Representatives of Elliptic Weyl Group Elements in Algebraic Groups

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Abstract. An element $w$ of a Weyl group $W$ is called elliptic if it has no eigenvalue 1 in the standard reflection representation. We determine the order of any representative $g$ in a semisimple algebraic group $G$ of an elliptic element $w$ in the corresponding Weyl group $W$. In particular if $w$ has order $d$ and $G$ is simple of type different from $C_n$ or $F_4$, then $g$ has order $d$ in $G$.

1. Introduction

An element $w$ of a Weyl group $W$ is called elliptic if it has no eigenvalue 1 in the standard reflection representation. It is well known that the Coxeter elements provide examples of such elements, but in general they are not the only examples [F, Proposition 8; H, Lemma 3.16]. If we think of $W$ not as the Weyl group of a root system but as the quotient $W = N_G(T)/T$ in some semisimple algebraic group $G$ with maximal torus $T$, the natural question arises whether representatives in $G$ of elliptic elements have any nice properties. In this paper we determine the order of any representative in $G$ of an elliptic element.

The classification of conjugacy classes in Weyl groups is provided in [C1], where they are essentially classified by certain admissible diagrams, which we will call Carter diagrams. These diagrams are particularly useful in the present context since they make it easy to single out elliptic elements. The question of determining the order of representatives of elliptic elements was analyzed in [F], with some substantial results in certain cases. The cases of $E_6$ and $E_7$, however, proved particularly troublesome in that paper, and in the classical cases the focus was on the case when $G$ is simple. In the present work, instead of analyzing the problem thinking of $G$ as a matrix group, we use Chevalley generators and relations to calculate the order of any representative of an elliptic element. One surprising result is that if $G$ is simple and $w$ is elliptic with order $d$, then representatives $g$ of $w$ almost always have order $d$, with the only counterexamples arising in $C_n$ and $F_4$. A summary of results is given in Table 1.3, see Definition 2.3 for an explanation of the terminology in the table.
Weyl group elements with no eigenvalue 1 are called \textit{elliptic} in [L] and \textit{generalized Coxeter elements} in [DW]. Here we will generally stick with “elliptic.” If \(w\) is elliptic we will also refer to the conjugacy class of \(w\) in \(W\) as “elliptic” since eigenvalues are conjugation invariant. Our main references for facts about root systems and semisimple groups are [C3] and [GLS]. We will use the numbering of the nodes of the Dynkin diagrams given in [C3]. (Note that the numbering for \(E_7\) and \(E_8\) is different than that given in [GLS].)

\[
\begin{align*}
A_{n-1} \quad & 1 \overset{2}{\cdots} n-2 \overset{n-1}{\cdots} n-1 \\
B_n \quad & 1 \overset{2}{\cdots} n-1 \overset{n}{\cdots} \\
C_n \quad & 1 \overset{2}{\cdots} n-1 \overset{n}{\cdots} \\
D_n \quad & 1 \overset{2}{\cdots} n-2 \overset{n-1}{\cdots} n \\
E_6 \quad & 1 \overset{2}{\cdots} 3 \overset{5}{\cdots} 6 \\
E_7 \quad & 1 \overset{2}{\cdots} 3 \overset{4}{\cdots} 6 \overset{7}{\cdots} \\
E_8 \quad & 1 \overset{2}{\cdots} 3 \overset{4}{\cdots} 5 \overset{7}{\cdots} 8 \\
F_4 \quad & 1 \overset{2}{\cdots} 3 \overset{4}{\cdots} \\
G_2 \quad & 1 \overset{2}{\cdots} \\
\end{align*}
\]

2. Preliminary results

Let \(\Phi\) be a reduced crystallographic root system with Weyl group \(W\), \(K\) an algebraically closed field, and \(G\) a semisimple algebraic \(K\)-group with root system \(\Phi\). Let \(G_u\) be the corresponding universal group and \(G_a\) the adjoint group, as in [GLS] Theorem 1.10.4]. Then we have epimorphisms \(G_u \to G \to G_a\) with \(\ker(G_u \to G_a) = Z(G_u)\) finite. In fact, \(G\) is always either \(G_a\) or \(G_u\) unless \(\Phi\) has type \(A_n\) or \(D_n\). Thus it is almost sufficient to just analyze \(G_a\) and \(G_u\).
We will need to think of $G$ in terms of Chevalley generators and relations, so we now establish some facts in that vein. Let $T$ be a maximal torus in $G$ and let $x_\alpha(\lambda)$ denote the standard Chevalley generators, where $\alpha \in \Phi$ and $\lambda \in K$. For each $\alpha \in \Phi$, $\lambda \in K^*$ define $m_\alpha(\lambda) := x_\alpha(\lambda)x_{-\alpha}(-\lambda^{-1})x_\alpha(\lambda)$ and $h_\alpha(\lambda) := m_\alpha(\lambda)m_\alpha(-1)$. Let $N := \langle m_\alpha(\lambda) \rangle$, and note that $T = \langle h_\alpha(\lambda) \rangle$ [GLS, Theorem 1.12.11]. It is a fact that $N/T \cong W$; see [S] Lemma 22). The following Chevalley relation, which we will need later, is established in the proof of [C2, Lemma 7.2.2].

(CR1): For $\alpha, \beta \in \Phi$, $m_\alpha(1)m_\beta(1)m_\alpha(1)^{-1} = m_{s_\alpha \beta}(c(\alpha, \beta))$ where $c(\alpha, \beta) = \pm 1$ is determined only by $\alpha$ and $\beta$.

This sign $c(\alpha, \beta)$ can sometimes be computed just from knowing the $\alpha$-chain of roots through $\beta$. As we will see in Lemma 5.1 if $\alpha$ and $\beta$ are orthogonal then $c(\alpha, \beta)$ is “usually” 1, and by orthogonality $s_\alpha \beta = \beta$ so then $m_\alpha(1)$ and $m_\beta(1)$ actually commute. Details are given in Lemma 5.1. An immediate corollary to (CR1) is the following, which does not depend on $c(\alpha, \beta)$:

(CR2): For $\alpha, \beta \in \Phi$, $m_\alpha(1)h_\beta(-1)m_\alpha(1)^{-1} = h_{s_\alpha \beta}(-1)$.

We now define $N_0$ to be $\langle m_\alpha(1) | \alpha \in \Phi \rangle$ and $T_0$ to be $\langle h_\alpha(-1) | \alpha \in \Phi \rangle$. It is easy to see that $N_0/T_0 \cong W$, by the same proof that $N/T \cong W$. See also [AZ, Lemma 4.2] and [GLS, Remark 1.12.11]. Since $T_0$ is abelian and all its elements square to 1, we immediately see that any Weyl group element $w$ of order $d$ has at least one representative $g_0$ of order either $d$ or $2d$.

For elliptic $w$, by [K, Theorem 1] and, independently, [AZ, Theorem 4.3], all representatives of $w$ in $N$ have the same order. In fact by the proof of [AZ, Theorem 4.3], for any representative $g$, $g^d = g_0^d$. Thus to determine the order of any representative $g$ of an elliptic Weyl group element $w$ with order $d$, it suffices to check whether $g_0^d = 1$ or not, for $g_0 \in N_0$ representing $w$. We encode this fact into the following proposition, which is proved in the sources mentioned above.

**Proposition 2.1.** Let $w \in W$ be an elliptic element with order $d$. Then all representatives of $w$ in $N$ have the same order. In particular they all have order $d$ if $g_0^d = 1$ and order $2d$ otherwise.

**Remark 2.2.** The converse of Proposition 2.1 is also true for most $K$; that is, if $w$ is not elliptic and if $K$ contains an element of infinite order, then $w$ has a representative of infinite order in $N$. This is proved in Theorem 4.3 in [AZ], but we will not need this fact here.

