BOUNDEDNESS IN A THREE-DIMENSIONAL KELLER-SEGEL-STOKES SYSTEM INVOLVING TENSOR-VALUED SENSITIVITY WITH SATURATION

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Abstract. This paper deals with a boundary-value problem for a coupled chemotaxis-Stokes system with logistic source

\[
\begin{aligned}
\partial_t n + u \cdot \nabla n &= \nabla \cdot (D(n) \nabla n) - \nabla \cdot (n \mathcal{S}(x, n, c) \cdot \nabla c) + \xi n - \mu n^2, & x \in \Omega, & t > 0, \\
\partial_t c + u \cdot \nabla c &= \Delta c - c + n, & x \in \Omega, & t > 0, \\
\partial_t u + \nabla P &= \Delta u + n \nabla \phi, & x \in \Omega, & t > 0, \\
\nabla \cdot u &= 0, & x \in \Omega, & t > 0,
\end{aligned}
\]

in three-dimensional smoothly bounded domains, where the parameters \( \xi \geq 0, \mu > 0 \) and \( \phi \in W^{1,\infty}(\Omega) \), \( D \) is a given function satisfying \( D(n) \geq C_D n^{m-1} \) for all \( n > 0 \) with \( m > 0 \) and \( C_D > 0 \). \( \mathcal{S} \) is a given function with values in \( \mathbb{R}^{3 \times 3} \) which fulfills

\[|\mathcal{S}(x, n, c)| \leq C_S (1 + n) - \alpha\]

with some \( C_S > 0 \) and \( \alpha > 0 \). It is proved that for all reasonably regular initial data, global weak solutions exist whenever \( m + 2\alpha > \frac{6}{5} \). This extends a recent result by Liu et al. (J. Diff. Eqns, 261 (2016) 967-999) which asserts global existence of weak solutions under the constraints \( m + \alpha > \frac{5}{4} \) and \( m \geq \frac{1}{4} \).

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1. Introduction. Chemotaxis is one particular mechanism responsible for some instances of such demeanor, where the organism like bacteria, adapts its movement according to the concentrations of a chemical signal. The chemotaxis-fluid Keller-Segel-Navier-Stokes model

\[
\begin{aligned}
&n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (T(\nabla n)), & x \in \Omega, & t > 0, \\
c_t + u \cdot \nabla c = \Delta c - c + n, & x \in \Omega, & t > 0, \\
u_t + \kappa(u \cdot \nabla u) = \Delta u - \nabla P + n \nabla \phi, & x \in \Omega, & t > 0, \\
\nabla \cdot u = 0, & x \in \Omega, & t > 0
\end{aligned}
\tag{1}
\]

arises in the modeling of bacterial populations, like Escherichia coli, in which the cells live in a viscous fluid so that cells and chemical substrates are transported with fluid, and that the motion of the fluid is under influence of gravitational forcing generated by aggregation of cells [40]. In addition, the model (1) is also introduced in [20, 21] to study the coral broadcast spawning, where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary. In this setting, \( n = n(x, t) \), \( c = c(x, t) \), \( u = u(x, t) \) and \( P = P(x, t) \) denote the cell population density, the chemical concentration, the fluid velocity and the associated pressure, respectively. The coefficient \( \kappa \) is related to the strength of nonlinear fluid convection. If the fluid flow is relatively slow, we can use the Stokes equation instead of the Navier-Stokes equation.

If all effects of fluid flow are ignored by letting \( u \equiv 0 \), model (1) can be reduced to quasilinear chemotaxis model

\[
\begin{aligned}
n_t = \nabla \cdot (D(n)\nabla n) - \nabla \cdot (nS(x, n, c)\nabla c), & x \in \Omega, & t > 0, \\
c_t = \Delta c - c + n, & x \in \Omega, & t > 0
\end{aligned}
\tag{2}
\]

which as an important variant of the classical chemotaxis Keller-Segel model [19] was proposed by Painter and Hillen [31] to model chemotaxis of cell populations. The signal is produced by the cells. The results about the chemotaxis model (2) appear to be rather complete, which concentrates on the problem (2) whether the solutions are global bounded or blow-up (see [1, 4, 5, 7, 8, 29, 11, 12, 13, 14, 15, 17, 18, 10, 30, 31, 41, 50, 51, 52, 54, 55] and references therein for detailed results). Moreover, for system (2) with logistic term \( f(n) = n - \mu n^2 \) and \( D(n) \equiv 1, S(x, n, c) \equiv \chi \), it is known that an arbitrarily small \( \mu > 0 \) can guarantee the boundedness of solutions in \( N = 2 \) [30], while for \( N > 2 \) solutions may blow up in finite time [10, 50]. In particular, for \( N > 2 \), an appropriately large \( \mu \) (as compared to the chemotactic coefficient \( \chi \)) can exclude unbounded solutions [52, 53]. It is known from these results that the main elements which determine the solution behavior are spatial dimension and the total mass of cells. In addition to considering the logistic dampening term, the nonlinear variants of chemotactic sensitivity \( S = S(n, c) \) ([2, 11, 12]) and diffusivity ([7, 22, 60]) have also been identified to prevent finite-time blow-up. From these works, it can be observed that the three different version of chemotaxis sensitivities: the signal-dependent sensitivity \( S = S(n, c) = \frac{C_S}{c} \) or \( \frac{C_S}{(1+\delta c^\delta)} \) with \( C_S > 0 \) and \( \delta > 0 \) (see [9, 54]) which reflects the inhibition of cell movement in the location of the high signal concentration [11, 23], the \( n \)-dependent sensitivity \( S = n^\delta \) or \( n(n+1)^\delta \) with \( q \in \mathbb{R} \) [14, 36] which shows the volume-filling effect in the process of chemotaxis [31] and the tensor-valued sensitivity \( S = (S_{ij})_{2 \times 2} \) with \( S_{ij} \in C^2(\Omega \times [0, \infty) \times [0, \infty)) \) for \( i, j \in (1, 2) \) [3, 24] which describes the rotational chemotactic migration happening close to the physical boundary of the domain [61, 62] have been deeply investigated by Cieslak and Stinner [5, 6], Tao and Winkler [36, 59] and Zheng et al. [64, 66]. An important discovery is that the system (2) with tensor-valued sensitivity loses some
energy structure as compared to that with scalar-valued sensitivities which leads to considerable difficulties in the mathematical analysis.

Recently, there have been increasing biological and mathematical interest in mathematical of the situation that the diffusion of bacteria (or, more generally, of cells) in a viscous fluid may be viewed like movement in a porous medium. Adjusting the above model accordingly, we shall subsequently consider the quasilinear Keller-Segel-Navier-Stokes system

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \nabla \cdot (D(n) \nabla n) - n \mathcal{S}(x, n, c) \nabla c + \xi n - \mu n^2, & x \in \Omega, & t > 0, \\
    c_t + u \cdot \nabla c &= \Delta c - c + n, & x \in \Omega, & t > 0, \\
    u_t + \kappa (u \cdot \nabla u) &= \Delta u - \nabla P + n \nabla \phi, & x \in \Omega, & t > 0, \\
    \nabla \cdot u &= 0,
\end{align*}
\]

(3)

where \( n, c, u, P, \phi \) and \( \kappa \) are mentioned as before. The function \( \mathcal{S} \) measures the chemotactic sensitivity which may depend on \( n \) and \( D(n) \) is the diffusion function. The bacteria may proliferate following a logistic law with \( \xi \geq 0 \) and \( \mu > 0 \). If \( D(n) \equiv 1 \) in system (3) without logistic source, the global boundedness of classical solutions to the Stokes-version of system (3) with the tensor-valued \( \mathcal{S} = \mathcal{S}(x, n, c) \) satisfying \( |\mathcal{S}(x, n, c)| \leq \frac{CS}{(1+n)^\alpha} \) with some \( CS > 0 \) and \( \alpha > 0 \) which implies that the effect of chemotaxis is weakened when the cell density increases has been proved for any \( \alpha > 0 \) in two dimensions [44] and for \( \alpha > \frac{1}{2} \) in three dimensions [45]. If the signal-dependent functions \( \mathcal{S} \) fulfilling \( \mathcal{S}(n, c) \leq \frac{CS}{(1+\beta c)} \) with \( \beta > 0 \), Liu et al. [27] proved that the two-dimensional Stokes-version of system (3) possesses a unique globally bounded classical solution and obtained a global weak solution for three-dimensional Navier-Stokes-version of system (3). For the system (3) with \( D(n) \equiv 1 \) and \( \xi, \mu > 0 \), Tao and Winkler [35] proved that the three-dimensional Stokes-version of system (3) with \( \mathcal{S}(x, n, c) \equiv CS > 0 \) and \( n \)–equation in (3) with a given external force \( g = g(x, t) \) possesses a globally classical solution for \( \mu > 23 \) and the solutions of system (3) decay to zero for the case \( \xi = 0 \), while for any \( \mu > 0 \) the analogous conclusion is obtained in the two-dimensional chemotaxis-Navier-Stokes-version of system (3) [34]. We further note that, with some exceptions such as [28, 65], the result on global boundedness and large time behavior properties for the variant of (3) with nonlinear diffusion and nonlinear cross-diffusion is absent. More related results are obtained (see [25, 26, 39, 43, 47, 68]).

