Probabilistic Motion Planning under Temporal Tasks and Soft Constraints

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Abstract—This paper studies motion planning of a mobile robot under uncertainty. The control objective is to synthesize a finite-memory control policy, such that a high-level task specified as a Linear Temporal Logic (LTL) formula is satisfied with a desired high probability. Uncertainty is considered in the workspace properties, robot actions, and task outcomes, giving rise to a Markov Decision Process (MDP) that models the proposed system. Different from most existing methods, we consider cost optimization both in the prefix and suffix of the system trajectory. We also analyze the potential trade-off between reducing the mean total cost and maximizing the probability that the task is satisfied. The proposed solution is based on formulating two coupled Linear Programs, for the prefix and suffix, respectively, and combining them into a multi-objective optimization problem, which provides provable guarantees on the probabilistic satisfiability and the total cost optimality. We show that our method outperforms relevant approaches that employ Round-Robin policies in the trajectory suffix. Furthermore, we propose a new control synthesis algorithm to minimize the probability of reaching a bad state while minimizing the cost. The resulting solution provides provable guarantees on the probabilistic satisfiability and the total cost optimality.

We validate the above schemes via both numerical simulations and experimental studies.

Index Terms—Markov Decision Process, Linear Temporal Logic, Chance Constrained Optimization, Motion Planning.

I. INTRODUCTION

In this paper we study the problem of robot motion planning under uncertainty and temporal task specifications. We consider uncertainty in the workspace properties, robot motion and actions, and outcome of task executions, which gives rise to a Markov Decision Process (MDP) to model the proposed system. MDPs have been used extensively to model motion and sensing uncertainty in robotics [1], [2] and then solve decision making problems that optimize a given control objective.

The most common objective is to reach a goal state from an initial state while minimizing the cost. The resulting solution is a policy that maps states to actions [2]. On the other hand, Linear Temporal Logic (LTL) provides a formal language to describe complex high-level tasks beyond the classic start-to-goal navigation. A LTL task formula is usually specified with respect to an abstraction of the robot motion within the allowed workspace [3], modeled by a deterministic finite transition system (FTS). Then a high-level discrete plan is found using off-the-shelf model-checking algorithms [4], which is then executed through low-level continuous controllers [3], [5]. This framework is extended to allow for both robot motion and actions in the task specification [6] and partially-known or dynamic workspaces in [7], [8].

Recently, there have been many efforts to address the problem of synthesizing a control policy for a MDP that satisfies high-level temporal tasks specified in various formal languages. Different classes of Probabilistic Computation Tree Logic (PCTL) formulas have been studied in [9] for abstraction and verification over Interval-valued Markov Chains. The work in [10] proposes a control policy for a mobile robot that maximizes the probability of satisfying a bounded linear temporal logic (BLTL) formula. Syntactical co-safe LTL formulas (sc-LTL) are considered in [11] for a deterministic robot that coexists with other robots whose behavior is modeled as a MDP. A FTS with time-varying rewards is controlled to satisfy a LTL formula and maximize the accumulated reward in [12]. A robust control policy for MDPs with uncertain transition probabilities is proposed in [8]. A verification toolbox is provided in [13] for probabilistic discrete-time or continuous-time Markov Chain (MC), under a wide variety of quantitative properties expressed in PCTL, LTL, CTL, and so on.

In this work, we study motion planning of a mobile robot under uncertainty in both robot motion and workspace properties. The goal is to synthesize a finite-memory control policy that generates robot trajectories that satisfy a high-level LTL task formula with desired high probability. At the same time, we optimize the total cost both in the prefix and suffix parts of the system trajectories. Our proposed approach is based on solving two coupled Linear Programs, one for the prefix and one for the suffix, over the occupancy measures of the product automaton introduced in [14]. Moreover, we explore cases where the probability of satisfying the LTL tasks is zero, so that an Accepting End Component (AEC) does not exist in the MDP, where most relevant work returns no solutions. To address such situations, we treat satisfaction of the tasks as soft constraints and propose a relaxed suffix plan that minimizes the frequency with which the system enters bad states that violate the task specifications. We show that our approach outperforms the widely-used Round-Robin policy, via both numerical simulations and experimental studies. We also compare our proposed method with the widely-used probabilistic model-checking tool PRISM [13].

Our work is related to literature on (i) policy synthesis for MDPs under multiple objectives; (ii) cost optimization within AECs in MDPs; and (iii) infeasible temporal tasks. We discuss below this literature and highlight our contributions.

Since we consider both temporal tasks and total-cost criteria over MDPs, this work is closely related to policy synthesis of MDPs under multiple objectives. The work in [14] proposes...
a framework with provable correctness to synthesize a control policy for MDPs under multiple constrained total-cost criteria. A survey on multi-objective decision-making for MDPs can be found in [15]. On the other hand, verification of MDPs under *multiple* high-level tasks is addressed in [16], where the probability of satisfying each subtask is lower-bounded by a given value. Moreover, a quantitative multi-objective verification scheme is proposed in [17], [18] for numerical queries over probabilistic reward predicates.

On the other hand, the seminal works [19], [20] consider MDPs with multi-dimensional weights under multi-percentile queries that may be conflicting. However, most of the above work does not address cost optimization over the suffix of the system trajectory within the AECs, neither does it address the case where no AECs can be found in the product automaton, which are the main contributions here.

The satisfaction of a LTL formula is associated with reaching the corresponding AECs. In particular, in [4], Chapter 10, a value iteration method is used to solve the maximal reachability problem towards the AECs to obtain a policy for the plan prefix. For planning within the AECs, [4], [17], [21] adopt the Round-Robin policy, which guarantees only correctness but not optimality. Optimal policies for the plan suffix that keeps the system within the AECs have been proposed in [22]–[25]. Specifically, in [22] the expected cost of satisfying instances of a desired property is minimized, while in [23] the minimal bottleneck cost is considered. Both approaches in [22], [23] require particular types of LTL formulas (such as “always eventually”). The work in [24], [25] considers MDPs with ω-regular specifications and quantitative resource constraints within the AECs. The work in [25] investigates the Pareto cost of a human-in-the-loop MDP measured by a given discounted cost function. Compared to this literature, the multi-objective optimization problem that we formulate to solve the control synthesis problem allows us to explicitly characterize the trade-off between prefix and suffix optimality. We then extend this methodology to the case where no AECs can be found.

Most aforementioned work [4], [17], [19]–[22], [27] relies on the assumption that the product automaton contains at least one AEC. However, in many situations this assumption does not hold so that the probability of satisfying the task under any policy is zero. In this case, it is still important to identify those policies that minimize the frequency with which the system will reach the bad states that violate the task specifications. Consequently, it is desirable to synthesize a policy with certain risk guarantees even when soft MDPs are considered that are only partially-feasible. To the best of our knowledge, there is no work on control synthesis for infeasible soft MDP task formulas defined on MDPs, especially when an AEC cannot be found in the resulting product automaton. For deterministic transition systems, a framework for robot motion planning in partially-known workspaces is proposed in [7] that can handle soft LTL task formulas whose satisfiability is improved over time; a least-violating control strategy is synthesized in [28] for a set of LTL safety rules. In the case of MDPs, a relevant formulation is considered in [29] where a MDP is controlled to satisfy an ω-regular formula. A policy is proposed to ensure that the MDP enters a failure state relatively late in the prefix. However, a multi-objective criterion of the control policy, especially in the plan suffix, is not considered there. Also, recent work in [30] proposes an approach to increase the satisfaction probability by modifying the task formula which, however, only considers co-safe LTL formulas without cost optimization constraints.

In summary, the main contribution of this work is three-fold: (i) a framework that optimizes the total cost both in the plan prefix and suffix, while ensuring that the tasks are satisfied with a desired high probability; (ii) a new algorithm to synthesize the control policies that have a high probability of satisfying the task over long time intervals, for cases where an AEC does not exist; and (iii) a new method that allows the system to recover from bad states and continue the task.

The rest of the paper is organized as follows. Section II introduces necessary preliminaries. In Section III we formalizes the considered problem. Section IV presents our solution in details, which includes four major parts. Section V demonstrates the feasibility of the results by numerical simulations. Section VI contains the experimental results. We conclude and discuss about future directions in Section VII.

II. Preliminaries

A. Transient MDP

A Markov Decision Process (MDP) is defined as a 6-tuple $M = (X, U, D, p_D, c_D, x_0)$, where $X$ is the finite state space; $U$ is the finite control action space (with a slight abuse of notation, $U(x)$ also denotes the set of control actions *allowed* at state $x \in X$); $D = \{(x, u) \mid x \in X, u \in U(x)\}$ is the set of possible state-action pairs; $p_D : X \times U \times X \rightarrow [0, 1]$ is the transition probability function so that $p_D(x, u, \bar{x})$ is the transition probability from state $x$ to state $\bar{x}$ via control action $u$ and $\sum_{\bar{x} \in X} p_D(x, u, \bar{x}) = 1$, $\forall (x, u) \in D$; $c_D : D \rightarrow \mathbb{R} \geq 0$ that $c_D(x, u)$ is the cost of performing action $u$ in $X$ at state $x \in X$; and $x_0 \in X$ is the initial state. Denote by $Post(x, u) \triangleq \{\bar{x} \in X \mid p_D(x, u, \bar{x}) > 0\}$, $\forall (x, u) \in D$.

The above MDP evolves by taking an action $u \in U(x)$ associated with every state $x \in X$. Denote by $R_T = x_0 u_0 x_1 u_1 \cdots x_T u_T$ the past run that is a sequence of previous states and actions up to time $T \geq 0$. As defined in [2], a control policy $\mu = \mu_0 \mu_1 \cdots$ is a sequence of decision rules $\mu_t$ at time $t \geq 0$. A control policy is stationary if $\mu_t = \mu, \forall t \geq 0$, where $\mu$ can be randomized so that $\mu : X \times U \rightarrow [0, 1]$ or deterministic so that $\mu : X \rightarrow U, \forall t \geq 0$. On the other hand, a policy is history dependent or finite-memory if $\mu_t : R_t \times U \rightarrow [0, 1]$, where $R_t$ is the past history until time $t \geq 0$.

