l} \) with \( 0 < l < 1 \) in (4.39), we obtain

\[
\int_0^T \int_\Omega \left( \frac{1 - \delta}{(1 + \vartheta \delta)} \right)^l \nabla \mathbf{u}_\delta : \nabla \mathbf{u}_\delta + \frac{\kappa(\vartheta \delta)}{(1 + \vartheta \delta)^{l+1}} |\nabla \vartheta \delta|^2 \psi \, dx \, dt \\
\leq \delta \int_0^T \int_\Omega \frac{\vartheta_\delta^2}{(1 + \vartheta \delta)} \psi \, dx \, dt - \int_0^T \int_\Omega (\delta + g_\delta) H(\vartheta \delta) \partial_t \psi \, dx \, dt \\
+ \int_0^T \int_\Omega \frac{\vartheta_\delta}{(1 + \vartheta \delta)} p_\delta (g_\delta) \text{div} \mathbf{u}_\delta \varphi \, dx \, dt,
\]

with \( H(\vartheta) = \int_0^\vartheta \frac{1}{(1+z)^l} \, dz \). Letting \( l \to 0 \) in (5.10) and combining with estimates (5.2)-(5.5) and (5.9), we have

\[
\int_0^T \int_\Omega \nabla \mathbf{u}_\delta \, dx \, dt \leq C,
\]

which with help of Proposition 3.3 and assumptions imposed on \( \mu(\vartheta) \) and \( \lambda(\vartheta) \) in Theorem 1.1, yields

\[
\| \mathbf{u}_\delta \|_{L^2(0,T;H_0^1(\Omega))} \leq C. \tag{5.11}
\]

Moreover, for fixed \( 0 < l < 1 \) in (5.10), by the growth restriction imposed on \( \kappa(\vartheta) \) (1.13), we obtain

\[
\| \vartheta_\delta^{3-l} \|_{L^2(0,T;H^1(\Omega))} \leq C(l), \tag{5.12}
\]

with the constant \( C(l) \) depending on \( l \in (0,1) \). Combining estimate (5.12) with (5.3) and thanks to the interpolation inequality, we deduce for a certain \( p > 1 \) and a small positive number \( \omega \)

\[
\vartheta_\delta^3 \text{ is bounded in } L^p(\{\vartheta_\delta(t,x) \geq \omega > 0\}) \tag{5.13}
\]

by a positive constant independent of \( \delta > 0 \).

5.2. Strong convergence of the temperature \( \vartheta_\delta \).

Following Chapter 7 in [9] and Chapter 5 in [10], to obtain the strong convergence of the temperature

\[
\vartheta_\delta \to \vartheta \quad \text{in } L^2(\{\vartheta > 0\}), \tag{5.14}
\]

by estimate (5.9), it suffices to show that

\[
\vartheta_\delta H(\vartheta_\delta) \to \vartheta H(\vartheta) \quad \text{in } L^2(0,T;W^{-1,2}(\Omega)). \tag{5.15}
\]

First, we introduce the following lemma, which can be regarded as a variant of the celebrated Aubin-Lions lemma (see Lemma 6.3 in [9]).

**Lemma 5.1.** Let \( \{v_n\}_{n=1}^{\infty} \) be a sequence of functions such that

- \( v_n \) is bounded in \( L^2(0,T;L^q(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \), with \( q > 6/5 \),
- furthermore, assume that \( \partial_t v_n \geq l_n \) in \( \mathcal{D}'((0,T) \times \Omega) \),

where

\[
l_n \quad \text{is bounded in } L^1(0,T;W^{-m,r}(\Omega))
\]
for a certain $m \geq 1$, $r > 1$.

Then $\{v_n\}_{n=1}^{\infty}$ contains a subsequence such that

$$v_n \to v$$

in $L^2(0,T; H^{-1}(\Omega))$.

Now we apply Lemma 5.1 to the sequence $(\delta + \varrho_\delta)H(\vartheta_\delta)$. By the temperature inequality (4.39), to obtain (5.15), it is enough to prove that

$$\|\vartheta_\delta\|_{L^3((0,T) \times \Omega)} \leq C,$$

which by (5.13) can be achieved provided that

$$\vartheta_\delta$$

is bounded in $L^3(\{\varrho_\delta(x,t) < \omega\})$ (5.16)

by a positive constant independent of $\delta > 0$, with $\omega$ being a sufficiently small positive number.

