EXPlicit Additive Decomposition of NORMS ON $\mathbb{R}^2$

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Abstract. A well-known result by Lindenstrauss is that any two-dimensional normed space can be isometrically imbedded into $L_1(0,1)$. We provide an explicit form of such an imbedding. The proof is elementary and self-contained. Applications are given concerning the following: (i) explicit representations of the moments of the norm of a random vector $X$ in terms of the characteristic function and the Fourier–Laplace transform of the distribution of $X$; (ii) an explicit and partially improved form of the exact version of the Littlewood–Khinchin–Kahane inequality obtained by Latała and Oleszkiewicz; (iii) an extension of an inequality by Buja–Logan–Reeds–Shepp, arising from a statistical problem.

1. Main result and discussion

Let $V$ be a vector space over $\mathbb{R}$ endowed with a norm $\| \cdot \|$, with the dual space $V^*$. For any $x \in V$ and $\ell \in V^*$, let $x\ell$ denote the value of the linear functional $\ell$ at $x$. Let us say that the norm $\| \cdot \|$ admits an additive decomposition if there exists a Borel measure $\mu$ on $V^*$ such that

$$\|x\| = \int_{V^*} |x\ell| \, d\mu(\ell) \quad \text{for all} \quad x \in V.$$  

Clearly, such a decomposition exists if $V$ is one-dimensional. In this note, an explicit decomposition of the form (1.1) will be given in the case when $V$ is two-dimensional.

It is well known that, in general, there is no such decomposition if the dimension of $V$ is greater than 2; cf. Remark 1.3 in the present note.

To state our main result, let us recall some basic facts about convex functions. Suppose that a function $f: \mathbb{R} \to \mathbb{R}$ is convex. Then $f$ is continuous and has finite nondecreasing right and left derivatives $f'_+$ and $f'_-$, which are right- and left-continuous, respectively. Moreover, the function $f' := (f'_+ + f'_-) / 2$ is non-decreasing as well. The Lebesgue–Stieltjes integral $\int_{\mathbb{R}} \varphi(t) \, df'(t)$ is the Lebesgue integral $\int_{\mathbb{R}} \varphi \, d\nu$, where $\nu$ is the Borel measure determined by the condition that $\nu((a, b]) = f'_+(b) - f'_-(a)$ for all real $a$ and $b$ such that $a < b$. The latter condition is equivalent to each of the following conditions: (i) $\nu([a, b)) = f'_-(b) - f'_-(a)$ for all real $a$ and $b$ such that $a < b$ and (ii) $\nu([^a,b)) + \nu([a,b]) = 2(f'(b) - f'(a))$ for all real $a$ and $b$ such that $a < b$.

Now we are ready to state the following explicit additive decomposition of an arbitrary norm in the case when $V = \mathbb{R}^2$.

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Theorem 1.1. Let \( \| \cdot \| \) be any norm on \( \mathbb{R}^2 \). Let \( N(u) := \|(u, 1)\| \) for all real \( u \). Then the function \( N \) is convex, the limit
\[
(1.2) \quad c := \lim_{u \to -\infty} \left( \|(u, 1)\| - 2\|(u, 0)\| + \|(u, -1)\| \right)
\]
exists and is finite and nonnegative, and
\[
(1.3) \quad \|(u, v)\| = \frac{c|v|}{2} + \frac{1}{2} \int_{\mathbb{R}} |u - tv| \, dN'(t) \quad \text{for all } (u, v) \in \mathbb{R}^2.
\]
Obviously, (1.3) is an explicit additive decomposition of the form (1.1), with \( \| \cdot \| \) defined by the condition that \( 2f \) is the limit of the sequence \( \frac{1}{2} \int_{\mathbb{R}} |u - tv| \, dN'(t) \). In these cases, the constant \( c \) in (1.3) is 0. A simple example is the case of the \( l^p \) norm on \( \mathbb{R}^2 \) with \( p > 1 \), are
\[
\|(u, v)\|_p = (|u|^p + |v|^p)^{1/p} = \frac{p-1}{2} \int_{\mathbb{R}} |u - tv| \, |t|^{p-2} (|t|^p + 1)^{1/p-2} \, dt
\]
when \( p \in (1, \infty) \) and
\[
\|(u, v)\|_\infty = \max(|u|, |v|) = \frac{1}{2} (|u + v| + |u - v|)
\]
for all \((u, v) \in \mathbb{R}^2\). In these cases, the constant \( c \) in (1.3) is 0. A simple case with a nonzero \( c \) is given by the formula \( \| (u, v) \| = \| (u, v) \|_1 = |u| + |v| \) for \((u, v) \in \mathbb{R}^2\), with \( c = 2 \).

