Dynamics of rimless wheel robots during collision

Yasushi Iwatani

Department of Science and Technology, Hirosaki University, Hirosaki, Japan

ABSTRACT
Rimless wheel robots are ground robots with high mobility on a variety of uneven terrains. Rimless wheels have no rims or tires but only spokes, and rimless wheel robots move along with collisions of spokes with the ground. Their equations of collision have been derived from a conservation law of angular momentum only for the simplest type of rimless wheel robots. It is uncertain when the conservation law is satisfied or violated, and what happens in the violated case. Complete equations of collision are unknown even for the simplest type of rimless wheel robots. In addition, this paper considers three types of rimless wheel robots including the simplest type and provides complete equations of collision for each type of rimless wheel robots. In addition, this paper describes properties of the dynamics during collision. Some of them reveal relationships between the dynamics of the three types of rimless wheel robots. Another shows a relationship between the dynamics of rimless wheels and circular wheels. They also clarify when the conservation law of angular momentum is satisfied or violated, and what happens in each case.

1. Introduction
Rimless wheel robots are ground robots whose wheels have no rims or tires but only spokes. Spokes are in contact with the ground instead of tires. The rimless wheel mechanism has been first introduced into robotics for theoretically analysing walking motions [1,2]. It has been then demonstrated that the rimless wheel and related mechanisms have high mobility on a variety of uneven terrains both indoors [3–9] and outdoors [10–12].

Rimless wheel robots move along with collisions of spokes with the ground. Although collisions necessarily occur in motion, equations of collision have not been carefully formulated for rimless wheel robots. In particular, equations of collision have been derived from a conservation law of angular momentum only for the simplest type of rimless wheel robots [13]. It is uncertain when the conservation law is satisfied or violated, and what happens in the violated case. Complete equations of collision are unknown even for the simplest type of rimless wheel robots.

This paper considers the following three types of rimless wheel robots: (i) a rimless wheel without a torso, which is the simplest type of rimless wheel robots, (ii) a rimless wheel with a torso connected by a backdrivable joint, and (iii) a rimless wheel with a torso connected by a non-backdrivable joint. They are the most fundamental among all rimless wheel mechanisms.

This paper provides complete equations of collision for each type of rimless wheel robots as follows. The dynamics during collision are considered in two cases: whether or not an impact force arises at the foot of the stance spoke. For each case, equations of collision are derived in a modelling framework for multi-body dynamics with impact formulated in [14,15]. In particular, they are given in closed form by the relationship between impulses of impact forces and change of momentum with constraints. Analysing physical validity of impact forces and foot velocities, we obtain a necessary and sufficient condition that the equations of collision hold for each of the two cases. It is shown that the two derived conditions are disjoint and cover the whole parameter space. The results described above were derived in a conference paper [16], but the conference paper just provided equations of collision.

As a further contribution, this paper describes properties of the dynamics during collision. They show that equations of collision derived in [16] are natural extensions of the existing equations of collision. In particular, some of them reveal relationships between the dynamics of the three types of rimless wheel robots. Another presents a relationship between the dynamics of rimless wheels and circular wheels. They also clarify when the conservation law of angular momentum is satisfied or violated, and what happens in each case. They are summarized in Section 5 which is an additional section.

In addition, as a minor contribution, this paper improves the proof of the main theorem that provides equations of collision. Although the proof in the conference paper explicitly treats only the case...
of clockwise rotations of wheels, this paper explicitly addresses both the cases of clockwise rotations and anticlockwise rotations. This improvement provides a uniform representation of closed-form solutions of equations of collision for both clockwise and anticlockwise rotations.

2. Rimless wheel robots

Let us consider a rimless wheel with \( n \) spokes for

\[ n \geq 3. \] (1)

The spokes are equal length and attached to the wheel centre at regular angular intervals. They are oriented in the radial direction. The centre of mass of the wheel is located at the wheel centre. This paper considers three types of rimless wheel robots illustrated in Figure 1. They are (i) the torso-less type: a rimless wheel without a torso, (ii) the backdrivable type: a rimless wheel with a torso connected by a backdrivable joint, and (iii) the non-backdrivable type: a rimless wheel with a torso connected by a non-backdrivable joint. This paper focuses only on the three types of rimless wheels, since they are the most fundamental among all rimless wheel mechanisms.

The outer end of a spoke is called a foot in this paper. Let us suppose that one foot is in contact with the ground, the wheel pivots on the foot, and then a next foot collides with the ground. The spoke whose foot is in contact with the ground before collision is called the stance spoke. The spoke whose foot collides with the ground is called the colliding spoke.

Notations used in this paper are summarized in Table 1. See also Figure 1. The \( y \) axis is directed in the normal direction to the slope. Note that the dynamics during collision are independent of the slope angle or an input torque at the joint in practice, since it is not affected by gravity or any continuous input torque \([14,15]\). The results obtained in this paper are available for non-flat surfaces such as curves and steps. If a wheel is on a non-flat surface, then the base line illustrated as the slope in Figure 1 is set to the straight line between the feet of the colliding spoke and the stance spoke.

This paper builds on the following assumptions.

**Assumption 2.1:** (1) One foot is in contact with the ground before collision.

(2) When a foot is in contact with the ground, no slips occur between the foot and the ground.

