MINIMAX FORMULA FOR THE REPLICA SYMMETRIC FREE ENERGY
OF DEEP RESTRICTED BOLTZMANN MACHINES

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Abstract. We study the free energy of a most used deep architecture for restricted Boltzmann machines, where the layers are disposed in series. Assuming independent Gaussian distributed random weights, we show that the error term in the so-called replica symmetric sum rule can be optimised as a saddle point. This leads us to conjecture that in the replica symmetric approximation the free energy is given by a min max formula, which parallels the one achieved for two-layer case.

MSC: 82D30, 49K35.

1. Introduction

The deep restricted Boltzmann machine (dRBM) is a widely studied generative model, first introduced in [1], in which many RBMs are piled up in a serial architecture. Indeed in many practical applications RBMs fail to model complex data distributions without a careful choice of the weight initialisation in the learning algorithm and augmenting the depth aims chiefly to increase the representational power of the model. Similarly to deep neural networks, the idea of dRBMs is that the multi-layer architecture increases the level of nonlinearity of the model and allows high-order representations in the hidden layers apt to capture higher-order correlations of the data.

Despite the underlying bipartite structure permitting Gibbs sampling to work efficiently, dRBMs are hard to train, which makes other generative models preferred in practice. The typical gradient-based algorithms as for instance contrastive divergence [2, 3] get more easily stuck in poorly representative local maxima of the log-likelihood, due to the more complex landscape. Many improvements of the standard algorithms have been proposed [4, 5], but the problem is substantially difficult due highly non-convex structure of the log-likelihood (a feature of RBMs, increasing with the depth). One major source of difficulty in the log-likelihood is the logarithm of the partition function of the model (physicist’s free energy, in this paper we adopt this terminology). Indeed its lack of convexity makes gradient ascent based optimisation algorithms and Monte Carlo approximation methods very sensitive to the initialisation.

The aim of this note is to obtain some more information of the convexity (or lack thereof) of the free energy. We study the replica symmetric (RS) sum rule of the free energy of dRBMs, assuming the weight matrix to be random, with independent standard Gaussian entries, as customary in the spin glass literature. Such a sum rule is obtained by Gaussian interpolation between the energy term of the dRBM and some independent random biases. The biases are taken to be centred Gaussian random variables, whose variances can be seen as Lagrange multipliers which one can optimise on. This optimisation is in fact the main focus of the paper: we show that it yields a min max variational formula similar the one proved for the standard RBM in [6, 7].

Next we introduce the objects we will deal with and state precisely the main result.
1.1. The model. The model is defined as follows. Let $N \in \mathbb{N}$ denote the total number of units and $\nu \geq 2$ the number of layers, indexed by $x = 1, \ldots, \nu$. The $x$-th layer has $N_x$ units, with $\sum_x N_x = N$ and $\alpha_x := \lim_N N_x/N \in (0,1)$, with $\sum_x \alpha_x = 1$. We will often use $\alpha_x$ to denote the ratio $N_x/N$ also at finite size with a small abuse of notation. A total configuration is indicated by $\sigma$ and the $x$-th layer configuration by $\sigma^{(x)}$.

We assume the units $\sigma^{(x)}$ for all $x \in [\nu]$ to be independent random variables with Bernoulli $\pm 1$ a priori distributions, which we bias independently by the entry-wise constant vectors $b^{(x)} := (b^{(x)}_1, \ldots, b^{(x)}_x) \in \mathbb{R}^{N_x}$, $b^{(x)}_x \in \mathbb{R}$ (with a little abuse of notations we indicate the vector and its constant components with the same symbol). Expectation values w.r.t. the $x$-th prior distribution is denoted by $\hat{E}_{\sigma^{(x)}}$. Mostly we will omit the biases in the notation.

Two consecutive layers $x, x + 1$ ($x \in [\nu - 1]$) interact via the RBM Hamiltonian

$$H_N^{(x)}(\sigma^{(x)}, \sigma^{(x+1)}) := - \sum_{i \in [N_x]} \sum_{j \in [N_{x+1}]} \frac{\epsilon^{(x)}_{ij}}{\sqrt{N}} \sigma_i^{(x)} \sigma_j^{(x+1)},$$

where the $\{\epsilon^{(x)}_{ij}\}_{x \in [\nu - 1]}$, $i \in [N_x]$ are i.i.d. quenched r.vs.

The multilayer model is defined by a combination of RBM Hamiltonians:

$$H_N(\sigma) = \sum_{x \in [\nu - 1]} H_N^{(x)}(\sigma^{(x)}, \sigma^{(x+1)}).$$

The Gibbs posterior distribution for any $\beta \geq 0$ reads

$$P_{\beta}(\sigma; \xi, b) := \frac{\exp \left( - \beta H_N(\sigma) + \sum_{x \in [\nu]} (b^{(x)}_x, \sigma^{(x)}_x) \right)}{\hat{E}_{\sigma^{(1)} \ldots \sigma^{(\nu)}} \left[ \exp \left( - \beta H_N(\sigma) + \sum_{x \in [\nu]} (b^{(x)}_x, \sigma^{(x)}_x) \right) \right]}$$

where we denote by $(\cdot, \cdot)$ the inner product of $\mathbb{R}^d$ (regardless of the dimension $d$). The free energy $A_N$ is defined as follows

$$A_N(\beta) := \frac{1}{N} \log \hat{E}_{\sigma^{(1)} \ldots \sigma^{(\nu)}} \left[ \exp \left( - \beta H_N(\sigma) + \sum_{x \in [\nu]} (b^{(x)}_x, \sigma^{(x)}_x) \right) \right].$$

The main interest is in computing the free energy in the thermodynamic limit, that is

$$N_1, \ldots, N_\nu \rightarrow \infty \quad \text{such that} \quad N_x/N \rightarrow \alpha_x \quad \forall x \in [\nu].$$

We indicate this limiting procedure simply as $\lim_N$. The existence of the limit of the free energy is at the moment an open mathematical problem. Assuming it exists however it must be self-averaging, by the standard argument using concentration of Lipschitz functions [8]. Therefore we can equivalently study the limit of the averages of $A_N$.

Central objects are the overlaps, i.e. normalised inner products, of each couple of configurations of the $x$-th layer $\sigma^{(x)}$, $\tau^{(x)}$

$$R^{(x)} = R^{(x)}(\sigma^{(x)}, \tau^{(x)}) := \frac{(\sigma^{(x)}_x, \tau^{(x)}_x)}{N_x}.$$

Taking two different points $\sigma, \tau \in \{-1,1\}^N$ let us compute the covariance of the energy in terms of the overlaps (assume for simplicity $b^{(x)} = 0$ for all $x \in [\nu]$). We have

$$E[H_N(\sigma)H_N(\tau)] = N \sum_{|x-y|=1} \alpha_x \alpha_y R^{(x)}R^{(y)}.$$
This quantity is not positive definite (the overlaps can take both positive and negative values), which is a major source of difficulties when comparing dRBMs to the Sherrington-Kirkpatrick model. As it will be clear later on, such a lack of positivity of the covariance of the Hamiltonian yields a non-convex variational principle for the free energy (see (1.6) and (1.11) below).

For simplicity we present our argument for $±1$ Bernoulli priors. We expect the extension to bounded units to be straightforward and that further extensions to sub-Gaussian units are possible, but technically more involved. Indeed two generic sub-Gaussian units coupled by the Hamiltonian (1.1) typically make the model ill defined for large $\beta$. Therefore one has to regularise the posterior distribution as discussed in [9], and similarly also the interpolating factors introduced below (as in [6]). These regularisations introduce a number of technicalities, irrelevant for the main message we aim to give here. Also, as long as the units are bounded, the choice of the Gaussian distribution for the random weights is not so stringent, as the same argument of [10] applies and universality of the free energy can be proven for a large class of independent random weights. Most interesting are extensions to dependent random weights [11], yet rather unexplored at the moment.

