On the Classification of Elliptic Fibrations modulo Isomorphism on K3 Surfaces with large Picard Number
— including a review on classifications of elliptic fibrations on K3 surfaces —

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Abstract
Motivated by a problem originating in string theory, we study elliptic fibrations on K3 surfaces with large Picard number modulo isomorphism. We give methods to determine upper bounds for the number of inequivalent K3 surfaces sharing the same frame lattice. For any given Neron–Severi lattice $S_X$, such a bound on the ‘multiplicity’ can be derived by investigating the quotient of the isometry group of $S_X$ by the automorphism group. The resulting bounds are strongest for large Picard numbers and multiplicities of unity do indeed occur for a number of K3 surfaces with Picard number 20. Under a few extra conditions, a more refined analysis is also possible by explicitly studying the embedding of $S_X$ into the even unimodular lattice $\Gamma_{1,25}$ and exploiting the detailed structure of the isometry groups of $S_X$ and $\Gamma_{1,25}$. We illustrate these methods in examples and derive bounds for the number of elliptic fibrations on Kummer surfaces of Picard numbers 17 and 20. As an intermediate step, we also discuss coarser classification schemes and review known results.
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1 Introduction and Summary

Let $X$ be a smooth complex K3 surface, and $(\pi_X, \sigma; X, \mathbb{P}^1)$ a set of data defining an elliptic fibration: $\pi_X : X \rightarrow \mathbb{P}^1$ is a morphism with the generic fibre a smooth curve of genus one and $\sigma : \mathbb{P}^1 \rightarrow X$ a section satisfying $\pi_X \cdot \sigma = \text{id}_{\mathbb{P}^1}$. One can think of classification problems of such data of elliptic fibrations by introducing various equivalence relations among them. One classification is by the type of elliptic fibration, and closely related is another classification by the isometry class of the frame lattice of an elliptic fibration. These two are referred to as $J^{(\text{type})}(X)$ and $J_2(X)$ classifications, respectively, in this article (see section 3 for more information). The main theme of this article is a classification of elliptic fibrations by isomorphism.

Definition: Two elliptic fibrations $(\pi_X, \sigma; X, \mathbb{P}^1)$ and $(\pi'_X, \sigma'; X, \mathbb{P}^1)$ are said to be isomorphic if there is a bijective morphism from $X$ to itself (automorphism) $f \in \text{Aut}(X)$ and a $g \in \text{Aut}(\mathbb{P}^1) = \text{PGL}(2; \mathbb{C})$ such that $\pi'_X = g \cdot \pi_X \cdot f$ and $\sigma = f \cdot \sigma' \cdot g$.

This introduces an equivalence relation among the data of elliptic fibration that $X$ admits, and individual equivalence classes are called isomorphism classes of elliptic fibrations on $X$. Let $J_1(X)$ denote the set of such isomorphism classes. The equivalence relation for $J_1(X)$ is smaller than that of $J_2(X)$, and hence the former classification is finer than the latter. In this article, mainly in section 5, we discuss how to derive upper bounds on the number of isomorphism classes of elliptic fibration that $X$ admits for $X$ with large Picard number $\rho_X$.

The second theme of this article is to provide a review on issues associated with elliptic fibrations on K3 surface in general, with contemporary string theorists in mind as our target readers. This purpose has been served by [Asp] for already more than a decade. More can be learned by consulting reviews purely dedicated to mathematics, such as [SS]. However, the problems faced by the present authors in [BKW], require more material to be covered. This article is therefore not only written to report new mathematical results on the issue mentioned above, but also to serve (when combined with sections 3–4.2 of [BKW]) as a review article supplementary to [Asp, SS].

Sections 2–4.2 should be regarded as review material for the most part. Section 3 explains four different classifications of elliptic fibrations on $X$—$J_0(X)$, $J_2(X)$, $J^{(\text{type})}(X)$ and $J_1(X)$—corresponding to different choices of equivalence relations. Exploiting the Torelli theorem for K3 surfaces, these classification problems are completely translated into the language of lattice theory. Although this subject is in principle considered to be well-known, i) we are interested in a version of this problem when the existence of a section $\sigma$ is required, and ii) a careful attention is paid to the choice of the modular group. Lemmas A and B in the logical chain, the proofs of which

\[1\] We included more review material than needed for an original article in mathematics, in order to make this article accessible to string theorists. Although jargon appears without explanation in this summary section, we have tried to provide explanations sufficient for string theorists in the following sections.
we have not found in the literature, are also provided. Sections 4.1 and 4.2 are devoted to a review of the Kneser–Nishiyama method \cite{Nish1, Nish2}, which makes it possible to work out the \( J_2(X) \) classification of elliptic fibrations on \( X \) systematically (for relatively large Picard numbers \( \rho_X \)).

The Kneser–Nishiyama method \cite{Nish1, Nish2} computes a negative definite rank \((26 - \rho_X)\) lattice \( T_0 \) from the transcendental lattice \( T_X \), and then uses \( T_0 \) to determine the \( J_2(X) \) classification of \( X \). Although this method has been used to determine the \( J_2(X) \) classification for some K3 surfaces with large Picard number, all studies that we are aware of are for cases where \( T_0 \) is either a direct sum of root lattices of \( A-D-E \) type, or an overlattice of a sum of root lattices \cite{Nish1, Nish2, BL}.

We have computed the lattice \( T_0 \) for twenty-four K3 surfaces with \( \rho = 20 \) (the companion article \cite{BKW} explains the motivation to study them), the results are found in Table 2. The \( T_0 \) lattice can be chosen as a sum of \( A-D-E \) root lattices for 11 K3 surfaces among them, while this is not the case for the other 13. One K3 surface, which we name \( X_{[3\ 0\ 2]} \), is chosen from the latter 13 K3 surfaces as an example for which \( T_0 \) is not a root lattice. Tables 3 and 5 in section 4.4 show how to carry out the \( J_2(X) \) classification for this K3 surface. These two tables contain only a part of the \( J_2(X) \) classification of \( X = X_{[3\ 0\ 2]} \), but they already exhausted all cases for which \( II^*, III^* \) or \( IV^* \)-type singular fibres appear. It turns out that there are 43 such entries in \( J_2(X) \) for \( X = X_{[3\ 0\ 2]} \).

The subject of section 5 is the modulo–isomorphism classification of elliptic fibrations, which we denote by \( J_1(X) \). The \( J_1(X) \) classification is built on top of the \( J_2(X) \) classification \cite{Og}. Since the former is finer than the latter, the \( J_1(X) \) classification can be described by specifying how many isomorphism classes are contained in a given equivalence class of the \( J_2(X) \) classification. We call this number the multiplicity or the number of isomorphism classes in this article. Oguiso completely worked out the \( J_2(X) \) and \( J_1(X) \) classifications for Kummer surfaces \( X = \text{Km}(E \times F) \) associated with the product of two elliptic curves \( E \) and \( F \) with generic complex structures \cite{Og}. By generalizing ideas of \cite{Og}, we first derive an upper bound on the multiplicity for a given K3 surface \( X \) which holds for any one of the equivalence classes in \( J_2(X) \) (Proposition C and Corollary D). The results recorded in Table 7 can be used with Proposition C and Corollary D to derive such upper bounds for thirty-four K3 surfaces \( X \) with \( \rho_X = 20 \). In particular, we found ten K3 surfaces with \( \rho_X = 20 \) where the multiplicity cannot be larger than 1, which means that \( J_1(X) = J_2(X) \).

It is also possible to derive a stronger upper bound on the multiplicity (Proposition E and Corollary F), by working on individual equivalence classes in \( J_2(X) \). This is done by studying orbits of a finite subgroup of the isometry group \( \text{Isom}(S_X) \) of the Neron–Severi lattice \( S_X \) of \( X \) acting on a divisor defining an elliptic fibration of a given equivalence class in \( J_2(X) \), as in \cite{Og}. For practical computations involving \( \text{Isom}(S_X) \), we use the theory of \cite{Bor2, Ko2}, which is applicable to K3 surfaces where \( T_0 \) lattice is a direct sum of root lattices of \( A-D-E \) type (or an overlattice of a direct sum of root lattices). Using Proposition E and Corollary F, we derive upper bounds on the multiplicity of elliptic fibrations individually for all the equivalence class of \( J_2(X) \) for two K3 surfaces: \( X = \text{Km}(A) \) (where \( \rho_X = 17 \)) and \( X = \text{Km}(E_\omega \times E_\omega) \) (where \( \rho_X = 20 \)). The results are shown in Table 8, Example G, Table 9, and Example J.

Appendices A.1 and A.3 are included in order to make this article self-contained even for readers with a physics background. In particular, the Appendices A.1 and A.3 explain basic material necessary for section 5, while Appendix A.2 will serve as an exercise problem that helps to digest the theory of \cite{Bor2, Ko2}. 

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2 Preliminaries

Lattice theory, the Torelli theorem for K3 surfaces and the structure theory of automorphism groups of K3 surfaces lie at the heart of all that is discussed in this article. For the convenience of readers with a physics background, sections 2.1 and 2.3 (along with section 3.1 of [BKW]) provide a minimum version of explanation concerning this material, and quote important theorems that will be used frequently in this article. We recommend the readers to refer to original articles such as [Nik1] (for lattice theory) and [P-SS] (for automorphism group) or textbooks/lecture notes such as [BHPV, H] for a more thorough treatment. In section 2.2, we also give a brief summary of notations used in expressing substructure of groups.

2.1 Basic Notions in Lattice Theory

2.1.1 Lattice, Primitive Embedding, Orthogonal Complement

**Definition** A lattice $L$ is a free abelian group of finite rank, i.e., $L \cong \mathbb{Z}^r$ for some $r \in \mathbb{N}$, with non-degenerate symmetric pairing $(\cdot, \cdot) : L \times L \to \mathbb{R}$. The value $(x, x) \in \mathbb{R}$ for an element $x \in L$ is called its norm or self-intersection and we sometimes abbreviate $(x, x)$ by $x^2$. If there is an isomorphism $\phi : L \to L'$ of Abelian groups between a pair of lattices $L$ and $L'$ and $\phi$ preserves the symmetric pairing, the two lattices $L$ and $L'$ are said to be isometric and $\phi$ is said to be an isometry or a lattice isomorphism.

**Definition** A lattice $L$ is said to be positive-definite (resp. negative-definite) if $(x, x) = x^2 > 0$ (resp. $x^2 < 0$) for all non-zero $x$ in $L$. A lattice that is neither positive definite nor negative definite is called indefinite. When a non-degenerate lattice $L$ has an intersection form (matrix representation of the symmetric pairing) with $r_+$ positive “eigenvalues” and $r_-$ negative “eigenvalues”, the pair of integers $(r_+, r_-)$ are said to be the signature of $L$. A lattice $L$ is said to be integral, if $(x, y) \in \mathbb{Z}$ for any $x, y \in L$. An Even lattice, or equivalently Type II lattice\(^2\) is an integral lattice $L$ with $x^2$ an even integer for all $x \in L$.

**Definition** For a lattice $L$, $\text{discr}(L)$—the discriminant of $L$—is the determinant of the intersection form (symmetric pairing) of $L$. An integral lattice $L$ is said to be unimodular when $\text{discr}(L) = \pm 1$.

**Definition** For a non-degenerate integral lattice $L$, its dual lattice $L^*$ is defined by

$$L^* := \left\{ y \in L \otimes \mathbb{Q} \mid (y, x) \in \mathbb{Z} \text{ for } \forall x \in L \right\}$$

as an Abelian group. Its symmetric pairing is given by naturally extending the symmetric pairing of $L$ to $L \otimes \mathbb{Q}$ first, and then by restricting it to $L^* \subseteq L \otimes \mathbb{Q}$. A lattice $L$ is said to be self-dual, if $L^*$ is isometric to $L$. From these definitions one can see that unimodularity and self-duality are equivalent, and the two words can be used interchangeably.

There is a strong classification theorem for even unimodular lattices modulo isometry. For a lattice $L$ with indefinite signature, i.e., $r_+ > 0$ and $r_- > 0$, an even unimodular lattice exists if $r_+ \equiv \ldots$\(^2\) Lattices that are not even are said to be odd, or equivalently Type I.
In this case, it is uniquely determined by the signature and its rank. Furthermore, \( L \) must be isometric to a direct product of \( U \), \( E_8 \) and \( E_8[-1] \). See section 2.1.3 for the definition of \( U \) and \( E_8 \) lattices. Lattices characterized uniquely (modulo isometry) by their signature \((r_+, r_-)\) are denoted by \( \Pi_{r_+, r_-} \):

\[
\Pi_{r_+, r_-} \cong U^{\oplus r_+} \oplus E_8^{\oplus m} \quad \text{if} \quad r_- = r_+ + 8m, \quad \Pi_{r_+, r_-} \cong U^{\oplus r_-} \oplus E_8[-1]^{\oplus m} \quad \text{if} \quad r_+ = r_- + 8m. \quad (2)
\]

For positive definite (or negative definite) even unimodular lattices, however, this uniqueness is lost. Although \( E_8 \) (root lattice) is the only rank-8 even unimodular lattice, there are two isometry classes of rank-16 even unimodular lattice and twenty-four rank-24 even unimodular lattices that are not mutually isometric \cite{CS}.

**Definition** A lattice \( M \) is said to be a sublattice of \( L \), when \( M \) can be identified with a free Abelian subgroup of \( L \) and the symmetric pairing on \( M \) is given by restricting the symmetric pairing of \( L \).

**Definition** An embedding \( \phi \) of a lattice \( M \) into \( L \) is an injective homomorphism \( \phi : M \to L \) of Abelian groups such that the lattice \( M \) and the sublattice \( \phi(M) \) of \( L \) are isometric under \( \phi \). A lattice embedding \( \phi : M \to L \) is said to be isomorphic to another embedding \( \phi' : M \to L' \) if there is an isometry \( f : L \to L' \) so that \( \phi' = f \cdot \phi : M \to L' \).

**Definition** A sublattice \( M \) in \( L \) is primitive when the quotient \( L/M \) becomes a torsion-free Abelian group. A lattice embedding \( \phi : M \to L \) is said to be primitive, if \( \phi(M) \) is a primitive sublattice of \( L \).

Suppose that \( M \) is a primitive sublattice of \( L \). Then the short exact sequence \( 0 \to M \to L \to L/M \to 0 \) of Abelian groups always splits and there exists an isomorphism (of Abelian groups)

\[
L \cong M \oplus (L/M). \quad (3)
\]

Note, however, that the symmetric pairing (intersection form) does not necessarily respect this direct sum decomposition.

It is useful to note that, for a lattice \( L \) and its sublattice \( M \), the two conditions

- \( M \) is primitive in \( L \),
- \((M \otimes \mathbb{Q} \subset L \otimes \mathbb{Q}) \cap L = M\),

are equivalent.

**Definition** For a sublattice \( M \) of a lattice \( L \), we can always define another sublattice \( M^\perp \) of \( L \):

\[
M^\perp := \{ x \in L | (x, y) = 0 \quad \forall y \in M \}. \quad (4)
\]

This new lattice \( M^\perp \subset L \) is called the orthogonal complement of \( M \) in \( L \). By definition, a sublattice of a lattice \( L \) obtained as the orthogonal complement of some sublattice of \( L \) is primitive in \( L \). The

\[\text{For a lattice } L, \text{ we denote a lattice that is isomorphic to } L \text{ as an Abelian group, but has a symmetric pairing which is } n \text{ times larger (with } n \in \mathbb{Z}) \text{ than that of } L \text{ by } L[n].\]
lattice homomorphism
\[ i : M \oplus M^\perp \hookrightarrow L \]  
(5)
is injective, and this embedding is always of finite index (i.e., \([L : i(M \oplus M^\perp)] < \infty\)). When a sublattice \(M\) of a lattice \(L\) is unimodular, then there even is an isometry (lattice isomorphism)
\[ L \simeq M \oplus M^\perp. \]  
(6)

**Example** Let \( L = \mathbb{Z} \langle e_1 \rangle \oplus \mathbb{Z} \langle e_2 \rangle \), with symmetric pairing \((e_i, e_j) = 2\delta_{ij}\). For a sublattice \(M\) generated by \((e_1 + e_2) \in L\), i.e., \(M = \mathbb{Z} \langle e_1 + e_2 \rangle \subset L\), the orthogonal complement is given by \(M^\perp = \mathbb{Z} \langle e_1 - e_2 \rangle\). \(M \oplus M^\perp\) generates a sublattice of \(L\), but \(L \neq M \oplus M^\perp\).

**Definition** A lattice \(L\) is said to be an overlattice of a lattice \(M\), when \(M\) is a sublattice of \(L\) with the index \([L : M]\) being finite. (So \(L\) and \(M\) have the same rank.) The dual lattice \(L^*\) is always an overlattice of \(L\), and \(L\) is an overlattice of \(M \oplus M^\perp\).

### 2.1.2 Discriminant Group and Discriminant Form

Although there is no strong classification theorem for lattices that are not unimodular, the discriminant group of a lattice and the discriminant form provide powerful tools in studying various properties associated with even non-unimodular lattices [Nik1].

**Definition** A discriminant group \(G_L\) of an even lattice \(L\) is a finite Abelian group given by
\[ G_L := L^*/L, \]  
(7)
which is non-trivial for lattices that are not unimodular. A discriminant form \(q_L\) on \(G_L\) is a quadratic form \(q_L : G_L \rightarrow \mathbb{Q}/2\mathbb{Z}\) defined by
\[ q_L : G_L \rightarrow \mathbb{Q}/2\mathbb{Z}, \quad x \mapsto x^2 \mod 2\mathbb{Z}. \]  
(8)
From this quadratic form, a bilinear form \(b_L : G_L \times G_L \rightarrow \mathbb{Q}/\mathbb{Z}\) can be defined by
\[ b_L(x, y) = \frac{1}{2} \{ q_L(x + y, x + y) - q_L(x, x) - q_L(y, y) \} \in \mathbb{Q}/\mathbb{Z} \]  
(10)
for \((x, y) \in G_L \times G_L\). Obviously \(q_L(x) = b_L(x, x)\).

Note that a finite Abelian group \(G_L\) can be written in the form \(\prod_i (\mathbb{Z}/p_i^{k_i} \mathbb{Z})\), where the \(p_i\)'s are prime numbers, and the \(k_i\)'s are positive integers. This is because \(\mathbb{Z}/m\mathbb{Z} \cong \prod_j (\mathbb{Z}/p_j^{k_j} \mathbb{Z})\), when \(m \in \mathbb{Z}_{\geq 0}\) is decomposed into primes as \(m = \prod_j p_j^{k_j}\). Let \(\sigma_i\) and \(\sigma_j\) be generators of the first two factors in \((\mathbb{Z}/p_i^{k_i} \mathbb{Z}) \times (\mathbb{Z}/p_j^{k_j} \mathbb{Z}) \times \cdots \subset G_L\), where \(p_i \neq p_j\). Then \(b_L(\sigma_i, \sigma_j) \equiv 0 \mod \mathbb{Z}\), because \(b_L(\sigma_i, \sigma_j) \in \mathbb{Q}/\mathbb{Z}\) has to simultaneously be an integral multiple of \(1/p_i^{k_i}\) and \(1/p_j^{k_j}\). When \(G_L\) contains only one factor of the form of \(\mathbb{Z}/p^k \mathbb{Z}\) for a given prime number \(p\), then the discriminant form on the factor \(\mathbb{Z}/p^k \mathbb{Z}\) is always in the form of \(q(\sigma) = b(\sigma, \sigma) = a/p^k\) with \(a \in \mathbb{Z}\), since
\[ b(p^k \sigma, \sigma) \equiv 0 \mod \mathbb{Z}. \] Such a discriminant form on \( G_L \cong (\mathbb{Z}/p^k \mathbb{Z}) \) may be denoted by \( q_a(p^k) \). When \( G_L \) contains more than one factor of subgroups of the form \( \mathbb{Z}/p^k \mathbb{Z} \) for a prime number \( p \), then the discriminant form can be more complicated.

Let \( \text{Isom}(G_L, q_L) \) or just simply \( \text{Isom}(q_L) \) be the group of isomorphisms of the Abelian group \( G_L \) preserving the discriminant form. There is a natural homomorphism

\[
p_L : \text{Isom}(L) \longrightarrow \text{Isom}(G_L, q_L).
\]

**Definition** Any non-zero primitive element\(^4\) \( x \) in a lattice \( L \) is called a root, if and only if the following reflection isometry

\[
s_x : L \otimes \mathbb{Q} \longrightarrow L \otimes \mathbb{Q}, \quad y \mapsto y - \frac{2(x, y)}{(x, x)} x,
\]

maps the subspace \( L \) of \( (L \otimes \mathbb{Q}) \) to \( L \) itself. If \( L \) is an even lattice, all the elements \( x \in L \) satisfying \( x^2 = -2 \) are roots, and hence they are called \((-2)\) roots. Roots that are not \((-2)\)-roots are called \((-n)\)-roots if \( x^2 = -n \neq -2 \). If \( L \) is an even unimodular lattice, then all the roots of \( L \) are \((-2)\)-roots.

The subgroup of \( \text{Isom}(L) \) generated by reflections is denoted by \( W(L) \). Its subgroup generated only by reflections associated with \((-2)\)-roots is denoted by \( W(2)(L) \). Whereas the \( W(2)(L) \) subgroup is contained in the kernel of the homomorphism \( p_L : \text{Isom}(L) \longrightarrow \text{Isom}(G_L, q_L) \) above, this is not necessarily true for \( W(L) \).

Here, we quote some results, mostly in [Nik1], that are quite important and will also be used in the rest of this article.

**Proposition** \(\alpha\) ([Nik1], Prop. 1.4.1): For an even lattice \( M \), and for an isotropic subgroup \( H \) of \( G_M \) (meaning that \( q_M|H = 0 \)), we can define a lattice \( (M; H) \)

\[
M; H := \{ m' \in M^* \mid [m' \mod M] \in H \subset G_M \},
\]

which is still an even lattice. \( M \) is a sublattice of \( (M; H) \), and \( (M; H) \) in turn, is a sublattice of \( M^* \). The discriminant group \( G_{(M; H)} \) of the new lattice \( (M; H) \) is given by

\[
G_{(M; H)} = \left\{ y \mod H \in G_M \mid b_L(y, m') = 0 \text{ for } \forall [m' \mod M] \in H \right\} \subset G_M/H.
\]

Conversely, for any even sublattice \( M' \) of \( M^* \), with \( M \) being a sublattice of \( M' \), a corresponding isotropic subgroup \( H \) of \( G_M \) can be found such that \( M' = M; H \).

**Proposition** \(\beta\) ([Nik1], Prop. 1.6.1): Let \( M \) and \( N \) be even lattices for which there is an isomorphism \( \gamma : G_M \cong G_N \) such that \( q_M = -q_N \cdot \gamma \). Then

\[
\Delta := \left\{ (m', \gamma(m')) \in G_M \times G_N \mid \forall m' \in G_M \right\}
\]

---

\(^4\) An element \( x \in L \) is called a primitive element, if \( \mathbb{Z} \langle x \rangle \) is a primitive sublattice of \( L \). This is precisely when \( x \) is one of the elements in \( \mathbb{Q}x \cap L \) closest to the origin.
is an isotropic subgroup of $G_M \times G_N$, the discriminant group of an even lattice $M \oplus N$. Thus, Proposition $\alpha$ above introduces an even lattice $L = (M \oplus N); \Delta$. This lattice $L$ is unimodular. $M$ and $N$ are primitive sublattices of $L$, and are mutually orthogonal complements in $L$. Conversely, for a primitive sublattice $M$ of an even unimodular lattice $L$, the orthogonal complement $N := [M^\perp \subset L]$ and $M$ have isomorphic discriminant groups, $\exists \gamma: G_M \cong G_N$, such that $q_M = -q_N : \gamma$.

**Proposition $\gamma$ ([Nik1], special case of Cor. 1.5.2):** Suppose that $M$ and $N$ are primitive sublattices of an even unimodular lattice $L$ and are mutually orthogonal complements, as in Proposition $\beta$ above. A pair of isometries $\varphi \in \text{Isom}(M)$ and $\psi \in \text{Isom}(N)$ can be lifted to an isometry of $L$ that restricts to $(\varphi, \psi)$ on $(M \oplus N) \subset L$, if and only if $\gamma \cdot p_M(\varphi) = p_N(\psi) \cdot \gamma$, where $\gamma: G_M \cong G_N$ is the isomorphism that appeared already in Proposition $\beta$.

**Theorem $\delta$ ([Nik1], Thm. 1.12.4):** This is a sufficient condition for existence of a primitive embedding of an even lattice $M$ of signature $(m_+, m_-)$ into an even unimodular lattice $L$ of signature $(l_+, l_-)$: A primitive embedding $\phi: M \hookrightarrow L$ exists if $\text{rank}(M) \leq (\text{rank}(L))/2$ and $l_+ \equiv l_- \pmod{8}$ as well as $m_+ \leq l_+$, $m_- \leq l_-.$

**Theorem $\epsilon$ ([Nik1], Thm. 1.14.4; [Mor], Thm. 2.8):** This is a sufficient condition for the existence and uniqueness (modulo isometry) of a primitive embedding of an even lattice $M$ into an even unimodular lattice $L$: $m_+ < l_+$, $m_- < l_-$ and $l(G_M) \leq \text{rank}(L) - \text{rank}(M) - 2$, where $l(G_M)$ is the smallest number of generators of $G_M$.

**Theorem $\zeta$ ([Mor], Cor. 2.10):** There exists a primitive embedding of an even lattice $T$ with signature $(2, 20 - \rho)$ into the even unimodular lattice $\Pi_{3, 19}$ if $\rho \geq 12$. Furthermore, such an embedding is unique modulo isometry of $\Pi_{3, 19}$. The uniqueness of this embedding guarantees that the orthogonal complement $S := [T^\perp \subset \Pi_{3, 19}]$ is also determined uniquely modulo isometry.

### 2.1.3 Miscellany

The rank-$2$ signature $(1, 1)$ even unimodular lattice is called hyperbolic plane and is denoted by $U$. It has a set of generators $U = \text{Span}_\mathbb{Z}\{f_1, f_2\}$ so that the intersection form is

$$
\begin{bmatrix}
(f_1, f_2) & (f_1, f_2) \\
(f_2, f_1) & (f_2, f_2)
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
$$

We also denote the quadratic form above with this choice of basis by $U$. When we refer to the hyperbolic plane lattice $U$, however, one is free to choose a set of generators different from $\{f_1, f_2\}$, and the symmetric pairing of the lattice may correspondingly be written as a matrix different from the one above. We do not expect confusion to arise from this abuse of notation. The lattice $U$ has an isometry group $\text{Isom}(U) \cong \mathbb{Z}/2\mathbb{Z}\langle \text{id}, \sigma_U \rangle \times \mathbb{Z}/2\mathbb{Z}\langle \sigma_U \rangle$, where

$$
(-\text{id}_U) : f_{1,2} \mapsto -f_{1,2}, \quad \sigma_U : f_{1,2} \mapsto f_{2,1}.
$$

The isometry group of $U \oplus U$ is studied and described in detail in [HSOY].

A lattice denoted by $(n)$ is a rank-$1$ lattice $\cong \mathbb{Z}\langle e \rangle$ whose symmetric pairing is given by $(e, e) = n.$
Root lattices of the $A_n$, $D_n$, and $E_n$ Lie algebras are also denoted by $A_n$, $D_n$, and $E_n$, respectively, in this article. The intersection form (= symmetric pairing) of these lattices are set to be the negative of their Cartan matrices, which means that the diagonal entries are all $(-2)$. They are negative definite and even integral lattices. The dual lattice of a root lattice of one of the $A$-$D$-$E$ types is its weight lattice, and the discriminant group is given by

\[ G_{A_n} \cong \mathbb{Z}/(n+1)\mathbb{Z}, \quad G_{D_n} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad (n \text{ even}), \quad \mathbb{Z}/4\mathbb{Z} \quad (n \text{ odd}), \]

\[ G_{E_6} \cong \mathbb{Z}/3\mathbb{Z}, \quad G_{E_7} \cong \mathbb{Z}/2\mathbb{Z}. \]  

(18)

For a set of simple roots $\alpha_i$ $(i = 1, \ldots, r)$ of a rank-$r$ Lie algebra of A–D–E type, a set of fundamental weights $\omega_i$ $(i = 1, \ldots, r)$ are the elements of the weight lattice satisfying $(\alpha_i, \omega_j) = -\delta_{ij}$. As summarized e.g. in §1 of [Nish1]:

- $G_{A_n}$ is generated by the mod-$A_n$ equivalence class of the weights of the “defining representation” of $\text{SU}(n+1)$, $[\omega^1]$. This generator is also denoted by $a_{n+1}$ in this article. For this generator, the value of the discriminant form is $q([\omega^1]) = -n/(n+1)$ modulo 2.

- $G_{D_n}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for an even $n$, where one of the two $\mathbb{Z}/2\mathbb{Z}$’s is generated by the mod-$D_n$ equivalence class of the weights of a spinor representation (denoted by $\text{sp}$ or $(1,0)$), and the other by that of the weights of the other spinor representation (denoted by $\overline{\text{sp}}$ or $(0,1)$). $\text{sp} + \overline{\text{sp}}$ in $G_{D_n} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is denoted by $v$. The discriminant bilinear form $b_L$ on this $\{\text{sp}, \overline{\text{sp}}\}$ basis is given by

\[ b_{D_n} = \begin{pmatrix} -n/4 & -(n-2)/4 \\ -(n-2)/4 & -n/4 \end{pmatrix}. \]

(19)

- $G_{D_n} \cong \mathbb{Z}/4\mathbb{Z}$ for an odd $n$ is generated by the mod-$D_n$ equivalence class of the weights of a spinor representation, denoted by $\text{sp}$ . $q(\text{sp}) = -n/4$ modulo 2.

- $G_{E_6} \cong \mathbb{Z}/3\mathbb{Z}$ is generated by the weights of a 27-dimensional representation of $E_6$, and hence the generator of $G_{E_6}$ is denoted by $27$. $q(27) = -4/3$ modulo 2.

- $G_{E_7} \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the weights of the 56-dimensional representation of $E_7$, and hence the generator of $G_{E_7}$ is denoted by $56$. $q(56) = -3/2$ modulo 2.

**Definition** A root lattice $L_{\text{root}}$ is the sublattice of a lattice $L$ generated by all the $(-2)$-roots of $L$.

### 2.2 Substructure of Groups

When there is a short exact sequence of groups

\[ 1 \longrightarrow N \longrightarrow G \longrightarrow O \longrightarrow 1, \]

(20)

where $N$ is a normal subgroup of $G$, we may write $G = N.O$ in a shorthand notation (but not $G = O.N$).

\[ ^6\text{It is often said that } G \text{ is an extension of } O \text{ by } N, \text{ but there are also authors with the opposite conventions saying that } G \text{ is an extension of } N \text{ by } O. \text{ We are not using any of these expressions, and, in order to avoid confusion, use } G = N.O \text{ instead. Reference [ATLAS] suggests to say either that } G \text{ is an upward extension of } N \text{ by } O \text{ or that } G \text{ is a downward extension of } O \text{ by } N. \]
If the above short exact sequence splits, i.e., there exists a homomorphism \( \phi : O \to G \) in addition to the short exact sequence
\[
1 \to N \to G \rightleftharpoons \psi O \to 1,
\]
and the condition \( \psi \circ \phi = \text{id}_O : O \to O \) is satisfied, we say that \( G \) is a semi-direct product of \( N \) and \( O \) (or \( O \) and \( N \)), and this situation is expressed in shorthand notations such as
\[
G = N \rtimes \phi(O), \quad G = \phi(O) \ltimes N \quad \text{(but not } N \rtimes \phi(O)), \quad G = N : O, \quad \text{and} \quad G = N : \text{Ad}_\phi O.
\]
(22)

When a group \( G \) has a structure \( G = N.O \), but not \( N : O \), one may write \( G = N'O \) to make this situation explicit.

6 All those notations above are also defined and explained, along with others, in section 5.2 of [ATLAS].

2.3 Automorphism Groups of K3 Surfaces

2.3.1 The Neron–Severi Lattice and the Transcendental Lattice

Throughout this article, let \( X \) denote a K3 surface: a smooth surface over \( \mathbb{C} \) with trivial canonical bundle and \( h^1(X, \mathcal{O}_X) = 0 \). As is well-known, the second cohomology group \( H^2(X; \mathbb{Z}) \) along with its intersection form is a lattice isometric to
\[
\Lambda_{\text{K3}} = U \oplus U \oplus U \oplus E_8 \oplus E_8 \cong \Pi_{3,19}.
\]
(23)

Whenever we refer to a K3 surface \( X \) in this article, we understand that it is equipped with a certain complex structure; if we are referring to a family of K3 surface, we will explicitly say so. Thus, a period vector
\[
[\Omega_X] \in \mathbb{P}\left[\{ \omega \in H^2(X; \mathbb{C}) \mid \omega \wedge \omega = 0, \omega \wedge \overline{\omega} > 0 \}\right]
\]
(24)
is fixed and given. For a period vector, and hence for \( X \), the Neron–Severi lattice\(^7\)—denoted by \( S_X \)—is given by
\[
S_X = \{ x \in H^2(X; \mathbb{Z}) \mid (x, \Omega_X) = 0 \}.
\]
(25)

Because of this characterization, \( S_X \) is a primitive sublattice of \( H^2(X; \mathbb{Z}) \). Its rank,
\[
\rho_X := \text{rank}(S_X),
\]
(26)
is called the Picard number, which ranges from 0 to 20, depending on the complex structure \( ([\Omega_X]) \) of \( X \). The Neron–Severi lattice \( S_X \) has signature \( (1, \rho_X - 1) \) and its discriminant group and discriminant form are denoted by \( G_{S_X} \) and \( q_{S_X} \), respectively.

\(^6\)An easy example of this situation is the case \( G = \mathbb{Z}/4\mathbb{Z} \) with \( N \) the subgroup generated by 2 mod 4.

\(^7\)The Neron–Severi group is defined to be the set of divisors (algebraic curves) on \( X \) modulo algebraic equivalence. Algebraic equivalence, however, is equivalent to numerical equivalence in the case of a K3 surface. Hence the Neron–Severi group does not have a torsion part, i.e., it is a free Abelian group. The intersection number between a pair of divisors introduces a symmetric pairing to this free Abelian group. On algebraic curves, the integral of the period vector \( \Omega_X \) vanishes.
The transcendental lattice of a K3 surface $X$—denoted by $T_X$—is defined to be the orthogonal complement of $S_X$ in $H^2(X; \mathbb{Z})$. It has signature $(2, 20 - \rho_X)$ and its discriminant group and discriminant form are denoted by $G_{T_X}$ and $q_{T_X}$, respectively. $T_X$ is a primitive sublattice of $H^2(X; \mathbb{Z})$, because of its definition. Proposition $\beta$ (quoted in section 2.1) guarantees that there always exists an isomorphism of Abelian groups $\gamma : G_{S_X} \cong G_{T_X}$ with $q_{T_X} \cdot \gamma = -q_{S_X}$ holds. The lattice $H^2(X; \mathbb{Z})$ can be regarded as a subset of $S_X^* \oplus T_X^*$ characterized by

$$\{ (s, \gamma(s)) \in G_{S_X} \times G_{T_X} \mid s \in G_{S_X} \} \subset G_{S_X} \times G_{T_X}. \quad (27)$$

The identification between $G_{T_X}$ and $G_{S_X}$ by $\gamma$ also establishes a canonical isomorphism

$$\text{Isom}(G_{T_X}, q_{T_X}) \cong \text{Isom}(G_{S_X}, q_{S_X}). \quad (28)$$

### 2.3.2 Automorphism Groups of K3 Surfaces

Automorphisms of a K3 surface $X$ form a group denoted by $\text{Aut}(X)$. Any automorphism $f \in \text{Aut}(X)$ induces $f_* : H_2(X; \mathbb{Z}) \to H_2(X; \mathbb{Z})$, which is an isometry of the lattice $H^2(X; \mathbb{Z})$. In particular, the homomorphism $\text{Aut}(X) \to \text{Isom}(H^2(X; \mathbb{Z}))$ is injective ([P-SS], §2 Prop. 2).

**Definition** For a K3 surface $X$, a *Hodge isometry* is an isometry $\phi : H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ that maps the complex line represented by $[\Omega_X] \in \mathbb{P}[H^2(X; \mathbb{C})]$ to itself. A *Hodge and effective isometry* of an algebraic K3 surface is a Hodge isometry that maps all the classes represented by effective curves to the same set of classes.

**Proposition $\eta$** ([P-SS], §7 Prop.): This is one of several different versions of the Global Torelli Theorem. It states that the image of the injective homomorphism $\text{Aut}(X) \to \text{Isom}(H_2(X; \mathbb{Z}))$ for an algebraic K3 surface $X$ is the group of Hodge and effective isometries. Thus, the automorphism group can be identified with this subgroup of $\text{Isom}(H_2(X; \mathbb{Z}))$.

**Definition** The *positive cone* of an algebraic surface $X$ is the subspace of $S_X \otimes \mathbb{R}$ given by

$$\text{Pos}_X := \{ x \in S_X \otimes \mathbb{R} \mid x^2 > 0 \} \quad (29)$$

Because $S_X$ has signature $(1, \rho_X - 1)$, it consists of two connected components. The connected component containing the classes of effective curves are denoted by $\text{Pos}_X^+$. From the definition of ample cone commonly adopted in algebraic geometry, it follows that the *ample cone* of an algebraic surface $X$ is a subspace of $S_X \otimes \mathbb{R}$ given by

$$\text{Amp}_X = \{ x \in \text{Pos}_X^+ \mid (x, C) > 0 \text{ for any curves (effective divisors) } C \}. \quad (30)$$

**Theorem** 1 of §6 in [P-SS] states that the index 2 subgroup of $\text{Isom}(S_X)$ that preserves $\text{Pos}_X^+$—called group of autochronos isometries of $S_X$ and denoted by $\text{Isom}^+(S_X)$—has the following structure, for a K3 surface $X$:

$$\text{Isom}^+(S_X) \cong W(2)(S_X) \rtimes \text{Isom}(S_X)^{\text{Amp}}. \quad (31)$$
Here, $W^{(2)}(S_X)$ is the subgroup of $\text{Isom}^+(S_X)$ generated by reflections associated with $(-2)$-curves, and $\text{Isom}(S_X)^{(\text{Amp})}$ is the subgroup of $\text{Isom}^+(S_X)$ preserving the ample cone. Note that the reflection hyperplane of an irreducible genus $g$ curve passes through $\text{Pos}_X^+$ only when $g = 0$. Thus, the ample cone is the fundamental region of the action of $W^{(2)}(S_X)$ on $\text{Pos}_X^+$.

