Self-forces from generalized Killing fields

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Abstract
A non-perturbative formalism is developed that simplifies the understanding of self-forces and self-torques acting on extended scalar charges in curved spacetimes. Laws of motion are locally derived using momenta generated by a set of generalized Killing fields. Self-interactions that may be interpreted as arising from the details of a body’s internal structure are shown to have very simple geometric and physical interpretations. Certain modifications to the usual definition for a center-of-mass are identified that significantly simplify the motions of charges with strong self-fields. A derivation is also provided for a generalized form of the Detweiler–Whiting axiom that pointlike charges should react only to the so-called regular component of their self-field. Standard results are shown to be recovered for sufficiently small charge distributions.

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1. Introduction
The detailed behavior of a compact body can depend on several kinds of interactions. These might include complicated internal contact stresses as well as the effects of long-range gravitational and electromagnetic fields. Restricting attention to only a few quantities like the center-of-mass acceleration often eliminates most of the dependence on an object’s internal details. Long-range fields largely determine the ‘bulk’ motion. Such effects can often be thought of as having two components. One is essentially imposed by the external universe, while the other arises from the body itself. There can be some physical ambiguity in this splitting, although there are many systems where it provides significant simplifications.

The discussion here focuses on the self-forces and self-torques affecting a body’s net linear and angular momenta. This problem has a very long history. One of its interesting aspects follows from the observation that a body’s own fields strongly depend on the details of its internal structure. Despite this apparent complication, there exists a regime where the motion remains relatively independent of that structure. Only a small portion of a body’s self-field directly influences its bulk motion (at least if inertial effects are excluded). This conclusion has
been reached using a number of calculations that derive approximate self-fields for extended bodies using perturbation theory [1–5]. Unfortunately, such methods are extremely tedious, specialized and not particularly enlightening. Similar results are much more easily obtained by writing down axioms for the behavior of point particles [6–10]. It is then assumed at the outset that only a particular portion of the self-field affects the motion. This has been an expectation rather than a prediction of the underlying theory.

One goal of this paper is to show that appropriately sharpened versions of these assumptions can be derived from first principles. An ignorable component of the self-field may be identified and removed in the full theory. It is not necessary to appeal to perturbation theory or the mathematical inconsistencies of point particles. This is done by considering which portions of the self-field satisfy an appropriate analog of Newton’s third law. Any such components cannot affect the net momenta, and may be discarded. Geometrically, this has the interpretation of considering how a particular Green function is deformed under the action of a generalized Poincaré group first discussed in [11]. This is the same group used to generate the quantities referred to as momenta in the absence of exact Killing fields.

The modern interest in self-force problems has mainly been motivated by the problem of extreme mass ratio binaries inspiralling under the action of gravitational radiation. Rather than considering this problem directly, the work here focuses on the model problem of a charge coupled to a massless scalar field in a fixed (though arbitrary) background spacetime. Some of the methods needed are first introduced in the context of Newtonian gravity in section 2. Various generalizations necessary to work in the relativistic case are then discussed in section 3. A general prescription for the self-force and self-torque acting on an extended charge is derived there, along with a non-perturbative notion for the effective field momentum. These results are finally applied in section 4 to obtain the equations of motion satisfied by charges much smaller than any significant timescale or curvature radius in the problem.

2. Newtonian self-interaction

It is instructive to review the nonrelativistic self-force problem before considering its generalizations. In a sense, this is trivial. Net self-forces and self-torques acting on bodies in both Newtonian gravity and ordinary electrostatics (with constant permittivity) always vanish. Their existence is forbidden by Newton’s third law. Despite this, several important features in the analysis of this simple problem persist even in the discussion of highly relativistic systems. The methods used in this section are more complicated than immediately necessary, although their unusual features are essential for subsequent generalizations.

Proving that Newtonian self-forces vanish first requires a precise definition for the field that generates them. This is, of course, meant to be the portion of the field produced by the object itself. Consider a compact body with a finite radius that interacts with the external universe purely via Newtonian gravity. The total gravitational potential $\phi$ is then determined by

$$\nabla^2 \phi(x, s) = 4\pi \rho(x, s),$$

(1)

where $\rho(x, s)$ represents the mass density at time $s$. This equation can be systematically solved for all reasonable mass distributions by introducing a symmetric Green function $G(x, x') = G(x', x)$ satisfying

$$\nabla^2 G(x, x') = 4\pi \delta(x, x').$$

(2)

This equation has a unique solution if $G$ is assumed to vanish when its arguments are infinitely separated. Once it is known, the gravitational potential produced by any mass distribution is
straightforward to compute. Denote the region occupied by the body in question at time \( s \) by \( \Sigma(s) \). It is then natural to let the self-field be given by

\[
\phi_{\text{self}}(x, s) = \int_{\Sigma(s)} \rho(x', s) G(x, x') \, dV'.
\]  

(3)

While this is a very common definition, it is not the only one. Some authors use the term in a purely perturbative sense indicating a difference between the total field with and without the body of interest \[12\]. While useful in some specialized contexts, interactions with external matter make it very difficult to derive any general properties of these difference fields. The net forces they generate do not necessarily vanish, for example. Those associated with (3) do, so they are all that will be considered here.

It is straightforward to explore the consequences of self-fields with this form. As is typical with self-force or radiation reaction problems, the focus will be on determining the bulk or ‘macroscopic’ aspects of a body’s motion. Intricate details of an object’s shape and internal composition are ignored as much as possible. The hope is that there exist a small number of state parameters that generically describe some interesting behavior in a large class of compact bodies. The typical example of such a parameter is the center-of-mass position \( \gamma(s) \). At least in certain limits, this couples very weakly to an object’s shape. Most state variables that are typically considered can be derived from a body’s net linear and angular momenta. There is an important reason for this. If the laws of motion which are to be derived are as general as hoped, the physics used to obtain them should be similarly generic. One might expect to make use of geometric structures in the background space and their effects on the laws of motion of arbitrary systems. The obvious examples derive from results like momentum conservation that are associated with underlying geometric symmetries.

The three-dimensional Euclidean space of Newtonian physics (as traditionally formulated) admits a six-parameter family of Killing fields. Given some closed system, each of these is associated with a conserved quantity built from the total linear and angular momenta. If the velocity field of the matter is denoted by \( u^a \), the conserved quantity associated with a Killing field \( K^a \) has the form

\[
P_{K}^{\text{tot}} = \int_{M} \rho u_a K^a \, dV,
\]  

(4)

where \( M \) denotes the entire space. Translational Killing fields generate components of the system’s linear momentum, while rotational Killing fields generate components of its angular momentum. Consider only the behavior of a particular body with finite radius. Its momenta are parametrized by an analog \( P_K \) of (4) obtained by integrating \( \rho u_a K^a \) over \( \Sigma(s) \subset M \). Such quantities are not usually conserved.

The time-dependence of each \( P_K \) is easily derived from the standard equations of continuum mechanics. A mass distribution with the stress tensor \( \Sigma_{ab} = \Sigma_{(ab)} \) generically satisfies

\[
\frac{\partial}{\partial s} (\rho u_a) + \nabla_b (\rho u_a u^b + \Sigma_a^\ b) = -\rho \nabla_a \phi.
\]  

(5)

This could partially describe the dynamics of some elastic solid. It is exactly Euler’s equation for a perfect fluid if the stress tensor is proportional to the metric. Regardless, the component of a body’s momentum generated by a Killing field \( K^a \) must evolve according to

\[
P_K = \frac{dP_K}{ds} = -\int_{\Sigma} \rho \mathcal{L}_K \phi,
\]  

(6)

where \( \mathcal{L}_K \) represents a Lie derivative along \( K^a \). The stress tensor does not appear explicitly in this equation, so the net forces and torques are approximately independent of the type of
material under consideration. This independence is not exact because $\Sigma_{ab}$ is still present implicitly in the equations governing changes in the mass density.

