Since January 2020 Elsevier has created a COVID-19 resource centre with free information in English and Mandarin on the novel coronavirus COVID-19. The COVID-19 resource centre is hosted on Elsevier Connect, the company’s public news and information website.

Elsevier hereby grants permission to make all its COVID-19-related research that is available on the COVID-19 resource centre - including this research content - immediately available in PubMed Central and other publicly funded repositories, such as the WHO COVID database with rights for unrestricted research re-use and analyses in any form or by any means with acknowledgement of the original source. These permissions are granted for free by Elsevier for as long as the COVID-19 resource centre remains active.
Global dynamics for a Filippov epidemic system with imperfect vaccination

Yunhu Zhang\textsuperscript{a,b}, Yanni Xiao\textsuperscript{a,*}

\textsuperscript{a} School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, 710049, PR China
\textsuperscript{b} School of Science, Lanzhou University of Technology, Lanzhou, 730050, PR China

\textbf{ARTICLE INFO}

\textbf{Article history:}
Received 13 October 2018
Received in revised form 20 November 2019
Accepted 2 June 2020
Available online 15 June 2020

\textbf{Keywords:}
Threshold policy
Susceptible–infected–vaccinated (SIV) epidemic model
Limit cycle
Sliding mode
Global stability

\textbf{ABSTRACT}

Given imperfect vaccination we extend the existing non-smooth models by considering susceptible and vaccinated individuals enhance the protection and control strategies once the number of infected individuals exceeds a certain level. On the basis of global dynamics of two subsystems, for the formulated Filippov system, we examine the sliding mode dynamics, the boundary equilibrium bifurcations, and the global dynamics. Our main results show that it is possible that the pseudo-equilibrium exists and is globally stable, or the pseudo-equilibrium, the disease-free equilibrium and the real equilibrium are tri-stable, or the pseudo-equilibrium and the real equilibrium are bi-stable, or the pseudo-equilibrium and disease-free equilibrium are bi-stable, which depend on the threshold value and other parameter values. The global stability of the disease-free equilibrium or pseudo-equilibrium reveals that we may eradicate the disease or maintain the number of infected individuals at a previously given value. Further, the bi-stability and tri-stability imply that whether the number of infected individuals tends to zero or a previously given value or other positive values depends on the parameter values and the initial states of the system. This emphasizes the importance of threshold policy and challenges in the control of infectious diseases if without perfect vaccines.

© 2020 Elsevier Ltd. All rights reserved.

\textbf{1. Introduction}

Mathematical models allow us to simulate the spread of diseases through different kinds of settings, model the possible interventions, and provide useful information to policymakers to quick response to curb the disease spread. There is a number of evidence showing that mathematical models play an essential role in investigating disease spread and identifying the key factors that significantly affect outbreaks [1–7]. It is well known that mathematical models have successfully helped understand and predict the trend of spread of diseases during the 2003 outbreak of Severe Acute Respiratory Syndrome (SARS) [8–11] and the 2009 novel A/H1N1 pandemic influenza [12–14]. In fact, the modeling processes are usually the approximation to the real world, and modelers have been looking for the balance between the formulation of sufficiently fine/realistic models and feasibly manipulated models.

Interventions are usually modeled by continuous processes and disease spread with interventions are often described by the ordinary differential equations. A common assumption for the existing models is that human exploitative activities occur continuously [15–24]. However, this is not how the thing looks like. Mostly, in the early stages of the outbreak of emergent infectious diseases, the public is less aware of information on the disease itself, the degree of crisis or
the necessary measures to protection and control, the disease spreads very quickly. As the disease further spreads or the number of infected individuals exceeds a certain number, the public then begin to implement some strategies to protect or control. Consequently, the threshold policy (TP) provides a natural description of such systems [7,25], and consequently Filippov systems become more realistic ones to represent density-dependent control strategies. These systems appear in almost every domain of applied sciences [26–30], and in particular, in population dynamics [31–33] and epidemiology [34–38].

Recently, a number of mathematical models have been proposed to investigate the effect of threshold policy on disease dynamics [31,34–37,39–42]. Existing approaches to model the impact of threshold policy have assumed that the threshold policy only influences on a certain type of population such that they implement control measures or change their behaviors. For example, only susceptible individuals enhance the protection strategies to let the incidence rate be reduced by a rate once the number of infected individuals exceeds a threshold level [35,36]. However, not only susceptible but also (imperfect) vaccinated individuals enhance protection and control measures to avoid to be infected. How effective the threshold policy implemented by not only one compartment remains, is therefore an issue of great importance for epidemics control, and quantifying this policy through a mathematical modelling framework falls within the scope of this study.

The purpose of this paper is to investigate the effect of threshold policy on disease dynamics, and further to study what particular threshold level can be used to guide to eradicate the infectious disease. We formulate a general piecewise susceptible–infectious–vaccinated (SIV) type of model with nonlinear incidence to investigate the effects of TP implemented by both susceptible and vaccinated individuals. In Section 2, we examine two subsystems and their own dynamical behaviors. In the following section, the sliding mode dynamics is proposed to find out the pseudo-equilibrium. We prove that the pseudo-equilibrium is stable when it is feasible. In Section 4, we investigate the types and stability of all possible equilibria, and global dynamic behaviors by considering several scenarios. Further, the boundary equilibrium bifurcation is examined in Section 5. Finally concluding marks are given in the last section.

2. The SIV model with threshold policy

First we divide the total population \( N \) into three compartments: susceptible (\( S \)), infected (\( I \)) and vaccinated (\( V \)) to establish the SIV model with threshold policy. In our model we assume that a fraction \( \phi \) of the susceptible class is vaccinated per unit time. Although vaccination can reduce the infection by protecting susceptibles, it may not be completely effective. We let a factor of parameter \( \sigma \) (\( 0 < \sigma < 1 \)) to measure the efficiency of the vaccine as a multiplier to the infection rate, here \( \sigma = 0 \) means that the vaccine is perfectly effective, while \( \sigma = 1 \) means the vaccine has no effect at all. Given that the vaccination is not permanent we let \( \theta \) be the rate of loss of immunity, then \( \frac{1}{\theta} \) is the immunity period after an effective vaccination. We divide each variable by \( N \) to normalize the variables, and hence in the following \( S, I, V \) represent the proportions of three compartments, then we get

\[
\begin{align*}
\frac{dS(t)}{dt} &= \mu - \beta(1 - \epsilon(t))SI - \mu S - \phi S + \gamma I + \theta V, \\
\frac{dI(t)}{dt} &= \beta(1 - \epsilon(t))SI + \sigma \beta(1 - \epsilon(t))VI - (\mu + \gamma)I, \\
\frac{dV(t)}{dt} &= \phi S - \sigma \beta(1 - \epsilon(t))VI - \mu V - \theta V.
\end{align*}
\]

(1)

with

\[
\epsilon(t) = \begin{cases} 
0, & H(I) = I(t) - I_c < 0, \\
1, & H(I) = I(t) - I_c > 0, 
\end{cases}
\]

(2)

here \( H(I) \) is a given threshold function which may depend on the number of infected individuals, \( \epsilon(t) \) is a piecewise function which is dependent on the sign of \( H(I) \). Constant \( \mu \) is the natural birth (death) rate of population, \( \beta \) is the basic transmission coefficient, and \( \gamma \) represents the rate of recovery/remove from infected class. All the parameters are assumed to be positive constants. Model (1) with (2) is a description of the normalized threshold policy (TP), which is referred to as an on-off control. We denote the structure without intervention \( (\epsilon(t) = 0) \) by the free system \( (S^1) \) and the structure with intervention \( \epsilon(t) = 1 \) by the control system \( (S^2) \). Here \( \epsilon(t) = 1 \) means that when the number of infected individuals exceeds a certain number \( (I(t) > I_c) \), people begin to enhance some precaution and control measures such as wearing face masks, hand-washing and avoiding crowded places, and consequently the baseline transmission rate \( \beta \) is then reduced to be \( \beta(1 - f) \). Positive constant \( f \) represents the intensity level of the implemented precaution and control strategies. There is a variable structure with two distinct structures with their own equilibrium points due to the threshold policy. The total population is supposed to be constant here, so we have \( (S + I + V)^\prime = 0 \). Substituting the variable \( 1 - I - V \) for \( S \) to yield the following two dimensional system

\[
\begin{align*}
\frac{dV(t)}{dt} &= \phi(1 - I - V) - \sigma \beta(1 - \epsilon(t))VI - \mu V - \theta V, \\
\frac{dI(t)}{dt} &= \beta(1 - \epsilon(t))(1 - I - V) + \sigma \beta(1 - \epsilon(t))VI - (\mu + \gamma)I.
\end{align*}
\]

(3)

It is easy to get the invariant region by using the similar method in [43]

\[\{(V, I) \in \mathbb{R}^2_+ \mid 0 < V + I < 1\} \equiv \Omega.\]
Let \( H(U) = I - I \) with the vector \( U = (V, I) \) for convenience. Then the discontinuous switching surface \( \Sigma \) can be defined as
\[
\Sigma = \{ U \in \mathbb{R}_+^2 | H(U) = 0 \},
\]
which divides the plane \( \mathbb{R}_+^2 \) into two separate parts
\[
G^1 = \{ U \in \mathbb{R}_+^2 | H(U) < 0 \}, \quad G^2 = \{ U \in \mathbb{R}_+^2 | H(U) > 0 \}.
\]
Here we characterize the right side of the equation in \( G^i (i = 1, 2) \) to the form of column vector \( F_i(U) \) with the elements \( F_y \) for convenient, therefore
\[
F_1(U) = \begin{bmatrix} F_{11} \\ F_{12} \end{bmatrix} = \begin{bmatrix} \phi(1 - I - V) - \sigma \beta VI - \mu V - \theta V \\ \beta(1 - I - V)I + \sigma \beta VI - (\mu + \gamma)I \end{bmatrix},
\]
\[
F_2(U) = \begin{bmatrix} F_{21} \\ F_{22} \end{bmatrix} = \begin{bmatrix} \phi(1 - I - V) - \sigma \beta(1 - f) VI - \mu V - \theta V \\ \beta(1 - f)(1 - I - V)I + \sigma \beta(1 - f) VI - (\mu + \gamma)I \end{bmatrix}.
\]
so we can formulate the Filippov system by the following form
\[
\dot{U} = \begin{cases} F_1(U), & U \in G^1, \\ F_2(U), & U \in G^2. \end{cases}
\]
Firstly, we will do the preliminary work to show the respective dynamics behavior for two subsystems.