(3) When an impact force arises at a foot in the positive \( y \) direction, the impact force resets the velocity of the foot after collision in the \( y \) direction to zero.

| Table 1. Notations used in this paper. |
|----------------------------------------|
| \( x_w, y_w, \theta_w \) | The position and the angle of the wheel centre. |
| \( x_t, y_t, \theta_t \) | The position and the angle of the centre of mass of the torso. |
| \( x_s, y_s, x_c, y_c \) | The position of the foot of the stance spoke or the colliding spoke. |
| \( \sigma, \ast \pm \) | Values of variable \( \sigma \) before and after collision. |
| \( f_{x_s}, f_{y_s} \) | Impulses of an impact force at \( (x_s, y_s) \) in the \( x \) and \( y \) directions. |
| \( m_w, m_t \) | Masses of the wheel and the torso. |
| \( J_w, J_t \) | Moments of inertia about the centre of mass for the wheel and the torso. |
| \( \ell_w, \ell_t \) | The radius of the wheel, and the length between the wheel centre and the centre of mass of the torso. |
| \( n \) | The number of the spokes. \( n \geq 3 \). |
| \( (X)_{ij}, (X)_{i:,} \) | For a given matrix \( X \), the \((i,j)\)-th element of \( X \). For a given matrix \( X \), the \( i\)-th row vector of \( X \). |
| \( \pm, \mp \) | The upper signs are used for \( \dot{\theta}_w > 0 \), and the lower ones for \( \dot{\theta}_w < 0 \). |

Assumption (3) is an interpretation of perfectly inelastic collisions for multibody systems. For the foot of the colliding spoke, it simply supposes that collisions are perfectly inelastic. It assumes the same for the foot of the stance spoke.
If collisions are not perfectly inelastic, then the colliding spoke bounces off the ground for every collision. This leads to unnatural locomotion, and it should be improved. The unnaturalness comes from material and/or structural factors, since the coefficient of restitution is a physical constant. It can not be improved by controllers. Therefore, it is only necessary to consider perfectly inelastic collisions for natural locomotion of rimless wheel robots.

3. Equations of collision

Theorem 3.1: Under Assumptions (1)–(3), the followings hold true for the three types of rimless wheel robots during collision:

1. Torso-less type:
   \[
   \begin{align*}
   \dot{\theta}_{w+} &= \alpha_1 \dot{\theta}_{w-}, & \text{if } \alpha_1 \geq 0, \\
   \dot{\theta}_{w+} &= 0, & \text{otherwise},
   \end{align*}
   \]
   where
   \[
   \alpha_1 = \frac{f_w + m_w \ell_w^2 \cos \frac{\pi}{n}}{f_w + m_w \ell_w^2 n}. \tag{3}
   \]

2. Backdrivable type:
   \[
   \begin{align*}
   \dot{\theta}_+ &= C_{21} \dot{\theta}_-, & \text{if } \alpha_2(\theta_i) \geq 0, \\
   \dot{\theta}_+ &= C_{22} \dot{\theta}_-, & \text{otherwise},
   \end{align*}
   \]
   where
   \[
   C_{21} = \begin{bmatrix}
   \alpha_2(\theta_i) \\
   \beta(\theta_i)
   \end{bmatrix},
   C_{22} = \begin{bmatrix}
   0 & 0 \\
   \gamma(\theta_i) & 1
   \end{bmatrix}, \tag{6}
   \]
   and \( \alpha_2(\theta_i) \), \( \beta(\theta_i) \) and \( \gamma(\theta_i) \) are defined later by (61), (62), and (77), respectively.

3. Non-backdrivable type:
   \[
   \begin{align*}
   \dot{\theta}_+ &= C_{31} \dot{\theta}_-, & \text{if } \alpha_3(\theta_i) \geq 0, \\
   \dot{\theta}_+ &= C_{32} \dot{\theta}_-, & \text{otherwise},
   \end{align*}
   \]
   where
   \[
   C_{31} = \begin{bmatrix}
   \alpha_3(\theta_i) \\
   \alpha_3(\theta_i) - 1
   \end{bmatrix},
   C_{32} = \begin{bmatrix}
   0 & 0 \\
   -1 & 1
   \end{bmatrix}, \tag{8}
   \]
   and \( \alpha_3(\theta_i) \) is defined later by (88).

Before proceeding to the proof, the theorem is explained.

For the torso-less type, (2) with \( \alpha_1 \geq 0 \) is same as the equation of collision derived from the conservation law of angular momentum under an assumption that the stance spoke instantaneously loses contact with the ground during collision [13]. This implies that the result in this study is consistent with the existing result. Equations of collision for the other case and the other types are obtained in this study, while only (2) with \( \alpha_1 \geq 0 \) has been derived in the existing work.

The coefficient \( \alpha_1 \) is independent of the state variables and depends only on the mechanical parameters. Note that if \( n \geq 4 \), then \( \alpha_1 > 0 \). The other coefficients \( \alpha_2 \), \( \beta \), \( \gamma \) and \( \alpha_3 \) are functions of \( \theta_i \).

It is immediate to show that, if \( \alpha_1 \geq 0 \), then \( \dot{\theta}_{w+} > 0 \) and \( y_{w+} > 0 \). Otherwise, or equivalently if \( \alpha_1 < 0 \), then \( \dot{\theta}_{w+} = 0 \) and \( y_{w+} = 0 \). This does not state that the wheel rotation stops when \( \alpha_1 < 0 \). The motion after collision depends on angular accelerations which are given by Lagrange’s equations of motion. If \( \dot{\theta}_{w+} = 0 \) and \( \dot{\theta}_{w+} \neq 0 \), then the wheel still rotates after collision. In addition, its rotational direction changes, if the signs of \( \theta_{w-} \) and \( \theta_{w+} \) are different.

