Pfaffian and Determinant Solutions to
A Discretized Toda Equation for $B_r$, $C_r$ and $D_r$

by

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Abstract

We consider a class of 2 dimensional Toda equations on discrete space-time. It has arisen as functional relations in commuting family of transfer matrices in solvable lattice models associated with any classical simple Lie algebra $X_r$. For $X_r = B_r, C_r$ and $D_r$, we present the solution in terms of Pfaffians and determinants. They may be viewed as Yangian analogues of the classical Jacobi-Trudi formula on Schur functions.

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1. Introduction

Consider the following systems of difference equations on $T_m^{(a)}(u)$ ($m \in \mathbb{Z}_0, u \in \mathbb{C}, a \in \{1, 2, \ldots, r\}$)

**$B_r$** ($r \geq 2$)

\[
T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_{m+1}^{(a+1)}(u) \quad (1a)
\]

\[
1 \leq a \leq r - 2,
\]

\[
T_m^{(r-1)}(u - 1)T_m^{(r-1)}(u + 1) = T_{m+1}^{(r-1)}(u)T_{m-1}^{(r-1)}(u) + T_m^{(r-2)}(u)T_{2m}^{(r)}(u), \quad (1b)
\]

\[
T_{2m}^{(r)}(u - \frac{1}{2})T_{2m}^{(r)}(u + \frac{1}{2}) = T_{2m+1}^{(r)}(u)T_{2m-1}^{(r)}(u)
+ T_m^{(r-1)}(u - \frac{1}{2})T_m^{(r-1)}(u + \frac{1}{2}), \quad (1c)
\]

\[
T_{2m+1}^{(r)}(u - \frac{1}{2})T_{2m+1}^{(r)}(u + \frac{1}{2}) = T_{2m+2}^{(r)}(u)T_{2m}^{(r)}(u) + T_m^{(r-1)}(u)T_{m+1}^{(r-1)}(u). \quad (1d)
\]

**$C_r$** ($r \geq 2$)

\[
T_m^{(a)}(u - \frac{1}{2})T_m^{(a)}(u + \frac{1}{2}) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_{m+1}^{(a+1)}(u) \quad (2a)
\]

\[
1 \leq a \leq r - 2,
\]

\[
T_{2m}^{(r-1)}(u - \frac{1}{2})T_{2m}^{(r-1)}(u + \frac{1}{2}) = T_{2m+1}^{(r-1)}(u)T_{2m-1}^{(r-1)}(u)
+ T_{2m}^{(r-2)}(u)T_m^{(r)}(u - \frac{1}{2})T_m^{(r)}(u + \frac{1}{2}), \quad (2b)
\]

\[
T_{2m+1}^{(r-1)}(u - \frac{1}{2})T_{2m+1}^{(r-1)}(u + \frac{1}{2}) = T_{2m+2}^{(r-1)}(u)T_{2m}^{(r-1)}(u)
+ T_{2m+1}^{(r-2)}(u)T_m^{(r)}(u)T_{m+1}^{(r)}(u), \quad (2c)
\]

\[
T_m^{(r)}(u - 1)T_m^{(r)}(u + 1) = T_{m+1}^{(r)}(u)T_{m-1}^{(r)}(u) + T_{2m}^{(r-1)}(u). \quad (2d)
\]

**$D_r$** ($r \geq 4$)

\[
T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_{m+1}^{(a+1)}(u) \quad (3a)
\]

\[
1 \leq a \leq r - 3,
\]

\[
T_m^{(r-2)}(u - 1)T_m^{(r-2)}(u + 1) = T_{m+1}^{(r-2)}(u)T_{m-1}^{(r-2)}(u)
+ T_m^{(r-3)}(u)T_m^{(r-1)}(u)T_{m}^{(r)}(u), \quad (3b)
\]

\[
T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_{m}^{(r-2)}(u) \quad a = r - 1, r. \quad (3c)
\]

$(T_m^{(0)}(u) = 1)$. We shall exclusively consider the initial condition $T_0^{(a)}(u) = 1$ for any $1 \leq a \leq r$. Then one can solve the systems $(1)$, $(2)$ and $(3)$ iteratively to express $T_m^{(a)}(u)$ in terms of $T_1^{(1)}(u + \text{shift}), \ldots, T_1^{(r)}(u + \text{shift})$. For example, $T_2^{(1)}(u) = T_1^{(1)}(u - 1)T_1^{(1)}(u + 1) - T_1^{(2)}(u)$.
from (1a). The purpose of this paper is to present the formulae that express an arbitrary $T_m^{(a)}(u) \ (m \geq 1)$ as a determinant or a Pfaffian of matrices with elements $0$ or $\pm T_1^{(b)}(u+\text{shift})$ $(0 \leq b \leq r)$.