The proof of Theorem 1.1 relies on

Lemma 1.2. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a convex function such that for some real \( k \) there exist finite limits
\[
(1.4) \quad d_+ := d_{f, k^+} := \lim_{u \to \infty} [f(u) - ku] \quad \text{and} \quad d_- := d_{f, k^-} := \lim_{u \to -\infty} [f(u) + ku].
\]
Then for all \( u \in \mathbb{R} \)
\[
(1.5) \quad f(u) = \frac{d_+ + d_-}{2} + \frac{1}{2} \int_{\mathbb{R}} |u - t| \, df'(t).
\]

Proof of Lemma 1.2. Since the function \( f' \) is nondecreasing, there exist limits \( k_+ := \lim_{x \to -\infty} f'(x) \in [-\infty, \infty] \). Moreover, for any real \( u > 0 \) one has \( f(u) - ku = f(0) + \int_0^u (f'(t) - k) \, dt \), which converges to a finite limit (as \( u \to \infty \)) only if \( k_+ = k \). Similarly, \( k_- = -k \). So, in view of (1.4), for any real \( u \)
\[
f(u) + ku = d_- + \int_{-\infty}^u (f'(z) + k) \, dz = d_- + \int_{-\infty}^u dz \int_{-\infty}^z df'(t)
\]
\[
= d_- + \int_{-\infty}^u df'(t) \int_t^u dz
\]
\[
= d_- + \int_{-\infty}^u \max(0, u - t) \, df'(t),
\]
so that
\[
f(u) + ku = d_- + \int_{-\infty}^u \max(0, u - t) \, df'(t).
\]
Similarly, \( f(u) - ku = d_+ + \int_\mathbb{R} \max(0, t - u) \, df'(t) \) for any real \( u \). Adding the last two identities, one obtains (1.5). \( \square \)
Proof of Theorem 1.1. That the function $N$ is convex follows immediately from the convexity of the norm. Note next that the limits $d_{f,k;\pm}$ in (1.4) exist in $[-\infty, \infty]$ for any convex function $f: \mathbb{R} \to \mathbb{R}$ and any real $k$. On the other hand, for all real $u$

\[ |N(u) - |u||1,0)|| = \|[(u,1)] - \|[(u,0)||| \leq \|(0,1)||, \]

by the norm inequality. So, the limits $d_{\pm} = d_{f,k;\pm}$ in (1.6) exist and are finite for $f = N$ and

\[ k = \|(1,0)||. \]

Therefore, by Lemma 1.2 holds with $f = N$ and $d_{\pm} = d_{N,\|(1,0)||;\pm}$. It follows that, with these $d_{\pm}$,

\[ 2\|\(u,v)\| = 2|v|\|[(u/v,1)]\| = 2|v|N(u/v) = (d_+ + d_-)|v| + \int_{\mathbb{R}} |u - tv| dN'(t) \]

for all real $u$ and all real $v \neq 0$. The last expression in (1.7) is continuous in $v \in \mathbb{R}$ by dominated convergence – because, by (1.7) with $(u,v) = (0,1)$, one has \[ \int_{\mathbb{R}} |t| dN'(t) = 2\|\(0,1)\|| - (d_+ + d_-) < \infty. \]

Thus, one has (1.3) – with $d_+ + d_-$ in place of $c$ – for all $(u,v) \in \mathbb{R}^2$.

