AN ATTRACTIVE BASIS FOR THE $q$–ONSAGER ALGEBRA

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Abstract. Let $A, A^*$ be the fundamental generators of the $q$–Onsager algebra. A linear basis for the $q$–Onsager algebra is known as the ‘zig-zag’ basis [T89]. In this letter, an attractive basis for the $q$–Onsager algebra is conjectured, based on the relation between the $q$–Onsager algebra and a quotient of the infinite dimensional algebra $A_q$ introduced in [BK05].

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1. Introduction

Introduced in the mathematical physics literature, the Onsager algebra (OA) admits two presentations. A first presentation is given in terms of two generators $A_0, A_1$ subject to a pair of relations, the so-called Dolan-Grady relations [DG82]:

$$ [A_0, [A_0, [A_0, A_1]]] = 16[A_0, A_1], \quad [A_1, [A_1, [A_1, A_0]]] = 16[A_1, A_0]. $$

A second presentation originates in Onsager’s work on the exact solution of the two-dimensional Ising model [OE4]. It is given in terms of generators $\{A_k, G_l|k, l \in \mathbb{Z}\}$ and relations:

$$ [A_k, A_l] = 4G_{k-l}, \quad [G_l, A_k] = 2A_{k+l} - 2A_{k-l}, \quad [G_k, G_l] = 0. $$

In the 90’s, the isomorphism between these first and second presentations was established [Dav91, Ro91], and generators $\{A_k, G_l|k, l \in \mathbb{Z}\}$ were systematically written as polynomials in $A_0, A_1$. For a review of the proof, see [EC12]. Note that the Onsager algebra has also a presentation as an invariant subalgebra of the loop algebra $\mathbb{C}[[t, t^{-1}]] \otimes sl_2$ by an involution $D$ [Dav91, Ro91].

The $q$–Onsager algebra ($q$–OA) has been introduced as a $q$–deformed analog of the presentation (1.1). In this presentation, the defining relations are given in Definition 4.1. It first appeared in the mathematical literature in the context of $P$– and $Q$–polynomial schemes, as a special case of the so-called tridiagonal algebra [T99]. It is now well-understood that the representation theory of tridiagonal algebras - in particular the $q$–OA - is intimately connected with the theory of Leonard pairs and tridiagonal pairs developed by Terwilliger and coauthors (see all the references citing [T99]). Moreover, the deformation scheme of the Onsager algebra to the $q$–Onsager algebra is given by Manin triples technique [BC12]. Independently, the $q$–OA also appeared in the context of quantum integrable systems [BD9, BK05], the spectral parameter dependent reflection equation algebra [BS09] and as a certain coideal subalgebra of $U_q(sl_2)$ [BD9].

By analogy with the OA, besides the first presentation associated with (1.1) it was expected that the $q$–OA admits a second presentation, that would be associated with an infinite number of generators satisfying certain $q$–deformed analogs of (1.2). In this direction, one strategy was to study in details the structure of more general (non-scalar) solutions of the reflection equation algebra, the so-called Sklyanin operators [Sk88]. An infinite dimensional algebra called $A_q$ with generators $\{W_{-k}, W_{k+1}, G_{k+1}, G_{k+1}|k \in \mathbb{Z}_+\}$ and relations (see Definition 3.1) was introduced in [BS09], based on the results of [BK05]. In particular, for the explicit examples of Sklyanin operators considered in [BK05] it was observed that the first generators $W_0, W_1$ satisfy the pair of relations (1.3). By a brute force calculation, it was also observed that the eight generators denoted $W_{-1}, W_0, W_1$, $G_0, G_1, G_2, G_3$ can be written as polynomials in $W_0, W_1$ only [BK05] see eqs. (49)]. This strongly suggested that the algebra $A_q$ or certain of its quotients are natural candidates for a second presentation of the $q$–OA.

In this letter, we investigate the relationship between the $q$–Onsager algebra and a class of quotients of the infinite dimensional algebra $A_q$, denoted $\tilde{A}_q$ (see Definition 6.1). It is shown that all generators

1 Another strategy is based on the braid group action of the $q$–Onsager algebra [BK16].
of $\tilde{A}^{(8)}_q$ are polynomials in $W_0, W_1$ (see Corollary 3.1), explicitly determined through recursive formulae (see Proposition 3.1 and Example 2). Then, we propose a Poincaré-Birkhoff-Witt (PBW) basis for the algebra $\tilde{A}^{(8)}_q$ (see Conjecture 11) in Section 4. Based on the existence of an homomorphism from $\tilde{A}^{(8)}_q$ to the $q$–Onsager algebra (see Proposition 4.1) as well as a detailed comparison between the PBW basis for $\tilde{A}^{(8)}_q$ and the so-called ‘zig-zag’ basis introduced by Ito and Terwilliger in [IT09], it is conjectured that the algebra $A^{(8)}_q$ and the $q$–Onsager algebra are isomorphic, see Conjecture 2. In other words, we claim that the algebra $\tilde{A}^{(8)}_q$ gives a second presentation for the $q$–Onsager algebra.

**Notation.** We fix a field $\mathbb{K}$ and a nonzero $q \in \mathbb{K}$ assumed not to be a root of unity. We denote $\mathbb{Z}_+$ the set of nonnegative integers. We denote $[X,Y]_q = qXY - q^{-1}YX$. In the text, $u$ and $v$ denote indeterminates. Let $n$ be a positive integer. We use the notation $[n]_q = (q^n - q^{-n})/(q - q^{-1})$.

## 2. The algebra $A_q$ and central elements

In this section, the infinite dimensional quantum algebra $A_q$ introduced in [BS09] is recalled through generators and relations. A generating function for central elements of $A_q$ is constructed, that will play a central role in the analysis of further sections.

**Definition 2.1** (See [BS09]). $A_q$ is an associative algebra with unit $1$, generators $\{W_{-k}, W_{k+1}, G_{k+1}, \tilde{G}_{k+1} | k \in \mathbb{Z}_+ \}$ and nonzero scalar $\rho \in \mathbb{K}$. Define $U = (qu^2 + q^{-1}u^{-2})/(q + q^{-1})$ and $V = (qu^2 + q^{-1}u^{-2})/(q + q^{-1})$.

