On the Current Carried by ‘Neutral’ Quasiparticles

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The current should be proportional to the momentum in a Galilean-invariant system of particles of fixed charge-to-mass ratio, such as an electron liquid in jellium. However, strongly-interacting electron systems can have phases characterized by broken symmetry or fractionalization. Such phases can have neutral excitations which can presumably carry momentum but not current. In this paper, we show that there is no contradiction: ‘neutral’ excitations do carry current in a Galilean-invariant system of particles of fixed charge-to-mass ratio. This is explicitly demonstrated in the context of spin waves, the Bogoliubov-de Gennes quasiparticles of a superconductor, the one-dimensional electron gas, and spin-charge separated systems in $2+1$ dimensions. We discuss the implications for more realistic systems, which are not Galilean-invariant.

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\section{I. INTRODUCTION.}

Conventional wisdom holds that, in a Galilean-invariant system of particles of fixed charge-to-mass ratio $e/m$, the local current density is proportional to the local momentum density, $\mathbf{J}(\mathbf{x}) = \frac{e}{m} \mathbf{P}(\mathbf{x})$. The conservation of total momentum then implies conservation of the total current, $\frac{\partial}{\partial t} \mathbf{J} = 0$. This is a stronger condition than charge conservation, $\frac{\partial}{\partial t} \mathbf{P} + \nabla \cdot \mathbf{J} = 0$, since it implies that the real part of the conductivity is given by $\sigma(\omega) = \frac{e^2}{m^2} \delta(\omega)$, where $n$ is the particle density. One might imagine that this hypothetical situation has some applicability to extremely clean real systems in which the effects of the lattice are unimportant because the Fermi surface is far from any nesting vector and the electron-phonon coupling is very weak. In such a case, one would be tempted to forget about impurities and the lattice of ions altogether and focus on the electrons, which have a fixed charge-to-mass ratio $e/m$.

On the other hand, we have become accustomed to quantum number fractionalization, particularly in the context of quasi-one-dimensional materials, \textsuperscript{2} the fractional quantum Hall effect,\textsuperscript{3} and theories of high-temperature superconductivity.\textsuperscript{4} Spin-charge separation is one possible pattern of quantum number fractionalization. It leads to charged, spinless quasiparticles – often called ‘holons’ – and neutral, spin-1/2 quasiparticles – often called ‘spinons’. Conventional wisdom would lead us to expect that the latter, being neutral, would carry no current, even when endowed with non-zero momentum. This is merely the most extreme and exotic case of a general phenomenon: the low-energy quasiparticles of a strongly-interacting system need not evince much resemblance to the underlying electron. This is true a fortiori if the low-temperature phase of the system exhibits fractionalization or broken symmetry. In particular, there is no reason why the quasiparticle charge-to-mass ratio should be $e/m$. A more familiar, but no less dramatic example is given by spin waves in a ferromagnet – neutral spin-1 excitations which carry momentum but, presumably, no current.

Clearly, there is some tension, if not an outright contradiction, between these two articles of conventional wisdom. The resolution, which we describe in this paper, is that ‘neutral’ quasiparticles do carry current according to $\mathbf{J}(\mathbf{x}) = \frac{e}{m} \mathbf{P}(\mathbf{x})$ in a Galilean-invariant system. However, even a small explicit breaking of Galilean invariance can have drastic consequences for this relation. As a result, even a small density of impurities or a weak periodic potential can result in a state in which ‘neutral’ quasiparticles carry momentum but no current and the DC conductivity is zero rather than infinity.

The current carried by neutral quasiparticles can be understood as arising from a Doppler shift interaction between them and the charge carriers. The latter are always gapless in a Galilean-invariant system, and they mediate the coupling between the electromagnetic field and the neutral quasiparticles. We will illustrate our thesis in a number of different contexts: spin waves, the Bogoliubov-de Gennes quasiparticles of a superconductor (which carry momentum but are not charge eigenstates), the one-dimensional electron gas, and spin-charge separated systems in $2+1$ dimensions. Finally, we will comment on our results and their applicability to realistic systems, which do not have Galilean invariance.