It is noted that the second equation in system (1) describes the situation where the chemical signal is consumed and also secreted by cells, while if the signal is only consumed by the cells and the diffusion of bacteria in a viscous fluid may be viewed like movement in a porous medium, then system (1) is transformed into

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \nabla \cdot (D(n) \nabla n) - \nabla \cdot (n \mathcal{S}(x, n, c) \nabla c), & x \in \Omega, & t > 0, \\
    c_t + u \cdot \nabla c &= \Delta c - n f(c), & x \in \Omega, & t > 0, \\
    u_t + \kappa (u \cdot \nabla u) &= \Delta u - \nabla P + n \nabla \phi, & x \in \Omega, & t > 0, \\
    \nabla \cdot u &= 0,
\end{align*}
\]

(4)

which was initially proposed by Tuval et al. [40] to describe the dynamics of the cell concentration, oxygen concentration and fluid velocity. To be more precise, they observed the large-scale convection patterns in a water drop through the fluid-air interface. Here \( c \) denotes the oxygen concentration, \( f(c) \) is the consumption rate of the oxygen by the cells and \( \phi \) is mentioned above. As to the mathematical analysis of system (4), numerous results on global existence and boundedness properties have been obtained for the variant of (4) obtained on assuming that...
\(\nabla \cdot (D(n) \nabla n)\) is linear diffusion \(\Delta n\) [16, 56, 57, 58]. Certain natural quasi-Lyapunov functional on the logarithmic entropy \(\int_{\Omega} n \ln n\) guarantees the global solvability of system (4) under some suitable structural hypothesis on \(\mathcal{S}\) and \(f\) [56, 57] and also allows for the construction that the solutions of the two-dimensional Navier-Stokes-version stabilize to the spatially homogeneous equilibrium \((\overline{n}_0, 0, 0)\) in the large time, where \(\overline{n}_0 := \frac{1}{\Omega} \int_{\Omega} n_0 > 0\) (see [58]). However, for the two-dimensional and three-dimensional (Navier-)Stokes-version system (4) with a tensor-valued sensitivity \(\mathcal{S}\) as mentioned before, the energy-based reasoning does not guarantee the existence of global solution. Therefore, the method of the combinational functional \(\int_{\Omega} n^p + \int_{\Omega} |\nabla c|^2\) are developed by relaxing its excessive dependence on the inflexible structural assumptions on \(\mathcal{S}\) and \(f\) such that \((\frac{p}{2})^\alpha \leq 0\) on \((0, \infty)\) for the two-dimensional system (4) with tensor-valued \(\mathcal{S}\) [16]. If the bacteria diffuse in a porous medium (i.e. \(D(n) = mn^{-m-1}\)), \(\mathcal{S}\) is the identity matrix \(I\) and \(f(c) = c\), it is shown that the three-dimensional Navier-Stokes-version of system (4) possesses at least one global weak solution for \(m \geq \frac{2}{3}\) [63] and that the three-dimensional chemotaxis-Stokes system (4) possesses at least one global weak solution [56]. As for the case of degenerate cell diffusion of porous medium type, Tao and Winkler [37, 38] proved that the two-dimensional degenerate-chemotaxis-Stokes system possesses a bounded global weak solution for \(m > 1\) and the three-dimensional degenerate-chemotaxis-Stokes has a locally bounded global weak solution for \(m > \frac{5}{4}\). Besides that, Ishida [16] proved that the two-dimensional (Navier-)Stokes-version system (4) with rotational flux \(\mathcal{S}(x, n, c)\) fulfilling \(|\mathcal{S}(x, n, c)| \leq S_0(c)|\) for all \((x, n, c) \in \Omega \times [0, \infty)\) with some nondecreasing \(S_0 : [0, \infty) \to \mathbb{R}\) admits a bounded global weak solution. Moreover, for general \(f(c)\), if \(D\) satisfies \(D \in C_{\text{loc}}^0((0, \infty))\) with some \(\alpha > 0\) as well as \(D(n) \geq C_D n^{-m-1}\) for all \(n \geq 0\) with \(m > \frac{2}{\alpha}\) and \(C_D > 0\), and \(\mathcal{S} \in C^2(\Omega \times [0, \infty)^2; \mathbb{R}^{3 \times 3})\) fulfilling \(|\mathcal{S}(x, n, c)| \leq S_0(c)|\) for all \((x, n, c) \in \Omega \times [0, \infty)\) with some nondecreasing \(S_0 : [0, \infty) \to \mathbb{R}\), Winkler [48] developed an alternative a priori estimates to obtain that the three-dimensional chemotaxis-Stokes system (4) possesses at least one bounded weak solution which stabilizes to the spatially homogeneous equilibrium \((\overline{n}_0, 0, 0)\) with \(\overline{n}_0\) as mentioned above as \(t \to \infty\). Very recently, it is shown [46] that the three-dimensional chemotaxis-Stokes system (4) with \(\mathcal{S}(x, n, c)\) fulfilling

\[
|\mathcal{S}(x, n, c)| \leq \frac{C_S}{(1 + n)^\alpha}
\]

with some \(C_S > 0\) and \(\alpha > 0\) possesses global weak solutions for \(m + \alpha > \frac{7}{6}\). More results about Cauchy problem are also obtained (see [32, 67]).

As compared to the Navier-Stokes-chemotaxis (3) with the linear diffusion and logistic source, Liu and Wang [28] recently obtained global boundedness and decay property for the following three-dimensional Keller-Segel-Stokes system involving tensor-valued sensitivity \(\mathcal{S}\) with some nondecreasing \(S\) and \(f(c) = c\):

\[
\begin{aligned}
\begin{cases}
\begin{aligned}
\n_t + u \cdot \nabla n = \nabla \cdot (D(n) \nabla n) - \nabla \cdot (nS(x, n, c) \cdot \nabla c) + \xi n - \mu n^2, \\
\c_t + u \cdot \nabla c = \Delta c - c + n, \\
\phi_t = \Delta \phi - \nabla P + n \nabla \phi, \\
\n \cdot u = 0, \\
n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x), \quad u(x, 0) = u_0(x),
\end{aligned}
\end{cases}
\end{aligned}
\]

for all \(x \in \Omega\), \(t > 0\).
Due to the presence of the tensor-valued sensitivity, the corresponding chemotaxis-Stokes system (6) loses some energy structure, which gives rise to considerable mathematical difficulties. The authors [28] derived a prior estimates for \( |n(\cdot, t)||_{L^p(\Omega)} \) and \( ||\nabla v(\cdot, t)||_{L^q(\Omega)} \) for all \( p, q > 1 \) by using an energy-like inequality. The main step is to estimate the two chemotaxis-related integral terms \( \int_{\Omega} n^{p+1-m-2\alpha}(\cdot, t)||\nabla c(\cdot, t)||^2dx \) and \( \int_{\Omega} n^2(\cdot, t)||\nabla c(\cdot, t)||^{2q-2}dx \) for all \( m > 0 \) and \( \alpha > 0 \). As compared to the methods in [28], we take a slightly different approach and apply a variant of the Gagliardo-Nirenberg inequality to control a part of the two chemotaxis-related integral terms as mentioned above by the integral term \( \int_{\Omega} n^{p+1}(\cdot, t)dx \) instead of the integral term \( \int_{\Omega} ||\nabla n||^{\alpha m+\alpha}(\cdot, t)dx \).