Introduce the formal power series:

\[
W_+(u) = \sum_{k \in \mathbb{Z}_+} W_{-k} U^{-k-1}, \quad W_-(u) = \sum_{k \in \mathbb{Z}_+} W_{k+1} U^{-k-1},
\]

\[
G_+(u) = \sum_{k \in \mathbb{Z}_+} G_{k+1} U^{-k-1}, \quad G_-(u) = \sum_{k \in \mathbb{Z}_+} \tilde{G}_{k+1} U^{-k-1}.
\]

The defining relations are:

\[
[W_\pm(u), W_\pm(v)] = 0,
\]

\[
[W_+(u), W_-(v)] + [W_-(u), W_+(v)] = 0,
\]

\[
(U - V)[W_\pm(u), W_\mp(v)] = \frac{(q - q^{-1})}{\rho(q + q^{-1})}(G_+(u)G_-(v) - G_-(v)G_+(u)) + \frac{1}{(q + q^{-1})}(G_+(u) - G_+(v) + G_-(v) - G_-(u)),
\]

\[
W_+(u)W_+(v) - W_+(v)W_+(u) + \frac{1}{\rho(q^2 - q^{-2})}[G_+(u), G_+(v)] + \frac{1 - UV}{U - V}(W_+(u)W_+(v) - W_+(v)W_+(u)) = 0,
\]

\[
U[G_+(v), W_\pm(u)] - V[G_+(u), W_\pm(v)] - (q - q^{-1})(W_+(u)G_+(v) - W_+(v)G_+(u)) + \rho(UW_\pm(u) - VW_\pm(v) - W_\pm(u) + W_\pm(v)) = 0,
\]

\[
U[G_+(u), G_\pm(v)] - V[G_+(v), G_\pm(u)] - (q - q^{-1})(W_\pm(u)G_+(v) - W_\pm(v)G_+(u)) + \rho(UW_\pm(u) - VW_\pm(v) - W_\pm(u) + W_\pm(v)) = 0,
\]

\[
[G_+(u), W_\pm(v)] + [W_\pm(u), G_+(v)] = 0, \quad \forall \epsilon = \pm,
\]

\[
[G_\pm(u), G_\pm(v)] = 0,
\]

\[
[G_+(u), G_-(v)] + [G_-(u), G_+(v)] = 0.
\]

**Remark 1.** There exists an automorphism $\Omega$ in $A_q$:

\[
\Omega(W_{-k}) = W_{k+1}, \quad \Omega(W_{k+1}) = W_{-k}, \quad \Omega(G_{k+1}) = \tilde{G}_{k+1}, \quad \Omega(\tilde{G}_{k+1}) = G_{k+1}.
\]

In $A_q$, an infinite number of central elements can be constructed using the explicit connection between the algebra $A_q$ and the reflection equation algebra introduced in [Cher84, Sk88]. Below we exhibit a generating function for those elements that commute with the currents $\tilde{A}^{(8)}_q$. 

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[References]:
[BS09]: Basilehac and Belliard
[Cher84]: Cher84
[Sk88]: Sk88
Proposition 2.1. The element

\[ \Delta(u) = -(q-q^{-1})(q^2 + q^{-2}) \left( \mathcal{W}_+(u) \mathcal{W}_+(uq) + \mathcal{W}_-(u) \mathcal{W}_-(uq) \right) - \frac{(q-q^{-1})}{\rho} \left( \mathcal{G}_+(u) \mathcal{G}_-(uq) + \mathcal{G}_-(u) \mathcal{G}_+(uq) \right) \\
+ (q-q^{-1})(u^2q^2 + u^{-2}q^{-2}) \left( \mathcal{W}_+(u) \mathcal{W}_-(uq) + \mathcal{W}_-(u) \mathcal{W}_+(uq) \right) \\
- \mathcal{G}_+(u) - \mathcal{G}_+(uq) - \mathcal{G}_-(u) - \mathcal{G}_-(uq) \]

satisfies \([\Delta(u), \mathcal{W}_\pm(v)] = [\Delta(u), \mathcal{G}_\pm(v)] = 0\). 

Proof. By [BS09 Theorem 1], the defining relations (2.3)-(2.11) with \(\rho = k_+k_- (q + q^{-1})^2\) can be written in the 'compact' form of a reflection equation [SK88].

\[ (2.12) \quad R(u/v) (K(u) \otimes I) R(uv) (I \otimes K(v)) = (I \otimes K(v)) R(uv) (K(u) \otimes I) R(u/v), \]

where \(I\) denotes the 2 \times 2 identity matrix,

\[ R(u) = \begin{pmatrix} uq - u^{-1}q^{-1} & 0 & 0 & 0 \\
0 & u - u^{-1} & q - q^{-1} & 0 \\
0 & q - q^{-1} & u - u^{-1} & 0 \\
0 & 0 & 0 & uq - u^{-1}q^{-1} \end{pmatrix} \]

and

\[ K(u) = \begin{pmatrix} uq\mathcal{W}_+(u) - u^{-1}q^{-1}\mathcal{W}_-(u) & \frac{1}{k_+(q+q^{-1})} \mathcal{G}_+(u) + \frac{k_+(q+q^{-1})}{(q-q^{-1})} \\
\frac{k_-}{(q+q^{-1})} \mathcal{G}_-(u) + \frac{k_-}{(q-q^{-1})} \mathcal{W}_+(u) - u^{-1}q^{-1}\mathcal{W}_-(u) & uq\mathcal{W}_+(u) - u^{-1}q^{-1}\mathcal{W}_-(u) \end{pmatrix}, \]

where \(k_\pm\) are nonzero scalars.

By [SK88 Proposition 5], the so-called quantum determinant:

\[ (2.15) \quad \Gamma(u) = \text{tr}(P_{12}^-(K(u) \otimes I) R(u^2q)(I \otimes K(uq))), \]

where \(P_{12}^-= (1 - P)/2\) with \(P = R(1)/(q - q^{-1})\), is commuting with the four entries of \(K\)-matrix (2.14):

\[ (2.16) \quad [\Gamma(u), (K(u))_{ij}] = 0 \quad \text{for any} \quad i, j. \]

Inserting (2.14) into (2.15), by straightforward calculations one identifies:

\[ \Gamma(u) = \frac{(u^2q^2 - u^{-2}q^{-2})}{2 (q - q^{-1})} \left( \Delta(u) - \frac{2\rho}{(q - q^{-1})} \right), \]

with \(\Delta(u)\) as given above. From (2.16), it follows \([\Delta(u), \mathcal{W}_\pm(v)] = [\Delta(u), \mathcal{G}_\pm(v)] = 0\). \(\square\)

From Proposition 2.1, \(\Delta(u)\) provides a generating function for elements that commute with all generators \(\{\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \mathcal{G}_{k+1}\}_{k \in \mathbb{Z}_+}\) of \(\mathcal{A}_q\). Explicit expressions are now derived.

