Sc-Smoothness, Retractions and New Models for Smooth Spaces

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1 Sc-Smoothness and M-Polyfolds

In paper [9], the authors have described a generalization of differential geometry based on the notion of splicings. The associated Fredholm theory in polyfolds, presented in [9, 10, 11], is a crucial ingredient in the functional analytic foundation of the Symplectic Field Theory (SFT). The theory also applies to the Floer theory as well as to the Gromov-Witten theory and quite generally should have applications in nonlinear analysis, in particular in studies of families of elliptic pde’s on varying domains, which can even change their topology.

A basic ingredient for the generalization of differential geometry is a new notion of differentiability in infinite dimensions, called sc-smoothness. The goal of this paper is to describe these ideas and, in particular, to provide some of the “hard” analysis results which enter the polyfold constructions.
in symplectic field theory (SFT). The advantage of the polyfold Fredholm theory can be summarized as follows.

- Many spaces, though they do not carry a classical smooth structure, can be equipped with a weak version of a smooth structure. The local models for the spaces, be they finite- or infinite-dimensional, can even have locally varying dimensions.

- Since the notion of the smooth structure is so weak, there are many charts so that many spaces carry a manifold structure in the new smoothness category.

- Finite-dimensional subsets in good position in these generalized manifold inherit an induced differentiable structure in the familiar sense.

- There is a notion of a bundle. Smooth sections of such bundles, which, under a suitable coordinate change, can be brought into a sufficiently nice form, are Fredholm sections. A Fredholm section looks nice (near a point) only in a very particular coordinate system and not necessarily in the smoothly compatible other ones. Since we have plenty of coordinate systems, many sections turn out to be Fredholm.

- The zero sets of Fredholm sections lie in the smooth parts of the big ambient space, so that they look smooth in all coordinate descriptions (systems). The invariance of the properties of solution sets under arbitrary coordinate changes is, of course, a crucial input for having a viable theory.

- There is an intrinsic perturbation theory, and moreover, a version of Sard-Smale’s theorem holds true. In applications, for example to a geometric problem, one might try to make the problem generic by perturbing auxiliary geometric data. As the Gromov-Witten and SFT-examples show, this is, in general, not possible and one needs to find a sufficiently large abstract universe, which offers enough freedom to construct generic perturbations. The abstract polyfold Fredholm theory provides such a framework.

- Important for the applications is a version of this new Fredholm theory for an even more general class of spaces, called polyfolds. In this case the generic solution spaces can be thought of locally as a finite union
of (classical) manifolds divided out by a finite group action. Moreover, the points in these spaces carry rational weights. Still the integration of differential forms can be defined for such spaces and Stokes' theorem is valid. This is used in order to define invariants. The Gromov-Witten invariants provide an example.

The current paper develops the analytical foundations for some of the applications of the theory described above. It also provides examples illustrating the ideas.

The organization of the paper is as follows.

The introductory chapter describes the new notions of smoothness for spaces and mappings leading, in particular, to novel local models of spaces, which generalize manifolds and which are called M-polyfolds. The general Fredholm theory in this analytical setting is outlined and an outlook to some applications is given, the proofs of which are postponed to chapter 3.

The second chapter is of technical nature and is devoted to detailed proofs of the new smoothness results which are crucial for many applications.

The third chapter illustrates the concepts by constructing M-polyfold structures on a set of mappings between conformal cylinders which break apart as the modulus tends to infinity. A strong bundle over this M-polyfold is constructed which admits the Cauchy-Riemann operator as an sc-smooth Fredholm section. Its zero-set consists of the holomorphic isomorphisms between cylinders of various sizes. Since the solution set carries a smooth structure, this has interesting functional analytic consequences for the behavior of families of holomorphic mappings.

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1.1 Sc-Structures on Banach Spaces

Sc-structures on Banach spaces generalize the smooth structure from finite dimensions to infinite dimensions. We first recall the definition of an sc-structure on a Banach space $E$ from [9]. In the following $\mathbb{N}_0$ stands for $\mathbb{N} \cup \{0\}$.

**Definition 1.1.** An sc-structure on a Banach space $E$ is a nested sequence
of Banach spaces \((E_m)_{m \in \mathbb{N}_0}\),
\[
E =: E_0 \supset E_1 \supset \ldots \supset E_{\infty} := \bigcap_{m \in \mathbb{N}_0} E_m,
\]
so that the following two conditions are satisfied:

1. The inclusion maps \(E_{m+1} \to E_m\) are compact operators.
2. The vector space \(E_{\infty}\) is dense in every \(E_m\).

Points in \(E_{\infty}\) are called smooth points. What just has been defined is a compact discrete scale of Banach spaces which is a standard object in interpolation theory for which we refer to [17]. Our interpretation as a generalization of a smooth structure on \(E\) seems to be new. The only sc-structure on a finite-dimensional vector space \(E\) is given by the constant structure \(E_m = E\) for all \(m\). If \(E\) is an infinite-dimensional Banach space, the constant structure is not an sc-structure because it fails property (1).

**Definition 1.2.** A linear map \(T : E \to F\) between the two sc-Banach spaces \(E\) and \(F\) is called an sc-operator if \(T(E_m) \subset F_m\) and if \(T : E_m \to F_m\) is continuous for every \(m \in \mathbb{N}_0\).

We shall need the notion of a partial quadrant \(C\) in an sc-Banach space \(E\).

**Definition 1.3.** A closed subset of an sc-Banach space \(E\) is called a partial quadrant if there are an sc-Banach space \(W\), a nonnegative integer \(k\), and a linear sc-isomorphism \(T : E \to \mathbb{R}^k \oplus W\) so that \(T(C) = [0, \infty)^k \oplus W\).

Given a partial quadrant \(C\) in an sc-Banach space \(E\), we define the degeneration index
\[
d_C : C \to \mathbb{N}_0
\]
as follows. We choose a linear sc-isomorphism \(T : E \to \mathbb{R}^k \oplus W\) satisfying \(T(C) = [0, \infty)^k \oplus W\). Hence for \(x \in C\), we have
\[
T(x) = (r_1, \ldots, r_k, w), \quad x \in C,
\]
where \((r_1, \ldots, r_k) \in [0, \infty)^k\) and \(w \in W\). Then we define the integer \(d_C(x)\) by
\[
d_C(x) = \sharp\{i \in \{1, \ldots, k\} | r_i = 0\}.
\]
It is not difficult to see that this definition is independent of the choice of an sc-linear isomorphism $T$.

Let $U$ be a relatively open subset of a partial quadrant $C$ in an sc-Banach space $E$. Then the sc-structure on $E$ induces the sc-structure on $U$ defined by the sequence $U_m = U \cap E_m$ equipped with the topology of $E_m$ and called the induced sc-structure on $U$. The points of $U_\infty = U \cap E_\infty$ are called smooth points of $U$. We adopt the convention that $U_k$ denotes the set $U_k \cap E_k$ equipped with the sc-structure $(U_k)_m := U_{k+m}$ for all $m \in \mathbb{N}_0$. If $U$ and $V$ are open subsets equipped with the induced sc-structures, we write $U \oplus V$ for the product $U \times V$ equipped with the sc-structure $(U_m \times V_m)_{m \in \mathbb{N}_0}$.

**Definition 1.4.** If $U$ is a relatively open subset of a partial quadrant $C$ in an sc-Banach space $E$, then its tangent $TU$ is defined by

$$TU = U^1 \oplus E.$$ 

**Example 1.5.** A good example which illustrates the concepts, and is also relevant for SFT, is as follows. We choose a strictly increasing sequence $(\delta_m)_{m \in \mathbb{N}_0}$ of real numbers starting with $\delta_0 = 0$. We consider the Banach spaces $E = L^2(\mathbb{R} \times S^1)$ and $E_m = H^{m,\delta_m}(\mathbb{R} \times S^1)$ where the space $H^{m,\delta_m}(\mathbb{R} \times S^1)$ consists of those elements in $E$ having weak partial derivatives up to order $m$ which if weighted by $e^{\delta_m |s|}$ belong to $E$. Using Sobolev’s compact embedding theorem for bounded domains and the assumption that the sequence $(\delta_m)$ is strictly increasing, one sees that the sequence $(E_m)_{m \in \mathbb{N}_0}$ defines an sc-structure on $E$. We take as the partial quadrant $C$ the whole space $E$ and let $B_E$ be the open unit ball centered at 0 in $E$. Then the tangent of $B_E$ is given by

$$TB_E = (B_E)^1 \oplus E = \{(u, h) | u \in H^{1,\delta_1}, |u|_{L^2} < 1, h \in L^2\}.$$

The sc-structure on $TB_E$ is defined by

$$(TB_E)_m = \{(u, h) | u \in H^{m+1,\delta_{m+1}}, |u|_{L^2} < 1, h \in H^{m,\delta_m}\}.$$

The notion of a continuous map $f : U \to V$ between two relatively open subsets of partial quadrants in sc-Banach spaces is as follows.

**Definition 1.6.** A map $f : U \to V$ is said to be sc$^0$ if $f(U_m) \subset V_m$ for all $m \in \mathbb{N}_0$ and if the induced maps $f : U_m \to V_m$ are continuous.
Example 1.7. An important example used later on is the shift-map. We consider the Hilbert space $E = L^2(\mathbb{R} \times S^1)$ equipped with the sc-structure $(E_m)_{m \in \mathbb{N}_0}$ introduced in Example 1.5. Then we define the map

$$
\Phi : \mathbb{R}^2 \oplus L^2(\mathbb{R} \times S^1) \to L^2(\mathbb{R} \times S^1), \quad (R, \vartheta, u) \mapsto (R, \vartheta) * u
$$

where

$$
((R, \vartheta) * u)(s, t) = u(s + R, t + \vartheta).
$$

The shift-map $\Phi$ is sc$^0$ as proved in Proposition 4.1. It is clearly not differentiable in the classical sense. However, in Proposition 4.2 we shall prove that the map $\Phi$ is sc-smooth for the new notion of smoothness which we shall introduce next. The shift map will be an important ingredient in later constructions and its sc-smoothness will be crucial.

1.2 Sc-Smooth Maps and M-Polyfolds

Having defined an appropriate notion of continuity we define what it means that the map is of class sc$^1$. This is the notion corresponding to a map being $C^1$ in our sc-framework.

Definition 1.8. Let $U$ and $V$ be relatively open subsets of partial quadrants $C$ and $D$ in sc-Banach spaces $E$ and $F$, respectively. An sc$^0$-map $f : U \to V$ is said to be sc$^1$ if for every $x \in U_1$ there exists a bounded linear operator $Df(x) : E_0 \to F_0$ so that the following holds:

1. If $h \in E_1$ and $x + h \in C$, then

$$
\lim_{|h|_1 \to 0} \frac{1}{|h|_1} \cdot |f(x + h) - f(x) - Df(x)h|_0 = 0.
$$

2. The map $Tf : TU \to TV$, called the tangent map of $f$, and defined by

$$(x, h) \mapsto (f(x), Df(x)h),$$

is of class sc$^0$.

In general, the map $U_1 \to L(E_0, F_0)$, $x \to Df(x)$ will not(!) be continuous if the space of bounded linear operators is equipped with the operator norm. However, if we equip it with the compact open topology it will be continuous.
The sc\(^1\)-maps between finite dimensional Banach spaces are the familiar C\(^1\)-maps.

Proceeding inductively, we define what it means for the map \( f \) to be sc\(^k\) or sc\(^\infty\). Namely, an sc\(^0\)-map \( f \) is said to be an sc\(^2\)-map if it is sc\(^1\) and if its tangent map \( Tf : TU \to TV \) is sc\(^1\). By Definition 1.8, the tangent map of \( Tf \),

\[
T^2 f := T(Tf) : T^2(U) = T(TU) \to T^2(V) = T(TV),
\]

is of class sc\(^0\). If the tangent map \( T^2 f \) is sc\(^1\), then \( f \) is said to be sc\(^3\), and so on. The map \( f \) is sc\(^\infty\), if it is sc\(^k\) for all \( k \).

Useful in our applications are the next two propositions which relate the sc-smoothness with the familiar notion of smoothness.

**Proposition 1.9** (Upper Bound). Let \( E \) and \( F \) be sc-Banach spaces and let \( U \) be a relatively open subset of a partial quadrant \( C \) in \( E \). Assume that \( f : U \to F \) is an sc\(^0\)-map so that for every \( 0 \leq l \leq k \) and every \( m \geq 0 \) the induced map

\[
f : U_{m+l} \to F_m
\]

is of class \( C^{l+1} \). Then \( f \) is sc\(^{k+1}\).

**Proposition 1.10** (Lower Bound). Let \( E \) and \( F \) be sc-Banach spaces and let \( U \) be a relatively open subset of a partial quadrant \( C \) in \( E \). If the map \( f : U \to F \) is sc\(^k\), then the induced map

\[
f : U_{m+l} \to F_m
\]

is of class \( C^l \) for every \( 0 \leq l \leq k \) and every \( m \geq 0 \).

The proofs of the two propositions will be carried out in section 2.1. In view of the following chain rule, the sc-smoothness is a viable concept.

**Theorem 1.11** (Chain Rule). Assume that \( U, V, \) and \( W \) are relatively open subsets of partial quadrants in sc-Banach spaces and let \( f : U \to V \) and \( g : V \to W \) be sc\(^1\). Then the composition \( g \circ f : U \to W \) is sc\(^1\) and

\[
T(g \circ f) = (Tg) \circ (Tf).
\]

The proof can be found in [9]. We would like to point out that the proof relies on the assumption that the inclusion operators between spaces in the nested sequence of Banach spaces are compact.

The next definition introduces the notions of an sc-smooth retraction and an sc-smooth retract. This will be the starting point for a differential geometry based on new local models.
**Definition 1.12.** Let $U$ be a relatively open subset of a partial quadrant $C$ in an sc-Banach space $E$. An sc-smooth map $r : U \to U$ is called an sc\(^{\infty}\)-retraction provided it satisfies

$$r \circ r = r.$$ 

A subset $O$ of a partial quadrant $C$ is called an sc-smooth or sc\(^{\infty}\)-retract (relative to $C$) if there exists a relatively open subset $U \subset C$ and an sc-smooth retraction $r : U \to U$ so that

$$O = r(U).$$

If $r : U \to U$ is an sc\(^{\infty}\)-retraction, then its tangent map $Tr : TU \to TU$ is also an sc\(^{\infty}\)-retraction. This follows from the chain rule. Next comes the crucial definition of the new local models of smooth spaces.

**Definition 1.13.** A local M-polyfold model is a triple $(O, C, E)$ in which $E$ is an sc-Banach space, $C$ is a partial quadrant of $E$, and $O$ is a subset of $C$ having the following properties:

1. There is an sc-smooth retraction $r : U \to U$ defined on a relative open subset $U$ of $C$ so that

$$O = r(U).$$

2. For every smooth point $x \in O_{\infty}$, the kernel of the map $(\text{id} - Dr(x))$ possesses an sc-complement which is contained in $C$.

3. For every $x \in O$, there exists a sequence of smooth points $(x_k) \subset O_{\infty}$ converging to $x$ in $O$ and satisfying $d_C(x_k) = d_C(x)$.

The choice of $r$ in the above definition is irrelevant as long as it is an sc-smooth retraction onto $O$ defined on a relatively open subset $U$ of $C$.

A special M-polyfold model has the form $(O, E, E)$. Such triples can be viewed as the local models for sc-smooth space $S$ without boundary whereas the more general triples are models for spaces with boundaries with corners. In the case without boundary the conditions (2) and (3) of Definition 1.13 are automatically satisfied.

In our applications the local sc-models $(O, C, E)$ quite often arise in the following way. We assume that we are given a partial quadrant $D$ in an sc-Banach space $W$ and a relatively open subset $V$ of $D$. Moreover, we assume that for every $v \in V$ we have a bounded linear projection

$$\pi_v : F \to F$$
into another sc-Banach space $F$. In general, the projection $\pi_v$ is not an sc-operator. We require that the map

$$V \oplus F \to F, \quad (v, f) \mapsto \pi_v(f)$$

is sc-smooth. Then we look at the sc-Banach space $E = W \oplus F$, the partial quadrant $C = D \oplus F$, and the relatively open subset $U = V \oplus F$ of $E$. Finally, we define the map $r : U \to U$ by

$$r(v, f) = (v, \pi_v(f)).$$

Then the map $r$ is an sc–smooth retraction and the set

$$K = \{(v, f) | \pi_v(f) = f\}$$

is an sc–smooth retract. We call this particular retraction, due to its partially linear character, a splicing. For more details on splicings we refer to [9].

**Lemma 1.14.** Let $(O, C, E)$ be an sc-smooth local model and assume that $r : U \to U$ and $r' : U' \to U'$ are two sc-smooth retractions defined on relatively open subsets $U$ and $U'$ of $C$ and satisfying $r(U) = r'(U') = O$. Then

$$Tr(TU) = Tr'(TU').$$

**Proof.** If $y \in U$, then there exists $y' \in U'$ so that $r(y) = r'(y')$. Consequently, $r' \circ r(y) = r' \circ r'(y') = r'(y') = r(y)$, and hence $r' \circ r = r$. Similarly, one sees that $r \circ r' = r'$. If $(x, h) \in Tr(TU)$, then $(x, h) = Tr(y, k)$ for a pair $(y, k) \in TU$. Moreover, $x \in O_1 \subset U'_1$ so that $(x, h) \in TU'$. From $r' \circ r = r$ it follows using the chain rule that

$$Tr'(x, h) = Tr' \circ Tr(y, k) = Tr(r' \circ r)(y, k) = Tr(y, k) = (x, h)$$

implying $Tr(TU) \subset Tr'(TU')$. Similarly one shows that $Tr'(TU') \subset Tr(TU)$ and the proof of the proposition is complete. 

The lemma allows us to define the tangent of a local M-polyfold model $(O, C, E)$ as follows.

**Definition 1.15.** The tangent of a local M-polyfold model $(O, C, E)$, denoted by $T(O, C, E)$, is defined as a triple

$$T(O, C, E) = (TO, TC, TE),$$

in which $TC = C^1 \oplus E$ is the tangent of the partial quadrant $C$ and $TO := Tr(TU)$, where $r : U \to U$ is any sc-smooth retraction onto $O$. 

10
As we already pointed out, the tangent map $Tr : TU \to TU$ of the retraction is an sc-smooth retraction. It is defined on the relatively open subset $TU$ of $TC$ and $TO = Tr(TU)$ is an sc$^\infty$-retract. Thus, the tangent $T(O,C,E)$ of a local M-polyfold model is also a local M-polyfold model.

It is clear what it means that the map $f : (O,C,E) \to (O',C',E')$ between two local M-polyfold models is sc$^0$. In order to define sc$^k$-maps between local models we need the following lemma.

**Lemma 1.16.** Let $f : (O,C,E) \to (O',C',E')$ be a map between two local M-polyfold models and let $r : U \to U$ and $s : V \to V$ be sc-smooth retractions onto $O$. Then the map $f \circ r : U \to E'$ is sc$^1$ if and only if the same holds true for the map $f \circ s : V \to E'$. Moreover, the map

$$T(f \circ r)|Tr(TU) : TO \to TO'$$

does not depend on the choice of an sc-smooth retraction $r$ as long as $r$ is an sc-smooth retraction onto $O$.

**Proof.** Assume that $f \circ r : U \to E'$ is sc$^1$. Since $s : V \to U \cap V$ is sc$^\infty$, the chain rule implies that the composition $f \circ r \circ s : V \to F$ is sc$^1$. Using the identity $f \circ r \circ s = f \circ s$, we conclude that $f \circ s$ is sc$^1$. Interchanging the role of $r$ and $s$, the first part of the lemma is proved. If $(x,h) \in TO$, then $(x,h) = Ts(x,h)$ and using the identity $f \circ r \circ s = f \circ s$ and the chain rule, we conclude

$$T(f \circ r)(x,h) = T(f \circ r)(Ts)(x,h) = T(f \circ r \circ s)(x,h) = T(f \circ s)(x,h)$$

Now take any sc–smooth retraction $q : W \to W$ defined on a relatively open subset $W$ of the the partial quadrant $C'$ in $E'$ satisfying $q(W) = O'$. Then $q \circ f = f$ so that $q \circ f \circ r = f \circ r$. Application of the chain rule yields the identity

$$T(f \circ r)(x,h) = T(q \circ f \circ r)(x,h) = Tq \circ T(f \circ r)(x,h)$$

for all $(x,h) \in Tr(TU)$. Consequently, $T(f \circ r)|Tr(TU) : TO \to TO'$ and this map is independent of the choice of an sc-smooth retraction onto $O$. ■

In view of the lemma, we define the map $f : (O,C,E) \to (O',C',E')$ between local models to be of class sc$^1$ if the composition $f \circ r : U \to E'$ is of class sc$^1$. If this is the case, we define the tangent map $Tf$ as

$$Tf = T(f \circ r)|Tr(TU),$$
where $r : U \to U$ is any sc-smooth retraction onto $O$. Similarly, $f : (O, C, E) \to (O', C', E')$ is of class sc$^k$ provided that the composition $f \circ r : U \to E'$ is of class sc$^k$ where $r : U \to U$ is any sc-smooth retraction defined on relatively open subset $U$ of $C$ satisfying $O = r(U)$.

In the following we simply write $O$ instead of $(O, C, E)$ for the local M-polyfold model, however, we always keep in mind that there are more data in the background.

With the above definition of sc$^1$-maps between local M-polyfold models, the next theorem is an immediate consequence of the chain rule stated in Theorem 1.11.

**Theorem 1.17** (General Chain Rule). Assume that $f : O \to O'$ and $g : O' \to O''$ are sc$^1$-maps between local M-polyfold models. Then the composition $g \circ f : O \to O''$ is an sc$^1$-map and

$$T(g \circ f) = Tg \circ Tf.$$ 

The degeneracy index $d_C : C \to \mathbb{N}_0$ introduced in Section 1.1 generalizes to local models as follows.

**Definition 1.18.** The degeneracy index $d : O \to \mathbb{N}_0$ of the local M-polyfold model $(O, C, E)$ is defined by

$$d(x) := d_C(x), \quad x \in O.$$ 

The next result shows that sc-diffeomorphisms recognize the difference between a straight boundary and a corner. Of course, this is true also for the usual notion of smoothness but not for homeomorphisms.

**Theorem 1.19** (Boundary Recognition). Consider local M-polyfold models $(O, C, E)$ and $(O', C', E')$ and let $f : O \to O'$ be an sc-diffeomorphism. Then

$$d_C(x) = d_{C'}(f(x))$$

at each point $x \in O$.

**Proof.** We slightly modify the argument in [9]. First, assuming that the theorem holds at smooth points $x \in O$, we show that it also holds at points on level 0. Indeed, take a point $x \in O$. By the condition (3) of Definition 1.13 we find a sequence of smooth points $(x_k) \subset O$ converging to $x$ in $O$ and satisfying $d_C(x_k) = d_C(x)$. That is, $d(x_k) = d(x)$. By assumption,
The same arguments apply at the point $x$, the decomposition $I$ of the map $f: \mathbb{R}^n \oplus W$ consisting of all vectors $(p, q)$ in the subspaces $1 \leq i \leq n$ consisting of vectors $(p, q)$ such that $a_i = 0$. In particular, $d(f) = \frac{\partial f}{\partial x}$. We denote by $\Sigma$ the subspace of $N$ consisting of vectors $(p, q) \in N$ with $p_i = 0$ for $i \in I$. From the decomposition $N = N_1 \oplus N_2$, we see that $\Sigma$ has codimension $\# I$ in $N$. The same arguments apply at the point $x' = (a', w') = f(x)$. Abbreviate by $I'$ the set of indices $1 \leq j \leq n'$ at which $a'_j = 0$. With the kernel $N'$ of the map $\text{id} - Dr(x')$ at $x'$, we let $\Sigma'$ be the codimension $\# I'$ subspace of $N'$ consisting of all vectors $(p', q')$ satisfying $p'_j = 0$ for $j \in I'$. We observe that the subspaces $N$ and $N'$ are precisely the tangent spaces $T_x O$ and $T_x O'$.

Take any smooth vector $(p, q) \in \Sigma$. Then $x + \tau (p, q) \in U \cap C$ for $0 < \tau < \varepsilon$ and sufficiently small $\varepsilon > 0$, so that $r(x + \tau (p, q)) \in O$ for $0 < \tau < \varepsilon$. Hence $f \circ r(x + \tau (p, q)) \in O'$ for all $0 < \tau < \varepsilon$.

By assumption, the map $f \circ r$ is sc-smooth. This implies that the map

$$[0, \varepsilon) \to E'_m, \quad \tau \mapsto f \circ r(x + \tau (q, p))$$

is sc-smooth at $x$. Then for $x' = f(x)$, we can find $(p, q)$ such that $x + \tau (p, q) \in \Sigma$ for all $0 < \tau < \varepsilon$. Hence $(p, q) \in \Sigma$ and $f \circ r(x + \tau (p, q)) \in O'$ for all $0 < \tau < \varepsilon$. Therefore $f \circ r$ is sc-smooth at $x'$.

Now we prove the equality for smooth points. Without loss of generality we may assume that $E = \mathbb{R}^n \oplus W$ and $C = [0, \infty)^n \oplus W$ and, similarly, $E' = \mathbb{R}^n' \oplus W'$ and $C' = [0, \infty)^n' \oplus W'$. Take a smooth point $x = (a, w) \in O$ and let $r : U \to U$ be an sc-smooth retract defined on the relatively open subset $U$ of $C$ satisfying $O = r(U)$. Abbreviate by $N$ the kernel of $\text{id} - Dr(x)$ at the point $x$. The kernel $N$ is a closed subspace of $\mathbb{R}^n \oplus W$ possessing, by the condition (2) of Definition 1.13, an sc-complement $M$ in $\{0\} \oplus W$. Then $N$ is the sc-direct sum of two closed sc-subspaces, namely,

$$N = (N \cap \{0\} \oplus W) \oplus \{(q, p) \in N \mid (0, p) \in M\} =: N_1 \oplus N_2.$$ 

Indeed, if $(p, q) \in N_1 \cap N_2$, then $p = 0$ implying that $(0, q) \in N \cap M = \{(0, 0)\}$, i.e., $q = 0$. If $(p, q) \in N$, then there is a unique decomposition $(p, 0) = (A, B) + (0, C) = (A, B + C)$ with $(A, B) \in N$ and $(0, C) \in M$ implying that $p = A$ and $B = -C$ and hence $(0, B) = (0, -C) \in M$. So,

$$(p, q) = (0, q - B) + (A, B)$$

with $(0, q - B) = (A, q) - (A, B) = (p, q) - (A, B) \in N \cap \{0\} \oplus W = N_1$ and $(A, B) \in N_2$.

For our smooth point $x = (a, w)$, we denote by $I$ be the set of indices $1 \leq i \leq n$ at which $a_i = 0$. In particular, $d(x) = \frac{\partial f}{\partial x}$. We denote by $\Sigma$ the subspace of $N$ consisting of vectors $(p, q) \in N$ with $p_i = 0$ for $i \in I$. From the decomposition $N = N_1 \oplus N_2$, we see that $\Sigma$ has codimension $\# I$ in $N$. The same arguments apply at the point $x' = (a', w') = f(x)$. Abbreviate by $I'$ the set of indices $1 \leq j \leq n'$ at which $a'_j = 0$. With the kernel $N'$ of the map $\text{id} - Dr(x')$ at $x'$, we let $\Sigma'$ be the codimension $\# I'$ subspace of $N'$ consisting of all vectors $(p', q')$ satisfying $p'_j = 0$ for $j \in I'$. We observe that the subspaces $N$ and $N'$ are precisely the tangent spaces $T_x O$ and $T_x O'$.
is smooth for every $m \geq 0$. Now for every $1 \leq j \leq n'$ we introduce the bounded linear functional $\lambda_j : \mathbb{R}^n + W' \to \mathbb{R}$ defined by $\lambda_j(b, y) = b_j$. Then we have

$$0 \leq \lambda_j(f \circ r(x + \tau(q, p))), \quad 0 < \tau < \varepsilon.$$ 

Fix $j \in I'$. Then, since $f \circ r(x + \tau(q, p)) = f(x) + \tau Df(x)(p, q) + o_m(\tau)$ where $\frac{o_m(\tau)}{\tau} \to 0$ as $\tau \to 0$ and $\lambda_j(f(x)) = 0$, we conclude that

$$0 \leq \frac{1}{\tau} \lambda_j(f(x) + \tau Df(x)(p, q) + o_m(\tau)) = \lambda_j\left(Df(x)(p, q) + \frac{o_m(\tau)}{\tau}\right)$$

which after letting $\tau \to 0$ gives

$$0 \leq \lambda_j(Df(x)(p, q)).$$

If $(p, q) \in \Sigma$, then $(-p, -q) \in \Sigma$ and the above inequality with $(-p, -q)$ replacing $(p, q)$ implies that $\lambda_j(Df(x)(p, q)) = 0$. Consequently,

$$\lambda_j(Df(x)(p, q)) = 0$$

for all $j \in I'$ and all vectors $(p, q) \in \Sigma$. At this point we have proved that $Tf(x)\Sigma \subset \Sigma'$. Recalling that $N$ and $N'$ are the tangent spaces $T_xO$ and $T_{x'}O'$ and that $Tf : T_xO \to T_{x'}O'$ is an sc–linear isomorphism, we see that the subspace $Tf(x)\Sigma$ has codimension $\sharp I$ in $N'$. It follows that $\sharp I' \leq \sharp I$. Repeating the same argument for the sc-diffeomorphism $f^{-1} : O' \to O$, we obtain the opposite inequality. Hence we conclude that $f$ preserves indeed the degeneracy index for smooth points and the proof of Theorem 1.19 is complete.

At this point it is clear that one can take potentially any recipe from differential geometry and construct new objects. We begin with the recipe for a manifold. Let $X$ be a metrizable space. A chart for $X$ is a triple $(\varphi, U, (O, C, E))$ in which $(O, C, E)$ is a local M-polyfold model, $U$ is an open subset of $X$, and $\varphi : U \to O$ is a homeomorphism. Two such charts

$$(\varphi, U, (O, C, E)) \quad \text{and} \quad (\varphi', U', (O', C', E'))$$

are called sc-smoothly compatible if the composition

$$\varphi' \circ \varphi^{-1} : O \to O'$$

is an sc-smooth map between local M-polyfold models.
Definition 1.20. An sc-smooth atlas for \( X \) consists of a collection of charts \((\varphi, U, (O, C, E))\) such that the associated open sets \( U \) cover \( X \) and the transition maps are sc-smooth. Two atlases are equivalent if the union is also an sc-smooth atlas. The space \( X \) equipped with an equivalence class of sc-smooth atlases is called an M-polyfold.

Observe that an M-polyfold \( X \) inherits a filtration

\[
X = X_0 \supset X_1 \supset \ldots \supset X_\infty = \bigcap_{m \in \mathbb{N}_0} X_i
\]

The same is true for subsets of \( X \). The tangent \( TX \) is defined by the usual recipe used in the infinite-dimensional situation. Consider tuples

\[
t = (x, \varphi, U, (O, C, E), h)
\]

in which \((\varphi, U, (O, C, E))\) is a chart, \( x \in U_1 \), and \((\varphi(x), h) \in TO \subset E^1 \oplus E\).

Two such tuples,

\[
(x, \varphi, U, (O, C, E), h) \quad \text{and} \quad (x', \varphi', U', (O', C', E'), h'),
\]

are called equivalent if \( x = x' \) and

\[
T(\varphi' \circ \varphi^{-1})(\varphi(x), h) = (\varphi'(x), h').
\]

An equivalence class \([t]\) of a tuple \( t \) is called a tangent vector at \( x \). The collection of all tangent vectors at \( x \) is denoted by \( T_x X \) and the tangent \( TX \) is defined by

\[
TX = \bigcup_{x \in X_1} \{x\} \times T_x X.
\]

One easily verifies that \( TX \) is an M-polyfold in a natural way with specific charts \( T\varphi \) given by

\[
T\varphi : TU \to TO, \quad T\varphi([x, \varphi, U, (O, C, E), h]) = (\varphi(x), h).
\]

Here \( TU \) is the union of all \( T_x X \) with \( x \in U_1 \),

\[
TU = \bigcup_{x \in U_1} \{x\} \times T_x X.
\]
Remark 1.21. For SFT we need a more general class of spaces called polyfolds. They are essentially a Morita equivalence class of ep-groupoids, where the latter is a generalization of the notion of an étale proper Lie groupoid to the sc-world. In a nutshell, this is a category in which the class of objects as well as the collection of morphisms are sets, which in addition carry $M$-polyfold structures. Further all category operations are sc-smooth and between any two objects there are only finitely many morphisms. A polyfold is a generalization of the modern notion of orbifold as presented in [16]. We note that the notion of proper has to be reformulated for our generalization as is explained in [11].

Next, we illustrate the previous concepts by an example. We shall construct a connected subset of a Hilbert space which is an sc–smooth retract and which has one- and two-dimensional parts. (By using the fact that the direct sum of two separable Hilbert spaces is isomorphic to itself one can use the following ideas to construct connected sc–smooth subsets which have pieces of many different finite dimensions.)

Example 1.22. We take a strictly increasing sequence of weights $(\delta_m)_{m \in \mathbb{N}_0}$ starting at $\delta_0 = 0$ and equip the Hilbert space $E = L^2(\mathbb{R})$ with the sc-structure given by $E_m = H^{m,\delta_m}(\mathbb{R})$ for all $m \in \mathbb{N}_0$. Next we choose a smooth compactly supported function $\beta: \mathbb{R} \to [0, \infty)$ having $L^2$-norm equal to 1, $$\int_{\mathbb{R}} \beta(s)^2 ds = 1.$$ Then we define a family of sc-operators $\pi_t: E \to E$ as follows. For $t \leq 0$, we put $\pi_t = 0$, and for $t > 0$, we define $$\pi_t(f) = \langle f, \beta(\cdot + e^{\frac{1}{t}}) \rangle_{L^2} \cdot \beta(\cdot + e^{\frac{1}{t}}), \quad f \in E.$$ In other words, for $t > 0$ we take the $L^2$-orthogonal projection onto the 1-dimensional subspace span by the function $\beta$ with argument shifted by $e^{\frac{1}{t}}$. Define $$r: \mathbb{R} \oplus E \to \mathbb{R} \oplus E, \quad r(t, f) = (t, \pi_t(f)).$$ Clearly, $r \circ r = r$. We shall prove below that the map $r$ is sc-smooth. Consequently, $r$ is an sc-smooth retraction and the image $r(\mathbb{R} \oplus E)$ of $r$, which is the subset $$\{(t, 0) \mid t \leq 0\} \cup \{(t, s \cdot \beta(\cdot + e^{\frac{1}{t}})) \mid t > 0, \ s \in \mathbb{R}\}$$
Figure 1: The retract in the example is homeomorphic to the set \((\mathbb{R}^- \times \{0\}) \cup (\mathbb{R}^+ \times \mathbb{R})^0\).

of \(\mathbb{R} \times E\), is an sc-smooth retract. Observe that the above retract is connected and consists of 1- and 2-dimensional parts.

Out of this example one can define more retractions which have parts of any finite dimension. The above example is enough to show that the subspace \(S\) of the plane obtained by taking the open unit disk and attaching a closed interval to it by mapping the end points to the unit circle in fact admits an sc-smooth atlas, i.e. is a generalization of a manifold in the sc-smooth world, see Figure 2.

Observe that \(S\) with this M-polyfold structure satisfies \(S_m = S\), i.e. the induced filtration is constant. This implies that \(S\) has a tangent space at every point. As a consequence of later constructions one could even construct a family of sc-projections

\[\rho_t : E \rightarrow E\]

where \(t \in \mathbb{R}\) and where \(E\) is the sc-Banach space from above, such that \(\rho_t = 0\) for \(t \leq 0\) and \(\rho_t\) has an infinite-dimensional image for \(t > 0\), and such that the map \((t, f) \mapsto (t, \rho_t(f))\) is sc-smooth. Hence the local jumps of dimension can be quite stunning. Of course, we can even combine the two previous examples in various ways.

We shall prove that the retraction \(r : V \oplus E \rightarrow V \oplus E\) is sc-smooth. We recall that taking a strictly increasing sequence \((\delta_m)_{m \geq 0}\) of real numbers starting with \(\delta_0 = 0\), the space \(E = L^2(\mathbb{R})\) is equipped with the sc-structure \((E_m)_{m \in \mathbb{N}_0}\) where \(E_m = H^{m, \delta_m}(\mathbb{R})\). We choose a smooth function
Figure 2: This figure shows a topological space obtained from the open unit disk by adding a closed arc. This space carries a smooth M-polyfold structure.

$\beta : \mathbb{R} \rightarrow [0, \infty)$ having support contained in the compact interval $[-A, A]$ and satisfying $|\beta|_{L^2} = 1$. Define the map $\Phi : \mathbb{R} \oplus E \rightarrow E$ by

$$
\Phi(t, u) = \begin{cases}
\langle u, \beta(\cdot + e_1^t) \rangle_{L^2} \cdot \beta(\cdot + e_1^t) & t > 0 \\
0 & t \leq 0.
\end{cases}
$$

Lemma 1.23. The map $\Phi$ is of class $sc^\infty$.

Proof. It is clear that the restriction $\Phi : (\mathbb{R} \setminus \{0\}) \oplus E_m \rightarrow E_m$ is smooth. Abbreviate $F(t, s) = \beta(s + e_1^t)$ for $s \in \mathbb{R}$ and $t > 0$. Observe that for the $k$-th derivative of $F$ with respect to $t$ we have the estimate,

$$
|F^{(k)}(t, s)| \leq p(t),
$$

where $p(t) = P(e_1^t, \frac{1}{t})$ is a polynomial of degree $k$ in the variable $e_1^t$ and of degree $2k$ in the variable $\frac{1}{t}$. It depends on $\beta$ and has nonnegative coefficients. In addition, the function $s \mapsto F^{(k)}(t, s)$ has its support contained in the interval $I_t := [-A - e_1^t, A - e_1^t]$. Note that $I_t \subset (-\infty, 0)$ if $t > 0$ is sufficiently small.
Then if $u \in E_m$ and if $t > 0$ is small we can estimate

$$
\langle u, F(t, \cdot) \rangle_{L^2}^2 = \int_{\mathbb{R}} |u(s)|^2 F(t, s)^2 \, ds
= \int_{I_t} |u(s)|^2 F(t, s)^2 e^{2\delta_m s} e^{-2\delta_m s} \, ds
\leq C e^{-2\delta_m e^{\frac{t}{t}}} \int_{I_t} |u(s)|^2 e^{-2\delta_m s} \, ds
$$

(2)

where we have used that

$$
\max_{s \in I_t} |F(s,t)|^2 e^{2\delta_m s} \leq C e^{-\delta_m e^{\frac{t}{t}}}
$$

Similarly one finds

$$
\langle u, F^{(j)}(t, \cdot) \rangle_{L^2}^2 = \int_{\mathbb{R}} |u|^2 F^{(j)}(t, s)^2 \, ds
= \int_{I_t} |u|^2 F^{(j)}(t, s)^2 e^{2\delta_m s} e^{-2\delta_m s} \, ds
\leq C p(t)^2 e^{-2\delta_m e^{\frac{t}{t}}} \int_{I_t} |u|^2 e^{-2\delta_m s} \, ds
$$

(3)

Using the estimate (2) we shall first show that the map $\Phi : \mathbb{R} \oplus E_m \to E_m$ is continuous at a point $(0, u_0)$. Take $h \in E_m$ satisfying $|h|_m < \varepsilon$ and set $u := u_0 + h$. By definition, $\Phi(0, u_0) = 0$ and so,

$$
|\Phi(t, u) - \Phi(0, u_0)|_m^2 = \langle u, F(t, \cdot) \rangle_{L^2} \sum_{i \leq m} \int_{I_t} |\partial_s^i F(t, s)|^2 e^{-2\delta_m s} \, ds
\leq C e^{-2\delta_m e^{\frac{t}{t}}} \left( \int_{I_t} |u|^2 e^{-2\delta_m s} \, ds \right) \cdot \left( \int_{I_t} e^{-2\delta_m s} \, ds \right)
\leq C \int_{I_t} |u|^2 e^{-2\delta_m s} \, ds \leq C \left( \int_{I_t} |u_0|^2 e^{-2\delta_m s} \, ds + \int_{I_t} |h|^2 e^{-2\delta_m s} \, ds \right)
\leq C \int_{I_t} |u_0|^2 e^{-2\delta_m s} \, ds + C \cdot \varepsilon.
$$

Since $\int_{I_t} |u_0(s)|^2 e^{-2\delta_m s} \, ds \to 0$ as $t \to 0^+$, one concludes that $\Phi(t, u) \to 0$ as $(t, u) \to (0, u_0)$ in $\mathbb{R} \oplus E_m$. So far we have proved that the map $\Phi : \mathbb{R} \oplus E \to E$ is $sc^0$. In order to prove that the map $\Phi$ is $sc^\infty$ we proceed by induction. Our inductive statements are as follows.
(Sk). The map $\Phi : \mathbb{R} \oplus E \to E$ is sc$^k$ and $T^k\Phi(t_1, u_1, \ldots, t_{2k}, u_{2k}) = 0$ if $t_1 \leq 0$. Moreover, if $\pi : T^kE \to E_m$ is a projection onto the factor $E_m$ of $T^kE$, then the composition $\pi \circ T^k\Phi$ at the point $(t_1, u_1, \ldots, t_{2k}, u_{2k})$ where $t_1 > 0$ is a linear combination of maps $\Gamma$ of the following type,

$$\Gamma : \mathbb{R}^{k+1} \oplus E_n \to E_m$$

$$(t_1, t_2, \ldots, t_{k+1}, u) \mapsto \langle u, F'(t_1, \cdot) \rangle_{L^2} \cdot t_2 \cdot \ldots \cdot t_{i+j}$$

where $n \geq m$ and $i + j \leq k$, and this holds for every $m \geq 0$.

We start with $k = 1$. The candidate for the linearization $D\Phi(t_1, u_1) : \mathbb{R} \oplus E_m \to E_m$ at a point $(t_1, u_1) \in \mathbb{R} \oplus E_{m+1}$ is given by

$$D\Phi(t_1 u_1) = 0, \quad \text{if } t_1 \leq 0$$

and if $t_1 > 0$, then the candidate is equal to

$$D\Phi(t_1, u_1)(t_2, u_2) = \langle u_2, F(t_1, \cdot) \rangle_{L^2} F(t_1, \cdot) \|

+ \langle u_1, F'(t_1, \cdot) \rangle_{L^2} F(t_1, \cdot) t_2 + \langle u_1, F(t_1, \cdot) \rangle_{L^2} F'(t_1, \cdot) t_2.$$  \hfill (4)

Clearly, $D\Phi(t_1, u_1) : \mathbb{R} \oplus E_m \to E_m$ is a bounded linear map. Moreover, for $t_1 \neq 0$, the linear map $D\Phi(t_1, u_1) : \mathbb{R} \oplus E_{m+1} \to E_m$ is the derivative of the map $\Phi : \mathbb{R} \oplus E_{m+1} \to E_m$ at the point $(t_1, u_1)$. We shall show that the derivative of $\Phi : \mathbb{R} \oplus E_{m+1} \to E_m$ at the point $(0, u_1)$ is the zero map. To do this, we estimate $|\Phi(t, u)|_m$ where $u \in E_n$ and $n > m$ using (2),

$$|\Phi(t, u)|^2_m = \langle u, F(t, \cdot) \rangle_{L^2} \sum_{i \leq m} \int_{I_t} |\partial^i_s F(t, s)|^2 e^{-2\delta m s} \, ds$$

$$\leq C e^{-2\delta n e^\frac{1}{2}} \left( \int_{I_t} |u|^2 e^{-2\delta n s} \, ds \right) \cdot \left( \int_{I_t} e^{-2\delta m s} \, ds \right)$$

$$\leq C e^{-2(\delta n - \delta m) e^\frac{1}{2}} \int_{I_t} |u|^2 e^{-2\delta n s} \, ds \leq C e^{-2(\delta n - \delta m) e^\frac{1}{2}} |u|^2_n.$$  \hfill (5)

For $n = m + 1$ we conclude, using that $\delta_{m+1} > \delta_m$,

$$\frac{1}{|\delta t| + |\delta u_1|_{m+1}} |\Phi(\delta t, u_1 + \delta u_1)|_m \leq C \cdot \frac{e^{-(\delta_{m+1} - \delta_m) e^\frac{1}{2}}}{|\delta t|} \cdot |u_1 + \delta u_1|_{m+1} \to 0$$

as $|\delta t| + |\delta u_1|_{m+1} \to 0$ proving that $D\Phi(0, u) = 0$. Moreover, the estimate (3) implies as $t_1 \to 0$ that

$$D\Phi(t_1, u_1)(t_2, u_2) \to 0 \quad \text{in } E_m$$

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where \((t_1, u_1) \in \mathbb{R} \oplus E_{m+1}\) and \((t_2, u_2) \in \mathbb{R} \oplus E_m\). Consequently, the tangent map

\[
T\Phi : T(\mathbb{R} \oplus E) \to TE
\]

\[
T\Phi(t_1, u_1, t_2, u_2) = (\Phi(t_1, u_1), D\Phi(t_1, u_1)(t_2, u_2))
\]
is sc\(^0\), and hence the map \(\Phi : \mathbb{R} \oplus E \to E\) is of class sc\(^1\). Moreover, in view of \([4]\), one sees that for \(t_1 > 0\) the composition \(\pi \circ T\Phi\) is of the form required in \(S_1\). We have proved that the statement \(S_1\) holds.

Assuming that we have proved the statement \(S_k\), we shall show that also the statement \(S_{k+1}\) holds true. To see this it suffices to consider the map \(\Gamma : \mathbb{R}^{k+1} \oplus E_n \to E_m\) defined by

\[
\Gamma(t_1, \ldots, t_{i+j}, u) = 0
\]

if \(t_1 \leq 0\), and if \(t_1 > 0\) it is defined as

\[
\Gamma(t_1, \ldots, t_{i+j}, u) = \langle u, F^{(i)}(t_1, \cdot)\rangle_{L^2} F^{(j)}(t_1, \cdot) \cdot t_2 \cdot \ldots \cdot t_{i+j}
\]

where \(n \geq m\) and \(i + j \leq k\). We have to show that \(\Gamma\) is of class sc\(^1\). This map is clearly smooth for \(t_1 \neq 0\). Take a point \((t_1, t_2, \ldots, t_{k+1}, u) \in \mathbb{R}^{k+1} \oplus E_{n+1}\).

If \(t_1 < 0\), then \(D\Gamma(t_1, t_2, \ldots, t_{k+1}, u) = 0\) and we claim that this holds also at \(t_1 = 0\). Indeed, we consider the quotient

\[
\frac{1}{|\delta t_1| + \ldots + |\delta t_{i+j}| + |u|_{n+1}} |\Gamma(\delta t_1, t_2 + \delta t_2, \ldots, t_{i+j} + \delta t_{i+j}, u + \delta u)|_m.
\]

If \(\delta t_1 \leq 0\), then \(\Gamma\) is equal to 0 and if \(\delta t_1 > 0\), one obtains for \(\Gamma\) an estimate similar to the one in \([5]\). Hence the quotient is bounded by

\[
\frac{|\Gamma(\delta t_1, t_2 + \delta t_2, \ldots, t_{k+1} + \delta t_{k+1}, u + \delta u)|_m}{|\delta t_1|} \leq C e^{-2(\delta_{n+1} - \delta_m) \epsilon \frac{1}{|\delta t_1|}} \cdot |u + \delta u|_{n+1} \to 0
\]
as \(\delta t_1 \to 0^+\). We have proved that \(D\Gamma(0, t_2, \ldots, t_{i+j}, u) = 0\). If \(t_1 > 0\), then the linearization \(D\Gamma(t_1, t_2, \ldots, t_{i+j}, u) : \mathbb{R}^{k+1} \oplus E_n \to E_m\) is equal to

\[
D\Gamma(t_1, \ldots, t_{i+j}, u)(\delta t_1, \ldots, \delta t_{i+j}, \delta u)
\]

\[
= \langle \delta u, F^{(i)}(t_1, \cdot)\rangle_{L^2} F^{(j)}(t_1, \cdot) \cdot t_2 \cdot \ldots \cdot t_{i+j}
\]

\[
+ \langle u, F^{(i+1)}(t_1, \cdot)\rangle_{L^2} F^{(j)}(t_1, \cdot) \cdot \delta t_1 \cdot t_2 \cdot \ldots \cdot t_{i+j}
\]

\[
+ \langle u, F^{(i)}(t_1, \cdot)\rangle_{L^2} F^{(j+1)}(t_1, \cdot) \cdot \delta t_1 \cdot t_2 \cdot \ldots \cdot t_{i+j}
\]

\[
+ \sum_{i=2}^{i+j} \langle u, F^{(j)}(t_1, \cdot)\rangle_{L^2} F^{(j)}(t_1, \cdot) \cdot t_2 \cdot \ldots \delta t_i \cdot \ldots \cdot t_{i+j}.
\]
Now one easily verifies that the remaining statements of $S_{k+1}$ are satisfied. Hence the map $\Phi$ is $sc^k$ for every $k$ and consequently $sc^\infty$.

**Remark 1.24.** Using the standard definition of Frechét differentiability a subset of a Banach space is a (classically) smooth retract of an open neighborhood if and only if it is a submanifold. So relaxing the notion of smoothness to $sc$-smoothness, which still is fine enough to detect boundaries with corners, we obtain new smooth spaces as the above example illustrates.

In order to carry out basic constructions known from differential geometry one quite often needs partitions of unity. It is known that Hilbert spaces always admit smooth partitions of unity, whereas there are Banach spaces which do not. However, as it turns out, there are many $sc$-Banach spaces for which we have $sc$-smooth partitions of unity. This will become clearer after the next section, particularly from Proposition 2.4 and Corollary 2.5.

We finish this section with a question.

**Question.** Assume that $X$ is a connected second countable paracompact $M$-polyfold without boundary built on $sc$-smooth retracts in separable Hilbert spaces equipped with $sc$-structures of Hilbert spaces. Is it true that there exists a local $sc$-model $(O, H, H)$ in which $H$ is a separable Hilbert space equipped with $sc$-structures of Hilbert spaces, so that $X$ is $sc$-diffeomorphic to $O$? In other words, can $X$ be covered by only one chart?

### 1.3 Sc-Smoothness Arising in Applications

We describe several constructions which will be used in the constructions of polyfolds (a generalization of $M$-polyfolds) in the Gromov-Witten theory and the Symplectic Field Theory (SFT). It is not our intention to prove the most general results. We just choose the examples in such a way that they cover the cases needed for the construction in SFT.

We denote by $D$ the closed unit disk in $\mathbb{C}$ and equip the Hilbert space $E = H^3(D, \mathbb{R}^N)$ of maps from $D$ into $\mathbb{R}^N$ with the $sc$-structure defined by the sequence $E_m = H^{3+m}(D, \mathbb{R}^N)$ for all $m \in \mathbb{N}_0$. We choose a map $u$ belonging to $E_\infty$, satisfying $u(0) = 0$ and, in addition, has a derivative $Du(0) : \mathbb{C} \to \mathbb{R}^N$ which is injective. By $H$ we abbreviate an algebraic complement of the image of the derivative $Du(0)$. Thus, $u$ intersects $H$ transversally at $0$. By the implicit function theorem, we find an open neighborhood $U$ of $u$ in $C^1(D, \mathbb{R}^N)$ and $0 < \varepsilon < 1$ so that for every $v \in U$, there exists a unique point $z_v$ in the
\[ \varepsilon\text{-disk around } 0 \text{ in } \mathbb{C} \text{ whose image under } v, v(z_v), \text{ lies in } H \text{ and such that the image } Dv(z_v) \text{ is transversal to } H. \] Moreover, the map

\[ U \rightarrow D, \quad v \mapsto z_v \]

is \( C^1 \). If we view this map as defined on \( U \cap C^k \), then it is actually of class \( C^k \).

**Theorem 1.25.** If \( U \cap E \) is equipped with the induced sc-structure, then the map

\[ U \cap E \rightarrow \mathbb{C}, \quad v \mapsto z_v \]

is sc-smooth.

**Proof.** The map \( U \cap E \rightarrow \mathbb{C} \) defined by \( v \mapsto z_v \) is of class \( C^{m+1} \) in view of the Sobolev embedding \( H^{m+3}(D, \mathbb{R}^N) \rightarrow C^{m+1}(D, \mathbb{R}^N) \). By Proposition 1.9 below, the map is sc-smooth. \( \square \)

In order to describe a typical application of Theorem 1.25 we consider a map \( u : S^2 \rightarrow M \) of class \( H^3 \) defined on the Riemann sphere \( (S^2, i) \) into a smooth manifold \( M \) and assume that \( H_1, H_2 \) and \( H_3 \) are three submanifolds of codimension 2 intersecting \( u \) transversally at the three different points \( z_1, z_2 \) and \( z_3 \) in \( S^2 \). In view of the previous result, under the map \( v \) which is \( H^3 \)-close to \( u \), the points \( z_i \) move to the points \( z_i(v) \) at which the map \( v \) intersects submanifolds \( H_i \) transversally. There exists a unique Möbius transformation on \( (S^2, i) \) mapping \( z_i \) to \( z_i(v) \). We denote this Möbius transformation by \( \phi_v \) and consider the composition

\[ \Phi : E \rightarrow E, \quad v \mapsto \Phi(v) := v \circ \phi_v \]

defined for maps \( v \) which are \( H^3 \)-close to \( u \). Note that \( \Phi(v) \) satisfies the transversal constraints at the original points \( z_i \). We shall prove that the map \( \Phi \) is sc-smooth. This is a consequence of the result we describe next.

We consider a family \( v \mapsto \phi_v \) of maps \( \phi_v : D \rightarrow \mathbb{C} \) parametrized by \( v \) belonging to some open neighborhood of the origin in \( \mathbb{R}^n \). Moreover, we assume that the map

\[ V \times D \rightarrow \mathbb{C}, \quad (v, x) \mapsto \phi_v(x) \]

is smooth, satisfies \( \phi_0(0) = 0 \) and

\[ \phi_v(x) \in D \quad \text{for all } x \in D \text{ and } v \in V, \]
so that the map
\[ \Phi : V \oplus E \to E, \quad (v, u) \mapsto u \circ \phi_v \]
is well-defined. The map \( \Phi \) is not even differentiable in the classical sense, due to the loss of one derivative, but it is smooth in the sense of \( \text{sc-smoothness} \), as the next result shows.

**Theorem 1.26.** The composition \( \Phi : V \oplus E \to E \) defined by \((v, u) \mapsto u \circ \phi_v \) is an \( \text{sc-smooth} \) map.

Theorem 1.26 will be proved in Section 2.2 below.

The concept of an \( \text{sc-smooth} \) retract is motivated by our gluing and anti-gluing constructions described next. The retract is a model for a smooth structure on a 2-dimensional family of Sobolev spaces of functions defined on finite cylinders which become longer and longer and eventually break apart into two half-cylinders. It also incorporates a twisting with respect to the angular variable. Such a situation arises in the study of function spaces on Riemann surfaces if the surfaces develop nodes.

We take a strictly increasing sequence \((\delta_m)_{m \in \mathbb{N}_0}\) starting with \(\delta_0 > 0\) and denote by \( E \) the space consisting of pairs \((u^+, u^-)\) of maps
\[ u^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^N \]
having the following properties. There is a common asymptotic limit \( c \in \mathbb{R}^N \) so that the maps
\[ r^\pm := u^\pm - c \]
have partial derivatives up to order 3 which if weighted by \( e^{\delta_0|s|} \) belong to \( L^2(\mathbb{R}^\pm \times S^1, \mathbb{R}^N) \). We equip the Hilbert space \( E \) with the sc-structure \((E_m)_{m \in \mathbb{N}_0}\) where \( E_m \) consists of those pairs \((u^+, u^-)\) in \( E \) for which \((r^+, r^-)\) are of Sobolev class \((3 + m, \delta_m)\). The \( E_m \)-norm of the pair \((u^+, u^-)\) is defined as
\[ \left| (\eta^+, \eta^-) \right|^2_{E_m} = |c|^2 + \left| r^+ \right|^2_{H^{3+m, \delta_m}} + \left| r^- \right|^2_{H^{3+m, \delta_m}}, \]
where
\[ \left| r^\pm \right|^2_{3+m, \delta_m} := \sum_{|\alpha| \leq 3+m} \int_{\mathbb{R}^\pm \times S^1} |D^\alpha r^\pm(s, t)|^2 e^{2\delta_m|s|} \, dsdt. \]

Our aim is the construction of an M-polyfold model \((O, \mathbb{C} \oplus E, \mathbb{C} \oplus E)\) where the set \( O \) is obtained by an \( \text{sc-smooth} \) retraction \( r : B^+_{1/2} \oplus E \to B^+_{1/2} \oplus E \) of the special form
\[ r(a, u^+, u^-) = (a, \pi_a(u^+, u^-)), \]
where $a \mapsto \pi_a$ is a family of linear and bounded projections parametrized by $a \in B_{1/2}$. The set $B_{1/2}$ is the open disk of radius $\frac{1}{2}$ in $\mathbb{C}$.

To proceed we need a gluing profile $\varphi$. By definition, this is a diffeomorphism $(0, 1] \to [0, \infty)$ with suitable growth properties. Convenient for our purposes is the exponential gluing profile defined by

$$\varphi(r) = e^r - e, \quad r \in (0, 1].$$

With the complex numbers $|a| < \frac{1}{2}$ we associate the abstract cylinders $C_a$ as follows. If $a = 0$, the cylinder $C_0$ is the disjoint union

$$C_0 = \mathbb{R}^- \times S^1 \bigsqcup \mathbb{R}^+ \times S^1.$$

We also use $Z_0$ to denote $C_0$. If $a \neq 0$ and $a = |a|e^{-2\pi i \theta}$ is its polar form, we set

$$R = \varphi(|a|).$$

We call $a$ the gluing parameter and $R$ obtained this way the gluing length. To define the cylinder $C_a$, we take the disjoint union of the half-cylinders $\mathbb{R}^+ \times S^1$ and $\mathbb{R}^- \times S^1$ and identify points $(s, t) \in [0, R] \times S^1$ with points $(s', t') \in [-R, 0] \times S^1$ if the following relations hold,

$$s = s' + R \quad \text{and} \quad t = t' + \theta.$$

On $C_a$ we have two sets of conformal coordinates, namely, $[s, t] \mapsto (s, t)$ obtained by extending the coordinates coming from $\mathbb{R}^+ \times S^1$ and $[s', t'] \mapsto (s', t')$ obtained by extending the coordinates coming from $\mathbb{R}^- \times S^1$. The infinite cylinder contains the finite subcylinder $Z_a$ consisting of the equivalence classes $[s, t]$ with $s \in [0, R]$.

Next, for $a \in B_{1/2}$, we define the sc-Hilbert space $G^a$. First, we introduce the space $H^{3, b_0}_c(C_a)$. To do this, we choose a smooth function $\zeta : \mathbb{R} \to [0, 1]$ satisfying $\zeta(s) = 1$ for $s \leq -1$, $\zeta'(s) < 0$ for $s \in (-1, 1)$, and $\zeta(s) + \zeta(-s) = 0$. The space $H^{3, b_0}_c(C_a)$ consists of those maps $u : C_a \to \mathbb{R}^N$ which belong to $H^3_{\text{loc}}(C_a)$ and there is a constant $c \in \mathbb{R}^N$ such that $u + \zeta_a \cdot c \in H^{3, b_0}_c(C_a)$, where

$$\zeta_a(s) := \zeta \left( s - \frac{R}{2} \right).$$

We note that if $u \in H^{3, b_0}_c(C_a)$ and $c$ is an asymptotic constant of $u$ at $\infty$, then the asymptotic constant at $-\infty$ is equal to $-c$. With the nested sequence
Figure 3: Glued finite and infinite cylinders $Z_a$ and $C_a$

$(H^{3+m,\delta_m}_c(C_a))_{m \in \mathbb{N}_0}$ of Hilbert spaces, the Hilbert space $H^{3,\delta_0}_c(C_a)$ becomes an sc-Hilbert space. The norms of these Hilbert spaces will be introduced in Section 2.5. We also equip the Hilbert space $H^3(Z_a)$ with the sc-structure given by $(H^{3+m}(Z_a))_{m \in \mathbb{N}_0}$.

Now, if $a = 0$, we set

$$G^0 = E \oplus \{0\},$$

and if $a \neq 0$, we define

$$G^a = H^3(Z_a) \oplus H^{3,\delta_0}_c(C_a).$$

The sc-structure of $G^a$ is given by the sequence $H^{3+m}(Z_a) \oplus H^{3+m,\delta_m}_c(C_a)$ for all $m \in \mathbb{N}_0$.

For every $a \in B_{\frac{1}{2}}$, we define the so-called total gluing map

$$\square_a = (\oplus_a, \ominus_a) : E \to G^a$$

as follows. We fix a smooth map $\beta : \mathbb{R} \to [0,1]$ satisfying

- $\beta(-s) + \beta(s) = 1$ for all $s \in \mathbb{R}$
- $\beta(s) = 1$ for all $s \leq -1$
- $\beta'(s) < 0$ for all $s \in (-1,1)$

(6)
Moreover, if $0 < |a| < \frac{1}{2}$ we abbreviate by $\beta_a$ or $\beta_R$ the translated function

$$\beta_a(s) = \beta_R(s) = \beta \left( s - \frac{R}{2} \right), \quad s \in \mathbb{R},$$

where $R = \varphi(a)$.

Figure 4: Graph of the function $\beta_a$ where $R = \varphi(|a|)$.

