ALGEBRAICALLY CONSTRUCTIBLE FUNCTIONS:
REAL ALGEBRA AND TOPOLOGY

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Abstract. Algebraically constructible functions connect real algebra with the topology of algebraic sets. In this survey we present some history, definitions, properties, and algebraic characterizations of algebraically constructible functions, and a description of local obstructions for a topological space to be homeomorphic to a real algebraic set.

More than three decades ago Sullivan proved that the link of every point in a real algebraic set has even Euler characteristic. Related topological invariants of real algebraic singularities have been defined by Akbulut and King using resolution towers and by Coste and Kurdyka using the real spectrum and stratifications.

Sullivan’s discovery was motivated by a combinatorial formula for Stiefel-Whitney classes. Deligne interpreted these classes as natural transformations from constructible functions to homology. Constructible functions have interesting operations inherited from sheaf theory: sum, product, pullback, pushforward, duality, and integral. Duality is closely related to a topological link operator. To study the topology of algebraic sets the authors introduced algebraically constructible functions. Using the link operator we have defined many local invariants which generalize those of Akbulut-King and Coste-Kurdyka.

Algebraically constructible functions are interesting from a purely algebraic viewpoint. From the theory of basic algebraic sets it follows that if a constructible function ϕ on an algebraic set of dimension d is divisible by $2^d$ then ϕ is algebraically constructible. Parusiński and Szafraniec showed that algebraically constructible functions are precisely those constructible functions which are sums of signs of polynomials. Bonnard has given a characterization of algebraically constructible functions using fans, and she has investigated the number of polynomials necessary to represent an algebraically constructible function as a sum of signs of polynomials. Pennaneac’h has developed a theory of algebraically constructible chains using the real spectrum.

In section 1 we briefly discuss the results of Sullivan, Akbulut-King, and Coste-Kurdyda. In the next section we define algebraically constructible functions and their operations. In section 3 we discuss the relations of algebraically constructible functions with real algebra. In the following section we describe how to generate our local topological invariants. In the final section we raise some questions for future research. Throughout we consider only algebraic subsets of affine space.

Related survey articles have been written recently by Coste [13], Bonnard [8], and McCrory [22]. We thank Michel Coste for his encouragement and insight.

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1. Akbulut-King Numbers

Let $X$ be a real semialgebraic set in $\mathbb{R}^n$, and let $x \in X$. Let $S(x, \varepsilon)$ be the sphere of radius $\varepsilon > 0$ in $\mathbb{R}^n$ centered at $x$. By the local conic structure lemma \cite{3} (9.3.6), for $\varepsilon$ sufficiently small the topological type of the space $S(x, \varepsilon) \cap X$ is independent of $\varepsilon$. This space is called the link of $x$ in $X$, and it is denoted by $\text{lk}(x, X)$.

Our starting point is Sullivan’s theorem \cite{31}:

**Theorem 1.1.** If $X$ is a real algebraic set in $\mathbb{R}^n$ and $x \in X$ then the Euler characteristic $\chi(\text{lk}(x, X))$ is even.

For example, the “theta space” $X \subset \mathbb{R}^2$,

$$X = \{(x, y) \mid x^2 + y^2 = 1\} \cup \{(x, y) \mid -1 \leq x \leq 1, y = 0\},$$

is not homeomorphic to an algebraic set, for the link of the point $(1, 0)$ (or the point $(-1, 0)$) in $X$ is three points, which has odd Euler characteristic.

Many proofs of Sullivan’s theorem have been published; see \cite{1}, \cite{17}, \cite{3} (3.10.4), \cite{3} (11.2.2), \cite{16} (4.4). Sullivan’s original idea was to use complexification. First he proved that the link of $x$ in the complexification $X_{\mathbb{C}}$ has Euler characteristic 0, and then he used that $\text{lk}(x, X)$ is the fixed point set of complex conjugation on $\text{lk}(x, X_{\mathbb{C}})$ to deduce that

$$\chi(\text{lk}(x, X)) \equiv \chi(\text{lk}(x, X_{\mathbb{C}})) \pmod{2}.$$

Mather \cite{21} (p. 221) gave a proof that the link $L$ of a point in a complex algebraic set has Euler characteristic 0 by constructing a tangent vector field on $L$ which integrates to a nontrivial flow of $L$.

The following result puts Sullivan’s theorem in a more general context (cf. \cite{3} (2.3.2)).

**Theorem 1.2.** If $X$ and $Y$ are real algebraic sets with $Y$ irreducible and $f : X \to Y$ is a regular map, there is an algebraic subset $Z$ of $Y$ with $\dim Z < \dim Y$ such that the Euler characteristic $\chi(f^{-1}(y))$ is constant mod 2 for $y \in Y \setminus Z$.

In other words, the Euler characteristic is generically constant mod 2 in every family of real algebraic sets. To deduce Sullivan’s theorem as a corollary let $Y = \mathbb{R}$, $x_0 \in X$, and $f(x) = (x - x_0)^2$. For $y < 0$ the fiber $f^{-1}(y)$ is empty, and for $y > 0$ sufficiently small, the fiber $f^{-1}(y)$ is $\text{lk}(x_0, X)$.

Benedetti-Dedò \cite{4} and Akbulut-King \cite{2} proved that Sullivan’s condition is not only necessary but also sufficient in low dimensions: If $X$ is a compact triangulable space of dimension less than or equal to 2, and the link of every point has even Euler characteristic, then $X$ is homeomorphic to a real algebraic set. (The link of a point in a triangulable space is the boundary of a simplicial neighborhood.) A triangulable space such that the link of every point has even Euler characteristic is called an Euler space.