As proved in [9, 10], the estimate (5.16) can be obtained by choosing the function

$$\varphi(t,x) = \psi(t)(\eta(t,x) - \eta), ~ 0 \leq \psi \leq 1, ~ \psi \in C^\infty_c(0,T),$$

where

$$\eta = \inf_{t \in [0,T], x \in \Omega} \eta,$$

and for each $t \in [0, T]$, $\eta = \eta_\delta$ is the unique solution of the following Neumann problem

$$\begin{align*}
\Delta \eta_\delta(t) &= B(\varrho_\delta(t)) - \frac{1}{|\Omega|} \int_\Omega B(\varrho_\delta(t))dx \text{ in } \Omega, \\
\nabla \varrho_\delta \cdot n &= 0 \text{ on } \partial \Omega, \\
\int_\Omega \eta_\delta(t)dx &= 0,
\end{align*}$$

with $B \in C^\infty(\mathbb{R})$ non-increasing satisfying

$$B(z) = \begin{cases} 0, & \text{if } z \leq \omega, \\
-1, & \text{if } z \geq 2\omega, \end{cases}$$

as a test function of the renormalized temperature inequality (4.39).

5.3. Strong convergence of the density $\varrho_\delta$.

In this subsection, our main goal is to prove

$$\varrho_\delta \to \varrho$$

in $L^1((0,T) \times \Omega)$. (5.17)

As in [10], this can be achieved by taking the quantity

$$\varphi(t,x) = \psi(t)\eta(x)\Delta^{-1}\partial_{x_i}[T_k(\varrho_\delta)]$$

with $\psi \in C^\infty_c(0,T)$, $\eta \in C^\infty_c(\Omega)$, and $T_k(\varrho)$ being cut-off functions

$$T_k(\varrho) = \min\{\varrho, k\}, ~ k \geq 1,$$

as a test function of the momentum equation (4.38).
5.4. **Passing the limit.**

Taking into account (5.11), (5.14) and (5.17), we can pass the limit $\delta \to 0$ for the approximate solutions $(\rho_\delta, u_\delta, \vartheta_\delta)$ constructed in Proposition 4.2, thus Theorem 1.1 is proved.

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**References**

[1] A.A. Amosov, The existence of global generalized solutions of the equations of one-dimensional motion of a real viscous gas with discontinuous data, Differ. Uravn. 36 (2000), 486-499; translation in Differ. Equ. 36 (2000), 540-558.

[2] R. Balescu, Equilibrium and Non-Equilibrium Statistical Mechanics, John Wiley and Sons, 1975.

[3] D. Bresch, B. Desjardins, Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model, Comm. Math. Phys. 238 (2003), 211-223.

[4] E. De Giorgi, Sulla differenziabilità e l’analiticità delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Natur. 3 (1957), 25-43.

[5] R.J. DiPerna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (1989), 511-547.

[6] B. Ducomet, Š. Nečasová, A. Vasseur, On global motions of a compressible barotropic and selfgravitating gas with density-dependent viscosities, Z. Angew. Math. Phys. 61 (2010), 479-491.

[7] B. Ducomet, Š. Nečasová, A. Vasseur, On spherically symmetric motions of a viscous compressible barotropic and selfgravitating gas, J. Math. Fluid Mech. 13 (2011), 191-211.

[8] E. Feireisl, A. Novotný, H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, J. Math. Fluid Mech. 3 (2001), 358-392.

[9] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford University Press, Oxford, 2004.

[10] E. Feireisl, On the motion of a viscous, compressible, and heat conducting fluid, Indiana Univ. Math. J. 53 (2004), 1705-1738.

[11] E. Feireisl, A. Novotný, Singular Limits in Thermodynamics of Viscous Fluids. Second edition. Advances in Mathematical Fluid Mechanics, Birkhäuser, Basel, 2017.

[12] E. Feireisl, J. Málek, On the Navier-Stokes equations with temperature-dependent transport coefficients, Differ. Equ. Nonlinear Mech., Art. ID 90616 (electronic) (2006), 1-14.

[13] Z.H. Guo, Q.S. Jiu, Z.P. Xin, Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients, SIAM J. Math. Anal. 39 (2008), 1402-1427.

[14] D. Hoff, Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data, Trans. Amer. Math. Soc. 303 (1987), 169-181.

[15] D. Hoff, Spherically symmetric solutions of the Navier-Stokes equations for compressible, isothermal flow with large, discontinuous initial data, Indiana Univ. Math. J. 41 (1992), 1225-1302.