Also, we deal with small biases, i.e. all the $b(x)$ are bounded by a fixed constant (see Theorem 1.2). Dealing with possibly large biases is in fact easier but it requires a procedure which is similar, yet not exactly the same as the one presented in the sequel. For clarity, we focus here only on the most interesting and challenging case.

In what follows we will conveniently separate the layers into two disjoint subsets. To fix the ideas let us set

\[ H := 2N \cap [\nu] \quad \text{and} \quad V := (2N - 1) \cap [\nu] \]

respectively the layers in an even or odd position in increasing order. Note that units within the layers in $V$ are conditionally independent w.r.t. $p_{\beta}(\sigma; \xi, b)$ given the layers of $H$ (and viceversa), reflecting the aforementioned bipartite structure of the dRBM.

Throughout $\eta$ (possibly labeled by one or more indices) denotes a $\mathcal{N}(b, \sigma)$ random variable, whose expectation is always denoted by $E_{b, \sigma}$, that is in the univariate case

\[ E_{b, \sigma}[f] = \int f(x) \frac{e^{-\frac{(x-b)^2}{2\sigma}}}{\sqrt{2\pi\sigma}} dx \]

(note that $\lim_{\sigma \to 0} E_{b, \sigma}[f] = f(b)$ for sufficiently regular $f$). When different $\eta$s are i.i.d. $\mathcal{N}(b, \sigma)$ we will use a unique symbol $E_{b, \sigma}$ to denote their joint expectation. Averages on the $\mathcal{N}(0, 1)$ quenched weights $\xi_{ij}^{(x)}$ are simply indicated by $\xi$.

1.2. Main result and RS approximation. The aim of this paper is to characterise the RS approximation of the free energy of deep RBMs in terms of a non-convex variational formula. The main motivation for that is to take a step towards the understanding of the free energy. Albeit its exact form will probably involve replica symmetry breaking, we believe that the structure described here should persist also in this more complicated scenario. For a further elaboration on that, see Conjecture 1.3 below and related discussion.

The starting point of our considerations is a sum-rule for the free energy as the one obtained for the Sherrington-Kirkpatrick model by Guerra in [12] and later developed in different contexts. Let $t \in [0,1]$, $q := (q_1, \ldots, q_\nu) \in [0,1]^\nu$. Let further for $x \in \nu \{ \eta^x_1 \}_{i \in [N_x]}$ be i.i.d. $\mathcal{N}(0,1)$. We set

\[ H_N'(\sigma) := - \sum_{x=1}^\nu \sum_{|y-z|=1} \alpha_y q_y \sum_{i=1}^{N_x} \eta^x_i \sigma_i^{(x)}, \quad (1.3) \]

\[ H_{N,t} := \sqrt{t} H_N(\sigma) + \sqrt{1-t} H_N'(\sigma). \quad (1.4) \]
We introduce the measures
\[ \langle \cdot \rangle_t := \frac{\hat{E}_{\sigma_t} \exp[-\beta H_N t]}{\hat{E}_{\sigma_t} \exp[-\beta H_N t]}, \quad \langle \cdot \rangle_{t=1} := \langle \cdot \rangle_{\beta, b, N}. \] (1.5)

and
\[ E_N[q] := \sum_{|x-y|=1} \int_0^1 dt E_{0,1} \exp \left( \left( (R^{(x)} - q_x)(R^{(y)} - q_y) \right) t \right) \] (1.6)
\[ \text{RS}(q) := \sum_{x=1}^\nu \alpha_x E_{0,1} \log \cosh \left( b^{(x)} + \beta \eta \sum_{|y-x|=1} \alpha_y q_y \right) \]
\[ + \frac{\beta^2}{2} \sum_{|x-y|=1} (\alpha_x - \alpha_x q_x)(\alpha_y - \alpha_y q_y). \] (1.7)

Then we have the following sum rule for the free energy

**Lemma 1.1.** For every \( q = (q_1, \ldots, q_\nu) \in [0, 1]^\nu \) we have
\[ E[A_N(\beta)] + \frac{\beta^2}{2} E_N[q] = \text{RS}(q). \] (1.8)

The proof is quite standard. For the model under consideration it can be found in [13, 14], but we give it anyway at the end of the paper for completeness.

The main contribution of this note is the following

**Theorem 1.2.** Assume \( |b^{(x)}| \leq \frac{1}{4} \log 2 + \sqrt{3} \) for any \( x \in [\nu] \). Then \( \text{RS}(q) \) has a unique stationary point \( \bar{q} \in [0, 1]^\nu \) satisfying
\[ \alpha_{x-1} q_{x-1} + \alpha_{x+1} q_{x+1} = E_{b^{(x)}, \beta \sigma x \nu} \tanh^2(q), \quad x = 1, \ldots, [\nu], \] (1.9)
and it holds
\[ \text{RS}(\bar{q}) = \min_{q_x : x \in V} \max_{q_x : x \in H} \text{RS}(q_1, \ldots, q_\nu). \] (1.10)

This \( \min \max \) formula marks an important difference between dRBMs and other better understood spin glass models, such as the Sherrington-Kirkpatrick model, for which the free energy (and its RS approximation) is given by a convex variational principle.

In view of (1.8) the last theorem can be reformulated as follows: there is a \( \check{q} \in [0, 1]^\nu \) independent on \( N \) such that
\[ E_N[\check{q}] = \min_{q_x : x \in V} \max_{q_x : x \in H} E_N[(q_1, \ldots, q_\nu)]. \] (1.11)

Theorem 1.2 leads us to the following conjecture

**Conjecture 1.3.** Assume there is \( \hat{q} = (\hat{q}_1, \ldots, \hat{q}_\nu) \in [0, 1]^\nu \) such that for some choice of the parameters \((\alpha_1, \ldots, \alpha_\nu, \beta, b^{(1)}, \ldots, b^{(\nu)}) \in \Omega_{RS} \subseteq (0, 1)^{2\nu} \times (0, \infty) \times \mathbb{R}^\nu\)
\[ \lim_N E [\langle q^{(x)} - R^{(x)} \rangle^2]_{\beta, b, N} = 0, \quad \forall x \in [\nu]. \] (1.12)

Then \( \hat{q} = \check{q} \) (\( \check{q} \) has been introduced in Theorem 1.2).

Equivalently we are conjecturing that if the overlaps are self-averaging to some value \( \check{q} \) then
\[ \lim_N A_N = \min_{q_x : x \in V} \max_{q_x : x \in H} \text{RS}(q_1, \ldots, q_\nu) = \text{RS}(\check{q}_1, \ldots, \check{q}_\nu) \] (1.13)
i.e. the optimum is attained in $\bar{q}$. Note that there is at least one case in which the conjecture in this latter form can be a posteriori verified to be true. Indeed if $b(x) = 0$ for all $x \in [\nu]$, we can check that $\bar{q} = 0$ satisfies (1.9). In this case RS(0) reduces to the annealed free energy and (1.13) holds in a certain region of parameters [14, Theorem 4.1]; one expects that the self-averaging of the overlaps around zero can be proved in this region by Talagrand’s exponential inequalities [15]. We also mention the work [16], in which the authors recover the free energy (1.10) on the Nishimori line, which enforces replica symmetry.

The rest of the paper is organised as follows. We finish the introduction discussing the connection of our work with the existing literature. In Section 2 we prove a number of auxiliary statements, which basically extend the so-call Guerra-Latala lemma. Then the proof of Theorem 1.2 is given in the Section 3. We generalise the optimisation procedure earlier introduced in [6, 7] to achieve the min max formula for the RS free energy of respectively Gaussian-Bernoulli and Bernoulli-Bernoulli RBMs. This optimisation is tricky: after a convenient change of coordinates, we avoid dealing with the Hessian of RS($q$), which turns out to be complicated, and proceed with an iterative nested optimisation of one variable at a time. To do so we will crucially make use of some extensions of the so-called Guerra-Latala Lemma [12, 15].