Akbulut and King \cite{3} showed that the situation in dimension 3 is more complicated. They defined four non-trivial topological invariants of a compact Euler space $Y$ of dimension at most 2, $a_i(Y) \in \mathbb{Z}/2$, $i = 0, 1, 2, 3$ (with $a_i(Y) = 0$ when $\dim Y < 2$). Let $\chi_2(Y)$ be the Euler characteristic mod 2. It is easy to see that if $X$ is an Euler space then the link of every point of $X$ is an Euler space.
Theorem 1.3. A compact 3-dimensional triangulable topological space \( X \) is homeomorphic to a real algebraic set if and only if, for all \( x \in X \), \( \chi_2(\text{lk}(x,X)) = 0 \) and \( a_i(\text{lk}(x,X)) = 0 \), \( i = 0, 1, 2, 3 \).

Akbulut and King’s invariants arise from a combinatorial analysis of the resolution of singularities of an algebraic set. The elementary definition of these Akbulut-King numbers and computations of examples can be found in Akbulut and King’s monograph [3], chapter VII, pages 190–197. (In the terminology of [3] (7.1.1), \( a_i(\text{lk}(x,X)) \) is the mod 2 Euler characteristic of the link of \( x \) in the characteristic subspace \( Z_i(X) \).) The depth of the method of resolution towers is shown by the remarkable result that the vanishing of the Akbulut-King numbers gives a sufficient condition for a triangulable 3-dimensional space to be homeomorphic to an algebraic set. Chapter I of [3] is an introduction to their methods, with informative examples.

Another descendant of Sullivan’s theorem is due to Coste and Kurdyka [14]:

Theorem 1.4. Let \( X \) be an algebraic set and let \( V \) be an irreducible algebraic subset. For \( x \in V \) the Euler characteristic of the link of \( x \) in \( X \) is generically constant mod 4: There is an algebraic subset \( W \) of \( V \) with \( \dim W < \dim V \) such that \( \chi(\text{lk}(x,X)) \) is constant mod 4 for \( x \in V \setminus W \).

This theorem was first proved by Coste [12] when \( \dim X - \dim V \leq 2 \) using chains of specializations of points in the real spectrum. The general case was proved topologically using stratifying families of polynomials. It can also be proved using Akbulut and King’s topological resolution towers ([3], exercise on p. 192).

Using the same techniques Coste and Kurdyka defined invariants mod \( 2^k \) associated to chains \( X_1 \subset X_2 \subset \cdots \subset X_k \) of algebraic subsets of \( X \) ([14], Theorem 4). Furthermore they used their mod 4 and mod 8 invariants to recover the Akbulut-King numbers. Using a relation between complex conjugation and the monodromy of the complex Milnor fibre of an ordered family of functions, the authors [23] reinterpreted and generalized the Coste-Kurdyka invariants as Euler characteristics of iterated links.

2. Constructible Functions

Algebraically constructible functions were introduced by the authors [24] as a vehicle for using the ideas of Coste and Kurdyka to generate new Akbulut-King numbers.

Let \( X \) be a real semialgebraic set. A constructible function on \( X \) is an integer-valued function

\[
\varphi : X \to \mathbb{Z}
\]

which can be written as a finite sum

\[
\varphi = \sum m_i 1_{X_i},
\]

where for each \( i \), \( X_i \) is a semialgebraic subset of \( X \), \( 1_{X_i} \) is the characteristic function of \( X_i \), and \( m_i \) is an integer.

The set of constructible functions on \( X \) is a ring under pointwise sum and product. If \( f : X \to Y \) is a semialgebraic map and \( \varphi \) is a constructible function on \( Y \), the pullback \( f^* \varphi \)
is the constructible function defined by

\begin{equation}
(2.2) \quad f^*\varphi(x) = \varphi(f(x)).
\end{equation}

The operations of pushforward and duality are defined using the Euler characteristic. If \( \varphi \) has compact support one may assume that the sets \( X_i \) in (2.1) are compact, and the Euler integral is defined by

\begin{equation}
(2.3) \quad \int_X \varphi \, d\chi = \sum m_i \chi(X_i).
\end{equation}

The Euler integral is additive, and it does not depend on the choice of representation (2.1) of \( \varphi \).

If \( f : X \to Y \) is a proper semialgebraic map and \( \varphi \) is a constructible function on \( X \), the pushforward, \( f_* \varphi \) is the constructible function on \( Y \) given by

\begin{equation}
(2.4) \quad f_* \varphi(y) = \int_{f^{-1}(y)} \varphi \, d\chi.
\end{equation}

Suppose that \( X \) is a semialgebraic set in \( \mathbb{R}^n \). If \( \varphi \) is a constructible function on \( X \), the link \( \Lambda \varphi \) is the constructible function on \( X \) defined by

\begin{equation}
(2.5) \quad \Lambda \varphi(x) = \int_{S(x,\varepsilon) \cap X} \varphi \, d\chi,
\end{equation}

for \( \varepsilon > 0 \) sufficiently small.

The dual \( D\varphi \) is defined by

\begin{equation}
(2.6) \quad D\varphi = \varphi - \Lambda \varphi.
\end{equation}

The operations sum, product, pullback, pushforward, and dual come from sheaf theory. Operations on constructible functions have been studied by Kashiwara and Schapira [18, 30] and by Viro [32].

Now suppose that \( X \) is a real algebraic set. A provisional definition of algebraically constructible functions would be to require the sets \( X_i \) in (2.1) to be algebraic subsets of \( X \). But the image of an algebraic set by a proper regular map is not necessarily algebraic, so this class of functions—which we call strongly algebraically constructible—is not preserved by the pushforward operation. To remedy this defect we make the following definition.

