Computational technology for improving the quality of difference schemes based on moving nodes

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Abstract. In our papers by us [11],[12] a new method of moving nodes (MMN) was introduced to obtain an approximate analytical solution and construct compact schemes for a one-dimensional convective-diffusion problem. It is proposed to improve the scheme in a three-point pattern. As an initial scheme, a counter flow with one-sided differences is taken. In the QUICK scheme [5], quadratic up flow interpolation is used to determine convective flow. Here we use the solution obtained by the upwind scheme based on MMN. In our above works, the improvement in the accuracy of difference schemes was obtained on the basis of using the method of multi point MMN. In this work, it is proposed to improve the accuracy of the constructed difference schemes using the three point MMN. This is achieved using a more accurate solution to the unknown function on the edge of the considered control volume, the algorithm of which is described in detail in this work.

1. Introduction
The problems of convection-diffusion are basic in modelling the problems of hydrodynamics and heat and mass transfer [1]. In connection with the extensive application of convective-diffusion transport processes, due attention is being paid to the numerical solution of differential equations describing them [2-9].

In the numerical solution of convection-diffusion transport equations, the main attention is paid to the approximation of convective terms. Discretization of convective terms of convective-diffusion transport equations has a decisive influence on the property of discrete equations [2],[3]. Widely used upwind schemes, hybrid schemes, upwind schemes of a high order. The main disadvantage of classical second-order approximation schemes using central differences is associated with stability violation [3-5].

Low-order schemes for the convection-diffusion equation relate unknown quantities at three nodal points (upwin schemes, schemes with central differences, a power law scheme, hybrid schemes) [3]. It is known that a scheme with a power law, which is obtained on the basis of an exact solution of convection-diffusion without a source, has errors comparable to the scheme against the flow for large Peclet grid numbers [4]. Higher order schemes will require nodes larger than three [6-7,9].

Starting with the work of Leonard [7], in order to improve the results of the numerical solution, attempts were made to improve the algorithm [8], which are built in a five-point pattern.

In all of the above schemes (except upwind scheme), the conditions of boundedness and non-negativity of the coefficients are violated.
In papers by us [11],[12] a new method of moving nodes (MMN) was introduced to obtain an approximate analytical solution and construct compact schemes for a one-dimensional convective-diffusion problem. It is proposed to improve the scheme in a three-point pattern. As an initial scheme, a counterflow with one-sided differences is taken. In the QUICK scheme [5], quadratic upflow interpolation is used to determine convective flow. Here we use the solution obtained by the upwind scheme based on MMN.

In our above works, the improvement in the accuracy of difference schemes was obtained on the basis of using the method of multipoint MMN. In this work, it is proposed to improve the accuracy of the constructed difference schemes using the three point MMN (see section 2.0.5). This is achieved using a more accurate solution to the unknown function on the edge of the considered control volume, the algorithm of which is described in detail in this work.

Proposed new MMN for some simple cases allows you to get an analytical representation of the solution between the nodal points of the boundary value problem. Based on this view, it is possible to build a better discrete circuit. In the case of a coarse grid (within the region there is one nodal point), an approximate analytical solution of the boundary value problem can be obtained. In the simplest cases, this solution is accurate. To clarify the solution, you can increase the number of nodes moved.

2. The influence of the choice of interpolation profile on the quality of the scheme

Here we give some classic schemes, as well as its improvement on the basis of MMN.

We analyze discretization schemes using a simple transfer example.

\[
\frac{d\Phi}{dx} = \frac{1}{Pe} \frac{d^2\Phi}{dx^2} + S(x),
\]

Here the unknown \( \Phi \) function, \( S(x) \) the source, \( Pe \) is the Peclet number. The equation is considered under the corresponding boundary conditions. We integrate equation (1) over the control volume.

\[
\Phi_e - \Phi_w = \frac{1}{Pe} \left( \frac{d\Phi}{dx} \right)_e - \frac{1}{Pe} \left( \frac{d\Phi}{dx} \right)_w + \int_e^w S(x)dx.
\]

Replacing the derivatives with difference relations, we have

\[
\Phi_e - \Phi_w = \frac{1}{Pe} \left( \Phi_E - \Phi_P \right) x_E - x_P - \frac{1}{Pe} \left( \Phi_P - \Phi_W \right) x_P - x_W + (x_e - x_w) f_P.
\]

Here \( f_P = \frac{1}{x_e - x_w} \int_w^e S(x)dx \). Depending on the type of function profile on the control volume, various schemes are obtained.

