Why Spectral Normalization Stabilizes GANs: Analysis and Improvements

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Abstract

Spectral normalization (SN) is a widely-used technique for improving the stability of Generative Adversarial Networks (GANs) by forcing each layer of the discriminator to have unit spectral norm [25]. This approach controls the Lipschitz constant of the discriminator, and is empirically known to improve sample quality in many GAN architectures. However, there is currently little understanding of why SN is so effective. In this work, we show that SN controls two important failure modes of GAN training: exploding and vanishing gradients. Our proofs illustrate a (perhaps unintentional) connection with the successful LeCun initialization technique [20], proposed over two decades ago to control gradients in the training of deep neural networks. This connection helps to explain why the most popular implementation of SN for GANs [25] requires no hyperparameter tuning, whereas stricter implementations of SN have poor empirical performance out-of-the-box [12, 8]. Unlike LeCun initialization which only controls gradient vanishing at the beginning of training, we show that SN tends to preserve this property throughout training. Finally, building on this theoretical understanding, we propose Bidirectional Spectral Normalization (BSN), a modification of SN inspired by Xavier initialization [9], a later improvement to LeCun initialization. Theoretically, we show that BSN gives better gradient control than SN. Empirically, we demonstrate that BSN outperforms SN in sample quality on several benchmark datasets, while also exhibiting better training stability.

1 Introduction

Generative adversarial networks (GANs) are state-of-the-art deep generative models, perhaps best known for their ability to produce high-resolution, photorealistic images [11]. The objective of GANs is to produce random samples from a target data distribution, given only access to an initial set of training samples. This is achieved by learning two functions: a generator $G$, which maps random input noise to a generated sample, and a discriminator $D$, which tries to classify input samples as either real (i.e., from the training dataset) or fake (i.e., produced by the generator). In practice, these functions are implemented by deep neural networks (DNNs), and the competing generator and
discriminator are trained in an alternating process known as adversarial training. Theoretically, given enough data and model capacity, GANs converge to the true underlying data distribution [11].

Although GANs have been very successful in improving the sample quality of data-driven generative models [17, 5], their adversarial training also contributes to the well-studied instability of GANs. That is, small hyperparameter changes and even randomness in the optimization can cause training to fail. Many approaches have been proposed for improving the stability of GANs, including different architectures [32, 17, 5], loss functions [1, 2, 13, 41], and regularizations/normalizations [25, 6, 35]. One of the most successful proposals to date is called spectral normalization (SN) [25, 12, 8]. SN forces each layer of the generator to have unit spectral norm during training. This has the effect of controlling the Lipschitz constant of the discriminator, which is empirically observed to improve the stability of GAN training [25]. However, there are many ways to control the Lipschitz constant of a function. To date, it remains unclear precisely why this specific normalization is so effective.

In this paper, we show that SN controls two important failure modes of GAN training: exploding gradients and vanishing gradients. These problems are well-known to cause instability in GANs [2, 5], leading either to bad local minima or stalled training prior to convergence. SN mitigates both problems by tightly bounding the magnitudes of gradients. The fact that SN upper bounds gradient magnitudes may be expected, since SN was initially proposed to bound the discriminator’s Lipschitz constant. However, there is no reason a priori for SN to also mitigate the gradient vanishing problem.

Although our results and analysis are not unique to GANs, we focus on them because SN seems to have a disproportionately beneficial effect on GANs [25]. In this work, we make three primary contributions:

1. Analysis of why SN avoids exploding gradients. Poorly-chosen architectures and hyperparameters, as well as randomness during training, can amplify the effects of large gradients on training instability, ultimately leading to generalization error in the learned discriminator. We show that SN imposes a strict upper bound on gradients during GAN training, mitigating these effects.

2. Analysis of why SN avoids vanishing gradients. Small gradients during training are known to cause GANs (and other DNNs) to converge to bad models [20, 2]. The well-known LeCun initialization, first proposed over two decades ago, mitigates this effect by carefully choosing the variance of the initial weights [20]. We prove theoretically that SN controls the variance of weights in a way that closely parallels LeCun initialization, despite little connection between the two ideas at first glance. Whereas LeCun initialization only controls the gradient vanishing problem at the beginning of training, we show empirically that SN preserves this nice property throughout training. Our analysis also explains why a strict implementation of SN [8] has poor out-of-the-box performance on GANs and requires additional hyperparameter tuning to avoid the gradient vanishing problem, whereas the implementation of SN in [25] requires no hyperparameter tuning.

3. Improving SN with Bidirectional Spectral Normalization. Based on our theoretical insights, we design an improved version of SN which we call Bidirectional Spectral Normalization (BSN). BSN is motivated by the so-called Xavier initialization, a refinement of LeCun initialization that controls not only the variances of internal outputs, but also the variance of backpropagated gradients [9]. We show that BSN achieves better or equal sample quality to SN on several benchmark datasets, including CIFAR10, STL10, CelebA, and ImageNet, while also exhibiting better stability during training.

2 Background and Preliminaries

The instability of GANs is believed to be predominantly caused by poor discriminator learning [1, 32]. We therefore focus in this work on the discriminator, and the effects of SN on discriminator learning. We adopt the same model as [25]. Consider a discriminator with L internal layers:

$$D_{\theta}(x) = a_L \circ l_{w_L} \circ a_{L-1} \circ l_{w_{L-1}} \circ \ldots \circ a_1 \circ l_{w_1}(x)$$

(1)

where x denotes the input to the discriminator and \(\theta = \{w_1, w_2, ..., w_L\}\) the weights; \(a_i (i = 1, ..., L - 1)\) is the activation function in the i-th layer, which is usually element-wise ReLU or leaky ReLU in GANs [11]; \(a_L\) is the activation function for the last layer, which is sigmoid for the vanilla GAN [11] and identity for WGAN-GP [13]; \(l_{w_i}\) is the linear transformation in i-th layer, which is usually fully-connected or a convolutional neural network [11, 32].

Lipschitz Regularization and Spectral Normalization. Prior work has shown that regularizing the Lipschitz constant of the discriminator \(\|D_{\theta}\|_{\text{Lip}}\) improves the stability of GANs [2, 15, 41]. For
example, WGAN-GP [13] adds a gradient penalty ($\| \nabla D_\theta(\tilde{x}) \| - 1$) to the loss function, where $\tilde{x} = \alpha x + (1 - \alpha) G(z)$ and $\alpha \sim \text{Uniform}(0, 1)$ to ensure that the Lipschitz constant of the discriminator is bounded by 1.

Spectral normalization (SN) takes a different approach. For fully connected layers (i.e., $l_w(x) = w_i x$), it regularizes the weights $w_i$ to ensure that spectral norm $\|w_i\|_{sp} = 1$ for all $i \in [1, L]$, where the spectral norm $\|w_i\|_{sp}$ is defined as the largest singular value of $w_i$. This bounds the Lipschitz constant of the discriminator since $\|D_\theta\|_{lip} \leq \prod_{i=1}^{L} \|w_i\|_{lip} \prod_{i=1}^{L} |a_i|_{lip} \leq \prod_{i=1}^{L} \|w_i\|_{sp} \prod_{i=1}^{L} |a_i|_{lip} \leq 1$, as $\|w_i\|_{lip} \leq \|w_i\|_{sp}$ and $|a_i|_{lip} \leq 1$ for networks with (leaky) ReLU as activation functions for the last layer [25]. Prior work has theoretically connected the generalization gap of neural networks to the product of the spectral norms of the layers [3, 27]. These insights led to multiple implementations of spectral normalization [8, 12, 42, 25], with the implementation of [25] achieving particular success on GANs. SN can be viewed as a special case of more general techniques for enhancing stability of neural network training by controlling the spectrum of the network’s input-output Jacobian [31], e.g., through techniques like Jacobian clamping [28], which constrains the values of the maximum and minimum singular values in the generator during training.

In practice, spectral normalization [8, 25] is implemented by dividing the weight matrix $w_i$ by its spectral norm: $\tilde{w}_i = \frac{w_i}{\|w_i\|_{sp}}$, where $u_j$ and $v_j$ are the left/right singular vectors of $w_j$ corresponding to its largest singular value. As observed by Gouk et al. [12], there are two approaches in the SN literature for instantiating the matrix $w_i$ for convolutional neural networks (CNNs). In a CNN, since convolution is a linear operation, convolutional layers can equivalently be written as a multiplication by an expanded weight matrix $\tilde{w}_i$ that is derived from the raw weights $w_i$. Hence in principle, spectral normalization should normalize each convolutional layer by $\|\tilde{w}_i\|_{sp}$ [12, 8]. We call this canonical normalization $SN_{\text{Conv}}$ as it controls the spectral norm of the convolution layer.

However, the spectral normalization that is known to outperform other regularization techniques and improve training stability for GANs [25], which we call $SN_w$, does not implement SN in a strict sense. Instead, it uses $\left\| w_i \cdot \prod_{c_{in}k_wk_h} \right\|_{sp}$; that is, it first reshapes the convolution kernel $w_i \in \mathbb{R}^{c_{out} \times c_{in} \times k_w \times k_h}$ into a matrix $\tilde{w}_i$ of shape $c_{out} \times (c_{in} k_w k_h)$, and then normalizes with the spectral norm $\|\tilde{w}_i\|_{sp}$, where $c_{in}$ is the number of input channels, $c_{out}$ is the number of output channels, $k_w$ is the kernel width, and $k_h$ is the kernel height. Miyato et al. showed that their implementation implicitly penalizes $w_i$ from being too sensitive in one specific direction [25]. However, this does not explain why $SN_w$ is more stable than other Lipschitz regularization techniques, and as observed in [12], it is unclear how $SN_w$ relates to $SN_{\text{Conv}}$. Despite this, $SN_w$ has empirically been immensely successful in stabilizing the training of GANs [5, 23, 44, 16, 43, 26, 22]. Even more puzzling, we show in [8, 4] that the canonical approach $SN_{\text{Conv}}$ has comparatively poor out-of-box performance when training GANs.

Hence, two questions arise: (1) Why is SN so successful at stabilizing the training of GANs? (2) Why is $SN_w$ proposed by [25] so much more effective than the canonical $SN_{\text{Conv}}$?

**Gradient explosion and vanishing.** In this work, we show that both questions are related to two well-known phenomena: vanishing and exploding gradients. These terms describe a problem in which gradients either grow or shrink rapidly during training [41, 29, 30], and they are known to be closely related to the instability of GANs [11, 5]. We provide an example to illustrate how vanishing or exploding gradients cause training instability in GANs in [App. I].

### 3 Exploding Gradients

In this section, we show that spectral normalization prevents gradient explosion by bounding the gradients of the discriminator. Moreover, we show that the common choice to normalize all layers equally achieves the tightest upper bound for a restricted class of discriminators. We use $\theta \in \mathbb{R}^d$ to denote a vector containing all elements in $\{w_1, \ldots, w_L\}$. In the following analysis, we assume linear transformations are fully-connected layers $l_w(x) = w_i x$ as in [25], though the same analysis can be applied to convolutional layers.
To highlight the effects of the spectral norm of each layer on the gradient and simplify the exposition, we will compute gradients with respect to \( w_i \) in the following discussion. In reality, gradients are computed with respect to \( w_i \); we defer this discussion to App. C where we show the relevant extension. The following proposition shows that under this simplifying assumption, spectral normalization controls the magnitudes of the gradients of the discriminator with respect to \( \theta \). Notice that simply controlling the Lipschitz constant of the discriminator (e.g., as in WGAN [11]) does not imply this property; it instead ensures small (sub)gradients with respect to the input, \( x \).

