Self-similar Gaussian Markov processes

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Abstract
We define a two-parameter family of Gaussian Markov processes, which includes Brownian motion as a special case. Our main result is that any centered self-similar Gaussian Markov process is a constant multiple of a process from this family. This yields short and easy proofs of some non-Markovianity results concerning variants of fractional Brownian motion (most of which are known). In the proof of our main theorem, we use some properties of additive functions, i.e. solutions of Cauchy’s functional equation. In an appendix, we show that a certain self-similar Gaussian process with asymptotically stationary increments is not a semimartingale.

1 Self-similar Gaussian Markov processes
For $H > 0$ and $c \leq -H$, the process
\[X_{t}^{H,c} := t^{2H+c}W(t^{-2H-2c}), \quad t \geq 0,\] (1.1)
where $W(\cdot)$ is a Brownian motion, is clearly a Gaussian Markov process. By Brownian scaling, it is easy to check that $X_{t}^{H,c}$ is $H$-self-similar. The main result of the present note is that this family of processes, augmented by the limiting case $c \to -\infty$, is not just a subset but equal to the class of self-similar centered Gaussian Markov processes (up to multiplication by constants). For the notion of self-similarity, we use the same definition as \cite{7} (pp. 1–3).

Definition 1.1. A stochastic process $(X_{t})_{t \geq 0}$ is $H$-self-similar with exponent $H > 0$, if it is stochastically continuous at zero, and for any $a > 0$ the process $(a^{H}X_{at})_{t \geq 0}$ has the same law as $(a^{H}X_{t})_{t \geq 0}$.

Self-similarity implies that $X_{0} = 0$ (see p. 2 in \cite{7}). In particular, the covariance function of a self-similar process satisfies $R(s,t) = 0$ for $s \wedge t = 0$.

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Definition 1.2. A symmetric function \( R : [0, \infty)^2 \to \mathbb{R} \) is positive definite if
\[
\sum_{k,l=1}^{d} a_k a_l R(t_k, t_l) \geq 0 \quad (1.2)
\]
for all \( d \geq 1 \), \( t_1, \ldots, t_d \geq 0 \) and \( a_1, \ldots, a_d \in \mathbb{R} \).

It is well known that positive definite functions are exactly the functions that occur as covariance functions of centered Gaussian processes \( (X_t)_{t \geq 0} \). The covariance function of \( X^{H,c} \) is
\[
R_{H,c}(s, t) := \begin{cases} \left(s \wedge t \right)^{2H+c}(s \wedge t)^{-c} & s \wedge t > 0, \\ 0 & s \wedge t = 0. \end{cases} \quad (1.3)
\]
The pointwise limit
\[
R_{H,-\infty}(s, t) := \lim_{c \to -\infty} R_{H,c}(s, t) = \begin{cases} t^{2H} & s = t, \\ 0 & s \neq t. \end{cases} \quad (1.4)
\]
is positive definite as well, and defines a centered Gaussian process \( X^{H,-\infty} \), which is obviously also self-similar and Markov. Note that putting \( H = \frac{1}{2} \) and \( c = -1 \) in (1.3) yields the covariance function \( s \wedge t \) of Brownian motion, the prime example of a self-similar Gaussian Markov process. We can now state our main result.

Theorem 1.3. Let \( H > 0 \) and \( X = (X_t)_{t \geq 0} \) be a one-dimensional, centered \( H \)-self-similar Gaussian Markov process. Then the covariance function of \( X \) is of the form
\[
R(s, t) = R(1,1)R_{H,c}(s, t), \quad s, t \geq 0, \quad (1.5)
\]
where \( c \in [\infty, -H] \) and \( R(1,1) \geq 0 \). Thus, \( X \) equals \( R(1,1)^{1/2}X^{H,c} \) in distribution.

Since comparing a given covariance function with the simple explicit function \( R_{H,c} \) is typically very easy, it seems that Theorem 1.3 essentially settles the problem of deciding whether a self-similar Gaussian process defined by an explicit covariance function is Markovian. We present several examples below, and note that Corollary 1.10 can be used to prove non-Markovianity without reading anything else in our paper.

It is well known that a centered real Gaussian process \( (X_t)_{t \geq 0} \) is Markov if and only if its covariance function \( R(s, t) = \mathbb{E}[X_sX_t] \) satisfies
\[
R(s, u)R(t, t) = R(s, t)R(t, u), \quad 0 \leq s \leq t \leq u. \quad (1.6)
\]
This is due to Doob; we refer to [13] for details and references. To prove Theorem 1.3 we use some facts about additive functions. For a proof of the following classical result, and further references, we refer to [4] (Theorem 1.1.7).

Theorem 1.4 (Ostrowski 1929). Suppose that \( f : \mathbb{R} \to \mathbb{R} \) satisfies Cauchy’s functional equation
\[
f(x + y) = f(x) + f(y), \quad x, y \in \mathbb{R}.
\]
Then either \( f(x) = cx \) for some constant \( c \), or else \( f \) is unbounded above and below on any set of positive measure.
We will also need the following continuation result.