\section{II. SPIN WAVES IN AN ELECTRON LIQUID}

As mentioned in the introduction, one might think that the paradox is already manifest in the context of spin waves (or other collective excitations) which can carry momentum but ought not – if we are to think of them as
neutral excitations – carry current. Since a spin wave is composed of an electron and a hole, it is, indeed, neutral. At a formal level, the creation operator for an \( S_z = 1 \) spin wave,

\[
S_+(\mathbf{x}, t) = c_1^\dagger(\mathbf{x}, t) c_1(\mathbf{x}, t)
\]

is invariant under a gauge transformation, \( c_\alpha(\mathbf{x}, t) \rightarrow e^{i\phi(\mathbf{x}, t)} c_\alpha(\mathbf{x}, t) \). Consequently, such an operator does not couple to the electromagnetic field through minimal coupling.

Nevertheless, a spin wave \emph{does} carry current. When the vector potential, \( \mathbf{A} \) vanishes, the current takes the form

\[
\mathbf{J} = \sum_k \frac{e}{m} \mathbf{k} c_\alpha^\dagger(\mathbf{k}) c_\alpha(\mathbf{k})
\]

The current operator has this form irrespective of the electron-electron interaction terms, so long as they are Galilean-invariant – i.e. so long as they are momentum-independent and translationally-invariant.

Consider the operator which creates a spin wave of momentum \( \mathbf{q} \):

\[
S_+(\mathbf{q}) = \sum_k c_1^\dagger(\mathbf{k} + \mathbf{q}) c_1(\mathbf{k})
\]

In so doing, it actually creates current as well, as may be seen by taking its commutator with the current operator

\[
[\mathbf{J}, S_+(\mathbf{q})] = \frac{e}{m} \mathbf{q} S_+(\mathbf{q})
\]

Hence, spin waves carry current. This is a purely kinematic statement which follows from the form of the current operator \( (2) \) which, in turn, follows from Galilean invariance. Our conclusion holds whether or not the electron liquid orders electronically.

However, it may be difficult to see how this electrical current appears in an effective field theory of spin waves in, for instance, the ferromagnetic state. Suppose we take our Galilean-invariant electronic Lagrangian,

\[
\mathcal{L} = c_\alpha^\dagger(i\partial_t - eA_t) c_\alpha + \frac{1}{2m} c_\alpha^\dagger(i\nabla - e\mathbf{A})^2 c_\alpha + \mathcal{L}_{\text{int}}
\]

and decouple \( \mathcal{L}_{\text{int}} \) with a Hubbard-Stratonovich field \( \mathbf{S} \) which couples linearly to \( c_\alpha^\dagger \sigma_{\alpha\beta} c_\beta \). We can integrate out the electrons and expand the resulting action about a ferromagnetic state which is ordered in the \( \hat{z} \) direction. On general grounds, we expect that the resulting effective action will be of the form

\[
\mathcal{L}_{\text{eff}} = S_+ i\partial_t S_- - D \nabla S_+ \cdot \nabla S_- + \ldots
\]

As we noted above, \( S_\pm \) is invariant under a gauge transformation, so it is hard to imagine how it can be coupled to the electromagnetic field, \( \mathbf{A} \). On the other hand, \( \mathbf{J} = \partial\mathcal{L}/\partial\mathbf{A} \), so there will be no current carried by \( S_\pm \) in the absence of such a coupling.

The resolution is that there is a coupling to \( \mathbf{A} \) hidden in the “…” in \( (6) \). If it is difficult to guess the form of this term, it is because we would be wrong in assuming that it is local. Since we have integrated out gapless fermionic degrees of freedom in obtaining \( (6) \) we should actually expect non-local terms. There are no non-local terms in the spin dynamics of \( (6) \) because the up- and down-spin Fermi wavevectors are different as a result of the development of ferromagnetic order; consequently spinful excitations of the Fermi surface have a minimum wavevector. However, the charged excitations extend down to \( \mathbf{q} = 0 \), and the coupling of \( S_\pm \) to \( \mathbf{A} \) is, indeed, non-local. It may be obtained by computing the diagrams of Fig. \ref{fig:1} and takes the form

\[
\mathcal{L}_A = \frac{e}{m} A^T \cdot S_+ i\nabla S_-
\]

In this equation, \( A^T \) denotes the transverse part of \( \mathbf{A} \), which is given in momentum space by:

\[
A^T(\mathbf{q}) = A(\mathbf{q}) - \mathbf{q} \cdot \frac{\mathbf{A}(\mathbf{q})}{q^2}
\]

This is both non-local and gauge-invariant since, a gauge transformation,

\[
\mathbf{A}(\mathbf{q}) \rightarrow \mathbf{A}(\mathbf{q}) + \mathbf{q} \phi(\mathbf{q})
\]

with \( \phi(\mathbf{q}) \) arbitrary, leaves \( A^T(\mathbf{q}) \) unchanged. Since \( S_+ \nabla S_- \) is also invariant under a gauge transformation, the entire term \( (7) \) is gauge-invariant, which is a cause for some relief.