1.3. Related literature. As already mentioned, a min max formula for the RS free energy of RBMs was derived in [6, 7] for Gauss-Bernoulli and Bernoulli-Bernoulli priors. Interesting enough, for RBMs with spherically symmetric sub-Gaussian priors, albeit their free energy admits also a min max formulation [17], a fully convex variational principle has been proven [18, 19, 9]. Also in problems with similar mathematical settings such as high dimensional linear inference, in which RS is enforced by the Nishimori line, a min max formula for the free energy can be achieved by similar methods (see e.g. [20]) and holds also in the spherical case [21]. Spherical models are special as they are typically RS and the prior allows diagonalisation of the energy (in the sense of principal values), so that a random matrix type analysis can be performed; we do not known whether this technical advantage is the sole responsible for convexity.

The mathematical study of multi-layered model initiated with the papers [22] and [23]. In [22] the authors analysed the Hamiltonian

$$(1.1) + \sum_{x \in [\nu]} \beta^{(x)} H^{(x)}_{SK},$$

where $H^{(x)}_{SK}$ denotes the Sherrington-Kirkpatrick Hamiltonian as a function of the $x$-th layer and $\beta^{(x)} \geq 0$. They proved an upper bound for the free energy under the assumption that $\beta^{(1)}, \ldots, \beta^{(\nu)}$ are large enough to enforce the positivity of the covariance of the energy in terms of the overlaps. A lower bound was proved in [23], independently on this assumption. These bounds are formulated in terms of a Parisi-like variational principle (therefore convex) and they match for the models studied in [22]. This caused a wide-spread belief that the lower bound of [23] should be optimal also in the non-convex case, supported by the results for the RS approximation for the two layer case [24, 7], but the question remains open.

The particular architecture of dRBMs (for Bernoulli priors) has been recently investigated in [13, 14] in which the authors are able to characterise the high temperature region and to generalise to dRBMs the lower bound of [7, 25] for the free energy (notably these are of the same type of the one of [23]). Furthermore they argue that replica symmetry is identified by a set of equations that we show here to determine the saddle point of RS($q$). In this sense our analysis extends and complements the work begun in [13, 14].

Finally we comment on the RS characterisation given here. A precise mathematical description of the RS phase has been one principal object of study at the early stage of the theory of disordered system, especially for the Sherrington-Kirkpatrick model in the attempt to disprove the replica symmetric formula of the free energy. The paradigm was to prove the following statement: if the
overlap is self-averaging then the free energy is equal to its RS expression. To prove that, the main point seems to be to select the right sequence of finite dimensional Gibbs measure w.r.t. which the overlap should be asymptotically self-averaging. In [26] the authors proved the statement working with a Gibbs measure biased by a very specific random vector vanishing as $N^{-\frac{1}{4}}$. This result was later revisited in [27] and in [28]: in either paper it is considered a Gibbs measure perturbed according to the so-called cavity method; the latest version of this proof can be found in [15, Section 1.6]. Finally from the approach of [12] it can be easily proven that if the interpolation error, i.e. the variance of the overlap w.r.t. the interpolating Gibbs measure, vanishes in some point, then the min of the RS functional must be attained only in its zero. This allows us to conclude that such a self-averaging property of the overlap implies the RS formula for the free energy. Once again for dRBMs the crucial difference is in their non-convex structure, which complicates things further, as the interpolation error is not positive definite.

1.4. Acknowledgements. This paper benefited greatly from the observations of an anonymous referee, who is gratefully acknowledged.

2. Guerra-Latala lemmas

The proof of our main theorem will heavily use the so-called Guerra-Latala lemma [12] [15]. In the next two lemmas we follow essentially [15, A.14], albeit with a different presentation, obtaining somewhat more general results.

We begin by the following formula, holding for instance for any $f \in C^2(\mathbb{R})$:

$$
\frac{d}{d\sigma} E_{b,\sigma}[f] = \frac{E_{b,\sigma}[(\eta - b)f'(\eta)]}{2\sigma}.
$$

(2.1)

The proof is simple:

$$
\frac{d}{d\sigma} E_{b,\sigma}[f(\eta)] = \frac{d}{d\sigma} E_{b,\sigma,1}[f(\eta\sqrt{\sigma} + b)] = \frac{1}{2\sqrt{\sigma}} E_{b,\sigma,1}[\eta f'(\eta\sqrt{\sigma} + b)] = \frac{E_{b,\sigma}[(\eta - b)f'(\eta)]}{2\sigma}.
$$

(2.2)

Let us now assume $\Phi \in C^2(\mathbb{R})$, odd, strictly increasing, with $E_{0,1}[|\Phi|]$ bounded, $\Phi(0) = 0$ and $x\Phi''(x) < 0$ for $x \neq 0$. We also assume that there is a number $L > 0$ such that $\Phi^2$ is convex for $|x| \leq L$. Later on we will specialise $\Phi(x) = \tanh(x)$, which clearly enjoys these properties (with $L = \log \sqrt{2 + \sqrt{3}}$).

We set for any $b \in \mathbb{R}$, $\sigma > 0$

$$
c_{b,\sigma}[\Phi] := \frac{E_{b,\sigma}[(\eta - b)\Phi(\eta)\Phi'(\eta)]}{E_{b,\sigma}[\Phi^2]}.
$$

(2.3)

Lemma 2.1. Let $\Phi$ and $L > 0$ defined as above. Then $c_{b,\sigma}[\Phi] \in (0, 1)$ for all $|b| \leq L/2$.

Proof. First we show the upper bound, in fact without any restriction on $b$. We use that $\Phi(z)\Phi'(z)$ is odd to have

$$
b E_{b,\sigma}[\Phi(\eta)\Phi'(\eta)] = \frac{b}{2} (E_{b,\sigma}[\Phi(\eta)\Phi'(\eta)] - E_{-b,\sigma}[\Phi(\eta)\Phi'(\eta)])
$$

$$
= be^{-\frac{b}{\sigma}} E_{b,\sigma}[\sinh(\frac{b}{\sigma})\Phi(\eta)\Phi'(\eta)] \geq 0,
$$

uniformly in $b$. Then

$$
c_{b,\sigma}[\Phi] \leq \frac{E_{b,\sigma}[\eta\Phi(\eta)\Phi'(\eta)]}{E_{b,\sigma}[\Phi^2]}.
$$

Now we show that

$$
E_{b,\sigma}[\Phi^2] > E_{b,\sigma}[\eta\Phi(\eta)\Phi'(\eta)].
$$
We study the function $x \mapsto x \Phi'(x) - \Phi(x)$ for $x \in \mathbb{R}$. We note that it vanishes at the origin and it decreases for $x \neq 0$. So it must be $x \Phi'(x) - \Phi(x) > 0$ for $x < 0$ and $x \Phi'(x) - \Phi(x) < 0$ for $x > 0$, whence
\[
\Phi(x)(x \Phi'(x) - \Phi(x)) < 0, \quad \forall \ x \neq 0.
\]

For the lower bound on $c_{b,\sigma}[\Phi]$ we consider for definiteness $b > 0$. As $\Phi(\eta)\Phi'(\eta)$ is an odd function then $\eta \Phi(\eta)\Phi'(\eta)$ is even, thus we have
\[
E_{b,\sigma}[(\eta - b)\Phi(\eta)\Phi'(\eta)] = e^{-\frac{\eta^2}{2b}} E_{0,\sigma}[(\eta - b)\Phi(\eta)\Phi'(\eta)e^{\frac{\eta^2}{2b}}]
\]
\[
= e^{-\frac{\eta^2}{2b}} E_{0,\sigma}[(\eta - b)\Phi(\eta)\Phi'(\eta)e^{\frac{\eta^2}{2b}}] + \frac{e^{-\frac{\eta^2}{2b}}}{2} E_{0,\sigma}[(\eta - b)\Phi(-\eta)\Phi'(\eta)e^{\frac{\eta^2}{2b}}]
\]
\[
= e^{-\frac{\eta^2}{2b}} E_{0,\sigma}[(\Phi(\eta)\Phi'(\eta)(\eta \cosh(\frac{\eta b}{\sigma})) - b \sinh(\frac{\eta b}{\sigma}))]
\]
\[
= 2e^{-\frac{\eta^2}{2b}} E_{0,\sigma}[(\eta \geq 0 \Phi(\eta)\Phi'(\eta) \cosh(\frac{\eta b}{\sigma}))(\eta - b \tanh(\frac{\eta b}{\sigma}))]
\]
\[
= \sum_{s = \pm 1} E_{sb,\sigma}[1_{\{\eta \geq 0\}} \Phi(\eta)\Phi'(\eta)(\eta - b \tanh(\frac{\eta b}{\sigma}))]. \quad (2.4)
\]