The interpretation of the scalars $P_K$ as components of momenta can be made more clear by directly introducing such objects as tensors at the mass center $\gamma(s)$. Using standard definitions for the linear momentum $p_a$ and angular momentum $S_{ab}$,

$$P_K = p_a K^a + \frac{1}{2} \epsilon_{abc} S^a \nabla^b K^c = p_a K^a + \frac{1}{2} S_{ab} \nabla^a K^b. \quad (7)$$

The second equality introduces the dual $S_{ab} = \epsilon_{abc} S^c$ to the usual angular momentum vector. This is the more fundamental quantity in the relativistic case, although the two objects are interchangeable in three dimensions. Appropriate choices for the Killing field in (7) can be used to extract any component of the linear or angular momenta. As an example, the translational vector field fixed by setting $\nabla_a K_b = 0$ and $K_a = \nabla_a x$ at some $\gamma(s)$ would recover the $x$-component of $p^a(s)$. Complete knowledge of the family $P_K$ for all possible Killing fields is equivalent to that of $p^a$ and $S_{ab}$. It is more than sufficient to extract the center-of-mass motion.

Now consider only the self-field’s effect on the momenta. Combining (3) with (6) shows that

$$\dot{P}_{self}^K = -\frac{1}{2} \int_{\Sigma} \int_{\Sigma} \rho \rho' L_K G(x, x'), \quad (8)$$

where $\rho = \rho(x, s)$ and $\rho' = \rho(x', s)$. Lie derivatives of two-point functions are defined to act independently on both of the subject’s arguments, so

$$L_K G(x, x') = K^a(x) \nabla_a G(x, x') + K^a(x') \nabla_a G(x, x'). \quad (9)$$

The derivation of (8) effectively replaced $K^a \nabla_a G$ with $L_K G/2$ by commuting integrals. While the mathematical justification for this is clear, it is interesting to mention its physical significance. The operation effectively averages ‘action–reaction pairs’ in the sense of Newton’s third law. It says that bulk self-field effects arise only if there are imbalances between the forces exerted by (say) mass in $dV$ on mass in $dV'$ versus the reverse. This is exactly what would be expected from intuitive considerations. Proving that Newtonian self-forces and self-torques vanish now requires only one more ingredient.

The Green function adopted here has been fixed by choosing it to vanish at infinity. The simplicity of this boundary condition together with the form of (2) implies that $G$ can only depend on the distance between its arguments. In anticipation of later generalizations, it may be thought of purely as a function of Synge’s world function $\sigma(x, x')$. This biscalar returns one-half of the geodesic distance between its arguments $[13, 14]$. Translating or rotating any two points by equivalent amounts does not change the distance between them, so

$$L_K \sigma(x, x') = 0, \quad (10)$$

for any Killing field $K^a$. Substituting this result into (8) immediately shows that $\dot{P}_{self}^K = 0$. This is the desired result: compact objects do not experience any self-force or self-torque in Newtonian gravity. It is a statement completely independent of a body’s shape or detailed structure. Essentially all that was used was the translational and rotational invariance of the Green function and the generic equation of motion (5). Invariance under translational Killing fields is equivalent to the weak form of Newton’s third law. Supplementing it with the rotational invariance recovers the strong form. An almost identical calculation leads to similar conclusions in ordinary electrostatics (with similarly simple boundary conditions) and many other theories. While this result could have been derived more directly, many aspects of the method presented here can now be generalized to analyze fully relativistic systems. Despite an apparent reliance on the symmetries of Euclidean space, geometric objects can
be defined that allow similar manipulations even in spacetimes admitting no Killing vectors at all.

Before discussing this, it should first be noted that there is a complementary method of understanding the Newtonian self-force problem. The approach just described effectively sums up the forces acting inside an extended body. Identical conclusions can also be obtained purely from the distant behavior of the gravitational field. Combining (1) and (6),

\[ \dot{P}_K = -\frac{1}{4\pi} \oint_{\partial \Sigma} \left[ \nabla^a \phi \nabla_a \phi - \frac{1}{2} \nabla^b \phi \nabla_b (K^a \phi) \right] dS_a. \]

The effect of the self-field may be found by substituting \( \phi \rightarrow \phi^{\text{self}} \) in this equation. The surface integral can then be evaluated over closed surfaces outside of \( \partial \Sigma \), if desired. It is convenient to consider spheres extending to infinity. The potentials are harmonic functions in this region, so they must fall off at least as fast as \( 1/r \) as \( r \rightarrow \infty \). Any term that does decrease this slowly cannot have any angular dependence. These two facts together with the properties of the Killing fields show that all surface integrals like (11) must vanish. It follows that \( \dot{P}_{K}^{\text{self}} = 0 \), as expected. Similar (though much more complicated) derivations can be applied in the relativistic self-force problem, although most of the discussion below takes the more local viewpoint embodied by the derivation of (8).

3. Relativistic scalar fields

The discussion just presented suggests that self-forces and self-torques could arise from local asymmetries in a field’s underlying Green function. Indeed, very small changes in the statement of the Newtonian self-interaction problem allows for the existence of nontrivial self-forces. Replacing the metric in the field equation (1) with one that is not maximally symmetric easily accomplishes this, for example. Using an elliptic differential operator constructed from non-geometric objects can have a similar effect. While (8) should not necessarily be blindly applied in such cases, it is clear that significant self-forces may arise.

These sorts of modifications are physically relevant in several contexts. Static systems involving a curved spacetime are often simplified with the use of a dimensional reduction procedure. Laplace operators constructed from non-Euclidean metrics then arise naturally in the field equations. Another interesting case is that of ordinary electrostatics in the presence of dielectric materials [15]. Even though the underlying space is very simple, the field equation need not be invariant under translations or rotations. Both of these systems allow self-fields to strongly affect the evolution of a body’s net linear and angular momenta. Relativistic extended bodies moving in curved spacetimes experience very similar effects.

Other mechanisms are also at work, however. The transition to a relativistic system involves fundamental changes in the character of the fields. Mathematically, they usually shift from solutions of an elliptic to a hyperbolic differential equation. This has several physical consequences. Newtonian potentials are uniquely determined by the instantaneous distribution of mass in the universe, for example. Relativistic potentials are not. The past history of a system is effectively remembered by the field in a complicated way. It acquires its own degrees of freedom, and may transport energy and momentum at a finite speed. These differences lead to the importance of radiation reaction and tail effects in self-force problems. While not completely independent of each other, the three mechanisms just described can all lead to significant self-forces.

A useful model system in which to illustrate these statements consists of a finite material body interacting with a scalar field \( \Phi(x) \). The spacetime will be assumed well behaved in a neighborhood of the body’s worldtube \( W \). Many of the results derived below do not make
any significant assumptions about the metric. Despite this, it would be difficult to draw useful conclusions from them if the geometry were significantly influenced by the presence of the charge. Effects related to gravitational self-interaction will therefore be ignored here. The metric is assumed to be known at the outset. This is essentially a test mass (but not a test charge) approximation. By contrast, the scalar sector here is treated exactly. Mild conditions ensuring the existence of standard center-of-mass constructions are assumed (see, e.g., [20]), although no perturbative restrictions need to be placed on the body’s internal structure beyond those required for its gravitational field to be negligible. Timescales in the system need not be large compared to a light-crossing time, for example.

The type of scalar field chosen is not particularly important as long as its field equation is linear. Still, some steps carried out below can be copied over from previous work if it is assumed that $\Phi$ is a massless minimally-coupled field satisfying

$$\Box \Phi(x) = -4\pi \rho(x).$$

(12)

$\rho$ represents the scalar charge density in this equation. It is straightforward to allow for a finite field mass or curvature coupling, although this is an unnecessary complication. In the Newtonian case, the field equation was needed mainly to define a Green function. The same is true here. Let

$$\Box G(x, x') = -4\pi \delta(x, x').$$

(13)

Solving this equation requires that certain boundary conditions be imposed. The physical self-field will be defined by those associated with the retarded Green function $G_{\text{ret}}$:

$$\Phi_{\text{self}}(x) = \int_W \rho(x') G_{\text{ret}}(x, x') \, dV'. $$

(14)

By construction, only points on the worldtube lying in the causal past of $x$ contribute to this integral.