In fact, such formulae had been partially conjectured in [KNS1], where a set of functional relations, $T$-system, was introduced for the commuting family of transfer matrices $\{T_m^{(a)}(u)\}$ for solvable lattice models associated with any classical simple Lie algebra $X_r$. In this context, eqs. (1), (2) and (3) correspond to $X_r = B_r, C_r$ and $D_r$ cases of the $T$-system, respectively. $T_m^{(a)}(u)$ denotes a transfer matrix (or its eigenvalue) with spectral parameter $u$ and “fusion type” labeled by $a$ and $m$ [KNS1]. Our result here confirms all of the determinant conjectures raised in section 5 of [KNS1]. Moreover it extends them to a full solution of (1), (2) and (3), which, in general, involves Pfaffians as well. In the representation theoretical viewpoint, this yields a Yangian analogue of the Jacobi-Trudi formula [Ma], i.e., a way to construct Yangian characters out of those for the fundamental representations [CP].

Beside the significance in the lattice model context [KNS2], the beautiful structure in these solutions indicate a rich content of the $T$-system allowing determinant and Pfaffian solutions at least for $X_r = A_r, B_r, C_r$ and $D_r$. See also the remarks in section 6 concerning the $T$-systems for twisted affine Lie algebras [KS].

The outline of the paper is as follows. In sections 2, 3 and 4, we present solutions to the $B_r, C_r$ and $D_r$ cases, respectively. Pfaffians are needed for $T_m^{(r)}(u)$ in $C_r$ and $T_m^{(r-1)}(u)$ and $T_m^{(r)}(u)$ in $D_r$. In section 5, we illustrate a proof for the $C_r$ case. The other cases can be verified quite similarly. Section 6 is devoted to summary and discussion.

Before closing the introduction, a few remarks are in order. Firstly, the original $T$-system [KNS1] had a factor $g_m^{(a)}(u)$ in front of the second term in the rhs of $T_m^{(a)}(u + \frac{1}{t_a})T_m^{(a)}(u - \frac{1}{t_a}) = \cdots$. Throughout this paper we shall set $g_m^{(a)}(u) = 1$. To recover the dependence on $g_m^{(a)}(u)$ is quite easy as long as the relation $g_m^{(a)}(u + \frac{1}{t_a})g_m^{(a)}(u - \frac{1}{t_a}) = g_{m+1}^{(a)}(u)g_{m-1}^{(a)}(u)$ is satisfied (cf [KNS1]). Secondly, $T$-systems (1) and (2) coincide for $r = 2$ under the exchange $T_m^{(1)}(u) \leftrightarrow T_m^{(2)}(u)$, which reflects the Lie algebra equivalence $B_2 \simeq C_2$. In this case (9) and (12) yield two alternative expressions for the same quantity. Thirdly, for $X_r = A_r$, the $T$-system $T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_m^{(a+1)}(u)T_m^{(a-1)}(u)$ $(1 \leq a \leq r, T_m^{(0)}(u) = T_m^{(r+1)}(u) = 1)$ is the so-called Hirota-Miwa equation. In the transfer matrix context, it has been proved in [KNS1] by using the determinant formula in [BR]. Finally, for $X_r = B_r$, a determinant solution different from (9) has been obtained in [KOS].
The relevant matrix there is not sparse as (7) and the matrix elements are not necessarily $T_1^{(a)}(u)$ but contain some quadratic expressions of $T_1^{(r)}(u)$ in general.