B. End Components

A sub-MDP of $M$ is a pair $(S, A)$ where $S \subseteq X$ and $A : S \rightarrow 2^U$ such that (i) $S \neq \emptyset$, $\emptyset \neq A(s) \subseteq U(s), \forall s \in S$; (ii) $Post(s, u) \subseteq S, \forall s \in S$ and $\forall u \in A(s)$. An End Component (EC) of $M$ is a sub-MDP $(S, A)$ such that the digraph $G_{S,A}$ induced by $(S, A)$ is strongly connected. An end component $(S, A)$ is called maximal if there is no other end component $(S', A')$ such that $(S, A) \neq (S', A'), S \subseteq S'$ and $A(s) \subseteq A'(s), \forall s \in S$. The set of Maximal End Components (MECs) of a MDP is finite and can be uniquely
C. LTL and DRA

The ingredients of a Linear Temporal Logic (LTL) formula are a set of atomic propositions \(AP\) and several Boolean and temporal operators. Atomic propositions are Boolean variables that can be either true or false. A LTL formula is specified according to the syntax [4]: \(\varphi \triangleq T \mid p \mid \varphi_1 \land \varphi_2 \mid \neg \varphi_1 \lor \varphi_2\), where \(T \triangleq \text{true}\), \(p \in AP\), \(\varphi_1 \land \varphi_2\) (next), \(\varphi_1 \lor \varphi_2\) (until) and \(\neg \triangleq \neg T\). For brevity, we omit the derivations of other operators like \(\Box\) (always), \(\Diamond\) (eventually), \(\Rightarrow\) (implication). The semantics of LTL is defined over the set of infinite words over \(2^{AP}\). Intuitively, \(p \in AP\) is satisfied in a word \(w = w(1)w(2)\ldots\) if it holds at \(w(1)\), i.e., if \(p \in w(1)\). Formula \(\Box \varphi\) holds true if \(\varphi\) is satisfied on the word suffix that begins in the next position w(2), whereas \(\varphi_1 \lor \varphi_2\) states that \(\varphi_1\) has to be true until \(\varphi_2\) becomes true. Finally, \(\Box \varphi\) and \(\Diamond \varphi\) are true if \(\varphi\) holds on \(w\) eventually and always, respectively. We refer the readers to Chapter 5 of [4] for the full definition.

The set of words that satisfy a LTL formula \(\varphi\) over \(AP\) can be captured through a Deterministic Rabin Automaton (DRA) \(A_\varphi\), defined as \(A_\varphi = (Q, 2^{AP}, \delta, q_0, \text{Acc}_A)\), where \(Q\) is a set of states; \(2^{AP}\) is the alphabet; \(\delta \subseteq Q \times 2^{AP} \times Q\) is a transition relation; \(q_0 \in Q\) is the initial state; and \(\text{Acc}_A \subseteq 2^Q \times 2^Q\) is a set of accepting pairs, i.e., \(\text{Acc}_A = \{(H^1_A, I^1_A), (H^2_A, I^2_A), \ldots, (H^N_A, I^N_A)\}\) where \(H^i_A, I^i_A \subseteq Q\), \(\forall i = 1, 2, \ldots, N\). An infinite run \(q_0q_1q_2\cdot \cdot \cdot\) of \(A\) is accepting if there exists at least one pair \((H^i_A, I^i_A) \in \text{Acc}_A\) such that for all \(n \geq 0\), it holds \(\forall m \geq n, q_m \notin H^i_A\) and \(\exists m \geq 0, q_n \in I^i_A\), where \(\exists\) stands for “existing infinitely many”. Namely, this run should intersect with \(H^i_A\) finitely many times while with \(I^i_A\) infinitely many times. There are translation tools [31] to obtain \(A_\varphi\) given \(\varphi\), which requires the process of translating firstly the LTL formula to the associated Non-deterministic Büchi Automaton (NBA), and then to the DRA with complexity \(2^{O(n \log n)}\), where \(n\) is the length of \(\varphi\). Our implementation of the Python interface for [31] can be found in [32]. Note that [31] allows for different levels of automata simplifications to be made regarding the size of \(A_\varphi\), and a simplified automation may result in loss of optimality.

III. Problem Formulation

A. Mathematical Model

In order to model uncertainty in both the robot motion and the workspace properties, we extend the definition of a MDP from Section II-A to include probabilistic labels, as the probabilistically-labeled MDP:

\[
\mathcal{M} = (X, U, D, p_D, (x_0, l_0), AP, L, p_L, c_D),
\]

where \(AP\) is a set of atomic propositions that capture the properties of interest in the workspace; \(L : X \rightarrow 2^{2^{AP}}\) contains the set of property subsets that can be true at each state; and \(p_L : X \times 2^{AP} \rightarrow [0, 1]\) specifies the associated probability. Particularly, \(p_L(x, l)\) denotes the probability that state \(x \in X\) satisfies the set of propositions \(l \subseteq AP\). Note that \(\sum_{l \subseteq AP} p_L(x, l) = 1\), \(\forall x \in X\). Moreover, \((x_0, l_0)\) contains the initial state \(x_0\) in \(X\) and the initial label \(l_0 \in L(x_0)\), while the rest of the notations in (1) are the same as defined in Section II-A. The probabilistic labeling function provides a way to consider time-varying and dynamic workspace properties. Moreover, there is a LTL task formula \(\varphi\) specified over the same set of atomic propositions \(AP\), as the desired behavior of \(\mathcal{M}\). We assume that the MDP \(\mathcal{M}\) in (1) is fully-observable due to the following assumption.

Assumption 1. At any stage \(t \geq 0\), the current robot state \(x_t \in X\) and its label \(l_t \in L(x_t)\) are fully-observable.

While the robot is moving within the workspace, it is capable of sensing an actual property and determine the label of the state it is located at. At stage \(T \geq 0\), the robot’s past path is given by \(X_T = x_0x_1\cdots x_T \in X^{(T+1)}\), the past sequence of observed labels is given by \(L_T = l_0l_1\cdots l_T \in (2^{AP})^{(T+1)}\) and the past sequence of control actions is \(U_T = u_0u_1\cdots u_T \in U^{(T+1)}\). It holds that \(p_D(x_t, u_t, x_{t+1}) > 0\) and \(p_L(x_t, l_t) > 0\), \(\forall t \geq 0\). These three sequences can be composed into the complete past run \(R_T = x_0l_0u_0x_1l_1u_1\cdots x_Tl_Tu_T\). Denote by \(X_T, L_T\) and \(R_T\) the set of all possible past sequences of states, labels, and runs up to stage \(T\). We set \(T = \infty\) for infinite sequences.

Definition 1. The mean total cost \(\text{Cost}(R_{\infty})\) of an infinite robot run \(R_{\infty}\) of \(\mathcal{M}\) is defined as

\[
\text{Cost}(R_{\infty}) = \lim_{n \to \infty} \inf_{n} \frac{1}{n} \sum_{t=0}^{n} c_D(x_t, u_t),
\]

where \(R_{\infty} = x_0l_0u_0x_1l_1u_1\cdots \in R_{\infty}\).

As discussed in [2], [20], [24], [33], the above mean total cost is called the mean-payoff function (or limit-average),
where the “lim” operator is needed as the limit-average might not exist for some runs, see [24, 33, 34].

Our goal is to find a finite-memory policy for $M$, denoted by $\mu = \mu_0 h_1 \cdots$. The control policy at stage $t \geq 0$ is given by $\mu_t : R_t \times U \to [0,1]$, where $R_t$ is the past run $R_t$, $\forall t \geq 0$. Denote by $\bar{\mu}$ the set of all such policies. Given a control policy $\mu \in \bar{\mu}$, the probability measure $Pr^\mu_M(\cdot)$ on the smallest $\sigma$-algebra, over all possible infinite sequences $R_\infty$ that contain $R_T$, is the unique measure [4] by

$$Pr^\mu_M(R_\infty) = \prod_{t=0}^{T} p_D(x_t, u_t, x_{t+1}) \cdot \mu_t(R_t, u_t),$$

where $\mu(R_t, u_t)$ is defined as the probability of choosing action $u_t$ given the past run $R_t$. Then we define the probability of $M$ satisfying $\varphi$ under policy $\mu$ by:

$$Pr^\mu_M(\varphi) = Pr^\mu_M\{R_\infty \mid L_0 = \varphi\},$$

where the satisfaction relation “$\models$” is introduced in Section II-C given an infinite word and a LTL formula. Accordingly, the risk is defined as the probability that the task formula $\varphi$ is not satisfied by $M$ under the policy $\mu$, namely, $Risk^\mu_M(\varphi) = 1 - Pr^\mu_M(\varphi)$.

**Problem 1.** Given the labeled MDP $M$ defined in [1] and the task specification $\varphi$, our goal is to solve:

$$\min_{\mu \in \bar{\mu}} \mathbb{E}^\mu_M\{\text{Cost}(R_\infty)\}$$

s.t. $Risk^\mu_M(\varphi) \leq \gamma,$

where $\gamma \geq 0$ is a pre-defined parameter as the allowed risk; the optimal policy minimizes the mean total cost and ensures that the risk of violating $\varphi$ remains bounded by $\gamma.$

Note that the traditional definition of un-discounted expected total cost over an infinite run from [2], [14] is not used here, as it is infinite except for the special case of transient MDPs defined in Section II-A. However, in this work, the model $M$ is not restricted to be transient. Moreover, the discounted total cost in [2] is not used here either due to two reasons: first, it is not obvious how to choose the discount factor for various control tasks $\varphi$ [25]; and second, we are more interested in optimizing the repetitive long-term behavior of the system, rather than the short-term one [20]. In-depth discussions on the optimization of infinite-horizon undiscounted or discounted total-cost criteria over MDPs with or without constraints can be found in [2].

**Remark 1.** Different from the maximal reachability problem addressed in [4], [21], a deterministic policy would not suffice here. Instead, randomization is required due to the mean total-cost criterion and the risk constraint, similar to [14].

**IV. Solution**

This section contains the three major parts of the proposed solution: (i) the construction of the product automaton and its AMECs; (ii) the algorithms to synthesize the optimal plan prefix and suffix, for both cases where the AMECs exist or not; (iii) the complete policy, and the online execution algorithm.

A. Product Automaton and AMECs

To begin with, we construct the DRA $A_\varphi$ associated with the LTL task formula $\varphi$ via the translation tools [31, 32]. Let it be $A_\varphi = (Q, 2A^P, \delta, q_0, Acc_A),$ where the notations are defined in Section II-C. Then we construct a product automaton between the robot model $M$ and the DRA $A_\varphi$.