Let \( X \) be a real algebraic set. An algebraically constructible function on \( X \) is an integer-valued function which can be written as a finite sum

\begin{equation}
(2.7) \quad \varphi = \sum m_i f_i* \mathbf{1}_{Z_i},
\end{equation}

where for each \( i \), \( Z_i \) is an algebraic set, \( \mathbf{1}_{Z_i} \) is the characteristic function of \( X_i \), \( f_i : Z_i \to X \) is a proper regular map, and \( m_i \) is an integer.

Clearly the sum of algebraically constructible functions is algebraically constructible. The product of algebraically constructible functions is algebraically constructible because the fiber product of algebraic sets over \( X \) is an algebraic set over \( X \): If \( f_1 : Z_1 \to X \) and \( f_2 : Z_2 \to X \)
are proper regular maps, then so is the fiber product \( f : Z_1 \times_X Z_2 \to X \),
\[
\begin{array}{ccc}
Z_1 \times_X Z_2 & \longrightarrow & Z_2 \\
\downarrow & & \downarrow f_2 \\
Z_1 & \longrightarrow & X \\
\end{array}
\]
where \( Z_1 \times_X Z_2 = \{(z_1, z_2) \mid f_1(z_1) = f_2(z_2)\} \) and \( f(z_1, z_2) = f_1(z_1) = f_2(z_2) \). Furthermore, for all \( x \in X \), \( f^{-1}(x) = f_1^{-1}(x) \times f_2^{-1}(x) \). Therefore \( (f_1 \ast Z_1)(f_2 \ast Z_2) = f_1 \ast Z \).

Similarly, the pullback (2.2) of an algebraically constructible function by a regular map is algebraically constructible. The pushforward (2.4) of an algebraically constructible function by a proper regular map is algebraically constructible, by the functoriality of pushforward \((g \circ f)_* = g_* \circ f_*\).

The fact that the link (2.5) of an algebraically constructible function is algebraically constructible follows from resolution of singularities and the fact that the link operator commutes with pushforward \((\Lambda f_*) = f_* \Lambda \). Resolution of singularities implies that all the algebraic sets \( Z_i \) in (2.7) may be taken to be smooth and irreducible. If \( Z \) is a smooth algebraic set of dimension \( d \), then
\[
\Lambda 1_Z = \begin{cases} 
2 \cdot 1_Z & d \text{ odd} \\
0 & d \text{ even}
\end{cases}
\]
Thus if \( \varphi = \sum m_i f_i \ast 1_{Z_i} \) as in (2.7),
\[
\Lambda \varphi = \Lambda \sum m_i f_i \ast 1_{Z_i} \\
= \sum m_i f_i \ast \Lambda 1_{Z_i} \\
= 2 \sum m_i f_i \ast 1_{Z_i},
\]
where the last sum is over all \( i \) such that \( \dim Z_i \) is odd. This argument actually gives the following stronger result [24].

**Theorem 2.1.** If \( \varphi \) is an algebraically constructible function on the algebraic set \( X \), then the values of \( \Lambda \varphi \) are even, and \( \frac{1}{2} \Lambda \varphi \) is an algebraically constructible function.

This theorem is the key to defining Akbulut-King numbers using constructible functions (section 4 below). We call \( \tilde{\Lambda} = \frac{1}{2} \Lambda \) the half-link operator.

The constructible function \( \varphi \) is Euler if all the values of \( \Lambda \varphi \) are even. The following examples show that the sets of constructible functions, Euler constructible functions, algebraically constructible functions, and strongly algebraically constructible functions are all different. (More examples can be found in [24].)

Let \( X = \mathbb{R}^2 \), and let \( Q \) be the closed first quadrant,
\[
Q = \{(x, y) \mid x \geq 0, y \geq 0\}.
\]
The constructible function \( 1_Q \) is not Euler, for \( \Lambda 1_Q = 1_O + 1_A + 1_B \), where \( O \) is the origin, \( A \) is the positive \( x \)-axis, and \( B \) is the positive \( y \)-axis.

The constructible function \( 2 \cdot 1_Q \) is Euler, since it is even, but it is not algebraically constructible. If it were algebraically constructible, then by Theorem 2.1 the half-link \( \tilde{\Lambda}(2 \cdot 1_Q) \)
would be algebraically constructible. But if $C$ is the $x$-axis, then $(1_C)(\Lambda 1_Q) = 1_O + 1_A$, which is not Euler and hence not algebraically constructible.

The constructible function $4 \cdot 1_Q$ is not strongly algebraically constructible, since the algebraic closure of the first quadrant is the plane. But $4 \cdot 1_Q$ is algebraically constructible,$

$$4 \cdot 1_Q = f_* 1_{\mathbb{R}^2} + g_* 1_{\mathbb{R}} + h_* 1_{\mathbb{R}} + 1_O,$$

where $f(x, y) = (x^2, y^2)$, $g(x) = (x^2, 0)$, and $h(x) = (0, x^2)$.

3. Real Algebra

The bridge between the topological and algebraic properties of algebraically constructible functions is given by the following theorem [26], [15], [27].

Theorem 3.1. Let $X$ be an algebraic subset of $\mathbb{R}^n$ and let $f : Z \to X$ be a regular map. Then there are real polynomials $g_1, \ldots, g_s \in \mathbb{R}[x_1, \ldots, x_n]$ such that for all $x \in X$,

$$\chi(f^{-1}(x)) = \sum_{i=1}^s \text{sgn} g_i(x).$$

(3.1)

For a polynomial, or more generally for a regular function $g$ on $X$, the projection $\pi : Z \to X$ from the double cover $Z = \{(x, t) \in X \times \mathbb{R} \mid t^2 = g(x)\}$ to $X$ satisfies $\text{sgn} g = \pi_* 1_Z - 1_X$. Thus the sign of $g$ and hence any finite sum of signs of polynomials on $X$ is algebraically constructible. This gives the following characterization of algebraically constructible functions.

