Exploiting geometric degrees of freedom in topological quantum computing

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In a topological quantum computer, braids of non-Abelian anyons in a (2+1)-dimensional space-time form quantum gates, whose fault tolerance relies on the topological, rather than geometric, properties of the braids. Here we propose to create and exploit redundant geometric degrees of freedom to improve the theoretical accuracy of topological single- and two-qubit quantum gates. We demonstrate the power of the idea using explicit constructions in the Fibonacci model. We compare its efficiency with that of the Solovay-Kitaev algorithm and explain its connection to the leakage errors reduction in an earlier construction [Phys. Rev. A 78, 042325 (2008)].

I. INTRODUCTION

Topological quantum computation is a rapidly developing subject in recent years [1, 2, 3, 4]. In this novel scheme of quantum computation, information is stored in topological quantum states and intrinsically protected from local noises, and manipulation of quantum information is achieved by topological operations. A prototypical topological quantum computer is envisaged to be a system of exotic quasiparticles called non-Abelian anyons, which are believed to exist in various two-dimensional quantum systems [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. A multiple of these anyons with fixed coordinates span a multi-dimensional Hilbert space, which can be used to construct qubits or encode quantum information [18, 19, 20]. Schemes have been proposed to control and move anyons microscopically [21, 22, 23]. The world-lines of these anyons intertwine in (2+1)-dimensional space-time forming braids, which are quantum gates for topological quantum computation.

In earlier studies [24, 25, 26, 27], researchers developed the method of brute-force search (and its variant) among braids within a given braid length (measured by the number of exchanges) to achieve a generic single-qubit quantum gate in the Fibonacci anyon model (which may be realizable in fractional quantum Hall systems [12, 28]), and then constructed controlled-rotation gates from single-qubit gates. These works explicitly demonstrated the equivalence between a specific theoretical realization of topological quantum computer and a universal quantum computer model [2]. In general, a single-qubit quantum gate can be represented by a $2 \times 2$ unitary matrix

$$G = e^{i\alpha} \begin{pmatrix} \sqrt{1 - b^2 e^{-i\beta}} & 0 \\ 0 & \sqrt{1 - b^2 e^{i\beta}} \end{pmatrix},$$

(1)

where $b, \alpha, \beta$ and $\gamma$ are real parameters. Apart from the overall phase factor $e^{i\alpha}$, one needs three parameters $b, \beta$ and $\gamma$ to specify the matrix. Within a given braid length, there are only a finite number of topological quantum gates, which form a discrete set in the $U(2)$ space, thus generic gates can only be realized with a distribution (wide on logarithmic scale) of error even in the ideal scenario (without technical or practical hindrance), due to the discrete nature of braid topology. This contrasts to many proposals of conventional quantum computation, where quantum gates can be realized by continuously tuning physical parameters so generic quantum gates are expected to be realized with only a narrow distribution of error (due to technical imperfections). On the other hand, the discreteness (thus error) in the realization of quantum gates with braids of finite length shares the same origin as the fault tolerance of topological quantum computation, as quantum states and quantum gates (braids) are topological and robust against local perturbations. This therefore poses an interesting question: How can we efficiently find the braid with finite length that approximates a desired quantum gate with error as small as possible?

In a recent work, the authors proposed a novel construction of low-leakage topological quantum computation based on the principle of error reduction by error introduction [26]. In topological quantum computation, the leakage errors in two-qubit gates one wants to minimize is often of topological origin, e.g., arising from the existence of noncomputational states. Nevertheless, one may find a class of equivalent braid constructions characterized by an additional geometric degree of freedom in braid segments, which can be said to correspond to a $U(1)$ symmetry of the construction. In practice, however, due to the discreteness of braids in the target space, such a symmetry is merely a pseudo symmetry. One finds that some of the constructions can have exponentially smaller errors than others – they are exactly what we want to find. The successful application of the principle in the Fibonacci anyon model led to the discovery of an exchange braid (with a length of 99) that exchanges anyons between two different qubits, which can be used to construct generic controlled-rotation gates with leakage error as small as $10^{-9}$. However, the idea can not be directly applied to construct single-qubit gates because apparently there is no such geometric freedom.