Proposition $\eta$ implies that the image of the injective homomorphism from $\text{Aut}(X)$ to $\text{Isom}(H^2(X;\mathbb{Z}))$ is contained in the following subgroup

$$\text{Isom}(T_X)^{(\text{Hodge})} \times \text{Isom}(S_X)^{(\text{Amp})} \subset \text{Isom}(H^2(X;\mathbb{Z})), \tag{32}$$

where $\text{Isom}(T_X)^{(\text{Hodge})}$ is the subgroup of $\text{Isom}(T_X)$ preserving the period $[\Omega_X] \in \mathbb{P}[T_X \otimes \mathbb{C}]$.

The image of $\text{Aut}(X)$ under the injective homomorphism into $\text{Isom}(T_X)^{(\text{Hodge})} \times \text{Isom}(S_X)^{(\text{Amp})}$ can be specified precisely by using the language of discriminant form (reviewed in section 2.1). Consider group homomorphisms

$$p_T : \text{Isom}(T_X) \to \text{Isom}(G_{T_X}, q_{T_X}), \tag{33}$$

$$p_S : \text{Isom}(S_X) \to \text{Isom}(G_{S_X}, q_{S_X}). \tag{34}$$

The latter homomorphism factors through $\text{Isom}(S_X)^{(\text{Amp})} \times \{\pm \text{id}_X\}$, because $W^{(2)}(S_X)$ is in the kernel of $p_S$. Proposition $\eta$ implies that $\text{Aut}(X)$ is characterized as the fibre product:

$$\text{Aut}(X) \xleftarrow{\text{Isom}(T_X)^{(\text{Hodge})}} \xrightarrow{\text{Isom}(S_X)^{(\text{Amp})}} \text{Isom}(G, q), \tag{35}$$

where the isomorphic groups $\text{Isom}(G_{T_X}, q_{T_X})$ and $\text{Isom}(G_{S_X}, q_{S_X})$ are simply denoted by $\text{Isom}(G, q)$. That is,

$$\text{Aut}(X) = \text{Isom}(T_X)^{(\text{Hodge})} \times_{\text{Isom}(G,q)} \text{Isom}(S_X)^{(\text{Amp})}. \tag{36}$$

See e.g. section 1.5 of [Vin1] for a proof of this characterization. The images of the projection

$$\pi_T : \text{Aut}(X) \to \text{Isom}(T_X)^{(\text{Hodge})}, \tag{37}$$

$$\pi_S : \text{Aut}(X) \to \text{Isom}(S_X)^{(\text{Amp})}, \tag{38}$$

associated with the fibre product (36) are denoted by $\text{Isom}(T_X)^{(\text{Hodge Amp})}$ and $\text{Isom}(S_X)^{(\text{Amp Hodge})}$, respectively.

Finally, let us list up a couple of relations among the various groups discussed above. First, it follows from the arguments above that the automorphism group $\text{Aut}(X)$ has the following structures:

$$\text{Aut}(X) = \text{Ker} \cdot \text{Isom}(S_X)^{(\text{Amp Hodge})}, \tag{39}$$

$$\text{Aut}(X) = \text{Aut}_N(X) \cdot \text{Isom}(T_X)^{(\text{Hodge Amp})}. \tag{40}$$

These relations are obtained from the projections $\pi_S$ and $\pi_T$, respectively, accompanied by the injectivity of the homomorphism from $\text{Aut}(X)$ to the group (32). Here, $\text{Ker}$ is the kernel of
The group of Hodge isometries of $H = \text{Isom}(\mathcal{S})$ is the subgroup of $\text{Aut}(X)$ given by the fibre class. The canonical frame sublattice $S_X$ is the fibre class and the class $\text{Aut}(X)$ is the kernel of $p_S : \text{Isom}(S_X)^{(\text{Amp Hodge})} \rightarrow \text{Isom}(q)$. It also follows that

$$\text{Isom}(S_X)^{(\text{Amp Hodge})} = \text{Aut}_N(X) \cdot p_T \left( \text{Isom}(T_X)^{(\text{Hodge Amp})} \right).$$

The group of Hodge isometries of $H_2(X; \mathbb{Z})$, $\text{Isom}(H_2(X; \mathbb{Z}))^{(\text{Hodge})}$, is isomorphic to

$$\text{Isom}(H_2(X; \mathbb{Z}))^{(\text{Hodge})} \cong \{ \pm 1 \} \times \left[ W(2)(S_X) \times \text{Aut}(X) \right].$$

### 3 Classifications of Elliptic Fibrations on a K3 Surface

#### 3.1 The $J_0(X)$ Classification

Consider a K3 surface $X$ with a given complex structure, so that $T_X$ and $S_X$ are already determined in $H_2(X; \mathbb{Z})$. Suppose there is an elliptic fibration $\pi : (X, \mathbb{P}^1)$. Then one can define a sublattice $U_* := \text{Span}_\mathbb{Z}\{[F], [\sigma + F]\}$. The canonical embedding $\Phi : S_X \rightarrow \mathbb{P}^2 \cong \text{Isom}(S_X)^{(\text{Amp Hodge})}$ provides a sublattice $S_X \cong \{ \pm 1 \} \times \left[ W(2)(S_X) \times \text{Aut}(X) \right]$.

For an elliptic fibration $(\pi_X, \sigma; S, C)$ of an algebraic surface, it is common to define the frame lattice $W_{\text{frame}}$ as follows:

$$W_{\text{frame}} := \left[ [F]^\perp \subset S_S \right] / \langle [F] \rangle,$$

where $S_S$ is the Neron–Severi lattice of $S$ and $[F]$ the fibre class. The canonical frame sublattice $W_{\text{frame}}$ in $S_X$ for a K3 surface $X$ is isomorphic to the $W_{\text{frame}}$ defined as above.

Let us call any embedding $\phi_* : U \rightarrow S_X$ of the hyperbolic plane lattice $U$ into $S_X$ of a K3 surface $X$ a canonical embedding when the following conditions are satisfied: for generators $f_1$ and $f_2$ of $U$ satisfying the symmetric pairing $\langle f_1, f_2 \rangle = 1$ and $\phi_*(f_2 - f_1)$ is a class of an irreducible $(\sigma - 2)$ curve. For any elliptic fibration $(\pi_X, \sigma; S, C)$, there is a canonical embedding $U$ by $\phi : f_1 \mapsto [F]$ and $f_2 - f_1 \mapsto [\sigma]$, so that $\phi : U \rightarrow U_* \subset S_X$. The converse is also true:

**Theorem** 3.1 (P-SS, §3 Thm. 1; Koizumi, Lemma 2.1): Whenever one finds a canonical embedding of $U$ into $S_X$, one finds that $h^0(X; \mathcal{O}_X(\phi(f_1))) = 2$, and an elliptic fibration morphism associated with the complete linear system of $\phi_*(f_1)$ is provided by $\Phi_\phi(f_1) : X \rightarrow \mathbb{P}^{(2-1)}$. Here, $\phi(f_1)$ becomes the fibre class and the class $\phi_*(f_2 - f_1)$ can be taken as a section of this fibration.

---

8This Aut$_N(X)$ subgroup of Isom$(S_X)^{(\text{Amp Hodge})}$ is also regarded as a subgroup of Aut$(X)$ given by the fibre product $\left[ \mathbb{P}^2 \right]$, and is called group of symplectic automorphisms of $X$. Another definition of the Aut$_N(X)$ subgroup of Aut$(X)$ is that of automorphisms acting trivially on the complex line $\Omega_X \subset H^2(X; \mathbb{C})$. These two definitions are equivalent because of [Nik2], Thm. 3.1.

9Throughout this article, we only consider elliptic fibrations that have a section $\sigma : \mathbb{P}^1 \rightarrow X$, even when we just refer to them as “elliptic fibrations”.

13
For a given K3 surface $X$, let $\mathcal{J}_0(X)$ be the set of all elliptic fibrations, $(\pi_X, \sigma; X, \mathbb{P}^1)$. Two elliptic fibration morphisms $\pi_X : X \to \mathbb{P}^1$ and $\pi'_X : X \to \mathbb{P}^1$ are regarded equivalent in the $\mathcal{J}_0(X)$ classification, if they are different only by $\pi'_X = g \cdot \pi_X$ for some $g \in \text{PGL}(2; \mathbb{C})$ on $\mathbb{P}^1$. Theorem $\theta$ implies that $\mathcal{J}_0(X)$ is characterized in terms of canonical embeddings of the hyperbolic plane lattice as follows:

$$\mathcal{J}_0(X) \cong \{ \text{canon. embedding } \phi : U \hookrightarrow S_X \}.$$  

(45)

The classification problem of $\mathcal{J}_0(X)$ can be translated into pure lattice theory language, so that the problem no longer involves geometric conditions such as $\phi(f_1)$ being in $\text{Amp}_X$ or $\phi_*(f_2 - f_1)$ being the class of an irreducible curve. Consider

$$\mathcal{J}_0(X) \to W^{(2)}(S_X) \setminus \{ \text{embedding } \phi : U \hookrightarrow S_X \} / \{ \pm \text{id}_U \}.$$  

(46)

This map is surjective. In fact

- [P-SS, §6 Thm. 1] For any embedding $\phi$, one can exploit the freedom to choose $\delta \in \{ \pm \text{id}_U \}$ to make sure that $\phi(\delta(f_1))$ is in $\text{Pos}_X^+$, and then there exists an element in $w \in W^{(2)}(S_X)$ such that $w \cdot \phi \cdot \delta(f_1)$ is in $\text{Amp}_X$.

- [P-SS, §6 Thm. 1; Ko1, Lemma 2.1]: When the choice $\delta$ and $w$ are made as above, the class $w \cdot \phi \cdot \delta(f_1)$ contains a smooth curve of genus 1, and $w \cdot \phi \cdot (f_2 - f_1)$ is an effective curve with self-intersection $-2$. Although this $-2$ curve is not necessarily irreducible, one can always find an irreducible component $C_0$ of $w \cdot \phi \cdot \delta(f_2 - f_1)$ such that $C_0$ is a class of smooth curve of genus 0, satisfies ($C_0, w \cdot \phi \cdot \delta(f_1)) = 1$, and the multiplicity of $C_0$ in $w \cdot \phi \cdot \delta(f_2 - f_1)$ is 1.

- [Lemma A (see below for a proof)] There exists an element in $w' \in W^{(2)}(S_X)$ that keeps $w \cdot \phi \cdot \delta(f_1)$ invariant, while it maps $w \cdot \phi \cdot \delta(f_2 - f_1)$ to $C_0$.

All of the above combined implies that for any embedding $\phi : U \hookrightarrow S_X$, one can always find an element $w' \cdot w \cdot \phi \cdot \delta$ in the orbit $W^{(2)}(S_X) \cdot \phi \cdot \{ \pm \text{id}_U \}$ that satisfies all the properties required for a canonical embedding $\phi : U \hookrightarrow S_X$.

**Proof of Lemma A:** We show that there is an algorithm of finding an appropriate $w' \in W^{(2)}(S_X)$.

Because of Theorem $\theta$, we can assume that there is an elliptic fibration $\pi : X \to \mathbb{P}^1$, where $w \cdot \phi \cdot \delta(f_1) = [F]$ is the fibre class, and $C_0$ is a section. Now, note that all the irreducible components of the effective divisor $w \cdot \phi \cdot \delta(f_2 - f_1)$ except $C_0$ are either the fibre class or irreducible components in the singular fibres of the elliptic fibration. Thus we can write

$$w \cdot \phi \cdot \delta(f_2 - f_1) = C_0 + \sum_I \left[ m(I)[F] + \lambda^{(I)} \right].$$  

(47)

where $I$ runs over all the singular fibres of $A-D-E$ type in the elliptic fibration, and $\lambda^{(I)}$ is an element of the $A-D-E$ type root lattice generated by irreducible components of the $I$-th fibre not meeting the section $C_0$. (We do not have to include the irreducible components *meeting* the section in the decomposition above, because they are given by linear combinations of elements in the root lattice and $[F]$.) Because $(f_2 - f_1)^2 = -2$, the coefficient $m(I)$ above should be chosen as $(\lambda^{(I)})^2 = -2m(I)$.

Let us now consider a subgroup of $W^{(2)}(S_X)$ generated by reflections associated with $\alpha^{(I)}$ and $[F] - \alpha^{(I)}$, where $\alpha^{(I)}$ are any roots in the $A-D-E$ root lattice of the $I$-th fibre. Under
this subgroup of $W(2)(S_X)$, the fibre class $[F]$ remains unchanged and $w \cdot \phi \cdot \delta(f_2 - f_1)$ also remains in the form of (47), except that $\lambda^{(I)}$ (and correspondingly $m^{(I)}$) may change. In the following, we show that $\lambda^{(I)}$ can be brought to zero by the subgroup of reflections we have just introduced, so that the assertion for the existence of $w' \in W(2)(S_X)$ is proved. This can be done separately for each $I$-th fibre.

To this end, it is sufficient to note that a reflection by $\alpha^{(I)}$ followed by another one associated with $[F] - \alpha^{(I)}$ turns (47) into

$$C_0 + \sum_I \left[ (m^{(I)} + (\alpha^{(I)}, \lambda^{(I)}) + 1) [F] + (\lambda^{(I)} - \alpha^{(I)}) \right],$$

which is to change $\lambda^{(I)}$ into $\lambda^{(I)'} := \lambda^{(I)} - \alpha^{(I)}$, and correspondingly $2m^{(I)} := -\lambda^{(I)}$ into $2(m^{(I)'} := -(\lambda^{(I)'})^2 = 2m^{(I)} + 2(\alpha^{(I)}, \lambda^{(I)}) + 2$. Since $\lambda^{(I)}$ is given by an integer linear combination of root vectors in the root lattice for the $I$-th fibre, $\lambda^{(I)'}$ becomes zero eventually. ■

**Lemma B**: The map (46) is also injective.

**Proof**: If there are two canonical embeddings $\phi_* : U \rightarrow S_X$ and $\phi_*' : U \rightarrow S_X$ that fall within a common orbit of $W(2)(S_X) : \phi \cdot \{ \pm \text{id}_U \}$ for some $\phi$, then there is an element $w'' \in W(2)(S_X)$ such that $w'' \cdot \phi_* = \phi_*'$. In the following, we prove that $w'' \cdot \phi_* = \phi_*'$ is the same embedding as $\phi_*'.

Let us first see that $\phi_*(f_1) = \phi_*(f_1)$, when $\phi_*$ and $\phi_*' = w'' \cdot \phi_*$ are both canonical embeddings of $U$ into $S_X$. This can be proved by contradiction as follows: assume that $\phi_*(f_1)$ and $\phi_*(f_1)$ are different and consider the following subset of $W(2)(S_X)$ for $n \in \mathbb{N}_{>0}$:

$$\mathcal{W}_n := \left\{ w \in W(2)(S_X) \mid w \cdot \phi_*(f_1) = \phi_*(f_1), \quad l(w) = n \right\}. \quad (49)$$

Let $n_0$ be the smallest number of $n$ where $\mathcal{W}_n$ is not empty, and choose any element $w_0 \in \mathcal{W}_{n_0}$. $w_0$ can be written as a successive application of $n_0$ simple reflections, $r_{i_{n_0}} \cdots r_{i_2} \cdot r_{i_1}$. Since we have assumed that $\phi_*(f_1) \neq \phi_*(f_1)$, at least one of the $n_0$ reflection planes does not contain $\phi_*(f_1)$. Let $H_{i_k}$ be the first reflection plane of that kind appearing in the sequence of simple reflections in $w_0$. Then both $\phi_*(f_1)$ and $\phi_*(f_1) = w_0 \cdot \phi_*(f_1)$ are on the same side of $H_{i_k}$ (because both $\phi_*$ and $\phi_*$ are a canonical embedding of $U$), yet the path connecting them passes through the other side of $H_{i_k}$ during the way. This means that this is a short-cut path; there is another $w \in W(2)(S_X)$ with $0 < l(w) < n_0$. This is a contradiction.

The element $w'' \in W(2)(S_X)$ therefore belongs to the stabilizer subgroup of $\phi_*(f_1) = \phi_*(f_1)$. This stabilizer subgroup is once again a reflection group. As a set of generators, we can take the reflections $s_\alpha$ associated with $(-2)$ curves $\alpha$ satisfying $(\alpha, \phi_*(f_1)) = 0$. In the present context, this means that we can take reflections associated with $\alpha^{(I)}$'s and

\[\text{Here, the group } W(2)(S_X) \text{ is generated by a set of reflections } \{ r_i \} \text{ (called simple reflections)} \text{ associated with a set of } (-2)-\text{roots}, \{ \alpha_\ell \} \text{ (called simple roots), and the fundamental chamber of this discrete reflection symmetry group is bounded by hyperplanes } \{ H_i \} \text{ corresponding to those reflections. Any element } w \in W(2)(S_X) \text{ is written as a product of simple reflections and } l(w) \text{ is the minimum number of simple reflections needed in obtain } w.\]

\[\text{This proof is an easy modification of the proof of simple transitivity of reflection groups. See e.g. the proof for Theorem 11.6 in } [33] \text{ or any other textbook on Coxeter groups.}\]
Hence this reason is, for a given \( W \) for a K3 surface \( \xrightarrow\phi \) has the following three properties: i) it is an even lattice, ii) it has signature \((0, \rho_X - 2)\), and finally iii) there exists an Abelian group isomorphism \( G_{W_{\text{frame}}} \cong G_{S_X} \) such that \( q_W = q_{S_X} \). This motivates to consider a set of lattices with these three properties modulo isometry for a K3 surface \( X \):

\[
\mathcal{J}_2(X) := \text{Isom} \left\{ W \mid \text{even } \text{sgn}(W) = (0, \rho_X - 2), \quad (G_W, q_W)^3 \cong (G_{S_X}, q_{S_X}) \right\}. \tag{51}
\]

A map \( \mathcal{J}_0(X) \longrightarrow \mathcal{J}_2(X) \) between the two classifications is given by

\[
\mathcal{J}_0(X) \ni (\phi : U \hookrightarrow S_X \cong \phi(U) \oplus W_{\text{frame}^*}) \quad \longmapsto \quad [W_{\text{frame}^*}] \in \mathcal{J}_2(X), \tag{52}
\]

\[
\mathcal{J}_0(X) \ni [\phi : U \hookrightarrow S_X] \quad \longmapsto \quad [(\phi(U))^\perp \subset S_X] \in \mathcal{J}_2(X), \tag{53}
\]

where the first line used the language of \( (45) \) and the second line that of \( (50) \).

The map \( \mathcal{J}_0(X) \longrightarrow \mathcal{J}_2(X) \) is not necessarily injective, but always factors through

\[
\mathcal{J}_2(X) := \text{Isom}^+(S_X) \setminus \{ \text{embedding } \phi : U \hookrightarrow S_X \} / \{ \pm \text{id}_U \} \cong \text{Isom}(S_X)^{\text{(Amp)}} \setminus \mathcal{J}_0(X), \tag{54}
\]

where we used \( (31) \) in order to obtain the last expression. That is,

\[
\mathcal{J}_0(X) \twoheadrightarrow \left[ \text{Isom}(S_X)^{\text{(Amp)}} \setminus \mathcal{J}_0(X) = \mathcal{J}_2(X) \right] \hookrightarrow \mathcal{J}_2(X). \tag{55}
\]

It is obvious that two elements of \( \mathcal{J}_0(X) \) identified in \( \mathcal{J}_2(X) \) provide frame lattices that are isometric to each other. Conversely, for two embeddings \( \phi : U \hookrightarrow S_X \) and \( \phi' : U \hookrightarrow S_X \), if there is an isometry \( \psi : W \rightarrow W' \), where \( W := [\phi(U)^\perp \subset S_X] \) and \( W' := [\phi'(U)^\perp \subset S_X] \), we can construct an isometry of \( S_X \) by \( (\phi' \cdot \phi^{-1}, \psi) : S_X \cong \phi(U) \oplus W \rightarrow \phi'(U) \oplus W' \cong S_X \).

Furthermore, any lattice \( W \) characterized as in \( \mathcal{J}_2(X) \) can be isomorphic to the frame lattice of some elliptic fibration in \( X \), when \( \text{rank}(T_X) \leq \text{rank}(S_X) - 2 \), or equivalently \( \rho_X \geq 12 \) \[\text{Nish1, SS}\]. Hence \( \mathcal{J}_2(X) \hookrightarrow \mathcal{J}_2^*(X) \) is also surjective for a K3 surface with \( \rho_X \geq 12 \), and

\[
\mathcal{J}_2(X) \cong \mathcal{J}_2^*(X). \tag{56}
\]

This is because, for a given \( W \) in \( \mathcal{J}_2(X) \), \( T_X \oplus (U \oplus W) \) can be embedded into the even unimodular lattice \( \Pi_{3,19} \cong \Lambda_{K3} \) \[\text{Proposition } \beta\], from which a primitive embedding \( T_X \hookrightarrow \Lambda_{K3} \) is given.
\( \zeta \) guarantees that such a primitive embedding is unique modulo \( \text{Isom}(\Lambda_{K3}) \) if \( \text{rank}(T_X) \leq \text{rank}(S_X) - 2 \). Thus there must be an isometry \( [T_X \subset \Lambda_{K3}] =: S_X \cong (U \oplus W) \), and this \( W \) is characterized as the orthogonal complement of some embedding \( U \hookrightarrow S_X \).

Note that both classifications \( J_2(X) \) in (54) and \( J'_2(X) \) in (51) are characterized purely in the language of lattice theory without reference to geometry. Note also that when two elliptic fibrations \( \pi_X \) and \( \pi'_X \) fall into the same element in the \( J_2(X) \mapsto J'_2(X) \) classification, the lattice isometry between the canonical frame lattices of \( \pi_X \) and \( \pi'_X \) is always lifted to an ample-cone preserving isometry of \( S_X \) such that \( U_* \) in \( U_* \oplus W_* \cong S_X \) (resp. \( W_* \) in \( S_X \)) for \( \pi_X \) is mapped to \( U'_* \) in \( U'_* \oplus W'_* \cong S_X \) (resp. \( W'_* \) in \( S_X \)).

Closely related to the \( J_2(X) \) classification of elliptic fibration is the \( J^{(\text{type})}(X) \) classification. For an elliptic fibration \((\pi_S, \sigma; S, C)\), its \textit{type} is the data of how many singular fibres of a given type in the Kodaira classification are present in \( \pi_S : S \to C \). This can be expressed in the form of \( n_1 I_1 + n_2 I_2 + \cdots + m_1 I_1^* + \cdots \). Two elliptic fibrations \((\pi_S, \sigma)\) and \((\pi'_S, \sigma')\) for a given \((S, C)\) are regarded to be \textit{of the same type}, when the type data of these two elliptic fibrations are the same and \( J^{(\text{type})} \) is the set of all possible types of elliptic fibration of a K3 surface \( X \).

The \( J_2(X) \) and \( J'_2(X) \) classifications are very close to the \( J^{(\text{type})}(X) \) classification, in that type data can almost be read out from the root lattice part of the frame lattice, \( W_{\text{root}} \). The root lattice \( W_{\text{root}} \) is in the form of \( \oplus_a R_a \), where \( R_a \) is one of \( A - D - E \) lattice. An \( I_{n+1} \) type singular fibre gives rise to an \( A_n \) component in \( W_{\text{root}} \), \( I_{n+4}^* \) type singular fibre to a \( D_n \) component, \( IV^* \) to \( E_6 \), \( III^* \) to \( E_7 \), \( II^* \) to \( E_8 \), III to \( A_1 \) and IV to \( A_2 \). On the one hand, the frame lattice hence misses certain information, because both \( I_2 \) and III type singular fibres appear as an \( A_1 \) component in \( W_{\text{root}} \), and both \( I_3 \) and IV type singular fibres as an \( A_2 \) component. It is not possible either, at least immediately, to read out the numbers of \( I_1 \) and II type singular fibres from the frame lattice. The frame lattice, on the other hand, contains more data—such as the Mordell–Weil group—than those we can read out from its sublattice \( W_{\text{root}} \subset W_{\text{frame}} \). In section 4.4 we will come across examples of mutually non-isometric frame lattices \( W_{\text{frame}} \) of a given K3 surface \( X \) sharing the same \( W_{\text{root}} \).

Apart from these subtle differences, the two classifications of elliptic fibrations \( J_2(X) \) and \( J^{(\text{type})}(X) \) are still quite close. For this reason, there are some authors where individual isometry classes of frame lattices in \( J_2(X) \) and \( J'_2(X) \) are referred to as types.

### 3.3 The \( J_1(X) \) Classification

It is definitely a question of mathematical interest to consider the classification of elliptic fibrations \((\pi_X, \sigma; X, \mathbb{P}^1)\) modulo automorphisms of \( X \). An equivalence relation is introduced to the set of all possible elliptic fibration data \((\pi_X, \sigma; X, \mathbb{P}^1)\) for a K3 surface \( X \), as defined at the beginning of the Introduction. The quotient space with respect to this equivalence relation is denoted by \( J_1(X) \). This modulo-isomorphism classification of elliptic fibrations is not just interesting as a problem in mathematics, but also the relevant classification scheme in F-theory compactification on a K3 surface \( X \) in the context of string theory [BKW].

Since the image of \( \pi_S : \text{Aut}(X) \to \text{Isom}(S_X)^{(\text{Amp})} \) is defined to be \( \text{Isom}(S_X)^{(\text{Amp Hodge})} \), the classification \( J_1(X) \) is equivalent to

\[
J_1(X) = \text{Isom}(S_X)^{(\text{Amp Hodge})} \backslash J_0(X),
\]

(57)
which can be phrased purely in lattice theory language\textsuperscript{12}:

\[ J_1(X) = \left[ W^{(2)}(S_X) \rtimes \text{Isom}(S_X)^{(\text{Amp Hodge})} \right] \setminus \{ \text{embed. } \phi : U \hookrightarrow S_X \}/\{ \pm \text{id}, U \}. \quad (58) \]

The quotient maps \( J_0(X) \to J_2(X) \) and \( J_0(X) \to J_1(X) \) already determine a map

\[ J_1(X) \to J_2(X) \tag{59} \]

automatically. The fibre of this map is the decomposition of a single orbit of \( \text{Isom}(S_X)^{(\text{Amp Hodge})} \) into orbits of the subgroup \( \text{Isom}(S_X)^{(\text{Amp Hodge})} \). Since \( \text{Isom}(S_X)^{(\text{Amp Hodge})} \) is not necessarily a normal subgroup of \( \text{Isom}(S_X)^{(\text{Amp})} \), however, \( J_2(X) \) is not necessarily regarded as a quotient of \( J_1(X) \).

4 The \( J_2(X) \) Classification

4.1 The \( J_2(X) \) Classification using Niemeier Lattices

(the Kneser–Nishiyama method)

In \cite{Nis1, Nis2} a systematic way to study the \( J_2(X) \) classification which is quite convenient in dealing with K3 surfaces with large Picard number was introduced. Nishiyama’s approach starts with a following observation:

For a K3 surface \( X \) with transcendental lattice \( T_X \), there exists a lattice \( T_0 \) with the three following properties: i) it is an even lattice, ii) its signature is \((0, 26 - \rho)\), and iii) \( (G_{T_0}, q_{T_0}) \cong (G_{T_X}, q_{T_X}) \). This is because Theorem \( \delta \) guarantees that there always exists a primitive embedding

\[ T_X[-1]^{(20-\rho, 2)} \hookrightarrow \left( E_8 \oplus U^\oplus(20-\rho) \right)^{(20-\rho, 28-\rho)}, \tag{60} \]

and the orthogonal complement of the \( T_X[-1] \) sublattice satisfies all the three properties of \( T_0 \) we mentioned above \cite{Nis2}.

Let us now pick one such a \( T_0 \) and consider the map

\[ \Pi^\alpha_{T_0} \left[ \text{Isom}(L(I)) \setminus \left\{ \text{prim. embed. } \phi_{T_0} : T_0 \hookrightarrow L(I) \right\} / \text{Isom}(T_0) \right] \to J_2(X), \tag{61} \]

where \( L(I) \) labelled by Greek letters \( I = \alpha, \beta, \cdots, \omega \) are even unimodular negative definite lattices of rank 24. As reviewed briefly in section 4.2, there are twenty-four mutually non-isometric lattices of that kind, and they are called Niemeier lattices. For any primitive embedding of \( T_0 \) into any one of the Niemeier lattices \( L(I) \), the orthogonal complement \( \left[ (\phi_{T_0}(T_0))^\perp \subset L(I) \right] \) satisfies the properties required for an element in \( J_2(X) \), see \[51\], and this is how the above map is determined. This map is well-defined and furthermore surjective (use Proposition \( \beta \)). This allows us to study the left-hand side of \( (61) \) instead in order to find all the elements in \( J_2(X) \) (see \cite{Nis1}, Lemma 6.3).

\textsuperscript{12} Such conditions as “ample-cone preserving” or “Hodge” on the subgroup \( \text{Isom}(S_X)^{(\text{Amp Hodge})} \) are already well-defined without invoking geometric intuition; the ample cone is one of the fundamental region of the reflection group \( W^{(2)}(S_X) \) acting on the lattice \( S_X \), and the \( \text{Isom}(T_X)^{(\text{Hodge})} \subset \text{Isom}(T_X) \) subgroup is characterized as the stabilizer of a complex plane \( [\Omega_X] \in \mathbb{P}[T_X \otimes \mathbb{C}] \).
One can also show that the map \([61]\) is injective if \(p_{T_0} : \text{Isom}(T_0) \rightarrow \text{Isom}(G_{T_0}, q_{T_0})\) is surjective. This condition is satisfied in many examples of singular K3 surfaces \((\rho = 20)\) that we will be interested in (see Table 2). Thus, it is not too inefficient to study the left-hand side of \((61)\) instead of \(J_2^\prime(X)\) itself.

Overall, the problem of finding the following embeddings and working out orthogonal complements,

\[
T_X^{(2,20-\rho)} \hookrightarrow H_2(X;\mathbb{Z})^{(3,19)}, \quad \phi_U : U^{(1,1)} \hookrightarrow S_{X}^{(1,\rho-1)} := [(T_X)^{\perp} \subset H_2(X;\mathbb{Z})]
\]

modulo isometry is now translated into the similar problem:

\[
T_X[-1]^{(20-\rho,2)} \hookrightarrow \left(E_8 \oplus U^{(20-\rho)}\right)^{(20-\rho,28-\rho)}, \quad \phi_{T_0} : [T_0^{(0,26-\rho)} : (T_X[-1])^{\perp}] \subset (L^{(1)})^{(0,24)}.
\]

The (isometry class of) frame lattice \(W_{\text{frame}}\) is obtained either as \([\phi_U(U)]^{\perp} \subset S_X\) or as \([\phi_{T_0}(T_0)]^{\perp} \subset L^{(1)}\). The latter problem is suitable for systematic calculations for various K3 surfaces with large Picard number. The target of the embedding \(\phi_{T_0}, L^{(1)}\), does not depend on individual choices of \(T_X\) and is also technically easier because of the (negative) definite symmetric pairing of \(L^{(1)}\).

### 4.2 A Brief Review on Niemeier Lattices

Even unimodular lattices with signature \((r_+, r_-)\) are known to be unique modulo isometry, if both \(r_+\) and \(r_-\) are non-zero, as reviewed in section 2.1. However, there can be more than one isometry classes of negative definite (i.e., \(r_+ = 0\)) even unimodular lattices of given rank \(r_- = 16, 24, \ldots\) (see [CS] for more information).

For example, there are two mutually non-isometric negative definite even unimodular lattices of rank \(r_- = 16\). One is \(L = E_8 \oplus E_8\). The other one, \(L'\), is characterized as an overlattice of \(D_{16}\). The discriminant group \(G_{D_{16}} = D_{16}/D_{16}\) is isomorphic to \(\mathbb{Z}/2\mathbb{Z}\langle \text{sp}\rangle \times \mathbb{Z}/2\mathbb{Z}\langle \text{sp}\rangle\). Both \(\mathbb{Z}/2\mathbb{Z}\) subgroups generated by the weights of the spinor/cospinor representations are isotropic subgroups of \((G_{D_{16}}, q_{D_{16}})\) (\(D_n\) with \(n\) divisible by 8). The lattice \(L' = D_{16}; \Delta = \mathbb{Z}/2\mathbb{Z}\langle \text{sp}\rangle\) or \(\Delta = \mathbb{Z}/2\mathbb{Z}\langle \text{sp}\rangle\) (see Proposition \(\alpha\)) becomes the other even unimodular negative definite lattice of rank-16. \(D_{16}; \Delta\) with the two different choices of \(\Delta\) are isometric. That is, \(L'\) is obtained by adding the weights of either the spinor or cospinor representation to the root lattice of \(D_{16}\).

Similarly, there are twenty-four isometry classes of negative definite even unimodular lattices of rank 24 that are called Niemeier lattices. Following [CS] we denote them by \(L^{(1)}\) with \(I = \alpha, \beta, \ldots, \psi, \omega\). Out of those twenty-four rank-24 lattices, twenty-three \((I \neq \omega)\) allow a description like the one just given for \(L'\). They can be specified by choosing of a rank-24 lattice given as the direct sum of \(A-D-E\) lattices \(\oplus_a R_a = L^{(1)}_{\text{root}}\), and an appropriate isotropic subgroup \(\Delta\) of \(G_{\oplus_a R_a}\).

For a fixed Niemeier lattice, the dual Coxeter numbers of all components \(R_a\) are equal and denoted by \(h^{(I)}\). This information is displayed in Table 3. There is one more rank-24 even unimodular negative definite lattice, which is called the Leech lattice and is denoted by \(L^{(I=\omega)} = \Lambda_{24}\), that does not have any norm \((-2)\) points and correspondingly cannot be described in a similar fashion to the other Niemeier lattices. More information on Leech lattice is provided in our review in the appendix A.1.
Table 1: The Niemeier lattices: out of the 24 Niemeier lattices $L(I)$ with $I = \alpha, \beta, \cdots, \psi, \omega$, twenty-three have a rank-24 root lattice, $L_{\text{root}}^{(I)} = \oplus \alpha R_\alpha$, while the root lattice for $L(I=\omega)$ is empty. The root lattices are given in the second column. The quotient $\Delta = L(I)/L_{\text{root}}^{(I)}$ must be an isotropic subgroup of the discriminant group $G_{\oplus \alpha R_\alpha}$ for all the cases except $I = \omega$. Generators of $\Delta \subset G_{\oplus \alpha R_\alpha}$ are specified in detail for the first twelve Niemeier lattices (in the left-hand side) in this table; subscripts attached to the generators, $m$ of $(g)_m$, carry the information that they generate a $\mathbb{Z}/m\mathbb{Z}$ $(g)$ factor in $\Delta$. For the next eleven Niemeier lattices, the limited space in this table is not enough to include detailed information on the generators, see Table 16.1 of [CS] for more information. In this table, we have introduced abbreviated notations $R_1 R_2 \cdots$ for $R_1 \oplus R_2 \oplus \cdots$, $\mathbb{Z}_m$ for $\mathbb{Z}/m\mathbb{Z}$, and $G_1 G_2 G_3 \cdots$ for $G_1 \times G_2 \times G_3 \times \cdots$, to save space.

4.3 Computing $T_0$

The first step of the program reviewed in section 4.1 (the Kneser–Nishiyama method) is to compute the lattice $T_0$ for a given K3 surface $X$ with the transcendental lattice $T_x$. This is done by first embedding $T_x[-1]$ primitively into $E_8 \oplus U^{(20-\rho)}$ and then using $T_0 = T_x[-1]$. Theorem $\epsilon$ does not guarantee uniqueness (modulo isometry of $E_8 \oplus U^{(20-\rho)}$) of primitive embeddings of $T_x[-1]$, but this is not a problem. The crucial point is that any $T_0$ satisfying the properties i)–iii) does the job of [61] for a given K3 surface $X$ [Nish1, Nish2]. Hence we only need to find one primitive embedding of $T_x[-1]$ and can then use the resulting $T_0$ for a given $X$. 
For some K3 surfaces, possible choices of $T_0$ can be found in the literature.\(^{13}\)

- Reference [Nish1] contains examples of $T_0$ for the four singular K3 surfaces ($\rho_X = 20$), $X = X_1, X_3, \text{Km}(E_i \times E_i)$ and $\text{Km}(E_\omega \times E_\omega)$, where $T_0 = D_6, E_6, A_3 \oplus A_3$ and $D_4 \oplus A_2$, respectively. The same reference also contains the result $T_0 = D_4 \oplus A_3$ for the slightly more general class of Kummer surfaces $X = \text{Km}(E \times E)$ (isogenous case), where $\rho_X = 19$ and $T_0 = D_4 \oplus D_4$ for the even more general class of Kummer surfaces $X = \text{Km}(E \times F)$ (product type), where $\rho_X = 18$.

- The most general class of Kummer surfaces, $X = \text{Km}(A)$, are associated with generic Abelian surfaces $A$, and $\rho_X = 17$. In this case, we can take $T_0$ as $[A_3 \oplus A_1^{\oplus 6}]H$ with the isotropic subgroup given by [Ko2]:

$$G_{A_3 \oplus A_1^{\oplus 6}} \cong (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^6 \supset H \cong \mathbb{Z}_2. \quad (64)$$

In this article, we are interested in K3 surfaces with large Picard number, particularly in singular K3 surfaces ($\rho_X = 20$). The present authors came up with a list of thirty-four singular K3 surfaces that can be used for a particular class of string compactifications [BKW]. Those thirty-four K3 surfaces are listed in the first column of Table 7. To explain the notation used in the table and elsewhere in this article, note that when the transcendental lattice $T_X$ is given an oriented basis\(^{14}\) \(\{f_2, f_1\}\), and the symmetric pairing of $T_X$ is given by\(^{14}\)

$$\begin{bmatrix}
  (f_1, f_1) & (f_1, f_2) \\
  (f_2, f_1) & (f_2, f_2)
\end{bmatrix} = \begin{bmatrix}
  2a & b \\
  b & 2c
\end{bmatrix}, \quad (65)$$

this determines a unique singular K3 surface $X$ [SI]. This K3 surface $X$ is denoted by $X[a \ b \ c]$ in this article, and the transcendental lattice of $X[a \ b \ c]$ may sometimes be abbreviated by a symbol $[a \ b \ c]$.