Corollary 3.2. Let $X$ be an algebraic set. Then $\varphi : X \to \mathbb{Z}$ is algebraically constructible if and only if $\varphi$ equals a finite sum of signs of polynomials on $X$.

Usually we suppose in the presentation (3.1) that none of polynomials $g_i$ vanishes identically on $X$. Suppose, moreover, that $X$ is irreducible. Then the product $g = \prod g_i$ does not vanish identically on $X$ and hence the zero set $W$ of $g$ is a proper algebraic subset of $X$. Since $X$ is irreducible, $\dim W < \dim X$. By (3.1), for $x \in X \setminus W$, the Euler characteristic $\chi(f^{-1}(x))$ is congruent mod 2 to $s$, the number of polynomials $g_i$. This gives an alternative proof of Theorem 1.2. We can go further and describe the Euler characteristic of fibres of $f$ mod 4: For $x \in X \setminus W$,

$$\chi(f^{-1}(x)) \equiv (s - 1) + \text{sgn} g(x) \pmod{4}.$$  

(3.2)

The existence of such $g$, called a discriminant of $f$, was proved by Coste and Kurdyka in [15]. Although the existence of a discriminant follows from (3.1), historically (3.2) was a prototype for Theorem 3.1. In general, a polynomial or a rational function $g$ on $X$ satisfying (3.2) is not unique. It is uniquely defined as an element of the multiplicative group of non-zero rational functions $\mathbb{R}(X) \setminus \{0\}$ on $X$ divided by the subgroup of those functions which are generically of constant sign on $X$, that is to say (by Hilbert’s seventeenth problem), by $\pm$ sums of squares of rational functions (cf. [15]).

Theorem 3.1 and Corollary 3.2 provide an abstract link to the theory of quadratic forms. Let us consider a more concrete example. Consider a finite regular map $f : Z \to \mathbb{R}^n$, where
$Z \in \mathbb{R}^n \times \mathbb{R}$ is the zero set of a polynomial without multiple factors,

$$P(x, z) = z^8 + \sum_{i=0}^{s-1} a_i(x)z^i,$$

$x \in \mathbb{R}^n, z \in \mathbb{R}$, and $f$ is induced by the projection on the first factor. Since $f$ is finite, $
 \chi(f^{-1}(x))$ equals the number of distinct real roots of the polynomial $P_x(z) = P(x, z)$ of one real variable. By a classical theorem of Hermite-Sylvester (see e.g. [6] Proposition 6.2.6), the number of distinct real roots of $P_x$ equals the signature of a symmetric matrix $Q$ of size $s$ with entries polynomials in the $a_i$. The determinant of $Q$ equals the discriminant $\Delta(x)$ of $P_x$. It is elementary to show that, if $\Delta(x)$ is non-zero, then the number of distinct real roots of $P_x$ is congruent to $s - 1 + \text{sgn} \Delta(x) \mod 4$. This justifies the name “discriminant” for $g$ satisfying (3.2). If we want to compute the signature of $Q$ we diagonalize it over the field of rational functions $\mathbb{R}(x_1, \ldots, x_n)$. The signature of the matrix obtained, and hence the signature of $Q$, equals the sum of signs of the elements on the diagonal, which are rational functions. The sign of the rational function $p/q$ equals the sign of the polynomial $pq$, in the complement of the zero set of the denominator. Thus we have shown the existence of polynomials $g_i$ that satisfy (3.1) generically; that is, for $x$ in the complement of a proper algebraic subset of $\mathbb{R}^n$.

In general, if $X$ is irreducible then we say that a function is defined or a property holds generically on $X$ if this is so in the complement of a proper algebraic subset of $X$. By Corollary 3.2, $\varphi$ is generically algebraically constructible on $X$ if and only if it is generically equal to the signature of a quadratic form over the field of rational functions $\mathbb{R}(X)$. Thus for any regular map $f : Z \to X$ there is a quadratic form $Q$ over $\mathbb{R}(X)$ such that $\chi(f^{-1}(x))$ generically equals the signature of $Q$. There are, of course, many quadratic forms with this property and we do not know whether, for arbitrary $f$, there is a natural choice of such a quadratic form. The proofs of Theorem 3.1 are all based on the construction of quadratic forms. In [13] the forms are constructed by means of Morse Theory, in [26] they are given by the Eisenbud-Levine Theorem, and in [27] by a modern version of the Hermite-Sylvester theorem.

Corollary 3.2 shows immediately that the sum and the product of algebraically constructible functions are algebraically constructible. This corollary was used in [26] to give another proof that the family of algebraically constructible functions is preserved by the half-link operator (Theorem 2.1), and by related topological operators such as specialisation (cf. [6], [24]). In general, Corollary 3.2 allows us to use algebraic methods to study algebraically constructible functions.

Recall that a basic open semialgebraic subset of a real algebraic set $X \subset \mathbb{R}^n$ is a subset of the form

$$U(g_1, \ldots, g_s) = \{x \in X \mid g_1(x) > 0, \ldots, g_s(x) > 0\},$$

where $g_i$ are polynomials on $X$. For a polynomial $g$, the function $2 \cdot 1_{U(g)} = \text{sgn} g + 1_{X} - 1_{g^{-1}(0)}$ is algebraically constructible, and hence so is $2^s1_{U(g_1, \ldots, g_s)} = \prod 2 \cdot 1_{U(g_i)}$. By a theorem of Bröcker and Scheiderer (cf. [3] section 6.5), every basic open semialgebraic subset of a real algebraic set of dimension $d$ can be defined by at most $d$ simultaneous strict polynomial
inequalities; that is, we may always choose \( s \leq d \). This gives, as shown in [24], the following result.

