Non-anticommutative ABJ Theory

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Abstract

In this paper we will discuss non-anticommutative deformations of the harmonic superspace. We will analyse the non-anticommutative deformation of the superspace that break the supersymmetry from \( \mathcal{N} = 3 \) supersymmetry to \( \mathcal{N} = 2 \) supersymmetry. We will then study the ABJ theory in this non-anticommutative superspace. This deformed ABJ theory will be shown to possess \( \mathcal{N} = 5 \) supersymmetry.

1 Introduction

In four dimensions \( \mathcal{N} = 2 \) supersymmetry has been studied in harmonic superspace \([1, 2]\), and this has been adopted for analysing \( \mathcal{N} = 3 \) supersymmetry in three dimensions \([3, 4, 5]\). The harmonic superspace variable parameterize the coset \( SU(2)/U(1) \) and are well suited for analysing theories with high amount of supersymmetry. Thus, harmonic superspace has been used for analysing the ABJM theory \([6]\), which is a superconformal Chern-Simons-matter theory with manifest \( \mathcal{N} = 6 \) supersymmetry \([7, 8, 11, 12]\). This theory is thought to be a low energy description of \( N \) M2-branes on \( C^4/Z_k \) orbifold because it coincides with the BLG theory for the only known example of a Lie 3-algebra \([13, 14, 15, 16]\). So, the supersymmetry of the ABJM theory is expected to be enhanced to full \( \mathcal{N} = 8 \) supersymmetry for \( k = 1, 2 \) \([17, 18]\). The gauge fields in the ABJM theory are governed by the Chern-Simons action and the matter fields live in the bifundamental representation of the gauge group \( U(N) \times U(N) \). A generalization of the ABJM theory to the case where the matter fields live in the bifundamental representation of gauge group \( U(M) \times U(N) \) with \( M \neq N \) has been made \([19, 20, 21, 22]\). This theory is called the ABJ theory and it also has \( \mathcal{N} = 6 \) supersymmetry. However, unlike the ABJM theory, non-planar corrections to the two-loop dilatation generator of ABJ theory mix states with positive and negative parity, and this mixing is proportional to \( M - N \) \([23]\). So, for \( M = N \), when the ABJ theory reduces to the ABJM theory, there is no mixing.

In string theory the \( NS \) background causes a noncommutative deformation between the spacetime coordinates \([26, 27, 28, 29]\), and a gravitino background causes a noncommutative deformation between the spacetime and Grassmann coordinates \([30, 31, 33, 34]\). All these deformations preserve the supersymmetry of the theory. However, the presence of a \( RR \) background causes a deformation
between the Grassmann coordinates and thus breaks the a certain amount of the supersymmetry of the theory \[35, 36, 37, 38, 39, 40\]. In four dimensions this can give rise to a fractional amount of supersymmetry like \( \mathcal{N} = 1/2 \) supersymmetry. Non-anticommutative deformation of harmonic superspace has also been analysed \[42, 43, 44, 45\].

As M-theory is dual to type II string theory, a deformation of the string theory side will also generate a deformation on the M-theory side. Thus, a noncommutative deformation of the M-theory will be dual to a noncommutative deformation of type II string theory caused by \( NS \) background. Similarly, a non-anticommutative deformation of the M-theory will be dual to a non-anticommutative deformation of type II string theory caused by \( RR \) background. In fact, this duality can be explicitly verified by using the novel Higgs mechanism \[46, 47, 48, 49\]. Thus, if we perform the higgsing of non-anticommutative M2-branes we will obtain non-anticommutative D2-branes. So, the non-anticommutative ABJ theory is dual to type II string theory deformed by a \( RR \) background. Noncommutative deformation of the M2-branes in \( \mathcal{N} = 1 \) superspace have been already studied \[12, 51\]. However, non-anticommutative deformation of the M2-branes has not been studied.