The scalar self-force problem now asks how such a field affects the bulk motion of the charge that sources it. As in the Newtonian case, it is reasonable to proceed by computing shifts in the body’s momenta. Appropriate analogs of the scalars (4) take the form

$$P_\xi(s) = \int_{\Sigma(s)} T^a_{\ b} \xi_b \, dS_a,$$

(15)

where $\xi^a$ is an as-yet unspecified vector field, $T^{ab}$ the body’s stress–energy tensor and $\Sigma(s)$ some spacelike hypersurface. The family of all such hypersurfaces is assumed to foliate the worldtube $W$. It is not generally possible to choose the generating vector fields in (15) to be Killing. Despite this, there should be some sense in which they come as close as possible to this ideal. A set of approximate Killing fields suggested in [11] will be adopted here. These exactly satisfy

$$\mathcal{L}_{\xi} g_{ab} |_\Gamma = \nabla_a \mathcal{L}_{\xi} g_{bc} |_\Gamma = 0,$$

(16)

where $\Gamma$ is a preferred timelike worldline involved in their construction. They are completely fixed throughout $W$ by the values of $\xi^a$ and $\nabla_a \xi_b = \nabla_{[a} \xi_{b]}$ at any point on this worldline. Each choice of initial data in this form defines a unique generalized Killing field$^1$ (GKF) $\xi^a$. Any genuine Killing fields that may exist are in this class. The set of all GKF form a generalization $GP$ of the Poincaré group. Like the standard Poincaré group, it has ten dimensions in four-dimensional spacetimes.

$^1$ The approximate symmetries defined in [11] took the form of vector fields with the property (among many others) that $\nabla_a \mathcal{L}_{\xi} g_{bc} |_\Gamma = 0$. These were called generalized affine collineations, or GACs. Some satisfy $\mathcal{L}_{\xi} g_{ab} |_\Gamma = 0$, and it is this subset of vector fields that properly generalize the Killing fields. They were referred to as Killing-type GACs before, although we now shorten this to generalized Killing fields (GKFs). They are all that will be used here.
This summary suggests an analog to (7). If \( p^a(s) \) and \( S_{ab} = S_{(ab)}(s) \) are the body’s linear and angular momenta represented as tensors at \( \gamma(s) = \Sigma(s) \cap \Gamma \), a relation of the following form should exist:
\[
P_\xi = p^a \xi_a + \frac{1}{2} S_{ab} \nabla_{(a} \xi_{b)}.
\] (17)
This may be taken as a definition. The resulting momenta are exactly those suggested by Dixon as being particularly useful for understanding the mechanics of extended bodies in curved spacetimes [11, 16–18]. Like the GKF, they depend on both \( \Gamma \) and \( \Sigma \). These objects will be assumed to be fixed using center-of-mass conditions [19, 20]. Varying over all possible GKF, \( P_\xi \) becomes a map from \( GP \times \mathbb{R} \to \mathbb{R} \). Knowledge of its behavior is completely equivalent to knowledge of \( p_a \) and \( S_{ab} \). These quantities are sufficient to determine a body’s mass, spin, center-of-mass worldline and so on. This includes almost all of the local parameters typically computed in self-force problems. There is also a sense in which it extracts all of the information that can be recovered purely from stress–energy conservation [17].

Rates of change of the scalar momenta \( P_\xi \) are easily related to more standard definitions for forces and torques. As discussed in [11], the GKF satisfies Killing transport equations on \( \Gamma \). It then follows from (17) that
\[
dP_\xi / ds = \left( p^a - \frac{1}{2} S^{bc} \dot{\gamma}^d R_{bcd} \right) \xi_a + \frac{1}{2} (S^{ab} - 2 p^a \dot{\gamma}^b) \nabla_{(a} \xi_{b)}.
\] (18)
Knowing the left-hand side allows the instantaneous force \( F^a \) and the torque \( N_{ab} = N_{(ab)} \) to be extracted. These objects are typically defined such that [16]
\[
\dot{p}^a = F^a + \frac{1}{2} S^{bc} \dot{\gamma}^d R_{bcd}^a
\] (19)
\[
\dot{S}_{ab} = N_{ab} + 2 p_{(a} \dot{\gamma}_{b)}.
\] (20)
Note that the Papapetrou equations hold if \( F^a = N_{ab} = dP_\xi / ds = 0 \).

Many consequences of \( \Phi^{\text{self}} \) cannot be determined from its direct effect on the momenta. Self-fields strongly influence the equilibrium shapes (and therefore the higher multipole moments) of highly charged objects, help resist tidal deformations, perturb distant matter and so on. These phenomena usually couple very weakly to \( P_\xi \). They are effectively ignored by the current formalism. This does not necessarily mean that they are negligible compared to the effects considered here. Especially in the context of gravitational self-interactions, self-forces defined in the present manner may be comparable to forces arising from shifts in the external field due to perturbations of distant masses. This is illustrated explicitly in [21], and is a standard problem. It can usually be made less severe in the scalar and electromagnetic self-force problems, so we will ignore it. The methods introduced here can at least be used to simplify a significant portion of the overall problem of motion.

The time-dependence of the momenta \( P_\xi \) follows from stress–energy conservation. If the only long-range field other than gravity is \( \Phi \), this requires that
\[
\nabla_a (T_{ab} + I_{ab}) = 0,
\] (21)
where the stress–energy tensor of the scalar field is
\[
I_{ab} = \frac{1}{4 \pi} \left( \nabla^a \Phi \nabla^b \Phi - \frac{1}{2} \delta^{ab} \nabla_c \Phi \nabla^c \Phi \right).
\] (22)
Combining these expressions with (12) gives
\[
\nabla_a T^a_{\ b} = \rho \nabla_b \Phi.
\] (23)
The Newtonian limit of this equation is essentially identical to (5). Its right-hand side represents the force density exerted by the scalar field on the matter distribution.
The evolution of the momenta is more convenient to analyze in terms of finite differences

\[ \delta P_\xi(s_2, s_1) = P_\xi(s_2) - P_\xi(s_1) \]  

(24)
rather than instantaneous rates of change. These might represent changes in a body’s mass or spin over the time interval \((s_1, s_2)\). Suppose that \(s_2 > s_1\), and that \(s\) increases monotonically as \(\gamma(s)\) extends into the future. If \(\Omega = \Omega(s_1, s_2)\) denotes the portion of \(W\) lying between the hypersurfaces \(\Sigma(s_1)\) and \(\Sigma(s_2)\), and \(T^{ab}|_{\partial W} = 0\), Gauss theorem shows that

\[ \delta P_\xi = \int_\Omega \left( \frac{1}{2} T^{ab} \mathcal{L}_\xi g_{ab} + \rho \mathcal{L}_\xi \Phi \right) dV. \]  

(25)
The first term here represents gravitational force and torque. It exists regardless of whether any scalar field is present. On each \(\Sigma(s)\), this portion of the integrand can be shown to be equivalent to a Christoffel symbol contracted with \(T^{ab}\) in a normal coordinate system based at \(\gamma(s)\) [18]. If the background geometry varies slowly throughout the body (both spatially and temporally), this gravitational term can be expanded in terms of the multipole moments of the stress–energy tensor. The lowest-order contribution comes from the quadrupole, and is relatively simple to take into account. Detailed examples of this exist in the literature [22, 23].

This leaves only the scalar field’s contribution to the momentum shift. It may be split into two parts. First consider the portion due to \(\Phi^{self}\). Let

\[ \delta P_\xi^{self} = \int_\Omega \rho \mathcal{L}_\xi \Phi^{self} dV. \]  

(26)
The remaining (external) component of the scalar field usually varies slowly inside the body. Its contribution to the motion may therefore be evaluated using another multipole expansion. Similar methods cannot be directly used to understand the self-force. As it stands, \(\Phi^{self}\) almost always varies rapidly over scales comparable to the body’s proper radius. Successive terms in a multipole expansion of (26) would therefore fail to decrease in magnitude. Such a series would not be useful.