**Theorem 3.3.** Let \( X \) be a real algebraic set of dimension \( d \). Then any constructible function on \( X \) divisible by \( 2^d \) is algebraically constructible.

For instance any constructible function on the plane \( \mathbb{R}^2 \) divisible by 4 is algebraically constructible. We do not know an elementary proof of this fact.

As we have mentioned before, the first version of Theorem 1.4 was proved by Coste [12] using the real spectrum. This approach was later developed by Bonnard, who introduced a fan criterion for algebraically constructible functions [7]. Fans, which are subsets of the real spectrum, were introduced by Bröcker in order to study quadratic forms.

The fan criterion has proved to be a very powerful tool. It gives (cf. [7]) the following “wall” criterion. Suppose that \( X \) is nonsingular and compact, \( \varphi: X \to \mathbb{Z} \), and \( W \) is an algebraic subset of \( X \) with normal crossings such that \( \varphi \) is constant on the connected components of \( X \setminus W \). For a smooth point \( w \in W \) we define \( \partial_W \varphi(w) \) as the average of the values of \( \varphi \) on \( X \setminus W \) computed on both sides of \( W \) at \( w \). We may extend \( \partial_W \varphi(w) \) arbitrarily to the singular part of \( W \). Then \( \varphi \) is generically algebraically constructible on \( X \) if and only if \( \partial_W \varphi(w) \) is generically algebraically constructible on \( W \). This allows one to use induction on dimension in working with generically algebraically constructible functions.

One may ask how many polynomials are necessary in order to describe a given algebraically constructible function as a sum of signs of polynomials. In [7] Bonnard gives two bounds: for a complete presentation and for a generic presentation.

**Theorem 3.4.** Let \( X \) be a real algebraic set of dimension \( d \) and let \( \varphi: X \to \mathbb{Z} \) be an algebraically constructible function such that \( \varphi(X) \subset [\delta - k, \delta + k] \), \( \delta \in \mathbb{Z} \), \( k \in \mathbb{N} \).

(i) Then \( \varphi \) can be written as the sum of signs of at most \( N'(d, k, \delta) \) polynomials, where \( N'(d, k, \delta) \) equals \( 2^{d - 1}3k + |\delta| \) for \( k \) even and \( 2^{d - 1}3(k - 1) + 2d + |\delta| \) for \( k \) odd.

(ii) Suppose, moreover, that \( X \) is irreducible. Then \( \varphi \) can be written generically as the sum of signs of at most \( N(d, k, \delta) \) polynomials, where \( N(d, k, \delta) \) equals \( 2^{d - 1}k + |\delta| \) for \( k \) even and \( 2^{d - 1}(k - 1) + 1 + |\delta| \) for \( k \) odd.

These bounds are sharp in the following sense. Given \( d, k, \text{and } \delta \) as above, there is an algebraic set \( X \) of dimension \( d \) and an algebraically constructible function \( \varphi' \), resp. \( \varphi \), on \( X \) such that \( N'(d, k, \delta) \), resp. \( N(d, k, \delta) \), is the minimal number of polynomials necessary to present \( \varphi' \), resp. to present \( \varphi \) generically, as a sum of signs of polynomials. Moreover, we may take \( X = \mathbb{R}^d \). Theorem 3.4 is proved using spaces of orderings.

For a fixed algebraically constructible function the bound given by Theorem 3.4 may not be sharp. Let \( X \) be irreducible and compact, and let \( \varphi \) be an algebraically constructible function on \( X \) constant on the connected components of \( X \setminus W \), where \( W \subset X \) is an algebraic subset with normal crossings. In [10] Bonnard gives a recursive method for effectively calculating the minimal number of polynomials representing \( \varphi \).

Using the real spectrum Pennaneac’h has defined algebraically constructible chains [28] and algebraically constructible homology of real algebraic varieties. An algebraically constructible \( k \)-chain of a real algebraic variety \( X \) is an oriented semialgebraic chain supported by a \( k \)-dimensional irreducible algebraic subset \( V \subset X \), and “weighted” by a generically algebraically
constructible function on \( V \). The boundary operator is given by half of the standard boundary. Pennaneac’h has proved that a constructible function is algebraically constructible if and only if its characteristic Lagrangian cycle is algebraically constructible.

4. Topological Invariants

Using algebraically constructible functions and the link operator we define local topological invariants which generalize the Akbulut-King numbers. The vanishing of these invariants gives necessary conditions for a topological space to be homeomorphic to an algebraic set.

Algebraic sets are necessarily triangulable (cf. \[6\], 9.3.2), and by definition a triangulable space is homeomorphic to a Euclidean simplicial complex, which is a semialgebraic set. So without loss of generality we assume that the spaces we consider are semialgebraic sets in Euclidean space.

If \( X \) is a semialgebraic set, let \( F(X) \) be the set of constructible functions on \( X \) \([2.1]\). The set \( F(X) \) is a commutative ring with identity \( 1_X \), and it is equipped with a linear operator \( \Lambda \), the link operator, and a linear integer-valued function \( \varphi \mapsto \int \varphi d\chi \), the Euler integral.

Of course \( F(X) \) is not a topological invariant of \( X \), but the identity element, the link operator, and the Euler integral are topological invariants in the following sense. Let \( h : X' \to X \) be a homeomorphism of semialgebraic sets. Then \( 1_{X'} = 1_X \circ h \). Let \( \varphi \in F(Y) \) be such that \( \varphi' = \varphi \circ h \in F(X') \). Let \( Y \subset X \) be a compact semialgebraic subset such that \( Y' = h^{-1}(Y) \) is also semialgebraic. Then

\[
\Lambda(\varphi') = (\Lambda \varphi) \circ h,
\]

\[
\int_{Y'} \varphi' d\chi = \int_Y \varphi d\chi.
\]

For the elementary proof see \[24\], appendix A.7. It follows that the subring of \( F(X) \) generated by \( 1_X \) and \( \Lambda \) is a topological invariant of \( X \).