2. $B_r$ Case

For any $k \in \mathbb{C}$, put

$$x_k^a = \begin{cases} T_1^{(a)}(u + k) & 1 \leq a \leq r, \\ 1 & a = 0. \end{cases}$$

(5)

We introduce the infinite dimensional matrices $T = (T_{ij})_{i,j \in \mathbb{Z}}$ and $E = (E_{ij})_{i,j \in \mathbb{Z}}$ as follows.

$$T_{ij} = \begin{cases} x_i^{i-1} + 1 & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i-j}{2} \in \{1,0,\ldots,2-r\}, \\ -x_i^{i+2r-2} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i-j}{2} \in \{1-r,-r,\ldots,2-2r\}, \\ -x_i^r & \text{if } i \in 2\mathbb{Z} \text{ and } j = i + 2r - 3, \\ 0 & \text{otherwise}, \end{cases}$$

(6)

$$E_{ij} = \begin{cases} \pm 1 & \text{if } i = j - 1 \pm 1 \text{ and } i \in 2\mathbb{Z}, \\ x_i^{-1} & \text{if } i = j - 1 \text{ and } i \in 2\mathbb{Z} + 1, \\ 0 & \text{otherwise}. \end{cases}$$

For example, for $B_3$, they read

$$\begin{pmatrix} x_0^1 & 0 & x_1^2 & 0 & -x_2^3 & 0 & -x_3^4 & 0 & -1 \\ 0 & 0 & 0 & 0 & -x_0^{5/2} & 0 & 0 & 0 & 0 \\ 1 & 0 & x_1^2 & 0 & x_2^3 & 0 & -x_3^4 & 0 & -x_4^5 \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_0^{9/2} & 0 & 0 \\ 0 & 0 & 1 & 0 & x_1^4 & 0 & x_2^5 & 0 & -x_3^6 \cdots \end{pmatrix},$$

(7a)

$$\begin{pmatrix} 0 & x_0^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \cdots \\ 0 & 0 & 0 & 0 & 0 & x_0^3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \end{pmatrix}.$$
Theorem 2.1. For \( m \in \mathbb{Z}_{\geq 1} \),

\[
T_m(a)(u) = \det \left( T_{m-1}(1, 1, x_{a-m+1}) + \mathcal{E}_{m-1}(1, 2, x_{r-m+r-a+\frac{1}{2}}) \right), \quad 1 \leq a < r,
\]

(9a)

\[
T_m(r)(u) = \left( -1 \right)^{m(m-1)/2} \det \left( T_m(1, 2, -x_{r-m-a+\frac{1}{2}}) + \mathcal{E}_{m}(1, 1, x_{r-a}) \right),
\]

(9b)

solves the \( B_r \) T-system (1).

Up to some conventional change, (9a) in the above had been conjectured in eq.(5.6) of [KNS1]. The formula (9b) is new.

3. Case \( C_r \)

Here we introduce the infinite dimensional matrix \( T \) by

\[
T_{ij} = \begin{cases} 
\frac{x_{i+j}^{j-i+1}}{2^{j-i}-1} & \text{if } i-j \in \{1, 0, \ldots, 1-r\}, \\
\frac{x_{i+j+2r+1}}{2^{i+j+2r+1}-1} & \text{if } i-j \in \{-1-r, -2-r, \ldots, -1-2r\}, \\
0 & \text{otherwise}.
\end{cases}
\]

(10)

For example, for \( C_2 \), it reads

\[
(T_{ij})_{i,j \geq 1} = \begin{pmatrix}
x_0^1 & x_1^2 & 0 & -x_3^{2/2} & -x_2^{1/2} & -1 & 0 & 0 \\
1 & x_1^1 & x_3^{3/2} & 0 & -x_5^{2/2} & -x_3^{1/2} & -1 & 0 & \cdots \\
0 & 1 & x_2^1 & x_5^{2/2} & 0 & -x_7^{2/2} & -x_4^{1/2} & -1 \\
0 & 0 & 1 & x_3^1 & x_7^{2/2} & 0 & -x_{9/2}^{2} & -x_5^{1/2} & \cdots \\
& & & & & & & & & \vdots
\end{pmatrix}.
\]

(11)

We keep the same notations (5) and \( T_m(i, j, \pm x_k^a) \) (1 \( \leq a \leq r \)) as in section 2. Note that \( T_m(1, 2, -x_k^a) \) is an anti-symmetric matrix for any \( m \). Our result in this section is

Theorem 3.1. For \( m \in \mathbb{Z}_{\geq 1} \),

\[
T_m(a)(u) = \det \left( T_{m}(1, 1, x_{a-r-a+\frac{1}{2}}) \right), \quad 1 \leq a < r,
\]

(12a)

\[
T_m(r)(u) = (-1)^{m(m-1)/2} \det \left( T_m(1, 2, -x_{r-m+1}) \right),
\]

(12b)
solves the $C_r$ $T$-system (2).

