Asymptotic analysis of first passage time in complex networks

Hon Wai Lau(a) and K. Y. Szeto(b)

Department of Physics, The Hong Kong University of Science and Technology, Hong Kong, China

PACS 05.40.Fb – Random walks and Levy flights
PACS 89.75.Hc – Networks and genealogical trees
PACS 87.10.-e – General theory and mathematical aspects

Abstract. - The first passage time (FPT) distribution for random walk in complex networks is calculated through an asymptotic analysis. For network with size \(N\) and short relaxation time \(\tau \ll N\), the computed mean first passage time (MFPT), which is inverse of the decay rate of FPT distribution, is inversely proportional to the degree of the destination. These results are verified numerically for the paradigmatic networks with excellent agreement. We show that the range of validity of the analytical results covers networks that have short relaxation time and high mean degree, which turn out to be valid to many real networks.

Network is important in many areas of science and engineering as a mathematical representation of the interaction of complex real systems [1, 2] such as the Internet, social network, etc. Some of the common topological properties for many real world networks, such as small-world and scale-free properties [1], and fractal scaling [3, 4] have been investigated. Recent research effort has begun to address the dynamical properties and critical phenomenon in complex network. Random walk problem, due to its simplicity, is therefore a key to the understanding of the propagation of dynamical quantities in network. The propagation of a random walker, which can represent the spreading of some signal in specific application, can be measured by the time needed to reach a selected node. In this regard, the first passage time (FPT) that measures the first time the random walker visiting a given destination, is critical to many events triggered by the walker, such as epidemics spreading in social network [5], neuron firing dynamics [6] and various target search process [7, 8]. Instead of the whole distribution, the mean first passage time (MFPT) is widely studied in different lattices with unbounded domain [9], and bounded domain with arbitrary shape of boundary \(\tau \ll N\). Although the general solution of MFPT has been found [10], the explicit relation between first passage time and structural properties of network is still unknown. A recent work has related MFPT to the distance between source and destination [11] for fractal networks, which use the analysis of pseudo Green function for finite lattice [9]. Nonetheless, it has been argued that this result should only be applicable to the deterministic fractal network with homogeneous degree distribution [12]. In comparison with most deterministic fractal with fixed degree, real world networks typically have strong inhomogeneity in degree distribution. Thus, it suggests that the degree of nodes should play an essential role for the random walk in complex network and a treatment differently from the analysis of fractal network should be used.

For many real networks, the number of nodes in a given data set is usually not big, so that a rigorous analysis on its properties is usually difficult. Hence, schemes to adjust the size of network have been proposed [3, 13], but it is difficult to preserve structural properties without knowing the underlying construction mechanism. Computing the MFPT for networks with a particular property can therefore be useful for real world network. Our method should be complementary to the pseudo Green function approach [11].

In this paper, we will provide an analytical expression for the asymptotic first passage time distribution function, from which all moments, including the mean, can be computed and compared with numerical results for real and paradigmatic networks. We focus on the study of random walk in complex networks with short relaxation time irrespective of the type of network. In contrast to the pseudo Green function approach, our asymptotic analysis can also reveal the local dynamical properties around the destination. In particular, it allows us to compute a good approximation of the decay rate of FPT distribution and...
For the random walk problem in a finite domain, there is a probability that the walker is located at node \( v \) at time \( t \). For a network with sink, we denote the probability that the walker has to visit many other nodes before the destination, resulting in a slow increase of FPT distribution.

MFPT for networks with short relaxation time. We further argue that these expressions are approximate bounds for other general complex networks. Since most real world networks have short relaxation time, we find that our analytical calculation produces excellent comparison to numerical results on the MFPT for paradigmatic as well as real world networks.

