Non-associative Hilbert scheme and Thom polynomials

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Abstract

Thom polynomial describes the cohomology class Poincaré dual to the locus of particular singularity of a generic holomorphic map. In this paper we derive a closed formula for the generating function of its coefficients. The method is based on a new construction of the embedding space of punctual Hilbert scheme that we call the non-associative Hilbert scheme. The efficiency of the method is demonstrated on explicit computation of a number of Thom polynomials, including those associated with singularities of Thom-Boardman types $\Sigma_{i,j}^1$, $\Sigma_{2,1}^1$, $\Sigma_{2,2}^1$, and $\Sigma_{2,2,2}^1$.

1 Introduction

Thom polynomial is associated with a local singularity type of holomorphic mappings. If $f : X \to Y$ is a generic holomorphic map between two nonsingular varieties then the Thom polynomial evaluated on the Chern classes of these varieties expresses the cohomology class Poincaré dual to the locus in $X$ of a given singularity type for $f$. The theory of Thom polynomials is deserved as a possible approach to some problems of enumerative projective algebraic geometry [1, 16]. Some interesting applications were discovered recently in another domains of mathematics such that representation theory and the study of moduli spaces [8,9,18,19].

The computation of Thom polynomials for particular singularity classes was initiated in 60-ies by R.Thom, and many researches contributed to this activity, see [28,20,25,12,24]. It was observed quite early [25,14] that in many cases, for example, for contact classification of singularities, the Thom polynomial is expressed in terms of the relative Chern classes $c_i = c_i(f^*TY - TX)$ of the map. Moreover, it is independent of the dimensions of the manifolds $X$ and $Y$ provided that the relative dimension $\ell = \dim Y - \dim X$ is fixed. If by the singularity type we mean the isomorphism class of the local algebra then it is possible to relate singularity classes for different values of the relative dimension $\ell$. The corresponding Thom polynomials differ dramatically, and even its degree depends on $\ell$.

However, it was noticed by Fehér and Rimányi in [6], that it is possible to represent the Thom polynomial in the form of an infinite series (aka Thom series)

$$T_{\eta} = \sum_{i_1,\ldots,i_\mu} \psi_{i_1,\ldots,i_\mu} c_{\ell+1+i_1} \cdots c_{\ell+1+i_\mu}$$

whose integer coefficients $\psi_{i_1,\ldots,i_\mu}$ are independent of $\ell$. Here the singularity type $\eta$ is assumed to be determined by the isomorphism class of the corresponding local algebra and $\mu$ is the dimension of the local algebra diminished by 1. First computations of Thom series can be found in [2,6,7,22,23].

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A new achievement in the development of the Thom polynomial theory was attained in [3]. In that paper the authors applied the Atiyah-Bott’s localization principle to the computation of the Thom polynomials for singularities of Thom-Boardman type $\Sigma^{1d} = \Sigma^{1,\ldots,1}$. The final formula obtained in that paper can be reformulated by saying that the coefficients of the Thom series are determined by a rational generation function whose terms are identified explicitly for $d \leq 6$ and up to certain undetermined coefficients for bigger $d$. The computations of this paper didn’t use the existence theorem for Thom polynomials and the very fact that the result is expressed in terms of the relative Chern classes is obtained by a lengthy direct computations with unexpected magic cancelations.

The localization methods was extended in [7] to the study of Thom polynomials for other kinds of singularities. It is established, in particular, that the whole Thom series can be recovered from a finite number of its initial terms. In other words, the knowledge of Thom polynomial for some $\ell$ big enough (in fact, for $\ell = \mu$) is sufficient to recover this polynomial for all $\ell$.

In the present paper we summarize the mentioned above results on the Thom series by presenting an explicit rational form of the generating function for its coefficients (Theorem 3.1). Our computation is direct and does not use an apriori existence theorem for Thom polynomials. This allows us to use minor assumptions on the chosen singularity type: it could be given by any algebraic family of finite dimensional local algebras. For the particular case of the singularity types $\Sigma^{1d}$ the formula of Theorem 3.1 specializes to that from [3]. However, our proof is essentially different.

The computation uses an old construction of partial resolution of the singularity locus by means of the punctual Hilbert scheme. The main novelty is a new geometric construction for the smooth embedding space of the Hilbert scheme that we call a nonassociative Hilbert scheme. The embedding of the Hilbert scheme to its ambient space can be identified as the solution locus of the associativity equation. The computational part of the proof of Theorem 3.1 is the application of the Gysin homomorphism for the constructed partial resolution. At this point we use the formalism developed in [17]. It allows one to keep the answer expressed in terms of the relative Chern classes at all intermediate steps. However, whenever the geometric construction is presented, the computation of the Gysin homomorphism is technical, and any other known ways of its computation, including the Schubert calculus or the localization formulas, would lead to the same answer.

All terms of the rational function of Theorem 3.1 are identified explicitly except certain polynomial $P_?\eta$ whose computation is not covered by the assertion of the theorem. We call this polynomial the local characteristic invariant of the singularity $\eta$. It is defined as the equivariant Poincaré dual class of certain explicitly given subvariety in a vector space. For the singularity determined by fixing the dimension vector of local algebra, it is given by the associativity equation.

At the moment, we have no regular efficient way to compute $P_?\eta$ in general, but first examples of its computation are also presented in this paper. It leads to a considerable list of closed formulas for Thom polynomials of different singularities. It includes also some cases for which the Thom polynomial was unknown before, in particular, those of Thom-Boardman types $\Sigma^{2,1,1}$, $\Sigma^{2,2,1}$, and $\Sigma^{2,2,2}$.

In the present paper, we use the topological language of complex geometry and cohomology. However, all results are also valid in the algebraic context of arbitrary algebraically closed ground field of zero characteristic and the Chow rings instead of cohomology. When necessary, we supply technical details allowing one to adopt our topological arguments to the algebraic setting.

The paper is organized as follows. In Section 2 we review the notions of Thom polynomials and equivariant Poincaré duals, compare topological and algebraic aspects of their definitions and revise the proof of the existence theorem for Thom polynomials. The main Theorem 3.1 expressing the rational generating function for the Thom series is formulated in section 3. The definition of the
The polynomial \( P_\eta \) used in the formulation of Theorem 3.1 is introduced in Sect. 4. In Section 5 we recall the resolution method for the computation of Thom polynomials. The main geometric construction of the paper—the nonassociative Hilbert scheme—is introduced in Sect. 6. The computation of Thom polynomial is completed in Section 7 by applying the Gysin homomorphism. The results on explicit computations based on Theorem 3.1 for particular singularity classes are put together in Sect. 8. In the final Section 9 we formulate some open problems.

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2 The notion of Thom polynomial

2.1 Equivariant Poincaré duals and multidegrees

Let a complex algebraic Lie group \( G \) act algebraically on a vector space \( V \). Then for any algebraic \( G \)-invariant subvariety \( \eta \subset V \) of pure codimension \( k \) there is a well defined its equivariant Poincaré dual cohomology class \( [\eta] \in H^{2c}_G(V) = H^{2k}_G(\text{pt}) = H^{2c}(BG) \).

If \( G = T^n = (\mathbb{C}^*)^n \) is the (complex) torus, then \( H^*(BG) = \mathbb{Z}[t_1, \ldots, t_n] \) with the (real) grading of the class \( t_i \) equal to 2 for all \( i = 1, \ldots, n \). The polynomial representing \( [\eta] \) in this case is also known in algebraic geometry as the Josef polynomial or multidegree. If \( f : V \to \mathbb{C} \) is a \( T^n \)-homogeneous function satisfying
\[
f(\lambda v) = \lambda_1^{a_1} \cdots \lambda_n^{a_n} f(v), \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in T^n,
\]
then the multidegree of its zero hypersurface is equal to
\[
[\{f = 0\}] = a_1t_1 + \cdots + a_nt_n.
\]

The multidegree of a (reduced) complete intersection given as the zero set of a collection of homogeneous functions is equal to the product of the corresponding linear factors. This can be applied, in particular, to coordinate subspaces. The multidegree of any \( T^n \)-invariant subvariety can be computed by deforming it in a flat family to a union of coordinate subspaces (probably, with multiplicities). Then its multidegree is equal to the corresponding linear combination of multidegrees of coordinate subspaces.

In the case when \( G = \text{GL}(n) \) we have \( BG = G_{n,\infty} \), the infinite Grassmann manifold, and \( H^*(BG) \) is the polynomial ring generated by the Chern classes \( c_i \) of the tautological vector bundle over the Grassmannian. Therefore, the equivariant Poincaré dual class \( [\eta] \) in this case is a polynomial with integer coefficients in \( c_1, \ldots, c_n \) of weighted degree \( k \) where we assume \( \deg(c_i) = i \). Restricting the action of \( \text{GL}(n) \) to the maximal torus \( T^n \subset \text{GL}(n) \) we can express \( [\eta] \) as a polynomial in the Chern roots \( t_1, \ldots, t_n \). This polynomial is symmetric and \( c_i \) corresponds to the \( i \)th elementary symmetric function in the variables \( t_1, \ldots, t_n \).

2.2 Thom polynomials

A Thom polynomial \( T_\eta \) is a special case of the equivariant Poincaré dual class when \( V = J^K_0(\mathbb{C}^m, \mathbb{C}^n) \) is the space of \( K \)-jets of map germs from \( \mathbb{C}^m \) to \( \mathbb{C}^n \) at the origin, \( G \) is the group of \( K \)-jets of changes of coordinates in the source and the target, respectively, and \( \eta \) is an arbitrary singularity type. By a ‘singularity type’ we mean here any \( G \)-invariant reduced irreducible algebraic
subvariety in \( J^K_0(\mathbb{C}^m, \mathbb{C}^n) \). The group \( G \) is homotopy equivalent to its subgroup \( \text{GL}(m) \times \text{GL}(n) \) consisting of linear changes. Therefore, we have, by definition,

\[
\text{Tp}_\eta = [\eta] \in H^*_G(V) = H^*_{\text{GL}(m) \times \text{GL}(n)}(\text{pt}) = H^*(B\text{GL}(m) \times B\text{GL}(n)) = \mathbb{Z}[c'_1, \ldots, c'_m, c''_1, \ldots, c''_n].
\]

We see, that from the very definition, \( \text{Tp}_\eta \) is a polynomial in two groups of variables, \( c'_j \) and \( c''_j \) corresponding to the source and the target, respectively.

The definition depends on a choice of the order \( K \) of jets. We say that the singularity \( \eta \subset J^K(\mathbb{C}^m, \mathbb{C}^n) \) is \( k \)-determined if it is of the form \( \eta = p^{-1}(\eta_k) \) where \( \eta_k \subset J^K_0(\mathbb{C}^m, \mathbb{C}^n) \) and \( p : J^K(\mathbb{C}^m, \mathbb{C}^n) \rightarrow J^K_0(\mathbb{C}^m, \mathbb{C}^n) \) is the natural projection. The Thom polynomial of a \( k \)-determined singularity is actually independent of a choice of \( K \geq k \).

The importance of the notion of Thom polynomial shows up in the following theorem \cite{23,13,15}. Let \( X \) and \( Y \) be manifolds of dimensions \( \dim X = m, \dim Y = n \). The jet bundle \( J^K(X,Y) \) is formed by all \( K \)-jets of maps from \( X \) to \( Y \) at all points \( x \in X \) and \( y \in Y \). It is the total space of a bundle over \( X \times Y \) with the fiber isomorphic to \( J^K_0(\mathbb{C}^m, \mathbb{C}^n) \). It follows that the projection \( J^K(X,Y) \rightarrow X \times Y \) is a homotopy equivalence and the induced homomorphism in cohomology is an isomorphism. Denote by \( \eta(X \times Y) \subset J^K(X,Y) \) the subvariety formed by jets of maps with prescribed singularity \( \eta \).

**Theorem 2.1.** The cohomology class

\[
[\eta(X \times Y)] \in H^*(J^K(X,Y)) = H^*(X \times Y)
\]

represented by the locus of the singularity type \( \eta \) is equal to the Thom polynomial \( \text{Tp}_\eta \) evaluated on the Chern classes of the tangent bundles \( c_i(TX) \) and \( c_j(TY) \) of the factors in \( X \times Y \).

**Corollary 2.2.** Let \( f : X \rightarrow Y \) be a holomorphic map, \( j^K f : X \rightarrow J^K(X,Y) \) be its jet extension, and \( \eta(f) = (j^K f)^{-1}\eta(X \times Y) \subset X \), the locus of the singularity \( \eta \) for the map \( f \). Then, if \( \eta(f) \) has expected dimension and reduced then its fundamental class in \( X \) represents the Thom polynomial \( \text{Tp}_\eta \) evaluated on the classes \( c_i(TX) \) and \( f^*c_j(TY) \). For an arbitrary map, \( \text{Tp}_\eta \) is represented by an algebraic cycle supported on \( \eta(f) \).