Moreover,

\[ d_+ + d_- = \lim_{u \to \infty} (N(u) - ku + N(-u) - ku) \]

\[ = \lim_{u \to \infty} (\|(u,1)|| - 2\|(1,0)||u + \|(-u,1)||) \]

\[ = \lim_{u \to \infty} (\|(u,1)|| - 2\|(u,0)|| + \|(u,-1)||) = c \geq 0, \]

by (1.2) and, again, the convexity of the norm.

This completes the proof of Theorem 1.1. \qed

From the proofs of Theorem 1.1 and Lemma 1.2, it follows that the nondecreasing function $N'$ tends to $\pm k$ as $x \to \pm \infty$, where $k$ is as in (1.6). So,

\[ F := \frac{1}{2} + \frac{1}{\mathcal{N}} N' \]

is a cumulative probability distribution function (cdf), regularized in the sense that $2F(u) = F(u+) + F(u-) \forall$ all real $u$. Let the function $F^{-1}: (0,1) \to \mathbb{R}$ be (the smallest, left-continuous generalized inverse to $F$,) defined by the condition

\[ F^{-1}(s) = \inf\{ u \in \mathbb{R} : F(u) \geq s \} \quad \text{for} \quad s \in (0,1). \]

A well-known fact is that, if $S$ is a random variable (r.v.) uniformly distributed on the interval $(0,1)$, then the regularized cdf of the r.v. $F^{-1}(S)$ is $F$. So, (1.3) can be rewritten as

\[ \|(u,v)|| = \int_0^1 |u\xi(s) + v\eta(s)||ds \]

for all $(u,v) \in \mathbb{R}^2$, where

\[ \xi(s) := \begin{cases} 1 & \text{if } 0 < s < \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \leq s < 1, \end{cases} \quad \eta(s) := \begin{cases} -F^{-1}(2s) & \text{if } 0 < s < \frac{1}{2}, \\ c & \text{if } \frac{1}{2} \leq s < 1. \end{cases} \]

Thus, the mapping $(u,v) \mapsto u\xi + v\eta$ is a linear isometric imbedding of $\mathbb{R}^2$ (endowed with the arbitrary norm $\| \cdot \|$) into $L_1(0,1)$.

That any two-dimensional normed space is isometric to a subspace of $L_1(0,1)$ was shown by Lindenstrauss [9, Corollary 2]. In distinction from that result, the
imbedding into \(L_1(0,1)\) given by formulas (1.8)–(1.9) is quite explicit. Another difference is that our method is elementary and the proof is self-contained. (Also, formula (1.3) is simpler than, and therefore in some situations may be preferable to, (1.8)–(1.9).) On the other hand, the study [9] contains a number of results that are more general than Corollary 2 therein.

It is well known that any Euclidean space is linearly isometric to a subspace of \(L_1\). Indeed, for all \(x \in \mathbb{R}^d\)

(1.10) \[ \|x\|_2 := \sqrt{x \cdot x} = \sqrt{\frac{\pi}{2} \int_{\mathbb{R}^d} |x \cdot t| \gamma_d(dt)}, \]

where \(\gamma_d\) is the standard Gaussian measure on \(\mathbb{R}^d\) and \(\cdot\) denotes the standard inner product on \(\mathbb{R}^d\). In place of \(\gamma_d\), one can similarly use any other spherically invariant measure \(\nu\) on \(\mathbb{R}^d\) such that \(\int_{\mathbb{R}^d} |x \cdot t| \nu(dt) \in (0, \infty)\) for some or, equivalently, any nonzero vector \(x\) in \(\mathbb{R}^d\).