Now, if $a = 0$, we define

$$\oplus_0(u^+, u^-) = (u^+, u^-) \quad \text{and} \quad \ominus_0(u^+, u^-) = 0 \in \{0\}.$$

If $a \neq 0$ and $a = |a|e^{-2\pi i \vartheta}$ is its polar form, we set $R = \varphi(|a|)$ and glue the two maps $u^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^N$ which are defined on positive respectively negative cylinder to a map defined on the glued cylinder $Z_a$ by means of the following convex sum

$$\oplus_a(u^+, u^-)([s, t]) = \beta_a(s) u^+(s, t) + (1 - \beta_a(s)) u^-(s - R, t - \vartheta).$$

for $[s, t] \in Z_a$ where $0 \leq s \leq R$. Further, we introduce the map $\ominus_a(u^+, u^-)$ on the glued infinite cylinder $C_a$ by the formula for $[s, t] \in C_a$,

$$\ominus_a(u^+, u^-)([s, t]) = -(1 - \beta_a(s)) \left[ u^+(s, t) - \text{av}_a(u^+, u^-) \right] + \beta_a(s) \left[ u^-(s - R, t - \vartheta) - \text{av}_a(u^+, u^-) \right]$$

where $s \in \mathbb{R}$ and where the average is defined by

$$\text{av}_a(u^+, u^-) := \frac{1}{2} \int_{S^1} \left[ u^+ \left( \frac{R}{2}, t \right) + u^- \left( -\frac{R}{2}, t \right) \right] dt$$

and $R = \varphi(|a|)$. We have used that $(1 - \beta_a(s)) = 0$ for $s \leq \frac{R}{2} - 1$ and $\beta_a(s) = 0$ if $s \geq \frac{R}{2} + 1$.

The map $\oplus_a(u^+, u^-)$ is called the glued map for the gluing parameter $a$ and the map $\ominus_a(u^+, u^-)$ is called the anti-glued map.
Theorem 1.27. For every \( a \in B_{\frac{1}{2}} \) the total gluing map

\[ \Box_a = (\oplus_a, \ominus_a) : E \to G^a \]

is a linear sc-isomorphism. In particular,

\[ E = (\ker \oplus_a) \oplus (\ker \ominus_a) \]

and for \( a \neq 0 \), the map

\[ \oplus_a : \ker \ominus_a \to H^3(Z_a) \]

is a linear sc-isomorphism. If \( a = 0 \), the map \( \oplus_0 \) is the identity map.

We postpone the proof until after the next theorem.

The gluing and anti-gluing maps determine the linear projection \( \pi_a : E \to E \) onto the kernel of the anti-glued map \( \ominus_a : E \to H^3_{c,\delta_0}(C_a) \) along the kernel of the glued map \( \oplus_a : E \to H^3(Z_a) \) by means of the following formulae for \( (h^+, h^-) \in E \),

\[
\begin{align*}
\oplus_a \circ (1 - \pi_a)(h^+, h^-) &= 0 \\
\ominus_a \circ \pi_a(h^+, h^-) &= 0.
\end{align*}
\]

Given the pair \((h^+, h^-) \in E\), we set

\[ \pi_a(h^+, h^-) = (\eta^+, \eta^-) \]

so that the pair \((\eta^+, \eta^-) \in E\) is determined by the two equations

\[
\begin{align*}
\oplus_a(\eta^+, \eta^-) &= \oplus_a(h^+, h^-) \\
\ominus_a(\eta^+, \eta^-) &= 0.
\end{align*}
\]

Explicitly, abbreviating \( \beta_a = \beta_a(s) \),

\[
\beta_a \cdot \eta^+(s, t) + (1 - \beta_a) \cdot \eta^-(s - R, t - \vartheta) = \beta_a \cdot h^+(s, t) + (1 - \beta_a) \cdot h^-(s - R, t - \vartheta)
\]

if \( 0 \leq s \leq R \), and

\[
-(1 - \beta_a) \cdot [\eta^+(s, t) - \text{av}_R(\eta^+, \eta^-)] + \beta_a \cdot [\eta^-(s - R, t - \vartheta) - \text{av}_R(\eta^+, \eta^-)] = 0
\]

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for all $s \in \mathbb{R}$. Observing that $\beta_a(R) = \frac{1}{2}$ and integrating the first equation at $s = \frac{R}{2}$ over the circle $S^1$, we obtain for the averages

$$\text{av}_a(\eta^+, \eta^-) = \text{av}_a(h^+, h^-)$$

so that the equations for $(\eta^+, \eta^-)$ become

$$
\begin{bmatrix}
\beta_a & 1 - \beta_a \\
-(1 - \beta_a) & \beta_a
\end{bmatrix}
\begin{bmatrix}
\eta^+ \\
\eta^-
\end{bmatrix} =
\begin{bmatrix}
\beta_a h^+ + (1 - \beta_a) h^- \\
(2\beta_a - 1)\text{av}_a(h^+, h^-)
\end{bmatrix}
$$

where we have abbreviated $\beta_a = \beta_a(s)$, $h^+ = h^+(s, t)$ and $h^- = h^-(s - R, t - \vartheta)$. Introducing the nowhere vanishing function $\gamma$ by

$$\gamma(s) := \beta(s)^2 + (1 - \beta(s))^2,$$

the determinant of the matrix on the left hand side is equal to $\gamma_a$ which is defined by

$$\gamma_a(s) := \gamma\left(s - \frac{R}{2}\right).$$

Hence, multiplying both sides by the inverse of this matrix, we arrive at the formula

$$
\begin{bmatrix}
\eta^+(s,t) \\
\eta^-(s-R,t-\vartheta)
\end{bmatrix} = \frac{1}{\gamma_a}
\begin{bmatrix}
\beta_a & -(1 - \beta_a) \\
1 - \beta_a & \beta_a
\end{bmatrix}
\begin{bmatrix}
\beta_a h^+ + (1 - \beta_a) h^- \\
(2\beta_a - 1)\text{av}_a(h^+, h^-)
\end{bmatrix}
$$

where we abbreviated $\gamma_a = \gamma_a(s)$. If we denote by $A$ the common asymptotic limit of $(h^+, h^-)$ so that

$$(h^+, h^-) = (A + r^+, A + r^-)$$

we arrive at the following explicit representation of the map $\pi_a$.

If $(h^+, h^-) = (A + r^+, A + r^-) \in E$ and $(\eta^+, \eta^-) = \pi_a(h^+, h^-)$, then

$$\eta^+(s,t) = A + \left(1 - \frac{\beta_a(s)}{\gamma_a(s)}\right)\text{av}_a(r^+, r^-)$$

$$+ \frac{\beta_a(s)}{\gamma_a(s)} [\beta_a(s) \cdot r^+(s,t) + (1 - \beta_a(s)) \cdot r^-(s - R, t - \vartheta)]$$

for all $s \geq 0$. There is a similar formula for $\eta^-(s', t')$. 29
One reads off that \((\eta^+, \eta^-) \in E\) so that the asymptotic limits of \(\eta^+\) and \(\eta^-\) coincide. Indeed, the asymptotic limits for \((\eta^+, \eta^-) = \pi_a(h^+, h^-)\) are the following,

\[
\lim_{s \to \infty} \eta^+(s, t) = \lim_{s \to -\infty} \eta^-(s, t) = \text{av}_a(h^+, h^-) = A + \text{av}_a(r^+, r^-),
\]

if \(h^\pm = A + r^\pm\), with the common limit \(A\) of \(h^\pm\). In addition,

\[
\text{av}_a(\eta^+, \eta^-) = \text{av}_a(h^+, h^-),
\]

so that the projection map \(\pi_a\) leaves the averages invariant. A computation shows that

\[
\pi_a \circ \pi_a(h^+, h^-) = \pi_a(h^+, h^-)
\]

so that the linear map \(\pi_a\) is indeed a projection.

**Theorem 1.28.** If \(\pi_a : E \to E\) denotes the above linear projection of \(E\) onto \(\ker \ominus_a\) along \(\ker \oplus_a\), then the map

\[
r : B^1 \oplus E \to B^1 \oplus E, \quad (a, u^+, u^-) \mapsto (a, \pi_a(u^+, u^-))
\]

is an sc-smooth retraction.

Theorem 1.28 will be proved in Section 2.4. At this point we shall prove Theorem 1.27.

**Proof of Theorem 1.27.** If \(a = 0\), there is nothing to prove. We assume that \(a \neq 0 \in B^1\), take \((\eta^+, \eta^-) \in E\), and set \(\oplus_a(\eta^+, \eta^-) = u\) and \(\ominus_a(\eta^+, \eta^-) = v\). Explicitly,

\[
\beta_a(s) \cdot \eta^+(s, t) + (1 - \beta_a(s)) \cdot \eta^-(s - R, t - \vartheta) = u(s, t)
\]

for \(0 \leq s \leq R\), and

\[
-(1 - \beta_a(s)) \left[\eta^+(s, t) - \text{av}_a(\eta^+, \eta^-)\right] + \beta_a(s) \left[\eta^-(s - R, t - \vartheta) - \text{av}_a(\eta^+, \eta^-)\right] = v(s, t)
\]

for all \(s \in \mathbb{R}\). Since \(\beta_a(s) = 0\) if \(s \geq \frac{R}{2} + 1\) and \(\beta_a(s) = 1\) if \(s \leq \frac{R}{2} - 1\), we conclude \(\lim_{s \to \infty} u(s, t) = \lim_{s \to \infty} \eta^+(s, t) + \text{av}_a(\eta^+, \eta^-)\) and \(\lim_{s \to -\infty} v(s, t) = \lim_{s \to -\infty} \eta^-(s, t) - \text{av}_a(\eta^-, \eta^-)\). Consequently, \((u, v) \in G^a\).
Conversely, given \((u, v) \in G^a\) we look for a solution \((\eta^+, \eta^-)\) of the two equations \(\oplus_a(\eta^+, \eta^-) = u\) and \(\ominus_a(\eta^+, \eta^-) = v\). Integrating the first equations at \(s = \frac{R}{2}\) over \(S^1\) and observing that \(\beta_a(\frac{R}{2}) = \beta(0) = \frac{1}{2}\), we obtain
\[
\mathcal{A}_a(\eta^+, \eta^-) = \int_{S^1} u\left(\frac{R}{2}, t\right) \, dt =: [u]
\]
and proceeding as above we arrive at the presentation
\[
\begin{bmatrix}
\eta^+(s, t) \\
\eta^-(s - R, t - \vartheta)
\end{bmatrix} = \frac{1}{\gamma_a} \begin{bmatrix}
\beta_a & -(1 - \beta_a) \\
1 - \beta_a & \beta_a
\end{bmatrix} \begin{bmatrix}
\eta^+(s, t) \\
\eta^-(s - R, t - \vartheta)
\end{bmatrix}
\]
where we have abbreviated \(\beta_a = \beta_a(s)\), \(\gamma_a = \gamma_a(s)\), \(u = u(s, t)\) and \(v = v(s, t)\). For the asymptotic limits we read off that \(\lim_{s \to \infty} \eta^+(s, t) = [u] - \lim_{s \to \infty} v(s, t)\) and \(\lim_{s \to \infty} \eta^-(s, t) = [u] + \lim_{s \to -\infty} v(s, t)\). Since the asymptotic limits of \(v\) have opposite signs, we conclude that \(\lim_{s \to \infty} \eta^+(s, t) = \lim_{s \to -\infty} \eta^-(s, t)\) and hence \((\eta^+, \eta^-) \in E\) as desired.

If \(\mathcal{O} = r(B_1 \oplus E)\) is the retract of the sc-smooth retraction guaranteed by Theorem 1.28, the map
\[
\mathcal{O} \to (\{0\} \times E) \bigcup \left( \bigcup_{a \in B_1} \{0\} \times H^3(Z_a) \right)
\]
declared by
\[(a, u^+, u^-) \mapsto (a, \oplus_a(u^+, u^-))\]
is, by construction, a bijection between \(\ker \ominus_a\) and \(E\) if \(a = 0\), respectively \(H^3(Z_a)\) if \(a \neq 0\). We equip the target space with the topology making this map a homeomorphism. Then the inverse of this map is a chart on the local model \(\mathcal{O}\). This way we obtain a construction describing a smooth structure for a suitable 2-dimensional family of function spaces on cylinders which have increasing modulus. We will study this smooth structure in detail in chapter 3.

There is another version of gluing which will be used in the proofs in Section 3. Taking the strictly increasing sequence \((\delta_m)_{m \in \mathbb{N}_0}\), we denote by \(F\) the sc–Hilbert space consisting of pairs \((u^+, u^-)\) of maps
\[
u^\pm: \mathbb{R}^\pm \times S^1 \to \mathbb{R}^N
\]
whose partial derivatives up to order 2 weighted by $e^{-\delta_0|s|}$ belong to $L^2(\mathbb{R}^+ \times S^1)$. We equip $F$ with the sc-structure defined by the sequence $F_m = H^{2+m,\delta_m}(\mathbb{R}^+ \times S^1) \oplus H^{2+m,\delta_m}(\mathbb{R}^- \times S^1)$. For $(u^+, u^-) \in F$, the glued map $\widehat{\oplus}_a(u^+, u^-)$ is defined by the same formula as $\oplus_a(u^+, u^-)$. However, the anti-glued map $\widehat{\ominus}_a$ takes the simpler form

$$
\widehat{\ominus}_a(u^+, u^-)([s, t]) = - (1 - \beta_a(s)) \cdot u^+(s, t) + \beta_a(s) \cdot u^-(s - R, t - \vartheta)
$$

for $[s, t] \in C_a$. The image of the anti-glued map $\widehat{\ominus}_a$ is $H^{2,\delta_0}(C_a)$. Proceeding as before we can define a projection $\widehat{\pi}_a : F \to F$ onto the kernel $\ker \widehat{\oplus}_a$ of the anti-gluing map $\widehat{\ominus}_a$ along the kernel $\ker \widehat{\oplus}_a$ of the gluing map $\widehat{\oplus}_a$.

We abbreviate by $\widehat{G}^a$ the following spaces. If $a = 0$, we set

$$
\widehat{G}^0 = F \oplus \{0\},
$$

and if $a \neq 0$, we define

$$
\widehat{G}^a = H^2(Z_a) \oplus H^{2,\delta_0}(C_a).
$$

The sc-structure of $G^a$ is given by the sequence $H^{2+m}(Z_a) \oplus H^{2+m,\delta_m}(C_a)$ for all $m \in \mathbb{N}_a$. The following theorem holds true.

**Theorem 1.29.**

1. If we equip the Hilbert space $H^2(Z_a) \oplus H^{2,\delta_0}(C_a)$ with the sc–structure $H^{2+m}(Z_a) \oplus H^{2+m,\delta_m}(C_a)$, then the total hat gluing map

$$
\widehat{\boxplus}_a = (\widehat{\oplus}_a, \widehat{\ominus}_a) : F \to \widehat{G}^a
$$

is an sc–smooth isomorphism for every $a \in B_{\frac{1}{2}}$. In particular, $E = (\ker \widehat{\oplus}_a) \oplus (\ker \widehat{\ominus}_a)$.

2. The map

$$
\widehat{\pi} : B_{\frac{1}{2}} \oplus F \to B_{\frac{1}{2}} \oplus F, \quad (a, u^+, u^-) \mapsto (a, \widehat{\pi}_a(u^+, u^-))
$$

is an sc-smooth retraction.

The proof is the same as that of Theorem 1.27 and Theorem 1.28.
Remark 1.30. The same result is true if we consider the projection on the space $H^{3,\delta_0}(\mathbb{R}^+ \times S^1) \oplus H^{3,\delta_0}(\mathbb{R}^- \times S^1)$ rather than on $F$. We choose $F$ here since this particular retraction $\tilde{r}$ occurs in the construction of bundles in SFT accompanying the constructions of base spaces which involve the previous retraction $r$.

Our constructions in Gromov-Witten theory and SFT makes use of the next result described in Theorem 1.31 below which is closely related to the previous constructions. Using Theorem 1.28 we have equipped the set

$$\mathcal{O} := (\{0\} \times E) \cup \bigcup_{a \in B_1 \setminus \{0\}} (\{a\} \times H^3(Z_a, \mathbb{R}^N))$$

with the structure of an M-polyfold. In fact, it is covered by a single chart which also defines the topology. We are interested in sc-smooth maps $\mathcal{O} \to \mathcal{O}$ which will arise in the construction of the polyfolds of SFT.

In order to describe these maps we start with the half-cylinders $\mathbb{R}^\pm \times S^1$ and assume that we are given two smooth families $v \mapsto j^\pm(v)$ of complex structures parameterized by $v$ which belongs to some open neighborhood $V$ of 0 in some finite-dimensional vector space. We require that the complex structures $j^\pm(v)$ away from the boundaries agree with the standard conformal structure. Suppose there exist two smooth families $v \mapsto p^\pm(v)$ of marked points on the boundaries $\partial (\mathbb{R}^\pm \times S^1)$. Given a sufficiently small gluing parameter $a$, we can construct the glued cylinder $Z_a$ equipped with the complex structure $j(a, v)$ induced from $j^\pm(v)$ and smooth families of induced marked points $p^\pm(a, v)$. We have defined the family

$$(a, v) \mapsto (Z_a, j(a, v), p^+(a, v), p^-(a, v))$$

of complex cylinders with marked points. We introduce a second family of finite cylinders as follows. We fix the special marked points $(0, 0)$ on the boundaries of the two standard cylinders. Then, given a gluing parameter $b$, we obtain the finite cylinder cylinder $Z_b$ equipped with the standard complex structure $i$ and the two marked points $p_b^\pm$. This way we obtain a second family

$$b \mapsto (Z_b, i, p_b^+, p_b^-)$$

of complex finite cylinders. It is a standard fact from the uniformization theorem that the cylinder $(Z_a, j(a, v))$ is biholomorphic to the cylinder $(\{0\} \times \mathbb{R}) \times
$S^1, i$) for a uniquely determined $R$. This biholomorphic map is unique up to rotation in the image (and reflection). If we require that the marked point $p^+(a, v)$ is mapped onto the point $p_b^+ = (0, 0)$, we find a uniquely determined complex number $b = b(a, v)$ such that $R = \varphi(|b|)$ and $p^-(a, v)$ is mapped to the marked point $p_b^-$. Thus given the pair $(a, v)$, there is precisely one gluing parameter $b := b(a, v)$ for which there exists a biholomorphic map

$$\Phi_{(a,v)} : (Z_a, j(a, v), p^+(a, v), p^-(a, v)) \rightarrow (Z_b, i, p_b^+, p_b^-).$$

If we use instead of the exponential gluing profile the logarithmic gluing profile $-\frac{1}{2\pi}\ln(r)$ it is well known that the map $(a, v) \mapsto b(a, v)$ is smooth. However, the same is true for the exponential gluing profile. This can be deduced from the result about the logarithmic gluing profile by means of a calculus exercise involving results from Section 4.2. We leave this approach to the reader. Here we shall derive this fact as a corollary from the following more general result. We consider the map which associates with the element $(v, a, w) \in V \oplus O$ for $(v, a)$ small the element $(b(a, v), w') \in O$ in which $w'$ is defined by

$$w' = w \circ \Phi_{(a,v)}^{-1}.$$

**Theorem 1.31.** Let $(\delta_m)_{m \geq 0}$ be a strictly increasing sequence satisfying

$$0 < \delta_m < 2\pi \quad \text{for all } m \geq 0.$$

Then the map

$$V \oplus O \rightarrow O, \quad (v, a, w) \mapsto (b(a, v), w \circ \Phi_{(a,v)}^{-1})$$

defined for $(a, v)$ small is an sc-smooth map for the M-polyfold structure on $O$.

The restriction that the sequence $(\delta_m)$ lies in the open interval $(0, 2\pi)$ is related to the behavior of the map $\Phi_{(a,v)}$ as $a \rightarrow 0$. In order to prove Theorem 1.31 it is necessary to understand the smoothness properties of the map

$$(a, v) \mapsto \Phi_{(a,v)}.$$

This will be part of the illustration in the Section 1.5.
Remark 1.32. There is no loss of generality in assuming that $p^\pm(v) = (0, 0)$ for all $v \in V$. This can be achieved by taking different complex structures $j^\pm(v)$. Indeed, we can choose a smooth family of diffeomorphisms which are supported near the boundary and map the points $p^\pm(v)$ to the point $(0, 0)$. Then we conjugate the original $j^\pm(v)$ with this family to obtain the new one. By the previous discussion, this family of diffeomorphisms acts sc-smoothly. So, Theorem 1.31 will follow once it is proved for the special case just described. We shall assume in the following that this reduction has been carried out.

1.4 Fredholm Theory

In this section we outline the Fredholm theory in M-polyfolds. We shall only describe the case of M-polyfolds without boundaries and refer the reader to [10, 11] for the general case.

We have already introduced the notion of an M-polyfold and now introduce the notion of a strong bundle in the case that the underlying base space does not have a boundary. Let $E$ and $F$ be sc-Banach spaces and let $U$ be an open subset of $E$. We define the nonsymmetric product $U \triangleleft F$ as follows. As a set the product $U \triangleleft F$ is equal to $U \times F$. However, it is equipped with the double filtration $(U \triangleleft F)_{m,k} = U_m \oplus F_k$ defined for all $m \in \mathbb{N}_0$ and all integers $0 \leq k \leq m + 1$. Given $U \triangleleft F$ we have, forgetting part of the structure, the underlying direct sum $U \oplus F$. We view $U \triangleleft F \to U$ as a bundle with base space $U$ and fiber $F$, where the double filtration has the interpretation that above a point $x \in U$ of regularity $m$ it makes sense to talk about the fiber regularity of a point $(x, h)$ up to order $k$ provided $k \leq m + 1$. The tangent space $T_d(U \triangleleft F)$ is defined by

$$T_d(U \triangleleft F) = (TU) \triangleleft (TF).$$

Observe that there is a difference in the order of the factors between $T_d(U \triangleleft F)$ and $T(U \oplus F)$. Indeed,

$$T_d(U \triangleleft F) = (U_1 \oplus E) \triangleleft (F_1 \oplus F)$$

while (for the underlying $U \oplus F$)

$$T(U \oplus F) = U_1 \oplus F_1 \oplus E \oplus F.$$
A map $f : U \triangleleft F \rightarrow V \triangleleft G$ between nonsymmetric products is an $\text{sc}^0$-map if for all $m \in \mathbb{N}_0$ and all $0 \leq k \leq m+1$, 

$$f(U_m \oplus F_k) \subset V_m \oplus G_k$$

and if the induced maps 

$$f : U_m \oplus F_k \rightarrow V_m \oplus G_k$$

are continuous. In addition, we require that the map $f$ is of the form 

$$f(u, h) = (f_0(u), \phi(u, h))$$

where $\phi(u, h)$ is linear in $h$.

We observe that the map $f$ induces $\text{sc}^0$-maps 

$$f : U \oplus F^i \rightarrow V \oplus G^i$$

for $i = 0, 1$. The map $f$ is $\text{sc}^1$ if the maps in (7) for $i = 0, 1$ are both of class $\text{sc}^1$. In this case the tangent maps $Tf : T(U \oplus F^i) \rightarrow T(V \oplus G^i)$ are defined as usual by 

$$Tf(x, h, y, k) = (f_0(x), \phi(x, h), Df_0(x)y, D\phi(x, h)(y, k)).$$

After reordering of factors of the domain and the target spaces, we obtain the $\text{sc}^0$- map 

$$T_\odot f : TU \triangleleft TF \rightarrow TV \triangleleft TG$$

defined by 

$$(x, h, y, k) \mapsto (f_0(x), Df_0(x)y, \phi(x, h), D\phi(x, h)(y, k)).$$

This reordering is consistent with the chain rule and one verifies that if $f$ and $g$ are $\text{sc}^1_\odot$-maps which can be composed, then also the composition $g \circ f$ is of class $\text{sc}^1_\odot$ and satisfies the chain rule, 

$$T_\odot(g \circ f) = (T_\odot g) \circ (T_\odot f).$$

Iteratively one defines the maps of class $\text{sc}^k_\odot$ for $k = 1, 2, \ldots$ and $\text{sc}_\odot$-smooth maps.
Definition 1.33. An $\text{sc}_\sigma$-smooth retraction is an $\text{sc}_\sigma$-smooth map

$$R : U \triangleleft F \to U \triangleleft F$$

satisfying $R \circ R = R$. The image $R(U \triangleleft F)$ is called an $\text{sc}$-smooth strong bundle retract.

It is implicitly required that the above retraction $R : U \triangleleft F \to U \triangleleft F$ is of the from $R(x,h) = (r(x), \rho(x,h))$ where $r : U \to U$ is an $\text{sc}$-smooth retraction and $\rho(x,h)$ is linear in $h$. We denote by $K = R(U \triangleleft F)$ the $\text{sc}$-smooth strong bundle retract and by $O = r(U)$ the associated $\text{sc}$-smooth retract. Then there is a canonical projection map

$$p : K \to O.$$

The set $K$ inherits the double filtration and $p$ maps points of regularity $(m,k)$ to points in $U$ of regularity $m$. The canonical projection $p : K \to O$ is our local model for spaces which we shall call strong $M$-polyfold bundles. Given $K$, we define the sets $K(i)$ for $i = 0, 1$ as follows,

$$K(i) = \{(u,h) \in U \oplus F^i \mid R(u,h) = (u,h)\}.$$ 

Clearly, the set $K(i)$ is an $\text{sc}$-smooth retract and the projection $p : K(i) \to O$ is $\text{sc}$-smooth for $i = 0, 1$.

Now we are in a position to define the notion of a strong bundle. We consider a surjective continuous map $p : W \to X$ between two metrizable spaces, so that for every $x \in X$ the space $p^{-1}(x) =: W_x$ comes with the structure of a Banach space. A strong bundle chart is the tuple $(\Phi, p^{-1}(U), K, U \triangleleft F)$ where $U \subset E$ is an open subset of an $\text{sc}$-Banach space, $K = R(U \triangleleft F)$ an $\text{sc}$-smooth strong bundle retract covering the smooth retraction $r : U \to O$. Moreover, $\Phi : p^{-1}(U) \to K$ is a homeomorphism covering a homeomorphism $\varphi : U \to O$, which between between every fiber is a bounded linear operator of Banach spaces. Two such charts are $\text{sc}_\sigma$-smoothly equivalent if the associated transition maps are $\text{sc}_\sigma$-smooth diffeomorphisms. We can introduce the notion of a strong bundle atlas and the notion of an equivalence between two such atlases. The continuous surjection $p : W \to X$, if equipped with an equivalence class of strong bundle atlases, is called a strong bundle.

Given two such local bundles $K \to O$ and $K' \to O'$ we can define the notion of a strong bundle map between them. Then, following the scheme
how we defined M-polyfolds, we can define strong bundle \( W \to X \) over an M-polyfold \( X \).

Next assume that \( p : W \to X \) is a strong M-polyfold bundle over the M-polyfold \( X \). We can distinguish two types of sections of \( p \). An \( \text{sc-smooth} \) section of \( p \) is a map \( f : X \to W \) with \( p \circ f = \text{id} \) satisfying \( f(x) \in W_{m,m} =: W(0)_m \) for \( x \in X_m \) so that \( f : X \to W(0) \) is sc-smooth. An \( \text{sc}^+ \)-section of \( p \) is a section which satisfies \( f(x) \in W_{m,m+1} =: W(1)_m \) for \( x \in X_m \) and the induced map \( f : X \to W(1) \) is sc-smooth. We shall denote these two classes of sections by \( \Gamma(p) \) and \( \Gamma^+(p) \), respectively.

Of special interest are the so-called (polyfold-) Fredholm sections. Their definition is much more general than that of classical Fredholm sections. The first property which we require is that such sections should have the regularizing property. This property models the outcome of elliptic regularity theory.

**Definition 1.34.** Let \( p : W \to X \) be a strong M-polyfold bundle over the M-polyfold \( X \). A section \( f \in \Gamma(p) \) is said to be regularizing provided that the following holds. If \( x \in X_m \) and \( f(x) \in W_{m,m+1} \), then \( x \in X_{m+1} \).

We observe that if \( f \in \Gamma(p) \) is regularizing and \( s \in \Gamma^+(p) \), then \( f + s \) is also regularizing.

Consider a strong local bundle \( K \to O \). Here \( K = R(U \triangleleft F) \) is the \( \text{sc-smooth} \) strong bundle retract associated to the \( \text{sc}_\triangleright \)-smooth retraction \( R : U \triangleleft F \to U \triangleleft F \) which covers the \( \text{sc-smooth} \) retraction \( r : U \to U \) defined on an open neighborhood \( U \) of \( 0 \) in the \( \text{sc-Banach} \) space \( E \). Moreover, \( O = r(U) \). In addition, we assume that \( 0 \in O = r(U) \). Then \( R(u,h) = (r(u), \phi(u)h) \) where \( \phi(u) : F \to F \) is linear.

We are interested in germs of sections \((f,0)\) of the strong local bundle \( K \to O \) which are defined near \( 0 \). We view \( f \) as a germ \( O(O,0) \to F \) identifying the local section with its principal part.

**Definition 1.35.** We say that the section germ \((f,0)\) has a filled version if there exists an \( \text{sc-smooth} \) section germ \((g,0)\) of the bundle \( U \triangleleft F \to U \), again viewed as a germ

\[
O(U,0) \to F,
\]

which extends \( f \) and has the following properties:

1. \( g(x) = f(x) \) for \( x \in O \) close to \( 0 \).
(2) If \( g(y) = \phi(r(y))g(y) \) for a point \( y \) in \( U \) near 0, then \( y \in O \).

(3) The linearisation of the map

\[
y \mapsto [\text{id} - \phi(r(y))]g(y)
\]

at the point 0, restricted to \( \ker Dr(0) \), defines a topological isomorphism

\[
\ker(Dr(0)) \rightarrow \ker(\phi(0)).
\]

In order to describe the significance of the three conditions in the above definition we assume that \( y \in U \) is a solution of the filled section \( g \) so that \( g(y) = 0 \). Then it follows from (2) that \( y \in O \) and from (1) that \( f(y) = 0 \).

We see that the original section \( f \) and the filled section \( g \) have the same solution set.

The requirement (3) plays a role if we compare the linearization \( Df(0) \) with the linearization \( Dg(0) \). It follows from the definition of a retract that \( \phi(r(y)) \circ \phi(r(y)) = \phi(r(y)) \). Hence, since \( y = 0 \in O \) we have \( r(0) = 0 \) and \( \phi(0) \circ \phi(0) = \phi(0) \) so that \( \phi(0) \) is a linear sc-projection in \( F \) and we obtain the sc-splitting

\[
F = \phi(0)F \oplus (\text{id} - \phi(0))F.
\]

Similarly, it follows from \( r(r(y)) = r(y) \) for \( y \in U \) that \( Dr(0) \circ Dr(0) = Dr(0) \) so that \( Dr(0) \) is a linear sc-projection in \( E \) which gives rise to the sc-splitting

\[
\alpha \oplus \beta \in E = Dr(0)E \oplus (\text{id} - Dr(0))E.
\]

We keep in mind that \( Dr(0)\alpha = \alpha \) and \( Dr(0)\beta = 0 \). The tangent space \( T_0O \) is equal to \( Dr(0)E \) and

\[
\phi(0) \circ Dg(0)|Dr(0)E = Df(0) : T_0O \rightarrow \phi(0)F.
\]

From the identity \( \phi(r(y))g(r(y)) = \phi(y)g(y) \) for all \( y \in O \) and the identity \( (\text{id} - \phi(r(y)))g(r(y)) = 0 \) for all \( y \in E \) we obtain by linearization at \( y = 0 \) that \( \phi(0)Dg(0)\beta = 0 \) and \( (\text{id} - \phi(0))Dg(0)\alpha = 0 \). Hence the matrix representation of \( Dg(0) : E \rightarrow F \) with respect to the splittings looks as follows,

\[
Dg(0) \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} Df(0) & 0 \\ 0 & (\text{id} - \phi(0)) \circ Dg(0) \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.
\]
In view of property (3), the linear map $\beta \mapsto (\text{id} - \phi(0))Dg(0)$ from the kernel of $Dr(0)$ onto the kernel of $\phi(0)$ is an isomorphism. Therefore, we conclude that

$$\text{kernel } Dg(0) = (\text{kernel } Df(0)) \oplus \{0\}.$$  

Moreover, $Df(0) : T_0O \to \phi(0)F$ is a Fredholm operator if and only if $Dg(0) : E \to E$ is a Fredholm operator and in this case they have the same indices. Clearly, the linearization $Df(0)$ is surjective if and only if $Dg(0)$ is surjective.

**Remark 1.36.** We see that, given a solution $x$ of $f(x) = 0$, the local study of the solution set $f(y) = 0$ for $y \in O$ near $x$ of the section $f$, is equivalent to the local study of the solution set $g(y) = 0$ for $y \in U$ near $x$ of the filled section $g$.

**Remark 1.37.** We assume that the section $(f, 0)$ has a filled version $(g, 0)$ and that $s$ is an sc+ -section of $K \to O$. If $t$ is the sc+ -section of $U \times F \to F$ defined by $t(y) := s(r(y))$, then $(g + t, 0)$ is a filled version of $(f + s, 0)$. Indeed, for $x \in O$ we compute $(g + t)(x) = g(x) + s(r(x)) = f(x) + s(x)$, which is property (1). From $(g + t)(y) = \phi(r(y))(g + t(y))$ we deduce that

$$g(y) = \phi(r(y))g(y) + \phi(r(y))s(r(y)) - s(r(y)) = \phi(r(y))g(y),$$

implying that $y \in O$ so that property (2) holds. Finally,

$$[\text{id} - \phi(r(y))](g(y) + t(y)) = [\text{id} - \phi(r(y))](g(y) + s(r(y))) = [\text{id} - \phi(r(y))]g(y),$$

so that the linearisation of the left-hand side at $0$ restricted to $Dr(0)$ satisfies the property (3) in view of the assumptions on $g$. Hence, if $(f, 0)$ has a filled version, so does the section $(f + s, 0)$ for every sc+ -section $s$ of the strong local bundle $K \to O$.

Next, we introduce a class of so-called basic germs denoted by $\mathcal{C}_{\text{basic}}$.

**Definition 1.38.** An element in $\mathcal{C}_{\text{basic}}$ is an sc-smooth germ

$$f : \mathcal{O}(\mathbb{R}^n \oplus W, 0) \to (\mathbb{R}^N \oplus W, 0),$$

where $W$ is an sc-Banach space, so that if $P : \mathbb{R}^N \oplus W \to W$ denotes the projection, then the germ $P \circ f : \mathcal{O}(\mathbb{R}^n \oplus W, 0) \to (W, 0)$ has the form

$$P \circ f(a, w) = w - B(a, w)$$

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for \((a, w)\) close to \((0, 0)\) \(\in \mathbb{R}^n \oplus W_0\). Moreover, for every \(\varepsilon > 0\) and \(m \geq 0\), we have the estimate

\[
|B(a, w) - B(a, w')|_m \leq \varepsilon \cdot |w - w'|_m
\]

for all \((a, w), (a, w')\) close to \((0, 0)\) on level \(m\).

We are in the position to define the notion of a Fredholm germ.

**Definition 1.39.** Let \(p : W \to X\) be a strong bundle, \(x \in X_\infty\), and \(f\) a germ of an sc-smooth section of \(p\) around \(x\). We call \((f, x)\) a Fredholm germ provided there exists a germ of \(sc^+\)-section \(s\) of \(p\) near \(x\) satisfying \(s(x) = f(x)\) and such that in suitable strong bundle coordinates mapping \(x\) to 0, the push-forward germ \(g = \Phi_* (f - s)\) around 0 has a filled version \(\mathcal{F}\) so that the germ \((\mathcal{F}, 0)\) is equivalent to a germ belonging to \(\mathcal{C}_{\text{basic}}\).

Let us observe that tautologically if \((f, x)\) is a Fredholm germ and \(s_0\) a germ of \(sc^+\)-section around \(x\), then \((f + s_0, x)\) is a Fredholm germ as well. Indeed, if \((f - s, 0)\) in suitable coordinates has a filled version, then \(((f - s_0) - (s - s_0), 0)\) has as well.

Finally, we can introduce the Fredholm sections of strong M-polyfold bundles.

**Definition 1.40.** Let \(p : W \to X\) be a strong M-polyfold bundle and \(f \in \Gamma(p)\) an sc-smooth section. The section \(f\) is called polyfold Fredholm section provided it has the following properties:

1. \(f\) is regularizing.
2. At every smooth point \(x \in X\), the germ \((f, x)\) is a Fredholm germ.

If \((f, x)\) is a Fredholm germ and \(f(x) = 0\), then the linearisation \(f'(x) : T_x X \to W_x\) is a linear sc-Fredholm operator. The proof can be found in [10]. If, in addition, the linearization \(f'(x) : T_x X \to W_x\) is surjective, then our implicit function theorem gives a natural smooth structure on the solution set of \(f(y) = 0\) near \(x\) as the following theorem from [10] shows.

**Theorem 1.41.** Assume that \(p : W \to X\) is a strong M-polyfold bundle and let \(f\) be a Fredholm section of the bundle \(p\). If the point \(x \in X\) solves \(f(x) = 0\) and if the linearization at this point \(f'(x) : T_x X \to W_x\) is surjective, there exists an open neighborhood \(U\) of \(x\) so that the solution set

\[
S_U := \{y \in U \mid f(y) = 0\}
\]
has in a natural way a smooth manifold structure induced from \( X \). In addition, \( S_U \subset X_\infty \).

### 1.5 An Illustration of the Concepts

In Section 3 we shall illustrate the polyfold concept by setting up a proof of Theorem 1.31 as a polyfold Fredholm problem. It illustrates the analytical and conceptual difficulties in the study of maps on conformal cylinders which break apart as the modulus tends to infinity. It also illustrates the notion of a strong bundle as well as that of a Fredholm section.

We denote by \( \Gamma \) the collection of pairs \((a,b)\) of complex numbers satisfying \( ab \neq 0 \) and \(|a|,|b| < \varepsilon\). The size of \( \varepsilon \) will be determined later. We denote by \( X \) the set consisting of all tuples \((a,b,w)\) in which \((a,b) \in \Gamma \) and the map \( w : Z_a \to Z_b \) is a \( C^1 \)-diffeomorphism of Sobolev class \( H^3 \) between the two cylinders and satisfying \( w(p_a^\pm) = p_b^\pm \). The points \( p_a^\pm \) and \( p_b^\pm \) are the points corresponding to the boundary points \((0,0) \in \partial(\mathbb{R}^+ \times S^1)\) before ‘plumbing’ the half-cylinders. Define the filtration \( (X_m)_{m \in \mathbb{N}_0} \) on \( X \) by declaring that \((a,b,w)\) belongs to \( E_m \) if \( w \) belongs to \( H^{3+m}(Z_a, Z_b) \).

**Proposition 1.42.** The space \( X \) carries in a natural way a second countable paracompact topology. For this topology the space is connected. Moreover, \( X \) carries in a natural way the structure of an \( M \)-polyfold built on local models which are open subsets of an sc-Hilbert space where the level \( m \) corresponds to the Sobolev regularity \( m + 3 \).

In other words the space \( X \) is locally homeomorphic to open subsets of an sc-Hilbert space so that the transition maps are sc-smooth.

In the next step we shall “complete” the space \( X \) to the space \( \overline{X} \) by adding elements corresponding to the parameter value \( a = b = 0 \). This new space \( \overline{X} \) will have as local models sc-smooth retracts. Moreover, \( \overline{X} \) will be connected and will contain \( X \) as an open dense subset.

We abbreviate by \( D = D^{3,\delta_0} \) the collection of pairs \((u^+, u^-)\) of \( C^1 \)-diffeomorphisms,

\[
u^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^\pm \times S^1,
\]

having the following properties. The maps \( u^\pm \) belong to the space \( H^3_{\text{loc}} \) and there are constants \((d^\pm, \vartheta^\pm) \in \mathbb{R} \times S^1 \) and maps \( r^\pm \in H^{3,\delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^2) \) so that

\[
u^\pm(s,t) = (s + d^\pm, t + \vartheta^\pm) + r^\pm(s,t).
\]
Moreover, we require that 
\[ u^\pm(0, 0) = (0, 0). \]

The sc-structure on \( D \) is defined by declaring that the \( m \)-level \( D_m \) consists of elements of regularity \( (m + 3, \delta_m) \). Then we define the set \( \overline{X} \) as the disjoint union
\[ \overline{X} = X \coprod \{(0, 0) \times D\}, \]
in which \( (0, 0) \) is the pair \( (a = 0, b = 0) \).

**Theorem 1.43.**

1. Fix \( \delta_0 \in (0, 2\pi) \). Then the space \( \overline{X} \) possesses a natural paracompact second countable topology. In this topology the set \( X \) is an open subset of \( \overline{X} \) and the induced topology on \( X \) coincides with the previously defined topology on \( X \). Moreover, \( \overline{X} \) is connected.

2. Fix a strictly increasing sequence \( (\delta_m)_{m \in \mathbb{N}_0} \) of real numbers starting at \( \delta_0 \) and satisfying \( 0 < \delta_m < 2\pi \) and fix the exponential gluing profile \( \varphi \) given by \( \varphi(r) = e^{\frac{1}{r}} - e \). Then there exists a natural M-polyfold structure on \( \overline{X} \) which induces on \( X \) the previously defined M-polyfold structure.

In our discussion of this theorem later on we shall describe some metric aspects related to this topology which gives a precise meaning of the convergence of elements \( (a, b, w) \) to \( (0, 0, (u^+, u^-)) \). The construction of the polyfold structure on \( \overline{X} \) involves a third kind of splicing which will be explained later. Variations of this kind of splicings will be used in SFT.

For the following we assume that the sequence \( (\delta_m) \) and the exponential gluing profile \( \varphi \) are fixed so that \( \overline{X} \) has a M-polyfold structure. We consider a smooth family
\[ v \mapsto j^\pm(v) \]
of complex structures on the half-infinite cylinders \( \mathbb{R}^\pm \times S^1 \) parametrized by \( v \) belonging to an open neighborhood of 0 in some finite-dimensional vector space \( V \). Moreover, we assume that \( j^\pm(v) = i \) outside of a compact neighborhood of the boundaries. It induces the complex structure \( j(a, v) \) on the finite cylinder \( Z_a \) if \( |a| \) is small enough. Clearly, \( V \times \overline{X} \) is an M-polyfold.
With points \((v, a, b, w) \in V \times \overline{X}\) satisfying \(a \neq 0\), we can associate maps 
\[ z \mapsto \phi(z) \]
defined on the cylinder \(Z_a\), whose images
\[
\phi(z) : (T_z Z_a, j(v)) \to (T_{w(z)} Z_b, i)
\]
are complex anti-linear and belong to the Sobolev space \(H^2\). If \(a = 0\) (and consequently \(b = 0\)), then \(Z_0\) is the disjoint union
\[
\mathbb{R}^+ \times S^1 \coprod \mathbb{R}^- \times S^1
\]
and here we consider two maps \(z \to \phi^\pm(z)\) defined on \(\mathbb{R}^\pm \times S^1\) whose complex anti-linear images
\[
\phi^\pm(z) : (T_z (\mathbb{R}^\pm \times S^1), j(v)) \to (T_{u^\pm(z)} (\mathbb{R}^\pm \times S^1), i)
\]
belong to \(H^2_{\text{loc}}\) on \(\mathbb{R}^\pm \times S^1\) where \(u^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^\pm \times S^1\) are the diffeomorphisms of the half cylinders introduced above. Moreover, the maps
\[
(s, t) \mapsto \phi^\pm(s, t) \frac{\partial}{\partial s}, \quad z = (s, t),
\]
belong to \(H^2_{\text{loc}}(\mathbb{R}^\pm \times S^1, \mathbb{R}^2)\) if the tangent space \(T_{w(z)} Z_b\) is identified with \(\mathbb{R}^2\).

The collection \(E\) of all \((v, a, b, w, \phi)\) in which \(\phi\) is as just described, possesses a projection map
\[
E \to V \times \overline{X}, \quad (v, a, b, w, \phi) \mapsto (v, a, b, w),
\]
whose fiber has in a natural way the structure of a Hilbert space. We can define a double filtration \(E_{m,k}\) of \(E\) where \(0 \leq k \leq m + 1\). An element \((v, a, b, w, \phi)\) belongs to \(E_{m,k}\) provided \((v, a, b, w) \in V \times \overline{X}_m\) and \(\phi\) is of class \((k + 2, \delta_k)\).

**Theorem 1.44.** Having fixed the exponential gluing profile \(\varphi\) and the increasing sequence \((\delta_m)_{m \in \mathbb{N}_0}\) of real numbers satisfying \(0 < \delta_m < 2\pi\), the space \(E\) admits in a natural way the structure of a strong bundle over \(V \times \overline{X}\).

Finally, we define the section \(\overline{\partial}\) of the bundle \(E \to V \times \overline{X}\) by its principal part
\[
\overline{\partial}(v, a, b, w) := \frac{1}{2} (T_w + i \circ (T_w) \circ j(a, v)),
\]
and observe that
\[
\overline{\partial}(v, a, b, w) = 0
\]
if and only if the map \( w : (Z_a, j(a, v), p_a^+, p_a^-) \to (Z_b, i, p_b^+, p_b^-) \) is biholomorphic.

We shall prove in section \( 3.4 \) that the Cauchy-Riemann section \( \partial \) is an sc-smooth Fredholm section of the strong M-polyfold bundle \( E \to V \oplus X \). By the uniformization theorem, there exists for every point \((v, a) = (v_0, 0)\) a unique pair \((u_0^+, u_0^-)\) of biholomorphic mappings

\[ u_0^\pm : (\mathbb{R}^\pm \times S^1, j^\pm(v_0)) \to (\mathbb{R}^\pm \times S^1, i) \]

preserving the boundary points \((0, 0) \in \partial(\mathbb{R}^\pm \times S^1)\) so that the special point \((v_0, 0, 0, u_0^+, u_0^-) \in V \oplus X\) is a solution of \( \partial(v_0, 0, 0, u_0^+, u_0^-) = 0 \).

As a consequence of the implicit function theorem for polyfold Fredholm sections in \([10]\), we shall establish near the reference solution biholomorphic mappings between finite cylinders for \((v, a)\) close to \((v_0, 0)\) and \(a \neq 0\). More precisely, we shall prove the following result.

**Theorem 1.45.** The Cauchy-Riemann section \( \partial \) of the strong bundle \( E \to V \times X \) is an sc-smooth polyfold Fredholm section. Its linearization at the reference solution \((v_0, 0, u_0^+, u_0^-) \in V \oplus X\) is surjective and there exists a uniquely determined sc-smooth map

\[ \Phi : B_\varepsilon(a_0) \oplus V \to V \oplus X \]

\[ (a, v) \mapsto (v, a, b(a, v), w(a, v)) \]

satisfying \((b(0, v_0), w(0, v_0)) = (0, u_0^+, u_0^-)\), and solving the Cauchy-Riemann equation

\[ \partial \Phi(v, a) = 0. \]

Moreover, these are the only solutions near the reference solution.

Theorem 1.45 describes, in particular, the smoothness properties of the family of biholomorphic maps between conformal cylinders which break apart. Unraveling the M-polyfold structure on \( X \), we shall obtain the following result, where the set \( D^{m, \varepsilon} \) is the space of diffeomorphisms \( u^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^\pm \times S^1 \) of the half-cylinders of the form

\[ u^\pm (s, t) = (s, t) + (d^\pm, \vartheta^\pm) + r^\pm (s, t) \]

where \( r^\pm \) have weak derivatives up to order \( m \) which weighted by \( e^{\varepsilon |s|} \) belong to \( L^2(\mathbb{R}^2 \times S^1, \mathbb{R}^2) \).
Theorem 1.46. Let \((v_0, 0) \in V \oplus \mathbb{C}\) be fixed and let
\[
\begin{align*}
(v, a) & \mapsto b(v, a) \quad \text{and} \quad (v, a) \mapsto w(v, a)
\end{align*}
\]
be the the germs of maps guaranteed by Theorem 1.45 and defined on some small open neighborhood \(O(v_0, 0)\) and where
\[
w(v, a): (Z_a, j(a, v), p^+_a, p^-_a) \to (Z_{b(v, a)}, i, p^+_a, p^-_a).
\]
is the associated biholomorphic map between the cylinders if \(a \neq 0\).

Then given a constant \(\Delta > 0\), the following holds on a possibly smaller neighborhood \(O(v_0, 0)\). There exists a map
\[
(a, v) \mapsto \tilde{w}(v, a) \in \bigcap_{m \geq 3, 0 < \varepsilon < 2\pi} D^{m, \varepsilon}
\]
having the following properties. For every \(m \geq 3\) and every \(0 < \varepsilon < 2\pi\) the map
\[
(v, a) \mapsto \tilde{w}(v, a) \in D^{m, \varepsilon}
\]
is smooth and satisfies
\[
\tilde{w}(v, a)(s, t) = w(v, a)(s, t)
\]
for all \((s, t) \in [0, \frac{1}{2} \varphi(|a|) + \Delta] \times S^1\). In addition,
\[
\tilde{w}(v, a)(s, t) = (s + d(v, a), t + \vartheta(v, a)),
\]
for \(s \geq s(a)\), that is, for large \(s\) where large depends on the gluing parameter \(a\).

2 Exploring Sc-Smoothness

In chapter 1 we have presented background material and some of the basic results used in the construction of the polyfold structures in the SFT. Their proofs in chapter 3 rely on the technical results which we shall prove now in chapter 2.
2.1 Basic Results about Abstract Sc-Smoothness

In this subsection we relate sc-smoothness with the more familiar notion of being \( C^\infty \) or \( C^k \). In particular, we shall prove Proposition 1.9 and Proposition 1.10. The first result relates the \( sc^1 \)-notion with that of being \( C^1 \) and provides an alternative definition of a map between sc-Banach spaces to be of class \( sc^1 \).

**Proposition 2.1.** Let \( E \) and \( F \) be sc-smooth Banach spaces and let \( U \) be a relatively open subset of a partial quadrant \( C \) in \( E \). Then an \( sc^0 \)-map \( f : U \to F \) is of class \( sc^1 \) if and only if the following conditions hold true:

1. For every \( m \geq 1 \), the induced map
   \[ f : U_m \to F_{m-1} \]
   is of class \( C^1 \). In particular, the derivative
   \[ df : U_m \to L(E_m, F_{m-1}), \quad x \mapsto df(x) \]
   is a continuous map

2. For every \( m \geq 1 \) and every \( x \in U_m \), the bounded linear operator \( df(x) : E_m \to F_{m-1} \) has an extension to a bounded linear operator \( Df(x) : E_{m-1} \to F_{m-1} \). In addition, the map
   \[ U_m \oplus E_{m-1} \to F_{m-1}, \quad (x, h) \mapsto Df(x)h \]
   is continuous.

**Proof.** It is clear that the conditions (1) and (2) imply that the map \( f \) is \( sc^1 \). The other direction is more involved and uses the compactness of the inclusions \( E_{m+1} \to E_m \) in a crucial way.

Assume that \( f : U \to F \) is of class \( sc^1 \). Then the induced map \( f : U_1 \to F \) is differentiable at every point \( x \) with the derivative \( df(x) = Df(x)|_{E_1} \in L(E_1, F) \). Hence the extension of \( df(x) : E_1 \to F \) to a continuous linear map \( E \to F \) is the postulated map \( Df(x) \). We claim that the derivative \( x \mapsto df(x) \) from \( U_1 \) into \( L(E_1, F) \) is continuous. Arguing indirectly, we find an \( \varepsilon > 0 \) and sequences \( x_n \to x \) in \( U_1 \) and \( h_n \) of unit norm in \( E_1 \) satisfying

\[ |df(x_n)h_n - df(x)h_n|_0 \geq \varepsilon. \]  

(8)
Taking a subsequence we may assume, in view of the compactness of the embedding $E_1 \to E_0$, that $h_n \to h$ in $E_0$. Hence, by the continuity property, 

$$df(x_n)h_n = Df(x_n)h_n \to Df(x)h$$ in $F_0$. Consequently,

$$df(x_n)h_n - df(x)h_n = Df(x_n)h_n - Df(x)h_n \to Df(x)h - Df(x)h = 0$$

in $F_0$, in contradiction to (8).

Next we prove that $f : U_{m+1} \to F_m$ is differentiable at $x \in U_{m+1}$ with derivative

$$df(x) = Df(x)|_{E_{m+1}} \in L(E_{m+1}, F_m)$$

so that the required extension of $df(x)$ is $Df(x) \in L(E_m, F_m)$. The map $f : U_1 \to F_0$ is of class $C^1$ and $df(x) = Df(x)|E_1$. Since, by continuity property (2), the map $(x, h) \mapsto Df(x)h$ from $U_{m+1} \oplus E_m \to F_m$ is continuous, we can estimate for $x \in U_{m+1}$ and $h \in E_{m+1}$,

$$\frac{1}{|h|_{m+1}} \cdot |f(x + h) - f(x) - Df(x)h|_m$$

$$= \frac{1}{|h|_{m+1}} \cdot \left| \int_0^1 [Df(x + \tau h) \cdot h - Df(x) \cdot h] d\tau \right|_m$$

$$\leq \int_0^1 \left| Df(x + \tau h) \cdot \frac{h}{|h|_{m+1}} - Df(x) \cdot \frac{h}{|h|_{m+1}} \right|_m d\tau.$$

Take a sequence $h \to 0$ in $E_{m+1}$. By the compactness of the embedding $E_{m+1} \to E_m$, we may assume that $\frac{h}{|h|_{m+1}} \to h_0$ in $E_m$. By the continuity property we now conclude that the integrand converges uniformly in $\tau$ to $|Df(x)h_0 - Df(x)h_0|_m = 0$ as $h \to 0$ in $E_{m+1}$. This shows that $f : U_{m+1} \to F_m$ is indeed differentiable at $x$ with derivative $df(x)$ being the bounded linear operator

$$df(x) = Df(x)|_{E_{m+1}} \in L(E_{m+1}, F_m).$$

The continuity of $x \mapsto df(x) \in L(E_{m+1}, F_m)$ follows by the argument already used above, so that $f : U_{m+1} \to F_m$ is of class $C^1$. This finishes the proof of the proposition. ■

As a consequence of Proposition 2.1 we obtain the following proposition.

**Proposition 2.2.** If $f : U \to V$ is an sc$^k$-map, then the induced map $f : U^1 \to V^1$ is also sc$^k$. 

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Proof. We prove the assertion by induction with respect to \( k \). Assume that \( k = 1 \) and that \( f : U \to V \) is \( \text{sc}^1 \). Equivalently, \( f \) satisfies parts (1) and (2) of Proposition 2.1 for all \( m \geq 1 \). This implies that, after replacing \( E \) and \( F \) by \( E^1 \) and \( F^1 \) and \( U \) by \( U^1 \), the map \( f : U^1 \to F^1 \) also satisfies the points (1) and (2) of Proposition 2.1. Applying Proposition 2.1 again, we conclude that \( f : U^1 \to F^1 \) is \( \text{sc}^1 \).

Now assume that the assertion holds for all \( \text{sc}^k \)-maps and let \( f : U \to V \) be an \( \text{sc}^{k+1} \)-map. This means that the tangent map \( Tf : TU \to TV \) is \( \text{sc}^k \). Then, by induction hypothesis, the tangent map \( Tf : (TU)^1 \to (TV)^1 \) is an \( \text{sc}^k \)-map. Since \( T(U^1) = (TU)^1 \) and \( T(V^1) = (TV)^1 \), we have proved that

\[
Tf : T(U^1) \to T(V^1)
\]

is an \( \text{sc}^k \)-map. But this precisely means that \( f : U^1 \to V^1 \) is an \( \text{sc}^{k+1} \)-map. The proof of the proposition is complete.

Next we study the relationship between the notions of being \( C^k \) and \( \text{sc}^k \).

**Proposition 2.3.** Let \( U \) and \( V \) be relatively open subsets of partial quadrants in \( \text{sc}\)-Banach spaces \( E \) and \( F \), respectively. If \( f : U \to V \) is \( \text{sc}^k \), then for every \( m \geq 0 \) the map \( f : U_{m+k} \to V_m \) is of class \( C^k \). Moreover, \( f : U_{m+l} \to V_m \) is of class \( C^l \) for every \( 0 \leq l \leq k \).

**Proof.** The last statement is a consequence of the former since an \( \text{sc}^k \)-map is also \( \text{sc}^l \)-map for \( 0 \leq l \leq k \). Now we prove the main statement. Note that it suffices to show that \( f : U_k \to F_0 \) is of class \( C^k \). Indeed, by Proposition 2.2, the map \( f : U^m \to V^m \) is of class \( \text{sc}^k \) and so we can repeat the argument of Proposition 2.2 for the map \( f : U^m \to V^m \) replacing the map \( f : U \to V \).

We prove the result by induction with respect to \( k \). If \( k = 0 \) the statement is trivially true and if \( k = 1 \) it is just the condition (1) of Proposition 2.1. Now we assume that result holds for all \( \text{sc}^k \)-maps and let \( f : U \to V \) be an \( \text{sc}^{k+1} \)-map. In particular, \( f \) is of class \( \text{sc}^k \) which, by induction hypothesis, implies that \( f : U_k \to F_0 \) is of class \( C^k \). Also, the tangent map \( Tf : TU \to TF \) is of class \( \text{sc}^k \) and therefore \( Tf : (TU)_k = U_{k+1} \oplus E_k \to TF \) is of class \( C^k \) as well. Denoting by \( \pi : TF = F^1 \oplus F \to F \) the projection onto the second factor, we consider the composition

\[
\Phi := \pi \circ Tf : U_{k+1} \oplus E_k \to F, \quad (x, h) \mapsto Df(x)h.
\]
We know that the map $\Phi$ is of class $C^k$. Taking $k$ derivatives but only with respect to $x$, we obtain a continuous map

$$U_{k+1} \oplus E_k \to L^k(E_{k+1}, \ldots, E_{k+1}; F), \quad (x, h) \mapsto (D^k_x \Phi)(x, h).$$

Observing that this map is linear in $h \in E_k$ and is continuous, we obtain the map

$$\Gamma : U_{k+1} \to L^{k+1}(E_{k+1}, \ldots, E_{k+1}; F)$$

defined by

$$\Gamma(x) : E_{k+1} \oplus \cdots \oplus E_{k+1} \to F, \quad (h_1, \ldots, h_k, h_n) \mapsto (D^k_x \Phi)(x, h)(h_1, \ldots, h_k).$$

We claim that $\Gamma$ is continuous. Indeed, arguing indirectly we find a point $x$ in $U_{k+1}$, a number $\varepsilon > 0$, and sequence of points $(x_n, h_{1,n}, \ldots, h_{k,n}, h_n) \in U_{k+1} \oplus E_{k+1} \oplus \cdots \oplus E_{k+1}$ so that $x_n \to x$ in $U_{k+1}$, all $h_{m,n}$ and $h_n$ have length 1 in the norm $|\cdot|_{k+1}$, and

$$|(\Gamma(x_n) - \Gamma(x))(h_{1,n}, \ldots, h_{k,n}, h_n)|_0 \geq \varepsilon > 0. \quad (9)$$

Since the inclusion $E_{k+1} \to E_k$ is compact, after perhaps taking a subsequence, we may assume that $h_n \to h$ in $E_k$. Then $(x_n, h_n) \to (x, h)$ in $U_{k+1} \oplus E_k$ and, by the continuity,

$$|D^k_x \Phi)(x_n, h_n) - (D^k_x \Phi)(x, h)|_{L^k(E_{k+1}, \ldots, E_{k+1}; F)} \to 0.$$

This, however, contradicts (9) since

$$|(\Gamma(x_n) - \Gamma(x))(h_{1,n}, \ldots, h_{k,n}, h_n)|_0 = |(D^k_x \Phi)(x, h_n) - (D^k_x \Phi)(x, h_n)(h_{1,n}, \ldots, h_{k,n})|_0 \leq |(D^k_x \Phi)(x_n, h_n) - (D^k_x \Phi)(x, h_n)|_{L^k(E_{k+1}, \ldots, E_{k+1}; F)}.$$

We shall now prove that $f : U_{k+1} \to F$ is of class $C^{k+1}$ by showing that the limit of

$$\frac{1}{|\delta x|_{k+1}}[D^k f(x + \delta x) - D^k f(x) - \Gamma(x)(\cdot, \delta x)]$$

in $L(E_{k+1}, \ldots, E_{k+1}; F)$ is equal to 0. For $x \in U_{k+1}$, $\delta x \in E_{k+1}$ small, and $t \in [0, 1]$, we consider the $C^k$-map

$$(t, \delta x) \mapsto D f(x + t \delta x) \delta x.$$
Integrating with respect to $t$, obtain the $C^k$-map

$$(x, \delta x) \mapsto f(x + \delta x) - f(x).$$

Differentiating this map $k$ times with respect to $x$, we find for $h_1, \ldots, h_k \in E_{k+1}$ that

$$D^k f(x + \delta x)(h_1, \ldots, h_k) - D^k f(x)(h_1, \ldots, h_k)$$

$$= D_x^k (f(x + \delta x) - f(x))(h_1, \ldots, h_k)$$

$$= D_x^k \left( \int_0^1 (Df(x + t\delta x)\delta x)dt \right) (h_1, \ldots, h_k)$$

$$= \int_0^1 D_x^k ((Df(x + t\delta x)\delta x)(h_1, \ldots, h_k)dt$$

$$= \int_0^1 \Gamma(x + t\delta x)(h_1, \ldots, h_k, \delta x)dt.$$ 

Hence

$$\frac{1}{|\delta x|_{k+1}} \cdot [(D^k f(x + \delta x) - D^k f(x))(h_1, \ldots, h_k) - \Gamma(x)(h_1, \ldots, h_k, \delta x)]$$

$$= \int_0^1 [(\Gamma(x + t\delta x) - \Gamma(x))(h_1, \ldots, h_k, \frac{\delta x}{|\delta x|_{k+1}})] dt.$$ 

Letting $\delta x \to 0$ in $E_{k+1}$ and using continuity of the map $\Gamma : U_{k+1} \to L(E_{k+1}, \ldots, E_{k+1}; F)$, we conclude that the left-hand side above converges to 0 in $L(E_{k+1}, \ldots, E_{k+1}; F)$. Consequently, $f : U_{k+1} \to F$ is of class $C^{k+1}$ and the proof of Proposition 2.3 is complete. 