Free system \( S^1 \) gives the following model equations
\[
\begin{cases}
\frac{dV}{dt} = \phi(1 - I - V) - \sigma \beta VI - \mu V - \theta V, \\
\frac{dI}{dt} = \beta(1 - I - V)I + \sigma \beta VI - (\mu + \gamma)I,
\end{cases}
\]
By employing the well-known next generation method in [44], we can derive the basic reproduction number
\[
R_0^1 = \frac{\beta}{\mu + \gamma} \frac{\mu + \theta + \sigma \phi}{\mu + \gamma + \phi}.
\]
It is easy to get the disease-free equilibrium \( E_0^1 = (\frac{\phi}{\mu + \theta + \phi}, 0) \). The endemic equilibria are solutions of the algebraic equations
\[
\begin{cases}
\phi(1 - I) = \sigma \beta VI + (\mu + \theta + \phi)V, \\
\beta(1 - I - (1 - \sigma)V) = \mu + \gamma.
\end{cases}
\]
One can get the quadratic equation for \( I \)
\[
G_1(I) = A_1 I^2 + B_1 I + C_1 = 0,
\]
with \( A_1 = \sigma \beta, B_1 = (\mu + \theta + \sigma \phi) + \sigma(\mu + \gamma) - \sigma \beta, C_1 = (\mu + \gamma) \frac{\mu + \theta + \phi}{\mu + \gamma - \sigma} - (\mu + \theta + \sigma \phi) \). So we have two possible positive roots
\[
I_1^1 = \frac{-B_1 + \sqrt{A_1}}{2A_1}, \quad I_2^1 = \frac{-B_1 - \sqrt{A_1}}{2A_1},
\]
when \( A_1 = B_1^1 - 4A_1C_1 > 0 \). When \( R_0^1 > 1 \) (i.e., \( C_1 < 0 \)), we can easily obtain \( I_1^1 > 0 \) and \( I_2^1 < 0 \). When \( R_0^1 < 1 \) and \( B_1 < 0 \), we have \( I_1^1 < I_2^1 < 0 \). Then the second equation of (7) gives \( V_1^1 = \frac{1}{1 + \sigma} \left( 1 - I_1^1 - \frac{\mu + \gamma}{\beta} \right) \) \( (i = 1, 2) \). It is not difficult to prove \( V_1^1 > 0 \) for \( R_0^1 > 1 \), and \( V_1^1 > V_1^1 > 0 \) for \( R_0^1 < 1 \). Hence we get the two equilibria \( E_1^1(V_1^1, I_1^1) \) and \( E_2^1(V_1^1, I_2^1) \) of subsystem \( S^1 \) when \( A_1 > 0, R_0^1 > 1 < 1 \), and \( B_1 < 0 \) are satisfied, and the unique positive equilibrium \( E_2^1(V_1^1, I_2^1) \) exists when \( A_1 > 0 \) and \( R_0^1 > 1 \) hold true.

In fact, system \( S^1 \) has been carefully studied in [43], and here we only introduce main results without giving detailed calculations. We easily know that there is a backward bifurcation at \( R_0^1 = 1 \) when the conditions \( B_1 < 0 \) and \( B_1^2 > 4A_1C_1 \) are satisfied. There are two positive endemiac equilibria when the parameter \( \beta \) changes from \( \beta \) to \( \beta_1 \), which are corresponding to \( B_1 = -2\sqrt{A_1C_1} \) and \( R_0^1 = 1 \) respectively. By computing the equation \( B_1^2 = 4A_1C_1 \), one can get
\[
\beta_1 = \frac{\sigma(\mu + \gamma) + 2\sqrt{\sigma(1 - \sigma) \beta(\mu + \gamma)}}{(\mu + \theta + \sigma \phi)} - (\mu + \theta + \sigma \phi).
\]
By referring [43], we have the following

**Proposition 1.** There is a critical value \( R_0^1 \) such that Eq. (8) has only one unique solution for \( R_0^1 = R_0^1 \), the critical basic reproduction number \( R_0^1 \) is given by
\[
R_0^1 = \frac{\mu + \theta + \sigma \phi \sigma(\mu + \gamma) + 2\sqrt{\sigma(1 - \sigma)(\mu + \gamma)\phi} - (\mu + \theta + \sigma \phi)}{\sigma(\mu + \gamma)}.
\]
Lemma 1. The disease-free equilibrium $E_0^1$ is a locally asymptotically stable (LAS) node when $R_0^1 < 1$ and unstable when $R_0^1 > 1$, while the endemic equilibrium $E_1^1 = (V_1^1, I_1^1)$ is globally asymptotically stable (GAS) if $R_0^1 > 1$. When $R_0^1 < R_1^1$, there are precisely two endemic equilibria for subsystem (6) provided $B_1 < 0$ and $\Delta_1 > 0$, and moreover $E_1^1$ is a locally stable node, while $E_1^0$ is an unstable saddle point. When $R_0^1 < R_1^1$, the disease-free equilibrium $E_0^1$ is GAS.

Control system $S^2$ gives

$$\begin{align*}
\frac{dV}{dt} &= \phi(1 - I - V) - \sigma \beta(1 - f)V - \mu V - \theta V, \\
\frac{dI}{dt} &= \beta(1 - f)(1 - I - V) + \sigma \beta(1 - f)V - (\mu + \gamma)I.
\end{align*}$$

(9)

The disease-free equilibrium is $E_0^2 = \left(\frac{\phi}{\mu + \theta + \phi}, 0\right)$. It is obviously that $E_0^1 = E_0^2 = E_0$. The basic reproduction number for control system $S^2$ is

$$R_0^2 = \frac{\beta(1 - f) \mu + \theta + \sigma \phi}{\mu + \gamma + \mu + \theta + \phi}.$$ 

Using the same method as free system $S^1$, we can derive the following quadratic equation for $I$

$$G_2(I) = A_2 I^2 + B_2 I + C_2 = 0,$$

with $A_2 = \sigma \beta(1 - f)$, $B_2 = (\mu + \theta + \sigma \phi) + \sigma (\mu + \gamma) - \sigma \beta(1 - f)$, $C_2 = \frac{\mu + \gamma}{\beta(1 - f)} - (\mu + \theta + \sigma \phi)$. There are two possible positive roots when $A_2 B_2 - 4 A_2 C_2 > 0$

$$I_1^2 = \frac{-B_2 + \sqrt{A_2}}{2 A_2}, \quad I_2^2 = \frac{-B_2 - \sqrt{A_2}}{2 A_2}.$$ 

It follows that

$$V_2^2 = \frac{1}{1 - \sigma} \left(1 - I_2^2 - \frac{\mu + \gamma}{\beta(1 - f)}\right), \quad V_1^2 = \frac{1}{1 - \sigma} \left(1 - I_1^2 - \frac{\mu + \gamma}{\beta(1 - f)}\right).$$

Hence, for subsystem $S^2$, we get the two equilibria $E_1^2(V_1^2, I_1^2)$ and $E_2^2(V_2^2, I_2^2)$ when $A_2 > 0$, $R_0^2 < 1$ and $B_2 > 0$ hold true, and only one feasible equilibrium $E_1^2$ when $A_2 > 0$ and $R_0^2 > 1$ hold. Using the same method as free system $S^1$, we get

Lemma 2. The disease-free equilibrium $E_0^2$ is a LAS node when $R_0^2 < 1$ and unstable when $R_0^2 > 1$, while the endemic equilibrium $E_1^2 = (V_1^2, I_1^2)$ is GAS if $R_0^2 > 1$. When $R_0^2 < R_1^2 < 1$, in which

$$R_1^2 = \frac{\mu + \theta + \sigma \phi (\mu + \gamma) + 2 \sqrt{(1 - \sigma)(\mu + \gamma)\phi - (\mu + \theta + \sigma \phi)}}{\sigma (\mu + \gamma)}$$

there are precisely two endemic equilibria for subsystem (6) provided $B_2 < 0$ and $\Delta_2 > 0$, then we conclude that $E_1^2$ is a locally stable node, while $E_2^2$ is an unstable saddle point. When $R_0^2 < R_1^2$, the disease-free equilibrium $E_0^2$ is GAS.

It is worth noting that both the two subsystems may have two positive equilibria, we can order them by the elementary of variable $I$ as $I_1^2 < I_1^1 < I_2^2 < I_1^1$, provided they are feasible (see detailed calculation in Appendix). Further, we note that $R_1^2$ and $R_1^1$, which are independent of transmission coefficient $\beta$, are the same, so we denote them by $R_\gamma$, that is, we use $R_1^1 = R_1^2 = R_\gamma$ in the following.

3. Sliding mode dynamics

Now we briefly recall the definitions of sliding segment and crossing segment, then calculate the pseudo-equilibrium and give the sufficient conditions for existence of the pseudo-equilibrium. By using the Filippov convex method [45], we let

$$h(U) = \langle H_U(U), F_1(U)\rangle/(H_U(U), F_2(U)),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product, $H_U(U) = (\partial h/\partial \mu, \partial h/\partial \mu)$ is the nonvanishing gradient on the discontinuity boundary $\Sigma$. It is well-known that the crossing segment $\Sigma_c \subset \Sigma$ is defined as

$$\Sigma_c = \{U \in \Sigma | h(U) > 0\},$$

which means that the vectors $F_1(U)$ and $F_2(U)$ have the same sign with nontrivial normal components [45].

A sliding mode ensures that both trajectories in the vicinity of vector fields along $\kappa$ are directing toward each other. When there are some subregions of the line $\Sigma$ such that both vectors of two subsystems $S^1$ and $S^2$ are directed toward each other, then the sliding segment appears on the boundary. We employ the equivalent control method as in [26] to examine the sliding domain and sliding mode dynamics of Filippov system (3). So the sliding domain can be defined as

$$\Sigma_s = \{U \in \Sigma | h(U) \leq 0\}.$$

(11)
Here $H = I - I_c$, which means $H(0) = (0, 1)$. Since $F_1(U) = (F_{11}(U), F_{12}(U))^T$, $F_2(U) = (F_{21}(U), F_{22}(U))^T$. When $h(U) = (H_0(U), F_1(U), H_0(U), F_2(U)) = F_{12}(U)F_{22}(U) ≤ 0$, note that $F_{12}(U) > F_{22}(U)$ for any $f > 0$, it is impossible to be $F_{12} ≤ 0$ and $F_{22} ≥ 0$. Therefore, (11) is equivalent to the following

$$\Sigma = \{(V, l) ∈ \Sigma | F_{12}(V, l) ≤ 0, F_{22}(V, l) ≤ 0\}.$$  

Let $A(V_l, I_c)$ and $B(V_R, I_c)$ (in which $V_l = \frac{\beta(1-l_c)(1-l_c)}{(1-\sigma)(1-\beta)}$, $V_R = \frac{\beta(1-l_c)(1+\gamma)}{(1-\sigma)(1-\beta)}$) denote the endpoints of the sliding segment. So we have the sliding domain of the Filippov system as follows

$$\Sigma = \{(V, l) ∈ R^2_+ | V_l ≤ V ≤ V_R, I = I_c\}.$$  

Now we investigate the dynamics of Filippov system on the boundary $\Sigma$. Let the second equation in (3) be 0, we have $\beta(1-\epsilon e) = \frac{\mu + \gamma}{1-\epsilon(1-\sigma)(1-\beta)}$. Substituting this $\epsilon$ and $H = I - I_c = 0$ into the first equation of (3), we will get the sliding mode dynamics