This non-anticommutative deformation of the M2-branes can occur due to the presence of a curved three form field \( C_{\mu \nu \tau} \). This is because the ABJM theory is the boundary gauge theory dual to the eleven dimensional supergravity on \( AdS_4 \times S_7/Z_k \). A deformation of this eleven dimensional supergravity on \( AdS_4 \times S_7/Z_k \) can be caused by a three form field \( C_{\mu \nu \tau} \). A constant three form field \( C_{\mu \nu \tau} \) is only expected to change the gauge group of the theory without breaking any supersymmetry. Thus, the ABJ theory can be viewed as a deformation of the ABJM theory. However, a deformation of the eleven dimensional supergravity on \( AdS_4 \times S_7/Z_k \) by a curved three form field \( C_{\mu \nu \tau} \) will change the geometry considerably and is expected to partially break the supersymmetry. The boundary theory dual to this deformed eleven dimensional supergravity will be a non-anticommutative ABJ theory.

Now, if \( H_{\mu \nu \tau} \) the field strength of this three form field, then we expect a non-anticommutative deformation proportional to \( \{ \theta_a, \theta_b \} \sim (\gamma^\mu \gamma^\nu \gamma^\tau H_{\mu \nu \tau})_{ab} \) to occur. However, it was not possible to study the non-anticommutative deformations of the ABJM theory in the \( \mathcal{N} = 1 \) superspace as that would break all the manifest supersymmetry of the superspace. It would thus be interesting to analyse the non-anticommutative deformations of this theory in superspace with higher amount of manifest supersymmetry. So, in this paper we analyse the non-anticommutative deformation of the ABJ theory in harmonic superspace. Non-anticommutative deformations break the total supersymmetry of this theory from \( \mathcal{N} = 6 \) supersymmetry to \( \mathcal{N} = 5 \) supersymmetry.

## 2 Harmonic superspace

In this section we will review harmonic superspace in three dimensions \[3, 4, 5, 52\]. These harmonic variable are parameterize by the coset \( SU(2)/U(1) \). So, the harmonic variables \( u^\pm \) are subjected to the constraints \( u^{+i}u^{-i} = 1 \), \( u^{+i}u^{+i} = u^{-i}u^{-i} = 0 \), and the superspace coordinates are given by

\[
z = (x^{ab}, \theta_a^{++}, \theta_a^{--}, \theta_a^0, u^\pm),
\]

(1)
where \( \theta^\pm_a = \theta^+_{ij} u_i^+ u_j^- \) and \( \theta^0_a = \theta^+_{ij} u_i^+ u_j^- \). In the harmonic superspace the following derivatives are constructed

\[
\begin{align*}
\partial^{++} &= u_i^+ \frac{\partial}{\partial u_i}, \\
\partial^{--} &= u_i^- \frac{\partial}{\partial u_i}, \\
\partial^0 &= u_i^+ \frac{\partial}{\partial u_i} - u_i^- \frac{\partial}{\partial u_i}.
\end{align*}
\] (2)

These derivatives are used to define the following derivatives

\[
\begin{align*}
D^{--}_a &= \frac{\partial}{\partial \theta^{++}_a} + 2i \theta^{--}_b \partial^A_{ab}, \\
D^0_a &= -\frac{1}{2} \frac{\partial}{\partial \theta^{0}_a} + i \theta^{0}_b \partial^A_{ab}, \\
D^{++}_a &= \frac{\partial}{\partial \theta^{--}_a}.
\end{align*}
\] (3)

Apart from these derivatives, the following derivatives are also constructed

\[
\begin{align*}
D^{++}_a &= \partial^{++}_a + 2i \theta^{++}_b \partial^A_{ab} + \theta^{++}_a \frac{\partial}{\partial \theta^{0}_a} + 2 \theta^{0}_a \frac{\partial}{\partial \theta^{--}_a}, \\
D^{--}_a &= \partial^{--}_a - 2i \theta^{--}_b \partial^A_{ab} + \theta^{--}_a \frac{\partial}{\partial \theta^{0}_a} + 2 \theta^{0}_a \frac{\partial}{\partial \theta^{++}_a}, \\
D^0_a &= \partial^0 + 2 \theta^{++}_a \frac{\partial}{\partial \theta^{++}_a} - 2 \theta^{--}_a \frac{\partial}{\partial \theta^{--}_a}.
\end{align*}
\] (4)