The next result is very useful in proving that a given map between sc-Banach spaces is sc-smooth.

**Proposition 2.4.** Let $E$ and $F$ be sc-Banach spaces and let $U$ be a relatively open subsets of partial quadrants in $E$. Assume that for every $m \geq 0$ and $0 \leq l \leq k$ the map $f : U \to V$ induces a map,

$$f : U_{m+l} \to F_m,$$

which is of class $C^{l+1}$. Then $f$ is sc$^{k+1}$. 

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Proof. The proof is by induction with respect to \( k \). In order that the induction runs smoothly we have to prove slightly more. Here is our inductive assumption:

\((S_k)\). If a map \( f : U \to F \) induces maps \( f : U_{m+l} \to F_m \) of class \( C^{l+1} \) for all \( m \in \mathbb{N}_0 \) and \( 0 \leq l \leq k \), then \( f \) is of class \( sc^{k+1} \). Moreover, if \( \pi : T^{k+1}F \to F^l \) denotes the projection onto a factor \( F^l \) of \( T^{k+1}F \), then the composition \( \pi \circ T^{k+1}f : T^{k+1}U \to F^l \) is a linear combination of maps of the following type,

\[
\Gamma : U^{k+1} \oplus E^{n_1} \oplus \ldots \oplus E^{n_j} \to F^l
\]

\[
(x_1, x_{k_1}, \ldots, x_{k_j}) \mapsto d^l(x_1)(x_{k_1}, \ldots, x_{k_j}),
\]

where \( x_1 \in U_{k+1}, x_{k_i} \in E_{n_i} \) and where the nonnegative indices \( j, k, l \) and \( n_i \) satisfy \( j + l - 1 \leq n_i \leq k \).

We start the proof with \( k = 0 \). By assumption, \( f : U_0 \to F_0 \) is of class \( C^1 \). For \( x_1 \in U_1 \), we define the map \( Df(x_1) : E_0 \to F_0 \) by \( Df(x_1)x_2 = df(x_1)x_2 \) where \( df(x_1) \) is the linearization of \( f : U_0 \to F_0 \) at the point \( x_1 \). Then \( Df(x_1) \in \mathcal{L}(E_0, F_0) \) and as a map from \( E_1 \to F_0 \), the map \( Df(x_1) \) is the derivative of \( f : U_1 \to F_0 \) at the point \( x_1 \). The tangent map \( Tf : TF \to TF \) is, by definition, given by

\[
Tf(x_1, x_2) = (f(x_1), df(x_1)x_2).
\]

Since the tangent map is continuous, we have proved that \( f \) is \( sc^1 \). If \( \pi \) is a projection onto any factor of \( TF \), then the composition \( \pi \circ Tf \) is either the map \( E^1 \to F^1 \) given by \( x_1 \mapsto f(x_1) \) or the map \( \pi \circ Tf : E^1 \oplus E \to F \) defined by \( \pi \circ Tf(x_1, x_2) = df(x_1)x_2 \). Both maps are of the required form and the indices satisfy the required inequalities. This finishes the proof of the statement \((S_0)\).

Next we assume that we have established \((S_k)\) and let \( f : U \to F \) be a map such that \( f : U_{m+l} \to F_m \) is of class \( C^{l+1} \) for all \( m \in \mathbb{N}_0 \) and \( 0 \leq l \leq k + 1 \). In particular, since the map \( f \) satisfies the statement \((S_k)\), it is of class \( sc^{k+1} \) and, in addition, the compositions \( \pi \circ T^{k+1}f : T^{k+1}E \to F^l \) are linear combinations of maps of types described in \((S_k)\). We have to show that \( f \) is \( sc^{k+2} \) and to do this we show that the tangent map \( T^{k+1}f \) is \( sc^1 \). It suffices to show that any of the terms making up the composition \( \pi \circ T^{k+1}f \) is \( sc^1 \).

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Hence we consider the map \( \Gamma : U^{k+1} \oplus E^{n_1} \oplus \ldots \oplus E^{n_j} \to F^l \) given by
\[
\Gamma(x_1, x_{k_1}, \ldots, x_{k_j}) = d^j(x_1)(x_{k_1}, \ldots, x_{k_j})
\]
with \( x_1 \in U_{k+1}, x_{k_i} \in E_{n_i} \) and the indices satisfying \( j + l - 1 \leq n_i \leq k \). Given \((x_1, x_{k_1}, \ldots, x_{k_j}) \in U^{k+2} \oplus E^{n_1+1} \oplus \ldots \oplus E^{n_j+1}, \) the candidate for the linearization
\[
D\Gamma(x_1, x_{k_1}, \ldots, x_{k_j}) : E^{k+1} \oplus E^{n_1} \oplus \ldots \oplus E^{n_j} \to F^l
\]
is the map defined by
\[
d^{j+1}f(x_1)(\delta x_1, x_{k_1}, \ldots, x_{k_j}) + \sum_{i=1}^{j} d^i f(x_1)(x_{k_1}, \ldots, \delta x_{k_i}, \ldots, x_{k_j}).
\]
The map in \( (\text{III}) \) is well-defined. Indeed, by assumption \( j + l - 1 \leq n_i \leq k \), so that the map \( f : U^{j+l-1} \to F^l \) is of class \( C^j \). Since \( E^k \subset E^{n_i} \subset U^{j+l-1} \), it follows that the map
\[
E^{n_i} \to F^l, \quad \delta x_{k_i} \mapsto d^{j+1}f(x_1)(x_{k_1}, \ldots, \delta x_{k_i}, \ldots, x_{k_j})
\]
is a bounded linear map. Similarly, since \( j + l - 1 \leq n_i \leq k \), we have \( j + l \leq n_i + 1 \leq k + 1 \) and that \( U^{k+2} \oplus E^{k+1} \oplus E^{n_1+1} \oplus \ldots \oplus E^{n_j+1} \subset U^{j+l} \oplus E^{j+l} \oplus \ldots \oplus E^{j+l} \). Since by our inductive assumption the map \( f : U^{j+l} \to F^l \) is of class \( C^j+1 \), it follows that for given \((x_{k_1}, \ldots, x_{k_j}) \in E^{n_1+1} \oplus \ldots \oplus E^{n_j+1}, \) the map \( U^{k+1} \to F^l \) defined by \( x_1 \mapsto d^j f(x_1)(x_{k_1}, \ldots, x_{k_j}) \) is of class \( C^1 \) which implies that the first term in \( (\text{III}) \) defines a bounded linear map
\[
E^{k+1} \to F^l, \quad \delta x_1 \mapsto d^{j+1}f(x_1)(\delta x_1, x_{k_1}, \ldots, x_{k_j}).
\]
Hence the map \( D\Gamma(x_1, x_{k_1}, \ldots, x_{k_j}) : E^{k+1} \oplus E^{n_1} \oplus \ldots \oplus E^{n_j} \to F^l \) defines a bounded linear operator and, as a map from \( E^{k+2} \oplus E^{n_1+1} \oplus \ldots \oplus E^{n_j+1} \to F^l, \) it is the derivative of \( \Gamma : U^{k+2} \oplus E^{n_1+1} \oplus \ldots \oplus E^{n_j+1} \to F^l. \) Moreover, the evaluation map
\[
D\Gamma : U^{k+2} \oplus E^{n_1+1} \oplus \ldots \oplus E^{n_j+1} \oplus E^{k+1} \oplus E^{n_1} \oplus \ldots \oplus E^{n_j} \to F^l
\]
\((x_1, x_{k_1}, \ldots, x_{k_j}, \delta x_1, \delta x_{k_1}, \ldots, \delta x_{k_j}) \mapsto D\Gamma(x_1, x_{k_1}, \ldots, x_{k_j}, \delta x_1, \delta x_{k_1}, \ldots, \delta x_{k_j}) \)
is continuous. So, the map \( \Gamma \) is \( sc^1 \) and hence the tangent map \( T^{k+1}f : T^{k+1}E \to T^{k+1}F \) is of class \( sc^1 \). We have proved that \( f : U \to F \) is of class \( sc^{k+2} \). The tangent map \( T^{k+2}f : T^{k+2}E \to T^{k+2}F \) is of the form
\[
T^{k+2}f(x_1, \ldots, x_{2k+2})
\]
\[
= (T^{k+1}f(x_1, \ldots, x_{2k+1}), DT^{k+1}f(x_1, \ldots, x_{2k+1})(x_{2k+1+1}, \ldots, x_{2k+2}).
\]
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We consider the composition \( \pi \circ T^{k+2}f \). If \( \pi \) is a projection onto the first \( 2^k \) summands we see the terms guaranteed by the induction hypothesis \((S_k)\) but now with the index of the spaces raised by one. If \( \pi \) is a projection onto any of the last \( 2^k \) factors, the previous discussion shows that they are linear combinations of the maps occurring in \((\Pi)\). Denote the new indices by \( j', k', l', n'_i \). Then in case of the map \( d^{j+1}f(x_1)(\delta x_1, x_{k_1}, \ldots, x_{k_j}) \) in \((\Pi)\) we have \( k' = k + 1, j' = j + 1, l' = l \) and \( n'_{i+1} = n_i + 1 \) for \( i = 1, \ldots, j \). For the map \( d^j f(x_1)(x_{k_1}, \ldots, x_{k_j}) \), we have \( k' = k + 1, j' = j, l' = l \) and \( n'_i = n_i \) for \( s \neq i \). One checks that the new indices satisfy the required inequalities. Hence \((S_{k+1})\) holds and the proof of Proposition 2.4 is complete.

There is a useful corollary to Proposition 2.4.

**Corollary 2.5.** Let \( U \subset C \subset E \) be a relatively open subset of a partial quadrant in the sc-Banach space \( E \). Assume that \( f : U \to \mathbb{R}^N \) is a map so that for some \( k \) and all \( 0 \leq l \leq k \) the map \( f : U_l \to \mathbb{R}^N \) belongs to \( C^{l+1} \). Then \( f \) is sc\(^{k+1} \).

### 2.2 Actions by Smooth Maps, Proof of Theorem 1.26

This section is devoted to the proof of Theorems 1.26. We recall the set-up. Denoting by \( D \) the closed unit-disk in \( \mathbb{C} \) and by \( V \) an open neighborhood of the origin 0 in \( \mathbb{R}^n \), we consider a smooth map

\[
V \times D \to D, \quad (v, x) \mapsto \phi_v(x)
\]
satisfying \( \phi_0(0) = 0 \). We equip the Hilbert space \( E = H^3(D, \mathbb{R}^N) \) with the sc-structure \( E_m = H^{3+m}(D, \mathbb{R}^N) \) and introduce the map

\[
\Phi : V \oplus E \to E, \quad (v, u) \mapsto u \circ \phi_v.
\]

Since \( \phi_v : D \to D \) is smooth, the map \( \Phi \) preserves the levels. In the following we shall refer to the map \( \Phi \) as to an “action by smooth maps”.

**Theorem 2.6.** The above map \( \Phi : V \oplus E \to E \) is sc-smooth.

**Proof.** We proceed by induction. Here is our inductive hypothesis:
(S_k). The map
\[ \Phi : V \oplus E \to E, \quad (v, u) \mapsto u \circ \phi_v \]
is sc^k and for every projection \( \pi : T^k F \to E^j \) onto a factor of \( T^k E \), the composition \( \pi \circ T^k \Phi \) is a finite linear combination of maps of the form
\[ V \oplus E^m \oplus (\mathbb{R}^n)^p \to E^j, \quad (v, h, a_1, \ldots, a_p) \mapsto \Phi(v, D^\alpha h) \cdot f(v, a_1, \ldots, a_p) \]
where \( f : V \oplus (\mathbb{R}^n)^p \times D \to \mathbb{R} \) is a smooth function which is linear in every variable \( a_i \). Moreover, \( |\alpha| \leq m - j \) and \( p \leq k \).

In the case \( k = 0 \), there is exactly one projection \( \pi : T^0 E = E \to E \), namely the identity map. So, the composition \( \pi \circ \Phi = \Phi \) has the required form with \( m = j = 0, \alpha = (0, 0) \), and \( f \equiv 1 \), where the empty product of \( a_i \)'s is defined as being 1. To prove that \( \Phi \) is sc^0, we fix a point \( (v_0, u_0) \in V \oplus E_m \), and, for given \( r > 0 \), we choose a smooth map \( w_0 : D \to \mathbb{R}^N \) satisfying \( |u_0 - w_0|_m \leq r \). Then we estimate
\[
|\Phi(v, u) - \Phi(v_0, u_0)|_m \leq |\Phi(v, u) - \Phi(v, w_0)|_m + |\Phi(v, w_0) - \Phi(v_0, w_0)|_m \\
+ |\Phi(v_0, w_0) - \Phi(v_0, u_0)|_m =: I + II + III.
\]
For \( v \) close enough to \( v_0 \), the map \( E_m \to E_m \) defined by \( h \mapsto h \circ \phi_v \) is a bounded linear operator with uniformly bounded norm by a constant \( C \) which only depends on an arbitrarily fixed small neighborhood of \( v_0 \). Consequently, we obtain the following estimates for the terms \( I \) and \( III \),
\[
I \leq C \cdot |u - w_0|_m \leq C \cdot |u - u_0|_m + C \cdot |u_0 - w_0|_m \leq C \cdot |u - u_0| + C \cdot r
\]
and
\[
III \leq C \cdot |u_0 - w_0|_m \leq C \cdot r.
\]
Since \( w_0 \) and \( (v, x) \mapsto \phi_v(x) \) are smooth, it follows immediately that \( D^\alpha (w_0 \circ \phi_v) \to D^\alpha (w \circ \phi_v) \) as \( v \to v_0 \). In particular, \( w_0 \circ \phi_v \to w \circ \phi_v \) in \( E_m \) which implies that
\[
II \to 0, \quad \text{as } v \to v_0.
\]
The number \( r \) can be chosen to be as small as we wish, so that our estimates show that \( \Phi \) is continuous on every level \( m \), that is, the map \( \Phi \) is of class sc^0. So, \( (S_0) \) holds.

To simplify the further steps, it turns out to be useful to first prove \( (S_1) \). We write \( s + it \) for the coordinates on \( \mathbb{C} \) and introduce the notation \( \phi_v = \)
\((A_v, B_v)\) where the maps \(V \times D \to \mathbb{R}, (v, x) \mapsto A_v(x)\) and \((v, x) \mapsto B_v(x)\) are smooth. The derivatives of \(A_v\) and \(B_v\) with respect to the variable \(v\) are denoted by \(DA_v\) and \(DB_v\), respectively.

The candidate for the linearization \(D\Phi(v, u) : V \oplus E \to E\) at the point \((v, u) \in V \oplus E^1\) is given by

\[
D\Phi(v, u)(a, h) = \Phi(v, h) + \Phi(v, u_s) \cdot DA_v \cdot a + \Phi(v, u_t) \cdot DB_v \cdot a \tag{12}
\]

where \((a, h) \in V \oplus E\).

Recalling that \(\Phi\) is \(sc^0\) and observing that the maps \(E^1 \to E\) defined by \(u \mapsto u_s, u_t\) are \(sc\)-operators and the functions \(V \times D \to \mathbb{R}\) defined by \((v, x) \mapsto DA_v(x) \cdot a, DB_v(x) \cdot a\) are smooth, we see that the map

\[
T(V \oplus E) \to TF, \quad (v, u, a, h) \mapsto (\Phi(v, u), D\Phi(v, u)(a, h)), \tag{13}
\]

is \(sc^0\) where \(D\Phi(v, u)(a, h)\) is defined by (12).

It remains to show that the right-hand side of (12) defines the linearization of \(\Phi\). With \((v, u, a, h) \in T(V \oplus E)\), we have

\[
\Phi(v + a, u + h) - \Phi(v, u) - D\Phi(v, u)(a, h)
= \Phi(v + a, h) - \Phi(v, h)
+ \int_0^1 [\Phi(v + \tau a, u_s) DA_{v + \tau a} \cdot a - \Phi(v, u_s) DA_v \cdot a] \, d\tau
+ \int_0^1 [\Phi(v + \tau a, u_t) DB_{v + \tau a} \cdot a - \Phi(v, u_t) DB_v \cdot a] \, d\tau
= I + II + III.
\]

We have used the formula

\[
\Phi(v + a, u) - \Phi(a, u) = \int_0^1 \frac{d}{d\tau} \Phi(v + \tau \cdot a, u) \, d\tau.
\]

We consider the term \(I\). Since \(\Phi\) is linear with respect to the second variable, we have

\[
\frac{1}{|a| + |h|} \cdot |\Phi(v + a, h) - \Phi(v, h)|_0 = \frac{|h|}{|a| + |h|} \cdot \left| \Phi\left( v + a, \frac{h}{|h|} \right) - \Phi\left( v, \frac{h}{|h|} \right) \right|_0
\]

for \(h \neq 0\). The inclusion \(E_1 \to E_0\) is compact and hence we may assume that \(\frac{h}{|h|} \to h_0\) in \(E_0\). Since \(\Phi\) is \(sc^0\), we conclude that

\[
\frac{1}{|a| + |h|} \cdot |\Phi(v + a, h) - \Phi(v, h)|_0 \to 0
\]
as $|a| + |h|_1 \to 0$. Next we consider the second term $II$. We have, for $a \neq 0$,

$$\frac{1}{|a| + |h|_1} \left| \int_0^1 \left[ \Phi(v + \tau a, u_s)DA_{v+\tau a} \cdot a - \Phi(v, u_s)DA_v \cdot a \right] d\tau \right|_0$$

$$\leq \frac{|a|}{|a| + |h|_1} \int_0^1 \left| \Phi(v + \tau a, u_s)DA_{v+\tau a} \cdot \frac{a}{|a|} - \Phi(v, u_s)DA_v \cdot \frac{a}{|a|} \right|_0 d\tau$$

Since $\Phi$ is sc$^0$ and $(v, x) \mapsto DA_v(x)$ is smooth, we conclude that the above expression converges to 0 as $|a| + |h|_1 \to 0$. The same holds for the term $III$. Thus,

$$\frac{1}{|a| + |h|_1} \cdot |\Phi(v + a, u + h) - \Phi(v, u) - D\Phi(v, u)(a, h)|_0 \to 0$$

as $|a| + |h|_1 \to 0$ so that the right-hand side of (12) indeed defines the linearization of $\Phi$ in the sense of Definition 1.8. Moreover, the tangent map $T\Phi : T(V + E) \to TE$ given by (13) is sc$^0$. Summing up, the map $\Phi$ is sc$^1$. From (12), it follows that the compositions of the tangent map $T\Phi$ with projections $\pi$ onto factors of $TE$ are linear combinations of maps of the required form. This completes the proof of $(S_1)$.

Next we assume that the assertion $(S_k)$ has been proved and claim that $(S_{k+1})$ also holds. It suffices to show that the compositions of the iterated tangent map $T^k\Phi : T^k(V + E) \to T^kE$ with the projections $\pi : T^kE \to E^j$ onto the factors of $T^kE$ are sc$^1$ and their linearizations have the required form. By induction hypothesis, $\pi \circ T^k\Phi$ is the linear combination of maps having the particular forms, and it suffices to show that our claim holds for each of these maps. Accordingly, we consider the map

$$\Psi : V \oplus E^m \oplus (\mathbb{R}^n)^p \to E^j, \quad (v, h, a) \mapsto \Phi(v, D^a h) \cdot f(v, a, \cdot)$$

where we have abbreviated $a = (a_1, \ldots, a_p)$. The function $f : V \times (\mathbb{R}^n)^p \times D \to \mathbb{R}$ is smooth and linear in each variable $a_i$. Moreover, $|\alpha| \leq m - j$ and $p \leq k$. Observe the the map $\Psi$ is the composition of the following maps. The map $E^m \to E^{m-|\alpha|}$ defined by $h \mapsto D^\alpha h$ is an sc-operator and hence sc-smooth, it is composed with the map

$$\Phi : V \oplus E^m \to E^j, \quad (v, u) \mapsto \Phi(v, u)$$

which we already know is of class sc$^1$. By the chain rule, this composition is at least of class sc$^1$. So, multiplication of this composition by a smooth
function $V \oplus (\mathbb{R}^n)^p \to \mathbb{R}$ defined by $(v, a) \mapsto f(v, a, \cdot)$ gives an $\text{sc}^1$-map. Having established that $\Psi$ is $\text{sc}^1$, it remains to show that the compositions $\pi \circ T\Psi$ of the tangent map $T\Psi : T(V \oplus E^m \oplus (\mathbb{R}^n)^p) \to T(F^j)$ with the projections onto factors of $T(F^j)$ are linear combinations of maps of the required form. The tangent map is given by

$$T\Psi(v, h, a, \delta v, \delta h, \delta a) = (\Psi(v, h, a), D\Psi(v, h, a)(\delta v, \delta h, \delta a))$$

where $(v, h, a) \in V \oplus E^{m+1} \oplus (\mathbb{R}^n)^p$ and $(\delta v, \delta h, \delta a) \in \mathbb{R}^n \oplus E^m \oplus (\mathbb{R}^n)^p$. If $\pi : T(E^j) = E^{j+1} \oplus E^j \to E^{j+1}$ is the projection onto the first factor, then $\pi \circ T\Psi = \Psi$ and this map, in view of inductive hypothesis, has the form as required in $(S_k)$ but with the indices $m$ and $j$ raised by 1. So, we consider the projection onto the second factor and the map $\pi \circ T\Psi = D\Psi$. Using the chain rule and the linearization of $\Phi$ given by (12), the linearization $D\Psi$ is a linear combination of the following four types of maps:

1. $V \oplus E^m \oplus (\mathbb{R}^n)^p \to F$, 

   $$(v, \delta h, a) \mapsto \Phi(v, D^a(\delta h)) \cdot f(v, a).$$

2. $V \oplus E^{m+1} \oplus (\mathbb{R}^n)^{p+1} \to F$, 

   $$(v, h, (\delta v, a)) \mapsto \Phi(v, D^{a+(1,0)}h) \cdot (DA_v \cdot \delta v) f(v, a)$$

   and 

   $$(v, h, (\delta v, a)) \mapsto \Phi(v, D^{a+(0,1)}h) \cdot (DB_v \cdot \delta v) f(v, a).$$

3. $V \oplus E^{m+1} \oplus (\mathbb{R}^n)^{p+1} \to F$, 

   $$(v, h, a) \mapsto \Phi(v, D^a h) \cdot D_v f(v, a) \cdot \delta v.$$ 

4. $V \oplus E^{m+1} \oplus (\mathbb{R}^n)^{p+1} \to F$, 

   $$(v, h, (a, \delta a_i)) \mapsto \Phi(v, D^a h) \cdot f(v, (a_1, \ldots, \delta a_i, \ldots, a_p))$$

   for every $1 \leq i \leq p$.

These types are all of the desired form. Having verified the statement $(S_{k+1})$, the proof of Theorem 1.26 is complete. $\blacksquare$
We mention a related result which has application in the constructions of SFT. We assume that $V$ is an open neighborhood of 0 in $\mathbb{R}^n$ and let the following data be given:

1. Smooth maps $c : V \to \mathbb{R}$ and $d : V \to S^1$.

2. A smooth map
   
   $$V \times (\mathbb{R}^+ \times S^1) \to \mathbb{R}^2, \quad (v, (s, t)) \mapsto r_v(s, t)$$

   where the function
   
   $$r_v : V \to H^{m,\varepsilon}(\mathbb{R}^+ \times S^1)$$

   is smooth for every $m \geq 3$ and every $\varepsilon \in (0, 2\pi)$.

3. If $v \in V$ and $(s, t) \in \mathbb{R}^+ \times S^1$, then
   
   $$(s + c(v), t + d(v)) + r_v(s, t) \in \mathbb{R}^+ \times S^1.$$ 

For every $v \in V$, define the map $\phi_v : \mathbb{R}^+ \times S^1 \to \mathbb{R}^+ \times S^1$ by

$$\phi_v(s, t) = (s + c(v), t + d(v)) + r_v(s, t).$$

Given a strictly increasing sequence $(\delta_m)_{m \in \mathbb{N}_0}$ of real numbers satisfying $0 < \delta_0 < \delta_m < 2\pi$, we equip the Banach space

$$E = H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N)$$

with the sc-structure defined by $E = H^{3+m,\delta_m}(\mathbb{R}^+ \times S^1, \mathbb{R}^N)$. We also define the Banach space $\hat{E}$ by

$$\hat{E} := \mathbb{R}^N + H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N).$$

A map $u : \mathbb{R}^+ \times S^1 \to \mathbb{R}^N$ belongs to $\hat{E}$ if it belongs to $H^3_{\text{loc}}$ and if there exists a constant $c \in \mathbb{R}^n$ satisfying $u - c \in H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N)$ so that $u$ can be written as $u = c + (u - c) \in \mathbb{R}^N + H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N)$.

**Theorem 2.7.** Let $\phi_v$ be as described above. Then the composition

$$\Psi : V \oplus E \to E, \quad (v, u) \mapsto u \circ \phi_v$$

is well-defined and sc-smooth. The same result is true if we replace $E$ by $\hat{E}$. 

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The proof is in its structure quite similar to the previous proof. It is clear that the result follows for $\tilde{E}$ once it is proved for $E$. One first shows that $\Psi$ is $sc^0$. Fixing an $m$ one should recall that compactly supported smooth maps are dense in $E_m$. For a compactly supported map $w_0$ the convergence $\Psi(v, w_0) \rightarrow \Psi(v_0, w_0)$ in $E_m$ as $v \rightarrow v_0$ is obvious. Then one easily verifies that for $v$ in a suitable open neighborhood of $v_0$ there is a uniform bound of the operator norm of $\Psi(v, \cdot)$. From this point on we can argue as in a previous proof to obtain continuity. Next one proves that $\Psi$ is $sc^1$ and proceeds by induction. We leave the details to the reader.

2.3 A Basic Analytical Proposition

We continue with our study of $sc$-smoothness. We denote by $\varphi$ the exponential gluing profile

$$\varphi(r) = e^{\frac{1}{r}} - e, \quad r > 0.$$ 

With the nonzero complex number $a$ (gluing parameter) we associate the gluing angle $\vartheta \in S^1$ and the gluing length $R$ via the formulae

$$a = |a|e^{-2\pi i \vartheta} \quad \text{and} \quad R = \varphi(|a|).$$

Note that $R \rightarrow \infty$ as $|a| \rightarrow 0$.

We denote by $L$ the Hilbert $sc$-space $L^2(\mathbb{R} \times S^1, \mathbb{R}^N)$ equipped the $sc$-structure $(L_m)_{m \in \mathbb{N}_0}$ defined by $L_m = H^{m, \delta_m}(\mathbb{R} \times S^1, \mathbb{R}^N)$, where $(\delta_m)$ is a strictly increasing sequence starting with $\delta_0 = 0$. Let us also introduce the $sc$-Hilbert spaces $F = H^{2, \delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N)$ with the $sc$-structure whose level $m$ corresponds to the Sobolev regularity $(m + 2, \delta_m)$. Here $\delta_0 > 0$ and $(\delta_m)$ is a strictly increasing sequence starting with $\delta_0$. Finally we introduce the $sc$-Hilbert space $E = H^{3, \delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N)$ whose level $m$ corresponds to the regularity $(m + 3, \delta_m)$ and the sequence $(\delta_m)$ is as in the $F$-case.

With these data fixed we prove the following proposition. The proposition has many applications. In particular, it will be used in Section 3.2 in order to prove that the transition maps between local M-polyfolds are $sc$-smooth.

**Proposition 2.8.** The following four maps

$$\Gamma_i : B_{\frac{1}{2}} \oplus G \rightarrow G, \quad i = 1, \ldots, 4,$$

where $G = L$, $G = F$ or $G = E$, are $sc$-smooth.
(1) Let \( f_1 : \mathbb{R} \to \mathbb{R} \) be a smooth function which is constant outside of a compact interval so that \( f_1(+\infty) = 0 \). Define

\[
\Gamma_1(a,h)(s,t) = f_1 \left( s - \frac{R}{2} \right) h(s,t)
\]

if \( a \neq 0 \) and \( \Gamma_1(0,h) = f(-\infty)h \) if \( a = 0 \).

(2) Let \( f_2 : \mathbb{R} \to \mathbb{R} \) be a compactly supported smooth function. Define

\[
\Gamma_2(a,h)(s,t) = f_2 \left( s - \frac{R}{2} \right) h(s-R,t-\vartheta)
\]

if \( a \neq 0 \) and \( \Gamma_2(0,h) = 0 \) if \( a = 0 \).

(3) Let \( f_3 : \mathbb{R} \to \mathbb{R} \) be a smooth function which is constant outside of a compact interval and satisfying \( f_3(\infty) = 0 \). Define

\[
\Gamma_3(a,h)(s',t') = f_3 \left( -s' - \frac{R}{2} \right) h(s',t')
\]

if \( a \neq 0 \) and \( \Gamma_3(0,h) = f_3(-\infty)h \) if \( a = 0 \).

(4) Let \( f_4 : \mathbb{R} \to \mathbb{R} \) be a smooth function of compact support and define

\[
\Gamma_4(a,h)(s',t') = f_4 \left( -s' - \frac{R}{2} \right) h(s'+R,t'+\vartheta)
\]

if \( a \neq 0 \) and \( \Gamma_4(0,h) = 0 \) if \( a = 0 \).

Let us first note that we only have to prove the proposition in the case \( G = L \), since the other cases are obtained by taking the sequence \((\delta_m)\) for the \( L \)-case and raising the index by 2 in the case \( G = F \) and by 3 in the case \( G = E \). The key point in the proof is the following. The gluing length \( R \) as well as the gluing angle \( \vartheta \) are functions of the gluing parameter \( a \). As long as \( a \neq 0 \) these functions are smooth. However, as \( a \to 0 \) their derivatives blow-up. To achieve \( sc \)-smoothness as stated in Proposition 2.3, it is important that the other terms occurring in the formulae have a sufficient decay behavior. Therefore we assume on exponential decays, as well as the filtration by levels comes in. We will only consider the maps \( \Gamma_1 \) and \( \Gamma_2 \), the proofs for the maps \( \Gamma_3 \) and \( \Gamma_4 \) are quite similar and left to the reader. The proof will require several steps and takes the rest of this section. In the first step we prove the \( sc\)-property.
Lemma 2.9. The maps $\Gamma_1$ and $\Gamma_2$ are $sc^0$.

Proof. Since, in view of Proposition 4.11, the shift operator is $sc^0$, the only difficulty can arise at $a = 0$. We begin with the map $\Gamma_1$. We may assume without loss of generality that $f_1(-\infty) = 1$ so that $\Gamma_1(0, h) = h$ for every $h \in L$. Fix a level $m$ and observe that

$$|\Gamma_1(a, h)|_m \leq C' \cdot \left[ \max_{0 \leq k \leq m} \sup_{\mathbb{R}} |f^{(k)}| \right] \cdot |h|_m = C|h|_m \quad \text{(14)}$$

with the constant $C = C' \cdot [\max_{0 \leq k \leq m} \sup_{\mathbb{R}} |f^{(k)}|]$ independent of $a$ and $h$.

The smooth compactly supported maps are dense in $L_m$ for every $m$. If $u_0$ a smooth compactly supported function and $|a|$ is sufficiently small, then $\Gamma_1(a, u_0) = u_0$. Given $h_0 \in L_m$ and $\varepsilon > 0$, we choose a smooth compactly supported function $u_0$ satisfying $|u_0 - h_0|_m \leq \varepsilon$. Then, recalling that $\Gamma_1(0, h_0) = h_0$ and using (14), we have, with $|a|$ sufficiently small, the following estimate,

$$|\Gamma_1(a, h) - \Gamma_1(0, h_0)|_m = |\Gamma_1(a, h) - h_0|_m$$
$$= |\Gamma_1(a, h) - \Gamma_1(a, h_0) + \Gamma_1(a, h_0) - \Gamma_1(a, u_0) + u_0 - h_0|_m$$
$$\leq |\Gamma_1(a, h) - \Gamma_1(a, h_0)|_m + |\Gamma_1(a, h_0) - \Gamma_1(a, u_0)|_m + |u_0 - h_0|_m$$
$$\leq C|h - h_0|_m + (C + 1)|u_0 - h_0|_m.$$  

So, if $|h - h_0|_m < \varepsilon$, then

$$|\Gamma_1(a, h) - \Gamma_1(0, h_0)|_m < (2C + 1)\varepsilon$$

which proves continuity of $\Gamma_1$ at $(0, h_0)$ on level $m$.

The $sc^0$-property of the map $\Gamma_2$ is more involved. Again, the difficulty arises at $a = 0$. We fix a level $m$ and first show that the norm of $\Gamma_2(a, h)$ is uniformly bounded with respect to $a$ close to 0. By assumption, the support of $f_2$ is contained in the interval $[-A, A]$. If $a$ is sufficiently small, then $[-A - \frac{R}{2}, A - \frac{R}{2}] \subset (-\infty, 0]$ and $[-A + \frac{R}{2}, A + \frac{R}{2}] \subset [0, \infty)$, respectively. Assuming $h \in L_m$, we estimate the norm $|\Gamma_2(a, h)|_m$. The square of the norm $|\Gamma_2(a, h)|_m$ is equal to the sum of the integrals

$$I_\alpha = \int_{\Sigma_R} \left| D^{\alpha} \left( f_2 \left( s - \frac{R}{2} \right) h(s - R, t - \vartheta) \right) \right|^2 e^{2\delta_m|s|} \, dsdt \quad \text{(15)}$$

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with $|\alpha| \leq m$ and where we have abbreviated $\Sigma_R = [-A + \frac{R}{2}, A + \frac{R}{2}] \times S^1$. Denoting by $C$ a generic constant independent of $a$ and $h$, we estimate the integral $I_\alpha$ in (15) as follows

\[
I_\alpha \leq C \int_{\Sigma_R} |D^\alpha h(s - R, t - \vartheta)|^2 e^{2\delta_m s} \, ds \, dt \\
\leq C \int_{\Sigma_R} |D^\alpha h(s, t)|^2 e^{2\delta_m (s + R)} \, ds \, dt \\
= C \int_{\Sigma_R} |D^\alpha h(s, t)|^2 e^{-2\delta_m s} e^{2\delta_m (2s + R)} \, ds \, dt \\
\leq e^{4\delta_m A} C \int_{\Sigma_R} |D^\alpha h(s, t)|^2 e^{-2\delta_m s} \, ds \, dt \\
\leq e^{4\delta_m A} C \cdot |h|^2_m.
\]

Hence

\[
|\Gamma_2(a, h)|_m \leq e^{2\delta_m A} C \cdot |h|^2_m
\]

where $C$ is a constant independent of $a$ and $h$. Now if $u_0$ is a smooth compactly supported map and $a$ is sufficiently small, then $\Gamma_2(a, u_0) = 0$. Given $h_0$ and $\varepsilon > 0$, we choose a smooth compactly supported map $u_0$ so that $|h_0 - u_0|_m < \varepsilon$. Using the estimate (16), we compute for $|a|$ small and $h \in L^m$ satisfying $|h - h_0|_m < \varepsilon$,

\[
|\Gamma_2(a, h) - \Gamma_2(0, h_0)|_m = |\Gamma_2(a, h)|_m \\
= |\Gamma_2(a, h) - \Gamma_2(a, h_0) + \Gamma_2(a, h_0) - \Gamma_2(a, u_0)|_m \\
\leq |\Gamma_2(a, h) - \Gamma_2(a, h_0)|_m + |\Gamma_2(a, h_0) - \Gamma_2(a, u_0)|_m \\
\leq Ce^{\delta_m A} (|h - h_0|_m + |h_0 - u_0|_m) < 2Ce^{\delta_m A} \varepsilon,
\]

showing that $\Gamma_2$ is continuous at $(0, h_0)$ on level $m$. This completes the proof of the lemma. ■

Next we derive decay estimates. The constants $d_{m+k,m}$ in the lemma are defined, for every pair of nonnegative integers $(m, k)$, as the differences

\[
d_{m+k,m} := \frac{1}{2}(\delta_{m+k} - \delta_m).
\]

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Lemma 2.10. For every pairs \((m,k)\) of nonnegative integers, there exists a constant \(C = C(m,k) \geq 0\) independent of \(h\) and \(a\) so that

\[
|h - \Gamma_1(a,h)|_m \leq C \cdot e^{-d_{m+k,m}R} \cdot |h|_{m+k} \\
|\Gamma_2(a,h)|_m \leq C \cdot e^{-d_{m+k,m}R} \cdot |h|_{m+k}
\]

for all \(h \in L_{m+k}\). Here \(R = \varphi(|a|)\).

Proof. We begin with the map \(\Gamma_1\). Again we assume that \(f_1(-\infty) = 1\). The function \(f = 1 - f_1\) satisfies \(f(s) = 0\) for \(s \ll 0\) and \(f(+\infty) = 1\) and we study the map \(\Gamma(a,h) = h - \Gamma_1(a,h)\) which is defined by

\[
\Gamma(a,h)(s,t) = f\left(s - \frac{R}{2}\right) h(s,t).
\]

The support of \(f(\cdot - \frac{R}{2})\) is contained in \([-A + \frac{R}{2}, \infty)\) and hence contained in \([0, \infty)\) if \(a\) is sufficiently small. For such an \(a\) and for \(h \in L_{m+k}\), the square of the norm \(|\Gamma(a,h)|_m\) is the sum of the integrals

\[
I_\alpha = \int_{\Sigma_R} |D^\alpha \left(f\left(s - \frac{R}{2}\right) h(s,t)\right)|^2 e^{2\delta_m s} \, dsdt
\]

with \(|\alpha| \leq m\). Here we have abbreviated \(\Sigma_R = [-A + \frac{R}{2}, \infty) \times S^1\). Then, with \(C\) denoting a generic constant independent of \(a\) and \(h\), we estimate

\[
I_\alpha \leq C \sum_{|\beta| \leq |\alpha|} \int_{\Sigma_R} |D^\beta h(s,t)|^2 e^{2\delta_m s} \, dsdt
\]

\[
= C \sum_{|\beta| \leq |\alpha|} \int_{\Sigma_R} |D^\beta h(s,t)|^2 e^{2\delta_{m+k}s - 4d_{m+k,m}s} \, dsdt
\]

\[
\leq Ce^{-2d_{m+k,m}R} \sum_{|\beta| \leq |\alpha|} \int_{\Sigma_R} |D^\beta h(s,t)|^2 e^{2\delta_{m+k}s} \, dsdt
\]

\[
\leq Ce^{-2d_{m+k,m}R} \cdot |h|_{m+k}^2
\]

Since \(|\Gamma(a,h)|_m^2 = \sum_{|\alpha| \leq m} I_\alpha\), we obtain the required estimate,

\[
|\Gamma(a,h)|_m \leq Ce^{-d_{m+k,m}R} \cdot |h|_{m+k}
\]

for \(a\) sufficiently small and \(h \in L_{m+k}\) with some constant \(C\) independent of \(a\) and \(h\). This is exactly the required estimate.
We turn to the map $\Gamma_2$. The support of $f_2$ is contained in $[-A,A]$ for some $A > 0$. Hence, the support of $f_2(\cdot - \frac{R}{2})$ is contained in the interval $[-A + \frac{R}{2}, A + \frac{R}{2}]$. Moreover, if $|a|$ is sufficiently small, then $[-A - \frac{R}{2}, A - \frac{R}{2}] \subset (-\infty, 0]$ and $[-A + \frac{R}{2}, A + \frac{R}{2}] \subset [0, \infty)$. We estimate the square of the norm $|\Gamma_2(a, h)|_m$ for $h \in L_{m+k}$ and sufficiently small $|a|$. The square of the norm $|\Gamma_2(a, h)|_m$ is equal to the sum of the integral expressions

$$I_\alpha = \int_{\Sigma_R} \left| D^\alpha \left( f_2 \left( s - \frac{R}{2} \right) h(s - R, t - \vartheta) \right) \right|^2 e^{2\delta_m s} \, dsdt$$

with $|\alpha| \leq m$ and where $\Sigma_R$ denotes the finite cylinder $[-A + \frac{R}{2}, A + \frac{R}{2}] \times S^1$. With $C$ denoting a generic constant not depending on $a$ and $h$, the integral $I_\alpha$ can be estimated as follows,

$$I_\alpha \leq C \int_{\Sigma_R} |D^\alpha h(s - R, t - \vartheta)|^2 e^{2\delta_m s} \, dsdt$$

$$= C \int_{\Sigma_R} |D^\alpha h(s, t)|^2 e^{2\delta_m (s + R)} \, dsdt$$

$$= C \int_{\Sigma_R} |D^\alpha h(s, t)|^2 e^{-2\delta_m s + 2(\delta_m + \delta_{m+k})s + 2\delta_m R} \, dsdt$$

$$\leq C \int_{\Sigma_R} |D^\alpha h(s, t)|^2 e^{-2\delta_m s} e^{2(\delta_m + \delta_{m+k})(A - \frac{R}{2}) + 2\delta_m R} \, dsdt$$

$$\leq Ce^{-2d_{m+k}mR} \int_{\Sigma_R} |D^\alpha h(s, t)|^2 e^{-2\delta_m s} \, dsdt$$

$$\leq Ce^{-2d_{m+k}mR} |h|_{m+k}^2.$$ 

Thus,

$$|\Gamma_2(a, h)|_m \leq Ce^{-d_{m+k}mR} |h|_{m+k}$$

for all $a$ sufficiently small and $h \in L_{m+k}$. The constant $C$ is independent of $a$ and $h$. The proof of Lemma 2.10 is complete.

From the sc-smoothness of the shift-map proved in Proposition 4.2 we conclude the following lemma.

**Lemma 2.11.** The maps $\mathbb{R} \oplus L \to L$, defined by

$$(R, u) \mapsto f_1(\cdot - \frac{R}{2})u$$

$$(R, v) \mapsto f_2(\cdot - \frac{R}{2})v(\cdot - R, \cdot - \vartheta),$$

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are sc-smooth.

We will need estimates for the derivative of the functions \( a \mapsto R(a) = e^{e^{-|a|}} - e \), where \( a = |a| \cdot e^{-2\pi i \varphi} \). In view of Lemma 4.6 proved in Appendix 4.2 we have the following estimates.

**Lemma 2.12.** For every multi-index \( \alpha = (\alpha_1, \alpha_2) \), there exists a constant \( C \) such that
\[
|D^\alpha R(a)| \leq C \cdot R(a) \cdot [\ln(R(a))]^{2|\alpha|}
\]
for \( 0 < |a| < \frac{1}{2} \).

Let us continue with the study of the map
\[
\Gamma_1 : B_{\frac{1}{2}} \oplus L \to L, \quad \Gamma_1(a, u) = f_1 \left( \cdot - \frac{R}{2} \right) \cdot u.
\]
As already done before we assume that \( f_1(-\infty) = 1 \) and study, rather than \( \Gamma_1 \), the map \( \Gamma(a, u) = u - \Gamma_1(a, u) \) which has the form
\[
\Gamma(a, u) = f \left( \cdot - \frac{R}{2} \right) \cdot u
\]
where \( f = 1 - f_1 \).

Let us denote by \( R(a) \), for \( a \neq 0 \), any product of derivatives of the kind
\[
R(a)(a_1, \ldots, a_n) = (D^{n_1}R(a)(a_1, \ldots, a_{n_1}) \cdots (D^{n_l}R(a)(a_{n_{l-1}+1}, \ldots, a_n)
\]
where \( n = n_1 + \ldots + n_l \) and \((a_1, \ldots, a_n) \in C^n \). We call the integer \( n \) the order of \( R(a) \). We define \( R(a) \) of order 0 to be the constant function equal to 1. To prove the sc-smoothness of \( \Gamma \), we need a structural statement about the form of \( T^k \Gamma \) for \( a \neq 0 \).

**Lemma 2.13.** Assume that \( \pi : T^k L \to L^j \) is the projection onto a factor of \( T^k L \). Then, for \( a \neq 0 \), the composition \( \pi \circ T^k \Gamma : T^k L \to L^j \) is a linear combination of maps
\[
A : (B_{\frac{1}{2}} \setminus \{0\}) \oplus C^n \oplus L^m \to L^j
\]
of the form
\[
(a, h, w) \mapsto R(a)(h) \cdot f^{(p)} \left( s - \frac{R}{2} \right) \cdot w,
\]
where \( h = (h_1, \ldots, h_n) \in C^n \) and \( R(a) \) has order \( n \). Moreover, the following inequalities hold
\[
p \leq m - j \quad \text{and} \quad n \leq k.
\]
Proof. We prove the lemma by induction with respect to $k$ starting with $k = 0$. In this case, the statement is trivially satisfied since $T^0 \Gamma = \Gamma$ so that $R(a) = 1$, $n = p = 0$, and $j = m = 0$. We assume that the statement has been proved for $k$ and verify that it holds for $k + 1$. If $\pi : T^{k+1}L \to L^j$ is a projection onto one of the first $2^k$ factors we know that terms of $\pi \circ T^{k+1} \Gamma$ have the required form by induction hypothesis. The only thing which is different is that the indices $m$ and $j$ are both raised by one (recall the definition of the tangent). If $\pi : T^{k+1}L \to L^j$ projects onto one of the last $2^k$ factors, the terms of $\pi \circ T^{k+1} \Gamma$ are the linear combinations of derivatives of maps guaranteed by the induction hypothesis. Hence we take a map of the form (17) and differentiate in the sc-sense. For $(a, h, w, \delta a, \delta h, \delta w) \in T((B_1^2 \setminus \{0\}) \oplus \mathbb{C}^n \oplus L^m) = (B_1^2 \setminus \{0\}) \oplus \mathbb{C}^n \oplus L^{m+1} \oplus \mathbb{C} \oplus \mathbb{C}^n \oplus L^m$ we obtain a linear combination of maps of the following four types:

(1) $B_1^2 \oplus \mathbb{C}^{n+1} \oplus L^{m+1} \to L^j$,

$$(a, (\delta a, h), w) \mapsto R'(a)(\delta a, h) f^{(p)} \left(s - \frac{R}{2}\right) \cdot w$$

where all the occurring $R'(a)$ have order $n' = n + 1$. Here $j' = j$, $m' = m + 1$, and $n' = n + 1$ so that $p' \leq m' - j'$ and $n' \leq k + 1$.

(2) $B_1^2 \oplus \mathbb{C}^n \oplus L^{m+1} \to L^j$,

$$(a, (h_1, \ldots, \delta h_i, \ldots, h_n), w) \mapsto A(a, (h_1, \ldots, \delta h_i, \ldots, h_n), w).$$

Here $j' = j$, $p' = p$, $m' = m + 1$ and $n' = n + 1$ so that $p' \leq m' - j'$ and $n' \leq k + 1$.

(3) $B_1^2 \oplus \mathbb{C}^{n+1} \oplus L^{m+1} \to L^j$,

$$(a, (\delta a, h), w) \mapsto R_1(a)(\delta a, h) \cdot f^{(p+1)} \left(s - \frac{R'}{2}\right) \cdot w$$

where $R_1(a)(\delta a, h) = (DR(a)(\delta a))R(a)(h)$ so that its order is equal to $n' = n + 1$. Moreover, $p' = p + 1$, $m' = m + 1$, and $j' = j$. Again, $p' \leq m' - j'$ and $n' \leq k + 1$.

(4) $B_1^2 \oplus \mathbb{C}^n \oplus L^m \to L^j$,

$$(a, h, \delta w) \mapsto A(a, h, \delta w).$$
Here $j' = j$, $m' = m$, $p' = p$, and $n' = n$ satisfy $p' \leq m' - j'$ and $n' \leq k + 1$.

This completes the proof of the lemma. ■

Let us observe that any map $A$ in Lemma 2.13 has a continuous extension to points $(0, h, w)$ by defining $A(0, h, w) = 0$. Indeed, if $m = j$ so that $p = 0$, we have $A(a, w) = \Gamma(a, w)$ as a map $B_{1/2} \oplus L^m \to L^m$ and we already know that this is $sc^0$. If, on the other hand, $m - j > 0$, then we combine the estimates in Lemmata 2.10 and 2.12 and obtain the estimate

$$|A(a, h, w)| \leq C e^{-d_{m,j} R} \cdot R^{3k} \cdot |h|^k \cdot |w|_m$$

with a constant $C$ depending on $m, j, p$, and $n$, but not on $a$. Recalling that $R = e^{1/|a|} - e$, the right-hand side converges to 0 as $|a| \to 0$ keeping $(h, w)$ bounded.

At this point we have proved the following lemma.

**Lemma 2.14.** The map $\Gamma : (B_{1/2} \setminus \{0\}) \oplus L \to L$ is $sc$-smooth. Moreover, its iterated tangent map $T^k \Gamma$ can be extended continuously by 0 over all points containing $a = 0$.

It remains to show the approximation property at points $(a, H) \in T^k(B_{1/2} \oplus L)$ where $a = 0 \in B_{1/2}$. Of course, the candidate is the 0 map. We have to show that, given $(0, H) \in T^k(B_{1/2} \oplus L)$,

$$\frac{1}{||(\delta a, \delta H)||_1} \cdot ||T^k \Gamma(\delta a, H + \delta H)||_0 \to 0 \quad \text{as} \quad ||(\delta a, \delta H)||_1 \to 0,$$

where $||\cdot||_0$ ($||\cdot||_1$) is the norm on the level 0 (1) of the iterated tangent $T^k(B_{1/2} \oplus L)$ and is equal to the sum of the norms on each of the factors of $(T^k(B_{1/2} \oplus L))_0 ((T^k(B_{1/2} \oplus L))_1)$.

We know from Lemma 2.13 that the maps $\pi \circ T^k \Gamma$ on the factors of $T^k(B_{1/2} \oplus L)$ are of particular forms. Hence it suffices to consider the maps defined in Lemma 2.13. More precisely, given a map

$$A : B_{1/2} \oplus \mathbb{C}^n \oplus L^m \to L^j, \quad (a, h, w) \mapsto (a, h, w) \mapsto R(a)(h) \cdot f^{(w)} \left( s - \frac{R}{2} \right) \cdot w$$

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and \((0, h, w) \in B_{\frac{1}{2}} \oplus \mathbb{C}^n \oplus L_{m+1}\) (recall that in order to linearize we have to raise the index \(m\) by 1), we have to show that

\[
\frac{1}{|\delta a| + |\delta h| + |\delta w|_{m+1}} |A(\delta a, h + \delta h, w + \delta w)|_j \to 0 \tag{19}
\]
as \(|\delta a| + |\delta h| + |\delta w|_{m+1} \to 0\). If \(\delta a = 0\), then \(A(\delta a, \cdot, \cdot) = 0\) and so we may assume that \(\delta a \neq 0\). In view of the estimate (18) in which \(R = \varphi(|\delta a|)\) for the exponential gluing profile \(\varphi\), the left-hand side of (19) is less or equal to

\[
\frac{C}{|\delta a| + |\delta h| + |\delta w|_{m+1}} e^{-d_{m,j}R} \cdot R^{3k} \cdot |h + \delta h|^k \cdot |w + \delta w|_m
\]
which converges to 0 as \(|\delta a| + |\delta h| + |\delta w|_{m+1} \to 0\). This proves the approximation property and completes the proof that \(\Gamma\), and hence \(\Gamma_0\), is sc-smooth.

Next we consider the map \(\Gamma_2\). Again we start with the prove of structural result about \(\pi \circ T_k \Gamma_2\). Before we do that, we state the estimate which we shall subsequently use. It follows immediately from Lemma 2.15.

**Lemma 2.15.** For all multi-indices \(\alpha\), there exists a constant \(C > 0\) so that

\[
|D^\alpha \vartheta(a)| \leq C \cdot |\ln R(a)||^{|\alpha|}
\]
if \(a \neq 0\).

Similar to \(R(a)\) we introduce for \(a \neq 0\) the expression \(\Theta(a)\) of products of derivatives of the form

\[
\Theta(a)(h_1, \ldots, h_k) = D^{k_1} \vartheta(a)(h_1, \ldots, h_{k_1}) \cdot \ldots \cdot D^{k_l} \vartheta(h_{k_{l-1}+1}, \ldots, h_k),
\]
where \(k = k_1 + \ldots + k_l\) is the order of \(\Theta(a)\). By Lemma 2.15

\[
|\Theta(a)(h_1, \ldots, h_k)| \leq C \cdot (\ln R)^k \cdot |h|^k
\]
for sufficiently small \(a \neq 0\) with a constant \(C > 0\) independent of \(a\).

Here is the necessary structural statement for the map \(\Gamma_2\).
Lemma 2.16. Assume that \( \pi : T^k L \to L^j \) is the projection onto a factor of \( T^k L \). Then, for \( a \neq 0 \), the composition \( \pi \circ T^k \Gamma_2 : T^k L \to L^j \) is a linear combination of maps

\[
A : (B_{\frac{1}{2}} \setminus \{0\}) \oplus \mathbb{C}^{p+\alpha_1} \oplus \mathbb{C}^{\alpha_2} \oplus L^m \to L^j
\]

of the form

\[
(a, h, e, w) \mapsto R(a)(h) \cdot \Theta(a)(e) \cdot f^{(p)} \left( -\frac{R}{2} \right) (D^a w)(\cdot - R, \cdot - \vartheta)
\]

where \( p + |\alpha| \leq m - j \) and \( |p| + |\alpha| \leq k \). Moreover, the sum of the orders of \( R(a) \) and \( \Theta(a) \) does not exceed \( k \).

Proof. Clearly the statement is true for \( k = 0 \). Assume it has been proved for \( k \). Consider the composition \( \pi \circ T^k+1 \Gamma_2 \) where \( \pi \) is the projection onto one of the first \( 2^k \) factors. Then the result follows from the induction hypothesis raising the indices in the domain and the target by 1. If \( \pi \) is a projection onto one of the last \( 2^k \) factors, then the composition \( \pi \circ T^k+1 \Gamma_2 \) is the linear combination of derivatives of terms guaranteed by the induction hypothesis. More precisely, assume that we consider a map \( A \) of the above form. If we differentiate \( A \) at a point where \( a \neq 0 \), then the linearization is a linear combination of the following types of maps:

1. \( (B_{\frac{1}{2}} \setminus \{0\}) \oplus \mathbb{C}^{n+\alpha_1+1} \oplus \mathbb{C}^{\alpha_2} \oplus L^{m+1} \to L^j \),

\[
(a, \delta a, h, e, w) \mapsto R'(a)(\delta a, h) \cdot \Theta(a)(e) \cdot f^{(p)} \left( -\frac{R}{2} \right) (D^a w)(\cdot - R, \cdot - \vartheta)
\]

where \( R'(a) \) is a linear combination of terms of the form \( R(a) \) of order \( n + \alpha_1 \).

2. This map is obtained by only differentiating \( R(a)(h) \) with respect to \( h \). This gives the map \( B_{\frac{1}{2}} \oplus \mathbb{C}^{n+\alpha_1} \oplus \mathbb{C}^{\alpha_2} \oplus L^{m+1} \to L^j \),

\[
(a, (h_1, \ldots, \delta h_i, \ldots, h_n), e, w) \mapsto A(a, (h_1, \ldots, \delta h_i, \ldots, h_n), e, w).
\]

3. \( (B_{\frac{1}{2}} \setminus \{0\}) \oplus \mathbb{C}^{n+\alpha_1} \oplus \mathbb{C}^{\alpha_2+1} \oplus L^{m+1} \to L^j \),

\[
(a, h, (\delta a, e), w) \mapsto R(a)(h) \cdot \Theta'(a)(\delta a, e) \cdot f^{(p)} \left( -\frac{R}{2} \right) (D^a w)(\cdot - R, \cdot - \vartheta)
\]
where $\Theta'(a)$ is a linear combination of terms of the form $\Theta(a)$ each of order $\alpha_1$ so that $\Theta'(a)$ is of order $\alpha'_2 = \alpha_2 + 1$.

(4) $B_{\frac{1}{2}} \oplus \mathbb{C}^{n+\alpha_1} \oplus \mathbb{C}^{\alpha_2} \oplus L^{m+1} \to L^j,
(a, h, (e_1, \ldots, \delta E_m, \ldots, e_{\alpha_2}), w) \mapsto A(a, h, (e_1, \ldots, \delta E_m, \ldots, e_{\alpha_2}), w)$.

(5) $(B_{\frac{1}{2}} \setminus \{0\}) \oplus \mathbb{C}^{1+n+\alpha_1} \oplus \mathbb{C}^{\alpha_2} \oplus L^{m+1} \to L^j,
(a, (\delta a, h), (e, w) \mapsto R_1(a)(\delta a, h) \cdot \Theta(a)(e) \cdot f^{(p+1)} \left( -\frac{R}{2} \right) (D^\alpha w)(-R, -\vartheta, -\vartheta)$
where $R_1(a)(\delta a, h) = (DR(a)\delta a)R_1(a)(h)$.

(6) If we differentiate with respect to $w$ we have to replace $w$ by $\delta w$ but with $\delta w \in L_m$. The gives the map $B_{\frac{1}{2}} \oplus \mathbb{C}^{n+\alpha_1} \oplus \mathbb{C}^{\alpha_2} \oplus L^{m} \to L^j,$
$(a, h, e, \delta w) \mapsto A(a, h, e, \delta w)$.

(7) Lastly, the map obtain by differentiating $(D^\alpha w)(s - R, t - \vartheta)$ with respect to $a$. This leads to the map which is a linear combination of the two following maps
$(B_{\frac{1}{2}} \setminus \{0\}) \oplus \mathbb{C}^{n+\alpha_1+1} \oplus \mathbb{C}^{\alpha_2} \oplus L^{m+1} \to L^j,$
$(a, (\delta a, h), e, w) \mapsto R_1(a)(\delta a, h) \cdot \Theta_1(a)(e) \cdot f^{(p+1)} \left( -\frac{R}{2} \right) (D^\alpha w)(-R, -\vartheta, -\vartheta)$
where $R_1(a)(\delta a, h) = (DR(a)\delta a)R_1(a)(h)$ and $\alpha' = \alpha + (1, 0)$, and
$(B_{\frac{1}{2}} \setminus \{0\}) \oplus \mathbb{C}^{n+\alpha_1} \oplus \mathbb{C}^{\alpha_2+1} \oplus L^{m+1} \to L^j,$
$(a, (\delta a, h), e, w) \mapsto R_1(a)(\delta a, h) \cdot \Theta_1(a)(e) \cdot f^{(p+1)} \left( -\frac{R}{2} \right) (D^\alpha w)(-R, -\vartheta, -\vartheta)$
where $\Theta_1(a)(\delta a, h) = (D\Theta(a)\delta a)\Theta(a)(e)$ and $\alpha'' = \alpha + (0, 1)$.

Hence this derivative is a finite linear combination of terms of the required form. Using the previous result and the previously derived exponential decay estimates, we see that every $T^k\Gamma_2$ can be extended in a $sc^0$-continuous way by 0 over points containing $a = 0$. Arguing as before (just after Lemma 2.14) we can also verify the approximation property. Hence the map $\Gamma_2$ is sc-smooth. This completes the proof of Proposition 2.8. ■
2.4 Gluing, Anti-Gluing and Splicings, Proof of Theorem 1.28

In this section we present the proof of Theorem 1.28. In order to do this we recall the formula for the projection map \( \pi_a : E \to E \). If \( a = 0 \), this projection is the identity since \( \ominus a \) is the zero map. So, we assume that \( 0 < |a| < \frac{1}{2} \) and set
\[
\pi_a(\xi^+, \xi^-) := (\eta^+, \eta^-).
\]
The pair \((\eta^+, \eta^-)\) is found by solving the following system of equations
\[
\begin{align*}
\oplus_a (\eta^+, \eta^-) &= \oplus_a (\xi^+, \xi^-) \\
\ominus_a (\eta^+, \eta^-) &= 0.
\end{align*}
\]
Recalling that \( \beta_a(s) = \beta_R(s) = \beta(s - \frac{R}{2}) \) and setting \( \gamma_a = \beta_a^2 + (1 - \beta_a)^2 \), we have derived in Section 1.3 the formula
\[
\eta^+(s, t) = \left(1 - \frac{\beta_a(s)}{\gamma_a(s)}\right) \cdot \frac{1}{2} \cdot \left([\xi^+]_R + [\xi^-]_{-R}\right)
\]
\[
+ \frac{\beta_a^2(s)}{\gamma_a(s)} \xi^+(s, t) + \frac{\beta_a(1 - \beta_a)}{\gamma_a(s)} \xi^-(s - R, t - \vartheta),
\]
where
\[
[\xi^+]_R = \int_{S^1} \xi^+ \left(\frac{R}{2}, t\right) \, dt \quad \text{and} \quad [\xi^-]_{-R} = \int_{S^1} \xi^- \left(-\frac{R}{2}, t\right) \, dt.
\]
A similar formula holds for \( \eta^- \). In order to study the sc-smoothness we consider the map
\[
(a, \xi^+, \xi^-) \mapsto \eta^+,
\]
the sc-smoothness of the map \((a, \xi^+, \xi^-) \mapsto \eta^-\) is verified the same way. If we write \( \xi^\pm = c + r^\pm \), where \( c \) is the common asymptotic constant, then the formula for \( \eta^+ \) takes the form
\[
\eta^+(s, t) = c + \frac{1}{2} \left(1 - \frac{\beta_a(s)}{\gamma_a(s)}\right) \cdot \left([r^+]_R + [r^-]_{-R}\right)
\]
\[
+ \frac{\beta_a^2(s)}{\gamma_a(s)} \cdot r^+(s, t) + \frac{\beta_a(1 - \beta_a)}{\gamma_a(s)} \cdot r^-(s - R, t - \vartheta)
\]
(20)
We shall study the following five mappings:
M1. The map
\[ H^{3,\delta}_c(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to \mathbb{R}^N, \quad \xi^+ \mapsto c \]
which associates with \(\xi^+\) its asymptotic constant \(c\).