Let $\eta^*$ denote the unique non-zero solution of $\eta = b \tanh(\frac{\eta b}{\sigma})$. It is easy to see that the origin is the unique solution of such equation iff $\sigma > b^2$. Moreover if $\sigma \leq b^2$ the function $\eta^*(\sigma) > 0$ is decreasing (as can be seen by implicit differentiation) from $\eta^*(0) = b$ to zero. Therefore the integrand in (2.4) is negative for $\eta < \eta^*$, otherwise it is positive. So if $\sigma \geq b^2$ then $\eta \geq b \tanh(\frac{\eta b}{\sqrt{2}b})$ and the whole expression (2.4) is positive.

Assuming now $\sigma \in [0, b^2]$, we split
\[
E_{sb,\sigma}[1_{\{\eta \geq 0\}} \Phi(\eta)\Phi'(\eta)(\eta - b \tanh(\frac{\eta b}{\sigma}))] = -E_{sb,\sigma}[1_{\{0 \leq \eta < \eta^*\}} \Phi(\eta)\Phi'(\eta)|\eta - b \tanh(\frac{\eta b}{\sigma})]]
\]
\[+ E_{sb,\sigma}[1_{\{\eta \geq \eta^*\}} \Phi(\eta)\Phi'(\eta)|\eta - b \tanh(\frac{\eta b}{\sigma})]]. \quad (2.5)
\]

We will use the estimate
\[
 b \tanh\left(\frac{\eta b}{\sigma}\right) \leq \eta^* + \frac{b^2 - \eta^2}{\sigma}(\eta - \eta^*)
\]
coming from Taylor expansion. We have
\[
E_{sb,\sigma}[1_{\{0 \leq \eta < \eta^*\}} \Phi(\eta)\Phi'(\eta)|\eta - b \tanh(\frac{\eta b}{\sigma})]]
\[
\leq \left(1 - \frac{b^2 - \eta^2}{\sigma}\right) \Phi(\eta^*)\Phi'(\eta^*) E_{sb,\sigma}[1_{\{0 \leq \eta < \eta^*\}} (\eta^* - \eta)] \quad (2.6)
\]
and
\[
E_{sb,\sigma}[1_{\{\eta \geq \eta^*\}} \Phi(\eta)\Phi'(\eta)|\eta - b \tanh(\frac{\eta b}{\sigma})]]
\[
\geq E_{sb,\sigma}[1_{\{\eta^* \leq \eta \leq 2\eta^*\}} \Phi(\eta)\Phi'(\eta)|\eta - b \tanh(\frac{\eta b}{\sigma})]]
\]
\[
\geq \left(1 - \frac{b^2 - \eta^2}{\sigma}\right) \Phi(\eta^*)\Phi'(\eta^*) E_{sb,\sigma}[1_{\{\eta^* \leq \eta \leq 2\eta^*\}} (\eta - \eta^*)]. \quad (2.7)
\]
We used in (2.6) and (2.7) that Φ′ is positive and increasing for \( x \in [0, 2b] \) (since \( b < L/2 \)); thus
\[
E_{b,\sigma}[\{\eta \geq 0\} \Phi(\eta)\Phi'(\eta)(\eta - b \tanh(\frac{\eta b}{\sigma}))]
\geq \left(1 - \frac{\eta^2}{\sigma^2}\right) \Phi(\eta^*)\Phi'(\eta^*) (E_{b,\sigma}[\{0 \leq \eta \leq 2b\} (\eta - \eta^*)]
= \left(1 - \frac{\eta^2}{\sigma^2}\right) \Phi(\eta^*)\Phi'(\eta^*) (E_{b,\sigma}[\{0 \leq \eta \leq 2b\} ((b - \eta^*)E_{b,\sigma}[\{0 \leq \eta \leq 2b\}])
= \left(1 - \frac{\eta^2}{\sigma^2}\right) \Phi(\eta^*)\Phi'(\eta^*)(b - \eta^*)E_{b,\sigma}[\{0 \leq \eta \leq 2b\}] > 0.
\]
(2.8)

On the other hand, neglecting completely the positive part, we have by (2.6)
\[
E_{-b,\sigma}[\{\eta \geq 0\} \Phi(\eta)\Phi'(\eta)(\eta - b \tanh(\frac{\eta b}{\sigma}))]
= - \left(1 - \frac{\eta^2}{\sigma^2}\right) \Phi(\eta^*)\Phi'(\eta^*)\sqrt{2}e^{-\frac{\eta^2}{\sigma^2}} \eta^*.
\]
Combining with (2.4) and (2.8) we get
\[
\sum_{s=\pm 1} E_{s,b,\sigma}[\{\eta \geq 0\} \Phi(\eta)\Phi'(\eta)(\eta - b \tanh(\frac{\eta b}{\sigma}))]
\geq \left(1 - \frac{\eta^2}{\sigma^2}\right) \Phi(\eta^*)\Phi'(\eta^*) ((b - \eta^*)E_{b,\sigma}[\{0 \leq \eta \leq 2b\}] - \sqrt{2}e^{-\frac{\eta^2}{\sigma^2}} \eta^*)
= \left(1 - \frac{\eta^2}{\sigma^2}\right) \Phi(\eta^*)\Phi'(\eta^*) ((b - \eta^*) (E_{0,1}[\{0 \leq \eta \leq \frac{L}{2}\}] + \sqrt{2}e^{-\frac{\eta^2}{\sigma^2}}) - b\sqrt{2}e^{-\frac{\eta^2}{\sigma^2}}). \tag{2.9}
\]
Noting that \( b - \eta^* = b(1 + e^{2b^2/\sigma^2})^{-1} \) it is immediate to verify that the expression (2.9) is non-negative for all \( \sigma \in [0, b^2] \), which ends the proof.

For any \( f \in C^2(\mathbb{R}^+) \) positive and increasing, by (2.1), (2.3) and Lemma 2.1 we have
\[
\frac{d}{dx} E_{b,f(x)}[\Phi^2] = \frac{f'(x)}{f(x)} E_{b,f(x)}[(\eta - b)\Phi(\eta)\Phi'(\eta)] = \frac{f'(x)c_{b,f(x)}[\Phi']E_{b,f(x)}[\Phi^2]}{f(x)} > 0,
\tag{2.10}
\]
\( i.e. \ E_{b,f(x)}[\Phi^2] \) is increasing w.r.t. \( x \) for any \( |b| \leq L/2 \). We will use much this fact in the sequel.

Lemma 2.2 (Guerra-Latala). Let \( b \in \mathbb{R} \) and \( f, g \) be differentiable, positive and increasing. The function \( \Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) defined by
\[
\Psi(x) := \frac{E_{b,f(x)}[\Phi^2]}{g(x)} \tag{2.11}
\]
is decreasing for those \( x \) such that
\[
g'(x)f(x) \geq c_{b,f(x)}[\Phi]f'(x)g(x). \tag{2.12}
\]
Otherwise it is increasing.