The superalgebra satisfied by these derivatives is given by

\[
\begin{align*}
\{D^{++}_a, D^{--}_b\} &= 2i \partial^A_{ab}, & \{D^0_a, D^0_b\} &= -i \partial^A_{ab}, \\
[D^{++}_a, D^{\pm}_b] &= 2D^{\pm}_a, & [D^0_a, D^{\pm}_b] &= \pm 2D^{\pm}_a, \\
\partial^0 &= [\partial^{++}, \partial^{--}], & [D^{++}, D^{--}] &= D^0, \\
\{D^{\pm}_a, D^0_b\} &= 0, & [D^{\pm}_a, D^0_b] &= D^{\pm}_a.
\end{align*}
\] (5)

The harmonic superspace has \( N = 3 \) supersymmetry in three dimensions. The generators of this \( N = 3 \) supersymmetry in three dimensions are given by

\[
\begin{align*}
Q^{++}_a &= u_i^+ u_j^- Q^{ij}_a, & Q^{--}_a &= u_i^- u_j^- Q^{ij}_a, \\
Q^0_a &= u_i^+ u_j^- Q^{ij}_a,
\end{align*}
\] (6)

where

\[
Q^{ij}_a = \frac{\partial}{\partial \theta^{ij}_a} - \theta^{ij} \partial_{ab}.
\] (7)

In harmonic superspace, the superfields which are independent of the \( \theta^{--}_a \) are called analytic superfields. Thus, these analytic superfields satisfy

\[
D^{++}_a \Phi_A = 0 \Rightarrow \Phi_A = \Phi_A(\zeta_A),
\] (8)

where the coordinates parameterizing the analytic subspace are given by

\[
\zeta_A = (x^{ab}_A, \theta^{++}_a, \theta^0_a, u_i^\pm).
\] (9)

Here \( x^{ab}_A \) is given by

\[
x^{ab}_A = (\gamma_m)^{ab} x^m_A = x^{ab} + i(\theta^{++}_a \theta^{--}_b + \theta^{++}_b \theta^{--}_a).
\] (10)
It is convenient to define

\[ d\zeta^{(-4)} = \frac{1}{4} d^3 x d u (D^{--})^2 (D^0)^2 \]

Furthermore, a conjugation in this superspace is defined by

\[ \tilde{u}^\pm = u^\pm, \quad \tilde{x}_m^A = x_m^A, \quad \tilde{\theta}^{\pm\pm}_a = \theta^{\pm\pm}_a, \quad \tilde{\theta}^{0_a} = \theta^{0_a}. \]

The analytic superspace measure is real \( \tilde{d}\zeta^{(-4)} = d\zeta^{(-4)} \) and the full superspace measure is imaginary \( d^\theta z = -d^\theta z \) because this conjugation is squared to \( -1 \) on the harmonics and to \( 1 \) on \( x_m^A \) and Grassmann coordinates.

### 3 Deformation of Harmonic Superspace

In this section we will analyse non-anticommutative deformation of the Harmonic superspace. Such deformation occurs due to a RR background in the string theory [35, 36]. Unlike the noncommutative deformation caused by a NS background or gravitino background, the non-anticommutative deformation a certain amount of the breaks the supersymmetry of the theory. In four dimensions this deformation can be used to break half the supersymmetry of a \( N = 1 \) supersymmetry theory to obtain a theory with \( N = 1/2 \) supersymmetry. This is because in four dimensions the supersymmetric generator \( Q_A \) can be split into \( Q_a \) and \( \dot{Q}_a \). So, it is possible to break the supersymmetry corresponding to one of these generators without breaking the other one. This is thus done by imposing the following anticommutator

\[ \{ \tilde{\theta}_a, \dot{\theta}_b \} = C_{ab}. \]