We have computed a possible choice for $T_0$ for twenty-four singular K3 surfaces among the thirty-four listed in Table 7, and the result is presented in Table 2. Four among them are the same already obtained in [Nish1].\(^{15}\) There are eleven singular K3 surfaces for which $T_0$ can be taken to be a direct sum of $A$–$D$–$E$ root lattice. For the other thirteen cases, the result of the computation is presented by describing a sublattice given by $(T_0)_{\text{root}}$ and its orthogonal complement in $T_0$ and supplying information of the quotient by this sublattice:

$$\left((T_0)_{\text{root}} \oplus [(T_0)_{\text{root}}] \subset T_0\right); H, \quad H := T_0/\left((T_0)_{\text{root}} \oplus [(T_0)_{\text{root}}] \subset T_0\right). \quad (66)$$

We find this way of describing lattices convenient (partially with application in physics in mind), and will use this also in Tables 4 and 5.

Let us briefly sketch how to compute $T_0$ in practice. The first step is to find a primitive embedding $\phi : T_X[-1] \hookrightarrow E_8$, which is not difficult for a singular K3 surface $X$ because the

\(^{13}\)The appendix [A3] provides a quick summary of various definitions, explanations and facts on Kummer surfaces that we need in this article.

\(^{14}\)We call \(\{f_2, f_1\}\) an oriented basis of $T_X = \text{Span}_\mathbb{Z}\{f_1, f_2\}$ if $\text{Im} \{\Omega_X, f_2/\Omega_X, f_1\} > 0$ for the complex structure of $X$, $[\Omega_X] \in \mathbb{P}[T_X \otimes \mathbb{C}]$.

\(^{15}\)We have adopted a convention of writing this matrix in the order of $f_1$ and $f_2$ (rather than $f_2$ and $f_1$) and parametrizing the matrix with $a$, $b$ and $c$ as in the main text, by following a literature in string theory [AK]; thus, $[a \ b \ c]$ in [AK] and also in this article should be read as $[c \ b \ a]$ in [SI].

\(^{16}\) $X[1 \ 0 \ 1] = X_4, X[1 \ 1 \ 1] = X_3, X[2 \ 0 \ 2] = \text{Km}(E_i \times E_i)$ and $X[3 \ 2 \ 2] = \text{Km}(E_\omega \times E_\omega)$.\[21\]
Table 2: The $T_0$ lattice (the 2nd column) for twenty-four singular K3 surfaces (the 1st column), with some additional information (the 3rd–5th columns) to be used in section 5. We use the same abbreviated notation as in Table 1. Out of the thirty-four singular K3 surfaces appearing in [BKW] (and also in Table 7), we have included all cases for which $T_0$ can be taken to be a root lattice.
transcendental lattice $T_X$ is just a rank-2 lattice in this case. The $E_8$ lattice can be described as:

$$E_8 \ni \left\{ \sum_{i=1}^{8} n_i L_i \mid n_i \in \mathbb{Z}, \sum_i n_i \equiv 0 \mod 2 \right\} \cup \left\{ \sum_{i=1}^{8} \left( \frac{1}{2} + m_i \right) L_i \mid m_i \in \mathbb{Z}, \sum_i m_i \equiv 0 \mod 2 \right\}. \tag{67}$$

In case 2a (or 2c) is 2, the images of $f_1$ (or $f_2$) should be chosen from the the $(-2)$-roots of $E_8$. If $f_1$ or $f_2$ are norm 4 vectors in $T_X$ (i.e., 2a or 2c is 4), then the image should be chosen from one of the vectors of the form

$$(\pm 2, 0^7), \quad ((\pm 1)^4, 0^4), \quad (\pm 3, (\pm 1)^7)/2. \tag{68}$$

Norm $(-6)$ elements in $E_8$ are of the form

$$(\pm 2, (\pm 1)^2, 0^5), \quad ((\pm 1)^6, 0^2), \quad ((\pm 3)^2, (\pm 1)^6)/2, \tag{69}$$

which can be used as the images of $f_{1,2}$ if either 2a or 2c is 6. To take the case $[a \ b \ c]=[5 \ 1 \ 1]$ as an example, we can take

$$\phi(f_2) = \frac{L_1 + \cdots + L_8}{2} = (1, \ldots, 1)/2,$$
$$\phi(f_1) = 2L_1 + (L_2 + L_3 + L_4) - (L_5 + L_6 + L_7) = (2, 1^3, (-1)^3, 0), \tag{70}$$

so that $(\phi(f_2), \phi(f_1)) = -1$. This embedding $\phi : T_X [-1] \hookrightarrow E_8$ is primitive, as $\phi(f_2)$ is chosen from the second half of $[67]$, while $\phi(f_1)$ from the first half. Note that $\phi(f_2)$ has coefficient $\pm 1/2$ for some $L_i$’s ($L_8$ in this case), while $\phi(f_1)$ has vanishing coefficient. Similar choices of $\phi(f_2)$ and $\phi(f_1)$ for other singular K3 surfaces easily produce primitive embeddings of $T_X [-1]$ into $E_8$.

The second step is to work out the orthogonal complement $T_0$, which is also almost straightforward. In the case of $[a \ b \ c]=[5 \ 1 \ 1]$ we first see that the five roots

$$(0, 1, -1, 0, 0^3, 0), \quad (0, 0, 1, -1, 0^3, 0), \quad (0, (-1)^2, 1, -1, 1^2, -1)/2, \quad (0, 0^3, 1, -1, 0, 0), \quad (0, 0^3, 0, 1, -1) \tag{71}$$

of $E_8$ are orthogonal to $\phi(T_X [-1])$ and generate an $A_5$ lattice. A primitive vector

$$\bar{\epsilon}_{114} := \frac{1}{2} (9, (-7)^3, (-1)^3, 15) \tag{72}$$

of $E_8$ is also in $[\phi(T_X [-1])^\perp \subset E_8]$, yet it is also orthogonal to $A_5$. The lattice $T_0 = [\phi(T_X [-1])^\perp \subset E_8]$ contains a rank-6 lattice $A_5 \oplus \mathbb{Z} \langle \bar{\epsilon}_{114} \rangle = A_5 \oplus (-114)$, but there are some elements left over. The lattice $T_0$ as the orthogonal complement of $\phi(T_X [-1])$ contains an element

$$E_8 \ni \frac{1}{2} (-1, 1^3, -1, 1^2, -3) = \frac{\alpha_1 + 2\alpha_2 + 3\alpha_3 - 2\alpha_4 - \alpha_5}{6} - \frac{1}{6} \bar{\epsilon}_{114}, \tag{73}$$

which is not within the sublattice $A_5 \oplus (-114)$; here, $\alpha_1, \ldots, 5$ are the five simple roots of $A_5$. Modulo $A_5 \oplus \mathbb{Z} \langle \bar{\epsilon}_{114} \rangle$, this element can be regarded as an isotropic element $18^{18} (-1, -19) \in G_5 \oplus (-114) \cong \mathbb{Z}/6\mathbb{Z} \langle \omega^1 \rangle \times \mathbb{Z}_{114} \langle \bar{\epsilon}_{114}/114 \rangle$, and generates a subgroup $\mathbb{Z}/6\mathbb{Z} \subset \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/114\mathbb{Z}$. This result is

---

\(^{17}\)The negative definite symmetric pairing in this $E_8$ lattice is given by $(L_i, L_j) = -\delta_{ij}$.

\(^{18}\) $q((-1, -19)) = q_{A_5}(-1) + q_{114}(-19) = (-5/6)(-1)^2 + (-1/114)(-19)^2 = -5/6 - 19/6 \equiv 0 \mod 2.$
already occur in Table 2. The remaining five are D type and studying their primitive embeddings into lattices. We can answer this question by compiling a list of all rank-6 lattices of this

It turns out from the computation of T₀, that the T₀ lattice for the singular K3 surface X₁₁₁ and that of X₁₂₂ are mutually isometric. This is enough to conclude that

\[ \mathcal{J}_2(X_{6 1 1}) = \mathcal{J}_2(X_{3 1 2}). \]  

(74)

One will also note from Table 2 that there exist two inequivalent embeddings of [3 2 2] into the root lattice of E₈. The first one was found with the technique explained above, whereas the second one, T₀ = A₄ ⊕ A₅/₂, can be found in the literature [Nish1]. In the second case, T₀ is a sum of A–D–E root lattices. The existence such a choice of T₀ is irrelevant for the \( \mathcal{J}_2 \) classification (except for maybe some practical simplifications), where having one T₀ satisfying the conditions spelled out at the beginning of Section 4.1 is sufficient. However, this question becomes relevant in the light of the results presented in Section 5.2 which are only applicable to cases in which T₀ is a sum of root lattices. This motivates to ask for all possibilities where T₀ can be chosen to be a sum of root lattices. We can answer this question by embedding by compiling a list of all rank-6 lattices of this type and studying their primitive embeddings into E₈. The resulting orthogonal complements then tell us about the corresponding Tₓ.

There are sixteen rank-6 lattices given by a direct sum of A–D–E root lattices, out of which 11 already occur in Table 2. The remaining five are \( D₁ A₁/², A₃ A₅/₃, A₅/₃, A₂ A₄/₄ \) and \( A₅/₆ \). It follows from Table 2.3 in [Nish1] that none of these allow a primitive embedding into E₈. Note that this also means that they can never have the same discriminant form as one of the \( Tₓ = \langle a \ b \ c \rangle \), as this would imply the existence of such an embedding. Furthermore, using results of [Nish1] one can show that all primitive embeddings of the eleven (sums of) A–D–E roots lattices appearing in Table 2 are unique, so that we have found all singular K3 surfaces for which T₀ can be a sums of A–D–E root lattices.

4.4 The \( \mathcal{J}_2(X) \) Classification for Singular K3 Surfaces: New and Known Results

The \( \mathcal{J}_2(X) \) classification of elliptic fibrations has already been worked out for some K3 surfaces \( X \) with large Picard number \( ρ_X \). To name a few,

- For \( X = \text{Km}(E × F) \), where \( ρ_X = 18 \), there are 11 entries in \( \mathcal{J}_2(X) \). See Ref. [Og] or Table 6 in the next section (which carries partial information of the results obtained in [Og]).
- For four singular K3 surfaces \( X = X₁, X₂, \text{Km}(E_i × E_i) \) and \( \text{Km}(E_ω × E_ω) \), and for \( \text{Km}(E × E) \), where \( ρ_X = 19 \). Ref. [Nish1] worked out the \( \mathcal{J}_2(X) \) classification through the procedure reviewed in section 4.1. \( \mathcal{J}_2(X) \) consists of 13, 6, 63 and 30 entries for these four singular K3 surfaces above, respectively, and \( |\mathcal{J}_2(X)| = 34 \) for \( X = \text{Km}(E × E) \).
- For \( X = \text{Km}(A) \), where \( ρ_X = 17 \), there are 25 entries in \( \mathcal{J}_2(X) \). See Ref. [Ku], or Table 8 in the next section of this article (which carries partial information of the results in [Ku]).
- For \( X = X_{2 0 1} \), one can choose \( T₀ ≃ D₅ ⊕ A₁ \), and \( |\mathcal{J}_2(X)| = 30 \) as worked out in [BL].

\footnote{In the case of \( \{a \ b \ c\}=\{5 1 1\} \), \( \text{discr}(T₀) = |\text{discr}(A₅)| × |\text{discr}(−114)|/|\mathbb{Z}/6\mathbb{Z}|² = (−6) × (−114)/6² = 19 \), which agrees with \( \text{discr}(Tₓ[−1]) = (−10)(−2) − (−1)² = 19 \).}

\footnote{There can be more than one primitive embeddings of this \( T₀ = [A₄ A₁(−230)]; Z₁₀ \) into \( E₈ \), since Theorem \( ε \) for uniqueness of primitive embeddings (modulo \( \text{Isom}(E₈) \)) cannot be applied to this case.}

\footnote{We expect that this should be extendable to cases where \( T₀ \) is an overlattice of a sum of A–D–E root lattices, but we do not discuss this here.}
Table 3: The $J_2(X)$ classification for $X_3$, quoted from \[\text{Nish1}\]. Each entry is obtained by embedding $T_0 \cong E_6$ primitively into one of the Niemeier lattices $L(I)$ specified in the second column; Greek letters correspond to those in Table 1. The last column indicates the type of ‘extremal elliptic K3’ classified in $[\text{SZ}]$. An extremal elliptic K3 surface is a K3 surface with elliptic fibration satisfying $\rho = 20$ and $\text{rank}(MW) = 0$.

Each entry of $J_2(X)$ for a given K3 surface $X$ is characterized by an isometry class of frame lattice $W_{\text{frame}}$. Once the frame lattice $W_{\text{frame}}$ is given, one can extract information about various objects. Its sublattice $W_{\text{root}}$ generated by all the norm $(-2)$ elements of $W_{\text{frame}}$ corresponds to the collection of singular fibres. The Mordell–Weil group of can be computed as the Abelian quotient group (see $[\text{Shi}]$, Thm. 1.3)

$$\text{MW} := W_{\text{frame}} / W_{\text{root}}.$$ (75)

Table 1.1 of $[\text{Nish1}]$—the result of $W_{\text{root}}$ and the Mordell–Weil group for the six distinct isometry classes of $W_{\text{frame}}$ in $J_2(X)$ for a singular K3 surface $X = X_3 = X_{[1 1 1]}$—is reproduced here as Table 3 for the convenience of the reader. With string theory applications in mind, however, more interesting objects are

$$W_{\text{gauge}} := W_{\text{root}} \oplus L(X), \quad L(X) := \left[(W_{\text{root}})^\perp \subset W_{\text{frame}} \right]$$ (76)

and the subgroup $(W_{\text{frame}}/W_{\text{gauge}}) \subset G_{\text{gauge}}$ $[\text{BKW}]$. For $X = X_3$, this information is summarized in section 4.1 of $[\text{BKW}]$. Here, $L(X)$ is called the essential lattice of an elliptic fibration on a K3 surface $X$. It is also isometric to $MW(X)^0[-1]$, where $MW(X)^0$ is the narrow Mordell–Weil lattice (see $[\text{Shi}]$, Thm. 8.9).

In all the cases referred to above, the lattice $T_0$ can be chosen to be either a direct sum of root lattices of A–D–E type, or an overlattice of a direct sum of root lattices of A–D–E type. Not all the $T_0$ lattices in Table 2 are of that form. However, even when $T_0$ does not contain a rank-6 root lattice, there is nothing preventing us from carrying out the procedure described in section 4.1.

As another example, let us consider the singular K3 surface $X_{[3 0 2]}$. As in Table 2, we can take $T_0 = A_5 \oplus (-4)$ for this singular K3 surface. The $A_5$ component has to be embedded primitively into the sublattice $(L(I))_{\text{root}} \subset L(I)$ of the Niemeier lattices, and the results of §4.1 of \[\text{Nish1}\] can be used for this part of the calculation. One still has to work out all possible embeddings of the generator of the $(-4)$ part of $T_0$, and make sure that the embedding into $L(I)$ is primitive.

Instead of carrying out the $J_2(X)$ classification for $X_{[3 0 2]}$ completely, we first carried out the part obtained from embedding into the Niemeier lattices $L(I)$ with $I = \alpha, \beta$ and $\gamma$; the results are found in Table 4. As a second part, we have studied all embeddings into the remaining Niemeier...
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$L^{(I)}$ & $A_5$ & $\lambda$ & $\alpha$ & $\beta$ & $\gamma$
\hline

$\leqslant 4\n v \in D_{18} \subset A_5^+$ & $\leqslant 4\n v \in D_{18} \subset A_5^+$ & $\lambda^4 v \in D_{16}$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$ & $E_8^{(1)}$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$ & $\lambda^4 v \in D_{10} \subset A_5^+$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$ & $E_8^{(1)}$ & $\lambda^4 v \in D_{10} \subset A_5^+$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$
\hline
sym$^2 v \in D_{18} \subset A_5^+$ & $\leqslant 4\n v \in D_{16}$ & $\lambda^4 v \in D_{16}$ & $A_1 A_2 D_{12} + \mathbb{Z}_2$ & $A_1 A_2 D_{12} + \mathbb{Z}_2$ & $E_8^{(1)}$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$ & $\lambda^4 v \in D_{10} \subset A_5^+$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$ & $E_8^{(1)}$ & $\lambda^4 v \in D_{10} \subset A_5^+$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$ & $A_2 A_1 + [A_3 D_{12}]; \mathbb{Z}_2$
\hline
\hline
\end{tabular}
\end{table}

Table 4: $\mathcal{J}_2(X)$ classification for a K3 surface $X = X_{[\beta \circ \gamma]}$. This table only shows the entries of $\mathcal{J}_2(X)$ obtained from primitive embeddings of $T_0 = A_5 \oplus (-4)$ into Niemeier lattices $L^{(I)}$ with $I = \alpha, \beta, \gamma$. The first column indicates which one of twenty-three Niemeier lattices $T_0$ is embedded into and the second and third columns specify the particular irreducible root lattices $R_a$ in $(L^{(I)})_{\text{root}} \cong \oplus_a R_a$ into which $A_5$ and $(-4)$ are embedded.
lattices for which the frame lattice contains a root lattice of type $E_6$, $E_7$ or $E_8$, the results are found in Table 5. Together, the two tables hence contain all the isometry classes of the frame lattice available in $X_{[3,0,2]}$ for which singular fibres of type $IV^*$, $III^*$ or $II^*$ occur. Overall, there are 43 distinct isometry classes in the $J_2(X_{[3,0,2]})$ classification containing either $E_6$, $E_7$ or $E_8$.

The resulting frame lattice $W_{\text{frame}}$ is expressed in Table 4 in the form of $W_{\text{gauge}}: (W_{\text{frame}}/W_{\text{gauge}})$, in order to make it easier to read out the information necessary in string theory applications.

Note that this list already contains some interesting subtleties. Let us focus on cases for which $W_{\text{root}} = A_1 D_8 E_7$; the 15th entry of Table 4 and the 12th and 18th entries of Table 5 share this property. Although the Mordell–Weil group is of rank 2 for all the three cases, the first one is without a torsion part, while the last two have $Z_2$ torsion. Furthermore, the last two cases share the same $W_{\text{root}}$ and the Mordell–Weil group, yet their frame lattices are not mutually isometric. The same relation holds between the 4th and 5th entries of Table 5.

5 The $J_1(X)$ Classification

5.1 Uniform Upper Bounds on Multiplicity

The classification $J_1(X)$ of elliptic fibrations $(\pi_X, \sigma; X, P^1)$ contains finer information than the $J_2(X)$ classification. For a given isometry class of frame lattice $[W] \in J_2(X)$, there may be more than one isomorphism classes of elliptic fibrations $(\pi_X, \sigma; X, P^1)$ in $J_1(X)$. In this section, we make an attempt at deriving upper bounds on this number, the number of isomorphism classes for a given $[W] \in J_2(X)$, which we call its multiplicity. The $J_1(X)$ classification is not only an interesting mathematical question, but also relevant to string theory applications [BKW].

The $J_1(X)$ classification has been worked out completely for $X = \text{Km}(E \times F)$ in [Og], the result is summarized in Table 6. The study of [Og] relies, to some extent, on things that are specific to the particular family of K3 surfaces $X = \text{Km}(E \times F)$, but there are also ideas and structures in [Og] that can also be applied to other K3 surfaces with large Picard number. In these cases we can apply them for deriving upper bounds on the multiplicities rather than for precisely determining them.

The first thing to notice is

**Proposition C**: for a K3 surface $X$, the number of isomorphism classes of elliptic fibration (multiplicity) is bounded uniformly from above by the number of elements of the coset space

\[
\text{Isom}(S_X)^{(\text{Amp Hodge})} \backslash \text{Isom}(S_X)^{(\text{Amp})} \cong \left[ W^{(2)}(S_X) \times \text{Isom}(S_X)^{(\text{Amp Hodge})} \right] \backslash \text{Isom}^+(S_X)
\]

(77)

for any isometry class $[W] \in J_2(X)$, because the difference between the two classifications $J_2(X)$ and $J_1(X)$ is only in the choice of the quotient group. ■

In practice, though, it is not easy to compute the coset space (77) for many different K3 surfaces without a great deal of knowledge about the geometry and the Neron–Severi lattice. Easier to use for K3 surfaces with large Picard number is

**Proposition C’**: for a K3 surface $X$, the number of isomorphism classes of elliptic fibration (multiplicity) is bounded uniformly from above by the number of elements of
| $L^{(1)}$ | $A_5$ | $W_{\text{frame}} = [T_0^\perp \subset L^{(1)}]$ | $W_{\text{root}}$ | MW |
|---|---|---|---|---|
| $\zeta$ | $A_{17}$ | $(0^{14}, 1^2, -1^2) \in A_{17}$ | $E_7A_7A_1A_1(-36)(-24); \mathbb{Z}_{12} \times \mathbb{Z}_4$ | $A_1A_7E_7$ | $\mathbb{Z} \times \mathbb{Z}$ |
| & | $g_{\zeta}$ | $E_6A_8A_2(-90)(-240); \mathbb{Z}_{90} \times \mathbb{Z}_3$ | $A_2A_3E_6$ | $\mathbb{Z} \times \mathbb{Z}$ |
| & | $2g_{\zeta} \mod A_{17} \oplus E_7$ | $E_7A_5A_5(-12); \mathbb{Z}_6$ | $A_5^{12}E_7$ | $\mathbb{Z}$ |
| $\eta$ | $D_{10}$ | $(0^6, 1^4) \in D_{10}$ | $E_7 + [A_3(-6)E_7]; \mathbb{Z}_2$ | $A_3E_7^2$ | $\mathbb{Z}$ |
| & | $(0^6, 2.0^3) \in D_{10}$ | $A_3(-6)E_7^2; \mathbb{Z}_2$ | $A_3E_7^3$ | $\mathbb{Z}$ |
| & | $g_{\eta 1}$ | $A_3E_6E_7(-12)(-48); \mathbb{Z}_4 \times \mathbb{Z}_6$ | $A_3E_6E_7$ | $\mathbb{Z} \times \mathbb{Z}$ |
| & | $D_{10} + E_7$ | $(-4)(-6)A_1^2D_6E_7; Z_2^3$ | $A_1^2D_6E_7$ | $\mathbb{Z} \times \mathbb{Z}$ |
| & | $(1^2, -1^2, 0^4) \in E_7$ | $D_4D_5A_1E_7(-6); (Z_2)^2$ | $A_1D_5E_7$ | $\mathbb{Z}$ |
| & | $g_{\eta 1} - g_{\eta 2}$ | $A_3(-6)(-12)(-48)E_6E_6; \mathbb{Z}_3^2 \times \mathbb{Z}_5^2$ | $A_3E_6E_6$ | $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ |
| $E_{7,1}$ | | $(1^2, -1^2, 0^4) \in D_{10}$ | $D_0A_3A_1(-6)E_7; \mathbb{Z}_2 \times \mathbb{Z}_2$ | $A_1A_3D_6E_7$ | $\mathbb{Z} \times \mathbb{Z}_2$ |
| & | $(2, 0^9) \in D_{10}$ | $D_9 + [A_1(-6)E_7]; \mathbb{Z}_2$ | $A_1D_9E_7$ | $\mathbb{Z} \times \mathbb{Z}_2$ |
| & | $D_{10} + E_7$ | $[A_1D_9E_7; \mathbb{Z}_2 + (-6)]; \mathbb{Z}_2 + (-4)$ | $A_1D_9E_7$ | $\mathbb{Z} \times \mathbb{Z}_2$ |
| & | $g_{\eta 1}$ | $A_6A_1E_7(-240); \mathbb{Z}_2 \times \mathbb{Z}_{10}$ | $A_1A_6E_7$ | $\mathbb{Z} \times \mathbb{Z}_2$ |
| & | $g_{\eta 2}$ | $A_9A_1(-6)E_6(-240); \mathbb{Z}_6 \times \mathbb{Z}_{10}$ | $A_1A_9E_6$ | $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ |
| & | $g_{\eta 1} - g_{\eta 2}$ | $D_9A_1E_6(-12)(-48); \mathbb{Z}_{24}$ | $A_1D_9E_6$ | $\mathbb{Z} \times \mathbb{Z}_2$ |
| $E_{7,2}$ | | $(1^2, -1^2, 0^4) \in D_{10}$ | $A_3D_9E_7; \mathbb{Z}_2 + A_2$ | $A_2A_3D_9E_7$ | $\mathbb{Z}_2$ |
| & | $(2, 0^9) \in D_{10}$ | $D_9A_2E_7$ | $A_2D_9E_7$ | $\{1\}$ |
| & | $D_{10} + (A_2 = A_5^3 \subset E_7)$ | $[A_1D_9E_7; \mathbb{Z}_2 + (-4)(-6)]; \mathbb{Z}_2$ | $A_1D_9E_7$ | $\mathbb{Z} \times \mathbb{Z}_2$ |
| & | $g_{\eta 2}$ | $A_2 + [A_6E_6(-240)]; \mathbb{Z}_{20}$ | $A_2A_6E_6$ | $\mathbb{Z}$ |
| $\lambda$ | $A_{11}$ | $(1^2, -1^2, 0^4) \in A_{11}$ | $A_5(-12)(-12)D_7E_6; \mathbb{Z}_2 \times \mathbb{Z}_{12}$ | $A_5^2D_7E_6$ | $\mathbb{Z} \times \mathbb{Z}$ |
| & | $(1, 1, -1, -1, 0^4) \in D_{7}$ | $A_5(-12)A_3^2E_6; \mathbb{Z}_{12}$ | $A_3^2E_6E_6$ | $\mathbb{Z}$ |
| & | $(2, 0^9) \in D_{7}$ | $A_5(-12)E_6; \mathbb{Z}_6 + D_6$ | $A_5D_6E_6$ | $\mathbb{Z}$ |
| & | $A_{11} + D_7$ | $A_3D_5A_1E_6(-12)(-4); \mathbb{Z}_{12} \times \mathbb{Z}_4$ | $A_1A_3D_5E_6$ | $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ |
| & | $3g_{\lambda} \mod E_6 \oplus D_7$ | $[A_2A_6(-18)(-4032) + E_6; \mathbb{Z}_3]; \mathbb{Z}_4 \mathbb{Z}_4 \mathbb{Z}_7$ | $A_5^2A_6E_6$ | $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_3$ |
| & | $6g_{\lambda} \mod A_{11} \oplus E_6 \oplus D_7$ | $[A_5D_6(-48)]; \mathbb{Z}_{12}$ | $A_5D_6E_6$ | $\mathbb{Z}$ |
| $D_7$ | $A_{11}$ | $A_5A_7; \mathbb{Z}_4 + (-24)(-6)(-4)E_6; \mathbb{Z}_{24}$ | $A_5A_7E_6$ | $\mathbb{Z}^2 \times \mathbb{Z}_2$ |
| & | $D_7$ | $(-6) + A_1E_6; \mathbb{Z}_3$ | $A_1E_6$ | $\mathbb{Z} \times \mathbb{Z}_3$ |
| & | $3g_{\lambda} \mod (E_6 \oplus D_7)$ | $[A_8A_2] \begin{bmatrix} -90 & 36 \\ 36 & -360 \end{bmatrix} + E_6; \mathbb{Z}_3]; \mathbb{Z}_{108}$ | $A_5^2A_6E_6$ | $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_3$ |
| & | $6g_{\lambda} \mod 11 \oplus E_6 \oplus D_7$ | $A_5A_6(-6)(-48); \mathbb{Z}_{12} \times \mathbb{Z}_3$ | $A_5E_6$ | $\mathbb{Z} \times \mathbb{Z}_3$ |
| $\mu$ | $E_6$ | $(0, 1^2, -1^2, 0^3) \in E_6^{(2)}$ | $A_1 + (-12)D_4E_6^2; \mathbb{Z}_6$ | $A_1D_4E_6^2$ | $\mathbb{Z}$ |
| & | $E_6^{(2)} + E_6^{(3)}$ | $(-4)A_1 + E_6A_5^2; \mathbb{Z}_2$ | $A_1A_5E_6^2$ | $\mathbb{Z} \times \mathbb{Z}_3$ |
| & | $A_1 + E_6^{(2)}$ | $(-4) + A_5E_6; \mathbb{Z}_3$ | $A_5E_6^2$ | $\mathbb{Z} \times \mathbb{Z}_3$ |

Table 5: $J_2(X)$ classification for a K3 surface $X = X_{13 \oplus 2}$. This table only shows the entries of $J_2(X)$ obtained from primitive embeddings of $T_0 = A_5 \oplus (-4)$ into Niemeier lattices (apart from those already considered in Table 4) such that $T_0^\perp$ contains one of the lattices $E_6$, $E_7$ or $E_8$. The columns are as in Table 4. The $g_I$’s appearing in the 3rd column are glue vectors of the indicated Niemeier lattices, i.e. generators of $\Delta \subset G_{L^{(1)}}$ shown in Table 1.
and hence \( t \in G \) to be in the same orbit of \( \{G \} \). This upper bound is generally weaker than that of Proposition C, but the two are the same if the homomorphism \( p_S : \text{Isom}^+(S_X) \to \text{Isom}(q) \) is surjective.

**Proof:** Let \( G^\text{tot} = \text{Isom}(q), G^{(s)} = p_S[\text{Isom}(S_X)^{\text{(Amp)}}], G^{(t)} = p_T[\text{Isom}(T_X)^{\text{(Hodge)}}], H = G^{(t)} \cap G^{(s)}, \) and \( G^{\text{relev}} \) be the subgroup of \( G^\text{tot} \) generated by all the elements in \( G^{(s)} \) and \( G^{(t)} \). First, note that the homomorphism \( p_S \) maps the coset space (77) one-to-one to another coset \( H \setminus G^{(s)} \) defined in \( G^\text{tot} \). Thus, the upper bound in Proposition C is given by \( |H \setminus G^{(s)}| \). We claim, then, that

\[
|H \setminus G^{(s)}| \leq |G^{(t)} \setminus G^{\text{relev}}| \leq |G^{(t)} \setminus G^\text{tot}|; \tag{79}
\]

since the last inequality is obvious, only the first inequality needs to be verified. Now, let \( \{s_i\}_{i \in I} \) be a set of representatives of the coset space \( H \setminus G^{(s)} \). If two representatives \( s_i \) and \( s_j \) \((i \neq j)\) were to be in the same orbit of \( G^{(t)} \) in \( G^{\text{relev}} \), i.e., \( \exists t \in G^{(t)} \) such that \( t \cdot s_i = s_j \), then \( t = s_j \cdot s_i^{-1} \in G^{(s)} \), and hence \( t \in H \); this is a contradiction. \( \blacksquare \)

In the case of \( X = \text{Km}(E \times F) \),

\[
\text{Isom}(q) \cong (S_3 \times S_3) \rtimes (\mathbb{Z}/2\mathbb{Z}), \quad p_T \left( \text{Isom}(T_X)^{\text{(Hodge)}} \right) = \{1\}, \tag{80}
\]

and \( p_S : \text{Isom}^+(S_X) \to \text{Isom}(q) \) is surjective \([\text{Og}]. \) Since the coset space (78) is a group of order 72, Proposition C \( ' \) implies that the number of isomorphism classes must be no more than 72 for any one of the eleven isometry classes of frame lattice \( \mathcal{J}_2(X) \) for \( X = \text{Km}(E \times F) \). From Table 6 this is indeed seen to be the case.

We computed the group \( \text{Isom}(q) \) and the coset space \( p_T(\text{Isom}(T_X)^{\text{(Hodge)}}) \setminus \text{Isom}(q) \) for the thirty-four singular K3 surfaces that showed up in the study of \([\text{BKWW}]. \) The result is summarized in Table 7. In the table,

\[
D_4(\tau, \sigma) := \mathbb{Z}_4 \langle r_1 \cdot s = \tau \rangle \rtimes \mathbb{Z}_2 \langle r_2 = \sigma \rangle \cong (\mathbb{Z}_2 \langle r_1 \rangle \times \mathbb{Z}_2 \langle r_2 \rangle) \rtimes \mathbb{Z}_2 \langle s \rangle \tag{81}
\]

is the dihedral group of order 8. We have,
Table 7: We study here thirty-four singular K3 surfaces \(X_{[a \; b \; c]}\) specified by \([a \; b \; c]\) in the first column. Their \(\text{Isom}(T_X^{(\text{Hodge})}) \cong \mathbb{Z}/m\mathbb{Z}\) and the discriminant group \(G_{T_X} \cong T_{\star}/T_X\) are shown in the 2nd and 3rd columns, respectively. In the 3rd column, expressions such as \(G_1 \times G_2\) and \((G_1 \times G_2)\) mean that the discriminant bilinear form is block diagonal in the former case, and it is not in the latter. \(\theta_m\) is the generator of \(\text{Isom}(T_X^{(\text{Hodge})}) \cong \mathbb{Z}/m\mathbb{Z}\) (angle \(2\pi/m\) rotation in \(T_X \otimes \mathbb{R}\)). The coset space \([78]\) in the last column is determined by the information in the 4th and 5th columns. The 5th column is left empty, when \(p_T\) is surjective.
Corollary D: For any one of the thirty-four singular K3 surfaces \( X_{[a \ b \ c]} \) studied in Table 7, the number of isomorphism classes of elliptic fibration (multiplicity) is bounded from above uniformly for any \([W] \in J_2(X)\) by the number of elements of the coset space in the last column of the table. In particular, there are ten singular K3 surfaces in the table,

\[
X_{[1 \ 0 \ 1]}, \ X_{[1 \ 1 \ 1]}, \ X_{[2 \ 0 \ 1]}, \ X_{[2 \ 1 \ 1]}, \ X_{[3 \ 0 \ 1]}, \\
X_{[3 \ 1 \ 1]}, \ X_{[4 \ 0 \ 1]}, \ X_{[5 \ 1 \ 1]}, \ X_{[6 \ 1 \ 1]}, \ X_{[3 \ 1 \ 2]},
\]

where \( J_1(X) = J_2(X) \). The multiplicity is at most 2 for any \([W] \in J_2(X)\) for ten other singular K3 surfaces \( X \) in the table. Even for the fourteen other singular K3 surfaces in the table, the multiplicity may only maximally be as large as \( |(D_4 \times D_4)/\mathbb{Z}_4| = 16 \) for some \([W] \in J_2(X)\), which happens in the case of \( X_{[6 \ 0 \ 0]} \). ■

We also remark here that the coset space \( \{78\} \) is not necessarily a group, because \( p_T(\text{Isom}(T_X)^{(\text{Hodge})}) \) is not always a normal subgroup of \( \text{Isom}(q) \). In fact, for singular K3 surfaces \( X = X_{[6 \ 0 \ 0]} \) and \( X_{[6 \ 6 \ 6]} \) in Table 7 the coset space \( \{78\} \) is not a group.

5.2 Type-dependent Upper Bounds on Multiplicity Using the II_{1,25} Lattice

Propositions C and C’ provide an upper bound on the number of isomorphism classes of elliptic fibrations (multiplicity) for a given K3 surface \( X \) which is applicable for any isometry class \([W] \in J_2(X)\). The number of isomorphism classes, however, can be different for different isometry classes \([W] \in J_2(X)\) for a given \( X \), and there must be room for deriving an upper bound on the number of isomorphism classes for individual isometry classes \([W] \in J_2(X)\). Furthermore, the universal bound derived in the last section is expected to be a relatively weak bound. In this section, we derive Proposition E and Corollary F by exploiting the theory on the structure of \( \text{Isom}(S_X) \) developed by [2] and [3].

The basic idea is this. First of all, since we consider the classification of elliptic fibrations \((\pi_X, \sigma; X, \mathbb{P}^1)\) modulo automorphism of \( X \), an explicit choice of the section \( \sigma \) is no longer important. When \( \sigma' \) is another section of \( \pi_X : X \rightarrow \mathbb{P}^1 \), \((\pi_X, \sigma; X, \mathbb{P}^1)\) and \((\pi_X, \sigma'; X, \mathbb{P}^1)\) are mutually isomorphic because the translation automorphism (the group law sum in the Mordell–Weil group [2] and [3]) and \((\pi_X, \sigma'; X, \mathbb{P}^1)\) are mutually isomorphic because the translation automorphism (the group law sum in the Mordell–Weil group extended also to singular fibres) maps one to the other \( \{78\} \). Thus, when an elliptic fibration is given by a primitive embedding of \( U \cong \text{Span}_\mathbb{Z}\{f_1, f_2\} \) into \( S_X \), \( f_2 \) does not play any other role than providing a divisor intersecting \( f_1 \) once. We call divisors satisfying \( f_1^2 = 0 \) which also span a lattice \( U \) with any other divisor in \( S_X \) elliptic divisors. \(^{22}\) When we study the quotient space \( \{58\} \), we only have to focus on elliptic divisors under the action of the groups \( \text{Isom}(S_X)^{\text{(Amp Hodge)}} \) and \( \text{Isom}(S_X)^{\text{(Amp)}} \).

We can always map an elliptic divisor for some isometry class \([W] \in J_2(X)\) into \( \text{Amp}_X \) by the \( W^{(2)}(S_X) \times \{\pm 1_{S_X}\} \) subgroup of \( \text{Isom}(S_X)^{\text{(PSS)}} \). For a given isometry class \([W] \in J_2(X)\), such elliptic divisors within \( \text{Amp}_X \) form a single orbit of \( \text{Isom}(S_X)^{\text{(Amp)}} \). This orbit is decomposed into a set of orbits of the subgroup \( \text{Isom}(S_X)^{\text{(Amp Hodge)}} \). The number of isomorphism classes for \([W] \in J_2(X)\) is the number of \( \text{Isom}(S_X)^{\text{(Amp Hodge)}} \) orbits within the single orbit of \( \text{Isom}(S_X)^{\text{(Amp)}} \). The trouble in formulating the problem in this way, however, is that the groups \( \text{Isom}(S_X)^{\text{(Amp Hodge)}} \) and \( \text{Isom}(S_X)^{\text{(Amp)}} \) may not be finite groups, and have complicated structures.