The expression (12a) is essentially the conjecture (5.10) in [KNS1]. The Pfaffian formula (12b) is new. In proving the theorem in section 5, we will also establish the relations

\[
T_m^{(r)}(u - \frac{1}{2})T_m^{(r)}(u + \frac{1}{2}) = \det (T_{2m}(1, 1, x_{r-m+1}^r)), \quad (13a)
\]
\[
T_m^{(r)}(u)T_{m+1}^{(r)}(u) = \det (T_{2m+1}(1, 1, x_{r-m}^r)). \quad (13b)
\]

4. $D_r$ Case

Here we define the infinite dimensional matrices $\mathcal{T}$ and $\mathcal{E}$ by

\[
\mathcal{T}_{ij} = \begin{cases} 
\frac{i-j+1}{2} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i-j}{2} \in \{1, 0, \ldots, 3 - r\}, \\
-\frac{i-j-1}{2} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i-j}{2} = \frac{5}{2} - r, \\
-\frac{i-j+3}{2} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i-j}{2} = \frac{3}{2} - r, \\
-\frac{i-j+2r-3}{2} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i-j}{2} \in \{1 - r, -r, \ldots, 3 - 2r\}, \\
0 & \text{otherwise},
\end{cases} \quad (14a)
\]
\[
\mathcal{E}_{ij} = \begin{cases} 
\pm 1 & \text{if } i = j - 2 \pm 2 \text{ and } i \in 2\mathbb{Z}, \\
x_{i-1}^r & \text{if } i = j - 3 \text{ and } i \in 2\mathbb{Z}, \\
x_{i-2}^r & \text{if } i = j - 1 \text{ and } i \in 2\mathbb{Z}, \\
0 & \text{otherwise.}
\end{cases} \quad (14b)
\]

For example, for $D_4$, they read

\[
(\mathcal{T}_{ij})_{i,j \geq 1} = \begin{pmatrix} 
x_1^1 & 0 & x_2^2 & -x_3^2 & 0 & -x_4^2 & -x_3^3 & 0 & -x_4^3 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & x_1^2 & 0 & x_2^2 & -x_3^2 & 0 & -x_4^2 & -x_3^3 & 0 & -x_4^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
\end{pmatrix}, \quad (15a)
\]
\[
(\mathcal{E}_{ij})_{i,j \geq 1} = \begin{pmatrix} 
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & x_1^4 & 0 & x_2^3 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & x_4^3 & 0 & x_3^3 & -1 & 0 & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
\end{pmatrix}. \quad (15b)
\]

We keep the same notations (5), $\mathcal{T}_m(i, j, \pm x_k^a)$ $(1 \leq a \leq r - 2)$ and $\mathcal{T}_m(i, j, -x_k^a), \mathcal{E}_m(i, j, x_k^a)$ $(a = r - 1, r)$ as in section 2. Our result in this section is

**Theorem 4.1.** For $m \in \mathbb{Z}_{\geq 1}$,

\[
T_m^{(a)}(u) = \det (T_{2m-1}(1, 1, x_{a-m+1}^a) + \mathcal{E}_{2m-1}(2, 3, x_{r-m-r+a+4}^r)), \quad 1 \leq a \leq r - 2, \quad (16a)
\]
\[
T_m^{(r-1)}(u) = pf(T_{2m}(2, 1, -x_{r-m+1}^r) + \mathcal{E}_{2m}(1, 2, x_{r-m+1}^r)), \quad (16b)
\]
\[
T_m^{(r)}(u) = (-1)^m pf(T_{2m}(1, 2, -x_{r-m+1}^r) + \mathcal{E}_{2m}(2, 1, x_{r-m+1}^r)), \quad (16c)
\]
solves the $D_r$ $T$-system (3).