The notions of Chern classes, equivariant Poincaré duals, and Thom polynomials make sense in the algebraic geometry context with Chow groups instead of cohomology. The assertions of the theorem and its corollary remain true in the algebraic setting. Since the proof of this generalization has never been published we reproduce it here. The topological counterpart of the proof uses the notions of classifying spaces and classifying maps for \( G \)-bundles. We will see that these notions can be adopted to the algebraic context, see also \cite{27}.

Let us first review the definition of the Chern classes of a given vector bundle \( E \) over a non-singular algebraic base \( X \). In topology, the Chern classes are certain distinguished elements in the cohomology of the Grassmannian that serves as the classifying space for vector bundles. Since the Chow ring of the Grassmannian is isomorphic to the cohomology, in order to define Chern classes, it is sufficient to construct a classifying map from \( X \) to the Grassmannian that induces the given vector bundle \( E \). Here is the construction of such classifying map adopted to the algebraic setting. Choose some integer \( N \gg 0 \) and define \( X_1 \) to be the total space of the bundle \( \text{Hom}(E, \mathbb{C}^N) \) over \( X \). Then obviously \( H^*(X) \cong H^*(X_1) \) and the same holds for Chow groups. The point of \( X_1 \) is a linear map from a fiber of \( E \) to \( \mathbb{C}^N \). Let \( X_2 \subset X_1 \) be the open subset consisting of injective linear maps. The complement \( X_1 \setminus X_2 \) has a very big codimension, growing together with \( N \), therefore, the inclusion \( X_2 \rightarrow X_1 \) induces an isomorphism in cohomology for any fixed grading, \( H^k(X_2) \cong H^k(X_1) \), if \( N \) is big enough, and a similar isomorphism holds in Chow groups.
The image of an injective linear map can be regarded as a point of the Grassmannian. This gives the required classifying map $\varphi : X_2 \to G_{n,N}$, $n = \text{rk} E$. Then we define the Chern classes $c_j(E)$ via the induced homomorphism $\varphi^* : H^{2j}(G_{n,N}) \to H^{2j}(X_2) \simeq H^{2j}(X_1) \simeq H^{2j}(X)$.

In the algebraic setting the Chern classes defined in this way (for a smooth base $X$) are classes of rational equivalence of $j$-codimensional subvarieties.

Now we present a similar construction for jet bundles. For given $m$, $n$, and $K$ chose some $N \gg 0$.

**Definition 2.3.** A $K$-jet of an $m$-dimensional submanifold in $\mathbb{C}^N$ is the $K$-jet of a map germ $(\mathbb{C}^m, 0) \to (\mathbb{C}^N, 0)$ with the derivative of full rank at the origin and considered up to a change of coordinates in the source $\mathbb{C}^m$. The variety of all $K$-jets of submanifolds is denoted by $\tilde{G}_{m,N}$.

Assignment of the tangent plane to a submanifold defines a natural mapping $\tilde{G}_{m,N} \to G_{m,N}$. This mapping can be represented as a tower of fibrations with affine fibers. This determines the structure of a nonsingular algebraic variety on $\tilde{G}_{m,N}$ and also shows an isomorphism between Chow groups and cohomology of the varieties $\tilde{G}_{m,N}$ and $G_{m,N}$.

**Definition 2.4.** A $K$-jet of an $n$-dimensional ‘quotient’ manifold of $\mathbb{C}^N$ is the $K$-jet of a map germ $(\mathbb{C}^N, 0) \to (\mathbb{C}^n, 0)$ with the derivative of full rank at the origin and considered up to a change of coordinates in the target $\mathbb{C}^n$. The variety of all $K$-jets of quotient manifolds is denoted by $\hat{G}_{N-n,N}$.

Similarly to the case of submanifolds, we have a natural projection $\hat{G}_{N-n,N} \to G_{N-n,N}$ that induces an isomorphism both in cohomology and in the Chow groups.

**Definition 2.5.** The classifying space of singularities is defined to be the product space

$$B = B_{m,n,N}^K = \tilde{G}_{m,N} \times \hat{G}_{N-n,N}. $$

Again, the natural projection $B \to G_{m,N} \times G_{N-n,N}$ induces an isomorphism both in cohomology and in the Chow groups.

A point $(f, g) \in B$ determines a $K$-jet of a map germ $g \circ f : \mathbb{C}^m \to \mathbb{C}^n$ defined up to a change of coordinates in the source and the target. Consider the subvariety $\eta(B) \subset B$ formed by points with the associated map of prescribed singularity type $\eta$. The following definition is equivalent to the preceding one.

**Definition 2.6.** The Thom polynomial of the singularity $\eta$ is the class represented by the subvariety $\eta(B) \subset B$ in either cohomology or the Chow ring of $B \sim G_{m,N} \times G_{N-n,N}$ and expressed in terms of the multiplicative generators of this ring for which we take traditionally the Chern classes of the tautological rank $m$ subbundle and the rank $n$ quotient vector bundles over the product of the Grassmannians.

It is easy to see that the Thom polynomial is independent of $N$ provided that $N$ is big enough.

Let now $X$ and $Y$ be manifolds of dimensions $m$ and $n$, respectively. Set

$$X_1 = J^K(X, \mathbb{C}^N), \quad Y_1 = J^K(\mathbb{C}^N, Y),$$

and consider open subsets $X_2 \subset X_1$ and $Y_2 \subset Y_1$ formed by jets of maps with the derivative of full rank.
On one hand, the composition of maps defines a natural surjective mapping
\[ \pi : X_2 \times Y_2 \to J^K(X, Y). \]

This mapping induces an isomorphism in the cohomology or the Chow groups of fixed grading provided that \( N \) is big enough. Therefore, we can replace the class of the subvariety \( \eta(X \times Y) \subset J^K(X, Y) \) participating in the statement of Theorem 2.1 by the class of its preimage \( \pi^{-1}(\eta(X \times Y)) \) in \( X_2 \times Y_2 \).

On the other hand, the classifying ‘tautological’ map \( \kappa : X_2 \times Y_2 \to B \) is well defined. This map respects singularity types, \( \kappa^{-1}(\eta(B)) = \pi^{-1}(\eta(X \times Y)) \) and the pullbacks of the basic Chern classes under \( \kappa^* \) are the Chern classes of tangent bundles of \( X \) and \( Y \), respectively, see the diagram.

\[
\begin{array}{ccc}
H^k(X \times Y) & \xrightarrow{\sim} & H^k(J^K(X, Y)) \\
\xrightarrow{\psi} & & \xleftarrow{\psi} \\
& [\eta(X \times Y)] & [\eta(B)] = Tp_\eta
\end{array}
\]

This proves both Theorem 2.1 and its algebraic geometry counterpart. \( \square \)

2.3 Thom polynomial of a contact singularity

A contact singularity is a particular case of a singularity type and a general definition of Thom polynomial is applicable in this case as well. There are, however, some details that we are going to clarify here. The local algebra of a map germ \( f : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0) \) is defined to be the quotient of the ring of functions \( \mathbb{C}[\![x_1, \ldots, x_m]\!] \) on the source modulo the ideal generated by the components \( f_1, \ldots, f_n \) of the mapping,

\[ Q_f = \frac{\mathbb{C}[\![x_1, \ldots, x_m]\!]}{(f_1, \ldots, f_n)}. \]

Two map germs are called contact equivalent, or \( K \)-equivalent, if their local algebras are isomorphic.

We will assume that \( \dim Q_f < \infty \). If \( n \geq m \), then the map germs with infinite dimensional local algebras form a subset of infinite codimension in the space of map germs. We do not consider such maps in this paper.

In a naive approach, a contact singularity type is the collection of map germs with a fixed isomorphism class of local algebras. This possible definition has a disadvantage. Consider, for example, the Porteous-Thom singularity \( \Sigma^r \) defined by the requirement that the derivative of the map has an \( r \)-dimensional kernel. It is natural to associate the local algebra \( Q_{\Sigma^r} = \mathbb{C}[\![x_1, \ldots, x_r]\!]/m^2 \) to this singularity, where \( m \) is the maximal ideal. The local algebra of a generic map germ with the singularity \( \Sigma^r \) is indeed isomorphic to \( Q_{\Sigma^r} \), if the relative dimension \( \ell = n - m \) of the map satisfies \( \ell \geq \frac{r(r+1)}{2} \). If \( \ell < \frac{r(r+1)}{2} \) then this local algebra is never realized since \( m + \frac{r(r+1)}{2} \) is the smallest number of generators of the ideal in \( \mathbb{C}[\![x_1, \ldots, x_m]\!] \) whose quotient algebra is isomorphic to \( Q_{\Sigma^r} \). However, it makes sense to consider the singularity \( \Sigma^r \) and its Thom polynomial for any \( \ell \geq -r \), even including some negative values of the relative dimension.

Let \( Q \) be a finite dimensional local algebra and \( f : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0) \) be a map germ.

**Definition 2.7.** We say that \( f \) has singularity type \( Q \) at the origin if there exist at most finitely many ideals \( I \) in \( \mathbb{C}[\![x_1, \ldots, x_m]\!] \) satisfying two requirements: 1) the quotient algebra \( \mathbb{C}[\![x_1, \ldots, x_m]\!]/I \) is isomorphic to \( Q \), and 2) the ideal generated by the components \( f_j \) of the germ \( f \)
is contained in $I$. The number of ideals satisfying these requirements is called the \textit{multiplicity} of the singularity $Q$ at the germ $f$.

By a \textit{contact singularity type} we mean now an isomorphism class of algebras or an (irreducible) algebraic family of local algebras of fixed dimension.

The \textit{Thom polynomial of a contact singularity} is the one associated with the closure of map germs of prescribed contact singularity type, multiplied by the multiplicity of its generic representative.

If $\ell$ is big enough (if $m + \ell$ is bigger than the minimal number of generators of an ideal $I \subset \mathbb{C}[x_1, \ldots, x_m]$ with the quotient algebra isomorphic to $Q$) then for a generic germ with all components in $I$ the components do generate the ideal $I$. In this case a generic germ with the singularity $Q$ has local algebra isomorphic to $Q$ and the multiplicity is equal to 1. In other words, a special account is required for the case of ‘small $\ell$’ only.

It is useful to note that the contact singularity class associated with any finite dimensional local algebra $Q$ is finitely determined, namely, it is at least $\mu$-determined where $\mu = \dim Q - 1$. This is a straightforward consequence of the fact that any ideal of finite codimension $\mu + 1$ in the local coordinate ring $\mathbb{C}[[x_1, \ldots, x_m]]$ contains the $(\mu + 1)$st power of the maximal ideal. It follows that the condition on a given map germ to have a contact singularity of type $Q$ is uniquely determined by the $\mu$-jet of the germ and that the corresponding Thom polynomial is independent of the order $K$ of jet space used in its definition provided that $K \geq \mu$.

The Thom polynomial of a contact singularity is expressed actually as polynomial in the \textit{quotient variables} $c_i$ defined by

$$1 + c_1 + c_2 + \cdots = \frac{1 + c_1' + \cdots + c_m''}{1 + c_1' + \cdots + c_m'}$$

For a given holomorphic map $f : X \to Y$ the quotient variables specialize to the relative Chern classes $c_i = c_i(f) = c_i(f^*TY - TX)$. Moreover, the Thom polynomial is independent of the dimensions $m$ and $n$ of the source and the target of the map provided that the relative dimension $\ell = n - m$ is fixed. This fact has a simple a priori explanation, see [25, 4, 15]. On the other hand, it is accounted automatically in the main formula of the next section for Thom polynomials of contact singularities.

\section{Main theorem}

It will be more convenient for us to encode contact singularities by isomorphism classes of finite dimensional \textit{nilpotent algebras}. The nilpotent algebra $N_f$ of a map germ $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ is the maximal ideal in the corresponding local algebra,

$$N_f = \frac{m_{x_1, \ldots, x_m}}{(f_1, \ldots, f_n)},$$

where $m_{x_1, \ldots, x_m}$ is the maximal ideal in the local coordinate ring $\mathbb{C}[[x_1, \ldots, x_n]]$. The nilpotent algebra carries the same information as the local one; the algebra $Q_f$ can be reconstructed from $N_f$ by just adding unity. In particular, $\dim Q_f = \dim N_f + 1$.