**Definition 2.** Denote by $P$ the product $M \times A_\varphi$ as a 7-tuple:

$$P = (S, U, E, p_E, c_E, s_0, Acc_P),$$

where: the state $S \subseteq X \times 2A^P \times Q$ is so that $(x, l, q) \in S$, $\forall x \in X, \forall l \in L(x)$ and $\forall q \in Q$; the action set $U$ is the same as in [1] and $U(s) = U(x)$, $\forall s = (x, l, q) \in S$; $E = \{(s, u) \mid s \in S, u \in U(s)\}$; the transition probability $p_E : S \times U \times S \to [0, 1]$ is so that

$$p_E((x, l, q), u, (\bar{x}, \bar{l}, \bar{q})) = p_D(x, u, \bar{x}) \cdot p_L(\bar{x}, \bar{l}),$$

where (i) $(x, l, q), (\bar{x}, \bar{l}, \bar{q}) \in S$; (ii) $(x, u) \in D$; and (iii) $\bar{q} \in \delta(q, l)$; the cost function $c_E : E \to \mathbb{R}^>0$ is so that $c_E((x, l, q), u) = c_D(x, u, \bar{q})$; (6)

$$\forall (x, l, q) \in S,$$ $\Delta$ the single initial state is $s_0 = (x_0, l_0, q_0) \in S$; the accepting pairs are defined as $Acc_P = \{(H^p_i, I^p_i) \mid i = 1, 2, \cdots, N\}$, where $H^p_i = (\{x, l, q) \in S \mid q \in H^p_i\}$ and $I^p_i = (\{x, l, q) \in S \mid q \in I^p_i\}$, $\forall i = 1, 2, \cdots, N$.

The product $P$ computes the intersection between all traces of $M$ and all words that are accepted by $A_\varphi$, to find all admissible robot behaviors that satisfy the task $\varphi$. It combines the uncertainty in robot motion and the workspace model by including both $x$ and $l$ in the states. The Rabin accepting condition of $P$ is defined as follows: An infinite path $R_P = s_0 s_1 \cdots$ of $P$ is accepting if for at least one pair $(H^p_i, I^p_i) \in Acc_P$ it holds that $R_P$ intersects with $H^p_i$ finitely often while with $I^p_i$ infinitely often. To transform this condition into equivalent graph properties, we need to compute the AMECs of $P$ associated with its accepting pairs $Acc_P$.

**Detailed definition of MECs is given in Section II-D**

In order to find the complete set of AMECs of $P$, for each pair $(H^p_i, I^p_i) \in Acc_P$, perform the following steps:

(i) Build the MDP $Z^H_{i - 1} \triangleq (S', U', E', p'_E)$, where $S' = S^H_{i - 1} \cup \{\nu\}$ is the set of states with $S^H_{i - 1} = S \setminus H^p_i$ and a trap state; $U' = U \cup \{\tau_0\}$ is the set of actions where $\tau_0$ is a pseudo action; $E' \subseteq S' \times S'$ is the set of transitions with the associated probability $p'_E$ which are defined by three cases:

(a) for the transitions within $S^H_{i - 1}$ it holds that $(s, u) \in E'$ and $p'_E(s, u, s') = p_D(s, u, s')$, $\forall (s, u) \in E$ where $s, s' \in S^H_{i - 1}$; (b) for the transitions from $S^H_{i - 1}$ to outside $S^H_{i - 1}$ it holds that $(s, u) \in E'$ and $p'_E(s, u, s') = \sum_{s \in S^H_{i - 1}} p_D(s, u, s')$, $\forall (s, u) \in E$ where $s \in S^H_{i - 1}$; and (c) the trap state is included in a self-loop such that $(\nu, \tau_0) \in E'$ and $p'_E(\nu, \nu, \nu) = 1$.

Simply speaking, all transitions from inside $S^H_{i - 1}$ to outside $S^H_{i - 1}$ are transformed to transitions to the trap state $\nu$.

(ii) Determine all MECs of $Z^H_{i - 1}$ above via Algorithm 47 in [4], which is based on splitting the strongly connected components (SCCs) of $Z^H_{i - 1}$ until the conditions of being an end component are fulfilled. Our implementation for this algorithm can be found in [32]. Denote
by $\Xi^i = \{(S^i_1, U^i_1), (S^i_2, U^i_2), \ldots (S^i_{C_i}, U^i_{C_i})\}$ the set of MECs, where $S^i_c \subseteq S^i$ and $U^i_c : S^i_c \rightarrow 2^{O^i_c}$, $\forall c = 1, 2, \ldots, C_i$. Note that $S^i_0 \cap S^i_{C_i} = \emptyset, \forall (S^i_c, U^i_c), (S^i_{C_i}, U^i_{C_i}) \in \Xi^i$.

(iii) Find $\{(S^i_0, U^i_0) \in \Xi^i\}$ that is accepting, i.e., it satisfies $\nu \notin S^i_0$ and $S^i_0 \cap I_{\\\overline{P}} \neq \emptyset$. Save the AMECs in $\Xi_{\\\overline{acc}}$. Since $\Xi_{\\\overline{acc}}$ is computed for each $(H^i_{\\\overline{P}}, I_{\\\overline{P}}) \in Acc_{\\\overline{P}}$, we denote by $\Xi_{\\\overline{acc}} = \{\Xi_{\\\overline{acc}}^i : i = 1, \ldots, N\}$ the complete set of AMECs of $P$.

**Remark 2.** A single state with a self-transition can be a MEC with a proper action set. Therefore, there exists at most $|S|^2$ MECs within $Z_{\\\overline{acc}}^i$, $\forall i = 1, \ldots, N$. Thus Step (ii) above has complexity $O(|S|^2)$, as shown in Lemma 10.126 of [4], while Steps (i) and (iii) have complexity linear with $|S^i|$.

### B. Plan Prefix and Suffix Synthesis

Given the complete set of AMECs $\Xi_{\\\overline{acc}}$ of $P$, in this section we show how to synthesize the control policy to drive the system towards $\Xi_{\\\overline{acc}}$ and furthermore remain inside $\Xi_{\\\overline{acc}}$ while satisfying the accepting condition. As mentioned in Section I most related work [4], [16], [17], [21] focuses on maximizing the probability of reaching the union of AMECs, i.e., $\cup_{(S^i_c, U^i_c) \in \Xi_{\\\overline{acc}}} S^i_c$, where dynamic programming techniques, such as value or policy iteration, can be applied to obtain the optimal policy. Furthermore, once the system enters any AMEC, e.g., $(S^i_c, U^i_c) \in \Xi_{\\\overline{acc}}$, it has probability 1 of staying within $S^i_c$ by following $U^i_c$ (see Lemma 10.119 of [4]). The Round-Robin policy is adopted in [4], [17], [21] that ensures all states in $S^i_c$ (including its nonempty intersection with $I_{\\\overline{P}}$) are visited infinitely often. As a result, the task $\varphi$ is satisfied by $P$ under this policy with the maximal probability.

The above solutions may suffice for verification problems that do not optimize cost or for tasks with trivial accepting conditions. However, for the purposes of plan synthesis and for general tasks, it is of practical interest to simultaneously satisfy the probability of reaching all the AMECs as well as optimize the mean cost of staying within any AMEC and fulfilling the accepting condition. Moreover, when no AECs can be found, instead of simply reporting failure, it is important to obtain a relaxed policy that guarantees high probability of satisfying the task over long time intervals thus minimizing the frequency of encountering bad events. In what follows we present a policy synthesis algorithm that consists of four parts:

- the **plan prefix** that drives the system from the initial state to all AMECs, while minimizing the expected cost and respecting the risk constraint; see Section IV-B1;
- the **plan suffix** that keeps the system within the AMEC it has reached, while satisfying the accepting condition and optimizing the expected suffix cost; see Section IV-B2;
- the **relaxed** prefix and suffix plans for the case where no AECs of $P$ can be found; see Section IV-B3 and IV-C1;
- the complete finite-memory policy for the original MDP $\mathcal{M}$; see Section IV-C1.

Before stating the solution, we introduce a partition of $S$ given the initial state $s_0$ and the set of AMECs $\Xi_{\\\overline{acc}}$. Let $S_r \subseteq S$ be the set of states within $S$ that can be reached from $s_0$, which can be derived via a simple graph search in $P$.

![Figure 2: Illustration of the partition of S in Definition 3](image-url)

**Definition 3.** Given $s_0$ and $\Xi_{\\\overline{acc}}$, $S$ is partitioned as $S = S_o \cup S_c \cup S_d \cup S_n$, where $S_o \triangleq S \setminus S_r$ is the set of states that can not be reached from $s_0$; $S_c$ is the union of all goal states in $\Xi_{\\\overline{acc}}$, i.e., $S_c \triangleq \cup_{(S^i_c, U^i_c) \in \Xi_{\\\overline{acc}}} S^i_c$, $S_d \subseteq S$ can be reached from $s_0$ but can not reach any state in $S_r$; and $S_n \triangleq S \setminus (S_c \cup S_d)$.

The set $S_d$ can be derived through a simple graph search, e.g., by reversing the directed graph associated with $P$, finding all reachable nodes of any state within each $(S^i_c, U^i_c) \in \Xi_{\\\overline{acc}}$ (as any AMEC is strongly connected) and finally computing its cross intersection with $S_r$; see [32] for implementation details. Roughly speaking, $S_n$ is the set of states related to the plan prefix, $S_c$ is the set of goal states related to the plan suffix, and $S_d$ is set of bad states to be avoided during the prefix. Since $S_o$ contains the states that can not be reached from $s_0$, it is neglected hereafter for the purpose of plan synthesis.

**Example 1.** This example illustrates the partition in Definition 3. Consider the toy product automaton $P$ in Figure 2. For state $s_0$, the set of reachable states is $S_r = \{s_0, s_1, s_2, s_3, s_5, s_6, s_7, s_8, s_10\}$, the set of unreachable states is $S_o = \{s_4, s_9\}$, the states within an AMEC are $S'_{c_1} = \{s_5, s_6, s_{10}\}$ and another AMEC $S'_{c_2} = \{s_7, s_8\}$, thus $S_c = S_{c_1} \cup S_{c_2} = \{s_5, s_6, s_7, s_8, s_{10}\}$, the states that can be reached from $s_0$ but can not reach $S_c$ are $S_d = \{s_1, s_3\}$, and the states that $s_0$ can reach outside $S_c \cup S_d$ are $S_n = \{s_0, s_2\}$.

1) **Plan Prefix:** Similar to [17], [18], we first construct a modified sub-MDP $Z_{\\\overline{pre}}$ of $P$ as $Z_{\\\overline{pre}} \triangleq (S_{\\\overline{pre}}, U_{\\\overline{pre}}, E_{\\\overline{pre}}, s_{\\\overline{pre}}, p_{\\\overline{pre}}, c_{\\\overline{pre}})$, where the set of states is given by $S_{\\\overline{pre}} = S_o \cup S_c$, with $S_o, S_c$ being defined in Definition 3. The set of actions is given by $U_{\\\overline{pre}} = U \cup \{\tau_0\}$ where $\tau_0$ is a self-loop action. The set of transitions $E_{\\\overline{pre}}$ is the subset of $E$ associated with $S_{\\\overline{pre}}$. Moreover, the transition probability $p_{\\\overline{pre}}$ is defined by $(i) p_{\\\overline{pre}}(s, u, s') = p_E(s, u, s'), \forall s, s' \in S_{\\\overline{pre}}$ and $u \in U(s)$; and $(ii) p_{\\\overline{pre}}(s, \tau_0, s) = 1, \forall s \in S_c$. Finally, the cost function $c_{\\\overline{pre}}$ is defined by $(i) c_{\\\overline{pre}}(s, u) = c_E(s, u), \forall s \in S_{\\\overline{pre}}$ and $u \in U(s)$; and $(ii) c_{\\\overline{pre}}(s, \tau_0) = 0, \forall s \in S_c$.