Proof. The assertion follows readily from
\[
\Psi'(x) = \Psi(x) \left( \frac{f'(x)c_{b,f(x)}[\Phi]}{f(x)} - \frac{g'(x)}{g(x)} \right), \tag{2.13}
\]
To prove it, we compute by (2.1)
\[
(g\Psi)' = \frac{f'}{f} E_{b,f(x)}[(\eta - b)\Phi(\eta)\Phi'(\eta)].
\]
Therefore by the Leibniz rule
\[
\frac{g(x)\Psi'(x)}{f(x)} = \frac{f'(x)}{f(x)} E_{b,f(x)}[(\eta - b)\Phi(\eta)\Phi'(\eta)] - \frac{g'(x)}{g(x)} E_{b,f(x)}[\Phi^2]
\]
\[
= E_{b,f(x)}[\Phi^2] \left( \frac{f'(x)c_{b,f(x)}[\Phi]}{f(x)} - \frac{g'(x)}{g(x)} \right),
\]
which proves (2.13).

The usual Guerra-Latala result corresponds to the special case \( f(x) = g(x) = x \). In turn the simpler bound
\[
g'f \geq f'g
\]
will be often sufficient to make \( \Psi \) decreasing thanks to Lemma 2.1. This amounts to assume \( f/g \) non-increasing. We state for clarity this simplified version of the above lemma as follows:

**Lemma 2.3.** Let \(|b| \leq L/2\) and \( f,g \) be differentiable, positive and increasing such that (2.15) holds. Then the function \( \Psi \) defined by (2.11) is decreasing.

The following cases are not covered by the previous lemmas.

**Lemma 2.4.** Let \( G : \mathbb{R}^+ \rightarrow \mathbb{R} \) be continuously differentiable and increasing. Assume \(|b| \leq L/2\) and that there is a unique \((x_0,y_0)\) such that
\[
G(x_0) = E_{b,y_0}[\Phi^2].
\]
Then there is a function \( \bar{y} \) uniquely defined by
\[
G(x) - E_{b,\bar{y}_0}( \Phi^2 ) = 0.
\]
Moreover \( \bar{y} \) and \( \bar{y}/G \) are increasing.

**Proof.** \( G(x) \) is increasing and \(-E_{b,\bar{y}}[\Phi^2]\) is decreasing. Therefore the implicit function theorem applies, giving the existence of a differentiable function \( \bar{y}(x) > 0 \), uniquely defined in a neighbourhood of \((x_0,y_0)\) by
\[
G(x) = E_{b,\bar{y}_0}( \Phi^2 ).
\]
Then \( \bar{y}(x) \) extends globally as a function \( \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) via (2.18) since the intersection point \((x_0,y_0)\) is unique. We compute the derivative via
\[
\bar{y}'(x) = \frac{\bar{y}G'(x)}{E_{b,\bar{y}}(\eta - b)\Phi'\Phi} \bigg|_{\eta = \bar{y}} = \frac{\bar{y}G'(x)}{c_{b,\bar{y}}[\Phi]G(x)} > 0,
\]
where we used (2.10). Thus \( \bar{y} \) is differentiable, positive and increasing as function of \( x \). Then, considering
\[
\frac{G(x)}{\bar{y}(x)} = \frac{E_{b,\bar{y}}[\Phi^2]}{\bar{y}(x)}
\]
we see that the r.h.s. is decreasing by Lemma 2.3 (with \( f = g = \bar{y} \)). Thereby \( \bar{y}(x)/G(x) \) is increasing.

**Lemma 2.5.** Let \( F : \mathbb{R}^+ \rightarrow \mathbb{R} \) be non-negative, continuously differentiable, with \( F'(y)y \geq F(y) \). Assume \(|b| \leq L/2\) and that there is a unique \((x_0,y_0)\) such that
\[
F(y_0) = E_{b,y_0+x_0}[\Phi^2].
\]
Then there is a unique function \( \bar{y} \) defined by
\[
F(y) - E_{b,\bar{y}+x}[\Phi^2] = 0, \quad x > 0.
\]
The function $\bar{y}$ is increasing and $\bar{y}(x)/x$ is decreasing. Moreover

$$\partial_y(F(y) - E_{b,y+x}[\Phi^2]) \bigg|_{y=\bar{y}} > 0. \quad (2.21)$$

**Proof.** By (2.10) $E_{b,y+x}[\Phi^2]$ is increasing in $x$ at fixed $y$ and in $y$ at fixed $x$. By Lemma 2.3 (with $f = y + x$ and $g = y$) we see that

$$\frac{E_{b,y+x}[\Phi^2]}{y}$$

is decreasing as a function of $y$. Therefore $E_{b,y+x}[\Phi^2]$ is increasing and concave, while by assumption $F(y)$ is increasing and convex. Thus the function

$$\frac{F(y)}{y} - \frac{E_{b,y+x}[\Phi^2]}{y}$$

is increasing. Therefore

$$\partial_y(F(y) - E_{b,y+x}[\Phi^2]) \bigg|_{(x,y)=(x_0,y_0)} > \frac{F(y)}{y} - \frac{E_{b,y+x}[\Phi^2]}{y} \bigg|_{(x,y)=(x_0,y_0)} = 0. \quad (2.22)$$

Hence by the implicit function theorem we can define a function $\bar{y}$ locally around $x_0$ such that $\bar{y}(x_0) = y_0$ and (2.21) holds. Combining with (2.10) we get

$$\bar{y}'(x) = \frac{\partial_x E_{b,y+x}[\Phi^2]}{\partial_y(F(y) - E_{b,y+x}[\Phi^2])} \bigg|_{y=\bar{y}} > 0, \quad (2.23)$$

so $\bar{y}$ is increasing in its domain. Moreover computing explicitly by (2.1) and using Lemma 2.1 yields

$$\bar{y}'(x) = \frac{c_{b,y+x}[\Phi]F(\bar{y})}{F'(\bar{y})x + F'(\bar{y}) - c_{b,y+x}[\Phi]F(\bar{y})} = \frac{c_{b,y+x}[\Phi]F(\bar{y})}{F'(\bar{y})x + (1 - c_{b,y+x}[\Phi])F(\bar{y})} < \frac{\bar{y}}{x}, \quad (2.24)$$

which implies $\bar{y}/x$ decreasing in its domain. \hfill \Box

**Lemma 2.6.** Assume $|b| \leq L/2$. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be non-negative, continuously differentiable, with $F'(y)y \geq F(y)$ and $y$ implicitly defined by

$$F(y) = E_{b,y+x}[\Phi^2], \quad x \geq 0. \quad (2.25)$$

Then

$$\Psi_1(x) := \frac{E_{b,y+x}[\Phi^2]}{x} \quad (2.26)$$

is decreasing.

**Proof.** We apply Lemma 2.3 with $f = y + x$ and $g = x$ and we see that condition (2.15) implying $\Psi_1$ decreasing reads $x\bar{y}'(x) \leq y(x) + x$. This is ensured for all $x > 0$ by Lemma 2.5. \hfill \Box
3. Proofs of Theorem 1.2 and Lemma 1.1

Here we prove Theorem 1.2. We rewrite the RS function we want to optimise
\[
\begin{align*}
\text{RS}(q) := \sum_{x=1}^\nu \alpha_x E_{0,1} & \left[ \log \cosh \left( b(x) + \beta \eta \sqrt{\sum_{|y-x|=1} \alpha_y q_y} \right) \right] + \frac{\beta^2}{2} \sum_{|x-y|=1} (\alpha_x - \alpha_y q_x)(\alpha_y - \alpha_x q_y). 
\end{align*}
\] (3.1)

Recall that we assumed the elements of $V$ to be the first $|V|$ odd numbers in $\{1, \ldots, \nu\}$. Note that if $x \in V$ then $\{y \in [\nu] : |x - y| = 1\} \subseteq H$. We set for $x \in V$
\[
Q_x := \sum_{|y-x|=1} \alpha_y q_y, \quad A_x := \sum_{|y-x|=1} \alpha_y.
\] (3.2)