M2. The map
\[ B_{\frac{1}{2}} \times H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to \mathbb{R}^N, \quad (a, r^+) \mapsto [r^+]_R. \]

M3. The map
\[ B_{\frac{1}{2}} \times H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N), \quad (a, r^+) \mapsto \frac{\beta_a}{\gamma_a}(\cdot)[r^+]_R. \]

M4. The map \( B_{\frac{1}{2}} \oplus H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N), \)
\[ (a, r^+) \mapsto \frac{\beta_a^2}{\gamma_a} \cdot r^+. \]

M5. The map \( B_{\frac{1}{2}} \oplus H^{3,\delta_0}(\mathbb{R}^- \times S^1, \mathbb{R}^N) \to H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N), \)
\[ (a, r^-) \mapsto \frac{\beta_a(1 - \beta_a)}{\gamma_a} r^- (\cdot - \bar{R}, \cdot - \bar{\vartheta}). \]

In view of the formula for the projection map \(\pi_a\) the sc-smoothness of the map \((a, (\xi^+, \xi^-)) \mapsto \pi_a(\eta^+, \eta^-)\) in Theorem 1.28 is a consequence of the following proposition.

**Proposition 2.17.** The maps M1-M5 listed above (and suitably defined at the parameter value \(a = 0\)) are sc-smooth in a neighborhood of \(a = 0\).

The proof of the proposition follows from a sequence of lemmata.

**Lemma 2.18.** The map \( H^{3,\delta_0}_c(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to \mathbb{R}^N, \quad \xi^+ \mapsto c \), which associates with \(\xi^+\) its asymptotic constant \(c\) is sc-smooth.

**Proof.** The map \(\xi^+ \mapsto c\) is an sc-projection and therefore sc-smooth. \(\blacksquare\)

**Lemma 2.19.** The map \(\Phi : B_{\frac{1}{2}} \oplus H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to \mathbb{R}^N\), defined by \(\Phi(0, h) = 0\) for \(a = 0\) and
\[ \Phi(a, h) = [h]_R = \int_{S^1} h \left( \frac{R}{2}, t \right) dt \]
for \(a \neq 0\), is sc-smooth.
Proof. We abbreviate in the proof $F = H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N)$. Using the Sobolev embedding theorem for bounded domains into continuously differentiable functions we see that the map

$$(0, \infty) \times F_m \to C^0(S^1, \mathbb{R}^N), \quad (R, h) \mapsto h \left( \frac{R}{2} \cdot \right)$$

is of class $C^{m+1}$ for every $m \geq 0$. In view of Corollary 2.5 this implies that the map

$$\hat{\Phi} : (0, \infty) \times F \to \mathbb{R}^N, \quad (R, h) \mapsto [h]_R$$

is sc-smooth. Since the map $a \mapsto R(a) := \varphi(|a|)$ is obviously smooth if $a \neq 0$, we conclude, using the chain rule for sc-smooth maps, that the map

$$\Phi : (B_{\frac{1}{2}} \setminus \{0\}) \oplus F \to \mathbb{R}^N, \quad (a, h) \mapsto [h]_R$$

is sc-smooth and we claim that the map $\Phi$ is sc at every point $(0, h) \in B_{\frac{1}{2}} \oplus F$. Indeed, assume that $(a_k, h_k) \in (B_{\frac{1}{2}} \setminus \{0\}) \oplus F$ is a sequence converging to $(0, h)$. Abbreviating $\Sigma_k = \left( \frac{R_k}{2} - 1, \frac{R_k}{2} + 1 \right) \times S^1$ where $R_k = \varphi(|a_k|)$, we show that $|\Phi(a_k, h_k)| = ||h_k|_{R_k}| \to 0$. By the Sobolev embedding theorem on bounded domains and using the bound $|e^{\delta \cdot \cdot h_k}|_{C^0(\Sigma_k)} \leq C|m|_{H^{m+3}(\Sigma_k)} \leq C''$. This implies

$$|[h_k]_{R_k}| \leq C'' \cdot e^{-\delta m R_k/2} \quad (21)$$

and the claim follows.

At this point we know that the map

$$\Phi : B_{\frac{1}{2}} \oplus F \to \mathbb{R}^N$$

is sc and, when restricted to $(B_{\frac{1}{2}} \setminus \{0\}) \oplus F$ it is sc. We shall denote points in $T^k(B_{\frac{1}{2}} \oplus F)$ by $(a, H)$ where $a \in B_{\frac{1}{2}}$. We shall prove inductively the following statements:

$(S_k)$. The map $\Phi : B_{\frac{1}{2}} \oplus F \to \mathbb{R}^N$ is of class sc and $T^k\Phi(0, H) = 0$ for every $(0, H) \in T^k(B_{\frac{1}{2}} \oplus F)$. Moreover, if $\pi : T^k(\mathbb{R}^N) \to \mathbb{R}^N$ is the projection onto a factor of $T^k\mathbb{R}^N$, then the composition $\pi \circ T^k\Phi$ is a linear combination of maps $\Gamma$ of the of the following types,

$$\Gamma : B_{\frac{1}{2}} \oplus \mathbb{C}^n \oplus F_m \to \mathbb{R}^N, \quad (a, b, v) \to \mathbb{R}(a)(b_1, \ldots, b_n) \cdot [\partial^k v]_R$$
for $a \neq 0$ and $\Gamma(0, b, v) = 0$. Here $j \leq m, n \leq k$, and $R(a)$ is the product of derivatives of the function $R(a) = e^{ \frac{a}{|a|^2} } - e$ of the form

$$R(a)(b_1, \ldots, b_n) = D^{n_1}R(a)(b_1, \ldots, b_{n_1}) \cdots D^{n_l}R(a)(b_{n_1+\ldots+n_{l-1}+1}, \ldots, b_n),$$

where the integer $n = n_1 + \ldots + n_l$ is called the order of $R(a)$. We set $R(a) = 1$ if $n = 0$.

We begin by verifying that $(S_0)$ holds. In this case, the projection $\pi : T^0\mathbb{R}^N = \mathbb{R}^N \to \mathbb{R}^N$ is the identity map, the indices $j, k, m$ and $n$ are equal to 0, and the composition $\pi \circ T^0\Phi$ is just the map $\Phi : B_1^2 \oplus F \to \mathbb{R}^N$ given by

$$(a, v) \to [v]_R.$$

The map has the required form with $R(a) = 1$ of order 0. With $\Phi(0, v) = 0$, we already know that $\Phi$ is $sc^0$. So, the assertion $(S_0)$ holds.

Assuming that $(S_k)$ holds, we show that $(S_{k+1})$ also holds. By induction hypothesis, the map $\Phi$ is $sc^k$, so that $T^k\Phi$ is $sc^0$. Moreover, $T^k\Phi(0, H) = 0$, $T^{k+1}\Phi$ is $sc$-smooth at points $(a, H)$ with $a \neq 0$, and $\pi \circ T^k\Phi$ can be written as a linear combination of maps of a certain form.

Setting $DT^k\Phi(0, H) = 0$, we prove the approximation property of $T^k\Phi$ at the points $(0, H) \in (T^k(B_1^2 \oplus F))^1$. That is, recalling that $T^k\Phi(0, H) = 0$, we show that

$$\frac{1}{\|((\delta a, \delta H))\|_1} |T^k\Phi(\delta a, H + \delta H)|_0 \to 0 \quad \text{as} \quad \|((\delta a, \delta H))\|_1 \to 0. \quad (22)$$

where the subscripts 0 and 1 refer to the levels of the iterated tangents. By the inductive assumption $(S_k)$, we know that the compositions $\pi \circ T^k\Phi$ with projections $\pi$ on different factors of $T^k\mathbb{R}^N$ are linear combinations of maps $A$ described in $(S_k)$. Hence to prove $(22)$ amounts to showing that at the point $(0, h, v) \in B_1^2 \oplus \mathbb{C}^n \oplus F^{m+1}$ we have

$$\frac{1}{|\delta a| + |\delta b| + |\delta v|_{m+1}} |A(\delta a, b + \delta b, v + \delta v)| \to 0 \quad (23)$$

as $|\delta a| + |\delta b| + |\delta v|_{m+1} \to 0$ for the maps

$$A : B_1^2 \oplus \mathbb{C}^n \oplus F^{m} \to \mathbb{R}^N, \quad (a, b, v) \mapsto R(a)(b_1, \ldots, b_n) \cdot [\partial^j v]_R$$

defined in $(S_k)$.
Using as above the Sobolev estimate on the bounded domain \( \Sigma_R = (\frac{R}{2} - 1, \frac{R}{2} + 1) \times S^1 \), we obtain
\[
|e^{\delta_{m+1}} \partial^j_s (v + \delta v)|_{C^0(\Sigma_R)} \leq C|e^{\delta_{m+1}} \partial^j_s (v + \delta v)|_{H^{m+3}(\Sigma_R)},
\]
where \( j \leq m \), and estimate
\[
[\partial^j_s (v + \delta v)]_R \leq Ce^{-\delta_{m+1} \frac{R}{2}} |v + \delta v|_{m+1}.
\]
Therefore, in view of the estimate of \( R(a) \) in Lemma 2.12
\[
|A(\delta a, b + \delta b, v + \delta v)| \leq Ce^{-\delta_{m+1} \frac{R}{2}} |R|^{3n} \cdot |b + \delta b|^n \cdot |v + \delta v|_{m+1}
\]
where \( R = \varphi(|\delta a|) \) and \( \delta a \neq 0 \). Consequently,
\[
\frac{|A(\delta a, b + \delta b, v + \delta v)|}{|\delta a| + |\delta h| + |\delta v|_{m+1}} \leq Ce^{-\delta_{m+1} \frac{R}{2}} |R|^{3n} \cdot |b + \delta b|^n \cdot |v + \delta v|_{m+1}. \tag{24}
\]
If \( R = \varphi(|\delta a|) \) is large (or \(|\delta a| \) is small), then \( 2R \geq 2 \ln R \geq \frac{1}{|\delta a|} \) so that the left hand-side of \( \tag{24} \) is smaller than
\[
Ce^{-\delta_{m+1} |\delta a|} |R|^{4n} \cdot |b + \delta b|^n \cdot |v + \delta v|_{m+1}
\]
which converges to 0 as \((\delta a, \delta b, \delta v) \to (0, 0, 0)\) in \( C \oplus C^n \oplus F^{n+1} \). Summing up our discussion so far, we have proved the approximation property for the map \( T^k \Phi \) and
\[
D(T^k \Phi)(0, H) = 0
\]
for all \((0, H) \in (T^k (B_2 \oplus F))_1\). To complete the proof, it remains to show that \( T^{k+1} \Phi \) is of class \( sc^0 \) (which will imply that \( \Phi \) is of class \( sc^k \)) and to show that the compositions \( \pi \circ T^{k+1} \Phi \) have the required form.

We consider \( \pi \circ T^{k+1} \Phi(a, H) \) where \( a \neq 0 \). If \( \pi \) is the projection onto one of the first \( 2^k \) factors, then \( \pi \circ T^{k+1} \Phi \) has the form of the map \( A \) in \((S_k)\). The only thing is that the indices are raised by 1. Denoting the new indices by \( j', m', \) and \( n' \), we have \( j' = j \), \( m' = m + 1 \), and \( n' = n \) which obviously satisfy \( j' \leq m' \leq k + 1 \), and \( n' \leq k + 1 \). If \( \pi \) is the projection onto one of the remaining \( 2^k \) factors, then \( \pi \circ T^{k+1} \Phi \) is equal to the sum of derivatives of maps in the induction hypothesis \((S_k)\). So, if the map \( A : B_2 \oplus C^n \oplus F^m \to \mathbb{R}^N \), given by
\[
(a, b, v) \mapsto R(a)(b_1, \ldots, b_j) \cdot [\partial^j_s v]_R
\]
for \( a \neq 0 \) and \( A(0, b, v) = 0 \), is one of the maps from \((S_k)\) and if we take the \( sc \)-derivative of \( A \) (which we already know exists at every point), we obtain a linear combination of maps of the following types:
(1) $B_1 \oplus \mathbb{C}^n \oplus F^{m+1} \to \mathbb{R}^N$ defined by
\[
(a, b_1, \ldots, b_i, \ldots, b_n, b) \mapsto R(a)(b_1, \ldots, \delta b_i, \ldots, b_j) \cdot [\partial_j v]_R
\]
for every $1 \leq i \leq n$.

(2) $B_1 \oplus \mathbb{C}^{n+1} \oplus F^{m+1} \to \mathbb{R}^N$ defined by
\[
(a, (\delta a, b, v)) \mapsto R'(a)(\delta a, b) \cdot [\partial_j v]_R
\]
and obtained by differentiation of $R(a)$ with respect to $a$.

(3) $B_1 \oplus \mathbb{C}^n \oplus F^m \to \mathbb{R}^N$ defined by
\[
(a, b, \delta v) \to R(a)(b) \cdot [\partial_j \delta v]_R
\]
and obtained by differentiating with respect to $v$.

(4) $B_1 \oplus \mathbb{C}^{n+1} \oplus F^{m+1} \to \mathbb{R}^N$ defined by
\[
(a, (\delta a, b, v)) \mapsto R_1(a)(\delta a, b) \cdot [\partial_{j+1} v]_R
\]
which is obtained by differentiating $R(a)$ in the term $[\partial_j v]_R$ with respect to $a$. Hence $R_1(a)(\delta a, b) = (DR(a)\delta a \cdot R(a)(b)$.

Note that in all of the above cases the new indices $j', m'$ and $n'$ stay the same or are raised by 1 so that we have $j' \leq m' \leq k + 1$ and $n' \leq k + 1$. We have verified that the statement $(S_{k+1})$ holds true. This completes the proof of Lemma 2.19.

Lemma 2.20. The map $\Psi : B_1 \oplus (H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \times H^{3,\delta_0}(\mathbb{R}^- \times S^1, \mathbb{R}^N)) \to H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N)$, defined by $\Phi(0, r^+, r^-) = 0$ at $a = 0$ and
\[
\Psi(a, r^+, r^-) = \left(1 - \frac{\beta_a}{\gamma_a}\right) \cdot ([r^+]_R + [r^-]_R),
\]
for $a \neq 0$, is sc-smooth.

Proof. We shall abbreviate $G = H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \times H^{3,\delta_0}(\mathbb{R}^- \times S^1, \mathbb{R}^N)$ and $F = H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N)$. We already know that the maps
\[
B_1 \oplus G \to \mathbb{R}^N, \quad (a, r^+, r^-) \mapsto [r^+]_R, \ [r^-]_R
\]
are sc-smooth. It suffices to consider only the map
\[ \Phi : B_{\frac{1}{2}} \oplus F \to F, \quad (a, r) \mapsto \frac{\beta_a}{\gamma_a} \cdot [r]_R \]
if \( a \neq 0 \) and \( \Psi(0, r) = 0 \) at \( a = 0 \). The similar map for \((a, r^-)\) can be dealt
with the same way. Clearly the map \( \Phi \) is sc-smooth on the set \((B_{\frac{1}{2}} \setminus \{0\}) \oplus F\)
and we shall prove the sc-smoothness at the points \((0, r) \in B_{\frac{1}{2}} \oplus F\). We set
\( \sigma := \frac{\beta}{\gamma} \) and \( \sigma_a = \sigma_R = \sigma(\cdot - \frac{R}{a}) \) where \( R = \varphi(|a|) \) for the exponential gluing
profile \( \varphi \). We shall prove the following statements \((S_k)\) by induction:

\((S_k)\). The map \( \Phi \) is of class \( sc^k \) and \( T^k \Phi(0, H) = 0 \). Moreover, if \( \pi : T^k (B_{\frac{1}{2}} \oplus F) \to \mathbb{R}^N \) is a projection onto a factor of \( T^k F \), then the composition \( \pi \circ T^k \Phi \)
is the linear combination of maps of the following type,
\[ A : B_{\frac{1}{2}} \oplus \mathbb{C}^n \oplus F^m \to F^j \]
\[(a, h, v) \to R(a)(h) \cdot [\sigma_a] \cdot [\partial^q v]_R \]
for \( a \neq 0 \) and \( A(0, h, v) = 0 \). In addition, the indices satisfy \( p + q = n, m \leq k \)
and \( l \leq m - j \).

We start with \((S_0)\). In this case there is only one projection \( \pi : T^0 F = F \to F \), namely, \( \pi = \text{id} \). Clearly \( \pi \circ \Phi = \Phi \) has the required form with \( R(a) = 1 \) of order 0 and all indices \( j, l, p \), and \( q \) equal to 0. Hence we only need to show that the map \( \Phi \) has \( sc^0 \)-property. This is clearly true at points
\((a, v) \in B_{\frac{1}{2}} \oplus F^m \) where \( a \neq 0 \). We carry out the proof of the \( sc^0 \)-property
for the map \( \Phi \) at \((0, v)\). We take a sequence \((a_k, v_k)\) converging to \((0, v)\)
in \( B_{\frac{1}{2}} \oplus F_m \) and we claim that \( \Phi(a_k, v_k) \to 0 \) in \( F_m \). Since \( \sigma_R \)
vanishes on \([\frac{R}{a} + 1, \infty)\), we can estimate
\[
|\Phi(a_k, v_k)|_m = |\sigma_R : [v_k]_{R_k}|^2 \frac{\beta_a}{\gamma_a^m} \int_{\mathbb{R}^+ \times S^1} |D^\alpha \sigma_R(s)|^2 e^{2\delta_m s} ds dt
\leq \sum_{|\alpha| \leq m} C_\alpha |[v_k]_{R_k}|^2 e^{2\delta_m \left(\frac{R}{a} + 1\right)}
\]
with constants \( C_\alpha \) depending only on \( \sigma \), the multi-index \( \alpha \), and \( m \). So,
to prove them claim we have to show that \([v_k]_{R_k} \to 0 \) in \( \mathbb{R}^N \). Abbreviate
\[\Sigma_k = \left[\frac{R_k}{2} - 1, \frac{R_k}{2} + 1\right] \times S^1.\]

By the Sobolev embedding theorem on bounded domains,
\[|e^{\delta m \cdot v}|_{C^0(\Sigma_k)} \leq C|e^{\delta m \cdot v}|_{H^m(\Sigma_k)} =: \varepsilon_k\]
with the constant \(C\) independent of \(v\) and \(k\). This shows that
\[|[v]_{R_k}| \leq \varepsilon_k \cdot e^{-\delta m R_k}.\] (25)

Also note that since \(v\) belongs to \(E_m\), the sequence \(\varepsilon_k\) converges to 0. Similarly, we have
\[|e^{\delta m \cdot (v_k - v)}|_{C^0(\Sigma_k)} \leq C|e^{\delta m \cdot (v_k - v)}|_{H^m(\Sigma_k)} \leq C|v_k - v|_m =: \varepsilon'_k\]
which implies that
\[|[v_k]_{R_k} - [v]_{R_k}| = |[v_k - v]_{R_k}| \leq \varepsilon'_k \cdot e^{-\delta m R_k}.\]

By assumption, \(|v - v_k|_m = \varepsilon_k \to 0\). Consequently,
\[|[v_k]_{R_k}| e^{\delta m R_k} \leq |[v_k]_{R_k} - [v]_{R_k}| \cdot e^{\delta m R_k} + |[v]_{R_k}| \cdot e^{\delta m R_k} \leq \varepsilon'_k + \varepsilon_k \to 0\]
which proves our claim. At this point we have proved the assertion (S0).

Now we assume that (S_k) has been established and prove that (S_{k+1}) holds. By induction hypothesis, the map \(\Phi\) is of class \(sc^k\), so that \(T^k \Phi\) is \(sc^0\), and \(T^k \Phi(0, H) = 0\). Moreover, \(\pi \circ T^k \Phi\) can be written as a linear combination of maps of a certain form. We also know that \(T^{k+1} \Phi\) is sc-smooth at points \((a, H)\) with \(a \neq 0\).

We begin by verifying the approximation property at points \((0, H)\). As before it suffices to do this for the maps \(A\) described in (S_k) as follows,
\[A : B_{\frac{1}{2}} \oplus \mathbb{C}^n \oplus F^m \to \mathbb{R}^N\]
\[(a, h, v) \to \mathbf{R}(a)(h) \cdot \sigma_a^{(p)} \cdot [\delta^j v]_R\]
for \(a \neq 0\) and \(A(0, h, v) = 0\). More precisely, we show that if \((0, h, v) \in B_{\frac{1}{2}} \oplus \mathbb{C}^n \oplus E^{m+1}\), then
\[\frac{1}{|\delta a| + |\delta h| + |\delta v|_{m+1}}|A(\delta a, h + \delta h, v + \delta v)|_j \to 0\]
as \(|\delta a| + |\delta h| + |\delta v|_{m+1} \to 0\) which will prove that \(A\) has the approximation property at \((0, h, v)\) with respect to the linearized map \(DA(0, h, v) = 0\).
Proceeding as in the proof of Lemma 2.19, one obtains the estimate

\[ \frac{|A(\delta a, h + \delta h, v + \delta v)|}{|\delta a| + |\delta h| + |\delta v|_{m+1}} \leq C e^{-\delta m+1} \frac{R^{3n}}{|\delta a|} |h + \delta h|^n |v + \delta v|_{m+1} \]

which converges to 0 as $|\delta a| + |\delta h| + |\delta v|_{m+1}$ converges to 0.

Finally, we need to show that $\pi \circ T^{k+1} \Phi$ is a linear combination of the maps of the required form and which have the required continuity properties at points with vanishing $a$. The terms making up $\pi \circ T^{k+1} \Phi$ are the terms guaranteed by $(S_k)$ provided $\pi$ is the projection onto one of the first $2^k$ factors. In this case the indices $m$ and $j$ are raised by one. If $\pi$ is the projection onto one of the last $2^k$ factors, then $\pi \circ T^{k+1} \Phi$ is a linear combination of derivatives of maps guaranteed by $(S_k)$. This leads to a case by case study quite similarly to that of the previous lemma and is left to the reader.

\[ \blacksquare \]

**Lemma 2.21.** The map $\Phi : B_{\frac{1}{2}} \oplus H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N)$, defined by $\Phi(0, r) = r$ if $a = 0$, and by

$\Phi(a, r) = \frac{\beta^2}{\gamma_a} \cdot r$

if $a \neq 0$, is $sc$-smooth.

**Proof.** We only have to prove the $sc$-smoothness at points $(0, r) \in B_{\frac{1}{2}} \oplus H^{3,\delta_0}(\mathbb{R}^+ \times S^1)$. Hence we may assume that $a$ is small. We choose a smooth function $\chi_1 : \mathbb{R}^+ \to [0, 1]$ satisfying $\chi_1(s) = 1$ for $s \in [0, 1]$ and $\chi_1(s) = 0$ for $s \geq 2$, and set $\chi_2 = 1 - \chi_1$. Then the map

$(a, r) \mapsto \frac{\beta^2}{\gamma_a} \cdot \chi_1 \cdot r$

is obviously $sc$-smooth since for $|a|$ small it is equal to the map

$(a, r) \mapsto \chi_1 \cdot r$

which is independent of $a$. It remains to deal with the map

$(a, r) \mapsto \frac{\beta^2}{\gamma_a} \cdot \chi_2 \cdot r.$
This map can be factored as follows. First, we apply the map
\[ B_{\frac{1}{2}} \oplus H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \rightarrow B_{\frac{1}{2}} \oplus H^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N), \quad (a, r) \mapsto (a, \chi_2 r) \]
which obviously is an sc-operator and hence sc-smooth. Then we compose this map with the map
\[ B_{\frac{1}{2}} \oplus H^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N) \rightarrow H^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N), \quad (a, u) \mapsto \frac{\beta_2^2}{\gamma_a} \cdot u \]
which is sc-smooth by Proposition 2.8. Finally, we take the restriction map
\[ H^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N) \rightarrow H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N), \]
which as an sc-operator is also sc-smooth. Since the composition of sc-smooth maps is an sc-smooth map, the proof is complete.

Lemma 2.22. The map \( \Phi : (B_{\frac{1}{2}} \oplus H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \rightarrow H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N), \)
defined by \( \Phi(0, r) = 0 \) if \( a = 0 \), and by
\[ (a, r) \mapsto \frac{\beta_2^2}{\gamma_a} \cdot r(\cdot - R, \cdot - \vartheta) \]
if \( a \neq 0 \), is sc-smooth.

Proof. Again we only have to study the map for \( a \) small. We choose a smooth map \( \chi : \mathbb{R}^+ \rightarrow [0, 1] \) satisfying \( \chi(s) = 1 \) for \( s \leq -1 \) and \( \chi(s) = 0 \) for \( s \in [-\frac{1}{2}, 0] \). If \( |a| \) is small, the map \( \Phi \) is the composition of the following three maps. The first map is defined by
\[ B_{\frac{1}{2}} \oplus H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \rightarrow B_{\frac{1}{2}} \oplus H^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N), \quad (a, u) \mapsto (a, \chi \cdot u). \]
It is an sc-operator and hence sc-smooth. The second map is defined by
\[ B_{\frac{1}{2}} \oplus H^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N) \rightarrow H^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N) \]
\[ (a, u) \mapsto \frac{\beta(1 - \beta_2)}{\gamma_a} \cdot u(\cdot - R, \cdot - \vartheta). \]
By Proposition 2.8, this map is sc-smooth. The last map is the restriction map
\[ H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \rightarrow H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \]
which is an sc-operator. This completes the proof of Lemma 2.22.

In view of the above lemmata 2.18, 2.22, the proof of Proposition 2.17 is finished. Hence the proof of Theorem 1.28 is complete.
2.5 Estimates for the Total Gluing Map

In section 1.3, we have introduced the space $E$ consisting of pairs $(\eta^+, \eta^-)$ of maps $\eta^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^N$ the Sobolev class $(3, \delta_0)$ having common asymptotic limits. Taking a strictly increasing sequence $(\delta_m)_{m \in \mathbb{N}_0}$ starting with $\delta_0 > 0$, we equip the Hilbert space $E$ with the sc-structure $(E_m)_{m \in \mathbb{N}_0}$ where $E_m$ consists of those pairs $(\eta^+, \eta^-)$ in $E$ of Sobolev class $(3 + m, \delta_m)$. We shall later impose boundary conditions, but this is not important for the moment.

If $(\eta^+, \eta^-)$ is a pair in $E$, then $\eta^\pm = c + r^\pm$ where $c$ is the common asymptotic limit and $r^\pm \in H^{3, \delta_0}((\mathbb{R}^\pm \times S^1), \mathbb{R}^N)$. The $E_m$-norm of the pair $(\eta^+, \eta^-)$ is defined as

$$|(\eta^+, \eta^-)|_{E_m}^2 = |c|^2 + |r^+|_{H^{3+m, \delta_m}}^2 + |r^-|_{H^{3+m, \delta_m}}^2.$$

For every gluing parameter $a \in B_2$, we introduce the space $G^a$ as follows. If $a = 0$, we set

$$G^0 = E \oplus \{0\},$$

and if $a \neq 0$, we define

$$G^a = Q^a \oplus P^a = H^3(Z_a) \oplus H^{3, \delta_0}_c(C_a).$$

The sc-structure of $G^a$ is given by the sequence $H^{3+m}(Z_a) \oplus H^{3+m, \delta_m}_c(C_a)$ for all $m \in \mathbb{N}_0$.

The total gluing map $\Box_a = (\oplus_a, \ominus_a) : E \to G^a$ is an sc-linear isomorphism for every $a \in B_2$ in view of Theorem 1.27.

For a pair $(k, \delta)$ in which $k$ is a non-negative integer and $\delta$ and a map $q : Z_a \to \mathbb{R}^N$ defined on the finite cylinder $Z_a$, we introduce the norm $||q||_{k, \delta}$ by

$$||q||_{k, \delta}^2 := \sum_{|\alpha| \leq k} \int_{[0,R] \times S^1} |D^\alpha q(s, t)|^2 \cdot e^{2\delta |s - R|} \ dsdt$$

where $q(s, t) := q([s, t])$.

We recall that the average of the map $q : Z_a \to \mathbb{R}^N$, denoted by $[q]_a$ or $[q]_R$, is defined as the integral over the middle loop,

$$[q]_a = \int_{S^1} q\left(\frac{R}{2}, t\right) \ dt.$$
If \( p : C_a \to \mathbb{R}^N \) is a map with vanishing asymptotic constants we set, with \( p(s, t) := p([s, t]) \),

\[
|p|^2_{m, \delta} = \sum_{|\alpha| \leq m} \int_{\mathbb{R} \times S^1} |(D^\alpha p)(s, t)|^2 e^{2\delta_m |s - \frac{B}{2}|} \, dsdt
\]

Observe that the center loop is located at \( s = \frac{B}{2} \), which explains the occurrence of the \( \frac{B}{2} \) in the exponential weight.

Now we define a norm for the pairs \((q, p) \in G^a\) as follows. With a map \( p \in H^{3,\delta_0}(C_a) \) which has the antipodal constants \( p_\infty = \lim_{s \to \infty} p(s, t) \) and \( p_{-\infty} = -p_\infty \), we associate the map \( \hat{\rho} : C_a \to \mathbb{R}^N \) defined by

\[
\hat{\rho}([s, t]) = p([s, t]) - (1 - 2 \cdot \beta_a(s))p_\infty.
\]

Then \( \lim_{s \to \pm \infty} \hat{\rho}(s, t) = 0 \) and \( \hat{\rho} \) belongs to \( H^{3,\delta_0}(C_a, \mathbb{R}^N) \). Now we introduce a norm on the level \( m \) of \( G^a = H^3(Z_a) \oplus H^{3,\delta_0}(C_a) \) by

\[
|\rho(p, q)|^2_{G^a_m} = |[q]_a - p_\infty|^2 + e^{\delta_m R} \cdot (\|q - [q]_a + p_\infty\|^2_{3+m, -\delta_m} + \|\hat{\rho}\|^2_{m+3, \delta_m}).
\]

To get a better understanding of the norm \(|(q, p)|^2_{G^a_m}\), we take the unique pair \((\eta^+, \eta^-) \in E\) satisfying \( q = \Theta_a(\eta^+, \eta^-) \) and \( p = \Theta_a(\eta^+, \eta^-) \) by using Theorem 1.21. We write \( \eta^\pm = c + r^\pm \) where \( c \) is the common asymptotic limit and \( r^\pm \in H^{3,\delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^N) \). Then

\[
q = \Theta_a(\eta^+, \eta^-) = c + \hat{\Theta}_a(r^+, r^-)
\]

\[
p = \Theta_a(\eta^+, \eta^-) = \hat{\Theta}_a(r^+, r^-) + (1 - 2\beta_a)av_a(r^+, r^-).
\]

The mean value \([q]_a\) of \( q \) is equal to \( c + av_a(r^+, r^-) \) and the positive asymptotic constant \( p_\infty \) of \( p \) is equal to \( p_\infty = av_a(r^+, r^-) \). Hence \([q]_a - p_\infty = c\) and \( q - [q]_a + p_\infty = \hat{\Theta}_a(r^+, r^-) \), and \( \hat{\rho} = p - (1 - 2\beta_a) \cdot p_\infty = \hat{\Theta}_a(r^+, r^-) \). Consequently, the \( G^a_m \)-norm of \((q, p)\) becomes,

\[
|(q, p)|^2_{G^a_m} = |[q]_a - p_\infty|^2 + e^{\delta_m R} \cdot (\|q - [q]_a + p_\infty\|^2_{3+m, -\delta_m} + \|\hat{\rho}\|^2_{m+3, \delta_m})
\]

\[
= |c|^2 + e^{\delta_m R} (\|\Theta_a(r^+, r^-)\|^2_{3+m, -\delta_m} + \|\hat{\Theta}_a(r^+, r^-)\|^2_{m+3, \delta_m}).
\]

**Theorem 2.23.** For every level \( m \) there exists a constant \( C_m > 0 \) independent of \( |a| < \frac{1}{2} \) so that the total gluing map \( \Box_a : E \to G^a \), defined by

\[
(\eta^+, \eta^-) \mapsto \Box_a(\eta^+, \eta^-) := (\Theta_a(\eta^+, \eta^-), \Theta_a(\eta^-, \eta^-))
\]

is an \( sc \)-isomorphism and satisfies the estimate

\[
C_m^{-1} \cdot |(\eta^+, \eta^-)|_{E_m} \leq |\Box_a (\eta^+, \eta^-)|_{G^a_m} \leq C_m \cdot |(\eta^+, \eta^-)|_{E_m}.
\]

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Proof. We take a pair \((\eta^+, \eta^-)\) belonging to the space \(E\) and represent it by \(\eta^\pm = c + r^\pm\) where \(c\) is the common asymptotic limit and \(r^\pm \in H^{3,\delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^N)\). Then we introduce \((q, p)\) by \((q, p) = (\oplus c(\eta^+, \eta^-), \ominus c(\eta^+, \eta^-))\) and compute the \(G_m^a\)-norm,

\[
|jqjp|_G^2 = |c|^2 + e^{\delta m R} \left( \|\hat{\oplus} c(r^+, r^-)\|_{3+m, -\delta m}^2 + \|\hat{\ominus} c(r^+, r^-)\|_{m+3, \delta m}^2 \right).
\]

In view of Lemma \[2.28\] below applied to \(r^+ = u, r^- = v\) and \(\hat{\oplus} c(r^+, r^-) = U\) and \(\hat{\ominus} c(r^+, r^-) = V\), there exists a constant \(C\) depending only on \(m\) and \(\delta_m\) so that

\[
\frac{1}{C} \cdot \|\hat{\oplus} c(r^+, r^-)\|_{3+m, -\delta m}^2 + \|\hat{\ominus} c(r^+, r^-)\|_{m+3, \delta m}^2 \leq e^{-\delta m R}[|r^+|_{H^{3+m, \delta m}}^2 + |r^-|_{H^{3+m, \delta m}}^2],
\]

and

\[
e^{-\delta m R}[|r^+|_{H^{3+m, \delta m}}^2 + |r^-|_{H^{3+m, \delta m}}^2] \leq C \cdot \|\hat{\oplus} c(r^+, r^-)\|_{3+m, -\delta m}^2 + \|\hat{\ominus} c(r^+, r^-)\|_{m+3, \delta m}^2.
\]

We have denoted by \(|\cdot|_{H^{3+m, \delta m}}\) our standard weighted Sobolev norms. Consequently, our desired estimate follows since the \(E_m\)-norm of the pair \((\eta^+, \eta^-)\) is defined by

\[
|jqjp|_{E_m}^2 = |c|^2 + |r^+|_{H^{3+m, \delta m}}^2 + |r^-|_{H^{3+m, \delta m}}^2.
\]

The proof of the proposition is complete.

---

We introduce the sc-Hilbert space \(\hat{E}\) consisting of pairs \((h^+, h^-)\) where \(h^\pm = h^\pm_\infty + r^\pm\) with \(r^\pm \in H^{3,\delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^N)\). We do not require that the asymptotic constants \(h^\pm_\infty\) are equal. In addition, we also impose the following boundary conditions,

\[
h^\pm(0, 0) = (0, 0) \quad \text{and} \quad h^\pm(0, t) \in \{0\} \times \mathbb{R}.
\]

Abbreviating the maps

\[
\tilde{h}^\pm = h^\pm + \frac{1}{2}(h^+ - h^-),
\]
we note that
\[ \tilde{h}^\pm = h_\infty^\pm + \frac{1}{2}(h_\infty^+ - h_\infty^-) + r^\pm = \frac{1}{2}(h_\infty^+ + h_\infty^-) + r^\pm, \]
so that the maps \( \tilde{h}^\pm \) have the same asymptotic limits equal to \( \frac{1}{2}(h_\infty^+ + h_\infty^-) \).
Consequently, the pair \( (\tilde{h}^+, \tilde{h}^-) \) belongs to the previously defined sc-Hilbert space \( E \).

The \( \hat{E}_m \)-norm of the pair \( (h^+, h^-) \) is defined by
\[ |(h^+, h^-)|^2_{\hat{E}_m} := |h_\infty^+|^2 + |h_\infty^-|^2 + |r^+|^2_{H^{3+m, \delta_m}} + |r^-|^2_{H^{3+m, \delta_m}}. \tag{26} \]

Considering \( \mathbb{R}^N \oplus E_m \) with the product norm, the norm of \( (h_\infty^+, h_\infty^-, (\tilde{h}^+, \tilde{h}^-)) \) is equal to
\[ |(h_\infty^+ - h_\infty^-, (\tilde{h}^+, \tilde{h}^-))|^2_{\mathbb{R}^N \oplus E_m} = |h_\infty^+ - h_\infty^-|^2 + |(\tilde{h}^+, \tilde{h}^-)|^2_{E_m}, \tag{27} \]
It follows from (26) and (27) that there exists a universal constant \( C \) so that
\[ \frac{1}{C} \cdot |(h^+, h^-)|^2_{\hat{E}_m} \leq |(h_\infty^+ - h_\infty^-, (\tilde{h}^+, \tilde{h}^-))|^2_{\mathbb{R}^N \oplus E_m} \leq C \cdot |(h^+, h^-)|^2_{\hat{E}_m}. \tag{28} \]

Now as a consequence of Theorem 2.23 we obtain the following corollary.

**Corollary 2.24.** For every level \( m \) there exists a constant \( C_m > 0 \) independent of the gluing parameter \( |a| < \frac{1}{2} \), so that for \( (h^+, h^-) \in \hat{E}_m \), the following estimate holds,
\[ C_m^{-1} \cdot |(h^+, h^-)|^2_{\hat{E}_m} \leq \left| |h_\infty^+ - h_\infty^-|^2 + |\Box_a (\tilde{h}^+, \tilde{h}^-)|^2_{G_m} \right| \leq C_m \cdot |(h^+, h^-)|^2_{\hat{E}_m}, \]
where \( \Box_a (\tilde{h}^+, \tilde{h}^-) = (\Box_a (\tilde{h}^+, \tilde{h}^-), \Box_a (\tilde{h}^+, \tilde{h}^-)) \).

**Proof.** Using (26), (27), (28), and Theorem 2.23, one obtains for a generic constant \( c_m \) depending on \( m \) and not on \( |a| < \frac{1}{2} \),
\[ |(h^+, h^-)|^2_{\hat{E}_m} \leq c_m \cdot |(h_\infty^+ - h_\infty^- (\tilde{h}^+, \tilde{h}^-)|^2_{\mathbb{R}^N \oplus E_m} \]
\[ = c_m \cdot \left| |h_\infty^+ - h_\infty^-|^2 + |(\tilde{h}^+, \tilde{h}^-)|^2_{E_m} \right| \]
\[ \leq c_m \cdot \left| |h_\infty^+ - h_\infty^-|^2 + |\Box_a (\tilde{h}^+, \tilde{h}^-)|^2_{G_m} \right| \]
\[ \leq c_m \cdot |(h^+, h^-)|^2_{\hat{E}_m}, \]
as claimed. \( \blacksquare \)
Remark 2.25. Later on we deal with the case \( N = 2 \) where the pair \((h^+, h^-) \in \hat{E}\) satisfies the boundary condition \( h^\pm(0,0) = (0,0) \) and \( h^\pm(0,t) \in \{0\} \times \mathbb{R} \). Then the map \( q = \oplus_a (\hat{h}^+, \hat{h}^-) : Z_a \to \mathbb{R}^2 \) will satisfy the following boundary conditions,

\[
q([0,0]) = -\frac{1}{2}(h^+_\infty - h^-_\infty), \quad q([0,0])' = \frac{1}{2}(h^+_\infty - h^-_\infty)
\]

and, in addition,

\[
q([0,t]) \in -\frac{1}{2}(h^+_\infty - h^-_\infty) + (\{0\} \times \mathbb{R}) \quad q([0,t])' \in \frac{1}{2}(h^+_\infty - h^-_\infty) + (\{0\} \times \mathbb{R}).
\]

Later on we will need the following variant variant of Theorem 2.23 with respect to the hat gluing and hat anti-gluing.

We denote by \( F \) the sc-Hilbert space \( F = H^{2,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \oplus H^{2,\delta_0}(\mathbb{R}^- \times S^1, \mathbb{R}^N) \) whose sc-structure is given by the sequence \( F_m = H^{2+m,\delta_m}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \oplus H^{2+m,\delta_m}(\mathbb{R}^- \times S^1, \mathbb{R}^N) \). The \( F_m \)-norm of the pair \((\xi^+, \xi^-) \in F\) is given by

\[
|((\xi^+, \xi^-))_F^2 := |\xi^+|_{H^{m+2,\delta_m}}^2 + |\xi^-|_{H^{m+2,\delta_m}}^2.
\]

We introduce the space \( \hat{G}_a \) as follows. If \( a = 0 \), set we \( \hat{G}_a = F \oplus \{0\} \) and if \( 0 < |a| < \frac{1}{2} \), then we define

\[
\hat{G}_a = \hat{Q}^a \oplus \hat{P}^a = H^2(Z_a, \mathbb{R}^N) \oplus H^{2,\delta_0}(C_a, \mathbb{R}^N).
\]

The sc-structure of \( \hat{G}^a \) is given by the sequence \( H^{2+m}(Z_a) \oplus H^{2+m,\delta_m}(C_a) \) for all \( m \in \mathbb{N}_0 \). The total gluing map \( \hat{\square}_a = (\hat{\square}_a, \hat{\square}_a) : F \to \hat{G}^a \) is an sc-linear isomorphism for every \( a \in B_+ \) in view of Theorem 1.29.

As in the case of the total gluing \( \square \) it is useful to introduce families of norms. We introduce the \( \hat{G}_m^a \)-norm of the pair \((q,p) \in \hat{G}_a\) by setting

\[
|((q,p))_{\hat{G}_m^a}^2 := e^{\delta_mR} \cdot \left[ \|q\|_{m+2,-\delta_m}^2 + \|p\|_{m+2,\delta_m}^2 \right]
\]

where these norms are defined above. Recall that if \((q,p) = \hat{\square}_a(\xi^+, \xi^-) = (\hat{\square}_a(\xi^+, \xi^-), \hat{\square}_a(\xi^+, \xi^-))\), then \((q,p)\) and \((\xi^+, \xi^-)\) are related as follows,

\[
\begin{bmatrix}
q(s,t) \\
p(s,t)
\end{bmatrix} = 
\begin{bmatrix}
\beta_a & 1 - \beta_a \\
-(1 - \beta_a) & \beta_a
\end{bmatrix}
\begin{bmatrix}
\xi^+(s,t) \\
\xi^-(s-R, t + \vartheta)
\end{bmatrix}
\]

where, as usual, \( \beta_a = \beta_a(s) \). Then, in view of the definition of the norm on \( \hat{G}_m \), one derives from the estimates of Lemma 2.28 the following theorem.
Theorem 2.26. Given the level \( m \) there exists a constant \( C_m \) not depending on \( a \in B_{ \frac{1}{2} } \) so that the following estimate holds,

\[
C_m^{-1} \cdot |(\xi^+, \xi^-)|_{F_m} \leq |\hat{\alpha}_a(\xi^+, \xi^-)|_{\hat{G}_m} \leq C_m \cdot |(\xi^+, \xi^-)|_{F_m}.
\]

Remark 2.27. The study of pairs \((\eta^+, \eta^-)\) in \( F \) on level \( m \) with respect to the norm \( |\cdot|_{F_m} \) is for every \( a \in B_{ \frac{1}{2} } \) completely equivalent (up to a multiplicative constant independent of \( a \)) to the study of the associated pairs \((q, p)\) on the level \( m \) with respect to the norm \( |\cdot|_{\hat{G}_m} \).

The two theorems can be deduced from the following lemma.

Lemma 2.28. There exists a constant \( C \) depending on \( m \) and \( \delta > 0 \) so that for maps \((U, V) \in H^m(Z_a, \mathbb{R}^N) \oplus H^{m, \delta}(C_a, \mathbb{R}^N)\) and \((u, v) \in H^{m, \delta}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \oplus H^{m, \delta}(\mathbb{R}^- \times S^1, \mathbb{R}^N)\) satisfying

\[
\begin{bmatrix}
U(s, t) \\
V(s, t)
\end{bmatrix} = \begin{bmatrix}
\beta_a & 1 - \beta_a \\
-(1 - \beta_a) & \beta_a
\end{bmatrix} \cdot \begin{bmatrix}
u(s, t) \\
v(s - R, t - \vartheta)
\end{bmatrix} = \begin{bmatrix} \hat{\alpha}_a(U, V) \\ \hat{\alpha}_a(U, V) \end{bmatrix},
\]

the following estimate holds true,

\[
\frac{1}{C} \left[ \|U\|_{m, -\delta}^2 + \|V\|_{m, \delta}^2 \right] \leq e^{-\delta R} \left[ \|u\|_{H^{m, \delta}}^2 + \|v\|_{H^{m, \delta}}^2 \right] \leq C \left[ \|U\|_{m, -\delta}^2 + \|V\|_{m, \delta}^2 \right].
\]

Proof. In view of (29) and recalling \( \beta_a(s) = 0 \) for \( s \geq \frac{R}{2} + 1 \) and \( 1 - \beta_a(s) = 0 \) for \( s \leq \frac{R}{2} - 1 \), it follows that \( \|U\|_{m, -\delta}^2 + \|V\|_{m, \delta}^2 \) is bounded above by a constant \( C \) times the sum of integrals of the following type:

- \( I_1 = \int_{[0, \frac{R}{2} + 1] \times S^1} |D^\alpha u(s, t)|^2 e^{-2\delta|s - \frac{R}{2}|}, \)
- \( I_2 = \int_{[\frac{R}{2} - 1, R] \times S^1} |D^\alpha v(s - R, t - \vartheta)|^2 e^{-2\delta|s - \frac{R}{2}|}, \)
- \( I_3 = \int_{[\frac{R}{2} - 1, \infty) \times S^1} |D^\alpha u(s, t)|^2 e^{-2\delta|s - \frac{R}{2}|}, \)
- \( I_4 = \int_{(-\infty, \frac{R}{2} + 1] \times S^1} |D^\alpha v(s - R, t - \vartheta)|^2 e^{-2\delta|s - \frac{R}{2}|}, \)

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where the multi-indices $\alpha$ satisfy $|\alpha| \leq m$. The constant $C$ depends on $m$ and the function $\beta$. We estimate each of the above integrals. To estimate the integrals $I_1$ and $I_3$ we use the fact that $-2\delta|s - \frac{R}{2}| - 2\delta s \leq -\delta R$ for all $s \in \mathbb{R}$. Then

$$I_1 = \int_{[0, \frac{R}{2} + 1] \times S^1} |D^\alpha u(s, t)|^2 e^{-2\delta|s - \frac{R}{2}|} = \int_{[0, \frac{R}{2} + 1] \times S^1} |D^\alpha u(s, t)|^2 e^{2\delta s} \cdot e^{-2\delta|s - \frac{R}{2}| - 2\delta s}$$

$$\leq e^{-\delta R} \int_{[0, \frac{R}{2} + 1] \times S^1} |D^\alpha u(s, t)|^2 e^{2\delta s} \leq e^{-\delta R} \cdot |u|_{H^m, \delta}^2$$

and

$$I_3 = \int_{[\frac{R}{2} - 1, \infty) \times S^1} |D^\alpha u(s, t)|^2 e^{-2\delta|s - \frac{R}{2}|} = \int_{[\frac{R}{2} - 1, \infty) \times S^1} |D^\alpha u(s, t)|^2 e^{2\delta s} \cdot e^{-2\delta|s - \frac{R}{2}| - 2\delta s}$$

$$\leq e^{-\delta R} \int_{[\frac{R}{2} - 1, \infty) \times S^1} |D^\alpha u(s, t)|^2 e^{2\delta s} \leq e^{-\delta R} \cdot |u|_{H^m, \delta}^2.$$

To estimate the integrals $I_2$ and $I_4$ we use $-2\delta|s + \frac{R}{2}| + 2\delta s \leq -\delta R$ for all $s$. Then abbreviating $\Sigma_R = [\frac{R}{2} - 1, R] \times S^1$ we obtain for the integral $I_2$,

$$I_2 = \int_{\Sigma_R} |D^\alpha v(s - R, t - \vartheta)|^2 e^{-2\delta|s - \frac{R}{2}|} = \int_{\Sigma_R} |D^\alpha v(s, t)|^2 e^{-2\delta|s + \frac{R}{2}|}$$

$$= \int_{\Sigma_{-R}} |D^\alpha v(s, t)|^2 e^{-2\delta s} e^{-2\delta|s + \frac{R}{2}| + 2\delta s} \leq e^{-\delta R} \int_{\Sigma_{-R}} |D^\alpha v(s, t)|^2 e^{-2\delta s}$$

$$\leq e^{-\delta R} \cdot |v|_{H^m, \delta}^2,$$

and abbreviating $\Sigma_R = (-\infty, \frac{R}{2} + 1] \times S^1$, the integral $I_4$ can be estimated

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as

\[
I_4 = \int_{\Sigma_R} |D^\alpha v(s - R, t - \vartheta)|^2 e^{-2\delta|s - \frac{R}{2}|} = \int_{\Sigma_R} |D^\alpha v(s, t)|^2 e^{-2\delta|s + \frac{R}{2}|} \\
= \int_{\Sigma_R} |D^\alpha v(s, t)|^2 e^{-2\delta s} \cdot e^{-2\delta|s + \frac{R}{2}| + 2\delta s} \\
\leq e^{-\delta R} \int_{\Sigma_R} |D^\alpha v(s, t)|^2 e^{-2\delta s} \leq e^{-\delta R} \cdot |v|_{H,m,\delta}^2.
\]

Summing up the estimates for the integrals \(I_1, \ldots, I_4\) for all multi-indices \(\alpha\) satisfying \(|\alpha| \leq m\), we get

\[
\|U\|_{m,-\delta}^2 + \|V\|_{m,\delta}^2 \leq C e^{-\delta R} \cdot \|u\|_{H,m,\delta}^2 + \|v\|_{H,m,\delta}^2
\]

as desired.

In order to estimate \(|u|_{m,\delta}^2 + |v|_{m,\delta}^2\) we multiplying both sides of (29) by the the inverse of the matrix and obtain

\[
\begin{bmatrix}
u(s - R, t - \vartheta) \\
\eta(s, t)
\end{bmatrix} = \frac{1}{\gamma_a} \begin{bmatrix}
\beta_a & -(1 - \beta_a) \\
1 - \beta_a & \beta_a
\end{bmatrix} \cdot \begin{bmatrix}
u(s, t) \\
\eta(s, t)
\end{bmatrix}
\]

where \(\gamma_a = \gamma_a(s)\) is the the determinant of the matrix. Hence

\[
v(s', t') = \frac{1 - \beta_a(s' + R)}{\gamma'(s' + R)} U(s' + R, t' + \vartheta) + \frac{\beta_a(s' + R)}{\gamma_a(s' + R)} V(s' + R, t' + \vartheta).
\]

Using the above relations and \(\beta_a(s' + R) = 0\) for \(s \geq -\frac{R}{2} + 1\) and \(1 - \beta_a(s' + R) = 0\) for \(s \leq -\frac{R}{2} - 1\), it follows that \(|u|_{m,\delta}^2 + |v|_{m,\delta}^2\) is bounded above by a constant \(C\) (depending only on \(m\) and the function \(\beta\)) times the sum of integrals of the following type:

\[
\begin{align*}
J_1 &= \int_{[0,\frac{R}{2} + 1] \times S^1} |D^\alpha U(s, t)|^2 e^{2\delta s}, \\
J_2 &= \int_{[\frac{R}{2} - 1, \infty) \times S^1} |D^\alpha V(s, t)|^2 e^{2\delta s}, \\
J_3 &= \int_{[-\frac{R}{2} - 1, 0] \times S^1} |D^\alpha U(s + R, t + \vartheta)|^2 e^{-2\delta s}, \\
J_4 &= \int_{[\frac{R}{2} - 1, \infty) \times S^1} |D^\alpha V(s + R, t + \vartheta)|^2 e^{-2\delta s},
\end{align*}
\]
\[ J_4 = \int_{(-\infty, -\frac{R}{2} + 1] \times S^1} |D^\alpha V(s + R, t + \vartheta)|^2 e^{-2\delta s}, \]

where the multi-indices \( \alpha \) satisfy \( |\alpha| \leq m \).

The integral \( J_1 \) is estimated as follows,

\[
J_1 = \int_{[0, \frac{R}{2} + 1] \times S^1} |D^\alpha U(s, t)|^2 e^{2\delta s} 
= \int_{[0, \frac{R}{2} + 1] \times S^1} |D^\alpha U(s, t)|^2 e^{-2\delta|s - \frac{R}{2}|} \cdot e^{2\delta|s - \frac{R}{2}| + 2\delta s} 
\leq e^{\delta(4 + R)} \int_{[0, \frac{R}{2} + 1] \times S^1} |D^\alpha U(s, t)|^2 e^{-2\delta|s - \frac{R}{2}|} \leq e^{4\delta + \delta R} \cdot \| U \|_{m, \delta}^2, 
\]

using \( 2\delta|s - \frac{R}{2}| + 2\delta s \leq \delta(4 + R) \) for all \( s \leq \frac{R}{2} + 1 \).

For the integral \( J_2 \) we obtain

\[
J_2 = \int_{[\frac{R}{2} - 1, \infty) \times S^1} |D^\alpha V(s, t)|^2 e^{2\delta s} 
= \int_{[\frac{R}{2} - 1, \infty) \times S^1} |D^\alpha V(s, t)|^2 e^{2\delta|s - \frac{R}{2}|} \cdot e^{-2\delta|s - \frac{R}{2}| + 2\delta s} 
\leq e^{4\delta} \int_{[\frac{R}{2} - 1, \infty) \times S^1} |D^\alpha V(s, t)|^2 e^{2\delta|s - \frac{R}{2}|} \leq e^{4\delta} \cdot \| V \|_{m, \delta}^2, 
\]

using \(-2\delta|s - \frac{R}{2}| + 2\delta s \leq \delta R \) for all \( s \in \mathbb{R} \).
For the integral \( J_3 \) we find

\[
J_3 = \int_{[-\frac{R}{2},-1,0] \times S^1} |D^\alpha U(s + R, t + \vartheta)|^2 e^{-2\delta s}
\]

\[
= \int_{[\frac{R}{2} - 1,R] \times S^1} |D^\alpha U(s, t)|^2 e^{-2\delta(s - R)}
\]

\[
= \int_{[\frac{R}{2} - 1,R] \times S^1} |D^\alpha U(s, t)|^2 e^{-2\delta|s - \frac{R}{2}|} \cdot e^{2\delta|s - \frac{R}{2}| - 2\delta(s - R)}
\]

\[
\leq e^{\delta(4 + R)} \int_{[\frac{R}{2} - 1,R] \times S^1} |D^\alpha U(s, t)|^2 e^{-2\delta|s - \frac{R}{2}|} \leq e^{\delta(4 + R)} \cdot \| U \|^2_{m, -\delta},
\]

using \( 2\delta|s - \frac{R}{2}| - 2\delta(s - R) \leq \delta(4 + R) \) for all \( s \geq \frac{R}{2} - 1 \).

The last integral can be estimated as follows

\[
J_4 = \int_{(-\infty,-\frac{R}{2} + 1] \times S^1} |D^\alpha V(s + R, t + \vartheta)|^2 e^{-2\delta s}
\]

\[
= \int_{(-\infty,-\frac{R}{2} + 1] \times S^1} |D^\alpha V(s, t)|^2 e^{-2\delta(s - R)}
\]

\[
= \int_{(-\infty,-\frac{R}{2} + 1] \times S^1} |D^\alpha V(s, t)|^2 e^{2\delta|s - \frac{R}{2}|} \cdot e^{-2\delta|s - \frac{R}{2}| - 2\delta(s - R)}
\]

\[
\leq e^{\delta R} \int_{(-\infty,-\frac{R}{2} + 1] \times S^1} |D^\alpha V(s, t)|^2 e^{2\delta|s - \frac{R}{2}|} \leq e^{\delta R} \cdot \| V \|^2_{m, \delta},
\]

in view of \( -2\delta|s - \frac{R}{2}| - 2\delta(s - R) \leq \delta R \) for all \( s \in \mathbb{R} \).

Summing up the estimates for the integrals \( J_1, \ldots, J_4 \) for all the multi-indices \( \alpha \) satisfying \( |\alpha| \leq m \), we obtain

\[
\| u \|^2_{H^{m, \delta}} + \| v \|^2_{H^{m, \delta}} \leq C e^{\delta R} \cdot \left[ \| U \|^2_{m, -\delta} + \| V \|^2_{m, \delta} \right],
\]

as claimed. The proof of Lemma 2.28 is complete. ■

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3 Long Cylinders and Gluing

In chapter 3 we are going to construct an M-polyfold structure on the set \( \mathcal{X} \) of diffeomorphisms between conformal cylinders which break apart. Using the technical results of chapter 2 we shall first prove Proposition 3.24 and Theorem 1.43. Then we shall illustrate the Fredholm theory outlined in chapter 1 by constructing a strong bundle over the M-polyfold \( \mathcal{X} \) proving Theorem 1.44. We shall show that this strong bundle admits the Cauchy-Riemann operator as an sc-Fredholm section. Finally, the sc-implicit function theorem for polyfold Fredholm sections provides the proofs of the Theorems 1.45 and 1.46.

We start with the introduction of another retraction used later on.

3.1 Another Retraction

We choose a strictly increasing sequence \((\delta_m)_{m \in \mathbb{N}_0}\) satisfying \(0 < \delta_m < 2\pi\). Recall that the Banach space \(\hat{E}\) consists of pairs \((h^+, h^-)\) of maps

\[ h^\pm \in H^{3,\delta_0}_c(\mathbb{R}^\pm \times S^1, \mathbb{R}^N) \]

for which there exist asymptotic constants \(c^\pm \in \mathbb{R}^N\) so that \(h^\pm - c^\pm \in H^{3,\delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^N)\). The sc-structure \((\hat{E}_m)\) is defined by the sequence

\[ \hat{E}_m = H^{3+m,\delta_m}_c(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \times H^{3+m,\delta_m}_c(\mathbb{R}^- \times S^1, \mathbb{R}^N). \]

Given a pair \((h^+, h^-) \in \hat{E}\), we denote the asymptotic constants of \(h^\pm\) by \(h^\pm_\infty\) and write

\[ h^\pm = h^\pm_\infty + r^\pm \]

so that \(r^\pm \in H^{3,\delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^N)\).

We point out that in contrast to Section 1.3 we do not require that the asymptotic constants \(h^+\) and \(h^-\) coincide.

We define for \(a \in \mathbb{C}\) satisfying \(0 \leq |a| < \frac{1}{2}\) the mapping

\[ \rho_a : \hat{E} \to \hat{E} \]

by

\[ \rho_a(h^+, h^-) = (h^+_\infty, h^-_\infty) + \pi_a(r^+, r^-), \]

where \(\pi_a\) is the projection defined by the formulae in Section 1.3. Since the projection \(\pi_0\) is equal to the identity map, also \(\rho_0(h^+, h^-) = (h^+, h^-)\) for any pair \((h^+, h^-) \in \hat{E}\). From \(\pi_a \circ \pi_a = \pi_a\) we derive the following lemma.
Lemma 3.1. The map $\rho_a : \hat{E} \to \hat{E}$ is a projection.

Proof. Given a $(h^+, h^-) \in \hat{E}$, we have to show that

$$\rho_a \circ \rho_a(h^+, h^-) = \rho_a(h^+, h^-).$$

We write $h^\pm = h^\pm_\infty + r^\pm$ and set $\pi_a(r^+, r^-) = (\eta^+, \eta^-)$. Then

$$\rho_a(h^+, h^-) = (h^+_\infty, h^-_\infty) + \pi_a(r^+, r^-) = (h^+_\infty, h^-_\infty) + (\eta^+, \eta^-). \quad (32)$$

From the formula for the projection $\pi_a$ in Section 1.3, we know that

$$(\eta^+, \eta^-) = (c, c) + (\xi^+, \xi^-)$$

where $c$ is the common asymptotic constant of the maps $\eta^+$ and $\eta^-$ which is equal to $\text{av}_R(r^+, r^-)$. Thus, continuing with (32),

$$\rho_a(h^+, h^-) = (h^+_\infty + c, h^-_\infty + c) + (\xi^+, \xi^-).$$

Applying $\rho_a$ to both sides, we obtain

$$\rho_a \circ \rho_a(h^+, h^-) = (h^+_\infty + c, h^-_\infty + c) + \pi_a(\xi^+, \xi^-). \quad (33)$$

Using $\pi_a \circ \pi_a = \pi_a$ and $\pi_a(r^+, r^-) = (\eta^+, \eta^-)$, we obtain $\pi_a(r^+, r^-) = \pi_a(\eta^+, \eta^-) = (c, c) + \pi_a(\xi^+, \xi^-)$. Therefore, the right-hand side of (33) is equal to

$$(h^+_\infty + c, h^-_\infty + c) + \pi_a(\xi^+, \xi^-) = (h^+_\infty, h^-_\infty) + \pi_a(r^+, r^-) = \rho_a(h^+, h^-).$$

This finishes the proof of the lemma. \[\blacksquare\]

From the already established sc-smoothness properties of $\pi_a$ in Section 1.3 we deduce the sc-smoothness of the projection $\rho_a$.

Theorem 3.2. The map

$$B^1 \oplus \hat{E} \to \hat{E}, \quad (a, (h^+, h^-)) \mapsto \rho_a(h^+, h^-)$$

is sc-smooth.
Proof. If $h^\pm = h^\pm_\infty + r^\pm$, then the map

$$(a, (h^+, h^-)) \mapsto (a, (r^+, r^-))$$

as well as the map

$$(a, (h^+, h^-)) \mapsto (h^\pm_\infty, h^-)$$

are sc–smooth. By composing the first map with the sc–smooth map $\pi : B_a \oplus E \to E$, $(a, (h^+, h^-)) \mapsto \pi(a)(h^+, h^-)$ and then adding to it the second map we obtain an sc–smooth map. \hfill \blacksquare

The geometric interpretation of the projection $\rho_a$ is the following.