Let $N$ be a finite dimensional nilpotent algebra. Choose a filtration on it, that is, a finite decreasing sequence of subalgebras

$$N = N_1 \supset N_2 \supset \cdots \supset N_{r+1} = 0$$
satisfying $N_i \cdot N_j \subset N_{i+j}$. The dimension of the subsequent quotients of this filtration is denoted by $d_k = \dim(N_k/N_{k+1})$. The sequence $\underline{d} = (d_1, \ldots, d_r)$ is called the dimension vector. Set also

$$\mu = d_1 + \cdots + d_r = \dim N.$$

A filtration on the nilpotent algebra always exists but it is usually not unique. A formula for the Thom polynomial presented below depends on a choice of the filtration (although the Thom polynomial itself is obviously independent of this choice). A particular filtration is given by the powers of the maximal ideal, $N_k = N^k$. In this case all components $d_i$ of the dimension vector are strictly positive. But in general, we allow for some components of the dimension vector to be vanishing. (I am grateful to R. Rimanyi who payed my attention to the possibility of using filtrations different from those formed by powers of the nilpotent algebra).

For an integer $i$ such that $d_1 + \cdots + d_{k-1} < i \leq d_1 + \cdots + d_k$, we define its weight to be $w(i) = k$ and the exponent to be $e(i) = d_1 + \cdots + d_k - i$. The weights and the exponents of the numbers $1, 2, \ldots, \mu$ form two sequences

$$w = (1, \ldots, 1, 2, \ldots, 2, \ldots, r, \ldots, r),$$

$$e = (d_1 - 1, \ldots, 1, 0, d_2 - 1, \ldots, 1, 0, \ldots, d_r - 1, \ldots, 1, 0).$$

Introduce the following rational function determined uniquely by the dimension vector

$$K_{\underline{d}}(t_1, \ldots, t_\mu) = \frac{\prod_{i=1}^\mu e(i)}{\prod_{1 \leq i < j \leq \mu} (t_j - t_i)} \prod_{1 \leq i \leq j \leq \mu \leq w(k)} (t_k - t_i - t_j).$$

(1)

Consider any contact singularity type $\eta$ given as a unique isomorphism class of nilpotent algebras or given by an irreducible algebraic family of nilpotent algebras with fixed dimension vector $\underline{d}$.

**Theorem 3.1.** The Thom polynomial of the singularity $\eta$ with the given dimension vector $\underline{d} = (d_1, \ldots, d_r)$ can be represented in the form of a series

$$T_p \eta = \sum_{i_1, \ldots, i_\mu} \psi_{i_1, \ldots, i_\mu} c_{\ell + 1 + i_1} \cdots c_{\ell + 1 + i_\mu},$$

(3)

in the quotient variables (the relative Chern classes) $c_1, c_2, \ldots$ whose coefficients $\psi_{i_1, \ldots, i_\mu}$ are independent of $\ell$ and are determined by the following rational generating function

$$\sum_{i_1, \ldots, i_\mu} \psi_{i_1, \ldots, i_\mu} t_1^{i_1} \cdots t_\mu^{i_\mu} = K_{\underline{d}}(t_1, \ldots, t_\mu) \cdot P_{\eta}(t_1, \ldots, t_\mu),$$

where $P_{\eta}$ is a polynomial defined below in Sect. 4.

By the power series expansion of a rational function we mean its expansion in the domain $t_1 \ll t_2 \ll \cdots \ll t_\mu$. It means the following. First, we expand the function as a Laurent series in $t_\mu^{-1}$ whose coefficients are rational functions in $t_1, \ldots, t_\mu$. Then we expand each of the coefficients as Laurent series in $t_\mu^{-1}$, and so on.
Equivalently, the relation of Theorem 3.1 can be represented formally in the form of iterated residue
\[ \text{res}_{t_1=\infty} \ldots \text{res}_{t_\mu=\infty} K_d(t) \cdot P_\eta(t) \prod_{i=1}^\mu \left( -C(1/t_i) t_i^\ell dt_i \right), \quad C(1/t) = \sum_{k=0}^\infty c_k t^{-k}. \]

The actual definition of the polynomial \( P_\eta \) is discussed in the next section. Even without knowing explicitly its coefficients we can formulate some general conclusions resulting from its existence. The fact that the Thom polynomial can be represented in a form of a series (3) was observed first by L. Fehér and R. Rimányi in [6]. Remark that the whole series is usually infinite but it has only finitely many nonzero terms for each particular value of \( \ell \). Every monomial in the Thom polynomial has homogeneous degree at most \( \mu \), and the change of the relative dimension \( \ell \) is accounted by a shift of the indices in the Chern classes. It follows that the codimension of the singularity \( \eta \) satisfies
\[ \text{codim} \eta = (\ell + 1) \mu + d \]
where \( d \) is independent of \( \ell \). We can regard \( d \) as the ‘virtual codimension’ of the singularity \( \eta \) in the case of the relative dimension \( \ell = -1 \).

It was proved in [7] that the whole Thom series can be formally reconstructed from finitely many its terms. The statement of the theorem makes this reconstruction procedure more explicit: we see that the series is uniquely determined by finitely many coefficients of the polynomial \( P_\eta \).

The language of generating series in presentation of characteristic classes was introduced in [17] to simplify some formal manipulations with them. For example, it allows one to pass easily from the basis of Chern monomials to the basis consisting of Schur functions. For any sequence of integers \( \lambda = (\lambda_1, \ldots, \lambda_\mu) \) we set
\[ \Delta_{\lambda_1, \ldots, \lambda_\mu} = \det |c_{\lambda_i-i+j}|_{1 \leq i,j \leq \mu}. \]
If the sequence \( \lambda \) is non-increasing, then this is the Schur function associated with the partition \( \lambda \). For any sequence, not necessarily non-increasing, this determinant is either zero or is equal to some Schur function, up to a sign. In terms of generating series passing from the basis of Chern monomials to the basis of Schur functions is accounted by a simple factor, see [17] for the proof.

**Corollary 3.2.** The Thom polynomial admits a Schur function expansion
\[ T_{P_\eta} = \sum \sum \tilde{\psi}_{i_1, \ldots, i_\mu} \Delta_{\ell+1+i_1, \ldots, \ell+1+i_\mu} \]
with the following generating function for its coefficients
\[ \sum \tilde{\psi}_{i_1, \ldots, i_\mu} t_1^{i_1} \ldots t_\mu^{i_\mu} = \tilde{K}_d(t_1, \ldots, t_\mu) \cdot P_\eta(t_1, \ldots, t_\mu), \]
where \( P \) is the same as in Theorem 3.1 and
\[ \tilde{K}_d = \frac{K_d}{\prod_{1 \leq i<j \leq \mu} (1 - t_i/t_j)} = \frac{\prod_{i=1}^\mu \epsilon(i)+i-1}{\prod_{1 \leq i<j \leq \mu} (t_k - t_i - t_j)}, \quad (4) \]
4 Definition of the polynomial \( P_\eta \) of Theorem 3.1

The polynomial \( P_\eta \) participating in Theorem 3.1 is the product of two factors,

\[
P_\eta = P'_\eta \cdot P''_\eta.
\]

We call these factors the local characteristic invariant of the singularity \( \eta \) and the normalizing factor, respectively. The definition of these factors is given below.

4.1 Associativity equation

Consider a flag of finite dimensional vector spaces

\[
N = N_1 \supset N_2 \supset \cdots \supset N_{r+1} = 0, \quad \dim N_k/N_{k+1} = d_k.
\]

A filtered commutative algebra structure on \( N \) is a linear mapping

\[
\psi : \text{Sym}^2 N \to N
\]

such that \( \psi(N_k \otimes N_m) \subset N_{k+m} \). The filtered commutative algebra structures on \( N \) form a vector space that we denote by \( \text{Alg}(\mathbf{d}) = \text{Alg}(d_1, \ldots, d_r) \).

Set \( \mu = \dim N = \sum_{i=1}^{r} d_i \) and choose a basis \( e_1, \ldots, e_\mu \) in \( N \) such that the vectors \( e_{d_1+\cdots+d_{k-1}+1}, \ldots, e_{d_1+\cdots+d_{k-1}+d_k} \) generate \( N_k/N_{k+1} \). Then the multiplication law is given by

\[
e_i \cdot e_j = \psi(e_i, e_j) = \sum_{k, w(k) \geq w(i) + w(j)} q^k_{i,j} e_k,
\]

where the weight sequence \( (w(1), \ldots, w(\mu)) \) is determined by the dimension vector \( (d_1, \ldots, d_r) \) as in (1). The coefficients

\[
q^k_{i,j} = q^k_{j,i}, \quad w(k) \geq w(i) + w(j),
\]

form the coordinate system in the vector space \( \text{Alg}(\mathbf{d}) \).

The multiplication operation does not satisfy automatically the associativity law. Denote by \( \text{Hilb}^{\text{loc}}_d \subset \text{Alg}(\mathbf{d}) \) the subvariety consisting of associative algebras. It is easy to see that it is algebraic. Moreover, its ideal is generated by quadratic equations of the form

\[
\sum_m q^m_{i,j} q^m_{i,k} - \sum_m q^m_{i,m} q^m_{j,k} = 0 \tag{5}
\]

for all quadruples \( (i, j, k, n) \) with \( w(i) + w(j) + w(k) \leq w(n) \). This identity expresses the equality of the \( n \)-components of the products \( (e_i \cdot e_j) \cdot e_k \) and \( e_i \cdot (e_j \cdot e_k) \).

Furthermore, if we are given a singularity type \( \eta \), denote by \( \text{Hilb}^{\text{loc}}_\eta \subset \text{Hilb}^{\text{loc}}_d \subset \text{Alg}(\mathbf{d}) \) the closure of the locus formed by the algebras isomorphic to one of the nilpotent algebras defining \( \eta \). Both subvarieties are invariant with respect to the natural action of the group \( \text{GL}(\mathbf{d}) \) of linear transformations of \( N \) preserving the flag \( N_\bullet \). This group is homotopy equivalent to the group \( \text{GL}(d_1) \times \cdots \times \text{GL}(d_r) \) and its ring of characteristic classes is the polynomial ring generated by the Chern classes of the \( \text{GL}(d_i) \)-invariant vector spaces \( N_i/N_{i+1}, \ i = 1, \ldots, r \) (this is true both in the topological and in the algebraic context). The maximal torus \( (\mathbb{C}^*)^\mu \subset \text{GL}(\mathbf{d}) \) acts on \( \text{Alg}(\mathbf{d}) \) by rescaling of the basic vectors \( e_i \) in \( N \). The multidegree of the coordinate \( q^k_{i,j} \) with respect to this torus action is equal to

\[
\deg q^k_{i,j} = t_k - t_i - t_j.
\]
Remark that the denominator of the function $K_\mathcal{A}(\mathcal{d})$ is nothing but the Euler class of $\text{Alg}(\mathcal{d})$, that is, the product of multidegrees of all its coordinates.

**Definition 4.1.** The local characteristic invariant $P'_\eta(t_1, \ldots, t_\mu)$ of the singularity $\eta$ is defined to be the multidegree of the subvariety $\text{Hilb}_\eta^{\text{loc}} \subset \text{Alg}(\mathcal{d})$, that is, its equivariant Poincaré dual cohomology class with respect to the natural action of the group $\text{GL}(d) \sim \prod_{i=1}^r \text{GL}(d_i) = \prod_{i=1}^r \text{GL}(N_i/N_{i+1})$ and expressed in terms of the Chern roots of that action.

**Example 4.2.** If $r \leq 2$, then the associativity equation is empty and we have $\text{Hilb}_\eta^{\text{loc}} = \text{Alg}(\mathcal{d})$ i.e. $P'_\eta = \text{Hilb}_\eta^{\text{loc}} = 1$ if the chosen singularity class consists of all algebras with the given dimension vector. The same is true if $r = 3$ and the dimension vector is $\mathcal{d} = (1, 1, 1)$.

In the case $r = 4$ and for the dimension vector $\mathcal{d} = (1, 1, 1, 1)$ we obtain a nontrivial associativity restriction

$$q^2_{1,1} \cdot q^4_{2,2} - q^4_{1,3} \cdot q^3_{1,2} = 0.$$  

Generic algebras from the hypersurface defined by this equation is isomorphic to $A_4 = \frac{\mathbb{C}[[x]]}{(x^4)}$. The multidegree of this hypersurface is equal to

$$P'_{A_4}(t_1, \ldots, t_4) = t_4 - t_2 - 2t_1.$$  

In more complicated cases the subvariety $\text{Hilb}_\eta^{\text{loc}} \subset \text{Alg}(\mathcal{d})$ is not a complete intersection and the polynomial $P'_\eta$ may contain nonlinear factors. The way of its computation depends on a particular choice of $\eta$.