**Proposition 1** (Upper bound of gradient’s Frobenius norm for spectral normalization). If \( \|w_i\|_{sp} \leq 1 \) for all \( i \in [1, L] \), then we have \( \| \nabla_w D_\theta(x) \|_F \leq \| x \| \sum_{i=1}^L \| a_i \|_{Lip} \), and the norm of the overall gradient can be bounded by \( \| \nabla_\theta D_\theta(x) \|_F \leq \sqrt{L} \| x \| \sum_{i=1}^L \| a_i \|_{Lip} \).

(Proof in App. A). Note that under the assumption that internal activation functions are ReLU or leaky ReLU, if the activation function for the last layer is identity (e.g., for vanilla GAN [11]), the above bounds can be simplified to \( \| \nabla_w D_\theta(x) \|_F \leq \| x \| \) and \( \| \nabla_\theta D_\theta(x) \| \leq \sqrt{L} \| x \| \), and if the activation for the last layer is sigmoid (e.g., for vanilla GAN [11]), the above bounds become \( \| \nabla_w D_\theta(x) \|_F \leq 0.25 \| x \| \) and \( \| \nabla_\theta D_\theta(x) \| \leq 0.25 \sqrt{L} \| x \| \). A comparable bound can also be found to limit the norm of the Hessian, which we defer to App. D due to space constraints.

![Figure 1: Gradient norms of different discriminator layers in MNIST.](image)

The bound in Prop. 1 has a significant effect in practice. [Fig. 1](image) shows the norm of the gradient for each layer of a WGAN trained on MNIST with and without spectral normalization. Without spectral normalization, some layers have extremely large gradients throughout training, which makes the overall gradient large. With spectral normalization, the gradients of all layers are upper bounded as shown in [Prop. 1](#). We see similar results in other datasets and network architectures (App. J).

**Optimal Spectral Norm Allocation** Common implementations of SN advocate setting the spectral norm of each layer to the same value [25][8]. However, the following proposition states that we can set the spectral norms of different layers to different constants, without changing the network’s behavior on the input samples, as long as the product of the spectral norm bounds is the same.

**Proposition 2.** For any discriminator \( D_\theta = a_L \circ l_w L \circ a_{L-1} \circ l_w_{L-1} \circ \ldots \circ a_1 \circ l_w_1 \) and \( D_\theta' = a_L \circ l_{c_L} \circ a_{L-1} \circ l_{c_{L-1}} \circ \ldots \circ a_1 \circ l_{c_1} \circ l_w_1 \) where the internal activation functions \( a_i \) satisfy that \( \prod_{i=1}^L c_i = 1 \), we have
\[
D_\theta(x) = D_\theta'(x) \quad \forall x \quad \text{and} \quad \frac{\partial^{n} D_\theta(x)}{\partial x^n} = \frac{\partial^{n} D_\theta'(x)}{\partial x^n} \quad \forall x, \forall n \in \mathbb{Z}^+ .
\]

(Proof in App. B). Given this observation, it is natural to ask if there is any benefit to setting the spectral norms of each layer equal. It turns out that the answer is yes, under some assumptions that appear to approximately hold in practice. Let
\[
\mathcal{D} \triangleq \left\{ D_\theta = a_L \circ l_{c_L} \circ a_{L-1} \circ l_{c_{L-1}} \circ \ldots \circ a_1 \circ l_{c_1} \circ l_w_1 : \frac{\| \nabla_w D_\theta(x) \|_F}{\| \nabla_w D_\theta'(x) \|_F} = \frac{\| w_i \|_{sp}}{\| w_i \|_{sp}} \quad a_i \in \{ \text{ReLU, leaky ReLU} \} \quad \forall i, j \in [1, L] \right\} .
\]

This intuitively describes the set of all discriminators for which scaling up the weight of one layer proportionally increases the gradient norm of all other layers; the definition of this set is motivated by our upper bound on the gradient norm (App. A). The following theorem shows that when optimizing over set \( \mathcal{D} \), choosing every layer to have the same spectral norm gives the smallest possible gradient norm, for a given set of parameters.

**Theorem 1.** Consider a given set of discriminator parameters \( \theta = \{ w_1, ..., w_L \} \). For a vector \( c = \{ c_1, ..., c_L \} \), we denote \( \theta_c = \{ c_i w_i \}_{i=1}^L \). Let \( \lambda_\theta = \prod_{i=1}^L \| w_i \|_{sp}^{1/L} \) denote the geometric mean of the spectral norms of the weights. Then we have
\[
\left\{ \frac{\lambda_\theta}{\| w_1 \|_{sp}}, ..., \frac{\lambda_\theta}{\| w_L \|_{sp}} \right\} = \arg \min_{c: \theta_c \in \mathcal{D}, \prod_{i=1}^L c_i = 1, c_i \in \mathbb{R}^+} \| \nabla_{\theta_c} D_\theta_c(x) \|_F
\]

(4)
Inverse ratio of spectral norm

Ratio of gradient norm

\[ y = \frac{x_{\text{layer 2}}}{x_{\text{layer 1}}} \]

\[ \frac{x_{\text{layer 3}}}{x_{\text{layer 1}}} \]

\[ \frac{x_{\text{layer 4}}}{x_{\text{layer 1}}} \]

Figure 2: Ratio of gradient norm v.s. ratio of spectral norm in MNIST.

(Proof in App. E). The key constraint in this theorem is that we optimize only over discriminators in set \( \mathcal{D} \) [Eq. (3)]. To show that this constraint is realistic (i.e. SN GAN discriminator optimization tends to choose models in \( \mathcal{D} \)), we trained a spectrally-normalized GAN with four hidden layers on MNIST, computing the ratios of the gradient norms at each layer and the ratios of the spectral norms, as dictated by Eq. (3). We computed these ratios at different epochs during training, as well as for different randomly-selected rescalings of the spectral normalization vector \( c \). Each point in Fig. 2 represents the results averaged over 64 real samples at a specific epoch of training for a given (random) \( c \). Vertical series of points are from different epochs of the same run, therefore their ratio of spectral norms is the same. The fact that most of the points are near the diagonal line suggests that training naturally favors discriminators that are in or near \( \mathcal{D} \); we confirm this intuition in other experimental settings in App. K. This observation, combined with [Thm. 1], suggests that it is better to force the spectral norms of every layer to be equal. Hence, existing SN implementations \([25, 8]\) chose the correct, uniform normalization across layers to upper bound discriminator’s gradients.

4 Vanishing Gradients

An equally troublesome failure mode of GAN training is vanishing gradients \([1]\). Prior work has proposed new objective functions to mitigate this problem \([1, 2, 13]\), but these approaches do not fully solve the problem (see Fig. 11). In this section, we show that SN also controls gradient vanishing.

Gradients tend to vanish for two reasons. First, gradients vanish when the objective function saturates \([20, 1]\), which is often associated with function parameters growing too large. Common loss functions (e.g., hinge loss) and activation functions (e.g., sigmoid, tanh) saturate for inputs of large magnitude. Large parameters tend to amplify the inputs to the activation functions and/or loss function, causing saturation. Second, gradients vanish when function parameters (and hence, internal outputs) grow too small. This is because backpropagated gradients are scaled by the function parameters (App. A).

These insights motivated the LeCun initialization technique \([20]\). The key idea is that to prevent gradients from vanishing, we must ensure that the outputs of each neuron do not vanish or explode. If the inputs to a neural unit are uncorrelated random variables with variance 1, then to ensure that the unit’s output also has variance (approximately) 1, the weight parameters should be zero-mean random variables with variance of one over the fan-in to the node (number of incoming connections) \([20]\).

Hence, LeCun initialization prevents gradient vanishing by controlling the variance of the individual parameters. In the following theorem, we show that SN enforces a similar condition.

**Theorem 2** (Variance of spectrally-normalized weights). For a matrix \( A \in \mathbb{R}^{m \times n} \) with i.i.d entries \( a_{ij} \) from a symmetric distribution (e.g. zero-mean Gaussian or uniform), we have

\[
\text{Var} \left( \frac{a_{ij}}{\|A\|_{sp}} \right) \leq \frac{1}{\max \{m, n\}}.
\]

Furthermore, if \( m, n \geq 2 \) and \( \max \{m, n\} \geq 3 \), and \( a_{ij} \) are from a zero-mean Gaussian, we have

\[
\frac{L}{\max \{m, n\} \log (\min \{m, n\})} \leq \text{Var} \left( \frac{a_{ij}}{\|A\|_{sp}} \right) \leq \frac{1}{\max \{m, n\}},
\]

where \( L \) is a constant which does not depend on \( m, n \).

(Proof in App. F). In other words, spectral normalization forces zero-mean parameters to have a variance that scales inversely with \( \max\{m, n\} \). The proof relies on a characterization of extreme values of random vectors drawn uniformly from the surface of a high-dimensional unit ball. Notice that this result holds regardless of the variance of the initial entries of \( A \), suggesting that SN is insensitive to initialization parameters. Many fully-connected, feed-forward neural networks have a fixed width across hidden layers, so \( \max\{m, n\} \) corresponds precisely to the fan-in of any neuron in a hidden layer, implying that SN has an effect like LeCun initialization.
In a CNN, the interpretation of $\max \{m, n\}$ depends on how SN is implemented. Recall that the implementation $\text{SN}_w$ by Miyato et al. [25] does not strictly implement SN, but a variant that normalizes by the spectral norm of $\hat{w}_i = w_i^t \cdot \text{Conv}(\cdot, k_w, h_b)$. In architectures like DCGAN [32], the larger dimension of $\hat{w}_i$ for hidden layers tends to be $c_h k_w h_b$, which corresponds to the fan-in. This means that SN gets the right variance for hidden layers in CNN. Our theoretical analysis only applies at initialization, when the parameters are selected randomly. However, unlike LeCun initialization which only controls the variance at initialization, we find empirically that Eq. (5) for SN appears to hold throughout training (Fig. 3). As a comparison, if trained without SN, the variance increases and the gradient decreases, which makes sample quality bad (App. L.2).

This explains why in practice GANs trained with SN are stable throughout training. This means that for the same discriminator using SN, the variance increases and the gradient decreases, which makes sample quality bad (App. L.2).

Figure 3: Parameter variances throughout training in CIFAR10.

Figure 4: Inception score of different SN variants in CIFAR10.

Figure 5: Gradient norms of different SN variants in CIFAR10.

Figure 6: Inception score of scaled SN in CIFAR10.

Perhaps surprisingly, we find empirically that the strict implementation $\text{SN}_{\text{Conv}}$ of [8] does not prevent gradient vanishing. Figs. 4 and 5 shows the gradients of $\text{SN}_{\text{Conv}}$ vanishing when trained on CIFAR10, leading to a comparatively poor inception score, whereas the gradients of $\text{SN}_w$ remain stable. To understand this phenomenon, recall that $\text{SN}_{\text{Conv}}$ normalizes by the spectral norm of an expanded matrix $\hat{w}_i$ derived from $w_i$. Thm. 2 does not hold for $\hat{w}_i$ since its entries are not i.i.d. (even at initialization); hence it cannot be used to explain this effect. However, Corollary 1 in [38] shows that $\|\hat{w}_i\|_{sp} \leq \|\hat{w}_i\|_{sp} \leq \|\hat{w}_i\|_{sp}$, where $\alpha$ is a constant only depends on kernel size, input size, and stride size of the convolution operation. This result has two implications:

1. $\|\hat{w}_i\|_{sp} \leq \alpha \|\hat{w}_i\|_{sp}$: Although $\text{SN}_w$ does not strictly normalize the matrix with the actual spectral norm of the layer, it does upper bound the spectral norm of the layer. Therefore, all our analysis in §3 still applies for $\text{SN}_w$ by changing the spectral norm constant from 1 to $\alpha \|\hat{w}_i\|_{sp}$. This means that $\text{SN}_w$ can still prevent gradient explosion.