**Lemma 1.5.** Suppose that \( f : [0, \infty) \to \mathbb{R} \) satisfies Cauchy’s functional equation

\[
f(x + y) = f(x) + f(y), \quad x, y \geq 0.
\]

Then \( f \) has an extension to \( \mathbb{R} \) that satisfies the equation, too.

**Proof.** We define \( f \) for \( x < 0 \) by

\[
f(x) := f(x + y) - f(y),
\]

where \( y \geq |x| \) is arbitrary. This is well-defined, because for \( y_1 > y_2 \geq |x| \) we have

\[
f(x + y_2) - f(y_2) = f(x + y_2) + f(y_1 - y_2) - (f(y_2) + f(y_1 - y_2)) = f(x + y_1) - f(y_1).
\]

By definition, this extension solves the equation for \( x < 0 \) and \( y \geq |x| \). Now let \( x < 0 \) and \( y < |x| \). Fix \( z_1 \) and \( z_2 \) with \( z_1 \geq |x| \) and \( z_2 \geq |y| \). Then we compute

\[
f(x + y) = f(x + y + z_1 + z_2) - f(z_1 + z_2)
= f(x + z_1) + f(y + z_2) - (f(z_1) + f(z_2))
= f(x + z_1) - f(z_1) + f(y + z_2) - f(z_2) = f(x) + f(y).
\]

We can now prove Theorem 1.3. The proof is an extension of the proof that fractional Brownian motion is not a Markov process, which is found in [10] and [15] (Theorem 2.3).

**Proof of Theorem 1.3.** By self-similarity and symmetry of \( R(\cdot, \cdot) \), it suffices to determine \( R(\cdot, 1) \) on \((0, 1]\). Using (1.6) (with \( u = 1 \)) and self-similarity, we obtain

\[
R(s, 1)R(1, 1) = R(s/t, 1)R(t, 1), \quad 0 \leq s < t \leq 1.
\]

Putting \( s = t^2 \), it follows that

\[
R(t^2, 1)R(1, 1) = R(t, 1)^2, \quad 0 \leq t \leq 1.
\]

If \( R(1, 1) = 0 \), then this equation implies that \( R(\cdot, 1) \) vanishes, hence \( X \) is identically zero. If \( R(1, 1) \neq 0 \), then (1.8) implies that \( R(\cdot, 1) \) is either non-negative or non-positive on \([0, 1]\). In the latter case, by putting \( u = 1 \) in (1.2), we see that \( R(\cdot, 1) \equiv 0 \). We conclude that \( R(\cdot, 1) \) is non-negative, and may assume from now on that \( R(1, 1) > 0 \). Define

\[
g(x) := \frac{R(e^{-x}, 1)}{R(1, 1)}, \quad x \geq 0.
\]

Writing \( s = e^{-x-y} \) and \( t = e^{-y} \), we see from (1.7) that \( g \) solves the functional equation

\[
g(x + y) = g(x)g(y), \quad x, y \geq 0.
\]

In particular,

\[
g(z) = g(z/2)^2, \quad z \geq 0.
\]
Clearly, $g(0) = 1 > 0$. By (1.9), if $x_0$ is a zero of $g$, then $g$ vanishes on $[x_0, \infty)$. Suppose that there is no interval $[0, \varepsilon)$ with $\varepsilon > 0$ on which $g$ is positive. Then, $g(x) = 0$ for $x > 0$, and thus $R = R(1, 1)R_{H, -\infty}$, the white-noise-type covariance function defined in (1.4). If, on the other hand, there is such an interval $[0, \varepsilon)$, then (1.10) yields

$$g(z) = g(z/2^k)^{2^k}, \ k \in \mathbb{N},$$

and by taking $k$ sufficiently large we get $g(z) > 0$ for any $z \geq 0$. Therefore, $\varphi := \log g$ is well-defined on $[0, \infty)$, and satisfies Cauchy’s functional equation

$$\varphi(x + y) = \varphi(x) + \varphi(y), \ x \geq y \geq 0.$$

By symmetry of the equation, it clearly holds for $0 \leq x < y$ as well. By Lemma 1.5, we can extend the solution $\varphi$ to the whole real line, which makes Theorem 1.4 applicable. Suppose for contradiction that $\varphi$ is unbounded above and below on any set of positive measure. This would imply

$$\sup \{ R(t_1/t_2, 1) : 1 \leq t_1 < t_2 \leq 2 \} = \infty.$$ 

We can thus pick $1 \leq t_1 < t_2 \leq 2$ such that

$$R(t_1/t_2, 1) > 2^{2H} R(1, 1).$$

Then positive definiteness is violated for $a_1 = 1, a_2 = -1$:

$$a_1^2 R(t_1, t_1) + 2a_1 a_2 R(t_1, t_2) + a_2^2 R(t_2, t_2) =
\begin{align*}
t_1^{2H} R(1, 1) - 2t_1^{2H} R(t_1/t_2, 1) + t_2^{2H} R(1, 1) \\
< 2^{2H} R(1, 1) - 2^{2H+1} R(1, 1) + 2^{2H} R(1, 1) \leq 0.
\end{align*}$$