4. Proof of Theorem 3.1

4.1. Proof of 1: the torso-less type

From Figure 1, we have
\[
\begin{align*}
   x_w &= x_s + \ell_w \sin \theta_w, \tag{9} \\
   y_w &= y_s + \ell_w \cos \theta_w, \tag{10} \\
   x_c &= x_w - \ell_w \sin \left( \theta_w + \frac{2\pi}{n} \right), \tag{11} \\
   y_c &= y_w - \ell_w \cos \left( \theta_w + \frac{2\pi}{n} \right). \tag{12}
\end{align*}
\]
Note that the upper and lower signs in (11) and (12) are used for \( \dot{\theta}_{w-} > 0 \) and \( \dot{\theta}_{w-} < 0 \), respectively, as described in Table 1.

We consider only motion during collision from here until the end of the paper, and henceforth each variable has a value during collision. Combining (9)–(12) with Assumptions (1)–(3), we obtain
\[
\begin{align*}
   \dot{\theta}_w &= \pm \frac{\pi}{n}, \tag{13} \\
   \dot{x}_s &= \dot{x}_{w-} - \ell_w \dot{\theta}_{w-} \cos \frac{\pi}{n} = 0, \tag{14} \\
   \dot{y}_s &= \dot{y}_{w-} - \ell_w \dot{\theta}_{w-} \sin \frac{\pi}{n} = 0, \tag{15} \\
   \dot{x}_c &= \dot{x}_{w+} - \ell_w \dot{\theta}_{w+} \cos \frac{\pi}{n} = 0, \tag{16} \\
   \dot{y}_c &= \dot{y}_{w+} - \ell_w \dot{\theta}_{w+} \sin \frac{\pi}{n} = 0. \tag{17}
\end{align*}
\]
during collision.

The proof is divided into two cases: (a) an impact force does not arise at the foot of the stance spoke, and (b) an impact force arises at the foot of the stance spoke.

Case (a): Suppose that an impact force does not arise at the foot of the stance spoke, or equivalently, that
\( f_{x_2} = f_{y_2} = 0 \). Then, an impact force at the foot of the colliding spoke changes linear and angular momenta \([14,15]\). The relationship is given by

\[
m_w \dot{x}_{w+} = m_w \dot{x}_{w-} + f_{x_2}, \tag{18}
\]

\[
m_w \dot{y}_{w+} = m_w \dot{y}_{w-} + f_{y_2}, \tag{19}
\]

\[
f_w \theta_{w+} = f_w \theta_{w-} - (y_w - y_c) f_{xc} + (x_w - x_c) f_{yc}, \tag{20}
\]

where (20) follows from the clockwise definition of \( \theta_{w-} \).

From the above equations, we obtain

\[
\begin{bmatrix}
m_w & 0 & 0 & -1 & 0 \\
0 & m_w & 0 & 0 & -1 \\
0 & 0 & J_w & \ell_w \cos \frac{\pi}{n} & + \pm \ell_w \sin \frac{\pi}{n} \\
-1 & 0 & \ell_w \cos \frac{\pi}{n} & 0 & 0 \\
0 & -1 & \pm \ell_w \sin \frac{\pi}{n} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_{w+} \\
\dot{y}_{w+} \\
\dot{\theta}_{w+} \\
f_{xc} \\
f_{yc}
\end{bmatrix}
\times

\begin{bmatrix}
m_w \ell_w \cos \frac{\pi}{n} \\
0 \\
0 \\
J_w \\
0
\end{bmatrix}

\begin{bmatrix}
\dot{x}_{w-} \\
\dot{y}_{w-} \\
\dot{\theta}_{w-} \\
f_{xc} \\
f_{yc}
\end{bmatrix}

= \begin{bmatrix}
m_w \ell_w \cos \frac{\pi}{n} \\
0 \\
0 \\
J_w \\
0
\end{bmatrix}

\begin{bmatrix}
\pm m_w \ell_w \sin \frac{\pi}{n} \\
\theta_{w-}, \tag{21}
\end{bmatrix}
\]

where the first equality follows from (16)–(20) with (11)–(13), and the second equality is given from (14) and (15). The matrix equation (21) can be written as

\[
A_{11} \lambda_{11} = b_{11} \theta_{w-}, \tag{22}
\]

where \( A_{11} \) is the 5 \times 5 matrix that appears in the left-hand side of (21), \( b_{11} \) is the 5 \times 1 vector in the right-hand side of (21), and

\[
\lambda_{11} = [\dot{x}_{w+} \quad \dot{y}_{w+} \quad \dot{\theta}_{w+} \quad f_{xc} \quad f_{yc}]^T. \tag{23}
\]

It is straightforward to see that

\[
det A_{11} = m_w \ell_w^2 + J_w \neq 0. \tag{24}
\]

Thus, from (22), we obtain

\[
\dot{\theta}_{w+} = \alpha_1 \dot{\theta}_{w-}, \tag{25}
\]

\[
2 \dot{\theta}_{w-} m_w \ell_w \left( J_w + m_w \ell_w^2 \cos \frac{\pi}{n} \right) \sin \frac{\pi}{n} = f_{yc}, \tag{26}
\]

\[
y_{s+} = y_{w+} \pm \ell_w \dot{\theta}_{w+} \sin \frac{\pi}{n} = \pm 2 \alpha_1 \dot{\theta}_{w-} \ell_w \sin \frac{\pi}{n}. \tag{27}
\]

Equation (22) is valid, if and only if all of \( \dot{y}_{c+}, \dot{y}_{s+}, f_{yc} \) and \( f_{y_2} \) are non-negative. The assumptions guarantee that \( \dot{y}_{c+} = 0 \) and \( f_{y_2} = 0 \). It is obvious from (1) and (26) that \( f_{yc} \) is always non-negative. It can be seen from (1) and (27) that \( y_{s+} \) is non-negative, if and only if \( \alpha_1 \geq 0 \). Thus, (25) is valid if \( \alpha_1 \geq 0 \).