Main results. Accordingly, the goal of the present work is mainly devoted to studying these questions for the three-dimensional Keller-Segel-Stokes system (6) with the tensor-valued sensitivity, and to give somewhat complete answer with regard to global existence, boundedness for general \( \xi, \mu > 0 \). The core step is still to obtain the upper bound of the functional

\[
y(t) := \int_{\Omega} n^p(\cdot, t) + \int_{\Omega} ||\nabla c(\cdot, t)||^{2q} \text{ for any } p > 1 \text{ and } q > 1,
\]

where \( n_e \) and \( c_e \) are components of the solutions to (17) below. In order to formulate our main results in this direction, let us specify the precise evolution problem addressed in the sequel by considering (6) along with the initial data

\[
n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x), \quad \text{and } u(x, 0) = u_0(x), \quad x \in \Omega \quad (8)
\]

and the boundary conditions

\[
\left( D(n)\nabla n - nS(x, n, c) \cdot \nabla c \right) \cdot \nu = 0, \quad \partial c / \partial \nu = 0 \text{ and } u = 0, \quad x \in \partial \Omega.
\]

Furthermore, we suppose that the diffusion coefficient \( D \) fulfills

\[
D \in C^\theta_{\text{loc}}((0, \infty)) \text{ for some } \theta > 0 \quad (10)
\]

as well as

\[
D(n) \geq C_D n^{m-1} \text{ for all } n > 0 \quad (11)
\]

with \( m > 0 \) and \( C_D > 0 \). As to the initial data, for simplicity we shall require throughout this paper that

\[
\begin{cases}
  n_0 \in C^\iota(\Omega) \text{ for certain } \iota > 0 \text{ with } n_0 \geq 0 \text{ in } \Omega, \\
  c_0 \in W^{1,\infty}(\Omega) \text{ fulfills } c_0 \geq 0 \text{ in } \Omega, \\
  u_0 \in D(A_r^\delta) \text{ for certain } \delta \in (\frac{3}{2}, 1) \text{ and any } r \in (1, \infty)
\end{cases}
\]

with \( A_r \) standing for the Stokes operator with domain \( D(A_r) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L^r_\sigma(\Omega) \), where \( L^r_\sigma(\Omega) := \{ \psi \in L^r(\Omega) | \nabla \cdot \psi = 0 \} \) for \( r \in (1, \infty) \).

Within the above framework, our main results concerning global existence and boundedness of solutions to (6) are stated as follows:

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary, and let \( \xi \geq 0 \) and \( \mu > 0 \). Assume that \( D \) and \( S \) fulfill (10)-(11) and (5) with \( m + 2\alpha > \frac{6}{\xi} \). Then for any \( (n_0, c_0, u_0) \) satisfying (12), (6) admits at least one global weak solution \( (n, c, u, P) \) in the sense of Definition 4.1 below. Also, this solution is bounded in \( \Omega \times (0, \infty) \) in the sense that

\[
||n(\cdot, t)||_{L^\infty(\Omega)} + ||c(\cdot, t)||_{W^{1,\infty}(\Omega)} + ||u(\cdot, t)||_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t > 0.
\]
with some positive constants $C$. In addition, $c$ and $u$ are continuous in $\bar{\Omega} \times (0, \infty)$, and $n$ as an $L^\infty(\Omega)$-valued function is continuous on $[0, \infty)$ with respect to the weak-* topology, i.e.,

$$n \in C^0_w([0, \infty); L^\infty(\Omega)).$$

**Remark 1.** We note that model (6) exists a global solution provided that $m + 2\alpha > \frac{6}{5}$, which implies that diffusivity and nonlinear variant of chemotactic sensitivity in (5) can prevent finite-time blow-up. Theorem 1.1 extends a recent result by Liu et al. [28] which asserts global existence of weak solutions under the constraints $m + \alpha > \frac{6}{5}$ and $m \geq \frac{1}{3}$.

The paper is organized as follows. After introducing the regularized system of (6) and collecting some basic estimates of the solutions in Section 2, we shall derive an upper bound for (7) by using a slightly different approach in Section 3. In Section 4, we construct the bounded global weak solution of (6) by passing to the limit in a standard manner on the basis of the previously established estimates.

**2. Preliminary.** In this section, we shall first deal with some boundary regularized approximate problem to overcome the difficulties brought by the nonlinear boundary condition in this section. As done in [24], we introduce an appropriate regularization in which $S$, defined below vanishes near the lateral boundary.

Next we approximate the diffusion coefficient function in (11) by a family $(D_\epsilon)_{\epsilon \in (0,1)}$ of functions

$$D_\epsilon \in C^2((0, \infty)) \text{ such that } D_\epsilon(n) \geq \epsilon \text{ for all } n > 0 \text{ and }$$

$$D(n) \leq D_\epsilon(n) \leq D(n) + 2\epsilon \text{ for all } n > 0 \text{ and } \epsilon \in (0, 1). \quad (15)$$

Next, we let $(\rho_\epsilon)_{\epsilon \in (0,1)} \subset C_0^\infty(\Omega)$ be a family of standard cut-off functions satisfying $0 \leq \rho_\epsilon \leq 1$ in $\Omega$ and $\rho_\epsilon \nearrow 1$ as $\epsilon \downarrow 0$ in $\Omega$, and define

$$S_\epsilon(x, n, c) = \rho_\epsilon(x)S(x, n, c), \quad x \in \bar{\Omega}, n \geq 0, c \geq 0 \quad (16)$$

for $\epsilon \in (0, 1)$ to approximate the sensitivity tensor $S$, which implies that $S_\epsilon(x, n, c) = 0$ on $\partial\Omega$ for each fixed $\epsilon \in (0, 1)$. Therefore, the regularized problem of (6) is presented as follows

\[
\begin{align*}
&n_{\epsilon t} + u_\epsilon \cdot \nabla n_\epsilon = \nabla \cdot (D_\epsilon(n_\epsilon)\nabla n_\epsilon) = \nabla \cdot (n_\epsilon S_\epsilon(x, n_\epsilon, c_\epsilon) \cdot \nabla c_\epsilon) + \xi n_\epsilon - \mu n_\epsilon^2, \\
&c_{\epsilon t} + u_\epsilon \cdot \nabla c_\epsilon = \Delta c_\epsilon - c_\epsilon + n_\epsilon, \\
&u_{\epsilon t} + \nabla P_\epsilon = \Delta u_\epsilon + n_\epsilon \nabla \phi, \\
&\partial_\nu c_\epsilon = 0, \quad \partial_\nu (n_\epsilon - \rho_\epsilon) = 0, \\
&n_\epsilon(x, 0) = n_0(x), \quad c_\epsilon(x, 0) = c_0(x), \quad u_\epsilon(x, 0) = u_0(x), \quad x \in \Omega.
\end{align*}
\]

Next we will state the local solvability of system (17), which can be proved by a straightforward adaption of the corresponding procedures in Lemma 2.1 of [33] to our current setting.

**Lemma 2.1.** Assume that $\epsilon \in (0, 1)$. Then, $T_{\text{max, } \epsilon} \in (0, \infty)$ and a classical solution $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ of (17) in $\Omega \times (0, T_{\text{max, } \epsilon})$ exist such that

\[
\begin{align*}
n_\epsilon &\in C^0([0, T_{\text{max, } \epsilon}) \cap C^{2,1}((0, T_{\text{max, } \epsilon})), \\
c_\epsilon &\in C^0([0, T_{\text{max, } \epsilon}) \cap C^{2,1}((0, T_{\text{max, } \epsilon})), \\
u_\epsilon &\in C^0([0, T_{\text{max, } \epsilon}) \cap C^{2,1}((0, T_{\text{max, } \epsilon})), \\
P_\epsilon &\in C^{1,0}([0, T_{\text{max, } \epsilon})).
\end{align*}
\]
by classically solving in $\Omega \times [0, T_{\text{max},\epsilon})$. Moreover, $n_\epsilon$ and $c_\epsilon$ are nonnegative in $\Omega \times (0, T_{\text{max},\epsilon})$, and

$$
\sup_{t \to T_{\text{max}}} \|n_\epsilon(\cdot, t)\|_{L^\infty(\Omega)} + ||c_\epsilon(\cdot, t)||_{W^{1,\infty}(\Omega)} + ||A^0 u_\epsilon(\cdot, t)||_{L^2(\Omega)} = \infty,
$$

(19)

where $\delta$ is defined in (12).