Then, we find a policy for $Z_{\\\overline{pre}}$ such that, starting from $s_0$, it can reach the set of goal states $S_c$ with a probability larger than $1 - \gamma$, while at the same time minimizing the expected total cost. Formally, consider the problem below:

**Problem 2.** Given the sub-MDP $Z_{\\\overline{pre}}$, compute an optimal
stationary prefix policy \( \pi_{\text{pre}}^* \in \overline{\pi} \) that solves the problem

\[
\min_{\pi \in \overline{\pi}} \left[ C_{\text{pre}}(S_c) \triangleq \mathbb{E}_{\pi_{\text{pre}}} \left\{ \sum_{t=0}^{\infty} c_p(s_t, u_t) \right\} \right]
\]  

(7)

s.t. \( P_{\pi_{\text{pre}}}(\cdot | S_c) \geq 1 - \gamma \),

where \( s_0 u_0 s_1 u_1 \cdots \) is a run of \( \mathcal{Z}_{\text{pre}}, \pi \) is the set of all stationary policies, the objective function is the expected total cost, \( \mathbb{E}_{\pi_{\text{pre}}} (\cdot | S_c) \) is the probability of reaching \( S_c \) from the initial state \( s_0 \), under the policy \( \pi \); and \( \gamma > 0 \) is from (4).

Note that the objective function in (7) is well-defined and finite due to the fact that \( \mathcal{Z}_{\text{pre}} \) is transient with respect to \( S_n \), and is equal to the expected total cost of reaching \( S_c \) since the cost of staying within \( S_n \) is zero. We omit the proof that \( \mathcal{Z}_{\text{pre}} \) is transient here and refer the interested readers to [2], [14].

Our proposed solution to Problem[2] is based on transforming it into a constrained optimization problem for MDPs, which can be then solved using linear programming. The approach is inspired by [14], [16], [17]. Particularly, denote by \( y_{s,u} \) the expected number of times over the infinite horizon that the system is at state \( s \) and action \( u \) is taken, \( \forall s \in S_n \) and \( \forall u \in U(s) \), which are often referred to as occupancy measures [14] as it holds \( y_{s,u} = \sum_{n=0}^{\infty} P_{\pi_{\text{pre}}}[s_t = s, u_t = u] \), where the probability is conditioned on a policy \( \pi \) and the initial state \( s_0 \).

Note that an occupancy measure is a sum of probabilities, but not a probability itself. Consider the linear program:

\[
\begin{align}
\min \quad & \sum_{(s,u)} \sum_{s' \in S_n} y_{s,u} p_p(s, u, s') c_p(s', u') \\
\text{s.t.} \quad & \sum_{(s,u) \in S_n} y_{s,u} p_p(s, u, \hat{s}) \geq 1 - \gamma; \\
& \sum_{u \in U(s)} y_{s,u} = \sum_{(s,u) \in S_n} y_{s,u} p_p(s, u, \hat{s}) + 1(\hat{s} = s_0), \forall s \in S_n; \\
& y_{s,u} \geq 0, \forall s \in S_n, \forall u \in U(s),
\end{align}
\]

(8a)

(8b)

(8c)

(8d)

where \( \sum_{(s,u)} \triangleq \sum_{s \in S_n} \sum_{u \in U(s)} \); the indicator function \( 1(\hat{s} = s_0) = 1 \) if \( \hat{s} = s_0 \) and \( 1(\hat{s} = s_0) = 0 \), otherwise. Denote by \( C_{\text{pre}}(S_c) \) the objective function associated with \( S_c \).

Let the solution of (8) be \( y_{\pi_{\text{pre}}}(s, u) = \{ y_{s,u}, s \in S_n, u \in U(s) \} \). Then the optimal stationary policy for the plan prefix, denoted by \( \pi_{\text{pre}}^* \), can be derived as follows: the probability of choosing action \( u \) at state \( s \) equals to \( \pi_{\text{pre}}^*(s, u) = y_{s,u}/(\sum_{u \in U(s)} y_{s,u}) \) if \( \sum_{u \in U(s)} y_{s,u} \neq 0 \); otherwise, the action at \( s \) can be chosen randomly, \( \forall s \in S_c \).

**Lemma 1.** Given an optimal solution \( y_{\pi_{\text{pre}}}(s, u) \) of (8), the associated policy \( \pi_{\text{pre}}^* \) ensures that \( P_{\pi_{\text{pre}}}(\cdot | S_c) \geq 1 - \gamma \).

**Proof.** First, \( y_{s,u} \) is finite and well-defined since \( \mathcal{Z}_{\text{pre}} \) is transient with respect to \( S_n \). The second part of the proof is similar to Lemma 3.3 of [10]. The summation \( \sum_{s,u} \sum_{s' \in S_n} y_{s,u} p_p(s, u, s') \) is the expected number of times that \( \mathcal{Z}_{\text{pre}} \) transitions from any state in \( S_n \) into \( S_c \) for the first time, under policy \( \pi_{\text{pre}}^* \), from the initial state \( s_0 \). Since the system remains within \( S_n \) once it enters \( S_c \), the summation equals the probability of eventually reaching the set \( S_c \), which is lower-bounded by \( 1 - \gamma \). This completes the proof.

**Example 2.** This example illustrates the important role of \( \gamma \) in the trade-off between reducing the expected total cost and minimizing the risk in Problem[2]. Consider the unicycle robot with action primitives illustrated in Figure 1 and defined in Section V. The robot moves within partitioned cells as shown in Figure 3, where the red cell has probability 0.9 to be occupied by an obstacle. Consider the task: \( \varphi = (\triangleright \square \diamond) \land (\square \diamond \bigtriangledown) \), i.e., to reach the yellow base without crossing any obstacle. In what follows, we solve (3) under risk factors \( \gamma = 0 \) and \( \gamma = 0.4 \) to derive two different optimal policies. Figure 3 shows a shorter trajectory with lower expected total cost of about 12.6 when a larger risk is allowed, compared with the right trajectory that avoids completely colliding with the obstacle, but with a much higher total cost of about 33.7.

2) Plan Suffix with AMECs: In this section, we present an algorithm to synthesize the plan suffix that minimizes the mean total cost within the AMECs, while ensuring that the system trajectory satisfies the accepting condition of \( \mathcal{P} \). Note that the plan prefix \( \pi_{\text{pre}}^* \) from the previous section guarantees that the system enters \( S_c \) from \( s_0 \) with probability higher than \( 1 - \gamma \). Recall also that \( S_c = \bigcup_{s \in S_n, u \in U(s)} \Xi_{\text{pre}}^c \). Thus it is possible that the system enters any set \( S_c' \) within \( \Xi_{\text{pre}}^c \) for this reason, we propose to treat each AMEC \( (S_c', U_c') \in \Xi_{\text{acc}} \) separately, as each \( S_c' \) is associated with different \( U_c' \) and thus a different accepting condition for \( S_c' \cap I_P \). Specifically, consider any AMEC \( (S_c', U_c') \in \Xi_{\text{acc}} \) and let \( I_c' \triangleq S_c' \cap I_P \), which is nonempty by the definition of an AMEC.

Once the system enters any AMEC, most related work [4], [17], [21] adopts the Round-Robin policy defined below:

**Definition 4.** For each state \( s_t \in S_c' \), create any ordered sequence of actions from \( U_c'(s_t) \), denoted by \( U_c(s_t) \) and its infinite repetition by \( U_c' \). Then at any stage \( t > 0 \), whenever the system reaches \( s_t \in S_c' \), the Round-Robin policy instructs the system to take the next action in \( U_c'(s_t) \), starting from the first action in \( U_c'(s_t) \).

**Lemma 5.** An accepting cyclic path of \( \mathcal{P} \), associated with \( S_c' \) and \( I_c' \), is a finite path that starts from any state \( s_f \in I_c' \) and ends in any state \( s_g \in I_c' \), while remaining within \( S_c' \).

Note that an accepting cyclic path does not necessarily start...
Remark 3. The initial distribution $y_0$ of $Z_{\text{out}}$ indicates how likely it is that the system controlled by the plan prefix $\pi_{\text{pre}}$ will enter the AMEC $(S'_c, U'_c)$ via each state inside $S'_c$. Let also $S'_c \triangleq S_c \setminus I_{in}$ and denote by $z_{s,u}$ the long-run frequency with which the system is at state $s$ and the action $u$ is applied, $\forall s \in S'_c$ and $\forall u \in U_c(s)$. Then, we can formulate the following linear program to solve Problem 3:

$$
\begin{align}
\min_{\{z_{s,u}\}} \ & \ C_{\text{out}}(S'_c, U'_c) \triangleq \sum_{(s,u) \in S'_c} \sum_{s' \in S'_c} z_{s,u} p_c(s, u, s') c_e(s, u) \\
\text{s.t.} \ & \ \sum_{(s,u) \in S'_c} \sum_{s' \in S'_c} z_{s,u} p_c(s, u, s') = \sum_{s' \in S'_c} y_0(s); \\
& \sum_{u \in V_e(s)} z_{s,u} = \sum_{s' \in S'_c} z_{s',u} p_c(s, u, s') + y_0(s), \forall s \in S'_c; \\
& z_{s,u} \geq 0, \forall s \in S'_c, \forall u \in U_c(s) \\
\end{align}
$$

where $\{z_{s,u}\}$ are the variables of (9). Furthermore, the transition probability $p_c$ is defined in cases below: (a) for transitions within $S'_c \setminus I_{in}'$, it holds that $p_c(s, u, s') = p_E(s, u, s')$, $\forall s, s' \in S'_c \setminus I_{in}'$, $\forall u \in U_c(s)$; (b) for transitions originated from $I_{out}$, it holds that $p_c(s_{out}, u, s) = p_E(s_{out}, u, s)$, $\forall s_{out} \in I_{out}$, $\forall u \in U_c(s_{out})$ and $\forall s \in S'_c$; (c) for transitions into $I_{in}$, it holds that $p_c(s, u, s_{in}) = p_E(s, u, s_{in})$, $\forall s \in S'_c \setminus U_c(s)$ and $\forall u \in U_c(s)$; (d) for transitions from $I_{out}$ to $I_{in}$, it holds that $p_c(s_{out}, u, s_{in}) = p_E(s_{out}, u, s_{in})$, $\forall s_{out} \in I_{out}$ and $\forall u \in U_c(s_{out})$; and (e) for transitions within $I_{in}$, it holds that $p_c(s_{in}, u, s_{in}) = p_E(s_{in}, u, s_{in})$, $\forall s_{in} \in I_{in}$, $\forall u \in U_c(s)$ and $\forall s \in S'_c \setminus I_{in}$.