**Case (b):** Suppose that an impact force arises at the foot of the stance spoke. The modelling framework in \([14,15]\) assumes that an impact force arises as a constraint force. This with Assumptions (2) and (3) implies that an impact force arises at the foot of the stance spoke, when the following constrains are imposed:

\[
\dot{x}_{s+} = x_{w+} - \ell_w \dot{\theta}_{w+} \cos \frac{\pi}{n} = 0, \tag{28}
\]

\[
y_{s+} = y_{w+} \pm \ell_w \dot{\theta}_{w+} \sin \frac{\pi}{n} = 0. \tag{29}
\]

Linear and angular momenta satisfy

\[
m_w \dot{x}_{w+} = m_w \dot{x}_{w-} + f_{xc} + f_{x_2}, \tag{30}
\]

\[
m_w \dot{y}_{w+} = m_w \dot{y}_{w-} + f_{yc} + f_{y_2}, \tag{31}
\]

\[
f_w \dot{\theta}_{w+} = f_w \dot{\theta}_{w-} - (y_w - y_c) f_{xc} - (y_w - y_s) f_{xc} + (x_w - x_c)f_{yc} + (x_w - x_s)f_{yc}. \tag{32}
\]

As similar to (21), we can write (16)–(17) and (28)–(32) in a matrix form as

\[
\dot{\tilde{A}}_{12} \tilde{x}_{12} = \tilde{b}_{12} \dot{\theta}_{w-}, \tag{33}
\]

where Equations (9)–(15) are used, and

\[
\begin{bmatrix}
m_w & 0 & 0 & -1 & -1 \\
0 & m_w & 0 & 0 & 0 \\
0 & 0 & J_w & \ell_w \cos \frac{\pi}{n} & + \pm \ell_w \sin \frac{\pi}{n} \\
-1 & 0 & \ell_w \cos \frac{\pi}{n} & 0 & 0 \\
-1 & 0 & \ell_w \cos \frac{\pi}{n} & 0 & 0 \\
0 & -1 & \pm \ell_w \sin \frac{\pi}{n} & 0 & 0 \\
0 & -1 & \pm \ell_w \sin \frac{\pi}{n} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_{w+} \\
\dot{y}_{w+} \\
\dot{\theta}_{w+} \\
f_{xc} \\
f_{yc}
\end{bmatrix}
\times

\begin{bmatrix}
m_w \ell_w \cos \frac{\pi}{n} \\
0 \\
0 \\
J_w \\
0
\end{bmatrix}

\begin{bmatrix}
\dot{x}_{w-} \\
\dot{y}_{w-} \\
\dot{\theta}_{w-} \\
f_{xc} \\
f_{yc}
\end{bmatrix}

= \begin{bmatrix}
m_w \ell_w \cos \frac{\pi}{n} \\
0 \\
0 \\
J_w \\
0
\end{bmatrix}

\begin{bmatrix}
\pm m_w \ell_w \sin \frac{\pi}{n} \\
\theta_{w-}, \tag{34}
\end{bmatrix}
\]

\[
\dot{\tilde{b}}_{12} = \begin{bmatrix}
m_w \ell_w \cos \frac{\pi}{n} + \pm m_w \ell_w \sin \frac{\pi}{n} \\
0 \\
0 \\
0 \\
0
\end{bmatrix}, \tag{35}
\]

\[
\tilde{x}_{12} = \begin{bmatrix}
\dot{x}_{w+} \\
\dot{y}_{w+} \\
\dot{\theta}_{w+} \\
f_{xc} \\
f_{yc}
\end{bmatrix}. \tag{36}
\]

It is obvious that

\[
det \tilde{A}_{12} = 0, \tag{37}
\]
since the fourth row of $\hat{A}_{12}$ is equal to the fifth row, and the forth column is equal to the fifth column. In order to avoid the singularity, we rewrite (33) as

$$A_{12} \lambda_{12} = b_{12} \dot{\theta}_{w-},$$

(38)

where

$$A_{12} = \begin{bmatrix}
  m_w & 0 & 0 & -1 \\
  0 & m_w & 0 & 0 \\
  0 & 0 & f_w & \ell_w \cos \frac{\pi}{n} \\
  -1 & 0 & \ell_w \cos \frac{\pi}{n} & 0 \\
  0 & 1 & \pm \ell_w \sin \frac{\pi}{n} & 0 \\
  0 & 1 & \mp \ell_w \sin \frac{\pi}{n} & 0 \\
  0 & 0 & 0 & 1 \\
  \pm \ell_w \sin \frac{\pi}{n} & \mp \ell_w \sin \frac{\pi}{n} & 0 & 0 \\
  0 & 0 & 0 & 0 \\
\end{bmatrix},$$

(39)

$$b_{12} = \begin{bmatrix}
  m_w \ell_w \cos \frac{\pi}{n} \pm m_w \ell_w \sin \frac{\pi}{n} \sin f_w 0 0 0 \end{bmatrix}^T,$$