Note that spin waves were empowered with the ability to carry a current by the gapless charge degrees of freedom with which they interact. In an insulating ferromagnet, spin waves will not carry a current proportional to their momentum. Since insulating behavior will only occur when a system is not translationally-invariant, there is no contradiction here.

![FIG. 1. The diagrams which contribute to the coupling between spin waves and the electromagnetic field.](image-url)
III. QUASIPARTICLES IN A SUPERCONDUCTOR

The Bogoliubov-de Gennes quasiparticles of a superconductor are coherent superpositions of electrons and holes. Hence, they do not have a well-defined charge. As the Fermi surface is approached, a Bogoliubov-de Gennes quasiparticle becomes an equal superposition of electron and hole; thus, one might be tempted to assign it zero charge in this limit. This is not an academic question in an unconventional superconductor such as one of $d_{x^2-y^2}$ symmetry – as the high-$T_c$ cuprates are believed to be – since, in the absence of a full gap, quasiparticles will be thermally excited down to zero temperature and their ability to carry current will have an impact on the superfluid density.

For the sake of concreteness, let us consider a two-dimensional $d_{x^2-y^2}$ superconductor and focus on its nodal quasiparticles. We assume that the system is Galilean-invariant so that the order parameter spontaneously breaks rotational symmetry when it chooses nodal directions. The effective action for a superconductor is of the form

$$S = \int \frac{d^2k}{(2\pi)^2} dt \Psi^\dagger(k,t) \left[ (i\partial_t - \tau^z eA_t) - \tau^x (\epsilon(k + \tau^z eA) - \mu) - \tau^x \Delta(k) - \tau^{-\Delta^\dagger}(k) \right] \Psi(k,t) \tag{10}$$

where we have used the Nambu-Gorkov notation:

$$\Psi_{aq}(\vec{k}) = \begin{bmatrix} c^+_{aq} \\ c^-_{aq} \\ c^+_{a\dagger} \\ -c^-_{aq} \end{bmatrix} \tag{11}$$

and the $\tau^i$ are Pauli matrices which act on the particle-hole index $a$. If we consider the four-component object as composed of two two-component blocks, the upper and lower blocks, then the $\tau^i$ mix the components within a block. There are also Pauli matrices $\sigma^i$ which act on the spin indices, $\alpha$, and mix the upper block with the lower block.

We will linearize this action about the nodes of $\Delta(k) = \Delta_0(\cos k_x a - \cos k_y a)$. We must retain two fermion fields, one for each pair of antipodal nodes, but these pairs of nodes are not coupled to each other in the low-energy limit, so we will often focus on just one. By linearizing about the nodes, we are approximating the momentum of an electron by $k_F$ and discarding the deviation from the Fermi surface. Hence, we will verify that the relation $\mathbf{J} = \pm \mathbf{P}$ is satisfied to this level of approximation, which means $\mathbf{J} = \mp N_{qp} \mathbf{k}_F$, where $N_{qp} = \Psi^\dagger \Psi$ is the difference between the number of electrons at one node and the number at the antipodal node. If we kept the full Galilean-invariant expression $\epsilon(k) = k^2/2m$, then we could verify $\mathbf{J} = \pm m \mathbf{P}$ exactly. We will do this in one-dimension, where it is particularly instructive. For now, we will content ourselves with a crude verification.

We align our coordinate system along the nodal direction and linearize the single-particle dispersion: $\epsilon(k) - \mu \approx k^2/(2m) k_x$, where $k_x$ is the momentum perpendicular to the Fermi surface, measured away from the node. A similar expression holds for the other pair of nodes, with $k_x$ replaced by $k_y$. We also linearize the gap about the nodes:

$$\Delta \tau^+ \approx v_\Delta \tau^+ + \frac{e e\varphi}{2} (-i\partial_y) e^{ie\varphi}/2 \tag{12}$$

where $e e\varphi$ is the phase of the superconducting order parameter. Some care was needed in obtaining the correct ordering of derivatives and $\varphi$s; for details, see Refs. 4, 13. Integrating out the electronic states far from the nodes and the fluctuations of the amplitude of the order parameter, we obtain the action

$$S = \int \Psi^\dagger \left[ (i\partial_t - \tau^z eA_t) + \tau^x \frac{e eA^\dagger}{m} (i\partial_x - \tau^z eA_x) + v_\Delta \tau^+ e^{ie\varphi}/2 i\partial_y e^{ie\varphi}/2 \right] \Psi^\dagger$$

$$+ \frac{1}{2} \rho_s \int \left[ \frac{1}{v_c^2} (\partial_t \varphi + 2A_t)^2 - (\partial_t \varphi + 2A_t) \right] \Psi^\dagger \Psi$$

$$+ \ldots \tag{13}$$

where $s = \pm$, and $\rho_s$ and $v_c$ are the bare superfluid density and velocity. The “...” includes the action for the other pair of nodes and higher-order terms, which we neglect.