It would be more convenient for us to regard below the algebra structure on $N$ as the coalgebra structure on the dual space $N^\vee$ defined by the adjoint filtered morphism $\psi^\vee : N^\vee \to \text{Sym}^2 N^\vee$. The space $N^\vee$ is equipped with the natural filtration

$$0 = D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_r = N^\vee, \quad D_k = \text{ann}(N_{k+1}) = (N/N_{k+1})^\vee,$$

and the comultiplication satisfies $\psi^\vee(D_k) \subset S_k$ where $S_k \subset \text{Sym}^2 N^\vee$ is spanned by the tensors $a \otimes b, a \in D_i, b \in D_j$, with $i+j \leq k$.

The subvariety $\text{Hilb}_\eta^{\text{loc}} \subset \text{Alg}(\mathcal{d})$ is determined by the associativity equation for this coalgebra and the subvariety $\text{Hilb}_\eta^{\text{loc}} \subset \text{Alg}(\mathcal{d})$ is identified by the condition that this coalgebra is isomorphic to the coalgebra induced on the dual space $N_f^\vee$ to the nilpotent algebra of one of the map germs $f$ representing a given singularity type.

### 4.2 Normalizing factor

**Definition 4.3.** A filtration on a given nilpotent algebra $N$ is called natural if any automorphism of the algebra preserves also the filtration.

For example, the filtration formed by the powers of the algebra is natural. There could be, however, different filtrations, both natural and not. For a natural filtration we set $P'_{\eta} = 1$.

If the filtration is not natural then there should exist a flag on $N$ forming a filtration different from the original one such that the algebra $N$ with this filtration is isomorphic to the original one as a filtered nilpotent algebra. Denote by $F_{\eta}$ the closure of all flags with this property in the flag variety in $N$.

**Definition 4.4.** If the singularity $\eta$ is defined by a unique isomorphism class of nilpotent algebras, we define $P''_{\eta}(t_1, \ldots, t_\mu)$ as an arbitrary polynomial combination of Chern classes of the
tautological vector bundles over the flag variety satisfying

\[ \int_{F_\eta} P_\eta^\mu(t_1, \ldots, t_\mu) = 1 \]

and expressed in terms of the Chern roots of these bundles. If \( \eta \) is given by a family of algebras, we apply the same definition to a generic member of the family.

By Chern roots we mean the formal substitution

\[ c(N_1/N_2) = \prod_{i=1}^{d_1} (1 + t_i), \ldots, c(N_r/N_{r+1}) = \prod_{i=d_1+\ldots+d_{r-1}+1} (1 + t_i). \]

A polynomial \( P''_\eta \) satisfying this property does exist since any nonempty algebraic subvariety in the flag variety has a nonzero homology fundamental class. Moreover, there is a big freedom in a choice of this polynomial: possible choices of \( P''_\eta \) form an affine hyperplane in the space of monomials of a fixed degree. This freedom can be used, for example, to achieve a cancelation with some factors in the denominator in the generating function of Theorem 3.1.

5 Hilbert scheme and partial resolution of the singularity locus

Let \( E \) and \( F \) be finite dimensional vector spaces. The given singularity type \( \eta \) constitutes a locus in the jet space of map germs between \( E \) and \( F \):

\[ \eta \subset J^K_0(E, F) \simeq \text{Hom} \left( \bigoplus_{i=1}^{K} \text{Sym}^i E, F \right) = (m/m^{K+1}) \otimes F \]

where \( m = m_E \) is the maximal ideal in the ring of function germs on \( E \) at the origin. We need to compute the cohomology class \( T_{p_\eta} = [\eta] \) represented by this locus in the \( (\text{GL}(E) \times \text{GL}(F)) \)-equivariant cohomology of the jet space. A step in the direction of this computation can be done using the following partial resolution of \( \eta \) the idea of which goes back to at least J. Damon [4].

Denote by \( \text{Hilb}_d(E) \) the variety parametrizing flags of ideals

\[ m = I_1 \supset I_2 \supset \cdots \supset I_{r+1} \]

such that \( \dim I_k/I_{k+1} = d_k \) for \( k = 1, \ldots, r \) and \( I_i \cdot I_j \subset I_{i+j} \) for all \( i \) and \( j \) with \( i + j \leq r + 1 \). We always have \( m^{r+1} \subset I_{r+1} \), therefore, the variety \( \text{Hilb}_d(E) \) can be embedded to an appropriate variety of flags in the finite dimensional vector space \( m/m^{K+1} \) for any \( K \geq r + 1 \). The variety \( \text{Hilb}_d(E) \) is equipped with the canonical subbundle \( I = I_{r+1}/m^{K+1} \) of the trivial vector bundle \( (m/m^{K+1}) \times \text{Hilb}_d(E) \rightarrow \text{Hilb}_d(E) \), the quotient bundle \( N = m/I_{r+1} \) of rank \( \mu \), and its dual that we denote by \( D = N^\vee \).

Denote by \( \text{Hilb}_\eta(E) \subset \text{Hilb}_d(E) \) the closure of the subvariety formed by flags of ideals with prescribed isomorphism types of the quotient filtered algebras. Both \( \text{Hilb}_\eta(E) \) and \( \text{Hilb}_d(E) \) are usually singular. Consider an embedding of \( \text{Hilb}_d(E) \) to some compact nonsingular variety \( M \) such that the action of \( \text{GL}(E) \) and the bundle \( D \) extend to \( M \).

**Proposition 5.1.** The Thom polynomial is given by the formula

\[ T_{p_\eta} = p_* (P'_\eta P''_\eta c_{\text{top}}(D, F)), \quad p : M \rightarrow \text{pt}, \quad (6) \]
where \( p_* \) is the push-forward, or Gysin homomorphism associated with the projection \( p \), \( P''_\eta \) is the cohomology class represented by the subvariety \( \text{Hilb}_\eta(E) \subset M \), \( P''_\eta \) is the normalizing factor of Definition 4.3, and \( c_{\text{top}} \) is the Euler class of the corresponding bundle.

The result is actually independent of the choice of \( M \). The traditional embedding space for the Hilbert scheme is the Grassmann (or flag) variety of subspaces of fixed dimension in the vector space \( \mathfrak{m}/\mathfrak{m}^{K+1} \) of jets of functions on \( E \). The main geometric idea of our approach is a construction of a new embedding space for the Hilbert scheme that we call the nonassociative Hilbert scheme. The advantage of this construction is that all ingredients of (6) in this case can be identified and computed explicitly.

**Proof.** Consider the variety parametrizing pairs of the form \((I,f)\) where \( I \subset (\mathfrak{m}/\mathfrak{m}^{K+1}) \) is an ideal of codimension \( \mu \) whose quotient algebra is isomorphic to one of the algebras defining the singularity \( \eta \), and \( f \) is the \( K \)-jet of a map whose all components \( f_i \) belong to \( I \). Denote by \( \overline{\eta} \) the closure of this variety in \( \text{Hilb}^{(\mu)}(E) \times J_0^K(E,F) \), where \( \text{Hilb}^{(\mu)}(E) \) is the punctual Hilbert scheme parametrizing all \( \mu \)-codimensional ideals in \( \mathfrak{m}/\mathfrak{m}^{K+1} \). The projection \( p_2 : \text{Hilb}^{(\mu)}(E) \times J_0^K(E,F) \to J_0^K(E,F) \) to the second factor is proper and the details in the discussion of Definition 2.7 show that we have the equality of classes of rational equivalence of subvarieties

\[
T_{p_\eta} \cap [J_0^K(E,F)] = p_{2*}(\overline{\eta}). \tag{7}\]

By definition, \( \overline{\eta} \) is the total space of a vector subbundle in the trivial bundle \( J_0^K(E,F) \times \text{Hilb}^{(\mu)}(E) \to \text{Hilb}^{(\mu)}(E) \) restricted to the subvariety in \( \text{Hilb}^{(\mu)}(E) \) formed by algebras of the given isomorphism type. The Euler class of the quotient bundle of this subbundle is equal to \( c_{\text{top}}(\text{Hom}(D,F)) \) where \( D = (\mathfrak{m}/I)^{\vee} \) is the dual of the canonical quotient bundle over the Hilbert scheme. The base of the bundle is of the form \( \pi(\text{Hilb}_\eta(E)) \) where \( \pi : \text{Hilb}_\eta(E) \to \text{Hilb}^{(\mu)} \) is the forgetful map that ignores the filtration on the quotient nilpotent algebra. Moreover, by the definition of the normalizing factor, we have

\[
[\pi(\text{Hilb}_\eta(E))] = \pi_* (P''_\eta \cap [\text{Hilb}_\eta(E)]).
\]

Therefore,

\[
[\overline{\eta}] = (\pi \times \text{Id})_* (P''_\eta c_{\text{top}}(\text{Hom}(D,F)) \cap [\text{Hilb}_\eta(E) \times J_0^K(E,F)]). \tag{8}\]

Combining (7) and (8) we obtain finally

\[
T_{p_\eta} \cap [J_0^K(E,F)] = \tilde{p}_{2*} (P''_\eta c_{\text{top}}(\text{Hom}(D,F)) \cap [\text{Hilb}_\eta(E) \times J_0^K(E,F)])
\]

where \( \tilde{p}_2 = p_2 \circ (\pi \times \text{Id}) : \text{Hilb}_\eta(E) \times J_0^K(E,F) \to J_0^K(E,F) \) is the projection to the second factor.

A cartesian product with a vector space induces isomorphisms both in cohomology and in the Chow groups. Therefore, the obtained equality is equivalent to the one of Proposition 5.1. \( \square \)

### 6 The nonassociative Hilbert scheme

Denote by \( \mathfrak{m} \) as above the maximal ideal in the ring of function germs at the origin of a vector space \( E \) (or \( K \)-jets of functions for some \( K \gg 0 \)). Let \( I \subset \mathfrak{m} \) be an ideal of finite codimension \( \mu \). Denote by \( N = \mathfrak{m}/I \) the corresponding nilpotent quotient algebra. Then we have two natural linear maps

\[
\psi_1 : E^{\vee} \to N, \quad \psi_2 : \text{Sym}^2 N \to N.
\]
The first one is the restriction of the natural projection $\mathfrak{m} \to \mathfrak{m}/I = N$ to the subspace $E^r \subset \mathfrak{m}$ consisting of linear functions. The second one is the structure morphism for the multiplication law in the algebra $N$. The following observation is obvious: the induced linear map $\psi_1 \oplus \psi_2 : E^r \oplus \text{Sym}^2 N \to N$ is surjective.

Indeed, consider a polynomial representing any element of $N^r$. This polynomial can be represented as a sum of linear terms which are in the image of $\psi_1$ and terms of order greater or equal to two which are in the image of $\psi_2$.

Conversely, let $N$ be a $\mu$-dimensional commutative associative nilpotent algebra with the structure morphism $\psi_2 : \text{Sym}^2 N \to N$, and $\psi_1 : E^r \to N$ an arbitrary linear map such that $\psi_1 \oplus \psi_2$ is surjective. Then $\psi_1$ extends uniquely to an algebra homomorphism $\tilde{\psi}_1 : \mathfrak{m} \to N$ for $E^r$ generates $\mathfrak{m}$. The homomorphism $\tilde{\psi}_1$ is surjective since $\psi_1 \oplus \psi_2$ is surjective. It follows that $N$ can be identified with $\mathfrak{m}/I$ where $I = \ker(\tilde{\psi}_1)$. We arrive at the following conclusion.

**Lemma 6.1.** There is a one-to-one correspondence between the set of $\mu$-codimensional ideals in $\mathfrak{m}$ and the set of isomorphism classes of pairs $(\psi_1, \psi_2)$ where $\psi_2 : \text{Sym}^2 N \to N$ is an associative commutative nilpotent algebra structure on a $\mu$-dimensional vector space $N$ and $\psi_1 : E^r \to N$ is a linear map such that $\psi_1 \oplus \psi_2$ is surjective. $\Box$

For the next definition we assume that the nilpotent algebra structure that we construct on a given vector space $N$ is compatible with a given filtration

$$N = N_1 \supset N_2 \supset \cdots \supset N_{r+1} = 0, \quad \dim N_k/N_{k+1} = d_k.$$  

Demote by $\tilde{M}_r \subset \text{Hom}(E^r \oplus \text{Sym}^2 N, N)$ the subset determined by the following two conditions: 1) the second component $\psi_2$ of $\psi_1 \oplus \psi_2 \in \tilde{M}_r$ satisfies $\psi_2(N_i \otimes N_j) \subset N_{i+j}$ for all $i$ and $j$; 2) the map $\psi_1 \oplus \psi_2$ is surjective. The first condition determines a vector subspace in $\text{Hom}(E^r \oplus \text{Sym}^2 N, N)$. The second one determines a Zariski open subset in that subspace. The group $\text{GL}(d)$ of preserving the flag $N_i$ linear transformations of $N$ acts naturally on $\tilde{M}_r$.

**Definition 6.2.** The nonassociative Hilbert scheme is the orbit space $M_r$ of the action of $\text{GL}(d)$ on $\tilde{M}_r$.