(40)

$$\lambda_{12} = \begin{bmatrix}
  \tilde{x}_{w+} - \tilde{y}_{w+} + f_x + f_s \\
  \tilde{y}_{w+} + f_{y+} + f_{y+} \\
  \alpha_1 (f_w + m_{w_2} \ell_w^2) \dot{\theta}_{w-} \\
\end{bmatrix}^T.$$

(41)

Although $f_x$ and $f_s$ can not be derived separately from (38), this causes no problem for obtaining $\dot{\theta}_{w+}$. It is straightforward to see that

$$\det A_{12} = -4 \ell_w^2 \sin^2 \frac{\pi}{n} \neq 0,$$

(42)

since $n \geq 3$. Thus, from (38), we obtain

$$\dot{\theta}_{w+} = 0,$$

(43)

$$f_{y+} = \alpha_1 (f_w + m_{w_2} \ell_w^2) \dot{\theta}_{w-},$$

(44)

Equation (38) is valid, if and only if $f_{y+}$ is positive and all of $\dot{y}_{c+}$, $\dot{y}_{t+}$, and $f_{y+}$ are non-negative. The assumptions guarantee that $\dot{y}_{c+} = 0$ and $\dot{y}_{t+} = 0$. It is obvious from (1) and (44) that $f_{y+}$ is always non-negative. It can be seen from (1) and (45) that $f_{y+}$ is positive, if and only if $\alpha_1 < 0$. Thus, (43) is valid if $\alpha_1 < 0$.

Summarizing the two cases, we conclude the proof.

### 4.2. Proof of 2: the backdrivable type

From geometry, we have

$$\dot{x}_t = x_w - \ell_t \sin \theta_t,$$

(46)

$$\dot{y}_t = y_w - \ell_t \cos \theta_t,$$

(47)

The joint constraint can be written by

\[
\dot{x}_{t-} = \dot{x}_{w-} - \ell_t \dot{\theta}_{t-} \cos \theta_t, \\
\dot{y}_{t-} = \dot{y}_{w-} + \ell_t \dot{\theta}_{t-} \sin \theta_t, \\
\dot{y}_{t+} = \dot{x}_{w+} - \ell_t \dot{\theta}_{t+} \cos \theta_t, \\
\dot{y}_{t+} = \dot{y}_{w+} + \ell_t \dot{\theta}_{t+} \sin \theta_t. 
\]

(48)

(49)

(50)

(51)

An impact force arises at the joint in this case. Without loss of generality, suppose that $f_{xw}$ and $f_{yw}$ are directed from the wheel to the torso, and their reactions are directed from the torso to the wheel. The rest of the proof is similar to 1.

**Case (a):** Suppose that an impact force does not arise at the foot of the stance spoke, or equivalently, that $f_{x1} = f_{y1} = 0$. Note that $f_{xw}$ and $f_{yw}$ are applied at the centre of mass of the wheel, and they do not affect the angular momentum of the wheel. Therefore, linear and angular momenta satisfy

$$m_w \dot{x}_{w+} = m_w \dot{x}_{w-} + f_{xw} - f_{xw},$$

(52)

$$m_w \dot{y}_{w+} = m_w \dot{y}_{w-} + f_{yw} - f_{yw},$$

(53)

$$J_w \dot{\theta}_{w+} = J_w \dot{\theta}_{w-} - (y_{w} - y_{c}) f_{xw} + (x_{w} - x_{c}) f_{yw},$$

(54)

$$m_w \dot{x}_t = m_w \dot{x}_{t-} + f_{xw},$$

(55)

$$m_w \dot{y}_t = m_w \dot{y}_{t-} + f_{yw},$$

(56)

$$J_t \dot{\theta}_t = J_t \dot{\theta}_{t-} - (y_t - y_{w}) f_{xw} + (x_t - x_{w}) f_{yw}.$$

(57)

Introducing matrices $A_{21} \in \mathbb{R}^{10 \times 10}$ and $B_{21} \in \mathbb{R}^{10 \times 2}$, we can write (16)–(17) and (50)–(57) in a matrix form as

$$A_{21} \lambda_{21} = B_{21} \dot{\theta}_-,$$

(58)

where

$$\lambda_{21} = \begin{bmatrix}
  \dot{x}_{w+} & \dot{y}_{w+} & \dot{\theta}_{w+} & \dot{x}_{t+} & \dot{y}_{t+} & \dot{\theta}_{t+} \\
  f_{x+} & f_{y+} & f_{xw} & f_{yw} \\
\end{bmatrix}^T.$$

(59)

It is straightforward to see that

$$\det A_{21} = (J_w + m_w \ell_w^2)(J_t + m_t \ell_t^2)$$

$$+ m_t \ell_t^2 \left[ J_t + m_t \ell_t^2 \sin^2 \left( \theta_t \pm \frac{\pi}{n} \right) \right] > 0.$$  

(60)

We here define

$$\alpha_2(\theta_t) = (A_{21}^{-1} B_{21})_{4,1},$$

(61)

$$\beta(\theta_t) = (A_{21}^{-1} B_{21})_{6,1}.$$  

(62)

The functions $\alpha_2(\theta_t)$ and $\beta(\theta_t)$ are represented in closed form as

$$\alpha_2(\theta_t) = \frac{\tilde{\alpha}_2(\theta_t)}{\det A_{21}},$$

(63)

$$\beta(\theta_t) = \frac{m_t \ell_t \ell_w \tilde{\beta}(\theta_t)}{\det A_{21}},$$