Lastly, the cost function satisfies $c_e(s, u) = c_E(s, u)$, $\forall s \in (S_c \setminus I_{in})$, $\forall u \in U_c(s)$, and $c_e(s_{in}, \tau_0) = 0$, $\forall s_{in} \in I_{in}$.

Lemma 2. If (11) has a solution, then the plan suffix $\pi_{\text{out}}$ solves Problem 3 for the chosen AMEC $(S'_c, U'_c) \in \Xi_{\text{acc}}$.

Proof. First, by Definition 5 the objective in (11) equals the mean cyclic cost of all accepting cyclic paths for $I_{in}'$. Moreover, by the definition of an AMEC, any path remains within $S'_c$ by choosing only actions within $U'_c(s)$ at each state $s \in S'_c$.

Lemma 3. Let $\tau_\mathcal{P}$ be the set of all accepting runs of $\mathcal{P}$ that enter $S'_c$ after a finite number of steps. If $\tau_\mathcal{P}$ is generated under $\pi_{\text{out}}$, then $\tau_\mathcal{P}$ satisfies the accepting condition of $\mathcal{P}$. Moreover, the mean total cost in (10) equals the mean cyclic cost in (11), i.e., $\mathbb{E}_{\tau_\mathcal{P}} \{\text{Cost}(\tau_\mathcal{P})\} = C_{\text{out}}(S'_c, U'_c)$.

Proof. By (11), any system trajectory of $\mathcal{P}$ under $\pi_{\text{out}}$ contains infinite occurrences of accepting cyclic paths. Since any

Definition 6. The total cost of a cyclic path $P_a = s_0 u_0 s_1 u_1 \cdots s_{N_a} u_{N_a}$ is defined as

$$
C_{\text{out}}(P_a) \triangleq \sum_{i=0}^{N_a} c_D(s_i, u_i)
$$

where $N_a \geq 1$ is the length of the path and $s_0, s_{N_a} \in I'_c$. Then its mean total cost is defined as $C_{\text{out}}(P_a) \triangleq \frac{1}{N_a} C_{\text{out}}(P_a)$.

Problem 3. Find a stationary suffix policy $\pi_{\text{out}}^*$ for $\mathcal{P}$ within $S'_c$ that minimizes the mean cyclic cost

$$
C_{\text{out}}(S'_c, U'_c) = E_{P_a \in \mathcal{P}_a} \{C_{\text{out}}(P_a)\},
$$

where $P_a$ is the set of all accepting cyclic paths associated with the AMEC $(S'_c, U'_c)$.
accepting cyclic path starts from and ends in $I'_c$ (which is finite), $\tau_P$ intersects with $I'_c$ infinitely often. Moreover, since any accepting cyclic path remains within $S'_c$, $\tau_P$ remains within $S'_c$ for all time after entering $S'_c$. In other words, $\tau_P$ intersects with $H_p$ a finite number of times before entering $S'_c$ and then intersects $I_p$ infinitely often after entering $S'_c$, which satisfies the Rabin accepting condition of $P$. To show the second part, notice that the product $P$ under $\pi_{acc}$ evolves as a Markov chain and the set of all accepting cyclic paths within $S'_c$ has a stationary distribution. By viewing any accepting run $\tau_P$ as the concatenation of an infinite number of cyclic paths, the mean total cost of $\tau_P$ defined in (4) over an infinite time horizon equals the mean cyclic cost in (10) of all cyclic paths contained in $\tau_P$. This result is important in showing the equivalence between Problems 1 and 3 later in Theorem 6.

Example 3. This example illustrates the difference between the plan suffix obtained by (11) and the Round-Robin policy. Consider the same robot model from Example 2 and the partitioned workspace in Figure 4. The task is to surveil three base stations in the corners, i.e. $\varphi = (\Box \varnothing b_1) \land (\Box \varnothing b_2) \land (\Box \varnothing b_3)$. The plan prefix is derived by solving (8) but two different plan suffixes are used: one using (11) and the Round-Robin policy. Figure 5 shows the simulated trajectory under these two policies. It can be seen that the trajectory under the optimal plan prefix approximates the shortest route to cross all base stations, while the trajectory under the Round-Robin policy exhibits a rather random behavior.

3) Plan Synthesis when AECs do Not Exist: The synthesis algorithms proposed in Sections IV-B1 and IV-B2 rely on the assumption that the set of AMECs $\Xi_{acc}$ of $P$ is nonempty which, however, might not hold in many scenarios. In this case, most existing techniques proposed in [4, 17, 21, 22] cannot be applied. In this section, we first provide a simple example where no AECs exist, and then propose an approach to synthesize a relaxed plan prefix and suffix.

Example 4. This example provides a robot model $M$ and its task $\varphi$ for which no AECs exist in the product automaton $P$. Consider the MDP $M$ in Figure 5 that transitions between two states $(S_1, S_2)$ with probability 1 using the action $f$. Note that $S_1$ has only probability 0.01 of being occupied by an obstacle and $S_2$ is the base station. The task is to surveil the base station while avoiding obstacles, i.e., $\varphi = (\Box \varnothing b) \land (\Box \varnothing \varnothing b)$. The associated DRA is shown in Figure 5. The resulting $P$ is shown in Figure 6 where the set of states $H_p$ to avoid in the suffix is in red and the set of states $I_p$ to intersect infinitely often in green. The reason that no AECs exist in $P$ is because by definition an AEC $(S'_c, \{f\})$ should include all successor states that are reachable by the single action $f$. Then, starting from any green state in $I_p$, the set of reachable states eventually intersect with the red states in $H_p$.

When no AECs exist in $P$, the probability of satisfying the task under any policy is zero. However, it is still important to identify those policies that ensure high probability of avoiding bad states over long time intervals. Consequently, we propose to use an accepting SCC (ASCC) of $P$ as the relaxed AMEC, due to the following lemma.

![Figure 5: The MDP $M$ (left) and DRA $A_c$ (right, derived via (31), (32)) described in Example 4, with one accepting pair $\{(s_2), \{0, 1\}\}$.](image)

![Figure 6: The product $P$ of $M$ and $A_c$ in Figure 5. The state and edge names are omitted as the structure is of importance here. At least one green state should be visited infinitely often while avoiding all red states. Note all transitions are driven by the action $f$.](image)

Lemma 4. Assume there exists one infinite path of $P$ that is accepting. Then, there exists at least one SCC of $P$ that intersects with $I_p$ but not with $H_p$, for at least one pair $(H_p, I_p) \in \text{Acc}_P$.

Proof. As mentioned before, an infinite path of $P$, denoted by $R_P$, is accepting if for at least one pair $(H_p, I_p) \in \text{Acc}_P$ it holds that $R_P$ intersects with all states in $H_p$ finitely often while with $I_p$ infinitely often. Since both $H_p$ and $I_p$ are finite, there exists a cyclic path $s_1 \cdot s_2 \cdot \cdots s_f \cdot \cdots s_k$ of $P$ that contains at least one $s_f \in I_p$ and does not contain any state within $H_p$. By definition, this cyclic path is a SCC of $P$ that intersects with $I_p$ but not with $H_p$. This completes the proof.
Therefore, the second step is to synthesize the relaxed plan prefix that keeps the system inside $S'_c$ to satisfy the accepting condition with the maximal probability. Define the set $I'_P = S'_c \cap I'_P$, which is not empty for an ASCC $S'_c$. Then, an accepting cyclic path of $\mathcal{P}$ associated with $I'_c$ and the cyclic cost associated with $S'_c$ and $I'_c$ can be defined similarly as in Definition 3. Formally, we consider the following problem:

**Problem 4.** Find a control policy for $\mathcal{P}$ that minimizes the mean cyclical cost associated with the ASCC $S'_c$: $\mathbb{E}_{\mathcal{P}_a \in \mathcal{P}_a} \{C_{\text{sufx}}(F_\alpha)\}$, where $\mathcal{P}_a$ is the set of all accepting cyclical paths associated with $S'_c$ and $C_{\text{sufx}}$ is defined as in Definition 3 while at the same time maximizing the probability that the cyclic paths stay within $S'_c$.

In Problem 4, the first objective of minimizing the mean cyclical cost corresponds to minimizing the mean total cost in $\mathcal{P}$ in Problem 1. The objective of maximizing the probability of staying within the system (the system staying within the ASCC $S'_c$) corresponds to minimizing the frequency with which the system will reach the bad states that violate the task specifications. It constitutes a relaxation of the risk constraint in Problem 1. To solve Problem 4, first we construct a modified MDP $Z_{\text{sufx}}$ over $S'_c$, which is similar to $Z_{\text{out}}$ in Section 4.B. The set $I'_c$ is split into two virtual copies: $I_{in}$, which only has incoming transitions and $I_{out}$ that has only outgoing transitions. Formally, we define $Z_{\text{sufx}} = (S'_c, U_c, E_c, y_0, p_c, c_c)$, where the set of states is $S'_c = (S'_c \setminus I'_c) \cup I_{in} \cup I_{out} \cup \{s_{\text{bad}}\}$, with $I_{in} = \{s_f^{in}, \forall s_f \in I'_c\}$ and $I_{out} = \{s_f^{out}, \forall s_f \in I'_c\}$ the two virtual copies of $I'_c$, and $s_{\text{bad}}$ is a virtual bad state. The set of control actions is given by $U_c = U \cup \{\tau_0\}$, where $\tau_0$ is a self-loop action. The set of transition is $E_c \subseteq S'_c \times U_c$ which satisfies that (i) $(s, u) \in E_c$, $s, u \in S'_c$ and $u \in U(s)$; (ii) $(s, \tau_0) \in E_c$, $s \in I_{in}$; and (iii) $(s_{\text{bad}}, \tau_0) \in E_c$. Moreover, $y_0$ is the initial distribution of states in $S'_c$ based on the transition from states in $S'_c$.