The matrices in (16b,c) are indeed anti-symmetric. Eq.(16a) is essentially the conjecture (5.15) in [KNS1]. The Pfaffian formulae (16b,c) are new. By using them one can show the relations

\[
\begin{align*}
T_m^{(r-1)}(u)T_m^{(r)}(u) &= (-1)^m \det(T_{2m}(1,1,-x_{-m+1}^{r-1}) + E_{2m}(2,2,x_{-m+1}^-)) \quad \text{(17a)} \\
T_m^{(r-1)}(u+1)T_m^{(r)}(u-1) &= (-1)^m \det(T_{2m}(1,1,-x_{-m}^-) + E_{2m}(2,2,x_{-m+2}^-)) \quad \text{(17b)} \\
T_{m+1}^{(r-1)}(u)T_m^{(r)}(u-1) &= (-1)^{m+1} \det(T_{2m+1}(1,1,-x_{-m}^{-r}) + E_{2m+1}(2,2,x_{-m}^-)) \quad \text{(17c)} \\
T_{m+1}^{(r-1)}(u+1)T_m^{(r)}(u) &= (-1)^m \det(T_{2m+1}(2,1,x_{-m+2}^{-r}) + E_{2m+1}(1,1,x_{-m}^-)) \quad \text{(17d)}
\end{align*}
\]

The proof of (17) is analogous to that of (13), which will be explained in the next section.

5. Proof of Theorem 3.1

Here we shall outline the proof of theorem 3.1, namely $C_r$ $T$-system (2) starting from (12). As it turns out, all of the three term relations in (2) reduce to Jacobi’s identity:

\[
D \begin{bmatrix} 1 \\ 1 \end{bmatrix} D \begin{bmatrix} n \\ n \end{bmatrix} = DD \begin{bmatrix} 1, n \\ 1, n \end{bmatrix} + D \begin{bmatrix} 1 \\ 1 \end{bmatrix} D \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad \text{(18)}
\]

Here $D$ is the determinant of any $n$ by $n$ matrix and $D[^{i_1,i_2,...}_{j_1,j_2,...}]$ denotes its minor removing the $i_k$’s rows and $j_k$’s columns.

Let us prove (13a) first. Taking its square and substituting (12b), we are to show

\[
\det(T_{2m}(1,2,-x_{-m+\frac{1}{2}}^r))\det(T_{2m}(1,2,-x_{-m+\frac{3}{2}}^r)) = \left(\det(T_{2m}(1,1,-x_{-m+\frac{1}{2}}^r))\right)^2. \quad \text{(19)}
\]

To see this we set

\[
D = \det(T_{2m+1}(1,2,-x_{-m+\frac{1}{2}}^r)) = \det\begin{pmatrix}
0 & -x_{-m+\frac{1}{2}}^r & \cdots & \cdots & -x_{-m+1}^{-r} \\
x_{-m+\frac{1}{2}}^r & 0 & \cdots & \cdots & -x_{-m+\frac{3}{2}}^{-r} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
x_{-m+\frac{1}{2}}^r & x_{-m+\frac{3}{2}}^{-r} & \cdots & \cdots & 0 \\
x_{-m+\frac{3}{2}}^{-r} & x_{-m+\frac{5}{2}}^{-r} & \cdots & \cdots & \cdots
\end{pmatrix} = 0, \quad \text{(20)}
\]

since this is an anti-symmetric matrix with odd size. From (20) it is easy to see

\[
\begin{align*}
D \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \det(T_{2m}(1,2,-x_{-m+\frac{1}{2}}^r)), \quad D \begin{bmatrix} 2m+1 \\ 2m+1 \end{bmatrix} = \det(T_{2m}(1,2,-x_{-m+\frac{3}{2}}^r)), \\
D \begin{bmatrix} 1 \\ 2m+1 \end{bmatrix} &= D \begin{bmatrix} 2m+1 \\ 1 \end{bmatrix} = \det(T_{2m}(1,1,x_{-m+\frac{1}{2}}^r)). \quad \text{(21)}
\end{align*}
\]

Thus (19) follows immediately from (21) and (18). In taking the square root of (19), the relative sign can be fixed by comparing the coefficients of $x_{-m+1/2}^r x_{-m+3/2}^r \cdots x_{m-1/2}^r$ on
both sides, which agrees with (13a). The relation (13b) can be shown similarly by setting
\( D = \det(T_{2m+2}(1, 2, -x^r_m)) \).