We deduce that the other possibility stated in Theorem 1.4 must hold, i.e. that there is a constant $c$ such that

$$\varphi(x) = cx, \ x \in \mathbb{R}.$$ 

By the definition of $\varphi$, we obtain

$$R(t, 1) = R(1, 1)t^{-c}, \ 0 < t \leq 1,$$

and, by self-similarity,

$$R(s, t) = (s \vee t)^{2H} R\left(\frac{s \wedge t}{s \vee t}, 1\right)
= R(1, 1)(s \vee t)^{2H+c}(s \wedge t)^{-c}, \ s, t > 0.$$ 

For $c > -H$, it is straightforward to check that $(s \vee t)^{2H+c}(s \wedge t)^{-c}$ does not satisfy the Cauchy-Schwarz inequality, hence it cannot be a covariance function. We conclude that $c \leq -H$, and that $R = R_{H,c}$. \(\square\)

An alternative argument for the case where $R$ is positive on $(0, \infty)^2$ is given in Appendix A.
Corollary 1.6. Consider covariance functions of the form
\[
R(s, t) = R(1, 1)(s \wedge t)^{2H l\left(\frac{|s-t|}{s \wedge t}\right)}, \quad s, t > 0,
\] (1.11)
where $H > 0$, $R(1, 1) > 0$, and $l : [0, \infty) \to \mathbb{R}$ satisfies $l(0) = 1$ and is not identically zero on $(0, \infty)$. Suppose that the associated centered Gaussian process, which is $H$-selfsimilar, is Markovian. Then, there exists $c \leq -H$ such that for all $\alpha \in (0, 1)$
\[
l(t^{-\alpha} - 1) = l^{2H+c-\alpha(2H+c)}, \quad t \geq 1.
\] (1.12)

Proof. By Theorem 1.3, we know that $R$ is of the form (1.5), for some $c \leq -H$. By our assumption on $l$, the number $c$ is finite. For $0 < s \leq t$, this implies
\[
t^{2H+c-s} = s^{2H l\left(\frac{t-s}{s}\right)}.
\]
Now put $s = t^{\alpha}$.

In each of the following examples, we will see that (1.12) cannot hold, by straightforward calculations concerning the left hand side for $t \uparrow \infty$. In all our examples, it actually suffices to consider $\alpha = \frac{1}{2}$. The expressions for the function $l$ can be found, e.g., in [20].

Example 1.7. For the Riemann-Liouville process $\Gamma(H+\frac{1}{2})^{-1} \int_0^t (t-s)^{H-1/2} dB_s$, where $B$ is a Brownian motion, we have $H > 0$, $R(1, 1) = (2H)^{-1} \Gamma(H + \frac{1}{2})^{-2}$, and
\[
l(u) = 2H \int_0^1 (v+u)^{H-1/2} dv = \frac{4H}{2H+1} u^{H-1/2} 2F_1\left(\frac{1}{2} - H, H + \frac{1}{2}; -\frac{1}{u}\right).
\]
Since the hypergeometric function $2F_1$ is analytic at zero and $2F_1(0) = 1$, for $\alpha \in (0, 1)$ we have
\[
l(t^{1-\alpha} - 1) \sim \frac{4H}{2H+1} t^{1-\alpha(H-1/2)}.
\]
But (1.12) can only hold if the constant factors agree on both sides, which implies $4H/(2H+1) = 1$, i.e. $H = \frac{1}{2}$. We conclude that the Riemann-Liouville process is not Markov for $H \in (0, \infty) \setminus \{\frac{1}{2}\}$. We are not aware of any proof of this in the existing literature.

For the following example, non-Markovianity has been shown in Theorem 3.1 of [14].

Example 1.8. Sub-fractional Brownian motion satisfies (1.11), with $H \in (0, 1)$, $R(1, 1) = 2 - 2^{2H-1}$, and
\[
l(u) = R(1, 1)^{-1} \left(1 + (1 + u)^{2H} - \frac{1}{4} ((2 + u)^{2H} + u^{2H})\right).
\]
For $H \neq \frac{1}{2}$, it is easy to see that
\[
l(u) = R(1, 1)^{-1} H(1 - 2H)u^{2H-2} + O(u^{2H-3}), \quad u \uparrow \infty.
\]
Similarly as in Example 1.7, if sfBm was Markov, then Corollary 1.10 would imply that

$$\frac{H(1 - 2H)}{2 - 2^{2H-1}} = 1,$$

but it is easily verified that the left hand side is smaller than $\frac{1}{2}$ for $H \in (0, 1)$, and so sfBm is not Markov for $H \in (0, 1) \setminus \{\frac{1}{2}\}$.

According to [17], for bi-fractional Brownian motion it is straightforward to check that (1.13) is not satisfied. We now give a proof using our theorem, which is also straightforward.