The elliptic elements are classified in [C1], and to each conjugacy class of elliptic elements is assigned an “admissible diagram” $\Gamma$, which we call a *Carter diagram*. For $w \in W = W(\Phi)$, we can always find linearly independent roots $\beta_1, \ldots, \beta_r$ such...
that $w = s_{\beta_1} \cdots s_{\beta_r}$, and $w$ is elliptic if and only if $r = n$ where $n$ is the rank of $\Phi$ \cite{C1}. In general $\Gamma$ is constructed by taking a node for each $\beta_i$ and connecting $\beta_i$ to $\beta_j$ with a certain number of edges given by the same rule as for Dynkin diagrams (that is, depending on the angle between $\beta_i$ and $\beta_j$). In particular if $\alpha_1, \ldots, \alpha_n$ are the simple roots then $w = s_{\alpha_1} \cdots s_{\alpha_n}$ is a Coxeter element and simply has Carter diagram equal to the Dynkin diagram of $\Phi$. Another important case is when $\Phi$ contains $n$ mutually orthogonal roots $\beta_1, \ldots, \beta_n$. In this case $w = s_{\beta_1} \cdots s_{\beta_n}$ is the negative identity element $-I$ in $W$, with Carter diagram $A_n^1$, i.e., $n$ unconnected nodes. It is possible that two elements in $W$ can have the same Carter diagram without being conjugate, but this will never happen for elliptic elements \cite{C1}.

At this point for the sake of brevity we introduce the following definitions:

**Definition 2.3.** Let $w \in W$ be elliptic with order $d$. If all representatives of $w$ in $G$ have order $d$ we say $w$ has spin 1. If all representatives of $w$ have order $2d$ we say $w$ has spin $-1$. Note that this is a property of $w$ and of $G$, not just of $w$. Thus we will often need to refer to $G$-spin, adjoint spin, or universal spin. Spin is of course preserved by conjugation, so we may also refer to the spin of a conjugacy class or Carter diagram. Furthermore, if $w \in W$ is elliptic with order $d$ and $g_0 \in N_0$ represents $w$, we will call $g_0^d \in T_0$ the spin signature of $w$. This doesn’t depend on the choice of $g_0$, and so is well defined.

Unlike spin, the spin signature may not be conjugation invariant. In practice we will often find that the spin signature of $w$ is central in $G$, in which case we can refer to the spin signature of a conjugacy class or Carter diagram. In Section 3 we will present a labeling of the Carter diagram of $w$ that helps to calculate the spin signature. First we establish a few results that simplify things considerably.

**Corollary 2.4.** Let $w \in W$ be elliptic with odd order $d$. Then $w$ has spin 1.

**Proof.** Let $g_0 \in N_0$ represent $w$, so $g_0^d \in T_0$. Since $(g_0g_0^d)^d = g_0^d$, in fact $g_0^d = 1$. But since $d$ is odd this means that $g_0$ cannot have order $2d$, and so has order $d$. \qed

**Lemma 2.5.** For any $w \in W$ and $r \in \mathbb{Z}$, if $w$ and $w^r$ are both elliptic then $w$ has the same spin and spin signature as $w^r$.

**Proof.** Say $w$ has order $d$ and spin 1. Then any representative $g$ of $w$ in $N$ has order $d$, so $g^r$ has the same order as $w^r$ implying that $w^r$ has spin 1. The spin $-1$ case follows by a parallel argument, and the fact that the spin signatures are the same is immediate. \qed

**Definition 2.6.** Let $w_1$ and $w_2$ be elements of $W$. If there exists $r \in \mathbb{Z}$ such that $w_1^r = w_2$ or $w_2^r = w_1$ then we will call $w_1$ and $w_2$ linked. Similarly we may refer
to the corresponding conjugacy classes as linked if there exist representatives from each class that are linked. The point is that linked classes have equal spins, and linked elements have equal spins and spin signatures.

To tell whether two elliptic classes are linked we will often make use of Table 3 in [C1], which lists the characteristic polynomials of elliptic elements. Knowing the eigenvalues of an elliptic element \( w \) allows us to easily check which powers \( w^r \) are elliptic, and to identify the conjugacy class of \( w^r \). For example if the eigenvalues of \( w \) are all primitive \( 2r \)th roots of unity, then \( w^r = -I \) and \( w \) is linked to \( -I \).

**Lemma 2.7.** Suppose \( -I \in W \). Then any representative \( g \) of \( -I \) in \( N \) satisfies \( g^2 \in Z(G) \). In particular if \( G \) is simple then \( -I \) has spin 1.

**Proof.** Since \( g^2 = g_0^2 \) for \( g_0 \in N_0 \) representing \( -I \), without loss of generality \( g \in N_0 \). By [C2, Lemma 7.2.1(i)], we thus have \( gx_\alpha(\lambda)g^{-1} = x_{-\alpha}(\epsilon_\alpha \lambda) \), where \( \epsilon_\alpha = \pm 1 \) depends on \( g \) and \( \alpha \) but not on \( \lambda \) or \( G \). Similarly \( g^2x_\alpha(\lambda)g^{-2} = x_\alpha(\epsilon_\alpha \epsilon_{-\alpha} \lambda) \). By [C2, Proposition 6.4.3; S, Lemma 19(a)] however, \( \epsilon_{-\alpha} = \epsilon_\alpha \), and so actually \( g^2x_\alpha(\lambda)g^{-2} = x_\alpha(\lambda) \). Since the \( x_\alpha(\lambda) \) generate \( G \), as explained in [GLS, Remark 1.12.3], indeed \( g^2 \in Z(G) \). \( \square \)

**Corollary 2.8.** If \( G \) is simple then any elliptic element \( w \) of \( W \) that is linked to \( -I \in W \) has spin 1.

We can calculate the spin of many elliptic conjugacy classes using just Corollaries 2.4 and 2.8. For those classes that cannot be dealt with using just these two corollaries, we need to do a bit of computation. To help with this we transcribe a version of Table 1.12.6 in [GLS], listing all elements of order 2 in \( Z(G_u) \). We will use the numbering of the simple roots \( \alpha_i \) given in Section 1. For each \( i = 1, \ldots, n \) let \( h_i := h_{\alpha_i}(-1) \).

**Table 1.** Central elements of order 2

| \( \Phi \) | elements of order 2 in \( Z(G_u) \) |
|-----------|----------------------------------|
| A\(n-1\) (\( n \) even) | \( h_1h_3\cdots h_{n-1} \) |
| A\(n-1\) (\( n \) odd) | none |
| B\(n\) | \( h_n \) |
| C\(n\) | \( h_1h_3\cdots h_k; \ k = 2 \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \) |
| D\(2\ell\) | \( h_1h_3\cdots h_{2\ell-1}, h_{2\ell-1}h_{2\ell}, h_1h_3\cdots h_{2\ell-3}h_{2\ell} \) |
| D\(2\ell+1\) | \( h_2h_{2\ell+1} \) |
| E\(6\) | none |
| E\(7\) | \( h_1h_3h_5 \) |

In all other cases \( Z(G_u) = 1 \)
3. Spin signatures of Coxeter elements

At this point we declare that we only consider fields $K$ with characteristic different than 2. If the characteristic is 2 then $T_0 = \{1\}$, and all elliptic elements have spin 1, so this case is trivial.

**Lemma 3.1.** Let $\alpha, \beta$ be orthogonal roots in a root system $\Phi$. If the $\alpha$-chain of roots through $\beta$ is just $\beta$, then $[m_\alpha, m_\beta] = 1$.