**Proposition 6.3.** The action of $\text{GL}(d)$ on $\tilde{M}_r$ is free. The variety $M_r$ is smooth and compact.

For the proof of this proposition we provide an independent construction of $M_r$ which is equivalent to that used in Definition 6.2. It is more convenient to use in this construction the dual coalgebra structure on $D = N^r$. Namely, we construct $M_r$ as the moduli space of flags

$$0 = D_0 \subset D_1 \subset \cdots \subset D_r = D, \quad \dim(D_k/D_{k-1}) = d_k,$$

equipped with an injective linear map $D \to E \oplus \text{Sym}^2 D$ such that $D_k \subset E \oplus S_k$ for $k = 1, \ldots, r$, where

$$S_k \subset \text{Sym}^2 D_{k-1} \subset \text{Sym}^2 D$$

is generated by subspaces of the form $D_i \otimes D_j$ with $i + j \leq k$.

The construction goes by induction in $r$. In the case $r = 1$ we have $S_1 = 0$, therefore, $D_1$ must be embedded to $E$. We define $M_1 = G_{d_1}(E)$, the Grassmann manifold with the tautological rank $d_1$ bundle $D_1$ over it.

Assume that the variety $M_{r-1}$ is already constructed with the corresponding tautological flag of bundles $D_1 \subset \cdots \subset D_{r-1}$ over it and with the embedding of subbundles $D_{r-1} \subset E \oplus S_{r-1}$. Remark that $S_r$ is determined by $D_1, \ldots, D_{r-1}$, therefore, $S_r$ can be regarded as a bundle over
$M_{r-1}$. The manifold $M_r$ should parameterize subspaces $D_r$ in $E \oplus S_r$ containing the subspaces $D_{r-1}$ constructed on the previous step. According to this, we define $M_r = G_d((E \oplus S_r)/D_{r-1})$, the corresponding Grassmann bundle over $M_{r-1}$.

By construction, $M_r$ is defined as the total space of a tower of fibrations with smooth compact fibers:

$$
M_r \xrightarrow{G_d(E_d/D_{r-1})} M_{r-1} \xrightarrow{G_d-1(E_{r-1}/D_{r-2})} \ldots \xrightarrow{G_1(E)} \text{pt}, \quad E_k = E \oplus S_k.
$$

This implies Proposition 6.3.

The just constructed manifold $M_r$ is equipped with the diagram of bundles and subbundles

$$
\begin{array}{cccc}
D_1 & \rightarrow & D_2 & \rightarrow & \cdots & \rightarrow & D_r \\
\downarrow & & \downarrow & & \cdots & & \downarrow \\
E = E_1 & \rightarrow & E_2 & \rightarrow & \cdots & \rightarrow & E_r, & E_k = E \oplus S_k.
\end{array}
$$

The projection of the subbundle $D_r \subset E \oplus S_k$ to the summands $E$ and $S_r$ determines a linear map $D_r \rightarrow E$ and a canonical filtered commutative coalgebra structure $D_r \rightarrow S_r$ on the fibers of $D_r$.

The dual bundle $N_r = D_r^\vee$ equipped, respectively, with a surjective bundle map $E^\vee \oplus S_r^\vee \rightarrow N_r$ defining a canonical linear map $E^\vee \rightarrow N_r$ and a canonical commutative filtered algebra structure on the fibers of $N_r$. The following assertion is a reformulation of Lemma 6.2.

**Proposition 6.4.** The Hilbert scheme $\text{Hilb}_d(E)$ is isomorphic to the sublocus in $M_r$ consisting of points for which the canonical commutative filtered algebra on the corresponding fiber of $N_r$ is associative. Its subvariety $\text{Hilb}_d(E)$ consists of the points of $M_r$ for which the canonical algebra on the corresponding fiber of $N_r$ is isomorphic to one of the algebras defining the singularity type $\eta$. □

7 The final computation of the Thom polynomial

We are able now to complete the computation of Thom polynomial applying Proposition 6.4 to the projection

$$p : M_r \rightarrow \text{pt} \quad (9)$$

where $M_r$ is the nonassociative Hilbert scheme constructed in the previous section. By proposition 6.4 the polynomial $P^\eta_{\ell}(t_1, \ldots, t_\mu)$ representing Poincaré dual of the locus $\text{Hilb}_d(E) \subset M_r$ is the local characteristic invariant of Definition 4.1 expressed in terms of the Chern roots $-t_1, \ldots, -t_\mu$ of the bundles $D_1/D_0$, $\ldots$, $D_r/D_{r-1}$ over $M_r$. It remains to compute the homomorphism $p_*$. Various alternative methods are known for the computation of the Gysin homomorphism. We prefer to use the machinery developed in [17] by two reasons. First, with this approach the formulas become quite simple. Second, this approach allows one to keep the Chern classes $c(E)$ and $c(F)$ in the relative combination $c(F-E)$ in all intermediate steps of computations. Here is the basic relation from [17].

**Proposition 7.1.** Let $E$ and $F$ be vector bundles of relative rank $\ell = \text{rk}F - \text{rk}E$ defined over a smooth base $X$. Denote by $D$ the canonical rank $d$ bundle over the total space of the Grassmann fibration $q : G_d(E) \rightarrow X$. Then the Gysin homomorphism $q_*$ acts by the following formula:

$$q_* : P(t_1, \ldots, t_d) c_{\text{top}}(\text{Hom}(D, F)) \mapsto P(t_1, \ldots, t_d) t_1^{\ell+d} t_2^{\ell+d-1} \cdots t_d^{\ell+1} \prod_{1 \leq i < j \leq d} (t_i - t_j).$$
Recall that $q_r$ is the Grassmann fibration $G_{d_r}/(E_r/D_{r-1})$ over $M_{r-1}$ where $E_r = E \oplus S_r$. By Proposition 7.1 in order to apply $q_{rs}$ we need to replace the factor $c_{top}(\text{Hom}(D_r/D_{r-1}))$ by the factor

$$s_{r}^{\ell_r + d_r} s_{2}^{d_r - 1} \cdots s_{d_r} s_{r-1} \prod_{1 \leq i < j \leq d_r} (s_i - s_j)$$

(10)

where we denote temporarily $s_i = t_{d_1 + \cdots + d_{r-1} + i}$ and $\ell_r = \text{rk} F - \text{rk} E_r/D_{r-1} = \ell + \text{rk} D_{r-1} - \text{rk} S_r$. The product of (10) with $P_{\eta} c_{top}(\text{Hom}(D_{r-1}, F))$ represents the class $q_{rs}(P_{\eta} c_{top}(\text{Hom}(D_r, F)))$ where $s_m^k$ in the monomial expansion denotes the class

$$c_k(F-E_r/D_{r-1}) = c_k(F-E + D_{r-1} - S_r).$$

We prefer to have an expression where $s_m^k$ would represent the class $c_k = c_k(F-E)$. By the Whitney formula we need to replace every monomial $s_m^k$ by the expression

$$s_m^k \sim s_m \sum_{j \geq 0} c_j (D_{r-1} - S_r) s_m^{-j} = s_m \frac{\prod_{i=1}^{d_1 + \cdots + d_{r-1}} (1 - t_i/s_m)}{\prod_{\lambda} (1 - \frac{t_\lambda}{s_m})}$$

where $\lambda$ runs over the list of Chern roots of the bundle $S_r$ expressed in terms of the Chern roots of $D_k$: $t_\lambda = t_i + t_j, 1 \leq i < j, w(i) + w(j) \leq r$. We see that passing from the presentation in terms of the Chern classes $c_k(F-E_r/D_{r-1})$ to the Chern classes in $c_k = c_k(F-E)$ is accounted by a factor. Multiplying all these factors for $m = 1, \ldots, d_r$ and multiplying by (10) we obtain that the actual action of $q_{rs}$ is correctly represented by the factor

$$\prod_{i=1}^{d_r} s_i^{\ell_i + d_i - 1} \prod_{j=1}^{d_r} s_j (s_j - t_i) \prod_{1 \leq i < j \leq d_r} (s_j - s_i) \prod_{k=1}^{d_r} s_k (s_k - t_\lambda)$$

where $s_i$ still denotes $t_{d_1 + \cdots + d_{r-1} + i}$. The next homomorphism $(q_{r-1})_*$ introduces another similar factor, and so on. Multiplying all these factors corresponding to $q_{rs}, \ldots, q_1$, we arrive at the function of Theorem 3.1 up to the factor $\prod_{i=1}^{\mu} t_i^{\ell_i + 1}$ which is accounted by the shift of the indices of the Chern classes $c_i$ in Eq. 3. Theorem 3.1 is proved. \[\Box\]
8 Examples and computations

In this Section we present some explicit computations based on the formula of Theorem 3.1. We use traditional notations for singularity types corresponding to isomorphism classes of nilpotent algebras, see [1]. An algebra can be characterized by a collection of generators of the ideal whose quotient is isomorphic to the given algebra. However, it is more demonstrative to use a different convention identifying algebras by their monomial bases. Namely, let a list of monomials in triangular brackets denote the quotient algebra over the monomial ideal spanned by the monomials which are not present in the list. Besides, we denote by \( m_{x,y,...} \) the maximal ideal in the local ring \( \mathbb{C}[[x, y, ...]] \) of infinite power series in the variables \( x, y, ... \). The discussion of this section involves the following nilpotent algebras:

| Notation | Algebra | dimension |
|----------|---------|-----------|
| \( \Sigma^\mu \) | \( \langle x_1, \ldots, x_\mu \rangle = \frac{m_{x_1 \ldots x_\mu}}{m_x} \) | \( \mu \) |
| \( A^\mu \) | \( \langle x, x^2, \ldots, x^\mu \rangle = \frac{m_x}{(x^\mu + 1)} \) | \( \mu \) |
| \( I_{a,b} \) | \( \langle x, x^2, \ldots, x^a, y, y^2, \ldots, y^b \rangle = \frac{m_{x,y}}{(x^a + x^b)} \) | \( a + b - 1 \) |
| \( II_{a,b} \) | \( \langle x, x^2, \ldots, x^{a-1}, y, y^2, \ldots, y^{b-1} \rangle = \frac{m_{x,y}}{(x^a + y^b)} \) | \( a + b - 2 \) |
| \( \Phi_{m,r} \) | \( \left\langle \frac{y_j^2 - y_k^2}{y_j^2 - y_k^2}, 1 \leq i \leq r, 1 \leq j, k \leq m - r \right\rangle \) | \( m + 1 \) |
| \( \Sigma^{a,b} \) | \( \langle x_i, y_k, x_i x_j, x_i y_k \rangle, 1 \leq i, j \leq b, 1 \leq k \leq a - b \) | \( \mu(a,b) \) |
| \( \Sigma^{j_1, \ldots, j_m} \) | \( \langle x_{i_1}, \ldots, x_{i_r} \rangle, 1 \leq r \leq m, i_1 \geq \cdots \geq i_r \geq 1, \mu(j_1, \ldots, j_m) \) | \( \mu(j_1, \ldots, j_m) \) |
| \( C_\lambda \) | \( \left\langle \frac{x, y, z, x^2, y^2, z^2, xy, xz, yz}{(y^2 + 2xz, 2yz, x^2 + 6xz + 3y^2, x^2 + 6xz + 3y^2)} \right\rangle \) | 6 |

The dimension \( \mu(j_1, \ldots, j_m) \) of the Thom-Boardman algebra \( \Sigma^{j_1, \ldots, j_m} \) is equal to the number of monomials in the basis from its description above. In particular, for \( m = 2 \) we have

\[
\mu(a,b) = a + \frac{b(b+1)}{2} + b(a-b) = a(b+1) - \frac{b(b-1)}{2}.
\]

8.1 Thom-Porteous formula

The locus of the Thom-Porteous singularity \( \Sigma^\mu \) for a given map \( f : X \to Y \) is the locus of points \( x \in X \) characterized by the condition that the derivative \( df : T_x X \to T_{f(x)} Y \) has kernel rank at least \( \mu \). The nilpotent algebra of this singularity has trivial multiplication. Therefore, it admits a natural filtration with the dimension vector \( d = (\mu) \) consisting of only one entry. By Corollary 3.2, the generating series for the Schur basis expansion of the Thom polynomial is just the monomial

\[
\tilde{K}_\mu(t_1, \ldots, t_\mu) = t_1^{\mu-1} t_2^{\mu-1} \cdots t_\mu^{\mu-1}
\]

which recovers the Porteous formula

\[
Tp_{\Sigma^\mu} = \Delta_{\ell+\mu, \ldots, \ell+\mu}.
\]
Remark that not only the answer for the singularity $\Sigma^\mu$ agrees with the Porteous formula but also the resolution used in the proof for this particular case coincides with the ‘standard’ one.