In this paper, we generalize the idea that errors in topological gate construction can be reduced by the introduction of an additional geometric redundancy (or symmetry) [26] and show that it can also be applied to the construction of generic single-qubit gates with unprece-
dented efficiency and accuracy in theory. We demonstrate this idea explicitly in the Fibonacci model, though it is applicable in generic models that support universal topological quantum computation. By introducing new degrees of freedom with unitary similarity transformation, we show a generic single-qubit gate can be approximated to a distance $2\alpha$ of the order $10^{-10}$ by a braid of length $\sim$300, which is more efficient than a direct application of the Solovay-Kitaev algorithm [22]. We also discuss the significant reduction of leakage error by error introduction in a parallel construction of two-qubit controlled-gates (see also Ref. [27, 30]) to demonstrate the power and generality of the idea of topological error reduction by exploiting redundant geometric degrees of freedom.

II. SINGLE-QUBIT GATE CONSTRUCTION

Let us first discuss the high-accuracy construction of a generic single-qubit gate $G$ represented by Eq. (1). To create additional degrees of freedom needed for error reduction, one is tempted to decompose the target gate as $G = G_1 G_2$, where

$$G_{1,2} = e^{i\alpha_{1,2}} \begin{bmatrix} 1 - b_{1,2}^2 & e^{-i\beta_{1,2}} & b_{1,2} e^{i\gamma_{1,2}} \\ -b_{1,2} e^{-i\gamma_{1,2}} & 1 - b_{1,2}^2 & e^{i\beta_{1,2}} \end{bmatrix}$$

(2)

are unitary matrices, or quantum gates. Unfortunately, the degrees of freedom of the two gates are dependent, as can be immediately seen from $G_1 = G G_2^\dagger$, i.e., one of the gate (e.g., $G_1$) is completely fixed by the other gate (e.g., $G_2$) up to an unimportant global phase factor. Therefore, one cannot use the new degrees of freedom for gate optimization. In fact, this decomposition of $G$ is the mathematical structure of the bidirectional search [22], which facilitate the search for twice longer braids without a significant increase in CPU power.

In fact, to create separable degrees of freedom with two gates, we can write $G = G_1 G_2 G_1^\dagger$ in a unitary similarity transformation, which creates geometric redundancy. It is easier to visualize the transformation in terms of rotation in three dimensions, thanks to the homomorphism between the groups $SO(3)$ and $SU(2)$. This means that a rotation around an arbitrary axis $\alpha$ by an angle $\beta$ on a Bloch sphere can be carried out by first rotating $\alpha$ to another direction $\alpha'$, then rotating around $\alpha'$ by an angle $\gamma$, and finally rotating $\alpha'$ back to $\alpha$. The geometric interpretation clearly indicates that the freedom in the choice of $\alpha'$ can be exploited to optimize the single-qubit gate $G$, because these apparently equivalent realizations use different braids to approximate the target gate with entirely different accuracy.

We can use a phase gate $P$ (a diagonal matrix) to illustrate the determination of $G_1$ and $G_2$ without loss of generality. This is because, according to the spectral theorem for normal matrices, any unitary matrix $G$ can be unitarily diagonalized as $G = S^* P S$, where $S$ is a unitary matrix. We can then contract $S$ with $G_1$, so that $P = G_1 G_2 G_1^\dagger$, where $G_1 = S G_1$.

For concreteness, let us assume $P = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}$,

(3)

which is a rotation around the $z$ axis by an angle $\beta$. The parameters $b_{1,2}, \beta_{1,2}$ and $\gamma_{1,2}$ of $G_1$ and $G_2$ that decompose $P$ must, therefore, satisfy

$$b_1 = \frac{1 - b_2^2}{\sqrt{2}} \cos \beta_2 = \cos \beta,$$

(4)

and

$$\begin{bmatrix} 1 & b_2 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} \sin \beta_2 & 0 \\ 0 & \sin \beta_2 \end{bmatrix},$$