Now we proceed to the proof of the \( T \)-system (2). To show (2a), it suffices to apply
(18) for \( D = \det(T_{m+1}(1, 1, x^a_m)) = T_{m+1}^{(a)}(u) \) and to note that \( D_{m+1}^{[1]} = T_{m+1}^{(a)}(u + \frac{1}{2}) \),
\( D_{m+1}^{[m+1]} = T_{m+1}^{(a)}(u - \frac{1}{2}) \). Similarly (2b) (resp. (2c)) can be derived by setting \( D = \det(T_{2m+2}(1, 1, x^r_m)) = T_{2m+2}^{(r)}(u) \)
(resp. \( D = \det(T_{2m+2}(1, 1, x^{r-1}_m)) = T_{2m+2}^{(r-1)}(u) \)) and using (13a) (resp. (13b)) to identify
\( D_{2m+1}^{[2m+1]} \) with \( T_{m}^{(r)}(u - \frac{1}{2})T_{m}^{(r)}(u + \frac{1}{2}) \) (resp. \( T_{m}^{(r)}(u)T_{m+1}^{(r)}(u) \)). Finally to show (2d), we put
\( D = \det(T_{2m+2}(1, 1, x^r_m)) \). Then from (12) and (13) we have

\[
D = T_{m}^{(r)}(u)T_{m+1}^{(r)}(u), \quad D_{m+1}^{[1]} = T_{m+1}^{(r)}(u)T_{m}^{(r)}(u),
\]

\[
D_{m+1}^{[2m+1]} = T_{2m+1}^{(r-1)}(u), \quad D_{2m+1}^{[2m+1]} = (T_{m}^{(r)}(u))^2.
\]

Substituting (22) into (18) (for \( n = 2m+1 \)) and cancelling out the common factor \( (T_{m}^{(r)}(u))^2 \),
we obtain (2d). This completes the proof of theorem 3.1.

6. Summary and discussion
In this paper we have considered the difference equations (1), (2) and (3), which may be viewed as 2 dimensional Toda equations on discrete space-time as argued in (4). They have arisen as the \( B_r \), \( C_r \) and \( D_r \) cases of the \( T \)-system, which are functional relations among commuting families of transfer matrices in the associated solvable lattice models. Under the initial condition \( T_0^{(a)}(u) = 1 \) (\( 1 \leq a \leq r \)), we have given the solutions (9), (12) and
(16) for \( T_m^{(a)}(u) \) with \( m \in \mathbb{Z}_{\geq 1} \). They are expressed in terms of Pfaffians or determinants
of the matrices (6), (10) and (14), which contain only \( \pm T_1^{(a)}(u + \text{shift}) \) or \( \pm 1 \) as their
matrix elements. This confirms the earlier conjectures [KNS1] and extends them to the full
solutions.

It will be interesting to extend a similar analysis to the \( T \)-system for the exceptional
algebras \( E_{6,7,8}, F_4, G_2 \) [KNS1] and also the twisted quantum affine algebras \( A_\mathfrak{n}^{(2)}, D_\mathfrak{n}^{(2)}, E_6^{(2)} \)
and \( D_4^{(3)} \) [KS]. In fact, the solutions to the \( A_\mathfrak{n}^{(2)}, D_\mathfrak{n}^{(2)} \) and \( D_4^{(3)} \) cases can be obtained just by
imposing the “modulo \( \sigma \) relations” ((3.4) in [KS]) on the corresponding non-twisted cases
\( A_\mathfrak{n}, D_\mathfrak{n} \) and \( D_4 \) treated in this paper. On the other hand, to deal with the exceptional cases,
it seems necessary to introduce matrices whose elements are some higher order expressions
in \( T_1^{(a)}(u) \)’s analogous to [KOS].

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