Nonperturbative $O(m_c^{-2})$ Effects in $B \to X_s \gamma$
from Heavy Quark Effective Theory

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Abstract

The nonperturbative contribution, suppressed by powers of the charm quark mass $m_c$, in the inclusive decay $B \to X_s \gamma$ is analyzed in the context of heavy quark effective field theory (HQEFT). According to previous analyses, the leading effects of $O(1/m_c^2)$ arise from an external gluon attached to a charm quark loop and can be expressed in terms of the chromomagnetic interaction of the $b$ quark inside the $B$ meson. This is also true at leading order in the HQEFT approach. However, the structure of higher-dimensional operators is different because the effects of external gluons alone do not give a complete set of operators. A systematic method to derive all the high-dimensional operators can be obtained in the HQEFT scheme using the operator product expansion.

Key words: nonperturbative effects, heavy quark effective theory, operator product expansion.

The inclusive radiative decay $B \to X_s \gamma$ has been considerably investigated experimentally and theoretically. The CLEO collaboration [1] reported the branching ratio for this decay to be $B(B \to X_s \gamma) = (2.32 \pm 0.57 \pm 0.35) \times 10^{-4}$. Theoretically this type of rare decay modes can be a sensitive probe for new physics beyond the standard model. And the photon energy spectrum is also interesting in understanding CP violation and new physics in $B$ decays [2]. In order to probe new physics, there should be a precise theoretical calculation of the branching ratio in the standard model. For the decay $B \to X_s \gamma$, the full next-to-leading order calculation in the standard model is completed combining matching conditions [3], matrix elements [4] and anomalous dimensions [5] in the effective Hamiltonian approach.

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Since the advent of the heavy quark effective field theory (HQEFT), the understanding of the decay of $B$ mesons has come to a more advanced level. Applying the HQEFT to $B$ mesons, the inclusive decay rates such as $B \rightarrow X_c \ell \nu$ and $B \rightarrow X_s \gamma$ can be obtained by simply calculating the corresponding quark decay rates at leading order [6]. The corrections to the leading result can be systematically incorporated in the scheme of the HQEFT. These corrections consist of a double series expansion both in the strong coupling constant $\alpha_s$ and in the inverse power of the $b$ quark mass, $m_b$. Systematic analyses of the nonperturbative effects suppressed by powers of $m_b$ have been considered extensively in various inclusive $B$ decays [7]. With all these ingredients the theoretical uncertainty is reduced to a 10% level. The object of this letter is to elaborate on the contribution of high-dimensional operators, especially of order $O(1/m_b^2)$ in the context of the HQEFT.

The theoretical framework to analyze the inclusive decay $B \rightarrow X_s \gamma$ is to use the effective weak Hamiltonian [8] which is given by

$$H_{\text{eff}} = \frac{4G_F}{\sqrt{2}} V_{cs}^* V_{cb} \sum_{i=1}^{8} C_i(\mu) O_i(\mu),$$

where $O_i$ are various operators. The operator $O_2$ is the four-quark operator given by $O_2 = (\bar{s}_L \alpha^\gamma \mu b_L) (\bar{c}_L \gamma^\mu c_L)$ in which $\alpha$, $\beta$ denote color indices. The operator $O_1$ differs from $O_2$ only in the way the color indices are contracted. Operators $O_3$ to $O_6$ are four-quark operators including all flavors below the renormalization scale $\mu$ of order $m_b$. The operator $O_7$ is given as $O_7 = (e/16\pi^2)m_b \bar{s}_L \sigma^{\mu\nu} F_{\mu\nu} b_R$, and $O_8$ is obtained from $O_7$ by replacing $eF_{\mu\nu}$ by $gG_{\mu\nu}$.

The contribution of these operators to the inclusive $B \rightarrow X_s \gamma$ decay rate is systematically calculable. In the leading log approximation, the matrix element of $C_7(\mu) O_7(\mu)$ dominates in evaluating the decay rate $B \rightarrow X_s \gamma$ for large enough photon energies. At next-to-leading order, other operators can contribute. In analyzing the contribution of other operators, when it is possible to use the operator product expansion (OPE), higher-order processes can be described by products of some coefficients which are calculable in perturbation theory, and local operators whose matrix elements represent long-distance effects.