Lemma 2.1. Let \(i, j, k, l \in \mathbb{Z}_+\) and denote \(\tilde{k} = 1\) (resp. 0) for \(k\) even (resp. odd) and \(\lfloor \frac{k}{2} \rfloor = \frac{k}{2}\) (resp. \(\frac{k-1}{2}\)) for \(k\) even (resp. odd). Define

\[ (2.17) \quad \Delta_{k+1} = \mathcal{G}_{k+1} + \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} d_l(\mathcal{G}_{2l+1}) \mathcal{W}_{l} \mathcal{W}_{l+1} + \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{i+j=2l+1-k} f_{ij}(\mathcal{W}_{i} \mathcal{W}_{j} + \mathcal{W}_{i+1} \mathcal{W}_{j+1}) + \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{i+j=2l+\tilde{k}} e_{ij} \mathcal{F}_{ij} \]

where

\[ \mathcal{G}_{i+1} = \mathcal{G}_{i+1} + \mathcal{G}_{i+1}, \]
\[ \mathcal{W}_{ij} = (q - q^{-1}) (\mathcal{W}_{i+1} \mathcal{W}_{j} + \mathcal{W}_{i+1} \mathcal{W}_{j}), \]
\[ \mathcal{F}_{ij} = (q - q^{-1}) (q^2 + q^{-2}) (\mathcal{W}_{i} \mathcal{W}_{j} + \mathcal{W}_{i+1} \mathcal{W}_{j+1}) + \frac{1}{\rho} (\mathcal{G}_{j+1} \mathcal{G}_{i+1} + \mathcal{G}_{j+1} \mathcal{G}_{i+1}), \]
and

\[
e_{ij}^{(k)} = -c_{k+1} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor + \lfloor \frac{j-i-k}{2} \rfloor} \sum_{i+j+k} w_i^j (1 + q^{2j-k}) G_{i+1} + \sum_{i+j+2(n+m)=k} (w_n^i w_m^j + w_n^j w_m^i) q^{-2j-4m+1} \mathbb{W}_{ij} \\
+ \sum_{i+j+2(n+m)=k-1} w_n^i w_m^j q^{-2j-4m-1} \mathbb{F}_{ij}
\]

with

\[
c_{k+1} = - \frac{(q + q^{-1})^{k+1} (q^{k+1} + q^{-k-1})}{q^{2k+2}}, \quad w_i^m = (-1)^m \frac{(m + i)!}{m!} (q + q^{-1})^{i+1} q^{-i-2m-1}.
\]

For any \( k \) and all \( X \in A_q \) we have \( [\Delta_{k+1}, X] = 0 \).

**Proof.** Insert (2.1), (2.2) in \( \Delta(u) \) as defined in Proposition 2.1. Observe that \( U \) admits a power series expansion about \( u = \infty \) or \( u = 0 \). About \( u = \infty \), \( \Delta(u) \) admits the power series expansion in \( u^{-1} \) of the form \( \sum_{k=0}^{\infty} u^{-2k-2} c_{k+1} \Delta_{k+1} \). The coefficients \( e_{ij}^{(k)}, f_{ij}^{(k)}, d_{ij}^{(k)} \) are determined as follows. Consider the power series expansion of \( U(u) \) about \( u = \infty \):

\[
U(u)^{-1-i} = \sum_{m=0}^{\infty} u_m^i u^{-2i-4k-2}.
\]

From the definition of \( \Delta(u) \) in Proposition 2.1 the expansion about \( u = \infty \) yields to:

\[
\Delta(u) = \sum_{k=0}^{\infty} u^{-2k-2} \left( \sum_{i+j+k} \left( G_{i+1} + \sum_{i+j+2(n+m)=k} \left( w_n^i w_m^j + w_n^j w_m^i \right) q^{-2j-4m+1} \mathbb{W}_{ij} + \sum_{i+j+2(n+m)=k-1} w_n^i w_m^j q^{-2j-4m-1} \mathbb{F}_{ij} \right) \right) \]

Then, it follows from Proposition 2.1 and (2.1), (2.2), that \( \Delta_{k+1} \) is central in \( A_q \). Note that the same central elements \( \Delta_{k+1} \) can be also derived by considering the power series expansion of \( \Delta(u) \) about \( u = 0 \).

**Example 1.** The leading terms in the power series expansion of \( \Delta(u) \) are given by:

\[
\begin{align*}
\Delta_1 &= G_1 + \tilde{G}_1 - (q - q^{-1}) (\mathbb{W}_0 \mathbb{W}_1 + \mathbb{W}_1 \mathbb{W}_0), \\
\Delta_2 &= G_2 + \tilde{G}_2 - \frac{(q^2 - q^{-2})}{(q^2 + q^{-2})} (q^{-1} \mathbb{W}_0 \mathbb{W}_2 + q \mathbb{W}_2 \mathbb{W}_0 + q^{-1} \mathbb{W}_1 \mathbb{W}_1 - q \mathbb{W}_1 \mathbb{W}_1) \\
&\quad + (q - q^{-1}) \left( \frac{(q^2 + q^{-2})}{\rho} (\mathbb{W}_0^2 + \mathbb{W}_1^2) + \frac{G_1 \tilde{G}_1 + \tilde{G}_1 \tilde{G}_1}{\rho} \right),
\end{align*}
\]
\[ \Delta_3 = G_3 + \tilde{G}_3 - \frac{(q-q^{-1})}{(q^2 + q^2 - 2 - 1)} (q^{-2} W_0 W_3 + q^2 W_3 W_0 + q^{-2} W_1 W_{-2} + q^2 W_{-2} W_1) \\
\quad - \frac{(q-q^{-1})}{(q^2 + q^2 - 2 - 1)} (W_2 W_{-1} + W_{-1} W_2) \\
\quad + \frac{(q-q^{-1})}{(q^2 + q^2 - 2 - 1)} ((q^2 + q^{-2})(W_0 W_{-1} + W_1 W_2) + \frac{G_1 + \tilde{G}_1}{\rho}) \\
\quad - \frac{1}{(q + q^{-1})^2} (G_1 + \tilde{G}_1 - (q-q^{-1})(W_0 W_1 + W_1 W_0)). \]

**Remark 2.** For all \( k \in \mathbb{Z}^+, \Omega(\Delta_{k+1}) = \Delta_{k+1} \).

### 3. A CLASS OF QUOTIENTS OF \( A_q \) AND THE POLYNOMIAL FORMULAE

In this section, we investigate a class of quotients of \( A_q \). A quotient in this class is denoted \( A_q^{(k)} \) (see Definition 3.1). In \( A_q^{(k)} \), we are going to show that any generator \( \{ W_{-k}, W_{k+1}, \tilde{G}_{k+1}, G_{k+1} | k \in \mathbb{Z}^+ \} \) is a certain polynomial of the two basic generators \( W_0, W_1 \) only. The polynomial formulae are obtained explicitly (see Proposition 3.1) and are unique modulo the relations (2.3)-(2.11).