2. $\|\hat{w}_i\|_{sp} \leq \|\hat{w}_i\|_{sp}$: This implies that $\text{SN}_{\text{Conv}}$ normalizes by a factor that is at least as large as $\text{SN}_w$. In fact, we observe empirically that $\|\hat{w}_i\|_{sp}$ is strictly larger than $\|\hat{w}_i\|_{sp}$ during training (App. L.3). This means that for the same $w_i$, a discriminator using $\text{SN}_{\text{Conv}}$ will have smaller outputs than the discriminator using $\text{SN}_w$. We hypothesize that the different scalings explain why $\text{SN}_{\text{Conv}}$ has vanishing gradients but $\text{SN}_w$ does not.

To confirm this hypothesis, for $\text{SN}_w$ and $\text{SN}_{\text{Conv}}$, we propose to multiply all the normalized weights by a scaling factor $s$, which is fixed throughout the training. Fig. 6 shows that $\text{SN}_{\text{Conv}}$ seems to be a shifted version of $\text{SN}_w$. $\text{SN}_{\text{Conv}}$ with $s = 1.75$ has similar inception score (Fig. 4) to $\text{SN}_w$, as well as similar gradients (Fig. 5) and parameter variances (App. L.4) throughout training. This, combined with Thm. 2, suggests that $\text{SN}_w$ inherently finds the correct scaling for the problem, whereas “proper” spectral normalization $\text{SN}_{\text{Conv}}$ requires additional hyperparameter tuning.
5 Bidirectional Spectral Normalization: Improving Spectral Normalization

In 2010, Glorot and Bengio [9] built on the intuition of LeCun [20] to design an improved initialization, commonly called Xavier initialization. Their key observation was that to limit gradient vanishing (and explosion), it is not enough to control only feed-forward outputs; we should also control the variance of backpropagated gradients. Let $n_i, m_i$ denote the fan-in and fan-out of layer $i$. (In fully-connected layers, $n_i = m_{i-1} = $ the width of layer $i$.) Whereas LeCun chooses initial parameters with variance $\frac{1}{n_i}$, Glorot and Bengio choose them with variance $\frac{2}{n_i + m_i}$, a compromise between $\frac{1}{n_i}$ (to control output variance) and $\frac{1}{m_i}$ (to control variance of backpropagated gradients).

We propose Bidirectional Spectral Normalization (BSN), which applies a similar intuition to improve the spectral normalization of Miyato et al. [25]. For fully connected layers, we keep the normalization the same as SN$_w$ [25]. For convolution layers, instead of normalizing by $\|w_{cin} \times (c_{in}, k_{w}, k_{h})\|_w$, we normalize by $\sigma_w = \sqrt{\|w_{cin} \times (c_{in}, k_{w}, k_{h})\|^2 + \|w_{cout} \times (c_{cout}, k_{w}, k_{h})\|^2}$, where $\|w_{cin} \times (c_{in}, k_{w}, k_{h})\|_w$ is the spectral norm of the reshaped convolution kernel of dimension $c_{in} \times k_{w} \times k_{h}$. For calculating these two spectral norms, we use the same power iteration method in [25]. The following theorem gives the theoretical explanation.

**Theorem 3** (Variance of Bidirectional Spectral Normalization). For a convolutional kernel $w \in \mathbb{R}^{c_{out} \times c_{in} \times k_{w} \times k_{h}}$ with i.i.d. entries $w_{ij}$ from a symmetric distribution (e.g. zero-mean Gaussian or uniform) where $k_{w} k_{h} \geq \max \{c_{out}, c_{in}\}$, and $\sigma_w$ defined as above, we have

$$\text{Var} \left( \frac{w_{ij}}{\sigma_w} \right) \leq \frac{2}{c_{in} k_{w} k_{h} + c_{out} k_{w} k_{h}}.$$

Furthermore, if $c_{in}, c_{out} \geq 2$ and $c_{in} k_{w} k_{h}, c_{out} k_{w} k_{h} \geq 3$, and $w_{ij}$ are from a zero-mean Gaussian distribution, there exists a constant $L$ that does not depend on $c_{in}, c_{out}, k_{w}, k_{h}$ such that

$$\frac{c_{in} k_{w} k_{h}}{c_{in} k_{w} k_{h} \log(c_{out}) + c_{out} k_{w} k_{h} \log(c_{in})} \text{Var} \left( \frac{w_{ij}}{\sigma_w} \right) \leq \frac{2}{c_{in} k_{w} k_{h} + c_{out} k_{w} k_{h}}.$$

*(Proof in Appendix G)*. Note that in convolution layers, $n_i = c_{in} k_{w} k_{h}$ and $m_i = c_{out} k_{w} k_{h}$. Therefore, BSN sets the variance of parameters to scale as $\frac{2}{n_i + m_i}$, as dictated by Xavier initialization. Moreover, BSN naturally inherits the benefits of SN discussed in §4 (e.g. insensitive to initialization parameters, controlling variance throughout the training).

5.1 Results

Since SN is widely regarded as one of the most successful normalization techniques for GANs [10,39,5] and our proposed BSN is meant to improve SN, we compare the performance of SN and BSN. In addition, we compare against two variants of SN proposed in the appendix of [25], which we denote “same $\gamma$” and “diff. $\gamma$” (details in Appendix M). The standard SN from [25] is denoted by “no $\gamma$”. We run experiments on CIFAR10, STL10, CelebA, and ImageNet, with two widely-used metrics for sample quality: inception score [34] and Frechet Inception Distance (FID) [15] (details in Appendix H). The code for reproducing all experiments can be found in https://github.com/fjxmlzn/BSN

**CIFAR10, STL10, and CelebA.** We use the network architecture from SN [25]. We controlled five hyperparameters (Table 7, Appendix N): $\alpha_g$ and $\alpha_d$, the generator/discriminator learning rates, $\beta_1, \beta_2$, Adam momentum parameters [13], and $n_{dis}$, the number of discriminator updates per generator update. Three hyperparameter settings are from [25], with equal discriminator and generator learning rates; the final two test unequal learning rates. More details are in Appendix N and O.

As in [25], we report the metrics from the best hyperparameter for each algorithm in Table 1. BSN outperforms the standard SN in all sample quality metrics except FID score on STL10, where their metrics are within standard error of each other. Regarding the SN variants, in CIFAR10 and STL10, they have worse performance than SN and BSN, same as reported in [25]. In CelebA, the SN variants have better performance for the best hyperparameter setting. But in general, these SN variants are very sensitive to hyperparameters (Appendix N to O), therefore they are not adopted in practice [25].

More importantly, the superiority of BSN is stable across hyperparameters. Figs. 8 and 9 show the inception scores of all the hyperparameters we tested on CIFAR10 and STL10. BSN has the best or
We found that BSN without any tuning has the same gradient vanishing problem we observe for Apps. N to P. which is the purpose of SN (and BSN). More results γ whereas “same” and “diff." are variants proposed in the appendix of [25].

Moreover, we see that as training proceeds, the sample quality of SN often drops, whereas the sample quality of BSN appears to monotonically increase (Fig. 7 more in Apps.). In most cases, BSN not only outperforms SN in final sample quality (after training), but also in peak sample quality. This means that BSN makes the training process more stable, which is the purpose of SN (and BSN). More results (generated images, training curves, FID plots) are in Apps. N to P.

**ImageNet.** To further test if BSN scales to large-scale dataset and larger network, we compare SN and BSN in ILSVRC2012 dataset. We take the same ResNet-based architectures [14] and hyperparameter settings as in [25], which we reproduced in App. Q.

We found that BSN without any tuning has the same gradient vanishing problem we observe for SN_{Conv} [5] in § 4. We hypothesize that this is because of the distinction between ResNet and our analysis in § 4 and [5], which does not account for shortcut connections. The optimal variance and scaling for BSN (and for SN) in ResNet-based architectures might be different from our results in § 4 and [5]. We defer theoretical analysis of ResNets to future work.

To verify that the gradient vanishing is caused by the wrong scaling, we apply the scaling techniques introduced in § 4 for BSN, and find that this successfully solves the gradient vanishing problem. Table 2 shows the results of SN and BSN with different scale parameters. We see that the sample quality of scaled BSN outperforms the SN without scaling by a large margin. To verify that this benefit is from the formulation of BSN instead of the scaling, we also apply scaling over SN. From the table we see that for all the scales that are larger than 1 we try, BSN always outperforms SN. The best BSN (with scale=1.4) outperforms the best SN (with scale=1.0/1.4) by a large margin. In fact, applying scaling on SN does not make noticeable improvement on the sample quality, and could even

| Hyper-parameters | Inception score | FID | Hyper-parameters | Inception score | FID | FID |
|------------------|----------------|-----|------------------|----------------|-----|-----|
| SN (no γ)        | 4.5            | 5.0 | SN (same γ)      | 4.0            | 5.0 | 5.0 |
| SN (different γ)| 4.0            | 5.0 | SN (no γ)        | 4.0            | 5.0 | 5.0 |
| BSN (ours)       | 4.5            | 5.0 | BSN (same γ)     | 4.0            | 5.0 | 5.0 |
| BSN (different γ)| 4.0            | 5.0 | BSN (no γ)       | 4.0            | 5.0 | 5.0 |

Table 1: Inception scores and FIDs on CIFAR10, STL10, and CelebA. Each experiment is conducted with 5 random seeds, with mean and standard error reported. We follow the common practice of excluding Inception Score in CelebA as the inception network is pretrained on ImageNet, which is very different from CelebA. “no γ” denotes the standard spectral normalization used in practice, whereas “same γ” and “diff. γ” are variants proposed in the appendix of [25].

![Figure 8: Inception score in CIFAR10. The results are averaged over 5 random seeds.](image)

![Figure 9: Inception score in STL10. The results are averaged over 5 random seeds.](image)

![Figure 7: Inception score in CIFAR10. The results are averaged over 5 random seeds.](image)
make it worse (e.g. when scale=1.6). This accords with the phenomenon we observed in §4. More results (e.g. samples, training curves) are in App. Q.

6 Discussion and Related Work

Our results suggest that SN stabilizes GANs by controlling exploding and vanishing gradients in the discriminator. However, our analysis applies to the training of any fully-connected, feed-forward neural network. This connection partially explains why SN helps train generators as well as discriminators [44][5], and why SN is more generally useful in training neural networks [8][12][42]. Formally extending this analysis to understand the effects of adversarial training is an interesting direction for future work. The other future direction is to propose an even better regularization technique where the variance is exactly \(\frac{2}{n_i + m_i}\), as our approach only gives a range for the variance.

A related result to our upper bound was shown in [36], which shows that batch normalization (BN) makes the scaling of the Hessian along the direction of the gradient smaller, thereby making gradients more predictive. Given Prop. 1, we can apply the reasoning from [36] to explain why spectrally-normalized GANs are robust to different learning rates as shown in [25]. However, our insights regarding the gradient vanishing problem are the more surprising result; this notion is not discussed in [36]. An interesting question for future work is whether BN similarly controls vanishing gradients.

| Scale | Approach | Inception score | FID         |
|-------|----------|----------------|-------------|
| 1.0   | SN       | 12.84 ± 0.33   | 75.06 ± 2.38 |
|       | BSN (ours) | 1.77 ± 0.13   | 265.20 ± 19.01 |
| 1.2   | SN       | 12.73 ± 0.13   | 74.10 ± 1.41 |
|       | BSN (ours) | 13.23 ± 0.19   | 71.53 ± 1.92 |
| 1.4   | SN       | 12.76 ± 0.17   | 73.21 ± 1.92 |
|       | BSN (ours) | 13.23 ± 0.16   | 69.04 ± 1.46 |
| 1.6   | SN       | 11.53 ± 0.49   | 81.33 ± 5.51 |
|       | BSN (ours) | 13.01 ± 0.12   | 71.56 ± 1.18 |

Table 2: Inception scores and FIDs on ILSVRC2012. SN with scale=1.0 is conducted with 5 random seeds, and all others are conducted with 3 random seeds, with mean and standard error reported. The bold font marks the best numbers between SN and BSN using a specific scale. The red color marks the best numbers among all runs.

