Abstract. First, we give a new example of silting-discrete algebras. Second, one explores when the algebra of triangular matrices over a finite dimensional algebra is $\tau$-tilting finite. In particular, we classify algebras over which triangular matrix algebras are $\tau$-tilting finite. Finally, we investigate when a triangular matrix algebra is silting-discrete.

1. Introduction

Silting objects play a central role in tilting theory to describe the structure of derived categories and control derived equivalences. One of the most crucial purposes is to clarify the whole picture of silting objects. To realize the goal, we first discuss when a triangulated category is silting-discrete; roughly speaking, the silting-discreteness is the finiteness of silting objects. A finite dimensional algebra is said to be silting-discrete if the perfect derived category is silting-discrete; for example, the following algebras are silting-discrete:

- representation-finite piecewise hereditary algebras [AI];
- representation-finite symmetric algebras [AI];
- Brauer graph algebras whose Brauer graphs have at most one odd cycle and none of even cycles [AAC];
- derived-discrete algebras with finite global dimension [BPP];
- symmetric preprojective algebras of Dynkin type [AM];
- algebras of dihedral, semidihedral and quaternion type [EBR];
- symmetric algebras of tubular type with nonsingular Cartan matrix [AHMW].

The first aim of this paper is to construct a new silting-discrete algebra from a given one. We denote by $\text{silt } \Lambda$ the set of isomorphism classes of basic silting objects of the perfect derived category for a finite dimensional algebra $\Lambda$. Here is the first main theorem.

**Theorem 1** (Theorem 2.1). Let $R$ and $\Lambda$ be a finite dimensional local and silting-discrete algebra over an algebraically closed field $K$, respectively. Put $\Gamma := R \otimes_K \Lambda$. Then we have a poset isomorphism $\text{silt } \Lambda \rightarrow \text{silt } \Gamma$. In particular, $\Gamma$ is also silting-discrete.

As an example, we consider the $n \times n$ (upper) triangular matrix algebra $T_n(R)$ over a local algebra $R$, which is actually isomorphic to $R \otimes_K \overrightarrow{K A_n}$. So, we get a corollary of this theorem (Proposition 3.7).

In the context of triangular matrix algebras $T_n(\Lambda)$ over an algebra $\Lambda$ (not necessarily local), it seems to be difficult to understand when $T_n(\Lambda)$ is silting-discrete. Thus, let us turn our attention to “two-term” silting objects. We say that an algebra is $\tau$-tilting...
finite if there are only finitely many two-term silting objects; see \[AIR\]. It is evident that a silting-discrete algebra is \(\tau\)-tilting finite. We also know that a representation-finite algebra is \(\tau\)-tilting finite. As an analogue of Auslander–Reiten results in \[AR\], we have the second main theorem of this paper.

**Theorem 2** (Proposition 3.1, Theorem 3.2, Proposition 3.5 and Theorem 3.3). Let \(\Lambda\) be a finite dimensional algebra over an algebraically closed field. Then the following hold:

1. Assume that \(\Lambda\) is representation-finite. Then \(T_2(\Lambda)\) is \(\tau\)-tilting finite if and only if so is the Auslander algebra of \(\Lambda\).
2. Assume that \(\Lambda\) is representation-finite and simply-connected. Then \(T_2(\Lambda)\) is \(\tau\)-tilting finite if and only if it is representation-finite.
3. The third triangular matrix algebra \(T_3^3(\Lambda)\) is not \(\tau\)-tilting finite.
4. If the separated quiver of \(\Lambda\) has a connected component which is not of type \(A_n\), then \(T_2(\Lambda)\) is \(\tau\)-tilting infinite.

We classify algebras \(\Lambda\) with \(T_n(\Lambda)\) \(\tau\)-tilting finite. Here is the third main theorem.