$$V' = φ(1-I_l - V) - σ \frac{μ + γ}{1-I_l(1-σ)V} V_l - (μ + θ)V \equiv g(V),$$

with $V_l ≤ V ≤ V_R.$

Let $V' = 0$, indicating that $g(V) = 0$, which is equivalent to

$$ψ(V) = A_0V^2 + B_0V + C_0 = 0,$$  

with

$$A_0 = (μ + θ + φ)(1-σ),$$  

$$B_0 = (-μ - θ - 2φ + σφ)(1-I_c) - σI_c(μ + γ),$$  

$$C_0 = φ(1-I_c)^2.$$

Firstly, we can show $Δ = B_0^2 - 4A_0C_0 > 0$, to prove this inequality, we let $E = (φ + μ + θ)(1-I_c) > 0$, $F = φ(1-I_c)(1-σ) > 0$, $G = σI_c(μ + γ) > 0$, and it follows that

$$Δ = B_0^2 - 4A_0C_0$$

$$= (φ + μ + θ)^2(1-I_c)^2 + φ^2(1-I_c)^2 + σ^2I_c^2(μ + γ)^2$$

$$+ 2φ(φ + μ + θ)(1-I_c)σI_c(μ + γ) + 2φ(φ + μ + θ)(1-I_c)(μ + γ)$$

$$= E^2 + F^2 + G^2 + 2FG + 2EG + 2FG - 4E$$

$$= E^2 + F^2 + 2EF + G^2 + 2EG + 2FG$$

$$= (E - F)^2 + 2G^2 + 2EG + 2FG > 0.$$  

Hence, there are two real roots for Eq. (12):

$$V_p^1 = \frac{-B_0 - \sqrt{B_0^2 - 4A_0C_0}}{2A_0},$$

$$V_p^2 = \frac{-B_0 + \sqrt{B_0^2 - 4A_0C_0}}{2A_0},$$

then two possible pseudo-equilibria are $E_p^1 = (V_p^1, I_c)$ and $E_p^2 = (V_p^2, I_c)$.

We now claim $V_p^2 > V_R$. In fact, we have

$$V_p^2 - V_R = \frac{-B_0 + \sqrt{B_0^2 - 4A_0C_0}}{2A_0} - \frac{\beta(1-I_c)(μ + γ)}{(1-σ)(1-β)}$$  

As $Δ = B_0^2 - 4A_0C_0 = (E - F + G)^2 + 4FG > (E - F + G)^2$, we have $\sqrt{Δ} > (E - F + G)$. Then, the right hand side of (13) satisfies

$$\frac{-B_0 + \sqrt{B_0^2 - 4A_0C_0}}{2A_0} - \frac{\beta(1-I_c)(μ + γ)}{(1-σ)(1-β)} > \frac{E + F + G + E - F + G}{2A_0} - \frac{\beta(1-I_c)(μ + γ)}{(1-σ)(1-β)}$$

$$= \frac{E + G}{2A_0} - \frac{\beta(1-I_c)(μ + γ)}{(1-σ)(1-β)}$$

$$= \frac{φ(φ + μ + θ)(1-I_c)(μ + γ)}{(φ + μ + θ)(1-σ)} - \frac{1-I_c}{1-σ} - \frac{μ + γ}{(1-σ)(1-β)}$$

$$= \frac{1}{1-σ}\left(1-I_c + σI_c(μ + γ) - (1-I_c)(μ + γ)\right)$$

$$= \frac{1}{1-σ}\left(σI_c(μ + γ) + μ + γ\right) > 0.$$  

So we get $V_p^2 > V_R$, which means that $V_p^2$ does not locate within the interval $[V_l, V_R]$. That is, pseudo-equilibria $E_p^2$ cannot locate in the sliding segment.
Secondly, we want to find the sufficient conditions under which $V_p^1 \in [V_L, V_K]$. For this purpose we get

$$
\psi(V_L) = (\mu + \theta + \phi + \sigma l_c(1 - \sigma) \frac{[1 - f(x_1 - k_c)(1 - \sigma)]}{(1 - \sigma) \beta |1 - f(x_1 - k_c)(1 - \sigma)|} - |\phi(1 - l_c)(1 - \sigma)|)
+ (\mu + \theta + \phi + \sigma l_c(1 - \sigma) + \sigma l_c(\mu + \gamma)) \frac{[1 - f(x_1 - k_c)(1 - \sigma)]}{(1 - \sigma) \beta |1 - f(x_1 - k_c)(1 - \sigma)|} + \phi(1 - l_c)^2
= \sigma l_c \frac{[\mu + \gamma]}{(1 - \sigma) \beta} + \frac{\mu + \gamma}{(1 - \sigma) \beta} \sigma \frac{[1 - f(x_1 - k_c)(1 - \sigma)]}{(1 - \sigma) \beta |1 - f(x_1 - k_c)(1 - \sigma)|} - |\phi(1 - l_c)(1 - \sigma)|)l_c
+ (\mu + \gamma) \frac{[\mu + \theta + \phi]}{(1 - \sigma) \beta |1 - f(x_1 - k_c)(1 - \sigma)|} - |\phi(1 - l_c)(1 - \sigma)|)l_c
\leq \frac{\mu + \gamma}{(1 - \sigma) \beta}(A_2 l_c^2 + B_2 l_c + C_2)
\leq \frac{\mu + \gamma}{(1 - \sigma) \beta} G_2(l_c).
$$

As $A_2 > 0$, the parabolic curve of $G_2$ is opening upward, then when

$$I_c < I_2^2 \text{ or } I_c > I_1^1,$$

we have $G_2(l_c) > 0$, and hence $\psi(V_L) > 0$. Moreover,

$$
\psi(V_K) = (\mu + \theta + \phi + \sigma l_c(1 - \sigma) \frac{[1 - f(x_1 - k_c)(1 - \sigma)]}{(1 - \sigma) \beta} - |\phi(1 - l_c)(1 - \sigma)|)
+ (\mu + \theta + \phi + \sigma l_c(1 - \sigma) + \sigma l_c(\mu + \gamma)) \frac{[1 - f(x_1 - k_c)(1 - \sigma)]}{(1 - \sigma) \beta |1 - f(x_1 - k_c)(1 - \sigma)|} + \phi(1 - l_c)^2
= \sigma l_c \frac{[\mu + \gamma]}{(1 - \sigma) \beta} + \frac{\mu + \gamma}{(1 - \sigma) \beta} \sigma \frac{[1 - f(x_1 - k_c)(1 - \sigma)]}{(1 - \sigma) \beta |1 - f(x_1 - k_c)(1 - \sigma)|} - |\phi(1 - l_c)(1 - \sigma)|)l_c
+ (\mu + \gamma) \frac{[\mu + \theta + \phi]}{(1 - \sigma) \beta |1 - f(x_1 - k_c)(1 - \sigma)|} - |\phi(1 - l_c)(1 - \sigma)|)l_c
\leq \frac{\mu + \gamma}{(1 - \sigma) \beta} G_1(l_c).
$$

As $A_1 > 0$, then when

$$I_2^1 < I_c < I_1^1,$$

it follows that $G_1(l_c) < 0$, and consequently we have $\psi(V_K) < 0$.

In order to make sure $V_p^1 \in [V_L, V_K]$, namely, $V_L < V_p^1 < V_K (< V_K^2)$, we need the sufficient condition to get $\psi(V_L) > 0$ and $\psi(V_K) < 0$. Therefore, when $I_c$ satisfies (14) and (15), i.e., $I_2^1 < I_c < I_2^2$ or $I_1^2 < I_c < I_1^1$, we have $\psi(V_K) < \psi(V_p^1) < \psi(V_L)$, then the pseudo-equilibrium $E_p^1$ locates in the segment $\{(V, I) \in R_1^2 | V_L \leq V \leq V_K, I = I_c\}$.

Note that $V' = g(V)$, then we get

$$g(V_p^1) = -\phi - \mu - \theta - \sigma (\mu + \gamma) l_c \frac{1}{(1 - l_c)(1 - \gamma)} < 0,$$

which means that $E_p^1$ is LAS, that is to say, the pseudo-equilibrium is locally stable when it is feasible. As $E_p^2$ does not locate in the sliding segment, may not be named as a pseudo-equilibrium, we now use $E_p$ to denote the only possible pseudo-equilibrium (when it exist) in the following (i.e., $E_p = E_p^1$).

4. Global dynamics of the switching system

In this section, we examine the global dynamics of the piecewise system. We initially analyze the nonexistence of all possible limit cycles, that is, we exclude the existence of three possible limit cycles, the canard limit cycles, which contain part of the sliding segment, the limit cycles that totally in the region $G(i = 1, 2)$ and the limit cycles that surround the sliding segment. By combining the type (i.e., regular/sliding equilibria) and local stability of equilibria, we can obtain the reachability of orbits theoretically and numerically. Then we can conclude the global stability of the equilibria. To the end, we classify all the types of equilibria, and discuss the global dynamics of the proposed system in terms of relations of $R_0$, $R_2$, $R_c$ and 1. According to the previous conclusion, we get $R_0^1 > R_2^2, R_1^1 = R_2^2 = R_c$.

4.1. Case (a): $1 < R_0^2 < R_0$

In such case we easily know that the endemic equilibria $E_1^1$ and $E_1^2$ are GAS for subsystem $S^1$ and $S^2$ according to Lemmas 1 and 2, respectively. In this case, $I_2^1(i = 1, 2)$ is monotonously increase with $\beta$ (the detailed proof is given in Appendix A.1), so we have $I_2^1 > I_2^2$ (shown in Fig. A.1(A)). According to Section 3, when the threshold value $I_c$ is between $I_2^2$ and $I_2^1$, the sliding mode with a pseudo-equilibrium appears. In the following, we will examine the global analysis of the switching system (3) with (2). Here we take case (a) as an example to show non-existence of limit cycle.

Note that the existence of limit cycle of other cases are similar to do, hence we omit the detailed proof for other cases.
There is no limit cycle which contains part of the sliding segment $AB$ in Case (a).

Lemma 5. There is no closed orbit surrounding the sliding segment $AB$ in Case (a).