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Appendix

A Proof of Prop. 1

The proposition makes use of the following observation: For the discriminator defined in (1), the norm of gradient for $w_t$ is upper bounded by

$$
\| \nabla_{w_t} D_\theta (x) \|_F \leq \| x \| \cdot \prod_{i=1}^{L} \| a_i \|_{\text{Lip}} \cdot \prod_{i=1}^{L} \| w_i \|_{\text{sp}} / \| w_t \|_{\text{sp}} \quad \text{for } \forall t \in [1, L] 
$$

(6)

To prove this, for simplicity of notation, let $o^t_a = a_t \circ l_{w_t} \circ \ldots \circ a_1 \circ l_{w_1}$, and $o^t_1 = l_{w_t} \circ a_{t-1} \circ \ldots \circ a_1 \circ l_{w_1}$. It is straightforward to show that the norm of each internal output of discriminator is bounded by

$$
\| o^t_a (x) \| \leq \| x \| \cdot \prod_{i=1}^{t} \| a_i \|_{\text{Lip}} \cdot \prod_{i=1}^{t} \| w_i \|_{\text{sp}} 
$$

(7)

and

$$
\| o^t_1 (x) \| \leq \| x \| \cdot \prod_{i=t+1}^{L} \| a_i \|_{\text{Lip}} \cdot \prod_{i=t+1}^{L} \| w_i \|_{\text{sp}} . 
$$

(8)

This holds because

$$
\| o^t_a (x) \| = \| a_t \circ o^t_1 (x) \| \leq \| a_t \|_{\text{Lip}} \cdot \| o^t_1 (x) \| 
$$

and

$$
\| o^t_1 (x) \| = \| l_{w_t} \circ o^{t-1}_a (x) \| \leq \| w_t \|_{\text{sp}} \cdot \| o^{t-1}_a (x) \| , 
$$

from which we can show the desired inequalities by induction.

Next, we observe that the norm of each internal gradient is bounded by

$$
\| \nabla o^t_a (x) D_\theta (x) \| \leq \prod_{i=t+1}^{L} \| a_i \|_{\text{Lip}} \cdot \prod_{i=t+1}^{L} \| w_i \|_{\text{sp}} 
$$

(9)

and

$$
\| \nabla o^t_1 (x) D_\theta (x) \| \leq \prod_{i=t+1}^{L} \| a_i \|_{\text{Lip}} \cdot \prod_{i=t+1}^{L} \| w_i \|_{\text{sp}} . 
$$

(10)

This holds because

$$
\| \nabla o^t_a (x) D_\theta (x) \| = \| w^T_{t+1} \nabla o^{t+1}_a (x) D_\theta (x) \| \leq \| w_{t+1} \|_{\text{sp}} \| \nabla o^{t+1}_a (x) D_\theta (x) \| 
$$

and

$$
\| \nabla o^t_1 (x) D_\theta (x) \| = \| \nabla o^t_1 (x) D_\theta (x), \left( a^t_1 (x) \right)_{x=o^t_1 (x)} \| \leq \| a_t \|_{\text{Lip}} \| \nabla o^t_a (x) D_\theta (x) \| , 
$$

from which we can show inequalities [Eqs. (9) and (10)] by induction.

Now we have that

$$
\| \nabla_{w_t} D_\theta (x) \|_F = \| \nabla o^t_1 (x) D_\theta (x) \cdot (o^{t-1}_a (x))^T \|_F = \| \nabla o^t_1 (x) D_\theta \|_F \cdot \| o^{t-1}_a (x) \| 
$$

$$
\leq \prod_{i=t}^{L} \| a_i \|_{\text{Lip}} \cdot \prod_{i=t}^{L} \| w_i \|_{\text{sp}} \cdot \| x \| \cdot \prod_{i=1}^{t-1} \| a_i \|_{\text{Lip}} \cdot \prod_{i=1}^{t-1} \| w_i \|_{\text{sp}}
$$

$$
= \| x \| \cdot \prod_{i=1}^{L} \| a_i \|_{\text{Lip}} \cdot \prod_{i=1}^{L} \| w_i \|_{\text{sp}} / \| w_t \|_{\text{sp}}
$$

where we use Eqs. (7) to (10) at the inequality. The upper bound of gradient’s Frobenius norm for spectrally-normalized discriminators follows directly.
Moreover, if the activation for the last layer is sigmoid (e.g., for vanilla GAN [11]), we have
\[ \text{Eq. (1)} \]

Proposition 3 (Upper bound of Hessian’s spectral norm)

Let \( H_{w_i}(D_\theta)(x) \) denote the Hessian of \( D_\theta \) at \( x \) with respect to the vector form of \( w_i \). If the internal activations are ReLU or leaky ReLU, the spectral norm of \( H_{w_i}(D_\theta)(x) \) is upper bounded by
\[
\| H_{w_i}(D_\theta)(x) \|_{sp} \leq \| H_{o_L}(x)D_\theta(x) \|_{sp} \cdot \| x \|_{2} \cdot \prod_{i=1}^{L} \| w_i \|_{sp} \cdot \| w_i \|_{sp}^2
\]

B Proof of Prop. 2

Proof. As \( l_w(x) \) is a linear transformation, we have \( l_{cw}(x) = c \cdot l_w(x) \), and \( l_w(cx) = c \cdot l_w(x) \). Moreover, since ReLU and leaky ReLU is linear in \( \mathbb{R}^+ \) and \( \mathbb{R}^- \) region, we have \( a_i(cx) = c \cdot a_i(x) \).

Therefore, we have
\[
D_\theta(x) = (a_L \circ l_{cw_1} \circ a_{L-1} \circ l_{cw_{L-1}} \circ \ldots \circ a_1 \circ l_{cw_1})(x)
\]
\[
= \prod_{i=1}^{L} c_i \cdot (a_L \circ l_{w_L} \circ a_{L-1} \circ l_{w_{L-1}} \circ \ldots \circ a_1 \circ l_{w_1})(x)
\]
\[
= D_\theta(x)
\]

C Additional Analysis of Gradient

In § 3, we discuss the gradients with respect to \( w_i' = \frac{w_i}{\| w_i \|_{sp}} \), where \( u_i, v_i \) are the singular vectors corresponding to the largest singular values. In this section we discuss the gradients with respect to the actual parameter \( w_i \). From Eq. (12) in [25] we know
\[
\nabla_{w_i} D_\theta(x) = \frac{1}{\| w_i \|_{sp}} \left( \nabla_{w_i'} D_\theta(x) - \left( \nabla_{o_L'(x)D_\theta(x)^T} o_L'(x) \right) \cdot u_i v_i^T \right)
\]

From App. A, we know that \( \| \nabla_{w_i'} D_\theta(x) \|_{lp} \), \( \| \nabla_{o_L'(x)} D_\theta(x) \|_{lp} \), and \( \| o_L'(x) \|_{lp} \) have upper bounds. Furthermore, \( \| u_i v_i^T \|_{lp} = 1 \). Therefore, \( \| \nabla_{w_i} D_\theta(x) \|_{lp} \) has an upper bound. From Theorem 1.1 in [37] we know that if \( w_i \) is initialized with i.i.d random variables from uniform or Gaussian distribution, \( \mathbb{E} \left( \| w_i \|_{sp} \right) \) is lower bounded away from zero at initialization. So \( \| \nabla_{w_i} D_\theta(x) \|_{lp} \) is upper bounded at initialization. Moreover, we observe empirically that \( \| w_i \|_{sp} \) is usually increasing during training. Therefore, \( \| \nabla_{w_i} D_\theta(x) \|_{lp} \) is typically upper bounded during training as well.

D Analysis of Hessian

The following proposition states that spectral normalization also gives an upper bound on \( \| H_{w_i}(D_\theta)(x) \|_{sp} \) for networks with ReLU or leaky ReLU internal activations.

Proposition 3 (Upper bound of Hessian’s spectral norm). Consider the discriminator defined in Eq. (11). Let \( H_{w_i}(D_\theta)(x) \) denote the Hessian of \( D_\theta \) at \( x \) with respect to the vector form of \( w_i \). If the internal activations are ReLU or leaky ReLU, the spectral norm of \( H_{w_i}(D_\theta)(x) \) is upper bounded by
\[
\| H_{w_i}(D_\theta)(x) \|_{sp} \leq \| H_{o_L}(x)D_\theta(x) \|_{sp} \cdot \| x \|_{2} \cdot \prod_{i=1}^{L} \| w_i \|_{sp} \cdot \| w_i \|_{sp}^2
\]

The proof is in App. D.1. Following Prop. 3 we can easily show the upper bound of Hessian’s spectral norm for spectral normalized discriminators.

Corollary 1 (Upper bound of Hessian’s spectral norm for spectral normalization). If the internal activations are ReLU or leaky ReLU, and \( \| w_i \|_{sp} \leq 1 \) for all \( i \in [1, L] \), then
\[
\| H_{w_i}(D_\theta)(x) \|_{sp} \leq \| H_{o_L}(x)D_\theta(x) \|_{sp} \cdot \| x \|_{2}^2
\]

Moreover, if the activation for the last layer is sigmoid (e.g., for vanilla GAN [17]), we have
\[
\| H_{w_i}(D_\theta)(x) \|_{sp} \leq 0.1 \cdot \| x \|_{2}^2
\]

If the activation function for the last layer is identity (e.g., for WGAN-GP [13]), we have
\[
\| H_{\theta}(D_\theta)(x) \|_{sp} = 0
\]
D.1 Proof of Prop. 3

Lemma 1. The spectral norm of each internal Hessian is bounded by

\[ \| H_{\alpha_t}(x) D_\theta(x) \|_{sp} \leq \| H_{\alpha_t'}(x) D_\theta(x) \|_{sp} \cdot \prod_{i=t+1}^{L} \| w_i \|_{sp}^2 \]

and

\[ \| H_{\alpha_t}(x) D_\theta(x) \|_{sp} \leq \| H_{\alpha_t'}(x) D_\theta(x) \|_{sp} \cdot \prod_{i=t+1}^{L} \| w_i \|_{sp}^2 \]

Proof. We have

\[ \| H_{\alpha_t}(x) D_\theta(x) \|_{sp} = \| w_{t+1}^T \cdot \nabla_{a_{t+1}}(x) D_\theta(x) \cdot w_{t+1} \|_{sp} \]

\[ \leq \| \nabla_{a_{t+1}}(x) D_\theta(x) \|_{sp} \cdot \| w_{t+1} \|_{sp}^2 . \]

We also have

\[ \| H_{\alpha_t}(x) D_\theta(x) \|_{sp} = \| \text{diag} \left( \{ a_{t}'(x) x = a_{t}'(x) \} \cdot H_{\alpha_{t+1}}(x) \cdot \text{diag} \left( \{ a_{t}'(x) x = a_{t}'(x) \} \right) \right) \|_{sp} \]

\[ \leq \| H_{\alpha_{t+1}}(x) D_\theta(x) \|_{sp} \]

where we use the property that ReLU or leaky ReLU is piece-wise linear. The desired inequalities then follow by induction.

Now let’s come back to the proof for Prop. 3.