The Thom-Porteous singularity is the only one for which the generating series is a polynomial. In all other cases the rational generating series has a non-trivial denominator leading to an infinite monomial expansion.

### 8.2 Example: Thom series for $I_{2,2}$ and $III_{2,3}$

Consider the 3-dimensional vector space $N$ with the basis $e_1, e_2, e_3$ and an algebra structure on it given by the multiplication law

$$e_1 \cdot e_1 = q_1^3 e_3, \quad e_1 \cdot e_2 = e_2 \cdot e_1 = q_{12}^3 e_3, \quad e_2 \cdot e_2 = q_{22}^3 e_3$$

and all other products of basic vectors being trivial. The complex coefficients $q_{11}^3, q_{12}^3, q_{22}^3$ form the 3-dimensional vector space $\text{Alg}(2, 1)$. The multiplication defines a quadratic form on $N^{(1)} = N/\mathbb{Z}_2 \simeq \mathbb{C}^2$ taking values in $\mathbb{N}_2 \simeq \mathbb{C}$. If this form is nondegenerate, then the algebra is isomorphic to $I_{2,2} = \frac{\mathbb{C}[x,y]}{(x^2, y^2)}$. It follows that the Thom polynomial for the singularity $I_{2,2}$ admits the generating series

$$I_{2,2} : \quad K_{2,1} = \frac{t_1(t_2 - t_1)(t_3 - t_1)(t_3 - t_2)}{(t_3 - 2t_1)(t_3 - t_1 - t_2)(t_3 - 2t_2)}$$

$$= t_1^2 t_3^{-1} - t_1^2 t_3^{-1} - 2t_1^2 t_3^{-2} + 2t_1^2 t_3^{-2} - 4t_1^4 t_3^{-3} + t_2^2 t_3^{-3}$$

$$+ (4t_1^3 t_3^{-3} - t_1^3 t_3^{-3}) - 8t_1^5 t_3^{-4} + 3t_1^2 t_3^{-4} + (8t_1^4 t_2^2 t_3^{-3} - 3t_1^4 t_2^2 t_3^{-4}) + \ldots$$

leading to the Thom polynomial

$$T_{I_{2,2}} = c_{\ell + 2} c_\ell - c_{\ell + 3} c_\ell c_\ell - c_{\ell + 4} c_\ell c_\ell c_\ell - 2c_{\ell + 2} c_\ell + c_\ell + 2 \ell + 2c_\ell c_\ell - 4c_{\ell + 5} c_\ell + c_\ell - 2 + c_{\ell + 3} c_\ell - 2$$

$$+ 3c_{\ell + 4} c_\ell c_\ell - 2 - 8c_{\ell + 6} c_\ell c_\ell + 3c_{\ell + 4} c_\ell c_\ell - 5c_{\ell + 5} c_\ell c_\ell - 3 + \ldots$$

where the dots denote the terms vanishing for $\ell > 3$. We see that different monomials of the generating series may contribute to one and the same monomial in the Chern classes. A consequence of this phenomenon is that the generating series is not unique. In particular, another choice of the filtration can lead to a different generating series for the same Thom polynomial. Consider, for example, the filtration with the dimension vector $(0, 1, 1, 0, 1)$. The generic algebra with this filtration is also isomorphic to $I_{2,2}$. One should take into account that such a filtration on $I_{2,2}$ is not natural in the sense of Definition 11.3, the choice of the subspace $N_3 \subset N$ corresponds to the choice of one of the two zero lines of the quadratic form mentioned above. By Theorem 11.1 we obtain an alternative generating series for the Thom polynomial $T_{I_{2,2}}$:

$$I_{2,2} : \quad \frac{1}{2} K_{0,1,1,0,1} = \frac{(t_2 - t_1)(t_3 - t_1)(t_3 - t_2)}{2(t_3 - 2t_1)(t_3 - t_1 - t_2)}.$$
Corollary \[\text{8.2}\] provides also a representation of the Thom polynomial in the Schur basis: expanding the function

\[
\frac{1}{2} K_{0,1,1,0,1} = \frac{t_2 t_3^2}{2(t_3 - 2t_1)(t_3 - t_1 - t_2)} = \sum_{k,j=0}^{\infty} \left( \sum_{i=0}^{k} \binom{i+j}{i} \right) t_1^j t_2^k t_3^{k-i-j}
\]

we get

\[
\text{Tp}_{I_2} = \sum_{k,j=0}^{\infty} \left( \sum_{i=0}^{k} \binom{i+j}{i} 2^{k-i-j} \right) \Delta_{\ell+1+k,\ell+2+j,\ell+1-i-j} = \Delta_{\ell+2,\ell+2,\ell} + 3\Delta_{\ell+3,\ell+2,\ell-1}
\]

\[
+ 3\Delta_{\ell+3,\ell+3,\ell-2} + 7\Delta_{\ell+4,\ell+2,\ell-2} + 10\Delta_{\ell+4,\ell+3,\ell-3} + 15\Delta_{\ell+5,\ell+2,\ell-3} + \ldots
\]

It is easy to check applying determinantal formulas for Schur functions that the two obtained expansions for the Thom polynomial of the singularity \(I_2\) agree.

Let us turn back to the space \(\text{Alg}(2,1)\). The discriminant has the equation \(q_{11}^3 q_{2,2}^3 - (q_{1,2}^3)^2 = 0\) of multidegree \(2(t_3 - t_2 - t_1)\). The algebras from this discriminant are isomorphic to \(III_{2,3}\). This leads to the following generating series for its Thom polynomial:

\[
III_{2,3} : \quad 2(t_3 - t_2 - t_1)K_{2,1} = \frac{2t_1(t_3 - t_2 - t_1)(t_2 - t_1)(t_3 - t_1)(t_3 - t_2)}{(t_3 - 2t_1)(t_3 - t_1 - t_2)(t_3 - 2t_2)}.
\]

It is interesting to observe that this algebra admits two more natural filtrations, those with the dimension vectors \((1,2)\) and \((0,1,1,1)\). Moreover, the algebras isomorphic to \(III_{2,3}\) form Zariski open subsets in both \(\text{Alg}(1,2)\) and \(\text{Alg}(0,1,1,1)\). This leads to two other possible choices of the generating series for the singularity \(III_{2,3}\):

\[
III_{2,3} : \quad K_{1,2} = \frac{t_2(t_2 - t_1)(t_3 - t_1)(t_3 - t_2)}{(t_3 - 2t_1)(t_2 - 2t_1)}, \quad K_{0,1,1,1} = \frac{(t_2 - t_1)(t_3 - t_1)(t_3 - t_2)}{t_3 - 2t_1}.
\]

The last function is certainly the simplest one among the three possible choices of the generating series for the \(III_{2,3}\) singularity presented above. It provides a particularly simple expression for the Thom polynomial in the Schur basis: expanding the series

\[
\tilde{K}_{0,1,1,1} = \frac{t_2 t_3^2}{t_3 - 2t_1} = t_2 t_3 + 2t_1 t_2 + 4t_1^2 t_2 t_3^{-1} + 8t_1^3 t_2 t_3^{-2} + \ldots
\]

we get (cf. \[\text{8.7}\])

\[
\text{Tp}_{III_{2,3}} = \sum_{i=1}^{\ell+2} 2^i \Delta_{\ell+1+i,\ell+2,\ell+2-i}.
\]

### 8.3 Summary on the computed Thom polynomials

An analysis of algebras corresponding to different choices of filtrations with relatively small number of nonzero entries in the dimension vector, similar to the one of the previous section, leads to the computation of generating series for Thom polynomials of particular classes of singularities. A part of these computations is presented in the tables below. In these tables, we denote by \(d\) the homogeneous degree of the generating function. It is related to the codimension of the singularity by the equality

\[
\text{codim } \eta = (\ell + 1) \mu + d.
\]
One may treat $d$ as the ‘virtual codimension’ of the singularity $\eta$ corresponding to the case $\ell = -1$. The number $c$ in the last column is the degree of the denominator (the number of its linear factors) that measures the ‘complexity’ of the corresponding rational function from the point of view of its power series expansion.

| $\eta$          | $\mu$ | $d$ | Generating series for $T_{p_\eta}$                          | $c$ |
|-----------------|-------|-----|-------------------------------------------------------------|-----|
| $A_1 = \Sigma^1$| 1     | 0   | $K_1$                                                      | 0   |
| $A_2 = \Sigma^{1,1}$ | 2     | 0   | $K_{1,1}$                                                  | 1   |
| $\Sigma^2$     | 2     | 2   | $K_2$                                                      | 0   |
| $A_3 = \Sigma^{1,1,1}$ | 3     | 0   | $K_{1,1,1}$                                                | 3   |
| $\Phi_{2,0} = I_{2,2}$ | 3     | 1   | $K_{2,1}$                                                  | 3   |
|                 |       |     | $K_{0,1,1,0,1} \cdot \frac{1}{2}$                         |     |
| $\Phi_{2,1} = III_{2,3}$ | 3     | 2   | $K_{2,1} \cdot 2(t_3 - t_1 - t_2)$                         | 2   |
|                 |       |     | $K_{1,2}$                                                  | 2   |
|                 |       |     | $K_{0,1,1,1,1}$                                            |     |
| $\Sigma^3$     | 3     | 6   | $K_3$                                                      | 0   |
| $A_4 = \Sigma^{1,1,1,1}$ | 4     | 0   | $K_{1,1,1,1} \cdot (t_4 - 2t_1 - t_2)$                     | 7   |
| $I_{2,3}$       | 4     | 1   | $K_{2,1,1} \cdot ((t_4 - 2t_1 - t_2)(t_4 - t_1 - 2t_2) - (t_4 - t_2 - t_3)(t_4 - t_1 - t_3))$ | 8   |
|                 |       |     | $K_{0,1,1,1,0,1}$                                          |     |
| $III_{3,3}$    | 4     | 2   | $K_{2,2}$                                                  | 6   |
|                 |       |     | $K_{0,0,1,1,0,1,0,1} \cdot \frac{1}{2}$                   |     |
| $III_{2,4}$    | 4     | 2   | $K_{1,2,1}$                                                | 5   |
|                 |       |     | $K_{1,1,2}$                                                | 5   |
|                 |       |     | $K_{0,1,0,1,1,1}$                                          | 4   |
|                 |       |     | $K_{0,0,1,0,1,1,0,0,1}$                                    | 4   |
| $\Sigma^{2,1}$ | 4     | 3   | $K_{2,2} \cdot 2(t_3 + t_4 - 2t_1 - 2t_2)$                 | 6   |
|                 |       |     | $K_{0,1,1,0,2}$                                            | 4   |
|                 |       |     | $K_{0,1,1,1,1}$                                            | 3   |
| $\Phi_{3,0}$   | 4     | 3   | $K_{3,1}$                                                  | 6   |
|                 |       |     | $K_{0,0,1,1,1,0,0,1} \cdot \frac{1}{2}(t_4 - 2t_1)$        | 3   |
| $\Phi_{3,1}$   | 4     | 4   | $K_{3,1} \cdot (3t_4 - 2t_1 - 2t_2 - 2t_3)$                | 6   |
|                 |       |     | $K_{0,2,1,1}$                                              | 3   |
|                 |       |     | $K_{0,0,1,1,1,0,1} \cdot \frac{1}{2}$                     | 2   |
| $\Phi_{3,2}$   | 4     | 6   | $K_{3,1} \cdot 4(t_4 - t_1 - t_2)(t_4 - t_2 - t_3)(t_4 - t_1 - t_3)$ | 3   |
|                 |       |     | $K_{0,1,2,1}$                                              | 1   |
| $\Sigma^4$     | 4     | 12  | $K_4$                                                      | 0   |
Generating series for $T_{p_{\eta}}$

| $\eta$ | $\mu$ | $d$ | Generating series |
|-------|-------|-----|-------------------|
| $\Sigma^{2,1,1}$ | 6 | 4 | $K_{0,1,1,1,1,1,1} \cdot (t_6 - 2t_1 - t_2)(t_5 - 2t_2)$ |
|       |      |     | $= K_{0,0,1,0,1,1,0,0,1} \cdot (t_6 - 2t_1 - t_2)$ |
| $\Sigma^{2,2,1}$ | 8 | 7 | $K_{0,0,1,1,1,1,1,1} \cdot (t_7 - 2t_1 - t_2)(t_8 - 2t_1 - t_2)(t_8 - t_1 - 2t_2)$ |
| $\Sigma^{2,2,2}$ | 9 | 9 | $K_{2,3,4} \cdot \prod_{k=6}^{2}(t_k - 2t_1 - t_2)(t_k - t_1 - 2t_2)$ |
| $C_{\gamma}$ | 6 | 4 | $K_{3,3} \cdot 4(t_4 + t_5 + t_6 - 2t_1 - 2t_2 - 2t_3)$ |
|       |      |     | $K_{0,2,1,0,3}$ |
|       |      |     | $K_{0,0,0,1,1,0,0,1,1,1} \cdot \frac{1}{2}$ |
|       |      |     | $K_{0,0,1,2,0,1,0,2}$ |
|       |      |     | $K_{0,0,0,0,1,1,1,0,1,0,1,0,1} \cdot \frac{1}{2}$ |

Every line of the presented tables can be regarded as an independent theorem. We will not provide complete proofs leaving most of the (quite elementary) checks to the reader. In the following sections we discuss the details of some most interesting cases manifesting particular phenomena appearing in the course of computations. But here we finish with just one useful observation.