**Definition 3.1.** Let \( \delta_{k+1}, k \in \mathbb{Z}^+ \) be fixed scalars in \( \mathbb{K} \). The algebra \( A_q^{(k)} \) is defined as the quotient of the algebra \( A_q \) by the ideal generated by the relations \{ \( \Delta_{k+1} = 2\delta_{k+1} \forall k \in \mathbb{Z}^+ \} \).

We start the analysis by considering the defining relations (2.3)-(2.11). Using the power series expansion (2.1) and (2.2), the defining relations of \( A_q \) can be written explicitly in terms of the generators \{ \( W_{-k}, W_{k+1}, \tilde{G}_{k+1}, G_{k+1} | k \in \mathbb{Z}^+ \} \), see [BSG] Definition 3.1. For the analysis below, it will be sufficient to stare at four of these relations. For all \( k \in \mathbb{Z}^+ \), they read:

(3.1) \[ [W_0, W_{k+1}] = [W_{-k}, W_1] = \frac{1}{(q + q^{-1})} (\hat{G}_{k+1} - \tilde{G}_{k+1}) \, , \]

(3.2) \[ [W_0, \tilde{G}_{k+1}] = \rho W_{k+1} - \rho W_{k+1} \, , \]

(3.3) \[ [G_{k+1}, W_1] = \rho W_{k+2} - \rho W_{k} \, , \]

(3.4) \[ [W_0, W_{-k}] = 0 \, , \quad [W_1, W_{k+1}] = 0 \, . \]

We first describe the straightforward consequences of the conditions \( \Delta_{k+1} = 2\delta_{k+1} \) for \( k = 0, 1 \) in addition to the subset of relations (3.1)-(3.4). Imposing \( \Delta_1 = 2\delta_1 \) in (2.18) and considering eqs. (3.1) for \( k = 0 \), the two equations read, respectively:

\[ G_1 + \tilde{G}_1 - (q - q^{-1})(W_0 W_1 + W_1 W_0) - 2\delta_1 = 0, \]

\[ G_1 - \tilde{G}_1 - (q + q^{-1})(W_0 W_1 - W_1 W_0) = 0. \]

It implies

(3.5) \[ G_1 = [W_1, W_0] + \delta_1 \quad \text{and} \quad \tilde{G}_1 = [W_0, W_1] + \delta_1. \]

Then, the r.h.s of (3.2), (3.3) determine uniquely the next two generators in terms of \( W_0, W_1, G_1 \):

(3.6) \[ W_{-1} = \frac{1}{\rho} [W_0, G_1] + W_1 \, , \quad W_2 = \frac{1}{\rho} [G_1, W_1] + W_0. \]

Note that the first equality in (3.1) is satisfied for \( k = 1 \). Indeed, using (3.5), (3.6) one finds \( [W_0, W_2] = [W_{-1}, W_1] \). Consider now the first and second eqs. of (3.1) for \( k = 1 \). Respectively, they read

(3.8) \[ [W_0, W_{-1}] = 0, \quad [W_1, W_2] = 0. \]

According to eqs. (3.5) (resp. (3.6)), we conclude that the generators \( G_1, \tilde{G}_1 \) (resp. \( W_{-1}, W_2 \)) are uniquely determined as polynomials of total degree 2 (resp. of degree less that 3) in \( W_0, W_1 \).

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2Note that inserting (3.5) into (3.1), one finds that \( W_0, W_1 \) satisfy the ‘\( q \)-Dolan-Grady relations’:

(3.7) \[ [W_0, [W_0, [W_0, W_1]_{q-1}]] - \rho [W_0, W_1], \quad [W_1, [W_1, [W_1, W_0]_{q-1}]] = \rho [W_1, W_0]. \]
Next, we impose $\Delta_2 = 2\delta_2$. Eqs. (2.19) and eqs. (3.1) for $k = 1$ become, respectively:

$$G_2 + \tilde{G}_2 = \frac{(q^2 - q^{-2})}{2(q^2 + q^{-2})} (q^{-1}W_0W_2 + qW_2W_0 + q^{-1}W_1W_{-1} + qW_{-1}W_1) + \frac{(q + q^{-1})}{2} [W_2, W_0]$$

$$- \frac{(q - q^{-1})}{2(q^2 + q^{-2})} ((q^2 + q^{-2}) (W_0^2 + W_1^2) + \frac{\tilde{G}_1G_1 + G_1\tilde{G}_1}{\rho}) - 2\delta_2 = 0,$$

where the relation $[W_0, W_2] = [W_{-1}, W_1]$ has been used. Also, one finds:

$$\tilde{G}_2 = \Omega(G_2).$$

Then, eqs. (3.3) determine uniquely the next two generators as polynomials of the ‘lower’ ones:

$$W_{-2} = \frac{1}{\rho} [W_0, G_2] + W_2,$n

In addition, the first and second eqs. of (3.3) for $k = 2$ read:

$$[W_0, W_{-2}] = 0, \quad [W_1, W_3] = 0.$$

Combining previous results, we conclude that, in $\tilde{A}_q^{(s)}$, the generators $G_2, \tilde{G}_2$ (resp. $W_{-2}, W_3$) are uniquely determined as polynomials of total degree less than 4 (resp. degree less that 5) in $W_0, W_1$. Above analysis is generalized in a straightforward manner for $k \geq 2$.