**Proof.** Since the $\alpha$-chain of roots through $\beta$ is just $\beta$, $c(\alpha, \beta) = 1$ by the proof of [C2, Proposition 6.4.3]. Since $\alpha, \beta$ are orthogonal, $s_\alpha \beta = \beta$, and so by (CR1) indeed $[m_\alpha, m_\beta] = 1$. □

This holds for example if $\alpha$ and $\beta$ are orthogonal long roots. Also, if $\Phi$ is simply laced then any two orthogonal roots will have this property. The next lemma is a version of [GLS, Theorem 1.12.1(e)], which is standard, and we will not prove it here.

**Lemma 3.2.** For any root $\alpha$, if $\alpha \langle \alpha, \alpha \rangle = \sum_{i=1}^{n} c_i \alpha_i \langle \alpha_i, \alpha_i \rangle$ then $h_\alpha(-1) = h_1^{c_1} \cdots h_n^{c_n}$.

Let $w \in W$ be elliptic with order $d$, and let $\Gamma$ be its Carter diagram. Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$ be a decomposition of $\Gamma$ into connected components. Note that roots labeling nodes of different components $\Gamma_i$ are orthogonal. We know that $w = w_1 \cdots w_r$ where each $w_i$ has Carter diagram $\Gamma_i$ and all the $w_i$ commute with each other (though note that the $w_i$ are not elliptic). Let $d_i$ denote the order of $w_i$, so $d$ is the least common multiple of the $d_i$.

**Definition 3.3.** If $\Gamma'$ is the Carter diagram of $w' \in W$ and $w'$ has order $d'$, define the content of $\Gamma'$ to be the power of 2 in the prime factorization of $d'$. If $\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$ and $w = w_1 \cdots w_r$ as above then any $\Gamma_i$ with the same content as $\Gamma$ will be called a relevant component. All other $\Gamma_i$ will be called irrelevant.

The point of this definition is that if $\Gamma_i$ is an irrelevant component, with $g_i \in N_0$ representing $w_i$, then

$$g_i^d = g_i^{d/d_i} = 1$$

since $g_i^{d_i} \in T_0$ and 2 divides $d/d_i$. Thus the irrelevant components in some sense do not contribute to the spin signature of $w$.

**Example 3.4.** Consider an elliptic element $w$ in $W = W(C_6)$ with Carter diagram $\Gamma = C_2 \times C_4$. (This exists by [C1, Proposition 24].) If $w = w_1 w_2$ is the corresponding decomposition, we see that $w_1$ has order 4 and $w_2$ has order 8, so $w$ has order 8 and the $C_2$ factor of $\Gamma$ is irrelevant. If $g_0 = g_1 g_2$ is the corresponding representative
of \( w = w_1w_2 \) in \( N_0 \) then \( g_1^8 = 1 \). It turns out the \( g_i \) commute (see Section 4.3), so \( g_0^8 = g_2^8 \).

To calculate \( g_0^d \) in general we need to find a way to calculate these powers of representatives of relevant components, and then combine them in the correct way. Let \( \Gamma_i \) be a relevant component that is itself a Dynkin diagram, and let \( w_i \) be as above. Without loss of generality we may assume \( w_i \) has order \( d \). Let \( \beta_1, \ldots, \beta_n \) be the roots labeling each node. For our representative of Lemma 3.5.

First suppose it is a single edge, so \( (s_j s_k)^2 = 1 \) in \( W' \), and we also have an immediate analogue to the relation \( s_j^2 = 1 \), namely \( m_j^2 = h_{\beta_j}(-1) \), which is just true by construction. This leaves the braid relations involving non-orthogonal roots.

Let \( \beta_j \) and \( \beta_k \) be non-orthogonal roots labeling nodes of \( \Gamma_i \). Since \( W' \) is a Weyl subgroup of \( W \), \( (s_j s_k) \) must have order either 3, 4, or 6. The order 6 case corresponds to a triple edge between the two nodes, which only appears if \( \Phi \) is \( G_2 \), and this case is easily covered in Section 4.3 using only Corollary 2.3. As such we can ignore this case, and assume the nodes have either a single edge or a double edge.

Lemma 3.5. With notation as above, if \( \beta_j \) and \( \beta_k \) label nodes connected by a single edge then \( (m_j m_k)^3 = 1 \), and if they label nodes connected by a double edge, with \( \beta_k \) the short root, then \( (m_j m_k)^4 = (m_k m_j)^4 = h_{\beta_k}(-1) \).

**Proof.** First suppose it is a single edge, so \( (s_j s_k) \) has order 3. Note that \( m_k m_j = m_{sk} \beta_j(c(\beta_k, \beta_j))m_k \) by (CR1). Also, by Proposition 6.4.3 in [C2], \( c(\beta_j, \beta_k) = -c(\beta_k, \beta_j) \) and \( c(\beta_k, \beta_j)(c(\beta_k, s_k) \beta_j) = -1 \).

Thus

\[
(m_j m_k)^3 = m_j m_{sk} \beta_j(c(\beta_k, \beta_j))m_j(c(\beta_k, \beta_j)(c(\beta_k, s_k) \beta_j))m_k^3 \\
= m_k(c(\beta_k, \beta_j)(c(\beta_k, s_k) \beta_j))m_j(c(\beta_k, \beta_j)(c(\beta_k, s_k) \beta_j))m_k^3 \\
= m_k m_j m_k^{-1} m_j^{-1} m_k^{-1} = 1.
\]

In other words, the braid relation \( s_j s_k s_j = s_k s_j s_k \) lifts to \( m_j m_k m_j = m_k^{-1} m_j^{-1} m_k^{-1} \) in \( N_0 \).
Now suppose it is a double edge, so \((s_js_k)s_k \) has order 4, and assume \(\beta_k \) is the short root. Proposition 6.4.3 in [C2] tells us that now \(c(\beta_k, \beta_j)c(\beta_k, s_j s_k) = 1 \), and that \(m_k(\lambda)m_{s_j \beta_k}(\mu) = m_{s_j \beta_k}(-\mu)m_k(\lambda) \). By repeated application of (CR1) we thus get that

\[
(m_j m_k)^4 = m_{s_j \beta_k}(c(\beta_j, \beta_k))m_k(c(\beta_j, \beta_k))m_{s_j \beta_k}(c(\beta_j, s_j \beta_k))m_k m_j^4
\]

\[
= m_{s_j \beta_k}(c(\beta_j, \beta_k))m_{s_j \beta_k}(-c(\beta_j, s_j \beta_k))m_k(c(\beta_j, \beta_k)c(\beta_j, s_j \beta_k))m_k
\]

\[
= h_{\beta_k}(-1).
\]

In other words, the braid relation \(s_j s_k s_j s_k = s_k s_j s_k s_j \) lifts to

\[
m_j m_k m_j m_k = h_{\beta_k}(-1)m_k^{-1}m_j^{-1}m_k^{-1}
\]

in \(N_0 \). Since \((m_k m_j)^4 \) just equals \(m_k (m_j m_k)^4 m_k^{-1} \) we also immediately get that \((m_k m_j)^4 = h_{\beta_k}(-1) \).

Note that these relations, plus (CR2), really are sufficient to calculate \(g_i^d \). This is because the corresponding relations in \(W' \) are sufficient to prove \(w_i^d = 1 \), and then (CR2) is enough to identify the correct element of \(T \). It is very important that these relations are completely local, i.e., they only depend on the roots involved and not on the global structure of \(\Phi \), and in particular don’t require us to know the sign of any \(c(\alpha, \beta) \). The only assumption we have made is that any \(m_j, m_k \) corresponding to orthogonal \(\beta_j, \beta_k \) should commute. The fact that these relations only depend on the roots means that, to calculate \(g_i^d \) for a relevant component \(\Gamma_i \) that is a Dynkin diagram, we can actually just calculate the spin signature of a Coxeter element in the Weyl group with \(\Gamma_i \) as its Dynkin diagram. This will work as long as we choose \(g_i \) correctly, i.e., as a product of \(m_\alpha(1) \) where \(\alpha \) ranges over \(\Gamma_i \). For example, it turns out that Coxeter elements in \(W(A_3) \) have spin signature \(h_1 h_3 \), and so if some relevant component \(\Gamma_i \) is of type \(A_3 \) and is labeled by \(\beta_1, \beta_2, \beta_3 \) (and if \(m_{\beta_1} \) and \(m_{\beta_3} \) commute, then \(g_i^d = h_{\beta_1}(-1)h_{\beta_3}(-1) \). We can then use Lemma 3.2 to express \(g_i^d \) as a product of \(h_j \).