The form of the self-force simplifies if another split is made. Let \(T^-\) denote the portion of \(W\) lying in the exclusive past of \(\Omega\); i.e. \(T^- = (J^-[\Omega]|\Omega) \cap W\) in the notation of [24]. Defining the retarded field sourced by charge in an arbitrary region \(\Lambda\) by

\[ \Phi^{ret}[\Lambda] = \int_\Lambda \rho' G^{ret} dV', \]  

(27)
it is trivially true that

\[ \Phi^{self}(x) = \Phi^{ret}[W] = \Phi^{ret}[\Omega] + \Phi^{ret}[T^-]. \]  

(28)
for any \(x \in \Omega\). The second term here will be left as-is for now. Its contribution to the self-force is reasonably well behaved even for a \(\delta\)-function source. The field due to charge in \(\Omega\) is more interesting. This is where most of the Coulomb and other quickly-varying components of the self-field arise.

Forces and torques exerted by \(\Phi^{ret}[\Omega]\) can be simplified by introducing regular and singular Green functions \(G_R\) and \(G_S\). For now, these objects will only be required to satisfy

\[ G^{ret} = G_R + G_S \]  

(29)
and the reciprocity relation \(G_S(x, x') = G_S(x', x)\). It will later be useful to also suppose that

\[ \square G_R = 0. \]  

(30)
This contradicts (13)—which applies for both \(G^{ret}\) and \(G_S\)—so it is something of a misnomer to call \(G_R\) a Green function. It is common to do so, however, and this practice will be followed.
here. These properties have been chosen so that $G_S$ is as close to a Newtonian Green function as possible. This presumably minimizes its influence on the body’s overall motion.

Many propagators with these properties exist, however. Perhaps the simplest derives from using a $G_S$ with the form

$$G_{S,D} = \frac{1}{2}(G_{\text{ret}} + G_{\text{adv}}),$$

where $G_{\text{adv}}$ is the advanced Green function. The regular or radiative Green function derived from this choice using (29) was central to Dirac’s classical electron model [6]. It will be referred to here as the Dirac Green function. Another possibility is to use the construction given by Detweiler and Whiting in connection with point particle self-force regularization in curved spacetimes [9, 13, 25]. Regardless of their specific definitions, the names given to these objects derive from their connection to point particle self-fields. The linearity of the field equation and (29) suggest that such fields may be split into singular and regular components respectively sourced by $G_S$ and $G_R$. For a point particle, the portion derived from the singular Green function diverges on its worldline. The remainder of the self-field remains bounded even at the source’s location. It is typically associated with radiation. Self-forces are often thought of (somewhat incompletely) as the local reaction to emitted radiation, so one would expect most of the self-interaction to arise from fields associated with $G_R$. Note that neither of the Green functions introduced here lead to any singular behavior for well-behaved extended charge distributions.

Regardless of which specific choices are made for $G_R$ and $G_S$, a relativistic analog of (8) is easily derived. Pair averaging is only meaningful for fields derived from Green functions that are symmetric in their arguments. This is one of the defining properties of $G_S$, so the averaging will only be applied on the singular portion of the self-field. It is then straightforward to show that (26) is equivalent to

$$\delta P^{\text{self}}_{\xi} = \int_{\Omega} dV \rho \left[ \mathcal{L}_{\xi} \left( \Phi_R[\Omega] + \Phi_{\text{ret}}[T_{\Omega}] \right) + \frac{1}{2} \int_{\Omega} dV' \rho' \mathcal{L}_{\xi} G_S \right].$$

(32)

$\Phi_R[\Lambda]$ is defined here by analogy to (27). The regular self-field affecting the momentum shift in this equation only depends on charge in $\Omega$. There are therefore no conceptual obstacles to adopting a $G_R$ with support in the chronological future of the field point.

This freedom has an undesirable consequence. $\delta P^{\text{self}}_{\xi}$ is trivially interpreted as the time average of $P_{\xi}^{\text{self}}$. The instantaneous force or torque is expected to depend only on the properties of the physical system and the choice of GKF. It is not affected by arbitrary parameters such as $s_1$ and $s_2$. Individual terms on the right-hand side of (32) might be expected to share this property. They do not. The various fields there are all derived from sources with sharp temporal boundaries. Their behavior always changes abruptly near these regions. This has no physical significance. The total motion is unaffected (as it must be), although it makes the interpretation of the various self-force contributions more difficult. Such effects can be separated out explicitly. Rewriting (32),

$$\delta P^{\text{self}}_{\xi} = \int_{\Omega} dV \rho \left( \mathcal{L}_{\xi} \Phi_R[W] + \frac{1}{2} \int_{W} dV' \rho' \mathcal{L}_{\xi} G_S + \mathcal{L}_{\xi} \Phi_S[W \setminus \Omega] \right) - \frac{1}{2} \int_{W \setminus \Omega} dV' \rho' \mathcal{L}_{\xi} G_S.$$  

(33)

The first two terms in parentheses here only depend on properties of the physical system. The remaining quantities are different. They are directly linked to the choice of $\Omega$. No matter how simple the charge distribution and external fields may be, this portion of the integrand always changes character near $\Sigma(s_1)$ and $\Sigma(s_2)$. Its contributions to the self-force would
be simplified if there was a sense in which they only contributed to the integral near these hypersurfaces.

This is accomplished by adding an additional axiom to those constraining the singular and regular Green functions. Although the specific Dirac form (31) for $G_S$ is very simple, it generically has support in the entire causal past and future of any field point. Computing terms like $\mathcal{L}_t \Phi_S[W\Omega]$ in (33) would then require knowing the entire past and future history of the system. There is no region in causal contact with $W$ where this quantity would be expected to vanish. Suppose that another singular Green function is chosen that always vanishes whenever its arguments are timelike separated. This is true of (31) only in flat spacetime. More generally, this assumption together with the original axioms constraining $G_S$ and $G_R$ uniquely specify the aforementioned Detweiler–Whiting Green functions [9, 13, 25].

Before specifying these objects explicitly, it is first useful to review the Hadamard decomposition for the singular Green function defined in (31). This has the form

$$G_{S,D}(x, x') = \frac{1}{2} \left[ \Delta^{1/2} \delta(\sigma) + V \Theta(-\sigma) \right](x, x').$$

(34)

There are two distinct contributions here. One—familiar from the study of massless fields in $(3+1)$-dimensional Minkowski spacetime—is concentrated entirely on the light cones of the field point $x$. The biscalar coefficient $\Delta$ is known as the van Vleck determinant. It may be expressed in terms of the first two derivatives of the world function $\sigma$ via [13]

$$\Delta(x, x') = -\det[-\nabla_a \nabla'_b \sigma(x, x')] \sqrt{-g} \sqrt{-g'}.$$  

(35)

In all reasonable cases of interest here, this is smooth, positive and reduces to unity as $x \to x'$. Its first derivatives also vanish at coincidence. The second (tail) term in (34) takes into account that disturbances in the field do not necessarily propagate only on null rays. $V(x, x')$ depends on the details of the spacetime, and is almost always nonzero. It does remain smooth, however. While $V$ is usually difficult to find, its coincidence limit is known to be [13]

$$\lim_{x' \to x} V(x, x') = \frac{1}{12} R(x).$$  

(36)

Both $\Delta$ and $V$ are symmetric in their arguments.

Using all of these definitions, the Detweiler–Whiting singular Green function may be shown to have the form [9, 13, 25]

$$G_{S,DW} = G_{S,D} - \frac{1}{2} V = \frac{1}{4} [\Delta^{1/2} \delta(\sigma) - V \Theta(\sigma)].$$

(37)

This generically has support everywhere but in the chronological past or future of either of its arguments. The regular Detweiler–Whiting Green function obtained from (29) has support everywhere except inside the future null cone of each field point. These propagators are therefore acausal. All derivations here have started by expanding retarded Green functions, so this has no unphysical consequences. Adopting the Detweiler–Whiting Green functions greatly simplifies the interpretation of (33). Each term in that equation might have initially appeared to involve knowledge of the body’s behavior in the infinite future. While the sum of all such contributions cancels out, it is not immediately obvious how this occurs. Setting $G_S = G_{S,DW}$ largely removes this problem. Knowledge of the system then appears to be required only out to times of order the body’s diameter beyond $s_2$. Such contributions still do not have direct physical consequences, although they are now much simpler to control and understand.