**Lemma 3.3.** The projection $\rho_a$ is the projection onto the kernel of the map $P^a_\infty : \hat{E} \to H^{3,\delta_0}_c(C_a)$, defined by

$$(h^+, h^-) \mapsto (\ominus_a(h^+ - h^+_{\infty}, h^- - h^-_{\infty}))$$

along the kernel of the map $P^a_\infty : \hat{E} \to \mathbb{R}^N \oplus H^3(Z_a)$, defined by

$$(h^+, h^-) \mapsto (h^+_{\infty} - h^-_{\infty}, \ominus_a(h^+, h^- + h^+_{\infty} - h^-_{\infty})).$$

**Proof.** We first show that $\hat{E} = \ker P^a_\infty \oplus \ker P^a_\infty$. To do this we consider the linear map

$$P : \hat{E} \to \mathbb{R}^N \oplus H^3(Z_a) \oplus H^{3,\delta_0}_c(C_a)$$

defined by

$$(h^+, h^-) \mapsto (h^+_{\infty} - h^-_{\infty}, \ominus_a(h^+, h^- + h^+_{\infty} - h^-_{\infty}), \ominus_a(h^+ - h^+_{\infty}, h^- - h^-_{\infty})).$$

Clearly, $P$ is linear and we claim that it is a bijection. Then the injectivity of $P$ will show that $\ker P^a_\infty \cap \ker P^a_\infty = \{(0, 0)\}$ while the surjectivity of $P$ will show that $\ker P^a_\infty + \ker P^a_\infty = \hat{E}$.

Now, if $(h^+, h^-)$ belongs to the kernel of the map $P$, then $h^+_{\infty} = h^-_{\infty}$ and consequently $\ominus_a(h^+, h^-) = 0$ and $\ominus_a(h^+, h^-) = 0$, and hence by Proposition 1.28, $h^+ = h^- = 0$. In order to prove that $P$ is a surjection we take $(\xi, \hat{u}, \hat{v}) \in \mathbb{R}^N \oplus H^3(Z_a) \oplus H^{3,\delta_0}_c(C_a)$. By Proposition 1.28 again, there is a unique pair $(u^+, u^-) \in E$ solving the equations

$$\ominus_a(u^+, u^-) = \hat{u} \quad \text{and} \quad \ominus_a(u^+, u^-) = \hat{v}$$

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where we have denoted by $E$ the subspace of $\widehat{E}$ consisting of pairs having matching asymptotic limits. If $c$ is the common asymptotic constant of the maps $u^\pm$, then $u^\pm = c + r^\pm$ and $\Theta_a(u^+, u^-) = \Theta_a(r^+, r^-) = \hat{v}$. Given $\xi$ we introduce the maps

$$h^+ = u^+ \quad \text{and} \quad h^- = -\xi + u^-$$

whose asymptotic constants are equal to $h^+_\infty = c$ and $h^-_\infty = -\xi + c$, so that $h^+_\infty - h^-_\infty = \xi$, $\Theta_a(h^+, h^- + h^+_\infty - h^-_\infty) = \Theta_a(u^+, u^-) = \hat{u}$, and $\Theta_a(h^+ - h^+_\infty, h^- - h^-_\infty) = \Theta_a(r^+, r^-) = \hat{v}$. We have proved that $P$ is surjective. This finishes the proof of our claim and hence $\widehat{E} = \ker P^+_a \oplus \ker P^-_a$.

We split a given pair $(h^+, h^-) \in \widehat{E}$ where $h^\pm = h^\pm_\infty + r \pm$ into the sum

$$(h^+, h^-) = (w^+, w^-) + (v^+, v^-)$$

in which

$$(w^+, w^-) = (h^+_\infty, h^-_\infty) + \pi_a(r^+, r^-)$$

and

$$(v^+, v^-) = (\text{id} - \pi_a)(r^+, r^-).$$

We claim that $(w^+, w^-) \in \ker \Theta_a$ and $(v^+, v^-) \in \ker \Theta_a$. Since $\text{id} - \pi_a$ is the projection onto $\ker \Theta_a$ we conclude that $\Theta_a(v^+, v^-) = 0$. Moreover, by the formula for the projection $\pi_a$ in Section 1.3, the pair $(v^+, v^-)$ has a common asymptotic limit equal to $c = -av_a(r^+, r^-)$. This means that $(v^+, v^-) \in E$ and

$$P^+_a(v^+, v^-) = (v^+_\infty - v^-_\infty, \Theta_a(v^+, v^- + v^+_\infty - v^-_\infty)$$

$$= (0, \Theta_a(v^+, v^-)) = (0, 0).$$

Hence $(v^+, v^-) \in \ker P^+_a$. Considering the term $(w^+, w^-)$, we note that $w^\pm_\infty = h^\pm_\infty + c$. Using the formula for the projection map $\pi_a$ and abbreviating $\pi_a(r^+, r^-) = (\eta^+, \eta^-)$, we have $\pi_a(r^+ - c, r^- - c) = (\eta^+ - c, \eta^- - c)$. Hence

$$P^+_a(w^+, w^-) = \Theta_a(w^+ - w^+_\infty, w^- - w^-_\infty) = \Theta_a(w^+ - h^+_\infty - c, w^- - h^-_\infty - c)$$

$$= \Theta_a(\eta^+ - c, \eta^- - c) = \Theta_a \circ \pi_a(r^+ - c, r^- - c) = 0,$$

since $\pi_a$ is the projection onto the kernel of the map $\Theta_a$. Consequently, $(w^+, w^-) \in \ker P^-_a$. This completes the proof of Lemma 3.3. \[\square\]
If \( a \neq 0 \) we abbreviate by \( F_a \) the kernel of the map \( P^+_a \). Lemma 3.3 implies that the restriction of the map \( P^+_a \) to the kernel,
\[
P^+_a : F_a \to \mathbb{R}^N \oplus H^3(Z_a, \mathbb{R}^N), \quad (h^+, h^-) \mapsto (h^+_\infty - h^-_\infty, \oplus_a (h^+, h^- + h^+_\infty - h^-_\infty))
\]
is a bijection. If \( a = 0 \), we put \( \rho_0 = \text{id} \). In this case, we set \( F_0 = \hat{E} \) and define the map \( P^+_0 : \hat{E} \to \mathbb{R}^N \oplus E \) by
\[
(h^+, h^-) \mapsto (h^+_\infty - h^-_\infty, h^+, h^- + h^+_\infty - h^-_\infty),
\]
where \( E \) consists of those pairs in \( \hat{E} \) which have matching asymptotic limits. This map is also a bijection.

Next we consider the case of dimension \( N = 2 \) and we restrict ourselves to the subspace \( \hat{E}_0 \) of \( \hat{E} \) consisting of all pairs \((h^+, h^-)\) of maps in \( H^3_{c_0}(\mathbb{R}^2 \times S^1, \mathbb{R}^2) \) satisfying
\[
h^\pm(0, 0) = (0, 0) \quad \text{and} \quad h^\pm(0, t) \in \{0\} \times \mathbb{R} \subset \mathbb{R}^2
\]
for all \( t \in S^1 \). We abbreviate the image of \( \hat{E}_0 \) under the projection \( \rho_a \) by
\[
G^a = \rho_a(\hat{E}_0) = \{(h^+, h^-) \in \hat{E}_0 | \ominus_a (h^+ - h^+_\infty, h^- - h^-_\infty) = 0\}
\]
where we have used Lemma 3.3. Observe that \( G^a \) is a subspace of \( \hat{E}_0 \).

The map \( \Delta : Z_a \to \mathbb{R}^2 \), defined by
\[
\Delta = \ominus_a (h^+, h^- + h^+_\infty - h^-_\infty),
\]
has the following properties:

1. \( \Delta(p^+_a) = \Delta([0, 0]) = (0, 0) \) and \( \Delta(p^-_a) = \Delta([0, 0']) = h^+_\infty - h^-_\infty \).

2. \( \Delta([0, t]) \in \{0\} \times \mathbb{R} \) and \( \Delta([R, t]) \in (h^+_\infty - h^-_\infty) + (\{0\} \times \mathbb{R}) \).

If \( 0 < |a| < \frac{1}{2} \), we denote by \( H_a \) the sc-subspace of \( H^3(Z_a) \) consisting of those maps \( u : Z_a \to \mathbb{R}^2 \) which satisfy \( u(p^+_a) = (0, 0) \) and
\[
u([0, t]) \in \{0\} \times \mathbb{R} \subset \mathbb{R}^2 \quad \text{and} \quad u([0, t']) \in u(p^-_a) + (\{0\} \times \mathbb{R}).
\]

If \( a = 0 \), we denote by \( H_0 \) the space consisting of all pairs \((h^+, h^-) \in \hat{E}_0 \) for which there exists a constant \( c \in \mathbb{R}^2 \) satisfying \((h^+, h^- + c) \in E\).
Lemma 3.4. For $|a| < \frac{1}{2}$, the map $\Phi_a : G^a \rightarrow H_a$, defined by

$$\Phi_a(h^+, h^-) = \begin{cases} \oplus_a(h^+, h^- + h^+_\infty - h^-\infty) & \text{if } a \neq 0, \\ (h^+, h^- + h^+_\infty - h^-\infty) & \text{if } a = 0, \end{cases}$$

is a linear sc-isomorphism.

Proof. We only consider the case $a \neq 0$ and begin by showing the injectivity of the map $\Phi$. We assume that $\oplus_a(h^+, h^- + h^+_\infty - h^-\infty) = 0$ for a pair $(h^+, h^-) \in G^a$. Hence the pair $(h^+, h^-) \in \hat{E}_a$ solves the system of two equations

$$\begin{align*} \oplus_a(h^+, h^- + h^+_\infty - h^-\infty) &= 0 \\ \ominus_a(h^+, h^- + h^+_\infty - h^-\infty) &= 0 \end{align*}$$

The system has a unique solution $h^+ = 0$ and $h^- + h^+_\infty - h^-\infty = 0$. Since $h^+ = h^+_\infty + r^+$ where $r^+ \in H^{3,\delta}(\mathbb{R}^+ \times S^1)$, we conclude that $h^+_\infty = 0$ and $r^+ = 0$. Hence $h^- = h^-\infty$. From $h^-(0,0) = 0$, we obtain that also the constant $h^-\infty = 0$ vanishes. Consequently, $(h^+, h^-) = (0,0)$ as claimed.

In order to prove the surjectivity, we choose a map $u : Z_a \rightarrow \mathbb{R}^2$ which belongs to $H_a$ and abbreviate $c = u(p^-_a) = u(R, \vartheta)$. We have to solve the system of two equations

$$\begin{align*} \oplus_a(h^+, h^- + h^+_\infty - h^-\infty) &= u \\ \ominus_a(h^+, h^- + h^+_\infty - h^-\infty) &= 0 \end{align*}$$

Integrating the first equation at $s = \frac{\vartheta}{2}$ over the circle $S^1$ one obtains $av_a(h^+, h^- + h^+_\infty - h^-\infty) = [u]$. These averages are defined in Section 1.3. Thus the solutions of the above system are given by the formulae

$$h^+(s, t) = [u] + \frac{\beta_a}{\gamma_a} \cdot (u - [u])$$

and

$$h^-(s - R, t - \vartheta) + h^+_\infty - h^-\infty = [u] + \frac{1 - \beta_a}{\gamma_a} \cdot (u - [u])$$

where, as usual, $\beta_a = \beta_a(s)$ and $\gamma_a = \beta_a^2 + (1 - \beta_a)^2$. Evaluating both sides of the first equation in (34) at the point $(R, \vartheta)$, we find

$$h^+_\infty - h^-\infty = u(R, \vartheta) =: c$$
so that $h^-$ in (36) becomes

$$h^-(s - R, t - \vartheta) = [u] - c + \frac{1 - \beta_a}{\gamma_a} : (u - [u]).$$

In view of the properties of the function $\beta_a$, the asymptotic constants $h^\pm_\infty$ are equal to $h^+_\infty = [u]$ and $h^-_\infty = [u] - c$.

It remains to show that $(h^+, h^-) \in \hat{E}_0$. Using (35) and the properties of the map $u$, we find $h^+(0, 0) = u(0, 0) = (0, 0)$ and $h^+(0, t) = u(0, t) \in \{0\} \times \mathbb{R}$. For the map $h^-$ we use (36) and find that $h^-(0, 0) = u(R, \vartheta) = c$ and $h^-(0, t) = u(R, \vartheta + t) \in c + \{0\} \times \mathbb{R}$. Hence $(h^+, h^-) \in \hat{E}_0$ and the proof of the lemma is finished. $\blacksquare$

**Remark 3.5.** We note that if $h^+$ and $h^-$ have matching asymptotic limits, then the glued map $\Delta$ takes the boundary values in $\{0\} \times \mathbb{R}$. The difference of asymptotic values makes $\Delta$ take boundary values in an affine subspace on the right-hand boundary of $Z_a$.

In order to understand the upcoming context of the above construction, we consider pairs $(u^+_0, u^-_0)$ of diffeomorphisms

$$u^+_0 : \mathbb{R}^+ \times S^1 \rightarrow \mathbb{R}^+ \times S^1$$

of the half-cylinders, which on the covering spaces are of the form $u^\pm = \text{id} + h^\pm$ with $h^\pm \in \hat{E}$ specified below. The diffeomorphisms leave the boundaries

$$\partial(\mathbb{R}^+ \times S^1) = \{0\} \times S^1$$

invariant and we require that they keep the distinguished points $(0, 0) \in \mathbb{R}^+ \times S^1$ fixed, so that

$$u^+_0(0, 0) = (0, 0).$$

On the covering spaces we shall represent the maps $u^+_0$ in the following form,

$$u^+_0(s, t) = (s, t) + (d^+_0, \vartheta^+_0) + r^+_0(s, t), \quad s \geq 0 \quad (37)$$

and

$$u^-_0(s', t') = (s', t') + (d^-_0, \vartheta^-_0) + r^-_0(s', t'), \quad s' \leq 0. \quad (38)$$

where $r^\pm_0 : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^2$ belong to $H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^2)$ and satisfy at the boundaries (where $s = 0$)
\[(d_0^\pm, \vartheta_0^\pm) + r_0^\pm(0, t) \in \{0\} \times \mathbb{R}.
\]

Our aim is to define for a gluing parameter \(a\) of sufficiently small modulus, a glued map

\[
\boxplus_a(u_0^+, u_0^-) : Z_a \to Z_b,
\]

which maps the glued finite cylinder \(Z_a\) introduced in Section 1.3 diffeomorphically onto the glued finite cylinder \(Z_b\) belonging to \(b = b(a, u_0^+, u_0^-)\). In order to define the glued map we associate with \(a \neq 0\), the pair \((R, \vartheta)\) defined by

\[
R = \varphi(|a|) \quad \text{and} \quad a = |a|e^{-2\pi i \vartheta},
\]

where, as before, \(\varphi\) is the exponential gluing profile \(\varphi(r) = e^{\frac{r}{2}} - e\). In view of the above representations of the diffeomorphisms \(u_0^\pm\) we define the pair \((R', \vartheta')\) by

\[
R' = R + d_0^+ - d_0^- \quad \text{and} \quad \vartheta' = \vartheta + \vartheta_0^+ - \vartheta_0^-.
\]

The pair \((R', \vartheta')\) is associated with the gluing parameter \(b = b(a, u_0^+, u_0^-)\).

**Definition 3.6.** If \(|a|\) is sufficiently small we define the map \(\boxplus_a(u_0^+, u_0^-) : Z_a \to Z_b\) as follows. If \(a = 0\), we set

\[
\boxplus_0(u_0^+, u_0^-) = (u_0^+, u_0^-),
\]

and if \(a \neq 0\), we define

\[
\begin{align*}
\boxplus_a(u_0^+, u_0^-) & ([s, t]) \\
& = [(s, t) + (d_0^+, \vartheta_0^+) + \boxplus_a(r_0^+, r_0^-)([s, t])] \\
& = [(s, t) + (d_0^+, \vartheta_0^+) + \beta_a(s) \cdot r_0^+(s, t) + (1 - \beta_a(s)) \cdot r_0^-(s - R, t - \vartheta)].
\end{align*}
\]

The coordinates in the domain of definition of \(\boxplus_a(u_0^+, u_0^-)\) are \([s, t]\) where \(s \in [0, R]\) and the coordinates in the target cylinder \(Z_b\) are \([S, T]\) where \(S \in [0, R']\) and where \(R' = R + d_0^+ - d_0^-\) as defined above.

Note that since \(\beta_a(s) = 1\) if \(s \leq \frac{R}{2} - 1\), we have \(\boxplus_0(u_0^+, u_0^-)([0, t]) = [0, T(t)]\) and since \(\beta_a(s) = 0\) if \(s \geq \frac{R}{2} + 1\), we have \(\boxplus_0(u_0^+, u_0^-)([R, t]) = [R + d_0^+ - d_0^-, t + \vartheta_0^+ + \vartheta_0^- + \hat{T}(t)].\)

The map \(\boxplus_a(u_0^+, u_0^-)\) involves the asymptotic data of the two maps \(u_0^\pm\) and incorporates them into the twist of the target cylinder. The choices involved in the construction are subject to further constraints later on.
If the gluing parameter $0 < |a|$ is sufficiently small (hence $R$ is sufficiently large), the map $\boxplus_a(u^+, u^-)$ defines a diffeomorphism between finite cylinders $Z_a$ and $Z_b$ mapping the marked points $p^+_a$ onto the marked points $p^+_b$. These are the points on $Z_a$ and $Z_b$ corresponding to the original boundary points $(0, 0)$.

Now recall that the subspace $\tilde{E}_0 \subset \tilde{E}$ consists of pairs $(h^+, h^-)$ of mappings in $H^3_{c,\delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^2)$ satisfying $h^+(0, 0) = (0, 0)$ and $h^+(0, t) \in \{0\} \times \mathbb{R} \subset \mathbb{R}^2$ for all $t \in S^1$. If

$$h^+_\infty = h^+_\infty + r^{\pm},$$

with the asymptotic constants $h^+_\infty$ of $h^\pm$, we obtain the mappings $u^+_0 + h^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^\pm \times S^1$ represented by

$$(u^+_0 + h^+)(s, t) = (s, t) + (d^+_0, \vartheta^+_0) + h^+_\infty + r^+_0(s, t) + r^+(s, t), \quad s \geq 0$$

and

$$(u^-_0 + h^-)(s', t') = (s', t') + (d^-_0, \vartheta^-_0) + h^-_\infty + r^-_0(s', t') + r^-(s', t'), \quad s' \leq 0.$$

If the norms of $r^\pm$ are sufficiently small, the maps $u^+_0 + h^\pm$ are still diffeomorphisms of the cylinders $\mathbb{R}^\pm \times S^1$ leaving the boundaries $\partial(\mathbb{R}^\pm \times S^1) = \{0\} \times S^1$ invariant and fixing the points $(0, 0)$. Therefore, if $|a|$ is sufficiently small, the glued map

$$\boxplus_a(u^+_0 + h^+, u^-_0 + h^-) : Z_a \to Z_b$$

for the gluing parameter $b = b(a, u^+_0 + h^+, u^-_0 + h^-)$, is a diffeomorphism between finite cylinders preserving the distinguished points. The gluing parameter $b(a, u^+_0 + h^+, u^-_0 + h^-)$ is associated with the pair $(R', \vartheta')$ and is given by

$$(R', \vartheta') = (R, \vartheta) + (d^+_0 - d^-_0, \vartheta^+_0 - \vartheta^-_0) + h^+_\infty - h^-_\infty.$$

**Lemma 3.7.** The map $B_2^{1} \oplus H^3_{c,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^2) \oplus H^3_{c,\delta_0}(\mathbb{R}^- \times S^1, \mathbb{R}^2) \to \mathbb{C}$, defined by

$$(a, h^+, h^-) \mapsto b(a, u^+_0 + h^+, u^-_0 + h^-),$$

is smooth on every level. In particular, it is sc-smooth.

**Proof.** If $a = |a|e^{-2\pi i\vartheta}$ and $b = |b|e^{-2\pi i\vartheta'}$, we have defined $R = \varphi(|a|)$ and $R' = \varphi(|b|)$, with the gluing profile $\varphi(r) = e^r - e$ for $0 < r \leq 1$. By
construction \((R', \vartheta') = (R, \vartheta) + (d^+ - d^-, \vartheta^+ - \vartheta^-) + (h^+_\infty - h^-_\infty)\). The maps 
\(h^\pm \mapsto h^\pm_\infty\) are linear projection and hence smooth on every level.

On the other hand, the function \(B\) defined by \(B(r, c) := \varphi^{-1}(\varphi(r) + c)\) if \(r > 0\) and \(B(0, c) = 0\) is smooth by Lemma 4.4 of the appendix. It follows that the map 
\((a, h^+ - h^-) \mapsto b(a, u^+_0 + h^+, u^-_0 + h^-)\) is indeed smooth on every level and, in particular, \(sc\)-smooth as desired.

Explicitly, the glued map is computed to be
\[
\Box_a (u^+_0 + h^+, u^-_0 + h^-)([s, t]) = \big([s, t] + (d^+_0, \vartheta^+_0) + h^+ + \Box_a(r^+_0 + r^-, r^-_0 + r^-)([s, t]) \big]
\]
\[
= \left[\Box_a (u^+_0, u^-_0)([s, t]) + h^+ + \Box_a(r^+, r^-)([s, t])\right]
\]
where \([, , ]\) denotes the equivalence class of coordinates in \(Z_a\) as well as in \(Z_b\).

Remark 3.8. The following remark will be made precise later on. If \(w_{a_0} : Z_{a_0} \to Z_{b_0}\) is the diffeomorphism constructed by means of \(w_{a_0} = \Box_{a_0} (u^+_0, u^-_0)\) and \(b_0 = b(a_0, u^+_0, u^-_0)\), then given \((a, b, w)\) close to \((a_0, b_0, w_{a_0})\) and abbreviating \(w_a = \Box_a (u^+_0, u^-_0)\) we shall solve later on the equation

\[
w_a + \Box_a(h^+, h^- + h^+_\infty - h^-_\infty) = w
\]
for \((h^+ - h^-) \in \hat{E}_0\) small and satisfying, in addition, \(\rho_a(h^+, h^-) = (h^+, h^-)\). As we shall see this problem has a unique solution which allows to construct M-polyfold charts for suitable spaces later on.

3.2 An M-Polyfold Construction, Proof of Theorem 1.43

The subsection 3.2 is devoted to the proofs of Proposition 1.42 and Theorem 1.43 about the M-polyfold structures on the set \(X\) and on its “completion” \(\hat{X}\).

In order to recall the definition of the set \(X\) we denote by \(\Gamma\) the set of all pairs \((a, b)\) of complex numbers satisfying \(a \cdot b \neq 0\) and \(|a|, |b| < \varepsilon\). The size of \(\varepsilon\) will be adjusted during the proof.
**Definition 3.9.** The set $X$ consists of all triples $(a, b, w)$ in which $a, b \in \Gamma$ and $w : Z_a \to Z_b$ is a $C^1$-diffeomorphism between the two finite glued cylinders belonging to the Sobolev class $H^3$ and satisfying $w(p_a^\pm) = p_b^\pm$.

In order to define a topology on $X$ we fix a point $(a_0, b_0, w_0) \in X$ and choose a family $a \mapsto \phi_a$ of diffeomorphisms $\phi_a : Z_a \to Z_{a_0}$ defined for gluing parameters $a$ close to $a_0$, mapping $p^\pm_a$ to $p^\pm_{a_0}$, and satisfying $\phi_{a_0} = \text{id}$. We assume that the family $a \mapsto \phi_a$ is smooth in the following sense. On the finite cylinder $Z_{a_0}$ we have the global coordinates $Z_{a_0} \ni [s, t] \mapsto (s, t) \in [0, R_0] \times S^1$ and $[s', t']' \mapsto (s', t') \in [-R_0, 0] \times S^1$. Similarly, we have two global coordinates on $Z_a$. If $a$ is close to $a_0$, we can express the map $\phi_a : Z_a \to Z_{a_0}$ with respect to four choices of coordinate systems, two in the domains and two in the target, namely

- $(s, t) \mapsto [s, t] \xrightarrow{\phi_a} [S, T] \mapsto (S, T)$
- $(s, t) \mapsto [s, t] \xrightarrow{\phi_a} [S', T]' \mapsto (S', T')$
- $(s', t') \mapsto [s', t']' \xrightarrow{\phi_a} [S, T] \mapsto (S, T)$
- $(s', t') \mapsto [s', t']' \xrightarrow{\phi_a} [S', T]' \mapsto (S', T')$.

The family $a \mapsto \phi_a$ is called smooth if all these coordinate expressions are smooth as maps of $(a, s, t)$, respectively of $(a, s', t')$, for $a$ close to $a_0$. Similarly, we can choose the second smooth family $b \mapsto \psi_b$ of diffeomorphisms $\psi_b : Z_{b_0} \to Z_b$ defined for gluing parameters $b$ close to $b_0$, mapping $p^\pm_{b_0}$ to $p^\pm_{b}$, and satisfying $\psi_{b_0} = \text{id}$. Given an open neighborhood $U(w_0)$ of the diffeomorphism $w_0$ in the space of diffeomorphisms $Z_{a_0} \to Z_{b_0}$ of class $H^3$, we introduce the set

$$U(a_0, b_0, w_0, U(w_0), \delta_0)$$

consisting of triples $(a, b, u)$ satisfying

$$|a - a_0| < \delta_0, \quad |b - b_0| < \delta_0, \quad u = \psi_b \circ w \circ \phi_a \text{ and } w \in U(w_0).$$
Clearly, the chosen triple \((a_0, b_0, w_0)\) belongs to the set \(U(a_0, b_0, w_0, U(w_0), \delta_0)\) so that the collection of the sets \(U(a_0, b_0, w_0, U(w_0), \delta_0)\) covers the set \(X\).

**Lemma 3.10.** We abbreviate

\[
U_0 = U(a_0, b_0, w_0, U(w_0), \delta_0) \quad \text{and} \quad U_1 = U(a_1, b_1, w_1, U(w_1), \delta_1).
\]

If \((a_2, b_2, w_2) \in U_0 \cap U_1\), then there exists a set \(U_2 = U(a_2, b_2, w_2, U(w_2), \delta_2)\) satisfying

\[U_2 \subset U_0 \cap U_1.\]

**Proof.** We consider the set of diffeomorphisms \(U_0 = U(a_0, b_0, w_0, U(w_0), \delta_0)\) consisting of points \((a, b, u)\) satisfying \(|a - a_0| < \delta_0, |b - b_0| < \delta_0\) and \(u = \psi_0^0 \circ w \circ \varphi_a^0\) for a diffeomorphism \(w \in U(w_0)\). Here \(\psi_0^0 : Z_{b_0} \to Z_b\) and \(\varphi_a^0 : Z_b \to Z_{a_0}\) are the associated smooth families of diffeomorphisms. Moreover, \(U(w_0)\) is an \(H^2\)-open neighborhood of \(C^1\)-diffeomorphisms of \(w_0 : Z_{a_0} \to Z_{b_0}\).

Similarly, we consider the second set \(U_1 = U(a_1, b_1, w_1, U(w_1), \delta_1)\) of points \((a, b, u)\) satisfying \(|a - a_1| < \delta_1, |b - b_1| < \delta_1\) and \(u = \psi_1^0 \circ w \circ \varphi_a^1\) for \(w \in U(w_1)\), where \(w_1 : Z_{a_1} \to Z_{b_1}\) is a \(C^1\)-diffeomorphism in \(H^3\). If

\[(a_2, b_2, w_2) \in U_0 \cap U_1\]

then

\[w_2 = \psi_2^0 \circ v_0 \circ \varphi_{a_2}^0 = \psi_2^1 \circ v_1 \circ \varphi_{a_2}^1\]

for two diffeomorphisms \(v_0 \in U(w_0)\) and \(v_1 \in U(w_1)\). In view of the remarks (Theorem 2.6) about the action by diffeomorphisms we find an open neighborhoods \(V(v_0) \subset U(w_0)\) and \(V(v_1) \subset U(w_1)\) and a number \(\delta_2 > 0\) sufficiently small so that

\[\psi_0^0 \circ V(v_0) \circ \varphi_a^0 \subset \psi_1^1 \circ V(v_1) \circ \varphi_a^1\]

(39)

for all \(|a - a_2| < \delta_2\) and \(|b - b_2| < \delta_2\). The smooth families of diffeomorphisms, defined by

\[
\tilde{\varphi}_a = (\varphi_{a_2}^0)^{-1} \circ \varphi_a^0 : Z_a \to Z_{a_2} \\
\tilde{\psi}_a = \psi_b^0 \circ (\psi_{b_2}^0)^{-1} : Z_{b_2} \to Z_b
\]

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for $|a - a_2| < \delta_2$ and $|b - b_2| < \delta_2$ satisfy $\tilde{\varphi}_{a_2} = \text{id}$ on $Z_{a_2}$ and $\tilde{\psi}_{b_2} = \text{id}$ on $Z_{b_2}$ and map the points $p_a^\pm$ onto $p_{a_2}^\pm$, respectively $p_{b_2}^\pm$ onto $p_b^\pm$. Next we define the open neighborhood in $H^3$ of the $C^1$-diffeomorphism $w_2 : Z_{a_2} \to Z_{b_2}$ by

$$U(w_2) = \psi_{b_2}^0 \circ V(v_0) \circ \varphi_{a_2}^0,$$

and introduce the set

$$U_2 = U(a_2, b_2, w_2, U(w_2), \delta_2) = \{(a, b, w) | |a - a_2| < \delta_2, |b - b_2| < \delta_2 \text{ and } w = \tilde{\psi}_b^0 \circ u \circ \tilde{\varphi}_a^0 \text{ for some } u \in U(w_2)\}.$$

If $(a, b, w) \in U_2$, then $w = \tilde{\psi}_b^0 \circ u \circ \tilde{\varphi}_a^0$ and $u \in U(w_2)$ and hence there exists $v \in V(v_0)$ satisfying

$$w = \tilde{\psi}_b^0 \circ \psi_{b_2}^0 \circ v \circ \varphi_{a_2}^0 \circ \tilde{\varphi}_a^0 = \psi_b^0 \circ v \circ \varphi_a^0.$$

Since $v \in V(v_0) \subset U(w_0)$, we conclude that $(a, b, w) \in U_0$. In view of (39), we also conclude that $w = \tilde{v}_b^1 \circ \tilde{v} \circ \tilde{\varphi}_a^1$ for some $\tilde{v} \in V(v_1) \subset U(v_1)$ and hence $(a, b, w) \in U_1$. Consequently, $U_2 \subset U_0 \cap U_1$ as claimed in Lemma 3.10. ■

Lemma 3.10 shows that the collection $\{U(a, b, w, U(w), \delta)\}$ defines the basis for the topology $\mathcal{T}$ on $X$. This topology is second countable and paracompact, and hence metrizable.

The construction used in the definition of the topology of $X$ allows to define $M$-polyfold charts on $X$ as follows.

We choose a point $(a_0, b_0, w_0) \in X$ where

$$w_0 : Z_{a_0} \to Z_{b_0}$$

is a $C^1$-diffeomorphism belonging to $H^3$ and mapping $p_{a_0}^\pm$ to $p_{b_0}^\pm$. There exists an $\varepsilon_0 > 0$ so that for given $h \in H^3(Z_{a_0}, \mathbb{R}^2)$ satisfying $h(p_{a_0}^\pm) = (0, 0)$ and $h([0, t]) \in \{0\} \times \mathbb{R}$, $h([0, t']) \in \{0\} \times \mathbb{R}$, and $|D^\alpha h([s, t])| < \varepsilon_0$ for $|\alpha| \leq 1$, the map

$$[s, t] \to w_0([s, t]) + h([s, t])$$

is still a $C^1$-diffeomorphism $Z_{a_0} \to Z_{b_0}$ between the finite glued cylinders. Let us denote by $\hat{H}^3(Z_{a_0}, \mathbb{R}^2)$ the closed subspace of $H^3(Z_{a_0}, \mathbb{R}^2)$ consisting of maps $h$ satisfying $h(p_{a_0}^\pm) = (0, 0)$, $h([0, t]) \in \{0\} \times \mathbb{R}$ and $h([0, t']) \in \{0\} \times \mathbb{R}$. Moreover, every diffeomorphism $Z_{a_0} \to Z_{b_0}$ in $H^3$ sufficiently close to $w_0$ in the $C^1$-norm can be written in such a way for a uniquely determined $h$.

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Remark 3.11. Let us note the following important fact. Take $\varepsilon_1 \in (0, \varepsilon_0)$. Then we find a diffeomorphism $w_1 : Z_{a_0} \to Z_{b_0}$ which is smooth and close to $w_0$ so that the previous discussion is valid for $w_1$ with $\varepsilon_0$ replaced by $\varepsilon_1$. In addition, $w_0 = w_1 + h_0$ for a suitable $h_0$ which is controlled by $\varepsilon_1$. Hence we may assume without loss of generality that in the triple $(a_0, b_0, w_0)$ chosen above, the map $w_0 : Z_{a_0} \to Z_{b_0}$ is a smooth diffeomorphism and the collection of triples $(a_0, b_0, w_0 + h)$ where $h$ satisfies the conditions described above contains an a-priori given triple in $X$.

Now, assuming that $w_0 : Z_{a_0} \to Z_{b_0}$ is a smooth diffeomorphism, we choose two smooth (in the sense explained after the Definition 3.9) families $a \mapsto \phi_a$ and $b \mapsto \psi_b$ of diffeomorphisms

$$
\phi_a : Z_a \to Z_{a_0} \quad \text{and} \quad \psi_b : Z_{b_0} \to Z_b
$$

defined for $a$ close to $a_0$ and $b$ close to $b_0$ and introduce the map

$$(a, b, h) \mapsto (a, b, \psi_b \circ (w_0 + h) \circ \phi_a) \in X$$

into the space $X$, defined for triples $(a, b, h)$ in which $(a, b)$ is close to $(a_0, b_0)$ and $h$ is varying in the subspace $\tilde{H}^3(Z_{a_0}, \mathbb{R}^2)$ of $H^3(Z_{a_0}, \mathbb{R}^2)$ as described above. The domain of definition of the map is a set in a neighborhood of the point $(a_0, b_0, 0)$ in the sc-Hilbert space

$$
\mathbb{C} \oplus \mathbb{C} \oplus \tilde{H}^3(Z_{a_0}, \mathbb{R}^2).
$$

By definition of the topology, this map is continuous and obviously a homeomorphism onto some open neighborhood of the point $(a_0, b_0, w_0) \in X$. Hence we may view it as the inverse of a chart.

We consider two such charts around the points $(a_0, b_0, w_0)$ and $(\tilde{a}_0, \tilde{b}_0, \tilde{w}_0) \in X$. The inverse of the second chart is the map

$$(\tilde{a}, \tilde{b}, \tilde{h}) \mapsto (\tilde{a}, \tilde{b}, \tilde{\psi}_b \circ (\tilde{w}_0 + \tilde{h}) \circ \tilde{\phi}_a)$$

with $\tilde{h} \in \tilde{H}^3(Z_{\tilde{a}_0}, \mathbb{R}^2)$. At the points where the two charts intersect we have $\tilde{a} = a$ and $\tilde{b} = b$ and hence $\psi_b \circ (w_0 + h) \circ \phi_a = \tilde{\psi}_b \circ (\tilde{w}_0 + \tilde{h}) \circ \tilde{\phi}_a$. Therefore, the transition map of the charts is of the form

$$(a, b, h) \mapsto (a, b, \tilde{h})$$

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where \( \tilde{h}(a, b, h) \in H^3(Z_{a_0}, \mathbb{R}^2) \) is defined by
\[
\tilde{h}(a, b, h) := (\psi_b^{-1} \circ \psi_b \circ (w_0 + h) \circ \phi_a \circ \tilde{\phi}_a^{-1}) - \tilde{w}_0.
\]
The map \( a \mapsto \phi_a \circ \tilde{\phi}_a^{-1} \) is a smooth family of diffeomorphisms \( Z_{a_0} \rightarrow Z_{a_0} \), and \( b \mapsto \psi_b^{-1} \circ \psi_b \) is a smooth family of diffeomorphisms \( Z_{b_0} \rightarrow Z_{b_0} \). Recalling the result about diffeomorphism actions we conclude from Theorem 1.26 together with the chain rule that the map \( (a, b, h) \mapsto \tilde{h}(a, b, h) \) is an sc-smooth map.

Having proved that the transition maps between the M-polyfold charts are sc-smooth, we have equipped the set \( X \) with the structure of an M-polyfold.

**Remark 3.12.** If we equip the (classical) Hilbert manifold of diffeomorphisms \( Z_{a_0} \rightarrow Z_{b_0} \) of class \( H^3 \) preserving the distinguished points, with the filtration for which the level \( m \) corresponds to the Sobolev regularity \( m + 3 \), we obtain an sc-manifold taking as charts the ones coming from exponential maps. We refer to [4] for the classical set-up. For this sc-manifold, the map
\[
(a, b, u) \mapsto (a, b, \psi_b \circ u \circ \phi_a)
\]
establishes an sc-diffeomorphism between an open neighborhood of the triple \( (a_0, b_0, u_0) \) in which \( u_0 \) is viewed as an element in the latter defined space, and an open neighborhood of the same triple in the former M-polyfold \( X \).

Let us consider two different points \( (a_1, b_1, w_1) \) and \( (a_2, b_2, w_2) \) in \( X \). We can connect \( (a_1, b_1, w_1) \) by a continuous path to an element of the form \( (a_2, b_2, w_3) \). We have to keep in mind that the space of end-point preserving diffeomorphisms is disconnected (think about Dehn twists). However keeping \( a_2 \) fixed we can vary \( b_2 \) and \( w_3 \) and connect it with \( (a_2, b_2, w_2) \). Therefore, the topological space \( X \) is connected. Since it carries a second countable paracompact topology, the proof of Theorem 1.42 is complete. \( \blacksquare \)

Compared to the general setting, the M-polyfold structure for \( X \) is quite special insofar as the local models are open subsets of sc-Hilbert spaces. This will change in the next step where we “complete” the space \( X \) to the space \( \overline{X} \) by adding the elements corresponding to the gluing parameter values \( a = 0 \) and \( b = 0 \).

We fix a number \( \delta_0 \in (0, 2\pi) \) and denote by \( D = D^{3, \delta_0} \) the space consisting of pairs \( (u^+, u^-) \) in which the maps \( u^\pm : \mathbb{R}^\pm \times S^1 \rightarrow \mathbb{R}^\pm \times S^1 \) are \( C^1 \)-diffeomorphisms belonging to \( H^3_{\text{loc}} \) and satisfying
\[
u^\pm(0, 0) = (0, 0).
\]
Moreover, we assume that there exist asymptotic constants \((d^\pm, \vartheta^\pm) \in \mathbb{R} \times S^1\), so that
\[
 u^\pm(s, t) = (s + d^\pm, t + \vartheta^\pm) + r^\pm(s, t),
\]
where the maps \(r^\pm\) belong to \(H^{3,0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^2)\). We define an \(sc\)-structure on \(D\) by declaring the level \(m\) to consist of elements of regularity \((m + 3, \delta^m)\) where \((\delta^m)\) is a strictly increasing sequence of real numbers contained in \((0, 2\pi)\) and starting with the previously chosen \(\delta_0\).

We recall from Section 1.5 that the set \(X\) is defined as the disjoint union
\[
 X = X \coprod \{(0,0) \times D\}.
\]

We shall construct charts around points of the form \((0,0, u^+, u^-)\), which are compatible with the \(M\)-polyfold structure already defined for \(X\) and, of course, are compatible among themselves. Before we do that we define the topology on \(\bar{X}\).

We consider a sufficiently small open neighborhood \(U(u^+_0, u^-_0)\) in \(D\) for the \(H^{3,0}\)-topology on \(D\). Then there exists \(\sigma_0 > 0\) so that for \(0 < |a| < \sigma_0\) the glued maps \(\boxplus_a(u^+, u^-)\) are diffeomorphisms \(Z_a \to Z_b\) between finite cylinders, where \((u^+, u^-) \in U(u^+_0, u^-_0)\) and \(b = b(a, u^+, u^-)\) and \(|a|, |b| < \varepsilon\).

We introduce for \(\sigma \in (0, \sigma_0)\) the set \(U_\sigma\) of triples
\[
 U_\sigma = \{(a, b(a, u^+, u^-), \boxplus_a(u^+, u^-)) \mid 0 \leq |a| < \sigma, \ (u^+, u^-) \in U(u^+_0, u^-_0)\}.
\]

We have already equipped the set \(X\) with a metrizable topology. The following lemma shows that the collection of subsets of type \(U_\sigma\) in \(\bar{X}\) as defined in (40), is compatible with the topology on \(X\).

**Proposition 3.13.** Let \(U = U_\sigma\) be as defined in (40). Then the set \(X \cap U\) is open in \(X\). Given two sets \(U\) and \(V\) as defined in (40) and a point \((0,0, u^+_0, u^-_0) \in \bar{X}\) which belongs to \(U \cap V\), there exists a third set \(W\) constructed according to the recipe (40) centered at \((0,0, u^+_0, u^-_0)\) so that
\[
 (0,0, u^+_0, u^-_0) \in W \subset U \cap V.
\]

Hence there exists a unique topology on \(\bar{X}\) for which the open sets in \(X\) and the new sets just introduced form a basis. Moreover, this topology is second countable and metrizable. Further, \(\bar{X}\) equipped with this topology is connected and \(X\) is open and dense in \(\bar{X}\).
Proof. We first consider the case that \((a_0, b_0, w_0) \in X \cap U_{\sigma}\), so that
\[
(a_0, b_0, w_0) = (a_0, b(a_0, u_0^+, u_0^-), \oplus_{a_0}(u_0^+, u_0^-)) \in U_{\sigma}
\]
where \(0 < |a_0| < \sigma\) and \(w_0 = \oplus_{a_0}(u_0^+, u_0^-)\) is a diffeomorphism \(Z_{a_0} \to Z_{b_0}\) for \(b_0 = b(a_0, u_0^+, u_0^-)\). We have to show that a small open neighborhood \(O\) of \((a_0, b_0, w_0)\) in \(X\) is contained in \(U_{\sigma}\). We recall that an open neighborhood \(O\) of the point \((a_0, b_0, w_0) \in X\) may be assumed of the form
\[
O = \{(a, b, \psi_b \circ u \circ \phi_a) \mid |a - a_0| < \varepsilon, |b - b_0| < \varepsilon \text{ and } u \in U(w_0)\}
\]
where \(\varepsilon > 0\) and where \(U(w_0)\) is a small \(H^3\)-neighborhood of \(w_0\) in the set of diffeomorphisms \(Z_{a_0} \to Z_{b_0}\) of Sobolev class \(H^3\) and fixing the distinguished points.

If \(a = a_0\) and \(b = b_0\), then \(u = w_0\) so that
\[
\psi_{b_0} \circ w_0 \circ \phi_{a_0} = w_0.
\]
We have used that \(\phi_{a_0} = \text{id}\) on \(Z_{a_0}\) and \(\psi_{b_0} = \text{id}\) on \(Z_{b_0}\).

We recall that the pair \((h^+, h^-) \in \hat{E}_0\) consists of maps \(h^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^2\) of the form
\[
h^\pm = h^\pm_{\infty} + r^\pm
\]
where \(r^\pm \in H^{3,\delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^2)\) and where \(h^\pm(0, 0) = (0, 0)\) and \(h^\pm(0, t) \in \{0\} \times \mathbb{R}\) for all \(t \in \mathbb{R}\).

If \(\varepsilon > 0\) in the definition of \(O\) is sufficiently small and \((a, b, \psi_b \circ u \circ \phi_a) \in O\) we look for a solution \((h^+, h^-) \in \hat{E}_0\) of the following equations
\[
\oplus_a (u_0^+ + h^+, u_0^- + h^-) = \psi_b \circ u \circ \phi_a \tag{41}
\]
and
\[
b(a, u_0^+ + h^+, u_0^- + h^-) = b \tag{42}
\]
so that \((a, b(a, u_0^+ + h^+, u_0^- + h^-), \oplus_a(u_0^+ + h^+, u_0^- + h^-)) \in U_{\sigma}\). We recall that \(u_0^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^\pm \times S^1\) are diffeomorphisms of the form
\[
\begin{align*}
u_0^+(s, t) &= (s, t) + (d_0^+, \tilde{d}_0^+) + r_0^+(s, t), \quad s \geq 0 \\
u_0^-(s', t') &= (s', t') + (d_0^-, \tilde{d}_0^-) + r_0^-(s', t'), \quad s' \leq 0
\end{align*}
\]
and \(r_0^\pm \in H^{3,\delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^2)\). From Section \ref{section} we know that
\[
\oplus_a(u_0^+ + h^+, u_0^- + h^-)((s, t)) = \left[\oplus_a(u_0^+, u_0^-)([s, t]) + \oplus_a(h^+, h^- + h^+_{\infty} - h^-_{\infty})([s, t])\right].
\]
Hence, abbreviating
\[ \ominus_a(u_0^+, u_0^-) = w_a, \]
we have to solve the equation
\[ \oplus_a(h^+, h^- + h^+ - h^-_\infty) = \psi_b \circ u \circ \phi_a - w_a \quad (43) \]
for the unknown maps \((h^+, h^-) \in \hat{E}_0\). The right hand side vanishes if \(a = a_{a_0}, \ b = b_0, \) and \(u = w_0\) since \(w_{a_0} = w_0\). It actually suffices to solve for \((q^+, q^-) \in E\) the equation
\[ \oplus_a(q^+, q^-) = \psi_b \circ u \circ \phi_a - w_a. \quad (44) \]
because the solution \((h^+, h^-) \in \hat{E}_0\) of (43) is then given by the formula
\[ (h^+, h^-) = (q^+, q^- + (h^-_\infty - h^+_\infty)). \]
Recall that \(h^-_\infty - h^+_\infty\) can be computed from \(a\) and \(b\), so that \((h^+, h^-)\) is uniquely determined. The equation (44) becomes uniquely solvable once we impose, in addition, the equation \(\ominus_a(q^+, q^-) = 0\). Abbreviating the right hand side (44) by
\[ g = \psi_b \circ u \circ \phi_a - w_a, \]
the two equations
\[ \begin{align*}
\oplus_a(q^+, q^-) &= g \\
\ominus_a(q^+, q^-) &= 0
\end{align*} \quad (45) \]
have the following unique solution. Explicitly, the equations (45) are represented by
\[ \beta_a(s) \cdot q^+(s, t) + (1 - \beta_a(s)) \cdot q^-(s - R, t - \vartheta) = g(s, t) \\
-(1 - \beta_a(s)) \cdot q^+(s, t) + \beta_a(s) \cdot q^-(s - R, t - \vartheta) = (2\beta_a(s) - 1)av_a(q^+, q^-). \]
Integrating the first equation at \(s = \frac{R}{2}\) over the circle \(S^1\), we find in view of \(\beta_a\left(\frac{R}{2}\right) = \frac{1}{2}\) that the average
\[ av_a(q^+, q^-) = \int_{S^1} g \left(\frac{R}{2}, t\right) dt = [g] \]
agrees with the mean value of the function \(g\). In matrix form the above two equations are now written as
\[ \begin{bmatrix}
\beta_a(s) & (1 - \beta_a(s)) \\
-(1 - \beta_a(s)) & \beta_a(s)
\end{bmatrix} \cdot \begin{bmatrix}
q^+(s, t) \\
q^-(s - R, t - \vartheta)
\end{bmatrix} = \begin{bmatrix}
g \\
(2\beta_a(s) - 1)[g]
\end{bmatrix}. \]
Denoting by $\gamma_a(s) = \beta_a(s)^2 + (1 - \beta_a(s))^2$ the determinant of the matrix, one arrives at the following formulae for the solution,

\[q^+(s, t) = [g] + \frac{\beta_a(s)}{\gamma_a(s)} \cdot (g(s, t) - [g]) \quad (46)\]

and

\[q^-(s - R, t - \vartheta) = [g] + \frac{1 - \beta_a(s)}{\gamma_a(s)} \cdot (g(s, t) - [g]). \quad (47)\]

The asymptotic constants of $q^\pm$ are both equal to $[g]$. In particular, we find for the asymptotic constant of $h^+ = q^+$ the value

\[h^+_{\infty} = [g]. \quad (48)\]

In order to solve the equation (42) we associate with the gluing parameter $b$ the pair $(R', \vartheta')$ consisting of the gluing length and the gluing angle as the solution of the equation

\[(R', \vartheta') = (R, \vartheta) + (d_0^+, -d_0^+, \vartheta_0^+ - \vartheta_0^-) + h^+_{\infty} - h^-_{\infty}, \quad (49)\]

then $b = b(a, u_0^+ + h^+, u_0^- + h^-_{\infty})$ as desired and we have proved that the equations (41) together with (42) and $\circ_a(h^+, h^- + h^+_{\infty} - h^-_{\infty}) = 0$ have a unique solution $(h^+, h^-) \in \widehat{E}$. Actually, $(h^+, h^-) \in \widehat{E}_0$ as the next lemma shows.

**Lemma 3.14.** The pair $(h^+, h^-)$ belongs to $\widehat{E}_0$.

**Proof.** We first calculate the values of $g = \psi_b \circ u \circ \phi_a - w_a$ at the points $(0, 0)$ and $(R, \vartheta)$ where $(R, \vartheta)$ are the gluing length and the gluing angle associated with the parameter $a$. We recall that $\phi_a : Z_a \to Z_{a_0}$ is a diffeomorphism mapping the distinguished points of the cylinder $Z_{a_0}$ onto the distinguished points of $Z_{a_0}$. The same holds true for the diffeomorphism $\psi_b : Z_{a_0} \to Z_b$ and the diffeomorphism $u : Z_{a_0} \to Z_{b_0}$ fixes the distinguished points. Hence, $\psi_b \circ u \circ \phi_a(0, 0) = (0, 0)$ and $\psi_b \circ u \circ \phi_a(R, \vartheta) = (R', \vartheta')$ where $(R', \vartheta')$ are the parameters associated with $b$. In addition, since the boundaries of $Z_a$ are mapped onto the corresponding boundaries of $Z_b$, we have $\psi_b \circ u \circ \phi_a(0, t) \in \{0\} \times \mathbb{R}$ and $\psi_b \circ u \circ \phi_a(R, \vartheta + t) \in \{(R', \vartheta')\} + \{(0) \times \mathbb{R}\}$ for all $t \in \mathbb{R}$.

To evaluate the diffeomorphism $w_a = \Xi_a(u_0^+, u_0^-)$ at the points $(0, 0)$ and $(R, \vartheta)$, we recall that the diffeomorphisms $u_0^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^\pm \times S^1$ are of the form $u_0^\pm(s, t) = (s, t) + (d_0^+, \vartheta_0^+) + r_0^+(s, t)$ and satisfy $u_0^+(0, 0) = (0, 0)$. Using
that \( \mathbb{E}_a(u^+_0, u^-_0)(s, t) = (s, t) + (d^+_0, \vartheta^+_0) + \beta_a r^-_0(s - R, t - \vartheta) \), we find \( w_a(0, 0) = (0, 0) \) and \( w_a(0, t) \in \{0\} \times \mathbb{R} \). Evaluating at \((R, \vartheta)\) we obtain, \( w_a(R, \vartheta) = (R, \vartheta) + (d^+_0, \vartheta^+_0) + r^-_0(0, 0) = (R, \vartheta) + (d^+_0 - d^-_0, \vartheta^+_0 - \vartheta^-_0) + u^-_0(0, 0) = (R, \vartheta) + (d^+_0 - d^-_0, \vartheta^+_0 - \vartheta^-_0) \) and similarly

\[
w_a(R, \vartheta + t) \in ((R, \vartheta) + (d^+_0 - d^-_0, \vartheta^+_0 - \vartheta^-_0)) + (\{0\} \times \mathbb{R}).
\]

Consequently, \( g(0, 0) = (0, 0) \) and \( g(R, \vartheta) = (R', \vartheta') - ((R, \vartheta) + (d^+_0 - d^-_0, \vartheta^+_0 - \vartheta^-_0)) \). In addition, \( g(0, t) \in \{0\} \times \mathbb{R} \) and

\[
g(R, \vartheta + t) \in ((R', \vartheta') - (R, \vartheta) - (d^+_0 - d^-_0, \vartheta^+_0 - \vartheta^-_0)) + (\{0\} \times \mathbb{R}).
\]

Now, since \( h^+ = q^+ \), it follows from (140) that \( h^+(0, 0) = q^+(0, 0) = g(0, 0) = (0, 0) \) and \( h^+(0, t) = g(0, t) \in \{0\} \times \mathbb{R} \) for all \( t \in \mathbb{R} \). Using (49) and (17) and \( h^- = q^- + h^-_0 - h^+_0 \), we find that \( h^-(0, 0) = q^-(0, 0) + h^-_0 - h^+_0 = g(R, \vartheta) + h^-_0 - h^+_0 = (R', \vartheta') - ((R, \vartheta) + (d^+_0 - d^-_0, \vartheta^+_0 - \vartheta^-_0)) + h^-_0 - h^+_0 = (0, 0) \) and \( h^-(0, t) = g(R, \vartheta) + h^-_0 - h^+_0 \in (\{0\} \times \mathbb{R}) \). Consequently, the pair \((h^+, h^-)\) belongs to \( \mathcal{E}_0 \) and the proof of the lemma is complete.

\[\square\]

We recall that if \( a = a_0, b = b_0 \) and \( u = w_0 \), then \( g = 0 \) and hence \( h^+ = 0 \) and \( h^- = 0 \) and we conclude from the above formulae that if \( \varepsilon > 0 \) is sufficiently small, then indeed \( \mathcal{O} \subset U_\sigma \).

It remains to consider the case in which \( a_0 = b_0 = 0 \). We claim that given two sets \( U_{\sigma_1} \) and \( U_{\sigma_2} \) as defined in (33) and a point \((0,0, u^+_0, u^-_0) \in \mathbb{X}\) which belongs to \( U_{\sigma_1} \cap U_{\sigma_2} \), then there exists a third set \( U_{\sigma_0} \) such that

\[
(0, 0, u^+_0, u^-_0) \in U_{\sigma_0} \subset U_{\sigma_1} \cap U_{\sigma_2}.
\]

Indeed, the sets \( U_{\sigma_1} \) and \( U_{\sigma_2} \) have the form

\[
U_{\sigma_1} := \{(a, b(a, u^+, u^-), \mathbb{E}_a(u^+, u^-)) : 0 \leq |a| < \sigma_1, (u^+, u^-) \in U(u^+_1, u^-_1)\}
\]

\[
U_{\sigma_2} := \{(a, b(a, u^+, u^-), \mathbb{E}_a(u^+, u^-)) : 0 \leq |a| < \sigma_2, (u^+, u^-) \in U(u^+_2, u^-_2)\}
\]

where \( U(u^+_1, u^-_1) \) is an open neighborhood of the pair \((u^+_1, u^-_1) \) in \textbf{D} for the \( H^{3,\delta_0} \)-topology on \textbf{D}, and similarly for \( U(u^+_2, u^-_2) \). Since \((0, 0, u^+_0, u^-_0) \in U_{\sigma_1} \cap U_{\sigma_2} \),

\[
(u^+_0, u^-_0) \in U(u^+_1, u^-_1) \cap U(u^+_2, u^-_2).
\]

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We choose an open neighborhood $U(u_0^+, u_0^-)$ of the pair $(u_0^+, u_0^-)$ in $\mathcal{D}$ for the $H^{3,\delta_0}$-topology satisfying

$$(u_0^+, u_0^-) \in U(u_0^+, u_0^-) \subset U(u_1^+, u_1^-) \cap U(u_2^+, u_2^-),$$

and set $\delta_0 = \min\{\delta_1, \delta_2\}$. Then the set $U_{\sigma_0}$, defined by

$$U_{\sigma_0} := \{(a, b(a, u^+, u^-), \oplus_a(u^+, u^-)) | |a| < \sigma_0, (u^+, u^-) \in U(u_0^+, u_0^-)\},$$

satisfies

$$(0, 0, u_0^+, u_0^-) \in U_{\sigma_0} \subset U_{\sigma_1} \cap U_{\sigma_2}$$

as claimed. The proof of Proposition 3.13 is complete.

Lemma 3.15. If the pair $(q^+, q^-) \in E$ is the unique solution of (45)

$$\oplus_a(q^+, q^-) = g$$
$$\ominus_a(q^+, q^-) = 0,$$

then the pair $(h^+, h^-) := (q^+, q^- + h^-_\infty - h^+_\infty) \in \hat{E}_0$ satisfies

$$\rho_a(h^+, h^-) = (h^+, h^-).$$

Proof. The condition $\ominus_a((q^+, q^-) = \ominus_a(h^+, h^- + h^+_\infty - h^-_\infty) = 0$ is the same as the condition $\ominus_a(h^+-h^+_\infty,h^-+h^-_\infty) = 0$ which is equivalent to the relation $\rho_a(h^+, h^-) = (h^+, h^-)$, in view of Lemma 3.3.

Lemma 3.16. We assume that $(u_0^+, u_0^-) \in \mathcal{D}^\infty$ are smooth diffeomorphisms of the half-cylinders $\mathbb{R}^\pm \times S^1$ and fix the smooth point $(a_0, b_0, w_0) \in X$ in which $w_0 = \oplus_{a_0}(u_0^+, u_0^-)$. Setting $g = \psi_a \circ u \circ \phi_a - w_a = w - w_a$, the mappings $X \rightarrow E$,

$$(a, b, w) \mapsto (q^+(a, b, w), q^-(a, b, w)) \in E,$$

defined on a neighborhood of $(a_0, b_0, w_0)$ in $X$ as the unique solution of (45), are sc-smooth.

Proof. We use the local sc-coordinates $(a, b, h) \mapsto (a, b, \psi_b \circ (w_0 + h) \circ \phi_a)$ of $X$ near $(a_0, b_0, w_0) \in X$, where $h$ varies in the sc-Hilbert space $\hat{H}^{3,\delta_0}(Z_{a_0}, \mathbb{R}^2)$ of functions on the fixed cylinder $Z_{a_0}$. Then $g = \psi_b \circ (w_0 + h) \circ \phi_a - w_a$ and the solutions $q^\pm$ of (45) are in these coordinates mappings $(a, b, h) \mapsto q^\pm(a, b, h)$. 

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Recalling the proof of Proposition 3.13, the map \( q^+ = d + r^+ \) is defined by
\[
d = \begin{bmatrix} g \end{bmatrix} \quad \text{and} \quad r^+ = \begin{bmatrix} \alpha_a(s) \gamma_a(s) (g - \begin{bmatrix} g \end{bmatrix}) \end{bmatrix}
\]
for \( s \geq 0 \). Since \( \beta_a(s) = 0 \) for \( s \geq \frac{R}{2} + 1 \), the map \( q^+ \) is obtained from the sole knowledge of \( g \) on the finite piece \([0, \frac{R}{2} + 1] \times S^1\). Clearly, \( w_a \) as a function of \( a \) near \( a_0 \) is sc-smoothly depending on \( a \) on the fixed cylinder \([0, \frac{R_0}{2} + 2] \times S^1\). In view of Theorem 1.26 about diffeomorphisms actions, the function \( g \) depends sc-smoothly on \((a, b, h)\) in the neighborhood of \((a_0, b_0, 0)\). The sc-smoothness of the map \((a, b, h) \mapsto q^+(a, b, h)\) now follows from the above formula for \( r^+(s, t) \) using Proposition 2.8 and the chain rule. Similar arguments show that also \((a, b, h) \mapsto q^-(a, b, h)\) is an sc-smooth function. \(\blacksquare\)

From Lemma 3.16 we deduce immediately the following result.

**Lemma 3.17.** Fix the smooth point \((a_0, b_0, w_0) \in X\). Then there exists an open neighborhood \(U \subset X\) of the point and an sc-smooth map
\[
\Gamma : U \to \hat{E}_0,
\]
defined by \( \Gamma(a, b, w) = (h^+, h^-) \) where \((h^+, h^-) = (q^+, q^- + h^+_\infty - h^-_\infty) \in \hat{E}_0\) is the unique solution of \((42)\) and \((45)\).

**Proof.** This follows immediately from Lemma 3.16 together with Lemma 2.19 applied to \(h^+_\infty = \begin{bmatrix} g \end{bmatrix}\) and Lemma 3.7 applied to \(h^-_\infty\). \(\blacksquare\)

**Lemma 3.18.** We assume that \((u_0^+, u_0^-) \in D^\infty\) are smooth diffeomorphisms of \(\mathbb{R}^\pm \times S^1\). Then the map
\[
U(a_0, 0, 0) \subset C \times \hat{E}_0 \to X,
\]
defined by
\[
(a, h^+, h^-) \mapsto (a, b(a, u_0^+ + h^+, u_0^- + h^-), \mathbb{R}_a(u_0^+ + h^+, u_0^- + h^-)) \in X
\]
in a small open neighborhood of \((a_0, h^+, h^-) = (a_0, 0, 0)\) where \(a_0 \neq 0\), is an sc-smooth map.
Proof. In the local sc-chart around the point \((a_0, b_0, w_0) \in X\) which is defined by \((a, b, h) \mapsto (a, b, \psi_b \circ (w_0 + h) \circ \phi_a)\), our map is represented by the formulae
\[
b = b(a, u_0^+ + h^+, u_0^- + h^-)
\]
and
\[
h = \psi_b \circ \ominus_a(u_0^+ + h^+, u_0^- + h^-) \circ \phi_a^{-1} - w_0.
\]
Arguing now as in the proof of Lemma \([3, 16]\) one verifies that \((a, h^+, h^-) \mapsto h\) is an sc-smooth map.

For the following it is important to observe that if \((a_0, b_0, \tilde{w}_0) \in X\) is a point in which \(\tilde{w}_0 = \ominus_{a_0}(\tilde{u}_0^+, \tilde{u}_0^-)\) for a pair \((\tilde{u}_0^+, \tilde{u}_0^-)\) of diffeomorphisms of \(\mathbb{R}^\pm \times S^1\) belonging to the space \(D^{3, \delta_0}\), then there exists a smooth point \((a_0, b_0, w_0) \in X\) in which \(w_0 = \ominus_{a_0}(u_0^+, u_0^-)\) for a pair \((u_0^+, u_0^-)\) of smooth diffeomorphisms of \(\mathbb{R}^\pm \times S^1\) belonging to the space \(D^\infty\), and an open neighborhood \(U \subset X\) of \((a_0, b_0, w_0)\), which contains the original point \((a_0, b_0, \tilde{w}_0)\).