**Theorem 3** (Theorem 3.10 and 3.12). Let \(\Lambda\) be a finite dimensional nonlocal algebra over an algebraically closed field whose Gabriel quiver has no loop and \(n \geq 3\). Then the following are equivalent:

1. \(T_n(\Lambda)\) is \(\tau\)-tilting finite;
2. One of the following cases holds:
   i. \(n = 4\) and \(\Lambda\) is the path algebra of type \(A_2\);
   ii. \(n = 3\) and \(\Lambda\) is a Nakayama algebra with precisely 2 simple modules;
   iii. \(n = 3\) and \(\Lambda\) is a Nakayama algebra with radical square zero.

This theorem tells us the fact that for a simply-connected algebra \(\Lambda\) and \(n \geq 3\), \(T_n(\Lambda)\) is \(\tau\)-tilting finite if and only if it is representation-finite (Corollary 3.13).

Finally, let us go back to the study on the silting-discreteness of \(T_n(\Lambda)\). Although it might be very hard to classify algebras \(\Lambda\) with \(T_n(\Lambda)\) silting-discrete in general, we try it for a radical-square-zero linear Nakayama algebra \(\Lambda\). Here is the last main theorem.

**Theorem 4** (Theorem 3.14). Let \(\Lambda\) be a radical-square-zero linear Nakayama algebra with \(r\) simple modules. Then \(T_n(\Lambda)\) is silting-discrete if and only if one of the following cases occurs: (i) \(n = 1\); (ii) \(r = 1\); (iii) \(n = 2\) and \(1 < r \leq 4\); (iv) \(1 < n \leq 4\) and \(r = 2\).

Throughout this paper, algebras are always assumed to be finite dimensional over an algebraically closed field \(K\), basic and indecomposable. Modules are finite dimensional and right modules. For an algebra \(\Lambda\), we denote by \(\text{mod} \Lambda\) (\(\text{proj} \Lambda\)) the category of (projective) modules over \(\Lambda\). The perfect derived category of \(\Lambda\) is denoted by \(K^b(\text{proj} \Lambda)\).

2. A NEW EXAMPLE OF SITTING-DISCRETE ALGEBRAS

In this section, we give a new construction of silting-discrete algebras.

Let \(\mathcal{T}\) be a triangulated category which is Krull–Schmidt, \(K\)-linear and Hom-finite. We say that an object \(T\) is **silting** if it satisfies \(\text{Hom}_{\mathcal{T}}(T,T[i]) = 0\) for any \(i > 0\) and \(\mathcal{T} = \text{thick} T\). Here, \(\text{thick} T\) stands for the smallest thick subcategory of \(\mathcal{T}\) containing \(T\).
It is known that the set \( \text{silt}\mathcal{T} \) of isomorphism classes of basic silting objects of \( \mathcal{T} \) has a partial order \( \geq \) and actions \( \mu^\pm \) of silting mutation; see [AI] for details.

A triangulated category \( \mathcal{T} \) is said to be silting-discrete if it admits a silting object \( T \), and for any \( n > 0 \) there are only finitely many (basic) silting objects \( U \) satisfying \( T \geq U \geq T[n] \). We obtain from [AI] Corollary 3.9 that if \( \mathcal{T} \) is silting-discrete, then the Hasse quiver of the poset \( \text{silt}\mathcal{T} \) is connected; namely, it is silting-connected.

When \( \mathcal{T} = \text{K}^b(\text{proj}\Lambda) \) for an algebra \( \Lambda \), we write \( \text{silt}\mathcal{T} \) by \( \text{silt}\Lambda \) and say that \( \Lambda \) is silting-discrete if \( \mathcal{T} \) is silting-discrete. Here is a new example of silting-discrete algebras.

**Theorem 2.1.** Let \( R \) and \( \Lambda \) be a local and silting-discrete algebra. Put \( \Gamma := R \otimes_K \Lambda \). Then we have a poset isomorphism \( \text{silt}\Lambda \to \text{silt}\Gamma \). In particular, \( \Gamma \) is also silting-discrete.