**Proposition 3.1.** Let $i, j, k, l \in \mathbb{Z}_+$ and denote $\tilde{k} = 1$ (resp. 0) for $k$ even (resp. odd) and $[k] = \frac{k}{2}$ (resp. $\frac{k-1}{2}$) for $k$ even (resp. odd). One has:

$$G_{k+1} = - \sum_{l=0}^{\frac{k}{2}} \sum_{i+j=2l+1-\tilde{k}} \frac{f_{ij}^{(k)}}{2} W_{ij} - \sum_{l=0}^{\frac{k}{2}-\tilde{k}} \sum_{i+j=2l+\tilde{k}} \frac{e_{ij}^{(k)}}{2} F_{ij} - \sum_{l=0}^{\frac{k-1}{2}} \frac{d_l^{(k)}}{2} G_{2(l+1)-\tilde{k}}$$

$$+ \frac{(q + q^{-1})}{2} [W_{k+1}, W_0] + \delta_{k+1}$$

where $W_{ij}, G_{ij}, F_{ij}, f_{ij}^{(k)}, e_{ij}^{(k)}$ and $d_l^{(k)}$ are given in Lemma 2.7 and

$$W_{-k-1} = \frac{1}{\rho} [W_0, G_{k+1}] + W_{k+1}.$$  

The expressions for the other generators are given by $W_{k+1} = \Omega(W_{-k}), \tilde{G}_{k+1} = \Omega(\tilde{G}_{k+1}).$

**Proof.** We use an inductive argument. Assume $\Delta_{l+1} = 2\delta_{l+1}$ for $l \leq k$. Then, (2.17) with (3.1) determine uniquely - modulo the relations (2.3)- (2.4) - the generators $G_{k+1}$ as polynomials of the generators of lower degree, given by (3.12). From Remark 1, one has $\tilde{G}_{k+1} = \Omega(\tilde{G}_{k+1})$. Given $W_{-k}, W_{k+1}$, $G_{k+1}$ for $k$ fixed, the relations (3.2) determine uniquely $W_{-k-1}$ as (3.13) and $W_{k+2} = \Omega(W_{-k-1}).$

Iterating the recursive formulae (3.12), (3.13) and using the automorphism $\Omega$, it follows that each generator of $\tilde{A}_q^{(s)}$ is a polynomial in $W_0, W_1$.

**Corollary 3.1.** The algebra $\tilde{A}_q^{(s)}$ is generated by $W_0, W_1$.

We now discuss more precisely how the original generators $W_{-k}, W_{k+1}, G_{k+1}, \tilde{G}_{k+1}$ are related to $W_0, W_1$. Let $d$ denote the total degree of a monomial in the elements $W_0, W_1$. According to (3.3), $G_1, \tilde{G}_1$ are polynomials in the generators $W_0, W_1$ of degree $d[G_1] = d[\tilde{G}_1] = 2$. Then, from (3.6) one finds $d[W_{-1}] = d[W_2] = 3$. By induction, according to Proposition 3.1 it follows immediately:
Corollary 3.2. The generators of \( \tilde{\mathcal{A}}^{(6)}_q \) are polynomials in \( \mathcal{W}_0, \mathcal{W}_1 \) of degree:

\[
d[\mathcal{W}_{-k}] = d[\mathcal{W}_{k+1}] = 2k + 1 \quad \text{and} \quad d[\tilde{\mathcal{G}}_{k+1}] = d[\tilde{\mathcal{G}}_{k+1}] = 2k + 2, \quad k \in \mathbb{Z}_+.
\]

Example 2. The first generators read:

\[
(3.15) \quad \mathcal{G}_1 = q\mathcal{W}_1\mathcal{W}_0 - q^{-1}\mathcal{W}_0\mathcal{W}_1 + \delta_1, \\
\mathcal{W}_{-1} = \frac{1}{\rho(q^2 + q^{-2})}(q^2 + q^{-2})\mathcal{W}_0\mathcal{W}_1\mathcal{W}_0 - \mathcal{W}_0\mathcal{V}_0\mathcal{V}_0 - \mathcal{W}_1\mathcal{W}_0\mathcal{V}_1 \quad \text{and} \quad \mathcal{W}_1 = \frac{q(1 - q)}{\rho(q^2 + q^{-2})}W_1 + \frac{\delta_1(q - q^{-1})}{\rho(q^2 + q^{-2})}W_0,
\]

Expressions of \( \mathcal{W}_{k+1} \) and \( \tilde{\mathcal{G}}_{k+1} \) are obtained using the automorphism \( \Omega \).

For simplicity, the explicit expressions for the generators \( \mathcal{W}_{-2}, \mathcal{W}_1, \mathcal{G}_3, \tilde{\mathcal{G}}_3 \) are reported in Appendix A.

Remark 3. In [BB10, Example 1], under certain assumptions polynomial formulae for the generators of \( \tilde{\mathcal{A}}_q \) were obtained as the unique solutions of a system of linear equations (see [BB10 Appendix]), through a detailed analysis of the defining relations \((2.8)-(2.17)\). For \( k = 0, 1, 2 \), we have checked that the polynomial formulae conjectured in [BB10] coincide with the polynomial formulae \((3.14)-(3.15)\) that holds in \( \tilde{\mathcal{A}}^{(6)}_q \).

4. Linear bases for the algebras \( \tilde{\mathcal{A}}^{(6)}_q \) and \( \mathcal{O}_q \)

In this section, a linear basis for the algebra \( \tilde{\mathcal{A}}^{(6)}_q \) is conjectured (Conjecture 1). We call it the WG–basis. Then, using the fact that the algebra \( \tilde{\mathcal{A}}^{(6)}_q \) is a homomorphic image of the \( q \)--Onsager algebra \( \mathcal{O}_q \), it is conjectured that the WG–basis gives a basis for the \( q \)--Onsager algebra. A comparison between the WG–basis and the ‘zig-zag’ basis of the \( q \)--Onsager algebra given by Ito-Terwilliger [IT09] gives a support for the conjecture.