This will break the supersymmetry corresponding to \( Q_a \) without breaking the supersymmetry corresponding to \( Q_a \). We could also break the supersymmetry corresponding to \( \dot{Q}_a \) without breaking breaking the supersymmetry corresponding to \( Q_a \). Now if we project the undeformed \( \mathcal{N} = 1 \) supersymmetry in four dimensions to three dimensions, then it will have \( \mathcal{N} = 2 \) supersymmetry. This is because in three dimensions both \( Q_a \) and \( \dot{Q}_a \) act as separate supercharges. Now the \( \mathcal{N} = 1/2 \) supersymmetry in four dimensions corresponded to \( \mathcal{N} = (1, 0) \) or \( \mathcal{N} = (0, 1) \) in three dimensions. It is not possible to obtain a \( N = 1/2 \) supersymmetry in three dimensions from this as there are enough degrees of freedom to do so.

The \( \mathcal{N} = 2 \) supersymmetric theory in four dimensions is generated by four supercharges. It is possible to break the supersymmetry with respect to any number of these supercharges. Thus, if the supercharges supercharges are denoted by \( Q_a^\pm \) and and \( Q_a \). Then it is possible to obtain a \( \mathcal{N} = 1/2 \) theory by breaking the supersymmetry with respect to three of these supercharges by imposing the following anti-commutators

\[ \{ \tilde{\theta}_a^+, \dot{\theta}_b^+ \} = C_{ab}^+, \quad \{ \tilde{\theta}_a^+, \dot{\theta}_b^- \} = C_{ab}^-, \]

\[ \{ \tilde{\theta}_a^-, \dot{\theta}_b^+ \} = C_{ab}^+, \quad \{ \tilde{\theta}_a^-, \dot{\theta}_b^- \} = C_{ab}^- \]

(14)
From a three dimensional perspective this corresponded to breaking a $\mathcal{N} = 4$ supersymmetric theory to $\mathcal{N} = ((1,0),(0,0))$. We could obtain similar deformations, $\mathcal{N} = ((0,1),(0,0)), \mathcal{N} = ((0,0),(1,0))$ and $\mathcal{N} = ((0,0),(0,1))$, depending upon which supercharges are left undeformed. It is also be possible to deform the four dimensional theory with $\mathcal{N} = 2$ supersymmetry as

$$\{\hat{\theta}^+_a, \hat{\theta}^+_b\} = C^+_{ab}, \quad \{\hat{\theta}^+_a, \hat{\theta}^+_b\} = C^+_{ab}.$$  \hspace{1cm} (15)

This will correspond to $\mathcal{N} = ((1,0),(1,0))$ in three dimensions. Now we can use generate other similar deformations like $\mathcal{N} = ((0,1),(1,0)), \mathcal{N} = ((0,1),(0,1)), \mathcal{N} = ((1,0),(0,1))$. Finally, we can break only one of the supercharges in the four dimensional theory and obtain a $\mathcal{N} = 2/3$ supersymmetric theory. Thus, if we impose

$$\{\hat{\theta}^+_a, \hat{\theta}^+_b\} = C^+_{ab},$$  \hspace{1cm} (16)

then we obtain a $\mathcal{N} = 2/3$ in four dimensions. This corresponded to $\mathcal{N} = ((1,0),(1,1))$ in three dimensions. Similarly, we can obtain $\mathcal{N} = ((0,1),(1,1)), \mathcal{N} = ((1,1),(0,1))$ and $\mathcal{N} = ((1,1),(1,0))$, in three dimensions.