**Example 1.9.** We use the following notation for bi-fractional Brownian motion, in order to keep the letter $H$ for the self-similarity exponent: $H = \bar{H} \tilde{K} \in (0, 1), \tilde{K} \in (0, 1], \bar{H} \in (0, 1)$. (Standard notation is $H, \tilde{K}$ instead of $H, \bar{K}.$) For bfBm, we have $R(1, 1) = 1$ and

$$l(u) = 2^{-\tilde{K}} \left((1 + (1 + u)^{2\bar{H}}) \tilde{K} - u^{2\bar{H} \tilde{K}}\right).$$

It is easy to check that, for $u \uparrow \infty$,

$$l(u) \sim 2^{-\tilde{K}} \tilde{K} u^{2\bar{H} \tilde{K} - 2\bar{H}}$$

$$l(u) = 2^{-\tilde{K}} (2\tilde{K} \bar{K}^{-1} + 2\tilde{K} (\tilde{K} - 1) u \bar{K}^{-2})$$

$$+ \frac{4}{3} \tilde{K} (\tilde{K} - 1) (\tilde{K} - 2) u \bar{K}^{-3} + O(u \bar{K}^{-4}), \quad \bar{H} = \frac{1}{2},$$

$$l(u) = 2^{-\tilde{K}} (2 \bar{H} \tilde{K} u^{2\bar{H} \tilde{K} - 1} + \bar{K} u^{2\bar{H} \tilde{K} - 2\bar{H}} + O(u^{2\bar{H} \tilde{K} - 2\bar{H}})), \quad \bar{K} \in (\frac{1}{2}, 1).$$

In the first case, (1.12) cannot hold, because $2^{-\tilde{K}} \bar{K} < 1$ cannot be equal to 1. If $\bar{H} = \frac{1}{2}$ and $\tilde{K} = 1$, then bfBm reduces to standard Brownian motion, which is Markov. For $\bar{H} = \frac{1}{2}$ and $\tilde{K} \in (0, 1)$, (1.13) implies

$$l(t^{1/2} - 1) = 2^{-\tilde{K}} (2\tilde{K} t^{\tilde{K} - 1/2} + \tilde{K} (\tilde{K} - 1) t^{\tilde{K} - 2\bar{H} - 3/2} + O(t^{\tilde{K} - 2})), \quad t \uparrow \infty.$$

Therefore, $l(t^{1/2} - 1)$ has an asymptotic expansion with at least two terms, which cannot be equal to the single power of $t$ on the right hand side of (1.12). Finally, if $\tilde{K} \in (\frac{1}{2}, 1)$, then

$$l(t^{1/2} - 1) = 2^{-\tilde{K}} (2 \bar{H} \tilde{K} t^{\tilde{K} - 1/2} + \bar{K} t^{\tilde{K} - \bar{H}} + O(t^{\tilde{K} - 1})), \quad t \uparrow \infty,$$

which again has two distinct powers of $t$ in its expansion. We conclude that bfBm is not Markovian, except for the trivial case when it reduces to Brownian motion.

We expect that the method used in the preceding example, based on the expansion of $R(t^{1/2}, t)$, will work mechanically for virtually any self-similar non-Markovian Gaussian process with explicit covariance function. For example, it also shows that the processes defined in [5, 18, 19] are not Markov. The method is subsumed in the following immediate consequence of Theorem 1.3. Note that $R_{t, -\infty}(1/2, t)$, defined in (1.4), is identically zero.

**Corollary 1.10.** Let $X$ be a centered self-similar one-dimensional Gaussian process such that the covariance $R(t^{1/2}, t)$ is not identically zero. If $R(t^{1/2}, t)$ has an expansion at infinity which contains more than one power of $t$, or a term that does not asymptotically equal a power of $t$ with coefficient 1, then $X$ is not Markov.
A Three additional proofs

In this appendix we give a variant of the proof of Theorem 1.3, show by a non-probabilistic argument that $R_{H,c}$ is positive definite, and discuss the Volterra representation of $X^{H,c}$.

A variant of the proof of Theorem 1.3. After settling the case $c = -\infty$, we may assume that $R$ is positive, except for $s \land t = 0$. There is an alternative argument to finish the proof in this case. Gaussian Markov covariance functions that have no zeros are always of the form

$$R(s, t) = G(s \land t)F(s \lor t); \quad \text{(A.1)}$$

see [6] and p. 11 in [11]. This result is not immediately applicable, because our covariance function vanishes on the axes. But it suffices to define $H(0) = 0$ in the proof on the first page of [6] ($H$ is our $F$) to see that (A.1) holds, with $G(0) = F(0) = 0$, $G, H > 0$ on $(0, \infty)$, and $G/H$ non-decreasing on $(0, \infty)$.