Using the imbedding-into-\(L_1\) formulas (1.10) and (1.8), it is straightforward to verify Hlawka’s inequality

\[ \|x + y + z\| + \|x\| + \|y\| + \|z\| \geq \|x + y\| + \|y + z\| + \|z + x\| \]

for all \(x, y, z\) in \(V\) when either the norm \(\| \cdot \|\) on \(V\) is Euclidean or \(V\) is two-dimensional. Another way to show that Hlawka’s inequality holds for any two-dimensional normed space was presented in [6].

Remark 1.3. In general, a normed space \(V\) of any given dimension greater than 2 is not linearly isometric to a subspace of \(L_1\). Indeed, otherwise Hlawka’s inequality would hold for all \(x, y, z\) in \(\mathbb{R}^3\). However, as pointed out e.g. in [4], Hlawka’s inequality fails to hold for some \(x, y, z\) in \(\mathbb{R}^3\) endowed with the supremum norm.

2. Applications

First here, one has the following representation of the expected norm of a random vector \(X\) in \(V\) in terms of the characteristic function (c.f.) \(V^* \ni \ell \mapsto E e^{it\ell X}\) of \(X\).

Corollary 2.1. If the additive decomposition (1.1) holds and \(X\) is any random vector in \(V\), then

(2.1) \[ E \|X\| = \frac{2}{\pi} \int_{V^* \times (0,\infty)} \frac{1 - \Re E e^{it\ell X}}{t^2} \mu(dt) \]

This follows immediately from (say) [10] Corollary 2]. A similar representation of \(E \|X\|\) in terms of the Fourier–Laplace transform of the distribution of \(X\) can be just as easily obtained based on [10] Theorem 1. In view of (1.3) and (1.10), these representations of \(E \|X\|\) are quite explicit if \(V\) two-dimensional or Euclidean.

Removing both instances of the expectation from (2.1) (that is, replacing \(X\) there with a nonrandom vector \(x \in V\)), then raising both sides to the \(j\)th power, and finally reapplying the expectation, one obtains

(2.2) \[ E \|X\|^j = \left( \frac{2}{\pi} \right)^j \int_{V^* \times (0,\infty)} \left( E \prod_{\alpha=1}^j \left( 1 - \Re e^{it_{\alpha} X_{\alpha}} \right) \right)^j \frac{\mu(dt_{\alpha}) dt_{\alpha}}{t_{\alpha}^2} \]
for any natural \( j \). Note that the expectation in (2.2) can be easily expressed in terms of the c.f. of \( X \), namely, as

\[
\sum_{(A,B)} \left( -\frac{1}{2} \right)^{|A \cup B|} \mathbb{E} \exp \left\{ iX \left( \sum_{\alpha \in A} t_{\alpha} \ell_{\alpha} - \sum_{\beta \in B} t_{\beta} \ell_{\beta} \right) \right\},
\]

where \( \sum_{(A,B)} \) denotes the summation over all ordered pairs \((A, B)\) of disjoint subsets of the set \( \{1, \ldots, j\} \) and \(|A \cup B|\) denotes the cardinality of the set \( A \cup B \).

Because of the multiplicativity property of the c.f. with respect to the convolution, representations such as (2.1) and (2.2) may be especially useful when \( X \) is the sum of independent random vectors.

Suppose now that \( x_1, \ldots, x_n \) are any vectors in any normed space \( V \) and \( \varepsilon_1, \ldots, \varepsilon_n \) are independent Rademacher r.v.’s, so that \( \mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2} \) for all \( i \). Latała and Oleszkiewicz [7] made an elementary but very ingenious argument to show that

\[
2 \left( \mathbb{E} \left\| \sum_{i} \varepsilon_i x_i \right\| \right)^2 \geq \mathbb{E} \left\| \sum_{i} \varepsilon_i x_i \right\|^2.
\]

Previously, Szarek [11] obtained this result in the case \( V = \mathbb{R} \), which had been a long-standing conjecture of Littlewood; see e.g. [5]. The constant factor 2 in (2.4) is the best possible, even for \( V = \mathbb{R} \) (take \( n = 2 \) and \( x_1 = x_2 \neq 0 \)).