When operators in $H_{\text{eff}}$ other than $O_7$ are included, if we try to calculate perturbatively the decay rate for $B \rightarrow X_s \gamma$, many problems arise because the decay rate receives contribution from those in which the photon couples to light quarks. Actually the amplitude for $b \rightarrow sg\gamma$ with a light quark loop is nonlocal. Therefore for light quarks coupled to a photon, the idea of the OPE is not applicable. That is, there is no OPE to parameterize nonperturbative effects from the photon coupling to light quarks in terms of $B$ meson matrix elements of local operators.
However there are nonperturbative effects arising from the photon coupling to the charm quark, and it is possible to have $B$ meson matrix elements of local operators which are suppressed by $(\Lambda_{\text{QCD}}/m_c)^2$ rather than $(\Lambda_{\text{QCD}}/m_b)^2$. We can calculate the nonperturbative effects suppressed by $1/m_c^2$ because the charm quark is heavy ($m_c \gg \Lambda_{\text{QCD}}$). It is exactly the reason why we can employ the HQEFT idea in order to obtain high-dimensional operators systematically.

Naively we expect that the high-dimensional operators, which result in nonperturbative effects, are suppressed by powers of $m_b$ since the only physically relevant scale in the inclusive decay $B \rightarrow X_s \gamma$ is the mass of the $b$ quark. However Voloshin [9] has observed that there are also nonperturbative effects which are suppressed by powers of $m_c$, the charm quark mass. This nonperturbative effect arises from the operator $O_2$ as shown in Fig. 1, which is produced by the quantum fluctuation of the $c\bar{c}$ pair emitting a hard photon, and an external soft gluon is attached to the charm quark loop. Here the blob represents the operator $O_2$.

![Feynman diagram for $b \rightarrow s\gamma g$ from $O_2$ in which a photon and a gluon are attached to the $c$ quark loop. The diagram with a photon and a gluon switched is omitted.](image)

To first order in the gluon momentum, the new effective Hamiltonian for $b \rightarrow s\gamma g$ is written as

$$H(b \rightarrow s\gamma g) = \frac{egQ_c}{48\pi^2m_c^2}G_F C_2 V_{cs} V_{cb} \left( \tilde{s}G_{\alpha\beta}\gamma_\mu(1-\gamma_5)b \right) \epsilon^{\lambda\mu\alpha\sigma} D^\beta F_{\sigma\lambda}. \quad (2)$$

Here $G_{\alpha\beta} = G_{\alpha\beta}^a T_a$ is the gluon field strength tensor, $F_{\sigma\lambda}$ is the electromagnetic field strength tensor, and $D^\alpha = \partial^\alpha + ieQ_c A^\alpha + igG^\alpha$ is the covariant derivative. $Q_c = 2/3$ is the electric charge of the charm quark. And the sign convention is...
\( \epsilon^{0123} = +1 \). There has been much effort to estimate the size of this new type of nonperturbative effects in various \( B \) decays such as \( B \rightarrow X_s \gamma \), \( B \rightarrow X_s \ell^+ \ell^- \) [10–13].

If we keep more powers of external gluon momentum, or if we attach more gluons to the charm quark loop, we can obtain higher-dimensional operators. These higher-dimensional operators are suppressed by more powers of \( m_c \) compared to the leading operator given in Eq. (2). These higher-order effects are quite small compared to the contribution of the leading operator, not because the expansion parameter \( m_b A_{QCD}/m_b^2 \) is small, but because the numerical coefficients of these operators are small [10,11].

Now we consider how we can implement the idea of the HQEFT and the OPE to derive high-dimensional operators suppressed by powers of \( m_c \) in a systematic way. In the heavy quark limit \( m_b \rightarrow \infty \), the \( b \) quark interacts with the light degrees of freedom, which is characterized by the residual momentum \( k \) of the \( b \) quark \( (p_b^\mu = m_b v^\mu + k^\mu) \). As the operator \( O_2 \) acts, the \( b \) quark turns into an \( s \) quark and a \( c\bar{c} \) pair. When the virtual \( c\bar{c} \) pair emits a hard photon and soft gluons, this process induces high-dimensional operators. Now let us also take the heavy quark limit \( m_c \rightarrow \infty \) while \( m_c/m_b \) held fixed. Then when the \( b \) quark turns into a \( c\bar{c} \) pair, the \( c\bar{c} \) pair is still immersed in the same light degrees of freedom from the \( b \) quark. Therefore the \( c \) quark also carries the residual momentum \( k \) of the original \( b \) quark. The fact that the charm quark should have a residual momentum is manifest if we consider the corresponding Feynman diagram in the full theory with \( W \) exchange.

From this process mentioned above, we can systematically obtain all the high-dimensional operators in powers of \( 1/m_c \) by expanding the residual momentum in the \( c \) quark and using the OPE. There are also other contributions such as the effect of the residual momentum in the \( s \) quark, but the operators obtained from the residual momentum of the \( s \) quark are suppressed by powers of \( m_b \). Since we are interested in the operators suppressed by powers of \( m_c \) here, we will neglect all the operators suppressed by \( m_b \). And at this leading order, the \( b \) field can be regarded as a heavy field \( b_v \) with momentum \( p_b^\mu = m_b v^\mu + k^\mu \) in the HQEFT.