Following Ref. 4, we can simplify this action by defining neutral quasiparticles, $\chi$, according to:

$$\chi = \exp(-ie\varphi \tau^z/2) \Psi \tag{14}$$

The action now takes the form:

$$S = \int \chi^\dagger \left[ i\partial_t + \tau^z \frac{e eA^\dagger}{m} i\partial_x + v_\Delta \tau^+ i\partial_y \right] \chi$$

$$- \frac{1}{2} \int \left[ e\chi^\dagger \chi \left( \partial_t \varphi + 2A_t \right) + e\chi^\dagger \chi \frac{e eA^\dagger}{m} \left( \partial_t \varphi + 2A_x \right) \right]$$

$$+ \frac{1}{2} \rho_s \int \left[ \frac{1}{v_c^2} (\partial_t \varphi + 2A_t)^2 - (\partial_t \varphi + 2A_t) \right] \chi^\dagger \chi$$

$$+ \ldots \tag{15}$$

The quasiparticle annihilation operator, $\chi$, is gauge-invariant since it is neutral, but $\varphi$, which is charged, is not. The action (15) is gauge-invariant because $\chi$ is only coupled to gauge-invariant quantities, such as the superfluid density and current, $\partial_t \varphi + 2A_t$.

These neutral excitations nevertheless carry current. By differentiating the Lagrangian of (15) with respect to $A_x$, we find that the current in the $x$-direction is

$$J_x = 2\rho_s (\partial_x \varphi + 2A_x) + \frac{e}{m} k_F \chi^\dagger \chi \tag{16}$$
The first term is the supercurrent; it derives from the final line of (15). The second term comes from the third line of (12), and it states that the quasiparticles carry a current which is $e/m$ times their momentum $k_F$:

$$ J_{x}^{\mu} = \frac{e}{m} k_F \chi^i \chi^j $$

(17)

By differentiating (15) with respect to $A_t$, we find that the corresponding charge density is $\rho = -2(\mu_v/c^2)(\partial_t \varphi + 2A_t) + e\chi^i \tau^i \chi$. The second term is the quasiparticle contribution. Although the quasiparticles are neutral in the sense of being gauge-invariant, they contribute to both the charge and current densities.

Suppose that we integrate out the fluctuations of the phase of the superconducting order parameter. What does the coupling between the quasiparticles and the electromagnetic field look like?

To integrate out $\varphi$, it is convenient to use the dual representation in which $\varphi$ is replaced by a dual gauge field, $a_\mu$. In this dual representation, (15) takes the form (for details, see ref. 4):

$$ S = \int \chi^i \left[ i \partial_t + \frac{k_F}{m} i \partial_\mu + v_\lambda \tau^{\mu} i \partial_\mu \right] \chi$$

$$ - \int \frac{1}{2} \rho_s (\epsilon_{\mu\nu\lambda} \partial_\mu a_\lambda)^2 + \left( 2A_\mu - \frac{1}{2} \rho_s J_{\mu}^{\mu} \right) \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda$$

$$ + \ldots $$

where we have chosen units with $v_c = 1$ to facilitate the use of ‘relativistic’ notation. The dual gauge field $a_\mu$ is related to the total current, $J_\mu$, by

$$ J_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda $$

(19)

It only enters the action in this transverse combination which is automatically conserved. Furthermore, this means that $\epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda$ is only coupled to the transverse parts of $A_\mu$ and $J_{\mu}^{\mu}$. Since it appears quadratically, we can now integrate it out, obtaining:

$$ S = \int \chi^i \left[ i \partial_t + \frac{k_F}{m} i \partial_\mu + v_\lambda \tau^{\mu} i \partial_\mu \right] \chi$$

$$ - \int A_\mu^T J_\mu^T \chi^i \chi + \ldots $$

(20)

The coupling between $A_\mu$ and $J_{\mu}^{\mu}$ is non-local because it only couples their transverse parts, as in (4). Again, since we have integrated out $a_\mu$ which is formally a gapless degree of freedom when $A_\mu$ is held fixed, we should not be surprised by the appearance of a non-local coupling between $A_\mu$ and $J_{\mu}^{\mu}$ through which only their gauge-invariant transverse components are coupled. Since $a_\mu$ does not couple to the longitudinal parts of $A_\mu$ and $J_{\mu}^{\mu}$, it is not possible to generate terms involving them.