A more concrete advantage of these special Green functions is that the meaning of the Detweiler–Whiting axiom can now be clarified. This states that the self-field derived from $G_{S,DW}$ exerts no force on a point particle [9, 13, 25]. Equations of motion obtained with this assumption are identical to those appearing in all other treatments of point particle motion. It
is therefore interesting to see how well it applies for a finite extended body. It cannot be exact, as it is known that the singular self-field contributes an effective mass to extended charges. This is a consequence of the fact that accelerating a particle requires accelerating both its matter and field components. It is a somewhat trivial effect in the sense that masses like those defined as

\[ m = \sqrt{-p^\mu p_\mu} \]  

(38)

are rarely measured directly. Doing so would require detailed knowledge of an object’s stress–energy tensor. More realistically, inertial masses are usually measured by observing an object’s motion under the application of known external forces. This method would recover a mass that included contributions from both \( p^\mu \) and the self-field. The interesting question is whether there are any effects on a body’s motion induced by its singular self-field that cannot be attributed merely to a (possibly time-dependent) mass shift. Calculations using perturbation theory in flat spacetime electromagnetism have found such phenomena even in cases where the Detweiler–Whiting axiom applied to a finite charge would imply the point particle equations of motion [5]. Interestingly, the methods introduced so far allow significant insight to be gained into this result without the use of any approximations.

First restrict attention to momentum shifts over times \( \delta s \) longer than the body’s light-crossing time \( D \). More precisely, assume that every point in \( \Sigma(s_2) \) is timelike separated from every point in \( \Sigma(s_1) \). Setting \( G_S = G_{S, DW} \), the last two terms in (33) may then be associated entirely with the boundary caps of \( \Omega \). Their contribution to the self-force and self-torque has the form \( \mathcal{E}_\xi(s_1) - \mathcal{E}_\xi(s_2) \), where

\[ \mathcal{E}_\xi = \frac{1}{2} \left( \int_{\Sigma^-} \rho \mathcal{L}_\xi \Phi^{S, DW}[\Sigma^-] \, dV - \int_{\Sigma^+} \rho \mathcal{L}_\xi \Phi^{S, DW}[\Sigma^+] \, dV \right). \]  

(39)

If \( \mathcal{E}_\xi = \mathcal{E}_\xi(s) \), the two regions of integration in this equation bisect the body’s worldtube. \( \Sigma^+(s) \) denotes the portion of \( W \) to the future of \( \Sigma(s) \), while \( \Sigma^-(s) \) represents the volume to its past. Both of these domains are unbounded, although the definition of the singular Green function used here effectively restricts them to small volumes extending over time intervals of \( D \) away from \( \Sigma(s) \).

Combining (33) and (39), all compact charge distributions are found to satisfy

\[ \delta(P^\xi_{self} + \mathcal{E}_\xi) = \int_{\Omega} dV \rho \left( \mathcal{L}_\xi \Phi_{R, DW}[W] + \frac{1}{2} \int_{\Omega} dV' \rho' \mathcal{L}_\xi G_{S, DW} \right). \]  

(40)

This result is exact as long as \( \delta s \) is not too small. Considering differences in the momenta between three times \( s_1, s_2, s_3 \) and \( s_1 \) satisfying \( s_3 - s_1 \gg D \) and \( s_2 - s_1 \gg D \) shows that it is actually correct for any time interval. It is therefore possible to consider an instantaneous form of (40). Restoring the forces directly due to the geometry and the external scalar field

\[ \frac{d}{ds}(P^\xi + \mathcal{E}_\xi) = \int_{\Sigma} \left[ \frac{1}{2} T^{ab} \mathcal{L}_\xi g_{ab} + \rho \mathcal{L}_\xi (\Phi^{ext} + \Phi_{R}^{self}) + \frac{1}{2} \int_{\Omega} dV' \rho' \mathcal{L}_\xi G_S \right] r^a dS_a. \]  

(41)

where \( r^a \) is the time evolution vector field for the foliation (\( \Sigma \)). It is also implicit here that \( \Phi_{R}^{self} = \Phi_{R}[W] \) and \( \Phi^{ext} = \Phi - \Phi^{self} \). This result holds for any \( G_S \) and \( G_R \). In general, though, individual terms will require knowledge of the system into the infinite future. It is only when the Detweiler–Whiting Green functions are used that this dependence is restricted to small times of order \( D \).

The third term on the right-hand side of (41) is expected from the Detweiler–Whiting axiom generalized for an extended body. There are two corrections to this. As already discussed, the term involving \( \mathcal{L}_\xi G_S \) arises from averaging action–reaction pairs in the sense
of Newton’s third law. It vanishes identically in Minkowski and de Sitter spacetimes. The calculations in section 4 also suggest that it generally contributes very little to the motion of a charge that is sufficiently small compared to the curvature scales of the background geometry.

The other interesting term in (41) involves \( \mathcal{E}_\xi \). Its presence suggests that there is a sense in which the momenta derived from (15) are incomplete. Extended charges seem to respond as though they had effective momenta

\[
\hat{p}_\xi = p_\xi + \mathcal{E}_\xi.
\]

The singular self-field is effectively conservative up to the term involving \( \mathcal{L}_\xi G \) in (41). Comparing these renormalized momenta with (39) and an expression analogous to (17) can be used to define \( \hat{\rho}_a \) and \( \hat{S}_{ab} \). Differences with their unhatted counterparts can be interpreted as being due to the momenta of the singular component of a particle’s self-field. It is straightforward to verify this identification in stationary systems. Assuming that all relevant quantities are time symmetric about some \( \Sigma \),

\[
\mathcal{E}_\xi = -\frac{1}{2} \int_\Sigma \rho \Phi_3[W] \xi^- dS_a.
\]

This is strongly reminiscent of the expression for a system’s self-energy. For a body in geodesic motion in flat spacetime, it has the explicit form

\[
\mathcal{E}_\xi = \frac{1}{2} \int d^3r d^3r' \rho(r) \rho(r') \left( \frac{\gamma^a [\xi_a(y) - r' \nabla_a \xi_i(y) ]}{|r - r'|} \right).
\]

Minkowski coordinates have been used here in the obvious way. The worldline defined by \( \mathbf{r} = 0 \) corresponds to one used to construct the GKF's. The net effect of \( \mathcal{E}_\xi \) on \( \hat{\rho}_a \) in this case is to add to the bare mass \( m \) a term equal to the (singular component of the) particle’s self-energy. The effective angular momentum may also be changed by \( \mathcal{E}_\xi \). This shift is purely orbital in character, and vanishes when the origin is placed at the center of the self-energy distribution. In general, this point will not coincide with the center-of-mass computed purely from \( T^{ab} \).

It is typical to define a center-of-mass frame by demanding that \( \Gamma \) be chosen such that

\[
(p^a S_{ab})^\Gamma = 0.
\]

Each \( \Sigma (s) \) is to be formed from the set of all geodesics passing through \( \gamma(s) \) orthogonally to \( p^a(s) \) \([19, 20]\). The existence of an effective momentum here suggests an alternative mass center \( \hat{\Gamma} \) that could be defined via

\[
(p^a \hat{S}_{ab})^\Gamma = 0,
\]

along with an appropriate condition for a foliation \( \{\hat{\Sigma}\} \). It is unclear which of these definitions more appropriately captures some sense of a charge’s average position. The laws of motion (41) are simpler for the effective rather than the bare momenta, so \( \hat{\Gamma} \) should have simpler evolution equations. This is not sufficient to justify assuming that the resulting worldline is preferable, although it is suggestive. Of course, there are many nontrivial cases where \( \Gamma = \hat{\Gamma} \), or where any differences are extremely small.