In view of the above discussion, we have establish the following result.

**Proposition 3.19.** If \((u_0^+, u_0^-) \in D^\infty\) is a pair of smooth diffeomorphisms of \(\mathbb{R}^\pm \times S^1\) and if \((a_0, b_0, w_0)\) is the smooth point in which \(w_0 = \ominus_{a_0}(u_0^+, u_0^-)\), then there exists an open neighborhood \(U \subset X\) of this point and an sc-smooth map
\[
\Gamma : U \rightarrow \hat{E}_0,
\]
defined by \(\Gamma(a, b, w) = (h^+, h^-)\) and having the following properties.

- \(\Gamma(a_0, b_0, w_0) = (0, 0)\)
- \(\ominus_a[(u_0^+, u_0^-) + \Gamma(a, b, w)] = w\).
- \(\rho_a(h^+, h^-) = (h^+, h^-)\).

In view of our observation above, it is sufficient to introduce chart maps of \(\hat{X}\) centered at smooth points \((0, 0, u_0^+, u_0^-) \in \hat{X} \setminus X\) where \((u_0^+, u_0^-) \in D^\infty\) are smooth diffeomorphisms of \(\mathbb{R}^\pm \times S^1\).

**Definition 3.20.** Given \((u_0^+, u_0^-) \in D^\infty\) we define the map
\[
\varphi : \{(a, h^+, h^-) \mid \rho_a(h^+, h^-) = (h^+, h^-), |a| < \sigma_0, (h^+, h^-) \in U\} \rightarrow \hat{X},
\]
in which \(U \subset \hat{E}_0\) is a small open neighborhood of \((0, 0) \in \hat{E}_0\), by
\[
\varphi(a, h^+, h^-) = (a, b(a, u_0^+ + h^+, u_0^- + h^-), \ominus_a(u_0^+ + h^+, u_0^- + h^-))
\]
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if $a \neq 0$, and by the formula,

$$\varphi(0, h^+, h^-) = (0, 0, u_0^+ + h^+, u_0^- + h^-)$$

in the case $a = 0$.

**Remark 3.21.** We would like to emphasize that due to the assumption $\rho_a(h^+, h^-) = (h^+, h^-)$ the map $\varphi$ is invertible. Indeed, since this assumption is equivalent to the requirement $\ominus_a(h^+ - h^+_{\infty}, h^- - h^-_{\infty}) = 0$, in view of Lemma 3.33 the injectivity of $\varphi$ follows by the arguments already used in the proof of Proposition 3.13. Namely, if

$$\varphi(a, h^+, h^-) = \varphi(\tilde{a}, k^+, k^-),$$

then $a = \tilde{a}$ and so $b(a, u_0^+ + h^+, u_0^- + h^-) = b(a, u_0^+ + k^+, u_0^- + k^-)$ and $\ominus_a(u_0^+ + h^+, u_0^- + h^-) = \ominus_a(u_0^+ + k^+, u_0^- + k^-)$ and the additional equations $\ominus_a(h^+ - h^+_{\infty}, h^- - h^-_{\infty}) = 0$ and $\ominus_a(k^+ - k^+_{\infty}, k^- - k^-_{\infty}) = 0$ imply that $h^+ = k^+$ and $h^- = k^-$. From the previous results we deduce the following result.

**Proposition 3.22.** The map $\varphi$ in Definition 3.20 restricted to triples $(a, h^+, h^-)$ satisfying $a \neq 0$ is an sc-diffeomorphism onto an open subset of the M-polyfold $X$.

The above maps of Definition 3.20 cover the set $X \setminus X$ and are compatible with the sc-msmooth structure of the M-polyfold $X$. Hence in order to establish an sc-smooth structure on $X$ it remains to show that the chart transformations are sc-smooth in a neighborhood of $a = 0$. We shall make use of the sc-smoothness results in chapter 2, in particular of Proposition 2.8 and of Lemma 2.18-2.21.

**Proposition 3.23.** If $\phi_1$ and $\phi_2$ are two chart maps of the topological space $X$ as introduced in Definition 3.20, then the chart transformation

$$\phi_2^{-1} \circ \phi_1$$

is an sc-smooth map.
Proof. We consider two chart maps \( \phi_1 \) and \( \phi_2 \) into \( \overline{X} \) according to Definition 3.20 which are defined in a neighborhood of \( a = 0 \). The first one is the map \( \phi_1 \)

\[
\{(a, h^+, h^-) \mid \rho_a(h^+, h^-) = (h^+, h^-), 0 \leq 0 \leq |a| < \sigma_1, (h^+, h^-) \in U_1 \} \rightarrow \overline{X},
\]

defined by

\[
\phi_1(a, h^+, h^-) := (a, b(a, u_1^+ + h^+, u_1^- + h^-), \Psi_a(u_1^+ + h^+, u_1^- + h^-))
\]

if \( a \neq 0 \), and by

\[
\phi_1(a, h^+, h^-) := (0, 0, u_1^+ + h^+, u_1^- + h^-)
\]

if \( a = 0 \). Here \( u_1^\pm : \mathbb{R}^\pm \times S^1 \rightarrow \mathbb{R}^\pm \times S^1 \) are two smooth diffeomorphisms of the positive resp. negative cylinder (in the covering spaces) of the form

\[
\begin{align*}
  u_1^+(s, t) &= (s, t) + (d_1^+(s, t), \vartheta_1^+(s, t)) + r_1^+(s, t), \\  s &\geq 0 \\
  u_1^-(s', t') &= (s', t') + (d_1^-(s', t'), \vartheta_1^-(s', t')) , \\  s' &\leq 0
\end{align*}
\]

where the pair \((r_1^+, r_1^-) \in D^\infty\) is a smooth point in the sc-Banach space \( H^{3, \delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^2) \times H^{3, \delta_0}(\mathbb{R}^- \times S^1, \mathbb{R}^2) \). In addition, \( U_1 \subset \widehat{E}_0 \) is an open neighborhood of \((0, 0) \in \widehat{E}_0\) consisting of pairs \((h^+, h^-)\) of maps \( h^\pm : \mathbb{R}^\pm \times S^1 \rightarrow \mathbb{R}^\pm \times S^1 \) of the form \( h^\pm(s, t) = h_\infty^\pm + r^\pm(s, t) \) and \( r^\pm \in H^{3, \delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^2) \). Similarly, the second chart map \( \phi_2 \),

\[
\{(a', k^+, k^-) \mid \rho_{a'}(k^+, k^-) = (k^+, k^-), 0 \leq |a'| < \sigma_2, (k^+, k^-) \in U_2 \} \rightarrow \overline{X}
\]

is defined by

\[
\phi_2(a', k^+, k^-) := (a', b'(a', u_2^+ + k^+, u_2^- + k^-), \Psi_{a'}(u_2^+ + k^+, u_2^- + k^-))
\]

if \( a' \neq 0 \), and by

\[
\phi_2(a', k^+, k^-) := (0, 0, u_2^+ + k^+, u_2^- + k^-)
\]

if \( a' = 0 \). The regions where the charts overlap are defined by

\[
\phi_1(a, h^+, h^-) = \phi_2(a', k^+, k^-).
\]
If \( a = 0 \), then it follows that also \( a' = 0 \), and hence \( b = 0 \) and \( b' = 0 \), so that
\[
\phi_1(0, h^+, h^-) = (0, 0, u_1^+ + h^+, u_1^- + h^-) = (0, 0, u_2^+ + k^+, u_2^- + k^-) = \phi_2(0, k^+, k^-).
\]
The map \( \phi_2^{-1} \circ \phi_1(0, h^+, h^-) = (0, k^+, k^-) \) is therefore given by
\[
k^+ = h^+ + (u_1^+ - u_2^+)
k^- = h^- + (u_1^- - u_2^-). \tag{52}
\]
If \( a \neq 0 \), then \( a' = a \) and hence \( b(a, u_1^++h^+, u_1^-+h^-) = b'(a, u_2^++k^+, u_2^-+k^-) \) and
\[
\bigoplus_a (u_1^+ + h^+, u_1^- + h^-) = \bigoplus_a (u_2^+ + k^+, u_2^- + k^-). \tag{53}
\]
In particular, the gluing lengths and gluing angles \((R_1, \vartheta_1)\) and \((R_2, \vartheta_2)\) associated with the gluing parameters \( a \) and \( a' \) are equal. Also the gluing lengths and gluing angles \((R'_1, \vartheta'_1)\) and \((R'_2, \vartheta'_2)\) associated with the gluing parameters \( b(a, u_1^+ + h^+, u_1^- + h^-) \) and \( b'(a, u_2^+ + k^+, u_2^- + k^-) \) are the same. Hence, in view of \((R'_1, \vartheta'_1) = (R_1, \vartheta_1) + (d_1^+ - d_1^-, \vartheta_1^+ - \vartheta_1^-) + h_\infty^+ - h_\infty^- \) and \((R'_2, \vartheta'_2) = (R_2, \vartheta_2) + (d_2^+ - d_2^-, \vartheta_2^+ - \vartheta_2^-) + k_\infty^+ - k_\infty^- \), we obtain the relation
\[
k_\infty^+ - k_\infty^- = (h_\infty^+ - h_\infty^-) + (d_1^+ - d_1^- - d_2^+ + d_2^-, \vartheta_1^+ - \vartheta_1^- - \vartheta_2^+ + \vartheta_2^-). \tag{54}
\]
Abbreviating
\[
w_a = \bigoplus_a (u_1^+, u_1^-) - \bigoplus_a (u_2^+, u_2^-),
\]
the equation \((53)\) becomes
\[
\bigoplus_a (k^+, k^- + k_\infty^+ - k_\infty^-) = \bigoplus_a (h^+, h^- + h_\infty^+ - h_\infty^-) + w_a. \tag{55}
\]
Recall that the pair \((k^+, k^-)\) satisfies \( \rho_a(k^+, k^-) = (k^+, k^-) \) which implies by Lemma \(3.3\) that \( \ominus_a(k^+ - k_\infty^+, k^- - k_\infty^-) = 0 \). Observing that the anti-gluing operation \( \ominus_a \) satisfies
\[
\ominus_a(v^+ + A, v^- + A) = \ominus_a(v^+, v^-)
\]
for every constant \( A \), we conclude that \( \ominus_a(k^+, k^- + k_\infty^+ - k_\infty^-) = 0 \). Setting \( q^+ = k^+ \) and \( q^- = k^- + k_\infty^+ - k_\infty^- \) and abbreviating
\[
g := \ominus_a(h^+, h^- + h_\infty^+ - h_\infty^-) + w_a, \tag{56}
\]
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we have therefore to solve the following system of equations for \((q^+, q^-)\)

\[
\oplus_a(q^+, q^-) = g \\
\ominus_a(q^+, q^-) = 0.
\]  

Integrating the first equation at \(s = \frac{R}{2}\) over the circle \(S^1\) and recalling that \(\beta_a(\frac{R}{2}) = \frac{1}{2}\), we find

\[
av_a(q^+, q^-) = [g].
\]  

In matrix form the system (57) is expressed as follows,

\[
\begin{bmatrix}
\beta_a & 1 - \beta_a \\
-(1 - \beta_a) & \beta \\
\end{bmatrix}
\begin{bmatrix}
q^+ \\
q^- \\
\end{bmatrix}
= \begin{bmatrix}
g \\
(2\beta_a - 1)av_a(q^+, q^-) \\
\end{bmatrix} = \begin{bmatrix}
g \\
(2\beta_a - 1)[g] \\
\end{bmatrix}.
\]

Denoting by \(\gamma_a = \beta + a^2 + (1 - \beta_a)^2\) the determinant of the matrix and abbreviating \(\gamma_a = \gamma_a(s)\) and \(\beta_a = \beta_a(s)\), the unique solution of (57) is given by

\[
q^+(s, t) = \left(1 - \frac{\beta_a}{\gamma_a}\right) \cdot [g] + \frac{\beta_a}{\gamma_a} \cdot g
\]

for \(s \geq 0\), and

\[
q^-(s - R, t - \vartheta) = \left(1 - \frac{1 - \beta_a}{\gamma_a}\right) \cdot [g] + \frac{1 - \beta_a}{\gamma_a} \cdot g
\]

for all \(s \leq R\).

In order to analyze the behavior at \(a = 0\) we write down the solutions in detail. To do this we represent \(h^\pm = h_\infty^\pm + r_3^\pm\) where \(r_3^\pm \in H^{3, \delta_0}(\mathbb{R}^\times S^1, \mathbb{R}^2)\) and use the explicit representations of

\[
g(s, t) = \oplus_a(h^+, h^- + h_\infty^+ - h_\infty^-)(s, t) + w_a(s, t)
\]

and

\[
w_a(s, t) = (s, t) + (d_1^+, \vartheta_1^+) + \beta_a \cdot r_1^+(s, t) + (1 - \beta_a) \cdot r_1^-(s - R, t - \vartheta)
\]

and

\[
= (d_1^+ - d_2^+, \vartheta_1^+ - \vartheta_2^+)
\]

and

\[
r_1^+(s, t) + (1 - \beta_a) \cdot (r_1^+ - r_2^+)(s - R, t - \vartheta),
\]

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so that

\[ g(s, t) = h^+_\infty + (d^+_1 - d^+_2, \vartheta^+_1 - \vartheta^+_2) + \beta_a \cdot r^+(s, t) + (1 - \beta_a) \cdot r^-(s - R, t - \vartheta) \]  

where we have abbreviated

\[ r^\pm = r^+_1 + r^+_2. \]  

The mean value \([g] = \int g \left( \frac{R}{2}, t \right) \, dt\) is computed to be

\[ [g] = h^+_\infty + (d^+_1 - d^+_2, \vartheta^+_1 - \vartheta^+_2) + a v_a(r^+, r^-). \]  

With (61) and (63) the solution \(q^+(s, t)\) for all \(s \geq 0\) is equal to

\[
q^+(s, t) = h^+_\infty + (d^+_1 - d^+_2, \vartheta^+_1 - \vartheta^+_2) + \left(1 - \frac{\beta_a}{\gamma_a}\right) \cdot a v_a(r^+, r^-) + \frac{\beta^2_a}{\gamma_a} \cdot r^+(s, t) + \frac{\beta_a(1 - \beta_a)}{\gamma_a} \cdot r^-(s - R, t - \vartheta),
\]  

where, as usual, we have abbreviated \(\beta_a = \beta_a(s)\). Since \(q^+ = k^+\) we read off the solution \(q^+(s, t)\) the asymptotic constant

\[ k^+_\infty = \lim_{s \to \infty} q^+(s, t) = h^+_\infty + (d^+_1 - d^+_2, \vartheta^+_1 - \vartheta^+_2) + a v_a(r^+, r^-) \]  

using that \(\beta_a(s) = 0\) for \(s \geq \frac{R}{2} + 1\).

In order to represent the solution \(q^-(s - R, t - \vartheta)\) for all \(s \leq R\) we introduce the variables \(s' = s - R\) and \(t' = t - \vartheta\). From \(\beta(s) = 1 - \beta(-s)\) one deduces \(\beta_a(-s') = \beta(-s' - \frac{R}{2}) = 1 - \beta(s' + \frac{R}{2}) = 1 - \beta_a(s' + R)\) and \(\gamma_a(-s') = \gamma_a(s' + R)\). Using this, the solution \(q^-(s', t')\) is represented by

\[
q^-(s', t') = k^-(s', t') + (k^+_\infty - k^-\infty) = h^-\infty + (d^-_1 - d^-_2, \vartheta^-_1 - \vartheta^-_2) + \left(1 - \frac{\beta_a(-s')}{\gamma_a(-s')}\right) \cdot a v_a(r^+, r^-) + \frac{1 - \beta_a(-s')}{\gamma_a(-s')} \cdot r^+(s' + R, t' + \vartheta) + \frac{\beta_a(-s')^2}{\gamma_a(-s')} \cdot r^-(s', t')
\]

for all \(s' \leq 0\). In view of the relation (54), the solution \(k^-(s', t')\) has the following representation,

\[
k^-(s', t') = h^-\infty + (d^-_1 - d^-_2, \vartheta^-_1 - \vartheta^-_2) + \left(1 - \frac{\beta_a(-s')}{\gamma_a(-s')}\right) \cdot a v_a(r^+, r^-) + \frac{\beta_a(-s')}{\gamma_a(-s')} \cdot r^+(s' + R, t' + \vartheta) + \frac{\beta_a(-s')^2}{\gamma_a(-s')} \cdot r^-(s', t').
\]

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The asymptotic constant \( k_\infty^- = \lim_{s' \to -\infty} k^-(s', t') \) of the solution \( k^- \) is equal to
\[
k^-_\infty = h^-\infty + (d^-_1 - d^-_2 \vartheta^-_1 - \vartheta^-_2) + av_a(r^+, r^-). \tag{67}
\]

From the expressions (65) and (67) one deduces using the Lemmata 2.18 and 2.19 that the asymptotic constants \( k^\pm_\infty \) depend sc-smoothly on \( (a, h^+, h^-) \).

To sum up our computations, we represent the chart transformation \( \Phi = \phi_2^{-1} \circ \phi_1 \) by the formula
\[
\Phi(a, h^+, h^-) = \begin{cases} 
(0, h^+ + (u^+_1 - u^+_2), h^- + (u^-_1 - u^-_2)) & \text{if } a = 0 \\
(0, k^+(a, h^+, h^-), k^-(a, h^+, h^-)) & \text{if } a \neq 0
\end{cases}
\]
where, for \( a \neq 0 \), the maps \( k^+(a, h^+, h^-) = q^+ \) and \( k^-(a, h^+, h^-) = q^- + k^-_\infty - k^+_\infty \) are defined by (64) and (66). We define the maps \( k^\pm \) at \( a = 0 \) by
\[
k^+(0, h^+, h^-) := h^+ + (u^+_1 - u^+_2)
\]
\[
k^-(0, h^+, h^-) := h^- + (u^-_1 - u^-_2),
\]
and observe that
\[
h^+ + (u^+_1 - u^+_2) = h^+_\infty + r^+_3 + (d^+_1 - d^+_2 \vartheta^+_1 - \vartheta^+_2) + (r^+_1 - r^+_2)
\]
\[
= h^+_\infty + (d^+_1 - d^+_2 \vartheta^+_1 - \vartheta^+_2) + r^+_3
\]
and
\[
h^- + (u^-_1 - u^-_2) = h^-\infty + (d^-_1 - d^-_2 \vartheta^-_1 - \vartheta^-_2) + r^-_3
\]
where as before \( r^\pm = r^+_3 + r^-_3 - r^\pm_2 \).

Checking every term in (64) and in (66), applying Proposition 2.28 and the Lemmata 2.18 2.21, one sees that the maps \( (a, h^+, h^-) \mapsto k^\pm(a, h^+, h^-) \) are sc-smooth in a neighborhood of \( a = 0 \).

Finally, in view of Lemma 3.7 also the map \( (a, h^+, h^-) \mapsto b(a, u^+_0 + h^+, u^-_0 + h^-) \) is sc-smooth.

The proof of the sc-smoothness of the chart transformations is complete.  

As a consequence we obtain the following result.

**Proposition 3.24.** The topological space \( \mathbf{X} \) has in a natural way the structure of an M-polyfold which induces on \( \mathbf{X} \) the previously defined M-polyfold structure.

With Proposition 3.24 the proof of Theorem 1.43 is complete.
3.3 A Strong Bundle, Proof of Theorem 1.44

Continuing with the illustration of the polyfold theory, we are going to construct a strong bundle over the M-polyfold $V \times \overline{X}$.

In order to define the Cauchy-Riemann operator as an sc-smooth Fredholm section we shall first equip the cylinders $Z_a$ with complex structures.

We assume that $\overline{X}$ has the M-polyfold structure defined above, using the gluing profile $\varphi(r) = e^{\frac{r}{2}} - e$ and the sequence $(\delta_m)$ of weights in the open interval $(0, 2\pi)$. We choose a smooth family

$$v \mapsto j^\pm(v)$$

of complex structures on the half cylinders $\mathbb{R}^\pm \times S^1$ parametrized by $v$ belonging to an open neighborhood $V$ of 0 in some finite dimensional vector space. We assume that $j^\pm(v) = i$ is the standard complex structure outside of a compact neighborhood of the boundaries $\partial(\mathbb{R}^\pm \times S^1)$. In order to arrange that the gluing of the half-cylinders $\mathbb{R}^\pm \times S^1$ takes place in a region where the complex structures $j^\pm(v)$ are the standard structures we shall not glue the half-cylinders along the pieces $[0, R] \times S^1$ resp. $[-R, 0] \times S^1$ as we did so far but along much shorter pieces and obtain the new finite cylinders $Z_a$ and the new infinite cylinders $C_a$, which we denote by the same letters because they are biholomorphically equivalent to the old cylinders we have considered so far.

We assume that $j^+(v) = i$ for $s \geq \frac{1}{2}s_0$ and $j^-(v) = i$ for $s \leq -\frac{1}{2}s_0$ and choose the gluing parameter $a$ so small that $R = \varphi(|a|)$ satisfies $R - s_0 > s_0$. We then identify the points $(s, t) \in [s_0, R - s_0] \times S^1$ of the cylinder $\mathbb{R}^+ \times S^1$ with the points $(s', t') \in [-R + s_0, -s_0] \times S^1$ of the negative cylinder $\mathbb{R}^- \times S^1$ if

$$s' = s - R$$
$$t' = t - \varphi,$$

as illustrated in Figure 5. We have to keep in mind that $Z_a$ possesses the distinguished points $p_a^\pm$.

Using the same identification we also redefine the infinite cylinders $C_a$ as illustrated in Figure 6.

The complex structures $j^\pm(v)$ induce the complex structures $j(a, v)$ on the glued finite cylinders $Z_a$ for sufficiently small gluing parameters $a$. We equip the glued infinite cylinder $C_a$ with the standard complex structure.
Figure 5: Glued finite cylinders $Z_a$

Figure 6: Glued infinite cylinders $C_a$
denoted by \( i \). It is clearly biholomorphic to the standard complex cylinder \( \mathbb{R} \times S^1 \).

With these new cylinders \( Z_a \) and \( C_a \), the gluing formula \( \oplus_a \) and the anti-gluing formula \( \ominus_a \) for the maps on the cylinders remain unchanged and all the definitions and corresponding results proved so far hold true also for the new cylinders with the identical proofs.

Clearly with \( X \), also \( V \times X \) is an M-polyfold. If the point \((v, a, b, w) \in V \times X\) satisfies \( a \neq 0 \) and \( b \neq 0 \), then \( w : Z_a \to Z_b \) is a diffeomorphism and associated with this point we consider mappings \( \xi \) which are defined on the cylinder \( Z_a \) and whose images

\[
\xi(z) : (T_z Z_a, j(a, v)) \to (T_{w(z)} Z_b, i)
\]

are complex anti-linear mappings belonging to the Sobolev space \( H^2 \). The complex anti-linearity requires that

\[
\xi(z) \circ j(a, v) = -i \circ \xi(z)
\]

for all \( z \in Z_a \). In the following we identify the tangent spaces of the cylinders \( Z_a \) and \( \mathbb{R}^+ \times S^1 \) with \( \mathbb{R}^2 \).

If \( a = 0 \) (and consequently \( b = 0 \)), then \( Z_0 \) is the disjoint union

\[
(\mathbb{R}^+ \times S^1) \coprod (\mathbb{R}^- \times S^1)
\]

and recalling the two diffeomorphisms \( u^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^\pm \times S^1 \), we are familiar with from the previous section, we associate with \( Z_0 \) two maps \( z \mapsto \xi^\pm(z) \) defined on the half cylinders \( \mathbb{R}^\pm \times S^1 \) whose images

\[
\xi^\pm(z) : (T_z \mathbb{R}^\pm \times S^1, j^\pm(v)) \to (T_{u^\pm(z)} \mathbb{R}^\pm \times S^1, i)
\]

are complex anti-linear and belong to \( H^{2, \delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^4) \).

The collection \( E \) of all multiplets \((v, a, b, w, \xi)\) in which \((v, a, b, w) \in V \times X\) and \( \xi \) is the associated complex anti-linear map, possesses the projection map

\[
E \to V \times X, \quad (v, a, b, w, \xi) \mapsto (v, a, b, w)
\]

where fibers (containing \( \xi \)) have in a natural way the structure of a Hilbert space.

On \( E \) we introduce the double-filtration \((E_{m,k})\) whose indices run over \( m \geq 0 \) and \( 0 \leq k \leq m + 1 \), as explained in Section 1.4. If \( ab \neq 0 \), then
an element \((v, a, b, w, \xi)\) belongs to \(E_{m,k}\) if \((v, a, b, w) \in V \times X_m\), where \(w\) belongs to the level \(m\) if it has the Sobolev regularity \((m + 3, \delta_m)\), and if \(\xi\) belongs to the class \((k + 2, \delta_k)\).

This subsection 3.3 is devoted to the proof of the following theorem announced in the introduction.

**Theorem 3.25.** Having fixed the exponential gluing profile \(\varphi\) and the increasing sequence \((\delta_m)_{m \in \mathbb{N}_0}\) of real numbers satisfying \(0 < \delta_m < 2\pi\), the set \(E\) admits in a natural way the structure of a strong bundle over the M-polyfold \(V \times X\).

**Proof.** In order to prove the theorem we have to define strong bundle charts. We first construct charts on \(E_{|V \oplus X}\). To do so, we fix a smooth point \((v_0, a_0, b_0, w_0) \in V \times X\) so that \(w_0 : Z_{a_0} \to Z_{b_0}\) is a smooth diffeomorphism preserving the marked points. We recall that a chart of \(X\) around \((v_0, a_0, b_0, w_0) \in V \times X\) has been previously constructed by the map \((a, b, h) \mapsto (a, b, \psi_b \circ (w_0 + h) \circ \phi_a)\) for \((a, b)\) close to \((a_0, b_0)\) and \(h \in H^3(Z_{a_0}, \mathbb{R}^2)\) satisfying \(h(p^\pm_{a_0}) = (0, 0)\) and \(h([0, t]) \in \{0\} \times \mathbb{R}\) and \(h([0, t]) \in \{0\} \times \mathbb{R}\). Moreover, the derivative \(|Dh([s, t])| < \varepsilon_0\) is so small that \(w_0 + h\) is still a diffeomorphism \(Z_{a_0} \to Z_{b_0}\). Then \(\psi_b \circ (w_0 + h) \circ \phi_a\) is a diffeomorphism \(Z_a \to Z_b\).

Given now a point \((v, a, b, \psi_b \circ (w_0 + h) \circ \phi_a) \in V \times X\), there is a one-to-one correspondence between complex anti-linear maps \(\xi(z) : (T_z Z_a, j(a, v)) \to (T_{\psi_b \circ (w_0 + h) \circ \psi_a}(z) Z_b, i)\) for \(z \in Z_a\), and elements of the Hilbert space \(H^2(Z_{a_0}, \mathbb{R}^2)\) on the fixed cylinder \(Z_{a_0}\), defined by the relation

\[
\eta(\phi_a(z)) = \xi(z) \cdot \frac{\partial}{\partial s}. \tag{68}
\]

This follows from the complex anti-linearity of \(\xi(z)\). Recall that \(\phi_a\) maps the cylinder \(Z_a\) diffeomorphically onto the cylinder \(Z_{a_0}\). The chart of \(E_{|V \oplus X}\) around the point \((v_0, a_0, b_0, w_0)\) is now defined by the map \(\Phi : (v, a, b, h, \eta) \mapsto (v, a, b, \psi_b \circ (w_0 + h) \circ \phi_a, \xi) \in E_{|V \oplus X}\) where \(\eta\) and \(\xi\) are related by \(\text{(68)}\). If

\[
\tilde{\Phi} : (\tilde{v}, \tilde{a}, \tilde{b}, \tilde{h}, \tilde{\eta}) \mapsto (\tilde{v}, \tilde{a}, \tilde{b}, \tilde{\psi}_b \circ (\tilde{w}_0 + \tilde{h}) \circ \tilde{\phi}_a, \tilde{\xi})
\]

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is a second such chart, then we have in the overlapping region $\tilde{v} = v$, $\tilde{a} = a$, and $\tilde{b} = b$, and hence $\tilde{\psi}_b \circ (\tilde{w}_0 + \tilde{h}) \circ \tilde{\phi}_a = \psi_b \circ (w_0 + h) \circ \phi_a$ and $\tilde{\xi} = \xi$ so that the formula for the chart transformation is as follows,

$$\tilde{\Phi}^{-1} \circ \Phi(v, a, b, h, \eta) = (v, a, b, \tilde{h}, \tilde{\eta})$$

where

$$\tilde{\eta} = \tilde{\psi}_b^{-1} \circ \psi_b \circ (\tilde{w}_0 + \tilde{h}) \circ \phi_a \circ \phi_a^{-1} - \tilde{w}_0$$

and

$$\tilde{\eta}(z) = \eta(\phi_a \circ \phi_a^{-1}(z))$$

for all $z \in Z_{\alpha_0}$.

Recalling from Section 1.4 the definition of an sc-smooth bundle map, we see from the above formulae that the chart transformation is an sc-smooth bundle map, in view of our results about the action by diffeomorphisms (Theorem 2.6).

We need to define strong bundle charts also for $E \to V \times \overline{X}$. To do so we first construct the local models for the bundle charts.

We take the collection $K'$ of all tuples $(v, a, h^+, h^-, \eta^+, \eta^-)$ in which $v \in V$ and $|a| < \varepsilon$, moreover, $(h^+, h^-) \in \hat{E}_0$ and $\eta^\pm \in H^{2,\delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^2) = F$. They satisfy the relation

$$\rho_a(h^+, h^-) = (h^+, h^-) \quad \text{and} \quad \hat{\pi}_a(\eta^+, \eta^-) = (\eta^+, \eta^-).$$

The projection $\rho_a : \hat{E}_0 \to \hat{E}_0$ has been introduced in Section 3.1 and the projection $\hat{\pi}_a : F \to F$ in Section 1.3. The Hilbert space $F$ is equipped with the sc-smooth structure $(F_m)$ defined by $F_m = H^{2+m,\delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^2)$.

By $O'$ we denote the collection of all tuples $(v, a, h^+, h^-, \eta^+, \eta^-)$ satisfying $v \in V$, $|a| < \varepsilon$, and $\rho_a(h^+, h^-) = (h^+, h^-)$. The natural projection

$$K' \to O', \quad (v, a, h^+, h^-, \eta^+, \eta^-) \mapsto (v, a, h^+, h^-)$$

defines a local model for a strong bundle. Indeed, if we define the map

$$R : (V \oplus B_{\varepsilon} \oplus \hat{E}_0) \circ F \to (V \oplus B_{\varepsilon} \oplus \hat{E}_0) \circ F$$

as

$$R(v, a, (h^+, h^-), (\eta^+, \eta^-)) = (v, a, \rho_a(h^+, h^-), \hat{\pi}_a(\eta^+, \eta^-)),$$
then the map $R$ is an sc-$\sigma$-smooth strong bundle retraction satisfying

$$K' = R((V \oplus B_\varepsilon \oplus \hat{E}_0) \circ \Phi)$$

in view of Theorem 1.27 and Theorem 1.29. Moreover, the map $R$ covers the retraction map $r : V \oplus B_\varepsilon \oplus \hat{E}_0 \rightarrow V \oplus B_\varepsilon \oplus \hat{E}_0$ defined by $r(v, a, (h^+, h^-)) = (v, a, \rho_a(h^+, h^-))$ whose image is the set

$$O' = r(V \oplus B_\varepsilon \oplus \hat{E}_0).$$

We next use these local models to define an atlas of charts of the bundle $E \rightarrow V \times \overline{X}$.

Recall that $E$ is the collection of tuples $(v, a, b, w, \xi)$ in which for $ab \neq 0$ the map $w : Z_a \rightarrow Z_b$ is a diffeomorphism and $\xi(z) : (T_z Z_a, j(a, v)) \rightarrow (T_{w(z)} Z_b, i)$ a complex anti-linear map defined for every $z \in Z_a$. If $a = 0$ and hence $b = 0$, the point in $E$ is defined as

$$(v, 0, 0, (u^+, u^-), (\xi^+, \xi^-))$$

where $u^\pm : \mathbb{R}^\pm \times S^1 \rightarrow \mathbb{R}^\pm \times S^1$ are two diffeomorphisms of the half cylinders satisfying $u^\pm(0, 0) = (0, 0)$. Moreover, $\xi^\pm(z) : (T_z(\mathbb{R}^\pm \times S^1), j^\pm(v)) \rightarrow (T_{u^\pm(z)}(\mathbb{R}^\pm \times S^1), i)$ are complex anti-linear maps defined for every $z \in \mathbb{R}^\pm \times S^1$.

In order to define strong bundle charts for $E \rightarrow V \times \overline{X}$ we start with the chart maps $\Psi : O \rightarrow V \oplus \overline{X}$ defined by

$$\Psi(v, a, h^+, h^-) = (v, a, b(a, u_0^+ + h^+, u_0^- + h^-), \bigoplus_a(u_0^+ + h^+, u_0^- + h^-)),$$

around a base pair $(u_0^+, u_0^-)$, if $a \neq 0$, and by

$$\Psi(v, 0, h^+, h^-) = (v, 0, 0, u_0^+ + h^+, u_0^- + h^-)$$

if $a = 0$. The domain of definition of the map $\Psi$ is the set

$$O =$$

$$\{(v, a, h^+, h^-) | \rho_a(h^+, h^-) = (h^+, h^-), v \in V, |a| < \sigma_0, \text{ and } (h^+, h^-) \in U\}$$

where $U \subset \hat{E}_0$ is an open neighborhood of $(0, 0) \in \hat{E}_0$. Now we observe that $O \subset O'$ is an open subset and define $K \subset K'$ as the subset of $K'$ lying above $O$. We shall show that $K \rightarrow O$ is a local model for the bundle $E \rightarrow V \oplus \overline{X}$. 126
We define the bundle chart \( \Gamma : K \to E \) which covers the chart \( \Psi \) of \( V \oplus X \) by the following map. If \( a \neq 0 \), we set

\[
\Gamma(v, a, h^+, h^-, \eta^+, \eta^-) = (\Psi(v, a, h^+, h^-), \xi) = (v, a, b(a, u_0^+ + h^+, u_0^- + h^-), \Xi_a(u_0^+ + h^+, u_0^- + h^-), \xi)
\]

where \((h^+, h^-) \in \hat{E}_0 \) satisfies \( \rho_a(h^+, h^-) = (h^+, h^-) \) and \((\eta^+, \eta^-) \in F \) satisfies \( \tilde{\pi}_a(\eta^+, \eta^-) = (\eta^+, \eta^-) \). Moreover, abbreviating the diffeomorphism \( w_a = \Xi_a(u_0^+ + h^+, u_0^- + h^-) : Z_a \to Z_b \), the fiber part \( \xi \) is the complex anti-linear map

\[
\xi(z) : (T_zZ_a, j(v, a)) \to (T_{w(z)}E, i)
\]

defined by

\[
\xi(z) \frac{\partial}{\partial s} = \tilde{\pi}_a(\eta^+, \eta^-)(z), \quad z \in Z_a.
\]

We recall that \( \tilde{\pi}_a(\eta^+, \eta^-)(s, t) = \beta_a(s) \cdot \eta^+(s, t) + (1 - \beta_a(s)) \cdot \eta^-(s - R, t - \vartheta) \) for \((s, t) \in [0, R] \times S^1 \) and \( \tilde{\pi}_a(\eta^+, \eta^-)(s, t) = -(1 - \beta_a(s)) \cdot \eta^+(s, t) + \beta_a(s) \cdot \eta^-(s - R, t - \vartheta) \) for \((s, t) \in \mathbb{R} \times S^1 \).

If \( a = 0 \), the map \( \Gamma : K \to E \) is defined as

\[
\Gamma(v, 0, h^+, h^-, \eta^+, \eta^-) = (\Psi(v, 0, u_0^+ + h^+, u_0^- + h^-), (\xi^+, \xi^-)) = (v, 0, 0, u_0^+ + h^+, u_0^- + h^-; (\xi^+, \xi^-))
\]

where the complex anti-linear maps

\[
\xi^\pm(z) : (T_z(\mathbb{R}^\pm \times S^1), j^\pm(v)) \to (T_{(u_0^\pm + h^\pm)}(\mathbb{R}^\pm \times S^1), i)
\]

are defined as

\[
\xi^\pm(z) \frac{\partial}{\partial s} = \eta^\pm(z)
\]

for all \( z = (s, t) \in \mathbb{R}^\pm \times S^1 \). It is easy to see that these charts are \( sc_a \)-smoothly equivalent. Namely, abbreviating two chart maps by \( \Gamma(v, a, h^+, h^-, \eta^+, \eta^-) = (\Psi(v, a, h^+, h^-), \xi) \) and \( \Gamma(v, a, h^+, h^-, \eta^+, \eta^-) = (\tilde{\Psi}(v, a, h^+, h^-), \tilde{\xi}) \) the chart transformation is given by

\[
\tilde{\Gamma}^{-1} \circ \Gamma(v, a, h^+, h^-, \eta^+, \eta^-) = (\tilde{\Psi}^{-1} \circ \Psi(v, a, h^+, h^-), \eta^+, \eta^-),
\]

if \( a \neq 0 \), and by

\[
\tilde{\Gamma}^{-1} \circ \Gamma(v, 0, h^+, h^-, \eta^+, \eta^-) = (v, 0, (u_0^+ - \bar{u}_0^+) + h^+, (u_0^- - \bar{u}_0^-) + h^-, \eta^+, \eta^-),
\]

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if $a = 0$. From the previous considerations we know that $\tilde{\Psi}^{-1} \circ \Psi$ is an $sc$-smooth diffeomorphism. In the fibers the transformation is the identity. Indeed, at a point of intersection we have $\xi = \tilde{\xi}$. It follows that $\tilde{\oplus}_a(\eta^+, \eta^-)(z) = \tilde{\oplus}_a(\eta^+, \eta^-)(z)$ for all $z \in Z_a$. Since, by definition, $\tilde{\oplus}_a(\eta^+, \eta^-) = 0$ and $\tilde{\oplus}_a(\eta^+, \eta^-) = 0$, we conclude from the uniqueness of the solutions of the system of two equations that $\eta^+ = \tilde{\eta}^+$ and $\eta^- = \tilde{\eta}^-$. We have proved that $\tilde{\Gamma}^{-1} \circ \Gamma$ is an $sc_\Delta$-bundle isomorphism between the local models of a strong bundle.

It remains to prove that the chart maps $\Gamma$ and $\Phi$ are $sc_\Delta$-smoothly equivalent. We assume that we are given two chart maps into $E$. The first one is

$$
\Phi(v, a, b, h, \eta) = (v, a, b, \psi_b \circ (w_0 + h) \circ \phi_a, \xi),
$$

defined for $(v, a, b)$ close to $(v_0, a_0, b_0)$ and $\eta$ and $\xi$ are related via $\xi(z) \frac{\partial}{\partial s} = \eta(\varphi_a(z))$. The second chart map is

$$
\Gamma(\tilde{v}, \tilde{a}, h^+, h^-, \eta^+, \eta^-) = (\tilde{v}, \tilde{a}, b(\tilde{a}, u_0^+ + h^+, u_0^- + h^-), \tilde{\oplus}_a(u_0^+ + h^+, u_0^- + h^-), \tilde{\xi})
$$

defined for $(\tilde{v}, \tilde{a})$ close to $(\tilde{v}_0, \tilde{a}_0)$ and where $\tilde{\xi}(z) : T_zZ_{\tilde{a}} \to T_wZ_{w'}$ is a complex anti-linear map uniquely defined by

$$
\tilde{\xi}(z) \frac{\partial}{\partial s} = \tilde{\oplus}_a(\eta^+, \eta^-)(z).
$$

Here $w = \oplus_a(u_0^+ + h^+, u_0^- + h^-)$, $b' = b(\tilde{a}, u_0^+ + h^+, u_0^- + h^-)$, and $z = (s, t)$. From $\Phi(v, a, b, h, \eta) = \Gamma(\tilde{v}, \tilde{a}, h^+, h^-, \eta^+, \eta^-)$ one concludes that $a = \tilde{a}$ and $b = b(\tilde{a}, u_0^+ + h^+, u_0^- + h^-)$ and, moreover,

$$
\psi_b \circ (w_0 + h) \circ \phi_a = \oplus_a(u_0^+ + h^+, u_0^- + h^-) \quad (69)
$$

and

$$
\xi = \tilde{\xi} \quad (70)
$$

The first equation can be solved for $h$ as map of $(a, h^+, h^-)$ resulting in

$$
h = \psi_b^{-1} \circ \oplus_a(u_0^+ + h^+, u_0^- + h^-) \circ \phi_a^{-1} - w_0
$$

where $b = b(a, u_0^+ + h^+, u_0^- + h^-)$. From the second equation $\xi = \tilde{\xi}$ we conclude that $\eta(\phi_a(z)) = \tilde{\oplus}_a(\eta^+, \eta^-)(z)$ where $z = [s, t] \in Z_a$. Consequently, the transition map $\Phi^{-1} \circ \Gamma$ has the following form

$$
\Phi^{-1} \circ \Gamma(v, a, h^+, h^-, \eta^+, \eta^-) = (v, a, h, \eta)
$$

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where
\[ b = b(a, u_0^+ + h^+, u_0^- + h^-) \]
\[ h = \psi_b^{-1} \circ \boxplus_a(u_0^+ + h^+, u_0^- + h^-) \circ \phi_a^{-1} - w_0 \]
\[ \eta(z) = \boxplus_a(\eta^+, \eta^-)(\phi_a^{-1}(z)), \quad z \in Z_{a0}. \]

For the transition map \( \Gamma \circ \Phi^{-1} \) we obtain
\[ \Gamma^{-1} \circ \Phi(v, a, b, h, \eta) = (v, a, h^+, h^-, \eta^+, \eta^-) \]
where the pair \((h^+, h^-)\) is obtained by solving \((69)\) for \((h^+, h^-)\) in terms of \((a, h)\). The formulae for \(h^\pm\) are given in the proof of Proposition 3.10. Namely,
\[ h^+ = q^+ \quad \text{and} \quad h^- = q^- + (h^- - h^+_{\infty}) \]
where
\[ q^+(s, t) = [g] + \frac{\beta_a(s)}{\gamma_a(s)} \cdot (g(s, t) - [g]) \]
\[ q^-(s - R, t - \vartheta) = [g] + \frac{1 - \beta_a(s)}{\gamma_a(s)} \cdot (g(s, t) - [g]) \]
and
\[ g = \psi_b \circ (w_0 + h) \circ \phi_a - w_a, \quad w_a = \boxplus_a(u_0^+, u_0^-). \]
The pair \((\eta^+, \eta^-)\) is obtained by solving the system
\[ \hat{\oplus}_a(\eta^+, \eta^-) = \eta \circ \phi_a \]
\[ \hat{\oplus}_a(\eta^+, \eta^-) = 0. \]
We have used that \( \hat{\pi}_a(\eta^+, \eta^-) = (\eta^+, \eta^-) \) is equivalent to \( \hat{\oplus}_a(\eta^+, \eta^-) = 0. \)
Solving the above system leads to the formulae
\[ \eta^+(s, t) = \frac{\beta_a(s)}{\gamma_a(s)} \cdot \eta \circ \phi_a(s, t), \quad s \geq 0 \]
\[ \eta^-(s - R, t - \vartheta) = \frac{1 - \beta_a(s)}{\gamma_a(s)} \cdot \eta \circ \phi_a(s - R, t - \vartheta), \quad s \leq R. \]

From the formulae for \( \Phi^{-1} \circ \Gamma \) and \( \Gamma^{-1} \circ \Phi \) we conclude, using Theorem 2.23 Proposition 2.8 and the Lemma 2.18--2.21, arguing as in the proof of Proposition 3.23, that the chart transformations are sc\(_c\)-smooth bundle isomorphisms.

Consequently, the bundle \( E \to V \oplus X \) admits the structure of a strong bundle over the M-polyfold \( V \oplus X \) and the proof of Theorem 1.44 is complete. \( \blacksquare \)
3.4 The Cauchy-Riemann Operator

We define the Cauchy-Riemann section $\overline{\mathcal{J}}$ of the M-polyfold bundle $E \to V \times X$ by

$$\overline{\mathcal{J}}(v, a, b, w) = (v, a, b, w; \overline{\partial}_{i,j(a,v)}w)$$

if $ab \neq 0$. Here $w : Z_a \to Z_b$ is a diffeomorphism, and the complex anti-linear map $\overline{\partial}_{i,j(a,v)}w(z) : (T_z Z_a, j(a, v)) \to (T_{w(z)} Z_b, i)$ is defined by

$$\overline{\partial}_{i,j(a,v)}w(z) := \frac{1}{2}[Tw + i \circ (Tw) \circ j(a, v)](z), \quad z \in Z_a.$$

If $a = 0$ and hence $b = 0$, the section $\overline{\mathcal{J}}$ is defined by

$$\overline{\mathcal{J}}(v, 0, 0, (u^+, u^-)) = (v, 0, 0, (u^+, u^-); (\overline{\partial}_{i,j(v)}u^+, \overline{\partial}_{i,j(-v)}u^-)).$$

Our aim is to prove that the section $\overline{\mathcal{J}}$ is a polyfold Fredholm section.

**Proposition 3.26.** The Cauchy-Riemann section $\overline{\mathcal{J}}$ is sc-smooth and regularizing.

**Proof.** In order to verify the regularizing property we assume that

$$\overline{\mathcal{J}}(v, a, b, w) \in E_{m,m+1}.$$

It then follows from elliptic regularity theory that $w \in H^{m+4}_{loc}$. Hence, if $a \neq 0$ we conclude that $(v, a, b, w) \in X_{m+1}$. Considering now the case $a = 0$ in which case also $b = 0$, we have $u = (u^+, u^-)$ with the two diffeomorphisms $u^\pm$ of $\mathbb{R}^+ \times S^1$ of the form $u^\pm(s, t) = (s, t) + (d^\pm, \vartheta^\pm) + r^\pm(s, t)$, where, by the assumption $r^\pm \in H^{3+m, \delta_m}(\mathbb{R}^+ \times S^1, \mathbb{R}^2)$. On $\mathbb{R}^+ \times S^1$, the complex structure is equal to $j(a, v)(s, t) = i$ if $s \geq s_0$ and hence we conclude from the assumption, that

$$\partial_\tau r^+ + i\partial_s r^+ \in H^{3+m, \delta_m+1}([s_0, \infty) \times S^1, \mathbb{R}^2).$$

Now we recall that the weights $\delta_m$ have been chosen in the open interval $(0, 2\pi)$. Since the asymptotic structure of the differential equation is $\frac{\partial}{\partial s} u = A u$, with the self-adjoint operator $A = -i\frac{d}{dt}$ in the Hilbert space $L^2(S^1)$ having the spectrum $2\pi \mathbb{Z}$, we therefore conclude by the standard arguments going back to Lockhardt and McOwen [15] that $r^+ \in H^{4+m, \delta_m+1}([s_0, \infty) \times S^1, \mathbb{R}^2)$.
By the same arguments, \( r^- \in H^{4+m,\delta_{m+1}}((-\infty, -s_0] \times S^1, \mathbb{R}^2) \) and hence \((u^+, u^-) \in X_{m+1}\).

In order to verify that the section \( \partial \) is sc-smooth, we have to study its coordinate representation. We recall the strong bundle chart \( \Gamma : K \to E \),
\[
\Gamma(v, a, h^+, h^-, \eta^+, \eta^-) = (v, a, b(a, u_0^++h^+, u_0^-+h^-), \Theta_a(u_0^++h^+, u_0^-+h^-); \xi)
\]
where, abbreviating the diffeomorphism \( w_a = \Theta_a(u_0^++h^+, u_0^-+h^-) : Z_a \to Z_b \), the complex anti-linear map \( \xi(z) : (T_2Z_a, j(a, v)) \to (T_{wa(z)}Z_b, i) \) is defined by \( \xi(z) = \Theta_a(\eta^+, \eta^-)(z) \) for \( z \in Z_a \).

Consequently, the Cauchy-Riemann section \( \partial \) is, in these local coordinates, represented by the map
\[
(v, a, h^+, h^-) \mapsto (v, a, h^+, h^-; \eta^+, \eta^-),
\]
where \((\eta^+, \eta^-)\) is the unique solution of the system
\[
\Theta_a(\eta^+, \eta^-) = (\partial_{\eta}j_{(a,v)}w_a) \frac{\partial}{\partial s},
\]
\[
\Theta_a(\eta^+, \eta^-) = 0.
\]
The solution \( \eta^+ \) is equal to
\[
\eta^+(s, t) = \frac{\beta_a}{\gamma_a} \Theta_a(\eta^+, \eta^-) \left( \frac{\partial}{\partial s} \right),
\]
where, as usual, we abbreviate \( \beta_a = \beta_a(s) = \beta(s - \frac{R}{2}) \) and \( \gamma_a = \beta_a^2 + (1 - \beta_a)^2 \).

Recalling that \( u_0^\pm(s, t) = (s, t) + (d_0^\pm, \vartheta_0^\pm) + r_0^\pm \) and representing \( h_\infty^\pm = h_\infty^\pm + r_1^\pm \) where \( r_0^\pm \) and \( r_1^\pm \) belong to \( H^{3,\delta_0}(\mathbb{R}^2 \times S^1, \mathbb{R}^2) \), the map \( w_a = \Theta_a(u_0^+ + h^+, u_0^- + h^-) : Z_a \to Z_b \) is equal to
\[
w_a(s, t) = (s, t) + (d_0^+, \vartheta_0^+) + h_\infty^+
+ \beta_a \cdot r^+(s, t) + (1 - \beta_a) \cdot r^-(s - R, t - \vartheta).
\]
for \( 0 \leq s \leq R \), where \( r^\\pm := r_0^\pm + r_1^\pm \). Observing that on \( Z_a \), in the coordinates \( s \geq 0 \), the complex structure is equal to \( j(a, v)(s, t) = j^+(v) \) while in the coordinates \( s' \leq 0 \) we have \( j(a, v)(s', t') = j^-(v) \), one obtains for the solution

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\[ \eta^+(s, t) \text{ for } (s, t) \in \mathbb{R}^+ \times S^1, \]
\[ \eta^+(s, t) = \frac{\beta_a}{\gamma_a} (\partial_{i,j}^+(v) \text{id}) \left( \frac{\partial}{\partial s} \right) (s, t) \]
\[ + \frac{\beta^2_a}{\gamma_a} \left[ \partial_{i,j}^+(v) r^+(s, t) \right] \left( \frac{\partial}{\partial s} \right) \]
\[ + \frac{\beta_a(1 - \beta_a)}{\gamma_a} \left[ \partial_{i,j}^-(v) r^-(s - R, t - \vartheta) \right] \left( \frac{\partial}{\partial s} \right) \]
\[ + \frac{\beta_a \beta'_a}{\gamma_a} [r^+(s, t) - r^-(s - R, t - \vartheta)]. \]

A similar formula holds for \( \eta^-(s', t') \) on \( \mathbb{R}^- \times S^1 \). Recalling that the complex structure \( j^+(v) \) is equal to the standard complex structure \( i \) for \( s \geq s_0 \), we see that \( (\partial_{i,j}^+(v) \text{id}) \left( \frac{\partial}{\partial s} \right) (s, t) = 0 \) for \( s \geq s_0 \). Recall that \( \frac{\beta_a(s)}{\gamma_a(s)} = 1 \) if \( s \leq \frac{R}{2} - 1 \). If \( |a| \) is so small that \( s_0 \leq \frac{R}{2} - 1 \), then for all \( (s, t) \in \mathbb{R}^+ \times S^1 \), the function
\[ \frac{\beta_a}{\gamma_a} (\partial_{i,j}^+(v) \text{id}) \left( \frac{\partial}{\partial s} \right) (s, t) = (\partial_{i,j}^+(v) \text{id}) \left( \frac{\partial}{\partial s} \right) (s, t) \]
has a compact support, hence belongs to \( H^{2,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^2) \) and is independent of \( a \). Hence as a function of \( a \) into \( H^{2,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^2) \), the map is constant. Applying the chain rule, Proposition 2.8, and the fact that the operators \( \partial_s, \partial_t : H^{3,\delta_0} \to H^{2,\delta_0} \) are sc-linear, one concludes that the remaining terms in the formula for \( \eta^+ \) define maps which depend sc-smoothly on \( (v, a, h^+, h^-) \). Consequently, the maps \( (v, a, h^+, h^-) \mapsto \eta^+(v, a, h^+, h^-) \) are sc-smooth and we have proved that the section \( \overline{J} \) is sc-smooth. \hfill \blacksquare

In the next step we shall prove that \( \overline{J} \) is a polyfold Fredholm section. By definition this means that around every smooth point \( (v, (a, b, w)) \in V \oplus X \) the Cauchy-Riemann operator has a germ which can be brought into a ‘nice’ form by a suitable coordinate change.

We consider two cases. The easy case concerns points in \( X \) where an open neighborhood is sc-diffeomorphic to an open subset of an sc-Hilbert space. The interesting case concerns points in \( X \setminus X \) whose local description is that of a nontrivial retract. Here the concept of a filler will play a decisive role.

The following lemma takes care of the easy case.
Lemma 3.27. Around a smooth point \((v, x) \in V \times X\) the germ \((\overline{\partial}, (v, x))\) of the Cauchy-Riemann section is an sc-Fredholm germ (in the sense of Section 1.4).

Proof. We choose the smooth point \((v_0, a_0, b_0, w_0) \in V \times X\). Then \(w_0 : Z_{a_0} \to Z_{b_0}\) is a diffeomorphism preserving the distinguished points at the boundaries. Around this point we choose a chart of \(V \times X\) as in Theorem 3.25 by means of the map 

\[
\Phi(v, a, b, h) = (v, a, b, \psi_b \circ (w_0 + h) \circ \phi_a)
\]

where \((v, a)\) belongs to an open neighborhood of the origin in \(V \times \mathbb{C} \times \mathbb{C}\) and where \(h\) belongs to the subspace \(\tilde{H}^3(Z_{a_0}, \mathbb{R}^2)\) of \(H^3(Z_{a_0}, \mathbb{R}^2)\), which consists of all \(h\) satisfying \(h(p^+_{a_0}) = (0, 0), h([0, t]) \in \{0\} \times \mathbb{R}\), and \(h([0, t']) \in \{0\} \times \mathbb{R}\). Recalling the diffeomorphisms

\[
\psi_b : Z_{b_0} \to Z_b,
\phi_a : Z_a \to Z_{a_0},
\]

the Cauchy-Riemann operator takes the form

\[
(v, a, b, h) \mapsto \overline{\partial}_{i,j(a,v)}(\psi_b \circ (w_0 + h) \circ \phi_a).
\]

On the cylinder \(Z_{b_0}\) we introduce the parameter depending complex structure \(k(b)\), defined by

\[
k(b) := (T\psi_b)^{-1} \circ i \circ (T\psi_b),
\]

and on the cylinder \(Z_{a_0}\) the complex structure

\[
\tilde{j}(a, v) := (T\phi_a) \circ j(a, v) \circ (T\phi_a)^{-1},
\]

so that the Cauchy-Riemann operator can be written as

\[
(v, a, b, h) \mapsto T\psi_b \circ \left(\frac{1}{2} \left[T(w_0 + h) + k(b) \circ T(w_0 + h) \circ \tilde{j}(a, v)\right]\right) \circ T\phi_a.
\]

Now we can introduce the obvious strong bundle coordinates in which the local expression of the Cauchy-Riemann section is as follows,

\[
(v, a, b, h) \mapsto \frac{1}{2}[T(w_0 + h) + k(b) \circ T(w_0 + h) \circ \tilde{j}(a, v)] \cdot \left(\frac{\partial}{\partial s}\right).
\]
As usual we identify the tangent spaces at points of the cylinder $Z_{b_0}$, as real vector spaces, with $\mathbb{R}^2$. Hence the above section associates with $(v, a, b, h)$ a function in $H^2(Z_{a_0}, \mathbb{R}^2)$.

In order to verify the Fredholm property of the Cauchy-Riemann section one needs to show near the smooth point $(v_0, a_0, b_0, w_0)$ the contraction germ property. This requires to study the map (71) for small data $(a, b, h)$.

Observing that for $h = 0$ the right hand side of (71) is an sc-smooth map, we define the sc+ -section $s$ of the strong local bundle by its principal part

$$s(v, a, b, h) = \frac{1}{2} [T w_0 + k(b) \circ T w_0 \circ \tilde{j}(a, v)] \cdot \left( \frac{\partial}{\partial s} \right).$$

Here again we identify the tangent fibers of of the tangent bundle $TZ_{b_0}$ of the cylinder with $\mathbb{R}^2$. Denoting the Cauchy-Riemann section by $f$ we now study the section $f - s$ whose principal part is given by

$$(f - s)(v, a, b, h) = \frac{1}{2} [T(h) + k(b) \circ T(h) \circ \tilde{j}(a, v)] \cdot \left( \frac{\partial}{\partial s} \right). \quad (72)$$

We shall abbreviate the parameters by $\lambda = (v, a, b)$. They vary in the finite dimensional vector space $\Lambda := V \times \mathbb{C} \times \mathbb{C}$ near its origin. Moreover, we abbreviate the sc-spaces $E = \tilde{H}^3(Z_{a_0}, \mathbb{R}^2)$ and $F = H^2(Z_{a_0}, \mathbb{R}^2)$ with the sc-structures $E_m = \tilde{H}^{3+m, \delta_m}(Z_{a_0}, \mathbb{R}^2)$ and $F_m = H^{2+m}(Z_{a_0}, \mathbb{R}^2)$ and denote the right hand side of (72) by $T_\lambda(h)$. Then $T_\lambda : E \to F$ is a family of bounded linear operators which are Fredholm operators $E_m \to F_m$ for all $m$ and depend in their operator norms smoothly on the parameters for all levels $m$. Setting $\lambda = \lambda_0 = (v_0, a_0, b_0)$, the sc-Fredholm operator $T_{\lambda_0} \equiv T_0 : E \to F$ defines the sc-splitings $E = Y \oplus K$ where $K$ is the kernel of $T_0$ and $F = R(T_0) \oplus C$ with the range $R(T_0)$ and the cokernel $C$ of $T_0$. Let

$$P : F = R(T_0) \oplus C \to R(T_0)$$

be the projection map and choose a linear isomorphism $\phi : \mathbb{R}^N \to C$ onto the cokernel of $T_0$. Then the restriction $T_0 | Y : Y \to R(T_0)$ is an sc-isomorphism. Moreover, the linear map

$$P T_0 + \phi : Y \oplus \mathbb{R}^N \to F,$$

defined by

$$(P T_0 + \phi)(y + r) = P T_0(y) + \phi(r)$$

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is an sc-isomorphism. Introducing the sc-bundle isomorphism
\[ \Phi : (\Lambda \oplus E) \triangleright F \to (\Lambda \oplus K \oplus Y) \triangleright (Y \oplus \mathbb{R}^N), \]
defined by \( \Phi(\lambda, k + y, f) = (\lambda, k, y; (PT_0 + \phi)^{-1}f) \) (for small \((\lambda, k + y)\)) and denoting by \( g(\lambda, \delta h) = T_\lambda(\delta h) \) the principal part of the Cauchy-Riemann section, we obtain for the push-forward section
\[ \Phi_* : \Lambda \oplus K \oplus Y \to Y \oplus \mathbb{R}^N \]
the expression \( \Phi_* (g)(\lambda, k, y) = (PT_0 + \phi)^{-1}T_\lambda(k + y) \).

With the projection \( Q : Y \oplus \mathbb{R}^N \to Y \),
the sc-smooth germ \((\lambda, k, y) \mapsto Q\Phi_* (g)(\lambda, k, y)\) satisfies, by construction
\[ Q\Phi_* (g)(0, 0, y) = y \]
and hence is of the form
\[ Q\Phi_* (g)(\lambda, k, y) = y - B(\lambda, k, y) \]
where \( B \) satisfies \( B(0, 0, y) = 0 \) and \( |B(\lambda, k, y) - B(\lambda, k, y')|_m \leq \varepsilon \cdot |y - y'|_m \)
for every \( m \geq 0 \) and every \( \varepsilon > 0 \) if only \((\lambda, k)\) is sufficiently small (depending on \( \varepsilon \) and \( m \)).

Having found coordinates in which the Cauchy-Riemann section near smooth points possesses the contraction germ property, the proof of Lemma 3.27 is complete.

Next we turn to the more interesting case and fix in \( V \times (\overline{X} \setminus X) \) the smooth point \((v_0, 0, 0, (u_0^+, u_0^-))\). We are going to construct a filler for the Cauchy-Riemann section \( \overline{\partial} \) near this point. The construction is based on the Cauchy-Riemann operator
\[ \overline{\partial}_0 : H^3_{e,\delta_0}(C_a, \mathbb{R}^2) \to H^2_{e,\delta_0}(C_a, \mathbb{R}^2) \]
defined by
\[ (\overline{\partial}_0 \xi)(s, t) = \frac{1}{2} \left( \frac{\partial}{\partial s} \xi + i \frac{\partial}{\partial t} \xi \right)(s, t) = \frac{1}{2} [T\xi + i \circ T\xi \circ i](s, t) \left( \frac{\partial}{\partial s} \right) \]
where we have used the coordinates \((s, t) \in \mathbb{R} \times S^1\) for the glued infinite cylinder \( C_a \), which is equipped with the standard complex structure \( i \). We recall
that the Hilbert space \( H^3_\delta_0 \) consists of all maps \( u \) in \( H^3_{\text{loc}}(C_a, \mathbb{R}^2) \) for which there exists a constant \( c \in \mathbb{R}^2 \) having the property that \( u(s, t) - c \) has weak partial derivatives up to order 3 which, if weighted by \( e^{\delta_0|s|} \) belong to the space \( L^2([0, \infty) \times S^1) \) and \( u(s, t) + c \) has the same properties with respect to \( L^2((-\infty, 0] \times S^1) \). Consequently, \( u \) converges at \( \pm \infty \) exponentially fast to the antipodal points \( c \) and \( -c \). We equip \( H^3_\delta_0 \) with the \( \text{sc} \)-structure for which the level \( m \) corresponds to the Sobolev regularity \((m + 3, \delta_m)\) and the \( \text{sc} \)-Hilbert space \( H^2_\delta_0 \) with the \( \text{sc} \)-structure for which the level \( m \) corresponds to the regularity \((2 + m, \delta_m)\). The norms are defined in Section 2.