For a semialgebraic set \( X \) let \( \hat{\Lambda}(X) \) be the subring of \( F(X) \otimes \mathbb{Q} \) generated by \( 1_X \) and the half-link operator \( \hat{\Lambda} = \frac{1}{2} \Lambda \). The ring \( \hat{\Lambda}(X) \) is a topological invariant of \( X \). Theorem \[2.1\] says that if \( X \) is an algebraic set then \( \hat{\Lambda}(X) \subset F(X) \). In other words, all of the functions obtained from \( 1_X \) by the arithmetic operations of sum, difference, and product, together with the half-link operator, are integer-valued.

So we have a method to produce topological obstructions for a semialgebraic set \( X \) to be homeomorphic to an algebraic set: Find expressions \( \varphi \) built from \( 1_X \) using the operations +, −, ×, and \( \hat{\Lambda} \) such that \( \varphi \) is not integer-valued. In \[24\] (see also \[13\]) this method is explicitly illustrated for Akbulut and King’s original example of a compact 3-dimensional Euler space which is not homeomorphic to an algebraic set. In \[24\] the authors show that this method reproduces the Akbulut-King invariants for sets of dimension at most 3.

In \[25\] we find all invariants produced by this method for sets of dimension at most 4. The number of independent invariants is enormous: \( 2^{29} - 29 \). But there is another surprise. It follows from Corollary \[8.2\] that for all algebraically constructible functions \( \varphi \), the function \( \frac{1}{2}(\varphi^4 - \varphi^2) \) is algebraically constructible \([25\], Lemma 4.1). So a necessary condition for \( X \) to be algebraic is that all of the functions obtained from \( 1_X \) by the arithmetic operations of sum, difference, and product, together with the half-link operator and the operator \( P(\varphi) = \)
\[ \frac{1}{2}(\varphi^4 - \varphi^2) \] are integer-valued. The total number of independent invariants produced by our method taking all of these operators into account is \(2^{43} - 43\).

The classification of these invariants is simplified by studying them locally. The function \(\varphi \in F(X)\) is Euler if and only if, for every \(x \in X\) and every \(\varepsilon = \varepsilon(x)\) sufficiently small, the restriction of \(\varphi\) to \(L_\varepsilon(x) = S(x, \varepsilon) \cap X\) has even Euler integral. Now \(1_X|L_\varepsilon(x) = 1_{L_\varepsilon(x)}\), and \(\tilde{\Lambda}(\varphi)L_\varepsilon(x) = \varphi[L_\varepsilon(x) - \tilde{\Lambda}(\varphi)L_\varepsilon(x)]\), by (2) 1.3(d). Thus all functions obtained from \(1_X\) by the operations +, −, ×, \(\tilde{\Lambda}\), and \(P\) are integer-valued if and only if, for all \(x \in X\), all functions obtained from \(1_{L_\varepsilon(x)}\) by these operations are integer-valued and have even Euler integral.

Akbulut and King obtain their numbers from cobordism invariants of resolution towers. Here we give a cobordism-style description of the Akbulut-King numbers which does not involve resolution towers.

Let \(n\) be a nonnegative integer, and let \(A_n\) be the set of homeomorphism classes of compact real algebraic sets of dimension at most \(n\). If \(X\) is a compact real algebraic set, we say that \(X\) is a boundary if there exists a real algebraic set \(W\) and a point \(w \in W\) such that \(X\) is homeomorphic to the link of \(w\) in \(W\). Let \(B_n\) be the subset of \(A_n\) consisting of homeomorphism classes of boundaries.

**Proposition 4.1.** For all compact algebraic sets \(X\), the disjoint union \(X \sqcup Y\) is a boundary.

**Proof.** Suppose \(X \subseteq \mathbb{R}^n\) is given by the polynomial equation \(f(x_1, \ldots, x_n) = 0\) of degree \(d\). Consider the homogeneous polynomial \(g(x_1, \ldots, x_n, x_{n+1})\) of degree \(d\) such that \(g(x_1, \ldots, x_n, 1) = f(x_1, \ldots, x_n)\). Let \(W \subseteq \mathbb{R}^{n+1}\) be given by \(g(x_1, \ldots, x_n, x_{n+1}) = 0\). Then \(X \sqcup Y\) is homeomorphic to the link of the origin in \(W\). \(\square\)

Let \(A'_n\) be the free abelian group on the set \(A_n\) modulo the subgroup generated by elements of the form \([X] + [Y] - [X \sqcup Y]\), and let \(B'_n\) be the subgroup of \(A'_n\) generated by \(B_n\). Let \(V_n = A'_n/B'_n\). By Proposition 4.1, \(V_n\) is a vector space over \(\mathbb{Z}/2\).

The Akbulut-King numbers are additive under disjoint union, so they define a linear map \(a : A_2 \rightarrow (\mathbb{Z}/2)^5\), \(a([X]) = (\chi_2(X), a_0(X), a_1(X), a_2(X), a_3(X))\). By Theorem 1.3, if \(\dim X \leq 2\) then \(X\) is a boundary if and only if \(a([X]) = 0\). (If \(a([X]) = 0\) then by Theorem 1.3 the cone on \(X\) is homeomorphic to an algebraic set.) Thus \(a\) induces an injective linear map

\[ a : V_2 \rightarrow (\mathbb{Z}/2)^5. \]

Akbulut and King show that \(a\) is an isomorphism by constructing 2-dimensional algebraic sets \(Y_0, Y_1, Y_2, \text{ and } Y_3\) such that \(a(\mathbb{RP}^2), a(Y_0), a(Y_1), a(Y_2), \text{ and } a(Y_3)\) are linearly independent (3, p. 195–197). Thus \((\mathbb{RP}^2, Y_0, Y_1, Y_2, Y_3)\) is a basis for \(V_2\).