\(^{22}\)One might further impose one more condition—\( f_1 \in \text{Amp}_X \) for a divisor \( f_1 \) to be called an elliptic divisor.
The theory of [Bor1, Bor2, Ko2] is a powerful tool in studying the structure of these two groups. The crucial part is an algorithm for setting a fundamental region under the action of these groups that is much smaller than Amp\(_X\)—the fundamental region of the reflection group \(W^{(2)}(S_X)\) generated only by \((-2)\) roots. In the following, we provide a brief review of the theory of [Bor2, Ko2], with a focus primarily on the aspect of setting a smaller fundamental region, while explaining notations and clarifying a set of sufficient conditions for the theory of [Bor1, Bor2, Ko2] to work.\(^{23}\)

Let us first assume that the lattice \(T_0 \subset E_8 \oplus U^{\oplus(20-\rho)}\) (introduced in [Nish1, Nish2] and reviewed in section 4.1) is a direct sum of root lattices of \(A-D-E\) type—(as-1)\(^{24}\) The isomorphism \(\gamma : G_{T_0} \cong G_{T_X}[-1] \cong G_{S_X}\) consistent with the discriminant form determines an embedding \(\phi(T_0, S_X)^{\ast} : T_0 \oplus S_X \hookrightarrow \Pi_{1,25}\). Upon restriction to \(T_0\), this is regarded as a primitive embedding of \(T_0 : \phi_{T_0}^* : T_0 \hookrightarrow \Pi_{1,25}\). The Neron–Severi lattice \(S_X\) is now regarded as the orthogonal complement \(\phi_{T_0}^*(T_0)^\perp \subset \Pi_{1,25}\). Theorem \(\epsilon\) also guarantees, as long as \(\rho_X \geq 12\), that for any primitive embedding \(\sigma_{T_0} : T_0 \hookrightarrow \Pi_{1,25}\), there exists an isometry \(f \in \text{Isom}(\Pi_{1,25})\) such that \(f \cdot \phi_{T_0} = \phi_{T_0}^*\). Thus, we can choose any primitive embedding \(\phi_{T_0} : T_0 \hookrightarrow \Pi_{1,25}\), and regard the orthogonal complement as \(S_X\), as long as \(\rho_X \geq 12\)—(as-2).

Let \(J\) denote the simple roots of \(T_0\) and \(W_J\) the Weyl group of \(T_0\). If we find a map \(\phi_{T_0} : J \rightarrow \Pi\) preserving the Coxeter matrix (the intersection form)\(^{25}\) then an embedding \(\phi_{T_0} : T_0 \hookrightarrow \Pi_{1,25}\) is obtained by extending the map \(\phi_{T_0} : J \rightarrow \Pi\) linearly. In an abuse of notation we use \(\phi_{T_0}\) for the embeddings of both \(J\) and \(T_0\). We focus our attention to primitive embeddings \(\phi_{T_0}\) satisfying this property.

Suppose that the homomorphism \(p_{T_0} : \text{Isom}(T_0) \rightarrow \text{Isom}(G_{T_0}, q_{T_0}) \cong \text{Isom}(G_{S_X}, q_{S_X})\) is surjective. Since \(\text{Isom}(T_0) \cong W_J \rtimes \text{Aut}(J)\), where \(\text{Aut}(J)\) is the group of automorphism of the Dynkin diagram of \(T_0\), this assumption—(as-3) is equivalent to the surjectiveness of

\[
p_{T_0} : \text{Aut}(J) \rightarrow \text{Isom}(q),
\]

where we use \(p_{T_0}\) also for this homomorphism. Hence all isomorphisms of \(S_X\) (resp. \(\text{Isom}^+(S_X)\)) are obtained by restricting certain isomorphism of \(\Pi_{1,25}\) (resp. \(\text{Isom}^+(\Pi_{1,25})\)). This assumption is satisfied by all the ten cases in Table 2 for which \(T_0\) is a direct sum of root lattices of \(A-D-E\) type.

The group of autochronous isomorphisms of \(\Pi_{1,25}\) preserving the two subspaces \(\phi_{T_0}(T_0) \) and \(S_X\) has a structure (see [Bor2, Lemma 2.1])

\[
\text{Isom}^+(\Pi_{1,25})(\phi(T_0), S_X) = W_{\phi(T_0)}(J) \rtimes W'_{\phi(T_0)}(J).
\]

\(^{23}\) The appendix [A.2] will serve as a side reader on this subject. It deals with this theory applied to a particular (and the simplest) example \(X = X_3 = X_{[1 \ 0 \ 1]}\). Example 5.3 of [Bor2]—a single-page dense description—is extended into a 7 page-long pedagogical presentation there, so that even those without a firm background in mathematics (including the present authors) can understand.

\(^{24}\) In the case discussed in [Ko2], \(T_0\) itself is not a direct sum of root lattices of \(A-D-E\) type, but contains such a lattice as an index 2 subspace. There is room for a choice of \(T_0\), more general than just being a direct sum of root lattices of \(A-D-E\) type for the theory of [Bor2, Ko2] to be applicable, but we do not try to present it in its most general form possible.

\(^{25}\) II is the set of Leech roots, a set of simple roots of \(\Pi_{1,25}\). See the appendix [A.1] for more.
Here, the Coxeter system \((W_{\phi T_0(J)}, \phi T_0(J))\) acts on \(\Pi_{1,25} \otimes \mathbb{R}\) and \(W'_{\phi T_0(J)}\) is the subgroup of \(\text{Isom}^+(\Pi_{1,25})\) mapping the fundamental chamber of the Coxeter group \(W_{\phi T_0(J)}\) to itself. Because the homomorphism \(\text{Aut}(J) \rightarrow \text{Isom}(q)\) is injective (which follows from the injectivity of \(\text{Aut}(J) \rightarrow \text{Isom}(q)\) for any one of the root lattices of \(A-D-E\) type), restriction of this subgroup on \(S_X\) induces an identification

\[
\text{Isom}^+(\Pi_{1,25})(T_0,S_X) \rightarrow W'_{\phi T_0(J)} \cong \text{Isom}^+(S_X). \tag{84}
\]

The \(W^{(2)}(S_X)\) subgroup of \(\text{Isom}^+(S_X)\) can also be regarded as a subgroup of \(W'_{\phi T_0(J)} \subset \text{Isom}^+(\Pi_{1,25})\). The fundamental chamber of the reflection group \(W^{(2)}(S_X)\) is the ample cone of \(X\), \(\text{Amp}_X \subset S_X \otimes \mathbb{R}\), but the approach of \([\text{Bor2}],\ [\text{Ko2}]\) is to exploit much a larger subgroup of \(\text{Isom}^+(S_X)\) in order to obtain a smaller fundamental region.

Let us now briefly explain a few concepts and introduce some notation in order to spell out the statements obtained from the works of \([\text{Bor2}],\ [\text{Ko2}]\). First, the Coxeter group \((W_H, \Pi)\) in

\[
W_H \subset \text{Isom}^+(\Pi_{1,25}) \cong W_H \times \text{Co}_\infty \tag{85}
\]

acts on \(\Pi_{1,25} \otimes \mathbb{R}\) as a reflection group and the fundamental chamber is denoted by \(C_H\):

\[
C_H = \left\{ x \in \Pi_{1,25} \otimes \mathbb{R} \mid (x, \lambda) > 0 \text{ for } \forall \lambda \in \Pi \right\}. \tag{86}
\]

The half of \(\{ x \in \Pi_{1,25} \otimes \mathbb{R} \mid x^2 > 0 \}\) containing \(C_H\) is called the **positive cone of \(\Pi_{1,25}\)**. The interior of

\[
(S_X \otimes \mathbb{R}) \cap \overline{C_H} \tag{87}
\]

in \(S_X \otimes \mathbb{R}\) is denoted by \(D'\). We choose an isometry between \([\phi T_0(J)]^\perp \subset \Pi_{1,25}\) and \(S_X\) so that this \(D'\) is contained in the ample cone \(\text{Amp}_X\) of \(S_X \otimes \mathbb{R}\).

Secondly, in the Coxeter diagram\(^{27}\) of the Coxeter system \((W_H, \Pi)\), any subdiagram is said to be **spherical** if it corresponds to one of the Dynkin diagrams of an \(A-D-E\) root system. For any one of the \(A-D-E\) root lattices, there is a unique element in the Weyl group that maps the Weyl chamber \(D\) to \(-D\) (note the simple transitive action on the chambers). This element is called **opposition involution** of a root lattice/system \(R = A_n, D_n, E_{6,7,8}\), and is denoted by \(\sigma_R\) or \(\sigma(R)\).

Finally, the following group

\[
\text{Aut}(D') := \{ g \in \text{Co}_\infty \mid g (\phi T_0(J)) = \phi T_0(J) \} = W'_{\phi T_0(J)} \cap \text{Co}_\infty \tag{88}
\]

plays an important role. Now, we are ready to spell out the following

**Theorem**\(^{26}\) (\([\text{Bor1}],\ [\text{Bor2}]; \ [\text{Ko2}],\ Lemma\ 7.3\)): One can find a subgroup \(N\) of \(\text{Isom}(S_X)^{\text{Amp Hodge}}\) acting on \(\text{Amp}_X\) so that the images of \(D'\) under \(N\) cover the entire \(\text{Amp}_X\), if, in addition to the assumptions \((\text{as}-1)\), \((\text{as}-2)\) and \((\text{as}-3)\) that are stated already, the two following assumptions are satisfied:

---

26 The representation of \(W_H\) on \(\Pi_{1,25} \otimes \mathbb{R}\) is regarded as the reduced representation of the canonical linear representation of \(W_H\) on \(V := \{ \sum_{\lambda \in \Pi} x_\lambda e_\lambda \mid x_\lambda \in \mathbb{R} \}\) in the sense of \([\text{Vin3}]\).

27 Nodes of the Coxeter diagram correspond to individual Leech roots in \(\Pi\). A pair of nodes corresponding to \(\lambda_i\) and \(\lambda_j\) in \(\Pi\) with \((\lambda_i, \lambda_j)_{1,25} = \delta m_{ij}\) are joined by a single line if \(\delta m_{ij} = +1\), by a thick line if \(\delta m_{ij} = +2\), and by a dotted line if \(\delta m_{ij} = +3, 4, \cdots\), following the conventions of \([\text{Vin2}]\).
• (as-4): for any spherical subdiagram of \( \Pi \) in the form of \( R' = R \cup r, R = \phi_{T_0}(J) \) for some \( r \in \Pi \), either \( \phi_{T_0}(J) \) is mapped to itself by \( \sigma_{R'} \cdot \sigma_R \), or there exists an element \( g' \in C_{0 \infty} \) such that it is mapped to itself by \( g' \cdot \sigma_{R'} \cdot \sigma_R \), and

• (as-5): the homomorphism \( \text{Aut}(D') \to \text{Aut}(J) \) is surjective.

When all of these assumptions are satisfied, it also follows that any element \( g \in \text{Isom}(S_X)^{\text{Amp}} \) (resp. \( g \in \text{Isom}(S_X)^{\text{Amp Hodge}} \)) can be written as \( g = g_d \cdot n \) for some \( n \in N \) and \( g_d \in \text{Aut}(D') \) (resp. \( g_d \in \text{Aut}(D')^{\text{Hodge}} \)). \( \text{Aut}(D')^{\text{Hodge}} \) is the inverse image of \( p_T(\text{Isom}(T_X)^{\text{Hodge}}) \subset \text{Isom}(q) \).

In the process of examining whether the assumptions (as-1)–(as-5) are all satisfied, one also has to carry out the following tasks:

I find an embedding \( \phi_{T_0} : J \to \Pi \) so that \( \phi_{T_0} : T_0 \to \Pi_{1,25} \) is primitive,

II list up Leech roots \( r \in \Pi \) where \( \phi_{T_0}(J) \cup r \) forms a spherical subdiagram of the Coxeter diagram of the Leech roots \( \Pi \),

III compute \( \sigma_{\phi_{T_0}(J) \cup r} \cdot \sigma_{\phi_{T_0}(J)} \) for the Leech roots \( r \) listed up in II, and

IV compute the group \( \text{Aut}(D') \).

.........................................................................................

Let us now return to the problem set at the beginning of this section. If all of the assumptions (as-1)–(as-5) are satisfied, then the theorem above implies for any isomorphism class of elliptic fibrations \( \{[\pi_X, \sigma; X, \mathbb{P}^1] \} \in \mathcal{J}_1(X) \), that its elliptic divisor (modulo \( \text{Aut}(X) \)) can be chosen not just within \( \text{Amp}_{T_0} \), but even within \( \overline{D'} \).

Let us focus on an isometry class \( [W] \in \mathcal{J}_2(X) \) and ask for its multiplicity in \( \mathcal{J}_1(X) \). The \( \text{Aut}(D') \) group acts on the set of all the elliptic divisors \( f \) in \( \overline{D'} \) for the class \( [W] \). Let

\[
\Pi_a F_a^{[W]} := \Pi_a \{ \text{Aut}(D') \cdot f_a \}
\]

(89)

denote the orbit decomposition of such elliptic divisors. The action of \( \text{Aut}(D') \) is not necessarily transitive (explicit examples of the non-transitive action are found in [Ku]), and the index \( a \) above is meant to label such different \( \text{Aut}(D') \) orbits.

**Proposition E:** Suppose that the assumptions (as-1)–(as-5) are satisfied for a K3 surface \( X \). Then for an isometry class \( [W] \in \mathcal{J}_2(X) \), each isomorphism class of elliptic fibrations in \( \mathcal{J}_1(X) \) falling into \( [W] \) has its elliptic divisor in any of the \( \text{Aut}(D') \) orbits, \( \{ F_a^{[W]} \} \).

Thus, for any \( \text{Aut}(D') \) orbit \( F_a^{[W]} \), its decomposition into the orbits of the action of the \( \text{Aut}(D')^{\text{Hodge}} \) subgroup is regarded as a complete set of representatives of the \( \mathcal{J}_1(X) \) classification for \( [W] \in \mathcal{J}_2(X) \).

**proof:** If the assumptions (as-1)–(as-5) are satisfied, we can find an elliptic divisor \( f \) in \( \overline{D'} \) for any isomorphism class in \( \mathcal{J}_1(X) \), as we have already seen above. This divisor \( f \) must be in one of the \( \text{Aut}(D') \)-orbits, say, \( F_{a'}^{[W]} \). For any other \( \text{Aut}(D') \)-orbit, say \( F_{a''}^{[W]} \) (if there is any), there must be a transformation \( \phi \in \text{Isom}(S_X)^{\text{Amp}} \) mapping \( f \) to \( F_{a''}^{[W]} \), because the elliptic divisors in \( F_{a'}^{[W]} \) belong to the same element \( [W] \in \mathcal{J}_2(X) \) as \( f \in F_{a'}^{[W]} \). Thanks to the assumption (as-5), there must
be a transformation \( g \in \text{Aut}(D') \) such that \( g \cdot \phi \in \text{Isom}(S_X)^{(\text{Amp Hodge})} \), which means that both the elliptic divisors \( f \) and \( g \cdot \phi(f) \) define elliptic fibrations in the same isomorphism class, i.e. the same element in \( J_1(X) \).

**Corollary F**: For an isometry class \([W] \in J_2(X)\), the number of isomorphism classes of elliptic fibrations (= multiplicity = the number of elements in the inverse image of \([W]\) in \( J_1(X) \)) is bounded from above by the number of \( \text{Aut}(D')^{(\text{Hodge})} \)-orbits within any one of \( F_{[W]}' \), and in particular, by the smallest one among them.

We remark that the complete sets of representatives in Proposition E are not necessarily minimal complete sets of representatives of isomorphism classes. In order to find a minimal complete set of representatives one would have to exploit more information of \( X \) as in the original work of \([Og]\) for \( X = \text{Km}(E \times F) \).

**5.3 An Example: \( X = \text{Km}(A) \)**

As the first application of Corollary F, we choose a \( \rho = 17 \) family of K3 surfaces \( X = \text{Km}(A) \), for which the tasks I–IV in section 5.2 have been carried out and all the assumptions (as-1)–(as-5) verified already \([Ko2]\). Furthermore, one representative elliptic divisor \( f_{[W]}'[1] \) has been determined for any of the \( \text{Aut}(D') \)-orbits \( F_{[W]}' \) of any \([W] \in J_2(X)\) in \([Ku]\) for this family of K3 surfaces.

There are 25 isometry classes in the \( J_2(X) \) classification, and the representative elliptic divisors obtained by \([Ku]\) are shown in the 2nd column of Table 8. We now only have to work out the action of \([Ko2]\)

\[
\text{Aut}(D') = (\mathbb{Z}/2\mathbb{Z})^5 \rtimes S_6, \quad \text{Aut}(D')^{(\text{Hodge})} = (\mathbb{Z}/2\mathbb{Z})^5
\]

(90)

on the elliptic divisors \( f_{[W]}'[1] \) to see how many \( \text{Aut}(D')^{(\text{Hodge})} \)-orbits individual \( \text{Aut}(D') \)-orbits \( F_{[W]}' \) are decomposed into. Necessary technical details on these two groups as well as \( S_X \) of this family of K3 surfaces are summarized in the appendix A.3. More interested readers may prefer to consult \([BHPV, \text{Nik3, K3, Ko2}]\).

We have studied the decomposition of each of the \( \text{Aut}(D') \)-orbits (each row in Table 8) into the orbits under the \( \text{Aut}(D')^{(\text{Hodge})} \) subgroup. Let us take the first isometry class in \( J_2(X) \), where \( W_{\text{root}} = D_4^2 \oplus A_1^6 \), as an example. It turns out that the \( \text{Aut}(D') \)-orbit, \( \text{Aut}(D') f_{[1]} \) consist of 240 elliptic divisors in \( D' \), which are grouped into 15 distinct \( \text{Aut}(D')^{(\text{Hodge})} \)-orbits, each one of which consists of 16 elliptic divisors. Similar computations have been carried out for all other elliptic divisors, and the result is presented in the last column of Table 8. Note that, in the case of \( X = \text{Km}(A) \), all the \( \text{Aut}(D')^{(\text{Hodge})} \)-orbits within a given \( \text{Aut}(D') \)-orbit consist of the same number of elliptic divisors because \( \text{Aut}(D')^{(\text{Hodge})} \) is a normal subgroup of \( \text{Aut}(D') \). The labels 0, A, B, \cdots refer to different elliptic divisors equivalent under \( \text{Aut}(X) \), as given in \([Ku]\). Interestingly, different bounds are established starting from different elliptic divisors.

**Example G**: The multiplicity is bounded from above by 15 (no.1), 360 (no.2), 90 (no.3), 60 (no.4), 90 (no.5), 45 (no.6), 15 (no.7), 180 (no.8), 360 (no.9), 60 (no.10), 180 (no.11), 360 (no.12), 60 (no.13), 10 (no.14), 720 (no.15), 720 (no.16), 360 (no.17), 360 (no.18), 45 (no.19), 360 (no.20), 180 (no.21), 72 (no.22), 360 (no.23), 360 (no.24), and 360 (no.25), respectively, for the twenty-five isometry classes in \( J_2(X) \) for \( X = \text{Km}(A) \). (We follow the numbering (no.1–25) on the isometry classes as used in \([Ku]\).)
| $[W]_{\text{root}}$ | Elliptic Divisor | # orbits |
|------------------|-----------------|---------|
| $D_{4}A_{1}^{6}$ | $f[1]$ $(-1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$ | 15 |
| $D_{6}D_{4}A_{4}^{4}$ | $f[2]$ $(-2, -1, 0, 0, 0, -1, 0, 0, 0, 0, -1, -1, 0, 0, 2)$ | 360 |
| $D_{6}A_{3}A_{1}^{6}$ | $f[3]$ $(-2, -1, 0, 0, -1, 0, 0, 0, 0, -1, 0, -1, 0, 0, 2)$ | 90 |
| $D_{4}A_{3}^{4}$ | $f[4]$ $(-2, -1, 0, 0, 0, 0, 0, 0, 0, -1, -1, 0, -1, 0, 2)$ | 60 |
| $A_{7}A_{3}^{2}$ | $f[5]$ $\frac{1}{2}(-3, -3, -1, -1, 0, 0, -1, -1, 0, -1, -2, 0, -2, 0, -1, 4)$ | 90 |
| $A_{7}A_{3}A_{1}^{4}$ | $f[6]$ $\frac{1}{2}(-4, -2, -1, 0, 0, -1, -1, 0, 0, -1, -1, 0, -2, -1, -1, -1, 4)$ | 45 |
| $A_{5}^{2}$ | $f[7]$ $\frac{1}{2}(-1, -1, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, -1, 2)$ | 15 |
| $D_{6}A_{2}^{2}$ | $f[8]$ $(-3, -2, -1, 0, 0, 0, 0, 0, -1, 0, -1, 0, -1, 0, 0, 3)$ | 360 |
| $D_{6}D_{5}A_{1}^{4}$ | $f[9]$ $(-3, -2, -1, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, -1, 0, 3)$ | 360 |
| $A_{7}A_{3}^{2}$ | $f[10]$ $\frac{1}{2}(-2, -1, 0, 0, -1, 0, -1, 0, -2, 0, -1, -1, 0, 3)$ | 15 |
| $D_{8}A_{1}^{6}$ | $f[11]$ $(-3, 2, 0, 0, -1, 0, 0, 0, -1, -1, -1, 0, -1, 0, 0, 3)$ | 360 |
| $D_{8}D_{4}A_{3}$ | $f[12]$ $\frac{1}{2}(-5, -4, 0, 0, -3, -1, -1, -1, 0, -1, -3, -2, -1, 0, -2, 6)$ | 360 |
| $E_{6}E_{4}$ | $f[13]$ $\frac{1}{2}(-6, -3, 0, 0, -3, 0, -1, -1, 0, -1, -2, -1, -2, -1, -1, 6)$ | 60 |
| $A_{5}^{2}$ | $f[14]$ $\frac{1}{2}(-3, -1, 0, 0, -1, -1, -1, -1, 0, 0, 0, -1, -1, -1, 0, 3)$ | 10 |
| $D_{8}D_{4}A_{3}$ | $f[15]$ $(-4, -2, 0, 0, -2, 0, 0, -1, -1, 0, -2, -1, -1, 0, 0, 4)$ | 720 |
| $E_{7}D_{4}A_{3}^{2}$ | $f[16]$ $\frac{1}{2}(-3, -3, 0, 0, -2, 0, -1, 0, 1, 0, 0, -2, 0, -2, 0, -1, 4)$ | 720 |
| $D_{7}D_{4}^{2}$ | $f[17]$ $(-4, -2, 0, 0, -1, 0, -1, 0, 0, 0, 0, -2, -2, 0, 0, -1, 4)$ | 360 |
| $E_{7}A_{3}A_{1}^{6}$ | $f[18]$ $(-3, -3, 0, 0, 0, -2, -1, -1, 0, 0, 0, -2, 0, 0, 0, 4)$ | 360 |
| $D_{10}A_{1}^{6}$ | $f[19]$ $\frac{1}{2}(-3, -3, -2, 0, 0, 0, 0, 0, 0, -2, -1, -1, -1, -1, -1, -1, 4)$ | 45 |
| $D_{10}^{2}$ | $f[20]$ $(-5, -2, 0, 0, -2, 0, 0, -1, -1, 0, -2, -1, -1, 0, 0, 5)$ | 720 |
| $D_{9}A_{1}^{6}$ | $f[21]$ $\frac{1}{2}(-4, -3, 0, 0, -1, -2, 0, 0, -1, 0, -1, -2, -3, 0, -1, 0, 5)$ | 180 |
| $D_{9}A_{1}^{6}$ | $f[22]$ $\frac{1}{2}(-5, -2, 0, -1, 0, -2, 0, -1, 0, -2, -1, -1, 0, 5)$ | 360 |
| $E_{8}A_{1}^{6}$ | $f[23]$ $(-5, -2, 0, 0, -2, 0, -1, 0, -1, 0, -3, 0, -2, -1, 0, 0, 5)$ | 360 |
| $E_{8}A_{1}^{6}$ | $f[24]$ $(-6, -5, 0, 0, -3, 0, 0, -1, 0, -2, -3, -3, -2, -1, 0, 7)$ | 720 |
| $E_{8}A_{1}^{6}$ | $f[25]$ $(-4, -5, 0, 0, -5, 0, -3, -1, 0, 0, -2, -4, -1, 0, 7)$ | 720 |
| $E_{8}A_{1}^{6}$ | $f[26]$ $(-6, -4, 0, 0, -2, 0, -2, -1, -1, 0, 0, -4, 0, 0, 0, 7)$ | 720 |
| $E_{8}A_{1}^{6}$ | $f[27]$ $(-3, -2, -4, 0, -3, -1, 0, -1, 0, -1, -7, 0, 0, 0, -2, 7)$ | 720 |
| $E_{8}A_{1}^{6}$ | $f[28]$ $\frac{1}{2}(-7, -5, -2, -2, 0, 0, 0, -4, -3, -3, -3, -1, -1, -1, 8)$ | 360 |

Table 8: The first two columns quote the results of $J_2(X)$ classification for $X = K\mu(A)$ from [Ku]. A representative elliptic divisor in each $\text{Aut}(D')$-orbit is described as linear combinations of divisors $N_{00}, N_{01}, \cdots, N_{05}, N_{12}, \cdots, N_{15}, N_{23}, \cdots, N_{25}, N_{34}, N_{35}, N_{45}$ and $H$ in $S_X$, and their 17 coefficients are shown in the 2nd column. The last column shows our computation on the number of $\text{Aut}(D')^{(\text{Hodge})}$-orbits within the individual $\text{Aut}(D')$-orbits.
Note that these upper bounds on the multiplicity for individual isometry classes \([W] \in \mathcal{J}_2(X)\) is stronger than the uniform upper bound obtained from Proposition C': \(|S_6| = 720\). Note also that the actual multiplicity for a given \([W] \in \mathcal{J}_2(X)\) may be smaller than the upper bound obtained here.

### 5.4 A Systematic Way to Find an Elliptic Divisor in \(\mathcal{D}'\)

In order to apply Corollary F to other K3 surfaces, we need to be able to find at least one elliptic divisor in \(\mathcal{D}'\) for each isometry class \([W] \in \mathcal{J}_2(X)\). In this subsection, we present a systematic procedure to find such an elliptic divisor, where one can combine both a) the Kneser–Nishiyama method for the \(\mathcal{J}_2(X)\) classification using Niemeier lattices (reviewed in section 4.1) and b) Borcherds and Kondo’s theory of handling the isometry group of \(S_X\) using the \(\Pi_{1,25}\) lattice (reviewed in section 5.2).

Because of the uniqueness of the even unimodular lattice of signature (1,25), there must be isometries among all the twenty-four lattices of the form \(U \oplus L(I)\), where \(I = \alpha, \beta, \cdots, \omega\) and \(L(I)\)'s are the twenty-three other Niemeier lattices. The isometry between the ones with \(I = \alpha, \beta, \cdots, \psi\) and the Leech lattice \(L(\omega)\) is denoted by

\[
\phi(I): L(I) \oplus U(I) \cong \Lambda_{24} \oplus U(\omega) \cong \Pi_{1,25}.
\]  

It is known ([CS], Chap. 26 Thm. 5) that, for a vector

\[
u(I) := (\vec{c}(I), 1, -(\vec{c}(I))^2/2) \in \left(\Lambda_{24} \oplus U(\omega)\right) \otimes \mathbb{Q},
\]

the twenty-three other Niemeier lattices \(L(I)\) are obtained by \([(u(I))^\perp \subset (\Lambda_{24} \oplus U(\omega))]/(u(I))\). Here, \(\vec{c}(I)\) is a centre of a deep hole of type \(I = \alpha, \beta, \cdots, \psi\), see appendix 5.1 for a brief review, or [CS] for extensive exposition. We start off by refining this theorem a little more to the level of constructing the isometry \(\phi(I)\) in (91) explicitly.

Let \((L(I))_{root} \cong \oplus_{a \in A} R_a\), where each one of the \(R_a\)'s is a root lattice of \(A-D-E\) type (see Table 1). Let \(\vec{c}(I) \in \Lambda_{24} \otimes \mathbb{Q}\) be a centre of a deep hole of type \(I\), and \(\vec{c}(I)^a(i = 0, \cdots, r_a = \text{rank}(R_a))\) the integral points of \(\Lambda_{24}\) surrounding it. With this data, an embedding \(\phi(I): (L(I))_{root} \hookrightarrow \Lambda_{24} \oplus U(\omega)\) is specified by mapping the simple roots of \(R_a\)

\[
\phi(I): \alpha_i^{(I)a} \mapsto \lambda_i^{(I)a} = \left(\vec{c}_i^{(I)a}, 1, -1 - \frac{(\vec{c}_i^{(I)a})^2}{2}\right)
\]

for \(i = 1, \cdots, r_a\) (\(i = 0\) is not included here).

We claim that this embedding of \((L(I))_{root} \hookrightarrow \Lambda_{24} \oplus U(\omega)\) can be lifted to an isometry (91). To see this, we only need to prove that the orthogonal complement of this embedding, \([\phi(I)(\oplus_{a \in A} R_a)^\perp \subset (\Lambda_{24} \oplus U(\omega))]\), is isometric to the hyperbolic plane lattice \(U(\omega)\). In order to prove (**), let us first see that the two vectors

\[
u_1(I) = \left(h(I)\vec{c}(I), h(I), -h(I)(\vec{c}(I))^2/2\right),
\]

\[
u_2(I) = \left(\vec{c}(I) + \frac{1}{h(I)}\vec{c}(I), 1, -(\vec{c}(I))^2/2 + \frac{1}{h(I)}(1 -(\vec{c}(I), \vec{c}(I)))\right),
\]

\footnote{An alternative will be to implement this problem on a computer and pick up candidates of elliptic divisors from the edges of \(\mathcal{D}'\), as in [Ko].}
are integral elements of $\Lambda_{24} \oplus U^{(\omega)}$. Here,
\[ \hat{\rho}^{(I)} := \sum_{a \in A} \hat{\rho}^{(I)a}, \quad \left( \hat{\rho}^{(I)a}, \hat{\rho}^{(I)b} - \hat{c}^{(I)} \right)_{\Lambda_{24} \otimes \mathbb{Q}} = -\delta^{ab} \quad (i = 1, \ldots, r_b); \]  \tag{96}
that is, $\hat{\rho}^{(I)a}$ is the Weyl vector of the (negative definite) root lattice $R_a$ contained in $(L^{(I)})_{\text{root}} \cong \oplus_{a \in A} R_a$. One can show after a little manipulation (using an arbitrary $a \in A$) that
\[ -\frac{h^{(I)}}{2} \left( \hat{c}^{(I)} \right)^2 = h^{(I)} \left( 1 + \frac{(\hat{c}^{(I)a})^2}{2} \right) - (\hat{c}^{(I)a}, h^{(I)} \hat{c}^{(I)}) \in \mathbb{Z}, \]  \tag{97}
\[ -\frac{(\hat{c}^{(I)})^2}{2} + \frac{1}{h^{(I)}} (1 - (\hat{\rho}^{(I)}, \hat{c}^{(I)}) = 2 + \frac{(\hat{c}^{(I)a})^2}{2} - \left( \hat{c}^{(I)a}, \hat{c} + \frac{1}{h^{(I)}} \hat{\rho}^{(I)} \right). \]  \tag{98}
Chap. 24, section 2 of [CS] guarantees that $\hat{c}^{(I)} + \frac{1}{h^{(I)}} \hat{\rho}^{(I)} \in \Lambda_{24}$, and hence both $u_1$ and $u_2$ are indeed integral elements in $\Lambda_{24} \oplus U^{(\omega)}$. Secondly, it is easy to see that these two vectors are orthogonal to $\phi^{(I)}(\oplus_{a} R_a)$, and hence $\text{Span}_{\mathbb{Z}} \{ u_1, u_2 \} \subset [\phi^{(I)}(\oplus_{a} R_a) \perp \subset (\Lambda_{24} \oplus U^{(\omega)})]$. Finally, the symmetric pairing on $\text{Span}_{\mathbb{Z}} \{ u_1, u_2 \}$ turns out to be
\[ (\hat{\rho}^{(I)})^2 = -2h^{(I)}(h^{(I)} + 1) \tag{99}\] is used to compute $(u_2^{(I)})^2$. This means that $\text{Span}_{\mathbb{Z}} \{ u_1, u_2 \}$ is isometric to $U$, and forms a primitive sublattice of $\Lambda_{24} \oplus U^{(\omega)}$. It thus follows that
\[ \left[ \phi^{(I)}(\oplus_{a} R_a) \perp \subset (\Lambda_{24} \oplus U^{(\omega)}) \right] = \text{Span}_{\mathbb{Z}} \{ u_1^{(I)}, u_2^{(I)} \} \cong U. \]  \tag{100}
The claim (**) is now proven, and we have

**Lemma H**: An isometry $\phi^{(I)}$ in [91] is obtained by mapping $L^{(I)}$ by (93) and embedding $U^{(I)}$ into the hyperbolic plane $\text{Span}_{\mathbb{Z}} \{ u_1^{(I)}, u_2^{(I)} + u_1^{(I)} \}$. Furthermore, the $u_1^{(I)}$ can be written as a positive coefficient sum of Leech roots under this isometry:
\[ u_1^{(I)} = \sum_{i=0}^{r_a} u_1^{(I)a}_i \lambda_i^{(I)a} \quad \text{for } a \in A, \]  \tag{101}
where $\lambda_i^{(I)a} := (\hat{c}^{(I)a}_0, 1, 1 - (\hat{c}^{(I)a})^2/2)$.

Suppose that the lattice $T_0$ for a K3 surface $X$ is a direct sum of root lattices of $A-D-E$ type, and a set of its simple roots is denoted by $J$, as in section 5.2. For an isometry class $[W] \in J_2(X)$ associated with a primitive embedding $\phi^{(I)}_{T_0} : T_0 \hookrightarrow L^{(I)}$, suppose that the primitive embedding $\phi_{T_0} : T_0 \hookrightarrow \Lambda_{24} \oplus U^{(\omega)} \cong \Pi_{1,25}$ is given by a combination of $\phi^{(I)}_{T_0} : J \longrightarrow (\text{simple roots of}) L^{(I)}_{\text{root}}$ and $\phi^{(I)}$ in Lemma H. Then the Neron–Severi lattice $S_X = [\phi_{T_0}(T_0) \perp \subset \Pi_{1,25}]$ is isomorphic to
\[ \text{Span}_{\mathbb{Z}} \{ u_1^{(I)}, u_2^{(I)} + u_1^{(I)} \} \oplus \phi^{(I)} \left( W = [\phi^{(I)}_{T_0}(T_0) \perp \subset L^{(I)}] \right). \]  \tag{102}
Therefore, $u_1^{(I)}$ and $u_2^{(I)} + u_1^{(I)}$ in $\Lambda_{24} \oplus U^{(\omega)} \cong \Pi_{1,25}$ can be regarded as elliptic divisors of this isometry class $[W] \in J_2(X)$.  

38
Lemma I: The elliptic divisor $u^{(I)}_1$ is in $\mathcal{D}$.

**proof:** It is in $S_X$, and also in $C_{11}$ because $(u^{(I)}_1, \lambda^{(I)}_a) = 0$ for all of $i = 0, \cdots, r_a$, $a \in A$, and $(u^{(I)}_1, \lambda)$ is positive for all other Leech roots $\lambda \in \Pi$. ■

### 5.5 Another Example: $X = \text{Km}(E_8 \times E_8)$

In this section, we take a singular K3 surface $X = \text{Km}(E_8 \times E_8) = X_{[2 \, 2 \, 2]}$ as an example, and apply Corollary F. The tasks I–IV in section 5.2 have been carried out and all the assumptions (as-1)–(as-5) are verified also for this K3 surface [KK]. The $\mathcal{J}_2(X)$ classification has also been worked out in [Nish] for this K3 surface, and there are 30 different isometry classes. However, we are not aware of an identification of the elliptic divisors, in particular, for these isometry classes in the literature. Therefore, we combine Lemma I and Corollary F with all that is known from [Nish] and [KK] to derive an upper bound on the number of isomorphism classes of elliptic fibrations for each isometry class $[W] \in \mathcal{J}_2(X)$ of $X = \text{Km}(E_8 \times E_8)$.

The lattice $T_0$ for this K3 surface is $D_4 \oplus A_2$ [Nish], and for all the thirty types in $\mathcal{J}_2(X)$, we stick to the following embedding of the simple roots of $D_4 \oplus A_2$ to the Leech roots. First, we define the following 6 vectors in $\Lambda_{24}$:

\[
\begin{align*}
\tilde{v}_1 &= \nu_\Omega + 4\nu_\infty =: X, \\
\tilde{v}_2 &= \tilde{0} =: Z, \\
\tilde{v}_3 &= \nu_\Omega + 4\nu_0 =: Y, \\
\tilde{v}_4 &= \nu_\Omega - 2\nu_{K_U} + 4\nu_{\{\infty,0,1\}} =: U, \\
\tilde{v}_p &= 4\nu_{\{0,\infty\}} =: P, \\
\tilde{v}_q &= \nu_\Omega - 4\nu_2 =: Q_2,
\end{align*}
\]

where $K_U$ is a codeword in $C_{24}(8)$ containing $\{\infty, 0, 1, 2\}$ as a subset. Leech roots that correspond to $\tilde{v}_{1,2,3,4}$ and $(\nu)_{p,q}$ in $\Lambda_{24}$ through the relation (154) are denoted by $\lambda_{1,2,3,4}$ and $\lambda_{p,q}$, respectively. The embedding $\phi_{T_0} : J \hookrightarrow \Pi$ of simple roots is given by assigning $\lambda_{1,2,3,4}$ to the simple roots $\omega_{1,2,3,4}$ of $D_4 \subset T_0$, and $\lambda_{p,q}$ to those of $A_2 \subset T_0$ [KK]. For $K_U$ we use the codeword given in (144). Once this embedding is fixed, then $S_X$ for $X = \text{Km}(E_8 \times E_8)$ is obtained as the orthogonal complement of $\phi_{T_0}(T_0)$ in $\Lambda_{24} \oplus U^{(\omega)} \cong \Pi_{1,25}$. $\text{Aut}(\mathcal{D})$ and $\text{Aut}(\mathcal{D})^{(\text{Hodge})}$ were determined for this set-up in [KK], whose result is quoted in appendix A.3.