2.0.1. Upwind scheme. Let the profile be piecewise constant in each control volume. Then assuming \( \Phi_e = \Phi_P, \Phi_w = \Phi_W \) we have a upwind scheme:

\[
\Phi_P - \Phi_W = \frac{1}{Pe} \left( \Phi_E - \Phi_P \right) x_E - x_P - \frac{1}{Pe} \left( \Phi_P - \Phi_W \right) x_P - x_W + (x_e - x_w) f_P.
\]

2.0.2. Central difference scheme. If the profile is linear between the nodes and the faces of the control volume are located in the middle between the nodal points \( \Phi_e = (\Phi_E + \Phi_P)/2, \Phi_w = (\Phi_P + \Phi_W)/2 \), we have a diagram with central differences.

2.0.3. Power Law scheme. This scheme [3] is obtained if we take a profile close to the analytical solution (1) for \( S(x) = 0 \). For a uniform step, this scheme has the form.

\[
a_P \Phi_P = a_E \Phi_E + a_W \Phi_W + hf_P
\]

where \( a_W = (1 - 0,1Rh)^5 + Rh, \ a_E = (1 - 0,1Rh)^5, \ a_P = a_W + a_E, \ Rh = h Pe\)
2.0.4. Upstream-Based Improvements. To improve the accuracy order of many schemes, the authors recommended various schemes [4-6,7]. All of these schemes are multipoint (more than three). A method for improving three-point nodes is provided here.

Applying schemes of type (4) for nodes \((x_W, x_w, x_P)\) and \((x_P, x_e, x_E)\) and we determine \(\Phi_e\) and \(\Phi_w\) with their help. The resulting expressions are used in (2). As a result, for a uniform step, we have

\[
\left[ \frac{R h^2}{4+Rh} + 2 \right] \Phi_P = \left[ 1 + \frac{(2+Rh)Rh}{4+Rh} \right] \Phi_W + \left[ 1 - \frac{2Rh}{4+Rh} \right] \Phi_E + \frac{h Rh f_P - h \cdot Rh^2}{2(4+Rh)} (f_e - f_w). 
\]

\(Rh < 4\) conditions are ensured by positive coefficients and stability of the scheme (6).

2.0.5. Improvement of the scheme using MMN. Using moved nodes, you can improve the quality of the scheme. We demonstrate this method based on the upwind scheme (3), writing as

\[
\frac{\Phi_P - \Phi_W}{x_P - x_W} = \frac{2}{Pe(x_E - x_W)} \left( \frac{\Phi_E - \Phi_P}{x_E - x_P} - \frac{\Phi - \Phi_W}{x_P - x_W} \right) + S(x_P). \tag{7}
\]

We write a scheme of type (7) for the segment \((x_W, x_P)\), taking an arbitrary point \(x \in (x_W, x_P)\).

We pass to the limit at \(x \to x_P\), considering the existence of the limit, we have

\[
\frac{\Phi_P - \Phi_W}{x_P - x_W} = \frac{2}{Pe(x_P - x_W)} \left( \frac{d \Phi_P}{dx_P} - \frac{\Phi_P - \Phi_W}{x_P - x_W} \right) + S(x_P).
\]

Here \(d \Phi_P^+/dx_P\) is the left-side derivative of an unknown function \(x \to x_P\) at a point \(x_P\).

Similarly, taking an arbitrary point \(x \in (x_P, x_E)\) and going to the limit \(x \to x_P\), we can get \(d \Phi_E^-/dx_P\).

Equating the flows \(d \Phi^+/dx = d \Phi^-/dx\), we get an improved scheme:

\[
c_P \Phi_P = a_P \Phi_W + b_P \Phi_E + d_P S(x_P) \tag{8}
\]

where \(a_P = \frac{2 + Pe(x_P - x_W)}{x_P - x_W}, b_P = \frac{2}{[2 + Pe(x_E - x_P)](x_E - x_P)}, c_P = a_P + b_P, d_P = \frac{Pe(x_E - x_P)}{2 + Pe(x_E - x_P)} + \frac{Pe(x_P - x_W)}{2} \).