On the other hand, the authors’ constructible function invariants (24) define a linear map \(b : A_2 \rightarrow (\mathbb{Z}/2)^5\), \(b([X]) = (\chi_2(X), b_1(X), b_2(X), b_3(X), b_4(X))\), where the \(b_i\) are the mod 2 reductions of the following Euler integrals. Let \(\tilde{\Omega}\) be the operator defined by \(\tilde{\Omega}(\varphi) = \varphi - \tilde{\Lambda}(\varphi)\). Let \(\alpha = \tilde{\Lambda}1_X\), \(\beta = \tilde{\Omega}(\alpha^2)\), and \(\gamma = \tilde{\Omega}(\alpha^3)\). Then \(b_1(X) = \int \alpha \beta d\chi\), \(b_2(X) = \int \alpha \gamma d\chi\), \(b_3(X) = \int \beta \gamma d\chi\), and \(b_4(X) = \int \alpha \beta \gamma d\chi\). The map \(b\) induces a linear isomorphism

\[ b : V_2 \rightarrow (\mathbb{Z}/2)^5. \]

The reader may verify that \(b\) is an isomorphism by computing \(b(Y_i), i = 0, 1, 2, 3\). (The computation of \(b_1(Y_2)\) is given in 22.)
From this viewpoint the authors’ results for 4-dimensional sets can be summarized as follows. The invariants of $\mathcal{A}'_3$ (Theorem 4.7) are additive under disjoint union, so they define a linear map $c: \mathcal{A}'_3 \to (\mathbb{Z}/2)^N$, $N = 2^{43} - 43$. If $X$ is a boundary of dimension at most 3, then $c([X]) = 0$. Thus $c$ induces a linear map

$$c: \mathcal{V}_3 \to (\mathbb{Z}/2)^N.$$  

We show that $c$ is surjective by constructing a large set $S$ of compact 3-dimensional Euler spaces with vanishing local Akbulut-King numbers. By Theorem 1.3 every space in $S$ is homeomorphic to a real algebraic set. We make $S$ so big that $\{c([X]) | X \in S\}$ spans $(\mathbb{Z}/2)^N$. Thus the dimension of $\mathcal{V}_3$ is at least $2^{43} - 43$. But we do not know whether $c$ is an isomorphism, because we do not know if $c([X]) = 0$ implies that $X$ is a boundary.

**Remark 4.2.** The one-point compactification of a real algebraic set is homeomorphic to a real algebraic set (cf. [1], [6] Proposition 3.5.3). Hence a locally compact topological space $X$ is homeomorphic to a real algebraic set if and only if its one-point compactification $\overline{X} = X \cup \{\infty\}$ is homeomorphic to a real algebraic set. Suppose $X$ is a closed semialgebraic subset of $\mathbb{R}^n$ and the local invariants constructed above vanish at every $x \in X$. Then they have to vanish at $\infty \in \overline{X}$. Indeed, suppose that $\hat{\Lambda}(X) \subset F(X)$. Of the operations that define $\hat{\Lambda}(X)$, that is $+,-,\times$, and $\hat{\Lambda}$, all but $\hat{\Lambda}$ preserve $F(\overline{X})$. Hence if, contrary to our claim, $\hat{\Lambda}(\overline{X}) \not\subset F(\overline{X})$ then there is $\varphi \in \hat{\Lambda}(\overline{X})$ such that $\hat{\Lambda}\varphi$ is not integer-valued. By our assumption the restriction of $\hat{\Lambda}\varphi$ to $X$ has to be integer-valued, and we may write $\hat{\Lambda}\varphi = \psi + \frac{1}{2}\mathbf{1}_\infty$ with $\psi \in F(\overline{X})$. Consequently the Euler integral $\int_X \hat{\Lambda}\varphi d\chi \not\in \mathbb{Z}$. This contradicts the identity

$$\int_X \hat{\Lambda} d\chi = 0,$$

which holds for any constructible function with compact support ([24] Corollary 1.3). The same argument can be applied to the extended invariants since the operator $P(\varphi) = \frac{1}{2}(\varphi^4 - \varphi^2)$ preserves integer-valued functions.

This remark can be also applied to Sullivan’s invariant and Akbulut and King’s numbers. This means that the one-point compactification of a semialgebraic Euler set is Euler, and that in Theorem 1.3 it suffices to suppose that $X$ is homeomorphic to a closed, not necessarily compact, semialgebraic subset of $\mathbb{R}^n$ or, equivalently, that $\overline{X}$ is triangulable.

**5. Generalizations and open questions**

The method of algebraically constructible functions may be extended to more general classes of sets. Fairly complete results have already been obtained for arc-symmetric semialgebraic sets. Much less is known for real analytic sets.