(64)
\[ \ddot{\theta}_i = f_x \left( J_x + m_t \ell_i^2 \right) + m_t^2 \ell_i^2 \theta_i \left( \cos^2 \frac{\pi}{n} - \cos^2 \theta_i \right) \]
\[ + \left\{ (m_w + m_t) J_l + m_t m_w \ell_i^2 \right\} \ell_i^2 \cos \frac{2\pi}{n}, \]
\[ \tilde{\beta}(\theta) = f_x \left( \cos \left( \theta \mp \frac{\pi}{n} \right) - \cos \left( \theta \mp \frac{\pi}{n} \right) \right) \]
\[ + \frac{(m_l + m_w) \ell_i^2}{2} \left\{ \cos \left( \theta_i \mp \frac{3\pi}{n} \right) \right\} \]
\[ - \cos \left( \theta_i \mp \frac{\pi}{n} \right). \]

Then, from (58), we have
\[ \theta_{w+} = C_{21} \theta_{w-}, \quad f_{y_+} = \pm \frac{2c_{21} \theta_{w-} - \ell_w \sin \frac{\pi}{n}}{\det A_{21}}, \quad \dot{y}_{w+} = \dot{y}_{w-} \pm \ell_w \theta_{w-} \sin \frac{\pi}{n} = \pm 2\alpha_2 \theta_{w-} \ell_w \sin \frac{\pi}{n}, \]
where
\[ c_{21} = J_x J_w (m_l + m_w) + (m_w + m_t \cos^2 \theta_i) J_w m_t \ell_i^2 \]
\[ + \left\{ J_l (m_l + m_w) + m_t m_w \ell_i^2 \right\} \times (m_l + m_w) \ell_i^2 \cos^2 \frac{\pi}{n} > 0. \]

Case (b): Suppose that an impact force arises at the foot of the stance. Then linear and angular momenta satisfy (55)–(57) and
\[ m_w \dot{x}_{w+} = m_w \dot{x}_{w-} + f_x \pm f_{x_+} - f_{x_+}, \quad m_w \dot{y}_{w+} = m_w \dot{y}_{w-} + f_y \pm f_{y_+} - f_{y_+}, \quad J_c \dot{\theta}_{w+} = J_c \dot{\theta}_{w-} - (y_w - y_{c+}) f_{x_+} - (x_w - x_{c+}) f_{y_+} - f_{\tau_w}, \]
\[ J_c \dot{\theta}_{w+} = J_c \dot{\theta}_{w-} + (y_w - y_{c-}) f_{x_+} + (x_w - x_{c-}) f_{y_+} + f_{\tau_w}, \]
Introducing matrices \( A_{22} \in \mathbb{R}^{11 \times 11} \) and \( B_{22} \in \mathbb{R}^{11 \times 2} \), we can write (16)–(17), (28)–(29), (50)–(51), (55)–(57), and (71)–(73) in a matrix form as
\[ A_{22} \lambda_{22} = B_{22} \dot{\theta}_-, \quad \text{det} A_{22} = -4c_{21}^2 \left( J_l + m_t \ell_i^2 \right) \sin^2 \frac{\pi}{n} < 0, \]
since \( n \geq 3 \). We here define
\[ \gamma(\theta_i) = \left( A_{22}^{-1} B_{22} \right)_{6,1} = -\frac{m_t \ell_i \ell_w \cos \left( \theta_i \mp \frac{\pi}{n} \right)}{J_l + m_t \ell_i^2}. \]

Then, from (74), we have
\[ \theta_{+} = C_{22} \theta_-, \quad f_{y+} = \pm \frac{2c_{22} \ell_w \sin \frac{\pi}{n}}{\det A_{22}} \dot{\theta}_{w-}, \quad f_{y+} = \pm \frac{2\alpha_2 \ell_w \left( \det A_{21} \right) \sin \frac{\pi}{n}}{\det A_{22}} \dot{\theta}_{w-}, \]
where
\[ c_{22} = (J_l + m_t \ell_i^2) (m_w + m_t \ell_i^2) + J_t m_t \ell_i^2 \]
\[ + m_t^2 \ell_i^2 \ell_w \left\{ 1 - \cos^2 \left( \theta_i \mp \frac{\pi}{n} \right) \right\} > 0. \]

The rest of the proof is omitted, since it is very similar to 1.

4.3. Proof of 3: the non-backdrivable type

The relative angular velocity between the wheel and the torso is invariant during collision due to the non-backdrivability. This yields
\[ \dot{\theta}_{t+} - \dot{\theta}_{w+} = \dot{\theta}_{t-} - \dot{\theta}_{w-}. \]

An impact force and an impact torque arise at the joint in this case. Without loss of generality, suppose that an impact torque \( \tau_w \) is applied at the joint from the wheel to the torso. The rest of the proof is similar to 2.