These high-dimensional operators include the operators obtained in Ref. [9–13], which correspond to the so-called “one-gluon matrix element” [7]. The operators from one-gluon matrix element are those in which an external gluon is attached to the charm quark loop. In order to make the resultant operator gauge invariant, we combine one gluon momentum operator and the external gluon field to form the gluon field strength tensor \( G_{\alpha \beta} \). Therefore these operators are proportional to \( G_{\alpha \beta} \), and its derivatives if there are additional gluon momenta at higher orders. If we attach more external gluons, we obtain high-dimensional operators in powers of \( G_{\alpha \beta} \). However, in the HQEFT, there
are also other operators which are not proportional to $G_{\alpha\beta}$ and its derivatives, and these operators can be obtained systematically using the HQEFT and the OPE. Interestingly enough, the leading contribution in the HQEFT is the same as the operator from the one-gluon matrix element. Therefore the numerical analysis for the contribution of the leading operator at $O(1/m_c^2)$ in Ref. [9–11] does not change. The contribution of the leading operator in Eq. (2) compared to the contribution of $O_7$ alone is about $\delta \Gamma / \Gamma \approx 3\%$ [13].

In order to systematically calculate high-dimensional operators suppressed by powers of $m_c$ in the HQEFT, consider the Feynman diagram shown in Fig. 2. Though the operator $O_2$ is shown in Fig. 2, note that the operator $O_1$ should be included also when there are two or more soft gluons are emitted. However the contribution of $O_1$ appears at next-leading order compared to that from $O_2$. The integral in evaluating the Feynman diagram in Fig. 2 can be written as

$$I^{\lambda\mu} = -ieQ_c \int \frac{d^4l}{(2\pi)^4} \text{tr} \left[ \frac{1}{l-q+k-m_c} \gamma^\lambda \frac{1}{l+k-m_c} \gamma^\mu (1-\gamma_5) \right],$$

(3)

where $q$ denotes the photon momentum.

Fig. 2. Feynman diagram for $b \rightarrow s\gamma$ to obtain high-dimensional operators from the expansion of the residual momentum in the charm quark loop using the OPE.

Now we expand $I^{\lambda\mu}$ in powers of the residual momentum $k$ and collect the terms which are suppressed by powers of $m_c$. We use the operator product expansion in order to obtain the corresponding operators. That is, we replace $k^\mu$ by $\Pi^\mu = iD^\mu$, where $D^\mu$ is the covariant derivative acting on the $b$ field. The operator $\Pi^\mu$ satisfies the commutation relation $[\Pi^\mu, \Pi^\nu] = -igG^{\mu\nu}$. The amplitude $M^\lambda$ for $b \rightarrow s\gamma + \text{many gluons}$ from Fig. 2 can be written in terms of $I^{\lambda\mu}$ as

$$iM^\lambda = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{cb} C_2 \left( \not\!\! \not\! \! \Pi^\lambda \gamma_\mu (1-\gamma_5) b \right).$$

(4)

At a first glance, the integral in Eq. (3) seems to be made independent of the residual momentum when we shift the loop momentum variable $l$ by $l+k$, hence
appear no high-dimensional operators. But this shift of the integration variable is not allowed in the integral because the integral as such is quadratically divergent. However, if we employ the dimensional regularization to regulate the integral, only the $\gamma_5$-dependent part of the integral should be treated carefully.

The $\gamma_5$-independent integral can be shifted to be independent of the residual momentum, hence producing no high-dimensional operators. Furthermore after the shift of the integration variable, the $\gamma_5$-independent integral vanishes for the on-shell photon. Therefore to all orders there are no high-dimensional operators from the $\gamma_5$-independent part. It means that high-dimensional operators suppressed by powers of $m_c$ in $B \rightarrow X_s\gamma$ should always include the Levi-Civita $\epsilon$ tensor which appears by taking the trace of Dirac matrices involving $\gamma_5$ in Eq. (3).

In order to calculate the remaining $\gamma_5$-dependent part which is suppressed by powers of $m_c$, we first expand the integrand in powers of $k$, and we can shift the integration variable only when the integral is guaranteed to be convergent. In the resultant expression, we get high-dimensional operators by replacing $k$ with the covariant derivative $iD$.