**Formulation.** - We consider a finite undirected network which consists of a set of nodes \( \{ v_1, v_2, ..., v_N \} \) that are connected by edges. The set of neighbors of a node \( v \) is denoted by Nei(\( v \)). Let \( \mathcal{P}(v, t) \) be the probability of a random walker located at node \( v \) at time \( t \). At each time step, the walker can move from the current node \( v_i \) to one of its nearest neighbor node \( v_j \) in Nei(\( v_i \)), with equal probability \( 1/k_i \). Hence, the master equation of random walk is

\[
\mathcal{P}(v_i, t+1) = \sum_{v_j \in \text{Nei}(v_i)} \frac{\mathcal{P}(v_j, t)}{k_j} \tag{1}
\]

For the random walk problem in a finite domain, there is a characteristic relaxation time \( \tau \) such that equilibrium is essentially reached when \( t \gg \tau \). We denote the equilibrium probability distribution by \( \mathcal{P}^\infty(v) \) for \( t \to \infty \). By the principle of detailed balance, the net flow of walker probability \( \mathcal{P}^\infty(v_i)/k_i - \mathcal{P}^\infty(v_j)/k_j \) is zero along each edge at equilibrium. Hence, the equilibrium probability is proportional to the node degree \( \mathcal{P}^\infty(v) \propto k_v \). After normalization, we get the Kac’s result [9], \( \mathcal{P}^\infty(v) = k_v/N \langle k \rangle \), where \( \langle k \rangle \) is the mean degree of network. Therefore, at equilibrium, the walker probability moving along each edge, in both directions, is equal to \( 1/N \langle k \rangle \).

Let \( \mathcal{F}(v_d, t|v_s) \) be the FPT distribution from a source node \( v_s \) to a destination node \( v_d \) that takes time \( t \). By adding a sink at the destination, the FPT distribution can be found by computing the probability of the walker trapped at the sink. For a network with sink, we denote the probability that the walker is located at node \( v \) at time \( t \) by \( W(v, t) \). Note that \( W(v, t) \) is the analog of \( \mathcal{P}(v, t) \), which addresses a network without sink. The FPT distribution for a sink at \( v_d \) can be computed by the following equations:

\[
\begin{align*}
W(v, 0) &= \delta_{vv}, \\
W(v_d, t) &= 0 \\
W(v, t) &= \sum_{u \in \text{Nei}(v)} \frac{W(u, t-1)}{k_u} \quad \forall t > 0 \\
\mathcal{F}(v_d, t|v_s) &= \sum_{u \in \text{Nei}(v_d)} \frac{W(u, t-1)}{k_u}
\end{align*} \tag{2}
\]

The first equation is the initial condition. The second equation is the absorbing boundary condition at the destination. The third equation is the random walk transition probability corresponding to (1). These three equations define the random walk problem with a sink located at \( v_d \). The last equation introduces the method to find the FPT distribution, which is given by the incoming probability flux flowing towards the destination. Moreover, removal of the walker at the sink decreases the total walker probability, or the survival probability \( W_{\text{total}}(t) \) defined by \( W_{\text{total}}(t) = \sum_{v \in V} W(v, t) \). Hence, the FPT distribution and the total walker probability are related by

\[
\mathcal{F}(v_d, t|v_s) = W_{\text{total}}(t-1) - W_{\text{total}}(t) \tag{3}
\]

as the random walker is absorbed by the destination node at time \( t \).

Now, we perform asymptotic analysis of FPT distribution. When \( t \gg \tau \), the initial information of the source node is washed away, so walker probability decreases uniformly for each node in the network. In this case, \( W(v, t) \) can be separated into two parts:

\[
W(v, t) \sim W(v)W_{\text{total}}(t), \quad t \gg \tau \tag{4}
\]

where \( W(v) \) is a time independent probability distribution that depends on network topology and the location of destination. By substituting Eq. (4) back into the last equation in (2), we can get a structural factor \( \beta_d \) that depends on the destination node \( v_d \):

\[
\beta_d = \sum_{u \in \text{Nei}(v_d)} \frac{W(u)}{k_u} \tag{5}
\]

and the asymptotic form of FPT distribution \( \mathcal{F}(v_d, t|v_s) \sim \beta_d W_{\text{total}}(t-1) \) for \( t \gg \tau \). Solving this equation along with Eq. (3), we have \( W_{\text{total}}(t) \sim (1 - \beta_d)W_{\text{total}}(t-1) \) and so \( W_{\text{total}}(t) \propto (1 - \beta_d)^t \) asymptotically. As we will show later, \( \beta_d \) is small, so that \( W_{\text{total}}(t) \propto \exp[-\beta_d t] \). With the known asymptotic form of \( W_{\text{total}}(t) \), we can conclude that the FPT distribution is \( \mathcal{F}(v_d, t|v_s) \sim A \exp[-\beta_d t] \). Thus, FPT distribution has an exponential tail with decay rate \( \beta_d \) as shown in Fig. 1. In addition, we now know a method to compute decay rate analytically by Eq. (5), provided that \( W(u) \) of all neighbors of destination \( v_d \) is known.