Again, the ability of quasiparticles to carry a current depends on their interaction with gapless charged degrees of freedom – in this case, a supercurrent. If the conductivity associated with this supercurrent (i.e. its Drude weight, not its superfluid density, see section V) is reduced, e.g. by the localization of some electrons at impurities, then Bogoliubov-DeGennes quasiparticles will carry a reduced current as well.

IV. SPIN-CHARGE SEPARATED
ONE-DIMENSIONAL ELECTRON GAS

The one-dimensional electron gas can be described completely in terms of its spin- and charge- collective modes. The electron itself is a combination of charge- and spin-carrying solitons – holons and spinons – in these collective modes. Because these collective modes have different velocities, the charge and spin of an electron move apart in time. Both the charge and spin modes can carry momentum, but one might assume that only the charged mode should couple to the electromagnetic field and carry current. Furthermore, the velocities $v_c$ and $v_s$ of these modes depend on the interaction strength; they are, in general, different from $k_F/m$, which might lead one to expect that even the charged mode will carry a current which is not equal to $e/k_F$ times its momentum. However, we have come, by now, to distrust such expectations.

The Hamiltonian density is often written in the bosonized form

$$ \mathcal{H} = \frac{1}{2} v_c \left[ K_c (\partial_x \varphi_c)^2 + \frac{1}{K_c} (\partial_x \theta_c)^2 \right] + \frac{1}{2} v_s \left[ K_s (\partial_x \varphi_s)^2 + \frac{1}{K_s} (\partial_x \theta_s)^2 \right] + v_F k_F \sqrt{\frac{2}{\pi}} \theta_c \theta_c $$

(21)

The final term is the Fermi energy (for a perfectly linear spectrum) multiplied by the electron number. This term is cancelled by the chemical potential, but we have retained it for purposes of comparison with the corresponding expression for the momentum density. If the system respects $SU(2)$ spin-rotational symmetry, then $K_c = 1$. If $K_c = 1$ as well, then the Hamiltonian describes free fermions. As $K_c$ is shifted away from 1 by the interactions, the charge of the fundamental charged soliton is also shifted away from $e$. $\varphi_c$ and $\theta_c$ are dual variables, $v_c \partial_x \varphi_c = K_c \partial_x \varphi_c$, as are $\varphi_s$ and $\theta_s$. They are symmetric and anti-symmetric combinations of left- and right-moving fields, $\theta_c = \phi_R - \phi_L$, $\varphi_c = \phi_R + \phi_L$. The charge and spin modes are symmetric and anti-symmetric combinations of up- and down-spin modes, $\theta_c = (\theta_1 + \theta_1)/\sqrt{2}$, $\varphi_c = (\theta_1 - \theta_1)/\sqrt{2}$, etc.

This Hamiltonian describes the physics of interacting fermions with a spectrum which is linearized about the Fermi surface, $\pm k_F$. The annihilation operator for a right-moving spin-up electron is

$$ \psi_R^+ = \frac{1}{\sqrt{2\pi a}} e^{-i \sqrt{2} (\varphi_c + \theta_c)} e^{-i \sqrt{2} (\varphi_s + \theta_s)} $$

(22)
where $a$ is a short-distance cutoff. Similar relations hold for down-spin, right-moving electrons and left-moving electrons of both spins. The right- and left-moving charge densities are:

$$
\rho_{R,L} = \frac{1}{\sqrt{2\pi}} \partial_x (\theta_r \pm \varphi_r)
$$

(23)

The right- and left-moving $S_z$ densities are given by a similar expression with $\theta_r, \varphi_r$ replaced by $\theta_s, \varphi_s$.