Define the grading in the ring $\mathbb{C}[x_1, \ldots, x_m]$ by assigning certain integer weights $\deg x_i = \omega_i$ to the variables. For an integer $K > 0$, denote by $I_{\omega_1, \ldots, \omega_m}^{(K)}$ the ideal generated by all monomials of degree greater than $K$ and set $N_{\omega_1, \ldots, \omega_m}^{(K)} = \frac{\mathbb{C}[x_1, \ldots, x_m]}{I_{\omega_1, \ldots, \omega_m}^{(K)}}$. This nilpotent algebra is graded, and we denote by $d_{\omega_1, \ldots, \omega_m}^{(K)}$ the dimension vector of the associated filtration.

**Proposition 8.1.** Algebras isomorphic to $N_{\omega_1, \ldots, \omega_m}^{(K)}$ form a Zariski open subset in the subvariety of $\text{Alg}(d_{\omega_1, \ldots, \omega_m}^{(K)})$ consisting of associative algebras. If the associativity equation is empty, then a generic algebra in $\text{Alg}(d_{\omega_1, \ldots, \omega_m}^{(K)})$ is isomorphic to $N_{\omega_1, \ldots, \omega_m}^{(K)}$.

Indeed, consider a structure of a commutative associative filtered algebra on a given vector space $N$. Chose generic vectors in the subspaces of filtrations $\omega_1, \ldots, \omega_m$ and denote these vectors by $x_1, \ldots, x_m$. Then monomials in these elements of quasihomogeneous degree smaller or equal to $K$ are generically linearly independent and span the whole $N$ while any monomial of degree greater than $K$ is equal to zero. It shows that the obtained algebra is isomorphic to $N_{\omega_1, \ldots, \omega_m}^{(K)}$. □

For a given isomorphism type of algebras we try to find a quasihomogeneous filtration to represent this algebra in the form $N_{\omega_1, \ldots, \omega_m}^{(K)}$ or as close to this form as possible. Thus, the funny
dimension vectors with many zeroes and ones in the tables of this section correspond to different choices of quasihomogeneity weights. For example,

\[ \Sigma^\mu \simeq N^{(1)}_{1,\mu}, \quad d = (\mu), \]
\[ A_\mu \simeq N^{(\mu)}_1, \quad d = (1_\mu), \]
\[ III_{2,k+1} \simeq N^{(k)}_{1,k}, \quad d = (1_{k-1}, 2), \]
\[ III_{2,k+1} \simeq N^{(2k)}_{2,2k-1}, \quad d = (0, 1, 0, 1, \ldots, 0, 1, 1, 1), \]
\[ \Phi_{m,m-1} \simeq N^{(4)}_{2,3m-1}, \quad d = (0, 1, m - 1, 1), \]
\[ \Sigma^{a,b} \simeq N^{(5)}_{2a,3a-b}, \quad d = (0, b, a - b, (b+1)/2, b(a - b)), \]
\[ \Sigma^{2,2,2} \simeq N^{(2)}_{1,1}, \quad d = (2, 3, 4), \]
\[ \Sigma^{2,2,2} \simeq N^{(15)}_{4,5}, \quad d = (0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1), \]
\[ \Sigma^{2,2,1} \simeq N^{(11)}_{3,4}, \quad d = (0, 0, 1, 1, 0, 1, 1, 1, 1). \]

### 8.4 Algebras of dimension at most 4

The table of the previous section contains the computed generating functions for Thom polynomials for all nilpotent algebras of dimension \( \leq 4 \). In fact, these Thom polynomials are computed in [7] by a different method (partially announced). In most cases we present several alternative forms of the generating series corresponding to different choices of possible filtrations. The classification of nilpotent algebras of small dimension is finite: there is a unique algebra \( \Sigma^1 = A_1 \) of dimension 1, two algebras, \( A_2 = \Sigma^{1,1} \) and \( \Sigma^2 \) of dimension 2, and also 4 isomorphism classes of algebras of dimension 3, and 9 classes of 4-dimensional algebras whose adjacencies are presented in the following diagram.

\[
\begin{array}{ccccccc}
\text{d} : & 0 & 1 & 2 & 3 & 4 & 6 & 12 \\
A_3 & \xrightarrow{} & I_{2,2} & \leftarrow & III_{2,3} & \longleftarrow & \Sigma^3 & \\
A_4 & \xrightarrow{} & I_{2,3} & \leftarrow & III_{3,3} & \Phi_{3,0} & \Phi_{3,1} & \Phi_{3,2} & \Sigma^4 & \\
& & & & & & & \\
III_{2,4} & \longleftarrow & \Sigma^{2,1} & \\
\end{array}
\]

These adjacencies are used in the classification of orbits in the spaces \( \text{Alg}(d) \) with \( \mu = \sum d_i \leq 4 \). It proves out, however, that the detailed study of orbits is redundant: in all cases except for \( A_4 \) an algebra of dimension \( \leq 4 \) admits a natural filtration such that the algebra is isomorphic to a generic algebra with this filtration. Therefore, the generating function for its Thom polynomial can be chosen in the form \( K_d \) for a suitable dimension vector \( d \).

In the case of the algebra \( A_4 \) there is a unique choice of the filtration. The dimension vector of this filtration is \((1, 1, 1, 1)\). A generic algebra with this filtration is isomorphic to \( A_4 \) provided that it is associative. This is the simplest case of a non-empty associativity condition. The associativity equation determines a hypersurface of multidegree \( t_4 - 2t_1 - t_2 \), see Example [4.2]. Therefore, we obtain a generating function for \( T P_{A_4} \) in the form \( K_{1,1,1,1}(t_4 - 2t_1 - t_2) \).
8.5 The singularities $A_\mu$

The Morin singularity is the one with the nilpotent algebra $m_{\Sigma^1(x^{\mu + 1})}$. It is denoted by $A_\mu = \Sigma^1 = \Sigma^{1,\ldots,1}$ ($\mu$ units). A natural filtration on this algebra is induced by the grading with $\deg x = 1$. The dimension vector of this filtration is $(1_\mu) = (1, \ldots, 1)$. It follows that the generating series for the Thom polynomial can be chosen in the form

$$A_\mu : \quad K_{1_\mu} \cdot R_\mu$$

where $R_\mu = P_{A_\mu}$ is the multidegree of the closure of algebras isomorphic to $A_\mu$ in $\text{Alg}(1_\mu)$. This presentation of the Thom polynomial for the singularity $A_\mu$ is essentially equivalent to the one found in [3] by a different approach. The degree of $R_\mu$ (that is, the codimension of the corresponding subvariety in $\text{Alg}(1_\mu)$) is shown in the following table.

| $\mu$ | 1  | 2  | 3  | 4  | 5  | 6  |
|-------|----|----|----|----|----|----|
| dim $\text{Alg}(1_\mu)$ | 0  | 1  | 3  | 7  | 13 | 22 |
| deg $R_\mu$    | 0  | 0  | 0  | 1  | 3  | 7  |

For $\mu \leq 3$ the algebras isomorphic to $A_\mu$ form a Zariski open subset in $\text{Alg}(1_\mu)$. Therefore, one has $R_\mu = 1$ for $\mu = 1, 2, 3$. In the case $\mu = 4$ the locus of $A_4$ in $\text{Alg}(1, 1, 1, 1)$ is the hypersurface determined by the associativity equation, as it is explained in Example [4.2].

For $\mu = 5$ the the locus $A_5$ is also determined by the associativity equations. This locus has codimension 3 but it is not a complete intersection, its ideal is generated by 4 (dependent) equations. They can be represented as the rank condition on a matrix whose entries are certain coordinates in $\text{Alg}(1_5)$. Thus, applying the Porteous formulas one computes the multidegree $R_5$ of this variety, see [3].

In the case $\mu = 6$ a new phenomenon occurs. The subvariety determined by the associativity equations in $\text{Alg}(1_6)$ is proved to be reducible. One of its components is the closure of the set of algebras isomorphic to $A_6$. But there is another component, of the same dimension, that corresponds to the ‘nets of conics’ singularity. The multidegree of the component $A_6$ is computed by means of computer algebra in [3].

For greater values of $\mu$ the situation is even more complicated. The variety of solutions of the associativity equations has many components, of even different dimensions, and there is no known regular method to single out the component $A_\mu$ and to compute its multidegree in a closed form.

8.6 The singularities $\Phi_{m,r}$

The singularities $\Phi_{m,r}$, $0 \leq r \leq m - 1$, are introduced and their Thom polynomials are computed in [7]. In our notations, they classify orbits in $\text{Alg}(m, 1)$. Consider a filtered nilpotent algebra $N$ of dimension $m + 1$ with the dimension vector $(m, 1)$. The multiplication defines a quadratic form on $N^{(1)} = N/N_2 \simeq \mathbb{C}^m$ with values in $N_2 \simeq \mathbb{C}$. We say that the algebra is of type $\Phi_{m,r}$ if this quadratic form is of rank $m - r$, that is, it has a kernel of dimension $r$.

It follows from the definition that the Thom series for this singularity can be chosen in the form

$$\Phi_{m,r} : \quad K_{m,1}Z_r$$

where $Z_r$ is the multidegree of the variety of quadratic forms of kernel rank $r$. An explicit form of the polynomial $Z_r$ is computed in the framework of the theory of symmetric degeneracy loci,
see [26][4][21][10][11]. On the other hand, there is a simpler and even more efficient way to compute the Thom polynomial in this case. Namely, the existence of an \( r \)-dimensional distinguished subspace (the kernel of the quadratic form) allows one to use a more detailed filtration in the algebra, the one with the dimension vector \( (0, m - r, r, 1) \). A generic algebra with this dimension vector is isomorphic to \( \Phi_{m,r} \). That means that \( K_{0,m-r,r,1} \) is a generating function for the Thom polynomial of this singularity.

### 8.7 The singularity \( \Sigma_{a,b} \)

The Thom-Boardman singularity \( \Sigma_{a,b} \), \( 1 \leq b \leq a \), is, by definition, the one associated with the quotient algebra of the coordinate ring \( \mathbb{C}[[x_1, \ldots, x_b, y_1, \ldots, y_{a-b}]] \) over the ideal spanned by the monomials of the form \( y_i y_j \) and also all monomials of degree greater than 2. A general algorithm for the computation of the Thom polynomial for these singularities was derived in [25]. A closed formula for the Thom series was found in [5] for particular cases and in [17] for general one. In the present approach the answer is obtained immediately with essentially no computations involved.

Consider a grading on the coordinate ring assigning the degrees of the variables to be \( \deg x_i = 2 \), \( \deg y_j = 3 \). Then the ideal is spanned by monomials of degree \( \geq 6 \). The filtration on the algebra \( \Sigma_{a,b} \) induced by this grading has the dimension vector \( (0, b, a - b, (b+1)/2, b(a-b)) \). Moreover, a generic algebra with this dimension vector is isomorphic to \( \Sigma_{a,b} \). It follows, that the function \( K_{0,b,a-b,(b+1)/2,b(a-b)} \) is the generating one for the Thom polynomial of this singularity.

### 8.8 Nets of conics

The Thom polynomials for the singularities discussed in this section above have been computed in one or another form by different authors before the method of this paper was developed. Now we present some computations in those cases for which the previous methods were not efficient enough.

The ‘net of conics’ singularity is the one with the local algebra \( \mathfrak{m}_{x,y,z}((S_1,S_2,S_3)+\mathfrak{m}^4) \) where \( S_1, S_2, \) and \( S_3 \) are generic quadratic forms in \( x, y, z \). The following canonical choice for these forms is suggested in [29]:

\[
C_\gamma = \frac{m_{x,y,z}}{(S_1,S_2,S_3)+\mathfrak{m}^4} : \quad S_1 = y^2 + 2xz, \quad S_2 = 2yz, \quad S_3 = x^2 + 6xz + \gamma z^2.
\]

where \( \gamma \in \mathbb{C} \) is a parameter (module). The algebras corresponding to different values of \( \gamma \) are non-isomorphic. This is the simplest example of a continuous family of non-isomorphic singularities. One of the ways to explain the appearance of a module is the following. The forms \( S_i \) span a three-dimensional subspace in \( \text{Sym}^2 \mathbb{C}^3 \). The intersection of this subspace with the discriminant consisting of degenerate conics defines a cubic projective plane curve, that is, an elliptic curve, whose analytic type is an obvious invariant of the net of conics. For the singularity \( C_\gamma \) the modular invariant of the discriminantal curve is

\[
j = -27 \frac{(\gamma - 1)^2}{\gamma(\gamma - 9)^2}.
\]

If \( \gamma \neq 0,9 \), then the discriminantal curve is smooth, while for \( \gamma = 0 \) or for \( \gamma = 9 \) it attains a singularity (a double point).