Note that $Q_x \leq A_x$. We shorten also $p_x := \alpha_x q_x$ for $x \in V$. Next we change coordinates and describe $\text{RS}(q)$ only in the variables $\{Q_x, p_x\}_{x \in V}$. Then (3.1) is rewritten as
\[
\begin{align*}
\text{RS}(p, Q) &= \sum_{x \in V} \alpha_x E_{0,1} \left[ \log \cosh \left( b(x) + \beta \eta \sqrt{Q_x} \right) \right] \\
&\quad + \sum_{x \in H} \alpha_x E_{0,1} \left[ \log \cosh \left( b(x) + \beta \eta \sqrt{\sum_{|y-x|=1} p_y} \right) \right] \\
&\quad + \frac{\beta^2}{2} \sum_{x \in V} (\alpha_x - p_x)(A_x - Q_x).
\end{align*}
\] (3.3)

With a small abuse of notation we will denote the different functions $\text{RS}(q, Q), \text{RS}(p, Q)$ with the same symbol; the meaning will always be clear by the context. We will prove that there is a unique stationary point $(\bar{q}, \bar{Q}) \in [0, 1]^{|V|} \times [0, 1]^{|V|}$ in which the min max is realised:
\[
\min_{q_1, \ldots, q_{|V|}} \max_{Q_1, \ldots, Q_{|V|}} \text{RS}(q, Q) = \text{RS}(\bar{q}, \bar{Q}).
\] (3.4)

Then, since the change of variables (3.2) is linear and injective, Theorem 1.2 follows.

In the proof below we shall use the lemmas of Section 2 always with $\Phi = \tanh$ and assuming that for all $x \in [\nu]$ it is $|b(x)| \leq \frac{1}{2} \log \sqrt{2 + \sqrt{3}}$.

**Proof of (3.4).** We will prove that the stationary points equations
\[
\partial_{Q_x} \text{RS} = 0, \quad \partial_{p_x} \text{RS} = 0, \quad x = 1, \ldots, \nu,
\] (3.5)

have a unique solution in which the minimax is attained.

We first take the derivatives w.r.t. $Q_x$ for all $x \in V$. We have
\[
\partial_{Q_x} \text{RS} = \frac{\beta^2}{2} \left( p_x - \alpha_x E_{b(x), \beta^2 Q_x} \left[ \tanh^2(\eta) \right] \right), \quad \forall x \in V,
\] (3.6)

where we used a Gaussian integration by parts. By (2.10) the function $Q_x \mapsto \alpha_x E_{b(x), \beta^2 Q_x} \left[ \tanh^2(\eta) \right]$ is increasing in $Q_x$ at fixed $p_x$. Hence there is a unique point $(p_x, 0, Q_x, 0)$ such that (3.6) vanishes. Then by Lemma 2.4 (with $G(x) = x, b = b(x)$) there is a unique function $Q_x(p_x)$ defined in a neighbourhood of the intersection point. We shorten
\[
B_x := \alpha_x \tanh^2(b(x)) \quad C_x := \alpha_x E_{b(x), \beta^2 A_x} \left[ \tanh^2(\eta) \right].
\] (3.7)

So the stationary point conditions (3.6) define uniquely the functions
\[
Q_x : [B_x, C_x] \mapsto [0, A_x]
\] (3.8)

via
\[
p_x = \alpha_x E_{b(x), \beta^2 Q_x(p_x)} \left[ \tanh^2(\eta) \right], \quad x \in V.
\] (3.9)
The functions $Q_x(p_x)$ are non-negative, with $Q_x(B_x) = 0$, $Q_x(C_x) = A_x$ and we can prolong them in the interval $[0, C_x]$ setting $Q_x = 0$ in $[0, B_x]$. Moreover again by Lemma 2.4 they are increasing and convex with

$$\frac{d}{dp_x} Q_x(p_x) > 0. \quad (3.10)$$

Therefore

$$\partial^2_{Q_x} \mathbb{RS} < 0, \forall x \in V,$$

and clearly

$$\partial^2_{Q_x, Q_x} \mathbb{RS} = 0, \forall x \neq y.$$ 

Now we set

$$\mathbb{RS}^{(1)}(p) := \max_{Q_1 \ldots Q_{|V|}} \mathbb{RS}(p, Q) = \mathbb{RS}(Q_1(p_1), Q_3(p_3) \ldots, Q_{2|V|-1}(p_{2|V|-1}), p_1, \ldots, p_{2|V|-1}). \quad (3.11)$$

We compute

$$\partial_{p_1} \mathbb{RS}^{(1)} = \frac{\beta^2}{2} (Q_1(p_1) - \alpha_2 E_{\gamma(z_2)} \beta_2(p_1 + p_3) [\tanh^2(\eta)]) . \quad (3.12)$$

First we study the stationary point equation

$$Q_1(p_1) - \alpha_2 E_{\gamma(z_2)} \beta_2(p_1 + p_3) [\tanh^2(\eta)] = 0. \quad (3.13)$$

We look at

$$z(p_1) := \frac{\alpha_2 E_{\gamma(z_2)} \beta_2(p_1 + p_3)}{Q_1(p_1)} [\tanh^2(\eta)] . \quad (3.14)$$

Since $Q_1(p_1)$ is monotone increasing (see (3.10)) and by Lemma 2.3 (with $f(p_1) = p_1 + \alpha_3$ and $g(p_1) = Q_1(p_1)$)

$$\frac{\alpha_2 E_{\gamma(z_2)} \beta_2(p_1 + p_3)}{Q_1(p_1)} [\tanh^2(\eta)]$$

is monotone decreasing, we have that $z(p_1)$ is decreasing from $\lim_{p_1 \to B_1} z(p_1) = \infty$ to $z(C_1)$. Therefore it exists a unique solution $(p_{3,0}, p_{1,0})$ to (3.13) provided

$$z_1(C_1) < 1, \quad (3.15)$$

which is always satisfied (since $Q_1(C_1) = A_1 = \alpha_2$). As $\frac{Q_1(p_1)}{p_1}$ is monotone increasing (see (3.10)) we can apply Lemma 2.5 (with $F = Q_1$, $(x, y) = (p_{3,0}, p_{1,0})$). This yields that the stationary point equation (3.13) defines uniquely a function $p_1(p_3)$ which satisfies

$$\frac{d}{dp_3} p_1(p_3) > 0, \quad \frac{d}{dp_3} p_1(p_3) < 0. \quad (3.16)$$

Moreover (see (2.21))

$$\partial_{p_1} (Q_1(p_1) - \alpha_2 E_{\gamma(z_2)} \beta_2(p_1 + p_3) [\tanh^2(\eta)]) \bigg|_{p_1 = p_1(p_3)} > 0, \quad (3.17)$$

Finally by Lemma 2.6 we get

$$\frac{d}{dp_3} E_{\gamma(z_2)} \beta_2(p_1 + p_3) [\tanh^2(\eta)] < 0. \quad (3.18)$$

The inequality (3.17) is equivalent to $\partial^2_{p_1} \mathbb{RS}^{(1)} \bigg|_{p_1 = p_1(p_3)} > 0$ which ensures that $p_1(p_3)$ is a minimum point. So we can set