**Proof.** By Wedderburn–Malcev decomposition, we have \( \Gamma \simeq R/\text{rad} R \otimes_K \Lambda \oplus \text{rad} R \otimes_K \Lambda \), which is isomorphic to \( \Lambda \oplus \text{rad} R \otimes_K \Lambda \) since \( R \) is local. Then, we obtain a triangle functor \( - \otimes_\Lambda \Gamma : \text{K}^b(\text{proj}\Lambda) \to \text{K}^b(\text{proj}\Gamma) \), which preserves the indecomposability of objects [AK] Proposition 3.4]. Since \( \Gamma \simeq \Lambda^n \) as a \( (\Lambda, \Lambda) \)-bimodule, this induces an injection \( \text{silt}\Lambda \to \text{silt}\Gamma \) preserving the partial order. Here, \( n := \dim_K R \).

We show that \( - \otimes_\Lambda \Gamma \) preserves approximations. Let \( f : X \to Y \) be a left \( \mathcal{Y} \)-approximation of \( X \) in \( \text{K}^b(\text{proj}\Lambda) \), where \( \mathcal{Y} \) is a full subcategory of \( \text{K}^b(\text{proj}\Lambda) \). Let \( g \) be a morphism \( X \otimes_\Lambda \Gamma \to Z \otimes_\Lambda \Gamma \) in \( \text{K}^b(\text{proj}\Gamma) \), where \( Z \) belongs to \( \mathcal{Y} \). Observing an isomorphism \( \text{Hom}_{\text{K}^b(\text{proj}\Gamma)}(X \otimes_\Lambda \Gamma, Z \otimes_\Lambda \Gamma) \simeq \text{Hom}_{\text{K}^b(\text{proj}\Lambda)}(X, Z)^n \), we see that \( g \) is given by \( n \) morphisms \( g_1, g_2, \ldots, g_n : X \to Z \) in \( \text{K}^b(\text{proj}\Lambda) \). If \( f \) is a left \( \mathcal{Y} \)-approximation of \( X \), there exist \( \alpha_1, \alpha_2, \ldots, \alpha_n : Y \to Z \) with \( g_i = \alpha_i \circ f \) for \( i = 1, 2, \ldots, n \). By a similar isomorphism above, we get \( \alpha : Y \otimes_\Lambda \Gamma \to Z \otimes_\Lambda \Gamma \) from \( \alpha_i \)’s. It is not hard to check that \( g = \alpha \circ (f \otimes_\Lambda \Gamma) \), whence \( f \otimes_\Lambda \Gamma \) is a left \( (\mathcal{Y} \otimes \Gamma) \)-approximation of \( X \otimes_\Lambda \Gamma \) in \( \text{K}^b(\text{proj}\Gamma) \).

Thus, it turns out that any arrow in \( \text{silt}\Lambda \) is also an arrow in \( \text{silt}\Gamma \) under the injection \( - \otimes_\Lambda \Gamma : \text{silt}\Lambda \to \text{silt}\Gamma \). Conversely, we obtain that all paths from/to \( \Gamma \) in \( \text{silt}\Gamma \) come from those from/to \( \Lambda \) in \( \text{silt}\Lambda \), because \( \text{K}^b(\text{proj}\Lambda) \) and \( \text{K}^b(\text{proj}\Gamma) \) have the same rank of the Grothendieck group.

Assume that there is a silting object \( U \) of \( \text{K}^b(\text{proj}\Gamma) \) with \( \Gamma \geq U \) which is out of \( \text{silt}\Lambda \) under \( - \otimes_\Lambda \Gamma \). By [AI] Proposition 2.36], we have a path \( \Gamma = : U_0 \to U_1 \to \cdots \to U_\ell \) in \( \text{silt}\Gamma \) with \( U_i \geq U \) for any \( i \), which admits an infinite length, contrary to the assumption of \( \Lambda \) being silting-discrete. Therefore, all silting objects of \( \Gamma \) smaller than \( \Gamma \) come from those of \( \Lambda \). Then, we derive from [AM] Theorem 2.4] that \( \Gamma \) is silting-discrete.

Finally, we see that the following are equivalent for any \( T, U \in \text{silt}\Lambda \):

(i) \( T \geq U \);
(ii) there exists a finite path from \( T \) to \( U \);
(iii) there is a finite path from \( T \otimes_\Lambda \Gamma \) to \( U \otimes_\Lambda \Gamma \);
(iv) \( T \otimes_\Lambda \Gamma \geq U \otimes_\Lambda \Gamma \).