It is useful to compare the results obtained so far to those derived (by very different methods) in \([5]\). There, the motions of a large class of extended electromagnetic charge distributions were studied in flat spacetime. This was done by assuming that the self-fields were derived from either retarded or regular Green functions. The momenta naturally associated with electromagnetically interacting bodies are more complicated than (15) \([16, 17, 26]\). There is an additional term involving the electromagnetic field and current distribution that does not have an analog in the scalar case considered here. Despite this, expressions very similar to (41)
might be expected to remain valid. Any term involving a Lie derivative of the electromagnetic Green function would vanish in flat spacetime, so the self-forces and self-torques might be expected to involve only the regular self-field and some analog of $\xi$. Considerable differences were found in [3] between the regular and retarded electromagnetic self-forces acting on the bare momenta. This was true even after obvious mass rescalings were taken into account. It therefore appears that changes in the self-momentum can have a nontrivial effect on $P_{\xi}$. Verifying this in the scalar case would require detailed calculations that will not be attempted here.

It was mentioned in section 2 that the motion of a Newtonian mass can be determined either by locally analyzing its internal forces or by studying the asymptotic structure of its self-field. This is also true in the relativistic case. We have focused on the local viewpoint so far. Alternatively, changes in the matter’s momenta may be viewed as arising from changes in the field momenta. Using the standard stress–energy tensor (22) for a scalar field, momenta can be associated with $\Phi_1$ or $\Phi_{self}$ just as they are with $T^{ab}$. Let

$$U_{\xi} = \int_{\Sigma} t^{ab} \xi_b dS_a.$$  \hfill (47)

The scalar field does not usually have compact support, so this quantity depends on the details of $\Sigma$ outside of $W$. Regardless, stress–energy conservation implies that changes in the total momentum $P_{\xi} + U_{\xi}$ satisfy

$$\frac{d}{ds} (P_{\xi} + U_{\xi}) = \frac{1}{2} \int_{\Sigma} (T^{ab} + t^{ab}) L_{\xi} s_{ab} t^a dS_a - \oint_{\partial \Sigma} t^{ab} \xi_a t^c dS_{bc}.$$  \hfill (48)

This is closely related to the Newtonian result (11). The first integral measures the degree to which momentum fails to be conserved in a curved spacetime. Such terms will usually become negligible if the body is sufficiently small and $\Sigma$ does not extend far outside of it. Unfortunately, the surface integral is simplest to evaluate very far away from $W$. Determining the optimal balance between these two competing influences is not trivial. It is also not simple to compute $dU_{\xi}/ds$. These difficulties are mentioned merely for completeness. They do not arise in the local description used to derive (41).

4. Small charges

Self-forces affecting the motion of arbitrary bodies can be extremely complicated. Internal oscillations might produce ‘radiation rockets,’ for example. Such effects can exist even in the absence of any external influences. There is little that can usually be said about these phenomena without considering specific models. It is therefore more typical to focus on self-interactions affecting small systems close to some stable equilibrium. Making this idea precise can be difficult. In general, standard radiation reaction effects are recovered by restricting a body’s spin, the position of its ‘center-of-charge’ with respect to its mass center, the magnitude of its self-energy, speeds of internal motions and many other parameters [5]. This procedure is very complicated, so it is common to ignore at the outset all effects related to a system’s internal structure.

4.1. Distributional sources

Naïvely, one might try to do this by analyzing the behavior of point charges. In the scalar case considered here, a charge density could be chosen with the form

$$\rho(x) = \int q(t) \delta(x, z(t)) dt.$$  \hfill (49)
This represents a particle with the charge $q(s)$ concentrated entirely on a worldline parametrized by $z(s)$. For simplicity, it is usually assumed that the dipole moment vanishes (meaning that $z = \gamma$) and the charge remains constant. It is well known that the self-field of such a source diverges like $1/r$ in normal coordinates centered on its worldline. Results like (26) then appear to be meaningless. A possible reaction to this is that point particles of the given type are unphysical. It is therefore not necessary for the standard laws of physics to be compatible with them. Despite this, many authors have introduced special regularization methods intended to force such objects into the theory [6–10].

In keeping with this tradition, it is interesting to discuss how point particles can be fit into the current formalism. The required assumption is surprisingly simple. First suppose that the self-force and self-torque are to be derived from (41). As it stands, this equation is not useful for a point particle. $E_\xi$ roughly involves the self-energy, so it diverges. This problem may be removed by only working with the effective momenta $\hat{P}_\xi$ defined in (42). If it is assumed that these quantities are always finite and that the particle’s worldline satisfies (46), standard results—first derived by Quinn [8]—follow when

$$\int_{\Sigma} dS_t r^a \int_W dV' \rho' L_\xi G_S = 0. \quad (50)$$

The main intention of this section is to demonstrate that this relation holds in all spacetimes smooth near the particle. The Detweiler–Whiting Green functions will be adopted here.

Before proving (50), note that the crucial step—assuming that $\hat{P}_\xi$ is finite—is very similar to a standard mass renormalization procedure. In that case, the force on a small charge is shown perturbatively to involve a term of the form $(\text{self-energy}) \times \dot{\gamma}^a$ [4, 27]. This effectively acts to shift the particle’s mass. Although the self-energy diverges as $D \to 0$ (with $q$ fixed), it is assumed that its combination with $m$ is finite. Here, the self-momentum was identified non-perturbatively, and can be absorbed into ‘observable parameters’ at the outset. It was never necessary to obtain explicit solutions of the field equation.

Trying to derive self-forces and self-torques from (32) instead of (41) leads to a slightly different point of view. The problematic self-energy $E_\xi$ was originally found to arise from the behavior of $\Phi_\Phi[\Theta]$ and $\Phi_{\text{rot}}[T_{\text{rot}}]$ very near $\Sigma(s_1)$ and $\Sigma(s_2)$. The relevant volume shrinks to zero in the point particle case. It is therefore very easy, in performing the various required integrations, to miss it entirely. There is sufficient ambiguity that it is not really clear that it should be there at all. A somewhat carefree application of (32) would find that no renormalization was necessary at all in order to obtain finite forces and torques on point particles. This is one kind of selective ignorance. Assuming that $\hat{P}_\xi$ is finite is another. The latter point of view will be adopted here. A fully consistent analysis would consider extended charge distributions whose mass, charge and radius all shrink to zero at appropriate rates. This will be discussed in section 4.2 below.

We now derive (50) for a point charge. Assume that $\dot{q} = 0$ and that the charge density is concentrated on a center-of-mass worldline $\Gamma$ satisfying (46). This is the worldline that will be used to construct the GFKFs. For notational convenience, hats will be omitted for the remainder of this section. It is implicit that all momenta and mass centers are associated with $\hat{P}_\xi$. The problem then reduces to evaluating

$$\lim_{s_1 \to s_2} \frac{1}{\delta s} \int_{s_1}^{s_2} dr \int_{-\infty}^{\infty} dr' G_S, DW(\gamma(t), \gamma(t')). \quad (51)$$

Given (37), the integrand here involves terms proportional to $\Theta(\sigma), \delta(\sigma)$ and $\delta'(\sigma)$. The first of these is manifestly finite, and scales like $(s_2 - s_1)^2$ in the limit $s_1 \to s_2$. It is therefore
irrelevant in (51). The potentially interesting quantities are

\[ \lim_{t' \to t} \left[ \Delta^{1/2} \mathcal{L}_\xi \ln \Delta/2 - V \mathcal{L}_\xi \sigma \right] \]

and

\[ \lim_{t' \to t} \left[ \frac{1}{\bar{\gamma}^a \sigma_a'} \frac{\partial}{\partial t'} \left( \Delta^{1/2} \mathcal{L}_\xi \sigma \right) \right]. \]

The standard notation \( \sigma_a' = \nabla_a \sigma \) has been used in these expressions. It generalizes in the obvious way for any combination of primed and unprimed indices.

The two limits here are easily computed using the properties of GKF\( \mathcal{L}_\xi \)s derived in [11]. First consider Lie derivatives of the world function evaluated on two nearby points on \( \Gamma \). These obviously vanish when the points coincide. What is needed is an estimate for precisely how fast they tend to zero as \( t \to t' \). It will be sufficient to note that on the center-of-mass worldline, (16) can be used to show that

\[ \xi^a \simeq -\sigma^a_a \left[ \xi^b + X^b \nabla_b \xi^a + \frac{1}{2} X^b X^c R^a_{bcd} \xi^d \right] + O(X^3). \]

(54)

\( X_a = -\sigma_a(\gamma, \gamma') \) acts like a separation vector between its two arguments. Using the antisymmetry of \( \nabla_b \xi^b \) on \( \Gamma \) together with the well-known identity [13, 14]

\[ \sigma_a = \sigma^a_a \]

(55)

shows that \( \mathcal{L}_\xi \sigma(\gamma, \gamma') \) decreases at least as fast as \( (t - t')^4 \) as these times approach each other. It is clear that \( \bar{\gamma}^a \sigma_a' \) scales like \( (t - t')^3 \) in the same limit. These two relations are sufficient to show that (53) always vanishes.