First, we exhibit a natural spanning set for the algebra \( \tilde{\mathcal{A}}^{(6)}_q \). From the relations \((2.3)\) and \((2.10)\) observe that \( \{\mathcal{W}_{-k} | k \in \mathbb{Z}_+\} \) (resp. \( \{\mathcal{W}_{k+1} | k \in \mathbb{Z}_+\}, \{\mathcal{G}_{k+1} | k \in \mathbb{Z}_+\}\) ) generate a commutative subalgebra of \( \tilde{\mathcal{A}}^{(6)}_q \). Let \( \{\alpha_i, \beta_i, \gamma_i, \lambda_i, \mu_i\} \) \( i \in \mathbb{Z}_+ \). Consider monomials of the form:

\[
(4.1) \quad \omega'_{\alpha, \beta, \gamma, \lambda, \mu} (\{\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1} | k \in \mathbb{Z}_+\}) = \mathcal{W}^{\alpha_1}_{-k_1} \cdots \mathcal{W}^{\alpha_N}_{-k_N} \mathcal{G}^{\beta_1}_{p_1+1} \cdots \mathcal{G}^{\beta_P}_{p_P+1} \mathcal{W}^{\gamma_1}_{l_1+1} \cdots \mathcal{W}^{\gamma_M}_{l_M+1} \mathcal{W}^{\mu_1}_{l_1+1}.
\]

According to Proposition 3.1 and Corollary 3.2 the monomial \((4.1)\) is a polynomial in \( \mathcal{W}_0, \mathcal{W}_1 \) of maximum degree:

\[
(4.2) \quad |\lambda'| = \sum_{i=1}^{N} \alpha_i(2k_i + 1) + \sum_{i=1}^{P} \beta_i(2p_i + 2) + \sum_{i=1}^{M} \gamma_i(2l_i + 1).
\]

For small values of \( |\lambda'| \), using the relations \((2.3)-(2.11)\) it is straightforward to identify the set of linearly independent monomials of the form \((4.1)\).

Example 3. Let \( d_{|\lambda|} \) denote the number of irreducible monomials of maximum degree \( |\lambda'| \). The irreducible monomials in \( \tilde{\mathcal{A}}^{(6)}_q \) of degree \( |\lambda'| \leq 6 \) are the set:

\[\text{Note that a more explicit form for the commutation relations between the generators } \{\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}\} \text{ has been used for this task, see [BS09, Definition 3.1 and Proposition 3.1].}\]
Let \( A \) be a nonzero scalar in \( \mathbb{K} \). The \( q \)-Onsager algebra is the associative algebra with unit and standard generators \( A, A^* \) subject to the so-called \( q \)--Dolan-Grady relations

\[
[A, A, [A, A^*]_{q^{-1}}] = 0, \quad [A^*, [A^*, A]_{q}] = 0.
\]

A linear basis for the \( q \)--Onsager algebra \( O_q \) has been constructed in [IT09]. It is the so-called ‘zig-zag’ basis. Let \( r \) denote a positive integer. Define \( \lambda_i = \{ \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^{r+1} | \lambda_0 \geq 0, \lambda_i \geq 1(1 \leq i \leq r) \} \). If there exists an integer \( i \) (\( 0 \leq i \leq r \)) such that

\[
\lambda_0 < \lambda_1 < \cdots < \lambda_i \geq \lambda_{i+1} \geq \cdots \geq \lambda_r,
\]

then \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r) \) is said to be irreducible. The sum \( |\lambda| = \lambda_0 + \lambda_1 + \cdots + \lambda_r \) is called the length of \( \lambda \). The following set is the zig-zag basis of the \( q \)--Onsager algebra \( O_q \) as a \( \mathbb{K} \)--vector space [IT09 Theorem 2.1] (see [IT03 Problem 3.4]):

\[
\{ \omega_\lambda(A, A^*) | \lambda \text{ is irreducible} \} \quad \text{where} \quad \omega_\lambda(A, A^*) = \begin{cases} A^{\lambda_0}A^{\lambda_1}\ldots A^{\lambda_r} & \text{if } r \text{ is even} \\ A^{\lambda_0}A^{\lambda_1}\ldots A^{\lambda_{r-1}} & \text{if } r \text{ is odd} \end{cases}
\]
We now review some results of [115] Section 4 that will be useful in the discussion below. For \(0 \leq |\lambda| < \infty\), let \(O_q|\lambda|\) denote the subspace of \(O_q\) spanned by the words of length at most \(|\lambda|\) in the generators \(A, A^*\). The sequence \(\{O_q|\lambda|\}_{\lambda \in \mathbb{Z}_+}\) is a filtration of \(O_q\). For \(n \in \mathbb{Z}_+\), consider the quotient \(K\)-vector space \(\overline{O}_q|\lambda| = O_q|\lambda|/O_q|\lambda|-1\) and introduce the formal direct sum \(\overline{O} = \sum_{|\lambda| \in \mathbb{Z}_+} \overline{O}_q|\lambda|\). The sequence \(\{\overline{O}_q|\lambda|\}_{\lambda \in \mathbb{Z}_+}\) is a \(\mathbb{Z}_+\)-grading of the algebra \(\overline{O}\). The algebra \(\overline{O}\) is called the graded algebra associated with the filtration \(\{O_q|\lambda|\}_{\lambda \in \mathbb{Z}_+}\). By construction, one has:

\[
\dim(O_q|\lambda|) = \dim(\overline{O}_q|\lambda|).
\]

Let \(U_q^+\) be the positive part of \(U_q(\mathfrak{sl}_2)\). By [115] Theorem 4.4, recall that the algebra \(\overline{O}\) is isomorphic to \(U_q^+\). It follows that the generating function of \(\dim(\overline{O}_q|\lambda|)\) is equal to the formal character of the Verma module for \(\mathfrak{sl}_2\) [IT04, eq. (40)]. From (4.7), for the \(q\)-Onsager algebra it follows.\(^4\)

**Theorem 2.** The generating function of the number of irreducible monomials in \(O_q\) is:

\[
\sum_{|\lambda|=0}^{\infty} \dim(O_q|\lambda|) v^{|\lambda|} = \prod_{m=1}^{\infty} (1 - v^{2m-1})^{-2}. \tag{4.8}
\]

**Example 4.** The irreducible monomials in \(O_q\) of length \(|\lambda| \leq 6\) are the set:

| \(|\lambda|\) | Zig-zag basis | \(\dim(O_q|\lambda|)\) |
|---|---|---|
| 0 | 1 | 1 |
| 1 | \(A, A^*\) | 2 |
| 2 | \(A^2, A^*A, AA^*, A^*A\) | 4 |
| 3 | \(A^3, A^*A^2, AA^*A, AA^*A, A^*AA^*, A^*AA^*\) | 8 |
| 4 | \(A^4, A^3A^*, A^2A^*A, A^*A^3, A^2A^*A^2, AA^*A, AA^*A, AA^*A, A^*A^2A^*\) | 14 |
| 5 | \(A^5, A^4A^*, A^3A^*A, A^*A^4, A^3A^*A, A^2A^*A^2, AA^*A, AA^*A, A^*A^2A^*\) | 24 |
| 6 | \(A^6, A^5A^*, A^4A^*A, A^*A^5, A^2A^*A^2, A^3A^*A, A^2A^*A^2, A^2A^*A^2, AA^*A^2, AA^*A, AA^*A, a^2A^*A^2, A^*A^2A^*\) | 40 |

There is a close relationship between the algebra \(\tilde{A}_q^{(s)}\) and the \(q\)-Onsager algebra \(O_q\). Indeed, following Corollary 3.1 the algebra \(\tilde{A}_q^{(s)}\) admits a presentation with two generators \(W_0, W_1\) subject to the relations (2.3)-(2.11) with (4.12), (4.13). Among the corresponding infinite family of polynomial relations satisfied by \(W_0, W_1\) inherited from (2.3)-(2.11), the relations of lowest degree are given by (1.5) with \(A \rightarrow W_0, A^* \rightarrow W_1\), see eqs. (4.7):

**Lemma 4.1.** The basic generators \(W_0, W_1\) of \(\tilde{A}_q^{(s)}\) satisfy the defining relations of the \(q\)-Onsager algebra.