This implies that the map \( - \otimes_\Lambda \Gamma \) is a poset isomorphism. \( \square \)

The trivial extension of an algebra \( \Lambda \) by a \( (\Lambda, \Lambda) \)-bimodule \( M \) is defined to be \( \Lambda \oplus M \) as a \( (\Lambda, \Lambda) \)-bimodule in which the composition of elements \( (a, m) \) and \( (b, n) \) is \( (ab, an + mb) \). Since the trivial extension of \( \Lambda \) by itself is isomorphic to \( K[x]/(x^2) \otimes_K \Lambda \), we immediately obtain the following corollary from Theorem 2.1.
Corollary 2.2. The trivial extension of $\Lambda$ by itself is silting-discrete if $\Lambda$ is so.

Remark 2.3. The trivial extension of $\Lambda$ by its $K$-dual is often called the trivial extension of $\Lambda$. Applying it frequently destroys the silting-discreteness of algebras. For instance, the algebra given by the quiver $\bullet \rightarrow \bullet$ with radical square zero is silting-discrete, but its trivial extension is neither silting-discrete nor even silting-connected [AGI].

3. The $\tau$-tilting finiteness of triangular matrix algebras

The first aim of this section is to develop the Auslander–Reiten results in [AR] to the $\tau$-tilting finiteness. We start with recalling important facts on $\tau$-tilting finite algebras.

Let $\Lambda$ be an algebra. We call a module $M$ over $\Lambda$ support $\tau$-tilting provided it is the 0th cohomology of a silting object $T$ in $\mathcal{K}^b(\text{proj} \, \Lambda)$ with $T_i = 0$ unless $i = 0, -1$. See [AIR] for more details. Our interest in this paper is when an algebra $\Lambda$ has only finitely many support $\tau$-tilting modules; so-called, $\Lambda$ is $\tau$-tilting finite. Evidently, if $\Lambda$ is silting-discrete, then it is $\tau$-tilting finite. We also know that any factor algebra of a $\tau$-tilting finite algebra is also $\tau$-tilting finite [DIRRT, Theorem 5.12(d)]. A module $M$ is said to be brick if $\text{End}_{\Lambda}(M)$ is isomorphic to $K$. It was shown that $\Lambda$ is $\tau$-tilting finite iff there are only finitely many bricks of $\Lambda$ [DIJ, Theorem 4.2].

A main algebra we study here is the $n \times n$ upper triangular matrix algebra $T_n(\Lambda)$, which is isomorphic to $\Lambda \otimes_K K^\gamma$ as $\Lambda$-modules. As is well-known, we can identify the category $\text{mod} \, T_2(\Lambda)$ with the category of homomorphisms in $\text{mod} \, \Lambda$; that is, the objects are triples $(M, N, f)$ of $\Lambda$-modules $M, N$ and a $\Lambda$-homomorphism $f : M \to N$.

For an additive category $\mathcal{C}$, we denote by $\text{mod} \mathcal{C}$ the full subcategory of the functor category of $\mathcal{C}$ consisting of finitely generated functors.

Let us recall an argument in [AR, Theorem 1.1]. It was shown that the functor $\Phi : \text{mod} \, T_2(\Lambda) \to \text{mod} \, \text{mod} \, \Lambda$ sending $(M, N, f) \to \text{Coker} \, \text{Hom}_\Lambda(-, f)$ is full and dense. Denote by $\mathcal{D}$ the full subcategory of $\text{mod} \, T_2(\Lambda)$ consisting of modules without indecomposable summands of the forms $(M, M, \text{id})$ and $(M, 0, 0)$, where $M$ is an indecomposable module of $\Lambda$. Then the restriction of $\Phi$ to $\mathcal{D}$ gives rise to an equivalence. Therefore, we have the following result.