Since each isometry class $[W] \in \mathcal{J}_2(X)$ is obtained in the form of $W := [\phi_{T_0}^{-1}(T_0) \subset L^{(I)}]$ for some primitive embedding into one of the Niemeier lattices, Lemma I can be used to determine an elliptic divisor for this $[W] \in \mathcal{J}_2(X)$, as we have stated prior to Lemma I. In practice, however, we need a centre $c^{(I)} \in \Lambda_{24} \otimes \mathbb{Q}$ of a deep hole of type $I$ (and vectors $v^{(I)}_{a,b}$'s around the centre) in order to determine the isometry $\phi^{(I)}$. It looks as if we can use the centre $c^{(I)}$ and vectors $v^{(I)}_{a,b}$ given explicitly in Chapt. 23 of [CS], but the lattice $T_0$ is now embedded into $\Pi_{1,25}$ through $\phi^{(I)} \cdot \phi_{T_0}^{-1}$, which is often different from the embedding determined by (103)–(108). It is one possibility to work

\[\text{This choice (144) for } K_U \text{ is not the same as } R_3 \text{ in } \text{KK}. \text{ We use } K_U \text{ for detailed computations in this section (and in the appendix A.3) only because some codewords in the MOG representation looks nice with (144), no other reasons. Our choice } K_U \text{ is equivalent to } R_3 \text{ in } \text{KK} \text{ modulo } M_{24}. \text{ In particular, the permutation in } M_{24} \text{ mapping } K_U \text{ to } R_3 \text{ is such that it leaves all other vectors except } \tilde{v}_i \text{ in (103)–(108) invariant. Hence our embedding is equivalent to the one used in } \text{KK}, \text{ and it is not necessary to repeat the Tasks I–IV and verify (as-4) independently.}
\]
out how the groups $\text{Aut}(D')$ and $\text{Aut}(D')^{(\text{Hodge})}$ act on the orthogonal complement of $\phi^{(I)} \cdot \phi_{0}^{(I)}(T_{0})$ for each isometry class $[W] \in \mathcal{J}_{2}(X)$. An alternative, however, is to find a centre of deep hole of type $I$ (and the vectors $\vec{v}_{i}^{(I)a}$ around it) for each $[W] = [\phi_{0}^{(I)}(T_{0})] \subset L^{(I)}$ in $\mathcal{J}_{2}(X)$ so that the vectors $[103, 108]$ are always included. We will take the latter approach in the following.

We have determined an elliptic divisor for each one of thirty isometry classes in $\mathcal{J}_{2}(X)$ for $X = \text{Km}(E_{\omega} \times E_{\omega})$. See Table 9 for the results. Nishiyama assigned numbers (type identification number) from 1 to 30 to the thirty isometry classes $[W] \in \mathcal{J}_{2}(X)$ in Table 1.3 of [Nish1]. We use the same type identification number in Table 9. Sections 5.5.1–5.5.4 provide detailed information of the determination process of elliptic divisors. Upper bounds on the multiplicities for individual isometry classes are discussed in section 5.5.5.

5.5.1 Determination of Elliptic Divisors: $D_{4} \hookrightarrow D_{n}$, $A_{2} \hookrightarrow$ other.

Let us begin with the first 13 types in $\mathcal{J}_{2}(X)$ of $X = \text{Km}(E_{\omega} \times E_{\omega})$ listed up in Table 9. For these 13 types the frame lattice $[W]$ is obtained as the orthogonal complement of a primitive embedding of $T_{0} \cong D_{4} \oplus A_{2}$ into $L^{(I)}$, as in section 4. In this subsection, we study cases where $D_{4}$ is embedded into a $D_{n}$ component of the irreducible decomposition of $(L^{(I)})_{\text{root}} \cong \oplus_{a} R_{a}$, and $A_{2} \subset T_{0}$ into another irreducible component. Some definitions of octads (codewords) used in this section can be found in the appendices A.1.2 and A.3.3.

The types [no. 29] and [no. 30] use an embedding into Niemeier lattices $L^{(I)}$ whose root lattices contain $D_{4}$ as an irreducible component. Thus, we need to find the centre of a deep hole of $\Lambda$ that is surrounded by $\vec{v}_{1,2,3,4}$ and one more vector $\vec{v}'$. From the conditions that $(\vec{v}' - \vec{v}_{1,2,3,4})^{2} = -4$, $(\vec{v}' - \vec{v}_{2})^{2} = -6$ and $(\vec{v}' - \vec{v}_{2})^{2} = -4$, we find that this vector should be of either one of the forms

\[
\begin{align*}
\vec{v}' & = \nu_{2} K + 4 \nu_{1,2}, & \nu_{i} & \in K \backslash \{\infty, 0, 1, 2\}, \\
\vec{v}' & = \nu_{2} K + 4 \nu_{1,2}, & K & = K_{\#},. \\
\vec{v}' & = \nu_{1,2} K + 4 \nu_{1,2}, & \{\infty, 0, 2\} & \subset K \cap K_{U}, |K \cap K_{U}| & = 4, 1 \notin K, i \notin K_{U}.
\end{align*}
\]

Let us take $\vec{v}'$ of the form (109) with $i = 11$ for now. Then there are 25 vectors $\vec{u} \in \Lambda_{24}$ satisfying $(\vec{u} - \vec{v}_{1,2,3,4})^{2} = -4$ and $(\vec{u} - \vec{v}'_{2})^{2} = -4$. It turns out that the diagram of these 25 vectors drawn under the rule explained in the appendix A.1 is the collection of five extended Dynkin diagrams of $D_{4}$. Hence a deep hole of type $\tau$ is surrounded by the vectors (see Table 1.), and the choice of $[103, 108]$ and $\vec{v}_{i}^{(I)a}$ corresponds to an embedding of $D_{4} \subset D_{4}^{(1)}$ and $A_{2} \subset D_{4}^{(2)}$ into $L_{\text{root}}^{(\tau)} = \oplus_{a=1}^{6} D_{4}^{(a)}$. This is the type [no. 29] in Table 1.3 of [Nish1]. The centre of this deep hole is

\[
\bar{c}_{(\tau)} = \frac{1}{h(\tau)} \left( 4 \nu_{K_{\#}} + 2 \nu_{K_{q}}, 2 \nu_{K_{q1}} + 2 \nu_{K_{q2}}, 2 \nu_{K_{q3}} + 2 \nu_{K_{q4}} \right) = \frac{1}{h(\tau)} 4 \begin{array}{ccccccc} 12 & 12 & 4 & 4 & 0 & 4 \\
12 & 12 & 4 & 4 & 0 & 4 \\
12 & 12 & 4 & 4 & 0 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4
\end{array}
\]

---

30 In the rest of this article, we refer to each isometry class $[W] \in \mathcal{J}_{2}(X)$ as each type of elliptic fibration for brevity, given the close relation between $\mathcal{J}_{2}(X)$ and $\mathcal{J}_{2}(X)^{(\text{Hodge})}$ classification.

31 The 25 vectors $\vec{u} \in \Lambda_{24}$ form five groups of 5 vectors, each one of which gives rise to an extended Dynkin diagram of $D_{4}$. The five groups of vectors consist of $P_{Q_{k+1}^{2}, 1, 1, 1, 1, 1}$, $2 \nu_{K}$ with $K \in \{K_{0}, K_{p}, K_{q1}, K_{q2}, K_{q3}\}$; $2 \nu_{K}$ with $K \in \{K_{0}, K_{p}, K_{q1}, K_{q2}, K_{q3}\}$; $2 \nu_{K}$ with $K \in \{K_{0}, K_{p}, K_{q1}, K_{q2}, K_{q3}\}$; and finally, $2 \nu_{K}$ with $K \in \{K_{0}, K_{p}, K_{q1}, K_{q2}, K_{q3}\}$.
The elliptic divisor is given by \( u_1^{(\tau)} \in \Pi_{1,25} \) in \([94]\), with \( h^{(I=\tau)} = 6 \). Using the 4 groups of \( \tilde{v}_1^{(\tau)a} \) forming an extended Dynkin diagram of \( D_4 \) (excluding those containing \([103,108]\)), we obtain

\[
\begin{align*}
u_1^{(\tau)} &= 2G_{11} + E_1 + F_1 + C_1 + D_1, \quad (113) \\
&= 2G_{24} + E_4 + F_2 + C_3 + D_3, \quad (114) \\
&= 2G_{43} + E_3 + F_4 + C_2 + D_2, \quad (115) \\
&= 2G_{32} + E_2 + F_3 + C_4 + D_4; \quad (116)
\end{align*}
\]

they describe 4 distinct singular fibres of \( I_0^* \) type, and are algebraically equivalent. One of these is used in Table 9. Choosing \( \tilde{v}'_4 \) of the form \([110]\) also implies an embedding of \( A_4 \oplus A_2 \) into \( L^{(\tau)} \), so we do not discuss this choice further.

Let us now take \( \tilde{v}'_4 \) of the form \([111]\) with

\[
K = \begin{array}{cc}
\ast & \ast \\
\ast & \ast \\
\ast & \ast \\
\ast & \ast \\
\end{array}
\]

(117)

instead. Then there are 24 vectors \( \tilde{u} \in A_{24} \) satisfying \((\tilde{u} - \tilde{v}_{1,2,3,4})^2 = -4 \) and \((\tilde{u} - \tilde{v}'_4)^2 = -4 \). The diagram of these 24 vectors turns out to be the collection of 4 extended Dynkin diagrams of \( A_5 \). The vectors \( P \) and \( Q_2 \), associated with the simple roots of \( A_2 \subset T_0 \), are part of one of the 4 \( A_5 \)'s. This is a deep hole of type \( \sigma \) (see Table 1), and correspond to the type \([\text{no. }30: D_4 \subset D_4, A_2 \subset A_5; A_5^{\oplus 3}] \) in Nishiyama’s \( J_2(X) \) classification. The centre of this deep hole is located at

\[
h^{(I=\sigma)}\tilde{c}^{(\sigma)} = (2\nu_{K_{a\rightarrow}} + 2\nu_{K_{\pi}} + 2\nu_{K_{\pi'}} + 2\nu_{K_{\gamma} + 2\nu_{K_{\pi'}}}),
\]

(118)

and the elliptic divisor in \( \overline{D'} \subset S_X \subset \Pi_{1,25} \) can be obtained by using Lemma I:

\[
\nu_1^{(\sigma)} = G_{11} + C_1 + G_{34} + F_3 + G_{33} + D_1, \quad (A_5, I_6) \quad (119)
\]

\[
= G_{24} + D_3 + G_{13} + C_4 + G_{21} + F_2, \quad (A_5, I_6) \quad (120)
\]

\[
= G_{43} + F_4 + G_{41} + D_4 + G_{14} + C_2. \quad (A_5, I_6), \quad (121)
\]

which describe three singular fibres of \( I_6 \) type.

Let us now turn to the types \([\text{no. }27]\) and \([\text{no. }28]\) in \( J_2(X) \), where the Niemeier lattice \( L^{(\tau)} \) is used, and \( D_4 \subset T_0 \) is embedded into an irreducible component \( D_5 \) in the root lattice of \( L^{(\tau)} \). Thus, we look for \( \tilde{v}_5 \), and choose

\[
\tilde{v}_5 = \nu_{12} - 2\nu_{K_L} + 4\nu_0 =: X_L, \quad \{0, 2\} \subset K_L, \quad \infty, 1 \notin K_L, \quad |K_L \cap K_U| = 4 .
\]

(122)

For \( K_L \in \mathcal{C}_{24}(8) \), we use the one in \([145]\). The centre of deep hole of \( I = \pi \) type, \( \tilde{c}^{(\pi)} \in A_{24} \oplus \mathbb{R} \), should be surrounded by \( \tilde{v}_1, \ldots, \tilde{v}_5 \) and one more vector \( \tilde{v}_5' \in A_{24} \). The vector \( \tilde{v}_5' \) should be of the same form as \( \tilde{v}_5 \) with \( K_L \in \mathcal{C}_{24}(8) \) for \( \tilde{v}_5 \) replaced by another \( K \in \mathcal{C}_{24}(8) \) satisfying the same conditions as \( K_L \) plus one more condition, \( |K \cap K_L| = 4 \).

For two different choices of this \( K \in \mathcal{C}_{24}(8) \) for \( \tilde{v}_5' \), say,

\[
K = \begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array}, \quad (123)
\]
the vectors $\vec{u} \in \Lambda_{24}$ satisfying $(\vec{u} - \vec{v}_1, \ldots, 5)^2 = -4$ and $(\vec{u} - \vec{v}_5)^2 = -4$ form the extended Dynkin diagram of $A_7 + A_7 + D_5$. The two vectors associated with the simple roots of $A_2 \subset T_0$, however, are contained in $D_5$, so that the first $K \in C_{24}(8)$ is used for $\vec{v}_5$, and they are in one of $A_7$ when the second $K \in C_{24}(8)$ is used. Therefore, these two cases correspond to the type [no.28] and [no.27] in Nishiyama’s classification, respectively. In type [no.28]: $D_4 \subset D_5$, $A_2 \subset D_5$; $A_7^2(2)$, the centre of the deep hole is given by

$$h(\gamma_\pi) = 2\nu_{K_{\gamma\pi}} + 2\nu_{K_{q\phi}} + 2\nu_{K_{\gamma\pi}} + 2\nu_{K_{q\pi}} + 2\nu_{K_{\gamma\phi}} + 2\nu_{K_{q\pi}} + 2\nu_{K_{\gamma\phi}} + 2\nu_{K_{q\pi}},$$  

and elliptic divisors are given by

$$u_{1(\pi)} = G_{21} + E_1 + G_{31} + F_3 + G_{33} + D_1 + G_{14} + C_4, \quad (A_7, I_8),$$

$$= G_{14} + F_1 + G_{12} + E_2 + G_{42} + C_1 + G_{23} + D_4. \quad (A_7, I_8).$$

For type [no.27]: $D_4 \subset D_5$, $A_2 \subset A_7$; $A_7 \oplus D_5$,

$$h(\gamma_\pi) = 2\nu_{K_{\gamma\pi}} + 2\nu_{K_{q\pi}} + 2\nu_{K_{\gamma\pi}} + 2\nu_{K_{q\pi}} + 2\nu_{K_{\gamma\phi}} + 2\nu_{K_{q\pi}} + 2\nu_{K_{\gamma\phi}} + 2\nu_{K_{q\pi}},$$

$$u_{1(\pi)} = G_{24} + D_3 + G_{31} + F_3 + G_{33} + D_1 + G_{14} + E_4, \quad (A_7, I_8)$$

$$= G_{43} + G_{42} + 2(D_2 + G_{12}) + F_1 + E_2. \quad (D_5, I_8).$$

There are 9 more types in $J_0(X)$ for $X = \text{Km}(E_8 \times E_8)$ by Nishiyama where the embedding $\phi': T_0 \hookrightarrow L^{(1)} \cong \oplus_{n \in A} R_n$ is given by sending $D_4 \subset T_0$ into $R_n = D_n$ and $A_2$ into another irreducible component $R_{a'}$. The first 9 entries of Table 9 are in this category. In order to determine the elliptic divisors for those types, the required task is to find a deep hole of type $I$ so that all the vectors [103–108] are among the surrounding vectors. This task can be carried out systematically, just like such a systematic approach was possible for a series of $L^{(1)}$ containing $D_n$ sublattice in Chapt. 23 of [CS]. Defining vectors in $\Lambda_{24}$ as follows,

$$\vec{v}_6 = 2\nu_{K_{\gamma\phi}}, \quad \vec{v}_6 = 2\nu_{K_{\gamma\pi}} \quad [\text{no.26}], \quad \vec{v}_6 = 2\nu_{K_{\gamma\phi}} \quad [\text{no.25}],$$

$$\vec{v}_7 = 2\nu_{K_{\gamma\phi}}, \quad \vec{v}_7 = 2\nu_{K_{\gamma\phi}} \quad [\text{no.22}], \quad \vec{v}_7 = 2\nu_{K_{\gamma\phi}} \quad [\text{no.20}],$$

$$\vec{v}_8 = 2\nu_{K_{q\phi}}, \quad \vec{v}_8 = 2\nu_{K_{q\phi}} \quad [\text{no.16}],$$

$$\vec{v}_9 = 2\nu_{K_{\gamma\pi}}, \quad \vec{v}_9 = 2\nu_{K_{\gamma\phi}} \quad [\text{no.17}],$$

$$\vec{v}_{10} = 2\nu_{K_{\gamma\phi}}, \quad \vec{v}_{10} = 2\nu_{K_{\gamma\phi}} \quad [\text{no.8}],$$

$$\vec{v}_{11} = 2\nu_{K_{\gamma\phi}},$$

$$\vec{v}_{12} = 2\nu_{K_{\gamma\phi}}, \quad \vec{v}_{12} = 2\nu_{K_{\gamma\phi}} \quad [\text{no.14}],$$

$$\vec{v}_{13} = 2\nu_{K_{\gamma\phi}},$$

$$\vec{v}_{14} = 2\nu_{K_{\gamma\phi}},$$

$$\vec{v}_{15} = 2\nu_{K_{\gamma\phi}},$$

$$\vec{v}_{16} = 2\nu_{K_{\gamma\phi}}, \quad \vec{v}_{16} = 2\nu_{K_{\gamma\phi}} \quad [\text{no.4}],$$

we see that the vectors $\vec{v}_1, \ldots, n$ along with one more vector $\vec{v}_n$ (or $\vec{v}_n'$) form an extended Dynkin diagram of $D_n$ for $n = 6, 7, 8, 9, 10, 12, 16$, where those vectors are arranged in the diagram as in Figure 1. By working out other irreducible components of the diagram, we see that they indeed correspond to an irreducible component of vectors surrounding a centre of deep hole of some type.
5.5.2 Determination of Elliptic Divisors: $D_4 \leftrightarrow E_n$, $A_2 \leftrightarrow$ other

Figure 2: A possible embedding of $D_4 \subset T_0$ into $E_n$. The vectors $\vec{v}_i$ are those given in \cite{131}. The extended Dynkin diagram of $D_n$ is formed by the skeleton made of $U$, $Y$, $Z$, $X$ and $\vec{v}_5, \ldots, n$ and one more $\vec{v}_n$ or $\vec{v}_n''$.

The corresponding type identification numbers in the $J_2(X)$ classification are already given above, and the type $I$ of Niemeier lattices/deep holes are found in Table \cite{6}. Elliptic divisors are determined for all those 9 types, as in the previous 4 types, and the results are shown in Table \cite{9}.

For types [no.4], [no.14] and [no.17], where $T_0$ is embedded into $L_\text{root}^{(I)} \cong D_{16} \oplus E_8$, $D_{12} \oplus D_{12}$ and $D_9 \oplus A_{15}$, respectively, there are only two irreducible components of the root lattice $L_\text{root}^{(I)}$. The elliptic divisors obtained in these cases inevitably have expressions containing $\lambda_p$, $\lambda_{q2}$. Of course $S_X$ is orthogonal to $T_0$ by construction, so this does not imply that these are fibre components. If we choose a basis of $T_0$ and $S_X$ without redundancy, the elliptic divisors can be solely written in terms of the basis of $S_X$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{A possible embedding of $D_4 \subset T_0$ into $D_n, n \geq 6$. The vectors $\vec{v}_i$ are those given in \cite{130}. The extended Dynkin diagram of $D_n$ is formed by the skeleton made of $U$, $Y$, $Z$, $X$ and $\vec{v}_5, \ldots, \alpha_n$ and one more $\vec{v}_n$ or $\vec{v}_n''$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{A possible embedding of $D_4 \subset T_0$ into $E_n$. The vectors in the diagram are defined in \cite{131} and below. The extended Dynkin diagram of a) $E_6$, b) $E_7$ and c) $E_8$ are formed by adding a) $\vec{v}_6$, b) $\{\vec{v}_7, \vec{v}_7\}$ and c) $\{\vec{v}_7, \vec{v}'_8\}$, respectively, on top of a common subdiagram that consists of the six vectors $\{U, X, Y, Z\}$ and $\vec{v}_{5,6}$ in \cite{131}.}
\end{figure}

Let us now move on to the second group in Table \cite{9} consisting of 8 types (no.2, 5, , 24) in $J_2(X)$ for $X = \text{Km}(E_\omega \times E_\omega)$. The frame lattice of the types in this group is obtained by embedding $D_4 \subset T_0$ into an irreducible component $E_n \subset (L^{(I)})_{\text{root}}$, and $A_2 \subset T_0$ into another
irreducible component of \((L^{(I)})_{\text{root}}\). For all of these types, we use
\[
\vec{v}_5 = \nu_\Omega - 2\nu_{K_L} + 4\nu_0 =: X_L,
\vec{v}_6 = \nu_\Omega - 2\nu_{K_R} + 4\nu_\infty =: Y_R,
\]
with \(K_L\) and \(K_R\) given in (145).

The types [no. 19], [no. 23] and [no. 24] in Table 9 involve an embedding of \(D_4 \subset T_6\) into an irreducible component \(E_6 \subset (L^{(I)})_{\text{root}}\). In order to find elliptic divisors for these types, we look for one more remaining vector \(\vec{v}_6 \in \Lambda_{24}\) which surrounds the centre of a deep hole of the corresponding type along with 6 other vectors \(\{X_L, X, Z, Y, Y_R, U\}\). This remaining vector should be of the form \(\vec{v}_6' = 2\nu_K\) for \(K \in \mathbb{C}_{24}(8)\). If we take
\[
\begin{array}{ccccccc}
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
\end{array}
\]

it turns out that \(\vec{v}_1, \ldots, \vec{v}_6\) indeed form an extended Dynkin diagram of \(E_6\), as shown in Figure 2.

Exchanging the rest of the diagram of vectors, we see that the three different choices of the codeword \(K \in \mathbb{C}_{24}(8)\) lead to three different embedding, [no.19: \(D_4 \subset E_8^{(1)}, A_2 \subset E_6^{(2)}; E_6^{(3)}\)], [no.23: \(D_4 \subset E_6, A_2 \subset A_{11}; D_7\)] and [no.24: \(D_4 \subset E_6, A_2 \subset D_7; A_{11}\)], respectively. Lemma I is used to determine elliptic divisors of those three types, and the results are recorded in Table 9.

For the types [no.9], [no.10] and [no.11], where an embedding \(D_4 \subset E_7\) is used, we take \(\vec{v}_7 = 2\nu_{K_{\alpha}}\). The centres of deep holes with an appropriate embedding of the vectors \(\{U, X, Y, Z, \vec{v}_5, 6, 7\}\) and \(\vec{v}_{p, q, 2}\) for these three types are found when we take one more vector \(\vec{v}''_7\) to complete the extended Dynkin diagram of \(E_7\) as
\[
\vec{v}_{p, q, 2} = 2\nu_{K_{\alpha}}\],
\[
\vec{v}''_7 = 2\nu_{K_{\alpha}},
\]
respectively.

The types [no. 2] and [no. 5] are associated with the embedding of \(D_4\) into \(E_8\). Along with \(\{U, X, Y, Z, \vec{v}_{5, 6, 7}\}\) we can take \(\vec{v}_8 = 2\nu_{K_{\alpha}}\) for both types, and complete the extended Dynkin diagram by adding
\[
\vec{v}_8 = 2\nu_{K_{\alpha}},
\]
respectively. Working out the other irreducible components of the diagram of the vectors around the deep hole, we see that these two types correspond to the embedding [no.2: \(D_4 \subset E_8^{(1)}, A_2 \subset E_8^{(2)}; E_8^{(3)}\)] and [no.5: \(D_4 \subset E_8, A_2 \subset D_{16}\)], as desired.

5.5.3 Determination of Elliptic Divisors: \(D_4 \oplus A_2 \rightarrow D_n, n \geq 7\)

We now discuss the third group in Table 9, type no.12, \(\ldots, 21\) of \(J_2(X)\), for which both \(D_4\) and \(A_2\) are embedded in a \(D_n\) component of \((L^{(I)})_{\text{root}}\). We start with the octad
\[
\begin{array}{cccc}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
\end{array}
\]

Examining the rest of the diagram of vectors, we see that the three different choices of the codeword \(K \in \mathbb{C}_{24}(8)\) lead to three different embedding, [no.19: \(D_4 \subset E_8^{(1)}, A_2 \subset E_6^{(2)}; E_6^{(3)}\)], [no.23: \(D_4 \subset E_6, A_2 \subset A_{11}; D_7\)] and [no.24: \(D_4 \subset E_6, A_2 \subset D_7; A_{11}\)], respectively. Lemma I is used to determine elliptic divisors of those three types, and the results are recorded in Table 9.

For the types [no.9], [no.10] and [no.11], where an embedding \(D_4 \subset E_7\) is used, we take \(\vec{v}_7 = 2\nu_{K_{\alpha}}\). The centres of deep holes with an appropriate embedding of the vectors \(\{U, X, Y, Z, \vec{v}_5, 6, 7\}\) and \(\vec{v}_{p, q, 2}\) for these three types are found when we take one more vector \(\vec{v}''_7\) to complete the extended Dynkin diagram of \(E_7\) as
\[
\vec{v}_{p, q, 2} = 2\nu_{K_{\alpha}}\],
\[
\vec{v}''_7 = 2\nu_{K_{\alpha}},
\]
respectively.

The types [no. 2] and [no. 5] are associated with the embedding of \(D_4\) into \(E_8\). Along with \(\{U, X, Y, Z, \vec{v}_{5, 6, 7}\}\) we can take \(\vec{v}_8 = 2\nu_{K_{\alpha}}\) for both types, and complete the extended Dynkin diagram by adding
\[
\vec{v}_8 = 2\nu_{K_{\alpha}},
\]
respectively. Working out the other irreducible components of the diagram of the vectors around the deep hole, we see that these two types correspond to the embedding [no.2: \(D_4 \subset E_8^{(1)}, A_2 \subset E_8^{(2)}; E_8^{(3)}\)] and [no.5: \(D_4 \subset E_8, A_2 \subset D_{16}\)], as desired.
systematically after this. The following vectors are used to complete the extended Dynkin diagram $D$ of $\vec{v}$ which allows us to define $\nu = 2\nu_{K_8};$ sitting in between $X$ and $P,$ see Figure 3. The 4 vectors $\{U, Y, Z, X\}$ for $D_4 \subset T_0,$ 2 vectors $\{P, Q_2\}$ for $A_2 \subset T_0$ and $S'$ already form the Dynkin diagram of $D_7.$ These 7 vectors form the stem for the 7 types of $J_2(X)$ studied in section 5.5.3 as well as for the 2 types to be covered in section 5.5.4.

The type [no.21] should be associated with the embedding $[(D_4 \oplus A_2) \subset D_7; E_6 \oplus A_{11}].$ For the other vector $\vec{v}_7'$ forming an extended Dynkin diagram of $D_7,$ we choose $\vec{v}_7' = \nu_{11} - 4\nu_{2} \equiv Q_k$ for $k \in \{11, 18, 8, 12\}$ (e.g., 11), see Figure 3. The other irreducible components of $L_{\nu_{11}}^2$ are made of vectors $\vec{u} \in \Lambda_{24}$ that are at norm $(-4)$ distance from all the seven vectors $U, Y, Z, X, S', P, Q_2$—(***)—here, the condition $(\vec{u} - U)^2 = (\vec{u} - Y)^2 = \cdots = -4$ is implied. There are 22 vectors satisfying (***)—they are shown in Figure 4. Note that the labels on the vertices are not $\vec{u} \in \Lambda_{24}$ but their corresponding Leech roots $\lambda = (\vec{u}, 1, -1 - (\vec{u})^2/2) \in S_X \subset \Lambda_{24} \oplus U(\omega).$ Among those 22 vectors, three vectors corresponding to $G_{43}, G_{24}$ and $G_{32}$ are not at the norm $(-4)$ distance from $\vec{v}_7.'$. Eliminating those three vertices from the diagram in Figure 4 we see that the extended Dynkin diagrams of $E_6$ and $A_{11}$ are left indeed. This is how we see that this choice of $\vec{v}_i^{(I)\alpha_i} \text{s}$ for a deep hole of type $I = \lambda,$ and for the embedding [no.21]: $(D_4 \oplus A_2) \subset D_7; E_7 \oplus A_{11}].$ Elliptic divisors can be computed by using Lemma I, as before.

Similarly to the study in sections 5.5.1 and 5.5.2 and to Chapt. 23 of [CS], we can proceed systematically after this. The following vectors are used to complete the extended Dynkin diagram...
Figure 4: Part of the Coxeter diagram of $S_X$ for $X = \text{Km}(E_\omega \times E_\omega)$. The 22 vertices are labelled using the identifications (240) and (241). The displayed subset is tailored towards the study of embeddings $D_4 \oplus A_2 \hookrightarrow D_n$ or $E_n$. As explained in the main text, this diagram allows to read off the vectors of the remaining components around deep holes for these cases.

of $D_n$ to which $T_0 = D_4 \oplus A_2$ is embedded:

$$
\begin{align*}
\vec{v}_8 &= 2\nu_{K,\heartsuit}, & \vec{v}_8' &= 2\nu_{K,\diamondsuit} \quad \text{[no. 15]}, \\
\vec{v}_9 &= 2\nu_{K,\heartsuit}, & \vec{v}_9' &= 2\nu_{K,\heartsuit} \quad \text{[no. 18]}, \\
\vec{v}_{10} &= 2\nu_{K,\heartsuit}, & \vec{v}_{10}' &= 2\nu_{K,\heartsuit} \quad \text{[no. 7]}, \\
\vec{v}_{11} &= 2\nu_{K,\heartsuit}, & \vec{v}_{11}' &= 2\nu_{K,\heartsuit}, \\
\vec{v}_{12} &= 2\nu_{K,\heartsuit}, & \vec{v}_{12}' &= 2\nu_{K,\heartsuit} \quad \text{[no. 13]}, \\
\vec{v}_{13} &= 2\nu_{K,\heartsuit}, & \vec{v}_{13}' &= 2\nu_{K,\heartsuit}, \\
\vec{v}_{14} &= 2\nu_{K,\heartsuit}, & \vec{v}_{14}' &= 2\nu_{K,\heartsuit}, \\
\vec{v}_{15} &= 2\nu_{K,\heartsuit}, & \vec{v}_{15}' &= 2\nu_{K,\heartsuit}, \\
\vec{v}_{16} &= 2\nu_{K,\heartsuit}, & \vec{v}_{16}' &= 2\nu_{K,\heartsuit} \quad \text{[no. 3]}, \\
\vec{v}_{17} &= 2\nu_{K,\heartsuit}, & \vec{v}_{17}' &= 2\nu_{K,\heartsuit}, \\
\vec{v}_{18} &= 2\nu_{K,\heartsuit}, & \vec{v}_{18}' &= 2\nu_{K,\heartsuit}, \\
\vec{v}_{19} &= 2\nu_{K,\heartsuit}, & \vec{v}_{19}' &= 2\nu_{K,\heartsuit}, \\
\vec{v}_{20} &= 2\nu_{K,\heartsuit}, & \vec{v}_{20}' &= 2\nu_{K,\heartsuit}, \\
\vec{v}_{21} &= 2\nu_{K,\heartsuit}, & \vec{v}_{21}' &= 2\nu_{K,\heartsuit}, \\
\vec{v}_{22} &= 2\nu_{K,\heartsuit}, & \vec{v}_{22}' &= 2\nu_{K,\heartsuit}, \\
\vec{v}_{23} &= 2\nu_{K,\heartsuit}, & \vec{v}_{23}' &= 2\nu_{K,\heartsuit}, \\
\vec{v}_{24} &= 2\nu_{K,\heartsuit}, & \vec{v}_{24}' &= 2\nu_{K,\heartsuit} \quad \text{[no. 12]},
\end{align*}
$$

With this data and Lemma I, one can compute the elliptic divisors for all of these types. An explicit identification with the Leech roots in $S_X \subset \Lambda_{24} \oplus U(\omega)$ and curves $\{G_{ij}, E_i, F_i, C_i, D_i\}$ (see appendix A.3) allows us to write those divisors in the form presented in Table 9.

Before moving on to section 5.5.4 let us explain how Figure 4 is used to read off the type of
deep holes and embedding of $T_0$ into Niemeier lattices, as well as to determine the vectors $\vec{v}_n$’s systematically for large $n$. In the following, we frequently use the identification among the vectors in $\Lambda_2$, listed above, their corresponding Leech roots, and elements in $S_X$.

As for type [no.15: $T_0 \subset D_8$; $D_8 \oplus D_8$], vectors corresponding to the two vertices $F_1$ and $F_2$ in Figure 4 are not at norm-(-4) distance from $\vec{v}_8$, and the vectors corresponding to $D_1$ and $D_4$ in the figure are not from $\vec{v}_8$. Removing these four vertices from Figure 4, the extended Dynkin diagram of $D_8$ appears twice.

For the type [no.18: $T_0 \subset D_9$; $A_{15}$], we keep the vertices $D_1$ and $D_4$ (because $\vec{v}_8$ is not in $D_9 \subset L_\text{root}^{(t)}$), but still excise $F_2$ and $F_1$. Furthermore, since $\vec{v}_0 = F_1$ and $\vec{v}_9 = F_2$, we also have to delete the nearest neighbours (not at norm-(-4) distance) of $F_2$ and $F_1$, namely the four vertices $G_{21}, G_{24}, G_{12}$ and $G_{13}$. The rest of the diagram in Figure 4 is in the shape of the extended Dynkin diagram of $A_{15} \subset (L^{(t)})_\text{root}$. For the type [no.7: $T_0 \subset D_{10}$; $E_7 \oplus E_7$], we have $\vec{v}_{10} = G_{13}$ and $\vec{v}_9 = G_{12}$ completing the extended Dynkin diagram of the vectors $D_{10}$. Hence we have to remove $F_2, F_1, G_{12}, G_{13}$ and two more vertices $E_2$ and $E_3$ (because they are not at norm-(-4) distance from $\vec{v}_{10}$ and $\vec{v}_9$, respectively) from Figure 4. As expected from the list of Niemeier lattices (the $D_{10}$ irreducible component is found only in $L^{(t)}$; see table 1), this reproduces twice the extended Dynkin diagram of $E_7$. Using (136), the remaining cases are treated in the same fashion.

Diagrams similar to Figure 4 can be used to determine the vectors (130) and types of embeddings systematically in sections 5.5.1 and 5.5.2.

### 5.5.4 Determination of Elliptic Divisors: $D_4 \oplus A_2 \hookrightarrow E_8$

Finally, we discuss the cases where $T_0 = (D_4 \oplus A_2)$ are embedded into $E_8$ (no other embeddings into lattices of type $E_n$ are possible). We use ($\vec{v}_6 = Y_{H}$), $Y, Z, X, S', P, Q_2$ and $U$ for $E_8$ and enlarge it to the extended diagram by adding a vector $\vec{v}_8$, as shown in Figure 3. The only two candidates for $\vec{v}_8$ are $2\nu_K$ with $K_\circ$ and $K_\bullet$. The two possible choices correspond to [no. 1], and [no. 6], respectively. This can be seen from Figure 3 as follows: first we note—through computations using codewords—that $\vec{v}_6$ are not at norm-(-4) distance from $D_4, G_{43}$ and $G_{34}$, so that these vertices must be removed from Figure 3 for both cases. In case [no. 1], where $\vec{v}_8 = 2\nu_{K_\circ}$, we furthermore have to remove $D_1$, so that twice the extended Dynkin diagram of $E_8$ appears. Hence this choice of $\vec{v}_8$ corresponds to the embedding $T_0 \hookrightarrow E_8 \hookrightarrow L^{(t)}_\text{root}$, type [no.1] of [NishI]. For case [no. 6], where $\vec{v}_8 = 2\nu_{K_\bullet}$, we need to remove $C_1$ instead of $D_1$ and find the extended Dynkin diagram of $D_{16} \subset L^{(t)}_\text{root}$. Elliptic divisors are computed by using Lemma I, and the results are presented in Table 9.

### 5.5.5 Multiplicities in $X = \Km(E_\omega \times E_\omega)$

The Aut($D'$) and Aut($D'^{(\text{Hodge})}$) group action on $S_X \subset \Pi_{125}$ is known in the literature for $X = \Km(E_\omega \times E_\omega) = X_{[2 \ 0 \ 2]}$, as summarized in the appendix A.3. Now that the study in sections 5.5.1-5.5.4 specified at least one elliptic divisor within $D'$ for individual isometry classes (type) in $\mathcal{J}_2(X)$, we are ready to use Corollary F to derive type-dependent upper bounds on the multiplicity. The results are found in the last column of Table 9.