Proof. We have

\[ \frac{\partial D_\theta}{\partial (w_t)}_{ij} \frac{\partial (w_t)}{\partial (w_t)_{kl}} = \left( H_{\alpha}(D_\theta)(x) \right)_{ik} \cdot \left( a_{t-1}(x) \right)_{j} \cdot \left( a_{t-1}(x) \right)_{l} . \]

Therefore,

\[ \| H_{w_t}(D_\theta)(x) \|_{sp} \leq \| H_{\alpha}(D_\theta)(x) \|_{sp} \cdot \| a_{t-1}(x) \|_{sp}^2 \leq \| H_{\alpha}(D_\theta)(x) \|_{sp} \cdot \| a_{t-1}(x) \|_{sp}^2 \]

Applying Eq. (7) and Lemma 1 we get

\[ \| H_{w_t}(D_\theta)(x) \|_{sp} \leq \| H_{\alpha}(D_\theta)(x) \|_{sp} \cdot \prod_{i=t+1}^{L} \| w_i \|_{sp}^2 \cdot \| x \|_{sp}^2 \cdot \prod_{i=1}^{t-1} \| w_i \|_{sp}^2 \]

\[ = \| H_{\alpha}(D_\theta)(x) \|_{sp} \cdot \| x \|_{sp}^2 \cdot \prod_{i=1}^{L} \| w_i \|_{sp}^2 \.

\]

E. Proof of Thm. 1

Proof. For any discriminator \( D_\theta = a_L \circ l_{w_L} \circ a_{L-1} \circ l_{w_{L-1}} \circ \ldots \circ a_1 \circ l_{w_1} \), consider \( \theta' = \{ w'_t = c_t w_t \}_{t=1}^{L} \) with the constraint \( \prod_{i=1}^{L} c_i = 1 \) and \( c_i \in \mathbb{R}^+ \). Let \( Q = \| \nabla_{w_t} D_\theta'(x) \|_F \| w_t' \|_{sp} \).
We have
\[
\|
\nabla_{\theta'} D_{\theta'}(x)\|_F = \sqrt{\sum_{i=1}^L \left\| \nabla_{w_i'} D_{\theta'}(x) \right\|_F^2}
\]  
\[
= \sqrt{\sum_{i=1}^L c_i^2 \|w_i\|^2_{sp}}
\]
\[
\geq \sqrt{L \left( \prod_{i=1}^L c_i^2 \|w_i\|^2_{sp} \right)^{1/L}}
\]
\[
= \sqrt{L} \cdot Q^{1/L} \cdot \left( \prod_{i=1}^L \|w_i\|_{sp} \right)^{-1/L}
\]
and the equality is achieved iff \(c_i^2 \|w_i\|^2_{sp} = c_j^2 \|w_j\|^2_{sp}, \ \forall i, j \in [1, L]\) according to AM-GM inequality. When \(c_i^2 \|w_i\|^2_{sp} = c_j^2 \|w_j\|^2_{sp}, \ \forall i, j \in [1, L]\), we have \(c_t = \prod_{i=1}^L \|w_i\|_{sp} / \|w_t\|_{sp}\).

F Proof of Thm. 2

Proof. Since \(a_{ij}\) are symmetric random variables, we know \(E \left( \frac{a_{ij}}{\|A\|_{sp}} \right) = 0\). Further, by symmetry, we have that for any \((i, j) \neq (h, \ell)\), \(E \left( \frac{a_{ij}^2}{\|A\|_{sp}} \right) = E \left( \frac{a_{h\ell}^2}{\|A\|_{sp}} \right)\). Therefore, we have
\[
\text{Var} \left( \frac{a_{ij}}{\|A\|_{sp}} \right) = E \left( \frac{a_{ij}^2}{\|A\|_{sp}} \right) = \frac{1}{mn} \cdot E \left( \sum_{i=1}^m \sum_{j=1}^n \frac{a_{ij}^2}{\|A\|_{sp}} \right) = \frac{1}{mn} \cdot E \left( \frac{\|A\|_{F}^2}{\|A\|_{sp}^2} \right)
\]
Our approach will be to upper and lower bound the quantity \(\frac{1}{mn} \cdot E \left( \frac{\|A\|_{F}^2}{\|A\|_{sp}^2} \right)\).

Upper bound Assume the singular values of \(A\) are \(\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min\{m, n\}}\). We have
\[
\frac{1}{mn} \cdot E \left( \frac{\|A\|_{F}^2}{\|A\|_{sp}^2} \right) = \frac{1}{mn} \cdot E \left( \frac{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}{\|A\|_{sp}^2} \right) \leq \min \{m, n\} \frac{\sigma_i^2}{\sigma_1^2} \leq \frac{1}{\max \{m, n\}},
\]
which gives the desired upper bound.

Lower bound Now for the lower bound, if \(a_{ij}\) are drawn from zero-mean Gaussian distribution and \(\max \{m, n\} \geq 3\), we have
\[
\frac{1}{mn} \cdot E \left( \frac{\|A\|_{F}^2}{\|A\|_{sp}^2} \right) \geq \frac{1}{mn} \cdot E \left( \frac{1}{\|A\|_{sp}^2} \|A\|_{F}^2 \right)
\]
\[
= \frac{1}{mn} \cdot E \left( \frac{1}{\|B\|_{sp}^2} \right)
\]
\[
= \frac{1}{mn} \cdot E \left( \frac{1}{\|B\|_{sp}^2} \right)
\]
where \(B \in \mathbb{R}^{m \times n}\) is uniformly sampled from the sphere of \(m \times n\)-dimension unit ball. We use the following lemma to lower bound (12).
Lemma 2 (Theorem 1.1 in [37]). Assume \( A \in \mathbb{R}^{m \times n} \) is uniformly sampled from the sphere of \( m \times n \)-dimension unit ball. When \( \max \{m, n\} \geq 3 \), we have

\[
\mathbb{E} \left( \|A\|_{sp}^2 \right) \leq K^2 \left( \mathbb{E} \left( \max_{1 \leq i \leq m} \|a_{i, \bullet}\|^2 \right) + \mathbb{E} \left( \max_{1 \leq j \leq n} \|a_{\bullet, j}\|^2 \right) \right),
\]

where \( K \) is a constant which does not depend on \( m, n \). Here \( a_{i, \bullet} \) denotes the \( i \)-th row of \( A \), and \( a_{\bullet, j} \) denotes the \( j \)-th column of \( A \).

We thus have that

\[
\frac{1}{mn} \cdot \frac{1}{\mathbb{E}(\|B\|_{sp}^2)} \geq \frac{1}{mn} \cdot K^2 \left( \mathbb{E} \left( \max_{1 \leq i \leq m} \|b_{i, \bullet}\|^2 \right) + \mathbb{E} \left( \max_{1 \leq j \leq n} \|b_{\bullet, j}\|^2 \right) \right).
\]

Hence, we need to upper bound \( \mathbb{E} \left( \max_{1 \leq i \leq m} \|b_{i, \bullet}\|^2 \right) \) and \( \mathbb{E} \left( \max_{1 \leq j \leq n} \|b_{\bullet, j}\|^2 \right) \). Let \( z \in \mathbb{R}^m \) be a vector uniformly sampled from the sphere of \( m \)-dimension unit ball. Observe that \( z = \sqrt{\|b_{1, \bullet}\|, \ldots, \|b_{n, \bullet}\|} \). The following lemma upper bounds the square of the infinity norm of this vector.

Lemma 3. Assume \( z = [z_1, z_2, \ldots, z_n] \) is uniformly sampled from the sphere of \( n \)-dimension unit ball, where \( n \geq 2 \). Then we have

\[
\mathbb{E} \left( \max_{1 \leq i \leq n} z_i^2 \right) \leq \frac{4 \log(n)}{n - 1}.
\]

(Proof in App. F.1)

Hence, when \( m, n \geq 2 \), we have

\[
\mathbb{E} \left( \max_{1 \leq i \leq m} \|b_{i, \bullet}\|^2 \right) \leq \frac{4 \log(m)}{m - 1}
\]

Similarly, we have

\[
\mathbb{E} \left( \max_{1 \leq j \leq n} \|b_{\bullet, j}\|^2 \right) \leq \frac{4 \log(n)}{n - 1}
\]

Therefore,

\[
\text{Var} \left( \frac{a_{i,j}}{\|A\|_{sp}} \right) \geq \frac{1}{mn} \cdot \frac{1}{K^2} \cdot \frac{4 \log(m)}{m - 1} + \frac{4 \log(n)}{n - 1}
\]

\[
\geq \frac{1}{8K^2} \cdot \frac{1}{\max \{m, n\}} \cdot \frac{1}{\log \left( \min \{m, n\} \right)}
\]

which gives the result.

\[\Box\]

Note that the original theorem in [37] requires that the entries of \( A \) be i.i.d. symmetric random variables, whereas in our case the entries are not i.i.d., as we require \( \|A\|_F = 1 \). However, the i.i.d. assumption in their proof is only used to ensure that \( A, S_{\sigma(1), \epsilon(1)}(A), \) and \( S_{\sigma(2), \epsilon(2)}(A) \) have the same distribution, where \( \sigma^{(t)} \) for \( t = 0, 1 \) are vectors of independent random permutations; \( \epsilon^{(t)} \) for \( t = 0, 1 \) are matrices of i.i.d. random variables with equal probability of being \( \pm 1 \); and \( S_{\sigma(1), \epsilon(1)}(A) = \begin{pmatrix} \epsilon_{ij}^{(1)} \cdot a_{ij}^{\sigma(1)(ij)} \end{pmatrix}_{i,j} \) and \( S_{\sigma(2), \epsilon(2)}(A) = \begin{pmatrix} \epsilon_{ij}^{(2)} \cdot a_{ij}^{\sigma(2)(ij)} \end{pmatrix}_{i,j} \). Our matrix \( A \) satisfies this requirement, and therefore the same theorem holds.
F.1 Proof of Lemma 3

Proof.

\[
\mathbb{E} \left( \max_{1 \leq i \leq n} z_i^2 \right) = \int_0^1 \mathbb{P} \left( \max_{1 \leq i \leq n} z_i^2 \geq \delta \right) d\delta 
\]

\[
\leq \int_0^1 \min \left\{ 1, n \cdot \mathbb{P} \left( z_1^2 \geq \delta \right) \right\} d\delta
\]

(13)

where (13) follows from the union bound. Next, we use the following lemma to upper bound \( \mathbb{P} \left( z_1^2 \geq \delta \right) \).

Lemma 4. Assume \( z = [z_1, z_2, \ldots, z_n] \) is uniformly sampled from the sphere of \( n \)-dimension unit ball, where \( n \geq 2 \). Then for \( \frac{1}{n} \leq \delta < 1 \) and \( \forall i \in [1, n] \), we have

\[
\mathbb{P} \left( z_i^2 \geq \delta \right) \leq e^{-\frac{n-1}{2} \delta + 1}.
\]

(Proof in App. F.2). This in turn gives

\[
\int_0^1 \min \left\{ 1, n \cdot \mathbb{P} \left( z_1^2 \geq \delta \right) \right\} d\delta \leq \int_0^1 \min \left\{ 1, \frac{2 \log(n) + 2}{n-1} \right\} 1 \cdot d\delta + \int_0^1 \min \left\{ 1, \frac{2 \log(n) + 2}{n-1} \right\} n \cdot e^{-\frac{n-1}{2} \delta + 1} \cdot d\delta
\]

(14)

\[
\leq \left\{ \frac{2 \log(n) + 2}{n-1} - \frac{2n}{n-1} e^{-\frac{n-3}{4}} + \frac{2}{n-1} \right\} (n \leq 6)
\]

\[
\leq \frac{4 \log(n)}{n-1} (n \geq 7)
\]

where Eq. (14) follows from Lemma 4.