Proposition 3.1. Let $\Lambda$ be a representation-finite algebra. Then the Auslander algebra of $\Lambda$ is $\tau$-tilting finite if and only if so is $T_2(\Lambda)$.

We observe that representation-finiteness and $\tau$-tilting finiteness of $T_2(\Lambda)$ may coincide.

Theorem 3.2. Let $\Lambda$ be a simply-connected algebra of finite representation type. Then $T_2(\Lambda)$ is $\tau$-tilting finite if and only if it is representation-finite.

Proof. Let us show the ‘only if’ part. As $T_2(\Lambda)$ is $\tau$-tilting finite, we see that it contains no subquiver of extended Dynkin type. Then, the simple-connectedness of $\Lambda$ (i.e. $\tilde{\Lambda} = \Lambda$ in the sense of [LSI]) implies that $T_2(\Lambda)$ is representation-finite by [LSI, Theorem 4].

Let $\Lambda$ be an algebra whose Gabriel quiver is $Q$. The separated quiver $Q^{\text{op}}$ of $\Lambda$ is defined as follows: The set of vertices consists of the vertices $i$ of $Q$ and their copies $i'$; we say that $i$ and $i'$ are the same character. We draw an arrow $i \to k$ if $i$ is a vertex of $Q$, $k = j'$ for some vertex $j$ of $Q$ and there is an arrow $i \to j$ in $Q$. (See [ARS].)
We know from [AR, Proposition 3.1] that if the separated quiver of an algebra \( \Lambda \) has a connected component which is not of type \( A_n \), then \( T_2(\Lambda) \) is representation-infinite. Here is a modification to \( \tau \)-tilting finiteness.

**Theorem 3.3.** Let \( \Lambda \) be an algebra. If the separated quiver of \( \Lambda \) has a connected component which is not of type \( A_n \), then \( T_2(\Lambda) \) is \( \tau \)-tilting infinite.

**Proof.** Assume that \( T_2(\Lambda) \) is \( \tau \)-tilting finite, then so are \( \Lambda \) and \( \Lambda / \text{rad}^2 \Lambda \). To derive a contradiction, suppose that there is a connected component \( C \) of the separated quiver of \( \Lambda \) which has a vertex \( v \) of degree at least 3. We divide the proof to two cases: (i) there is no loop at \( v \) in the (Gabriel) quiver \( Q \) of \( \Lambda \); (ii) otherwise.

(i) In the case, it is seen that the 4 points around \( v \) in \( C \) are different characters. This implies that the separated quiver of \( T_2(\Lambda) \) contains the diagram of type \( \tilde{E}_6 \) whose vertices are distinct characters, but this contradicts the assumption of \( T_2(\Lambda) \) being \( \tau \)-tilting finite. Therefore, we observe that every point of \( C \) has degree at most 2, which says that \( C \) is of type \( A_n \) or \( \tilde{A}_n \). In the case where \( C \) is of type \( \tilde{A}_n \), the (Gabriel) quiver \( Q \) of \( \Lambda \) admits a subquiver of type \( \tilde{A}_5 \) with distinct characters, whence it is not \( \tau \)-tilting finite. Consequently, we conclude the fact that \( C \) is of type \( A_n \).

(ii) This case occurs when \( Q \) has \( 2 \rightarrow 1 \rightarrow 3 \) or \( 2 \rightarrow 1 \rightarrow 3 \) as a subquiver, where \( \nu = 1 \). Let us consider the former case; the other can be handled similarly. Truncating by idempotents and factoring by ideals, we focus on the algebra \( \Gamma \) given by the quiver with radical square zero. As \( T_2(\Lambda) \) is \( \tau \)-tilting finite, it follows that \( T_2(\Gamma) \) is also \( \tau \)-tilting finite. Since \( \Gamma \) is representation-finite, we obtain from Theorem 3.1 that the Auslander algebra of \( \Gamma \) is \( \tau \)-tilting finite. Observing the Auslander–Reiten quiver of \( \Gamma \), the quiver of the Auslander algebra of \( \Gamma \) has the form

![Quiver Diagram]

as a subquiver, whence the Auslander algebra of \( \Gamma \) is not \( \tau \)-tilting finite, which is contrary. Thus, it turns out that this case does not happen. \( \square \)

We denote by \( \tilde{A}_n \) the linearly oriented \( A_n \)-quiver \( 1 \rightarrow 2 \rightarrow \cdots \rightarrow n \).