Due to the presence of logistic source, some useful estimates for $n_\epsilon$, $c_\epsilon$ and $u_\epsilon$ can be derived from Lemma 2.2-Lemma 2.6 in [35].

**Lemma 2.2.** Let $(n_\epsilon, c_\epsilon, u_\epsilon, \rho_\epsilon)$ be the solution of (17). Then there exists a positive constant $C > 0$ independent of $\epsilon$ such that

$$
\|n_\epsilon(\cdot, t)\|_{L^1(\Omega)} + ||\nabla u_\epsilon(\cdot, t)||_{L^2(\Omega)} + ||c_\epsilon(\cdot, t)||_{W^{1,2}(\Omega)} \leq C
$$

(20)

for all $t \in (0, T_{\text{max},\epsilon})$.

**Proof.** By Lemma 2.2-Lemma 2.6 in [35], we can find a positive constant $C_1$ such that

$$
\int_\Omega n_\epsilon(\cdot, t) + \int_\Omega |\nabla u_\epsilon(\cdot, t)|^2 + \int_\Omega c_\epsilon(\cdot, t) + \int_\Omega |\nabla c_\epsilon(\cdot, t)|^2 \leq C_1
$$

for all $t \in (0, T_{\text{max},\epsilon})$,

thus there exists a constant $C_2 > 0$ such that $\int_\Omega c_\epsilon^2(\cdot, t) \leq C_2$ for all $t \in (0, T_{\text{max},\epsilon})$, which is a direct consequence of the Poincaré inequality. This proves (20). \qed

According to Corollary 3.4 in [48], we can establish the $\|\cdot\|_{L^\infty(\Omega)}$ estimates for $Du_\epsilon$ in the sequel. Here we only state it as a lemma, while for the detailed proof readers can refer to [48].

**Lemma 2.3.** Let $p \in [1, \infty)$ and $r \in [1, \infty)$ be such that

$$
\begin{cases}
  r < \frac{3p}{p+1} & \text{if } p \leq 3, \\
  r \leq \infty & \text{if } p > 3.
\end{cases}
$$

(21)

Then for all $K > 0$ there exists $C = C(p, r, K, u_0, \phi)$ such that if for some $\epsilon \in (0, 1)$ we have

$$
\|n_\epsilon\|_{L^r(\Omega)} \leq K \text{ for all } t \in (0, T_{\text{max},\epsilon}),
$$

(22)

then

$$
\|Du_\epsilon\|_{L^r(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max},\epsilon}).
$$

(23)

3. **A priori estimates.** In this section, we shall propose some regularity estimates for $n_\epsilon$ and $c_\epsilon$ by tracking the time evolution of a certain combinational functional of them.

In order to establish some estimates for the coupled functional in (7), we first recall the following two lemmas proved in [28].

**Lemma 3.1.** Let $p > 1$. Then for all $\epsilon \in (0, 1)$,

$$
\frac{1}{p} \int_\Omega \frac{d}{dt} \int_\Omega n_\epsilon^p + \frac{(p - 1)CD}{2} \int_\Omega n_\epsilon^{p+m-3}|\nabla n_\epsilon|^2 \leq \frac{(p - 1)C_S^2}{2CD} \int_\Omega n_\epsilon^{p+1-m-2\alpha}|\nabla c_\epsilon|^2 \\
+ \xi \int_\Omega n_\epsilon^p - \mu \int_\Omega n_\epsilon^{p+1}
$$

(24)

for all $t \in (0, T_{\text{max},\epsilon})$, where $C_S$ and $C_D$ are defined in (5) and (11), respectively.
Lemma 3.2. Let $q > 1$. Then for all $\epsilon \in (0, 1)$, there exists a positive constant $C_1$ independent of $\epsilon$ such that

$$
\frac{1}{2q} \int_\Omega |\nabla c_\epsilon|^{2q} + \frac{q - 1}{q^2} \int_\Omega |\nabla |\nabla c_\epsilon|^q|^2 + \frac{1}{2} \int_\Omega |\nabla c_\epsilon|^{2q-2}|D^2 c_\epsilon|^2
\leq \frac{(2q - 2 + \sqrt{3})^2}{2} \int_\Omega n_\epsilon^2 |\nabla c_\epsilon|^{2q-2} + \int_\Omega |\nabla c_\epsilon|^{2q} |Du_\epsilon| + C_1 \quad \text{for all } t \in (0, T_{\max})
$$

(25)

Next we can estimate the integrals on the right-hand sides of (24) and (25) by taking a slightly different approach from [28].

Lemma 3.3. Assume that $p > 1$ and $q = \frac{p+1}{m+2\alpha} > 1$. Then for all $\eta_1 > 0$, we can find a constant $C_5 := C_5(p, q, \eta_1) > 0$ such that for all $\epsilon \in (0, 1)$

$$
\int_\Omega n_\epsilon^{p+1-m-2\alpha} |\nabla c_\epsilon|^2 \leq \eta_1 \int_\Omega n_\epsilon^{p+1} + \eta_1 \int_\Omega |\nabla |\nabla c_\epsilon|^q|^\frac{\lambda_1}{2} + C_9
$$

(26)

for all $t \in (0, T_{\max}).$

Proof. Picking $\frac{p+1}{p+1-m-2\alpha}$ and $q = \frac{p+1}{m+2\alpha}$ as exponents, we employ the Hölder inequality once more and then

$$
\int_\Omega n_\epsilon^{p+1-m-2\alpha} |\nabla c_\epsilon|^2 \leq \left( \int_\Omega n_\epsilon^{p+1} \right)^{\frac{p+1-m-2\alpha}{p+1}} \left( \int_\Omega |\nabla c_\epsilon|^{2q} \right)^{\frac{1}{q}}
$$

(27)

for all $t \in (0, T_{\max})$. Then we can apply the Gagliardo-Nirenberg inequality [49] and (20) to find positive constants $C_i = C_i(q) (i = 2, 3, 4)$ such that

$$
\left( \int_\Omega |\nabla c_\epsilon|^{2q} dx \right)^{\frac{1}{q}} = |||\nabla c_\epsilon|||^{\frac{2}{q}}_{L^2(\Omega)}
\leq C_2 |||\nabla |\nabla c_\epsilon|^q|||^{\frac{2\lambda_1}{L^2(\Omega)}}|||\nabla c_\epsilon|||^\frac{1}{q} |||\nabla c_\epsilon|||^\frac{2(1-\lambda_1)}{L^2(\Omega)} + C_2 |||\nabla c_\epsilon|||^{\frac{2\lambda_1}{L^2(\Omega)}} + C_4
$$

(28)

for all $t \in (0, T_{\max})$, where $\lambda_1 = \frac{\frac{2}{q} + \frac{1}{2}}{\frac{2}{q} + \frac{1}{2}} < 1$. Since $\lambda_1 \in (0, 1)$, we know that

$$
\frac{p+1-m-2\alpha}{p+1} + \frac{\lambda_1}{q} < 1.
$$

(29)

Thus (28) along with (27) warrants that

$$
\int_\Omega n_\epsilon^{p+1-m-2\alpha} |\nabla c_\epsilon|^2 \leq \left( \int_\Omega n_\epsilon^{p+1} \right)^{\frac{p+1-m-2\alpha}{p+1}} \left( C_3 |||\nabla |\nabla c_\epsilon|^q|||^{\frac{2\lambda_1}{L^2(\Omega)}} + C_4 \right)
$$

$$
= C_3 \left( \int_\Omega n_\epsilon^{p+1} \right)^{\frac{p+1-m-2\alpha}{p+1}} \left( \int_\Omega |\nabla |\nabla c_\epsilon|^q|^\frac{\lambda_1}{q} \right) + C_4 \left( \int_\Omega n_\epsilon^{p+1} \right)^{\frac{p+1-m-2\alpha}{p+1}}
$$

(30)
for all \( t \in (0, T_{\text{max}, \varepsilon}) \). Next we may use the version of Young’s inequality (see [38, Lemma 2.5]) to see that given \( \eta_2 > 0 \) we can find \( C_5 = C_5(p, q, \eta_2) > 0 \) fulfilling
\[
\int_{\Omega} n^{p+1-m-2\alpha} |\nabla c_\varepsilon|^2 \leq \eta_1 \int_{\Omega} n^{p+1} + \eta_2 \int_{\Omega} |\nabla|\nabla c_\varepsilon|\|^2 + C_5
\]
for all \( t \in (0, T_{\text{max}, \varepsilon}) \), this establishes (26).