The momentum can be obtained from the energy-momentum tensor, $T_{\mu\nu}$. While the Hamiltonian density is the $t\bar{t}$ component, $H = T_{tt}$, the momentum density is given by $P = T_{tx}$.

$$
P = k_F \sqrt{\frac{2}{\pi}} \partial_x \varphi_c + [(\partial_x \varphi_c) (\partial_x \theta_c) + (\partial_x \varphi_s) (\partial_x \theta_s)]
$$

(24)

Note that this takes a somewhat different form than is usual for relativistic scalar fields since excitations about the ground state are centered at $\pm k_F$; the first term would not be present in an ordinary relativistic system at zero-density, where low-energy excitations are centered about $k = 0$. It is the counterpart to the final term in (21); it assigns momentum $\pm k_F$ to each right- or left-mover. The second term accounts for possible changes in the local value of $k_F$.

In order to determine the current operator, we modify the Hamiltonian via minimal coupling, which replaces $\partial_x \varphi_c$ with $\partial_x \varphi_c - e \sqrt{2/\pi} A_x$. We now differentiate with respect to $A_x$ to obtain $J_x = -\partial H/\partial A_x$. This coupling is dictated by the fact that $\varphi_c \rightarrow \varphi_c - e \sqrt{2/\pi} \chi$ when $\psi_{R,L} \rightarrow e^{i\varphi_c} \psi_{R,L}$, $A_x \rightarrow A_x - \partial_x \chi$. Since $\psi_c \partial_x \theta_c = K_c \partial \varphi_c$, it does not couple to $A_x$.

However, before we do this, we need to exercise some care with regards to Galilean invariance. We would like to consider only momentum-independent interactions. Hence, the interaction terms cannot have independent coefficients $\lambda_{RR}$ and $\lambda_{RL}$ for the $\rho_R \rho_R + \rho_L \rho_L$ interaction and the $\rho_R \rho_L$ interaction. The only allowed local interaction between charge densities is a simple density-density interaction of the form

$$
\lambda \rho \rho = \lambda (\rho_R \rho_R + \rho_L \rho_L) + 2 \lambda \rho_R \rho_L = 2 \frac{\lambda}{\pi} (\partial_x \theta_c)^2
$$

(25)

i.e. $\lambda_{RL} = 2 \lambda_{RR}$. If $\lambda_{RL} \neq 2 \lambda_{RR}$, the Hamiltonian will contain a term of the form $(\rho_R - \rho_L)^2$, which is proportional to the total momentum squared, in which case the Hamiltonian is not Galilean-invariant. This is the case for the edge states of a quantum Hall bar or quantum Hall line junction. Hence, when we look at the charged sector of the Hamiltonian (21), which arises by combining the free and interaction terms,

$$
H_{\text{charge}} = \frac{1}{2} \frac{k_F}{m} \left[ (\partial_x \varphi_c)^2 + (\partial_x \theta_c)^2 \right] + \frac{2 \lambda}{\pi} (\partial_x \theta_c)^2
$$

$$
+ \frac{1}{2} v_c \left[ K_c (\partial_x \varphi_c)^2 + \frac{1}{K_c} (\partial_x \theta_c)^2 \right]
$$

(26)

we see that $v_c K_c = k_F/m$. In other words, in a Galilean-invariant system, the change in the charge velocity is precisely compensated by the change in the soliton charge so that their product, which will determine the current, is the same as the free fermion value, $k_F/m$.

The second point which requires some care is the linearization of the Hamiltonian. By linearizing our Hamiltonian about the Fermi surface, we are approximating our system by a ‘relativistic’ one. In a relativistic system, the current density and momentum density cannot be proportional to each other since the former is the spatial component of a vector, $J_\mu$, and the other is a component of a tensor $T_{\mu\nu}$ (the total momentum is the spatial component of vector, but this is obtained by integrating the momentum density over the entire system); a relation of the form $J_x = \frac{1}{2} T_{tx}$ would break ‘relativistic’ invariance. Hence, we need to retain the terms which break ‘relativistic’ invariance and contain the information about Galilean invariance. While the linearized terms in the Hamiltonian are of the form $v_F (k - k_F)$, the terms which ‘know’ about Galilean invariance are of the form $(k - k_F)^2/2m$. These terms actually couple the spin and charge modes, thereby resulting in an electrical current carried by spinons.