**Decay rate of FPT distribution**. - After deriving the exponential tail of the FPT distribution, our next task is to compute the decay rate \( \beta_d \) for the network with short relaxation time \( \tau \) such that \( \tau \ll 1/\beta_d \). Physically, \( \tau \) represents the time to reach equilibrium in a network.
without sink, and $1/\beta_d$ is the approximate time scale for the decrease in total walker probability of the network when there is a sink to absorb the random walker. For network with short relaxation time, the removal of walker probability around the destination can be accomplished rather quickly. On the other hand, the total probability decreases with a much slower rate and this result in a quasi-equilibrium state. This physical picture implies that the quantity $W(u) = \lim_{t \to \tau} W(u,t)/W_{\text{total}}(t)$ is approximately equal to $P^\infty(u)$. This can be observed for the Erdos-Renyi (ER) network and Barabasi-Albert (BA) network which have very short relaxation time (see Table 1). One of them is illustrated numerically in Fig. 2a. In the simplest case, it is reasonable to assume that $W(u) = P^\infty(u)$ for $d(u,v_d) \geq 1$. While this assumption is substituted back to Eq. (5), the decay rate is $\beta_d = k_d/N \langle k \rangle$. It scales linear with the degree of destination and scales inversely with the size of the network. Thus, for a large network, $\beta_d$ can be very small.

Next, we proceed to a better approximation for the dependence of $W(u)$ on the $k_u$ around the destination. For a nearest neighbor $u$ of the destination, $W(u)$ has to take a smaller value than $P^\infty(u)$ to compensate the flow of walker toward the sink. Hence, it is natural to expect that $W(u) \leq P^\infty(u)$. Rather than assuming $W(u) = P^\infty(u)$ for nodes other than destination, a refined assumption can be made for the next nearest neighbors:

$$W(u,t) = P^\infty(u)W_{\text{total}}(t), \quad d(u,v_d) \geq 2$$

With this assumption, $W(u)$ for $u \in \text{Nei}(v_d)$ has to be found before computing the decay rate. Here, nodes with distance two from the destination are treated as the reservoir of walker probability. We thus assume that each edge of node $v$, that are of distance greater than 1 away from the destination, $d(v,v_d) \geq 2$, has probability $1/N \langle k \rangle$ moving out at each time step.

Now, let us focus on a nearest neighbor node $u$ of the destination. Since there is no probability flowing from $v_d$ to $u$, therefore, only $k_u - 1$ neighbors of $u$ have probability flowing into $u$. If there are no edges connecting between neighbors of $v_d$, i.e., assuming zero clustering coefficient of $v_d$, then each edge contributes $1/N \langle k \rangle$ to $u$ from the second neighbor of $v_d$, or $W(u) = (k_u - 1)/N \langle k \rangle$. If there are some edges connecting neighbors of $v_d$, i.e., for the case of non-zero clustering coefficient, then some neighbors of $u$ are also neighbors of $v_d$. In this case, nodes in $\text{Nei}(v_d)$ have value less than $P^\infty$ and the probability moving out has value less than or equal to $1/N \langle k \rangle$. The exact value can be found by solving $W(u)$ for all $u \in \text{Nei}(v_d)$ simultaneously and it is upper bounded by

$$W(u) \leq \left(\frac{k_u - 1}{k_u}\right) P^\infty(u), \quad d(u,v_d) = 1$$

As shown in Fig. 2b for the BA model, this result fits well with the simulation of $W(u)$ and the equality sign holds approximately. In general, this upper bound can be reached for networks with very short relaxation time. Now we put Eq. (7) into Eq. (5) to get

$$\beta_d \leq \frac{k_d}{N \langle k \rangle} \left(1 - \frac{1}{k_d} \sum_{u \in \text{Nei}(v_d)} \frac{1}{k_u}\right)$$