To illustrate that we can calculate \(g_i^d \) without knowing the various \(c(\alpha, \beta) \), we do the example of \(A_3 \) here (with \(d = 4 \)).

**Example 3.6.** Suppose \(\beta_1, \beta_2, \beta_3 \) are roots labeling nodes in \(\Gamma \) such that

\[
\beta_1 \quad \beta_2 \quad \beta_3
\]

is a connected component of \(\Gamma \). Let \(m_i = m_{\beta_i}(1) \) and assume that \(m_1 \) and \(m_3 \) commute. We now show that \((m_1 m_2 m_3)^4 = h_{\beta_1}(-1)h_{\beta_3}(-1) \), using only the relations \(m_i^2 = h_{\beta_i}(-1) \), \(m_1 m_2 m_1 = m_2^{-1}m_1^{-1}m_2^{-1} \), \(m_2 m_3 m_2 = m_3^{-1}m_2^{-1}m_3^{-1} \),
For brevity we will write $h_i$ for $h_{\beta_i}(-1)$, but note that $\beta_i$ probably is different than the simple root $\alpha_i$ in $\Phi$.

$$\begin{align*}
(m_1m_2m_3)^4 & = (m_1m_2m_3m_2m_3)^2 \\
& = (m_2^{-1}m_1^{-1}m_2^{-1}m_3^{-1}m_2^{-1})^2 \\
& = m_2^{-1}m_1^{-1}h_2m_3^{-1}h_2m_1^{-1}h_2m_3^{-1}m_2^{-1} \\
& = h_1(h_1h_2h_3)m_2^{-1}h_1h_3m_2^{-1} \\
& = h_2(h_1h_2)(h_2h_3)h_2 = h_1h_3
\end{align*}$$

The last two lines made repeated use of (CR2) and Lemma [3.2]

We could theoretically develop an algorithm to calculate the spin signatures of Coxeter elements for any $\Phi$ in this way, but this would not be a realistic way to calculate the spin of an arbitrary elliptic element. The point is that since any such calculations depend only on the roots in the diagram and not on $\Phi$, we don’t have to do this, provided we can calculate the spin signatures of Coxeter elements some other way. We can in fact do this for $A_n$, $B_n$, $C_n$, and $E_7$, and this turns out to be sufficient.

3.1. Coxeter elements in $A_{n-1}$. The results for this case are well-known but we present them for completeness. By Proposition 23 in [31], the Coxeter elements are the only elliptic elements in $W = W(A_{n-1})$. Thinking of $W$ as $S_n$, these are precisely the $n$-cycles. If $n$ is odd then these all have odd order and thus spin $1$ by Corollary 2.4. Suppose now that $n$ is even. By [38, Theorem 1.10.7(a)] we know that $G$ is a quotient of $\text{SL}_n(K)$ by a central subgroup $Z'$. Let $w$ be a Coxeter element and $g_0$ a representative in $N_0$. Since $w$ is an odd permutation, and $g_0$ has determinant $1$, we see that an odd number of entries of $g_0$ are $-1$. Thus $g_0^n = -I_n$, and so $w$ has spin $1$ if $-I_n \in Z'$ and spin $-1$ if $-I_n \notin Z'$. In particular all elliptic elements of $W(A_{n-1})$ have universal spin $-1$ if $n$ is even. Also note that when $n$ is even, $-I_n$ is the unique element of order $2$ in $Z(G_n)$, so by Table [4] the spin signature $g_0^n$ must equal $h_1h_3\cdots h_{n-1}$. In particular in the $A_3$ case we get $h_1h_3$, as referenced earlier.

3.2. Coxeter elements in $B_n$. Let $w \in W = W(B_n)$ be a Coxeter element, with order $2n$. Since $w$ has characteristic polynomial $t^n + 1$, $w$ is linked to $-I$, and any representative of $w$ raised to the $2n$ will equal $g_0^2$, where $g_0$ represents $-I$ in $N_0$. It thus suffices to calculate $g_0^2$. Let $\{\beta_1, \ldots, \beta_n\}$ be the orthonormal basis of roots given in [33, Section 8.3], so $-I = s_{\beta_1}\cdots s_{\beta_n}$ and $g_0 = m_1\cdots m_n$ where $m_i := m_{\beta_i}(1)$. Note that for any $i \neq j$, the $\beta_j$-chain of roots through $\beta_i$ is $\beta_i - \beta_j, \beta_i, \beta_i + \beta_j,$
and \( \beta_i, \beta_j \) are orthogonal. Thus by the proof of Proposition [C2 Proposition 6.4.3], \( m_i m_\beta(e) m_i^{-1} = m_\beta(-e) \) for \( e = \pm 1 \). Moreover, it is straightforward to calculate that for any \( i \), \( h_\beta_i(-1) = h_n \). We can now calculate \( g_0^2 \).

\[
g_0^2 = m_1 \cdots m_n m_1 \cdots m_n = m_1 m_1 ((-1)^{n-1}) m_2 m_2 ((-1)^{n-2}) \cdots m_n m_n ((-1)^0) = h_n^k
\]

where \( k = \left\lfloor \frac{n+1}{2} \right\rfloor \). Since \( h_n \in Z(G_n) \) by Table I this tells us that \(-I\), and thus any Coxeter element, has adjoint spin 1. In the universal case the spin is 1 if and only if \( n \) is congruent to 0 or 3 modulo 4.

3.3. Coxeter elements in \( C_n \). As in the \( B_n \) case, we need to calculate \( g_0^2 \), where \( g_0 \) represents \(-I\). We claim that \( g_0^2 = h_1 h_2 \cdots h_k \) where \( k = 2 \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \). Let \( \beta_1, \ldots, \beta_n \) denote the orthonormal basis of \( (\Phi)_{\mathbb{R}} \) given in [C3 Section 8.4], so \(-I = s_2 \beta_1 \cdots s_2 \beta_n \) and \( g_0 = m_1 \cdots m_n \) where \( m_i = m_2 \beta_i(1) \). Since the \( 2 \beta_i \) are all long and are mutually orthogonal, they have trivial root chains through each other and so the \( m_i \) all commute by Lemma 3.1. Thus \( g_0^2 = h_{2 \beta_1}(-1) \cdots h_{2 \beta_n}(-1) \). Now, for each \( i \), \( 2 \beta_i = 2 \alpha_i + 2 \alpha_{i+1} + \cdots + 2 \alpha_{n-1} + \alpha_n \). By Lemma 3.2 then, \( h_{2 \beta_i}(-1) = h_i h_{i+1} \cdots h_n \). The result now follows immediately. As a consequence we see that \(-I\), and thus all Coxeter elements in \( W(C_n) \), have universal spin \(-1\), and adjoint spin 1 by Table I.