Next, in quite a similar manner, we can also estimate the first term on the right hand of (25).

**Lemma 3.4.** Let \( m + 2\alpha > \frac{6}{5}, \quad p > 1 \) and \( q = \frac{p+1}{m+2\alpha} \geq 2 \) such that
\[
p > \max\left\{ 2 - \frac{m + 2\alpha}{2}, \frac{6 - (m + 2\alpha)}{5(m + 2\alpha) - 6} \right\}.
\]
Then for all \( \eta_2 > 0 \) there exists \( C_{13} = C_{13}(p, q, \eta_2) > 0 \) such that for some \( \varepsilon \in (0, 1) \) we have
\[
\int_{\Omega} n^2 |\nabla c_\varepsilon|^{2q-2} \leq \eta_2 \int_{\Omega} n^{p+1} + \eta_2 \int_{\Omega} |\nabla|\nabla c_\varepsilon|\|^2 + C_{13}
\]
for all \( t \in (0, T_{\text{max}, \varepsilon}) \).

**Proof.** By virtue of the Holder’s inequality applied with exponents with \( \frac{p+1}{2} \) and \( \eta = \frac{p+1}{p-1} \), we have
\[
\int_{\Omega} n^2 |\nabla c_\varepsilon|^{2q-2} \leq \left( \int_{\Omega} n^{p+1} \right)^{\frac{p}{p+1}} \left( \int_{\Omega} |\nabla c_\varepsilon|^{2(q-1)\eta} \right)^{\frac{1}{\eta}}.
\]
Here we observe that the inequality \( p > 2 - \frac{m + 2\alpha}{2} \) in particular ensures that
\[
6(q(p-1) - 2(q-1)(p+1)) = \frac{6(p+1)(p-1) - 2(p+1-m-2\alpha)(p+1)}{m+2\alpha} = \frac{(p+1)(4p-8+2m+4\alpha)}{m+2\alpha} > 0
\]
and hence \( \frac{2(q-1)(p+1)}{q(p-1)} < 6 \). In view of the Gagliardo-Nirenberg inequality (see [49]) for a version and the boundedness of \( \nabla c_\varepsilon \) with respect to \( L^2 \)-norm, we can find some constants \( C_i = C_i(q) > 0 \) \( (i = 6, 7, 8) \) such that
\[
\left( \int_{\Omega} |\nabla c_\varepsilon|^{2(q-1)\eta} \right)^{\frac{1}{\eta}} = \left\| |\nabla c_\varepsilon|\right\|^{\frac{2q-2}{q} L_{\frac{\alpha}{q-1}(q-1-\lambda_2)}(\Omega)} \\
\leq C_6 \left\| |\nabla|\nabla c_\varepsilon|\right\|^{\frac{2(q-1)\lambda_2}{2q-2} L_{\frac{\alpha}{q-1}(q-1-\lambda_2)}(\Omega)} \\
\leq C_7 \left\| |\nabla|\nabla c_\varepsilon|\right\|^{\frac{2(q-1)\lambda_2}{2q-2} L_{\frac{\alpha}{q-1}(q-1-\lambda_2)}(\Omega)} + C_8 \left\| |\nabla c_\varepsilon|\right\|^{\frac{2q-2}{q} L_{\frac{\alpha}{q-1}(q-1-\lambda_2)}(\Omega)}
\]
for all \( t \in (0, T_{\text{max}, \varepsilon}) \).
where \( \lambda_2 = \frac{q-1}{p+1} \) and thus \( \frac{2}{p+1} + \frac{(q-1)\lambda_2}{q} < 1 \). In light of \((33)-(35)\), we may use the version of Young's inequality (see [38, Lemma 2.5]) once more to see that for any \( \eta > 0 \) we can find \( C_1 = C(\eta) > 0 \) \((i = 9, 10)\) fulfilling

\[
\int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2q-2} \leq \left( \int_\Omega n_\varepsilon^{p+1} \right)^{\frac{1}{p+1}} \left( C_7 ||\nabla|\nabla c_\varepsilon|^{q}||_{L^2(\Omega)} \right)^{\frac{q-1}{q}} + C_8 \\
\leq C_9 \left( \int_\Omega n_\varepsilon^{p+1} \right)^{\frac{1}{p+1}} \left( \int_\Omega |\nabla|\nabla c_\varepsilon|^{q}||^{\frac{q-1}{q}} \right)^{\frac{q-1}{q}} \\
+ C_9 \left( \int_\Omega n_\varepsilon^{p+1} \right)^{\frac{1}{p+1}} \\
\leq \eta \int_\Omega n_\varepsilon^{p+1} + \eta \int_\Omega |\nabla|\nabla c_\varepsilon|^{q}||^{\frac{q-1}{q}} + C_{10}
\]

for all \( t \in (0, T_{\text{max},\varepsilon}) \). The claimed inequality \((32)\) thus results from \((37)\). \( \square \)

**Lemma 3.5.** Assuming that \( q > 1 \). If

\[
||Dc_\varepsilon(\cdot, t)||_{L^2(\Omega)} \leq K \text{ for all } t \in (0, T_{\text{max},\varepsilon}).
\]

Then for all \( \eta > 0 \), the solution of \((17)\) satisfies

\[
\int_\Omega |Dc_\varepsilon| |\nabla c_\varepsilon|^{2q} \leq \eta \int_\Omega |\nabla|\nabla c_\varepsilon|^{q}||^{2} + C_{15} \text{ for all } t \in (0, T_{\text{max},\varepsilon}),
\]

where a positive constant \( C_{15} \) depends on \( q \) and \( \eta \).

**Proof.** By virtue of the Hölder inequality and \((38)\), we have

\[
\int_\Omega |\nabla c_\varepsilon|^{2q} \cdot |Dc_\varepsilon| \leq \left( \int_\Omega |Dc_\varepsilon|^{2} \right)^{\frac{1}{2}} \left( \int_\Omega |\nabla c_\varepsilon|^{4q} \right)^{\frac{1}{2}} \\
\leq K ||\nabla c_\varepsilon||_{L^{2q}(\Omega)}^{2q}.
\]

for all \( t \in (0, T_{\text{max},\varepsilon}) \). Since \( 4q \in [2, 6q] \), we apply the Gagliardo-Nirenberg inequality [49] and the boundedness of \( ||\nabla c_\varepsilon||_{L^2(\Omega)} \) in Lemma 2.2 to find \( C_i = C_i(q) > 0 \) \((i = 11, 12, 13)\) such that

\[
||\nabla c_\varepsilon||_{L^{2q}(\Omega)}^{2q} \leq C_{11} \left( ||\nabla|\nabla c_\varepsilon|^{q}||_{L^2(\Omega)}^{\frac{12q}{12q-6}} \ ||\nabla c_\varepsilon||_{L^2(\Omega)}^{\frac{2q}{12q-6}} + ||\nabla c_\varepsilon||_{L^2(\Omega)}^{\frac{2q}{12q-6}} \right)^{\frac{6q-3}{12q-6}} + C_{13}
\]

\[
\leq C_{12} \left( \int_\Omega |\nabla|\nabla c_\varepsilon|^{q}||^{2} \right)^{\frac{6q-3}{12q-6}} + C_{13}
\]
for all $t \in (0, T_{\text{max}, \epsilon})$, which together with the Young inequality provides a constant $C_{14} = C_{14}(q, \eta_3) > 0$ such that for any $\eta_3 > 0$

$$\|\nabla c_t\|_{L^q(\Omega)}^2 \leq \frac{\eta_3}{K} \int_{\Omega} |\nabla|\nabla c_t|^q|^2 + C_{14} \text{ for all } t \in (0, T_{\text{max}, \epsilon}).$$

(42)

Thus, (40) along with (42) yields that (39) holds.