where the last term is the mean of inverse degree of neighbors that can be approximated by mean field. Let’s recall that the probability of selecting one of nearest neighbors with degree $k$ for uncorrelated network is $P(k)k/\langle k \rangle$ [14], where $P(k)$ is the degree distribution of the network. Similar to the computation of mean degree of neighbor in Ref. [15], the mean of inverse degree of neighbor is $\int_0^\infty (P(k)k/\langle k \rangle) \frac{1}{k} dk = \frac{1}{\langle k \rangle}$, so

$$\beta_d \leq \frac{k_d}{N \langle k \rangle} \left(1 - \frac{1}{\langle k \rangle}\right)$$

which suggests that the actual decay rate obtained in simulation is slower than this upper bound. Note that this result is consistent with our basic hypothesis $\tau \ll 1/\beta_d$ if $\tau \ll N$. We find that this theoretical decay rate is in excellent agreement with the simulation result for the ER and BA network (see Table 1).
Mean first passage time. – Since FPT distribution is normalized for a finite undirected network, $\int_0^\infty F(v_d, t|v_s)dt = 1$, it can be split asymptotically as the sum of short time scale and long time scale as:

$$\int_0^c F(v_d, t|v_s)dt + \int_c^\infty Ae^{-\beta_d t}dt \sim 1, \quad c \gg \tau$$  \hspace{1cm} (10)

where $c$ is a cutoff value. In a finite domain, FPT distribution always rise from 0 at $t = 0$ and decreases with exponential tail [8] when $t \to \infty$ (see Fig. 1). For a destination node far from the source node, $F(v_d, t|v_s)$ is bounded above by $Ae^{-\beta_d t}$ within $0 < t < c$. Hence, for short relaxation time $\tau \ll c \ll 1/\beta_d$, the first term is bounded by $\int_0^c Ae^{-\beta_d t}dt \sim 0$ which contributes negligibly small value to the sum. By extending the domain from $c$ to 0 for the dominant second term, we obtain $A \sim \beta_d$ by computing $\int_0^\infty Ae^{-\beta_d t}dt \sim 1$. Similar analysis can be performed for MFPT $\langle T \rangle = \int_0^\infty tF(v_d, t|v_s)dt$ and the result is $\langle T \rangle \sim 1/\beta_d$. Using the decay rate derived in Eq. (9), the MFPT is given by

$$\langle T(v_d) \rangle \geq \frac{N \langle k \rangle}{k_d} \left( \frac{1}{1 - \langle k \rangle^{-\tau}} \right)$$  \hspace{1cm} (11)

which is a lower bound for the MFPT. We can get a similar approximate result for $\langle T(k_d) \rangle$ of MFPT averaged over all node pairs with same destination degree because this average is dominated by nodes separated far away. Networks with short relaxation time $\tau \ll N$ such as the ER and BA model have diameter of order $O(\ln N)$ and the number of nodes grow exponential as the distance increase. So the MFPT is dominated by the nodes with separation of order $O(\ln N)$. We see excellent agreement of our theory with numerical results for the ER and BA networks in Fig. 3. However, for networks with fractal scaling [9], the diameter scales as $O(N^\alpha)$ for some $\alpha$, so that more details about the structure are required to compute MFPT. Hence, we do not expect our result is good when applied to those fractal networks. Note that Eq. (11) is compatible with a previous numerical result [16] suggesting that $\langle T(k_d) \rangle \sim 1/k_d$, in which we have also computed the explicit form and shown that it is the lower bound of MFPT of networks with short relaxation time. With the method we used, higher moments of FPT distribution can also be computed as $\langle T^m(k_d) \rangle \approx m!\beta_d^{-m}$. Note also that our derivation depends on the property of short relaxation time regardless of the size of network. In addition, the mean first passage time $\langle T \rangle_G$ of the whole network $G$ can be computed by taking weighted average over the degree distribution as $\langle T \rangle_G = \sum P(k) \langle T(k) \rangle$. Since $P(k) \to 0$ as $k \to \infty$ and $\langle T(k) \rangle \propto 1/k$ for the networks considered in this paper, we have $\langle T \rangle_G \propto N \langle k \rangle$ which grows linear in the network size.

Real world networks. – We now test the applicability of our theory to real world networks which may have long relaxation time and structural inhomogeneity. The networks we examined are the Coauthorship network (astro-ph, cond-mat, hep-th) [17], C. Elegans neural network [19], E. Coli. Metabolic network [20], Yeast protein interaction [21] and Western State Power Grid [22]. In the simulation, only the largest connected component of each undirected network is considered.