Understanding the remaining limit (52) requires knowing how fast \( \mathcal{L}_\xi \ln \Delta \) decreases as \( t \to t' \). It is shown in [11] that

\[ \mathcal{L}_\xi \ln \Delta = -H^a_a \left( \xi^b \sigma^a_{ba} + \xi^b \sigma^a_{a'b} \right), \]

(56)

where

\[ H^a_a = \left[ -\sigma^a_a \right]^{-1}. \]

(57)

The ‘−1’ on the right-hand side of this equation denotes a matrix inverse. It is assumed here that \( H^a_a \) exists in all regions of interest. It reduces to the identity when its arguments coincide. A straightforward application of Synge’s rule [13, 14] and other standard results of bitensor analysis shows that both \( \mathcal{L}_\xi \ln \Delta \) and its first covariant derivatives vanish in a similar limit. Such Lie derivatives therefore scale like \( (t - t')^3 \) as \( t \to t' \). Deriving this result actually does not require any properties of the GKF\( \mathcal{L}_\xi \)s. It holds for all smooth vector fields \( \xi^a \). This scaling relation together with the previously-discussed one for \( \mathcal{L}_\xi \sigma \) imply that (52) always vanishes. It follows that (50) holds, as originally claimed. This is effectively equivalent to stating that the singular self-field always satisfies Newton’s third law in sufficiently small regions near \( \Gamma \). Although it did not require any external assumptions, this result can be thought of as an effective renormalization of the point particle self-field. The degree to which the generalized Killing fields live up to their name has removed any singularities that might have been expected to arise from the field sourced by \( G_S \). Note, however, that this procedure cannot be applied to charges with arbitrary distributional structures. It would also fail if the point charge was not concentrated on the same worldline used to define the GKF\( \mathcal{L}_\xi \)s.

Equations of motion for a point charge can now be derived from the behavior of its momenta. Assume that the body’s stress–energy tensor has the standard form

\[ T^{ab}(x) = m \int \bar{\gamma}^a \bar{\gamma}^b \delta(x, y(t)) \, dt, \]

(58)
so that the mass and charge are concentrated on the same worldline. It then follows from (16)
that \( T^{ab} \xi_{(s) g_{ab}} = 0 \). Applying (41),
\[
dP/\xi \xi = q \xi \xi \nabla \xi (\Phi^{\text{ext}} + \Phi^{\text{self}_{R,DW}})_{\xi_{(s)}}.
\]
Comparing with (18) shows that the torque vanishes. The force is therefore
\[
F_{\xi} = q \xi \nabla \xi (\Phi^{\text{ext}} + \Phi^{\text{self}_{R,DW}})_{\xi_{(s)}}.
\]
This does not completely determine the evolution of the particle’s linear momentum. It could
still couple to the angular momentum.

Using (20), one finds that
\[
\dot{S}_{ab} = 2p_{a\xi \gamma b}.
\]
A center-of-mass condition must be placed on \( \Gamma \) in order to solve this equation. It is possible
to use (46) to derive an expression for the difference \( \ddot{\gamma}^{a} - p^{a} / m \) assuming that \( p^{a} \gamma_{a} = -m \)
[18, 19]. Suppose that the spin vanishes at least instantaneously. Given that \( N_{ab} = 0 \), one
then finds that (unsurprisingly)
\[
p^{a} = m \dot{\gamma}^{a}.
\]
Substituting this into (61) shows that \( \dot{S}_{ab} = 0 \). We have derived in a rather pedantic way the
fact that the angular momentum of a point particle vanishes for all time if it does so at any
instant.

Taking advantage of this, the motion of a nonspinning body is completely determined
by (60) and (62). The gradient of the regular self-field was derived in [8]. Substituting
appropriately,
\[
D \frac{d}{ds}(m \dot{\gamma}^{a}) = q \nabla /\Phi^{\text{ext}} + q \left[ \frac{1}{3} h^{a}_{b} \left( \dot{\gamma}^{a} + \frac{1}{2} R_{bc} \dot{\gamma}^{c} \right) \right.
\]
\[\left. - \frac{1}{12} R \dot{\gamma}^{a} + \lim_{\epsilon \to 0} \int_{-\infty}^{\epsilon} \nabla G_{\text{ext}}(\gamma', \gamma) \, d\gamma' \right].
\]
Equations governing \( \ddot{\gamma}^{a} \) and \( m \) are easily extracted using the projection operator
\[
h^{a}_{b} = \delta^{a}_{b} + \delta^{a}_{b} \dot{\gamma}^{b}.
\]
This result is standard, and has been found in the past using several different methods
[8–10]. The derivation here starts from a formalism that is exact for any finite body. The
only external assumption required to include point particles was that the momentum could
be renormalized via (42). The portion of the singular self-field not taken into account with
this procedure was shown to be irrelevant to the body’s motion. These results were obtained
without any detailed calculations of the singular self-field. This is convenient, as its structure
is usually much more complicated than that of \( \Phi_{R}^{\text{self}} \).

4.2. A scaling limit

Given the definitions (15) and (39) for \( P_{\xi} \) and \( E_{\xi} \), it is not completely consistent to assume
that the effective momenta \( P_{\xi} \) remain finite for distributional sources like (49). The laws of
motion in section 3 were derived under the assumption that \( \rho \) is well behaved. It is not clear
that they can be used to discuss the behavior of singular charge distributions. Furthermore, the
direct use of point charges often loses all sense of mathematical meaning when considering
couplings to fields that satisfy nonlinear wave equations. This problem is particularly well
known when trying to discuss gravitational self-forces [28].
Despite these remarks, point particles are introduced in practice (with some special rules) in order to simplify calculations. They are intended to represent the behavior of ‘sufficiently small’ extended charges in an appropriate sense. Understanding this equivalence and its limits is difficult, although it is relatively straightforward to comment on a particular class of extended charges whose behavior approaches that of a point particle. The techniques already developed in section 4.1 generalize fairly easily to the discussion of a scaling limit.

Consider a one-parameter family of charge distributions \( \rho(x; \lambda) \) with diameters proportional to \( \lambda \). These objects shrink into \( \Gamma \) as \( \lambda \to 0 \). The total charge cannot remain fixed in this limit if \( E_\xi \) is to remain well defined. Suppose that

\[
\rho = \lambda^{-\alpha} \rho(r/\lambda, s_0 + s/\lambda),
\]

where \( r \) and \( s \) represent Fermi normal coordinates constructed using the center-of-mass as an origin. An appropriate choice for the constant \( \alpha \) is not obvious, so it will be left free for now.

Assume that the stress–energy tensor shrinks like the charge density, but that its magnitude is made proportional to \( \lambda^{-\beta} \) rather than \( \lambda^{-\alpha} \). Whatever \( \beta \) happens to be, the bare mass scales like \( \lambda^{3-\beta} \) to leading order.

Each contribution to the laws of motion affecting this family of charges scales differently as \( \lambda \to 0 \). If the external scalar field remains finite in this limit, the force that it exerts satisfies

\[
\int \rho \mathcal{L}_\xi \Phi_{ext} \, dS \sim O(\lambda^{3-\alpha}).
\]

This is assumed to be the dominant influence on a body’s motion when it is sufficiently small. One additional power of \( \lambda \) appears if the GKF vanishes at the appropriate point on \( \Gamma \). The torque therefore scales like \( \lambda^{4-\alpha} \).