To see this, consider a term in the Hamiltonian which gives a quadratic spectrum, $(k - k_F)^2/2m$, and its bosonized form:

$$
\psi_{R,L} \frac{1}{2m} (i \partial_\chi)^2 \psi_{R,L}^\dagger = \frac{1}{3} \frac{1}{2\pi} \frac{1}{2m} \left[ \partial_x \sqrt{\pi} \left( \varphi_c + \theta_c + \varphi_s + \theta_s \right) \right]^3 + \text{total derivative terms}
$$

(27)

Hence, summing over both spins and over right- and left-movers, we have

$$
\psi_{R,L} \frac{1}{2m} (i \partial_\chi)^2 \psi_{R,L} = \psi_{R,L} \frac{1}{2m} \partial_\chi^2 \psi_{R,L}^\dagger + \psi_{R,L} \frac{1}{2m} \partial_\chi^2 \psi_{R,L}^\dagger = \frac{1}{2m} \left[ \left( \partial_x \varphi_c \right)^2 (\partial_x \theta_c) + 2 (\partial_x \varphi_c) (\partial_x \varphi_s) (\partial_x \theta_s) \right] + \text{terms which do not contain } \varphi_c
$$

(28)

In a Galilean-invariant system, with single-particle kinetic energy $k^2/2m$, these are the only other terms which we must add.

Hence, going beyond linearization about the Fermi points and retaining the quadratic single-particle spectrum of a Galilean-invariant system, we have the following Hamiltonian:

$$
H = \frac{1}{2} \frac{k_F}{m} (\partial_x \varphi_c)^2 + \frac{v_c}{K_c} (\partial_x \theta_c)^2
$$

$$
+ \frac{1}{2} v_s \frac{K_s}{\pi} (\partial_x \varphi_s)^2 + \frac{1}{K_s} (\partial_x \theta_s)^2
$$

(26)
spin and charge are confined to obtain \( \Phi \), by the Meissner or diamagnetic response, which vanishes. Hence, upon integrating out \( a_\mu \) and \( \Phi_{hc/e} \), we obtain the same induced coupling between spinons and the electromagnetic field, \( A^{T}_{\mu} j^{\text{spinon}}_{\mu} \), that we obtained for Bogoliubov-de Gennes quasiparticles.

However, even infinitesimal translational-symmetry breaking, such as that caused by a small density of impurities, will pin the holon Wigner crystal. Consequently, the system will be an insulator and \( a_\mu \) will be gapped. The coupling between spinons and the electromagnetic field will now be of the form

\[
S_{\text{coupling}} = \int A_{\mu} (\partial^{2} j_{\mu}^{\text{spinon}} - \partial_{\nu} j_{\nu}^{\text{spinon}})
\]

In other words, spinons will be truly neutral since they do not carry a current proportional to their momentum density, in contrast to the merely ‘neutral’ spinons that do.

Note that holons are not necessarily bosonic. A bosonic holon can form a bound state with an uncondensed \( hc/2e \) vortex, or ‘vison’, thereby becoming fermionic. In this case, the holon Wigner crystal is not the only possible non-superconducting ground state because the holons could form a perfectly conducting Fermi liquid. If the spinons pair and form a spin gap, then a spin-gapped metallic state can result, in which spinons carry a current proportional to their momentum density.

**VI. DISCUSSION**

The basic form of the interaction between ‘neutral’ and ‘charged’ quasiparticles is \( j^{\text{neutral}}_{\mu} j^{\text{charged}}_{\mu} \). It can be interpreted as a ‘Doppler shift’ by which the motion of the neutral quasiparticles brings the charged ones along for the ride. As a result, the relation \( \mathbf{J}(\mathbf{x}) = \frac{e}{m} \mathbf{P}(\mathbf{x}) \)
satisfied even in a system with formally ‘neutral’ quasiparticles. The facility with which the charge carriers can move along with the neutral quasiparticles is, of course, a consequence of Galilean invariance.

Even a mild violation of Galilean invariance can have dramatic consequences for the relationship between current and momentum and, hence, for the conductivity. In the case of spin-charge separation in \(2 + 1\) dimensions, we saw that infinitesimal translation symmetry breaking can make a perfect conductor into an insulator; as a consequence, ‘neutral’ quasiparticles which carry a current proportional to their momentum become truly neutral quasiparticles carrying no current. Similarly, spin waves in a Galilean-invariant electron system carry current, but spin waves in an insulating ferromagnet on a lattice do not carry current. Thus the lattice has a large effect on the electrical properties of spin waves, even though it does not seem to be particularly important for the magnetic properties of the ferromagnet phase. Even in \(2 + 1\) dimensions, in those situations in which the effects of the ionic lattice are otherwise mild because the Fermi surface is far from nested, the relation between current and momentum can be strongly violated as a result of the effect of the lattice on interaction parameters and ‘small’ corrections to the band dispersion.