In (2), we use the profile obtained on the basis of (8). Acting in a similar way as in the derivation of (7), for a uniform step we obtain the following scheme:

\[
\frac{(4+Rh)^2}{2[(4+Rh)^2 + 16]} \Phi_P = \frac{1}{Rh} \left[ \frac{16}{(4+Rh)^2 + 16} \right] \Phi_E + \left[ \frac{(4+Rh)^2}{(4+Rh)^2 + 16} + \frac{1}{Rh} \right] \Phi_W + \frac{h S(P)}{2[(4+Rh)^2 + 16]} \cdot (S(x_w) - S(x_e)). \tag{9}
\]

3. Numerical experiments

3.0.1. Model problem 1 Consider the equation

\[
\frac{du}{dx} = \frac{1}{Pe} \frac{d^2u}{dx^2} + \sin \pi x.
\]

with boundary conditions \(u(0) = u(1) = 0\). Table 1 shows the maximum absolute differences of the schemes calculated at the nodal points \((u-\) the exact solution to the problem, \(u_1-\) the solution obtained according to the scheme against the flow, \(u_2-\) according to the power law, \(u_3-\) according to the Leonard scheme, \(u_4\) according to (6) and \(u_5\) - according to the scheme (9).
Table 1. Comparison difference schemes.

| $P_e$ | $Rh$ | $\max |u - u_1|$ | $\max |u - u_2|$ | $\max |u - u_3|$ | $\max |u - u_4|$ | $\max |u - u_5|$ |
|-------|------|----------------|----------------|----------------|----------------|----------------|
| 100   | 10   | 0.0526         | 0.03770        | 0.1801         | 0.03701        | 0.00077        |
| 1000  | 100  | 0.0470         | 0.0464         | 0.0273         | 0.01607        | 0.00927        |

3.0.2. Model problem 2

Consider the equation

\[
\frac{du}{dx} = \frac{1}{Pe} \frac{d^2 u}{dx^2} + s(x),
\]

with boundary conditions $u(0) = 0$, $u(1) = 1$, with source

\[
s(x) = \begin{cases} 
10 - 50x, & 0 \leq x \leq 0.3, \\
50x - 20, & 0.3 < x < 0.4, \\
0, & 0.4 < x \leq 1.
\end{cases}
\]

Figure 1. Comparison of various schemes

Fig. 1 a) shows that, scheme (9) gives the best results. Leonard’s scheme gives an incorrect solution near the right border. Scheme (6) also exhibits a slight non-monotony. This is due to the fact that scheme (6) is stable at $Rh < 4$.

Fig. 1 b) shows that for large Peclet grid numbers, the upstream and Patancar schemes give similar results. Scheme (9) gives the best results. This can also be seen in table 2, which shows a comparison of the considered schemes (CDS - the central scheme). The solid line in fig.1 and fig.2 is the exact solution, the circle is the upwind scheme, the solid circle is the Patancar scheme, the asterisk is the Leonard scheme, + is the scheme (6), the box is according to (9)

3.0.3. Model problem 3

Two-dimensional case. Consider the equation

\[
\frac{\partial g}{\partial x} = \frac{1}{Pe} \left( \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right) + s(x, y).
\]
Table 2. Comparison of the difference schemes.

| scheme | Re  | h  | Rh | max | $\sum |u_i - (u_p)|$ | $\sum |u_i|$ |
|--------|-----|----|----|-----|----------------|----------|
| upwind | 100 | 1/40 | 2.5 | 0.1627 | 0.2116 | |
|        | 100 | 1/20 | 5   | 0.3258 | 0.4224 | |
|        | 500 | 1/20 | 25  | 0.3650 | 0.4101 | |
| Power law | 100 | 1/40 | 2.5 | 0.0833 | 0.1057 | |
|         | 100 | 1/20 | 5   | 0.2385 | 0.3025 | |
|         | 500 | 1/20 | 25  | 0.3454 | 0.3868 | |
| (6)    | 100 | 1/40 | 2.5 | 0.0164 | 0.0169 | |
|        | 100 | 1/20 | 5   | 0.0460 | 0.0401 | |
|        | 500 | 1/20 | 25  | 0.0531 | 0.0398 | |
| (9)    | 100 | 1/40 | 2.5 | 0.0129 | 0.00840 | |
|        | 100 | 1/20 | 5   | 0.0452 | 0.0358 | |
|        | 500 | 1/20 | 25  | 0.0571 | 0.0404 | |
| QUICK  | 100 | 1/40 | 2.5 | 0.0700 | 0.0701 | |
|        | 100 | 1/20 | 5   | 0.2231 | 0.1931 | |
|        | 500 | 1/20 | 25  | 0.3653 | 0.2055 | |
| CDS    | 100 | 1/40 | 2.5 | 0.1237 | 0.0062 | |
|        | 100 | 1/20 | 5   | 0.3033 | 0.0467 | |
|        | 500 | 1/20 | 25  | 0.5136 | 0.1355 | |