We could also start from the harmonic superspace in three dimensions and impose the following deformation

$$\{\hat{\theta}^{++}_a, \hat{\theta}^{++}_a\} = C^+_{ab}. $$  \hspace{1cm} (17)

This will break the supersymmetry corresponding to $Q^+_a$ without breaking the supersymmetry corresponding to $Q^-_a$ and $Q^0_a$. Thus, we will obtain $\mathcal{N} = 2$ supersymmetry in three dimensions. This we could have similarly broken the supersymmetry with respect to $Q^-_a$ or $Q^0_a$ and left the remaining two intact to obtain $\mathcal{N} = 2$ supersymmetry in three dimensions. We can also break the supersymmetry with respect to any two supercharges say $Q^{++}_a$ and $Q^{--}_a$ and leave the supersymmetry with respect to $Q^0_a$ intact by imposing the following deformations

$$\{\hat{\theta}^{++}_a, \hat{\theta}^{++}_a\} = C^+_{ab}, \quad \{\hat{\theta}^{--}_a, \hat{\theta}^{--}_a\} = C^-_{ab}. $$ \hspace{1cm} (18)

This way we will obtain a theory with $\mathcal{N} = 1$ supersymmetric. We could have similarly have left either $Q^{++}_a$ or $Q^{--}_a$ intact to obtain a theory with $\mathcal{N} = 1$ supersymmetric. It may be noted that $\mathcal{N} = 3$ supersymmetry in three dimensions corresponded to $\mathcal{N} = 3$ supersymmetry in two dimensions. This is because each of the supercharges splits into two independent supercharges $Q^{++}_a, Q^{--}_a$ or $Q^0_a$. Thus, $\mathcal{N} = 2$ supersymmetry in three dimensions will correspond to one of the following, $((1,1), (1,1), (0,0)), ((0,0), (1,1), (1,1))$ and $((1,1), (0,0), (1,1))$, in two dimensions. Similarly, $\mathcal{N} = 1$ supersymmetry in three dimensions will correspond to one of the following, $((1,1), (0,0), (0,0)), (0,0), (1,1), (0,0))$ and $((0,0), (0,0), (1,1))$, in two dimensions.

## 4 Deformed ABJ Theory

In this section we will analyse non-anticommutative deformation of the ABJ theory in the harmonic superspace with one of generators of the supersymmetry broken due to the deformation. Other deformations can be analysed in a
similar way. So, to start with defining a vector field $V^{++}$ in the harmonic superspace. We now deform the harmonic superspace by breaking the supersymmetry generated by $Q^{++}_a$ by imposing the following relations,

$$\{\hat{\theta}^{++}_a, \hat{\theta}^{++}_b\} = C^{++}_{ab}.$$  

(19)

We now use Weyl ordering and express the Fourier transformation of a superfield on this deformed superspace as,

$$\hat{V}^{++}(\hat{z}) = \int dp V^{++}(p) \exp(ip\hat{z}),$$  

(20)

where

$$\exp(ip\hat{z}) = \exp(-ik\hat{x} - \pi^{++}_a \hat{\theta}^{++}_a - \pi^{--}_a \hat{\theta}^{--}_a - \pi^0_0 \hat{\theta}^0_a),$$

$$dp = \delta^{3kd_2} \pi^{++}_a \delta^{2d_2} \pi^{--}_a \delta^{2d_2} \pi^0_a,$$

$$V^{++}(p) = V^{++}(k, \pi^{++}_a, \pi^{--}_a, \pi^0_a, u^\pm).$$  

(21)

Thus, we obtain a one to one map between a function of $\hat{z}$ to a function of ordinary superspace coordinates $z$ via

$$V^{++}(z) = \int dp V^{++}(p) \exp(ipz).$$  

(22)

We can express the product of two fields $\hat{V}^{++}(\hat{z})\hat{V}^{++}(\hat{z})$ on this deformed superspace as

$$\hat{V}^{++}(\hat{z})\hat{V}^{++}(\hat{z}) = \int dp_1 dp_2 \exp(i(p_1 + p_2)\hat{z}) \exp(i\Delta)V^{++}(p_1)V^{++}(p_2),$$  

(23)

where

$$\exp(i\Delta) = \exp(-\frac{1}{2} \left( C^{++}_{ab} \partial^{++}_a \partial^{++}_b + C^{--}_{ab} \partial^{--}_a \partial^{--}_b \right)).$$  