The converse of Theorem 3.3 does not necessarily hold.

**Example 3.4** (See also Theorem 3.2). Let \( \Lambda := K\tilde{A}_n^{\rightarrow} \). Observe that \( T_2(\Lambda) \) is the commutative ladder of degree \( n \); see [AHMW, EH, LS1]. Then the following are equivalent: (i) \( n \leq 4 \); (ii) \( T_2(\Lambda) \) is representation-finite; (iii) it is \( \tau \)-tilting finite.

Combining this observation and Proposition 3.1 we recover [IX, Corollary 4.8]; that is, the following are equivalent: (i) \( n \leq 4 \); (ii) the Auslander algebra of \( \Lambda \) is representation-finite; (iii) it is \( \tau \)-tilting finite.
In the paper [AR], it was also discussed that the third triangular matrix algebra \( T_2^3(\Lambda) \) over an algebra \( \Lambda \) is not representation-finite [AR, Theorem 3.4]. To see this, we consider the triangular matrix algebra \( T_2^3(\Lambda/\text{rad} \Lambda) \). It is because this is a factor algebra of \( T_2^3(\Lambda) \), since \( T_2^3(\Lambda) / I \cong T_2^3(\Lambda/\text{rad} \Lambda) \). Here, \( I \) stands for the ideal \((\text{rad} \Lambda, \text{rad} \Lambda, 0, \text{rad} \Lambda)\). As \( T_2^3(\Lambda/\text{rad} \Lambda) \) is the direct product of some copies of \( T_2^3(K) \), the next step is to observe \( T_2^3(K) \). We see that \( T_2^3(K) \) is presented by the quiver

\[
\begin{array}{ccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
5 & \leftarrow & 6 & \leftarrow & 7 & \leftarrow & 8 \\
\end{array}
\]

whose separated quiver contains the connected component

\[
\begin{array}{ccccccc}
2 & \leftarrow & 3 & \leftarrow & 4 & \leftarrow & 5 & \leftarrow & 6 \\
\end{array}
\]

This implies that \( T_2^3(K) / \text{rad}^2 T_2^3(K) \) is representation-infinite. Consequently, it turns out that \( T_2^3(\Lambda) \) is of infinite representation type.

We can apply this argument to obtain the following result; cf. [Ad].

**Proposition 3.5.** The triangular matrix algebra \( T_2^3(\Lambda) \) is not \( \tau \)-tilting finite.

The following corollary is an immediate consequence of this proposition.

**Corollary 3.6.** For nonlocal algebras \( \Lambda, \Gamma \) and \( \Sigma \), \( \Lambda \otimes_K \Gamma \otimes_K \Sigma \) is \( \tau \)-tilting infinite.

**Proof.** By assumption, there is an algebra epimorphism from the algebra to \( K \overrightarrow{A_2} \otimes_K K \overrightarrow{A_2} \cong T_2^3(K) \). Then apply Proposition 3.5. \( \square \)

We might expect that there is an upper bound of \( n \) such that the \( n \times n \) triangular matrix algebra over a nonsemisimple algebra is \( \tau \)-tilting finite, but one has the following observation; cf. [LS2, Theorem 6.1].

**Proposition 3.7.** If \( \Lambda \) is local, then \( T_n(\Lambda) \) is silting-discrete. Hence, it is \( \tau \)-tilting finite.

**Proof.** The algebra is isomorphic to \( \Lambda \otimes_K \overrightarrow{K A_n} \), and then apply Theorem 2.1. \( \square \)

Let us consider the converse of this proposition; that is, what happens if the triangular matrix algebra is \( \tau \)-tilting finite. Here is a first observation.

**Lemma 3.8.** Let \( \Lambda \) be a nonlocal algebra. If \( T_n(\Lambda) \) is \( \tau \)-tilting finite, then we have \( n \leq 4 \).