In the case when the norm admits an additive decomposition, the lower bound \( \mathbb{E} \left\| \sum_{i} \varepsilon_i x_i \right\|^2 \) in (2.4) can be improved:

**Corollary 2.2.** If decomposition (1.1) holds, then

\[
2 \left( \mathbb{E} \left\| \sum_{i} \varepsilon_i x_i \right\| \right)^2 \geq \left( \int_{V^*} \sqrt{\sum_{\ell}(x_{i\ell})^2} \mu(d\ell) \right)^2.
\]

This follows immediately from (1.1) and Szarek’s result. Moreover, if (1.1) holds, then one can rewrite the lower bounds in (2.4) and (2.5) respectively as

\[
\int_{V^* \times V^*} \mathbb{E} \left| \sum_{i} \varepsilon_i x_i \right| \left| \sum_{j} \varepsilon_j x_j \right| \mu(d\ell) \mu(dm) \quad \text{and} \quad \int_{V^* \times V^*} \sqrt{\sum_{\ell}(x_{i\ell})^2 \sum_{m}(x_{j\ell})^2} \mu(d\ell) \mu(dm).
\]

So, by the Cauchy–Schwarz inequality, the lower bound in (2.5) is no less than that in (2.4). Moreover, the bound in (2.5) may be of simpler structure and easier to compute (without having to use the expectation), especially in the two-dimensional case, when one has the explicit additive decomposition (1.3) of the norm. On the other hand, (2.4) holds for any norm and any dimension.

In conclusion, consider the inequality

\[
\mathbb{E} \|X - Y\|_2 \leq \mathbb{E} \|X + Y\|_2,
\]

where \( X \) and \( Y \) are independent identically distributed (iid) random vectors in \( \mathbb{R}^d \) and \( \| \cdot \|_2 \) is the Euclidean norm, as in (1.10). This inequality was obtained in [3]. As noted in [8], in the case \( d = 1 \), (2.6) follows immediately from the identity

\[
\mathbb{E} |X + Y| = \mathbb{E} |X - Y| + 2 \int_0^\infty \mathbb{P}(X > r) - \mathbb{P}(X < -r) r^2 dr.
\]

Now the \( L_1 \)-imbedding formula (1.8) immediately yields
Corollary 2.3. For any two-dimensional normed space $V$ and any iid random vectors $X$ and $Y$ in $V$,

$$\mathbb{E} \|X - Y\| \leq \mathbb{E} \|X + Y\|. \tag{2.7}$$

As shown by Johnson [2], for each natural $d \geq 3$ inequality (2.7) fails to hold for $V = \mathbb{R}^d$ in general. Indeed, define the norm on $\mathbb{R}^d$ by the formula

$$\|x\| := \max\{|x_i| \vee |x_i - x_j| : i, j = 1, \ldots, d\}$$

for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Let $(e_1, \ldots, e_d)$ be the standard basis of $\mathbb{R}^d$. Let $X$ and $Y$ be iid random vectors in $\mathbb{R}^d$. For any natural $d \geq 4$, suppose that the random vector $X$ is such that $P(X = e_i) = \frac{1}{d}$ for each $i = 1, \ldots, d$. Then $\mathbb{E} \|X - Y\| = 2\frac{d-1}{d} > \frac{d-1}{d} = \mathbb{E} \|X + Y\|$. In the remaining case when $d = 3$, suppose that the random vector $X$ is such that $P(X = e_1) = P(X = e_2) = P(X = e_3) = \frac{1}{3}$, $P(X = -\frac{1}{2}(e_1 + e_2 + e_3)) = \frac{1}{3}$. Then $\mathbb{E} \|X - Y\| = \frac{21}{16} > \frac{19}{16} = \mathbb{E} \|X + Y\|$. 

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