(24)

This motivates the definition of the star product between ordinary vector fields, which is now defined as

$$V^{++}(z) \star V^{++}(z) = \exp(-\frac{1}{2} \left( C^{++}_{ab} \partial^{++}_a \partial^{++}_b + C^{--}_{ab} \partial^{--}_a \partial^{--}_b \right)) \times V^{++}(z_1)V^{++}(z_2) |_{z_1 = z_2 = z}. $$  

(25)

If we impose the following deformation

$$\{\hat{\theta}^{++}_a, \hat{\theta}^{++}_b\} = C^{++}_{ab}, \quad \{\hat{\theta}^{--}_a, \hat{\theta}^{--}_b\} = C^{--}_{ab},$$  

(26)

and proceed in a similar way, we obtain the following definition of the star product between ordinary vector fields

$$V^{++}(z) \star V^{++}(z) = \exp(-\frac{1}{2} \left( C^{++}_{ab} \partial^{++}_a \partial^{++}_b + C^{--}_{ab} \partial^{--}_a \partial^{--}_b \right)) \times V^{++}(z_1)V^{++}(z_2) |_{z_1 = z_2 = z}. $$  

(27)

However, we will only analyse the deformation corresponding to Eq. (19) in this paper. So, in this paper the deformed ABJ model will have manifest $\mathcal{N} = 2$ supersymmetry. It is now possible to write $V^{--}$ in-terms of $V^{++}$ as

$$V^{--}(z, u) = \sum_{n=1}^{\infty} (-1)^n \int du_1 \ldots du_n E^{++},$$  

(28)
where

\[ E^{++} = \frac{V^{++}(z, u_1) \ast V^{++}(z, u_2) \ast \ldots \ast V^{++}(z, u_n)}{(u_1^+ u_1^-)(u_2^+ u_2^-) \ldots (u_n^+ u_n^-)}. \]  

(29)

Now we can write the action for the deformed ABJ theory. This theory is invariant under the gauge group \( U(N) \times U(M) \). In this theory the matter fields are denoted by \((q^+)_{L}^{A}\) and \((\bar{q}^+)_{L}^{\alpha}\) and the gauge fields corresponding to \( U(M) \) and \( U(N) \) are denoted by \((V^{++}_L)_{AB}^{\alpha}\) and \((V^{++}_R)_{\alpha}^{\beta}\), respectively. Here the underlined indices refer to the right \( U(M) \) gauge group. The covariant derivatives for the matter fields in the deformed ABJ theory can be written as

\[
\begin{align*}
\nabla^{++} q^+ & = D^{++} q^+ + V^{++}_L \ast q^+ - q^+ \ast V^{++}_R, \\
\nabla^{++} \bar{q}^+ & = D^{++} \bar{q}^+ - \bar{q}^+ \ast V^{++}_L + V^{++}_R \ast \bar{q}^+, 
\end{align*}
\]

(30)

We can write the action for the ABJ theory in the deformed harmonic superspace as,

\[ S = S_{CS,k}[V^{++}_L]_s + S_{CS,-k}[V^{++}_R]_s + S_M[q^+, \bar{q}^+], \]

(31)

where

\[
\begin{align*}
S_{CS,k}[V^{++}_L]_s & = \frac{ik}{4\pi} tr \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \int d^3x d\theta du_1 \ldots du_n H^{++}_L, \\
S_{CS,-k}[V^{++}_R]_s & = -\frac{ik}{4\pi} tr \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \int d^3x d\theta du_1 \ldots du_n H^{++}_R, \\
S_M[q^+, \bar{q}^+]_s & = tr \int d^3x d\xi \int_{-4}^{4} \bar{q}^+ \ast \nabla^{++} \ast q^+, 
\end{align*}
\]

(32)