**Example J:** Among the 30 different types of elliptic fibrations in $\mathcal{J}_2(X)$ for $X = \Km(E_\omega \times E_\omega)$, there are at least 15 types where there is a unique isomorphism class of elliptic...
| no. | $I$ | $D_4$ | $A_2$ | elliptic divisor | n |
|-----|-----|-------|-------|------------------|---|
| 4   | $\beta$ | $D_{16}$ | $E_8$ | $D_4 + 2G_{23} + 3E_3 + 4G_{13} + 5\lambda_{q_{11}} + 6\lambda_p + 3\lambda_{q_{15}} + 4\lambda_{q_2} + 2\lambda_K,\bullet$ | 2 |
| 14  | $\epsilon$ | $D_{12}^{(1)}$ | $D_{12}$ | $D_1 + C_4 + \lambda_{q_{18}} + \lambda_{q_2} + 2(\lambda_p + \lambda_{q_{11}} + G_{43} + E_3 + G_{23} + D_4 + G_{14} + E_4 + G_{44})$ | 2 |
| 8   | $\eta$ | $D_{10}$ | $E_7$ | $E_1 + D_4 + 2(G_{21} + G_{14} + D_1) + 3(C_4 + E_4) + 4G_{44}$ | 2 |
| 17  | $\theta$ | $D_9$ | $A_{15}$ | $\lambda_p + \lambda_{q_2} + \lambda_K,\bullet + F_3 + G_{31} + E_1 + G_{21} + C_4 + G_{44} + E_4$ | 2 |
| 16  | $\iota$ | $D_{8}^{(1)}$ | $D_{8}^{(2)}$ | $E_4 + D_1 + D_3 + F_3 + 2(G_{44} + C_4 + G_{21} + E_1 + G_{31})$ | 2 |
| 20  | $\lambda$ | $D_7$ | $A_{11}$ | $G_{14} + G_{42} + G_{43} + 2(D_4 + C_1 + E_3) + 3G_{23}$ | 2 |
| 22  | $\lambda$ | $D_7$ | $E_6$ | $G_{14} + D_4 + G_{23} + C_1 + G_{42} + D_3 + E_1 + G_{31} + C_4 + G_{21} + E_4 + G_{44}$ | 1 |
| 25  | $\xi$ | $D_6^{(1)}$ | $D_{6}^{(2)}$ | $E_1 + F_3 + 2(G_{31} + D_3 + G_{42}) + C_1 + E_2$ | 1 |
| 26  | $\nu$ | $D_6$ | $A_9$ | $G_{43} + D_2 + G_{21} + C_4 + G_{44} + E_4 + G_{14} + D_4 + G_{23} + E_3$ | 2 |
| 27  | $\pi$ | $D_5$ | $A_7$ | $G_{24} + D_3 + G_{31} + F_3 + G_{33} + D_1 + G_{44} + E_4$ | 1 |
| 28  | $\pi$ | $D_5^{(1)}$ | $D_{5}^{(2)}$ | $G_{14} + F_1 + G_{12} + E_2 + G_{42} + C_1 + G_{23} + D_4$ | 1 |
| 29  | $\tau$ | $D_4^{(1)}$ | $D_{4}^{(2)}$ | $2G_{11} + E_1 + F_1 + C_1 + D_1$ | 1 |
| 30  | $\sigma$ | $D_4$ | $A_5$ | $G_{11} + C_1 + G_{34} + F_3 + G_{33} + D_1$ | 1 |
| 2   | $\gamma$ | $E_8$ | $E_8^{(2)}$ | $G_{23} + 2(E_3 + F_1) + 3(G_{33} + G_{24}) + 4(D_1 + G_{14}) + 5G_{44} + 6E_4$ | 2 |
| 5   | $\beta$ | $E_8$ | $E_8$ | $D_1 + E_3 + F_1 + D_2 + 2(G_{33} + F_3 + G_{31}) + E_1 + G_{41} + \lambda_{q_{18}} + \lambda_p + \lambda_{q_2} + \lambda_K,\bullet + C_1 + G_{42} + E_2 + G_{12}$ | 2 |
| 9   | $\eta$ | $E_7$ | $D_{10}$ | $2G_{24} + G_{33} + 2D_1 + 3G_{44} + 4E_4 + 3G_{14} + 2F_1 + G_{12}$ | 1 |
| 10  | $\eta$ | $E_7^{(1)}$ | $E_{7}^{(2)}$ | $\lambda_p + G_{53} + 2(\lambda_{q_2} + E_3 + G_{42}) + 3(\lambda_K,\bullet + G_{23}) + 4C_1$ | 1 |
| 11  | $\zeta$ | $E_7$ | $A_{17}$ | $C_1 + \lambda_K,\bullet + \lambda_{q_2} + \lambda_p + \lambda_{q_{18}} + G_{11} + E_1 + G_{21} + D_2 + G_{12}$ | 2 |
| 19  | $\mu$ | $E_6^{(1)}$ | $E_{6}^{(2)}$ | $G_{31} + G_{23} + G_{44} + 2(F_3 + E_3 + D_1) + 3G_{33}$ | 1 |
| 23  | $\lambda$ | $E_6$ | $A_{11}$ | $G_{24} + G_{14} + F_3 + E_3 + 2(E_4 + G_{44} + D_1 + G_{33})$ | 2 |
| 24  | $\lambda$ | $E_6$ | $D_7$ | $\lambda_{q_{18}} + \lambda_{q_2} + G_{44} + G_{14} + 2(\lambda_p + \lambda_{q_{11}} + G_{24} + E_4)$ | 1 |
| 12  | $\alpha$ | $D_{24}$ | $D_{24}$ | $\lambda_1 + \lambda_4 + G_{21} + G_{31} + 2(\lambda_2 + \lambda_3 + 2\nu_K,\bullet + \lambda_p + \lambda_{q_2} + \lambda_K,\bullet + F_1 + G_{13} + E_3 + G_{33} + D_1 + G_{44} + E_4 + G_{34} + C_1 + G_{42} + E_2 + G_{32} + D_4 + G_{41} + E_1)$ | 2 |
| 3   | $\beta$ | $D_{16}$ | $D_{16}$ | $G_{42} + 2(C_2 + E_2) + 3(G_{32} + G_{21}) + 4(G_{31} + D_1) + 5G_{41} + 6E_1$ | 2 |
| 13  | $\epsilon$ | $D_{12}^{(1)}$ | $D_{12}$ | $G_{24} + G_{44} + G_{21} + G_{31} + 2(E_4 + G_{34} + C_1 + G_{42} + E_2 + G_{32} + D_4 + G_{41} + E_1)$ | 1 |
| 7   | $\eta$ | $D_{10}$ | $D_{10}$ | $G_{33} + G_{42} + 2(D_1 + C_1 + G_{24}) + 3(G_{44} + G_{34}) + 4E_4$ | 1 |
| 18  | $\theta$ | $D_9$ | $D_9$ | $E_1 + G_{41} + D_4 + C_2 + E_2 + G_{42} + C_1 + G_{34}$ | 1 |
| 15  | $\iota$ | $D_{8}^{(1)}$ | $D_{8}^{(2)}$ | $E_4 + G_{44} + D_1 + G_{33} + E_3 + G_{43} + C_2 + G_{31}$ | 1 |
| 21  | $\lambda$ | $D_7$ | $D_7$ | $D_4 + F_2 + C_2 + 2(G_{41} + G_{21} + G_{31}) + 3E_1$ | 1 |
| 1   | $\gamma$ | $E_8^{(1)}$ | $E_{8}^{(1)}$ | $G_{33} + 2E_3 + 3G_{13} + 4F_1 + 5G_{12} + 6E_2 + 4G_{42} + 2C_1 + 3G_{32}$ | 2 |
| 6   | $\beta$ | $E_8$ | $E_8$ | $G_{31} + G_{41} + G_{42} + G_{32} + 2(E_1 + G_{21} + F_2 + G_{24} + E_4 + G_{44} + D_1 + G_{33} + E_4 + G_{13} + F_1 + G_{12} + E_2)$ | 2 |

Table 9: Type-dependent upper bounds on the multiplicity for the thirty types for $J_3(X)$ of $X = \text{Km}(E_\omega \times E_\omega)$ (shown in the last column). The type id. number in the first column is that of Table 1.3 in [Nishi1]. In the second column, the Niemeier lattice which $T_0$ is embedded into is specified. The third and fourth columns give more details of this embedding. In the fifth column, we give a possible choice of an elliptic divisor as an element of $\Pi_{1,25}$, followed by the upper bound on the multiplicity in the last column.
fibration. They are
no.7 \([W_{\text{root}} = A_3 E_7^{22}]\), no.9 \([A_1^{(2)} D_7 E_7]\), no.10 \([A_1^{(2)} A_5 D_{10}]\), no.13 \([D_5 D_{12}]\), no.15 \([D_8^{(2)}]\),
no.18 \([A_1^{(2)} A_3 D_6^{22}]\), no.19 \([A_1^{(2)} E_6^{22}]\), no.21 \([A_1^{(2)} A_3 A_5 D_{10}]\), no.22 \([A_1^{(2)} A_3 A_5 D_{10}]\),
no.24 \([A_1^{(2)} A_3 A_5 D_{10}]\), no.25 \([A_1^{(2)} A_3 D_6^{22}]\), no.26 \([A_1^{(2)} A_3 D_6^{22}]\), no.27 \([A_4 A_7 D_5]\), no.28 \([A_1^{(2)} A_3 D_6^{22}]\), no.29 \([A_1^{(2)} A_3 D_6^{22}]\), no.30 \([A_1^{(2)} A_3 D_6^{22}]\).

The remaining 15 types have at most two isomorphism classes of elliptic fibrations.

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A Appendix

A.1 The Leech Lattice and the Leech Roots

Here, we provide a review on Leech lattice \(\Lambda_{24}\), which we use extensively in section 5 in this article.
Conway–Sloan’s textbook \([CS]\) is the ideal reference for this subject, but this review will explain the
minimum prerequisite in reading this article, and at the same time, at least serve for the purpose
of setting notation to be used in this article.

A.1.1 The Leech Lattice

The Leech lattice \(\Lambda_{24}\) is uniquely characterized as the even unimodular negative definite lattice of
rank 24 which does not have any element of norm \((-2)\).

In order to construct Leech lattice \(\Lambda_{24}\), we begin by describing its free Abelian group of rank-24
as a subset of \(\mathbb{R}^{24} = \{\sum_{i \in \Omega} x_i \nu_i \mid x_i \in \mathbb{R}\}\), where \(\nu_i\)'s labelled by \(\Omega := \{0, 19, 15, 5, \infty, 3, 6, 9, 1, 20, 14, 21, 11, 4, 16, 13, 2, 10, 17, 7, 22, 18, 8, 12\}\) \((137)\)
are vectors in \(\mathbb{R}^{24}\), mutually linearly independent over \(\mathbb{R}\). It is convenient to introduce a notation
\(\nu_S := \sum_{i \in S} \nu_i \in \mathbb{R}^{24}\) for a subset \(S \subset \Omega\). With this notation, \(\Lambda_{24}^{\text{even}}\) is given by
\[
\Lambda_{24}^{\text{even}} = \left\{2\nu_C + \sum_{i \in \Omega} 4n_i \nu_i \mid n_i \in \mathbb{Z}, \sum_i n_i \equiv 0 \text{ mod } 2\right\} \quad (138)
\]
where \(C\) can be any one of codewords of the extended binary Golay code \(C(24)\) (see below). \(\Lambda_{24}^{\text{odd}}\)
consists of
\[
\nu_\Omega + 2\nu_C + \sum_{i \in \Omega} 4m_i \nu_i \mid m_i \in \mathbb{Z}, \sum_i m_i \equiv 1 \text{ mod } 2\right\}. \quad (139)
\]

\(^{32}\)The ordering used here anticipates the introduction of the miracle octad generator, see \((143)\).
$\Lambda_{24} = \Lambda_{24}^{\text{even}} \cup \Lambda_{24}^{\text{even}}$ forms a free Abelian group of rank 24 under the ordinary vector sum in $\mathbb{R}^{24}$.

A little bit of explanation on the Golay code is in order here. Let $P(\Omega)$ denote a set that consists of any subset of $\Omega$. $P(\Omega)$ consists of $2^{24}$ elements, as it can be identified with a vector space $(\mathbb{F}_2)^{24}$ over the field $\mathbb{F}_2$. The extended binary Golay code of length 24, $C_{24}$, is a specific choice of a subset of $P(\Omega)$. Instead of explaining its construction, we restrict ourselves to record some properties relevant to this article, see [CS] for a detailed treatment.

- $C_{24}$ is a $\mathbb{F}_2$-linear 12-dimensional subspace of $P(\Omega) \cong (\mathbb{F}_2)^{24}$. Thus, $|C_{24}| = 2^{12}$.
- $\emptyset \in C_{24}$, $\Omega \in C_{24}$.
- $C_{24}$ is decomposed into five subsets $\{\phi\} \cup \{\Omega\} \cup C_{24}(8) \cup C_{24}(12) \cup C_{24}(16)$, and all the elements of $C_{24}(8)$ (resp. $C_{24}(12)$, $C_{24}(16)$) are 8-element (resp. 12-element, 16-element) subsets of $\Omega$.

Any codeword in $C_{24}(8)$ is called an octad.

- $|C_{24}(8)| = |C_{24}(16)| = 759$ and $|C_{24}(12)| = 2576$.
- If one chooses four arbitrary elements from $\Omega$, i.e., $P \subseteq \Omega$, $|P| = 4$, then there are five (and not more than five) codewords $K_{1,2,3,4,5} \in C_{24}(8)$ that contain the specified 4 elements $P$. $K_i \cap K_j$ consists precisely of the 4 elements $P$, when $i \neq j$. Thus, for a given 4-element subset $P \subseteq \Omega$, there is a unique way to decompose $\Omega$ into six 4-element subsets; $\Omega = \Pi_{i=1}^5 (K_i \setminus P) \cup P$. Such a decomposition is called sextet decomposition, see below for explicit examples. For an explicit algorithm of finding out this decomposition, [CS] Chapter 11 will be the best reference to look at.

- A sextet decomposition may be described as $\Omega = \Pi_{a=0,\ldots,5} \Xi_a$. It is a sextet decomposition of 4-element subsets $\Xi_a \subseteq \Omega$ with $a = 1, \ldots, 5$ as much as for the 4-element subset $\Xi_0 \subseteq \Omega$.
- In a given sextet decomposition, $\Pi_a \Xi_a$, any 8-element subset of the form $K_{ab} = \Xi_a \cup \Xi_b$ ($a \neq b$) is a codeword in $C_{24}(8)$.
- If one chooses arbitrary 5 elements out of $\Omega$, then there is a unique codeword in $C_{24}(8)$ that contains the five elements.

The free Abelian group $\Lambda_{24}$ becomes the Leech lattice with the symmetric pairing $(\nu_i, \nu_j) = -\delta_{ij}/8$. It is known that the Leech lattice defined in this way is an even unimodular negative definite lattice of rank 24.

There is no norm $(-2)$ elements in this lattice. Norm $(-4)$ elements of $\Lambda_{24}$ are of the form
\begin{equation}
(2^8, 0^{16})^T, \quad (3, 1^{23})^T, \quad (4^2, 0^{22})^T
\end{equation}
in the component description $(x_0, x_{19}, x_{15}, \ldots, x_8, x_{12})^T$ modulo signs of each entries and ordering.

Norm $(-6)$ elements of $\Lambda_{24}$ are of the form
\begin{equation}
(2^{12}, 0^{12})^T, \quad (3^3, 1^{21})^T, \quad (4, 2^8, 0^{15})^T, \quad (5, 1^{23})^T.
\end{equation}

The isometry group of this lattice, Isom($\Lambda_{24}$), is often denoted by $\langle 0 \rangle$, or $Co_0$. Its Affine transformation group is denoted by $\langle -\infty \rangle$ or $Co_{\infty}$:
\begin{equation}
1 \rightarrow \mathbb{Z}^{24} \rightarrow Co_{\infty} \rightarrow Co_0 \rightarrow 1.
\end{equation}
A.1.2 More on $C_{24}(8)$

It proves convenient in dealing with codewords of $C_{24}$ to describe subsets of $\Omega$ (i.e., elements of $\mathbb{P}(\Omega)$) as follows. Any subset, let’s say $K$, of $\Omega$ is specified by whether individual elements are contained in it. So, when the 24 elements in $\Omega$ are laid out as in

$$\Omega = \begin{pmatrix}
0 & \infty & 1 & 11 & 2 & 22 \\
19 & 3 & 20 & 4 & 10 & 18 \\
15 & 6 & 14 & 16 & 17 & 8 \\
5 & 9 & 21 & 13 & 7 & 12
\end{pmatrix} \quad (143)$$

a subset $K_U = \{0, \infty, 1, 11, 2, 18, 8, 12\}$, for example, can be denoted by

$$\begin{array}{cccc}
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{array} \quad (144)$$

We will also use the two following codewords in $C_{24}(8)$ in the main text of this article:

$$K_L = \begin{array}{cccc}
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{array}, \quad K_R = \begin{array}{cccc}
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{array} \quad (145)$$

The layout of the 24 elements of $\Omega$ in (143) follows so called “standard MOG labelling” (p.309 Fig. 11.7 of [CS]).

Because we will use them in the main text of this article, we leave a list of some of the sextet decompositions we referred to above. The sextet decomposition for the following five 4-element subsets of $K_U$ (and hence subsets of $\Omega$), $\Xi_\epsilon = \\{\infty, 0, 1, 2\}$, $\Xi_\alpha = \\{\infty, 0, 1, 11\}$, $\Xi_\beta = \\{\infty, 0, 1, 18\}$, $\Xi_\gamma = \\{\infty, 0, 1, 8\}$, $\Xi_\delta = \\{\infty, 0, 1, 12\}$ are given as follows:

$$\begin{array}{ccc}
\Pi_{a \in A} \Xi_\epsilon & = & \begin{array}{cccc}
\clubsuit & \Diamond & \spadesuit & \heartsuit \\
\spadesuit & \heartsuit & \spadesuit & \heartsuit \\
\heartsuit & \spadesuit & \heartsuit & \spadesuit \\
\spadesuit & \heartsuit & \spadesuit & \heartsuit
\end{array} \\
\Pi_{a \in A} \Xi_\alpha & = & \begin{array}{cccc}
\bigstar & \bigstar & \bigstar & \bigstar \\
\spadesuit & \spadesuit & \spadesuit & \spadesuit \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit \\
\diamondsuit & \diamondsuit & \diamondsuit & \diamondsuit
\end{array} \\
\Pi_{a \in A} \Xi_\beta & = & \begin{array}{cccc}
\heartsuit & \spadesuit & \bigstar & \bigstar \\
\spadesuit & \spadesuit & \spadesuit & \bigstar \\
\heartsuit & \heartsuit & \heartsuit & \bigstar \\
\diamondsuit & \diamondsuit & \diamondsuit & \bigstar
\end{array} \\
\Pi_{a \in A} \Xi_\gamma & = & \begin{array}{cccc}
\ddiamondsuit & \diamondsuit & \spadesuit & \spadesuit \\
\spadesuit & \spadesuit & \spadesuit & \spadesuit \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit \\
\spadesuit & \spadesuit & \spadesuit & \spadesuit
\end{array} \\
\Pi_{a \in A} \Xi_\delta & = & \begin{array}{cccc}
\bigstar & \bigstar & \bigstar & \bigstar \\
\spadesuit & \spadesuit & \spadesuit & \spadesuit \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit \\
\spadesuit & \spadesuit & \spadesuit & \spadesuit
\end{array}
\end{array} \quad (146)$$

We followed the explanations in Chapter 11 of [CS] in order to determine these sextet decompositions in $C_{24}$. 

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where \( a \in A = \{ \ast, \circ, \bigtriangledown, \blacklozenge, \blacklozenge, \Diamond \} \). From these five sextet decompositions, we can find out all the codewords \( K \) in \( \mathcal{C}_{24}(8) \) satisfying \( \{0, \infty, 1\} \subset K \) and \( |K \cap K_U| = 4 \):

\[
\begin{align*}
K_{\ast a} & = \Xi^{(\ast)} I \Xi^{(\ast)}_a, & a \in \{ \bigtriangledown, \blacklozenge, \blacklozenge, \Diamond \} \\
K_{\circ a} & = \Xi^{(\circ)} I \Xi^{(\circ)}_a, & K_{\beta a} = \Xi^{(\beta)} I \Xi^{(\beta)}_a, \\
K_{\blacklozenge a} & = \Xi^{(\blacklozenge)} I \Xi^{(\blacklozenge)}_a, & K_{\delta a} = \Xi^{(\delta)} I \Xi^{(\delta)}_a.
\end{align*}
\]

(147) - (149)

For example, \( K_{\ast 0} = \{0, \infty, 1, 4, 16, 13, 2, 22\} \subset \Omega \). Overall, there are twenty codewords \( K \in \mathcal{C}_{24}(8) \) satisfying \( \{0, \infty, 1\} \subset K \) and \( |K \cap K_U| = 4 \).

### A.1.3 The Centres of Deep Holes in \( \Lambda_{24} \)

All that we state in this appendix can also be found in Chapt. 23 in [CS]. We only quote the results that we need in the main text of this article. For more systematic exposition on this subject, we recommend to consult [CS].

For any \( \vec{u} \in (\Lambda_{24} \otimes \mathbb{R}) \setminus \Lambda_{24} \), it is known that

\[
\text{Min} \left[ -(\vec{u} - \vec{v})^2 \mid \vec{v} \in \Lambda_{24} \right] \leq 2. \tag{150}
\]

Points in \( \Lambda_{24} \otimes \mathbb{R} \) saturating this inequality—points in \( \Lambda_{24} \otimes \mathbb{R} \) that can be as far away as possible from integral points \( \Lambda_{24} \)—are called centres of deep holes in \( \Lambda_{24} \). The \( C_{\infty} \) symmetry group of \( \Lambda_{24} \), with its action naturally extended linearly to \( \Lambda_{24} \otimes \mathbb{R} \), acts on these centre of deep holes. It is known that the centres of deep holes form twenty-three distinct orbits of \( C_{\infty} \), and are labelled by \( I = \alpha, \beta, \cdots, \phi, \chi, \psi \), as in Table 1. Centre of deep holes that belong to the label-\( I \) orbit are said to be type \( I \) in this article.

It is due to the following reason that the twenty-three different labels \( I = \alpha, \beta, \cdots \) for Niemeier lattices can also be used to distinguish \( C_{\infty} \)-inequivalent deep holes. For the centre of a deep hole \( \vec{c}^{(I)} \in \Lambda_{24} \otimes \mathbb{R} \) of type \( I \), consider all of \( \vec{v} \in \Lambda_{24} \) satisfying \( (\vec{v} - \vec{c}^{(I)})^2 = -2 \). A graph is determined for this set of points \( \{ \vec{v} \} \), by assigning one node for each \( \vec{v} \) in this set, and by drawing, between a pair of nodes for \( \vec{v}_i \) and \( \vec{v}_j \), two lines if \( (\vec{v}_i - \vec{c}, \vec{v}_j - \vec{c}) = +2 \), one line if \( (\vec{v}_i - \vec{c}, \vec{v}_j - \vec{c}) = +1 \), and no line if \( (\vec{v}_i - \vec{c}, \vec{v}_j - \vec{c}) = 0 \). It is known that the graph becomes the collection of extended Dynkin diagrams of A–D–E root system in the combination specified in one of the twenty-three \( (I = \alpha, \beta, \cdots, \psi) \) entries in Table 1 (see [CS] Chapt. 23). The type label \( I \) for the \( C_{\infty} \)-orbits of the centres of deep holes is assigned through the correspondence with the classification of Niemeier lattices. The points \( \vec{v} \in \Lambda_{24} \) satisfying \( (\vec{v} - \vec{c}^{(I)})^2 = -2 \) for a given deep hole of type \( I \) are denoted by \( \vec{v}_{i}^{(I)a} \), where \( a = 1, 2, \cdots \) label irreducible components of the A–D–E root systems in \( (L^{(I)})_{\text{root}} = \oplus_a R_a \), and \( i \) runs from 0 to the rank \( r_a \) of the \( a \)-th component \( R_a \). For example, for \( I = \beta \), \( R_{a=1} \) and \( R_{a=2} \) correspond to \( D_{16} \) and \( E_8 \), respectively, and \( i = 0, 1, \cdots, r_1 = 16 \) for \( a = 1 \) and \( i = 1, \cdots, r_2 = 8 \) for \( a = 2 \). It is further known that

\[
h_{I}(\vec{c}^{(I)}) = \sum_{i=0}^{r_1} n_{i}^{(I)a=1}, v_{i}^{(I)a=1}
\]

(151)

holds for any \( a \). Here, \( n_{i}^{(I)a} \) are integers (sometimes referred to as Kac label or Dynkin label) assigned to individual nodes of the A–D–E extended Dynkin diagram (\( n_0 = 1 \) for any A–D–E root system),

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so that the maximal root is given by the linear combination of simple roots $\alpha_i^0$ as $\sum_{i=1}^{n} n_i^0 \alpha_i$. The number $h(t)$ on the left-hand side is the dual Coxeter number of the A–D–E root system, which is common for all irreducible components $R_a$ in the individual entries of Table 1, i.e. there is one $h(t)$ for any Niemeier lattice.

It is useful for practical computations to note that, 

$$
(\vec{v}_i - \vec{v}_j)^2 = \begin{cases} 
-8 & \quad \text{if } i = j \\
-6 & \quad \text{if } i \neq j \\
-4 & \quad \text{if } i = 1 \\
0 & \quad \text{if } i \neq 1
\end{cases} = (\vec{v}_i - \vec{c}, \vec{v}_j - \vec{c}), \quad (152)
$$

respectively, under the condition that $(\vec{v} - \vec{c})^2 = -2$. Thus, in particular, the condition that $\vec{v}_i - \vec{c}$ and $\vec{v}_j - \vec{c}$ are orthogonal is equivalent to the norm-(-4) distance between the two vectors $\vec{v}_i$ and $\vec{v}_j$ around a deep hole.

### A.1.4 Leech Roots and the II_{1,25} Lattice

The rank-26 lattice $\Lambda_{24} \oplus U$ is an even (i.e., type II) unimodular lattice of signature $(1, 25)$. From the classification theorem for indefinite even unimodular lattices, mentioned in Section 2.1, it follows that it is unique modulo isometry. This lattice—denoted by $\Pi_{1,25}$—can be described as an Abelian group as 

$$
\Pi_{1,25} \cong \Lambda_{24} \oplus U = \{ (\vec{v}, m, n) \mid \vec{v} \in \Lambda_{24}, m, n \in \mathbb{Z} \}. \quad (153)
$$

The reflection symmetry group of this even unimodular lattice, $W(\Pi_{1,25}) = W(2)(\Pi_{1,25})$, can be generated by reflections associated with simple roots, $\Pi$, which are called Leech roots in this case. It is possible to take a fundamental chamber of this reflection symmetry group, $C_\Pi$, so that the simple roots (i.e., Leech roots) are of the form

$$
\Pi = \left\{ \lambda = \left( \vec{v}, 1, -1 - \frac{(\vec{v})^2}{2} \right) \in \Pi_{1,25} \mid \vec{v} \in \Lambda_{24} \right\}, \quad (154)
$$

where $(\vec{v})^2$ is the norm of $\vec{v}$ in $\Lambda_{24}$. For this reason, there is a one-to-one correspondence between the Leech roots $\Pi$ and $\Lambda_{24}$. Note that all these Leech roots are of norm $(-2)$ in $\Pi_{1,25}$, and that all of them satisfy $\langle w, \lambda \rangle = 1$ for $w = (0, 0, 1) \in \Pi_{1,25}$. It is also useful to note that

$$
\langle \lambda, \lambda' \rangle_{\Pi_{1,25}} = (\vec{v}, \vec{v}') - 2 - \frac{(\vec{v})^2 + (\vec{v}')^2}{2} = -2 - \frac{(\vec{v} - \vec{v}')^2}{2}, \quad (155)
$$

which takes the values $-2, 0, +1, +2, \cdots$.

The isometry group of $\Pi_{1,25}$—$\text{Isom}(\Pi_{1,25}) \cong \text{Isom}^+(\Pi_{1,25}) \times \{ \pm 1 \}$—has the following structure: $\text{Isom}^+(\Pi_{1,25}) = W_\Pi \times \text{Isom}(\Pi_{1,25})^{(Cn)}$, where $\text{Isom}(\Pi_{1,25})^{(Cn)}$ is the group of isometries of $\Pi_{1,25}$ that map $C_\Pi$ to itself. Since the Leech roots correspond to the boundary walls (reflection hyperplanes) of the chamber $C_\Pi$, isometries in $\text{Isom}(\Pi_{1,25})^{(Cn)}$ map Leech roots $\Pi$ to themselves, and upon identification between $\Pi$ and $\Lambda_{24}$, the group $\text{Isom}(\Pi_{1,25})^{(Cn)}$ can be regarded as $C_{\infty}$ on $\Lambda_{24}$.

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34Such a vector is sometimes called a Weyl vector.
A.2  The Neron–Severi Lattice of $X_3$

A.2.1  Tasks I–IV for $X_3$

In this section, we see how the tasks I–IV introduced in section 5.2 are carried out and how the assumptions (as-1)–(as-5) verified in practice by working on a specific example in detail. We choose a singular K3 surface $X = X_{3} = X_{[1 1 1]}$ where the symmetric pairing of $T_X$ is given by \[
\begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\]
and the structure of the Neron–Severi lattice and its isometry group have been studied very well (e.g., [Vin1, Bor2]). The following discussion in appendices A.2.1 and A.2.2 is a kind of calculation note that fills the gaps in these references which are not obvious for non-experts.

In this example, $T_0$ is the $E_6$ root lattice, so that the assumptions (as-1) and (as-2) are satisfied. Let $J = \{\alpha_1, \cdots, \alpha_6\}$ be the simple roots of $T_0 = E_6$, see Figure 5. The Aut($J$) ∼ $\mathbb{Z}/2\mathbb{Z}$ group is generated by the left-right flip of the Dynkin diagram (Figure 5), which exchanges the weights of the 27 and 27 representations. Thus Aut($J$) ∼ $\mathbb{Z}/2\mathbb{Z}$ is mapped to Isom($q$) ∼ $\mathbb{Z}/2\mathbb{Z}$ of the discriminant group $G_T \cong G_{T_X} \cong \mathbb{Z}/3\mathbb{Z}$ (Table 2) and the assumption (as-3) is also satisfied.

[task I] The embedding $\phi_{T_0} : J \rightarrow \Pi$ is given as follows. First, we define 6 points in $\Lambda_{24}$ as follows,

\[
\begin{align*}
\vec{v}_1 &:= 4\nu_{(0,1)} =: P', \\
\vec{v}_2 &:= 2\nu_{K_L'}, \\
\vec{v}_3 &:= \nu_{1} + 4\nu_{1} =: X', \\
\vec{v}_4 &:= 0 =: Z, \\
\vec{v}_5 &:= \nu_{1} + 4\nu_{0} =: Y, \\
\vec{v}_6 &:= \nu_{1} - 2\nu_{K_U'} + 4\nu_{0}.
\end{align*}
\]

$K_{L'}$ and $K_{U'}$ are codewords in $C_{24}(8)$ satisfying the following conditions:

\[
0 \in K_{L'}, K_{U'}, \quad 1 \notin K_{L'}, K_{U'}, \quad |K_{L'} \cap K_{U'}| = 2.
\]

The example $X = X_3$ is chosen because it is the easiest example in carrying out the tasks I–IV and verify assumptions 1–5 in practice. Apart from this exercise purpose, though, there is not much meaning in calculating Aut($D'$) for this particular example $X = X_3$, because we know already that $p_T : \text{Isom}(T_X)^{(\text{Hodge})} \rightarrow \text{Isom}(q)$ is surjective (Table 7 2nd row). We can conclude from Proposition C without any detailed knowledge about the structure of Aut($D'$) that there is only one isomorphism class of elliptic fibrations in $\mathcal{J}_1(X)$ for $X = X_3$ for any isometry class $[W] \in \mathcal{J}_2(X)$. 

\[54\]
As an example, we can take

\[
K_L' = \begin{bmatrix}
* & * \\
* & * \\
* & * \\
\end{bmatrix}, \quad K_U' = \begin{bmatrix}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
\end{bmatrix} \tag{163}
\]

where we follow the conventions explained in appendix [A.1]. Noting that \((\vec{v}_i - \vec{v}_{i+1})^2 = -6\) for all \(i = 1, \ldots, 4\) and \((\vec{v}_3 - \vec{v}_6)^2 = -6\), but \((\vec{v}_i - \vec{v}_j)^2 = -4\) for \(i \neq j\) otherwise, we see that the six simple roots \(\alpha_1, \ldots, 6\) of \(T_0\) can be embedded into the six Leech roots related to the six points \(\vec{v}_i \in \Lambda_{24}\) through \([154] [155]\). This embedding \(J \rightarrow \Pi\) as well as its linear extension \(T_0 \hookrightarrow \Pi_{1,25}\) is denoted by \(\phi_{T_0}\).

In order to see that \(\phi_{T_0} : T_0 \hookrightarrow \Pi_{1,25}\) is a primitive embedding, it is sufficient to make sure that only \(T_0 \cong E_6\) within \(T_0' = E_6^*\) are mapped to integral points in \(\Pi_{1,25}\). The discriminant group \(G_{E_6} \cong \mathbb{Z}/3\mathbb{Z}\) is generated by a weight of the 27-dimensional representation of \(E_6\), \((4\alpha_1 + 5\alpha_2 + 6\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6)/3\). Since \(\phi_{T_0}\) maps this weight to \(\Pi_{1,25} \otimes \mathbb{Q}\), but not to \(\Pi_{1,25}\), this embedding \(\phi_{T_0} : T_0 \hookrightarrow \Pi_{1,25}\) is indeed primitive and the [task I] is completed.

[task II] Let \(R := \phi_{T_0}(J)\). Since \(R = E_6\) in this example, the only possible spherical subdiagrams \(R' = R \cup \lambda\) of the Coxeter diagram of \(\Pi\) are of the form of either type 0) \(R' = E_6 + A_1\) or type 1) \(R' = E_7\).

Leech roots \(\lambda\) of type 0), namely \(\{\lambda \in \Pi| (\lambda, \lambda_i) = 0\text{ for } i = 1, \ldots, 6\}\), correspond to

\[
\{\vec{v} \in \Lambda_{24} | (\vec{v} - \vec{v}_i)^2 = -4\text{ for } i = 1, \ldots, 6\}. \tag{164}
\]

With the embedding given by \([156] [161]\), the condition for \(i = 3\) implies that \(\vec{v}\) must be a norm \((-4)\) element of \(\Lambda_{24}\), which is always in the form of \([140]\). Further imposing the conditions for \(i = 1, 2, 4, 5, 6\), it turns out that \(\vec{v} \in \Lambda_{24}\) should be of the form of \(2\nu_K\) for \(K \in C_{24}(8)\) satisfying

\[
\{0, 1\} \subset K, \quad |K \cap K_L'| = 4, \quad |K \cap K_U'| = 2. \tag{165}
\]

The corresponding Leech roots of type 0) are of the form \((2\nu_K, 1, 1)\). There are 24 codewords \(K \in C_{24}(8)\) satisfying these conditions, and hence there are 24 Leech roots of type 0). Among them, six involve the codewords \(K \in C_{24}(8)\) listed explicitly in the appendix [A.1]

\[
\lambda_{22} = (2\nu_{K^H}, 1, 1), \quad \lambda_6 = (2\nu_{K^H}, 1, 1), \quad \lambda_{15} = (2\nu_{K^H}, 1, 1), \tag{166}
\]

\[
\lambda_{23} = (2\nu_{K^H}, 1, 1), \quad \lambda_3 = (2\nu_{K^H}, 1, 1), \quad \lambda_{12} = (2\nu_{K^H}, 1, 1). \tag{167}
\]

The remaining 18 other codewords \(K\) are also given in Appendix [A.2.2]. Because all the Leech roots of type 0) are also norm \((-2)\) roots of \(S_X\), the Coxeter diagram of the twenty-four Leech roots of type 0) becomes a subgraph of the Coxeter diagram of \(W(S_X)\). This graph for \(X_3\) is given in Figure [6] which reproduces the result of [Vin1].

Leech roots of type 1) correspond to vectors \(\vec{u} \in \Lambda_{24}\) satisfying either

- \((\vec{u} - \vec{v}_i)^2 = -4\) for \(i = 1, 2, 3, 4, 6\), but \((\vec{u} - \vec{v}_5)^2 = -6\), or
- \((\vec{u} - \vec{v}_i)^2 = -4\) for \(i = 2, 3, 4, 5, 6\), but \((\vec{u} - \vec{v}_1)^2 = -6\).
The vectors of the first form have to be
\[ \vec{u}_{k'} = \nu_\Omega - 2\nu_{K_{k'}} + 4\nu_1 \] (168)
for a codeword \( K_{k'} \in C_{24}(8) \) satisfying
\[ |K_{k'} \cap K_{U'}| = 4, \quad 1 \in K_{k'}, \quad \text{and} \quad K_{k'} \cap K_{U'} = \phi. \] (169)
There are six codewords satisfying this set of conditions, and they are labelled by \( k' \in K_{U'} \setminus \{0, \infty\} = \{11, 2, 22, 20, 14, 21\} \). The explicit form of those codewords is given in Appendix A.2.2. The vectors \( \vec{u}_{k''} \in \Lambda_{24} \) of the second form have to be
\[ \vec{u}_{k''} = \nu_\Omega - 4\nu_{k''} \] (170)
for \( k'' \in K_{U'} \setminus \{0, \infty\} \), so that there are also six choices of \( k'' \in \{20, 14, 21, 11, 2, 22\} \). Overall, there are twelve Leech roots of type 1), reproducing the result of [Vin1]. The [task II] is now completed.

Section 2 of [Bor2] contains all the information necessary in carrying out the [task III]. For type 0) roots, \( \sigma_{R'} \cdot \sigma_R \) preserves individual simple roots or \( R = E_6 \), and for type 1) roots, \( \sigma_{R'} \cdot \sigma_R \) maps the simple roots of \( R = \phi_{T_0}(J) \) to itself as a whole, but with the left-right flip. This also means that the assumption (as-4) is automatically satisfied in the case \( X = X_3 \).

The [task IV] is to compute the Aut(\( D' \)) group, which is carried out by following the logic of the latter half of the proof of Lemma 4.5 in [Ko2]. For the argument in Lemma 4.5 of [Ko2] to work, it is sufficient that the following two conditions are satisfied:

(as-6) Leech roots of type 0) generate \( S_X \) with \( \mathbb{Z} \) coefficients,
(as-7) \( w' \)—the orthogonal projection of the Weyl vector \( w \in \Pi_{1,25} \) onto \( S_X \)—can be written in the form of a linear combination of the type 0) Leech roots that is manifestly invariant under the symmetry of the Coxeter graph of the type 0) Leech roots.
In the case of $X = X_3$, we will see explicitly in the appendix A.2.2 that these two conditions are satisfied. They also hold in the case of $X = \text{Km}(A)$ studied in [Ko2] and $X = \text{Km}(E \times F)$, $\text{Km}(E \times E)$, $\text{Km}(E_g \times E_g)$ and $\text{Km}(E \times E)$ in [KK]. Under the condition (as-6), the symmetry group $\text{Aut}(\Gamma)$ of the Coxeter graph of the type 0) Leech roots $\Gamma$ generates isometries of $S_X$ and is identified with a subgroup of $\text{Isom}(S_X)$. The condition (as-7) further implies that it is a subgroup of $\text{Aut}(D')$. Conversely, any $g \in \text{Aut}(D')$ maps type 0) Leech roots to themselves, inducing a symmetry transformation of the graph $\Gamma$. Therefore,

\[ \text{Aut}(\Gamma) \cong \text{Aut}(D'). \tag{171} \]

In the case of $X = X_3$, the symmetry of the Coxeter graph of the type 0) Leech roots, i.e., Figure 6, is $(S_3(\Delta) \times S_3(\nabla)) \rtimes \mathbb{Z}_2 \langle \sigma_\Delta \rangle$, where $S_3(\Delta) \langle \tau_\Delta, \sigma_\Delta \rangle$ is the permutation of the three trivalent vertices $\{\lambda_22, \lambda_6, \lambda_{15}\}$, and $S_3(\nabla) \langle \tau_\nabla, \sigma_\nabla \rangle$ that of the other three trivalent vertices $\{\lambda_23, \lambda_{12}, \lambda_3\}$.

\[ \tau_\Delta : \lambda_22 \rightarrow \lambda_6 \rightarrow \lambda_{15} \rightarrow \lambda_22, \quad \tau_\nabla : \lambda_23 \rightarrow \lambda_{12} \rightarrow \lambda_3 \rightarrow \lambda_23, \]

\[ \sigma_\Delta : \lambda_6 \leftrightarrow \lambda_{15}, \quad \lambda_22 \text{ remains invariant}, \]

\[ \sigma_\nabla : \lambda_3 \leftrightarrow \lambda_{12}, \quad \lambda_23 \text{ remains invariant}. \tag{173} \]

The generator $\sigma_\Delta$ is the anti-podal transformation of the Coxeter diagram Figure 6, $\sigma_\nabla : \lambda_{22} \leftrightarrow \lambda_{23}$, $\lambda_3 \leftrightarrow \lambda_{15}$, $\lambda_6 \leftrightarrow \lambda_{12}$. With (171), now the [task IV] is completed.