**Lemma 3.28.** The operator \( \overline{\partial}_0 \) is an \( \text{sc} \)-isomorphism.

Identifying the cylinder with the Riemann sphere with two antipodal points removed, the lemma follows from the results about the Cauchy-Riemann operator acting on maps on the Riemann sphere and discussed for example in [14], and from the asymptotic study of the operator near the ends.

In order to introduce a filler for the Cauchy-Riemann section \( \overline{\partial} \) we first recall the local model for the strong M-polyfold bundle and start with the bundle

\[
(V \oplus B_\varepsilon \oplus \hat{E}_0) \triangleleft F \to V \oplus B_\varepsilon \oplus \hat{E}_0
\]

where \( F \) is the \( \text{sc} \)-Hilbert space consisting of pairs \((\eta^+, \eta^-)\) of functions in \( H^{2,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^2) \oplus H^{2,\delta_0}(\mathbb{R}^- \times S^1, \mathbb{R}^2) \) equipped, as usual, with the \( \text{sc} \)-structure \( F_m = H^{2+m,\delta_m}(\mathbb{R}^+ \times S^1, \mathbb{R}^2) \oplus H^{2+m,\delta_m}(\mathbb{R}^- \times S^1, \mathbb{R}^2) \).

The retraction \( R : (V \oplus B_\varepsilon \oplus \hat{E}_0) \triangleleft F \to (V \oplus B_\varepsilon \oplus \hat{E}_0) \triangleleft F \) is defined by

\[
R(v, a, (h^+, h^-), (\eta^+, \eta^-)) = (v, a, \rho_a(h^+, h^-), \hat{\pi}_a(\eta^+, \eta^-))
\]

and covers the retraction \( r : V \oplus B_\varepsilon \oplus \hat{E}_0 \to V \oplus B_\varepsilon \oplus \hat{E}_0 \), defined by

\[
r(v, a, (h^+, h^-)) = (v, a, \rho_a(h^+, h^-)).
\]

With the retracts \( K := R((V \oplus B_\varepsilon \oplus \hat{E}_0) \triangleleft F) \) and \( O = r(V \oplus B_\varepsilon \oplus \hat{E}_0) \) we obtain the local model

\[
K \to O
\]

of the strong M-polyfold bundle. In these local coordinates the Cauchy-Riemann section \( \overline{\partial} : O \to K \) near the point \((v, a, h^+, h^-) = (v_0, 0, 0, 0)\) becomes

\[
\overline{\partial}(v, a, h^+, h^-) = (v, a, h^+, h^-; \eta^+, \eta^-)
\]

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where \((\eta^+, \eta^-)\) is a solution of

\[
\begin{align*}
\widehat{\oplus}_a(\eta^+, \eta^-) &= (\bar{\partial}_{i,j(a,v)}(w)) \cdot \left( \frac{\partial}{\partial s} \right) \\
\widehat{\ominus}_a(\eta^+, \eta^-) &= 0
\end{align*}
\]

in the case \(a \neq 0\), where \(w = \bigoplus_a(u^+_0 + h^+, u^-_0 + h^-) : Z_a \to Z_b\) is a diffeomorphism and \(b = b(a, u^+_0 + h^+, u^-_0 + h^-)\). If \(a = 0\), then

\[
\bar{\partial}(v, 0, h^+, h^-) =
\]

\[
(v, 0, h^+, h^-; \bar{\partial}_{i,j(v)}(u^+_0 + h^+), \bar{\partial}_{i,j(v)}(u^-_0 + h^-)).
\]

A filler for \(\bar{\partial}\) is an extension of the local section \(\bar{\partial} : O \to K\) to a section

\[
g : V \oplus B_\varepsilon \oplus \hat{E}_0 \to (V \oplus B_\varepsilon \oplus \hat{E}_0) \triangleleft F
\]

of the original bundle which is a section defined on an open set in an sc-Hilbert space. It is defined as follows. If \(a = 0\), we set \(g(v, 0, h^+, h^-) = \bar{\partial}(v, 0, h^+, h^-)\). If \(a \neq 0\), we define the section

\[
g(v, a, h^+, h^-) = (v, a, h^+, h^-; \eta^+, \eta^-)
\]

the following way. We take \((h^+, h^-) \in \hat{E}_0\) and construct the associated diffeomorphism \(\bigoplus_a(u^+_0 + h^+, u^-_0 + h^-) : Z_a \to Z_b\) between the glued cylinders where \(b = b(a, u^+_0 + h^+, u^-_0 + h^-)\). In addition, we define the function \(\xi \in H^{2,\delta_0}(C_a, \mathbb{R}^2)\) by

\[
\xi = \bigoplus_a(h^+ - h^+_\infty, h^- - h^-\infty).
\]

In view of Theorem 1.29, we have the sc-isomorphism

\[
(\widehat{\oplus}_a, \widehat{\ominus}_a) : F \to H^2(Z_a, \mathbb{R}^2) \oplus H^{2,\delta_0}(C_a, \mathbb{R}^2)
\]

and define the principal part of the section \(g\) as the unique solution of the two equations

\[
\widehat{\oplus}_a(\eta^+, \eta^-) = (\bar{\partial}_{i,j(a,v)}(w)) \cdot \left( \frac{\partial}{\partial s} \right)
\]

\[
\widehat{\ominus}_a(\eta^+, \eta^-) = \partial_0 \xi.
\]
Lemma 3.29. The above section \( g \) whose principal part is given by

\[
(v, 0, h^+, h^-) \mapsto (\eta^+, \eta^-),
\]

and which is defined on an open neighborhood of \((v_0, 0, (0,0))\) in \( V \oplus C \oplus \hat{E}_0 \) and has its image in \( F \), is a filled version of the local Cauchy-Riemann section \( \overline{\partial} : O \to K \) near the point \((v_0, 0, u_0^+, u_0^-) \in V \times (X \setminus X)\).

Proof. As in Proposition 3.26 one sees that the section is sc-smooth. Recalling Definition 1.35 of a filler in Section 1.4, we have to verify the three defining properties.

If we restrict \( g \) to the sc-smooth retract \( O \), then, in view of Lemma 3.3, it follows from \( \rho_a(h^+, h^-) = (h^+, h^-) \) that

\[
\xi = \Theta_a(h^+ - h_\infty, h^- - h_\infty) = \Theta_a(h^+, h^- + h_\infty - h_\infty) = 0,
\]

and therefore \( \Theta_a(\eta^+, \eta^-) = 0 \) so that \((v, a, h^+, h^-; \eta^+, \eta^-) \in K\). Consequently, \( g|O \) is a section of the bundle \( K \to O \). By construction, it is the local coordinate representation of the Cauchy-Riemann section \( \overline{\partial} : O \to K \) near the originally given point. This proves the property (1) of the requirements to be a filled version.

In order to verify property (2) of a filler, we assume that

\[
g(y) = \phi(r(y)) \cdot g(y)
\]

for a point \( y = (v, a, h^+, h^-) \in V \oplus B_\varepsilon \oplus \hat{E}_0 \) close to \((v_0, 0, 0, 0)\). We conclude that

\[
(\eta^+, \eta^-) = \hat{\pi}_a(\eta^+, \eta^-)
\]

and hence \( \hat{\Theta}_a(\eta^+, \eta^-) = 0 \). Since \( \overline{\partial}_0 \) is an isomorphism in view of Lemma 3.28 we obtain \( \Theta_a(h^+ - h_\infty, h^- - h_\infty) = \Theta_a(h^+, h^- + h_\infty - h_\infty) = 0 \) so that by Lemma 3.3 again, \( \rho_a(h^+, h^-) = (h^+, h^-) \) and consequently, \((v, a, h^+, h^-) \in O\). We have verified that the section \( g \) satisfies property (2) of a filler.

Finally, the third property of a filler is easily verified. At the point \( y_0 = (v_0, 0, (0,0)) \in V \oplus C \oplus \hat{E}_0 \), the derivative of the retraction \( Dr(y_0) = \text{id} \) is the identity map so that kernel \( Dr(y_0) = \{0\} \). Since \( \phi(y_0) = \hat{T}_0 = \text{id} \) we also have that kernel \( \phi(y_0) = \{0\} \). Consequently, the linearization of the map \( y \mapsto [\text{id} - \phi(r(y))] \cdot g(y) \) at the point \( y_0 \) restricted to ker \( Dr(y_0) \) defines trivially an isomorphism ker \( Dr(y_0) \to \ker \phi(y_0) = \{0\} \). The proof of Lemma 3.29 is complete.

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We are going to verify the Fredholm property of the section \( g \) near the smooth point \((v_0, 0, 0, u^+_0, u^-_0) \in V \times \overline{X}_\infty\), where

\[
u^+_0(s, t) = (s, t) + (d^+_0, \vartheta^+_0) + r^+_0(s, t)
\]

are diffeomorphisms of the half-cylinders \( \mathbb{R}^+ \times S^1 \) and \( \mathbb{R}^- \times S^1 \) which satisfy \( r^+_0 \in \bigcap_{m \geq 0} H^{3+m; \beta_m}(\mathbb{R}^+ \times S^1, \mathbb{R}^2)\).

For this purpose we introduce, in our local coordinates near the above distinguished point, the \( sc^+ \)-section

\[
g_0(v, a, h^+, h^-) = g(v, a, 0, 0) \in F_\infty
\]

for \((v, a, h^+, h^-) \) near \((v_0, 0, 0, 0)\).

**Lemma 3.30.** There exists \( \tau \in (0, \frac{1}{2}) \) which depends only on the size of the support of \( j^\pm(v) \) defined by the parameter \( s_0 > 0 \) for which \( j^+(v) = i \) on \([s_0, \infty) \times S^1 \) for \( v \in V \), and \( j^-(v) = i \) on \((-\infty, -s_0] \times S^1 \), so that the following holds. If \( 0 < |a| < \tau \), then the principal part \((v, a, h^+, h^-) \mapsto (g - g_0)(v, a, h^+, h^-) \) \( \in F \) of the \( sc^- \)-section \( g - g_0 \) is given by the formula,

\[
(g - g_0)(v, a, h^+, h^-) = \left[ \frac{1}{2} [Th^+ + i \circ Th^+ \circ j^+(v)] \left( \frac{\partial}{\partial s} \right) \right] g_0
+ \frac{\beta'_a}{\gamma_a} \left[ \begin{array}{cc} \beta_a & \beta_a - 1 \\ 1 - \beta_a & \beta_a \end{array} \right] \left[ \begin{array}{c} h^+ - h^- - h^+_\infty + h^-_\infty \\ h^+ + h^- - 2av_a(h^+, h^-) \end{array} \right]
\]

where, as usual, \( \beta_a = \beta_a(s) = \beta(s - \frac{R}{2}) \) and \( h^+ = h^+(s, t) \) and \( h^- = h^-(s - R, t - \vartheta) \) and \( 0 \leq s \leq R \). Moreover, \( \beta'_a \) stands for the derivative of the function \( \beta_a \).

If \( a = 0 \), then the principal part \((v, 0, h^+, h^-) \mapsto (g - g_0)(v, 0, h^+, h^-) \) \( \in F \) of the \( sc^- \)-section \( g - g_0 \) is given by the formula,

\[
(g - g_0)(v, 0, h^+, h^-) = \left[ \frac{1}{2} [Th^+ + i \circ Th^+ \circ j^+(v)] \left( \frac{\partial}{\partial s} \right) \right] g_0
+ \left[ \frac{1}{2} [Th^- + i \circ Th^- \circ j^-(v)] \left( \frac{\partial}{\partial s} \right) \right] g_0.
\]

**Proof.** In the following proof we use the abbreviated notations

\[
\overline{\partial}_v w \equiv \overline{\partial} w = \frac{1}{2} [Tw + i \circ Tw \circ j(a, v)] \left( \frac{\partial}{\partial s} \right) w
\]

\[
\overline{\partial}_0 w := \frac{1}{2} [Tw + i \circ Tw \circ i] \left( \frac{\partial}{\partial s} \right) w
\]

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for the Cauchy-Riemann operators. Recalling that the section $g$ has the principal part $(\eta^+, \eta^-) \in F$ defined by the equations

\[
\hat{\oplus}_a(\eta^+, \eta^-) = \overline{\partial}(w)
\]
\[
\hat{\ominus}_a(\eta^+, \eta^-) = \partial_0(\ominus_a(h^+ - h^- - h^-_\infty)),
\]
\[
w = \bigstar_a(u^+ + h^+ + u^- + h^-) : Z_a \to Z_b,
\]
the principal part of the $\text{sc}^+$-section $g_0(a, v, h^+, h^-) = (\eta^+_0, \eta^-_0)$ is determined by the equations

\[
\hat{\oplus}_a(\eta^+_0, \eta^-_0) = \overline{\partial}(w_0)
\]
\[
\hat{\ominus}_a(\eta^+_0, \eta^-_0) = 0
\]
\[
w_0 = \bigstar_a(u^+_0, u^-_0).
\]

Note that $(\eta^+_0, \eta^-_0)$ is a function of $(a, v)$ which we suppress in the notation. Also $\overline{\partial}$ is depending on $(a, v)$. Observing that $w - w_0 = \bigstar_a(u^+_0 + h^+ + u^-_0 + h^-) - \bigstar_a(u^+_0, u^-_0) = \ominus_a(h^+ + h^- + h^+ - h^-_\infty)$ and recalling the definitions of $\ominus_a$ and $\hat{\ominus}_a$ and $\hat{\ominus}_a = \ominus_a$, we see that the principal part $(\eta^+ - \eta^+_0, \eta^- - \eta^-_0)$ of the section $g - g_0$ is characterized by the equations

\[
\left[ \hat{\oplus}_a(\eta^+ - \eta^+_0, \eta^- - \eta^-_0) \right] - \left[ \hat{\ominus}_a(\eta^+ - \eta^+_0, \eta^- - \eta^-_0) \right] = \left[ \hat{\ominus}_a(\overline{\partial}h^+, \overline{\partial}h^-) \right] - \left[ \hat{\ominus}_a(\overline{\partial}h^+_0, \overline{\partial}h^-_0) \right] + \beta'_a \cdot \left[ \frac{h^+ - h^- - h^+ + h^-_\infty}{h^+ + h^- - 2av_a(h^+, h^-)} \right].
\]

In view of the definition of $\beta_a$ and $j(a, v)$ we conclude that

\[
\hat{\ominus}_a(\overline{\partial}h^+, \overline{\partial}h^-) = \hat{\ominus}_a(\overline{\partial}h^+_0, \overline{\partial}h^+_0)
\]

and hence obtain

\[
\left[ \frac{\eta^+ - \eta^+_0}{\eta^-_0 - \eta^-_0} \right] = \left[ \frac{\overline{\partial}h^+}{\overline{\partial}h^-} \right] + \beta'_a \cdot \left[ \frac{\beta_a - 1}{\gamma_a} \right] \cdot \left[ \frac{h^+ - h^- - h^+ + h^-_\infty}{h^+ + h^- - 2av_a(h^+, h^-)} \right].
\]

Since $j(a, v) = j^+(v)$ on $\mathbb{R}^+ \times S^1$, the lemma is proved.  
\[\blacksquare\]
Let us define $L(a,v)$ by (73) via

$$\begin{aligned}
(a,v,h^+, h^-) \rightarrow L(a,v)(h^+, h^-) := (g - g_0)(a,v, h^+, h^-).
\end{aligned}$$

In view of the above lemma, the section $g - g_0 : V \oplus B_\varepsilon \oplus \hat{E}_0 \rightarrow F$ is given by a family $L(\lambda) : \hat{E}_0 \rightarrow F$ of linear operators (not continuous as a family of operators on any level) parametrized by $\lambda = (v,a)$ and of the form

$$L(\lambda) = D_v + \Delta_a,$$

where $D_v : \hat{E}_0 \rightarrow F$ is the Cauchy-Riemann operator. From the formulae in the above lemma, we see that at $a = 0$ we have $\Delta_0(h^+, h^-) = 0$ and that the map $(a, h^+, h^-) \mapsto \Delta_a(h^+, h^-)$ is the sc-smooth in view of the Lemmata 2.18,2.22. Hence we have to study for data $(\lambda, h^+, h^-)$ near $(\lambda_0, 0, 0)$ the sc-smooth map

$$(\lambda, h^+, h^-) \rightarrow L(\lambda)(h^+, h^-).$$

(75)

For the later discussion we note the following formula

$$\begin{aligned}
(\hat{\lambda}_a, \hat{\lambda}_a)(L(a,v)(h^+, h^-))
\end{aligned}$$

$$(\hat{\lambda}_a, \hat{\lambda}_a)(L(a,v)(h^+, h^-)) = (\overline{\partial}_v(\ominus_a(h^+ - h_\infty^+)), \overline{\partial}_0(\ominus_a(h^+ - h_\infty^+, h^- - h_\infty^-)))$$

(76)

which follows immediately from the proof of the previous lemma.

**Remark 3.31.** The Cauchy-Riemann operator $D_v$ is a classical Fredholm operator between $(\hat{E}_0)_m$ and $F_m$ for every $m \geq 0$. One verifies readily that $D_v$ is bijective. Indeed, this follows from the classical fact that the standard Cauchy-Riemann operator acting on $\mathbb{C}$-valued functions on the disk with real boundary conditions is surjective with a 1-dimensional kernel. Then in view of the boundary conditions of the functions in the domain $\hat{E}_0$, the kernel of $D_v$ is equal to $\{0\}$ and hence $D_v$ is an isomorphism. The linear operator $\Delta_a$ is compact and hence the operators $L(\lambda)$ are all Fredholm operators of index 0.

**Proposition 3.32.** We consider the sc-smooth map $(\lambda, h^+, h^-) \mapsto L(\lambda)(h^+, h^-)$ defined in (75) in a neighborhood of the parameter value $\lambda_0 = (v_0, a_0) = (v_0, 0)$. There exists a constant $\sigma > 0$ so that the following holds.

1. If $\lambda = (v, a)$ satisfies $|\lambda - \lambda_0| < \sigma$, then the linear operator $L(\lambda) : \hat{E}_0 \rightarrow F$ is an sc-isomorphism.
(2) For every \( m \geq 0 \) there exists a constant \( C_m \) independent of \( \lambda \) so that the norm of the inverse operator

\[
L(\lambda)^{-1} : F_m \to (\hat{E}_0)_m
\]

is bounded by \( C_m \) for every \( |\lambda - \lambda_0| < \sigma \).

We postpone the proof of this nontrivial proposition to the appendix and use it in order to verify that the section \( g - g_0 \) is a Fredholm germ. Let \( B_\sigma(\lambda_0) \) be the open ball in \( V \oplus \mathbb{C} \) centered at \( \lambda_0 = (v_0, 0) \) of radius \( \sigma \). In view of Proposition 3.32, the map \( B_\sigma(\lambda_0) \oplus \hat{E}_0 \to (V \oplus \mathbb{C}) \oplus F \), defined by,

\[
(\lambda, (h^+, h^-)) \mapsto (\lambda, L(\lambda)(h^+, h^-)),
\]

satisfies the assumptions of Proposition 4.8 in the appendix, from which we conclude that the inverse map \( B_\sigma(\lambda_0) \oplus F \to B_\sigma(\lambda_0) \oplus \hat{E}_0, \)

\[
(\lambda, (\xi^+, \xi^-)) \to L(\lambda)^{-1}(\xi^+, \xi^-),
\]
is sc-smooth. This allows to introduce the local strong bundle coordinate change

\[
\Phi : (B_\sigma(\lambda_0) \oplus U) \triangleleft F \to (B_\sigma(\lambda_0) \oplus U) \triangleleft \hat{E}_0,
\]
defined by

\[
(\lambda, (h^+, h^-), (\xi^+, \xi^-)) \mapsto (\lambda, (h^+, h^-), L(\lambda)^{-1}(\xi^+, \xi^-)).
\]

Since the map \( (\lambda, \xi^+, \xi^-) \mapsto L(\lambda)^{-1}(\xi^+, \xi^-) \) between \( B_\sigma(\lambda_0) \oplus F \) and \( \hat{E}_0 \) is sc-smooth, it is also sc-smooth as a map between \( B_\sigma(\lambda_0) \oplus F^1 \) and \( \hat{E}^1_0 \), in view of Proposition 2.2. Therefore, the map \( \Phi \) is \( sc_\sigma \)-smooth.

Now we consider the push-forward section \( \Phi_\ast(g - g_0) \). By construction, its principal part is the map \( \Phi_\ast(g - g_0) : (B_\sigma \oplus U) \to \hat{E}_0, \) given by

\[
(\lambda, (h^+, h^-)) \mapsto (h^+, h^-).
\]

Obviously, it is an \( sc^0 \)-contraction germ where the contraction term

\[
B(\lambda, h^+, h^-)
\]
vanishes identically.
To sum up, we have studied the Cauchy-Riemann section
\[ f : (v, a, h^+, h^-) \mapsto (v, a, h^+, h^-, \eta^-, \eta^-) \]
in local coordinates near the smooth point \((v_0, 0, 0, 0)\) and have constructed a filled section \(g\) of \(f\). Moreover, we have established an \(sc^+\)-section \(g_0\) satisfying \((g - g_0)(v, 0, 0, 0) = 0\) and found a bundle isomorphism \(\Phi\) so that the push-forward \(\Phi_*(g - g_0)\) is a germ belonging to \(\mathcal{C}_{\text{basic}}\). This shows that \(f\) is a Fredholm germ at the point \((v_0, 0, 0, 0)\).

The proof that the Cauchy-Riemann section \(\overline{\partial}\) of the strong polyfold bundle \(E \to V \oplus X\) is a polyfold Fredholm section is complete.

### 3.5 Application of the Sc-Implicit Function Theorem

We are going to prove Theorems 1.45 and Theorem 1.46. The previous sections demonstrated that the Cauchy-Riemann operator
\[ \overline{\partial} : V \oplus X \to E \]
is an \(sc\)-smooth Fredholm section of the strong M-polyfold bundle \(E \to V \oplus X\). Therefore, the \(sc\)-implicit function theorem from [10] can be applied to our situation, and it follows that if the smooth point \(x_0 = (v_0, a_0, b_0, w_0) \in V \oplus X\) is a solution of
\[ \overline{\partial}(v_0, a_0, b_0, w_0) = 0 \]
and if the linearized map at this point is surjective, then the solution set \(\overline{\partial}(x) = 0\) nearby is a smooth manifold of the dimension of the kernel of the linearized map at the reference solution \(x_0\).

We now consider the distinguished solution
\[ \overline{\partial}(v_0, 0, 0, u^+_0, u^-_0) = 0 \]
at \(a = 0\) where
\[ u^+_0 : (\mathbb{R}^\pm \times S^1, j^+(v_0)) \to (\mathbb{R}^\pm \times S^1, i) \]
are the unique biholomorphic mappings of the half-cylinders which fix the boundary points \((0, 0) \in \partial(\mathbb{R}^\pm \times S^1)\). These biholomorphic mappings are guaranteed by the uniformization theorem. Indeed, the maps \(z = e^{2\pi(s+it)}\) and \(z = e^{-2\pi(s+it)}\) are diffeomorphisms between the half-cylinders \(\mathbb{R}^\pm \times S^1\).
and the closed unit disc $D \setminus \{0\}$ with the origin removed. Now there is a unique biholomorphic map $(D, j) \to (D, i)$ leaving a boundary point and the origin fixed. By assumption on the complex structures $j^\pm(v_0)$, the induced complex structure $j$ in $D$ agrees in an open neighborhood of the origin with the standard structure $i$. Therefore, since $h(0) = 0$ and hence $h(z) = az + \cdots$, near the origin, the corresponding maps of the half-cylinders are of the form

$$u_0^\pm(s, t) = (s, t) + (d_0^\pm, \vartheta_0^\pm) + r_0^\pm(s, t)$$

where the maps $r_0^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^2$ decay with all their derivatives to 0 at every rate bounded by $C e^{-\varepsilon |s|}$ for every $0 \leq \varepsilon < 2\pi$.

Around the distinguished point $p_0 = (v_0, 0, 0, u^+_0, u^-_0) \in V \oplus \overline{X}$ at $a = 0$ we take our chart $\Psi$ of the M-polyfold,

$$\Psi(v, a, h^+, h^-) = (v, a, b(a, u^+_0 + h^+, u^-_0 + h^-), [a(u^+_0 + h^+, u^-_0 + h^-)])$$

where $\rho_a(h^+, h^-) = (h^+, h^-)$, and take the associated bundle chart. In these local coordinates, the principal part of the Cauchy-Riemann section $\overline{\partial}$ is expressed by the map

$$f(v, a, h^+, h^-) = (\eta^+, \eta^-) \in F$$

where

$$\widehat{\partial}_a(\eta^+, \eta^-) = \overline{\partial}_{i,j(a,v)}(\oplus_a(u^+_0 + h^+, u^-_0 + h^-))$$

$$\widehat{\text{r}}_a(\eta^+, \eta^-) = 0.$$  

In Lemma 3.30 we have computed the linearization $D(g - g_0)(p_0)$ where $g$ is the filled section of $f$. It is a surjective Fredholm operator. The partial linearizations with respect to the variables $(h^+, h^-)$, denoted by $D_2(g - g_0)(p_0)$, is an sc-isomorphisms from $E$ onto $F$. The same holds true for the linearization $Dg(p_0)$. In view of the definition of the filler in section 1.4 one concludes that the linearization $Df(p_0)$ of the section $f$ is a surjective Fredholm operator and $D_2f(p_0)$ an sc-isomorphism.

From the sc-implicit function theorem, Theorem 4.6 in [10], we therefore conclude that there exists a unique sc-smooth map defined near $(v_0, 0),

$$\sigma(v, a) = (v, a, h^+(v, a), h^-(v, a))$$

satisfying $\sigma(v_0, 0) = (v_0, 0, 0, 0)$ and

$$f(v, a, h^+(v, a), h^-(v, a)) = 0.$$
This means for the section $\delta$ on the M-polyfold $V \oplus X$, that
\[
\delta(\Phi \circ \sigma(v,a)) = 0
\]
for the solutions near the reference solution. Moreover, these are all the solutions of the Cauchy-Riemann equations near $(v_0, 0, 0, u_0^+, u_0^-) \in V \oplus X$. This completes the proof of Theorem 1.45. ■

Rather than taking a chart around the reference solution $p_0 = (v_0, 0, 0, u_0^+, u_0^-)$ we can take a chart around the smooth point $(v_0, 0, 0, w_0^+, w_0^-)$ nearby so that the chart contains the solution $p_0$ and satisfies, moreover,
\[
w_0^\pm(s, t) = (s + d_0^\pm, t + \vartheta_0^\pm)
\]
for all large $|s| \geq s_0$. Applying the sc-implicit function theorem to the reference solution $p_0$ we obtain in the new coordinates the sc-smooth map
\[
\tilde{\sigma}(v, a) = (v, a, \hat{h}^+(v, a), \hat{h}^-(v, a))
\]
near $(v_0, 0)$ and the associated biholomorphic maps between the finite cylinders
\[
(Z_{a, j}(a, v), p_a^+, p_a^-) \rightarrow (Z_{b(a, v)}, i, p_{b(a, v)}^+, p_{b(a, v)}^-)
\]
given by
\[
w(v, a) = \Box_a(w_0^+ + \hat{h}^+(v, a), w_0^- + \hat{h}^-(v, a)).
\]
In view of the uniqueness of the solutions near $(v_0, 0)$ we know, in particular, that at $a = 0$
\[
u_0^\pm = w_0^\pm + \hat{h}^\pm(0, v_0).
\]
It follows for $(v, a)$ close to $(v_0, 0)$ that the mappings
\[
w_0^\pm + \hat{h}^\pm(a, v) : \mathbb{R}^\pm \times S^1 \rightarrow \mathbb{R}^\pm \times S^1
\]
are diffeomorphisms. By definition of the chart,
\[
\rho_a(\hat{h}^+(a, v), \hat{h}^-(a, v)) = (\hat{h}^+(a, v), \hat{h}^-(a, v))
\]
and hence we derive from the representation formula for $\rho_a$ the following asymptotics
\[
\hat{h}^+(a, v)(s, t) = \hat{h}^+(a, v)_\infty \quad \text{if } s \geq \frac{R}{2} + 1
\]
\[
\hat{h}^-(a, v)(s', t') = \hat{h}^-(a, v)_\infty \quad \text{if } s' \leq -\frac{R}{2} - 1,
\]
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from which Theorem 1.46 follows in the case \( \Delta \leq 1 \).

In order to obtain the general case we proceed as follows. By construction,

\[
\begin{align*}
    w(a, v) &= h_a(w_0^+, w_0^-) + h^+(a, v) + a(h^+(a, v), \hat{r}^-(a, v)).
\end{align*}
\]

We define for \(|a|\) small enough the diffeomorphism

\[
\begin{align*}
    \tilde{w}(a, v) : \mathbb{R}^+ \times S^1 &\to \mathbb{R}^+ \times S^1 \\
    \tilde{w}(a, v)(s, t) &= \begin{cases} \\
        w_0^+(s, t) + \hat{h}_\infty^+(a, v) + \beta \left( s - \frac{R}{2} - \Delta - 1 \right) \cdot a(h^+(a, v), \hat{r}^-(a, v)). & (77) \\
    \end{cases}
\end{align*}
\]

Observe that for \(s \in [0, R^2 + \Delta]\) if \(|a|\) is small enough,

\[
    \begin{align*}
        w_0^+(s, t) &= h_a(w_0^+, w_0^-)([s, t]).
    \end{align*}
\]

Therefore,

\[
    \tilde{w}(a, v)(s, t) = w(a, v)([s, t])
\]

for all \(s \in [0, R^2 + \Delta]\). Finally, we note that

\[
    (a, v) \mapsto \tilde{w}(a, v)
\]

defines a smooth map into every \(\mathcal{D}^m,\varepsilon\) for all \(m \geq 2\) and \(\varepsilon \in (0, 2\pi)\). Since in our set-up for the implicit function theorem we can take any sequence \((\delta_m)\) as long as it is strictly increasing and stays below \(2\pi\), we see that the maps

\[
    (a, v) \mapsto \hat{h}^\pm(a, v)
\]

as maps from an open neighborhood of \((0, v_0)\) into \(H^m,\varepsilon(\mathbb{R}^\pm \times S^1, \mathbb{R}^2)\) are smooth. The same is true, for \(\hat{r}^\pm(a, v)\). Now using Proposition 2.8 we see that the map

\[
    (a, v) \mapsto \beta \left( s - \frac{R}{2} - \Delta - 1 \right) \cdot a(h^+(a, v), \hat{r}^+(a, v))(s, t).
\]

is ss-smooth for every choice of the sequence of \((\delta_m)\) as described above. Choosing \((\delta_m) \in (\varepsilon, 2\pi)\) we deduce that the map

\[
    (a, v) \mapsto \tilde{w}(a, v),
\]

which is defined near \((0, v_0)\), is smooth into every space \(\mathcal{D}^m,\varepsilon\). This completes the proof of Theorem 1.46.
4 Appendix

In the appendix we shall prove the \(\text{sc}\)-smoothness of the shift-map. Moreover, we collect informations about the gluing profile and provide proofs of several technical results about families of \(\text{sc}\)-isomorphisms and estimates for the Cauchy-Riemann section which are used in our constructions.

4.1 The Shift-Map

Fixing a strictly increasing sequence \(\delta_m \geq 0\) of real numbers starting with \(\delta_0 = 0\), we consider the Hilbert space \(E = L^2(\mathbb{R} \times S^1)\) equipped with the sc-structure defined by the sequence \(E_m = H^{m,\delta_m}\) for all \(m \geq 0\). The shift-map \(\Phi : \mathbb{R}^2 \oplus E \to E\) is defined as

\[
\Phi : (R, \vartheta, u) \to (R, \vartheta) * u := u(s + R, t + \vartheta).
\]

Thus,

\[
|\Phi(R, \vartheta, u)|_m \leq e^{\delta_m |R|} |u|_m.
\]

Our first result concerns the \(\text{sc}^0\)-property of \(\Phi\).

**Proposition 4.1.** The shift-map \(\Phi\) is \(\text{sc}^0\)-continuous.

**Proof.** Fix a level \(m\) and take \(u \in E_m\). We estimate the norm \(|(R, \vartheta) * u|_m\) as follows,

\[
|(R, \vartheta) * u|_m^2 = \sum_{|\alpha| \leq m} \int |(D^\alpha u)(s + R, t + \vartheta)|^2 e^{2\delta_m |s|} dsdt \leq \sum_{|\alpha| \leq m} \int |(D^\alpha u)(s + R, t + \vartheta)|^2 e^{2\delta_m |s+R|} e^{2\delta_m |R|} dsdt = e^{2\delta_m |R|} |u|_m^2.
\]

Thus,

\[
|((R, \vartheta) * u)|_m \leq e^{\delta_m |R|} |u|_m.
\]

Since the set of smooth compactly supported functions \(\mathbb{R} \times S^1 \to \mathbb{R}\) is dense in \(H^m(\mathbb{R} \times S^1)\), it is also dense in \(E_m\). We claim that \(\Phi\) is continuous at the point \(((0, 0), u_0) \in \mathbb{R}^2 \times E_m\). To see this, first note that if \(v\) is smooth and compactly supported, then \(\Phi(R, \vartheta, v) \to v\) in \(C^\infty\) as \((R, \vartheta) \to 0\), which immediately implies the convergence in the \(m\)-norm,

\[
\lim_{(R, \vartheta) \to (0, 0)} |\Phi(R, \vartheta, v) - v|_m = 0.
\]

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Now, if \( u \in E_m \) and if \( v \) is a compactly supported smooth function, we estimate,
\[
|((R, \vartheta) \circ u) - u_0|_m = |((R, \vartheta) \circ (u - v) + ((R, \vartheta) \circ v - v) + (v - u_0))|_m
\]
\[
\leq e^{\delta_m |\varepsilon|} \cdot |u - v|_m + |(R, \vartheta) \circ v - v|_m + |v - u_0|_m
\]
Given \( \varepsilon > 0 \), we chose \( v \) so that \( |v - u_0|_m < \varepsilon \). Then for all \( u \in E_m \) satisfying \( |u - u_0|_m < \varepsilon \) and \((R, \vartheta)\) close to \((0, 0)\), we have
\[
|u - v|_m < 2\varepsilon, \quad |(R, \vartheta) \circ v - v|_m < \varepsilon, \quad e^{\delta_m |\varepsilon|} < 9,
\]
so that the above estimate gives
\[
|(R, \vartheta) \circ u - u_0|_m < 10\varepsilon,
\]
proving the continuity at the point \((0, 0, u_0) \in \mathbb{R}^2 \oplus E_m\). Since for fixed \((R_0, \vartheta_0)\) the map \( E \to E \) defined by \( u \mapsto (R_0, \vartheta_0) \circ u \) is clearly an \( sc^0 \)-operator and since
\[
(R, \vartheta) \circ u - (R_0, \vartheta_0) \circ u_0 = (R - R_0, \vartheta - \vartheta_0) \circ ((R_0, \vartheta_0) \circ u) - (R_0, \vartheta_0) \circ u_0,
\]
the previous discussion shows that \( \Phi \) is continuous at \((R_0, \vartheta_0) \circ u_0 \in \mathbb{R}^2 \oplus E_m\). Consequently, \( \Phi \) is \( sc^0 \) as claimed. \( \blacksquare \)

Having proved that the shift-map is of class \( sc^0 \), we show that it is \( sc \)-smooth.

**Proposition 4.2.** If \( E = L^2(\mathbb{R} \times S^1) \) is equipped with the \( sc \)-structure \((E_m)_{m \in \mathbb{N}_0}\) as described before, the shift-map \( \Phi : \mathbb{R}^2 \oplus E \to E \) is \( sc \)-smooth.

**Proof.** By Proposition 4.1, the map \( \Phi \) is \( sc^0 \) and we first show that it is \( sc^1 \). Take a point \((R, \vartheta, u) \in \mathbb{R}^2 \oplus E_1\). We want to find a linear bounded operator
\[
D\Phi(R, \vartheta, u) : \mathbb{R}^2 \oplus E_0 \to E_0
\]
satisfying points (1) and (2) of Definition 1.8. Our candidate for the linearization \( D\Phi(R, \vartheta, u) \) of \( \Phi \) at the point \((R, \vartheta, u)\) is the formal derivative
\[
\Psi : (\mathbb{R}^2 \oplus E)^1 \oplus (\mathbb{R}^2 \oplus E) \to E
\]
defined as the map
\[
\Psi(R, \vartheta, u)(R_1, \vartheta_1, v) = (R, \vartheta) \circ (R_1 \cdot u_s + \vartheta_1 \cdot u_t + v)
\]
\[
= \Phi(R, \vartheta, R_1 \cdot u_s + \vartheta_1 \cdot u_t + v).
\]

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We note that the map $\Psi$ is sc$^0$. Indeed, we already know that the shift-map $\Phi$ is sc$^0$ and the two maps $E^1 \to E$ defined by $u \mapsto u_s, u_t$ are sc$^0$-operators. Also, a scalar multiplication $\mathbb{R} \oplus E \to E$ sc$^0$. Hence, the map $\Psi$ can be written as the composition of sc$^0$-maps and, therefore, is sc$^0$. It remains to prove the approximation property. To do so we have to show that

$$\frac{|\Phi(R + R_1, \partial + \partial_1, u + v) - \Phi(R, \partial, u) - \Phi(R, \partial, R_1 \cdot u_s + \partial_1 \cdot u_t + v)|_0}{|(R_1, \partial_1, v)|_1}$$

converges to 0 as $|(R_1, \partial_1, v)|_1 \to 0$. Once this is proved, we will know by our earlier discussion that $\Phi$ is sc$^1$ and its linearization at the point $(R, \partial, u) \in R^2 \oplus E$ is given by

$$D\Phi(R, \partial, u)(R_1, \partial_1, v) = \Phi(R, \partial, R_1 \cdot u_s + \partial_1 \cdot u_t + v).$$

We compute,

$$\Phi(R + R_1, \partial + \partial_1, u + v) - \Phi(R, \partial, u) - \Phi(R, \partial, R_1 \cdot u_s + \partial_1 \cdot u_t + v)$$

$$= \Phi(R + R_1, \partial + \partial_1, u) - \Phi(R, \partial, u) - \Phi(R, \partial, R_1 \cdot u_s + \partial_1 \cdot u_t)$$

$$+ \Phi(R + R_1, \partial + \partial_1, v) - \Phi(R, \partial, v).$$

We first show that

$$\frac{|\Phi(R + R_1, \partial + \partial_1, u) - \Phi(R, \partial, u) - \Phi(R, \partial, R_1 \cdot u_s + \partial_1 \cdot u_t)|_0}{|(R_1, \partial_1, v)|_1}$$

converges to 0 as $|(R_1, \partial_1, v)|_1 \to 0$. Since the map $\Phi(R, \partial, \cdot) : E \to E$ is an isometry on level 0, it suffices to show that

$$\lim_{|(R_1, \partial_1)| \to 0} \frac{|\Phi(R_1, \partial_1, u) - u - R_1 \cdot u_s + \partial_1 \cdot u_t|_0}{|(R_1, \partial_1)|} = 0$$

If $u$ is compactly supported smooth function, then we have

$$\Phi(R_1, \partial_1, u) - u - R_1 \cdot u_s + \partial_1 \cdot u_t$$

$$= \int_0^1 [\Phi(\tau R_1, \tau \partial_1, R_1 \cdot u_s + \partial_1 \cdot u_t) - R_1 \cdot u_s + \partial_1 \cdot u_t] \, d\tau.$$ 

Now if $u \in E_1$, then we find a sequence $(u_n)$ of compactly supported smooth functions so that $|u_n - u|_1 \to 0$. The above equality holds for each of the
functions $u_n$ and letting $n \to \infty$ we find that it also holds for $u$. Moreover, note that

$$|\Phi(\tau R_1, \tau \vartheta_1, R_1 \cdot u_s + \vartheta_1 \cdot u_t) - R_1 \cdot u_s + \vartheta_1 \cdot u_t|_0$$

$$\leq |R_1| |\Phi(\tau R_1, \tau \vartheta_1) u_s|_0 + |\vartheta_1| |\Phi(\tau R_1, \tau \vartheta_1, u_t) - u_t|_0$$

and that each summand of the right hand side divided by $|(R_1, \vartheta_1)|$ converges uniformly for $\tau \in [0,1]$ to 0 as $|(R_1, \vartheta_1)| \to 0$. Consequently,

$$\frac{|\Phi(R_1, \vartheta_1, u) - u - R_1 \cdot u_s - \vartheta_1 \cdot u_t|_0}{|(R_1, \vartheta_1)|}$$

$$= \left| \int_0^1 \frac{\Phi(\tau R_1, \tau \vartheta_1, R_1 \cdot u_s + \vartheta_1 \cdot u_t) - R_1 \cdot u_s + \vartheta_1 \cdot u_t}{|(R_1, \vartheta_1)|} \, d\tau \right|_0$$

$$\leq \int_0^1 \frac{|\Phi(\tau R_1, \tau \vartheta_1, R_1 \cdot u_s + \vartheta_1 \cdot u_t) - R_1 \cdot u_s + \vartheta_1 \cdot u_t|_0}{|(R_1, \vartheta_1)|} \, d\tau$$

and the right hand side converges to 0 as $|(R_1, \vartheta_1)| \to 0$. Next we show that

$$\frac{|\Phi(R + R_1, \vartheta + \vartheta_1, v) - \Phi(R, \vartheta, v)|_0}{|(R_1, \vartheta_1, v)|_1}$$

as $|(R_1, \vartheta_1, v)|_1 \to 0$. This is more tricky and relies on the compactness of the inclusion $E_1 \to E_0$.

Arguing indirectly we find an $\varepsilon > 0$ and a sequence $(R_1^n, \vartheta_1^n, v_n) \to (0,0,0)$ in $\mathbb{R}^2 \oplus E_1$ satisfying

$$\frac{|\Phi(R + R_1^n, \vartheta + \vartheta_1^n, v^n) - \Phi(R, \vartheta, v^n)|_0}{|(R_1^n, \vartheta_1^n, v_n)|_1} \geq \varepsilon. \quad (78)$$

The sequence

$$w^n = \frac{v^n}{|(R_1^n, \vartheta_1^n, v_n)|_1}$$

is bounded in $E_1$ and we may assume, using that the inclusion $E_1 \to E_0$ is compact, that $w^n \to w$ in $E_0$. By the already established $c^0$-continuity of $\Phi$ we conclude that

$$\frac{\Phi(R + R_1^n, \vartheta + \vartheta_1^n, v^n) - \Phi(R, \vartheta, v^n)}{|(R_1^n, \vartheta_1^n, v_n)|_1}$$

$$= \Phi(R + R_1^n, \vartheta + \vartheta_1^n, w^n) - \Phi(R, \vartheta, w^n)$$

$$\to \Phi(R, \vartheta, w) - \Phi(R, \vartheta, w) = 0,$$
At this point we have proved that $\Phi$ is of class $sc^1$ and its tangent map

$$T\Phi : T(\mathbb{R}^2 \oplus E) = \mathbb{R}^2 \oplus E^1 \oplus R^2 \oplus E \rightarrow TE = E^1 \oplus E$$

is given by

$$T\Phi(R, \vartheta, u, R_1, \vartheta_1, v) = (\Phi(R, \vartheta, u), \Phi(R, \vartheta, R_1u_s + \vartheta_1u_t + v)). \quad (79)$$

This will allow us to give an inductive argument to show that $\Phi$ is of class $sc^k$. We prove the following statements by induction:

1. The map $\Phi$ is of class $sc^k$ and for every projection $\pi : T^kE \rightarrow E^j$ onto one of the factors of $T^kE$, the composition $\pi \circ T^k\Phi$ is a linear combination of maps of the form

   $$A : \mathbb{R}^2 \oplus E^m \oplus \mathbb{R}^{[\alpha]} \rightarrow E^j, \quad (R, \vartheta, u, h) \mapsto \Phi(R, \vartheta, h_1 \cdot \ldots \cdot h_{|\alpha|} \cdot D^\alpha u)$$

   with a multi-index $\alpha$ satisfying $|\alpha| \leq m - j$.

2. We already know that $\Phi$ is $sc^1$. If $\pi : TE = E^1 \oplus E \rightarrow E^1$ is the projection onto the first factor, then, by (79), the map $\pi \circ T\Phi$ is given by

   $$\mathbb{R}^2 \oplus E^1 \rightarrow E^1, \quad (R, \vartheta, v) \mapsto \Phi((R, \vartheta), v).$$

   If $\pi : TE = E^1 \oplus E^0 \rightarrow E^0$ is the projection onto the second factor, then the composition $\pi \circ T\Phi$ is given by

   $$\mathbb{R}^2 \oplus E^1 \oplus \mathbb{R} \rightarrow E^0$$

   $$(R, \vartheta, u, R_1, \vartheta_1, v) \mapsto \Phi((R, \vartheta), R_1u_s + \vartheta_1u_t + v)$$

   which can be written as a sum of the maps of the following types,

   $$\mathbb{R}^2 \oplus E \rightarrow E, \quad (R, \vartheta, v) \mapsto \Phi(R, \vartheta, v)$$

   and

   $$\mathbb{R}^2 \oplus E^1 \oplus \mathbb{R} \rightarrow E, \quad (R, \vartheta, u, h) \mapsto \Phi((R, \vartheta, hD^\alpha u),$$

   where $\alpha = (1, 0)$ and $\alpha = (0, 1)$. Hence the assertion $(S_1)$ holds.
Now assume that we have proved \((S_k)\). We shall prove \((S_{k+1})\). We first 
show that the map \(T^k \Phi : T^k(\mathbb{R}^2 \oplus E) \rightarrow T^k E\) is \(sc^1\). It suffices to show that maps described in \((S_k)\) are all of class \(sc^1\). So, we consider the map 

\[ A : \mathbb{R}^2 \oplus E^m \oplus \mathbb{R}^{|\alpha|} \rightarrow E^j, \quad (R, \vartheta, u, h) \mapsto \Phi(R, \vartheta, h_1 \cdot \ldots \cdot h_{|\alpha|} \cdot D^\alpha u), \]

where \(|\alpha| \leq m - j\).

We observe that this map is a composition of the following maps. The 

first one, defined by 

\[ \mathbb{R}^2 \oplus E^m \oplus \mathbb{R}^{|\alpha|} \rightarrow \mathbb{R}^2 \oplus E^j, \quad (R, \vartheta, u, h) \mapsto (R, \vartheta, h_1 \cdot \ldots \cdot h_{|\alpha|} D^\alpha u) \]

is clearly \(sc^\infty\). The second map is our shift-map 

\[ \Phi : \mathbb{R}^2 \oplus E^j \rightarrow E^j. \]

By induction hypothesis the shift-map \(\Phi : \mathbb{R}^2 \oplus E \rightarrow E\) is \(sc^k\) so that by 

Proposition 2.2 the map \(\Phi : \mathbb{R}^2 \oplus E^j \rightarrow E^j\) is, in particular, \(sc^1\). Applying 

the chain rule, we conclude that the map \(A\) is \(sc^1\). At this point we know 

that \(\Phi\) is \(sc^{k+1}\) and it remains to show that the iterated tangent map \(T^{k+1} \Phi : T^{k+1}(\mathbb{R}^2 \oplus E) \rightarrow T^{k+1} E\) satisfies the remaining statements of \((S_{k+1})\).

We consider the composition \(\pi \circ T^{k+1} \Phi\) where \(\pi\) is the projection onto one 

of the factors of \(T^{k+1} E\). If \(\pi\) is a projection onto one of the first \(2^k\) factors, 

then \(\pi \circ T^{k+1} \Phi\) is a sum of maps obtained from the maps \(A\) described in \((S_k)\) 

by raising the index of the domain by 1. If \(\pi\) is a projection onto one of the 

last \(2^k\) factors, then \(\pi \circ T^{k+1} \Phi\) is a linear combination of derivatives of maps 

\(A\) in \((S_k)\). Hence we have to consider the map 

\[ A : \mathbb{R}^2 \oplus E^m \oplus \mathbb{R}^{|\alpha|} \rightarrow E^j, \quad (R, \vartheta, u, h) \mapsto \Phi(R, \vartheta, h_1 \cdot \ldots \cdot h_{|\alpha|} \cdot D^\alpha u). \]

Its derivative after raising the index of the domain is the map 

\[ \mathbb{R}^2 \oplus E^{m+1} \oplus \mathbb{R}^{|\alpha|} \oplus \mathbb{R}^2 \oplus E^m \oplus \mathbb{R}^{|\alpha|} \rightarrow E^j, \]

defined by 

\[
(R, \vartheta, u, h, \delta R, \delta \vartheta, \delta u, \delta h) \mapsto \Phi(R, \vartheta, u, h_1 \cdot \ldots \cdot h_{|\alpha|} \cdot D^\alpha \delta u) \\
+ \sum_{i=1}^{|\alpha|} \Phi(R, \vartheta, u, h_1 \cdot \ldots \cdot \delta h_i \cdot \ldots \cdot h_{|\alpha|} \cdot D^\alpha u) \\
+ \Phi(R, \vartheta, u, h^1 \cdot \ldots \cdot h_{|\alpha|} \cdot (\delta R \cdot \partial_x D^\alpha u + \delta \vartheta \cdot \partial_t D^\alpha u)).
\]
This is the sum of maps of the form:
\[ \mathbb{R}^2 \oplus E^m \oplus \mathbb{R}^{\lvert \alpha \rvert} \to E^j, \quad (c, d, v, h) \to \Phi(c, d, h_1 \cdot \ldots \cdot h_{\lvert \alpha \rvert} \cdot D^\alpha v) \]
with \( \lvert \alpha \rvert \leq m - j \), and
\[ \mathbb{R}^2 \oplus E^{m+1} \oplus \mathbb{R}^{\lvert \alpha \rvert+1} \to E^j \]
\[ (c, d, u, (h_1, \ldots, \delta h, \ldots h_{\lvert \alpha \rvert})) \to \Phi(c, d, h_1 \cdot \ldots \cdot \delta h \cdot \ldots \cdot h_{\lvert \alpha \rvert} D^\alpha u) \]
with \( \lvert \alpha \rvert \leq m - j \leq m + 1 - j \), and finally
\[ \mathbb{R}^2 \oplus E^{m+1} \oplus \mathbb{R}^{\lvert \alpha \rvert+1} \to E^j, \quad (c, d, u, (h, \gamma)) \to \Phi(c, d, h_1 \cdot \ldots \cdot h_{\lvert \alpha \rvert} \gamma D^\beta u), \]
where \( \beta = \alpha + (1, 0) \) or \( \beta = \alpha + (0, 1) \). Then \( \lvert \beta \rvert = \lvert \alpha \rvert + 1 \leq m + 1 - j \) and we have verified that the map \( \Phi \) satisfies \((S_{k+1})\). The proof of Proposition 4.2 is complete.

4.2 The Gluing Profile

Definition 4.3. A gluing profile is a smooth diffeomorphism
\[ \varphi : (0, 1] \to [0, \infty). \]

See Figure 7 for the graph of a gluing profile. In order to construct a polyfold structure on our moduli spaces, the gluing profile has to satisfy additional properties, which hold true for the special profile \( \varphi(x) = e^{1/x} - e \) we have chosen.

Lemma 4.4. We consider the gluing profile
\[ \varphi(x) = e^{1/x} - e \]
and define the function \( B : [0, r) \times [-R, R] \to \mathbb{R} \) for sufficiently small \( 0 < r < 1 \) by
\[ B(x, c) = \begin{cases} \varphi^{-1}[\varphi(x) + c] & \text{if } x \in (0, r) \\ 0 & \text{if } x = 0. \end{cases} \]
Then \( B \) is smooth and satisfies
\[ D_x B(0, c) = 1 \quad \text{and} \quad D^\alpha B(0, c) = 0 \]
for all multi-indices \( \alpha = (\alpha_1, \alpha_2) \) with \( \alpha_1 \geq 2 \) and \( \alpha_2 \geq 0 \).
Proof. The inverse of the function \( \varphi(x) = e^{1/x} - e \), defined on the domain \((0, 1]\), is the function

\[
\varphi^{-1}(y) = \frac{1}{\ln[e + y]}.
\]

Hence the function \( B \) satisfies

\[
B(x, c) = \frac{1}{\ln[e^{1/x} + c]}
\]

for \( x > 0 \). To prove our claim we have to show that

\[
B(x, c) \to 0, \quad D_x B(x, c) \to 1 \quad \text{and} \quad D^{n,m} B(x, c) \to 0 \quad (80)
\]

as \( x \to 0 \) uniformly in \( c \), for all \( n \geq 2 \). Writing \( \ln[e^{1/x} + c] = \ln[e^{1/x} \cdot (1 + c \cdot e^{-1/x})] = \frac{1}{x} + \ln[1 + c \cdot e^{-1/x}] \), we represent \( B(x, c) \) as

\[
B(x, c) = x \cdot \frac{1}{1 + x \ln[1 + c \cdot e^{-1/x}]} = x \cdot f(x, c)
\]

where

\[
f(x, c) = \frac{1}{1 + x \ln[1 + c \cdot e^{-1/x}]}.
\]

Clearly, \( f(x, c) \to 1 \) as \( x \to 0 \), uniformly in \( c \). Defining the function \( g \) by

\[
f(x, c) = \frac{1}{1 + g(x, c)}
\]

it suffices to show that \( D^\alpha g(x, c) \to 0 \) for \( |\alpha| \geq 1 \) uniformly in \( c \) as \( x \to 0 \). Since \( g(x, c) = x \ln[1 + c e^{-1/x}] \), this follows from the fact that the function \( h \), defined by \( h(x, c) = c e^{-1/x} \) satisfies \( D^\alpha h(x, c) \to 0 \), uniformly in \( c \), as \( x \to 0 \).

In order to prove the second assertion for \( B \), we observe that a derivative of order \( n \) of \( e^{1/x} \) is a product of \( e^{1/x} \) with a polynomial in the variable \( 1/x \) from which the desired assertion follows. \( \blacksquare \)

The function \( B(x, c) \) defined in Lemma 4.4 meets the assumptions of the next statement.
Lemma 4.5. Let $B : [0, 1) \times \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

$$B(0, c) = 0, \quad D_1 B(0, c) = 1 \quad \text{and} \quad D^n_1 B(0, c) = 0 \quad \text{for} \quad n \geq 2.$$ 

Then the function $f : D \setminus \{0\} \times \mathbb{R} \to \mathbb{C}$, defined by

$$f(z, c) = B(|z|, c)rac{z}{|z|},$$

is smooth and satisfies as $z \to 0$, uniformly in $c$ belonging to a bounded subset of $\mathbb{R}$,

$$f(z, c) \to 0$$

$$D_1 f(z, c) \to \text{Id}$$

$$D^n_1 f(z, c) \to 0 \quad \text{for} \quad n \geq 2.$$ \hspace{1cm} (81)

In particular, the function $f$ extends smoothly over $\{z = 0\} \times \mathbb{R}$.

Proof. If $g = g(s, c)$ we shall write $g^{(n)}$ for the $n$th derivative with respect to $s$. We begin with the following simple calculus observation. Assuming that $g(s, c)$ is a smooth function on $[0, 1] \times \mathbb{R}$ whose derivatives with respect to $s$ at $s = 0$ all vanish so that $g^{(n)}(0, c) = 0$ for all $n \geq 0$, the function $h(s, c)$, defined by $h(s, c) = g(s, c)/s$ for $s > 0$ and $h(0, c) = 0$, is smooth and
all its derivatives at $s = 0$ vanish so that $h^{(n)}(0, c) = 0$ for all $n \geq 0$. Indeed, we may represent $g(s)$ in the form

$$ g(s, c) = sR(s, c) \quad \text{with} \quad R(s, c) = \int_0^1 g'(\tau s, c) d\tau. $$

Observing that $R(s, c)$ is smooth for $s \geq 0$ and satisfies $R^{(n)}(0, c) = 0$ for all $n \geq 0$, we conclude that $h(s, c) = R(s, c)$ is smooth and satisfies $h^{(n)}(0, c) = 0$ for all $n \geq 0$.

We use this observation to obtain the following conclusions. Set $C_0(s, c) = B(s, c)/s$ for $s > 0$ and $C_0(0, c) = 1$, and define a sequence of functions $C_n(s, c) = C_{n-1}'(s, c)/s$ for $n \geq 1$ and $s > 0$ where the prime stands for the first derivative. We claim that every function $C_n(s, c)$ for $n \geq 1$ smoothly extends over $s = 0$ and the derivatives $C^{(k)}_n(0, c) = 0$ vanish for all $k \geq 0$. To see this we represent $B(s, c)$ using the assumptions as

$$ B(s, c) = s + s^2 \tilde{R}(s, c) \quad \text{with} \quad \tilde{R}(s) = \frac{1}{2} \int_0^1 (1 - \tau)^2 B^{(3)}(\tau s, c) d\tau. $$

Then $C_0(s, c) = 1 + s^2 \tilde{R}(s, c)$ and its first derivative is given by $C'_0(s, c) = 2s \tilde{R}(s, c) + s^2 \tilde{R}'(s, c)$. Hence $C_1(s, c) = 2\tilde{R}(s, c) + s \tilde{R}'(s, c)$ showing that $C_1(s, c)$ is smooth for $s \geq 0$ and $C^{(k)}_1(0, c) = 0$ for all $k \geq 0$. Our claim follows by applying this procedure successively to all the functions $C_n(s, c)$ for $n \geq 2$.

With the above definition of $C_0(s, c)$, the function $f(z, c)$ is defined as

$$ f(z, c) = B(|z|, c) \frac{z}{|z|} = C_0(|z|, c)z. $$

Differentiating this expression at $z \neq 0$ we obtain

$$ D_1f(z, c)h_1 = \frac{C_0'(|z|, c)}{|z|} \langle z, h_1 \rangle + C_0(|z|, c)h_1 = C_1(|z|, c)\langle z, h_1 \rangle + C_0(|z|, c)h_1. $$

Using the properties of $C_0(s, c)$ and $C_1(s, c)$ we conclude that $D_1f(z, c) \to \text{Id}$ as $z \to 0$. In general, the higher derivatives of $f(z, c)$ are of the form

$$ D^{(2n)}_1 f(z, c)(h_1, \ldots, h_{2n}) = C_{2n}(|z|, c)\langle z, h_1 \rangle \cdots \langle z, h_{2n} \rangle z \quad \text{for all} \quad n \geq 1 $$

$$ + \sum_{j=0}^{n-1} C_{n+j}(|z|, c)M_{2n,2j+1}(z, c, h_1, \ldots, h_{2n}) $$
\[D_{1}^{(2n+1)} f(z, c)(h_1, \ldots, h_{2n+1}) = C_{2n+1}(|z|, c) \langle z, h_1 \rangle \cdots \langle z, h_{2n+1} \rangle z + \sum_{j=0}^{n} C_{n+j}(|z|, c) M_{2n+1,2j}(z, c, h_1, \ldots, h_{2n+1}),\]

where \(M_{2n,2j+1}(z, c, \cdots)\) and \(M_{2n+1,2j}(z, c, \cdots)\) are \(2n\)- and \((2n + 1)\)-linear maps which are smoothly depending on \((z, c)\) and satisfying

\[|M_{2n,2j+1}(z, c, h_1, \ldots, h_{2n})| \leq |z|^{2j+1}|h_1| \cdots |h_{2n}|\]

and

\[|M_{2n+1,2j}(z, c, h_1, \ldots, h_{2n})| \leq |z|^{2j}|h_1| \cdots |h_{2n+1}|.\]

Consequently, in view of the properties of the functions \(C_n(s, c)\), the derivatives \(D_{1}^{(n)} f(z, c)\) tend to 0 for all \(n \geq 2\) as \(z \to 0\), and the proof of the lemma is complete. 

In our constructions we have to study the gluing length \(R\) and the gluing angle \(\vartheta\) associated to a nonzero gluing parameter \(a \in B \setminus \{0\}\) via the gluing profile \(\varphi\) defined by \(\varphi(x) = e^{\frac{x^2}{2}} - e\). The pair \((R, \vartheta)\) is defined by

\[R = \varphi(|a|) \quad \text{and} \quad a = |a| \cdot e^{-2\pi i \vartheta}.\]

It is important to have estimates for the map \(a \mapsto (R(a), \vartheta(a))\). In order to derive the appropriate estimates, we use the identification \(a = x + iy = (x, y)\). If \(\beta = (\beta_1, \beta_2)\) is a multi-index, we write

\[a^\beta = x^{\beta_1} \cdot y^{\beta_2}.\]

We shall prove the following statement.