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1 For reviews, see, e.g., V. Emery, in Highly- Conducting One-Dimensional Solids (Plenum, New York, 1979), eds. J.T. Devreese, R.P. Evard, and V.E. van Doren; A.O. Gogolin, A.A. Nersesyan, and A.M. Tsvelik, Bosonization and Strongly Correlated Systems (Cambridge 1998); E. Fradkin, Field Theories of Condensed Matter Systems (Addison-Wesley, 1991).

2 See, e.g., The Quantum Hall effect, edited by R. Prange and S. M. Girvin (Springer-Verlag, New York, 1987); Perspectives in quantum Hall effects : novel quantum liquids in low-dimensional semiconductor structures, edited by S. Das Sarma and A. Pinczuk, (Wiley, New York, 1997); and references therein.

3 P.W. Anderson, The Theory of Superconductivity in the High-Tc Cuprates (Princeton University Press, Princeton, 1997) and references therein; T. Senthil and M.P.A. Fisher, Phys. Rev. B 62, 7850 (2000); V.J. Emery, S.A. Kivelson, and O. Zachar, Phys. Rev. B 56, 6120 (1997).

4 L. Balents, M. P. A. Fisher, and C. Nayak, Int. J. Mod. Phys. B 12, 1033 (1998).

5 L. Balents, M. P. A. Fisher, and C. Nayak, Phys. Rev. B 60, 1654 (1999); L. Balents, M. P. A. Fisher, and C. Nayak, Phys. Rev. B 61, 6307 (2000).

6 For other perspectives on fractionalization, see Ref. 12–22.

7 X. Yang and C. Nayak, in preparation.

8 The field \(\chi\) is multi-valued in the presence of a flux \(hc/e\) vortex, about which \(\chi\) winds by \(2\pi\). This can be incorporated by introducing Chern-Simons gauge fields which mediate the interaction between vortices and quasiparticles.

9 A. Mitra and S. M. Girvin, Phys. Rev. B (in press for May 15, 2001), cond-mat/0101214.

10 D. Scalapino, S. White, and S. Zhang, Phys. Rev. Lett. 68, 2830 (1992).

11 N. Read and B. Chakraborty, Phys. Rev. B 40, 7133 (1989); S. A. Kivelson, Phys. Rev. B 39, 259 (1989); A. For a more recent discussion, see. E. Demler, C. Nayak, H.-Y. Kee, Y.-B. Kim, and T. Senthil, in preparation, and references therein.

12 S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Phys. Rev. B 35, 8865 (1987).

13 D.S. Rokhsar and S. Kivelson Phys. Rev. Lett. 61, 2376 (1988);

14 N. Read and S. Sachdev, Phys. Rev. Lett. 66, 1773 (1991); S. Sachdev and N. Read, Int. J. Mod. Phys. B5, 219 (1991).

15 R. Jalabert and S. Sachdev, Phys. Rev. B 44, 686 (1991).

16 X.G. Wen and Q. Niu, Phys. Rev. B41, 9377 (1990); X.G. Wen, Phys. Rev. B44, 2664 (1991).

17 T. Senthil and Matthew P.A. Fisher, cond-mat/0008084; cond-mat/0006431.

18 T. Senthil and Matthew P.A. Fisher, cond-mat/0008084.

19 X.G. Wen, Phys. Rev. B44, 2664 (1991).

20 G. Baskaran, Z. Zou, and P. W. Anderson, Solid State Commun. 63, 973 (1987); J. B. Marston and I. Affleck, Phys. Rev. B 39, 11538 (1989); G. Baskaran and P. W. Anderson, Phys. Rev. B 37, 580 (1988); G. Kotliar and Liu Jialin, Phys. Rev. B 38, 5142 (1988); N. Read and B. Chakraborty, Phys. Rev. B 40, 7133 (1989); L. B. Ioffe and A. I. Larkin, Phys. Rev. B 39, 8988 (1989); P. A. Lee and N. Nagaosa, Phys. Rev. B 46, 5621 (1992); A. M. Tinkham and R. B. Laughlin, Phys. Rev. B 50, 10165 (1994); B. L. Altshuler, L. B. Ioffe, and A. J. Millis, Phys. Rev. B 53, 415 (1996); P.A. Lee and X.G. Wen, J. Phys. Chem. Solids 59, 1723 (1998).

21 E. Fradkin and S. A. Kivelson, Mod. Phys. Lett. B4, 225 (1990).

22 R. Moessner, S. L. Sondhi, and E. Fradkin, cond-mat/0103306.