The random walk properties and MFPT for different networks are presented in Table 1 and Fig. 4. The relaxation time $\tau$ is computed by $\tau = -1/\ln(|\lambda_2|)$, where $\lambda_2$ is the second largest eigenvalue of the time evolution operator Eq. (1), or the random walk matrix (see [23]). From Table 1, we can draw the following observations. (1) The decay rates are bounded by our theoretical value for all networks tested ($\beta_d/\beta_d$ is always less than 1). This observation strongly suggests that our result of decay rate can be applied to real world networks. (2) The inverse of the decay rate is approximately equal to the network size for networks with short relaxation time, in agreement with Eq. (9) (3) For BA and ER networks, the relaxation time $\tau$ is short, of the order $O(\ln N)$. In fact, the relaxation time of a BA network is shorter than the corresponding ER network with the same size and mean degree, which implies that the propagation in the BA network is faster than the ER network. This is caused by the shorter diameter of the BA network $O(\ln N/\ln \ln N)$, while for ER networks are presented in Table 1 and Fig. 4. The relaxation time $\tau$ is computed by $\tau = -1/\ln(|\lambda_2|)$, where $\lambda_2$ is the second largest eigenvalue of the time evolution operator Eq. (1), or the random walk matrix (see [23]). From Table 1, we can draw the following observations. (1) The decay rates are bounded by our theoretical value for all networks tested ($\beta_d/\beta_d$ is always less than 1). This observation strongly suggests that our result of decay rate can be applied to real world networks. (2) The inverse of the decay rate is approximately equal to the network size for networks with short relaxation time, in agreement with Eq. (9) (3) For BA and ER networks, the relaxation time $\tau$ is short, of the order $O(\ln N)$. In fact, the relaxation time of a BA network is shorter than the corresponding ER network with the same size and mean degree, which implies that the propagation in the BA network is faster than the ER network. This is caused by the shorter diameter of the BA network $O(\ln N/\ln \ln N)$, while for ER networks are presented in Table 1 and Fig. 4. The relaxation time $\tau$ is computed by $\tau = -1/\ln(|\lambda_2|)$, where $\lambda_2$ is the second largest eigenvalue of the time evolution operator Eq. (1), or the random walk matrix (see [23]). From Table 1, we can draw the following observations. (1) The decay rates are bounded by our theoretical value for all networks tested ($\beta_d/\beta_d$ is always less than 1). This observation strongly suggests that our result of decay rate can be applied to real world networks. (2) The inverse of the decay rate is approximately equal to the network size for networks with short relaxation time, in agreement with Eq. (9) (3) For BA and ER networks, the relaxation time $\tau$ is short, of the order $O(\ln N)$. In fact, the relaxation time of a BA network is shorter than the corresponding ER network with the same size and mean degree, which implies that the propagation in the BA network is faster than the ER network. This is caused by the shorter diameter of the BA network $O(\ln N/\ln \ln N)$, while for ER networks are presented in Table 1 and Fig. 4. The relaxation time $\tau$ is computed by $\tau = -1/\ln(|\lambda_2|)$, where $\lambda_2$ is the second largest eigenvalue of the time evolution operator Eq. (1), or the random walk matrix (see [23]). From Table 1, we can draw the following observations. (1) The decay rates are bounded by our theoretical value for all networks tested ($\beta_d/\beta_d$ is always less than 1). This observation strongly suggests that our result of decay rate can be applied to real world networks. (2) The inverse of the decay rate is approximately equal to the network size for networks with short relaxation time, in agreement with Eq. (9) (3) For BA and ER networks, the relaxation time $\tau$ is short, of the order $O(\ln N)$. In fact, the relaxation time of a BA network is shorter than the corresponding ER network with the same size and mean degree, which implies that the propagation in the BA network is faster than the ER network. This is caused by the shorter diameter of the BA network $O(\ln N/\ln \ln N)$, while for ER networks are presented in Table 1 and Fig. 4. The relaxation time $\tau$ is computed by $\tau = -1/\ln(|\lambda_2|)$, where $\lambda_2$ is the second largest eigenvalue of the time evolution operator Eq. (1), or the random walk matrix (see [23]). From Table 1, we can draw the following observations. (1) The decay rates are bounded by our theoretical value for all networks tested ($\beta_d/\beta_d$ is always less than 1). This observation strongly suggests that our result of decay rate can be applied to real world networks. (2) The inverse of the decay rate is approximately equal to the network size for networks with short relaxation time, in agreement with Eq. (9) (3) For BA and ER networks, the relaxation time $\tau$ is short, of the order $O(\ln N)$. In fact, the relaxation time of a BA network is shorter than the corresponding ER network with the same size and mean degree, which implies that the propagation in the BA network is faster than the ER network. This is caused by the shorter diameter of the BA network $O(\ln N/\ln \ln N)$, while for ER networks are presented in Table 1 and Fig. 4. The relaxation time $\tau$ is computed by $\tau = -1/\ln(|\lambda_2|)$, where $\lambda_2$ is the second largest eigenvalue of the time evolution operator Eq. (1), or the random walk matrix (see [23]). From Table 1, we can draw the following observations. (1) The decay rates are bounded by our theoretical value for all networks tested ($\beta_d/\beta_d$ is always less than 1). This observation strongly suggests that our result of decay rate can be applied to real world networks. (2) The inverse of the decay rate is approximately equal to the network size for networks with short relaxation time, in agreement with Eq. (9) (3) For BA and ER networks, the relaxation time $\tau$ is short, of the order $O(\ln N)$. In fact, the relaxation time of a BA network is shorter than the corresponding ER network with the same size and mean degree, which implies that the propagation in the BA network is faster than the ER network. This is caused by the shorter diameter of the BA network $O(\ln N/\ln \ln N)$, while for
it is of order $O(\ln N)$ [24]. Our theory of FPT distribution works very well for the ER and BA networks, as they both have short relaxation times. (4) For those real world networks with relaxation time $\tau$, simulation decay rate $\beta^*_u$ and theoretical decay rate $\beta_\ast$ given by Eq. (9). The method to compute $\tau$ is described in text and the $\beta^*_u$ can be found numerically after a long run by using Eq. (5).