**Proof.** Suppose there is a limit cycle $L = L_1 + L_2$ surrounding $AB$, which intersects with switching line $I = I_2$ at point $P$ and $Q$, as shown in Fig. 1(B). We add two auxiliary lines $I = I_2 - \epsilon$ and $I = I_2 + \epsilon$ in the form of a dotted line ($\forall \epsilon > 0$). The two lines intersect with $L$ at points $P_1, Q_1$ and $P_2, Q_2$ respectively. Denote the region delimited by $L_1$ and $P_1Q_1$ by $K_1$ in the lower part, while the region delimited by $L_2$ and $P_2Q_2$ by $K_2$ in the higher part. The directed boundary $P_1L_1Q_1 + Q_1P_1$ and $Q_2L_2P_2 + P_2Q_2$ are clockwise. Moreover, suppose that the abscissas of the points $P, Q, P_1, Q_1, P_2, Q_2$ are $V_1, V_2, V_1 + \delta_1(\epsilon), V_2 - \delta_2(\epsilon), V_1 + \delta_3(\epsilon), V_2 - \delta_4(\epsilon)$, respectively, where $\delta(\epsilon)$ satisfying $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$ are continuous with
respect to $\epsilon$ ($i = 1, 2, 3, 4$). Let the Dulac function be $B = \frac{1}{V}$, we can get the following results by the Green’s theorem.

$$
\int_{K_1} \left[ \frac{\partial B}{\partial x_1} + \frac{\partial B}{\partial x_2} \right] \, dV = \oint_{\Gamma_1} B (F_1 \, dx_1 - F_2 \, dx_2)
$$

$$
= \int_{\Gamma_1} B (F_1 \, dx_1 - F_2 \, dx_2) \, dt = \int_{\Gamma_1} BF_1 \, dx_2 - \int_{\Gamma_1} BF_2 \, dx_1
$$

$$
= - \int_{V_{1+\delta_1}}^{V_{1+\delta_2}} \left[ \frac{1}{V} \right] \beta (1 - I - V) \, dV - \sigma \beta V I (\mu + \gamma) \, dV
$$

$$
= - \int_{V_{2-\delta_2}}^{V_{2-\delta_1}} \left[ \frac{1}{V} \right] \beta (1 - I - (\sigma - 1) \beta - \frac{\mu + \gamma}{V}) \, dV.
$$

and

$$
\int_{K_2} \left[ \frac{\partial B}{\partial x_1} + \frac{\partial B}{\partial x_2} \right] \, dV = \oint_{\Gamma_2} B (F_1 \, dx_1 - F_2 \, dx_2)
$$

$$
= \int_{\Gamma_2} B (F_1 \, dx_1 - F_2 \, dx_2) \, dt = \int_{\Gamma_2} BF_1 \, dx_2 - \int_{\Gamma_2} BF_2 \, dx_1
$$

$$
= - \int_{V_{1+\delta_1}}^{V_{1+\delta_2}} \left[ \frac{1}{V} \right] \beta (1 - (1 - I - V) + \sigma \beta V I (\mu + \gamma) \, dV
$$

$$
= - \int_{V_{2-\delta_2}}^{V_{2-\delta_1}} \left[ \frac{1}{V} \right] \beta (1 - I + (\sigma - 1) \beta - \frac{\mu + \gamma}{V}) \, dV.
$$

Sum up the both sides of the two equations above, we obtain the following

$$
\int_{K_1} \left[ \frac{\partial B}{\partial x_1} + \frac{\partial B}{\partial x_2} \right] \, dV + \int_{K_2} \left[ \frac{\partial B}{\partial x_1} + \frac{\partial B}{\partial x_2} \right] \, dV
$$

$$
= - \int_{V_{1+\delta_1}}^{V_{1+\delta_2}} \left[ \frac{1}{V} \right] \beta(1 - I) + (\sigma - 1) \beta - \frac{\mu + \gamma}{V} \, dV
$$

$$
= \int_{V_{1+\delta_1}}^{V_{1+\delta_2}} \left[ \frac{1}{V} \right] \beta(1 - f)(1 - I) + (\sigma - 1) \beta(1 - f) - \frac{\mu + \gamma}{V} \, dV.
$$

On the limit of $\epsilon \to 0$, we have $\lim_{\epsilon \to 0} \delta_1(\epsilon) = 0$, and get

$$
\lim_{\epsilon \to 0} \left( \int_{K_1} \left[ \frac{\partial B}{\partial x_1} + \frac{\partial B}{\partial x_2} \right] + \int_{K_2} \left[ \frac{\partial B}{\partial x_1} + \frac{\partial B}{\partial x_2} \right] \right) \, dV
$$

$$
= - \int_{V_{1+\delta_1}}^{V_{1+\delta_2}} \left[ \frac{1}{V} \right] \beta(1 - I) - (\sigma - 1) \beta \, dV - \int_{V_{1+\delta_1}}^{V_{1+\delta_2}} \left[ \frac{1}{V} \right] \beta(1 - f)(1 - I) - (\sigma - 1) \beta f \, dV
$$

$$
= \int_{V_{1+\delta_1}}^{V_{1+\delta_2}} \left[ \frac{1}{V} \right] \beta(1 - I - (1 - \sigma)V) \, dV.
$$

On the switching line, we have $I = I_c$ and $\frac{dI}{dt} = 0$, which mean

$$
(1 - I - V) + \sigma V = (1 - I) - (1 - \sigma)V = \frac{1}{\beta(1 - f)}(\mu + \gamma)I > 0.
$$

Therefore, we have

$$
\int_{V_{1+\delta_1}}^{V_{1+\delta_2}} \left[ \frac{1}{V} \right](1 - I - (1 - \sigma)V) \, dV > 0. \tag{16}
$$

On the other hand, $\frac{1}{\beta - \frac{\phi(1 - f)}{V^2}} < 0$ in $K_1$, $-\frac{\phi(1 - f)}{V^2} < 0$ in $K_2$, hence we obtain

$$
\int_{K_1} \left[ \frac{\partial B}{\partial x_1} + \frac{\partial B}{\partial x_2} \right] \, dV + \int_{K_2} \left[ \frac{\partial B}{\partial x_1} + \frac{\partial B}{\partial x_2} \right] \, dV
$$

$$
= \int_{K_1} \left[ \frac{\partial B}{\partial x_1} + \frac{\partial B}{\partial x_2} \right] \, dV + \int_{K_2} \left[ \frac{\partial B}{\partial x_1} + \frac{\partial B}{\partial x_2} \right] \, dV
$$

$$
< 0.
$$

This is an obvious contradiction with Eq. (16). So we have the conclusion that there is no limit cycle surrounding the sliding segment $AB$, the proof is then completed.

### 4.1.2. Global stability of the endemic equilibrium

In the following, we will investigate the global stability of endemic equilibria in terms of relationship of $I_1^1, I_1^2$ and $I_c$. Here, three subcases are considered according to this relationship.

**Subcase (a1): assume $I_c > I_1^1$.** In this subcase, the endemic equilibrium $E_1^1$ is real, while $E_1^2$ is virtual. We know that the sliding mode does exist but there is not a pseudo-equilibrium according to the previous derivation. We conclude that $E_1^1$ is GAS in the following.

**Theorem 1.** The endemic equilibrium $E_1^1 = (V_1^1, I_1^1)$ is a GAS node if $1 < R_0^1 < R_0$ and $I_c > I_1^1$.

**Proof.** According to Lemma 1, the endemic equilibrium $E_1^1$ is real and is a LAS node for subsystem $S^1$. And we know that any trajectory, once touching the sliding mode, moves from the left to the right along the sliding segment $AB$ by the procedure of proof of Lemma 4.
Fig. 2. (A) Backward bifurcation diagram for the two subsystems of case (a). The parameter values are \( \mu = 0.1, \sigma = 0.2, \phi = 3, \gamma = 12, \theta = 0.5, f = 0.2 \). Note that subsystem 2 was plotted on this diagram with \( R_2^0 = 0.8 * R_1^0 \). (B)-(D), Phase portraits of the switching system when the switching line is \( I_c = 0.65, 0.56, 0.43 \), \( \beta = 36.3 \). (Here the parameter \( \beta \) is actually \( \beta N \) in the original model).

Note that there is no limit cycle totally located in the region \( G^1 \) or \( G^2 \) according to Lemma 3. Moreover, we know that there is no limit cycle which contains part of the sliding line or surrounding the sliding line \( AB \) by Lemmas 4 and 5. Hence, trajectories initiating from region \( G^1 \) will either go toward \( E_1^1 \) directly or hit the sliding line and slide along this line from the left side to the right endpoint \( B \), then go toward \( E_1^1 \) ultimately. Trajectories starting from region \( G^2 \) will either cross over the switching line to get into the region \( G^1 \) or touch the sliding domain and then move along this line from left side to right in order to approach \( E_1^1 \) (shown in Fig. 2(B)). Thus, all trajectories will tend to equilibrium \( E_1^1 \) ultimately. So the endemic equilibrium \( E_1^1 \) is GAS. This completes the proof.

Subcase (a2): assume \( I_c^2 < I_c < I_1^1 \). In such subcase both \( E_1^1 \) and \( E_2^1 \) are virtual, the sliding mode does exist, and the pseudo-equilibrium \( E_p \) does exist and is LAS. Now we want to show that \( E_p \) is GAS in the following.

**Theorem 2.** The pseudo-equilibrium \( E_p = (V_p^1, I_c) \) is GAS if \( 1 < R_2^0 < R_1^0 \) and \( I_c^2 < I_c < I_1^1 \).

**Proof.** In such case we easily know that the virtual equilibrium \( E_1^1 \) is in region \( G^2 \) while \( E_2^1 \) is in region \( G^1 \). So trajectories go toward the opposite regions in order to approach their own equilibrium. Consequently, all the trajectories collide with the switching line. In view of Section 3, \( E_p \) is locally stable. Hence, some trajectories which intersect with the switching line at the sliding segment move to the pseudo-equilibrium \( E_p \) along the switching line, while others which intersect with the switching line at the crossing segment enter into the opposite region. Further, there is no limit cycle according to Lemmas 3–5, thus all the trajectories touch the sliding segment finally (shown in Fig. 2(C)). Hence, we obtain that the pseudo-equilibrium \( E_p \) is GAS.
4.4. Case pseudo-equilibrium $E_p$

Suppose $R_0 < R^*_0$, they touch the switching line, and either slide along the switching line or enter the region $GAS$ (shown in Fig. 2(D)). Then we get the following conclusion:

**Theorem 5.** The endemic equilibrium $E_1^2 = (V^*_1, I^*_1)$ is a GAS node if $1 < R^*_0 < R_0^1$ and $I_c < I^*_1$.

4.2. Case (b): $R_c < R^*_0 < 1 < R_0^1$

In such case, there is only one globally stable endemic equilibrium $E^*_1$ for free subsystem $S^1$, while there are two endemic equilibria $E^*_1$ and $E^*_2$ for control subsystem $S^2$. In view of Lemmas 1 and 2, $E^*_1$ and $E^*_2$ are LAS nodes, while $E^*_1$ is a unstable saddle point, and the disease-free equilibrium $E_0$ is unstable for subsystem $S^1$ as $R_0^1 > 1$. There is a backward bifurcation at $R^*_0 = 1$ (i.e., $C_2 = 0$) for control system $S^2$, then we have the order $I^*_2 < I^*_1 < I^*_1$ according to Fig. 3(B), and see detailed calculation in the Appendix. Again, we discuss the asymptotic behavior in terms of order of $I^*_1, I^*_2, I^*_3$ and $I_c$. 