[16] D. Hoff, Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids, Arch. Rational Mech. Anal. 139 (1997), 303-354.

[17] D. Hoff, Compressible flow in a half-space with Navier boundary conditions, J. Math. Fluid Mech. 7 (2005), 315-338.

[18] D. Hoff, J. Smoller, Non-formation of vacuum states for compressible Navier–Stokes equations, Comm. Math. Phys. 216 (2001), 255-276.

[19] X.P. Hu, D.H. Wang, Global solutions to the three-dimensional full compressible magnetohydrodynamic flows, Comm. Math. Phys. 283 (2008), 255-284.

[20] S. Jiang, Global smooth solutions of the equations of a viscous, heat-conducting, one-dimensional gas with density-dependent viscosity, Math. Nachr. 190 (1998), 169-183.
[21] S. Jiang, P. Zhang, On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations, Comm. Math. Phys. 215 (2001), 559-581.

[22] S. Jiang, Z.P. Xin, P. Zhang, Global weak solutions to 1D compressible isentropic Navier–Stokes equations with density-dependent viscosity, Methods Appl. Anal. 12 (2005), 239-251.

[23] Q.S. Jiu, Y. Wang, Z.P. Xin, Global well-posedness of the Cauchy problem of two-dimensional compressible Navier–Stokes equations in weighted spaces, J. Differential Equations 255 (2013), 351-404.

[24] Q.S. Jiu, Y. Wang, Z.P. Xin, Global well-posedness of 2D compressible Navier-Stokes equations with large data and vacuum, J. Math. Fluid Mech. 16 (2014), 483-521.

[25] Q.S. Jiu, Y. Wang, Z.P. Xin, Global classical solution to two-dimensional compressible Navier-Stokes equations with large data in $\mathbb{R}^2$, Phys. D 367 (2018), 180-194.

[26] J.I. Kanel, A model system of equations for the one-dimensional motion of a gas, Differ. Uravn. 4 (1968), 721-734 (in Russian).

[27] A.V. Kazhikhov, V.V. Shelukhin, Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas, J. Appl. Math. Mech. 41 (1977), 273-282; translated from Prikl. Mat. Meh. 41 (1977), 282-291 (in Russian).

[28] J. Li, Z.P. Xin, Global existence of weak solutions to the barotropic compressible Navier-Stokes flows with degenerate viscosities, preprint, arXiv:1504.06826

[29] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. 1: Incompressible Models, Oxford University Press, Oxford, 1996.

[30] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. 2: Compressible Models, Oxford University Press, Oxford, 1998.

[31] T.P. Liu, J. Smoller, On the vacuum state for the isentropic gas dynamics equations, Adv. Appl. Math. 1 (1980), 345-359.

[32] T.P. Liu, Z.P. Xin, T. Yang, Vacuum states for compressible flow, Discrete Contin. Dynam. Systems 4 (1998), 1-32.

[33] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ. 20 (1980), 67-104.

[34] A. Matsumura, S. Yanagi, Uniform boundedness of the solutions for a one-dimensional isentropic model system of compressible viscous gas, Comm. Math. Phys. 175 (1996), 259-274.

[35] A. Mellet, A. Vasseur, On the barotropic compressible Navier-Stokes equation, Comm. Partial Differential Equations 32 (2007), 431-452.

[36] A. Mellet, A. Vasseur, A bound from below for the temperature in compressible Navier-Stokes equations, Monatsh. Math. 157 (2009), 143-161.

[37] A. Mellet, A. Vasseur, $L^p$ estimates for quantities advected by a compressible flow, J. Math. Anal. Appl. 355 (2009), 548-563.

[38] R.C. Reid, J.M. Prausnitz, B.E. Poling, The Properties of Gases and Liquids, McGraw-Hill Book Company, 1987.

[39] V.A. Vaigant, A.V. Kazhikhov, On the existence of global solutions of two-dimensional Navier-Stokes equations of a compressible viscous fluid, Sibirsk. Mat. Zh. 36 (1995), 1283-1316; translated from Siberian Math. J. 36 (1995), 1108-1141.

[40] A. Vasseur, C. Yu, Existence of global weak solutions for 3D degenerate compressible Navier-Stokes equations, Invent. math. 206 (2016), 935-974.

[41] T. Yang, Some recent results on compressible flow with vacuum, Taiwanese J. Math. 4 (2000), 33-44.

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