| Networks                                      | $N$   | $\langle k \rangle$ | $\tau$ | $\frac{1}{\langle k \rangle}$ | $\langle \beta^*_u \rangle$ |
|-----------------------------------------------|-------|----------------------|--------|------------------------------|----------------------------|
| ER model                                      | 10000 | 12                   | 1.7    | 10996                        | 0.99                       |
| BA model                                      | 10000 | 12                   | 1.6    | 10085                        | 1.00                       |
| BA model                                      | 10000 | 8                    | 2.3    | 11624                        | 0.99                       |
| BA model                                      | 10000 | 4                    | 5.6    | 14457                        | 0.98                       |
| Coauthorship network, astro-ph [17, 18]       | 14845 | 16.1                 | 304    | 18435                        | 0.82                       |
| C. Elegans neural network                     | 297   | 14.5                 | 4.6    | 323                          | 0.99                       |
| Coauthorship network, cond-mat [17, 18]      | 13861 | 6.4                  | 139    | 23001                        | 0.70                       |
| E. Coli. Metabolic network [20]               | 2268  | 5.0                  | 16     | 3190                         | 0.92                       |
| Coauthorship network, hep-th [17, 18]         | 5835  | 4.7                  | 179    | 10055                        | 0.69                       |
| Yeast protein interaction [21]                | 1458  | 2.7                  | 120    | 3398                         | 0.83                       |
| Western State Power Grid [22]                 | 4941  | 2.7                  | 3689   | 30008                        | 0.33                       |

Finally, we have also observed a correlation between the random walk centrality [10] and the measured decay rate. In fact, our numerical results show that the random walk centrality is approximately equal to the measured decay rate. This will be investigated further in future research.