Case (a): Suppose that an impact force does not arise at the foot of the stance, or equivalently, that \( f_{x+} = 0 \). Then angular momenta satisfy
\[ J_c \dot{\theta}_{w+} = J_c \dot{\theta}_{w-} - (y_w - y_{c+}) f_{x_+} - (x_w - x_{c+}) f_{y_+} - f_{\tau_w}, \]
\[ J_c \dot{\theta}_{t+} = J_c \dot{\theta}_{t-} + (y_t - y_{c+}) f_{x_+} + (x_t - x_{c+}) f_{y_+} + f_{\tau_w}, \]
and linear momenta are given by (52)–(53) and (55)–(56). Introducing matrices \( A_{31} \in \mathbb{R}^{11 \times 11} \) and \( B_{31} \in \mathbb{R}^{11 \times 2} \), we can write (16)–(17), (50)–(53), (55)–(56), and (82)–(84) in a matrix form as
\[ A_{31} \lambda_{31} = B_{31} \dot{\theta}_-, \quad \text{det} A_{31} = -J_l - J_w - (m_t \ell_w - \ell_i)^2 - m_w \ell_w \]
\[ -2m_t \ell_i \ell_w \left\{ 1 - \cos \left( \theta_i \mp \frac{\pi}{n} \right) \right\} < 0. \]
We here define
\[ \alpha_3(\theta_t) = \left( A_{31}^{-1} B_{31} \right)_{33} = -\frac{\tilde{\alpha}_3(\theta_t)}{\det A_{31}}, \]  
(88)
\[ \tilde{\alpha}_3(\theta_t) = J_t + J_w + m_t \ell_t^2 + (m_w + m_t) \ell_w^2 \cos \frac{2\pi}{n} 
- 2m_t \ell_t \ell_w \cos \theta_t \cos \frac{\pi}{n}. \]  
(89)

Then, from (85), we have
\[ \dot{\theta}_+ = C_{31} \dot{\theta}_-, \]
(90)
\[ f_{\chi} = \mp \frac{2c_{31} \theta_{w-} \ell_w \sin \frac{\pi}{n}}{\det A_{31}}, \]
(91)
\[ y_{+w} = y_{w+} \pm \ell \dot{\theta}_{w+} \sin \frac{\pi}{n} = \pm 2\alpha_3 \dot{\theta}_w \ell_w \sin \frac{\pi}{n}, \]
(92)

where
\[ c_{31} = (J_t + J_w)(m_t + m_w) + m_t m_w \ell_t^2 
+ \left( m_t \ell_t \cos \theta_t - (m_t + m_w) \ell_w \cos \frac{\pi}{n} \right)^2 > 0. \]
(93)

Case (b): Suppose that an impact force arises at the foot of the stance spoke. Then, angular momenta satisfy (84) and
\[ J_w \dot{\theta}_{w-} = J_w \dot{\theta}_{w+} = (\dot{y}_w - y_\chi) f_{\chi} - (\dot{x}_w - x_\chi) f_{\chi} 
+ (\dot{x}_w - x_\chi) f_{\chi} + (\dot{x}_w - x_\chi) f_{\chi} - \tau_w, \]
(94)

and linear momenta are given by (55)–(56) and (71)–(72). Introducing matrices \( A_{32} \in \mathbb{R}^{12 \times 12} \) and \( B_{32} \in \mathbb{R}^{12 \times 2} \), we can write (16)–(17), (28)–(29), (50)–(51), (55)–(56), (71)–(72), (82), (84) and (94) in a matrix form as
\[ A_{32} \lambda_{32} = B_{32} \dot{\lambda}_-, \]
(95)

where
\[ \lambda_{32} = \begin{bmatrix} x_{w+} & y_{w+} & \dot{x}_{w+} & \dot{y}_{w+} & x_t+ & y_t+ & \dot{x}_t+ & \dot{y}_t+ & \ell_t+ \\ f_{\chi} & f_{\chi} & f_{\chi} & f_{\chi} & f_{\chi} & f_{\chi} & f_{\chi} & f_{\chi} & \tau_w \end{bmatrix}^\top. \]
(96)

It is straightforward to see that
\[ \det A_{32} = 4 \ell_w^2 \sin^2 \frac{\pi}{n} > 0, \]
(97)

since \( n \geq 3 \). From (95), we have
\[ \dot{\lambda}_+ = C_{32} \dot{\lambda}_-, \]
(98)
\[ f_{\chi} = \pm \frac{c_{32}}{2 \ell_w \sin \frac{\pi}{n}} \theta_{w-}, \]
(99)
\[ f_{\chi} = \pm \frac{2\alpha_3 \ell_w (\det A_{31}) \sin \frac{\pi}{n}}{\det A_{32}} \theta_{w-}, \]
(100)

where
\[ c_{32} = J_t + J_w + m_t \ell_t^2 + m_t (\ell_t - \ell_w)^2 
+ 2m_t \ell_t \ell_w \left\{ 1 - \cos \left( \theta_t \mp \frac{\pi}{n} \right) \right\} > 0. \]
(101)

The rest of the proof is omitted, since it is very similar to 1.

5. Properties of the dynamics

This section contains four corollaries that describe properties of the dynamics. Proofs of the first three corollaries are omitted. They are immediate from closed-form solutions of equations of collision shown in the previous section, although they are not trivial.

**Corollary 5.1:** The coefficients \( \alpha_1, \alpha_2, \alpha_3, \beta \) and \( \gamma \) are bounded with respect to \( n \) and \( \theta_t \).

Corollary 5.1 confirms the validity of the mathematical models in Theorem 3.1. It guarantees bounded change of angular velocities during collision. Conversely, an unbounded model lacks validity, since it allows an infinitely large amount of velocity change. In [7], a mathematical model has been derived for a rimless wheel robot with a torso, although its modelling process is unclear. The model in [7] does not have the boundedness.

**Corollary 5.2:** The coefficients \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) go to 1 for any \( \theta_t \) as \( n \) approaches to infinity. In addition, the matrices \( C_{31} \) and \( C_{32} \) tend to the identity matrix for any \( \theta_t \), as \( n \) approaches to infinity.