\[
y_0(s) = \sum_{(s, u)} p(s, u, s) y_{\text{prox}}(s, u), \forall s \in (S'_c \setminus I'_c) \cup I_{out},
\]

where $\sum_{(s, u)} = \sum_{s \in S'_c} \sum_{u \in U(s)}$ and $y_{\text{prox}}(s, u)$ are the variables solutions from the synthesis of the relaxed plan preffix, and $y_0(s_{\text{bad}}) = 0$. Furthermore, the transition probability $p_c$ is defined in seven cases below: (a) for transitions within $S'_c \setminus I'_c$, it holds that $p_c(s, u, s) = p_E(s, u, s)$, $\forall s, s \in S'_c \setminus I'_c$, $\forall u \in U_c(s)$; (b) for transitions originated from $I_{out}$, it holds that $p_c(s_{out}, u, s) = p_E(s_f, u, s)$, $\forall s_f^{out} \in I_{out}$, $\forall u \in U(s_f^{out})$ and $\forall s \in S'_c \setminus I'_c$; (c) for transitions into $I_{in}$, it holds that $p_c(s, u, s_f^{in}) = p_E(s, u, s_f)$, $\forall s, s_f, u \in U(s)$, $\forall s_f^{in} \in I_{in}$; (d) for transitions from $I_{out}$ to $I_{in}$, it holds that $p_c(s_{out}, u, s_f^{in}) = p_E(s_f^{out}, u, s_f)$, $\forall s_f^{out} \in I_{out}$ and $\forall s_f^{in} \in I_{in}$; (e) for transitions into the bad state $s_{\text{bad}}$, it holds that $p_c(s, u, s_{\text{bad}}) = p_E(s, u, s_{\text{bad}})$, $\forall s, u \in U_c(s)$ and $u \in U(s)$; (f) each state within $I_{in}$ is included in a self-loop such that $p_c(s_{\text{in}}, \tau_0, s_{\text{in}}) = 1$, $\forall s_{\text{in}} \in I_{in}$; (g) the bad state is included in a self-loop such that $p_c(s_{\text{bad}}, \tau_0, s_{\text{bad}}) = 1$. Finally, the cost function $c_c$ is defined in two cases: (i) $c_c(s, u, c_E(s, u), \forall s \in S'_c, \forall u \in U_c(s)$ and (ii) $c_c(s, \tau_0, 0) = 0$, $\forall s_{\text{in}} \in I_{in}$ and $c_c(s_{\text{bad}}, \tau_0) = 0$.

**Remark 5.** Note that $E_{\alpha}$ contains all actions for each state in $S'_c$, compared with $E_\alpha$ as allowed by the AMEC.

Let $S'_c \triangleq S'_c \setminus (I_{in} \cup \{s_{\text{bad}}\})$ and $C_{\text{sufx}}(S'_c, d)$. We can also show that $Z_{\text{sufx}}$ above is $S'_c$-transient. Then, to solve Problem 4, we rely on a technique proposed in [35] to deal with dead ends in Stochastic Shortest Path (SSP) problems. First we introduce a large positive penalty for reaching the dead state, denoted by $d > 0$. Then, we modify (11) as follows: denote by $z_{s,u}$ the long-run frequency with which the system is at state $s$ and the action $u$ is taken, $\forall s \in S'_c$ and $\forall u \in U_c(s)$. We want to minimize the mean total cost of reaching $I_{in}$ from $I_{out}$, while minimizing the probability of leaving $S'_c$. In particular, we consider the following optimization:

\[
\min_{\{z_{s,u}\}} \left[ C_{\text{sufx}}(S'_c, d) \right] \triangleq \sum_{(s, u)} \left( \sum_{s \in S'_c} \eta(s, u, s) c_c(s, u) + \eta(s, u, s_{\text{bad}}) d \right)
\]

s.t. \[
\sum_{u \in U_c(s)} z_{s,u} = \sum_{(s, u)} \eta(s, u, s) + y_0(s), \forall s \in S'_c;
\]

\[
\sum_{(s, u)} \sum_{s \in I_{in}} \eta(s, u, s) + \eta(s, u, s_{\text{bad}}) = \sum_{s \in S'_c} y_0(s);
\]

\[
z_{s,u} \geq 0, \forall s \in S'_c, \forall u \in U_c(s);
\]

where the notation $\sum_{(s, u)} \triangleq \sum_{s \in S'_c} \sum_{u \in U_c(s)}$, the variables satisfy that $\eta(s, u, s) = z_{s,u} p_c(s, u, s), \eta(s, u, s_{\text{bad}}) \triangleq z_{s,u} p_c(s, u, s_{\text{bad}})$, $C_{\text{sufx}}(S'_c, d)$ denotes the objective function as the summation of the mean cost of reaching $I_{in}$ and the expected penalty of reaching $s_{\text{bad}}$. The first constraint balances the incoming and outgoing flow at each state, while the second constraint ensures that $I_{in} \cup \{s_{\text{bad}}\}$ are eventually reached. Let the optimal solution of (12) be $z_{s,u}^* = \{z_{s,u}^*, s \in S'_c \setminus I_{in}, u \in U_c(s)\}$. Then, the optimal stationary policy for the relaxed plan prefix, denoted by $\pi_{\text{sufx}}$, can be defined as follows: for states in $S'_c$, the optimal policy is given by $\pi_{\text{sufx}}(s, u) = z_{s,u}^*/(\sum_{u \in U_c(s)} z_{s,u}^*)$ if $z_{s,u}^* \neq 0$; otherwise the action at $s$ is chosen randomly, $\forall s \in S'_c$. Note that $p_{\pi_{\text{sufx}}}(s, u, s_f) = p_{\pi_{\text{sufx}}}(s_f^{out}, u), \forall s_f^{out} \in I_{out}$ and $\forall u \in U(s_f)$.

**Lemma 5.** Under the relaxed plan prefix $\hat{\pi}_{\text{sufx}}$, the probability of $Z_{\text{sufx}}$ reaching $I_{in}$ from $I_{out}$ while staying within $S'_c$ over an infinite horizon, is lower bounded by $1 - \gamma_{\text{sufx}}(d)$, where $\gamma_{\text{sufx}}(d) \triangleq \sum_{s \in S'_c} \sum_{u \in U_c(s)} \pi_{\text{sufx}}(s, u, s_{\text{bad}}) p_c(s, u, s_{\text{bad}})$.\[\]

**Proof.** The proof is a simple inference from (12c).

**Remark 6.** A lower bound can be enforced on $\gamma_{\text{sufx}}$ as in (8). However, this bound is hard to estimate and a large bound can yield the problem infeasible. In contrast, (12) always has a solution and $\gamma_{\text{sufx}}(d)$ is tunable by varying $d$.

**C. The Complete Policy**

In this section, we present how to combine the stationary plan prefix and plan suffix of $\mathcal{P}$ into the complete finite-memory policy of the original MDP $\mathcal{M}$. Furthermore, we show how to execute this finite-memory policy online.
Combining the Plan Prefix and Suffix: When AMECs of $\mathcal{P}$ exist, we can combine the plan prefix synthesis and the plan suffix synthesis for each AMEC into one Linear Program:

$$\min_{\{y_{s,u}, z_{s,u}\}} \beta \cdot C_{\text{pre}}(S_c) + (1 - \beta) \sum_{(S'_c, U'_c) \in \Xi_{\text{acc}}} C_{\text{suf}}(S'_c, U'_c),$$

(13)

s.t. 

Constraints (8b)–(8d) and (11c)–(11d),

where $C_{\text{pre}}(S_c)$ and $C_{\text{suf}}(S'_c, U'_c)$ are defined in (8a) and (11a), respectively, the variables $\{y_{s,u}\}$ satisfy the constraints (8b)–(8d) and (11c), and the variables $z_{s,u} \triangleq \{z_{s,u}(S'_c), \forall S'_c \in \Xi_{\text{acc}}\}$, where $z_{s,u}(S'_c)$ satisfy the constraints (11c)–(11d) for the AMEC $(S'_c, U'_c) \in \Xi_{\text{acc}}$. The parameter $0 \leq \beta \leq 1$ captures the importance of minimizing the expected total cost to reach $S_c$ versus stay in $S_c$. Note that the initial conditions $y_0$ in (11c) for each state in the suffix are expressed over the variables $\{y_{s,u}\}$. In other words, the initial conditions of each AMEC are now optimized to solve the combined objective function (13). It can be solved via any Linear Programming solver, e.g., “Gurobi” [36] and “CPLEX”.

As in Section IV-B2, we can combine the relaxed plan prefix and relaxed plan suffix for each ASCC into one Linear Program:

$$\min_{\{y_{s,u}, z_{s,u}\}} \beta \cdot C_{\text{pre}}(S_c) + (1 - \beta) \sum_{S'_c \in \Omega_{\text{acc}}} C_{\text{suf}}(S'_c, d),$$

(14)

s.t. 

Constraints (8b)–(8d) and (12b)–(12d),

where $C_{\text{pre}}(S_c)$ and $C_{\text{suf}}(S'_c, d)$ are defined in (8a) and (12a), respectively, the variables $\{y_{s,u}\}$ satisfy the constraints (8b)–(8d) and (12b), and the variables $z_{s,u} \triangleq \{z_{s,u}(S'_c), \forall S'_c \in \Omega_{\text{acc}}\}$, where $z_{s,u}(S'_c)$ satisfy the constraints (12b)–(12d) for the ASCC $S'_c \in \Omega_{\text{acc}}$. The parameter $0 \leq \beta \leq 1$ captures the importance of minimizing the expected total cost to reach $S_c$ versus stay in $S_c$. Similar to the previous case, the initial conditions $y_0$ in (12b) for each state in the ASCCs are expressed over the variables $\{y_{s,u}\}$. Thus the initial conditions are now optimized to solve the combined objective function (14). Again, it can be solved via any Linear Programming solver. Once the optimal $\{y_{s,u}^*, z_{s,u}^*\}$ is obtained, the optimal relaxed plan prefix $\pi_{\text{pre}}^*$ and relaxed plan suffix $\pi_{\text{suf}}^*$ can be constructed as described in Section IV-B3.

Note that the size of both Linear Programs in (13) and (14) is linear with respect to the number of transitions in $\mathcal{P}$ and can be solved in polynomial time [37]. Note also that the multi-objective costs introduced in (13) and (14) provide a balance between optimizing the plan prefix and suffix. Compared to only optimizing the plan suffix, i.e., for $\beta = 0$ as required to solve Problems 3 and 4, increasing slightly the value of $\beta$ can lead to a significant decrease in the total plan prefix, without sacrificing much the optimality in the plan suffix.

Observe that the optimal policy derived above only includes the states within $S_n \cup S_c$. Thus no policy is specified for the bad states in $S_d$. Once the system reaches any bad state, it has violated the formula $\varphi$ and can not satisfy it anymore. Thus, it is common practice to stop the system once that happens [4], [21]. We propose here a new method that allows the system to recover from the bad state in $S_d$ and continue performing the task, which could be useful for partially-feasible tasks with soft constraints, as discussed in [7].