F.2 Proof of Lemma 4

Proof. Due to the symmetry of \( z_i \), we only need to prove the inequality for \( i = 1 \) case. Let \( x = [x_1, \ldots, x_n] \sim \mathcal{N}(0, I_n) \), where \( I_n \) is the identity matrix in \( n \) dimension. We know that

\[
\mathbb{P} \left( \frac{x_1^2}{\sum_{i=2}^n x_i^2} \geq \delta \right) = \mathbb{P} \left( \frac{x_1^2}{\sum_{i=2}^n x_i^2} \geq \delta \right) = \mathbb{P} \left( \frac{x_1^2}{\sum_{i=2}^n x_i^2} \geq \frac{(n-1) \delta}{(n-1) - \delta} \right)
\]

Note that \( x_1^2 \) and \( \sum_{i=2}^n x_i^2 \) are two independent chi-squared random variables, therefore, we know that

\[
\frac{x_1^2}{\sum_{i=2}^n x_i^2} \sim F(1, n-1),
\]

where \( F(1, n-1) \) is the central F-distribution. Therefore,

\[
\mathbb{P} \left( \frac{x_1^2}{\sum_{i=2}^n x_i^2} \geq \frac{(n-1) \delta}{(n-1) - \delta} \right) = 1 - I_{1-\delta} \left( \frac{n-1}{2}, \frac{1}{2} \right)
\]

\[
= I_{1-\delta} \left( \frac{n-1}{2}, \frac{1}{2} \right)
\]

\[
= \frac{B_{1-\delta} \left( \frac{n-1}{2}, \frac{1}{2} \right)}{B \left( \frac{n-1}{2}, \frac{1}{2} \right)}
\]

(15)

where \( I_{1-\delta} (a, b) \) is the regularized incomplete beta function, \( B_{1-\delta} (a, b) \) is the incomplete beta function, and \( B(a, b) \) is beta function.
For the ease of computation, we take the log of Eq. (15). The numerator gives
\[
\log \left( B_{1-\delta} \left( \frac{n-1}{2}, \frac{1}{2} \right) \right) \\
= \log \left( \frac{(1-\delta)^{(n-1)/2}}{(n-1)/2} _2F_1 \left( \frac{n-1}{2}, \frac{1}{2}; \frac{n+1}{2}; 1-\delta \right) \right) \\
= \frac{n-1}{2} \log (1-\delta) - \log(n-1) + \log \left( _2F_1 \left( \frac{n-1}{2}, \frac{1}{2}; \frac{n+1}{2}; 1-\delta \right) \right) + \log(2) ,
\] (16)
where \(_2F_1 (\cdot)\) is the hypergeometric function. Let \((q)_i = \begin{cases} 1 & (i = 0) \\ q(q+1)\ldots(q+i-1) & (i > 0) \end{cases}\), we have
\[
_2F_1 \left( \frac{n-1}{2}, \frac{1}{2}; \frac{n+1}{2}; 1-\delta \right) \\
= \sum_{i=0}^{\infty} \frac{(n-1)_i}{(n+1)_i} \frac{(1-\delta)^i}{i!} \\
\leq \sum_{i=0}^{\infty} \frac{(1/2)^i}{i!} (1-\delta)^i \\
= \frac{1}{\delta}.
\] (17)
Substituting it into Eq. (16) gives
\[
\log \left( B_{1-\delta} \left( \frac{n-1}{2}, \frac{1}{2} \right) \right) \leq \frac{n-1}{2} \log (1-\delta) - \log(n-1) - \frac{1}{2} \log (\delta) + \log(2) .
\] (18)

The log of the denominator of Eq. (15) is
\[
\log \left( B \left( \frac{n-1}{2}, \frac{1}{2} \right) \right) \\
= \log \left( \frac{\Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{n}{2} \right)} \right) \\
\geq \log \left( \sqrt{\pi} \cdot \left( \frac{n+1}{2} \right)^{-\frac{1}{2}} \right) \\
= -\frac{1}{2} \log(n+1) + \frac{1}{2} \log(2) + \frac{1}{2} \log(\pi) .
\] (19)
where \(\Gamma\) denotes the Gamma function and we use the Gautschi’s inequality: \(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} < (x+1)^{\frac{1}{2}}\) for positive real number \(x\).

Combining Eq. (15), Eq. (18) and Eq. (19) we get
\[
\log \left( \mathbb{P} \left( \sum_{i=2}^{X^2_1} \left( \frac{x_i^2}{(n-1)} \geq \frac{(n-1)\delta}{1-\delta} \right) \right) \right) \\
\leq \frac{n-1}{2} \log (1-\delta) - \log(n-1) + \frac{1}{2} \log(n+1) - \frac{1}{2} \log (\delta) + \frac{1}{2} \log(2/\pi) \\
\leq \frac{n-1}{2} \log (1-\delta) - \frac{1}{2} \log(n-1) - \frac{1}{2} \log(\delta) + \frac{1}{2} \log(6/\pi) \\
\leq \frac{n-1}{2} \log (1-\delta) - \frac{1}{2} \log \left( \frac{n-1}{n} \right) + \frac{1}{2} \log(6/\pi) \\
\leq \frac{n-1}{2} \log (1-\delta) + \frac{1}{2} \log \frac{12}{\pi} \\
\leq -\frac{n-1}{2} \cdot \delta + 1
\]
Therefore, we have

$$\mathbb{P} \left( z_i^2 \geq \delta \right) \leq e^{-\frac{n-1}{2} \delta + 1}$$

\[ \square \]

**G  Proof of Thm. 3**

*Proof.* Let $s_w = c_{in}c_{out}k_wk_h$. Since $w_{ij}$ are symmetric random variables, we know $\mathbb{E} \left( \frac{w_{ij}}{\sigma_w} \right) = 0$. Therefore, we have

$$\text{Var} \left( \frac{w_{ij}}{\sigma_w} \right) = \mathbb{E} \left( \frac{w_{ij}^2}{\sigma_w^2} \right) = \frac{1}{s_w} \cdot \mathbb{E} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{w_{ij}^2}{\sigma_w^2} \right) = \frac{1}{s_w} \cdot \mathbb{E} \left( \frac{\|w\|_F^2}{\sigma_w^2} \right)$$

Note that

$$\frac{1}{s_w} \cdot \mathbb{E} \left( \frac{\|w\|_F^2}{\sigma_w^2} \right) \leq \left[ \frac{\|w\|_F^2}{\sigma_w^2} \cdot \mathbb{E} \left( \|w\|_F^2 \cdot \frac{\|w\|_F^2}{\sigma_w^2} \right) \right] \leq \left[ \frac{\|w\|_F^2}{\sigma_w^2} \cdot \mathbb{E} \left( \|w\|_F^2 \right) \right]$$

Assume the singular values of $w^{c_{out} \times (c_{in}k_wk_h)}$ are $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{c_{out}}$, and the singular values of $w^{c_{in} \times (c_{out}k_wk_h)}$ are $\sigma'_1 \geq \sigma'_2 \geq \ldots \geq \sigma'_{c_{in}}$. We have

$$\frac{\|w\|_F^2}{\sigma_w^2} \cdot \mathbb{E} \left( \|w\|_F^2 \cdot \frac{\|w\|_F^2}{\sigma_w^2} \right) \leq \frac{1}{s_w} \cdot \mathbb{E} \left( \|w\|_F^2 \right) \leq \frac{2}{c_{in}k_wk_h + c_{out}k_wk_h}$$

which gives the desired upper bound.

As for the lower bound, observe that

$$\frac{2}{s_w} \cdot \mathbb{E} \left( \frac{\|w\|_F^2}{\sigma_w^2} \right) = \frac{2}{s_w} \cdot \mathbb{E} \left( \|w\|_F^2 \right) \leq \frac{1}{s_w} \cdot \mathbb{E} \left( \|w\|_F^2 \right)$$

Then we can follow the same approach in App. F for bounding $\mathbb{E} \left( \frac{\|w^{c_{out} \times (c_{in}k_wk_h)}\|_2^2}{\|w\|_F^2} \right)$ and $\mathbb{E} \left( \frac{\|w^{c_{in} \times (c_{out}k_wk_h)}\|_2^2}{\|w\|_F^2} \right)$, which gives the desired lower bound. \[ \square \]
H Datasets and Metrics

H.1 Datasets

**MNIST** \[21\] We use the training set for our experiments, which contains 60000 images of handwritten digits of shape $28 \times 28 \times 1$. The pixels values are normalized to $[0, 1]$ before feeding to the discriminators.

**CIFAR10** \[19\] We use the training set for our experiments, which contains 50000 images of shape $32 \times 32 \times 3$. The pixels values are normalized to $[-1, 1]$ before feeding to the discriminators.

**STL10** \[7\] We use the unlabeled set for our experiments, which contains 100000 images of shape $96 \times 96 \times 3$. Following \[25\], we resize the images to $48 \times 48 \times 3$ for training. The pixels values are normalized to $[-1, 1]$ before feeding to the discriminators.

**CelebA** \[24\] This dataset contains 202599 images. For each image, we crop the center $128 \times 128$, and resize it to $64 \times 64 \times 3$ for training. The pixels values are normalized to $[-1, 1]$ before feeding to the discriminators.

**ImageNet (ILSVRC2012)** \[33\] The dataset contains 1281167 images. Following \[25\], for each images, we crop the central square of the images according to min(width, height), and then reshape it to $128 \times 128 \times 3$ for training. The pixels values are normalized to $[-1, 1]$ before feeding to the discriminators.

H.2 Metrics

**Inception score** \[34\] Following \[25\], we use 50000 generated images and split them into 10 sets for computing the score.

**FID** \[15\] Following \[25\], we use 5000 real images and 10000 generated images for computing the score.

I Gradient Explosion and Vanishing in GANs

I.1 Results

To illustrate that gradient explosion and vanishing are closely related to the instability in GANs, we trained a WGAN \[13\] on the CIFAR10 dataset with different hyperparameters leading to stable training, exploding gradients, and vanishing gradients over 40,000 training iterations (more experimental details in App. I.2). Fig. 10 shows the resulting inception scores for each of these runs, and Fig. 11 shows the corresponding magnitudes of the gradients over the course of training. Note that the stable run has improved sample quality and stable gradients throughout training. This phenomenon has also been observed in prior literature \[1\] \[5\]. We will demonstrate that by controlling these gradients, SN (and SNw in particular) is able to achieve more stable training and better sample quality.

![Figure 10: Inception score over the course of training. The “gradient vanishing” inception score plateaus as training is stalled.](image1)

![Figure 11: Norm of gradient with respect to parameters during training. The vanishing gradient collapses after 200k iterations.](image2)
\[ z \in \mathbb{R}^{128} \sim \mathcal{N}(0, I) \]

- Fully connected \((M_g \times M_g \times 512)\), BN, ReLU.
- Deconvolution \((c = 256, k = 4, s = 2)\), BN, ReLU.
- Deconvolution \((c = 128, k = 4, s = 2)\), BN, ReLU.
- Deconvolution \((c = 64, k = 4, s = 2)\), BN, ReLU.
- Deconvolution \((c = 3, k = 3, s = 1)\), Tanh.

Deconvolution \((c = 256, k = 4, s = 2)\), BN, ReLU.

Deconvolution \((c = 128, k = 4, s = 2)\), BN, ReLU.

Deconvolution \((c = 64, k = 4, s = 2)\), BN, ReLU.

Deconvolution \((c = 3, k = 3, s = 1)\), Tanh.

\[ x \in \mathbb{R}^{M \times M \times 3} \]

- Convolution \((c = 64, k = 3, s = 1)\), Leaky ReLU (0.1).
- Convolution \((c = 64, k = 4, s = 2)\), Leaky ReLU (0.1).
- Convolution \((c = 128, k = 3, s = 1)\), Leaky ReLU (0.1).
- Convolution \((c = 128, k = 4, s = 2)\), Leaky ReLU (0.1).
- Convolution \((c = 256, k = 3, s = 1)\), Leaky ReLU (0.1).
- Convolution \((c = 256, k = 4, s = 2)\), Leaky ReLU (0.1).
- Convolution \((c = 512, k = 3, s = 1)\), Leaky ReLU (0.1).

Fully connected (1).