Relying on the estimates established in Lemma 3.3-Lemma 3.5, we can obtain a boundedness for the functional in (7).

**Lemma 3.6.** Assume that $m + 2\alpha > \frac{6}{5}$ and $q = \frac{p+1}{m+2\alpha}$. Then for sufficiently large $p \geq 1$ satisfying $p > 2(m + 2\alpha) - 1$, one can find a constant $C > 0$ such that for all $\epsilon \in (0, 1)$

$$\|n_\epsilon(\cdot, t)\|_{L^p(\Omega)} + \|\nabla c_\epsilon(\cdot, t)\|_{L^{2q}(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}, \epsilon}).$$

(43)

**Proof.** If $m + 2\alpha > \frac{6}{5}$, an elementary calculation ensures that $2(m + 2\alpha) - 1 > 1$ and

$$2(m + 2\alpha) - 1 > \max \left\{ \frac{2 - m + 2\alpha}{2}, \frac{6 - (m + 2\alpha)}{5(m + 2\alpha) - 6} \right\}.$$  

(44)

Then we can choose sufficiently large $p \geq 1$ such that $p > 2(m + 2\alpha) - 1$. Now by taking all $\eta_i$ ($i = 1, 2, 3$) in Lemmas 3.3-3.5 sufficiently close to zero such that

$$\mu p - \left( \frac{p(p-1)C_5^2}{2C_D} \eta_1 + q(2q - 2 + \sqrt{3})^2 \eta_2 \right) > 0$$

(45)

and

$$\frac{2(q-1)}{q} - \left( \frac{p(p-1)C_5^2}{2C_D} \eta_1 + q(2q - 2 + \sqrt{3})^2 \eta_2 + 2q\eta_3 \right) > 0$$

(46)

hold for any $\mu > 0$ and substituting those inequalities (26), (32) and (39) into Lemmas 3.1-3.2, the solution of (17) satisfies

$$\frac{d}{dt} \left( \int_{\Omega} n_\epsilon^p + \int_{\Omega} |\nabla c_\epsilon|^{2q} \right) + \left( \int_{\Omega} n_\epsilon^p + \int_{\Omega} |\nabla c_\epsilon|^{2q} \right)$$

$$+ \frac{p(p-1)C_D}{2} \int_{\Omega} n_\epsilon^{p+m-3}|\nabla n_\epsilon|^2 + \left( \frac{2(q-1)}{q} - \mu_2 \right) \int_{\Omega} |\nabla|\nabla c_\epsilon|^q|^2$$

$$+ q \int_{\Omega} |\nabla c_\epsilon|^{2q-2} |D^2 c_\epsilon|^2 + \left( \mu p - \mu_1 \right) \int_{\Omega} n_\epsilon^{p+1}$$

$$\leq \left( \xi p + 1 \right) \int_{\Omega} n_\epsilon^p + \int_{\Omega} |\nabla c_\epsilon|^{2q} + C_{16} \text{ for all } t \in (0, T_{\text{max}, \epsilon})$$

(47)

with some constants $C_{16} > 0$, where $\mu_1 := \frac{p(p-1)C_5^2}{2C_D} \eta_1 + q(2q - 2 + \sqrt{3})^2 \eta_2$ and $\mu_2 := \frac{p(p-1)C_5^2}{2C_D} \eta_1 + q(2q - 2 + \sqrt{3})^2 \eta_2 + 2q\eta_3$. In view of (20), we can make use of the Gagliardo-Nirenberg inequality and the version of Young’s inequality [38, Lemma 2.5] to obtain some constants $C_i(q) > 0$ ($i = 17, 18$) fulfilling

$$\int_{\Omega} |\nabla c_\epsilon|^{2q} dx \leq C_{17} \left[ \int |\nabla|\nabla c_\epsilon|^q\right]^{\frac{q-6}{q-2}} + C_{17} \left[ \int |\nabla c_\epsilon|_{L^2(\Omega)} \right]^{\frac{4q}{q-2}}$$

$$\leq \eta_4 \int_{\Omega} |\nabla|\nabla c_\epsilon|^q|^2 + C_{18} \text{ for all } t \in (0, T_{\text{max}, \epsilon})$$

(48)
for all $\eta_4 > 0$. Furthermore, we can make use of the Young inequality to obtain
\[(\xi p + 1) \int_\Omega n_\epsilon^p \leq (\mu p - \mu_1) \int_\Omega n_\epsilon^{p+1} + C_{19}(\delta)\] (49)
for any $\delta > 0$ with some $C_{19}(\delta) > 0$. Choosing properly small $\eta_4 > 0$, we substitute (48)-(49) into (47) and can find $C_{20} > 0$ such that
\[\frac{d}{dt} \left( \int_\Omega n_\epsilon^p + \int_\Omega |\nabla c_\epsilon|^2 \right) + \left( \int_\Omega n_\epsilon^p + \int_\Omega |\nabla c_\epsilon|^2 \right) \leq C_{20}\] (50)
for all $t \in (0, T_{\text{max}, \epsilon})$, and thereby (50) implies (43) by an ODE comparison argument. We complete the proof.

With all above regularization properties of each component $n_\epsilon, c_\epsilon, u_\epsilon$ at hand, we can obtain the following boundedness results by invoking a Moser-type iteration (see [37, Lemma A.1]) and standard parabolic regularity arguments.

**Lemma 3.7.** Let $m + 2\alpha > \frac{6}{5}$. Moreover, $\delta$ is supposed to be as in (12). Then one can find $C > 0$ independent of $\epsilon \in (0, 1)$ such that the solutions of (17) fulfill
\[||n_\epsilon(\cdot,t)||_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}, \epsilon})\] (51)
and
\[||c_\epsilon(\cdot,t)||_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}, \epsilon})\] (52)
as well as
\[||u_\epsilon(\cdot,t)||_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}, \epsilon}).\] (53)
Moreover, we also have
\[||A^\delta u_\epsilon(\cdot,t)||_{L^2(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}, \epsilon}).\] (54)

**Proof.** In light of (23) with $p > 3$, we can infer from Lemma 2.3 with $r := \infty$ that $Du_\epsilon$ is bounded in $L^\infty(\Omega \times (0, T_{\text{max}, \epsilon}))$ and thereby (53) is valid. Taking the results of Lemma 3.6 with properly large $p$ and $q$ as a starting point, we apply a Moser-type iteration to the $n_\epsilon$-equation in (17) and then get (51). Next we can apply the well-known arguments from parabolic regularity theory to the $c_\epsilon$-equation in (17) to obtain (52) on the basis of (40) and (42) (see the reasoning of [14, Lemma 4.1]). Finally, similar to the proof of [27, Lemma 3.9], we can also obtain (54).

In view of (19) and Lemma 3.7, the local-in-time solution can be extended to the global-in-time solution.

**Proposition 1.** Let $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)_{\epsilon \in (0, 1)}$ be the classical solutions of (17) constructed in Lemma 2.1 on $[0, T_{\text{max}, \epsilon}]$. Then the solution is global on $[0, \infty)$. Proposition 1 allows for an extension of the outcome in Lemma 3.7 from $[0, T_{\text{max}, \epsilon})$ to $[0, \infty)$. Next we will state the lemma.

**Lemma 3.8.** Let $m + 2\alpha > \frac{6}{5}$. In addition, $\delta$ is supposed to be as in (12). Then one can find $C > 0$ independent of $\epsilon \in (0, 1)$ such that
\[||n_\epsilon(\cdot,t)||_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, \infty)\] (55)
and
\[||c_\epsilon(\cdot,t)||_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t \in (0, \infty)\] (56)
as well as
\[ \|u_\epsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t \in (0, \infty). \]  \hfill (57)

Moreover, we also have
\[ \|A^3 u_\epsilon(\cdot, t)\|_{L^2(\Omega)} \leq C \text{ for all } t \in (0, \infty). \]  \hfill (58)

According to Lemma 3.18 and Lemma 3.19 in Winkler [48], we can also use the uniform bound (55)-(58) and the standard parabolic regularity theory to obtain the following uniform Hölder continuity for \( c_\epsilon, \nabla c_\epsilon \) and \( u_\epsilon \).