Self-similarity yields

$$G(xs)F(xt) = x^{2H}G(s)F(t), \quad 0 \leq s \leq t, \quad x \geq 0. \quad \text{(A.2)}$$

Define $A(y) := \log G(e^y)$ and $B(y) := \log F(e^y)$, $y \in \mathbb{R}$. Then (A.2) implies

$$A(y + u) + B(y + v) = 2Hy + A(u) + B(v), \quad u \leq v, \quad y \in \mathbb{R}. \quad \text{(A.3)}$$

The function $A - B$ is non-decreasing and hence almost everywhere differentiable. Setting $v = y + u$ we obtain

$$A(y + u) - B(y + u) = 2Hy + A(u) - B(2y + u), \quad u \in \mathbb{R}, \quad y \geq 0,$$

which shows that $B$, and hence also $A$, is almost everywhere differentiable. For $n \in \mathbb{N}$ choose $v_n > n$ such that $B$ is differentiable at each $v_n$. From (A.3) we have

$$A(y + u) - A(u) = 2Hy - (B(y + v_n) - B(v_n))$$

for $u < v_n$. This implies

$$A'(u) = 2H - B'(v_n),$$

and hence $A$ has constant derivative for $u < v_n$. As this holds for all $n \in \mathbb{N}$, the function $A$ has constant derivative on the whole real line. Thus, $A$ is linear, and hence $G$ is a power function on $(0, \infty)$, say $G(x) = G(1)x^{-c}$ with $c \in \mathbb{R}$. Since

$$G(x)F(x) = x^{2H}F(1)G(1),$$

$F(x) = F(1)x^{2H+c}$ is a power function on $(0, \infty)$ as well, and we obtain $R = R(1, 1)R_{H,c}$. $\square$

The following theorem is a special case of the main theorem in [12]. For background on lattices and Möbius functions (which we do not require here), we refer to chapter 3 in [21].
Theorem A.1 (Lindström 1969). Let \( \{x_1, \ldots, x_d\} \) be a set of real numbers in increasing order, \( x_1 \leq x_2 \leq \cdots \leq x_d \). For functions \( f_i \) defined on \( \{x_1, \ldots, x_i\} \), \( i = 1, \ldots, d \), the identity

\[
\det (f_i(x_i \wedge x_j))_{i,j=1,\ldots,d} = \prod_{i=1}^{d} \sum_{j=1}^{d} f_i(x_j) \mu(x_j, x_i) \tag{A.4}
\]

holds, where \( \mu \) is the Möbius function of the chain \( \{x_1, \ldots, x_d\} \), i.e.

\[
\mu(x_j, x_i) = \begin{cases} 
1 & i = j, \\
-1 & i = j + 1, \\
0 & \text{otherwise.} 
\end{cases} \tag{A.5}
\]

Another proof that \( R_{H,c} \) is positive definite. We show that the function

\[
(s, t) \mapsto \begin{cases} 
(t^\alpha s \wedge t^\beta) & s \wedge t > 0, \\
0 & s \wedge t = 0, 
\end{cases} \tag{A.6}
\]

is positive definite if and only if \( \alpha + \beta \leq 0 \). Let \( 0 < t_1 \leq t_2 \leq \cdots \leq t_d \) and define

\[
f_i(t_j) := \left( \frac{t_i}{t_j} \right)^{\alpha+\beta} \left( \frac{t_i \wedge t_j}{t_i \vee t_j} \right)^{\alpha}, \quad j \leq i.
\]

We then compute

\[
\det \left( \left( \frac{t_i \vee t_j}{t_i \wedge t_j} \right)^{\alpha} \right) = (t_1 \cdots t_d)^{\alpha-\beta} \det \left( \left( \frac{t_i^\beta (t_i \vee t_j)^\alpha}{(t_i \wedge t_j)^{\alpha+\beta}} \right) \right)
\]

\[
= (t_1 \cdots t_d)^{\alpha-\beta} \prod_{i=1}^{d} \sum_{j=1}^{d} f_i(t_j) \mu(t_j, t_i)
\]

\[
= t_d^{\alpha-\beta} \prod_{i=2}^{d} t_{i-1}^{\alpha-\beta} \left( 1 - \frac{t_i^{\alpha+\beta}}{t_{i-1}^{\alpha+\beta}} \right)
\]

\[
= t_d^{\alpha-\beta} \prod_{i=1}^{d-1} t_i^{\alpha+\beta} - t_d^{\alpha+\beta} t_{d-1}^{\alpha+\beta} = t_d^{\alpha-\beta} \prod_{i=1}^{d-1} \frac{t_i^{\alpha+\beta}}{t_d^{\alpha+\beta}}
\]

where we have used (A.3) and (A.5). The statement about (A.6) now follows from the fact that a matrix is positive semidefinite if and only if all its principal minors have non-negative determinant.

As noted above, it is clear that the limiting covariance function \( R_{H,-\infty} \) is also positive definite. This can be easily checked directly as well: Let \( \{s_1, \ldots, s_m\} = \{t_1, \ldots, t_d\} \), where \( t_1, \ldots, t_d \geq 0 \), and the \( s_i \) are distinct. Then (1.2) holds, since

\[
\sum_{k,d=1}^{d} a_k a_l R_{H,-\infty}(t_k, t_l) = \sum_{i=1}^{m} \sum_{k,l=1}^{d} a_k a_l s_i^{2H}
\]

\[
= \sum_{i=1}^{m} s_i^{2H} \left( \sum_{k=1}^{d} a_k \right)^2 \geq 0.
\]
For $H > 0$ and $c < -H$, the process \([X_t^{H,c}]_{t \geq 0}\) has the Volterra representation
\[
(X_t^{H,c})_{t \geq 0} \overset{d}{=} \left( \int_0^t K_{H,c}(s,t)dB_s \right)_{t \geq 0},
\]
where $B$ is a Brownian motion, and
\[
K_{H,c}(s,t) := \sqrt{-2(c+H)t^{H-1/2}/s - c - H - 1/2} / (s/t), \quad 0 \leq s \leq t.
\]
Indeed, it is easily checked that this kernel satisfies
\[
R_{H,c}(s,t) = \int_0^{s \wedge t} K_{H,c}(u,s)K_{H,c}(u,t)du, \quad s, t \geq 0.
\]