3.4. Coxeter elements in \( E_7 \). In type \( E_7 \), the eigenvalues of a Coxeter element are the primitive 18th roots of unity and \(-1\), so a Coxeter element to the 9th power equals \(-I\). Since linked elements have the same spin, as before we actually want to calculate the spin of \(-I\). Let \( e_1, \ldots, e_8 \) be an orthonormal basis of \( \mathbb{R}^8 \), with the simple roots given by \( \alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4, \alpha_4 = e_4 - e_5, \alpha_5 = e_5 - e_6, \alpha_6 = e_5 + e_6, \alpha_7 = \frac{1}{2}(e_1 + \cdots + e_8) \). (This is as in [C3 Section 8.7], though we use different notation.) If we then let \( \beta_1 = e_1 - e_2, \beta_2 = e_1 + e_2, \beta_3 = e_3 - e_4, \beta_4 = e_3 + e_4, \beta_5 = e_5 - e_6, \beta_6 = e_5 + e_6, \beta_7 = -e_7 - e_8 \), the \( \beta_i \) are mutually orthogonal so \(-I = s_{\beta_1} \cdots s_{\beta_7} \).

Let \( g_0 = m_1 \cdots m_7 \) represent \(-I\) in \( N_0 \), where \( m_i = m_\beta_i(1) \). Since \( E_7 \) is simply laced, by Lemma 3.1 \( g_0^2 = h_{\beta_1}(-1) \cdots h_{\beta_7}(-1) \). Now, \( \beta_1 = \alpha_1, \beta_2 = \alpha_1 + 2 \alpha_2 + 2 \alpha_3 + 2 \alpha_4 + \alpha_5 + \alpha_6, \beta_3 = \alpha_3, \beta_4 = \alpha_3 + 2 \alpha_4 + \alpha_5 + \alpha_6, \beta_5 = \alpha_5, \beta_6 = \alpha_6, \) and \( \beta_7 = \alpha_1 + 2 \alpha_2 + 3 \alpha_3 + 4 \alpha_4 + 2 \alpha_5 + 3 \alpha_6 + 2 \alpha_7 \), and so by Lemma 3.2 \( g_0^2 = h_1 h_3 h_5 \). By Table I this is precisely the non-trivial element of the center. So \(-I\) has universal spin \(-1\) and adjoint spin 1.
4. Orders and spin signatures of elliptic elements

Let $w \in W$ be elliptic with order $d$, and let $\Gamma$ be its Carter diagram. Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$ be a decomposition of $\Gamma$ into connected components. Since we only care about the roots labeling the nodes of the relevant components inasmuch as they yield a certain product of $h_j$ according to Lemma 3.2, we devise the following convenient way to label Carter diagrams, which we will call a spin labeling. If a node is labeled with the root $\alpha$, we re-label it with the tuple $(i_1, \ldots, i_k)$ such that $h_\alpha(-1) = h_{i_1} \cdots h_{i_k}$. If $\alpha = \alpha_i$ is simple, we just maintain the original “$i$” label. We will only need to worry about $\Gamma_i$ that are Dynkin diagrams, so the spin signature $g^d_i$ is just the product of $h_j$ where $j$ ranges in the appropriate way over the spin labeling of $\Gamma_i$. For instance if $\Gamma_i$ is $(1,3) 2 (2,3,4)$ and relevant, then $g^d_i = (h_1 h_3)(h_2 h_3 h_4) = h_1 h_2 h_4$, since $\Gamma_i$ has type $A_3$.

These spin labelings are really only useful when $\Gamma$ is the Dynkin diagram of a Weyl subgroup of $W$, which is equivalent to saying that $\Gamma$ is cycle-free [C1, Lemma 8]. Luckily, it will turn out that for all examples where $\Gamma$ has cycles, we can find its spin just using Corollaries 2.4 and 2.8. Since Carter diagrams that are Dynkin diagrams all arise by an iterated process of removing nodes from extended Dynkin diagrams, we are especially interested in the spin labeling of nodes corresponding to $-\tilde{\alpha}$, where $\tilde{\alpha}$ is a highest root. In the $B_n$ case it will be convenient to instead use the highest short root $\tilde{\alpha}_s$. We collect here the decompositions of negative highest roots into simple roots, only for the cases we will actually use.

| $B_n$  | $-\alpha_s = -\alpha_1 - \alpha_2 - \cdots - \alpha_n$ |
|-------|----------------------------------------------------------|
| $C_n$  | $-\tilde{\alpha} = -2(\alpha_1 + \cdots + \alpha_{n-1}) - \alpha_n$ |
| $E_6$  | $-\tilde{\alpha} = -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6$ |
| $E_7$  | $-\tilde{\alpha} = -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 2\alpha_5 - 3\alpha_6 - 2\alpha_7$ |
| $E_8$  | $-\tilde{\alpha} = -2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 5\alpha_4 - 6\alpha_5 - 3\alpha_6 - 4\alpha_7 - 2\alpha_8$ |
| $F_4$  | $-\tilde{\alpha} = -2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4$ |

The spin labeled extended Dynkin diagrams that we will need later can now be found using Lemma 3.2 and are exhibited below. Our general reference for the extended Dynkin diagrams is the Appendix in [C3]. Note that the diagram we need for $B_n$ actually has the negative highest short root added.

$B_n \ (n > 1) \quad \begin{array}{c|c|c|c|c} \multicolumn{5}{c}{n} \\ \hline \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & \cdots & \multicolumn{1}{c|}{n-1} & \multicolumn{1}{c}{n} \end{array}$
It is certainly possible that a given Carter diagram $\Gamma$ could have more than one spin labeling. For example the Carter diagram $C_2 \times A_1$ in type $C_3$ could have either of the spin labelings below.

\[
\begin{array}{c}
(1, 2, 3) \quad 1 \quad 3 \\
(1, 2, 3) \quad 2 \quad 3
\end{array}
\]

The first is obtained by removing the node labeled “2” from the extended Dynkin diagram, and the second by removing the node labeled “1.”

Luckily, as we will see, the spin signature is almost always central and so different spin labelings will still produce the same spin signature. The example given here is one of the few for which different spin labelings produce different spin signatures, namely $h_1$ and $h_2$, as seen in Section 4.3. In any case, to at least calculate the spin it doesn’t matter which spin labeling we pick for our Carter diagram.

We can now calculate the spin and spin signature of all elliptic elements in any Weyl group. Let $w \in W$ be elliptic with order $d$, and let $\Gamma$ be its Carter diagram. Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$ be a decomposition of $\Gamma$ into connected components. We know that $w = w_1 \cdots w_r$ where each $w_i$ has Carter diagram $\Gamma_i$ and all the $w_i$ commute with each other. Let $d_i$ denote the order of $w_i$, so $d$ is the least common multiple of the $d_i$.

4.1. The $A_{n-1}$ case. All elliptic elements are Coxeter elements, and so we already calculated their spin and spin signature in Section 3.1.

4.2. The $B_n$ case. In [F] it is shown that all elliptic elements in $W = W(B_n)$ have adjoint spin 1. We now have the tools to calculate the universal spin of any elliptic element, with the adjoint case as a corollary. Note that these are the only
two cases since \(|Z(G_u)| = 2\). By [C1] Proposition 24, each \(\Gamma\) is a Dynkin diagram of type \(B_n\), for some \(n_i\), and \(n_1 + \cdots + n_r = n\). Here \(B_1\) is identified with \(\tilde{A}_1\), a single node corresponding to a short root. By Proposition 24 and Table 2 in [C1] each \(\Gamma\) arises by an iterated process of attaching a node for the negative highest short root and removing a node. As seen in the previous section, these new nodes will all have spin labeling “\(n\)” If \(g_i\) is the usual representative of \(w_i\) in \(N_0\), it is thus easy to calculate \(g_i^d\) using Section 3.2. The problem though is that the result of Lemma 3.1 does not hold, since the negative highest short roots introduced do not have trivial root chains through each other. Luckily by Table 1 \(h_n\) is central, and so it is not too difficult to calculate the spin signature of \(w\) explicitly.