In order to confirm that the assumption (as-5) is satisfied, we need to know the homomorphism $p_S : \text{Aut}(D') \rightarrow \text{Isom}(q)$. In particular, we need to know the kernel of this map, which is denoted by $\text{Aut}(D')_0$. Since $\text{Aut}(D')_0$ is the subgroup of $\text{Co}_{\infty}$ preserving all $\vec{v}_1, \ldots, \vec{v}_6$ in (156, 161), any elements of $\text{Aut}(D')_0$ should preserve $Z = \vec{0}$. This means that $\text{Aut}(D')_0 \subset \text{Co}_{\infty}$. Furthermore, because any element has to preserve $P' + Y - X = 8\nu_0$, $\text{Aut}(D')_0 \subset 2^{12}M_{24}$ (Ref. [CS], Chapt. 10 Thm.26). With a little more thought, one can see that $\text{Aut}(D')_0 \subset M_{22}$. This great deal of simplification follows from choosing the vectors (156, 161) so that $X', Y, Z$ and $P'$ are included. This method, introduced in [Ko2] and exploited also in [KK], can be used for any $\nu_0$ containing $A_3 \oplus A_1$ as a sublattice.

In the case of $X = X_3$, $\text{Aut}(D')$ has to be a subgroup of $(S_6 \times S_6 \times S_9) \cap M_{22}$, where the first two $S_6$'s act as permutation on $K_{Lu} \{0, \infty\}$ and $K_{Lu'} \{0, \infty\}$, and the last $S_9$ acts on $\Omega \backslash (K_{Lu} \cup K_{Lu'} \cup \{1\})$. The $\text{Aut}(D')_0$ group is also a subgroup of $\text{Aut}(D') \cong \text{Aut}(\Gamma)$. This is enough to determine $\text{Aut}(D')_0$. It turns out that

\[ ((\mathbb{Z}/3\mathbb{Z}) \langle \tau_\Delta \rangle \times (\mathbb{Z}/3\mathbb{Z}) \langle \tau_\nabla \rangle) \rtimes (\mathbb{Z}/4\mathbb{Z}) \langle \sigma_\Delta \cdot \sigma_\nabla \rangle, \tag{176} \]

which is an index 2 subgroup of $\text{Aut}(D') \cong (S_3 \times S_3) \rtimes (\mathbb{Z}/2\mathbb{Z})$. It thus follows that

\[ p_S : \text{Aut}(D') \rightarrow \text{Aut}(D')/\text{Aut}(D')_0 \cong \mathbb{Z}/2\mathbb{Z} \tag{177} \]

is a surjective homomorphism to $\text{Isom}(q) \cong \text{Aut}(J) \cong \mathbb{Z}/2\mathbb{Z}$. Now the assumption (as-5) is verified for $X = X_3$.

As a side remark, $\text{Aut}(D')(\text{Hodge})$ is also the same as $\text{Aut}(D')$, because $p_T : \text{Isom}(T_X)(\text{Hodge}) \rightarrow \text{Isom}(q)$ is surjective in this example (see Table 7).

It is also possible to work out explicitly the subgroup of $\text{Aut}(D')$ that preserves $R' = \phi_{\nu_0}(J) \cup \lambda$ for a given Leech root $\lambda$ of type 1). Let us take $\vec{u}_{11\nu}$ as an example. Such a subgroup should preserve
\{\lambda_{24}, \lambda_{25}, \lambda_{26}\} separately from all the other Leech roots of type 0), because these three type 0) Leech roots are not orthogonal to the type 1) Leech root associated with \(\vec{u}_{11'}\). The symmetry group of the three type 0) Leech roots above, and of the graph of the remaining 21 Leech roots is a diagonal subgroup of \(S_3^{(\alpha)}\) and \(S_3^{(\gamma)}\). The coset space \(\text{Aut}(\Gamma)/S_3\) contains 12 elements, which is the number of type 1 Leech roots. This calculation roughly reproduces the discussion in [Bor2].

### A.2.2 Supplementary Details

Here we present some details that have been omitted in the appendix A.2.1.

When the Neron–Severi lattice of \(S_X\) for \(X = X_3\) is realized as the orthogonal complement \(\phi_{T_0}(J)^\perp \subset \Pi_{1,25}\) for the embedding using \([156,161,154]\), all the Leech roots of type 0) are of the form \((2 \nu K, 1, 1)\) with some codewords \(K \in C_{24}^{(8)}\). There are 24 codewords satisfying the conditions \([165]\), six of which (those in \([166,167]\) corresponding to the trivalent vertices in the Coxeter graph) have already been specified in Appendix A.2.1. The remaining 18 codewords correspond to the remaining 18 type 0) Leech roots, which are located in between the trivalent vertices of the graph in Figure 6.

Let \(K_{(n)}\) denote the codeword corresponding to the Leech root \(\lambda_n = (2 \nu K_{(n)}, 1, 1)\). The Leech roots in between the trivalent vertex \(\lambda_{22}\) and \(\lambda_3\) [resp. \(\lambda_{12}\) or \(\lambda_{23}\)] are \{\lambda_1, \lambda_2\} [resp. \{\lambda_{10}, \lambda_{11}\} or \{\lambda_{19}, \lambda_{20}\}]. The Leech roots \{\lambda_5, \lambda_4\}, \{\lambda_8, \lambda_9\} and \{\lambda_7, \lambda_{24}\} lie in between the trivalent vertex \(\lambda_6\) and three other trivalent vertices \(\lambda_3, \lambda_{23}\) and \(\lambda_{12}\) (Figure 6), and the corresponding codewords are as follows:

Finally, for the type 0) Leech roots located in between the trivalent vertex \(\lambda_{15}\) and three other
trivalent vertices $\lambda_{12}$, $\lambda_{23}$ and $\lambda_3$,

$$K_{(14)} = \begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ * & * & * \end{array} \quad K_{(17)} = \begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ * & * & * \end{array} \quad K_{(16)} = \begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ * & * & * \end{array}$$

$$K_{(13)} = \begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ * & * & * \end{array} \quad K_{(18)} = \begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ * & * & * \end{array} \quad K_{(25)} = \begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ * & * & * \end{array}$$

Among the twelve Leech roots of type 1), those of the first type are given by the codewords $K_{k'}$ satisfying the conditions \((169)\). There are six of them: three are the codewords $K_{k'}$ for $k' \in \{11, 22, 22\}$, which are the collection of the two columns of 1 and $k'$ within $\Omega$ in the MOG presentation (see below). The remaining three codewords are $K_{k'}$ for $k = \{20, 14, 21\}$, which consist of the two rows of 1 and $k'$ in the centre and right blocks of the MOG presentation. $K_{k'=11'}$ and $K_{k'=20'}$, for example, are

$$K_{11'} = \begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ * & * & * \end{array} \quad K_{20'} = \begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ * & * & * \end{array}$$

The $\sigma_R' \cdot \sigma_R$ transformation for a type 1) Leech root $\lambda$ of $X_3$ is contained in $W_{\phi_0(\cdot)} \subset W_{\Pi}$ and is hence regarded as an isometry of $S_X$. This isometry maps $\lambda'$ (the orthogonal projection onto $S_X \otimes \mathbb{Q}$) to $-\lambda'$. Thus it acts on $S_X$ as a reflection and can hence be regarded as a part of the generators of the reflection symmetry group $W(S_X)$. With the information above, it is straightforward to compute the Coxeter diagram of the reflection symmetry group generated by the type 0) and type 1) Leech roots. Such a computation reproduces the result of [Vin1].

Let us now verify that the conditions (as-6) and (as-7) in Appendix A.2.1 are indeed satisfied in the example $X = X_3$. As for the condition (as-6), note first that $S_X \cong U^{(\gamma)} \oplus E_8^{(1)} \oplus E_8^{(2)} \oplus A_2$. Let $U^{(\gamma)} = \text{Span}_{\mathbb{Z}}\{u_1^{(\gamma)}, \bar{u}_1^{(\gamma)}\}$ with the symmetric pairing $\langle u_1, \bar{u}_1 \rangle = 1$, $\langle u_1, u_1 \rangle = \langle \bar{u}_1, \bar{u}_1 \rangle = 0$. The simple roots of $E_8^{(a=1,2)}$ are denoted by $\alpha_1^{(a)}$ (see Figure 7), and the negative of the maximal root by $\alpha_\gamma$. The simple roots of $A_2$ are denoted by $\alpha_{1,3}^{(3)}$ and $\alpha_{3}^{(3)}$, because this $A_2$ and an $E_6$ subalgebra generated by $\alpha_{i=3,\cdots,8}^{(3)}$ form a maximal subalgebra of $E_8^{(3)}$. The twenty-four type 0) Leech roots

$\bar{u}_1^{(\gamma)}$ here corresponds to $u_2^{(\gamma)} + u_1^{(\gamma)}$ in section 5.4

59
((-2)-roots of $S_X$) $\lambda_1, \ldots, \lambda_{21, \ldots, 25}$ are written in this basis of $S_X$ as

$$
\begin{align*}
\lambda_1 & \leftrightarrow u_1^{(\gamma)} + \alpha^{(1)}_6, \\
\lambda_2 & \leftrightarrow u_1^{(\gamma)} + \alpha^{(1)}_6, \\
\lambda_{10} & \leftrightarrow u_1^{(\gamma)} + \alpha^{(2)}_{-6}, \\
\lambda_{11, \ldots, 15} & \leftrightarrow \alpha^{(2)}_{-6}, \\
\lambda_{17,18} & \leftrightarrow \alpha^{(2)}_{6}, \\
\lambda_{16} & \leftrightarrow \alpha^{(2)}_{8}, \\
\lambda_{19} & \leftrightarrow u_1^{(\gamma)} + \alpha^{(3)}_{-6}, \\
\lambda_{20} & \leftrightarrow \alpha^{(3)}_1, \\
\lambda_{22} & \leftrightarrow u_1^{(\gamma)} - u_1^{(\gamma)}.
\end{align*}
$$

(188)

Thus, it is easy to see that all the generators $\alpha^{(a=1,2,3)}_{i=\ldots,8}, \alpha^{(3)}_{-6,1}, u_1^{(\gamma)}$ and $\bar{u}_1^{(\gamma)}$ of $S_X$ are written as linear combinations of the type 0) Leech roots with integer coefficients. Now (as-6) is verified.

As for the condition (as-7), note that the Weyl vector $w := (0,0,1) \in \Lambda_{24} \oplus U^{(\omega)} \cong \Pi_{1,25}$ is expressed in the language of Bor3

$$
\begin{align*}
(\phi^{(\gamma)})^{-1} : \Pi_{1,25} \cong U^{(\gamma)} \oplus L^{(\gamma)} = U^{(\gamma)} \oplus E_8^{(1)} \oplus E_8^{(2)} \oplus E_8^{(3)}
\end{align*}
$$

(191)

as

$$
\begin{align*}
w = -\rho^{(\gamma)1} - \rho^{(\gamma)2} - \rho^{(\gamma)3} + 30u_1^{(\gamma)} + 31\bar{u}_1^{(\gamma)},
\end{align*}
$$

(192)

where $\rho^{(\gamma)a}$ for $a = 1, 2, 3$ are the Weyl vectors of the root lattice $E_8^{(a)}$. This means that $(\rho^{(\gamma)a}, \alpha^{(b)}_{i}) = -\delta_{ab}$ for any $i = 1, \ldots, 8$. To be more explicit, $\rho^{(\gamma)a} = (29\alpha_1 + 57\alpha_2 + 84\alpha_3 + 110\alpha_4 + 135\alpha_5 + 91\alpha_6 + 46\alpha_7 + 68\alpha_8)^{(a)}$. When $T_0 \cong E_6$ is embedded into $\Pi_{1,25}$ by mapping $\alpha_1, \ldots, \alpha_6 \in J$ to $\alpha^{(3)}_3, \ldots, \alpha^{(3)}_8$, $w'$ (the projection of the Weyl vector $w$ to $S_X \otimes \mathbb{Q}$) is given by

$$
\begin{align*}
w' = w + (8\alpha_3 + 15\alpha_4 + 21\alpha_5 + 15\alpha_6 + 8\alpha_7 + 11\alpha_8)^{(3)}.
\end{align*}
$$

(193)

It is now a straightforward computation to see that this $w'$ is equal to the sum of $4 \times (\lambda_{22} + \lambda_6 + \lambda_{15} + \lambda_{23} + \lambda_{12} + \lambda_3)$—a sum over all the type 0) Leech roots at the trivalent vertices in the Coxeter diagram of Figure 6—and $3 \times (\lambda_1 + \lambda_2 + \lambda_4 + \cdots)$—a sum over all the other 18 Leech roots of type 0). This linear combination is manifestly invariant under the action of $\text{Aut}(\Gamma) \cong (S_3 \times S_3) \rtimes (\mathbb{Z}/2\mathbb{Z})$, the symmetry of the graph in Figure 6. Now (as-7) is also verified.

A.2.3 Elliptic Divisors for the Six Different Types on $X_3$

The singular K3 surface $X_3$ admits 6 different types of elliptic fibrations. Due to Corollary D, they also form the moduli-isomorphism classification of elliptic fibration on $X_3$. We have learned from Lemma I and sections 5.5.1, 5.5.4 that elliptic divisors can be determined by eliminating some

37 For the record, we also write down the three remaining Leech roots of type 0).

$$
\begin{align*}
\lambda_{23} & \leftrightarrow 2(\alpha_1^{(\gamma)} + \bar{u}_1^{(\gamma)}) + \alpha^{(3)}_{-6} - (2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 8\alpha_4 + 10\alpha_5 + 7\alpha_6 + 4\alpha_7 + 5\alpha_8)^{(1)}, \\
\lambda_{24} & \leftrightarrow 3(\alpha_1^{(\gamma)} + \bar{u}_1^{(\gamma)}) + 2\alpha^{(3)}_{-6} + \alpha^{(3)}_1 - (3\alpha_1 + 6\alpha_2 + 9\alpha_3 + 12\alpha_4 + 15\alpha_5 + 10\alpha_6 + 5\alpha_7 + 8\alpha_8)^{(1)}, \\
\lambda_{25} & \leftrightarrow 3(\alpha_1^{(\gamma)} + \bar{u}_1^{(\gamma)}) + 2\alpha^{(3)}_{-6} + \alpha^{(3)}_1 - (3\alpha_1 + 6\alpha_2 + 9\alpha_3 + 12\alpha_4 + 15\alpha_5 + 10\alpha_6 + 5\alpha_7 + 8\alpha_8)^{(2)},
\end{align*}
$$

(185)

(186)

(187)

60
of the vertices from the Coxeter diagram of $W^{(2)}(S_X)$. Particularly in the case of $X = X_3$, it is obvious from Figure which vertex to eliminate. The result is recorded in the following.

Type no.1 is associated with the embedding $[E_6 \subset E_8; E_8 \oplus E_8]$, and $W_{\text{root}} = A_2E_8E_8$. The elliptic divisors is given by

$$u_1^{(\gamma)} = \lambda_1 + 2\lambda_2 + 3\lambda_3 + 4\lambda_4 + 5\lambda_5 + 6\lambda_6 + 3\lambda_7 + 4\lambda_8 + 2\lambda_9, \quad (E_8, \Pi^*) \quad (194)$$
$$u_1^{(\gamma)} = \lambda_1 + 2\lambda_2 + 3\lambda_3 + 4\lambda_4 + 5\lambda_5 + 6\lambda_6 + 3\lambda_7 + 4\lambda_8 + 2\lambda_9. \quad (E_8, \Pi^*) \quad (195)$$

They correspond to two distinct singular fibres of type $\Pi^*$ and are hence algebraically equivalent.

Type no.2 is obtained by the embedding $[E_6 \subset E_7; D_{16}]$, and hence $W_{\text{root}} = A_2D_{16}$.

$$u_1^{(\beta)} = \lambda_2 + \lambda_5 + 2(\lambda_3,4,5,6,8,9 + \lambda_23 + \lambda_18,17,15,13,12) + \lambda_24 + \lambda_11. \quad (D_{16}, I_7^{12}) \quad (196)$$

Type no.3 is for $[E_6 \subset E_7; D_7 \oplus E_7]$, where $W_{\text{root}} = D_{10}E_7$. The elliptic divisor is

$$u_1^{(\eta)} = \lambda_7 + \lambda_5 + 2(\lambda_6,8,9 + \lambda_23 + \lambda_18,17,15) + \lambda_14 + \lambda_16. \quad (D_{10}, I_6^*) \quad (197)$$
$$u_1^{(\eta)} = \lambda_7 + \lambda_5 + 2(\lambda_6,8,9 + \lambda_23 + \lambda_18,17,15) + \lambda_14 + \lambda_16. \quad (D_{10}, I_6^*) \quad (198)$$

Type no.4 is obtained by $[E_6 \subset E_7; A_{17}]$, where $W_{\text{root}} = A_{17}$.

$$u_1^{(\zeta)} = \lambda_22 + \lambda_1,6,8,9 + \lambda_23 + \lambda_18,17,15,\ldots,10. \quad (199)$$

For type no.5 based on $[E_6 \subset E_6; E^{\oplus 3}/E_6]$ with $W_{\text{root}} = E_6^{\oplus 3}$,

$$u_1^{(\theta)} = 3\lambda_22 + 2\lambda_{1,10,19} + \lambda_{2,11,20}, \quad (E_6, IV^*) \quad (200)$$
$$u_1^{(\theta)} = 3\lambda_6 + 2\lambda_{5,8,7} + \lambda_{4,9,24}, \quad (E_6, IV^*) \quad (201)$$
$$u_1^{(\theta)} = 3\lambda_{15} + 2\lambda_{14,17,16} + \lambda_{13,18,25}. \quad (E_6, IV^*) \quad (202)$$

Finally, for type no.6 from $[E_6 \subset E_6; A_{11} \oplus D_7]$, the elliptic divisor is given by

$$u_1^{(\lambda)} = \lambda_{1,6,8,9} + \lambda_{23} + \lambda_{20,19} + \lambda_{22}, \quad (A_{11}, I_1^{12}) \quad (203)$$
$$u_1^{(\lambda)} = \lambda_{24} + \lambda_{11} + 2\lambda_{12,\ldots,15} + \lambda_{17} + \lambda_{16}. \quad (D_7, I_3^*) \quad (204)$$

### A.3 Kummer surfaces

**Definition** A Kummer surface is the minimal resolution of the sixteen $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$ singularity points in $A/[-id.]$, where $A$ is an Abelian surface. Such a Kummer surface $X$ is denoted by $\text{Km}(A)$.

#### A.3.1 Abelian varieties

**Definition** An Abelian variety of dimension-$d$ is a complex $d$-dimensional torus that has a projective embedding. Abelian varieties of $d = 1$ are called elliptic curves, and those of $d = 2$ Abelian surfaces.

A general treatment of Abelian varieties can be found in the classic textbook by Griffith and Harris [GH]. Here, we only highlight those aspects which are relevant to the further discussion.

A complex torus of dimension $d$ is given by $T = \mathbb{C}^d/\Lambda$, where the lattice $\Lambda \subset \mathbb{C}^d$ is defined in terms of $\Lambda = \text{Span}_\mathbb{Z}\{\ell_1, \ell_2, \cdots, \ell_{2d}\}$. The generators of the lattice are specified by their coordinates
in $\mathbb{C}^d$, $\ell_i = (z_1^i, z_2^i, \ldots, z_d^i)$. These $\ell_i$ form a basis of $H_1(T; \mathbb{Z})$. Their dual basis in $H^1(T; \mathbb{Z})$ in the form of $\mathbb{R}$-valued 1-forms are denoted by $\{\lambda^i\}_{i=1,2,\ldots,2d}$ and satisfy $\langle \lambda^i, \ell_j \rangle = \delta^i_j$.

Complex valued 1-forms $dz^a$ can be written as linear combinations of these 1-forms, $dz^a = \sum_i C^a_i \lambda^i$. By taking appropriate linear combinations of the holomorphic coordinates and by properly changing the basis $\{\lambda^1\}$ (and $\{\ell_i\}$ accordingly), one can always write

$$
\begin{align*}
    dz^1 &= \lambda^1 + \tau^{11} \lambda^d + \cdots + \tau^{1d} \lambda^{2d}, \\
    dz^2 &= \lambda^2 + \tau^{21} \lambda^d + \cdots + \tau^{2d} \lambda^{2d}, \\
    &\vdots \\
    dz^d &= \lambda^d + \tau^{d1} \lambda^d + \cdots + \tau^{dd} \lambda^{2d}.
\end{align*}
$$

(205)

**Theorem** The complex torus $T$ specified by the $\mathbb{C}$-valued $d \times d$ matrix $(\tau^{ab})$ is an Abelian variety (i.e. has a projective embedding) if and only if $\tau^{ab}$ is symmetric and the associated $\mathbb{R}$-valued matrix $H^{ab} = \text{Im}(\tau^{ab})$ is positive definite.

**Example:** In the one-dimensional case $\tau^{ab}$ is a $1 \times 1$ matrix and the condition to be an abelian variety is $\text{Im}(\tau^{11}) > 0$. This is the well-known case of an elliptic curve. Here, elliptic functions provide a projective embedding $T \ni u \mapsto [\wp(u) : \wp'(u) : 1] \in \mathbb{P}^2$.

**Example:** Another class of examples is given by the Jacobian varieties. Consider a curve $\Sigma_g$ of genus $g$. By integrating the $g$ independent harmonic $(1,0)$ forms over an element of $H_1(\Sigma_g, \mathbb{Z})$ we obtain a vector in $\mathbb{C}^g$. The set of all such vectors defines a lattice $\Lambda \subset \mathbb{C}$. $\text{Jac}(\Sigma_g) = \mathbb{C}/\Lambda$ is an example of an Abelian variety of dimension $d = g$. Once the complex structure of $\Sigma_g$ is given, the corresponding Abelian variety $\text{Jac}(\Sigma_g) = \text{Pic}^0(\Sigma_g)$ is specified. While the moduli space of genus $g$ curves is of dimension $(3g - 3)$ for $g \geq 2$, which grows only linearly in $g$, the moduli space of $d$-dimensional Abelian varieties has dimension $d(d + 1)/2$, which grows quadratically in $d$. Thus $\text{Jac}(\Sigma_g)$ form a very special class of Abelian varieties for large $g$.

In the case of $d = g = 2$, however, any Abelian surface can be obtained as the Jacobian variety of a curve $\Sigma_2$ of genus two. In this case, the moduli spaces of both the genus two curve and Abelian surfaces are three-dimensional. When we let $\tau^{12} = \tau^{21} = 0$ (while having $\text{Im}(\tau^{11}) > 0$ and $\text{Im}(\tau^{22}) > 0$), the Abelian variety becomes a product of elliptic curves $E \times F$. Such an Abelian variety corresponds to the Jacobian variety of $\Sigma_2$ that degenerates into the two elliptic curves $E$ and $F$.

As Abelian varieties are obtained as the quotient of $\mathbb{C}^n$ by a lattice, they inherit the additive group structure of $\mathbb{C}^n$. Hence, as their name suggests, they are manifolds which are equipped with the structure of an abelian group. For one-dimensional abelian varieties, this abelian group is nothing but the classic group law of elliptic curves.

### A.3.2 Algebraic Cycles in Abelian Surfaces

The algebraic cycles of an Abelian surface form its Nloron–Severi lattice, $S_A = H^{1,1}(A; \mathbb{R}) \cap H^2(A; \mathbb{Z})$. As for K3 surfaces, its rank is denoted by $\rho_A$. The minimal resolution of $A/[-\text{id}]$ introduces 16 independent classes, so that the rank of $S_X$ for a Kummer surface $X = \text{Km}(A)$ is $\rho_X = 16 + \rho_A$. By definition, Abelian varieties allow for a projective embedding, so that there is at least one integral
(1,1)-form on $A$ and we generically have $\rho_X = 17$. For special choices of $A$, $\rho_A$ and hence the rank of $S_X$ can be enhanced, which is what we discuss in the following.

In an Abelian variety $A$ of dimension $d = 2$, we can take

$$
(e_1, e^1, e_2, e^2, e_3, e^3)^T := (\lambda^3 \wedge \lambda^2, \lambda^1 \wedge \lambda^4, \lambda^1 \wedge \lambda^2, -\lambda^3 \wedge \lambda^4, \lambda^1 \wedge \lambda^3, \lambda^2 \wedge \lambda^4)^T
$$

(206)
as a basis of $H^2(A; \mathbb{Z})$. Let us denote the dual basis of $H_2(A; \mathbb{Z})$ by

$$(C_{32}, C_{14}, C_{12}, C_{43}, C_{13}, C_{24}).$$

(207)

Let us take $\lambda^1 \wedge \lambda^3 \wedge \lambda^2 \wedge \lambda^4$ as the orientation of the 4-cycle $[A]$, i.e. the volume form on $A$. As $(e_i, e^j) = \delta_i^j$, the intersection form on $H_2(A; \mathbb{Z})$ then becomes $U \oplus U \oplus U$ in this basis.

These cycles $C_{ij}$ in $A$ are mapped into cycles $\bar{C}_{ij}$ of $\text{Km}(A)$. Since $(C_{ij}, \bar{C}_{kl}) = 2(C_{ij}, C_{kl})$, their intersection form on $\text{Km}(A)$ is $U[2] \oplus U[2] \oplus U[2]$.

The period vector $\pi_A \in (H_2(A; \mathbb{Z}))^* \otimes \mathbb{C}$ is obtained by expanding $dz^1 \wedge dz^2$ in the basis (206):

$$

\begin{align*}
dz^1 \wedge dz^2 & = \tau_{11} \lambda^3 \wedge \lambda^2 + \tau_{22} \lambda^1 \wedge \lambda^4 + \lambda^1 \wedge \lambda^2 + \det(ab)\lambda^3 \wedge \lambda^4 + \tau_{21} \lambda^1 \wedge \lambda^3 - \tau_{12} \lambda^2 \wedge \lambda^4, \\
\pi_A & = (\tau_{11}, \tau_{22}, 1, -\det(\tau), \tau_{21}, -\tau_{12}),
\end{align*}
$$

(208)

where the choice of basis in (206) is assumed in the component description in the last line.

Because $A$ is an Abelian surface (and not just a complex torus), $\tau_{12} = \tau_{21}$. Hence the integral cycle $(C_{13} + C_{24}) \leftrightarrow (0, 0, 0, 1, 1)^T \in H_2(A; \mathbb{Z})$ is part of $S_A$ for any Abelian surface, so that we generically have $\rho_A = 1$.

The transcendental lattice of a generic Abelian surface $T_A$ is the orthogonal complement of $(C_{13} + C_{24})$ in $H_2(A; \mathbb{Z})$, which is generated by the first four basis vectors in (207) and $(C_{13} - C_{24})$. Thus we find that $T_A = U \oplus U \oplus (-2)$. From this it follows that $T_X = U[2] \oplus U[2] \oplus (-4)$ for a Kummer surface $X = \text{Km}(A)$ of a generic Abelian surface $A$.

When the Abelian surface has the special form, $A = E \times F$ (product type) for some pair of elliptic curves $E$ and $F$, we can take $\tau_{12} = \tau_{21} = 0$. As each of the two elliptic curves $E$ and $F$ allow for a projective embedding, $\rho_A = 2$. This can be seen explicitly by noting that both $C_{13}$ and $C_{24}$ are orthogonal to the period vector $\pi_A$. Correspondingly, the transcendental lattices of $A$ is $T_A = U \oplus U$, so that $\rho_X = 18$ and $T_X = U[2] \oplus U[2]$ for the associated Kummer surface.

Next we consider the case when the Abelian surface has the even more special form of being the product of two mutually isogenous curves, $A = E \times E'$. Two elliptic curves $E$ and $E'$ are said to be isogenous when their complex structure parameters $\tau$ and $\tau'$ in the upper half plane are related by a $GL(2; \mathbb{Q})$ transformation, $\tau = (a\tau' + b)/(c\tau' + d)$ where $a, b, c, d \in \mathbb{Z}$ do not necessarily satisfy $ad - bc = 1$. In this case, we may exploit the modular group action on each of the two elliptic curves to choose $\tau_{11} = \tau$ and $\tau_{22} = \tau'$ such that

$$

\tau' = \frac{n_2}{n_1} \tau,
$$

(209)

with $n_1, n_2 \in \mathbb{Z}$. In this case, also $(n_2C_{32} - n_1C_{14}) \leftrightarrow (n_2, -n_1, 0, 0, 0, 0)^T$ is orthogonal to $\pi_A$ and hence algebraic. Now $\rho_A = 3$, and hence $\rho_X = 19$. $T_A$ is generated by $n_2C_{32} + n_1C_{14}, C_{12}$.
and $C_{43}$, and the intersection form is $U \oplus (2n_1n_2)$. Thus for the corresponding Kummer surface,
$$X = \text{Km}(E \times E'), \quad T_X = U[2] \oplus (4n_1n_2).$$

Finally, let us consider the case where $A = E \times E'$, with $E$ and $E'$ isogenous, and furthermore suppose that they are mutually coprime integers $r, p, q$ such that $r\tau^2 + p\tau + q = 0$ holds for the complex structure parameter $\tau$ of $E$. This implies that $\tau^2$ can be written as a $\mathbb{Q}$-coefficient linear combination of $\tau$ and 1. After using (209), we may thus write
$$\pi_A = (\tau, n_2/n_1, 1, -n_2/n_1\tau^2, 0, 0),$$
so that now $(pn_2C_{32} + qn_2C_{12} - rn_1C_{43}) \leftrightarrow (pn_2, 0, qn_2, -rn_1, 0, 0)^T$ is an algebraic cycle. From this it follows that now $\rho_A = 4$ and $\rho_X = 20$. The intersection form of $T_A$ can be obtained from this information, and that of $T_X$ for the corresponding Kummer surface is then again obtained by simply multiplying by 2.

Consider the example $\tau = \tau' = i$, for which $\tau^2 + 1 = 0$. In this case $C_{12} - C_{43}$ is an algebraic cycle of $A$, and the rank-2 transcendental lattice $T_A$ is generated by $C_{12} + C_{43}$ and $-(C_{32} + C_{14})$. $T_A = (2) \oplus (2)$, and $T_X = (4) \oplus (4)$.

As another example, consider $\tau = \tau' = \omega$, with $\omega = e^{2\pi i/3}$. Now we have $\tau^2 + \tau + 1 = 0$, $(C_{14} + C_{12} - C_{43})$ is an algebraic cycle of $A$, and the rank-2 lattice $T_A$ is generated by $C_{12} + C_{43}$ and $-C_{32} - C_{14} + C_{43}$. Hence
$$T_A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad T_X = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}.$$  

A.3.3 $T_X$, $S_X$ and their Symmetries for $\rho_X = 17$ Kummer Surfaces

In this section, we provide a brief summary of known results on the symmetries of the Neron–Severi and transcendental lattices of Kummer surfaces that we use in this article.

Let us begin with the Kummer surface $\text{Km}(A)$ of an Abelian surface $A$ with generic complex structure. As already stated above, the transcendental lattice $T_X$ is
$$T_X = U[2] \oplus U[2] \oplus (-4)$$
generated by $\{C_{32}, C_{14}\}$, $\{C_{12}, C_{43}\}$ and $C_{13} - C_{24}$. Isom$(T_X)^{(\text{Hodge})} = \{\pm \text{id.}\} \cong \mathbb{Z}/2\mathbb{Z} \langle -\text{id.}\rangle$. It is much easier to compute the discriminant group $G_{T_X} \cong G_{S_X}$ from $T_X$ than from $S_X$. As an Abelian group,
$$G_{T_X} \cong (\mathbb{Z}/2\mathbb{Z})^4 \times (\mathbb{Z}/4\mathbb{Z})$$
generated by $C_{32}/2$, $C_{14}/2$, $C_{12}/2$, $C_{43}/2$ and $C_{13} - C_{24}/4$. Based on an explicit computation, one can derive that
$$\text{Isom}(q) \cong \mathbb{Z}/2\mathbb{Z} \langle -4 \rangle \times S_6.$$  

Here, the $S_6$ acts on both the $(\mathbb{Z}/2\mathbb{Z})^4$ and the $\mathbb{Z}/4\mathbb{Z}$ factor and $\langle -4 \rangle$ reverses the sign of the $\mathbb{Z}/4\mathbb{Z}$. Since $\rho_T : (-\text{id.}) \mapsto -4$,
$$p_T \left[ \text{Isom}(T_X)^{(\text{Hodge})} \right] \cong \mathbb{Z}/2\mathbb{Z} \langle -4 \rangle.$$  

---

38 Elliptic curves with this property are said to have complex multiplication. See [GV] and references therein for use of elliptic curves with complex multiplication in string theory.

39 One can also see from [209] that $E'$ also satisfies the same property when $E$ does, and vice versa.
Let us now move on and provide a description of the Neron–Severi lattice $S_X$ for $X = \text{Km}(A)$. The involution acting on the Abelian surface has 16 fixed points, which give rise to 16 singularities of type $A_1$. These fixed points are located at the sixteen 2-torsion points\(^{40}\) of $A$, 

$$
\mu_{\vec{s}} = \frac{1}{2} \vec{s} \vec{s} = \sum_{i=1}^{4} \frac{1}{2} s_i \ell_i \in A,
$$

(216)

where $\vec{s} = (s_1, s_3, s_2, s_4) \in (F_2)^4$. The exceptional cycles of $X = \text{Km}(A)$ obtained after the minimal resolution of the $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$ singularities at these 2-torsion points are denoted by $G_{\vec{s}}$ for $\vec{s} \in (F_2)^4$. They are algebraic and hence elements of the Neron–Severi lattice $S_X$. There is one more independent divisor class in $S_X$, which corresponds to the algebraic cycle $(C_{13} + C_{24})$ in $A$ and is denoted by $H$. $G_{\vec{s}}$’s and $H$ combined can be chosen as a set of $\mathbb{Q}$-coefficient generators of $S_X \otimes \mathbb{Q}$. The symmetric pairing on $S_X$ (and $S_X \otimes \mathbb{Q}$ also) is determined by $(G_{\vec{s}}, G_{\vec{s}'}) = -2\delta_{\vec{s},\vec{s}'}$, $(H, H) = 4$, and $(H, G_{\vec{s}}) = 0$. The signature of $S_X$ is $(1, 16)$.

Let $L$ denote (temporarily) the lattice generated by $H$ and $G_{\vec{s}}$; $L = \text{Span}_\mathbb{Z}\{H, G_{\vec{s}}\}$. It must be an index $2^6$ sublattice of $S_X$, because $\text{discr}(L) = 2^{18}$, whereas $\text{discr}(S_X) = |\text{discr}(T_X)| = 2^6$. There are two equivalent ways to describe which elements of $G_L = L^*/L$ should be added to $L$ in order to obtain $S_X$, i.e. which rational linear combinations of the $G_{\vec{s}}$ and $H$ are integral cycles of $X = \text{Km}(A)$.

One way to do this is to say that the $(\mathbb{Z}/2\mathbb{Z})^6 \subset G_L$ elements to be added are given by $\mathbb{Z}/2\mathbb{Z}$ generated by

$$
T_{0000} = \frac{1}{2} \left( H - G_{0000} - G_{1000} - G_{1100} - G_{0110} - G_{0111} - G_{0101} \right)
$$

(217)

modulo $L$, and by $(\mathbb{Z}/2\mathbb{Z})^5$ (Nik3 Cor.5) in the form of

$$
[D^{(\vec{k})}] := \left[ \frac{1}{2} \sum_{\vec{s} \in (F_2)^4} (\vec{k} \cdot \vec{s} + k_0)G_{\vec{s}} \right] \in G_L.
$$

(218)

for any $\vec{k} := (\vec{k}, k_0) = (k_1, k_3, k_2, k_4, k_0) \in (F_2)^5$.

The other way to describe the elements of $L^*/L$ corresponding to $S_X/L$ deals with $T_{0000}$ in \((217)\) in a way similar to how the sixteen $G_{\vec{s}}$’s are treated. Let $I(0, 0, 0, 0)$ denote the six elements $\vec{s} \in (F_2)^4$ that appear in the linear combination of \((217)\). Now we can rewrite \((217)\) as follows:

$$
T_{0000} = \frac{1}{2} \left( H - \sum_{\vec{s} \in I(0, 0, 0, 0)} G_{\vec{s}} \right).
$$

(219)

Similarly, we define fifteen other subsets of $(F_2)^4$ by

$$
I(\vec{s}) := \{ \vec{s} + \vec{r} | \vec{r} \in I(0, 0, 0, 0) \}, \quad \text{for } \vec{s} \in (F_2)^4,
$$

(220)

and define fifteen other $\mathbb{Q}$-coefficient linear combinations of the generators of the lattice $L$ by:

$$
T_{\vec{s}} = \frac{1}{2} \left( H - \sum_{\vec{r} \in I(\vec{s})} G_{\vec{r}} \right).
$$

(221)

\(^{40}\)The 2-torsion points of an abelian variety $A$ are those points $p$ which give the unit element, i.e. the origin, when added to themselves under the group law of $A$. 

65
The second description of the integral components of $S_X$ for $X = \text{Km}(A)$ is to say that $S_X$ is generated by the sixteen $G\tilde{s}$'s and sixteen $T\tilde{s}$'s with $\mathbb{Z}$-coefficients; $G\tilde{s}$'s are called \textit{nodes}, and $T\tilde{s}$'s \textit{tropes} in the literature.

The second description of $S_X$ is known to be equivalent to the first one using (217) and (218). Symmetries can be made manifest in the second one, while practical computation may be easier in the first one.