(5)

$$\begin{bmatrix} \beta_1 + \gamma_1 = \gamma_2 + (k + 1/2) \pi, \end{bmatrix}$$

(6)

where the integer $k$ is even for positive $\sin \beta$ or odd for negative $\sin \beta$ (we exclude the case $\sin \beta = 0$ when the corresponding gate is proportional to the identity). From Eq. (4) we can see that $G_2$ is a rotation of the same angle as that of $P$ but around a new axis, which is related to the original rotation axis of $P$ by Eqs. (5) and (6). Nevertheless, when $G_2$ is fixed, $G_1$ is only partially determined by $G_2$ and $P$ and still has a degree of freedom (between $\beta_1$ and $\gamma_1$). In other words, the similarity transformation $G_1 G_2 G_1^\dagger$ has an $SU(2)$ symmetry, from which we have three free parameters to choose. $G_1$, the rotation of $z$-axis to a new axis, has a $U(1)$ symmetry (i.e., one free parameter), while $G_2$, fixing the direction of the new axis, has a symmetry of $SU(2)/U(1) \sim S^2$ (i.e., two free parameters). Hence, we can successfully separate the three degrees of freedom into two parts in $G_1$ and $G_2$, which allow us to efficiently search for high-accuracy single-qubit gates.

As an explicit demonstration of the algorithm, we construct a phase gate

$$P_1 = e^{i7\pi/5} \begin{bmatrix} e^{-i2\pi/5} & 0 \\ 0 & e^{i2\pi/5} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & e^{-i\pi/5} \end{bmatrix},$$

(7)

in the Fibonacci anyon model (please refer to Refs. [26, 31] for details of this model), where there are two types of anyons with topological charges 0 (vacuum) and 1 (Fibonacci anyon) satisfying a nontrivial fusion rule $1 \times 1 = 0 + 1$. We use two pairs of Fibonacci anyons with total charge 0 to encode one bit of quantum information. The basis states are chosen as $|0\rangle = ((|1\rangle_0 |1\rangle_0 |0\rangle_0 |0\rangle)$ and $|1\rangle = (|1\rangle_1 |1\rangle_1 |1\rangle_0 |0\rangle)$, where the subscripts specify the fusion results (or total topological charges) of the anyons in the preceding brackets. Therefore, four-strand braids can be generated by the elementary braids with representation

$$\sigma_1 = \sigma_3 = \begin{bmatrix} e^{-i4\pi/5} & 0 \\ 0 & -e^{-i2\pi/5} \end{bmatrix},$$

(8)

$$\sigma_2 = \begin{bmatrix} -e^{-i\pi/5} & \sqrt{2} e^{i2\pi/5} \\ -\sqrt{2} e^{i2\pi/5} & -7 \end{bmatrix},$$

(9)
and their inverses, where $\tau = (\sqrt{5} - 1)/2$. We find a set of $G_1 = a_2a_3a_2^{-1}a_3^{-1}a_2^{-1}a_3a_2^{-1}a_3^{-1}a_2^{-1}$ and $G_2 = a_2a_3a_2^{-1}a_3^{-1}a_2^{-1}a_3a_2^{-1}a_3^{-1}a_2^{-1}a_3$. One can verify $G_1G_2G_1^{-1}$ with 280 interchanges [32], approximates $P_1$ with a distance $\sim 4 \times 10^{-10}$. In general, such a precision can be achieved by a braid of length $\sim 300$ for a generic single-qubit gate with the algorithm specified above. A braid of similar accuracy is expected to exist at a length of as short as 150 [29], but the search for it is exponentially harder.

III. TWO-QUBIT GATE CONSTRUCTION

The single-qubit construction scheme echoes the earlier low-leakage two-qubit construction scheme [28], in which one exchanges a two-anyon composite in the control qubit with the neighboring anyon in the target qubit and performs single-qubit operations on the new target qubit, which translate into controlled rotations in the original two-qubit system, before one exchanges the anyons back to their original locations. There is, however, a notable difference: the $SU(2)$ symmetry in the single-qubit construction is broken in the two-qubit construction due to leakage error. As discussed in the introduction, only a $U(1)$ symmetry exists in the two-qubit construction when we enforce leakage errors to be negligible. This suggests that the requirement of zero (or ultralow) leakage errors in the two-qubit construction eats two degrees of freedom. It becomes clear in an alternative high-accuracy ($\sim 10^{-10}$) implementation presented in the following, in which the construction of the two-qubit gates are based on a mapping from two qubits to one qubit in the four-anyon encoding scheme (see also [27, 30]), which also demonstrates the generality of the idea of topological error reduction by redundant geometric degree of freedom.