The algebra \( C_\gamma \) is 6-dimensional. Its square defines a filtration with the dimension vector \( (3, 3) \). The algebra structure is determined by a linear map

\[
\psi : \text{Sym}^2 N^{(1)} \to N_2, \quad N^{(1)} = N_1/N_2, \quad \dim N^{(1)} = \dim N_2 = 3.
\]
We obtain that the Thom polynomial of the singularity $C_\gamma$ (for a fixed value of $\gamma$) is given by the generating series

$$C_\gamma : \quad K_{3,3} \cdot Z_\gamma,$$

where $Z_\gamma$ is the multidegree of the hypersurface $Z_\gamma \subset \text{Alg}(3,3)$ consisting of algebras with fixed value of the parameter $\gamma$. It is clear that neither $Z_t$ nor the Thom polynomial of $C_\gamma$ depend on $\gamma$. Therefore, it is sufficient to compute them for any distinguished parameter value, say, for $\gamma = 0$ or for $\gamma = 9$.

For $\gamma = 9$ the singularity of the discriminantal curve is attained at the point corresponding to a quadratic form of rank 1. It follows that the algebra $C_9$ is characterized by the property that $N^{(1)}$ has a one-dimensional subspace $L$ such that $L \cdot L = 0$. The multidegree of the subvariety $Z_9$ is computed as

$$Z_9 = \int_{PN^{(1)}} c_3(\text{Hom}(O(-2), N_2))$$

For $\gamma = 0$ the singularity of the discriminantal curve is attained at a form of rank 2. The simplest way to characterize the locus $Z_0$ is to apply duality. Let $K \subset \text{Sym}^2 N^{(1)}$ be the kernel of the structure map $\psi$ of the algebra $C_\gamma$. By genericity assumption, $\dim K = 3$ and $K^\vee \simeq \text{Sym}^2 N^{(1)\vee} / N_2^\vee$. Then we have also the restriction map

$$\psi^* : \text{Sym}^2 N^{(1)\vee} \to K^\vee.$$ 

This map can be treated as the structure homomorphism of the algebra $C_{\gamma^*}$ for another value of the parameter. This defines an involution on the parameter space and one can check that $\gamma^* = 9 - \gamma$. In particular, if $\gamma = 0$ then $\gamma^* = 9$ and by above arguments we get

$$Z_0 = \int_{PN^{(1)\vee}} c_3(\text{Hom}(O(-2), \text{Sym}^2 N^{(1)\vee} / N_2^\vee))$$

In both approaches we get

$$Z_0 = Z_9 = 4(c_1(N_2) - 2c_1(N^{(1)})) = 4(t_4 + t_5 + t_6 - 2t_1 - 2t_2 - 2t_3).$$

We obtained, therefore, a generating series for the net of conics singularity in the form

$$C_\gamma : \quad K_{3,3} \cdot 4(t_4 + t_5 + t_6 - 2t_1 - 2t_2 - 2t_3).$$

The considerations above show that the algebras $C_9$ and $C_0$ posses a reach additional structures. This structures can be used to define more delicate filtrations that lead to more efficient generating series. One can see, for example, that the algebra $C_9$ is generic for the dimension vector $(0,2,1,0,3)$ or $(0,0,0,1,1,1,0,0,1,1,1)$, and the algebra $C_0$ is generic for the dimension vector $(0,0,1,2,0,1,0,2)$ or $(0,0,0,0,1,1,1,0,1,0,1,0,1,0,1)$. This leads to several more possible generating functions for the Thom polynomial of the singularity $C_\gamma$:

$$C_\gamma : \quad K_{0,2,1,0,3}, \quad \frac{1}{2}K_{0,0,0,1,1,1,0,0,1,1,1},$$

$$K_{0,0,1,2,0,1,0,2}, \quad \frac{1}{2}K_{0,0,0,0,1,1,1,0,0,1,0,1}.$$

The factors $\frac{1}{2}$ in two cases are due to the fact that the corresponding filtrations are not natural in a sense of Definition 4.3; they depend on a choice of one of the two branches of the discriminantal curve at its singular point.
Applying Corollary 3.2 to any of these functions we obtain an expansion for the Thom polynomial
\[
T_{P\Sigma} = 4\Delta_{\ell+3,\ell+3,\ell+3,\ell+1,\ell} + 8\Delta_{\ell+4,\ell+3,\ell+3,\ell+3,\ell,\ell}
+ 18\Delta_{\ell+4,\ell+3,\ell+3,\ell+1,\ell-1} + 32\Delta_{\ell+4,\ell+3,\ell+3,\ell,\ell-1} + 40\Delta_{\ell+4,\ell+3,\ell+3,\ell+1,\ell-1,\ell-1}
+ 80\Delta_{\ell+4,\ell+4,\ell+4,\ell-1,\ell-1} + 32\Delta_{\ell+5,\ell+3,\ell+3,\ell,\ell-1} + 20\Delta_{\ell+5,\ell+3,\ell+3,\ell+1,\ell-1,\ell-1}
+ 120\Delta_{\ell+5,\ell+4,\ell+3,\ell-1,\ell-1} + 160\Delta_{\ell+5,\ell+3,\ell+3,\ell-1,\ell-1,\ell-1} + 80\Delta_{\ell+5,\ell+5,\ell+3,\ell+1,\ell-1,\ell-1}
+ 40\Delta_{\ell+6,\ell+3,\ell+3,\ell-1,\ell-1} + 112\Delta_{\ell+6,\ell+4,\ell+3,\ell-1,\ell-1,\ell-1} + 16\Delta_{\ell+7,\ell+3,\ell+3,\ell-1,\ell-1,\ell-1} + \cdots
\]
where the dots denote the terms vanishing for \( \ell > 1 \). For the case \( \ell = 0 \) this Thom polynomial is computed in [7]. It corresponds to the first two terms of the above expansion.

We assumed in the above consideration that the parameter \( \gamma \) of the net of conics is fixed. But one can also consider the singularity type \( C_\gamma \) as defined in the union of the singularities \( C_\gamma \) for all \( \gamma \). Equivalently, the singularity \( C_\gamma \) is identified as a generic one whose nilpotent algebra admits the dimension vector \((3,3)\). The Thom polynomial of the singularity \( C_\gamma \) is determined by the following generating series
\[
C_\gamma : \quad K_{3,3}.
\]

### 8.9 Singularities determined by the third-order decomposition

In this section we explain the computation of Thom polynomials for the Thom-Boardman singularities
\[
\Sigma^{2,1,1} = \frac{m_{x,y}}{(y^2) + m^4}, \quad \Sigma^{2,2,1} = \frac{m_{x,y}}{(y^3) + m^4}, \quad \Sigma^{2,2,2} = \frac{m_{x,y}}{m^4}.
\]
These are probably the most interesting examples manifesting the efficiency of the method introduced in this paper. With the previous approaches the problem of finding Thom polynomials for these singularities was considered as a very difficult one or even hopeless. With our approach the answer is obtained almost immediately, with no complicated computations. Consider the gradings on these algebras corresponding to the following choice of the weights of the variables:

- \( \Sigma^{2,1,1} \): \( \deg x = 3 \), \( \deg y = 5 \), \( \mathbf{d} = (0,0,1,0,1,1,0,1,0,1) \),
- \( \Sigma^{2,2,1} \): \( \deg x = 3 \), \( \deg y = 4 \), \( \mathbf{d} = (0,0,1,1,0,1,1,1,1,1) \),
- \( \Sigma^{2,2,2} \): \( \deg x = \deg y = 1 \), \( \mathbf{d} = (2,3,4) \),
- \( \Sigma^{2,2,2} \): \( \deg x = 4 \), \( \deg y = 5 \), \( \mathbf{d} = (0,0,0,1,0,0,1,1,1,1,1,1,1,1) \).

**Proposition 8.2.** For each of the listed cases, the subvariety in \( \text{Alg}(\mathbf{d}) \) of associative algebras is a complete intersection with a unique component, and a generic algebra from this variety is isomorphic to the corresponding algebra \( \Sigma^{2,1,1} \).

The fact that the collection of the associativity equations (5) form a regular sequence for each of the discussed cases follows from their explicit form. This proofs the first assertion of the proposition. The second assertion for the singularities \( \Sigma^{2,2,1} \) and \( \Sigma^{2,2,2} \) are particular cases of Proposition 8.1. That proposition does not cover the case of the singularity \( \Sigma^{2,1,1} \) and the dimension vector \((0,0,1,0,1,1,0,1,1,0,1)\) since the list of monomials of degree \( \leq 11 \) in \( x \) and \( y \) with \( \deg x = 3 \), \( \deg y = 5 \) contains also \( y^2 \) which is zero in the algebra \( \Sigma^{2,1,1} \). The arguments in the proof of Proposition 8.1 show that a generic associative algebra with this dimension vector can be written in the form
\[
N = \frac{\langle x, y, x^2, xy, x^3, y^2, x^2y \rangle}{(y^2 + ax^2y)}
\]
for some $a \in \mathbb{C}$. Applying the change of coordinates $y = \tilde{y} - \frac{1}{2}a x^2$ we can bring the relation $y^2 + ax^2y = 0$ to the form $\tilde{y}^2 = 0$. It proves that the algebra $N$ is isomorphic to $\Sigma^{2,1,1}$ for any $a$. □

The proposition implies the formulas for the generating functions of Thom polynomials for these singularities from the table of Sect. [5,3]. An extra factor $t_1$ in the case of the dimension vector $(0,0,1,1,0,1,1,0,1,1,1,1)$ for the singularity $\Sigma^{2,2,2}$ is due to the fact that a filtration with this dimension is not natural, its choice is determined by a choice of the direction of the $y$-axis in the plane of coordinates $x$ and $y$. Possible filtrations form the projective line $\mathbb{CP}^1$ and the normalizing monomial $t_1$ is chosen by the requirement $\int_{\mathbb{CP}^1} t_1 = 1$ in accordance with the Definition [4,3].

One can check the obtained Thom polynomials by restricting them to some special cases. For example, setting $\ell = -1, 1 + c_1 + c_2 + \cdots = \frac{1+\beta}{(1+\alpha_1)(1+\alpha_2)}$ in the Thom polynomial for $\Sigma^{2,2,2}$ we get its expected value for the case of the 2-dimensional source and 1-dimensional target of the mapping,

\[ Tp_{\Sigma^{2,2,2}} \bigg|_{m=2,n=1} = c_{\text{top}}(J_0^3(\mathbb{C}^2, \mathbb{C})) = \prod_{1 \leq i+j \leq 3} (\beta - i\alpha_1 - j\alpha_2). \]

9 Final remarks and open questions

A complete classification of singularities is so wild and irregular that there is no hope to obtain the Thom polynomial for any given in advance singularity in a closed form. I have even doubts whether the Thom polynomial of the Morin singularity, $A_\mu$, will ever be computed for all $\mu$, without speaking about generic Thom-Boardman singularities $\Sigma^{j_1,\ldots,j_r}$. On can extend the list of singularities with known Thom polynomials, say, by classifying singularities with nilpotent algebras of small dimension, or those determined by small order of jets. One should notice, however, that the new classes of singularities have a relatively big codimension. Any explicit computations for them would require considerable computer resources which makes them less attractive from the viewpoint of applications.

One of the possible extensions of the presented approach is to the multisingularity theory [16]. In this theory, the class of a given multisingularity type of a mapping is expressed in terms of the so called residual polynomials in the relative Chern classes of the mapping. First experiments show that the residual polynomials of multisingularities experience a stabilization with the growth of the relative dimension $\ell$ similar to that for Thom polynomials of monosingularities, but there is no general theorem explaining this stabilization.

The non-uniqueness of the generating series brings up a number of interesting questions on the algebra of generating functions. Let us call two rational functions in $t_1, \ldots, t_\mu$ equivalent if they lead to equal expansions in the Chern classes. Is there a simple way to determine whether two rational functions are equivalent? What is the smallest number of linear factors in the denominator of rational functions equivalent to the given one? These questions will be addressed in further studies.

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