Definition 7. The projected distance of a bad state $s_d = (x, l, q) \in S_d$ onto $S_c \cup S_n$ via $u \in U(s_d)$ is defined as:

$$\kappa(s_d, u) \triangleq \sum_{\ell, \chi, q} \frac{D(l, \chi, q)}{|\chi(q, \hat{q})|} p_{E}(x, u, \hat{x}) \cdot p_{L}(\hat{x}, \hat{\ell}) \cdot \frac{1}{|\chi(q, \hat{q})|} p_{E}(x, u, \hat{x}) \cdot p_{L}(\hat{x}, \hat{\ell}),$$

(15)

where $\hat{s} \triangleq (\hat{x}, \hat{\ell}, \hat{q})$ and function $D : 2^{AP} \times 2^{AP} \to \mathbb{N}$ returns the distance between an element $\ell \in 2^{AP}$ and a set $\chi \subseteq 2^{AP}$ was firstly introduced in [7] and restated below.

Simply speaking, $\kappa(s_d, u)$ evaluates how much the product automaton $\mathcal{P}$ is violated on the average if the bad state $s_d \in S_d$ is projected into the set of good states $S_c \cup S_n$ using action $u \in U(s_d)$. Function $D(\ell, \chi) = 0$ if $\ell \in \chi$, and $D(\ell, \chi) = \min_{E \in \chi} |\{a \in AP | a \in E, a \notin \ell'\}|$, otherwise. Namely, it returns the minimal difference between $\ell$ and any element in $\chi$. Given $\kappa(\cdot)$, the policy at $s_d \in S_d$ is given by

$$\pi^*(s_d, u) = \begin{cases} 1 & \text{for } u = \arg\min_{u \in U(s_d)} \kappa(s_d, u), \\ 0 & \text{otherwise}. \end{cases}$$

(16)

which chooses the single action that minimizes (15). Combining (13), (14) and (16) provides the complete policy for $\mathcal{P}$. The above discussions are summarized in Algorithm 1.

Algorithm 1: Complete Policy Synthesis

Input: $\mathcal{P}$ by Definition 2, $\gamma$, $\beta$

Output: the complete policy $\pi^*$, $\mu^*$

if $\Xi_{\text{acc}} \neq \emptyset$ then

1. Construct $Z_{\text{pre}}$ and $Z_{\text{suf}}$ for each $(S'_c, U'_c) \in \Xi_{\text{acc}}$.
2. Derive $\pi^*$ via solving (13), and (16).
else

1. Construct $Z_{\text{pre}}$ and $Z_{\text{suf}}$ for each $(S'_c, U'_c) \in \Omega_{\text{acc}}$.
2. Derive $\pi^*$ via solving (14), and (16).
3. Construct $\mu^*$ from $\pi^*$ by (17).

2) Mapping $\pi^*$ to $\mu^*$: Lastly, we need to map the optimal stationary policy $\pi^*$ of $\mathcal{P}$ above to the optimal finite-memory policy $\mu^*$ of $\mathcal{M}$. Starting from stage $t = 0$, the initial state $s_0 = (x_0, l_0, q_0) \in S_n$ and the optimal action to take is given by the distribution $\pi^*(s_0)$. Assume that $u \in U(s_0)$ is taken. Then at stage $t = 1$, the robot observes its resulting state $x_1$ and the label $l_1$. Thus the subsequent state in $\mathcal{P}$ is $s_1 = (x_1, l_1, q_1)$, where $q_1 = \delta(q_0, l_0)$ is unique as $A_\varphi$ is deterministic. The optimal action to take now is given by the distribution $\pi^*(s_1)$. This process repeats itself indefinitely. Denote by $s_t \in S$ the reachable state at stage $t \geq 0$ which is always unique given the robot’s past sequence of states $X_t = x_0 x_1 \cdots x_t$ and labels $L_t = l_0 l_1 \cdots l_t$. Thus the optimal policy $\mu^*$ at stage $t \geq 0$ given $X_t$ and $L_t$ is

$$\mu^*(X_t, L_t) = \pi^*(s_t),$$

(17)
i.e., the control policy at the reachable state $s_t$ in $P$ is the best control policy in $M$ at stage $t$, $\forall t \geq 0$. Last but not least, if the system reaches a bad state at stage $t - 1$, i.e., $s_{t-1} \in S_d$, according to policy $\pi$, the robot will take action $u^*$ and more importantly the next reachable state is set to be $s_t \triangleq (x_t, l_t, q_t^*) \in (S_r \cup S_b)$, where $x_t$, $l_t$ are the observed robot location and label at stage $t$ and $q_t^* \triangleq \arg\min_{\tilde{q} \in \text{Post}(q_{t-1})} \{l_t-1 \cdot \chi(q_{t-1}, \tilde{q})\}$.

**Theorem 6.** Algorithm 7 solves Problem 7 if AECs of $P$ exist and $\beta = 0$. Otherwise, if no AECs of $P$ exist, then Problem 7 has no solution. In this case, Algorithm 7 provides a relaxed policy that minimizes the relaxed suffix cost $C_{\text{suffix}}(S', d)$ defined in (12). Moreover, given any finite run $S_T = s_0s_1 \cdots s_T$ of $P$ under the optimal policy $\pi^*$, the probability that $S_T$ does not intersect with the set of bad states $S_d$ for all time $t \in [0, T]$ is bounded as

$$Pr(s_t \notin S_d, \forall t \in [0, T]) \geq (1 - \gamma_{\text{prefix}}) \cdot (1 - \gamma_{\text{suffix}}(d))^N_s,$$

where $N_s \geq 0$ is the number of accepting cyclic paths contained in $S_T$ that depends on $T$.

**Proof.** To show the first part of this theorem, similar to Lemma 1, the constraints of (8b)–(8d) ensures that the total probability of reaching any state of all AMECs is lower-bounded by $1 - \gamma$. Moreover, the first part of Lemma 2 shows that any infinite run $\tau_P$ of $P$ would satisfy $\varphi$ once it enters any AMEC $(S_r', U_r') \subseteq \Omega_{\text{ACC}}$ by following the plan suffix. The fact that $\pi^*$ also minimizes the mean total cost in (11) when $\beta = 0$ in (13) can be shown as follows: as discussed in [24], [33], [34], the mean payoff objective depends on how the system suffix behaves within the AMECs. The second part of Lemma 3 guarantees that the derived plan suffix $\pi^*_{\text{suffix}}$ minimizes the mean total cost of staying within any of the AMECs, while satisfying the accepting condition.

To show the second part of this theorem, no solution to Problem 1 exists regardless of the choice of $\gamma$, as the probability of satisfying the task is zero. Instead, when $\beta = 0$, the optimal policy $\pi^*$ obtained by Algorithm 1 minimizes the relaxed suffix cost $C_{\text{suffix}}(S', d)$. At the same time, due to the constraints in (8) that are also present in (13), the plan prefix $\pi^*_{\text{prefix}}$ ensures that all runs stay within $S_n$ with at least probability $(1 - \gamma_{\text{prefix}})$ before entering any ASCC $S_r' \subseteq \Omega_{\text{ACC}}$, while the relaxed plan suffix $\pi^*_{\text{suffix}}$ ensures that the runs stay within $S_d'$ with at least probability $(1 - \gamma_{\text{suffix}}(d))$ for one execution of any accepting cyclic path. Consequently, if the finite run contains $N_s$ accepting cyclic paths, the probability of avoiding $S_d$ is lower bounded by $(1 - \gamma_{\text{prefix}}) \cdot (1 - \gamma_{\text{suffix}}(d))^N_s$. Even though this probability approaches zero as $N_s$ approaches infinity, this result still ensures that the frequency of visiting bad states over finite intervals is minimized.

**Algorithm 2: Policy Execution**

**Input:** $M$, $\varphi$, observed state $s_t$ and label $l_t$ at stage $t \geq 0$

**Output:** $\mu^*$ and $u_t$ at stage $t \geq 0$

1. **Offline:** Construct $P$ and synthesize $\pi^*$ by Alg. 1.
2. At $t = 0$: set $s_0 = (x_0, l_0, q_0)$ and apply $u_0 \sim \pi^*(s_0)$.
3. **while** $t = 1, 2, \cdots$ **do**
   - observe $x_t$ and $l_t$.
   - if $s_{t-1} \notin S_d$ then
     - Set $s_t = (x_t, l_t, q_t^*)$, where $q_t = \delta(q_{t-1}, l_{t-1})$.
   - else
     - Set $s_t = (x_t, l_t, q_t^*) \in (S_n \cup S_c)$.
   - Apply action $u_t \sim \pi^*(s_t)$.

V. SIMULATION RESULTS

In this section, we present simulation results to validate the scheme. All algorithms are implemented in Python 2.7 and available online [32]. All simulations are carried out on a laptop (3.06GHz Duo CPU and 8GB of RAM).

A. Model Description

We consider a partitioned $10 \times 10$ workspace as shown in Figure 8 where each cell is a $2 \times 2$ area. The properties of interest are $\{\text{Obs}, b_1, b_2, b_3, \text{Sp1}\}$. The properties satisfied at each cell are probabilistic: three cells at the corners satisfy $b_1$, $b_2$, and $b_3$, respectively with probability one. Four cells at $(1, 5m)$, $(5m, 3m)$, $(9m, 5m)$, $(5m, 9m)$ satisfy $\text{Sp1}$ with probabilities ranging from 0.2 to 0.8, modeling the likelihood that a supply appears at that particular cell. One cell at $(5m, 5m)$ satisfies $\text{Obs}$ with probability 0.7. Other obstacles will be described later upon different task scenarios.

The robot motion follows the unicycle model, i.e., $x = v \cos(\theta)$, $y = v \sin(\theta)$, $\dot{\theta} = \omega$, where $p(t) = (x(t), y(t)) \in \mathbb{R}^2$, $\theta(t) \in (-\pi, \pi]$ are the robot’s position and orientation at time $t \geq 0$. The control input is $u(t) = (v(t), \omega(t))$ and contains the linear and angular velocities. Due to actuation noise and drifting, the robot’s motion is subject to uncertainty.

The action primitives and the associated uncertainties are shown in Figure 1 and described below: action “FR” means driving forward for $2m$ by setting $v(t) = v_0$ and $\omega(t) = 0$, $\forall t \in [0, 2/v_0]$. This action has probability 0.8 of reaching $2m$ forward and probability 0.1 of drifting to the left or right by $2m$, respectively; action “BK” can be defined analogously to “FR”; action “TR” means turning right by an angle of $\pi/2$ by setting $v(t) = 0$ and $\omega(t) = -\omega_0$, $\forall t \in [0, \pi/2(2\omega_0)]$. This action has probability 0.9 of turning to the right by $\pi/2$, probability 0.05 of turning less than $\pi/4$ due to undershoot and probability 0.05 of turning more than $3\pi/4$ due to overshoot; action “TL” can be defined analogously to “TR”; finally, action “ST” means staying still by setting $v(t) = \omega(t) = 0$, $\forall t \in [0, T_0]$ where $T_0$ is the chosen waiting time. It has probability 1.0 of staying where it is. The cost of each action is given by $\{2, 4, 3, 3, 1\}$, respectively, where the cost of “ST” is set to 1 as it consumes time to wait at one cell.