Table 3: Generator network architectures for CIFAR10, STL10, and CelebA experiments (from [25]). For CIFAR10, \(M_g = 4\). For STL10, \(M_g = 6\). For CelebA, \(M_g = 8\). BN stands for batch normalization. \(c\) stands for number of channels. \(k\) stands for kernel size. \(s\) stands for stride.

Table 4: Discriminator network architectures for CIFAR10, STL10, and CelebA experiments (from [25]). For CIFAR10, \(M = 32\). For STL10, \(M = 48\). For CelebA, \(M = 64\). \(c\) stands for number of channels. \(k\) stands for kernel size. \(s\) stands for stride.

I.2 Experimental Details

The network architectures are shown in Tables 3 and 4. The dataset is CIFAR10. All experiments are run for 400k iterations. Batch size is 64. The optimizer is Adam. Let \(\lambda\) be the WGAN’s gradient penalty weight [13]. For the stable run, \(\alpha_g = 0.0001, \alpha_d = 0.0002, \beta_1 = 0.5, \beta_2 = 0.999, \lambda = 10, n_{dis} = 1\). For the gradient explosion run, \(\alpha_g = 0.001, \alpha_d = 0.001, \beta_1 = 0.5, \beta_2 = 0.999, \lambda = 10, n_{dis} = 1\). For the gradient vanishing run, \(\alpha_g = 0.001, \alpha_d = 0.001, \beta_1 = 0.5, \beta_2 = 0.999, \lambda = 50, n_{dis} = 1, \) and the activation functions in the discriminator are changed from leaky ReLU to ReLU.

J Experimental Details and Additional Results on Gradient Norms

J.1 Experimental Details

For the MNIST experiment, the network architectures are shown in Tables 5 and 6. All experiments are run for 100 epochs. Batch size is 64. The optimizer is Adam. \(\alpha_g = 0.0001, \alpha_d = 0.0001, \beta_1 = 0.5, \beta_2 = 0.999, n_{dis} = 1\). For the CIFAR10 experiment, the network architectures are shown in Tables 3 and 4. All experiments are run for 400k iterations. Batch size is 64. The optimizer is Adam. \(\alpha_g = 0.0001, \alpha_d = 0.0001, \beta_1 = 0.5, \beta_2 = 0.999, n_{dis} = 1\).

Let \(\lambda\) be the WGAN’s gradient penalty weight [13]. For the runs without SN, \(\lambda = 10\). For the runs with SN, we use the strict SN implementation [8] in order to verifying the theoretical results (the popular SN implementation [25] only gives a loose bound on the actual spectral norm of layers, see § 4). Since it already ensures that the Lipschitz constant of the discriminator is no more than 1, we discard the gradient penalty loss from training.

For all the results, the gradient norm only considers the weights and excludes the biases (if exist), so as to be consistent with the theoretical analysis.

21
\[ z \in \mathbb{R}^{100} \sim \text{Uniform}(-1, 1) \]

- Fully connected \((7 \times 7 \times 128)\). Leaky ReLU \((0.2)\). BN.
- Deconvolution \((c = 64, k = 5, s = 2)\). Leaky ReLU \((0.2)\). BN.
- Deconvolution \((c = 1, k = 5, s = 2)\). Sigmoid.

Table 5: Generator network architectures for MNIST experiments. BN stands for batch normalization. \(c\) stands for number of channels. \(k\) stands for kernel size. \(s\) stands for stride.

\[ x \in \mathbb{R}^{28 \times 28 \times 1} \]

- Convolution \((c = 64, k = 5, s = 2)\). No bias. Leaky ReLU \((0.2)\).
- Convolution \((c = 128, k = 5, s = 2)\). No bias. Leaky ReLU \((0.2)\).
- Convolution \((c = 256, k = 5, s = 2)\). No bias. Leaky ReLU \((0.2)\).
- Fully connected \((1)\). No bias.

Table 6: Discriminator network architectures for MNIST experiments. \(c\) stands for number of channels. \(k\) stands for kernel size. \(s\) stands for stride.

### J.2 Additional Results

Figs. 12 and 13 show the gradient norms of each discriminator layer in MNIST and CIFAR10. Despite the difference on the network architecture and dataset, we see the similar phenomenon: when training without SN, some layers have extremely large gradient norms, which causes the overall gradient norm to be large; when training with SN, the gradient norms are much smaller and are similar across different layers.

### K Experimental Details and Additional Results for Confirming Eq. (3)

#### K.1 Experimental Details

For the MNIST experiment, the network architectures are shown in Tables 5 and 6. All experiments are run for 100 epochs. Batch size is 64. The optimizer is Adam. \(\alpha_g = 0.001, \alpha_d = 0.001, \beta_1 = 0.5, \beta_2 = 0.999, n_{\text{dis}} = 1\). We use WGAN loss with the strict SN implementation [8]. Since it already ensures that the Lipschitz constant of the discriminator is no more than 1, we discard the gradient penalty loss from training. The random scaling are selected in a way the geometric mean of spectral norms of all layers equals 1.

For the CIFAR10 and STL10 experiments, the network architectures are shown in Tables 3 and 4. All experiments are run for 400k iterations. Batch size is 64. The optimizer is Adam. \(\alpha_g = 0.0001, \alpha_d = 0.0001, \beta_1 = 0.5, \beta_2 = 0.999, n_{\text{dis}} = 1\). We use hinge loss [25] with the strict SN implementation [8]. The random scaling are selected in a way the geometric mean of spectral norms of all layers equals 1.75, which avoids the gradient vanishing problem as seen in § 4.

Figure 12: Gradient norms of each discriminator layer in MNIST at epoch 50.

Figure 13: Gradient norms of each discriminator layer in CIFAR10 at iteration 10000.
K.2 Additional Results

Figures 14 and 15 show the ratios of the gradient norms at each layer and the inverse ratios of the spectral norms in CIFAR10 and STL10. Generally, we see that the most of the points are near the diagonal line, which means that the assumption in Eq. (3) is reasonably true in practice. However, we note that the last layer (layer 8) somehow has slightly smaller gradient, as the points of “layer 8 / layer 1” are slightly lower than the diagonal line. This could result from the fact that layer 8 is a fully connected layer whereas all other layers are convolutional layers. We defer the more detailed analysis of this phenomenon to future work.

L Experimental Details and Additional Results on Vanishing Gradient

L.1 Experimental Details

The network architectures are shown in Tables 3 and 4. The dataset is CIFAR10. All experiments are run for 400k iterations. Batch size is 64. The optimizer is Adam. $\alpha_g = 0.0001, \alpha_d = 0.0001, \beta_1 = 0.5, \beta_2 = 0.999, n_{dis} = 1$. We use hinge loss [25].

L.2 Parameter Variance With and Without SN

Figures 16 and 17 show the parameter variance of each layer without and with SN. Note that Fig. 17 is just collecting the empirical lines in Fig. 3 for the ease of comparison here. Figures 18 and 19 show the gradient norm and inception score.

We can see that when training with SN, the parameter variance is stable throughout training (Fig. 17), and the magnitude of gradient is also stable (Fig. 18). However, when training without SN, the parameter variance tends to increase throughout training (Fig. 16), which causes a quick decrease in the magnitude of gradient in the beginning of training (Fig. 18) because of the saturation of hinge loss §4. Because SN promotes the stability of the variance and gradient throughout training, we see that SN improves the sample quality significantly (Fig. 19).

L.3 Comparing Two Variants Spectral Norms

Figures 20 and 21 show the ratio between two versions of spectral norm [25, 8] throughout the training of the popular SN [25] and the strict SN [8]. $\|\tilde{w}\|_{sp}$ denotes the spectral norm of the expanded matrix $\|\tilde{w}\|_{sp}$ used in [8]. $\|w\|_{sp}$ denotes the spectral norm of reshaped matrix $\|\hat{w}\|_{sp}$ used in [25]. The theoretical lower and upper bound are calculated according to Corollary 1 in [38]. We can see that no matter in which architecture, $\|\tilde{w}\|_{sp}$ is usually strictly larger than $\|\hat{w}\|_{sp}$. Note that the reason why in some cases the ratio exceeds the upper bound in Fig. 20 is because the spectral norms are calculated using power iteration [25, 8] which has approximation error.
L.4 Parameter Variance of Scaled SN

Figure 22 shows the parameter variance of scaled SN for both SN versions [25, 8]. We can see that when scale=1.75, the product of parameter variances for SN\textsubscript{Conv} [8] is similar to the one of SN\textsubscript{w} [25]. Moreover, by comparing Fig. 22 and Fig. 6 we can see that when the products of variances of two SN variants are similar, the sample quality is also similar. This confirms the intuition from LeCun initialization [20] that the magnitude of variance plays an important role on the performance of neural network, and it should not be too large nor too small.

M Details on SN Variants

In Appendix E of [25], a variant of SN is introduced. Instead of strictly setting the spectral norm of each layer, the idea of this approach is to release the constraint by multiplying each spectral normalized weights with a trainable parameter $\gamma$. However, this would make the gradient of discriminator arbitrarily large, which violates the original motivation of SN. Therefore, the approach incorporates gradient penalty [13] for setting the Lipschitz constant of discriminator to 1. The gradient penalty weights are set to 10 in all experiments.

However, from the description in [25], it is unclear if all layers have the same or separated $\gamma$. Therefore, we try both versions in our experiments. “Same $\gamma$” denotes that version where all layers share the same $\gamma$. “Diff. $\gamma$” denotes the version where each layer has a separate $\gamma$.

N Experimental Details and Additional Results on CIFAR10

N.1 Experimental Details

The network architectures are shown in Tables 3 and 4. All experiments are run for 400k iterations. Batch size is 64. The optimizer is Adam. We use the five hyper-parameter settings listed in Table 7. We use hinge loss with the popular SN implementation [25].
Figure 20: The ratio of two spectral norms throughout the training of the popular SN [25] in CIFAR10.

Figure 21: The ratio of two spectral norms throughout the training of the strict SN [8] in CIFAR10.

Figure 22: The parameter variance of scaled SN in CIFAR10.

N.2 FID Plot

Fig. 23 shows the FID score in CIFAR10 dataset. We can see that BSN has the best performance in all 5 hyper-parameter settings.

N.3 Training Curves

From § 5.1 we can see that SN (no $\gamma$) and BSN generally have the best performance. Therefore, in this section, we focus on comparing these two algorithms with the training curves. Figs. 7 and 24 to 32 show the inception score and FID of these two algorithms during training. Generally, we see that BSN converges slower than SN at the beginning of training. However, as training proceeds, the sample quality of SN often drops (e.g. Figs. 7 and 24 to 30), whereas the sample quality of BSN always increases and then stabilizes at the high level. In most cases, BSN not only outperforms SN at the end of training, but also outperforms the peak sample quality of SN during training (e.g. Figs. 7 and 24 to 30). From these results, we can conclude that BSN improves both the sample quality and training stability over SN.

| $\alpha_g$ | $\alpha_d$ | $\beta_1$ | $\beta_2$ | $n_{dis}$ |
|------------|------------|-----------|-----------|----------|
| 0.0001     | 0.0001     | 0.5       | 0.9       | 5        |
| 0.0001     | 0.0001     | 0.5       | 0.999     | 1        |
| 0.0002     | 0.0002     | 0.5       | 0.999     | 1        |
| 0.0001     | 0.0002     | 0.5       | 0.999     | 1        |
| 0.0002     | 0.0001     | 0.5       | 0.999     | 1        |

Table 7: Hyper-parameters tested in CIFAR10 and STL10 experiments. The first three settings are from [25, 13, 40, 32]. $\alpha_g$ and $\alpha_d$: learning rates for generator and discriminator. $\beta_1, \beta_2$: momentum parameters in Adam. $n_{dis}$: number of discriminator updates per generator update.
Figure 23: FID in CIFAR10. The results are averaged over 5 random seeds.