**Lemma 4.6.** If \(0 < |a| < 1\), the partial derivative \((D^\alpha R)(a)\) of order \(|\alpha| \geq 1\) is a linear combination of terms of the form

\[e^{|a|} \cdot \frac{1}{|a|^k} a^\beta\]

with the integers \(k\) and the multi-indices \(\beta\) satisfying \(k \leq 3 \cdot |\alpha|\) and \(|\beta| \leq |\alpha|\).
Proof. In the case $|\alpha| = 1$, we may take without loss of generality $\alpha = (1, 0)$. Then, in view of $a = x + iy \equiv (x, y)$,

$$
\frac{\partial}{\partial x} R(a) = -e^{\frac{i}{|a|}} \cdot \frac{1}{|a|^2} \cdot \frac{x}{|a|} = -e^{\frac{i}{|a|}} \cdot \frac{x}{|a|^3}.
$$

Hence $\beta = (1, 0)$ and $3 \leq 3|(1, 0)|$ and $|\beta| \leq |\alpha|$. The same holds for the partial derivative with respect to $y$. Assuming the result to hold for all multi-indices $\alpha$ with $|\alpha| \leq l$, we consider the partial derivative $D^\alpha R$ for a multi-index $\alpha$ of order $|\alpha| = l + 1$. Without loss of generality we assume that $\alpha = \alpha_0 + (1, 0)$. We know that $D^{\alpha_0} R$ is the linear combination of terms of the form

$$
e^{\frac{i}{|a|}} \frac{1}{|a|^k} a^\beta
$$

where $|\beta| \leq l$ and $k \leq 3l$. Applying $\frac{\partial}{\partial x}$, we obtain

$$
\frac{\partial}{\partial x} \left( e^{\frac{i}{|a|}} \frac{1}{|a|^k} a^\beta \right) = e^{\frac{i}{|a|}} \cdot \left( -\frac{1}{|a|^2} \cdot \frac{x}{|a|} \cdot \frac{1}{|a|^k} a^\beta - \frac{k}{|a|^{k+1}} \cdot \frac{x}{|a|} a^\beta + \beta_1 \cdot \frac{1}{|a|^k} a^\beta' \right).
$$

Here $a^\beta' = 0$ if $\beta_1 = 0$ and otherwise $\beta' = \beta - (1, 0)$. This derivative is a linear combination of at most three terms, namely

$$
-e^{\frac{i}{|a|}} \cdot \frac{1}{|a|^{k+3}} \cdot a^{\beta+(1, 0)}, \quad -ke^{\frac{i}{|a|}} \cdot \frac{1}{|a|^{k+2}} \cdot a^{\beta+(1, 0)}, \quad \text{and} \quad e^{\frac{i}{|a|}} \cdot \frac{1}{|a|^k} a^\beta'.
$$

We see that $\beta$ is increased by at most one order and $k$ is increased by at most 3 so that our statement is proved.

Next we consider for $a \neq 0$, the function $\vartheta(a)$ defined by

$$
\frac{a}{|a|} = e^{-2\pi i \vartheta(a)}.
$$

**Lemma 4.7.** For every multi-index $\alpha$ satisfying $|\alpha| \geq 1$, the partial derivative $D^\alpha \vartheta(a)$ at a point $a \neq 0$, is a linear combination of terms of the form $a^\beta |a|^k$ with $k \leq 2|\alpha|$ and $|\beta| \leq |\alpha|$.

Proof. First we assume that $\alpha$ has order one. As long as $a = x + iy$ satisfies $x \neq 0$ we have $2\pi \cdot \vartheta(a) = \arctan(\frac{y}{x})$ so that

$$
2\pi \cdot \frac{\partial \vartheta}{\partial x}(a) = -\frac{y}{|a|^2} \quad \text{and} \quad 2\pi \cdot \frac{\partial \vartheta}{\partial y}(a) = \frac{x}{|a|^2}.
$$
This has the required form with \( k = 2 \) and \( \beta = (0, 1) \) or \( \beta(1, 0) \). We assume that the assertion has been proved for all \( \alpha \) of order \( l \) and compute the derivatives of order \( l + 1 \). Without loss of generality we may assume that \( \alpha = \alpha_0 + (1, 0) \). The derivative

\[
\frac{\partial}{\partial x} \left( \frac{a^\beta}{|a|^k} \right)
\]

is equal to

\[
\frac{\partial}{\partial x} \left( \frac{a^\beta}{|a|^k} \right) = \beta_1 \cdot \frac{a^{\beta'}}{|a|^{k+2}} - k \cdot \frac{a^\beta}{|a|^{k+2}}x = \beta_1 \cdot \frac{a^{\beta'}}{|a|^k} - k \cdot \frac{a^{\beta+(1,0)}}{|a|^{k+2}}.
\]

where we put \( a^{\beta'} = 0 \) if \( \beta_1 = 0 \) and otherwise \( \beta' = \beta - (0, 1) \). This shows that \( \beta \) increased at most by order one and \( k \) in the denominator by 2. This proves the assertion.

\[\blacksquare\]

### 4.3 Families of Sc-Isomorphisms

We assume that \( V \) is an open subset of a finite-dimensional vector space \( H \) and \( E \) and \( F \) are sc-Banach spaces and consider a family \( v \mapsto L(v) \) of linear operators parametrized by \( v \in V \) such that \( L(v) : E \rightarrow F \) are sc-isomorphisms having the following properties.

1. The map \( \hat{L} : V \oplus E \rightarrow F \), defined by
   \[
   \hat{L}(v, h) := L(v)h,
   \]
   is sc-smooth.

2. There exists for every \( m \) a constant \( C_m \) such that
   \[
   |L(v)h|_m \geq C_m \cdot |h|_m
   \]
   for all \( v \in V \) and all \( h \in E_m \).

Let us note that \( L(v) \) is not assumed to be continuously depending on \( v \) as an operator. Since the map \( L(v) : E \rightarrow F \) is an sc-isomorphism, the equation

\[
L(v)h = k
\]

has for every pair \( (v, k) \in V \oplus F \) a unique solution \( h \) denoted by

\[
h = L(v)^{-1}k =: f(v, k),
\]

so that \( \hat{L}(v, f(v, k)) = k \).
Proposition 4.8. The map $f : V \oplus F \to E$ defined above is sc-smooth.

Proof. We start by proving that $f$ is sc-continuous. We fix a level $m$ and recall that $|L(v)h|_m \geq C_m \cdot |h|_m$ for all $v \in V$ and $h \in E_m$. We take a point $(v_0, k_0) \in V \oplus F_m$ and a sequence $(v_n, k_n) \in V \oplus F_m$ converging to $(v_0, k_0)$. Setting $f(v_n, k_n) = h_n$ and $f(v_0, k_0) = h_0$ we compute,

$$L(v_n)(h_n - h_0) = k_n - L(v_n)h_0$$

$$= k_n - L(v_0)h_0 + L(v_0)h_0 - L(v_0)h_0$$

$$= k_n - k_0 + L(v_0)h_0 - L(v_n)h_0 =: \delta_n.$$

Since the map $(v, h) \mapsto L(v)h$ is sc-smooth, it follows that $\delta_n \to 0$ in $F_m$, and using property (2) above,

$$|f(v_n, k_n) - f(v_0, k_0)|_m = |h_n - h_0|_m$$

$$\leq \frac{1}{C_m} \cdot |L(v_n)(h_n - h_0)|_m = \frac{1}{C_m} \cdot |\delta_n|_m \to 0.$$

Consequently, $f$ is sc-continuous.

Next we shall show that $f$ is a map of class sc$^1$. In order to define the candidate for the linearization $Df(v, k) : H \oplus F \to E$ of the map $f$ at the point $(v, k) \in V \oplus F_1$, we formally differentiate the equation

$$\hat{L}(v, f(v, k)) = k$$

and obtain,

$$\delta k = D_1\hat{L}(v, f(v, k)) \cdot \delta v + D_2\hat{L}(v, f(v, k)) \cdot Df(v, k) \cdot [\delta v, \delta k]$$

$$= D_1\hat{L}(v, f(v, k)) \cdot \delta v + \hat{L}(v, Df(v, k) \cdot [\delta v, \delta k])$$

$$= D_1\hat{L}(v, f(v, k)) \cdot \delta v + L(v)Df(v, k) \cdot [\delta v, \delta k]$$

where $[\delta v, \delta k] \in H \oplus F_1$. Hence

$$Df(v, k) \cdot [\delta v, \delta k] = L(v)^{-1}(\delta k - D_1\hat{L}(v, f(v, k)) \cdot \delta v)$$

$$= f(v, \delta k - D_1\hat{L}(v, f(v, k)) \cdot \delta v)$$

for all $[\delta v, \delta k]$ in $H \oplus E_1$. Observe that for fixed $(v, k) \in V \oplus F_1$, the map $(\delta v, \delta k) \mapsto \delta k - D_1\hat{L}(v, f(v, k))$ defines a bounded linear operator between $H \oplus F_0$ and $F_0$. Then, since $L(v) : E_0 \to F_0$ is a linear isomorphism, the
right-hand side in the first line of the identity above defines a bounded linear operator between $H \oplus F_0$ and $E_0$. In addition, since $f$ is an sc$^0$-map, it follows that the map $(v, k, \delta v, \delta k) \in V \oplus F_{m+1} \oplus H \oplus F_m \to E_m$, given by

$$(v, k, \delta v, \delta k) \mapsto Df(v, k) \cdot [\delta v, \delta k],$$

is continuous so that $Df$ is sc$^0$.

It remains to verify the approximation property. We take $(v, k) \in V \oplus F_1$ and $(\delta v, \delta k) \in H \oplus F_1$ and abbreviate

$$\delta h = f(v + \delta v, k + \delta k) - f(v, k) - f(v, \delta k - D_1 \tilde{L}(v, f(v,k)) \cdot \delta v) \in E_0.$$ 

Since $C_0|\delta h|_0 \leq |L(v + \delta v)\delta h|_0$, it suffices to show that

$$\frac{1}{|\delta v| + |\delta k|_1}|L(v + \delta v)\delta h|_0 \to 0.$$ 

In order to prove this, we compute at the point $(v, k) \in V \oplus F_1$ for $(\delta v, \delta k) \in H \oplus F_1$,

$$L(v + \delta v)\left[f(v + \delta v, k + \delta k) - f(v, k) - f(v, \delta k - D_1 \tilde{L}(v, f(v,k)) \cdot \delta v)\right] = k + \delta k - L(v + \delta v) \cdot f(v, k) - L(v + \delta v) \cdot f(v, \delta k - D_1 \tilde{L}(v, f(v,k)) \cdot \delta v)$$

$$= \delta k - D_1 \tilde{L}(v, f(v,k)) \cdot \delta v - [L(v + \delta v) \cdot f(v, k) - L(v) \cdot f(v, k) - D_1 \tilde{L}(v, f(v,k)) \cdot \delta v]$$

$$= L(v + \delta v) \cdot f(v, \delta k - D_1 \tilde{L}(v, f(v,k)) \cdot \delta v)$$

$$= -[L(v + \delta v) \cdot f(v, k) - L(v) \cdot f(v, k) - D_1 \tilde{L}(v, f(v,k)) \cdot \delta v]$$

$$= \frac{1}{|\delta v| + |\delta k|_1} \cdot |(L(v + \delta v) \cdot f(v, k) - L(v) \cdot f(v, k) - D_1 \tilde{L}(v, f(v,k)) \cdot \delta v)|_0 \to 0,$$

as $|\delta v| + |\delta k|_1 \to 0$. We next consider the second term. The map $f(v, k)$ is
linear with respect to the variable $k$ so that for fixed $(v, k)$

$$
\frac{1}{|\delta v| + |\delta k|_1} \left| [L(v + \delta v) - L(v)] \cdot f(v, \delta k - D_1 \tilde{L}(v, f(v, k)) \cdot \delta v) \right|_0 \\
\leq \left| [L(v + \delta v) - L(v)] \cdot f \left(v, \frac{\delta k}{|\delta v| + |\delta k|_1} \right) \right|_0 \\
+ \left| [L(v + \delta v) - L(v)] \cdot f \left(v, D_1 \tilde{L}(v, f(v, k)) \frac{\delta v}{|\delta v| + |\delta k|_1} \right) \right|_0 \\
= I_1 + I_2
$$

Since the embedding $E_1 \to E_0$ is compact and $\delta k \in F_1$, we may assume without loss of generality that $\frac{\delta k}{|\delta v| + |\delta k|_1}$ converges in $F_0$ and since the maps $f$ is $sc^0$ and $(v, h) \mapsto \tilde{L}(v, h)$ is $sc$-smooth, we conclude that $I_1 \to 0$ as $|\delta v| + |\delta k|_1 \to 0$.

Since $\delta v$ is an element of the finite-dimensional space $H$, we may assume that $\frac{\delta v}{|\delta v| + |\delta k|_1}$ converges in $F_0$. Using again that $f$ is $sc^0$ and $(v, h) \mapsto \tilde{L}(v, h)$ is $sc$-smooth, we conclude that also the second term $I_2$ converges to 0 as $|\delta v| + |\delta k|_1 \to 0$. Summing up, we have proved that

$$
\frac{1}{|\delta v| + |\delta k|_1} |L(v + \delta v)\delta h|_0 \to 0.
$$

Consequently, the map $f : V \oplus F \to E$ is of class $sc^1$.

We have also proved that the tangent map $Tf : V \oplus E_1 \oplus H \oplus E_0 \to F_1 \oplus F_0$ has the form

$$
Tf(v, k, \delta v, \delta k) = (f(v, k), f(v, \delta k - D_1 \tilde{L}(v, f(v, k)) \cdot \delta v))
$$

which is an $sc^0$-map.

To prove that $f$ is of class $sc^2$, it suffices to show that $f$ is of class $sc^2$ on $V' \oplus F$ for every subset $V' \subset V$ having compact closure in $V$. We introduce the family $(v, \delta v) \mapsto L^1(v, \delta v)$, parametrized by $(v, \delta v) \in TV$, of $sc$-operators $L^1(v, \delta v) : TE \to TF$ defined by

$$(h, \delta h) \mapsto (L(v)h, D_1 \tilde{L}(v, h) \cdot [\delta v, \delta h]) = (L(v)h, L(v)\delta h + D_1 \tilde{L}(v, h) \cdot \delta v).$$

It follows from the properties of the family $v \mapsto L(v)$ that the map $L^1(v, \delta v)$ is an $sc$-isomorphism and the map

$$
TV \oplus TE \to TF, \quad (v, \delta v, h, \delta h) \mapsto L^1(v, \delta v)(h, \delta h)
$$
is sc-smooth.

We fix an open subset $V' \subset V$ having compact closure in $V$. Since by our assumption the map $\hat{L} : V' \oplus E \to F$ is sc-smooth, it follows that also the map

$$V' \oplus H \oplus E \to F, \quad (v, \delta v, h) \mapsto D_1 \hat{L}(v, h) \cdot \delta v$$

is sc-smooth. We conclude, by a compactness argument, that for every level $m$ there exists a positive constant $d_m$ so that

$$|D_1 \hat{L}(v, h) \cdot \delta v|_m \leq d_m \cdot |h|_{m+1} \cdot |\delta v|_m$$

for $v \in V'$, $\delta v \in H$, and $h \in F_{m+1}$. This implies that given level $m$ there exists a constant $C'_m$ such that

$$|L^1(v, \delta v)(h, \delta h)|_m \geq C'_m \cdot |(h, \delta h)|_m$$

(82)

for all $v \in V'$ and all $\delta v \in H$ satisfying $|\delta v| < 1$. Indeed, if $0 < \varepsilon \leq C_m$, where $C_m$ is the constant required in property (2) at the beginning of this section, then

$$|L^1(v, \delta v)(h, \delta h)|_m = |L(v)h|_{m+1} + |L(v)\delta h + D_1 \hat{L}(v, h)\delta v|_m$$

$$\geq C_{m+1} \cdot |h|_{m+1} + C_m \cdot |\delta h + L(v)^{-1}D_1 \hat{L}(v, h)\delta v|_m$$

$$\geq C_{m+1} \cdot |h|_{m+1} + \varepsilon \cdot |\delta h + L(v)^{-1}D_1 \hat{L}(v, h)\delta v|_m$$

$$\geq C_{m+1} \cdot |h|_{m+1} + \varepsilon \cdot |\delta h|_m - \varepsilon \cdot |L(v)^{-1}D_1 \hat{L}(v, h)\delta v|_m$$

$$\geq C_{m+1} \cdot |h|_{m+1} + \varepsilon \cdot |\delta h|_m - \varepsilon \cdot C_m \cdot d_m \cdot |h|_{m+1}$$

for all $v \in V'$ and all $\delta v \in H$ satisfying $|\delta v| < 1$. Hence, taking $\varepsilon > 0$ small enough we obtain the desired estimate (82). Now our previous discussion applied to $L^1$ shows that the map

$$f^1 : V' \oplus \{\delta v \in H| \ |\delta v| < 1\} \oplus TF \to TE,$$

defined as the solution of

$$L^1(v, \delta v)f^1(v, \delta v, k, \delta k) = (k, \delta k),$$

is of class sc$^1$. Now we observe that

$$f^1(v, \delta v, k, \delta k) = (f(v, k), f(v, \delta k - D_1 \hat{L}(v, f(v, k)) \cdot \delta v)).$$

This shows that the tangent map $Tf$ is of class sc$^1$ implying that $f$ is of class sc$^2$. The result now follows by induction.

The above result remains true if $V$ is relatively open in the partial quadrant of a finite-dimensional vector space.
4.4 Cauchy-Riemann Operators

The crucial point in the discussion of the Fredholm property of our Cauchy-Riemann operator is the behavior of the operator under the gluing of the half-cylinders. We start with the linear Cauchy-Riemann operator and first recall some standard facts. As usual we use the symbol $Z_a$ to denote the finite glued cylinders and $C_a$ to denote the infinite glued cylinders.

For the first result we work on the sc- Hilbert space $H^3_{c,\delta_0}(C_a, \mathbb{R}^2)$ which consists of all maps $u : C_a \to \mathbb{R}^2$ in $H^3_{loc}$ so that there exists a constant $c \in \mathbb{R}^2$ for which the map $u - c$ has weak partial derivatives up to order 3 which weighted by $e^{\delta_0|s|}$ belong to $L^2((0, \infty) \times S^1)$ while the weak partial derivatives up to order 3 of the map $u + c$ belong to $L^2((\infty, 0] \times S^1)$. The level $m$ of the sc-Hilbert space $H^3_{c,\delta_0}(C_a, \mathbb{R}^2)$ corresponds to the Sobolev regularity $(m + 3, \delta_m)$ and the level $m$ of $H^2_{c,\delta_0}(C_a, \mathbb{R}^2)$ corresponds to the regularity $(2 + m, \delta_m)$ where $(\delta_m)_{m \geq 0}$ is a strictly increasing sequence satisfying $0 < \delta_m < 2\pi$. The norms of these Hilbert spaces are defined in Section 2.5.

Lemma 4.9. The Cauchy-Riemann operator

$$\overline{\partial}_0 : H^3_{c,\delta_0}(C_a, \mathbb{R}^2) \to H^2_{c,\delta_0}(C_a, \mathbb{R}^2), \quad \xi \mapsto \overline{\partial}_0 \xi$$

is an sc-isomorphism. In particular, for every $m \geq 0$ there exists a constant $C_m$ such that

$$\frac{1}{C_m} \cdot |\xi|_{H^{3+m,\delta_m}} \leq |\overline{\partial}_0 \xi|_{H^{2+m,\delta_m}} \leq C_m \cdot |\xi|_{H^{3+m,\delta_m}}.$$

We also need the Cauchy Riemann operator in a different set-up. Assume that we have two copies of the cylinder $\mathbb{R} \times S^1$, which we denote by $\Sigma^\pm$. Then viewing $\mathbb{R}^+ \times S^1 \subset \Sigma^+$ and $\mathbb{R}^- \times S^1 \subset \Sigma^-$ the original gluing construction carried out for the half-cylinders results as before in $C_a$, besides $Z_a$ we also have an infinite cylinder denoted by $Z_a^*$, as illustrated in Figure 8. It is important that $Z_a^*$ and $C_a$ both contain $Z_a$ in a natural way and the two different holomorphic coordinates extend to $Z_a^*$ as well as to $C_a$.

The Hilbert spaces $H^{3+m,-\delta_m}(Z_a^*, \mathbb{R}^2)$ for $m \geq 0$ consist of maps $u : Z_a^* \to \mathbb{R}^2$ for which the associated maps $v : \mathbb{R} \times S^1 \to \mathbb{R}^2$, defined by $(s, t) \mapsto u([s, t])$ have partial derivatives up to order $3 + m$ which, if weighted by $e^{-\delta_m|s-\frac{\pi}{2}|}$ belong to $L^2(\mathbb{R} \times S^1)$. The spaces $H^{2+m,-\delta_m}(Z_a^*, \mathbb{R}^2)$ are defined
Figure 8: The extended cylinder $Z^*_a$.

analogously. The norms are denoted by $\|u\|_{3+m,-\delta_m}$, respectively $\|u\|_{2+m,-\delta_m}$. The spaces $H^{3+m,-\delta_m}(Z_a, \mathbb{R}^2)$ and $H^{2+m,-\delta_m}(Z_a, \mathbb{R}^2)$ are defined the same way. Their norms are denoted by $\|u\|_{3+m,-\delta_m}$, respectively by $\|u\|_{2+m,-\delta_m}$. We would like to point out that there is no sc-structures on $H^{3,0}(Z^*_a, \mathbb{R}^2)$, respectively $H^{2,0}(Z^*_a, \mathbb{R}^2)$, where the level $m$ corresponds to the regularity $(3+m,-\delta_m)$, respectively $(2+m,-\delta_m)$. We denote by $[u]_a$ the average of a map $u : \mathbb{R} \times S^1 \to \mathbb{R}^2$ over the circle at $[R/2, t]$, defined by

$$[u]_a := \int_{S^1} u\left[\frac{R}{2}, t\right] dt.$$

**Lemma 4.10.** The Cauchy-Riemann operator

$$\partial_0 : H^{3+m,-\delta_m}(Z^*_a, \mathbb{R}^2) \to H^{2+m,-\delta_m}(Z^*_a, \mathbb{R}^2), \quad \xi \mapsto \partial_0 u$$

is a surjective Fredholm operator of real Fredholm index equal to 2 for all $m \geq 0$. The kernel consists of the constant functions. Moreover, there exists a constant $C_m > 0$ such that

$$\frac{1}{C_m} \|u - [u]_a\|_{m+3,-\delta_m}^* \leq \|\partial_0(u - [u]_a)\|_{m+2,-\delta_m}^* \leq C_m \cdot \|u - [u]_a\|_{m+3,-\delta_m}^*.$$
4.5 Proof of Proposition 3.32

Recalling Remark 3.31 and the definition of the family $\lambda \rightarrow L(\lambda)$ from Section 3.4, we shall prove the Proposition 3.32.

We recall that the space $\widehat{E}$ consists of pairs $(h^+, h^-)$ with $h^\pm \in H^3_{\delta_0}(\mathbb{R}^\pm \times S^1, \mathbb{R}^2)$ equipped with the obvious sc-structure and $\widehat{E}_0$ is the closed subspace of $\widehat{E}$ consisting of those pairs$(h^+, h^-)$ which satisfy $h^+(0, 0) = (0, 0)$ and $h^\pm(0, t) \in \{0\} \times \mathbb{R}$. Using the decomposition $h^\pm = h^\pm_\infty + r^\pm$ in which $h^\pm_\infty$ are the asymptotic constants and $r^\pm \in H^{3+m, \delta_m}(\mathbb{R}^\pm \times S^1)$, the $\widehat{E}_m$-norm of $(h^+, h^-)$ is defined as

$$|(h^+, h^-)|^2_{\widehat{E}_m} = |h^\pm_\infty|^2 + |h^-_\infty|^2 + |r^+|^2_{H^{3+m, \delta_m}} + |r^-|^2_{H^{3+m, \delta_m}}. \quad (83)$$

We also recall the space $F$ consists of pairs $(\eta^+, \eta^-) \in H^{2, \delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^2) \oplus H^{2, \delta_0}(\mathbb{R}^- \times S^1, \mathbb{R}^2)$ equipped with the sc-structure $H^{2+m, \delta_m}(\mathbb{R}^+ \times S^1, \mathbb{R}^2) \oplus H^{2, \delta_0}(\mathbb{R}^- \times S^1, \mathbb{R}^2)$. The $F_m$-norm of the pair $(\eta^+, \eta^-) \in F$ is given by

$$|((\eta^+, \eta^-))|^2_{F_m} = |\eta^+_m|^2_{H^{m+2, \delta_m}} + |\eta^-|^2_{H^{m+2, \delta_m}}. \quad (84)$$

**Proposition 3.32** If $(\lambda, h^+, h^-) \rightarrow L(\lambda)(h^+, h^-)$ and $\lambda = (v, a)$ is the sc-smooth map defined in (75) in a neighborhood of $\lambda_0 = (v_0, 0)$, then there exists a constant $\sigma > 0$ so that the following holds.

1. If $\lambda = (v, a)$ satisfies $|\lambda - \lambda_0| < \sigma$, then the linear operator $L(\lambda) : \widehat{E}_0 \rightarrow F$ is an sc-isomorphism.

2. For every $m \geq 0$ there exists a constant $C_m$ independent of $\lambda$ so that the norm of the inverse operators

$$L(\lambda)^{-1} : F_m \rightarrow (\widehat{E}_0)_m$$

is bounded by $C_m$ for every $|\lambda - \lambda_0| < \sigma$.

**Proof.** We already know from Remark 3.31 that for $\lambda_0 = (v_0, 0)$ the operator $L(\lambda_0) : (\widehat{E}_0) \rightarrow F$ is an sc-isomorphism. We also know that $\lambda \mapsto L(\lambda)$ is a family of Fredholm operators of index 0,

$$\text{ind } L(\lambda) = \text{ind } L(\lambda_0) = 0.$$

The family is not continuous in the operator norm. Therefore, our main task is to prove the following injectivity estimate.
(*) There exists a constant $\sigma > 0$ independent of $m$, so that for every level $m$ there exists a constant $C_m$ such that

$$|L(\lambda)(h^+, h^-)|_{F_m} \geq C_m |(h^+, h^-)|_{\tilde{E}_m}$$  \hspace{1cm} (84)

for all $(h^+, h^-) \in (\tilde{E}_0)_m$ and all $\lambda$ satisfying $|\lambda - \lambda_0| < \sigma$.

We claim that (*) is a consequence of the following statement (**).

(**) There exists an open neighborhood $U$ of the point $\lambda_0 = (v_0, 0)$ in $H \times \mathbb{C}$ such that the following holds: Given a level $m$, a point $\lambda \in U$, and two sequence $\lambda_k \to \lambda$, and $(h_k^+, h_k^-) \in (\tilde{E}_0)_m$ satisfying $|(h_k^+, h_k^-)|_{\tilde{E}_m} = 1$ and

$$|L(\lambda_k)(h_k^+, h_k^-)|_{\tilde{E}_m} \to 0,$$

the sequence $((h_k^+, h_k^-))$ has a convergent subsequence in $(\tilde{E}_0)_m$.

Let us show that the statement (***) implies the statement (*). Assuming that (**) holds, we first consider the level 0 and claim that there are positive constants $\sigma'$ and $C_0$ such that

$$|L(\lambda)(h^+, h^-)|_{F_0} \geq C_0 |(h^+, h^-)|_{\tilde{E}_0}$$  \hspace{1cm} (85)

for all $|\lambda - \lambda| < \sigma'$ and all $(h^+, h^-) \in (\tilde{E}_0)_0$. Indeed, otherwise, we find sequences $\lambda_k \to \lambda_0$ and $(h_k^+, h_k^-) \in (\tilde{E}_0)_0$ satisfying $|(h_k^+, h_k^-)|_{\tilde{E}_0} = 1$ and $L(\lambda_k)(h_k^+, h_k^-) \to 0$ in $F_0$. Applying (**), we may assume that the sequence $(h_k^+, h_k^-)$ converges in $\tilde{E}_0$ to a pair $(h^+, h^-) \in (\tilde{E}_0)_0$ whose $\tilde{E}_0$-norm is equal to 1. Consequently,

$$L(\lambda_k)(h_k^+, h_k^-) \to L(\lambda_0)(h^+, h^-) = (0, 0).$$

This is impossible since $L(\lambda_0)$ is an isomorphism and so our claimed is proved. Since $\text{ind } L(\lambda) = \text{ind } L(\lambda_0) = 0$, it follows from (85) that the linear operators $L(\lambda) : \tilde{E}_0 \to F$ are sc-isomorphism for all $|\lambda - \lambda_0| < \sigma'$. This proves the statement (1) of Proposition 3.3.2.

Next we fix a level $m$ and a parameter $\lambda$ satisfying $|\lambda - \lambda_0| < \sigma'$. Arguing as above and using that $L(\lambda) : (\tilde{E}_0)_m \to F_m$ is an isomorphism for this $\lambda$, we find an open neighborhood $U_{\lambda,m}$ of $\lambda$ in $H \times \mathbb{C}$ and a positive constant $c_{\lambda,m}$ such that

$$|L(\lambda')(h^+, h^-)|_{F_m} \geq c_{\lambda,m} |(h^+, h^-)|_{\tilde{E}_m}$$  \hspace{1cm} (86)
for all $\lambda' \in U_{\lambda,m}$ and $(h^+, h^-) \in (\tilde{E}_0)_m$.

We choose $0 < \sigma < \sigma'$. Since the closed ball $\overline{E}_\sigma(\lambda_0)$ is compact in a finite dimensional space $H \times \mathbb{C}$, we find finitely many open sets $U_{\lambda_1,m}, \ldots, U_{\lambda_k,m}$ covering $B_\sigma(\lambda_0)$ such that the estimate \((86)\) holds for all $\lambda \in U_{\lambda_j,m}$ with constants $c_{\lambda_j,m}$ replacing $c_{\lambda,m}$. Choosing $C_m := \min\{c_{\lambda_1,m}, \ldots, c_{\lambda_k,m}\}$, we conclude

$$|L(\lambda')(h^+, h^-)|_{F_m} \geq C_m|(h^+, h^-)|_{\tilde{E}_m}$$

for all $\lambda \in B_\sigma(\lambda_0)$ and $(h^+, h^-) \in (\tilde{E}_0)_m$. Since the constant $\sigma$ is independent of level $m$, we have proved that \((\ast\ast)\) indeed implies \((\ast)\).

It remains to prove \((\ast\ast)\). We first define the set $U \subset H \times \mathbb{C}$. We recall that on the half-cylinders $\mathbb{R}^+ \times S^1$ we are given smooth families $v \mapsto j^+(v)$ of complex structures satisfying $j^+(v) = i$ on $[s_0 - 1, \infty) \times S^1$ and $j^-(v) = i$ on $(-\infty, -s_0 + 1] \times S^1$. We recall the abbreviations

$$\overline{\partial}_v h = \frac{1}{2}[Th + i \circ Th \circ j(v)] \frac{\partial}{\partial s}$$

$$\overline{\partial}_0 h = \frac{1}{2}[Th + i \circ Th \circ i] \frac{\partial}{\partial s}$$

of the Cauchy-Riemann operators.

In view of the standard elliptic estimates we have

$$|\overline{\partial}_v h|_{H^{2+m}} \geq C|h|_{H^{3+m}}$$

for some positive constant $C$ and all $h \in H^{3+m}([0, s_0] \times S^1)$ having compact supports in $[0, s_0] \times S^1$ and satisfying $h({0}) \times S^1 \subset {0} \times \mathbb{R}$. Observe that since $0 < \delta_m < 2\pi$, the norms on the Sobolev spaces $H^{2+m}([0, s_0] \times S^1)$ and $H^{2+m, \delta_m}([0, s_0] \times S^1)$ are equivalent so that the above estimate can be restated as

$$|\overline{\partial}_v h|_{H^{2+m, \delta_m}} \geq C|h|_{H^{3+m, \delta_m}}.$$

Since

$$\overline{\partial}_v h = \overline{\partial}_0 h + \frac{1}{2}[i \circ (Th) \circ (j^+(v) - j^+(v_0)] \frac{\partial}{\partial s}$$

and the family $v \mapsto j^+(v)$ of complex structures on the half-cylinder $\mathbb{R}^+ \times S^1$ is smooth, we conclude that

$$|\overline{\partial}_v h|_{H^{2+m, \delta_m}} \geq |\overline{\partial}_0 h|_{H^{2+m, \delta_m}} - c(v)|h|_{H^{3+m, \delta_m}}$$
where \( c(v) \) is a function converging to 0 as \( v \to v_0 \). Consequently, we may choose positive constants \( C_m \) and \( \rho \) so that
\[
|\overline{\partial}_v h|_{H^{2+m,\delta_m}} \geq C_m |h|_{H^{3+m,\delta_m}}
\]
(87)
for all \( v \in B_\rho(v_0) \) and all maps \( h \in H^{3+m}([0, s_0] \times S^1) \) having compact supports in \([0, s_0] \times S^1\) and satisfying \( h(\{0\} \times S^1) \subset \{0\} \times \mathbb{R} \). Taking, if necessary, smaller constants \( C_m \) and \( \rho \), we have also the estimate
\[
|\overline{\partial}_v h|_{H^{2+m,\delta_m}} \geq C |h|_{H^{3+m,\delta_m}}
\]
for the Cauchy-Riemann operator \( \overline{\partial}_v \) acting on maps \( h \in H^{3+m}([-s_0, 0] \times S^1) \) having compact supports in \([-s_0, 0] \times S^1\) and satisfying \( h(\{0\} \times S^1) \subset \{0\} \times \mathbb{R} \) for all \( v \in B_\rho(v_0) \). Having defined \( \rho \), we choose a sufficiently small number \( \tau > 0 \) satisfying \( 2s_0 + 4 < \varphi(\tau) = e^\tau - e \) and set
\[
U = B_\rho(v_0) \times B_\tau(0) \subset H \times \mathbb{C}.
\]

With this choice of the set \( U \) we are ready to prove the statement (**) We fix a point \( \lambda = (v, a) \in U \), a level \( m \), and take two sequences \((h_k^+, h_k^-) \in (E_0)_m \) and \( \lambda_k = (v_k, a_k) \in U \) satisfying \(|(h_k^+, h_k^-)|_{E_m} = 1 \) and \( \lambda_k \to \lambda = (v, a) \) and
\[
|L(\lambda_k)(h_k^+, h_k^+)|_{E_m} \to 0.
\]
(88)
We shall show that the sequence \(((h_k^+, h_k^-))\) has a converging subsequence in \((E_0)_m \). Since
\[
1 = |(h_k^+, h_k^-)|_{E_m} = |h_{k,\infty}^+|^2 + |h_{k,\infty}^-|^2 + |r_k^+|^2_{H^{3+m,\delta_m}} + |r_k^-|^2_{H^{3+m,\delta_m}},
\]
where \( h_k^\pm = h_{k,\infty}^\pm + r_k^\pm \) in which \( h_{k,\infty}^\pm \) are the asymptotic constants and \( r_k^\pm \in H^{3+m,\delta_m}(\mathbb{R}^\pm \times S^1) \), the sequences \( h_{k,\infty}^+ \) and \( h_{k,\infty}^- \) are bounded in \( \mathbb{R}^2 \) so that we can assume without loss of generality the convergence \((h_{k,\infty}^+, h_{k,\infty}^-) \to (h_\infty^+, h_\infty^-) \). In addition, the compact embedding theorem implies that there are subsequences again denoted by \( h_k^+ \) and \( h_k^- \) converging in \( H^{2+m,\delta_m} \) on \([0, s_0 + 1] \times S^1 \) and \([-s_0 - 1, 0] \times S^1 \).

Next we choose a smooth function \( \alpha_+ : \mathbb{R} \to [0, 1] \) having derivative \( \alpha'_+ \leq 0 \) and satisfying \( \alpha_+(s) = 1 \) on \((-\infty, s_0 - 1]\) and \( \alpha(s) = 0 \) on \([s_0, \infty) \). We define \( \alpha_-(s) = \alpha_+(s) \) and set \( \gamma_\pm = 1 - \alpha_\pm \).
In order to prove our claim we shall show, using the decomposition
\( h_k^\pm = h_{k,\infty}^\pm + r_k^\pm \), that the sequences \((\alpha_+ r_k^+, \alpha_- r_k^-)\) and \((\gamma_+ r_k^+, \gamma_- r_k^-)\) posses subsequences satisfying

\[
(\alpha_+ r_k^+, \alpha_- r_k^-) \rightarrow (f^+, f^-) \quad \text{and} \quad (\gamma_+ r_k^+, \gamma_- r_k^-) \rightarrow (g^+, g^-)
\]  

(89)
in \( \hat{E}_m \) for two pairs \((f^+, f^-)\) and \((g^+, g^-)\) belonging to \( \hat{E}_m \). We already know that the sequence \((h_{k,\infty}^+, h_{k,\infty}^-)\) of asymptotic constants converges to \((h_\infty^+, h_\infty^-)\).

Then assuming the convergence in (89) and setting \( h_k^\pm = h_{k,\infty}^\pm + \alpha_\pm r_k^\pm + \gamma_\pm r_k^\pm \), we conclude, using \( h_k^\pm = h_{k,\infty}^\pm + \alpha_\pm r_k^\pm + \gamma_\pm r_k^\pm \), that

\[
(h_k^+, h_k^-) \rightarrow (h^+, h^-)
\]
in \( \hat{E}_m \), as claimed.

We begin with the sequence \((\alpha_+ r_k^+, \alpha_- r_k^-)\), and abbreviate

\[
L(\lambda_k)(h_k^+, h_k^-) = (\eta_k^+, \eta_k^-).
\]  

(90)

It follows from formula (73) for the operator \( L(\lambda) \) that

\[
\begin{bmatrix}
\eta_k^+ \\
\eta_k^-
\end{bmatrix} = \begin{bmatrix}
\overline{\partial_v h_k^+} \\
\overline{\partial_v h_k^-}
\end{bmatrix} + \begin{bmatrix}
\Phi_{\alpha k}^+(h_k^+, h_k^-) \\
\Phi_{\alpha k}^-(h_k^+, h_k^-)
\end{bmatrix}
\]

where the maps \( \Phi_{\alpha k}^+(h_k^+, h_k^-) \) vanish outside of the finite cylinders \([\frac{R_k}{2} - 1, \frac{R_k}{2} + 1] \times S^1 \) and \([-\frac{R_k}{2} - 1, -\frac{R_k}{2} + 1] \times S^1 \), respectively. We recall that \(|a_k| < \tau \).

Since \( 2s_0 + 4 < \phi(\tau) \), it follows that \( 2s_0 + 4 < R_k \) for all \( k \). Since the functions \( \alpha_\pm \) vanish for \( s \geq s_0 \) and \( s \leq -s_0 \), respectively, we conclude that

\[
\alpha_+ \eta_k^+ = \alpha_+ \overline{\partial_v h_k^+} \quad \text{and} \quad \alpha_- \eta_k^- = \alpha_- \overline{\partial_v h_k^-}.
\]

Our assumption \( L(\lambda_k)(h_k^+, h_k^-) = (\eta_k^+, \eta_k^-) \rightarrow (0, 0) \) in \( F_m \) implies the convergence \((\alpha_+ \eta_k^+, \alpha_- \eta_k^-) \rightarrow (0, 0) \) in \( F_m \). Consequently,

\[
(\alpha_+ \overline{\partial_v h_k^+}, \alpha_- \overline{\partial_v h_k^-}) \rightarrow (0, 0) \quad \text{in} \quad F_m.
\]

Then using

\[
\overline{\partial_v h_k^+} = \overline{\partial_v h_k^+} + \frac{1}{2} [i \circ Th_k^+ \circ (j^+(v) - j^+(v_k))] \circ \frac{\partial}{\partial s}
\]

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and the fact that the family \( v \mapsto j^+(v) \) is smooth and that \( |h^+_k|_{H^{3+m,\delta_m}} \leq 1 \), we find also that

\[
(\alpha_+ \overline{\partial}_v h^+_k, \alpha_- \overline{\partial}_v h^-_k) \to (0,0) \quad \text{in } F_m.
\]  

(91)

Since \( \alpha'_+ \) vanishes outside of the interval \([s_0 - 1, s_0]\) and \( |h^+_k|_{H^{3+m,\delta_m}} \leq 1 \), the sequence \((\alpha_+ h^+_k)\) is bounded in the \(H^{3+m,\delta_m}\)-norm and, by the Sobolev compact embedding theorem, we may assume that the sequence converges in the \(H^{2+m,\delta_m}\)-norm. Now the equality \( \overline{\partial}_v (\alpha_+ h^+_k) = \alpha_+ \overline{\partial}_v h^+_k + \alpha'_+ h^+_k \) for all \( k \) and the estimate \([85]\) applied to \( \alpha_+ h^+_k - \alpha_+ h^+_n \) show that

\[
C_m |\alpha_+ h^+_k - \alpha_+ h^+_n|_{H^{3+m,\delta_m}} \leq |\overline{\partial}_v (\alpha_+ h^+_k - \alpha_+ h^+_n)|_{H^{2+m,\delta_m}} \\
\leq |\alpha_+ \overline{\partial}_v h^+_k - \alpha_+ \overline{\partial}_v h^+_n|_{H^{2+m,\delta_m}} \\
+ |\alpha'_+ h^+_k - \alpha'_+ h^+_n|_{H^{2+m,\delta_m}}.
\]

Since we already know that the sequences \((\alpha'_+ h^+_k)\) and \((\alpha_+ \overline{\partial}_v h^+_k)\) converge in \(H^{2+m,\delta_m}\) on \([0, s_0 + 1] \times S^1\), we conclude that the right-hand side converges to 0. Hence the sequence \((\alpha_+ h^+_k)\) is a Cauchy sequence and so converges in the \(H^{3+m,\delta_m}\)-norm. The same argument applied to the sequence \((\alpha_- h^-_k)\) shows that also the sequence \((\alpha_- h^-_k)\) converges in \(H^{3+m,\delta_m}\) and we conclude the convergence \((\alpha_+ h^+_k, \alpha_- h^-_k) \to (f^+, f^-)\) in \(E_m\) to some \((\tilde{f}^+, \tilde{f}^-) \in E_m\). Moreover, \(\tilde{f}^+(0, 0) = (0, 0)\) and \(\tilde{f}^-(\{0\} \times S^1) \subset \{0\} \times \mathbb{R}\). Since \(\alpha_\pm r^\pm_k = \alpha_\pm h^\pm_k - \alpha_\pm h^\pm_{k,\infty}\) and \(h^\pm_{k,\infty} \to h^\pm\) and \(\alpha_\pm\) is equal to 0 if \(s \geq s_0\) and \(s \leq -s_0\), respectively, we conclude that also the sequence \((\alpha_+ r^+_k, \alpha_- r^-_k)\) converges to \((f^+, f^-) = (\tilde{f}^+ - \alpha_- h^-_{\infty}, \tilde{f}^- - \alpha_- h^-_{\infty})\) in \(E_m\). This finishes the proof of the first convergence in \([89]\).

Next we prove the convergence of the sequence \((\gamma_+ r^+_k, \gamma_- r^-_k)\). In order to do this, we consider the maps \(\oplus a_k(r^+_k, r^-_k)\) and \(\ominus a_k(r^+_k, r^-_k)\). Recall the convergence

\[
L(\lambda_k)(h^+_k, h^-_k) = (\eta^+_k, \eta^-_k) \to (0, 0) \quad \text{in } F_m.
\]

Hence \((\gamma_+ \eta^+_k, \gamma_- \eta^-_k) \to 0\) in \(F_m\), and the estimate for the total hat-gluing in Theorem 2.26 leads to

\[
e^{\delta m} \frac{\eta_k}{h^+_k} \|\oplus a_k(\gamma_+ \eta^+_k, \gamma_- \eta^-_k)\|_{m+2,\delta_m} \to 0
\]

and

\[
e^{\delta m} \frac{\eta_k}{h^+_k} \|\ominus a_k(\gamma_+ \eta^+_k, \gamma_- \eta^-_k)\|_{m+2,\delta_m} \to 0.
\]

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so that
\[
\| \widehat{\psi}_{a_k}(\gamma^+_k, \gamma^-_k) \|_{\mathcal{M}^{2m, -\delta_m}} \leq \varepsilon_k e^{-\delta_m \frac{R}{m}} \tag{92}
\]
and
\[
\| \widehat{\Theta}_{a_k}(\gamma^+_k, \gamma^-_k) \|_{\mathcal{M}^{2m, -\delta_m}} \leq \varepsilon_k e^{-\delta_m \frac{R}{m}} \tag{93}
\]
for a sequence \(\varepsilon_k\) of positive numbers converging to 0.

Now, in view of the formula (76), the maps \(\oplus_{a_k}(r^+_k, r^-_k)\) and \(\ominus_{a_k}(r^+_k, r^-_k)\) solve the equations
\[
\overline{\partial} v_k \oplus_{a_k} (r^+_k, r^-_k) = \ominus_{a_k}(\eta^+_k, \eta^-_k) \tag{94}
\]
\[
\overline{\partial} 0 \ominus_{a_k} (r^+_k, r^-_k) = \widehat{\Theta}_{a_k}(\eta^+_k, \eta^-_k). \tag{95}
\]
Introducing the functions \(\gamma_k(s) = \oplus_{a_k}(\gamma^+_s, \gamma^-_s) = \beta_{a_k}(s) \cdot \gamma^+_s + (1 - \beta_{a_k}(s)) \cdot \gamma^-_s\), we note that \(\gamma_k\) is equal to 0 on the interval \([s_0 - 1, R_k - s_0 + 1]\) and equal to 1 on \([s_0, R_k - s_0]\). Hence \(\gamma_k \oplus_{a_k}(r^+_k, r^-_k) = \ominus_{a_k}(\gamma^+_r, \gamma^-_r)\) and the same identity holds for the hat gluing. Now, multiplying (94) by \(\gamma_k\), we obtain
\[
\widehat{\psi}_{a_k}(\gamma^+_k, \gamma^-_k) = \gamma_k \widehat{\psi}_{a_k}(\eta^+_k, \eta^-_k)
= \gamma_k \overline{\partial} v_k \oplus_{a_k} (r^+_k, r^-_k)
= \overline{\partial} v_k (\gamma_k \oplus_{a_k} (r^+_k, r^-_k)) = \gamma_k \overline{\partial} v_k \oplus_{a_k} (r^+_k, r^-_k)
= \overline{\partial} v_k (\gamma_k + r^+_k, \gamma^-_k) - \ominus_{a_k}(\gamma^+_r, \gamma^-_r).
\]
Since \(\gamma_k = 0\) outside of \([s_0 - 1, R_k - s_0 + 1]\) and \(j^+(v_k) = i\) on \([s_0 - 1, \infty) \times S^1\) and \((-\infty, -s_0 + 1] \times S^1\), the above identity becomes
\[
\overline{\partial} 0 \oplus_{a_k} (\gamma^+_k, \gamma^-_k) = \oplus_{a_k}(\gamma^+_k, \gamma^-_k) + \ominus_{a_k}(\gamma^+_r, \gamma^-_r). \tag{96}
\]
The maps \(\oplus_{a_k}(\gamma^+_k, \gamma^-_k)\) and \(\widehat{\Theta}_{a_k}(\gamma^+_k, \gamma^-_k)\) : \(Z_{a_k}^* \rightarrow \mathbb{R}^2\) have compact supports contained in the finite cylinders \([s_0 - 1, R_k - s_0 + 1]\) \times S^1\) and hence we can view them as defined on the infinite cylinders \(Z_{a_k}^*\). By Lemma 4.10, there are unique maps \(\xi_k \in H^{3+m,-\delta_m}(Z_{a_k}^*, \mathbb{R}^2)\) having mean-values \(\{\xi_k\}_{a_k} = 0\) and satisfying
\[
\overline{\partial} 0 \xi_k = \oplus_{a_k}(\gamma^+_k, \gamma^-_k). \tag{97}
\]
In addition, by (92) and Lemma 4.10
\[
\| \xi_k \|_{3+m, -\delta_m} \leq C_m \varepsilon_k e^{-\delta_m \frac{R}{m}} \tag{98}
\]
where the constant $C_m$ is independent of $k$. Then, in view of (96),

\[
\mathcal{D}_0[\oplus a_k(\gamma^+_r r_k^+; \gamma^-_r r_k^-) - \xi_k] = \oplus a_k(\gamma^+_r r_k^+; \gamma^-_r r_k^-). \tag{99}
\]

Abbreviating

\[ q_k = \oplus a_k(\gamma^+_r r_k^+; \gamma^-_r r_k^-), \]

we claim that $q_k - \xi_k - [q_k - \xi_k]_{a_k}$ belongs to $H^{3+(m+1),-\delta_{m+1}}(Z^*_{a_k})$. To see this, we first estimate the $H^{3+m,-\delta_{m+1}}(Z^*_{a_k})$-norms of the maps $\oplus a_k(\gamma^+_r r_k^+; \gamma^-_r r_k^-)$. The derivatives $\gamma'_+$ and $\gamma'_-$ vanish outside of the intervals $[s_0 - 1, s_0]$ and $[-s_0, -s_0 + 1]$ so that the square of the $H^{3+m,-\delta_{m+1}}(Z^*_{a_k})$-norm of $\oplus a_k(\gamma^+_r r_k^+; \gamma^-_r r_k^-)$ is equal to a constant times the sum of the following integrals

\[
\int \Sigma |D^\alpha r_k^+|^2 e^{-2\delta_{m+1}|s - R_k/2|} \quad \text{and} \quad \int \Sigma |D^\alpha r_k^- (s - R_k, t - \vartheta_k)|^2 e^{-2\delta_{m+1}|s - R_k/2|}
\]

where the sum is taken over all multi-indices $\alpha$ of length $|\alpha| \leq 3 + m$ and where $\Sigma = [s_0 - 1, s_0] \times S^1$.

Recalling that $s_0 + 2 < R_k^2$ and $|r_k^-|_{H^{3+m,-\delta_{m}}} \leq 1$, the first integral can be estimated as follows,

\[
\int_{[s_0 - 1, s_0] \times S^1} |D^\alpha r_k^+|^2 e^{-2\delta_{m+1}|s - R_k/2|} = \int_{[s_0 - 1, s_0] \times S^1} |D^\alpha r_k^+|^2 e^{2\delta_{m} s} e^{-2\delta_{m+1}|s - R_k/2|} - 2\delta_{m} s
\]

\[
\leq e^{-\delta_{m} R_k} e^{2(\delta_{m+1} - \delta_{m}) s_0} \int_{[s_0 - 1, s_0] \times S^1} |D^\alpha r_k^+|^2 e^{2\delta_{m} s} \leq e^{-\delta_{m+1} R_k} e^{2(\delta_{m+1} - \delta_{m}) s_0}.
\]

Since the same estimate holds for the second integral,

\[
\int_{[R_k - s_0, R_k - s_0 + 1] \times S^1} |D^\alpha r^- (s - R_k, t - \vartheta_k)|^2 e^{-2\delta_{m+1}|s - R_k/2|} \leq e^{-\delta_{m+1} R_k} e^{2(\delta_{m+1} - \delta_{m}) s_0},
\]

we obtain

\[
\widehat{\oplus a_k}(\gamma^+_r r_k^+; \gamma^-_r r_k^-) \leq C_m e^{-\delta_{m+1} R_k}. \tag{100}
\]

Since the map $\widehat{\oplus a_k}(\gamma^+_r r_k^+; \gamma^-_r r_k^-)$ belongs to $H^{3+m,-\delta_{m+1}}(Z^*_{a_k})$, it follows from Lemma (10) that there exists a unique map $f_k \in H^{3+(m+1),-\delta_{m+1}}(Z^*_{a_k})$ having mean-value $[f_k]_{a_k} = 0$ and satisfying

\[
\mathcal{D}_0 f_k = \widehat{\oplus a_k}(\gamma^+_r r_k^+; \gamma^-_r r_k^-). \tag{101}
\]

Moreover, by (100), the following estimate holds,

\[
\|f_k\|_{3+(m+1),-\delta_{m+1}} \leq C_m e^{-\delta_{m+1} R_k}. \tag{102}
\]

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Now, observe that both maps $f_k$ and $q_k - \xi_k - [q_k - \xi_k]_{ak}$ have their averages equal to 0 and satisfy equations \eqref{eq99} and \eqref{eq101} . Moreover, $q_k - \xi_k - [q_k - \xi_k]_{ak} \in H^{3+m,\delta_m}(Z^*_{ak}) \subset H^{3+m,\delta_{m+1}}(Z^*_{ak})$ and $f_k \in H^{3+(m+1),-\delta_{m+1}}(Z^*_{ak}) \subset H^{3+m,\delta_{m+1}}(Z^*_{ak})$. Since $\Theta_{ak}(\gamma+r_k^+, \gamma-r_k^-) \in H^{3+m,\delta_{m+1}}(Z^*_{ak}) \subset H^{2+m,-\delta_{m+1}}(Z^*_{ak})$, it follows from Lemma 4.10 applied to the Cauchy-Riemann operator $\partial_0 : H^{3+m,\delta_{m+1}}(Z^*_{ak}) \to H^{2+m,-\delta_{m+1}}(Z^*_{ak})$ that $f_k = q_k - \xi_k - [q_k - \xi_k]_{ak}$, as claimed. In addition, in view of the inequality \eqref{eq102}, the following estimate holds,

$$
\|q_k - \xi_k - [q_k - \xi_k]_{ak}\|_{3+(m+1),-\delta_{m+1}} \leq C_m e^{-\delta_{m+1} \frac{R_k}{\varepsilon}}
$$

(103)

where the constant $C_m$ is independent of $k$.

We will also need estimates for $\Theta_{ak}(\gamma+r_k^+, \gamma-r_k^-)$. We first observe that $\Theta_{ak}(\eta_k^+, \eta_k^-) = \Theta_{ak}(\gamma+\eta_k^+, \gamma-\eta_k^-)$ and $\Theta_{ak}(r_k^+, r_k^-) = \Theta_{ak}(\gamma+r_k^+, \gamma-r_k^-)$ so that the equation \eqref{eq95} can be written as

$$
\partial_0 \Theta_{ak}(\gamma+r_k^+, \gamma-r_k^-) = \Theta_{ak}(\gamma+\eta_k^+, \gamma-\eta_k^-).
$$

Then, abbreviating

$$
\zeta_k := \Theta_{ak}(\gamma+r_k^+, \gamma-r^-)
$$

and using $\Theta_{ak}(\gamma+\eta_k^+, \gamma-\eta^-) \in H^{2+m,\delta_m}(C_a)$, it follows from \eqref{eq104} and Lemma 4.10 that $\zeta_k$ is a unique solution in $H^{3+m,\delta_m}(C_a)$ of the equation

$$
\partial_0 \zeta_k = \Theta_{ak}(\gamma+\eta_k^+, \gamma-\eta_k^+).
$$

Moreover, in view of \eqref{eq93}, its $H^{3+m,\delta_m}$-norm is estimated as

$$
|\zeta_k|_{H^{3+m,\delta_m}} = \left[|\zeta_{k,\infty}|^2 + \|\widehat{\zeta}_k\|_{3+m,\delta_m}^2\right]^{1/2} \leq C_m \varepsilon_k e^{-\delta_m \frac{R_k}{\varepsilon}}
$$

(105)

where $\widehat{\zeta}_k = \zeta_k - (1 - 2\beta_{ak})\zeta_{k,\infty}$ and $\zeta_{k,\infty} = \lim_{s \to \infty} \zeta_k(s, t) = \text{av}_{a_k}(r_k^+, r_k^-)$. Hence

$$
\partial_0[\Theta_{ak}(\gamma+r_k^+, \gamma-r_k^-) - \zeta_k] = 0.
$$

We denote the restrictions of $q_k$ and $\xi_k$ to the finite cylinder $Z_{ak}$ by $\overline{q}_k$ and $\overline{\xi}_k$ and observe that the restriction of $q_k - \xi_k$ to $Z_{ak}$ is equal to $\overline{q}_k - \overline{\xi}_k$ and $\overline{[q_k - \xi_k]}_{ak} = [q_k]_{ak}$ since $\overline{\xi_k}_{ak} = 0$. In view of the estimates \eqref{eq98} and \eqref{eq103}, we have

$$
\|\overline{\xi}_k\|_{3+m,-\delta_m} \leq C_m \varepsilon_k e^{-\delta_m \frac{R_k}{\varepsilon}}
$$

(106)
and

\[ \|\mathcal{F}_k - \xi_k - [\mathcal{E}_k - \xi_k]_{ak}\|_{3+(m+1)-\delta_{m+1}} \leq C_m e^{-\delta_{m+1}}. \quad (107) \]

Now, using Theorem 1.27 we find a unique pair \((p_k^+, p_k^-)\) in \(\hat{E}_m\) satisfying

\[
\oplus_{ak}(p_k^+, p_k^-) = \xi_k \\
\ominus_{ak}(p_k^+, p_k^-) = \zeta_k
\]

and we claim that the sequence of maps \((p_k^+, p_k^-)\) converges to \((0, 0)\) in \(\hat{E}_m\). In view of the estimate for the total gluing map in Theorem 2.23 it suffices to prove the convergence

\[ |(\xi_k, \zeta_k)|^2_{C_{m,k}^{\alpha}} \to 0. \]

Indeed, using (105) and (106) and \([q_k]_{ak} = 0\) and the estimate

\[ |\zeta_{k,\infty}|_{3+m, -\delta_m}^2 = \int_{[0,R_k] \times S^1}|\zeta_{k,\infty}|^2 e^{-\delta_m s - \frac{s}{\alpha}} \leq \frac{|\zeta_{k,\infty}|^2}{\delta_m}, \]

we obtain

\[
|(\xi_k, \zeta_k)|^2_{C_{m,k}^{\alpha}} = |\zeta_{k,\infty}|^2 + e^{\delta_m R_k} \left[ |\xi_k + \zeta_{k,\infty}|_{3+m, -\delta_m}^2 + |\mathcal{F}_k|_{m+3, -\delta_m}^2 \right] \\
\leq C_m e^{\delta_m R_k} \left[ |\zeta_{k,\infty}|^2 + |\xi_k + \zeta_{k,\infty}|_{3+m, -\delta_m}^2 + |\mathcal{F}_k|_{m+3, -\delta_m}^2 \right] \\
\leq C_m \varepsilon_k^2
\]

as claimed.

From (108) and \(\oplus_{ak}(\gamma^+ r_k^+, \gamma^- r_k^-) = \mathcal{F}_k\) and \(\ominus_{ak}(\gamma^+ r_k^+, \gamma^- r_k^-) = \ominus_{ak}(p_k^+, p_k^-) = \zeta_k\), we have the following equations,

\[
\oplus_{ak}(\gamma^+ r_k^+ - p_k^+, \gamma^- r_k^- - p_k^-) = q_k - \xi_k \\
\ominus_{ak}(\gamma^+ r_k^+ - p_k^+, \gamma^- r_k^- - p_k^-) = 0.
\]

We claim that the sequence \((\gamma^+ r_k^+ - p_k^+, \gamma^- r_k^- - p_k^-)\) is bounded in \(\hat{E}_m\). By Theorem 2.23 is suffices to show that the \(C_{m+1}\)-norm of \((q_k - \xi_k, 0)\) of maps on the right-hand side is bounded. Using (103) and the fact that \([\xi_k]_{ak} = 0\), we find that

\[
|\langle q_k - \xi_k, 0 \rangle|^2_{C_{m+1}^{\alpha}} \\
= |[q_k - \xi_k]_{ak}|^2 + e^{\delta_m + R_k} \cdot \|q_k - \xi_k - [q_k - \xi_k]_{ak}\|_{3+(m+1)-\delta_{m+1}}^2 \quad (109) \\
= |[q_k]_{ak}|^2 + e^{\delta_m + R_k} \cdot \|q_k - \xi_k - [q_k]_{ak}\|_{3+(m+1)-\delta_{m+1}}^2 \\
\leq |[q_k]_{ak}|^2 + C_m^2
\]

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and it remains to estimate $|\langle q_k \rangle_{a_k}|$. Recall that $\langle q_k \rangle_{a_k} = \alpha v_{a_k}(r_k^+, r_k^-)$ since $q_k = \oplus_{a_k} (\gamma_+ r_k^+, \gamma_- r_k^-)$. Using the Sobolev embedding theorem on bounded domains and the bound $|r_k^+|_{H^{3+m, \delta m}} \leq 1$, we estimate

$$
|e^{\delta m} \cdot r_k^+|_{C^0(\Sigma_k)} \leq C_m |e^{\delta m} \cdot r_k^+|_{H^{m+3}(\Sigma_k)} \leq C_m
$$

where $\Sigma_k = [\frac{R_k}{2} - 1, \frac{R_k}{2} + 1] \times S^1$. This implies

$$
|\langle q_k \rangle_{a_k}| \leq C_m e^{-\delta m R_k / 2}
$$

and, similarly, $|\langle r_k \rangle_{a_k}| \leq C_m e^{-\delta m R_k / 2}$. Consequently,

$$
|\langle q_k \rangle_{a_k}| = |\alpha v_{a_k}(r_k^+, r_k^-)| = \frac{1}{2} |\langle r_k^+ \rangle_{a_k} + \langle r_k^- \rangle_{a_k}| \leq C_m e^{-\delta m \frac{R_k}{2}}.
$$

This implies that $|\langle (\overline{q}_k - \overline{r}_k, 0) \rangle_{a_k}^{2, \infty} \subset E_{m+1}$ are bounded and, in view by Theorem 2.23 that

$$
|\langle (\gamma_+ r_k^+ - p_k^+, \gamma_- r_k^- - p_k^+) \rangle_{E_{m+1}}| \leq C_m.
$$

Now, using the compact embedding $E_{m+1} \to \hat{E}_m$, we may assume after taking a subsequence that

$$
(\gamma_+ r_k^+ - p_k^+, \gamma_- r_k^- - p_k^+) \to (g^+, g^-) \text{ in } \hat{E}_m
$$

for some pair $(g^+, g^-) \in \hat{E}_m$. We have already proved that $(p_k^+, p_k^-) \to (0, 0)$ in $\hat{E}_m$. Hence

$$
(\gamma_+ r_k^+, \gamma_- r_k^-) \to (g^+, g^-) \text{ in } \hat{E}_m.
$$

This proves the second convergence in (89).

Summing up, we have proved that sequences $(\alpha_+ r_k^+, \alpha_- r_k^-)$ and $(\gamma_+ r_k^+, \gamma_- r_k^-)$ posses subsequences which converge to some elements $(f^+, f^-)$ and $(g^+, g^-)$, respectively, in $\hat{E}_m$. Since the sequence of asymptotic constants $(h_k^+, h_k^-)$ converges to $(h_k^+, h_k^-)$, it follows that $(h_k^+, h_k^-)$ converges in $\hat{E}_m$ to the element $(h_k^+, h_k^-) = (h_k^+, f^+ + g^+, h_k^+, f^- + g^-)$. We also have that $f^+(0, 0) = (0, 0)$ and $f^\pm(\{0\} \times S^1) \subset \{0\} \times \mathbb{R}$. The proof of the proposition is complete. ■

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