Corollary 5.2 shows that the collisional effect vanishes as \( n \) becomes infinite. Note that circular wheels generate no collision. Combining these two facts, we see that the dynamics of rimless wheels tend to those of circular wheels, as \( n \) goes to infinity. This is consistent, since the shapes of rimless wheels tend to circles, as the number of spokes approaches to infinity. Corollary 5.2 presents a relationship between the dynamics of rimless wheels and circular wheels.

**Corollary 5.3:** It holds that
\[ C_{ij} = M_{ij}^{-1} N_{ij}, \]
(102)
for \( i = 2, 3 \) and \( j = 1, 2 \), where
\[ M_{21} = \begin{bmatrix} J_w + (m_t + m_w) \ell_w^2 
- m_t \ell_t \ell_w \cos \theta_t \pm \frac{\pi}{n} \\ -m_t \ell_t \ell_w \cos \left( \theta_t \pm \frac{\pi}{n} \right) \\ J_t + m_t \ell_t^2 \end{bmatrix}, \]
(103)
The above discussion shows that $M_{ij}$ and $N_{ij}$ can be chosen as (113) and (114) are satisfied. Equations (113) and (114) are exploited in the proof of the next corollary.

Finally, the following corollary shows a relationship between the dynamics of the three types of rimless wheel robots. It also clarifies a common physical property in their dynamics. Furthermore, it gives a physical meaning to (113) for the backdrivable type of rimless wheel robots.

**Corollary 5.4:**

- For any $n$ and any $\theta_i$, both $\alpha_2$ and $\alpha_3$ tend to $\alpha_1$, as $m_l$ and $J_t$ approach to zero.
- For each type of rimless wheel robots, if $\alpha_i \geq 0$, then the total angular momentum around the foot of the colliding spoke is conserved during collision.
- For the backdrivable type of rimless wheel robots, the angular momentum of the torso around the joint is conserved during collision.

**Proof:** The first part is immediate from closed-form solutions of equations of collision.

The second part with $i = 1$ has already been discussed in Section 3. Consider a rimless wheel robot with a torso. The total angular momenta around the foot of the colliding spoke before and after collision are given by

$$L_{a-} = J_w \dot{\theta}_{w-} + m_w(y_w - y_c)x_{w-} - m_w(x_w - x_c)\dot{y}_{w-} + J_t \dot{\theta}_{t-} + m_t(y_t - y_c)x_{t-} - m_t(x_t - x_c)\dot{y}_{t-},$$

(115)

$$L_{a+} = J_w \dot{\theta}_{w+} + m_w(y_w - y_c)x_{w+} - m_w(x_w - x_c)\dot{y}_{w+} + J_t \dot{\theta}_{t+} + m_t(y_t - y_c)x_{t+} - m_t(x_t - x_c)\dot{y}_{t+},$$

(116)

respectively. Substituting (9)–(17) and (46)–(51) to the above equations yields

$$L_{a-} = (N_{31})_1 \dot{\theta}_{-},$$

(117)

$$L_{a+} = (M_{31})_1 \dot{\theta}_{+},$$

(118)

where $M_{31}$ and $N_{31}$ are defined in (107) and (108), respectively. From Corollary 5.3 and (114)–(118), we obtain the conclusion of the second part.

Let $L_{w-}$ and $L_{w+}$ denote the angular momenta of the torso around the joint before and after collision, respectively. Then, we have

$$L_{w-} = J_t \dot{\theta}_{t-} + m_t(y_t - y_c)x_{t-} - m_t(x_t - x_c)\dot{y}_{t-} = (N_{21})_2 \dot{\theta}_{-} = (N_{22})_2 \dot{\theta}_{-},$$

(119)

$$L_{w+} = J_t \dot{\theta}_{t+} + m_t(y_t - y_c)x_{t+} - m_t(x_t - x_c)\dot{y}_{t+} = (M_{21})_2 \dot{\theta}_{+} = (M_{22})_2 \dot{\theta}_{+},$$

(120)

as similar to (115)–(118). They conclude the third part.
6. Concluding remarks

In an existing research, equations of collision have been derived from a conservation law of angular momentum only for the torso-less type of rimless wheel robots. It was uncertain when the conservation law is satisfied or violated, and what happens in the violated case. This paper provided complete equations of collision for three fundamental types of rimless wheel robots. The derived equations of collision clarify when the conservation law of angular momentum is satisfied or violated, and what happens in each case. This paper also reveals dynamical properties of rimless wheel robots during collision.

The results in this paper are useful for analysis, control and design of rimless wheel robot in future studies. This paper focused only on two-dimensional motion of single-wheel systems with perfectly inelastic collisions. The modelling process in this paper follows from a general-purpose modelling framework for multi-body dynamics with impact formulated in [14,15]. The framework covers three-dimensional motion, inelastic collisions, and perfectly elastic collisions. Therefore, the modelling process in this paper may be extended to such the cases and multi-wheel systems, which is left as future work.

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Notes on contributor

Yasushi Iwatani received his B.S., M.S., and Ph.D. degrees from the Tokyo Institute of Technology, in 1999, 2001 and 2004, respectively. He was a research fellow of the Japan Society for the Promotion of Science from 2003 to 2005, and a faculty member at Tohoku University from 2005 to 2009. In 2009, he joined Hirosaki University, where he is currently an Associate Professor of the Department of Science and Technology. His current research interests are in robotics and its applications to biology.

ORCID

Yasushi Iwatani https://orcid.org/0000-0003-1446-8176

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