Let \(g_0 \in N_0\) represent \(w\). Let \(f\) be the number of relevant \(\Gamma_i\) such that \(n_i \equiv 1, 2\). If \(d \equiv 4\) 0 or \(r \equiv 4\) 1 then set \(e := f\). If \(d \equiv 2\) 0 and \(r \equiv 4\) 2, 3 then set \(e := f + 1\). Note that \(d\) is even so these are the only possibilities.

**Theorem 4.1.** With the above setup, \(g_0^d = h_n^e\). In particular all elliptic \(w\) have adjoint spin 1.

**Proof.** For each \(i\) let \(g_i \in N_0\) be the standard representative given by the product of \(m_\alpha(1)\) as \(\alpha\) ranges over \(\Gamma_i\). Without loss of generality \(g_0 = g_1 \cdots g_r\). By Section 3.2 and the fact that all nodes of \(\Gamma\) corresponding to short roots have spin labeling “\(n\),” it is immediate that \(g_i^d \cdots g_r^d = h_n^f\). We now claim that for any \(i \neq j\), \(g_ig_jg_j^{-1} = g_jh_n^f\). Indeed, if \(\Gamma_i\) is labeled by the roots \(\beta_1, \ldots, \beta_{n_i}\) and \(\Gamma_j\) by \(\gamma_1, \ldots, \gamma_{n_j}\) (with \(\beta_n\) and \(\gamma_n\) short), then by Lemma 3.1 \(m_{\beta_k}(1)\) commutes with \(m_{\gamma_l}(1)\) for all \((k, \ell) \neq (n_i, n_j)\). Also, the \(\beta_n\)-chain of roots through \(\gamma_n\) is \(\gamma_{n_j} - \beta_{n_j}, \gamma_{n_j}, \gamma_{n_j} + \beta_{n_j}\), so \(m_{\beta_{n_j}}(1)m_{\gamma_{n_j}}(1)m_{\beta_{n_j}}(1)^{-1} = m_{\gamma_{n_j}}(-1)\). Since \(m_{\gamma_{n_j}}(-1) = m_{\gamma_{n_j}}(1)h_{\gamma_{n_j}}(-1)\) and \(h_{\gamma_{n_j}}(-1) = h_n\), in fact \(m_{\beta_{n_j}}(1)m_{\gamma_{n_j}}(1)m_{\beta_{n_j}}(1)^{-1} = m_{\gamma_{n_j}}(1)h_n\). This proves our claim that \(g_ig_jg_j^{-1} = g_jh_n\).

It is now a straightforward exercise to calculate \((g_1 \cdots g_r)^d\) in terms of \(g_1^d, \ldots, g_r^d\). Since \(h_n\) is central and \(T_0\) is abelian, we get the following:

\[
g_0^d = (g_r h_n^{d-1} g_r h_n^{2(r-1)} \cdots g_r h_n^{d(r-1)}) \cdots (g_2 h_n^2 g_2 h_n^2 \cdots g_2 h_n^d) g_1^d
\]

\[
= g_1^d \cdots g_r^d h_n^d/2h_n^{2d/2} \cdots h_n^{(r-1)d/2}
\]

\[
= h_n^f h_n^{d/4(1+2+\cdots+(r-1))}
\]

\[
= h_n^{f + \frac{d(r+1)}{4}}
\]

If \(d \equiv 4\) 0 or \(r \equiv 4\) 0, 1, we see that \(g_0^d = h_n^f = h_n^e\). If \(d \equiv 2\) 0 and \(r \equiv 4\) 2, 3 then \(g_0^d = h_n^f h_n = h_n^e\).

\(\square\)
Corollary 4.2. Let \( w \in W(B_n) \) be elliptic with characteristic polynomial \( (t^{n_1} + 1) \cdots (t^{n_r} + 1) \). Then \( w \) has spin signature \( h^e \) where \( e \) is as in Theorem 4.1.

Proof. By [C1, Proposition 24] \( w \) has Carter diagram of type \( B_{n_1} \times \cdots \times B_{n_r} \), and the result is immediate from Theorem 4.1. \( \square \)

In summary, all elliptic elements in \( W(B_n) \) have adjoint spin 1, and we can calculate the universal spin just knowing the characteristic polynomial of \( w \). Many conjugacy classes have universal spin 1, and many have universal spin \(-1\). We illustrate this with a few examples.

Example 4.3. Let \( w \in W(B_7) \) have characteristic polynomial \( (t^3 + 1)(t^3 + 1)(t + 1) \). The spin labeled Carter diagram we use is

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
7 & 1 & 2 & 7 & 4 & 5 \quad 7 \\
6 & & & & & \\
\end{array}
\]

All three components are relevant since they all have content 2. Since \( n_1 = n_2 = 3 \) and \( n_3 = 1 \), we have \( f = 1 \). Also, since \( d = 6 \) and \( r = 3 \), we have \( e = f + 1 = 2 \). Thus \( w \) has spin signature \( h^2 \), and so has universal spin 1. In the language of algebraic groups, any representative of \( w \) in \( SO_{15} \) has order 6, and even in \( Spin_{15} \), any representative has order 6.

Example 4.4. Let \( w \in W(B_7) \) have characteristic polynomial \( (t^6 + 1)(t + 1) \). The spin labeled Carter diagram we use is

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
7 & 1 & 2 & 3 & 4 & 5 \quad 7 \\
6 & & & & & \\
\end{array}
\]

Only the first component is relevant, and \( n_1 = 6 \), so \( f = 1 \). Also, since \( d = 12 \) and \( r = 2 \) we have \( e = f = 1 \). Thus \( w \) has spin signature \( h_7 \), and so has universal spin \(-1\). In particular any representative of \( w \) in \( SO_{15} \) has order 12 but any representative in \( Spin_{15} \) actually has order 24.

Remark 4.5. The name “spin” is slightly justified now. Indeed, it in some sense measures the tendency of representatives in \( SO_m \) of elliptic \( w \) to pick up an extra “twist” when lifting to \( Spin_m \), that is, the order doubles. As we have seen not every \( w \) has this property, but we can tell which ones do just based on their characteristic polynomials, so this really is an inherent property of \( w \).

4.3. The \( C_n \) case. The universal case is covered in [F], with the conclusion that all elliptic elements have universal spin \(-1\). While we could realize \( G_u \) and \( G_u \) explicitly as \( Sp_{2n} \) and \( PSP_{2n} \), we find that to cover the general case it is convenient to just deal directly with Carter diagrams. The result we find is the following.
Theorem 4.6. Let \( w \in W(C_n) \) be elliptic. Then \( w \) has universal spin \(-1\), and has adjoint spin 1 if and only if \( w' = -I \in W \) for some \( r \).

As in the \( B_n \) case, the Carter diagrams for elements of \( W = W(C_n) \) all arise by removing nodes from extended Dynkin diagrams. This time though, we will use the negative highest roots instead of the negative highest short roots. Now each \( \Gamma_i \) is \( C_n \), for some \( n_i \), and \( n_1 + \cdots + n_r = n \) [C1 Proposition 24]. (We identify \( C_1 \) with \( A_1 \).) Since we only ever introduce long roots, every short root corresponding to a node of \( \Gamma \) must actually be one of the simple roots \( \alpha_1, \ldots, \alpha_{n-1} \). We claim that if two roots \( \alpha, \beta \) corresponding to nodes of \( \Gamma \) are orthogonal, then the \( \alpha \)-chain through \( \beta \) is just \( \beta \). This is clear if either \( \alpha \) or \( \beta \) is long. Also, if both roots are short, they are both simple, and orthogonal simple roots satisfy this property. In any case, if \( \alpha \) and \( \beta \) are orthogonal then by Lemma 3.1, \( [m_{\alpha}(1), m_{\beta}(1)] = 1 \).