Table 3. Comparison of the difference schemes.

| scheme | $Pe = 100$, $n = 5$, $h = 0,1$ | $Pe = 500$, $n = 5$, $h = 0,1$ | $Pe = 1000$, $n = 10$, $h = 0,1$ |
|--------|-------------------------------|---------------------------------|----------------------------------|
|        | max $|g - g_p|$ | $\sum |g - g_p|$ | max $|g - g_p|$ | $\sum |g - g_p|$ | max $|g - g_p|$ | $\sum |g - g_p|$ |
| upwind | 0.150 | 0.074 | 0.169 | 0.074 | 0.186 | 0.129 |
| CDS    | 0.074 | 0.023 | 0.035 | 0.018 | 0.470 | 0.382 |
| Power Law | 0.130 | 0.061 | 0.165 | 0.071 | 0.184 | 0.127 |
| QUICK  | 0.063 | 0.017 | 0.020 | 0.0051 | 0.097 | 0.016 |
| (6)    | 0.035 | 0.019 | 0.013 | 0.008 | 0.057 | 0.023 |
| (9)    | 0.033 | 0.016 | 0.008 | 0.005 | 0.060 | 0.015 |
| VONOS  | 0.055 | 0.016 | 0.019 | 0.005 | 0.073 | 0.015 |

Exact solution $g = 6g^{10}(1 - y^{10})(1 - x^3) + 6x^3y(1 - y)$ . The equations are solved in the field $[0, 1] \times [0, 1]$. The source term is defined so that this function is a solution to the equation. The boundary conditions were determined based on the exact solution. In the table. 3 shows the results of calculations according to the schemes.

From table 3 it is clear that the proposed schemes show the best results.

4. MMN-based analytical solution

To obtain an approximate solution to the boundary value problem, we use a node with one movable node. If the area where the solution of the control is determined represents from one control volume or one moving unit, then it is possible to obtain a simple approximate analytical solution of the control. To clarify the approximate solution, you can add roaming nodes. Let’s look at some examples.
4.0.1. The flow in a flat pipe

The viscous fluid flow in a flat pipe in a one-dimensional formulation is described by the equation [18]

\[
\frac{d^2 u}{dy^2} = \frac{1}{\mu} \frac{\Delta p}{l} \tag{10}
\]

Where \( u \) is the fluid velocity, \( y \) is the vertical coordinate perpendicular to the flow, \( \Delta p/l \) is the pressure drop (const), \( \mu \) is the viscosity. Let the \( y = 0 \) and \( y = h \) motionless walls. On fixed walls, adhesion conditions are specified, i.e. \( u(0) = 0, u(h) = 0 \). For segments \( y \in [0, h] \), we replace the second derivative with the difference relation:

\[
\frac{d^2 u}{dy^2} \approx \frac{2}{h} \left[ \frac{v(h) - v(y)}{h - y} - \frac{v(y) - v(0)}{y - 0} \right]
\]

Then we replace equation (10) by the difference equation with a moving node and solving the resulting equation, we obtain

\[
v(y) = \frac{1}{2\mu} \frac{\Delta p}{l} y(h - y)
\]

For this problem, an approximate solution obtained using relocatable nodes coincides with an exact solution.

4.0.2. Flow in an ellipsoidal tube

The equation describing the one-dimensional flow in an ellipsoidal tube of a viscous fluid has the form [17]:

\[
\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{\Delta p}{\mu l}.
\]

Here \( u \) is the fluid velocity, \( \mu \) is the viscosity, and \( \Delta/l \) is the pressure drop. The equation is considered in the field \( \frac{y^2}{a^2} + \frac{z^2}{b^2} \leq 1 \) (cross section of an ellipsoidal tube). Using the method of a moving node, we replace the partial derivatives with difference relations, taking into account the boundary conditions (adhesion conditions), we can obtain

\[
u_1 = \frac{a^2 b^2}{2(a^2 + b^2)} \left( 1 - \frac{y^2}{a^2} - \frac{z^2}{b^2} \right) \frac{\Delta p}{\mu l}
\]

matching the exact solution.

5. Conclusion

The proposed schemes were applied to test problems. For comparison, various schemes were used: the upwind scheme, the Patancar scheme, QUICK, VONOS [10]. The calculation results, for various grid Peclet numbers, showed the grid convergence of the approximate solution. The proposed schemes have demonstrated an advantage over other schemes.

Thus, the proposed schemes allow one to obtain better numerical results.

MMN can also be successfully applied for the approximate solution of applied problems.

6. Reference

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