$$\mathbb{RS}^{(3)}(p_3, \ldots, p_{|V|}) := \min_{p_1} \mathbb{RS}^{(1)}(p_1, \ldots, p_{|V|}) = \mathbb{RS}^{(1)}(p_1(p_3), p_3, \ldots, p_{|V|}). \quad (3.19)$$
Now we compute
\[
\frac{\partial p_3 }{ \partial p_3 } \overline{\text{RS}}^{(3)} = \frac{\beta^2}{2} \left( Q_3(p_3) - \alpha_2 E_{b^{(2)}, \beta^2(p_1, p_3) + p_3} \left[\tanh^2(\eta)\right] - \alpha_4 E_{b^{(2)}, \beta^2(p_1, p_3) + p_3} \left[\tanh^2(\eta)\right] \right) \tag{3.20}
\]
So the stationary point equation reads as
\[
Q_3(p_3) - \alpha_2 E_{b^{(2)}, \beta^2(p_1, p_3) + p_3} \left[\tanh^2(\eta)\right] - \alpha_4 E_{b^{(2)}, \beta^2(p_1, p_3) + p_3} \left[\tanh^2(\eta)\right] = 0. \tag{3.21}
\]
We set
\[
\begin{aligned}
z_3(p_3) := & \frac{\alpha_2 E_{b^{(2)}, \beta^2(p_1, p_3) + p_3} \left[\tanh^2(\eta)\right] + \alpha_4 E_{b^{(2)}, \beta^2(p_1, p_3) + p_3} \left[\tanh^2(\eta)\right]}{Q_3(p_3)},
\end{aligned}
\tag{3.22}
\]
By (3.10) and by the first inequality of (3.18) we have
\[
\frac{Q_j'(p_3)}{Q_3(p_3)} \geq \frac{1}{p_3} \geq \frac{p_j'(p_3)}{p_1(p_3)}.
\]
Then applying Lemma 2.3 for each summand, that is with \( f = p_3 + \alpha_5 \) or \( f = p_1(p_3) + p_3 \) and \( g = Q_3(p_3) \) we have that \( z_3(p_3) \) is decreasing. Therefore (3.21) yields a unique intersection point \((p_{5,0}, p_{3,0})\) provided
\[
z_3(C_3) \leq 1, \tag{3.23}
\]
which holds true for all the values of the parameters.

Let us now set
\[
F_3(p_3) := Q_3(p_3) - \alpha_2 E_{b^{(2)}, \beta^2(p_1, p_3) + p_3} \left[\tanh^2(\eta)\right].
\]
Note that \( F_3(p_3)/p_3 \) is monotone increasing: this follows by (3.10) and (3.18). Therefore by Lemma 2.5 (with \( F = F_3 \) and \((x, y) = (p_{5,0}, p_{3,0})\)) the stationary point equation (3.21) singles out a unique function \( p_3(p_5) \) which satisfies
\[
\frac{d}{dp_5} p_3(p_5) > 0, \quad \frac{d}{dp_5} \frac{p_3(p_5)}{p_5} < 0. \tag{3.24}
\]
Therefore \( p_3(p_5) \) is increasing. Moreover (see (2.21))
\[
\begin{aligned}
\partial p_3 \left( F_3(p_3) - \alpha_4 E_{b^{(2)}, \beta^2(p_3, p_3) + p_3} \left[\tanh^2(\eta)\right] \right) & \bigg|_{p_3 = p_3(p_5)} > 0, \tag{3.25}
\end{aligned}
\]
Finally by Lemma 2.6 we get
\[
\begin{aligned}
\frac{d}{dp_5} E_{b^{(2)}, \beta^2(p_3, p_3) + p_3} \left[\tanh^2(\eta)\right] < 0. \tag{3.26}
\end{aligned}
\]
The inequality (3.25) is equivalent to \( \partial x \overline{\text{RS}}^{(3)} \bigg|_{p_3 = p_3(p_5)} > 0 \), that is in \( p_3(p_5) \) a minimum is attained. So we can set
\[
\overline{\text{RS}}^{(5)}(p_5, \ldots, p_{|\nu|}) := \min_{p_3} \overline{\text{RS}}^{(3)}(p_1(p_3), p_3, \ldots, p_{|\nu|}) = \overline{\text{RS}}^{(3)}(p_1(p_3(p_5)), p_3(p_5), p_5, \ldots, p_{|\nu|}). \tag{3.27}
\]
In general, for \( x \in (2N - 1) \cap [\nu] \setminus \{1\} \) at \( x \)-th step we have already proven
\[
\begin{aligned}
\frac{d}{dp_x} p_{x-2}(p_x) > 0, \quad \frac{d}{dp_x} \frac{p_{x-2}(p_x)}{p_x} < 0, \quad \frac{d}{dp_x} E_{b^{(x-1)}, \beta^2(p_{x-2}, p_x) + p_x} \left[\tanh^2(\eta)\right] < 0. \tag{3.28}
\end{aligned}
\]
We shorten in what follows \( p_{x-2}(p_x) = p_{x-2} \). We want to optimise only w.r.t. \( p_x \) the function \( \overline{\text{RS}}^{(x-1)}(p_x, \ldots, p_{|\nu|}) \) defined by
\[
\overline{\text{RS}}^{(x)} := \min_{p_{x-2}} \overline{\text{RS}}^{(x-2)}. \tag{3.29}
\]
The stationary point equation reads as

\[ Q_x(p_x) - \alpha_{x-1} E_{b(x-1), \beta^2(p_x) + p_x} \left[ \tanh^2(\eta) \right] - \alpha_{x+1} E_{b(x+1), \beta^2(p_x - p_{x+2}) + p_x} \left[ \tanh^2(\eta) \right] = 0. \quad \text{(3.30)} \]

We set

\[ z_x(p_x) := \frac{\alpha_{x-1} E_{b(x-1), \beta^2(p_x) + p_x} \left[ \tanh^2(\eta) \right] + \alpha_{x+1} E_{b(x+1), \beta^2(p_x - p_{x+2}) + p_x} \left[ \tanh^2(\eta) \right]}{Q_x(p_x)}. \quad \text{(3.31)} \]

By (3.10) and by the second inequality of (3.28) we have

\[ \frac{Q_x'(p_x)}{Q_x(p_x)} > \frac{1}{p_x} > \frac{p_{x-2}(p_x)}{p_{x-2}(p_x)}. \]

Then applying Lemma 2.3 for each summand, that is with \( f = p_x + \alpha_{x+2} \) or \( f = p_{x-2}(p_x) + p_x \) and \( g = Q_x(p_x) \) we have that \( z_x(p_x) \) is decreasing. Therefore (3.30) yields a unique intersection point \((p_x, 0, p_x, 0)\) provided

\[ z_x(C_x) \leq 1, \quad \text{(3.32)} \]

which is always satisfied.

We set now

\[ F_x(p_x) := Q_x(p_x) - \alpha_{x-1} E_{b(x-1), \beta^2(p_x) + p_x} \left[ \tanh^2(\eta) \right] \quad \text{(3.33)} \]

and see that \( F_x(p_x) / p_x \) is increasing thanks to the combination of (3.10) and (3.28). Therefore we can apply Lemma 2.5 (with \( F = F_x \) and \( (x, y) = (p_{x+2}, 0, p_x, 0) \)) to prove that the stationary point equation (3.30) singles out a unique function \( p_x(p_x + 2) \) with

\[ \frac{d}{dp_{x+2}} p_x(p_{x+2}) > 0, \quad \frac{d}{dp_{x+2}} p_x(p_{x+2}) < 0. \quad \text{(3.34)} \]

Moreover (see (2.21))

\[ \partial_{p_x} \left( F_x(p_x) - \alpha_{x+1} E_{b(x+1), \beta^2(p_x + p_{x+2}) + p_x} \left[ \tanh^2(\eta) \right] \right) \bigg|_{p_x = p_x(p_x + 2)} > 0, \quad \text{(3.35)} \]

Finally by combining (3.30), (3.28) and Lemma 2.6 we get

\[ \frac{d}{dp_{x+2}} p_x(p_{x+2}) < 0, \quad \frac{d}{dp_{x+2}} E_{b(x+1), \beta^2(p_x + p_{x+2}) + p_x} \left[ \tanh^2(\eta) \right] < 0. \quad \text{(3.36)} \]

The inequality (3.35) is equivalent to \( \partial^2_{p_x} RS(x) \bigg|_{p_x = p_x(p_x + 2)} > 0 \), that is in \( p_x(p_x + 2) \) a minimum is attained. So we can set

\[ RS(x+2) := \min_{p_x} RS(x) \quad \text{(3.37)} \]

and iterate.