**Proof.** Suppose that \( n \geq 5 \). Since \( \Lambda \) is nonlocal, we obtain algebra epimorphisms \( T_n(\Lambda) \rightarrow T_5(K \overrightarrow{A_2}) \cong T_2(\overrightarrow{K A_5}) \), whose target is not \( \tau \)-tilting finite by Example 3.3, contrary. \( \square \)
We treat radical-square-zero Nakayama algebras, which play a role in our goal.

**Lemma 3.9.** Let $\Lambda$ be a nonlocal Nakayama algebra with radical square zero. If $\Lambda \neq K\bar{A}_2$, then $T_4(\Lambda)$ is $\tau$-tilting infinite.

**Proof.** Assume that $\Lambda$ is linear Nakayama with at least 3 simple modules. Since $T_4(\Lambda)$ is strongly simply-connected and representation-infinite by [LS2, Theorem 6.2], we obtain from [W, Theorem 2.6] that it is $\tau$-tilting infinite.

If $\Lambda$ is cyclic Nakayama with at least 3 simple modules, then there is an algebra epimorphism $\Lambda \rightarrow \Gamma$, which induces $T_4(\Lambda) \rightarrow T_4(\Gamma)$. Here, $\Gamma := K\bar{A}_3/\text{rad}^2 K\bar{A}_3$. As above, this implies that $T_4(\Lambda)$ is $\tau$-tilting infinite.

We show that $T_4(\Lambda)$ is not $\tau$-tilting finite if $\Lambda$ is a radical-square-zero cyclic Nakayama algebra with precisely 2 simple modules. Then one sees from the Happel–Vossieck List [HV] that it has a tame concealed factor algebra of type $\tilde{E}_7$ as follows:

```
\[
\begin{array}{c}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]
```

Hence, it turns out that $T_4(\Lambda)$ is not $\tau$-tilting finite. □

We solve the problem for the case that given algebras have at least 3 simple modules.

**Theorem 3.10.** Let $\Lambda$ be an algebra given by a quiver $Q$ which has no loops and at least 3 vertices. Let $n \geq 3$. Then the following are equivalent:

1. $T_n(\Lambda)$ is $\tau$-tilting finite;
2. It is representation-finite;
3. $n = 3$ and $\Lambda$ is a Nakayama algebra with radical square zero.

**Proof.** It is trivial that (2) implies (1). It follows from [LS2, Theorem 6.1] that the implications $(2) \Leftrightarrow (3)$ hold true.

We show that (1) implies (3). Assume that $Q$ has $\cdot \leftarrow \cdot \rightarrow \cdot \rightarrow \cdot$ or $\cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot$ as a subquiver. Then, we see that there is an algebra epimorphism $T_n(\Lambda) \rightarrow T_3(A)$, where $A$ is the path algebra of $\cdot \rightarrow \cdot \rightarrow \cdot$ or $\cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot$. By the Happel–Vossieck List [HV], we observe that a tame concealed algebra of type $\tilde{E}_7$ appears as a factor algebra of $T_3(A)$, which is $\tau$-tilting infinite, and hence, so is $T_n(\Lambda)$. Thus, we find out that $\Lambda$ is a Nakayama algebra. As a similar argument above, we deduce the fact that $Q$ does not admit $\cdot \rightarrow \cdot \rightarrow \cdot$ without zero relation, which implies that $\Lambda$ has radical square zero. Finally, apply Lemma 3.8 and 3.9 to get $n = 3$. □

Let us turn to the case where a given algebra has precisely 2 simple modules. We prepare a lemma to reduce the length.

**Lemma 3.11.** Let $\Lambda$ be a cyclic Nakayama algebra with precisely 2 simple modules. Then we have a poset isomorphism $\text{s}\tau\text{-tilt} T_n(\Lambda) \simeq \text{s}\tau\text{-tilt} T_n(\Lambda/\text{rad}^2 \Lambda)$.