**Proof.** Insert the polynomial expressions of \(W_{-1}\) (resp. \(W_2 = \Omega(W_{-1})\)) given in (3.15) in the commutation relations (3.4) for \(k = 0, l = 1\). \(\square\)

It follows:

**Proposition 4.1.** The map \(\Psi : O_q \rightarrow \tilde{A}_q^{(s)}\) such that

\[
\Psi(A) = W_0, \quad \Psi(A^*) = W_1 \tag{4.9}
\]

is a surjective homomorphism.

\(^4\)See also [115] Note 4.7.
We now compare the irreducible set of monomials in the $W_G$–basis (4.3) to the ones in the zig-zag basis (4.6). First, it is elementary to check that both generating functions (4.4) and (4.8) coincide. Secondly, using the data in Example 3 and Example 4, the transition matrix from one basis to another is considered up to $|\lambda| = |\lambda'| = 4$, see Appendix B. It is found that this matrix of size $29 \times 29$ is invertible.

**Conjecture 2.** The map $\Psi$ in Proposition 4.1 is a bijection.

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APPENDIX A. THE GENERATORS $W_{-2}, W_3, G_3, \tilde{G}_3$

Recall that the generators $G_1, W_{-1}, G_2$ are given explicitly in terms of $W_0, W_1$ in Example 3 and $G_1 = \Omega(G_1), W_{-1} = \Omega(W_2), \tilde{G}_2 = \Omega(G_2)$. In addition, one finds:

$$W_{-2} = w_1 W_0^2 W_1^2 + w_2 W_0 W_1 W_0^3 W_1 + w_3 (W_1 W_0^2 W_0 + W_0^2 W_1 W_0) + w_4 W_0 W_1 W_0 W_1 W_0$$

$$+ w_5 W_0^3 W_1^2 + w_6 W_0 W_1^2 + w_7 (W_0^2 W_1 + W_1 W_0^2) + w_8 W_0 W_1 W_0 + w_9 W_1 W_0 + w_{10} W_0 + w_{11} W_1$$

where

$$w_1 = \frac{1}{\rho}, \quad w_2 = -\frac{[2]_q [8]_q}{\rho^2 [4]_q^2}, \quad w_3 = -\frac{[4]_q}{\rho^4 [2]_q}, \quad w_4 = \frac{1}{\rho^2} \left( \frac{[2]_q [3]_q [8]_q}{[4]_q^2} + 1 \right),$$

$$w_5 = \frac{1}{\rho^2 [3]_q}, \quad w_6 = -\frac{[2]_q [8]_q}{\rho^2 [3]_q [4]_q}, \quad w_7 = \frac{[2]_q^2}{\rho^2 [3]_q [4]_q}, \quad w_8 = -\frac{1}{\rho}, \quad w_9 = -\frac{1}{\rho [3]_q},$$

$$w_{10} = \frac{(q-q^{-1})^2}{\rho^2} \delta_1, \quad w_{11} = \frac{1}{\rho} \left( \frac{[2]_q [3]_q [8]_q}{[4]_q^2} + 1 \right), \quad w_{12} = \frac{(q-q^{-1})^2 [4]_q \delta_1}{\rho^2 [2]_q},$$

$$w_{13} = \frac{1}{\rho} \left( \frac{q-q^{-1}}{[4]_q} \right), \quad w_{14} = 1 - \frac{(q-q^{-1})^2 [2]_q \delta_1^2 + (q-q^{-1}) \delta_2}{\rho^2 [4]_q}, \quad w_{15} = \frac{(q-q^{-1}) \delta_1}{\rho}$$

and $W_3 = \Omega(W_{-2})$.

$$G_3 = g_1 W_0^3 W_1^2 + g_2 W_0 W_1^2 W_0 W_1 + g_3 W_0^2 W_1^3 W_0 - g_4 W_0 W_1 - g_5 W_0 W_1 W_0 W_1$$

$$+ g_6 W_0^2 W_1^2 W_0 + g_7 W_0 W_1^2 W_0 W_1 + g_8 W_0 W_1 W_0 W_1 W_0 + g_9 W_0 W_1 W_0 W_1$$

where

$$g_1 = -\frac{q^{-3} [2]_q}{\rho^2 [6]_q}, \quad g_2 = 2 \left( q^{-7} + q^{-3} + q^{-1} \right) [2]_q [3]_q \frac{[2]_q}{\rho^2 [4]_q [6]_q},$$

$$g_3 = \frac{q^{-1} [4]_q (q^2 - q^{-2} - 1)}{\rho^2 [6]_q}, \quad g_4 = \frac{(q^5 + q^3 - q - q^{-3} - q^{-5} - q^{-7}) [2]_q^2}{\rho^2 [4]_q [6]_q},$$

$$g_5 = \frac{(q-q^{-1}) [3]_q^2 [4]_q}{\rho^2 [6]_q}, \quad g_6 = q^{-1} [2]_q [8]_q, \quad g_7 = \frac{(q^{-7} + q^{-1} + q^{-3} + q^{-5}) [2]_q^2 [3]_q}{\rho^2 [4]_q [6]_q},$$

$$g_8 = \frac{q^3 [2]_q}{\rho^2 [6]_q}, \quad g_9 = -\frac{2 (q^7 + q^3 + q) [2]_q [3]_q}{\rho^2 [4]_q [6]_q},$$

$$g_{10} = \frac{-q [4]_q (q^2 - q^{-2} - 1)}{\rho^2 [6]_q}, \quad g_{11} = \frac{(q^7 + q^5 + q^3 + q^{-1} - q^{-3} - q^{-5}) [2]_q^2}{\rho^2 [4]_q [6]_q},$$