Arc-symmetric sets were introduced by Kurdyka in [19] and applied in [20] to show that any injective regular self-map of a real algebraic variety is surjective. A semialgebraic subset $A$ of a real algebraic set $X$ is called arc-symmetric if for each real analytic arc $\gamma : (-\varepsilon, \varepsilon) \to X$ such that $\gamma((-\varepsilon,0)) \subset A$, the entire image $\gamma((-\varepsilon,\varepsilon))$ is contained in $A$. Then, by the curve selection lemma, $A$ is closed in $X$. For instance a connected component of a real algebraic subset of $X$ is arc-symmetric. For more examples see [19].
Let $X$ be a real algebraic set. A *Nash constructible function* on $X$ is an integer-valued function which can be written as a finite sum

$$\varphi = \sum m_i f_i 1_{Z'_i},$$

where for each $i$, $Z'_i$ is a connected component of an algebraic set $Z_i$, $f_i : Z_i \to X$ is a proper regular map, and $m_i$ is an integer. By [25], $A \subset X$ is arc-symmetric if and only if $A$ is closed in $X$ and the characteristic function of $A$ is Nash constructible. The half-link of a Nash constructible function is Nash constructible and hence for $A$ arc-symmetric $\tilde{\Lambda}(A) \subset F(A)$. This shows that every arc-symmetric set is Euler and Akbulut and King’s invariants vanish at each point of a 3-dimensional arc-symmetric set. Thus, by Theorem 1.3 and Remark 4.2, every 3-dimensional arc-symmetric semialgebraic set is homeomorphic to a real algebraic set.

Suppose, moreover, that $X$ is compact. Then, as shown by Bonnard in [9], most of the algebraic results presented in section 3 above have their analogs for Nash constructible functions. For instance, a function on $X$ is Nash constructible if and only if it is a finite sum of signs of blow-Nash functions. An analytic function on $X$ is called *Nash* if its graph is semialgebraic. It is called *blow-Nash* if it can be made Nash after composing with a finite sequence of blowings-up. There is a fan criterion for Nash constructible functions, and the minimum number of blow-Nash functions required to represent a Nash constructible function admits the same bounds as in Theorem 3.4. These bounds are again sharp.

If $X$ is compact then for every arc-symmetric semialgebraic subset $A \subset X$ of dimension 4, all $2^{43} - 43$ local invariants constructed in section 4 vanish at each $x \in A$. Without the assumption of compactness of $X$ we do not know whether the operator $P(\varphi) = \frac{1}{2}(\varphi^4 - \varphi^2)$ preserves Nash constructible functions on $X$, since we do not have a similar representation as finite sums of signs. We may conclude only that the $2^{29} - 29$ invariants constructed with the help of $\tilde{\Lambda}(A)$ vanish.

So far there has not been a similar study for real analytic sets. One natural possibility would be to define analytically constructible functions on a real analytic set $X$ as in (2.7) with the $f_i$ proper real analytic maps. The set of such functions forms a ring. By resolution of singularities this ring is preserved by the half-link operator $\tilde{\Lambda}$. Hence we conclude that real analytic sets are Euler, a result due to Sullivan [31]. Akbulut and King’s local invariants vanish on real analytic sets, as well as the $2^{29} - 29$ invariants constructed in section 4. Again, we do not know whether the operator $P(\varphi) = \frac{1}{2}(\varphi^4 - \varphi^2)$ preserves this family of constructible functions.

*Question 1.* Does there exist a topological proof that the operator $P(\varphi) = \frac{1}{2}(\varphi^4 - \varphi^2)$ preserves algebraically constructible functions? Does this follow from the resolution of singularities, for instance by the method of resolution towers?

All of the constraints on the topology of real algebraic, arc-symmetric, or real analytic sets obtained by our constructions are local, and we do not know whether any constraints of a global character exist.

*Question 2.* Let $X$ be a compact triangulable topological space and suppose that for each $x \in X$ the link $\text{lk}(x, X)$ is homeomorphic to a link of a real algebraic set. Is $X$ homeomorphic to a real algebraic set?
If $X$ is of dimension $\leq 3$ then the answer is positive by the results of Benedetti-Dedò and Akbulut-King (see section [1]).

**Question 3.** Let $X$ be a compact triangulable topological space of dimension 4 such that for each $x \in X$ the topological invariants of $\text{lk}(x, X)$ constructed in section [4] vanish. Is $X$ homeomorphic to a real algebraic set?

In other words we ask whether the vanishing of our invariants is sufficient for a set to be homeomorphic to a real algebraic set.

**Question 4.** Let $X$ be a compact real algebraic set of dimension 3 and suppose that all the topological invariants of $X$ constructed in section [4] vanish, i.e. $c([X]) = 0$. Is then $X$ homeomorphic to a link of a real algebraic set?

Equivalently, we ask whether the linear map $c : A'_4 \to (\mathbb{Z}/2)^N$, introduced section [4], is injective. A positive answer to Question 3 answers positively Question 4. These two questions are equivalent if we suppose a positive answer to Question [2] for sets of dimension 4.

One possible way to approach these questions is to reinterpret our invariants in terms of Akbulut and King’s resolution towers [3].

**Question 5.** Let $X$ be a compact triangulable topological space of dimension 4 such that for each $x \in X$ the topological invariants of $\text{lk}(x, X)$ constructed in section [4] vanish. Does $X$ admit a topological resolution tower in the sense of Akbulut and King [3]? What extra properties does such a resolution tower satisfy?

Akbulut and King [3] show that a resolution tower can be made algebraic if it satisfies some additional properties. These properties are stronger that those given by the resolution of singularities of real algebraic sets. Thus a positive answer to Question 5 would not necessarily guarantee a solution to Question 3. Previte [29] has done interesting work on obstructions for a 4-dimensional space to have an algebraic resolution tower.

Following the original idea of Sullivan one may study the links of real algebraic sets using complexification. The vanishing of Akbulut and King’s numbers on real algebraic sets can be proved using a relation between complex conjugation and the monodromy of the complex Milnor fibre of an ordered family of functions [23].

**Question 6.** Can one construct our $2^{43} - 43$ topological invariants in dimension 4 by means of complexification?

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