The magnitude of the scalar self-force arising from the body’s regular self-field can be estimated from its point particle expansion. This was used in (63). If the result there can be considered approximately valid inside a slowly-evolving extended charge distribution (as has been verified directly in electromagnetism [5]), \( \mathcal{L}_\xi \Phi_{self}^{\text{eff}} \) depends on \( \bar{q} r^a \) and \( q/R^2 \), where \( R \) is the curvature radius. Given that the background geometry is assumed to be independent of the charge’s existence, \( R \) does not depend on \( \lambda \). The rate of change of acceleration can be estimated from (66). These two contributions sometimes scale differently, so

\[
\int \rho \mathcal{L}_\xi \Phi_{self}^{\text{eff}} \, dS \sim O(\lambda^{6+\beta-3\alpha} + \lambda^{6-2\alpha}).
\]

Which of these estimates dominates depends on whether or not \( \alpha \) is larger than \( \beta \). The center-of-mass acceleration remains finite as \( \lambda \to 0 \) when \( \beta \geq \alpha \). It is reasonable to suppose that this is always true, in which case the self-force due to \( \Phi_{self}^{\text{eff}} \) is always proportional to \( \lambda^{6-2\alpha} \).

There is also a self-force due to the singular component of the self-field. Slightly generalizing results used in the point particle case,

\[
\mathcal{L}_\xi \sigma \sim O(\lambda^4), \quad \mathcal{L}_\xi \ln \Delta \sim O(\lambda^2).
\]

It follows that

\[
\int \mathcal{L}_\xi \sigma \, dS \sim \int \mathcal{L}_\xi \ln \Delta \sim O(\lambda^{7-2\alpha}).
\]

This will always decrease faster than the regular component of the self-force as \( \lambda \to 0 \). The Detweiler–Whiting axiom is therefore satisfied for all sufficiently small and slowly varying charge distributions.
This does not guarantee that \( \Gamma \) will evolve like the worldline of a point particle with the appropriate mass and charge. Strictly adhering to the point particle equations of motion requires placing several restrictions on the two scaling parameters \( \alpha \) and \( \beta \). One of these comes from demanding that any deviations from \( \dot{p}^a \approx D(m\gamma^a)/ds \) be small compared to the regular component of the self-force. Such terms can scale like \( \lambda^{4-\alpha} \), so let
\[
\alpha > 2. \tag{70}
\]
This same condition also arises if the interaction of the charge’s dipole moment with the external field is assumed to be negligible.

The magnitude of the gravitational force arising from \( T^{ab}\mathcal{L}_\xi g_{ab} \) must also be addressed. The gravitational force can be estimated by
\[
\int \Sigma T^{ab}\mathcal{L}_\xi g_{ab} p^a d\Sigma_a \sim O(\lambda^{5-\beta}). \tag{71}
\]
This follows from the fact that \( \mathcal{L}_\xi g_{ab} \) and its first derivatives always vanish on the central worldline. It is reasonable to suppose that the effective momenta are mainly determined by the body’s stress–energy tensor (rather than its self-field), so \( q^2/mD \) should not increase as \( \lambda \to 0 \). This means that
\[
\beta \geq 2(\alpha - 1). \tag{72}
\]
If the mass is assumed to remain bounded—meaning that \( \beta \leq 3 \)—(70) and (72) are more than sufficient to guarantee that the gravitational force is negligible compared to the self-force.

One last detail is the Papapetrou spin-curvature coupling in (19). The body’s angular momentum generically scales like \( \lambda^{4-\beta} \). The spin force shares this same behavior, and can only be small compared to the regular self-force when \( \beta < 2(\alpha - 1) \). This contradicts (72). The one-parameter families of charges considered here have the property that either the Papapetrou force is important or the mass has very little to do with \( T^{ab} \). The latter possibility seems difficult to accept, as any objects whose inertia was dominated by their self-energy would probably be unstable or at least experience rapid internal oscillations. It is also not clear if the center-of-mass conditions are meaningful in such cases. As a compromise, the equality in (72) might be assumed to hold. This means that \( 2 < \alpha \leq 5/2 \). It implies that the fractional self-energy remains finite as \( \lambda \to 0 \). Self-forces have effects comparable to those of the angular momentum, so one cannot be included without the other. Initial conditions might be chosen such that the angular momentum can be ignored, although it is not clear how long this condition could be kept consistent.

In conclusion, it is difficult to arrange all extended-body effects to be negligible compared to those arising from the self-field. This is especially true if the mass and charge densities are demanded not to diverge as \( \lambda \to 0 \). Such a condition might be required in order to maintain the test mass approximation that has been assumed. Despite all of these remarks, it is rather trivial to modify (63) to include charge dipole or mass quadrupole effects. The main point of this discussion is really that the ‘extended body self-force’ (69) does generically become small compared to the regular self-force (67). This is the main content of the Detweiler–Whiting axiom applied to extended charges.

5. Discussion

Following [11, 17, 18], approximate Killing fields have been used to define the linear and angular momenta of extended scalar charge distributions in curved spacetimes. These quantities were shown to evolve according to (41). The various terms in that equation all have simple interpretations. A charge’s behavior is seen to have five distinct components. Two of
these are standard test body interactions with the background geometry and the external scalar field. The remaining three contributions to the momentum evolution decompose the self-force and self-torque in a particular way. The portion due to the regular self-field is essentially as expected.

More interesting are the effects of the singular self-field. One consequence of its presence is the introduction of what appear to be effective linear and angular momenta connected to the scalars $\xi_a$. This is at least qualitatively an expected result. Effective masses are usually found when introducing specific charge distributions and using perturbation theory to approximate their self-fields [4, 27]. By contrast, the definition (39) for the self-momentum obtained here required only straightforward manipulations in the full theory. It includes a number of effects more complicated than simple mass shifts. These would probably not be obvious from an inspection of approximate forces and torques. A related issue is that the effective momenta introduce a possible ambiguity in determining a charge’s motion. Centers-of-mass might be defined using only the bare momenta defined in terms of the stress–energy tensor, or using the full momenta $\hat{P}_\xi = P_\xi + E_\xi$. These two possibilities generically lead to different worldlines. It is not clear which—if either—is more appropriate for charges with very large self-fields. The laws of motion simplify when the full momenta are used to define a body’s mass center. Many of the unexpected results in [5] regarding the behavior of electromagnetic charges in flat spacetime can probably be attributed to failing to fully apply this simplification.

The singular self–self field affects the force and torque more directly as well. This arises from a term in (41) involving $L_\xi G_\xi$. It has the physical interpretation of measuring the degree to which the singular self-field fails to satisfy Newton’s third law in the direction defined by $\xi_a$. It is also related to the failure of this field to be conservative. Comparison with (8) shows that such effects would exist even if the field equation were elliptic. It therefore should not be thought of as a reaction to emitted radiation. In general, the Lie derivative of a singular Green function always satisfies

$$\Box L_\xi G_\xi(x, x') = [\nabla^a \nabla^b G_\xi + 2\pi \delta(x, x')g^{ab}]L_\xi g_{ab} + \left[ \nabla^b L_\xi g_{ab} - \frac{1}{2} \nabla_a (g^{bc} L_\xi g_{bc}) \right] \nabla^a G_\xi.$$  (73)

The degree to which $L_\xi g_{ab}$ remains small determines how large the source terms on the right-hand side can be. By construction, they always decrease near the worldline used to construct the GKF.

The results derived here provide a simple framework within which to generalize the Detweiler–Whiting axiom for extended charge distributions. Let this mean that the singular component of a body’s self-field—as defined by the Green function (37)—has no explicit effect on the evolution of the full momenta $\hat{P}_\xi$. It is equivalent to demanding (50) or an approximate equivalent. All real Killing fields are also GKF, so this result is exact for any charge distribution in the maximally symmetric Minkowski or de Sitter spacetimes. If only one or a few Killing fields exist, it is also exact for linear combinations of the momenta with the form $\hat{P}_a K_a + \hat{S}^{ab} \nabla_a K_b/2$. The results of section 4 extend the Detweiler–Whiting axiom to be approximately valid in all spacetimes when a charge’s diameter is much smaller than the local curvature scales. Extensions of these ideas to electromagnetic and gravitational self-forces will be explored in future papers.

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