Proof of Theorem 4.6. First note that the characteristic polynomial of \( w_i \) is \( t^{n_i} + 1 \), and so \( w \) is linked to \(-I\) if and only if every \( \Gamma_i \) is relevant. If \( w \) is linked to \(-I\) then by Section 3.3 \( w \) has spin signature \( h_1 h_3 \cdots h_k \), so \( w \) has adjoint spin 1 and universal spin \(-1\). Now suppose \( w \) is not linked to \(-I\). We know that \( g^d_0 \) is the product of the \( g^d_i \) ranging over all \( i \) such that \( \Gamma_i \) is relevant. Also, for each relevant \( \Gamma_i \), \( n_i \) must be even since otherwise all \( w_j \) would have order congruent to 2 mod 4, implying that all \( \Gamma_i \) are relevant and \( w \) in fact is linked to \(-I\). By Section 3.3 \( g^d_i \) is a product of \( h_\alpha(-1) \) where \( \alpha \) ranges over every other root of \( \Gamma_i \), beginning with the terminal short root. Also, since \( n_i \) is even for relevant \( \Gamma_i \), all such \( \alpha \) are short roots and thus simple roots. This tells us that \( g^d_0 \) is a product of \( h_\alpha(-1) \) as \( \alpha \) ranges over every simple root contained in a relevant \( \Gamma_i \). Such an \( i \) exists, and so immediately we see that \( g^d_0 \neq 1 \), and \( w \) has universal spin \(-1\). By Table I it now suffices to show that for some \( j = 1, 3, \ldots, k \), the simple root \( \alpha_j \) is not a node in any relevant \( \Gamma_i \).

Indeed, since \( w \) is not linked to \(-I\) we know there exists some irrelevant \( \Gamma_i \). The only way \( \Gamma_i \) can avoid containing a node \( \alpha_j \) for odd \( j \) is if \( n_i = 2 \) and the two nodes of \( \Gamma_i \) are a long root and some \( \alpha_\ell \) for even \( \ell \). But then one of \( \alpha_{\ell+1} \) or \( \alpha_{\ell-1} \) must have been removed from the graph, or else \( \Gamma_i \) would not be a connected component of \( \Gamma \). We conclude that \( g^d_0 \) cannot equal \( h_1 h_3 \cdots h_k \), and so \( w \) has adjoint spin \(-1\). \( \square \)

Remark 4.7. The last paragraph of the proof does not explicitly calculate the spin signature \( g^d_0 \), and indeed since the spin signatures are non-central, conjugate Weyl group elements may have different spin signatures.
Example 4.8. In $C_6$, consider the conjugacy class with Carter diagram $C_2 \times C_4$. Any corresponding element $w$ has order 8. One spin labeling of the Carter diagram is

$$(1, \ldots, 6) \begin{array}{cccccc}
1 & 3 & 4 & 5 & 6 \\
\end{array}$$

and only the $C_4$ component is relevant, so $g_0^8 = h_3h_5$, which is not central. Thus $w$ has spin $-1$, even in the adjoint case.

Example 4.9. In $C_8$, consider an element $w$ with Carter diagram $C_2 \times C_6$ with spin labeling

$$(1, \ldots, 8) \begin{array}{ccccccccc}
1 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}$$

and order 12. Now both components are relevant, so $g_0^{12} = h_1h_3h_5h_7$, which is a nontrivial element of $Z(G_u)$. Thus $w$ has adjoint spin 1 and universal spin $-1$.

It turns out the $C_n$ case provides the only source of elliptic Weyl group elements with adjoint spin $-1$, except for one conjugacy class in $F_4$. The $C_n$ case is also the only case where every elliptic element has universal spin $-1$.

4.4. The $D_n$ case. Certain cases are essentially done in [F], though there are no results there for the universal case. Unfortunately, type $D_n$ is the only classical type in which not every elliptic Carter diagram arises by removing nodes from extended Dynkin diagrams, and applying our present approach to diagrams with cycles would be very difficult. However, having completely handled the $B_n$ case we can now just use the natural embeddings $W(D_n) \leq W(B_n)$ and $G_u(D_n) \leq G_u(B_n)$ to figure out the spin of any $w \in W(D_n)$. Indeed, if $w \in W(D_n)$ is elliptic then it is also an elliptic element in $W(B_n)$, so we know its spin and spin signature in $G_u(B_n)$ just from its characteristic polynomial. Then since all the spin signatures are central they are independent of the choice of representative $g_0$, and we can choose a representative in $G_u(D_n)$, which tells us the spin and spin signature in $G_u(D_n)$, though we have to use Lemma 3.2 to express the spin signature in the correct notation. As an example we show the case of Coxeter elements in $W(D_n)$.

Example 4.10. Let $w \in W = W(D_n)$ be a Coxeter element. Then $w$ has characteristic polynomial $(t^{n-1} + 1)(t + 1)$, and so as an element of $W' = W(B_n)$, $w$ has spin labeled Carter diagram

$$_n \begin{array}{cccccc}
1 & 2 & \cdots & n-1 & n \\
\end{array}$$

So as not to confuse central elements of $G_u(B_n)$ and $G_u(D_n)$ we will use $\tilde{h}_n$ for the central element of $G_u(B_n)$. Direct calculation shows that $e$ equals 0, 1, 2, or 3, if
$n$ is congruent modulo 4 to 1, 3, 0, or 2, respectively. Thus the spin signature of $w$ in $G_u(B_n)$ is either 1 or $\tilde{h}_n$, if $n \equiv 0, 1$ or $n \equiv 2, 3$, respectively.

Now to figure out the spin signature of $w$ in $G_u(D_n)$ we need to calculate $\tilde{h}_n$ in terms of the $h_i$. We know that $\Phi(D_n)$ is the subroot system of $\Phi(B_n)$ consisting of the long roots, with fundamental roots $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, -\tilde{\alpha}$. The root $\alpha_n$ equals $(-\tilde{\alpha}-\alpha_1-2(\alpha_2+\cdots+\alpha_{n-1}))/2$, and so by Lemma 3.2 $\tilde{h}_n = \tilde{h}_{-\tilde{\alpha}}(-1)^{\tilde{h}_1}$. Converting to the standard numbering of fundamental roots in $\Phi(D_n)$, this equals $h_{n-1}h_n$, one of the central elements in $G_u(D_n)$.

**Corollary 4.11.** Coxeter elements in $W(D_n)$ have adjoint spin 1, and have universal spin 1 or -1, if $n \equiv 0, 1$ or $n \equiv 2, 3$, respectively. Moreover, if the universal spin is -1 then the spin signature is $h_{n-1}h_n$. □

**Remark 4.12.** Note that if $n$ is even, $G_u$ has two central elements of order 2 other than $h_{n-1}h_n$, but they will never appear as spin signatures of elliptic elements.

4.5. **The $G_2$ case.** The $G_2$ case was completely dealt with in [F] using a different method, but we will present it for completeness. The Weyl group $W = W(G_2)$ is just the dihedral group of order 12. The Coxeter element $w$ is the rotation of order 6, and a complete list of elliptic elements is given by $w, w^2, w^3, w^4, w^5$; in particular they are all linked. Since $w^2$ has order 3 it has spin 1 by Corollary 2.4, and so by Lemma 2.5 all elliptic elements have spin 1.