**Proposition A.2.** Let $H > 0$ and $c = -H$. There is no function $F \in L^2(\mathbb{R}_+)$ such that
\[
(X_t^{H,-H})_{t \geq 0} \overset{d}{=} \left( \int_0^t t^{H-1/2}F(u/t)dB_u \right)_{t \geq 0}.
\]

**Proof.** This result follows from applying Theorem 2.2 and Remark 2.3 in [22] to the degenerate process $X_t^{H,-H} = t^H W(1)$, but can be easily proved directly as well: Suppose there is such a function $F$. By Ito’s isometry, we have
\[
(st)^H = R_{H,-H}(s,t) = \int_0^s t^{H-1/2}u^{H-1/2}F(u/t)F(u/s)du, \quad s \leq t,
\]
and thus
\[
\int_0^1 F(v)F(xv)dv = \frac{1}{\sqrt{x}}, \quad x > 0.
\]
For $x = 1$, we obtain $\int_0^1 F(v)^2dv = 1$. From this and the Cauchy-Schwarz inequality, we get
\[
\frac{1}{x} = \left( \int_0^1 F(v)F(xv)dv \right)^2 \leq \int_0^1 F(xv)^2dv \int_0^1 F(v)^2dv = \int_0^1 F(xv)^2dv.
\]
Substituting $u = xv$ yields
\[
\int_0^x F(u)^2du \geq 1, \quad x \in (0,1],
\]
and so $\int_0^1 F(u)^21_{\{u > x\}}du = 0$. By monotone convergence,
\[
0 = \lim_{x \downarrow 0} \int_0^1 F(u)^21_{\{u > x\}}du = \int_0^1 F(u)^2du,
\]
which contradicts $\int_0^1 F(v)^2dv = 1$. \qed
B Self-similarity, asymptotically stationary increments and the semimartingale property

The standard argument to show that fBM is not a semimartingale uses stationarity of its increments, self-similarity, the Birkhoff-Khinchin ergodic theorem and well-known properties of the quadratic variation of a semimartingale; see [15, 16] for details. In this appendix we extend this approach, using a refined ergodic theorem (Corollary 7.10 in [8]) that requires only asymptotically stationary increments, which covers the Riemann-Liouville process and related processes. (It is known that the RL process itself is not a semimartingale; see below for references.)

Lemma B.1. Let $Z$ be a centered Gaussian process such that $Z_{t+1} - Z_t$ converges in law (equivalently, in $L^2$) to a (Gaussian) random variable $J$ as $t \to \infty$. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is a measurable function such that $f(J)$ is integrable. Then we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(Z_{k+1} - Z_k) = \mathbb{E}[f(J)] \quad \text{a.s.}$$

Proof. Define the sequence of discrete difference processes

$$(R^n_k)_{k \in \mathbb{N}} = (Z_{n+k+1} - Z_{n+k})_{k \in \mathbb{N}}, \quad n \in \mathbb{N}.$$ 

Each process defines a probability measure on the product space $(\mathbb{R}^N, \mathcal{A}^\otimes N)$ where $\mathcal{A}$ is the Borel $\sigma$-algebra. Denote by $\mu_n$ the image measure of $R^n$. Since the finite dimensional distributions of $\mu_n$ converge weakly, we have that $\mu_n$ converges weakly to some measure $\nu$ (see p. 19 in [3]). Define the shift operator $\tau: (\mathbb{R}^N, \mathcal{A}^\otimes N) \to (\mathbb{R}^N, \mathcal{A}^\otimes N)$ by

$$\tau: (x_k)_{k \in \mathbb{N}} \mapsto (x_{k+1})_{k \in \mathbb{N}}$$

and set $\mathcal{I} := \{C \in \mathcal{A}^\otimes N| \tau^{-1}(C) = C\}$. Now set $\mu := \mu_0$ and notice that $\mu_n = \mu \circ \tau^{-n}$. Then for any $C \in \mathcal{A}^\otimes N$ we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(\tau^{-k}(C)) = \nu(C).$$

Hence $\nu$ is the stationary mean of $\mu$. Let $X_0$ be the projection map

$$X_0: (x_k)_{k \in \mathbb{N}} \mapsto x_0,$$

and set $X_k = X_0 \circ \tau^k$. Notice that $\mathcal{I}$ is trivial since for any non-empty $C \in \mathcal{I}$ we have

$$X_k(C) = X_k(\tau^{-k-1}(C)) = X_0(\tau^{-1}(C)) = \mathbb{R},$$

and hence $C \in \cap_{k \in \mathbb{N}} X_k^{-1}(\mathbb{R}) = \{\mathbb{R}\}$. Additionally let $f$ be such that $f \circ X_0$ is $\nu$-integrable. Corollary 7.10 in [3] implies

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) = \mathbb{E}_\nu[f(X_0)| \mathcal{I}] = \mathbb{E}_\nu[f(X_0)] \quad \mu\text{-a.s.}$$
Then we conclude

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(Z_{k+1} - Z_k) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_0(R^k)) \]

\[ = \mathbb{E}_\nu[f(X_0)] = \mathbb{E}[f(J)] \quad \text{a.s.} \]

**Theorem B.2.** Let \( Z \) be as in Lemma B.1 and assume that \( Z \) is \( H \)-self-similar for some \( H \in (0, 1) \setminus \{ \frac{1}{2} \} \). Then \( Z \) is not a semimartingale.