Subcase (b1): assume $I_c > I^*_1$. The equilibrium $E^*_1$ is real and LAS, while $E^*_2$ is virtual but stable in this subcase. It is noticeable that the sliding mode exist but there is no pseudo-equilibrium. As we can preclude the existence of limit cycle using the same method as case (a), by analyzing the trend of any trajectory (similar to case (a)) we can obtain that $E^*_1$ is GAS (shown in Fig. 3(A)).

Subcase (b2): assume $I^*_2 < I_c < I^*_1$. All the equilibrium $E^*_1, E^*_2$ and $E^*_3$ are virtual. So all trajectories initiating from $G^1$ go upward in order to approach to $E^*_1$, while trajectories initiating from $G^2$ go downward in order to approach the equilibrium $E^*_1$. As there is no limit cycle, the sliding mode happens and the pseudo-equilibrium appears (shown in Fig. 3(B)), then the pseudo-equilibrium $E_p$ is GAS.

Subcase (b3): assume $I^*_2 < I_c < I^*_2$. In this subcase, $E^*_2$ is a real and a stable equilibrium, while $E^*_1$ and $E^*_3$ are virtual. The sliding mode without pseudo-equilibrium exists. On the basis of non-existence of limit cycle for the whole system, we can confirm that $E^*_2$ is the only GAS equilibrium (shown in Fig. 3(C)).

Subcase (b4): assume $0 < I_c < I^*_2$. Both $E^*_2$ and $E^*_3$ are real equilibria, while $E^*_1$ is virtual. Trajectories starting above $I^*_2$ (the stable manifold of $E^*_2$, the pink line in Fig. 3(D)) in region $G^2$ tend to equilibrium $E^*_2$, while trajectories initiating below $I^*_2$ in $G^2$ will tend to equilibrium $E_0$. Further, when trajectories starting from $G^1$ intent do go upward to approach the equilibrium $E^*_1$, they collide with the switching line, and consequently the locally stable pseudo-equilibrium $E_p$ appears. Hence, both $E^*_2$ and $E_p$ are bi-stable (shown in Fig. 3(D)). We summarize the above conclusion as the following:

**Theorem 4.** Suppose $R_c < R^*_0 < 1 < R_0^1$, we conclude that the endemic equilibrium $E^*_1$ is GAS for $I_c > I^*_1$; the pseudo-equilibrium $E_p$ is GAS for $I^*_1 < I_c < I^*_1$; $E^*_1$ is GAS for $I^*_2 < I_c < I^*_1$; and both $E^*_2$ and $E_p$ are bi-stable for $I_c < I^*_2$.

4.3. Case (c): $R^*_0 < R_c < 1 < R_0^1$

In such case endemic equilibrium $E^*_1$ of system $S^1$ is GAS in $G^1$, while system $S^2$ has no positive equilibrium. We consider the following two subcases in terms of relations of $I_c$ and $I^*_1$.

Subcase (c1): assume $I_c > I^*_1$. In such subcase, $E^*_1$ is a real equilibrium and is stable in $G^1$, while the disease-free equilibrium $E_0$ is unstable for subsystem $S^1$. Since subsystem $S^2$ has no endemic states, its disease-free equilibrium $E^*_2$ is stable in $G^2$. Any trajectory initiating from $G^2$ intends to approach to the disease-free equilibrium, and then go down to the region $G^1$ by crossing switching line $I = I_c$. As the limit cycles are ruled out, we can derive that the equilibrium $E^*_1$ is GAS (shown in Fig. 4(A)).

Subcase (c2): assume $0 < I_c < I^*_1$. In this subcase, $E^*_1$ is in $G^2$ and is virtual. Thus, all trajectories initiating from $G^1$ intend to go upward in order to approach $E^*_1$, while trajectories initiating from $G^2$ intend go downward in order to approach the equilibrium $E_0$ ($E^*_2$). Then two types of trajectories collide at the switching line, the sliding mode with a pseudo-equilibrium appears. Using the same method as before we rule out the existence of limit cycles, we can easily derive that the pseudo-equilibrium $E_p$ is GAS (shown in Fig. 4(B)). Then we have the following conclusion:

**Theorem 5.** Suppose $R^*_0 < R_c < 1 < R_0^1$, we conclude that the endemic equilibrium $E^*_1$ is GAS for $I_c > I^*_1$; while the pseudo-equilibrium $E_p$ is GAS for $I_c < I^*_1$.

4.4. Case (d): $R_c < R^*_0 < R_0^1 < 1$

In such case, both the two subsystems have backward bifurcations for $R^*_0 = 1(i = 1, 2)$, respectively. As the order of the four biological roots are determined (see Fig. A.1(C)), that is, $I^*_2 < I^*_2 < I^*_1 < I^*_1$, we can classify all the types of phase portraits in term of the relation of $I^*_1, I^*_2, I^*_3, I^*_2$ and $I_c$.

Subcase (d1): assume $I_c > I^*_1$. In such subcase, the switching line $I = I_c$ is higher than all the endemic equilibria, $E^*_1$ and $E^*_2$ are real, while $E^*_1$ and $E^*_2$ are virtual. When trajectories starting from $G^2$ intend to go downward to approach $E^*_2$, they touch the switching line, and either slide along the switching line or enter the region $G^1$. While trajectories initiating
Fig. 3. Phase portraits of the switching system of case (b) when the switching line are selected as (A) $I_c = 0.6$, (B) $I_c = 0.46$, (C) $I_c = 0.2$, (D) $I_c = 0.02$, the parameter values are $\mu = 0.1, \sigma = 0.2, \phi = 3, \gamma = 12, \theta = 0.5, f = 0.05, \beta = 36.3$. Here * represent stable equilibrium, o represent unstable equilibrium.

Fig. 4. Phase portraits of the switching system of subcase (c) when the switching line are selected as (A) $I_c = 0.6$, (B) $I_c = 0.46$, the parameter values are $\mu = 0.1, \sigma = 0.2, \phi = 3, \gamma = 12, \theta = 0.5, f = 0.2, \beta = 36.3$. 
from $G^1$ approach either $E_1^1$ or $E_0$, depending on the starting points. In particular, when the starting point is above the stable manifold of $E_2^1$ (the pink line $Γ_1$ in Fig. 5(A)), the trajectories move upward to tend to $E_1^1$, when the starting point is below the stable manifold of $E_2^1$, the trajectories go downward to $E_0$. Hence both $E_1^1$ and $E_0$ are LAS in their own attraction domain (shown in Fig. 5(A)), and then they are bi-stable in this subcase.

Subcase (d2): assume $l_0^2 < l_0 < l_1^2$. Here we know $E_2^1$ is real, while $E_1^1$, $E_1^2$ and $E_2^2$ are virtual. Trajectories starting from $G^2$ intend to go downward to approach $E_1^2$, after a slide or refraction, then enter into the region $G^1$. While the trajectories initiating above the stable manifold of $E_2^1$ in $G^1$ intend to go upward to approach to $E_1^1$. Then two types of trajectories touch the switching line, and slide to the pseudo-equilibrium. The trajectories initiating below the stable manifold $Γ_1$ of $E_1^1$ in region $G^1$ intend to go downward to tend to $E_0$ locally. Hence, both $E_0$ and $E_2$ are bi-stable (shown in Fig. 5(B)).

Subcase (d3): assume $l_0^2 < l_0 < l_1^2$. The endemic equilibria $E_1^1$ and $E_2^2$ are virtual, while $E_1^2$ and $E_2^1$ are real. Then trajectories starting from $G^2$ intend to tend to $E_1^2$, while the trajectories starting from the region above $Γ_1$ (the stable manifold of $E_2^1$) in $G^1$ intend to go upward to tend to $E_1^1$, after a slide or refraction at the switching line, move upward to $E_1^1$ following the vector field for $S^2$ system. Trajectories starting from the region below the stable manifold $Γ_1$ of $E_2^1$ in $G^1$ go downward to approach $E_0$ locally (shown in Fig. 5(C)). Therefore, the real equilibrium $E_1^1$ and the disease-free equilibrium $E_0$ are bi-stable in this subcase.

Subcase (d4): assume $0 < l_0 < l_1^2$. Then we get $E_1^1$ is virtual, while $E_1^2$, $E_2^1$ and $E_2^2$ are real. We divide the whole region into four parts according to the attraction domain separatrix and the switching line. The trajectories starting above $Γ_2$ (the stable manifold of $E_2^1$ in $G^2$ of subsystem 2) intend to tend to $E_1^2$ as the real equilibrium $E_1^2$ is locally stable, while trajectories starting below the stable manifold of $E_2^1$ in $G^2$ intend to go downward to $E_0$. Further, trajectories starting above $Γ_1$ (the stable manifold of $E_1^1$) in $G^1$ intend to go upward to approach $E_1^1$. These two types of trajectories touch the switching line, the sliding mode with locally stable pseudo-equilibrium appears. Trajectories initiating from the region below $Γ_1$ in $G^1$ intend to go downward to $E_0$, then $E_0$ is locally stable in this situation. As is shown in Fig. 5(D), the real equilibrium $E_1^1$, the pseudo-equilibrium $E_0$, and the disease-free equilibrium $E_0$ are locally stable. Hence three equilibria are tri-stable in such subcase.

Subcase (d5): assume $0 < l_0 < l_1^2$. Here we get that $E_1^1$ and $E_2^1$ are virtual, while $E_1^2$ and $E_2^2$ are real. Trajectories starting above $Γ_2$ (the stable manifold of $E_2^1$) in $G^2$ intend to tend to equilibrium $E_1^2$. Trajectories initiating below $Γ_2$ in $G^2$ will tend to $E_0$, and it is notable that when they go across the switching line, there is a slide or refraction on this line, and then these trajectories move to $E_0$ according to the vector field for subsystem $S^1$. Meanwhile, it is obviously that trajectories starting from $G^1$ tend to $E_0$ directly. Hence, both $E_1^2$ and $E_0$ are bi-stable (shown in Fig. 5(E)). Then we conclude the following:

**Theorem 6.** Suppose $R_c < R_0^1 < R_0^0 < 1$, we conclude that equilibria $E_1^1$ and $E_0$ are bi-stable for $l_0 > l_1^1$; equilibria $E_p$ and $E_0$ are bi-stable for $l_0^2 < l_0 < l_1^1$; equilibria $E_1^1$ and $E_0$ are bi-stable for $l_0^2 < l_0 < l_1^1$; equilibria $E_1^2$, $E_p$ and $E_0$ are tri-stable for $l_0^2 < l_0 < l_1^1$, both $E_1^2$ and $E_0$ are bi-stable for $l_0 < l_1^1$.

4.5. Case (e): $R_0^0 < R_c < R_0^0 < 1$

In such case, subsystem $S^1$ has two endemic equilibria, while subsystem $S^2$ have no biological equilibrium. So we consider the following three subcases in term of the relation of $l_0^1$, $l_0^2$ and $l_0$.