4.6. **The $F_4$ case.** The $F_4$ case is partially covered in [F], in particular it is shown that any elliptic power of a Coxeter element has spin 1. Here we show that one elliptic conjugacy class has spin $-1$ and all others have spin 1. First note that $-I \in W = W(F_4)$ and $G$ is simple, so by Corollary 2.8 any elliptic $w$ linked to $-I$ will have spin 1. By Carter’s classification in [C1], there are 9 elliptic conjugacy classes in $W$, and inspecting Tables 3 and 8 in [C1] it is clear that 7 of these are linked to $-I$. The two remaining classes have Carter diagram $A_2 \times \tilde{A}_2$ and $A_3 \times \tilde{A}_1$, where a tilde indicates the roots labeling the nodes are short. Elements corresponding to the first diagram have odd order and thus spin 1 by Corollary 2.4. This leaves the single class with diagram $A_3 \times \tilde{A}_1$ having unknown spin. Let $w$ be an element of this class, so $w$ has order 4 and spin labeled diagram

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(2, 4) 1 2 4
```

Inspecting the root system for $F_4$, it is clear that the $\alpha_4$-chain of roots through any of $\alpha_1, \alpha_2, -\tilde{\alpha}$ consists of a single root. The same is true of $\alpha_2$ and $-\tilde{\alpha}$, and so the conclusion of Lemma 3.1 holds. Since the $\tilde{A}_1$ component is irrelevant, this implies that $g_0^4 = (h_2h_4)h_2 = h_4$. We conclude that $w$ actually has spin $-1$ in this case.
Also note that different spin labelings may yield different (through still non-trivial) spin signatures.

4.7. The $E_6$ case. This is the first case for which no results were found in [F]. However, we can now completely handle this case, with the result that all elliptic elements have spin 1. By Carter’s classification in [C1], there are 5 elliptic conjugacy classes in $W = W(E_6)$, and inspecting Tables 3 and 9 in [C1] it is clear that 4 of these are linked to the class with Carter diagram $A_3^2$. Elements of this class have order 3, and so have spin 1 by Corollary [24]. This leaves only the class with diagram $A_1 \times A_5$ having unknown spin. Let $w$ be a representative of this class, so $w$ has order 6. A spin labeling of the Carter diagram is

$$\begin{array}{cccccc}
1 & 2 & 3 & 5 & 6 \\
\circ & \circ & \circ & \circ & \circ \\
\end{array}$$

Since $E_6$ is simply laced and both components are relevant, $g_6^6 = h_1 h_3 h_6 h_3 h_6 = 1$.

Thus all elliptic elements in $W(E_6)$ have spin 1, in both the adjoint and universal case.

4.8. The $E_7$ case. Like $E_6$, no results were found in [F] for the $E_7$ case. Since $|Z(G_u)| = 2$, $G$ must be either $G_a$ or $G_u$. In Table 10 of [C1] the elliptic conjugacy classes in $W = W(E_7)$ are classified, and in Table 3 in [C1] the corresponding characteristic polynomials are given, so we can tell which elliptic elements are linked to each other. If $w_1, \ldots, w_{12}$ denote choices of representatives of each elliptic conjugacy class, in the order given in [C1], then $w_1 = -I$ is linked to $w_i$ for $i = 5, 6, 7, 8, 9, 11, 12,$ and $w_2$ is linked to $w_{10}$. By Section [34], then, $w_i$ has universal spin $-1$ and adjoint spin 1 for $i = 1, 5, 6, 7, 8, 9, 11, 12$. We now determine the spin of $w_i$ for $i = 2, 3, 4, 10$.

First consider $w_2$, which has order 4, Carter diagram $A_3^2 \times A_1$, and spin labeling

$$\begin{array}{ccccccc}
1 & 2 & 3 & 5 & 6 & 7 \\
\circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}$$

The $A_1$ component is irrelevant, so $g_6^4$ equals $h_1 h_3 h_6 (h_1 h_3 h_6)$, which is 1. Thus $w_2$ (and consequently $w_{10}$) has spin 1.

Next consider $w_3$, with order 6, Carter diagram $A_5 \times A_2$, and spin labeling

$$\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 7 \\
\circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}$$
The $A_2$ component is irrelevant, so $g^6_0 = h_1 h_3 h_5$. But this is precisely the non-trivial element of $Z(G_u)$, by Table 1. So $w_3$ has universal spin $-1$ and adjoint spin 1.

Lastly consider $w_4$, with order 8, Carter diagram $A_7$, and spin labeling

$$\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 6 & 7 & (1, 3, 6) \\
\end{array}$$

Then $g^8_0 = h_1 h_3 h_6 (h_1 h_3 h_6) = 1$, so $w_4$ has spin 1.

In conclusion, $w_2$, $w_4$ and $w_{10}$ always have spin 1, and all other elliptic elements have adjoint spin 1 and universal spin $-1$.

4.9. The $E_8$ case. As with the $F_4$ case, in [F] it is indicated that powers of Coxeter elements have spin 1. Here we show that all elliptic elements have spin 1. Note that $-I \in W = W(E_8)$ and $G$ is simple, so by Corollary 2.8 any elliptic $w$ linked to $-I$ will have spin 1. By Carter’s classification in [C1], there are 30 elliptic conjugacy classes in $W$. Choose representatives for each class, denoted $w_1, \ldots, w_{30}$.

Inspecting Table 3 in [C1] it is clear that $w_i$ is linked to $w_1$ for all $i$ except for $i = 2, 3, 4, 5, 6, 7, 11, 17, 18, 20, 29$. Since $w_1 = -I$ the classes linked to $w_1$ all have spin 1. Moreover for $i = 2, 4, 7$, $w_i$ has odd order so $w_i$ has spin 1, and for $i = 17, 18, 29$, $w_i$ is linked to $w_2$ and so has spin 1. Also, the $w_i$ for $i = 3, 11, 20, 29$ are all linked to $w_3$, and $w_{29}$ has spin 1 so all these $w_i$ do too. This leaves $w_5$ and $w_6$ as the only remaining cases, which we can handle using spin labelings.

First consider $w_5$, with Carter diagram $A_5 \times A_1 \times A_2$ and spin labeling

$$\begin{array}{cccccccc}
(2, 4, 6) & 1 & 2 & 3 & 4 & 6 & 7 & 8 \\
\end{array}$$

The $A_2$ component is irrelevant, so $g^6_5 = (h_2 h_4 h_6) h_2 h_4 h_6 = 1$ and $w_5$ has spin 1.

Lastly consider $w_6$, with Carter diagram $A_7 \times A_1$ and order 8 and spin labeling

$$\begin{array}{cccccccc}
(2, 4, 6) & 1 & 2 & 3 & 4 & 5 & 6 & 8 \\
\end{array}$$

The $A_1$ component is irrelevant, so $g^8_6 = (h_2 h_4 h_6) h_2 h_4 h_6 = 1$ and $w_6$ has spin 1.

It is remarkable that outside some cases in $C_n$ and one case in $F_4$, every elliptic conjugacy class has adjoint spin 1. We also see that universal spin $-1$ occurs all the time in $C_n$, half the time in $A_{n-1}$, never in $E_6$, most of the time in $E_7$, and quite often in $B_n$ and $D_n$. It seems possible that these results could be proved without appealing the classification at all, but at present there is no general method that can handle every case. The following table summarizes our results:
Table 3. Spins of elliptic elements

| $\Phi$ | $\Gamma$ | adjoint spin | universal spin |
|--------|--------|-------------|----------------|
| $A_{n-1}$ | $A_{n-1}$ | 1 | $(-1)^{n-1}$ |
| $B_n$ | any | 1 | see Section 4.2 |
| $C_n$ | linked to $A_n^1$ | 1 | -1 |
| $C_n$ | all others | -1 | -1 |
| $D_n$ | any | 1 | see Section 4.4 |
| $G_2$ | any | 1 | 1 |
| $F_4$ | $A_3 \times A_1$ | -1 | -1 |
| $F_4$ | all others | 1 | 1 |
| $E_6$ | any | 1 | 1 |
| $E_7$ | $A_1 \times A_3^2$ | 1 | 1 |
| $E_7$ | $A_7$ | 1 | 1 |
| $E_7$ | $E_7(a_2)$ | 1 | 1 |
| $E_7$ | all others | 1 | -1 |
| $E_8$ | any | 1 | 1 |

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