This theory is invariant under the following infinitesimal gauge transformations

\[
\begin{align*}
\delta q^+ & = \Lambda_L \ast q^+ - q^+ \ast \Lambda_R, \\
\delta \bar{q}^+ & = \Lambda_R \ast \bar{q}^+ - \bar{q}^+ \ast \Lambda_L, \\
\delta V^{++}_L & = -D^{++} \Lambda_L - [V^{++}_L, \Lambda_L], \\
\delta V^{++}_R & = -D^{++} \Lambda_R - [\Lambda_R, V^{++}_R]. 
\end{align*}
\]

(34)

The deformation of the ABJ theory breaks the supersymmetry from \( \mathcal{N} = 3 \) supersymmetry to \( \mathcal{N} = 2 \) supersymmetry. However, the original ABJ theory had \( \mathcal{N} = 6 \) supersymmetry. We have broken manifest the supersymmetry corresponding to \( Q^{++}_a \), so we should still be left with \( \mathcal{N} = 5 \) supersymmetry. Now as we have manifest \( \mathcal{N} = 2 \) supersymmetry generated by to the supercharges \( Q^+_a \) and \( Q^-_a \), we should have an additional \( \mathcal{N} = 3 \) supersymmetry to generate \( \mathcal{N} = 5 \) supersymmetry. This is achieved by the following supersymmetric
transformations,
\[
\begin{align*}
\delta_q^+ &= \imath e^a \hat{\nabla}_a^0 q^+ , \\
\delta_{\bar{q}}^+ &= \imath e^a \hat{\nabla}_a^0 \bar{q}^+ , \\
\delta V_{L}^{++} &= \frac{8\pi}{k} e^a \theta_0^a q^+ , \\
\delta V_{R}^{++} &= \frac{8\pi}{k} e^a \theta_0^a \bar{q}^+ \bar{q}^+ ,
\end{align*}
\]
(35)

where
\[
\begin{align*}
\hat{\nabla}_a^0 q^+ &= \nabla_a^0 q^+ + \vartheta_a^- (W_{L}^{++} q^+ - q^+ W_{L}^{++} ) , \\
\nabla_a^0 q^+ &= D_a^0 q^+ + V_{L, a}^0 q^+ - q^+ V_{R, a}^0 , \\
V_{L, a}^0 &= -\frac{1}{2} D_{a}^{++} V_{L}^{--} , \\
V_{R, a}^0 &= -\frac{1}{2} D_{a}^{++} V_{R}^{--} .
\end{align*}
\]
(36)

Here $\hat{\nabla}_a^0 q^+$ and $\nabla_a^0 q^+$ are obtained via conjugation and the field strengths $W_{R}^{++}$ and $W_{L}^{++}$ are defined by
\[
\begin{align*}
W_{L}^{++} &= -\frac{1}{4} D_{a}^{++} D_{a}^{++} V_{L}^{--} , \\
W_{R}^{++} &= -\frac{1}{4} D_{a}^{++} D_{a}^{++} V_{R}^{--} .
\end{align*}
\]
(37)

These field strengths satisfies
\[
\begin{align*}
D^{++} W_{L}^{++} + [V_{L}^{++}, W_{L}^{++}]_* &= 0 , \\
D^{++} W_{R}^{++} + [V_{R}^{++}, W_{R}^{++}]_* &= 0 .
\end{align*}
\]
(38)

Now using Fierz rearrangement, we get $-\delta_q S_M[q^+, \bar{q}^+]_* = \delta_\epsilon S_{CS, \epsilon}[V_{L}^{++}]_* + \delta_\epsilon S_{CS, -\epsilon}[V_{R}^{++}]_*$, and so we have
\[
\delta_\epsilon S = 0 ,
\]
(39)

and thus the action is invariant under $N = 5$ supersymmetry. Thus, unlike the undeformed ABJ theory which is invariant under $N = 5$ supersymmetry, the deformed ABJ theory is only invariant under $N = 5$ supersymmetry.