Explicitly, the mapping scheme is the following. For clarity, we label the anyons in the target qubit $a_{1-4}$ and those in the control qubit $a_5-a_8$ as in Fig. 1(a). We can treat the two pairs of anyons in each qubit as two composite anyons, which have a total topological charge 0. Then we have a mapping from two qubits of Fibonacci anyons to one qubit of composite anyons, which we label $A_1$-4 as in Fig. 1(a). The computational basis states are chosen as

$$
|00\rangle = |((11)_0(11)_0)(11)_0(11)_0\rangle = |((00)_0(00)_0)\rangle,
|01\rangle = |((11)_0(11)_0)(11)_1(11)_1\rangle = |((00)_0(11)_0)\rangle,
|10\rangle = |((11)_1(11)_1)(11)_0(11)_0\rangle = |((11)_0(00)_0)\rangle,
|11\rangle = |((11)_1(11)_1)(11)_1(11)_1\rangle = |((11)_1(11)_0)\rangle,
$$

(10)

where $\hat{0}$ and $\hat{1}$ denote the topological charge of composite anyons. In fact, the composite qubit is not a qubit in the normal encoding scheme, because the composite anyons $A_i$ can have charge 0. Each pair of composite anyons (e.g., $A_1$ and $A_2$) always have total charge 0, unless leakage error occurs so each of the original qubits has total charge 1. Therefore, the class of braids that we look for to manipulate the the composite qubit as in Fig. 1(b) without introducing leakage errors are the ones that realize phase gates, for example,

$$
P_2 = e^{i\alpha_2}
\begin{array}{cc}
e^{-i\beta_2} & 0 \\
0 & e^{i\beta_2}
\end{array},
$$

(11)

We see immediately that two degrees of freedom in $SU(2)$ disappear due to the requirement of zero leakage errors and we are left with a $U(1)$ symmetry only. In practice, we restrict ourselves to move the composite anyon $A_2$ to braid with the composite anyons $A_3$ and $A_4$ and return $A_2$ back to the original position at the end of the braid; this is known as a weave [28]. When the composite qubit is in the state $|((11)_0(11)_0)\rangle$, this braid will introduce a phase factor $e^{i(\alpha_2-\beta_2)}$ to the two-qubit system (e.g., a phase factor -1 for the braid approximating $P_1$ in Eq. (7)). While if the composite qubit is originally in the other computational states in Eq. (10), either the topological charge of the composite anyon $A_2$ or the topological charges of the composite anyons $A_3$ and $A_4$ are 0. Since the braid between an anyon with topological charge 0 and another anyon with topological charge either 0 or 1 does not change the state of the system, the braid will bring only a trivial phase factor 1 to the system. Thus a braid approximating the single-qubit phase gate $P_2$ in Eq. (11) corresponds to a controlled-phase gate, e.g., a controlled-Z gate for the braid approximating $P_1$ in Eq. (7).

A scheme to construct an arbitrary controlled-rotation gate, parallel to the single-qubit gate construction, is illustrated in Fig. 2 (see also Fig. 3 in Ref. 30 for an $SU(2)_3$ construction). In particular, we need to apply a single-qubit gate $G_3$ on the target qubit after the controlled-phase gate and its inverse $G_3^{-1} = G_3^+$ before the controlled-phase gate. This is in the same spirit as the similarity transformation in the single-qubit case, except that the controlled-phase gate is defined on the composite qubit, not the target qubit. For completeness, we
need to introduce another single-qubit phase gate

$$P_3 = e^{i\alpha_3} \begin{bmatrix} e^{-i\beta_3} & 0 \\ 0 & e^{i\beta_3} \end{bmatrix} ,$$