Figure 24: FID in CIFAR10. The results are averaged over 5 random seeds. The hyper-parameters are: \( \alpha_g = 0.0001 \), \( \alpha_d = 0.0001 \), \( n_{dis} = 1 \).

**N.4 Generated Images**

Figs. 33 to 36 show the generated images from the run with the best inception score for each algorithm.
Figure 25: Inception score in CIFAR10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0002$, $n_{dis} = 1$.

Figure 26: FID in CIFAR10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0002$, $n_{dis} = 1$.

Figure 27: Inception score in CIFAR10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0002$, $\alpha_d = 0.0001$, $n_{dis} = 1$.

Figure 28: FID in CIFAR10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0002$, $\alpha_d = 0.0001$, $n_{dis} = 1$.

Figure 29: Inception score in CIFAR10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0002$, $\alpha_d = 0.0002$, $n_{dis} = 1$.

Figure 30: FID in CIFAR10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0002$, $\alpha_d = 0.0002$, $n_{dis} = 1$. 
Figure 31: Inception score in CIFAR10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 5$.

Figure 32: FID in CIFAR10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 5$.

Figure 33: Generated samples from the best run of SN (same $\gamma$) in CIFAR10. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 5$. Inception score is 6.64. FID is 41.01.
Figure 34: Generated samples from the best run of SN (diff. $\gamma$) in CIFAR10. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 5$. Inception score is 6.55. FID is 41.18.

Figure 35: Generated samples from the best run of SN (no $\gamma$) in CIFAR10. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0002$, $n_{dis} = 1$. Inception score is 7.56. FID is 28.64.
Figure 36: Generated samples from the best run of BSN in CIFAR10. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0002$, $n_{dis} = 1$. Inception score is 7.70. FID is 25.96.
Figure 37: FID in STL10. The results are averaged over 5 random seeds.

Figure 38: Inception score in STL10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 1$.

Figure 39: FID in STL10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 1$.

O Experimental Details and Additional Results on STL10

O.1 Experimental Details

The network architectures are shown in Tables 3 and 4. Batch size is 64. The optimizer is Adam. We use the five hyper-parameter settings listed in Table 7. We use hinge loss with the popular SN implementation [25].

SN (no $\gamma$) and BSN under $n_{dis} = 1$ settings are run for 800k iterations as we observe that they need longer time to converge. All other experiments are run for 400k iterations.

O.2 FID Plot

Fig. 37 shows the FID score in STL10 dataset. We can see that BSN has the best or competitive performance in most of the hyper-parameter settings. Again, the only exception is $n_{dis} = 5$ setting.

O.3 Training Curves

From § 5.1 we can see that SN (no $\gamma$) and BSN generally have the best performance. Therefore, in this section, we focus on comparing these two algorithms with the training curves. Figs. 38 to 47 show the inception score and FID of these two algorithms during training. Generally, we see that BSN converges slower than SN at the beginning of training. However, as training proceeds, BSN finally has better metrics in most cases. Note that unlike CIFAR10, SN seems to be more stable in STL10 as its sample quality does not drop in most hyper-parameters. But the key conclusion is the same: in most cases, BSN not only outperforms SN at the end of training, but also outperforms the peak sample quality of SN during training (e.g. Figs. 38 to 45). The only exception is the $n_{dis} = 5$ setting, where both SN and BSN has instability issue: the sample quality first improves and then significantly drops. The problem with BSN seems to be severer. We discussed about this problem in § 5.1.
O.4 Generated Images

Figs. 48 to 51 show the generated images from the run with the best inception score for each algorithm.

Figure 40: Inception score in STL10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0002$, $n_{dis} = 1$.

Figure 41: FID in STL10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0002$, $n_{dis} = 1$.

Figure 42: Inception score in STL10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0002$, $\alpha_d = 0.0001$, $n_{dis} = 1$.

Figure 43: FID in STL10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0002$, $\alpha_d = 0.0001$, $n_{dis} = 1$. 
Figure 44: Inception score in STL10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0002$, $\alpha_d = 0.0002$, $n_{dis} = 1$.

Figure 45: FID in STL10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0002$, $\alpha_d = 0.0002$, $n_{dis} = 1$.

Figure 46: Inception score in STL10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 5$.

Figure 47: FID in STL10. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 5$.

Figure 48: Generated samples from the best run of SN (same $\gamma$) in STL10. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 5$. Inception score is 8.96. FID is 53.94.
Figure 49: Generated samples from the best run of SN (diff. $\gamma$) in STL10. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 5$. Inception score is 8.88. FID is 56.14.

Figure 50: Generated samples from the best run of SN (no $\gamma$) in STL10. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0002$, $n_{dis} = 1$. Inception score is 9.26. FID is 44.38.
Figure 51: Generated samples from the best run of BSN in STL10. The hyper-parameters are: $\alpha_g = 0.0001, \alpha_d = 0.0002, n_{dis} = 1$. Inception score is 9.46. FID is 42.78.
P Experimental Details and Additional Results on CelebA

P.1 Experimental Details

The network architectures are shown in Tables 3 and 4. All experiments are run for 400k iterations. Batch size is 64. The optimizer is Adam. We use the five hyper-parameter settings listed in Table 7. We use hinge loss with the popular SN implementation [25].

P.2 FID Plot

Fig. 52 shows the FID score in CelebA dataset. We can see that BSN outperforms the standard SN in all 5 hyper-parameter settings.

P.3 Training Curves

From § 5.1 we can see that SN (no $\gamma$) and BSN generally have the best performance. Therefore, in this section, we focus on comparing these two algorithms with the training curves. Figs. 53 to 57 show the FID of these two algorithms during training. Generally, we see that BSN converges slower than SN at the beginning of training. However, as training proceeds, BSN finally has better metrics in all cases. Note that unlike CIFAR10, SN seems to be more stable in CelebA as its sample quality does not drop in most hyper-parameters. But the key conclusion is the same: in most cases, BSN not only outperforms SN at the end of training, but also outperforms the peak sample quality of SN during training (e.g. Figs. 53 to 56). The only exception is the $n_{dis} = 5$ setting, where both SN and BSN has instability issue: the sample quality first improves and then significantly drops. But even in this case, BSN has better final performance than the standard SN.

P.4 Generated Images

Figs. 58 to 61 show the generated images from the run with the best FID for each algorithm.
Figure 54: FID in CelebA. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0002$, $n_{dis} = 1$.

Figure 55: FID in CelebA. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0002$, $\alpha_d = 0.0001$, $n_{dis} = 1$.

Figure 56: FID in CelebA. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0002$, $\alpha_d = 0.0002$, $n_{dis} = 1$.

Figure 57: FID in CelebA. The results are averaged over 5 random seeds. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 5$. 
Figure 58: Generated samples from the best run of SN (same $\gamma$) in CelebA. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 5$. FID is 7.40.

Figure 59: Generated samples from the best run of SN (diff. $\gamma$) in CelebA. The hyper-parameters are: $\alpha_g = 0.0001$, $\alpha_d = 0.0001$, $n_{dis} = 5$. FID is 7.29.
Figure 60: Generated samples from the best run of SN (no $\gamma$) in CelebA. The hyper-parameters are: $\alpha_g = 0.0002$, $\alpha_d = 0.0001$, $n_{dis} = 1$. FID is 8.34.

Figure 61: Generated samples from the best run of BSN in CelebA. The hyper-parameters are: $\alpha_g = 0.0002$, $\alpha_d = 0.0001$, $n_{dis} = 1$. FID is 8.06.
\[ z \in \mathbb{R}^{128} \sim \mathcal{N}(0, I) \]

- Fully connected (4 \times 4 \times 1024).
- ResNet-up (c = 1024).
- ResNet-up (c = 512).
- ResNet-up (c = 256).
- ResNet-up (c = 128).
- ResNet-up (c = 64).

\[ \text{BN. ReLU. Convolution (c = 3, k = 3, s = 1). Tanh} \]

Table 8: Generator network architectures for ILSVRC2012 experiments (from [25]). BN stands for batch normalization. \( c \) stands for number of channels. \( k \) stands for kernel size. \( s \) stands for stride.

\[ \text{Direct connection} \]

- BN. ReLU. Unpooling(2). Convolution (k = 3, s = 1).
- BN. ReLU. Convolution (k = 3, s = 1).

\[ \text{Shortcut connection} \]

- Unpooling(2). Convolution (k = 1, s = 1).

Table 9: ResNet-up network architectures for ILSVRC2012 experiments (from [25]). BN stands for batch normalization. \( k \) stands for kernel size. \( s \) stands for stride.

**Q Experimental Details and Additional Results on ILSVRC2012**

**Q.1 Experimental Details**

The network architectures are shown in Table 8 to 13. All experiments are run for 500k iterations. Discriminator batch size is 16. Generator batch size is 32. The optimizer is Adam. \( \alpha_g = 0.002, \alpha_d = 0.002, \beta_1 = 0.9, \beta_2 = 0.9, n_{dis} = 5 \) We use hinge loss with the popular SN implementation [25].

**Q.2 Training Curves**

Figs. 62 and 63 show the inception score and FID of SN and BSN during training. For SN, we can see that the runs with scale=1.0/1.2/1.4 have similar performance throughout training. When scale=1.6, the performance is much worse. For BSN, the runs with scale=1.2/1.4 perform better than SN runs throughout the training. When scale=1.6, BSN has similar performance as SN at the early stage of training, and is slightly better at the end. When scale=1.0, the performance is very bad as there is gradient vanishing problem.

**Q.3 Generated Images**

Figs. 64 to 71 show the generated images from the run with the best inception score for SN and BSN with different scale parameters.

\[ x \in \mathbb{R}^{128 \times 128 \times 3} \]

- ResNet-first (c = 64).
- ResNet-down (c = 128).
- ResNet-down (c = 256).
- ResNet-down (c = 512).
- ResNet-down (c = 1024).
- ResNet (c = 1024).

\[ \text{ReLU. Global pooling. Fully connected (1).} \]

Table 10: Discriminator network architectures for ILSVRC2012 experiments (from [25]). BN stands for batch normalization. \( c \) stands for number of channels. \( k \) stands for kernel size. \( s \) stands for stride.
Table 11: ResNet-down network architectures for ILSVRC2012 experiments (from [25]). \( k \) stands for kernel size. \( s \) stands for stride.

Table 12: ResNet-first network architectures for ILSVRC2012 experiments (from [25]). \( k \) stands for kernel size. \( s \) stands for stride.

Table 13: ResNet network architectures for ILSVRC2012 experiments (from [25]). \( k \) stands for kernel size. \( s \) stands for stride.

Figure 62: Inception score in ILSVRC2012. The results are averaged over 5 random seeds.

Figure 63: FID in ILSVRC2012. The results are averaged over 5 random seeds.
Figure 64: Generated samples from the best run of SN (scale=1.0) in ILSVRC2012. Inception score is 13.50. FID is 72.18.

Figure 65: Generated samples from the best run of SN (scale=1.2) in ILSVRC2012. Inception score is 13.04. FID is 72.51.
Figure 66: Generated samples from the best run of SN (scale=1.4) in ILSVRC2012. Inception score is 13.04. FID is 69.12.

Figure 67: Generated samples from the best run of SN (scale=1.6) in ILSVRC2012. Inception score is 12.62. FID is 70.36.
Figure 68: Generated samples from the best run of BSN (scale=1.0) in ILSVRC2012. Inception score is 2.07. FID is 242.51.

Figure 69: Generated samples from the best run of BSN (scale=1.2) in ILSVRC2012. Inception score is 13.55. FID is 71.30.
Figure 70: Generated samples from the best run of BSN (scale=1.4) in ILSVRC2012. Inception score is 13.63. FID is 70.88.

Figure 71: Generated samples from the best run of BSN (scale=1.6) in ILSVRC2012. Inception score is 13.24. FID is 69.06.