**Proof.** By assumption, $\Lambda$ is given by the quiver $1 \xrightarrow{x/y} 2$. Then it is seen that $z := xy + yx$ belongs to the center and the radical of $\Lambda$, whence $zI$ is in those of $T_n(\Lambda)$. Here, $I$ is the
identity matrix. We observe that the factor algebra of $T_n(\Lambda)$ by the ideal generated by $zI$ is isomorphic to $T_n(\Lambda/\text{rad}^2\Lambda)$, which completes the proof by [EJR, Theorem 11].

Now, we totally realize our goal.

**Theorem 3.12.** Let $\Lambda$ be an algebra whose quiver has precisely 2 vertices and no loops. Let $n \geq 3$. Then the following are equivalent:

1. $T_n(\Lambda)$ is $\tau$-tilting finite;
2. $n = 3$ and $\Lambda$ is a Nakayama algebra, or $n = 4$ and $\Lambda = K\overrightarrow{A}_2$.

**Proof.** If $T_n(\Lambda)$ is $\tau$-tilting finite, then it is observed that $n \leq 4$ by Lemma 3.8, and $\Lambda$ is also $\tau$-tilting finite, which implies that $\Lambda$ has no double arrow, so it is Nakayama.

Let $n = 4$. By Lemma 3.11 we can suppose that $\Lambda$ has radical square zero, whence $\Lambda = K\overrightarrow{A}_2$ by Lemma 3.9.

Let us show that the implication (2)$\Rightarrow$(1) holds true. By Example 3.4, we have only to check the case where $n = 3$ and $\Lambda$ is cyclic Nakayama. From Lemma 3.11 one obtains $\text{sr-tilt} T_3(\Lambda) \simeq \text{sr-tilt} T_3(\Lambda/\text{rad}^2\Lambda)$, which is a finite set because $T_3(\Lambda/\text{rad}^2\Lambda)$ is representation-finite by [LS2, Theorem 6.1]. Thus, we have done.

As a corollary of Theorem 3.10 and 3.12, we get the following.

**Corollary 3.13.** Let $\Lambda$ be a simply-connected algebra and $n \geq 3$. Then $T_n(\Lambda)$ is $\tau$-tilting finite if and only if it is representation-finite.

Finally, we give a complete list of positive integers $n$ and $r$ such that $T_n(\Lambda)$ is silting-discrete for $\Lambda := K\overrightarrow{A}_r/\text{rad}^2 K\overrightarrow{A}_r$.

**Theorem 3.14.** Let $\Lambda$ be a radical-square-zero linear Nakayama algebra with $r$ simple modules. Then $T_n(\Lambda)$ is silting-discrete if and only if one of the following cases occurs:

1. $n = 1$;  
2. $r = 1$;  
3. $n = 2$ and $1 < r \leq 4$;  
4. $1 < n \leq 4$ and $r = 2$.

**Proof.** It is well-known that $\Lambda$ is derived equivalent to $K\overrightarrow{A}_r$, and so $T_n(\Lambda)$ is derived equivalent to $T_n(K\overrightarrow{A}_r)$, which is $\tau$-tilting infinite if $n \geq 3$ and $r \geq 3$ by Theorem 3.10. In the case, it is not silting-discrete.

We already know that $T_n(K\overrightarrow{A}_2) \simeq T_2(K\overrightarrow{A}_n)$ is not silting-discrete for $n \geq 5$; see Example 3.4. For $n = 1, 2, 3$ and 4, we have the ADE-chain $A_2, D_4, E_6$ and $E_8$, respectively. This means that $T_n(K\overrightarrow{A}_2)$ is derived equivalent to the path algebra of each type $\text{LP}$, which is silting-discrete. This completes the proof.

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Department of Mathematics, Tokyo Gakugei University, 4-1-1 Nukuikita-machi, Koganei, Tokyo 184-8501, Japan

Email address: aihara@u-gakugei.ac.jp

Department of Mathematics, Tokyo University of science, 1-3 Kagurazaka, Shinjuku, Tokyo 162-8601, Japan

Email address: 1119704@ed.tus.ac.jp