With this in mind, let us consider $I^{\lambda\mu}$ in Eq. (3). When we first replace $k$ by the covariant derivative $\Pi = iD$, and expand $I^{\lambda\mu}$ in powers of $\Pi$, we get

\[
I^{\lambda\mu} = i e Q_c \int \frac{d^4 l}{(2\pi)^4} \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n+1}}{(l^2 - m_c^2)^{m+1}} \frac{1}{(l^2 - m_c^2)^{n+1}} \\
\times \text{tr} \left[ (I - \not{q} + \not{\Pi} + m_c) \left( 2\Pi \cdot (l - q) + \Pi^2 - \frac{g}{2} G_{\alpha\beta} \sigma^{\alpha\beta} \right)^m \right] \\
\times \gamma^\lambda (I + \not{\Pi} + m_c) \left( 2\Pi \cdot l + \Pi^2 - \frac{g}{2} G_{\rho\tau} \sigma^{\rho\tau} \right)^n \gamma^\mu (1 - \gamma_5),
\]

(5)

where $\sigma^{\alpha\beta} = i[\gamma^\alpha, \gamma^\beta]/2$. Here the appearance of $G_{\alpha\beta}$ comes from the $\not{\Pi}$ term when we expand the denominator in Eq. (3) with the identity

\[
\not{\Pi} = \Pi^2 - \frac{g}{2} G_{\alpha\beta} \sigma^{\alpha\beta}.
\]

(6)

From Eq. (5) we can systematically obtain high-dimensional operators by a series expansion with respect to $m$ and $n$. Here I replace the residual momentum by the covariant derivative first and treat the order of the products carefully as in Eq. (6). Because of this procedure, there appears $G_{\alpha\beta}$ in Eq. (5).

In the literature [7], there is another way of calculating the integral. First we expand the diagram in powers of the residual momentum $k$, regarding it as a number, i.e., $\not{\Pi} = \Pi^2$. Then there appears no $G_{\alpha\beta}$ in the expansion. In order to include the effect from the gluon field strength tensor $G_{\alpha\beta}$, we attach an
external gluon and extract the term proportional to $G_{\alpha\beta}$. It is called a “one-gluon matrix element”. However it turns out that these two approaches are equivalent.

It is complicated to obtain a simple and general form for the series expansion in Eq. (5). However we can calculate the first few high-dimensional operators explicitly in order to see the structure of them. The zeroth-order term ($m = n = 0$) does not contribute to the high-dimensional operators suppressed by $m_\epsilon$. Also there is no term linear in $\Pi$. At second order in $\Pi$, there are no symmetric terms quadratic in $\Pi$ such as $\Pi^\alpha\Pi^\beta + \Pi^\beta\Pi^\alpha$ or $\Pi^2$. The only surviving terms up to quadratic terms in $\Pi$ are those with $G_{\alpha\beta}$. These terms come from the series expansion in Eq. (5) with $m + n = 1$ and 2 respectively. The result is written as

$$\frac{egQ_c}{24\pi^2m_\epsilon^2}\epsilon^{\lambda\mu\sigma\nu}q_\lambda q_\sigma G_{\alpha\beta}. \tag{7}$$

Note that all the quadratic contributions which do not involve $G_{\alpha\beta}$ explicitly in Eq. (5) cancel out, leaving only the one-gluon matrix element. The reason why the term with $G_{\alpha\beta}$ alone survives is clear when we look at the diagram in Fig. 2. If a single gluon is attached to the loop, the loop has the structure of a vector-vector-axial triangle and only the term with the Levi-Civita $\epsilon$ tensor and $G_{\alpha\beta}$ survives. The effective operator from Eq. (7) can be derived in a straightforward way, and is given in Eq. (2). Therefore the leading term of order $1/m_\epsilon^2$ coincides with the result by Voloshin [9], which is obtained by attaching a soft gluon to the $c$ quark loop.

Now let us consider higher-order corrections to the leading term given in Eq. (7). The operators of $O(1/m_\epsilon^4)$ are obtained from the terms with $m + n = 2$ and 3 respectively in Eq. (5). The result is written as

$$\frac{egQ_c}{180\pi^2m_\epsilon^4}\epsilon^{\lambda\mu\sigma\nu}q_\lambda q_\sigma [q \cdot \Pi, G_{\alpha\beta}]. \tag{8}$$