Subcase (e1): assume $l_0 > l_1^1$. Under this condition, both $E_1^1$ and $E_2^1$ are real, $E_0$ is real too. All trajectories initiating above the stable manifold $Γ_1$ of $E_2^1$ in $G^1$ intend to go upward to $E_1^1$, while trajectories initiating below $Γ_1$ in $G^1$ go downward to $E_0$. And trajectories starting from $G^2$ intend to go downward to approach to $E_0$, then they collide at the switching line. As there is no closed orbit, these trajectories will either slide along the switching line or refract at this line, and then continue to move according to vector field of system $S^1$. Therefore, both $E_1^1$ and $E_0$ are bi-stable (shown in Fig. 6(A)).

Subcase (e2): assume $l_0^1 < l_0 < l_1^1$. Here we have $E_1^1$ is virtual, while $E_1^2$ is real. All trajectories starting from $G^2$ intend to go downward to approach $E_0$, trajectories initiating above the stable manifold $Γ_1$ of $E_1^2$ in $G^1$ go upward to approach $E_1^1$. These two types of trajectories then collide at the switching line, the sliding mode with pseudo-equilibrium appears. We know the pseudo-equilibrium $E_p$ is LAS. Meanwhile, trajectories initiating below $Γ_1$ in $G^1$ go downward to $E_0$, indicating $E_0$ is LAS. Hence equilibria $E_0$ and $E_p$ are bi-stable (shown in Fig. 6(B)).

Subcase (e3): assume $0 < l_0 < l_1^2$. Both $E_1^1$ and $E_2^1$ are virtual, while $E_0$ is real. It is obviously that all the trajectories intend to go downward to tend to $E_0$; moreover we note that trajectories initiating from $G^2$ slide from left to right side along the switching line, or refract at the line. Therefore, $E_0$ is GAS (shown in Fig. 6(C)). Then we conclude the following:

**Theorem 7.** Suppose $R_0^0 < R_c < R_0^1 < 1$, we conclude that both $E_1^1$ and $E_0$ are bi-stable for $l_0 > l_1^1$; equilibria $E_p$ and $E_0$ are bi-stable for $l_0^1 < l_0 < l_1^1$; $E_0$ is GAS for $l_0 < l_1^2$.

4.6. Case (f): $R_0^0 < R_0^1 < R_c < 1$

In such case, there is no biological endemic state for subsystem $S^1$ or $S^2$, the disease-free equilibrium $E_0 = E_0^1 = E_0^2$ is the only equilibrium for two subsystems. It is obvious that $E_0$ is always located in the $G^1$ region. Thus it is always real for
Fig. 5. Phase portraits of the switching system of subcase (d) when the switching line are selected as (A) $l_c = 0.6$, (B) $l_c = 0.34$, (C) $l_c = 0.2$, (D) $l_c = 0.1$, (E) $l_c = 0.01$. The parameter values are $\mu = 0.1; \sigma = 0.2; \phi = 3; \gamma = 12; \theta = 0.5; f = 0.05; \beta = 36.3$.

subsystem $S^1$ but virtual for subsystem $S^2$. As there is no other locally stable equilibrium or limit cycle, simply analyzing the trend of orbits can yield that the disease-free equilibrium is GAS for this scenario (shown in Fig. 6(D)).
Fig. 6. Phase portraits of the switching system of subcase (e) when the switching line are selected as (A) $I_c = 0.5$, (B) $I_c = 0.2$, (C) $I_c = 0.03$, and subcase (f) as (D) $I_c = 0.4$. The other parameter values are $\mu = 0.1; \sigma = 0.2; \phi = 3; \gamma = 12; \theta = 0.5; \beta = 36.3$, (A-C) $f = 0.2$, (D) $f = 0.05$.

In summary, we discuss the types and stability of all possible equilibria for Filippov system (3), and obtain the global asymptotic behavior for all cases. We then summarize the stability of all possible equilibria in Table 1. It is worth noting that the idea of proving global stability of Filippov system (3) is based on the local stability of equilibria, reachability of orbits (i.e., regular/sliding equilibria) and nonexistence of limit cycles. For this purpose we carefully analyze the types of various equilibria and trend of orbits in phase plane under different cases.

5. Boundary equilibrium bifurcation analysis

Now we investigate the boundary equilibrium bifurcation of the switching system. The readers can find the detailed definitions for the boundary equilibrium and the tangent point in Appendix.

Let $E_B$ be a boundary equilibrium, namely, $E_B$ satisfies the following equations:

$$
\begin{align*}
\phi(1 - I - V) - \sigma \beta(1 - f \epsilon) VI &- \mu V - \theta V = 0, \\
\beta(1 - f \epsilon)(1 - I - V) + \sigma \beta(1 - f \epsilon) VI - (\mu + \gamma) I &- 0, \\
I - I_c &- 0.
\end{align*}
$$

(17)

So we can get two possible boundary equilibria by solving the above equations in (17)

$$
E_B^1 = \left(\frac{1}{1-\sigma} \left(1 - I_c - \frac{\mu + \gamma}{\beta}\right), I_c\right), \quad E_B^2 = \left(\frac{1}{1-\sigma} \left(1 - I_c - \frac{\mu + \gamma}{\beta(1-f)}\right), I_c\right)
$$
Existence and stability of the equilibria for the switching system (3) with (2).

| Cases | Threshold value | $E_1^1$ ($V_1^1, T_1^1$) | $E_2^1$ ($V_2^1, T_2^1$) | $E_1^2$ ($V_1^2, T_1^2$) | $E_2^2$ ($V_2^2, T_2^2$) | $E_p$ ($V_p, L_c$) | $E_0$ ($\phi_0 = 0$) | Global stability |
|-------|----------------|------------------------|------------------------|------------------------|------------------------|-------------------|-------------------|------------------|
| Case a | $l_c > l_1^1$ | Real Stable | Virtual Stable | - | - | - | Real Unstable | $E_1^1$ GAS |
| | $l_1^2 < l_c < l_1^1$ | Virtual Stable | Virtual Unstable | - | - | - | Stable | $E_p$ GAS |
| | $l_c < l_1^2$ | Virtual Stable | Real Stable | - | - | - | Real Unstable | $E_1^1$ GAS |
| Case b | $l_c > l_1^1$ | Real Stable | Virtual Stable | Virtual Unstable | - | - | Real Unstable | $E_1^1$ GAS |
| | $l_1^2 < l_c < l_1^1$ | Virtual Stable | Virtual Unstable | - | - | - | Stable | $E_p$ GAS |
| | $l_2^2 < l_c < l_1^2$ | Virtual Stable | Real Unstable | - | - | - | Real Unstable | $E_1^1$ GAS |
| | $l_c < l_2^2$ | Virtual Stable | Real Stable | Real Unstable | - | - | Stable | $E_1^1$ and $E_p$ bi-stable |
| Case c | $l_c > l_1^1$ | Real Stable | - | - | - | - | Real Unstable | $E_1^1$ GAS |
| | $l_1^1 < l_c$ | Virtual Stable | - | - | - | Stable | Real Unstable | $E_p$ GAS |
| Case d | $l_c > l_1^1$ | Real Stable | Virtual Stable | Virtual Unstable | Real Unstable | - | Real Stable | $E_1^1$ and $E_0$ bi-stable |
| | $l_1^2 < l_c < l_1^1$ | Virtual Stable | Virtual Unstable | Real Unstable | Stable | Real stable | $E_p$ and $E_0$ bi-stable |
| | $l_2^2 < l_c < l_1^2$ | Virtual Stable | Real Unstable | Real Unstable | - | - | Real Stable | $E_1^1$ and $E_0$ bi-stable |
| | $l_c < l_2^2$ | Virtual Stable | Real Stable | Real Unstable | Virtual Stable | Stable | Real Stable | $E_1^1$, $E_0$ and $E_p$ tri-stable |
| Case e | $l_c > l_1^1$ | Real Stable | - | - | - | - | Real Stable | $E_1^1$ and $E_0$ bi-stable |
| | $l_1^2 < l_c < l_1^1$ | Virtual Stable | - | - | - | Real Unstable | Real Stable | $E_p$ and $E_0$ bi-stable |
| | $l_c < l_2^1$ | Virtual Stable | - | - | Virtual Unstable | - | Real Stable | $E_0$ GAS |
| Case f | $l_c > 0$ | - | - | - | - | - | Real Stable | $E_0$ GAS |

The equilibrium does not exist; GAS: globally asymptotically stable.

$E_p = E_1^1$; $E_0 = E_1^2 = E_2^2$.

which are corresponding to $\epsilon = 0$ or $\epsilon = 1$, respectively. Here $l_c$ satisfies $G_1(l_c) = 0$ or $G_2(l_c) = 0$, so we have $l_c = l_1^1$ or $l_2^1$ for $G_1(l_c) = 0$, and $l_c = l_1^2$ or $l_2^2$ for $G_2(l_c) = 0$. Then we get four possible boundary equilibria.

$$E_{1a} = \left( \frac{1}{1 - \sigma} \left( 1 - T_1^1 - \frac{\mu + \gamma}{\beta} \right), l_1^1 \right), \quad E_{2a} = \left( \frac{1}{1 - \sigma} \left( 1 - T_2^1 - \frac{\mu + \gamma}{\beta} \right), l_1^1 \right),$$

$$E_{1b} = \left( \frac{1}{1 - \sigma} \left( 1 - T_1^1 - \frac{\mu + \gamma}{\beta(1 - f)} \right), l_1^2 \right), \quad E_{2b} = \left( \frac{1}{1 - \sigma} \left( 1 - T_2^1 - \frac{\mu + \gamma}{\beta(1 - f)} \right), l_1^2 \right).$$

Let $T$ be a tangent point, according to the definition we can get the following equations:

$$\begin{aligned}
\beta(1 - f)(1 - I - V)l + \sigma \beta(1 - f)e\gamma l - (\mu + \gamma)l &= 0, \\
l - l_c &= 0.
\end{aligned} \tag{18}$$

It follows that the possible tangent points are

$$T_1 = \left( \frac{\beta(1 - l_c) - (\mu + \gamma)}{(1 - \sigma)\beta}, l_c \right), \quad T_2 = \left( \frac{\beta(1 - f)(1 - l_c) - (\mu + \gamma)}{(1 - \sigma)\beta(1 - f)}, l_c \right),$$

which are the solutions of Eq. (18) corresponding to $\epsilon = 0$ and $\epsilon = 1$, respectively.
With the variation of the threshold value $I_c$, the boundary equilibrium bifurcation occurs when the regular equilibrium collides with the tangent point and the boundary equilibrium. Fig. 7 illustrates a series of the boundary equilibrium bifurcations for case (d), in which each subsystem has two positive equilibria: a stable node and a saddle point. The real and stable node $E_1^1$ coexists with the visible tangent point $T_2$ for $I_c > I_1^1$ (shown in Fig. 7(A)). As $I_c$ decreases from $I_1^1$ to $I_1^2$, $E_1^1$ collides with $T_2$ (shown in Fig. 7(B)). As threshold $I_c$ continues to decrease to $I_2^1 < I_c < I_1^1$, the stable pseudo-equilibrium $E_p$ appears and $T_2$ becomes an invisible tangent point (shown in Fig. 7(C)). This bifurcation shows how a stable pseudo-equilibrium appears. Moreover, another boundary bifurcation occurs when $I_c$ passes through the critical value $I_2^1$. The tangent point $T_1$, the real node $E_2^1$ and the pseudo-equilibrium collide when $I_c = I_1^2$ (shown in Fig. 7(D)), then the pseudo-equilibrium $E_p$ disappears, and stable node $E_2^1$ becomes the locally attractor (shown in Fig. 7(E)). When $I_c$ continuously decreases to $I_2^2$, the third boundary bifurcation occurs, the visible tangent point $T_1$ collides with the saddle point $E_2^2$ (shown in Fig. 7(F)).

