Simulating at Realistic Quark Masses: Light quark masses

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We present new results for light quark masses. The calculations are performed using two flavours of $O(a)$ improved Wilson fermions. We have reached lattice spacings as small as $a \sim 0.07\text{fm}$ and pion masses down to $m_\pi \sim 340\text{MeV}$ in our simulations. This gives us significantly better control on the chiral and continuum extrapolations.

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1. Introduction

The ‘running’ of the renormalised quark mass as the scale $M$ is changed is controlled by the $\beta$ and $\gamma$ functions in the renormalisation group equation, defined by

$$\beta^S(g^S(M)) \equiv \frac{\partial g^S(M)}{\partial \log M} \bigg|_{\text{bare}},$$

$$\gamma^S_m(g^S(M)) \equiv \frac{\partial \log Z^S_m(M)}{\partial \log M} \bigg|_{\text{bare}},$$

where the bare parameters are held constant. These functions are given perturbatively as power series expansions in the coupling constant,

$$\beta^S(g) = -b_0 g^3 - b_1 g^5 - b_2 g^7 - b_3 g^9 - \ldots,$$

$$\gamma^S_m(g) = d_{m0} g^2 + d_{m1} g^4 + d_{m2} g^6 + d_{m3} g^8 + \ldots.$$  

The first two coefficients of the $\beta$-function and first coefficient of the $\gamma$ function are scheme independent,

$$b_0 = \frac{1}{(4\pi)^2} \left( 11 - \frac{2}{3} n_f \right), \quad b_1 = \frac{1}{(4\pi)^4} \left( 102 - \frac{38}{3} n_f \right).$$

and

$$d_{m0} = -\frac{8}{(4\pi)^2},$$

while all others depend on the scheme chosen.

We may immediately integrate eq. (1.1) to obtain

$$\frac{M}{\Lambda^S} = b_0 g^S(M)^2 \frac{\exp \left[ \frac{1}{2b_0 g^S(M)^2} \right]}{\exp \left\{ \int_0^{g^S(M)} d\xi \left[ \frac{1}{\beta^S(\xi)} + \frac{1}{b_0 \xi} - \frac{b_1}{b_0 \xi} \right] \right\}}.$$

The renormalisation group invariant (RGI) quark mass\(^1\) is defined from the renormalised quark mass as

$$m^\text{RGI}_q = \Delta Z^S_m(M) m^S(M) = \Delta Z^S_m(M) Z^S_m(M) m^\text{bare}_q = Z^S_m m^\text{bare}_q,$$

where

$$[\Delta Z^S_m(M)]^{-1} = 2b_0 g^S(M)^2 \frac{d_{m0}}{b_0^2} \exp \left\{ \int_0^{g^S(M)} d\xi \left[ \frac{\gamma^S_m(\xi)}{\beta^S(\xi)} + \frac{d_{m0}}{b_0 \xi} \right] \right\},$$

and so the integration constant upon integrating eq. (1.1) is given by $\Lambda^S$, and similarly from eq. (1.2) the integration constant is $m^\text{RGI}_q$. $\Lambda^S$ and $m^\text{RGI}_q$ are thus independent of the scale. (Note that although the functional form of $\Delta Z^S_m(M)$ is fixed, the absolute value is not; conventions vary for its definition.) Also for a scheme change $S \rightarrow S'$ (it is now sufficient to take them at the same scale) given by

$$g^S = G(g^S) = g^S (1 + \frac{1}{2} t_1 (g^S)^2 + \ldots),$$

\(^1\)Analogous definitions hold for other quantities which depend on the scheme and scale chosen.
$m_q^{\text{RI}}$ remains invariant, while $\Lambda$ changes as $\Lambda^S = \Lambda^S \exp(t_1/(2b_0))$. Note also that analytic expressions for the integrals in eq. (1.7) or eq. (1.8) can be found for low orders, for example to two loops we have

$$\Delta Z_m^S(M) = \left[ 2b_0^2 (g^S(M))^2 \right]^{\frac{a_0}{2b_0}} \left[ 1 + \frac{b_1}{b_0} (g^S(M))^2 \right]^{\frac{a_1}{2b_0}}. \tag{1.10}$$

Thus we have a convenient splitting of the problem into two parts: a number, $m_q^{\text{RI}}$, which involves a non-perturbative computation, and is the goal of this paper and, if desired, an evaluation of $\Delta Z_m^S$ which allows the running quark mass to be given in a renormalisation scheme $S$.