Recall that the covariant derivative operator $\Pi$ is applied to the $b$ field. Then the commutator in Eq. (8) can be written as

$$[q \cdot \Pi, G_{\alpha\beta}]b = (q \cdot \Pi G_{\alpha\beta})b, \tag{9}$$

where the covariant derivative $\Pi$ in the right-hand side only applies to $G_{\alpha\beta}$. Of course, it should be understood that this holds only for the derivative in the covariant derivative $\Pi$. Using the identity Eq. (9), Eq. (8) can be written
as
\[
\frac{egQ_c}{180\pi^2m_c^4}\epsilon^{\lambda\alpha\mu\sigma}q_{\sigma}q^\beta(q \cdot \Pi G_{\alpha\beta}),
\]
where \(\Pi\) applies only to the gluon field strength tensor. This is where the HQEFT result is different from the result by attaching soft gluons to the charm quark loop. To be more quantitative, the HQEFT result is half the higher-dimensional contribution derived from the one-gluon matrix element and its derivatives, which makes the contribution of higher-dimensional operators of \(O(1/m_c^4)\) smaller than estimated before [10,11].

There are also other operators which cannot be obtained by attaching external gluons. These operators can be obtained in the HQEFT scheme. For example, there are higher-dimensional operators of order \(\Pi^3/m_c^2\), though their contributions are suppressed by \(\Lambda/m_b\) compared to the operators of order \(\Pi^2/m_c^2\). These terms arise from \(m+n = 1, 2, 3\) respectively. After a long but straightforward calculation, the result is given by

\[
\frac{eQ_c}{16\pi^2m_c^2}
\left[
\frac{8}{3}\epsilon^{\alpha\lambda\mu\rho}q_\rho\Pi_\alpha\Pi^2 - \frac{g}{6}\epsilon^{\alpha\beta\lambda\mu}[g \cdot \Pi, G_{\alpha\beta}] - \frac{g}{3}[\Pi_\delta, G_{\alpha\beta}]
\left(6g^{\alpha_\delta\epsilon\beta\lambda\mu}\epsilon_\rho - 4g^{\alpha_\mu}\epsilon^{\beta\lambda\rho}\epsilon_\rho + \epsilon^{\beta\lambda\epsilon\mu}\epsilon_\rho - 4g^{\lambda_\mu}\epsilon^{\alpha\beta\delta\rho}\epsilon_\rho - 2q^{\mu}\epsilon^{\alpha\beta\delta\lambda}\right)\right].
\]

In evaluating the contribution of higher-dimensional operators of order \(O(\Pi^3)\), Eqs. (8) and (11), to the decay rate, there is one more caveat that we have to include the contribution of both the operators \(O_1\) and \(O_2\). For the leading term, given in Eq. (7), with one gluon field strength tensor, only the operator \(O_2\) contributes since it has the correct color structure. However, as mentioned before, when we include the emission of two or more gluons, the contribution from the operator \(O_1\) should also be considered. We have to separate these higher-dimensional operators into the color-octet part for \(O_2\) and the color-singlet part for \(O_1\). But since these contributions are numerically insignificant, it will not be pursued here any more.

Implementing the HQEFT idea to the charm quark which is immersed in the same light degrees of freedom of the decaying \(b\) quark, we can systematically obtain a complete set of high-dimensional operators suppressed by powers of \(m_c\). The effect of attaching external gluons gives the correct leading term, but it does not include all the higher-dimensional operators. In other words, attaching external gluons corresponds to the expansion in powers of the gluon field strength tensor \(G_{\alpha\beta}\) and its derivatives. The resultant operators for \(n\) attached gluons are symbolically of the form \(G_{\alpha\beta}^n\) with some derivatives. In
order to obtain a complete set of high-dimensional operators, the HQEFT idea can be applied to the decay $B \to X_s \gamma$.

The numerical estimate of the higher-dimensional operators are negligible compared to the leading operator in Eq. (2), which already contributes about 3% to the decay rate. As pointed out in Refs. [10,11,13], it is not because the expansion parameter $m_b \Lambda_{QCD}/m_c^2$ is small, but because the coefficients are small. Even though higher-dimensional operators are numerically insignificant, the idea of the HQEFT offers a systematic way of obtaining a complete set of high-dimensional operators.

Conceptually there is an advantage in considering the nonperturbative contributions using the HQEFT idea. The effect of the operators suppressed by powers of $m_c$ is nonperturbative. The reason why we could systematically classify high-dimensional operators is because the charm quark is heavy compared to the QCD scale $\Lambda_{QCD}$. If the charm quark is not heavy, it is impossible to analyze the $1/m_c$ effects using the OPE, and this effect belongs to a purely nonperturbative regime. And it is not surprising to have nonperturbative contributions governed by the scale $m_c$ as well as $m_b$ in $B$ decays since these effects arise simply because the $c$ quark is also heavy. In the same line of reasoning, it is impossible to analyze the contribution of the up quark loop from a similar process in inverse powers of the up quark mass using the OPE since this contribution is genuinely nonperturbative.

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