**Lemma 3.9.** Let \( m + 2\alpha > \frac{6}{5} \). Then there exists \( \zeta \in (0, 1) \) such that for some \( C > 0 \) we have
\[ \|c_\epsilon\|_{C^{\zeta}([t, t+1])} \leq C \text{ for all } t \geq 0 \]  \hfill (59)
as well as
\[ \|u_\epsilon\|_{C^{\zeta}([t, t+1])} \leq C \text{ for all } t \geq 0 \]  \hfill (60)
and such that for each \( \tau > 0 \) we can find \( C(\tau) > 0 \) such that
\[ \|\nabla c_\epsilon\|_{C^{\zeta}([t, t+1])} \leq C \text{ for all } t \geq \tau. \]  \hfill (61)

To derive strong compactness properties, we shall need an appropriate boundedness property of the time derivative of certain power \( n_\epsilon \). On time intervals of a fixed finite length, this can be achieved by making use of the priori bounds derived above.

**Lemma 3.10.** Suppose that \( m + 2\alpha > \frac{6}{5} \), and let \( \gamma > m \) satisfy \( \gamma \geq 2(m - 1 + \alpha) \). Then for all \( T > 0 \) there exists \( C(T) > 0 \) such that
\[ \int_0^T \|\partial_t n_\epsilon^\gamma(\cdot, t)\|_{W^{2,\gamma}(\Omega)}, dt \leq C(T) \text{ for all } \epsilon \in (0, 1). \]  \hfill (62)

**Proof.** To estimate the integrals on the right of (66) below appropriately, we first apply Lemma 3.8 to fix constants \( C_i > 0 \) (\( i = 1, 2, 3 \)) such that
\[ n_\epsilon \leq C_1, \quad |\nabla c_\epsilon| \leq C_2 \text{ and } |u_\epsilon| \leq C_3 \text{ in } \Omega \times (0, \infty) \text{ for all } \epsilon \in (0, 1), \]  \hfill (63)
which along with (5), (10), (11), \( n_\epsilon \geq 0 \) and \( D_\epsilon \leq D + 2\epsilon \) in \( \Omega \times (0, \infty) \) yields that
\[ D_\epsilon(n_\epsilon) \leq C_4 := \|D\|_{L^\infty(\Omega)} + 2 \text{ in } \Omega \times (0, \infty) \text{ for all } \epsilon \in (0, 1) \]  \hfill (64)
and
\[ |S_\epsilon(x, n_\epsilon, c_\epsilon)| \leq \frac{C_S}{(1 + n_\epsilon)^{\alpha}} \leq C_S \text{ in } \Omega \times (0, \infty) \text{ for all } \epsilon \in (0, 1). \]  \hfill (65)
For any fixed \( \varphi \in C_0^\infty(\Omega) \), we test the first equation in (17) by \( n_\epsilon^{\gamma-1}\varphi \) and then get
\[
\frac{1}{\gamma} \int_\Omega \partial_t n_\epsilon^\gamma(\cdot, t) \cdot \varphi = \int_\Omega n_\epsilon^{\gamma-1} \left( \nabla \cdot (D_\epsilon(n_\epsilon) \nabla n_\epsilon) - \nabla \cdot (n_\epsilon S_\epsilon(x, n_\epsilon, c_\epsilon) \cdot \nabla c_\epsilon) \right) \cdot \varphi \\
- \int_\Omega n_\epsilon^{\gamma-1} u_\epsilon \cdot \nabla n_\epsilon \cdot \varphi + \int_\Omega n_\epsilon^{\gamma-1}(\xi_n \epsilon - \mu_n^2) \varphi \\
= - (\gamma - 1) \int_\Omega n_\epsilon^{\gamma-2} D_\epsilon(n_\epsilon) \nabla n_\epsilon |^2 \varphi - \int_\Omega n_\epsilon^{\gamma-1} D_\epsilon(n_\epsilon) \nabla n_\epsilon \cdot \nabla \varphi \\
+ (\gamma - 1) \int_\Omega n_\epsilon^{\gamma-1} \nabla n_\epsilon \cdot \left( S_\epsilon(x, n_\epsilon, c_\epsilon) \cdot \nabla c_\epsilon \right) \varphi.
\]
Let \( p := \gamma - m + 1 \), then \( \gamma > m \) and \( \gamma \geq 2(m-1+\alpha) \) yield \( p > 1 \) and \( p \geq m-1+2\alpha \).

In view of (63), we integrate (24) with respect to \( t \) over \((0, T)\) for some fixed \( T > 0 \) and then have

\[
\int_0^n n_i^\gamma \bigg( S_i(x, n, c) \cdot \nabla c \bigg) \cdot \nabla \varphi + \frac{1}{\tau} \int_\Omega n_i^\gamma u \cdot \nabla \varphi + \frac{1}{\gamma} \int_\Omega n_i^\gamma \varphi - \mu \int_\Omega n_i^{\gamma+1} \varphi \quad \text{for all } t > 0.
\]

(66)

Now using (63)-(65) and the Young inequality, we can estimate the rest of the integrals on the right-hand side of (66) as follows:

\[
\int_0^T \int_\Omega n_i^{-2} |\nabla n_i|^2 dx = \int_0^T \int_\Omega n_i^{p+3+\gamma} |\nabla n_i|^2 dx \leq C_0 T + C_7.
\]

(68)

Now using (63)-(65) and the Young inequality, we can estimate the rest of the integrals on the right-hand side of (66) as follows:

\[
\bigg| - (\gamma - 1) \int_\Omega n_i^{\gamma-2} D_i(n) \nabla n_i \cdot \nabla \varphi \bigg| \leq (\gamma - 1) C_4 \left( \int_\Omega n_i^{\gamma-2} |\nabla n_i|^2 dx \right) \cdot \| \varphi \|_{L^\infty(\Omega)}
\]

(69)

for all \( t \in (0, T_{\text{max}}) \). Combining (63)-(64) and Young’s inequality, it yields that

\[
\bigg| - \int_\Omega n_i^{\gamma-1} D_i(n) \nabla n_i \cdot \nabla \varphi \bigg| \leq C_4 \cdot \left( \int_\Omega n_i^{\gamma-1} |\nabla n_i| \right) \cdot \| \nabla \varphi \|_{L^\infty(\Omega)}
\]

\[
\leq C_8 \cdot \left( \int_\Omega n_i^{\gamma-2} |\nabla n_i|^2 + \int_\Omega n_i^{\gamma} \right) \cdot \| \nabla \varphi \|_{L^\infty(\Omega)}
\]

\[
\leq C_9 \left( \int_\Omega n_i^{\gamma-2} |\nabla n_i|^2 + C_i^2 \right) \cdot \| \nabla \varphi \|_{L^\infty(\Omega)}
\]

(70)

with some \( C_i > 0 \) \( (i = 8, 9) \) and, similarly,

\[
\bigg| (\gamma - 1) \int_\Omega n_i^{\gamma-1} \nabla n_i \cdot \left( S_i(x, n, c) \cdot \nabla c \right) \varphi \bigg|
\leq (\gamma - 1) \cdot \left( \int_\Omega n_i^{\gamma-1} |\nabla n_i| dx \right) \cdot C_5 C_2 \| \varphi \|_{L^\infty(\Omega)}
\]

\[
\leq (\gamma - 1) C_5 C_{10} \cdot \left( \int_\Omega n_i^{\gamma-2} |\nabla n_i|^2 + C_i^2 \right) \cdot \| \nabla \varphi \|_{L^\infty(\Omega)}
\]

(71)

for all \( t \in (0, T_{\text{max},e}) \) with \( C_{10} > 0 \). By means of Young inequality, (63) and (65) we can estimate

\[
\left| \int_\Omega n_i^\gamma \left( S_i(x, n, c) \cdot \nabla c \right) \cdot \nabla \varphi \right| \leq C_1 C_5 C_2 \| \nabla \varphi \|_{L^\infty(\Omega)}
\]