### 2. Simulation

We have estimated the light quark masses in the $\overline{\text{MS}}$ scheme at 2GeV by first using the axial Ward identity (AWI) to determine the lattice quark mass. This is renormalised using the RI’−MOM scheme [1] (for our variation on the method see [3]), converted to a RGI form as described in section 1 and after the continuum limit has been taken rewritten in the $\overline{\text{MS}}$ scheme. Further details and results are given in [3]. We perform our simulations with two flavours of non-perturbatively clover-improved dynamical Wilson fermions and Wilson glue. Using these actions, the QCDSF and UKQCD collaborations have generated gauge field configurations with the parameters given in Table 1. We also use configurations generated by the DIK collaboration which have been made available through the ILDG. This large set of lattices enables us to extrapolate to the chiral and the continuum limit.

### 3. Quark masses

As a first check we perform the chiral extrapolation for pseudoscalar mass. In Fig. 1 we plot $(am_{ps})^2$ against $am_q^{\text{AWI}}$ together with the fit result. Our data shows that $(am_{ps})^2$ goes to 0 at $am_q^{\text{AWI}}=0$. This means that even at our lightest quark mass the data is not seriously affected by either an Aoki phase or weak 1st order phase transition.

![Figure 1: $(am_{ps})^2$ versus $am_q^{\text{AWI}}$ for $\beta=5.29$.](image-url)
We use the next to leading order (NLO) chiral perturbation theory ($\chi$PT) to estimate the quark masses,

\[
\begin{align*}
    r_0 m^{RQ}_s &= c^{RQ}_a \left[ (r_0 m_{K^+})^2 + (r_0 m_{K^0})^2 - (r_0 m_{\pi^+})^2 \right] \\
        &\quad + (c^{RQ}_b - c^{RQ}_d) \left[ (r_0 m_{K^+})^2 + (r_0 m_{K^0})^2 \right] (r_0 m_{\pi^+})^2 \\
        &\quad + \frac{1}{2} (c^{RQ}_c + c^{RQ}_d) \left[ (r_0 m_{K^+})^2 + (r_0 m_{K^0})^2 \right]^2 \\
        &\quad - (c^{RQ}_b + c^{RQ}_c) (r_0 m_{\pi^+})^4 \\
        &\quad - c^{RQ}_d \left[ (r_0 m_{K^+})^2 + (r_0 m_{K^0})^2 \right] \left[ (r_0 m_{K^+})^2 + (r_0 m_{K^0})^2 - (r_0 m_{\pi^+})^2 \right] \\
        &\quad \times \ln \left( (r_0 m_{K^+})^2 + (r_0 m_{K^0})^2 - (r_0 m_{\pi^+})^2 \right) \\
        &\quad + c^{RQ}_d (r_0 m_{\pi^+})^4 \ln (r_0 m_{\pi^+})^2.
\end{align*}
\]

The fit function to determine $c^{RQ}_a$ and $c^{RQ}_i$, $i = b, c, d$ is

\[
\begin{align*}
    \frac{r_0 m^{RQ}_q}{(r_0 m_{ps})^2} &= c^{RQ}_a + c^{RQ}_b (r_0 m_{ps})^2 + c^{RQ}_c (r_0 m_{ps})^2 + c^{RQ}_d \left( (r_0 m_{ps}^S)^2 - 2(r_0 m_{ps})^2 \right) \ln (r_0 m_{ps})^2,
\end{align*}
\]

where $m_{ps}, m_{ps}^S$ are the valence and sea pseudoscalar masses respectively (both using mass degenerate quarks, since we found the relevant quantities $am_{ps}$ and $am_q$ to differ by $\lesssim 1\%$ between the degenerate quarks case and the non-degenerate quarks case). The first term is the leading order,
LO, result in $\chi^*$PT while the remaining terms come from the next non-leading order, NLO, in $\chi^*$PT. We note that to NLO, we can determine $c_{a}^{R \xi}$ and $c_{i}^{R \xi}$, $i = b, c, d$ using mass degenerate quarks and then simply substitute them in eqs. (3.1, 3.2).