When $I_c$ further passes through $I_2^2$ to $I_1^1 < I_c < I_2^2$, the locally stable pseudo-equilibrium $E_p$ appears (Fig. 7(G)), and the tri-stable phenomenon ($E_p$, the disease-free equilibrium $E_0$ and real node $E_2^1$) emerges. The fourth boundary equilibrium occurs when $I_c$ passes through $I_2^2$, the tangent point $T_2$ collides with saddle point $E_2^1$ and the pseudo-equilibrium $E_p$ for $I_c = I_2^2$ (Fig. 7(H)). When the threshold continues to decrease to be lower than $I_1^1$, the pseudo-equilibrium $E_p$ disappears and tangent point $T_2$ becomes invisible (Fig. 7(I)), and consequently the disease-free equilibrium $E_0$ and the node point $E_2^1$ are bi-stable in this situation. Moreover, it is notable that the first two boundary equilibrium bifurcations are boundary node bifurcations, while the last two are boundary saddle bifurcations.
6. Discussion and conclusion

It has been observed that interventions, such as quarantine, isolation, treatment and vaccination, play a significant role in controlling the emerging and reemerging infectious diseases. However, not all interventions are implemented from the beginning of the outbreak or during the entire outbreak. Then the threshold policy can be naturally described by the Filippov system [35,37,39]. In this study, we extend the existing SIV-type model by including the extra control strategies once the number of infected individuals exceeds a certain level. In particular, based on the SIV-type model we consider susceptible and imperfect vaccinated individuals to enhance protection and control measures conditional upon relatively high prevalence.

The global dynamics of the Filippov system is fully investigated for different cases. It is interesting to note that two subsystems can undergo backward bifurcations, and consequently one endemic state and disease-free equilibrium for free or control system may be bistable when its specific reproduction number is less than one. The switching system may stabilize at regular equilibrium $E^*_1$ or $E^*_2$, or the pseudo-equilibrium $E_p$, which depends on the threshold value $I_c$. The global stability of the pseudo-equilibrium indicates that when we choose the threshold value properly, the number of infected individuals can stabilize at a previously given value. These results are similar to those obtained by Wang and Xiao [25,37]. For cases (a), (b) and (c), we observed some new phenomena. The disease-free equilibrium $E_0$ and $E_2^*$, or $E_0$ and $E_p$ may be bistable (as shown in Fig. 5(B,C,D)). Especially, the disease-free equilibrium $E_0$, $E_p$ and $E_2^*$ may be tri-stable, and the disease-free equilibrium $E_0$ may be GAS. These results imply that the dynamic behaviors of the system depend not only on the parameter values but also on the initial values of the system. Moreover, the threshold policy makes the whole system show more complex dynamical behaviors. Further, we analyze the boundary equilibrium bifurcation of the case (d). The four boundary bifurcations are illustrated with the variation of threshold value $I_c$. All the evolutionary processes are displayed when $I_c$ passes across the four particular values $I_1^1$, $I_2^1$, $I_2^2$ and $I_1^2$.

It is worth noticing that the idea of proving global stability of the Filippov system (3) is based on the local stability of equilibria, reachability of orbits (i.e., regular/sliding equilibria) and nonexistence of limit cycles. For this purpose, we carefully analyze the types of various equilibria and trend of orbits in phase plane under different cases. Consequently, the proof of the global stability is a bit descriptive instead of using mathematical notations, which has also been intensely applied to investigating the global dynamics for other Filippov systems [27,37,38,42]. Note that the proof of global stability can also be given by using mathematical notations like attractor, limit set and etc. We choose this way of description seems more readable and simple.

The proposed switching system exhibits rich dynamics, implying that three equilibria including pseudo-equilibrium may be tri-stable or bi-stable. In particular, the pseudo-equilibrium or disease-free equilibrium may be GAS for particular conditions and suitable threshold level $I_c$, which means that the disease can be eradicated or stabilize to a previously given level. Hence the modeling approach and main results give us a new understanding to realize the disease eradication and epidemic control.

CRediT authorship contribution statement

Yunhu Zhang: Conceptualization, Formal analysis, Writing - orginal draft. Yanni Xiao: Conceptualization, Funding acquisition, Review & editing.

Acknowledgments

The work was supported by the National Natural Science Foundation of China (NSFC 11631012(YX)).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

In this appendix, we initially give some definitions of the various type of equilibria for the switching system [25,45]

**Definition 1 (The Real (Virtual) Equilibrium).** Let $U^*$ be such that $F_i(U^*) = 0$ with threshold policy (2). Then $U^*$ is called a real equilibrium of system (5) with (4) if it belongs to $\mathcal{G}$, and a virtual equilibrium if it belongs to $\mathcal{G}'$, where $i,j = 1,2$, ($i \neq j$).

**Definition 2 (The Pseudo-equilibrium).** A point $U^*$ is said to be a pseudo-equilibrium of system (5) with (2) if $U^*$ is an equilibrium of the sliding mode of system (5) with (2), i.e., $\lambda F_1(U^*) + (1 - \lambda)F_2(U^*) = 0$, $H(U) = 0$, where

$$\lambda = \frac{\langle H_0(U), F_{20}(U) \rangle}{\langle H_0(U), F_{20}(U) - F_{10}(U) \rangle}.$$
Definition 3 (Boundary Equilibrium). The boundary equilibrium $E_B$ is defined as $F_1(E_B) = 0, H(E_B) = 0$ or $F_2(E_B) = 0, H(E_B) = 0$.

Definition 4 (Tangent Point). The tangent point $T$, which refers to the point at which $F_i(T) \neq 0 (i = 1, 2)$, and the trajectory of $S^i (i = 1, 2)$ tangents to the switching line $\Sigma$. That is to say, the tangent points satisfy $(F_1(T), H_0(T)) = 0$ or $(F_2(T), H_0(T)) = 0$.

Definition 5 (Visible (Invisible) Tangent Point). If the orbit of $\dot{U} = F_1(U)$ starting at $T$ belongs to $S^1$ for all sufficiently small $|t| \neq 0$, we say that this tangent point is visible. While if this orbit belongs to $S^2$, this tangent point is invisible. Similar definitions hold for the vector field $F_2(U)$.

We now give detailed calculation for the order of arrangement of the four roots $I_1^1, I_1^2, I_2^2$, and $I_2^2$. Our purpose is to show the order of the four biological roots are determined when all the four or two or three of them are feasible.

A.1. $R_0^1 > R_0^2^1 > 1$

Firstly, we want to show $G_2(I_1^1) > 0$ when $R_0^1 > 1$.

\[ G_2(I_1^1) = A_2(I_1^1)^2 + B_2 I_1^1 + C_2 \]
\[ = \frac{A_2}{r_1}(B_1 I_1^1 + C_1) + B_2 I_1^1 + C_2 \]
\[ = \left[-(1-f)B_1 + B_2 I_1^1 \right] + \left[-(1-f)C_1 + C_2 \right] \]
\[ = f((\mu + \theta + \sigma \phi) + \sigma(\mu + \gamma))l_1^1 + (f - 1)\frac{(\mu + \theta + \sigma \phi)}{\beta} \leq \frac{(\mu + \gamma)(\mu + \theta + \phi)}{\beta} + f(\mu + \theta + \sigma \phi). \]

To simplify the expression, let

\[ M = (\mu + \theta + \sigma \phi) + \sigma(\mu + \gamma), \quad N = (\mu + \gamma)(\mu + \theta + \phi), \quad K = \mu + \theta + \sigma \phi, \]

then we need to show

\[ G_2(I_1^1) = fMi_1^1 + N \frac{2-f}{\beta} f - fK \]
\[ = fM \frac{-B_1 + \sqrt{B_1^2 - 4A_1 C_1}}{2A_1} + N \frac{2-f}{\beta} f - fK \]
\[ > 0. \]

The both two sides of the inequation are multiplied by $\frac{\alpha \beta}{f}$ at the same time

\[ G_2(I_1^1) > 0 \iff M \frac{-B_1 + \sqrt{B_1^2 - 4A_1 C_1}}{2A_1} + \sigma N \frac{2-f}{\beta} f - \sigma \beta K > 0. \quad (A.1) \]

The proof is classified into two cases in terms of the relation of $R_0^1$ and $\frac{2-f}{\beta}$.

Case 1. When $1 < R_0^1 < \frac{2-f}{\beta}$, then $C_1 < 0 \Rightarrow \sqrt{B_1^2 - 4A_1 C_1} > -B_1$, then we have $-B_1 + \sqrt{B_1^2 - 4A_1 C_1} > 0$. Besides, note that $\sigma N \frac{2-f}{\beta} f - \sigma \beta K = \sigma N \left(\frac{2-f}{\beta} - R_0^1\right) > 0$. It is obviously that (A.1) holds.

Case 2. When $R_0^1 > \frac{2-f}{\beta}$, we have

\[ (A.1) \iff M \sqrt{B_1^2 - 4A_1 C_1} > MB_1 + 2 \sigma \beta K - 2 \sigma N \frac{2-f}{\beta} \]
\[ \iff M^2(B_1^2 - 4A_1 C_1) > (MB_1 + 2 \sigma \beta K)^2 - 4(MB_1 + 2 \sigma \beta K)\sigma N \frac{2-f}{\beta} + 4\sigma^2 N^2 \frac{2-f}{\beta} \]
\[ \iff MB_1\sigma N \frac{2-f}{\beta} + 2 \sigma^2 \beta KN \frac{2-f}{\beta} > M^2 A_1 C_1 + MB_1 \sigma \beta K + \sigma^2 \beta^2 K^2 + \sigma^2 N^2 \frac{2-f}{\beta} \]

Note that $A_1 = \sigma \beta, R_0^1 = \frac{MK}{N} > \frac{2-f}{\beta}$, divided by $\sigma^2 N^2$ at each side of the inequation, one obtains

\[ (A.1) \iff \frac{M^2}{\sigma N} \frac{1}{1-f} + \frac{R_0^1 B M}{N} + \frac{2-f}{\beta} R_0^1 > \frac{\sigma M^2 \frac{2-f}{\beta}}{N} + (R_0^1)^2 + \left(\frac{2-f}{\beta}\right)^2 \]
\[ \iff \left(R_0^1 - \frac{2-f}{\beta}\right)^2 < \frac{\sigma M^2}{N} \left(R_0^1 - \frac{2-f}{\beta}\right) + \frac{M^2}{\sigma N} \frac{1}{1-f} \]
\[ \iff R_0^1 - \frac{2-f}{\beta} < R_0^1 + \frac{\sigma}{N} (\mu + \gamma) + \frac{1}{R_0^1} \frac{M^2}{\sigma N} \frac{1}{1-f} \]
\[ \iff -\frac{2-f}{\beta} < \frac{\sigma}{N} (\mu + \gamma) + \frac{1}{R_0^1} \frac{M^2}{\sigma N} \frac{1}{1-f}. \]

The left side is negative, while the right side is positive, the inequality is evidently valid so that (A.1) holds.