To conclude we add the following

**Proof of Lemma 1.1.** Let \( t \in [0, 1] \) and for any \( x \in [\nu] \) \( \{\eta^x_j\} \) be i.i.d. \( \mathcal{N}(0, 1) \). Recall

\[ H'_N := -\sum_{x=1}^{\nu} \sqrt{\sum_{|y-x|=1} \sum_{i=1}^{N_x} \alpha_i y_{x} \sigma_{x}^i}, \quad \text{(3.38)} \]

\[ H_{N, t} := \sqrt{t} H_N + \sqrt{1-t} H'_N. \quad \text{(3.39)} \]

Define the interpolating function

\[ \phi_N(t) := \frac{1}{N} EE_{0, 1}[\log \hat{E}_{\sigma(x)} e^{-\beta H_{N, t}}]. \quad \text{(3.40)} \]
We have
\[
\begin{aligned}
\phi_N(0) &= \sum_{z=1}^N \alpha_z E_{0,1} \left[ \log \cosh \left( \beta(\epsilon_z) + \beta \eta \sqrt{\sum_{y \neq z} q_y} \right) \right], \\
\phi_N(1) &= A_N(\beta).
\end{aligned}
\] (3.41)

Recalling the definition (1.5) we compute
\[
\frac{d}{dt} \phi_N(t) = \frac{\beta}{2\sqrt{tN}} \langle H_N \rangle_t - \frac{\beta}{2\sqrt{1-tN}} \langle H'_N(t) \rangle_t,
\]
and by Gaussian integration by parts we get
\[
\langle H_N \rangle_t = \frac{\beta \sqrt{t}}{N} \sum_{|x-y|=1} N_x N_y (1 - \langle R^{(x)} R^{(y)} \rangle_t),
\]
\[
\langle H'_N \rangle_t = 2\beta \sqrt{1-t} \sum_{|x-y|=1} N_x N_y q_y (1 - \langle R^{(x)} \rangle_t).
\]

Therefore
\[
\frac{d}{dt} \phi_N(t) = \frac{\beta^2}{2} \sum_{|x-y|=1} \alpha_x \alpha_y (1 - q_x)(1 - q_y) - \frac{\beta^2}{2} \sum_{|x-y|=1} \alpha_x \alpha_y \left( \langle q_x - R^{(x)} \rangle \langle q_y - R^{(y)} \rangle \right)_t. \tag{3.42}
\]

So we can write
\[
\phi_N(1) - \phi_N(0) = \int_0^1 \frac{d}{dt} \phi_N(t) \, dt
= \frac{\beta^2}{2} \sum_{|x-y|=1} \alpha_x \alpha_y (1 - q_x)(1 - q_y)
- \frac{\beta^2}{2} \sum_{|x-y|=1} \alpha_x \alpha_y \int_0^1 \left( \langle q_x - R^{(x)} \rangle \langle q_y - R^{(y)} \rangle \right)_t \, dt. \tag{3.43}
\]

Then (1.8) follows just noting
\[
\text{RS}(q) = \phi_N(0) + \frac{\beta^2}{2} \sum_{|x-y|=1} \alpha_x \alpha_y (1 - q_x)(1 - q_y).
\]

\[\square\]

References

[1] R. Salakhutdinov, G. Hinton, *Deep Boltzmann machines*. In Proceedings of the 24th International Conference of Artificial Intelligent and Statistics, AISTATS 2009, 448-455, (2009).

[2] G. Hinton, *Training products of experts by minimizing contrastive divergence*. Neural computation 14.8 (2002): 1771-1800.

[3] T. Tieleman, *Training restricted Boltzmann machines using approximations to the likelihood gradient*. Proceedings of the 25th international conference on Machine learning, (2008).

[4] R. Salakhutdinov, Ruslan, H. Larochelle. *Efficient learning of deep Boltzmann machines* Proceedings of the thirteenth international conference on artificial intelligence and statistics. 2010.

[5] I. J. Goodfellow, A. Courville, Y. Bengio. *Joint Training of Deep Boltzmann Machines* for Classification stat 1050 (2013): 1.

[6] A. Barra, G. Genovese, F. Guerra, The Replica Symmetric Behaviour of the Analogical Neural Network, J. Stat. Phys. 142, 654, (2010).

[7] A. Barra, G. Genovese, F. Guerra, *Equilibrium statistical mechanics of bipartite spin systems*, J. Phys. A: Math. Theor. 44, 245002 (2011).

[8] B. S. Tsirelson, I. A. Ibragimov, V. N. Sudakov. *Norms of Gaussian sample functions*, Proceedings of the Third Japan-USSR Symposium on Probability Theory. Springer, Berlin, Heidelberg, (1976).

[9] G. Genovese, D. Tantari *Legendre Equivalences of Spherical Boltzmann Machines*, in Journal Physics A, special issue Machine learning and statistical physics, theory, inspiration, application, Ed. E. Agliari, A. Barra, P. Sollich, L. Zdeborova, (2020).

[10] G. Genovese, *Universality in Bipartite Mean Field Spin Glasses*, J. Math. Phys. 53, 123304, (2012).
[11] M. Mezard, Mean-field message-passing equations in the Hopfield model and its generalizations, Phys. Rev. E 95, 022117 (2017).
[12] F. Guerra, Sum rules for the free energy in the mean field spin glass model, in Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects, Fields Institute Communications 30, 161 (2001).
[13] D. Alberici, A. Barra, P. Contucci, E. Mingione Annealing and Replica-Symmetry in Deep Boltzmann Machines, J. Stat. Phys., 179, 1-15, (2020).
[14] D. Alberici, P. Contucci, E. Mingione Deep Boltzmann machines: rigorous results at arbitrary depth, Annales Henri Poincaré 22, 2619-2642 (2021).
[15] M. Talagrand, Mean Field Models for Spin Glasses, Vol. 1, Springer-Verlag Berlin Heidelberg (2011).
[16] D Alberici, F. Camilli, P. Contucci, E. Mingione, The solution of the deep Boltzmann machine on the Nishimori line, Comm. Math. Phys. 387, 1191-1214 (2021).
[17] G. Genovese, A remark on the spherical bipartite spin glass, Math. Phys. Anal. Geom. 25, 14 (2022).
[18] A. Auffinger, W.-K. Chen, Free energy and complexity of spherical bipartite models, J. Stat. Phys. 157, 40-59, (2014).
[19] J. Baik, J. O. Lee Free energy of bipartite spherical Sherrington-Kirkpatrick model, Ann. Inst. H. Poincaré Probab. Statist. 56(4): 2897-2934 (2020).
[20] N. Macris, J. Barbier. The adaptive interpolation method: a simple scheme to prove replica formulas in Bayesian inference Probab. Theory Related. Fields 174, 1133?1185 (2019).
[21] C. Luneau, N. Macris, J. Barbier. High-dimensional rank-one nonsymmetric matrix decomposition: the spherical case IEEE International Symposium on Information Theory (ISIT) IEEE (2020).
[22] A. Barra, P. Contucci, E. Mingione, D. Tantari, Multi-Species Mean Field Spin Glasses. Rigorous Results, Ann. H. Poincaré 16, 691-708, (2015).
[23] D. Panchenko, The Free Energy in a Multi-Species Sherrington-Kirkpatrick Model, Ann. Prob. 43, 3494-3513 (2015).
[24] R. Brunetti, G. Parisi, F. Ritort. Asymmetric Little spin-glass model Physical Review B 46.9 (1992): 5339.
[25] A. Barra, G. Genovese, F. Guerra, D. Tantari, How glassy are neural networks?, J. Stat. Mech. P07009 (2012).
[26] L. Pastur, M.V. Shcherbina, Absence of self-averaging of the order parameter in the Sherrington-Kirkpatrick model, Journal of Statistical Physics 62, 1-19 (1991)
[27] M.V. Shcherbina, More about the absence of selfaverageness of order parameter in SK-model, CARR Reports in Mathematical Physics, n. 3/91, Department of Mathematics, University of Rome “La Sapienza” (1991)
[28] M. Talagrand. The Sherrington-Kirkpatrick model: A challenge for mathematicians, Probability theory and related fields 110.2 (1998): 109-176.

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