To reduce the total error on the result, it proved advantageous to use eq. (3.1) to eliminate $c_{a}^{R \xi}$ from eq. (3.3) in terms of

$$c_{a}^{R \xi} = \frac{r_{0}m_{K}^{R \xi}}{(r_{0}m_{K}^{+})^{2} + (r_{0}m_{K}^{-})^{2} - (r_{0}m_{\pi}^{+})^{2}}.$$ (3.4)

This results in a modified fit function of the form

$$r_{0}m_{ps}^{R \xi} = c_{a}^{R \xi} + c_{b}^{R \xi} [(r_{0}m_{ps}^{S})^{2} - d_{b}] + c_{c}^{R \xi} [(r_{0}m_{ps}^{S})^{2} - d_{c}]$$

$$+ c_{d}^{R \xi} \left[ ((r_{0}m_{ps}^{S})^{2} - 2(r_{0}m_{ps})^{2}) \ln(r_{0}m_{ps})^{2} - d_{d} \right],$$ (3.5)

where $d_{i}$ ($i = b, c, d$) can be read-off from eq. (3.1) and have the effect of shifting the various terms in the fit function by a constant.

![Figure 2](image-url)

Figure 2: $c_{a}^{R \xi}$, $c_{i}^{R \xi}$ ($i = b, c, d$) and $c_{a}^{R \xi}$ versus $(a/r_{0})^{2}$. The open symbols represent the values of $c_{a}^{R \xi}$, $c_{i}^{R \xi}$ ($i = b, c, d$) and $c_{a}^{R \xi}$ in the continuum limit.

In Fig. 2 we plot $c_{a}^{R \xi}$, $c_{i}^{R \xi}$ ($i = b, c, d$) and $c_{a}^{R \xi}$ against $(a/r_{0})^{2}$. The coefficients of NLO are small compared to the LO coefficient. Finally, we find

$$m_{ud}^{\overrightarrow{E}}(2 \text{GeV}) = \begin{cases} 121(2)(3)(6) \text{MeV} & \text{for } r_{0} = 0.5 \text{fm} \\ 115(2)(3)(6) \text{MeV} & \text{for } r_{0} = 0.467 \text{fm} \end{cases}$$ (3.6)

where the first error is statistical and the second is systematic $\approx 3$ MeV. We have determined it from the effect on $c_{i}^{R \xi}$ by changing the fit interval $(r_{0}m_{ps})^{2} < 5$ to $(r_{0}m_{ps})^{2} < 4$ or 6 or $\infty$, i.e. include all the data. Furthermore the additional third (systematic) error is due to the uncertainty with which value to identify $r_{0}$.

For the light quark mass, we find that corrections from LO to NLO $\chi^*$PT are negligibly small. We shall just quote the LO result of

$$m_{ud}^{\overrightarrow{E}}(2 \text{GeV}) = \begin{cases} 4.57(05)(07)(23) \text{MeV} & \text{for } r_{0} = 0.5 \text{fm} \\ 4.34(05)(07)(23) \text{MeV} & \text{for } r_{0} = 0.467 \text{fm} \end{cases}$$ (3.7)
Table 2: Comparison of the updated and previous values of $m_s^{\text{MS}}(2\text{GeV})$, $m_{ud}^{\text{MS}}(2\text{GeV})$ and $m_s^{\text{MS}}(2\text{GeV})/m_{ud}^{\text{MS}}(2\text{GeV})$.

| previous          | new                        |
|-------------------|-----------------------------|
| $m_s^{\text{MS}}(2\text{GeV})$ | 111(6)(4)(6)MeV | 115(2)(3)(6)MeV |
| $m_{ud}^{\text{MS}}(2\text{GeV})$ | 4.08(23)(19)(23)MeV | 4.34(05)(07)(23)MeV |
| $m_s^{\text{MS}}(2\text{GeV})/m_{ud}^{\text{MS}}(2\text{GeV})$ | 27.2(3.2) | 26.6(1.8) |

where again the second and third errors are systematic. Finally, we see that the ratio

$$\frac{m_s^{\text{MS}}(2\text{GeV})}{m_{ud}^{\text{MS}}(2\text{GeV})} = 26.6(1.8). \quad (3.8)$$

4. Conclusion

We have updated our estimate for the light quark masses using results at smaller lattice spacing and smaller quark masses data. In Table 2, we compare the updated and previously published values [3]. Our results are in rough agreement with other group’s results. In order to improve the precision and accuracy of analysis, simulations of smaller quark masses and lattice spacing and 2+1 flavours are needed.

Acknowledgments

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