With the above model, we can abstract the robot state by the cell coordinate in which it belongs, namely, $(x_c, y_c) \in$
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\gamma$ & Total Cost & Failure & Success & Unfinished \\
\hline
0 & 132.2 & 0 & 910 & 90 \\
0.1 & 118.1 & 99 & 872 & 29 \\
0.2 & 110.5 & 219 & 770 & 11 \\
0.3 & 104.6 & 308 & 692 & 0 \\
0.4 & 98.3 & 417 & 583 & 0 \\
\hline
\end{tabular}
\caption{Statistics of 1000 Monte Carlo simulations of 500 time steps, under different $\gamma$ for task [18]} \label{tab:stats}
\end{table}

\end{document}
It took around 16s to construct the product automaton that has 4224 states, 41344 transitions and 1 accepting pair. Since two AMECs exist in the product, we synthesize the optimal policy using Algorithm 1 via solving (13) under $\gamma = 0$ and $\beta = 0.1$, which took around 0.2s given the complexity of task (20). Notice that the optimal plan sometimes requires the robot to wait at a cell marked by $\text{Spl}$ by taking action “$\text{ST}$”, since the expected cost of traveling to another cell with supply might be higher than waiting there for the supply to appear. Figure 8 compares the simulated trajectories under the optimal policy and the Round-robin policy. Based on 1000 Monte Carlo simulations, the total cost of accepting cyclic paths is much lower under the optimal policy than the Round-robin policy (70 versus 550). Furthermore, Figure 9 shows the average number of supplies received at each base under these two policies. It can be seen that much more supplies are received at each base station under the optimal policy than the Round-robin policy. Simulation videos of both cases can be found in [38]. Lastly, to show how the choice of $\beta$ in (13) affects the optimal prefix and suffix cost, we repeat the above procedure for different $\beta$ and the results are summarized in Table III. In the table, the prefix cost equals to $C_{\text{pre}}(S_c)$, the mean suffix cost equals to $\sum_{(S_c', U_c') \in \Xi_{\text{acc}}} C_{\text{surf}}(S_c', U_c')$ from (13). The total suffix cost is computed based on (9) in order to magnify the changes in the suffix cost. It can be noticed that for small non-zero values of $\beta$, less 0.2, the optimal prefix cost is reduced dramatically (from 180.7 to 66.1), without increasing much the optimal suffix cost (from 66.1 to 65.2).

In order to demonstrate scalability and computational complexity of the proposed algorithm, we repeat the policy synthesis under the same task (20) but for workspaces of various sizes. Particularly, we increase the number of cells from $5^2$ to $9^2$, $15^2$, $19^2$, $25^2$, $29^2$. The size of resulting $\mathcal{M}$, $\mathcal{P}$, $\Xi_{\text{acc}}$ and the time taken to compute them are shown in Table IV where we also list the complexity of the LP (13), which consists of (8) and (11), and the time taken to solve (13). It can be seen from Table IV that solving (13) requires a small fraction of total time, compared to the construction of $\mathcal{M}$, $\mathcal{P}$ and $\Xi_{\text{acc}}$.

### E. Surveillance with Clustered Obstacles

In this case, we demonstrate how the relaxed plan prefix and suffix can be synthesized under scenarios where no AECs can be found. In particular, we consider the surveillance task in (19) but more obstacles are placed in the workspace as shown in Figure 10. The center cell $(5m, 5m)$ has probability 0.9 of being occupied by an obstacle and the four cells above and on the left have probability 0.01 of being occupied by an obstacle. Thus, $b_1$ is surrounded by possible obstacles around it, even though the probability is very low.

The resulting product automat on has 1184 states, 13888 transitions, and 1 accepting pair. It can be verified that no AECs exist in $\mathcal{P}$ and thus the second case of Algorithm 1 is activated, where the optimal solution is derived by solving (14). We synthesize the relaxed optimal policy under different $\gamma_{\text{prex}}$ and $d$, as shown in Table IV. It took in average 37s to synthesize the complete policy for $\beta = 0.1$ and any chosen $\gamma_{\text{prex}}$ and $d$ in this case. Recall that $d$ is a large penalty for entering the set of bad states in (12).

The resulting product automat on has 1184 states, 13888 transitions, and 1 accepting pair. It can be verified that no AECs exist in $\mathcal{P}$ and thus the second case of Algorithm 1 is activated, where the optimal solution is derived by solving (14). We synthesize the relaxed optimal policy under different $\gamma_{\text{prex}}$ and $d$, as shown in Table IV. It took in average 37s to synthesize the complete policy for $\beta = 0.1$ and any chosen $\gamma_{\text{prex}}$ and $d$ in this case. Recall that $d$ is a large penalty for entering the set of bad states in (12).

In particular, we first choose $\gamma_{\text{prex}} = 0.1$ and $d = 300$. Two simulated trajectories under the derived policy are shown in Figure 10. Furthermore, we perform 1000 Monte Carlo simulation under the $\gamma_{\text{prex}}$ and $d$ listed in Table IV where we compare the number of times that the robot fails the
task by colliding with obstacles (the failure), the number of times that the robot successfully reaches the set of ASCC \( S_c \) (the prefix success), and the number of times that the robot successfully executes one accepting cyclic path associated with \( S_c \) and \( I' \) of one ASCC (the suffix success). It can be seen that \((1 - (1 - \gamma_{\text{prex}})(1 - \gamma_{\text{sufx}})), (1 - \gamma_{\text{prex}}) \) and \((1 - \gamma_{\text{sufx}})\) matches very well the probability of failure, the prefix success, and the suffix success, respectively, as discussed in Theorem 6.

Also, it can be seen that the system can recover from the bad states and continue executing the task if the recovery policy proposed in [16] is activated. It can also be seen that increasing \( \gamma_{\text{prex}} \) leads to a lower prefix success rate and decreasing \( d \) leads to a lower suffix success rate.

To demonstrate scalability and computational complexity of the proposed algorithm when AMECs do not exist, we repeat the policy synthesis under the same task (19) but for different workspaces of various sizes, as in Section V-D. We set \( \gamma = 0.3 \), \( d = 300 \) and \( \beta = 0.1 \). The size of resulting \( \mathcal{M}, \mathcal{P}, \Omega_{\text{acc}} \) and the time taken to compute them are shown in Table V where we also list the complexity of the (14), which consists of (8) and (12), and the time taken to solve (14). It can be seen above that solving (14) now requires a larger fraction of total time, compared to the construction of \( \mathcal{M}, \mathcal{P} \) and \( \Omega_{\text{acc}} \). However, it requires much less time to compute the set of ASCCs \( \Omega_{\text{acc}} \) than the set of AMECs \( \Xi_{\text{acc}} \). For instance, in the case of 29^2 cells in the workspace, it took around 23.1 seconds to construct \( \mathcal{P} \) (which has approximately \( 2.8 \times 10^4 \) states and \( 2.9 \times 10^5 \) transitions) and 19.6 seconds to construct its ASCCs (compared with 160 minutes in Table II). Once (14) is constructed, it took around 2.5 minutes to solve it.

**F. Comparison with PRISM**

In this section, we compare the proposed algorithm to the widely-used model checking tool PRISM [13]. The following results were obtained using PRISM 4.3.1, where Linear Programming is chosen as the solution method. First, since PRISM does not take the probabilistically-labeled MDP in (1) as inputs, we translate the product automaton in (5) into PRISM language and verify its Rabin accepting condition directly. Implementation details can be found in [32]. For tasks (18), (19) and (20), PRISM verifies that the probability of satisfying each of them is 1.0, within time 0.46s, 0.38s and 6.4s, respectively. The difference in computation time is likely due to the difference in the LP solvers. Second, in order to test different values of \( \gamma \), we use the “multi-objective property” to find the minimal cumulative reward while ensuring the risk of violating the task is bounded by \( \gamma \). Note that the associated model has to be the modified product model \( Z_{\text{pre}} \) defined in Section IV-B1 as PRISM does not currently support multi-objective property with the “F target” operator (i.e., \( \Diamond \text{S} \)). The computation time is approximately the same as in the previous cases. Last, the current PRISM version does not support the mean-payoff optimization in the AMECs, nor does it generate the relaxed control policy for the case where no AMECs exist in the product automaton. In fact, PRISM will simply return that the maximal probability of satisfying the task is 0. The MultiGain tool recently proposed in [34] can handle multiple mean-payoff constraints but does not allow the tuning of the satisfaction probability \((1 - \gamma)\).

**VI. EXPERIMENTAL STUDY**

In this section, we present an experimental study. We use a differential-driven “iRobot” whose position we track in real-time via an Optitrack motion capture system. The communication among the planning module, the robot actuation module, and the Optitrack is handled by the Robot Operating System (ROS). The software implementation for this experiment is available in [39]. The experiment videos are online [40].

**A. Model Description**

Consider the 2.5m × 1.5m experiment workspace as shown in Figure 11 with three base stations located at the corners and one obstacle region. It consists of \( 5 \times 3 \) square cells of dimension 0.5m × 0.5m each. The robot’s motion within the workspace is abstracted similarly as in Section V-A. The resulting MDP has 60 states and 456 edges.
Lastly, to demonstrate the proposed scheme for much larger workspaces and more complex tasks, particularly when no AMECs can be found in the product automaton, we create a virtual experiment platform based on V-REP \cite{41}, which is available in \cite{32}. A snapshot is shown in Figure \ref{fig:11}. The user can easily change the configuration of the workspace and the robot task specification. Once the control policy is synthesized via Algorithm \ref{alg:1} and saved, the user can perform any number of test runs in this environment. Demonstration videos are online \cite{40} where we replicate the surveillance task with clustered obstacles from Section \ref{sec:V-E}. It can be seen that the relaxed control policy can ensure high probability of avoiding bad states over long time intervals.

### VII. Conclusion and Future Work

In this paper, we propose a plan synthesis algorithm for probabilistic motion planning, subject to high-level LTL task formulas and risk constraints. Uncertainties in both the robot movement and the workspace properties are considered. We obtain optimal policies that optimize the total cost both in the prefix and suffix of the system trajectory. We also address the case where no AECs exist in the product automaton in which case the probability of satisfying the task is zero. The proposed solution provides provable guarantees on the probabilistic satisfiability and the mean total-cost optimality, and is verified via both numerical simulations and experimental studies. Future work involves extensions to multi-robot systems.

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