**Summary and discussion.** – In conclusion, our asymptotic analysis of FPT distribution allows us to compute an accurate expression of the decay rate of FPT distribution (9) for networks with short relaxation time $\tau \ll 1/\beta_\ast$. We have also shown that the decay rate of real world networks has similar order of magnitude as $P^\infty(v_u) = k_u/N \langle k \rangle$ and is upper bounded by this value. This upper bound can be explained by a physical picture that local neighbors around the destination, at the quasi-equilibrium state, should take the walker probability less than $P^\infty(u)$. Moreover, we have computed the MFPT (11) for destination far from the source. A similar expression of MFPT that depends on destination degree is argued to be correct for network with $\tau \ll N$. These theoretical results on the relationship between MFPT and destination degree have been verified by numerical simulations on both paradigmatic and real world networks. Our analysis can also be readily extended to other variants of random walk problem, such as random walk in weighted networks and biased random walk [25], with short relaxation time.
it predicts the MFPT to be

\[ \langle T(k) \rangle \gtrsim \frac{1}{P^\infty(k)} \] (12)

where \( P^\infty(k) \) is the equilibrium probability for the corresponding random walk problem.

Since \( 1/\beta_d \) also scales linearly in \( N \), the condition for networks with short relaxation time then becomes \( \tau \ll N \).

Assuming that this condition of short relaxation time is met, we have shown numerically that the relaxation times of many real world networks are actually short. Therefore, our analytical results are actually applicable to these real networks. We also argued that real networks with high mean degree should have short relaxation time.

Finally, we should note that although paradigmatic networks with random global links, such as the ER and BA network, show extremely short relaxation time \([26]\) and our theory fits very well with the numerical data on these networks, we expect more complex dynamical properties from real networks, which deserve further study.

K. Y. Szeto acknowledges the support of grant CERG 602506 and 602507.

References

[1] Albert R. and Barabási A. L., Rev. Mod. Phys., 74 (2002) 47.
[2] Dorogovtsev S. N., Goltsev A. V. and Mendes J. F. F., Rev. Mod. Phys., 80 (2008) 1275.
[3] Song C., Havlin S. and Makse H. A., Nat Phys, 2 (2006) 275.
[4] Kim J. S., Gor K.-I., Saavi G., Oh E., Kahng B. and Kim D., Phys. Rev. E, 75 (2007) 016110.
[5] Frasca M., Buscarino A., Rizzo A., Fortuna L. and Boccaletti S., Phys. Rev. E, 74 (2006) 036110.
[6] Tuckwell H., Introduction to theoretical neurobiology (Cambridge University Press) 1988.
[7] Hughes B., Random walks and random environments: Volume 1: Random Walks (Oxford University Press, USA) 1995.
[8] Redner S., A guide to first-passage processes (Cambridge University Press) 2001.
[9] Condamin S., Bénichou O. and Moreau M., Phys. Rev. E, 75 (2007) 021111.
[10] Noh J. D. and Rieger H., Phys. Rev. Lett., 92 (2004) 118701.
[11] Condamin S., Bénichou O., Tejedor V., Vortureux R. and Klafter J., Nature, 450 (2007) 77.
[12] Hawkes C. P. and Roberts A. P., Phys. Rev. Lett., 103 (2009) 020601.
[13] Gfellner D. and Rios P. D. L., Phys. Rev. Lett., 100 (2008) 174104.
[14] Newman M. E. J., Strogatz S. H. and Watts D. J., Phys. Rev. E, 64 (2001) 026118.
[15] Ma C. W. and Szeto K. Y., Phys. Rev. E, 73 (2006) 041701.

Baronchelli A. and Lören V., Phys. Rev. E, 73 (2006) 026103.
[16] Newman M. E. J., Phys. Rev. E, 64 (2001) 016131.
[17] Newman M. E. J., Phys. Rev. E, 64 (2001) 016132.
[18] White J. G., Southgate E., Thomson J. N. and Brenner S., Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences, 314 (1986) 1.
[19] Jeong H., Tombor B., Albert R., Oltvai Z. N. and Barabási A.-L., Nature, 407 (2000) 651.
[20] Jeong H., Mason S. P., Barabási A.-L. and Oltvai Z. N., Nature, 411 (2001) 41.
[21] Watts D. J. and Strogatz S. H., Nature, 393 (1998) 440.
[22] Kim D. and Kahng B., Chaos, 17 (2007) 036115.
[23] Cohen R. and Havlin S., Phys. Rev. Lett., 90 (2003) 058701.
[24] Franscak A. and Franscak P., Phys. Rev. E, 80 (2009) 016107.
[25] Samukhin A. N., Dorogovtsev S. N. and Mendes J. F. F., Phys. Rev. E, 77 (2008) 036115.