Summing up the above two cases, we have $G_2(I_1^1) > 0$. On the other hand, it is obviously that $G_1(I_1^1) = 0$, as the two parabolic curves of $G_1$ and $G_2$ are opening upward, so we have the relation $I_1^1 > I_1^2$ (see Fig. A.1(A)).
A.2. $R_0^1 > 1 > R_0^2 > R_c$

Secondly, when $R_0^2 < 1$, $C_2 > 0$, so $I_2^2 > 0$. We want to prove the inequality $G_1(I_1^2) < 0$ holds when $R_0^1 > 1 > R_0^2$.

$$G_1(I_1^2) = A_1(I_1^2)^2 + B_1I_1^2 + C_1 = -\frac{A_1}{R_0^2}(2B_1I_1^2 + C_2) + B_1I_1^2 + C_1$$

$$= (-\frac{1}{1-f})B_2 + B_1I_1^2 + (-\frac{1}{1-f})C_1 + C_2$$

$$= -\frac{1}{1-f}N I_1^2 + \frac{-N+(1-f)^2N}{\beta(1-f)^2} + \frac{K(1-f)}{1-f}$$

It is easy to see that $-\frac{1}{1-f}N I_1^2 < 0$, note that $R_0^2 = \frac{dR}{N}$, we have

$$\frac{-N+(1-f)^2N}{\beta(1-f)^2} + \frac{K(1-f)}{1-f} = \frac{dR}{1-f} \left( \frac{f}{1-f} \right) + 1$$

$$= \frac{dR}{1-f} \left( \frac{2+f}{K_0} - 1 \right)$$

$$< 0 \quad (R_0^2 < 1).$$

Consequently, it is obviously $G_1(I_1^2) < 0$. Using the same method to get $G_1(I_2^2) < 0$. Combine these two inequalities with $G_1(I_1^2) = 0$, we have the relation $I_1^1 > I_1^2 > I_2^2$ (see Fig. A.1(B)).

Besides, when $R_0^1 < 1$, $C_1 > 0$, we can get the relation $I_1^1 > I_1^2 > I_2^2 > I_2^1$ by using the similar method (see sketch map in Fig. A.1(C)).

References

[1] Z. Ma, Y. Zhou, J. Wu, Modeling and Dynamics of Infectious Diseases, Higher Education Press & World Scientific, 2009.

[2] R.M. Anderson, R.M. May, Infectious Diseases of Humans, Dynamics and Control, Oxford university press, 1992.

[3] H. Cao, Y. Xiao, Z. Ma, Bifurcation analysis of a discrete SIS model with bilinear incidence depending on new infection, Math. Biosci. Eng. 10 (2013) 1399–1417.

[4] M. Shen, Y. Xiao, L. Rong, L.A. Meyers, S.E. Bellan, The cost-effectiveness of oral HIV pre-exposure prophylaxis and early antiretroviral therapy in the presence of drug resistance among men who have sex with men in San Francisco, BMC Med. 16 (2018) 58.

[5] P. Song, Y. Xiao, Global hopf bifurcation of a delayed equation describing the lag effect of media impact on the spread of infectious disease, J. Math. Biol. 76 (2018) 1249–1267.

[6] Q. Zhang, B. Tang, S. Tang, Vaccination threshold size and backward bifurcation of SIR model with state-dependent pulse control, J. Theoret. Biol. 455 (2018) 75–85.

[7] W. Qin, S. Tang, C. Xiang, Y. Yang, Effects of limited medical resource on a Filippov infectious disease model induced by selection pressure, Appl. Math. Comput. 283 (2016) 339–354.

[8] S. Riley, C. Fraser, C.A. Donnelly, A.C. Ghani, L.J. Abu-Raddad, Transmission dynamics of the etiological agent of SARS in Hong Kong: impact of public health interventions, Science 300 (2003) 1961.

[9] Z. Zhang, The outbreak pattern of SARS cases in China as revealed by a mathematical model, Ecol. Model. 204 (2007) 420–426.

[10] W. Chen, W. Lu, W. Chen, Mathematical model and prediction of epidemic trend of SARS, China Trop. Med. 3 (2003) 421–426.

[11] H. Li, X. Ren, S. Liu, Analysis of the efficiency of the preventing and Isolating Treatments of SARS based on mathematical model, J. Biomath. 19 (2004) 72–76.

[12] J. Zhang, Y. Xiao, Modelling strategies for controlling H1N1 Outbreaks in China, Int. J. Biomath. 05 (2012) 11001593.

[13] S. Tang, Y. Xiao, L. Yuan, R.A. Cheke, J. Wu, Campus quarantine (Fengxiao) for curbing emergent infectious diseases: Lessons from mitigating A/H1N1 in Xi’an, China, J. Theor. Biol. 295 (2012) 47–58.

[14] S. Tang, Y. Xiao, Y. Yang, Y. Zhou, J. Wu, Community-based measures for mitigating the 2009 H1N1 Pandemic in China, Plos One 5 (6) (2010) e10911.

[15] S. Gao, Y. Liu, J.J. Nieto, H. Andrade, Seasonality and mixed vaccination strategy in an epidemic model with vertical transmission, Math. Comput. Simulation 84 (2011) 1855–1868.
[16] B. Tang, Y. Xiao, S. Tang, R. Cheke, A feedback control model of comprehensive therapy for treating immunogenic tumours, Int. J. Bifurcation Chaos 26 (2016) 1650039.
[17] X. Meng, L. Chen, B. Wu, A delay SIR epidemic model with pulse vaccination and incubation times, Nonlinear Anal. RWA 11 (2010) 88–98.
[18] Z. Qiu, Z. Feng, Transmission dynamics of an influenza model with vaccination and antiviral treatment, Bull. Math. Biol. 72 (2010) 1–33.
[19] C. Sun, W. Yang, Global results for an SIRS model with vaccination and isolation, Nonlinear Anal. RWA 11 (2010) 4223–4237.
[20] J.M. Tchuenche, C.T. Bauch, Dynamics of an infectious disease where media coverage influences transmission, ISRN Biomath. 2012 (2012).
[21] R. Liu, J. Wu, H. Zhu, Media/psychological impact on multiple outbreaks of emerging infectious diseases, Comput. Math. Methods Med. 8 (2007) 153–164.
[22] J. Cui, X. Tao, H. Zhu, An SIS infection model incorporating media coverage, Rocky Mountain J. Math. 38 (2008) 1323–1334.
[23] J. Cui, Y. Sun, H. Zhu, The impact of media on the spreading and control of infectious disease, J. Dynam. Differential Equations 20 (2008) 31–53.
[24] J.M. Tchuenche, N. Dube, C.P. Bhunu, R.J. Smith, C.T. Bauch, The impact of media coverage on the transmission dynamics of human influenza, BMC Public Health 11 (2011) S5.
[25] A. Wang, Y. Xiao, A Filippov system describing media effects on the spread of infectious diseases, Nonlinear Anal. Hybrid Syst. 11 (2014) 84–97.
[26] V.I. Utkin, Sliding Modes in Control and Optimization, Springer Science & Business Media, 1992, p. 2013.
[27] C. Chen, C. Li, Y. Kang, Modelling the effects of cutting off infected branches and replanting on fire-blight transmission using Filippov systems, J. Theoret. Biol. 439 (2018) 127–140.
[28] V.I. Utkin, Sliding Mode Control in Electro-Mechanical Systems, second ed., CRC press, 2009.
[29] N.H. Hyng, V.A. Utkin, Control of DC electric motor, Autom. Remote Control 67 (2006) 767–782.
[30] R. Wagnerov, Using Sliding modes in control theory, Acta Montan. Slovaca 13 (2008) 170–173.
[31] Y. Xiao, S. Tang, J. Wu, Media impact switching surface during an infectious disease outbreak, Sci. Rep. 5 (2015) 7838.
[32] V.V. Krylov, Optimal Foraging and Predator-Prey dynamics, Theor. Popul. Biol. 49 (1996) 265–290.
[33] T. Zhao, Y. Xiao, R.J. Smith, Non-smooth plant disease models with economic thresholds, Math. Biosci. 241 (1) (2013) 34–48.
[34] Y. Xiao, X. Xu, S. Tang, Sliding mode control of outbreaks of emerging infectious diseases, Bull. Math. Biol. 74 (2012) 2403–2422.
[35] A. Wang, Y. Xiao, Sliding bifurcation and global dynamics of a Filippov epidemic model with vaccination, Int. J. Bifurcation Chaos 23 (2013) 1350144.
[36] W. Zhou, Y. Xiao, R.A. Cheke, A threshold policy to interrupt transmission of West Nile Virus to birds, Appl. Math. Model. 40 (2016) 8794–8809.
[37] C. Chen, N.S. Chong, R. Smith, A filippov model describing the effects of media coverage and quarantine on the spread of human influenza, Math. Biosci. 296 (2018) 98–112.
[38] Y. Xiao, T. Zhao, S. Tang, Dynamics of an infectious diseases with media/psychology induced non-smooth incidence, Math. Biosci. Eng. 10 (2013) 445–461.
[39] A. Wang, Y. Xiao, H. Zhu, Dynamics of a Filippov epidemic model with limited Hospital Beds, Math. Biosci. Eng. 15 (2018) 739–764.
[40] A. Wang, Y. Xiao, R.A. Cheke, Global dynamics of a piece-wise epidemic model with switching vaccination strategy, Discrete Contin. Dyn. Syst. Ser. B 19 (2014) 2915–2940.
[41] Y. Xiao, T. Zhao, Multiscale system for environmentally-driven infectious disease with threshold control strategy, Int. J. Bifurcation Chaos 28 (05) (2018) 1850064.
[42] F. Brauer, Backward bifurcations in simple vaccination models, J. Math. Anal. Appl. 298 (2004) 418–431.
[43] P. Van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci. 180 (2002) 29–48.
[44] Y.A. Kuznetsov, S. Rinaldi, A. Gragnani, One-parameter Bifurcations in Planar Filippov Systems, Int. J. Bifurcation Chaos 13 (2003) 2157–2188.