$$g_{12} = g_5, \quad g_{13} = \frac{q [2]_q [8]_q}{\rho^2 [4]_q}, \quad g_{14} = \frac{(q^3 + q^{-7} + 2 q^5 + q^3 + 3 q) [2]_q [3]_q}{\rho^2 [4]_q [6]_q},$$

$$g_{15} = \frac{(q^{-1} + q^{-3} + q^{-5} + q^{-7}) [2]_q^2 [3]_q}{\rho^2 [4]_q [6]_q}.$$
\[ g_{15} = -\frac{q^{-1}(2q^6 + q^4 + 2q^2 + 1 + 4q^{-2} + 2q^{-4} + 2q^{-6})[2]_q}{\rho[3]_q[4]_q[2]_q}, \]
\[ g_{16} = \frac{2q[6]_q}{\rho[3]_q[4]_q}, \quad g_{17} = -q(q^6 + 2q^4 + 3q^2 + q^{-2} - q^{-4} - q^{-6} - q^{-8})[2]_q^2, \]
\[ g_{18} = \frac{q^{-2}(q - q^{-1})[2]_q^2}{\rho^2[4]_q}, \quad g_{19} = -\frac{(q - q^{-1})^2[2]_q[3]_q\delta_1}{\rho[2]_q[4]_q}, \]
\[ g_{20} = \frac{-q^{-3}(q - q^{-1})(q^2 + [3]_q)[2]_q\delta_1}{\rho^2[4]_q}, \]
\[ g_{21} = \frac{q(2q^6 + 2q^4 + 4q^2 + 1 + 2q^{-2} + q^{-4} + 2q^{-6})[2]_q}{\rho[3]_q[4]_q[6]_q}, \]
\[ g_{22} = -\frac{2q^{-1}[6]_q}{\rho[3]_q[4]_q}, \quad g_{23} = \frac{-q^{-1}(q^8 + q^6 + q^4 - q^2 - 3q^{-2} - 2q^{-4} - q^{-6})[2]_q^2}{\rho[3]_q[4]_q[6]_q}, \]
\[ g_{24} = -\frac{q^2(q - q^{-1})[2]_q^2}{\rho^2[4]_q}, \quad g_{25} = g_{19}, \]
\[ g_{26} = \frac{q^2(q - q^{-1})(q^{-2} + [3]_q)[2]_q^2}{\rho^2[4]_q}, \]
\[ g_{27} = g_{28} = \frac{(q - q^{-1})^2[2]_q[3]_q\delta_1}{\rho[4]_q}, \]
\[ g_{29} = \frac{-q^{-1}(q - q^{-1})}{\rho^2} \left( \rho\delta_2 - \frac{(q - q^{-1})[2]_q\delta_1}{[4]_q} \right) + \frac{q^3(q^4 - q^8 - 2q^{-6})[2]_q^2}{[3]_q[4]_q[6]_q}, \]
\[ g_{30} = \frac{q(q - q^{-1})}{\rho^2} \left( \rho\delta_1 - \frac{(q - q^{-1})[2]_q\delta_3}{[4]_q} \right) + \frac{q^{-3}(q^6 - q^{-4} + 2q^8)[2]_q^2}{[3]_q[4]_q[6]_q}, \]
\[ g_{31} = \delta_3 + \left( \frac{1}{[3]_q[4]_q[6]_q} - \frac{(q - q^{-1})^2[3]_q[2]_q}{\rho[6]_q} \right) \delta_1 \delta_2, \]

and \( \tilde{G}_3 = \Omega(G_3) \).
Appendix B. Transition coefficients between the zig-zag basis and $WG-$basis for $|\lambda| = |\lambda| \leq 4$

In this Appendix, the non-vanishing entries of the transition matrix from the zig-zag basis with elements $1, A, A^2, \ldots$ to the $WG-$basis with elements $1, W_0, W_1, \ldots$ are displayed for $|\lambda| = |\lambda| \leq 4$ in the table below. For instance, Proposition 4.1 together with the polynomial formulae (3.5) give the combination: $qAA^*A - q^{-1}A^2A^* + \delta_1 A \rightarrow W_0G_1$.

| $\lambda$ | $A^*$ | $A^{*2}$ | $AA^*$ | $A^3A^*$ | $A^2A^*$ | $A^4A^*$ | $AA^{*5}$ | $A^2A^*A$ | $A^3A^*$ | $A^2A^2$ | $A^4A^*$ | $A^2AA^*$ | $A^4A^*$ | $A^3AA^*$ | $A^2A^*A$ | $A^4A^*$ | $A^3AA^*$ | $A^2A^*A$ | $A^4A^*$ |
|----------|-------|-------|-------|---------|---------|---------|--------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $W_0$    | 1     | 1     | $-q^{-1}$ | $q$     | $-\frac{\rho}{\rho}$ | $\frac{\rho}{\rho}$ | $-\frac{1}{\rho}$ | $\frac{1}{\rho}$ | $-\frac{1}{\rho}$ | $-\frac{1}{\rho}$ | $-\frac{1}{\rho}$ | $-\frac{1}{\rho}$ | $-\frac{1}{\rho}$ | $-\frac{1}{\rho}$ | $-\frac{1}{\rho}$ | $-\frac{1}{\rho}$ | $-\frac{1}{\rho}$ | $-\frac{1}{\rho}$ | $-\frac{1}{\rho}$ |
| $W_1$    | 1     | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ |
| $G_1$    | 1     | $-q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ |
| $G_2$    | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ | $q^{-1}$ |

Here we define the scalars: $a = \frac{\delta_1(q-q^{-1})}{\rho}$, $b = \frac{[q]_q}{[2]_q}$, $c = \frac{q^{-2}[2]_q^2}{[4]_q^2}$, $d = \frac{q^2[2]_q^2}{[4]_q^2}$, $e = (q-q^{-1})\frac{[2]_q}{[4]_q}$, $f = \frac{(q^{-5}+q^{-3}+2q^{-1})[2]_q^2}{[4]_q^2}$, $g = \frac{(q^5+q^3+2q)\rho}{[4]_q^2}$, $h = -q^{-1}\frac{[2]_q[3]_q}{[4]_q}$, $i = -\frac{q^{-2}[2]_q^2}{[4]_q^2}$, $j = \delta_2 - \delta_1(q-q^{-1})\frac{[2]_q}{[4]_q}$. 

$\lambda = q\rho = \lambda q^3$.
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