(72)

for all \( t \in (0, T_{\text{max},e}) \) and

\[
\left| \frac{1}{\gamma} \int_\Omega n_i^\gamma u \cdot \nabla \varphi \right| \leq \frac{1}{\gamma} C_1^2 C_3 \| \nabla \varphi \|_{L^\infty(\Omega)}
\]

(73)
for all $\epsilon \in (0, 1)$. Here by (63)

$$\xi \int_{\Omega} n_0^\gamma \varphi - \mu \int_{\Omega} n_0^{\gamma+1} \varphi \leq C_1^\gamma |\Omega| (\xi + \mu C_1) ||\varphi||_{L^\infty(\Omega)}$$

(74)

for all $\epsilon \in (0, 1)$. Employing (69)-(74) and the embedding $W^{3,2}_0(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ in three-dimensional space, we furthermore obtain that

$$\left|\left|\partial_t n_0^\gamma(\cdot, t)\right|\right|_{(W^{3,2}_0(\Omega))'} \leq C_{11} \cdot \left\{ \int_{\Omega} n_0^{\gamma-2} |\nabla n_0| \right\}^2 + 1 \right\) \text{ for any } \epsilon \in (0, 1)$$

(75)

for all $t > 0$ with some $C_{11} > 0$. Thereupon, estimates (75) and (68) imply finally

$$\int_0^T \left|\left|\partial_t n_0^\gamma(\cdot, t)\right|\right|_{(W^{3,2}_0(\Omega))'} dt \leq C_{11} \left( C_6 T + C_7 + T \right)$$

(76)

for all $\epsilon \in (0, 1)$, which immediately leads to our conclusion. \qed

4. Proof of Theorem 1.1.

Our generalized solution concept reads as follows.

**Definition 4.1.** Let $T > 0$ and $(n_0, c_0, u_0)$ fulfills (12). Then a triple of functions $(n, c, u)$ is called a weak solution of (1) if the following conditions are satisfied

$$\begin{cases} 
  n \in L^1_{\text{loc}}(\Omega \times [0, T]), \\
  c \in L^\infty_{\text{loc}}(\Omega \times (0, T)) \cap L^1_{\text{loc}}([0, T]; W^{1,1}(\Omega)), \\
  u \in L^1_{\text{loc}}((0, T); W^{1,1}(\Omega)),
\end{cases}$$

(77)

where $n \geq 0$ and $c \geq 0$ in $\Omega \times (0, T)$ as well as $\nabla \cdot u = 0$ in the distributional sense in $\Omega \times (0, T)$. Moreover,

$$\mathcal{H}(n), \ n|\nabla c| \text{ and } n|u| \text{ belong to } L^1_{\text{loc}}(\Omega \times [0, T]),$$

and

$$-\int_0^T \int_{\Omega} n \psi_t - \int_{\Omega} n_0 \psi(\cdot, 0) = \int_0^T \int_{\Omega} \mathcal{H}(n) \Delta \psi + \int_0^T \int_{\Omega} n \left( S(x, n, c) \cdot \nabla c \right) \cdot \nabla \psi$$

$$+ \int_0^T \int_{\Omega} n u \cdot \nabla \psi + \xi \int_0^T \int_{\Omega} n \cdot \psi - \mu \int_0^T \int_{\Omega} n^2 \cdot \psi$$

for any $\psi \in C^\infty_0(\Omega \times [0, T])$ satisfying $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial \Omega \times (0, T)$ as well as

$$-\int_0^T \int_{\Omega} c \psi_t - \int_{\Omega} c_0 \psi(\cdot, 0) = -\int_0^T \int_{\Omega} \nabla c \cdot \nabla \psi + \int_0^T \int_{\Omega} n \cdot \psi$$

$$- \int_0^T \int_{\Omega} c \cdot \psi + \int_0^T \int_{\Omega} u \cdot \nabla \psi$$

for an $\psi \in C^\infty_0(\Omega \times [0, T])$ and

$$-\int_0^T \int_{\Omega} u \cdot \psi_t - \int_{\Omega} u_0 \psi(\cdot, 0) = -\int_0^T \int_{\Omega} \nabla u \cdot \nabla \psi + \int_0^T \int_{\Omega} n \nabla \phi \cdot \psi$$

for any $\psi \in C^\infty_0(\Omega \times [0, T]; \mathbb{R}^3)$ fulfilling $\nabla \cdot \psi \equiv 0$ in $\Omega \times (0, T)$, where we let

$$\mathcal{H}(s) := \int_0^s D(p)dp \text{ for } s \geq 0.$$

If $(n, c, u) : \Omega \times (0, \infty) \rightarrow \mathbb{R}^5$ is a weak solution of (6) in $\Omega \times (0, T)$ for any $T > 0$, it is said that $(n, c, u)$ is a global weak solution of (6).
In this framework, (6) is indeed globally solvable. This can be seen by making use of the above a priori estimates and extracting suitable subsequences in a standard manner.

**Lemma 4.2.** Suppose that \( m + 2\alpha > \frac{6}{5} \). Then one can find \((\epsilon_j)_{j \in \mathbb{N}} \subset (0, 1)\) such that \( \epsilon_j \searrow 0 \) as \( j \to \infty \) and that

\[
\begin{align*}
n_e &\to n \text{ a.e. in } \Omega \times (0, \infty), \quad (78) \\
n_e &\rightharpoonup^* n \text{ in } L^\infty(\Omega \times (0, \infty)), \quad (79) \\
n_e &\to n \text{ in } C^0_{\text{loc}}([0, \infty); (W^{2,2}_0(\Omega))^*), \quad (80) \\
c_e &\to c \text{ in } C^0_{\text{loc}}(\overline{\Omega} \times [0, \infty)), \quad (81) \\
\nabla c_e &\to \nabla c \text{ in } C^0_{\text{loc}}(\overline{\Omega} \times [0, \infty)), \quad (82) \\
\nabla c_e &\to \nabla c \text{ in } L^\infty(\Omega \times [0, \infty)), \quad (83) \\
u_e &\to u \text{ in } C^0_{\text{loc}}(\overline{\Omega} \times [0, \infty)), \quad (84)
\end{align*}
\]

as well as

\[
Du_e \rightharpoonup Du \text{ in } L^\infty(\Omega \times (0, \infty)), \quad (85)
\]

where the triple of limit function \((n, c, u)\) is global weak solution of (1) in the sense of Definition 4.1. Furthermore, \( n \) fulfills

\[
n \in C^0_{\text{w-loc}}([0, \infty); L^\infty(\Omega)). \quad (86)
\]

**Proof.** Firstly, We fix \( p > 1 \) and \( \gamma = \frac{m+p-1}{2} \). Given \( T > 0 \), (68) ensures that there is \( C > 0 \) such that \( \int_0^T \int_{\Omega} |\nabla n_\epsilon|^2 \leq C \) for any \( \epsilon \in (0, 1) \) and we infer that \((n_\epsilon^\gamma)_{\epsilon \in (0,1)}\) is bounded in \( L^2(0,T; W^{1,2}(\Omega)) \). Furthermore, Lemma 3.10 implies that \((c_\epsilon^\gamma)_{\epsilon \in (0,1)}\) is bounded in \( L^1((0,T); (W^{3,2}(\Omega))^*) \). So we can obtain that \((n_\epsilon^\gamma)_{\epsilon \in (0,1)}\) is a relatively compact subset of the space \( L^2(0,T; L^2(\Omega)) \) according to the embedding \( W^{1,2} \hookrightarrow L^2(\Omega) \hookrightarrow (W^{3,2}(\Omega))^* \) and the Aubin-Lions compactness Lemma, whence there exists a sequence of numbers \( \epsilon = \epsilon_j \searrow 0 \) such that \( n_\epsilon^\gamma \rightharpoonup n^\gamma \) in \( L^2(0,T; L^2(\Omega)) \) along this sequence with some function \( n^\gamma \) in this space, which, along a subsequence, ensures (78). Next Lemma 3.8 warrants that for certain \( n \in L^\infty(\Omega \times (0, \infty)) \) (79) holds. As for (80)-(86), we can achieve their validity by the same reasoning as in the proof of [49, Lemma 4.1], so we can obtain the desired results.

**Proof of Theorem 1.1.** A combination of Lemma 3.8 and Lemma 4.1 directly leads to Theorem 1.1.

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