(12)

to adjust the phase of the control qubit such that the resulting gate is

$$e^{i(\alpha_3 - \beta_3)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & R \end{bmatrix} ,$$

(13)

where $R$ is related to $G_3$ by

$$e^{i\left(\frac{2\alpha_2 - \beta_2 + 3\beta_3}{2}\right)}G_3 \begin{bmatrix} e^{-i(\alpha_3 - \beta_3)/2} & 0 \\ 0 & e^{i(\alpha_3 - \beta_3)/2} \end{bmatrix} G_3^+ .$$

(14)

Given a target $R$, we should search for a braid realizing the corresponding $G_3$. As $R$ can be diagonalized by a similarity transformation $SRS^+$, the constraint on $G_3$ is that $SG_3$ should be a single-qubit phase gate with an arbitrary phase, which again allows a redundant $U(1)$ degree of freedom.

In fact, we intentionally designed our presentation so that the braid realization of $P_1$ in Eq. 17 is the same for $P_2$ in order to construct a controlled-NOT (CNOT) gate. Correspondingly, we find the sequence for one instance of $G_3$ as

$$\sigma_2^4 \sigma_3^2 \sigma_2^2 \sigma_3^2 \sigma_2^{-2} \sigma_3^{-2} \sigma_2^4 \sigma_3^{-2} \sigma_2^{-2} \sigma_3^2 \sigma_2^2 \sigma_3^2 \sigma_2^{-2} \sigma_3^{-2} \sigma_2^4 \sigma_3^{-2} \sigma_2^{-2} \sigma_3^2 \sigma_2^{-2} \sigma_3^{-2} .$$

The total braid for the CNOT gate, with an error of $5 \times 10^{-10}$, contains 280 exchanges of double braids and 208 exchanges of single braids. Note that $P_3$ is trivial for the CNOT gate. We would like to point out that this construction is conceptually interesting but technically less efficient, because an exchange of two double braids is, in fact, four exchanges of single braids. The new single-qubit construction combined with the two-qubit construction in the earlier proposal [20], which also involves exchanges of a single braid and a double braid, can achieve a similar error within 1000 exchanges of single braids.

IV. CONCLUSION AND DISCUSSION

In conclusion, we proposed the idea of exploiting redundant geometric degrees of freedom in topological quantum computation to reduce the topological errors due to discreteness of gates realized by finite-length braids. This is possible because we can separate the redundant degrees of freedom into (partially) independent parts, which allows topological quantum gate construction to be more efficient. We also established the intriguing connection between the sacrifice of two such degrees of freedom and the minimization of two-qubit leakage errors.

We can understand the error reduction from a different angle. In the three-dimensional space of unitary matrices, a target gate is just a zero-dimensional point. The introduction of geometric redundancies transforms the target into a one- or higher-dimensional object, thereby allows an efficient deeper search. The algorithm is practically useful as computational errors can be reduced exponentially at all length scales by the introduction of redundant degrees of freedom as shown, e.g., in Fig. 4 of Ref. [26]. The increase in braid length by a factor of roughly three is thus well compensated by the exponential suppression in error.

Finding optimal braids belongs to the generic question of approximating an arbitrary unitary operation by a set of discrete gates (or matrices) relevant to, e.g., constructing quantum circuits in generic quantum computation, for which a remarkable rate of convergence can be achieved by the Solovay-Kitaev algorithm [33]. The Solovay-Kitaev algorithm is based on the principle of error cancellation by a group commutator structure of $ABA^{-1}B^{-1}$ factor (given an initial $\epsilon$-net). It has been implemented in the context of topological quantum computation by Hormozi et al., who achieved a gate with a distance $\simeq 4.2 \times 10^{-5}$ to $iX$ with a braid of length 220 in one iteration [25]. Therefore, the algorithm presented here can achieve comparable accuracy to that from applying one iteration of the Solovay-Kitaev algorithm, albeit with braids that are about 40% shorter than those obtained from the Solovay-Kitaev algorithm. An iterable modification of the algorithm, as well as its performance comparison with the Solovay-Kitaev algorithm, is presented elsewhere [35].

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