**Proof.** Let \( f(x) = |x|^p \) with \( p \in [1, \infty) \). By the self-similarity of \( Z \) and Lemma B.1 we have

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(n^H (Z_{k+1}/n - Z_k/n)) = \mathbb{E}[f(J)]. \]

This implies

\[ \sum_{k=0}^{n-1} |Z_{(k+1)/n} - Z_{k/n}|^p \to \begin{cases} 0 & \text{if } p > \frac{1}{H} \\ \mathbb{E}[|J|^p] & \text{if } p = \frac{1}{H} \\ +\infty & \text{if } p < \frac{1}{H}. \end{cases} \] (B.1)

The rest of the proof is the same as that of Theorem 2.2 in [15].

**Definition B.3.** Denote by \( G \) the set of functions \( g : [0, 1) \to \mathbb{R} \) of the form

\[ g(x) = \frac{G(1/(1-x))}{1-x}, \]

where \( G : \mathbb{R}^+ \to \mathbb{R} \) is continuously differentiable, and \(|G(x)| \) and \(|xG'(x)| \) grow slower than any power function \( x \mapsto x^\varepsilon \), \( \varepsilon > 0 \), as \( x \to \infty \).

**Example B.4.** Define the \( H \)-self-similar centered Gaussian process

\[ Z_{t}^{H,\beta,g} := t^{-1/2} \int_{0}^{t} (1 - s/t)^\beta g(s/t)dB_s, \] (B.2)

where \( H \in (0, \frac{1}{2}), \beta > 0, g \in G, \) and \( B \) is a Brownian motion. This process is not a semimartingale. Indeed, the following lemmas show that the increments of \( Z_{t}^{H,\beta,g} \) satisfy the asymptotic stationarity property required in Lemma B.1 and Theorem B.2.

We expect that asymptotic stationarity of the increments extends to \( \beta > \frac{1}{2} \), but this might be slightly tedious. Then, the Riemann-Liouville process becomes a special case of (B.2), by putting \( g \equiv 1 \) and \( \beta = H - \frac{1}{2} \). Note that Theorem 3.1 in [1] implies that the RL process is not a semimartingale for \( H \in (0, 1) \setminus \{ \frac{1}{2} \} \). This can also be shown using the proof of Lemma 2.4 in [9], which shows that (B.1) holds for the RL process. We also note that the literature contains quite general criteria for Gaussian processes to be semimartingales, see [2] and the references therein, but it seems not straightforward to apply them to (B.2).

**Lemma B.5.** Let \( Z = Z_{t}^{H,\beta,g} \) as in Example B.4 and define \( F(x) := (1 - x)^\beta g(x) \). Then

\[ \lim_{t \to \infty} \mathbb{E}[(Z_{t+1} - Z_t)^2] = \int_{0}^{1} F(s)^2 ds. \]
Proof. We have

\[ Z_{t+1} = \int_0^t (t + 1)^{H-1/2} F(s/(t + 1)) dB_s + \int_t^{t+1} (t + 1)^{H-1/2} F(s/(t + 1)) dB_s. \]

Now \( I_2 \) is independent of \( Z_t \) and hence

\[ \mathbb{E}[(Z_{t+1} - Z_t)^2] = \mathbb{E}[(I_1 - Z_t)^2] + \mathbb{E}[I_2^2]. \]

From the Ito isometry we have

\[ \mathbb{E}[(Z_{t+1} - Z_t)^2] = \int_0^t \left[ (t + 1)^{H-1/2} F(s/(t + 1)) - t^{H-1/2} F(s/t) \right]^2 ds + \int_t^{t+1} \left[ (t + 1)^{H-1/2} F(s/(t + 1)) \right]^2 ds. \]

By making appropriate substitutions this equals

\[ (t + 1)^{2H} \int_0^1 F(s)^2 ds - 2(t + 1)^{H-1/2} t^{H+1/2} \int_0^1 F((1 - 1/(t + 1))s) F(s) ds \]

\[ + t^{2H} \int_0^1 F(s)^2 ds. \quad (B.3) \]

Notice that

\[ 2(t + 1)^H t^H \int_0^1 F(s)^2 ds - 2(t + 1)^{H-1/2} t^{H+1/2} \int_0^1 F((1 - 1/(t + 1))s) F(s) ds \]

\[ = 2(t + 1)^H t^H \int_0^1 F(s) \left[ F(s) - \sqrt{\frac{t}{1 + t}} F((t/(t + 1))s) \right] ds, \]

and hence \((B.3)\) is equal to

\[ \int_0^1 F(s)^2 ds + 2(t + 1)^H t^H \int_0^1 F(s) \left[ F(s) - \sqrt{\frac{t}{1 + t}} F((1 - 1/(t + 1))s) \right] ds. \quad (B.4) \]

We will show that

\[ \lim_{t \to \infty} t \int_0^1 F(s) \left[ F(s) - \sqrt{\frac{t}{1 + t}} F((1 - 1/(t + 1))s) \right] ds = 0. \quad (B.5) \]

Since \( H \leq \frac{1}{2} \), this implies that the second summand in \((B.4)\) converges to zero.