## 5 Conclusion

In this paper we analysed the non-anticommutative deformation of the ABJ theory in harmonic superspace. This theory is dual to multiple D2-brane in $RR$ background. The full multiple D2-brane action includes couplings to the background fields of type II string theory. For a single brane this would be the pull back of $C_{\mu\nu\tau}$ to the world volume but for the non-Abelian multiple D2-brane action it must include further dielectric couplings to all of the $RR$ form fields. M-theory contains a background three form field and its dual field. These should reduce to the $RR$ to fields of string theory. The D-branes have been studied in various backgrounds. So, it will be interesting to analyse the
coupling of the three form field explicitly to the multiple M2-brane action in harmonic superspace.

As the RR background partially break the supersymmetry in type II string theory, the dual deformations of it will also partially break the supersymmetry on the M-theory side. We analyse the M2-branes in harmonic superspace. This harmonic superspace initially had manifest $\mathcal{N} = 3$ supersymmetry. However, after the non-anticommutative deformation, it only had $\mathcal{N} = 2$ supersymmetry. This was because the supersymmetry corresponding to $Q^{++}_a$ was broken by the imposition of the non-anticommutative deformation of the superspace. Thus, the total supersymmetry of the ABJ theory was reduced from $\mathcal{N} = 6$ to $\mathcal{N} = 5$ supersymmetry. We also discussed the deformations of the harmonic superspace that break the supersymmetry corresponding to two of the supercharges, namely $Q^{++}_a$ and $Q^{-+}_a$, respectively. This deformation only leaves manifest $\mathcal{N} = 1$ supersymmetry unbroken. Thus a similar analysis of the ABJ theory in this superspace will only have $\mathcal{N} = 4$ supersymmetry. There are other type of deformations that occur in superspace. These occur due to non-vanishing values of commutators between spacetime and Grassmann coordinates and physically correspond to a deformation generated by a gravitino background. It will be interesting to study the ABJ theory with this kind of deformations in harmonic superspace. The interesting thing about these deformations is that they do not break any amount of supersymmetry. Thus, the ABJ theory with deformed superspace, where the deformations are caused by a gravitino background will preserve all of the $\mathcal{N} = 6$ supersymmetry.

Chern-Simons-matter theories also have important applications in condensed matter physics. This is because of their relevance to the fractional quantum Hall effect, which is based on the concept of statistical transmutation. \cite{53, 54, 55, 56}. Recently, supersymmetric generalisation of the fractional quantum Hall effect has also been investigated \cite{57, 58, 59, 60}. In particular, physical properties of the topological excitations in the supersymmetric quantum Hall liquid were discussed in a dual supersymmetric Chern-Simons theory \cite{61}. Furthermore, a close connection between the fractional quantum Hall noncommutativity of the spacetime has been discovered \cite{62, 63, 64, 65}. Thus, the results of this paper can have interesting condensed matter applications. This is because we can analyse the non-anticommutative deformation of the supersymmetric fractional quantum Hall effect. This can change the behavior of fractional condensates and thus have important consequences for the transport properties in supersymmetric quantum hall systems.

It may be noted that the BRST symmetry of the ABJM theory has been analysed in deformed $\mathcal{N} = 1$ superspace \cite{11, 12, 50}. So, it will be interesting to analyse the BRST symmetry of ABJ theory in deformed harmonic superspace. We can also use the BRST symmetry of this theory to show the unitarity of the $S$-matrix. It is possible to reduce the ABJM action to a $\mathcal{N} = 8$, super-Yang-Mills theory describing N D2-branes by using the novel Higgsing mechanism \cite{46, 47, 48, 49}. In this Higgsing mechanism the gauge group of the ABJM theory is spontaneously broken down to its diagonal subgroup. This analysis has also been performed in $\mathcal{N} = 1$ superspace \cite{12, 51}, and it will interesting to repeat this analysis in harmonic superspace. By doing that we will be able to analyse this Higgsing mechanics for the ABJM theory with non-anticommutative deformations. This will give us a better understanding of the existence of these non-anticommutative deformation in the M-theory, as we will be able to relate
it to the familiar objects in the string theory.

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