The discriminant group $G_{S_X} \cong S_X^*/S_X$ should be isomorphic to the one calculated from $T_X$. In the language of $S_X$, $G_{S_X} \cong (\mathbb{Z}/2\mathbb{Z})^4 \times (\mathbb{Z}/4\mathbb{Z})$, because $S_X \cong D_8 \oplus D_8 \oplus (4)$ [Ke]. A set of generators is explicitly given in [Ke]:

$$B_1 = (G_{1100} + G_{0100} + G_{0110} + G_{1110})/2, \quad B_2 = (G_{1000} + G_{0100} + G_{0111} + G_{1011})/2,$$
$$B_3 = (G_{1100} + G_{0100} + G_{0111} + G_{1111})/2, \quad B_4 = (G_{1000} + G_{0100} + G_{0110} + G_{1010})/2,$$
$$B_0 = H/4 + (G_{0000} + G_{1000} + G_{1100} + G_{0100})/2.$$

In this basis, the discriminant bilinear form is $b = \left[ \begin{array}{cc} 0 & 1/2 \\ 1/2 & 0 \end{array} \right] \oplus (1/4)$, where diagonal (resp. off-diagonal) entries are evaluated mod $2\mathbb{Z}$ (resp. $\mathbb{Z}$) [Ke]. The discriminant form is isomorphic to the one obtained from $T_X$, (213).

We will now describe a certain subgroup of the isometry group of this Neron–Severi lattice. First, there is a subgroup $(\mathbb{Z}/2\mathbb{Z})^4 \subset \text{Isom}(S_X)$ consisting of translations. For any $\vec{s}_0 \in (\mathbb{F}_2)^4$, a translation is given by

$$\tau_{\vec{s}_0} : \begin{cases} G_{\vec{s}} \mapsto G_{\vec{s} + \vec{s}_0}, \\ T_{\vec{s}} \mapsto T_{\vec{s} + \vec{s}_0}, \\ H \mapsto H. \end{cases}$$

There is another subgroup $\mathbb{Z}/2\mathbb{Z} \langle \text{sw} \rangle \subset \text{Isom}(S_X)$, the generator of which,

$$\text{sw} : \begin{cases} G_{\vec{s}} \mapsto T_{\vec{s}}, \\ T_{\vec{s}} \mapsto G_{\vec{s}}, \\ H \mapsto 3H - \sum_{\vec{s} \in (\mathbb{F}_2)^4} G_{\vec{s}}. \end{cases}$$

is called switch. The switch and translations commute, and they form a $(\mathbb{Z}/2\mathbb{Z})^5$ subgroup.

Another subgroup $S_6 \subset \text{Isom}(S_X)$ is understood better, when we introduce a different notation that involves more geometric aspects of Kummer surfaces. For this reason, we take a moment here to make a little digression (see [Ke]). As we have already remarked, Abelian surfaces can be realized as Jacobians of a genus 2 curve. Choose a curve of genus 2 $\Sigma_{q=2}$ appropriately, so that $A = \text{Jac}(\Sigma_{q=2})$. All genus 2 curves are known to be hyperelliptic, and we take its expression of the form $y^2 = \prod_{a=0}^{5} (x - x_a)$. The points $p_a = \{(x, y) = (x_a, 0) \in \Sigma_2\}$ for $a = 0, \cdots, 5$ are called \textit{Weierstrass points}. The Abel–Jacobi map

$$\mu : \Sigma_2 \ni p \mapsto (z_1, z_2) = \left( \int_{p_0}^{p} \omega^1, \int_{p_0}^{p} \omega^2 \right) \in \mathbb{C}^2/\text{Span}_{\mathbb{Z}} \{\ell_1, \ell_2, \ell_3, \ell_4\} \in A$$

sends the six Weierstrass points $p_a$’s to six of the sixteen 2-torsion points of $A$. Here, $\omega^1$ and $\omega^2$ are independent holomorphic 1-forms on $\Sigma_2$ normalized so that $(\omega^i, \alpha_j) = \delta_{ij}$ for 1-cycles $\alpha_1$ and $\alpha_2$. Two other independent 1-cycles of $\Sigma_2$, $\beta_1$ and $\beta_2$ are chosen so that $(\alpha_i, \beta_j) = -(\beta_j, \alpha_i) = \delta_{ij},$
Figure 8: A curve of genus 2 with a symplectic choice of basis for $H_1(Σ_2; ℤ)$. We have furthermore displayed the locations of the branch points (Weierstrass) $p_a$ which occur in the realization of the genus two curve as a hyperelliptic curve.

$$\alpha_i, \beta_i = 0 \text{ and } (\beta_i, \beta_i) = 0,$$

It is easy to see from the figure that the six 2-torsion points are as follows:$$p_0 \mapsto \int_{p_0}^{p_0} \omega = \mu_{0000} =: \mu_{00}, \quad p_1 \mapsto \int_{p_0}^{p_1} \omega = \mu_{1000} =: \mu_{01}, \quad p_2 \mapsto \int_{p_0}^{p_2} \omega = \mu_{1100} =: \mu_{02},$$
$$p_3 \mapsto \int_{p_0}^{p_3} \omega = \mu_{0110} =: \mu_{03}, \quad p_4 \mapsto \int_{p_0}^{p_4} \omega = \mu_{0110} =: \mu_{04}, \quad p_5 \mapsto \int_{p_0}^{p_5} \omega = \mu_{0101} =: \mu_{05}.$$

The remaining ten 2-torsion points correspond to period integrals $\int_{p_{a}}^{p_{b}} \omega$, so that they can be denoted also by $\mu_{ab}$ using the indices $a, b = 1, \cdots, 5$, $a \neq b$. Note that $\mu_{ab} = \mu_{ba}$ in $A$. Since one can also regard $\mu_{a}$'s for $a = 1, \cdots, 5$ as $\mu_{00} = \mu_{00}$, and $\mu_{0}$ as $\mu_{00} = \mu_{11} = \cdots = \mu_{55}$, all the sixteen 2-torsion points can be labelled by using $(ab)$ with $a, b = 0, \cdots, 5$. The sixteen nodes $X = \text{Km}(A)$ associated with $\mu_{\bar{s}}$ for $\bar{s} \in (ℤ_2)^4$—$G_{\bar{s}}$'s—can also be labelled by a pair of $a, b$; the following notation is introduced (as in [Ke]):

$$N_{00} = G_{0000} \quad N_{01} = G_{1000} \quad N_{02} = G_{1100} \quad N_{03} = G_{0110}$$
$$N_{04} = G_{0111} \quad N_{05} = G_{0101} \quad N_{12} = G_{0100} \quad N_{13} = G_{1110}$$
$$N_{14} = G_{1111} \quad N_{15} = G_{1101} \quad N_{23} = G_{1101} \quad N_{24} = G_{1011}$$
$$N_{25} = G_{1001} \quad N_{34} = G_{0001} \quad N_{35} = G_{0011} \quad N_{45} = G_{0010}$$

The sixteen tropes $T_{\bar{s}}$ of $X = \text{Km}(A)$ can also be relabelled by $a, b = 0, \cdots, 5$ by using the same dictionary between $\bar{s} \in (ℤ_2)^4$ and $(ab)$; $T_{00} = T_{0000}, T_{01} = T_{1000}, T_{12} = T_{0100}$ etc. With this new

---

[^1]: They are contained in the image of the Abel–Jacobi map, $\mu(Σ_2) \subset A$. These six 2-torsion points precisely correspond to the exceptional cycles that appeared in the linear combination of $(217)$. $T_{0000}$ in $S_X$ corresponds to the curve $\mu(Σ_2) \subset A$. Similarly, $T_{\bar{s}}$'s for other $\bar{s} \in (ℤ_2)^4$ in $S_X$ correspond to curves in $A$ obtained by shifting $\mu(Σ_2)$ by $\mu_{\bar{s}}$.}
notation, the $S_6$ subgroup of $\text{Isom}(S_X)$ acts as:

\[
S_6 \supseteq \sigma : \begin{cases} 
N_{ab} \mapsto N_{\sigma(a)\sigma(b)} \\
H \mapsto H,
\end{cases}
\]

where $S_6$ acts as permutation on the indices $a, b \in \{0, \ldots, 5\}$.

The $(\mathbb{Z}/2\mathbb{Z})^5$ and $S_6$ subgroups of $\text{Isom}(S_X)$ as a whole form a subgroup $(\mathbb{Z}/2\mathbb{Z})^5 \rtimes S_6$, or, equivalently $(\mathbb{Z}/2\mathbb{Z})^5 : S_6$. This subgroup of $\text{Isom}(S_X)$ is characterized as the one acting as permutation on the nodes $(T_x)$ and tropes $(T_y)$ while preserving intersection numbers among them [Nik4]. Reference [Ko2] Lemma 4.5 further proved that this $(\mathbb{Z}/2\mathbb{Z})^5 : S_6$ subgroup of $\text{Isom}(S_X)$ does correspond to the $\text{Aut}(D') \subset \text{Isom}(S_X)\text{^{(Amp)}}$ associated with the realization of $S_X$ for $X = \text{Km}(A)$ using the embedding of $(T_0)_{\text{root}} = A_3 \oplus A_4$ into $\text{II}_{1,25}$.

The homomorphism $p_S : \text{Isom}(S_X) \rightarrow \text{Isom}(G_{S_X}, q)$ maps this subgroup $\text{Aut}(D') \cong (\mathbb{Z}/2\mathbb{Z})^5 \rtimes S_6$ to $\text{Isom}(G_{S_X}, q)$. It is known that the kernel is the $(\mathbb{Z}/2\mathbb{Z})^4$ subgroup corresponding to translations, $p_S(\text{sw}) = (-1) \in \text{Isom}(q)$, and the $S_6$ subgroup of $\text{Aut}(D')$ becomes the $S_6$ subgroup of $\text{Isom}(q)$. In particular, $p_S : \text{Aut}(D') \rightarrow \text{Isom}(q)$ is surjective. Within the $\text{Aut}(D')$ subgroup of $\text{Isom}(S_X)\text{^{(Amp)}}$, only the $(\mathbb{Z}/2\mathbb{Z})^5$ subgroup—translations and switch—fall into the image of $\text{Isom}(T_X)\text{^{(Hodge)}}$, so that they can be lifted to automorphisms of $X = \text{Km}(A)$.

### A.3.4 The Neron–Severi Lattice of $X = X_{[2, 2, 2]} = \text{Km}(E_\omega \times E_\omega)$

In this appendix, we give a minimal description of the Neron–Severi lattice of $X = \text{Km}(E_\omega \times E_\omega) = X_{[2, 2, 2]}$ so notations to be used in the main text are explained. For a more systematic study, see [KK].

Let us begin with a more general K3 surface, $X = \text{Km}(E \times F)$, where the Abelian surface is of product type $A = E \times F$, yet the elliptic curves $E = (\mathbb{C}/\text{Span}\mathbb{Z}\{\ell_1, \ell_3\}) = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ and $F = (\mathbb{C}/\text{Span}\mathbb{Z}\{\ell_2, \ell_4\}) = \mathbb{C}/(\mathbb{Z} + \tau'\mathbb{Z})$ are not necessarily isogenous and their complex structure parameters $\tau, \tau'$ are generic.

Let $\{p_1, \ldots, p_4\}$ be the four 2-torsion points in $E$, and $\{q_1, \ldots, q_4\}$ be those in $F$:

\[
\begin{align*}
p_1 &= [0] \in E, & p_2 &= [\ell_1/2] \in E, & p_3 &= [\ell_3/2] \in E, & p_4 &= [\ell_1/2 + \ell_3/2] \in E, \\
q_1 &= [0] \in F, & q_2 &= [\ell_2/2] \in F, & q_3 &= [\ell_4/2] \in F, & q_4 &= [\ell_2/2 + \ell_4/2] \in F,
\end{align*}
\]

$E_j$ for $j = 1, \ldots, 4$ are $(-2)$ curves of $X = \text{Km}(E \times F)$ corresponding to $E \times q_j$, and $F_i$ for $j = 1, \ldots, 4$ to $p_i \times F$. The exceptional curve at $(p_i, q_j)$ is now denoted by $G_{ij}$ instead of $G_{x}$; $G_{0000} = G_{11}$, $G_{1000} = G_{21}$, $G_{0001} = G_{13}$ and $G_{0111} = G_{34}$, for example. Under the symmetric pairing of $S_X$,

\[
(G_{ij}, G_{kl}) = -2\delta_{ik}\delta_{jl}, \quad (G_{ij}, E_i) = \delta_{ij}, \quad (F_k, G_{ij}) = \delta_{ik}.
\]

The $S_6$ subgroup of $\text{Isom}(S_X)$ also acts as permutation on the sixteen tropes, but not in a way that is as simple as on the the sixteen nodes. $S_6$ is generated by binary permutations of the form $(a, b)$ with $1 \leq a < b \leq 5$ and $(0, a)$ with $a \neq 0$, and under such permutations,

\[
\begin{align*}
(a, b) : & \quad T_{0a} \leftrightarrow T_{0b}, & T_{ae} \leftrightarrow T_{be}, & T_{00}, T_{0c}, T_{cd} \ \text{inv.}, \\
(0, a) : & \quad T_{00} \leftrightarrow T_{0a}, & T_{bc} \leftrightarrow T_{dc}, & T_{0b}, T_{ab} \ \text{inv.},
\end{align*}
\]

where $1 \leq a, b, c, d, e \leq 5$ are all different.

The switch and $S_6$ do not commute in $\text{Isom}(S_X)$, but images under $p_S$ do commute in $\text{Isom}(q)$.
All of
\[ 2E_j + \sum_{i=1}^{4} G_{ij} \] (233)
are equivalent in \( S_X \), as they correspond to the four singular fibres (all of type \( I_0 \)) of one of the two Kummer pencils (elliptic fibrations) \( \pi_X : \text{Km}(E \times F) \rightarrow F/(\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{P}^1 \). For the other Kummer pencil, \( \pi : \text{Km}(E \times F) \rightarrow E/(\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{P}^1 \), all of the four singular fibres of type \( I_0 \),
\[ 2F_i + \sum_{j=1}^{4} G_{ij} \] (234)
are similarly equivalent in \( S_X \).

The \( 16 + 4 + 4 = 24 \) (\( -2 \)) curves, \( \{ G_{ij}, E_j, F_i \} \), can be chosen as a set of integer-coefficient generators of \( S_X \) for \( X = \text{Km}(E \times F) \) with non-isogenous elliptic curves \( E \) and \( F \) and generic complex structure parameters \( \tau, \tau' \), where the symmetry of \( S_X \) is manifest. The set of 24 curves, however, is redundant as a \( \mathbb{Q} \)-coefficient basis of the rank-18 \( S_X \). We can choose \( \{ G_{ij}, E_1, F_1 \} \) as a basis of \( S_X \), though some of the symmetries of \( S_X \) are not manifest in this basis.

Let us now turn to the case \( X = \text{Km}(E_\omega \times E_\omega) \). This is a special case of \( X = \text{Km}(E \times F) \), in that \( E \) and \( F \) are isogenous and both have the complex structure \( \omega = e^{2\pi i/3} \), so that the Picard number is enhanced to 20.

All of the twenty-four (\( -2 \))-curves \( \{ G_{ij}, E_j, F_i \} \) remain in the \( S_X \) lattice. As we consider the case where \( E \) and \( F \) are isogenous curves, however, the “diagonal subset” of \( A = E \times F \) and its images under the translations are also algebraic cycles of \( A \), and their images in \( S_X \) also remain algebraic. Denoting the coordinates of \( A = E_\omega \times E_\omega \) by \((z^1, z^2) \in \mathbb{C}/(\mathbb{Z} + \omega \mathbb{Z}) \times \mathbb{C}/(\mathbb{Z} + \omega \mathbb{Z})\), we embed a torus [resp. rational curve] (with complex coordinate \( z \) [resp. \( x/(\omega) \)]) in \( A \) [resp. \( X = A/(\omega) \)] by mapping \( z \) to
\[
D_1 : (z, z) \quad \quad \quad \quad \quad \quad D_2 : (\frac{1}{2} + z, z) \\
D_3 : \left( \frac{1}{2} \omega + z, z \right) \quad \quad \quad \quad \quad \quad D_4 : \left( \frac{1}{2}(1 + \omega) + z, z \right). \quad (235)
\]
Furthermore, due to the special complex structure \( \tau = \omega \), we may also embed rational curves (by exploiting complex multiplication) as
\[
C_1 : (\omega z, -\omega^2 z) \quad \quad \quad \quad \quad \quad C_2 : (\omega(z - \frac{1}{2}), -\omega^2 z) \\
C_3 : (\omega(z - \frac{1}{2} \omega), -\omega^2 z) \quad \quad \quad \quad \quad \quad C_4 : (\omega(z - \frac{1}{2}(1 + \omega)), -\omega^2 z). \quad (236)
\]
The thirty-two (\( -2 \)) curves of \( X = \text{Km}(E_\omega \times E_\omega) \) are grouped into \( \{ G_{ij} \} \) and \( \{ C_i, D_i, E_i, F_i \} \); there are no mutual intersection between the sixteen elements of \( \{ C_i, D_i, E_i, F_i \} \) and also between the sixteen elements \( \{ G_{ij} \} \). Just like each one of \( E_i \)'s or \( F_i \)'s finds four (\( -2 \)) curves (and precisely four) within the sixteen \( \{ G_{ij} \} \)'s to intersect, each one of \( C_i \)'s and \( D_i \)'s also have four of \( \{ G_{ij} \} \)'s to intersect.

---

\(^{44}\)When \( F_1 \) is chosen as the zero section, \( W_{\text{root}} = \oplus_j D_4^{(j)} = \oplus_j \text{Span}_\mathbb{C}\{ E_j, G_{2j}, G_{3j}, G_{4j} \} \), and \( MW = W_{\text{frame}}/W_{\text{root}} \cong \mathbb{Z}/2\mathbb{Z} \langle [F_2 - F_1] \rangle \times \mathbb{Z}/2\mathbb{Z} \langle [F_3 - F_1] \rangle \).
intersect:
\[
C_1 \cdot G_{11}, G_{23}, G_{34}, G_{42} = 1, \quad D_1 \cdot G_{11}, G_{22}, G_{33}, G_{44} = 1,
\]
\[
C_2 \cdot G_{14}, G_{22}, G_{31}, G_{43} = 1, \quad D_2 \cdot G_{12}, G_{21}, G_{34}, G_{43} = 1,
\]
\[
C_3 \cdot G_{12}, G_{24}, G_{33}, G_{41} = 1, \quad D_3 \cdot G_{13}, G_{31}, G_{24}, G_{42} = 1,
\]
\[
C_4 \cdot G_{13}, G_{21}, G_{32}, G_{44} = 1, \quad D_4 \cdot G_{14}, G_{41}, G_{23}, G_{32} = 1.
\] (237)

All other intersection numbers between \((C_i, G_{kl})\) and \((D_i, G_{kl})\) vanish.

These thirty-two \((-2)\) curves are known to be a set of integer-coefficient generators of the Neron–Severi lattice \(S_X\). This set respects the symmetry of \(S_X\), but is redundant as a \(\mathbb{Q}\)-basis for the rank-20 lattice \(S_X\). As such a basis, we can take \(\{G_{ij}, C_1, D_1, E_1, F_1\}\), for example. This is because there are linear equivalence relations such as
\[
2D_1 + (G_{11} + G_{22} + G_{33} + G_{44}) \sim 2D_2 + (G_{21} + G_{12} + G_{34} + G_{43})
\] (238)
for \(C_i\)’s and \(D_i\)’s, just like those for \(E_i\)’s and \(F_i\)’s in (233, 234).

Let us now describe the identification between \(S_X\) for \(X = \text{Km}(E_\omega \times E_\omega)\) and the orthogonal complement of a primitive embedding \(\phi_{T_0} : T_0 = D_4 \oplus A_2 \hookrightarrow \Pi_{1,25}\) defined by \((103, 108, 144, 154)\).

**Type 0 Leech root** is a Leech root \(\lambda \in \Pi\) that is orthogonal to any one of \(\{\lambda_1, 2, 3, 4, \lambda_9, \lambda_{17}\}\), which are the \(\phi_{T_0}\)-images of the simple roots of \(D_4 \oplus A_2\). Under the choice of the embedding \((103, 108)\), all the type 0 Leech roots are of the form \(\lambda = (2\nu_K, 1, 1)\) for a codeword \(K \in \mathcal{C}_{24}(8)\), and the codeword \(K\) satisfies the following conditions \([KK]\):

\[
\{0, \infty\} \subset K \quad \text{but} \quad 2 \not\in K, \quad \text{and either} \quad \begin{cases} 1 \in K \quad \text{and} \quad |K \cap K_U| = 4, \quad \text{or} \\ 1 \not\in K \quad \text{and} \quad |K \cap K_U| = 2. \end{cases}
\] (239)

Under the choice of \(K_U\) in (144), there are sixteen codewords \(K \in \mathcal{C}_{24}(8)\) of the first type, \(1 \in K\) and \(|K \cap K_U| = 4\). They are constructed from the sext decompositions in (146) in the form of \(K_{\alpha a}, K_{\beta a}, K_{\gamma a}\) and \(K_{\delta a}\) with \(a \in \{\heartsuit, \spadesuit, \clubsuit, \diamondsuit\}\). We identify the sixteen type 0 Leech roots with the \(\{G_{ij}\}\)’s. To be more explicit, the correspondence between the sixteen \(\{G_{ij}\}\)’s and the sixteen codewords are as follows:

\[
\begin{align*}
G_{24} &\leftrightarrow K_{\alpha \heartsuit}, & G_{43} &\leftrightarrow K_{\alpha \spadesuit}, & G_{11} &\leftrightarrow K_{\alpha \clubsuit}, & G_{32} &\leftrightarrow K_{\alpha \diamondsuit}, \\
G_{13} &\leftrightarrow K_{\beta \heartsuit}, & G_{41} &\leftrightarrow K_{\beta \spadesuit}, & G_{34} &\leftrightarrow K_{\beta \clubsuit}, & G_{22} &\leftrightarrow K_{\beta \diamondsuit}, \\
G_{12} &\leftrightarrow K_{\gamma \heartsuit}, & G_{14} &\leftrightarrow K_{\gamma \spadesuit}, & G_{33} &\leftrightarrow K_{\gamma \clubsuit}, & G_{21} &\leftrightarrow K_{\gamma \diamondsuit}, \\
G_{44} &\leftrightarrow K_{\delta \heartsuit}, & G_{31} &\leftrightarrow K_{\delta \spadesuit}, & G_{12} &\leftrightarrow K_{\delta \clubsuit}, & G_{23} &\leftrightarrow K_{\delta \diamondsuit}. 
\end{align*}
\] (240)

There are also sixteen codewords satisfying the second type of condition—\(1 \not\in K\) and \(|K \cap K_U| = 2\)—in (239). Under the choice of \(K_U\) in (144), they are

\[
\begin{array}{|c|c|}
\hline
K_p & K_q 1 \\
\hline
* & * & * & * & * & * \\
\hline
\end{array}
\]
We set up the identification between the sixteen \((-2)\)-curves \(\{C_i, D_i, E_i, F_i\}\) in \(S_X\) and these sixteen codewords as follows:

\[
\begin{align*}
E_1 &\leftrightarrow K_p, & F_1 &\leftrightarrow K_{q\overline{\omega}}, & C_1 &\leftrightarrow K_{q1}, & D_1 &\leftrightarrow K_{q\overline{\omega}}, \\
E_2 &\leftrightarrow K_{r\overline{\omega}}, & F_2 &\leftrightarrow K_{s1}, & C_2 &\leftrightarrow K_{s\overline{\omega}}, & D_2 &\leftrightarrow K_{r\overline{\omega}}, \\
E_3 &\leftrightarrow K_{r\overline{\omega}}, & F_3 &\leftrightarrow K_{\overline{p}\overline{\omega}}, & C_3 &\leftrightarrow K_{f1}, & D_3 &\leftrightarrow K_{s1}, \\
E_4 &\leftrightarrow K_{r1}, & F_4 &\leftrightarrow K_{s\overline{\omega}}, & C_4 &\leftrightarrow K_{s\overline{\omega}}, & D_4 &\leftrightarrow K_{s\overline{\omega}}.
\end{align*}
\]

As a check, we can recover the intersection form by noting that two Leech roots associated to an
octad have intersection number one if they share two entries in the MOG and intersection number zero if they share four entries in the MOG.

As explained in section 5.5, we can embed the Néron–Severi lattice into \( \mathbb{II}_{1,25} \) and obtain the root lattice \( D_4 \oplus A_2 \) as its orthogonal complement. Restricting the closure of the fundamental chamber \( \overline{C}_{\mathbb{II}} \) of the reflection group of \( \mathbb{II}_{1,25} \) to the positive cone of \( S_X \), we find \( D' \) which is bounded by a finite number of faces. The group of automorphisms of of \( D' \) has the following structure:

\[
\text{Aut}(D') \cong (\mathbb{Z}/2\mathbb{Z})^4.A_4. \tag{242}
\]

The kernel of \( p_S : \text{Aut}(D') \to \text{Isom}(q) \cong (S_3 \times \mathbb{Z}/2\mathbb{Z}) \) is

\[
\text{Aut}(D')_0 \cong (\mathbb{Z}/2\mathbb{Z})^4.A_4, \tag{243}
\]

and \( p_S \) is surjective \cite{KK}.

In the case of \( X = \text{Km}(E_\omega \times E_\omega) \), tasks I–IV and the verification of assumptions (as-1)–(as-7) were carried out in \cite{KK}, specifically in Lemma 3.1 (task I), Lemma 3.2 (task II type 0)), Lemma 3.6 (task II type non-0), Lemma 3.4 for (as-3), §4.3 (task III and (as-4)), Lemma 2.1 for (as-6), Lemma 3.4 for (as-7), and Lemma 3.5 (task IV).

Since we would like to use the action of \( \text{Aut}(D') \) on \( S_X \) and compute the orbit of elliptic divisors under the action of this group, let us extract more details of the action of the group from \cite{KK}.

As before, we may consider translations on \( A = E_\omega \times E_\omega \) by any of the two-torsion points \( \mu_{ij} \) which generate the normal subgroup \( (\mathbb{Z}/2\mathbb{Z})^4 \) of \( \text{Aut}(D') \). These will not mix the \( C, D, E, F, G \) but leave each class of cycles separate. We use the following names for the translations:

\[
t_1 \sim \mu_{21} = (\frac{1}{2}, 0) \quad t_2 \sim \mu_{31} = (\frac{3}{2}, 0) \quad t_3 \sim \mu_{12} = (0, \frac{1}{2}) \quad t_4 \sim \mu_{13} = (0, \frac{3}{2}) \tag{244}
\]

They act in the following way on the generators of the Néron–Severi lattice:

\[
\begin{align*}
t_1 : & \\
E_i & \leftrightarrow E_i & C_1 & \leftrightarrow C_4 & E_i & \leftrightarrow E_i & C_1 & \leftrightarrow C_2 \\
C_2 & \leftrightarrow C_3 & & & C_3 & \leftrightarrow C_4 & & \\
F_1 & \leftrightarrow F_2 & D_1 & \leftrightarrow D_2 & F_1 & \leftrightarrow F_3 & D_1 & \leftrightarrow D_3 \\
F_3 & \leftrightarrow F_1 & D_3 & \leftrightarrow D_4 & F_2 & \leftrightarrow F_4 & D_2 & \leftrightarrow D_4 \\
G_{1j} & \leftrightarrow G_{2j} & G_{3j} & \leftrightarrow G_{4j} & G_{1j} & \leftrightarrow G_{3j} & G_{2j} & \leftrightarrow G_{4j} \\
\end{align*}
\]

\[
\begin{align*}
t_2 : & \\
E_i & \leftrightarrow E_i & C_1 & \leftrightarrow C_4 & E_i & \leftrightarrow E_i & C_1 & \leftrightarrow C_2 \\
C_2 & \leftrightarrow C_4 & & & C_3 & \leftrightarrow C_4 & & \\
F_1 & \leftrightarrow F_2 & D_1 & \leftrightarrow D_2 & F_1 & \leftrightarrow F_3 & D_1 & \leftrightarrow D_3 \\
F_3 & \leftrightarrow F_1 & D_3 & \leftrightarrow D_4 & F_2 & \leftrightarrow F_4 & D_2 & \leftrightarrow D_4 \\
G_{1j} & \leftrightarrow G_{2j} & G_{3j} & \leftrightarrow G_{4j} & G_{1j} & \leftrightarrow G_{3j} & G_{2j} & \leftrightarrow G_{4j} \\
\end{align*}
\]

\[
\begin{align*}
t_3 : & \\
F_i & \leftrightarrow F_i & C_1 & \leftrightarrow C_3 & F_i & \leftrightarrow F_i & C_1 & \leftrightarrow C_4 \\
C_2 & \leftrightarrow C_4 & & & C_2 & \leftrightarrow C_3 & & \\
E_1 & \leftrightarrow E_2 & D_1 & \leftrightarrow D_2 & E_1 & \leftrightarrow E_3 & D_1 & \leftrightarrow D_3 \\
E_3 & \leftrightarrow E_4 & D_3 & \leftrightarrow D_4 & E_2 & \leftrightarrow E_4 & D_2 & \leftrightarrow D_4 \\
G_{i1} & \leftrightarrow G_{i2} & G_{i3} & \leftrightarrow G_{i4} & G_{i1} & \leftrightarrow G_{i3} & G_{i2} & \leftrightarrow G_{i4} \\
\end{align*}
\]

\[
\begin{align*}
t_4 : & \\
F_i & \leftrightarrow F_i & C_1 & \leftrightarrow C_3 & F_i & \leftrightarrow F_i & C_1 & \leftrightarrow C_4 \\
C_2 & \leftrightarrow C_4 & & & C_2 & \leftrightarrow C_3 & & \\
E_1 & \leftrightarrow E_2 & D_1 & \leftrightarrow D_2 & E_1 & \leftrightarrow E_3 & D_1 & \leftrightarrow D_3 \\
E_3 & \leftrightarrow E_4 & D_3 & \leftrightarrow D_4 & E_2 & \leftrightarrow E_4 & D_2 & \leftrightarrow D_4 \\
G_{i1} & \leftrightarrow G_{i2} & G_{i3} & \leftrightarrow G_{i4} & G_{i1} & \leftrightarrow G_{i3} & G_{i2} & \leftrightarrow G_{i4} \\
\end{align*}
\]

\[\]
Using the embedding of the Neron–Severi lattice into Leech roots, we may realize these maps as transformations in $M_{24} \subset \text{Co}_0 \cong \text{Isom}(\Lambda_{24})$. Written in MOG form they are given by

$$
\begin{align*}
t_1 &: \begin{pmatrix}
    c & e & f & e & h \\
    d & c & b & a \\
    d & g & f & g & h
\end{pmatrix} \\
t_2 &: \begin{pmatrix}
    a & d & f & d & h \\
    b & c & f & c & h \\
    a & b & e & g
\end{pmatrix} \\
t_3 &: \begin{pmatrix}
    e & d & b & c & h \\
    c & e & a & d & h \\
    a & b & f & f & g
\end{pmatrix} \\
t_4 &: \begin{pmatrix}
    g & e & d \\
    c & h & a \\
    b & h & d & e & g
\end{pmatrix}
\end{align*}
$$

Here, any two entries which share a letter are exchanged under the $\mathbb{Z}/2\mathbb{Z}$, all others are left fixed.

The action by the alternating group $A_4$ is generated by the automorphisms on $A/(−\text{id.})$:

$$
\begin{align*}
g_1 &: (z^1, z^2) \mapsto (z^2, −z^1 + z^2) \\
g_2 &: (z^1, z^2) \mapsto (\omega(z^2 − z^1), −\omega^2 z^2)
\end{align*}
$$

(248)

Given the explicit expression for the various algebraic curves, we can work out the action of $g_1$: ($i = 2, 3, 4$)

$$
\begin{align*}
G_{i1} & \quad \leftarrow \quad G_{i2} \quad \leftarrow \quad G_{i3} \quad \leftarrow \quad G_{i4} \\
G_{ii} & \quad \leftarrow \quad G_{1i} \quad \leftarrow \quad G_{23} \quad \leftarrow \quad G_{34} \quad \leftarrow \quad G_{32} \quad \leftarrow \quad G_{43}
\end{align*}
$$

(249)

Each of the generators $g_1$ and $g_2$ also satisfy $(g_1)^3 = \text{id}.$ and $(g_2^{-1})^3 = \text{id}.$ on $A/(−\text{id.})$, and furthermore, $(g_2^{-1} \cdot g_1) : (z^1, z^2) \mapsto (\omega z^1 + z^2, \omega(z^1 − z^2))$ is an involution, $(g_2^{-1} \cdot g_1)^2 = \text{id}.$ Thus, $g_1$ and $g_2^{-1}$ satisfy the same set of relations as the generators $a$ and $b$ of the group $A_4$.

---

45 When we see $A_4$ as a subgroup of $S_4 \langle \sigma_{12}, \sigma_{23}, \sigma_{34} \rangle$, it is

$$
A_4 = \langle a := (\sigma_{12} \cdot \sigma_{23}), b := (\sigma_{23} \cdot \sigma_{34}) \mid a^3 = b^3 = 1, (ab)^2 = 1 \rangle.
$$

(247)

Each of the generators $g_1$ and $g_2$ also satisfy $(g_1)^3 = \text{id}.$ and $(g_2^{-1})^3 = \text{id}.$ on $A/(−\text{id.})$, and furthermore, $(g_2^{-1} \cdot g_1) : (z^1, z^2) \mapsto (\omega z^1 + z^2, \omega(z^1 − z^2))$ is an involution, $(g_2^{-1} \cdot g_1)^2 = \text{id}.$ Thus, $g_1$ and $g_2^{-1}$ satisfy the same set of relations as the generators $a$ and $b$ of the group $A_4$.
Similarly, the action of $g_2$ is

\[
\begin{array}{cccc}
G_{11} & G_{22} & G_{33} \\
G_{44} & G_{21} & G_{32} \\
G_{42} & G_{13} & G_{41} & G_{31} & G_{43} & G_{24} \\
E_1 & E_2 & F_1 \\
E_1 & E_3 & E_4 & D_1 & C_1 \\
F_2 & F_3 & F_4 \\
D_4 & C_2 & D_2 & C_3 & D_3 & C_4
\end{array}
\]

(250)

This $A_4$ subgroup of $\text{Aut}(D')_0$ can also be realize as transformations in $M_{24}$. In the MOG representation,

\[
g_1: \begin{array}{|ccc|}
 b_1 & a_2 & c_2 \\
 e_3 & c_3 & b_2 \\
 c_1 & e_1 & a_1 \\
 a_3 & b_3 & d_2
\end{array} \quad \quad \quad g_2: \begin{array}{|ccc|}
 a_1 & b_1 & c_1 \\
 a_2 & b_2 & c_2 \\
 a_3 & b_3 & c_3 \\
 d_1 & f_1 & f_2 & f_3
\end{array}
\]

Here, the maps $g_1$ and $g_2$ act by sending $\xi_1 \to \xi_2 \to \xi_3 \to \xi_1$ for any letter.

The remaining maps in $\text{Aut}(D')$ are $S_3 \times \mathbb{Z}/2\mathbb{Z} \cong \text{Aut}(D')/\text{Aut}(D')_0$. Let $S_3 = \langle s_1, s_2 \mid (s_1)^3 = 1, (s_2)^2 = 1, s_2 \cdot s_1 \cdot s_2 = (s_1)^2 \rangle$. The order-3 element $s_1$ can be represented by an automorphism

\[
s_1 : (z^1, z^2) \mapsto (\omega z^1, \omega z^2),
\]

which acts on $\{C_i, D_i, E_i, F_i, G_{ij}\}$ such that it leaves the index 1 fixed and cyclically permutes all other indices:

\[
\begin{array}{c}
2 \\
4 \leftrightarrow 3 \\
\end{array}
\]

(252)

The other generator $s_2$ is represented by an isometry of $S_X$,

\[
s_2 : \begin{array}{cccc}
 E_2 & E_3 & F_3 & F_4 \\
 G_{32} & G_{43} & G_{33} & G_{42} \\
 C_1 & D_1 & C_2 & D_2 \\
 G_{12} & G_{13} & G_{22} & G_{23} \\
 G_{31} & G_{41} & G_{34} & G_{44} \\
 C_3 & D_3 & C_4 & D_4
\end{array}
\]

(253)
which leaves $E_1, E_4, F_1, F_2, G_{11}, G_{14}, G_{21}, G_{24}$ fixed.
Let $\sigma$ be the generator of the remaining $\mathbb{Z}/2\mathbb{Z}$ factor; it acts on $S_X$ as follows:

\[
\begin{array}{cccc}
E_1 & \leftrightarrow & G_{24} & \\
E_2 & \leftrightarrow & G_{43} & \\
E_3 & \leftrightarrow & G_{32} & \\
E_4 & \leftrightarrow & G_{11} & \\
F_1 & \leftrightarrow & G_{44} & \\
F_2 & \leftrightarrow & G_{31} & \\
F_3 & \leftrightarrow & G_{23} & \\
F_4 & \leftrightarrow & G_{12} & \\
D_1 & \leftrightarrow & G_{14} & \\
D_2 & \leftrightarrow & G_{42} & \\
D_3 & \leftrightarrow & G_{21} & \\
D_4 & \leftrightarrow & G_{33} & \\
C_1 & \leftrightarrow & G_{34} & \\
C_2 & \leftrightarrow & G_{22} & \\
C_3 & \leftrightarrow & G_{41} & \\
C_4 & \leftrightarrow & G_{13}.
\end{array}
\]

(254)

Since the image of $\text{Isom}(T_X)^{(\text{Hodge})} \cong \mathbb{Z}/6\mathbb{Z} \langle \theta_6 \rangle$ under $p_T$ is $\mathbb{Z}/2\mathbb{Z} \langle \sigma \rangle \times \mathbb{Z}/3\mathbb{Z} \langle s_1 \rangle \subset \text{Isom}(q)$
the subgroup $\text{Aut}(D')^{(\text{Hodge})}$ is given by

\[
\text{Aut}(D')^{(\text{Hodge})} \cong (\mathbb{Z}/2\mathbb{Z})^4 \cdot A_4 \cdot (\mathbb{Z}/3\mathbb{Z} \langle s_1 \rangle \times \mathbb{Z}/2\mathbb{Z} \langle \sigma \rangle).
\]

(255)
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