We have

\[ t \int_0^1 F(s) \left[ F(s) - \sqrt{\frac{t}{1 + t}} F((1 - 1/(t + 1))s) \right] ds \]

\[ = t \int_0^1 F(s) [F(s) - F((1 - 1/(t + 1))s)] ds \]

\[ + t \left( 1 - \sqrt{\frac{t}{1 + t}} \right) \int_0^1 F(s) F((1 - 1/(t + 1))s) ds. \]

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By Lemma B.6 below, the first summand on the right hand side converges to $-\frac{1}{2} \int_0^1 F(s)^2 ds$. For the second summand, note that

$$\lim_{t \to \infty} \int_0^1 F(s)F((1 - 1/(1 + t))s) ds = \int_0^1 F(s)^2 ds,$$

since the integrand is bounded, and that

$$\lim_{t \to \infty} t \left(1 - \sqrt{\frac{t}{1 + t}}\right) = 1/2.$$

Thus, (B.5) is established.

Lemma B.6. With $F$ as in Lemma B.5, we have

$$\lim_{t \to \infty} t \int_0^1 F(s) [F(s) - F((1 - 1/t)s)] ds = -\frac{1}{2} \int_0^1 F(s)^2 ds.$$

Proof. Using the mean value theorem we obtain

$$t \int_0^1 F(s) [F(s) - F((1 - 1/t)s)] ds = \int_0^1 sF(s)F'(c_s) ds$$

for some $c_s(t)$ with $s - s/t \leq c_s(t) \leq s$. For notational simplicity we will write $c_s = c_s(t)$. Substituting $F(x) = (1 - x)^{\beta - 1}g(x)$ yields

$$\int_0^1 s(1-s)^{\beta}g(s) \left[\beta(1-c_s)^{\beta-1}g(c_s) + (1-c_s)^{\beta}g'(c_s)\right] ds.$$

Set

$$I_1' = \beta \int_0^1 s(1-s)^{\beta} (1-c_s)^{\beta-1} g(s) g(c_s) ds$$

and

$$I_2' = \int_0^1 s(1-s)^{\beta} (1-c_s)^{\beta} g(s) g'(c_s) ds.$$

Choose $0 < \varepsilon < \beta$. We have

$$I_1' = \beta \int_0^1 s(1-s)^{\beta-\varepsilon} (1-c_s)^{\beta-1-\varepsilon} \left[(1-s)^{\varepsilon} g(s)(1-c_s)^{\varepsilon} g(c_s)\right] ds.$$

Clearly, $(1-s)^{\varepsilon} g(s)$ is bounded on $[0, 1]$. If $\beta > 1$, then we can choose $\varepsilon$ sufficiently small such that $\beta - 1 - \varepsilon \geq 0$, and hence the entire integrand is bounded. Otherwise, since $c_s \geq s$ we have

$$s(1-s)^{\beta-\varepsilon} (1-c_s)^{\beta-1-\varepsilon} \left[(1-s)^{\varepsilon} g(s)(1-c_s)^{\varepsilon} g(c_s)\right] \leq s(1-s)^{2\beta-1-2\varepsilon} \left[(1-s)^{\varepsilon} g(s)(1-c_s)^{\varepsilon} g(c_s)\right],$$
which is integrable as $2\beta - 1 - 2\varepsilon > -1$. Applying dominated convergence yields
\[
\lim_{t \to \infty} I^1_t = \beta \int_0^1 s(1-s)^{2\beta-1} g(s)^2 ds.
\]
For $I^2_t$ we have
\[
I^2_t = \int_0^1 s(1-s)^{3-\varepsilon}(1-c_s)^{\beta-1-\varepsilon} \left[(1-s)^\varepsilon g(s)(1-c_s)^{1+\varepsilon} g'(c_s)\right] ds.
\]
Since $x^{1+\varepsilon}g'(x)$ is bounded on $[0,1)$, using the same arguments as for $I^1_t$ yields
\[
\lim_{t \to \infty} I^2_t = \int_0^1 s(1-s)^{2\beta-1} g(s)g'(s) ds.
\]
It follows that
\[
\lim_{t \to \infty} \int_0^1 F(s) [F(s) - F((1-1/t)s)] ds = \int_0^1 s F(s) F'(s) ds.
\]
Noticing that $\frac{d}{dx} F(x)^2 = 2 F(x) F'(x)$ and applying integration by parts yields the result.

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